1. Introduction.

The application of the Legendre transformation to a hyperregular Lagrangian system results in a Hamiltonian vector field generated by a Hamiltonian defined on the phase space of the mechanical system. The Legendre transformation in its usual interpretation cannot be applied to homogeneous Lagrangians found in relativistic mechanics. The dynamics of relativistic systems must be formulated in terms of implicit differential equations in the phase space and not in terms of Hamiltonian vector fields. The constrained Hamiltonian systems introduced by Dirac [1] are not general enough to cover some important cases. We formulate a geometric framework which permits Lagrangian and Hamiltonian descriptions of the dynamics of a wide class of mechanical systems. This framework extends the applicability of earlier definitions of the Legendre transformation [2], [3]. Lagrangians and Hamiltonians are presented as families of functions. The Legendre transformation and the inverse Legendre transformation are described as transitions between these families. Two simple examples are given.

2. Iterated tangent and cotangent functors.

For each differential manifold \( Q \) the fibrations

\[
\begin{array}{ccc}
TQ & \tau_Q \downarrow & T^*Q \\
\downarrow & \pi_Q & \\
Q & & Q
\end{array}
\]  

(1)

are a dual pair of vector fibrations. We have operations

\[
+ : TQ \times_Q T \to TQ,
\]

(2)

\[
\cdot : \mathbb{R} \times TQ \to TQ,
\]

(3)

\[
+ : T^*Q \times_Q T^*Q \to T^*Q,
\]

(4)

\[
\cdot : \mathbb{R} \times T^*Q \to T^*Q,
\]

(5)

and the canonical pairing

\[
\langle , \rangle : T^*Q \times_Q TQ \to \mathbb{R}.
\]

(6)

Substituting the tangent bundle \( TQ \) for the differential manifold \( Q \) we obtain the fibrations
with operations

\[ +: \mathsf{T} \mathsf{T} \times \mathsf{T} \rightarrow \mathsf{T}, \]
\[ \cdot: \mathbb{R} \times \mathsf{T} \rightarrow \mathsf{T}, \]
\[ +: \mathsf{T} \times \mathsf{T} \rightarrow \mathsf{T}, \]
\[ \cdot: \mathbb{R} \times \mathsf{T} \rightarrow \mathsf{T}, \]

and the canonical pairing

\[ \langle , \rangle: \mathsf{T} \times \mathsf{T} \rightarrow \mathbb{R}. \]

Sets \( \mathsf{T} \times \mathsf{T}, \mathsf{T} \times \mathsf{T}, \mathsf{T} \times \mathsf{T} \times \mathsf{T}, \) and \( \mathsf{T} \times \mathsf{T} \times \mathsf{T} \times \mathsf{T} \) have the following definitions:

\[ \mathsf{T} \times \mathsf{T} = \{ (u,v) \in \mathsf{T} \times \mathsf{T} : \mathsf{T}(u) = \mathsf{T}(v) \}, \]
\[ \mathsf{T} \times \mathsf{T} = \{ (f,g) \in \mathsf{T} \times \mathsf{T} : \mathsf{T}(f) = \mathsf{T}(g) \}, \]

and

\[ \mathsf{T} \times \mathsf{T} = \{ (f,u) \in \mathsf{T} \times \mathsf{T} : \mathsf{T}(f) = \mathsf{T}(u) \}. \]

By applying the tangent functor to the fibrations (1) we obtain a dual pair of vector fibrations

\[ \mathsf{T} \mathsf{T} \times \mathsf{T} \rightarrow \mathsf{T}, \]
\[ \mathsf{T} \times \mathsf{T} \rightarrow \mathsf{T}, \]
\[ \mathsf{T} \times \mathsf{T} \times \mathsf{T} \rightarrow \mathsf{T}, \]
\[ \mathsf{T} \times \mathsf{T} \times \mathsf{T} \rightarrow \mathsf{T}, \]

with operations

\[ +: \mathsf{T} \times \mathsf{T} \rightarrow \mathsf{T}, \]
\[ \cdot: \mathbb{R} \times \mathsf{T} \rightarrow \mathsf{T}, \]
\[ +: \mathsf{T} \times \mathsf{T} \rightarrow \mathsf{T}, \]
\[ \cdot: \mathbb{R} \times \mathsf{T} \rightarrow \mathsf{T}, \]

and the pairing

\[ \langle , \rangle: \mathsf{T} \times \mathsf{T} \rightarrow \mathbb{R}. \]

The operations \( \cdot \) and the pairing \( \langle , \rangle \) are extracted from the tangent operations

\[ \mathsf{T} \cdot: \mathbb{R} \times \mathsf{T} \rightarrow \mathsf{T}, \]
\[ \mathsf{T} \cdot: \mathbf{R} \times \mathsf{T} \rightarrow \mathsf{T}, \]
\[ \mathsf{T} \cdot: \mathbf{R} \times \mathsf{T} \rightarrow \mathsf{T}, \]
\[ \mathsf{T} \cdot: \mathbf{R} \times \mathsf{T} \rightarrow \mathsf{T}, \]

and the tangent pairing

\[ \mathsf{T} \langle , \rangle: \mathsf{T} \times \mathsf{T} \rightarrow \mathbb{R}. \]

Sets \( \mathsf{T} \times \mathsf{T}, \mathsf{T} \times \mathsf{T}, \mathsf{T} \times \mathsf{T} \times \mathsf{T}, \) and \( \mathsf{T} \times \mathsf{T} \times \mathsf{T} \times \mathsf{T} \) in formulae (2), (4), and (6) are defined by

\[ \mathsf{T} \times \mathsf{T} = \{ (u,v) \in \mathsf{T} \times \mathsf{T} : \mathsf{T}(u) = \mathsf{T}(v) \}. \]
\[ TT^*Q \times_{TTQ} TT^*Q = \{(x,y) \in TT^*Q \times TT^*Q; \ T\pi_Q(x) = T\pi_Q(y)\}, \]  
(26)

and

\[ TT^*Q \times_{TTQ} TTQ = \{(x,u) \in TT^*Q \times TTQ; \ T\pi_Q(x) = T\pi_Q(u)\}. \]  
(27)

Pairings (12) and (6) permit the introduction of the vector fibration isomorphism

\[
\begin{array}{ccc}
TT^*Q & \xrightarrow{\alpha_Q} & T^*TQ \\
\downarrow \text{T\pi_Q} & & \downarrow \text{\pi_{TTQ}} \\
TTQ & \xrightarrow{\pi_{TTQ}} & TTQ
\end{array}
\]  
(28)

dual to the vector fibration isomorphism

\[
\begin{array}{ccc}
TTQ & \xrightarrow{\kappa_Q} & TTQ \\
\downarrow \text{T\tau_Q} & & \downarrow \text{\tau_{TTQ}} \\
TTQ & \xrightarrow{\tau_{TTQ}} & TTQ
\end{array}
\]  
(29)

constructed with the canonical involution

\[ \kappa_Q: TTQ \rightarrow TTQ. \]  
(30)

For each manifold \( Q \) we have the mapping

\[ \chi_{TQ}: TQ \times_Q TQ \rightarrow TTQ \\
\colon (v,v') \mapsto t\gamma(0), \]  
(31)

where \( \gamma \) is the curve

\[ \gamma: \mathbb{R} \rightarrow TQ \\
\colon s \mapsto v + sv' \]  
(32)

and the symbol \( t\gamma(0) \) is used to denote the vector tangent to \( \gamma \). There is also the mapping

\[ \chi_{T^*Q}: T^*Q \times_Q T^*Q \rightarrow TT^*Q \\
\colon (p,p') \mapsto t\gamma(0), \]  
(33)

where \( \gamma \) is the curve

\[ \gamma: \mathbb{R} \rightarrow T^*Q \\
\colon s \mapsto p + sp'. \]  
(34)

Mappings \( \chi_{TQ} \) and \( \chi_{T^*Q} \) are two examples of a general mapping \( \chi_E \) defined for each vector fibration

\[
\begin{array}{ccc}
E & \downarrow \varepsilon & TQ \\
\downarrow \twoheadrightarrow & & \twoheadrightarrow \\
Q & \Downarrow & TQ
\end{array}
\]  
(35)
by
\[ \chi_E: E \times_Q E \to E \]
\[ : (e, e') \mapsto t \gamma(0), \]
where \( \gamma \) is the curve
\[ \gamma: \mathbb{R} \to T^*Q \]
\[ : s \mapsto e + se'. \]

3. Derivations.

Let \( P \) be a differential manifold. We denote by \( \Phi(P) \) and \( \Phi(TP) \) the exterior algebras of differential forms on \( P \) and \( TP \) respectively. A derivation of degree \( k \) relative to \( \tau_P: TP \to P \) is a linear operator \( a: \Phi(P) \to \Phi(TP) \) such that
\[ \text{(1) degree}(a \mu) = \text{degree}(\mu) + k \]
\[ \text{for each form } \mu \text{ on } P, \]
\[ \text{(2) } a(\mu \wedge \nu) = a \mu \wedge \tau_P^* \nu + (-1)^{k \text{degree}(\mu)} \tau_P^* \mu \wedge a \nu \]
\[ \text{for all forms } \mu \text{ and } \nu \text{ on } P. \]
The derivation \( a \) is said to be of type \( i_* \) if \( af = 0 \) for each function \( f \) on \( P \). The derivation \( a \) is said to be of type \( d_* \) if \( ad - (-1)^k da = 0 \).

We associate a derivation \( i_X \) of type \( i_* \) and degree \( k \) and a derivation \( d_X \) of type \( d_* \) and degree \( k + 1 \) with each mapping
\[ X: \wedge^{k+1} TP \to TP \]
such that
\[ \tau_P(X(v^1 \wedge v^2 \wedge \ldots \wedge v^{k+1})) = \tau_P(\tau_{TP}(v^1)). \]
The derivation \( i_X \) is characterized by the relation
\[ \langle i_X df, v^1 \wedge v^2 \wedge \ldots \wedge v^{k+1} \rangle = \langle X(v^1 \wedge v^2 \wedge \ldots \wedge v^{k+1}), df \rangle \]
for each function \( f \) on \( P \) and each sequence of vectors \( v^1, v^2, \ldots, v^{k+1} \) in \( TP \) such that \( \tau_{TP}(v^1) = \tau_{TP}(v^2) = \ldots = \tau_{TP}(v^{k+1}) \). The derivation \( d_X \) is defined by
\[ d_X = i_X d - (-1)^k d i_X. \]
The mapping (38) is called an infinitesimal deformation of the projection \( \tau_P: TP \to P \).

Let \( T: TP \to TP \) denote the identity mapping of \( TP \) interpreted as an infinitesimal deformation of \( \tau_P: TP \to P \). The derivations \( i_T \) and \( d_T \) of degrees \(-1 \) and \( 0 \) respectively are used in the definitions of Lagrangian systems and Hamiltonian systems. If \( f \) is a function on \( P \), then \( i_T df = d_T f \) is the function
\[ i_T df: TP \to \mathbb{R} \]
\[ : v \mapsto (df, v). \]

4. Special symplectic structures.

Let \((P, \omega)\) be a symplectic manifold. A special symplectic structure for \((P, \omega)\) is a diagram
\[ (P, \theta) \]
\[ \pi \]
\[ Q \]
\[ \text{(43)} \]
where $\pi: P \to Q$ is a vector fibration and $\vartheta$ is a vertical one-form on $P$ such that $d\vartheta = \omega$. An additional requirement is the existence of a vector fibration morphism

$$
\begin{array}{c}
P \xrightarrow{\psi} T^*Q \\
\pi \\
\pi_Q \\
Q \rightarrow Q
\end{array}
$$

(44)

such that $\vartheta = \psi^*\vartheta_Q$. We denote by $\vartheta_Q$ the Liouville one-form on $T^*Q$.

Let $Q$ be differential manifold. The diagram

$$
\begin{array}{c}
(T^*Q, \vartheta_Q) \\
\pi_Q \\
Q
\end{array}
$$

(45)

is a special symplectic structure for $(T^*Q, d\vartheta_Q)$.

A special symplectic structure morphism is a diagram

$$
\begin{array}{c}
(P_1, \vartheta_1) \\
\pi_1 \\
\varphi \\
Q_1 \rightarrow Q_2
\end{array}
\xrightarrow{\psi}
\begin{array}{c}
(P_2, \vartheta_2) \\
\pi_2 \\
Q_2
\end{array}
$$

(46)

where

$$
\begin{array}{c}
P_1 \xrightarrow{\psi} P_2 \\
\pi_1 \\
\varphi \\
Q_1 \rightarrow Q_2
\end{array}
$$

(47)

is a vector fibration morphism and $\vartheta_1 = \psi^*\vartheta_2$. A special symplectic structure morphism is necessarily an isomorphism and the mapping $\psi$ is a symplectomorphism from $(P_1, d\vartheta_1)$ to $(P_2, d\vartheta_2)$.

The definition of a special symplectic structure postulates the existence of a special symplectic structure morphism

$$
\begin{array}{c}
(P, \vartheta) \\
\pi \\
\pi_Q \\
Q \rightarrow Q
\end{array}
\xrightarrow{\psi}
\begin{array}{c}
(T^*Q, \vartheta_Q) \\
\pi_Q \\
Q
\end{array}
$$

(48)

for each special symplectic structure (43).

**Proposition 1.** The special symplectic structure morphism (48) is unique for each special symplectic structure (43).
Proof: For each \(w \in TP\) we have
\[
\langle \vartheta, w \rangle = \langle \psi^* \vartheta_Q, w \rangle = \langle \vartheta_Q, T\psi(w) \rangle = \langle \tau_{\pi^*} (T\psi(w)), T\pi_Q (T\psi(w)) \rangle = \langle \psi(\tau_P(w)), (T\pi_Q \circ \psi)(w) \rangle = \langle \psi(\tau_P(w)), T\pi(w) \rangle
\]
(49)
it follows that the mapping \(\psi: P \to T^*Q\) is completely characterized by
\[
\langle \psi(p), v \rangle = \langle \vartheta, w \rangle,
\]
(50)where \(v \in T_{\pi(p)}Q\) and \(w\) is any vector in \(T_pP\) such that \(T\pi(w) = v\). Hence, the mapping \(\psi\) is unique.

The unique special symplectic structure morphism (48) is said to be the canonical special symplectic structure morphism for the special symplectic structure (43).

Let
\[
(P_1, \vartheta_1) \xrightarrow{\pi_1} Q_1
\]
and
\[
(P_2, \vartheta_2) \xrightarrow{\pi_2} Q_2
\]
be special symplectic structures for symplectic manifolds \((P_1, \omega_1)\) and \((P_2, \omega_2)\) respectively together with their canonical special symplectic structure morphisms. If
\[
(P_1, \vartheta_1) \xrightarrow{\psi} (P_2, \vartheta_2)
\]
(53)
is a special symplectic structure morphism, then
\[
\psi = \psi_2^{-1} \circ T^* \varphi^{-1} \circ \psi_1.
\]
(54)

5. Examples of special symplectic structures.

In the following series of examples each special symplectic structure is presented together with its canonical special symplectic structure morphism.

Example 1. If
\[
(P, \vartheta) \xrightarrow{\pi} Q
\]
(55)
is a special symplectic structure for a symplectic manifold \((P,\omega)\), then the diagrams

\[
\begin{align*}
(P, \vartheta) & \quad \xrightarrow{\pi} \quad (\mathbb{T}^* Q, \vartheta_Q) \\
\end{align*}
\]

represent a special symplectic structure for the symplectic manifold \((P, -\omega)\).

**Example 2.** Let

\[
\begin{align*}
(P_1, \vartheta_1) & \quad \xrightarrow{\pi_1} \quad (\mathbb{T}^* Q_1, \vartheta_{Q_1}) \\
\end{align*}
\]

and

\[
\begin{align*}
(P_2, \vartheta_2) & \quad \xrightarrow{\pi_2} \quad (\mathbb{T}^* Q_2, \vartheta_{Q_2}) \\
\end{align*}
\]

represent special symplectic structures for symplectic manifolds \((P_1, \omega_1)\) and \((P_2, \omega_2)\) respectively. We denote by \(\omega_2 \oplus \omega_1\) the two-form on \(P_2 \times P_1\) defined by

\[
\omega_2 \oplus \omega_1 = pr_2^* \omega_2 + pr_1^* \omega_1,
\]

where \(pr_1: P_2 \times P_1 \to P_1\) and \(pr_2: P_2 \times P_1 \to P_2\) are the canonical projections. The symbol \(\vartheta_2 \oplus \vartheta_1\) will denote the one-form on \(P_2 \times P_1\) defined by

\[
\vartheta_2 \oplus \vartheta_1 = pr_2^* \vartheta_2 + pr_1^* \vartheta_1.
\]

We introduce a mapping

\[
\psi_2 \oplus \psi_1: P_2 \times P_1 \to \mathbb{T}^* (Q_2 \times Q_1)
\]

characterized by

\[
\langle \psi_2 \oplus \psi_1(p_2, p_1), w \rangle = \langle \psi_2(p_2), \mathbb{T}pr_2(w) \rangle + \langle \psi_1(p_1), \mathbb{T}pr_1(w) \rangle
\]

for each \(w \in \mathbb{T}(Q_2 \times Q_1)\) such that \(\tau_{Q_2 \times Q_1}(w) = (\pi_2(p_2), \pi_1(p_1))\). Symbols \(pr_1\) and \(pr_2\) this time denote the canonical projections \(pr_1: Q_2 \times Q_1 \to Q_1\) and \(pr_2: Q_2 \times Q_1 \to Q_2\). Diagrams

\[
\begin{align*}
(P \times P_1, \vartheta_2 \oplus \vartheta_1) & \quad \xrightarrow{\pi_2 \times \pi_1} \quad (\mathbb{T}^* (Q_2 \times Q_1), \vartheta_{Q_2 \times Q_1}) \\
\end{align*}
\]
represent a special symplectic structure for the symplectic manifold \((P_2 \times P_1, \omega_2 \oplus \omega_1)\). Let 
\[ \omega_2 \oplus \omega_1 = \omega_2 \oplus (-\omega_1), \quad \vartheta_2 \oplus \vartheta_1 = \vartheta_2 \oplus (-\vartheta_1), \] and 
\[ \psi_2 \oplus \psi_1 = \psi_2 \oplus (-\psi_1). \]

Diagrams

\[ (P_2 \times P_1, \vartheta_2 \oplus \vartheta_1) \xrightarrow{\pi_2 \times \pi_1} (P_2 \times P_1, \vartheta_2 \oplus \vartheta_1) \xrightarrow{\pi_{Q_2 \times Q_1}} (T^* (Q_2 \times Q_1), \vartheta_{Q_2 \times Q_1}) \] (64)

represent a special symplectic structure for the symplectic manifold \((P_2 \times P_1, \omega_2 \oplus \omega_1)\).

**Example 3.** If

\[ (P, \vartheta) \xrightarrow{\pi} (P, \vartheta) \xrightarrow{\psi} (T^* Q, \vartheta_Q) \]

represents a special symplectic structure for a symplectic manifold \((P, \omega)\), then the diagrams

\[ (TP, d_T \vartheta) \xrightarrow{T \pi} (TP, d_T \vartheta) \xrightarrow{T \pi} (T^* TQ, \vartheta_{TQ}) \]

represent a special symplectic structure for the symplectic manifold \((TP, d_T \omega)\).

**Example 4.** Let \((P, \omega)\) be a symplectic manifold. The diagrams

\[ (TP, i_T \omega) \xrightarrow{\tau_P} (TP, i_T \omega) \xrightarrow{\beta(p, \omega)} (T^* P, \vartheta_P) \]

represent a special symplectic structure for the symplectic manifold \((TP, d_T \omega)\). The mapping \(\beta(p, \omega): TP \rightarrow T^* P\) is characterized by

\[ \langle \beta(p, \omega)(u), v \rangle = \langle \omega, u \wedge v \rangle \] (68)

with \(u \in TP\) and \(v \in TP\) such that \(\tau_P(u) = \tau_P(v)\).

**6. Families of functions.**

Let

\[ R \]
\[ \rho \]
\[ X \]

be a differential fibration. We denote by \(VR\) the *bundle of vertical vectors* defined by

\[ VR = \{ v \in TR; \ T\rho(v) = 0 \} \] (70)

8
We denote by $V^oR$ the polar of the vertical bundle defined by

$$V^oR = \{s \in T^*R; \forall v \in VR^o\tau_R(v) = \pi_R(s) \Rightarrow \langle s, v \rangle = 0\} \quad (71)$$

A function $F: R \rightarrow \mathbb{R}$ can be considered a family of functions defined on fibres of the fibration $\rho$. We will represent a family of functions by a diagram

$$
\begin{array}{c}
R \\
\rho \downarrow \\
X
\end{array}
\xrightarrow{F} \mathbb{R}
$$

(72)

The critical set for a family of functions $F: R \rightarrow \mathbb{R}$ is the set

$$S(F, \rho) = \{r \in R; \forall v \in VR^oF, (dF, v) = 0\}. \quad (73)$$

At each point $r \in S(F, \rho)$ we define a bilinear mapping

$$W(F, r): VR^oR \times T_rR \rightarrow \mathbb{R}$$

$$: (v, w) \mapsto D^{(1,1)}(F \circ \chi)(0, 0), \quad (74)$$

where $\chi$ is a mapping from $\mathbb{R}^2$ to $R$ such that $v = t\chi(\cdot, 0)(0)$ and $w = t\chi(0, \cdot)(0)$. The family $F$ is said to be regular if the rank of $W(F, r)$ is the same at each $r \in S(F, \rho)$. The family $F$ is called a Morse family if the rank of $W(F, r)$ is maximal at each $r \in S(F, \rho)$.

It is known that if $F$ is a regular family, then the set

$$N = \{p \in T^*X; \exists r \in R \rho(r) = q = \pi_X(p) \}
\forall v \in T_qX \forall w \in T_rR T\rho(w) = v \Rightarrow \langle p, v \rangle = \langle dF, w \rangle\} \quad (75)$$

is an immersed Lagrangian submanifold of the symplectic space $(T^*X, \theta_X)$. This Lagrangian submanifold is said to be generated by the family (72).

For each regular family $F$ we have the mapping $\kappa: S(F, \rho) \rightarrow T^*X$ characterized by

$$\langle \kappa(r), v \rangle = \langle dF, w \rangle \quad (76)$$

for each $v \in TX$ such that $\tau_X(v) = \rho(r)$ and each $w \in TR$ such that $T\rho(w) = v$. The mapping $\kappa$ is a submersion. If $F$ is a Morse family, then $\kappa$ is an immersion. The immersed Lagrangian submanifold (75) is the image $\text{im}(\kappa)$ of the mapping $\kappa$.

There is a surjective submersion $\lambda: V^oR \rightarrow T^*X$ characterized by

$$\langle \lambda(q), v \rangle = \langle q, w \rangle \quad (77)$$

for each $v \in TX$ such that $\tau_X(v) = \rho(\pi_R(q))$ and each $w \in TR$ such that $T\rho(w) = v$. The intersection $\text{im}(dF) \cap V^oR$ is clean if $F$ is a regular family. This intersection is transverse if $F$ is a Morse family. The immersed Lagrangian submanifold (75) is the set $\lambda(\text{im}(dF) \cap V^oR)$.

If the fibration $\rho$ is the identity morphism $1_X: X \rightarrow X$, then the family of functions is a function

$$F: X \rightarrow \mathbb{R} \quad (78)$$

and the Lagrangian submanifold $N$ is the set

$$N = \left\{ p \in T^*X; \pi_X(p) = q, \forall v \in T_qX \langle p, v \rangle = \langle dF, v \rangle \right\}
= \text{im}(dF). \quad (79)$$
7. Generating objects.

Let

\[(P, \vartheta) \xrightarrow{\psi} (\pi_Q \circ T^*Q, \vartheta_Q)\]

be a special symplectic structure for a symplectic manifold \((P, \omega)\) with its canonical special symplectic structure morphism, let

\[\iota_X : X \to Q\]

be the canonical injection of a submanifold \(X \subset Q\), let

\[R \xrightarrow{\rho} X\]

be a differential fibration, let

\[N \subset T^*X\]

be the immersed Lagrangian submanifold generated by this family. We denote by \(\xi\) the mapping from \(T^*Q = \pi_Q^{-1}(X)\) to \(T^*X\) characterized by

\[\langle \xi(p), v \rangle = \langle p, v \rangle\]

for each \(v \in TQ\). Let \(N' = \psi^{-1}(N)\) be a Lagrangian submanifold of \((T^*Q, d\vartheta_Q)\) and the set

\[\{ p \in \psi^{-1}(N) \}

is a Lagrangian submanifold of \((T^*Q, d\vartheta_Q)\) and the set

\[A = \psi^{-1}(N')\]

is a Lagrangian submanifold of \((P, d\vartheta)\). The diagram
is called a generating object for the immersed Lagrangian submanifold $A$.

The diagram

\[ (P, \vartheta) \xrightarrow{\pi} R \xrightarrow{F} \mathbb{R} \]

\[ Q \xrightarrow{\ell_X} X \]

is the simplest case of a generating object. The Lagrangian submanifold

\[ A = \left\{ p \in P; \forall w \in T_p P, \langle \vartheta, w \rangle = \langle dF, \pi(w) \rangle \right\} \]

generated by this object is the image of the section $\psi^{-1} \circ dF$ of the fibration $\pi$.

If $X$ is a submanifold of $Q$ and the fibration $\rho$ is the identity morphism $1_X : X \rightarrow X$, then we have a generating object

\[ (P, \vartheta) \xrightarrow{\pi} Q \xrightarrow{\ell_X} X \xrightarrow{F} \mathbb{R} \]

and the Lagrangian submanifold generated by this object is the set

\[ A = \left\{ p \in P; q = \pi(p) \in X, \forall w \in T_p P, \forall v \in T_q X, T_q (w) \subset T_q (v) \Rightarrow \langle \vartheta, w \rangle = \langle dF, v \rangle \right\} \]

Another special case is obtained with $X = Q$ and a non trivial fibration $\rho$. The generating object has the form

\[ (P, \vartheta) \xrightarrow{\pi} R \xrightarrow{F} \mathbb{R} \]

\[ Q \xrightarrow{\rho} Q \]

and the Lagrangian submanifold generated by this object is the set

\[ A = \left\{ p \in P; \exists r \in R, \rho(r) = \pi(p), \forall w \in T_p P, \forall z \in T_r R, T \rho(z) = T \pi(w) \Rightarrow \langle \vartheta, w \rangle = \langle dF, z \rangle \right\} \]

8. Reduction of generating objects.

Let

\[ R \xrightarrow{F} \mathbb{R} \]

\[ \rho \]

\[ X \]

be a family of functions generating a Lagrangian submanifold $N$ of $(T^* X, d\vartheta_x)$. 11
Proposition 2. If the family (94) satisfies conditions

(1) the image $\tilde{X} = \rho(S(F,\rho))$ of the critical set $S(F,\rho)$ by the projection $\rho$ is a submanifold of $X$,
(2) the mapping

$$\tilde{\rho}: S(F,\rho) \to \tilde{X}$$

$$r \mapsto \rho(r)$$

induced by $\rho$ is a differential fibration with connected fibres,

then there is a function $\tilde{F}: \tilde{X} \to \mathbb{R}$ such that $F|S(F,\rho) = \tilde{F} \circ \tilde{\rho}$ and $N \subset \tilde{N}$, where $\tilde{N}$ is the Lagrangian submanifold of $(T^*X, d\vartheta_X)$ generated by the generating object

$$\pi_X$$

$$\tilde{\rho}$$

$$\tilde{F}$$

$$\mathbb{R}$$

(95)

PROOF: The function $\tilde{F}$ can be defined since $F$ is constant on fibres of $\tilde{\rho}$. We have

$$N = \left\{ p \in T^*X; \exists r \in R, \rho(r) = \pi_X(p), \forall z \in T_rR \langle p, T\rho(z) \rangle = \langle dF, z \rangle \right\}$$

and

$$\tilde{N} = \left\{ p \in T^*X; q = \pi_X(p) \in \tilde{X}, \forall v \in T_q\tilde{X} \subset T_qX \langle p, v \rangle = \langle \tilde{dF}, v \rangle \right\}.$$ (101)

Let $p \in N$ and let $q = \pi_X(p)$. There is a point $r \in R$ such that $\rho(r) = q$ and $\langle p, T\rho(z) \rangle = \langle dF, z \rangle$ for each $z \in T_rR$. If $T\rho(z) = 0$, then $\langle dF, z \rangle = 0$. Hence, $r \in S(F,\rho)$ and $q = \pi_X(p) \in \tilde{X}$. If $v \in T_q\tilde{X} \subset T_qX$, $z \in T_rR$ and $T\rho(z) = v$, then $z \in T_rS$ and

$$\langle p, v \rangle = \langle \tilde{dF}, z \rangle$$

$$= \langle \tilde{d}(F|S(F,\rho)), z \rangle$$

$$= \langle \tilde{d}(\tilde{F} \circ \tilde{\rho}), z \rangle$$

$$= \langle \tilde{\rho}^*d\tilde{F}, z \rangle$$

$$= \langle \tilde{d}\tilde{F}, v \rangle.$$ (99)

It follows that $p \in \tilde{N}$.

Proposition 3. If a Lagrangian submanifold $N$ of $(T^*X, d\vartheta_X)$ is generated by a family (94) such that the critical set $S(F,\rho)$ is the image of a section $\sigma: X \to R$ of $\rho$, then $N$ is generated by the function

$$X \xrightarrow{\tilde{F}} \mathbb{R}$$

with $\tilde{F} = F \circ \sigma$.

PROOF: Let $\tilde{N}$ be the Lagrangian submanifold generated by the function $\tilde{F}$. For each $q \in X$ we have

$$N_q = N \cap T_qX$$

$$= \left\{ p \in T^*X; \exists r \in R_q, \forall z \in T_rR \langle p, T\rho(z) \rangle = \langle dF, z \rangle \right\}.$$ (100)
and

\[ \tilde{N}_q = \tilde{N} \cap T_q X = \left\{ p \in T^* X ; \forall_{v \in T_q X} \langle p, v \rangle = \langle dF, v \rangle \right\}. \]  \quad (102)

The inclusion \( N_q \subset \tilde{N}_q \) is proved as in Proposition 2. The set \( N_q \) is not empty and \( \tilde{N} = \left\{ d\tilde{F}(q) \right\} \). Hence, \( N_q = \tilde{N} \) for each \( q \in X \). It follows that \( N = \tilde{N} \). ■

The following two propositions are easily derivable from Proposition 2 and Proposition 3.

**Proposition 4.** If \( N \) is generated by a family (94) such that

1. the fibration \( \rho \) is the composition \( \rho'' \circ \rho' \) of fibrations \( \rho' : R \rightarrow R' \) and \( \rho'' : R' \rightarrow X \),
2. the images \( \tilde{R} = \rho'(S(F, \rho')) \) and \( \tilde{X} = \rho(S(F, \rho')) \) of the critical set \( S(F, \rho') \) are submanifolds of \( R' \) and \( X \) respectively,
3. the induced mapping

\[ \tilde{\rho} : \tilde{R} \rightarrow \tilde{X} \]

\[ : r' \mapsto \rho''(r') \]  \quad (103)

is a differential fibration

4. the mapping

\[ \tilde{\theta} : S(F, \rho') \rightarrow \tilde{R} \]

\[ : r \mapsto \rho(r) \]  \quad (104)

is a differential fibration with connected fibres,

then there is a function \( \tilde{F} : \tilde{R} \rightarrow \mathbb{R} \) such that \( F|S(F, \rho') = \tilde{F} \circ \tilde{\theta} \) and \( N \subset \tilde{N} \), where \( \tilde{N} \) is generated by the generating object

\[ (T^* X, \vartheta_X) \]

\[ \xymatrix{ & \tilde{R} \ar[r]^{\tilde{F}} & \mathbb{R} \ar[d]^\tilde{\rho} \ar[dl]_{\pi_X} \ar[d]_{\pi_X} \\
X \ar[r]_{\iota_{\tilde{X}}} & \tilde{X} & X}
\]  \quad (105)

with \( \tilde{F} = F \circ \sigma \).

**Proposition 5.** If \( N \) is generated by a family (94) such that the fibration \( \rho \) is the composition \( \rho \circ \rho' \) of fibrations \( \rho' : R \rightarrow R \) and \( \tilde{\rho} : \tilde{R} \rightarrow X \) and the critical set \( S(F, \rho') \) is the image of a section \( \sigma : \tilde{R} \rightarrow R \) of \( \rho' \), then \( N \) is generated by the generating object

\[ (T^* X, \vartheta_X) \]

\[ \xymatrix{ & \tilde{R} \ar[r]^{\tilde{F}} & \mathbb{R} \ar[d]^\tilde{\rho} \ar[dl]_{\pi_X} \\
X & X}
\]  \quad (106)

with \( \tilde{F} = F \circ \sigma \).
The last four propositions have the following obvious counterparts in the more general setting obtained by replacing the family of functions (94) by a generating object

\[(P, \vartheta) \xrightarrow{\pi} \pi \xrightarrow{\rho} \pi \xrightarrow{t_X} X\]  

(107)

**Proposition 6.** If a Lagrangian submanifold \(A\) of \((P, d\vartheta)\) is generated by a generating object (107) and

1. the image \(\tilde{X} = \rho(S(F, \rho))\) of the critical set \(S(F, \rho)\) by the projection \(\rho\) is a submanifold of \(X\),
2. the mapping

\[\tilde{\rho}: S(F, \rho) \to \tilde{X} \quad : r \mapsto \rho(r)\]  

(108)

induced by \(\rho\) is a differential fibration with connected fibres,

then \(A \subset \tilde{A}\), where \(\tilde{A}\) is the Lagrangian submanifold of \((P, d\vartheta)\) generated by the generating object

\[(P, \vartheta) \xrightarrow{\pi} \pi \xrightarrow{\tilde{\rho}} \pi \xrightarrow{t_{\tilde{X}}} \tilde{X} \xrightarrow{\tilde{F}} \tilde{R} \xrightarrow{\rho} \tilde{R} \xrightarrow{t_{\tilde{X}}} \tilde{X}\]  

(109)

**Proposition 7.** If a Lagrangian submanifold \(A\) of \((P, d\vartheta)\) is generated by a generating object (107) and the critical set \(S(F, \rho)\) is the image of a section \(\sigma: X \to R\) of \(\rho\), then \(A\) is generated by the generating object

\[(P, \vartheta) \xrightarrow{\pi} \pi \xrightarrow{t_{\tilde{X}}} \tilde{X} \xrightarrow{\tilde{F}} \tilde{R} \xrightarrow{\rho} \tilde{R}\]  

(110)

with \(\tilde{F} = F \circ \sigma\).

**Proposition 8.** If \(A\) is generated by a generating object (107) such that

1. the fibration \(\rho\) is the composition \(\rho'' \circ \rho'\) of fibrations \(\rho': R \to R'\) and \(\rho'': R' \to X\),
2. the images \(\tilde{R} = \rho'(S(F, \rho'))\) and \(\tilde{X} = \rho(S(F, \rho'))\) of the critical set \(S(F, \rho')\) are submanifolds of \(R'\) and \(X\) respectively,
3. the induced mapping

\[\tilde{\rho}: \tilde{R} \to \tilde{X} \quad : r' \mapsto \rho''(r')\]  

(111)

is a differential fibration,
4. the mapping

\[\tilde{\varphi}: S(F, \rho') \to \tilde{R} \quad : r \mapsto \rho(r)\]  

(112)

is a differential fibration with connected fibres,
then there is a function $\tilde{F}: \tilde{R} \to \mathbb{R}$ such that $F|S(F, \rho') = \tilde{F} \circ \varphi$ and $A \subset \tilde{A}$, where $\tilde{A}$ is generated by the generating object

\[
\begin{array}{ccc}
(P, \vartheta) & \tilde{R} & \tilde{F} \to \mathbb{R} \\
\pi & \varphi & \\
Q & \tilde{X} & \\
\end{array}
\]

with $\tilde{F} = F \circ \sigma$.

**Proposition 9.** If $A$ is generated by a generating object (107) such that the fibration $\rho$ is the composition $\varphi \circ \rho'$ of fibrations $\rho': R \to \tilde{R}$ and $\varphi: \tilde{R} \to X$, and the critical set $S(F, \rho')$ is the image of a section $\sigma': \tilde{R} \to R$, then $A$ is generated by the generating object

\[
\begin{array}{ccc}
(P, \vartheta) & \tilde{R} & \tilde{F} \to \mathbb{R} \\
\pi & \varphi & \\
Q & \tilde{X} & X \\
\end{array}
\]

with $\tilde{F} = F \circ \sigma'$.

**9. Composition of generating objects.**

The Lagrangian submanifold $A_{21}$ generated by a generating object

\[
\begin{array}{ccc}
(P_2 \times P_1, \vartheta_2 \oplus \vartheta_1) & R_{21} & F_{21} \to \mathbb{R} \\
\pi_2 \times \pi_1 & \rho_{21} & \\
Q_2 \times Q_1 & t_{X_{21}} & X_{21} \\
\end{array}
\]

is interpreted as the graph of a symplectic relation $\Pi_{21}$ from $(P_1, d\vartheta_1)$ to $(P_2, d\vartheta_2)$. Let $A_1$ be a Lagrangian submanifold of $(P_1, d\vartheta_1)$ generated by a generating object

\[
\begin{array}{ccc}
(P_1, \vartheta_1) & R_1 & F_1 \to \mathbb{R} \\
\pi_1 & \rho_1 & \\
Q_1 & t_{X_1} & X_1 \\
\end{array}
\]

If the image

\[
\Pi_{21}(A_1) = \{p_2 \in P_2; \exists p_1 \in A_1, (p_2, p_1) \in A_{21}\}
\]

of $A_1$ by the relation $\Pi_{21}$ is a Lagrangian submanifold $A_2$ of $(P_2, d\vartheta_2)$, then it is generated by the generating object

\[
\begin{array}{ccc}
(P_2, \vartheta_2) & R_2 & F_2 \to \mathbb{R} \\
\pi_2 & \rho_2 & \\
Q_2 & t_{X_2} & X_2 \\
\end{array}
\]

described below.
We introduce the sets
\[ X_2 = X_{21} \circ X_1 \]
\[ = \left\{ q_2 \in Q_2; \exists q_1 \in X_1 (q_2, q_1) \in X_{21} \right\}, \] (119)
\[ X_{2[11]} = \left\{(q_2, q_1, q'_1) \in Q_2 \times Q_1 \times Q_1; (q_2, q_1) \in X_{21}, q'_1 = q_1 \in X_1\right\}, \] (120)
and the projection
\[ p^{2[11]}: X_{2[11]} \to X_2 \]
\[ : (q_2, q_1, q'_1) \mapsto q_2. \] (121)

We define the set \( R_2 \), the fibration \( \rho_2: R_2 \to X_2 \), and the family of functions \( F_2 : R_2 \to \mathbb{R} \) by

\[ R_2 = (\rho_{21} \times \rho_1)^{-1}(X_{2[11]}), \] (122)
\[ \rho_2: R_2 \to X_2 \]
\[ : (r_{21}, r_1) \mapsto p^{2[11]}(\rho_{21}(r_{21}), \rho_1(r_1)), \] (123)
and

\[ F_2: R_2 \to \mathbb{R} \]
\[ : (r_{21}, r_1) \mapsto F_{21}(r_{21}) + F_1(r_1). \] (124)

10. Lagrangian systems.

Let \( Q \) be the configuration manifold of a mechanical system. The special symplectic structure

\[ (TT^*Q, d_T \vartheta_Q) \]
\[ \text{generated by the Lagrangian system (126) is called the dynamics of the mechanical system.} \] (128)

is called the Lagrangian special symplectic structure. A generating object

\[ (TT^*Q, d_T \vartheta_Q) \]
\[ \text{is called a Lagrangian system. The family of functions} \]
\[ \begin{array}{ccc}
  \text{Y} & \overset{L}{\rightarrow} & \mathbb{R} \\
  \text{TQ} & \overset{\eta}{\downarrow} & \text{C} \\
  \text{TQ} \end{array} \] (126)

is called a Lagrangian family and the immersed Lagrangian submanifold

\[ D = \left\{ w \in TT^*Q; \forall v \in T\pi_Q(w) \in C, \exists \nu \in Y_\nu, \right. \]
\[ \left. \forall x \in TT^*Q, \forall z \in T\eta(z) T\pi_Q(x) \Rightarrow \right\} \] (128)

generated by the Lagrangian system (126) is called the dynamics of the mechanical system.
11. Hamiltonian systems.

Let \((P, \omega)\) be a symplectic manifold representing the phase space of a mechanical system. The special symplectic structure represented by the diagrams

\[
\begin{array}{ccc}
(TP, i_T \omega) & \xrightarrow{\beta(P, \omega)} & (T^*P, \theta_P) \\
\tau_P & \downarrow & \pi_P \\
P & \xrightarrow{\iota_K} & P
\end{array}
\]

is called the Hamiltonian special symplectic structure. A generating object

\[
\begin{array}{ccc}
(TP, i_T \omega) & \xrightarrow{Z} & \mathbb{R} \\
\tau_P & \downarrow & \zeta \\
P & \xrightarrow{\iota_K} & K
\end{array}
\]

is called a Hamiltonian system. The family of functions

\[
\begin{array}{ccc}
Z & \xrightarrow{H} & \mathbb{R} \\
\zeta & \downarrow & \\
K & & 
\end{array}
\]

is called a Hamiltonian family and the immersed Lagrangian submanifold

\[
D = \left\{ w \in TP; \ p = \tau_P(w) \in K, \exists z \in Z_p \ \forall x \in T_wTP \ \forall u \in T_zZ \ T\zeta(u) = T\tau_P(x) \Rightarrow \langle i_T\omega, x \rangle = -(dH, u) \right\}
\]

(132)
generated by the Hamiltonian system (130) is called the dynamics of the mechanical system.

A Hamiltonian system

\[
\begin{array}{ccc}
(TP, i_T \omega) & \xrightarrow{\iota_K} & \mathbb{R} \\
\tau_P & \downarrow & \\
P & \xrightarrow{\iota_K} & K
\end{array}
\]

is called a Dirac system. A more general Hamiltonian system (130) is called a generalized Dirac system.

The phase space of a mechanical system is usually the cotangent bundle \(T^*Q\) of the configuration manifold with its canonical symplectic structure \(d\theta_Q\). The Hamiltonian special symplectic structure is in this case represented by

\[
\begin{array}{ccc}
(TT^*Q, i_{T^*}d\theta_Q) & \xrightarrow{\beta(T^*Q, d\theta_Q)} & (T^*T^*Q, \theta_{T^*Q}) \\
\tau_{T^*Q} & \downarrow & \pi_{T^*Q} \\
T^*Q & \xrightarrow{\pi_{T^*Q}} & T^*Q
\end{array}
\]

(134)
and a Hamiltonian system is a generating object

\[
\begin{align*}
(TT^*Q, i_Td\vartheta_Q) & \xrightarrow{\tau_{T^*Q}} \tau_{T^*Q} \\
Z & \xrightarrow{H} \mathbb{R} \\
T^*Q & \xrightarrow{\iota_K} K
\end{align*}
\]

The dynamics of the mechanical system is the set

\[
D = \{ w \in TT^*Q; p = \tau_{T^*Q}(w) \in K, \exists z \in z_p \forall x \in T_w TT^*Q,\forall u \in T_z Z \tau_{T^*Q}(x) = \tau_{T^*Q}(x) = -\langle dH, u \rangle \}.
\]

12. The Legendre transformation.

Let \( Q \) be the configuration manifold of a mechanical system. The phase space of the system is the symplectic manifold \((T^*Q, d\vartheta_Q)\). There are two canonical special symplectic structures for the symplectic manifold \((TT^*Q, dT^*d\vartheta_Q)\). We have the Lagrangian special symplectic structure

\[
\begin{align*}
(TT^*Q, d_T\vartheta_Q) & \xrightarrow{\alpha_Q} (T^*Q, \vartheta_Q) \\
T\pi_Q & \xrightarrow{\pi_{TQ}} TQ \\
TQ & \xrightarrow{\pi_{TQ}} TQ
\end{align*}
\]

and the Hamiltonian special symplectic structure

\[
\begin{align*}
(TT^*Q, i_Td\vartheta_Q) & \xrightarrow{\beta_{(TT^*Q, d\vartheta_Q)}} (T^*T^*Q, \vartheta_{T^*Q}) \\
\tau_{T^*Q} & \xrightarrow{\pi_{T^*Q}} T^*Q \\
T^*Q & \xrightarrow{\pi_{T^*Q}} T^*Q
\end{align*}
\]

The identity morphism \(1_{TT^*Q}\) is a symplectic relation generated by the generating object

\[
\begin{align*}
(TT^*Q \times TT^*Q, i_Td\vartheta_Q \ominus d_T\vartheta_Q) & \xrightarrow{\tau_{T^*Q} \times T\pi_Q} \\
T^*Q \times TQ & \xrightarrow{\iota_{T^*Q} \times \vartheta_{T^*Q}} T^*Q \times Q \xrightarrow{-\langle \cdot, \cdot \rangle} \mathbb{R}
\end{align*}
\]

and by the generating object

\[
\begin{align*}
(TT^*Q \times TT^*Q, dT\vartheta_Q \ominus i_Td\vartheta_Q) & \xrightarrow{T\pi_Q \times \tau_{T^*Q}} \\
TQ \times T^*Q & \xrightarrow{\iota_{TQ} \times \vartheta_{T^*Q}} TQ \times T^*Q \xrightarrow{\langle \cdot, \cdot \rangle^\sim} \mathbb{R}
\end{align*}
\]

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where $\langle , \rangle^\sim$ is the mapping
\[
\langle , \rangle^\sim: TQ \times T^*Q \to \mathbb{R} \\
(v, p) \mapsto \langle p, v \rangle.
\] (141)

Starting with a Lagrangian system
\[
\begin{array}{ccc}
(TT^*Q, d\tau d\vartheta_Q) & \quad & Y \xrightarrow{L} \mathbb{R} \\
\downarrow \tau \pi_Q & \quad & \downarrow \eta \\
TQ & \leftarrow \iota_C & C
\end{array}
\] (142)

we construct a Hamiltonian system
\[
\begin{array}{ccc}
(TT^*Q, i_T d\vartheta_Q) & \quad & Z \xrightarrow{-H} \mathbb{R} \\
\downarrow \tau_T^*Q & \quad & \downarrow \zeta \\
T^*Q & \leftarrow \iota_K & K
\end{array}
\] (143)

with
\[
K = \pi_Q^{-1}(\tau_Q(C)) \\
= \{ p \in T^*Q : \exists v \in C, \pi_Q(p) = \tau_Q(v) \},
\] (144)
\[
Z = \{ (p, v, y) \in T^*Q \times TQ \times Y : v \in C, \eta(y) = v \},
\] (145)
\[
\zeta: Z \to K \\
: (p, v, y) \mapsto p,
\] (146)

and
\[
H: Z \to \mathbb{R} \\
: (p, v, y) \mapsto \langle p, v \rangle - L(y).
\] (147)

This Hamiltonian system is obtained by composing the Lagrangian system with the generating object (139). The two systems generate the same dynamics since the generating object (139) generates the identity relation. The passage from the Lagrangian system (142) to the Hamiltonian system (143) is called the Legendre transformation.

Conversely, given a Hamiltonian system
\[
\begin{array}{ccc}
(TT^*Q, i_T d\vartheta_Q) & \quad & Z \xrightarrow{-H} \mathbb{R} \\
\downarrow \tau_T^*Q & \quad & \downarrow \zeta \\
T^*Q & \leftarrow \iota_K & K
\end{array}
\] (148)

we construct a Lagrangian system
\[
\begin{array}{ccc}
(TT^*Q, d\tau d\vartheta_Q) & \quad & Y \xrightarrow{L} \mathbb{R} \\
\downarrow \tau \pi_Q & \quad & \downarrow \eta \\
TQ & \leftarrow \iota_C & C
\end{array}
\] (149)
with
\[ C = \tau_Q^{-1}(\pi_Q(K)) \]
\[ = \left\{ v \in T^*Q; \exists p \in K \pi_Q(p) = \tau_Q(v) \right\}, \tag{150} \]
\[ Y = \{(p, v, z) \in T^*Q \times Q \times Z; p \in K, \zeta(z) = p \}, \tag{151} \]
\[ \eta: Y \to C \]
\[ :(p, v, z) \mapsto v, \tag{152} \]
and
\[ L: Y \to \mathbb{R} \]
\[ :(p, v, z) \mapsto \langle p, v \rangle - H(z) \tag{153} \]
by composing the Hamiltonian system with the generating object (140). This passage is called the inverse Legendre transformation.

In addition to the Legendre transformations we have Legendre relations. Given a Lagrangian system (142) we have the first Legendre relation
\[ \Lambda_1(L): Y \to T^*Q, \tag{154} \]
whose graph is the set
\[ \text{graph}(\Lambda_1(L)) = \left\{ (p, y) \in T^*Q \times Y; y \in S(L, \eta), \pi_Q(p) = q = \tau_Q(\eta(y)) \right\} \]
\[ \forall u \in C \cap T^*q \quad \forall z \in T^*Y \quad T\eta(z) = \chi_Q(\eta(y), u) \Rightarrow \langle p, u \rangle = \langle dL, z \rangle \} \tag{155} \]
and the second Legendre relation
\[ \Lambda_2(L): TQ \to T^*Q, \tag{156} \]
whose graph is the set
\[ \text{graph}(\Lambda_2(L)) = \left\{ (p, v) \in T^*Q \times TQ; \exists y \in Y, \eta(y) = v, (p, y) \in \text{graph}(\Lambda_1(L)) \right\}. \tag{157} \]
If \( D \subset T^*Q \) is the dynamics generated by the Lagrangian system, then
\[ \text{graph}(\Lambda_2(L)) = (\tau_{T*Q}, T\pi_Q)(D). \tag{158} \]
A Lagrangian system is said to be hyperregular if the second Legendre relation \( \Lambda_2(L) \) is a diffeomorphism.

Given a Hamiltonian system (143) we introduce the first inverse Legendre relation
\[ \Omega_1(H): Z \to TQ, \tag{159} \]
whose graph is the set
\[ \text{graph}(\Omega_1(H)) = \left\{ (v, z) \in TQ \times Z; z \in S(H, \zeta), \tau_Q(v) = q = \pi_Q(\zeta(z)) \right\} \]
\[ \forall a \in K \cap T^*_z Q \quad \forall u \in T_z Z \quad T\zeta(u) = \chi_{T^*Q}(\zeta(z), a) \Rightarrow \langle a, v \rangle = \langle dH, u \rangle \} \tag{160} \]
and the second inverse Legendre relation

\[ \Omega_2(H) : T^*Q \to TQ, \quad (161) \]

whose graph is the set

\[ \text{graph}(\Omega_2(H)) = \{(v, p) \in TQ \times T^*Q; \exists z \in \mathbb{Z} \zeta(z) = p, (v, z) \in \text{graph}(\Omega_1(W))\}. \quad (162) \]

If the Hamiltonian system is obtained from the Lagrangian system (142) by the Legendre transformation, then the second inverse Legendre relation \( \Omega_2(H) \) is the transpose \( \Lambda_2(L)^t \) of the second Legendre relation \( \Lambda_2(L) \). The transpose of a relation \( \Phi : A \to B \) is the relation \( \Phi^t : B \to A \) with

\[ \text{graph}(\Phi^t) = \{(a, b) \in A \times B; (b, a) \in \text{graph}(\Phi)\}. \quad (163) \]

13. Homogeneous families of functions.

Let

\[ R \xrightarrow{F} \mathbb{R} \]

be a family of functions and let

\[ \mathbb{R}_+ \times R \xrightarrow{\nu} R \]

be a group action of the group \((\mathbb{R}_+, \cdot)\) by fibration isomorphisms. The family (164) is said to be homogeneous with respect to the group action (165) if

\[ F(\nu(k, r)) = kF(r) \quad (166) \]

for each \( r \in R \) and each \( k \in \mathbb{R}_+ \).

For each \( k \in \mathbb{R}_+ \) we have the diffeomorphism \( \nu(k, \cdot): R \to R \) and for each \( r \in R \) there is the linear isomorphism \( T_r(\nu(k, \cdot)): T_rR \to T_{\nu(k, \cdot)}R \). A vector \( z \in T_rR \) is vertical if and only if \( T_r(\nu(k, \cdot))(z) \) is vertical. If (164) is a homogeneous family, then

\[ \nu(k, \cdot)^*F = F \circ \nu(k, \cdot) = kF \quad (167) \]

and

\[ \nu(k, \cdot)^*dF = d\nu(k, \cdot)^*F = kdF. \quad (168) \]

Hence,

\[ \langle dF, T_r(\nu(k, \cdot))(z) \rangle = \langle \nu(k, \cdot)^*dF, z \rangle = k\langle dF, z \rangle \quad (169) \]

for each \( z \in T_rR \).

A set \( S \subset R \) is said to be homogeneous if \( r \in S \) implies \( \nu(k, r) \in S \) for each \( k \in \mathbb{R}_+ \).

**Proposition 10.** The critical set \( S(F, \rho) \) for a homogeneous family (164) is homogeneous.
Proof: Let $r \in S(F, \rho)$ and let $z \in T_{\nu(k, r)}R$ be a vertical vector. We have

$$
\langle dF, z \rangle = \langle dF, T_r(\nu(k, *))((T_r(\nu(k^{-1}, *)))(z)) \rangle \\
= k \langle dF, T_r(\nu(k^{-1}, *))((z)) \rangle \\
= 0 \quad (170)
$$

since $T_r(\nu(k^{-1}, *))((z))$ is a vertical vector in $T_xR$ and $r$ is a critical point. It follows that $\nu(k, r)$ is a critical point. □

The group action $\mu: \mathbb{R}_+ \times X \to X$ is lifted to the action

$$
\begin{array}{ccc}
\mathbb{R}_+ \times T^*X & \xrightarrow{\overline{\mu}} & T^*X \\
1_{\mathbb{R}_+} \times \pi_X & \downarrow & \pi_X \\
\mathbb{R}_+ \times X & \xrightarrow{\mu} & X
\end{array}
$$

(171)

with $\overline{\mu}: \mathbb{R}_+ \times T^*X \to T^*X$ defined by

$$
\overline{\mu}(k, f) = kT^*\mu(k^{-1}, f) \quad (172)
$$

for each $k \in \mathbb{R}_+$ and each $f \in T^*X$.

The set $N$ generated by a homogeneous family of functions is homogeneous.

14. Homogeneous generating objects.

Let

$$
\begin{array}{ccc}
(P, \vartheta) & \xrightarrow{\psi} & (T^*Q, \vartheta_Q) \\
\pi & \downarrow & \pi \\
Q & \xrightarrow{\pi_Q} & Q
\end{array}
$$

(173)

be a special symplectic structure with its canonical special symplectic structure morphism. A group action

$$
\mu: \mathbb{R}_+ \times Q \to Q \quad (174)
$$

is lifted to the action

$$
\begin{array}{ccc}
\mathbb{R}_+ \times P & \xrightarrow{\widehat{\mu}} & P \\
1_{\mathbb{R}_+} \times \pi & \downarrow & \pi \\
\mathbb{R}_+ \times Q & \xrightarrow{\mu} & Q
\end{array}
$$

(175)

with $\widehat{\mu}: \mathbb{R}_+ \times P \to P$ defined by

$$
\widehat{\mu}(k, \cdot) = \psi^{-1} \circ \overline{\mu} \circ \psi \quad (176)
$$

and

$$
\overline{\mu}(k, f) = kT^*\mu(k^{-1}, f) \quad (177)
$$

for each $k \in \mathbb{R}_+$ and each $f \in T^*Q$. Let

$$
\iota_X: X \to Q \quad (178)
$$
be the canonical injection of a homogeneous submanifold \( X \subset Q \) and let
\[
\mu: \mathbb{R}_+ \times X \to X
\]
be the restriction to \( X \) of the group action (174) in the sense that
\[
\begin{align*}
\begin{array}{c}
\mathbb{R}_+ \times X \\
\mathbb{R}_+ \times Q
\end{array}
\xrightarrow{egin{array}{c}
\mu \\
\mu
\end{array}}
\begin{array}{c}
X \\
Q
\end{array}
\end{align*}
\]
\[
\begin{array}{c}
\mathbb{R}_+ \times X \\
\mathbb{R}_+ \times Q
\end{array}
\xrightarrow{egin{array}{c}
1_{\mathbb{R}_+} \times \iota_X \\
1_{\mathbb{R}_+} \times \iota_X
\end{array}}
\begin{array}{c}
X \\
Q
\end{array}
\]
\]
is a commutative diagram. Let
\[
\begin{array}{c}
R \\
X
\end{array}
\xrightarrow{\rho}
\begin{array}{c}
\mathbb{R} \\
X
\end{array}
\]
be a family of functions homogeneous with respect to a group action
\[
\begin{align*}
\begin{array}{c}
\mathbb{R}_+ \times R \\
\mathbb{R}_+ \times X
\end{array}
\xrightarrow{egin{array}{c}
\nu \\
\mu
\end{array}}
\begin{array}{c}
R \\
X
\end{array}
\end{align*}
\]
\[
\begin{array}{c}
\mathbb{R}_+ \times R \\
\mathbb{R}_+ \times X
\end{array}
\xrightarrow{egin{array}{c}
1_{\mathbb{R}_+} \times \rho \\
1_{\mathbb{R}_+} \times \iota_X
\end{array}}
\begin{array}{c}
R \\
X
\end{array}
\]
of the group \((\mathbb{R}_+, \cdot)\). The generating object
\[
\begin{align*}
\begin{array}{c}
(P, \vartheta) \\
Q
\end{array}
\xrightarrow{\pi}
\begin{array}{c}
R \\
X
\end{array}
\end{align*}
\]
is said to be homogeneuos with respect to the group actions (174) and (182).

The immersed Lagrangian submanifold \( A \) generated by a homogeneous generating object is homogeneous.

15. Reduction of homogeneous generating objects.

Reduction of a generating object
\[
\begin{align*}
\begin{array}{c}
(P, \vartheta) \\
Q
\end{array}
\xrightarrow{\pi}
\begin{array}{c}
R \\
X
\end{array}
\end{align*}
\]
is possible if
1. the fibration \( \rho \) is the composition \( \rho'' \circ \rho' \) of fibrations \( \rho': R \to R' \) and \( \rho'': R' \to X \),
2. the images \( \bar{R} = \rho'(S(F, \rho')) \) and \( \bar{X} = \rho(S(F, \rho')) \) of the critical set \( S(F, \rho') \) are submanifolds of \( R' \) and \( X \) respectively.
(3) the induced mapping

\[ \tilde{\rho}: \tilde{R} \rightarrow \tilde{X} \]
\[ : r' \mapsto \rho''(r') \] (185)

is a differential fibration,

(4) the mapping

\[ \overline{\pi}: S(F, \rho') \rightarrow \tilde{R} \]
\[ : r \mapsto \rho(r) \] (186)

is a differential fibration with connected fibres.

If the generating object is homogeneous with respect to group actions

\[ \mu: \mathbb{R}_+ \times Q \rightarrow Q \] (187)

and

\[ \begin{array}{ccc}
\mathbb{R}_+ \times R & \xrightarrow{\nu} & R \\
1_{\mathbb{R}_+} \times \rho & \downarrow & \rho \\
\mathbb{R}_+ \times X & \xrightarrow{\mu} & X 
\end{array} \] (188)

and there is an action

\[ \nu': \mathbb{R}_+ \times R' \rightarrow R' \] (189)

such that diagrams

\[ \begin{array}{ccc}
\mathbb{R}_+ \times R & \xrightarrow{\nu} & R \\
1_{\mathbb{R}_+} \times \rho' & \downarrow & \rho' \\
\mathbb{R}_+ \times R' & \xrightarrow{\nu'} & R' 
\end{array} \] (190)

and

\[ \begin{array}{ccc}
\mathbb{R}_+ \times R' & \xrightarrow{\nu'} & R' \\
1_{\mathbb{R}_+} \times \rho'' & \downarrow & \rho'' \\
\mathbb{R}_+ \times X & \xrightarrow{\mu} & X 
\end{array} \] (191)

are commutative, then the result of the reduction process is a homogeneous object. It can be proved as in Proposition 10 that the critical set \( S(F, \rho') \) is homogeneous. It follows that \( \tilde{R} \) and \( \tilde{X} \) are homogeneous. Consequently, we have a group action

\[ \begin{array}{ccc}
\mathbb{R}_+ \times \tilde{R} & \xrightarrow{\overline{\nu}} & \tilde{R} \\
1_{\mathbb{R}_+} \times \tilde{\rho} & \downarrow & \tilde{\rho} \\
\mathbb{R}_+ \times \tilde{X} & \xrightarrow{\overline{\mu}} & \tilde{X} 
\end{array} \] (192)
and the function $\tilde{F}: \tilde{R} \rightarrow \mathbb{R}$ satisfies the homogeneity condition $\tilde{F}(\tilde{\nu}(k,r')) = k\tilde{F}(r')$ for each $r' \in \tilde{R}$ and each $k \in \mathbb{R}_+$. The reduced generating object

$$
\begin{array}{ccc}
(P, \vartheta) & \tilde{R} & \tilde{X} \\
\pi & \tilde{\rho} & \tilde{F} \\
Q & \tilde{X}
\end{array}
$$

is homogeneous.

16. Composition of homogeneous generating objects.

Let

$$
\begin{array}{ccc}
(P_1, \vartheta_1) & \pi_1 & (P_2, \vartheta_2) & \pi_2 \\
Q_1 & \rho_2 & Q_2
\end{array}
$$

be special symplectic structures and let

$$\mu_1: \mathbb{R}_+ \times Q_1 \rightarrow Q_1$$

and

$$\mu_2: \mathbb{R}_+ \times Q_2 \rightarrow Q_2$$

be group actions of the group $(\mathbb{R}_+, \cdot)$. We consider a symplectic relation $\Pi_{21}$ from $(P_1, \xi \vartheta_1)$ to $(P_1, \xi \vartheta_1)$, whose graph $A_{21}$ is generated by a generating object

$$
\begin{array}{ccc}
(P_2 \times P_1, \vartheta_2 \ominus \vartheta_1) & \rho_{21} & \tilde{R}_{21} \\
\pi_2 \times \pi_1 & \rho_{21} & \tilde{F}_{21} \rightarrow \mathbb{R} \\
Q_2 \times Q_1 & \tilde{X}_{21} & X_{21}
\end{array}
$$

homogeneous with respect to a group action

$$
\begin{array}{ccc}
\mathbb{R}_+ \times R_{21} & \rho_{21} & R_{21} \\
1_{\mathbb{R}_+} \times \rho_{21} & \rho_{21} & \rho_{21} \\
\mathbb{R}_+ \times X_{21} & \mu_{21} & X_{21}
\end{array}
$$

where

$$\mu_{21}: \mathbb{R}_+ \times X_{21} \rightarrow X_{21}$$

is the restriction of the action

$$\mu_{21}: \mathbb{R}_+ \times Q_2 \times Q_1 \rightarrow Q_2 \times Q_1$$

$$(k, q_2, q_1) \mapsto (\mu_2(k, q_2), \mu_1(k, q_1))$$

(200)

to the submanifold $X_{21}$. Let $A_1$ be a Lagrangian submanifold of $(P_1, d\vartheta_1)$ generated by a generating object.
17. Homogeneous Lagrangian systems.

Let $Q$ be a differential manifold and let $\kappa$ be the action

$$\kappa: \mathbb{R}_+ \times TQ \to TQ$$

$$\kappa: (k, v) \mapsto kv.$$  \hfill (204)

Let

$$\tilde{T}Q = \{ v \in TQ; v \neq 0 \}$$  \hfill (205)

be the tangent bundle of $Q$ with the image of the zero section removed. The set $\tilde{T}Q$ is an homogeneous open submanifold of $TQ$.

A Lagrangian system

$$\left( T^* \mathbb{T} Q, d \mathbb{T} \vartheta Q \right) \xrightarrow{Y} \mathbb{R}$$

is said to be homogeneous with respect to an action

$$\left( \mathbb{R}_+ \times Y \right) \xrightarrow{\rho} Y$$

if $C$ is an homogeneous submanifold of $\tilde{T}Q$,

$$\kappa: \mathbb{R}_+ \times C \to C$$

$$\kappa: (k, v) \mapsto kv.$$  \hfill (208)

is the restriction of the action $\kappa$ to $C$, and

$$L(\rho(k, y)) = kL(y)$$  \hfill (209)
for each $y \in Y$ and each $k \in \mathbb{R}_+$. The Lagrangian family

\[ \begin{array}{ccc}
Y & \xrightarrow{L} & \mathbb{R} \\
\eta & \downarrow & \\
C & \end{array} \] (210)

is said to be a **homogeneous Lagrangian family**.

The action $\kappa$ is lifted to the action

\[ \begin{array}{ccc}
\mathbb{R}_+ \times T\Gamma^*Q & \xrightarrow{\tilde{\kappa}} & T\Gamma^*Q \\
1_{\mathbb{R}_+} \times T\pi_Q & \downarrow & T\pi_Q \\
\mathbb{R}_+ \times TQ & \xrightarrow{\kappa} & TQ \\
\end{array} \] (211)

with the action

\[ \tilde{\kappa}: \mathbb{R}_+ \times T\Gamma^*Q \to T\Gamma^*Q \] (212)

defined by

\[ \tilde{\kappa}(k, \cdot) = \alpha^{-1}_Q \circ \kappa(k, \cdot) \circ \alpha_Q \] (213)

for each $k \in \mathbb{R}_+$. The relation

\[ \tilde{\kappa}(k, w) = kw \] (214)

holds for each $w \in T\Gamma^*Q$ and each $k \in \mathbb{R}_+$.

The dynamics $D$ generated by a homogeneous Lagrangian system is homogeneous.

18. **Homogeneous Hamiltonian systems.**

Let $(P, \omega)$ be a symplectic manifold representing the phase space of a mechanical system

\[ \begin{array}{ccc}
(TP, i_T\omega) & \xrightarrow{-H} & \mathbb{R} \\
\tau_P & \downarrow & \zeta \\
P & \xrightarrow{i_K} & K \\
\end{array} \] (215)

be a Hamiltonian system. Let

\[ \sigma: \mathbb{R}_+ \times Z \to Z \] (216)

be an action of the group $(\mathbb{R}_+, \cdot)$ such that $\zeta \circ \sigma(k, \cdot) = \zeta$ for each $k \in \mathbb{R}_+$. The Hamiltonian system (215) is said to be **homogeneous** with respect to the group action $\sigma$ if

\[ H(\sigma(k, z)) = kH(z) \] (217)

for each $z \in Z$ and each $k \in \mathbb{R}_+$. The Hamiltonian family

\[ \begin{array}{ccc}
Z & \xrightarrow{H} & \mathbb{R} \\
\zeta & \downarrow & \\
K & \end{array} \] (218)

is said to be a **homogeneous Hamiltonian family**.

The same concepts of homogeneity apply to a Hamiltonian system.
based on a phase space \((T^*Q, d\vartheta_Q)\).

The dynamics \(D\) generated by a homogeneous Hamiltonian system is homogeneous.

19. **Homogeneous Legendre transformations.**

The generating objects

\[
(TT^*Q \times TT^*Q, i_T d\vartheta_Q \ominus d_T \vartheta_Q) \xrightarrow{\tau_{TT^*Q} \times T\pi_Q} T^*Q \times TQ \xrightarrow{T^*Q \times \langle , \rangle} \mathbb{R}
\]  

(220)

and

\[
(TT^*Q \times TT^*Q, d_T \vartheta_Q \ominus i_T d\vartheta_Q) \xrightarrow{T\pi_Q \times T_{TT^*Q}} TQ \times T^*Q \xrightarrow{TQ \times \langle , \rangle^\sim} \mathbb{R}
\]  

(221)

are homogeneous with respect to the group action

\[
\kappa: \mathbb{R}_+ \times TQ \to TQ
\]

\[(k, v) \mapsto kv\]  

(222)

combined with the trivial action of \(\mathbb{R}_+\) in \(T^*Q\) since

\[
\langle p, kv \rangle = k \langle p, v \rangle
\]  

(223)

for each \(k \in \mathbb{R}_+\). It follows that a homogeneous Lagrangian system will be transformed by the Legendre transformation in a homogeneous Hamiltonian system and a homogeneous Hamiltonian system will be transformed by the inverse Legendre transformation in a homogeneous Lagrangian system. The graphs of Legendre relations and inverse Legendre relations are homogeneous with respect to the relevant group actions.

20. **Examples of homogeneous systems.**

Two examples of homogeneous system in the affine space-time of special relativity will be discussed.

An **affine space** is a triple \((Q, V, \sigma)\), where \(Q\) is a set, \(V\) is a real vector space of finite dimension and \(\sigma\) is a mapping \(\sigma: Q \times Q \to V\) such that

1. \(\sigma(q_3, q_2) + \sigma(q_2, q_1) + \sigma(q_1, q_3) = 0\);  
2. the mapping \(\sigma(\cdot, q): Q \to V\) is bijective for each \(q \in Q\).

The tangent bundle \(TQ\) of an affine space is identified with the product \(Q \times V\) and the cotangent bundle \(T^*Q\) is identified with \(Q \times V^*\). The space \(\overset{\circ}{V}\) is identified with \(Q \times \overset{\circ}{V}\), where

\[
\overset{\circ}{V} = \{v \in V; v \neq 0\}
\]  

(224)
Spaces $TT^*Q$ and $TTT^*Q$ are identified with $Q \times V^* \times V \times V^*$ and $Q \times V^* \times V \times V^* \times V \times V^* \times V \times V^*$ respectively. The evaluations of the forms $d_T\vartheta$ and $i_{T}d\vartheta$ on an element $(q,p,\dot{q},\dot{p},\delta q,\delta p,\delta \dot{q},\delta \dot{p})$ is represented by

$$\langle d_T\vartheta, (q,p,\dot{q},\dot{p},\delta q,\delta p,\delta \dot{q},\delta \dot{p}) \rangle = \langle \dot{p}, \delta q \rangle + \langle p, \delta \dot{q} \rangle \quad (225)$$

and

$$\langle i_{T}d\vartheta, (q,p,\dot{q},\dot{p},\delta q,\delta p,\delta \dot{q},\delta \dot{p}) \rangle = \langle \dot{p}, \delta q \rangle - \langle \delta p, \dot{q} \rangle. \quad (226)$$

**Example 5. Dynamics of a relativistic particle.** Let $(Q,V,\sigma,g)$ be the affine space-time of special relativity. The triple $(Q,V,\sigma)$ is an affine space and $g:V \rightarrow V^*$ is the Minkowski metric. The dynamics of a free particle of mass $m$ is generated by the Lagrangian system

$$\left( TT^*Q, d_T\vartheta Q \right) \quad (227)$$

where

$$C = Q \times \{ \dot{q} \in V; \langle g(\dot{q}), \dot{q} \rangle > 0 \} \quad (228)$$

and

$$L: C \rightarrow \mathbb{R} : (q,\dot{q}) \mapsto m\sqrt{\langle g(\dot{q}), \dot{q} \rangle}. \quad (229)$$

The dynamics is the set

$$D = \left\{ (q,p,\dot{q},\dot{p}) \in TT^*Q; \langle g(\dot{q}), \dot{q} \rangle > 0, p = \frac{mg(\dot{q})}{\sqrt{\langle g(\dot{q}), \dot{q} \rangle}}, \dot{p} = 0 \right\}. \quad (230)$$

The set $C$ is a homogeneous submanifold of $TQ$ and $L(q,k\dot{q}) = kL(q,\dot{q})$. It follows that the Lagrangian system (227) is homogeneous.

The Legendre transformation applied to the Lagrangian system (227) leads to a Hamiltonian system

$$\left( TT^*Q, i_{T}d\vartheta Q \right) \quad (231)$$

with

$$Z = T^*Q \times \{ v \in V; \langle g(v), v \rangle > 0 \}, \quad (232)$$

$$\zeta: Z \rightarrow T^*Q \quad (233)$$

and

$$H: Z \rightarrow \mathbb{R} : (q,p,v) \mapsto \langle p,v \rangle - m\sqrt{\langle g(v), v \rangle}. \quad (234)$$
The Hamiltonian system (231) can be simplified. The fibration \(\zeta\) can be represented as a composition \(\zeta'' \circ \zeta'\) of fibrations

\[
\zeta': Z \rightarrow Z' \\
: (q, p, v) \mapsto (q, p, \|v\|),
\]

(235)

and

\[
\zeta'': Z' \rightarrow T^*Q \\
: (q, p, \lambda) \mapsto (q, p),
\]

(236)

where

\[
Z' = T^*Q \times \mathbb{R}_+ 
\]

(237)

and

\[
\|v\| = \sqrt{\langle g(v), v \rangle}.
\]

(238)

Equating to zero the derivative of \(H\) along the fibres of \(\zeta'\) we obtain the relation

\[
p = \mu g(v)
\]

(239)

valid for some values of the Lagrange multiplier \(\mu\). It follows from this relation that the covector \(p\) is in the set

\[
\{p \in V^*; \langle p, g^{-1}(p) \rangle > 0\}
\]

(240)

and that

\[
\{(q, p, v) \in Z; \|v\|p = \pm \|p\|g(v)\}
\]

(241)

is the critical set \(S(H, \zeta')\). We have denoted by \(\|p\|\) the norm \(\sqrt{\langle p, g^{-1}(p) \rangle}\) of \(p\). The images \(\tilde{K} = \zeta(S(H, \zeta'))\) and \(\tilde{Z} = \zeta'(S(H, \zeta'))\) of the critical set \(S(H, \zeta')\) are the sets

\[
\tilde{K} = Q \times \{p \in V^*; \langle p, g^{-1}(p) \rangle > 0\}
\]

(242)

and

\[
\tilde{Z} = \tilde{K} \times \mathbb{R}_+.
\]

(243)

The mapping

\[
\tilde{\zeta}: \tilde{Z} \rightarrow \tilde{K} \\
: (q, p, \lambda) \mapsto (q, p).
\]

(244)

is a differential fibration. The critical set \(S(H, \zeta')\) is the union of images of two local sections

\[
\xi_+: \tilde{Z} \rightarrow Z \\
: (q, p, \lambda) \mapsto \left(q, p, \frac{\lambda}{\|p\|} g^{-1}(p)\right)
\]

(245)

and

\[
\xi_-: \tilde{Z} \rightarrow Z \\
: (q, p, \lambda) \mapsto \left(q, p, -\frac{\lambda}{\|p\|} g^{-1}(p)\right)
\]

(246)

of the fibration \(\zeta'\). We have two families of functions.
The functions

\[ \tilde{H}_- : \tilde{Z} \to \mathbb{R} \]
\[ : (q, p, \lambda) \mapsto -\lambda (\|p\| + m) \]  

and

\[ \tilde{H}_+ : \tilde{Z} \to \mathbb{R} \]
\[ : (q, p, \lambda) \mapsto \lambda (\|p\| - m) \]  

are the compositions \( H \circ \xi_- \) and \( H \circ \xi_+ \) respectively. It is easily seen that the family \((247)\) generates an empty set. The dynamics of the particle is generated by the reduced Hamiltonian system

\[ (TT^*Q, i_Td\theta_Q) \]
\[ \xrightarrow{\tau_{T^*Q}} \tilde{Z} \]
\[ \tilde{\zeta} \]
\[ \tilde{K} \]
\[ \tilde{H}_+ \]
\[ \mathbb{R} \]

\[ (TT^*Q, i_Td\theta_Q) \]
\[ \xrightarrow{\tau_{T^*Q}} \tilde{Z} \]
\[ \tilde{\zeta} \]
\[ \tilde{K} \]
\[ \tilde{H}_+ \]
\[ \mathbb{R} \]

\[ \tau_{T^*Q} \]
\[ T^*Q \]
\[ \tilde{\zeta} \]
\[ \tilde{K} \]

\[ (TT^*Q, i_Td\theta_Q) \]
\[ \xrightarrow{\tau_{T^*Q}} \tilde{Z} \]
\[ \tilde{\zeta} \]
\[ \tilde{K} \]
\[ \tilde{H}_+ \]
\[ \mathbb{R} \]

\[ \tau_{T^*Q} \]
\[ T^*Q \]
\[ \tilde{\zeta} \]
\[ \tilde{K} \]
\[ \tilde{H}_+ \]
\[ \mathbb{R} \]

\[ \tau_{T^*Q} \]
\[ T^*Q \]
\[ \tilde{\zeta} \]
\[ \tilde{K} \]
\[ \tilde{H}_+ \]
\[ \mathbb{R} \]

\[ \tau_{T^*Q} \]
\[ T^*Q \]
\[ \tilde{\zeta} \]
\[ \tilde{K} \]
\[ \tilde{H}_+ \]
\[ \mathbb{R} \]

\[ \tau_{T^*Q} \]
\[ T^*Q \]
\[ \tilde{\zeta} \]
\[ \tilde{K} \]
\[ \tilde{H}_+ \]
\[ \mathbb{R} \]

\[ \tau_{T^*Q} \]
\[ T^*Q \]
\[ \tilde{\zeta} \]
\[ \tilde{K} \]
\[ \tilde{H}_+ \]
\[ \mathbb{R} \]

\[ \tau_{T^*Q} \]
\[ T^*Q \]
\[ \tilde{\zeta} \]
\[ \tilde{K} \]
\[ \tilde{H}_+ \]
\[ \mathbb{R} \]

\[ \tau_{T^*Q} \]
\[ T^*Q \]
\[ \tilde{\zeta} \]
\[ \tilde{K} \]
\[ \tilde{H}_+ \]
\[ \mathbb{R} \]

\[ \tau_{T^*Q} \]
\[ T^*Q \]
\[ \tilde{\zeta} \]
\[ \tilde{K} \]
\[ \tilde{H}_+ \]
\[ \mathbb{R} \]

\[ \tau_{T^*Q} \]
\[ T^*Q \]
\[ \tilde{\zeta} \]
\[ \tilde{K} \]
\[ \tilde{H}_+ \]
\[ \mathbb{R} \]

\[ \tau_{T^*Q} \]
\[ T^*Q \]
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\[ \tilde{K} \]
\[ \tilde{H}_+ \]
\[ \mathbb{R} \]

\[ \tau_{T^*Q} \]
\[ T^*Q \]
\[ \tilde{\zeta} \]
\[ \tilde{K} \]
\[ \tilde{H}_+ \]
\[ \mathbb{R} \]

\[ \tau_{T^*Q} \]
\[ T^*Q \]
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\[ \tilde{K} \]
\[ \tilde{H}_+ \]
\[ \mathbb{R} \]

\[ \tau_{T^*Q} \]
\[ T^*Q \]
\[ \tilde{\zeta} \]
\[ \tilde{K} \]
\[ \tilde{H}_+ \]
\[ \mathbb{R} \]

\[ \tau_{T^*Q} \]
\[ T^*Q \]
\[ \tilde{\zeta} \]
\[ \tilde{K} \]
\[ \tilde{H}_+ \]
\[ \mathbb{R} \]

\[ \tau_{T^*Q} \]
\[ T^*Q \]
\[ \tilde{\zeta} \]
\[ \tilde{K} \]
\[ \tilde{H}_+ \]
\[ \mathbb{R} \]

\[ \tau_{T^*Q} \]
\[ T^*Q \]
\[ \tilde{\zeta} \]
\[ \tilde{K} \]
\[ \tilde{H}_+ \]
\[ \mathbb{R} \]

\[ \tau_{T^*Q} \]
\[ T^*Q \]
\[ \tilde{\zeta} \]
\[ \tilde{K} \]
\[ \tilde{H}_+ \]
\[ \mathbb{R} \]

\[ \tau_{T^*Q} \]
\[ T^*Q \]
\[ \tilde{\zeta} \]
\[ \tilde{K} \]
\[ \tilde{H}_+ \]
\[ \mathbb{R} \]

\[ \tau_{T^*Q} \]
\[ T^*Q \]
\[ \tilde{\zeta} \]
\[ \tilde{K} \]
\[ \tilde{H}_+ \]
\[ \mathbb{R} \]

\[ \tau_{T^*Q} \]
\[ T^*Q \]
\[ \tilde{\zeta} \]
\[ \tilde{K} \]
\[ \tilde{H}_+ \]
\[ \mathbb{R} \]
\[
\begin{array}{c}
\text{(TT}^*Q, d_T \vartheta_Q) \xrightarrow{\eta} \tilde{Y} \xrightarrow{\tilde{L}} \mathbb{R} \\
\begin{array}{c}
T_{\pi_Q} \xrightarrow{\xi \tilde{C}} \tilde{C} \\
TQ \xrightarrow{\eta} \tilde{C}
\end{array}
\end{array}
\]

with

\[
\tilde{C} = TQ, \quad \tilde{Y} = \left\{ (q, \dot{q}, r, \lambda) \in Q \times V \times V^* \times \mathbb{R}_+; \; \langle r, g^{-1}(r) \rangle > 0 \right\}
\]

\[
\tilde{\eta}: \tilde{Y} \to \tilde{C} \\
(q, \dot{q}, r, \lambda) \mapsto (q, \dot{q})
\]

and

\[
\tilde{L}: \tilde{Y} \to \mathbb{R} \\
(q, \dot{q}, r, \lambda) \mapsto \langle r, \dot{q} \rangle - \lambda (\|r\| - m).
\]

The image \(\tilde{\eta}(S(\tilde{L}, \tilde{\eta}))\) of the critical set

\[
S(\tilde{L}, \tilde{\eta}) = \left\{ (q, \dot{q}, r, \lambda) \in \tilde{Y}; \; \|r\| = m, m\dot{q} = \lambda g^{-1}(r) \right\}
\]

is the set

\[
C = Q \times \{ \dot{q} \in V; \; \langle g(\dot{q}), \dot{q} \rangle > 0 \}.
\]

The critical set is the image of the local section

\[
\xi: C \to \tilde{Y} \\
(q, \dot{q}) \mapsto \left( q, \dot{q}, m \frac{\dot{q}}{\|\dot{q}\|} g(\dot{q}), \|\dot{q}\| \right)
\]

The composition \(\tilde{L} \circ \xi\) is the function

\[
L: C \to \mathbb{R} \\
(q, \dot{q}) \mapsto m \sqrt{\langle g(\dot{q}), \dot{q} \rangle}.
\]

The reduced Lagrange system is the original Lagrange system (227).

The Hamiltonian systems (231) and (250) are homogeneous. The graph of the second Legendre relation \(\Lambda_2(L)\) is the homogeneous set

\[
\text{graph}(\Lambda_2(L)) = \left\{ (q, p, q', \dot{q}) \in T^*Q \times TQ; \; q' = q, p = \frac{mg(\dot{q})}{\sqrt{\langle g(\dot{q}), \dot{q} \rangle}} \right\}.
\]

**Example 6. Space-time formulation of geometric optics.** The dynamics of light rays in affine Minkowski space-time \((Q, V, \sigma, g)\) is generated by the Lagrangian system

\[
\begin{array}{c}
(TT^*Q, d_T \vartheta_Q) \xrightarrow{\eta} Y \xrightarrow{L} \mathbb{R} \\
\begin{array}{c}
T_{\pi_Q} \xrightarrow{\xi C} C \\
TQ \xrightarrow{\eta} C
\end{array}
\end{array}
\]
with

\[ C = \mathcal{T} Q, \quad \text{(266)} \]
\[ Y = C \times \mathbb{R}_+, \quad \text{(267)} \]

\[ \eta: Y \to C \]
\[ : (q, \dot{q}, \mu) \mapsto (q, \dot{q}), \quad \text{(268)} \]

and

\[ L: Y \to \mathbb{R} \]
\[ : (q, \dot{q}, \mu) \mapsto \frac{1}{2\mu} \langle g(\dot{q}), \dot{q} \rangle. \quad \text{(269)} \]

The dynamics is the set

\[ D = \left\{ (q, p, \dot{q}, \dot{p}) \in \mathcal{T}^* Q; \quad \langle g(\dot{q}), \dot{q} \rangle = 0, \exists \mu \in \mathbb{R}_+ p = \frac{1}{\mu} g(\dot{q}), \dot{p} = 0 \right\}. \quad \text{(270)} \]

The Lagrangian system (265) is homogeneous.

The result of the Legendre transformation applied to the Lagrangian system (265) is the Hamiltonian system

\[
\begin{array}{ccc}
\mathcal{T}^* Q & \xrightarrow{\tau_{\mathcal{T}^* Q}} & Z \\
\mathcal{T}^* Q & \xrightarrow{\zeta} & \mathcal{T}^* Q \\
& \xrightarrow{\zeta'} & \mathcal{T}^* Q \\
& \xrightarrow{\zeta''} & \mathcal{T}^* Q \\
& \xrightarrow{H} & \mathbb{R}
\end{array}
\quad \text{(271)}
\]

with

\[ Z = \mathcal{T}^* Q \times \overset{\circ}{V} \times \mathbb{R}_+, \quad \text{(272)} \]

\[ \zeta: Z \to \mathcal{T}^* Q \\
\quad : (q, p, v, \mu) \mapsto (q, p), \quad \text{(273)} \]

and

\[ H: Z \to \mathbb{R} \\
\quad : (q, p, v, \mu) \mapsto \langle p, v \rangle - \frac{1}{2\mu} \langle g(v), v \rangle. \quad \text{(274)} \]

The Hamiltonian system (271) can be simplified as in the case of a relativistic particle with mass \( m > 0 \). The fibration \( \zeta \) is be represented as a composition \( \zeta'' \circ \zeta' \) of fibrations

\[ \zeta': Z \to Z' \\
\quad : (q, p, v, \mu) \mapsto (q, p, \mu), \quad \text{(275)} \]

and

\[ \zeta'': Z' \to \mathcal{T}^* Q \\
\quad : (q, p, \mu) \mapsto (q, p), \quad \text{(276)} \]
where
\[ Z' = \mathbb{T}^*Q \times \mathbb{R}_+. \] (277)

We have
\[ S(H, \zeta') = \{(q, p, v, \mu) \in Z; \mu p = g(v)\} \] (278)

and
\[ \tilde{Z} = \zeta'(S(H, \zeta')) = \{(q, p, \mu) \in Z'; p \neq 0\}. \] (279)

The critical set \( S(H, \zeta') \) is the image of the local section
\[ \zeta: \tilde{Z} \to Z \]
\[ : (q, p, \mu) \mapsto (q, p, \mu g^{-1}(p), \mu) \] (280)

of \( \zeta' \). The reduced system is the Hamiltonian system
\[ \left( \mathbb{T}^*Q, i_{\pi_{\mathbb{T}^*Q}} d\vartheta_{\zeta} \right) \]
\[ \tilde{Z} \xrightarrow{\tilde{H}} \mathbb{R} \] (281)

with
\[ \tilde{\zeta}: \tilde{Z} \to \tilde{K} \]
\[ : (q, p) \mapsto (q, p) \] (282)

and
\[ \tilde{H}: \tilde{Z} \to \mathbb{R} \]
\[ : (q, p, \mu) \mapsto \frac{\mu}{2} (p, g^{-1}(p)). \] (283)

No further simplification is possible.

The application of the inverse Legendre transformation to the Hamiltonian system (281) results in the original Lagrangian system (265).

The Hamiltonian systems (271) and (281) are homogeneous. The graph of the first Legendre relation \( \Lambda_1(L) \) is the homogeneous set
\[ \text{graph}(\Lambda_1(L)) = \{(q, p, \dot{q}, \dot{\mu}) \in \mathbb{T}^*Q \times Y; \dot{q} = g(\dot{q}), \dot{\mu} = \mu p = g(\dot{q})\}. \] (285)

The graph of the second Legendre relation \( \Lambda_2(L) \) is the set
\[ \text{graph}(\Lambda_2(L)) = \{(q, p, \dot{q}, \dot{\mu}) \in \mathbb{T}^*Q \times \mathbb{T}Q; \dot{q} = q, g(\dot{q}), \dot{\mu} = \mu p = g(\dot{q})\}. \] (286)

The graph of the first inverse Legendre relation \( \Omega_1(H) \) is the set
\[ \text{graph}(\Omega_1(H)) = \{(q, p, \dot{q}, \dot{\mu}) \in \mathbb{T}Q \times \tilde{Z}; \dot{q} = q, \dot{p}, g^{-1}(p) = 0, \dot{\mu} = \mu g^{-1}(p)\}. \] (287)

21. References.

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