(E, F)-MULTIPLIERS AND APPLICATIONS

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Abstract. For two given symmetric sequence spaces E and F we study the (E, F)-multiplier space, that is the space all of matrices M for which the Schur product M * A maps E into F boundedly whenever A does. We obtain several results asserting continuous embedding of (E, F)-multiplier space into the classical (p, q)-multiplier space (that is when E = l_p, F = l_q). Furthermore, we present many examples of symmetric sequence spaces E and F whose projective and injective tensor products are not isomorphic to any subspace of a Banach space with an unconditional basis, extending classical results of S. Kwapień and A. Pełczyński [14] and of G. Bennett [2] for the case when E = l_p, F = l_q.

1. Introduction

For an infinite scalar-valued (real or complex) matrix A = (a_{ij})_{i,j=1}^\infty and n \in \mathbb{N} we set

\[ T_n(A) := (t_{ij}^{(n)})_{i,j=1}^\infty, \quad \text{where} \quad t_{ij}^{(n)} = a_{ij}, \quad \text{for} \quad 1 \leq j \leq i \leq n \]

and t_{ij}^{(n)} = 0 otherwise. The operator T_n is called the n-th main triangle projection. S. Kwapień and A. Pełczyński studied in [14] the norms of the operators (T_n)_{n \geq 1} acting on the space of all bounded linear operators B(l_p, l_q) and obtained that

\[ \sup_n \|T_n\|_{B(l_p, l_q) \rightarrow B(l_p, l_q)} = \infty \]

for 1 \leq q \leq p \leq \infty, q \neq \infty, p \neq 1. Moreover, as an application, they have established that for 1 < p \leq \infty, 1 < q \leq \infty and \( \frac{1}{p} + \frac{1}{q} \leq 1 \) (respectively, 1 \leq p < \infty, 1 \leq q < \infty and \( \frac{1}{p} + \frac{1}{q} \geq 1 \)) projective (respectively, injective) tensor product of the spaces l_p and l_q is not isomorphic to any subspace of a Banach space with an unconditional basis. In the same paper [14] the question (Problem 1) whether the sequence \( (\|T_n\|_{B(l_p, l_q) \rightarrow B(l_p, l_q)})_{n \geq 1} \) is bounded for 1 < p < q < \infty was stated. The positive answer to that question was obtained by G. Bennett in his article [1], where he established that the main triangle projection T defined on an element A = (a_{ij})_{i,j=1}^\infty \in B(l_p, l_q) by T(A) := (t_{ij})_{i,j=1}^\infty where t_{ij} = a_{ij} for 1 \leq j \leq i and t_{ij} = 0 otherwise is bounded for 1 < p < q < \infty.

G. Bennett obtained his result on the operator T in the framework of the general theory of Schur multipliers on B(l_p, l_q) (briefly, (p, q)-multipliers). For a deep study and applications of this notion in analysis and operator theory we refer to [1, 2, 19].

The classical Banach spaces l_p, (1 \leq p \leq \infty) is an important representative of the class of symmetric sequence spaces (see e.g. [15]). The present paper extends results from [14] and [2] to a wider class of symmetric sequence spaces satisfying certain convexity conditions. In particular, we present sufficient conditions in terms of p-convexity and q-concavity of symmetric sequence spaces guaranteeing that their

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projective and injective tensor products are not isomorphic to any subspace of a Banach space with an unconditional basis (Section 6, Theorem 6.5). Our methods are based on the study of general Schur multipliers on the space $B(E, F)$ (briefly $(E, F)$-multipliers) extending and generalizing several results from [2]. In particular, we establish a number of results concerning the embedding of an $(E, F)$-multiplier space into a $(p, q)$-multiplier space and their coincidence (Section 4, Theorems 4.12 and 4.16).

An important technical tool used in this paper is the theory of generalized Köthe duality (Section 3), which (to our best knowledge) was firstly introduced by Hoffman [7] and presented in a detailed manner in [17] (see also recent papers [4] and [6]).

In the final section (Section 6), we present an extension of Kwapień and Pełczyński results for $l_p$-spaces to a wide class of Orlicz-Lorentz sequence spaces (Theorem 6.7).

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2. Preliminaries and notation

Let $c_0$ be a linear space of all converging to zero real sequences. For every $x \in c_0$, by $|x|$ we denote the sequence $(|x_i|)_{i=1}^{\infty}$ and by $x^*$ the non-increasing rearrangement of $|x|$, that is $x^* = (x^*_i)_{i=1}^{\infty} \in c_0$, where

$$x^*_i = |x_{n_i}| \quad (i = 1, 2, ...),$$

where $(n_i)_{i=1}^{\infty}$ is a such permutation of natural numbers, that the sequence $(|x_{n_i}|)$ is non-increasing.

In this paper, we work with symmetric sequence spaces which are a 'close relative' of the classical $l_p$-spaces, $1 \leq p \leq \infty$ (see [15, 16]).

Recall that a linear space $E \subset c_0$ equipped with a Banach norm $\| \cdot \|$ is said to be a symmetric sequence space, if the following conditions hold:

(i) if $x, y \in E$ and $|x| \leq |y|$, then $\|x\| \leq \|y\|$;

(ii) if $x \in E$, then $x^* \in E$ and $\|x^*\| = \|x\|$.

Without loss of generality we shall assume that $\|(1, 0, 0, ...)\| = 1$.

A symmetric sequence space $E$ is said to be $p$-convex ($1 \leq p \leq \infty$), respectively, $q$-concave ($1 \leq q \leq \infty$) if

$$\left\| \left( \sum_{k=1}^{n} |x_k|^p \right)^{1/p} \right\| \leq C \left( \sum_{k=1}^{n} \|x_k\|_E^p \right)^{1/p},$$

respectively,

$$\left\| \left( \sum_{k=1}^{n} \|x_k\|_E^q \right)^{1/q} \right\| \leq C \left( \sum_{k=1}^{n} |x_k|^q \right)^{1/q},$$

(with a natural modification in the case $p = \infty$ or $q = \infty$) for some constant $C > 0$ and every choice of vectors $x_1, x_2, \ldots, x_n$ in $X$. The least such constant is denoted by $M^{(p)}(E)$ (respectively, $M^{(q)}(E)$) (see e.g. [16]).

Remark 2.1. Any symmetric sequence space is 1-convex and $\infty$-concave with constants equal to 1.

The following proposition links $p$-convex and $q$-concave sequence spaces to classical $l_p$-spaces.
Proposition 2.2. [16] p. 132] If a symmetric sequence space $E$ is $p$-convex and $q$-concave, then

$$l_p \subset E \subset l_q$$

and

$$\| \cdot \|_{q/M(q)}(E) \leq \| \cdot \| \leq M^{(p)}(E) \| \cdot \|_p.$$  

Without loss of generality we shall assume that the embedding constants in (2.2) are both equal to 1 [16] Proposition 1.d.8].

Below, we restate the result given in [16, Proposition 1.d.4 (iii)] for the case of symmetric sequence spaces. (Here, by $E^*$ we denote the Banach dual of $E$.)

Proposition 2.3. Let $1 \leq p, q \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. A separable symmetric sequence space $E$ is $p$-convex (concave) if and only if the space $E^*$ is $q$-concave (convex).

Remark 2.4. Let $E$ be $q$-concave. If $E$ is not separable, then $E$ does not have order-continuous norm. It follows from [12] Chapter 10, §4 that there exists a pairwise disjoint sequence $(z_n)_n \subset E$ such that $z_n \geq 0$ and $\|z_n\|_E = 1, n = 1, 2, \ldots$ and $x = \sum_{n=1}^{\infty} z_n \in E$, and this contradicts to $q$-concavity of $E$. So if $E$ is $q$-concave, then $E$ is separable and then $E^*$ is a symmetric sequence space.

Given a symmetric sequence space $E$ and $0 < p < \infty$ we denote

$$E^p := \{ x \in c_0 : |x|^p \in E \}, \quad \|x\|_{E^p} := (\| \cdot \|^p_E)^{1/p}.$$  

The space $(E^p, \| \cdot \|_{E^p})$ is called the $p$-convexification of $E$ if $p > 1$ and $p$-concavification of $E$ if $p < 1$ (see e.g. [16] Chapter 1.d).

Remark 2.5. (i) If $1 \leq p < \infty$, then $(E^p, \| \cdot \|_{E^p})$ is a Banach space (for the proof see [12] Proposition 1].) It is also clear from the definition that $(E^p, \| \cdot \|_{E^p})$ is a symmetric sequence space.

(ii) Generally speaking, the space $((E)^{1/p}, \| \cdot \|_{E^{1/p}})$ is not a Banach space, but if $E$ is $p$-convex, then $(E)^{1/p}$ is a Banach space and so it is a symmetric sequence space (for details see [16] Chapter 1, p. 54). Furthermore, this is not difficult to check that if $E$ is $p$-convex, then the space $((E)^{1/p})^p$ is isometrically isomorphic to $E$.

3. Generalized Köthe duality

For a symmetric sequence space $E$ by $E^\times$ we denote its Köthe dual, that is

$$E^\times := \{ y \in l_\infty : \sum_{n=1}^{\infty} |x_n y_n| < \infty \text{ for every } x \in E \},$$

and for $y \in E^\times$ we set

$$\|y\|_{E^\times} := \sup \{ \sum_{n=1}^{\infty} |x_n y_n| : \|x\|_E \leq 1 \}.$$  

The space $(E^\times, \| \cdot \|_{E^\times})$ is a symmetric sequence space (see [13] Chapter II, §3].)

We say that $\| \cdot \|_E$ is a Fatou-norm if given $x \in E$ and a sequence $0 \leq x_n \in E$ such that $x_n \uparrow x$, it follows that $\|x_n\|_E \uparrow \|x\|_E$. It is known (see [12] Part I, Chapter X, §4, Theorem 7]) that $\| \cdot \|_E$ is a Fatou-norm if and only if $\|x\|_E = \|x\|_{E^\times \times}$ for every $x \in E$, where $(E^\times)^\times = E^{\times \times}$.  


For a pair of sequences \( x = (x_n)_{n=1}^{\infty}, y = (y_n)_{n=1}^{\infty} \in l_\infty \) by \( x \cdot y \) we denote the sequence \( (x_n y_n)_{n=1}^{\infty} \).

For any two symmetric sequence spaces \((E, \| \cdot \|_E)\) and \((F, \| \cdot \|_F)\), we set
\[
E^F := \{ x \in c_0 : x \cdot y \in F, \text{ for every } y \in E \}
\]
and for \( x \in E^F \)
\[
\| x \|_{E^F} := \sup_{\| y \|_E \leq 1} \| x \cdot y \|_F.
\]

**Remark 3.1.** Another suggestive notation for the space \( E^F \) introduced above would be \( F : E \) (see e.g. [7]). We use the notation \( E^F \) since it is in line with the notations from [17], [4] which are widely used in this section.

The fact that the supremum in (3.2) above is finite for every \( x \in E^F \) is explained below.

**Remark 3.2.** Any element \( x \in E^F \) can be consider as a linear bounded operator from \( E \) into \( F \) and \( x \in E^F \) if and only if for every \( y \in F \) the following inequality holds
\[
\| x \cdot y \|_F \leq \| x \|_{E^F} \| y \|_E.
\]

So the fact that the supremum in Equation (3.2) is finite can be proved via the closed graph theorem considering the operator \( x(y) = x \cdot y \) for every \( y \in E \).

The proof of the following proposition is routine and is therefore omitted.

**Proposition 3.3.** [3, Theorem 4.4] \((E^F, \| \cdot \|_{E^F})\) is a symmetric sequence space.

Analyzing definitions of the Köthe dual and generalized Köthe dual spaces, it is not difficult to see that the spaces \( E^1 \) and \( E^\infty \) coincide (see also [17]).

In the following proposition we collect a number of known results from [17].

**Proposition 3.4.** (i) [17, p. 326, item (f)] \( l_1^E = E \);

(ii) [17, Proposition 3] if \( 1 \leq r \leq p \leq \infty \) and \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} \), then \( l_r^p = l_q^r \);

(iii) [17, Theorem 2] if \( 1 \leq p < r \leq \infty \), then \( l_r^p = l_\infty^r \).

It is known (see e.g. [17, Theorem 2]) that in the general setting of Banach function spaces the space \( E^F \) can be trivial, that is \( E^F = \{ 0 \} \). The following proposition shows that this is not the case in the setting of symmetric sequence spaces.

**Proposition 3.5.** \( E^F \supset F \).

**Proof.** Let \( x \in l_\infty^F \) and \( y \in E \), then obviously \( y \in l_\infty \) and from Remark 3.2 we have
\[
\| x \cdot y \|_F \leq \| x \|_l_\infty^E \| y \|_E.
\]
In particular \( x \in E^F \) and \( \| x \|_{E^F} \leq \| x \|_l_\infty^E \). That is \( E^F \supset l_\infty^E \).

Since \( F = l_\infty^E \) (see Proposition 3.3 (i)), the claim follows.

The following proposition explains the connection between generalized Köthe duality and \( p \)-convexification.

**Proposition 3.6.** [17, Example 1] \((E^p)^l^p = (E^\infty)^p \).

Let \( E, F \) be sets of sequences, then we will denote
\[
E \cdot F := \{ x = y \cdot z \mid y \in E, z \in F \}.
\]
Proposition 3.7. [5, Theorem 1] $E \cdot E^\times = \ell_1$.

When $E$ is $p$-convex, we can extend the result of Proposition 3.7 as follows.

Proposition 3.8. If $E$ is $p$-convex, then $E \cdot (E)^{lp} = \ell_p$.

Proof. By the definition of the space $E^{lp}$ we have that $E \cdot (E)^{lp} \subset \ell_p$.

We claim that $E \cdot (E)^{lp} \supset \ell_p$.

Since $E$ is $p$-convex, the space $(E)^{1/p}$ is a symmetric sequence space and we have that $((E)^{1/p})^p = E$ (see Remark 2.3). Denoting $Y = (E)^{1/p}$, we have $Y^p = E$ and $(Y^\times)^p = E^{lp}$ (see Proposition 3.4). Thus our claim is that $Y^p \cdot (Y^\times)^p \supset \ell_p$.

If $z \in \ell_p$, then $|z|^p \in \ell_1$. By Proposition 3.7 we have that exist $y_1 \in Y$ and $y_2 \in Y^\times$ such that $|z|^p = y_1 \cdot y_2$. Hence, $|z| = (y_1 y_2)^{1/p} = |y_1|^{1/p} |y_2|^{1/p}$, and $z = \text{sgn}(z) |y_1|^{1/p} |y_2|^{1/p}$.

Since $y_1 \in Y$, we have $\text{sgn}(z) |y_1|^{1/p} \in Y^p$. Similarly we can obtain that $|y_2|^{1/p} \in (Y^\times)^p$. Thus the inclusion $z \in Y^p \cdot (Y^\times)^p$ holds and the claim is established. \hfill $\Box$

The second generalized Köthe dual is defined by $E^{EF} := (E^F)^F$. The following proposition is taken from [4] (see there Theorem 3.4 and Proposition 5.3, respectively).

Proposition 3.9. (i) If $E$ is $q$-concave for $1 < q \leq \infty$, then $i^q_{EF} = \ell_q$.

(ii) If $E$ is $p$-convex for $1 \leq p < \infty$, then $E^{lp} = E$.

Remark 3.10. Since any symmetric sequence space $E$ is a solid subspace of $\ell_\infty$, we have that $\ell_\infty \cdot E = E$.

4. Schur multipliers.

Let $E$ and $F$ be symmetric sequence spaces. For every $A \in B(E, F)$, we set for brevity

$$\|A\|_{E,F} := \|A\|_{B(E,F)} = \sup_{\|x\|_E \leq 1} \|A(x)\|_F$$

and

$$\|A\|_{1,F} := \|A\|_{B(1,F)}, \quad \|A\|_{E,\infty} := \|A\|_{B(E,\ell_\infty)}.$$ 

Any such operator $A$ can be identified with the matrix $A = (a_{ij})_{i,j=1}^\infty$, whose every row represents an element from $E^\times$ and every column represents an element from $F$. For a sequence $x = (x_n)_{n \geq 1} \in E$, we have $A(x) = \{ \sum_j a_{ij} x_j \}_i \in F$.

Proposition 4.1. If $E$ and $F$ are symmetric sequence spaces and $F$ has a Fatou-norm, then $\|A\|_{E,F} = \|A^T\|_{F^\times, E^\times}$, where $A^T$ is the transpose matrix for $A$.

Proof.

$$\|A\|_{E,F} = \sup_{\|x\|_E \leq 1} \|A(x)\|_F = \sup \{ \langle A(x), y \rangle : \|x\|_E \leq 1, \|y\|_{F^\times} \leq 1 \} =$$

$$\sup \{ \langle x, A^T(y) \rangle : \|x\|_E \leq 1, \|y\|_{F^\times} \leq 1 \} = \sup_{\|y\|_{F^\times} \leq 1} \|A^T(y)\|_{E^\times} = \|A^T\|_{F^\times, E^\times}. \hfill \Box$$

The following preposition presents the formulae for computing the norm of $A = (a_{ij})_{i,j=1}^\infty \in B(E, F)$ in some special cases.
Remark 4.4. Let $M$ be a matrix, then
\[ \|M\| \geq \|M\|_{\text{max}}. \]

**Proof.** By definition we have
\[ \|A\|_{1,E} = \sup_{\|x\|_{1} \leq 1} \|A(x)\|_{E}. \]

Hence, if $x = e_j$, where $e_j = (e_j^k)_{k} \in c_0$ such that $e_j^k = 1$ for $j = k$ and $e_j^k = 0$ for $j \neq k$, then $\|A\|_{1,E} \geq \| \{a_{ij}\}_i \|_E$ for $j = 1, 2, \ldots$, that is $\|A\|_{1,E} \geq \sup_{j} \| \{a_{ij}\}_i \|_E$.

Using the triangle inequality for the norm, we obtain the converse inequality
\[ \|A(x)\|_{E} = \| \{ \sum_j a_{ij} x_j \}_i \|_E \leq \sum_j |x_j| \| \{a_{ij}\}_i \|_E \leq \sup_j \| \{a_{ij}\}_i \|_E \sum_j |x_j| = \sup_j \| \{a_{ij}\}_i \|_E \|x\|_1. \]

Hence, $\|A\|_{1,E} \leq \sup_{j} \| \{a_{ij}\}_i \|_E$.

(ii) Applying Propositions 4.1 and 4.2 (i) we obtain $\|A\|_{\infty} = \|A^{T}\|_{1,E} = \sup_{i} \| \{a_{ij}\}_j \|_{E\ast}$.

If $A = (a_{ij})$ and $B = (b_{ij})$ are matrices of the same size (finite or infinite), their Schur product is defined to be the matrix of elements-wise products $A \ast B = (a_{ij}b_{ij})$.

**Definition 4.3.** An infinite matrix $M = (m_{ij})$ is called $(E, F)$-multiplier if $M \ast A \in B(E, F)$ for every $A \in B(E, F)$.

The set of all $(E, F)$-multipliers is denoted by
\[ \mathcal{M}(E, F) := \{ M : M \ast A \in B(E, F), \forall A \in B(E, F) \}. \]

The collection $\mathcal{M}(E, F)$ is a normed space with respect to the norm
\[ \|M\|_{(E,F)} := \sup_{\|A\|_{E,F} \leq 1} \|M \ast A\|_{E,F}. \]

(when $E = l_p$ and $F = l_q$, we use the notation $\mathcal{M}(p,q)$ for (4.1) and $\|M\|_{(p,q)}$ for (4.2)).

**Remark 4.4.** (i) Viewing $M \in \mathcal{M}(E, F)$ as a linear operator $M : B(E, F) \to B(E, F)$, one easily checks that the supremum in (4.2) is finite via the closed graph theorem.

(ii) Since $\|M\|_{(E,F)} = \sup_{\|A\|_{E,F} \leq 1} \|M \ast A\|_{E,F} \geq \|M \ast u_{jk}\|_{E,F} = |m_{jk}|$ for every $j, k = 1, 2, \ldots$, where $u_{jk} = (u_{jk}^{nm})_{nm}$ such that $u_{jk}^{nm} = 1$ if $n = j, m = k$ and $u_{jk}^{nm} = 0$ otherwise, for $j, k, n, m = 1, 2, \ldots$, we have that $\|M\|_{(E,F)} \geq \sup_{j,k} |m_{jk}|$.

The proofs of Theorem 4.5 and Lemma 4.6 below are routine and incorporated here for convenience of the reader.

**Theorem 4.5.** The normed space $(\mathcal{M}(E, F), \| \cdot \|_{(E,F)})$ is complete.
Proof. Since \( M(E, F) \subseteq B(B(E, F)) \), all we need to see is that \( M(E, F) \) is closed. Take a sequence of \( M_n \in M(E, F), \ n \geq 1. \) Let

\[
M_n = \left( m_{jk}^{(n)} \right)_{j,k=1}^\infty, \quad n \geq 1.
\]

Assume that \( \lim_{n \to \infty} \| M_n - T \|_{B(B(E, F))} = 0 \), for some \( T \in B(B(E, F)) \).

Fix \( j, k \geq 0 \) and fix a matrix unit \( u_{jk} \in B(E, F) \). The sequence \( (M_n)_{n \geq 1} \) is Cauchy in \( B(B(E, F)) \). Consequently, the sequence \( (M_n (u_{jk}))_{n \geq 1} \) is Cauchy in \( B(E, F) \). Thus, the sequence \( (m_{jk}^{(n)})_{n \geq 0} \) is Cauchy in \( \mathbb{R} \). Thus, for every \( j, k = 1, 2, \ldots \) there is a number \( m_{jk} \) such that

\[
\lim_{n \to \infty} \left| m_{jk}^{(n)} - m_{jk} \right| = 0.
\]

Since

\[
\| (M_n (u_{jk}) - m_{jk} u_{jk}) (x) \|_F = \left\| (m_{jk}^{(n)} u_{jk} - m_{jk} u_{jk}) (x) \right\|_F = \left| m_{jk}^{(n)} - m_{jk} \right| \| x \|_E
\]

for every \( x \in E \), we have

\[
\| M_n (e_{jk}) - m_{jk} e_{jk} \|_{E,F} \leq \left| m_{jk}^{(n)} - m_{jk} \right|.
\]

Hence

\[
\lim_{n \to \infty} \| M_n (u_{jk}) - m_{jk} u_{jk} \|_{E,F} = 0.
\]

On the other hand, the assumption

\[
\lim_{n \to \infty} \| M_n - T \|_{B(B(E, F))} = 0
\]

implies that

\[
\lim_{n \to \infty} \| M_n (u_{jk}) - T (u_{jk}) \|_{E,F} = 0.
\]

Combining \( \mathbf{1.3} \) with \( \mathbf{1.4} \) yields

\[
T (u_{jk}) = m_{jk} u_{jk} \quad \text{for every } j, k = 1, 2, \ldots.
\]

That is, \( T \) is a Schur multiplier.

\[ \square \]

Lemma 4.6. Let \( A \in B(E, F) \) and \( M = (m_{ij}) \) be such that \( \sup_{i,j} |m_{ij}| < \infty \). If \( M \ast A \) maps \( E \) into \( F \), then \( M \ast A \in B(E, F) \).

Proof. Since \( A = (a_{ij}) \in B(E, F) \), we have that

\[
\| (a_{ij}) \|_{E^*} < +\infty, \quad \text{for all } i = 1, 2, \ldots.
\]

Assume that \( x = (x_i), x_n = (x_i^{(n)}) \in E, y = (y_i) \in F \) are such that \( x_n \to x \) and \( (M \ast A) (x_n) \to y \). The claim of the lemma follows from the closed graph theorem if we show that \( y = (M \ast A) (x) \). To that end, using Holder inequality, we have that

\[
\left| \sum_j m_{ij} a_{ij} (x_j^{(n)} - x_j) \right| \leq \sum_j |m_{ij}| |a_{ij}| |x_j^{(n)} - x_j| \leq
\]

\[
\sup_{i,j} |m_{ij}| \sum_j |a_{ij}| |x_j^{(n)} - x_j| \leq \sup_{i,j} |m_{ij}| \|x_n - x\|_E \| (a_{ij})_j \|_{E^*} \to 0,
\]
for every $i = 1, 2, \ldots$. It follows that

$$y_i = \lim_{n \to \infty} \sum_j m_{ij}a_{ij}x_j^{(n)} = \sum_j m_{ij}a_{ij}x_j, \quad \text{for every } i = 1, 2, \ldots$$

So we obtain that $y = (M \ast A)(x)$. \hfill \boxdot

**Proposition 4.7.** If $E$ and $F$ are symmetric sequence spaces, $F$ has a Fatou-norm and $M \in \mathcal{M}(E, F)$, then $\|M\|_{(E,F)} = \|M^T\|_{(E^*,F^*)}$. 

**Proof.** Using Proposition 4.1, we obtain that

$$\|M\|_{(E,F)} = \sup_{\|A\|_{E,F} \leq 1} \|M \ast A\|_{E,F} = \sup_{\|A\|_{F^*,E^*} \leq 1} \|M \ast A\|_{F^*,E^*} =$$

$$\sup_{\|A\|_{F^*,E^*} \leq 1} \|M^T \ast A^T\|_{F^*,E^*} = \sup_{\|B\|_{F^*,E^*} \leq 1} \|M^T \ast B\|_{F^*,E^*} = \|M^T\|_{(E^*,F^*)}.$$ 

\hfill \boxdot

**Proposition 4.8.** For every symmetric sequence space $E$, the following equations hold

(i) $\|M\|_{(1,E)} := \sup_{i,j} |m_{ij}|;$

(ii) $\|M\|_{(E,\infty)} := \sup_{i,j} |m_{ij}|.$

**Proof.** (i) As we have seen above $\|M\|_{(1,E)} \geq \sup_{i,j} |m_{ij}|$ (Remark 4.4). Let us prove the converse inequality. For every operator $A = (a_{ij}) \in B(E,F)$, using Proposition 4.2 (i), we have

$$\|M \ast A\|_{1,E} = \sup_{j} \|(m_{ij}a_{ij})_i\|_E \leq \sup_{j} \|(a_{ij})_i\|_E \sup_{j} |m_{ij}| \|a_{ij}\|_E$$

$$\leq \sup_{j} \sup_{i} |m_{ij}| \sup_{j} \|(a_{ij})_i\|_E = \sup_{i,j} |m_{ij}| \|M\|_{1,E}.$$ 

Hence, $\|M\|_{(1,E)} \leq \sup_{i,j} |m_{ij}|$. 

(ii) The claim follows from Proposition 4.7 and (i) above. \hfill \boxdot

**Corollary 4.9.** The multiplier spaces $\mathcal{M}(1,E)$ and $\mathcal{M}(F,\infty)$ are isometrically isomorphic and do not depend on a choice of the spaces $E$ and $F$.

Now, we are well equipped to consider the question of embedding of the $(E,F)$-multiplier space into an $(p,q)$-multiplier space. We start by recalling the following result from [2].

Analyzing the proof of [2, Theorem 6.1], we restate its result as follows.

**Theorem 4.10.** (i) If $1 \leq p_2 \leq p_1 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$, then $\mathcal{M}(p_1, q_1) \subseteq \mathcal{M}(p_2, q_2)$;

(ii) If $1 \leq q_1, q_2 \leq \infty$ and $p_1 = p_2 = 1$, then $\mathcal{M}(p_1, q_1) = \mathcal{M}(p_2, q_2)$;

(iii) If $1 \leq p_1, p_2 \leq \infty$ and $q_1 = q_2 = \infty$, then $\mathcal{M}(p_1, q_1) = \mathcal{M}(p_2, q_2)$;

(iv) If $1 \leq p_1, q_1 \leq \infty$ and $q_2 \leq 2 \leq p_2$, then $\mathcal{M}(p_2, q_2) \subseteq \mathcal{M}(p_1, q_1)$.

The following corollary follows immediately from Theorem 4.10 (i) and (iv).

**Corollary 4.11.** $\mathcal{M}(2,2) = \mathcal{M}(\infty, 1)$.

The following theorem extends Theorem 4.10 and is the main result of this section.
Theorem 4.12. Let $1 \leq p,q \leq \infty$ be given. If $E$ is $p$-convex and $F$ is $q$-concave, then $\mathcal{M}(E,F) \subseteq \mathcal{M}(p,q)$.

Proof. Let $A = (a_{ij})_{i,j=1}^{\infty} \in B(l_p,l_q)$ and $M \in \mathcal{M}(E,F)$. To prove the claim of the Theorem 4.12, we need to show that $M \ast A \in B(l_p,l_q)$.

For $x \neq 0 \in E^p$, $y \neq 0 \in l^p$ consider the operator $yAx$ which acts on the element $z \in E$ as follows:

$$(yAx)(z) = y \cdot (A(x \cdot z)) = \left( \sum_n y_m a_{mn} x_n z_n \right)_m.$$ 

Since $z \in E$, $x \in E^p$, by the definition of the space $E^p$, we have that $x \cdot z \in l_p$. By the assumption $A \in B(l_p,l_q)$, we have $A(x \cdot z) \in l_q$. Since $y \in l^p$, we obtain $y \cdot (A(x \cdot z)) \in F$. That is, the operator $yAx$ maps the space $E$ into the space $F$. Moreover, for $z \in E$, we have

$$\|yAx\|_F = \|y \cdot (A(x \cdot z))\|_F \leq \|y\|_{l^p} \|A(x \cdot z)\|_q \leq \|y\|_{l^p} \|A\|_{p,q} \|x \cdot z\|_p \leq \|y\|_{l^p} \|A\|_{p,q} \|z\|_F \|x\|_E.$$ 

Hence, $yAx \in B(E,F)$. By the assumption we have $M \in \mathcal{M}(E,F)$, and so we obtain $M \ast (yAx) \in B(E,F)$. It is easy to see that $M \ast (yAx) = y(M \ast A)x$. Therefore $y(M \ast A)x \in B(E,F)$.

The next step is to prove that the operator $M \ast A$ maps $l_p$ into $l_q$. Let $x \in l_p$ be given. Since $F$ is $q$-concave, by Proposition 3.9 (i) we have that if $y \cdot z \in F$ for every $y \in l^p_q$, then $z \in E$. We claim that $y \cdot ((M \ast A)(x)) \in F$ for every $y \in l^p_q$. Since $E$ is $p$-convex and $x \in l_p$, there exist $x_1 \in E^p$ and $x_2 \in E$ such that $x = x_1 \cdot x_2$ (see Proposition 3.8). Since $y(M \ast A)x \in B(E,F)$ for every $x \in E^p$ and $y \in l^p_q$, we have

$$y \cdot ((M \ast A)(x)) = y \cdot ((M \ast A)(x_1 \cdot x_2)) = (y(M \ast A)x_1)(x_2) \in F.$$ 

We conclude $(M \ast A)(x) \in l_q$, that is, $M \ast A : l_p \to l_q$. Since $M \in \mathcal{M}(E,F)$, we have that $\sup_{i,j} |m_{ij}| < \infty$ (Remark 2.1 (ii)). By Lemma 4.10 we obtain that $M \in \mathcal{M}(p,q)$ (see also 21 Lemma 2)).

Since any symmetric sequence space is 1-convex and $\infty$-concave (Remark 2.1), the following corollary follows immediately from Theorem 4.12.

Corollary 4.13. For every pair of symmetric sequence spaces $E$ and $F$, we have $\mathcal{M}(E,F) \subseteq \mathcal{M}(1,\infty)$.

Corollary 4.14. If $E$ is $p$-convex, $F$ is $q$-concave, then $\mathcal{M}(E,F) \subseteq \mathcal{M}(p_1,q_1)$ for every $1 \leq p_1 \leq p \leq \infty$ and $1 \leq q \leq q_1 \leq \infty$.

Proof. By the assumptions of the Corollary and by Theorem 4.12 we have that $\mathcal{M}(E,F) \subseteq \mathcal{M}(p,q)$. From Theorem 4.10 (i) it follows that $\mathcal{M}(p,q) \subseteq \mathcal{M}(p_1,q_1)$. Hence, the claim follows.

The proof of the following theorem is similar to the proof of Theorem 4.12. We need to apply Proposition 3.8 (i) with $p = 1$ and Remark 3.10 instead of Propositions 3.9 (i) and 3.8, respectively. We supply details for completeness.

Theorem 4.15. For symmetric sequence spaces $E$ and $F$ the embedding $\mathcal{M}(\infty,1) \subseteq \mathcal{M}(E,F)$ holds.
Proof. Let $A = (a_{ij})_{i,j=1}^{\infty} \in B(E, F)$ and $M \in \mathcal{M}(\infty, 1)$. To prove the claim of the Theorem 4.12, we need to show that $M \ast A \in B(E, F)$.

For $x \neq 0 \in l_\infty^E$, $y \neq 0 \in F^{l_1}$ consider the operator $yAx$ which acts on the element $z \in l_\infty$ as follows:

$$(yAx)(z) = y \cdot (A(x \cdot z)) = \left( \sum_n y_m a_{mn} x_n z_n \right)_m.$$

Since $z \in l_\infty$, $x \in l_\infty^E$, by the definition of the space $l_\infty^E$, we have that $x \cdot z \in E$. By the assumption $A \in B(E, F)$, we have $A(x \cdot z) \in F$. Since $y \in F^{l_1}$, we obtain $y \cdot (A(x \cdot z)) \in l_\infty$. That is, the operator $yAx$ maps the space $l_\infty$ into the space $l_1$.

Moreover, for $z \in l_\infty$, using (3.3), we have

$$\|yAx(z)\|_1 = \|y \cdot (A(x \cdot z))\|_1 \leq \|y\|_{l_1^E} \|A(x \cdot z)\|_F \leq \|y\|_{l_1^E} \|A\|_{E,F} \|x\|_{l_\infty^E} \|z\|_E.$$ 

Hence, $yAx \in B(l_\infty, l_1)$. By the assumption we have $M \in \mathcal{M}(\infty, 1)$, and so we obtain $M \ast (yAx) \in B(l_\infty, l_1)$. It is easy to see that $M \ast (yAx) = y(M \ast A)x$.

Therefore $y(M \ast A)x \in B(l_\infty, l_1)$.

The next step is to prove that the operator $M \ast A$ maps $E$ into $F$. Let $x \in E$ be given. By Proposition 3.10 (ii) with $p = 1$ we have that if $y \cdot z \in l_1$ for every $y \in F^{l_1}$, then $z \in F$. We claim that $y \cdot ((M \ast A)(x)) \in l_1$ for every $y \in F^{l_1}$. Since $x \in E$, there exist $x_1 \in l_\infty^E$ and $x_2 \in l_\infty$ such that $x = x_1 \cdot x_2$ (see Remark 3.10). Since $y(M \ast A)x \in B(l_\infty, l_1)$ for every $x \in l_\infty^E$ and $y \in F^{l_1}$, we have

$$y \cdot ((M \ast A)(x)) = y \cdot ((M \ast A)(x_1 \cdot x_2)) = (y(M \ast A)x_1)(x_2) \in l_1,$$

We conclude $(M \ast A)(x) \in l_\infty$, that is, $M \ast A : E \to F$. Since $M \in \mathcal{M}(\infty, 1)$, we have that $\sup_{i,j} |m_{ij}| < \infty$ (Remark 4.3 (ii)). By Lemma 4.6 we obtain that $M \in \mathcal{M}(E, F)$. 

Note that Corollary 4.13 and Theorem 4.15 imply the existence of the maximal and minimal multiplier spaces, that is we have

$$\mathcal{M}(\infty, 1) \subseteq \mathcal{M}(E, F) \subseteq \mathcal{M}(1, \infty).$$

The following theorem gives sufficient conditions on $E$ and $F$ guaranteeing the equality $\mathcal{M}(\infty, 1) = \mathcal{M}(E, F)$.

**Theorem 4.16.** If $E$ is 2-convex, $F$ is 2-concave and $1 \leq q \leq 2 \leq p \leq \infty$, then

$$\mathcal{M}(E, F) = \mathcal{M}(p, q).$$

**Proof.** It is sufficient to prove the assertion for the case $p = q = 2$. Indeed, for $q \leq 2 \leq p$ embeddings $\mathcal{M}(p, q) \subseteq \mathcal{M}(2, 2)$ and $\mathcal{M}(2, 2) \subseteq \mathcal{M}(p, q)$ follow then from Theorem 4.10 (i) and (iv), respectively.

By Theorem 4.12 we have $\mathcal{M}(E, F) \subseteq \mathcal{M}(2, 2)$. Using Theorem 4.10 (iv) we obtain $\mathcal{M}(2, 2) \subseteq \mathcal{M}(\infty, 1)$. Theorem 4.15 yields the converse embedding. The proof is completed.

The following corollaries follow immediately from [19] Theorem 5.1 (i) and Theorem 4.16.
Corollary 4.17. Let $E$ be 2-convex and $F$ be 2-concave. A matrix $M = (m_{ij})_{i,j=1}^\infty$ is an element of the space $\mathcal{M}(E, F)$ if and only if there is a Hilbert space $H$ and families $(y_i)_{i=1}^\infty$, $(x_j)_{j=1}^\infty$ of elements of $H$ such that $m_{ij} = \langle y_i, x_j \rangle$, for every $(i, j) \in \mathbb{N} \times \mathbb{N}$ and $\sup_j \|y_i\| \sup_j \|x_j\| < \infty$.

By $E \otimes F$ we shall denote the algebraic tensor product of $E$ and $F$. We introduce the tensor norm $\gamma_2^*$ defined as follows. For all $u \in E \otimes F$ we define

$$\gamma_2^*(u) := \inf \{(\sum_j \|x_j\|_E^2)^{1/2} (\sum_i \|\xi_i\|_F^2)^{1/2}\},$$

where the infimum runs over all finite sequences $(x_j)_{j=1}^n$ in $E$ and $(\xi_i)_{i=1}^n$ in $F$ such that $u = \sum_{i=1}^n x_i \otimes \xi_i$. It is not difficult to check that $\gamma_2^*$ is a norm on $E \otimes F$. We will denote by $E \hat{\otimes} \gamma_2^* F$ the completion of $E \otimes F$ with respect to that norm.

The next result follows from Theorem 4.10 and [19, Theorem 5.1 (ii) and Theorem 5.3].

Corollary 4.18. Let $E$ be 2-convex and $F$ be 2-concave. Then

$$\mathcal{M}(E, F) = (l_1 \hat{\otimes} \gamma_2^* l_1)^*.$$

We complete this section with the following observation.

Proposition 4.19. The embeddings in Theorems and Corollaries 4.10 through 4.16 are continuous.

Proof. For example, we will prove the claim for Theorem 4.10 (i). Let $p_1, q_1$ be the same as in Theorem 4.10 (i) and $I : \mathcal{M}(p_1, q_1) \to \mathcal{M}(p_2, q_2)$ is an operator of embedding, that is $I(M) = M$ for every $M \in \mathcal{M}(p_1, q_1)$. Let the sequence $(M_n)_{n \geq 1} \subset \mathcal{M}(p_1, q_1)$ be such that $M_n \to 0$ in the space $\mathcal{M}(p_1, q_1)$ for $n \to \infty$ and $I(M_n) = M_n \to M$ in the space $\mathcal{M}(p_2, q_2)$ for $n \to \infty$. Using the notations from the proof of Proposition 4.2, we have

$$\langle M_n(u_{jk}), e_k \rangle = m^{(n)}_{jk} e_j$$

and

$$\langle M(u_{jk}), e_k \rangle = m_{jk} e_j$$

for every $j, k, n = 1, 2, \ldots$. Since $M_n \to 0$ in the space $\mathcal{M}(p_1, q_1)$, we obtain that $m^{(n)}_{jk} \to 0$ for $n \to \infty$ and $j, k = 1, 2, \ldots$. By the assumption $M_n \to M$ in the space $\mathcal{M}(p_2, q_2)$ we have that $m_{jk} \to m_{jk}$. Hence, we obtain $m_{jk} = 0$ for all $j, k, n = 1, 2, \ldots$, that is $M = 0$. Thanks to Theorem 4.5 we may apply the closed graph theorem and conclude that the operator $I$ is bounded.

5. The main triangle projector

As before, the $n$-th main triangle projection is denoted by $T_n$ ($n \in \mathbb{N}$). The question when the sequence $(\|T_n\|_{\mathcal{B}(l_p, l_q) \to \mathcal{B}(l_p, l_q)})_{n \geq 1}$ is (un)bounded was completely answered in [14] and [15].

Proposition 5.1. [14] Proposition 1.2] Let $p \neq 1$, $q \neq \infty$, $q \leq p$. Then we have

$$\|T_n\|_{(p, q)} \geq C(p, q) \ln n,$$

where $C(p, q)$ is a constant dependent only on $p$ and $q$. 


The following proposition extends the result of Proposition [5.1] to a wider class of symmetric sequence spaces.

**Proposition 5.2.** Let $E$ and $F$ be symmetric sequence spaces. If $E$ is $p$-convex and $F$ is $q$-concave for $p \neq 1$, $q \neq \infty$ and $q \leq p$, then

$$
\|T_n\|_{(E,F)} \geq C(p,q) \ln n,
$$

where $C(p,q)$ is a constant dependent only on $p$ and $q$.

**Proof.** Consider the space of multipliers $\mathcal{M}(E,F)$. Since the operator $T_n$ is a finite rank operator, we have that $T_n \in \mathcal{M}(E,F)$ and also $T_n \in \mathcal{M}(p,q)$. Since $E$ is $p$-convex and $F$ is $q$-concave, by Theorem [4.12] we have that $\mathcal{M}(E,F) \subset \mathcal{M}(p,q)$.

By Proposition [4.19] there exists a constant $C_1$ such that

$$
\|T_n\|_{(E,F)} \geq C_1\|T_n\|_{(p,q)}, \quad \forall n \geq 1.
$$

Applying Proposition [5.1] we conclude

$$
\|T_n\|_{(E,F)} \geq C_1\|T_n\|_{(p,q)} \geq C_1C(p,q) \ln n, \quad \forall n \geq 1.
$$

\[\square\]

6. Projective and injective tensor products of symmetric sequence spaces

We briefly recall some notions and notations from [14]. Let $\mathcal{M}_0$ be the set of scalar-valued (real or complex) infinite matrices, such that if $A = (a_{ij}) \in \mathcal{M}_0$, then $a_{ij} \neq 0$ for all but finitely many $(i, j) \in \mathbb{N} \times \mathbb{N}$.

A non-negative function $\| \cdot \|_{\mathcal{M}_0}$ on $\mathcal{M}_0$ is called a **matrix norm**, if it satisfies the following conditions:

(i) for every $A, B \in \mathcal{M}_0$ and for any scalar $\alpha$

$$
\|A\|_{\mathcal{M}_0} = 0 \text{ if } A = 0;
$$

$$
\|\alpha A\|_{\mathcal{M}_0} = |\alpha| \|A\|_{\mathcal{M}_0};
$$

$$
\|A + B\|_{\mathcal{M}_0} \leq \|A\|_{\mathcal{M}_0} + \|B\|_{\mathcal{M}_0};
$$

(ii) $\|u_{jk}\|_{\mathcal{M}_0} = 1$, $\forall j, k \geq 1$ (see the definition of the matrix unit $u_{jk}$ in Remark [4.3]).

(iii) $\|P_{nm}(A)\|_{\mathcal{M}_0} \leq \|A\|_{\mathcal{M}_0}$ for all $A \in \mathcal{M}_0$, $n, m = 1, 2, ..., $ where $P_{nm}$ is projector on the first $n$ lines and $m$ columns.

A matrix norm is called **unconditional** if

(iv) $\|A\|_{\mathcal{M}_0} = \|(x_{ij}a_{ij})\|_{\mathcal{M}_0}$, for all $A \in \mathcal{M}_0$, where $x_{ij} = \pm 1$, $i, j = 1, 2, ...$.

An unconditional matrix norm is called **symmetric** if

(v) $\|A\|_{\mathcal{M}_0} = \|(a_{\varphi(i)\psi(j)})_{ij}\|_{\mathcal{M}_0}$ for all $A \in \mathcal{M}_0$ and for all permutations $\varphi, \psi$ of positive integers.

If $\| \cdot \|_{\mathcal{M}_0}$ is a matrix norm, then the conjugate norm defined by

$$
\|A\|_{\mathcal{M}_0}^* := \sup\{\sum_{i,j} a_{ij}b_{ij} : B \in \mathcal{M}_0, \|B\|_{\mathcal{M}_0} \leq 1\}
$$

We have $\|A\|_{\mathcal{M}_0}^{**} = \|A\|_{\mathcal{M}_0}$.

We denote

$$
\|T_n\|_{(\mathcal{M}_0)} := \sup\{\|T_n(A)\|_{\mathcal{M}_0} : \|A\|_{\mathcal{M}_0} \leq 1\}.
$$

It is known that (see [13] Equation (1.1))

$$
\|T_n\|_{(\mathcal{M}_0)} := \sup\{\|T_n(A)\|^{*} : A \in \mathcal{M}_0, \|A\|^{*} \leq 1\} = \|T_n\|_{(\mathcal{M}_0)}.
$$

(6.1)
The following theorem reflects the connection between the boundedness of the norms of the main triangle projections and the question concerning a possibility of embedding of a matrix space into a Banach space with an unconditional basis.

**Theorem 6.1.** ([14], Theorem 2.3) Let \( \| \cdot \|_{\mathcal{M}_0} \) be a such symmetric matrix norm. If the sequence \( \{ \| T_n \|_{(\mathcal{M}_0)} \}_{n} \) is unbounded, then the space \( (\mathcal{M}_0, \| \cdot \|_{\mathcal{M}_0}) \) is not isomorphic to any subspace of a Banach space with an unconditional basis.

In the present paper, we consider only two types of matrix spaces, projective and injective tensor products. Recall the definition of the spaces (see e. g. [20]).

Let \( (E, \| \cdot \|_E) \), \( (F, \| \cdot \|_F) \) be Banach spaces over the field \( \mathbb{K} \) (real or complex numbers). By \( E \otimes F \) we denote the algebraic tensor product of \( E \) and \( F \).

For every \( u \in E \otimes F \) we define the *projective tensor norm*

\[
\pi(u) := \inf \left\{ \sum_{i=1}^{n} \| x_i \|_E \| y_i \|_F : u = \sum_{i=1}^{n} x_i \otimes y_i \right\}
\]

(respectively, *injective tensor norm*

\[
\varepsilon(u) := \sup \left\{ \sum_{i=1}^{n} \varphi(x_i)\psi(y_i) : \varphi \in E^*, \| \varphi \|_{E^*} \leq 1, \psi \in F^*, \| \psi \|_{F^*} \leq 1 \right\}.
\]

The completion of \( E \otimes F \) with respect to the norm \( \pi \) (respectively, \( \varepsilon \)) we shall denote by \( \hat{E} \otimes \hat{F} \) (respectively, \( \hat{E} \overset{\pi}{\otimes} \hat{F} \)) and call by projective (respectively, injective) tensor product of Banach spaces \( E \) and \( F \).

For convenience, we denote the norm \( \pi \) (respectively, \( \varepsilon \)) on the space \( E \otimes F \) by \( \pi_{E,F} \) (respectively, \( \varepsilon_{E,F} \)).

Let \( c_0 \) be the linear space of all finitely supported sequences. The tensor product \( c_0 \otimes c_0 \) can be identified with the space of matrices \( \mathcal{M}_0 \) on \( \mathbb{K} \). The tensor product basis \( \{ e_j \otimes e_k \}_{j,k=1}^{\infty} \) corresponds to the standard basis in \( \mathcal{M}_0 \) (see [20, §1.5] and [14, §3]).

If \( E \) is separable and \( p \)-convex, \( F \) is \( q \)-concave symmetric sequence spaces, then the spaces \( E^* \) and \( F^* \) are symmetric spaces too (see Remark 24), furthermore their dual spaces coincide with \( E^\times \) and \( F^\times \), respectively (see [12, Part I, Chapter X, §4, Theorem 1]). Therefore \( \varepsilon_{E,F} \) and \( \pi_{E,F} \) are symmetric matrix norm on the space \( c_0 \otimes c_0 \). For this reason, below we shall only consider separable symmetric sequence spaces.

The following proposition explains the connection between tensor product norms and the operator norm in \( B(E,F) \) (see [20, §2.2 and 3.1]).

**Proposition 6.2.** (i) The norm \( \varepsilon_{E,F} \) coincides with the operator norm on the space \( B(E^\times, F) \);

(ii) the conjugate norm for the norm \( \pi_{E,F} \) coincides with the operator norm on the space \( B(E, F^\times) \).

**Remark 6.3.** In particular, Proposition 6.2 (i) shows that for \( A = (a_{ij}) \in B(E^\times, F) \), we have

\[
\| A \|_{\varepsilon_{E,F}} = \sup \left\{ \sum_{i,j} a_{ij} x_i y_j : \| x \|_{E^\times} \leq 1, \| y \|_{F^\times} \leq 1 \right\}.
\]

Another important observation is...
Proof. (i) Since the norm

\[ \|T_n\|_{(E,F)} = \|T_n\|_{(E \otimes \hat{F}, E \times F)} \quad \text{for every} \quad n \geq 1. \]

We can reformulate Proposition 6.2 as follows:

**Proposition 6.4.** Let \( E \) and \( F \) be symmetric sequence spaces.

(i) If \( E \) is \( p \)-concave and \( F \) is \( q \)-concave for \( p \neq \infty \), \( q \neq \infty \) and \( q \leq p^* \), then

\[ \|T_n\|_{(E,F)} \geq C(p, q) \ln n. \]

(ii) If \( E \) is \( p \)-convex and \( F \) is \( q \)-convex for \( p \neq 1 \), \( q \neq 1 \) and \( p^* \leq q \) then

\[ \|T_n\|_{(E,F)} \geq C(p, q) \ln n. \]

**Proof.** (i) Since the norm \( \varepsilon_{E,F}(\cdot) \) coincides with the norm \( \| \cdot \|_{E \times F} \) (see Proposition 6.2(i)) and \( E^\times \) is \( p^* \)-convex (see Proposition 2.3), by Proposition 5.2 we have that

\[ \|T_n\|_{(E,F)} = \|T_n\|_{(E^\times,F)} \geq C(p, q) \ln n. \]

(ii) Applying (6.2) and (6.1), we obtain that

\[ \|T_n\|_{(E,F)} = \|T_n\|_{(E \otimes \hat{F}, E \times F)} = \|T_n\|_{(E^\times,F)}. \]

Since \( E \) (respectively, \( F \)) is \( p \)-convex (respectively, \( q \)-convex), we have \( E^\times \) (respectively, \( F^\times \)) is \( p^* \)-concave (respectively, \( q^* \)-concave) (see Proposition 2.3). By the item (i) above, we have

\[ \|T_n\|_{(E \times F)} \geq C(p, q) \ln n, \quad \forall n \geq 1 \]

whenever \( p^* \neq \infty \), \( q^* \neq \infty \) and \( q^* \leq p \). Applying (6.3) and (6.4), we obtain that

\[ \|T_n\|_{(E,F)} = \|T_n\|_{(E \times F)} \geq C(p, q) \ln n, \quad \forall n \geq 1 \]

for \( p \neq 1 \), \( q \neq 1 \) and \( p^* \leq q \).

\[ \square \]

The following theorem is the main result of the section.

**Theorem 6.5.** Let \( E \) and \( F \) be symmetric sequence spaces.

(i) If \( E \) is \( p \)-concave and \( F \) is \( q \)-concave for \( p \neq \infty \), \( q \neq \infty \) and \( q \leq p^* \), then the tensor product \( \hat{E} \otimes \hat{F} \) is not isomorphic to any subspace of a Banach space with an unconditional basis.

(ii) If \( E \) is \( p \)-convex and \( F \) is \( q \)-convex for \( p \neq 1 \), \( q \neq 1 \) \( p^* \leq q \), then the tensor product \( \hat{E} \otimes \hat{F} \) is not isomorphic to any subspace of a Banach space with an unconditional basis.

**Proof.** (i) By Proposition 6.4(i), with \( p \neq \infty \), \( q \neq \infty \) and \( q \leq p^* \), we have that the sequence \( \{\|T_n\|_{(E,F)}\}_n \) is unbounded. By Theorem 6.1 we obtain that the space \( c_{00} \otimes \hat{c}_{00} \) with the norm \( \varepsilon_{E,F} \) is not isomorphic to any subspace of a Banach space with an unconditional basis. Since \( (c_{00} \otimes \hat{c}_{00}, \varepsilon_{E,F}) \) is a linear subspace in \( \hat{E} \otimes \hat{F} \), we conclude that the space \( \hat{E} \otimes \hat{F} \) is not isomorphic to any subspace of a Banach space with an unconditional basis.

(ii) Similarly to the item (i), using Proposition 6.4(ii) instead of Proposition 6.4(i).
Now we consider a class of symmetric sequence spaces, called Orlicz-Lorentz sequence spaces, generalizing the class of $l_p$-spaces. For detailed studies of this class of spaces we refer to [3], [9] and [10].

We recall that $G : [0, \infty) \to [0, \infty)$ is an Orlicz function, that is, a convex function which assumes value zero only at zero) and $w = (w_k)$ is a weight sequence, a non-increasing sequence of positive reals such that $\sum_{k=1}^{\infty} w_k = \infty$.

The Orlicz-Lorentz sequence space $\lambda_{w,G}$ is defined by

$$\lambda_{w,G} := \{ x = (x_k) : \sum_{k=1}^{\infty} G(\lambda x_k^*) w_k < \infty \text{ for some } \lambda > 0 \}.$$ 

It is easy to check that $\lambda_{G,w}$ is a symmetric sequence space, equipped with the norm

$$\|x\|_{w,G} := \inf\{ \lambda > 0 : \sum_{k=1}^{\infty} G(\lambda x_k^*) w_k \leq 1 \}.$$ 

Two Orlicz functions $G_1$ and $G_2$ are said to be equivalent if there exist such a constant $c < \infty$ that $G_1(c^{-1} t) \leq G_2(t) \leq G_1(ct)$, for every $t \in [0, \infty)$.

The following theorem indicates sufficient conditions under which the space $\lambda_{w,G}$ is $p$-convex or $q$-concave (see also [11]).

**Theorem 6.6.** [15, Theorem 5.1] Let $G$ be an Orlicz function, $w = (w_k)$ be a weight sequence and $1 < p, q < \infty$. Then the following claims hold:

(i) If $G \circ t^{1/p}$ is equivalent to a convex function and $\sum_{k=1}^{n} w_k$ is concave, then the space $\lambda_{w,G}$ is $p$-convex;

(ii) If $G \circ t^{1/q}$ is equivalent to a concave function and $\sum_{k=1}^{n} w_k$ is convex, then the space $\lambda_{w,G}$ is $q$-concave.

According to Theorem 6.6 we can reformulate Theorem 6.5 for Orlicz-Lorentz sequence spaces as follows.

**Theorem 6.7.** Let $G_1$ and $G_2$ be Orlicz functions and $w_1 = (w_k^{(1)})$, $w_2 = (w_k^{(2)})$ be weight sequences such that the spaces $\lambda_{w_1,G_1}$ and $\lambda_{w_2,G_2}$ are separable. If $G_1 \circ t^{1/p}$ and $G_2 \circ t^{1/q}$ are equivalent to concave (convex, respectively) functions for $p \neq \infty$, $q \neq \infty$ and $q \leq p^*$ ($p \neq 1$, $q \neq 1$ and $p^* \leq q$, respectively), and $\sum_{k=1}^{n} w_k^{(1)}$, $\sum_{k=1}^{n} w_k^{(2)}$ are convex (concave, respectively) functions, then the tensor product $\lambda_{w_1,G_1} \hat{\otimes} \lambda_{w_2,G_2}$ is not isomorphic to any subspace of a Banach space with an unconditional basis.

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