On the $N$-extended Euler system: generalized Jacobi elliptic functions

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1 Introduction

We are interested in the real functions $\omega_l(v)$ which are solutions of the integrable system of differential equations

$$\frac{d\omega_l}{dv} = \alpha_i \prod_{j \neq i} \omega_j(v), \quad (1 \leq i, j \leq N),$$

(1)

with coefficients and initial conditions $\omega_l(0) \in \mathbb{R}$. Our study is based on the quadratic expressions

$$C_{ij}(v) = \alpha_i \omega_j(v)^2 - \alpha_j \omega_i(v)^2$$

(2)

which are integrals of the system (1). Initial conditions (IC) will be denoted $\omega^0 \equiv \omega(0) = (\omega_1(0), \ldots, \omega_N(0))$. To simplify expressions, we will use as notation $\omega_l(0)$ and $\omega'_l \equiv d\omega_l/dv$.

From the geometric point of view, the integrals (2) tell us that the flow defined by (1) is the result of the intersection of quadrics in dimension $N$, more precisely, elliptic and hyperbolic cylinders. Thus, the $N$-EES family belongs to a larger family where the paraboloids are also included, as well as the degenerate cases defined by the hyperplanes. Its Poisson character is obtained as a Nambu nested structure,
and it is defined by a determinant built on the gradients of the independent integrals, i.e., the Casimirs, see for further details Appendix 1. When $N = 3$, the classic mixed product is precisely the determinant: One of the integrals is the Casimir and the other the Hamiltonian.

The interest of the study of system (1) is justified by distinguishing several cases. Nevertheless, the unified treatment of the solutions for arbitrary dimension enriches the knowledge of these functions showing their common characteristics, but also the differences among them. One of the features of the system (1) is that it allows, from a dynamical system point of view, to deal with a large family of functions in the real domain in a unified way. It ranges from trigonometric functions (harmonic oscillator) to elliptic functions (pendulum and free rigid body), including also rational functions (for unbounded trajectories), etc. We will learn that different systems will allow us to introduce the same functions. For instance, the hyperbolic functions may be introduced with $N = 2$, but also appear when $N = 3$ and two of the coefficients are equal. The case $N = 4$, first proposed by Mahler in [13], is special as we show below. The solutions of this case were introduced years later by Abdel-Salam [1], where the author shows that these solutions are related to phenomena in applied mathematics, physics and engineering described by nonlinear partial differential equations. The interest of the study of the generic system $N > 4$ lies in the fact that we face then hyperelliptic integrals and their inverses, a well established theory of special functions of complex variable made in XIX century which, nowadays, is in a revival in several branches of science, particularly in mechanics. But, although the theory is ‘at hand,’ nevertheless its application results a nontrivial task, because of the number of parameters involved in the definition of the functions, solutions of an IVP.

1.1 On Euler system, Jacobi functions and 3-EES

In this paper, our program is to generalize Jacobi elliptic functions. Thus, within the dynamical system point of view we have adopted, let us remember how all this started. The history of the $N$-EES begins with the well-known Euler system of nonlinear differential equations in three dimensions [19], giving the reduced dynamics of the free rigid body problem (the dynamics of the angular momentum vector $\mathbf{\Pi}$ in the moving frame)

$$
\Pi_1' = \alpha_1 \Pi_2 \Pi_3, \quad \Pi_2' = \alpha_2 \Pi_1 \Pi_3, \quad \Pi_3' = \alpha_3 \Pi_1 \Pi_2,$n

(3)

such that $\sum \alpha_i = 0$, where $\alpha_i$ is function of the principal moments of inertia.

Associated with (3), the second fundamental system, known as the Jacobi system, is given by

$$
\omega_1' = \omega_2 \omega_3, \quad \omega_2' = -\omega_1 \omega_3, \quad \omega_3' = -m \omega_1 \omega_2, \quad (4)
$$

with $\omega(0) = (0, 1, 1)$. The functions solution of (4), denoted as $\omega_1 \equiv \text{sn}, \omega_2 \equiv \text{cn}$ and $\omega_3 \equiv \text{dn}$, are called Jacobi elliptic functions. Then, the solution of (3) is given by means of those functions, using the method of undetermined coefficients. For some readers could be useful to consult our paper [5] where we have studied the extended Euler system

$$
\omega_1' = \alpha_1 \omega_2 \omega_3, \quad \omega_2' = \alpha_2 \omega_1 \omega_3, \quad \omega_3' = \alpha_3 \omega_1 \omega_2, \quad (5)
$$

i.e., the (1) for $N = 3$, considering generic values for coefficients $\alpha_i$ and initial conditions defining the system.

Relying on the work of Tricomi [25], Hille [13] and Meyer [20] dedicated to system (4), we have shown in a straightforward manner how Jacobi and Weierstrass elliptic functions in the real domain are connected with this system [5], although the tradition is to treat them separately because of the their intrinsic differences in the complex domain (see for instance Whittaker and Watson [26], Byrd and Friedman [3] and Lawden [17]). Here we will apply the same approach to the system in $N$-dimensions. More precisely, we will present the generalization of both types of functions, where the $N$-Weierstrass function relates with the norm of the vector defined by the functions $\omega_i$.

1.2 Integrals, functions and regularization

Moreover, as an alternative to confront directly with hyperelliptic functions, we propose to experiment with reparametrizations starting from low dimensions. More precisely, we extend the regularization $d\nu^* = \omega_3 d\nu$, already studied for the case $N = 3$ by Molero et al. [21]. This way of proceeding seems to be an open line of work. The fact that elliptic and hyperelliptic functions are ’naturally’ introduced within the context of complex functions may explain why we have not found references. It is due to the consideration of those
functions in a dynamical systems context, in the real domain, that the regularization enters on the scene. More precisely, we focus on 'regularizations' of the type \(d\omega^* = g(\omega_1)dv\), a technique well known in classical fields such as Celestial Mechanics (where they are used for studies ranging from collisions to efficient numerical integration schemes). We will see that the new variable is a generalization of the Jacobi amplitude. This procedure, based on the symmetry of the system, alleviates the manipulation of the hyperelliptic functions involved, which are relegated to only one quadrature (the \textit{regularization equation}), separating it from the geometry (it is part of our research, knowing more on how generic this procedure is).

This research has two parts. Part I, which makes the content of this paper, works in detail the cases \(N = 4, 5\). The key aspect associated with this case is that for each IVP we deal with two or three parameters. In Sect. 2, we briefly refers to the equilibria as well as particular solutions such as the rectilinear. After that we fix the dimension considering the case 4-EES. In Sect. 3, we present a basic feature related to the coefficients of the functions. In Sect. 4.2, we focus in a biparametric system, which we dubbed as Mahler system. In Sect. 5, we apply to our system the regularization technique. We identify that the new variable is a 'generalized amplitude.' In Sect. 6, we provide with the addition formulas associated with the Mahler system. Using them we propose extending the work of Bulirsch and Fukushima, and we introduce some formulas related to the numerical evaluation of a 4-EES. In Sect. 7, we approach the system for \(N = 5\), focusing in one of the particular cases, showing its connection with the previous dimension.

For the benefit of the reader, we include two Appendices which contain properties of \(\theta_i\) and elliptic Jacobi functions. There is a Part II, devoted to generic features of (1) from the geometric point of view, and to the numeric evaluation of the Mahler system, following the steps of Bulirsch and Fukushima. This will be published elsewhere.

We ought to close the Introduction pointing out that this paper does not contain a complete analysis of the relative role of the parameters involved in the defined functions. Some transformations related to the range of those parameters are required, similar to the well-known transformations for the elliptic modulus of the Jacobi functions. That analysis is still in progress.

2 Some basic features of \(N\text{-EES}\)

We have mentioned in the Introduction that our interest in this paper focuses on the study of some systems (1) of low dimension. Nevertheless, as in any dimension common features are present, it is worth to briefly refer to some of them.

2.1 On the Poisson structure

The \(N\text{-EES}\) may be expressed into the Nambu formalism. It is a generalization of the Hamiltonian formalism proposed in [23], which is based on the Poisson bracket generalization by the multiple operation named as Nambu bracket. For the interested reader in this subject, we have included a brief summary in the Appendix 1 (see also Crespo et al. [6]). \(N - 1\) hyper-manifolds in \(\mathbb{R}^N\), the Nambu–Hamilton equations of motion are interpreted as a parametrization of their intersection curves. The \(N\text{-extended Euler system} \) illustrates this geometric interpretation. Let us consider the trajectory \(C\) described by a point constrained to be in the intersection of (2). This is a elliptic curve in the three-dimensional case; generally hyperelliptic for \(N > 4\). The set of hyperquadrics given by \(C_1 = \{C_{11}, \ldots, C_{1N-1}\}\) is functionally independent, and their intersection gives \(C\). Therefore, the hyperelliptic curves are the trajectories of the Nambu–Hamilton equations of motion

\[
\dot{\omega}_i = \{C_{11}, \ldots, C_{1N-1}, \omega_i\}_N, \quad i = 1, \ldots, N, \tag{6}
\]

where \(\{\ldots\}_N\) is the standard Nambu brackets in \(\mathbb{R}^N\). In addition, taking into account the subordinated Nambu structures of lower degree and fixing \(N - 2\) integrals, we are left with the following Poisson bracket

\[
\{F, G\}_P = \{C_{11}, \ldots, C_{1N-2}, F, G\}_N, \\
\forall F, G \in \mathcal{C}^\infty(\mathbb{R}^N). \tag{7}
\]

Thus, the bilinear operation \(\{,\}_P\) is a Poisson bracket with the set of Casimirs \(\{C_{11}, \ldots, C_{1N-2}\}\), and the Hamiltonian equations associated with the Hamiltonian function \(C_{1N-1}\) are given by

\[
\dot{\omega}_i = \{C_{1N-1}, \omega_i\}_P, \quad i = 1, \ldots, N, \tag{8}
\]

which are equivalent to (6). After some straightforward computations, one can see that these equations are exactly the ones given in (1) of the \(N\text{-EES}\).
2.2 On particular solutions: equilibria and straight lines through the origin

- Before we start our analysis of the IVP, a first question is to identify the equilibria of the system (1). Denoting $P = (p_1, p_2, \ldots, p_n)$ an equilibrium point, we easily check that the system has the following set of equilibria:
  - Origin $P = 0 \in \mathbb{R}^n$,
  - For $n \geq 3$, the points: $P_1 = (0, \ldots, p_i, \ldots, 0)$, $1 \leq i \leq n$, functions of the initial conditions.
  - For $n \geq 4$, planes $\Pi_{i_1, i_2} = (0, \ldots, p_{i_1}, \ldots, p_{i_2}, \ldots, 0)$, $1 \leq i_1 < i_2 \leq n$, functions of the initial conditions.
  - For $n \geq 5$, the hyperplanes
    $$\Pi_{i_1, i_2, \ldots, i_{n-2}} = (0, \ldots, p_{i_1}, \ldots, p_{i_2}, \ldots, p_{i_{n-2}}, \ldots, 0),$$
    $1 \leq i_1 < i_2 < i_{n-2} \leq n$.

Thus, associated with these equilibria hyperplanes, we have the study of their invariant manifolds and their connections, generalizing the heteroclinic trajectories in three dimensions. This is out of the scope of the present paper.

- **Straight lines through the origin.** Meanwhile in the generic study of the quadratures associated with our system (see Sect. 2.3), an assumption is commonly made; namely, the roots of the polynomials involved are different; when considering an IVP we may be under a scenario where we have multiple roots. This is precisely the case with **straight lines through the origin**. Then, instead of requiring the use of special functions, the solutions are expressed by means of **elementary functions**, different for each dimension.

2.3 Reduction to quadratures: generalized Weierstrass function

Taking into account the integrals (2) and proceeding like in the classic case $N = 3$, we may reduce the system to a fundamental differential equation in two forms. The first one, after choosing one of the functions, say $\omega_i$, it leads to the differential equation

$$\left(\frac{d\omega_i}{dv}\right)^2 = \alpha_i^{3-N} \left[ \prod_{j \neq i}^{N} (\alpha_j \omega_i^{2} + C_j^{i}) \right].$$

or, by separation, the corresponding quadrature

$$\alpha_i^{(3-N)/2} v = \int \left[ \prod_{j \neq i}^{N} (\alpha_j \omega_i^{2} + C_j^{i}) \right]^{1/2}. \quad (10)$$

As an alternative, if we introduce the **square of the norm**

$$\mathcal{N}(v) \equiv \omega(v)^2 = \sum_{i=1}^{N} \omega_i(v)^2, \quad (11)$$

after some straightforward computations we obtain

$$\left(\frac{d\mathcal{N}}{dv}\right)^2 = 4 \sum_{i=1}^{N} \left(\mathcal{N} - b_i\right), \quad \sum_{i=1}^{N} b_i = 0, \quad (12)$$

a differential equation whose solution $\mathcal{N}(v)$ may be seen as the generalized Weierstrass function $\wp(v)$. Following either way, we confront generically hyperelliptic integrals.

2.4 On the normalized $N$-EES

Associated with a generic $N$-EES (1), i.e., assuming that $\sum \alpha_i \neq 0$, we consider the **square norm function** (11) that satisfies

$$\frac{d\omega}{dv} = \left(\sum_{i=1}^{N} \alpha_i\right) \frac{1}{\omega} \prod_{i=1}^{N} \omega_i. \quad (13)$$

Thus, introducing the functions

$$\tilde{\omega}_i = \frac{\omega_i}{\omega},$$

we have

$$\frac{d}{dv} \left(\frac{\omega_i}{\omega}\right) = \left[ \alpha_i \omega^2 - \left(\sum_{i=1}^{N} \alpha_i\right) \omega_j^2 \right] \frac{1}{\omega^3} \prod_{j \neq i}^{N} \omega_j. \quad (14)$$

which may be written also as

$$\frac{d\tilde{\omega}_i}{dv} = c_i \prod_{j \neq i}^{N} \tilde{\omega}_j \omega^{N-4}, \quad (15)$$
where the coefficients
c_i = \alpha_i \omega^2 - \left( \sum \alpha_i \right) \omega_i^2 \quad (16)

are integrals of the flow, whose values are determined for each IVP by the initial conditions. In other words, carrying out the reparametrization \( v \rightarrow v^* \) given by
\[
dv^* = \omega^{N-4} dv,
\]
(17)

associated with (1) we have the normalized system
\[
\frac{d\tilde{\omega}_i}{dv^*} = c_i \prod_{j \neq i} \tilde{\omega}_j,
\]
(18)

with initial conditions
\[
\tilde{\omega}_i(0) = \omega_i(0)/\omega(0), \quad \omega(0)^2 = \sum \omega_i(0)^2, \quad (19)
\]
i.e., the flow (18) lives in \( S^{N-1} \) and, like the differential system satisfied by the angular momentum in 3-D, we have \( \sum c_i = 0 \). Note that to deal with the system (18) versus (1) will bring advantages, at least from the numerical point of view.

With (18) integrated, we have \( \tilde{\omega}_i = \tilde{\omega}_i(v^*) \). Then, we still have to implement the quadrature associated with the regularization (17) in order to recover the relation with the original variable. For instance, considering the first integral \( c_1 \) we obtain
\[
dv = \omega^{4-N} dv^* = \left( \frac{c_1 - \left( \sum \alpha_i \right) \tilde{\omega}_1(v^*)^2}{\alpha_1} \right)^{\frac{4-N}{2}} dv^*, \quad (20)
\]
whose quadrature gives the parametrization relation, solved generically by numeric methods. Note that, the case \( N = 4 \) is special, because we do not need to do regularization.

Moreover, we will not pursue here with the study of the normalized system (18).

Let us close this section pointing out another basic feature of this system; we refer to it as the scaling factor. If the functions \( \omega_i(v), (i = 1, \ldots, N) \) are a set of solutions, then taking a constant \( c \), the functions \( u_i(v) = c \omega_i(c^{N-2}v) \) satisfy the same system with the corresponding IC given by \( u_i(0) = c \omega_i(0) \). We will make use of this property along the paper.

2.5 Intrinsic symmetries to the system

The \( N \)-EES provides its solutions with several symmetries properties. For the case \( N = 3 \), when suitable coefficients and initial conditions are chosen, they imply the well-known symmetry properties of the Jacobi elliptic functions.

**Proposition 1** Let \( A \) and \( B \) be to sets of integers satisfying that \( A \cup B = \{1, \ldots, N\} \) and let
\[
\omega(s) = (\omega_1(s), \ldots, \omega_N(s))
\]
be a solution of the initial value problem given by (1). Then

(i) If \( \text{Car}(A) \) is odd, we have that \( Y(s) = (y_1(s), \ldots, y_N(s)) \), defined by
\[
y_i(s) = -\omega_i(-s), \quad i \in A,
y_j(s) = \omega_j(-s), \quad j \in B,
\]
(21)

is also a solution of the original system

(ii) If \( \text{Car}(B) \) is even, we have that \( Z(s) = (z_1(s), \ldots, z_N(s)) \), defined by
\[
z_i(s) = \omega_i(s), \quad i \in A,
z_j(s) = -\omega_j(s), \quad j \in B,
\]
(22)

is also a solution of the original system

**Proof** It is a straightforward computation to check that those \( N \)-tuples also satisfy the original system. \( \square \)

From now on, we study solutions assuming that the initial conditions satisfy that \( \omega_1^0, \ldots, \omega_N^0 \) always be positive numbers or zero. Any other arrangement could be studied by applying the above proposition.

**Corollary 1** Let us assume the notation of Proposition 1 and let \( \omega(s) \) be a solution with the initial condition \( \omega(0) = (\omega_1^0, \ldots, \omega_N^0) \), such that \( \omega_i^0 = 0 \) for \( i \in A \) and \( \omega_j^0 \neq 0 \) for \( j \in B \). Then \( \omega_l(s) \) is an odd function if \( l \in A \) and even for \( l \in B \).

**Proof** Let \( (\omega_1(s), \ldots, \omega_N(s)) \) be the solution of system (1) satisfying that \( \omega_i^0 = 0 \) for \( i \in A \) and \( \omega_j^0 \neq 0 \) for \( j \in B \). By Proposition 1, we have that \( (y_1(s), \ldots, y_N(s)) \) is also a solution for system (1) and it also satisfies the same initial condition. Therefore, the result follows by the uniqueness theorem for ODE. Namely
\[
\omega_i(s) = y_i(s) = -\omega_i(-s), \\
\omega_j(s) = y_j(s) = \omega_j(-s),
\]
(23)
and we have that \( \omega_i(s) \) are odd functions and \( \omega_j(s) \) are even functions. Where \( i \in A \) and \( j \in B \). \( \square \)
2.6 Stability analysis

The $N$-EES is endowed with several equilibria. A straightforward computation shows that any solution having two zeros among the initial conditions provides an equilibrium. The stability study of the equilibria having just two zero coordinates may be tackled using the linearization of the system (unstable equilibria) and the Energy-Casimir method (stable equilibria). Difficulties arise when the number of zero coordinates is greater than two, since the eigenvalues of the linearized system and the functions involved in the Energy-Casimir method vanish. Thus, an alternative method is needed to study these equilibria. Nevertheless, the simplicity and the symmetry of the integrals defining the solution trajectories allow to study stability by means of the next geometric criterion.

In what follows, it is convenient to introduce the following notation. Given an initial condition $\omega = (\omega_1, \ldots, \omega_N)$ and a set of independent integrals \( \{H_1, \ldots, H_{N-1}\} \), we denote by \( h(\omega) = (h_1^{\omega}, \ldots, h_{N-1}^{\omega}) \) the vector of \( \mathbb{R}^{N-1} \) given by the evaluation of the integrals at \( \omega \), i.e., \( h_i^{\omega} = H_i(\omega) \) and let \( H_{h(\omega)} = \{H_1^{h(\omega)}, \ldots, H_{N-1}^{h(\omega)}\} \) be the set of hypersurfaces defined by the level sets \( H_i = h_i^{\omega} \). Then, a particular trajectory of the $N$-EES for a given initial condition $\omega$ is given by one of the connected components of the set $\gamma_{h(\omega)}$, which is obtained as the following intersection

\[
\gamma_{h(\omega)} = \bigcap_{i=1}^{N-1} H_i^{h(\omega)}.
\]

In what follows, the set $\gamma_{h(\omega)}$ is dubbed as the energy level at $\omega$. It is also satisfied that any solution starting at a connected component of $\mathcal{C}$ remains in that component for all $s$ in the domain of the solution. Thus, by means of the continuation theorem for differential equations [12], pp. 16–17, we obtain that solutions starting in a connected component of $\mathcal{C}$ have to traverse all of it.

**Theorem 1** (Stability geometric criterion) Let $\omega$ be an equilibrium of system (1). Then, it is stable if and only if the energy level at $\omega$ is a finite set of isolated points.

**Proof** Any equilibrium point $\omega$ satisfies that it has, at least, two zero coordinates. Thus, since the integrals are the hyperquadrics given in (2), solving the algebraic system that they define, we have that there are only two different kinds of energy level sets at $\omega$. Namely, $\gamma_{h(\omega)}$ is a finite set of isolated points or a union of curves that intersect at several points. They correspond, respectively, to the cases:

- An isolated set of stable equilibria.
- A union of unstable equilibria and homoclinic or heteroclinic orbits associated with them. □

3 The case $N = 4$: Relying on Jacobi elliptic functions?

We focus now on the 4-EES case. For each IVP, with some abuse of notation, we refer to the functions solutions generically with $\omega_1$. Later, referring to some specific systems, we will introduce new notations.

At this point, perhaps some readers would like to know the original motivation of our interest in 4-EES case. The reason is connected with an observation about the classical way in which the study of the rigid body dynamics is developed, based on Jacobi elliptic functions. Meanwhile, those functions depend on one parameter (elliptic modulus) and appear naturally tied to problems like the pendulum or the measure of an arc of ellipse; when we apply them to the rigid body problem, we need to consider a second parameter (the characteristic, a function of the principal moments of inertia). In other words, the first and third Legendre elliptic integrals are involved. Since Jacobi, the way to proceed has been: (i) to introduce complementary functions $Z$ and $\Theta$ and (ii) to make use of the addition formulas of elliptic functions, dealing with the second parameter as an amplitude, etc. Here we search for an alternative to such approach considering a generalization of Jacobi elliptic functions with two parameters.

Thus, we start with the 4-EES

\[
\begin{align*}
\omega_1' &= \alpha_1 \omega_2 \omega_3 \omega_4, \\
\omega_2' &= \alpha_2 \omega_1 \omega_3 \omega_4, \\
\omega_3' &= \alpha_3 \omega_1 \omega_2 \omega_4, \\
\omega_4' &= \alpha_4 \omega_1 \omega_2 \omega_3,
\end{align*}
\]

with given initial conditions $\omega^0$, and the corresponding six quadratic first integrals (2), of which three are independent (Fig. 1 shows a graph of the solution of the system (25)). Although by scaling and a change of variables, we could get rid of two of the coefficients $\alpha_i$, for our purpose it is convenient here to maintain all of them.
we think the case to approach both aspects, apart from its own interest, to discuss and (ii) to introduce again regularizations. In order were defined. Note that represent a drastic reduction in the case of higher dimension, i.e., in the reign of hyperelliptic methodologies to follow when dealing with systems of the case of the 4-EES given by the integrals $C^j_i = \alpha_i\omega^2_j - \alpha_j\omega^2_i$.

**Proof** It is straightforward making use of the definition of the 4-EES

**Remark 1** From the previous Proposition 2, readers familiar with the expressions of Jacobi elliptic functions, and their computation by means of Jacobi theta functions $\theta_i(x)$, may wonder what the relation between those functions and the $\omega_j(v)$ might be. We have gathered some of those systems in an Appendix. In fact the reader will find in Lawden (Chp 1) a number of properties of $\theta_i$ functions which are also satisfied by the $\omega_i$. Perhaps, the simple fact that $\theta'_i(0) = \theta_2(0)\theta_3(0)\theta_4(0)$ is satisfied for the 4-EES when we take $\alpha_1 = 1$, is one of the most surprising. We will come back to this below.

**Remark 2** Note that there is the possibility to take a slight different version of the ratios, namely to work with $u'_j = c'_j \omega_j/\omega_j$, with coefficients $c'_j$ still to be determined, in order to simplify some expressions, adjust constants in applications, etc. We do not follow this alternative in this paper.

**Proposition 3** For suitable IC the 4-EES (25), has as solution the bounded functions $\omega_i(v) \equiv \omega_i(v; \alpha_i, \omega_i(0))$ given by

$$\omega_1(v) = \tilde{C}^4_1 \frac{\text{sn}(av|m_1)}{\sqrt{1 - n_1\text{sn}^2(av|m_1)}}$$

$$\omega_2(v) = \tilde{C}^4_2 \frac{\text{cn}(av|m_1)}{\sqrt{1 - n_1\text{sn}^2(av|m_1)}}$$

$$\omega_3(v) = \tilde{C}^4_3 \frac{\text{dn}(av|m_1)}{\sqrt{1 - n_1\text{sn}^2(av|m_1)}}$$

$$\omega_4(v) = \tilde{C}^4_4 \frac{1}{\sqrt{1 - n_1\text{sn}^2(av|m_1)}}$$

where $\text{sn}(av|m_1)$, $\text{cn}(av|m_1)$, $\text{dn}(av|m_1)$, etc are the Jacobi elliptic functions, and the constants $\tilde{C}^4_i$, $a$, $m_1$ and $n_1$ are functions of $\alpha_i$ and $\omega_i(0)$. 

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![Graphical solution of the previous system (25) for \(\alpha_1 = 1\); \(\alpha_2 = -1\); \(\alpha_3 = 2\); \(\alpha_4 = -0.5\)

Fig. 1]
Proof Let us assume IC \( \omega^0 = (\omega_1^0, \ldots, \omega_4^0) \) such that \( \omega_j \neq 0 \) in its domain of definition. According to the previous Proposition, we consider the ratios and the reciprocals \( 1/\omega_j \) that we denote
\[
\frac{u_i}{\omega_j}, \quad i \neq j, \quad u_j = \frac{1}{\omega_j},
\]
in the domain where \( \omega_j \) is defined. Without loss of generality, we assume we refer to the case \( j = 4 \), with IC such that \( \omega_4 > 0 \). Moreover, we still simplify a bit more the notation writing \( u_i^4 = u_i \).

Then, according to Proposition 2 it results for the functions \( u_i, i = 1, 2, 3 \) and we have the following system
\[
\begin{align*}
    u_1' &= C_4^2 u_2 u_3, \\
    u_2' &= C_2^3 u_3 u_1, \\
    u_3' &= C_3^2 u_1 u_2,
\end{align*}
\]
with IC \( u_i(0) = u_i^0 = \omega_i^0/\omega_j^0 \). Moreover, from the first integral
\[
\alpha_1 \omega_4^2 - \alpha_4 \omega_1^2 = C_1^4
\]
we may write
\[
u_4^2 = \frac{1}{C_1^4} (\alpha_1 - \alpha_4 u_1^4).
\]

Because the functions \( u_i, i = 1, 2, 3 \) satisfy (32), they belong to the set of functions defined by the 'Jacobi elliptic functions’ sn, cn, dn and their ratios. Then, following Crespo and Ferrer [5], we know our system corresponds to one of the four possible cases (Glashier systems), depending on the sign of the integrals. Here, to continue our reasoning on the system (25), we focus on the case where the sign of \( C_1^4 \) is different of \( C_2^4 \) and \( C_3^4 \) (the other cases are treated likewise). This means that \( u_i, i = 1, 2, 3 \) are of the form, say
\[
\begin{align*}
    u_1(v) &= \delta_1 \text{sn}(av, m_1), \\
    u_2(v) &= \delta_2 \text{cn}(av, m_1), \\
    u_3(v) &= \delta_3 \text{dn}(av, m_1).
\end{align*}
\]

Proceeding by the method of undetermined coefficients, replacing (35) in (32) we identify that the constants \( \delta_i, a \gamma m_1 \) satisfy a system of algebraic equations whose solution is
\[
\begin{align*}
    \delta_2 &= u_2^0, \\
    \delta_3 &= u_3^0, \\
    \delta_1 &= \sqrt{-\alpha_1/\alpha_2 \delta_2}, \\
    a &= \alpha_1 \delta_2 \delta_3/\delta_1, \\
    m_1 &= \alpha_3 \delta_2^2/\alpha_2 \delta_3^2
\end{align*}
\]
(for details see for instance Lawden [17], p. 132).

Summarizing, according to (31) and (32) we have \( \omega_i = u_i/u_4 \), where \( u_i(i = 1, 2, 3) \) are the Jacobi elliptic functions and \( u_4 \) is given by (34). From those expressions, we obtain the functions (27)–(30), where
\[
\begin{align*}
    C_4^4 &= \sqrt{C_1^4/\alpha_4}, \\
    \tilde{C}_4^4 &= \delta_4/\tilde{C}_4^4, \\
    n_1 &= \alpha_4 \delta_4^2/\alpha_1
\end{align*}
\]
and, as stated in the Proposition, initial conditions still have to be chosen such that \( n_1 < 1 \). □

Before we continue it is convenient to formulate the previous Proposition in a ‘complementary form,’ where we make more transparent the role played by coefficients and initial conditions.

Proposition 4 The functions \( \omega_i(v), i = 1, \ldots, 4 \), given by
\[
\begin{align*}
    \omega_1(v) &= \frac{\omega_2(0) \omega_3(0) \omega_4(0)}{a} \frac{\text{sn}(av|m_1)}{\sqrt{1 + n_1 \text{sn}^2(av|m_1)}}, \\
    \omega_2(v) &= \omega_2(0) \frac{\text{cn}(av|m_1)}{\sqrt{1 + n_1 \text{sn}^2(av|m_1)}}, \\
    \omega_3(v) &= \omega_3(0) \frac{\text{dn}(av|m_1)}{\sqrt{1 + n_1 \text{sn}^2(av|m_1)}}, \\
    \omega_4(v) &= \omega_4(0) \frac{1}{\sqrt{1 + n_1 \text{sn}^2(av|m_1)}},
\end{align*}
\]
satisfy a differential system of the type (25) given by
\[
\begin{align*}
    \omega_1' &= \omega_2 \omega_3 \omega_4, \\
    \omega_2' &= -(1 + n_1) \frac{a^2}{\omega_2(0) \omega_3(0)} \omega_1 \omega_3 \omega_4, \\
    \omega_3' &= -(m_1 + n_1) \frac{a^2}{\omega_2(0) \omega_4(0)} \omega_1 \omega_2 \omega_4, \\
    \omega_4' &= -n_1 \frac{a^2}{\omega_2(0) \omega_3(0)} \omega_1 \omega_2 \omega_3,
\end{align*}
\]
with \( \omega = (0, \omega_2(0), \omega_3(0), \omega_4(0)) \) as initial conditions.

Proof It is a straightforward exercise by computing derivatives. □

Remark 3 In particular, choosing \( \omega_i(0) = 1 \) (\( i = 2, 3, 4 \)) and \( a = 1 \), join with \( n_1 = n \) and \( m_1 = m - n \) in Proposition 4, we have the Jacobi elliptic functions
\[
\begin{align*}
    \text{sn}(v) &= \frac{\omega_1(v)}{\omega_4(v)}, \\
    \text{cn}(v) &= \frac{\omega_2(v)}{\omega_4(v)}, \\
    \text{dn}(v) &= \frac{\omega_3(v)}{\omega_4(v)}
\end{align*}
\]
with elliptic modulus \( m_1 = m - n \), where \( \omega_i(v; m, n) \) satisfy the system

\[
\begin{align*}
\omega_1' &= \omega_2 \omega_3 \omega_4, \\
\omega_2' &= -(1 + n) \omega_1 \omega_3 \omega_4, \\
\omega_3' &= -m \omega_1 \omega_2 \omega_4, \\
\omega_4' &= -n \omega_1 \omega_2 \omega_3,
\end{align*}
\] (39)

with integrals

\[
\begin{align*}
\omega_2^2 + (1 + n) \omega_1^2 &= 1, \\
\omega_3^2 + m \omega_1^2 &= 1, \\
\omega_4^2 + n \omega_1^2 &= 1.
\end{align*}
\] (40)

If \( 0 < n < m < 1 \), we have \(-1/\sqrt{1 + n} \leq \omega_1 \leq 1/\sqrt{1 + n}, -1 \leq \omega_2 \leq 1, \sqrt{1 - m/(1 + n)} \leq \omega_3 \leq 1 \) and \( \sqrt{1 - n/(1 + n)} \leq \omega_4 \leq 1 \).

More details on the system (39) will not be given in the rest of this paper.

4 Studying two 4-EES systems

Looking for the generalization of Jacobi elliptic functions, we now focus on two cases of (25):

- One-parameter (\( \theta_i \) similar) family in Sect. 4.1 and
- Two-parameter family (Mahler system) in Sect. 4.2.

It is worth noting that the first two equations in both systems (see (45) and (46)) are equal, with the consequence that one of the integrals is \( \omega_1^2 + \omega_2^2 = 1 \), which is not the case for the previous system (39).

In relation to both, before we continue, a comment on notation is due. In what follows, it is convenient to redefine some of the constants which appear in the previous expressions. More precisely, in Sect. 4.1 we write \( m_1 \equiv k^2 \), and we will find that \( a \) and \( n_1 \) are functions of \( k \). Likewise, in Sect. 4.2 we fix all initial conditions and coefficients except two of them, denoted by \(-m\) and \(-n\).

4.1 One-parameter \( \omega_i(v) \) functions, ‘similar’ to Jacobi \( \theta_i \) functions

We look here for functions \( \omega_i \), solutions of our differential system (25), similar to Jacobi \( \theta_i \) functions. What we mean by that should be made more precise: (i) Coefficients and IC of the 4-EES have to be dependent only of one parameter: \( \alpha_i = \alpha_i(k), \omega_i^0 = \omega_i^0(k) \); (ii) Moreover those functions \( \omega_i(v; k) \) ought to be found imposing that they verify properties defining \( \theta_i \) de Jacobi.

Such search does not appear straightforward because, we remember, \( \theta_i \) functions are defined as one-parameter Fourier series solving the heat equation. Our way of proceeding will be to take into account those properties of \( \theta_i \) which could be imposed on the differential system: Both the ratios and the identities satisfied by \( \theta_i(0) \) are essential for us.

**Proposition 5** (\( \omega_i \): ‘similar Jacobi \( \theta_i \) functions’)

Choosing initial conditions as functions of the elliptic modulus

\[
\omega_1(0) = 0, \quad \omega_2(0) = \sqrt{a \cdot k}, \quad \omega_3(0) = \sqrt{a \cdot k'},
\]

join with

\[
a = \frac{2K}{\pi}, \quad n_1 = k' - 1, \quad m_1 = k^2
\]

where \( k' = \sqrt{1 - k^2} \), then we may write

\[
\begin{align*}
v_1(\omega_3^2(0)z) &= \frac{\omega_3(0) \omega_1(z)}{\omega_2(0) \omega_4(z)}, \\
v_2(\omega_3^2(0)z) &= \frac{\omega_2(0) \omega_4(z)}{\omega_4(0) \omega_3(z)}, \\
v_3(\omega_3^2(0)z) &= \frac{\omega_3(0) \omega_4(z)}{\omega_3(0) \omega_4(z)}
\end{align*}
\] (43)

in other words, we express the Jacobi elliptic functions as ratios of the \( \omega_i(v) \), in a similar way as Jacobi gave them with respect to the \( \theta_i \) functions.

**Proof** It is a straightforward exercise replacing the previous values (41) and (42) in Proposition 4. The result is that the functions are

\[
\begin{align*}
\omega_1(z, k) &= \sqrt{a \cdot k} \cdot \frac{\text{sn}(u)}{\sqrt{1 - (1 - k') \cdot \text{sn}^2(u)}}, \\
\omega_2(z, k) &= \sqrt{a \cdot k} \cdot \frac{\text{cn}(u)}{\sqrt{1 - (1 - k') \cdot \text{sn}^2(u)}}, \\
\omega_3(z, k) &= \sqrt{a} \cdot \frac{\text{dn}(u)}{\sqrt{1 - (1 - k') \cdot \text{sn}^2(u)}}, \\
\omega_4(z, k) &= \sqrt{a \cdot k'} \cdot \frac{1}{\sqrt{1 - (1 - k') \cdot \text{sn}^2(u)}},
\end{align*}
\] (44)

join with \( u = az \).
Thus the system (38) given by
\begin{align*}
\omega_1' &= \omega_2 \omega_3 \omega_4, \\
\omega_2' &= -\omega_3 \omega_4 \omega_1, \\
\omega_3' &= -\frac{1 - k'}{k} \, \omega_4 \, \omega_1 \, \omega_2, \\
\omega_4' &= \frac{1 - k'}{k} \, \omega_1 \, \omega_2 \, \omega_3,
\end{align*}
with initial conditions (41) is the IVP which we were looking for. Fig. 2 shows an example of a graph of this set of functions.

It is an exercise to check that the functions (44) verify identical relations to the linear combinations satisfied by the square of Jacobi \( \theta_i \) functions [see Lawden [17], formulae (1.4.49)–(1.4.52), p. 11].

4.2 Mahler system: a biparametric 4-EES

As a second distinguished 4-EES we consider now a ‘biparametric’ case we call Mahler system. It is an IVP which defines the functions \( \omega_i(v; m, n) \), solutions of (25) depending on two parameters, such that

- coefficients \( \alpha = (1, -1, -m, -n) \)
- initial conditions \( \omega^0 = (0, 1, 1, 1) \).

When \( n = 0 \) then \( \omega_3(v) \) are the Jacobi elliptic functions and \( \omega_4(v) \equiv 1 \).

Note that this represents some abuse of notation, because \( n \) has already been used to denote the last component of an \( N \)-dimension system. Nevertheless, we think by the context it will become clear when is a coefficient: \( n \in \mathbb{R} \), although in some occasions \( n \) might be used as a counter (ordinal number: \( n \in \mathbb{N} \)).

**Proposition 6** (Mahler system) The 4-EES given by
\begin{align*}
\omega_1' &= \omega_2 \omega_3 \omega_4, \\
\omega_2' &= -\omega_3 \omega_4 \omega_1, \\
\omega_3' &= -m \, \omega_1 \omega_2 \omega_4, \\
\omega_4' &= -n \, \omega_1 \omega_2 \omega_3,
\end{align*}
where \( n < m < 1 \), with IC \( \omega(0) = (0, 1, 1, 1) \), has the functions
\begin{align*}
\omega_1 &= A \frac{\text{sn}(av|m_1)}{\sqrt{1 - n_1 \, \text{sn}^2(av|m_1)}}, \\
\omega_2 &= \frac{\text{cn}(av|m_1)}{\sqrt{1 - n_1 \, \text{sn}^2(av|m_1)}}, \\
\omega_3 &= \frac{\text{dn}(av|m_1)}{\sqrt{1 - n_1 \, \text{sn}^2(av|m_1)}}, \\
\omega_4 &= \frac{1}{\sqrt{1 - n_1 \, \text{sn}^2(av|m_1)}},
\end{align*}
as solution, with values \( a, A, m_1, n_1 \) given by
\begin{align*}
a &= \sqrt{1 - m}, \quad A = 1/\sqrt{1 - n}, \\
n_1 &= \frac{n}{n - 1}, \quad m_1 = \frac{n - m}{n - 1}.
\end{align*}

**Proof** Let us consider the system (46) as an IVP with \( \omega(0) = (0, \omega_2(0), \omega_3(0), \omega_4(0)) \), \( \omega_3(0) \neq 0 \), \( i = 2, 3, 4 \) dependent of two parameters \( (m, n) \). It admits as solution the functions
\begin{align*}
\tilde{\omega}_1(v) &= A \frac{\text{sn}(av|m_1)}{\sqrt{1 - n_1 \, \text{sn}^2(av|m_1)}}, \\
\tilde{\omega}_2(v) &= \omega_2(0) \frac{\text{cn}(av|m_1)}{\sqrt{1 - n_1 \, \text{sn}^2(av|m_1)}}, \\
\tilde{\omega}_3(v) &= \omega_3(0) \frac{\text{dn}(av|m_1)}{\sqrt{1 - n_1 \, \text{sn}^2(av|m_1)}}, \\
\tilde{\omega}_4(v) &= \omega_4(0) \frac{1}{\sqrt{1 - n_1 \, \text{sn}^2(av|m_1)}},
\end{align*}
where \( a, A, m_1, n_1 \) are given by
\begin{align*}
a &= \omega_3(0) \sqrt{\omega_2^2(0) - n \, \omega_2^2(0)}, \\
A &= \frac{\omega_2(0) \omega_4(0)}{\sqrt{\omega_2^2(0) - n \, \omega_2^2(0)}}, \\
n_1 &= \frac{\omega_2^2(0) \, n \, \omega_2^2(0) - \omega_2^2(0)}{\omega_2^2(0) \, (n \, \omega_2^2(0) - m \, \omega_2^2(0))}, \\
m_1 &= \frac{\omega_2^2(0) \, (n \, \omega_2^2(0) - m \, \omega_2^2(0))}{\omega_2^2(0) \, (n \, \omega_2^2(0) - m \, \omega_2^2(0))}
\end{align*}
and the derivatives at the origin satisfy
\begin{align*}
\tilde{\omega}_1(0)' &= \omega_2(0) \, \omega_3(0) \, \omega_4(0), \quad \tilde{\omega}_1(0)' = 0,
\end{align*}
where \( i = 2, 3, 4 \). Then, choosing as IC the quantities \( \omega(0) = (0, 1, 1, 1) \) and replacing them in (49), we readily obtain the values (48) for those parameters. \( \blacksquare \)
Remark 4 In particular, the case $n=0$ leads to: $a=1$, $A=1$, $n_1=0$ and $m_1=m$, i.e., the Jacobi elliptic functions. We have another special case when $m=0$. As we have assumed $n<m$, in this case $n<0$ and the differential system (46) corresponds again to a Jacobi system, but now with negative parameter (there is a transformation to reduce it to the normal case, see Appendix 2, Sect. 1). For more on particular cases, see Section 5.4. We leave for the reader to work out the other particular cases defined by special values of the pair $(m, n)$.

5 Regularization and ‘generalized amplitudes’ for the Mahler system

We have just solved the system $N=4$ in the standard way: making use of known functions (Jacobi elliptic functions). In what follows, we are going to proceed making use of the regularization. To do that, we start remembering in Sect. 5.1 the recent proposal of the authors for $N=3$ (see Molero et al. [21]), which is intrinsically connected with the Jacobi amplitude. After that we develop the same approach for the $N=4$ case. That proposal entails to study, at least, two possible regularizations $v \rightarrow v^*$ given by

$$-\frac{dv^*}{dv} = \omega_4, \quad -\frac{dv^*}{dv} = \omega_3 \omega_4,$$

which we gather in Sects. 5.2 and 5.3. Let us proceed one by one. But, before, we remember in Sect. 5.1 how this has been done for the 3-EES.

5.1 Preliminaries: 3-EES and regularization

Let us consider the 3-EES (5) with initial conditions $\omega^0 \equiv \omega(0) = (\omega_1(0), \omega_2(0), \omega_3(0))$, whose values we choose below. This system has the integrals

$$\alpha_1 \omega_2^2 - \alpha_2 \omega_1^2 = C_1^2, \quad \alpha_1 \omega_3^2 - \alpha_3 \omega_1^2 = C_1^3. \quad (51)$$

Let us assume $\alpha_i$ and IC such that $\omega_3(v) > 0$. Then, making use of the parametrization

$$\frac{dv^*}{dv} = \omega_3, \quad (52)$$

the system (5) reduces to

$$\frac{d\omega_1}{dv^*} = \alpha_1 \omega_2, \quad \frac{d\omega_2}{dv^*} = \alpha_2 \omega_1, \quad (53)$$

join with the quadrature defined by (52). Choosing the coefficients $\alpha_1 = 1, \alpha_2 = -1$ and IC $(\omega_1(0), \omega_2(0)) = (0, 1)$, the system (53) defines the trigonometric (circular) functions:

$$\sin(v^*), \cos(v^*). \quad (54)$$

(with other conditions, by a change of variables we may reduce it to this case), Then, keeping in mind (51), the regularization (52) takes the form

$$\frac{dv^*}{dv} = \sqrt{C_1^3 + \alpha_3 \omega_1^2}. \quad (55)$$

Motivated by the dynamical system defining the simple pendulum$^1$, it is chosen $\omega_3(0)$ join with $\alpha_3 = -k^2$, where $k^2 < 1$. Thus, replacing in (55) we have

$$dv = \frac{dv^*}{\sqrt{1 - k^2 \sin^2 v^*}}, \quad (56)$$

whose quadrature and inversion lead us to the Jacobi “am” function:

$$v^* = \text{am}(v, k). \quad (57)$$

Finally, replacing in (54) we have the Jacobi functions

$$\sin(v^*(v)) = \sin(\text{am}(v, k)), \quad \cos(v^*(v)) = \cos(\text{am}(v, k)), \quad (58)$$

which today, following Gudermann, are denoted in the form

$$\text{sn}(v; k) \equiv \text{am}(v, k)), \quad \text{cn}(v; k) \equiv \cos(\text{am}(v, k)).$$

Completing our set of functions $\omega_3$ is given by

$$\omega_3(v) \equiv \text{dn}(v; k) = \sqrt{1 - k^2 \text{sn}^2(v; k)}. \quad (59)$$

$^1$ This lead us to an interpretation of the regularization: $v \equiv t$ and $v^* \equiv \phi$, in other words ‘time’ and ‘angle’. Angle in the 1-2 plane; arc through the integral $\omega_1^2 + \omega_2^2 = 1$, a circle projection of the integral which is a cylinder.
Summarizing, using the previous notation, the integrals (51) lead us to the well-known expressions relating these functions

\[
\sin^2 + \cos^2 = 1, \quad \frac{\partial^2}{\partial x^2} + k^2 \sin^2 = 1.
\] (60)

Finally, replacing in (5) we write what some authors refer as “derivation rules” of Jacobi functions:

\[
\sin' = \cos, \quad \cos' = -\sin, \quad \cos' = -k^2 \sin \cos.
\] (61)

5.2 The \( \frac{dv^*}{dv} = \omega_4 \) regularization

Proceeding as in the previous Section, we treat now the case \( N = 4 \) by means of the regularization

\[
\frac{dv^*}{dv} = \omega_4.
\] (62)

Remark 5 Remember the comment above in relation to notation; although there is some abuse using again \( v^* \) for denoting the new independent parameter, from the context we distinguish it from the one studied in the previous section.

As a consequence, the system (25) is reduced to

\[
\frac{\partial \omega_1}{\partial v^*} = \alpha_1 \omega_2 \omega_3, \quad \frac{\partial \omega_2}{\partial v^*} = \alpha_2 \omega_1 \omega_3, \quad \frac{\partial \omega_3}{\partial v^*} = \alpha_3 \omega_1 \omega_2,
\]

and \( \omega_4(\nu^*) \) which will be obtained using one of the integrals, after we have solved the previous system.

We focus on the case \( \alpha_1 = 1, \alpha_2 = -1 \) and \( \alpha_3 = -m \) because, as we have said before, we plan to generalize Jacobi elliptic functions. Thus, we have

\[
\omega_1 = \sin(v^*; m_1), \quad \omega_2 = \cos(v^*; m_1), \quad \omega_3 = \sin(v^*; m_1)
\] (63)

and for the differential relation using the integral \( n \omega_1^2 + \omega_2^2 = C_4^i \) and the initial conditions, we may write

\[
v = \int \frac{dv^*}{\sqrt{1 - n \sin^2(v^*; m_1)}}.
\] (64)

5.3 \( N = 4 \): the regularization \( \frac{dv^*}{dv} = \omega_3 \omega_4 \)

Proceeding the same way as for \( N = 3 \), we treat now the case \( N = 4 \) by means of the regularization

\[
\frac{dv^*}{dv} = \omega_3 \omega_4.
\] (65)

As a consequence, the system (25) reduces to

\[
\frac{\partial \omega_1}{\partial v^*} = \alpha_1 \omega_2, \quad \frac{\partial \omega_2}{\partial v^*} = \alpha_2 \omega_1,
\]

and two quadratures associated with \( \omega_3 \) and \( \omega_4 \). In fact, they are not needed because the integrals gives us

\[
\omega_i^2 = C_i^2 - \alpha_i \omega_1^2, \quad (i = 3, 4).
\]

Note that \( C_i^1 \) are constants which depend on the initial conditions.

Without loss of generality we will assume our system is made of bounded functions. Then, by a change of variables, our system (66) reduces to \( \alpha_1 = 1, \alpha_2 = -1 \); thus, it results

\[
\omega_1(\nu^*) = \sin \nu^*, \quad \omega_2(\nu^*) = \cos \nu^*,
\] (67)

Considering the previous integrals, we may write (65) as follows

\[
v = \int \frac{dv^*}{\sqrt{1 - \beta_1 \sin^2 \nu^*}}
\] (68)

or in a slightly different form

\[
\lambda dv = \frac{dv^*}{\sqrt{(1 - \beta_1 \sin^2 \nu^*)(1 - \beta_2 \sin^2 \nu^*)}}
\] (69)

where \( \beta_i \) and \( \lambda \) are functions of \( C_1^i \) and \( \alpha_i \).

In what follows, with the Mahler system in mind as the basic 4-EES, it is convenient to take the associated notation:

\[
\beta_1 \equiv n, \quad \beta_2 \equiv m, \quad \lambda \equiv 1.
\]

In other words, the differential relation (69) reads

\[
v = \int \frac{dv^*}{\sqrt{(1 - n \sin^2 \nu^*)(1 - m \sin^2 \nu^*)}}
\] (70)
The quadrature takes the form
\[ v = G(v^*, n, m) = \int_0^{v^*} \frac{d\vartheta}{\sqrt{(1-n \sin^2 \vartheta)(1-m \sin^2 \vartheta)}}. \quad (71) \]

Thus, we define the period as the two-parameters function
\[ G(\pi/2, n, m) = \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{(1-n \sin^2 \vartheta)(1-m \sin^2 \vartheta)}}. \quad (72) \]

Thus, when \((n, m) = (0, 0)\), we have \(G(0, 0) = \pi/2\), and when \((n, m) = (1, 1)\), we have \(G(1, 1) = \infty\).

When \((m, n)\) are small, if we carry out the Taylor expansion of the integrand, after the evaluation of the quadratures, \(G(n, m)\) may be approximated in the form
\[ G(n, m) \approx \frac{\pi}{2} \left[ 1 + \frac{m}{4} + \frac{9m^2}{64} + \frac{25m^3}{256} + \frac{1225m^4}{16384} \right. \]
\[ + \frac{n}{4} \left( 1 + \frac{3m}{8} + \frac{15m^2}{64} + \frac{175m^3}{1024} + \frac{2205m^4}{16384} \right) \]
\[ + \frac{9n^2}{64} \left( 1 + \frac{5m}{12} + \frac{35m^2}{128} + \frac{105m^3}{512} + \frac{2695m^4}{16384} \right) \]
\[ + \frac{25n^3}{256} \left( 1 + \frac{7m}{16} + \frac{189m^2}{640} + \frac{231m^3}{1024} + \frac{3003m^4}{16384} \right) \]
\[ + \frac{1225n^4}{16384} \left[ 1 + \frac{9m}{20} + \frac{99m^2}{320} + \frac{429m^3}{1792} + \frac{6435m^4}{32768} \right] \]
\[ + \text{h.o.t.} \]

although the previous expression may be written in different form making more explicit its symmetric character with respect to \(m\) and \(n\).

Now we define the \textit{generalized amplitude} \(\text{amg}\) as the inverse function
\[ v^* = \text{amg}(v; n, m). \quad (73) \]

Thus, considering the expressions (67), we have
\[ \sin v^* = \sin \text{amg}(v, n, m) = \text{sng}(v, n, m) \quad (74) \]
and
\[ \cos v^* = \cos \text{amg}(v, n, m) = \text{cnng}(v, n, m) \quad (75) \]

- There is an alternative way of proceeding. If we consider the change of variable \(\sin \vartheta = x\), it allows to follow the steps of Jacobi for the case \(N = 3\).

Then, the differential relation (70) takes the form
\[ dv = \frac{dx}{\sqrt{(1-x^2)(1-m x^2)}} \quad (76) \]
or inverting the expression
\[ \frac{dx}{dv} = \sqrt{(1-x^2)(1-m x^2)}. \quad (77) \]

In other words, we define the function \(\text{sng}\)
\[ x = x(v; n, m) = \text{sng}(v; n, m) \quad (78) \]
as the two-parameters function (whose range is made more precise below), solution of the differential equation
\[ \left( \frac{dx}{dv} \right)^2 = (1-x^2)(1-m x^2). \quad (79) \]

In this paper, we will restrict to a range \(n \leq m \leq 1\).
- Then, associated with \(\text{sng}\) we propose the following functions
\[ \text{cnng}(v; n, m) = \pm \sqrt{1 - \text{sng}^2(v; n, m)}, \quad (80) \]
\[ \text{dnng}(v; n, m) = \sqrt{1 - m \text{sng}^2(v; n, m)}, \quad (81) \]
\[ \text{fnng}(v; n, m) = \sqrt{1 - n \text{sng}^2(v; n, m)}. \quad (82) \]

To simplify notation, we will write \(\text{sng}(v; n, m) \equiv \text{sng}\), etc. Examples of the graph of these new functions can be seen in Figs. 3 and 4.

Due to the process we have followed, we immediately check that these functions \(\text{sng}\), etc verify the following IVP

![Fig. 3](image1)

Fig. 3 Mahler \(n = 0.1, m = 0.8\)

![Fig. 4](image2)

Fig. 4 Mahler \(n = -2, m = 0.5\)
Taylor expansions of $\text{sng}$, $\text{cng}$, $\text{dn}$ and $\text{fn}$ near the origin. As a direct application of the definition of those functions by the differential system (83), we may easily compute to any order the Taylor expansion of the previous functions:

\[
\frac{\text{d} \text{sn}}{\text{d} \text{v}} = \text{cn} \text{d} \text{ng} \text{fn},
\]
\[
\frac{\text{d} \text{cn}}{\text{d} \text{v}} = -\text{sn} \text{d} \text{ng} \text{fn},
\]
\[
\frac{\text{d} \text{ng}}{\text{d} \text{v}} = -\text{m} \text{sn} \text{cn} \text{fn},
\]
\[
\frac{\text{d} \text{fn}}{\text{d} \text{v}} = -\text{n} \text{sn} \text{cn} \text{dn},
\]

with initial conditions $(0, 1, 1, 1)$. The integrals, as we have mentioned before, lead to the following expressions

\[
\text{cn}^2 + \text{sn}^2 = 1, \quad \text{dn}^2 + m \text{sn}^2 = 1,
\]
\[
\text{fn}^2 + n \text{sn}^2 = 1.
\]

Thus, from the functions solution of the Mahler system, the Jacobi functions are given by

\[
\text{sn}(\text{av}; m_1) = \frac{1}{A} \text{sn}(\text{v}; m, n),
\]
\[
\text{cn}(\text{av}; m_1) = \frac{\text{cn}(\text{v}; m, n)}{\text{fn}(\text{v}; m, n)},
\]
\[
\text{dn}(\text{av}; m_1) = \frac{\text{dn}(\text{v}; m, n)}{\text{fn}(\text{v}; m, n)},
\]

Remark 6 The interest of these expansions is connected with the computation of these functions. By extension of the process followed by Bulirsch and Fukushima computing Jacobi elliptic functions (see Appendix). Nevertheless, there is still work to be done comparing that scheme with the possible advantages of using regularization.

5.4 Particular cases

- $n = 0$. In this case, due to the choice of the initial conditions, we have $\text{fn}(\text{v}) \equiv 1$. Moreover we have $\text{sn}(\text{v}; 0, m) = \text{sn}(\text{v}, m)$, etc, i.e., the Jacobi elliptic functions with elliptic modulus $m$.
- $m = 0$. Here, based on the initial conditions, we have $\text{dn}(\text{v}) \equiv 1$. Moreover $\text{sn}(\text{v}; n, 0) = \text{sn}(\text{v}, n)$, etc, i.e., the Jacobi elliptic functions have an elliptic modulus $n$ (que es negativo, thus we still needs to make a transformation; see (48) leading to $m_1$).
- $m = 1$. In this case, the differential equation is

\[
\frac{\text{d} \text{x}}{\text{d} \text{v}} = (1 - \text{x}^2)\sqrt{1 - n \text{x}^2}.
\]

For this quadrature, we obtain

\[
v = \frac{1}{2\sqrt{1 - n}} \ln \left( \frac{1 + \text{x}}{1 - \text{x}} \right) \left( 1 + n \text{x} + \sqrt{(1 - n)(1 - n\text{x}^2)} \right),
\]

whose inversion is possible, because it is injective.
- $m = n$. Now the differential equation is

\[
\frac{\text{d} \text{x}}{\text{d} \text{v}} = (1 - m\text{x}^2)\sqrt{1 - \text{x}^2}.
\]

We obtain

\[
v = \frac{1}{\sqrt{1 - m}} \text{ArcTan}\left( \frac{\sqrt{1 - m} \text{x}}{\sqrt{1 - \text{x}^2}} \right).
\]

Again the inversion is possible because it is injective

\[
\tan(\sqrt{1 - m} \text{v}) = \sqrt{1 - m} \frac{\text{x}}{\sqrt{1 - \text{x}^2}}.
\]
6 Addition formulas

In order to alleviate the notation, we introduce the following convention

\[
\begin{align*}
\text{sng}(a x; m, n) &= s_{ax}, \\
\text{cng}(a x; m, n) &= c_{ax}, \\
\text{dng}(a x; m, n) &= d_{ax}, \\
\text{fng}(a x; m, n) &= f_{ax}.
\end{align*}
\]

**Theorem 2** (Addition-Subtraction formulae for the 4-Mahler functions) The addition and subtraction formulae for the 4-Mahler functions are given next.

\[
\text{sng}(x \pm y; m, n) = \frac{A (s_{ax} c_{ay} d_{ay} f_{ax} \pm s_{ay} c_{ax} d_{ax} f_{ay})}{\sqrt{(f_{ax}^2 f_{ay}^2 - m_1 s_{ax}^2 s_{ay}^2)^2 - n_1 (s_{ax} c_{ay} d_{ay} f_{ax} \pm s_{ay} c_{ax} d_{ax} f_{ay})^2}}
\]

\[
\text{cng}(x \pm y; m, n) = \frac{c_{ax} c_{ay} f_{ax} \pm s_{ax} s_{ay} d_{ax} d_{ay}}{\sqrt{(f_{ax}^2 f_{ay}^2 - m_1 s_{ax}^2 s_{ay}^2)^2 - n_1 (s_{ax} c_{ay} d_{ay} f_{ax} \pm s_{ay} c_{ax} d_{ax} f_{ay})^2}}
\]

\[
\text{dng}(x \pm y; m, n) = \frac{d_{ax} d_{ay} f_{ax} \pm s_{ax} s_{ay} c_{ax} c_{ay}}{\sqrt{(f_{ax}^2 f_{ay}^2 - m_1 s_{ax}^2 s_{ay}^2)^2 - n_1 (s_{ax} c_{ay} d_{ay} f_{ax} \pm s_{ay} c_{ax} d_{ax} f_{ay})^2}}
\]

\[
\text{fng}(x \pm y; m, n) = \frac{f_{ax}^2 f_{ay}^2 - m_1 s_{ax}^2 s_{ay}^2}{\sqrt{(f_{ax}^2 f_{ay}^2 - m_1 s_{ax}^2 s_{ay}^2)^2 - n_1 (s_{ax} c_{ay} d_{ay} f_{ax} \pm s_{ay} c_{ax} d_{ax} f_{ay})^2}}
\]

where \(A, a, m_1\) and \(n_1\) are given in formula (43) (en la proposicion 5).

**Proof** Let us prove the formula corresponding to \(\text{sng}(x \pm y; m, n)\), the remaining ones are analogous. By Proposition 5, we have that

\[
\text{sng}(x \pm y; m, n) = A \frac{\text{sn}(ax + ay; m_1)}{\sqrt{1 - n_1 \text{sn}^2(ax + ay; m_1)}}
\]

Thus, using the addition and subtraction formulae for the Jacobi elliptic sine (see Appendix 2) and assuming the following convention

\[
\text{sn}(ax; m_1) = s_x, \quad \text{cn}(ax; m_1) = c_x, \quad \text{dn}(ax; m_1) = d_x,
\]

we obtain

\[
\text{sng}(x \pm y; m, n) = A \frac{s_x c_y d_y \pm s_y c_x d_x}{\sqrt{(1 - m_1 s_x^2 s_y^2)^2 - n_1 (s_x c_y d_y \pm s_y c_x d_x)^2}}.
\]
simplifying denominators

\[
sng(x \pm y; m, n) = A \frac{s_x c_y d_y \pm s_y c_x d_x}{\sqrt{(1 - m_1 s_x^2 s_y^2) - n_1 (s_x c_y d_y \pm s_y c_x d_x)^2}}.
\]

Finally, recalling that

\[
s_x = \frac{1}{A} sng(ax; m, n) \quad c_x = \frac{cng(ax; m, n)}{fng(ax; m, n)} \quad d_x = \frac{sng(ax; m, n)}{fng(ax; m, n)},
\]

and likewise for \(s_y, c_y, d_y\), if we multiply numerator and denominator in (96) by \(fng^2(ax; m, n)\) and \(fng^2(ay; m, n)\) we obtain (95) after algebraic simplifications. \(\square\)

**Corollary 2** The formulae for the double angle of the 4-Mahler functions are given by

\[
sng(2x; m, n) = \frac{2A s_{ax} c_{ax} d_{ax} f_{ax}}{\sqrt{(f_{ax}^4 - m_1 s_{ax}^4)^2 - n_1 (2 s_{ax} c_{ax} d_{ax} f_{ax})^2}}
\]

\[
cng(2x; m, n) = \frac{c_{ax}^2 f_{ax}^2 \mp s_{ax}^2 d_{ax}^2}{\sqrt{(f_{ax}^4 - m_1 s_{ax}^4)^2 - n_1 (2 s_{ax} c_{ax} d_{ax} f_{ax})^2}}
\]

\[
dng(2x; m, n) = \frac{d_{ax}^2 f_{ax}^2 \mp c_{ax}^2 s_{ax}^2}{\sqrt{(f_{ax}^4 - m_1 s_{ax}^4)^2 - n_1 (2 s_{ax} c_{ax} d_{ax} f_{ax})^2}}
\]

\[
fng(2x; m, n) = \frac{f_{ax}^4 - m_1 s_{ax}^4}{\sqrt{(f_{ax}^4 - m_1 s_{ax}^4)^2 - n_1 (2 s_{ax} c_{ax} d_{ax} f_{ax})^2}}.
\]

**Corollary 3** The formulae for the half-angle of the 4-Mahler functions are given by

\[
sng \left(\frac{x}{2}; m, n\right) = A \sqrt{\frac{f_{ax} - c_{ax}}{f_{ax} + d_{ax} - n_1 (f_{ax} - c_{ax})}}
\]

\[
cng \left(\frac{x}{2}; m, n\right) = \sqrt{\frac{d_{ax} + c_{ax}}{f_{ax} + d_{ax} - n_1 (f_{ax} - c_{ax})}}
\]

\[
dng \left(\frac{x}{2}; m, n\right) = \sqrt{\frac{(c_{ax} + d_{ax})(f_{ax} + d_{ax})}{(f_{ax} + c_{ax})(f_{ax} + d_{ax}) - n_1 (f_{ax}^2 - c_{ax}^2)}}
\]

\[
fng \left(\frac{x}{2}; m, n\right) = \sqrt{\frac{f_{ax} + d_{ax}}{f_{ax} + d_{ax} - n_1 (f_{ax} - c_{ax})}}
\]

6.1 On the numerical computation of \(\omega_i\) functions by extending Bulirsch–Fukushima method

As we know Jacobi elliptic functions are defined by some ratios of \(\theta_i\) Jacobi functions. This way of handling the Jacobi elliptic functions is convenient due to the fast convergency of those series. Nevertheless, at present, fast numeric codes compete with this classic analytic approach. More precisely, in order to implement those codes addition formulas that compute Jacobi elliptic functions are basic expressions in that process (see Fukushima \cite{9,10}).

We can extend those expressions to the \(\omega_i\) functions. Thus, as Fukushima explains, the algorithm is made of three steps:

(i) the forward transformation defined by (Corollary 2, \(\omega_1^\prime\) arguments formulas: (98) reducing the values of \(\omega_i\) by a number of iterations;

(ii) evaluation of the Mac-Laurin series expansions given by (86) and;

(iii) the backward transformation (Corollary 1: Double arguments formulas (97)) as many times as the forward transformation.

7 On the case \(N = 5\)

As we have pointed out in the introduction, hyperelliptic integrals appear in (1) when \(N \geq 5\). Thus, it is convenient to see in some detail the case \(N = 5\), the lower system belonging to this category.

Thus, as before, we start keeping the notation used in lower dimension

\[
\omega_1 = \alpha_1 \omega_2 \omega_3 \omega_4 \omega_5,
\]

\[
\omega_2 = \alpha_2 \omega_1 \omega_3 \omega_4 \omega_5,
\]

\[
\omega_3 = \alpha_3 \omega_1 \omega_2 \omega_4 \omega_5.
\]

\[
\omega_4 = \alpha_4 \omega_1 \omega_2 \omega_5 \omega_5,
\]

\[
\omega_5 = \alpha_5 \omega_1 \omega_2 \omega_5 \omega_4,
\]

with given initial conditions \(\omega(0)\). As examples in Figs. 7 and 8, we present two set of functions of the 5-EES family.

We will proceed as in the lower dimensions \(N = 3, 4\), considering alternative procedures to the classic solution based on the direct reduction to hyperelliptic integrals. In other words:
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Fig. 7 5-Mahler system graphs for $p = 0.2, n = 0.4, m = 0.7$

Fig. 8 5-Mahler system graphs for $p = -2, n = -1, m = 0.4$

(i) We introduce the functions $u_i^j(v)$, (where we maintain the notation) ratios of the $\omega_i$

$$u_i^j = \frac{\omega_i}{\omega_j}, \quad i \neq j, \quad u_i^j = \frac{1}{\omega_j}, \quad (100)$$

in the domain of definition of $\omega_j$.

(ii) In the rest of the section we will study the effect of the introduction of some possible regularizations, namely two of them

- $dv^* = \omega_5\, dv$.
- $dv^* = \omega_3 \omega_4 \omega_5 \, dv$.

Again, we have to keep in mind that with the notation used in the above regularizations, the new variable $v^*$ is different from one case to the other.

7.1 The $dv^*/dv = \omega_5$ regularization

Then, associated with the ratios $u_i$, if we carry out the regularization

$$dv = u_5 \, dv^*. \quad (101)$$

we have the following regularized differential system

$$\frac{du_1}{dv^*} = C_1^5 u_2 u_3 u_4, \quad \frac{du_2}{dv^*} = C_2^5 u_1 u_3 u_4, \quad \frac{du_3}{dv^*} = C_3^5 u_1 u_2 u_4, \quad \frac{du_4}{dv^*} = C_4^5 u_1 u_2 u_3, \quad (102)$$

with IC $u_i(0) = u_i^0 = \omega_i^0/\omega_j^0, \quad i = 1, \ldots, 4.$

Thus, dividing the integral $\alpha_1 \omega_5^2 - \alpha_5 \omega_1^2 = C_1^5$ by $\omega_5^2$ we write: $u_5^2 = (\alpha_1 - \alpha_5 u_1^3)/C_1^5$. Then, we obtain

$$v = \frac{\alpha_1}{C_1^5} \sqrt{1 - n_2 [u_1(v^*)]^2} \, dv^*, \quad (103)$$

where $n_2 = \alpha_5/\alpha_1$ and $u_1(v^*)$ are a function solution of the system (102); quadrature which will be solved numerically. As we know that the solution of (102) can be obtained by undetermined coefficients, making use of the 4-Mahler functions defined by the system (83), but in the variable $v^*$. In other words, the previous form of the solution represents an alternative to the use of hyperelliptic integrals for solving (1) for $N = 5$; or, in a more precise form, we have separated geometry from dynamics. The trajectory is expressed by Jacobi or Mahler functions; meanwhile, the quadrature of the parametrization (103) will lead generically to a hyperelliptic integral.

7.2 The $dv^*/dv = \omega_3 \omega_4 \omega_5$ regularization

Let us consider again the system 5-EES (99). Now we try the regularization

$$\frac{dv^*}{dv} = \omega_3 \omega_4 \omega_5. \quad (104)$$

in a domain where $\omega_3 \omega_4 \omega_5 \neq 0$. This means that the system reduces to

$$\frac{d\omega_1}{dv^*} = \alpha_1 \omega_2, \quad \frac{d\omega_2}{dv^*} = \alpha_2 \omega_1, \quad (105)$$

and three quadratures associated with $\omega_3, \omega_4$ and $\omega_5$. In fact, they are not needed because the integrals allow to write $\omega_i^2 = C_i^1 - \alpha_i \omega_i^2, (i = 3, 4, 5)$. Remember that $C_i^1$ are constants, functions of the initial conditions.

Assuming the bounded case, we can always choose, by scaling and transformation of functions, that $\alpha_1 = 1, \quad \alpha_2 = -1$. In other words, we have

$$\omega_1(v^*) = \sin v^*, \quad \omega_2(v^*) = \cos v^*, \quad (106)$$

Then, the quadrature (104), taking into account the previous mentioned integrals, we have
\[ \lambda v = \int \frac{dv^*}{\sqrt{\prod_{i=3}^{5}(1 - \beta_i \sin^2 v^*)}} \] (107)

where \( \beta_i \) and \( \lambda \) are functions of \( C_i^1 \) and \( \alpha_i \). This lead us, in the generic case, to a hyperelliptic quadrature.

Dealing with the 5-Mahler System In what follows, we choose as the basic system in \( \mathcal{N} = 5 \) a Mahler type system

\[
\begin{align*}
\omega_1 &= \omega_2 \omega_3 \omega_4 \omega_5, \\
\omega_2 &= -\omega_1 \omega_3 \omega_4 \omega_5, \\
\omega_3 &= -m \omega_1 \omega_2 \omega_4 \omega_5, \\
\omega_4 &= -n \omega_1 \omega_2 \omega_3 \omega_5, \\
\omega_5 &= -p \omega_1 \omega_2 \omega_3 \omega_4,
\end{align*}
\] (108)

with initial conditions \((0, 1, 1, 1, 1)\).

Moreover, apart from adjusting coefficients, an alternative form of dealing with (107) is to make a change of variable \( \sin v^* = x \). Then, the corresponding new expression for the regularization is given by

\[ \lambda \, dv = \frac{dx}{\sqrt{(1 - x^2)(1 - m x^2)(1 - n x^2)(1 - p x^2)}}. \] (109)

Denoting

\[ w = \lambda \, v \]

we define by \( \text{Amg} \) (generalized amplitude) the inverse function

\[ v^* = \text{Amg}(w; p, m, n). \] (110)

Then, by analogy with the notation introduced in lower dimensions, we propose to write

\[ \sin v^* = \sin \text{Amg}(w; p, n, m) \equiv \text{Sng}(w; p, n, m) \] (111)

In other words, we define \( \text{Sng} \)

\[ x = x(w; p, n, m) = \text{Sng}(w; p, n, m) \] (112)

as the three-parameter function solution of the differential equation

\[ \left( \frac{dx}{dw} \right)^2 = (1 - x^2)(1 - p x^2)(1 - n x^2)(1 - m x^2). \] (113)

In the rest of this paper, we restrict ourselves to the domain of parameters \( \Delta = \{(p, n, m) \in [0, 1] \times [0, 1] \times [0, 1]\} \).

Then, associated with \( \text{Sng} \) we introduce the following functions

\[
\begin{align*}
\text{Cng}(w; p, n, m) &= \pm \sqrt{1 - \text{Sng}^2(w; p, n, m)}, \\
\text{Dng}(w; p, n, m) &= \sqrt{1 - m \text{Sng}^2(w; p, n, m)}, \\
\text{Fng}(w; p, n, m) &= \sqrt{1 - n \text{Sng}^2(w; p, n, m)}, \\
\text{Hng}(w; p, n, m) &= \sqrt{1 - p \text{Sng}^2(w; p, n, m)}. \\
\end{align*}
\] (114)

To simplify the notation, we will write in some expressions

\[
\begin{align*}
\text{Sng}(w; p, n, m) &= \equiv \text{Sng}, \quad \text{Cng}(w; p, n, m) \equiv \text{Cng}, \\
\text{Dng}(w; p, n, m) &= \equiv \text{Dng}, \quad \text{Fng}(w; p, n, m) \equiv \text{Fng}, \\
\text{Hng}(w; p, n, m) &= \equiv \text{Hng}.
\end{align*}
\]

Then, we write again (108) as the following IVP

\[
\begin{align*}
\frac{d\text{Sng}}{dw} &= \text{Cng} \text{Dng} \text{Fng} \text{Hng}, \\
\frac{d\text{Cng}}{dw} &= -\text{Sng} \text{Dng} \text{Fng} \text{Hng} \\
\frac{d\text{Dng}}{dw} &= -m \text{Sng} \text{Cng} \text{Fng} \text{Hng} \\
\frac{d\text{Fng}}{dw} &= -n \text{Sng} \text{Cng} \text{Dng} \text{Hng}, \\
\frac{d\text{Hng}}{dw} &= -p \text{Sng} \text{Cng} \text{Dng} \text{Fng}
\end{align*}
\] (115)

with initial conditions \((0,1,1,1,1)\). Note that in agreement with (114), the integrals take the following form

\[
\begin{align*}
\text{Cng}^2 + \text{Sng}^2 &= 1, \quad \text{Dng}^2 + m \text{Sng}^2 = 1, \\
\text{Fng}^2 + n \text{Sng}^2 &= 1, \quad \text{Hng}^2 + p \text{Sng}^2 = 1. \\
\end{align*}
\] (116)

We are not going to deal with the generic study of our system (115). It is out of the scope of this paper. In the last section, we will restrict to analyze some particular cases

\section*{8 $\mathcal{N} = 5$: some particular cases}

Like in previous dimensions, we consider two particular cases
8.1 The case $p = 0$

Now, according to (114), we have $\text{Hng} \equiv 1$. This corresponds to the previous studied case: 4-Mahler system.

8.2 The case $p = n$

As we have just pointed out, a particular case of (107) we will consider now two of the $\beta_i$ equals. According to the notation introduced, we write

$$\lambda \nu = \int_0^{\tilde{\nu}^*} \frac{d\vartheta}{(1 - n \sin^2 \vartheta)\sqrt{1 - m \sin^2 \vartheta}},$$  

(117)

**Remark 7** In relation to the quadrature (117), the reader will remember that this is precisely the Legendre third elliptic integral $\Pi(\tilde{\nu}^*; m, n)$. Thus, for the particular cases $n = 0$ and $n = m$, we encounter the other Legendre elliptic integrals:

$$F(\varphi, m) = \int_0^\varphi \frac{d\vartheta}{\sqrt{1 - m \sin^2 \vartheta}} = \Pi(\varphi, 0, m),$$

$$E(\varphi, m) = \int_0^\varphi \sqrt{1 - m \sin^2 \vartheta} \, d\vartheta, = (1 - m) \Pi(\varphi, m, m) + m \frac{\sin(2\varphi)}{2\sqrt{1 - m \sin^2 \varphi}}.$$  

Denoting

$$w = \lambda \nu,$$

we define as $\text{Amg}$ (generalized amplitude) the inverse function

$$\tilde{\nu}^* = \text{Amg}(w; n, n, m).$$  

(118)

Then, by analogy with the notation introduced in lower dimensions, we propose to write

$$\sin \tilde{\nu}^* = \sin \text{Amg}(w; n, n, m) \equiv \text{Sng}(w; n, m)$$  

(119)

For later use, we also include here the expression for our particular case of (109)

$$dw = \frac{dx}{(1 - n x^2)\sqrt{(1 - x^2)(1 - m x^2)}}.$$  

(120)

From our initial conditions, we have $\text{Hng} \equiv \text{Fng}$. Then, from (108) we immediately obtain that $\omega_4 \equiv \omega_5$, and that these functions satisfy the following IVP

$$\frac{d\text{Sng}}{dw} = \text{Cng Dng Fng}^2,$$

$$\frac{d\text{Cng}}{dw} = -\text{Sng Dng Fng}^2,$$

$$\frac{d\text{Dng}}{dw} = -m \text{Sng Cng Fng}^2,$$

$$\frac{d\text{Fng}}{dw} = -n \text{Sng Cng Dng Fng},$$  

(121)

transforms (121) in a regularized system which is a 4-Mahler system in the new variable.

After we have solved the regularized system, we still need to compute the quadrature associated with the differential relation (122). Explicitly we have

$$dw = \int \frac{d\tilde{\nu}}{\sqrt{1 - n \text{Sng}^2(w(\tilde{\nu}))}}.$$  

(123)

We will give details of this process, both from the analytical and from numerical point of view, in a forthcoming paper.

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**Appendix 1: On the ratios of Jacobi $\theta_i$ functions as solutions of 3-EES**

From Lawden [17] (Chp. 1) we borrow the following 3-EES differential systems satisfied by the ratios of the Jacobi $\theta_i$ functions
\[
\frac{d}{dv} \left( \frac{\theta_1}{\theta_4} \right) = \frac{\theta_2}{\theta_4} \frac{\theta_3}{\theta_4}, \quad (124)
\]

\[
\frac{d}{dv} \left( \frac{\theta_2}{\theta_4} \right) = -\frac{\theta_3^2}{\theta_4} \frac{\theta_1}{\theta_4}, \quad (125)
\]

\[
\frac{d}{dv} \left( \frac{\theta_3}{\theta_4} \right) = -\frac{\theta_2^2}{\theta_4} \frac{\theta_1}{\theta_4}, \quad (126)
\]

etc. We find convenient to introduce the notation \( x_{ij} = \theta_j / \theta_i \) and the reparametrization \( v \to \tau \) given by

\[
d\tau = \sqrt{2K/\pi} dv, \text{ with } x_{ij} = dx_{ij}/d\tau. \text{ Thus, taking into account the values of } \theta_i(0), \text{ where } k^2 = m, \quad k^2 + k'^2 = 1 \text{ and } K(m) \text{ are the complete Legendre first elliptic integral, we write those IVP systems as follows. Note that, as was pointed out in Crespo and Ferrer [5], considering the sign of the coefficients, we may distinguish }

- Two bounded systems:

\[
x'_{41} = k' x_{42} x_{43},
\]

\[
x'_{42} = -x_{41} x_{43},
\]

\[
x'_{43} = -k x_{41} x_{42}, \quad (0, \sqrt{k/k'}, 1/\sqrt{k'})
\]

and

\[
x'_{31} = x_{32} x_{34},
\]

\[
x'_{32} = -k' x_{31} x_{34},
\]

\[
x'_{34} = k x_{31} x_{32}, \quad (0, \sqrt{k}, \sqrt{k'})
\]

- Two unbounded systems:

\[
x'_{21} = k x_{23} x_{24},
\]

\[
x'_{23} = k' x_{21} x_{24},
\]

\[
x'_{24} = x_{21} x_{23}, \quad (0, 1/\sqrt{k}, \sqrt{k'/k})
\]

and

\[
x'_{12} = -k x_{13} x_{14},
\]

\[
x'_{13} = -x_{12} x_{14},
\]

\[
x'_{14} = -k' x_{12} x_{13}, \quad (1, \sqrt{(k'+1)/k}, \sqrt{(k'+1)/k})
\]

Then, we may express those ratios as functions the Jacobi elliptic functions and their Glashier ratios.

**Appendix 2: Transformations and addition formulas for Jacobi elliptic functions**

For the benefit of the reader, we bring here some well-known transformations involving the elliptic modulus. They may be found in any handbook of elliptic functions (remember that, depending on the authors, two notations are used: ‘modulus’ or ‘parameter’ related by \( k^2 \equiv m \), and their complementaries). Those formulas should be used for the reduction to the normal case of some of the particular cases mentioned along the paper.

- **Negative parameter**

Let \( m \) be a positive number and write

\[
\mu = \frac{m}{1+m}, \quad \mu_1 = \frac{1}{1+m}, \quad v = \frac{u}{\sqrt{\mu_1}}. \quad (127)
\]

Then,

\[
\sin(u ; -m) = \sqrt{\mu_1} \frac{\sin(v ; \mu)}{\text{dn}(v ; \mu)},
\]

\[
\cos(u ; -m) = \frac{\cos(v ; \mu)}{\text{dn}(v ; \mu)},
\]

\[
\text{dn}(u ; -m) = \frac{1}{\text{dn}(v ; \mu)}.
\]

Thus, elliptic functions with negative parameter may be expressed by elliptic functions with a positive parameter. Note that \( 0 < \mu < 1 \).

A final comment related to the complete elliptic integral of first kind is due here. Unlike Maple, the software Mathematica yields the following result

\[
\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-m \sin^2 \phi}} = \frac{1}{\sqrt{1-m}} K \left( \frac{m}{m-1} \right) \quad (128)
\]

for \( \forall m \leq 1 \), instead of the expected result \( K(m) \). By applying the previous change (127), we have that, being \( m \) a positive number,

\[
K(-m) = \frac{1}{\sqrt{1+m}} K \left( \frac{m}{1+m} \right) = \sqrt{\mu_1} K(\mu) \quad (129)
\]

which is exactly the same result given by Mathematica for \( m < 0 \).

- **Reciprocal parameter**

Denoting now \( v = \sqrt{\mu} u \), we have

\[
\sin(u ; m) = \frac{1}{\sqrt{m}} \sin(v ; m^{-1}),
\]

\[
\cos(u ; m) = \frac{1}{\sqrt{m}} \cos(v ; m^{-1}),
\]

\[
\text{dn}(u ; m) = \cos(v ; m^{-1}).
\]

This is Jacobi’s real transformation. If \( m > 1 \), then \( m^{-1} < 1 \); thus, elliptic functions whose parameter is
greater than 1 are related to the ones whose parameter
is less than 1. In short, there is no loss of generality
assuming 0 ≤ m ≤ 1.

- Decrease of parameter

\[ \mu = \left(1 - \sqrt{m_1}\right)^2, \quad v = \frac{u}{1 + \sqrt{\mu}}. \] (130)

\[ \text{sn}(u : m) = \frac{(1 + \sqrt{\mu})\text{sn}(v ; \mu)}{1 + \mu \text{sn}^2(v ; \mu)}, \]
\[ \text{cn}(u : m) = \frac{\text{cn}(v ; \mu) \text{dn}(v ; \mu)}{1 + \sqrt{\mu} \text{sn}^2(v ; \mu)}, \]
\[ \text{dn}(u : m) = \frac{1 - \sqrt{\mu} \text{sn}^2(v ; \mu)}{1 + \mu \text{sn}^2(v ; \mu)}. \]

This is Gauss transformation or the descending Landen transformation, which makes elliptic functions to
depend on functions with a smaller parameter.

Note that, making use of the double angle, we may also write

\[ \text{dn}(u : m) = \frac{\sqrt{\mu} \text{cn}(2v ; \mu) + \text{dn}(2v ; \mu)}{1 + \sqrt{\mu}}. \] (131)

There are analogous expressions for the increase of parameter. For a recent study where generalized for-
mules are given, see [16].

- Addition formulae

Complementing previous transformations, we collect also here the addition formulae

\[ \text{sn}(\alpha + \beta) = \frac{\text{sn} \alpha \text{cn} \beta \text{dn} \beta + \text{sn} \beta \text{cn} \alpha \text{dn} \alpha}{1 - m \text{sn}^2 \alpha \text{sn}^2 \beta}, \]
\[ \text{cn}(\alpha + \beta) = \frac{\text{cn} \alpha \text{cn} \beta - \text{sn} \alpha \text{sn} \beta \text{dn} \alpha \text{dn} \beta}{1 - m \text{sn}^2 \alpha \text{sn}^2 \beta}, \]
\[ \text{dn}(\alpha + \beta) = \frac{\text{dn} \alpha \text{dn} \beta - m \text{sn} \alpha \text{sn} \beta \text{cn} \alpha \text{cn} \beta}{1 - m \text{sn}^2 \alpha \text{sn}^2 \beta}, \]

which we have generalized for the new functions; more precisely, this has been done for the 4-EES Mahler sys-
tem.

A précis on Generalized Nambu dynamics

With the aim of making this work as self-contained as possible, we review here, without giving a proof,
some of the basics concepts and facts relative to Nambu
dynamics. All of them and further details may be found in [23,24] and the references therein.

In 1973, Yoichiro Nambu introduced a generalization of the Hamiltonian dynamics by introducing a new
bracket called the Nambu bracket [23]. This new structure generalizes the Poisson bracket and enables to
define the Nambu-Poisson manifolds (see [24]). Here we recall some of the basic ideas about this generalization.

Hamiltonian dynamics takes place on a Poisson manifold, i.e., a pair made of a smooth manifold M
dowered with a bilinear operation \{, \} on \( \mathcal{F}(M) = C^\infty(M) \) satisfying skew-symmetry, the Jacobi identity
and the Leibniz rule. Nambu’s generalization hinges on the introduction of the Nambu-Poisson manifold of
order N. It is a smooth manifold M endowed with a N-multilinear skew-symmetric operation on \( \mathcal{F}(M) \) called
the Nambu bracket of order N, satisfying the fundamental identity, which generalizes the Jacobi identity
and one more property extending the Leibniz rule

(i) The Nambu bracket

\[ \{ , \} : \mathcal{F}(M) \otimes^N \rightarrow \mathcal{F}(M) \] (132)

is a N-multilinear operation.

(ii) Skew-symmetry,

\[ \{ f_1, \ldots, f_N \} = \epsilon(i_1, \ldots, i_N)\{f_{i_1}, \ldots, f_{i_N}\}. \] (133)

Where \( \epsilon \) is the N-dimensional Levi-Civita symbol.

(iii) The fundamental identity holds

\[ \{\{ f_1, \ldots, f_{N+1}, \ldots, f_{2N-1}\} \}
\[ +\{ f_N, \{ f_1, \ldots, f_{N-1}, f_{N+1}\}, f_{N+2}, \]
\[ \ldots, f_{2N-1}\} + \cdots + \{ f_N, \ldots, f_{2N-2}, \]
\[ \{ f_1, \ldots, f_{N-1}, f_{2N-1}\}\}
\[ = \{ f_1, \ldots, f_{N-1}, \{ f_{N}, \ldots, f_{2N-2}\}\}. \] (134)

(iv) \{ , \} Leibniz rule is satisfied in each factor

\[ \{ f_1 f_2, \ldots, f_{N+1}\} = f_1\{ f_2, \ldots, f_{N+1}\} \]
\[ +f_2\{ f_1, \ldots, f_{N+1}\}, \forall f_1, \ldots, f_{N+1} \in \mathcal{F}(M). \] (135)

According to the above properties, Nambu bracket may be rendered in a geometrical sense as a smooth
section of the vector space of exterior N-forms on the
tangent bundle of M, i.e., \( \wedge^N TM \). In other words,
the Nambu bracket is realized as the $N$-contravariant tensor

$$\{f_1, \ldots, f_N\} \equiv \beta(df_1, \ldots, df_N), \quad (136)$$

where $\beta \in \bigwedge^N TM$ is called the Nambu tensor. It is given, in local coordinates $x = (\omega_1, \ldots, \omega_N)$, by the following expression

$$\beta = \sum_{i_1, \ldots, i_N = 1}^N \beta_{i_1, \ldots, i_N}(x) \frac{\partial}{\partial \omega_{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial \omega_{i_N}}, \quad (137)$$

Notice that $\beta$ may be expressed, for suitable local coordinates, by $\beta_{i_1, \ldots, i_N}(x) = \epsilon(i_1, \ldots, i_N)$, where $\epsilon$ is the Levi-Civita tensor. In what follows, we name the $N$-contravariant tensor given by Levi-Civita tensor as the standard Nambu bracket on $\mathcal{F}(M)$, which will be denoted by the usual bracket $\{,\}$. That is to say, the standard Nambu bracket is given by the determinant of the gradients of the functions involved.

Even though Nambu is a generalization of Hamiltonian dynamics, there are also fundamental differences between them. For example, in [15], it is proven that every Nambu-Poisson bracket with $N \geq 3$ is essentially a determinant. This is not true for Poisson ones.

This bracket allows to study the variation in $f \in \mathcal{F}(M)$ on Nambu-Poisson manifolds when it is restricted to be in the intersection of $N - 1$ hypermanifolds $H_i$

$$\frac{df}{dt} = \{f, H_1, \ldots, H_{N-1}\}. \quad (138)$$

In this vein, the Nambu formulation has been applied to Hamiltonian systems with constraints (see [14] and the references therein). It is straightforward to extend the above formalism to the dynamics of points $\omega = (\omega_1, \ldots, \omega_N)$ in the phase space $M$ by means of the Nambu–Hamilton equations of motion as they were first given in [23]

$$\frac{d\omega_i}{dt} = (\omega_i, H_1, \ldots, H_{N-1}) = \sum_{i_1, i_{N-1} \neq i} \epsilon(i, i_1, \ldots, i_{N-1}) \frac{\partial H_{i_1}}{\partial \omega_i} \ldots \frac{\partial H_{N-1}}{\partial \omega_{i_{N-1}}}, \quad (139)$$

where $H_i$ are called the Hamiltonian functions.

Next we gather below some basic features of the Nambu structures, which will be of high relevance in the subsequent development. All of them, except the Remark 8, can be found in [24].

**Theorem 3** (Nambu nested structure) The set of all possible Nambu structures on $M$ is isomorphic to the Grassmann algebra $\bigwedge T M$. Since it is a graded associative algebra, every Nambu structure is considered as an $N$-degree element of $\bigwedge T M$ and by fixing $f_1, \ldots, f_k$ in (136), with $k \leq N - 2$; we are left with a Nambu structure of order $N - k$. More precisely for the case $k = N - 2$, the Nambu structure obtained is a Poisson structure and the fixed integrals $f_1, \ldots, f_{N-2}$ are the Casimirs.

**Theorem 4** ($SL(N, \mathbb{R})$ Nambu bracket invariance) The Nambu bracket is invariant under the action (left or right) of the special linear group $SL(N, \mathbb{R})$. That is to say, let $\phi$ be the action of $SL(N, \mathbb{R})$ on $\mathcal{F}(M) \otimes^N$ given by

$$\phi : SL(N, \mathbb{R}) \times \mathcal{F}(M) \otimes^N \rightarrow \mathcal{F}(M) \otimes^N,$$

$$(A, F) \rightarrow F',$$

where $A \in SL(N, \mathbb{R})$ and $F, F' \in \mathcal{F}(M) \otimes^N$ are the $N$-tuples given by $F = (f_1, \ldots, f_N)$ and $F' = A F = (f'_1, \ldots, f'_N)$. Thus, the following identity holds

$$\{f_1, \ldots, f_N\} = \{f'_1, \ldots, f'_N\} \quad (141)$$

**Theorem 5** (Liouville Condition) The corresponding phase flow on the phase space of the Nambu–Hamilton equations of motion is divergence-free and preserves the standard volume form $d\omega_1 \wedge \ldots \wedge d\omega_N$.

However, the reciprocal of Theorem 5 is not true, i.e., divergence-free systems can be written into the Nambu formalism. Such a statement was made in [11], but the error is shown in [4], see [22] for further details.

**Remark 8** (Geometric interpretation) Let us consider $\mathbb{R}^N$ together with the standard Nambu bracket and $N - 1$ hypermanifolds $M^i_{M_1}$ given by the level sets $H_i = h_i$ of the functions $H_i \in \mathcal{F}(M)$ for $i = 1, \ldots, N - 1$. Then, the Nambu–Hamilton equations of motion given in (138) may be interpreted as a parametrization of the intersection curves resulting from $\bigcap_{i=1}^{N-1} M^i_{M_1}$.
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