Noncommutative Projective Geometry

J. T. Stafford

Abstract

This article describes recent applications of algebraic geometry to noncommutative algebra. These techniques have been particularly successful in describing graded algebras of small dimension.

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1. Introduction

In recent years a surprising number of significant insights and results in noncommutative algebra have been obtained by using the global techniques of projective algebraic geometry. This article will survey some of these results.

The classical approach to projective geometry, where one relates a commutative graded domain \( C \) to the associated variety \( X = \text{Proj} C \) of homogeneous, nonirrelevant prime ideals, does not generalize well to the noncommutative situation, simply because noncommutative algebras do not have enough ideals. However, there is a second approach, based on a classic theorem of Serre: If \( C \) is generated in degree one, then the categories \( \text{coh}(X) \) of coherent sheaves on \( X \) and \( \text{qgr} C \) of finitely generated graded \( C \)-modules modulo torsion are equivalent.

Surprisingly, noncommutative analogues of this idea work very well and have lead to a number of deep results. There are two strands to this approach. First, since \( X \) can be reconstructed from \( \text{coh}(X) \) \[\text{[21]}\] we will regard \( \text{coh}(X) \) rather than \( X \) as the variety since this is what generalizes. Thus, given a noncommutative graded \( k \)-algebra \( R = \bigoplus R_i \) generated in degree one we will consider \( \text{qgr} R \) as the corresponding “noncommutative variety” (the formal definitions will be given in a moment). In particular, we will regard \( \text{qgr} R \) as a noncommutative curve, respectively surface, if \( \dim_k R_i \) grows linearly, respectively quadratically. This analogy works well, since there are many situations in which one can pass back and forth

\*The author is supported in part by the National Science Foundation under grant DMS-9801148.
\†Department of Mathematics, The University of Michigan, Ann Arbor, MI 48109-1109, USA. E-mail: jts@umich.edu.
between $R$ and qgr $R$ and, moreover, substantial geometric techniques can be applied to study qgr $R$. A survey of this approach may be found in [25].

The second strand is more concrete. In order to use algebraic geometry to study noncommutative algebras we need to be able to create honest varieties from those algebras. This is frequently possible and such an approach will form the basis of this survey. Once again, the idea is simple: when $R$ is commutative, the points of Proj $R$ correspond to the graded factor modules $M = R/I = \bigoplus_{i \geq 0} M_i$ for which $\dim_k M_i = 1$ for all $i$. These modules are still defined when $R$ is noncommutative and are called point modules. In many circumstances the set of all such modules is parametrized by a commutative scheme and that scheme controls the structure of $R$.

This article surveys significant applications of this idea. Notably:

- If $R = \bigoplus R_i$ is a domain such that $\dim_k R_i$ grows linearly, then qgr $R \simeq \text{coh}(X)$ for a curve $X$ and $R$ can be reconstructed from data on $X$. Thus, noncommutative curves are commutative (see Section 4).
- The noncommutative analogues qgr $R$ of the projective plane can be classified. In this case, the point modules are parametrized by either $\mathbb{P}^2$ (in which case qgr $R \simeq \mathbb{P}^2$) or by a cubic curve $E \subset \mathbb{P}^2$, in which case data on $E$ determines $R$ (see Section 2).
- For strongly noetherian rings, as defined in Section 5, the point modules are always parametrized by a projective scheme. However there exist many noetherian algebras $R$ for which no such parametrization exists. This has interesting consequences for the classification of noncommutative surfaces.

We now make precise the definitions that will hold throughout this article. All rings will be algebras over a fixed, algebraically closed base field $k$ (although most of the results actually hold for arbitrary fields). A $k$-algebra $R$ is called connected graded (cg) if $R$ is a finitely generated $\mathbb{N}$-graded $k$-algebra $R = \bigoplus_{i \geq 0} R_i$ with $R_0 = k$. Note that this forces $\dim_k R_i < \infty$ for all $i$. Usually, we will assume that $R$ is generated in degree one in the sense that $R$ is generated by $R_1$ as a $k$-algebra. If $R = \bigoplus_{i \in \mathbb{N}} R_i$ is a right noetherian cg ring then define gr $R$ to be the category of finitely generated, $\mathbb{Z}$-graded right $R$-modules, with morphisms being graded homomorphisms of degree zero. Define the torsion subcategory, tors $R$, to be the full subcategory of gr $R$ generated by the finite dimensional modules and write qgr $R = \text{gr} R / \text{tors} R$. We write $\pi$ for the canonical morphism gr $R \to \text{qgr} R$ and set $\mathcal{R} = \pi(R)$.

One can—and often should—work more generally with all graded $R$-modules and all quasi-coherent sheaves of $\mathcal{O}_X$-modules, but two categories are enough.

In order to measure the growth of an algebra we use the following dimension function: For a cg ring $R = \bigoplus_{i \geq 0} R_i$, the Gelfand-Kirillov dimension of $R$ is defined to be $\text{GKdim} R = \inf \{ \alpha \in \mathbb{R} : \dim_k (\sum_{i = 0}^n R_i) \leq n^\alpha \text{ for all } n \gg 0 \}$. Basic facts about this dimension can be found in [17]. If $R$ is a commutative cg algebra then $\text{GKdim} R$ equals the Krull dimension of $R$ and hence equals $\dim \text{Proj} R + 1$. Thus a noncommutative curve, respectively surface, will more formally be defined as qgr $R$ for a cg algebra $R$ with $\text{GKdim} R = 2$, respectively 3.
2. Historical background

We begin with a historical introduction to the subject. It really started with the work of Artin and Schelter [2] who attempted to classify the noncommutative analogues $R$ of a polynomial ring in three variables (and therefore of $\mathbb{P}^2$). The first problem is one of definition. A “noncommutative polynomial ring” should obviously be a cg ring of finite global dimension, but this is too general, since it includes the free algebra. One can circumvent this problem by requiring that $\dim_k R_i$ grows polynomially, but this still does not exclude unpleasant rings like $k\{x, y\}/(xy)$ that has global dimension two but is neither noetherian nor a domain. The solution is to impose a Gorenstein condition and this leads to the following definition:

**Definition 1** A cg algebra $R$ is called AS-regular of dimension $d$ if $\text{gl dim } R = d$, $\text{GKdim } R < \infty$ and $R$ is AS-Gorenstein; that is, $\text{Ext}^i(k, R) = 0$ for $i \neq d$ but $\text{Ext}^d(k, R) = k$, up to a shift of degree.

One advantage with the Gorenstein hypothesis, for AS-regular rings of dimension 3, is that the projective resolution of $k$ is forced to be of the form

$$0 \rightarrow R \rightarrow R^n \rightarrow R^n \rightarrow R \rightarrow k \rightarrow 0$$

for some $n$ and, as Artin and Schelter show in [2], this gives strong information on the Hilbert series and hence the defining relations of $R$. In the process they constructed one class of algebras that they were unable to analyse:

**Example 2** The three-dimensional Sklyanin algebra is the algebra

$$\text{Skl}_3 = \text{Skl}_3(a, b, c) = k\{x_0, x_1, x_2\}/(ax_0x_{i+1} + bx_i + cx_{i+2}^2 : i \in \mathbb{Z}_3),$$

where $(a, b, c) \in \mathbb{P}^2 \setminus F$, for a (known) set $F$.

The original Sklyanin algebra $\text{Skl}_4$ is a 4-dimensional analogue of $\text{Skl}_3$ discovered in [23]. Independently of [2], Odesskii and Feigin [18] constructed analogues of $\text{Skl}_4$ in all dimensions and coined the name Sklyanin algebra. See [13] for applications of Sklyanin algebras to another version of noncommutative geometry.

In retrospect the reason $\text{Skl}_3$ is hard to analyse is because it depends upon an elliptic curve and so a more geometric approach is required. This approach came in [6] and depended upon the following simple idea. Assume that $R$ is a cg algebra that is generated in degree one. Define a point module to be a cyclic graded (right) $R$-module $M = \bigoplus_{i\geq 0} M_i$ such that $\dim_k M_i = 1$ for all $i \geq 0$. The notation is justified by the fact that, if $R$ were commutative, then such a point module $M$ would be isomorphic to $k[x]$ and hence equal to the homogeneous coordinate ring of a point in $\text{Proj } R$. Point modules are easy to analyse geometrically and this provides an avenue for using geometry in the study of cg rings.

We will illustrate this approach for $S = \text{Skl}_3$. Given a point module $M = \bigoplus M_i$ write $M_i = n_i k$ for some $n_i \in M_i$ and suppose that the module structure is defined by $m_{ij}x_i = \lambda_{ij}n_{i+1}$ for some $\lambda_{ij} \in k$. If $f = \sum f_{ij}x_i x_j$ is one of the relations for $S$, then necessarily $m_0 f = (\sum f_{ij} \lambda_{0i} \lambda_{1j}) m_2$, whence $\sum f_{ij} \lambda_{0i} \lambda_{1j} = 0$. 
This defines a subvariety $\Gamma \subseteq \mathbb{P}(S_1^e) \times \mathbb{P}(S_1^e) = \mathbb{P}^2 \times \mathbb{P}^2$ and clearly $\Gamma$ parametrizes the truncated point modules of length three: cyclic $R$-modules $M = M_0 \oplus M_1 \oplus M_2$ with $\dim M_i = 1$ for $0 \leq i \leq 2$. A simple computation (see [6] Section 3 or [26] Section 8]) shows that $\Gamma$ is actually the graph of an automorphism $\sigma$ of an elliptic curve $E \subset \mathbb{P}^2$. It follows easily that $\Gamma$ also parametrizes the point modules. As a morphism of point modules, $\sigma$ is nothing more than the shift functor $M = \bigoplus M_i \mapsto M_{i+1} = M_1 \oplus M_2 \oplus \cdots$.

The next question is how to use $E$ and $\sigma$ to understand $\text{Sk}_3$. Fortunately, one can create a noncommutative algebra from this data that is closely connected to $\text{Sk}_3$. This is the twisted homogeneous coordinate ring of $E$ and is defined as follows. Let $X$ be a $k$-scheme, with a line bundle $\mathcal{L}$ and automorphism $\sigma$. Set \( \mathcal{L}_n = \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}} \), where $\mathcal{L}^\tau = \tau^* \mathcal{L}$ denotes the pull-back of $\mathcal{L}$ along an automorphism $\tau$. Then the twisted homogeneous coordinate ring is defined to be the graded vector space $B = B(X, \mathcal{L}, \sigma) = k + \bigoplus_{n \geq 1} B_n$ where $B_n = H^0(X, \mathcal{L}_n)$. The multiplication on $B = B(Y, \mathcal{L}, \sigma)$ is defined by the natural map

\[
B_n \otimes_k B_m \cong H^0(X, \mathcal{L}_n) \otimes_k \sigma^n H^0(X, \mathcal{L}_m) \\
\cong H^0(X, \mathcal{L}_n) \otimes_k H^0(X, \mathcal{L}_m^\sigma) \xrightarrow{\phi} H^0(X, \mathcal{L}_{n+m}) = B_{n+m}.
\]

The ring $B$ has two significant properties. First, it has been constructed ensures that the natural isomorphism $S_1 \cong H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \cong B_1$ induces a ring homomorphism $\phi : S \rightarrow B$. With a little more work using the Riemann-Roch theorem one can even show that $B \cong S/gS$ for some $g \in S_3$. Secondly—and this will be explained in more detail in the next section—$\text{qgr} \ B \cong \text{coh}(E)$. The latter fact allows one to obtain a detailed understanding of the structure of $B$ and the former allows one to pull this information back to $S$.

To summarize, the point modules over the Sklyanin algebra $\text{Sk}_3$ are determined by an automorphism of an elliptic curve $E$ and the geometry of $E$ allows one to determine the structure of $\text{Sk}_3$. As is shown in [6] this technique works more generally and this leads to the following theorem.

\textbf{Theorem 3} [6, 20, 27] The AS-regular rings $R$ of dimension 3 are classified. They are all noetherian domains with the Hilbert series of a weighted polynomial ring $k[x, y, z]$; thus the $(x, y, z)$ can be given degrees $(a, b, c)$ other than $(1, 1, 1)$.

Moreover, $R$ always maps homomorphically onto a twisted homogeneous coordinate ring $B = B(X, \mathcal{L}, \sigma)$, for some scheme $X$. Thus $\text{coh}(X) \simeq \text{qgr} \ B \simeq \text{qgr} \ R$.

In this result, Artin, Tate and Van den Bergh [6] classified the algebras generated in degree one, while Stephenson [26, 27] did the general case.

There are strong arguments (see [11] or [25] Section 11]) for saying that the noncommutative analogues of the projective plane are precisely the categories $\text{qgr} \ R$, where $R$ is an AS-regular ring with the Hilbert series $1/(1-t^3)$ of the unweighted polynomial ring $k[x, y, z]$. So consider this class, which clearly includes the Sklyanin algebra. The second paragraph of the theorem can now be refined to say that either $X = \mathbb{P}^2$, in which case $\text{qgr} \ R \simeq \text{coh}(\mathbb{P}^2)$, or $X = E$ is a cubic curve in $\mathbb{P}^2$. Thus, the theorem can be interpreted as saying that noncommutative projective planes are either equal to $\mathbb{P}^2$ or contain a commutative curve $E$. 
3. Twisted homogeneous coordinate rings

The ideas from [6] outlined in the last section have had many other applications, but before we discuss them we need to analyse twisted homogeneous coordinate rings in more detail. The following exercise may give the reader a feel for the construction.

**Exercise 4** Perhaps the simplest algebra appearing in the theory of quantum groups is the quantum (affine) plane $k_q[x, y] = k[[x, y]]/(xy - qyx)$, for $q \in k^*$. Prove that $k_q[x, y] \cong B(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1), \sigma)$ where $\sigma$ is defined by $\sigma(a : b) = (a : qb)$, for $(a : b) \in \mathbb{P}^1$.

For the rest of the section, fix a $k$-scheme $X$ with an invertible sheaf $L$ and automorphism $\sigma$. When $\sigma = 1$, the homogeneous coordinate ring $B(X, L) = B(X, L, 1)$ is a standard construction and one has Serre’s fundamental theorem: If $L$ is ample then $\text{coh}(X) \cong \text{qgr}(B)$. As was hinted in the last section, this does generalize to the noncommutative case, provided one changes the definition of ampleness. Define $L$ to be $\sigma$-ample if, for all $F \in \text{coh}(X)$, one has $H^q(X, F \otimes L^n) = 0$ for all $q > 0$ and all $n \gg 0$. The naïve generalization of Serre’s Theorem then holds.

**Theorem 5** (Artin-Van den Bergh [7]) Let $X$ be a projective scheme with an automorphism $\sigma$ and let $L$ be a $\sigma$-ample invertible sheaf. Then $B = B(X, L, \sigma)$ is a right noetherian cg ring such that $\text{qgr}(B) \cong \text{coh}(X)$.

This begs the question of precisely which line bundles are $\sigma$-ample. A simple application of the Riemann-Roch Theorem shows that

$$\text{if } X \text{ is a curve, then any ample invertible sheaf is } \sigma\text{-ample,} \tag{3.1}$$

and the converse holds for irreducible curves. This explains why Theorem 5 could be applied to the factor of the Sklyanin algebra in the last section.

For higher dimensional varieties the situation is more subtle and is described by the following result, for which we need some notation. Let $X$ be a projective scheme and write $A_{\text{num}}^1(X)$ for the set of Cartier divisors of $X$ modulo numerical equivalence. Let $\sigma$ be an automorphism of $X$ and let $P_\sigma$ denote its induced action on $A_{\text{num}}^1(X)$. Since $A_{\text{num}}^1(X)$ is a finitely generated free abelian group, $P_\sigma$ may be represented by a matrix and $P_\sigma$ is called quasi-unipotent if all the eigenvalues of this matrix are roots of unity.

**Theorem 6** (Keeler [15]) If $\sigma$ be an automorphism of a projective scheme $X$ then:

1. $X$ has a $\sigma$-ample line bundle if and only if $P_\sigma$ is quasi-unipotent. If $P_\sigma$ is quasi-unipotent, then all ample line bundles are $\sigma$-ample.
2. In Theorem 5, $B$ is also left noetherian.

There are two comments that should be made about Theorem 5. First, it is standard that $\text{GKdim } B(X, L) = 1 + \dim X$, whenever $L$ is ample. However, it can happen that $\text{GKdim } B(X, L, \sigma) > 1 + \dim X$. Secondly, one can still construct $B(X, L, \sigma)$ when $L$ is ample but $P_\sigma$ is not quasi-unipotent, but the resulting algebra is rather unpleasant. Indeed, possibly after replacing $L$ by some $L^\otimes n$, $B(X, L, \sigma)$ will be a non-noetherian algebra of exponential growth. See [15] for the details.
4. Noncommutative curves and surfaces

As we have seen, twisted homogeneous coordinate rings are fundamental to the study of noncommutative projective planes. However, a more natural starting place would be to algebras of Gelfand-Kirillov dimension two since, as we suggested in the introduction, these should correspond to noncommutative curves. Their structure is particularly simple.

**Theorem 7** [4] Let $R$ be a cg domain of GK-dimension 2 generated in degree one. Then there exists an irreducible curve $Y$ with automorphism $\sigma$ and ample invertible sheaf $L$ such that $R$ embeds into the twisted homogeneous coordinate ring $B(Y, L, \sigma)$ with finite index. Equivalently, $R_n \cong H^0(Y, L \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma n-1})$ for $n \gg 0$.

By (3.1) we may apply Theorem 5 to obtain part (1) of the next result.

**Corollary 8** Let $R$ be as in Theorem 7. Then:

1. $R$ is a noetherian domain with $\text{qgr~}R \cong \text{coh}(Y)$. In particular, $\text{qgr~}R \cong \text{qgr~}C$ for the commutative ring $C = B(Y, L, \text{Id})$.
2. If $|\sigma| < \infty$ then $R$ is a finite module over its centre. If $|\sigma| = \infty$, then $R$ is a primitive ring with at most two height one prime ideals.

If $R$ is not generated in degree one, then the analogue of Theorem 7 is more subtle, since more complicated algebras appear. See [4] for the details. One should really make a further generalization by allowing $R$ to be prime rather than a domain and to allowing $k$ to be arbitrary (since this allows one to consider the projective analogues of classical orders over Dedekind domains). Theorem 7 and Corollary 8 do generalize appropriately but the results are more technical. The details can be found in [5].

Although these results are satisfying they are really only half of the story. As in the commutative case one would also like to define noncommutative curves abstractly and then show that they can indeed be described by graded rings of the appropriate form. Such a result appears in [19] but to state it we need a definition.

Let $\mathcal{C}$ be an Ext-finite abelian category of finite homological dimension with derived category of bounded complexes $D^b(\mathcal{C})$. Recall that a cohomological functor $H : D^b(\mathcal{C}) \to \text{mod}(k)$ is of finite type if, for $A \in D^b(\mathcal{C})$, only a finite number of the $H(A[n])$ are non-zero. The category $\mathcal{C}$ is saturated if every cohomological functor $H : D^b(\mathcal{C}) \to \text{mod}(k)$ of finite type is of the form $\text{Hom}(A, -)$ (that is, $H$ is representable). If $X$ is a smooth projective scheme, then $\text{coh}(X)$ is saturated [10], so it is not unreasonable to use this as part of the definition of a “noncommutative smooth curve.”

**Theorem 9** (Reiten-Van den Bergh [19, Theorem V.1.2]) Assume that $\mathcal{C}$ is a connected saturated hereditary noetherian category. Then $\mathcal{C}$ has one of the following forms:

1. $\text{mod}(\Lambda)$ where $\Lambda$ is an indecomposable finite dimensional hereditary algebra.
2. $\text{coh}(\mathcal{O})$ where $\mathcal{O}$ is a sheaf of hereditary $\mathcal{O}_X$-orders over a smooth connected projective curve $X$. 

It is easy to show that the abelian categories appearing in parts (1) and (2) of this theorem are of the form $qgr R$ for a graded ring $R$ with $\text{GKdim } R \leq 2$, and so this result can be regarded as a partial converse to Theorem 7. A discussion of the saturation condition for noncommutative algebras may be found in [12].

If one accepts that noncommutative projective curves and planes have been classified, as we have argued, then the natural next step is to attempt to classify all noncommutative surfaces and this has been a major focus of recent research. This program is discussed in detail in [25] Sections 8–13 and so here we will be very brief. For the sake of argument we will assume that an (irreducible) noncommutative surface is $qgr R$ for a noetherian cg domain $R$ with $\text{GKdim } R = 3$, although the precise definition is as yet unclear. For example, Artin [1] demands that $qgr R$ should also possess a dualizing complex in the sense of Yekutieli [30]. Nevertheless in attempting to classify surfaces it is natural to mimic the commutative proof:

(a) Classify noncommutative surfaces up to birational equivalence; equivalently classify the associated graded division rings of fractions for graded domains $R$ with $\text{GKdim } R = 3$. Artin [1, Conjecture 4.1] conjectures that these division rings are known.

(b) Prove a version of Zariski’s theorem that asserts that one can pass from any smooth surface to a birationally equivalent one by successive blowing up and down. Then find minimal models within each equivalence class.

Van den Bergh has created a noncommutative theory of blowing up and down [28, 29] and used this to answer part (b) in a number of special cases. A key fact in his approach is that (after minor modifications) each known example of a noncommutative surface $qgr R$ contains an embedded commutative curve $C$, just as $qgr(\text{Skl}_3) \ra \text{coh}(E) = E$ in Section 2. This is important since he needs to blow up points on that subcategory. In general, define a point in $qgr R$ to be $\pi(M)$ for a point module $M \in \text{gr } R$. Given such a point $p$, write $p = \pi(R/I) = R/I$. Mimicking the classical situation we would like to write

$$B = R \oplus I \oplus I^2 \oplus \cdots ,$$

and then define the blow-up of $qgr R$ to be the category $qgr B$ of finitely generated graded $B$-modules modulo those that are right bounded. However, there are two problems. A minor one is that $I$ needs to be twisted to take into account the shift functor on $qgr R$. The major one is that $I$ is only a one-sided ideal of $R$, and so there is no natural multiplication on $B$. To circumvent these problems, Van den Bergh [28] has to define $B$ in a more subtle category so that it is indeed an algebra. It is then quite hard to prove that $qgr B$ has the appropriate properties.

5. Hilbert schemes

Since point modules and twisted homogeneous coordinate rings have proved so useful, it is natural to ask how generally these techniques can be applied. In particular, one needs to understand when point modules, or other classes of modules, can be parametrized by a scheme. Indeed, even for point modules over surfaces the
The best positive result is due to Artin, Small and Zhang [3, 9], for which we need a definition. A $k$-algebra $R$ is called strongly noetherian if $R \otimes_k C$ is noetherian for all noetherian commutative $k$-algebras $C$.

**Theorem 10** (Artin-Zhang [9, Theorems E4.3 and E4.4]) Assume that $R$ is a strongly noetherian, cg algebra and fix $h(t) = \sum h_i t^i \in k[[t]]$. Let $C$ denote the set of cyclic $R$-modules $M = R/I$ with Hilbert series $h_M(t) = \sum \dim_k(M_i) t^i$ equal to $h(t)$. Then:

1. $C$ is naturally parametrized by a (commutative) projective scheme.
2. There exists an integer $d$ such that, if $M = R/I \in C$, then $I$ is generated in degrees $\leq d$ as a right ideal of $R$.

In particular, if $R$ is a strongly noetherian cg algebra generated in degree one, then the set of point modules is naturally parametrized by a projective scheme $\mathcal{P}$.

Although we have concentrated on point modules, more general classes of modules are also important. An example where line modules (modules $M$ with the Hilbert series of $k[[x, y]]$) are needed in a classification problem appears in [22].

How strong is the strongly noetherian hypothesis? Certainly most of the standard examples of noetherian cg algebras (including the Sklyanin algebras) are strongly noetherian (see [3, Section 4]) and so one might hope that this is always the case. But in fact, as Rogalski [20] has shown, cg noetherian algebras that are not strongly noetherian exist in profusion.

These examples are constructed as subrings of $B = B(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \sigma)$ for an appropriate automorphism $\sigma$. Given $\sigma \in \text{Aut}(\mathbb{P}^n)$, pick $c \in \mathbb{P}^n$ and set $\mathcal{C} = \{c_i = \sigma^{-i}(c) : i \in \mathbb{N}\}$. Then $\mathcal{C}$ is called critically dense if, for any infinite subset $D \subseteq \mathcal{C}$, the Zariski closure of $D$ equals $\mathbb{P}^n$. This is not a particularly stringent condition, since it holds for a generic set of $(\sigma, c) \in \text{Aut}(\mathbb{P}^n) \times \mathbb{P}^n$. Corresponding to $c$ one has the point module $M = B/VB$ for some codimension one subspace $V = V(c) \subseteq B_1$. Rogalski’s example is then simply $S(\sigma, c) = k\langle V \rangle \subset B$, and it has remarkable properties:

**Theorem 11** (Rogalski [20]) Keep the above notation. Assume that $\sigma \in \text{Aut}(\mathbb{P}^n)$ and $c \in \mathbb{P}^n$ for $n \geq 2$ are such that $\mathcal{C}$ is critically dense. Then:

1. $S = S(\sigma, c)$ is always noetherian but never strongly noetherian.
2. The point modules for $S$ are not naturally parametrized by a projective scheme.
3. $S$ satisfies the condition $\chi_1$ but not the condition $\chi_2$, as defined below. Moreover, $\text{qgr } S$ has finite cohomological dimension.
Some comments about the theorem are in order. First, the point modules for 
S = S(\sigma, c) are actually parametrized by an “infinite blowup of \mathbb{P}^n" in the sense 
that they are parametrized by \mathbb{P}^n except that for each p \in \mathcal{P} one has a whole family 
\mathcal{P}_p of point modules parametrized by \mathbb{P}^{n-1}. In contrast, the points in qgr S are 
actually parametrized by \mathbb{P}^n since, if M, N \in \mathcal{P}_p, then \pi(N) \cong \pi(M) in qgr S.

The conditions \chi_i in part (3) are defined as follows: A cg ring R satisfies \chi_n if, 
for each 0 \leq j \leq n and each M \in \text{gr} R, one has \dim_k \text{Ext}^1_R(k, M) < \infty. The 
significance of \chi_1 is that, by [5, Theorem 4.5], one can reconstruct S = S(\sigma, c) from 
qgr S and so the peculiar properties of S are reflected in qgr S. In particular, part 
(4) implies that S does not satisfy \chi_2. The significance of part (4) is that, for all 
the algebras R considered until now, Serre’s finiteness theorem holds in the sense 
that H^i(F) is finite dimensional for all F \in qgr R and all i.

Here is the simplest example of S(\sigma, c). Pick algebraically independent elements 
p, q \in k and define \sigma \in \text{Aut}(\mathbb{P}^2) by \sigma(a:b:c) = (pa:qb:c). If c = (1:1:1) \in \mathbb{P}^2 
then C is critically dense and an argument like that of Exercise 4 shows that

\[ B = k\langle x, y, z \rangle / (zx - pxz, zy - qyz, yx - pq^{-1}xy) \quad \text{and} \quad S(\sigma, c) = k\langle y - x, z - x \rangle. \]

This example was first considered by Jordan [14] who was able to parametrize the 
point modules for S(\sigma, c) but was unable to determine if the ring was noetherian.

Rogalski’s examples show that, even for surfaces, the picture is much more 
complicated than the discussion of the last section would suggest. Yet even these 
examples appear in a geometric framework; indeed they can be constructed as blow-
ups of \mathbb{P}^n if one uses the naïve approach of [4].

This works as follows. As before, assume that (\sigma, c) \in \text{Aut}(\mathbb{P}^n) \times \mathbb{P}^n for n \geq 2 is 
such that C is critically dense. In coh(\mathbb{P}^n) let \mathcal{L}_c denote the ideal sheaf corresponding 
to the point c. If L is a coherent module over \mathcal{O} = \mathcal{O}_{\mathbb{P}^n}, we form a bimodule \mathcal{L}_\sigma 
such that as a left module, \mathcal{L}_\sigma \cong L but the right action is twisted by \sigma: if s \in \mathcal{L}_\sigma(U) 
and a \in \mathcal{O}_{\mathbb{P}^n}(\sigma(U)), then sa \in \mathcal{L}_\sigma(U) is defined by the formula sa = a^c s. See [4] 
pp. 252-3 for a more formal discussion. Now set \mathcal{J} = \mathcal{L}_c \otimes \mathcal{O}(1)_{\sigma} \subseteq \mathcal{O}(1)_{\sigma} and let 
\mathcal{B} = \mathcal{B}(\sigma, c) = \mathcal{O} \oplus \mathcal{J} \oplus \mathcal{J}^2 \oplus \cdots, where \mathcal{J}^n is the image of \mathcal{J}^\otimes n in \mathcal{O}(1)_{\sigma}^\otimes \cong \mathcal{O}(n)_{\sigma}. 
This does not define a sheaf of rings in the usual sense since we are “playing a game 
of musical chairs with the open sets [7, p.252].” Nevertheless \mathcal{B} does have an natural 
graded algebra structure and so we can form qgr \mathcal{B} in the usual way. If \sigma = 1 then 
qgr \mathcal{B} is simply coh(X), where X is the blow-up of \mathbb{P}^n at c. In contrast, Keeler, 
Rogalski and the author have recently proved:

**Theorem 12** [14] Pick (\sigma, c) \in \text{Aut}(\mathbb{P}^n) \times \mathbb{P}^n for n \geq 2 such that C is critically 
dense. Then \mathcal{B} = \mathcal{B}(\sigma, c) is noetherian. Moreover qgr(\mathcal{B}) \simeq qgr S(\sigma, c).

Thus, qgr S(\sigma, c) is nothing more than the (noncommutative) blow-up of \mathbb{P}^n at 
a point! The differences between this blow-up and Van den Bergh’s are illustrative. 
Van den Bergh had to work hard to ensure that the analogue of the exceptional 
divisor really looks like a curve. Indeed much of his formalism is required for just
this reason. In contrast, in Theorem 12 the analogue of the exceptional divisor (which in this case equals $B/(I_{e-1})B$) is actually a point. This neatly explains the structure of the points in $qgr S(\sigma, c)$; they are indeed parametrized by $\mathbb{P}^n$ although the point corresponding to $c$ (and hence the shifts of this point, which are nothing more than the points corresponding to the $c_i$) are distinguished.

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