THE SPACE OF STABILITY CONDITIONS FOR $A_3$-QUIVER

TAKAHIISA SHINA

Abstract. The author studied in [Shi11] the covering map property of the local homeomorphism associating to the space of stability conditions over the $n$-Kronecker quiver. In this paper, we discuss the covering map property for stability conditions over the Dynkin quiver of type $A_3$. The local homeomorphism from a connected component of stability conditions over $A_3$ to 3-dimensional complex vector space becomes a covering map when we restrict it to the complement of six codimension one subspaces.

1. Introduction

T. Bridgeland introduced the notion of stability conditions on triangulated categories ([Bri07]). The idea comes from Douglas’s work on $\pi$-stability for D-branes in string theory ([Dou02]). Bridgeland defined a topology (generalized metric) on the set of all (locally finite) stability conditions, $\text{Stab}(\mathcal{F})$, on a triangulated category $\mathcal{F}$. Each connected component $\Sigma \subset \text{Stab}(\mathcal{F})$ is equipped with a local homeomorphism $\mathcal{Z}$ to a certain topological vector space $V(\Sigma)$ ([Bri07, Theorem 1.2]). Moreover Bridgeland showed that $\Sigma$ is a (possibly infinite-dimensional) complex manifold.

It is an important problem to show simply connectivity of $\Sigma$, connectivity of $\text{Stab}(\mathcal{F})$, or (universal) covering map property of the local homeomorphism $\mathcal{Z} : \Sigma \to V(\Sigma)$. It have been studied intensively, for example, stability conditions over K3 surfaces, over curves, and over Kleinian singularities are studied [Bri09, Bri08, BT11, IUU10, Mac07, ST01, Tho06].

In [Shi11], the author studied the space of stability conditions $\text{Stab}(P_n)$ constructed on the bounded derived category $D^b(P_n)$ of the finite dimensional representations over $n$-Kronecker quiver $P_n$ – the quiver with two vertices and $n$-parallel arrows. More precisely, we discussed the covering map property of the local homeomorphism

$$\mathcal{Z} : \text{Stab}(P_n) \to \text{Hom}_Z(K(P_n), \mathbb{C}).$$

As we only refer to the 1-Kronecker quiver $P_1$, i.e. the Dynkin quiver of type $A_2$, the restriction of $\mathcal{Z}$ onto the complement of three lines,

$$\mathcal{Z}|_{\mathcal{Z}^{-1}(X)} : \mathcal{Z}^{-1}(X) \to \text{Hom}_Z(K(P_1), \mathbb{C}) \setminus (L_0 \cup L_1 \cup L_2) = X,$$

is a covering map. Here $L_i = \{ Z \in \text{Hom}_Z(K(P_1), \mathbb{C}) \mid Z(E_i) = 0 \}$ and $E_0, E_1, E_2$ are well-known exceptional objects of $D^b(P_1)$.

In this paper, we discuss the covering map property for stability conditions, $\text{Stab}(A_3)$, over the Dynkin quiver of type $A_3$. Following is the Main Theorem:

**Theorem 1.1.** Let $\Sigma(A_3)$ be a connected component of $\text{Stab}(A_3)$ associating to exceptional collections of $D^b(A_3)$. The restriction of the local homeomorphism $\mathcal{Z} : \Sigma(A_3)$ to the open set $\Sigma(A_3) \setminus (L_0 \cup L_1 \cup L_2) = X$ is a covering map.
\[\Sigma(A_3) \to \text{Hom}_\mathbb{Z}(K(A_3), \mathbb{C}) \text{ becomes a covering map when we remove, at least, six codimension one subspaces from base space, i.e.} \]

\[Z : Z^{-1} \to \text{Hom}_\mathbb{Z}(K(A_3), \mathbb{C}) \setminus (L_1 \cup L_2 \cup L_3 \cup L_4 \cup L_5 \cup L_6)\]

is a covering map, where \(L_i = \{ Z \in \text{Hom}_\mathbb{Z}(K(A_3), \mathbb{C}) | Z(E_i) = 0 \}\) and \(E_i\)'s are well-known exceptional objects of \(D^b(A_3)\).

The organization of this paper is as follows: In Section 2 we prepare basic definitions of stability conditions by Bridgeland (2.1) and useful results by Macrì relating to stability conditions on triangulated categories generated by finitely many exceptional objects (2.2). Then we prove the Main Theorem in Section 3, including basic properties of \(D^b(A_3)\) and connectivity of \(\Sigma(A_3)\).

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## 2. Preliminaries

In this section we recall definitions and properties for Bridgeland’s stability conditions on triangulated categories, and recall Macrì’s works for the space of stability conditions on triangulated categories generated by exceptional collection. See [Bri07] and [Mac07] for more details.

### 2.1. Stability conditions

Let \(k\) be a field and \(\mathcal{T}\) be a \(k\)-linear triangulated category. The Grothendieck group \(K(\mathcal{T})\), is the quotient group of the free abelian group generated by all isomorphism classes of objects in \(\mathcal{T}\) modulo the subgroup generated by the elements of the form \([A]+[B]-[C]\) for each distinguished triangle \(A \to C \to B\) in \(\mathcal{T}\).

A stability condition on \(\mathcal{T}\) consists of a central charge and a slicing. A central charge is a group homomorphism \(Z : K(\mathcal{T}) \to \mathbb{C}\). A slicing \(P\) is a family of full additive subcategories \(P(\phi)\) of \(\mathcal{T}\) indexed by real numbers \(\phi\) satisfying that \(\text{Hom}_{\mathcal{T}}(A_1, A_2) = 0\) if \(A_1 \in P(\phi_1)\) and \(\phi_1 > \phi_2\), \(P(\phi + 1) = P(\phi)[1]\) for all \(\phi \in \mathbb{R}\) and for each nonzero object \(X\) there are sequence of maps \(X_0 \to X_1 \to \cdots \to X_n\) in \(\mathcal{T}\) and sequence of real numbers \(\phi_1 > \cdots > \phi_n\) such that \(X_0 = X, X_n = X\) and \(A_i \in P(\phi_i)\) for all \(i\) which are objects fitting into the distinguished triangle \(X_{i-1} \to X_i \to A_i\). A pair of a central charge and a slicing, \(\sigma = (Z, P)\), called a stability condition if \(Z(A) = m(A) \exp(i\pi \phi)\) for any nonzero object \(A \in P(\phi)\) and some \(m(A) > 0\). A nonzero object of \(P(\phi)\) is called semistable of phase \(\phi\) and a simple object of \(P(\phi)\) stable. A stability condition \((Z, P)\) is called locally finite if there exists \(\varepsilon > 0\) such that \(P(\phi - \varepsilon, \phi + \varepsilon)\) is Artinian and Noetherian.

Let \(\text{Stab}(\mathcal{T})\) be the set of all locally-finite stability conditions on \(\mathcal{T}\). A generalized metric on \(\text{Stab}(\mathcal{T})\) is defined by

\[d(\sigma_1, \sigma_2) = \sup_{0 \neq E_0 \in \mathcal{T}} \left\{ |\phi_{\sigma_2}^- (E) - \phi_{\sigma_1}^- (E)|, |\phi_{\sigma_2}^+ (E) - \phi_{\sigma_1}^+ (E)|, \left| \log \frac{m_{\sigma_2}(E)}{m_{\sigma_1}(E)} \right| \right\} \in [0, \infty)\]

for \(\sigma_1, \sigma_2 \in \text{Stab}(\mathcal{T})\). Here \(\phi_{\sigma}^- (E)\) is the lowest number \(\phi_n\) and \(\phi_{\sigma}^+ (E)\) is the greatest number \(\phi_1\) in the sequence of \(E\) associated to \(\sigma\) and \(m_\sigma(E) = \sum_{i=1}^{\infty} |Z(A_i)|\).

When we equip a well-defined linear topology on \(\text{Hom}_\mathbb{Z}(K(\mathcal{T}), \mathbb{C})\), Bridgeland showed that there is a natural local homeomorphism.
Theorem 2.1. [Bri07] Theorem 1.2] For each connected component \( \Sigma \subset \text{Stab}(\mathcal{F}) \) there are a linear subspace \( V(\Sigma) \subset \text{Hom}_{\mathbb{Z}}(K(\mathcal{F}), \mathbb{C}) \) and a local homeomorphism \( \mathcal{Z} : \Sigma \to V(\Sigma) \) which maps a stability condition \((Z, \mathcal{P})\) to its central charge \(Z\).

The additional important structure on \( \text{Stab}(\mathcal{F}) \) is the right action of \( \text{GL}^+(2, \mathbb{R}) \), the universal covering of \( \text{GL}^+(2, \mathbb{R}) \), and the left action of the autoequivalences of \( \mathcal{F} \) (see [Bri07]).

2.2. Triangulated categories generated by exceptional objects. An object \( E \) in \( \mathcal{F} \) is exceptional if \( \text{Hom}^k_{\mathcal{F}}(E, E) = \mathbb{C} \) if \( k = 0 \) and \( = 0 \) otherwise. For two exceptional objects \( E \) and \( F \), we write \( \mathcal{L}_E F \) to be a left mutation of \( F \) by \( E \) and \( \mathcal{R}_E F \) a right mutation of \( E \) by \( F \), which are objects fitting into distinguished triangles: \( \mathcal{L}_E F \to \text{Hom}^*(E, F) \otimes E \to F \) and \( E \to \text{Hom}^*(E, F)^* \otimes F \to \mathcal{R}_E F \).

An exceptional collection is a sequence \( \mathcal{E} = (E_1, E_2, \ldots, E_n) \) of exceptional objects such that \( \text{Hom}^k_{\mathcal{F}}(E_i, E_j) = 0 \) for all \( k \) and \( i > j \). An exceptional collection is called complete if \( \{E_i\} \) generates \( \mathcal{F} \) by shifts and extensions and is called Ext-exceptional if \( \text{Hom}^{<0}_{\mathcal{F}}(E_i, E_j) = 0 \) for all \( i \neq j \). A left mutation \( \mathcal{L}_i \mathcal{E} \) and a right mutation \( \mathcal{R}_i \mathcal{E} \) of \( \mathcal{E} \) are defined by

\[
\mathcal{L}_i \mathcal{E} = (E_1, \ldots, E_{i-1}, E_{i+1}, E_i, E_{i+2}, \ldots, E_n)
\]

and

\[
\mathcal{R}_i \mathcal{E} = (E_1, \ldots, E_{i-1}, E_{i+1}, E_i, E_{i+2}, \ldots, E_n).
\]

The mutation of a (complete) exceptional collection becomes again a (complete) exceptional collection. The operations \( \mathcal{L}_i \) and \( \mathcal{R}_i \) are invertible each other: \( \mathcal{L}_i \mathcal{R}_i = \mathcal{R}_i \mathcal{L}_i = \text{id} \) for each \( i \), and they satisfy the braid relation: \( \mathcal{R}_i \mathcal{R}_{i+1} \mathcal{R}_i = \mathcal{R}_{i+1} \mathcal{R}_i \mathcal{R}_{i+1} \) and \( \mathcal{L}_i \mathcal{L}_{i+1} \mathcal{L}_i = \mathcal{L}_{i+1} \mathcal{L}_i \mathcal{L}_{i+1} \).

Macrì showed how to construct a stability condition from complete exceptional collections. The key rules are following two theorem: To give a stability condition is equivalent to giving a bounded t-structure and a stability function on its heart [Bri07, Proposition 5.3]; Taking a sequence of integers \( p = (p_1, \ldots, p_n) \) with \( \mathcal{E}_p = (E_1[p_1], \ldots, E_n[p_n]) \) is an Ext-exceptional collection, the smallest extension-closed full subcategory generated by \( \mathcal{E}_p \), \( Q_p \), is a heart of a bounded t-structure [Mac07, Lemma 3.2]. By fixing \( n \)-points \( z_1, \ldots, z_n \) in \( \{ m \exp(i \pi \phi) \mid m > 0, 0 < \phi \leq 1 \} \) and defining \( Z_p : K(\mathcal{F}) \to \mathbb{C} \) by \( Z_p(E_i[p_i]) = z_i \), the pair \( (Z_p, Q_p) \) occurs a stability condition.

We write \( \Theta_\mathcal{E} \) to be the set of all stability conditions defined from \( \mathcal{E} \) via above process up to the action of \( \text{GL}^+(2, \mathbb{R}) \). Macrì proved that \( \Theta_\mathcal{E} \) is homeomorphic to

\[
\{ (m_1, \ldots, m_n, \phi_1, \ldots, \phi_n) \in \mathbb{R}^{2n} \mid m_i > 0 \text{ for all } i \text{ and } \phi_i < \phi_j + \alpha_{i,j} \text{ for } i < j \}
\]

where

\[
\alpha_{i,j} = \min_{i < t_1 < t_2 < \cdots < t_s < j} \{ k_{i,t_1} + k_{i,t_2} + \cdots + k_{i,t_s} - s \}
\]

and

\[
k_{i,j} = \begin{cases} +\infty & \text{if } \text{Hom}^k(E_i, E_j) = 0 \text{ for all } k, \\ \min \{ k \mid \text{Hom}^k(E_i, E_j) \neq 0 \} & \text{otherwise.} \end{cases}
\]

A stability condition \( \sigma = (Z, \mathcal{P}) \) in \( \Theta_\mathcal{E} \) corresponds to \( (m_1, \ldots, m_n, \phi_1, \ldots, \phi_n) \) by \( m_i = |Z(E_i)| \) and \( \phi_i = \phi_\sigma(E_i) \). Moreover, all \( E_i \)'s are stable in \( \sigma \in \Theta_\mathcal{E} \) (Mac07 Lemma 3.16) and \( \Theta_\mathcal{E} \) is an open, connected and simply connected \((n+1)\)-dimensional submanifold (Mac07 Lemma 3.19).
The union of the open subsets $\Theta_{\mathcal{F}}$ over all iterated mutations $\mathcal{F}$ of $\mathcal{E}$ is denoted as $\Sigma_{\mathcal{E}}$. It is also an open and connected $(n + 1)$-dimensional submanifold ([Mac07, Corollary 3.20]). When all exceptional collections of $\mathcal{F}$ can be obtained, up to shifts, by iterated mutations of a single exceptional collection, the previous open subset is denoted as $\Sigma(\mathcal{F})$ and such $\mathcal{F}$ is called constructible.

3. Proof of the Main Theorem

First we recall some basic facts of the Dynkin quiver of type $A_3$: $1 \to 2 \to 3$. We denote by $\mathbb{C}A_3$ the path algebra of $A_3$ over the ground field $\mathbb{C}$, denote by $\text{mod} \mathbb{C}A_3$ the category of finitely generated $\mathbb{C}A_3$-modules, denote by $D^b(\mathbb{C}A_3)$ the bounded derived category of $\text{mod} \mathbb{C}A_3$ and denote by $\text{Stab}(\mathbb{C}A_3)$ the space of stability conditions on $D^b(\mathbb{C}A_3)$. Note that the category of finitely generated modules over a path algebra of a quiver is equivalent to the category of finite dimensional representations over the one.

The Auslander-Reiten quiver of $\text{mod} \mathbb{C}A_3$ is well-known, and in this paper we write it as follows:

![Auslander-Reiten quiver diagram]

where $S_1$, $S_2$, and $S_3$ are simple objects relating to vertices of $A_3$.

The Grothendieck group, $K(\mathbb{C}A_3)$, of $D^b(\mathbb{C}A_3)$ is a free abelian group generated by isomorphism classes of $S_1$, $S_2$, and $S_3$.

The set of all equivalent classes of complete exceptional collections on $D^b(\mathbb{C}A_3)$ is shown in [Ara13] in which two exceptional collections $(E_1, \ldots, E_n)$ and $(F_1, \ldots, F_n)$ are called equivalent if there exists a permutation $\sigma$ and integers $l_1, \ldots, l_n$ such that $F_i = E_{\sigma(i)}[l_i]$ for every $i$. There are 12 representatives of complete exceptional collections on $D^b(\mathbb{C}A_3)$, listed in Table 1 in our notation. In the Table, each components

\[(X) \quad S_x \xleftarrow{a} S_y \xrightarrow{b} S_z \xleftarrow{c} S_x\]

means that $X$ is the exceptional collection $(S_x, S_y, S_z)$ such that

\[
a = \min \left\{ k \mid \text{Hom}^k_{D^b(\mathbb{C}A_3)}(S_x, S_y) \neq 0 \right\},
\]

\[
b = \min \left\{ k \mid \text{Hom}^k_{D^b(\mathbb{C}A_3)}(S_x, S_z) \neq 0 \right\} \quad \text{and}
\]

\[
c = \min \left\{ k \mid \text{Hom}^k_{D^b(\mathbb{C}A_3)}(S_y, S_z) \neq 0 \right\},
\]

however we omit the arrow if $\text{Hom}^k_{D^b(\mathbb{C}A_3)}(S_x, S_y) = 0$ for all $k$.

Note that $\text{Hom}^k_{D^b(\mathbb{C}A_3)}(S_x, S_y) \neq 0$ for at most one $k$ and all $x, y$, and that $\text{Hom}^1_{D^b(\mathbb{C}A_3)}(S_{12}, S_{23}) = \mathbb{C}$ and $\text{Hom}^0_{D^b(\mathbb{C}A_3)}(S_{23}, S_{12}) = \mathbb{C}$.

Any two exceptional collections of $D^b(\mathbb{C}A_3)$ are transitive by iterated mutations, as illustrated in Figure 1 in which solid arrows mean $R_1$, the right mutation between left and center, and dotted arrows mean $R_2$, between center and right. Applying
Table 1. Representatives of exceptional collections on $D^b(A_3)$

|   |   |   |
|---|---|---|
| (A) $S_1 S_2 S_3$ | (B) $S_2 S_1 S_3$ | (I) $S_{12} S_1 S_3$ |
| (C) $S_2 S_3 S_{123}$ | (D) $S_3 S_{23} S_{123}$ | (J) $S_{23} S_2 S_{123}$ |
| (E) $S_3 S_{123} S_1$ | (F) $S_{123} S_1 S_2$ | (K) $S_2 S_{123} S_{12}$ |
| (G) $S_{123} S_1 S_2$ | (H) $S_1 S_{23} S_2$ | (L) $S_3 S_1 S_{23}$ |

$\mathcal{R}_2$ (resp. $\mathcal{R}_1$) to $I$ or $J$ (resp. $K$ or $L$) occurs an exceptional collection equivalent to itself.

3.1. **Applying Macri’s method to $D(A_3)$**. Let $X = (E_1, E_2, E_3)$ be one of the exceptional collection in the following list:
According to Macrì, $\Theta_\mathcal{E}$ is homeomorphic to
\[
\{ (m_1, m_2, m_3, \phi_1, \phi_2, \phi_3) \in \mathbb{R}^6 \mid m_i > 0 \text{ and } \phi_i < \phi_j + \alpha_{i,j} \ (i < j) \},
\]
where $\alpha_{i,j}$'s and $k_{i,j}$'s are defined as follows:

|   | I  | II | III | IV |
|---|----|----|-----|----|
| $k_{1,2}$ | $a$ | $a$ | $a$ | $+\infty$ |
| $k_{2,3}$ | $b$ | $b$ | $+\infty$ | $b$ |
| $k_{1,3}$ | $+\infty$ | $c$ | $c$ | $c$ |
| $\alpha_{1,2}$ | $a$ | $a$ | $a$ | $+\infty$ |
| $\alpha_{2,3}$ | $b$ | $b$ | $+\infty$ | $b$ |
| $\alpha_{1,3}$ | $a + b - 1$ | $\min\{a + b - 1, c\}$ | $c$ | $c$ |

Applying this to exceptional collections in Table 1, we obtain relations of $\phi_i$ for each exceptional collection $X$ as Table 2 in which we simply denote $S_i$ instead of its phase $\phi_i$. And underlying figures illustrate these relations graphically, where solid, dashed or dotted arrows mean $Z(S_i)$ in the complex plane and curved arrows mean that the phase of $S_x$ on arrow-tail is lower than the one of $S_y$ on arrow-head. For example, we have:

\[
\Theta_A \cong \left\{ (m_1, m_2, m_3, \phi_1, \phi_2, \phi_3) \in \mathbb{R}^6 \mid \begin{array}{l}
m_i > 0, \\
\phi_1 < \phi_2 + 1, \\
\phi_2 < \phi_3 + 1, \\
\phi_1 < \phi_3 + 1
\end{array} \right\}
\]

\[
m_i = |Z(S_i)|, \quad \phi_i = \phi_{\sigma}(S_i) \ (= S_i \text{ for simply})
\]

It is easy to show that $\Theta_X = \Theta_Y$ if $X$ is equivalent (in the sense of Araya [Ara13]) to $Y$.

Since $D^b(A_3)$ is constructible, we denote $\Sigma(A_3)$ to be the union of all $\Theta_X$ ($X = A, \ldots, L$). It is an open and connected 3-dimensional submanifold, and is also a connected component of $\text{Stab}(A_3)$ (we will see the proof in Appendix).

3.2. **Main Theorem.** Let $L_i = \{ Z \in \text{Hom}_\mathbb{Z}(K(A_3), \mathbb{C}) \mid Z(S_i) = 0 \} \ (i = 1, 2, 3)$, $L_4 = \{ Z(S_{12}) = 0 \}$, $L_5 = \{ Z(S_{23}) = 0 \}$ and $L_6 = \{ Z(S_{123}) = 0 \}$. The subject of this paper is that the local homeomorphism $Z : \Sigma(A_3) \to \text{Hom}_\mathbb{Z}(K(A_3), \mathbb{C})$ becomes a covering map if it is restricted to the inverse image of $\text{Hom}_\mathbb{Z}(K(A_3), \mathbb{C}) \setminus (L_1 \cup \cdots \cup L_6)$.

3.3. **The image of the local homeomorphism.** First we show that the image of $Z : \Sigma(A_3) \to \text{Hom}_\mathbb{Z}(K(A_3), \mathbb{C})$ is $\text{Hom}_\mathbb{Z}(K(A_3), \mathbb{C}) \setminus \{0\}$. It is clear that $\text{Im}(Z)$ does not contain 0.

When a central charge $Z : K(A_3) \to \mathbb{C}$ vanishes at most two $S_i$ ($i = 1, 2, 3$), there exists a stability condition $(Z, 0)$ by appropriate choice of phases of $S_x$ and $X = A, \ldots, L$.

For example, let $Z$ be as the left one in Figure 2. Taking phases as $S_1 = \alpha$, $S_2 = \beta + 2$ and $S_3 = \gamma + 2$, $(Z, A_{(-1,-2,-2)})$ is in $\Theta_A$. Remark that $(Z, A_{(-1,-2,-2)})$ is a stability condition given by the pair of a stability function and an additive full subcategory generated by Ext-exceptional collection $(S_1[-1], S_2[-2], S_3[-2])$. Otherwise, taking $S_2 = \beta$, $S_3 = \gamma$ and $S_{123} = \gamma + 2$, $(Z, C_{(0,0,-2)})$ is in $\Theta_C$. 

\[
\begin{array}{c|c|c|c|c}
(I) & E_1 & E_2 & E_3 \\
\hline (III) & E_1 & E_2 & E_3 \\
(II) & E_1 & E_2 & E_3 \\
(IV) & E_1 & E_2 & E_3
\end{array}
\]
Proof of the covering map property.

Table 2. $\phi_i < \phi_j + \alpha_{i,j}$ for $\Theta_X$

| $X$ | $\phi_{1, \phi_2}$ | $\phi_{2, \phi_3}$ | $\phi_1, \phi_3$ |
|-----|--------------------|--------------------|-----------------|
| A   | $S_1 < S_2 + 1$    | $S_2 < S_1 + 1$   | $S_1 < S_3 + 1$ |
| B   | $S_2 < S_{12}$     | $S_{12} < S_3 + 1$| $S_2 < S_3$    |
| C   | $S_2 < S_3 + 1$    | $S_3 < S_1_{23}$  | $S_2 < S_1_{23}$|
| D   | $S_1 < S_{23}$     | $S_{23} < S_{123}$| $S_3 < S_{123} - 1$ |
| E   | $S_3 < S_{123}$    | $S_{123} < S_1$   | $S_3 < S_1 - 1$|
| F   | $S_{123} < S_1$    | $S_{123} < S_1$   | $S_{123} < S_1 - 1$ |
| G   | $S_{123} < S_1$    | $S_1 < S_2 + 1$   | $S_{123} < S_2$|
| H   | $S_1 < S_{23} + 1$ | $S_{23} < S_2$    | $S_1 < S_2$    |
| I   | $S_{23} < S_2$     | $S_{23} < S_2$    | $S_{23} < S_2$ |
| J   | $S_{123} < S_{12}$ | $S_2 < S_{12}$    | $S_{123} < S_2$|
| K   | $S_{123} < S_1$    | $S_{123} < S_1$   | $S_{123} < S_2$|
| L   | $S_{123} < S_2$    | $S_1 < S_{23} + 1$| $S_1 < S_{23}$ |

Let $Z$ be as the center one in Figure 2. Taking phases as $S_2 = \beta$, $S_{12} = \beta + 2$ and $S_3 = \gamma + 2$, $(Z, B_{(0,-2,-2)})$ is in $\Theta_B$.

Let $Z$ be as the right one in Figure 2. Taking phases as $S_3 = \gamma$, $S_{23} = \gamma + 2$ and $S_{123} = \gamma + 4$, $(Z, D_{(0,-2,-4)})$ is in $\Theta_D$.

3.4. **Proof of the covering map property.** Next we show that if $Z$ is in $L_i$ ($i = 1, \ldots, 6$) then any open neighborhood $U$ of $Z$ has a non-homeomorphic inverse image. Although, it is almost clear. For instance, when $Z$ satisfies $Z(S_1) = 0$, $Z(S_2) \neq 0$ and $Z(S_3) \neq 0$, $Z^{-1}(U) \cap \Theta_A$ has a connected component $V$ such that $Z(V)$ covers over $U$ except $Z$ many times.
Finally, we finish the proof of Main Theorem.
Suppose that $Z$ does not vanish all $S_x$ ($x \in \{1, 2, 3, 12, 23, 123\}$) and $Z(S_x) \neq \lambda Z(S_y)$ for all nonzero real number $\lambda$. In this case, it is easy to see that $Z^{-1}(U) \cap \Theta_X$ consists of distinct open sets, which homeomorphic to $U$, for any $X = A, \ldots, L$.

The case we should concern is the neighborhood $U$ of $Z$ which lies on a boundary of $\Theta_X$. For example, $Z$ with $Z(S_1) = 1$, $Z(S_2) = -2$ and $Z(S_3) = i$ lie on the boundary of $\Theta_A$, that is, there is a connected component $V_A$ of $Z^{-1}(U) \cap \Theta_A$ such that it does not surjective on $U$. Table\textsuperscript{3} show that each boundary of $\Theta_X$ is included in the interior of some $\Theta_Y$. For example, the inverse image of $\{Z(S_2) = -\lambda Z(S_1) (\lambda > 0), Z(S_3) \neq 0\}$ is a boundary of $\Theta_A$ when we define the phase of $S_2$ is equal to the phase of $S_1$ minus 1 and the phase of $S_3$ is grater than it, and it also is a boundary of $\Theta_B$. However it is included in the interior of $\Theta_I$.

Now we conclude that the $Z^{-1}(U)$ is a disjoint union of connected components homeomorphic to $U$ for enough small open neighborhood of $Z$ in $\text{Hom}_\mathbb{Z}(\mathcal{K}(A_3), \mathbb{C})\setminus \{L_1, \ldots, L_6\}$.

Appendix A.

We prove that $\Sigma(A_3)$ is a connected component of $\text{Stab}(A_3)$. It is enough to show that $\Sigma(A_3) = \Sigma(A_3)$ since $\Sigma(A_3)$ is an open submanifold of $\text{Stab}(A_3)$.

Let $\sigma = (Z, \mathcal{P})$ lie in the boundary of $\Sigma(A_3)$. There is a sequence $\{\sigma_i = (Z_i, \mathcal{P}_i)\}_{i=1,2,3,\ldots}$ consisting of stability conditions in $\Sigma(A_3)$. We can assume all $\sigma_i$ belongs to $\Theta_{\mathcal{E}}$ for some exceptional collection $\mathcal{E}$ since $\Sigma(A_3)$ is a union of finitely many $\Theta_{\mathcal{E}}$'s. Put $\mathcal{E} = (E_1, E_2, E_3)$. Then all $E_j$'s are $\sigma_i$-stable for all $i$.

When we denote $X$ to be a subset of $\text{Stab}(A_3)$ consisting of all stability conditions in which $E_i$ ($i = 1, 2, 3$) are semistable, $X$ is a closed subset because of a subset consisting of all stability conditions which has an object as semistable is closed (\textsuperscript{[Bri07]} Section 5). Thus $\Theta_{\mathcal{E}}$ is included in $X$ and $\overline{\Theta_{\mathcal{E}}} \subset \overline{X} = X$. Therefore $\sigma$ is included in $X$, so $\sigma$ has all $E_i$'s as semistable.

Now we have $Z(E_i) \neq 0 (i = 1, 2, 3)$ and phases $\phi_{E_i}$ are defined. Because $\sigma$ lies in the boundary of $\Theta_{\mathcal{E}}$, $\phi_{\sigma}(E_i) = \phi_{\sigma}(E_j) + \alpha_{i,j}$ for one or two of $(i, j) = (1, 2), (1, 3), \text{or} (2, 3)$ and $m_{\sigma}(E_i) \neq m_{\sigma}(E_j)$ if $\alpha_{i,j} = k_{i,j}$.

However all possible stability conditions which belongs to the boundary of $\Theta_{\mathcal{E}}$ are listed in Table\textsuperscript{3} and it shows each such stability condition is included in some $\Theta_{\mathcal{E}}$. This shows that $\Sigma(A_3) \subset \Sigma(A_3)$.

References

\cite{Ara} Tokuji Araya. Exceptional sequences over path algebras of type $A_n$ and non-crossing spanning trees. \textit{Algebr. Represent. Theory}, 16(1):239–250, 2013.
Table 3.

| Diagram 1 | Diagram 2 | Diagram 3 |
|-----------|-----------|-----------|
| ![Diagram 1](image1.png) | ![Diagram 2](image2.png) | ![Diagram 3](image3.png) |

[1] Tom Bridgeland. Stability conditions on triangulated categories. *Ann. of Math.* (2), 166(2):317–345, 2007.

[2] Tom Bridgeland. Stability conditions on $K3$ surfaces. *Duke Math. J.*, 141(2):241–291, 2008.
[Bri09] Tom Bridgeland. Stability conditions and Kleinian singularities. *Int. Math. Res. Not. IMRN*, (21):4142–4157, 2009.

[BT11] Christopher Brav and Hugh Thomas. Braid groups and Kleinian singularities. *Math. Ann.*, 351(4):1005–1017, 2011.

[Dou02] Michael R. Douglas. Dirichlet branes, homological mirror symmetry, and stability. In *Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002)*, pages 395–408, Beijing, 2002. Higher Ed. Press.

[IUU10] Akira Ishii, Kazushi Ueda, and Hokuto Uehara. Stability conditions on $A_n$-singularities. *J. Differential Geom.*, 84(1):87–126, 2010.

[Mac07] Emanuele Macrì. Stability conditions on curves. *Math. Res. Lett.*, 14(4):657–672, 2007.

[Shi11] Takahisa Shiina. The space of stability conditions for quivers with two vertices. *arXiv:1105.2999* (To appear.)

[ST01] Paul Seidel and Richard Thomas. Braid group actions on derived categories of coherent sheaves. *Duke Math. J.*, 108(1):37–108, 2001.

[Tho06] R. P. Thomas. Stability conditions and the braid group. *Comm. Anal. Geom.*, 14(1):135–161, 2006.

Academic Support Center, Kogakuin University, Tokyo, Japan

E-mail address: kt13423@ns.kogakuin.ac.jp