SBV regularity for Burgers-Poisson equation

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Abstract

The SBV regularity of weak entropy solutions to the Burgers-Poisson equation for initial data in $L^1(\mathbb{R})$ is considered. We show that the derivative of a solution consists of only the absolutely continuous part and the jump part.

Keywords: Burgers-Poisson equation, entropy weak solution, SBV regularity

1 General setting

The Burgers-Poisson equation is given by the balance law obtained from Burgers’ equation by adding a nonlocal source term

$$u_t + \left(\frac{u^2}{2}\right)_x = [G \ast u]_x.$$  \hspace{1cm} (1.1)

Here, $G(x) = -\frac{1}{\pi}e^{-|x|}$ is the Poisson Kernel such that

$$[G \ast f](x) = \int_{-\infty}^{+\infty} G(x - y) \cdot f(y) \, dy$$

solves the Poisson equation

$$\varphi_{xx} - \varphi = f.$$  \hspace{1cm} (1.2)

Equation (1.1) has been derived in [16] as a simplified model of shallow water waves and admits conservation of both momentum and energy. For sufficiently regular initial data $u_0$, the local existence and uniqueness of solutions of (1.1) has been established in [9]. Additionally, their analysis of traveling waves showed that the equation features wave breaking in finite time. More generally, it has been demonstrated that (1.1) does not admit a global smooth solution ([12]). Hence, it is natural to consider entropy weak solutions.

**Definition 1.1.** A function $u \in L^1_{loc}([0, \infty) \times \mathbb{R}) \cap L^\infty_{loc}([0, \infty), L^\infty(\mathbb{R}))$ is an entropy weak solution of (1.1) if $u$ satisfies the following properties:
(i) the map \( t \mapsto u(t, \cdot) \) is continuous with values in \( L^1(\mathbb{R}) \), i.e.,
\[
\|u(t, \cdot) - u(s, \cdot)\|_{L^1(\mathbb{R})} \leq L \cdot |t - s| \quad \text{for all } 0 \leq s \leq t
\]
for some constant \( L > 0 \).

(ii) For any \( k \in \mathbb{R} \) and any non-negative test function \( \phi \in C^1_c([0, \infty[ \times \mathbb{R}, \mathbb{R}) \) one has
\[
\int \int \left[ |u - k| \phi_t + \text{sign}(u - k) \left( \frac{u^2}{2} - \frac{k^2}{2} \right) \phi_x + \text{sign}(u - k) [G_* u(t, \cdot)](x) \phi \right] dx dt \geq 0.
\]

Based on the vanishing viscosity approach, the existence result for a global weak solution was provided for \( u_0 \in BV(\mathbb{R}) \) in [9]. However, this approach cannot be applied to the more general case with initial data in \( L^1(\mathbb{R}) \). Moreover, there are no uniqueness or continuity results for global weak entropy solutions of (1.1) established in [9]. Recently, the existence and continuity results for global weak entropy solutions of (1.1) were established for \( L^1(\mathbb{R}) \) initial data in [10]. The entropy weak solutions are constructed by a flux-splitting method. Relying on the decay properties of the semigroup generated by Burgers equation and the Lipschitz continuity of solutions to the Poisson equation, approximating solutions satisfy an Oleinik-type inequality for any positive time. As a consequence, the sequence of approximating solutions is precompact and converges in \( L^1_{\text{loc}}(\mathbb{R}) \). Moreover, using an energy estimate, they show that the characteristics are Hölder continuous, which is used to achieve the continuity property of the solutions. The Oleinik-type inequality gives that the solution \( u(t, \cdot) \) is in \( BV_{\text{loc}}(\mathbb{R}) \) for every \( t > 0 \). In particular, this implies that the Radon measure \( Du(t, \cdot) \) is divided into three mutually singular measures
\[
Du(t, \cdot) = D^a u(t, \cdot) + D^j u(t, \cdot) + D^s u(t, \cdot)
\]
where \( D^a u(t, \cdot) \) is the absolutely continuous measure with respect to the Lebesgue measure, \( D^j u(t, \cdot) \) is the jump part which is a countable sum of weighted Dirac measures, and \( D^s u(t, \cdot) \) is the non-atomic singular part of the measure called the Cantor part. For a given \( w \in BV_{\text{loc}}(\mathbb{R}) \), the Cantor part of \( Dw \) does not vanish in general. A typical example of \( D^s w \) is the derivative of the Cantor-Vitali ternary function. If \( D^s w \) vanishes then we say the function \( w \) is locally in the space of special functions of bounded variation, denoted by \( SBV_{\text{loc}}(\mathbb{R}) \). The space of \( SBV_{\text{loc}} \) functions was first introduced in [11] and plays important role in the theory of image segmentation and with variational problems in fracture mechanics. Motivated by results on \( SBV \) regularity for hyperbolic conservation laws ([2], [15], [4], [13]), we show that

**Theorem 1.2.** Let \( u : [0, \infty[ \times \mathbb{R} \to \mathbb{R} \) be the unique locally \( BV \)-weak entropy solution of (1.1) with initial data \( u_0 \in L^1(\mathbb{R}) \). Then there exists a countable set \( T \subset \mathbb{R}^+ \) such that
\[
u(t, \cdot) \in SBV_{\text{loc}}(\mathbb{R}) \quad \text{for all } t \in \mathbb{R}^+ \setminus T.
\]

As a consequence, the slicing theory of \( BV \) functions and the chain rule of Vol’pert [3] implies that the weak entropy solution \( u \) is in \( SBV_{\text{loc}}([0, +\infty[ \times \mathbb{R}) \). This is the first example of the \( SBV \) regularity for scalar conservation laws with nonlocal source term. A common theme in the proofs of recent results on \( SBV \) regularity involve an appropriate
geometric functional which has certain monotonicity properties and jumps at time \( t \) if \( u(t, \cdot) \) does not belong to \( \text{SBV} \) (see e.g. in [2]). More precisely, let \( \mathcal{J}(t) \) be the set of jump discontinuities \( \mathcal{J}(t) \) of \( u(t, \cdot) \). For each \( x_j \in \mathcal{J}(t) \), there are minimal and maximal backward characteristics \( \xi^{-}_j(s) \) and \( \xi^{+}_j(s) \) emanating from \( (t, x_j) \) which define a nonempty interval \( I_j(s) := [\xi^{-}_j(s), \xi^{+}_j(s)] \) for any \( s < t \). In this case, the functional \( F_s(t) \) defined as the sum of the measures of \( I_j(s) \) is monotonic and bounded. Relying on a careful study of generalized characteristics, one shows that if the measure \( Du(t, \cdot) \) has a non-vanishing Cantor part then the function \( F_s \) “jumps” up at time \( t \) which implies that the Cantor part is only present at countably many \( t \). Due to the nonlocal source, \( u(t, \cdot) \) does not necessarily have compact support. Thus, we approach the domain by first looking at compact sets and then “glue” the sections together to recover the full domain.

2 Preliminaries

2.1 \( BV \) and \( SBV \) functions

Let us now introduce the concept of functions of bounded variation in \( \mathbb{R} \). We refer to [3] for a comprehensive analysis.

**Definition 2.1.** Given an open set \( \Omega \subseteq \mathbb{R} \), let \( w \) be in \( L^1(\Omega) \). We say that \( w \) is a function of bounded variation in \( \Omega \) (denoted by \( w \in BV(\Omega) \)) if the distributional derivative of \( w \) is representable by a finite Radon measure \( Du \) on \( \Omega \), i.e.,

\[-\int \Omega w \cdot \varphi' \, dx = \int \Omega \varphi \, dDw \quad \text{for all} \quad \varphi \in C_\infty^\infty(\Omega) \]

with total variation (denoted by \( \|Dw\| \)) given by

\[\|Dw\| (\Omega) = \sup \left\{ \int \Omega w \cdot \varphi' \, dx : \varphi \in C_\infty^\infty(\Omega), \|\varphi\|_{L^\infty} \leq 1 \right\}.\]

Moreover, \( w \) is of locally bounded variation on \( \Omega \) (denoted by \( w \in BV_{loc}(\Omega) \)) if \( w \in L^1_{loc}(\Omega) \) and \( w \) is in \( BV(U) \) for all \( U \subset \subset \Omega \).

Given \( w \in BV_{loc}(\mathbb{R}) \), we split \( Dw \) into the absolutely continuous part \( D^a w \) and singular part \( D^s w \) provided by the Radon-Nikodým theorem (see e.g. [3, Theorem 1.28]). In the 1-D case, the singular part is concentrated on the \( L^1 \)-negligible set

\[ S_w = \left\{ t \in \mathbb{R} \mid \lim_{\delta \to 0} \frac{|Dw|(t-\delta, t+\delta)}{|\delta|} = +\infty \right\}. \]

We can further decompose \( D^s w \) by isolating the set of atoms \( A_w = \left\{ t \in \mathbb{R} \mid Dw(\{t\}) \neq 0 \right\} \), contained in \( S_w \). Hence, we can consider two mutually singular measures

\[ D^j w := D^s w \llcorner A_w \quad \text{and} \quad D^c w := D^s w \llcorner (S_w \setminus A_w) \]

respectively called the jump part of the derivative and the Cantor part of the derivative. Furthermore, we have the following structure result (see e.g. [3, Theorem 3.28])
Proposition 2.2. Let $\Omega \subseteq \mathbb{R}$ and $w \in BV(\Omega)$. Then, for any $x \in A_w$, the left and right hand limits of $w(x)$ exist and
\[
D^j w = \sum_{x \in A_w} (w(x+) - w(x-)) \delta_x
\]
where $w(x\pm)$ denote the one-sided limits of $w$ at $x$. Moreover, $D^c w$ vanishes on any sets which are $\sigma$-finite with respect to $\mathcal{H}^0$.

Definition 2.3. Let $w$ be in $BV_{loc}(\mathbb{R})$ then $w$ is a special function of bounded variation (denote by $w \in SBV$) if the Cantor part $D^c w$ vanishes.

We want to show that the weak entropy solutions of (1.1) belong to $SBV$.

2.2 Oleinik-type inequality and non-crossing of characteristics

The global existence and $BV$-regularity of (1.1) was studied extensively in [10]. For convenience, we recall their main results here.

Theorem 2.4. The Cauchy problem (1.1)-(1.2) with initial data $u_0 = u(0, \cdot) \in L^1(\mathbb{R})$ admits a unique solution $u(t, x)$ such that for all $t > 0$ the following hold:

(i) the $L^1$-norm is bounded by
\[
\|u(t, \cdot)\|_{L^1(\mathbb{R})} \leq e^t \cdot \|u_0\|_{L^1(\mathbb{R})}; \tag{2.1}
\]

(ii) the solution satisfies the following Oleinik-type inequality
\[
u(t, y) - u(t, x) \leq K_t \cdot (y - x) \quad \text{for all } y > x \tag{2.2}
\]
with $K_t = 1 + 2t + 2t^2 + 4t^2 e^t \cdot \|u_0\|_{L^1(\mathbb{R})}$;

(iii) the $L^\infty$-norm is bounded by
\[
\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \sqrt{\frac{2K_t}{t}} \|u(t, \cdot)\|_{L^1(\mathbb{R})} \leq \sqrt{\frac{2K_t e^t}{t}} \|u_0\|_{L^1(\mathbb{R})}. \tag{2.3}
\]

In particular, this implies that for all $t > 0$, $u(t, \cdot)$ is in $BV_{loc}(\mathbb{R})$ and satisfies
\[
u(t, x-) \geq u(t, x+) \quad \text{for all } x \in \mathbb{R}. \tag{2.4}
\]

We recall the definition and theory of generalized characteristic curves associated to (1.1). For a more in depth theory of generalized characteristics, we direct the readers to [7].

Definition 2.5. For any $(t, x) \in [0, +\infty[ \times \mathbb{R}$, an absolutely continuous curve $\xi_{(t,x)}(\cdot) \in [0, +\infty[ \times \mathbb{R}$ is called a backward characteristic curve starting from $(t, x)$ if it is a solution of differential inclusion
\[
\dot{\xi}_{(t,x)}(s) \in [u(s, \xi(s,t)) (+), u(s, \xi(s,t)) (-)] \tag{2.5}
\]
a.e. $s \in [0, t]$ with $\xi_{(t,x)}(t) = x$. If $s \in [t, +\infty[ \in (2.5)$ then $\xi$ is called a forward characteristic curve, denoted by $\xi^{(t,x)}(\cdot)$. The characteristic curve $\xi$ is called genuine if $u(t, \xi(t)-) = u(t, \xi(t)+)$ for almost every $t$. 

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The existence of backward (forward) characteristics was studied by Fillipov. As in [7] and [15], the speed of the characteristic curves are determined and genuine characteristics are essentially classical characteristics:

**Proposition 2.6.** Let $\xi : [a, b] \to \mathbb{R}$ be a characteristic curve for the Burgers-Poisson equation \((1.1)\), associated with an entropy solution $u$. Then for almost every time $t \in [a, b]$, it holds that

$$
\dot{\xi}(t) = \begin{cases} 
  u(t, \xi(t)) & \text{if } u(t, \xi(t)^+) = u(t, \xi(t)^-), \\
  \frac{u(t, \xi(t)^+) + u(t, \xi(t)^-)}{2} & \text{if } u(t, \xi(t)^+) < u(t, \xi(t)^-).
\end{cases}
$$

(2.6)

In addition, if $\xi$ is genuine on $[a, b]$, then there exists $v(t) \in C^1([a, b])$ such that

$$
u(t, \xi(t)^-) = v(t) = u(t, \xi(t)^+) \quad \text{for all } t \in ]a, b[,$$

and $(\xi(\cdot), v(\cdot))$ solve the system of ODEs

$$
\begin{cases}
  \dot{\xi}(t) = v(t) \\
  \dot{v}(t) = [G \ast u(t, \cdot)](\xi(t))
\end{cases}
$$

(2.7)

Backward characteristics $\xi(t, x)(\cdot)$ are confined between a maximal and minimal backward characteristics, as defined in [7] (denoted by $\xi(t, x^+)(\cdot)$ and $\xi(t, x^-)(\cdot)$). Relying on the above proposition and (2.4), we can obtain properties of generalized characteristics, associated with entropy solutions of the Burgers-Poisson equation, including the non-crossing property of two genuine characteristics:

**Proposition 2.7.** Let $u$ be an entropy solution to \((1.1)\). Then for any $(t, x) \in ]0, +\infty[ \times \mathbb{R}$, the following holds:

(i) The maximal and minimal backward characteristics $\xi(t, x^\pm)$ are genuine and thus the function $u(\tau, \xi(t, x^\pm)(\tau))$ solves (2.7) for $\tau \in [0, t]$ with initial data $u(t, \xi(t, x^\pm)(t))$.

(ii) [Non-crossing of genuine characteristics] Two genuine characteristics may intersect only at their endpoints.

(iii) If $u(t, \cdot)$ is discontinuous at a point $x$, then there is a unique forward characteristic $\xi(t, x)$ which passes through $(t, x)$ and

$$
u(\tau, \xi(t, x)(\tau^-)) > u(\tau, \xi(t, x)(\tau^+)) \quad \text{for all } \tau \geq t.$$

Throughout this paper, we shall denote by $\mathcal{J}(t) = \{ x \in \mathbb{R} : u(t, x^-) > u(t, x^+) \}$, the jump set of $u(t, \cdot)$ for any $t > 0$. For any $x \in \mathcal{J}(t)$, the base of the backward characteristic cone starting from $(t, x)$ at time $s \in [0, t]$ is

$$
I_{(t, x)}(s) := \left[ \xi(t, x^-)(s), \xi(t, x^+)(s) \right].
$$

(2.8)

By the non-crossing property, for any $T > 0$ and $z_1 < z_2 \in \mathbb{R} \setminus \mathcal{J}(T)$, the set

$$
A_{[z_1, z_2]}^T := \bigcup_{s \in [0, T]} A_{[z_1, z_2]}^s \quad \text{with} \quad A_{[z_1, z_2]}^T(s) := \left[ \xi(T, z_1)(s), \xi(T, z_2)(s) \right],
$$

(2.9)
yielding (2.11). The later statement follows directly.

Then the followings hold:

\[ I_{\tau,x}^{T} = \bigcup_{x \in A_{\tau,x}^{T} \cap \mathcal{J}(\tau)} I_{(\tau,x)}(s). \quad (2.10) \]

Due to the no-crossing property of two genuine backward characteristics and the uniqueness of forward characteristics in Proposition 2.7, the following holds:

**Corollary 2.8.** Given \( T > 0 \) and \( z_1 < z_2 \in \mathbb{R} \setminus \mathcal{J}(T) \), the map \( \tau \mapsto I_{\tau,z_1,z_2}^{T}(s) \) is increasing in the interval \([s,T]\) in the following sense

\[ I_{\tau_1,z_1,z_2}^{T}(s) \subseteq I_{\tau_2,z_1,z_2}^{T}(s) \quad \text{for all} \quad 0 \leq s \leq \tau_1 \leq \tau_2 \leq T. \quad (2.11) \]

Moreover, for any \( x \in A_{\tau_1,z_1,z_2}^{T}(\tau_1) \setminus I_{\tau_1,z_1,z_2}^{T}(\tau_1) \) with \( 0 < \tau_1 < \tau_2 < t \), the unique forward characteristic \( \xi(\tau_1,x) \) passing through \((\tau_1,x)\) is genuine in \([\tau_1,\tau_2]\).

**Proof.** Let \( x \in \mathcal{J}(\tau_1) \cap A_{\tau_1,z_1,z_2}^{T}(\tau_1) \) and let \( \chi(\cdot) \) be the unique forward characteristic emanating from \((\tau_1,x)\). By property (iii) of Proposition 2.7 for a fixed \( \tau_2 \in [\tau_1,T] \) we have that \( \chi(\tau_2) \in \mathcal{J}(\tau_2) \) and by the non-crossing property, \( \chi(\tau_2) \in A_{\tau_1,z_1,z_2}^{T}(\tau_2) \). Since the backward characteristics that form the base of a characteristic cone are genuine, the non-crossing property implies that

\[ I_{\tau_1,x}^{T}(s) \subseteq I_{\tau_2,\chi(\tau_2)}^{T}(s) \subseteq A_{\tau_1,z_1,z_2}^{T}(s) \quad \text{for all} \quad s \in [0,\tau_1] \]

yielding (2.11). The later statement follows directly. \( \square \)

### 3 \text{ SBV-regularity}

Throughout this section, let \( u : [0,\infty[ \times \mathbb{R} \to \mathbb{R} \) be the unique locally BV-weak entropy solution of (1.1) for some initial data \( u_0 \in L^1(\mathbb{R}) \). The section aims to prove Theorem 1.2. For simplicity, denote the jump and Cantor parts of \( Du(t,\cdot) \) by

\[ \nu_t = D^J u(t,\cdot) \quad \text{and} \quad \mu_t = D^c u(t,\cdot) \quad \text{for any} \quad t \in ]0,\infty[ \]

which, by (2.2), are both non-positive. We will show that \( \mu_t(\mathbb{R}) < 0 \) for at most countable positive times \( t > 0 \). In order to do so, let us first establish some basic bounds on backward characteristics.

**Lemma 3.1.** For any given \( 0 < t_0 < t \) and \( x_1 \leq x_2 \), let \( \xi_i(\cdot) \) be a genuine backward characteristic starting from \((t,x_i)\) and

\[ v_i(s) = u(s,\xi_i(s)) \quad \text{for all} \quad s \in [0,t], \ i \in \{1,2\}. \]

Then the followings hold:

\[ |v_2(s) - v_1(s)| + |\xi_2(s) - \xi_1(s)| \leq c_i(s) \cdot (|v_2(t) - v_1(t)| + |\xi_2(t) - \xi_1(t)|) \quad (3.1) \]
for all \( s \in [0, t] \) and

\[
\xi_2(t_0) - \xi_1(t_0) \geq \frac{x_2 - x_1 + (v_1(t_0) - v_2(t_0)) \cdot (t - t_0)}{\Gamma_{[t_0, t]}}
\]  

(3.2)

with

\[
\begin{cases}
    c_t(s) = \exp \left\{ 2 \cdot \left( \sqrt{2K_te^t \|u_0\|_{L^1(\mathbb{R})}} + (e^t \|u_0\|_{L^1(\mathbb{R})} + 1) \cdot \sqrt{t} \right) \cdot (\sqrt{t} - \sqrt{s}) \right\}, \\
    \Gamma_{[t_0, t]} = 1 + \left( \frac{2K_t e^t \|u_0\|_{L^1(\mathbb{R})} + e^t \|u_0\|_{L^1(\mathbb{R})}}{t_0} \right) \cdot e^{K_t(t - t_0)} \cdot (t - t_0)^2.
\end{cases}
\]  

(3.3)

Proof. 1. Let’s first proof (3.1). From Proposition 2.6, it holds that

\[
\begin{align*}
    \dot{\xi}_i(s) &= v_i(s) \\
    \dot{v}_i(s) &= [G \ast u(s, \cdot)]_x(\xi_i(s))
\end{align*}
\]  

for all \( s \in [0, t] \). (3.4)

In particular, this implies that

\[
\frac{d}{ds} \left| \xi_2(s) - \xi_1(s) \right| \geq - |v_2(s) - v_1(s)|
\]

and

\[
\frac{d}{ds} \left| v_2(s) - v_1(s) \right| \geq - \left| [G \ast u(s, \cdot)]_x(\xi_2(s)) - [G \ast u(s, \cdot)]_x(\xi_1(s)) \right|
\]

Since \( \xi_2(s) \geq \xi_1(s) \) for all \( s \in [0, t] \), we estimate

\[
\begin{align*}
    &\left| [G \ast u(s, \cdot)]_x(\xi_2(s)) - [G \ast u(s, \cdot)]_x(\xi_1(s)) \right| \\
    \leq &\frac{1}{2} \cdot \int_{-\infty}^{\xi_1(s)} |u(s, z)| \cdot |e^{z-\xi_2(s)} - e^{z-\xi_1(s)}| \, dz \\
    + &\frac{1}{2} \cdot \int_{\xi_1(s)}^{\xi_2(s)} |u(s, z)| \cdot |e^{z-\xi_2(s)} + e^{\xi_1(s)-z}| \, dz \\
    + &\frac{1}{2} \cdot \int_{\xi_2(s)}^{+\infty} |u(s, z)| \cdot |e^{\xi_1(s)-z} - e^{\xi_2(s)-z}| \, dz \\
    \leq &\frac{1}{2} \cdot \left( 1 - e^{\xi_1(s)-\xi_2(s)} \right) \int_{\mathbb{R}} |u(s, z)| \, dz + \int_{\xi_1(s)}^{\xi_2(s)} |u(s, z)| \, dz \\
    \leq &\left( \frac{1}{2} \cdot \|u(s, \cdot\|_{L^1(\mathbb{R})} + \|u(s, \cdot\|_{L^\infty(\mathbb{R})}) \right) \cdot \left| \xi_2(s) - \xi_1(s) \right|.
\end{align*}
\]

Hence, (2.1) and (2.3) imply that

\[
\left| [G \ast u(s, \cdot)]_x(\xi_2(s)) - [G \ast u(s, \cdot)]_x(\xi_1(s)) \right|
\leq \left( \sqrt{\frac{2K_t e^t}{s} \|u_0\|_{L^1(\mathbb{R})} + e^t \|u_0\|_{L^1(\mathbb{R})}} \right) \cdot \left| \xi_2(s) - \xi_1(s) \right|.
\]  

(3.5)

Setting \( M_t = \sqrt{2K_t e^t \|u_0\|_{L^1(\mathbb{R})} + (e^t \|u_0\|_{L^1(\mathbb{R})} + 1) \cdot \sqrt{t}} \), we have

\[
\frac{d}{ds} \left( \left| \xi_2(s) - \xi_1(s) \right| + \left| v_2(s) - v_1(s) \right| \right) \geq - \frac{M_t}{\sqrt{s}} \cdot \left( \left| \xi_2(s) - \xi_1(s) \right| + \left| v_2(s) - v_1(s) \right| \right),
\]

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for all $s \in [0, t]$, and Grönwall’s inequality yields (3.1).

2. To prove (3.2), we first apply (2.2) to (3.4) to get

$$
\dot{\xi}_2(s) - \dot{\xi}_1(s) = u(s, \xi_2(s)) - u(s, \xi_1(s)) \leq \frac{K_t}{s} \cdot (\xi_2(s) - \xi_1(s)),
$$

and this implies

$$
\xi_2(s) - \xi_1(s) \leq \frac{e^{K_t s}}{t_0} \cdot (\xi_2(t_0) - \xi_1(t_0)) \leq \frac{e^{K_t t}}{t_0} \cdot (\xi_2(t_0) - \xi_1(t_0)) \quad \text{for all } s \in [t_0, t]. \hspace{1cm} (3.6)
$$

Therefore, from (3.4) and (3.5), it holds for $s \in [t_0, t]$ that

$$
v_2(s) - v_1(s) = v_2(t_0) - v_1(t_0) + \int_{t_0}^{s} \left[ G * u(\tau, \cdot) \right]_{x}(\xi_2(\tau)) - \left[ G * u(\tau, \cdot) \right]_{x}(\xi_1(\tau)) \, d\tau
\leq v_2(t_0) - v_1(t_0) + \int_{t_0}^{s} \left( \sqrt{\frac{2K_t e^{t}}{t_0} \| u_0 \|_{L^1(\mathbb{R})} + e^{t} \| u_0 \|_{L^1(\mathbb{R})}} \right) \cdot (\xi_2(\tau) - \xi_1(\tau)) \, d\tau
\leq v_2(t_0) - v_1(t_0) + \gamma_{[t_0, t]} \cdot (\xi_2(t_0) - \xi_1(t_0))
$$

with

$$
\gamma_{[t_0, t]} = \left( \sqrt{\frac{2K_t e^{t}}{t_0} \| u_0 \|_{L^1(\mathbb{R})} + e^{t} \| u_0 \|_{L^1(\mathbb{R})}} \right) \cdot \frac{e^{K_t t}}{t_0} \cdot (t - t_0).
$$

Integrating the first equation in (3.4) over $[t_0, t]$, we get

$$
\xi_2(t) - \xi_1(t) = \xi_2(t_0) - \xi_1(t_0) + \int_{t_0}^{t} v_2(\tau) - v_1(\tau) \, d\tau
\leq (v_2(t_0) - v_1(t_0)) \cdot (t - t_0) + (1 + \gamma_{[t_0, t]} \cdot (t - t_0)) \cdot (\xi_2(t_0) - \xi_1(t_0))
$$

and this yields (3.2). □

As a consequence, we obtain the following two corollaries. The first one provides an upper bound on the base of characteristic cone $C_{(t, x)}$ at time $s \in [0, t]$ for every $x \in J(t)$.

**Corollary 3.2.** For any $(t, x) \in [0, +\infty[ \times J(t)$, it holds that

$$
|I_{(t, x)}(s)| \leq -c_t(s) \cdot \nu_t(\{x\}) \quad \text{for all } s \in [0, t]. \hspace{1cm} (3.7)
$$

**Proof.** Since $x \in J(t)$, the inequality (2.1) implies that

$$
\nu_t(\{x\}) = u(t, x^+) - u(t, x^-) < 0.
$$

Thus, recalling (3.1), we obtain

$$
|\xi_{(t, x^+)}(s) - \xi_{(t, x^-)}(s)| \leq c_t(s) \cdot |u(t, x^+) - u(t, x^-)|
$$

and this yields (3.7). □
In the next corollary, we show that two distinct characteristics are separated for all positive time; moreover, the distance between them is proportional to the difference in the values of the solution along the characteristics.

**Corollary 3.3.** Given \( x_1 < x_2 \) and \( \sigma \) and \( t \), such that \( 0 < \sigma < t \leq T \), let \( \xi_i(\cdot) \) be a genuine backward characteristic starting from \((t, x_i)\) and

\[
v_i(s) = u(s, \xi_i(s)) \quad \text{for all } s \in [0, t], \ i \in \{1, 2\}.
\]

Then it holds that

\[
\xi_2(\sigma/2) - \xi_1(\sigma/2) \geq \frac{v_1(t) - v_2(t)}{t - \sigma/2}.
\]  \hspace{1cm} (3.8)

where

\[
\kappa_{[\sigma, T]} = \frac{\sigma}{2} \left[ \Gamma_{[\sigma/2, T]} + \left( \frac{4K^T e^T}{\sigma} \|u_0\|_{L^1(\mathbb{R})} + e^T \|u_0\|_{L^1(\mathbb{R})} \right) \cdot e^{K^T T} \cdot (T - \sigma/2) \right]^{-1}.
\]

**Proof.** Integrating the second equation in (2.7) over \([\sigma/2, t]\) yields

\[
v_1(t) - v_2(t) = v_1(\sigma/2) - v_2(\sigma/2) + \int_{\sigma/2}^{t} \left[ G \ast u(\tau, \cdot) \right]_x (\xi_1(\tau)) - \left[ G \ast u(\tau, \cdot) \right]_x (\xi_2(\tau)) \ d\tau
\]

\[
\leq v_1(\sigma/2) - v_2(\sigma/2) + \int_{\sigma/2}^{t} \left| \left[ G \ast u(\tau, \cdot) \right]_x (\xi_2(\tau)) - \left[ G \ast u(\tau, \cdot) \right]_x (\xi_1(\tau)) \right| \ d\tau
\]

and by (3.5) and (3.6) it holds that

\[
v_1(t) - v_2(t) \leq v_1(\sigma/2) - v_2(\sigma/2) + \left( \frac{4K^T e^T}{\sigma} \|u_0\|_{L^1(\mathbb{R})} + e^T \|u_0\|_{L^1(\mathbb{R})} \right) \cdot \frac{2e^{K^T T}}{\sigma} \cdot (T - \sigma/2) \cdot (\xi_2(\sigma/2) - \xi_1(\sigma/2)).
\]  \hspace{1cm} (3.9)

On the other hand, by (3.2) we have that

\[
v_1(\sigma/2) - v_2(\sigma/2) \leq \frac{\Gamma_{[\sigma/2, t]}}{t - \sigma/2} \cdot (\xi_2(\sigma/2) - \xi_1(\sigma/2)) \leq \frac{2\Gamma_{[\sigma/2, T]}}{\sigma} \cdot (\xi_2(\sigma/2) - \xi_1(\sigma/2)).
\]

which, when applied to (3.9), implies (3.8). \( \square \)

The next lemma shows that, for a certain positive time \( s \), if \( u(s, \cdot) \) is not in \( SBV \), then at future times \( s + \varepsilon \) the Cantor part of \( u(s, \cdot) \) gets transformed into jump singularities. Following the main idea in \([2, 15]\), for any \( s \in [0, T[ \) and \( z_1 < z_2 \in \mathbb{R} \setminus \mathcal{J}(T) \), let us consider the set of points \( E^T_{[z_1, z_2]}(s) \) in \( A^T_{[z_1, z_2]}(s) \) where the Cantor part of \( D_x u(s, \cdot) \) prevails, i.e.,

\[
E^T_{[z_1, z_2]}(s) = \left\{ x \in A^T_{[z_1, z_2]}(s) : \lim_{\eta \to 0^+} \frac{\eta + |D_x u(s, \cdot) - \mu_s([x - \eta, x + \eta])|}{\mu_s([x - \eta, x + \eta])} = 0 \right\} \quad (3.10)
\]

Besicovitch differentiation theorem \([3]\) gives that \( \mu_s \left( A^T_{[z_1, z_2]}(s) \setminus E^T_{[z_1, z_2]}(s) \right) = 0 \) and

\[
\lim_{\eta \to 0^+} \frac{u^-(s, x - \eta) - u^+(s, x + \eta)}{-\mu_s([x - \eta, x + \eta])} = 1 \quad \text{for all } x \in E^T_{[z_1, z_2]}(s). \quad (3.11)
\]
Moreover, for $\mu_s$-a.e. $x$ in $E^T_{[z_1, z_2]}(s)$, it holds that
\[
\lim_{\eta \to 0} \frac{u(s, x + \eta) - u(s, x)}{\eta} = -\infty. \tag{3.12}
\]

**Lemma 3.4.** Let $0 < s < t \leq T$ and $z_1 < z_2 \in \mathbb{R} \setminus \mathcal{J}(T)$ be fixed. Then, it holds for $\mu_s$-a.e. $x \in A^T_{[z_1, z_2]}(s)$ that
\[
|x - \eta x, x + \eta x| \subset \mathcal{I}^T_{[z_1, z_2]}(s) \quad \text{for some } \eta_x > 0.
\]

**Proof.** Since $\mathcal{I}^T_{[z_1, z_2]}(s)$ is open, it is sufficient to prove that every point $x \in E^T_{[z_1, z_2]}(s) \setminus \mathcal{J}(s)$ satisfying \((3.12)\) is in $\mathcal{I}^T_{[z_1, z_2]}(s)$. Assume by a contradiction that
\[
x \in A^T_{[z_1, z_2]}(s) \setminus \mathcal{I}^T_{[z_1, z_2]}(s) \bigcup \partial(\mathcal{I}^T_{[z_1, z_2]}(s)).
\]

1. If $x \in A^T_{[z_1, z_2]}(s) \setminus \mathcal{I}^T_{[z_1, z_2]}(s)$ then
\[
|x - \eta_0, x + \eta_0| \bigcap \mathcal{I}^T_{[z_1, z_2]}(s) = \emptyset \quad \text{for some } \eta_0 > 0. \tag{3.13}
\]

Given any $\eta \in [0, \eta_0]$, let $\xi^\eta_1(\cdot)$ and $\xi^\eta_2(\cdot)$ be the unique forward characteristics emanating from $x - \eta$ and $x + \eta$ at time $\tau_0$. From Corollary 2.8 both $\xi^\eta_1(\cdot)$ and $\xi^\eta_2(\cdot)$ are genuine in $[t_0, t]$ and
\[
\xi^\eta_2(\tau) - \xi^\eta_1(\tau) \geq 0 \quad \text{for all } \tau \in [s, t]. \tag{3.14}
\]

Thus, \((3.2)\) in Lemma 3.1 implies
\[
2\eta = \xi^\eta_2(s) - \xi^\eta_1(s) \geq \frac{\xi^\eta_2(t) - \xi^\eta_1(t) + (u(s, x - \eta) - u(s, x + \eta))(t - s)}{\Gamma_{[s, t]}} \geq -\frac{(u(s, x + \eta) - u(s, x - \eta))(t - s)}{\Gamma_{[s, t]}},
\]
which yields a contradiction to \((3.12)\) when $\eta$ is sufficiently small.

2. Suppose that $x \in \partial(\mathcal{I}^T_{[z_1, z_2]}(s))$. In this case, $\xi(s, x)(\cdot)$ is either a minimal or maximal backward characteristic in $[s, t]$. Moreover, for every $\eta > 0$ there exists $x_\eta \in [x - \eta, [\bigcup]x, x + \eta]$ such that $x_\eta \notin \mathcal{I}^T_{[z_1, z_2]}(s)$ and the unique forward characteristics $\xi(s, x_\eta)(\cdot)$ emanating from $x_\eta$ at time $s$ is genuine and does not cross $\xi(s, x)(\cdot)$ in the time interval $[s, t]$. With the same computation in the previous step, we get
\[
\frac{u(s, x_\eta) - u(s, x)}{x_\eta - x} \geq -\frac{\Gamma_{[s, t]}}{t - s},
\]
and this also yields a contradiction to \((3.12)\) when $\eta$ is sufficiently small. \(\Box\)
We are now ready to prove our first main theorem.

**Proof of Theorem 1.2** The proof is divided into two steps:

**Step 1.** Fix \( T > 0 \) and \( z_1, z_2 \in \mathbb{R} \setminus \mathcal{J}(T) \) with \( z_1 < z_2 \) and, recalling (2.9) and (2.10) let
\[
\mathcal{A} = \mathcal{A}^T_{[z_1, z_2]}, \quad A_t = A^T_{[z_1, z_2]}(t) \quad \text{and} \quad I^t(s) = I^t_{[z_1, z_2]}(s)
\]
for all \( 0 < s < t \leq T \). We claim that the set
\[
\mathcal{T}_{[z_1, z_2]} := \{ t \in [0, T] : \mu_t(A_t) \text{ does not vanish} \}
\]
is at most countable.

(i). Fix \( \sigma \in ]0, T[ \). By Proposition 2.6 and (2.3), one has
\[
|A_t| \leq |z_2 - z_1| + 2 \sqrt{\frac{2K_T e^T}{\sigma}} \|u_0\|_{L^1(\mathbb{R})} \cdot T \quad \text{for all } t \in [\sigma, T],
\]
and the Oleinik-type inequality (2.2) yields
\[
|Du(t, \cdot)| (A_t) \leq M^T_{\sigma} \quad \text{for all } t \in [\sigma, T]
\]
with
\[
M^T_{\sigma} = 2 \sqrt{\frac{2K_T e^T}{\sigma}} \|u_0\|_{L^1(\mathbb{R})} + \frac{2K_T}{\sigma} \left( |z_2 - z_1| + 2 \sqrt{\frac{2K_T e^T}{\sigma}} \|u_0\|_{L^1(\mathbb{R})} \cdot T \right).
\]
Let the geometric functional \( F_\sigma : [\sigma, T] \to [0, \infty[ \) be defined by
\[
F_\sigma(t) = \left| \bigcup_{x \in \mathcal{J}(t) \cap A_t} I_{(t, x)}(\sigma/2) \right| = \sum_{x \in \mathcal{J}(t) \cap A_t} |I_{(t, x)}(\sigma/2)| \quad \text{for all } t \in [\sigma, T]
\]
where the second equality follows by the non-crossing property. By Corollaries 2.8 and 3.2 the map \( t \mapsto F_\sigma(t) \) is non-decreasing in \([\sigma, T]\) and uniformly bounded
\[
\sup_{t \in [\sigma, T]} F_\sigma(t) \leq c_T(\sigma/2) \cdot \sup_{t \in [\sigma, T]} (|\nu_t|(A_t)) \leq c_T(\sigma/2) \cdot M^T_{\sigma}
\]
with \( c_T(\sigma/2) \) defined in (3.3).

(ii). Assume that a Cantor part is present in \( \mathcal{A} \) at time \( t \in ]\sigma, T[ \), i.e.,
\[
\mu_t(A_t) \leq -\alpha \quad \text{for some } \alpha > 0,
\]
which by (3.10) is concentrated on \( E_t := E^T_{[z_1, z_2]}(t) \). We will show that
\[
F_\sigma(t+\varepsilon) - F_\sigma(t) \geq \frac{\kappa_{[\sigma, T]}}{2} \cdot \alpha
\]
where \( \kappa_{[\sigma, T]} \) is defined in Corollary 3.3. It is sufficient to prove that
\[
F_\sigma(t + \varepsilon) - F_\sigma(t) = |I^{t+\varepsilon}(\sigma/2) \setminus I^t(\sigma/2)| \geq \frac{\kappa_{[\sigma, T]}}{2} \cdot \alpha
\]
for any given \( \varepsilon \in ]0, T - t[. By Lemma \ref{lemma:countableDiscontinuities}, for \( \mu_t \)-a.e. \( x \in E_t \) there exists \( \eta_x > 0 \) such that
\[
]x - \eta_x, x + \eta_x[ \subset I^{t+\varepsilon}(t) \tag{3.20}\]
On the other hand, given \( x \in E_t \) and \( \eta > 0 \), we denote the interval
\[
J_{x,\eta}^{\sigma/2} = ]\xi_{(t,x-\eta)}(\sigma/2), \xi_{(t,x+\eta)}(\sigma/2)[,
\]
and Corollaries \ref{corollary:countableDiscontinuities} and \ref{corollary:coveringLemma} imply that
\[
\left| J_{x,\eta}^{\sigma/2} \setminus I^{\sigma/2}(\sigma/2) \right| = \xi_{(t,x+\eta)}(\sigma/2) - \xi_{(t,x-\eta)}(\sigma/2) - \left| J_{x,\eta}^{\sigma/2} \cap I^{\sigma/2}(\sigma/2) \right| \geq \kappa_{[\sigma,T]} \cdot (u(t, x - \eta) - u(t, x + \eta)) + c_T(\sigma/2)\mu_t(]x - \eta, x + \eta[).
\]
Furthermore, by \ref{inequality:monotonicity} and the definition of \( E_t \), there exists \( \eta_0 > 0 \) such that
\[
\left| J_{x,\eta}^{\sigma/2} \setminus I^{\sigma/2}(\sigma/2) \right| \geq - \frac{\kappa_{[\sigma,T]}}{2} \mu_t(]x - \eta, x + \eta[) \quad \text{for all } \eta \in ]0, \eta_0[. \tag{3.21}\]
By the Besicovitch covering lemma, we can cover \( \mu_t \)-a.e. \( E_t \) with countably many pairwise disjoint intervals \([x_j - \eta_j, x_j + \eta_j]\) where \( \eta_j \) is chosen such that both \ref{inequality:boundaryConditions} and \ref{inequality:DisjointIntervals} hold. Proposition \ref{proposition:monotonicity} (ii) implies that the intervals \( J_{x_j,\eta_j}^{\sigma/2} \) are pairwise disjoint and by \ref{inequality:boundaryConditions} we have that \( J_{x_j,\eta_j}^{\sigma/2} \) is contained in \( A_{\sigma/2} \). Therefore, it holds that
\[
F_\sigma(t + \varepsilon) - F_\sigma(t) = \left| I^{t+\varepsilon}(\sigma/2) \setminus I^{\sigma/2}(\sigma/2) \right| \geq \sum_j \left| J_{x_j,\eta_j}^{\sigma/2} \setminus I^{\sigma/2}(\sigma/2) \right| .
\]
Applying \ref{inequality:coveringLemma} and then \ref{inequality:monotonicity} to the above inequality yields
\[
F_\sigma(t + \varepsilon) - F_\sigma(t) \geq - \frac{\kappa_{[\sigma,T]}}{2} \sum_j \mu_t(]x_j - \eta_j, x_j + \eta_j[) \geq - \frac{\kappa_{[\sigma,T]}}{2} \mu_t(E_t) \geq \frac{\kappa_{[\sigma,T]}}{2} \alpha ,
\]
and therefore \ref{inequality:sumOfIntervals} holds.

(iii). By the monotonicity of \( F_\sigma \) and \ref{inequality:discontinuitySet}, \( F_\sigma \) has at most countable many discontinuities on \([\sigma, T] \). Thus, for any given \( \sigma \in ]0, T[ \), \ref{inequality:discontinuitySet} imply that the set
\[
\bigcup_{n \in \mathbb{N}} \{ t \in [\sigma, T] : \mu_t(A_t) \leq -2^{-n} \} = \{ t \in [\sigma, T] : \mu_t(A_t) < 0 \}
\]
is at most countable and therefore,
\[
\bigcup_{n \in \mathbb{N}} \{ t \in [2^{-n}, T] : \mu_t(A_t) < 0 \} = T_{[z_1, z_2]} \text{ is countable.}
\]

**Step 2.** To complete the proof, it is sufficient to show that for any given \( T > 0 \), there exists an at most countable subset \( T_T \) of \([0, T] \) such that
\[
\forall t \in [0, T] \\setminus T_T \quad u(t, \cdot) \in SBV_{\text{loc}}(\mathbb{R}) \quad \text{for all } t \in [0, T] \setminus T_T . \tag{3.22}\]
For any \( k \in \mathbb{Z} \), we pick a point \( \bar{z}_k \in [k, k + 1] \setminus \mathcal{J}(T) \). Let \( \xi_k(\cdot) \) be the unique genuine backward characteristic starting at point \((T, \bar{z}_k)\) for every \( k \in \mathbb{Z} \) and define

\[
A^T_k = A^T_{[\bar{z}_k, \bar{z}_{k+1}]} \cup \{ (\xi_k(t), t) : t \in [0, T] \} \quad \text{and} \quad A^T_k(t) = A^T_{[\bar{z}_k, \bar{z}_{k+1}]}(t) \cup \{ \xi_k(t) \}.
\]

Due to the no-crossing property of two genuine backward characteristics in Proposition 2.7, it holds that

\[
\bigcup_{k \in \mathbb{Z}} A^T_k = [0, T] \times \mathbb{R} \quad \text{and} \quad \bigcup_{k \in \mathbb{Z}} A^T_k(t) = \mathbb{R} \quad \text{for all} \ t \in [0, T].
\]

From Step 1, it holds that, for every \( k \in \mathbb{Z} \), the set

\[
\{ t \in [0, T] : \mu_t(A^T_k(t)) \neq 0 \}
\]

is countable.

Hence,

\[
\mathcal{T}_T = \{ t \in [0, T] : \mu_t(A^T_k(t)) \neq 0 \quad \text{for some} \ k \in \mathbb{Z} \}
\]

is also countable. and this yields (3.22).

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