The Role of Baryon Number Conservation in Measurements of Fluctuations

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I discuss the role and impact of net-baryon number conservation in measurements of net proton fluctuations in heavy-ion collisions. I show that the magnitude of the fluctuations is entirely determined by the strength of two particle correlations. At LHC and top RHIC energy, this implies the fluctuations are proportional to the integral of the balance function (BF), $B^{pp}$ of protons and anti-protons, while in the context of the RHIC beam energy scan (BES), one must also account for correlations of “stopped” protons. The integral of $B^{pp}$ measured in a 4π detector depends on the relative cross-sections of processes yielding $pp$ and those balancing the proton baryon number via the production of other anti-baryons. The accepted integral of $B^{pp}$ further depends on the shape and width of the BF relative to the width of the acceptance. The magnitude of the measured second order cumulant of net proton fluctuations thus has much less to do with QCD susceptibilities than with the creation/transport of baryons and anti-baryons in heavy-ion collisions, and most particularly the impact of radial flow on the width of the BF. I thus advocate that net-proton fluctuations should be studied by means of differential BF measurements rather integral correlators.

Keywords: net-baryon fluctuations, balance functions, correlations, QGP, heavy-ion collisions

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I. INTRODUCTION

Lattice QCD calculations (LQCD) with physical quark masses suggest that at RHIC top energy and LHC energy, the matter produced in heavy-ion collisions consists of a state of matter known as Quark Gluon Plasma (QGP) \textsuperscript{1,2}. LQCD also indicates that for vanishing baryon chemical potential ($\mu_B$), the transition from the QGP to a hadron gas phase (HGP) is of crossover type \textsuperscript{3}, while at large baryon chemical potential, it should be of first order. This implies the existence of a critical point (CP). Theoretical considerations further suggest that within the vicinity of the CP, one should expect sizable changes in the matter’s correlation length and that divergent net-charge ($\Delta Q$), net-strangeness ($\Delta S$), or net-baryon ($\Delta B$) fluctuations should occur \textsuperscript{4}. Away from the CP, in the cross-over region, some trace of critical behavior might also remain \textsuperscript{3,5}. There is thus a strong interest in mapping the magnitude of $\Delta Q$, $\Delta S$, $\Delta B$ with $\mu_B$ and temperature ($T$). This can be accomplished, in principle, by measuring second, third, and fourth order cumulants of these quantities as a function of beam energy ($\sqrt{S_{NN}}$). However, a number of caveats must be considered. First, LQCD predicts the magnitude of $\Delta B$ fluctuations in a finite coordinate space volume, $V$, but, experimentally, in heavy-ion collisions, these are measured based on a specific volume, $\Omega$, in momentum space. It is ab initio unclear how charge transport (e.g., flow, diffusion, etc), within the QGP produced in heavy-ion collisions map $V$ onto $\Omega$ and how this mapping shall affect the fluctuations observed in momentum space \textsuperscript{6}. Second, $\Delta B$ fluctuations in $V$ are not globally constrained by net-baryon number conservation while those in $\Omega$ intrinsically are \textsuperscript{7}. Third, it is not obvious that a measurement of proton vs. anti-proton fluctuations is sufficient to make a statement about baryon number fluctuations. What is indeed the effect of the unobserved baryons, i.e., anti-neutron ($\bar{n}$), anti-lambda ($\bar{\Lambda}$), etc? A host of other questions may also be considered, including whether the produced system has time to thermalize globally and whether, consequently, it is meaningful to invoke the notion of susceptibility in this context.

In this paper, I focus the discussion on fluctuations of conserved charges, more specifically the net-baryon number $\Delta B$, and examine the impact of baryon number conservation on measurements of the second cumulant $\kappa_2(\Delta B)$. I also consider the effects of a partial measurement of baryon fluctuations based on fluctuations of the net proton number.

In the context of the Grand Canonical Ensemble (GCE), fluctuations of $\Delta B$ are related to the reduced susceptibility $\hat{\chi}_2^B$ according to \textsuperscript{7,8}

$$\hat{\chi}_2^B = \frac{1}{V T^3} \kappa_2(\Delta B),$$

where $V$ is the volume of the system, $T$ its temperature, and $\kappa_2(\Delta B)$, the second order cumulant of $\Delta B$. The second order cumulants amounts to the variance and is calculated according to

$$\kappa_2(\Delta B) = \langle (\Delta B)^2 \rangle - \langle \Delta B \rangle^2,$$

where $\langle \Delta B \rangle$ and $\langle (\Delta B)^2 \rangle$ are the first and second moments, measured over an ensemble of events, of the net-baryon number $\Delta B = N_B - N_{\bar{B}}$. The variables $N_B$ and $N_{\bar{B}}$ represent multiplicities of baryon and anti-baryons, respectively, within the volume $V$ in a particular instance of the system (collision). Averages are computed over all
been reported \[12\]. While second, third, and fourth or-
ergy scan (BES I) \[10, 11\]. Cross-cumulants have also
RhIC, in particular, in the context of the first beam en-
terprises, so-called volume fluctuations, are displayed in
-10, 11\], which demonstrates a limited role for the
or related to higher order suscepti-
reducible dependence. Higher cumulants are deemed of
higher power dependence and are of great interest because of their higher power dependence
on the correlation length \(\xi\). This dependence should be taken into account when interpreting cumulants.

Measurements of second, third, and fourth order cu-
mulants of \(\Delta Q\), \(\Delta S\), and \(\Delta B\) have been conducted at
RHIC, in particular, in the context of the first beam en-
ergy scan (BES I) \[11\]. Cross-cumulants have also
been reported \[12\]. While second, third, and fourth or-
der \(\Delta Q\) and \(\Delta S\) are observed to have either modest
or monotonic dependence on the beam energy, the third
and fourth cumulant of the net proton number exhibit
monotonic behaviors vs. \(\sqrt{s_{\text{NN}}}\), with what appears
to be a statistically significant minimum near
\(\sqrt{s_{\text{NN}}} = 20\) GeV. Interestingly, this energy is also the locus of a mini-
um in the magnitude of directed flow, \(v_1\), observed in
Au–Au collisions at \(\sqrt{s_{\text{NN}}} = 200\) GeV. The existence
of these two minima at the same energy has been inter-
preted as an indicator of the presence of the CP in
this vicinity \[14\]. However, the observed non-monotonic
behavior and minimum have received a variety of other
interpretations \[15\]. Indeed, several caveats may impact
the interpretation of the existing results, as well as those
of future experiments. Primary among these are concerns
associated with the role of baryon number conservation.

The total baryon number of an isolated system is a
conserved quantity. This implies that the net-baryon
number of all particles produced in a given \(A-A\) colli-
sion should add to the sum of the baryon numbers of
the incoming nuclei. However, fluctuations of the net-
baryon number, \(\Delta B\), shall be observed when measur-
ing baryon production in a fiducial acceptance limited
to central rapidities. This much is true. Furthermore,
it is generally assumed that the measured magnitude of
\(\kappa_n(\Delta B)\) shall inform us about the susceptibilities
\(\chi_n^B\). It is argued, in particular, that great care has to be
accepted to the choice of the width of the rapidity ac-
ceptance used in measurements of \(\kappa_n(\Delta B)\): too narrow
an acceptance should lead to Poisson fluctuations of \(\Delta B\)
while too wide an acceptance should greatly suppress the
fluctuations because the net baryon number of the entire
system must be conserved. Moreover, it is often stated
that for an acceptance of about one to two units of rapid-
ity, such as those of the STAR and ALICE experiments,
the effect of baryon number conservation should be neg-
ligible and only small corrections need to be applied to
interpret \(\kappa_n(\Delta B)\) measurements in terms of susceptibil-
ities. Unfortunately, these assertions are factually incor-
rect as I shall demonstrate in this paper: at LHC and
top RHIC energies, the non-trivial part of the cumulant
\(\kappa_2(\Delta B)\) is entirely determined by baryon number con-
servation and the width of the experimental acceptance,
while at lower energies of the RHIC Beam-Energy-Scan
(BES), one must account for fluctuations in the proton
yield associated with baryon stopping and collision ge-
ometry. The good news, however, is that local baryon
number conservation applies both in infinite static mat-
ter and within a system (heavy-ion collision) undergoing
fast longitudinal and radial expansion. The only impor-
tant consideration then is how radial and longitudinal ex-
pansion affect the fraction of conserved baryons focused
within the experimental acceptance, on average. While
such a fraction cannot be measured directly by means
of cumulants, it can be assessed and extrapolated, in
principle, from measurements of balance functions. It
is my goal, in this paper, to demonstrate that second cu-
mulants of the net-baryon number are intrinsically and
determined by baryon number conservation, ra-
dial flow, and the width of the acceptance. I further
show that while integral correlators, such as \(\kappa_2(\Delta B)\),
are sensitive to radial flow, they do not allow easy discrim-
ation between effects of radial flow and the width of
the acceptance in transverse momentum, \(p_T\), and pseudo-
drapidity, \(\eta\). However, differential correlation functions
in the form of balance function (BF) offer a much better
method to assess the interplay between finite acceptance,
radial flow, and baryon number conservation.

In order to demonstrate these assertions, I first need
to express the second order cumulant of net-baryon (pro-
ton) fluctuations, measured within a specific acceptance,
in terms of second order (pair) factorial cumulants. I will
then show that these are related to the \(\nu_{\text{dyn}}\) correlation
observable, which in turn, is proportional to the integral,
within the same acceptance, of the baryon balance func-
tion. I will show how the integral of the balance function
is determined by the hadro-chemistry of the collision sys-
tem and that the shape and width of the balance function
are largely determined by longitudinal and radial flow.

This paper is divided as follows. Section \[II\] defines mo-
moment, cumulant, factorial moment, and factorial cu-
mulant notations used in the remainder of the paper. The
Poisson limit of fluctuations and the relation between
\(\kappa_2\) and the \(\nu_{\text{dyn}}\) correlator are discussed in sec. \[III\].
The connection between \(\nu_{\text{dyn}}\) and the balance function, and
the role of baryon number conservation at LHC and top
RHIC energy are discussed in sec. \[IV\] while the impact of
baryon stopping and a net excess of baryons in the fidu-
cial volume of the measurement are addressed in sec. \[V\].
Conclusions are summarized in sec. \[VI\].
II. DEFINITIONS AND NOTATIONS

For simplicity, protons and anti-protons are assumed to be measured in the same fiducial momentum acceptance $\Omega$. Measured proton and anti-proton multiplicities, to be measured in the same fiducial momentum acceptance $\Omega$, are denoted $N_p$ and $N_{\bar{p}}$, respectively. The net-proton number is defined as $\Delta N_p = N_p - N_{\bar{p}}$.

Theoretically, the fluctuations may be described in terms of a joint probability $P(N_p, N_{\bar{p}}|\Omega, C)$ determined by the acceptance $\Omega$ and the centrality $C$ of the heavy-ion collisions of interest. Experimentally, fluctuations may be characterized in terms of moments of multiplicities calculated as event ensemble averages denoted $\langle \cdot \rangle$. First and second moments of the proton and anti-proton multiplicities are defined according to

$$m_1^\alpha = \langle N_\alpha \rangle = \sum_{i=0}^{\infty} N_\alpha P(N_p, N_{\bar{p}}|\Omega, C),$$

$$m_2^{\alpha,\beta} = \langle N_\alpha N_\beta \rangle = \sum_{i=0}^{\infty} N_\alpha N_\beta P(N_p, N_{\bar{p}}|\Omega, C),$$

where $N_\alpha$ and $N_\beta$ stand for either of $N_p$ or $N_{\bar{p}}$. Cumulants of multiplicities $N_\alpha$ and $N_\beta$ are written

$$\kappa_1^\alpha = m_1^\alpha,$$
$$\kappa_2^{\alpha,\beta} = m_2^{\alpha,\beta} - m_1^\alpha m_1^\beta.$$  

The cumulants $\kappa_2^{\alpha,\beta}$, with $\beta \neq \alpha$, correspond to the variance of $N_\alpha$ and the covariance of $N_\alpha$ and $N_\beta$, respectively.

Experimentally, particle losses associated with the detection and event reconstruction modify these moments and cumulants. Corrections for such losses are most straightforward when carried out for single particles and pairs of particles. It is thus convenient to introduce factorial moments of the multiplicities $N_\alpha$ and $N_\beta$ as

$$f_1^\alpha = \langle N_\alpha \rangle = \tilde{m}_1^\alpha,$$
$$f_2^{\alpha,\beta} = \langle N_\alpha N_\beta - \delta_{\alpha,\beta} N_\alpha \rangle = m_2^{\alpha,\beta} - \delta_{\alpha,\beta} m_1^\alpha.$$  

Given factorial moments of measured multiplicities $n_\alpha$ and $n_\beta$, corrected factorial moments are obtained as

$$f_1^\alpha = \tilde{f}_1^\alpha / \varepsilon_\alpha,$$
$$f_2^{\alpha,\beta} = \tilde{f}_2^{\alpha,\beta} / (\varepsilon_\alpha \varepsilon_\beta),$$

where $\tilde{f}_1^\alpha$ and $\tilde{f}_2^{\alpha,\beta}$ represent raw (or uncorrected) factorial moments, while $\varepsilon_\alpha$ and $\varepsilon_\beta$ are detection efficiencies for particle species $\alpha$ and $\beta$, respectively. Note that best experimental precision may require one accounts for dependences of these quantities on the transverse momentum, the azimuth angle, and the pseudorapidity of the particles [16–17]. By construction, these factorial moments are determined by the single and pair densities of produced particles according to

$$f_1^\alpha = \int_\Omega \rho_1^\alpha(\vec{p}) d^3 p,$$
$$f_2^{\alpha,\beta} = \int_\Omega \rho_2^{\alpha,\beta}(\vec{p}_1, \vec{p}_2) d^3 p_1 d^3 p_2,$$

where $\rho_1^\alpha(\vec{p})$ is the single particle density of particle species $\alpha$, and $\rho_2^{\alpha,\beta}(\vec{p}_1, \vec{p}_2)$ is the pair-density of particle species $\alpha$ and $\beta$.

Factorial moments (corrected for efficiency losses) are combined to obtain factorial cumulants according to

$$F_1^\alpha = f_1^\alpha = \kappa_1^\alpha = m_1^\alpha,$$
$$F_2^{\alpha,\beta} = f_2^{\alpha,\beta} - f_1^\alpha f_1^\beta,$$
$$= m_2^{\alpha,\beta} - \delta_{\alpha,\beta} m_1^\alpha - m_1^\alpha m_1^\beta.$$  

Factorial cumulants $F_2^{\alpha,\beta}$ are, by construction, true measures of pair correlations: they vanish identically in the absence of particle correlations and take finite values, either negative or positive, in the presence of such correlations. However, null $F_2^{\alpha,\beta}$ values are not a sufficient condition to conclude measured particles are uncorrelated. Using the above definitions of first and second order factorial cumulants, one verifies second order cumulants may be written

$$\kappa_2^{\alpha,\beta} = \delta_{\alpha,\beta} F_1^\alpha + F_2^{\alpha,\beta}.$$  

It is convenient to introduce normalized factorial cumulants defined according to

$$R_2^{\alpha,\beta} = \frac{F_2^{\alpha,\beta}}{F_1^\alpha F_1^\beta} - 1 = \frac{R_2^{\alpha,\beta}}{F_1^\alpha F_1^\beta},$$

as well as the following linear combination of normalized two-cumulants:

$$\nu_{\text{dyn}}^{\alpha,\beta} = R_2^{\alpha,\beta} + R_2^{\beta,\alpha} - 2R_2^{\alpha,\beta},$$

where $\alpha \neq \beta$ represent two distinct types of particles. The correlator $\nu_{\text{dyn}}$ was originally introduced to search for the suppression of net-charge fluctuations in heavy-ion collisions [18–22]. It is of practical interest because it is experimentally robust, impervious to statistical fluctuations, and singles out dynamical fluctuations involved in particle production [20]. Its use has since been extended to study fluctuations of the relative yields of several types of particle species at RHIC and LHC energies [23–20].

III. MOMENTS OF NET PROTON DISTRIBUTION AND SKELLAM LIMIT

The net proton number is defined as $\Delta N_p = N_p - N_{\bar{p}}$. One straightforwardly verifies that its first and second cumulants are

$$\kappa_1(\Delta N_p) = \kappa_1^p - \kappa_1^{\bar{p}},$$
$$\kappa_2(\Delta N_p) = \kappa_2^{p,p} + \kappa_2^{\bar{p},\bar{p}} - 2\kappa_2^{p,\bar{p}},$$

where the first and second cumulants of proton and anti-proton multiplicities, denoted by the indices $p$ and $\bar{p}$, respectively, are defined according to Eqs. (6–7). Using Eq. (16), these may alternatively be written

$$\kappa_1(\Delta N_p) = F_1^p - F_1^{\bar{p}},$$
$$\kappa_2(\Delta N_p) = F_1^p + F_1^{\bar{p}} + F_2^{p,p} + F_2^{\bar{p},\bar{p}} - 2F_2^{p,\bar{p}}.$$
One finds that the second cumulant of the net-proton number involves two parts, the first being determined by the average multiplicities of protons and anti-protons and a more interesting part driven by two-particle correlations.

As stated above, in the absence of two-particle or higher order particle correlations, the factorial moments $F_{alpha}^{2,\beta}$ vanish. The Poisson limit of the second order cumulant, often labelled Skellam, is thus simply

$$\kappa_2^{\text{Skellam}}(\Delta N_p) = F_1^p + F_1^\bar{p}. \quad (23)$$

It is convenient to consider the ratio, $r_\Delta$, of a measured cumulant $\kappa_2(\Delta N_p)$ and its Skellam limit. Using Eqs. \ref{eq:22} \ref{eq:23}, one gets

$$r_\Delta = \frac{\kappa_2(\Delta N_p)}{\kappa_2^{\text{Skellam}}(\Delta N_p)} = 1 + \frac{F_2^{p,p} + F_2^{\bar{p},\bar{p}} - 2F_2^{p,\bar{p}}}{F_1^p + F_1^\bar{p}}. \quad (24)$$

This may also be written as

$$r_\Delta = 1 + \frac{(F_1^p)^2 R_2^{p,p} + (F_1^{\bar{p}})^2 R_2^{\bar{p},\bar{p}} - 2F_1^p F_1^{\bar{p}} R_2^{p,\bar{p}}}{F_1^p + F_1^{\bar{p}}}. \quad (25)$$

where I inserted normalized factorial cumulants defined according to Eq. \ref{eq:17}.

**IV. LHC AND TOP RHIC ENERGY**

At LHC and top RHIC energy, one has $\langle N_p \rangle \approx \langle N_\bar{p} \rangle$. The ratio $r_\Delta$ is thus approximately

$$r_\Delta = 1 + \frac{F_1^p}{2} \left[ R_2^{p,p} + R_2^{\bar{p},\bar{p}} - 2R_2^{p,\bar{p}} \right], \quad (26)$$

$$= 1 + \frac{1}{4} \langle N_T \rangle \nu_{\text{dyn}}^{p,\bar{p}}, \quad (27)$$

where $\langle N_T \rangle = \langle N_p \rangle + \langle N_\bar{p} \rangle$ is formally defined as

$$\langle N_T \rangle = \int_{\Omega} \rho_1^p(p)d^3p + \int_{\Omega} \rho_1^{\bar{p}}(\bar{p})d^3\bar{p}. \quad (28)$$

But given the densities $\rho_1^p$ and $\rho_1^{\bar{p}}$ are approximately constant at central rapidities, one can write $\langle N_T \rangle = dN_T/d\eta \times \Delta \eta$, where $\Delta \eta$ represents the longitudinal width of the experimental acceptance. The ratio $r_\Delta$ may thus be written as

$$r_\Delta = 1 + \frac{1}{4} \Delta \eta \frac{dN_T}{d\eta} \nu_{\text{dyn}}^{p,\bar{p}}. \quad (29)$$

As I discuss below, net-baryon number conservation implies that $\nu_{\text{dyn}}^{p,\bar{p}}$ is negative with an absolute magnitude that depends on the width $\Delta \eta$ of the fiducial acceptance. Neglecting this dependence, one would expect the ratio $r_\Delta$ to have a trivial, approximately linear, dependence on the width of the acceptance \cite{22}:

$$r_\Delta \approx 1 - a \Delta \eta, \quad (30)$$

where $a = \frac{1}{4} dN_T/d\eta |\nu_{\text{dyn}}^{p,\bar{p}}|$. But given that $\nu_{\text{dyn}}^{p,\bar{p}}$ should depend on $\Delta \eta$, the above is likely to be a poor approximation of the actual dependence of $r_\Delta$ on $\Delta \eta$. Given $r_\Delta \rightarrow 1$ in the limit $\Delta \eta \rightarrow 0$, one might be tempted to conclude that fluctuations of the net-proton number are Poissonian (Skellam) in that limit. That is actually incorrect. The true measure of correlations is given by $\nu_{\text{dyn}}^{p,\bar{p}}$ and it can be non-zero (and in general is) even in the limit $\Delta \eta \rightarrow 0$.

It is next of interest to assess how the value of $\nu_{\text{dyn}}^{p,\bar{p}}$ may depend on the acceptance of the measurement. This is readily achieved with the introduction of a proton balance function. Balance functions (BF) are differential correlations functions that contrast the strength of like-sign (in the context of this paper, same baryon number) and unlike-sign (opposite charge or opposite baryon number) particles correlations \cite{28, 29}. For $p$ and $\bar{p}$, a balance function can be written

$$B^{p,\bar{p}}(\Delta y) = \frac{1}{2} \left\{ \frac{\rho_1^p R_2^{p,p}(\Delta y) - \rho_1^{\bar{p}} R_2^{p,\bar{p}}(\Delta y)}{\rho_1^p} - \frac{\rho_2^p R_2^{\bar{p},\bar{p}}(\Delta y) - \rho_2^{\bar{p}} R_2^{p,\bar{p}}(\Delta y)}{\rho_1^p} \right\} . \quad (31)$$

where $\rho_1^p$ and $\rho_1^{\bar{p}}$ are single particle densities of protons and anti-protons at central rapidity, respectively, and $\rho_2^p, \rho_2^{\bar{p}}(\Delta y)$ are pair densities for species $\alpha$ and $\beta$. In this context, particle $\alpha$ is considered as the “trigger” or “given” particle, while particle $\beta$ is regarded as the “associate”. The ratios $\rho_2^{\alpha,\beta}(\Delta y)/\rho_1^\alpha$ are conditional densities expressing the number of particles of species $\beta$ at a separation $\Delta y$ from a particle of species $\alpha$. A BF can also be written in terms of differential normalized cumulants according to

$$B^{p,\bar{p}}(\Delta y) = \frac{1}{2} \left\{ \frac{\rho_1^p R_2^{p,p}(\Delta y) - \rho_1^{\bar{p}} R_2^{p,\bar{p}}(\Delta y)}{\rho_1^p} + \frac{\rho_1^{\bar{p}} R_2^{\bar{p},\bar{p}}(\Delta y) - \rho_1^p R_2^{p,\bar{p}}(\Delta y)}{\rho_1^p} \right\} . \quad (32)$$

where

$$R_2^{\alpha,\beta}(\Delta y) = \frac{\rho_2^{\alpha,\beta}(\Delta y)}{\rho_1^\alpha \otimes \rho_1^\beta} - 1 = \frac{F_2^{\alpha,\beta}(\Delta y)}{F_1^\alpha \otimes F_1^\beta}(\Delta y), \quad (33)$$

in which $F_2^{\alpha,\beta}(\Delta y) = \rho_2^{\alpha,\beta}(\Delta y) - \rho_1^\alpha \otimes \rho_1^\beta$ are differential factorial cumulants with an explicit dependence on the pair separation. In A–A collisions and in the limit $\langle N_p \rangle = \langle N_\bar{p} \rangle$, one has $\rho_1^p = \rho_1^{\bar{p}}$ and $R_2^{p,p}(\Delta y) = R_2^{p,\bar{p}}(\Delta y)$. The BF simplifies to

$$B^{p,\bar{p}}(\Delta y) = -\frac{\Delta \eta dN_T}{4 d\eta} \left\{ R_2^{p,p}(\Delta y) + R_2^{\bar{p},\bar{p}}(\Delta y) - 2R_2^{p,\bar{p}}(\Delta y) \right\}. \quad (34)$$

Integration of $F_2^{\alpha,\beta}(\Delta y)$ across the $\Delta y$ acceptance yields the integral factorial cumulant $F_2^{\alpha,\beta}$ defined by Eq. \ref{eq:15}. The integral of the BF can thus be written

$$I_{p,\bar{p}}(\Omega) = -\frac{1}{4} \langle N_T \rangle \nu_{\text{dyn}}^{p,\bar{p}}(\Omega). \quad (35)$$
We conclude that at high-energy, i.e., in the limit \( N_p = N_{\bar{p}} \), the deviation of the Skellam ratio from unity is identically equal to the integral of the BF. Let us thus next examine what determines the magnitude of this integral.

Neglecting the effect of incoming and stopped protons from incoming projectiles, the shape and amplitude of the BF reflect how (where) baryon conserving pairs balancing of protons and anti-protons are created and transported in the aftermath of A–A collisions. If only an anti-proton \((Q = -1, B = -1)\) could balance the production of a proton \((Q = 1, B = 1)\), then, by construction, the balance function would integrate to unity over the full phase space of particle production.

However, baryon number conservation can be satisfied by the production of other anti-baryons. It is thus of interest to consider the balance function \( B \) by the production of other anti-baryons. It is thus of phase space of particle production.

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However, baryon number conservation can be satisfied by the production of other anti-baryons. It is thus of interest to consider the balance function \( B^{p,\bar{B}}(\Delta y) \) corresponding to the anti-baryon number density at \( \Delta y = y_{\bar{B}} - y_p \), given a proton is observed at \( y_p \). An anti-baryon of some kind must accompany the production of a proton. This balance function thus integrates to unity over the full particle production phase space:

\[
I_{p,\bar{B}}^{4\pi} = 1,
\]  

where \( 4\pi \) denotes that the integral is extending over all rapidities and transverse momenta. Indeed, besides anti-protons, the balancing of the baryon number of a proton \((B = 1)\) may be accomplished by the production of an anti-neutron \((\bar{n})\), an anti-lambda \((\bar{\Lambda})\), an anti-sigma \((\bar{\Sigma}^-)\), or any other heavier anti-baryons. It is thus useful to consider corresponding balance functions \( B^{p,\bar{n}}(\Delta y) \), \( B^{p,\bar{\Lambda}}(\Delta y) \), etc, and their respective integrals \( I_{p,\bar{n}}(\Omega) \), \( I_{p,\bar{\Lambda}}(\Omega) \), etc.

The production of pairs \( p\bar{p}, p\bar{n}, p\bar{\Lambda}, p\bar{\Sigma}^- \), etc, have probabilities determined by their relative cross-sections. These, in turn, must be equal to integrals of their respective balance functions. One can then write

\[
1 \equiv I_{p,\bar{B}}^{4\pi} = I_{p,\bar{p}}^{4\pi} + I_{p,\bar{n}}^{4\pi} + I_{p,\bar{\Lambda}}^{4\pi} + \cdots = \sum_{\beta} I_{p,\bar{\beta}}^{4\pi},
\]

where, once again, \( 4\pi \) denotes that the integrals are extending over all rapidities and transverse momenta, and \( \sum_{\beta} \) represents a sum over all anti-baryons \((B = -1)\).

In this context, the functions \( I_{p,\bar{\beta}}^{4\pi} \) can be considered as probabilities of the respective baryon number balancing processes determined by their cross-sections. The \( p\bar{p} \) balance function integral is one of many components of the full \( p\bar{B} \) BF. Its value is thus smaller than unity.

Experimentally, however, particles are measured within limited (pseudo)rapidity and transverse momentum ranges. The probabilities \( I_{p,\bar{\beta}}^{4\pi} \) are thus not directly measurable. Figures (a,b) presents schematic examples of Gaussian balance functions \( B^{p,\bar{B}}(\Delta y) \) of width \( \sigma \) and corresponding values of the integral \( I_{p,\bar{\beta}}^{4\pi} \) as a function of the width \( \sigma \) for a nominal acceptance of \(-1 < Y < 1\). Clearly, the integral \( I_{p,\bar{\beta}}^{4\pi} \) depends on the width \( \sigma \) relative to the measurement acceptance \(-1 < Y < 1\). This is further illustrated in Fig. (c) which displays the integral of a Gaussian of width \( \sigma = 0.5 \) as a function of the width \( \Delta Y \) of the experimental acceptance. Given the baryon
number balancing of the proton may be achieved with several distinct anti-baryon species, one must then consider the evolution of integrals $I_{p,\bar{\beta}}$ for all species $\bar{\beta}$ as a function of the measurement acceptance $\Omega$, as illustrated schematically in Fig. 2.

$$\text{FIG. 2: Schematic dependence of the integral of balance functions } B^{p,\bar{\beta}}(\Delta p) \text{ vs. the width of the experimental acceptance } \Omega. \text{ The colored bands schematically illustrate contributions from distinct baryon number balancing anti-baryons.}$$

The integral of the balance function is proportional to $\nu_{\text{dyn}}^{p,\bar{\beta}}$ which, as we saw in Eq. (36), is also proportional to $1 - r_{\Delta}$. The magnitude of $\kappa_2(\Delta p)$, at high energy, is thus entirely determined by the integral of the balance function across the fiducial acceptance. The integral of the balance function, in turn, is determined by baryon number conservation and the chemistry of the collision, i.e., what fraction of protons are accompanied by an anti-proton. If protons were balanced exclusively by anti-proton, the integral of the balance function over the entire phase space would yield unity. With finite ranges in $p_T$ and $Y$, the integral is determined by the width of these ranges. The larger they are, the closer the integral gets to saturation (unity if only anti-protons balance protons). The measured values of $\kappa_2(\Delta p)$ at LHC and top RHIC energy are thus determined ab-initio by baryon number conservation and the width of the balance function relative to that of the acceptance.

It is well established that the shape and width of the balance function of charge particles exhibit a significant narrowing with increasing collision centrality [25, 30, 31]. This narrowing is understood to result largely from radial flow and was successfully modeled with the blast wave model: the more central collisions are, the faster is the radial flow [32]. The value of $1 - r_{\Delta}$ is thus determined in large part by the magnitude of radial flow and the width of the acceptance and much less by the full coverage integral $I_{p,\bar{\beta}}^{k\nu}$.

Nominally, if effects of radial flow were invariant with collision centrality, the multiplicity $\langle N_T \rangle_{AA}$ measured in A–A collisions would scale in proportion to its value in pp collisions $\langle N_T \rangle_{pp}$ according

$$\langle N_T \rangle_{AA} = \langle n_s \rangle \langle N_T \rangle_{pp}, \quad (39)$$

where $\langle n_s \rangle$ is the effective number of sources involved, on average, in a given A–A centrality range. In contrast, one also expects that, in the absence of re-scattering of secondaries, that $\nu_{\text{dyn}}^{p,\bar{\beta}(AA)}$ measured in A–A should scale as

$$\nu_{\text{dyn}}^{p,\bar{\beta}(AA)} = \frac{1}{\langle n_s \rangle} \nu_{\text{dyn}}^{p,\bar{\beta}(pp)}, \quad (40)$$

relative to the value $\nu_{\text{dyn}}^{p,\bar{\beta}(pp)}$ measured in pp collisions [20]. Such scaling is in fact essentially observed in Au–Au and Pb–Pb collisions [21, 22, 24]. In this context, the ratio $r_{\Delta}$ would then be invariant with A–A collision centrality. But the radial flow velocity is known to increase in more central collisions thereby leading to a narrowing of the balance function [30]. This consequently leads to an increase of the integral $I_{p,\bar{\beta}}$ within the experimental acceptance. The centrality dependence of $r_{\Delta}$ shall then be driven primarily by the evolution of radial flow with collision centrality and it might have essentially nothing to do with the chemistry of the system and its susceptibility $\hat{\chi}_B^2$.

The width of the net-charge balance function is also observed to increase monotonically with decreasing beam energy ($\sqrt{s_{NN}}$) [33]. This can be in part understood as a result of slower radial flow profile with decreasing beam energy. Should the $p\bar{p}$ balance function behave in a similar fashion, one would expect the integral $I_{p,\bar{\beta}}$ to reduce monotonically with decreasing beam energy because the fraction of the BF within the acceptance shrinks as its width increases. Once again, one expects the magnitude of $\kappa_2(\Delta p)$ to change with beam energy for reasons completely independent of the susceptibility $\hat{\chi}_B^2$.

However, the ratio $\langle N_{\bar{p}} \rangle / \langle N_p \rangle$ is also known to fall rapidly with decreasing beam energy. The $\langle N_{\bar{p}} \rangle = \langle N_p \rangle$ hypothesis used to derive Eqs. (29,36) is thus indeed invalid at the low energy end of the BES. One must thus examine the effect of baryon stopping on the fluctuations.

V. NET PROTONS FLUCTUATIONS IN THE PRESENCE OF NUCLEAR STOPPING

In order to model the effect of baryon stopping, I will assume, as in [34], that one can partition the measured protons into two subsets: the first, denoted $i$, corresponding to “stopped” protons, and the second, denoted $p$, corresponding to protons produced by $p\bar{p}$ pair creation. All anti-protons are assumed produced by pair production and I will neglect, for simplicity, the impact of annihilation.

I thus consider Eq. (24) with the following substitutions for the first and second order factorial cumulants of
protons and anti-protons:

\[ F_i^p \rightarrow F_i^+ + F_i^- = \langle N_i \rangle + \langle N_p \rangle \]
\[ F_i^\bar{p} \rightarrow F_i^{-} = \langle N_\bar{p} \rangle \]
\[ F_{2}^{p,p} \rightarrow F_{2}^{p,i} + F_{2}^{i,p} + F_{2}^{p,i} + F_{2}^{p,p} \]
\[ F_{2}^{p,\bar{p}} \rightarrow F_{2}^{i,\bar{p}} + F_{2}^{\bar{p},i} \]
\[ F_{2}^{\bar{p},\bar{p}} \rightarrow F_{2}^{\bar{p},\bar{p}}. \]

In symmetric A–A collisions, one must have \( F_{2}^{i,p} = F_{2}^{p,i} \), \( F_{2}^{i,\bar{p}} = F_{2}^{\bar{p},i} \), and \( F_{2}^{p,\bar{p}} = F_{2}^{\bar{p},p} \). Neglecting annihilation, one can also write \( F_i^p = F_1^p \), \( F_2^{p,p} = F_2^{p,\bar{p}} \), and \( F_2^{\bar{p},p} = F_2^{\bar{p},\bar{p}} \). Introducing \( \langle N_T \rangle = \langle N_i \rangle + 2 \langle N_p \rangle \) and \( \xi = \langle N_i \rangle / \langle N_T \rangle = F_1^p / (F_1^p + 2F_2^\bar{p}) \), one gets

\[ r_\Delta = 1 + \frac{F_{2}^{i,i} + 2F_{2}^{p,p} + F_{2}^{\bar{p},p} + F_{2}^{i,\bar{p}} - 2F_{2}^{p,\bar{p}} - 2F_{2}^{i,p} - 2F_{2}^{p,p}}{F_{1}^{p} + F_{2}^{\bar{p}}}, \]

\[ = 1 + 2\xi^2 \langle N_T \rangle R_{2}^{i,i} + \frac{1}{4} (1 - \xi)^2 \langle N_T \rangle \nu_{\text{dyn}}^{p,\bar{p}}. \]

The second term, proportional to \( R_{2}^{i,i} \), is a measure of the correlation strength of stopped protons, while the third term, proportional to \( \nu_{\text{dyn}}^{p,\bar{p}} \) corresponds to the pair creation component found in the high-energy limit, Eq. (29). Experimentally, it has been observed that nucleons from the projectile and target lose, on average, approximately two units of rapidity in nuclear collisions. At LHC and top RHIC energy, this leads to a vanishing net-baryon density in the central rapidity region but for decreasing \( \sqrt{s_{\text{NN}}} \), and particularly at the low end of RHIC the beam energy scan, this yields a large net proton excess at central rapidity. Given the production of \( pp \) pairs is a logarithmic function of \( \sqrt{s_{\text{NN}}} \), one expects the term proportional to \( R_{2}^{i,i} \) should largely dominate at the low end of the BES range while the term proportional to \( \nu_{\text{dyn}}^{p,\bar{p}} \), driven by baryon number conservation, should dominate at LHC and top RHIC energy. Equation (42) thus tells us that the beam energy evolution of \( r_\Delta - 1 \) should be determined by the interplay of baryon stopping and net-baryon conservation, the former and the latter dominating at low and high \( \sqrt{s_{\text{NN}}} \), respectively. One may thus anticipate that \( r_\Delta - 1 \) might exhibit a non-monotonic behavior with \( \sqrt{s_{\text{NN}}} \). Such non monotonic behavior, however, has little to do with the properties of nuclear matter near equilibrium and more to do with dynamic considerations including nuclear stopping power and radial flow resulting from large inside out pressure gradients.

VI. SUMMARY

I showed there is straightforward connection between the fluctuations of net-baryon number measured at central rapidities in A–A collisions in terms of second order cumulants of the net-baryon number and the strength of two-particle correlations factorial cumulants. I further showed that in the high-energy limit, corresponding to a vanishing net-baryon number, fluctuations are entirely determined by the strength and width of the \( pp \) balance function relative to the width of the acceptance. By contrast, at low energy, the fluctuations of the net-baryon number are more likely dominated by proton-proton correlations resulting from nuclear stopping. Overall, one can expect the fluctuations to display a smooth evolution with \( \sqrt{s_{\text{NN}}} \) between these two extremes but nowhere can one expect the magnitude of the fluctuations to be trivially sensitive to the nuclear matter baryon susceptibility \( \chi_B^{\rho} \).

I here focused the discussion on second order cumulants of the net-baryon number but it is clear that the same line of argument can be extended to higher cumulants. Measurements of fluctuations by STAR at RHIC have used the magnitude of the second order cumulant of the net-baryon number as a reference to factor out the ill defined notion of volume involved in relations between cumulants and susceptibilities. This would make sense if the susceptibilities determined the magnitude of the cumulants. But, as I have shown, the magnitude of \( \kappa_2(\Delta p) \) is in fact determined largely by the width of the acceptance of the measurement relative to the width of the balance function at high-energy and by proton-proton correlations associated with nuclear stopping at low energy. The use of \( \kappa_2(\Delta p) \) thus does not provide a sound basis to cancel out volume effects and normalize the magnitude of higher cumulants. All is not lost, however. It might be possible with measurements of momentum dependent balance functions to quantitatively assess the role of both baryon number conservation and nuclear stopping, and henceforth obtain sensitivity to QCD matter susceptibilities.

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