THE ZAPPASZÉP PRODUCT OF A FELL BUNDLE AND A GROUPOID

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Abstract. We define the Zappa-Szép product of a Fell bundle by a groupoid, which turns out to be a Fell bundle over the Zappa-Szép product of the underlying groupoids. Under certain assumptions, every Fell bundle over the Zappa-Szép product of groupoids arises in this manner. We then study the representation associated with the Zappa-Szép product Fell bundle and show its relation to covariant representations. Finally, we study the associated universal \( C^\ast \)-algebra, which turns out to be a \( C^\ast \)-blend, generalizing an earlier result about the Zappa-Szép product of groupoid \( C^\ast \)-algebras. In the case of discrete groups, the universal \( C^\ast \)-algebra of a Fell bundle embeds injectively inside the universal \( C^\ast \)-algebra of the Zappa-Szép product Fell bundle.

1. Introduction

The Zappa-Szép product originated as a generalization of the semi-direct product of groups. For groups \( G \) and \( H \), in addition to encoding an \( H \)-action on \( G \) in the semi-direct product, the Zappa-Szép product encodes a \( G \)-restriction map on \( H \). This results in a two-way interaction between \( G \) and \( H \) in their Zappa-Szép product.

The crossed product construction, in the realm of operator algebras, closely resembles that of a semi-direct product: given a \( C^\ast \)-algebra \( A \) and a group \( H \) acting on \( A \) by automorphisms, one can define the algebraic crossed product \( A \rtimes^\alpha H \) in a similar fashion to the semi-direct product. There are various ways to take the closure of the algebraic crossed product that could result in different \( C^\ast \)-algebras that have drawn much research interest.

Naturally, one may wonder if we can similarly construct a Zappa-Szép product of a \( C^\ast \)-algebra. This is the main motivation behind this paper. To define a Zappa-Szép product of a \( C^\ast \)-algebra \( A \), one must encode both an \( H \)-action on the \( C^\ast \)-algebra and an \( A \)-restriction on the group \( H \). One possible approach is to put a \( G \)-grading on \( A \), by dissembling \( A \) into linearly independent subspaces \( \{ A_x \}_{x \in G} \) whose direct sum is dense in \( A \) (for example, the notion of a graded \( C^\ast \)-algebra considered by Exel [8]). With this approach, the elements in \( A_x \) induce a restriction map on \( H \) in a similar manner as the \( G \)-restriction map in a Zappa-Szép product. Algebraically,
one can define a Zappa-Szép type product of the subspaces \( \{ A_x \} \) and \( \mathcal{H} \). However, this approach faces a key challenge: it is difficult, sometimes even impossible to reassemble the individual pieces back to a \( C^* \)-algebra, and even if that is possible, the resulting \( C^* \)-algebra might have lost much of the information of the original \( C^* \)-algebra \( A \).

To avoid these difficulties, instead of decomposing a \( C^* \)-algebra \( A \), we start directly with a collection of fibres \( \{ A_x \} \) that can be reasonably assembled into different \( C^* \)-algebras. The approach to study the fibres instead of the \( C^* \)-algebra as a whole leads us to the notion of Fell bundles.

Suppose \( \mathcal{G} \) and \( \mathcal{H} \) are étale groupoids that have a Zappa-Szép product groupoid \( \mathcal{G} \bowtie \mathcal{H} \), which is known to be étale as well \[3\]. Given a Fell bundle \( \mathcal{B} \) over \( \mathcal{G} \), we first define the notion of a \((\mathcal{G}, \mathcal{H})\)-compatible \( \mathcal{H} \)-action \( \beta \) on the Fell bundle (Definition \[3.1\]) that allows us to construct a Zappa-Szép-type product \( \mathcal{B} \bowtie \beta \mathcal{H} \). This product turns out to be a Fell bundle over the Zappa-Szép product of the underlying groupoids \( \mathcal{G} \bowtie \mathcal{H} \) (Theorem \[3.8\]). Conversely, certain Fell bundles over \( \mathcal{G} \bowtie \mathcal{H} \) arise in this manner (Theorem \[3.15\]).

We then study representations of the Zappa-Szép product Fell bundle. We define (Definition \[4.1\]) the notion of covariant representations of the system \((\mathcal{B}, \beta)\), in parallel to the classical notion of covariant representations of a \( C^* \)-dynamical system. We prove an integration theorem that every covariant representation gives rise to an \( I \)-norm decreasing \( * \)-representation of \( \Gamma_c(\mathcal{G} \bowtie \mathcal{H}; \mathcal{B} \bowtie \beta \mathcal{H}) \) (Theorem \[4.7\]). Conversely, under the assumption that \( \mathcal{B} \) is unital, we also prove a disintegration theorem that every nondegenerate \( I \)-norm decreasing \( * \)-representation of \( \Gamma_c(\mathcal{G} \bowtie \mathcal{H}; \mathcal{B} \bowtie \beta \mathcal{H}) \) is equivalent to the integrated form of some covariant representation (Theorem \[4.10\]).

Finally, we study the universal \( C^* \)-algebra associated with the Zappa-Szép product Fell bundle. We first prove that the \( C^* \)-algebra of the Zappa-Szép product Fell bundle \( C^*(\mathcal{B} \bowtie \beta \mathcal{H}) \) is a \( C^* \)-blend of \( C^*(\mathcal{B}) \) and the groupoid \( C^* \)-algebra \( C^* \mathcal{H} \) when \( \mathcal{B} \) is unital (Theorem \[5.4\]). This generalizes an earlier result \[3\] that the groupoid \( C^* \)-algebra of the Zappa-Szép product groupoid \( C^*(\mathcal{G} \bowtie \mathcal{H}) \) is a \( C^* \)-blend of \( C^*(\mathcal{G}) \) and \( C^*(\mathcal{H}) \). Finally, in the case of discrete groups, we prove that any representation of the Fell bundle \( \mathcal{B} \) induces a covariant representation of the Zappa-Szép product Fell bundle (Lemma \[6.5\]), in analogy to the classical regular representation of \( C^* \)-dynamical systems (for example, \[23\] Example 2.14]). As a result, \( C^*(\mathcal{B}) \) embeds injectively inside \( C^*(\mathcal{B} \bowtie \beta \Lambda) \) for \( \Lambda \) a discrete group (Theorem \[6.6\]).

In recent years, the Zappa-Szép product has attracted much attention in the study of operator algebras. For example, recent studies include the \( C^* \)-algebra of self-similar groups \[20\]; the Zappa-Szép product of right LCM semigroups \[11, 22\] and étale groupoids \[3\]; self-similar group actions on graphs \[11\] and on \( k \)-graphs \[17, 18\]; self-similar groupoid actions on \( k \)-graphs \[1\]. Our hope is that this paper brings a new perspective into this line of research.
2. Preliminary

2.1. Zappa-Szép Product of Groupoids. In group theory, the Zappa-Szép product provides a way to construct a group from certain interactions between two groups. It is a natural generalization of the semi-direct product $G \rtimes H$ of groups $G$ and $H$, which encodes an $H$-action on the group $G$ by defining the multiplication

$$(x, h)(y, k) = (x(h \cdot y), hk).$$

However, it is possible that $G$ also acts on $H$. This is known as the $G$-restriction map $(x, h) \mapsto h|_x$ in the Zappa-Szép product, and the multiplication in the Zappa-Szép product encodes this two-way action by setting

$$(x, h)(y, k) = (x(h \cdot y), h|_y k).$$

For other algebraic structures, one can often define an analogous version of their Zappa-Szép product: for example, the case of right LCM semigroups is considered in [4]. For two groupoids $G$ and $H$, one can define a similar notion of Zappa-Szép product when their unit spaces $G(0), H(0)$ match [2], which we will recall below. One may refer to [21] for a detailed introduction of groupoids and to [3] for a detailed discussion on their Zappa-Szép products.

Let $G$ and $H$ be groupoids. For $x \in G$, define its source by $s_G(x) = x^{-1}x$ and its range by $r_G(x) = xx^{-1}$; similarly, define $s_H$ and $r_H$ for $H$. We assume that $G$ and $H$ have the same unit space $G(0) = H(0)$, so that the ranges of the maps $s_G, s_H, r_G, r_H$ are all the same set. Assume further that there exist a continuous right-action of $G$ on $s_H: H \to G(0)$, and a continuous left-action of $H$ on $r_G: G \to H(0)$. Following [3], we denote these by

$$(H \curvearrowright G): H \times_{r_G} G \to G, \quad (h, x) \mapsto h \cdot x,$$

and

$$(H \curvearrowleft G): H \times_{s_H} G \to H, \quad (h, x) \mapsto h|_x,$$

where we used the following notation for the fibered product:

$$(2.1) \ Y \times_X p X := \{(y, x) \in Y \times X : q(y) = p(x)\} \text{ whenever } Y \xrightarrow{q} Z \xleftarrow{p} X.$$

We call $\cdot$ the $H$-action map and $|$ the $G$-restriction map. Recall the following properties, intrinsic to groupoid actions:

- $(ZS1)$ $(h_1h_2) \cdot x = h_1 \cdot (h_2 \cdot x)$,
- $(ZS2)$ $r_G(h \cdot x) = r_H(h)$,
- $(ZS3)$ $r_G(x) \cdot x = x$,
- $(ZS4)$ $h_{|_{xy}} = (h_{|_{x}})_{|_{y}}$,
- $(ZS5)$ $s_H(h|_x) = s_G(x)$,
- $(ZS6)$ $h|_{s_H(h)} = h$.

We further assume that these actions satisfy the following compatibility conditions:

- $(ZS7)$ $s_G(h \cdot x) = r_H(h|_x)$,
- $(ZS8)$ $h \cdot (xy) = (h \cdot x)(h|_x \cdot y)$,
- $(ZS9)$ $(h_1h_2)|_x = h_1|_{h_2 \cdot x}h_2|_x$. 

**Definition 2.1** ([2], Definition 1.1). A pair of topological groupoids \((\mathcal{G}, \mathcal{H})\) equipped with a continuous \(\mathcal{H}\)-action map and continuous \(\mathcal{G}\)-restriction map which satisfy \((\text{ZS1})\) \((\text{ZS9})\) is called a matched pair.

The following lemma is taken from [3, Lemma 4] and is especially useful in later calculations.

**Lemma 2.2.** For any \(h \in \mathcal{H}\) and \(x \in \mathcal{G}\) with \(s_\mathcal{H}(h) = r_\mathcal{G}(x)\), we have

\begin{align*}
(\text{ZS10}) & \quad h \cdot s_\mathcal{H}(h) = r_\mathcal{H}(h), \\
(\text{ZS11}) & \quad r_\mathcal{G}(x)|_x = s_\mathcal{G}(x), \\
(\text{ZS12}) & \quad (h \cdot x)^{-1} = h|_x \cdot x^{-1}, \\
(\text{ZS13}) & \quad (h|_x)^{-1} = h^{-1}|_{h \cdot x}.
\end{align*}

We can now define the Zappa-Szép product groupoid as follows.

**Definition 2.3.** Suppose \((\mathcal{G}, \mathcal{H})\) is a matched pair. Define

\[ \mathcal{G} \rtimes \mathcal{H} := \mathcal{G} \times_{r_\mathcal{H}} \mathcal{H} = \{(x, h) : x \in \mathcal{G}, h \in \mathcal{H}, s_\mathcal{G}(x) = r_\mathcal{H}(h)\}, \]

with multiplicable pairs

\[ (\mathcal{G} \rtimes \mathcal{H})^2 = \{((x, h), (y, g)) : r_\mathcal{G}(y) = s_\mathcal{H}(h)\}. \]

Define multiplication

\[ (x, h)(y, g) = (x(y), h|_y g), \]

and inverse map

\[ (x, h)^{-1} = (h^{-1} \cdot x^{-1}, h^{-1}|_{x^{-1}}). \]

Then \(\mathcal{G} \rtimes \mathcal{H}\) is a groupoid [3], called the Zappa-Szép product of \(\mathcal{G}\) and \(\mathcal{H}\).

We point out that the set of idempotents \((\mathcal{G} \rtimes \mathcal{H})^0\) of \(\mathcal{G} \rtimes \mathcal{H}\) can be identified with \(\mathcal{G}^0 = \mathcal{H}^0\) since one can prove

\[ s_{\mathcal{G} \rtimes \mathcal{H}}(x, h) = (s_\mathcal{H}(h), s_\mathcal{G}(x)) \quad \text{and} \quad r_{\mathcal{G} \rtimes \mathcal{H}}(x, h) = (r_\mathcal{G}(x), r_\mathcal{H}(h)), \]

using Conditions \((\text{ZS8})\) \((\text{ZS4})\) \((\text{ZS5})\) and \((\text{ZS10})\) for the first and \((\text{ZS1})\) \((\text{ZS9})\) \((\text{ZS3})\) \((\text{ZS11})\) for the second equality.

Any self-similar group action on a groupoid gives rise to a Zappa-Szép product as follows.

**Example 2.4.** Let \(\mathcal{G}\) be an étale groupoid, and let \(\sigma : H \to \text{Aut}(\mathcal{G})\) be an action of a discrete group \(H\) on \(\mathcal{G}\). Denote \(\sigma(h)(x)\) by \(h \ast x\). The action is called self-similar if there exists another map \(H \times \mathcal{G} \ni (h, x) \mapsto h \bullet x \in H\) that satisfies:

1. For all \(v \in \mathcal{G}^0\), \(h \in H\), and \(x \in \mathcal{G}\), we have \(h \bullet v = h = e_H \bullet x = e_H\).
2. For all \((x, y) \in \mathcal{G}^2\) and \(h \in H\), we have \(h \bullet (xy) = (h \bullet x) \bullet y\).
3. For all \((x, y) \in \mathcal{G}^2\) and \(h \in H\), we have \(h \ast (xy) = (h \ast x)((h \bullet x) \ast y)\).
4. For all \(g, h \in H\) and \(x \in \mathcal{G}\), we have \((gh) \bullet x = (g \bullet (h \ast x))(h \bullet x)\).

The self-similar groupoid is then defined as \(\mathcal{G}^H = \{(x, h) : x \in \mathcal{G}, h \in H\}\) with multiplication given by \((x, h)(y, k) = (x(h \ast y), (h \bullet y)k)\) whenever \(h \ast r(y) = s(x)\), and with inverse \((x, h)^{-1} = (h^{-1} \ast x^{-1}, h^{-1} \bullet x^{-1})\).

Let \(\mathcal{H} = \mathcal{G}^0 \rtimes H = \{(u, h) : u \in \mathcal{G}^0, h \in H\}\) be the transformation groupoid of the \(H\)-action restricted to \(\mathcal{G}^0\); its multiplication is given by

\[ (u, h)(v, k) = (u\ast v, h \bullet k). \]
The Zappa-Szép product of a Fell bundle and a groupoid

\[(u, h)(v, k) = (u, hk)\] whenever \(v = h^{-1} \ast u\), and its inverses are \((u, h)^{-1} = (h^{-1} \ast u, h^{-1})\). One can verify that \((\mathcal{G}, \mathcal{H})\) is a matched pair of étale groupoids where the \(\mathcal{H}\)-action on \(\mathcal{G}\) is given by \((u, h) \cdot x := h \ast x\) and the \(\mathcal{G}\)-restriction on \(\mathcal{H}\) is given by \((u, h)|_x = (h \ast s(x), h \cdot x)\). The resulting Zappa-Szép product groupoid \(\mathcal{G} \bowtie \mathcal{H}\) coincides with the self-similar groupoid \(\mathcal{G}^\mathcal{H}\).

Note further that, when \(\mathcal{G}\) and \(\mathcal{H}\) are étale groupoids, their Zappa-Szép product is again étale.

**Proposition 2.5** (see [3, Proposition 9]). When \(\mathcal{G} \bowtie \mathcal{H}\) is endowed with the relative product topology on \(\mathcal{G} \times \mathcal{H}\), it is étale if and only if both \(\mathcal{G}\) and \(\mathcal{H}\) are étale, and both the \(\mathcal{H}\)-action map and the \(\mathcal{G}\)-restriction maps are continuous.

2.2. Fell Bundles. Fell bundles over groups were first introduced and studied by Fell [12], under the notion of \(C^\ast\)-algebraic bundle. It is a powerful device in the study of graded \(C^\ast\)-algebras, and many well-known \(C^\ast\)-algebras are naturally graded. Instead of studying the graded \(C^\ast\)-algebra as a whole, Fell bundles focus on the fibres from the grading and provide a general framework to reassemble fibres back to various graded \(C^\ast\)-algebras. Here, we give a brief introduction to Fell Bundles over étale groupoids. One may refer to [10] for Fell bundles over discrete groups and its connection with partial dynamical systems; and to [25, 16] for a more detailed discussion of Fell bundles over groupoids.

**Definition 2.6** (see [5, Definition 2.1]). Suppose \(\mathcal{G}\) is a locally compact Hausdorff étale groupoid, and \(B\) is a topological space together with a continuous, open surjection \(p : B \to \mathcal{G}\). We call \(B = (B, p)\) an upper semi-continuous Banach bundle if its fibres \(B_x := p^{-1}(x)\) have the structure of complex Banach spaces and if

- (USC1) the map \(B \to \mathbb{R}_{\geq 0}, b \mapsto \|b\|\), is upper semi-continuous,
- (USC2) when \(B_p \times_p B = \{(a, b) \in B \times B : p(a) = p(b)\}\)
  is equipped with the subspace topology; then the map \(B_p \times_p B \to B, (a, b) \mapsto a + b\), is continuous,
- (USC3) for each \(\lambda \in \mathbb{C}\), the map \(B \to B, b \mapsto \lambda b\), is continuous and
- (USC4) if \((b_i)_i\) is a net in \(B\) such that \(p(b_i)\) converges to \(x \in \mathcal{G}\) and \(\|b_i\| \to 0\), then \((b_i)_i\) converges to \(0 \in \mathcal{B}_x\) in \(B\).

We note that (USC2) and (USC3) are to be understood with the Banach space structure of \(\mathcal{B}_{p(b)}\) in mind. By a standard abuse of notation, we will often write \(\mathcal{B} = \{\mathcal{B}_g\}_{g \in \mathcal{G}}\) for the bundle, omitting explicit referencing of the topological space \(B\) and the map \(p\).

**Definition 2.7** (see [5, Definition 2.8]). An upper semi-continuous Banach bundle \(\mathcal{B} = (B, p)\) is called Fell bundle (or \(C^\ast\)-algebraic bundle) if it comes with continuous maps

\[\cdot : \mathcal{B}^{(2)} := \{(a, b) \in B \times B : (p(a), p(b)) \in \mathcal{G}^{(2)}\} \to B\] and \(* : B \to B\)
such that:
Suppose \( \mathcal{G} \).

**Definition 2.11** (cf. [13, Definition II.13.8] and [14, Definition VIII.3.3])

If \( (b, c) \in \mathcal{B}(\mathcal{G}) \), then \( \| b \cdot c \| \leq \| b \| \cdot \| c \| \), where the norm is the Banach norm on the respective fibre.

**F2** The multiplication is bilinear.

**F3** The multiplication is associative, whenever it is defined.

**F4** If \( (b, c) \in \mathcal{B}(\mathcal{G}) \), then \( \| b \cdot c \| \leq \| b \| \cdot \| c \| \), where the norm is the Banach norm on the respective fibre.

**F5** For any \( x \in \mathcal{G} \), \( \mathcal{B}_x \subset \mathcal{B}_x^{-1} \).

**F6** The involution map \( b \mapsto b^* \) is conjugate linear.

**F7** If \( (b, c) \in \mathcal{B}(\mathcal{G}) \), then \( (b \cdot c)^* = c^* \cdot b^* \).

**F8** For any \( b \in B \), \( b^{**} = b \).

**F9** For any \( b \in B \), \( \| b^* \cdot b \| = \| b \|^2 = \| b^* \|^2 \).

We will often write \( bc \) for \( b \cdot c \), and \( s_\mathcal{G} \) resp. \( r_\mathcal{G} \) for \( s_\mathcal{G} \circ p \) resp. \( r_\mathcal{G} \circ p \).

We note that \([F9]\) makes sense because of \([F5]\). Moreover, whenever \( x \in \mathcal{G}^{(0)} \) is an idempotent, \( \mathcal{B}_x \) is in fact a C*-algebra. Since \( b^* b \in \mathcal{B}_{s(\mathcal{G})} \) for \( b \in \mathcal{B}_y \) by \([F1]\) and \([F4]\), we can understand the positivity in \([F10]\) within the C*-algebra \( \mathcal{B}_{s(\mathcal{G})} \).

**Lemma 2.8** (cf. [6] Lemma 3.30]). Suppose \( \mathcal{G}^{(0)} \) is discrete. A Fell bundle \((B, p)\) in the above sense is automatically continuous, i.e. the norm \( b \mapsto \| b \| \) is continuous on \( B \).

**Proof.** Let \( \{ b_i \}_{i \in I} \) be a net in \( B \) converging to \( b \). By the assumptions that involution and multiplication are continuous, \( b_i^* b_i \to b^* b \). Since \( p : B \to \mathcal{G} \) is continuous, \( p(b_i^* b_i) \to p(b^* b) \). Because \( \mathcal{G}^{(0)} \) is discrete, we may assume without loss of generality that \( p(b_i^* b_i) = p(b^* b) \) for all \( i \). Therefore, \( \{ b_i^* b_i \} \) is a net converging to \( b^* b \) in the C*-algebra \( p(b^* b) = \mathcal{B}_{s(\mathcal{G})} \). Consequently, \( \| b_i \|^2 = \| b_i^* b_i \| \to \| b^* b \| = \| b \|^2 \). Therefore, \( b \mapsto \| b \| \) is continuous on \( B \). \( \Box \)

**Example 2.9.** Given an étale groupoid \( \mathcal{G} \), one can define the groupoid Fell bundle \( \mathcal{B}(\mathcal{G}) := \mathbb{C} \times \mathcal{G} = \{(a, x) : a \in \mathbb{C}, x \in \mathcal{G}\} \) as follows. A fibre \( \mathcal{B}(\mathcal{G})_x = \mathbb{C} \times \{x\} \) naturally inherits its norm from \( \mathbb{C} \). Multiplication is given by \( (a, x)(b, y) = (ab, xy) \) whenever \((x, y) \in \mathcal{G}^{(2)}\), and involution is given by \( (a, x)^* = (\overline{a}, x^{-1}) \). One can easily verify that \( \mathcal{B}(\mathcal{G}) \) is a Fell bundle over \( \mathcal{G} \).

**Example 2.10.** Let \( G \) be a discrete group. Exel [10] Definition 16.2] defined the notion of C*-grading that is closely related to Fell bundles over \( G \): For a C*-algebra \( A \), a C*-grading is a collection of linearly independent subspaces \( \{ A_g \}_{g \in G} \) such that \( \bigoplus_{g \in G} A_g \) is dense in \( A \), \( A_g A_h \subset A_{gh} \), and \( A_g^* \subset A_{g^{-1}} \).

Given such a grading, \( \mathcal{B} = \{ A_g \}_{g \in G} \) defines a Fell bundle over \( G \), where the multiplication and involution are inherited from the underlying C*-algebra.

One has to be cautious that by passing from the C*-algebra \( A \) to the Fell bundle \( \{ A_g \} \), one may lose much information of the original C*-algebra \( A \). As pointed out by Exel [10] Remark 16.3], there may be multiple ways of completing \( \bigoplus_{g \in G} A_g \), some of which may not recover \( A \).

**Definition 2.11** (cf. [13] Definition II.13.8] and [14] Definition VIII.3.3).

Suppose \( \mathcal{B} = (B, p) \) and \( \mathcal{B}' = (B', p') \) are two Fell bundles over groupoids
\( \mathcal{G} \) and \( \mathcal{G}' \) respectively, and let \( f : \mathcal{G} \to \mathcal{G}' \) be a continuous groupoid homomorphism. A continuous map \( \phi : B \to B' \) between the total spaces is called a homomorphism \( \mathcal{B} \to \mathcal{B}' \) of Fell bundles covariant with \( f \) if

(H1) \( \phi(\mathcal{B}_x) \subseteq \mathcal{B}'_{f(x)} \) for all \( x \in \mathcal{G} \) and each \( \phi|_{\mathcal{B}_x} \) is a linear map between the Banach spaces \( \mathcal{B}_x \) and \( \mathcal{B}'_{f(x)} \).

(H2) \( \phi \) is multiplicative, i.e., if \( (b, c) \in \mathcal{B}'^{(2)} \), then \( \phi(bc) = \phi(b)\phi(c) \), and

(H3) \( \phi \) is \( * \)-preserving, i.e., \( \phi(b^*) = \phi(b)^* \) for all \( b \in \mathcal{B} \).

If, moreover, \( \| \phi(b) \| = \| b \| \) for all \( b \in \mathcal{B} \), we call \( \phi \) isometric. If \( \phi \) is bijective (and isometric), \( \mathcal{G} = \mathcal{G}' \), and \( f \) is the identity, then we say that \( \phi \) is an (isometric) isomorphism of Fell bundles.

3. Zappa-Szép Product of a Fell Bundle and a Groupoid

Suppose \( (\mathcal{G}, \mathcal{H}) \) is a matched pair of étale groupoids and \( \mathcal{B} = (B, p) \) is a Fell bundle over \( \mathcal{G} \). The goal of this section is to define a Zappa-Szép product of \( \mathcal{B} \) by the groupoid \( \mathcal{H} \) and show that this Zappa-Szép product is a Fell bundle over the Zappa-Szép product \( \mathcal{G} \bowtie \mathcal{H} \). The Fell bundle \( \mathcal{B} \) defines a ‘\( \mathcal{B} \)-restriction map’ on \( \mathcal{H} \) quite easily: it can simply inherit the \( \mathcal{G} \)-restriction map on \( \mathcal{H} \) from its \( \mathcal{G} \)-grading. However, we need to additionally assume a certain type of \( \mathcal{H} \)-action on the Fell bundle.

**Definition 3.1.** Assume \( (\mathcal{G}, \mathcal{H}) \) is a matched pair of étale groupoids as described in Definition 2.1 and assume \( \mathcal{B} = (B, p) \) is a Fell bundle over \( \mathcal{G} \). For \( r_\mathcal{B} := r_\mathcal{G} \circ p : B \to \mathcal{G}^{(0)} = \mathcal{H}^{(0)} \), let \( \mathcal{H}_{s_\mathcal{H} \times r_\mathcal{B}} B \) be defined as in Equation (2.1), equipped with the subspace topology of \( \mathcal{H} \times B \).

A \( (\mathcal{G}, \mathcal{H}) \)-compatible \( \mathcal{H} \)-action on \( \mathcal{B} \) is a continuous map

\[
\mathcal{H}_{s_\mathcal{H} \times r_\mathcal{B}} B \to B
\]

satisfying the following conditions:

(A1) For any \( (h, x) \in \mathcal{H}_{s_\mathcal{H} \times r_\mathcal{B}} \mathcal{G} \), the map \( \beta_h := \beta(h, \unlhd) \) maps \( \mathcal{B}_x \) into \( \mathcal{B}_{x,\unrhd} \) and is linear.

(A2) For any \( (g, h) \in \mathcal{H}^{(2)} \), \( \beta_{gh} = \beta_g \circ \beta_h \).

(A3) For any \( u \in \mathcal{H}^{(0)} \), \( \beta_u \) is the identity map.

(A4) For any \( (b, c) \in \mathcal{B}^{(2)} \) such that \( (h, bc) \in \mathcal{H}_{s_\mathcal{H} \times r_\mathcal{B}} B \), we have

\[
\beta_h(bc) = \beta_h(b)\beta_{h|_{p(b)}}(c).
\]

(A5) For any \( b \in \mathcal{B}_x \) with \( r_\mathcal{G}(x) = s_\mathcal{H}(h) \), we have

\[
\beta_h(b)^* = \beta_{h|_x}((b)^*).
\]

Using both (F1)–(F10) and (ZS1)–(ZS13), one can check that the above conditions make sense.

**Proposition 3.2.** For any \( h \in \mathcal{H} \), the restricted map \( \beta_h : \mathcal{B}_{s_\mathcal{H}(h)} \to \mathcal{B}_{r_\mathcal{H}(h)} \) is an isometric \( * \)-isomorphism of \( C^* \)-algebras.

**Proof.** For any \( h \in \mathcal{H} \), take \( a \in \mathcal{B}_{s_\mathcal{H}(h)} \). By (A1), \( \beta_h(a) \in \mathcal{B}_{h \cdot s_\mathcal{H}(h)} = \mathcal{B}_{r_\mathcal{H}(h)} \). For any \( a, b \in \mathcal{B}_{s_\mathcal{H}(h)} \),

\[
\beta_h(ab) = \beta_h(a)\beta_{h|_{s_\mathcal{H}(h)}}(b) = \beta_h(a)\beta_h(b).
\]
Moreover, by (A5),
\[ \beta_h(a)^* = \beta_{h|s_h(a)}(a)^* = \beta_h(a^*). \]
Therefore, \( \beta_h : B_{s_H(h)} \to B_{r_H(h)} \) is a *-homomorphism. By (A2) and (A3),
\[ \beta_{h^{-1}} = \beta_{r_H(h)} \] is the identity on \( B_{s_H(h)} \) and \( \beta_h \beta_{h^{-1}} = \beta_{r_H(h)} \) is the identity on \( B_{r_H(h)} \); thus, \( \beta_h \) is a bijective *-isomorphism. Since \( B_{s_H(h)} \) and \( B_{r_H(h)} \) are \( C^* \)-algebras, it is automatic that \( \beta_h \) is isometric on \( B_{s_H(h)} \).

**Corollary 3.3.** For any \( h \in H \) and \( x \in G \) with \( r_G(x) = s_H(h) \), \( \beta_h \) is isometric from \( B_x \) to \( B_{h \cdot x} \). That is, for any \( a \in B_x \),
\[ \| \beta_h(a) \| = \| a \|. \]

**Proof.** By Proposition 3.2, \( \beta_h \) is isometric on \( B_{s_H(h)} \). Consider \( aa^* \in B_{r_G(x)} = B_{s_H(h)} \), we have \( \| \beta_h(aa^*) \| = \| aa^* \| = \| a \|^2 \). On the other hand, by (A4) and (A5),
\[ \| \beta_h(aa^*) \| = \| \beta_h(a)\beta_h(a^*) \| = \| \beta_h(a)\beta_h(a)^* \| = \| \beta_h(a) \|^2. \]
Therefore, \( \| \beta_h(a) \| = \| a \| \), as desired. \( \square \)

From an \( H \)-action \( \beta \) on the Fell bundle \( (B, p) \), we now construct a Fell bundle \( (C, q) \) over the locally compact Hausdorff étale groupoid \( G \bowtie H \) as follows.

\( C \) As a topological space, let
\[ C := B_{s_B} \times_{r_B} H = \{(b, h) \in B \times H : s_B(b) = r_B(h)\}, \]
equipped with the subspace topology. The constraint we put on elements of \( C \) enables us to define the following map with values in the Zappa-Szép product:
\[ q : C \to G \bowtie H, \quad q(b, h) = (p(b), h). \]

Let \( C := (C, q) \).

\( C \) We define a multiplication \( \bullet : C^{(2)} \to C \) by
\[ (a, g) \bullet (b, h) := (a \beta_g(b), g|_{p(b)} h), \]
where
\[ C^{(2)} := \{(a, g) \in C \times C : (q(a, g), q(b, h)) \in (G \bowtie H)^{(2)}\}, \]
as defined before for \( B = (B, p) \).

\( C \) We define an involution \( \ast : C \to C \) by
\[ (b, h)^* = (\beta_{h^{-1}}(b^*), h^{-1}|_{p(b^{-1})}). \]

**Remark 3.4.** The fibre \( C_{(x, h)} := q^{-1}(x, h) \) of \( C \) is canonically isomorphic to the fibre \( B_x \) of \( B \), making \( C_{(x, h)} \) a complex Banach space. In particular, for \( u \in G^{(0)} = H^{(0)} \), the fibre \( C_u = C_{(u, u)} \) is a \( C^* \)-algebra. An element \( (b, u) \in C_u \) is positive if and only if \( b \in B_u \) is positive.

**Proposition 3.5.** The pair \( C = (C, q) \) is an upper semi-continuous Banach bundle over the Zappa-Szép product \( G \bowtie H \).
Proof. The map \( q \) is clearly a continuous open surjection, since \( p \) is.

As \( \mathcal{C}_{(x,h)} := q^{-1}(x,h) \) inherits its structure of a complex Banach space from \( \mathcal{B} \), we have, for \((a,h),(b,g)\) \( \in \mathcal{C} \times_{q} \mathcal{C} \) and \( \lambda \in \mathbb{C} \), that \( h = g \) and \( \lambda(a,h) + (b,g) = (\lambda a + b,h) \). It is now clear that addition on \( C \times_q C \) and multiplication by a scalar \( \lambda \) on \( C \) are continuous since this is the case for \((B,p)\), proving \([\text{USC2}]\) and \([\text{USC3}]\) of Definition \(2.6\).

Since \( b \) over \( s \) satisfies \([\text{USC4}]\) and is all in all an upper semi-continuous Banach bundle Lemma \(3.6\).

The multiplication \( \circ \) commutes, continuity of the coordinate projection \( \text{pr}_1 \) and upper semi-continuity of \( \mathcal{B} \) imply upper semi-continuity of \( \mathcal{C} \), i.e. \([\text{USC1}]\) holds.

Lastly, assume \((b_i,h_i)\) is a net in \( C \) with \( q(b_i,h_i) \) converging to \( (x,h) \in \mathcal{G} \sqsupsetneq \mathcal{H} \) and \( \|b_i\| \to 0 \), i.e. \( p(b_i) \to x \), \( h_i \to h \), and \( \|b_i\| \to 0 \). Since \((B,p)\) satisfies \([\text{USC4}]\), it follows that \((b_i)\) converges to \( 0 \in \mathcal{B} \) in \( B \), so that \((b_i,h_i)\) converges to \((0,h) = 0 \in \mathcal{C}_{(x,h)} \) in \( C \). This shows that \( \mathcal{C} \) satisfies \([\text{USC4}]\) and is all in all an upper semi-continuous Banach bundle over \( \mathcal{G} \sqsupsetneq \mathcal{H} \).

\[ \square \]

Lemma 3.6. The multiplication \( \bullet \) on \( \mathcal{C} \) is well-defined and continuous.

Proof. Let \((a,g),(b,h)\) \( \in \mathcal{C} \) such that \((g(a,g),q(b,h))\) \( \in (\mathcal{G} \sqsupsetneq \mathcal{H})^{(2)} \), i.e. \( s_{\mathcal{H}}(g) = r_{\mathcal{B}}(b) = r_{\mathcal{G}}(p(b)) \). This means, first of all, that \( x := p(b) \in \mathcal{G} \) can act on \( g \), yielding \( g \cdot x \in \mathcal{H} \). Moreover, it means that \((g,b)\) is in \( \mathcal{H} \times_{s_{\mathcal{H}}} \mathcal{B} \), the domain of \( \beta \), so that \( \beta_{g}(b) \) is defined. As \((a,g)\) \( \in \mathcal{C} \), we have \( s_{\mathcal{B}}(a) = r_{\mathcal{H}}(g) \), so that 

\[ r_{\mathcal{H}}(h) = s_{\mathcal{G}}(x) s_{\mathcal{H}}(g \cdot x), \]

i.e. \((a,\beta_{g}(b))\) \( \in \mathcal{B}^{(2)} \), so that their product \( a\beta_{g}(b) \) is defined. Similarly, since \((b,h)\) \( \in \mathcal{C} \), we have 

\[ s_{\mathcal{B}}(a\beta_{g}(b)) = s_{\mathcal{G}}(p(\beta_{g}(b))) = s_{\mathcal{G}}(g \cdot x) r_{\mathcal{H}}(g \cdot x), \]

so that \( g \cdot x \in \mathcal{G}^{(2)} \), i.e. their product \( g \cdot x \) is defined in \( \mathcal{G} \). All in all, it makes sense to define

\[ (a,g) \bullet (b,h) := (a\beta_{g}(b),g \cdot x), \]

and it remains to show that it is an element of \( C \). To this end, we note that \( p(a\beta_{g}(b)) = p(a)p(\beta_{g}(b)) \) by \([\text{F1}]\) and \( p(\beta_{g}(b)) = g \cdot p(b) = g \cdot x \) by \([\text{A1}]\) so that 

\[ s_{\mathcal{B}}(a\beta_{g}(b)) = s_{\mathcal{G}}(p(\beta_{g}(b))) = s_{\mathcal{G}}(g \cdot x) = r_{\mathcal{H}}(g \cdot x), \]

Thus,

\[ (a,g) \bullet (b,h) \in B s_{\mathcal{B}} s_{\mathcal{H}} = C. \]

It is now obvious that \( \bullet \) is continuous, as the \( \mathcal{H} \)-action \( \beta \) on \( B \), the multiplication on \( B \), the bundle map \( p \), the \( \mathcal{G} \)-restriction on \( \mathcal{H} \), and the multiplication on \( \mathcal{H} \) are all continuous maps.

\[ \square \]
Lemma 3.7. The involution $*$ on $C$ is well-defined and continuous.

Proof. Let $(b, h) \in C$ with $x := p(b)$, i.e. $s_{\mathcal{B}}(b) = r_{\mathcal{H}}(h)$. As $p(b^*) = x^{-1}$ by (F5) we have

$$s_{\mathcal{H}}(h^{-1}) = r_{\mathcal{H}}(h) = s_{\mathcal{B}}(b) = s_{\mathcal{G}}(x) = r_{\mathcal{G}}(x^{-1}) = r_{\mathcal{B}}(b^*).$$

This shows both that $(h^{-1}, x^{-1}) \in \mathcal{H}_{s_{\mathcal{H}} \times r_{\mathcal{G}}}$, so that $h^{-1}|_{x^{-1}}$ is defined, and that $(h^{-1}, b^*)$ is in $\mathcal{H}_{s_{\mathcal{H}} \times r_{\mathcal{B}}} B$, the domain of $\beta$, so that $\beta_{h^{-1}}(b^*)$ is defined. All in all, it makes sense to define

$$(b, h)^* := (\beta_{h^{-1}}(b^*), h^{-1}|_{x^{-1}}),$$

and it remains to show that it is an element of $C$. To this end, we compute

$$s_{\mathcal{B}}(\beta_{h^{-1}}(b^*)) s_{\mathcal{H}}(h^{-1} \cdot p(b^*)) = s_{\mathcal{G}}(h^{-1} \cdot x^{-1}) r_{\mathcal{H}}(h^{-1}|_{x^{-1}}).$$

Thus,

$$(b, h)^* \in B_{s_{\mathcal{B}} \times r_{\mathcal{H}}} H = C.$$

It is now obvious that $*$ is continuous, as the $\mathcal{H}$-action $\beta$ on $B$, involution on $B$, the bundle map $p$, the $\mathcal{G}$-restriction on $H$, and inversion on $\mathcal{H}$ are all continuous maps.

\[\square\]

Theorem 3.8. The bundle $C$ defined in (C1) together with the multiplication defined in (C2) and the involution defined in (C3) is a Fell bundle over the Zappa-Szép product $\mathcal{G} \bowtie \mathcal{H}$. We will denote $C$ by $\mathcal{B} \bowtie_{\beta} \mathcal{H}$, and we call it the Zappa-Szép product of the Fell bundle $\mathcal{B}$ by $\mathcal{H}$.

Proof. We have already seen in Proposition 3.5 that $C$ is an upper semi-continuous Banach bundle over $\mathcal{G} \bowtie \mathcal{H}$, and in Lemmas 3.6 and 3.7 that $\bullet$ resp. $*$ are well-defined continuous $C$-valued maps. It remains to show that $C$ satisfies conditions (F1) through (F10).

For (F1) Take $(a, h) \in C_{(x, h)}$ and $(b, g) \in C_{(y, g)}$ such that $r_{\mathcal{G}}(y) = s_{\mathcal{H}}(h)$, so that

$$(a, h)(b, g) = (a\beta_h(b), h|_{y} g).$$

By (A1) $\beta_h(b) \in \mathcal{B}_{x,y}$. By (ZS2), $r_{\mathcal{G}}(h \cdot y) = r_{\mathcal{H}}(h) = s_{\mathcal{G}}(x)$, and thus $(x, h \cdot y) \in G^{(2)}$. Therefore, by (F1) for $\mathcal{B}$, $a\beta_h(b) \in \mathcal{B}_{x(h,y)}$. Hence, the product $(a\beta_h(b), h|_{y} g)$ is an element of $C_{(x(h,y), h|_{y} g)} = C_{(x(h,y), h|_{y} g)}$.

For (F2) The multiplication on $B$ is bilinear and $\beta_h$ is linear, so it is clear that the multiplication on $C$ is bilinear.

For (F3) Take $(a, h) \in C_{(x, h)}$, $(b, g) \in C_{(y, g)}$, and $(c, k) \in C_{(z, k)}$, such that $r_{\mathcal{G}}(y) = s_{\mathcal{H}}(h)$ and $r_{\mathcal{G}}(z) = s_{\mathcal{H}}(g)$. By definition,

$$((a, h)(b, g))(c, k) = (a\beta_h(b)\beta_{h|_{y} g}(c), (h|_{y} g)|_{z} k).$$

On the other hand,

$$a, h \left((b, g)(c, k)\right) = (a\beta_h(b\beta_g(c)), h|_{y(g z)} g|_{z} k).$$

By (ZS4) and (ZS3)

$$(h|_{y} g)|_{z} k = (h|_{y}|_{g z} g|_{z} k = h|_{y(g z)} g|_{z} k.$$

By (A4)

$$a\beta_h(b\beta_g(c)) = a\beta_h(b)\beta_{h|_{y} g}(\beta_g(c)) = a\beta_h(b)\beta_{h|_{y} g}(c).$$
Therefore, the multiplication is associative.

For \([F4]\) For \((a, h) \in \mathcal{C}_{(x, h)}\) and \((b, g) \in \mathcal{C}_{(y, g)}\) such that \(r_G(y) = s_H(h)\), we have

\[
\|(a, h)(b, g)\| = \|(a\beta_h(b), h|_y g)\| \\
= \|a\beta_h(b)\| \\
\leq \|a\|\|\beta_h(b)\| \\
= \|a\|\|b\| = \|(a, h)\|(b, g)\|.
\]

Here, we applied Corollary 3.3 which stated that \(\|\beta_h(b)\| = \|b\|\).

For \([F5]\) For \((a, h) \in \mathcal{C}_{(x, h)}\), we have

\[
(a, h)^* = (\beta_{h^{-1}}(a^*), h^{-1}|_{x^{-1}}) \in \mathcal{C}_{(h^{-1}, x^{-1}, h^{-1}|_{x^{-1}})} = \mathcal{C}_{(x, h)}^{-1}.
\]

For \([F6]\) Since \(\beta_{h^{-1}}\) is linear, we have

\[
(a + \lambda b, h)^* = (\beta_{h^{-1}}(a^* + \lambda b^*), h^{-1}|_{x^{-1}}) = (a, h)^* + \lambda(b, h)^*.
\]

For \([F7]\) Take \((a, h) \in \mathcal{C}_{(x, h)}\) and \((b, g) \in \mathcal{C}_{(y, g)}\) such that \(r_G(y) = s_H(h)\).

One can compute:

\[
((a, h)(b, g))^* = (\beta_{h|_y g}^{-1}(\beta_h(b)^* a^*), (h|_y g)^{-1}|_{(x, y)^{-1}}),
\]

and

\[
(b, g)^*(a, h)^* = \left(\beta_{g^{-1}}(b^*)\beta_{g^{-1}|_{y^{-1}}}(\beta_h^{-1}(a^*)), (g^{-1}|_{y^{-1}})h^{-1}_{x^{-1}}h^{-1}_{x^{-1}}\right).
\]

By \([A5]\) \(\beta_h(b)^* = \beta_{h|_y}(b^*)\). By Lemma 2.2 \((h|_y)^{-1} = h^{-1}|_{h, y}\) and \((h \cdot y)^{-1} = h|_y y^{-1}\). Therefore,

\[
\beta_{(h|_y g)^{-1}}(\beta_h(b)^* a^*) = \beta_{(h|_y g)^{-1}}(\beta_{h|_y}(b^*) a^*) \\
= \beta_{(h|_y g)^{-1}}(\beta_{h|_y}(b^*) a^*) \\
= \beta_{g^{-1}}(b^*) \beta_{g^{-1}|_{y^{-1}}}((h|_{h|_y})^{-1} h^{-1}_{y^{-1}} h^{-1}_{x^{-1}}) a^* \\
= \beta_{g^{-1}}(b^*) \beta_{g^{-1}|_{y^{-1}}}((h|_{h|_y})^{-1} h^{-1}_{y^{-1}} h^{-1}_{x^{-1}}) h^{-1}_{x^{-1}} a^* \\
= \beta_{g^{-1}}(b^*) \beta_{g^{-1}|_{y^{-1}}}((h|_{h|_y})^{-1} h^{-1}_{y^{-1}} h^{-1}_{x^{-1}}) h^{-1}_{x^{-1}} a^*.
\]

Moreover,

\[
(h|_y g)^{-1}|_{(x, y)^{-1}} = g^{-1}|_{(h|_{h|_y})^{-1}, (x, y)^{-1}} (h|_{y^{-1}})^{-1} (x, y)^{-1}
\]

\[
= g^{-1}|_{(h|_{h|_y})^{-1}, (h|_{y^{-1}})^{-1} x^{-1}} (h^{-1}|_{h|_y}) (h|_{y^{-1}})^{-1} x^{-1}
\]

\[
= g^{-1}|_{y^{-1}(h|_{h|_y})^{-1} x^{-1} h^{-1}} x^{-1}
\]

\[
= g^{-1}|_{y^{-1}(h|_{h|_y})^{-1} h^{-1}} x^{-1}
\]

\[
= (g^{-1}|_{y^{-1}})h^{-1}_{x^{-1}} x^{-1}.
\]

Therefore,

\[
((a, h)(b, g))^* = (b, g)^*(a, h)^*.
\]
For (F8) Take any \((a, h) \in C_{x,h}\), one can compute that 
\[(a, h)^{**} = \left( \beta_{h^{-1}|_{x^{-1}}}^{-1}(\beta_{h^{-1}}(a^*)^*), (h^{-1}|_{x^{-1}})^{-1}|_{(h^{-1}|_{x^{-1}})^{-1}} \right).\]

By (A5) 
\[\beta_{h^{-1}}(a^*)^* = \beta_{h^{-1}|_{x^{-1}}}^{-1}(a).\]

Therefore, 
\[\beta_{h^{-1}|_{x^{-1}}}^{-1}(\beta_{h^{-1}}(a^*)^*) = \beta_{h^{-1}|_{x^{-1}}}^{-1}(\beta_{h^{-1}}(a)) = \beta_{r_{G}(x)}(a) = a.\]

Moreover, 
\[(h^{-1}|_{x^{-1}})^{-1}|_{(h^{-1}|_{x^{-1}})^{-1}} = (h|_{h^{-1}|_{x^{-1}}})|_{(h^{-1}|_{x^{-1}})^{-1}} = h.\]

Hence, \((a, h)^{**} = (a, h)\).

For (F9) For \((a, h) \in C_{x,h}\), we have 
\[(a, h)^{*}(a, h) = (\beta_{h^{-1}}(a^*)\beta_{h^{-1}|_{x^{-1}}}(a), s_{\mathcal{H}}(h)) = (\beta_{h^{-1}}(a^*a), s_{\mathcal{H}}(h)).\]

Note that \(a^*a \in \mathcal{B}_{s_{\mathcal{G}(x)}} = \mathcal{B}_{r_{\mathcal{G}}(h)} = \mathcal{B}_{s_{\mathcal{H}}(h^{-1})}\). By Proposition 3.2, \(\beta_{h^{-1}}\) is isometric on \(\mathcal{B}_{s_{\mathcal{H}}(h^{-1})}\). Therefore, 
\[\| (a, h)^{*}(a, h) \| = \| \beta_{h^{-1}}(a^*a) \| = \| a^*a \| = \| (a, h) \|^2.\]

Moreover, 
\[\| a^*a \| = \| a a^* \| = \| (a, h)^{**}(a, h)^{*} \| = \| (a, h)^{*} \|^2.\]

Finally, for (F10) We have shown that 
\[(a, h)^{*}(a, h) = (\beta_{h^{-1}}(a^*a), s_{\mathcal{H}}(h)).\]

Since \(\beta_{h^{-1}}\) is a \(*\)-automorphism on \(\mathcal{B}_{s_{\mathcal{H}}(h^{-1})}\), we have \(\beta_{h^{-1}}(a^*a) \geq 0\) and thus by Remark 3.4 
\[(a, h)^{*}(a, h) \geq 0.\]

\begin{proof}

Example 3.9. If \(\mathcal{H} = \mathcal{G}^{(0)}\) is trivial, so that \(\mathcal{G} \rtimes \mathcal{G}^{(0)} \cong \mathcal{G}\) via \((x, s_{\mathcal{G}}(x)) \mapsto x\), then the trivial \(\mathcal{G}^{(0)}\)-action on some Fell bundle \(\mathcal{B}\) over \(\mathcal{G}\), defined by \(\beta(r_{\mathcal{G}}(b), b) = b\) for all \(b \in \mathcal{B}\), is clearly \((\mathcal{G}, \mathcal{G}^{(0)})\)-compatible, and \(\mathcal{B} \rtimes \mathcal{G}^{(0)} \cong \mathcal{B}\) via \((b, s_{\mathcal{G}}(b)) \mapsto b\).

Example 3.10. On the other hand, if \((\mathcal{G}, \mathcal{H})\) is a matched pair of étale groupoids, then the groupoid Fell bundle \(\mathcal{B}(\mathcal{G}) := (\mathbb{C} \times \mathcal{G}, pr_{\mathcal{G}})\) carries a canonical \(\mathcal{H}\)-action \(\beta\): for \(h \in \mathcal{H}\) and \(x \in \mathcal{G}\) with \(r_{\mathcal{G}}(x) = s_{\mathcal{H}}(h)\), define \(\beta(h, (z, x)) = (z, h \cdot x)\). This is clearly linear and continuous, and one can easily verify that \(\beta\) satisfies conditions (A1) through (A5).

By Theorem 3.8 the bundle \(C\) with fibres 
\[C_{(z, h)} = \{(z, x, h) : (z, x) \in \mathcal{B}_x, r_{\mathcal{H}}(h) = s_{\mathcal{G}}(x)\}\]
for \((x, h) \in \mathcal{G} \rtimes \mathcal{H}\), with multiplication 
\[((z_1, x), h)((z_2, y), g) = ((z_1 z_2, x h \cdot y), h|_{g}g),\]
and with involution 
\[((z, x), h)^{*} = ((z, h^{-1} \cdot x^{-1}), h^{-1}|_{x^{-1}}),\]
defines an isometric isomorphism between the Fell bundle $\mathcal{C}$. Observe that the map $((z, x), h) \mapsto (z, (x, h))$ defines an isometric isomorphism between the Fell bundle $\mathcal{C} = \mathcal{B}(\mathcal{G}) \rtimes_\beta \mathcal{H}$ and the groupoid Fell bundle $\mathcal{B}(\mathcal{G} \rtimes \mathcal{H})$.

Example 3.11. Let $\mathcal{G}$ be a locally compact Hausdorff groupoid and $H$ a locally compact group. Suppose there is a continuous left action of $H$ on $\mathcal{G}$, denoted by $(h, x) \mapsto h \ast x$. Recall the semi-direct product groupoid $\mathcal{G} \rtimes H$ considered by Kaliszewski, Muhly, Quigg, and Williams in [15],

$$\mathcal{G} \rtimes H = \{(x, h) : x \in \mathcal{G}, h \in H\},$$

with multiplication $(x, h)(y, k) = (x(h \ast y), hk)$ whenever $s_\mathcal{G}(x) = h \ast r_\mathcal{G}(y)$ and with inverse $(x, h)^{-1} = (h^{-1} \ast x^{-1}, h^{-1})$ (the reader should compare this construction to that of the self-similar groupoid in Example 2.14). The notion of a semi-direct product of a Fell bundle $\mathcal{B} = (B, p)$ over $\mathcal{G}$ by a group $H$ in [15] is related to our construction as follows.

In [15], an $H$-action on $\mathcal{B}$ is defined as a homomorphism $\alpha : H \to \text{Aut}(\mathcal{B})$ such that $H \times B \ni (h, b) \mapsto \alpha_h(b) \in B$ is continuous and $p(\alpha_h(b)) = h \ast p(b)$ for all $h \in H$ and $b \in B$. They form the semi-direct product Fell bundle

$$\mathcal{B} \rtimes_\alpha H = \{(b, h) : b \in B, h \in H\}$$

with multiplication $(b, h)(c, k) = (b \alpha_h(c), hk)$ whenever $s_\mathcal{B}(b) = r_\mathcal{B}(h \ast p(c))$ and with involution $(b, h)^* = (\alpha_{h^{-1}}(b)^*, h^{-1})$. They proved that $\mathcal{B} \rtimes_\alpha H$ is a Fell bundle over the semi-direct product groupoid $\mathcal{G} \rtimes H$ [15, Proposition 6.2].

In the special case when $\mathcal{G}$ is étale and $H$ is discrete, one can verify that this construction coincides with ours in the following way. Let $\mathcal{H} = \mathcal{G}^{(0)} \rtimes H = \{(u, h) : u \in \mathcal{G}^{(0)}, h \in H\}$ be the transformation groupoid of the $H$-action restricted to $\mathcal{G}^{(0)}$ with multiplication $(u, h)(v, k) = (u, hk)$ whenever $v = h^{-1} \ast u$ and with inverse $(u, h)^{-1} = (h^{-1} \ast u, h^{-1})$. Then $(\mathcal{G}, \mathcal{H})$ is a matched pair of étale groupoids where the $\mathcal{H}$-action on $\mathcal{G}$ is given by $(u, h) \cdot x := h \ast x$ and the $\mathcal{G}$-restriction on $\mathcal{H}$ is given by $(u, h)|_x := (h \ast s(x), h)$. Moreover, the Zappa-Szép product groupoid $\mathcal{G} \rtimes_\beta \mathcal{H}$ coincides with the semi-direct product groupoid $\mathcal{G} \rtimes H$. The $H$-action $\alpha$ can be lifted to a $(\mathcal{G}, \mathcal{H})$-compatible $\mathcal{H}$-action $\beta$ by setting $\beta_{(u, h)}(b) = \alpha_h(b)$. Furthermore, the Zappa-Szép product Fell bundle $\mathcal{B} \rtimes_\beta \mathcal{H}$ is isomorphic to the semi-direct product Fell bundle $\mathcal{B} \rtimes_\alpha H$.

We note that, in general, the Fell bundle $\mathcal{B}$ is preserved isometrically inside $\mathcal{B} \rtimes_\beta \mathcal{H}$, but that we must make an additional assumption in order for it to also contain a copy of $\mathcal{B}(\mathcal{H})$.

**Proposition 3.12.** Define $\Phi : B \to B \rtimes_\beta \mathcal{H}$ by $\Phi(b) = (b, s_\mathcal{B}(b))$. Then $\Phi$ is an isometric homomorphism $\mathcal{B} \to \mathcal{B} \rtimes_\beta \mathcal{H}$ of Fell bundles, covariant with the embedding $\mathcal{G} \to \mathcal{G} \rtimes \mathcal{H}$. Furthermore, $\Phi(\mathcal{B}_x) = (\mathcal{B} \rtimes_\beta \mathcal{H})(x, s_\mathcal{G}(x))$.

If we assume that $\mathcal{B}_u$ is unital for all $u \in \mathcal{G}^{(0)}$, then we may further define $\Psi : C \times \mathcal{H} \to B \rtimes_{\beta} \mathcal{H}$ by $\Psi(z, h) = (z1_u, h)$ where $u = r_\mathcal{H}(h)$. Then $\Psi$ is an isometric homomorphism $\mathcal{B}(\mathcal{H}) \to \mathcal{B} \rtimes_\beta \mathcal{H}$ of Fell bundles, covariant with the embedding $\mathcal{H} \to \mathcal{G} \rtimes \mathcal{H}$. 


Proof. Let $\mathcal{C} := \mathcal{B} \rtimes_\beta \mathcal{H}$. Since $r_\mathcal{H}(s_\mathcal{B}(b)) = s_\mathcal{B}(b)$, $\Phi$ indeed takes values in $\mathcal{B} s_\mathcal{B} \times r_\mathcal{H} \mathcal{H}$. Clearly $\Phi$ is continuous and restricts to an isometric isomorphism between the Banach spaces $\mathcal{B}_x$ and $\mathcal{C}(x,s_\mathcal{B}(x))$, since the norm on the latter is given by

$$\|(b, s_\mathcal{B}(b))\|_{\mathcal{C}(x,s_\mathcal{B}(b))} = \|b\|_{\mathcal{B}(x)}.$$  

Using $\text{(ZS8)}$, $\text{(ZS9)}$, $\text{(ZS11)}$, and the fact that $\beta_u$ is the identity map, one can easily verify that $\Phi$ is $*$-preserving and multiplicative.

The map $\Psi$ is continuous and by construction covariant with the embedding. Since the fibre $\mathcal{C}(u,h)$ inherits its Banach space structure from $\mathcal{B}_u$, $\Psi$ restricts to a linear and isometric map between the fibres. To see that $\Psi$ is multiplicative, let $((z,h),(w,k)) \in \mathcal{B}(\mathcal{H})^{(2)}$, i.e. $v := s_\mathcal{H}(h) = r_\mathcal{H}(k)$. If $u := r_\mathcal{H}(h)$, then

$$(\Psi(z,h),\Psi(w,k)) = ((z1_u, h), (w1_v, k))$$

is in $\mathcal{C}^{(2)}$, and according to Condition $\text{(C2)}$, their product is given by

$$\Psi(z,h) \cdot \Psi(w,k) = (z1_u \beta_h(w1_v), h1_{(p(w1_v))}).$$

Since $\beta_h \colon \mathcal{B}_v \to \mathcal{B}_u$ is automorphic on $\mathcal{B}_r(h)$ and thus unital and since $h1_{(p(w1_v))} = h1 = h$ by $\text{(ZS6)}$, we see that

$$\Psi(z,h) \cdot \Psi(w,k) = ((zw)1_u, hk) = \Psi((z,h)(w,k)).$$

For the same reasons, we also have $\Psi(z,h)^* = \Psi(z,h^{-1}) = \Psi((z,h)^*)$.  

We would like to analyze under which conditions a Fell bundle $\mathcal{C} = (C,q)$ over the Zappa-Szép product groupoid $\mathcal{K} = \mathcal{G} \Join \mathcal{H}$ can be decomposed as a Zappa-Szép product $\mathcal{B} \Join_\beta \mathcal{H}$ of some Fell bundle $\mathcal{B}$ over $\mathcal{G}$ with $(\mathcal{G},\mathcal{H})$-compatible $\mathcal{H}$-action $\beta$. First, the map $\iota \colon \mathcal{G} \to \mathcal{K} = \mathcal{G} \Join \mathcal{H}$ given by $\iota(x) = (x,s_\mathcal{G}(x))$ defines a continuous groupoid homomorphism, and thus the pullback bundle $\mathcal{B} = \iota^*(\mathcal{C}) = \{(x,c) \in \mathcal{G} \times C : \iota(x) = q(c)\}$ with the map $p \colon \mathcal{B} \to \mathcal{G}$, $p(x,c) = x$ is a Fell bundle over $\mathcal{G}$ (\cite[Remark 2.6]{H}). Here, each fibre $\mathcal{B}_x$ can be identified as $\mathcal{C}(x,s_\mathcal{G}(x))$.  

Definition 3.13. Let $\mathcal{C} = (C,q)$ be a Fell bundle over $\mathcal{K}$ and $\mathcal{H}$ a wide subgroupoid of $\mathcal{K}$, let $j \colon \mathcal{H} \to \mathcal{K}$ denote the inclusion. We call a continuous section $u \colon \mathcal{H} \to j^*(\mathcal{C})$ of the pullback bundle $j^*(\mathcal{C})$ an $\mathcal{H}$-unitary family in $\mathcal{C}$ if

1. For any $(h,k) \in \mathcal{H}^{(2)}$, $u_hu_k = u_{hk}$ and $u_h^* = u_{h^{-1}}$.
2. For each $v \in \mathcal{H}^{(0)}$, $u_v = 1_v$ is the identity on $\mathcal{C}_v$.

Proposition 3.14. Let $\mathcal{C} = (C,q)$ be a Fell bundle over $\mathcal{K} = \mathcal{G} \Join \mathcal{H}$ and suppose that there exists an $\mathcal{H}$-unitary family $u$ (where we identify $h \in \mathcal{H}$ with $(r_\mathcal{H}(h),h) \in \mathcal{K}$). Let $\mathcal{B} = (B,p) = \iota^*(\mathcal{C})$ be the pullback bundle along the inclusion $\iota \colon \mathcal{G} \to \mathcal{G} \Join \mathcal{H}$. For any $(h,x) \in \mathcal{H} s_\mathcal{H} \times r_\mathcal{B} B$, define $\beta_h \colon \mathcal{B}_x \to \mathcal{B}_{h,x}$ by

$$\beta_h(a) = u_hau_{h|_x}^*.$$  

Then $\beta$ is a $(\mathcal{G},\mathcal{H})$-compatible $\mathcal{H}$-action on $\mathcal{B}$.  

Proof. First of all, let us verify that \( \beta_h \) is well-defined: by assumption, \( u_h \) is an element of \( C(r_H(h), h) \), so that \( u_h^{*}|_{x} = u_{(h|x)}{^{-1}} \in C(s_H(h|x), (h|x))^{-1} \). Since \( s_H(h) = r_G(x) \) and \( s_H(h|x) = s_G(x) \) by [ZS2], we have for \( a \in \mathcal{B}_x = C(x, s_G(x)) \) that \( u_h a u_h^{*}|_{x} \) is well-defined and is an element of the fibre of \( C \) over \((r_H(h), h)(x, s_G(x)), (h|x)\)^{-1} = (h \cdot x, h|x)\)^{-1} = (h \cdot x, s_G(h \cdot x)) \).

Let us now verify that \( \beta \) satisfies Definition 3.1. For [A1] since \( h \mapsto u_h \) is continuous and multiplication is continuous, we have that \( \beta : (h, a) \mapsto u_h a u_h^{*}|_{x} \) is continuous.

For [A2] pick any \((g, h) \in H(2)\), any \( x \in G \) with \( s_H(h) = r_G(x) \), and any element \( b \in \mathcal{B}_x \). Then
\[
\beta_{gh}(b) = u_{gh} b u_{gh}^{*} = u_g u_h b u_h^{*} = u_g \beta_h(b) u_h^{*} = \beta_g(\beta_h(b)).
\]

For [A3] for each \( v \in H(0) \), we have \( u_v = 1_v \), and so \( \beta_v \) is clearly the identity map.

For [A4] for any \((b, c) \in \mathcal{B}(2)\) such that \((h, bc) \in H \times s_H \times B \), we have
\[
\beta_h(bc) = u_h(b) u^*_h|_{p(b) \circ p(c)} u_{h|_{p(b)}} u_{h|_{p(b)}} u_{h|_{p(c)}} = \beta_h(b) \beta_h(c).
\]

For [A5] for any \( b \in \mathcal{B}_x \) with \( r_G(x) = s_H(h) \), we have
\[
\beta_h(b)^{*} = (u_h b u_h^{*})^{*} = u_h b^{*} u_h.
\]
Since \( b^{*} \in \mathcal{B}_{x^{-1}} \) and \((h|_{x^{-1}})|_{x^{-1}} = h|_{r_G(x)} = h \), we conclude that
\[
\beta_h(b)^{*} = u_{h|_{x^{-1}}} b^{*} u_{h|_{x^{-1}}} = \beta_h(b^{*}). \]

We now prove that the Zappa-Szép product bundle \( \mathcal{B} \bowtie \beta \mathcal{H} \) is isometrically \( * \)-isomorphic to the original bundle \( \mathcal{C} \) over \( \mathcal{G} \bowtie \mathcal{H} \).

Theorem 3.15. Let \( \mathcal{C} = (C, q) \) be a Fell bundle over \( \mathcal{G} \bowtie \mathcal{H} \) and suppose that there exists an \( \mathcal{H} \)-unitary family \( u \). With \( \mathcal{B} = (B, p) = \iota^{*}(\mathcal{C}) \) the pull back bundle along \( \iota : \mathcal{G} \rightarrow \mathcal{G} \bowtie \mathcal{H} \), define \( \Theta : B \bowtie \mathcal{H} \rightarrow C \) by \( \Theta(a, h) = au_h \). Then \( \Theta \) is an isometric isomorphism \( \mathcal{B} \bowtie \mathcal{H} \rightarrow \mathcal{C} \) of Fell bundles over \( \mathcal{G} \bowtie \mathcal{H} \), where \( \beta \) is as defined in Proposition 3.14.

Proof. First of all, \( \Theta \) clearly preserves fibres and is fibrewise linear. It is further continuous, by definition of the topology on the pullback bundle \( \mathcal{B} \), by the assumption that \( h \mapsto u_h \) is continuous, and since multiplication on \( \mathcal{C} \) is continuous. To see that \( \Theta \) is multiplicative, take any \((a, h), (b, g) \in \mathcal{B} \bowtie \mathcal{H}(2)\). By definition, \((a, h)(b, g) = (a \beta_h(b), h|_{p(b)} g)\). On the other hand,
\[
\Theta((a, h)) \Theta((b, g)) = au_h b u_g
= au_h b u_h^{*}|_{p(b)} u_{h|_{p(b)}} u_{g}
= a \beta_h(b) u_{h|_{p(b)} g} = \Theta((a, h)(b, g)).
\]
To see that \( \Theta \) is \( * \)-preserving, take \((a, h) \in \mathcal{B} \bowtie \mathcal{H} \) with \( a \in \mathcal{B}_x \), so that \((a, h)^{*} = (\beta_h^{-1}(a^*), h|_{x^{-1}}^{\circ})\). We have,
\[
\Theta((a, h)^{*}) = u_{h^{-1}} a^{*} = u_{h^{-1}} a^{*} u_{h^{-1}|_{x^{-1}}} u_{h^{-1}|_{x^{-1}}} = \beta_h^{-1}(a^{*}) u_{h^{-1}|_{x^{-1}}} = \Theta((a, h)^{*}).
\]
To see that $\Theta$ is isometric,

$$
\|\Theta((a, h))\| = \|au_h\| = \|au_h a^*a^*\|^{1/2} = \|aa^*\|^{1/2} = \|a\| = \|(a, h)\|.
$$

In particular, $\Theta$ is automatically injective. Finally, for any $c \in C(x, h)$,

$$
cu^*_h \in C(x, h) = C(x, s_h(h), h^{-1}) = C(x, sh^{-1}) = C(x, s_\Theta(x)) \cong B_x.
$$

Therefore,

$$
c = cu^*_h u_h = \Theta((cu^*_h, h)).
$$

As a result, $\Theta$ is surjective. \(\square\)

As an immediate corollary, we obtain the following version of the internal Zappa-Szép product of Fell bundles.

**Corollary 3.16.** Let $C$ be a Fell bundle over $K$, and let $G, H$ be subgroupoids of $K$. Let $B$ be the pullback bundle along the inclusion $G \to K$, and suppose that there exists an $H$-unitary family $u$. Suppose for each $c \in C$, there exist unique $b \in B$ and $h \in H$ with $c = bu_h$. Then $K = G \bowtie H$ and $C$ is isometrically isomorphic to $B \bowtie_\beta H$ for some $(G, H)$-compatible $H$-action $\beta$.

**Proof.** We first claim that for any $k \in K$, there exists a unique $g \in G$ and $h \in H$ such that $(g, h) \in K(2)$ and $k = gh$. To see there exists at least one such pair $(g, h)$, pick any $c \in C_k$. By assumption, there exists a unique $b \in B$ and $h \in H$ such that $c = bu_h$. As $b$ is an element of some fibre $B_g$, then $k = gh$. To see that such a pair is unique, suppose $k = g'h'$ for some $g' \in G$ and $h' \in H$. Then $cu^*_h \in C_{g'h'}^{-1} = C_g \cong B_g$. In this case we have $bu_h = c = (cu^*_h)u_{h'}$, so the uniqueness of such decomposition implies $h' = h$ and hence $g' = g$.

By the internal Zappa-Szép product for groupoids \[3\text{, Proposition 3.4}], we conclude $K = G \bowtie H$. The rest of the proof follows from Theorem 3.15. \(\square\)

4. $C^*$-algebras of Fell bundles

For a Fell bundle $B$ over an étale groupoid $G$, one can define a universal $C^*$-algebra $C^*(B)$ with respect to certain $*$-representations of the $*$-algebra $\Gamma_c(G; B)$ of continuous compactly supported sections of the bundle $B$. If we use the standard notation $G^v := r_1^{-1}(\{v\})$, $G_v := s_1^{-1}(\{v\})$ for an idempotent $v$ of $G$, then the $*$-algebra structure on $\Gamma_c(G; B)$ is given by

$$
(\sigma \square \tau)(x) = \sum_{y \in G^v} \sigma(y) \cdot \tau(y^{-1}x), \text{ and } \sigma^*(x) = \sigma(x^{-1})^*,
$$

where $x \in G^v$ and where the multiplication $\cdot$ is to be understood in $B$. Let

$$
\|\sigma\|_{I,v} := \sup_{v \in G^v} \left( \sum_{x \in G^v} \|\sigma(x)\| \right) \quad \text{and} \quad \|\sigma\|_{I,s} := \sup_{v \in G^v} \left( \sum_{x \in G^v} \|\sigma(x)\| \right).
$$

The $I$-norm on the $*$-algebra $\Gamma_c(G; B)$ is given by

$$
\|\sigma\|_I := \max(\|\sigma\|_{I,v}, \|\sigma\|_{I,s}).
$$
Definition 4.1 (see [19, Example 4.8]). A $\ast$-homomorphism $L$ of $\Gamma_c(\mathcal{G}; \mathcal{B})$ into the bounded operators $\mathbb{B}(H)$ on some Hilbert space $H$ is called a representation if it is continuous when $\Gamma_c(\mathcal{G}; \mathcal{B})$ has the inductive limit topology and $\mathbb{B}(H)$ the weak operator topology.

Note that $L$ is continuous in this sense if it is $I$-norm decreasing, i.e. $\|L(\sigma)\| \leq \|\sigma\|_I$ for all $\sigma \in \Gamma_c(\mathcal{G}; \mathcal{B})$.

We point out that strict representations of $\mathcal{B}$ can be integrated to representations of $\Gamma_c(\mathcal{G}; \mathcal{B})$ [19, Prop. 4.10], and that the converse holds also: every nondegenerate representation of $\Gamma_c(\mathcal{G}; \mathcal{B})$ is the integrated form of a strict representation [19 Thm 4.13] (Note that [19] ask for nondegeneracy when they talk about representations; we will explicitly mention when we assume nondegeneracy.)

We define $C^*(\mathcal{B})$ to be the completion of $\Gamma_c(\mathcal{G}; \mathcal{B})$ with respect to the universal norm

$$\|\sigma\| := \sup\{\|L(\sigma)\| : L \text{ is an } I\text{-norm decreasing } \ast\text{-representation}\}.$$

For a more detailed description, one may refer to [10, Definition 16.25] for the case when $\mathcal{G}$ is a discrete group, and [25] or [19] for the case when $\mathcal{G}$ is an étale groupoid.

Lemma 4.2. For $\sigma \in \Gamma_c(\mathcal{G}; \mathcal{B})$ and $(x, h) \in \mathcal{G} \bowtie \mathcal{H}$, define

$$i(\sigma)(x, h) := \begin{cases} (\sigma(x), h) = \Phi(\sigma(x)), & \text{if } h = r_H(h) \text{ is an idempotent,} \\ (0_x, h), & \text{otherwise,} \end{cases}$$

where we wrote $0_x$ for zero in the Banach space $\mathcal{B}_x$. Then $i(\sigma)$ defines an element of $\Gamma_c(\mathcal{G} \bowtie \mathcal{H}; \mathcal{B} \bowtie_\beta \mathcal{H})$.

Proof. Let $\mathcal{C} := \mathcal{B} \bowtie_\beta \mathcal{H}$, with total space $C := B \ast_\mathcal{G} \times_{r_H} \mathcal{H}$, and let $\mathcal{K} := \mathcal{G} \bowtie \mathcal{H}$. Because $\sigma$ is a section, $i(\sigma)$ takes values in $\mathcal{C}$. Indeed,

$$s_\mathcal{G}(\sigma(x)) = s_\mathcal{G}(p(\sigma(x))) = s_\mathcal{G}(x) \text{ resp. } s_\mathcal{B}(0_x) = s_\mathcal{G}(p(0_x)) = s_\mathcal{G}(x),$$

both of which equal $r_H(h)$ since $(x, h) \in \mathcal{G} \bowtie \mathcal{H}$. It is, moreover, a section of the bundle $\mathcal{C} = (C, q)$, as

$$q(i(\sigma)(x, h)) = \begin{cases} (p(\sigma(x)), h) & \text{if } h \in \mathcal{H}^{(0)} \\ (p(0_x), h) & \text{if } h \notin \mathcal{H}^{(0)} \end{cases} = (x, h).$$

To see that $i(\sigma)$ is continuous, assume that $(x_i, h_i)i_i$ is a net in $\mathcal{G} \bowtie \mathcal{H}$ which converges to $(x, h)$. If $h$ is not an idempotent, then since $\mathcal{H}^{(0)}$ is closed, there exists $i_0$ such that $h_i \notin \mathcal{H}^{(0)}$ for $i \geq i_0$, so that $i(\sigma)(x_i, h_i) = (0_x, h_i)$. By (USC4) of Definition 2.6 for $\mathcal{B}$, we know that $p(0_x) = x_i \to x = p(0_x)$ in $\mathcal{G}$ implies $0_x \to 0_x$ in $B$, and so $i(\sigma)(x_i, h_i) \to (0_x, h) = i(\sigma)(x, h)$, since $h_i \to h$ also.

On the other hand, if $h = v \in \mathcal{H}^{(0)}$, then since $\mathcal{H}$ is étale so that $\mathcal{H}^{(0)}$ is also open, there exists $i_1$ such that for all $i \geq i_1$, we have $h_i = v_i \in \mathcal{H}^{(0)}$ also. Thus,

$$i(\sigma)(x_i, h_i) = (\sigma(x_i), v_i) \to (\sigma(x), v) = i(\sigma)(x, h),$$

proving continuity of $i(\sigma)$.
To see that $i(\sigma)$ is compactly supported, note that

$$\text{supp}(i(\sigma)) \subset \text{supp}(\sigma) \times s_\sigma(\text{supp}(\sigma)) \subset \mathcal{G} \times \mathcal{H}.$$ 

Since $\text{supp}(\sigma)$ is compact, so is $\text{supp}(\sigma) \times s_\sigma(\text{supp}(\sigma))$. Since $\mathcal{G} \bowtie \mathcal{H}$ is closed in $\mathcal{G} \times \mathcal{H}$, this implies that the closed set $\text{supp}(i(\sigma))$ is contained in a compact subset of $\mathcal{G} \bowtie \mathcal{H}$, making it compact.

**Proposition 4.3.** The map

$$i : \Gamma_c(\mathcal{G}; \mathcal{B}) \to \Gamma_c(\mathcal{G} \bowtie \mathcal{H}; \mathcal{B} \bowtie_\beta \mathcal{H}),$$

with $i(\sigma)$ as defined in Lemma 4.2, is a $\ast$-algebra homomorphism and extends to a $\ast$-homomorphism $i : C^*(\mathcal{B}) \to C^*(\mathcal{B} \bowtie_\beta \mathcal{H})$.

**Proof.** We have shown in Lemma 4.2 that $i(\sigma)$ is an element of $\Gamma_c(\mathcal{G} \bowtie \mathcal{H}, \mathcal{B} \bowtie_\beta \mathcal{H})$, and $\sigma \mapsto i(\sigma)$ is clearly linear.

To see that the map is $\ast$-preserving, we compute for $(x, h) \in \mathcal{G} \bowtie \mathcal{H}$,

$$i(\sigma)^*(x, h) \overset{(4.2)}{=} [i(\sigma)((x, h)^{-1})]^* \overset{(ZS6)}{=} [i(\sigma)(h^{-1} \cdot x^{-1}, h^{-1}|_{x^{-1}})]^*.$$ 

First note that $h^{-1}|_{x^{-1}} \in \mathcal{H}^{(0)}$ if and only if $h^{-1} \in \mathcal{H}^{(0)}$: we know from (ZS11) that the restriction of a unit is a unit, and

$$h^{-1}|_{s_\mathcal{H}(h^{-1})} = h^{-1}|_{s_\mathcal{G}(x)} \overset{(ZS4)}{=} (h^{-1}|_{x^{-1}})|_{x}.$$ 

So we may first assume that $h^{-1}|_{x^{-1}}, h^{-1}$ are both elements of $\mathcal{H}^{(0)}$. By (ZS5) and (ZS3), we have $h^{-1}|_{x^{-1}} = r_\mathcal{G}(x)$ and $h^{-1} \cdot x^{-1} = x^{-1}$, respectively. Therefore,

$$i(\sigma)^*(x, h) \overset{(4.2)}{=} (\sigma(h^{-1} \cdot x^{-1}), r_\mathcal{G}(x))^* = (\sigma(x^{-1}), r_\mathcal{G}(x))^*$$

$$\overset{(C3)}{=} (\beta_{r_\mathcal{G}(x)}(\sigma(x^{-1})^*), r_\mathcal{G}(x)^{-1}|_{p(\sigma(x^{-1}))^{-1}})$$

$$\overset{(A3)}{=} (\sigma(x^{-1})^*, r_\mathcal{G}(x)|_{x})$$

$$\overset{(ZS11)}{=} (\sigma(x^{-1})^*, s_\mathcal{G}(x)) \overset{(4.1)}{=} (\sigma^*(x), r_\mathcal{H}(h)) = i(\sigma^*)(x, h).$$

On the other hand, assume $h$ and $h^{-1}|_{x^{-1}}$ are not idempotents. Since $i(\sigma)^*$ and $i(\sigma^*)$ are both sections, we do not need to keep track of the subscript of the zero-element; instead, we can just compute (using that $\beta$ is linear and hence sends 0 to 0)

$$i(\sigma)^*(x, h) \overset{(4.2)}{=} (0, (h^{-1}|_{x^{-1}})^{-1}|_{(h^{-1}, x^{-1})^{-1}})$$

$$\overset{(2.2)}{=} (0, (h|_{h^{-1}, x^{-1}})^{-1}|_{(h^{-1}, x^{-1})^{-1}})$$

$$\overset{(ZS4)}{=} (0, h) = i(\sigma^*)(x, h).$$
Lastly, we need to see that \( \sigma \mapsto i(\sigma) \) is multiplicative, so let \( \tau \) be another element of \( \Gamma_c(\mathcal{G}; \mathcal{B}) \). We compute for \((x, h) \in \mathcal{G} \bowtie \mathcal{H}, \) using (4.1)

\[
(\sigma \boxtimes i(\tau))(x, h) = \sum_{(y, k) \in \mathcal{G} \bowtie \mathcal{H}^{(x, h)}} i(\sigma)(y, k) \cdot i(\tau)((y, k)^{-1}(x, h)).
\]

Recall from Equation (2.3) that the range of \((x, h) \in \mathcal{G} \bowtie \mathcal{H}^{(0)}\) is given by \( r_\mathcal{G}(x) \). Note that, if \((y, k)\) has \( k \notin \mathcal{H}^{(0)} \), then \( i(\sigma)(y, k) = (0, y, k) \). We get by the definition in (C2)

\[
(0, k) \cdot (b, h) = (0, k |_{\rho(b)} h) \quad \text{and} \quad (y, k) \cdot (0, h) = (0, k |_{\rho(b)} h).
\]

We see that in Equation (4.3), only summands of the form \((y, k) = (y, s_\mathcal{B}(y))\) might not vanish. Using the equality

\[
(y, s_\mathcal{B}(y))^{-1}(x, h) = (y^{-1}x, h)
\]

in \( \mathcal{G} \bowtie \mathcal{H} \), this all in all yields

\[
(i(\sigma) \boxtimes i(\tau))(x, h) = \sum_{y \in G^{(x)}} i(\sigma)(y, s_\mathcal{B}(y)) \cdot i(\tau)(y^{-1}x, h)
\]

\[
= \begin{cases} (0, h) & \text{if } h \notin \mathcal{H}^{(0)}, \\ \sum_{y \in G^{(x)}} (\sigma(y), s_\mathcal{B}(y)) \cdot (\tau(y^{-1}x), s_\mathcal{B}(x)) & \text{otherwise}. \end{cases}
\]

On the other hand, \( i(\sigma \boxtimes \tau)(x, h) \) is also zero if \( h \notin \mathcal{H}^{(0)} \), and otherwise

\[
i(\sigma \boxtimes \tau)(x, h) = ((\sigma \boxtimes \tau)(x), s_\mathcal{B}(x)) = \sum_{y \in G^{(x)}} (\sigma(y) \cdot (\tau(y^{-1}x), s_\mathcal{B}(x)),
\]

which coincides with \( (i(\sigma) \boxtimes i(\tau))(x, h) \) by definition, see (C2).

It remains to show that \( i \) extends. Since \( \| (b, h) \|_{\mathcal{C}} = \| b \|_{\mathcal{B}} \) for \( \mathcal{C} := \mathcal{B} \bowtie_\beta \mathcal{H} \), we have

\[
\| i(\sigma) \|_{1, s} = \sup_{v \in K^{(0)}} \left( \sum_{x \in K_v} \| i(\sigma)(x) \| \right) = \sup_{v \in K^{(0)}} \left( \sum_{(x, s_\mathcal{G}(x)) \in K_v} \| (\sigma(x), v) \| \right)
\]

\[
\leq \sup_{v \in K^{(0)}} \left( \sum_{x \in G_v} \| \sigma(x) \| \right) = \| \sigma \|_{1, s},
\]

and similarly

\[
\| i(\sigma) \|_{1, f} = \sup_{v \in K^{(0)}} \left( \sum_{x \in K_v} \| i(\sigma)(x) \| \right) = \sup_{v \in K^{(0)}} \left( \sum_{(x, s_\mathcal{G}(x)) \in K_v} \| (\sigma(x), s_\mathcal{G}(x)) \| \right)
\]

\[
\leq \sup_{v \in K^{(0)}} \left( \sum_{x \in G_v} \| \sigma(x) \| \right) = \| \sigma \|_{1, f},
\]

which implies \( \| i(\sigma) \|_I \leq \| \sigma \|_I \). Thus, every \( I \)-norm decreasing representation \( L \) of \( \Gamma_c(K; \mathcal{B}) \) gives rise to an \( I \)-norm decreasing representation \( L \circ i \).
of $\Gamma_c(\mathcal{G}; \mathcal{B})$. In particular,

\[
\|i(\sigma)\|_{C^*(\mathcal{E})} = \sup\{\|L(i(f))\| : L \text{ norm decreasing } \ast\text{-rep. } L \text{ of } \Gamma_c(\mathcal{K}; \mathcal{C})\} \\
\leq \sup\{\|L'(\sigma)\| : L' \text{ norm decreasing } \ast\text{-rep } L' \text{ of } \Gamma_c(\mathcal{G}; \mathcal{B})\},
\]

i.e. $\|i(\sigma)\|_{C^*(\mathcal{E})} \leq \|\sigma\|_{C^*(\mathcal{B})}$, which proves that $i$ extends. \hfill \square

**Definition 4.4.** Assume $(\mathcal{G}, \mathcal{H})$ is a matched pair of étale groupoids and let $\mathcal{B} = (B, \rho)$ be a Fell bundle over $\mathcal{G}$ with a $(\mathcal{G}, \mathcal{H})$-compatible $\mathcal{H}$-action $\beta$. Let $\mathcal{U} := \mathcal{G}^{(0)} = \mathcal{H}^{(0)}$. A covariant representation of $(\mathcal{B}, \beta)$ is a quadruple $(\mu, \mathcal{U} \ast \mathcal{H}, \hat{\pi}, \hat{M})$ consisting of

- (R1) a Radon measure $\mu$ on $\mathcal{U}$ which is quasi-invariant with respect to the Haar system of counting measures on $\mathcal{G} \bowtie \mathcal{H}$,
- (R2) a Borel Hilbert bundle $\mathcal{U} \ast \mathcal{H}$ over $\mathcal{U}$, where we write $\mathcal{H} = \{H(v)\}_{v \in \mathcal{U}}$,
- (R3) a Borel $\ast$-functor $\hat{\pi} : \mathcal{B} \to \text{End}(\mathcal{U} \ast \mathcal{H})$, and
- (R4) a Borel homomorphism $\hat{M} : \mathcal{H} \to \text{Iso}(\mathcal{U} \ast \mathcal{H})$ such that $\hat{M}_h = (r_\mathcal{H}(h), M, s_\mathcal{H}(h))$ for some unitary operator $M_h : H(s_\mathcal{H}(h)) \to H(r_\mathcal{H}(h))$, such that

\[(4.4) \quad \hat{M}_h \hat{\pi}(b) = \hat{\pi}(\beta(h, b))\hat{M}_{h|_{\beta(b)}}\]

for all $(h, b) \in \mathcal{H} \times_{r_{\mathcal{B}}} \mathcal{B}$.

We let $B(\mathcal{U} \ast \mathcal{H})$ denote the Borel sections as defined in [23, Definition F.1], where the interested reader can also find a precise definition of (R2). We refer the reader further to [19, Definition 4.5] for the definition of (R3) and to [24, Definition 3.37] for the definition of (R4). We point out that $\hat{\pi}$ being a $\ast$-functor in particular allows us to write $\hat{\pi}(b) = (r_{\mathcal{B}}(b), \pi(b), s_{\mathcal{B}}(b))$ for some operator $\pi(b) : H(s_{\mathcal{B}}(b)) \to H(r_{\mathcal{B}}(b))$ (so we could have written Equation (4.4) without the hats).

**Example 4.5** (see Example 3.9). If $\mathcal{H} = \mathcal{G}^{(0)}$, so that $\mathcal{G} \bowtie \mathcal{H} \cong \mathcal{G}$ and $\mathcal{B} \bowtie_{\text{triv}} \mathcal{H} \cong \mathcal{B}$, then $\mu$ as in Condition (R1) is quasi-invariant with respect to the Haar system of counting measures on $\mathcal{G}$. Furthermore, since $\hat{M}$ of Condition (R4) is a homomorphism, it is just (fibre-wise) the identity on $H(v)$ for each $v \in \mathcal{H} = \mathcal{G}^{(0)}$, so Equation (4.4) becomes vacuous. It follows that covariant representations of $\mathcal{B}$ equipped with the trivial $\mathcal{G}^{(0)}$-action are exactly strict representations of $\mathcal{B}$ in the sense of [19, Definition 4.9].

**Example 4.6** (see Example 3.10). Conversely, it was shown in [19, Appendix B] that, if a Borel $\ast$-functor $\hat{\pi}$ for the trivial line bundle $\mathcal{B}(\mathcal{G})$ gives rise to a nondegenerate $\ast$-representation of $C_c(\mathcal{G})$, then $\hat{\pi}$ can be viewed as a unitary representation $\hat{N}$ of $\mathcal{G}$ by defining

\[
\hat{N}_x := (r_{\mathcal{G}}(x), \pi(1, x), s_{\mathcal{G}}(x)).
\]

Thus, in that case, Definition 4.4 boils down to a choice of quasi-invariant measure $\mu$ on $\mathcal{G} \bowtie \mathcal{H}$ and two unitary representations $\hat{N}$ of $\mathcal{G}$ and $\hat{M}$ of $\mathcal{H}$ on the same Borel Hilbert bundle $\mathcal{U} \ast \mathcal{H}$ satisfying

\[(4.5) \quad M_h N_x = N_{h^{-1}x} M_{h|_{\beta(x)}}, \quad \text{if } s_\mathcal{H}(h) = r_\mathcal{B}(x)\]
If we let $K_{(x,h)} := N_x M_h$ for $(x,h) \in G \rtimes H$, then the above equation makes $K$ a homomorphism, so that $(\mu, U \ast \mathcal{H}, \hat{K})$ is a unitary representation of $G \rtimes H$.

As in [24, F.2], we let $L^2(U \ast \mathcal{H}, \mu)$ be the direct integral of the Hilbert bundle, i.e. the normed vector space formed by the quotient of

$$L^2(U \ast \mathcal{H}, \mu) := \{ \xi \in B(U \ast \mathcal{H}) \text{ s.t. } v \mapsto \|\xi(v)\|^2 \text{ is } \mu\text{-integrable} \}$$

where functions agreeing $\mu$-almost everywhere are being identified. Furthermore, let $\Delta := dv/d\nu^{-1}$ be the Radon-Nykodym derivative of $\nu = \mu \circ \lambda$ and its pushforward $\nu^{-1}$. Since $G$ and $H$ are assumed to be second countable and locally compact (so that $G \rtimes H$ is as well), we know by [24, Prop. 7.9] that $\Delta : G \rtimes H \to (\mathbb{R}^+, \times)$ can be chosen to be a Borel homomorphism.

For $\varepsilon = (x,h)$ some element of $G \rtimes H$ and $\sigma$ a section of $\beta \rtimes H$, we will write $\sigma_{\beta}(\varepsilon) := \operatorname{pr}_2(\sigma(\varepsilon)) \in \beta_x$ and $\hat{M}_\varepsilon := M_h$. Note that we have to be careful with the latter notation; for example, if $(\varepsilon, \varphi) \in (G \rtimes H)^{(2)}$ for some $\varphi = (y,k)$, then

$$\varepsilon \varphi = (x(h \cdot y), h|_y k), \text{ so that } M_{h|_y \varphi} = M_{h|_y} M_k = M_{h|_y k} = M_{\varepsilon \varphi}. \tag{4.6}$$

**Theorem 4.7.** Given a covariant representation $(\mu, U \ast \mathcal{H}, \hat{\pi}, \hat{M})$ of $(\beta, \beta)$, define for a section $\sigma \in \Gamma_c(G \rtimes H; \beta \rtimes H)$, any $\xi \in L^2(U \ast \mathcal{H}, \mu)$, and $v \in U$,

$$(L(\sigma)\xi)(v) = \sum_{\varepsilon \in (G \rtimes H)^v} \pi(\sigma_{\beta}(\varepsilon)) M_\varepsilon (\xi(s(\varepsilon))) \Delta(\varepsilon)^{-\frac{1}{2}}.$$

Then $L$ is a $1$-norm decreasing $*$-representation of $\Gamma_c(G \rtimes H; \beta \rtimes H)$ on $L^2(U \ast \mathcal{H}, \mu)$.

Similarly to [19, Remark 4.12], we emphasize that $L$ need not be nondegenerate and thus not a representation in the sense of [19, Definition 4.7]. Furthermore, we point out that the modular function $\Delta$ in the formula for $L$ is needed to account for the fact that we defined our involution formula on the $\ast$-algebra of sections without $\Delta$.

**Remark 4.8.** When $\mathcal{H} = \mathcal{H}^{(0)}$, so that $G \rtimes H \simeq G$ and $\beta \rtimes H \simeq \beta$ (see also Examples 3.9 and 4.5), then the above theorem recovers [19, Proposition 4.10].

Conversely, assume $\beta = \beta(G)$ and that we are in the situation of Example 4.6 so that the covariant representation $(\mu, U \ast \mathcal{H}, \hat{\pi}, \hat{M})$ of $\beta(G)$ can be viewed instead as a unitary representation $(\mu, U \ast \mathcal{H}, \hat{K})$ of $G \rtimes H$, where $K_{(x,h)} := \pi(1_x) M_h$. In this case, the above theorem recovers a slightly weakened version of [24, Proposition 7.12] in which the assumption of an ‘almost everywhere unitary representation’ in said proposition is made stronger by dropping the word ‘almost’.

**Proof of Theorem 4.7** For the duration of this proof, let $K := G \rtimes H$, and let $s$ and $r$ denote its range and source maps. Since $\hat{\pi}$ is a Borel $\ast$-functor and $\hat{M}$ a Borel homomorphism, $\varepsilon \mapsto \hat{\pi}(\sigma_{\beta}(\varepsilon)) \hat{M}_\varepsilon$ is a Borel map from $K$ to
\[ F : K \to \mathbb{C}, \, \varepsilon \mapsto \left\langle \pi(\sigma_\varepsilon(\varepsilon))M_\varepsilon\xi(s(\varepsilon)) \right| \zeta(r(\varepsilon)) \right\rangle \Delta(\varepsilon)^{-\frac{1}{2}} \]

is also Borel. We will use an argument from the proof of [24, Proposition 1.29] to show that this implies that the map

\[ f : \mathcal{U} \to \mathbb{C}, \, v \mapsto \sum_{\varepsilon \in \mathcal{K}^v} \left\langle \pi(\sigma_\varepsilon(\varepsilon))M_\varepsilon\xi(s(\varepsilon)) \right| \zeta(v) \right\rangle \Delta(\varepsilon)^{-\frac{1}{2}}, \]

is Borel also. Since \( K \) is étale, we can cover the compact support of \( \sigma \) by finitely many open bisections. If we take a continuous partition of unity subordinate to that cover, then we see that we may without loss of generality assume that \( \sigma \) is supported in a bisection, say \( U \subset K \). If we let \( t : \mathcal{U} \to K \) be the continuous local inverse of the range map, then \( f(v) = F(t(v)) \) is Borel as composition of a Borel and a continuous function.

We next claim that it is \( \mu \)-integrable. To show this, we first compute

\[ \int_{\mathcal{U}} |f(v)| \, d\mu(v) \leq \int_{K} \|\pi(\sigma_\varepsilon(\varepsilon))M_\varepsilon(\xi(s(\varepsilon)))\| \|\zeta(r(\varepsilon))\| \Delta(\varepsilon)^{-\frac{1}{2}} \, d\nu(\varepsilon). \]

Note that the \( * \)-functor \( \hat{\pi} \) is norm-decreasing [19, Remark 4.6]. Since the codomain of \( M \) is the isomorphism groupoid, \( M_\varepsilon \) is a unitary, so in particular \( \|M_\varepsilon\| = 1 \). Both combined yield

\[ \|\pi(\sigma_\varepsilon(\varepsilon))M_\varepsilon(\xi(s(\varepsilon)))\| \leq \|\sigma_\varepsilon(\varepsilon)\| \|\xi(s(\varepsilon))\| = \|\sigma(\varepsilon)\| \|\xi(s(\varepsilon))\|. \]

Next, we will use a trick from [19, Proposition 4.7] resp. [24, Proposition 7.12], attributed to Renault: Using Cauchy–Schwarz for \( \nu \) in the second of the following inequalities, we get

\[ \left( \int_{\mathcal{U}} |f(v)| \, d\mu(v) \right)^2 \leq \left( \int_{K} \|\sigma(\varepsilon)\| \|\xi(s(\varepsilon))\| \|\zeta(r(\varepsilon))\| \Delta(\varepsilon)^{-\frac{1}{2}} \, d\nu(\varepsilon) \right)^2 \]

\[ \leq \left( \int_{K} \|\sigma(\varepsilon)\| \|\xi(s(\varepsilon))\|^2 \Delta(\varepsilon)^{-1} \, d\nu(\varepsilon) \right) \cdot \left( \int_{K} \|\sigma(\varepsilon)\| \|\zeta(r(\varepsilon))\|^2 \, d\nu(\varepsilon) \right). \]

For the first factor, we compute

\[ \int_{\mathcal{U}} \sum_{\varepsilon \in \mathcal{K}^v} \|\sigma(\varepsilon)\| \|\xi(s(\varepsilon))\|^2 \Delta(\varepsilon)^{-1} \, d\mu(v) = \int_{\mathcal{U}} \sum_{\varepsilon' \in \mathcal{K}^v} \|\sigma(\varepsilon')\| \|\xi(s(\varepsilon'))\|^2 \, d\mu(v) \]

\[ \leq \int_{\mathcal{U}} \|\sigma\|_{I,s} \|\xi(v)\|^2 \, d\mu(v) = \|\sigma\|_{I,s} \|\xi\|^2_2. \]

Similarly one gets for the second factor

\[ \int_{K} \|\sigma(\varepsilon)\| \|\zeta(r(\varepsilon))\|^2 \, d\nu(\varepsilon) \leq \|\sigma\|_{I,r} \|\zeta\|^2_2. \]
Both combined yield
\[
\left( \int_{U^i} |f(v)| \, d\mu(v) \right)^2 \leq (\|\sigma\|_{L^s} \|\xi\|_2^2)(\|\sigma\|_{L^r} \|\zeta\|_2^2) \leq \|\sigma\|_{L}^2 \|\xi\|_2^2 \|\zeta\|_2^2.
\]
Since
\[
\langle L(\sigma)\xi \mid \zeta \rangle = \int_K \langle \pi(\sigma, (x, h)) M_h(\xi(s_H(h))) \mid \zeta(v) \rangle \Delta(x, h)^{-\frac{1}{2}} \, dv(x, h)
\]
we have thus proved that \( L(\sigma)\xi \) is an element of \( L^2(U \ast H, \mu) \) if \( \xi \) is. We point out that this also shows that \( L \) is \( I \)-norm decreasing.

We next check that \( L \) is multiplicative, so let \( \sigma, \tau \) be two sections. Using the definition of \( L \) twice, we get
\[
(L(\sigma)L(\tau)\xi)(v) = \sum_{\varepsilon \in K^v} \pi(\sigma, (\varepsilon, \varphi)) M_{\varepsilon} [L(\tau)\xi(s_H(h))] \Delta(\varepsilon)^{-\frac{1}{2}}
\]

\[
= \sum_{(\varepsilon, \varphi) \in K^{(2)}} \pi(\sigma, (\varepsilon, \varphi)) M_{\varepsilon} \left[ \pi(\tau, (\varphi)) M_{\varphi} (\xi(s(\varphi))) \Delta(\varphi)^{-\frac{1}{2}} \right] \Delta(\varepsilon)^{-\frac{1}{2}}.
\]

If \( \varepsilon = (x, h) \) and \( \varphi = (y, k) \), then by the covariance condition and the fact that \( p(\tau, (y, k)) = y \) since \( \tau \) is a section, we have
\[
M_h \pi(\tau, (y, k)) = \pi(\beta(h, \tau, (y, k))) M_{h|y}.
\]
Using Equation \((4.6)\), we arrive at
\[
M_{\varepsilon} \pi(\tau, (\varphi)) M_{\varphi} = \pi(\beta(h, \tau, (\varphi))) M_{\varepsilon, \varphi}.
\]
Since \( s(\varphi) = s(\varepsilon \varphi) \), since \( \Delta \) is a homomorphism, and since \( \hat{\pi} \) is a \(*\)-functor, we conclude: if \( \beta_\varepsilon := \beta_h \) for \( \varepsilon = (x, h) \) \( \in K \), then
\[
(L(\sigma)L(\tau)\xi)(v) = \sum_{\varepsilon \in K^v} \sum_{\varphi \in K^v} \pi(\sigma, (\varepsilon, \varphi)) M_{\varepsilon, \varphi} \xi(s(\varphi)) \Delta(\varphi)^{-\frac{1}{2}}
\]

\[
= \sum_{\varepsilon \in K^v} \sum_{\varphi \in K^v} \pi(\sigma, (\varepsilon, \varphi)) M_{\varepsilon} \xi(s(\varphi)) \Delta(\varphi)^{-\frac{1}{2}},
\]
where we used ‘left-invariance’ of the counting measure in the last step. On the other hand,
\[
(L(\sigma \Box \tau)\xi)(v) = \sum_{\varphi \in K^v} \pi((\sigma \Box \tau), (\varphi)) M_{\varphi} (\xi(s(\varphi))) \Delta(\varphi)^{-\frac{1}{2}}.
\]
Using \((C2)\) (the definition of multiplication in \( \beta \Box H \)), we compute for \( \varphi \in K^v \)
\[
(\sigma \Box \tau)(\varphi) = \sum_{\varepsilon \in K^v} \sigma(\varepsilon) \beta_\varepsilon (\tau(\varepsilon^{-1} \varphi)).
\]
Thus, if we exchange the order of the (finite) summation, we see that indeed $L(\sigma)L(\tau) = L(\sigma \Box \tau)$.

To see that $L$ is $*$-preserving, let us write $\sigma^* := (\sigma^*)_{\mathcal{B}} = \text{pr}_{\mathcal{B}} \circ (\sigma^*)$ and consider

$$(L(\sigma^*)\zeta)(v) = \sum_{\varepsilon \in (\mathcal{G} \otimes \mathcal{H})^v} \pi(\sigma^*_{\mathcal{B}}(\varepsilon))M_{\varepsilon}(\zeta(s(\varepsilon)))\Delta(\varepsilon)^{-\frac{1}{2}}.$$

If $\varepsilon = (x, h)$, then $\varepsilon^{-1} = (h^{-1} \cdot x^{-1}, h^{-1}|_{x^{-1}})$ and thus

$$\sigma^*_{\mathcal{B}}(\varepsilon) = \text{pr}_{\mathcal{B}}(\sigma(\varepsilon^{-1})^*) = \text{pr}_{\mathcal{B}}(\sigma(\varepsilon^{-1})) = [\sigma_{\mathcal{B}}(h^{-1} \cdot x^{-1}, h^{-1}|_{x^{-1}}), h^{-1}|_{x^{-1}}]^*.$$

If we let $k = (h^{-1}|_{x^{-1}})^{-1}$, so that $h = k|_{k^{-1}, x}$, then the above together with the covariance condition gives

$$\pi(\sigma^*_{\mathcal{B}}(\varepsilon^{-1}))M_h = \pi(\beta_k(\sigma_{\mathcal{B}}(\varepsilon^{-1})^*))M_{k|_{k^{-1}, x}} = M_k\pi(\sigma_{\mathcal{B}}(\varepsilon^{-1})^*),$$

where we used that $\pi(\sigma_{\mathcal{B}}(\varepsilon^{-1})^*) = (h^{-1} \cdot x)^{-1} = k^{-1} \cdot x$ because $\sigma$ is a section. Since $M_k = M_{k^{-1}} = M_{\varepsilon^{-1}}$, since $\pi$ is a $*$-functor, and since $\Delta$ is a homomorphism, we conclude all in all

$$\langle \xi | L(\sigma^*)\zeta \rangle = \int_{\mathcal{K}} \sum_{\varepsilon \in (\mathcal{G} \otimes \mathcal{H})^v} \langle \xi(v) | M_{\varepsilon^{-1}}\pi(\sigma_{\mathcal{B}}(\varepsilon^{-1}))^*(\zeta(s(\varepsilon))) \rangle \Delta(\varepsilon)^{-\frac{1}{2}} d\mu(v)$$

$$= \int_{\mathcal{K}} \sum_{\varepsilon \in (\mathcal{G} \otimes \mathcal{H})^v} \langle \pi(\sigma_{\mathcal{B}}(\varepsilon^{-1}))M_{\varepsilon^{-1}}(\xi(s(\varepsilon))) \rangle \Delta(\varepsilon)^{-\frac{1}{2}} d\mu(v)$$

$$= \int_{\mathcal{K}} \langle \pi(\sigma_{\mathcal{B}}(\varepsilon^{-1}))M_{\varepsilon^{-1}}(\zeta(s_{\mathcal{B}}(\varepsilon^{-1}))) \rangle \Delta(\varepsilon)^{-\frac{1}{2}} d\mu(v)$$

$$\overset{(s)}{=} \int_{\mathcal{K}} \langle \pi(\sigma_{\mathcal{B}}(\varphi))M_{\varepsilon}(\zeta(s(\varphi))) \rangle \Delta(\varepsilon)^{-\frac{1}{2}} d\mu(v)$$

where $(s)$ holds by construction of $\Delta$.

Since we have already seen that $L$ is $I$-norm decreasing, this concludes our proof.

\begin{definition}
A covariant representation $(\mu, U \ast \mathcal{H}, \hat{\pi}, \hat{M})$ of $(\mathcal{B}, \beta)$ induces, by Theorem 4.7, a (not necessarily nondegenerate) $*$-representation of $C^*(\mathcal{B} \rtimes_{\beta} \mathcal{H})$. We denote it by $\hat{\pi} \rtimes \hat{M}$ and call it the integrated form of $(\mu, U \ast \mathcal{H}, \hat{\pi}, \hat{M})$.
\end{definition}

\begin{theorem}
Suppose $\mathcal{B}_u$ is unital for all $u \in \mathcal{U}$, and let $L$ be an nondegenerate $I$-norm decreasing $*$-representation of $\Gamma_c(\mathcal{G} \rtimes \mathcal{H}; \mathcal{B} \rtimes_{\beta} \mathcal{H})$. Then there exists a covariant representation $(\mu, U \ast \mathcal{H}, \hat{\pi}, \hat{M})$ of $(\mathcal{B}, \beta)$ such that $L$ is equivalent to the integrated form of $(\mu, U \ast \mathcal{H}, \hat{\pi}, \hat{M})$.
\end{theorem}
Proof. By the disintegration theorem for representations of Fell bundles [19, Theorem 4.13], there exists a strict representation \( (\mu, \mathcal{U} \ast \mathcal{H}, \hat{\psi}) \) of \( \mathcal{B} \bowtie_\beta \mathcal{H} \) such that for all \( \sigma \in \Gamma_c(\mathcal{G} \bowtie \mathcal{H}; \mathcal{B} \bowtie_\beta \mathcal{H}) \), \( \xi \in L^2(\mathcal{U} \ast \mathcal{H}, \mu) \), and \( v \in \mathcal{U} \),

\[
\tilde{L}(\sigma)\xi(v) = \sum_{(x,h) \in (\mathcal{G} \bowtie \mathcal{H})^v} \psi(\sigma(x,h))\xi(s_{\mathcal{H}}(h))\Delta(x,h)^{-\frac{1}{2}},
\]

and \( L \) is unitarily equivalent to this integrated form \( \tilde{L} \). For each \( b \in \mathcal{B} \), define \( \pi(b) : H(s_{\mathcal{H}}(b)) \to H(r_{\mathcal{H}}(b)) \) by \( \pi(b) = \psi(b, s_{\mathcal{H}}(b)) \). For each \( h \in \mathcal{H} \), define \( M_h : H(s_{\mathcal{H}}(h)) \to H(r_{\mathcal{H}}(h)) \) by \( M_h = \psi(1_{r_{\mathcal{H}}(h)}, h) \), where \( 1_{r_{\mathcal{H}}(h)} \) is the unit in the unital \( C^* \)-algebra \( \mathcal{B}_{r_{\mathcal{H}}(h)} \). We first prove that \( (\mu, \mathcal{U} \ast \mathcal{H}, \hat{\pi}, \hat{M}) \) is a covariant representation of \( (\mathcal{B}, \beta) \), so let us verify that this quadruple satisfies all the conditions in Definition [4.3].

For Conditions [R1] and [R2] since \( (\mu, \mathcal{U} \ast \mathcal{H}, \psi) \) is a strict representation of \( \mathcal{B} \bowtie_\beta \mathcal{H} \), a Fell bundle over \( \mathcal{G} \bowtie \mathcal{H} \), we automatically have that \( \mu \) is quasi-invariant and \( \mathcal{U} \ast \mathcal{H} \) is a Borel \( \ast \)-functor, showing Condition [R3]. For each \( h \in \mathcal{H}, M_h = \psi(1_{r_{\mathcal{H}}(h)}, h) \) is a map from \( H(s_{\mathcal{H}}(h)) \) to \( H(r_{\mathcal{H}}(h)) \), and \( M_h \) is clearly unitary with inverse \( M_{h^{-1}} \). For any \( h, k \in \mathcal{H}, 
\[
M_hM_k = \psi(1_{r_{\mathcal{H}}(h)}, h)\psi(1_{r_{\mathcal{H}}(k)}, k) = \psi(1_{r_{\mathcal{H}}(h)}\beta_{r_{\mathcal{H}}(h)}(1_{r_{\mathcal{H}}(k)}), hk) = \psi(1_{r_{\mathcal{H}}(h)}, hk) = \psi(1_{r_{\mathcal{H}}(hk)}, hk) = M_{hk},
\]

which shows Condition [R4]. Finally, we have to check that Equation [4.4] holds, so take any \( (h, b) \in \mathcal{H} \times_{r_{\mathcal{H}}} \mathcal{B} \) and compute

\[
M_h\pi(b) = \psi(1_{r_{\mathcal{H}}(h)}, h)\psi(b, s_{\mathcal{H}}(b)) = \psi(\beta(h, b), h|_{p(b)}) = \psi(\beta(h, b), s_{\mathcal{H}}(\beta(h, b)))\psi(1_{r_{\mathcal{H}}(h)|_{p(b)}}, h|_{p(b)}) = \pi(\beta(h, b))M_{h|_{p(b)}},
\]

Therefore, \( (\mu, \mathcal{U} \ast \mathcal{H}, \hat{\pi}, \hat{M}) \) is a covariant representation.

Now, by the definition of \( \mathcal{B} \bowtie_\beta \mathcal{H}, 
\[
\sigma(x, h) = (\sigma_{\mathcal{F}}(x, h), h) = (\sigma_{\mathcal{F}}(x, h), r_{\mathcal{H}}(h))(1_{r_{\mathcal{H}}(h)}, h).
\]

Since \( \hat{\psi} \) is a Borel \( \ast \)-functor, we thus have that

\[
\psi(\sigma(x, h)) = \psi(\sigma_{\mathcal{F}}(x, h), r_{\mathcal{H}}(h))\psi(1_{r_{\mathcal{H}}(h)}, h) = \pi(\sigma_{\mathcal{F}}(x, h))M_h.
\]

Therefore, by Equation [4.7],

\[
\tilde{L}(\sigma)\xi(v) = \sum_{(x,h) \in (\mathcal{G} \bowtie \mathcal{H})^v} \pi(\sigma_{\mathcal{F}}(x,h))M_h\xi(s_{\mathcal{H}}(h))\Delta(x,h)^{-\frac{1}{2}} = (\hat{\pi} \bowtie \hat{M})(\sigma)\xi(v).
\]

\[\square\]
This shows that the integrated form of \((\mu, U \ast \mathcal{H}, \hat{\pi}, \hat{M})\) is exactly \(\hat{L}\), which is unitarily equivalent to \(L\).

5. \(C^*\)-blend

In the case of the Zappa-Szép product of étale groupoids and their \(C^*\)-algebras, it is known that one can find \(*\)-homomorphisms \(i: C^*(G) \to C^*(G \bowtie \mathcal{H})\) and \(j: C^*(\mathcal{H}) \to C^*(\mathcal{B} \bowtie \mathcal{H})\) such that \((C^*(G), C^*(\mathcal{H}), i, j, C^*(G \bowtie \mathcal{H}))\) is a \(C^*\)-blend \([3, \text{Theorem 13}]\) in the sense of Exel \([9]\). Notice that the groupoid \(C^*\)-algebra \(C^*(G)\) is the same as the universal \(C^*\)-algebra of the groupoid Fell bundle \(C^*(G \bowtie \mathcal{H})\) for an étale groupoid \(G\) (see \([19, \text{Appendix B}]\)). We have shown in Example \(3.10\) that the Zappa-Szép product of the groupoid Fell bundle \((\mathcal{C} \times G) \bowtie \mathcal{H}\) is the same as the groupoid Fell bundle of the Zappa-Szép product \(\mathcal{C} \times (G \bowtie \mathcal{H})\). This alludes to a generalization of the result of Brownlowe et al to Zappa-Szép products of Fell bundles.

Recall the definition of \(C^*\)-blend \([9]\):

\textbf{Definition 5.1.} A \(C^*\)-blend is a quintuple \((A_1, A_2, i, j, X)\) where

\begin{enumerate}
  \item \(A_1, A_2, X\) are \(C^*\)-algebras.
  \item \(i: A_1 \to \mathcal{M}(X)\) and \(j: A_2 \to \mathcal{M}(X)\) are \(*\)-homomorphisms.
  \item Define linear maps \(i \circ j: A_1 \otimes_C A_2 \to \mathcal{M}(X)\) and \(j \circ i: A_2 \otimes_C A_1 \to \mathcal{M}(X)\) on the algebraic tensor products by
    \[
    i \circ j(a \otimes b) = i(a)j(b); j \circ i(b \otimes a) = j(b)i(a).
    \]
\end{enumerate}

Then the ranges of \(i \circ j\) and \(j \circ i\) are both dense in \(X\).

As pointed out in \([9]\), the range of \(i \circ j\) is dense if and only if the range of \(j \circ i\) is dense, because their ranges are adjoints of each other.

Assume \((G, \mathcal{H})\) is a matched pair of étale groupoids and let \(\mathcal{B} = (B, p)\) be a Fell bundle over \(G\) with a \((G, \mathcal{H})\)-compatible \(\mathcal{H}\)-action \(\beta\). By Proposition \(4.3\) one can build a \(*\)-homomorphism \(i: C^*(\mathcal{B}) \to C^*(\mathcal{B} \bowtie_\beta \mathcal{H})\). We would like to find a \(*\)-homomorphism \(j: C^*(\mathcal{H}) \to C^*(\mathcal{B} \bowtie_\beta \mathcal{H})\). In order to embed \(\mathcal{H}\) in the Zappa-Szép product bundle \(\mathcal{B} \bowtie_\beta \mathcal{H}\), we assume that \(\mathcal{B}_u\) is a unital \(C^*\)-algebra for all \(u \in \mathcal{G}^{(0)}\).

\textbf{Lemma 5.2.} Assume that \(\mathcal{B}_u\) is unital for all \(u \in \mathcal{G}^{(0)}\). We define for \(f \in C_c(\mathcal{H})\) and \((x, h) \in \mathcal{G} \bowtie \mathcal{H}\),

\[
    j(f)(x, h) = \begin{cases}
    (f(h)1_x, h) & \text{if } x = r_\mathcal{H}(h), \\
    (0_x, h) & \text{else}.
    \end{cases}
\]

Then \(j(f)\) is an element of \(\Gamma_c(\mathcal{G} \bowtie \mathcal{H}; \mathcal{B} \bowtie_\beta \mathcal{H})\).

\textbf{Proof.} Clearly, \(j(f)\) is a section. To see that it is compactly supported, note that \(\text{supp}(f)\) is contained in \((s_\mathcal{H}(\text{supp}(f))) \times \text{supp}(f)) \cap \mathcal{G} \bowtie \mathcal{H}\), which is compact since \(\text{supp}(f) \subset \mathcal{H}\) is compact and since \(s_\mathcal{H}\) is continuous. To see that \(j(f)\) is continuous, we use the same argument as in the proof of Proposition \(4.3\) so suppose the net \((x_i, h_i)\) converges to \((x, h)\) in \(\mathcal{G} \bowtie \mathcal{H}\). If \(x = r_\mathcal{H}(h)\), then openness of \(\mathcal{G}^{(0)}\) implies that \(x_i \in \mathcal{G}^{(0)}\) for large \(i\), in which
case Condition \([\text{USC4}]\) for the upper semi-continuous bundle \(B \triangleright_{\beta} \mathcal{H}\) implies that
\[
j(f)(x_i, h_i) = (f(h_i)1_{x_i}, h_i) \text{ converges to } (f(h)1_x, h) = j(f)(x, h).
\]
If \(x \neq r_{\mathcal{H}}(h)\), then closedness of \(\mathcal{G}^{(0)}\) implies that \(x_i \notin \mathcal{G}^{(0)}\) for large \(i\), in which case Condition \([\text{USC4}]\) again implies that
\[
j(f)(x_i, h_i) = (0, h_i) \text{ converges to } (0, h) = j(f)(x, h).
\]
\(\square\)

**Proposition 5.3.** Assume that \(B_u\) is unital for all \(u \in \mathcal{G}^{(0)}\). The map \(f \mapsto j(f)\) defined in Lemma 5.2 is a \(*\)-algebra homomorphism and extends to \(j : C^*(\mathcal{H}) \rightarrow C^*(B \triangleright_{\beta} \mathcal{H})\).

**Proof.** Clearly, \(j\) is linear. To see that it is multiplicative, we point out that for \((x, h) \in \mathcal{K} := \mathcal{G} \bowtie \mathcal{H}\) with \(v = r_\mathcal{G}(x), u = s_\mathcal{G}(x)\), we have
\[
j(f_1) * j(f_2)(x, h) = \sum_{(y, k) \in (B \bowtie \mathcal{H})^v} j(f_1)(y, k) j(f_2)((y, k)^{-1}(x, h)).
\]
Only if \(y = v\), the factor \(j(f_1)(y, k)\) does not vanish, in which case \((y, k)^{-1}(x, h) = (x, k^{-1}h)\). The factor \(j(f_2)((y, k)^{-1}(x, h))\) vanishes unless we also have \(x = u\), so that
\[
j(f_1) * j(f_2)(x, h) = \begin{cases} \sum_{k \in \mathcal{H}^v} (f_1(k)1_u, k) (f_2(k^{-1}h)1_u, k^{-1}h), & \text{if } x = u \\ (0, h), & \text{else.} \end{cases}
\]
By \([\text{C2}]\) we have
\[
(f_1(k)1_u, k) (f_2(k^{-1}h)1_u, k^{-1}h) = (f_1(k)1_u \beta_k(f_2(k^{-1}h)1_u), k|_{\pi(u)}(k^{-1}h))
\]
\[
= (f_1(k)f_2(k^{-1}h)1_u, h),
\]
so that
\[
j(f_1) * j(f_2)(x, h) = \begin{cases} ((f_1 * f_2)(h)1_u, h), & \text{if } x = u \\ (0, h), & \text{else.} \end{cases} = j(f_1 * f_2)(x, h).
\]
To see that \(j\) is \(*\)-preserving, recall that
\[
j(f)^*((x, h)) \overset{\text{[4.1]}}{=} [j(f)((x, h)^{-1})]^* \overset{\text{[2.2]}}{=} [j(f)(h^{-1} \cdot x^{-1}, h^{-1}|_{x^{-1}})]^*.
\]
Note that \(x \notin \mathcal{G}^{(0)}\) if and only if \(h^{-1} \cdot x^{-1} \notin \mathcal{G}^{(0)}\) (cf. to our argument after Equation \([4.2]\)), in which case we have \(j(f)^*(x, h) = (0, h) = j(f^*)(x, h)\), simply because \(j(f^*)\) and \(j(f)^*\) are both sections. On the other hand, if \(x = r_{\mathcal{H}}(h) = u \in \mathcal{G}^{(0)}\), then
\[
j(f)^*(x, h) = [j(f)(h^{-1} \cdot u, h^{-1}|_u)]^* \overset{\text{[ZS10]}}{=} [j(f)(r_{\mathcal{H}}(h^{-1}), h^{-1})]^*
\]
\[
= (f(h^{-1})1_{r_{\mathcal{H}}(h^{-1}), h^{-1}}) \overset{\text{[C3]}}{=} (f(h^{-1})1_{h \cdot r_{\mathcal{H}}(h^{-1}), h})
\]
\[
\overset{\text{[ZS10]}}{=} (f(h^{-1})1_{r_{\mathcal{H}}(h), h}) = j(f^*)(x, h).
\]
It remains to see that \(j\) extends. Since
\[
\|(f(h)1_v, h)\| = |f(h)| \cdot \|1_v\| = |f(h)|,
\]
one can argue, mutatis mutandis, as in Proposition 4.3 that \( \| j(f) \|_I \leq \| f \|_I \)
and thus \( \| j(f) \|_{C^*(\mathcal{E})} \leq \| f \|_{C^*(\mathcal{H})} \), which proves that \( j \) extends. □

**Theorem 5.4.** The quintuple \( (C^*(\mathcal{B}), C^*(\mathcal{H}), i, j, C^*(\mathcal{B} \Join_\beta \mathcal{H})) \) is a \( C^* \)-blend, where \( i \) is given by Proposition 4.3, and \( j \) is given by Proposition 5.3.

By Propositions 4.3 and 5.3, \( i \) and \( j \) are \( * \)-homomorphisms to \( C^*(\mathcal{B} \Join_\beta \mathcal{H}) \), and we want to show that they satisfy (3) of Definition 5.1. For the proof, we will need the following helpful lemma.

**Lemma 5.5.** Suppose \( \mathcal{E} = (C, q) \) is an upper semi-continuous Banach bundle over some locally compact Hausdorff space \( X \), and let \( \Gamma \subset \Gamma^0(X; \mathcal{E}) \) and \( \mathfrak{X} \subset C_0(X) \) be subspaces such that

1. \( \mathfrak{X} \) is uniformly dense,
2. if \( \mathfrak{x} \in \mathfrak{X} \) and \( s \in \Gamma \), then their (pointwise) product \( \mathfrak{x}s \) is in \( \Gamma \), and
3. for each \( x \in X \), the set \( \Gamma(x) := \{ s(x) \mid s \in \Gamma \} \) is dense in \( \mathcal{E}_x \).

Then \( \Gamma \) is uniformly dense in \( \Gamma^0(X; \mathcal{E}) \).

The authors would like to thank Dana Williams for pointing them to [23, Prop. C.24].

**Proof.** Let \( t \in \Gamma^0(X; \mathcal{E}) \) and \( \epsilon > 0 \) be arbitrary. Because of Assumption (3) and upper semi-continuity of \( \mathcal{E} \), we can use the proof of [23, Prop. C.24] to find \( s_1, \ldots, s_n \in \Gamma \) and \( \rho_1, \ldots, \rho_n \in C_0(X) \) such that

\[
\left\| t - \sum_{i=1}^{n} \rho_i s_i \right\|_\infty < \frac{\epsilon}{2}.
\]

By Assumption (1) we can find \( \psi_1, \ldots, \psi_n \in \mathfrak{X} \) such that

\[
\| \rho_i - \psi_i \|_\infty < \frac{\epsilon}{2n \| s_i \|_\infty + 1}
\]

for each \( 1 \leq i \leq n \). This yields

\[
\left\| t - \sum_{i=1}^{n} \psi_i s_i \right\|_\infty \leq \left\| t - \sum_{i=1}^{n} \rho_i s_i \right\|_\infty + \sum_{i=1}^{n} \| \rho_i - \psi_i \|_\infty \| s_i \|_\infty
\]

\[
< \frac{\epsilon}{2} + \sum_{i=1}^{n} \frac{\epsilon}{2n \| s_i \|_\infty + 1} \| s_i \|_\infty < \epsilon.
\]

By Assumption (2) and since \( \Gamma \) is a subspace, we have \( \sum_i \psi_i s_i \in \Gamma \), so that we have approximated the arbitrary element \( t \) by an element of \( \Gamma \). □

**Lemma 5.6.** In the setting of Theorem 5.4, the range of \( i \circ j \) is contained in \( \Gamma_c(\mathcal{G} \Join \mathcal{H}; \mathcal{B} \Join_\beta \mathcal{H}) \) and it is uniformly dense in \( \Gamma^0(\mathcal{G} \Join \mathcal{H}; \mathcal{B} \Join_\beta \mathcal{H}) \).

**Proof.** Let \( \mathcal{E} = (C, q) \) be the Fell bundle \( \mathcal{B} \Join_\beta \mathcal{H} \), and \( \mathcal{K} := \mathcal{G} \Join \mathcal{H} \). Let us first show that the range of \( i \circ j \) is contained in \( \Gamma_c(\mathcal{K}; \mathcal{E}) \), so take
σ ∈ Γ_c(G; ℌ) and f ∈ C_c(H). If ε ∈ K, then by definition of the product in Γ_c(K; ℌ) (see Equation (4.11)), we have

\[ i \circ j(\sigma \otimes_C f)(\varepsilon) = \sum_{\varphi; r_K(\varphi) = r_K(\varepsilon)} i(\sigma)(\varphi) \cdot j(f)(\varphi^{-1} \varepsilon), \]

where • refers to multiplication in ℌ. Only if \( \varphi \) is of the form \((y, s_G(y))\) for some \( y \in G \) does \( i(\sigma)(\varphi) \) not necessarily vanish, in which case \( \varphi^{-1} = (y^{-1}, r_G(y)) \) and \( \varphi^{-1} \varepsilon = (y^{-1}x, h) \) where \( \varepsilon = (x, h) \). Similarly, \( j(f)(y^{-1}x, h) \) vanishes if \( y^{-1}x \notin G^{(0)} \), i.e. \( y = x \). We conclude

\[ i \circ j(\sigma \otimes_C f)(x, h) = i(\sigma)(x, s_G(x)) \cdot j(f)(r_H(h), h). \]

Let \( v = s_G(x) = r_H(h) \). Using the definition of \( i \) resp. \( j \), we see that

\[ i(\sigma)(x, v) = (\sigma(x), v) \quad \text{and} \quad j(f)(v, h) = (f(h)1_v, h). \]

By definition of • (see [C2]) and since \( \beta_v = \text{id} \) since \( v \in H^{(0)} \), we have

\[ (5.1) \quad i \circ j(\sigma \otimes_C f)(x, h) = (\sigma(x), v) \cdot (f(h)1_v, h) = (\sigma(x)f(h), h) \]

where we used that \( f(h) \) is just a scalar. Thus, \( i \circ j(\sigma \otimes_C f) \) is clearly continuous and, by construction, a section. The above shows furthermore that

\[ \text{supp}(i \circ j(\sigma \otimes_C f)) = (\text{supp}(\sigma) \times \text{supp}(f)) \cap K, \]

which is compact since \( K \) is closed in \( G \times H \). This implies that the range of \( i \circ j \) is a subspace of \( \Gamma_c(K; \mathcal{G}) \).

To see that the range is dense, we will employ Lemma 5.5. Let \( \Gamma \) be the range of \( i \circ j \) and let \( X \) be the linear span of those functions \( F \) in \( C_0(K) \) such that \( F(x, h) = f_1(x)f_2(h) \) for some \( f_1 \in C_0(G) \) and \( f_2 \in C_0(H) \); we will write \( f_1 \Join f_2 := F \) for the duration of this proof. Clearly, \( X \) is a *-subalgebra of \( C_0(K) \) that separates points, and for each \( (x, h) \in K \), we may find \( f_1, f_2 \) with \( f_1(x) \neq 0 \neq f_2(h) \), i.e. \( f_1 \Join f_2(x, h) \neq 0 \). Thus, by the Stone–Weierstrass Theorem, \( X \) is dense in \( C_0(K) \), i.e. Assumption 1 of Lemma 5.5 holds.

Furthermore, we can rewrite the pointwise product of any \( f_1 \Join f_2 \) with elements in the range of \( i \circ j \) as follows:

\[ f_1 \Join f_2[i \circ j(\sigma \otimes_C f)] = i \circ j((f_1\sigma) \otimes_C (f_2f)), \]

since \( \Gamma_c(G; \mathcal{B}) \) is a \( C_0(G) \)-module and \( C_c(H) \) is a \( C_0(H) \)-module. This shows that the element \( f_1 \Join f_2[i \circ j(\sigma \otimes_C f)] \) is in the range of \( i \circ j \), i.e. \( \Gamma \) satisfies Assumption 2 of Lemma 5.5.

Next fix \((x, h) \in K\), any element \( b \) in \( \mathcal{B}_x \cong \mathcal{C}(x, h) \), and \( \varepsilon > 0 \). Since \( \mathcal{B} \) has enough continuous cross-sections (see [12, Theorem 12]), i.e. \( \Gamma_c(G; \mathcal{B})(x) = \mathcal{B}_x \), we can find \( \sigma \in \Gamma_c(G; \mathcal{B}) \) with \( \|b - \sigma(x)\| < \varepsilon \). If \( f \in C_c(H) \) with \( f(h) = 1 \), then by Equation (5.1),

\[ \|b, h) - i \circ j(\sigma \otimes_C f)(x, h)\| = \|b - \sigma(x), h\| < \varepsilon, \]

which proves that \( \Gamma(x, h) \) is dense in \( \mathcal{C}(x, h) \), i.e. Assumption 3 of Lemma 5.5 is also satisfied. It follows that the range \( \Gamma \) of \( i \circ j \) is uniformly dense in \( \Gamma_0(K; \mathcal{G}) \).
Proof of Theorem 5.4. We only have to show that \( i \circ j \) has dense range, so fix an arbitrary \( t \in \Gamma_c(K; C) \), where we again write \( C \) for the bundle \( B \rtimes_\alpha H \) and \( K \) for \( G \rtimes_\beta H \). By a standard ‘partition of unity’ argument, using the compact support of \( t \), we can without loss of generality assume that \( \text{supp}^t(t) \) is contained in a basic open set \( W \).

Note that the topology of \( K \) has a basis of open sets consisting of \( U \rtimes V := (U \times V) \cap K \), where \( U \subset G \), \( V \subset H \) are basic open sets. In particular, we can assume that \( W = U \rtimes V \), where \( U \) and \( V \) are small bisections, i.e. bisections which are precompact and whose closure is contained in another open bisection, say \( \overline{U} \subset U' \) and \( \overline{V} \subset V' \) (See [7, Lemma 5.1]).

Fix \( \epsilon > 0 \). By Lemma 5.6 we know that there exists \( s \) in the range of \( i \circ j \) such that \( \|t - s\|_\infty < \epsilon \). Let \( f_1 \in C_c(G) \) be a \([0, 1]\)-valued function such that \( f_1|_U \equiv 1 \) and \( f_1 \) vanishes off of \( U' \). Similarly, let \( f_2 \in C_c(H) \) be such that \( f_2|_V \equiv 1 \), and \( f_2 \) vanishes off of \( V' \). We have seen in the proof of Lemma 5.6 that \( s' : (x, h) \mapsto f_1(x)f_2(h)s(x, h) \) is also in the range of \( i \circ j \).

Since \( t \) vanishes off of \( U \rtimes V \) and since \( f_1, f_2 \) are \([0, 1]\)-valued, we see that

\[
\|t - s'\|_\infty \leq \max \left( \sup_{\varepsilon \in U \rtimes V} \|(t - s')(\varepsilon)\|, \sup_{\varepsilon \in K \setminus (U \rtimes V)} \|s(\varepsilon)\| \right).
\]

Since \( f_1|_U \equiv 1 \) and \( f_2|_V \equiv 1 \), the right-hand side is exactly \( \|t - s\|_\infty \), which is smaller than \( \epsilon \) by choice of \( s \). Note that \( t - s' \) is supported in the open bisection \( U' \rtimes V' \). We may thus use the computation in [3, Lemma 4.4] to conclude that \( \|t - s'\|_J = \|t - s'\|_\infty < \epsilon \). Since the \( J \)-norm dominates the full \( C^* \)-norm, this proves that \( s' \), an element in the range of \( i \circ j \), approximates \( t \) in \( C^*(C) \).

\[\Box\]

6. Embedding of \( C^*(\mathcal{B}) \) into \( C^*(\mathcal{B} \rtimes_\alpha \Lambda) \)

It is a well known fact that for a \( C^* \)-dynamical system \((A, \Lambda, \alpha)\) where \( \Lambda \) is a discrete group, there exists an injective \(*\)-homomorphism \( i : A \to A \rtimes_\alpha \Lambda \) that embeds \( A \) inside the crossed product \( C^* \)-algebra \( A \rtimes_\alpha \Lambda \), since \( A \) embeds injectively inside the dense \(*\)-subalgebra \( \Gamma_c(\Lambda, A) \) via \( a \mapsto au_\alpha \). We note that the crossed product can be viewed as the \( C^* \)-algebra of a Zappa-Szép product of Fell bundle, as illustrated in the following example.

Example 6.1. Let \( A \) be a \( C^* \)-algebra, \( \Lambda \) a discrete group, and \((A, \Lambda, \alpha)\) a \( C^* \)-dynamical system in the classical sense. If we think of \( A \) as a Fell bundle \( \mathcal{A} \) over the trivial group \( \{e\} \), then \( A = C^*(\mathcal{A}) \). Since the \( \Lambda \)-action \( \alpha \) satisfies all conditions in Definition 3.1, we can form the Zappa-Szép product \( \mathcal{A} \rtimes_\alpha \Lambda \), a Fell bundle over \( \{e\} \rtimes \Lambda \cong \Lambda \). One can verify that its \( C^* \)-algebra is canonically isomorphic to the crossed product \( C^* \)-algebra \( A \rtimes_\alpha \Lambda \). In particular, when \( A \) is unital, this implies that the inclusion map \( i : A = C^*(\mathcal{A}) \to C^*(\mathcal{A} \rtimes_\alpha \Lambda) \) defined in Proposition 4.3 is injective.

The above example motivates the question whether the inclusion map \( i : C^*(\mathcal{B}) \to C^*(\mathcal{B} \rtimes_\alpha \Lambda) \) from Proposition 4.3 is always injective when \( \Lambda \) is discrete. Throughout this section, we assume that \( \Gamma \) and \( \Lambda \) are discrete.
groups, so that \( \mathcal{U} = \Gamma^{(0)} = \Lambda^{(0)} \) is merely \( \{ e \} \). We further assume that the \( C^* \)-algebra \( \mathscr{B}_e \) is unital, with its unit denoted by 1.

In this setting, a covariant representation \((\mu, \mathcal{U} \ast \mathcal{L}, \pi, \hat{M})\) of \((\mathcal{B}, \beta)\) collapses to a \( * \)-representation \( \pi: \mathcal{B} \to \mathbb{B}(H) \) and a unitary representation \( M: \Lambda \to \mathbb{U}(H) \) on some Hilbert space \( H \) which are covariant in the sense of Equation (4.4).

**Proposition 6.2.** Assume \((\Gamma, \Lambda)\) is a matched pair of discrete groups and let \( \mathcal{B} = (B, p) \) be a Fell bundle over \( \Gamma \) with a \((\Gamma, \Lambda)\)-compatible \( \Lambda \)-action \( \beta \). If \( \pi \) is a strict representation of \( \mathcal{B} \) on \( H \), let \( \tilde{H} := \ell^2(\Gamma \bowtie \Lambda) \otimes H \) and for \( b \in \mathcal{B} \), \((x, h) \in \Gamma \bowtie \Lambda \), and \( \xi \in H \), define

\[
\Pi_0(b)(\delta_{(x,h)} \otimes \xi) = \delta_{(p(b)x,h)} \otimes \pi(\beta(h^{-1}|_{(p(b)x)^{-1}}, b))(\xi).
\]

Then \( \Pi_0(b) \) extends to a bounded linear map \( \Pi(b) \in \mathbb{B}(\tilde{H}) \), and \( \Pi: \mathcal{B} \to \mathbb{B}(\tilde{H}) \) is a \( * \)-representation.

**Proof.** To see that \( \Pi_0(b) \) extends, we compute for \( f \in c_{00}(\Gamma \bowtie \Lambda, H) \subset \tilde{H} \):

\[
\| \Pi_0(b)(f) \|_2^2 = \sum_{(x,h) \in \Gamma \bowtie \Lambda} |\Pi_0(b)(f)(x,h)| \leq \sum_{(x,h) \in \Gamma \bowtie \Lambda} \| \pi(\beta(h^{-1}|_{x^{-1}}, b)) [f(p(b)^{-1}x, h)] \| \leq \sum_{(x,h) \in \Gamma \bowtie \Lambda} \| \pi(\beta(h^{-1}|_{x^{-1}}, b)) \| \| f(p(b)^{-1}x, h) \|.
\]

Since \( \pi \) is a representation of \( \mathcal{B} \) on \( H \), it is norm decreasing. Furthermore, each \( \beta_b \) is isometric by Corollary 5.5 so we have

\[
\| \pi(\beta(h^{-1}|_{x^{-1}}, b)) \| \leq \| \beta(h^{-1}|_{x^{-1}}, b) \| = \| b \|.
\]

Thus,

\[
\| \Pi_0(b)(f) \|_2^2 \leq \| b \|^2 \sum_{(x,h) \in \Gamma \bowtie \Lambda} | f(p(b)^{-1}x, h) |^2 \leq \| b \|^2 \sum_{(y,h) \in \Gamma \bowtie \Lambda} | f(y, h) |^2 = \| b \|^2 \| f \|_2^2,
\]

where \((\ast)\) follows from the left-invariance of counting measure on the discrete group \( \Gamma \bowtie \Lambda \). This shows that \( \Pi_0(b) \) extends to a map \( \Pi(b) \in \mathbb{B}(\tilde{H}) \).

Let us now prove that \( \Pi \) is a \( * \)-representation. To see that \( \Pi \) is fibrewise linear, let \( b, c \in \mathcal{B} \) be in the same fibre, so that \( y := p(b) = p(c) = p(\lambda b + c) \) for any \( \lambda \in \mathbb{C} \). Then for \( f \in c_{00} \),

\[
\Pi_0(\lambda b + c)(f)(x, h) = \pi(\beta(h^{-1}|_{x^{-1}}, \lambda b + c)) [f(y^{-1}x, h)].
\]

Since \( \beta \) was assumed to be fibrewise linear, and since \( \pi \) is a \( * \)-homomorphism, we have

\[
\pi(\beta(h^{-1}|_{x^{-1}}, \lambda b + c)) = \lambda \pi(\beta(h^{-1}|_{x^{-1}}, b)) + \pi(\beta(h^{-1}|_{x^{-1}}, c)),
\]

\[
\Pi_0(\lambda b + c)(f)(x, h) = \lambda \Pi_0(b)(f)(x, h) + \Pi_0(c)(f)(x, h).
\]

Therefore, \( \Pi \) is fibrewise linear. Moreover, \( \Pi \) is a \( * \)-representation, as desired.
which proves $\Pi(\lambda b + c) = \lambda \Pi(b) + \Pi(c)$. Similarly, to see that $\Pi$ is fibre-wise multiplicative, let $(b, c) \in \mathcal{B}^{(2)}$. Then $p(b \cdot c) = p(b)p(c)$ by \textbf{[F1]} and \( \beta(h^{-1}|_{x-1}, b \cdot c) = \beta(h^{-1}|_{x-1}, b)\beta(h^{-1}|_{x-1}p(b), c) \) by \textbf{[A4]} Thus
\[
\pi(\beta(h^{-1}|_{x-1}, b \cdot c)) = \pi(\beta(h^{-1}|_{x-1}, b)\beta(h^{-1}|_{x-1}p(b), c)) = \pi(\beta(h^{-1}|_{x-1}, b))\pi(\beta(h^{-1}|_{x-1}p(b), c)),
\]
so that
\[
\Pi_0(b \cdot c)(f)(x, h) = \pi(\beta(h^{-1}|_{x-1}, b \cdot c)) [f(p(b \cdot c)^{-1}x, h)] = \pi(\beta(h^{-1}|_{x-1}, b))\pi(\beta(h^{-1}|_{x-1}p(b), c)) [f(p(c)^{-1}p(b)^{-1}x, h)] = \pi(\beta(h^{-1}|_{x-1}, b)) [\Pi_0(c)(f)(p(b)^{-1}x, h)] = \Pi_0(b)(\Pi_0(c)(f))(x, h),
\]
proving that $\Pi(b \cdot c) = \Pi(b)\Pi(c)$. Lastly, to see that $\Pi$ is fibre-*-preserving, note that $p(b^*) = p(b)^{-1}$ by \textbf{[F5]} and also $\beta(h^{-1}|_{y^{-1}}, b^*) = \beta(h^{-1}|_{y^{-1}p(b)^{-1}}, b)^*$ by \textbf{[A5]} Thus
\[
\pi(\beta(h^{-1}|_{x-1}, b^*)) = \pi(\beta(h^{-1}|_{(p(b)x)^{-1}}, b))^*,
\]
so with $g \in c_{00}(\Gamma \bowtie \Lambda, H)$, and $(y, h) \in \Gamma \bowtie \Lambda$, we have
\[
\Pi_0(b^*)(g)(y, h) = \pi(\beta(h^{-1}|_{y^{-1}}, b^*)) [g(p(b^*)^{-1}y, h)] = \pi(\beta(h^{-1}|_{(p(b)y)^{-1}}, b))^* [g(p(b)y, h)].
\]
Therefore, if $f \in c_{00}(\Gamma \bowtie \Lambda, H)$, then
\[
\left\langle \Pi_0(b^*)(g) \right| f \right\rangle^H = \sum_{(y, h) \in \Gamma \bowtie \Lambda} \left\langle \Pi_0(b^*)(g)(y, h) \right| f(y, h) \right\rangle^H = \sum_{(y, h) \in \Gamma \bowtie \Lambda} \left\langle \pi(\beta(h^{-1}|_{(p(b)y)^{-1}}, b))^* [g(p(b)y, h)] \right| f(y, h) \right\rangle^H = \sum_{(y, h) \in \Gamma \bowtie \Lambda} \left\langle g(p(b)y, h) \right| \pi(\beta(h^{-1}|_{(p(b)y)^{-1}}, b))[f(y, h)] \right\rangle^H \overset{(*)}{=} \sum_{(x, h) \in \Gamma \bowtie \Lambda} \left\langle g(x, h) \right| \pi(\beta(h^{-1}|_{x^{-1}}, b))[f(p(b)^{-1}x, h)] \right\rangle^H = \left\langle g \right| \Pi_0(b)f \right\rangle^\Lambda = \left\langle \Pi_0(b)^*(g) \right| f \right\rangle^\Lambda,
\]
where $(*)$ follows, again, from the left-invariance of counting measure. This proves that $\Pi(b^*) = \Pi(b)^*$, and all in all that $\Pi$ is a *-homomorphism. \(\square\)

**Definition 6.3.** In the setting of Proposition \textbf{6.2} the properties we have proved yield that $\Pi$ is another strict representation of $\mathcal{B}$. We will call it the **twisted amplification** of $\pi$.

For a fixed $h \in \Lambda$, if $\xi \in H$ and $(x, k) \in \Gamma \bowtie \Lambda$, we define
\[
M_h(\delta_{(x,k)} \otimes \xi) = \delta_{(c,h)(x,k)} \otimes \xi.
\]

**Lemma 6.4.** The map $M_h$ extends to a unitary map on $\tilde{H} = l^2(\Gamma \bowtie \Lambda) \otimes H$. Moreover, $M : \Lambda \to \U(\tilde{H})$ is a unitary representation of $\Lambda$. 


\textbf{Proof.} To see that $M_h$ extends, we compute for $f \in c_{00}(\Gamma \rtimes \Lambda, H) \subset \tilde{H}$
\[
\|M_h(f)\|_2^2 = \sum_{(x,k) \in \Gamma \rtimes \Lambda} \|f((e, h)^{-1}(x, k))\|^2 = \sum_{(y,l) \in \Gamma \rtimes \Lambda} \|f(y, l)\|^2 = \|f\|^2,
\]
where $(\ast)$ holds because the counting measure on $\Gamma \rtimes \Lambda$ is left-invariant. Therefore, $M_h$ extends to an isometric linear map on $\ell^2(\Gamma \rtimes \Lambda) \otimes H$. For any $g, h \in \Lambda$ and $f$ as above, one can compute
\[
M_gM_h(f)(x, k) = M_h(f)((e, g)^{-1}(x, k))
= f((e, h)^{-1}(e, g)^{-1}(x, k))
= f((e, gh)^{-1}(x, k))
= M_{gh}(f)(x, k).
\]
Therefore, $M_gM_h = M_{gh}$ and $M : \Lambda \to \mathbb{U}(\ell^2(\Gamma \rtimes \Lambda) \otimes H)$ is a homomorphism. We have that $M_{h^{-1}}M_h = M_hM_{h^{-1}} = M_e = I$, and thus each $M_h$ is unitary and $M$ is a unitary representation. \hfill \square

\textbf{Lemma 6.5.} Let $\pi$ and its twisted amplification $\Pi$ be as in Proposition 6.2 and $M$ be as in Lemma 6.4. Then,
\[
M_h\Pi(b) = \Pi(\beta(h, b))M_{h|_\beta(b)}
\]
for all $h \in \Lambda$ and $b \in \mathcal{B}$. In particular, $(\Pi, M)$ is a covariant representation of $(\mathcal{B}, \beta)$ in the sense of Definition 4.3.

\textbf{Proof.} It only remains to check the covariance condition in Equation 4.4 so let $p(b) = y \in \Gamma$ and take any $f \in c_{00}(\Gamma \rtimes \Lambda) \otimes H$. By definition,
\[
(M_h\Pi(b)f)(x, k) = (\Pi(b)f)(h^{-1} \cdot x, h^{-1}|_x k)
= \pi(\beta(\cdot k^{-1}|_{x^{-1}}(h^{-1} \cdot x)^{-1}, b))f(y^{-1}h^{-1} \cdot x, h^{-1}|_x k),
\]
and
\[
(\Pi(\beta(h, b))M_{h|_\beta}f)(x, k) = \pi(\beta(k^{-1}|_{x^{-1}}\cdot \beta(h, b))(M_{h|_\beta}f)(p(\beta(h, b))^{-1}x, k)
= \pi(\beta(k^{-1}|_{x^{-1}}\cdot \beta(h, b)))f((e, h|_y)^{-1}p(\beta(h, b))^{-1}x, k).
\]
By (A2),
\[
\beta(k^{-1}|_{x^{-1}}\cdot \beta(h, b)) = \beta(k^{-1}|_{x^{-1}}h, b)).
\]
On the other hand,
\[
(h^{-1}|_x k)^{-1}|_{(h^{-1} \cdot x)^{-1}} = \sum_{(h^{-1} \cdot x)^{-1}} (h^{-1}|_x k)^{-1}|_{(h^{-1} \cdot x)^{-1}}
= k^{-1}|_{(h^{-1} \cdot x)^{-1}}h^{-1}|_{x^{-1}h}.
\]
Therefore,
\[
\pi(\beta((h^{-1}|_x k)^{-1}|_{(h^{-1} \cdot x)^{-1}}, b)) = \pi(\beta(k^{-1}|_{x^{-1}}\cdot \beta(h, b)))).
\]
Now by (A1), \( p(\beta(h, b)) = h \cdot p(b) = h \cdot y \). We can compute that

\[
(e, h|_y)^{-1} (p(\beta(h, b))^{-1} x, k) = ((h|_y)^{-1} \cdot ((h \cdot y)^{-1} x), (h|_y)^{-1}(h \cdot y)^{-1} x, k)
\]

\[
= ((h|_y)^{-1} \cdot ((h \cdot y)^{-1} x), (h^{-1}|_{h \cdot y}) (h \cdot y)^{-1} x, k)
\]

\[
= (y^{-1}(h|_y)^{-1} |_{h|_y^{-1}} \cdot x, h^{-1} x, k)
\]

\[
= (y^{-1}(h^{-1}|_{h|_y}) (h \cdot y)^{-1} \cdot x, h^{-1} x, k)
\]

\[
= (y^{-1}(h^{-1} \cdot x, h^{-1} x, k).
\]

Therefore, we have

\[
(M_h \Pi(b)f)(x, k) = (\Pi(\beta(h, b)) M_{h|_y} f)(x, k),
\]

which implies that

\[
M_h \Pi(b) = \Pi(\beta(h, b)) M_{h|_y}
\]

for all \( h \in \Lambda \) and \( b \in \mathcal{B} \). \(\square\)

**Theorem 6.6.** The canonical map \( i: C^*(\mathcal{B}) \to C^*(\mathcal{B} \rtimes_\beta \Lambda) \) is an injective *-homomorphism.

**Proof.** Denote \( \Omega = \Gamma \rtimes \Lambda \) and \( \mathcal{C} = \mathcal{B} \rtimes_\beta \Lambda \). Let \( \pi^u: \mathcal{B} \to C^*(\mathcal{B}) \) be the universal *-representation of \( \mathcal{B} \), where \( C^*(\mathcal{B}) \) is understood as a concrete \( C^* \)-algebra inside some \( \mathcal{B}(H) \). Let \( \Pi^u: \mathcal{B} \to \mathcal{B}(\ell^2(\Gamma \rtimes \Lambda) \otimes H) \) be its twisted amplification.

By Lemma 6.5, we can construct a unitary representation \( M \) of \( \Lambda \) such that \( (\Pi^u, M) \) is a covariant representation of \((\mathcal{B}, \beta)\). Let \( L_{\Pi^u, M} \) be its integrated form, as constructed in Theorem 4.7.

Let \( i: C^*(\mathcal{B}) \to C^*(\mathcal{B} \rtimes_\beta \Lambda) \) be the map from Proposition 4.3. Assume that \( i(\sigma) = 0 \) for some \( \sigma \in \Gamma_c(\Gamma, \mathcal{B}) \); we have to show that \( \sigma = 0 \). So let \( \tau = \sigma \square \sigma^* \) and note that \( i(\tau) = i(\sigma)i(\sigma^*) = 0 \), since \( i \) is a *-homomorphism. As

\[
\|i(\tau)\| = \sup\{\|L(i(\tau))\| : L \text{ norm decreasing } *-\text{rep. of } \Gamma_c(\Omega; \mathcal{C})\},
\]

it follows that \( L_{\Pi^u, M}(i(\tau)) = 0 \), since \( L_{\Pi^u, M} \) is an I-norm decreasing *-representation of \( \Gamma_c(\Omega; \mathcal{C}) \).

For any \( \xi \in H \), let \( f_\xi := \delta_{(e,e)} \otimes \xi \in \ell^2(\Omega, H) \). For \( \eta \in H \), we have

\[
\langle L_{\Pi^u, M}(i(\tau)) f_\xi | f_\eta \rangle = \left\langle \sum_{(x,h) \in \Gamma \rtimes \Lambda} \Pi^u(i(\tau)) \beta(x, h)) M_h f_\xi | f_\eta \right\rangle.
\]
Since \( i(\tau)(x, h) = 0 \) if \( h \neq e \) and \( i(\tau)(x, h) = (\tau(x), h) \) if \( h = e \), we have that

\[
\langle L_{\Pi^u,M}(i(\tau))f_\xi \mid f_\eta \rangle = \left\langle \sum_{x \in \Gamma} \Pi^u(\tau(x))f_\xi \mid f_\eta \right\rangle 
= \sum_{x \in \Gamma} \sum_{(y,h) \in \Gamma \rtimes \Delta} \langle (\Pi^u(\tau(x))f_\xi)(y, h) \mid f_\eta(y, h) \rangle 
\overset{(*)}{=} \sum_{x \in \Gamma} \langle (\Pi^u(\tau(x))f_\xi)(e, e) \mid \eta \rangle 
= \sum_{x \in \Gamma} \langle \pi^u(\tau(x))f_\xi(x^{-1}, e) \mid \eta \rangle.
\]

Here, \((*)\) follows since \( f_\eta(y, h) = 0 \) for \((y, h) = (e, e)\) and \( f_\eta(e, e) = \eta \). Similarly, \( f_\xi(x^{-1}, e) = 0 \) for \( x \neq e \) and \( f_\xi(e, e) = \xi \), so we have that

\[
\langle \pi^u(\tau(e))\xi \mid \eta \rangle = \langle L_{\Pi^u,M}(i(\tau))f_\xi \mid f_\eta \rangle = 0.
\]

Since \( \pi^u \) is the universal representation, \( \pi^u \) is faithful on \( \mathcal{B} \) and thus we have \( \tau(e) = 0 \). One can compute that \( \tau(e) = \sum_{x \in \Gamma} \sigma(x)\sigma(x)^* \). This implies that

\[
\tau(e) = \sum_{x \in \Gamma} \sigma(x)\sigma(x)^* = 0,
\]

and thus \( \sigma(x) = 0 \) for all \( x \in \Gamma \) and \( \sigma = 0 \). Therefore, \( i \) is injective as desired. \( \square \)

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