Applications of closed models defined by counting to graph theory and topology.

Abstract.

In this paper, we introduce the notion of closed models defined by counting and we compute their homotopy categories. We apply this construction to various categories of graphs. We show that there does not exist a closed model defined in the category of undirected graphs which characterizes the Ihara Zeta function. Finally, we apply our construction to Galoisian complexes and dessins d’enfant.

1. Introduction.

The theory of closed models defined by Quillen in the setting of category theory provides the foundations of homotopy theory in various settings. Thus, ideas inherited from topology can be applied successfully to others fields of mathematics which are endowed with suitable closed models. To teach this important idea, it is necessary to have in hands examples which are easy to understand, are not trivial, and can be presented after a short introduction. A good framework to find such models is graph theory. In our papers [1] and [2] written in collaboration with Bisson, we have defined such Quillen models in the category of directed graphs; one has the virtue to characterize the Zeta function and the other is adapted to symbolic dynamic. It is an interesting question to ask whether such a similar model which classifies the Ihara Zeta function exists in the category of undirected graphs. In this paper, we answer negatively to this question. The models defined in [1] and [2] are particular examples of a model obtained by counting a family of objects $(X_i)_{i \in I}$ in a topos $C$; that is, its class of weak equivalences is the subclass $W$ of the class of maps of $C$, such that for every $f : X \to Y \in W$, and every $i \in I$, the natural morphism $c_f : \text{Hom}_C(X_i, X) \to \text{Hom}_C(X_i, Y)$ defined by $c_f(h) = f \circ h$ is bijective. We start this paper by presenting properties of closed models defined by counting, in particular, we characterize their homotopy category. We define models in subcategories of the category of undirected graphs obtained by counting which characterize the Ihara Zeta function of the objects of a large subclass of their class of objects. A
particular interesting example amongst these models is defined in the category $BC_n$ whose objects are $n$-colored graphs; this category is equivalent to the category of $G_n$-sets where $G_n$ is the group generated by $a_1, ..., a_n$ such that $a_i^2 = 1$, $i = 1, ..., n$. This category is studied by many others authors, we can quote for example Ladegaillerie [15] who has established an equivalence between $BC_n$ and the category of Galoisian $n$-complexes, we deduce the existence of Quillen models in these categories and in particular in the category of Galoisian 2-complexes which is related to dessins d’enfants.

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2. Quillen models.

In this section, we are going to present the basic properties of Quillen models defined by counting; we start by the following definitions:

**definitions 2.1.**

A class $W$ of morphisms of a category $C$ satisfies the $2 \rightarrow 3$ property if and only if for every morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ of $C$, if two morphisms of the triple $(f, g, g \circ f)$ is in $W$ then the third is also in $W$.

We say that the morphism $g : Y \rightarrow T$ has the right lifting property with respect to $h : X \rightarrow Z$, and that $h$ has the left lifting property with respect to $g$ if and only if for every commutative diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{p} & Y \\
h \downarrow & & \downarrow g \\
Z & \xrightarrow{q} & T
\end{array}
$$

there exists a morphism $l : Z \rightarrow Y$ such that $l \circ h = p$ and $g \circ l = q$.

Let $G$ and $H$ be two classes of morphisms of $C$, we say that $(G, H)$ is a weak factorization system if and only if:

- Every morphism $f \in C$ can be written $f = g \circ h$ where $g \in G, h \in H$;
- $G$ is the class of maps which has the right lifting property in respect of $H$;
- $H$ is the class of maps which has the left lifting property in respect of $G$.

Let $C$ be a category stable by limits and colimits; we say that $C$ is endowed with a closed model or equivalently a Quillen model if and only if there exists three classes of morphisms $(Fib, Cof, W)$ such that:

- $W$ satisfies the $2 \rightarrow 3$ property,
- Let \( \text{Fib}' = \text{W} \subseteq \text{Fib} \), \((\text{Fib}', \text{Cof})\) is a weak factorization system
- Let \( \text{Cof}' = \text{W} \subseteq \text{Cof} \), \((\text{Fib}, \text{Cof}')\) is a weak factorization system.

We start by the following general example:

**Proposition 2.2.**

Let \( C \) be a category stable by limits and colimits, let \( W \) be a class of maps of \( C \) which satisfies the \( 2 - 3 \) property. Suppose that there exists a class of morphisms \( \text{Cof} \) of \( C \) such that \((W, \text{Cof})\) is a weak factorization system; there exists a Quillen model on \( C \) whose class of weak equivalences is \( W \), the class of cofibrations is \( \text{Cof} \), the class of weak cofibrations \( \text{Cof}' \) is the class \( \text{Iso}(C) \) of isomorphisms of \( C \), the class of fibrations \( \text{Fib} \) is the class \( \text{Hom}(C) \) of all maps of \( C \) and the class of weak fibrations \( \text{Fib}' \) is \( W \).

**Proof.**

We have: \((\text{Fib}, \text{Cof}') = (\text{Hom}(C), \text{Iso}(C))\) and \((\text{Fib}', \text{Cof}) = (W, \text{Cof})\) are weak factorization systems. We also have \( \text{Fib} \cap W = \text{Hom}(C) \cap W = W = \text{Fib}' \) and \( \text{Cof} \cap W = \text{Iso}(C) \). Since \((W, \text{Cof})\) is a factorization system. We deduce that that \((\text{Hom}(C), W, W)\) defines a closed Quillen model on \( C \).

### 3. Quillen model and counting.

We are going to apply the previous result to define Quillen models to count objects in categories. Let \( C \) be a category stable by limits and colimits whose initial object is denoted by \( \phi \); for every objects \( X \) and \( Y \) of \( C \), we denote by \( X + Y \) the sum of \( X \) and \( Y \). Let \( (X_i)_{i \in I} \) be a family of objects of \( C \) and \( l_i : \phi \to X_i \) the canonical morphism. There exists morphisms \( j_1^i : X_i \to X_i + X_i \) and \( j_2^i : X_i \to X_i + X_i \) such that for every morphisms \( f : X_i \to Z \) and \( g : X_i \to Z \), there exists a unique morphism \( m(f, g) : X_i + X_i \to Z \) such that \( m(f, g) \circ j_1^i = f \) and \( m(f, g) \circ j_2^i = g \). We set \( m_i = m(Id_{X_i}, Id_{X_i}) \). Such a morphism is often called a folding map. We suppose that the class of maps \( l_i, m_i \in I \) admits the small element argument (see [11] 12.4.13). We denote by \( W_I \) the class of morphisms which are right orthogonal to the family of maps \( l_i \) and \( m_i \).

**Proposition 3.1.**

A morphism \( f : X \to Y \) of \( C \) is an element of \( W_I \) if and only if for every \( i \in I \), the map \( c_f^i : \text{Hom}_C(X_i, X) \to \text{Hom}_C(X_i, Y) \) defined by \( c_f^i(h) = f \circ h \) is bijective. We deduce that \( W_I \) satisfies the \( 2 - 3 \)-property and there exists a Quillen model on \( C \) whose class of weak equivalences is \( W_I \).

**Proof.**

Let \( f : X \to Y \) be a morphism of \( C \), suppose that \( f \) is orthogonal to \( l_i \) and \( m_i \) for every \( i \in I \). Let \( h, h' \in \text{Hom}_C(X_i, X) \) such that \( f \circ h = f \circ h' \). The following diagram commutes:

\[
\begin{array}{ccc}
X_i + X_i & \xrightarrow{h + h'} & X \\
\downarrow m_i & & \downarrow f \\
X_i & \xrightarrow{f \circ h} & Y
\end{array}
\]
Since $f$ is right orthogonal to $m_i$, we deduce the existence of a morphism $l : X_i \to X$ such that $l \circ m_i = h + h'$. We have $l \circ m_i \circ j_i^1 = l \circ m_i \circ j_i^2 = l$. We deduce that $h = (h + h') \circ j_i^1 = l \circ m_i \circ j_i^1 = l \circ m_i \circ j_i^2 = (h + h') \circ j_i^2 = h'$.

Let $h : X_i \to Y$ be any morphism, the following diagram commutes:

$$
\begin{array}{ccc}
\phi & \longrightarrow & X \\
\downarrow & & \downarrow f \\
X_i & \xrightarrow{h} & Y
\end{array}
$$

thus it has a filler $p : X_i \to X$ such that $f \circ p = h$. This implies that $c_f^i$ is bijective.

We show now that $W_I$ satisfies the 2-3 property. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of $C$, $c_{g \circ f} = c_g^i \circ c_f^i$. Since $c_f^i, c_g^i$ and $c_{g \circ f}$ are morphisms of sets, we deduce that if two morphisms between the triple $(c_f^i, c_g^i, c_{g \circ f})$ is bijective, so does the third.

Let $\text{cell}(I)$ be the class of maps of $C$ which are retracts of transfinite compositions of pushouts of $l_i, m_i, i \in I$, the propositions 12.4.14 and 12.4.20 of [11] imply that $(\text{cell}(I), W_I)$ is a factorization system. We deduce from the proposition 2.2 the existence of a Quillen model on $C$ whose class of weak equivalences is $W_I$.

Remarks.
Recall that a Quillen model $(\text{Fib}, \text{Cof}, W)$ on a category is cofibrantly generated (see [11] 13.2.2.) if there exists sets of morphisms $(f_i)_{i \in I}$ and $(g_j)_{j \in J}$ both which allow the small element argument such that the class of fibrations is the right orthogonal of $(g_j)_{j \in J}$ and the class of weak fibrations is the right orthogonal of $(f_i)_{i \in I}$. Thus, a Quillen model defined by counting is cofibrantly generated.

Let $K$ be a subset of $I$, we denote by $X_K$ the sum of $(X_k)_{k \in K}$. The morphism $\phi \to X_K$ is a cofibration, since it is transfinite composition of pushouts of elements of $(l_k)_{k \in K}$.

**The homotopy category of a Quillen model defined by counting.**

One of the main purpose of the theory of Quillen models is to find a proper framework to localize classes of morphisms. In this perspective, we are going to compute the homotopy category of a Quillen model defined by counting. We start by remarking the fact that since every morphism is a fibration in a Quillen model defined by counting, every object is fibrant. Let us determine now the cofibrant replacement of an object:

**Proposition 3.2.**

Let $C$ be a $U$-category endowed with a Quillen model defined by counting a set of objects $(X_i)_{i \in I}$ where $I$ is an $U$-set. Let $Z$ be an object of $C$, then $Z$ has a cofibrant replacement $QZ$, isomorphic to a transfinite composition of a subset of $l_i, m_i, i \in I$.

**Proof.**
We denote by Let $I_Z = \{ f \in \text{Hom}_C(X_i, Z), i \in I \}$. It is an $U$-set. We can define the sum of elements of $I_Z$, $d_Z : X_{I_Z} \to Z$, we denote by $i_f : X_i \to X_{I_Z}$ defined by $d_Z \circ i_f = f$. For every $i \in I$, $\text{Hom}_C(X_i, X_{I_Z}) \to \text{Hom}_C(X_i, Z)$ is surjective since if $h \in \text{Hom}_C(X_i, Z)$, we have $d_Z \circ i_h = h$. Consider $L(Z)$ the $U$-set whose objects are morphisms $p_V : X_{I_Z} \to V$ such that $p_V$ is a transfinite composition of pushouts of a subset of $m_i, i \in I$, and there exists a morphism $f_V : V \to Z$ such that $f_V \circ p_V = d_Z$. There exist a relation of order define on $L(Z)$ such that $p_V \geq p_W$ if and only if there exists a morphism $h_{V,W} : W \to V$ such that $p_V = h_{V,W} \circ p_W$. Let $(p_{V_j})_{j \in J}$ be an ordered family of $L(Z)$; $\lim_{\longrightarrow} j p_{V_j}$ is a boundary of $(p_{V_j})_{j \in J}$. The Zorn lemma implies then that $L(Z)$ has a maximal $c_Z : QZ \to Z$ which is a weak equivalence.

**Remark.**
Let $V$ and $W$ be objects of $C$, every morphism $f : V \to W$ induces a morphism $d(f) : X_{I_V} \to X_{I_W}$ by composition with $f$; $d(f)$ also induces a morphism $Qf : QV \to QW$ such that $f$ is a weak equivalence if and only if $Qf$ is an isomorphism. (See also [6] Lemma 5.1).

**Definitions 3.3.**
A path object of $Z$ is an object $Z'$ such that there exists a weak equivalence $i_Z : Z \to Z'$, a morphism $p_Z : Z' \to Z \times Z$ such that $(id_Z, id_Z) = p_Z \circ i_Z$.

Two morphisms $f, g : Y \to Z$ are right homotopic if and only if there exists a path object $Z'$ and a morphism $H : Y \to Z'$ such that $(f, g) = p_Z \circ H$.

**Proposition 3.4.**
Let $C$ be a category endowed with a closed model defined by counting and $Y$ a cofibrant object of $C$. Two morphisms of $C$ $f, g$ are right homotopic if and only if they are equal.

**Proof.**
If $Y$ is cofibrant and $f, g$ are right homotopic, we can suppose that there exists a path object $Z'$ such that $i_Z : Z \to Z'$ is an acyclic cofibration and $(f, g) = p_Z \circ H$ (See [6] Lemma 4.15). Since $i_Z$ is an acyclic cofibration of a model defined by counting, it is an isomorphism. We can thus suppose that $Z' = Z$ and $p_Z = (id_Z, id_Z)$. This implies that $f = p_1 \circ p_Z \circ H = p_2 \circ p_Z \circ H = g$ where $p_1, p_2 : Z \times Z \to Z$ are the projections on the first and second factors.

**Remark.**
Let $Y$ and $Z$ be two objects of $C$, we denote by $C(Y)$ (resp. $C(Z)$) the cofibrant replacement of $Y$ (resp. $Z$). The objects $C(Y)$ and $C(Z)$ are also fibrant. The homotopy category of $C$ is the category which have the same class of objects than $C$, and the set $\text{Hom}_{\text{Haut}}(Y, Z)$ of morphisms of the homotopy category between the objects $Y$ and $Z$ is the set $\pi(X, Y)$ whose elements are right homotopy classes of maps between $C(Y)$ and $C(Z)$. (See Dwyer and Spalinski 4.22 and definition 5.6) We deduce:

**Proposition 3.5.**
Let $C$ be a category endowed with a closed model defined by counting, for every objects $X$ and $Y$ of $C$, we have $\text{Hom}_{\text{Haut}}(Y, Z) = \pi(C(Y), C(Z)) = \text{Hom}_C(C(Y), C(Z))$. 

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4. Counting Quillen models in the categories of directed graphs and undirected graphs.

We will define now various Quillen models in different categories of graphs. We start by the category of directed graphs.

Let $\text{CD}$ be the category which has two objects that we denote 0, 1. We suppose that $\text{Hom}_{\text{CD}}(0, 1)$ contains two elements $s, t$, $\text{Hom}_{\text{CD}}(0, 0)$ and $\text{Hom}_{\text{CD}}(1, 1)$ are reduced to their identity maps.

**Definition 4.1.**
A directed graph is a presheaf defined on $\text{C}$. Let $X$ be such a presheaf; $X$ is defined by two sets $X(0)$ and $X(1)$, two maps $X(s), X(t) : X(1) \to X(0)$.

The set $X(0)$ is called the space of nodes and the set $X(1)$ the space of directed arcs.

**Definition 4.2.**
A morphism $f : X \to Y$ between the graphs $X$ and $Y$ is a natural transformation between the presheaves $X$ and $Y$; thus $f$ is defined by a morphism $f_0 : X(0) \to Y(0)$, $f_1 : X(1) \to Y(1)$ such that $f_0 \circ X(s) = Y(s) \circ f_1$, $f_0 \circ X(t) = Y(t) \circ f_1$.

We denote by $\text{Gph}$ the category of directed graphs. The categories of directed graphs is stable by limits and colimits since it is a topos.

Examples of directed graphs are: the directed dot graph $D$. It is the graph such that $D(0)$ is a singleton and $D(U)(1)$ is empty.

The directed arc graph $A$ is the graph defined by $A(0) = \{0, 1\}$, $A(1) = \{a\}$ and $A(s)(a) = 0$, $A(t)(a) = 1$.

Let $p$ be a strictly positive integer. We denote by $c_p$ the graph whose set of nodes is $\mathbb{Z}/p\mathbb{Z}$, let $[n]$ be the class of the integer $n$ in $\mathbb{Z}/p$, there exists a unique arc $a_n$ such that $c_p(s)(a_n) = [n]$ and $c_p(t)(a_n) = [n + 1]$.

We can define on $\text{Gph}$ the Quillen model obtained by counting the elements of the set $\text{Cyl} = \{c_p, p \in N - \{0\}\}$.

**Remark.**
The Quillen model obtained here is similar to the Quillen model defined in [1] but they are not identical; their classes of weak cofibrations are different, but they have the same class of weak equivalences.

**Definition 4.3.**
Let $X$ be a directed graph, for every non zero integer $p$, we denote by $n_p(X)$ the cardinality of $\text{Hom}_{\text{Gph}}(c_p, X)$. Suppose that for every strictly positive integer $p$, $n_p(X)$ is finite. The Zeta serie $Z_X(t)$ of $X$ is:

$$\exp\left(\sum_{p=1}^{p=\infty} n_p(X)\frac{t^p}{p}\right).$$

**Proposition 4.4.**
Let $X$ and $Y$ be two finite directed graphs, $Z_X(t) = Z_Y(t)$ if and only if there exists an isomorphism $f$ in $\text{Hom}_{\text{Hot}}(X, Y)$.

**Proof.**

Let $X$ and $Y$ be two finite graphs; the proposition 3.5 implies that there exists an isomorphism in $\text{Hom}_{\text{Hot}}(X, Y)$ if and only if there exists an isomorphism of graphs $f : C(X) \to C(Y)$; this is equivalent to saying that $Z_X(t) = Z_{C(X)}(t) = Z_{C(Y)}(t) = Z_Y(t)$.

**Undirected graphs.**

It is natural to try to generalize this Quillen model to others categories of graphs, unfortunately straightforward generalizations do not have the same natural properties, for example we do not obtain the same characterization of the weak equivalences with the corresponding Zeta series. We will consider the category $U\text{Gph}$ of undirected graphs.

Let $C_U$ be the category which has two objects that we denote by 0 and 1. We suppose that $\text{Hom}_{C_U}(0, 1)$ contains two elements $s, t$, $\text{Hom}_{C_U}(0, 0)$ is reduced to the identity element and $\text{Hom}_{C_U}(1, 1)$ contains the identity and an involution $i$ such that $i \circ s = t$.

**Definition 4.5.**

An undirected graph is a presheaf defined on $C_U$. Let $X$ be such a presheaf, $X$ is defined by two sets $X(0)$ and $X(1)$, two maps $X(s), X(t) : X(1) \to X(0)$ and an involution $X(i)$ of $X(1)$ such that $X(s) \circ X(i) = X(t)$.

The set $X(0)$ is called the space of nodes, and the space $X(1)$ the space of half-arcs. For an half-arc $a \in X(1)$, $X(s)(a)$ is the source of $a$ and $X(t)(a)$ is the target of $a$.

Remark that $X(i)$ is an involution of $X(1)$ and the source of the half-arc $a$ is the target of $X(i)(a)$ since $i \circ s = t$.

We have not assume that $X(i)$ acts freely, this implies the existence of undirected graphs $X$ with degenerated loops; these are half-arcs fixed by $X(i)$.

An arc of the graph $X$ is defined by a couple $(u, X(i)(u))$ where $u \in X(1)$. We denote by $\text{Arc}(X)$ the space of arcs of the graph $X$. The source or the target of $u$ will often be called an end of the arc $(u, X(i)(u))$.

Geometrically, if there does not exist a degenerated loop, an undirected graph can be represented by a set of points corresponding to its nodes and an arc $(u, X(i)(u))$ is an unoriented segment connecting $X(s)(u)$ and $X(t)(u)$.

**Definition 4.6.**

A morphism $f : X \to Y$ between the graphs $X$ and $Y$ is a natural transformation between the presheaves $X$ and $Y$; thus $f$ is defined by a morphism $f_0 : X(0) \to Y(0)$, $f_1 : X(1) \to Y(1)$ such that $f_0 \circ X(s) = Y(s) \circ f_1$, $f_0 \circ X(t) = Y(t) \circ f_1$ and $f_1 \circ X(i) = Y(i) \circ f_1$.

The morphism of graphs $f : X \to Y$ induces a morphism $a(f) : \text{Arc}(X) \to \text{Arc}(Y)$. If there is no confusion, we will often denote $a(f)$ by $f_1$. 


Examples of undirected graphs are: the undirected dot graph $D_U$. It is the graph such that $D_U(0)$ is a singleton and $D_U(1)$ is empty.

The undirected arc graph $A_U$ is the graph defined by $A_U(0) = \{u_1, u_2\}$, $A_U(1) = \{a_1, a_2\}$ such that $A_U(i)(a_1) = a_2$, $A_U(s)(a_1) = u_1$ and $A_U(t)(a_1) = u_2$.

The graph $V_U$ is the graph defined by $V_U(0) = \{v_1, v_2, v_3\}$, $V_U(1) = \{b_1, b_2, c_1, c_2\}$ such that $V_U(s)(b_1) = V_U(s)(c_1) = v_1$, $V_U(t)(b_1) = v_2$, $V_U(t)(c_1) = v_3$, $V_U(i)(b_1) = b_2$ and $V_U(i)(c_1) = c_2$.

The path graph $P_n$ is the graph whose set of nodes is $\{0, ..., n-1\}$, $P_n(1) = \{p^+, p^-, p = 0, ..., n-2\}$ such that $P_n(s)(p^+) = p$, $P_n(t)(p^+) = p+1$, $P_n(i)(p^+) = p^-$.

There is a natural morphism $f : V_U \to A_U$ such that $f_0(v_1) = u_1$, $f_0(v_2) = f_0(v_3) = u_2$, $f_1(b_1) = f_1(c_1) = a_1$. This map is called the elementary folding.

Let $p$ be a strictly positive integer. We denote by $c_p^U$ the undirected graph whose set of nodes is $\mathbb{Z}/p\mathbb{Z}$, let $[n]$ be the class of the integer $n$. We have $c_p^U(1) = \{[n]^+, [n]^-, [n] \in \mathbb{Z}/p\mathbb{Z}\}$, $c_p^U(s)([n]^+) = [n]$, $c_p^U(s)([n]^-) = [n+1]$, $c_p^U(t)([n]^+) = [n+1]$ and $c_p^U(t)([n]^-) = [n]$, $c_p^U(i)([n]^+) = [n]^+$ and $c_p^U(i)([n]^-) = [n]^-$.

Thus there exists a unique arc $([n]^+, [n]^-) \in c_p^U$ between $[n]$ and $[n+1]$. The graph $c_p^U$ is called the undirected $p$-cycle.

Let $X$ and $Y$ two undirected graphs isomorphic to the 1-cycle $c_1^U$. Consider two morphisms $f : D_U \to X$ and $g : D_U \to Y$ defined by identifying $D_U$ respectively to the node of $X$ and to the node of $Y$. The pushout of $f$ and $g$ is the eight graph. Geometrically, it corresponds to two 1-circles glued at their node.

**Definition. 4.7.**

Let $X$ be an undirected graph a $p$-cycle of $X$ is a morphism $f : c_p^U \to X$. We say that the $p$-cycle $f$ have backtracking if and only if there exists an integer $n$ such that $f_1([n+1]^+) = f_1([n]^-)$. We denote by $\text{Cycl}_p(X)$ the set of $p$-cycles of the undirected $X$ without backtrackings.

We denote by $W_U$ the class of morphisms of $\mathbf{UGph}$ such that for every $f : X \to Y$ in $W_U$ the map $c_p(f) : \text{Hom}(c_p^U, X) \to \text{Hom}(c_p^U, Y)$ which sends the morphism $h$ to $f \circ h$ induces a bijection on cycles without backtracking. The class of maps $W_U$ satisfies the $2 - 3$-property.

Let $X$ be a finite undirected graph, we denote by $c_p(X)$ the cardinality of the set of morphisms $c_p^U \to X$ without backtracking. The Ihara zeta function of $X$ is defined by:

$$\exp\left(\sum_{p \geq 1} \frac{c_p(X)}{p} t^p\right)$$

Remark that if $f : X \to Y$ is a morphism in $W_U$ the the graphs $X$ and $Y$ have the same Ihara zeta function. We want to find a Quillen model for which $W_U$ is the class of weak equivalences. We will see that such a model does not exist.
Remark.

We can naively adapt the previous Quillen model defined in the category of directed graphs to the category of undirected graphs: so, we define the Quillen model obtained by counting elements of the family \((c^p_U)_{p \in \mathbb{N} - \{0\}}\). A morphism \(f : X \to Y\) of \(UGph\) is a weak equivalence of this Quillen model if and only for every strictly positive integer, the map \(\text{Hom}(c^p_U, X) \to \text{Hom}(c^p_U, Y)\) induced by \(f\) is bijective. Thus \(f\) induces a bijection between the \(p\)-cycles of \(X\) and the \(p\)-cycles of \(Y\) for every strictly positive integer, but the image of a cycle without backtracking by \(f\) is not necessarily a cycle without backtracking. This Quillen model is essentially trivial as shows the following result:

**Proposition 4.8.**

Let \(f : X \to Y\) be a weak equivalence between two undirected finite and connected graphs for the closed model defined by counting cycles in \(UGph\), then \(f\) is an isomorphism.

**Proof.**

Firstly, we show that \(f_1 : X(1) \to Y(1)\) is injective. Let \(a, b\) be two distinct arcs such that \(f_1(a) = f_1(b)\). There exists morphisms \(h_i : c^2_U \to X, i = 1, 2\) such that the image of \(h_1\) is \(a\) and the image of \(h_2\) is \(b\), the square diagram:

\[
\begin{array}{ccc}
    c^2_U + c^2_U & \xrightarrow{h_1 + h_2} & X \\
    \downarrow j_2 & & \downarrow f \\
    c^2_U & \xrightarrow{f \circ h_1} & Y
\end{array}
\]

is commutative and does not have a filler this is contradiction.

We show now that \(f_0\) is surjective. Let \(a\) be an arc of \(Y\), there exists an arc \(h : c^2_U \to Y\) whose image is \(a\). Since \(f\) is right orthogonal to \(i_2 : \phi \to c^2_U\), we deduce the existence of a morphism \(h' : c^2_U \to X\) such that \(h = f \circ h'\). This implies that \(f_1\) is surjective on arcs.

We show now that \(f_0 : X(0) \to Y(0)\) is injective. Let \(x\) and \(y\) be two distinct nodes of \(X\) such that \(f_0(x) = f_0(y)\). Since \(X\) is connected, there exists a path \(h\) between \(x\) and \(y\), \(f_1(h)\) is a \(p\)-cycle. Since \(f\) induces a bijection on \(p\)-cycles, we deduce the existence of a \(p\)-cycle \(c\) such that \(f_1(c) = f_1(h)\). This is a contradiction with the fact that \(f\) is bijective on arcs.

We show now that \(f_0\) is surjective.

Let \(y\) be a node of \(Y\) since \(Y\) is connected, there exists an arc \(b\) of \(Y\) which has \(y\) as an end. Since \(f_1\) is bijective, we deduce the existence of an arc \(a\) of \(X\) such that \(b = f_1(a)\). This implies that \(a\) has an end \(x\) such that \(f_0(x) = y\).

**Theorem 4.9.**

There does not exist a Quillen model on \(UGph\) whose class of weak equivalences is the class \(W_U\).

**Proof.**

Suppose that such a Quillen model exists, then the elementary folding \(f : V_U \to A_U\) would be a weak equivalence, and we can write \(f = g \circ h\) where
$h : V_U \to X$ is a weak cofibration and $g : X \to A_U$ is a fibration. The $2-3$ property implies here that $g$ is a weak fibration.

Suppose that $h_1(b_1) = h_1(c_1)$, consider the morphism $l : V_U \to c_U^2$ defined by $l_0(v_1) = [0], l_0(v_2) = l_0(v_3) = [1], l_1(b_1) = [0]^+$ and $l_1(c_1) = [1]^-$, We can define the pushout diagram:

$$
\begin{array}{ccc}
V_U & \xrightarrow{h} & X \\
l & \downarrow & \downarrow \ p \\
c_U^2 & \xrightarrow{q} & Z \\
\end{array}
$$

Remark $X$ does not have any cycles without backtracking since $h$ is a weak equivalence and $V$ does not have cycles without backtrackings. We deduce that $Z$ does not have cycles without backtrackings since $Z$ is isomorphic to $X$. This implies that $q$ is not a weak equivalence. This is a contradiction with the fact that in a Quillen model, the pushout of a weak cofibration is a weak cofibration. Thus $h_1(b_1)$ is distinct of $h_1(c_1)$. Remark that the image of $V_U$ by $h$ cannot be isomorphic to a 2-cycle since $h$ is a weak equivalence; we deduce that this image is isomorphic to $V_U$.

Now consider the morphism $m : c_U^2 \to A_U$ defined by $m_0([0]) = u_1, m_0([1]) = u_2$ and $m_1([0]^+) = m_1([1]^-) = a_1$. Consider the pullback diagram:

$$
\begin{array}{ccc}
U & \xrightarrow{p'} & X \\
q' & \downarrow & \downarrow \ g \\
c_U^2 & \xrightarrow{m} & A_U \\
\end{array}
$$

The morphism $q' : U \to c_U^2$ must be a weak equivalence since in a Quillen model the pullback of a weak fibration must be a weak fibration. But, there exists a subgraph of $U$ isomorphic to the pullback of the elementary folding by $m : c_U^2 \to A_U$. Such a subgraph is isomorphic to the eight graph. So $q'$ cannot be a weak equivalence. This is a contradiction.

5. Counting models in the category of undirected colored graphs.

We are going to define a Quillen model on some subcategories of $UGph$, and in particular in the category of undirected colored graphs.

**Definitions 5.1.**

Let $X$ be an undirected graph, for every node $x$ of $X$, we denote by $X(x, *)$ the set of arcs $(u, X(i)(u))$ such that $X(s)(u) = x$ or $X(t)(u) = x$.

A morphism $f : X \to Y$ between undirected graphs is a covering if and only if for every $x \in X_0$, the morphism $f_x : X(x, *) \to Y(f_0(x), *)$ is bijective. Remark that for every undirected graph, the morphism $\phi \to X$ is a covering.

Let $X$ be an undirected graph, we denote by $C_X$ the category whose objects are coverings $f : Y \to X$, a morphism between the objects $f : Y \to X$ and $g : Z \to X$ is a covering morphism $h : Y \to Z$ such that $g \circ h = f$.

**Proposition 5.2.**

Limits and colimits exists in $C_X$.  

10
Proof.

The class \( \text{Cov} \) of covering morphisms of \( UGph \) is the class of morphisms which are right orthogonal to the elementary folding \( f_e : U \to A_U \) and \( i_U : D_U \to A_U \). This implies that \( \text{Cov} \) is stable by pullbacks. Since the products in \( C_X \) are pullbacks in \( \text{Cov} \), we deduce that products and pullbacks exists in \( C_X \), and henceforth that limits exists in \( C_X \). (See SGA 4 proposition 2.3).

Let \((f_i : Y_i \to X)_{i \in I}\) be a family of elements of \( C_X \). The morphism \( f : \sum_{i \in I} Y_i \to X \) whose restriction to \( Y_i \) is \( f_i \) is a covering; this implies that sums exist in \( C_X \).

We show now that pushouts exist in \( C_X \). Let \( h : Z \to X \) and \( h' : Z' \to X \) be two elements of \( C_X \); consider an element \( p : Y \to X \) and \( f : Y \to Z \) \( g : Y \to Z' \) two morphisms of \( C_X \). Without restricting the generality, we can suppose that \( Y, Z \) and \( Z' \) are connected. The maps \( f \) and \( g \) are surjective on nodes and arcs since they are coverings and the pushout of \( f \) and \( g \) is defined by the graph \( L \) such that \( L_0 \) is the quotient of \( Z + Z' \) by the equivalence relation generated by: let \( x \in Z_0 \) and \( x' \in Z'_0 \), \( x \simeq x' \) if and only if there exists \( x'' \in Y_0 \) such that \( f_0(x'') = x \) and \( g_0(x'') = x' \); \( L_1 \) is the quotient of \( Z_1 + Z'_1 \) by the equivalence relation generated by: let \( a \in Z_1 \) and \( b \in Z'_1 \), \( a \simeq b \) if and only if there exists \( c \in Y_1 \) such that \( f_1(c) = a \) and \( g_1(c) = b \). We denote by \( p_1 : Y \to L \) the quotient map. There exists a morphism \( l : L \to X \) such that \( p = l \circ p_1 \) since \( p = h \circ f = h' \circ g \). The morphism \( l \) is a covering since \( f \) and \( g \) are elements of \( C_X \); it is the pushout of \( f \) and \( g \).

We consider \( R^p_X \) the set of graphs in \( C_X \) such that an element of \( R^p_X \) is obtained by attaching a forest to a \( p \)-cycle. We define on \( C_X \) the Quillen model obtained by counting elements of \( R_X = \{ R^p_X , p \in N - \{0\} \} \). Thus a morphism \( f : X \to Y \) of \( C_X \) is a weak equivalence if and only if for every for every \( p > 0 \), for every \( U_p \in R^p_X \), \( \text{Hom}_{C_X}(U_p, X) \to \text{Hom}_{C_X}(U_p, Y) \) is bijective.

Proposition 5.3.

Let \( f : Y \to X \) be an object of \( C_X \) such that \( Y \) does not have loops, then for every \( p \)-cycle \( h : c^p_U \to Y \), \( p > 1 \), without backtracking there exists an element \( U \) of \( R^p_X \) and a morphism \( g : U \to Y \) of \( C_X \) whose restriction to the \( p \)-cycle is \( h \).

Proof.

Let \( h : c^p_U \to Y \) be a \( p \)-cycle. Since \( Y \) does not have loops, for every integer \( n \), \( h_1([n]^+) \) is distinct from \( h_1([n+1]^+) \). This implies that we can attach a tree to every node of \( c^p_U \) to obtain a graph \( U \) for which there exists a covering \( g : U \to Y \) whose restriction to \( c^p_U \) is \( h \).

Remark.

The previous proposition is not true if there are 1-cycle in \( Y \). Consider the following example: \( X \) is the 1-cycle. Consider the morphism of defined by \( h : c^p_U \to X \) such that \( h_1([0]^+) = h_1([1]^+) \). This cycle does not have backtracking, but it is impossible extend \( h \) to an element of \( R^2_X \) such that it becomes a covering, since the restriction of \( h_1 \) to \( X([1], *) \) is not injective.
Proposition 5.4.

Let \( f : Y \to X \) and \( g : Z \to X \) be objects of \( C_X \) such that \( Y \) and \( Z \) does not have loops and their Zeta function are defined. If there exists a weak equivalence between \( f \) and \( g \), then \( Y \) and \( Z \) have the same Ihara Zeta function.

Proof.

A weak equivalence between \( f : Y \to X \) and \( g : Z \to X \) is defined by a covering \( h : Y \to Z \) such that \( g \circ h = f \). We are going to show that \( h \) induces a bijection on \( p \)-cycles without backtracking. Let \( u, u' : c^p_U \to Y \) be two \( p \)-cycles of \( Y \) without backtracking such that \( h \circ u = h \circ u' \). Since \( Z \) does not have loops, there exists an element \( v : V \to X \in R^p_X \) and a morphism between \( v \) and \( g \) whose restriction to the \( p \)-cycle of \( V \) coincide with \( h \circ u \). Since \( h \) is a covering, we can lift \( v \) to morphisms \( v_1 : V \to Y \) (resp \( v_2 : V \to Y \)) whose restriction the \( p \)-cycle is \( u \) (resp. \( u' \)) and such that \( v_1, v_2 \) are morphisms of \( C_X \) respectively between \( v \) and \( f \). Since \( h \) is a weak equivalence, we deduce that \( v_1 = v_2 \) and henceforth that \( u = u' \). Thus \( h \) is injective on \( p \)-cycle. The fact that \( f \) is surjective on \( p \)-cycles results from the fact that for every \( p \)-cycle without backtracking \( l : c^p_U \to Z \) there exists an element \( v : V \to X \in R^p_X \) and a morphism \( d \) of \( C_X \) between \( v \) and \( g \) whose restriction to the \( p \)-cycle of \( V \) is \( l \).

We can lift \( d : V \to Z \) to a morphism \( d' : V \to Y \) since \( h \) is a weak equivalence, the restriction of \( d' \) to the \( p \)-cycle of \( V \) is a preimage of \( l \).

Remarks.

Let \( f : Y \to X \) be an object of \( C_X \) without loops, for every \( p \)-cycle \( u : c^p_U \to Y \) without backtracking, consider the element \( V_u \) of \( R^p_X \) such that there exists a morphism \( v_u : V_u \to Y \) of \( C_X \) whose restriction to the cycle of \( V_u \) coincide with \( u \). Let \( V \) be the direct summand of the graphs \( V_u \), there exists a covering \( c : V \to Y \) whose restriction to \( V_u \) is \( v \). The morphism \( c \) is not necessarily a weak equivalence; but we can find a subset \( H \) of \( \{ v_u : V_u \to Y, u \in L_V \} \) such that the restriction \( c' \) to the direct sum \( c(Y) \) of \( \{ v_u, v_u \in H \} \) is a weak equivalence. Remark that the morphism \( \phi \to c(Y) \) is a cofibration, thus \( c(Y) \) is a cofibrant replacement of \( Y \).

Let \( B_n \) be the undirected graph which has one node \( * \) and \( n \) undirected loops. Let \( X \) be an undirected graph, there exists a covering \( f : X \to B_n \) if and only if \( X \) is a \( n \)-regular graph and the edges of \( X \) can be colored by \( n \)-distinct colors. The proposition 5.4 implies that there exists a Quillen model on the category of \( n \)-regular graphs whose edges can be colored by \( n \) distinct colors such that if two finite graphs without loops in this category are weak equivalent then they have the same Zeta function.

Proposition 5.5.

Suppose that \( X = B_n \), let \( Y \to X \) and \( Z \to X \) be two objects of \( C_X \) such that \( Y \) and \( Z \) are finite and do not have loops; moreover suppose that \( Y \) and \( Z \) have the same Zeta function and there exists isomorphisms \( H_p : \text{Cycl}_p(Y) \to \text{Cycl}_p(Z) \) such that for every element \( c \in \text{Cycl}_p(Y) \) the color induced by \( Y \) on \( c \) is the colors induced by \( Z \) on \( H_p(c) \). There exists a graph \( L \), coverings \( p : L \to Y \) and \( p' : L \to Z \) which are weak equivalences.
Proof.
If $Y \to X$ and $Z \to X$ are two objects of $C_X$ such that the Zeta series of $Y$ and $Z$ are equal, then any isomorphism between their respective set of cycles which respects their colors as described above induces an isomorphism between their cofibrant replacements since they are $n$-colored graphs.

Remarks.
Let $X$ be an undirected graph; $X$ is $n$-regular if and only if for every node $x$ of $X$, the cardinal of $X(x, *)$ is $n$. Remark that a $n$-colored graph is a $n$-regular but not every $n$-regular graph is colored as shows the snark graphs. The Quillen model that we have just defined in the category of $n$-regular colored graphs cannot be naively extended to the category whose objects are $n$-regular graphs and the morphisms are the coverings morphisms, since the pushouts do not exist in this category.

We are going to relate $n$-colored graphs to Cayley graphs. Let $G$ be a group and $S$ a set of generators of $G$, for every $G$-set $X$, the Cayley graph $C(X, S, G)$ is the directed graph whose set of nodes is $X$, and for every elements $x$ and $y$ of $X$, the set of arcs between $x$ and $y$ is in bijection with $\{s \in S, s(x) = y\}$.

Let $G_n$ be the group generated by $S_n = \{a_0, ..., a_n\}$ such that $a_i^2 = 1$ for every $i = 0, ..., n$. For every $G_n$-set $X$, the Cayley graph $C(X, S_n, G_n)$ is endowed with the structure of an undirected graph; let $x$ be a node of $X$, the symmetric of the half-edge between $x$ and $a_i(x)$ defined by the relation $a_i(x) = y$ is the half-edge between $a_i(x)$ and $x$ defined by $a_i(a_i(x)) = x$. We denote by $UC_{G_n}$ the category of $G_n$-sets.

Proposition 5.6.
The correspondence which associates to $X, C(X, S_n, G_n)$ induces an isomorphism between $UC_{G_n}$ and the category of $n$-regular colored graphs $C_B$.

Proof.
We have only to construct the inverse of $C(S_n, G_n)$. Let $X$ be an $n$-colored graph. We assume that the colors are labeled by $a_0, ..., a_n$. We associate to $X$ the $G_n$-set $X_0$, if $x \in X_0$ and there exists an arc between $x$ and $y$ colored by $a_i$, we set $a_i(x) = y$.

Remarks.
The Quillen model defined on $C_{B_n}$ induces a Quillen model on $UC_{G_n}$.

Let $LUC_{G_n}$ be the full subcategory of $UC_{G_n}$ such that for every object $X$ of $UC_{G_n}$, every $x \in X$ and for every $i = 0, ..., n$, $a_i(x) \neq x$. The functor $C(S_n, G_n)$ establishes an isomorphism between $LUC_{G_n}$ and the category of $n$-regular colored graphs without loop.

Quillen models can also be defined in others interesting comma categories of $UGph$, here is an example:

Definition 5.7.
The category of bipartite $BUGph$ graphs is the comma category $UGph/A_U$.

Thus a bipartite graph is a morphism $f : X \to A_U$. A morphism between the
objects $f : X \to A_U$ and $g : Y \to A_U$ of $UGph_u/A_U$ is a morphism $h : X \to Y$ such that $f = g \circ h$.

Consider the undirected graph $D_n$ which has two nodes 0 and 1, and such that there exists $n$-arcs $a_1, \ldots, a_n$ between 0 and 1. The category $BC_n = C_{D_n}$ of coverings of $D_n$ is the category of graphs which are bipartite and $n$-colored. We deduce the existence of a Quillen model on $BC_n$ obtained by counting objects of $BC_n$ obtained by attaching a forest to a cycle.

6. Quillen models by counting and topology.

We are going to use the Quillen model defined on $BC_n$ to study the Galoisian complexes introduced by Ladegaillerie [15]:

**Definitions 6.1.**

Let $S^+_n$ be the oriented standard affine $n$-simplex whose vertices are labeled $A_0, \ldots, A_n$. We denote by $S^-_n$ the corresponding simplex with the opposite orientation. Let $I$ be a set, and $(S^+_i)_{i \in I}$ a set of examples of $S^+_n$ and $(S^-_i)_{i \in I}$ the corresponding set of examples of $S^-_n$. The elements of $(S^+_i)_{i \in I}$ are called the direct simplexes, and the elements of $(S^-_i)_{i \in I}$ are called the undirect simplexes.

A Galoisian $n$-complex $C$ is obtained by gluing elements of $(S^+_i)_{i \in I}$ with elements of $(S^-_i)_{i \in I}$ such that the gluing respect the labeling, affine structures and inverse orientations. Moreover, we suppose that each face of $C$ belongs to exactly two simplexes; one direct and the other undirect.

A morphism $f : X \to Y$ between two Galoisian $n$-complexes is a continuous map which sends a direct simplex to a direct simplex, an undirect simplex to an undirect simplex and respects the labelings and the affine structures. This defines the category $CG_n$ whose objects are Galoisian $n$-complexes and the morphisms are morphisms between Galoisian $n$-complexes.

Let $X$ be a Galoisian $n$-complex. We denote by $\Omega^+_n(X)$ the union of elements of $(S^+_i)_{i \in I}$ by $\Omega^-_n(X)$ the union of $(S^-_i)_{i \in I}$ and by $\Omega^+_n(X)$ and $\Omega^-_n(X)$. We can define $s_j, j = 0, \ldots, n$ the involution of $\Omega(X)$ such that for an element $S^+_i$ of $\Omega^+_n(X)$, $s_j(S^+_i)$ is the unique undirected simplex of $\Omega^-_n(X)$ whose $j$-face is identified with the $j$-face of $S^+_i$. If $G_n$ is the group generated by $\{a_0, \ldots, a_n\}$ with the relations $a_j^2 = 1, j = 0, \ldots, n$, Ladegaillerie [15] shows that the correspondence $\Omega_n$ between $CG_n$ and the category of $G_n$-sets which sends a Galoisian $n$-complex $X$ to the $G_n$-set $\Omega_n(X)$ endowed with the action just defined is an equivalence of $CG_n$ onto its image in the category of $G_n$-sets. Remark that the image of the composition of $C(S_n, G_n) \circ \Omega_n$ is the category of undirected $n$-colored bipartite graphs $BC_n$. This shows the existence of a Quillen model on $CG_n$. We will denote by $L_n(X)$ the Cayley graph of the $G_n$-set $\Omega_n(X)$, defined by the generators of $G_n$, $a_0, \ldots, a_n$.

**Remarks.**

Let $X$ and $Y$ finite Galoisian complexes, since $L_n(X)$ is a bipartite graph, it does not have loops, we deduce that if $L_n(X)$ and $L_n(Y)$ are weakly equivalent then they have the same Ihara Zeta function.
Question.

Is it possible to provide a geometric interpretation of the coefficient of $L_n(X)$?

The Galoisian complex $X^n_0$ defined by two elements $S^+_n$ and $S^-_n$ is homeomorphic to the $n$-sphere $S^n$. For every complex $X$ defined by $\Omega^n_+(X) \cup \Omega^n_-(X)$, there exists a morphism of Galoisian complexes $p : X \to X^n_0$ which identifies the elements of $\Omega^n_+(X)$ to $S^+_n$ and the elements of $\Omega^n_-(X)$ to $S^-_n$. This map is ramified on the $(n-2)$-subcomplex of $C$.

Proposition 6.2.

Let $X$ be an $n$-Galoisian complex, every cycle of $L_n(X)$ has an even length. Let $c$ be a cycle of $L_n(X)$ of length $2p$ colored by $m$ colors, if $n - m > 0$, then the intersection $I(c)$ of the vertices of $c$ is not empty, and the projection $p_X : X \to S^2$ is ramified at $p_X(I(c))$ with ramification degree $p$.

Proof.

Since the graph $L_n(X)$ is a bipartite then every of its cycle has an even length. Locally, $I(c)$ corresponds to the intersection of $m$-faces of an $n$-simplex; thus $I(c)$ it is not empty if and only if $n > m$. Suppose that $I(c)$ is not empty, let $Y$ be the union of the vertices of $c$. The restriction of $p_X$ to $Y - I(c)$ is a covering of degree $p$ on its image, and the restriction of $p_X$ to $I(c)$ is injective. This shows that $p_X(I(c))$ has ramification degree $p$ since $Y$ is a neighborhood of $I(c)$.

The subgroup of $G_n$ generated by $a_i a_n$ is isomorphic to the free subgroup generated by $n$ elements, $F_n$. Let $X$ be a Galoisian $n$-complex defined by $\Omega(X) = \Omega^n_+(X) \cup \Omega^n_-(X)$. The previous action of $G_n$ on $\Omega(X)$ induces an action of $F_n$ on $\Omega^n_+(X)$. Ladegaillerie [15] shows this action induces an equivalence of categories between $CG_n$ and the category of $F_n$-sets.

For every $F_n$-set $X$, we can define the Cayley graph $L_n^+(X)$ defined by the set of generators $a_0 a_1, \ldots, a_{n-1} a_n$. The equivalence of categories between $CG_n$ and the category of $F_n$-sets $E_{F_n}$ induces a closed Quillen model on $E_{F_n}$. Others Quillen models can be defined on $E_{F_n}$. In the next section we are going to present a general construction to transfer the Quillen model of $Gph$ to the category of $G$-sets for any group $G$.

Quillen model and $G$-sets.

Let $G$ be a group, consider the category $C_G$ which has only one object that we denote by $*$, we suppose that $\text{Hom}_{C_G}(*, *) = G$. An object of the category $\mathcal{C}_G$, of presheaves over $C_G$ is a set $E$, endowed with an action of $G$. Thus, $\mathcal{C}_G$ is the category of $G$-sets. We are going to transfer the Quillen model defined at the section 4 in the category of directed graphs to the category of $G$-sets. On this purpose, we define firstly some canonical functors.

The category $C_D$ (see section 4) can be imbedded in $\hat{C}_D$ by using the Yoneda embedding as follows: to 0, we associate the presheaf $\hat{0}$ defined by $\hat{0}(0) = \text{Hom}_{C_D}(0, 0), \hat{0}(1) = \text{Hom}_{C_D}(1, 0)$: We see that $\hat{0}$ is $D_D$ the dot graph. To 1, we
associate the presheaf \( \hat{1} \) defined by \( \hat{1}(0) = \text{Hom}_{C_D}(0, 1), \hat{1}(1) = \text{Hom}_{C_D}(1, 1) \). We remark that \( \hat{1} \) is the arc graph \( A_D \). Let \( X \) be a directed graph, we denote by \( C_D/X \) the category whose objects are morphisms of presheaves between the objects of \( C_D \) and \( X \). The objects of \( C_D/X \) are morphisms \( D_D \to X \) and \( A_D \to X \). Thus the class of objects of \( C_D/X \) can be identified with the union of the set of nodes of \( X \) and its set of arcs. Let \( f : U \to X \) and \( g : V \to X \) be two objects of \( C_D/X \), a morphism between \( f \) and \( g \) is a morphism \( h : U \to V \) such that \( f = g \circ h \). Let \( a \) be an arc of \( X \), there exists a morphism \( f_a : A_D \to X \) whose image is \( a \). We also have morphisms \( s_a : D_D \to s(a) \to X \) and \( t_a : D_D \to t(a) \to X \) where \( s(a) \) and \( t(a) \) are respectively the source and the target of \( a \). The source and the target morphisms \( D_D \to A_D \) induces morphisms of \( C_D/X \) between \( s_a \) and \( f_a \) and between \( t_a \) and \( f_a \). Remark that if \( a \) is the loops these two morphisms are distinct.

Let \( A \) be a set of generators of \( G \). For every \( G \)-set \( S \), recall that we have denoted by \( C(S, A, G) \) the Cayley graph of \( S \) associated to \( G \) and \( A \).

Let \( S_A \) be the final element of the category of \( G \)-sets \( E_G; S_A \) is the \( G \)-set which has a unique element \( n_A \). We denote by \( B_A \) the Cayley graph of \( S_A \) defined by \( A \). The objects of the category \( C_D/B_A \) are the unique morphism \( i_A : D_D \to B_A \) and the morphisms \( c_a : A_D \to B_A \) which sends \( A_D \) to the loop of \( B_A \) corresponding to \( a \). The morphisms of \( C_D/B_A \) are the isomorphisms and the morphisms \( \hat{s}_a : i_A \to c_a \) induced by \( \hat{s} : \hat{0} \to \hat{1} \) and \( \hat{t}_a : i_A \to c_a \) induced by \( \hat{t} \). We have a functor \( F_A : C_D/B_A \to Gph/B_A \) such that \( F_A(i_A) = F_A(c_a) = * \), \( F_A(\hat{s}_a) = \text{Id} \), \( F_A(\hat{t}_a) = a \). In SGA4 p. 33, Grothendieck defines an equivalence of categories. \( e_{B_A} : C_D/B_A \to Gph/B_A \).

**Proposition 6.3.**

The composition: \( e_{B_A} \circ \hat{F}_A : E_G \to Gph/B_A \) is the map which associates to a \( G \)-set \( X \), the canonical map \( C(X, A, G) \to B_A \) and henceforth, the Cayley graph functor \( C(A, G) \) has \( C_A \) has a left adjoint.

**Proof.**

Let \( E^G \) be a \( G \)-set, we denote by \( E \) the image of \( E^G \) by the forgetful functor \( G \to \text{Sets} \to \text{Set} \). We have: \( \hat{F}_A(E^G)(i_A) = \hat{F}_A(E^G)(c_a) = E \). The construction in SGA 4.1 5.10.1 shows that:

\[
e_{B_A} \circ \hat{F}_A(E^G)(0) = E
\]

\[
e_{B_A} \circ \hat{F}_A(E^G)(1) = \bigcup_{a \in A} \hat{F}_A(E^G(c_a)) = \bigcup_{a \in A} E_a
\]

where \( E_a = E \). This shows that the set of nodes (resp. the set of arcs) of the Cayley graph of \( E^G \) coincide with the set of nodes (resp. the set of arcs) of \( e_{B_A} \circ \hat{F}_A(E) \).

The restriction of \( e_{B_A} \circ \hat{F}_A(E^G)(s) \) to \( E_a \) is the identity, and the restriction of \( e_{B_A} \circ \hat{F}_A(E^G)(t) \) to \( E_a \) is the multiplication by \( a \). This shows that the Cayley graph of \( E^G \) is \( e_{B_A} \circ \hat{F}_A(E^G) \).
We deduce that the Cayley $C(A,G)$ has a left and right adjoint since $e_{B_A}$ is an equivalence of categories and $F_A$ has left and right adjoint see SGA 4.1 proposition 5.1.

**Remark.**
The existence of a left adjoint of $C(A,G)$ can be shown directly, by showing that $C(A,G)$ commutes with limits. The functor $E_G \to Gph$ which sends a $G$-set to its Cayley graph does not have always a left adjoint since it does not commutes always with limits.

We present now the transfer theorem that we are going to use see [5] theorem 3.3. Let $C$ and $D$ be categories in which limits and colimits exist. Suppose that $C$ is endowed with a Quillen model. We denote by $W_C$ the class of weak equivalences, $Cof_C$ the class of cofibrations and $Fib_C$ the class of fibrations of this Quillen model. Suppose that there exists a functor $F : C \to D$ which has a right adjoint functor $G$.

We denote $W_D$ the class of morphisms of $D$ such that for every morphism $f \in W_D$, $G(f)$ is a weak equivalence.

We denote $Fib_D$ the class of morphisms of $D$ such that for every $f \in D$, $G(f)$ is a fibration.

We denote by $Fib'_D$ the intersection of $W_D$ and $Fib_D$.

An arrow of $D$ is a cofibration if and only if its has the left lifting property with respect to $Fib'_D$. We denote by $Cof_D$ the class of cofibrations.

**Theorem 6.4.**
With the notations above, suppose that $D$ allows small object argument. Suppose that for every morphism $d$ of $D$ which is a transfinite composition of pushouts of coproducts of morphisms $F(c)$ where $c$ is a weak cofibration, $G(d)$ is a weak equivalence. Then there exists a Quillen model on $D$ whose class of weak equivalences is $W_D$, its class of fibrations is $Fib_D$ and its class of cofibrations is $Cof_D$.

We can deduce the following result:

**Corollary 6.5.**
Let $G$ be a group, $A$ a set of generators of $G$ and $E_G$ the category of $G$-sets. There exists a Quillen model on $E_G$ such that the morphism of $G$-sets $f : X \to Y$ is a weak equivalence if and only if $C(A,G)(f) : C(X, A, G) \to C(Y, A, G)$ is a weak equivalence of directed graphs.

**Proof.**
The category of $G$-sets allow small objects argument. Let $d$ be a morphism of $E_G$ which is a transfinite composition of pushouts of coproducts of morphisms $F(c)$, where $c$ is a weak cofibration. Since weak cofibrations in $Gph$ are isomorphisms, we deduce that $d$ and $C(A,G)(d)$ are isomorphisms.

We are going to apply this construction to the free group $F_n$. Let $A_n = \{a_1, ..., a_n\}$ be a set of generators of $F_n$; we denote by $R_n$ the set of $F_n$-sets such that for every $F_n$-set $X$ in $R_n$, $C(X, A_n, F_n)$ is obtained by attaching a forest to a sum of cycles.
Proposition 6.6.
A $F_n$-set $X$ is cofibrant for the Quillen model obtained by transferring the Quillen model of $Gph$ to $F_n$-sets with the Cayley functor if and only if it is an element of $R_n$.

Proof.
Let $X$ be a cofibrant $F_n$-set. The cofibrant replacement $X'$ of $C(X, A_n, F_n)$ is the sum $\sum_{i \in I} X'_i$ of cycles. Let $x'_1, \ldots, x'_i$ be the nodes of $X'_i$. We suppose that there exists an arc between $x'_j$ and $x'_{j+1}$ if $j < n$ and an arc between $x'_n$ and $x'_1$. The nodes of $X'_i$ are elements of $X$, and there exists a generators $a_{i,j}$ such that $a_{i,j}(x'_j) = x'_{j+1}$, $j < n$ and $a_{i,n}(x'_n) = x'_1$. We deduce that for every cycle $X'_i$, there exists a $F_n$-set $X_i$ endowed with a morphism $h_i : X_i \rightarrow X$ such that $C(X_i, A_n, F_n)$ is a graph obtained by attaching a tree to $X'_i$. The sum $h = \sum_{i \in I} h_i$ is a weak equivalence.

Consider the commutative diagram:

$$
\begin{array}{ccc}
\phi & \rightarrow & \sum_{i \in I} X_i \\
\downarrow & & \downarrow \sum_{i \in I} h_i \\
X & \xrightarrow{id_X} & X \\
\end{array}
$$

Since $\sum_{i \in I} h_i$ is a weak equivalence or equivalently a weak fibration, we deduce that the existence of a morphism $f : X \rightarrow \sum_{i \in I} X_i$ which fills the previous commutative diagram. This implies that $X$ is an element of $R_n$.

Conversely, let $X$ be an element of $R_n$, let $f : Y \rightarrow Z$ be a weak equivalence or equivalently a weak fibration such that there exists a commutative diagram:

$$
\begin{array}{ccc}
\phi & \rightarrow & Y \\
\downarrow & & \downarrow f \\
X & \xrightarrow{g} & Z \\
\end{array}
$$

Since $C(X, A_n, F_n)$ is a union of cycles, and $f$ is a weak equivalence the morphism $C(A_n, F_n)(g)$ can be lifted to a morphism $C(A_n, F_n)(h) : C(X, A_n, F_n) \rightarrow C(Y, A_n, F_n)$ which is the image by the functor $C(A_n, F_n)$ of a morphism $h : X \rightarrow Y$ which makes the previous diagram commutes.

Remarks.
Let $X$ and $Y$ be $F_n$-sets, if $c(X)$ is a cofibrant replacement of $X$, it is also a fibrant replacement of $X$ since every morphism of $F_n$-sets is a fibration. There exists a functor $c : F_n$-sets $\rightarrow F_n$-sets such that $c(X)$ is a cofibrant replacement of $X$. To construct $c(X)$, consider a cofibrant replacement $c(X)$ of $X$ and suppose that every connected component of $C(c(X), A_n, F_n)$ is not isomorphic to a tree.

Proposition 6.7.
Let $X$ and $Y$ be $F_n$-sets, the set of morphisms $\text{Hom}_{Hot}(X, Y)$ between $X$ and $Y$ in the homotopy category is $\text{Hom}_{F_n\text{-sets}}(c(Y), c(Y))$ where $c(Y)$ and $c(Y)$ are respectively cofibrant replacements of $X$ and $Y$.

Proof.
We are going to show that the category $\text{Hot}_n$ whose objects are $F_n$-sets and such that for every objects $X$ and $Y$ of $\text{Hot}_n$, $\text{Hom}_{\text{Hot}_n}(X, Y) = \text{Hom}_{F_n}(c(X), c(Y))$ is a localization of $F_n$-sets by the class of weak equivalences. There result follows from [6] p.29. Let $f : X \rightarrow Y$ be a weak equivalence between $F_n$-sets. The morphism $c(f) : c(X) \rightarrow c(Y)$ is also a weak equivalence. This is equivalent to saying that $C(F_n, A_n)(c(f))$ is a weak equivalence. Since $c(X)$ and $c(Y)$ are sum of cycles, we deduce that $C(F_n, A_n)(c(f))$ is an isomorphism. This implies that $c(f)$ is also an isomorphism. Since the canonical localization functor defined from the category of $F_n$-sets to the homotopy category factors by $\text{Hot}_n$, we deduce that $\text{Hot}_n$ is the homotopy category.

Quillen model and Dessins d’enfants.

In this section, we are going to recall the definition of a dessin d’enfant and see how it is a particular case of the construction above.

Let $\text{FCG}_2$ be the category of finite Galoisian 2-complexes, and $X$ an object of $\text{FCG}_2$. The map $p_X : X \rightarrow S^2$ is a covering of the 2-sphere ramified at three elements that we denote $A_0, A_1$ and $A_2$. We can identify $S^2 - \{A_0, A_1, A_2\}$ with $R^2 - \{0, 1\}$. Let $[0, 1]$ be the segment draw between 0 and 1 in the plan for every 2-complex $C$ endowed with the structure map $p : C \rightarrow R^2 - \{0, 1\}$, $X_C = p^{-1}([0, 1])$ is an undirected graph. Remark that $p$ induces a morphism $X_C \rightarrow [0, 1]$; thus $X_C$ is a bipartite graph.

Let $U^0_X = p^{-1}(0)$ and $U^1_X = p^{-1}(1)$. We have $F_2 = \pi_1(R^2 - \{0, 1\})$. This group has two generators $s_0$ and $s_1$ such that for each $x \in p^{-1}(0)$, the monodromy of $s_0$ induces an action on $X_C(x, \ast)$ and for every $x \in p^{-1}(1)$ the monodromy of $s_1$ induces an action on $X_C(x, \ast)$. This action is nothing but the restriction of the action of $F_2$ on $\Omega^+(X)$ (see [17]). A bipartite graph endowed with such an action of $F_2$ is called a dessin d’enfant. Conversely, any finite $F_2$-set define a dessin d’enfant. Let $FS_2$ be the category of finite $F_2$-sets; the functor $F : \text{FCG}_2 \rightarrow FS_2$ which associates to a finite Galoisian 2-complex the $F_2$-set defined by its dessin d’enfant induces an isomorphism between $\text{FCG}_2$ and $FS_2$ (see [17]).

A finite dessin d’enfant is an algebraic curve defined over the algebraic closure $\bar{Q}$ of the field of rational integers. This induces the action of the Galois group $\text{Gal}(\bar{Q}/Q)$ on the category of dessins d’enfant (see [17]).

Let $\text{Hot}(FS_2)$ be the homotopy category of the Quillen model defined on $FS_2$ obtained by transferring on $FS_2$ the Quillen model defined on $Gph$.

Remark.

Let $D_0$ be the Dessin d’enfant whose underlying graph is $A_U$ and such that $F_2$ acts trivially on the nodes. The Cayley graph $\text{Cal}(D_0)$ associated to this action is $B_2$: the graph which has one node $n$ an two loops $a, b$.

We denote by $D_1$ the dessin d’enfant whose underlying graph is $P_2$. Let $s_1$ and $s_2$ be the generators of $F_2$. We suppose that $s_1$ acts trivially on the arcs of $P_2$ and $s_2$ defines a non trivial involution on them. The Cayley graph $\text{Cal}(D_1)$ associated to this action is a directed graph which has two nodes $x$ and $y$, there
exists one directed arc between $u$ and $y$, one loop $u_x$ at $x$, one directed arc $v$ between $y$ and $x$ and one loop $v_y$ at $y$. The morphism $f : \text{Cal}(D_1) \to \text{Cal}(D_0)$ defined by $f_0(x) = f_0(y) = n$, $f_1(u) = f_1(u_x) = a$, $f_1(v) = f_1(v_y) = b$ is a weak equivalence for the Quillen model defined on $Gph$ by counting the cycles but not an isomorphism. This implies that $D_0$ and $D_1$ are weak equivalent.

Questions.
Is the action of $\text{Gal}(\overline{Q}/Q)$ on the category of dessins d’enfant induces an action of $\text{Gal}(\overline{Q}/Q)$ on the image of $FS_2$ in the homotopy category of the Quillen model defined on the category of $F_2$-sets by transferring the Quillen model defined on $Gph$?

Remark.
We have constructed a closed model in $CG_2$ induced by the Quillen model defined on the category of colored 3-regular graphs. We can also ask whether the action of $\text{Gal}(\overline{Q}/Q)$ on dessins d’enfant induces an action on the image of the category of finite Galoisian 2-complexes in the homotopy category of this Quillen model.

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