Green-Schwarz, Nambu-Goto Actions, 
and Cayley’s Hyperdeterminant

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Abstract

It has been recently shown that Nambu-Goto action can be re-expressed in terms of Cayley’s hyperdeterminant with the manifest \( SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \) symmetry. In the present paper, we show that the same feature is shared by Green-Schwarz \( \sigma \)-model for \( N = 2 \) superstring whose target space-time is \( D = 2+2 \). When its zweibein field is eliminated from the action, it contains the Nambu-Goto action which is nothing but the square root of Cayley’s hyperdeterminant of the pull-back in superspace \( \sqrt{\det (\Pi_{\alpha i})} \) manifestly invariant under \( SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \). The target space-time \( D = 2+2 \) can accommodate self-dual supersymmetric Yang-Mills theory. Our action has also fermionic \( \kappa \)-symmetry, satisfying the criterion for its light-cone equivalence to Neveu-Schwarz-Ramond formulation for \( N = 2 \) superstring.

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1. Introduction

Cayley’s hyperdeterminant [1], initially an object of mathematical curiosity, has found its way in many applications to physics [2]. For instance, it has been used in the discussions of quantum information theory [3][4], and the entropy of the STU black hole [5][6] in four-dimensional string theory [7].

More recently, it has been shown [8] that Nambu-Goto (NG) action [9][10] with the \( D = 2 + 2 \) target space-time possesses the manifest global \( SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \equiv [SL(2, \mathbb{R})]^3 \) symmetry. In particular, the square root of the determinant of an inner product of pull-backs can be rewritten exactly as a Cayley’s hyperdeterminant [1] realizing the manifest \( [SL(2, \mathbb{R})]^3 \) symmetry.

It is to be noted that the space-time dimensions \( D = 2 + 2 \) pointed out in [8] are nothing but the consistent target space-time of \( N = 2 \) \(^3\) NSR superstring [16][17][18][19][13][14][15]. However, the NSR formulation [16][17] has a drawback for rewriting it purely in terms of a determinant, due to the presence of fermionic superpartners on the 2D world-sheet. On the other hand, it is well known that a GS formulation [12] without explicit world-sheet supersymmetry is classically equivalent to a NSR formulation [11] on the light-cone, when the former has fermionic \( \kappa \)-symmetry [20][15]. From this viewpoint, a GS \( \sigma \)-model formulation in [14] of \( N = 2 \) superstring [16][17][13] seems more advantageous, despite the temporary sacrifice of world-sheet supersymmetry. However, even the GS formulation [14] itself has an obstruction, because obviously the kinetic term in the GS action is not of the NG-type equivalent to a Cayley’s hyperdeterminant.

In this paper, we overcome this obstruction, by eliminating the zweibein (or 2D metric) \( \text{via} \) its field equation which is \textit{not} algebraic. Despite the \textit{non}-algebraic field equation, such an elimination is possible, just as a NG action [9][10] is obtained from a Polyakov action [21]. Similar formulations are known to be possible for Type I, heterotic, or Type II superstring theories, but here we need to deal with \( N = 2 \) superstring [16] with the target space-time

\(^3\) The \( N = 2 \) here implies the number of world-sheet supersymmetries in the Neveu-Schwarz-Ramond (NSR) formulation [11]. Its corresponding Green-Schwarz (GS) formulation [12][13][14] might be also called ‘\( N = 2 \)’ GS superstring in the present paper. Needless to say, the number of world-sheet supersymmetries should not be confused with that of space-time supersymmetries, such as \( N = 1 \) for Type I superstring, or \( N = 2 \) for Type IIA or IIB superstring [15].
$D = 2 + 2$ instead of 10D. We show that the same global $[SL(2, \mathbb{R})]^3$ symmetry [8] is inherent also in $N = 2$ GS action in [14] with $N = (1, 1)$ supersymmetry in $D = 2 + 2$ as the special case of [13], when the zweibein field is eliminated from the original action, re-expressed in terms of NG-type determinant form.

As is widely recognized, the quantum-level equivalence of NG action [9][10] to Polyakov action [21] has not been well established even nowadays [22]. As such, we do not claim the quantum equivalence of our formulation to the conventional $N = 2$ NSR superstring [16][17] or even to $N = 2$ GS string [13] itself. In this paper, we point out only the existence of fermionic $\kappa$-symmetry and the manifest global $[SL(2, \mathbb{R})]^3$ symmetry with Cayley’s hyperdeterminant as classical-level symmetries, after the elimination of 2D metric from the classical GS action [14] of $N = 2$ superstring [16][17].

As in $N = 2$ NSR superstring [16][17], the target $D = (2, 2; 2, 2)^4$ superspace [19] of $N = 2$ GS superstring [14] can accommodate self-dual supersymmetric Yang-Mills (SDSYM) multiplet [18][19] with $N = (1, 1)$ space-time supersymmetry [13][19][14], which is supersymmetric generalization of purely bosonic YM theory in $D = 2 + 2$ [23]. The importance of the latter is due to the conjecture [24] that all the bosonic integrable or soluble models in dimensions $D \leq 3$ are generated by self-dual Yang-Mills (SDYM) theory [23]. Then it is natural to ‘supersymmetrize’ this conjecture [24], such that all the supersymmetric integrable models in $D \leq 3$ are generated by SDSYM in $D = 2 + 2$ [18][19], and thereby the importance of $N = 2$ GS $\sigma$-model in [14] is also re-emphasized.

In the next two sections, we present our total action of $N = 2$ GS $\sigma$-model [14] whose target superspace is $D = (2, 2; 2, 2)$ [19], and show the existence of fermionic $\kappa$-symmetry [20] as well as $[SL(2, \mathbb{R})]^3$ symmetry, due to the Cayley’s hyperdeterminant for the kinetic terms in the NG form. We next confirm that our action is derivable from the $N = 2$ GS $\sigma$-model [14] which is light-cone equivalent to $N = 2$ NSR superstring [16][17], by elimi-
nating a zweibein or a 2D metric.

2. Total Action with \([SL(2,\mathbb{R})]^3\) Symmetry

We first give our total action with manifest global \([SL(2,\mathbb{R})]^3\) symmetry, then show its fermionic \(\kappa\)-symmetry \([20]\). Our action has classical equivalence to the GS \(\sigma\)-model formulation \([14]\) of \(N = 2\) superstring \([16][17]\) with the right \(D = (2,2;2,2)\) target superspace that accommodates self-dual supersymmetric YM multiplet \([17][19][18][14]\). In this section, we first give our total action of our formulation, leaving its derivation or justifications for later sections.

Our total action \(I \equiv \int d^2\sigma L\) has the fairly simple lagrangian

\[
L = + \sqrt{-\det (\Gamma_{ij})} + \epsilon^{ij} \Pi_i^A \Pi_j^B B_{BA} \quad (2.1a)
\]

\[
= + \sqrt{+\det (\Pi_{i\alpha})} (1 + 2 \Pi_i^A \Pi_j^B B_{BA}) \equiv L_{NG} + L_{WZNW} \quad (2.1b)
\]

where respectively the two terms \(L_{NG}\) and \(L_{WZNW}\) are called ‘NG-term’ and ‘WZNW-term’. The indices \(i, j, \ldots = 0, 1\) are for the curved coordinates on the 2D world-sheet, while \(+, -\) are for the light-cone coordinates for the local Lorentz frames, respectively defined by the projectors

\[
P_{(i)}^{(j)} \equiv \frac{1}{2}(\delta_{(i)}^{(j)} + \epsilon_{(i)}^{(j)}) \quad , \quad Q_{(i)}^{(j)} \equiv \frac{1}{2}(\delta_{(i)}^{(j)} - \epsilon_{(i)}^{(j)}) \quad (2.2)
\]

where \((i), (j), \ldots = (0), (i), \ldots\) are used for local Lorentz coordinates, and \((\eta_{(i)(j)}) = \text{diag.} (+, -)\).

Note that \(\delta_{++}^+ = \delta_{--}^+ = +1, \epsilon_{++}^+ = -\epsilon_{--}^+ = +1, \eta_{++} = \eta_{--} = 0, \eta_{+-} = \eta_{-+} = 1\). Whereas \(\Pi_i^A\) is the superspace pull-back, \(\Gamma_{ij}\) is a product of such pull-backs:

\[
\Pi_i^A \equiv (\partial_i Z^M) E_M^A \quad , \quad (2.3a)
\]

\[
\Gamma_{ij} \equiv \eta_{\underline{a}\underline{b}} \Pi_i^{\underline{a}} \Pi_j^{\underline{b}} = \Pi_i^{m} \Pi_j^{\underline{m}} \quad (2.3b)
\]

for the target superspace coordinates \(Z^M\). The \((\eta_{\underline{a}\underline{b}}) = \text{diag.}(+,-,-,-)\) is the \(D = 2 + 2\) space-time metric. We use the indices \(\underline{a}, \underline{b}, \ldots = 0, 1, 2, 3\) (or \(m, n, \ldots = 0, 1, 2, 3\)) for the bosonic local Lorentz (or curved) coordinates. The \(E_M^A\) is the flat background vielbein \([25]\) for \(D = (2,2;2,2)\) target superspace \([19][14]\). Its explicit form is

\[
(E_M^A) = \left(\begin{array}{cc}
\delta_{m}^{\underline{a}} & 0 \\
-i\frac{1}{2} (\sigma^2 \theta)_{m}^{\underline{a}} & \delta_{\underline{a}}^{\underline{a}}
\end{array}\right) \quad , \quad (E_A^M) = \left(\begin{array}{cc}
\delta_{\underline{a}}^{m} & 0 \\
+i\frac{1}{2} (\sigma^2 \theta)_{\underline{a}}^{m} & \delta_{m}^{m}
\end{array}\right) \quad (2.4)
\]
We use the underlined Greek indices: \( \underline{\alpha} \equiv (\alpha, \dot{\alpha}), \underline{\beta} \equiv (\beta, \dot{\beta}) \), ... for the pair of fermionic indices, where \( \alpha, \beta, ... = 1, 2 \) are for chiral coordinates, and \( \dot{\alpha}, \dot{\beta}, ... = 1, 2 \) are for anti-chiral coordinates [19]. The indices \( \underline{\mu}, \underline{\nu}, ... = 1, 2, 3, 4 \) are for curved fermionic coordinates. Similarly to the superspace for the Minkowski space-time with the signature \((+,-,-,-)\) [25], a bosonic index is equivalent to a pair of fermionic indices, e.g., \( \Pi_{\underline{a}} \equiv \Pi_{i, \alpha} \). In (2.4), we use the expressions like \( (\sigma^a \theta)_{\underline{\alpha}} \equiv - (\sigma^a)_{\alpha \beta} \theta^\beta \) for the \( \sigma \)-matrices in \( D = 2 + 2 \) [26][19]. Relevantly, the only non-vanishing supertorsion components are [19][14]

\[
T_{\underline{\alpha} \underline{\beta} \underline{c}} = i(\sigma^c)_{\underline{\alpha} \underline{\beta}} = \begin{cases} +i(\sigma^c)_{\alpha \beta} & , \\ -i(\sigma^c)_{\dot{\alpha} \dot{\beta}} & , \\ +i(\sigma^c)_{\dot{\alpha} \beta} & = +i(\sigma^c)_{\beta \dot{\alpha}} & . 
\end{cases} (2.5)
\]

The antisymmetric tensor superfield \( B_{AB} \) has the superfield strength

\[
G_{ABC} \equiv \frac{1}{2} \nabla_{[AB} B_{BC]} - \frac{1}{2} T_{[AB]} \bar{D}_{B,C]} . (2.6)
\]

Our anti-symmetrization rule is such as \( M_{[AB]} \equiv M_{AB} - (-1)^{AB} M_{BA} \) without the factor 1/2. The flat-background values of \( G_{ABC} \) is [19][14]

\[
G_{\underline{\alpha} \underline{\beta} \underline{c}} = +i(\sigma^c)_{\underline{\alpha} \underline{\beta}} = \begin{cases} +i(\sigma^c)_{\alpha \beta} & , \\ +i(\sigma^c)_{\dot{\alpha} \dot{\beta}} & = +i(\sigma^c)_{\beta \dot{\alpha}} & . 
\end{cases} (2.7)
\]

In our formulation, the lagrangian (2.1a) needs the ‘square root’ of the matrix \( \Gamma_{ij} \), analogous to the zweibein \( e_{i}^{(j)} \) as the ‘square root’ of the 2D metric \( g_{ij} \), defined by

\[
\gamma_{i}^{(k)} \gamma_{j}^{(k)} = \Gamma_{ij} , \quad \gamma_{(k)}^{i} \gamma_{(k)}^{(j)} = \Gamma^{ij} , \quad (2.8a)
\]

\[
\gamma_{i}^{(k)} \gamma_{(k)}^{j} = \delta_{i}^{j} , \quad \gamma_{(i)}^{k} \gamma_{(j)}^{(k)} = \delta_{(i)}^{(j)} . \quad (2.8b)
\]

Relevantly, we have \( \gamma = \sqrt{-\Gamma} \) for \( \Gamma \equiv \det (\Gamma_{ij}) \) and \( \gamma \equiv \det (\gamma_{i}^{(j)}) \). We define \( \Pi^{A} \equiv \gamma^{1} \Pi_{i}^{A} \) for the \( \pm \) local light-cone coordinates. For our formulation with (2.1), we always use the \( \gamma \)'s to convert the curved indices \( i, j, ... = 0, 1 \) into local Lorentz indices \( (i), (j), \) ... = (0), (1).

From (2.8), it is clear that we can always define the ‘square root’ of \( \Gamma_{ij} \) of (2.3b) just as we can always define the zweibein \( e_{i}^{(j)} \) out of a 2D metric \( g_{ij} \). In fact, (2.8) determines \( \gamma_{i}^{(j)} \) up to 2D local Lorentz transformations \( O(1, 1) \), because (2.8) is covariant under arbitrary \( O(1, 1) \). However, (2.8) has much more significance, because if the curved
indices $i,j$ of $\Gamma_{ij}$ are converted into 'local' ones, then it amounts to

$$\Gamma_{(i)(j)} = \gamma(i)^k\gamma(j)^l \Gamma_{kl} = \gamma(i)^k\gamma(j)^l (\gamma_k^m\gamma_l^m)$$

$$= (\gamma(i)^k\gamma_k^m)(\gamma(j)^l\gamma_l^m) = \delta_{(i)(j)} = \eta_{(i)(j)} \implies \Gamma_{(i)(j)} = \eta_{(i)(j)} \ . \quad (2.9)$$

In terms of light-cone coordinates, this implies formally the Virasoro conditions [27]

$$\Gamma_{++} \equiv \Pi_{++}^2 \Pi_{++} = 0 \ , \quad \Gamma_{--} \equiv \Pi_{--}^2 \Pi_{--} = 0 \ , \quad (2.10)$$

because $\eta_{++} = \eta_{--} = 0$. The only caveat here is that our $\gamma_{(j)}$ is not exactly the zweibein $e_{(j)}$, but it differs only by certain factor, as we will see in (4.6).

The result (2.10) is not against the original results in NG formulation [9][10]. At first glance, since the NG action has no metric, it seems that Virasoro condition [27] will not follow, unless a 2D metric is introduced as in Polyakov formulation [21]. However, it has been explicitly shown that the Virasoro conditions follow as first-order constraints, when canonical quantization is performed [10]. Naturally, this quantum-level result is already reflected at the classical level, i.e., the Virasoro condition (2.10) follows, when the $ij$ indices on $\Gamma_{ij} \equiv \Pi_{i\alpha}^2 \Pi_{j\alpha}$ are converted into 'local Lorentz indices' by using the $\gamma$'s in (2.8).

Most importantly, $\text{Det} (\Pi_{i\alpha\alpha})$ in (2.1b) is a Cayley’s hyperdeterminant [1][8], related to the ordinary determinant in (2.1a) by

$$\text{Det} (\Pi_{i\alpha\alpha}) = -\frac{1}{2} \epsilon_{ij}^k \epsilon^{\alpha\beta} \epsilon_{\gamma\delta}^\cdot \epsilon_{ij}^\cdot \Pi_{i\alpha\alpha} \Pi_{j\beta\beta} \Pi_{k\gamma\gamma} \Pi_{l\delta\delta} = -\text{det} (\Gamma_{ij}) \ , \quad (2.11a)$$

$$\Gamma_{ij} \equiv \Pi_{i\alpha}^2 \Pi_{j\alpha} = \Pi_{i\alpha\alpha}^\cdot \Pi_{j\alpha\alpha} = \epsilon^{\alpha\beta} \epsilon_{ij}^\cdot \Pi_{i\alpha\gamma} \Pi_{j\beta\delta} \ . \quad (2.11b)$$

The global $[SL(2,\mathbb{R})]^3$ symmetry of our action $I$ is more transparent in terms of Cayley’s hyperdeterminant, because of its manifest invariance under $[SL(2,\mathbb{R})]^3$. For other parts of our lagrangian, consider the infinitesimal transformation for the first factor group$^5$ of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ with the infinitesimal real constant traceless 2 by 2 matrix parameter $p$ as

$$\delta_p \Pi_{i}^{A} = p_i^j \pi_j^{A} \ , \quad \delta_p \gamma_{(i)}^j = -p_i^j \gamma_{(i)}^k \quad (p_i^i = 0) \ . \quad (2.12)$$

$^5$ In a sense, this invariance is trivial, because $SL(2,\mathbb{R}) \subset GL(2,\mathbb{R})$, where the latter is the 2D general covariance group.
The latter is implied by the definition of $\Gamma_{ij} \equiv \Pi_{i}^{\alpha} \Pi_{j}^{\dot{\alpha}}$ and $\gamma^{i\dot{j}}$ in (2.8). Eventually, we have $\delta_{\mu} \Pi_{(i)}^{A} = 0$, while $\mathcal{L}_{\text{WZNW}}$ is also invariant, thanks to $\delta_{\mu} \Pi_{(i)}^{A} = 0$. This concludes $\delta_{\mu} \mathcal{L} = 0$.

The second and third factor groups in $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ act on the fermionic coordinates $\alpha$ and $\dot{\alpha}$ in $D = (2, 2; 2, 2)$, which need an additional care. We first need the alternative expression of $\mathcal{L}_{\text{WZNW}}$ by the use of Vainberg construction [28][29]:

$$\mathcal{L} = +\sqrt{-\text{Det} (\Pi_{i\alpha}^{\dagger})} + i \int \bar{\sigma} \epsilon^{i\dot{j}k} \tilde{\Pi}_{i\alpha} \tilde{\Pi}_{j}^{\alpha} \tilde{\Pi}_{k}^{\dot{\alpha}} .$$

We need this alternative expression, because superfield strength $G_{AB}$ is less ambiguous than its potential superfield $B_{AB}$ avoiding the subtlety with the indices $\alpha$ and $\dot{\alpha}$. In the Vainberg construction [28][29], we are considering the extended 3D ‘world-sheet’ with the coordinates $(\tilde{\sigma}^{i}) \equiv (\sigma^{i}, y)$ $(i = 0, 1, 2)$, where $\tilde{\sigma}^{2} \equiv y$ is a new coordinate with the range $0 \leq y \leq 1$. Relevantly, $\tilde{\epsilon}^{i\dot{j}k}$ is totally antisymmetric constant, and $\tilde{\epsilon}^{2i\dot{j}} = \epsilon^{ij}$. All the hatted indices and quantities refer to the new 3D. Any hatted superfield as a function of $\tilde{\sigma}^{i}$ should satisfy the conditions [28], e.g.,

$$\tilde{Z}^{M}(\sigma, y = 1) = Z^{M}(\sigma), \quad \tilde{Z}^{M}(\sigma, y = 0) = 0 .$$

Consider next the isomorphism $\text{SL}(2, \mathbb{R}) \approx \text{Sp}(1)$ [30] for the last two groups in $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \approx \text{SL}(2, \mathbb{R}) \times \text{Sp}(1) \times \text{Sp}(1)$. These two $\text{Sp}(1)$ groups are acting respectively on the spinorial indices $\alpha$ and $\dot{\alpha}$. The contraction matrices $\epsilon_{\alpha\beta}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$ are the metrics of these two $\text{Sp}(1)$ groups, used for raising/lowering these spinorial indices. Now the infinitesimal transformation parameters of $\text{Sp}(1) \times \text{Sp}(1)$ can be 2 by 2 real constant symmetric matrices $q_{\alpha\beta}$ and $r_{\dot{\alpha}\dot{\beta}}$ acting as

$$\delta_{q} \tilde{\Pi}_{i\alpha} = -q^{\alpha\beta} \tilde{\Pi}_{i}^{\beta}, \quad \delta_{q} \tilde{\Pi}_{i\dot{\alpha}} = q_{\dot{\alpha}\gamma} \tilde{\Pi}_{i}^{\gamma} , \quad \delta_{r} \tilde{\Pi}_{i}^{\dot{\alpha}} = -r^{\dot{\alpha}\dot{\beta}} \tilde{\Pi}_{i}^{\beta}, \quad \delta_{r} \tilde{\Pi}_{i\dot{\alpha}} = r_{\dot{\alpha}\gamma} \tilde{\Pi}_{i}^{\gamma} ,$$

where $q^{\alpha\beta} \equiv \epsilon^{\alpha\gamma} q_{\gamma\beta}$, $r^{\dot{\alpha}\dot{\beta}} \equiv \epsilon^{\dot{\alpha}\dot{\gamma}} r_{\dot{\gamma}\dot{\beta}}$, etc. Then it is easy to confirm for $\mathcal{L}_{\text{WZNW}}$ that

$$\delta_{q} \left( \tilde{\Pi}_{i\alpha\dot{\alpha}} \tilde{\Pi}_{j}^{\alpha} \tilde{\Pi}_{k}^{\dot{\alpha}} \right) = 0 , \quad \delta_{r} \left( \tilde{\Pi}_{i\alpha\dot{\alpha}} \tilde{\Pi}_{j}^{\alpha} \tilde{\Pi}_{k}^{\dot{\alpha}} \right) = 0 .$$
because of \( q_\alpha^\gamma = +q^\gamma_\alpha \) and \( r_\alpha^\gamma = +r^\gamma_\alpha \). We thus have the total invariances \( \delta_q \mathcal{L} = 0 \) and \( \delta_r \mathcal{L} = 0 \). Since \( \delta_p \mathcal{L} = 0 \) has been confirmed after (2.12), this concludes the \([SL(2, \mathbb{R})]^3\)-invariance proof of our action (2.1).

It was pointed out in ref. [8] that ‘hidden’ discrete symmetry also exists in NG-action under the interchange of the three indices for \([SL(2, \mathbb{R})]^3\). In our system, however, this hidden triality seems absent. This can be seen in (2.1b), where the Cayley’s hyperdeterminant or \( \mathcal{L}_{\text{NG}} \) indeed possesses the discrete symmetry for the three indices \( i \alpha \hat{\alpha} \), while it is lost in \( \mathcal{L}_{\text{WZNW}} \). This is because the mixture of \( \Pi_{i\alpha} \) and \( \Pi^{i\alpha} \) or \( \Pi^{i\hat{\alpha}} \) via the non-zero components of \( B_{AB} \) breaks the exchange symmetry among \( i \alpha \hat{\alpha} \), unlike Cayley’s hyperdeterminant.

3. Fermionic Invariance of our Action

We now discuss our fermionic \( \kappa \)-invariance. Our action (2.1) is invariant under

\[
(\delta_\kappa Z^M) E_M^{\alpha} = +i(\sigma_\alpha)_{\alpha\beta}^{\beta} \kappa_{-\beta} \Pi_{+ \beta} \equiv +i(\Pi_{+\kappa-})^{\alpha} , \tag{3.1a}
\]

\[
(\delta_\kappa Z^M) E_M^{\hat{\alpha}} = 0 , \tag{3.1b}
\]

\[
\delta_\kappa \Gamma_{ij} = +[\kappa_{-\alpha}(\sigma_\alpha \sigma_\gamma)_{\alpha\beta}^{\beta} \Pi_{+ \beta}] \Pi_{+ \alpha} \equiv + (\Pi_{+ \kappa-} \Pi_{+ \alpha} \Pi_{+ \beta}) . \tag{3.1c}
\]

The \( \kappa_{- \alpha} \) is the parameter for our fermionic symmetry transformation, just as in the conventional Green-Schwarz superstring [12][20]. Since \( Z^M \) is the only fundamental field in our formulation, (3.1c) is the necessary condition of (3.1a) and (3.1b).

We can confirm \( \delta_\kappa I = 0 \) easily, once we know the intermediate results:

\[
\delta_\kappa \mathcal{L}_{\text{NG}} = +\sqrt{-\Gamma} (\Pi_{-} \Pi_{+} \Pi_{(i)} \Pi_{(j)}) , \tag{3.2a}
\]

\[
\delta_\kappa \mathcal{L}_{\text{WZNW}} = -\epsilon^{ij} (\Pi_{-} \Pi_{+} \Pi_{i} \Pi_{j}) . \tag{3.2b}
\]

By using the relationships, such as \( \sqrt{-\Gamma} \epsilon^{(k)(l)} = +\epsilon^{ij} \gamma_{i}^{(k)} \gamma_{j}^{(l)} \), with the most crucial equation (2.10), we can easily confirm that the sum (3.2a) + (3.2b) vanishes:

\[
\delta_\kappa \mathcal{L} = \delta_\kappa (\mathcal{L}_{\text{NG}} + \mathcal{L}_{\text{WZNW}}) = +2\sqrt{-\Gamma} (\Pi_{-} \Pi_{+} \Pi_{+} \Pi_{-}) = 0 . \tag{3.3}
\]

Thus the fermionic \( \kappa \)-invariance \( \delta_\kappa I = 0 \) works also in our formulation, despite the absence of the 2D metric or zweibein. The existence of fermionic \( \kappa \)-symmetry also guarantees the light-cone equivalence of our system to the conventional \( N = 2 \) GS superstring [14].
4. Derivation of Lagrangian and Fermionic Symmetry

In this section, we start with the conventional GS $\sigma$-model action [14] for $N = 2$ superstring [16][17], and derive our lagrangian (2.1) with the fermionic transformation rule (3.1). This procedure provides an additional justification for our formulation.

The $N = 2$ GS action $I_{GS} \equiv \int d^2 \sigma \mathcal{L}_{GS}$ [14] which is light-cone equivalent to $N = 2$ NSR superstring [16][17] has the lagrangian

$$
\mathcal{L}_{GS} = +\frac{1}{2}\sqrt{-g}g^{ij}\Pi_i^a\Pi_j^a + \epsilon^{ij}\Pi^A_i\Pi^B_jB_{BA} \\
= +e\Pi_+^a\Pi_-^a + 2e\Pi_-^A\Pi_+^B B_{BA},
$$

(4.1)

where $g \equiv \text{det} (g_{ij})$ is for the 2D metric $g_{ij}$, while $e \equiv \text{det} (e_i^{(j)}) = \sqrt{-g}$ is for the zweibein $e_i^{(j)}$. The action $I_{GS}$ is invariant under the fermionic transformation rule [20][15]

$$
\delta_\lambda E^a = +i(\sigma^a_i)^j_i\lambda^i\Pi_i^a = +i(\Pi_i^a\lambda^i)^a ,
$$

(4.2a)

$$
\delta_\lambda E^a = 0 ,
$$

(4.2b)

$$
\delta_\lambda e_+^i = -(\lambda_+^a\Pi_+^a) e_+^i \equiv -\tilde{(\lambda_+^a\Pi_+^a)} e_+^i ,
$$

(4.2c)

$$
\delta_\lambda e_-^i = 0 ,
$$

(4.2d)

where $\lambda$ has only the negative component: $\lambda_{(ij)}^a \equiv Q_{(ij)}^{(j)}\lambda_{(j)}^a$. Only in this section, the local Lorentz indices are related to curved ones through the zweibein as in $\Pi_{(ij)}^A \equiv e_{(i)}^j\Pi_j^A$, instead of $\gamma_{(ij)}$ in the last section. In the routine confirmation of $\delta_\lambda \mathcal{L}_{GS} = 0$, we see its parallel structures to $\delta_\kappa \mathcal{L} = 0$.

We next derive our lagrangians $\mathcal{L}_{NG}$ and $\mathcal{L}_{WZNW}$ from $\mathcal{L}_{GS}$ in (4.1). To this end, we first get the 2D metric field equation from $I_{GS}$

$$
g_{ij} \doteq +2(g^{kl}\Pi_k^a\Pi_l^a)^{-1}(\Pi_i^a\Pi_j^a) \equiv 2\Omega^{-1}\Gamma_{ij} \equiv h_{ij} ,
$$

(4.3a)

$$
\Omega \equiv g^{ij}\Pi_i^a\Pi_j^a = g^{ij}\Gamma_{ij} .
$$

(4.3b)

As is well-known in string $\sigma$-models, this field equation is not algebraic for $g_{ij}$, because the r.h.s. of (4.3) again contains $g^{ij}$ via the factor $\Omega$. Nevertheless, we can formally delete the

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6) We use the parameter $\lambda$ instead of $\kappa$ due to a slight difference of $\lambda$ from our $\kappa$ (Cf. eq. (4.8)).

7) We use the symbol $\doteq$ for a field equation to be distinguished from an algebraic one.
metric from the original lagrangian, using a procedure similar to getting NG string [9][10] from Polyakov string [21], or NG action out of Type II superstring action [12], as

\[
\frac{1}{2} \sqrt{-g} g^{ij} \Gamma_{ij} = \frac{1}{2} \sqrt{-g} \Omega = \frac{1}{2} \sqrt{-\det (h_{ij})} \Omega = \frac{1}{2} \sqrt{-\det (2 \Omega^{-1} \Gamma_{ij})} \Omega = \Omega^{-1} \sqrt{-\det (\Gamma_{ij})} \Omega = \sqrt{-\Gamma} = \mathcal{L}_{\text{NG}}. \tag{4.4}
\]

Thus the metric disappears completely from the resulting lagrangian, leaving only \(\sqrt{-\Gamma}\) which is nothing but \(\mathcal{L}_{\text{NG}}\) in (2.1). As for \(\mathcal{L}_{\text{WZNW}}\), since this term is metric-independent, this is exactly the same as the second term of (4.1).

We now derive our fermionic transformation rule (3.1) from (4.2). For this purpose, we establish the on-shell relationships between \(e_i^{(j)}\) and our newly-defined \(\gamma_i^{(j)}\). By taking the ‘square root’ of (4.3a), we get the \(e_i^{(j)}\)-field equation expressed in terms of the \(\Pi\)’s, that we call \(f_i^{(j)}\) which coincides with \(e_i^{(j)}\) only on-shell:

\[
e_i^{(j)} = f_i^{(j)} = f_i^{(j)}(\Pi_k^A), \tag{4.5a}
\]

\[
f_{i(k)} f_j^{(k)} = h_{ij}, \quad f^{(k)i} f_{(k)}^j = h^{ij}, \quad f_{i(k)}^j f_{(k)}^j = \delta_i^j, \quad f_{i(k)}^j f_k^{(j)} = \delta_{(i)}^{(j)}. \tag{4.5b}
\]

Note that the \(f\)’s is proportional to the \(\gamma\)’s by a factor of \(\sqrt{\Omega/2}\), as understood by the use of (4.3), (4.5) and (2.8):

\[
e_i^{(j)} = f_i^{(j)} = \sqrt{\frac{2}{\Omega}} \gamma_i^{(j)}, \quad e_{(i)}^j = f_{(i)}^j = \sqrt{\frac{\Omega}{2}} \gamma_{(i)}^j. \tag{4.6}
\]

Recall that the factor \(\Omega\) contains the 2D metric or zweibein which might be problematic in our formulation, while \(\gamma_i^{(j)}, \gamma_{(i)}^j\) are expressed only in terms of the \(\Pi_i^A\)’s. Fortunately, we will see that \(\Omega\) disappears in the end result.

Our fermionic transformation rule (3.1a) is now obtained from (4.2a), as

\[
\delta_{\lambda} E^\alpha = i(\Pi_i^A \lambda^i)^\alpha = if^{(i)(j)}(\Pi_j^A \lambda_{(i)})^\alpha = i\sqrt{\frac{\Omega}{2}} {\gamma_{(i)}}^{(j)}(\Pi_j^A \lambda_{(i)})^\alpha
\]

\[
= i\gamma_{(i)}^{(j)} \left[ \Pi_j^A \left( \sqrt{\frac{\Omega}{2}} \lambda_{(i)} \right) \right]^\alpha = i(\Pi_i^A \kappa_{(i)})^\alpha = \delta_{\kappa} E^\alpha, \tag{4.7}
\]

where \(\lambda\) and \(\kappa\) are proportional to each other by

\[
\kappa_{(i)} \equiv \sqrt{\frac{\Omega}{2}} \lambda_{(i)}. \tag{4.8}
\]
Such a re-scaling is always possible, due to the arbitrariness of the parameter \( \lambda \) or \( \kappa \).

As an additional consistency confirmation, we can show the \( \kappa \)-invariance of (2.10), using the convenient lemmas

\[
(\delta_{\kappa} \gamma_+^i) \gamma_i^+ = (\delta_{\kappa} \gamma_-^i) \gamma_i^- = \frac{1}{2} \Omega^{-1} \delta_{\kappa} \Omega \quad , \quad (\delta_{\kappa} \gamma_+^i) \gamma_i^- = (\delta_{\kappa} \gamma_-^i) \gamma_i^+ = - (\pi_- \Pi_-) . \quad (4.9)
\]

Combining these with (3.1c), we can easily confirm that \( \delta_{\kappa} \Gamma_{++} = 0 \) and \( \delta_{\kappa} \Gamma_{--} = 0 \), as desired for consistency of the ‘built-in’ Virasoro condition (2.10).

The complete disappearance of \( \Omega \) in our transformation rule (3.1) is desirable, because \( \Omega \) itself contains the metric that is not given in a closed algebraic form in terms of \( \Pi_i^A \). If there were \( \Omega \) involved in our transformation rule (3.1), it would pose a problem due to the metric \( g_{ij} \) in \( \Omega \). To put it differently, our action (2.1) and its fermionic symmetry (3.1) are expressed only in terms of the fundamental superfield \( Z^M \) via \( \Pi_i^A \) with no involvement of \( g_{ij} \), \( e_1^{(j)} \) or \( \Omega \), thus indicating the total consistency of our system. This concludes the justification of our fermionic \( \kappa \)-transformation rule (3.1), based on the \( N = 2 \) GS \( \sigma \)-model [14] light-cone equivalent to \( N = 2 \) NSR superstring [16][17].

5. Concluding Remarks

In this paper, we have shown that after the elimination of the 2D metric at the classical level, the NG-action part \( I_{NG} \) of GS \( \sigma \)-model action [14] for \( N = 2 \) superstring [16][17] is entirely expressed as the square root of a Cayley’s hyperdeterminant with the manifest \([SL(2, \mathbb{R})]^3\) symmetry. In particular, this is valid in the presence of target superspace background in \( D = (2, 2; 2, 2) \) [19]. From this viewpoint, \( N = 2 \) GS \( \sigma \)-model [14] seems more suitable for discussing the \([SL(2, \mathbb{R})]^3\) symmetry via a Cayley’s hyperdeterminant. We have seen that the \([SL(2, \mathbb{R})]^3\) symmetry acts on the three indices \( i, a, \dot{a} \) carried by the pull-back \( \Pi_{i\alpha\dot{a}} \) in \( \det (\Pi_{i\alpha\dot{a}}) \) in \( D = (2, 2; 2, 2) \) superspace [19][14]. The hidden discrete symmetry pointed out in [8], however, seems absent in \( N = 2 \) string [17][19][14] due to the WZNW-term \( \mathcal{L}_{WZNW} \).

We have also shown that our action (2.1) has the classical invariance under our fermionic \( \kappa \)-symmetry (3.1), despite the elimination of zweibein or 2D metric. Compared with the
original $I_{GS}$ [14], our action has even simpler structure, because of the absence of the 2D metric or zweibein. Due to its fermionic $\kappa$-symmetry, we can also regard that our system is classically equivalent to NSR $N = 2$ superstring [16][17], or $N = 2$ GS superstring [13]. As an important by-product, we have confirmed that the Virasoro condition (2.10) are inherent even in the NG reformulation of $N = 2$ GS string [14] at the classical level. This is also consistent with the original result that Virasoro condition is inherent in NG string [9][10].

One of the important aspects is that our action (2.1) and the fermionic transformation rule (3.1) involve neither the 2D metric $g_{ij}$, the zweibein $e_i^{(j)}$, nor the factor $\Omega$ containing these fields. This indicates the total consistency of our formulation, purely in terms of superspace coordinates $Z^M$ as the fundamental independent field variables.

In this paper, we have seen that neither the 2D metric $g_{ij}$ nor the zweibein $e_i^{(j)}$, but the superspace pull-back $\Pi_{i\dot{\alpha}}$ is playing a key role for the manifest symmetry $[SL(2, \mathbb{R})]^3$ acting on the three indices $i, \alpha$. In particular, the combination $\Gamma_{ij} \equiv \Pi_i^{a\dot{a}} \Pi_j^{b\dot{b}}$ plays a role of ‘effective metric’ on the 2D world-sheet. This suggests that our field variables $Z^M$ alone are more suitable for discussing the global $[SL(2, \mathbb{R})]^3$ symmetry of $N = 2$ superstring [16][17][14].

As a matter of fact, in $D = 2 + 2$ unlike $D = 3 + 1$, the components $\alpha$ and $\dot{\alpha}$ are not related to each other by complex conjugations [26][18][19]. Additional evidence is that the signature $D = 2 + 2$ seems crucial, because $SO(2, 2) \approx SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ [30], while $SO(3, 1) \approx SL(2, \mathbb{C})$ for $D = 3 + 1$ is not suitable for $SL(2, \mathbb{R})$. Thus it is more natural that the NG reformulation of $N = 2$ GS superstring [14] with the target superspace $D = (2, 2; 2, 2)$ is more suitable for the global $[SL(2, \mathbb{R})]^3$ symmetry acting on the three independent indices $i, \alpha$ and $\dot{\alpha}$.

It seems to be a common feature in supersymmetric theories that certain non-manifest symmetry becomes more manifest only after certain fields are eliminated from an original lagrangian. For example, in $N = 1$ local supersymmetry in 4D, it is well-known that the $\sigma$-model Kähler structure shows up, only after all the auxiliary fields in chiral multiplets are eliminated [31]. This viewpoint justifies to use a NG-formulation with the 2D metric
eliminated, instead of the original \( N = 2 \) GS formulation \([13][14]\), in order to elucidate the global \( [SL(2, \mathbb{R})]^3 \) symmetry of the latter, via Cayley’s hyperdeterminant.

It has been well known that the superspace \( D = (2, 2; 2, 2) \) is the natural background for SDYM multiplet \([17][18][19][14]\). Moreover, SDSYM theory \([18][19][14]\) is the possible underlying theory for all the (supersymmetric) integrable systems in space-time dimensions lower than four \([24]\). All of these features strongly indicate the significant relationships among Cayley’s hyperdeterminant \([1][8]\), \( N = 2 \) superstring \([16][17]\), or \( N = 2 \) GS superstring \([13][14]\) with \( D = (2, 2; 2, 2) \) target superspace \([19][14]\), its NG reformulation as in this paper, the STU black holes \([5][6]\), SDSYM theory in \( D = 2 + 2 \) \([18][19][14]\), and supersymmetric integrable or soluble models \([24][17][19][14]\) in dimensions \( D \leq 3 \).

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