Applications of a new P-Q modular equation of degree two

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Abstract. At scattered places in his first notebook, Ramanujan recorded the values for 107 class invariants or irreducible monic polynomials satisfied by them. On pages 294-299 in his second notebook, he gave a table of values for 77 class invariants $G_n$ and $g_n$ in his second notebook. Traditionally, $G_n$ is determined for odd values of $n$ and $g_n$ for even values of $n$. On pages 338 and 339 in his first notebook, Ramanujan defined the remarkable product of theta-functions $a_{m,n}$. Also, he recorded eighteen explicit values depending on two parameters, namely, $m$, and $n$, where these are odd integers. In this paper, we initiate to study explicit evaluations of $G_n$ for even values of $n$. We establish a new general formula for the explicit evaluations of $G_n$ involving class invariant $g_n$. For this purpose, we derive a new P-Q modular equation of degree two. Further application of this modular equation, we establish a new formula to explicit evaluation of $a_{m,2}$. Also, we compute several explicit values of class invariant $g_n$ and singular moduli $\alpha_n$.

1. Introduction

The following definitions of theta functions \[ \varphi, \psi, \text{ and } f \text{ with } |q| < 1 \] are classical:

\[
\varphi(q) = f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_{\infty}^2 (q^2; q^2)_{\infty},
\]

\[
\psi(q) = f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = (q^2; q^2)_{\infty} (q^2; q^2)_{\infty}^{-1},
\]

\[
f(-q) = f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/4} = (q; q)_{\infty},
\]

where, $(a; q)_{\infty} = \prod_{n=0}^{\infty} (1 - aq^n)$.

For $q = e^{-\pi \sqrt{n}}$, Weber-Ramanujan class invariants \[ \text{[2] p.183, (1.3)} \] are defined by

\[
G_n = 2^{-1/4} q^{-1/24} \chi(q) \quad ; \quad g_n = 2^{-1/4} q^{-1/24} \chi(-q),
\]

where, $n$ is a positive rational number and $\chi(q) = (-q; q^2)_{\infty}$. Ramanujan \[ \text{[2] Entry 2.1, p.187} \] recorded simple formula relating these class invariants as follows:

\[
g_{4n} = 2^{1/4} g_n G_n.
\]

Ramanujan evaluated a total of 116 class invariants \[ \text{[2] p.189-204} \]. Traditionally, $G_n$ is determined for odd values of $n$ and $g_n$ for even values of $n$. These have been proved by

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various authors using techniques such as modular equations, Kronecker limit formula, and empirical process (established by Watson) [2, Chapter 34].

The ordinary or Gaussian hypergeometric function is defined by

$$2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n, \quad |z| < 1$$

where, $a, b, c$ are complex numbers such that $c \neq 0, -1, -2, \ldots$, and $(a)_0 = 1, (a)_n = a(a + 1)(a + 2)\ldots(a + n - 1)$ for any positive integer $n$.

Now, we shall recall the definition of modular equation from [1]. The complete elliptic integral of the first kind $K(k)$ of modulus $k$ is defined by

$$K(k) = \int_{0}^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})^n}{(n!)^2} k^{2n} = \frac{\pi}{2} \varphi^2 \left( e^{-\pi \sqrt{k^2}} \right), \quad (0 < k < 1) \quad (1.3)$$

and let $K' = K(k')$, where $k' = \sqrt{1 - k^2}$ is represented as the complementary modulus of $k$. Let $K, K', L, L'$ denote the complete elliptic integrals of the first kind associated with the moduli $k, k', l, l'$ respectively. In case, the equality

$$n \frac{K'}{K} = \frac{L'}{L} \quad (1.4)$$

holds for a positive integer $n$, then a modular equation of degree $n$ is the relation between the moduli $k, l$, which is implied by equation (1.4). Ramanujan defined his modular equation involving $\alpha, \beta$, where, $\alpha = k^2$, and $\beta = l^2$. Then we say $\beta$ is of degree $n$ over $\alpha$.

Ramanujan recorded 23 $P-Q$ modular equations in terms of their theta function in his notebooks [6]. All those proved by Berndt et al. by employing the theory of theta functions and modular forms.

If, as usually quoted in the theory of elliptic functions, $k = k(q)$ denotes the modulus, then, the singular moduli $k_n$ is defined by $k_n = (e^{-\pi \sqrt{\alpha}})$, where $n$ is a positive integer. In terms of Ramanujan, set $\alpha = k^2$ and $\alpha_n = k_n^2$, he hypothesized the values of over 30 singular moduli in his notebooks. On page 82 of his first notebook, Ramanujan stated three additional theorems for calculating $\alpha_n$ for even values of $n$. Particularly, he offered formulae for $\alpha_{4p}, \alpha_{8p}$, and $\alpha_{16p}$. Moreover, he recorded several values of $\alpha_n$ for odd values of $n$ in his first and second notebook. All these results have proved by Berndt et al. by employing Ramanujan’s class invariants $G_n$ and $g_n$. Also we observed that representation for $\alpha_n$ in terms of theta function. This is given by [1] Entry 12 (i),(iv) Ch.17, p.124

$$\alpha_n = \left( \frac{f(q)}{2^{1/2} q^{1/8} f(-q^4)} \right)^{-8}, \quad \text{where}, \quad q = e^{-\pi \sqrt{\alpha}}. \quad (1.5)$$

On page 338 in his first notebook [6], Ramanujan defined

$$a_{m,n} = \frac{nq^{(n-1)/4} \psi^2 (q^n) \varphi^2 (-q^{2n})}{\psi^2 (q) \varphi^2 (-q^2)} \quad (1.6)$$
where, \( q = e^{-\pi \sqrt{m/n}} \) and \( m, n \) are positive rationals then, on page 338 and 339, he offered a list of 18 particular values. All those 18 values have proved by Berndt, Chan, and Zhang [3]. Recently, Prabhakaran, and Ranjith Kumar [5] have established a new general formula for the explicit evaluations of \( a_{3m,3} \) and \( a_{m,9} \) by using \( P - Q \) mixed modular equations, and the values for certain class invariant of Ramanujan. Also they have calculated some new explicit values of \( a_{3m,3} \) for \( m = 2, 7, 13, 17, 25, 37 \), and \( a_{m,9} \) for \( m = 17, 37 \).

Naika, and Dharmendra [4] have given alternative form of (1.6) as follows:

\[
a_{m,n} = \frac{nq^{(n-1)/4} \psi^2(-q^n) \varphi^2(q^n)}{\psi^2(-q) \varphi^2(q)}.
\]  

(1.7)

They have proved some general theorems to calculate explicit values of \( a_{m,n} \).

The organisation of the present study is as follows. In Section 2, we collect some identities which are useful in proofs of our main results. In Section 3, we derive a new \( P - Q \) modular equation of degree two. Applying this modular equation, we establish new general formulae for the explicit evaluations of class invariant \( G_n \) for even values of \( n \), and the Ramanujan’s remarkable product of theta functions \( a_{m,2} \) along with class invariant \( g_n \). By using these formulae, we compute several explicit values of class invariant \( G_n \), and \( a_{m,2} \). Also, we evaluate several explicit values of class invariant \( g_n \), and singular moduli \( \alpha_n \). These are presented in Section 4.

2. Preliminaries

We list a few identities which are useful in establishing our main results.

**Lemma 2.1.** [1] Entry 24(iii) p. 39] We have

\[
f(q)f(-q^2) = \psi(-q)\varphi(q).
\]

(2.1)

**Lemma 2.2.** [1] Entry 12(i),(iii) Ch.17, p.124] We have

\[
f(q) = \sqrt{z2^{-1/6}} \frac{(\alpha(1 - \alpha)/q)^{1/24}}{\varphi(1 - \alpha + \beta)};
\]

\[
f(-q^2) = \sqrt{z2^{-1/3}} \frac{(\alpha(1 - \alpha)/q)^{1/12}}{\varphi(1 - \alpha + \beta)}.
\]

(2.2)

(2.3)

**Lemma 2.3.** [1] Entry 24(ii), p. 214] If \( \beta \) is of degree 2 over \( \alpha \), then,

\[
m\sqrt{1 - \alpha} + \sqrt{\beta} = 1,
\]

\[
m^2\sqrt{1 - \alpha} + \beta = 1.
\]

(2.4)

(2.5)

where, \( m = z_1/z_2 = \varphi^2(q)/\varphi^2(q^2) \).

**Lemma 2.4.** [7] Theorem 3.5.1] If \( P = \frac{f(-q)}{q^{1/8}f(-q^4)} \) and \( Q = \frac{f(-q^2)}{q^{1/4}f(-q^8)} \), then,

\[
(PQ)^4 + \left(\frac{4}{PQ}\right)^4 = \left(\frac{Q}{P}\right)^{12} - 16 \left(\frac{P}{Q}\right)^4 - 16 \left(\frac{Q}{P}\right)^4.
\]

(2.6)
3. GENERAL FORMULAE FOR THE EXPLICIT EVALUATIONS OF \( G_{2n}, G_{n/2}, \) AND \( a_{m,2} \)

In this section, we derive a new P-Q modular equation of degree two. As application of this modular equation, we establish some general formulae for the explicit evaluations of \( G_{2n}, G_{n/2}, \) and \( a_{m,2} \) in term of the class invariant \( g_n. \)

**Theorem 3.1.** If \( P = \frac{f(q)}{q^{1/24} f(q^2)} \) and \( Q = \frac{f(-q^2)}{q^{1/12} f(-q^4)}, \) then,

\[
Q^{16} - P^4 Q^{14} + 8 P^4 Q^2 - 4 P^8 = 0.
\]

**Proof.** Transcribing \( P, \) and \( Q \) using (2.2), and (2.3), then simplifying, we arrive at

\[
P = \sqrt{z_1} \left( \frac{\alpha(1-\alpha)}{\beta(1-\beta)} \right)^{1/24}; \quad Q = \sqrt{z_2} \left( \frac{\alpha(1-\alpha)}{\beta(1-\beta)} \right)^{1/12},
\]

where, \( \beta \) is of degree 2 over \( \alpha. \) It follow that

\[
\frac{Q}{P} = \left( \frac{\alpha(1-\alpha)}{\beta(1-\beta)} \right)^{1/24}; \quad m = \frac{P^4}{Q^2}.
\]

Now isolating \( \alpha, \) and \( \beta \) from (2.4), and (2.5), we deduce that

\[
\alpha = \frac{4(m-1)}{m^2}; \quad \beta = (m-1)^2.
\]

By (3.4), we observe that

\[
\alpha(1-\alpha) = \frac{4(m-1)(m-2)^2}{m^4}; \quad \beta(1-\beta) = -m(m-1)^2(m-2).
\]

Employing (3.5) in first term of (3.3), and simplifying, we arrive at

\[
(m-1)(m-2) \left( m^6 Q^{24} - m^5 Q^{24} + 4 m P^{24} - 8 P^{24} \right) = 0.
\]

We observe that the last factor of the above equation vanish for \( q \to 0, \) whereas, the other factors does not vanish for that specific value. Thus, we obtain that

\[
m^6 Q^{24} - m^5 Q^{24} + 4 m P^{24} - 8 P^{24} = 0.
\]

Now applying the value of \( m \) in the above equation, we complete the proof. \( \Box \)

**Theorem 3.2.** If \( n \) is any positive rational, and

\[
\Lambda = \frac{g_{2n}^{12} + g_{2n}^{-12}}{2},
\]

then,

\[
\frac{G_{2n}}{G_{n/2}} = \frac{1}{g_{2n}} \left( \sqrt{\Lambda} + \sqrt{\Lambda - 1} \right)^{1/4},
\]

\[
\left( \sqrt{2} G_{2n} G_{n/2} \right)^{12} - 16 \left( \left( \sqrt{2} G_{2n} G_{n/2} \right)^4 + \left( \sqrt{2} G_{2n} G_{n/2} \right)^{-4} \right)
= 16 g_{2n}^{12} \left( \sqrt{\Lambda} + \sqrt{\Lambda - 1} \right) + 16 g_{2n}^{-12} \left( \sqrt{\Lambda} - \sqrt{\Lambda - 1} \right).
\]
Proof. Solving (3.1) for \( P/Q \), and choosing the appropriate root, we obtain that
\[
\frac{P}{Q} = \frac{Q}{2^{3/4}} \left( \sqrt{Q^{12} + \frac{64}{Q^{12}}} - \sqrt{Q^{12} + \frac{64}{Q^{12}} - 16} \right)^{1/4}.
\]
(3.8)

We observed that some representations for \( G_n \) and \( g_n \) in terms of \( f(q) \) and \( f(-q) \) by Entry 24(iii) [1, p.39] as follow:
\[
G_n = \frac{f(q)}{2^{1/4}q^{1/24}f(-q^2)}; \quad g_n = \frac{f(-q)}{2^{1/4}q^{1/24}f(-q^2)}.
\]
(3.9)

Employing (3.9) in (3.8) along with \( q = e^{-\pi \sqrt{m/2}} \), we arrive at (3.6). Now, setting \( q = e^{-\pi \sqrt{m/2}} \) in Lemma 2.4 and employing the definition of \( g_n \), we obtain that
\[
PQ = 2g_{2n}g_{n/2}g_{8n}; \quad \frac{P}{Q} = \frac{g_{n/2}}{g_{8n}}.
\]
(3.10)

Now applying (1.2) in (3.10), we deduce that
\[
PQ = 2g_{4n}^2G_{2n}G_{n/2}; \quad \frac{P}{Q} = \frac{1}{\sqrt{2G_{2n}G_{n/2}}}.
\]
(3.11)

By (3.6), we observe that
\[
g_{2n}^4G_{2n}/G_{n/2}^3 = g_{2n}^3 \left( \frac{g_{2n}^{12} + g_{2n}^{-12}}{2} + \sqrt{\frac{g_{2n}^{12} + g_{2n}^{-12}}{2} - 1} \right)^{1/4}.
\]
(3.12)

Now applying (3.11) and (3.12) in (2.6), we obtain (3.7). \(\square\)

Theorem 3.3. If \( m \) is any positive rational, then
\[
a_{m,2} = \frac{1}{g_{4m}^6} \left( \sqrt{\frac{g_{2m}^{12} + g_{2m}^{-12}}{2}} + \sqrt{\frac{g_{2m}^{12} + g_{2m}^{-12}}{2} - 1} \right)^{1/2}.
\]

Proof. Solving (3.1) for \( P^4Q^4 \), and choosing the appropriate root, we arrive at
\[
P^4Q^4 = \frac{Q^{12}}{8} \left( \sqrt{Q^{12} + \frac{64}{Q^{12}}} - \sqrt{Q^{12} + \frac{64}{Q^{12}} - 16} \right).
\]
(3.13)

Let \( q = e^{-\pi \sqrt{m/2}} \), then the identity (1.7), becomes
\[
a_{m,2} = \frac{2q^{1/4}g^2(-q^2)\varphi^2(q^2)}{q^2(-q)\varphi^2(q)}.
\]
(3.14)

Now applying (2.1) in (3.14), we conclude that
\[
a_{m,2} = \frac{2q^{1/4}f^2(q^2)f(-q^4)}{f^2(q)f^4(-q^2)}.
\]
(3.15)

Employing the second term of (3.9) in (3.13) along with \( q = e^{-\pi \sqrt{m/2}} \), then it follow that reporting in (3.15), we arrive at desired result. \(\square\)
4. Explicit evaluations

In section, we compute several explicit evaluations of class invariant $G_n$ for even values of $n$, and $a_{m,2}$ by using Theorem 3.2 and Theorem 3.3 respectively. After obtaining class invariant $G_n$, then we evaluate several explicit evaluations of class invariant $g_n$, and singular moduli $\alpha_n$.

**Theorem 4.1.** We have

$$G_{46} = \frac{1}{2^{1/8}} \left(78\sqrt{2} + 23\sqrt{23}\right)^{1/16} \left(5 + \frac{\sqrt{23}}{\sqrt{2}}\right)^{1/8} \left(\sqrt{\frac{3\sqrt{2} + 8}{4}} + \sqrt{\frac{3\sqrt{2} + 4}{4}}\right)^{1/2}$$

$$\times \left(\frac{6\sqrt{2} + 11}{4} - \frac{6\sqrt{2} + 7}{4}\right)^{1/4}.$$

**Proof.** From the table in Chapter 34 of Ramanujan’s notebooks [2, p.201], we have

$$g_{46} = \sqrt{\frac{3 + \sqrt{2} + \sqrt{7} + 6\sqrt{2}}{2}}.$$

It follows that

$$g_{46}^{12} + g_{46}^{-12} = 2646 + 1872\sqrt{2}. \quad (4.1)$$

Employing (4.1) in (3.6) with $n = 23$, we conclude that

$$\frac{G_{46}}{G_{23/2}} = \left(2645 + 1872\sqrt{2} + \sqrt{14004792 + 9902880\sqrt{2}}\right)^{1/8}$$

$$\times \left(\sqrt{\frac{6\sqrt{2} + 11}{4}} - \sqrt{\frac{6\sqrt{2} + 7}{4}}\right)^{1/2}. \quad (4.2)$$

By (9.5) [1, p.284], observe that

$$\sqrt{14004792 + 9902880\sqrt{2}} = 552\sqrt{23} + 390\sqrt{46}. \quad (4.3)$$

Reporting (4.3) in (4.2), and further simplification, we obtain that

$$\frac{G_{46}}{G_{23/2}} = \left(78\sqrt{2} + 23\sqrt{23}\right)^{1/8} \left(5 + \frac{\sqrt{23}}{\sqrt{2}}\right)^{1/4} \left(\sqrt{\frac{6\sqrt{2} + 11}{4}} - \sqrt{\frac{6\sqrt{2} + 7}{4}}\right)^{1/2}. \quad (4.4)$$

Applying (4.1) in (3.7), and after a straightforward, lengthy calculation, we deduce that

$$h^{32} - 32h^{24} - \left(4356352 + 3080448\sqrt{2}\right)h^{20} + 224h^{16} + \left(69701632 + 49287168\sqrt{2}\right)h^{12}$$

$$- \left(3587934720 + 2537054208\sqrt{2}\right)h^{8} + \left(69701632 + 49287168\sqrt{2}\right)h^{4} + 256 = 0,$$
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where, \( h = \sqrt{2}G_{46}G_{23/2} \). Now isolating the terms involving \( \sqrt{2} \) on one side of the above equation, squaring both sides, and simplifying, we deduce that

\[
(h^{16} - 208h^{12} + 456h^8 - 832h^4 + 16) (h^{48} + 208h^{44} + 42744h^{40} + 84032h^{36} - 1838096h^{32} + 33126912h^{28} + 104902912h^{24} - 1089853440h^{20} - 761938176h^{16} + 10947629056h^{12} - 7091652608h^8 + 2230665216h^4 + 4096) = 0.
\]

A numerical calculation show that \( h \) is not a root of the second factor. Since the first factor has positive roots, and it follows that

\[
h^{16} - 208h^{12} + 456h^8 - 832h^4 + 16 = 0,
\]

or equivalently,

\[
\left( h^4 + \frac{4}{h^4} \right)^2 - 208 \left( h^4 + \frac{4}{h^4} \right) + 448 = 0.
\]

Solving the above quadric equation, and choosing the convenient root, we deduce that

\[
h^4 + \frac{4}{h^4} = 104 + 72\sqrt{2}.
\]

(4.5)

Again solving (4.5) for \( h \), and \( h > 1 \), we obtain that

\[
G_{46}G_{23/2} = \frac{1}{2^{1/4}} \left( 26 + 18\sqrt{2} + \sqrt{1323 + 936\sqrt{2}} \right)^{1/4}.
\]

(4.6)

Now we apply Lemma 9.10 [2] p.292 with \( r = 26 + 18\sqrt{2} \). Then \( t = \left( 3\sqrt{2} + 6 \right) / 4 \), and so

\[
26 + 18\sqrt{2} + \sqrt{1323 + 936\sqrt{2}} = \left( \sqrt{3\sqrt{2} + 8} / 4 + \sqrt{3\sqrt{2} + 4} / 4 \right)^4.
\]

(4.7)

Now combining (4.4), (4.6), and (4.7), we arrive at the desired result. \( \square \)

**Theorem 4.2.** We have

\[
G_{14} = \frac{1}{2^{1/8}} \left( 2\sqrt{2} + \sqrt{7} \right)^{1/16} \left( 3 + \sqrt{7} \right)^{1/8} \left( \sqrt{2\sqrt{2} + 3} / 2 + \sqrt{2\sqrt{2} + 1} / 2 \right)^{1/4} \times \left( \sqrt{2\sqrt{2} + 3} / 4 - \sqrt{2\sqrt{2} - 1} / 4 \right)^{1/4},
\]

\[
G_{22} = \frac{1}{2^{1/8}} \left( \sqrt{2} - 1 \right)^{1/4} \left( 3\sqrt{11} + 7\sqrt{2} \right)^{1/8} \left( \sqrt{11} + 3 \right)^{1/4} / \sqrt{2}.
\]
\[ G_{34} = \frac{1}{2^{1/8}} \left( \sqrt{2} + 1 \right)^{1/4} \left( 3\sqrt{2} + \sqrt{17} \right)^{1/8} \left( \frac{\sqrt{17} + 5}{4} + \frac{\sqrt{17} + 1}{4} \right) \]
\[ \times \left( \frac{3\sqrt{17} + 13}{8} - \frac{3\sqrt{17} + 5}{8} \right)^{1/4} \]

\[ G_{58} = \frac{1}{2^{1/8}} \left( \sqrt{2} + 1 \right)^{3/4} \left( 13\sqrt{58} + 99 \right)^{1/8} \left( \frac{\sqrt{29} - 5}{2} \right)^{1/4} \]

\[ G_{70} = \frac{1}{2^{1/8}} \left( \sqrt{2} - 1 \right)^{1/4} \left( 2\sqrt{2} + \sqrt{7} \right)^{1/4} \left( 3\sqrt{14} + 5\sqrt{5} \right)^{1/8} \left( \frac{\sqrt{7} + \sqrt{5}}{\sqrt{2}} \right)^{1/4} \]
\[ \times \left( \frac{3 + \sqrt{7}}{\sqrt{2}} \right)^{1/4} \left( \frac{\sqrt{5} - 1}{2} \right)^{1/2} \]

\[ G_{82} = \frac{1}{2^{1/8}} \left( \sqrt{2} + 1 \right)^{1/2} \left( \sqrt{82} + 9 \right)^{1/8} \left( \frac{\sqrt{41} + 7}{2} + \frac{\sqrt{41} + 5}{2} \right)^{1/2} \]
\[ \times \left( \frac{\sqrt{41} + 13}{8} - \frac{\sqrt{41} + 5}{8} \right)^{1/2} \]

\[ G_{130} = \frac{1}{2^{1/8}} \left( \sqrt{2} + 1 \right)^{1/2} \left( \sqrt{10} + 3 \right)^{1/4} \left( \sqrt{26} + 5 \right)^{1/4} \left( 5\sqrt{130} + 57 \right)^{1/8} \]
\[ \times \left( \frac{\sqrt{5} - 1}{2} \right)^{3/4} \left( \frac{\sqrt{13} - 3}{2} \right)^{1/4} \]

\[ G_{142} = \frac{1}{2^{1/8}} \left( 287\sqrt{71} + 1710\sqrt{2} \right)^{1/16} \left( \frac{59 + 7\sqrt{71}}{\sqrt{2}} \right)^{1/8} \left( \sqrt{27\sqrt{2} + 40} + \sqrt{27\sqrt{2} + 36} \right)^{1/2} \]
\[ \times \left( \sqrt{\frac{90\sqrt{2} + 131}{4}} - \sqrt{\frac{90\sqrt{2} + 127}{4}} \right)^{1/4} \]

\[ G_{190} = \frac{1}{2^{1/8}} \left( \sqrt{10} - 3 \right)^{3/4} \left( 2\sqrt{5} + \sqrt{19} \right)^{1/4} \left( \sqrt{19} + 3\sqrt{2} \right)^{1/4} \left( 37\sqrt{19} + 51\sqrt{10} \right)^{1/8} \]
\[ \times \left( \frac{3\sqrt{19} + 13}{\sqrt{2}} \right)^{1/4} \left( \frac{\sqrt{5} - 1}{2} \right)^{3/4} \]

**Proof.** Employing class invariant \( g_n \) for \( n = 14, 22, 34, 58, 70, 82, 130, 142, \) and 190 [2, p. 200-203] in Theorem 3.2, we obtain all the above values. Since the proof is analogous to the previous theorem, so we omit the details. \( \square \)
Theorem 4.3. We have

\[ g_{56} = 2^{1/8} \left( 2\sqrt{2} + \sqrt{7} \right)^{1/16} \left( \frac{3 + \sqrt{7}}{\sqrt{2}} \right)^{1/8} \left( \sqrt{\frac{2\sqrt{2} + 3}{2}} + \sqrt{\frac{2\sqrt{2} + 1}{2}} \right)^{1/4} \times \left( \sqrt{\frac{2\sqrt{2} + 3}{4}} + \sqrt{\frac{2\sqrt{2} - 1}{4}} \right)^{1/4} \]

\[ g_{88} = 2^{1/8} \left( \sqrt{2} + 1 \right)^{1/4} \left( 3\sqrt{11} + 7\sqrt{2} \right)^{1/8} \left( \frac{\sqrt{11} + 3}{\sqrt{2}} \right)^{1/4} \]

\[ g_{136} = 2^{1/8} \left( \sqrt{2} + 1 \right)^{1/4} \left( 3\sqrt{2} + \sqrt{17} \right)^{1/8} \left( \sqrt{\frac{17}{4}} + \sqrt{\frac{17 + 1}{4}} \right)^{1/2} \times \left( \sqrt{\frac{3\sqrt{17} + 13}{8}} + \sqrt{\frac{3\sqrt{17} + 5}{8}} \right)^{1/4} , \]

\[ g_{184} = 2^{1/8} \left( 78\sqrt{2} + 23\sqrt{23} \right)^{1/16} \left( \frac{5 + \sqrt{23}}{\sqrt{2}} \right)^{1/8} \left( \sqrt{\frac{3\sqrt{2} + 8}{4}} + \sqrt{\frac{3\sqrt{2} + 4}{4}} \right)^{1/2} \times \left( \sqrt{\frac{6\sqrt{2} + 11}{4}} + \sqrt{\frac{6\sqrt{2} + 7}{4}} \right)^{1/4} , \]

\[ g_{232} = 2^{1/8} \left( \sqrt{2} + 1 \right)^{3/4} \left( 13\sqrt{58} + 99 \right)^{1/8} \left( \frac{\sqrt{29} + 5}{2} \right)^{1/4} \]

\[ g_{280} = 2^{1/8} \left( \sqrt{2} + 1 \right)^{1/4} \left( 2\sqrt{2} + \sqrt{7} \right)^{1/4} \left( 3\sqrt{14} + 5\sqrt{5} \right)^{1/8} \left( \frac{\sqrt{7} + \sqrt{5}}{\sqrt{2}} \right)^{1/4} \times \left( \frac{3 + \sqrt{7}}{\sqrt{2}} \right)^{1/4} \left( \frac{\sqrt{5} + 1}{2} \right)^{1/2} , \]

\[ g_{328} = 2^{1/8} \left( \sqrt{2} + 1 \right)^{1/2} \left( \sqrt{82} + 9 \right)^{1/8} \left( \sqrt{\frac{41}{2}} + \sqrt{\frac{41 + 7}{2}} \right)^{1/2} \times \left( \sqrt{\frac{41 + 13}{8}} + \sqrt{\frac{41 + 5}{8}} \right)^{1/2} , \]

\[ g_{520} = 2^{1/8} \left( \sqrt{2} + 1 \right)^{1/2} \left( \sqrt{10} + 3 \right)^{1/4} \left( \sqrt{26} + 5 \right)^{1/4} \left( 5\sqrt{130} + 57 \right)^{1/8} \times \left( \frac{\sqrt{5} + 1}{2} \right)^{3/4} \left( \frac{\sqrt{13} + 3}{2} \right)^{1/4} . \]
\[ g_{568} = 2^{1/8} \left( 287\sqrt{71} + 1710\sqrt{2} \right)^{1/16} \left( \frac{59 + 7\sqrt{71}}{\sqrt{2}} \right)^{1/8} \left( \sqrt{\frac{27\sqrt{2} + 40}{4}} + \sqrt{\frac{27\sqrt{2} + 36}{4}} \right)^{1/2} \]
\times \left( \sqrt{\frac{90\sqrt{2} + 131}{4}} + \sqrt{\frac{90\sqrt{2} + 127}{4}} \right)^{1/4},
\]
\[ g_{760} = 2^{1/8} \left( \sqrt{10} + 3 \right)^{1/4} \left( 2\sqrt{5} + \sqrt{19} \right)^{1/4} \left( \sqrt{19} + 3\sqrt{2} \right)^{1/4} \left( 37\sqrt{19} + 51\sqrt{10} \right)^{1/8} \]
\times \left( \sqrt{\frac{3\sqrt{19} + 13}{\sqrt{2}}} \right)^{1/4} \left( \frac{\sqrt{5} + 1}{2} \right)^{3/4}.
\]

**Proof.** Employing pervious theorems in (1.2), we obtain all the above values. □

**Theorem 4.4.** We have

\[ \alpha_{14} = \left( 2\sqrt{2} - \sqrt{7} \right)^2 \left( \frac{3 - \sqrt{7}}{\sqrt{2}} \right)^2 \left( \sqrt{\frac{2\sqrt{2} + 3}{2}} - \sqrt{\frac{2\sqrt{2} + 1}{2}} \right)^4, \]
\[ \alpha_{22} = \left( 3\sqrt{11} - 7\sqrt{2} \right)^2 \left( \frac{\sqrt{11} - 3}{\sqrt{2}} \right)^4, \]
\[ \alpha_{34} = \left( \sqrt{2} - 1 \right)^4 \left( 3\sqrt{2} - \sqrt{17} \right)^2 \left( \sqrt{\frac{\sqrt{17} + 5}{4}} - \sqrt{\frac{\sqrt{17} + 1}{4}} \right)^8, \]
\[ \alpha_{46} = \left( 78\sqrt{2} - 23\sqrt{23} \right)^2 \left( \frac{5 - \sqrt{23}}{\sqrt{2}} \right)^2 \left( \sqrt{\frac{3\sqrt{2} + 8}{4}} - \sqrt{\frac{3\sqrt{2} + 4}{4}} \right)^8, \]
\[ \alpha_{58} = \left( \sqrt{2} - 1 \right)^{12} \left( 13\sqrt{58} - 99 \right)^2, \]
\[ \alpha_{70} = \left( 2\sqrt{2} - \sqrt{7} \right)^4 \left( 3\sqrt{14} - 5\sqrt{5} \right)^2 \left( \sqrt{\frac{\sqrt{7} - \sqrt{5}}{\sqrt{2}}} \right)^4 \left( \frac{3 - \sqrt{7}}{\sqrt{2}} \right)^4, \]
\[ \alpha_{82} = \left( \sqrt{2} - 1 \right)^8 \left( \sqrt{82} - 9 \right)^2 \left( \sqrt{\frac{\sqrt{41} + 7}{2}} - \sqrt{\frac{\sqrt{41} + 5}{2}} \right)^8, \]
\[ \alpha_{130} = \left( \sqrt{2} - 1 \right)^8 \left( \sqrt{10} - 3 \right)^4 \left( \sqrt{26} - 5 \right)^4 \left( 5\sqrt{130} - 57 \right)^2, \]
\[ \alpha_{142} = \left( 287\sqrt{71} - 1710\sqrt{2} \right)^2 \left( \frac{59 - 7\sqrt{71}}{\sqrt{2}} \right)^2 \left( \sqrt{\frac{27\sqrt{2} + 40}{4}} - \sqrt{\frac{27\sqrt{2} + 36}{4}} \right)^8, \]
\[ \alpha_{190} = \left( 2\sqrt{5} - \sqrt{19} \right)^4 \left( \sqrt{19} - 3\sqrt{2} \right)^4 \left( 37\sqrt{19} - 51\sqrt{10} \right)^2 \left( \frac{3\sqrt{19} - 13}{\sqrt{2}} \right)^4. \]
Proof. Employing (3.9) in (1.5), we obtain $\alpha_n = (G_n g_{4n})^{-8}$. Applying theorems 4.1-4.3 in the identity, we obtain all the above values. □

**Theorem 4.5.** We have

\[
a_{2,2} = \left(\frac{1}{2^{7/8}} (\sqrt{2} + 1)^{1/2}\right),
\]

\[
a_{7,2} = \left(2\sqrt{2} + \sqrt{7}\right)^{1/4} \left(\frac{3 + \sqrt{7}}{\sqrt{2}}\right)^{1/2} \left(\sqrt{\frac{2\sqrt{2} + 3}{4}} - \sqrt{\frac{2\sqrt{2} - 1}{4}}\right)^3,
\]

\[
a_{11,2} = \left(\sqrt{2} - 1\right)^3 \left(3\sqrt{11} + 7\sqrt{2}\right)^{1/2},
\]

\[
a_{15,2} = \left(\sqrt{10} - 3\right) \left(\sqrt{6} + \sqrt{3}\right)^{1/2} \left(\frac{\sqrt{5} + \sqrt{3}}{\sqrt{2}}\right) \left(\frac{\sqrt{5} - 1}{2}\right)^3,
\]

\[
a_{17,2} = \left(\sqrt{\frac{17 + 5}{4}} + \sqrt{\frac{17 + 1}{4}}\right)^2 \left(\sqrt{\frac{3\sqrt{17} + 13}{8}} - \sqrt{\frac{3\sqrt{17} + 5}{8}}\right)^3,
\]

\[
a_{21,2} = \left(\sqrt{2} + 1\right) \left(2\sqrt{2} - \sqrt{7}\right) \left(\frac{\sqrt{3} + 1}{\sqrt{2}}\right)^2 \left(\frac{\sqrt{7} - \sqrt{3}}{2}\right)^3,
\]

\[
a_{23,2} = \left(78\sqrt{2} + 23\sqrt{23}\right)^{1/4} \left(\frac{5 + \sqrt{23}}{\sqrt{2}}\right)^{1/2} \left(\sqrt{\frac{6\sqrt{2} + 11}{4}} - \sqrt{\frac{6\sqrt{2} + 7}{4}}\right)^3,
\]

\[
a_{29,2} = \left(\sqrt{2} + 1\right)^3 \left(\frac{\sqrt{29} - 5}{2}\right)^3,
\]

\[
a_{35,2} = \left(\sqrt{2} - 1\right)^3 \left(3\sqrt{14} + 5\sqrt{5}\right)^{1/2} \left(\frac{3 + \sqrt{7}}{\sqrt{2}}\right) \left(\frac{\sqrt{5} - 1}{2}\right)^6,
\]

\[
a_{39,2} = \left(\sqrt{26} - 5\right) \left(\sqrt{13} + 2\sqrt{3}\right) \left(3\sqrt{3} + \sqrt{26}\right)^{1/2} \left(\frac{\sqrt{13} - 3}{2}\right)^3,
\]

\[
a_{41,2} = \left(\sqrt{\frac{41 + 7}{2}} + \sqrt{\frac{41 + 5}{2}}\right)^2 \left(\sqrt{\frac{41 + 13}{8}} - \sqrt{\frac{41 + 5}{8}}\right)^6,
\]

\[
a_{51,2} = \left(\sqrt{2} - 1\right)^3 \left(\sqrt{3} + \sqrt{2}\right)^2 \left(3\sqrt{2} - \sqrt{17}\right)^2 \left(\sqrt{51} + 5\sqrt{2}\right)^{1/2},
\]

\[
a_{65,2} = \left(\sqrt{10} + 3\right) \left(\sqrt{26} + 5\right) \left(\frac{\sqrt{13} - 3}{2}\right)^3 \left(\frac{\sqrt{5} - 1}{2}\right)^9,
\]

\[
a_{71,2} = \left(287\sqrt{71} + 1710\sqrt{2}\right)^{1/4} \left(\frac{59 + 7\sqrt{71}}{\sqrt{2}}\right)^{1/2} \left(\sqrt{\frac{90\sqrt{2} + 131}{2}} - \sqrt{\frac{90\sqrt{2} + 127}{2}}\right)^3,
\]
\[ a_{95,2} = \left( \sqrt{10} - 3 \right)^3 \left( 2\sqrt{5} + \sqrt{19} \right) \left( 37\sqrt{19} + 51\sqrt{10} \right)^{1/2} \left( \frac{\sqrt{5} - 1}{2} \right)^9. \]

**Proof.** We set \( m = 2, 7, 11, 15, 17, 21, 23, 29, 35, 39, 41, 51, 65, 71, \) and 95 in Theorem 3.3 and use the corresponding values of \( g_{2m} \) from [7, Theorem 4.1.2 (i)], and [2, p. 200-203] to complete the proof. □

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