Certifying quantum memories with coherence

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Quantum memories are an important building block for quantum information processing. We present a general method to characterize and test these devices based on their ability to preserve coherence. We introduce a quality measure for quantum memories and characterize it in detail for the qubit case. The measure can be estimated from sparse experimental data and may be generalized to characterize other building blocks, such as quantum gates or teleportation schemes.

I. INTRODUCTION

In order to work, quantum computers need reliable and well-characterized routines and devices. The loss of quantum coherence, however, is one of the major obstacles on the way to a scalable platform for quantum computing and the suppression of decoherence is known as one of the DiVincenzo criteria for quantum computers [1]. One main ingredient in any computing architecture is the memory. Quantum computers are no exception and furthermore, quantum memories play a central role in the development of quantum repeaters [2, 3]. Consequently, the search for reliable systems that store quantum states for a reasonable amount of time preserving quantum properties is an active area of research [4–10].

A possible way to verify the proper functioning of quantum gates and quantum memories is to completely characterize their behavior via quantum process tomography [11, 12]. This, however, requires an effort exponentially increasing in the size of the system. More importantly, such a full characterization is not directly linked to the behavior of physical properties, such as entanglement and coherence, under the prescribed time evolution. Therefore, it is desirable to characterize devices directly by their effect on physical phenomena. In this direction, several methods have been suggested to characterize quantum memories. This can be done by their effect on entanglement and quantum steering, or based on resource theoretic arguments [13–16]. Still, their usage requires many and well characterized test states as inputs, or many measurements on the output.

In this paper we introduce a way to characterize and verify quantum memories using the phenomenon of coherence. The key idea is that an ideal quantum memory preserves the coherence in any basis. Based on that, we introduce a quality measure for memories, and show that it obeys several natural properties. The measure can be used to prove that a memory preserves entanglement, moreover, it can be estimated with few measurements, without the need of well characterized input states. Our concept can be generalized to characterize also other quantum primitives such as teleportation schemes and, using generalized notions of coherence [17, 18], also to multi-particle quantum gates.

II. MEMORY QUALITY MEASURES

To start, let us study what physical properties a measure for the quality of a quantum memory should have. As non-classical properties are essential for many quantum algorithms, the storage should preserve as many of these properties as possible. A perfect quantum memory is given by the identity channel. In practice, however, this is rather difficult to achieve. Contrary to that, measure-and-prepare schemes (or entanglement-breaking channels) can be easily simulated using only classical storage. One just performs measurements on the input state and stores the result. Based on that, one then prepares a quantum state on demand.

These two examples already show that if one wants to quantify the quality of a quantum memory, this measure should have two natural properties: First, it should be maximal for memories that output the input perfectly. As we assume that we can perform unitary rotations, we also allow the memory to apply a known and fixed unitary rotation to the input. Second, the measure should have a non-maximal quality for the measure-and-prepare schemes described above, certifying genuine quantum storage.

Formally, such measure-and-prepare channels can always be written as [19]

\[
M(\rho) = \sum_\lambda \text{Tr}(E_\lambda \rho_\lambda),
\]

where the set \( \{E_\lambda\} \) forms a positive operator values measure (POVM) and the \( \rho_\lambda \) are density matrices. In the following, let \( B \subset \mathbb{C}^{D \times D} \) denote the set of density matrices of dimension \( D \), and \( M : B \to B \) a quantum channel, i.e., a completely positive, trace preserving map (CPTP). We denote by \( \mathcal{T} \) the set of all CPTP maps on \( B \). We can now formulate our criteria for the quality measures.

Definition 1. A map \( Q : \mathcal{T} \to [0, 1] \) is called memory quality measure, if it satisfies

\[
M1: Q(M) = 1 \text{ if } M(\rho) = V \rho V^\dagger \text{ for some unitary } V,
\]

\[
M2: Q(M) \leq c \text{ for some constant } c \in [0, 1] \text{ if } M \text{ is a measure-and-prepare channel}.
\]

A memory quality measure is called sharp, if it additionally fulfills
$M1': Q(M) = 1 \Leftrightarrow M(\rho) = V \rho V^\dagger$ for some unitary $V$.

Obviously, the condition $M1'$ implies that the identity channel has unit quality. Note that, for continuous sharp measures, $M1'$ implies $M2$. This is due to the compactness of the set of measure-and-prepare channels [20].

### III. DEFINITION OF THE MEASURES

We define a physically motivated quality measure from the following considerations: Given a quantum channel $M$, there is a “most classical” basis, in which even the most robust maximally coherent state with respect to that basis is mapped to a state with small coherence. This basis is identified by our proposed measure, and the conserved coherence in this basis defines the quality.

As a measure of coherence in a fixed basis (defined by some unitary $U$ such that $|b_i⟩ := U |i⟩$) we use the normalized $l_1$-measure [21]

$$C_U(\rho) := \frac{1}{D-1} \sum_{i\neq j} |⟨b_i|\rho|b_j⟩|.$$  \hspace{1cm} (2)

We stress, however, that our results, except for those yielding numerical values, are valid for any continuous and convex coherence measure with the property that the only states maximizing the measure for a fixed basis $U$ are given by

$$|\Psi_U^\alpha⟩ := \frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} e^{i\alpha_j} |b_j⟩ = U Z_\alpha |+⟩,$$  \hspace{1cm} (3)

where $\alpha$ is some $D$-dimensional vector of phases and $Z_\alpha$ is a diagonal unitary matrix with entries $e^{i\alpha_j}$, acting on $|+⟩ := \frac{1}{\sqrt{D}} \sum_i |i⟩$. Note that these states are the only states from which any state can be reached using incoherent operations only, thus they are also maximally coherent in a resource theoretic sense. In fact, the set of maximally coherent states of any valid coherence measure includes these states, and for many prominent coherence measures such as the $l_1$-measure, the relative entropy measure and the coherence of formation, it is identical to this set [21] [22].

**Definition 2.** For a quantum channel $M$, the quality $Q_0$ is given by

$$Q_0(M) := \min_{U} \frac{1}{D} \sum_{i\neq j} |⟨b_i|\rho|b_j⟩|.$$  \hspace{1cm} (4)

Here and in the following, we write $M(|\Psi_U^\alpha⟩)$ instead of $M(|\Psi_U^\alpha⟩|\Psi_U^\alpha⟩)$ for convenience. If $Q_0(M) = 1$, then this means that in any basis at least one maximally coherent state is preserved. Later, we show that this already implies that $M$ is unitary.

Despite the clear physical interpretation of this measure, there are related quantities which turn out to be useful for the discussion. Therefore, we introduce two additional parameters, which provide an upper and lower bound on $Q_0$. First, we consider the minimal coherence left in any basis of the most robust maximally coherent states if one minimizes over their bases:

**Definition 3.** For a quantum channel $M$, the quantity $Q_-$ is defined by

$$Q_-(M) := \min_{U,U'} \max_{\alpha} C_{U,U'}[M(|\Psi_U^\alpha⟩)],$$  \hspace{1cm} (5)

Note that, in contrast to $Q_0$, the basis of coherence is varied independently of the basis of the maximally coherent states. Due to this fact, we have that $Q_-(M) \leq Q_0(M)$. Second, as an upper bound to $Q_0$, we consider the minimal coherence in any basis maximized over all states in the range:

**Definition 4.** For a quantum channel $M$, the quantity $Q_+$ is defined by

$$Q_+(M) := \min_{\rho} \max_{U} C_{\rho,U}[M(\rho)]$$  \hspace{1cm} (6)

$$= \min_{\rho} \max_{|ψ⟩} C_{\rho,ψ}[M(|ψ⟩)],$$

where the equality is due to the convexity of the coherence measure and linearity of $M$.

Here, in contrast to $Q_0$, the maximization is not limited to maximally coherent states. Hence, we have that

$$Q_-(M) \leq Q_0(M) \leq Q_+(M).$$  \hspace{1cm} (7)

Note that due to the minimization over all bases $U$ (and $U'$ for $Q_-$), for all channels $M$ and unitary channels $V$ with $V(\rho) = V \rho V^\dagger$ where $V$ is any unitary, we have the following identities:

$$Q_+(M) = Q_+(V \circ M) = Q_+(M \circ V),$$  \hspace{1cm} (8)

$$Q_0(M) = Q_0(V \circ M \circ V^{-1}).$$

Interestingly, the quantities $Q_\pm$ are completely invariant under prior and subsequent rotations, whereas $Q_0$ is only invariant under joint rotations. As such, the quantities $Q_\pm$ are useful to obtain bounds on $Q_0$. However, their physical interpretation is not so clear: On the one hand, since coherence is a basis-dependent concept, the individual minimization over the bases in the definition of $Q_-$ is counter-intuitive. On the other hand, the definition of $Q_+$ also accounts for an increase of coherence and therefore does not admit an interpretation of how well coherence is preserved.

Finally, note that all measures are continuous in the space of quantum channels:

**Lemma 5.** The quantities $Q_\pm$ and $Q_0$ are continuous.

**Proof.** Note that the expression $C_{U,U'}[M(|\Psi_{U'}^\alpha⟩)]$ is continuous in $\alpha, U'$ and $U''$, as well as in $M$, which can be seen by using the Choi-Jamiołkowski isomorphism [23] [24]. As the maximizations’ and minimizations’ domains can be chosen compact, the measures $Q_-$ and $Q_0$ are continuous in $M$ as well [23]. For $Q_+$, a similar argument holds. \hfill $\square$
IV. PROPERTIES OF $Q_-$, $Q_0$ AND $Q_+$

We will now show that the quantities $Q_-$, $Q_0$ and $Q_+$ are sharp memory quality measures. We start by showing property M1.

**Lemma 6.** The measures $Q_\pm$ and $Q_0$ fulfill property M1, i.e. $Q(\mathcal{V}) = 1$ for all unitary channels $\mathcal{V}$.

**Proof.** As $Q_-(\mathcal{M}) \leq Q_0(\mathcal{M}) \leq Q_+(\mathcal{M})$, it suffices to show the property for $Q_-$. Furthermore, as $Q_-$ is invariant under unitary rotations, it suffices to consider only the identity channel $\mathcal{I}$. Note that

$$Q_-(\mathcal{I}) = \min_{U, U'} \max_{\tilde{a}, \tilde{b}} C_{U}(U Z_{\tilde{a}} \Psi |+\rangle) = 1,$$

where $Z_{\tilde{a}}$ is a diagonal matrix with phases $e^{i\alpha_i}$ as entries, is equivalent to the statement that for all bases $U$ and $U'$, there exists a maximally coherent state in $U$ that is also maximally coherent in $U'$. Formally,

$$\forall U, U' \exists \tilde{a}, \tilde{b} : U Z_{\tilde{a}} |+\rangle = U' Z_{\tilde{b}} |+\rangle$$

$$\Leftrightarrow \forall U \exists \tilde{a}, \tilde{b} : Z_{\tilde{a}}^\dagger U Z_{\tilde{b}} |+\rangle = |+\rangle,$$

which is equivalent to the statement that the sets of maximally coherent states w.r.t. two different bases always have non-empty intersection. This interesting geometrical question has been investigated and answered positively recently; it was shown that any unitary operator $\mathcal{U}$ can be decomposed as

$$U = Z_1 X Z_2,$$

where $Z_1$ and $Z_2$ are diagonal unitaries with the upper left entry equal to one and $X$ is a unitary matrix where the elements of each row and each column sum to one. Inserting this decomposition into Eq. (10) shows that choosing $\tilde{a}$ and $\tilde{b}$ such that $Z_{\tilde{a}} = Z_1^\dagger$ and $Z_{\tilde{b}} = Z_2$ yields the desired equality, as $|+\rangle$ is an eigenstate of $X$.

Now we prove the converse statement:

**Theorem 7.** $Q_\pm$ and $Q_0$ fulfill property M1', i.e., if $Q(\mathcal{M}) = 1$, then $\mathcal{M}$ is a unitary channel.

**Proof.** To prove the theorem, it is sufficient to consider $Q_+(\mathcal{M}) = 1$, as $Q_-(\mathcal{M}) \leq Q_0(\mathcal{M}) \leq Q_+(\mathcal{M}) \leq 1$. If $Q_+(\mathcal{M}) = 1$, then for all unitaries $U$ it holds that

$$\max_{\psi} C_{U}(\mathcal{M}(\psi)) = 1.$$

This implies that for all $U$, there exists a state $|\Psi\rangle$ and a maximally coherent state $|\Psi\rangle$ w.r.t. $U$ such that

$$\mathcal{M}(|\Phi\rangle) = |\Psi\rangle.$$

To prove the statement, we show the following three facts: (1) If $Q_+(\mathcal{M}) = 1$, then we find a basis $\{|\Psi_i\rangle\}$ that is mapped to a basis $\{|\Psi_i\rangle\}$ by $\mathcal{M}$. (2) In the range of $\mathcal{M}$, there exist vectors $\{|\Psi_{ij}\rangle = \sum_{i=1}^{D} \beta_{ij}^{(j)} |\Psi_{ij}\rangle\}_{j=1}^{D}$ with the property $\beta_{ij}^{(j)} \neq \beta_{ij}^{(j)}$ for all $j$. (3) From the existence of the $|\Psi_{ij}\rangle$ and $|\Psi_{ij}\rangle$, it follows that $\mathcal{M}$ is unitary.

For the first fact, in order to find the state $|\Psi_{1}\rangle$, we simply choose a random basis and obtain a pure (maximally coherent) state in the range of $\mathcal{M}$ due to the property $Q_+(\mathcal{M}) = 1$. For the second state $|\Psi_{2}\rangle$, we choose a basis with $|\Psi_{1}\rangle$ as a basis state. The corresponding maximally coherent state has an overlap of $|\langle \Psi_{2}\rangle| = \frac{1}{\|\tilde{D}\|}$ and is therefore linearly independent. All other states $|\Psi_{2}\rangle$ can be found step by step: Let us assume that we have already found $|\Psi_{1}\rangle, \ldots, |\Psi_{m}\rangle$ linearly independent states. We construct an orthonormal set of states spanning the same subspace and extend it to an orthonormal basis. The corresponding maximally coherent state has non-vanishing overlap with the space orthogonal to $\text{span}\{|\Psi_{2}\rangle, \ldots, |\Psi_{m}\rangle\}$ and is therefore also linearly independent.

With this procedure we obtain the non-orthonormal basis $\{|\Psi_{1}\rangle\}$. The corresponding preimages also form a basis, as from the Kraus decomposition (see also below) it follows that the dimension of their span must be equal to $D$ as well.

For the second fact, we have to show the existence of the vectors $\{|\Psi_{1}\rangle\}$ with the properties mentioned above. It suffices to show the existence of $|\Psi_{12}\rangle$; the proof for the other $|\Psi_{1j}\rangle$ is analogous.

Given the basis $\{|\Psi_{1}\rangle\}$, we consider the normalized dual basis $\{|\gamma_{i}\rangle\}$ with the property $\langle \gamma_{i}|\Psi_{1}\rangle = c_{i} b_{i}$ for some $c_{i} > 0$ [27]. In this basis, $\beta_{ij}^{(j)} = c_{i}^{-1} \langle \gamma_{i}|\Psi_{j}\rangle$ holds. Now we search for a vector $|\Psi_{12}\rangle$ in the range of $\mathcal{M}$ with the properties $\langle \gamma_{1}|\Psi_{12}\rangle \neq 0 \neq \langle \gamma_{2}|\Psi_{12}\rangle$, as from these conditions the presence of the desired coefficients $\beta_{1}^{(2)}$ and $\beta_{2}^{(2)}$ follows.

To that end, consider the orthonormal basis $|b_{1}\rangle = |\gamma_{1}\rangle, |b_{2}\rangle \propto |\gamma_{2}\rangle - |\gamma_{1}\rangle$ and the other $|b_{i}\rangle$ arbitrary. The maximally coherent state $|\Psi\rangle$ in the range of $\mathcal{M}$ in this basis can be written as $|\Psi\rangle = \frac{1}{\sqrt{2}} \sum_{i=1}^{D} e^{i\phi_{i}} |b_{i}\rangle$.

The overlaps are given by

$$\langle \gamma_{1}|\Psi\rangle \propto e^{i\phi_{1}} \neq 0,$$

$$\langle \gamma_{2}|\Psi\rangle \propto e^{i\phi_{1}} + e^{i\phi_{2}},$$

$$\langle \gamma_{2}|\Psi\rangle \propto e^{i\phi_{1}} - e^{i\phi_{2}} \neq 0.$$

If $|\langle \gamma_{2}|\Psi\rangle| = |\langle \gamma_{2}|\Psi\rangle|$, $|\Psi_{12}\rangle = |\Psi\rangle$ satisfies the desired properties.

Otherwise, we instead choose the basis $|b_{1}'\rangle = a |b_{1}\rangle + be^{i\theta}|b_{2}\rangle$, $|b_{2}'\rangle = b |b_{1}\rangle - ae^{i\theta}|b_{2}\rangle$, with $a > b > 0$. Now, the maximally coherent state $|\Psi\rangle$ is $\frac{1}{\sqrt{a}} \sum_{i=1}^{D} e^{i\phi_{i}} |b_{i}\rangle$, with respect to the basis $|b_{i}'\rangle$, has the overlaps

$$\langle \gamma_{1}|\Psi\rangle \propto ae^{i\phi_{1}} + be^{i\phi_{2}} \neq 0,$$

$$\langle \gamma_{2}|\Psi\rangle \propto (ae^{i\phi_{1}} + be^{i\phi_{2}}) \langle \gamma_{2}|b_{1}\rangle$$

$$+ (be^{i\theta} - ae^{i\theta}) e^{i\theta} \langle \gamma_{2}|b_{2}\rangle.$$

As in this case $|\langle \gamma_{2}|b_{1}\rangle| \neq |\langle \gamma_{2}|b_{2}\rangle|$, we can choose $\theta$ such that $\langle \gamma_{2}|b_{1}\rangle = e^{i\theta} \langle \gamma_{2}|b_{2}\rangle$, and hence, the right hand
side of Eq. (16) does not vanish. Thus, in this case we choose |Ψ_{12}⟩ = |Ψ⟩.

Finally, concerning the third fact, as ℳ is a quantum channel, it admits a Kraus representation, i.e. \( ℳ(ρ) = \sum_{l=1}^{r} K_l p K_l^\dag \) with \( \sum_l K_l^\dag K_l = 1 \). Using the fact that the \( |Φ_i⟩ \) are mapped to pure states, we have for all \( l = 1, \ldots, r \) that

\[ K_l |Φ_i⟩ = μ_{li} |Ψ_i⟩ \]  

for \( i = 1, \ldots, D \), and

\[ K_l |Φ_{lj}⟩ = κ_{lj} |Ψ_{lj}⟩ \]  

for some \( |Φ_{lj}⟩ = \sum_{k=1}^{D} α_{kj}^{(j)} |Φ_k⟩ \) and \( j = 2, \ldots, D \).

Decomposing the right hand side of Eq. (16) in terms of the basis \( \{|Ψ_i⟩\} \) and using linearity on the left hand side, we have

\[ \kappa_{lj} \sum_{k=1}^{D} β_{jk}^{(j)} |Ψ_k⟩ = \sum_{k=1}^{D} μ_{lk} α_{kj}^{(j)} |Ψ_k⟩ \]  

(19)

for all \( l \). Thus, for all \( l, j \) and \( k \),

\[ \kappa_{lj} β_{jk}^{(j)} = μ_{lk} α_{kj}^{(j)}. \]  

(20)

For a fixed \( j \), consider the two equations for \( k = 1 \) and \( k = j \), where the corresponding \( β_{jk}^{(j)} \) do not vanish by assumption. If \( α_{1j}^{(j)} \) or \( α_{jj}^{(j)} \) were zero, \( κ_{lj} = 0 \) for all \( l \) would follow. This would imply that \( ℳ(|Φ_{lj}⟩) = 0 \), which cannot be true if \( ℳ \) is a channel.

Otherwise, if \( κ_{lj} \) was zero for one \( l \), then this would imply \( μ_{li} = μ_{lj} = 0 \) for this \( l \). However, \( μ_{li} = 0 \) implies that \( κ_{lj'} = 0 \) for all \( j' \) which in turn implies that \( μ_{lj'} = 0 \) for all \( j' \). Thus, \( K_l \) would map a whole basis to zero and therefore vanishes, and can be neglected from the decomposition of the channel.

Thus, we have that \( κ_{lj} \neq 0 \) and from that \( μ_{li} \neq 0 \neq μ_{lj} \). Then, the ratio

\[ \frac{μ_{li}}{μ_{lj}} = \frac{β_{lj}^{(j)} α_{1j}^{(j)}}{β_{lj}^{(j)} α_{1j}^{(j)}} \]  

(21)

is independent of \( l \). As this holds for all \( j \), it follows from Eq. (17) that the \( K_l \) must be proportional to each other, i.e. \( K_l \propto K_l' \). Using now that \( ℳ \) is trace preserving, i.e. \( \sum_l K_l^\dag K_l = 1 \), leads to \( K_l^\dag K_l \propto 1 \). Thus, all Kraus operators have to be proportional to the same unitary \( V \) and hence, \( ℳ(ρ) = V ρ V^\dag \).

As noted before, the continuity of \( Q_+ \) and \( Q_0 \) together with property \( M1' \) implies that property \( M2 \) is fulfilled as well. Thus:

**Corollary 8.** The quantities \( Q_+ \) and \( Q_0 \) are sharp memory quality measures.

![FIG. 1. Top: The image of the Bloch sphere (red) of single-qubit maps is an ellipsoid (blue) with semi-axes \( λ_i \), displaced by \( k \). Bottom left: Projection of the ellipsoid in 1-2-direction to obtain upper bounds on the measures. The red dots indicate the points of the image of maximally coherent states in some basis which touch the boundary of the projected ellipse. Bottom right: Projection of the ellipsoid in the direction of \( k \). The semi-axes of the projection are bounded by the semi-axes of the ellipsoid.](image)

The proof above does not allow to infer a specific numerical value for the bound on measure-and-prepare channels. In the case of single-qubit channels, however, we can find tight bounds (see Theorem 12 in the next section).

Apart from the properties listed above, the quality measure \( Q_+ \) satisfies another useful property: pre-processing a quantum state before storing it in a quantum memory cannot improve the quality of the extended storage process.

**Lemma 9.** The quality measure \( Q_+ \) cannot increase by pre-processing of the input, i.e. \( Q_+(ℳ \circ ℵ) \leq Q_+(ℳ) \) for all quantum channels \( ℳ \) and \( ℵ \).

**Proof.** This can be easily seen by recalling that \( Q_+(ℳ) = \min_{U} \max_{|ψ⟩} C_U(ℳ(|ψ⟩)) \), hence, we have that

\[ Q_+(ℳ \circ ℵ) = \min_{U} \max_{|ψ⟩} C_U(ℳ(ℵ(|ψ⟩))) \leq \min_{U} \max_{|ψ⟩} C_U(ℳ(|ψ⟩)) = Q_+(ℳ), \]  

(22)

which proves the lemma.

For \( Q_- \), we can prove a similar statement for the case of unital single-qubit channels (see Lemma 13 in the next section).

### V. THE SINGLE-QUBIT CASE

Single-qubit channels, i.e. channels with \( D = 2 \), have additional useful properties. In particular, we can find
tight bounds for the quality $Q_0(\mathcal{M})$ if $\mathcal{M}$ is a measure-and-prepare channel.

Single-qubit channels are special in the sense that their action can be well understood in the Bloch picture. The Bloch decomposition of a qubit state is given by $\rho = \frac{1}{2}(1 + \vec{v} \cdot \vec{\sigma})$, where $\vec{v} \in \mathbb{R}^3$ is required to have length equal or smaller than 1 in order for $\rho$ to be positive semi-definite, and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)^T$ with $\sigma_i$ being the Pauli matrices.

Any trace-preserving quantum channel corresponds to an affine transformation $\vec{v} \mapsto \Lambda \vec{v} + \vec{\kappa}$ with a real matrix $\Lambda$ and a displacement vector $\vec{\kappa}$ defined by displacement vector $\vec{\kappa}$ and transformation matrix $\Lambda$ with singular values $\lambda_1 \leq \lambda_2 \leq \lambda_3$. Let $\vec{\kappa} = (\kappa_1, \kappa_2, \kappa_3)^T$ in the bases where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$. Then, $Q_-(\mathcal{M}) \leq \min(\sqrt{\kappa_1^2 + \kappa_2^2 + \lambda_1}, \lambda_2)$ and $Q_0(\mathcal{M}) \leq Q_+(\mathcal{M}) \leq \min(\sqrt{\kappa_1^2 + \kappa_2^2 + \lambda_2}, \lambda_3)$.

Proof. Instead of minimizing over all bases, we restrict the minimization to a discrete set to obtain an upper bound. For both $Q_-(\mathcal{M})$ and $Q_+(\mathcal{M})$, we consider the axes along $\vec{\kappa}$ and along the largest singular value of $\Lambda$.

To obtain an upper bound on $Q_+(\mathcal{M})$, we simply take into account all states on the surface of the ellipsoid. The largest possible distance to the axis along $\vec{\kappa}$ clearly is $\lambda_3$ since the axis goes through the center of the ellipsoid (see bottom right in Fig.1). Similarly, the distance from the axis along $\lambda_3$ is the distance to the center, which is given by $\sqrt{\kappa_1^2 + \kappa_2^2}$ plus at most $\lambda_2$ since the axis is parallel to $\lambda_3$ (see bottom left in Fig.1). Because of the minimization over all bases, an upper bound is then given by $\min(\sqrt{\kappa_1^2 + \kappa_2^2 + \lambda_2}, \lambda_3)$.

In the case of $Q_-$, we can additionally choose the set of maximally coherent states. Since the channel $\mathcal{M}$ corresponds to an affine transformation of $\vec{v}$, any ellipse on the surface of the ellipsoid with the same center as the ellipsoid is the image of a great circle on the surface of the Bloch sphere. Each of these circles is the set of maximally coherent states with respect to some basis. Hence, we can choose any ellipse on the surface of the ellipsoid and determine the maximal distance to the chosen axis to obtain an upper bound. For the axis along $\vec{\kappa}$, we choose the ellipse with semi-axes $\lambda_1$ and $\lambda_2$. Then, the maximal distance is at most $\lambda_2$ since the axis goes through the center of the ellipse. In the case of the axis along $\lambda_3$, the ellipse with semi-axes $\lambda_1$ and $\lambda_3$ limits the maximal distance to $\sqrt{\kappa_1^2 + \kappa_2^2} + \lambda_1$ (see bottom left in Fig.1). Again, the minimum of the cases considered gives an upper bound on $Q_-(\mathcal{M})$.

One can also find lower bounds on the quantities, which will be useful for later applications.

Lemma 11. Let $\mathcal{M}$ be a single-qubit quantum channel defined by displacement vector $\vec{\kappa}$ and transformation matrix $\Lambda$ with singular values $\lambda_1 \leq \lambda_2 \leq \lambda_3$. Then, $Q_0(\mathcal{M}) \geq Q_-(\mathcal{M}) \geq \lambda_1$ and $Q_+(\mathcal{M}) \geq \lambda_2$. If $\mathcal{M}$ is unital ($\vec{\kappa} = 0$), equality holds.

Proof. In order to find lower bounds, we have to show the bound in all coherence bases.

For $Q_+$, we have to consider – for every coherence basis – the maximal distance to the center of the projection of the ellipsoid onto the plane perpendicular to the coherence direction. This projection is an ellipse with semi-axes $\mu_1 \geq \lambda_1$ and $\mu_2 \geq \lambda_2$, displaced by some vector from the center. If the displacement is zero, the maximal distance is given by $\mu_2$ and therefore at least $\lambda_2$. For non-vanishing displacement, the maximal distance can only increase, yielding the lower bound for $Q_+$.

For $Q_-(\mathcal{M})$, we additionally have to minimize the maximal distance to the axis of two opposite points on this ellipse, due to the additional minimization over the input coherent states. This is in any case larger than $\mu_1$ and therefore larger than $\lambda_1$.

Finally, if the channel is unital, note that the minimum over the coherence bases is attained in the direction of $\lambda_3$, where for $Q_-(\mathcal{M})$, we consider the states mapped to an ellipse along the $\lambda_1$-$\lambda_3$-axes, giving a maximum distance of $\lambda_1$. For $Q_+(\mathcal{M})$, the maximum distance of the non-displaced ellipsoid in this basis is given by $\lambda_2$.

We now use the upper bounds on the quality measures to obtain tight bounds for measure-and-prepare qubit channels.
Theorem 12. Let $\mathcal{M}$ be a single-qubit measure-and-prepare quantum channel. Then, it holds that $Q_0(\mathcal{M}) \leq Q_+(\mathcal{M}) \leq \frac{1}{\sqrt{2}}$ and $Q_-(\mathcal{M}) \leq \frac{1}{\sqrt{2}}$. Additionally, if $\mathcal{M}$ is unital ($\vec{\kappa} = 0$), $Q_0(\mathcal{M}) \leq Q_+(\mathcal{M}) \leq \frac{1}{2}$ and $Q_-(\mathcal{M}) \leq \frac{1}{2}$. All of these bounds are tight.

Proof. Let $\mathcal{M}$ be defined by displacement vector $\vec{\kappa}$ and transformation matrix $\Lambda$ with singular values $\lambda_1 \leq \lambda_2 \leq \lambda_3$. Since we only consider $Q_+$ and $Q_-$, we can assume w.l.o.g. that $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$. Let $\vec{X} = (\lambda_1, \lambda_2, \lambda_3)^T$. Complete positivity of a single-qubit channel $\mathcal{M}$ is equivalent to $\rho_M \geq 0$ where $\rho_M$ is the Choi matrix of $\mathcal{M}$ [23, 24, 25]. Using Descartes’ rule of signs [29] on the characteristic polynomial of the Choi matrix $\rho_M$, complete positivity of the channel is equivalent to the following set of inequalities

\begin{align}
|\vec{\kappa}|^2 + |\vec{X}|^2 &\leq 3, \quad (23) \\
|\vec{\kappa}|^2 + |\vec{X}|^2 - 2\lambda_1\lambda_2\lambda_3 &\leq 1, \quad (24) \\
(1 - |\vec{\kappa}|^2)^2 - 2(1 - |\vec{\kappa}|^2)|\vec{X}|^2 - \frac{1}{2}|\vec{X}|^4 + 8\lambda_1\lambda_2\lambda_3 + \frac{1}{2} \sum_i D_i^2 - 4\vec{K} \cdot \vec{L} &\geq 0, \quad (25)
\end{align}

where $D_i = \sum_{j=1}^3(-1)^{i+j}\lambda_j^2$, $\vec{K} = (\kappa_1^2, \kappa_2^2, \kappa_3^2)^T$ and $\vec{L} = (\lambda_1^2, \lambda_2^2, \lambda_3^2)^T$. Similarly, single-qubit channels are measure-and-prepare channels if and only if $\frac{1}{2} I - \rho_M$ is positive semi-definite [25]. This yields the same set of equations with $\lambda_i \leftrightarrow -\lambda_i$. In the following, we apply these restrictions to Lemma 10. Clearly, the bounds from Lemma 10 only become worse if $\vec{\kappa}$ is rotated such that $\vec{\kappa} = (|\kappa_i|, 0, 0)^T$. However, rotating a measure-and-prepare channel in such a way always leads to another measure-and-prepare channel as can be seen from Eqs. (23) to (25). Thus, we can restrict ourselves to this type of channels. For these channels, the eigenvalues can be evaluated analytically and maximization of the bounds over these channels for $Q_-$ results in the channel

$$
\mathcal{M}_-(\rho) = \frac{1}{2} \left[ I + \frac{1}{\sqrt{2}} (\sigma_x + \text{Tr}(\rho\sigma_y)\sigma_y + \text{Tr}(\rho\sigma_z)\sigma_z) \right].
$$

(26)

It is visualized in the Bloch picture in Fig. 2 and has a quality of $Q_-(\mathcal{M}_-) = \frac{1}{\sqrt{2}}$. For $Q_+$, the optimization of the bounds over the channels yields

$$
\mathcal{M}_+(\rho) = \frac{1}{2} \left[ I + \frac{1}{\sqrt{2}} (\sigma_x + \text{Tr}(\rho\sigma_y)\sigma_y + \text{Tr}(\rho\sigma_z)\sigma_z) \right],
$$

(27)

with $Q_0(\mathcal{M}_+) = Q_+(\mathcal{M}_+) = \frac{1}{\sqrt{2}}$. The channel is visualized in Fig. 2.

For unital channels, i.e. $\vec{\kappa} = 0$, the condition for separability reads $\sum_i |\lambda_i| \leq 1$ [19]. Maximizing under this constraint yields for $Q_-$ the depolarizing channel

$$
\mathcal{M}_-^\prime(\rho) = \frac{1}{3} \rho + \frac{1}{3} I
$$

(28)

with $Q_-(\mathcal{M}_-^\prime) = \frac{1}{3}$. For $Q_0$ and $Q_+$, we obtain the planar channel

$$
\mathcal{M}_+^\prime(\rho) = \frac{1}{2} \left[ I + \frac{1}{2} (\text{Tr}(\rho\sigma_y)\sigma_y + \text{Tr}(\rho\sigma_z)\sigma_z) \right]
$$

(29)

with $Q_0(\mathcal{M}_+^\prime) = Q_+(\mathcal{M}_+^\prime) = \frac{1}{2}$.

It should be noted that all of these qualities reach the upper bounds in Lemma 11 and hence, the bounds given in the Lemma are tight as well.

Finally, note that we have a statement similar to Lemma 1 for $Q_-$ if the channel is unital:

Lemma 13. Let $\mathcal{M}$ and $\mathcal{N}$ be unital channels acting on single qubits ($D = 2$). Then, it holds that $Q_-(\mathcal{M} \circ \mathcal{N}) \leq Q_-(\mathcal{M})$.

Proof. First, note that the composition of unital channels is again a unital channel. As shown in Lemma 11, the quality measure $Q_-(\mathcal{M})$ for a unital channel $\mathcal{M}$ is given by the minimal singular value of the matrix $\Lambda_{\mathcal{M}}$, i.e. $\lambda_1(\Lambda_{\mathcal{M}})$. With this, we have that

$$
Q_-(\mathcal{M} \circ \mathcal{N}) = \lambda_1(\Lambda_{\mathcal{M} \circ \mathcal{N}}) \\
\leq \lambda_1(\Lambda_{\mathcal{M}})\lambda_3(\Lambda_{\mathcal{N}}) \\
\leq \lambda_1(\Lambda_{\mathcal{M}}) = Q_-(\mathcal{M}).
$$

(30)

For the first inequality, we used that $\Lambda_{\mathcal{M} \circ \mathcal{N}} = \Lambda_{\mathcal{M}}\Lambda_{\mathcal{N}}$ and Theorem 3.3.16 from Ref. [30]. The second inequality follows from the fact that for channels, all the singular values of the matrix $\Lambda$ have to be smaller or equal to 1. \qed
VI. EXAMPLES OF SINGLE-QUBIT CHANNELS

In the following, we will consider several well-known channels and derive their quality in terms of $Q_\pm$ and $Q_0$.

A. The phase-flip channel $P$

The matrix $A$ for the unital (i.e., $\tilde{\kappa} = 0$) phase-flip channel $P$, is given by $\text{diag}(1-p, 1-p, 1-p)$ with $0 \leq p \leq 1$. It can be realized by a measure-and-prepare scheme for $p = 1$ only.

Using the result from Lemma 11 for unital channels, we have that $Q_-(P) = Q_0(P) = Q_+(P) = 1 - p$. It should be noted that any bit-flip or bit-phase-flip channel is related to a phase-flip channel with the same error probability $p$ via a transformation of the form $V \circ P \circ V^{-1}$ where $V(\rho) = V \rho V^\dagger$ is a unitary channel. Hence, the quality measure $Q_\pm$ and $Q_0$ for these channels with same error probability coincide. Note that $Q_-$ excludes unitary measure-and-prepare schemes for $p < \frac{2}{3}$, while $Q_+$ and $Q_0$ exclude them for $p < \frac{1}{2}$.

B. The amplitude-damping channel $A$

The matrix $A$ for the amplitude-damping channel $A$ is given by $\text{diag}(\sqrt{1-p}, \sqrt{1-p}, 1-p)$ and $\tilde{\kappa} = (0, 0, p)^T$, where $0 \leq p \leq 1$. This channel can again be implemented by measure-and-prepare schemes only if $p = 1$. Considering the maximal coherence of the states in the image of this channel with respect to the computational basis shows that $Q_+(A) \leq \sqrt{1-p}$. Using that $\lambda_1 \leq Q_- \geq Q_0 \leq Q_+ = \sqrt{1-p}$. Thus, $Q_-$ excludes measure-and-prepare schemes for $p < \frac{2}{3}$, whereas $Q_+$ and $Q_0$ exclude them for $p < \frac{1}{2}$.

C. The planar channel $L$

The planar channel $L$ is a unital channel with $\Lambda = \text{diag}(0, s, q)$ that can always be implemented using a measure-and-prepare scheme. We will assume w.l.o.g. that $s \leq q$, as this can be achieved by transformations of the type $V \circ P \circ V^{-1}$ where $V$ is a unitary channel. Complete positivity then requires that $s \leq 1 - q$. Using again Lemma 11, we obtain $Q_-(L) = 0$ and $Q_+(L) = s$. In order to determine $Q_0$, we note that we can obtain a lower bound if we choose the value of $\delta$ for each basis $U$ such that $L(U(\lambda))$ has no contribution in $\sigma_3$-direction. Note that any specific choice of $\alpha$ (as a function of $U$) leads to a lower bound on $Q_0$. For our choice, the minimization over $U$ then yields, for each $U$, $L(U(\lambda)) = s$ which, together with $Q_+(L) = s$, implies that $Q_0(L) = s$.

VII. EXPERIMENTAL ESTIMATION OF THE QUALITY OF A QUANTUM MEMORY

In this section, we explain how to determine a lower bound on the quality measures from experimental data for qubit systems. These lower bounds will be non-trivial if the memory is sufficiently close to the identity channel. This situation is of major interest as a perfect quantum memory corresponds exactly to the identity channel.

Obviously, it is possible to obtain (lower bounds on) the quality measures by performing process tomography of the channel and then using the obtained characterization. However, process tomography requires the ability to prepare a set of input states with high precision as well as many and well characterized measurements [11, 12]. Here, we only assume that one can prepare three different states $\{\rho_i\}_{i=1}^3$ for which one can certify a lower bound $c_i \in [0, 1]$ on the following coherences (see e.g. 31) for possible methods to obtain such bounds

$$
C_{U_1}(\rho_1) \geq c_1, \quad C_{U_2}(\rho_1) \geq c_1, \\
C_{U_1}(\rho_2) \geq c_2, \quad C_{U_2}(\rho_2) \geq c_2, \\
C_{U_1}(\rho_3) \geq c_3, \quad C_{U_2}(\rho_3) \geq c_3,
$$

(31)

where the $U_j$ correspond to the usual $x, y$ and $z$ direction on the Bloch sphere (i.e., $U_j = \sqrt{i\sigma_j} = e^{i\sigma_j \pi/4}$ for $j = x, y, z$). Note that we only assume a bound on the coherence of the output of the quantum memory, nothing additional is assumed on the input- or output states.

For simplicity, we consider the case where $c := c_1 = c_2 = c_3$. As the smallest semi-axis is a lower bound on $Q_-$, one can then determine the channel that shows the smallest possible $\lambda_1$ compatible with the observed data. In particular, it is required that the image of the channel contains states for which the bounds given in Eqs. (31) are fulfilled. For $c > \sqrt{2/3} \approx 0.82$, there must be at least three different states close to the boundary of the Bloch sphere. Numerically optimizing over all compatible
channels leads to the lower bounds depicted in Fig. 5. Hence, for values of \( c \gtrsim 0.82 \) it is possible to obtain lower bounds on the quality measure \( Q_- \) (and hence, also on \( Q_0 \) and \( Q_+ \)) by having access only to a few lower bounds on the coherences of three different states. Measure-and-prepare channels can be excluded with certainty if \( Q_- > \sqrt{1-p} \approx 0.45 \), which is given for \( c \gtrsim 0.9 \).

As an example, consider the amplitude-damping channel \( \mathcal{A} \) from above. One can find states for which \( c = \sqrt{1-p} \), and thus exclude measure-and-prepare channels for \( p \lesssim 0.19 \).

**VIII. CONCLUSIONS**

We introduced a physically motivated measure \( Q_0 \) that characterizes quantum memories by their ability to preserve coherence. Using the upper and lower bound \( Q_\pm \) we proved that the measure fulfills all the desirable properties for such a quantifier. For a single-qubit quantum memory, the measure can be evaluated for many scenarios, even if only restricted experimental data are available. In contrast to full process tomography, our scheme does not require the precise preparation of states but only demands the certification of (sufficiently high) lower bounds on certain coherences of three unknown states.

For future work, it would be desirable to extend the method to characterize and verify other basic elements of quantum information processing. A simple extension is the case of quantum teleportation, here the results can be directly applied. More interesting is an application to two-qubit gates. The fact that a two-qubit gate generates entanglement, can be seen as the property that a certain two-level coherence increases \([17, 15]\). In this sense, our method may be extended to characterize the entangling capability of multi-qubit quantum gates.

*Note added:* While finishing this work, we became aware of a similar approach \([32]\), also introducing a measure on quantum channels using coherence. Instead of considering the most robust or maximally coherent states, the authors are interested in the average coherence preserved over all states.

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