Parameter-estimation Biases for Eccentric Supermassive Binary Black Holes in Pulsar Timing Arrays: Biases Caused by Ignored Pulsar Terms

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Abstract

The continuous nanohertz gravitational waves (GWs) from individual supermassive binary black holes (SMBBHs) can be encoded in the timing residuals of pulsar timing arrays (PTAs). For each pulsar, the residuals actually contain an Earth term and a pulsar term, but usually only the Earth term is considered as a signal and the pulsar term is dropped. This leads to parameter-estimation biases (PEBs) for the SMBBHs, and currently there are no convenient evaluations of the PEBs. In this article, we formulate the PEBs for a SMBBH with an eccentric orbit. In our analyses, the unknown phases of pulsar terms are treated as random variables obeying the uniform distribution $U(0, 2\pi)$, due to the fact that pulsar distances are generally poorly measured. Our analytical results are in accordance with the numerical work by Zhu et al. at 1.5$\sigma$ level. Additionally, we find that the biases $\Delta \varphi$ and $\Delta e/e$ for two parameters—that is, Earth-term phase $\varphi$ and orbital eccentricity $e$—monotonically decrease as $e$ increases, which partly confirms a hypothesis in our previous work. Furthermore, we also calculate the PEBs caused by the recently observed common-spectrum process (CSP). We find that if the strain amplitude of the continuous GW is significantly stronger (three times larger, in our cases) than the stochastic GW background, then the PEBs from pulsar terms are larger than those from the CSP. Our formulae of the PEBs can be conveniently applied in the future PTA data analyses.

Unified Astronomy Thesaurus concepts: Gravitational waves (678); Pulsar timing method (1305); Astronomy data analysis (1858)

1. Introduction

Pulsar timing arrays (PTAs) provide a promising tool to search for the nanohertz band ($10^{-9}−10^{-6}$ Hz) gravitational waves (GWs) from inspiralling supermassive binary black holes (SMBBHs), via precise measurements of the time of arrival (TOA) of radio pulses from arrays of millisecond pulsars (MSPs) (Sazhin 1978; Detweiler 1979; Foster & Backer 1990). Currently, the PTA observations are mainly carried out by the following three regional collaborations: the Parkes PTA (PPTA; Hobbs 2013; Manchester et al. 2013), the North American Nanohertz Observatory for GWs (NANO-Grav; McLaughlin 2013; Ransom et al. 2019), and the European PTA (EPTA; Kramer & Champion 2013), and they have combined as the International PTA (IPTA) collaboration (Hobbs et al. 2010; Manchester 2013). The IPTA has accumulated timing residual data for more than 10 yr, offering nearly 100 stable MSPs with white noise below 300 ns (Desvignes et al. 2016; Perera et al. 2019; Kerr et al. 2020; Alam et al. 2021a, 2021b). Furthermore, all of these collaborations have recently detected a common-spectrum process (CSP), whose origin is still unclear. A potential explanation is the signal from the stochastic GW background (SGWB) (Arzoumanian et al. 2020; Chen et al. 2021; Goncharov et al. 2021; Antoniadis et al. 2022). Additionally, the Indian PTA (InPTA) (Joshi et al. 2018) has also joined the IPTA recently, and the Chinese PTA (CPTA) (Lee et al. 2016) is preparing to join the IPTA. The next-generation radio telescopes, such as the Five-hundred-meter Aperture Spherical Telescope (FAST) (Nan et al. 2011) and the Square Kilometre Array (SKA) (Smits et al. 2009), are expected to enhance the number of the well-timed MSPs (noise rms $\lesssim$ 100 ns) to $O(10^5)$ (Wang & Mohanty 2017, 2018; Hobbs et al. 2019; Feng et al. 2020; Weltman et al. 2020).

The timing residuals for each MSP in a PTA are actually caused by both the GWs at Earth (hereafter, Earth term) and at the MSP (pulsar term). The pulsar term has a phase earlier than the Earth term, which is proportional to the distance of the MSP from Earth (Jenet et al. 2004; Burke-Spolaor et al. 2019). Hence, the Earth terms for all MSPs in the PTA are the same and can be coherently added to enlarge the total signal-to-noise ratio (S/N), while the pulsar terms have individual phases and tend to cancel each other (Corbin & Cornish 2010; Lee et al. 2011; Burke-Spolaor et al. 2019). In addition, the pulsar distances are generally poorly measured, with uncertainties that are much larger than the wavelengths of nanohertz-GWs ($10^{-1}−10$ ly) (Sesana & Vecchio 2010; Verbiest et al. 2012), so it is difficult to fix those phases of pulsar terms. Furthermore, if we include both Earth term and pulsar terms in the signal templates, and take all of the pulsar phases as free parameters, namely adopting a full-signal search (Ellis et al. 2012a, 2012b; Zhu et al. 2016), then the number of parameters will be larger than the number of MSPs in the data analyses, and this will make the parameter estimations complicated and computationally expensive (Sesana & Vecchio 2010; Wang et al. 2014, 2015). Therefore, in most of the current PTA data analyses, only the Earth term is deemed as a coherent GW signal, and the pulsar terms are usually treated as a kind of self-noise and ignored. This scheme is termed the Earth-term-only
search (Sesana & Vecchio 2010; Ellis et al. 2012a, 2012b; Zhu et al. 2016).

However, the Earth-term-only search will lead to biases on parameter estimations because it drops all pulsar terms in the timing residual templates. For instance, the works by Ellis et al. (2012a) and Zhu et al. (2016), which use synthetic data sets, have demonstrated that the sky localizations of GW sources are biased in the Earth-term-only search. In addition, the biases in Zhu et al. (2016) are not smaller than the localization uncertainties due to the injected white noise of the MSPs, even for a strong signal case (the network S/N = 100). Besides the sky localizations, other parameters can also be biased in the Earth-term-only search (Corbin & Cornish 2010). Meanwhile, the parameter-estimation biases (PEBs) from the ignored pulsar terms cannot be systematically estimated. Hence, in this article we will investigate the issue and propose a technique to calculate the PEBs.

Due to the recent detection of the aforementioned CSP and the studies in Rosado et al. (2015), Mingarelli et al. (2017), Taylor et al. (2020), Pol et al. (2021), and Ali-Haïmoud et al. (2021), one can expect that the SGWB may be detected earlier than the continuous GWs from individual sources. Even if continuous GWs can also be detected in the future, the CSP will still have considerable impacts on them (Bécsey & Cornish 2020) and lead to PEBs for the individual SMBBHs if the CSP is not properly considered in the noise model. Hence, it is worthwhile to calculate the PEBs arising from the CSP and compare them with the PEBs from pulsar terms. In this article, we focus on the data analyses of a single SMBBH and we assume that the SMBBH can be detected, even in the case that its continuous GW is weaker than the SGWB.1 The current research can serve as a foundation for further studies on the PEBs for multiple sources (Qian et al. 2021; Songsheng et al. 2021) and even for SGWB because SGWB in the astrophysical scenario is generally deemed to be a composition of multiple unresolved SMBBHs (Phinney 2001).

Furthermore, the SMBBHs in PTA observations are allowed to have large orbital eccentricities (Taylor et al. 2016), and some candidate sources are reported to contain non-negligible eccentricities (Feng et al. 2019); for example, OJ 287 has an eccentricity $e \approx 0.7$ (Lehto & Valtonen 1996; Dey et al. 2019) and NGC 4151 has $e \approx 0.4$ (Bon et al. 2012). In addition, in our previous work (Chen & Zhang 2018) we hypothesized that for the SMBBH with a large eccentricity (i.e., $e \gtrsim 0.5$), the PEBs due to pulsar terms should be smaller than those for a quasicircular SMBBH. We will test this hypothesis in this work. Consequently, our analyses in this article will be based on an eccentric SMBBH, but we will also give the results for a circular SMBBH as a specific case.

This paper is structured as follows. In Section 2, we revisit the template of timing residual from an eccentric SMBBH. In Section 3, we derive the PEBs from the ignored pulsar terms by using the noise-projection technique (Cutler & Flanagan 1994; Cutler & Harms 2006; Harms et al. 2008). In Section 4, the formulae of the PEBs are derived. In Section 5, we test the formulae of PEBs by comparing them with the numerical results given by Zhu et al. (2016). In addition, Chen & Zhang’s (2018) hypothesis will also be tested in Section 5. Moreover, the PEBs from CSP will be calculated and compared with the PEBs from pulsar terms in Section 6. Finally, we summarize in Section 7.

2. Template of Timing Residual Signals

The GWs fluctuate the TOAs of the pulses emitted from the MSPs in a PTA, and lead to timing residuals. For the $i$th MSP in a PTA ($i = 1, 2, \ldots, N_p$, with $N_p$ being the number of MSPs), its residual signal induced by the GWs at the time $t_i$ ($i = 1, 2, \ldots, N_t$, with $N_t$ being the number of data points for the $i$th MSP) is

\[ s_i(t_i) = s_i^E(t_i) - s_i^P(t_i), \]

where $s_i^E$ denotes the Earth-term signal, $s_i^P$ the pulsar-term signal, and hereafter the superscripts “$E$” and “$P$” will denote “Earth term” and “pulsar term,” respectively. Both the Earth term and pulsar term can be written as (Taylor et al. 2016)

\[ s_i^{E,P}(t_i) = \langle 2 \psi F_{1+} - \sin 2 \psi F_{1x} \rangle s_i^{E,P}(t_i) \\
+ \langle 2 \psi F_{1x} + \cos 2 \psi F_{1+} \rangle s_i^{E,P}(t_i), \]

where the subscripts “+$\times$” and “+$\times$” represent two GW polarization components respectively, $\psi$ is the polarization angle, and $\langle \rangle$ denotes the Earth-term signal, $\langle \rangle$ the pulsar-term signal,

\[ F_{1+} = \frac{(1 + \sin^2 \delta) \cos^2 \delta \cos(2\alpha - 2\alpha_p^0) - \sin 2\delta \sin 2\delta_p \cos(\alpha - \alpha_p^0) + \cos^2 \delta (2 - 3 \cos^2 \delta_p^0)}{4[1 - \cos \delta \cos \delta_p \cos(\alpha - \alpha_p^0) - \sin \delta \sin \delta_p^0]}, \]

\[ F_{1x} = \frac{-\sin \delta \cos^2 \delta_p \cos(2\alpha - 2\alpha_p^0) + \cos \delta \sin 2\delta_p \cos(\alpha - \alpha_p^0)}{2[1 - \cos \delta \cos \delta_p \cos(\alpha - \alpha_p^0) - \sin \delta \sin \delta_p^0]} \]

are the antenna pattern functions, with $\alpha$ and $\delta$ being the R.A. (R.A.) and decl. (decl.) of the SMBBH, respectively, and $\alpha_p^0$ and $\delta_p^0$ being the R.A. and decl. of the MSP, respectively (Wahlquist 1987; Zhu et al. 2016).

In general, the pulsar term corresponds to the GW at an earlier stage, so it should have different amplitude and frequency (or period) from the Earth term due to the orbital reduction of the SMBBH (Lee et al. 2011). However, for simplicity, we adopt a non-evolving assumption here—namely, the Earth term and pulsar term have the same period and amplitude. For an eccentric SMBBH, two polarization components for both the Earth term and pulsar term in Equation (2) can be written as (Taylor et al. 2016)

\[ s_{1+}^{E,P}(t) = A_m \sum_{n=1}^{\infty} \left(1 + c_n^2\right)(X_n \cos 2\phi_p S_n^{E,P} - Y_n \sin 2\phi_p C_n^{E,P}) \\
+ 2(1 - c_n^2)Z_n S_n^{E,P}, \]

\[ s_{1x}^{E,P}(t) = -2A_m c_n \sum_{n=1}^{\infty} (X_n \sin 2\phi_p S_n^{E,P} + Y_n \cos 2\phi_p C_n^{E,P}), \]
where $A_m$ is the characteristic amplitude of the timing residual signal, $c_i$ stands for the cosine of the inclination angle, $\phi_p$ denotes the orbital phase of the periastron, and

$$
X_n = J_{n-2}(en) - 2enJ_{n-1}(en) + \frac{2}{n}J_n(en) + 2eJ_{n+1}(en) - J_{n+2}(en),
$$

$$
Y_n = \sqrt{1 - e^2} [-J_{n-2}(en) + 2enJ_{n-1}(en) - J_{n+2}(en)],
$$

$$
Z_n = \frac{1}{n}J_n(en),
$$

(5)

are time-independent coefficients, with $e$ being the orbital eccentricity, $J_n$ being the first-kind Bessel function of order $n$; $S_{n}^{E}$ and $C_{n}^{E}$ are time dependent functions

$$
S_{n}^{E} = \sin \left( \frac{2n\pi t}{T} + n\phi_E \right),
$$

$$
C_{n}^{E} = \cos \left( \frac{2n\pi t}{T} + n\phi_E \right),
$$

$$
S_{n}^{P} = \sin \left( \frac{2n\pi t}{T} + n\phi_P \right),
$$

$$
C_{n}^{P} = \cos \left( \frac{2n\pi t}{T} + n\phi_P \right),
$$

(6)

de where $t=0$ denotes the initial observational time, $T$ is the orbital period of the SMBBH, $\phi_E$ denotes the phase of Earth term, and $\phi_P$ represents the phase of pulsar term (pulsar phase, hereafter).

Given the above, if we only detect the Earth-term signal (i.e., taking an Earth-term-only search), then nine parameters should be considered and they can be included in a nine-dimensional vector

$$
\{ \lambda^a \} = \{ T, A_m, \alpha, \delta, \psi, c_i, \phi_E, \phi_P, e \},
$$

(7)

with $a = 1, 2, \ldots, 9$. Moreover, if we detect pulsar terms for all MSPs (i.e., taking a full-signal-search), then all of the phases $\phi_P$ (for $i = 1, 2, \ldots, N_p$) should also be considered and the total number of parameters in this case is $9 + N_p$. The complexity grows as the number of unknown parameters enlarges, so the full-signal search is not applied in most of the current PTA data analyses (Sesana & Vecchio 2010; Ellis et al. 2012a, 2012b).

Specifically, in the circular limit $e \rightarrow 0$, the coefficients $X_n$, $Y_n$, $Z_n$ in Equation (5) approach $\delta_{n,2} - \delta_{n,2}$ and 0, respectively, with $\delta_{n,2}$ being the Kronecker delta function, so only the mode $n=2$ leaves and the GW in this case is monochromatic, with the GW frequency being twice the orbital frequency $f_{sw} = 2/T$. In addition, in the circular case, the eccentricity $e$ vanishes in the template and the orbital phase of the periastron $\phi_p$ can be absorbed into the phase parameters $\phi_E$ and $\phi_P$. As a result, for a circular SMBBH, one needs to consider seven parameters in the Earth-term-only search (the vector $\{ \lambda^a \}$ in Equation (7) with $a = 1, 2, \ldots, 7$) (Lee et al. 2011; Wang et al. 2014, 2015), and $7 + N_p$ parameters in the full-signal search (Sesana & Vecchio 2010).

As mentioned earlier, template (4) is based on a non-evolving assumption. If we relax the assumption and consider the evolution of the SMBBH, then the number of parameters will become larger in the full-signal search. Specifically, if the periastron precession is considered, then parameter $\phi_p$ in each pulsar term should be different, so the total number of parameters is $9 + 2N_p$ in this case. If the orbital reduction due to gravitational radiation is also taken, then amplitude $A_m$, orbital period $T$, and eccentricity $e$ should vary from different pulsar terms, so $9 + 5N_p$ parameters should be included in this case. If the spins of the SMBBH are also contained, then the direction of the orbital plane generally changes (Kidder 1995), and the parameters $c_i$ and $\psi$ in each pulsar term are different, so the $9 + 7N_p$ parameter should be taken into account in this general case. In conclusion, the full-signal search will become extremely complicated in these generalized cases.

### 3. Parameter-estimation Biases from Pulsar Terms

Before deriving the PEBs, let us first introduce the method of parameter estimation. In this article, we adopt the Generalized Likelihood Ratio Test (GLRT) method (see Wang et al. 2014, 2015 for details), in which we estimate parameters by the detection statistic as follows

$$
\text{GLRT} (\{ d \}) = \frac{L (\{ d \}; H (\lambda^a))}{L (\{ d \}; H_0)} = \frac{\mathcal{L} (\{ d \}; H (\lambda^a))}{\mathcal{L} (\{ d \}; H_0)},
$$

(8)

where $L$ denotes likelihood, $\{ d \} = \{ d_{i} \}$ (with $i = 1, 2, \ldots, N_p$) is the observational data set of the PTA, $t_i$ is the TOA of the $i$th data point, $\{ \lambda^a \}$ is the parameter vector given by Equation (7), $\{ \lambda^a \}$ represents the best fitted parameters from the data, $H (\lambda^a)$ denotes the hypothesis that the data contains signal with the parameter set $\{ \lambda^a \}$, and $H_0$ is the hypothesis that the data only contains noise.

The log-likelihood ratio can be expressed as (Ellis et al. 2012b; Wang et al. 2014, 2015)

$$
\log \frac{\mathcal{L} (\{ d \}; H (\lambda^a))}{\mathcal{L} (\{ d \}; H_0)} = \sum_{i=1}^{N_p} \Lambda_i (d_i; \{ \lambda^a \}),
$$

(9)

with

$$
\Lambda_i (d_i; \{ \lambda^a \}) = \langle d_i | T_i (\{ \lambda^a \}) \rangle - \frac{1}{2} \langle T_i (\{ \lambda^a \}) | T_i (\{ \lambda^a \}) \rangle, \quad (10)
$$

where $T_i$ is the template of timing residual induced by the GW for the $i$th MSP, and $\langle \ldots \rangle$ denotes the noise weighted inner product, which is defined as

$$
\langle x | y \rangle = \sum_{t_1}^{t_2} x(t_1) y(t_1) \tilde{s}_i(t_1),
$$

with $\tilde{s}_i(t_1)$ being the auto-covariance matrix of the noise for the $i$th MSP (Wang et al. 2014). In this article, we assume that the noises are dominated by stationary white Gaussian noises with standard deviation $\sigma_t$, and in this case the inner product reduces to

$$
\langle x | y \rangle = \langle y | x \rangle = \frac{1}{\sigma_t^2} \sum_{i=1}^{N_p} x(t_i) y(t_i) \approx \frac{\int x(t) y(t) dt}{\sigma_t^2 \delta t},
$$

(11)

where $\delta t_i$ is the cadence between two data points for the $i$th MSP. Taking the partial differentiation of Equation (8), one
obtains
\[
\frac{\partial \Lambda}{\partial \lambda^a} = \sum_{i=1}^{N_p} \left( d_i - T_i \right) \frac{\partial T_i}{\partial \lambda^a} = 0,
\]
\[
\{ \lambda^a \} = \{ \hat{\lambda}_k^a \},
\]
(12)
with \( \lambda^a \) being the \( a \)th element in the parameter vector \( \{ \lambda^a \} \) given by Equation (7).

In the following, we will study the full-signal and Earth-term-only searches in the frame of GLRT, respectively. In the full-signal search, both the Earth term and the pulsar terms are included in the template, i.e., \( T_i(t_i) = s_F^T(t_i) - s_p^T(t_i) \), and from (12) one obtains
\[
\frac{\partial N^g}{\partial \lambda^a} = \sum_{i=1}^{N_p} \left( d_i - s_F^T + s_p^T \left| \frac{\partial s_p^T}{\partial \lambda^a} \right| \right)_i = 0,
\]
\[
\{ \lambda^a \} = \{ \hat{\lambda}_k^a \},
\]
(13)
where the super-/subscript “fs” denotes the full-signal search. Similarly, in the Earth-term-only search, the template only includes the Earth term \( T_i(t_i) = s_F^T(t_i) \), and one yields
\[
\frac{\partial N_{eto}}{\partial \lambda^a} = \sum_{i=1}^{N_p} \left( d_i - s_F^T \right) \left| \frac{\partial s_F^T}{\partial \lambda^a} \right|_i = 0,
\]
\[
\{ \lambda^a \} = \{ \hat{\lambda}_k^a \},
\]
(14)
where the super-/subscript “eto” denotes the Earth-term-only search. It is clear that the difference between these two best fitted values \( \delta \lambda^a = \hat{\lambda}_k^a - \hat{\lambda}_{eto}^a \) is the PEBs caused by the ignored pulsar terms, which will be derived in the following.

3.2. Derivations of the Parameter-estimation Biases

By combining Equations (13) and (14), one yields
\[
\frac{\partial N^g}{\partial \lambda^a} - \frac{\partial N_{eto}}{\partial \lambda^a} = \xi_a,
\]
(15)
with
\[
\xi_a = \sum_{i=1}^{N_p} \left( d_i - s_F^T + s_p^T \left| \frac{\partial s_p^T}{\partial \lambda^a} \right| \right)_i + \left( s_p^T \left| \frac{\partial s_p^T}{\partial \lambda^a} \right| \right)_i.
\]
Note that the observational data \( d_i \) actually consists of Earth term, pulsar term, and white noise \( n_i \)—that is, \( d_i(t_i) = s_F^T(t_i) - s_p^T(t_i) + n_i(t_i) \)—and we can assume that the white noise is independent from the pulsar terms \( \langle n_i | \partial s_p^T / \partial \lambda^a \rangle = 0 \). 2 Therefore, one obtains
\[
\xi_a = \sum_{i=1}^{N_p} \left( s_p^T \left| \frac{\partial s_p^T}{\partial \lambda^a} \right| \right)_i.
\]
(16)
In the following, we will derive the PEBs and the analyses will be based on the Earth-term-only search. At the best fitted

\[\Delta \varphi_l \equiv \varphi_l - \varphi_F = 2\pi \left( 1 - \cos \mu_l \right) L_l / cT,\]
where \( \mu_l \) is the opening angle between the SMBBH and the MSP, \( L_l \) denotes the distance from the MSP to Earth, and \( c \) denotes light speed. For most MSPs, their distances \( L_l \) are poorly measured, with uncertainties much larger than \( cT \) (about

value given by the full-signal search, Equation (15) reduces to
\[
\xi_a = - \frac{\partial N_{eto}}{\partial \lambda^a}, \{ \lambda^a \} = \{ \hat{\lambda}_k^a \}.
\]
(17)

We then expand Equation (17) as \( \hat{\lambda}_k^a = \hat{\lambda}_{eto}^a + \delta \lambda^a \) and assume that the PEBs are small \( \delta \lambda^a \ll \hat{\lambda}_{eto}^a \). As a result, the leading order of the left-hand side of (17) is
\[
\xi_a \big|_{\{ \lambda^a \} = (\hat{\lambda}_{eto}^a)} + \mathcal{O}(\delta \lambda^a),
\]
and the right-hand side is
\[
\frac{\partial N_{eto}}{\partial \lambda^a} \delta \lambda^a = -\xi_a, \{ \lambda^a \} = \{ \hat{\lambda}_{eto}^a \}.
\]
(18)
Furthermore, from Equation (14), we have
\[
\frac{\partial^2 N_{eto}}{\partial \lambda^a \partial \lambda^b} \delta \lambda^a = -\sum_{i=1}^{N_p} \left( \frac{\partial s_p^T}{\partial \lambda^a} \left| \frac{\partial s_p^T}{\partial \lambda^b} \right| \right)_i.
\]
(19)
It is worth noting that in the analyses based on the Earth-term-only search, the pulsar term \( s_p^T \) is ignored—that is, \( d_i - s_F^T = n_i \), so the first term on the right-hand side of (19) should vanish, and one obtains
\[
\frac{\partial^2 N_{eto}}{\partial \lambda^a \partial \lambda^b} \delta \lambda^a = -\sum_{i=1}^{N_p} \left( \frac{\partial s_p^T}{\partial \lambda^a} \left| \frac{\partial s_p^T}{\partial \lambda^b} \right| \right)_i = -\Gamma_{ab},
\]
(20)
where \( \Gamma_{ab} \) is the so-called Fisher information matrix (Sesana & Vecchio 2010). As a result, we obtain the PEBs due to the ignored pulsar terms
\[
\delta \lambda^a = \Gamma_{ab} \xi_b = \Gamma_{ab} \sum_{i=1}^{N_p} \left( s_p^T \left| \frac{\partial s_p^T}{\partial \lambda^a} \right| \right)_i, \{ \lambda^a \} = \{ \hat{\lambda}_{eto}^a \},
\]
(21)
where \( \Gamma_{ab} \) is the inversed Fisher information matrix. Note that the above analytical method is the so-called noise-projection technique (Cutler & Flanagan 1994; Cutler & Harms 2006; Harms et al. 2008), and \( \xi_a \) in Equation (16) is termed as the noise-projection operator with the noises taken as \( s_p^T \).

3.3. Statistical Properties of the PEBs

From Equation (21), one sees that the PEBs \( \delta \lambda^a \) are functions of pulsar terms \( s_p^T \), which contain unknown parameters \( \varphi^p_f \), with \( i = 1, 2, \ldots, N_p \). In the following, we will take these unknown pulsar phases \( \varphi^p_f \) as random variables, and analyze the statistical signatures of \( \delta \lambda^a \).

For the \( j \)th MSP, its pulsar phase \( \varphi^p_f \) satisfies the relation
\[
\Delta \varphi_l \equiv \varphi_l - \varphi_F = 2\pi \left( 1 - \cos \mu_l \right) L_l / cT,
\]
(21)
10^{-1}-10 \text{ ly} \ (\text{Verbiest et al.} \ 2012), \text{so the phase difference } \Delta \phi_p \text{ generally has uncertainty much larger than } 2\pi. \text{ For convenience in calculation, we will restrict the value of } \Delta \phi_p \text{ within the span [0, 2\pi); that is, taking the transformation }
\Delta \phi_p \rightarrow 2\pi \left[ \frac{\Delta \phi_p}{2\pi} - \mathcal{N} \left( \frac{\Delta \phi_p}{2\pi} \right) \right], \quad (22)

where \( \mathcal{N}(x) \) denotes the integer part of \( x \). After this transformation, \( \Delta \phi_p \) can be considered as a random variable with a uniform distribution function \( \Delta \phi_p \in [0, 2\pi) \). Equivalently, the pulsar-term phase \( \phi_p \equiv \Delta \phi_p + \phi^{E} \), if taken the same transformation as (22), should also follow the uniform distribution \( \phi_p \in [0, 2\pi) \).

The value of each PEB \( \delta \lambda^a \), as a summation of \( N_p \) random terms according to Equation (21), should also be a random variable, and it is reasonable to assume that \( \delta \lambda^a \) follows a Gaussian distribution function when \( N_p \gg 1 \), due to the central limit theorem. Hence, if we calculate the expected value \( \langle \delta \lambda^a \rangle \) and the covariance matrix \( \Pi^{ab} \equiv \langle \delta \lambda^a \delta \lambda^b \rangle \), then we can completely fix its probability density function (PDF)
\[ \mathcal{L}(\{ \delta \lambda^a \}) \propto \exp \left( -\frac{1}{2} \langle \delta \lambda^a \rangle \Pi^{-1}_{ab} \langle \delta \lambda^b \rangle \right), \quad (23) \]

with \( \Pi^{-1}_{ab} \) being the inverse of \( \Pi^{ab} \).

It is not difficult to show that \( \delta \lambda^a \) has a vanished expected value, under the assumption \( \Delta \phi_p \in [0, 2\pi) \). From Equation (21), one obtains that
\[ \langle \delta \lambda^a \rangle = \Gamma_{ab} \sum_{i=1}^{N_p} \left\langle \left. s_p^a \right| \partial \phi^E - \frac{\partial s_p^E}{\partial \lambda^b} \right\rangle, \]

where \( \langle x \rangle = \int d\phi^E \cdot x \cdot U[0, 2\pi) = \int d(\Delta \phi_p) \cdot x \cdot U[0, 2\pi) \) denotes the average of \( x \) over \( \phi^E \) or \( \Delta \phi_p \). According to Equations (4) and (6), the pulsar term \( s_p^a \) consists of a linear combination of \( S_n^a \) and \( C_n^a \), and they satisfy \( \langle S_n^a \rangle = \langle C_n^a \rangle = 0 \). As a result, one has \( \langle \delta \lambda^a \rangle = 0 \).

Therefore, in the following, we only need to express the covariance matrix
\[ \Pi^{ab} \equiv \langle \delta \lambda^a \delta \lambda^b \rangle = \Gamma_{ac} \Gamma_{bd} \langle \xi_a \xi_b \rangle. \quad (24) \]

It is seen that to obtain the covariance matrix \( \Pi^{ab} \), we first need to calculate all of the elements of the Fisher information matrix \( \Gamma_{ab} \) and the mean-squared noise-projection matrix \( \langle \xi_a \xi_b \rangle \). The computation details are given in Section 4.

4. Computation of the Parameter-estimation Biases

From Equations (16) and (20), one can see that the expressions of \( \Gamma_{ab} \) and \( \langle \xi_a \xi_b \rangle \) should contain terms such as \( s_p^a \) and \( \partial s_p^E / \partial \lambda^b \). Hence, for convenience, we first rewrite the template (2) as
\[ s_p^{E,P} = \sum_{n=1}^{N_p} A_n S_n^{E,P} + B_n C_n^{E,P}, \quad (25) \]

with
\[ A_n = A_n \{ (\cos 2\psi F_{1s} - \sin 2\psi F_{1c}) \} (1 + c_{2}^{2}) X_n \cos 2\phi_p + 2(1 - c_{2}^{2}) Z_n - 2(\sin 2\psi F_{1s} + \cos 2\psi F_{1c}) \} X_n \sin 2\phi_p, \]
\[ B_n = A_n \{ (\cos 2\psi F_{1s} + \sin 2\psi F_{1c}) Y_n \} \sin 2\phi_p - 2(\sin 2\psi F_{1s} + \cos 2\psi F_{1c}) \} Y_n \cos 2\phi_p \].

Note that the coefficients \( A_n \) and \( B_n \) actually vary from MSPs, and should have the subscript “,” but we ignore this point for clarity without confusion. Furthermore, it is seen that, unlike in template (4), the summation in (25) has a maximum mode \( N_{f} = N_{f}(T/2 \delta t_{f}) \), which corresponds to the Nyquist frequency (Bracewell 2000), and the higher-frequency modes \( n > N_{f} \) cannot be detected by PTA with observational cadence \( \delta t_{f} \).

The partial derivative of the template can be expressed as
\[ \frac{\partial s_p^{E,P}}{\partial \lambda^a} = \sum_{n=1}^{N_p} D_{a,n} S_n^{E} + E_{a,n} C_n^{E}, \quad (27) \]

with the parameter \( \lambda^a \) given in Equation (7), and the coefficients \( D_{a,n} \) and \( E_{a,n} \) (for \( a = 1, 2, ... , 9 \)) listed in Appendix A.

In the following, we will derive the matrices \( \Gamma_{ab} \) and \( \langle \xi_a \xi_b \rangle \), respectively.

4.1. Fisher Information Matrix \( \Gamma_{ab} \)

From Equation (27), one can expand the Fisher matrix as
\[ \Gamma_{ab} = \sum_{i=1}^{N_p} \left\langle \left. \frac{\partial s_p^{E,P}}{\partial \lambda^a} \right| \partial \phi^E - \frac{\partial s_p^{E,P}}{\partial \lambda^b} \right\rangle = \sum_{i=1}^{N_p} \sum_{n=1}^{N_n} \sum_{m=1}^{N_m} \langle \xi_a \xi_b \rangle \langle \xi_a \xi_b \rangle \langle \xi_a \xi_b \rangle \langle \xi_a \xi_b \rangle \]

From the formulae of \( D_{a,n} \) and \( E_{a,n} \) given in Appendix A, it is seen that for the case \( a \neq 1 \), the coefficients \( D_{1,n} \) and \( E_{1,n} \) are time independent, and can be moved out of \( \langle \xi_a \xi_b \rangle \). However, for the case \( a = 1 \), both coefficients \( D_{1,n} \) and \( E_{1,n} \) are proportional to the time \( t \), and it should be left inside of \( \langle \xi_a \xi_b \rangle \). As a result, the Fisher matrix should be divided into three parts, as follows: the “11”-component, “1a” (or “a1”)-components (with \( a \neq 1 \)) and “ab”-components (with both \( a = 1 \) and \( b \neq 1 \)). The resulting formulae are
\[ \Gamma_{11} = \frac{4\pi^2}{T^2} \sum_{n=1}^{N_n} \sum_{m=1}^{N_m} \sum_{l=1}^{N_l} \sum_{j=1}^{N_j} \langle \xi_a \xi_b \rangle \langle \xi_a \xi_b \rangle \langle \xi_a \xi_b \rangle \langle \xi_a \xi_b \rangle \]

\[ \Gamma_{1a} = \frac{2\pi}{T^2} \sum_{n=1}^{N_n} \sum_{m=1}^{N_m} \sum_{l=1}^{N_l} \sum_{j=1}^{N_j} \langle \xi_a \xi_b \rangle \langle \xi_a \xi_b \rangle \langle \xi_a \xi_b \rangle \langle \xi_a \xi_b \rangle \]

\[ \Gamma_{ab} = \Gamma_{1a} = \Gamma_{11} \]

(28)
for $a \neq 1$, and

$$
\Gamma_{ab} = \sum_{l=1}^{N_n} \sum_{m=1}^{N_n} \sum_{n=1}^{N_n} \{D_{a,n}D_{b,n} \langle S_n^E | S_{m}^E \rangle_l + D_{a,n}E_{b,n} \langle S_n^E | C_{m}^E \rangle_l + E_{a,n}D_{b,n} \langle C_n^E | S_{m}^E \rangle_l + E_{a,n}E_{b,n} \langle C_n^E | C_{m}^E \rangle_l \}.
$$

(30)

for both $a \neq 1$ and $b \neq 1$.

Since $S_n^E$ and $C_n^E$ are trigonometric functions of time, their inner products can be analytically calculated from Equation (11) if the white noise $\sigma_n$ and cadence $\delta t$ are time independent. The formulae of all the inner-product terms in Equations (28)–(30) are listed in Appendix B.

### 4.2. Mean-squared Noise-projection Matrix $\langle \xi_a \xi_b \rangle$

From Equation (16), we can express the matrix as

$$
\langle \xi_a \xi_b \rangle = \sum_{i=1}^{N_n} \sum_{f=1}^{N_n} \langle s_f^p \rho \delta t \partial \delta \xi \rangle_i \langle s_f^p \rho \delta t \partial \delta \xi \rangle_j.
$$

(31)

For the case $l' = l$, the pulsar terms $s_f^p$ and $s_f^p$ should be independent from each other, so the second term on the right-hand side of (31) (in the 2nd row) should vanish, and one obtains a simplified formula

$$
\langle \xi_a \xi_b \rangle = \sum_{l=1}^{N_n} \langle s_f^p \rho \delta t \partial \delta \xi \rangle_i \langle s_f^p \rho \delta t \partial \delta \xi \rangle_l.
$$

(32)

Furthermore, taking Equations (25) and (27) into (32), we obtain

$$
\langle \xi_a \xi_b \rangle = \sum_{l=1}^{N_n} \sum_{m=1}^{N_n} \sum_{n=1}^{N_n} \{A_{a,n}S_n^p \rho \delta t \partial \delta \xi \} \langle D_{a,n}S_m^p \rho \delta t \partial \delta \xi \rangle_l + \{A_{a,n}C_n^p \rho \delta t \partial \delta \xi \} \langle E_{a,n}S_m^p \rho \delta t \partial \delta \xi \rangle_l + \{B_{a,n}S_n^p \rho \delta t \partial \delta \xi \} \langle D_{a,n}C_m^p \rho \delta t \partial \delta \xi \rangle_l + \{B_{a,n}C_n^p \rho \delta t \partial \delta \xi \} \langle E_{a,n}C_m^p \rho \delta t \partial \delta \xi \rangle_l \}.
$$

(33)

For the same reason as we calculate $\Gamma_{ab}$ in Section 4.1, the matrix $\langle \xi_a \xi_b \rangle$ should also be divided into three cases, and the resulting formulae are shown in Equations (D1)–(D3).

In the following, we will calculate the averaged terms $\langle \cdot \rangle$ in Equations (D1)–(D3). Recall that $\langle x \rangle = \int d(\Delta \phi_f) x \cdot U[0, 2\pi]$, so one should first express $S_n^E$ and $C_n^E$ as functions of $S_n^E$, $C_n^E$ and $\Delta \phi_f$. From Equation (6), one has

$$
S_n^p = \cos(n\Delta \phi_f)S_n^E + \sin(n\Delta \phi_f)C_n^E,
$$

$$
C_n^p = \cos(n\Delta \phi_f)C_n^E - \sin(n\Delta \phi_f)S_n^E,
$$

(34)

so the parameter $\Delta \phi_f$ in Equations (D1)–(D3) can be moved out of $\langle \cdot \rangle$, for example

$$
\langle S_n^p | S_m^E \rangle_l = \cos(n\Delta \phi_f)\langle S_n^E | S_m^E \rangle_l + \sin(n\Delta \phi_f)\langle C_n^E | S_m^E \rangle_l,
$$

and all the other inner-product terms in (D1)–(D3) are rewritten in Equation (C1). One can see that the matrix $\langle \xi_n \xi_m \rangle$ has a quadratic form of the trigonometric functions of $\Delta \phi_f$, and their expected values can easily be calculated

$$
\langle \sin(n_1\Delta \phi_f) \sin(n_3\Delta \phi_f) \rangle = \langle \cos(n_1\Delta \phi_f) \cos(n_3\Delta \phi_f) \rangle = \frac{1}{2} \delta_{n_1,n_3},
$$

$$
\langle \sin(n_1\Delta \phi_f) \cos(n_3\Delta \phi_f) \rangle = 0.
$$

(35)

After calculating all the averaged terms in Equations (D1)–(D3) (see Appendix C for details), we finally obtain the elements of the matrix listed in (D4)–(D6).

In conclusion of this part, we can calculate the mean-squared noise-projection matrix $\langle \xi_a \xi_b \rangle$ from Equations (D4)–(D6), where the coefficients $A_{a,n}$, $B_{a,n}$ are given by (26), $D_{a,n}$ and $E_{a,n}$ listed in Appendix A, and all of the inner-product terms are shown in Appendix B.

### 4.3. Simplification: Leading-order Expressions

With the Fisher matrix $\Gamma_{ab}$ given by Equations (28)–(30) and the mean-squared noise-projection matrix by (D4)–(D6), the covariance matrix of the PEBs $\Pi_{ab}$ can be computed from Equation (24). However, one can see that the formulae of the inner-product terms in Appendix B are very complicated. Consequently, in the following we will simplify these results for the efficient evaluation of $\Pi_{ab}$ by keeping them up to leading orders, in the limit $t \gg T$.

For example, from Equation (B1), we can see that $\langle S_n^E | S_m^E \rangle_l$ contains a linearly growing term ($\propto t$, when $n_1 = n_2$), oscillating terms (e.g., $\propto \sin(\pi n_1 \pm n_2 \Delta \phi_f)$), and time-independent terms (e.g., $\langle S_n^E | S_m^E \rangle_l \propto \sin(n_1 \pm n_2 \Delta \phi_f)$). In the limit $t \gg T$, the growing term dominates the results, so all the other terms can be dropped and one obtains $\langle S_n^E | S_m^E \rangle \propto 1$. As a result, we can express all the simplified formulae of the inner products in Appendix B as follows

$$
\langle S_n^E | S_m^E \rangle_{l=1} = \langle C_n^E | C_m^E \rangle_{l=1} = \frac{1}{2\sigma_f} \delta_{n_1,n_2},
$$

$$
\langle S_n^E | C_m^E \rangle_{l=1} = \langle C_n^E | S_m^E \rangle_{l=1} = 0,
$$

$$
\langle S_n^E | tC_m^{E \lead} \rangle_{l=1} = \langle C_n^E | tS_m^{E \lead} \rangle_{l=1} = \frac{t^2}{4\sigma_f^2} \delta_{n_1,n_2},
$$

$$
\langle S_n^E | tS_m^{E \lead} \rangle_{l=1} = \langle C_n^E | tC_m^{E \lead} \rangle_{l=1} = 0,
$$

$$
\langle tS_n^E | tC_m^{E \lead} \rangle_{l=1} = \langle tC_n^E | tS_m^{E \lead} \rangle_{l=1} = \frac{t^3}{6\sigma_f^2} \delta_{n_1,n_2},
$$

$$
\langle tS_n^E | tS_m^{E \lead} \rangle_{l=1} = 0.
$$

(36)

It is worth noting that the vanished terms in (36) actually have non-zero results: $\langle S_n^E | C_m^{E \lead} \rangle \rightarrow \text{const}$, $\langle S_n^E | tC_m^{E \lead} \rangle \propto t$, and $\langle S_n^E | tS_m^{E \lead} \rangle \propto t^2$, and we ignore them because they grow more slowly than their congeners $\langle S_n^E | S_m^E \rangle_{l=1}$, $\langle S_n^E | tS_m^{E \lead} \rangle_{l=1}$ and $\langle tS_n^E | tS_m^{E \lead} \rangle_{l=1}$, respectively.
Taking Equation (36) into Equations (28)–(30), we obtain the simplified expressions for the Fisher matrix

\[
\begin{align*}
\Gamma_{11}^{\text{lead}} &= \sum_{i=1}^{N_p} \sum_{j=1}^{N_f} \frac{8\pi^2}{3T^2\tau^2} (A_n^2 + B_n^2), \\
\Gamma_{1u}^{\text{lead}} &= \sum_{i=1}^{N_p} \sum_{j=1}^{N_f} \frac{\pi T^2}{T^2\tau^2} (B_n D_{a,n} - A_n E_{a,n}), \\
\Gamma_{ab}^{\text{lead}} &= \sum_{i=1}^{N_p} \sum_{j=1}^{N_f} \frac{t}{2\tau^2} (D_{a,n} D_{b,n} + E_{a,n} E_{b,n}), \quad a \neq 1 \& b \neq 1,
\end{align*}
\]

and taking (36) into Equations (D4)–(D6), we yield the simplified mean-squared-noise-projection matrix

\[
\begin{align*}
\langle \xi_1 \xi_1 \rangle^{\text{lead}} &= \frac{1}{2} \sum_{i=1}^{N_p} \sum_{j=1}^{N_f} \left( \frac{\pi T^2}{T^2\tau^2} \right)^2 (A_n^2 + B_n^2)^2, \\
\langle \xi_1 \xi_a \rangle^{\text{lead}} &= \frac{1}{2} \sum_{i=1}^{N_p} \sum_{j=1}^{N_f} \left( \frac{\pi T^2}{T^2\tau^2} \right)^2 (A_n^2 + B_n^2) (B_n D_{a,n} - A_n E_{a,n}), \\
\langle \xi_a \xi_b \rangle^{\text{lead}} &= \frac{1}{2} \sum_{i=1}^{N_p} \sum_{j=1}^{N_f} \left( \frac{t}{2\tau^2} \right)^2 (A_n^2 + B_n^2) (D_{a,n} D_{b,n} + E_{a,n} E_{b,n}), \quad a \neq 1 \& b \neq 1.
\end{align*}
\]

Finally, the leading-order formula of covariance matrix of the PEBs \( \Gamma^{ab} \) is

\[
(\Gamma^{ab})^{\text{lead}} = (\Gamma^{ac})^{\text{lead}} (\Gamma^{bd})^{\text{lead}} \langle \xi_c \xi_d \rangle^{\text{lead}},
\]

where \( (\Gamma^{ab})^{\text{lead}} \) is the inverse of \( \Gamma^{ab} \).

4.4. Circular Case

In this part, we will formulate the PEBs for the circular case \( e = 0 \).

As was mentioned in Section 2, the orbital phase of the periastron \( \phi_p \) can be absorbed by the phases \( \varphi^p \) or \( \varphi_p^c \) in this case, so we can take \( \phi_p = 0 \) in the following analyses. Therefore, one needs to consider only seven parameters in the Earth-term-only search; that is, the parameter vector \( \{ \lambda^n \} \) given by (7) should be considered only for \( a = 1, 2, \ldots, 7 \).

Furthermore, only the \( n = 2 \) mode survives in the formulae when \( e = 0 \), so the template in Equations (25) and (27) reduces to

\[
\begin{align*}
\tilde{s}_f^{E,P} &= AS^{E,P} + BC^{E,P}, \\
\frac{\partial \tilde{s}_f^E}{\partial \lambda^a} &= D_a S^E + E_a C^E,
\end{align*}
\]

where \( A, B, D_a, E_a, S^{E,P} \), and \( C^{E,P} \) denote \( A_2, B_2, D_{a,2}, E_{a,2}, S_2^{E,P} \), and \( C_2^{E,P} \) respectively, and we default the subscript “2” in the analyses of the circular case. The coefficients \( A, B, D_a \) and \( E_a \) are formulated in Appendix E.

As a result, the Fisher information matrix \( \Gamma_{ab} \) in Equations (28)–(30) reduces to

\[
\begin{align*}
\Gamma_{11} &= \frac{16\pi^2}{T^4} \sum_{i=1}^{N_p} B^2 (tS_f^E) tS_f^E | tC_f^E | + AB (tS_f^E) tC_f^E | tC_f^E |, \\
&\quad - AB (tC_f^E) tS_f^E | tC_f^E | + A^2 (tC_f^E) tC_f^E), \\
\Gamma_{1a} &= \frac{4\pi^2}{T^2} \sum_{i=1}^{N_p} BD_a (tS_f^E) tS_f^E | tC_f^E | - AD_a (tS_f^E) tC_f^E), \\
&\quad + BE_a (tC_f^E) tS_f^E | tC_f^E | - AE_a (tC_f^E) tC_f^E | tC_f^E |,
\end{align*}
\]

for \( a \neq 1 \) and \( b \neq 1 \), and the mean-squared-noise-projection matrix \( \langle \xi_a \xi_b \rangle \) given by (D4)–(D6) reduces to Equations (G1)–(G3). Given the above, we can calculate the matrices \( \Gamma_{ab} \) and \( \langle \xi_a \xi_b \rangle \) for the circular case from Equations (42)–(44) and (G1)–(G3), with all the inner-product terms therein given in Appendix F.

Furthermore, the leading-order formulae of \( \Gamma_{ab} \) and \( \langle \xi_a \xi_b \rangle \) in (37) and (38) reduce to

\[
\begin{align*}
\Gamma_{11}^{\text{lead}} &= \sum_{i=1}^{N_p} \frac{8\pi^3}{3T^4\tau^2} (A^2 + B^2), \\
\Gamma_{1a}^{\text{lead}} &= \sum_{i=1}^{N_p} \frac{\pi T^2}{T^2\tau^2} (BD_a - AE_a), \quad a \neq 1, \\
\Gamma_{ab}^{\text{lead}} &= \sum_{i=1}^{N_p} \frac{t}{2\tau^2} (D_a D_b + E_a E_b), \quad a \neq 1 \& b \neq 1,
\end{align*}
\]

and

\[
\begin{align*}
\langle \xi_1 \xi_1 \rangle^{\text{lead}} &= \frac{1}{2} \sum_{i=1}^{N_p} \left( \frac{\pi T^2}{T^2\tau^2} \right)^2 (A^2 + B^2)^2, \\
\langle \xi_1 \xi_a \rangle^{\text{lead}} &= \frac{1}{2} \sum_{i=1}^{N_p} \left( \frac{\pi T^2}{T^2\tau^2} \right)^2 (A^2 + B^2) (BD_a - AE_a), \quad a \neq 1, \\
\langle \xi_a \xi_b \rangle^{\text{lead}} &= \frac{1}{2} \sum_{i=1}^{N_p} \left( \frac{t}{2\tau^2} \right)^2 (A^2 + B^2) (D_a D_b + E_a E_b), \quad a \neq 1 \& b \neq 1.
\end{align*}
\]
The time evolutionary behaviors of standard deviations of the PEBs $\Delta \lambda^a \equiv \sqrt{\Pi^a}$ (for $a = 1, 2, \ldots, 7$) are illustrated in Figure 1, and both the full results (from Equations (42)–(44) and (G1)–(G3)) and the leading-order results (from Equations (45)–(46)) are plotted as a comparison. It is seen that after an orbital period $t > T$ (3 yr in this case), the full and leading-order results almost overlap. This implies that the leading-order approximation is quite accurate. Furthermore, we can conclude that $\Delta T = 0$, and the other PEBs $\Delta \lambda^a (a \neq 1)$ approach constants when $t \gg T$. This result indicates that pulsar terms do not practically affect the measurement of $T$ when the

Figure 1. The top left-hand panel shows the sky locations of the injected GW source (black dot) and 10 MSPs (green stars), and the remaining panels illustrate the evolutionary behaviors of $\Delta \lambda^a$ (for $a = 1, 2, \ldots, 7$ respectively), with the orange curves representing the results given by the full formulae (42)–(44) and (G1)–(G3), and the black-thick lines standing for the results from leading-order approximation (45)–(46). The white noise for the 10 MSPs is 100 ns, and the injected GW parameters are: $T = 3$ yr, $A_\alpha = 10$ ns, $\psi = -45^\circ$, $\alpha_i = 0.3$, and $\varphi^e = 1.0$ rad. Note that, the $\Delta T$ curve on the top right-hand panel actually represents the numerical error, and one can effectively yield $\Delta T = 0$. 

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observational time is long enough, but may have strong impacts on the measurements of the other parameters. For example, the PEB of the amplitude in Figure 1 is \( \Delta A_m \approx 3.766 \) ns, with the true amplitude being \( A_m = 10 \) ns, so the corresponding relative deviation is as large as \( \Delta A_m/A_m \approx 37.66\% \). For the result \( \Delta T = 0 \), we infer that this happens because we have taken a non-evolving assumption in our analyses, namely the pulsar terms have the same orbital period as the Earth term, so they have no bias in the measurement of \( T \). We predict that if the non-evolving assumption is eliminated, then one will have \( \Delta T \neq 0 \) (as will be shown in Section 6). In addition, we argue that even if \( \Delta T \neq 0 \), its value should decay soon. This happens because from (24), one yields

\[
\Delta T = \sqrt{\Gamma_{11}^1\Gamma_{11}^1\langle \xi_1 \xi_1 \rangle + \Gamma_{11}^1\Gamma_{11}^2\langle \xi_1 \xi_2 \rangle + \Gamma_{11}^1\Gamma_{11}^3\langle \xi_1 \xi_3 \rangle},
\]

for both \( a = 1 \) and \( b 
eq 1 \), where \( (\Gamma_{11}^1)_{\text{lead}} \propto t^{-3} \), \( (\Gamma_{11}^2)_{\text{lead}} \propto t^{-2} \), \( (\Gamma_{11}^3)_{\text{lead}} \propto t^{-1} \), \( \langle \xi_1 \xi_2 \rangle_{\text{lead}} \propto t^4 \), \( \langle \xi_1 \xi_3 \rangle_{\text{lead}} \propto t^3 \), and \( \langle \xi_2 \xi_3 \rangle_{\text{lead}} \propto t^2 \) according to (45)–(46), so finally we obtain a decaying result \( \Delta T_{\text{lead}} \propto t^{-1} \). Similarly, we have

\[
\Delta \lambda^a = \sqrt{\Gamma_{41}^a\Gamma_{41}^a\langle \xi_1 \xi_1 \rangle + \Gamma_{41}^a\Gamma_{41}^{ab}\langle \xi_1 \xi_2 \rangle + \Gamma_{41}^a\Gamma_{41}^{ac}\langle \xi_1 \xi_3 \rangle},
\]

for \( a = 1 \), \( b = 1 \) and \( c = 1 \), and from (45)–(46), we can obtain \( \Delta \lambda^a \rightarrow \text{const} \) (for \( a = 1 \)), as is shown earlier.

5. Tests and Applications

5.1. Sky Localization Biases of a Circular SMBBH

In Section 4, we presented the formulae of the PEBs caused by pulsar terms. In this section, we will test the validity of the formulae. As mentioned earlier, Zhu et al. (2016) used synthetic data to demonstrate that the ignored pulsar terms can lead to biased localization of the SMBBH. In Figure 2, we show the estimated sky locations by Zhu et al. (2016) for both Earth-term-only and full-signal searches. It is seen that the estimated locations in two schemes (red and black regions respectively) do not overlap, which implies that the PEBs due to pulsar terms are no smaller than the measurement errors due to the white noise (the sizes of the black and red regions). Hereafter, we want to apply our formulae to calculate sky localization biases due to pulsar terms and we compare our results with those given by Zhu et al. (2016). Note that to convincingly test the formulae, more results given by other authors (Corbin & Cornish 2010; Ellis et al. 2012a) should be considered. However, here we only illustrate the test using the case in Zhu et al. (2016), due to the data availability.

We adopt the same parameters used in Zhu et al. (2016)—the SMBBH has a circular orbit with (injected values): \( T = 6.34 \) yr (the GW frequency \( f_0 = 10 \) nHz in the paper), \( A_m = 213.3 \) ns (\( A_m = h_0 T / 4 \pi \), with GW amplitude \( h_0 = 1.34 \times 10^{-14} \) for the strong signal case \( S/N = 100 \) in that paper), \( \alpha = 3.2594 \) rad, \( \delta = 0.2219 \) rad, \( c_i = 0.88 \), \( \psi = 0.5 \), \( \varphi = 2.89 \). Furthermore, we

![Figure 2](image-url)
assume that all parameters except $\alpha_0$ and $\delta_0$ have been accurately measured in both Earth-term-only and full-signal searches ($\lambda_{et0}^a = \lambda_{et1}^a$, for $a = 3, 4$), and the best fitted location given by the Earth-term-only search are $\delta_{et0} = 3$ and $\delta_{eto} = 0.305$ (i.e., $\cos(\delta_{et0} - \pi/2) = 0.3$ in that paper). Note that, here we ignore the PEBs of the other parameters because we intend to recover the case in Zhu et al. (2016), and the results may change if they are considered, as will be shown in Section 6. The 12 MSPs chosen in Zhu et al. (2016) are: J0437-1475, J1600-3053, J1640+2224, J1713+0747, J1741+1351, J1744-1134, J1909-3744, J1939+2134, J2017+0603, J2043+1711, J2241-5236, and J2317+1439, respectively, with their sky locations ($\{\alpha_0^I \delta_0^I\}$ with $I = 1, \ldots, 12$) given by the ATNF pulsar Catalogue\(^3\) (Manchester et al. 2005), and the rms of their white noises ($\sigma_0$) are: 58, 202, 158, 116, 233, 203, 102, 104, 238, 170, 300, and 267 ns, respectively. In addition, the total observational time is $t = 15$ yr and the cadence is taken as $\delta t = 2$ weeks for all MSPs.

We will calculate the marginalized likelihood with respect to the sky locations. Since we have assumed that all parameters follow Gaussian distributions in Equation (23), the marginalized likelihood is obtained simply by dropping the irrelevant variables

\[
\begin{split}
\mathcal{L} \propto & \exp \left( -\frac{1}{2} \delta \lambda^a \Pi_{ab} \delta \lambda^b \right) \\
= & \exp \left( -\frac{1}{2} \left( \lambda_{et0}^a - \bar{\lambda}_{et0}^a \right)^2 \Pi_{ab} \left( \lambda_{et0}^b - \bar{\lambda}_{et0}^b \right) \right),
\end{split}
\]

where with $a = 3, 4$, and $b = 3, 4$, and the covariance matrix being

\[
\begin{align*}
\Pi_{33} &= \Gamma_{33} \Gamma_{33} + 2 \Gamma_{34} \Gamma_{34} \langle \xi_3 \xi_4 \rangle + \Gamma_{44} \Gamma_{34} \langle \xi_4 \xi_3 \rangle \\
\Pi_{34} &= \Gamma_{34} \Gamma_{34} (\langle \xi_3 \xi_4 \rangle + \langle \xi_4 \xi_3 \rangle) + \Gamma_{34} \Gamma_{44} \langle \xi_4 \xi_4 \rangle + \Gamma_{34} \Gamma_{44} \langle \xi_3 \xi_3 \rangle \\
\Pi_{44} &= \Gamma_{44} \Gamma_{44} \langle \xi_4 \xi_4 \rangle + 2 \Gamma_{44} \Gamma_{34} \langle \xi_3 \xi_3 \rangle + \Gamma_{44} \Gamma_{44} \langle \xi_4 \xi_4 \rangle.
\end{align*}
\]

Note that because $t > T$ is satisfied in this example, we can use the leading-order approximations in Equations (45)–(46) to calculate the matrices $\Gamma_{ab}$ and $\langle \xi_a \xi_b \rangle$ (for $a, b = 3, 4$) above. The results are shown in Figure 2, and we can see that the sky localization biases given by Zhu et al. (2016) are in accordance with our formulae at 1.5$\sigma$ level. This implies that our formulae are effective, at least as an estimation of the order of magnitude.

\footnote{http://www.atnf.csiro.au/research/pulsar/psrcat/}

5.2. PEBs as Functions of Eccentricity

In our previous work (Chen & Zhang 2018), we proposed the following hypothesis. For a highly eccentric SMBBH (say, $e \geq 0.5$), its GW in each orbital period is strong only in a short duration around the periastron, so the waveforms (for both Earth term and pulsar terms) should have periodic-burst-like profiles. As a result, the “bursts” of Earth term and pulsar terms have small probabilities of overlapping. In this case, the Earth term and pulsar terms can be distinguished clearly, so pulsar terms will have small impact on the detection of Earth-term signals; namely, the PEBs due to pulsar terms for a highly eccentric SMBBH should be smaller than those for a quasi-circular SMBBH.

To test this hypothesis, here we calculate the standard deviations $\Delta \lambda^a$ (for $a = 2, 3, \ldots, 9$),\(^4\) to see whether their values decrease as the eccentricity $e$ increases for various parameters. The parameters of the GW/SMBBH in our calculation are chosen as $T = 1$ yr, $A_m = 10$ ns, $a = 0$, $\beta = 0$, $\psi$, $c_i$, $\varphi^E$, and $\varphi_p$ are randomly taken by $\psi \in U[0, \pi]$, $c_i \in U[0, 1]$, $\varphi^E \in U[0, 2\pi]$, and $\varphi_p \in U[0, 2\pi]$. For each case of these parameters, we consider $e = \{0, 0.05, 0.1, \ldots, 0.95\}$. In addition, we simulate 20 MSPs to detect the GW, with their sky locations randomly distributed in a region around the SMBBH $\{\alpha_0 \in U[-0.1, 0.1] \& \sin \delta_0 \in U[-0.1, 0.1]\}$, their white noise is taken as $\sigma_0 = 100$ ns, the sampling cadence $\delta t = 2$ weeks, and the total observational time $t = 15$ yr for all MSPs ($I = 1, 2, \ldots, 20$). Because the observational time is much larger than the orbital period $t \gg T$, we can use the simplified formulae (37) and (38) to calculate the PEBs.

By repeating this calculation, we find that for only two parameters, $\varphi^E$ and $e$, do their PEBs (or functions of PEBs), $\Delta \varphi^E$ and $\Delta e/e$, present monotonically decreasing relations with the eccentricity $e$ for various parameters, as is shown in Figure 3. We argue that the two parameters $\varphi^E$ and $e$ have more direct relationships with the profile of the waveform than with the other parameters. In detail, $\varphi^E$ determines the moments of “bursts” and $e$ determines the widths of the “bursts.” This is the reason why their PEBs (or functions of PEBs) present obvious correlations with $e$, while the other parameters do not. Furthermore, from Figure 3, one can see that the decreasing relationships are only valid when $e \leq 0.7$. The anomaly for $e \geq 0.7$ can be explained because for the GWs with $T = 1$ yr and $e \geq 0.7$, the widths of “bursts” are smaller

\footnote{Since the non-evolving assumption is taken, one has $\Delta \lambda^1 = \Delta T = 0$, as is stated in Section 4.4.}
than the cadence $\delta t_l = 2$ weeks; namely, the super-Nyquist modes $n > N_l = N(T/2\delta t_l)$ have important contributions in this case, and they cannot be detected by the PTA. This implies that to detect highly eccentric GWs by PTA, high-cadence observations or staggered samplings (Wang et al. 2021) are required in PTA. Note that there have been a few MSPs with high-cadence (daily or higher) observations, such as J1939+2143 (B1937+21) (Yi et al. 2014) and J1713+0747 (Dolch et al. 2016; Perera et al. 2018) and so on. This may help the search for highly eccentric SMBBHs in the future.

In conclusion, we find that two PEBs, $\Delta \varphi_{E}$ and $\Delta e/e$, monotonically decrease as the eccentricity $e$ increases, which partly confirms our hypothesis in Chen & Zhang (2018).

6. Comparison Between PEBs from Pulsar Terms and from CSP

As mentioned in Section 1, multiple collaborations have recently detected a CSP, which also leads to PEBs for the individual SMBBH if the CSP is not properly considered in the noise model. In this part, we will calculate the PEBs arising from the CSP, and compare them with the PEBs caused by pulsar terms.

Although the origin of the CSP is still unclear, it is usually considered as an inking of the SGWB with a characteristic strain

$$h_{\text{SGWB}}(f) = A_{\text{SGWB}} \left( \frac{f}{f_{\nu_i}} \right)^{\beta},$$

(48)

with $A_{\text{SGWB}} \approx 2 \times 10^{-15}$, $\beta = -2/3$ and $f_{\nu_i} = 31.7$ nHz (Phinney 2001; Arzoumanian et al. 2020; Chen et al. 2021; Goncharov et al. 2021; Antoniadis et al. 2022). Hence, the time domain gravitational waveform of the SGWB for the $i$th MSP in the PTA can be written as

$$h_i(t) = \sum_{f} h_{i,f} \cos(2\pi ft + \phi_{t,f}),$$

where the amplitude $h_{i,f}$ is a random variable with a standard deviation $h_{\text{SGWB}}(f)$ given by (48), and the phase $\phi_{t,f} \in U[0, 2\pi)$. According to the NANOGrav 12.5 yr results (Arzoumanian et al. 2020), the five lowest frequencies ($f \in [1, 2, 3, 4, 5] \times f_{\nu_i}/12.5$) contribute 99.98% of the total S/N. Hence, we will only consider these five frequencies in the following computations. Furthermore, if the CSP truly arises from the SGWB, then the amplitudes $h_{i,f}$ among different MSPs should be correlated to ensure that the overlap reduction function of the timing residuals follows the Hellings–Downs (HD) correlation (Hellings & Downs 1983). However, the reported overlap reduction functions by various collaborations (Arzoumanian et al. 2020; Chen et al. 2021; Goncharov et al. 2021; Antoniadis et al. 2022) tend to be uncorrelated when compared with the HD correlation.5 Therefore, in our approach, the amplitudes can be taken as random variables independently from MSPs $h_{i,f} \in N[0, h_{\text{SGWB}}(f)^2]$. As a result, the timing residual corresponding to the CSP can be expressed as

$$s_{i}^{\text{CSP}}(t) = \int h_i(t') dt' = \sum_{f} h_{i,f} \frac{1}{2\pi f} \sin(2\pi ft + \phi_{t,f}),$$

(49)

and hereafter the superscript "(CSP)" denotes signals or PEBs arising from the CSP.

In the following, we still take the circular case used in Zhu et al. (2016) and in Section 5.1. In this case, the PEBs$^C_{\text{CSP}}$ can also be evaluated by the noise-projection technique (Cutler & Flanagan 1994; Cutler & Harms 2006; Harms et al. 2008)

$$\delta \chi^2(\text{CSP}) = \Gamma_{\text{CSP}} \sum_{i=1}^{N_p} \left[ \frac{\partial \delta \lambda^{\text{CSP}}}{\partial \lambda^i} \right]_t^2,$$

(50)

By combining Equations (48)–(50), it is clear that the PEBs$^{C}_{\text{CSP}}$ are proportional to $A_{\text{SGWB}}$. Additionally, the ratios between the PEBs$^C_{\text{CSP}}$ and the PEBs from pulsar terms (hereafter, noted by the superscript "(PT)" depends on the relative strength between the SGWB and the continuous GW $h_{\text{SGWB}}(f_0)/h_0$, with $h_0$ and $f_0$ being the amplitude and frequency of the continuous GW, respectively. To illustrate, we will allow a variation of $h_0$ in the following, considering all the strong ($h_0 = 1.34 \times 10^{-14}$ or $S/N = 100$), moderate ($h_0 = 4.03 \times 10^{-15}$ or $S/N = 30$), and weak ($h_0 = 1.07 \times 10^{-15}$ or $S/N = 8$) cases in Zhu et al. (2016) (see the first panel in Figure 4). Note that, the strong case in Zhu et al. (2016) has actually been ruled out by Zhu et al. (2014), Babak et al. (2015), Aggarwal et al. (2020), and Taylor et al. (2020), and here we consider it only for the purpose of illustrating the analyses. Furthermore, we expect that the formulae of the covariance matrix of PEBs$^{C}_{\text{CSP}}$ would be more complicated than Equations (45) and (46) for the PEB (PT). Consequently, numerical simulations will be applied to calculate $\delta \chi^2$ (CSP) for $a = 1, 2, ..., 7$, and obtain their PDFs and standard deviations $\Delta \chi^2_{\text{a(CSP)}}$.

The resulting PDFs of the PEBs$^{C}_{\text{CSP}}$ for the strong case are shown in Figure 4. First, it is seen that, unlike the pulsar terms, the CSP affects the estimation of the period; that is, $\delta \tau^{\text{CSP}} \neq 0$. This happens because the pulsar terms are assumed to have the same frequency as the Earth term, while the CSP/SGWB generally contains components with different frequencies. Hence, this result confirms our inference in Section 4.4. Furthermore, we note that the localization biases from pulsar terms ($\Delta \chi^2(\text{PT}) = 0.34$ and $\Delta \chi^2(\text{PT}) = 0.31$) are different from the results in Figure 2 ($\Delta \chi^2(\text{PT}) = 0.23$ and $\Delta \chi^2(\text{PT}) = 0.33$). This happens because the results in Section 5.1 are based on the assumption that all of the other parameters are precisely measured, which is removed in this case: $\Delta \chi^2_{\text{CSP}}/\Delta \chi^2_{\text{m}} \approx 4.57$, $\Delta \chi^2(\text{PT}) \approx 18.3$, $\Delta \chi^2(\text{PT}) \approx 4.26$, and $\Delta \chi^2(\text{PT}) \approx 18.4$. Additionally, in this strong case $h_0/h_{\text{SGWB}}(f_0) \approx 3$, the localization biases ($\delta \phi$ and $\delta \delta$) given by pulsar terms and by the CSP are nearly the same, while the PEBs$^{C}_{\text{CSP}}$ for the other parameters (except the period $T$) are considerably smaller than PEBs$^{(PT)}$. In this sense, the CSP will have larger impacts on the localization of the SMBBH than the other parameters (except $T$). To better compare PEBs$^{C}_{\text{CSP}}$ and PEBs$^{(PT)}$, we present the ratios between their standard deviations in Table 1, which indicate that $\Delta \chi^2_{\text{a(CSP)}}/\Delta \chi^2_{\text{a(PT)}}$ (for $a = 2, 3, ..., 7$) are approximately inversely proportional to $h_0/h_{\text{SGWB}}(f_0)$. Finally, we can conclude that to ensure that PEBs$^{C}_{\text{CSP}}$ are smaller than PEBs$^{(PT)}$, the continuous GW is required to be significantly stronger than the SGWB, $h_0/h_{\text{SGWB}}(f_0) \gtrsim 3$, at least for the cases investigated in this work.

5 For simplicity, we do not consider the complicated cases that include multiple components of uncorrelated, monopolar, dipolar, and quadrupolar (HD) correlations.
Figure 4. The top left-hand panel shows the strain of the SGWB (blue-dashed curve) given by (48) and the continuous GWs discussed in Zhu et al. (2016), with the red, magenta, and green stars representing the strong, moderate, and weak cases, respectively. The remaining panels illustrate the PDFs of the PEBs (CSP) and PEBs (PT) for the strong case, where the light-pink histograms illustrate the PDFs of $\lambda_{\text{CSP}}$ from 1000 realizations, the red curves are the fitted Gaussian PDFs of $\lambda_{\text{CSP}}$, and the blue curves represent the PDFs of $\lambda_{\text{PT}}$. 
7. Summary and Further Discussions

In this article, we have presented an analytical approach to estimate the PEBs caused by the ignored pulsar terms in PTA data analyses. Our formulae can be applied conveniently, as long as a GW event is announced and the measured parameters based on an Earth-term-only search are released by a PTA collaboration in the future. The analyses based on a single SMBBH in this work will be extended to the cases of multiple SMBBHs, or even the SGWB, in the future.

Our formulae of sky localization biases are in accordance with the numerical results given by Zhu et al. (2016) at 1.5σ level. This implies that our results are effective, at least as an estimation of the order of magnitude for this case. Testing the formulae through more numerical simulations will be one of the subjects of our future works.

We also investigate the PEBs in eccentric cases, and find that \( \Delta \varphi^E \) and \( \Delta e \) monotonically decrease as the eccentricity \( e \) increases. The decreasing relations are helpful to project future PTA observations. For example, the PTA observations are usually planned with averaged cadence \( \delta t_\text{f} = \text{const} \), but this is not a very good strategy for detecting the GWs from an eccentric SMBBH. One can expect that if the data points are centralized in the “bursts” of the Earth-term waveform (with the total number of data points \( N_f \) fixed), then the PEBs from pulsar terms (or other GW sources) will decrease. In addition, more frequent observations around the “bursts” (\( \delta t_\text{f} < 2 \text{ weeks} \)) will increase the Nyquist Frequency, so higher-frequency modes can be well detected. Therefore, if the residuals from a highly eccentric SMBBH are observed (i.e., measure \( \varphi^E \) and \( e \) with small biases), then we will know the time and durations of the following “bursts” accurately. We can then re-arrange the subsequent observational time to ensure that the data points are around the “bursts” to decrease the PEBs.

Moreover, we also numerically calculate the PEBs arising from the CSP and find that the PEBs(PT) are larger when the continuous GW is significantly stronger than the SGWB, and otherwise the PEBs(CSP) are larger. To better understand the properties of the PEBs(CSP), we plan to derive their formulae in the same manner as we treat the PEBs(PT) in the future. Additionally, we intend to extend the analyses to the case considering a joint search for both the continuous GWs and SGWB. Note that Bécsy & Cornish (2020) have considered the case using Bayesian methods, finding that the SGWB decreases the significance/Bayes factors for low-frequency continuous GWs \( f_0 \leq 20 \text{ nHz} \). Hence, we hope to compare our analytical results with Bécsy & Cornish (2020) in the future.

Currently, the distances of MSPs are generally poorly measured, so we have treated the pulsar-term phases \( \varphi_f^P \) as random variables obeying \( U(0, 2\pi) \) for all the MSPs. However, note that there are still a few MSPs with precisely measured distances; for example, PSR J0437-4715 has a distance \( 156.79 \pm 0.25 \text{ pc} \) (Reardon et al. 2015), with the uncertainty \( 0.25 \text{ pc} \) comparable or even smaller than the typical GW wavelengths of PTA (0.1–10 ly). As a result, when these MSPs are contained in the PTA, the PDF \( \varphi_f^P \in U(0, 2\pi) \) no longer holds and the statistical results of \( \delta \lambda^a \) should change (e.g., the expected value \( \langle \delta \lambda^a \rangle = 0 \) is no longer satisfied). Therefore, it is worth studying the PEBs for PTA with these MSPs.

Furthermore, from Equations (28)–(30) and (D1)–(D3), we see that \( \Gamma^{ab} \propto N_p^{-1} \) and \( \langle \xi_j \xi_k \rangle \propto N_p \), implying \( \delta \lambda^a \propto N_p^{-1/2} \); namely, the PEBs decay as the MSP number increases. This implies that in the future PTA based on SKA or FAST, as more MSPs are expected to be detected, it is possible that the PEBs will be small enough and the Earth term can be detected accurately. For example, as we estimated, for SMBBHs with various parameters, the typical relative PEB of the amplitude is \( \Delta A_m/A_m \approx 15\% \) at the current level (\( N_p \sim \mathcal{O}(10^5) \)) and it can achieve the accuracy \( \approx 5\% \) in the SKA or FAST era (\( N_p \sim \mathcal{O}(10^5) \)). If the Earth term has been accurately measured and a few MSPs have high individual S/Ns \( \rho_l = \langle s_f^P | s_f^P \rangle \approx \langle s_f^P | s_f^P \rangle \gg 1 \), then we can further measure their pulsar phases \( \varphi_f^P \) precisely (i.e., detect their pulsar terms). This scheme seems simpler than the full-signal research, requiring fewer parameters in the data analyses, and merits further study in our follow-up work. Precisely measured pulsar terms can serve as a powerful tool for astrophysical and cosmological research, such as improving the pulsar distance measurements to sub-parsec precision (Lee et al. 2011), probing the long-time (\( \gtrsim 10^3 \text{ yr} \)) evolutionary histories of GW sources (Mingarelli et al. 2012; Chen & Zhang 2018), yielding a standard siren with purely GW measurement (D’Orazio & Loeb 2021), and so on.

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| Case   | \( \Delta \Gamma^{\text{CSP}}/T \) | \( \Delta \Gamma^{\text{CSP}}/\Delta \mu^P \) | \( \Delta \epsilon^{\text{CSP}}/\Delta \mu^P \) | \( \Delta \epsilon^{\text{CSP}}/\Delta \mu^P \) | \( \Delta \epsilon^{\text{CSP}}/\Delta \mu^P \) | \( \Delta \epsilon^{\text{CSP}}/\Delta \mu^P \) |
|--------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| Strong | 0.09                           | 0.77                           | 0.94                           | 1.01                           | 0.71                           | 0.75                           |
| Moderate | 0.32                         | 2.53                           | 3.16                           | 3.28                           | 2.43                           | 2.46                           | 2.44                           |
| Weak   | 1.14                           | 9.74                           | 13.0                           | 12.8                           | 9.55                           | 9.48                           | 9.57                           |
Appendix A

Coefficients $D_{a,n}$ and $E_{a,n}$

The coefficients $D_{a,n}$ and $E_{a,n}$ for $a = 1, 2,...,9$ in Equation (27) are as follows

\begin{align}
D_{1,n} &= \frac{2\pi n B_{a,n}}{T^2}, \quad E_{1,n} = -\frac{2\pi n A_{a,n}}{T^2}, \tag{A1} \\
D_{2,n} &= \frac{A_n}{A_m}, \quad E_{2,n} = \frac{B_n}{A_m} \tag{A2}
\end{align}

\begin{align}
D_{3,n} &= A_m \{ \cos 2\psi F_{3}^{(3)} + 2(1 - c_7^2)X_n \cos 2\phi_p + 2(1 - c_7^2)Z_n \}, \\
E_{3,n} &= A_m \{ -\cos 2\psi F_{3}^{(3)} + 2(1 + c_7^2)Y_n \sin 2\phi_p + 2(\sin 2\psi F_{3}^{(3)} + \cos 2\psi F_{3}^{(3)})c_1 Y_n \sin 2\phi_p \}, \tag{A3}
\end{align}

with

\begin{align}
F_{3}^{(3)} &= \frac{1}{8[1 - \cos \delta \cos \delta_F^p \cos(\alpha - \alpha_F^p) - \sin \delta \sin \delta_F^p]}
\{ -4 \cos^3 \delta \cos \delta_F^p \sin(\alpha - \alpha_F^p) \\
&+ \cos \delta \cos^3 \delta_F^p \sin(\alpha - \alpha_F^p) \} + \cos 2\delta \sin 2\delta_F^p \cos(\alpha - \alpha_F^p) + \cos 2\delta \sin 2\delta_F^p \cos(\alpha - \alpha_F^p) + \cos 2\delta \sin 2\delta_F^p \cos(\alpha - \alpha_F^p) + \cos 2\delta \sin 2\delta_F^p \cos(\alpha - \alpha_F^p) \},
\end{align}

\begin{align}
F_{3}^{(3)} &= \frac{1}{2[1 - \cos \delta \cos \delta_F^p \cos(\alpha - \alpha_F^p) - \sin \delta \sin \delta_F^p]}
\{ 2 \sin \cos^2 \delta_F^p \cos(\alpha - \alpha_F^p) \} \sin \sin \delta_F^p \\
&+ \cos \delta \cos \sin(\alpha - \alpha_F^p) - 1 \} + \cos \delta \sin \sin \delta_F^p \cos \delta_F^p \\
&+ \cos(\alpha - \alpha_F^p) \sin(\alpha - \alpha_F^p) - \sin \delta \sin \delta_F^p \cos \delta_F^p + \sin \delta \sin \delta_F^p \cos \delta_F^p \}, \tag{A4}
\end{align}

\begin{align}
D_{4,n} &= A_m \{ \cos 2\psi F_{4}^{(4)} + 2(1 - c_7^2)X_n \cos 2\phi_p + 2(1 - c_7^2)Z_n \}, \\
E_{4,n} &= A_m \{ -\cos 2\psi F_{4}^{(4)} + 2(1 + c_7^2)Y_n \sin 2\phi_p + 2(\sin 2\psi F_{4}^{(4)} + \cos 2\psi F_{4}^{(4)})c_1 Y_n \sin 2\phi_p \}, \tag{A5}
\end{align}

with

\begin{align}
F_{4}^{(3)} &= \frac{1}{4[1 - \cos \delta \cos \delta_F^p \cos(\alpha - \alpha_F^p) - \sin \delta \sin \delta_F^p]}
\{ 2 \cos(\alpha - \alpha_F^p) + \sin \sin \delta_F^p \\
&- \cos \delta \cos \sin(\alpha - \alpha_F^p) - \sin \sin \delta_F^p \} \sin \sin \delta_F^p \\
&- \cos \delta \sin \sin \delta_F^p \cos(\alpha - \alpha_F^p) + \cos \delta \sin \sin \delta_F^p \cos(\alpha - \alpha_F^p) \},
\end{align}

\begin{align}
F_{4}^{(3)} &= \frac{1}{2[1 - \cos \delta \cos \delta_F^p \cos(\alpha - \alpha_F^p) - \sin \delta \sin \delta_F^p]}
\{ -\cos \delta \cos \sin(\alpha - \alpha_F^p) - \sin \sin \delta_F^p \} \sin \sin \delta_F^p \\
&+ \sin \sin \delta_F^p \cos(\alpha - \alpha_F^p) - \cos \sin \sin \delta_F^p \cos(\alpha - \alpha_F^p) + \cos \sin \sin \delta_F^p \cos(\alpha - \alpha_F^p) \}, \tag{A6}
\end{align}

\begin{align}
D_{5,n} &= 2A_m \{ \cos 2\psi F_{5}^{(3)} + 2(1 - c_7^2)X_n \cos 2\phi_p + 2(1 - c_7^2)Z_n \}, \\
E_{5,n} &= 2A_m \{ \sin 2\psi F_{5}^{(3)} + 2(1 + c_7^2)Y_n \sin 2\phi_p \}, \tag{A7}
\end{align}

\begin{align}
D_{6,n} &= A_m \{ 2c_1 \cos 2\psi F_{6}^{(3)} + 2(1 + c_7^2)Y_n \sin 2\phi_p \}, \\
E_{6,n} &= A_m \{ 2c_1 \cos 2\psi F_{6}^{(3)} + 2(1 - c_7^2)Z_n \}, \tag{A8}
\end{align}

\begin{align}
D_{7,n} &= nB_n, \quad E_{7,n} = -nA_n. \tag{A9}
\end{align}

\begin{align}
D_{8,n} &= 2A_m \{ \cos 2\psi F_{7}^{(3)} + 2(1 + c_7^2)X_n \cos 2\phi_p \}, \\
E_{8,n} &= 2A_m \{ \sin 2\psi F_{7}^{(3)} + 2(1 - c_7^2)Y_n \sin 2\phi_p \}, \tag{A10}
\end{align}

\begin{align}
D_{9,n} &= A_m \{ \cos 2\psi F_{9}^{(3)} + 2(1 + c_7^2)X_n \cos 2\phi_p \}, \\
E_{9,n} &= A_m \{ \sin 2\psi F_{9}^{(3)} + 2(1 - c_7^2)Y_n \sin 2\phi_p \}, \tag{A11}
\end{align}

and

\begin{align}
D_{9,n} &= A_m \{ \cos 2\psi F_{9}^{(3)} + 2(1 + c_7^2)X_n \cos 2\phi_p + 2(1 - c_7^2)Z_n \}, \\
E_{9,n} &= A_m \{ \sin 2\psi F_{9}^{(3)} + 2(1 - c_7^2)Y_n \sin 2\phi_p \}, \tag{A11}
\end{align}
with

\[
X_n^{(e)} = \frac{n}{2} J_{n-3}(en) - en J_{n-2}(en) - \frac{2 + n}{2} J_{n-1}(en) + 2en J_n(en) + \frac{2 - n}{2} J_{n+1}(en) - en J_{n+2}(en) + \frac{n}{2} J_{n+3}(en),
\]

\[
Y_n^{(e)} = -\frac{\sqrt{1 - e^2}}{2} J_{n-3}(en) + \frac{e}{\sqrt{1 - e^2}} J_{n-2}(en) + \frac{3\sqrt{1 - e^2}}{2} J_{n-1}(en) - \frac{2e}{\sqrt{1 - e^2}} J_n(en) - \frac{3\sqrt{1 - e^2}}{2} J_{n+1}(en) + \frac{e}{\sqrt{1 - e^2}} J_{n+2}(en) + \frac{\sqrt{1 - e^2}}{2} J_{n+3}(en),
\]

\[
Z_n^{(e)} = J_{n-1}(en) - J_{n+1}(en).
\]

(A12)

**Appendix B**

**Analytical Results of the Inner-product Terms**

From Equation (11), we can analytically solve the inner-product terms in (D1)–(D3) that

\[
\langle S_{n_1}^{E} | S_{n_2}^{E} \rangle = \begin{cases} \frac{1}{\sigma^2_{1}\delta t_1} \left[ \frac{t}{2} - \frac{T S_{2n_1}^E}{8\pi n_1} + \frac{T \sin(2n_1 \varphi_E)}{8\pi n_1} \right] & , \quad n_1 = n_2, \\ \frac{T}{4\pi \sigma^2_{1}\delta t_1} \left[ \frac{S_{n_1-n_2}^E}{n_1-n_2} - \frac{\sin[(n_1-n_2) \varphi_E]}{n_1-n_2} - \frac{S_{m+n_2}^E}{n_1+n_2} + \frac{\sin[(n_1+n_2) \varphi_E]}{n_1+n_2} \right] & , \quad n_1 \neq n_2, \end{cases}
\]

(B1)

\[
\langle C_{n_1}^{E} | C_{n_2}^{E} \rangle = \begin{cases} \frac{1}{\sigma^2_{1}\delta t_1} \left[ \frac{t}{2} + \frac{T S_{2n_1}^E}{8\pi n_1} - \frac{T \sin(2n_1 \varphi_E)}{8\pi n_1} \right] & , \quad n_1 = n_2, \\ \frac{T}{4\pi \sigma^2_{1}\delta t_1} \left[ \frac{S_{n_1-n_2}^E}{n_1-n_2} - \frac{\sin[(n_1-n_2) \varphi_E]}{n_1-n_2} + \frac{S_{m+n_2}^E}{n_1+n_2} - \frac{\sin[(n_1+n_2) \varphi_E]}{n_1+n_2} \right] & , \quad n_1 \neq n_2, \end{cases}
\]

(B2)

\[
\langle S_{n_1}^{E} | C_{n_2}^{E} \rangle = \begin{cases} -\frac{T C_{2n_1}^E}{8\pi n_1 \sigma^2_{1}\delta t_1} + \frac{T \cos(2n_1 \varphi_E)}{8\pi n_1 \sigma^2_{1}\delta t_1} & , \quad n_1 = n_2, \\ \frac{T}{4\pi \sigma^2_{1}\delta t_1} \left[ \frac{C_{n_1-n_2}^E}{n_1-n_2} + \frac{\cos[(n_1-n_2) \varphi_E]}{n_1-n_2} - \frac{C_{m+n_2}^E}{n_1+n_2} + \frac{\cos[(n_1+n_2) \varphi_E]}{n_1+n_2} \right] & , \quad n_1 \neq n_2, \end{cases}
\]

(B3)

\[
\langle S_{n_1}^{E} | t S_{n_2}^{E} \rangle = \begin{cases} \frac{1}{\sigma^2_{1}\delta t_1} \left[ \frac{t^2}{4} - \frac{t T S_{2n_1}^E}{8\pi n_1} - \frac{T^2 C_{2n_1}^E}{32\pi^3 n_1^3} + \frac{T^2 \cos(2n_1 \varphi_E)}{32\pi^3 n_1^3} \right] & , \quad n_1 = n_2, \\ \frac{T^2}{8\pi \sigma^2_{1}\delta t_1} \left[ \frac{2\pi \tau S_{n_1-n_2}^E}{(n_1-n_2) T} - \frac{2\pi \tau S_{n_1+n_2}^E}{(n_1+n_2) T} + \frac{C_{m-n_2}^E}{(n_1-n_2)^2} \right] - \frac{\cos[(n_1-n_2) \varphi_E]}{(n_1-n_2)^2} + \frac{C_{m+n_2}^E}{(n_1+n_2)^2} + \frac{\cos[(n_1+n_2) \varphi_E]}{(n_1+n_2)^2} \right] & , \quad n_1 \neq n_2, \end{cases}
\]

(B4)

\[
\langle C_{n_1}^{E} | t C_{n_2}^{E} \rangle = \begin{cases} \frac{1}{\sigma^2_{1}\delta t_1} \left[ \frac{t^2}{4} + \frac{t T S_{2n_1}^E}{8\pi n_1} + \frac{T^2 C_{2n_1}^E}{32\pi^3 n_1^3} - \frac{T^2 \cos(2n_1 \varphi_E)}{32\pi^3 n_1^3} \right] & , \quad n_1 = n_2, \\ \frac{T^2}{8\pi \sigma^2_{1}\delta t_1} \left[ \frac{2\pi \tau S_{n_1-n_2}^E}{(n_1-n_2) T} + \frac{2\pi \tau S_{n_1+n_2}^E}{(n_1+n_2) T} - \frac{C_{m-n_2}^E}{(n_1-n_2)^2} \right] + \frac{\cos[(n_1-n_2) \varphi_E]}{(n_1-n_2)^2} + \frac{C_{m+n_2}^E}{(n_1+n_2)^2} - \frac{\cos[(n_1+n_2) \varphi_E]}{(n_1+n_2)^2} \right] & , \quad n_1 \neq n_2, \end{cases}
\]

(B5)
\[ \langle S_n^E \mid tS_{n_2}^E \rangle = \langle C_{n_1}^E \mid tC_{n_2}^E \rangle = \begin{cases} \frac{1}{\sigma^2 t} \left[ \frac{t^3 T E_{2n_1}}{27n_1^3} + \frac{T^2 S_{2n_1}^E}{54n_1^3} - \frac{T^2 \sin(2n_1 \phi_E)}{32n_1^3} \right], & n_1 = n_2, \\ \frac{T}{8\pi^2 \sigma^2 t} \left[ -2\pi t C_{m-n_2}^E - \frac{2\pi t C_{m+n_2}^E}{(n_1 - n_2)^3} + \frac{2\pi t C_{m+n_2}^E}{(n_1 + n_2)^3} - \frac{T^2 \sin[(n_1 - n_2) \phi_E]}{(n_1 - n_2)^3} \right], & n_1 \neq n_2 \end{cases} \] (B6)

\[ \langle tS_n^E \mid tS_{n_2}^E \rangle = \begin{cases} \frac{1}{\sigma^2 t} \left[ \frac{t^3 T E_{2n_1}}{27n_1^3} + \frac{T^2 S_{2n_1}^E}{54n_1^3} - \frac{T^2 \sin(2n_1 \phi_E)}{32n_1^3} \right], & n_1 = n_2, \\ \frac{T}{8\pi^2 \sigma^2 t} \left[ -2\pi t C_{m-n_2}^E - \frac{2\pi t C_{m+n_2}^E}{(n_1 - n_2)^3} + \frac{2\pi t C_{m+n_2}^E}{(n_1 + n_2)^3} - \frac{T^2 \sin[(n_1 - n_2) \phi_E]}{(n_1 - n_2)^3} \right], & n_1 \neq n_2 \end{cases} \] (B7)

\[ \langle tC_n^E \mid tC_{n_2}^E \rangle = \begin{cases} \frac{1}{\sigma^2 t} \left[ \frac{t^3 T E_{2n_1}}{27n_1^3} + \frac{T^2 S_{2n_1}^E}{54n_1^3} - \frac{T^2 \sin(2n_1 \phi_E)}{32n_1^3} \right], & n_1 = n_2, \\ \frac{T}{8\pi^2 \sigma^2 t} \left[ -2\pi t C_{m-n_2}^E - \frac{2\pi t C_{m+n_2}^E}{(n_1 - n_2)^3} + \frac{2\pi t C_{m+n_2}^E}{(n_1 + n_2)^3} - \frac{T^2 \sin[(n_1 - n_2) \phi_E]}{(n_1 - n_2)^3} \right], & n_1 \neq n_2 \end{cases} \] (B8)

and

\[ \langle tS_n^E \mid tC_{n_2}^E \rangle = \begin{cases} \frac{1}{\sigma^2 t} \left[ \frac{t^3 T E_{2n_1}}{27n_1^3} + \frac{T^2 S_{2n_1}^E}{54n_1^3} - \frac{T^2 \sin(2n_1 \phi_E)}{32n_1^3} \right], & n_1 = n_2, \\ \frac{T}{8\pi^2 \sigma^2 t} \left[ -2\pi t C_{m-n_2}^E - \frac{2\pi t T E_{2n_1}}{32n_1^3} - \frac{T^2 \sin[(n_1 - n_2) \phi_E]}{(n_1 - n_2)^3} \right], & n_1 \neq n_2 \end{cases} \] (B9)

Appendix C

Calculation of the Expected Values

Taking Equation (34) into Equations (D1)–(D3), all of the inner-product terms are expanded as

\[ \langle S_n^E \mid S_{n_2}^E \rangle = \cos(n_1 \Delta \phi) \langle S_n^E \mid S_{n_2}^E \rangle + \sin(n_1 \Delta \phi) \langle C_n^E \mid S_{n_2}^E \rangle, \]

\[ \langle S_n^E \mid C_{n_2}^E \rangle = \cos(n_1 \Delta \phi) \langle S_n^E \mid C_{n_2}^E \rangle + \sin(n_1 \Delta \phi) \langle C_n^E \mid C_{n_2}^E \rangle, \]

\[ \langle C_n^E \mid S_{n_2}^E \rangle = \cos(n_1 \Delta \phi) \langle C_n^E \mid S_{n_2}^E \rangle - \sin(n_1 \Delta \phi) \langle S_n^E \mid S_{n_2}^E \rangle, \]

\[ \langle C_n^E \mid C_{n_2}^E \rangle = \cos(n_1 \Delta \phi) \langle C_n^E \mid C_{n_2}^E \rangle - \sin(n_1 \Delta \phi) \langle S_n^E \mid C_{n_2}^E \rangle, \]

\[ \langle S_n^E \mid tS_{n_2}^E \rangle = \cos(n_1 \Delta \phi) \langle S_n^E \mid tS_{n_2}^E \rangle + \sin(n_1 \Delta \phi) \langle tC_n^E \mid S_{n_2}^E \rangle, \]

\[ \langle S_n^E \mid tC_{n_2}^E \rangle = \cos(n_1 \Delta \phi) \langle S_n^E \mid tC_{n_2}^E \rangle + \sin(n_1 \Delta \phi) \langle tC_n^E \mid tC_{n_2}^E \rangle, \]

\[ \langle C_n^E \mid tS_{n_2}^E \rangle = \cos(n_1 \Delta \phi) \langle C_n^E \mid tS_{n_2}^E \rangle - \sin(n_1 \Delta \phi) \langle S_n^E \mid tS_{n_2}^E \rangle, \]

\[ \langle C_n^E \mid tC_{n_2}^E \rangle = \cos(n_1 \Delta \phi) \langle C_n^E \mid tC_{n_2}^E \rangle - \sin(n_1 \Delta \phi) \langle tC_n^E \mid tC_{n_2}^E \rangle. \] (C1)
Subsequently, from Equation (35), we can obtain all the expected terms in Equations (D1)–(D3) that

\[
\langle (S_{n}^{p} | S_{n}^{E} | S_{n}^{E}) \rangle = \frac{1}{2} \delta_{n,m} \langle (S_{n}^{p} | S_{n}^{E} | S_{n}^{E}) \rangle + \langle (S_{n}^{p} | S_{n}^{E} | S_{n}^{E}) \rangle - \langle (S_{n}^{p} | S_{n}^{E} | S_{n}^{E}) \rangle - \langle (S_{n}^{p} | S_{n}^{E} | S_{n}^{E}) \rangle
\]

and

\[
\langle (S_{n}^{p} | tS_{n}^{E} | tS_{n}^{E}) \rangle = \frac{1}{2} \delta_{n,m} \langle (S_{n}^{p} | tS_{n}^{E} | tS_{n}^{E}) \rangle + \langle (S_{n}^{p} | tS_{n}^{E} | tS_{n}^{E}) \rangle - \langle (S_{n}^{p} | tS_{n}^{E} | tS_{n}^{E}) \rangle - \langle (S_{n}^{p} | tS_{n}^{E} | tS_{n}^{E}) \rangle
\]
Finally, by inserting the above results into Equations (D1)–(D3), one obtains the resulting Equations (D4)–(D6).

Appendix D

Components of the Matrix $\langle \xi_a \xi_b \rangle$

By expanding Equation (33), all of the elements of the matrix $\langle \xi_a \xi_b \rangle$ are written as

$$\langle \xi_1 \xi_1 \rangle = \frac{4 \pi^2}{T^4} \sum_{i=0}^{N_1} \sum_{j=0}^{N_1} \sum_{k=0}^{N_1} \sum_{l=0}^{N_1} n_2 n_4 \times \left[ A_{n_1} B_{n_1} A_{n_1} B_{n_1} \langle S_{n_1} P | t S_{n_1} E \rangle \langle S_{n_1} E | t S_{n_1} E \rangle \right]$$

$$\langle \xi_1 \xi_2 \rangle = \frac{2 \pi}{N_2} \sum_{i=0}^{N_2} \sum_{j=0}^{N_2} \sum_{k=0}^{N_2} \sum_{l=0}^{N_2} n_2 \times \left[ A_{n_2} B_{n_2} A_{n_2} B_{n_2} \langle S_{n_2} P | t S_{n_2} E \rangle \langle S_{n_2} E | t S_{n_2} E \rangle \right]$$

$$\langle \xi_2 \xi_2 \rangle = \frac{2 \pi}{N_2} \sum_{i=0}^{N_2} \sum_{j=0}^{N_2} \sum_{k=0}^{N_2} \sum_{l=0}^{N_2} n_2 \times \left[ A_{n_2} B_{n_2} A_{n_2} B_{n_2} \langle S_{n_2} P | t S_{n_2} E \rangle \langle S_{n_2} E | t S_{n_2} E \rangle \right]$$

$$\langle \xi_1 \xi_3 \rangle = \frac{2 \pi}{N_2} \sum_{i=0}^{N_2} \sum_{j=0}^{N_2} \sum_{k=0}^{N_2} \sum_{l=0}^{N_2} n_2 \times \left[ A_{n_2} B_{n_2} A_{n_2} B_{n_2} \langle S_{n_2} P | t S_{n_2} E \rangle \langle S_{n_2} E | t S_{n_2} E \rangle \right]$$

$$\langle \xi_3 \xi_3 \rangle = \frac{2 \pi}{N_2} \sum_{i=0}^{N_2} \sum_{j=0}^{N_2} \sum_{k=0}^{N_2} \sum_{l=0}^{N_2} n_2 \times \left[ A_{n_2} B_{n_2} A_{n_2} B_{n_2} \langle S_{n_2} P | t S_{n_2} E \rangle \langle S_{n_2} E | t S_{n_2} E \rangle \right]$$

$$\langle \xi_2 \xi_4 \rangle = \frac{2 \pi}{N_2} \sum_{i=0}^{N_2} \sum_{j=0}^{N_2} \sum_{k=0}^{N_2} \sum_{l=0}^{N_2} n_2 \times \left[ A_{n_2} B_{n_2} A_{n_2} B_{n_2} \langle S_{n_2} P | t S_{n_2} E \rangle \langle S_{n_2} E | t S_{n_2} E \rangle \right]$$

$$\langle \xi_4 \xi_4 \rangle = \frac{2 \pi}{N_2} \sum_{i=0}^{N_2} \sum_{j=0}^{N_2} \sum_{k=0}^{N_2} \sum_{l=0}^{N_2} n_2 \times \left[ A_{n_2} B_{n_2} A_{n_2} B_{n_2} \langle S_{n_2} P | t S_{n_2} E \rangle \langle S_{n_2} E | t S_{n_2} E \rangle \right]$$

$$\langle \xi_3 \xi_4 \rangle = \frac{2 \pi}{N_2} \sum_{i=0}^{N_2} \sum_{j=0}^{N_2} \sum_{k=0}^{N_2} \sum_{l=0}^{N_2} n_2 \times \left[ A_{n_2} B_{n_2} A_{n_2} B_{n_2} \langle S_{n_2} P | t S_{n_2} E \rangle \langle S_{n_2} E | t S_{n_2} E \rangle \right]$$

$$\langle \xi_4 \xi_4 \rangle = \frac{2 \pi}{N_2} \sum_{i=0}^{N_2} \sum_{j=0}^{N_2} \sum_{k=0}^{N_2} \sum_{l=0}^{N_2} n_2 \times \left[ A_{n_2} B_{n_2} A_{n_2} B_{n_2} \langle S_{n_2} P | t S_{n_2} E \rangle \langle S_{n_2} E | t S_{n_2} E \rangle \right]$$

$$\langle \xi_2 \xi_3 \rangle = \frac{2 \pi}{N_2} \sum_{i=0}^{N_2} \sum_{j=0}^{N_2} \sum_{k=0}^{N_2} \sum_{l=0}^{N_2} n_2 \times \left[ A_{n_2} B_{n_2} A_{n_2} B_{n_2} \langle S_{n_2} P | t S_{n_2} E \rangle \langle S_{n_2} E | t S_{n_2} E \rangle \right]$$

$$\langle \xi_3 \xi_3 \rangle = \frac{2 \pi}{N_2} \sum_{i=0}^{N_2} \sum_{j=0}^{N_2} \sum_{k=0}^{N_2} \sum_{l=0}^{N_2} n_2 \times \left[ A_{n_2} B_{n_2} A_{n_2} B_{n_2} \langle S_{n_2} P | t S_{n_2} E \rangle \langle S_{n_2} E | t S_{n_2} E \rangle \right]$$

$$\langle \xi_2 \xi_4 \rangle = \frac{2 \pi}{N_2} \sum_{i=0}^{N_2} \sum_{j=0}^{N_2} \sum_{k=0}^{N_2} \sum_{l=0}^{N_2} n_2 \times \left[ A_{n_2} B_{n_2} A_{n_2} B_{n_2} \langle S_{n_2} P | t S_{n_2} E \rangle \langle S_{n_2} E | t S_{n_2} E \rangle \right]$$

$$\langle \xi_4 \xi_4 \rangle = \frac{2 \pi}{N_2} \sum_{i=0}^{N_2} \sum_{j=0}^{N_2} \sum_{k=0}^{N_2} \sum_{l=0}^{N_2} n_2 \times \left[ A_{n_2} B_{n_2} A_{n_2} B_{n_2} \langle S_{n_2} P | t S_{n_2} E \rangle \langle S_{n_2} E | t S_{n_2} E \rangle \right]$$

$$\langle \xi_3 \xi_4 \rangle = \frac{2 \pi}{N_2} \sum_{i=0}^{N_2} \sum_{j=0}^{N_2} \sum_{k=0}^{N_2} \sum_{l=0}^{N_2} n_2 \times \left[ A_{n_2} B_{n_2} A_{n_2} B_{n_2} \langle S_{n_2} P | t S_{n_2} E \rangle \langle S_{n_2} E | t S_{n_2} E \rangle \right]$$

$$\langle \xi_4 \xi_4 \rangle = \frac{2 \pi}{N_2} \sum_{i=0}^{N_2} \sum_{j=0}^{N_2} \sum_{k=0}^{N_2} \sum_{l=0}^{N_2} n_2 \times \left[ A_{n_2} B_{n_2} A_{n_2} B_{n_2} \langle S_{n_2} P | t S_{n_2} E \rangle \langle S_{n_2} E | t S_{n_2} E \rangle \right]$$
for $a \neq 1$, and

$$\langle \xi_a \xi_b \rangle = \sum_{l_1} \sum_{l_2} \sum_{m_1} \sum_{m_2} \sum_{n_1} \sum_{n_2} \sum_{o_1} \sum_{o_2} \sum_{p_1} \sum_{p_2} \{ A_{a_m n} D_{a_m n} A_{n o n} (\langle S_{m_1}^E \rangle | t S_{m_2}^E ) | t S_{n_1}^E ) + \langle C_{m_1}^E \rangle | t C_{m_2}^E ) | t C_{n_1}^E ) \} + \langle C_{m_1}^E | t C_{m_2}^E | t C_{n_1}^E \rangle$$

for both $a \neq 1$ and $b \neq 1$.

After calculating all the averaged terms in Equations (D1)–(D3) (see Appendix C for details), we finally obtain the elements of the matrix as follows

$$\langle \xi_a \xi_b \rangle = \frac{2 \pi^2}{T} \sum_{l_1} \sum_{l_2} \sum_{m_1} \sum_{m_2} \sum_{n_1} \sum_{n_2} \sum_{o_1} \sum_{o_2} \sum_{p_1} \sum_{p_2} \sum_{q_1} \sum_{q_2} \sum_{r_1} \sum_{r_2} \sum_{s_1} \sum_{s_2} \{ A_{a_m n} B_{b_n o} (\langle S_{m_1}^E \rangle | t S_{m_2}^E ) | t S_{n_1}^E ) + \langle C_{m_1}^E | t C_{m_2}^E | t C_{n_1}^E \rangle$$

(D4)
for both $a \neq 1$ and $b \neq 1$.

\[ (E, A, B, D_a, E_a) \text{ for the Circular Case} \]

The coefficients $A$ and $B$ in the template \((40)\) are, respectively

\[
A = A_{m}(1 + c_i^{2})(\cos 2\psi F_{1} - \sin 2\psi F_{1}),
\]

\[
B = 2A_{m}c_i(\sin 2\psi F_{1} + \cos 2\psi F_{1}).
\]

The coefficients $D_a$ and $E_a$ over $a = 1, 2, ..., 7$ in Equation \((41)\) are, respectively

\[
D_1 = \frac{4\pi B}{T^2}, \quad E_1 = -\frac{4\pi A}{T^2},
\]

\[
D_2 = \frac{A}{A_m}, \quad E_2 = \frac{B}{A_m},
\]

\[
D_3 = A_{m}(1 + c_i^{2})(\cos 2\psi F_{1}^{(3)} - \sin 2\psi F_{1}^{(3)}), \quad E_3 = 2A_{m}c_i(\sin 2\psi F_{1}^{(3)} + \cos 2\psi F_{1}^{(3)}),
\]

with $F_{1}^{(3)}$ and $F_{1}^{(4)}$ given by Equation \((A4)\),

\[
D_4 = A_{m}(1 + c_i^{2})(\cos 2\psi F_{1}^{(4)} - \sin 2\psi F_{1}^{(4)}), \quad E_4 = 2A_{m}c_i(\sin 2\psi F_{1}^{(4)} + \cos 2\psi F_{1}^{(4)}),
\]
with \( F_{1+}^{(4)} \) and \( F_{1+}^{(4)} \) given by Equation (A6),

\[
D_5 = -2A_m(1 + c_1^2)(\sin 2\psi F_{1+} + \cos 2\psi F_{1+}), \quad E_5 = 4A_m c_1(\cos 2\psi F_{1+} - \sin 2\psi F_{1+}),
\]

(E6)

\[
D_6 = 2A_m \cdot c_1(\cos 2\psi F_{1+} - \sin 2\psi F_{1+}), \quad E_6 = 2A_m(\sin 2\psi F_{1+} + \cos 2\psi F_{1+}),
\]

(E7)

and

\[
D_7 = 2B, \quad E_7 = -2A.
\]

(E8)

### Appendix F

**Analytical Results of the Inner-product Terms for the Circular Case**

From Equation (11), we can solve the following inner-product terms in Equations (42)–(44) and (G1)–(G3):

\[
\langle S^E | S^E \rangle_l = \frac{1}{\tilde{t}} \left[ -\frac{T \sin \left( \frac{8\pi}{T} + 4\varphi^E \right)}{16\pi} + \frac{T \sin 4\varphi^E}{16\pi} \right],
\]

(F1)

\[
\langle C^E | C^E \rangle_l = \frac{1}{\tilde{t}} \left[ \frac{T \sin \left( \frac{8\pi}{T} + 4\varphi^E \right)}{16\pi} - \frac{T \sin 4\varphi^E}{16\pi} \right],
\]

(F2)

\[
\langle S^E | C^E \rangle_l = \langle C^E | S^E \rangle_l = \frac{1}{\tilde{t}} \left[ \frac{T \cos \left( \frac{8\pi}{T} + 4\varphi^E \right)}{16\pi} - \frac{T \cos 4\varphi^E}{16\pi} \right],
\]

(F3)

\[
\langle S^E | tS^E \rangle_l = \frac{1}{\tilde{t}^2} \left[ \frac{t^2 T \sin \left( \frac{8\pi}{T} + 4\varphi^E \right)}{16\pi} \right] - \frac{T^2 \cos \left( \frac{8\pi}{T} + 4\varphi^E \right)}{128\pi^2} + \frac{T^2 \cos 4\varphi^E}{128\pi^2},
\]

(F4)

\[
\langle C^E | tC^E \rangle_l = \frac{1}{\tilde{t}^2} \left[ \frac{t^2 T \sin \left( \frac{8\pi}{T} + 4\varphi^E \right)}{16\pi} \right] - \frac{T^2 \cos \left( \frac{8\pi}{T} + 4\varphi^E \right)}{128\pi^2} - \frac{T^2 \cos 4\varphi^E}{128\pi^2},
\]

(F5)

\[
\langle S^E | tC^E \rangle_l = \langle C^E | tS^E \rangle_l = \frac{1}{\tilde{t}^2} \left[ \frac{-T \cos \left( \frac{8\pi}{T} + 4\varphi^E \right)}{16\pi} \right] + \frac{T \sin \left( \frac{8\pi}{T} + 4\varphi^E \right)}{128\pi^2} - \frac{T \sin 4\varphi^E}{128\pi^2},
\]

(F6)

\[
\langle tS^E | tS^E \rangle_l = \frac{1}{\tilde{t}^3} \left[ \frac{t^3 T \sin \left( \frac{8\pi}{T} + 4\varphi^E \right)}{16\pi} \right] - \frac{T^3 \cos \left( \frac{8\pi}{T} + 4\varphi^E \right)}{64\pi^2} + \frac{T^3 \sin \left( \frac{8\pi}{T} + 4\varphi^E \right)}{512\pi^3} - \frac{T^3 \sin 4\varphi^E}{512\pi^3},
\]

(F7)

\[
\langle tC^E | tC^E \rangle_l = \frac{1}{\tilde{t}^3} \left[ \frac{t^3 T \sin \left( \frac{8\pi}{T} + 4\varphi^E \right)}{16\pi} \right] + \frac{T^3 \cos \left( \frac{8\pi}{T} + 4\varphi^E \right)}{64\pi^2} - \frac{T^3 \sin \left( \frac{8\pi}{T} + 4\varphi^E \right)}{512\pi^3} + \frac{T^3 \sin 4\varphi^E}{512\pi^3},
\]

(F8)

and

\[
\langle tS^E | tC^E \rangle_l = \langle tC^E | tS^E \rangle_l = \frac{1}{\tilde{t}^3} \left[ \frac{-t^3 \cos \left( \frac{8\pi}{T} + 4\varphi^E \right)}{16\pi} \right] - \frac{T^3 \sin \left( \frac{8\pi}{T} + 4\varphi^E \right)}{64\pi^2} + \frac{T^3 \cos \left( \frac{8\pi}{T} + 4\varphi^E \right)}{512\pi^3} - \frac{T^3 \cos 4\varphi^E}{512\pi^3},
\]

(F9)
In the circular case $e = 0$, the mean-squared noise-projection matrix $\langle \xi_a \xi_b \rangle$ given by (D4)–(D6) reduces to

$$\langle \xi_i \xi_i \rangle = \frac{8\pi^2}{T^2} \sum_{t=0}^{N_t} \left[ A^2 B^2 \langle \langle S^E \rangle^2 \rangle \langle \langle t S^E \rangle^2 \rangle + \langle \langle C^E \rangle^2 \rangle \langle \langle t C^E \rangle^2 \rangle \right],$$

$$\langle \xi_i \xi_j \rangle = \frac{2\pi}{T} \sum_{t=0}^{N_t} \left[ A^2 B^2 \langle \langle S^E \rangle^2 \rangle \langle \langle t S^E \rangle^2 \rangle + \langle \langle C^E \rangle^2 \rangle \langle \langle t C^E \rangle^2 \rangle \right],$$

$$\langle \xi_i \xi_j \rangle = \frac{1}{2} \sum_{t=1}^{N_t} \left[ A^2 D^2 \langle \langle S^E \rangle^2 \rangle \langle \langle t S^E \rangle^2 \rangle + \langle \langle C^E \rangle^2 \rangle \langle \langle t C^E \rangle^2 \rangle \right]$$

for $a \neq 1$ and $b \neq 1$.

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**Appendix G**

Matrix $\langle \xi_a \xi_b \rangle$ for the Circular Case

In the circular case $e = 0$, the mean-squared noise-projection matrix $\langle \xi_a \xi_b \rangle$ given by (D4)–(D6) reduces to

$$\langle \xi_i \xi_i \rangle = \frac{8\pi^2}{T^2} \sum_{t=1}^{N_t} \left[ A^2 B^2 \langle \langle S^E \rangle^2 \rangle \langle \langle t S^E \rangle^2 \rangle + \langle \langle C^E \rangle^2 \rangle \langle \langle t C^E \rangle^2 \rangle \right],$$

$$\langle \xi_i \xi_j \rangle = \frac{2\pi}{T} \sum_{t=1}^{N_t} \left[ A^2 B^2 \langle \langle S^E \rangle^2 \rangle \langle \langle t S^E \rangle^2 \rangle + \langle \langle C^E \rangle^2 \rangle \langle \langle t C^E \rangle^2 \rangle \right],$$

$$\langle \xi_i \xi_j \rangle = \frac{1}{2} \sum_{t=1}^{N_t} \left[ A^2 D^2 \langle \langle S^E \rangle^2 \rangle \langle \langle t S^E \rangle^2 \rangle + \langle \langle C^E \rangle^2 \rangle \langle \langle t C^E \rangle^2 \rangle \right]$$

for $a \neq 1$ and $b \neq 1$. 

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