Partial Encryption after Encoding for Security and Reliability in Data Systems

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Abstract—We consider the problem of secure and reliable communication over a noisy multipath network. Previous work considering a noiseless version of our problem proposed a hybrid universal network coding cryptosystem (HUNCC). By combining an information-theoretically secure encoder together with partial encryption, HUNCC is able to obtain security guarantees, even in the presence of an all-observing eavesdropper. In this paper, we propose a version of HUNCC for noisy channels (N-HUNCC). This modification requires four main novelties. First, we present a network coding construction which is jointly, individually secure and error-correcting. Second, we introduce a new security definition which is a computational analogue of individual security, which we call individual indistinguishability under chosen-ciphertext attack (individual IND-CCA1), and show that N-HUNCC satisfies it. Third, we present a noise based decoder for N-HUNCC, which permits the decoding of the encoded-then-encrypted data. Finally, we discuss how to select parameters for N-HUNCC and its error-correcting capabilities.

I. INTRODUCTION

We consider the problem of secure and reliable communication over a noisy multipath network. A transmitter, Alice, wishes to transmit confidential messages to a legitimate receiver, Bob, over multiple noisy communication links, in the presence of two types of possible eavesdropper’s, Eve. Weak Eve can obtain noiseless information transmitted over a subset of the paths, while strong Eve can obtain information over all paths, as illustrated in Figure 1. While Bob’s links are noisy, we assume Eve may obtain noiseless observation of Alice transmissions. Thus, in contrast to the techniques utilized in physical layer security [1], we do not rely on the noise in the network for any security purposes.

In [2], a noiseless version of our setting was studied. In that setting, the authors proposed a hybrid universal network coding cryptosystem (HUNCC) which combines information-theoretic security with a computationally secure cryptosystem (see [3], [4] for a comparison with other approaches). HUNCC works by first premixing the data using a particular type of secure network coding scheme [5] and then partially encrypting the mixed data before transmitting it across the (noiseless) untrusted multipath network. Thus, obtaining information-theoretic security against a weak eavesdropper which does not observe all communication links while still guaranteeing computational security against a strong eavesdropper which observes all communication.

In this paper, we introduce a variation of HUNCC for noisy channels (N-HUNCC), i.e., we modify HUNCC so that it can be applied to noisy communication links. This requires four main novelties. First, we present a secure network coding scheme, which extends [5] into a code with both security and error-correcting capabilities against weak Eve. We note that this construction is of independent interest to those working on network information-theoretic security. Second, against a strong Eve, we introduce a new stronger notion of security than the individual computational security proposed in [2], which we call individual indistinguishability under chosen-ciphertext attack (Individual IND-CCA1). This stronger notion can also be readily applied to the settings in [2], [6]. Third, we provide a novel joint decryption-decoding scheme that combine error correction using an efficient Guessing Random Additive Noise Decoding (GRAND) [7], with decryption in an intermediate stage of GRAND decoding algorithm as illustrated in Figure 2. Finally, we discuss how to select parameters for N-HUNCC and its error-correcting capabilities.

The structure of this work is as follows. In Section II we describe the system model. The security notations we use in this work are defined in Section III. The proposed secure and reliable N-HUNCC scheme with our main results are presented in Section IV. Section V describes the partial encryption scheme against strong Eve with the security analysis. The joint decryption encoded data is described in Section VI. In Appendix A we present the construction of the individual secure scheme. Finally, we conclude this work in Section VII.

II. SETTING

We consider a setting where a transmitter, Alice, wishes to transmit confidential messages \( M = [M_1, \ldots, M_{k_u}] \in \mathbb{F}_q^{k_u} \) to a legitimate receiver, Bob, over \( \ell \) communication links, in the presence of an eavesdropper, Eve. While Bob’s links are noisy, we assume Eve may obtain noiseless observation of
Algorithms:

Definition 2. A public-key cryptosystem consists of three algorithms:

- A key generation algorithm $Gen(\kappa)$ which takes as input a security parameter $\kappa$ and generates a public key $p_k$ and a secret key $s_k$.
- An encryption algorithm $Enc(m, p_k)$ which takes as input a message $m$ belonging to some set of messages $M$ and the public key $p_k$ and then outputs a ciphertext $c$ belonging to some set of ciphertexts $C$.
- A polynomial time decryption algorithm $Dec(c, s_k)$ which takes as input a ciphertext $c = Enc(m, p_k)$ and the secret key $s_k$ and outputs the original message $m$.

The encryption algorithm may be probabilistic and indeed must, in order to satisfy the following security constraint.

Definition 3 (IND-CCA1). Indistinguishability under chosen ciphertext attack (IND-CCA1) is defined by the following game between an adversary and a challenger.

1) The challenger generates a key pair $(p_k, s_k)$ for some security parameter $\kappa$ and shares the public key $p_k$ with the adversary.
2) The adversary may send a polynomial amount of ciphertexts to the challenger and receive back their decryptions. They may also perform a polynomial amount of operations.
3) The adversary chooses two challenge messages $m_1$ and $m_2$, and sends them to the challenger.
4) The challenger chooses $i \in \{1, 2\}$ uniformly at random.
5) The challenger sends the challenge ciphertext $c^* = Enc(m_i, p_k)$ to the adversary.
6) The adversary performs a polynomial amount of operations before outputting a guess for whether $b = 1$ or $b = 2$ and sends it to the challenger. If the adversary guesses correctly, they win.

The cryptosystem is indistinguishability under chosen ciphertext attack if any adversary has only a negligible advantage over a uniformly random guess of $b$, i.e. if they win the game with probability $\frac{1}{2} + \varepsilon(\kappa)$, where $\varepsilon(\kappa)$ is such that for every positive integer $d$, there exists an integer $\kappa_d$ such that for all $\kappa > \kappa_d$, it holds that $\varepsilon(\kappa) < \frac{1}{\kappa^d}$. Such a function $\varepsilon(\kappa)$ is called a negligible function.

We now present a new notion of security which combines Definitions 1 and 3.
Definition 4 (Individual IND-CCA1). Let the set of messages be $\mathcal{M} = F_q^k$. Thus, each message $m = (m_1, \ldots, m_k)$. We refer to each $m_i$ as an individual message. Then, individual indistinguishability under chosen ciphertext attack (Individual IND-CCA1) is defined by the following game between an adversary and a challenger.

1) The challenger generates a key pair $\text{Gen}(\kappa) = (pk_\kappa, sk_\kappa)$ for some security parameter $\kappa$ and shares the public key $pk_\kappa$ with the adversary.

2) The adversary may send a polynomial amount of ciphertexts to the challenger and receive back their decryptions. They may also perform a polynomial amount of operations.

3) The adversary chooses an index $j^* \in \{1, \ldots, k_u\}$ and two challenge individual messages $m^1_{j^*}$ and $m^2_{j^*}$, and sends them to the challenger.

4) The challenger chooses $i \in \{1, 2\}$ uniformly at random.

5) The challenger chooses $k_u - 1$ individual messages $m_j$, for $j \in \{1, \ldots, k_u\} - \{j^*\}$ uniformly at random and then constructs the message $m = (m_1, \ldots, m_{k_u})$, where $m_j = m^j_{j^*}$.

6) The challenger sends the challenge ciphertext $c^* = \text{Enc}(m, pk_\kappa)$ to the adversary.

7) The adversary performs a polynomial amount of operations before outputting a guess for whether $b = 1$ or $b = 2$ and sends it to the challenger. If the adversary guesses correctly, they win.

The cryptosystem is individually indistinguishable under chosen ciphertext attack if any adversary has only a negligible advantage over a uniformly random guess of $b$, i.e. if they win with probability $\frac{1}{2} + \epsilon(\kappa)$, where $\epsilon(\kappa)$ is a negligible function.

Individual IND-CCA1 is thus a computational analogue of individual security. It guarantees that an adversary can only learn a negligible amount of information about any individual message. It is thus suited for the same settings where individual security can be applied, but where Eve might observe all communication, rendering individual security useless.

IV. JOINT SECURE-RELIABLE CODING CRYPTO SYSTEM (MAIN RESULTS)

In this section, we present our scheme N-HUNCC and our main results. In Figure 2, we illustrate how the scheme operates on a noisy multipath network with $\ell$ communication links. The noise we consider at Bob is, for each link, an independent BSC with a sum rate, for each channel transmission, of at most $\nu = \sum_{i=1}^\ell H(p)$. We note, however, that we allow Eve to obtain noiseless observations of the transmissions over the links, i.e., no noise in the channel is used for security purposes.

N-HUNCC follows the main ideas proposed in [2] in which one can encrypt only part of the data transmitted. However, unlike [2], here we assume that Bob may obtain noise observation of the data transmitted. We present four main novelties: 1) a new random code design with error correction, 2) that N-HUNCC is individually IND-CCA1 secure, 3) a novel joint decryption-decoding scheme, 4) a discussion on parameter selection and error-correcting capabilities. We detail those in Appendix A and Sections V and VI, respectively. We note that the individual IND-CCA1 security proof can be readily applied to the settings in [2, 6]. We now provide a high level description of our proposed scheme.

We start by looking at the encoding process at Alice. The messages are encoded using an $(\ell, w)$-individual secure random code as given in Appendix A. The number of individually secret messages against weak Eve is given by $k_u = \ell - \nu - w - 2k_u\epsilon$. In the encoding process, using a random code, Alice encodes each of the $i$-th columns in the messages matrix $M$ independently. Thus the encoder is given by

$$E : M(i) \in F_q^{k_u} \rightarrow X(i) \in F_q^\ell,$$

which maps the $i$-th column $M(i)$ in the message matrix to the $i$-th column $X(i)$ in the codeword matrix. Then, each row in the codeword matrix is transmitted in the $\ell$ independent links.

Using this secure coding scheme against a weak Eve, our first main result is the following achievability theorem.

Theorem 1. Assume a noisy BSC multipath communication $(\ell, \nu, w)$. N-HUNCC’s encoder delivers, with high probability, $k_u \leq \ell - \nu - \epsilon$ messages at the legitimate decoder, while keeping a weak eavesdropper which observes $w < \ell - \nu$ noiseless links ignorant with respect to any set of $k_u \leq k_u - w - 2k_u\epsilon$ messages individually, such that $I(M^w : Z^w) \leq \epsilon\ell$, whenever $\nu \leq \sum_{i=1}^\ell H(p)$ and $k_u \epsilon = o(k_u)$.

The construction of the individual secure network code follows almost directly from [5, Section IV] considering carefully the increased size of the codewords required due to channel noise. Due to the space limitation the construction together with the proofs of reliability and individual secrecy are deferred to [10].

Now, in order to obtain security against a strong Eve, as opposed to the traditional approach where all communication links must be encrypted, we show that we only need to encrypt a portion $c = \ell - w$ of the communication links, where $\ell = k_u + \nu + w + 2k_u\epsilon$. Without loss of generality, we let the links indexed by 1, . . . , $c$ to be the encrypted ones. The encryption by the cryptosystem at each $i$-th column of the $c$ links is given by

$$\text{Crypt}_1 : X(i) \in F_q^{\ell-w} \rightarrow Y(i) \in F_q^{\ell-w+r},$$

which by adding an extra $r$ symbols (depending on the encryption used) encrypts the $i$-th column $X(i)$ in the codeword matrix into row $Y(i)$ in the matrix obtained at Bob. The extra $r$ bits at the outcome of the cryptosystem per column are concatenated to be transmitted over the $c$ encrypted links. We note that, because we encrypt after encoding, the encryption can be performed at any stage in the system as long as it occurs before a strong Eve’s observation. This is opposed to traditional schemes which require the encryption to essentially be done at Alice, or some equivalent of her, since all messages are needed for the error correction encoding which is usually applied after encryption.
Our next main result shows that N-HUNCC is secure against a computationally bounded strong eavesdropper.

**Theorem 2.** In the setting of Theorem 1 let Crypt be a IND-CCA1 secure cryptosystem used as described in [1]. Then, N-HUNCC is individually IND-CCA1 secure.

The proof of the theorem is given in Section V.

At Bob, the joint decryption-decoding scheme is given by

\[ D : [Y \in \mathbb{F}^{e+r \times k_s} ; Y \in \mathbb{F}^{e-c \times k_s}] \rightarrow \tilde{M} \in \mathbb{F}^{q \times k_s}, \]

which maps the outcome noisy channel \( Y \) to \( \tilde{M} \).

To decode the messages, Bob utilizes a modified version of the GRAND decoder [7]. Given the noisy channel outcome \( Y \), Bob orders the noise sequences from most likely to least likely. He then goes through the list, subtracting the noise from \( Y \) and then preforming the decryption on the first \( e \) rows. After this, he checks if all columns of the decrypted matrix are elements of the codebook. The first time this occurs Bob decodes that message. As shown in [7], this procedure is a Maximum Likelihood (ML) decoder. We discuss this decoder more in detail in Section VI.

We now discuss the parameter selection for N-HUNCC in order to be reliable on a BSC with error probability \( p \). The main challenge is to deal with how the encryption affects the error correction capabilities of the code. If the output of the encryption is uniformly random, we have the following result.

**Theorem 3.** In the setting of Theorem 2, suppose the output of Crypt is uniformly distributed. Then, if \( \frac{k_u + r_0}{\ell + r} \) is less than the capacity of the BSC channel, N-HUNCC can asymptotically transmit at arbitrarily low probability of errors at a rate of \( \frac{k_u}{\ell + r} \). Here, \( r_0 \) denotes the amount of randomness in Crypt.

The proof of the theorem is given in Section VI.

In practice it might be hard to enforce a uniform output on the encryption. However, there are cryptosystems with outputs which are computationally indistinguishable from a uniform distribution [17], [18]. Thus, we expect using such cryptosystems as an inner crypto function Crypt should make N-HUNCC perform as described in Theorem 3. Indeed, most, if not all (to the best of our knowledge), practical implementations of random codes are actually pseudorandom.

We finish this section with the following remarks.

**Remark 1.** The security guarantee in Theorem 2 can be readily applied to noise-less HUNCC in order to obtain individual IND-CCA1 security for the settings in [2], [6], for example.

**Remark 2.** The arguments in Theorem 3 can be readily applied to scenarios in which one encodes data for reliability before using any encryption with an output which is indistinguishable from uniformly random. The Advanced Encryption Standard (AES), for example, consistently passes output uniformity tests [19], [20].

**V. PARTIAL ENCRYPTION AGAINST A STRONG EVE**

In this section we provide the proposed partial encryption scheme on \( \ell - w \) links using a cryptosystem after encoding as given in Appendix A. The encryption process and the security analysis against a strong Eve are detailed in Subsections V-A and V-B, respectively.

**A. Encrypting Encoded Data**

The encoding construction in N-HUNCC, (see Appendix A), starts with an encoding function \( \text{Enc} : \mathbb{F}^{\ell + r_0} \rightarrow \mathbb{F}^{\ell + r} \). To satisfy the Individual IND-CCA1 security as given in Definition 4 by only encrypting \( \ell - w \) links, we consider an IND-CCA1 cryptosystem. Furthermore, for reliability, discussed in Section VI, we consider such cryptosystems with pseudorandom output, e.g., as given in [17].

In order to satisfy IND-CCA1 security, the cryptosystem adds randomness (possibly through some form of padding) of size \( r_0 \) and ultimately increases the output by \( r \geq r_0 \). This encryption is given by the deterministic injective function Crypt : \( \mathbb{F}^{\ell + r_0 - w} \rightarrow \mathbb{F}^{\ell + w + r} \), where \( r_0 \leq r \) represents the randomized part of the cryptosystem algorithm and is therefore, not shared with Bob. We may also refer (by abuse of notation) to the cryptosystem as the probabilistic function Crypt : \( \mathbb{F}^{\ell - w} \rightarrow \mathbb{F}^{\ell - w + r} \) where the randomness \( r_0 \) is implicit in the function.

N-HUNCC, then, consists in the cryptographic scheme

\[ \text{Crypt}_2 = (\text{Crypt}_1 \circ \pi_1 \circ \text{Enc}) \times (\pi_2 \circ \text{Enc}) : \mathbb{F}^{q \times k_u + r_0 - w} \rightarrow \mathbb{F}^{q \times r_0}, \]

where \( \pi_1 : \mathbb{F}^{\ell + r_0 - w} \rightarrow \mathbb{F}^{\ell + r_0} \) is the projection of the first \( \ell + r_0 - w \) entries and \( \pi_2 : \mathbb{F}^{\ell} \rightarrow \mathbb{F}^{w} \) is the projection of the last \( w \) entries. In other words, N-HUNCC consists in taking the message with \( k_u \) symbols, encoding it with the individual code described in [16, Appendix A] to obtain \( \ell \) symbols. It then inputs the first \( \ell - w \) of these symbols into Crypt together with \( r_0 \) randomly chosen symbols needed for the cryptosystem, obtaining \( \ell - w + r \) encrypted symbols. The output of Crypt is then the concatenation of the \( \ell - w + r \) encrypted symbols with the \( w \) unencrypted ones.

**B. Security Against A Strong Eve (A Proof For Theorem 2)**

Here we show that Crypt and thus N-HUNCC is individually IND-CCA1 secure as given in Definition 4.

For each \( i \)-th column in the massage matrix \( M \), the encoding function of the individual secure random code is given by

\[ \text{Enc}(M_1(i), \ldots, M_{k_u}(i)), \]

where the first \( k_u \) coordinates determine the bin \( b(i) \) in the codebook and the last \( k_u - k_s \) determine the position within the bin \( e(i) \) the codeword is selected from, as described in [16, Appendix A]. We now play the security game described in Definition 4. We assume that the adversary chooses a \( j \in [1, k_s] \) (the other case will follow analogously but for position instead of bin). The adversary then chooses two messages \( M_1^j(i) \) and \( M_2^j(i) \).

We show the stronger statement that even if we give the adversary the other entries \( M_1(i), \ldots, M_{k_u}(i) \), he is not able to distinguish between the two bins.

\[ (f \times g)(x, y) = (f(x), g(y)). \]
\( b_1(i) = (M_1(i), \ldots, M_{1}^k(i), \ldots, M_k(i)) \) and \( b_2(i) = (M_1(i), \ldots, M_{2}^k(i), \ldots, M_k(i)) \). The challenger still chooses \( e(i) = (M_{k+1}(i), \ldots, M_{k_n}(i)) \) uniformly at random and does not share it with the adversary. The challenger then chooses \( j \in \{1, 2\} \) uniformly at random and sends back the ciphertext \( c_j = \text{Crypt}_2(b_j(i), e(i)) \) to the adversary. Both crypto bins \( b_1(i) \) and \( b_2(i) \) have a total of \( 2^{w+k_u \varepsilon} \) codewords each of size \( \ell + r \) as shown in [16], of which the first \( \ell + r - w \) symbols are encrypted by \( \text{Crypt}_2 \) and the last \( w \) symbols are not. From knowing the last \( w \) symbols of \( c_j \) the adversary is able to reduce the number of the possible codewords in the bins to \( B_1 \) and \( B_2 \) respectively, with each cypher having probability \( \frac{1}{B_1 + B_2} \). As shown in [5] Section IV.B, the probability that the actual number of the possible codewords in the bins deviates from the average by more than \( \varepsilon' \) is bounded for sufficient large \( k_u \), so that, for any bin \( b_j, j \in \{1, \ldots, 2^{k_u - w}\} \), it holds that \( (1 - \varepsilon')2^{k_u \varepsilon} \leq B_j \leq (1 + \varepsilon')2^{k_u \varepsilon} \) with high probability.

The last step now consists on the adversary trying to distinguish the messages via the information leaked by the encrypted part. The best possible case for the adversary (and worse for the challenger) is the case where bin 1 has as high probability as possible, which corresponds to \( B_1 \geq B_2 \) with as much difference as possible, and such that the distinguishability from the \( \text{Crypt}_1 \) is as high as possible. Thus, we consider the case where all the ciphertexts in bin 1 have probability \( p_c + \varepsilon_c \) and the ciphertexts in bin 2 have probability \( p_c - \varepsilon_c \), thus aligning the probabilities so that they concentrate in bin 1. Since \( B_1(p_c + \varepsilon_c) + B_2(p_c - \varepsilon_c) = 1 \), it follows that

\[
 p_c = \frac{1 - (B_1 - B_2)\varepsilon_c}{B_1 + B_2} \tag{2}
\]

From the IND-CCA1 security of \( \text{Crypt}_1 \) it follows that for every \( d \in \mathbb{N} \), there exists an \( k_u \) such that \( k_u \geq k_d \) implies in

\[
 p_c + \varepsilon_c \leq \frac{2(k_d(B_1 + B_2) + 2(B_1 - B_2))}{k_d} \tag{3}
\]

Hence, substituting (2) in (3) we have

\[
 \varepsilon_c \leq \frac{2}{k_d(B_1 + B_2) + 2(B_1 - B_2)} \tag{4}
\]

Now we show that the difference between the probability of the correct bin being bin 1 and \( \frac{1}{2} \) is negligible. Indeed,

\[
 \Pr[\text{bin 1}] - \frac{1}{2} = B_1(p_c + \varepsilon_c) \leq \frac{k_d(B_1 - B_2) + 2(B_1 + B_2)}{2k_d(B_1 + B_2) + 4(B_1 - B_2)} \tag{5}
\]

We select the bins with the highest deviation possible of codewords in the bins, as described above, to analyze the worst case. Thus, for \( \varepsilon' = \frac{1}{k_d} \) and any \( t \geq 2 \), we have

\[
 \frac{B_1}{B_2} = \frac{1 + \varepsilon'}{1 - \varepsilon'} = \frac{k_t + 1}{k_t - 1}. \tag{6}
\]

Hence, substituting (6) in (5) we obtain

\[
 \Pr[\text{bin 1}] - \frac{1}{2} \leq \frac{1}{k_t} + \frac{1}{k_t} - \frac{1}{2k_t} \leq \frac{1}{2k_t^2} + \frac{1}{k_t} \tag{7}
\]

which for every \( d' \) can be made smaller than \( \frac{1}{2k_t^2} \) for large enough \( k_u \) by choosing an appropriate \( d' \) and taking \( t \) to grow more than a constant, e.g. \( t = \log(k_u) \).

VI. DECRYPTING ENCODED DATA
(A PROOF FOR THEOREM [4])

In this section, we analyze the proposed joint decryption-decoding scheme as presented in Section IV using GRAND [7]. GRAND algorithms operate by sequentially inverting putative noise effects from the demodulated received sequence and querying if what remains is in the code-book [7], [21]–[23]. If those noise effects are queried in order from most-likely to least likely, the first instance where a code-book member is found is an ML decoding [7]. If the code is unstructured and stored in a dictionary, a code-book query corresponds to a tree-search with a complexity that is logarithmic in the code-length. If the code is linear in any finite field, code-book membership can be determined by a matrix multiplication and comparison. For encrypted, encoded data, only one extra step is required: the effect of each putative noise sequence is removed from the encrypted data, which is then decrypted and the resulting sequence tested for code-book membership.

Our encoding construction starts with a random code \( C_0 = \text{Enc}(F_{q^{k}}) \subseteq F_{q}^{d} \) of size \( q^{k_u} \). When considering the randomness from \( \text{Crypt}_1 \) we obtain a code \( C_1 = C_0 \times F_{q}^{d} \subseteq F_{q}^{d+r} \) of size \( q^{k_u+r} \). That is, every original message \( m \in F_{k_u} \) corresponds to \( q^{r} \) possible codewords in \( C_1 \). Finally, after applying \( \text{Crypt}_2 \) we obtain a code \( C_2 \subseteq F_{q}^{d+r} \) with the same size \( q^{k_u+r} \) of \( C_1 \). The last \( w \) symbols of a codeword of \( C_2 \) are uniformly distributed. The first \( \ell - w \) symbols are the output of \( \text{Crypt}_1 \). If this output is uniform, then we have a random code \( C_2 \subseteq F_{q}^{d+r} \) of size \( q^{k_u+r} \) with the property that multiple (more precisely \( q^{r} \)) codewords decode to the same message. Let us suppose that we want the code to act as a regular code, i.e. treating each codeword of the \( q^{r} \) same-message codewords as if they corresponded to distinct messages. In this worse scenario, the code would asymptotically transmit at arbitrarily low probability of errors if [7], [23]

\[
 \frac{k_u + r_0}{\ell + r} < \text{Capacity of the BSC channel}. \tag{7}
\]

However, when setting the parameters to satisfy (7), the effective rate of the code is given by \( \frac{k_u}{\ell + r} \).

Moreover, we can compute an error exponent for our joint decryption-decoding scheme. Consider a communication on one of the coded-packets, \( Y(i) \), expressed as a binary string \( Y^b(i) \). It is transmitted over a BSC and impacted by a binary noise effect \( N^b(i) \) resulting in a received signal \( Y^b(i) = Y^b(i) + N^b(i) \), where addition is \( F_2 \) and \( N^b(i) \) is a string of independent Bernoulli \( p \) random variables. Let \( Y^b(i) \) be the GRAND-decoding estimate of \( Y^b(i) \). If a code-book of rate \( R \leq \frac{k_u + r_0}{\ell + r} \) is selected uniformly, then the likelihood of an erroneous decoding decays exponentially in \( n = \ell + r \) with Gallager’s error exponent [7][Proposition 1]. That is,

\[
 \lim_{n \to \infty} \frac{1}{n} \log P \left( Y^b(i) \neq Y^b(i) \right) = -\varepsilon(R, p),
\]
where the exact form for $\varepsilon(R, p)$ can be identified as follows. Define the Rényi entropy of the BSC noise process with bit flip probability $p$ and parameter $\alpha \in (0, 1) \cup (1, \infty)$ to be

$$H_\alpha = \frac{1}{1 - \alpha} \log \left( p^\alpha + (1 - p)^\alpha \right),$$

with $H_1$ being the Shannon entropy and min entropy denoted

$$H_{\min} = -\log(\max(p, 1 - p)).$$

Then defining

$$A_N^{\alpha}(\alpha) = \begin{cases} \alpha H_1/(1 + \alpha) & \text{for } \alpha \in (-1, 1) \\ -H_{\min} & \text{for } \alpha \leq -1, \end{cases}$$

$$I_N^{\alpha}(x) = \sup_{\alpha \in \mathbb{R}} (\alpha x - A_N^{\alpha}(\alpha))$$

and $d/dx I_N^{\alpha}(x)|_{x=x^*} = 1$, then the error exponent can be identified as

$$\varepsilon(R, p) = \begin{cases} 1 - R - H_1/2 & \text{if } R \in (0, 1 - x^*) \\ I_N^{\alpha}(1 - R) & \text{if } R \in [1 - x^*, 1 - H(p)) \end{cases},$$

and the error exponent for the full system follows from an application of the principle of the largest term [25, Lemma 1.2.15].

**Lemma 1.** The error exponent for the likelihood that one or more of the decodings is erroneous is given by

$$\lim_{n \to \infty} \frac{1}{n} \log P \left( \bigcup_{i=1}^\ell \{ y^i \neq y^b(i) \} \right) = -\varepsilon(R, p).$$

**VII. CONCLUSIONS**

In this work, we suggest a noisy hybrid universal network coding cryptosystem that can be applied to noisy communications systems. The proposed cryptosystem is secure against a strong eavesdropper under a new security notion we introduce of Individual IND-CCA1. This notion of security can be readily applied to other HUNCC solutions offered in the literature with partial encryption [2, 6]. Finally, we present a joint decryption-decoding scheme that combines error correction using GRAND with decryption in an intermediate stage.

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**APPENDIX A**

**SECURE RANDOM CODES AGAINST A WEAK EVE**

(A PROOF FOR THEOREM I)

Here we provide the individual secure binary random code used at Alice to encode the message matrix $M$. We propose a secure code for weak Eve which can observe the information of at most $w = \ell - \nu - k_u - 2k_u \varepsilon$ links. That is, we assume a degraded channel at weak Eve, with $p(y, z|x) = p(y|x)p(z|y)$. The individual security is obtained for $k_u \leq k_u - w - 2k_u \varepsilon$ messages form the $k_u$ messages decoded at Bob correctly with high probability for $\nu \leq \sum_{i=1}^\ell H(p)$. We may now turn to the detailed construction and analysis.

- a) **Codebook Generation:** Set $\Delta = 2^{\ell + k_u \varepsilon}$ and $k_u = \ell - \nu - \varepsilon$. Let $P(x) \sim \text{Bernoulli}(1/2)$. Using a distribution $P(X^i) = \prod_{j=1}^\ell P(x_j)$, for each possible column

$M_1(i), \ldots, M_k(i)$ in the message matrix, that is, $2^{k_u-(w+k_u \varepsilon)}$
possibilities, generate $\Delta$ independent and identically distributed codewords $x^\ell(e)$, $1 \leq e \leq \Delta$. Thus, we have $2^{k_u - (w + k_u \varepsilon)}$ bins, each of size $2^{w + k_u \varepsilon}$. Note that the length of the columns in the bins is $\ell$, thus the codebook matrix is increased by $\nu + \varepsilon$ compared to the message matrix $M$.

b) Encoding: The encoder selects, for each column $i$ of bits $M_1(i); \ldots; M_{k_u}(i)$, one codeword, $x^\ell(e(i))$, from the bin indexed by $M_1(i); \ldots; M_{k_u}(i)$, where $e(i) = M_{k_u + 1}(i); \ldots; M_{k_u}(i)$. That is, $k_x = k_u - (w + k_u \varepsilon)$ bits of the column choose the bin, and the remaining $w - \varepsilon$ bits choose the codeword within the bin.

c) Reliability: Bob maps $Y_s$ back to $M_s$, as per column $1 \leq i \leq c$, the index of the bin in which the codeword $Y_s(i)$ resides, using for example GRAND decoder for noise $\nu$ [7]. $M_1(i); \ldots; M_{k_u}(i)$ and the index of the codeword location in that bin is $M_{k_u + 1}(i); \ldots; M_{k_u}(i)$. The analysis on the probability of successfully decoding $M(i)$ from $Y(i)$ is a direct consequence using standard analysis of random coding [1, Section 3.4].

As for the information leakage at the weak eavesdropper, to show that $k_x$-individual security constraint is met, the proof is follows directly form [5, Section IV.B]. That is, for each column $i \in \{1, \ldots, k_u\}$, as long we choose $k_u \varepsilon$ to be an integer, $I(M^{k_u}(i); Z^w(i)) = O(k_u^{-t+1})$ for any $t \geq 2$. 