Node-Initiated Byzantine Consensus Without a Common Clock

Danny Dolev, Hebrew University of Jerusalem, dolev@cs.huji.ac.il
Christoph Lenzen, Massachusetts Institute of Technology, clenzen@csail.mit.edu

Abstract

The majority of the literature on consensus assumes that protocols are jointly started at all nodes of the distributed system. We show how to remove this problematic assumption in semi-synchronous systems, where message delays and relative drifts of local clocks may vary arbitrarily within known bounds. Our framework is self-stabilizing and efficient both in terms of communication and time; more concretely, compared to a synchronous start in a synchronous model of a non-self-stabilizing protocol, we achieve a constant-factor increase in the time and communicated bits to complete an instance, plus an additive communication overhead of $O(n \log n)$ broadcasted bits per time unit and node. The latter can be further reduced, at an additive increase in time complexity.

1 Introduction

Consensus is a fundamental fault-tolerance primitive in distributed systems, which has been introduced several decades ago [15]. Given a system of $n$ nodes, some of which may be faulty and disobey the protocol in an arbitrary fashion, the problem can be concisely stated as follows. Each node $v$ is given some input $i(v)$ from some set of possible inputs $I$. A consensus protocol resilient to $f$ faults must—under the constraint that at most $f$ nodes are faulty—satisfy that

**Termination:** every correct node eventually terminates and outputs a value $o(v) \in I$;

**Agreement:** $o(v) = o(w)$ for correct nodes $v, w$ (we thus may talk of the output of the protocol);

**Validity:** if $i(v) = i(w)$ for all correct $v, w$, this is also the output value.

The main optimization criteria are the resilience $f$ ([15]), the running time, i.e., the time until all nodes terminate ([12]), and the number of messages and bits sent by correct nodes ([8]).

A Motivating Example

Two major international banks, $A$ and $B$, are long-standing rivals. The evil CEO of bank $B$ hires a professional infiltration specialist and hacker to cause maximal damage to bank $A$. She offers her services to bank $A$ to inspect the robustness of their systems. Having full access, she learns that bank $A$ has been very thorough: All data is replicated at multiple locations, and any changes are done using a state-of-the-art consensus algorithm; if a customer or employee commits any value, it is encrypted and sent to the different locations via secure channels, where it serves as input for the consensus routine.

Consequently, she looks for a way to bring the system down. The consensus algorithm is Byzantine fault-tolerant, i.e., resilient to arbitrary behavior of a minority of the data centers. However, she knows that such algorithms are costly in terms of computation and communication. Closer examination reveals that the bank solved this by restricting the frequency at which consensus is run and using a synchronous protocol. Every second, an instance of the algorithm is started that
commits the batch of recent changes. Due to these choices, the infrastructure is capable of dealing with an influx of operations well beyond the peak loads.

Nonetheless, the spy now knows how to break the system. She takes control of the external sources the data centers obtain their timing information from—which are outside bank A’s control—and feeds conflicting time values to the different data centers. For most operations, the result is denial of service, but in some instances the lack of synchrony leads to inconsistent commits. Despite bank A’s excellent IT security, the damage is dramatic.

An Alternative Approach

Could bank A have averted its fate? Clearly, relying on a trusted external time reference is chancy. On the other hand, dropping timing conditions entirely would necessitate to use asynchronous consensus protocols, which offer substantially worse trade-offs between efficiency and reliability. A third option is to make use of a time reference under control of the bank. However, to avoid introducing a new point of vulnerability to the system, it needs to be replicated as well. An obvious choice here is to equip each node of the system (a.k.a. data center) with its own clock.

There are two possibilities for leveraging such hardware clocks: (i) using local timing conditions at each node in a consensus algorithm, or (ii) running a clock synchronization algorithm to compute synchronized logical clocks and executing synchronous algorithms driven by these clocks.

Regarding (ii), repetitive consensus (or other, tailored solutions, e.g., [18]) can be used to maintain a common clock despite faults. However, how can initial synchronization be established, without relying on some external means? Moreover, if a node rejoins the network (after maintenance or a transient fault), or the system is to recover from a partition, is this possible without external intervention? The notion of self-stabilization [5] covers all these scenarios: a self-stabilizing algorithm must eventually establish a valid state, no matter the initial state it is started from.

Byzantine self-stabilizing clock synchronization algorithms [6, 9, 13] provide solutions for (ii). There is a close relation to consensus in general and (i) in particular. To the best of our knowledge, all known algorithms for (ii) employ techniques commonly found in consensus algorithms, and most of them make explicit use of consensus protocols as subroutines.

With respect to (i), the question arises on how to decide on when to run consensus. Without any agreement on a global time among the nodes, some other global reference needs to be established in order to jointly start an instance of the consensus protocol at all nodes. While a broadcast of a single node could establish such a reference, this is problematic if the respective node is corrupted. Even if we add redundancy by allowing for multiple nodes initiating instances, a mechanism is required to prevent that corrupt nodes overwhelm the system with too many instances or initialize runs inconsistently.

In particular the latter issue is not be taken lightly, as it entails to establish agreement among the nodes on whether consensus is to be run or not! Considering that the vast majority of the literature on consensus assumes that all nodes in unison start executing an algorithm at a specific time\(^1\) we consider it both surprising and alarming that this issue is not addressed by existing work.

Further Related Work

The issues that surface when running consensus in practice have been studied extensively. Researchers distinguished between “Atomic Broadcast” that may be repeatedly executed and “Con-

\(^{1}\)Observe that even asynchronous models assume that all nodes fully participate at the invoked consensus once they “wake-up”. This can be logically mapped to all nodes participating unconditionally from time 0 on, much like the unison start of the synchronous algorithms.
sensus" ([17]) that was considered as a “single shot”. Running a synchronous protocol in a semi-synchronous environment was studied comprehensively [1, 2, 3, 4, 10, 16, 14]. Lower bounds, upper bounds and failure models were presented, and complexity measures were analyzed. But all previous work explicitly or implicitly assumes that when consensus is invoked, every node has an input and executes the protocol to reach the target value that is determined by the set of inputs. Solving the question we consider using previous work translates to continuously running consensus on whether to run consensus or not. Thus, such an approach enables the faulty nodes to cause the correct nodes to get involved in an unbounded number of invocations.

Contribution

In this work, we provide a generic solution to (i), where the hardware clock $H_v$ of node $v$ may run at rates that vary arbitrarily within $[1, \vartheta]$ and message delays may vary arbitrarily within $(0, d)$.

**Theorem 1.1.** Suppose $P$ is a synchronous consensus protocol that tolerates $f < n/3$ faults, runs for $R \in \text{polylog}(n)$ rounds, and guarantees that no correct node sends more than $B$ bits. Then for each $T \geq 2\vartheta^2 d$, there are a value $S \in O(R + T)$ and an algorithm with the following properties.

- Each correct node can initiate an instance of $P$ at any time $t \geq S$, provided that it has not done so at any time $t' < t$ for which $H_v(t) - H_v(t') \leq T$.
- For any instance that terminates at a time larger than $S$, it holds that nodes determine their inputs according to their local view of the system during some interval $[t_1, t_1 + O(1)]$, and terminate during some interval $[t_2, t_2 + O(1)]$, where $t_2 \in t_1 + \Theta(R)$.
- If a correct node initiates an instance at time $t \geq S$, then $t_1 = t$.
- Each instance for which $t_2 \geq S$ satisfies termination, agreement, and validity.
- Each correct node sends at most $O(n^2 \log n + nBR/T)$ amortized bits per time unit.
- The above guarantees hold in the presence of $f$ faulty nodes and for arbitrary initial states.

This statement can be extended to randomized algorithms that satisfy agreement and validity with high probability, and accepting that inputs are determined within a time window of size $O(T)$ enables to decrease the amortized bit complexity per node to $O((n^2 \log n + nBR)/T)$ (see Section 7).

We remark that our results bear the promise of improved solutions to (ii).

2 Model and Problem

For the purpose of our analysis, we assume that there is a global reference time. Whenever we talk of a time, it will refer to this reference time from $\mathbb{R}_0^+$, which is unknown to the nodes.

Distributed System

We model the distributed system as a finite set of $n$ nodes $V = \{1, \ldots, n\}$ that communicate via message passing. Each such message is subject to a delay from the range $(0, d)$, where $d \in \mathcal{O}(1)$. Every node can directly communicate to every other node. The sender of a message can be identified by the receiver. Up to $f < n/3$ nodes are Byzantine faulty, i.e., exhibit arbitrary behavior. We denote the set of correct nodes (i.e., those that are not faulty) by $G$. Initially, correct nodes’ states are arbitrary, and the communication network may deliver arbitrary messages prior to time $d$.

Each node $v \in V$ is equipped with a hardware clock $H_v : \mathbb{R}_0^+ \to \mathbb{R}_0^+$. (We will show in Section 5 that bounded and discrete clocks are sufficient, but use the unbounded and continuous abstraction

\footnote{I.e., probability at least $1 - 1/n^c$, where $c$ is an arbitrary constant that is chosen upfront.}
throughout our proofs.) Clock rates are within \([1, \vartheta]\) (with respect to the reference time), where \(\vartheta - 1\) is the (maximal) clock drift. For any times \(t < t'\) and correct node \(v \in G\), it holds that \(t' - t \leq H_v(t') - H_v(t) \leq \vartheta (t' - t)\). Since hardware clocks are not synchronized, nodes typically use them to approximately measure timespans by timeouts. A timeout can be either expired or not expired. When node \(v\) resets a timeout of duration \(T \in \mathbb{R}^+\) at time \(t\), it is not expired during \([t, t')\), where \(t'\) is the unique time satisfying that \(H_v(t') - H_v(t) = T\). The timeout expires when it has not been reset within the last \(T\) local time (i.e., the last \(T\) units of time according to \(H_v\)).

**Algorithms and Executions.** Executions are event-based. An event can be a node’s hardware clock reaching a certain value, a timeout expiring, or the reception of a message. Upon an event at time \(t\), a node may read its hardware clock \(H_v(t)\), perform local computations, store values, send messages, and reset timeouts. For simplicity, all these operations require 0 time and we assume that no two events happen at the same time.

**Problem Formulation**

We will solve a slightly weaker problem than stated in the abstract, from which the claimed results readily follow. We are given a deterministic, synchronous, binary \(R\)-round consensus protocol \(P\) resilient to \(f < n/3\) faults. The goal of the initiation problem is to enable correct nodes to initiate independent executions of instances of \(P\). More precisely:

1. Each instance carries a label \((v, H_v) \in V \times \mathbb{R}_+^t\).
2. If node \(v \in G\) decides to initiate an instance, this instance has label \((v, H_v(t))\).
3. For each instance, each node \(w \in G\) decides whether it participates in the instance at some time \(t_w\). If it does, we assume that it has access to some appropriate input \(i_w(v, H_v, t_w) \in \{0, 1\}\) (which it may or may not use as the input for this instance), and we require that it will eventually terminate the instance and outputs some value \(o_w(v, H_v) \in \{0, 1\}\).
4. If \(v \in G\) initializes instance \((v, H_v(t))\) at time \(t\), each \(w \in G\) decides to participate, \(t_w \in t + \Theta(d)\), and it will terminate this instance at some time from \(t + \Theta(R)\).
5. If \(w, w' \in G\) participate in instance \((v, H_v)\), then \(o_w(v, H_v) = o_{w'}(v, H_v)\).
6. If all nodes in \(G\) participate in an instance and use the same input \(b\), they all output \(b\).
7. If \(o_u(v, H_v) \neq 0\) for some \(w \in G\), then all nodes in \(G\) participate in this instance, terminate within a time window \([t^-, t^+]\) of constant size, and some \(u \in G\) satisfies that \(i_u(v, H_v, t_u) = o_{w}(v, H_v)\) for some \(t_u \in t^+ - \Theta(R)\).

Compared to “classic” consensus, property 4 corresponds to termination, property 5 to agreement, and property 6 to validity. Note that validity is replaced by a safety property in case not all correct nodes participate: property 7 states that non-zero output is feasible only if no correct node is left out. Finally, property 8 makes sure that all nodes participate in case a non-faulty node initializes an instance, therefore ensuring validity for such instances.

Put simply, these rules ensure that each instance \((v, H_v(t))\) initiated by a correct node behaves like a “classic” consensus instance with inputs \(i_w(v, H_v(t), t_w)\), where \(t_w \approx t\), which terminates within \(\Theta(R)\) time, and roughly simultaneously at all correct nodes. If a faulty node initializes an instance, the timing conditions are guaranteed only if some non-faulty node outputs a non-zero value; in this case we are also ensured that there has been some corresponding input \(\Theta(R)\) time in the past, i.e., the computed output is valid.

Assuming that the “fallback” output 0 never causes any damage, this is certainly acceptable. In particular, we can use the initiation problem to agree on whether an instance of an arbitrary (possibly non-binary) consensus protocol should be jointly started by all correct nodes upon terminating the instance, at roughly the same time. Setting the inputs \(i_v \equiv 1\) (in the initiation problem) will then ensure that the result is a solution to the task stated in the abstract and the introduction.
3 Algorithm

In this section, we present a simplified version of our algorithm; in particular, it is not self-stabilizing and has unbounded communication complexity. We will address these issues in Section 5.

Our algorithm consists of three main components beside the employed consensus protocol \( P \). The first provides each node with an estimate of each other node’s clock, with certain consistency guarantees that apply also to clocks of faulty nodes. The second uses this shared timing information to enforce clean initialization of consensus instances within a small window of time. Finally, the third component provides a wrapper for the consensus protocol that simulates synchronous execution. Before presenting the protocol, however, we need to introduce an additional property the employed consensus protocol \( P \) must satisfy.

Silent Consensus

We call a consensus protocol silent, if in any execution in which all correct nodes have input 0, correct nodes send no messages and output 0. Observe that even if not all correct nodes participate, a silent protocol will end up running correctly and output 0 at all participating nodes if no correct node has non-zero input. We show that any consensus protocol can be transformed into a silent one.

Lemma 3.1. Any synchronous consensus protocol \( P \) in which nodes sent at most \( B \) bits can be transformed into a silent synchronous binary consensus protocol \( P_s \) with the same properties except that it runs for two more rounds, during which each node may perform up to \( 2 \) 1-bit broadcasts.

Proof. The new protocol can be seen as a “wrapper” protocol that manipulates the inputs and then each node may or may not participate in an instance of the original protocol. The output of the original protocol, \( P \), will be taken into account only by correct nodes that participate throughout the protocol, as specified below. In the first round of the new protocol, \( P_s \), each participating node broadcasts its input if it is not 0 and otherwise sends nothing. If a node receives fewer than \( n - f \) times the value 1, it sets its input to 0. In the second round, the same pattern is applied.

Subsequently, \( P \) is executed by all nodes that received at least \( f + 1 \) messages in the first round (where any missing messages from nodes that do not participate are set to an arbitrary valid message by the receiver). If in the execution of \( P \) a node would have to send more bits than it would have according to the known bound \( B \), it (locally) aborts the execution of \( P \). Likewise, if the running time bound of \( P \) would be violated, it aborts as well. Finally, a node outputs 0 in the new protocol if it did not participate in the execution of \( P \), aborted it, or received \( f \) or less messages in the second round, and it outputs the result according to the run of \( P \) otherwise.

We first show that the new protocol, \( P_s \), is a consensus protocol with the same resilience as \( P \) and the claimed bounds on communication complexity and running time. We distinguish two cases. First, suppose that all correct nodes participate in the execution of \( P \) at the beginning of the third round. As all nodes participate, the bounds on resilience, communication complexity, and running time that apply to \( P \) hold in this execution, and no node will quit executing the protocol before termination. To establish agreement and validity, again we distinguish two cases. If all nodes output the outcome of the execution of \( P \), these properties follow right away since \( P \) satisfies them; here we use that although the initial two rounds might affect the inputs of nodes, a node will change its input to 0 only if there is at least one correct node with input 0. On the other hand, if some node outputs 0 because it received \( f \) or less messages in the second round of \( P_s \), no node received more than \( 2f < n - f \) messages in the second round. Consequently, all nodes executed
Algorithm 1: Actions of node $v \in V$ at time $t$ that relate to maintaining clock estimates.

1 if $H_v(t) \mod 2\theta d = 0$ then
2     for $w \in \{1, \ldots, n\} \setminus \{v\}$ do
3         if $H_v(t) - R_w^v > (2\theta^2 + \theta)d$ then
4             $M_{vuw}$ := ⊥
5             broadcast update($M_v$)
6     end if
7 if received update($M_{w11}, \ldots, M_{wnn}$) from node $w$ at time $t$ then
8         if $H_v(t) - R_w^v < d$ or $M_{uwv} - M_{vuw} \neq 2\theta d$ then
9             $M_{vuw}$ := ⊥
10            else
11                $M_{uw} := (M_{w11}, \ldots, M_{wnn})$
12                if $|\{u \in V | |M_{uwv} - M_{vuw}| \leq (2\theta^2 + 4\theta)d\}| < n - f$ then
13                    $M_{vuw}$ := ⊥
14                else
15                    $R_w^v$ := ⊥
16                end if
17            end if
18        end if
19    end for
20 end if

$P$ with input 0 and computed output 0 by the agreement property of $P$, implying agreement and validity of the new protocol.

The second case is that some correct node does not participate in the execution of $P$. Thus, it received at most $f$ messages in the first round of $P$, implying that no node received more than $2f < n - f$ messages in this round. Consequently, correct nodes set their input to 0 and will not transmit in the second round. While some nodes may execute $P$, all correct nodes will output 0 no matter how $P$ behaves. Since nodes abort the execution of $P$ if the bounds on communication or time complexity are about to be violated, the claimed bounds for the new protocol hold.

It remains to show that the new protocol is silent. Clearly, if all correct nodes have input 0, they will not transmit in the first two rounds. In particular, they will not receive more than $f$ messages in the first round and not participate in the execution of $P$. Hence correct nodes do not send messages at all, as claimed.

Distributing Clocks

The first step towards simulating round-based protocols is to establish a common timeline for each individual node. This can be easily done by a broadcast, however, such a simple mechanism would give the adversary too much opportunity to fool correct nodes. Therefore, we require nodes to continuously broadcast their local clock values, and keep updating the other nodes on the values they receive. By accepting values only if a sufficient majority of nodes supports them, we greatly diminish the ability of faulty nodes to introduce inconsistencies. In addition, nodes check whether clock updates occur at a proper frequency, as it is known that correct nodes send them regularly. This approach is more robust than the timing tests used in [7] to eliminate untimely messages sent by faulty nodes.

Nodes exchange their clock values at a regular frequency and relay to others the values they have received. To this end, node $v$ maintains memory entries $M_{uwv}$, $w, u \in V$, where $M_{uwv} = H$ is to be understood as “$u$ told me that $w$ claimed to have clock value $H$”. At any time $t$, node $v$ will either trust the clock value node $w$ claims to have, i.e., its estimate of $H_w(t)$ is $M_{uwv}(t)$, or it
Algorithm 2: Actions of node \(v \in V\) at time \(t\) that relate to initiating consensus instances. \(\Delta\) is a sufficiently large constant that will be fixed later.

1. if \(v\) initiates consensus at time \(t\) then
2. broadcast \(\text{init}(H_v(t))\)
3. if \(\text{received} \ \text{init}(H_w) \text{ from } w \in V \text{ at time } t \) and \(|H_w - M_{vuw}(t)| \leq 3\delta\) then
4. broadcast \(\text{echo}(w, H_w)\)
5. if \(\text{received} \ \text{echo}(w, H_w) \text{ from node } u \text{ at time } t \) and \(|H_w - M_{vuw}(t)| \leq \Delta\) then
6. store \((u, \text{echo}(w, H_w))\)
7. if \(|\{u \in V | (u, \text{echo}(w, H_w)) \text{ stored}\}| \geq f + 1 \) and \(E^w_v(H_w)\) is expired then
8. reset \(E^w_v(H_w)\)
9. if \(E^w_v(H_w)\) expires at time \(t\) then
10. if \(|\{\text{stored tuples } (\cdot, \cdot, \text{echo}(w, H_w))\}| \geq n - f\) then
11. participate in \((w, H_w)\) with input \(i_v(w, H_w, t)\)
12. else
13. participate in \((w, H_w)\) with input 0

will not trust \(w\). The latter we express concisely by \(M_{vuw}(t) = \perp\), i.e., any comparison involving \(M_{vuw}(t)\) will fail. Note that if that happens at any time, it cannot be undone; since \(w\) proved to be faulty and we are not concerned with self-stabilization here, \(v\) will just ignore \(w\) in the future. For simplicity, we set \(M_{vuw}(t) := H_v(t)\) for all times \(t\) and, to avoid initialization issues, assume that \(M_{vuw}(0) = H_w(0)\) for all \(u, v, w \in G\). Finally, \(v\) stores its local time when it received a clock update from \(w\) in the variable \(R^w_v\), in order to recognize \(w\) violating the timing constraints on update messages. The actions \(v\) takes in order to maintain accurate estimates of other’s clocks are given in Algorithm 1. The “broadcast” in the protocol means sending to all nodes.

Initiating Consensus

Algorithm 1 forces Byzantine nodes to announce consistent clock values to most of the correct nodes or be revealed as faulty. In particular, it is not possible for a Byzantine node to convince two correct nodes to accept significantly different estimates of its clock.

However, timestamps alone are insufficient to guarantee the consistency of every execution of the consensus protocol. Even if correct nodes know that a node claiming to initiate consensus is faulty, they might be forced to participate in the respective instance because unsuspecting nodes require the assistance of all correct nodes to overcome \(f < n/3\) faults. Ironically, it would require to solve agreement in order for all correct nodes to either participate or not. This chicken-and-egg problem can be avoided using a gradecast-like technique, cf. [11]. If at least \(n - f\) nodes send an echo message (supposedly in response to an initiate message) in a timely fashion (corresponding to confidence level 2 in gradecast), the initiating node might be correct. Hence the receiver \(w\) participates in the respective instance, with input determined by \(i_w\). If between \(f + 1\) and \(n - f - 1\) echo messages are received (confidence level 1), the node participates (as there might be a correct node that fully trusts in the instance), but defaults its input value to “0”. Finally, if \(f\) or less echo messages are received (confidence level 0), it is for sure that no correct node participates with non-zero input and it is safe to ignore the instance.

For every \(w \in V \setminus \{v\}\), \(v\) has a timeout \(E^w_v(H)\), \(H \in \mathbb{R}_0^+\), of duration \(2\delta d\), which serves to delay the start of an instance until all nodes had time to make their decision. Algorithm 2 gives

\(^3\text{Since the full algorithm is self-stabilizing, we do not need to worry about initialization in our simplified setting.}\)
Algorithm 3: Actions of $v \in V$ at time $t$ that relate to running instance $(w, H_w)$ invoked at time $t_v$. $C$ is a sufficiently large constant that will be fixed later.

1. if $H_v(t) = H_v(t_v)$ then
   2. $H_v^{(1)} := H_v(t_v) + C$
3. if received message $(m, i)$ from $u \in V \setminus \{v\}$ at time $t$ and no tuple $(u, m, i)$ stored then
   4. store $(u, m, i)$
   5. if $|\{(u, m, i) \mid (u, m, i) \text{ stored}\}| \geq n - f$ and $H_v^{(i+1)} = \perp$ then
   6. $H_v^{(i+1)} := H_v(t) + 2\vartheta d$
   7. if $|\{(u, m, i) \mid (u, m, i) \text{ stored}\}| \geq f + 1$ and $(H_v^{(i)} = \perp \text{ or } H_v^{(i)} > H_v(t))$ then
   8. $H_v^{(i)} := H_v(t)$
9. if $H_v(t) = H_v^{(1)}$ then
   10. compute $M_v^{(1)}$ based on input
11. if $H_v(t) = H_v^{(i+1)}$ for $i \leq R - 1$ then
   12. compute $M_v^{(i+1)}$, where $\exists$ stored tuple $(u, m, i)$ with $m \neq \emptyset \Leftrightarrow$ received $m$ from $u$ in round $i$
13. if $H_v(t) = H_v^{(i)}$ for $i \leq R$ then
   14. for $w \in V$ do
   15.  if $\exists (m, w) \in M_v^{(i)}$ then
   16.    send $(m, i)$ to $w$
   17.  else
   18.    send $(\emptyset, i)$ to $w$
19. if $H_v(t) = H_v^{(R+1)}$ then
   20. compute output, where $\exists$ stored tuple $(u, m, R)$ with $m \neq \emptyset \Leftrightarrow$ received $m$ from $u$ in round $R$

the pseudocode of the subroutine. We will choose $\Delta$ sufficiently large such that each correct node waits for all correct nodes’ echoes before deciding which input to use.

Running Consensus

Denote by $t_v$ the time when $v$ decides to participate in instance $(w, H_w)$, and by $M_v^{(i)}$, $i \in \{1, \ldots, R\}$, the messages it needs to send in round $i$ of the protocol. Note that since Algorithm 2 also specifies node $v$’s input, it can compute $M_v^{(1)}$ (the messages to send in the first round of the simulated consensus algorithm) by time $H_v^{(1)} := H_v(t_v) + C$, where $C$ is a suitable constant that will be specified later. All messages of the instance are labelled by $(w, H_w)$ in order to distinguish between instances. For ease of notation, we omitted these labels in Algorithm 3.

Essentially, the algorithm runs the fault-tolerant synchronization algorithm from [18] to ensure that the clock drift does not separate the nodes’ estimates of the progression of time during the execution by too much. If a node can be sure that some correct node performed round $i$ (because it received $f + 1$ corresponding messages), it knows that it can safely do so himself. To progress to the next round, nodes wait for $n - f$ nodes. Of these $n - 2f \geq f + 1$ must be correct and will make sure that others catch up. A timeout of $2\vartheta d$ guarantees that this information spreads and all messages of round $i$ can be received before round $i + 1$ actually starts. The “non-messages” $\emptyset$ are explicitly sent to compensate for missing messages.
Note that if not all correct nodes participate, the timing bounds stated above may become violated. However, since the employed protocol is silent and we made sure that all inputs are 0 if not all correct nodes participate, interpreting missing messages as no message being received is sufficient to ensure a consistent execution outputting 0 at all nodes in this case.

4 Analysis

Distributing Clocks

As mentioned earlier, we do not have to worry about correct initialization here, since the ultimate goal is a self-stabilizing algorithm. To simplify the following analysis, we may thus assume that at time 0 each node sends two consecutive (imagined) zero-delay update messages. This avoids issues in the proof logic when referring to previous such messages.

First, we show that correct nodes maintain trusted and accurate clock estimates of each other.

Lemma 4.1. If \( v, w \in G \), then at any time \( t \) it holds that \( H_w(t) \geq M_{vwu}(t) \geq H_w(t) - 3\delta d \).

Proof. Node \( w \) sends a clock update at least every \( 2\delta \) local time. Since messages are delayed by at most \( d \) time units, the clock of \( w \) will proceed by at most \( \delta d \) until such a message is received. Recall that we assume that \( M_{vwu}(0) = H_w(0) \). Thus, it is sufficient to show that \( w \) never sets \( M_{vwu} := \bot \), implying that it always sets \( M_{vwu} \) to a value from \( (H_w(t) - 3\delta d, H_w(t)) \) before \( M_{vwu}(t) = H_w(t) - 3\delta d \) becomes satisfied.

Assume for contradiction that \( t \) is the minimal time when some node \( v \in G \) sets \( M_{vwu} := \bot \) for some node \( w \in G \). The clock of \( v \) proceeds by at most \( (2\delta + 1)d \) between consecutive updates from \( w \). Together with the assumption that \( R_u(0) = H_w(0) \), this shows \( v \) cannot execute Line 4 of Algorithm 1 at time \( t \). Similarly, since nodes send clock updates every \( 2\delta d \) local time (i.e., at most \( 2d \) real time apart) and messages are delayed by at most \( d \), \( v \) cannot set \( M_{vwu} := \bot \) according to Line 7 of the algorithm at time \( t \). This leaves Line 12 as remaining possibility. We claim that \( |M_{vwu} - M_{vwu}| \leq H_w(t) - H_w(0) \leq (2\delta^2 + 4\delta)d \) for all \( u \in G \). Given that \( |G| \geq n - f \), from this claim we can conclude that \( v \) does not execute this line at time \( t \) either, resulting in a contradiction.

Consider the most recent update message (before time \( t \)) \( w \) received from a node \( u \in G \). It has been sent at a time \( t_u \geq t - 2\delta d + d \), as otherwise the next update message would already have arrived. Since \( t > t_u \) is minimal, we have that \( H_u(t_u) \geq M_{vwu}(t_u) \geq H_u(t_u) - 3\delta d \). We conclude that

\[
|M_{vwu}(t) - M_{vwu}(t)| \leq |H_w(t) - H_w(t_u)| + |H_w(t_u) - M_{vwu}(t_u)| \leq (2\delta^2 + 4\delta)d,
\]
as claimed. By the previous observations, this completes the proof.

The next lemma shows that the employed consistency checks force faulty nodes to present reasonably similar clock estimates to different correct nodes.

Lemma 4.2. Suppose that \( v, w \in G \), \( u \in V \), \( t_v \geq t_v \), and \( M_{vwu}(t_v) \neq \bot \neq M_{uwv}(t_w) \). Then it holds that \( M_{vwu}(t_v) - M_{vwu}(t_v) \in [2(t_v - t_v) - (2\delta + 3) - \mathcal{O}(d), 2\delta(t_v - t_v) + \mathcal{O}(d)] \).

Proof. Consider the most recent update messages \( v \) and \( w \) received until time \( t_v \), at times \( t_v, t_w \in (t_u - (2\delta + 1)d, t_v) \). Due to the prerequisites that \( M_{vwu}(t_v) \neq \bot \neq M_{uwv}(t_w) \), neither does \( v \) set \( M_{vwu} := \bot \) at time \( t_v \) nor does \( w \) set \( M_{uwv} := \bot \) at time \( t_w \). Hence,

\[
\exists X_v \subseteq V : |X_v| \geq n - f \land \forall x \in V \ : |M_{vwu}(t_v) - M_{vxu}(t_v)| \leq (2\delta^2 + 4\delta)d,
\]

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and there is a set $X_w$ satisfying the same condition for $w$ at time $t'_w$. Clearly, $|X_v \cap X_w| \geq n - 2f \geq f + 1$. Hence, there is a correct node $g \in X_v \cap X_w \cap G$.

Denote by $t'_v, t'_w \in (t_v - (2\vartheta + 1)d, t_v)$ the receiving times of the latest update messages from $g$ that $v$ and $w$ received until time $t_v$ and by $t'_s, t'_w \in (t_v - (2\vartheta + 1)d, t_v)$, respectively, their sending times. Note that there never is more than one update message from $g$ in transit. Therefore, either $t'_s = t'_w$ or one of the messages received by $v$ and $w$ directly precedes the other one. Thus, $|H_g(t'_v) - H_g(t'_w)| \leq 2\vartheta d$. Within $2\vartheta d$ time, $g$ receives at most $[2\vartheta]_d$ update messages from $u$, each of which must increases its estimate $M_{gua}$ of $u$’s clock by exactly $2\vartheta d$, as otherwise it would set $M_{gua} := \bot$. We conclude that $|M_{gua}(t'_v) - M_{gua}(t'_w)| \in \mathcal{O}(d)$, yielding

$$
|M_{vu}(t'_v) - M_{wu}(t'_w)| \\
\leq |M_{vu}(t'_v) - M_{gu}(t'_v)| + |M_{gu}(t'_v) - M_{gu}(t'_w)| + |M_{wu}(t'_v) - M_{wu}(t'_w)| \in \mathcal{O}(d).
$$

It remains to bound the progress of the estimates $M_{vu}$ and $M_{wu}$ during $[t'_v, t_v]$ and $[t'_w, t_w]$, respectively. Again, $v$ must not set $M_{vu} := \bot$ during $[t'_v, t_v]$ and $w$ must not set $M_{wu} := \bot$ during $[t'_w, t_w]$. Due to the fact that $v$ and $w$ accept update messages without losing trust in $w$ only if they arrive at least $d$ and at most $(2\vartheta^2 + 3\vartheta)d$ time apart, we can bound

$$
\frac{2(t_v - t'_v)}{2\vartheta + 3} - 2\vartheta d \leq M_{vu}(t_v) - M_{wu}(t'_v) \leq 2\vartheta(t_v - t'_v) + 2\vartheta d
$$

and

$$
\frac{2(t_w - t'_w)}{2\vartheta + 3} - 2\vartheta d \leq M_{wu}(t_w) - M_{vu}(t'_w) \leq 2\vartheta(t_w - t'_w) + 2\vartheta d.
$$

Putting all bounds together, we obtain

$$
M_{wu}(t_w) - M_{vu}(t_v) = M_{wu}(t_w) - M_{wu}(t'_v) + M_{wu}(t'_v) - M_{wu}(t'_w) - M_{vu}(t'_w) + M_{vu}(t_v) - M_{vu}(t'_v) \\
\in \left[\frac{2(t_w - t_v)}{2\vartheta + 3} - \mathcal{O}(d), 2\vartheta(t_v - t'_v) + \mathcal{O}(d)\right],
$$

concluding the proof. \( \square \)

**Initiating Consensus**

Having set up the bounds on the differences of clock estimates among correct nodes, we can discuss their mutual support in invoking consensus. First, we show that correct nodes can initiate instances unimpaired by the consistency checks of Algorithm 2.

**Lemma 4.3.** If $v \in G$ initiates a consensus instance at time $t$ and $\Delta \geq 3\vartheta d$, then each node $w \in G$ participates at some time $t_w \in [t + 2d, t + \mathcal{O}(d)]$ with input value $i_w(v, H_v(t), t_w)$.

**Proof.** By Lemma 4.1, we have for all times $t' \in [t, t + 2d]$ and nodes $w \in G$ that $H_w(t) - 3\vartheta d \leq M_{uvv}(t) \leq M_{wuv}(t) \leq H_v(t) + 2\vartheta d$. Each node $w \in G$ will receive the init($H_v(t)$) message from $v$ at some time $t' \in [t, t + d]$ and, as by the above bound the condition in Line 4 is met, broadcast an echo($v, H_v(t)$) message. These messages will be received at times $t' \in [t, t + 2d]$ and, as the condition in Line 5 is met, be stored by nodes $w \in G$.

Since only faulty nodes may send an echo($v, H_v(t)$) message earlier than time $t$ and $|G| \geq n - f > f + 1$, the condition for resetting $E_w^*(H_v(t))$ will be met at each $w \in G$ at some time during $[t, t + 2d]$. Therefore, each such node participates in the instance ($v, H_v(t)$) at some time $t_w \in [t + 2d, t + \mathcal{O}(d)]$. By this time, $w$ will have received all echo($v, H_v(t)$) messages from nodes in $G$. Thus, the condition in Line 11 is met at time $t_w$ and it will use input $f_w(t_w)$. \( \square \)
The following statement summarizes how the guarantees of the clock estimates control faulty nodes’ ability to feed inconsistent information to correct nodes by timing violations.

**Corollary 4.4.** If at times \( t_v, t_w \in \mathbb{R}_0^+ \) nodes \( v, w \in G \) send \( \text{echo}(u, H_u) \), then \( |t_v - t_w| \in \mathcal{O}(d) \).

**Proof.** By Line 4 of Algorithm 2, we have that \( |M_{vuw}(t_v) - M_{wuu}(t_w)| \leq 6d \). By Lemma 4.2, \( |M_{vuw}(t_v) - M_{wuu}(t_w)| \in \Omega(|t_v - t_w|) \). Hence, \( |t_v - t_w| \in \mathcal{O}(d) \). \( \square \)

This entails that correct nodes use non-zero input only when all correct nodes participate.

**Lemma 4.5.** Suppose that \( \Delta \in \mathcal{O}(d) \) is sufficiently large. If for any \( u \in V \), \( v \in G \) participates in a consensus instance labeled \((u, H_u)\) with an input value different from 0 at time \( t_v \), then each node \( w \in G \) participates at some time \( t_w \in [t^-, t^+] \), where \( t^+ - t^- \in \mathcal{O}(d) \).

**Proof.** Since \( v \) participates in the instance with non-zero input, it stores at least \( n - f \) tuples \((x, \text{echo}(u, H_u))\). At least \( n - 2f \geq f + 1 \) of these correspond to \( \text{echo}(u, H_u) \) messages sent by correct nodes. Since \( v \) participates at time \( t_v \), it received one of these messages at some time \( t_v - \Theta(d) \). By Corollary 4.4, all such messages sent by correct nodes must have been sent (and thus received) within an interval \([t_v - \mathcal{O}(d), t_v + \mathcal{O}(d)]\). We conclude that (i) no correct node will join the instance earlier than time \( t_v - \mathcal{O}(d) \), (ii) all correct nodes will receive at least \( f + 1 \) \( \text{echo}(u, H_u) \) messages from different sources by time \( t_v + \mathcal{O}(d) \), (iii) as \( \Delta \) is sufficiently large, at all correct nodes the condition in Line 5 of Algorithm 2 will be met when receiving these messages, and therefore (iv) all correct nodes join the instance by time \( t_v + \mathcal{O}(d) \). \( \square \)

**Running Consensus**

The silence property of the employed consensus protocol deals with all instances without a correct node with non-zero input. **Lemma 4.5** shows that all correct nodes participate in any other instance. Hence, we need to show that any instance in which all correct nodes participate successfully simulates a synchronous execution of the consensus protocol.

**Lemma 4.6.** Suppose that \( \Delta, C \in \mathcal{O}(d) \) are sufficiently large and that some node from \( G \) participates in instance \((v, H_v)\) at time \( t_0 \) with input value different from 0. Then each node \( w \in G \) computes an output for the instance (Line 20 of Algorithm 3) at some time \( t_w \in [t^-, t^+] \), where \( t^+ - t^- \in \mathcal{O}(d) \) and \( t^- - t^+ \in t_0 + \Theta(R) \). These outputs are the result of some synchronous run of \( \mathcal{P}_s \) with the inputs the nodes computed when joining the instance.

**Proof.** We will denote for each node \( w \in G \) and each \( i \in \{1, \ldots, R + 1\} \) by \( t_w^{(i)} \) the time satisfying that \( H_w(t_w^{(i)}) = H_w^{(i)}(t_w^{(i)}) \); we will show by induction that these times exist and are unique. Define \( t^{(i)} := \min_{w \in G}\{t_w^{(i)}\} \). The induction will also show that all nodes \( w \in G \) compute and send their messages, as well as receive and store all messages from other nodes in \( G \) for rounds \( j < i \), \( i \in \{2, \ldots, R + 1\} \), of the protocol (i.e., execute Lines 10 or 12 and 3 of Algorithm 3) at times smaller than \( t^{(i)} \). Note that these properties show that the progression of Algorithm 3 can be mapped to a synchronous execution of \( \mathcal{P}_s \) and the messages \( M_w^{(i)} \) can indeed be computed according to \( \mathcal{P}_s \). Finally, the induction will show that \( t_w^{(i)} \in t_0 + \Theta(id) \) for all \( w \in G \) and \( i \in \{2, \ldots, R + 1\} \); the stated time bounds on \( t^- \) and \( t^+ \) follow. As the messages \( M_w^{(1)} \) the nodes compute in Line 10 are based on the inputs the node compute when joining the instance, completing the induction will thus also complete the proof.

Before we perform the induction, let us make a few observations. The only way to manipulate \( H_w(t) \neq \bot \) at some time \( t \) is to set it to \( H_w(t) \), provided it was larger than that (Line 8). Thus, once
defined, $H_w^{(i)}(\cdot)$ is non-increasing, and can never be set to a value smaller than $H_w(t)$. In particular, the times $t_w^{(i)}$ are unique (if they exist). Furthermore, the conditions for computing and sending messages are checked after this line, implying that the lines in which messages are computed and sent are indeed performed at the unique time $t_w^{(i)}$. Therefore, each node $u \in G$ sends (at most) one message $(\cdot, i)$ to each node $w \in G$, which will be received and stored a time from $(t_w^{(i)}, t_w^{(i)} + d)$, assuming that the receiver already joined the instance. The latter can be seen as follows. We apply Lemma 4.5 to see that each node $w \in G$ participates in the instance at some time $t_w^{(0)} \in t_0 + \Theta(d)$. Thus, if $C \in \mathcal{O}(d)$ is sufficiently large, each node $w \in G$ has joined the instance and computed $H_w^{(1)} = H_w(t_w^{(0)}) + C$ before time $t^{(1)}$ (which exists because $H_w^{(1)}$ has been set to some value).

We now perform the induction step from $i \in \{1, \ldots, R\}$ to $i + 1$. First, let us show that the times $t_w^{(i+1)}$ exist. Since each node $G$ sends some message $(m, i)$ to each other node in $G$ at some time from $t_0 + \Theta(id)$, each node $w \in G$ will execute Line 6 for $i$ at some time $t \in t_0 + \Theta(id)$, setting $H_w^{(i+1)} := H_w(t) + 2\delta d$. We conclude that the times $t_w^{(i+1)}$ exist. Clearly, no node in $G$ can execute Line 8 before time $t^{(i+1)}$, as until then no messages $(\cdot, i+1)$ are sent by any nodes in $G$. Hence, $t_w^{(i+1)} \in t_0 + \Theta((i+1)d)$ for all $w \in G$. Now suppose that $t_{i+1}$ is minimal with the property that some node $w \in G$ executes Line 6, defining $H_w^{(i+1)}$. At this time, it stores $n - f$ tuples $(u, m, i)$ for $u \in V$, at least $n - 2f \geq f + 1$ of which satisfy that $u \in G$. For each such $u \in G$, it holds that $t_u^{(i)} < t_{i+1}$, implying that at each node $x \in G$ the first part of the condition for executing Line 8 for index $i$ will be satisfied at some time smaller than $t_{i+1} + d$. Consequently, $t_u^{(i)} < t_{i+1} + d$, and all messages from nodes in $G$ corresponding to round $i$ will be sent by time $t_{i+1} + d$ and received by time $t_{i+1} + 2d \leq t^{(i+1)}$. By induction hypothesis, the same holds for all messages to and from nodes in $G$ for rounds $j < i$. Thus, all claimed properties are satisfied for step $i + 1$, completing the induction and hence the proof.

We conclude that Algorithms 1–3 together solve the initialization problem.

Theorem 4.7. Each consensus instance $(v, H_v)$ can be mapped to a synchronous execution of $\mathcal{P}_s$. If the instance has output $o \neq 0$, all nodes in $G$ output $o$ within $O(d)$ time of each other. Moreover, there is a node in $w \in G$ satisfying that $f_w(t) = o$ for some time $t \in t_w - \Theta(Rd)$, where $t_w$ is the time when it outputs $o$. Finally, if $v$ is correct and $t_0$ is the time when it initialized the instance, all nodes in $w \in G$ compute their inputs as $f_w(t_w)$ at some time $t_w \in t_0 + \Theta(d)$.

Proof. Assume first that no node in $G$ participates in the instance with an input different from 0. Then no node in $G$ will send a message $(m, i)$ for any $i$ with $m \neq 0$ for this instance: $\mathcal{P}_s$ is silent, and Algorithm 3 interprets any “missing” message as having received no message from the respective node in Lines 12 and 20; in particular, all nodes in $G$ will output 0.

Next, suppose that some correct node has input different from 0. In this case, the claimed properties follow from Lemma 4.6 and the properties of $\mathcal{P}_s$.

Finally, assume that $v \in G$ and $t_0$ is the time when $v$ initializes the instance. Lemma 4.3 shows that each node $w \in G$ participates in the instance at some time $t_w \in t_0 + \Theta(d)$ with input $f_w(t_w)$.

With the initialization problem being solved, it is straightforward to derive an algorithm that enables consistent initialization of arbitrary consensus protocols.

Corollary 4.8. Given any $R$-round synchronous consensus algorithm $\mathcal{P}$ tolerating $f < n/3$ faults, there is an algorithm with the following guarantees.

- Each (correct) node can initiate an instance of $\mathcal{P}$ at any time $t$.
For any instance (also those initiated by faulty nodes) it holds that nodes determine their inputs according to their local view of the system during some interval \([t_1, t_1 + O(1)]\), and terminate during some interval \([t_2, t_2 + O(1)]\), where \(t_2 \in t_1 + \Theta(R)\).

- If a correct node initiates an instance at time \(t\), then \(t_1 = t\).
- Each instance satisfies termination, agreement, and validity.
- The above guarantees hold in the presence of \(f\) faulty nodes.

**Proof.** We run algorithms Algorithms 1–3 in the background, with input functions always returning 1 and \(P_s\) (the derived silent protocol from Lemma 3.1) as the utilized silent consensus protocol. Whenever a node wants to initiate an instance of \(P\) at a time \(t\), it first initiates an instance of \(P_s\) using our framework. When reaching the threshold of echo messages to participate in the instance (at some time from \((t, t + O(d))\)), correct nodes store the input they will use if this call leads to an actual run of \(P\), according to their current view of the system.

Provided that a correct node initiates an instance, by Theorem 4.7 all correct nodes will compute output 1 for the associated instance of \(P_s\) (by validity). This is mapped to starting an associated run of \(P\) with the inputs memorized earlier, where a copy of Algorithm 3 is used to run \(P\). Note that, since all correct nodes participate, Lemma 4.6 shows that we can map the execution of Algorithm 3 to a synchronous execution of \(P\) with the inputs determined upon initialization, where each correct node terminates during an interval \([t', t' + O(d)]\) for some \(t' \in t + \Theta(R)\).

On the other hand, output 0 is mapped to taking no action at all. For instances of \(P_s\) that output 1, Theorem 4.7 shows that all nodes terminate within \(O(d)\) time off each other. Previous arguments also show that the inputs to the resulting run of \(P\) have been determined \(\Theta(R)\) time earlier, as desired. We conclude that all claimed properties are satisfied.

5 Self-Stabilization and Bounded Communication Complexity

In this section, we discuss how the previous results can be generalized to Theorem 1.1. We will add self-stabilization first, then argue how to use discrete and bounded clocks, and finally control the rate at which consensus instances can be initiated.

Self-Stabilization

Within \(d\) time, the links deliver all spurious messages from earlier times; afterwards, each message received from a correct node will be sent in accordance with the protocol.

We take a look at the individual components of the algorithm. Algorithm 1 is not self-stabilizing, because the loss of trust in a node cannot be reversed. This is straightforward to rectify, by nodes starting to forward received claimed clock values if their senders are well-behaving for sufficient time, and subsequently starting to trust a node again if receiving consistent reports on its clock from \(n - f\) nodes for sufficiently long. This is detailed in Section 6, where we present Algorithm 4, a self-stabilizing variant of Algorithm 1.

As Algorithm 4 will operate correctly after \(O(R)\) time, it is not hard to see how to make Algorithm 2 self-stabilizing. We know that a “correct” execution for a given label will start with a “clean slate” (i.e., no tuples stored at any correct node). All related messages sent and received by correct nodes as well as possibly joining the instance are confined within a time window of length \(\tau \in O(d)\). Hence, we can add timeouts deleting stored tuples from memory \(\vartheta\tau\) local time after they have been written to memory, without disturbing the operation of the algorithm.4

4 Note that this can be done in a self-stabilizing way by memorizing the local times when they have been stored; if such a time lies in the future or more than \(\vartheta\tau\) time in the past (according to the current value of the hardware
the time to regain trust in a (faulty) node’s clock (distributed by Algorithm 4) larger than \( \theta \tau \), we can guarantee that memory will be wiped before the faulty node can “reuse” the same label at a later time (by “setting its hardware clock back”). This modification ensures that Algorithm 2 will stabilize within \( O(d) \) time once Algorithm 4 does.

Similar considerations apply to Algorithm 3. We know that a “correct” execution of the algorithm progresses to the next simulated round of \( P_s \) within \( \tau \in O(d) \) time (all correct nodes participate) or correct nodes do not send any messages in the simulated execution of \( P_s \) and output 0 (by silence). Adding a timeout of \( \theta \tau \) (locally) terminating the instance with output 0 if no progress is made thus guarantees termination within \( O(R) \) time. Naturally, this may entail that correct nodes “leave” an instance prematurely, but this may happen if the instance was not initialized correctly (i.e., nodes have lingering false memory entries from time 0) or the instance is silent (i.e., there is no need to send messages and the output is 0 at all correct nodes) only. Similar to Algorithm 2, this strategy guarantees that false memory entries can be safely wiped within \( \theta \tau R \) rounds; increasing the timeout to regain trust in Algorithm 4 to \( \theta^2 \tau R \) thus guarantees that Algorithm 3 will stabilize within \( O(R) \) rounds once Algorithm 4 and 2 have, in the sense that to its future outputs the arguments and bounds from Section 4 apply.

Finally, we note that when calling Algorithm 3 for protocol \( P \) in Corollary 4.8, always all nodes participate. Hence, the same arguments apply and a total stabilization time of \( O(R) \) follows.

### Discrete and Bounded Clocks

In practice, clocks are neither continuous nor unbounded; moreover, we need clock values to be bounded and discrete to encode them using few bits. Discretizing clocks with a granularity of \( \Theta(d) \) will asymptotically have no effect on the bounds: We simply interpret the discrete clocks as readings of continuous clocks with error \( O(d) \). It is not hard to see that this can be mapped to a system with exact readings of continuous clocks and larger maximal delay \( d' \in O(d) \), where all events at node \( v \) happen at times when \( H_v(t) \in \mathbb{N} \).

As shown in Corollary 6.2, choosing \( B \in \Theta(R) \) in Algorithm 4 guarantees the following. For each sufficiently large time \( t \geq t_0 \in \Theta(R) \), all correct nodes from \( G \) trusting some node \( v \in V \) at time \( t \) received clock values from \( v \) that increased at constant rate for \( \Theta(R) \) time and differed at most by \( O(d) \). From this it follows that using clocks modulo \( M \in \Theta(R) \) is sufficient: Choosing \( M \) sufficiently large, we can make sure that for any label \( (v, H) \), every \( \Theta(R) \) time there will be a period of at least \( \theta^2 \tau R \) (\( \tau \) as above) time during which all correct node reject initialization messages labeled \( (v, H) \). This ensures that memory will be wiped before the next messages are accepted and the previous arguments for self-stabilization apply.

### Bounding the Communication Complexity

Using bounded and discrete clocks and assuming that \( R \) is polynomially bounded in \( n \), each clock estimate (and thus each label) can be encoded by \( O(\log n) \) bits. Hence, each correct node will broadcast \( O(n \log n) \) bits in \( \Theta(d) \) time when executing Algorithm 4, for a total of \( O(n^2 \log n) \) bits per node and time unit.

However, so far each node may initiate an instance at any time, implying that faulty nodes could initiate a large number of instances with the goal of overloading the communication network. Hence, we require that correct nodes wait for at least \( T \geq 2 \theta d \) local time between initializing instances. Under this constraint, it is feasible that correct nodes ignore any init message from \( v \in V \) that is clock), the entries need to be deleted.

5This entails that timeouts are integer, which also clearly does not affect the asymptotic bounds.
received less than $T/\theta - d$ local time after the most recent init message from $v$. As a result, no node will broadcast more than $O(n \log n)$ bits within $T$ time due to executing Algorithm 2.

Moreover, now there cannot be more than one instance per node $v$ and $\bar{T} = (T/\theta - d)/\theta$ time such that some correct node participates with non-zero input due to messages sent at times greater than 0 alone (i.e., not due to falsely memorized echo messages at time 0): this requires the reception of $n - 2f$ corresponding echo messages from correct nodes, which will not send echo messages for another instance labeled $(v, \cdot)$ for $T/\theta$ time. Such an instance runs for $O(R)$ time. There are at most $|G| = n - f$ other instances with label $(v, \cdot)$ a node may participate in within $\bar{T}$ time ($f + 1$ received messages imply one was from a correct node), all of which terminate within 2 simulated rounds with “empty” messages $(\emptyset, 1)$ or $(\emptyset, 2)$ only.

For any $v \in V$, this leads to the following crucial observations: (i) If a node memorizes that it participates in more than $k_1 \in O(R/\bar{T})$ instances labeled $(v, \cdot)$ which did not terminate by the end of round 2 or sent other messages than $(\emptyset, 1)$ or $(\emptyset, 2)$, its memory content is inconsistent; (ii) if a node memorizes that it participates in more than $k_2 \in O(n/\bar{T})$ instances $(v, \cdot)$, its memory content is inconsistent; (iii) as memorized echo messages and memory associated with an instance of Algorithm 3 is cleared within $O(R + T)$ time, (i) or (ii) may occur at times $t \in O(R + T)$ only; and (iv) if a node $w \in G$ detects (i) or (ii) at time $t$ and deletes at time $t + d$ all memorized echo messages, forces all timeouts $E_v(\cdot)$ into the expired state, and clears all memory entries corresponding to Algorithm 3, (i) or (ii) cannot happen again at this node.

Hence, we add the rule that a node detecting (i) or (ii) stops sending any messages corresponding to Algorithm 3 for $\theta d$ local time and then clears memory according to observation (iv). By (iii), this mechanism will stop interfering with stabilization after $O(R + T)$ time; afterwards, the previous arguments apply. Furthermore, (i) and (ii) imply that a node never concurrently participates in more than $k_1$ instances for which it sends non-empty messages, and sends at most $O(n^2 \log n)$ bits ($O(n)$ broadcasted round numbers and labels) in $O(\bar{T}) = O(T)$ time due to other instances.

Hence, it remains to control the number of bits sent by the at most $k_1$ remaining instances. Recall that the messages sent by Algorithm 3 are of the form $(m, i)$, where $m$ is a message sent by $P_x$ and $i$ is a round number. We know that in a correct simulated execution of such an instance, the node sends up to $B + O(rn \log n)$ bits within $rd$ time: $B$ is the maximal number of bits sent by a node in an execution $P$, the additional two initial round of $P_x$ require nodes to broadcast single-bit messages, and $\log R \in O(\log n)$ broadcasted bits are required to encode round numbers and labels. Therefore, a node can safely locally terminate any instance violating these bounds and output, say, 0. Such a violation may only happen if the instance has not been properly initialized; since any instance terminates within $O(R)$ time and Algorithm 4 and subsequently Algorithm 2 will stabilize within $O(R + T)$ time, we can conclude that, again, this mechanism will not interfere with stabilization once $O(R + T)$ time has passed.

In summary, we have shown the following.

- We can modify the algorithm from Corollary 4.8 such that it self-stabilizes in $O(R)$ time.
- We can further modify it to operate with bounded and discrete hardware clocks.
- For $T \geq 2\theta d$, additional modifications ensure that, for each correct node, the amortized number of bits sent per time unit is $O(n^2 \log n + k_1 n B) = O(n^2 \log n + nBR/T)$; this increases the stabilization time to $O(R + T)$ and entails that correct nodes wait at least $T$ local time between initializing instances.

The resulting statement is exactly Theorem 1.1.
6 Self-Stabilizing Clock Distribution

Algorithm 4, the self-stabilizing variant of Algorithm 1, is essentially identical, except that the loss of trust upon detecting an inconsistency is only temporary. To this end node \( v \in V \) maintains timeouts \( A^w_v \) and \( B^w_v \) for each node \( w \in V \), of durations \( 2\vartheta d \) and \( B \), respectively. The clock estimate \( v \) of \( w \) then is \( M_{vw}(t) \) at times \( t \) when \( B^w_v \) is expired and \( \perp \) otherwise. Timeout \( A^w_v \) is reset whenever \( w \) announces clock values to \( v \) that violate the timing constraints, i.e., an update message is sent too soon or too late after the previous, or it does not have contain the previous value increased by \( 2\vartheta d \). Whenever \( A^w_v \) is not expired, \( v \) will report \( \perp \) as the “clock value” it received from \( w \) to others, expressing that there has been an inconsistency; at other times, it reports the most recent value received. If a node keeps sending values in accordance with the timing constraints, eventually all correct nodes will be reporting these values (as their \( A \)-timeouts expire). Subsequently the check in Line 17 will always be passed, which resets \( B^w_v \) whenever there is insufficient support from others for the clock value \( w \) claims to \( v \). Eventually, \( B^w_v \) will expire, and \( w \)’s trust in \( v \) is restored.

**Algorithm 4**: Actions of node \( v \in V \) at time \( t \) that relate to maintaining self-stabilizing clock estimates.

```c
if H_v(t) mod 2\theta d = 0 then
    for w \in \{1, \ldots, n\} \setminus \{v\} do
        if H_v(t) - R^w_v > (2\vartheta^2 + \vartheta)d then
            reset A^w_v and B^w_v
        if A^w_v = 1 then
            \( \hat{M}_{vw} := M_{vw} \)
        else
            \( \hat{M}_{vw} := \perp \)
        M_v := (\hat{M}_{v1}, \ldots, \hat{M}_{vn})
    broadcast update(M_v)
else
    if received update(M_{w1}, \ldots, M_{wn}) from node w at time t then
        if H_v(t) - R^w_v < d or M_{vw} - M_{vw} \neq 2\theta d then
            reset A^w_v and B^w_v
        M_v := (M_{w1}, \ldots, M_{wn})
        for x \in \{1, \ldots, n\} \setminus \{v\} do
            if \(|\{u \in V | |M_{vxx} - M_{vxx}| \leq (2\vartheta^2 + 4\vartheta)d\}| < n - f then
                reset B^x_v
        R^w_v := H_v(t)
```

Note that it is straightforward to adapt the algorithm to bounded clocks modulo some value \( M \gg B \). As we just argued why the algorithm stabilizes in the sense that correct nodes eventually trust each other, the following analogon to Lemma 4.1 is immediate.

**Corollary 6.1.** Suppose that \( t_0 \in O(d + B) \) is sufficiently large. If \( v, w \in G \), then at any time \( t \geq t_0 \) it holds that \( H_v(t) \geq M_{vw}(t) \geq H_v(t) - 3\theta d \).

**Lemma 4.2** is translated in a similar fashion.

**Corollary 6.2.** Suppose that \( v, w \in G, u \in V, t_v \geq t_0 \) for a sufficiently large \( t_0 \in O(d) \), \( t_w \in [t_v, t_v + B/\vartheta - (2\vartheta + 1)d] \), \( B^v_w \) is expired at time \( t_v \), and \( B^w_v \) is expired at time \( t_w \). Then \( M_{wuw}(t_w) - M_{vw}(t_v) \in [2(t_w - t_v)/(2\vartheta + 3) - O(d), 2\vartheta(t_w - t_v) + O(d)] \).
Proof. The requirement that \( t_v \geq t_0 \) ensures that all spurious messages in the communication network at time 0 have been received and, afterwards, all correct nodes sent and received at least two update messages from each other correct node.

Consider the most recent update messages \( v \) and \( w \) received until time \( t_v \), at times \( t'_v, t'_w \in (t_v - (2\theta + 1)d, t_v] \). Due to the prerequisites that \( B_v^w \) is expired at time \( t_v \) and \( B_v^w \) is expired at time \( t_w \), neither does \( v \) set \( M_{vwu} := \bot \) at time \( t'_v \) nor does \( w \) set \( M_{uwv} := \bot \) at time \( t'_w \). Hence,

\[
\exists X_v \subseteq V : |X_v| \geq n - f \land \forall x \in X_v : |M_{vwu}(t'_v) - M_{vexu}(t'_v)| \leq (2\theta^2 + 4\theta)d,
\]

and there is a set \( X_w \) satisfying the same condition for \( w \) at time \( t'_w \). Clearly, \( |X_v \cap X_w| \geq n - 2f \geq f + 1 \). Hence, there is a correct node \( g \in X_v \cap X_w \cap G \).

From here we proceed analogously to the proof of Lemma 4.1, noting that \( A_g^w \) being of duration \( 2\theta d \) guarantees that \( |M_{guu}(t'_v) - M_{guu}(t'_w)| \in \mathcal{O}(d) \) for two consecutive update messages sent by \( g \) at times \( t'_v \) and \( t'_w \). \[ \square \]

7 Further Results

One can reduce the bit complexity from Theorem 1.1 further by reducing the frequency at which clock estimates are updated. The loss in accuracy however comes at the cost of increasing the time interval during which input values are determined.

Corollary 7.1. Suppose \( \mathcal{P} \) is a synchronous consensus protocol tolerating \( f < n/3 \) faults, runs for \( R \in \text{polylog}(n) \) rounds, and guarantees that no correct node sends more than \( B \) bits. For each \( T \geq 2\theta d \), there is a value \( S \in \mathcal{O}(T + R) \) and an algorithm with the following properties.

- Each correct node \( v \) can initiate an instance of \( \mathcal{P} \) at any time \( t \geq S \), provided that it has not done so at any time \( t' < t \) for which \( H_v(t) - H_v(t') \leq T \).
- For any instance that terminates at a time larger than \( S \), it holds that nodes determine their inputs according to their local view of the system during some interval \( [t_1, t_1 + \mathcal{O}(T)] \), and terminate during some interval \( [t_2, t_2 + \mathcal{O}(1)] \), where \( t_2 \in t_1 + \Theta(T + R) \).
- If a correct node \( v \) initiates an instance at time \( t \geq S \), then \( t_1 = t \).
- Each instance for which \( t_2 \geq S \) satisfies termination, agreement, and validity.
- Each correct node sends at most \( \mathcal{O}((n^2 \log n + RBn)/T) \) amortized bits per time unit.
- The above guarantees hold in the presence of \( f \) faulty nodes and for arbitrary initial states.

Proof. We apply our reasoning for \( d' \in \Theta(T) \), except that Algorithm 3 still progresses at one simulated round within \( \Theta(d) \) time.\(^6\) In other words, nodes send clock updates every \( \Theta(T) \) time, implying that the clock estimates are accurate up to \( \Theta(T) \), and instances of Algorithm 3 are joined within a time window of \( \Theta(T) \) by correct nodes. Algorithm 3 thus terminates within \( \mathcal{O}(C + R) = \mathcal{O}(T + R) \) rounds, so we can choose the timeouts for regaining trust in Algorithm 4 and the maximal clock value in \( \Theta(T + R) \) as well; this ensures that the stabilization time remains \( \mathcal{O}(T + R) \).

With these modifications, we have a bit complexity of \( \mathcal{O}(n^2 \log n) \) per node and \( T \) time for Algorithms 1 and 2. The bound of \( \mathcal{O}((n^2 \log n + Bn)/T) \) amortized bits per node and time unit for Algorithm 3 holds as before, resulting in a total of \( \mathcal{O}((n^2 \log n + Bn)/T) \) bits per node and time unit for the compound algorithm. \[ \square \]

Since our framework is deterministic, it can operate in any adversarial model. What is more, we make use of the agreement and validity properties of \( \mathcal{P} \) only in executions simulating a synchronous execution of the protocol in which all nodes participate. This happens only polynomially often in \( n \).

\(^6\)Note that we have to set \( C \in \Theta(d') = \Theta(T) \), though.
Hence, we can also plug randomized consensus algorithms in our framework that satisfy agreement and validity w.h.p. only. A randomized consensus protocol terminating within $R$ rounds satisfies the following properties.

**Termination:** Every correct node terminates within $R$ rounds and outputs a value $o(v) \in I$. 

**Agreement:** With high probability, $o(v) = o(w)$ for correct nodes $v, w$. 

**Validity:** If $i(v) = i(w)$ for all correct $v, w$, with high probability this is also the output value. 

Note that, typically, agreement and validity are required to hold deterministically, while termination is only satisfied probabilistically. It is simple to translate such an algorithm in one that satisfies the above criteria by forcing termination after $R$ rounds, where $R$ is sufficiently large to guarantee termination w.h.p.\(^7\) For suitable randomized algorithms, the following corollary is immediate.

**Corollary 7.2.** Suppose $P$ is a synchronous randomized consensus protocol tolerating $f < n/3$ faults that terminates in $R \in \text{polylog}(n)$ rounds and guarantees that no correct node sends more than $B$ bits w.h.p. Then there is a value $S \in O(R)$ and an algorithm with the following properties.

- Each correct node $v$ can initiate an instance of $P$ at any time $t \geq S$, provided that it has not done so at any time $t' < t$ for which $H_v(t) - H_v(t') \leq R$.
- For any instance that terminates at a time larger than $S$, it holds that nodes determine their inputs according to their local view of the system during some interval $[t_1, t_1 + O(R)]$, and terminate during some interval $[t_2, t_2 + O(1)]$, where $t_2 \in t_1 + \Theta(R)$. 
- If a correct node initiates an instance at time $t \geq S$, then $t_1 = t$. 
- Each instance for which $t_2 \geq S$ satisfies agreement and validity w.h.p.\(^8\) 
- Each correct node sends at most $O(n^2 \log n + Bn)$ bits within $R$ time. 
- The above guarantees hold in the presence of $f$ faulty nodes and for arbitrary initial states.

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\(^7\)Frequently, running time bounds are shown to hold in expectation only. To the best of our knowledge, in all these cases an additional factor of $O(\log n)$ is sufficient to obtain a bound that holds w.h.p.

\(^8\)This statement holds per instance; during superpolynomially large time intervals, some instances may fail.
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