THE NAVARRO REFINEMENT OF THE MCKAY CONJECTURE FOR
FINITE GROUPS OF LIE TYPE IN DEFINING CHARACTERISTIC

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Abstract. In this paper we verify Navarro’s refinement of the McKay conjecture for quasi-simple groups of Lie type $G$ and the prime $p$, where $p$ is the defining characteristic of $G$. Navarro’s refinement takes into account the action of specific Galois automorphisms on the characters present in the McKay conjecture [11]. Our proof of this special case of the conjecture relies on a character correspondence which Maslowski constructed [10] in the context of the inductive McKay conditions by Isaacs, Malle and Navarro [7].

1. Introduction

For a finite group $G$, a prime $p$ and a Sylow $p$-subgroup $P$ of $G$ the McKay conjecture asserts that there exists a bijection between the set of $p'$-degree characters $\text{Irr}_{p'}(N_G(P))$ of $N_G(P)$ and the set of $p'$-degree characters $\text{Irr}_{p'}(G)$ of $G$. However, Navarro suggests that there should exist a bijection between these sets of characters which is compatible with certain Galois automorphisms. He proposes the following refinement of the McKay conjecture, see [11, Conjecture A].

Conjecture 1.1. Let $\sigma \in \text{Gal}(\mathbb{Q}|G|/\mathbb{Q})$ be an $(e,p)$-Galois automorphism for a nonnegative integer $e$, see Definition 2.1. Then there exists a bijection between the $\sigma$-invariant characters of $\text{Irr}_{p'}(N_G(P))$ and the $\sigma$-invariant characters of $\text{Irr}_{p'}(G)$.

If this conjecture is true it implies several consequences which are of independent interest. One of consequences was proved by Navarro, Tiep and Turull, see [13, Theorem A].

For the original McKay conjecture a reduction theorem was proved by Isaacs, Malle and Navarro, see [7, Theorem B]. This theorem asserts that the McKay conjecture is true for all finite groups and the prime $p$ if all nonabelian simple groups satisfy the so-called inductive McKay conditions for the prime $p$. In view of a possible reduction theorem of Conjecture 1.1 extending the earlier reduction theorem of Isaacs, Malle and Navarro, it is important to study and verify Conjecture 1.1 for quasi-simple groups. We contribute to this program by proving the following theorem.

Theorem 1.2. Let $G$ be a simple algebraic group of simply connected type defined over an algebraic closure of $\mathbb{F}_p$. Let $F$ be a Frobenius endomorphism of $G$ and $\sigma \in \text{Gal}(\mathbb{Q}|G^F|/\mathbb{Q})$ be an $(e,p)$-Galois automorphism. Suppose that $p$ is a good prime for $G$ in the sense of [2, Section 1.14]. Then there exists a bijection

$$f : \text{Irr}_{p'}(B^F)^\sigma \to \text{Irr}_{p'}(G^F)^\sigma.$$
Moreover, for every central character $\lambda \in \text{Irr}(\mathbb{Z}(G^F))$ the map $f$ restricts to a bijection $\text{Irr}_{p'}(B^F \mid \lambda) \rightarrow \text{Irr}_{p'}(G^F \mid \lambda)$.

Observe that in the situation of the theorem, $B^F$ is exactly the normalizer of a Sylow $p$-subgroup of $G^F$. We also remark that we can prove Theorem 1.2 for bad primes $p$ if the center of $G^F$ is trivial, see Corollary 5.13 for a precise statement.

The proof of Theorem 1.2 is based on Maslowski’s work on the inductive McKay-conditions for simple groups of Lie type in defining characteristic. Maslowski constructs a certain automorphism-equivariant bijection for the $p'$-characters of the universal covering group of a finite simple group of Lie type defined over a field of characteristic $p$. Later Späh used Maslowski’s result to prove that these groups satisfy the inductive McKay conditions for the prime $p$, see [15, Theorem 1.1].

The structure of our paper is as follows. In Section 2 we introduce some notation. In Section 3 we recall some basic facts about the representation theory of finite groups of Lie type. We discuss the action of Galois automorphisms on Lusztig series and recall a description of irreducible $p'$-characters originally due to Green, Lehrer and Lusztig. In Section 4 we recall the McKay bijection due to Maslowski [10]. In Section 5 we use the results of the two previous sections to provide a proof of Theorem 1.2.

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**2. Notation**

2.1. **Rings and Fields.** Let $p$ be a prime and $q$ be an integral power of $p$. We let $k$ be an algebraic closure of $\mathbb{F}_p$. Let $\ell$ be a prime different from $p$ and denote by $K$ an algebraic closure of the field of $\ell$-adic integers. Denote by $(\mathbb{Q}/\mathbb{Z})_{p'}$ the subgroup of elements of the abelian group $\mathbb{Q}/\mathbb{Z}$ whose order is not divisible by $p$. We fix once and for all an isomorphism $k^\times \rightarrow (\mathbb{Q}/\mathbb{Z})_{p'}$ and an injective morphism $k^\times \rightarrow K^\times$.

2.2. **Characters of finite groups.** If $Y$ is a finite group we denote by $\text{Irr}(Y)$ the set of irreducible $K$-valued characters of $Y$. For $K$-valued characters we follow the notation of [8]. Moreover, if $X$ is a normal subgroup of $Y$ and $\vartheta \in \text{Irr}(X)$ we denote by $\text{Irr}(Y \mid \vartheta)$ the set of irreducible characters of $Y$ lying above $\vartheta$. Similarly, if $\chi \in \text{Irr}(Y)$ we mean by $\text{Irr}(X \mid \chi)$ the set of irreducible characters of $X$ lying below $\chi$.

2.3. **Groups of Lie type.** Let $G$ be a connected reductive group defined over $\mathbb{F}_q$ via a Frobenius endomorphism $F: G \rightarrow G$. We fix a maximal $F$-stable torus $T$ of $G$ contained in an $F$-stable Borel subgroup $B$. Let $U$ be the unipotent radical of $B$. We denote by $\Phi$ the root system of $G$ with respect to the torus $T$ and by $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ the set of simple roots of $\Phi$ with respect to $T \subseteq B$. We let $\Phi^+$ be the set of positive roots and $\Phi^\vee$ the set of coroots.

The action of the Frobenius endomorphism $F$ induces an automorphism $\tau$ of finite order on the character group $X(T)$. We let $w$ the order of the automorphism $\tau$. We say that $F$ is a standard Frobenius and write $F = F_q$ if the action of $\tau$ on $X(T)$ is trivial, i.e., $w = 1$. 


2.4. Finite groups and Galois automorphisms. Let $Y$ be a finite group and $m = |Y|$. We denote by $\mathbb{Q}_m$ the $m$-th cyclotomic field. By Brauer’s theorem the Galois group $\text{Gal}(\mathbb{Q}_m/\mathbb{Q})$ acts on the set of irreducible characters $\text{Irr}(Y)$. Let $\sigma \in \text{Gal}(\mathbb{Q}_m/\mathbb{Q})$ be a Galois automorphism. For a generalized character $\chi \in Z\text{Irr}(Y)$ we let $\chi^\sigma \in Z\text{Irr}(Y)$ be the generalized character defined by $\chi^\sigma(y) = \sigma(\chi(y))$, for $y \in Y$. If $M \subseteq \text{Irr}(Y)$ we let $M^\sigma$ be the set of characters of $M$ stabilized by the action of $\sigma$. Navarro considers the following class of Galois automorphisms:

**Definition 2.1.** Let $e$ be a nonnegative integer and $p$ be a prime number. Then a Galois automorphism $\sigma \in \text{Gal}(\mathbb{Q}_m/\mathbb{Q})$ is called $(e, p)$-Galois automorphism if $\sigma$ sends any $p'$-root of unity $\zeta \in \mathbb{Q}_m$ to $\zeta^{p^e}$.

3. Representation theory of groups of Lie type

3.1. Lusztig series and Galois automorphisms. Let $T'$ be a maximal $F$-stable torus of $G$. We denote by $R^\mathbb{G}_T(\theta)$ the Deligne–Lusztig character associated to the character $\theta \in \text{Irr}(T'^F)$, see [1] Definition 11.1.

We let $(G^*, T^*, F^*)$ be a pair in duality with $(G, T, F)$ as in [1] Definition 13.10. This together with the choices made in [2] gives rise to a bijection between the set of $G^F$-conjugacy classes of pairs $(T', \theta)$ where $T'$ is an $F$-stable maximal torus of $G$ and $\theta \in \text{Irr}(T'^F)$ with the set of $G^{F^*}$-conjugacy classes of pairs $(T'^*, s)$ where $s \in G^{F^*}$ is a semisimple element and $T'^*$ is an $F^*$-stable maximal torus with $s \in T'^*$, see [1] Proposition 13.13.

As usually, we write $R^\mathbb{G}_{T'}(s)$ for the character $R^\mathbb{G}_T(\theta)$ if $(T', \theta)$ is in duality with $(T'^*, s)$. If $s \in G^{F^*}$ is a semisimple element we denote by $(s)$ its $G^{F^*}$-conjugacy class. We denote by $\mathcal{E}(G^F, (s)) \subseteq \text{Irr}(G^F)$ its rational Lusztig series. We have the following lemma, see the proof of [12] Lemma 9.1.

**Lemma 3.1.** Let $\sigma \in \text{Gal}(\mathbb{Q}_m/\mathbb{Q})$ with $m = |G^F|$ and $\sigma(\xi) = \xi^k$ for a primitive $m$-th root of unity $\xi \in \mathbb{Q}_m$. Then $R^\mathbb{G}_T(\theta) = R^\mathbb{G}_T(\theta^k)$ for any $\theta \in \text{Irr}(T'^F)$. In particular, we have $\sigma(\mathcal{E}(G^F, (s))) = \mathcal{E}(G^F, (s^k))$.

To each semisimple conjugacy class $(s)$ of $G^{F^*}$ we associate a character $\chi(s) \in \mathbb{Z}_{\geq 0} \text{Irr}(G^F)$, as in [1] Definition 14.40. If the center of $G$ is connected then $\chi(s)$ is an irreducible character by [1] Corollary 14.47(a) and we have $\chi(s) \in \mathcal{E}(G^F, (s))$.

3.2. The duality functor. We consider the Alvis–Curtis duality map $D_G : \mathbb{Z}\text{Irr}(G^F) \rightarrow \mathbb{Z}\text{Irr}(G^F)$, see [1] Chapter 8]. Note that $D_G$ is an involution and an isometry on the space $\mathbb{Z}\text{Irr}(G^F)$, see [1] Corollary 8.14] resp. [1] Proposition 8.10.

3.3. Gelfand–Graev characters. In order to introduce the Gelfand–Graev characters of $G$ we shall proceed as in the proof of [3] Theorem 2.4]. The Frobenius endomorphism $F$ of $G$ induces an automorphism $\tau$ of the character group $X(T)$. Since $T$ is a maximally split torus it follows by [3] Proposition 22.2] that $\tau$ stabilizes the set of positive roots $\Phi^+$ and the set of simple roots $\Delta$. Hence, $\tau$ acts naturally on the index set of $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. Thus, we have a partition

$$\{1, \ldots, n\} = A_1 \cup \cdots \cup A_r$$

of the index set of $\Delta$ into its $\tau$-orbits. For each $A_i$ we fix a representative $a_i \in A_i$. For $\alpha \in \Phi$ we denote by $U_\alpha$ the root subgroup of $G$ associated to the root $\alpha \in \Phi$. We denote by $U_{A_i}$,
$i = 1, \ldots, r$, the product in $U/[U,U]$ of the root subgroups $U_{\alpha_j}$, $j \in A_i$. By Lemma 3.2 we have

$$U^F/[U,U]^F \cong \prod_{i=1}^r U_{A_i}^F.$$ 

For each $\alpha \in \Phi$ there is an isomorphism $x_\alpha : (k,+) \to U_\alpha$ with $F(x_\alpha(a)) = x_{\tau(\alpha)}(a^q)$ for all $a \in k$ and all $\alpha \in \Phi$. These maps induce an isomorphism $x_i : (\mathbb{F}_{q^{A_i}}, +) \to U_{A_i}^F$ given by

$$x_i(a) = \prod_{k=0}^{|A_i|-1} x_{r_k \alpha_i}(a^{q^k}).$$

Now fix a character $\phi_0 \in \text{Irr}((\mathbb{F}_{qN}, +))$, where $|A_i|$ divides $N$ for all $i = 1, \ldots, r$, such that the restriction of $\phi_0$ to $(\mathbb{F}_q, +)$ is nontrivial. Then any character $\phi_i \in \text{Irr}(U_{A_i}^F)$ is given by $\phi_i(x_i(a)) = \phi_0(c_i a)$ for all $a \in \mathbb{F}_{q^{A_i}}$ and some $c_i \in \mathbb{F}_{q^{A_i}}$. Consequently, any irreducible character $\phi \in \text{Irr}((U/[U,U])^F)$ is of the form $\phi = \prod_{i=1}^r \phi_i$.

**Lemma 3.2.** Let $\gamma : \text{Irr}((U/[U,U])^F) \to \prod_{i=1}^r \mathbb{F}_{q^{A_i}}$ be the map given by

$$\phi = \prod_{i=1}^r \phi_i \mapsto (c_1, \ldots, c_r),$$

where the $c_i \in \mathbb{F}_{q^{A_i}}$ are such that $\phi_i(x_i(a)) = \phi_0(c_i a)$ for all $a \in \mathbb{F}_{q^{A_i}}$. Then $\gamma$ is a bijection.

Let $\iota : G \hookrightarrow \tilde{G}$ be an extension of $G$ by a central torus such that $\tilde{G}$ is a connected reductive group. One possible way to construct $\tilde{G}$ is described after the proof of Corollary 14.47. We have a natural isomorphism $H^1(F, Z(G)) \cong \tilde{G}^F/G^F Z(\tilde{G})^F$ by Corollary 1.2 and Proposition 1.5. Let $\psi_1 = \gamma^{-1}(1, \ldots, 1)$ or more concretely,

$$\psi_1 = \prod_{i=1}^r \phi_i,$$

with $\phi_i(x_i(a)) = \phi_0(a)$ for $a \in \mathbb{F}_{q^{A_i}}$.

Denote $\Gamma_1 = \psi_1^{-1} G^F$. For $z \in H^1(F, Z(G)) \cong \tilde{G}^F/G^F Z(\tilde{G})^F$ we take a representative $g_z \in \tilde{G}^F$ and define the Gelfand–Graev character associated to the class $z$ by $\Gamma_z = g_z \Gamma_1$.

**3.4. The $p'$-characters of $\tilde{G}^F$ and $G^F$.** We want to describe the $p'$-characters of $\tilde{G}^F$. Since the center of $\tilde{G}$ is connected it follows that $\tilde{G}$ has a unique Gelfand–Graev character which we will denote by $\Gamma$. If the prime $p$ is good for $G$ then the $p'$-characters of $\tilde{G}^F$ are precisely the irreducible constituents of $D_G(\Gamma)$, see Proposition 8.3.4. The following lemma is a consequence of this fact.

**Lemma 3.3.** Let $\tilde{G}$ be a connected reductive algebraic group with connected center. Let $p$ be a good prime for the group $\tilde{G}$. Then $\chi \in \text{Irr}(\tilde{G}^F)$ is a $p'$-character if and only if $\chi = \varepsilon D_{\tilde{G}}(\chi(\tilde{s}))$ for some semisimple conjugacy class $\tilde{s}$ of $\tilde{G}^{k^{F^*}}$ and some sign $\varepsilon \in \{\pm 1\}$. 

Proof. Since $\tilde{G}$ has connected center its unique Gelfand–Graev character $\Gamma$ can be written as $\Gamma = \sum_{(\bar{s})} \chi(\bar{s})$, where $(\bar{s})$ runs over the semisimple conjugacy classes of $\tilde{G}^{F^*}$. This is a decomposition of the Gelfand–Graev character into distinct irreducible characters, see [4, Corollary 14.47]. By applying the duality map $D_{\tilde{G}}$, we obtain $D_{\tilde{G}}(\Gamma) = \sum_{(\bar{s})} D_{\tilde{G}}(\chi(\bar{s}))$. Since $\chi(\bar{s})$ is irreducible and $D_{\tilde{G}}$ is an isometry it follows that $D_{\tilde{G}}(\chi(\bar{s}))$ is an irreducible character up to a sign. By [3, Proposition 8.3.4] it follows that the irreducible constituents of $D_{\tilde{G}}(\Gamma)$ are precisely the characters of $p'$-degree. □

The embedding $\iota : G \hookrightarrow \tilde{G}$ gives rise to a dual morphism $\iota^* : \tilde{G}^* \to G^*$ which is surjective and has central kernel, see [1, Section 15.1]. If $(\bar{s})$ is a semisimple conjugacy class of $\tilde{G}^{F^*}$, it follows that $(s)$ is a semisimple conjugacy class of $G^{F^*}$, where $s := \iota^*(\bar{s})$. By [3, Proposition 3.10] the characters $\Gamma_z$ and $\chi(s)$ have a unique common irreducible constituent which we denote by $\chi(s,z)$.

Lemma 3.4. Let $G$ be a connected reductive group and $p$ a good prime for $G$. Let $\iota : G \hookrightarrow \tilde{G}$ and $\iota^* : \tilde{G}^* \to G^*$ be as above. Let $\chi = \varepsilon D_{\tilde{G}}(\chi(\bar{s})) \in \text{Irr}_{p'}(\text{G}^F)$. Then the irreducible constituents of $\chi_{G^F}$ are precisely the characters $\varepsilon D_{G}(\chi(s),z)$ for $s = \iota^*(\bar{s})$ and $z \in H^1(F, Z(G))$.

Proof. By [3, 3.15.1] we have

$$\chi(s)_{G^F} = \chi(s) = \sum_z \chi(s,z),$$

where the sum is over a complete set of distinct characters $\chi(s,z)$ with $z \in H^1(F, Z(G))$. Since the duality functor commutes with restriction we conclude that

$$\chi_{G^F} = (\varepsilon D_{\tilde{G}}(\chi(\bar{s})))_{G^F} = \varepsilon D_{G}(\chi(s)) = \sum_z \varepsilon D_{G}(\chi(s,z)).$$

Using the fact that $D_{G}$ is an isometry it follows that this is a decomposition into distinct irreducible characters. □

4. A McKay-type bijection

We fix an indecomposable root system $\Phi$ of rank $n$. From now on $G$ will denote a simple algebraic group of simply connected type with root system $\Phi$ defined over a field of characteristic $p$.

4.1. A regular embedding. The center of $G$ is a finite group. We let $d_p$ be its minimal number of generators. For our fixed root system $\Phi$ we let $d$ be the maximal $d_p$ occurring for any prime $p$. We have $d = 2$ if the root system of $G$ is of type $D_n$ and $n$ is even. In the remaining cases we either have $d = 1$ or $d = 0$.

Let $S = (k^\times)^d$ be a torus of rank $d$. Let $\rho : Z(G) \hookrightarrow S$ be an injective group homomorphism and define a group $\tilde{G}$ by

$$\tilde{G} = G \times_p S = (G \times S)/\{(z, \rho(z)^{-1}) | z \in Z(G)\}.$$
4.2. Generators of the torus. Let us define an integer \( \bar{d} \) by

\[
\bar{d} = \begin{cases} 
1 & \text{if } (G, F) \text{ is of type } D_n, \text{ } n \text{ even and } w = 2, \\
0 & \text{if } (G, F) \text{ is of type } D_n, \text{ } n \text{ even and } w = 3, \\
d & \text{otherwise}.
\end{cases}
\]

With this notation, \( Z(\hat{G}^F) \) is generated by \( \bar{d} \) elements for \( p \) large enough. If \( \bar{d} = 0 \) we define \( t_0 = 1 \). If \( \bar{d} = 1 \) we let \( t_0 \) be the generator of \( Z(\hat{G}^F) \) as in [10, Section 10]. If \( \bar{d} = 2 \) we let \( t_{q(1)}, t_{q(2)} \) be the generators of \( Z(\hat{G}^F) \) as in [10, Section 10]. In this case, we mean by \( t_0 \) both elements \( t_{q(1)} \) and \( t_{q(2)} \).

Recall from 3.3 that the integer \( r \) denotes the number of \( \tau \)-orbits of \( \Delta \). We let \( t_1, \ldots, t_r \in \hat{T}^F \) be as introduced in [10, Proposition 8.1] resp. [10, Proposition 10.2]. Then the torus \( \hat{T}^F \) is generated by the elements \( t_0, t_1, \ldots, t_r \).

4.3. The linear characters of \( U^F \). Let us from now on assume that \( G^F \) is not of type \( B_2(2), F_4(2) \) or \( G_2(3) \). In this case, we have \([U, U]^F = [U^F, U^F] \) by [5, Lemma 7]. By Lemma 3.2 we obtain a bijection

\[
\gamma : \text{Irr}(U^F/[U^F, U^F]) \rightarrow \prod_{i=1}^r (\mathbb{F}_{q^{|A_i|}}, +).
\]

Let \( S \) be a subset of \( \{1, \ldots, r\} \). We denote \( S^c = \{0, 1, \ldots, r\} \setminus S \). Define the character \( \phi_S \) of \( U^F/[U^F, U^F] \) to be \( \phi_S = \gamma^{-1}(c_1, \ldots, c_r) \) with

\[
c_i = \begin{cases} 
0 & \text{if } i \notin S, \\
1 & \text{if } i \in S.
\end{cases}
\]

For simplicity we identify \( \phi_S \in \text{Irr}(U^F/[U^F, U^F]) \) with its inflation to \( U^F \). Note that with this notation the linear character \( \psi_1 \) introduced in 3.3 coincides with \( \phi_{\{1, \ldots, r\}} \).

The action of \( \hat{T}^F \) on the characters of \( U^F \) can be described explicitly and one obtains the following result.

**Proposition 4.1.** The characters \( \{\phi_S \in \text{Irr}(U^F) \mid S \subseteq \{1, \ldots, r\}\} \) form a complete set of representatives for the \( \hat{B}^F \)-orbits on the linear characters of \( U^F \). Moreover any character \( \phi_S \in \text{Irr}(U^F) \) extends to its inertia group \( I_{\hat{B}^F}(\phi_S) = \{t_i \mid i \in S^c\}U^F \).

**Proof.** This is [10, Proposition 8.4] and [10, Proposition 8.5]. \( \square \)

As a consequence of the previous lemma we can describe the action of Galois automorphisms on linear characters of \( U^F \).

**Lemma 4.2.** Let \( \sigma \in \text{Gal} (\mathbb{Q}(\mathbb{G}_F)/\mathbb{Q}) \). Then \( \phi_S^\sigma = \phi_{S^\sigma}^\bar{\sigma} \) for some \( \bar{\sigma} \in \hat{T}^F \).
Proof. By the uniqueness statement of Proposition 4.1 we have $\phi_S^g = \phi_S^{\tilde{t}}$ for some $S' \subseteq \{1, \ldots, r\}$ and some $\tilde{t} \in \tilde{T}^F$. Recall that the subgroups $U_{A_i}^F$ are stabilized by the $\tilde{T}^F$-action. Thus, we have

$$\prod_{i \in S'} \phi_i^{\tilde{t}} = \phi_{S'}^g = \prod_{i \in S} \phi_i^g.$$ 

Since the characters $\phi_i \in \text{Irr}(U_{A_i}^F)$ are nontrivial this implies $S = S'$ and $\phi_S^g = \phi_S^\tilde{t}$.

\[\square\]

4.4. A labeling for the local characters. We can now parametrize the $p'$-characters of $\tilde{B}^F$. Let $\psi \in \text{Irr}_{p'}(\tilde{B}^F)$. Since $U^F$ is a normal $p'$-subgroup of $\tilde{B}^F$ and $\psi$ has $p'$-degree it follows by Clifford’s theorem that $\psi$ lies above a linear character of $U^F$. Hence, by Proposition 4.1 there exists a uniquely determined subset $S \subseteq \{1, \ldots, r\}$ such that $\psi$ lies above $\phi_S \in \text{Irr}(U^F)$. By Clifford correspondence there exists a unique character $\lambda \in \text{Irr}(\tilde{B}^F | \phi_S)$ with $\lambda^{\tilde{B}^F} = \psi$. Note that $\text{Irr}(\tilde{B}^F | \phi_S) = \{t_i \mid i \in S'\}U^F$ by Proposition 4.1. We define the map $\tilde{f}_{\text{loc}} : \text{Irr}_{p'}(\tilde{B}^F) \to (K^\times)^d \times K^r$ by

$$\tilde{f}_{\text{loc}}(\psi)_0 = \begin{cases} \lambda(t_0) & \text{if } \tilde{d} \leq 1, \\ \left(\lambda(t_0^{(1)}), \lambda(t_0^{(2)})\right) & \text{if } \tilde{d} = 2, \end{cases}$$

and

$$\tilde{f}_{\text{loc}}(\psi)_i = \begin{cases} \lambda(t_i) & \text{if } i \in S' \setminus \{0\}, \\ 0 & \text{if } i \in S, \end{cases},$$

where $\lambda$ is determined by $\psi$ as above. For $i \in S'$ the values $\lambda(t_i)$ of the linear character $\lambda$ are $(q^{w} - 1)$-th roots of unity, where $w$ is as defined in 3.3. Under the embedding $k^\times \to K^\times$ chosen in 2.1 we may consider $(\tilde{f}_{\text{loc}}(\psi))_i$ as an element of $F_{q^e}$, and we obtain a map $\tilde{f}_{\text{loc}} : \text{Irr}_{p'}(\tilde{B}^F) \to (F_{q^e})^d \times F_{q^e}$. Let $A \subseteq (F_{q^e})^d \times F_{q^e}$ be the image of the map $\tilde{f}_{\text{loc}}$. By \cite{10} Theorem 10.8 the map $\tilde{f}_{\text{loc}} : \text{Irr}_{p'}(\tilde{B}^F) \to A$ is a bijection.

4.5. The dual group. We give an explicit construction of the dual algebraic group of $\tilde{G}$, following the construction in \cite{10} Section 7. It is similar to the construction in 4.1. Let $G^\vee$ be a simple algebraic group of simply connected type with root system $\Phi^\vee$. We fix a maximal torus $T^\vee$ of $G^\vee$ and identify the root system of $G^\vee$ relative to the torus $T^\vee$ with the coroot system $\Phi^\vee$.

We let $S^\vee = (k^\times)^d$, where $d$ is as in 4.1, and we choose an injective group homomorphism $\rho^\vee : Z(G^\vee) \to S^\vee$ as in \cite{10} Section 7. Denote by $\tilde{G}^*$ the resulting linear algebraic group $\tilde{G}^* = G^\vee \times_{\rho^\vee} S^\vee$ with maximal torus $\tilde{T}^* := T^\vee S^\vee$. By the results of \cite{10} Section 7 there exists a Frobenius endomorphism $F^*$ of $\tilde{G}^*$ such that $(\tilde{G}, \tilde{T}, F)$ is dual to the triple $(\tilde{G}^*, \tilde{T}^*, F^*)$.

4.6. Fundamental weights. Since $G^\vee$ is a simple algebraic group of simply connected type its character group $X(T^\vee)$ has a basis given by the fundamental weights. More precisely, let $\beta_1, \ldots, \beta_n \in X(T^\vee)$ be a basis of the root system $\Phi^\vee$ (corresponding to $\alpha_1^\vee, \ldots, \alpha_n^\vee$ under the identification of the root system of $G^\vee$ with $\Phi^\vee$). Denote by $\langle , \rangle : X(T^\vee) \times Y(T^\vee) \to \mathbb{Z}$ the canonical pairing. Then there exist roots $\omega_i \in X(T^\vee)$ satisfying $\langle \omega_i, \beta_j^\vee \rangle = \delta_{ij}$ for all $i, j = 1, \ldots, n$. Moreover, we let $\tilde{\omega}_i \in X(\tilde{T}^*)$ be the unique extension of $\omega_i$ to $\tilde{T}^*$ which acts trivially on $S^\vee$. 

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4.7. The determinant map. Write $g \in \tilde{G}^*$ as $g = xz$ with $x \in G^\vee$ and $z \in S^\vee$. We define the determinant map $\det : \tilde{G}^* \to S^\vee$ to be the map with $\det(xz) = z^l$ where $l$ is the exponent of the fundamental group of the root system $\Phi$.

Note that the map $\det$ is a well-defined homomorphism of algebraic groups, see the remark below [10, Definition 7.2]. Furthermore we denote by $\det_i : \tilde{G}^* \to k^\times$ the $i$-th component of the determinant map.

4.8. The modified Steinberg map. Following Maslowski [10, Section 14], we introduce the modified Steinberg map which separates the semisimple conjugacy of $\tilde{G}^*$.

By a theorem of Chevalley, see [9, Theorem 15.17], there exists a rational irreducible $kG^\vee$-module $V_i$ which is a highest weight module of highest weight $\omega_i \in Y(T^\vee)$. Let $\pi_i : G^\vee \to k^\times$ denote the trace function of the representation associated to the $kG^\vee$-module $V_i$. We define the Steinberg map

$$\pi : G^\vee \to k^n : g \mapsto (\pi_1(g), \ldots, \pi_n(g))$$

as the product map of these trace functions. A fundamental property of the Steinberg map is that two semisimple elements of $G^\vee$ are $G^\vee$-conjugate if and only if they have the same image under the Steinberg map, see [16, Corollary 6.7].

We need a further property of the Steinberg map which is a slight generalization of [10, Lemma 14.1] and follows from the proof given there.

Lemma 4.3. Let $s \in G^\vee$ be a semisimple element. Then $\pi_i(s^p) = \pi_i(s)^p$.

We can write any element $\tilde{g} \in \tilde{G}^*$ (not necessary unique) as $\tilde{g} = xz$ with $x \in G^\vee$ and $z \in S^\vee$. In [10, Section 14] Maslowski defines the map $\tilde{\pi} : \tilde{G}^* \to (k^\times)^d \times k^n$ by

$$\tilde{g} = xz \mapsto (\det(xz), (\pi_1(x)\omega_1(z), \ldots, \pi_n(x)\omega_n(z))).$$

Based on the result of Steinberg mentioned above, Maslowski shows in [10, Proposition 14.2] that the map $\tilde{\pi}$ separates semisimple conjugacy classes of $\tilde{G}^*$. Moreover, if $F = F_q$, then the semisimple conjugacy classes $(xz)$ of $\tilde{G}^*$ with image $\tilde{\pi}(xz)$ in $(F_q^\times)^d \times F_q^n$ are precisely the $(q - 1)^d q^n$ different $F_q^\times$-stable semisimple conjugacy classes of $\tilde{G}^*$.

4.9. A labeling for the global characters. We now describe a labeling for the $p'$-characters of $\tilde{G}^F$. Let $\chi \in \text{Irr}_{p'}(\tilde{G}^F)$ be a $p'$-character. Then there exists a conjugacy class $\langle \tilde{s} \rangle$ of $\tilde{G}^*\tilde{F}^*$ such that $\chi \in \mathcal{E}(\tilde{G}^F, \langle \tilde{s} \rangle)$.

We first consider the case that $F = F_q$. In this case, we define the label of $\chi$ by $\tilde{\pi}(\tilde{s}) = (b_0, (b_1, \ldots, b_n)) \in (F_q^\times)^d \times F_q^n$.

Now suppose that $F$ is not a standard Frobenius map. Since $\tilde{G}^*\tilde{F}^* \subseteq \tilde{G}^*\tilde{F}^*_{\tilde{q}^w}$ we have $\tilde{s} \in \tilde{G}^*\tilde{F}^*_{\tilde{q}^w}$. In particular it holds $\tilde{\pi}(\tilde{s}) = (b_0, (b_1, \ldots, b_n)) \in (F_q^\times)^d \times F_q^n$. We define the label of $\chi$ by $(b_{a_1}, (b_{a_1^w}, \ldots, b_{a_w})) \in (F_q^\times)^d \times F_q^n$ where $a_i \in A_i$ are the fixed representatives of the orbits of the $\tau$-action and $b_{a_{i^w}}$ is the first component of $b_0 \in (F_q^w)^d$.

In any case, the possible labels which occur consist precisely of the elements $\mathcal{A}$, where $\mathcal{A}$ is defined as in [4.3]. We shall denote by $\tilde{f}_{\text{glo}} : \text{Irr}_{p'}(\tilde{G}^F) \to \mathcal{A}$ the map which maps a character to its label.
4.10. **The Maslovksi bijection and its properties.** From now on we often write $H = H^F$ for the group of fixed points under $F$ of an $F$-stable subgroup $H$ of $\tilde{G}$. In most cases, the map $\tilde{f}_{glo} : \text{Irr}_{p'}(\tilde{G}^F) \to A$ is known to be bijective.

**Theorem 4.4.** Suppose that $(G, F)$ is not contained in the following table.

| type | Frobenius map |
|------|---------------|
| $D_n$ | $q = 2, w = 2$ |
| $B_n, C_n, D_n, G_2, F_4$ | $q = 2, w = 1$ |
| $G_2$ | $q = 3, w = 1$ |

Then the map $\tilde{f}_{glo} : \text{Irr}_{p'}(\tilde{G}) \to A$ is a bijection. Consequently the map $\tilde{f} = \tilde{f}_{glo} \circ \tilde{f}_{loc} : \text{Irr}_{p'}(\tilde{B}) \to \text{Irr}_{p'}(\tilde{G})$ is a bijection. Moreover, for every central character $\lambda \in \text{Irr}(Z(\tilde{G}))$ the bijection $\tilde{f}$ restricts to a bijection $\text{Irr}_{p'}(\tilde{B} \mid \lambda) \to \text{Irr}_{p'}(\tilde{G} \mid \lambda)$

**Proof.** This is [10] Theorem 15.3.

We shall keep the assumptions of Theorem 4.4 for the remainder of this article. An important step towards the proof of Theorem 4.4 is the following important theorem.

**Theorem 4.5.** Any $p'$-character of $\tilde{B}$ restricts multiplicity free to $B$. Analogously, any $p'$-character of $\tilde{G}$ restricts multiplicity free to $G$.

**Proof.** This is [10] Proposition 11.3] in the local situation and [11] Theorem 15.11] in the global situation.

The following result from [13] Theorem 3.5(b)] shows that for characters in $\text{Irr}(\tilde{G} \mid 1_G)$ the map $\tilde{f}^{-1}$ in Theorem 4.4 is just restriction of characters. A proof of this result can be found in [11] Corollary 3.20].

**Theorem 4.6.** Let $\lambda \in \text{Irr}(\tilde{G} \mid 1_G)$ and $\chi \in \text{Irr}_{p'}(\tilde{G})$. Then $\tilde{f}^{-1}(\chi \lambda) = \tilde{f}^{-1}(\chi)\lambda_B$.

The previous two theorems imply the following result, see [10] Theorem 15.4].

**Theorem 4.7.** There exists a bijection $f : \text{Irr}_{p'}(B) \to \text{Irr}_{p'}(G)$ such that for every $\lambda \in \text{Irr}(Z(G))$ the map $f$ restricts to a bijection $\text{Irr}_{p'}(B \mid \lambda) \to \text{Irr}_{p'}(G \mid \lambda)$.

**Proof.** Let $\psi \in \text{Irr}_{p'}(\tilde{B})$ and $\chi = \tilde{f}(\psi)$. Furthermore we let $\vartheta \in \text{Irr}(B \mid \psi)$ and $\phi \in \text{Irr}(G \mid \chi)$. By Theorem 4.5 we have $\text{Irr}(\tilde{B} \mid \vartheta) = \{\psi \xi \mid \xi \in \text{Irr}(\tilde{B} \mid 1_B)\}$ and analogously $\text{Irr}(\tilde{G} \mid \phi) = \{\chi \lambda \mid \lambda \in \text{Irr}(G \mid 1_G)\}$. Together with Theorem 4.6 this implies that $\tilde{f}$ restricts to a bijection $\tilde{f} : \text{Irr}(\tilde{B} \mid \vartheta) \to \text{Irr}(\tilde{G} \mid \phi)$. By Clifford theory we deduce

$$|\{\phi \tilde{b} \mid \tilde{b} \in \tilde{B}\}| = |\text{Irr}(\tilde{B} \mid \vartheta)| \text{ and } |\{\phi \tilde{g} \mid \tilde{g} \in \tilde{G}\}| = |\text{Irr}(\tilde{G} \mid \phi)|.$$

Thus it is possible to define a bijective map $\text{Irr}(B \mid \psi) \to \text{Irr}(G \mid \chi)$ by sending the set $\{\phi \tilde{b} \mid \tilde{b} \in \tilde{B}\}$ bijectively to the set $\{\phi \tilde{g} \mid \tilde{g} \in \tilde{G}\}$. This then gives rise to a bijection $f : \text{Irr}_{p'}(B) \to \text{Irr}_{p'}(G)$. The compatibility of $f$ with central characters follows from the corresponding statement about central characters in Theorem 4.4.

We need to recall a fact from Clifford theory. For this let us introduce some notation. Let $X$ be a normal subgroup of a finite group $Y$ such that $Y/X$ is abelian. Then the character group $\text{Irr}(Y \mid 1_X)$ acts by multiplication on the characters in $\text{Irr}(Y)$. For a character $\chi \in \text{Irr}(Y)$ we shall denote by $\text{Stab}_{\text{Irr}(Y/X)}(\chi)$ the stabilizer of $\chi$ under the action of $\text{Irr}(Y \mid 1_X)$. 


Lemma 4.8. Under the assumptions as above, let $\chi \in \text{Irr}(Y)$ be an irreducible character which restricts multiplicity free to $X$. Then for $\vartheta \in \text{Irr}(X | \chi)$ it holds that

$$I_Y(\vartheta) = \bigcap_{\lambda \in \text{Stab}_{\text{Irr}(Y/X)}(\chi)} \text{Kern}(\lambda).$$

Proof. Let us abbreviate $I = I_Y(\vartheta)$ and let $y \in Y$. Since $Y/X$ is abelian it follows that $y \in I$ if and only if $\lambda(y) = 1$ for all $\lambda \in \text{Irr}(Y | 1_I)$. By assumption there exists a character $\psi \in \text{Irr}(I)$ which extends $\vartheta$ and such that $\psi^Y = \chi$. Let $\lambda \in \text{Irr}(Y | 1_X)$ be arbitrary. By Clifford correspondence we have $\lambda \chi = \chi$ if and only $\lambda I \vartheta = \vartheta$. By Gallagher’s theorem the latter is equivalent to $\lambda I = 1_I$ since $Y/X$ is abelian. □

The previous lemma allows us to show that the inertia groups in $\tilde{B}$ of $p'$-characters which correspond to each other under the map $\tilde{f}$ are the same.

Corollary 4.9. Let $\vartheta \in \text{Irr}_{p'}(B)$ and $\phi = f(\vartheta)$. Then we have $I_B(\phi) = I_{\tilde{G}}(\vartheta) \cap \tilde{B}$.

Proof. Let $\psi \in \text{Irr}(\tilde{B} | \vartheta)$ and $\chi = \tilde{f}(\psi)$. By construction of $\tilde{f}$ we deduce that $\chi \in \text{Irr}(\tilde{G} | \phi)$. Let $\tilde{t} \in \tilde{B}$. By Lemma 4.8 we have $\tilde{t} \in I_{\tilde{B}}(\phi)$ if and only if $\lambda(\tilde{t}) = 1$ for all $\lambda \in \text{Stab}_{\text{Irr}(\tilde{G}/\tilde{G})}(\chi)$. By Theorem 4.7 $\lambda \chi = \chi$ is equivalent to $\lambda_{\tilde{B}} \psi = \psi$. Since the inclusion map $\tilde{B} \to \tilde{G}$ induces an isomorphism of the factor groups $\tilde{G}/\tilde{G}$ and $\tilde{B}/B$ it follows that the restriction map $\text{Irr}(\tilde{G} | 1_{\tilde{G}}) \to \text{Irr}(B | 1_B)$ is bijective. Thus, we conclude that $\tilde{t} \in I_{\tilde{B}}(\phi)$ if and only if $\xi(\tilde{t}) = 1$ for all characters $\xi \in \text{Stab}_{\text{Irr}(\tilde{B}/B)}(\psi)$. Now apply Lemma 4.8 again. □

5. The McKay Conjecture and Galois automorphisms

This section is divided in four parts. In 5.1 we show that the bijection $\tilde{f} : \text{Irr}_{p'}(\tilde{B}) \to \text{Irr}_{p'}(\tilde{G})$ is equivariant for $(e,p)$-Galois automorphisms. In 5.2 and 5.3 we relate the $p'$-characters of $\tilde{B}$ (resp. of $\tilde{G}$) with the $p'$-characters of $B$ (resp. of $G$). In 5.4 we use these results to provide a proof of Theorem 1.2 from the introduction.

We shall keep the assumptions of Theorem 4.4.

5.1. Compatibility of the character bijection with Galois automorphisms. For a positive integer $m$ we write $m_p$ for the highest $p$-power dividing $m$ and $m_{p'}$ for the $p'$-part of $m$. The following properties of $(e,p)$-Galois automorphisms are elementary and easy to prove.

Lemma 5.1. Let $\sigma \in \text{Gal}(\mathbb{Q}_m/\mathbb{Q})$ and $k$ be an integer such that $\sigma(\xi) = \xi^k$ for a primitive $m$-th root of unity $\xi \in \mathbb{Q}_m$.

(a) Then $\sigma$ is an $(e,p)$-Galois automorphism if and only if $k \equiv p^e \text{ mod } m_{p'}$.

(b) Let $\bar{m}$ be a multiple of $m$. Then any $(e,p)$-Galois automorphism $\sigma \in \text{Gal}(\mathbb{Q}_{\bar{m}}/\mathbb{Q})$ extends to an $(e,p)$-Galois automorphism $\tilde{\sigma} \in \text{Gal}(\mathbb{Q}_m/\mathbb{Q})$.

Note that part (b) of Lemma 5.1 implies that any $(e,p)$-Galois automorphism $\sigma \in \text{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$ extends to an $(e,p)$-Galois automorphism $\tilde{\sigma} \in \text{Gal}(\mathbb{Q}_{|\tilde{G}|}/\mathbb{Q})$. This means that if we want to prove Conjecture 1.1 for the finite group $G$ and an $(e,p)$-Galois automorphism $\sigma \in \text{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$, we may (and we will) assume without loss of generality that $\sigma \in \text{Gal}(\mathbb{Q}_{|\tilde{G}|}/\mathbb{Q})$.

Now we show that the bijection $\tilde{f} : \text{Irr}_{p'}(\tilde{B}) \to \text{Irr}_{p'}(\tilde{G})$ is equivariant for $(e,p)$-Galois automorphisms. In the proof we freely use the notation introduced in Section 4.
Theorem 5.2. Let $\sigma \in \text{Gal}(\mathbb{Q}(\overline{g}/\mathbb{Q})$ be an $(e, p)$-Galois automorphism. Then the bijection $
abla : \text{Irr}_p(\overline{B}) \to \text{Irr}_p(\overline{G})$ satisfies $\hat{f}(\psi) = \hat{f}(\psi)^\sigma$ for any character $\psi \in \text{Irr}_p(\overline{B})$. Moreover, if we denote the label of $\psi$ by $\hat{f}_\text{loc}(\psi) = (c_0, (c_1, \ldots, c_r)) \in (\mathbb{F}_q^\times)^d \times \mathbb{F}_q^w$ then the label of $\psi^\sigma$ is given by $\hat{f}_\text{loc}(\psi^\sigma) = (c_{\sigma 0}, (c_{\sigma 1}, \ldots, c_{\sigma r}))$.

Proof. Let $\psi \in \text{Irr}_p(\overline{B})$ with label $\hat{f}_\text{loc}(\psi) = (c_0, (c_1, \ldots, c_r))$. Denote by $\chi = \hat{f}(\psi)$ the $p'$-character of $\overline{G}$ which has the same label as $\psi$.

We proceed in several steps. In a first step we compute the label $g(\psi^\sigma)$ of the character $\psi^\sigma$. In a second step we prove that $\hat{f}_\text{glob}(\chi^\sigma) = \hat{f}_\text{loc}(\psi^\sigma)$ which implies $\hat{f}(\psi^\sigma) = \chi^\sigma$ since $\hat{f} = \hat{f}_\text{loc} \circ \hat{f}_\text{glob}$.

First step: By Proposition 4.1 there exists a unique set $S \subseteq \{1, \ldots, r\}$ such that the character $\psi \in \text{Irr}_p(\overline{B})$ lies above the character $\phi_S \in \text{Irr}(U)$. By Clifford correspondence there exists a unique character $\lambda \in \text{Irr}(\text{Irr}(U))$ such that $\lambda \overline{B} = \psi$. Since $\psi$ lies above $\phi_S$ it follows that $\psi^\sigma$ lies above the character $\phi_S$. By Lemma 4.2 we have $\phi_S = \phi_S^\sigma$ for some $\overline{\sigma} \in \overline{T}$. The character $\lambda^\sigma$ lies above the character $\phi_S = \phi_S^\sigma$. Since the factor group $\overline{B}/U \cong \overline{T}$ is abelian we have $(\lambda^\sigma)^{\overline{\sigma}^{-1}} \in \text{Irr}(\text{Irr}(U))$. Consequently, $(\lambda^\sigma)^{\overline{\sigma}^{-1}}$ lies above the character $\phi_S$ and $((\lambda^\sigma)^{\overline{\sigma}^{-1}})^\overline{B} = \psi^\sigma$. Let $m = |\overline{G}|$ and $\xi$ be a primitive $m$-th root of unity. Furthermore we let $k$ be an integer such that $\sigma(\xi) = \xi^k$. Since $\lambda$ is linear we have $\lambda^\sigma = \lambda^k$. Thus we obtain

$$(\hat{f}_\text{loc}(\psi^\sigma))_i = (\lambda^k)^{\overline{\sigma}^{-1}} (t_i) = (\lambda^k)(\overline{\sigma}^{-1}t_i) = \lambda^k(t_i).$$

By definition of the map $\hat{f}_\text{loc}$ we have $(\hat{f}_\text{loc}(\psi^\sigma))_i = c_i^k$ for every $i \in S^c$ and $(\hat{f}_\text{loc}(\psi^\sigma))_i = 0$ for $i \in S$. Consequently, the label of $\psi^\sigma$ is given by $\hat{f}_\text{loc}(\psi^\sigma) = (c_0^k, c_1^k, \ldots, c_r^k)$.

By Lemma 5.1 we have $k \equiv p^e \mod m_{p'}$ since $\sigma \in \text{Gal}(\mathbb{Q}_m/\mathbb{Q})$ is an $(e, p)$-Galois automorphism. By [9] Table 24.1 it follows that $(q^w - 1)$ divides $m_{p'}$, which implies that $k \equiv p^e \mod (q^w - 1)$. Since $c_i \in \mathbb{F}_q^w$ for all $i$, we have

$$\hat{f}_\text{loc}(\psi^\sigma) = (c_0^k, (c_1^k, \ldots, c_r^k)) = (c_0^{p^e}, (c_1^{p^e}, \ldots, c_r^{p^e})).$$

Second step: We have $\chi \in \mathcal{E}(\tilde{G}^F, (\tilde{s}))$ for some semisimple conjugacy class $(\tilde{s})$ of the dual group $\tilde{G}^{*s_{p^e}}$. By Lemma 3.3 we have $\chi^\sigma \in \mathcal{E}(\tilde{G}^F, (\tilde{s}^k))$. We have $m = |\tilde{G}^{*s_{p^e}}|$ since $(\tilde{G}, F)$ and $(\tilde{G}^{*}, F^*)$ are in duality, see [2] Proposition 4.4.4. Thus, the order of the semisimple element $\tilde{s}$ is a divisor of $m_{p'}$. Since $k \equiv p^e \mod m_{p'}$ by Lemma 5.1 this shows $\tilde{s}^{p^e} = \tilde{s}^k$. Hence, we have $\chi^\sigma \in \mathcal{E}(\tilde{G}^F, (\tilde{s}^{p^e}))$ by Lemma 3.1.

First we assume that $F = F_q$ is a standard Frobenius map. We may write $\tilde{s} \in \tilde{G}^s$ as $\tilde{s} = xz$ where $x \in \tilde{G}^V$ and $z \in S^V$. The label of the character $\chi^\sigma \in \mathcal{E}(\tilde{G}^F, (\tilde{s}^{p^e}))$ is given by

$$\tilde{\pi}(\tilde{s}^{p^e}) = \tilde{\pi}((xz)^{p^e}) = (\text{det}((xz)^{p^e}), (\pi_1(x)^{p^e})\tilde{\omega}_1(z^{p^e}), \ldots, \pi_n(x)^{p^e})\tilde{\omega}_n(z^{p^e})).$$

Note that $\text{det}((xz)^{p^e}) = \text{det}(xz)^{p^e}$ since $\text{det}$ is multiplicative. Recall that $\tilde{\pi}_i((xz)^{p^e}) = \pi_i(x)^{p^e}\tilde{\omega}_i(z^{p^e})$ by definition of the modified Steinberg map. By Lemma 4.3 we have $\pi_i(x)^{p^e} = \pi_i(x)^{p^e}$. Moreover, we have $\tilde{\omega}_i(z^{p^e}) = \tilde{\omega}_i(z)^{p^e}$ since $\tilde{\omega}_i \in X(\tilde{T}^*)$. Therefore we obtain $\tilde{\pi}_i((xz)^{p^e}) = \pi_i(x)^{p^e}\tilde{\omega}_i(z)^{p^e} = \pi_i(\tilde{s})^{p^e}$. Since $\pi((xz)) = (c_0, (c_1, \ldots, c_n))$ we have $\tilde{\pi}((xz)^{p^e}) = (c_0^{p^e}, (c_1^{p^e}, \ldots, c_n^{p^e}))$ and therefore the label of $\chi^\sigma$ is given by $\hat{f}_\text{glob}(\chi^\sigma) = (c_0^{p^e}, (c_1^{p^e}, \ldots, c_n^{p^e})) = \hat{f}_\text{loc}(\psi^\sigma)$ and we have $\hat{f}(\psi) = \hat{f}(\psi^\sigma)$, as required.
Let us now assume that \( F \) is not a standard Frobenius endomorphism. Let \( \tilde{\pi}(\tilde{s}) = (b_0, (b_1, \ldots, b_n)) \in (\mathbb{F}_q^\times)^d \times \mathbb{F}_q^r \) be the image of \( \tilde{s} \in \tilde{G}^e_{\mathcal{F}} \subseteq \tilde{G}^e_{\mathcal{F}_\psi} \) under the modified Steinberg map. As we have shown above, the image of \((\tilde{s})^e\) under the modified Steinberg map is given by \( \tilde{\pi}(\tilde{s}) = (b_0^e, (b_1^e, \ldots, b_n^e)) \). Let \( b_0(\tilde{s}) \) be the first component of \( b_0 \in (\mathbb{F}_q^\times)^d \). The label of the character \( \chi \in \mathcal{E}(\tilde{G}^F, (\tilde{s})) \) is given by \( \tilde{f}_{\text{glo}}(\chi) = (b_0(\tilde{s}), (b_1(\tilde{s}), \ldots, b_n(\tilde{s})) \). We have \( \tilde{f}_{\text{loc}}(\chi) = (c_0, (c_1, \ldots, c_r)) = (b_0(\tilde{s}), (b_1(\tilde{s}), \ldots, b_n(\tilde{s})) \). Thus, we conclude that
\[
\tilde{f}_{\text{loc}}(\chi) = (c_0^e, (c_1^e, \ldots, c_r^e)) = (b_0^e(\tilde{s}), (b_1^e(\tilde{s}), \ldots, b_n^e(\tilde{s})) \).\]
This shows \( \tilde{f}(\psi) = \chi^e \), as desired. \( \square \)

Theorem 5.2 gives us Theorem 1.2 in the case that \( Z(G) \) is trivial.

**Corollary 5.3.** Let \( \sigma \in \text{Gal}( \mathbb{Q}(\tilde{G}) / \mathbb{Q} ) \) be an \((e, p)\)-Galois automorphism. Suppose that \( Z(G) = 1 \). Then there exists a \( \sigma \)-equivariant bijection \( f : \text{Irr}_p(B) \rightarrow \text{Irr}_p(G) \).

**Proof.** By Theorem 4.1 and Theorem 5.2 we obtain a \( \sigma \)-equivariant bijection \( \text{Irr}_{\psi}(\tilde{B} \mid \lambda) \rightarrow \text{Irr}_{\psi}(\tilde{G} \mid \lambda) \) for any central character \( \lambda \in Z(\tilde{G}) \). Since \( Z(G) = 1 \) we have \( \tilde{G} \cong G \times Z(\tilde{G}) \). By Theorem 5.2 we have a bijection \( \tilde{f} : \text{Irr}_{\psi}(\tilde{B} \mid \lambda)^{\sigma} \rightarrow \text{Irr}_{\psi}(\tilde{G} \mid \lambda)^{\sigma} \) for every central character \( \lambda \in Z(\tilde{G}) \). So in particular, for \( \lambda = 1 \) we obtain a bijection \( f : \text{Irr}_{\psi}(B) \rightarrow \text{Irr}_{\psi}(G) \). \( \square \)

### 5.2. Relating the \( p' \)-characters of \( \tilde{B} \) and \( B \).

**Lemma 5.4.** Let \( \sigma \in \text{Gal}( \mathbb{Q}(\tilde{G}) / \mathbb{Q} ) \) be a Galois automorphism, where \( m = |\tilde{G}| \). Let \( \psi \in \text{Irr}_{\psi}(\tilde{B}) \) and \( \vartheta \in \text{Irr}(B \mid \psi) \). Furthermore, we let \( S \) be the unique subset of \( \{1, \ldots, r\} \) such that \( \psi \in \text{Irr}(\tilde{B} \mid \phi_S) \). Then \( \vartheta \) is \( \sigma \)-invariant if and only if \( \psi_B \) is \( \sigma \)-invariant and there exists an element \( t \in B \) such that \( \phi^\sigma_S = \phi^t_S \).

**Proof.** We first assume that \( \vartheta \) is \( \sigma \)-invariant. By Clifford’s theorem all irreducible constituents of \( \psi_B \) are \( \sigma \)-invariant. This implies that \( \psi_B \) is \( \sigma \)-invariant as well. The character \( \phi_S \) is a constituent of some \( \tilde{B} \)-conjugate of \( \vartheta \), such that we may assume that \( \phi_S \) is below \( \vartheta \). Since \( \vartheta \) is \( \sigma \)-invariant we conclude that \( \phi^\sigma_S \) is below \( \vartheta \). By Clifford’s theorem it follows that \( \phi^\sigma_S \) is \( B \)-conjugate to \( \phi_S \).

Now assume conversely that \( \psi_B \) is \( \sigma \)-invariant and that there exists some \( t \in B \) such that \( \phi^\sigma_S = \phi^t_S \). We abbreviate \( I = I_B(\phi_S) \) and \( \tilde{I} = I_B(\phi_S) \). We let \( \lambda \in \text{Irr}(\tilde{I} \mid \phi_S) \) be the character such that \( \lambda^{\tilde{B}} = \psi \). By Clifford correspondence we conclude that \( (\lambda^I)^B \) is an irreducible character of \( B \). Straighforward calculation show that the character \( (\lambda^I)^B \) lies below \( \psi \).

Since all irreducible constituents of \( \psi_B \) are \( \tilde{B} \)-conjugate we may assume \( \vartheta = (\lambda^I)^B \). Since \( \psi_B \) is \( \sigma \)-invariant, we have \( \psi^\sigma \in \text{Irr}(\tilde{B} \mid \vartheta) \). By Theorem 4.5 there exists a character \( \eta \in \text{Irr}(\tilde{B} \mid 1_B) \) such that \( \psi^\sigma = \psi \eta \). Since \( \lambda \) lies above \( \phi_S \), it follows that \( \lambda^\sigma \) lies above \( \phi^\sigma_S = \phi^t_S \). Thus, the character \( (\lambda^\sigma)^{t^{-1}} \) lies above \( \phi_S \). This implies that
\[
((\lambda^\sigma)^{t^{-1}})^\tilde{B} = \psi^\sigma = \psi \eta = (\lambda^\sigma)^{t^{-1}} \tilde{B}.
\]
By Clifford correspondence we deduce \( (\lambda^\sigma)^{t^{-1}} = \lambda^\sigma \). Since \( \eta \in \text{Irr}(\tilde{B} \mid 1_B) \) this implies \( \lambda^\sigma = \lambda^t \). This shows that \( \vartheta = (\lambda^I)^B = (\lambda^t)^B = \vartheta^\sigma \). \( \square \)
Lemma 5.5. Let \( \vartheta \in \text{Irr}_{p'}(B) \) and \( S \subseteq \{1, \ldots, r\} \) such that \( \vartheta \in \text{Irr}(B \mid \phi_S) \). Then \( I_B(\phi_S)B = I_B(\vartheta) \).

Proof. Let \( \xi \in \text{Irr}(I_B(\phi_S) \mid \phi_S) \) be such that \( \xi^B = \vartheta \). Since \( \xi \) is an extension of \( \phi_S \) it follows that \( I_B(\phi_S) = I_B(\xi) \). Moreover, it is clear that \( I_B(\xi) \subseteq I_B(\vartheta) \). This shows \( I_B(\phi_S)B \subseteq I_B(\vartheta) \).

For the converse direction let \( \tilde{t} \in I_B(\vartheta) \). Then \( \vartheta \) is a character below \( \tilde{t} \). By Clifford’s theorem there exists some \( t \in B \) such that \( \phi_t^\vartheta = \phi_{\tilde{t}}^\vartheta \). We have \( t\tilde{t}^{-1} \in I_B(\phi_S) \) and hence \( \tilde{t} \in I_B(\phi_S)B \).

Let \( \sigma \in \text{Gal}(\mathbb{Q}/\mathbb{G}/\mathbb{Q}) \). Recall that there exists some \( \tilde{t} \in \tilde{T} \) such that \( \psi_1^\sigma = \psi_{\tilde{t}}^1 \), see Lemma 5.2. As a consequence of Lemma 5.5 we obtain the following proposition.

Proposition 5.6. Let \( \sigma \in \text{Gal}(\mathbb{Q}/\mathbb{G}/\mathbb{Q}) \) be a Galois automorphism and let \( \tilde{t} \in \tilde{T} \) such that \( \psi_1^\sigma = \psi_{\tilde{t}}^1 \). Let \( \psi \in \text{Irr}_{p'}(\tilde{B}) \) and \( \vartheta \in \text{Irr}(B) \) be a character below \( \psi \). Then the following are equivalent:

(i) \( \vartheta \) is \( \sigma \)-invariant.
(ii) \( \psi_B \) is \( \sigma \)-invariant and \( \tilde{t} \in I_B(\vartheta) \).

Proof. Let \( S \subseteq \{1, \ldots, r\} \) such that \( \phi_S \) is below \( \psi \). Since the conjugation action of \( \tilde{T}^F \) stabilizes the subgroups \( U^F_{\mathbb{G}} \), we obtain \( \phi_S^\vartheta = \phi_{\tilde{t}}^\vartheta \). By Lemma 5.4 the character \( \vartheta \) is \( \sigma \)-invariant if and only if \( \psi_B \) is \( \sigma \)-invariant and \( \phi_S^\vartheta = \phi_{\tilde{t}}^\vartheta \) for some \( t \in B \). This is equivalent to saying that \( t\tilde{t}^{-1} \in I_B(\phi_S) \). The latter statement is equivalent to \( \tilde{t} \in I_B(\phi_S)B \). Our proposition follows now with Lemma 5.5.

Example 5.7. With Proposition 5.6 we are (at least theoretically) able to compute the cardinality of \( \text{Irr}_{p'}(B)^\sigma \) for an \((e, p)\)-Galois automorphism \( \sigma \in \text{Gal}(\mathbb{Q}/\mathbb{G}/\mathbb{Q}) \). In [14 Example 3.15] we have considered the case when \( \mathbb{G} \) is of type \( C_n \) and \( q \) is odd.

Let \( k \) be an integer such that \( \sigma(\xi) = \xi^k \) for a primitive \( m \)-th root of unity \( \xi \in \mathbb{Q}_m \). Let \( q = p^f \) and let \( s = \gcd(e, f) \). Let \( \mu \) be a primitive \((q - 1)\)-th root of unity. We let \( c \) be an integer such that \( \mu^c = k \in \mathbb{F}_q^\times \). If \( c \) is even then

\[
|\text{Irr}_{p'}(B)^\sigma| = p^{sn} + 3p^{s(n-1)}.
\]

If \( c \) is odd we obtain

\[
|\text{Irr}_{p'}(B)^\sigma| = p^{sn} - p^{s(n-1)}.
\]

Let us single out a special case. If \( \sigma = \text{id} \) then \( s = f \) and \( c \equiv 0 \mod 2 \). This implies that

\[
|\text{Irr}_{p'}(B)| = p^{fn} + 3p^{f(n-1)} = q^n + 3q^{n-1}
\]

and we recover as a special case the result obtained by Maslowski in [10 Example 11.7].

5.3. Relating the \( p' \)-characters of \( \tilde{G} \) and \( G \). We assume from now on that \( p \) is a good prime for \( G \). In particular, the assumptions of Theorem 4.4 are satisfied.

The following lemma describes the action of Galois automorphisms on the Alvis–Curtis dual of the Gelfand–Graev characters.

Lemma 5.8. Let \( \sigma \in \text{Gal}(\mathbb{Q}/\mathbb{G}/\mathbb{Q}) \) and \( \tilde{t} \in \tilde{T} \) such that \( \psi_1^\sigma = \psi_{\tilde{t}}^1 \). Then \( D_G(\Gamma_1)^\sigma = D_G(\Gamma_1)^{\tilde{t}} \).

Proof. Recall that the functor \( D_G \) is defined using Harish–Chandra induction and restriction which is compatible with the action of Galois automorphisms. On the other hand the Levi subgroups and parabolic subgroups which occur in the definition of \( D_G \) can be chosen such that they are stabilized by the \( \tilde{T} \)-conjugation. So we deduce that \( D_G(\Gamma_1^\sigma) = D_G(\Gamma_1)^{\tilde{t}} \) and
\[ D_G(\Gamma_1^\varphi) = D_G(\Gamma_1)^\varphi. \] Since \( \psi_1^\varphi = \psi_1^\varphi \) we conclude \( \Gamma_1^\varphi = \Gamma_1^\varphi \). Consequently, we have \( D_G(\Gamma_1)^\sigma = D_G(\Gamma_1)^\varphi \).

\[ \square \]

The following proposition can be seen as an analog of Proposition 5.6 for the global characters.

**Proposition 5.9.** Let \( \varphi \in \tilde{T} \) such that \( \psi_1^\varphi = \psi_1^\varphi \). Let \( \chi \in \text{Irr}(\tilde{G}) \) and \( \vartheta \in \text{Irr}(G \mid \chi) \). Then the following are equivalent:

(i) \( \vartheta \) is \( \sigma \)-invariant

(ii) \( \chi_G \) is \( \sigma \)-invariant and \( \varphi \in I_B(\vartheta) \).

**Proof.** By Clifford’s theorem the condition that \( \chi \) is \( \sigma \)-invariant is necessary for the irreducible constituents of \( \chi_G \) to be \( \sigma \)-invariant. Thus, we may assume that \( \chi_G \) is \( \sigma \)-invariant.

Recall from Lemma 5.3 that \( \chi = \varepsilon D_G(\chi(s)) \) for some semisimple conjugacy class \( (s) \) of \( \tilde{G}^s \) and some \( \varepsilon \in \{ \pm 1 \} \). Moreover, by Lemma 3.4 we know that the characters \( \varepsilon D_G(\chi(s), \varepsilon) \), \( \varepsilon \in H^1(F, Z(G)) \), are precisely the irreducible constituents of \( \chi_G \). Note that \( \vartheta := \varepsilon D_G(\chi(s), 1) \) is the unique common irreducible constituent of \( D_G(\Gamma_1) \) and \( \chi \), see remark before Lemma 3.4. Then \( \vartheta^\sigma \) is the unique common irreducible constituent of \( D_G(\Gamma_1)^\sigma \) and \( \chi^\sigma \). By Lemma 5.8 we have \( D_G(\Gamma_1)^\sigma = D_G(\Gamma_1)^\varphi \). Since \( \chi_G \) is \( \sigma \)-invariant we conclude that \( \vartheta^\sigma = \varphi^\sigma \).

\[ \square \]

5.4. Proof of the main result.

**Lemma 5.10.** Let \( \sigma \in \text{Gal}(\mathbb{Q}_1/G) \) be an \( (e, p) \)-Galois automorphism. Let \( \psi \in \text{Irr}(\hat{B}) \) and \( \chi = \hat{f}(\psi) \). Then the character \( \psi_B \) is \( \sigma \)-invariant if and only if \( \chi_G \) is \( \sigma \)-invariant.

**Proof.** Suppose that \( \psi_B \) is \( \sigma \)-invariant. Then \( \psi^\sigma = \psi \eta \) for some \( \eta \in \text{Irr}(\hat{B} \mid 1_B) \) by Theorem 4.5. Using Theorem 4.6 we obtain \( \hat{f}(\psi^\sigma) = \hat{f}(\psi) \hat{f}(\eta) \) with \( \hat{f}(\eta) \in \text{Irr}(\hat{G} \mid 1_G) \). By Theorem 5.2 we have \( \hat{f}(\psi^\sigma) = \hat{f}(\psi) \). Therefore

\[ \chi^\sigma_G = (\hat{f}(\psi)_G)^\sigma = \hat{f}(\psi^\sigma)_G = \hat{f}(\psi)_G \hat{f}(\eta)_G = \hat{f}(\psi)_G = \chi_G, \]

which shows that \( \chi_G \) is \( \sigma \)-invariant. An analogous argument shows that if \( \chi_G \) is \( \sigma \)-invariant then \( \psi_B \) is \( \sigma \)-invariant.

\[ \square \]

**Corollary 5.11.** Let \( \sigma \in \text{Gal}(\mathbb{Q}_1/G) \) be an \( (e, p) \)-Galois automorphism. Let \( \psi \in \text{Irr}(\hat{B}) \) and let \( \chi = \hat{f}(\psi) \). All irreducible constituents of \( \psi_B \) are \( \sigma \)-invariant if and only if all irreducible constituents of \( \chi_G \) are \( \sigma \)-invariant.

**Proof.** By Lemma 5.10 it follows that \( \psi_B \) is \( \sigma \)-invariant if and only if \( \chi_G \) is \( \sigma \)-invariant. Let \( \vartheta \in \text{Irr}(B \mid \psi) \) and \( \phi \in \text{Irr}(G \mid \chi) \). By Corollary 4.9 we obtain \( I_B(\psi) = I_G(\phi) \cap \hat{B} \). Now the claim follows from Proposition 5.6 and Proposition 5.9.

\[ \square \]

Now we can prove Theorem 5.12 from the introduction.

**Theorem 5.12.** Let \( \sigma \in \text{Gal}(\mathbb{Q}_G/F) \) be an \( (e, p) \)-Galois automorphism. Suppose that \( p \) is a good prime for \( G \). Then the bijection constructed in Theorem 4.7 restricts to a bijection

\[ f : \text{Irr}(B)^\sigma \rightarrow \text{Irr}(G)^\sigma. \]

Moreover, for every central character \( \lambda \in \text{Irr}(Z(G)) \) the map \( f \) restricts to a bijection \( \text{Irr}(B \mid \lambda)^\sigma \rightarrow \text{Irr}(G \mid \lambda)^\sigma \).

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Proof. Let \( f : \text{Irr}_p'(B) \to \text{Irr}_p'(G) \) be the bijection constructed in the proof of 4.7. Thus, it is sufficient to see that \( \vartheta \) is \( \sigma \)-invariant if and only if \( f(\vartheta) \) is \( \sigma \)-invariant for \( \vartheta \in \text{Irr}_p'(B) \). Let \( \psi \in \text{Irr}_p'(\check{B} \mid \vartheta) \). By construction of \( f \) it follows that \( f(\vartheta) \in \text{Irr}(G \mid \check{f}(\psi)) \). By Corollary 5.11 it follows that the character \( \vartheta \) is \( \sigma \)-invariant if and only if \( f(\vartheta) \) is \( \sigma \)-invariant. The statement about compatibility with central characters follows from the statement about central characters in Theorem 4.7. □

Using Corollary 5.3 we can prove Theorem 5.12 for some more cases.

**Corollary 5.13.** Let \( \sigma \in \text{Gal}(\mathbb{Q} \mid G \mid \mathbb{Q}) \) be an \((e,p)\)-Galois automorphism. Suppose that \((G,F)\) is not contained in the table of Theorem 4.4. Assume in addition that \( p \neq 2 \) if \((G,F)\) is of type \((E_6,F_q)\) and \( p \neq 3 \) if \((G,F)\) is of type \((E_7,F_q)\). Then the bijection constructed in Theorem 4.7 restricts to a bijection

\[ f : \text{Irr}_p'(B)^\sigma \to \text{Irr}_p'(G)^\sigma. \]

Moreover, for every central character \( \lambda \in \text{Irr}(Z(G)) \) the map \( f \) restricts to a bijection \( \text{Irr}_p'(B \mid \lambda)^\sigma \to \text{Irr}_p'(G \mid \lambda)^\sigma \).

**Proof.** If \( p \) is a good prime then this is precisely Theorem 5.12. Now suppose that \( p \) is a bad prime. By an inspection of [1, Table 13.11] we observe that \( Z(G) = 1 \) unless \( p = 2 \) and \((G,F)\) is of type \((E_6,F_q)\) or \( p = 3 \) and \((G,F)\) is of type \((E_7,F_q)\). If \( Z(G) = 1 \) we can apply Corollary 5.3. □

**References**

[1] M. Cabanes and M. Enguehard. *Representation Theory of Finite Reductive Groups*. Cambridge University Press, Cambridge, new edition, 2004.

[2] R. W. Carter. *Finite Groups of Lie Type – Conjugacy Classes and Complex Characters*. John Wiley & Sons, Ltd., Chichester, 1985.

[3] F. Digne, G. I. Lehrer, and J. Michel. The characters of the group of rational points of a reductive group with nonconnected centre. *J. Reine Angew. Math.*, 425:155–192, 1992.

[4] F. Digne and J. Michel. *Representations of Finite Groups of Lie Type*. Cambridge University Press, Cambridge, 1991.

[5] R. B. Howlett. On the degrees of Steinberg characters of Chevalley groups. *Math. Z.*, 135:125–135, 1973/74.

[6] I. M. Isaacs. *Character Theory of Finite Groups*. Courier Corporation, New York, 2013.

[7] I. M. Isaacs, G. Malle, and G. Navarro. A reduction theorem for the McKay conjecture. *Invent. Math.*, 170(1):33–101, 2007.

[8] G. I. Lehrer. On the value of characters of semisimple groups over finite fields. *Osaka J. Math.*, 15(1):77–99, 1978.

[9] G. Malle and D. Testerman. *Linear Algebraic Groups and Finite Groups of Lie Type*. Cambridge University Press, Cambridge, 2011.

[10] J. Maslowski. *Equivariant Character Bijections in Groups of Lie Type*. PhD thesis, TU Kaiserslautern, 2010.

[11] G. Navarro. The McKay conjecture and Galois automorphisms. *Ann. of Math. (2)*, 160(3):1129–1140, 2004.

[12] G. Navarro and P. H. Tiep. Rational irreducible characters and rational conjugacy classes in finite groups. *Trans. Amer. Math. Soc.*, 360(5):2443–2465, 2008.

[13] Gabriel Navarro, Pham Huu Tiep, and Alexandre Turull. \( p \)-rational characters and self-normalizing Sylow \( p \)-subgroups. *Represent. Theory*, 11:84–94, 2007.

[14] L. Ruhstorfer. The Navarro Refinement of McKay’s Conjecture for Groups of Lie Type. *http://www2.math.uni-wuppertal.de/~ruhstorf/files/master.pdf*, 2016.

[15] B. Späth. Inductive McKay condition in defining characteristic. *Bull. Lond. Math. Soc.*, 44(3):426–438, 2012.
R. Steinberg. Regular elements of semisimple algebraic groups. *Inst. Hautes Études Sci. Publ. Math.*, (25):49–80, 1965.

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