Scaling and memory in the return intervals of energy dissipation rate in three-dimensional fully developed turbulence

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We study the statistical properties of return intervals \( r \) between successive energy dissipation rates above a certain threshold \( Q \) in three-dimensional fully developed turbulence. We find that the distribution function \( P_Q(r) \) scales with the mean return interval \( R_Q \) as \( P_Q(r) = R_Q^{-1} f(r/R_Q) \) except for \( r = 1 \), where the scaling function \( f(x) \) has two power-law regimes. The return intervals are short-term and long-term correlated and possess multifractal nature. The Hurst index of the return intervals decays exponentially against \( R_Q \), predicting that rare extreme events with \( R_Q \rightarrow \infty \) are also long-term correlated with the Hurst index \( H_\infty = 0.639 \).

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Extreme events are ubiquitous in nature and society and understanding their dynamics is of crucial importance.1, 2 However, extreme events are usually rare, which makes it difficult to investigate their occurrence properties. Recently, there is an increasing interest in the study of return intervals or reoccurrence times \( r \) between successive events above (or below) some threshold \( Q \), aiming at unveiling the laws governing the occurrence of extreme events by studying the statistics of the return intervals for increasing threshold \( Q \).

Recent studies show that the long-term correlation structure has essential influence on the statistics of return intervals.3 For long-term power-law correlated monofractal records with exponent \( \gamma \), numerical analysis illustrates that the distribution density function of the return intervals follows a stretched exponential \( P_Q(r) \sim \exp[-b(r/R_Q)^\gamma] \) with the same exponent \( \gamma \) and the return intervals are also long-term correlated, again with the same exponent \( \gamma \), where \( R_Q \) is the mean return interval associated with threshold \( Q \). Un correlated records with \( \gamma = 1 \) is a special case, whose return intervals are exponentially distributed. When \( 0 < \gamma \leq 1 \), theoretical analysis with certain approximation shows that the distribution of return interval is Weibull: \( P_Q(r) \sim r^{\gamma-1} \exp(-c r^\gamma) \). This seems consistent with recent numerical results.3

For multifractal records in the presence or absence of linear correlations, extensive simulations based on the multiplicative random cascade (MRC) model10 and the multifractal random walk (MRW) model11 unveil that the return intervals have a power-law decay in the distribution and are long-term correlated governed by power laws whose exponents depend explicitly on the threshold \( Q \), and the conditional return intervals increase as a power-law function of the previous return interval.12, 13, 14. These results are of particular interest since a variety of time series exhibit multifractal nature. For instance, the returns of common stocks can be well modeled by the multifractal random walk13, 10, and the statistics of the associated return intervals are found to comply with the numerical prediction12, 14, which can be used to significantly improve risk estimation12.

Another important issue of multifractal records concerns the possible scaling behavior of the return interval distributions \( P_Q(r) \) over different thresholds \( Q \). Numerical simulations of MRC and MRW time series find no evidence of such scaling12, 13, 14. However, empirical return interval analysis of financial volatility gives miscellaneous results. Several studies reported that there is a scaling law in the return interval distributions17, 18, 19, 20, 21, while others argued that the cumulative distributions of return intervals had systematic deviations from scaling and showed multiscaling behaviors.22, 23, 24, 25.

In this Letter, we perform return interval analysis of the energy dissipation rate in three-dimensional fully developed turbulence based on a high-Reynolds turbulence data set collected at the S1 ONERA wind tunnel by the Grenoble group from LEGI21. The size of the velocity time series \( \{v_i : i = 1, 2, \ldots, N\} \) is about \( 1.73 \times 10^7 \). Using Taylor’s frozen flow hypothesis which replaces a spatial variation of the fluid velocity by a temporal variation measured at a fixed location, the rate of kinetic energy dissipation at position \( i \) is \( \epsilon_i \sim (v_{i+1} - v_i)/\delta t \), where \( \delta t \) is the resolution (translated in spatial scale) of the measurements. The energy dissipation rate time series exhibit multifractal nature27 and its Hurst index is \( H = 1 - \gamma/2 = 0.81 \). Contrary to previous studies,
we find that the return interval distributions show two power-law regimes and collapse onto a single curve for different thresholds \( Q \). The scaling phenomenon is also observed for conditional interval distributions.

We have calculated the return interval time series \( r_i \) for different thresholds \( Q \), which can be mapped nonlinearly to the mean return intervals \( R_Q \). Logarithmic binning is adopted to construct the distribution density functions \( P_Q(r) \). In order to ensure that the bins cover the whole \( r \)-axis, we use the following procedure. First, the interval \([1, \max(r_i)]\) is partitioned logarithmically into \( n-1 \) subintervals whose edges are \( x_1 < x_2 < \cdots < x_n \). Then we obtain the sequence \( y_i = [x_i] \), where \([x_i]\) is a round function of \( x_i \). We discard duplicate integers in the \( y_i \) sequence and obtain a new sequence \( w_j \). The edge sequence of the bins are determined by \( \{e_i\} = \{0.5, \{w_i + 0.5\}\} \). For each bin \( (e_i, e_i+1]\), the empirical density function can be calculated by

\[
P_Q(r_i) = \frac{\#(e_i < r < e_i+1)}{\#(r > 0)} \frac{1}{e_i+1 - e_i},
\]

where \( r_i = (e_i + e_{i+1})/2 \) and \( \#() \) is the number of return intervals that satisfies the condition in the parenthesis. The empirical distribution of the return intervals is depicted in Fig. 1 for \( R_Q = 50, 150 \) and \( 500 \). We find that the three distributions collapse onto a single curve except for \( r = 1 \), in remarkable contrast to the simulation results for MRC and MRW time series. Two power-law regimes are observed

\[
R_Q P_Q(r) \sim \begin{cases} 
A_1 (r/R_Q)^{-\delta_1} & \text{if } 1 < r < r_c \\
A_2 (r/R_Q)^{-\delta_2} & \text{if } r > r_c
\end{cases},
\]

where \( A_1 = 0.107 \) and \( A_2 = 33.4 \) are prefactors, the crossover return interval \( r_c \approx 7R_Q \), and \( \delta_1 = 0.987 \pm 0.013 \) and \( \delta_2 = 3.88 \pm 0.09 \). For the shuffled data, the \( R_Q P_Q(r) \) curves collapse to a same exponential curve.

In risk estimation, a quantity of great interest is the probability \( W_Q(\Delta t; t) \) that an extreme event occurs after a short time \( \Delta t \ll t \) from now on, conditioned that the time elapsed \( t \) after the occurrence of the previous extreme event \[12\]:

\[
W_Q(\Delta t|t) = \frac{\int_{t}^{t+\Delta t} P_Q(t)\,dt}{\int_{t}^{\infty} P_Q(t)\,dt}.
\]

When \( t > r_c \), simple algebraic manipulation leads to

\[
W_Q(\Delta t|t) \approx (\delta_2 - 1)\Delta t/t.
\]

The probability \( W_Q(\Delta t|t) \) is found to be proportional to \( \Delta t \) and inversely proportional to \( t \). An intriguing feature is that \( W_Q(\Delta t|t) \) is independent of the threshold \( Q \), which is a direct consequence of the scaling behavior of

\[
R_Q P_Q(r) \text{ shown in Fig. 1. When } t < r_c, \text{ we obtain that}
\]

\[
W_Q(\Delta t|t) \approx \frac{(\delta_1 - 1) \left( \frac{t}{R_Q} \right)^{-\delta_1} \frac{\Delta t}{R_Q}}{\left( \frac{t}{R_Q} \right)^{1-\delta_1} + A_1 A_2 t^{-1} \left( \frac{t}{R_Q} \right)^{1-\delta_2} - \left( \frac{t}{R_Q} \right)^{1-\delta_1} - 1.0308}
\]

We find that \( W_Q(\Delta t|t) \) is proportional to \( \Delta t \). However, \( W_Q(\Delta t|t) \) also depends on \( R_Q \).

In order to test the memory effects of the return intervals, we first investigate the conditional PDF \( P_Q(r|r_0) \), which is the distribution of return intervals immediately after \( r_0 \). To gain better statistics, we study \( P_Q(r|r_0) \) for a range of \( r_0 \) rather than individual \( r_0 \) values. For each threshold \( Q \) or \( R_Q \), the return intervals sequence are sorted in an increasing order and then divided into eight groups \( G_1, \cdots, G_8 \) with approximately equal size. An empirical conditional distribution is determined for each \( r_0 \) group. Figure 2 shows \( P_Q(r|r_0) \) for \( r_0 \in G_1 \) and \( r_0 \in G_8 \). For each group, the three distributions collapse onto a single curve, indicating evident scaling behavior. The figure shows that the probability of finding small (large) \( r \) in \( G_1 \) is enhanced (decreased) compared with \( G_8 \). This discrepancy in the two groups of distributions unveils the memory effect that large (small) return intervals tend to follow large (small) return intervals. This is true since the distributions associated with different \( G_i \) should not exhibit significant discrepancy if there is no memory in the return intervals \[17\].

The memory effect in the conditional distribution \( P_Q(r|r_0) \) can also be illustrated by the mean conditional return interval \( \langle r|r_0 \rangle \). If there is no memory in the return
Figure 3 plots the mean conditional return interval rate, which does not depend on the conditional return interval of the shuffled energy dissipation empirical studies. Also shown in Fig. 3 is the mean conditional return interval as a function of \( r/R_Q \) for certain values and the power-law exponent decreases with \( R_Q \), which is consistent with many simulational and empirical studies. Also shown in Fig. 3 is the mean conditional return interval of the shuffled energy dissipation rate, which does not depend on \( r_0 \).

We now study the long-term correlation in the return intervals using the multifractal detrended fluctuation analysis (MF DFA), which is able to extract long-term power-law correlation in non-stationary time series [28, 29, 30]. The MF DFA considers the cumulative time series \( R_i = \sum_{i=1}^{m}(r_i - \langle r \rangle) \), which is partitioned into \( N_s \) disjoint boxes with the same size \( s \). In each box \( k \), the local trend is removed from the subseries by a polynomial function and the local rms fluctuation \( f_k(s) \) is determined. The overall detrended fluctuation is calculated by

\[
F_q(s) = \left\{ \frac{1}{N_s} \sum_{k=1}^{N_s} [f_k(s)]^q \right\}^{1/q}
\]

By varying the value of \( s \), one can expect the detrended fluctuation function \( F_q(s) \) scales with the size \( s \):

\[
F_q(s) \sim s^{h(q)},
\]

where \( h(q) \) is the generalized Hurst index. The return interval series possesses multifractal nature if and only if \( h(q) \) is a nonlinear function of \( q \). When \( q = 2 \), \( h(2) \) is nothing but the Hurst index \( H \) and the MF DFA reduces to the DFA. The Hurst index \( H \) is related to the autocorrelation exponent \( \gamma \) by \( \gamma = 2 - 2H \).

We first investigate the linear long-term correlation property of the return intervals using DFA. The dependence of the fluctuation function \( F_2 \) is drawn in Fig. 4 against \( s \) for \( R_Q = 50, 150 \), and 500. In all cases, we find nice power-law relation and the scaling range decreases with the increase of \( R_Q \). The inset shows the dependence of the Hurst index with respect to \( R_Q \), which has an exponential decay:

\[
H = H_\infty + e^{-R_Q/R_c} = 0.639 + 0.158e^{-R_Q/69.9},
\]

where \( R_c = 69.9 \) is the characteristic scale. For extreme events with very large \( Q \), \( R_Q \) tends to infinity, and the Hurst index can be predicted as \( H = H_\infty = 0.639 \). This implies that the return intervals of those extreme events also exhibit long-term memory.
The $F_q(s)$ function becomes more noisy, especially for negative $q$. The slopes of the straight lines are the linear least-squares estimates of the generalized Hurst indexes $h(q)$, which are drawn in Fig. 5(b). The mass scaling exponent function $\tau(q) = q h(q) - 1$ and the multifractal spectrum $f(\alpha)$ calculated according to the Legendre transform of $\tau(q)$ are illustrated respectively in Fig. 5(c) and Fig. 5(d). The sound nonlinearity in $h(q)$ and $\tau(q)$ is a hallmark of multifractality in the return intervals, whose singularity strength increases with $R_Q$.

![Graphs](image)

**FIG. 5:** (color online). Multifractal detrended fluctuation analysis of the return interval time series for different $R_Q$. (a) MFDF A fluctuation function for different $q$ when $R_Q = 50$. (b) Generalized Hurst indexes $h(q)$. (c) Mass scaling exponents $\tau(q)$. (d) Multifractal singularity spectra $f(\alpha)$.

In summary, we have studied the statistical properties of return intervals of the energy dissipation rate in three-dimensional fully developed turbulence. The interval distribution is found to exhibit scaling behavior across different $R_Q$ and two power-law regimes, except for intervals $r = 1$. We found that the conditional interval distributions also collapse onto a single curve for same $r_0$, but deviate for different $r_0$, and the mean conditional interval increases as a power law of $r_0$, which indicates the presence of short memory in the return intervals. The long-term memory and the multifractal nature in the return intervals are confirmed by the DFA and MFDF A. The Hurst index of the return intervals decays exponentially against $R_Q$, which allows us to predict the asymptotic Hurst index of return intervals between rare extreme events as $H_\infty = 0.639$. Our results signify discrepancy with the numerical results using MRC and MRW models [12], implying that the energy dissipation process can not be modeled using these models.

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