Partitions of the complete hypergraph $K_6^3$ and a determinant-like function

Mihai D. Staic $^{1,2}$ · Steven R. Lippold $^1$

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Abstract
In this paper, we introduce a determinant-like map $\det^{S_3}$ and study some of its properties. For this, we define a graded vector space $\Lambda^{S_3} \mathcal{V}$ that has similar properties with the exterior algebra $\Lambda \mathcal{V}$ and the exterior GSC-operad $\Lambda^{S_2} \mathcal{V}$ from Staic. When $\dim(\mathcal{V}_2) = 2$, we show that $\dim_k(\Lambda^{S_3} \mathcal{V}_2[6]) = 1$, which gives the existence and uniqueness of the map $\det^{S_3}$. We also give an explicit formula for $\det^{S_3}$ as a sum over certain 2-partitions of the complete hypergraph $K_6^3$.

Keywords
Exterior algebra · Partitions of hypergraphs

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1 Introduction

The exterior algebra $\Lambda \mathcal{V}$ is the quotient of the tensor algebra $\mathcal{T} \mathcal{V}$ by the ideal generated by all elements $u \otimes u$ where $u \in \mathcal{V}$. It is well known that if $\dim(\mathcal{V}_d) = d$, then $\dim_k(\Lambda \mathcal{V}_d[d]) = 1$, and that one gets the determinant of a linear transformation $T : \mathcal{V} \to \mathcal{V}$ as the unique constant $\det(T) = \Lambda(T) : \Lambda \mathcal{V}_d[d] \to \Lambda \mathcal{V}_d[d]$. Equivalently, the determinant map is the unique (up to a constant) nontrivial linear function $\det : \mathcal{V}_d \otimes^d \to k$ with the property $\det(\bigotimes_{1 \leq i \leq d}(v_i)) = 0$ if there exists $1 \leq x < y \leq d$ such that $v_x = v_y$.

Mihai D. Staic
mstaic@bgsu.edu
Steven R. Lippold
steverl@bgsu.edu

1 Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, OH 43403, USA
2 Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, 70700 Bucharest, Romania
One of the applications of the exterior algebra is in the Hochschild–Kostant–Rosenberg (HKR) theorem. This states that for a smooth algebra $A$ there is an isomorphism between $\Lambda \Omega_{A,k}$ the exterior algebra of the module of Kähler differentials and $H_\bullet(A, A)$ the Hochschild homology of $A$ (see [11]). In [6], Pirashvili defined a generalization of the Hochschild homology, known as higher-order Hochschild homology $HH^X_\bullet(A, A)$, where $X$ is a simplicial set and $A$ is a commutative $k$-algebra. When $X = S^1$, one recovers the classical Hochschild homology.

Inspired by the HKR theorem, Pirashvili’s higher-order Hochschild homology [6], and Voronov’s Swiss-Cheese operads [10], the first author introduced in [7] the exterior Graded-Swiss-Cheese (GSC) operad $\Lambda^S V$. More precisely, if we let $T^V = \bigoplus_{n \geq 0} V_{n} \otimes \Lambda_0^{n-1}$, the exterior GSC-operad $\Lambda^S V$ was defined as a quotient of the tensor GSC-operad $T^V$ by the ideal $\mathcal{E}^S V$ generated by elements of the form

$$
\begin{pmatrix}
1 & u & u \\
1 & u & \\
\otimes & & 1
\end{pmatrix}
$$

where $u \in V$. We would direct the reader to Sect. 2 of this paper as well as [7] and [5] for the details of this construction.

The exterior GSC-operad $\Lambda^S V$ has many properties similar to the exterior algebra $\Lambda V$. For example, if we consider $V_d$ to be a $k$-vector space of dimension $d$, it was proved in [7] that $\dim_k(\Lambda^S V_3[4]) = 1$, and in [5] that $\dim_k(\Lambda^S V_3[6]) = 1$. In particular, if $d = 2$ (or $d = 3$), we have a determinant-like function $\det^S : V_d^{\otimes (2d-1)} \to k$ that is nontrivial and unique (up to a constant) with the property that $\det^S(\otimes 1 \leq i < j \leq 2d(v_i, j)) = 0$ if there exists $1 \leq x < y < z \leq 2d$ such that $v_{x,y} = v_{x,z} = v_{y,z}$. It was conjectured in [7] that a similar nontrivial function $\det^S : V_d^{\otimes (2d-1)} \to k$ exists and is unique up to a constant for any $d$. The results from [5] were obtained by exploring a connection between $\Lambda^S V_3[2d]$ and the set of homogeneous cycle-free edge partitions of the complete graph $K_{2d}$.

Using $\Lambda V$ and $\Lambda^S V$ as models, in this paper we consider a similar construction denoted $\Lambda^S S$. The main result is that when $\dim_k(V_2) = 2$, we have $\dim_k(\Lambda^S S_3[6]) = 1$. In particular, we get the existence and uniqueness of a nontrivial determinant-like function $\det^S : V_2^{\otimes 20} \to k$ with the property that $\det^S(\otimes 1 \leq i < j < k \leq 6(v_i, j, k)) = 0$ if there exists $1 \leq x < y < z < t \leq 6$ such that $v_{x,y,z} = v_{x,y,t} = v_{x,z,t} = v_{y,z,t}$. The map $\det^S$ is invariant under the actions of the group $SL_2(k)$ and of the symmetric group $S_6$.

The paper is organized as follows: In Sect. 2, we recall a few results about $\Lambda V$ and $\Lambda^S V$. We also give some definitions and examples of hypergraphs. In Sect. 3, we introduce $\Lambda^S S = \bigoplus_{n \geq 0} \Lambda^S S_3[n]$ as the quotient of $T^S \Lambda V$ by a certain subspace generated by elements similar to $\left(\begin{array}{ccc}
u & u & u \\
u & u \\
u & \end{array}\right)$ for all $u \in V$ (see 3.1 for an explanation of this notation). We exhibit a connection between $T^S \Lambda V_3[n]$ and the set of $d$-partitions of the $3$-uniform hypergraph $K_3^n$ and give a presentation with generators and relations for $\Lambda^S S$. 

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Section 4 deals with the case \( \dim(V_2) = 2 \). We prove the main result of this paper, namely that \( \dim_k \left( \Lambda^3 V_2[6] \right) = 1 \), which gives the existence and uniqueness of the \( \det^3 \) map. We study properties of the map \( \det^3 \) and compute \( \dim_k(\Lambda_1^* S_3 V_2[6]) = 1 \), which gives the existence and uniqueness of the \( \det S_3 \) map. We study properties of the map \( \det S_3 \) and compute \( \dim_k(\Lambda_1^* S_3 V_2[n]) \) for all \( n \geq 0 \). In Sect. 5, we discuss a few related problems and generalizations.

Using the fact that \( \det^3 \) is invariant under the action of \( SL_2(k) \), in “Appendix” we give an explicit presentation for \( \det S_3 \) as a sum of products of determinants of \( 2 \times 2 \) matrices. This gives an alternative proof for the existence of the map \( \det S_3 \).

We also present a classification of a certain class of homogeneous 2-partitions of complete hypergraph \( K^3_6 \) under the action of the group \( S_6 \times S_2 \). (This was obtained using MATLAB.)

2 Preliminary

2.1 Generalizations of the exterior algebra

In this paper, \( k \) is an infinite field such that \( \text{char}(k) \neq 2 \) and \( \text{char}(k) \neq 3 \). The tensor product \( \otimes \) is over the field \( k \). For a vector space \( V_d \) of dimension \( d \), we fix a basis \( B_d = \{e_1, \ldots, e_d\} \).

In order to provide some context and motivation for our construction and notations, in this subsection we recall a few results about the exterior algebra \( \Lambda^* V_d \), the determinant, the exterior GSC operad \( \Lambda^* S_2 V_d \), and the \( \det S_2 \) map. First, recall that the tensor algebra \( T V_d \) is the graded algebra defined as

\[
T V_d = \bigoplus_{n \geq 0} T V_d[n],
\]

where \( T V_d[n] = V_d^\otimes n \), with the product given by concatenation of tensors.

The exterior algebra of \( V_d \), denoted \( \Lambda^* V_d \), is the quotient of \( T V_d \) by the graded ideal \( J \) generated by simple tensors of the form \( e_i \otimes e_j + e_j \otimes e_i \), where \( 1 \leq i \leq j \leq d \).

The determinant is the unique (up to a constant) nontrivial function \( \det: V_d^\otimes d \to k \) such that \( \det(\otimes_{1 \leq i \leq d}(v_i)) = 0 \) if there exists \( 1 \leq x < y \leq d \) such that \( v_x = v_y \).

Focusing on the determinant, notice that the symmetric group \( S_d \) can be identified with \( \mathcal{P}_d^h(\{1, \ldots, d\}) \) the set of homogeneous, ordered \( d \)-partitions of the set \( \{1, \ldots, d\} \). More precisely to the permutation \( \sigma \in S_n \), we associate \( \pi(\sigma) = (\{\sigma(1)\}, \{\sigma(2)\}, \ldots, \{\sigma(d)\}) \in \mathcal{P}_d^h(\{1, \ldots, d\}) \). One can easily see that this map is
bijective. With this notation, if we let $v_i = (v_{i1}, \ldots, v_{id}) \in V_d$, then the usual formula for the determinant of a matrix $A = [v_1, \ldots, v_d] = [v_j^T]_{1 \leq i,j \leq d}$ can be rewritten as

$$\det(A) = \sum_{\sigma \in S_d} \varepsilon(\sigma)v_1^{\sigma(1)}v_2^{\sigma(2)}\ldots v_d^{\sigma(d)} = \sum_{\pi \in P_d(\{1,\ldots,d\})} \varepsilon(\pi)M_{\pi}(v_1, \ldots, v_d), \quad (2.1)$$

where $M_{\pi(\sigma)}(v_1, \ldots, v_d) = v_1^{\sigma(1)}v_2^{\sigma(2)}\ldots v_d^{\sigma(d)}$ is the corresponding monomial expression associated with $A$ and the $d$-partition $\pi(\sigma)$. We will see later in the paper how this equivalent presentation of the determinant fits into more general settings.

Next, recall from [5] the construction of $\Lambda^{S_2}_1 V_d$. For every $n \geq 0$, we denote $T^{S_2}_V[n] := V \otimes^{n(n-1)/2}$. A simple tensor in $T^{S_2}_V[n]$ will be denoted by $\otimes_{1 \leq i < j \leq n}(v_i^j)$ where $v_i^j \in V$. Alternatively, we can present a simple tensor in $T^{S_2}_V[n]$ as an upper triangular tensor matrix

$$\left(\begin{array}{cccccc}
1 & v_{1,2} & v_{1,3} & \cdots & v_{1,n-2} & v_{1,n-1} & v_{1,n} \\
& 1 & v_{2,3} & \cdots & v_{2,n-2} & v_{2,n-1} & v_{2,n} \\
& & \ddots & \vdots & \vdots & \vdots & \vdots \\
& & & 1 & v_{n-2,n-1} & v_{n-2,n} & v_{n-1,n} \\
\otimes & & & & 1 & v_{n-1,n-1} & 1
\end{array}\right) \in V \otimes^{n(n-1)/2},$$

where $v_i^j \in V$ (for similar notations see [2, 3], or [7]). A general element in $T^{S_2}_V[n]$ is a sum of simple tensors.

**Remark 2.1** The grading that we use in this paper is different from the one from [7]; more precisely, the relation between the two gradings is

$$T^{S_2}_V[n] = T^{S_2}_V(n+1) = V \otimes^{n(n-1)/2}.$$ 

This convention is more intuitive and is consistent with the usual grading on the exterior algebra $\Lambda_V$.

Next, we consider $E^{S_2}_V[n]$ the subspace of $V \otimes^{n(n-1)/2}$ that is linearly generated by all the simple tensor $\otimes_{1 \leq i < j \leq n}(v_i^j) \in T^{S_2}_V[n]$, with the property that there exist $1 \leq x < y < z \leq n$ such that $v_{x,y} = v_{x,z} = v_{y,z}$.

**Definition 2.2** Let $V$ be a $k$ vector space. We define $\Lambda^{S_2}_V$ as the graded vector space with the component in degree $n$ defined as the quotient vector space

$$\Lambda^{S_2}_V[n] := \frac{E^{S_2}_V[n]}{\Lambda^{S_2}_V(n+1)}.$$
\[ \Lambda_{V}^{S^2}[n] = \frac{\mathcal{T}_{V}^{S^2}[n]}{\mathcal{E}_{V}^{S^2}[n]} \]

for every \( n \geq 0 \).

**Remark 2.3** It was shown in [7] that \( \mathcal{T}_{V}^{S^2} \) has a Graded-Swiss-Cheese (GSC) operad structure. In that setting, \( \mathcal{E}_{V}^{S^2} \) is an (GSC-operad) ideal in \( \mathcal{T}_{V}^{S^2} \), and so we get a GSC-operad structure on \( \Lambda_{V}^{S^2} \). Our results in this paper do not rely on that structure, so a definition of \( \Lambda_{V}^{S^2} \) in terms of a graded vector space suffices.

We recall a few results about \( \Lambda_{V_d}^{S^2} \).

**Proposition 2.2** ([5, 7]) Let \( V_d \) be a vector space of dimension \( d \) with a basis \( B_d = \{e_1, \ldots, e_d\} \).

1. \( \mathcal{E}_{V_d}^{S^2} \) is generated by \( \left\{ \begin{pmatrix} e_i & e_j \\ \otimes & 1 \end{pmatrix} + \begin{pmatrix} e_i & e_k \\ 1 & e_j \end{pmatrix} + \begin{pmatrix} e_i & e_j \\ 1 & e_k \end{pmatrix} + \begin{pmatrix} e_k & e_i \\ 1 & e_j \end{pmatrix} + \begin{pmatrix} e_k & e_j \\ 1 & e_i \end{pmatrix} \right\} \) for all \( 1 \leq i \leq j \leq k \leq d \).

2. If \( n > 2d \), then \( \dim_k (\Lambda_{V_d}^{S^2}[n]) = 0 \).

3. If \( \dim_k(V_2) = 2 \) or \( \dim_k(V_3) = 3 \), then \( \dim_k (\Lambda_{V_d}^{S^2}[2d]) = 1 \).

4. If \( d = 2 \) or \( d = 3 \), the map \( \det^{S^2} \) is the unique nontrivial function, up to a constant, \( \det^{S^2} : V_d^{\otimes (2d-1)} \to k \) such that \( \det^{S^2}(\otimes_{1 \leq i < j \leq 2d}(v_i,j)) = 0 \) if there exists \( 1 \leq x < y < z \leq 2d \) such that \( v_{x,y} = v_{x,z} = v_{y,z} \).

Notice the parallels between Propositions 2.1 and 2.2. Based on the results summarized in Proposition 2.2, it was conjectured in [7] that if \( \dim_k(V_d) = d \), then \( \dim_k (\Lambda_{V_d}^{S^2}[2d]) = 1 \).

In Proposition 2.2, a connection between a particular basis for \( \mathcal{T}_{V_d}^{S^2}[n] \) and the set of \( d \)-partitions of the complete graph \( K_n \) was used in [5]. We recall that setting since it will provide motivation for some of the notations and results in Sect. 3.

Let \( B_d = \{e_1, e_2, \ldots, e_d\} \) be a basis for \( V_d \). Take the basis of \( V_d^{\otimes \binom{n(n-1)}{2}} \) given by

\[ \mathcal{G}_{B_d}^{S^2}[n] = \left\{ \begin{pmatrix} 1 & v_{1,2} & \cdots & v_{1,n-1} & v_{1,n} \\ & 1 & \cdots & v_{2,n-1} & v_{2,n} \\ & & \ddots & \vdots & \vdots \\ & & & 1 & v_{n-1,n} \\ & & & & 1 \end{pmatrix} \in V_d^{\otimes \binom{n(n-1)}{2}} | v_{i,j} \in B_d \right\} \]

Recall that a \( d \)-partition of the complete graph \( K_n \) is an ordered collection \( \Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_d) \) of subgraphs of \( K_n \) such that \( E(\Gamma_i) \cap E(\Gamma_j) = \emptyset \) for \( i \neq j \), and \( \bigcup_{i=1}^{n} E(\Gamma_i) = E(K_n) \).
Fig. 1 (Γ₁, Γ₂, Γ₃) the 3-partition of K₆ associated with E₃

It was shown in [5] that the basis $G_{B_d}^S[n]$ is in bijection with $P_d(K_n)$ the set of ordered $d$-partitions of the complete graph $K_n$. More precisely, to $\omega = \bigotimes_{1 \leq i < j \leq n} (v_i, j) \in G_{B_d}^S[n]$, we can associate the subgraphs $\Gamma_i(\omega)$ of $K_n$ by taking $V(\Gamma_i(\omega)) = \{1, 2, \ldots, n\}$, and $E(\Gamma_i(\omega)) = \{(a, b) | v_{a, b} = e_i\}$. From the definition of $G_{B_d}^S[n]$, it follows that this map is a bijection. As an example, let $E_3$ be the element $E_3 = \begin{pmatrix} 1 & e_1 & e_2 & e_3 & e_1 \\ 1 & e_1 & e_2 & e_3 \\ 1 & e_2 & e_3 & e_2 \\ 1 & e_2 & e_3 \\ \bigotimes & 1 \end{pmatrix} \in G_{B_3}^S[6].$

Since we have the entry $e_1$ in the positions $(1, 2)$, $(1, 4)$, $(1, 6)$, $(2, 3)$ and $(2, 5)$, we get $\Gamma_1$ the leftmost graph in Fig. 1. We could likewise do this construction for $e_2$ and $e_3$ to obtain $\Gamma_2$ and $\Gamma_3$, respectively, given in Fig. 1.

Notice that if $\dim_k \left( \Lambda_{V_d}^2[2d] \right) = 1$, then we get the existence of a unique (up to a constant) nontrivial linear map

$\det^S : V_d^\otimes (2d-1) \rightarrow k,$

with the property that $\det^S(\otimes (v_i, j))_{1 \leq i < j \leq 2d} = 0$ if there exist $1 \leq x < y < z \leq 2d$ such that $v_{x, y} = v_{x, z} = v_{y, z}$. When $d = 2$ or $d = 3$, it was shown in [5] that such a map exists and it has the expression

$$\det^S(\otimes_{1 \leq i < j \leq 2d} (v_i, j)) = \sum_{(\Gamma_1, \ldots, \Gamma_d) \in P_d^{h,cf}(K_{2d})} \varepsilon^S_{d, \Gamma} ((\Gamma_1, \ldots, \Gamma_d)) M(\Gamma_1, \ldots, \Gamma_d) \times (\otimes_{1 \leq i < j \leq 2d} (v_i, j)).$$

(2.2)

Here, the sum is taken over $P_d^{h,cf}(K_{2d})$ the set of cycle-free, homogeneous $d$-partitions $(\Gamma_1, \ldots, \Gamma_d)$ of the complete graph $K_{2d}$, the map $\varepsilon^S_{d, \Gamma} : P_d^{h,cf}(K_{2d}) \rightarrow \{1, -1\}$ is a sign map on the set of homogeneous, cycle-free $d$-partitions of $K_{2d}$, and $M(\Gamma_1, \ldots, \Gamma_d)(\otimes_{1 \leq i < j \leq 2d} (v_i, j))$ is a certain monomial associated with the partition.
(Γ₁, . . . , Γₖ) and to the element ⊗₁≤i<j≤2d(vᵢ,j) ∈ Vₙ⊗d(2d−1). For example, if d = 3 and vᵢ,j = αᵢ,j e₁ + βᵢ,j e₂ + γᵢ,j e₃, then

\[ M(Γ₁,Γ₂,Γ₃)(⊗₁≤i<j≤6(vᵢ,j)) = \prod_{(u₁,v₁)∈E(Γ₁)} α_{u₁,v₁} \prod_{(u₂,v₂)∈E(Γ₂)} β_{u₂,v₂} \prod_{(u₃,v₃)∈E(Γ₃)} γ_{u₃,v₃}. \]

Notice the similarities between formulas (2.1) and (2.2), as well as between \( M_π \) and \( M(Γ₁,...,Γₖ) \).

**Remark 2.4** When \( d = 2 \) or \( d = 3 \), the condition of \( \det^S(⊗(vᵢ,j)₁≤i<j≤2d) = 0 \) has a geometrical interpretation that was discussed in [8]. It is interesting to notice that the case \( d = 2 \) is essentially equivalent with an old result of Pappus of Alexandria (see [4]).

In this paper, we will deal with a similar construction \( \Lambda^S_V \) and the associated determinant-like function \( \det^S \). This will give us results analogous to Propositions 2.1 and 2.2, as well as a formula for \( \det^S \) analogous to formulas (2.1) and (2.2).

### 2.2 Partition of hypergraphs

We recall from [1] a few definitions and examples of hypergraphs that will be used later in this paper.

**Definition 2.5** A hypergraph \( \mathcal{H} = (V,E) \) consists of two finite sets \( V = \{v₁,v₂,\ldots,vₙ\} \) called the set of vertices, and \( E = \{E₁,E₂,\ldots,Eₘ\} \) a family of subsets of \( V \) called the hyperedges of \( \mathcal{H} \).

If every hyperedge of \( \mathcal{H} \) is of size \( r \), then \( \mathcal{H} \) is called an \( r \)-uniform hypergraph. For \( 2 ≤ r ≤ n \), we define the complete \( r \)-uniform hypergraph to be the hypergraph \( K_n^r = (V,E) \) for which \( V = \{1,2,\ldots,n\} \), and \( E \) is the family of all subsets of \( V \) of size \( r \).

A 2-uniform hypergraph is nothing else but a graph, and \( K_n^2 \) is the complete graph \( K_n \). In this paper, we are interested in 3-uniform hypergraphs. A hyperedge of a 3-uniform hypergraph will be called a face.

**Definition 2.6** Let \( \mathcal{H} \) be a hypergraph and \( k ≥ 2 \) be a natural number. A \( k \)-partition of \( \mathcal{H} \) is an ordered collection \( \mathcal{P} = (\mathcal{H}_₁,\mathcal{H}_₂,\ldots,\mathcal{H}_k) \) of sub-hypergraphs \( \mathcal{H}_i \) of \( \mathcal{H} \) such that:

(i) \( V(\mathcal{H}_i) = V(\mathcal{H}) \) for all \( 1 ≤ i ≤ k \),
(ii) \( E(\mathcal{H}_i) \cap E(\mathcal{H}_j) = \emptyset \) for all \( i ≠ j \),
(iii) \( ∪₁≤i≤kE(\mathcal{H}_i) = E(\mathcal{H}) \).

We say that the partition \( \mathcal{P} = (\mathcal{H}_₁,\mathcal{H}_₂,\ldots,\mathcal{H}_k) \) is homogeneous if \( |E(\mathcal{H}_i)| = |E(\mathcal{H}_j)| \) for all \( 1 ≤ i < j ≤ k \).

We will denote by \( \mathcal{P}_d(\mathcal{H}) \) the set of \( d \)-partitions of the hypergraph \( \mathcal{H} \), and with \( \mathcal{P}_d^h(\mathcal{H}) \) the set of homogeneous \( d \)-partitions of the hypergraph \( \mathcal{H} \).
Since we are only interested in 3-uniform hypergraphs, we will draw each hyperedge as a triangle (face) that connects three vertices. In order to avoid drawing three-dimensional pictures, we will draw a projection in the plane, allowing the possibility to draw the same vertex several times in our picture. Finally, because in this paper we are interested mostly in 2-partitions of $K^3_n$, when we draw a partition $(\mathcal{H}_1, \mathcal{H}_2)$, we will shade the hyperedges in $\mathcal{H}_1$ and do not shade the hyperedges in $\mathcal{H}_2$.

**Example 2.7**

(i) Consider the complete hypergraph $K^3_4$. Take $E(\mathcal{H}_1) = \{\{1, 2, 4\}, \{1, 3, 4\}\}$, and $E(\mathcal{H}_2) = \{\{1, 2, 3\}, \{2, 3, 4\}\}$. Then, $(\mathcal{H}_1, \mathcal{H}_2)$ is a homogeneous, 2-partition for $K^3_4$ (see Fig. 2).

(ii) Consider the complete graph $K^3_4$. Take $E(\mathcal{L}_1) = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}\}$, and $E(\mathcal{L}_2) = \{\{2, 3, 4\}\}$. Then, $(\mathcal{L}_1, \mathcal{L}_2)$ is a 2-partition for $K^3_4$ that is not homogeneous (see Fig. 3).

**Remark 2.8**

Notice that if $\mathcal{H}$ is a sub-hypergraph of $K^3_n$ and $\sigma \in S_n$, then $\sigma \cdot \mathcal{H}$ is also a sub-hypergraph of $K^3_n$, where $V(\sigma \cdot \mathcal{H}) = \{\sigma(v) | v \in V(\mathcal{H})\}$ and $E(\sigma \cdot \mathcal{H}) = \{\{\sigma(a), \sigma(b), \sigma(c)\} | \{a, b, c\} \in E(\mathcal{H})\}$.

With this notation, one can see that on $\mathcal{P}^h_d(K^3_{3d})$ there is an action of the group $S_{3d} \times S_d$ given by

$$(\sigma, \tau) \cdot (\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_d) = (\sigma \cdot \mathcal{H}_{\tau^{-1}(1)}, \sigma \cdot \mathcal{H}_{\tau^{-1}(2)}, \ldots, \sigma \cdot \mathcal{H}_{\tau^{-1}(d)})$$

Later in the paper, we will use the case $d = 2$. 
3 Generators and relations for $\Lambda^S_V$

In this section, we define $\Lambda^S_V$ for a vector space $V$ and discuss connections with partitions of hypergraphs.

Take

$$T^S_V = \bigoplus_{n \geq 0} T^S_V [n],$$

where

$$T^S_V [n] = V \otimes \frac{n(n-1)(n-2)}{6}.$$

A simple tensor in $T^S_V [n]$ is denoted by $\otimes_{1 \leq i < j < k \leq n} (v_{i,j,k})$ where $v_{i,j,k} \in V$. A general element in $T^S_V [n]$ is a sum of simple tensors. One should think about $T^S_V$ as a generalization of the tensor algebra $T(V)$, or of the tensor GSC-operad $T^S_V$.

When convenient we will also use a tensor matrix notation similar to the ones from [2], or [7],

$$\omega = \otimes_{1 \leq i < j < k \leq n} (v_{i,j,k}) = \begin{pmatrix} v_{1,2,3} & v_{1,2,4} & v_{1,3,4} & \cdots \\ v_{1,2,5} & v_{1,3,5} & v_{1,4,5} & \cdots \\ v_{2,3,4} & v_{2,3,5} & v_{2,4,5} & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ v_{1,2,n} & \cdots & v_{1,n-2,n} & v_{1,n-1,n} \\ \cdots & \cdots & \cdots & \cdots \\ v_{n-2,n-1,n} & \cdots & v_{2,n-2,n} & v_{2,n-1,n} \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \in V \otimes \frac{n(n-1)(n-2)}{6}.$$

(3.1)

**Definition 3.1** Take $E^S_V [n]$ to be the subspace of $T^S_V [n]$ generated by tensors $\otimes_{1 \leq i < j < k \leq n} (v_{i,j,k})$ with the property that there exists $1 \leq x < y < z < t \leq n$ such that $v_{x,y,z} = v_{x,y,t} = v_{x,z,t} = v_{y,z,t}$. We define

$$\Lambda^S_V [n] = \frac{T^S_V [n]}{E^S_V [n]},$$

and

$$\Lambda^S_V = \bigoplus_{n \geq 0} \Lambda^S_V [n].$$
Again, one can think of $\Lambda V^3$ as a generalization of the exterior algebra, or of the exterior GSC-operad $\Lambda V^2$.

The image of the element $\omega = \bigotimes_{1 \leq i < j < k \leq n} (v_{i,j,k}) \in T^3 V$ from 3.1 in $\Lambda V^3 [n]$ will be denoted as $\hat{\omega} = \wedge_{1 \leq i < j < k \leq n} (v_{i,j,k})$, or as

$$
\hat{\omega} = \begin{pmatrix}
v_{1,2,3} & \\
v_{1,2,4} & v_{1,3,4} & \\
v_{1,2,5} & v_{1,3,5} & v_{1,4,5} & \\
v_{2,3,5} & v_{2,4,5} & \\
\vdots & \vdots & \vdots & \\
v_{1,2,n} & \ldots & v_{1,n-2,n} & v_{1,n-1,n} & \\
\ldots & \ldots & v_{2,n-2,n} & v_{2,n-1,n} & \\
\ldots & & \ldots & \\
v_{n-2,n-1,n} & \\
\end{pmatrix} \in \Lambda V^3 [n].
$$

Let’s see a few examples of identities in $\Lambda V^3 [4]$.

**Example 3.2** Take $v = \alpha e_1 + \beta e_2 \in V_2$, using linearity we have

$$
0 = \begin{pmatrix} v \wedge v \wedge v \\ v \end{pmatrix} = \begin{pmatrix} (\alpha e_1 + \beta e_2) \wedge (\alpha e_1 + \beta e_2) \\ \alpha e_1 + \beta e_2 \end{pmatrix} = \alpha^4 \begin{pmatrix} e_1 \wedge e_1 \\ e_1 \end{pmatrix} + \beta^4 \begin{pmatrix} e_2 \wedge e_2 \\ e_2 \end{pmatrix} + \alpha^3 \beta \left( \begin{pmatrix} e_1 \wedge e_1 \\ e_1 \end{pmatrix} + \begin{pmatrix} e_1 \wedge e_2 \\ e_2 \end{pmatrix} + \begin{pmatrix} e_2 \wedge e_1 \\ e_2 \end{pmatrix} + \begin{pmatrix} e_2 \wedge e_2 \\ e_2 \end{pmatrix} \right) + \alpha^2 \beta^2 \left( \begin{pmatrix} e_1 \wedge e_1 \\ e_2 \end{pmatrix} + \begin{pmatrix} e_1 \wedge e_2 \\ e_2 \end{pmatrix} + \begin{pmatrix} e_2 \wedge e_1 \\ e_2 \end{pmatrix} + \begin{pmatrix} e_2 \wedge e_2 \\ e_2 \end{pmatrix} \right) + \alpha \beta^3 \left( \begin{pmatrix} e_1 \wedge e_1 \\ e_2 \end{pmatrix} + \begin{pmatrix} e_1 \wedge e_2 \\ e_2 \end{pmatrix} + \begin{pmatrix} e_2 \wedge e_1 \\ e_2 \end{pmatrix} + \begin{pmatrix} e_2 \wedge e_2 \\ e_2 \end{pmatrix} \right).
$$

Since this is true for all $\alpha$ and $\beta$, and $k$ is infinite, we get the following identities

$$
\begin{pmatrix} e_1 \wedge e_1 \\ e_1 \end{pmatrix} + \begin{pmatrix} e_1 \wedge e_2 \\ e_2 \end{pmatrix} + \begin{pmatrix} e_2 \wedge e_1 \\ e_1 \end{pmatrix} + \begin{pmatrix} e_2 \wedge e_2 \\ e_2 \end{pmatrix} = 0 \in \Lambda V^3 [4], \quad (3.2)
$$
and

\[
\left( e_1 \wedge e_1 \right) + \left( e_2 \wedge e_2 \right) + \left( e_1 \wedge e_2 \right) + \left( e_2 \wedge e_1 \right) = 0 \in \Lambda_{V_2}^3[4].
\]

(3.3)

More generally, we have the following result.

**Proposition 3.1** Let \( V_d \) be a vector space of dimension \( d \), and \( B_d = \{e_1, \ldots, e_d\} \) a basis for \( V_d \). Then, \( E_{V_d}^{S_3}[4] \) is the subspace of \( T_{V_d}^{S_3}[4] \) linearly generated by the following elements:

\[
\begin{pmatrix}
    e_i \\
    e_i e_i
\end{pmatrix},
\]

(3.4)

for all \( 1 \leq i \leq d \),

\[
\begin{pmatrix}
    e_i \\
    e_i e_j
\end{pmatrix} + \begin{pmatrix}
    e_i \\
    e_j e_i
\end{pmatrix} + \begin{pmatrix}
    e_i \\
    e_j e_j
\end{pmatrix},
\]

(3.5)

for all \( 1 \leq i \neq j \leq d \),

\[
\sum_{\sigma \in S_2(j,k)} \begin{pmatrix}
    e_\sigma(j) \\
    e_\sigma(k)
\end{pmatrix} + \begin{pmatrix}
    e_\sigma(j) \\
    e_\sigma(k)
\end{pmatrix},
\]

(3.7)

for all \( 1 \leq i \leq d \), \( 1 \leq j < k \leq d \), \( i \neq j \), \( i \neq k \) with the sum taken over all permutations of the set \( \{j, k\} \),

\[
\sum_{\sigma \in S_4(i,j,k,l)} \begin{pmatrix}
    e_\sigma(i) \\
    e_\sigma(j) e_\sigma(k) e_\sigma(l)
\end{pmatrix},
\]

(3.8)

for all \( 1 \leq i < j < k < l \leq d \), where \( \sigma \) runs over all permutations of the set \( \{i, j, k, l\} \).
Proof In the definition of $E^3/V_d[n]$, take $v = \alpha e_i + \beta e_j + \gamma e_k + \delta e_l$. Using linearity, we get several terms with coefficients homogeneous monomials of total degree 4 in $\alpha, \beta, \gamma$ and $\delta$. The expressions corresponding to $\alpha^4, \alpha^3\beta, \alpha^2\beta^2, \alpha^2\beta\gamma$, and $\alpha\beta^2\gamma\delta$ are, respectively, relations 3.4, 3.5, 3.6, 3.7, and 3.8. \qed

Remark 3.3 One can notice that if $d = 2$, only relations 3.4, 3.5 and 3.6 make sense. If $d = 3$, we can add 3.7, while for $d \geq 4$ all five relations make sense.

Remark 3.4 Even if $n \geq 4$, the above relations still give a set of generators for $E^3/V_d[n]$ as a vector space. More precisely, for $n \geq 4$ and $1 \leq x < y < z < t \leq n$ we can obtain an element in $E^3/V_d[n]$ by considering a generic element $\otimes_{1 \leq i < j < k < n}^n(v_{i,j,k}) \in V_d^{\otimes \frac{n(n-1)(n-2)}{6} - 4}$ that has all the entries $v_{i,j,k} \in \{e_1, \ldots, e_d\}$, and empty spots in the positions $(x, y, z), (x, y, t), (x, z, t)$ and $(y, z, t)$. In order to get an element in $E^3/V_d[n]$, one fills the empty positions with any of the five relations 3.4, 3.5, 3.6, 3.7, or 3.8.

For example, if we take $n = 6, (x, y, z, t) = (1, 3, 4, 6)$, we consider a generic element that is missing entries in the positions $(1, 3, 4), (1, 3, 6), (1, 4, 6)$, and $(3, 4, 6)$

$$
\begin{pmatrix}
 v_{1,2,3} & v_{1,2,4} & \square \\
 v_{1,2,5} & v_{1,3,5} & v_{1,4,5} \\
 v_{2,3,5} & v_{2,4,5} & v_{3,4,5} \\
 v_{1,2,6} & \square & \square \\
 v_{2,3,6} & v_{2,4,6} & v_{2,5,6} & \square \\
 v_{3,5,6} & v_{4,5,6}
\end{pmatrix} \otimes
\begin{pmatrix} \in V_d^{\otimes (20-4)} \end{pmatrix}
$$

and $v_{i,j,k} \in \{e_1, \ldots, e_d\}$. If we use relation 3.5, then we get the following equality
for all $1 \leq i \neq j \leq d$. One can easily see that in this way we get all the relations in $\Lambda_{V_d}[n]$.

Next we exhibit a system of generators for $\Lambda_{V_d}[n]$ that is indexed by $d$-partitions of the complete hypergraph $K_n^3$. This is similar to the result from [5] which gives a relation between a set of generators for $\Lambda_{V_d}[n]$ and the set of edge $d$-partitions of $K_n$ (see also Sect. 2).

Let $B_d = \{e_1, \ldots, e_d\}$ be a basis for $V_d$, we define

$$G_{B_d}[n] = \{ \otimes_{1 \leq i < j < k \leq n}(v_{i,j,k}) \in T_{V_d}^3[n] \mid v_{i,j,k} \in B_d\}.$$  

Because of linearity, it is obvious that $G_{B_d}[n]$ is a basis for $T_{V_d}^3[n]$, and so its image in $\Lambda_{V_d}[n]$ will be a system of generators.

Moreover, there exists a bijection between the elements in $G_{B_d}[n]$ and the set of $d$-partitions of the hypergraph $K_n^3$. Indeed to every element in $\omega = \otimes_{1 \leq i < j < k \leq n}(v_{i,j,k}) \in G_{B_d}[n]$, we associate the partition $\mathcal{P}_{\omega} = (\mathcal{H}_1, \ldots, \mathcal{H}_d)$ where the hyperedge $\{a, b, c\} \in \mathcal{H}_i$ if and only if $v_{a,b,c} = e_i$. It is easy to see that this map is a bijec-

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig4.png}
\caption{$\mathcal{P}^{(1)} = \mathcal{P}_\omega$ the 2-partition of $K_6^3$ associated with $\omega$}
\end{figure}
tation between $G_{S^3_Bd}[n]$ and $P_d(K_n^3)$. For a $d$-partition $P \in P_d(K_n^3)$, the corresponding element in $G_{S^3_Bd}[n]$ will be denoted by $\omega_P$.

Notice the similarities with the bijection between $G_{S^2_Bd}[n]$ and $P_d(K_n)$ described in Sect. 2.1.

**Example 3.5** Consider the element:

$$\omega = \begin{pmatrix} e_1 & \otimes \\ e_1 e_1 & e_2 \\ e_1 e_2 & e_2 e_2 & e_1 \\ e_1 e_2 e_2 e_1 \\ e_2 e_1 e_1 e_2 \\ e_2 e_1 e_2 e_1 \\ e_1 e_2 e_2 \\ e_1 e_2 e_1 \\ e_1 e_2 \\ e_1 \end{pmatrix} \in G_{S^3_Bd}[6].$$

The corresponding partition $P_\omega$ is presented in Fig. 4.

**Example 3.6** Using the above dictionary between 2-partitions of $K_n^3$ and elements from $G_{S^3_Bd}[n]$, one can translate relations 3.5 and 3.6 into relations among partitions as in Fig. 5, respectively, Fig. 6.

If $d \geq 3$, there are similar pictures for 3.7, and 3.8, but since in this paper we are mainly interested in the case $d = 2$, we only present these two pictures.

**Remark 3.7** Notice that a more natural indexing set for the $\binom{n}{3}$ positions of a tensor in $\bigotimes_{i=1}^{n} (V_{\otimes i})$ (i.e., $(i, j, k)$ with $1 \leq i < j < k \leq n$) is the set of set of hyperedges of $K_n^3$ (i.e., $(i, j, k)$ where $1 \leq i, j, k \leq n$ with $i \neq j \neq k \neq i$). However, instead of using the notation $v_{[i,j,k]}$ we will use the convention $v_{i,j,k} = v_{i,k,j} = v_{j,i,k} = v_{k,i,j} = v_{j,k,i} = v_{k,j,i}$ for all $1 \leq i < j < k \leq n$.

## 4 Main result

In this section, we consider the case $\dim_k(V_2) = 2$ and compute the dimension of $\Lambda_{V_2}^{S^3}[n]$ for all $n \geq 0$. In particular, we show that $\dim_k(\Lambda_{V_2}^{S^3}[6]) = 1$, which implies
We will use induction. When

\[ 1 \leq n \leq 4 \]

Lemma 4.1 Let \( n \geq 4 \) and take \( \omega = \bigotimes_{1 \leq i < j < k \leq n} (v_{i,j,k}) \in G_{B_2}^{S^3} [n] \subseteq V_2^{(3)} \) (i.e., \( v_{i,j,k} \in \{e_1, e_2\} \)). Assume that there are at least \( \binom{n-1}{2} + 1 \) entries equal to \( e_1 \) among the vectors \( v_{i,j,k} \), then \( \hat{\omega} = 0 \in \Lambda_{V_2}^{S^3} [n] \).

**Proof** We will use induction. When \( n = 4 \), we have that \( \binom{4-1}{2} + 1 = 4 \) and so all entries in \( \omega \) are equal to \( e_1 \), which means that \( \omega \in \mathcal{E}_{V_2}^{S^3} [4] \), and so \( \hat{\omega} = 0 \in \Lambda_{V_2}^{S^3} [4] \).

The plan is to show that we can change \( \omega \) with a sum of elements \( \omega_p \in G_{B_2}^{S^3} [n] \), such that each \( \omega_p \) is either zero in \( \Lambda_{V_2}^{S^3} [n] \), or it has at most \( n - 2 \) entries equal to \( e_1 \) in the \( n^{th} \) slice (i.e., among the entries \( \{v_{i,j,n}\}_{1 \leq i < j < n-1} \)). And so, each of these elements \( \omega_p \) has at least \( \binom{n-1}{2} + 1 - (n-2) = \binom{n-2}{2} + 1 \) entries equal to \( e_1 \) among the entries \( \{v_{i,j,k}\}_{1 \leq i < j < k < n-1} \) of \( \omega_p \). By induction, we would get that \( \hat{\omega} = 0 \in \Lambda_{V_2}^{S^3} [n] \).

Consider the \( n^{th} \)-slice of \( \omega \), which consists of all the entries in the positions \( (i, j, n) \) where \( 1 \leq i < j \leq n - 1 \). First notice that if there are more than \( n - 2 \) entries equal to \( e_1 \) in the \( n^{th} \) slice, then there exists a cycle \( (i_1, i_2, \ldots, i_q) \) where \( 1 \leq q \leq n - 1 \), \( 1 \leq i_s < n - 1 \), and \( i_s \neq i_t \) for \( s \neq t \) such that \( v_{i_1,i_2,n} = v_{i_2,i_3,n} = \cdots v_{i_{q-1},i_q,n} = v_{i_q,i_1,n} = e_1 \). In such a situation, we will say that the \( n^{th} \) slice has an \( e_1 \)-cycle of length \( q \). We will do a second induction over \( q \).

Assume that \( n > 4 \) and \( q = 3 \), then we have \( v_{i_1,i_2,n} = v_{i_2,i_3,n} = v_{i_3,i_1,n} = e_1 \) for some distinct integers \( 1 \leq i_1, i_2, i_3 \leq n - 1 \) (i.e., we have an \( e_1 \)-cycle of length 3). If \( v_{i_1,i_2,i_3} = e_1 \) then \( \omega \in \mathcal{E}_{V_2}^{S^3} [n] \) and so \( \hat{\omega} = 0 \). If \( v_{i_1,i_2,i_3} = e_2 \), then using identity 3.5 on the positions \( (i_1, i_2, i_3), (i_1, i_2, n), (i_1, i_3, n) \) and \( (i_2, i_3, n) \) we can move one of the \( e_1 \)

\[ \odot \text{ Springer} \]
entries from the $n^{th}$ slice to a lower slice which decrease the numbers of entries equal to $e_1$ in the $n^{th}$ slice. More precisely, looking only at the entries

\[
\begin{pmatrix}
v_{i_1,i_2,i_3} \\ v_{i_1,i_2,n} \\ v_{i_2,i_3,n}
\end{pmatrix},
\]

we have the identity

\[
\begin{pmatrix}
e_1 ∧ e_1 \\
e_1 ⊗ e_2 \\
e_2 \\
e_1 ⊗ e_1
\end{pmatrix}
\]

\[
\begin{pmatrix}
e_1 ∧ e_1 \\
e_1 ⊗ e_2 \\
e_2 \\
e_1 ⊗ e_1
\end{pmatrix} + \begin{pmatrix}
e_1 ∧ e_1 \\
e_1 ⊗ e_2 \\
e_2 \\
e_1 ⊗ e_1
\end{pmatrix} + \begin{pmatrix}
e_2 ∧ e_2 \\
e_2 ⊗ e_1 \\
e_1 \\
e_2 ⊗ e_1
\end{pmatrix} = 0,
\]

where the boxed tensor matrix corresponds to our initial element. Notice that all the other tensor matrix in the above expression has the entry in the position $(i_1, i_2, i_3)$ equal to $e_1$, and so fewer entries equal to $e_1$ in the $n^{th}$ slice.

Next suppose that $q = 4$, i.e., we have an $e_1$-cycle $(i_1, i_2, i_3, i_4)$ of length 4, which means that $v_{i_1,i_2,n} = v_{i_2,i_3,n} = v_{i_3,i_4,n} = v_{i_4,i_1,n} = e_1$, and $v_{i_1,i_3,n} = v_{i_2,i_4,n} = e_2$. (If any of these two entries is $e_1$, then we have a shorter $e_1$-cycle in the $n^{th}$ slice.) If $v_{i_1,i_2,i_3} = v_{i_2,i_3,i_4} = v_{i_3,i_4,i_1} = v_{i_4,i_1,i_2} = e_1$, then $\omega \in \mathcal{E}_{\mathcal{V}_2}^g$ and so $\omega$ is trivial.

So, we can assume that one of the entries $v_{i_1,i_2,i_3}, v_{i_2,i_3,i_4}, v_{i_3,i_4,i_1}, v_{i_4,i_1,i_2}$ is equal to $e_2$. Then, we will use identity 3.6 to either move one of the $e_1$ entries from the $n^{th}$ slice to a lower slice or to get a $e_1$-cycle of length 3 inside the $n^{th}$ slice. (And so we reduce our problem to the case $q = 3$.) Indeed, for example if $v_{i_1,i_2,i_3} = e_2$, then looking only at the entries

\[
\begin{pmatrix}
v_{i_1,i_2,i_3} \\ v_{i_1,i_2,n} \\ v_{i_1,i_3,n} \\ v_{i_2,i_3,n}
\end{pmatrix},
\]

from equation 3.6 we have the following identity

\[
\begin{pmatrix}
e_1 ∧ e_1 \\
e_1 ⊗ e_2 \\
e_2 \\
e_1 ⊗ e_1
\end{pmatrix} + \begin{pmatrix}
e_1 ∧ e_1 \\
e_1 ⊗ e_2 \\
e_2 \\
e_1 ⊗ e_1
\end{pmatrix} + \begin{pmatrix}
e_2 ∧ e_2 \\
e_2 ⊗ e_1 \\
e_1 \\
e_2 ⊗ e_1
\end{pmatrix} + \begin{pmatrix}
e_2 ∧ e_2 \\
e_2 ⊗ e_1 \\
e_1 \\
e_2 ⊗ e_1
\end{pmatrix} = 0,
\]

where the boxed tensor matrix corresponds to our initial element. Notice that all the other tensor matrix in the above expression has either the entry in the position $(i_1, i_2, i_3)$ equal to $e_1$ (and so less entries equal to $e_1$ in the $n^{th}$ slice) or the entry in the position $(i_1, i_3, n)$ equal to $e_1$ (and so the $e_1$-cycle $(i_1, i_3, i_4)$ of length 3 in the $n^{th}$ slice). The other cases are similar.

Finally, suppose that we have an $e_1$-cycle $(i_1, i_2, \ldots, i_q)$ in the $n^{th}$ slice with $q \geq 5$, which means that $v_{i_1,i_2,n} = v_{i_2,i_3,n} = v_{i_3,i_4,n} = v_{i_4,i_5,n} = \cdots = v_{i_q,i_1,n} = e_1$ and $v_{i_1,i_3,n} = v_{i_2,i_4,n} = v_{i_4,i_1,n} = e_2$. (If any of these three are equal to $e_1$, then there is a shorter $e_1$-cycle in the $n^{th}$ slice.)

If $v_{i_1,i_2,i_3} = e_2$ or $v_{i_2,i_3,i_4} = e_2$, then, just like above, we can use relation 3.6 to either move an $e_1$ to a lower slice or to get an $e_1$-cycle of length at most $q − 1$ in the $n^{th}$ slice. So, we can assume that $v_{i_1,i_2,i_3} = v_{i_2,i_3,i_4} = e_1$.

If $v_{i_1,i_3,i_4} = e_2$, then we can use relation 3.5 to either move an $e_1$ to a lower slice or to get a shorter $e_1$-cycle in the $n^{th}$ slice. Indeed, looking only at the entries
Let \( G \) be a group acting on \( S \), we have the following identity

\[
\left( v_{1,i_1,i_4} \otimes v_{1,i_3,n} v_{1,i_4,n} v_{1,i_3,n} \right),
\]

where the boxed tensor matrix corresponds to our initial element. Notice that all the other tensor matrix in the above expression has either the entry in the position \((i_1, i_3, i_4)\) equal to \(e_1\) (and so less entries equal to \(e_1\) in the \(n^{th}\) slice) or the entry in the position \((i_1, i_3, n)\) or \((i_1, i_4, n)\) equal to \(e_1\) (and so a shorter \(e_1\)-cycle inside the \(n^{th}\) slice). The case \(v_{1,i_2,i_4} = e_2\) is similar.

To summarize, we have that \(v_{1,i_2,i_3} = v_{1,i_2,i_4} = v_{1,i_3,i_4} = v_{1,i_3,i_4} = e_1\) and so \(\hat{\omega} = 0\). This means that we may assume that the \(n\)-th slice has at most \(n - 2\) entries equal to \(e_1\), and so by induction we get our claim. \(\square\)

The following is the \(S^3\)-version of second statement of Proposition 2.2.

**Corollary 4.2** If \(\dim_k(V_2) = 2\), then \(\dim_k(\Lambda_{V_2}^{S^3}[n]) = 0\) for all \(n \geq 7\).

**Proof** Take \(n \geq 7\) and consider \(\omega = \otimes_{1 \leq i < j < k \leq n} (v_{i,j,k}) \in G_{B_2}^{S^3}[n] \subseteq V_2^{\otimes(\binom{n}{3})}\) (i.e., \(v_{i,j,k} \in \{e_1, e_2\}\)). Since \(n \geq 7\), we have that \(\binom{n}{3} > 2\binom{n-1}{2}\) and so, without loss of generality, we may assume that \(\omega\) has at least \(\binom{n-1}{2} + 1\) entries equal to \(e_1\). From Lemma 4.1, we know that \(\hat{\omega} = 0 \in \Lambda_{V_2}^{S^3}[n]\) is trivial. \(\square\)

**Corollary 4.3** Let \(\omega = \otimes_{1 \leq i < j < k \leq 6} (v_{i,j,k}) \in G_{B_2}^{S^3}[6] \subseteq V_2^{\otimes 20}\), and assume that there are at least 11 entries equal to \(e_1\) among the \(v_{i,j,k}\), then \(\hat{\omega} = 0 \in \Lambda_{V_2}^{S^3}[6]\).

**Proof** It follows directly from Lemma 4.1 for \(n = 6\). \(\square\)

**Remark 4.4** Notice that if \(\omega \in G_{B_2}^{S^3}[6]\) such that \(\hat{\omega} \neq 0 \in \Lambda_{V_2}^{S^3}[6]\), then by Corollary 4.3 the corresponding partition \(\mathcal{P}_\omega = (\mathcal{H}_1, \mathcal{H}_2)\) of \(K^{S^3}_d\) must be homogeneous (i.e., \(\omega\) has ten entries equal to \(e_1\) and one entry equal to \(e_2\)).

**Lemma 4.5** (i) There is an action of the symmetric group \(S_n\) on \(\Lambda_{V_2}^{S^3}[n]\) given by

\[
\sigma \cdot \Lambda_{1 \leq i < j < k \leq n}^{S^3}(v_{i,j,k}) = \Lambda_{1 \leq i < j < k \leq n}^{S^3}(v_{\sigma^{-1}(i), \sigma^{-1}(j), \sigma^{-1}(k)}).
\]

(ii) There is an action of the group \(GL_d(k)\) on \(\Lambda_{V_2}^{S^3}[n]\) given by

\[
T \cdot \Lambda_{1 \leq i < j < k \leq n}^{S^3}(v_{i,j,k}) = \Lambda_{1 \leq i < j < k \leq n}^{S^3}(T(v_{i,j,k})).
\]
If $P$ is a homogeneous 2-partition of $K^3_6$ and $T \in GL_2(k)$, then

$$T \cdot \hat{\varpi} P = \det(T)^{10} \hat{\varpi} P.$$ 

In particular, there exists an action of the group $S_6 \times S_2$ on $\Lambda^{S^3_6}_V[6]$, where the $S_6$ action is the one described in 1), and $S_2$ is the subgroup on $GL_2(k)$ generated by $\tau : V_2 \rightarrow V_2$, $\tau(e_1) = e_2$ and $\tau(e_2) = e_1$.

**Proof** The first two statements follow directly from the definition of $\Lambda^{S^3_6}_S V_d[6]$. For statement (iii), first notice that it is enough to check it for diagonal and elementary transformations. Take $T_1, T_2 : V_2 \rightarrow V_2$,

$$T_1(e_s) = \begin{cases} \lambda e_1 & s = 1 \\ e_2 & s = 2, \end{cases}$$

$$T_2(e_s) = \begin{cases} e_1 & s = 1 \\ e_2 + \lambda e_1 & s = 2, \end{cases}$$

and consider $P = (\mathcal{H}_1, \mathcal{H}_2)$ a homogeneous partition of $K^3_6$. Then,

$$T_1 \cdot \hat{\varpi}_P = \lambda^{10} \hat{\varpi}_P = \det(T_1)^{10} \hat{\varpi}_P$$

where the first equality is true because the tensor product is linear in each component, and because there are exactly ten entries equal to $e_1$ in $\varpi_P$.

Next we have

$$T_2 \cdot \hat{\varpi}_P = \sum_{j=1}^{2^{10}} \lambda^{j} \hat{\varpi}_{P^j},$$

where the sum is taken over all 2-partitions $P^j = (\mathcal{H}^j_1, \mathcal{H}^j_2)$ of $K^3_6$ with the property that $E(\mathcal{H}_1) \subseteq E(\mathcal{H}^j_1)$ and $k_j = |E(\mathcal{H}^j_1)| - 10$. Obviously, the only homogeneous partition among the $P^j$’s is our initial partition $P$, and so from Corollary 4.3 we get

$$T_2 \cdot \hat{\varpi}_P = \hat{\varpi}_P = \det(T_2)^{10} \hat{\varpi}_P,$$

which completes our proof. \qed

As mentioned above, if $\omega \in \mathcal{G}^{S^3_6}_{B_2}[6]$ whose image in $\Lambda^{S^3_6}_V[6]$ is nonzero, then its corresponding partition $P_\omega$ must be homogeneous. There are exactly $\binom{20}{10} = 184756$ homogeneous 2-partitions of $K^3_6$, but not all of them give nonzero elements in $\Lambda^{S^3_6}_V[6]$.

For example, any partition $P = (\mathcal{H}_1, \mathcal{H}_2)$ for which we can find $1 \leq x < y < z < t \leq 6$ such that \{x, y, z\}, \{x, y, t\}, \{x, z, t\}, and \{y, z, t\} $\in E(\mathcal{H}_1)$ have the property that $\hat{\varpi}_P = 0 \in \Lambda^{S^3_6}_V[6]$. More generally, we have the following.
Lemma 4.6 Let $\mathcal{P} = (\mathcal{H}_1, \mathcal{H}_2)$ be a homogeneous 2-partition of $K_6^3$. If $\mathcal{P}$ satisfies one of the following conditions, then $\omega_{\mathcal{P}} = 0 \in \Lambda_{V_2}^{S_3}[6]$.

(i) There exist four distinct integers $1 \leq x, y, z, t \leq 6$ and $1 \leq i \leq 2$ such that $\{x, y, z\}, \{x, y, t\}, \{x, z, t\}$, and $\{y, z, t\} \in E(\mathcal{H}_i)$.

(ii) There exist five distinct integers $1 \leq x, y, z, u, v \leq 6$ and $1 \leq i \leq 2$ such that $\{x, y, z\}, \{x, y, t\}, \{x, z, t\}, \{y, z, u\}, \{y, t, u\}$, and $\{z, t, u\} \in E(\mathcal{H}_i)$.

(iii) There exist six distinct integers $1 \leq x, y, z, u, v, y \leq 6$ and $1 \leq i \leq 2$ such that $\{x, y, u\}, \{x, y, v\}, \{y, z, u\}, \{x, z, t\}, \{y, t, v\}, \{y, z, t\}$, and $\{x, z, t\} \in E(\mathcal{H}_i)$.

(iv) There exist six distinct integers $1 \leq x, y, z, t, u, v \leq 6$ and $1 \leq i \leq 2$ such that $\{x, y, z\}, \{x, y, t\}, \{x, z, t\}, \{x, y, u\}, \{y, t, u\}, \{x, y, v\}, \{y, z, v\}, \{z, t, v\}$, and $\{t, u, v\} \in E(\mathcal{H}_i)$.

Proof We only give details for (ii), and the other statements are similar. Using relation 3.4 for $(x, y, z, t)$, we get that

$$\omega_{\mathcal{P}} = -\omega_{\mathcal{P}(1)} - \omega_{\mathcal{P}(2)} - \omega_{\mathcal{P}(3)},$$

where $\mathcal{P}(i) = (\mathcal{H}_1^{(i)}, \mathcal{H}_2^{(i)})$ are distinct homogeneous 2-partitions of $K_6^3$ such that $\{y, z, t\} \in E(\mathcal{H}_i^{(j)})$, and $\mathcal{P}(j)$ coincide with $\mathcal{P}$ everywhere except maybe on the hyperedges $\{x, y, z\}, \{x, y, t\}, \{x, z, t\}$, and $\{y, z, t\}$.

Notice that $\{y, z, u\}, \{y, t, u\}, \{z, t, u\}$, and $\{y, z, t\} \in E(\mathcal{P}(i))$ for $1 \leq j \leq 3$, and so by (i) we get that $\omega_{\mathcal{P}(i)} = 0$ for $1 \leq j \leq 3$ which proves our statement. \qed

Remark 4.7 There are 184 756 homogeneous 2-partitions of $K_6^3$. One can use Lemma 4.6 and MATLAB to sort out the trivial partitions. After this process, we are still left with 13 644 nontrivial homogeneous 2-partitions of $K_6^3$, which we will denote by $\mathcal{P}_h^{h,nt}(K_6^3)$ (i.e., those partitions that are not listed in Lemma 4.6).

Recall that on $\mathcal{P}_h^h(K_6^3)$ there is a natural action of the group $S_6 \times S_2$. It is obvious that the action of $S_6 \times S_2$ restricts to $\mathcal{P}_h^{h,nt}(K_6^3)$. Using MATLAB, one can give a classification of the elements in $\mathcal{P}_h^{h,nt}(K_6^3)$ under this action and obtain 20 equivalence classes. The details are presented in “Appendix.”

Definition 4.8 Let $\mathcal{P} = (\mathcal{H}_1, \mathcal{H}_2) \in \mathcal{P}_h^h(K_6^3)$ and $1 \leq x < y < z < t \leq 6$. We denote by $\text{Pair}(\mathcal{P}, (x, y, z, t))$ the set of all homogeneous 2-partitions of $K_6^3$ that coincide with $\mathcal{P}$ except maybe on the hyperedges $\{x, y, z\}, \{x, y, t\}, \{x, z, t\}$, and $\{y, z, t\}$.

Remark 4.9 We have one of the following three cases.

(i) $\text{Pair}(\mathcal{P}, (x, y, z, t))$ has one single element only if all the hyperedges $\{x, y, z\}, \{x, y, t\}, \{x, z, t\}$, and $\{y, z, t\}$ belong to $\mathcal{H}_1$, or all of them belong to $\mathcal{H}_2$.

(ii) If three of the hyperedges $\{x, y, z\}, \{x, y, t\}, \{x, z, t\}$, and $\{y, z, t\}$ belong to $\mathcal{H}_1$, and one to $\mathcal{H}_2$ (or the other way around), then $\text{Pair}(\mathcal{P}, (x, y, z, t))$ has four elements.

(iii) If two of the hyperedges $\{x, y, z\}, \{x, y, t\}, \{x, z, t\}$, and $\{y, z, t\}$ belong to $\mathcal{H}_1$ and two belong to $\mathcal{H}_2$, then $\text{Pair}(\mathcal{P}, (x, y, z, t))$ has six elements.
With these notations, we have the following result.

**Lemma 4.10** There exists a unique map \( \varepsilon^{S_3}_2 : \mathcal{P}_2^h(K_6^3) \to \{-4, -1, 0, 1\} \) such that

(i) if \( \mathcal{P} \in \mathcal{P}_2^h(K_6^3) \) and \( 1 \leq x < y < z < t \leq 6 \) then

\[
\sum_{\mathcal{K} \in \text{Pair}(\mathcal{P},(x,y,z,t))} \varepsilon^{S_3}_2(\mathcal{K}) = 0,
\]

(ii) \( \varepsilon^{S_3}_2 \) takes value 1 on the partition \( \mathcal{P}^{(1)} \) from Fig. 4.

**Proof** Using relations 3.4, 3.5, 3.6 and Remark 4.9, one can write a system of linear equation that gives all the solutions for condition (i). Using MATLAB, one can show that the corresponding matrix has co-rank equal to 1, and so because of condition (ii), we get a unique solution. \( \square \)

**Remark 4.11** It is rather interesting to notice that \( \varepsilon^{S_3}_2(\mathcal{P}) \neq 0 \) if and only if \( \mathcal{P} \in \mathcal{P}_{2,nt}^h(K_3^3) \). This is somehow similar to the results from [5] where \( \varepsilon^{S^2}_d((\Gamma_1, \Gamma_2, \ldots, \Gamma_d)) \neq 0 \) if and only if \( (\Gamma_1, \Gamma_2, \ldots, \Gamma_d) \) was cycle free. One can see that the partitions listed in Lemma 4.6 have a copy of \( S^2 \) (made of hyperedges/faces) either in \( \mathcal{H}_1 \) or in \( \mathcal{H}_2 \). However, unlike \( \varepsilon^{S^2}_d \), the map \( \varepsilon^{S_3}_2 \) takes also the value \(-4\) which is a rather unexpected fact. A table of values of \( \varepsilon^{S_3}_2 \) on all elements in \( \mathcal{P}_{2,nt}^h(K_6^3) \) is given in “Appendix.”

The map \( \varepsilon^{S_3}_2 \) plays a role similar to the signature of a permutation, and with the \( \varepsilon^{S^2}_d \) map from [5]. It allows us to define a determinant-like function \( \det^{S_3} : V_2^{\otimes 20} \to k \).

More precisely, take \( v_{i,j,k} = \alpha_{i,j,k}e_1 + \beta_{i,j,k}e_2 \in V_2 \) for \( 1 \leq i < j < k \leq 6 \). For a 2-partition \( (\mathcal{H}_1, \mathcal{H}_2) \in \mathcal{P}_{2,nt}^h(K_6^3) \), define

\[
M^{S_3}_{(\mathcal{H}_1, \mathcal{H}_2)}((v_{i,j,k})_{1 \leq i < j < k \leq 6}) = \prod_{\underline{u},v_1,w_1 \in E(\mathcal{H}_1)} \alpha_{u_1,v_1,w_1} \prod_{\underline{u}_2,v_2,w_2 \in E(\mathcal{H}_2)} \beta_{u_2,v_2,w_2}.
\]

Next, take

\[
\text{Det}^{S_3} : V_2^{20} \to k
\]

determined by

\[
\text{Det}^{S_3}((v_{i,j,k})_{1 \leq i < j < k \leq 6}) = \sum_{(\mathcal{H}_1, \mathcal{H}_2) \in \mathcal{P}_{2,nt}^h(K_6^3)} \varepsilon^{S_3}_2(\mathcal{H}_1, \mathcal{H}_2) M^{S_3}_{(\mathcal{H}_1, \mathcal{H}_2)} \times \mathcal{P}^{(1)}_{2,nt}(K_6^3) \quad (4.1)
\]

Notice that \( \text{Det}^{S_3} \) is multi-linear and so we get a linear map \( \det^{S_3} : V_2^{\otimes 20} \to k \). The following theorem is the main result of this paper.
Theorem 4.12 The map $\det^S : V_2^{\otimes 20} \to k$ is the unique (up to a constant) nontrivial linear map on $V_2^{\otimes 20}$ with the property that $\det^S(\otimes_{1 \leq i < j < k \leq 6}(v_i, j, k)) = 0$ if there exist $1 \leq x < y < z < t \leq 6$ such that $v_{x,y,z} = v_{x,y,t} = v_{x,z,t} = v_{y,z,t}$. In particular, we have a unique (up to a constant) nontrivial linear map $\det^S : \Lambda_{V_2}^S[6] \to k$.

Proof This follows directly from Proposition 3.1 and Lemma 4.10.

Remark 4.13 An alternative proof (which does not use MATLAB) for the existence of the $\det^S$ map is given in “Appendix.” That approach is based on the fact that $\det^S$ is invariant under the action of $SL_2(k)$. However, that approach does not prove the uniqueness.

Remark 4.14 It is worth noticing the similarities between formulas 2.1, 2.2, and 4.1. This suggests a generalization to $S^r$ for any $r$, which will be discussed in the next section.

Corollary 4.15 If $\dim_k(V_2) = 2$, then $\dim_k(\Lambda_{V_2}^{S_3}[6]) = 1$.

Proof It follows from Theorem 4.12 and the definition of $\Lambda_{V_2}^{S_3}[6]$.

To conclude this section, we have the following complete list of $\dim_k(\Lambda_{V_2}^{S_3}[n])$.

Proposition 4.1 Let $V_2$ be a vector space of dimension two. Then, we have:

(i) $\dim_k(\Lambda_{V_2}^{S_3}[0]) = \dim_k(\Lambda_{V_2}^{S_3}[1]) = \dim_k(\Lambda_{V_2}^{S_3}[2]) = 1$,
(ii) $\dim_k(\Lambda_{V_2}^{S_3}[3]) = 2$,
(iii) $\dim_k(\Lambda_{V_2}^{S_3}[4]) = 11$,
(iv) $\dim_k(\Lambda_{V_2}^{S_3}[5]) = 62$,
(v) $\dim_k(\Lambda_{V_2}^{S_3}[6]) = 1$,
(vi) $\dim_k(\Lambda_{V_2}^{S_3}[n]) = 0$ if $n \geq 7$.

Proof Most of the results are either trivial or were already covered in this section. The only interesting cases are $n = 4$ and $n = 5$.

Recall that $\mathcal{G}_{D_2}^{S_3}[n]$ is in bijection with $\mathcal{P}_2(K_n^3)$. We denote by $\mathcal{P}_2(K_n^3)^{p,q}$ those 2-partitions $(\mathcal{H}_1, \mathcal{H}_2)$ of $K_n^3$ with the property that $|E(\mathcal{H}_1)| = p$ and $|E(\mathcal{H}_2)| = q$. (Obviously, we must have that $p+q = \binom{n}{3}$.) We denote by $\Lambda_{V_2}^{S_3}[n]^{p,q}$ the corresponding subspace in $\Lambda_{V_2}^{S_3}[n]$.

Since the relations in $\Lambda_{V_2}^{S_3}[n]$ are homogeneous in the number of $e_1$’s and $e_2$’s, it follows that if $\omega_{p,q} \in \Lambda_{V_2}^{S_3}[n]^{p,q}$, and $\sum \omega_{p,q} = 0 \in \Lambda_{V_2}^{S_3}[n]$, then we must have that $\omega_{p,q} = 0$ for all $p, q$.

Let’s first consider the case $n = 4$. If $\mathcal{P} \in \mathcal{P}_2(K_4^3)^{4,0}$, or $\mathcal{P} \in \mathcal{P}_2(K_4^3)^{0,4}$, then obviously $\hat{x}_\mathcal{P} = 0 \in \Lambda_{V_2}^{S_3}[4]$.

There are four elements $\mathcal{P} \in \mathcal{P}_2(K_4^3)^{3,1}$, and the only relation among them is listed in Fig. 5. This means that we get three linearly independent vectors that generate $\Lambda_{V_2}^{S_3}[4]^{3,1}$. The case $\Lambda_{V_2}^{S_3}[4]^{1,3}$ is similar.
There are six elements \( P \in \mathcal{P}_2(K^3_4) \), and the only relation among them is listed in Fig. 6. This means that we get five linearly independent vectors that generate \( \Lambda^3_{V_2}[4] \). To conclude, we have

\[
\dim_k(\Lambda^3_{V_2}[4]) = \dim_k(\Lambda^3_{V_2}[4]^{3,1}) + \dim_k(\Lambda^3_{V_2}[4]^{1,3}) + \dim_k(\Lambda^3_{V_2}[4]^{2,2}) = 3 + 3 + 5 = 11.
\]

The case \( n = 5 \) is similar but computationally heavier, so we had to use MATLAB. The interesting cases are \((p, q) \in \{(6, 4), (5, 5), (4, 6)\}\). One can show that \( \dim_k(\Lambda^3_{V_2}[5]^{6,4}) = 15 = \dim_k(\Lambda^3_{V_2}[5]^{4,6}) \) and \( \dim_k(\Lambda^3_{V_2}[5]^{5,5}) = 32 \). This gives that \( \dim_k(\Lambda^3_{V_2}[5]) = 15 + 15 + 32 = 62 \).

\[ \square \]

5 Some remarks

In the previous section, we computed \( \dim_k\left( \Lambda^3_{V_2}[n] \right) \) for all \( n \geq 0 \). It would be interesting to understand how \( \Lambda^3_{V_d}[n] \) behaves for any \( d \). Based on the results from this paper, we have the following question.

**Question 5.1** Suppose \( \dim_k(V_d) = d \). Is it true that \( \dim_k\left( \Lambda^3_{V_d}[n] \right) = 0 \) for \( n > 3d \)? Is it true that \( \dim_k\left( \Lambda^3_{V_d}[3d] \right) = 1 \)?

Note that this question is in the spirit of the results from [7]. The particular challenge here is that even for the simplest case \( d = 3 \), the computation is less feasible, as there are on the order of \( 1.17 \times 10^{38} \) possible homogeneous 3-partitions for the hypergraph \( K^3_9 \), and our proof for Theorem 4.15 is computational. A different, more theoretical approach is necessary in order to solve this problem.

**Remark 5.2** Recall from [5] that if \( \Gamma = (\Gamma_1, \ldots, \Gamma_d) \) is a homogeneous \( d \)-partitions of \( K_{2d} \) such that \( \text{f} \Gamma \neq 0 \in \Lambda^3_{V_d}[2d] \), then \( \Gamma \) must be cycle-free. It would be interesting to find a similar result for elements in \( \mathcal{O}^3_{B_d}[3d] \). More precisely, let \( \mathcal{P} = (\mathcal{H}_1, \ldots, \mathcal{H}_d) \) be a homogeneous \( d \)-partition of the hypergraph \( K^3_{3d} \), find a combinatorial property of \( \mathcal{P} \) such that \( \text{f} \mathcal{P} \neq 0 \).

**Remark 5.3** Since \( \text{det}^3 \) is invariant under the action of \( SL_2(k) \), we know from general theory of invariant functions (see [9]) that condition \( \text{det}^3(\otimes_{1 \leq i < j < k \leq 6}(v_{i,j,k})) = 0 \) must have a geometrical interpretation. It would be interesting to find an explicit description similar to the results from [8] for the \( \text{det}^2 \) map.

One can try to generalize the construction from this paper to any sphere \( S' \) as follows. Take

\[
T^S_{S'}[n] = V^{\otimes(\cdot)}
\]
and define $E^S V [n]$ to be the subspace of $T^S V [n]$ generated by simple tensors

$$\otimes 1 \leq i_1 < i_2 < \cdots < i_r \leq n (v_{i_1}, i_2, \ldots, i_r)$$

with the property that there exists $1 \leq x_1 < x_2 < \cdots < x_r + 1 \leq n$ such that

$$v_{x_1, x_2, \ldots, x_r} = v_{x_1, x_2, \ldots, x_r-1, x_r+1} = \cdots = v_{x_1, x_3, \ldots, x_r+1} = v_{x_2, x_3, \ldots, x_r+1}.$$

**Definition 5.4** With the above notations, we define

$$\Lambda^S V [n] = T^S V [n] / E^S V [n],$$

and

$$\Lambda^S V = \bigoplus_{n \geq 0} \Lambda^S V [n].$$

**Question 5.5** Suppose $\dim(V_d) = d$. Is it true that $\dim_k \left( \Lambda^S V_d [n] \right) = 0$ for $n > rd$?

Is it true that $\dim_k \left( \Lambda^S V_d [rd] \right) = 1$?

**Remark 5.6** As we recalled in introduction, $\Lambda_V$ has algebra structure on it, and $\Lambda^S V$ is a GSC-operad. It is natural to ask whether there is more structure on $\Lambda^S V$. As far as we can tell there is no obvious algebra, or GSC-operad structure on it. We expect that some operad-like structure exists on $\Lambda^S V$, and we plan to investigate this problem in a follow-up paper.

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**Data availability** The datasets generated during the current study are available from the corresponding author on reasonable request.

**Appendix A: An explicit formula for $\det^S 3$**

It follows from Lemma 4.5 that the map $\det^S 3$ is invariant under the action of $SL_2(k)$ on $V_2$. From the general results of invariant theory (see [9]), it follows that $\det^S 3 (\otimes 1 \leq i < j < k \leq 6 (v_{i, j, k}))$ can be written as sum of product of determinants of two by two matrices with columns consisting of the vectors $v_{i, j, k}$. In this section, we give an explicit formula for $\det^S 3$. This is also an alternative proof for the existence of the map $\det^S 3$. 
For two vectors \( u = ae_1 + be_2 \) and \( v = ce_1 + de_2 \), we denote by \([u, v]\) the determinant of the matrix that has the first column equal to \(u\) and the second column equal to \(v\), that is,

\[
[u, v] = ad - bc.
\]

**Proposition 5.1** Let \( v_{i,j,k} \in V_2 \) for all \( 1 \leq i < j < k \leq 6 \), then we have

\[
\text{det}^S \left( \bigotimes_{1 \leq i < j < k \leq 6} (v_{i,j,k}) \right) = [v_{1,2,3}, v_{2,3,4}][v_{1,2,4}, v_{2,4,5}][v_{1,2,5}, v_{2,5,3}][v_{1,2,6}, v_{2,6,5}][v_{1,3,4}, v_{3,4,6}]
\]

\[
+ [v_{1,3,5}, v_{3,5,4}][v_{1,3,6}, v_{3,6,2}][v_{1,4,5}, v_{4,5,6}][v_{1,4,6}, v_{4,6,2}][v_{1,5,6}, v_{5,6,3}]
\]

\[
+ [v_{1,2,3}, v_{2,3,5}][v_{1,2,4}, v_{2,4,3}][v_{1,2,5}, v_{2,5,4}][v_{1,2,6}, v_{2,6,5}][v_{1,3,4}, v_{3,4,6}]
\]

\[
+ [v_{1,3,5}, v_{3,5,6}][v_{1,3,6}, v_{3,6,2}][v_{1,4,5}, v_{4,5,3}][v_{1,4,6}, v_{4,6,2}][v_{1,5,6}, v_{5,6,4}]
\]

\[
+ [v_{1,2,3}, v_{2,3,5}][v_{1,2,4}, v_{2,4,3}][v_{1,2,5}, v_{2,5,4}][v_{1,2,6}, v_{2,6,4}][v_{1,3,4}, v_{3,4,5}]
\]

\[
+ [v_{1,3,5}, v_{3,5,6}][v_{1,3,6}, v_{3,6,2}][v_{1,4,5}, v_{4,5,3}][v_{1,4,6}, v_{4,6,5}][v_{1,5,6}, v_{5,6,2}]
\]

\[
+ [v_{1,2,3}, v_{2,3,4}][v_{1,2,4}, v_{2,4,5}][v_{1,2,5}, v_{2,5,3}][v_{1,2,6}, v_{2,6,4}][v_{1,3,4}, v_{3,4,6}]
\]

\[
+ [v_{1,3,5}, v_{3,5,6}][v_{1,3,6}, v_{3,6,2}][v_{1,4,5}, v_{4,5,3}][v_{1,4,6}, v_{4,6,5}][v_{1,5,6}, v_{5,6,2}]
\]

\[
+ [v_{1,2,3}, v_{2,3,4}][v_{1,2,4}, v_{2,4,5}][v_{1,2,5}, v_{2,5,3}][v_{1,2,6}, v_{2,6,3}][v_{1,3,4}, v_{3,4,5}]
\]

\[
+ [v_{1,3,5}, v_{3,5,6}][v_{1,3,6}, v_{3,6,2}][v_{1,4,5}, v_{4,5,3}][v_{1,4,6}, v_{4,6,5}][v_{1,5,6}, v_{5,6,2}]
\]

\[
+ [v_{1,2,3}, v_{2,3,4}][v_{1,2,4}, v_{2,4,6}][v_{1,2,5}, v_{2,5,6}][v_{1,2,6}, v_{2,6,3}][v_{1,3,4}, v_{3,4,5}]
\]

\[
+ [v_{1,3,5}, v_{3,5,6}][v_{1,3,6}, v_{3,6,2}][v_{1,4,5}, v_{4,5,3}][v_{1,4,6}, v_{4,6,5}][v_{1,5,6}, v_{5,6,3}]
\]

\[
+ [v_{1,2,3}, v_{2,3,6}][v_{1,2,4}, v_{2,4,3}][v_{1,2,5}, v_{2,5,6}][v_{1,2,6}, v_{2,6,4}][v_{1,3,4}, v_{3,4,5}]
\]

\[
+ [v_{1,3,5}, v_{3,5,2}][v_{1,3,6}, v_{3,6,5}][v_{1,4,5}, v_{4,5,3}][v_{1,4,6}, v_{4,6,5}][v_{1,5,6}, v_{5,6,3}]
\]

\[
+ [v_{1,2,3}, v_{2,3,6}][v_{1,2,4}, v_{2,4,3}][v_{1,2,5}, v_{2,5,4}][v_{1,2,6}, v_{2,6,4}][v_{1,3,4}, v_{3,4,6}]
\]

\[
+ [v_{1,3,5}, v_{3,5,2}][v_{1,3,6}, v_{3,6,5}][v_{1,4,5}, v_{4,5,3}][v_{1,4,6}, v_{4,6,5}][v_{1,5,6}, v_{5,6,2}]
\]

\[
+ [v_{1,2,3}, v_{2,3,4}][v_{1,2,4}, v_{2,4,6}][v_{1,2,5}, v_{2,5,4}][v_{1,2,6}, v_{2,6,3}][v_{1,3,4}, v_{3,4,5}]
\]

\[
+ [v_{1,3,5}, v_{3,5,2}][v_{1,3,6}, v_{3,6,5}][v_{1,4,5}, v_{4,5,3}][v_{1,4,6}, v_{4,6,3}][v_{1,5,6}, v_{5,6,2}]
\]

\[
+ [v_{1,2,3}, v_{2,3,4}][v_{1,2,4}, v_{2,4,6}][v_{1,2,5}, v_{2,5,3}][v_{1,2,6}, v_{2,6,3}][v_{1,3,4}, v_{3,4,6}]
\]

\[
+ [v_{1,3,5}, v_{3,5,4}][v_{1,3,6}, v_{3,6,5}][v_{1,4,5}, v_{4,5,2}][v_{1,4,6}, v_{4,6,5}][v_{1,5,6}, v_{5,6,2}]
\]

\[
+ [v_{1,2,3}, v_{2,3,6}][v_{1,2,4}, v_{2,4,3}][v_{1,2,5}, v_{2,5,4}][v_{1,2,6}, v_{2,6,4}][v_{1,3,4}, v_{3,4,5}]
\]

\[
+ [v_{1,3,5}, v_{3,5,4}][v_{1,3,6}, v_{3,6,4}][v_{1,4,5}, v_{4,5,2}][v_{1,4,6}, v_{4,6,5}][v_{1,5,6}, v_{5,6,3}]
\]
Theorem 3.1 implies that $\lambda(v) = 0$ because 

$$\mathcal{H}(v) \subseteq \{0\}.$$ 

Proof We denote by $B(\otimes_{1 \leq i < j < k \leq 6}(v_i, j, k))$ the right-hand side of the above equality. In order to prove $B = \det^{S^3}$, one can use the universality property of the $\det^{S^3}$ map. It is easy to check that $B(\omega_{P(1)}) = 1$.

Next, we want to show that if $\otimes_{1 \leq i < j < k \leq 6}(v_i, j, k) \in V_2^{\otimes 20}$ such that there exist $1 \leq x < y < z < t \leq 6$ with the property that $v_{x,y,z} = v_{x,y,t} = v_{x,z,t} = v_{y,z,t}$, then $B(\otimes_{1 \leq i < j < k \leq 6}(v_i, j, k)) = 0$.

First, notice that $B$ is invariant under the action of the subgroup $G \subseteq S_6$ that is generated by $(2,3), (3,4), (4,5)$, and $(5,6)$. Because of this fact, it is enough to check the universality property for $(x, y, z, t) = (1, 2, 3, 4)$, and for $(x, y, z, t) = (2, 3, 4, 5)$.

Case I If $(x, y, z, t) = (1, 2, 3, 4)$, then $v_{1,2,3} = v_{1,2,4} = v_{1,3,4} = v_{2,3,4}$. It is easy to see that all the terms in $B(\otimes_{1 \leq i < j < k \leq 6}(v_i, j, k))$ are equal to 0; for example, the first term is equal to zero because $[v_{1,2,3}, v_{2,3,4}] = 0$.

Case II If $(x, y, z, t) = (2, 3, 4, 5)$, then $v_{2,3,4} = v_{2,3,5} = v_{2,4,5} = v_{3,4,5}$. In this situation, all the terms will cancel in pairs. For example, if we look to the first term and thirty-seventh term, we have the following expressions, respectively:

$$[v_{1,2,3}, v][v_{1,2,4}, v][v_{1,2,5}, v][v_{1,2,6}, v][v_{1,3,4}, v][v_{1,3,5}, v][v_{1,3,6}, v][v_{3,6,2}, v][v_{4,5,6}, v][v_{4,6,2}, v][v_{5,6,2}, v].$$
and

$$-[v_{1,2,3}, v][v_{1,2,4}, v][v_{1,2,5}, v][v_{1,3,4}, v][v_{1,3,5}, v][v_{1,3,6}, v][v_{1,4,5}, v][v_{1,4,6}, v][v_{1,5,6}, v][v_{1,6,7}, v].$$

So, we get these terms to sum up to zero. The rest of the matching pairs are given in the following table:

| Term | Matching term |
|------|---------------|
| 1    | 37            |
| 2    | 43            |
| 3    | 31            |
| 4    | 44            |
| 5    | 32            |
| 6    | 38            |
| 7    | 47            |
| 8    | 41            |
| 9    | 26            |
| 10   | 55            |
| 11   | 25            |
| 12   | 56            |
| 13   | 45            |
| 14   | 35            |
| 15   | 28            |

| Term | Matching term |
|------|---------------|
| 16   | 57            |
| 17   | 27            |
| 18   | 58            |
| 19   | 39            |
| 20   | 33            |
| 21   | 30            |
| 22   | 59            |
| 23   | 29            |
| 24   | 60            |
| 25   | 11            |
| 26   | 9             |
| 27   | 17            |
| 28   | 15            |
| 29   | 23            |
| 30   | 21            |

| Term | Matching term |
|------|---------------|
| 31   | 3             |
| 32   | 5             |
| 33   | 20            |
| 34   | 51            |
| 35   | 14            |
| 36   | 53            |
| 37   | 1             |
| 38   | 6             |
| 39   | 19            |
| 40   | 49            |
| 41   | 8             |
| 42   | 54            |
| 43   | 2             |
| 44   | 4             |
| 45   | 13            |

| Term | Matching term |
|------|---------------|
| 46   | 50            |
| 47   | 7             |
| 48   | 52            |
| 49   | 40            |
| 50   | 46            |
| 51   | 34            |
| 52   | 48            |
| 53   | 36            |
| 54   | 42            |
| 55   | 10            |
| 56   | 12            |
| 57   | 16            |
| 58   | 18            |
| 59   | 22            |
| 60   | 24            |

One can see that the pairs of terms have opposite signs. So, all the terms will cancel in pairs, which proves our statement. \( \square \)
Appendix B: Equivalence classes of 2-partitions under the $S_6 \times S_2$ Action

As discussed earlier in the paper, there are 184,756 homogeneous 2-partitions of the hypergraph $K_6^3$. Using MATLAB, one can show that 13,644 of them are nontrivial (see Lemma 4.6). We also know that there is an action of the group $S_6 \times S_2$ on $\mathcal{P}_2^{h,n} (K_6^3)$. In this section, we present the 20 equivalence classes of $\mathcal{P}_2^{h,n} (K_6^3)$ under the action of $S_6 \times S_2$ and the value of $\varepsilon_2^{S_3}$ on each element in $\mathcal{P}_2^{h,n} (K_6^3)$. All results in this section were obtained using MATLAB.

A summary of this information is presented in the following table, where $\mathcal{P}^{(i)}$ for $1 \leq i \leq 20$ are representatives of the 20 equivalence classes:

| Partition $\mathcal{P}^{(i)}$ | $\varepsilon_2^{S_3} (\mathcal{P}^{(i)})$ | Orbit size |
|-------------------------------|-------------------------------------|------------|
| $\mathcal{P}^{(1)}$           | 1                                   | 1440       |
| $\mathcal{P}^{(2)}$           | $-1$                                | 240        |
| $\mathcal{P}^{(3)}$           | $-1$                                | 1440       |
| $\mathcal{P}^{(4)}$           | 1                                   | 360        |
| $\mathcal{P}^{(5)}$           | $-1$                                | 1440       |
| $\mathcal{P}^{(6)}$           | 1                                   | 1440       |
| $\mathcal{P}^{(7)}$           | $-1$                                | 720        |
| $\mathcal{P}^{(8)}$           | 1                                   | 1440       |
| $\mathcal{P}^{(9)}$           | $-1$                                | 1440       |
| $\mathcal{P}^{(10)}$          | 1                                   | 360        |

In Figs. 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25 and 26, we present 20 partitions that are representatives for the equivalences classes under the action of $S_6 \times S_2$. To understand these pictures better, we consider Fig. 7 and give an explicit description of the hyperedges in $\mathcal{P}^{(1)}$. Here, we have

$$E(\mathcal{H}_1^{(1)}) = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 2, 5\}, \{3, 4, 5\}, \{1, 5, 6\}, \{2, 4, 6\}, \{2, 5, 6\}, \{3, 4, 6\}, \{4, 5, 6\}\}$$

and

$$E(\mathcal{H}_2^{(1)}) = \{\{1, 2, 6\}, \{1, 3, 5\}, \{1, 3, 6\}, \{1, 4, 5\}, \{1, 4, 6\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 3, 6\}, \{2, 4, 5\}, \{3, 5, 6\}\}$$

given by which triangles in the picture are shaded and not shaded, respectively. The other representatives can be obtained from the corresponding pictures similarly.
**Fig. 7** $\mathcal{P}^{(1)}$ of orbit size 1440 and sign $\varepsilon_2^{S^3}(\mathcal{P}^{(1)}) = 1$

**Fig. 8** $\mathcal{P}^{(2)}$ of orbit size 240 and sign $\varepsilon_2^{S^3}(\mathcal{P}^{(2)}) = -1$

**Fig. 9** $\mathcal{P}^{(3)}$ of orbit size 1440 and sign $\varepsilon_2^{S^3}(\mathcal{P}^{(3)}) = -1$

**Fig. 10** $\mathcal{P}^{(4)}$ of orbit size 360 and sign $\varepsilon_2^{S^3}(\mathcal{P}^{(4)}) = 1$
Fig. 11  $\mathcal{P}^{(5)}$ of orbit size 1440
and sign $\varepsilon_2^{S^3}(\mathcal{P}^{(5)}) = -1$

Fig. 12  $\mathcal{P}^{(6)}$ of orbit size 1440
and sign $\varepsilon_2^{S^3}(\mathcal{P}^{(6)}) = 1$

Fig. 13  $\mathcal{P}^{(7)}$ of orbit size 720
and sign $\varepsilon_2^{S^3}(\mathcal{P}^{(7)}) = -1$

Fig. 14  $\mathcal{P}^{(8)}$ of orbit size 1440
and sign $\varepsilon_2^{S^3}(\mathcal{P}^{(8)}) = -1$
Fig. 15  $\mathcal{P}^{(9)}$ of orbit size 1440  
and sign $\varepsilon_2^{S_3^2}(\mathcal{P}^{(9)}) = 1$

Fig. 16  $\mathcal{P}^{(10)}$ of orbit size 360  
and sign $\varepsilon_2^{S_3^2}(\mathcal{P}^{(10)}) = 1$

Fig. 17  $\mathcal{P}^{(11)}$ of orbit size 720  
and sign $\varepsilon_2^{S_3^2}(\mathcal{P}^{(11)}) = -1$

Fig. 18  $\mathcal{P}^{(12)}$ of orbit size 720  
and sign $\varepsilon_2^{S_3^2}(\mathcal{P}^{(12)}) = 1$
Fig. 19 $\mathcal{P}^{(13)}$ of orbit size 360
and sign $\varepsilon_2^S (\mathcal{P}^{(13)}) = 1$

Fig. 20 $\mathcal{P}^{(14)}$ of orbit size 360
and sign $\varepsilon_2^S (\mathcal{P}^{(14)}) = -1$

Fig. 21 $\mathcal{P}^{(15)}$ of orbit size 72
and sign $\varepsilon_2^S (\mathcal{P}^{(15)}) = -1$

Fig. 22 $\mathcal{P}^{(16)}$ of orbit size 360
and sign $\varepsilon_2^S (\mathcal{P}^{(16)}) = 1$
Fig. 23 \( \mathcal{P}^{(17)} \) of orbit size 120 and sign \( \varepsilon_2^{S^3}(\mathcal{P}^{(17)}) = 1 \)

Fig. 24 \( \mathcal{P}^{(18)} \) of orbit size 360 and sign \( \varepsilon_2^{S^3}(\mathcal{P}^{(18)}) = -1 \)

Fig. 25 \( \mathcal{P}^{(19)} \) of orbit size 240 and sign \( \varepsilon_2^{S^3}(\mathcal{P}^{(19)}) = 1 \)

Fig. 26 \( \mathcal{P}^{(20)} \) of orbit size 12 and sign \( \varepsilon_2^{S^3}(\mathcal{P}^{(20)}) = -4 \)

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