CLASSICAL AFFINE W-SUPERALGEBRAS VIA GENERALIZED DRINFELD-SOKOLOV REDUCTIONS AND RELATED INTEGRABLE SYSTEMS

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Abstract. The purpose of this article is to investigate relations between W-superalgebras and integrable super-Hamiltonian systems. To this end, we introduce the generalized Drinfel’d-Sokolov (D-S) reduction associated to a Lie superalgebra $g$ and its even nilpotent element $f$, and we find a new definition of the classical affine W-superalgebra $\mathcal{W}(g, f, k)$ via the D-S reduction. This new construction allows us to find free generators of $\mathcal{W}(g, f, k)$, as a differential superalgebra, and two independent Lie brackets on $\mathcal{W}(g, f, k)/\partial \mathcal{W}(g, f, k)$. Moreover, we describe super-Hamiltonian systems with the Poisson vertex algebras theory. A W-superalgebra with certain properties can be understood as an underlying differential superalgebra of a series of integrable super-Hamiltonian systems.

1. Introduction

Classical affine W-algebras have been studied in the theory of integrable systems since 1980’s, when Drinfeld-Sokolov ([12]) discovered relations between a finite dimensional simple Lie algebra $g$ and a sequence of integrable systems.

The main idea of Drinfel’d and Sokolov in [12] is considering Lax operators associated to a Lie algebra $g$. Precisely, such Lax operators have the form of

\begin{equation}
L = \frac{\partial}{\partial x} + q(x) + \Lambda
\end{equation}

where (i) $q$ is a differentiable function whose value is in a borel subalgebra $n_+ \oplus \mathfrak{h} \subset g$ (ii) $\Lambda = f + zs \in g[z]$ for the principal nilpotent element $f \in n_-$ and $s \in \ker(\text{ad} n_+)$. Here, $n_+ \oplus \mathfrak{h} \oplus n_-$ is a triangular decomposition of $g$. On the phase space $\mathcal{F}_g$ consisting of functions $q$ in Lax operators, Drinfel’d-Sokolov defined gauge transformations. As a consequence, the W-algebra $\mathcal{W}(g)$ associated to $g$ was introduced as a set of gauge invariant functions.

Furthermore, a Lax operator (1.1) gives rise to a bi-Poisson structure on $\mathcal{W}(g)$, which does an important role to find related integrable systems ([12]). A bi-Poisson structure $(\{ , \}_K, \{ , \}_H)$ consists of a couple of linearly independent local Poisson brackets which involve delta distributions, that is,

\[ \{u(x), v(y)\}_X = \sum_{n \in \mathbb{Z}_+ \cup \{0\}} W_n(y) \frac{\partial}{\partial n} \delta(x - y) \quad \text{for} \quad u, v, W_n \in \mathcal{W}(g), \quad X = K, H \]

satisfy skew symmetries, Jacobi identities and Leibniz rules. After all, a systematic algorithm of getting a sequence of Hamiltonian integrable systems on $\mathcal{W}(g)$ was discovered. In this algorithm,

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they used the Lenard-Magri scheme and the bi-Poisson structure, i.e. there are $k_i \in W(g)$ for $i \in \mathbb{Z}_{\geq 0}$ such that

$$\frac{d\phi(x,t)}{dt} = \int \{k_i(x), \phi(y)\} H dy = \int \{k_{i+1}(x), \phi(y)\} K dy, \text{ for } \phi \in W(g),$$

are all distinct integrable systems.

In an algebraic point of view, classical affine W-algebras are Poisson vertex algebras. A Poisson vertex algebra (PVA) is a differential algebra endowed with a Poisson $\lambda$-bracket structure, denoted by $\{ \lambda \}$. Here, a $\lambda$-bracket can be understood as an algebraic interpretation of a local Poisson bracket. On the other hand, Poisson vertex algebras are closely related to vertex algebras since the quasi-classical limit of a certain family of vertex algebras is a Poisson vertex algebra. As one can expect, there is a vertex algebra called a quantum affine W-algebra which is a quantization of a classical affine W-algebra.

The quantum affine W-(super)algebra associated to $g$ and $f \in g$ is introduced via the quantum affine BRST complexes (or the quantum Drinfeld-Sokolov reduction), provided that $g$ is a finite dimensional simple Lie superalgebra with a non-degenerate even supersymmetric bilinear form and $f$ is an even nilpotent element with an $sl_2$-triple $(e,h,f)$ \([5, 13, 19, 20]\). In \([24, 25]\), the author proved that the quasi-classical limit of the quantum affine W-(super)algebra associated to $g$ and $f$ is

$$W(g, f, k) = (P/I)^{ad_\lambda},$$

where $P = S(\mathbb{C}[\partial] \otimes g)$ is the affine PVA, and the Lie subalgebra $n$ of $g$ and the differential algebra ideal $I$ of $P$ are determined by $f$. Here the $ad_\lambda n$-action is induced from the $\lambda$-bracket on the affine PVA $P$ and $k \in \mathbb{C}$ is the constant involved in the $\lambda$-bracket of $P$. These results imply that classical affine W-(super)algebras have properties analogous to those of finite W-(super)algebras \([14]\). Note that the W-algebra $W(g)$ introduced by Drinfeld and Sokolov in \([12]\) is just $W(g, f, k)$, when $f = f_{\text{princ}}$ is a principal nilpotent element and $k = 1$ (see also \([7]\)).

Since $W(g)$ is a special case of W-algebras associated to $g$, there have been many attempts to understand algebraic structures of $W(g, f, k)$ and to find integrable systems associated to $W(g, f, k)$ for any nilpotent element $f$ (see \([1, 4]\) for instance). Regarding these topics, there are plenty of considerable articles, provided that $g$ is a Lie algebra. In \([11, 28]\), using the definition \((1.2)\), De Sole, Kac and Valeri succeeded in explaining generators of $W(g, f, k)$ and the $\lambda$-bracket relations between them. Moreover, integrable systems on $W(g, f, k)$ are discovered in \([3, 6, 8, 9]\). However, in the case when $g$ is a Lie superalgebra, there still remain many open problems.

For algebraic structures of W-algebras associated to Lie superalgebras (W-superalgebras), we refer \([22, 23, 27, 29]\), where structures of finite W-superalgebras are considered, and \([20, 26]\), where the algebraic structures of affine W-superalgebras associated to minimal nilpotent elements are given. On the other hand, integrable systems has not been yet explored in precise connections with W-superalgebras \([12]\) via PVA structures, to the best of the author’s knowledge. There are some articles which investigated integrable systems on noncommutative algebras (see \([10, 17, 16, 21]\) and the references therein). In particular, in \([17, 16, 21]\), the authors described relations between integrable systems and Lie superalgebras, for instance spo$(2|1)$ and sl$(n|n)$. However, it is not clear if these integrable systems can be explained by PVA structures of W-superalgebras \([12]\).
In this context, a natural question is whether a $W$-superalgebra can be related to a sequence of integrable systems. In this paper, as the first step toward answering this question, we construct $W$-superalgebras using Lax operators, mainly inspired by the important papers [1, 4, 7, 12]. The key idea is to consider Lax operators in algebraic languages:

$$L = k\partial + q - \Lambda \otimes 1 \in \mathbb{C}\partial \ltimes (g[z] \otimes \mathcal{P}/\mathcal{I})_0$$

with even parities, where $\mathcal{P}/\mathcal{I}$ is the differential algebra in (1.2) (see Definition 3.1). In other words, we assume that the entire phase space

$$\mathcal{F}_g = \{ q \mid L = k\partial + q - \Lambda \otimes 1 \text{ is even Lax operator} \} \subset (g[z] \otimes \mathcal{P}/\mathcal{I})_0$$

is even. By considering even gauge transformations, we prove that the construction of $W$-superalgebras via Lax operators is equivalent to (1.2) (see Theorem 3.11). This new construction is particularly useful to find free generators of $W$-superalgebras as differential algebras (see Proposition 3.14).

Recall that if $g$ is a Lie algebra and $f \in g$ is a nilpotent element, then the $W$-algebra $W(g, f, k)$ is endowed with a pair of Poisson $\lambda$-brackets, and these brackets play crucial roles to describe integrable Hamiltonian systems associated to $W(g, f, k)$. Therefore, in order to find integrable systems associated to $W$-superalgebras in an analogous approach, we need to understand the followings:

- the definition of super-Hamiltonian evolution equations using Poisson $\lambda$-brackets of Poisson vertex algebras;
- how to describe two linearly independent Lie (super)brackets on $W(g, f, k)/\partial W(g, f, k)$ with Lax operators and variational derivatives;
- Lenard-Magri scheme associated to super-Hamiltonian integrable systems.

To describe two linearly independent Lie (super)brackets on $W(g, f, k)/\partial W(g, f, k)$, we consider the special operator called the sign twisted universal Lax operator,

$$L_{\text{univ}}^\sigma = k\partial + q_{\text{univ}}^\sigma - \Lambda \otimes 1 \in \mathbb{C}\partial \ltimes g[z] \otimes \mathcal{P}/\mathcal{I}.$$

(see Proposition 3.18 for the precise definition). Employing this operator, we describe the Lie brackets on $W(g, f, k)/\partial W(g, f, k)$ which are needed to use the Lenard-Magri scheme. Indeed, under the assumption that $\Lambda$ is semisimple in $g((z^{-1}))$, we show that there are super-Hamiltonian integrable systems associated to classical affine $W$-superalgebras (see Theorem 5.14). As a simplest example, we show one of super-Hamiltonian integrable systems associated to $W(\text{spo}(2|1), f, k)$ is equivalent to the super KdV equation, which appears in [21], up to constant factors.

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1The main reason we consider sign twisted operators is explained in Remark 4.16.

2This is a quite strong assumption (see Remark 5.5). An important open question in this field is to find integrable systems associated to arbitrary $W$-superalgebras.
2. Classical affine W-algebras

In this section, we review some known facts about classical affine W-algebras. For the notions about Poisson vertex algebras, we refer to [2, 18]. Properties about W-algebras and Lax operators denoted in this section can be found in [7, 12].

2.1. Poisson vertex algebras.

A vector superspace is a vector space $V$ with the $\mathbb{Z}/2\mathbb{Z}$-graded decomposition $V = V_0 \oplus V_1$. For $i = 0, 1$, we denote the parity by $p(a) = i$ for a homogeneous element $a \in V_i$. The superalgebra $\text{End}(V) = \text{End}(V)_0 \oplus \text{End}(V)_1$ is a vector superspace such that $F \in \text{End}(V)_i$ then $F(V_j) \subset V_{i+j}$.

The supersymmetric algebra $A$ is a superalgebra with the supersymmetry $ab = (-1)^{p(a)p(b)}ba$, for homogeneous elements $a$ and $b$.

Definition 2.1. A Lie conformal algebra (LCA) $R$ is a $\mathbb{C}[\partial]$-module with a $\mathbb{C}$-linear $\lambda$-bracket $[[\lambda]] : R \otimes R \to R[\lambda]$ satisfying the following properties:

• (sesquilinearity) $[a_{\lambda} \partial b] = (\lambda + \partial)[a_{\lambda} b]$, $[\partial a_{\lambda} b] = -\lambda[a_{\lambda} b]$,
• (skewsymmetry) $[a_{\lambda} b] = -(-1)^{p(a)p(b)}[b_{-\lambda-\partial} a]$,
• (Jacobi identity) $[a_{\lambda} [b_{\lambda+\mu} c]] = [[a_{\lambda} b_{\lambda} c] + (-1)^{p(a)p(b)}[b_{\mu} [a_{\lambda} c]]]$.

Here, we assume $\partial$ is an even operator on $R$.

Remark 2.2. Let $R$ be a LCA.

1. The sesquilinearity implies $\partial$ is a derivation for the $\lambda$-bracket on $R$, i.e., $\partial[a_{\lambda} b] = [\partial a_{\lambda} b] + [a_{\lambda} \partial b]$.
2. For any $a, b \in R$, we denote by $[a_{\lambda} b] = \sum_{n \geq 0} a_{(n)} b_{(n)} \lambda^n$ for $a_{(n)} b \in R$. Here, $a_{(n)} b \in R$ is called the $n$-th product of $a$ and $b$.

Definition 2.3.

1. A differential (super)algebra $D$ is an associative (super)algebra with an even operator $\partial : D \to D$ called derivation such that

\begin{equation}
\label{2.1}
\partial(AB) = \partial(A)B + A\partial(B) \quad \text{for } A, B \in D.
\end{equation}

In other words, the derivation $\partial$ defined on a generating set of $D$ can be extended using the Leibniz rule (2.1).

2. The (super)algebra of differential polynomials

$\mathbb{C}[\partial][w_i \mid i \in I_0 \cup I_1]$ generated by even elements in $\{w_i\}_{i \in I_0}$ and odd elements $\{w_i\}_{i \in I_1}$ is the algebra isomorphic to the tensor products of a symmetric algebra and an exterior algebra

$S(V_0) \otimes \wedge(V_1)$

where $V_0 := \text{Span}_C(\partial^n w_i \mid i \in I_0, n \in \mathbb{Z}_{\geq 0})$ and $V_1 := \text{Span}_C(\partial^n w_i \mid i \in I_1, n \in \mathbb{Z}_{\geq 0})$. 
Remark 2.4. For the simplicity of notations, we denote by $S(V)$ the supersymmetric algebra generated by the vector superspace $V = V_0 \oplus V_1$. In other words,

$$S(V) := S(V_0) \otimes \Lambda(V_1).$$

Using the notion of a LCA and a differential algebra, we can introduce Poisson vertex algebras.

Definition 2.5. A quintuple $(P, 1, \{\lambda\}, \partial, \cdot)$ is a Poisson vertex algebra (PVA) if

1. $(P, \{\lambda\}, \partial)$ is a Lie conformal algebra,
2. $(P, 1, \cdot, \partial)$ is a supersymmetric differential algebra with the derivation $\partial$ and the unity 1,
3. the Leibniz rule holds:

$$\{A_\lambda BC\} = (-1)^{p(A)p(B)} B\{A_\lambda C\} + (-1)^{p(C)(p(A)+p(B))} C\{A_\lambda B\}.$$

Example 2.6. Let $g$ be a finite simple Lie superalgebra with the even supersymmetric bilinear form $(\cdot | \cdot)$. The affine LCA of $g$ is $R = \mathbb{C}[\partial] \otimes g$ with the $\lambda$-bracket defined by

$$[a_\lambda b] = [a, b] + \lambda k(a|b), \quad \text{for } a, b \in g \text{ and } k \in \mathbb{C},$$

and sesquilinearity. The affine PVA of $g$ is the (super)symmetric algebra $S(R)$ generated by $R$ endowed with the $\lambda$-bracket induced from the bracket of $R$ and the Leibniz rule.

Proposition 2.7. Let $P$ be a PVA and $\partial P$ be the subspace $\{\partial p | p \in P\}$ of $P$. Then the quotient space $P/\partial P = \{p + \partial P | p \in P\}$ endowed with the bracket

$$[a + \partial P, b + \partial P] := \{a_\lambda b\}_{\lambda=0} + \partial P$$

for $a, b \in P$ is a well-defined Lie superalgebra.

Proof. By the sesquilinearity of $\lambda$-brackets, we can see $[\partial a, b]$ and $[a, \partial b]$ are in $\partial P$. Hence it is a well-defined bilinear map $P/\partial P \times P/\partial P \to P/\partial P$. Skew-symmetry and Jacobi identity of $[,]$ follow from those properties of $\{\lambda\}$. □

Definition 2.8. Let $P$ be a PVA and $H : P \to P$ be a diagonalizable operator. Denote by $\Delta_a$ the eigenvalue of homogenous element $a \in P$ with respect to the operator $H$. If

$$\Delta_1 = 0, \quad \Delta_{\partial a} = \Delta_a + 1, \quad \Delta_{ab} = \Delta_a + \Delta_b, \quad \Delta_{a_{(n)}b} = \Delta_a + \Delta_b - n - 1,$$

for any homogenous elements $a, b \in P$ then $H$ is called a Hamiltonian operator and $\Delta_a$ is called the conformal weight of $a$.

Remark 2.9. Let $L$ be an element of a PVA $P$. If (i) $L_{(0)} = \partial$, (ii) $L_{(1)}$ is a diagonalizable, and

(iii) $\{L_\lambda L\} = (\partial + 2\lambda)L + c\lambda^3$, \quad for $c \in \mathbb{C}$

then $L$ is called an energy momentum field of $P$. By sesquilinearities and Leibniz rules, $L_{(1)}$ is a Hamiltonian operator. By convention, we denote by $L_n = L_{(n+1)}$ for $n \geq -1$.

The operator $L_0 = L_{(1)}$ is called the Hamiltonian operator induced from the energy momentum field $L$.

Hamiltonian operators are useful to describe relations between PVAs (VAs) and Poisson algebras (associative algebras).
2.2. Classical affine W-algebras.

There are some ingredients to construct a classical affine W-algebra. From now on, we fix notations to indicate them.

Setup 2.10. Let \( \mathfrak{g} \) be a finite simple Lie superalgebra and \( (e, h, f) \) be an even \( \mathfrak{sl}_2 \)-triple in \( \mathfrak{g} \). Suppose \( (|) \) is the even supersymmetric bilinear form on \( \mathfrak{g} \) such that \( (e|f) = \frac{1}{2}(h|h) = 1 \) and \( \mathfrak{g} = \bigoplus_{i=-d}^{d} \mathfrak{g}(i) \) is the \( \text{ad}_{\mathfrak{g}}^{2} \) eigenspace decomposition. We consider two Lie subalgebras \( n \) and \( m \):

\[
\mathfrak{n} = \bigoplus_{i>0} \mathfrak{g}(i) = \bigoplus_{i\geq 1/2} \mathfrak{g}(i) \quad \subset \quad \mathfrak{m} = \bigoplus_{i\geq 1} \mathfrak{g}(i).
\]

Recall that the grading by \( \text{ad} \mathfrak{g} \) is called a Dynkin grading, which is an example of a good grading. Hence we have the following properties:

1. \( \text{ad} f : \bigoplus_{i\geq 1/2} \mathfrak{g}(i) \to \bigoplus_{i\geq -1/2} \mathfrak{g}(i) \) is injective,
2. \( \text{ad} f : \bigoplus_{i\leq 1/2} \mathfrak{g}(i) \to \bigoplus_{i\leq -1/2} \mathfrak{g}(i) \) is surjective.

By (1) and (2), we have the bijection \( \text{ad} f : \bigoplus_{i=1/2} \mathfrak{g}(i) \to \bigoplus_{i=-1/2} \mathfrak{g}(i) \).

Using notations in Setup 2.10 we define classical affine W-algebras.

Definition 2.11. Let \( \mathcal{I} \) be the differential algebra ideal of the affine PVA \( \mathcal{P} = S(\mathbb{C}[\partial] \otimes \mathfrak{g}) \) generated by \( m + (f|m) \) for \( m \in \mathfrak{m} \). The classical affine W-(super)algebra \( \mathcal{W}(\mathfrak{g}, f, k) \) associated to \( \mathfrak{g} \), \( f \) and \( k \in \mathbb{C} \) is

\[
\mathcal{W}(\mathfrak{g}, f, k) = (\mathcal{P}/\mathcal{I})^{\text{ad}\mathfrak{n}},
\]

where \( \text{ad}_{\mathfrak{n}} \)-action on \( \mathcal{P}/\mathcal{I} \) is induced from the \( \lambda \)-bracket on \( \mathcal{P} \) in Example 2.6. The W-algebra \( \mathcal{W}(\mathfrak{g}, f, k) \) is a PVA with the \( \lambda \)-bracket induced from that of \( \mathcal{P} \).

Proposition 2.12 ([7]). Suppose there is an even element \( s \in \mathfrak{g}(d) \), where \( d \) is the largest integer such that \( \mathfrak{g}(d) \neq \{0\} \). (See Remark 2.14) The W-(super)algebra \( \mathcal{W}(\mathfrak{g}, f, k) \) in Definition 2.11 is endowed with another Poisson \( \lambda \)-bracket which is induced from the bracket on the affine PVA \( \mathcal{P} := S(\mathbb{C}[\partial] \otimes \mathfrak{g}) \) defined by

\[
\{a_\lambda b\}_2 = (s|[a,b]).
\]

Proof. Consider the one parameter family of Poisson \( \lambda \)-brackets on \( S(\mathbb{C}[\partial] \otimes \mathfrak{g}) \) defined by

\[
\{a_\lambda b\}_t = [a,b] + t\lambda(a|b) + t(s|[a,b]), \quad t \in \mathbb{C}.
\]

Observe that \( [s,n] = 0 \) and \( \{n_\lambda A\} = \{nA\}^0 \), where \( \{\lambda\} \) is the Poisson \( \lambda \)-bracket on the affine PVA in Example 2.6. Hence \( \{n_\lambda A\} = \{nA\} \). Thus, for the ideal \( \mathcal{I} \) in Definition 2.11 we have

\[
\mathcal{W}(\mathfrak{g}, f, k) := \{A \in \mathcal{P}/\mathcal{I} \mid \{n_\lambda A\} \in \mathcal{I}\} \simeq \mathcal{W}(\mathfrak{g}, f, k), \quad A \mapsto A,
\]

as differential algebras. One can also check that \( \mathcal{W}(\mathfrak{g}, f, k) \) is a PVA endowed with the \( \lambda \)-bracket induced from (2.3) and extended via Leibniz rules. For \( A, B \in \mathcal{W}(\mathfrak{g}, f, k) \), we have

\[
\{A_\lambda B\}_2 := \{A_\lambda B\}_t^{t+1} - \{A_\lambda B\}_t^{t+1} \in \mathcal{W}(\mathfrak{g}, f, k)
\]

which defines another \( \lambda \)-bracket on \( \mathcal{W}(\mathfrak{g}, f, k) \). The well-definedness of the bracket \( \{\lambda\}_2 \) can be shown by the master formula, or Proposition 4.4. This bracket can be understood as the bracket induced from (2.2).

Remark 2.13. In order to distinguish two \( \lambda \)-brackets on \( \mathcal{W}(\mathfrak{g}, f, k) \), we denote by \( \{\lambda\}_1 \) or \( \{\lambda\}_2 \) the bracket in Definition 2.11 and by \( \{\lambda\}_2 \) the bracket in Proposition 2.12.
**Remark 2.14.** A nonzero even element $s \in \mathfrak{g}(d)$ exists for a subalgebra of $\mathfrak{gl}(m|n)$. In [15], Hoyt showed that Dynkin grading on $\mathfrak{g}_0$ can be extended to $\mathfrak{g}$. For example, in $\mathfrak{sl}(m|n)$ case, it can be shown as follows. A Dynkin grading of $\mathfrak{gl}(m|n)$ corresponds to a pair $(\lambda|\mu)$ of partitions of $m$ and $n$. If $\lambda = (p_1, p_2, \ldots, p_{\rho_2})$ and $\mu = (q_1, q_2, \ldots, q_{\rho_1})$ are decreasing sequences then the largest numbers $d_0$ and $d_1$ such that $\mathfrak{g}_0(d_0) \neq \{0\}$ and $\mathfrak{g}_1(d_1) \neq \{0\}$ satisfy
\[ d_0 = \max\{2(p_1 - 1), 2(q_1 - 1)\}, \quad d_1 = (p_1 - 1) + (q_1 - 1). \]
Hence, for $d = \max\{d_0, d_1\} = d_0$, there exists a nonzero even element $s \in \mathfrak{g}(d)$. Similar argument works in $\text{spo}(m|n)$ cases.

**Proposition 2.15 (7).** Let $\{u_i\}_{i \in I}$ and $\{v^i\}_{i \in I}$ be dual bases of $\mathfrak{g}$ with respect to the bilinear form $(\cdot|\cdot)$ and let $\{v_i\}_{i \in I_{1/2}}$ and $\{v^i\}_{i \in I_{1/2}}$ be the dual bases of $\mathfrak{g}_{1/2}$ with respect to the bilinear form $\omega(\cdot|\cdot)$ on $\mathfrak{g}_{1/2}$ defined by
\[ \omega(a, b) = (f[[a, b]]). \]
Then
\[ L = \sum_{i \in I} \frac{1}{2k} u^i u_i + \sum_{i \in I_{1/2}} \frac{1}{2} \partial(v^i) v_i + \frac{\partial h}{2} \]
is an energy momentum field of the affine PVA of $\mathfrak{g}$ with the $\lambda$-bracket
\[ \{a, b\} = [a, b] + k \lambda(a|b), \quad \text{for } a, b \in \mathfrak{g}. \]
The conformal weight $\Delta_a$ of $a \in \mathfrak{g}(j_a)$ is $1 - j_a$. Moreover, the element $L \in \mathcal{W}(\mathfrak{g}, f, k)$ which is the quotient of $L$ in (2.4) is an energy momentum field of $\mathcal{W}(\mathfrak{g}, f, k)$.

About algebraic structures of classical affine W-algebras, the following proposition can be found in [20]. Also we note that, in [3], there is the analogous result for quantum affine W-algebras.

**Proposition 2.16 (4, 20).** Let $L$ be the energy momentum field of $\mathcal{W}(\mathfrak{g}, f, k)$ in (2.4). For the Hamiltonian operator $L_0$, let $\Delta_a$ be the conformal weight of the homogenous element $a \in \mathcal{W}(\mathfrak{g}, f, k)$.

1. As differential algebras $\mathcal{W}(\mathfrak{g}, f, k) \simeq S(\mathbb{C}[\partial] \otimes \mathfrak{g})_f$, where $\mathfrak{g}_f = \ker(\text{ad} f) \subset \mathfrak{g}$.
2. Let $B_f$ be a basis of $\mathfrak{g}_f$. There is $\Phi_f = \{ \phi_g | g \in B_f \} \subset \mathcal{W}(\mathfrak{g}, f)$ such that
\[ \phi_g = g + \psi_g, \]
where $\Delta_{\psi_g} = \Delta_g$ and $\psi_g$ is an element of the algebra of differential polynomials generated by $\bigoplus_{i > 1 - \Delta_g} \mathfrak{g}(i)$. Note that $g \in \mathfrak{g}(1 - \Delta_g)$.

Moreover,
\[ \mathcal{W}(\mathfrak{g}, f, k) = \mathbb{C}_{\text{diff}}[\phi_g | g \in B_f]. \]

**Example 2.17.** Let $\mathfrak{g} = \mathfrak{sl}_2$. Then $n = m = \mathbb{C}e$ and the ideal $\mathcal{I}$ of $S(\mathbb{C}[\partial] \otimes \mathfrak{g})$ is generated by $e + 1$. As differential algebras,
\[ \mathcal{W}(\mathfrak{g}, f, k) = \mathbb{C}_{\text{diff}}[\phi_f], \]
where $\phi_f = f - \frac{1}{2} x^2 - k \partial x$ for $x = \frac{b}{2}$. We can check that
\[ \{ \phi_f, \lambda \phi_f \} = -k(\lambda + 2 \partial) \phi_f - \frac{k^3}{2} \lambda^3. \]
Example 2.18. Let \( g = \text{spo}(2|1) \subset \mathfrak{gl}(2|1) \). Then the even part \( g_0 \) is generated by an \( \mathfrak{sl}_2 \)-triple \((e_{ev}, h, f_{ev})\) and the odd part \( g_1 \) is generated by \( e_{od} \) and \( f_{od} \). As matrix forms,

\[
h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_{ev} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_{ev} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
e_{od} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad f_{od} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}.
\]

Consider the even supersymmetric invariant bilinear form \((\cdot | \cdot)\) such that \((h|h) = 2(e_{ev}|f_{ev}) = 2\) and \((e_{od}|f_{od}) = -2\). There are two elements

\[
\phi_{od} := f_{od} - \frac{1}{2} e_{od} h - k \partial e_{od}, \quad \phi_{ev} := f_{ev} + \frac{1}{2} f_{od} e_{od} - \frac{1}{4} h^2 + k \frac{1}{4} e_{od} \partial e_{od} - k \frac{1}{2} \partial h,
\]

which satisfy

\[
\text{ad}_\lambda e_{ev}(\phi_{od}) = \text{ad}_\lambda e_{od}(\phi_{od}) = \text{ad}_\lambda e_{ev}(\phi_{ev}) = \text{ad}_\lambda e_{od}(\phi_{ev}) = 0 + I.
\]

Hence

\[
\mathcal{W}(g, f_{ev}, k) = \mathbb{C}_\text{diff}[\phi_{od}, \phi_{ev}]
\]

as a differential algebra. By direct computations, we can check that the \( \lambda \)-bracket of \( \mathcal{W}(g, f_{ev}, k) \) is defined as follows:

\[
\{ \phi_{od} \lambda \phi_{od} \} = -2 \phi_{ev} - 2k^2 \lambda^2,
\]

\[
\{ \phi_{ev} \lambda \phi_{od} \} = -k(\partial + \frac{3}{2} \lambda) \phi_{od},
\]

\[
\{ \phi_{ev} \lambda \phi_{ev} \} = -k(\partial + 2 \lambda) \phi_{ev} - \frac{k^3}{2} \lambda^3.
\]

2.3. Generalized Drinfeld-Sokolov reductions.

In Section 2.3, we recall the construction of classical \( \mathcal{W} \)-algebras associated to Lie algebras via Drinfeld-Sokolov reductions in PVA theories. For the purpose, we assume \( g \) is a finite dimensional simple Lie algebra (without odd part), in this subsection.

For a symmetric differential algebra \( \mathcal{V} \), the vector space \( g \otimes \mathcal{V} \) is the Lie algebra endowed with the bracket

\[
[a \otimes F, b \otimes G] = [a, b] \otimes FG \quad \text{for} \quad a, b \in g, \quad F, G \in \mathcal{V}.
\]

The bilinear form \((\cdot | \cdot) : g \times g \to \mathbb{C}\) on \( g \) can be extended to the map \((\cdot | \cdot) : g \otimes \mathcal{V} \times g \otimes \mathcal{V} \to \mathcal{V}\) by

\[
(a \otimes F | b \otimes G) = (a|b)FG.
\]

The derivation \( \partial : \mathcal{V} \to \mathcal{V} \) on \( \mathcal{V} \) can be extended to an endomorphism on \( g \otimes \mathcal{V} \) such that

\[
\partial(a \otimes F) = a \otimes \partial F.
\]

Consider \( \mathbb{C}\partial \) as the trivial one dimensional Lie algebra. Then \( \mathbb{C}\partial \ltimes g \otimes \mathcal{V} \) is the semidirect product of Lie algebras \( \mathbb{C}\partial \) and \( g \otimes \mathcal{V} \) endowed with the bracket

\[
[ c_1 \partial + a \otimes F, c_2 \partial + b \otimes G ] = c_1 (b \otimes \partial G) - c_2 (a \otimes \partial F) + [a \otimes F, b \otimes G].
\]
Let us consider the subspace (2.6)
\[ p = \bigoplus_{i < 1} g(i) = \bigoplus_{i \leq 1/2} g(i) \]
of \( g \). Then we have

(i) \( g = m \oplus p \), (ii) \( p \cong m^\perp \) by the bilinear form \((|)\).

**Definition 2.19.** Let \( p \) be defined as (2.6) and let \( V(p) := S(\mathbb{C}[\partial] \otimes p) \).

(1) Let \( F_{p,f} \) be the set of elements
\[
q = \sum_{i \in I_p} q_i \otimes P_i \in m^\perp \otimes V(p)
\]
for a basis \( \{ q_i \mid i \in I_p \} \) of \( m^\perp \) and a subset \( \{ P_i \mid i \in I_p \} \subset V(p) \). The set \( F_{p,f} \) is called the phase space associated to \( g \) and \( f \).

(2) For a given \( p \) and \( k \in \mathbb{C} \), the operator \( L \) of the form
\[
L = k\partial + q - f \otimes 1 \in \mathbb{C}\partial \times g \otimes V(p).
\]
is called a Lax operator.

(3) The gauge transformation of \( q \in F_{p,f} \) with \( A \in n \otimes V(p) \) is \( q^A \in F_{p,f} \) where
\[
e^{adA}(k\partial + q - f \otimes 1) = k\partial + q^A - f \otimes 1.
\]
On the other hand, for \( q' \in F_{p,f} \), if there is \( A \in n \otimes V(p) \) such that \( q' = q^A \) then we say they are gauge equivalent and write \( q \sim q' \). Note that it is not hard to check that the equivalence relation is well-defined in \( F_{p,f} \).

Elements in the differential algebra \( V(p) \) can be identified with functions \( F_{p,f} \rightarrow V(p) \) as follows:
\[
p(a \otimes F) = (p|a|)F, \quad PQ(a \otimes F) = P(a \otimes F)Q(a \otimes F), \quad \partial P(a \otimes F) = \partial(P(a \otimes F)),
\]
for \( p \in p \), \( P, Q \in V(p) \) and \( a \otimes F \in F_{p,f} \). An element \( P \in V(p) \) is said to be a gauge invariant function if \( P(q) = P(q') \) whenever \( q \sim q' \) for \( q, q' \in F_{p,f} \).

**Proposition 2.20** ([7]). The set \( W \) of gauge invariant functions in \( V(p) \) is a differential subalgebra of \( V(p) \). Moreover, \( W \) is isomorphic to the classical affine \( W \)-algebra \( W(g, f, k) \) associated to \( g \) and \( f \) as differential algebras.

**Remark 2.21.** In [12], a pair of local Poisson structures in \( W \) is described by a Lax operator. The Poisson structures are equivalent to the PVA structures on the classical affine \( W \)-algebra \( W(g, f, k) \) which are induced from those in the affine PVA \( S(\mathbb{C}[\partial] \otimes g) \). (See Definition 2.11 and Proposition 2.14.)

The construction of \( W \)-algebras in Proposition 2.20 allows to compute generators of the algebras (see Theorem 2.23).

**Lemma 2.22** ([7]). Let \( V \) be a subspace of \( m^\perp \) such that \( m^\perp = [n, f] \oplus V \). Take a basis \( \{ v_i \}_{i \in I_p} \) of \( m^\perp \) such that \( \{ v_i \}_{i \in J \subset I_p} \) is a basis of \( V \) and \( \{ v_i \}_{i \in I_p \setminus J} \) is a basis of \( [n, f] \). If \( \{ v^i \}_{i \in I_p} \) is the dual basis of \( p \) then \( \{ v^i \}_{i \in J} \) is a basis of \( g^f := \ker(ad f) \).
Theorem 2.23. Let $\{q_i\}_{i \in \mathbb{I}}$ and $\{q^i\}_{i \in \mathbb{I}_p}$ be bases of $\mathfrak{m}^\perp$ and $\mathfrak{p}$ such that $(q_i|q^j) = \delta_{ij}$. Denote by

$$q_{\text{univ}} = \sum_{i \in \mathbb{I}} q_i \otimes q^i \quad \text{and} \quad \mathcal{L} = k\partial + q_{\text{univ}} - f \otimes 1.$$  

Then there is unique $X \in \mathfrak{n} \otimes \mathcal{V}(\mathfrak{p})$ such that $q_{X,\text{univ}}^X \in \mathcal{V} \otimes \mathcal{V}(\mathfrak{p})$ satisfies

$$e^{ad X} \mathcal{L} = k\partial + q_{X,\text{univ}}^X - f \otimes 1.$$  

(1) As in Lemma 2.22 let $\{q_i\}_{i \in \mathbb{I}_p}$ be a basis of $V$. If

$$q_{X,\text{univ}}^X = \sum_{i \in \mathbb{I}_p} q_i \otimes w_i$$

for $q_{X,\text{univ}}^X$ in (1) then $w_i$ are gauge invariant functions in $\mathcal{V}(\mathfrak{p})$. Moreover, by Lemma 2.22 we have $w_i = v^i + (\text{degree} \geq 2 \text{ part}).$

(2) The set of gauge invariant functions in $\mathcal{V}(\mathfrak{p})$ is the algebra of differential polynomials

$$\mathbb{C}_{\text{diff}}[w_i | i \in \mathbb{I}].$$

Remark 2.24. We have the differential algebra isomorphism $\mathcal{V}(\mathfrak{p}) \simeq S(\mathbb{C}[\partial] \otimes \mathfrak{g})\mathcal{I} =: \mathcal{V}_\mathcal{I}(\mathfrak{p}), A \mapsto A$, where $\mathcal{I}$ is the ideal defined in Definition 2.11. Due to the isomorphism,

(i) we can consider a Lax operator $L$ an element in $\mathbb{C}[\partial] \times \mathfrak{m}^\perp \otimes \mathcal{V}_\mathcal{I}(\mathfrak{p})$,

(ii) since the W-algebra $\mathcal{W}(\mathfrak{g}, f, k)$ is a subalgebra of $\mathcal{V}_\mathcal{I}(\mathfrak{p})$, we prefer to regard $\mathcal{W}(\mathfrak{g}, f, k)$ as a set of functions from $\mathcal{F}_{\phi,f}$ to $\mathcal{V}_\mathcal{I}(\mathfrak{p})$.  

Theorem 2.25. The W-algebra $\mathcal{W}(\mathfrak{g}, f, k)$ is the set of gauge invariant functions in $\mathcal{V}_\mathcal{I}(\mathfrak{p})$. Hence we can find free generators by Theorem 2.25.

The following is the simplest example of classical affine W-algebras.

Example 2.26. Let $\mathfrak{g} = \mathfrak{sl}_2$. Then $q_{\text{univ}} = e \otimes f + h \otimes x$ for $x = \frac{h}{2}$ and

$$\mathcal{L} = k\partial + q_{\text{univ}} - f \otimes 1.$$  

If we take $X = e \otimes x$ then $q_{X,\text{univ}}^X = e \otimes (f - x^2 - k\partial x)$. Hence $\mathbb{C}_{\text{diff}}[f - x^2 - k\partial x]$ is the set of gauge invariant functions. Indeed, we can check that $\phi_f := f - x^2 - k\partial x$ is a gauge invariant function as follows:

Let $Y = e \otimes r$ for $r \in \mathcal{V}_\mathcal{I}(\mathfrak{p})$. Then $q_Y^Y = h \otimes (x-r) + e \otimes (f - k\partial r - 2rx + r^2)$ and

$$\phi_f(q_Y^Y) = (f - k\partial r - 2rx + r^2) - (x-r)^2 - k\partial(x-r) = f - x^2 - k\partial x$$

which is independent on $r$. Also, we can check that the algebra of differential polynomials $\mathbb{C}_{\text{diff}}[\phi_f]$ is isomorphic to the W-algebra $\mathcal{W}(\mathfrak{g}, f, k)$ in Example 2.17.

2.4. Integrable Hamiltonian systems associated to W-algebras.

Integrable Hamiltonian systems can be investigated by Poisson vertex algebras theories (3). In this subsection, we briefly review basic notions related to integrable Hamiltonian systems.

Definition 2.27. Let $\mathcal{P}$ be an (even) algebra of differential polynomials with a PVA structure.

(i) An evolution equation is called a Hamiltonian system on $\mathcal{P}$ if there is $h \in \mathcal{P}$ such that

$$\frac{du}{dt} = \{h_\lambda u\}_{\lambda = 0} \quad \text{for} \quad u \in \mathcal{P}.$$  

(ii) Consider the quotient map $\int : \mathcal{P} \rightarrow \mathcal{P}/\partial \mathcal{P}$. The image $\int f$ of $f \in \mathcal{P}$ is called a local functional.
(3) A Hamiltonian system is called an \textit{integrable system} if there are infinitely many linearly independent integrals of motion $\int h_i, \ i \in \mathbb{Z}_{\geq 0}$. Here, an integral of motion $\int h_i$ is a local functional such that $\int \frac{dh_i}{dt} = 0$.

In the rest of this subsection, consider the Laurent series $g((z^{-1}))$ with the Lie bracket $[az^n, bz^m] = [a, b]z^{n+m}$ for $a, b \in g$.

Recall $\mathcal{W}(g, f, k)$ is endowed with a bi-Poisson $\lambda$-bracket which is induced from that on $S(C[\partial] \otimes g)$:

(12.13) \[ \{a, b\}_1 = [a, b] + k\lambda(a[b], \quad \{a, b\}_2 = (s)[a, b]. \]

Remark 2.28 (Lenard-Magri Scheme). Let $\mathcal{P}$ be a PVA with the bi-Poisson $\lambda$-bracket $(\{ \lambda \}_1, \{ \lambda \}_2)$. Suppose there is a sequence of linearly independent local functionals $\int h_i \in \mathcal{P}/\partial \mathcal{P}, \ i = 0, 1, 2, \cdots$ such that

(i) $\{h_0, \mathcal{P}\}_2|_{\lambda = 0} = 0$, \quad (ii) $\{h_i, \mathcal{P}\}_1|_{\lambda = 0} = \{h_{i+1}, \mathcal{P}\}_2|_{\lambda = 0}$ for $i \geq 0$ and $p \in \mathcal{P}$.

Then $\frac{du}{dt} = \{h_i, u\}_K|_{\lambda = 0}$ for $i = 0, 1, 2, \cdots$ are Hamiltonian integrable systems.

\textbf{Theorem 2.29}. Suppose $\Lambda := f + sz \in g((z^{-1}))$ is semisimple for $s \in \ker(\text{ad } n)$. There is a sequence of integrable systems on $\mathcal{W}(g, f, k)$ which satisfies the assumptions of Lenard-Magri scheme. More specifically, consider

$\mathcal{L}(\Lambda) := k\partial + q_{\text{univ}} - \Lambda \otimes 1 = \mathcal{L} - zs \otimes 1 \in C\partial \times g((z^{-1})) \otimes \mathcal{V}_T(p)$

and take $h(z) \in (\ker(\text{ad } \Lambda) \cap g[[z^{-1}]]) \otimes \mathcal{V}_T(p)$ such that $e^{adS(z)}\mathcal{L}(\Lambda) = k\partial + h(z) + \Lambda \otimes 1$ for some $S(z) \in g[[z^{-1}]]) \otimes \mathcal{V}_T(p)$. Then $h_i = (z^i \Lambda \otimes 1| h(z))$ is an element in $\mathcal{W}(g, f, k)$ and the Hamiltonian equation

$\frac{du}{dt} = \{h_i, u\}_K|_{\lambda = 0}$

is an integrable system on $\mathcal{W}(g, f, k)$. Here, the bilinear form on $g((z^{-1})) \otimes \mathcal{V}_T(p)$ is defined by

$(az^n \otimes F | bz^m \otimes G) = (a[b]FG\delta_{n+m,0}$ for $a, b \in g$ and $F, G \in \mathcal{V}_T(p)$.

Remark 2.30. (2) It is a natural question to ask that if we can find a semisimple element $\Lambda = f + zs$ for a given nilpotent element $f$. In the case when $g = sl_n$, if $f$ corresponds to one of the following partitions $\lambda$ of $n$ then we can find such a semisimple element $\Lambda$.

(1) $\lambda = (r, r, \cdots, r, 1, 1, \cdots, 1)$, \quad (2) $\lambda = (r, r - 1, r, r - 1, \cdots, r, r - 1, 1, 1, \cdots, 1)$.

3. CLASSICAL AFFINE W-SUPERALGEBRAS AND GENERALIZED DRYNFELD-SOKOLOV REDUCTIONS

In this section, we shall show a set of generators of a classical affine W-superalgebra as a superalgebra of differential polynomials can be obtained by an analogous method to the generalized Drinfeld-Sokolov reduction.

Let $g$ be a simple Lie superalgebra with a nondegenerate invariant even supersymmetric bilinear form $(\cdot | \cdot)$ and let $\mathcal{V}$ be a supersymmetric differential superalgebra. The vector superspace $g \otimes \mathcal{V}$ is endowed with the Lie bracket and the bilinear form defined by

$[a \otimes F, b \otimes G] = (-1)^{p(b)p(F)}[a, b] \otimes FG, \quad (a \otimes F | b \otimes G) = (-1)^{p(b)p(F)}(a[b]FG$

for the homogeneous elements $a, b \in g$ and $F, G \in \mathcal{V}$. 


Due to the invariance of the bilinear form on $\mathfrak{g}$, we get the invariance of the bilinear form $\langle \cdot | \cdot \rangle$ on $\mathfrak{g} \otimes \mathcal{V}$

\[
\langle a \otimes F | [b \otimes G, c \otimes H] \rangle = \langle [a \otimes F, b \otimes G] | c \otimes H \rangle,
\]

for $a, b, c \in \mathfrak{g}$ and $F, G, H \in \mathcal{V}$.

Let us consider an even derivation $\partial : \mathcal{V} \to \mathcal{V}$ on $\mathcal{V}$. Then it can be extended to the map on $\mathfrak{g} \otimes \mathcal{V}$ by $\partial(a \otimes F) = a \otimes \partial F$. The Lie superalgebra

\[
\mathbb{C}\partial \ltimes (\mathfrak{g} \otimes \mathcal{V})
\]
is the semidirect product of the trivial Lie algebra $\mathbb{C}\partial$ and the Lie superalgebra $\mathfrak{g} \otimes \mathcal{V}$.

Suppose the Lie superalgebra $\mathfrak{g}$ has an $\text{sl}_2$-triple $(e, h, f)$ with the even supersymmetric bilinear form $\langle \cdot | \cdot \rangle$ such that $\langle e | f \rangle = \frac{1}{2} \langle h | h \rangle = 1$. As in Section 2.3, let $\mathfrak{m} = \bigoplus_{i \geq 1} \mathfrak{g}(i)$, $\mathfrak{m}^\perp = \bigoplus_{i < 0} \mathfrak{g}(i)$, where $\mathfrak{g}(i)$ is the $\text{ad}_f^i$ eigenspace with the eigenvalue $i$. Recall that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{p}$ and $\mathfrak{p} \simeq \mathfrak{m}^\perp$ as vector superspaces via the bilinear form $\langle \cdot | \cdot \rangle$ on $\mathfrak{g}$. For the superspace $\mathfrak{m}^\perp = \mathfrak{m}_0^\perp \oplus \mathfrak{m}_1^\perp$, there is a basis $\{ q_i | i \in I := I_0 \cup I_1 \}$ of $\mathfrak{m}^\perp$ such that

\[
\langle q_i | i \in I_0 \rangle \text{ is a basis of } \mathfrak{m}_0^\perp, \quad (ii) \langle q_i | i \in I_1 \rangle \text{ is a basis of } \mathfrak{m}_1^\perp.
\]

**Definition 3.1.** Let $V(p) := S(\mathbb{C}[\partial] \otimes \mathfrak{p})$ be the differential superalgebra generated by the vector superspace $\mathfrak{p}$. A Lax operator $L$ is an even element in $\mathfrak{g} \otimes V(p)$ such that

\[
L = k\partial + \sum_{i \in I_0} q_i \otimes P_i + \sum_{i \in I_1} q_i \otimes P^i - f \otimes 1 \in \mathfrak{g} \otimes (V(p))_0,
\]

where $q = \sum_{i \in I_0} q_i \otimes P_i + \sum_{i \in I_1} q_j \otimes P^j \in (\mathfrak{m}^\perp \otimes V(p))_0$.

**Remark 3.2.** Let $\mathcal{I}$ be the differential superalgebra ideal of $V(\mathfrak{g})$ generated by $\{ m^+(f|m)|m \in \mathfrak{m} \}$. Denote $V_T(p) := V(\mathfrak{g})/I$. Then we can check the following facts:

1. $V(p) \simeq V_T(p)$ as differential superalgebras by the canonical isomorphism $\iota : V(p) \to V_T(p)$. If there is no danger of confusion then we denote $\iota(P) \in V_T(p)$ by $P$.
2. We can regard Lax operators as elements in $\mathfrak{g} \otimes (\mathfrak{m}^\perp \otimes V_T(p))_0$ via the isomorphism $\iota$ in (1).

Recall that the W-superalgebra $W(\mathfrak{g}, f, k)$ is a subset of $V_T(p)$. In order to see the relation between W-superalgebras and Lax operators, we use (2) in Remark 3.2.

A Lax operator $L$ acts on $\mathfrak{g} \otimes V_T(p)$ by

\[
L(a \otimes F) := [L, a \otimes F].
\]

Consider the phase space

\[
\mathcal{F}_{g,f} := (\mathfrak{m}^\perp \otimes V_T(p))_0.
\]

Then for any $q \in \mathcal{F}_{g,f}$, there is the corresponding Lax operator $L = k\partial + q - f \otimes 1$. Note that, for a Lax operator $L$ and an element $X \in (n \otimes V_T(p))_0$, there is $q^X \in \mathcal{F}_{g,f}$ such that

\[
e^{\text{ad}X}L = e^{\text{ad}X}(k\partial + q - f \otimes 1) = k\partial + q^X - f \otimes 1.
\]

Hence $e^{\text{ad}X}L$ is again a Lax operator.

**Definition 3.3.**

1. Let $q \in \mathcal{F}_{g,f}$ and $X \in (n \otimes V_T(p))_0$. Then $q^X \in \mathcal{F}_{g,f}$ defined as in (3.2) is said to be the gauge transformation of $q \in \mathcal{F}_{g,f}$ by $X$. 

Remark 3.4. Since the bilinear form $(\cdot, \cdot)$ defined on the Lie superalgebra $\mathfrak{g}$ is even, the universal Lax operator in (3.3) is even. Hence it is a Lax operator.

Lemma 3.8. Also, a Lax operator $L$ is defined on the Lie superalgebra $\mathfrak{g}$ and $f$ is

(3.3) \[ \mathcal{L} = k\partial + q_{\text{univ}} - 1 \otimes f = k\partial + \sum_{i \in I = I_0 \cup I_1} q_i \otimes q^i - f \otimes 1, \]

where $\{q_i\}_{i \in I}$ and $\{q^i\}_{i \in I}$ are bases of $\mathfrak{m}^\perp$ and $\mathfrak{p}$, such that $(q_i|q^j) = \delta_{ij}$.

Remark 3.5. If $\mathfrak{g}$ is even then $p(a \otimes F) := (p|a)F = (a|p)F$. Hence (3.4) and (3.5) define the same functions as those in (2.10). If $\mathfrak{g}$ is not even and $a \otimes F$ is an even element in $\mathfrak{g} \otimes V_I(p)$ then

(3.4) \[ p(\partial) = 0, \quad c(q) = c, \quad p(a \otimes F) = F(a|p) = (a|p)F, \]

for $c \in \mathbb{C}, p \in \mathfrak{p} \subset V_I(p)$ and $a \otimes F \in (\mathfrak{g} \otimes V_I(p))_0$. For $P, Q \in V_I(p)$, we have

(3.5) \[ PQ(a \otimes F) = P(a \otimes F)Q(a \otimes F), \quad \partial P(a \otimes F) = \partial(P(a \otimes F)). \]

Definition 3.6. A function $P \in V_I(p)$ is said to be gauge invariant if $P(q) = P(q')$ for any gauge equivalent elements $q$ and $q'$ in $\mathcal{F}_{g,f}$.

Proposition 3.7. The subset of $V_I(p)$ consisting of gauge invariant functions is a differential superalgebra.

Proof. It is clear that if $P, Q \in V_I(p)$ are gauge invariant then $PQ$ and $\partial P$ are also gauge invariant.

We note that

(3.6) \[ P(\mathcal{L}) = P \quad \text{for} \quad P \in V_I(p), \]

since an element $p \in \mathfrak{p} \subset V_I(p)$ has properties $(f|p) = 0$ and $p(\partial) = 0$ so that

\[ p(\mathcal{L}) = p(q_{\text{univ}}) = \sum_{i \in I = I_0 \cup I_1} q^i(q_i|p) = p. \]

Also, a Lax operator $L = k\partial + Q - f \otimes 1$ with $Q = \sum_{i \in I} q_i \otimes Q^i \in \mathcal{F}_{g,f}$ satisfies

(3.7) \[ P(Q) = P(L) = P(\mathcal{L})|_{q^i = Q^i} \quad \text{for} \quad P \in V_I(p). \]

Here, the subscript $q^i = Q^i$ means that we substitute $q^i$ by $Q^i$.

Now, the following lemma is useful to see detailed computations in the proof of Proposition 3.8.

Lemma 3.8. Let $X = n \otimes r \in (\mathfrak{n} \otimes V_I(p))_0$ and $p \in V_I(p)$. Then we have

\[ p([X, \mathcal{L}]) = -k\partial r(n|p) - r[n, p] \in V_I(p). \]
Proof. We have
\[(3.8) \quad [X, \mathcal{L}] = -n \otimes k \partial r + [n \otimes r, q_{\text{univ}} - f \otimes 1] = -n \otimes k \partial r - \sum_{i \in I} [q_i, n] \otimes r q^i - [n, f] \otimes r\]
for any \((3.14)\)
\[(3.9) \quad p([X, \mathcal{L}]) = -k \partial r(n[p]) - r q^i([g, n] \mid p) + r([f, n] \mid p) = -k \partial r(n[p]) - r[n, p] \in \mathcal{V}_k(p).\]

\[
\square
\]

Proposition 3.9. If \(W \in \mathcal{V}_k(p)\) is a gauge invariant function then \(W \in \mathcal{W}(g, f, k)\).

Proof. Let \(X = n \otimes r \in (n \otimes \mathcal{V}_k(p))_0\) and \(\epsilon \in \mathbb{C}\). For \(W \in \mathcal{V}_k(p)\), we denote
\[(3.10) \quad W(\mathcal{L} + \epsilon [X, \mathcal{L}] + \frac{1}{12} [X, [X, \mathcal{L}]] + \cdots) = \sum_{t \geq 0} \epsilon^t P_t^{W}\]
for some \(P_t^{W} \in \mathcal{V}_k(p)\).

If \(W\) is a gauge invariant function, we have
\[(3.11) \quad W(e^{\text{ade}X}(\mathcal{L})) = W\]
for any \(\epsilon \in \mathbb{C}\). Since \((3.11)\) implies \(P_t^{W} = 0\) for any \(t \geq 1\), it is enough to show that \(P_1^{W} = 0\) implies \(W \in \mathcal{W}(g, f, k)\).

In order to show that \(P_1^{W} = 0\), let us denote by \([A \lambda B]_I \in \mathcal{V}_k(p)[\lambda]\), for \(A, B \in \mathcal{V}(g)\), the image of \([A \lambda B] \in \mathcal{V}(g)[\lambda]\). In other words,
\[(3.12) \quad [A \lambda B]_I = [A \lambda B] + I[\lambda].\]

Note that \([\lambda]_I\) induces the well-defined \(\lambda\)-bracket on \(\mathcal{V}(g, f, k)\) since
\[(3.13) \quad [n \lambda P(m + (f|m))]_I = 0 + I[\lambda] \quad \text{for} \quad P \in \mathcal{V}(g) = S(\mathbb{C}[\partial] \otimes g) \text{ and } m \in m\]
so that \([n \lambda]_I = I[\lambda]\).

By the definition of \(\mathcal{W}(g, f, k)\), an element \(W \in \mathcal{V}_k(p)\) is in \(\mathcal{W}(g, f, k)\) if and only if \([n \lambda \tilde{W}]_I = 0\), where \(\tilde{W}\) is an element in \(\mathcal{V}(p)\) such that \(\iota(\tilde{W}) = W\) for the map \(\iota\) in Remark 3.2. Thus, it is enough to show that
\[(3.14) \quad P_1^{W} = -\sum_{t \geq 0} (\partial^t r) P_t^{W'} \quad \text{if and only if} \quad [n \lambda \tilde{W}]_I = \sum_{t \geq 0} \lambda^t \cdot P_t^{W}.\]

Now, observe the following facts.

(i) If we substitute \(W\) with \(p \in p \subset \mathcal{V}(p)\) in \((3.10)\), then, by Lemma 3.8, we have \(P_1^{W} = -k \partial r(n[p]) - r[n, p] \in \mathcal{V}(p)\). On the other hand, take \(\tilde{W} \in \mathcal{V}(p)\) such that \(\iota(\tilde{W}) = p\). Since \([n \lambda \tilde{W}]_I = [n, p] + k \lambda(n[p]) \in \mathcal{V}(p)[\lambda]\), we have \((3.14)\).

(ii) If \(W = \partial^m p\) then \(P_1^{W} = -\partial^m (k \partial r(n[p]) - r[n, p])\). In this case, the sesquilinearity \([n \lambda \partial^m p] = (\lambda + \partial)^m [n \lambda p]\) implies \((3.14)\).

(iii) Suppose \(W = BC\) for homogeneous elements \(B, C \in \mathcal{V}_k(p)\). Since
\[W(e^{\text{ade}X}(\mathcal{L})) = B(e^{\text{ade}X}(\mathcal{L})) \cdot C(e^{\text{ade}X}(\mathcal{L}))\]
we have \(P_1^{W} = BP_1^C + P_1^B C\). Denote \(P_1^B = \sum_{t \geq 0} (\partial^t r) P_t^B\) and \(P_1^C = \sum_{t \geq 0} (\partial^t r) P_t^C\). Then
\[(3.15) \quad P_1^{W} = \sum_{t \geq 0} \left[(-1)^{p(B)p(B)}(\partial^t r)B P_t^C + (\partial^t r)P_t^B C\right].\]

On the other hand, by the Leibniz rule, we have
\[(3.16) \quad [n \lambda \tilde{W}]_I = [n \lambda \tilde{B} C]_I = (-1)^{p(B)p(B)} B[n \lambda \tilde{C}]_I + [n \lambda \tilde{B}]_I C\]
\[= -\sum_{t \geq 0} \left[(-1)^{p(B)p(B)}B \lambda^t P_t^C + \lambda^t P_t^B C\right],\]
for \( \tilde{B}, \tilde{C} \in \mathcal{V}(\mathfrak{p}) \) such that \( \iota(\tilde{B}) = B \) and \( \iota(\tilde{C}) = C \). It is not hard to see (3.15) and (3.16) imply (3.14).

By (i), (ii), (iii) and the induction, we prove the proposition.

**Proposition 3.10.** If \( W \in W(\mathfrak{g}, f, k) \) then \( W \) is a gauge invariant function in \( \mathcal{V}_\mathfrak{g}(\mathfrak{p}) \).

**Proof.** Let \( W \) be an element in \( W(\mathfrak{g}, f, k) \). Note that

(i) in the proof of Proposition 3.9 we showed that \( W \in W(\mathfrak{g}, f, k) \) if and only if \( W([X, \mathcal{L}]) = 0 \) for any \( X \in (n \otimes \mathcal{V}(\mathfrak{p}))_0 \),

(ii) for \( P \in \mathcal{V}_\mathfrak{g}(\mathfrak{p}) \) and a Lax operator \( L = k\partial + \sum_{i \in I} q_i \otimes Q^i - f \otimes 1 \), we have

\[
P([X, \mathcal{L}]) = P([X, \mathcal{L}])|_{q^i=Q^i}.
\]

Hence \( W([X, \mathcal{L}]) = 0 \) for any \( X \in (n \otimes \mathcal{V}(\mathfrak{p}))_0 \) and any Lax operator \( L \). Moreover, since \( \text{ad}^{n-1}X(L) \) is a Lax operator for \( n \geq 1 \), we have \( W(\text{ad}^nX(L)) = 0 \). Thus,

\[
W(e^{\text{ad}X}(L)) = W(L),
\]

which means that \( W \) is gauge invariant.

**Theorem 3.11.** The set of gauge invariant functions in \( \mathcal{V}_\mathfrak{g}(\mathfrak{p}) \) is the classical affine \( W \)-superalgebra associated to \( \mathfrak{g} \) and \( f \).

**Proof.** It directly follows from Proposition 3.9 and Proposition 3.10.

Now the following propositions are useful to find generators of \( W \)-superalgebras.

**Proposition 3.12.** Let us fix an \( \text{ad}h \)-invariant complementary subspace \( V_f \subset \mathfrak{g} \) of \( [f, n] \) in \( m^\perp \):

\[
m^\perp = V_f \oplus [f, n].
\]

Consider a Lax operator \( L = k\partial + Q - f \otimes 1 \) for \( Q \in \mathcal{F}_{\mathfrak{g}, f} \). Then there exists unique \( Q^\text{can} \in V_f \otimes (\mathcal{V}_\mathfrak{g}(\mathfrak{p}))_0 \) and unique \( X \in (n \otimes \mathcal{V}(\mathfrak{p}))_0 \) such that

\[
e^{\text{ad}X}L = k\partial + Q^\text{can} - f \otimes 1.
\]

**Proof.** We can write \( Q = \sum_{i>\frac{1}{2}} Q_i \) where \( Q_i \in \mathfrak{g}(i) \otimes \mathcal{V}(\mathfrak{p}) \). Similarly, let \( X = \sum_{i \geq \frac{1}{2}} X_i \) and \( Q^\text{can} = \sum_{i \geq 0} Q_i^\text{can} \) for \( X_i, Q_i^\text{can} \in \mathfrak{g}(i) \otimes \mathcal{V}(\mathfrak{p}) \). Then the \( \frac{1}{2} \)-decomposition of (3.17) implies the following equalities:

\[
Q_{-\frac{1}{2}} + [X_{\frac{1}{2}}, -f \otimes 1] = 0,
\]

\[
Q_0 + [X_1, -f \otimes 1] + [X_{\frac{1}{2}}, Q_{-\frac{1}{2}}] = Q_0^\text{can},
\]

\[
Q_{\frac{1}{2}} + [X_{\frac{3}{2}}, -f \otimes 1] + [X_1, Q_{-\frac{1}{2}}] + [X_{\frac{1}{2}}, \partial + Q_0] = Q_{1/2}^\text{can},
\]

\[...
\]

Then we can determine \( X_{1/2} \) uniquely by the first equation in (3.18) and it is even. Also, \( X_1 \) and \( Q_0^\text{can} \) can be uniquely determined by \( X_{1/2} \) and the second equation in (3.18). Since \( [X_{1/2}, Q_{-1/2}] + Q_0 \) is even, both \( X_1 \) and \( Q_0^\text{can} \) are even. The Inductively, even elements \( X_{i+1} \) and \( Q_i^\text{can} \) are determined uniquely by \( X_{i+1}, Q_j^\text{can} \) for \( j < i \) and (3.18).

**Lemma 3.13.** Let us take the universal Lax operator \( L = k\partial + q_{\text{univ}} - 1 \otimes f \) in Proposition 3.12 and let \( \{ q_i \mid i \in \mathcal{J} \} \) be a basis of \( V_f \). If we denote \( q_{\text{univ}} = \sum_{i \in \mathcal{J}} q_i \otimes w^i \) then we have the following properties.
Example 3.15. \( q_{\text{can}} = \sum_{i \in \mathcal{I}} q_i \otimes (w^i(q)) \).

(2) Let \( P \in \mathcal{V}_T(p) \) and \( q \in \mathcal{F}_{\otimes f} \). Then we have \( P(q) = P(q_{\text{can}}) \) if and only if \( P(q_{\text{univ}}) = P(q_{\text{can}}) \).

Proof. (1) For any \( q \in \mathcal{F}_{\otimes f} \) and the basis \( \{ q^i \}_{i \in I} \) of \( p \) such that \( (q_i|q^j) = \delta_{ij} \), the Lax operator \( L = k\partial + q - f \otimes 1 = k\partial + \sum_{i \in I} q_i \otimes (q^i(q)) = f \otimes 1 \).

Also, for \( X = n \otimes r \in n \otimes \mathcal{V}_T(p) \) such that \( e^{adX} \mathcal{L} = k\partial + q_{\text{can}} - f \otimes 1 \), if we let \( X_q := n \otimes (r(q)) \) then \( e^{adX_q}(L) = k\partial + q_{\text{can}} - f \otimes 1 \) is obtained from \( e^{adX}(L) \) by substituting \( q^i \) in \( q_{\text{can}} \) by \( q^i(q) \).

In other words, \( q^X_q \in V_q \otimes \mathcal{V}_T(p) \) and
\[
q^X_q = q_{\text{can}} = \sum_{i \in \mathcal{I}} q_i \otimes (w^i(q)).
\]

(2) It is enough to show that \( P(q_{\text{univ}}) = P(q_{\text{can}}) \) implies \( P(q) = P(q_{\text{can}}) \) for any \( q \in \mathcal{F}_{\otimes f} \).

Suppose \( P(q_{\text{univ}}) = P(q_{\text{can}}) \). Then, by (1), we have
\[
P(q_{\text{can}}) = P\left( \sum_{i \in \mathcal{I}} q_i \otimes w^i | q^i = q^i \right) = P(q_{\text{univ}})|_{q^i = q^i} = P(q_{\text{univ}})|_{q^i = q^i} = P(q)
\]
where \( |q^i = q^i \) denotes we substitute \( q^i \) by \( q^i \).

\[\square\]

Proposition 3.14. For \( q_{\text{can}} = \sum_{i \in \mathcal{I}} q_i \otimes w^i \in V_f \otimes \mathcal{V}_T(p) \) in Lemma 3.13 we have
\[
\mathcal{W}(g, f, k) = C_{\text{diff}}[w^i | i \in \mathcal{I}] \subset \mathcal{V}_T(p)
\]
as differential superalgebras.

Proof. Note that we have
\[
(i) \ w^i = w^i(q_{\text{univ}}) \text{ since } q_{\text{univ}} = \sum_{i \in I} q_i \otimes q^i, \quad \text{ (ii) } w^i = q^i(q_{\text{can}}).
\]

Consider the subset \( \{ q^i | i \in \mathcal{I} \} \) of the basis \( \{ q^i \}_{i \in I} \) of \( p \) and take an element \( \Phi \in \mathcal{W}(g, f, k) \). Then
\[
\Phi = \Phi(q_{\text{univ}}) = \Phi(q_{\text{can}}) \in \mathbb{C}_{\text{diff}}[q^i(q_{\text{can}}) | i \in \mathcal{I}] = \mathbb{C}_{\text{diff}}[w^i | i \in \mathcal{I}]
\]
and \( \Phi \in \mathbb{C}_{\text{diff}}[w^i | i \in \mathcal{I}] \). Hence, by Lemma 3.13, we have \( \mathcal{W}(g, f, k) \subset \mathbb{C}_{\text{diff}}[w^i | i \in \mathcal{I}] \).

Conversely, for \( q, q' \in \mathcal{F}_{\otimes f} \) such that \( q \sim q' \), we have \( q_{\text{can}} = q_{\text{can}} \). Since \( w^i(q) = q^i(q_{\text{can}}) = q^i(q_{\text{can}}) = w^i(q_{\text{can}}) \), we get \( \mathbb{C}_{\text{diff}}[w^i | i \in \mathcal{I}] \subset \mathcal{W}(g, f, k) \).

By Proposition 3.14 we can find generators of \( \mathcal{W}(\text{spo}(2|1), f, k) \) as follows (cf. Example 2.13).

Example 3.15. Let \( g = \text{spo}(2|1) \) and let
\[
h = e_{11} - e_{22}, \quad e = e_{12}, \quad f = e_{21}, \quad e_{od} = e_{11} + e_{12}, \quad f_{od} = e_{21} - e_{11}
\]
where \( e_{ij} \) is the matrix in \( \mathfrak{g}(2|1) \) which has 1 in the \( ij \)-entry and 0 in other entries. Then \( h, e, f \) are even elements and \( e_{od}, f_{od} \) are odd elements. Lie brackets between generators are
\[
[h, e_{od}] = e_{od}, \quad [h, f_{od}] = -f_{od}, \quad [e_{od}, f_{od}] = [f_{od}, e_{od}] = -h, \quad [e_{od}, f] = -f_{od},
\]
\[
[f_{od}, e] = -e_{od}, \quad [e_{od}, e_{od}] = 2e, \quad [f_{od}, f_{od}] = -2f
\]
The even supersymmetric bilinear form we consider satisfies
\[
(h|h) = 2(e|f) = 2, \quad (e_{od}|f_{od}) = -2.
\]
Take the Lax operator
\[ \mathcal{L} = k\partial + \frac{1}{2} f_{od} \otimes e_{od} + \frac{1}{2} h \otimes h - \frac{1}{2} e_{od} \otimes f_{od} + e \otimes f - f \otimes 1 \]
and \( V_f = \mathbb{C}e \oplus \mathbb{C}e_{od} \). We can check that \( m = V_f \oplus [n, f] \). Suppose \( X = e_{od} \otimes r_{od} + e \otimes r \) for \( r_{od} \in V_f(p)_1 \) and \( r \in V_f(p)_0 \) satisfies
\[ e^{adX} \mathcal{L} = \mathcal{L}^\text{can} \in \mathbb{C} \otimes V_f \otimes V_f(p). \]
The terms with \( adh \)-eigenvalue \(-1\) in (3.21) are
\[ \frac{1}{2} f_{od} \otimes e_{od} - [e_{od}, f] \otimes r_{od} = 0. \]
Hence \( r_{od} = -\frac{1}{2} e_{od} \). The terms with \( adh \)-eigenvalue \( 0 \) in (3.21) are
\[ \frac{1}{2} h \otimes h + h \otimes (-\frac{1}{2} e_{od} r_{od} - r) + \frac{1}{2} [e_{od}, f] \otimes r^2_{od} = 0. \]
Since \( e_{od} r_{od} = r^2_{od} = 0 \), we have \( r = \frac{1}{2} h \). Hence \( X = -\frac{1}{2} e_{od} \otimes e_{od} + \frac{1}{2} e \otimes h \) and, by direct computations,
\[ e^{adX} \mathcal{L} = k\partial + e_{od} \otimes ( -\frac{1}{2} f_{od} + \frac{k}{2} \partial e_{od} + \frac{1}{4} h e_{od} ) + e \otimes ( f + \frac{1}{2} f_{od} e_{od} - \frac{1}{4} h^2 + \frac{k}{4} e_{od} \partial e_{od} - \frac{k}{2} \partial h ) - f \otimes 1. \]
If we denote \( \phi_{od} = -\frac{1}{2} f_{od} + \frac{k}{2} \partial e_{od} + \frac{1}{4} h e_{od} \) and \( \phi_{ev} = f + \frac{1}{2} f_{od} e_{od} - \frac{1}{4} h^2 + \frac{k}{4} e_{od} \partial e_{od} - \frac{k}{2} \partial h \) then \( \mathcal{W}(g, f, k) = \mathbb{C}_{\text{diff}}[\phi_{od}, \phi_{ev}] \).

**Example 3.16.** Let \( g = \mathfrak{sl}(2|1) \) and take \( e = e_{12} \) and \( f = e_{21} \). Consider the Lax operator
\[ \mathcal{L} = k\partial + Q - f \otimes 1 = k\partial + e \otimes f + e_{11} \otimes e_{11} - e_{12} \otimes e_{21} + \frac{1}{2} h \otimes h - \frac{1}{2} \tau \otimes \tau - f \otimes 1, \]
where \( h = [e, f] = e_{11} - e_{22} \) and \( \tau = e_{11} + e_{22} + 2e_{12} \in \ker ad f \). Fix
\[ V_f = \mathbb{C}e_{12} \oplus \mathbb{C}e_{11} \oplus \mathbb{C}e_{22} \oplus \mathbb{C} \tau \]
so that \( [n, f] \oplus V_f = m \). Let us take
\[ X = e_{11} \otimes X_{11} + e_{12} \otimes X_{12} + e \otimes X_{12} \]
such that \( e^{adX} \mathcal{L} \in \partial \otimes V_f \otimes V_f(p) \). In order to vanish degree \( \frac{1}{2} \)-part of \( e^{adX} \mathcal{L} \), which is
\[ -e_{11} \otimes e_{11} + e_{21} \otimes e_{12} + e_{21} \otimes X_{11} - e_{11} \otimes X_{12}, \]
set \( X_{11} = -e_{12} \) and \( X_{12} = -e_{11} \). Moreover, we want degree 0-part of \( e^{adX} \mathcal{L} \) lies in \( \partial \otimes \mathbb{C} \tau \otimes V_f(p) \). In other words,
\[
\mathcal{L}_0 = \left[ X_{1/2}, \mathcal{L}_{-1/2} \right] + \left[ X_{1/2}, [X_{1/2}, f] \right] = k\partial + \frac{1}{2} h \otimes h - \frac{1}{2} \tau \otimes \tau - \frac{1}{2} (h + \tau) \otimes e_{11} X_{11} - \frac{1}{2} (h - \tau) \otimes e_{12} X_{12} - h \otimes X_{12} - \frac{1}{2} [e_{12}, [e_{11}, f]] \otimes X_{11} X_{12} - \frac{1}{2} [e_{11}, [e_{12}, f]] \otimes X_{12} X_{11} = k\partial + h \otimes \left( \frac{1}{2} h - X_{12} \right) + \tau \otimes (-\frac{1}{17} + \frac{1}{2} e_{11} e_{12}) \in \partial \otimes \mathbb{C} \tau \otimes V_f(p). \]
Here, \( L_l \) and \( X_l \) denote the degree \( l \)-parts of \( L \) and \( X \), respectively. Hence, we fix \( X_{12} = \frac{1}{2} h \). In other words, we have

\[
X = e_{12} \otimes \frac{1}{2} h - e_{12} \otimes e_{11} - e_{11} \otimes e_{12}.
\]

Now, we compute degree 1/2-part and 1-part of \( e^{\text{ad}X} L \) as follows. For degree 1/2-part, we have

\[
L_{1/2} = e_{11} \otimes e_{11} - e_{12} \otimes e_{21},
\]

\[
[X_{1/2}, L_0] = e_{12} \otimes (k\partial e_{11} + \frac{1}{2} e_{11}(h - \tau)) + e_{11} \otimes (k\partial e_{12} + \frac{1}{2} e_{12}(h + \tau)),
\]

\[
[X_1, L_{-1/2}] = e_{12} \otimes \frac{1}{2} e_{11}h + e_{11} \otimes \frac{1}{2} e_{12}h,
\]

\[
\frac{1}{2} [X_{1/2}, [X_{1/2}, L_{-1/2}]] + \frac{1}{2} [X_{1/2}, [X_1, L_{-1}]] = -\frac{1}{4} e_{12} \otimes he_{11} - \frac{1}{4} e_{11} \otimes he_{12},
\]

By adding all the formulas, we get 1/2-degree part of \( e^{\text{ad}X} L \):

\[
e_{11} \otimes (e_{11} + k\partial e_{12} + \frac{1}{2} e_{12}(h + \tau)) + e_{12} \otimes (-e_{21} + k\partial e_{11} + \frac{1}{2} e_{11}(h - \tau)).
\]

Similarly, degree 1-part of \( e^{\text{ad}X} L \) is obtained by adding the following formulas:

\[
L_1 = e \otimes f,
\]

\[
[X_{1/2}, L_1] = e \otimes (-e_{11}e_{11} + e_{21}e_{12}),
\]

\[
[X_1, L_0] = e \otimes (-\frac{k}{2} \partial h - \frac{1}{2} h^2),
\]

\[
\frac{1}{2} ([X_{1/2}, [X_{1/2}, L_0]] + [X_{1/2}, [X_1, L_{-1/2}]] = e \otimes \frac{1}{2} (-k\partial e_{11}e_{12} - k\partial e_{12}e_{11} + \tau e_{11}e_{12}),
\]

\[
\frac{1}{2} ([X_1, [X_{1/2}, L_{-1/2}]] + [X_1, [X_1, L_{-1}]] = e \otimes \frac{1}{4} h^2,
\]

\[
\frac{1}{6} ([X_{1/2}, [X_{1/2}, [X_{1/2}, L_{-1/2}]] + [X_{1/2}, [X_{1/2}, [X_1, L_{-1}]]] + [X_{1/2}, [X_1, [X_{1/2}, L_{-1}]]] + [X_1, [X_{1/2}, [X_{1/2}, L_{-1}]]] = 0.
\]

Hence the degree 1 part of \( e^{\text{ad}X} L \) is

\[
e \otimes (f - e_{11}e_{11} + e_{21}e_{12} - \frac{1}{4} h^2 - \frac{k}{2} \partial h - \frac{k}{2} \partial e_{11}e_{12} - \frac{k}{2} \partial e_{12}e_{11} + \frac{1}{2} \tau e_{11}e_{12}).
\]

If we denote

\[
\phi_t = -\frac{1}{2} \tau + \frac{1}{2} e_{11}e_{12},
\]

\[
\phi_{11} = e_{11} + k\partial e_{12} + \frac{1}{2} e_{12}(h + \tau),
\]

\[
\phi_{12} = -e_{21} + k\partial e_{11} + \frac{1}{2} e_{11}(h - \tau),
\]

\[
\phi_e = f - e_{11}e_{11} + e_{21}e_{12} - \frac{1}{4} h^2 - \frac{k}{2} \partial h - \frac{k}{2} \partial e_{11}e_{12} - \frac{k}{2} \partial e_{12}e_{11} + \frac{1}{2} \tau e_{11}e_{12}
\]

then

\[
\mathcal{W}(g, f, k) = \mathcal{C}_{\text{diff}}[\phi_t, \phi_{11}, \phi_{12}, \phi_e].
\]
Here, we note that
\[ \phi_e = -kL - 2\phi_T^2, \]
for the energy momentum field \( L \), which is defined in Proposition 3.10.

Now, we introduce another equivalent way to get a generating set of a \( W \)-superalgebra, which will be used in the later sections to describe bi-Poisson structures via Lax operators. We consider the \textit{sign twisting} linear map
\[
(3.24) \quad \sigma : \mathfrak{g} \otimes \mathcal{V}_\mathcal{T}(\mathfrak{p}) \rightarrow \mathfrak{g} \otimes \mathcal{V}_\mathcal{T}(\mathfrak{p}), \quad a \otimes F \mapsto (a \otimes F)^\sigma,
\]
where \((a \otimes F)^\sigma \equiv (-1)^{p(a)p(F)} a \otimes F\).

\textbf{Remark 3.17.} One can check that the map \((3.24)\) induces a Lie algebra automorphism on \((\mathfrak{g} \otimes \mathcal{V}_\mathcal{T}(\mathfrak{p}))_0\). Moreover, if we consider the Lie algebra \((\mathcal{V}_\mathcal{T}(\mathfrak{p}) \otimes \mathfrak{g})_0\) with the bracket \([F \otimes a, G \otimes b] = (-1)^{p(a)p(b)} FG \otimes [a, b]\) then there is a natural isomorphism
\[
\phi : (\mathfrak{g} \otimes \mathcal{V}_\mathcal{T}(\mathfrak{p}))_0 \rightarrow (\mathcal{V}_\mathcal{T}(\mathfrak{p}) \otimes \mathfrak{g})_0
\]
defined by \(\phi((a \otimes F)^\sigma) = F \otimes a\). In other words, applying the map \(\sigma\) to \((\mathfrak{g} \otimes \mathcal{V}_\mathcal{T}(\mathfrak{p}))_0\) is equivalent to consider the space \((\mathcal{V}_\mathcal{T}(\mathfrak{p}) \otimes \mathfrak{g})_0\).

\textbf{Proposition 3.18.} For a Lax operator \( L = k\partial + \sum_{i \in I} q_i \otimes Q^i - f \otimes 1, \) we denote
\[
L^\sigma = k\partial + \sum_{i \in I} (-1)^{p(i)} q_i \otimes Q^i - f \otimes 1 \in (\mathfrak{g}[e] \otimes \mathcal{V}_\mathcal{T}(\mathfrak{p}))_0,
\]
where \(p(i)\) denotes the parity \(p(q_i)\).

1. Two elements \(q, q' \in \mathcal{F}_{g,f}\) are gauge equivalent if and only if \(q^\sigma, q'^\sigma\) are gauge equivalent.
2. Recall that we denote the universal Lax operator by \( \mathcal{L} = k\partial + q_{univ}^\sigma - f \otimes 1 \) and the operator gives rise to generators of \( W(\mathfrak{g}, f, k) \) (see Proposition 3.14). By (1), we can find generators of \( W(\mathfrak{g}, f, k) \) using sign twisted universal Lax operator
\[
\mathcal{L}^\sigma = k\partial + q_{univ}^\sigma - f \otimes 1.
\]

\textbf{Proof.} The proof follows from the fact that
\[
e^{\text{ad}(n\otimes r)}(L^\sigma) = L'^\sigma \quad \text{if and only if} \quad e^{\text{ad}(n\otimes r)}(L) = L'.
\]

For (2), we note that
\[
q_{univ}^\text{can} = \sum_{i \in \mathcal{J}} q_i \otimes w^i \quad \text{if and only if} \quad (q_{univ}^\sigma)^\text{can} = \sum_{i \in \mathcal{J}} p(i) q_i \otimes w^i.
\]
Since \(C_{\text{diff}}[w^i | i \in \mathcal{J}] = C_{\text{diff}}[p(i)w^i | i \in \mathcal{J}]\), we proved (2). \(\square\)

\section*{4. Lax operators and Lie brackets on \( W(\mathfrak{g}, f, k) / \partial W(\mathfrak{g}, f, k) \)}

\subsection*{4.1. Derivatives on a differential superalgebra.}

Fix the superalgebra of differential polynomials
\[
\mathcal{P} = C_{\text{diff}}[u_i | i \in I].
\]
The index set \( I = I_0 \cup I_1 \) consists of two subindex sets \( I_0 \) and \( I_1 \) such that \( u_i \) is even if \( i \in I_0 \) and is odd if \( i \in I_1 \).
Definition 4.1. Take \( j_1, j_2, \ldots, j_k \in I_1 \) and \( n_1, n_2, \ldots, n_k \in \mathbb{Z}_{\geq 0} \) such that \((j_k, n_k) \neq (j_2, n_2)\) for distinct numbers \( k_1, k_2 \in \{1, \ldots, k\} \). Consider an element
\[
\phi_1 = u_j^{(n_1)} u_j^{(n_2)} \cdots u_j^{(n_k)} \in \mathcal{C}_{\text{diff}}[u_i | i \in I_1],
\]
where \( u_j^{(n)} = \partial^n u_j \) and a monomial
\[
\phi = \phi_0 \phi_1 \in \mathcal{P}, \text{ for } \phi_0 \in \mathcal{C}_{\text{diff}}[u_i | i \in I_0].
\]

(1) The (left) derivative of \( \phi \) with respect to \( u_j^{(n)} \) for \( t = 1, \ldots, k \) is
\[
\frac{\partial \phi}{\partial u_j^{(n)}} = (-1)^{t-1} \phi_0 \cdot u_j^{(n_1)} u_j^{(n_2)} \cdots u_j^{(n_t-1)} u_j^{(n_{t+1})} \cdots u_j^{(n_{k-1})} u_j^{(n_k)}.
\]
If \( u_j^{(n)} \neq u_j^{(n_1)} \) is an odd element then \( \frac{\partial \phi}{\partial u_j^{(n)}} = 0 \). If \( j \in I_0 \), we let \( \frac{\partial \phi}{\partial u_j^{(n)}} = \frac{\partial \phi_0}{\partial u_j^{(n)}} \cdot \phi_1 \).

(2) The right derivative of \( \phi \) with respect to \( u_j^{(n)} \) for \( t = 1, \ldots, k \) is
\[
\frac{\partial_R \phi}{\partial u_j^{(n)}} = (-1)^{k-t} \phi_0 \cdot u_j^{(n_1)} u_j^{(n_2)} \cdots u_j^{(n_{t-1})} u_j^{(n_{t+1})} \cdots u_j^{(n_{k-1})} u_j^{(n_k)}.
\]
If \( u_j^{(n)} \neq u_j^{(n_1)} \) is an odd element then \( \frac{\partial_R \phi}{\partial u_j^{(n)}} = 0 \). If \( j \in I_0 \), we let \( \frac{\partial_R \phi}{\partial u_j^{(n)}} = \frac{\partial \phi_0}{\partial u_j^{(n)}} \).

(3) The (left) variational derivative and right variational derivative of \( \phi \) with respect to \( u_i \) are
\[
\frac{\delta \phi}{\delta u_i} = \sum_{n \in \mathbb{Z}_{\geq 0}} (-\partial)^n \frac{\partial \phi}{\partial u_i^{(n)}}, \quad \frac{\delta_R \phi}{\delta u_i} = \sum_{n \in \mathbb{Z}_{\geq 0}} (-\partial)^n \frac{\partial_R \phi}{\partial u_i^{(n)}}.
\]

(4) The variational derivative of \( \phi \) with respect to \( \{ u_i | i \in I \} \) is
\[
\frac{\delta \phi}{\delta u_i} = \sum_{i \in I} u_i \otimes \frac{\delta \phi}{\delta u_i}.
\]
The derivatives defined in (1), (2), (3), (4) have linearity so that they are well-defined on the set of differential algebra \( \mathcal{P} = \mathcal{C}_{\text{diff}}[u_i | i \in I] \).

Remark 4.2. We have
\[
\frac{\delta \phi}{\delta u_i} = (-1)^{p(u_i)} p(\phi; u_i) \frac{\delta_R \phi}{\delta u_i}.
\]
Hence, for the map \( \sigma \) in (3.24),
\[
\left( \frac{\delta \phi}{\delta u_i} \right)^{\sigma} = \sum_{i \in I} \left( u_i \otimes \frac{\delta \phi}{\delta u_i} \right)^{\sigma} = \sum_{i \in I} u_i \otimes \frac{\delta_R \phi}{\delta u_i}.
\]

Remark 4.3. If \( u_i \in \mathcal{P} \) is an odd element then \( \frac{\partial}{\partial u_i^{(n)}} \) for any \( n \in \mathbb{Z}_{\geq 0} \) is an odd derivation, i.e.
\[
\frac{\partial}{\partial u_i^{(m)}} (FG) = \left( \frac{\partial}{\partial u_i^{(m)}} F \right) \cdot G + (-1)^p(F) \cdot \left( \frac{\partial}{\partial u_i^{(m)}} G \right).
\]

Proposition 4.4. For homogenous elements \( f, g \in \mathcal{P} \), we have
\[
\{ f \ast g \} = \sum_{i, j \in I} C_{i, j}^{f, g} \frac{\partial_R g}{\partial u_i^{(m)}} (\lambda + \partial)^n \{ u_i \lambda + \partial u_j \} \ast (-\lambda - \partial)^m \frac{\partial f}{\partial u_i^{(m)}}
\]
where \( C_{i, j}^{f, g} = (-1)^{p(f)p(g)+p(i)p(j)} \). Note that
\[
\{ a \lambda + \partial b \} \ast c := \sum_{i \in \mathbb{Z}_{\geq 0}} \frac{1}{m!} a_{(i)} b (\lambda + \partial)^n c \quad \text{for} \quad a, b, c \in \mathcal{P}.
\]
Proof. The formula (4.2) follows from sesquilinearities and Leibniz rules of λ-brackets. The only part we have to be careful is the constant factor $C_{i,j}^{I,g}$ in (4.2). One can see that

$$C_{i,j}^{I,g} = (-1)^{|p(i) + p(f)}(p(j) + p(g)) \cdot C_{i,j}^{I} \cdot C_{j,i}^{g}$$

for $C_{i,j}^{I} = (-1)^{|p(i) + p(f)}(p(j) + p(g))$ and $C_{j,i}^{g} = (-1)^{|p(g) + p(j)}(p(i) + p(g))$. Note that switching the position of $\frac{\partial f}{\partial u_i(m)}$ and $\frac{\partial g}{\partial u_j(m)}$ gives rise to the $(-1)^{|p(i) + p(f)}(p(j) + p(g))$ in $C_{i,j}^{I,g}$ and switching the position of $u_j$ and $\frac{\partial g}{\partial u_i(m)}$ (resp. $u_i$ and $\frac{\partial f}{\partial u_j(m)}$) gives rise to the constant factor $C_{i,j}^{I}$ (resp. $C_{j,i}^{g}$).

\[\square\]

**Proposition 4.5.**

(1) For any variable $u_i^{(n)}$ in $\mathcal{P}$, we have

$$\left(\frac{\partial}{\partial u_i^{(n)}}, \partial \right) = \left(\frac{\partial}{\partial u_i^{(n-1)}}, \partial \right).$$

(2) Let $\phi \in \mathcal{P}$. Then $\frac{\delta}{\delta u_i} \phi = 0$ for any $i \in I$.

Proof. (1) For any $j \in I$ and $m \in \mathbb{Z}_{\geq 0}$, we can check that

$$\left(\frac{\partial}{\partial u_j^{(m)}}, \partial \right) (u_j^{(m)}) = \left(\frac{\partial}{\partial u_j^{(m-1)}}, \partial \right) (u_j^{(m)})$$

Suppose we get the same element in $\mathcal{P}$ when we apply $F$ (resp. $G$) to the LHS and RHS of the equation (4.3). If $i \in I_1$ then

$$\left(\frac{\partial}{\partial u_i^{(n)}}, \partial \right) (FG) = \left(\frac{\partial}{\partial u_i^{(n)}}, \partial \right) G + (-1)^{|p(F)\partial(F) + p(G)\partial(G)} + (-1)^{|p(F)\partial(G) + p(G)\partial(F)}$$

By (1), we have

$$\sum_{i \in \mathbb{Z}_{\geq 0}} (-\partial)^n \frac{\partial}{\partial u_i^{(n)}} \partial = -\sum_{i \in \mathbb{Z}_{\geq 0}} (-\partial)^{n+1} \frac{\partial}{\partial u_i^{(n-1)}} + \sum_{i \in \mathbb{Z}_{\geq 1}} (-\partial)^n \frac{\partial}{\partial u_i^{(n-1)}} = 0$$

\[\square\]

4.2. Lie brackets on $\mathcal{W}(g, f, k) / \partial \mathcal{W}(g, f, k)$.

Now we are ready to find Lie superalgebra structures on a quotient space of $W$-algebra via Lax operators. In this section, we assume $g$ is a finite dimensional simple Lie superalgebra with even supersymmetric bilinear invariant form $(\lambda)$.

Let us denote by $g((z^{-1})) := g[z] \oplus z^{-1}g[[z^{-1}]]$ the Lie superalgebra endowed with the bracket $[az^m, bz^n] = [a, b] z^{m+n}$ where $[a, b]$ is the Lie bracket on $g$. The vector superspace $g((z^{-1})) \otimes \mathcal{V}_2(p)$ is a Lie superalgebra endowed with the bracket such that

$$[az^m \otimes F, bz^n \otimes G] = (-1)^{|p(b)\partial(F)[az^m, bz^n] \otimes FG}$$

for $a, b \in g$, $n, m \in \mathbb{Z}$ and $F, G \in \mathcal{V}_2(p)$. We extend the Lie bracket to that on $\mathbb{C} \partial \times g((z^{-1})) \otimes \mathcal{V}_2(p)$ by considering $\mathbb{C} \partial$ as the trivial Lie algebra.
Consider the universal Lax operator associated to \( g \) and \( \Lambda \):

\[
L(\Lambda) = k \partial + q_{\text{univ}} - \Lambda \otimes 1 \in \mathbb{C} \partial \ltimes g((z^{-1})) \otimes \mathcal{V}_z(p),
\]

where \( q_{\text{univ}} \) is that in (3.3) and \( \Lambda = f + zs \) for a nonzero even element \( s \) in \( g(d) \). Here, \( d \) is the largest eigenvalue such that \( g(d) \neq 0 \). The operator acts on \( g((z^{-1})) \otimes \mathcal{V}_z(p) \) by the adjoint action. Note that we assume the even part of \( g(d) \neq 0 \) (see Remark 2.14).

**Remark 4.6.** For the operator \( L \) associated to \( g \) and \( f \in \mathfrak{g} \), we have \( L(\Lambda) = L - zs \otimes 1 \). Denote by

\[
L_{[1]}(\Lambda) := L \quad \text{and} \quad L_{[2]}(\Lambda) := -s \otimes 1.
\]

The term \( L_{[2]}(\Lambda) \) do the crucial role to define bi-Poisson structures on \( \mathcal{W}(g, f, k) \). However, this part is not important when we find generators of \( \mathcal{W}(g, f, k) \).

Also, using the map (3.24), we denote the *sign twisted universal Lax operator* by

\[
L^\sigma(\Lambda) = k \partial + q_{\text{univ}} - \Lambda \otimes 1 \in \mathbb{C} \partial \ltimes g((z^{-1})) \otimes \mathcal{V}_z(p).
\]

Now, via \( L^\sigma(\Lambda) \), we aim to define Lie brackets on \( \mathcal{W}(g, f, k)/\partial \mathcal{W}(g, f, k) \) for the subspace \( \partial \mathcal{W}(g, f, k) := \{ \partial W \mid W \in \mathcal{W}(g, f, k) \} \subset \mathcal{W}(g, f, k) \). (In Remark 4.10 we explain why we consider \( L^\sigma(\Lambda) \) instead of \( L(\Lambda) \).)

The following notion is analogous to the notion in (2) of Definition 2.27.

**Definition 4.7.** Let \( \mathcal{V} \) be a differential superalgebra with the derivation \( \partial \) and denote \( \partial \mathcal{V} := \{ \partial V \mid V \in \mathcal{V} \} \). For the map

\[
\int : \mathcal{V} \to \mathcal{V}/\partial \mathcal{V}, \quad V \mapsto V + \partial \mathcal{V} := \int V,
\]

we call \( \int V \) the local functional of \( V \in \mathcal{V} \).

Let the bilinear form \( (\cdot, \cdot) : g((z^{-1})) \otimes \mathcal{V}_z(p) \times g((z^{-1})) \otimes \mathcal{V}_z(p) \to \mathcal{V}_z(p) \) be defined by

\[
(a z^m \otimes F) \cdot (b z^n \otimes G) = (-1)^{p(b)p(F)}(a | b) \delta_{m,n,0} F G.
\]

Define two bilinear brackets \( \{,\} : \mathcal{V}_z(p) \times \mathcal{V}_z(p) \to \mathcal{V}_z(p) \), for \( i = 1, 2 \), by

\[
\{ \phi, \psi \}_1 = \left( \delta \phi \bigg| \frac{\delta}{\delta q} \bigg| L^\sigma(\Lambda) \right), \quad \{ \phi, \psi \}_2 = -\left( \delta \phi \bigg| \frac{\delta}{\delta q} \bigg| z^{-1} L^\sigma(\Lambda) \right) = -\left( \delta \phi \bigg| \bigg| \frac{\delta}{\delta q} \bigg| L^\sigma(\Lambda) \right),
\]

where \( q = (q^i) \in \mathfrak{g} \) is the basis of \( \mathfrak{g} \) in (3.3) and \( z^{-1}(a z^m \otimes F) := a z^{m-1} \otimes F \in g((z^{-1})) \otimes \mathcal{V}(p) \).

By Proposition 4.5 (2), we can consider the induced bilinear brackets on \( \mathcal{W}_z(p)/\partial \mathcal{W}_z(p) \)

\[
\{,\} : \mathcal{V}_z(p)/\partial \mathcal{V}_z(p) \times \mathcal{V}_z(p)/\partial \mathcal{V}_z(p) \to \mathcal{V}_z(p)/\partial \mathcal{V}_z(p), \quad i = 1, 2
\]

such that \( [\int \phi, \int \psi]_i := \int \{ \phi, \psi \}_i \).

**Lemma 4.8.** Let \( q^i, r \in \mathcal{F}_g, f = (m^0 \otimes \mathcal{V}_z(p))_0 \). For \( \epsilon \in \mathbb{C} \), we have

\[
\left. \frac{\int \phi(q^i + \epsilon r) - \int \phi(q^i)}{\epsilon} \right|_{\epsilon = 0} = \int \left( r^a \bigg| \frac{\delta \phi(q^i)}{\delta q} \right).
\]

**Proof.** If \( r \in (m^0)_0 \otimes (\mathcal{V}_z(p))_0 \) then the proof is similar to the non-super algebras cases in \( \mathbb{N} \). Suppose \( r = q_i \otimes r^i \) for \( q_i \in (m^+) \). Then

\[
\left( r^a \bigg| \frac{\delta \phi(q^i)}{\delta q} \right) = \left( -q_i \otimes r^i \bigg| \frac{\delta \phi(q^i)}{\delta q} \right) = r^i (q_i \otimes 1 | q^i \otimes 1) \frac{\delta \phi(q^i)}{\delta q} = r^i \frac{\delta \phi(q^i)}{\delta q}.
\]
Lemma 4.10. For \( \phi = \partial^{n_1}(q^{i_1}) \partial^{n_2}(q^{i_2}) \cdots \partial^{n_t}(q^{i_t}) \), if we denote the first \( k - 1 \) terms in \( \phi \) by \( \phi_k = \partial^{n_1}(q^{i_1}) \partial^{n_2}(q^{i_2}) \cdots \partial^{n_k-1}(q^{i_k-1}) \) and the last \( t - k \) terms in \( \phi \) by \( \psi_k = \partial^{n_{k+1}}(q^{i_{k+1}}) \partial^{n_{k+2}}(q^{i_{k+2}}) \cdots \partial^{n_t}(q^{i_t}) \) for \( k = 0, 1, \ldots, t \) then

\[
\phi = \phi_k \cdot \partial^{n_k} q^{i_k} \cdot \psi_k
\]

for any \( k \). Now, for such \( \phi \), let \( r = q_i \otimes r^i \) and \( q' = \sum_{j \in I} q_j \otimes Q^j \). Then we have

\[
\frac{\phi(q'+r)-\phi(q')}{r} \bigg|_{r=0} = \sum_{k=0}^{t} \phi_k(q') \cdot \partial^{n_k} q^{i_k}(r) \cdot \psi_k(q')
\]

(4.11)

The last equality holds since \( \left(q_i|q^j\right) = \delta_{ij} \). Since \( \int \partial(F)G = \int -F \partial(G) \), the lemma is proved by (4.10) and (4.11).

Proposition 4.9. The brackets \( [\cdot, \cdot] \) for \( i = 1, 2 \) satisfy skew-symmetry.

Proof. For \( \phi = a \otimes F \) and \( \psi = b \otimes G \) in \( g \otimes V_T(p) \), we have

\[
\int (a \otimes F)[b \otimes G, \partial] = \int (a \otimes F)[b \otimes \partial G] = - \int (-1)^{p(\phi)p(\psi)} (b \otimes \partial G|a \otimes F) = - \int (-1)^{p(\phi)p(\psi)} (b \otimes G|a \otimes \partial F).
\]

Hence, by invariance and skew-symmetry of the bilinear form on \( g \otimes V_T(p) \), we have

\[
\int \phi, \int \psi \iota = \int \{\phi, \psi\} \iota = -(-1)^{p(\phi)p(\psi)} \int \{\psi, \phi\} \iota = -(-1)^{p(\phi)p(\psi)} \int \psi, \int \phi \iota, \quad i = 1, 2.
\]

Lemma 4.10. For \( \psi \in V_T(p) \), we have:

\[
\int q^i, \int \psi \iota_1 = \sum_{j \in I} \int ([q^i, q^j] - (q^i|q^j) k \partial) \frac{\delta_{m^j}}{\delta q^j} \psi,
\]

\[
\int q^i, \int \psi \iota_2 = \sum_{j \in I} \int ([q^i, q^j]|s) \frac{\delta_{m^j}}{\delta q^j} \psi,
\]

where \( \{q^j|j \in I\} \) is a basis of \( p \).

Proof. By expanding the LHS of \( \int \{q^i, \psi\} \iota_1 \), we obtain

(4.12)

\[
\int q^i, \int \psi \iota_1 = \sum_{j \in I} \int ([q^i \otimes \frac{\delta_{m^j}}{\delta q^j} \psi, k \partial + p(j')q_{j'} \otimes q^j - f \otimes 1])
\]

\[
= \sum_{j \in I} \int (q^i|q^j)(-k \partial) \frac{\delta_{m^j}}{\delta q^j} \psi + \sum_{j, j' \in I} \int q^j(q_{j'}([q^i, q^j]|f)) \frac{\delta_{m^j}}{\delta q^j} \psi - \sum_{j \in I} ([q^i, q^j]|f) \frac{\delta_{m^j}}{\delta q^j} \psi,
\]

where \( \{q_j|j \in I\} \) is the basis of \( m^+ \) such that \( (q_j|q^j') = \delta_{j, j'} \) and \( p(j') \) is the parity of \( q^j' \). In \( V_T(p) \), we have

\[
\sum_{j \in I} ([q^i, q^j]|f) \frac{\delta_{m^j}}{\delta q^j} \psi = \sum_{j, j' \in I} \int ([q^i, q^j]|q^j') q^j' \frac{\delta_{m^j}}{\delta q^j} \psi - \sum_{j \in I} ([q^i, q^j]|f) \frac{\delta_{m^j}}{\delta q^j} \psi.
\]

Hence the first equality is proved. The second equality also can be proved similarly.

Proposition 4.11. We have the following equations:

(4.13)

\[
\int \phi, \int \psi \iota_1 = \sum_{i, j \in I} \int \frac{\delta_{m^j}}{\delta q^j} \phi([q^i, q^j] - (q^i|q^j) k \partial) \frac{\delta_{m^j}}{\delta q^j} \psi,
\]

\[
\int \phi, \int \psi \iota_2 = \sum_{i, j \in I} \int \frac{\delta_{m^j}}{\delta q^j} \phi([q^i, q^j]|s) \frac{\delta_{m^j}}{\delta q^j} \psi.
\]
Proof. Observe that
\[
\left( \frac{\delta}{\delta g} \right) | a \otimes F = \sum_{i \in I} \frac{\delta g_i}{\delta q_i} (q_i^t \otimes 1) a \otimes F
\]
for any \( a \otimes F \in g \otimes \mathcal{V}_\mathcal{Z}(p) \). Since \( \frac{\delta}{\delta q} q^t = q^t \otimes 1 \), we have
\[
[\phi, \psi]_t = \int \sum_{i \in I} \frac{\delta g_i}{\delta q_i} \left( q_i^t \otimes 1 \left[ \frac{\delta}{\delta q} \psi, L^\sigma_{[i]}(\Lambda) \right] \right) = \int \sum_{i \in I} \frac{\delta g_i}{\delta q_i} (q_i^t, \psi)_t
\]
for \( t = 1, 2 \). Now, by the proof of Lemma 4.10, we can see (4.13).

By previous propositions, we know how to compute the brackets \([ , ]_i\), \((i = 1, 2)\), defined on \(\mathcal{V}_\mathcal{Z}(p)/\partial \mathcal{V}_\mathcal{Z}(p)\). The next thing we want to show is that the brackets can be understood as brackets on \(\mathcal{W}(g, f, k)/\partial \mathcal{W}(g, f, k):\)
\[
[ , ] : \mathcal{W}(g, f, k)/\partial \mathcal{W}(g, f, k) \times \mathcal{W}(g, f, k)/\partial \mathcal{W}(g, f, k) \rightarrow \mathcal{W}(g, f, k)/\partial \mathcal{W}(g, f, k).
\]
We provide proofs in two different ways. One (Proposition 4.13) is by the definition \(W\)-algebras in Section [3] and the other one (Proposition 4.14) is purely algebraic. Note that the first proof is inspired from [1] and the second one is inspired form [7].

Lemma 4.12. Let \(\mathcal{V}\) be a superalgebra of differential polynomials with both even generators and odd generators. If \(G, G' \in \mathcal{V}\) satisfy
\[
\int FG = \int FG' \quad \text{or} \quad \overline{FG} = \overline{FG'} \quad \text{in} \ \mathcal{V}/\partial \mathcal{V}
\]
for all \(F \in \mathcal{V}_0\) or all \(F \in \mathcal{V}_1\) then \(G = G'\).

Proof. Let us denote \(\mathcal{V} = \mathcal{V}_0 \otimes \mathcal{V}_1\), where \(\mathcal{V}_0 = \mathbb{C}_{\text{diff}}[u_i | i = 1, 2, \cdots, k_0]\) and \(\mathcal{V}_1 = \mathbb{C}_{\text{diff}}[v_i | i = 1, 2, \cdots, k_1]\) for even variables \(u_i\) and odd variables \(v_i\).

It is enough to show that if \(\int u_i^{(n)} G = 0\) (resp. \(\int v_i^{(n)} G = 0\)) for any even (resp. odd) variable \(u_i^{(n)}\) (resp. \(v_i^{(n)}\)) then \(G = 0\).

Let us first show that if \(\int u_i^{(n)} G = 0\) for any even variable \(u_i^{(n)}\) then \(G = 0\). If \(G \in \mathbb{C}^x\) then \(u_1 G \not\in \partial \mathcal{V}\). Suppose \(G\) is not a constant. Take an integer \(m \in \mathbb{Z}_{\geq 0}\) such that no monomial in \(G\) has terms with \(u_i^{(n)}\) for \(n \geq m\). Then \(u_1^{(m+1)} G \not\in \partial \mathcal{V}\). That is because if \(\partial G_1 = u_1^{(m+1)} G\) then \(G_1\) should have the term \(u_1^{(m+1)} G_2\) for a nonconstant element \(G_2\). Hence \(\partial G_1 = u_1^{(m+1)} G\) has the term with \(u_1^{(m+2)}\). This is a contradiction to our assumption and \(G\) should be 0.

Suppose \(\int v_i^{(n)} G = 0\) for any odd variable \(v_i^{(n)}\) then \(G = 0\). We have \(G \not\in \mathbb{C}^x\) since otherwise \(v_1 G \not\in \partial \mathcal{V}\). Similarly to the previous case, for a nonconstant element \(G\), let us take an integer \(m \in \mathbb{Z}_{\geq 0}\) such that no monomial in \(G\) has terms with \(v_i^{(n)}\) for \(n \geq m\). Then \(v_1^{(m+2)} G\) is not in \(\partial \mathcal{V}\). Here we note that \(v_1^{(m+1)} G\) can be in \(\partial \mathcal{V}\), for example \(\partial (v_1^{(m)} v_1^{(m-1)}) = v_1^{(m+1)} v_1^{(m-1)}\). Hence \(G = 0\).

Proposition 4.13. The brackets \([4, 4]\) induce brackets on \(\mathcal{W}(g, f, k)/\partial \mathcal{W}(g, f, k)\).

Proof. It is enough to show that for \(\phi, \psi \in \mathcal{W}(g, f, k)\), we have \(\{\phi, \psi\}_1\) and \(\{\phi, \psi\}_2\) are also in \(\mathcal{W}(g, f, k)\).

Let \(X ∈ (n \otimes \mathcal{V}_\mathcal{Z}(p))_0\) and two elements \(q, q' \in F_{g, f}\) be gauge equivalent by \(X\), i.e.
\[
k\partial + q^\sigma - f \otimes 1 = e^{adX^\sigma} (k\partial + q^\sigma - f \otimes 1).
\]
Note that, since \([s, n] = 0\), we have

\[
(4.15) \quad k\partial + q'^\sigma - \Lambda \otimes 1 = e^{adX^\sigma}(k\partial + q'^\sigma - \Lambda \otimes 1).
\]

If \(r'^\sigma := e^{-adX^\sigma}r'^\sigma \in \mathcal{F}_{g,f}\) for some \(r' \in \mathcal{F}_{g,f}\), we have

\[
\phi(q) = \phi(q') \quad \text{and} \quad \phi(q' + er') = \phi(q + er) \quad \text{for} \quad e \in \mathbb{C}.
\]

Hence, by Lemma 4.8, the equation \(\phi(q + r) - \phi(q) = \phi(q' + r') - \phi(q')\) implies

\[
(4.16) \quad \int \left( r'^\sigma \left| \frac{\delta\phi(q')}{\delta q} \right| \right) = \int \frac{\phi(q' + er') - \phi(q')}{e} = \int \frac{\phi(q + er) - \phi(q)}{e} = \int \left( r'^\sigma \left| e^{adX^\sigma} \frac{\delta\phi(q)}{\delta q} \right| \right).
\]

By Lemma 4.12 and (4.16), we have \(\frac{\delta\phi(q')}{\delta q} = e^{adX^\sigma} \frac{\delta\phi(q)}{\delta q}\). It is obvious that \(\psi\) has the same property. Hence we have

\[
(4.17) \quad \{\phi, \psi\}_1(q') = \left( \frac{\delta\phi(q')}{\delta q}, k\partial + q'^\sigma - 1 \otimes \Lambda \right) = \left( e^{adX^\sigma} \frac{\delta\phi(q)}{\delta q}, e^{adX^\sigma}(k\partial + q'^\sigma - 1 \otimes \Lambda) \right) = \{\phi, \psi\}_1(q).
\]

On the other hand, since \(e^{adX^\sigma}(s \otimes 1) = s \otimes 1\), we can prove that \(\{\phi, \psi\}_2(q') = \{\phi, \psi\}_2(q)\) by the same argument. \(\square\)

The classical W-superalgebra \(\mathcal{W}(g, f, k)\) is a PVA endowed with the \(\lambda\)-brackets induced from those on the affine PVA \(S(\mathbb{C}[\partial] \otimes g)\) such that

\[
\{a, b\}_1 = [a, b] + k\lambda(a|b), \quad \{a, b\}_2 = (s|a, b) \quad \text{for} \quad a, b \in g.
\]

Hence there are Lie brackets \([, ]_i, i = 1, 2\), on \(\mathcal{W}(g, f, k)/\partial\mathcal{W}(g, f, k)\) induced by the \(\lambda\)-brackets on \(\mathcal{W}(g, f, k)\). More precisely,

\[
(4.18) \quad [\int W_1, \int W_2]_i = \int \{W_{1\lambda}W_{2\lambda}\}_i|_{\lambda=0}.
\]

**Proposition 4.14.** Brackets \([, ]_i\) in (4.18) and \([, ]_i\) in (4.8) for \(i = 1, 2\), defined on \(\mathcal{W}(g, f, k)/\partial\mathcal{W}(g, f, k)\) are same.

**Proof.** By Proposition 4.4 for \(\phi, \psi \in \mathcal{W}(g, f, k)\), we have

\[
(4.19) \quad \{\phi, \psi\}_i = \sum \sum_{m,n \in \mathbb{Z}_{\geq 0}} C_{i,j}^{\phi, \psi} \frac{\partial\phi}{\partial q^m}(\lambda + \partial)^n([q^i, q^j] + (q^i|q^j)k(\lambda + \partial))(-\lambda - \partial)^m \frac{\partial\psi}{\partial q^m}(\partial)^n
\]

for the sign consideration \(C_{i,j}^{\phi, \psi}\). If we apply \(\lambda = 0\) to (4.19) then

\[
\{\phi, \psi\}_1|_{\lambda=0} = \sum \sum_{m,n \in \mathbb{Z}_{\geq 0}} C_{i,j}^{\phi, \psi} \frac{\partial\phi}{\partial q^m}(\partial)^n([q^i, q^j] + (q^i|q^j)k\partial)(-\partial)^m \frac{\partial\psi}{\partial q^m}(\partial)^n
\]
and

\[
[f \phi, f \psi]_1 = \int \{\phi, \psi\}_1 \mid_{\lambda = 0} = \sum_{i,j \in I, m,n \in \mathbb{Z}_{\geq 0}} \int C_{i,j}^{\phi, \psi} \left( \left( -\partial \right)^n \frac{\partial q^i}{\partial R^q} \left( q^j, q^m \right) \left( q^i, q^n \right) \right) \left( -\partial \right)^m \frac{\partial \phi}{\partial q^i} \mid_{\lambda = 0} = \sum_{i,j \in I, m,n \in \mathbb{Z}_{\geq 0}} \int \left( -\partial \right)^m \frac{\partial \phi}{\partial q^i} \mid_{\lambda = 0} \left( q^i, q^n \right) \left( -\partial \right)^n \frac{\partial \psi}{\partial q^j} \mid_{\lambda = 0} + \sum_{i,j \in I, m,n \in \mathbb{Z}_{\geq 0}} \int \left( -\partial \right)^m \frac{\partial \phi}{\partial q^i} \mid_{\lambda = 0} \left( q^i, q^n \right) \left( -\partial \right)^n \frac{\partial \psi}{\partial q^j} \mid_{\lambda = 0}.
\]

(4.20)

Hence \([f \phi, f \psi]_1 = [f \phi, f \psi]_1 = [f \phi, f \psi]_1\). By same arguments, we have \([f \phi, f \psi]_2 = [f \phi, f \psi]_2\). □

**Theorem 4.15.** Brackets [, ] and [ , ] are Lie brackets on \([W, f, k] := W(g, f, k)/\partial W(g, f, k)\).

**Proof.** We know that if \(\{\lambda\}\) is a PVA bracket on \(\mathcal{P}\) then \(\{\lambda\}\mid_{\lambda = 0}\) is a Lie algebra bracket on \(\mathcal{P}/\partial \mathcal{P}\). Hence the theorem directly follows from Proposition 4.14. □

**Remark 4.16.** If we consider \(\mathcal{L}(\Lambda)\) instead of \(\mathcal{L}^\prime(\Lambda)\), we have

\[
[f \phi, f \psi]_{\mathcal{L}(\Lambda)} := \int \left( \frac{\delta \phi}{\delta \lambda} \mid_{\lambda = 0} \left[ \frac{\delta \psi}{\delta \lambda} \mid_{\lambda = 0} \mathcal{L}(\Lambda) \right] \right) = \sum_{i,j \in I, m,n \in \mathbb{Z}_{\geq 0}} \left( -1 \right)^{p(i) + p(j)} \int \frac{\delta \phi}{\delta \lambda} \mid_{\lambda = 0} \left[ \frac{\delta \psi}{\delta \lambda} \mid_{\lambda = 0} \right] \left( q^i, q^n \right) \left( q^j, q^m \right) \left( -\partial \right)^m \frac{\partial \phi}{\partial q^i} \mid_{\lambda = 0} \left( -\partial \right)^n \frac{\partial \psi}{\partial q^j} \mid_{\lambda = 0} + \sum_{i,j \in I, m,n \in \mathbb{Z}_{\geq 0}} \left( -1 \right)^{p(i) + p(j)} \int \frac{\delta \phi}{\delta \lambda} \mid_{\lambda = 0} \left[ \frac{\delta \psi}{\delta \lambda} \mid_{\lambda = 0} \right] \left( q^i, q^n \right) \left( q^j, q^m \right) \left( -\partial \right)^m \frac{\partial \phi}{\partial q^i} \mid_{\lambda = 0} \left( -\partial \right)^n \frac{\partial \psi}{\partial q^j} \mid_{\lambda = 0}.
\]

(4.21)

for bases \(\{q_i\}_{i \in I}\) and \(\{q^i\}_{i \in I}\) of \(p\) and \(p^\perp\) such that \((q_i, q^i) = \delta_{ij}\). In this article, we want to discuss integrable systems associated to a W-superalgebra whose PVA structure induces \([\ , \ ]_1\) more than \([\ , \ ]_1\). Hence we prefer to use \(\mathcal{L}^\prime(\Lambda)\) than \(\mathcal{L}(\Lambda)\) (see also the Remark 5.15).

5. super-Hamiltonian equations and Poisson vertex algebras

Let us introduce super-Hamiltonian equations via Poisson vertex algebras. Recall that infinite dimensional Hamiltonian equation on the even differential algebra \(\mathcal{P} = \mathbb{C}_{\text{diff}}[u_i \mid i \in I]\) is an evolution equation of the form

\[
\frac{du}{dt} = H(\partial) \frac{\delta h}{\delta u}
\]

(5.1)

where

1. the Poisson operator \(H(\partial) = \{H_{ij}(\partial)\}_{i,j \in I, n \in \mathbb{Z}_{\geq 0}}\) is an \(|I| \times |I|\) matrix operator such that

   (i) \(H_{ij}(\partial) = \sum_{n=0}^{N} H_{ij,n} \partial^n\) for \(H_{ij,n} \in \mathbb{C}[\partial^n u_i \mid i \in I, n \in \mathbb{Z}_{\geq 0}]\),

   (ii) if the \(\lambda\)-bracket on \(\mathcal{P}\) is defined by \(\{u_i, \lambda \} = \sum_{n=0}^{N} H_{ij,n} \lambda^n\) then the differential algebra \(\mathcal{P}\) with the induced \(\lambda\)-bracket \(\{ \lambda \} = \sum_{n=0}^{N} H_{ij,n} \lambda^n\) is a Poisson vertex algebra.

2. the Hamiltonian \(h\) is an element in \(\mathcal{P}\).

Moreover, the equation (5.1) can be written with the \(\lambda\)-bracket \(\{ \lambda \} = \sum_{n=0}^{N} H_{ij,n} \lambda^n\) as follows:

\[
\frac{du_i}{dt} = \{h, \lambda \} \mid_{\lambda = 0} \quad i \in I.
\]

For details, we refer to the paper [3].

Analogously, we define super-Hamiltonian systems and integrable systems when \(\mathcal{P}\) is a differential superalgebra.
Definition 5.1. Let \( \mathcal{P} = \mathbb{C}_{\text{diff}}[u_i \mid i \in I] \) be the superalgebra of differential polynomials and let \( \mathcal{P}_0 \) and \( \mathcal{P}_1 \) be the even and odd subspaces of \( \mathcal{P} \).

1. A super-Hamiltonian evolution equation on the differential superalgebra on \( \mathcal{P} \) is an evolution equation of the form
\[
\frac{d\phi}{dt} = \{ h_\lambda \phi \}\big|_{\lambda=0}, \quad \phi \in \mathcal{P}
\]
for some \( h \in \mathcal{P}_0 \).

2. An integral of motion of \( \mathcal{P} \) is the local functional \( \int f \in \int \mathcal{P} \) such that
\[
\int \frac{df}{dt} = 0.
\]

3. If \( \mathcal{P} \) has infinitely many linearly independent integrals of motion then it is called an integrable system.

From now on, we let \( \mathcal{P} = \mathbb{C}_{\text{diff}}[u_i \mid i \in I] \) be the differential superalgebra and \( u_i \) be homogeneous variables of \( \mathcal{P} \), that is \( I = I_0 \cup I_1 \) and \( u_i \) for \( i \in I_0 \) (resp. \( I_1 \)) are even (resp. odd.)

Remark 5.2. For \( f \in \mathcal{P} \), we let
\[
\frac{df}{dt} = \sum_{n \in \mathbb{Z}_{\geq 0}, i \in I} \left( \partial_n \frac{du_i}{dt} \right) \frac{\partial f}{\partial u_i^{(n)}},
\]
inspired from chain rules. Then
\[
\left[ \frac{du_i}{dt} = \{ h_\lambda u_i \}\big|_{\lambda=0} \text{ for any } i \in I \right] \text{ iff } \left[ \frac{df}{dt} = \{ h_\lambda f \}\big|_{\lambda=0} \text{ for any } f \in \mathcal{P} \right].
\]

For the following proposition, recall if \( \mathcal{P} \) is a PVA with a \( \lambda \)-bracket \( \{ \lambda \} \) then the superspace \( \mathcal{P}/\partial \mathcal{P} \) is a Lie superalgebra endowed with the bracket \( \{ f, g \} := \{ f g \}\big|_{\lambda=0} \).

Proposition 5.3 (generalized Lenard-Magri scheme). Suppose \( \mathcal{P} \) is endowed with two compatible \( \lambda \)-brackets \( \{ \lambda \}_H \) and \( \{ \lambda \}_K \). If there are linearly independent even elements \( \int h_i, i \in \mathbb{Z}_{\geq 0}, \) in \( \int \mathcal{P} \) such that
\[
[\int h_m, \int u_i]_H = [\int h_{m+1}, \int u_i]_K \quad \text{for } m \in \mathbb{Z}_{\geq 0} \text{ and } i \in I,
\]
then \( \frac{d\phi}{dt} = \{ h_\lambda \phi \}\big|_{\lambda=0} \) for \( m \in \mathbb{Z}_{\geq 0} \) are integrable systems and \( \int h_m, m' \in \mathbb{Z}_{\geq 0} \) are integrals of motion.

Proof. If we assume \( m > n \) then
\[
[\int h_m, \int h_n]_H = [\int h_m, \int h_{n+1}]_K \quad \text{and} \quad [\int h_m, \int h_n]_K = [\int h_{m-1}, \int h_n]_H.
\]
Inductively, we can prove that if \( m - n \) is odd then \( [\int h_m, \int h_n]_H = [\int h_l, \int h_l]_K = 0 \) (resp. \( [\int h_m, \int h_n]_K = [\int h_l, \int h_l]_K = 0 \)) for some \( l \) with \( m \geq l > n \) (resp. \( m > l > n \)). If \( m - n \) is even then \( [\int h_m, \int h_n]_H = [\int h_l, \int h_l]_H = 0 \) (resp. \( [\int h_m, \int h_n]_K = [\int h_l, \int h_l]_K = 0 \)) for some \( l \) with \( m > l > n \).

Remark 5.4. The Lenard-Magri scheme in Proposition 5.3 does not give any clue of finding odd integrals of motion.

Consider the adjoint map \( \text{ad}(\Lambda) : \mathfrak{g}((z^{-1})) \to \mathfrak{g}((z^{-1})) \) such that \( \text{ad}(\Lambda)(A) = [\Lambda, A] \) for \( A \in \mathfrak{g}((z^{-1})) \). In the rest of this section, we assume that \( \Lambda \) is semisimple so that
\[
\mathfrak{g}((z^{-1})) = \text{ker}(\text{ad}(\Lambda)) \oplus \text{im}(\text{ad}(\Lambda)).
\]
Remark 5.5. The assumption that \( \Lambda \) is a semisimple element is quite a big constraint. Such \( \Lambda \) does not always exist for any nilpotent element \( f \). However, when \( g = \mathfrak{sl}(m | n) \) and the nilpotent element \( f \) corresponds to the pair of partitions \( \lambda \) and \( \mu \) of \( m \) and \( n \) which has the form of one of followings:

\[
\begin{align*}
(1) \quad \lambda &= (r^{m_r}, 1^{n_1}), \quad \mu = (r^{m_r}, 1^{n_1}), \\
(2) \quad \lambda &= ((r, r-1)^{pr-r-1}, 1^{p_1}), \quad \mu = ((r, r-1)^{pr-r-1}, 1^{q_1}),
\end{align*}
\]

we can find an even element \( s \) such that \( \Lambda \) is semisimple. Here, \( n_r, n_1, m_r, m_1, p_r, r-1, p_1, q_r, r-1, q_1 \) are all nonnegative integers. Note that this remark directly follows from \( g = \mathfrak{sl}_n \) case (see [7] and Remark 2.30).

Take the gradation on \( g((z^{-1})) \) defined by

\[
\deg(z) = -d - 1 \quad \text{and} \quad \deg(g) = j/2 \quad \text{if} \quad g \in g \quad \text{and} \quad [h/2, g] = (j/2)g
\]

and denote by \( g((z^{-1})) \) consisting of elements with degree \( k \).

Proposition 5.6. Let \( L(\Lambda) = k\partial + q - \Lambda \otimes 1 \) be a Lax operator. For a subspace \( V \subset g((z^{-1})) \), denote \( V_t = \mathfrak{gl}(\mathbb{C}, \mathbb{C}) \). There exist unique even element \( S(q) \in \bigoplus_{t > 0} \ker(\text{ad} \Lambda_t) \otimes \mathcal{V}_t(p) \) and unique even element \( h(q) \in \bigoplus_{t > 0} \ker(\text{ad} \Lambda_t) \otimes \mathcal{V}_t(p) \) such that

\[
L_0^t(\Lambda) := e^{ad S(q)^t} L_0(\Lambda) = k\partial + h(q)^t - \Lambda \otimes 1.
\]

Proof. We can decompose (5.8) via the gradation on \( g((z^{-1})) \). Then degree \(-\frac{1}{2}\) part of (5.8) is

\[
q^t_{-1/2} + [S^t_{1/2}, -\Lambda \otimes 1] = h^t_{-1/2},
\]

where \( q^t_{-1/2}, S^t_{1/2} \) and \( h^t_{-1/2} \) are degree \(-1/2, 1/2 \) and \(-1/2 \) part of \( q^t, S(q)^t \) and \( h(q)^t \). By (5.6), \( S^t_{1/2} \) and \( q^t_{-1/2} \) are uniquely determined. Also, since \( q^t_{-1/2} \) and \( \Lambda \) are even, both \( S^t_{1/2} \) and \( q^t_{-1/2} \) are even. Inductively, \( S^t_n \) and \( h^t_{m-1} \) are uniquely determined for \( n \geq 1 \). More precisely, if \( S^t_m \) and \( h^t_{m-1} \) are known for \( m < n \) then degree \( n-1 \) part of (5.8) is

\[
Q^t_{n-1} + [S^t_n, -\Lambda \otimes 1] = h^t_{n-1},
\]

where \( Q^t_{n-1} \) is then degree \( n-1 \) part determined by \( S^t_m \)'s for \( m < n \). Now, by (5.6), \( S^t_n \) and \( q^t_{-1} \) are uniquely determined. \( \square \)

Remark 5.7. By Proposition 5.6 and its proof, we can see that for an operator \( L(\Lambda) = k\partial + q - \Lambda \otimes 1 \), there is unique \( h(q) \in \bigoplus_{t > 0} \ker(\text{ad} \Lambda_t) \otimes \mathcal{V}_t(p) \) such that \( L_0^t(\Lambda) := e^{ad S(q)^t} L_0(\Lambda) = k\partial + h(q)^t - \Lambda \otimes 1 \) for some \( S(q) \in \bigoplus_{t > 0} \mathcal{V}_t(p) \) (which is not necessarily unique).

For \( L(\Lambda) = k\partial + q_{\text{univ}} - \Lambda \otimes 1 \), consider

\[
H_n = (h(q_{\text{univ}})^n) z^n \Lambda \otimes 1 \in \mathcal{V}_t(p), \quad n \in \mathbb{Z}_{\geq 0}.
\]

and fix

\[
H_{-1} = ((h(q_{\text{univ}})^n - \Lambda \otimes 1) z^{-1} \Lambda \otimes 1) = (-\Lambda \otimes 1) z^{-1} \Lambda \otimes 1) \in \mathbb{C}.
\]

Remark 5.8. Note that \( (h(q_{\text{univ}})^n) z^n \Lambda \otimes 1 \) \( \neq 0 \) only if \( h(q_{\text{univ}}) \in g_0 \otimes \mathcal{V}_t(p) \). Hence

\[
H_n = (h(q_{\text{univ}})^n) z^n \Lambda \otimes 1 \in \mathcal{V}_t(p).
\]

Lemma 5.9. For \( H_n \in \mathcal{V}_t(p), \quad n \in \mathbb{Z}_{\geq -1} \), we have \( H_n \in \mathcal{W}(g, f, k) \).
Let us define $H_n \in \mathcal{V}_T(p) = (h(q_{\text{univ}})|z^n \Lambda \otimes 1) \in \mathcal{W}(g, f, k)$.

**Lemma 5.10.** Let $S$ be the same as in (5.8) and $r \in (m^\perp \otimes \mathcal{V}_T(p))_0$. Then we have

$$(5.11) \quad \int \left( r^\sigma \left| \frac{\partial H_n}{\partial q} \right| \right) = \int \left( r^\sigma \left| e^{-adS^\sigma} \right| (z^n \Lambda \otimes 1) \right), \quad n \in \mathbb{Z}_{\geq 1},$$

where $q = (q_i)_{i \in I}$ for a basis $q_i$ of $p$.

**Proof.** If $n = -1$, we have

$$e^{-adS^\sigma} (z^n \Lambda \otimes 1) \in g[z]z \otimes \mathcal{V}_T(p)$$

so that $\left( r^\sigma \left| e^{-adS^\sigma} \right| (z^n \Lambda \otimes 1) \right) = 0$. Since $\left. \frac{\partial H_n}{\partial q} \right|_{\sigma = 0} = 0$, the both sides of (5.11) are 0.

To show the proposition when $n \geq 0$, recall that

$$\int \left. \frac{H_n(q+\varepsilon r) - H_n(q)}{\varepsilon} \right|_{\varepsilon = 0} = \int \left( r^\sigma \left| \frac{\partial H_n}{\partial q} \right| \right)$$

for $r \in (m^\perp \otimes \mathcal{V}(p))_0$ and $q = \sum_{i \in I} q_i \otimes Q_i$. Let us define $L_0(\varepsilon)$ and $S(\varepsilon)$ by

$$L_0^\varepsilon(\varepsilon) = e^{adS(\varepsilon)\rho(\partial + (q^\sigma + \varepsilon r^\sigma) - \Lambda \otimes 1)} = \partial + h(q + \varepsilon r)^\sigma - \Lambda \otimes 1.$$

Then

$$\left( \frac{d}{d\varepsilon} L_0^\varepsilon(\varepsilon) \right) |_{\varepsilon = 0} = \left. \frac{H_n(q+\varepsilon r) - H_n(q)}{\varepsilon} \right|_{\varepsilon = 0}.$$

By the process finding $S(\varepsilon)$ in Proposition 5.6, we have

$S(\varepsilon) = S_0 + \varepsilon S_1 + \varepsilon^2 S_2 + \cdots \in \text{im}(ad\Lambda)((z^{-1}) \otimes \mathcal{V}_T(p))[\varepsilon],$

for $S_0, S_1, \cdots \in g((z^{-1})) \otimes \mathcal{V}_T(p)$. Also, we have

$$(5.12) \quad \frac{d}{d\varepsilon} L_0^\varepsilon(\varepsilon) |_{\varepsilon = 0} = e^{adS(\varepsilon)r^\sigma} |_{\varepsilon = 0} + \frac{d}{d\varepsilon} (e^{adS(\varepsilon)^\sigma} L^\sigma(0)) |_{\varepsilon = 0}$$

where $L^\sigma(0) = \partial + q^\sigma - \Lambda \otimes 1$ and $e^{adS_0^\sigma} L^\sigma(0) = L_0^\sigma(0) = \partial + h^\sigma(q) - \Lambda \otimes 1$. In order to investigate the last term in (5.12), we need the following facts:

(i) $\frac{d}{d\varepsilon} (e^{adS(\varepsilon)^\sigma} L^\sigma(0)) |_{\varepsilon = 0} = \sum_{n \in \mathbb{Z}_{\geq 1}} \frac{1}{n!} \sum_{m=0}^{n-1} \text{ad}^m S_0^\sigma (\text{ad}^{n-1-m} S_0^\sigma (L^\sigma(0)))$

(ii) $\frac{1}{n!} \sum_{m=0}^{n-1} \text{ad}^m S_0^\sigma \left( \text{ad}^{n-1-m} S_0^\sigma (L^\sigma(0)) \right)$

$= \sum_{m=0}^{n-1} \left[ \frac{1}{(n-m)!} \text{ad}^{n-1-m} S_0^\sigma (S_0^\sigma) \right.$

$+ \frac{1}{m} \text{ad}^m S_0^\sigma (L^\sigma(0)) \left. \right]$

(iii) $\sum_{n \in \mathbb{Z}_{\geq 1}} \sum_{m=0}^{n-1} \left[ \frac{1}{(n-m)!} \text{ad}^{n-1-m} S_0^\sigma (S_0^\sigma) \right.$

$+ \frac{1}{m} \text{ad}^m S_0^\sigma (L^\sigma(0)) \left. \right]$

$$= \sum_{l \in \mathbb{Z}_{\geq 0}} \left[ \frac{1}{(l+1)!} \text{ad}^l S_0^\sigma (S_0^\sigma), L_0^\sigma(0) \right].$$

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By (5.12) and (5.13), we have
\[
\left. \frac{d}{dc} \left( e^{adS(\epsilon)\sigma} L^0(0) \right) \right|_{c=0} = \sum_{l \in \mathbb{Z}_{\geq 0}} \left. \frac{1}{(l+1)!} \right| e^{adL^0_0(S^0)} (S^0_I, L^0_0(0)) .
\]

Observe that
\[
(5.14) \quad \int \left( r^\sigma \left| \frac{\partial H_\sigma}{\partial q} \right| \right) = \int \left( \frac{d}{dc} L^0_0(\epsilon) \right| z^\Lambda \otimes 1 \right|_{c=0}
\]
and
\[
(5.15) \quad \left. \frac{d}{dc} L^0_0(\epsilon) \right| z^\Lambda \otimes 1 \right|_{c=0} = \left( e^{adS(\epsilon)\sigma} r^\sigma \right| z^\Lambda \otimes 1 \right|_{c=0} + \sum_{l \in \mathbb{Z}_{\geq 0}} \frac{1}{(l+1)!} \left. \left[ \frac{\partial L^0_0(S^0)}{\partial q} \right]|_{c=0} \right) + \sigma (z^\Lambda \otimes 1) \right|_{c=0}
\]

The last term in (5.15) is in \( \partial H_\sigma \). Hence, for \( S(0)^\sigma = S^\sigma \), we have
\[
\int \left( r^\sigma \left| \frac{\partial H_\sigma}{\partial q} \right| \right) = \int \left( r^\sigma \left| e^{-adS(\epsilon)} (z^\Lambda \otimes 1) \right| \right).
\]

\[\Box\]

**Proposition 5.11.** Recall that \( g = m \oplus p \) and suppose \( \{ q^i | i \in I \} \) is a basis of \( p \) and \( \{ q^i | i \in I' \} \) is a basis of \( m \). For \( Q = \sum_{i \in I} q^i \otimes Q_i + \sum_{i \in I'} q^i \otimes Q_i \in g \otimes \mathcal{V}(p) \), we denote \( Q|_p = \sum_{i \in I} q^i \otimes Q_i \in p \otimes \mathcal{V}(p) \). Then we have
\[
\frac{\delta H_\sigma}{\delta q} = e^{-adS(\epsilon)} (z^\Lambda \otimes 1) \right|_p .
\]

**Proof.** It directly follows from Lemma 5.10. \[\Box\]

**Proposition 5.12.** Let \( \phi \in \mathcal{W}(g, f, k) \) and \( a_m \otimes F \in (m \otimes \mathcal{V}(p))_0 \). Then we have
\[
\int \left( \delta_\phi \left[ (a_m \otimes F)^\sigma, \mathcal{L}^\sigma_1(\Lambda) \right] \right) = \int \left( \delta_\phi \left[ z^{-1}[(a_m \otimes F)^\sigma, zs \otimes 1] \right] \right) = 0.
\]

**Proof.** Let us denote \( S = a_m \otimes F \in (m \otimes \mathcal{V}(p))_0 \) and denote
\[
\mathcal{L}^\sigma(\epsilon) := e^{adS(\epsilon)} \mathcal{L}^\sigma(\Lambda).
\]
Then \( \mathcal{L}^\sigma(\epsilon) = \partial + e^{S(\epsilon)}(\epsilon) - \Lambda \otimes 1 \) for \( q^\sigma(\epsilon) = \theta(a_m + e[S^\sigma, \mathcal{L}^\sigma(\Lambda)] + o(\epsilon^2) \). Since \( \phi \) is gauge invariant, we have
\[
0 = \int \frac{d\phi(g(q(\epsilon)))}{d\epsilon} \left|_{\epsilon=0} = \int \left( \delta_\phi \left[ (a_m \otimes F)^\sigma, \mathcal{L}^\sigma(\Lambda) \right] \right)
\]
for any \( F \) which has the same parity as \( a_m \). Hence \( \int \left( \delta_\phi \left[ (a_m \otimes F)^\sigma, \mathcal{L}^\sigma_1(\Lambda) \right] \right) = 0. \)

Also, the second equality follows from \( [a_m, s] = 0. \) \[\Box\]

**Proposition 5.13.** Let \( S \) be the same as in (5.14) and let \( \phi \in \mathcal{W}(g, f, k) \). Then we have
\[
(5.16) \quad \int \left( \delta_\phi \left[ \left| \frac{\partial H_\sigma}{\partial q} \right|, \mathcal{L}^\sigma_1(\Lambda) \right] \right) = \int \left( \delta_\phi \left[ e^{-adS^\sigma}(z^\Lambda \otimes 1), \mathcal{L}^\sigma_1(\Lambda) \right] \right)
\]
and
\[
\int \left( \delta_\phi \left[ \left| \frac{\partial H_\sigma}{\partial q} \right|, s \otimes 1 \right] \right) = \int \left( \delta_\phi \left[ e^{-adS^\sigma}(z^\Lambda \otimes 1), s \otimes 1 \right] \right).
\]

**Proof.** It follows from Proposition 5.11 and Proposition 5.12. \[\Box\]
Recall that the brackets 

**Proposition 3.14** has the summand  

\[ \frac{du}{dt} = \{H_{i\lambda}u\}_1|_{\lambda=0}, \quad u \in \mathcal{W}(g, f, k) \text{ and } i \in \mathbb{Z} \]  

has linearly independent integrals of motion \( \int H_j \) for \( j \in \mathbb{Z}_{\geq 0} \). Hence (5.17) is an integrable system. 

**Proof.** In order to use Lenard scheme, we aim to show that  

\[ \int H_i, \int u \]  

(5.18) \[ = \int H_{i+1}, \int u \], \quad i \in \mathbb{Z}_{\geq -1}. \]  

Recall that the brackets \( \{H, \int u\}_1 \) and \( \{H, \int u\}_2 \) are defined by  

\[ \{H, \int u\}_1 = \frac{\delta H}{\delta q} \left[ \left[ \frac{\delta H}{\delta q}, \mathcal{L}_1^\sigma(\Lambda) \right] \right] = -\frac{\delta u}{\delta q} \left[ \left[ \frac{\delta H}{\delta q}, \mathcal{L}_1^\sigma(\Lambda) \right] \right], \]  

\[ \{H, \int u\}_2 = \frac{\delta H}{\delta q} \left[ \left[ \frac{\delta H}{\delta q}, s \otimes 1 \right] \right] = -\frac{\delta u}{\delta q} \left[ \left[ \frac{\delta H}{\delta q}, s \otimes 1 \right] \right]. \]  

Denote \( \frac{\delta H(z)}{\delta q} \sum_{i \in \mathbb{Z}} \delta H_i z^{-i} \). Then  

\[ \frac{\delta H(z)}{\delta q} \left[ \mathcal{L}_1^\sigma(\Lambda) \right] = \sum_{i \in \mathbb{Z}} \left( \frac{\delta H}{\delta q} \left[ \mathcal{L}_1^\sigma(\Lambda) \right] - \frac{\delta H}{\delta q} \left[ s \otimes 1 \right] \right) z^{-i}. \]  

By (5.19) and (5.20), (5.18) is equivalent to  

\[ \int \left( \frac{\delta H(z)}{\delta q} \right) \mathcal{L}_1^\sigma(\Lambda) \]  

(5.21)  

\[ = \int \left( \frac{\delta H}{\delta q} \right) \left[ e^{-adS^r}(z^iA \otimes 1), \mathcal{L}_1^\sigma(\Lambda) \right] + \int \left( \frac{\delta u}{\delta q} \right) \left[ z^{-1}e^{-adS^r}(z^{i+1}A \otimes 1), zs \otimes 1 \right] \]  

\[ = \int \left( \frac{\delta H}{\delta q} \right) \left[ e^{-adS^r}(z^iA \otimes 1), \mathcal{L}_1^\sigma(\Lambda) \right] = 0. \]

for any \( \phi \in \mathcal{W}(g, f, k) \) and \( i \in \mathbb{Z} \). Here, we used Proposition 5.13 for the first equality. Hence we proved (5.19) by the following fact:  

\[ \left[ e^{-adS^r}(z^iA \otimes 1), \mathcal{L}_1^\sigma(\Lambda) \right] = e^{-adS^r} \left[ z^iA \otimes 1, e^{adS^r} \mathcal{L}_1^\sigma(\Lambda) \right] = 0. \]

In particular, since \( H_{-1} \) is constant, we have \( \{H_{-1}, \int u\}_1 = \{H_0, \int u\}_1 = 0 \). 

Now, the only thing to prove is that \( \{H_j\}_{j \in \mathbb{Z}_{\geq 0}} \) is linearly independent. Since, for given \( H, u \in \mathcal{W}(g, f, k) \) such that \( \{H_{j\lambda}u\}_1|_{\lambda=0} \neq 0 \), the total degree of \( \{H_{j\lambda}u\}_1 \) is greater than the total degree of \( \{H_{\lambda\lambda}u\}_2|_{\lambda=0} \) in the algebra of polynomials \( \mathbb{C}\langle q^i \rangle | i \in I, n \in \mathbb{Z}_{\geq 0} \rangle \), where \( \{q^i | i \in I\} \) is a basis of \( p \). Hence, if we can show \( \{H_{0\lambda} \mathcal{W}(g, f, k)\}|_{\lambda=0} \neq 0 \) then the linear independence of \( \{H_j\}_{j \in \mathbb{Z}_{\geq 0}} \) follows. Indeed, this can be proved as above. Suppose \( d_{univ} \) in Proposition 5.13 has the summand \( f \otimes \phi_e \). Then \( H_0 = \phi_e \) so that \( \{H_0, \mathcal{W}(g, f, k)\}|_{\lambda=0} \neq 0 \). 

**Remark 5.15.** Recall that the formula (5.5) \( e^{adS(q)}L^\sigma(\Lambda) \) is same as \( (e^{S(q)}L(\Lambda))^\sigma \). Also, since \( \Lambda \) is even, we have  

\[ (h(q_{univ}))^\sigma|z^{-n}\Lambda \otimes 1) = (h(q_{univ}))(z^{-n}\Lambda \otimes 1). \]

Hence we can use \( L(\Lambda) \) instead of \( L(\Lambda)^\sigma \) to compute \( H_n \).

**Example 5.16.** As in Example 5.15, the Lie superalgebra \( g = spo(2\mid 1) \) is generated by \( e, e_{ad}, h, f_{ad} \) and \( f \). For \( \Lambda = f + ze \) and \( K := -f + ze \), we can see that \( g((z^{-1})) \) is the \( \mathbb{C}((z^{-1})) \)-module
generated by $e_{od}, f_{od}, h = 2x, \Lambda, K$. The subspace $\text{im}(\text{ad}\Lambda)$ is generated by $e_{od}, f_{od}, h, K$ and $\ker(\text{ad}\Lambda)$ is generated by $\Lambda$. Note that the Lie brackets between generators are

$$[x, \Lambda] = K, \quad [f_{od}, \Lambda] = -ze_{od}, \quad [e_{od}, \Lambda] = -f_{od}, \quad [K, \Lambda] = 2zh = 4zx, \quad [x, K] = \Lambda.$$  

Consider the operator,

$$L(\Lambda) = k\partial + e_{od} \otimes \phi_{od} + e \otimes \phi_{ev} - \Lambda \otimes 1$$

for the generators $\phi_{od} = -\frac{1}{2}f_{od} + \frac{1}{4}he_{od}$ and $\phi_{ev} = f + \frac{1}{2}f_{od}e_{od} - \frac{1}{4}h^2 + \frac{1}{4}e_{od}\theta e_{od} - \frac{1}{2}\partial h$ of the algebra in Example 5.15. We want to find $h \in \bigoplus_{t \geq -1} \ker(\text{ad}\Lambda)_t \otimes \mathcal{V}_I(P)$ such that

$$L_0(\Lambda) := e^{\text{ad}\Lambda} L(\Lambda) = \partial + h - \Lambda \otimes 1.$$  

for some $S \in \bigoplus_{t \geq 0} \ker(\text{ad}\Lambda)_t \otimes \mathcal{V}_I(P)$. Let us denote $U = \sum_{t \geq 0} U_t$ for $U_t \in \mathcal{V}_I(P) \otimes \mathcal{g}((z^{-1}))_t$ and $h = \sum_{t \geq -1} h_t$ for $h_t = \mathcal{V}_I(P) \otimes \mathcal{g}((z^{-1}))_t \cap \ker(\text{ad}\Lambda))$. Since $z^4\Lambda$ has degree $-1 - 2i$, we have $h = \sum_{t \in \mathbb{Z}, t \geq -1} h_{2t+1}$.  

By comparing degree $\frac{1}{2}$ part of (5.22), we have $S_{1/2} = S_1 = 0$ and 

$$e_{od} \otimes \phi_{od} - [S_{3/2}, \Lambda \otimes 1] = 0.$$  

Hence $S_{3/2} = -z^{-1}e_{od} \otimes \phi_{od}$. By comparing degree 1 part of (5.22), we have 

$$e \otimes \phi_{ev} - [S_2, \Lambda \otimes 1] = h_1$$  

Hence $S_2 = \frac{1}{2}z^{-1}x \otimes \phi_{ev}$ and $h_1 = \frac{1}{2}z^{-1}\Lambda \otimes \phi_{ev}$ so that 

$$H_0 = (h_1|\Lambda \otimes 1) = \phi_{ev}.$$  

By degree $\frac{3}{2}$ and 2 parts of (5.22), we have 

$$-[S_{5/2}, \Lambda \otimes 1] + [S_{3/2}, k\partial] = 0 \quad \text{and} \quad S_{5/2} = -z^{-1}e_{od} \otimes k\partial \phi_{od};$$  

$$-[S_3, \Lambda \otimes 1] + [S_{2}, k\partial] + [S_{3/2}, e_{od} \otimes \phi_{od}] - [S_{3/2}, [S_{3/2}, \Lambda \otimes 1]] = 0$$  

and $S_3 = -\frac{1}{8}z^{-2}K \otimes k\partial \phi_{ev}$. By degree 3 part of (5.22), we have 

$$-[S_4, \Lambda \otimes 1] + [S_3, k\partial] + [S_{5/2}, e_{od} \otimes \phi_{od}] + [S_2, e \otimes \phi_{ev}]$$  

$$-\frac{1}{2}[S_{2}, [S_2, \Lambda \otimes 1]] - \frac{1}{2}[S_{3/2}, [S_{5/2}, \Lambda \otimes 1]] - \frac{1}{2}[S_{5/2}, [S_{3/2}, \Lambda \otimes 1]]$$  

$$+ \frac{1}{2}[S_{3/2}, [S_3/2, k\partial]] = h_3$$  

(5.23)

Since the LHS of (5.23) is 

$$-[S_4, \Lambda \otimes 1] + z^{-2} \otimes \left(\frac{1}{8}\phi_{ev}^2 + \frac{k}{2}\partial \phi_{od} \phi_{od}\right) + z^{-2}K \otimes \left(\frac{k^2}{8}\phi_{ev}^2 + \frac{1}{4}\phi_{ev}^2 + \frac{k}{2}\partial \phi_{od} \phi_{od}\right)$$  

and $K$ is in the image of $\text{ad}\Lambda$, we can conclude $h_3 = z^{-2}(\frac{1}{8}\phi_{ev}^2 + \frac{k}{2}\partial \phi_{od} \phi_{od})$ so that 

$$H_1 = (h_3|z\Lambda \otimes 1) = \frac{1}{4}\phi_{ev}^2 + k\partial \phi_{od} \phi_{od}.$$  

Note that $h_3$ is the first integral of motion which gives rise to a nonlinear Hamiltonian equation. Now we get

$$\begin{align*}
\frac{d\phi_{ev}}{dt} &= \{H_1, \lambda \phi_{ev}\}_H|_{\lambda = 0} = -\frac{k^3}{4}\phi_{ev}^3 - 3k^2\partial \phi_{ev} \phi_{ev} + 3k^2 \phi_{od} \partial^2 \phi_{od}, \\
\frac{d\phi_{od}}{dt} &= \{H_1, \lambda \phi_{od}\}_H|_{\lambda = 0} = k^3\partial^3 \phi_{od} - \frac{3k^2}{4}\partial \phi_{od} \phi_{ev} - \frac{3k}{4}\phi_{od} \partial \phi_{ev},
\end{align*}$$
which is same as super-KdV equation in [21] up to constant factors.

**Example 5.17.** Let $g = \mathfrak{sl}(2|1)$ and $f = e_{21}$. With the notations in Example 3.16 we consider the operator

$$L(\Lambda) = k\partial + \tau \otimes \phi_r + e_{11} \otimes \phi_{11} + e_{12} \otimes \phi_{12} + e \otimes \phi_e - \Lambda \otimes 1.$$ 

Recall that $\phi_e = -kL - 2\phi_r^2$ for the energy momentum field $L$. It is not hard to check that

$$\{\phi_r \lambda \phi_r\} = -\frac{1}{2} k\lambda, \quad \{\phi_r \lambda \phi_{11}\} = -\frac{1}{2} \phi_{11}, \quad \{\phi_r \lambda \phi_{12}\} = \frac{1}{2} \phi_{12}, \quad \{\phi_{11} \lambda \phi_{11}\} = \{\phi_{12} \lambda \phi_{12}\} = 0.$$ 

Also, 

$$\{\phi_{11} \lambda \phi_{12}\} = kL - k\partial \phi_r - 2k\lambda \phi_r - k^2 \lambda^2, \quad \{\phi_{12} \lambda \phi_{11}\} = kL + k\partial \phi_r + 2k\lambda \phi_r - k^2 \lambda^2,$$

for the energy momentum field $L$ in Proposition 2.15. Recall that $\phi_e = -kL - 2\phi_r^2$ and $\lambda$-brackets between $\phi_e$ and elements in $\mathcal{W}(g, f, k)$ can be computed using

$$\{L_\lambda \phi_r\} = (\partial + \lambda) \phi_r, \quad \{L_\lambda \phi_{11}\} = (\partial + \frac{3}{2} \lambda) \phi_{11}, \quad \{L_\lambda \phi_{12}\} = (\partial + \frac{3}{2} \lambda) \phi_{12},$$

and

$$\{L_\lambda L\} = (\partial + 2\lambda) L - \frac{1}{2} k\lambda^2.$$

Now, let us consider $S \in \bigoplus_{t > 0} \text{im}(\text{ad}\Lambda)_t \otimes \mathcal{V}_t(p)$ and $h \in \bigoplus_{t > 1} \ker(\text{ad}\Lambda)_t \otimes \mathcal{V}_t(p)$ such that

$$L_0(\Lambda) = k\lambda + h - \Lambda \otimes 1 = e^{\text{ad}S} L(\Lambda).$$

By equating degree $\leq 1/2$-parts of the both sides of (5.24), we get

$$S_{1/2} = S_1 = 0, \quad \text{and} \quad S_{3/2} = -z^{-1} e_{21} \otimes \phi_{11} + z^{-1} e_{11} \otimes \phi_{12}.$$ 

By equating degree 1-parts of the both sides of (5.24):

$$[S_2, \Lambda \otimes 1] + h_1 = e \otimes \phi_e = \frac{1}{2} z^{-1} \Lambda \otimes \phi_e + \frac{1}{2} z^{-1} K \otimes \phi_e,$$

for $K = ze - f$, we get

$$S_2 = \frac{1}{2} z^{-1} x \otimes \phi_e, \quad h_1 = \frac{1}{2} z^{-1} \Lambda \otimes \phi_e, \quad H_0 = (h_1 | \Lambda \otimes 1) = \phi_e.$$ 

Hence

$$\begin{align*}
\frac{d\phi_r}{dt} &= \{\phi_e \lambda \phi_r\}|_{\lambda=0} = k(\lambda + \partial) \phi_r|_{\lambda=0} = k\partial \phi_r, \\
\frac{d\phi_{11}}{dt} &= \{\phi_e \lambda \phi_{11}\}|_{\lambda=0} = -k(\partial + \frac{3}{2} \lambda) \phi_{11} + 2\phi_{11} \phi_r|_{\lambda=0} = -k\partial \phi_{11} + 2\phi_{11} \phi_r, \\
\frac{d\phi_{12}}{dt} &= \{\phi_e \lambda \phi_{12}\}|_{\lambda=0} = -k(\partial + \frac{3}{2} \lambda) \phi_{12} - 2\phi_{12} \phi_r|_{\lambda=0} = -k\partial \phi_{12} - 2\phi_{12} \phi_r, \\
\frac{d\phi_e}{dt} &= \{\phi_e \lambda \phi_e\}|_{\lambda=0} = k^2(\partial + 2\lambda) L - \frac{k^3}{2} \lambda|_{\lambda=0} = k^2 \partial L = -k\partial(\phi_e + 2\phi_r^2),
\end{align*}$$

is the simplest integrable system associated to $\mathfrak{sl}(2|1)$.

The degree 2-part of (5.24) is

$$[S_{5/2}, \Lambda \otimes 1] = [S_{3/2}, k\partial + \tau \otimes \phi_r]$$

$$= z^{-1} e_{21} \otimes (k\partial \phi_{11} - \phi_r \phi_{11}) + z^{-1} e_{11} \otimes (-k\partial \phi_{12} - \phi_r \phi_{12}).$$
Hence \( S_{3/2} = z^{-1}e_{11} \otimes (-k\partial \phi_{11} + \phi_r \phi_{11}) + z^{-1}e_{12} \otimes (-k\partial \phi_{12} - \phi_r \phi_{12}) \).

The degree 2-part of \((5.24)\) is
\[
[S_3, \Lambda \otimes 1] + h_2
\]
\[
(5.27)
\]
\[ = [S_2, k\partial + \tau \otimes \phi_r] + [S_{3/2}, e_{11} \otimes \phi_{11} + e_{12} \otimes \phi_{12}] - \frac{1}{2}[S_{3/2}, [S_{3/2}, \Lambda \otimes 1]].
\]

Hence \( S_3 = z^{-2}K \otimes -\frac{1}{2}k\partial \phi_e \) and \( h_2 = z^{-1}\tau \otimes \frac{1}{2} \phi_{11} \phi_{12} \).

The degree 3-part of \((5.24)\) is
\[
[S_4, \Lambda \otimes 1] + h_3
\]
\[
(5.28)
\]
\[ = [S_3, k\partial + \tau \otimes \phi_r] + [S_{3/2}, e_{11} \otimes \phi_{11} + e_{12} \otimes \phi_{12}] + \frac{1}{2}[S_{3/2}, [S_{3/2}, k\partial + \tau \otimes \phi_r]]
\]
\[- \frac{1}{2}[S_2, [S_2, \Lambda \otimes 1]] - \frac{1}{2}[S_{3/2}, [S_{3/2}, \Lambda \otimes 1]] - \frac{1}{2}[S_{3/2}, [S_{3/2}, \Lambda \otimes 1]].
\]

Hence \( S_4 = z^2h \otimes \frac{1}{2} \left( \frac{k^2}{8} \partial^2 \phi_e + \frac{1}{2} \phi_r \phi_{12} \phi_{11} + \frac{k}{2} \partial \phi_{12} \phi_{11} - \frac{k}{2} \phi_{12} \partial \phi_{11} \right) \) and
\[
h_3 = z^{-1}\Lambda \otimes \left( -\frac{1}{8} \phi_c^2 + \frac{1}{2} \phi_r \phi_{12} \phi_{11} + \frac{k}{2} \partial \phi_{12} \phi_{11} - \frac{k}{2} \phi_{12} \partial \phi_{11} \right)
\]
so that
\[
H_1 = (h_3|z\Lambda \otimes 1) = -\frac{1}{4} \phi_c^2 + \phi_r \phi_{12} \phi_{11} + k\partial \phi_{12} \phi_{11} - k\phi_{12} \partial \phi_{11}.
\]

Hence the second integrable system associated to \( sl_2 \), getting by the Hamiltonian \( 4H_1 \), is
\[
\left\{
\begin{aligned}
\frac{d\phi_r}{dt} &= 6k(\partial \phi_{11} \phi_{12} + \phi_{11} \partial \phi_{12}), \\
\frac{d\phi_{11}}{dt} &= \phi_{11}(8\phi_c^2 - 8\phi_e \phi_r - 12k\phi_r \partial \phi_r - 3k\partial \phi_3 + 4k^2 \partial^2 \phi_r), \\
&+ \partial \phi_{11}(6k\phi_e - 24k^2 \phi_r + 16k^2 \partial \phi_r) + 20k^2 \partial^2 \phi_{11} \phi_r + 8k^3 \partial^3 \phi_{11}, \\
\frac{d\phi_{12}}{dt} &= \phi_{12}(-8\phi_c^3 - 8\phi_e \phi_r - 12k\phi_r \partial \phi_r + 3k\partial \phi_e - 4k^2 \partial^2 \phi_r), \\
&+ \partial \phi_{12}(10k\phi_e - 24k^2 \phi_r - 16k^2 \partial \phi_r) - 12k^2 \partial^2 \phi_{12} \phi_r - 8k^3 \partial^3 \phi_{12}, \\
\frac{d\phi_e}{dt} &= -12k\partial H_1 + k^3 \phi_e + 24k \phi_r \phi_{12} \phi_{11} - 24k \phi_r \phi_{11} \phi_{12}.
\end{aligned}
\right.
(5.29)
\]

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