Abstract. In this paper, we first introduce the concepts of $G$-systems, quotient $G$-systems and isomorphism theorems on $G$-systems of $n$-ary semihypergroups. Also we consider the Green’s equivalences on $G$-systems and further investigate some of their properties. A number of $n$-ary semihypergroups are constructed and presented as examples in this paper.

Keywords: $n$-Ary semihypergroup, $G$-Systems, Greens relations.

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1. Introduction

The concept of Green relation introduced by Green [9] and have played a fundamental role in the development of semigroup theory. The Green’s relations provide the necessary tools for using similar arguments on the monoid. The Green’s relations are well known, and presented in deep detail in several places. The concept of an $n$-group was introduced by Dörnte in [7]. This concept is a natural generalization of a group. Since then there are numbers of papers concerning various $n$-ary algebras in the literature. It is noted that the algebraic hyperstructures are suitable generalizations of the classical algebraic structures. In a classical algebraic structure, the composition of two elements
is an element, while in an algebraic hyperstructure, the composition of two elements is not necessarily an element but is a system. The notion of the hypergroup was introduced in 1934 by Marty [13] at the 8th Congress of Scandinavian Mathematicians. He then published some notes on hypergroups, using them in different contexts such as algebraic functions, rational actions, non commutative groups. Since then, hundreds of research papers and several monographs have been published in this topic and several kind of hypergroups have been particularly studied, such as regular hypergroups, reversible regular hypergroups, canonical hypergroups, cogroups, cyclic hypergroups, associativity hypergroups, for example see ([2] and [3]). (the monograph by P. Corsini and V. Leoreanu)

A recent monograph on hyperstructures [5] points out on their applications in fuzzy and rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs. Moreover, Davvaz and Vougiouklis [6] have established a connection between the two do mains in the form of an extension of the concept of n-ary groups to the concept of n-ary hypergroups. They determined some connections between this hypergroupoid and Ivo Rosenberg’s hypergroupoid associated with a binary relation. In [4] Cristea and Stefanescu, associated a hypergroupoid \( (H, \otimes, \rho) \) with an n-ary relation \( \rho \) defined on a non-empty set \( H \). The n-ary hyperoperations studied by many researchers, for example, see [2, 3, 4, 6, 10, 11, 12, 14, 15].

These Green’s relations on a semigroup were first defined and studied by Green [9] dated back to 1951. These Green’s relations on a semigroup played a fundamental role in the development of semigroup theory. In particular, Chinram and Siammai have considered and studied extensively the Green’s relations on the \( \Gamma \)-semigroups and reductive \( \Gamma \)-semigroups [1]. Also, the Green’s relations and congruences on \( n \)-ary semigroups were studied and are investigated by Sioson in 1967 [16].

In this paper, we define the left and right \( G \)-systems in the context of \( n \)-ary semihypergroups and introduce the concept of regular relation on the \( G \)-systems. Also, we consider the Green’s equivalence relations on the \( G \)-systems and find some of their properties.

2. Basic definition

In this section, we present some definitions concerning the \( n \)-ary semihypergroups as a generalization of the \( n \)-ary semigroups and semigroups.

Let \( G \) be a non-empty set and \( f \) a mapping \( f : G \times G \rightarrow \varphi^*(G) \), where \( \varphi^*(G) \) is the set of all non-empty subsets of \( G \). Then, we call \( f \) is a binary hyperoperation on the set \( G \). We denote by \( G^n \) the Cartesian product \( G \times G \times \cdots \times G \), where \( G \) appears \( n \) times. The couple \( (G, f) \) is now called a hypertroupoid. For
any two non-empty subsets $G_1$ and $G_2$ of $G$, we define

$$G_1 \circ G_2 = \bigcup_{g_i \in G_1, g_2 \in G_2} g_1 \circ g_2.$$ 

A hypergroupoid $(G, f)$ is called a semihypergroup if for all $g_1, g_2, g_3$ of $G$, we have $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$.

In general, $f : G^n \rightarrow \varphi^*(G)$ is called an $n$-ary hyperoperation on $G$ and we call $(G, f)$ an $n$-ary hypergroupoid (see [3]).

Let $G_1, G_2, \ldots, G_n$ be a non-empty subsets of $G$. Then, we define

$$f(G_1, G_2, \ldots, G_n) = \bigcup \{ f(g_1, g_2, \ldots, g_n) : g_i \in G_i, i \in \{1, 2, \ldots, n\} \}.$$ 

The sequence $g_i, g_{i+1}, \ldots, g_{i+1}$ will be denoted by $g_j^i$. For $j < i$, $g_j^i$ is a empty set.

**Definition 2.1.** [8] The $n$-ary hypergroupoid $(G, f)$ is called an $n$-ary semihypergroup if for any $i, j \in \{1, 2, \ldots, n\}$ and $g_1^{2n-1}$,

$$f(g_1^{i-1}, f(g_j^{i+1-1}, g_n^{2n-i-1})) = f(g_1^{i-1}, f(g_j^{n+i-1}, g_n^{2n-i-1})).$$

We call $G$ an $n$-ary semihypergroup with identity if there is an element $e \in G$ such that

$$x \in f\left(e^{(i-1)}, x, e^{(n-i)}\right).$$

Let $(G, f)$ be an $n$-ary semihypergroup and $H$ be a non-empty subset of $G$. Then, $H$ is an $n$-ary subhypergroup of $G$ if it is closed under the $n$-ary hyperoperation $f$, i.e., for every $(h_1, h_2, \ldots, h_n) \in H^n$ implies that $f(h_1, h_2, \ldots, h_n) \subseteq H$.

The $n$-ary semihypergroup $(G, f)$ with the equation $g \in f \left(g_1^{i-1}, x_i, g_i^{n+1}\right)$ has the solution $x_i \in G$ for any $g_1^{i-1}, x_i, g_i^{n+1} \in G$ and $1 \leq i \leq n$, is called an $n$-ary hypergroup.

An $n$-ary semihypergroup $(G, f)$ is commutative if for all $g_i^n \in G$ and for any permutation $\sigma$ of $\{1, 2, \ldots, n\}$, we have

$$f(g_1^n) = f(g_{\sigma(1)}, g_{\sigma(2)}, \ldots, g_{\sigma(n)}).$$

Let $(G_1, f_1)$ and $(G_2, f_2)$ be two $n$-ary semihypergroups. Then, a mapping $\varphi : G_1 \rightarrow G_2$ is called a homomorphism if for all $x_i^n \in G_1$ we have

$$\varphi(f_1(x_1, x_2, \ldots, x_n)) = f_2(\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_n)).$$

when $G_1$ and $G_2$ are $n$-ary semihypergroups with a scaler identity, $\varphi(e_1) = e_2$.

The following examples are easy examples of $n$-ary hypergroup.
Example 2.2. Let \((G, +)\) be a semihypergroup and \(f\) an \(n\)-ary hyperoperation on \(G\) defined by
\[
f(g_1^n) = \sum_{i=1}^{n} g_i; \quad \forall g_i^n \in G.
\]
Then, we can verify that \((G, f)\) is an \(n\)-ary semihypergroup.

Example 2.3. Let \(G\) be a group and \(< x, y >\) a subgroup generated by \(x, y\). Define
\[
f(g_1, g_2, ..., g_n) = < g_1, g_2, ..., g_n >,
\]
on \(G\). Then, one can verify that \((G, f)\) is an \(n\)-ary hypergroup.

Example 2.4. Let \(G\) be a semigroup and \(N\) a normal subsemigroup of \(G\). For all \(g_1^n \in G\), define \(f(g_1, g_2, ..., g_n) = g_1g_2...g_nH\). Then, obviously, \((G, f)\) is an \(n\)-ary semihypergroup.

Example 2.5. Let \(D\) be an integral domain and \(F\) be its field of fractions. Denote by \(U\) the group of the invertible elements of \(D\). Define the following \(n\)-ary hyperoperation on \(F/U\). For all \(g_i^n\), with \(1 \leq i \leq n\),
\[
f(g_1^n, g_2^n, ..., g_n^n) = \{ g : \exists u_i^n \in U^n, g = u_1g_1 + u_2g_2 + ... + u_ng_n \}.
\]
Then, we can easily verify that \((F/U, f)\) is an \(n\)-ary semihypergroup.

We now construct the following non-trivial \(n\)-ary semihypergroups.

Example 2.6. Let \(V\) be a vector space over an ordered field \(F\) and \(x_1, x_2, ..., x_n \in V\). Then, we define
\[
f(x_1, x_2, ..., x_n) = \{ \lambda_1x_1 + \lambda_2x_2 + ... + \lambda_nx_n : \lambda_i > 0, \Sigma_{i=1}^{n}\lambda_i = 1 \}.
\]
Hence, \((V, f)\) is an \(n\)-ary semihypergroup.

Example 2.7. Let \(G\) be a semigroup and \(\{ A_g \}_{g \in G}\) be a collection of non-empty distinct sets and \(S = \bigcup_{g \in G} A_g\). For every \(x_1, x_2, ..., x_n \in S\), we define
\[
f(x_1, x_2, ..., x_n) = A_{g_1}g_2...g_n,
\]
where \(x_i \in A_{g_i}\) for some \(1 \leq i \leq n\). Then, \(S\) is an \(n\)-ary semihypergroup.

3. G-systems and regular relations

In this section, we define the \(G\)-systems, the regular relations and prove the isomorphism theorems for \(G\)-systems.

Let \((G, f)\) be an \(n\)-ary semihypergroup with identity and \(X\) be a non-empty set. We say that \(X\) is a left \(G\)-system if there is an action \(h : G^{n-1} \times X \rightarrow X\) with the properties:
\[
h(g_1^{n-1}, h(u_1^{n-1}, x)) = h(f(g_1^{n-1}, u_1), f(g_1^{n-1}, u_2), ..., f(g_1^{n-1}, u_{n-1}), x)
\]
\[
h(e^{n-1}, x) = x,
\]
for every $x \in X$ and $g_1^{2n-1}, u_1^{n-1} \in G$.

Dually, we call a non-empty set $X$ is a right $G$-system if there is an action $X \times G^{n-1} \rightarrow X$,

$$h \left( h \left( x, u_1^{n-1} \right), g_1^{n-1} \right) = h \left( x, f \left( u_1^{n-1}, g_1 \right), f \left( u_1^{n-1}, g_2 \right), \ldots, f \left( u_1^{n-1}, g_{n-1} \right) \right).$$

$$h \left( x, e^{n-1} \right) = x.$$

We first state the $G$-systems and discuss the $G$-systems.

**Example 3.1.** Let $G$ be an $n$-ary semihypergroup with identity and $X$ be an $n$-ary sub-semi hypergroup of $G$. We define

$$h : G^{n-1} \times X \rightarrow X,
\left( g_1^{n-1}, x \right) \mapsto e,$$

where $e$ is an identity and $g_1^{n-1} \in G$ and $x \in X$. Then, $X$ is a $G$-system.

**Example 3.2.** Let $G = \bigcup_{n \geq 0} A_n$, $A_0 = \{0\}$, $A_n = [n, n+1)$ and $X = \mathbb{Z}^+$. Then, we define

$$f : G^n \rightarrow P^*(G)
\left( g_i^n \right) \mapsto A_t,$$

where $t = \max\{m_1, m_2, \ldots, m_n\}$ and $g_i \in A_{m_i}$. Then, $(G, f)$ is an $n$-ary semi hypergroup. Also,

$$h : G^{n-1} \times X \rightarrow X
\left( g_1^{n-1}, x \right) \mapsto \max\{m_1, m_2, \ldots, m_{n-1}, x\}.$$

Then, $X$ is a $G$-system.

Let $G$ and $H$ be $n$-ary semihypergroups. Then, we say that $X$ is a $(G, H)$-system if it is a left $G$-system by action $h_1 : G^{n-1} \times X \rightarrow X$ and a right $H$-system by action $h_2 : X \times H^{n-1} \rightarrow X$ and

$$h_2(h_1(g_1^{n-1}, x), t_1^{n-1}) = h_1(g_1^{n-1}, h_2(x, t_1^{n-1})),
\text{where } g_1^{n-1} \in G^{n-1}, t_1^{n-1} \in H^{n-1} \text{ and } x \in X.$$

Let $X$ and $Y$ be left $G$-systems and $h_1 : G^{n-1} \times X \rightarrow X$ and $h_2 : G^{n-1} \times Y \rightarrow Y$. Then, a map $\varphi : X \rightarrow Y$ is a morphism when

$$\varphi \left( h_1 \left( g_1^{n-1}, x \right) \right) = h_2 \left( g_1^{n-1}, \varphi(x) \right).$$

Let $\text{Mor}(X, Y)$ be the set of all $G$-morphism from $X$ into $Y$ and $X, Y$ are left $G$-systems such that $h_1 : G^{n-1} \times X \rightarrow X$ and $h_2 : G^{n-1} \times Y \rightarrow Y$. Then, we define

$$h : G^{n-1} \times \text{Mor}(X, Y) \rightarrow \text{Mor}(X, Y)
\left( g_1^{n-1}, \varphi \right) \mapsto \varphi.$$
where \( \overline{\varphi} : X \rightarrow Y \) and \( \overline{\varphi}(x) = h_2(g_1^{n-1}, \varphi(x)) \). Hence
\[
\overline{\varphi}(h_1(g_1^{n-1}, x)) = h_2(g_1^{n-1}, \varphi(h_1(g_1^{n-1}, x))) = h_2(g_1^{n-1}, h_2(g_1^{n-1}, \varphi(x))) = h_2(g_1^{n-1}, \overline{\varphi}(x)).
\]
This implies that \( \overline{\varphi} \in Mor(X, Y) \). Moreover, we have the following equalities:
\[
h(g_1^{n-1}, h(k_1^{n-1}, \varphi))(x) = h_2(k_1^{n-1}, h(g_1^{n-1}, \varphi)(x)) = h_2(k_1^{n-1}, h_2(g_1^{n-1}, \varphi(x))) = h_2(f(k_1^{n-1}, g_1), \ldots, f(k_1^{n-1}, g_{n-1}), \varphi(x)).
\]
Then,
\[
h(g_1^{n-1}, h(k_1^{n-1}, \varphi)) = h(f(k_1^{n-1}, g_1), \ldots, f(k_1^{n-1}, g_{n-1}), \varphi)
\]
Hence \( Mor(X, Y) \) is a left \( G \)-system.

It is clear that the cartesian product \( X \times Y \) of a left \( G_1 \)-system \( X \) and a right \( G_2 \)-system \( Y \) become \((G_1, G_2)\)-system by the following definitions:
\[
\overline{h_1}(g_1^{n-1}, (x, y)) = (h_1(g_1^{n-1}, x), y),
\overline{h_2}((x, y), t_1^{n-1}) = (x, h_2(y, t_1^{n-1}),
\]
where \( x \in X, y \in Y, g_i \in G_1 \) and \( t_j \in G_2 \), for \( 1 \leq i \leq n \) and \( 1 \leq j \leq n \).

Let \( G \) be an \( n \)-ary semihypergroup and \( X \) be a left \( G \)-system. A relation \( \rho \) on the left \( G \)-system is called left regular if
\[
\forall x_1, x_2 \in X, g_1^{n-1} \in G, \ x_1 \rho x_2 \Rightarrow h(g_1^{n-1}, x_1) \rho h(g_1^{n-1}, x_2),
\]
when \( X \) is a right \( G \)-system and
\[
\forall x_1, x_2 \in X, g_1^{n-1} \in G, \ x_1 \rho x_2 \Rightarrow h(x_1, g_1^{n-1}) \rho h(x_2, g_1^{n-1}),
\]
then, \( X \) is a right regular relation. Now, we simply call a relation \( \rho \) is a regular relation if it is both a left and a right regular relation.

Let \( X \) be a left \( G \)-system and \( \rho \) be a regular relation on \( X \). Then, \([X : \rho] = \{ \rho(x) : x \in X \}\) is a left \( G \)-system by the following map:
\[
\overline{h} : G^{n-1} \times [X : \rho] \rightarrow [X : \rho],
(g_1^{n-1}, \rho(x)) \mapsto \rho(h(g_1^{n-1}, x)).
\]
Let \( X_1 \) and \( X_2 \) be left \( G \)-systems and \( \varphi : X_1 \rightarrow X_2 \) be a morphism. Then, we call a relation \( ker(\varphi) = \{(x_1, x_2) \in X \times X : \varphi(x_1) = \varphi(x_2)\} \) the kernel of \( \varphi \). This relation is obviously a left regular relation.
In the following theorem, we describe the morphism on a left $G$-system.

**Theorem 3.3.** Let $G$ be an $n$-ary semihypergroup, $X$ be a left $G$-system and \( \varphi : X_1 \to X_2 \) be a morphism. Then, \( [X : (\ker\varphi)] \cong \text{Im}\varphi \).

**Proof.** Suppose that $K = \ker(\varphi)$ and $\psi : [X : K] \to \text{Im}\varphi$ defined by $\psi(K(x)) = \varphi(x)$. Then the map $\psi$ is well-defined and one to one. Indeed, we have the following equalities:

\[
K(x) = K(y) \iff \varphi(x) = \varphi(y) \iff \psi(K(x)) = \psi(K(y)).
\]

On the other hand, we have

\[
\psi(h(g^{n-1}_1, K(x)) = \psi(K(h(g^{n-1}_1, x))) = \varphi(h(g^{n-1}_1, x)) = h(g^{n-1}_1, \varphi(x)) = h(g^{n-1}_1, \psi(K(x))).
\]

This completes the proof. \( \square \)

The following lemma is a crucial lemma of the morphisms on left $G$-systems.

**Lemma 3.4.** Let $\rho$ be a regular relation on a left $G$-system $X$ and $\varphi : X \to Y$ be a morphism such that $\rho \subseteq \ker(\varphi)$. Then, there is a unique morphism $\overline{\varphi} : [X : \rho] \to Y$ such that $\text{Im}(\overline{\varphi}) = \text{Im}(\varphi)$ and $\overline{\varphi} \circ \pi = \varphi$, where $\pi : X \to [X : \rho]$ is a natural morphism.

**Proof.** Suppose that $\overline{\varphi} : [X : \rho] \to Y$ defined by $\overline{\varphi}(\rho(x)) = \varphi(x)$, where $x \in X$. Then, $\overline{\varphi}$ is well-defined, since, for all $x_1, x_2 \in X$

\[
\rho(x_1) = \rho(x_2) \implies (x_1, x_2) \in \rho \subseteq \ker\varphi \implies \varphi(x_1) = \varphi(x_2).
\]

Also $\overline{\varphi} \circ \pi = \varphi$. The uniqueness of $\overline{\varphi}$ is clear. \( \square \)

Let $\rho_1$ and $\rho_2$ be regular relations on the left $G$-system $X$, where $\rho_1 \subseteq \rho_2$. By Theorem 3.4, there is a morphism $\overline{\varphi} : [X : \rho_1] \to [X : \rho_2]$, such that $\overline{\varphi} \circ \pi_1 = \pi_2$, where $\pi_1 : X \to [X : \rho_1]$ and $\pi_2 : G \to [X : \rho_2]$ are natural maps. The morphism $\overline{\varphi}$ is given by

\[
\overline{\varphi}(\rho_1(x)) = \rho_2(x), \ x \in X.
\]

Also, the regular relation $\ker(\overline{\varphi})$ on $[X : \rho_1]$ given by

\[
\ker(\overline{\varphi}) = \{(\rho_1(x_1), \rho_1(x_2)) \in [X : \rho_1] \times [X : \rho_1] : (x_1, x_2) \in \rho_2\}.
\]

For the regular relations on the left $G$-system $X$, we have the following theorem.
Theorem 3.5. Let \( \rho_1 \) and \( \rho_2 \) be regular relations on a left \( G \)-system \( X \) such that \( \rho_1 \subseteq \rho_2 \). Then,
\[
[\rho_1 : \rho_2] = \{ (\rho_2(a_1), \rho_2(a_2)) \in [X : \rho_2] \times [X : \rho_2] : (a_1, a_2) \in \rho_1 \},
\]
is regular on \([X : \rho_1]\) and \([[[X : \rho_1] : [\rho_1 : \rho_2]]] \cong [X : \rho_2]\).

Proof. The proof is straightforward and is hence omitted. 

Suppose that \( X \) is a \((G, G)\)-system such that \( h_1 : G^{n-1} \times X \rightarrow X, h_2 : X \times G^{n-1} \rightarrow X \) and \( \rho \) is an equivalence relation on \( X \). We define \( \rho^* = \{ (h_1 (g_1^{n-1}, h_2(x, t_1^{n-1})), h_1 (g_1^{n-1}, h_2 (y, t_1^{n-1})))) : g_1^{n-1}, t_1^{n-1} \in G, (x, y) \in \rho \} \).

For the \((G, G)\)-systems, we have the following propositions.

Proposition 3.6. Let \( G \) be an \( n \)-ary semigroup and \( X \) be a \((G, G)\)-system such that \( \rho \) is an equivalence relation on \( X \). Then, \( \rho^* \) is the smallest regular relation containing \( \rho \).

Proof. It is clear that \( \rho \subseteq \rho^* \). To show that \( \rho^* \) is a left regular, suppose that \((\overline{x}, \overline{y}) \in \rho^* \) and \( k_1^{n-1} \in G \). Hence there are \( g_1^{n-1}, t_1^{n-1} \in G \) and \( x, y \in X \) such that
\[
\overline{x} = h_1 (g_1^{n-1}, h_2(x, t_1^{n-1})), \quad \overline{y} = h_1 (g_1^{n-1}, h_2(y, t_1^{n-1})).
\]
Also, we deduce the following equalities:
\[
h_1 (k_1^{n-1}, \overline{x}) = h_1 (k_1^{n-1}, h_1 (g_1^{n-1}, h_2(x, t_1^{n-1}))) = h_1 (f(k_1^{n-1}, g_1), ..., f(k_1^{n-1}, g_{n-1}), h_2(x, t_1^{n-1}))
\]
and
\[
h_1 (k_1^{n-1}, \overline{y}) = h_1 (k_1^{n-1}, h_1 (g_1^{n-1}, h_2(y, t_1^{n-1}))) = h_1 (f(k_1^{n-1}, g_1), ..., f(k_1^{n-1}, g_{n-1}), h_2(y, t_1^{n-1})).
\]
This implies that \((h_1(k_1^{n-1}, x), h_1(k_1^{n-1}, y)) \in \rho^* \).

By using the same arguments, we obtain the following equalities:
\[
h_2 (\overline{x}, k_1^{n-1}) = h_2 (h_1 (g_1^{n-1}, h_2 (x, t_1^{n-1})), k_1^{n-1}) = h_2 (h_2 (h_1 (g_1^{n-1}, x), t_1^{n-1}), k_1^{k-1}) = h_2 (h_1 (g_1^{n-1}, x), f(t_1^{n-1}, k_1), f(t_1^{n-1}, k_2), ..., f(t_1^{n-1}, k_{n-1})).
\]
Let \( \rho_1 \) and \( \rho_2 \) be equivalence relations on \( X \). Then,

1. \( \rho_1 \subseteq \rho_2 \) implies that \( \rho_1^* \subseteq \rho_2^* \),
2. \( (\rho_1 \cup \rho_2)^* = \rho_1^* \cup \rho_2^* \).

Proof. The proof is straightforward and is hence omitted.

**Proposition 3.8.** Let \( \rho \) be an equivalence on \( X \) and \( G \) be an \( n \)-ary semigroup. Then,

\[
\rho^* = \{(x, y) \in X \times X : \forall g_1^{n-1}, t_1^{n-1} \in G, (h_1(g_1^{n-1}, h_2(x, t_1^{n-1})), h_1(g_1^{n-1}, h_2(y, t_1^{n-1}))) \in \rho\},
\]

is the largest regular relation on \( X \) contained in \( \rho \).

Proof. Suppose that \( (x, y) \in \rho^* \) and \( k_1^{n-1} \in G \). Then for every \( g_1^{n-1}, t_1^{n-1} \in G \), we have

\[
(h_1(g_1^{n-1}, h_2(x, t_1^{n-1})), h_1(g_1^{n-1}, h_2(y, t_1^{n-1}))) \in \rho.
\]

Hence

\[
h_1(g_1^{n-1}, h_2(h_1(k_1^{n-1}, x), t_1^{n-1})) = h_1(g_1^{n-1}, h_1(k_1^{n-1}, h_2(x, t_1^{n-1})))
\]

\[
= h_1(f(g_1^{n-1}, k_1), ..., f(g_1^{n-1}, k_{n-1}), h_2(x, t_1^{n-1})).
\]

In a similar way,

\[
h_1(g_1^{n-1}, h_2(h_1(k_1^{n-1}, y), t_1^{n-1})) = h_1(g_1^{n-1}, h_1(k_1^{n-1}, h_2(y, t_1^{n-1})))
\]

\[
= h_1(f(g_1^{n-1}, k_1), ..., f(g_1^{n-1}, k_{n-1}), h_2(y, t_1^{n-1})).
\]

This implies that \( (h_1(k_1^{n-1}, x), h_1(k_1^{n-1}, y)) \in \rho^* \). In a similar way, we can see

\[
(h_2(x, k_1^{n-1}), h_2(y, k_1^{n-1})) \in \rho^*.
\]

Let \( R \) be a regular relation on \( X \) contained in \( \rho \). Then,

\[
(x, y) \in \rho \implies \forall g_1^{n-1}, t_1^{n-1} \in G, (h_2(x, t_1^{n-1}), h_2(y, t_1^{n-1})) \in R
\]

\[
\implies (h_1(g_1^{n-1}, h_2(x, t_1^{n-1})), h_1(g_1^{n-1}, h_2(y, t_1^{n-1}))) \in R.
\]
This implies that $\rho^o \in R$.  

4. The Green’s Relations

In this section, we introduce the Green’s relations on a $G$-system and prove some properties.

Let $G$ be an $n$-ary semihypergroup and $X$ a $(G,G)$-system. Then, we define the relations $L$ and $R$ on the $G$-system $G$ as follows:

\[(x,y) \in L \iff \exists g_1^{n-1}, t_1^{n-1} \in G : h_1(g_1^{n-1}, x) = y, h_1(t_1^{n-1}, y) = x.\]

\[(x,y) \in R \iff \exists k_1^{n-1}, s_1^{n-1} \in G : h_2(y, k_1^{n-1}) = x, h_2(x, s_1^{n-1}) = y.\]

The following Proposition is a basic result of the regular relations on a $(G,G)$-systems.

**Proposition 4.1.** Let $G$ be an $n$-ary semihypergroup and $X$ be a $(G,G)$-system. Then, the relation $R$ is a left regular relation and the relation $L$ is a right regular relation.

**Proof.** Suppose that $(x,y) \in L$ and $k_1^{n-1} \in G$. Then,

\[h_1(g_1^{n-1}, x) = y, h_1(t_1^{n-1}, y) = x,\]

where $g_1^{n-1}, t_1^{n-1} \in G$. We have the following equalities:

\[h_2(x, k_1^{n-1}) = h_2(h_1(t_1^{n-1}, y), k_1^{n-1}) = h_1(t_1^{n-1}, h_2(y, k_1^{n-1})).\]

In a same way, we can see $h_2(y, k_1^{n-1}) = h_3(g_1^{n-1}, h_2(y, k_1^{n-1})).$

These results lead that $L$ is a left regular relation. By using similar arguments, we can show that $R$ is a left regular relation.  

**Corollary 4.2.** Let $G$ be a commutative $n$-ary semihypergroup. Then, $R$ and $L$ are regular relations.

For the $n$-ary semihypergroups, we have the following proposition.

**Proposition 4.3.** Let $G$ be an $n$-ary semihypergroup and $X$ be a $(G,G)$-system. Then, $L \circ R = R \circ L$.

**Proof.** Suppose that $G$ be an ary-semihypergroup and $(x,y) \in L \circ R$. Then, there exists $z \in X$ such that $(x,z) \in L$ and $(z,y) \in R$. Hence there exist $g_1^{n-1}, t_1^{n-1}, k_1^{n-1}, s_1^{n-1} \in G$ such that

\[h_1(g_1^{n-1}, x) = z, \ h_1(t_1^{n-1}, z) = x,\]

\[h_2(z, k_1^{n-1}) = y, \ h_2(y, s_1^{n-1}) = z.\]
Let \( w = h_2 \left( t_1^{n-1}, z, k_1^{n-1} \right) \). Then,

\[
\begin{align*}
  h_2 \left( w, s_1^{n-1} \right) &= h_2 \left( h_2 \left( h_1 \left( t_1^{n-1}, z, k_1^{n-1} \right), s_1^{n-1} \right) \right) \\
  &= h_2 \left( h_1 \left( t_1^{n-1}, h_2 \left( z, k_1^{n-1} \right), s_1^{n-1} \right) \right) \\
  &= h_2 \left( h_1 \left( t_1^{n-1}, y, s_1^{n-1} \right) \right) \\
  &= h_1 \left( t_1^{n-1}, h_2 \left( y, s_1^{n-1} \right) \right) \\
  &= h_1 \left( t_1^{n-1}, z \right) = z.
\end{align*}
\]

and

\[
\begin{align*}
  h_2 \left( x, k_1^{n-1} \right) &= h_2 \left( h_1 \left( t_1^{n-1}, z, k_1^{n-1} \right) \right) = w.
\end{align*}
\]

Hence \( xRw \). Moreover, we have

\[
\begin{align*}
  h_1 \left( t_1^{n-1}, y \right) &= h_1 \left( t_1^{n-1}, h_2 \left( z, k_1^{n-1} \right) \right) = h_2 \left( h_1 \left( t_1^{n-1}, z, k_1^{n-1} \right) \right) = w.
\end{align*}
\]

and

\[
\begin{align*}
  h_1 \left( g_1^{n-1}, w \right) &= h_1 \left( g_1^{n-1}, h_2 \left( h_1 \left( t_1^{n-1}, z, k_1^{n-1} \right) \right) \right) \\
  &= h_1 \left( g_1^{n-1}, h_2 \left( x, k_1^{n-1} \right) \right) \\
  &= h_2 \left( h_1 \left( g_1^{n-1}, x \right), k_1^{n-1} \right) \\
  &= h_2 \left( z, k_1^{n-1} \right) = y.
\end{align*}
\]

Hence \( wLy \). We deduce that \((x, y) \in R \circ L \). This implies that \( L \circ R \subseteq R \circ L \). The reverse inclusion follows in a similar way. \( \square \)

Let \( G \) be an \( n \)-ary semihypergroup and \( X \) be a \((G, G)\)-system and \( D = L \circ R \). We define a relation \( \zeta \) on \( X \) as follows:

\[
(x, y) \in \zeta \iff \exists g_1^{n-1}, t_1^{n-1}, k_1^{n-1}, s_1^{n-1} \in G : \\
  h_1 \left( g_1^{n-1}, h_2 \left( h_1 \left( t_1^{n-1}, z, k_1^{n-1} \right) \right) \right) = y, 
  h_1 \left( k_1^{n-1}, h_2 \left( y, s_1^{n-1} \right) \right) = x.
\]

It is easy to see that \( L \subseteq \zeta \) and \( R \subseteq \zeta \). Hence \( D = L \circ R \subseteq \zeta \). It is easy to see that \( D \) is the smallest equivalence containing \( L \) and \( R \).

Finally, we state the main theorem of this paper.

**Theorem 4.4.** Let \( G \) be an \( n \)-ary semihypergroup, \( X \) be a \((G, G)\)-system and \( a, b \in X \) such that \((a, b) \in R \) and \( g_1^{n-1}, t_1^{n-1} \in G \) such that \( h_2 \left( a, g_1^{n-1} \right) = b \) and \( h_2 \left( b, t_1^{n-1} \right) = a \). Then, there exist \( \rho_{g_1^{n-1}} : X \rightarrow X \) and \( \rho_{t_1^{n-1}} : X \rightarrow X \) such that \( \rho_{g_1^{n-1}} \mid_{i(a)} \) and \( \rho_{t_1^{n-1}} \mid_{i(b)} \) are mutually inverse \( R \)-classes preserving from \( L(a) \) onto \( L(b) \) and \( L(b) \) onto \( L(a) \).

**Proof.** Suppose that \((a, b) \in R \). Then, by the definition of \( R \), there exists \( g_1^{n-1}, t_1^{n-1} \in G \) such that \( h_2 \left( a, g_1^{n-1} \right) = b \) and \( h_2 \left( b, t_1^{n-1} \right) = a \). We define

\[
\begin{align*}
  \rho_{g_1^{n-1}} : X &\rightarrow X \\
  x &\mapsto h_2 \left( x, g_1^{n-1} \right).
\end{align*}
\]
Hence, $\rho_{g_1^{n-1}}(a) = h_2(a, g_1^{n-1}) = b$. Since $L$ is regular, this implies that for every $x \in R(a)$,

$$h_2(x, g_1^{n-1}) \preceq h_2(a, g_1^{n-1}) = b.$$ 

Hence $\rho_{g_1^{n-1}}(L(a)) \subseteq L(b)$. In a similar way, we can define

$$\rho_{t_1^{n-1}} : X \longrightarrow X \quad x \longmapsto h_2(x, t_1^{n-1})$$

and $\rho_{t_1^{n-1}}(b) = h_2(b, t_1^{n-1}) = a$. Thus, for every $x \in L(a)$,

$$h_2(x, t_1^{n-1}) \preceq h_2(b, t_1^{n-1}) = b.$$ 

This implies that $\rho_{t_1^{n-1}}(L(b)) \subseteq L(a)$ and $\rho_{t_1^{n-1}} \circ \rho_{g_1^{n-1}} : L(a) \longrightarrow L(a)$. Let $x \in L(a)$. Then, there exists $u_1^{n-1} \in G$ such that $x = h_1(u_1^{n-1}, a)$. Hence

$$\rho_{t_1^{n-1}} \circ \rho_{g_1^{n-1}}(x) = \rho_{t_1^{n-1}}(h_2(x, g_1^{n-1})) = h_2(h_2(x, g_1^{n-1}), t_1^{n-1}) = h_2(h_2(u_1^{n-1}, a, g_1^{n-1}), t_1^{n-1}) = h_1(u_1^{n-1}, h_2(a, g_1^{n-1})), t_1^{n-1}) = h_2(h_1(u_1^{n-1}, b), t_1^{n-1}) = h_1(u_1^{n-1}, h_2(b, t_1^{n-1})), t_1^{n-1}) = h_1(u_1^{n-1}, a) = x.$$ 

Thus, $\rho_{t_1^{n-1}} \circ \rho_{g_1^{n-1}}$ is the identity map and we can show in a closely similar way that $\rho_{g_1^{n-1}} \circ \rho_{t_1^{n-1}}$ is the identity map on $L(b)$. This completes the proof. □

5. Conclusion

The Green’s relations provide the necessary tools for using similar arguments on the monoid. When working in language theory using automata, several tools comes naturally into play. A typical example is the use of the decomposition of the graph of the automaton into strongly connected components, and the use of the connected components for driving an induction in a proof. Since the Green relation used in automata theory we introduced this concept on $n$-ary semihypergroup. In future works, we consider and used of $G$-systems and Green relation for solving automata related questions.

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