Inverting weak random operators. *

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Abstract

We analyze two weak random operators, initially motivated from processes in random environment. Intuitively speaking these operators are ill-defined, but using bilinear forms one can deal with them in a rigorous way. This point of view can be found for instance in the work Skorohod [14], and it remarkably helps to carry out specific calculations. In this paper, we find explicitly the inverse of such weak operators, by providing the forms of the so-called Green kernel. We show how this approach helps to analyze the spectra of the operators. In addition, we provide the existence of strong operators associated to our bilinear forms. Important tools that we use are the Sturm-Liouville theory and the stochastic calculus.

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1 Introduction

There are plenty of examples of probabilistic models where there is an operator that resembles differential operator with coefficients given in terms of the derivative of the Brownian motion. For instance, the so-called stochastic heat equation or the random Schrödinger equations are well known cases studied in the literature. In this paper we work with two examples of

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random operators defined in a weak sense using bilinear forms. One of our aims is to find
the inverse, in a suitable sense, of such weak random operators.

This kind of models are instances of the so-called Schrödinger operators with random
potential. They have been important in theoretical physics, in particular in the theory of
disorder systems, e.g. [10]. The importance of these models is well documented, see for
instance [2].

Let us mention two important examples. In [6] it is consider the Schrödinger operator
with random potential informally given by the expression

\[ Lf(t) = -f''(t) + W'(t)f(t), \quad t \in [0, 1], \]

where \( W' \) is white noise and it can be thought as the derivative of the Brownian motion.
One very first task is to give a proper meaning of the operator \( L \). As shown in [4] such
operator has a discrete spectrum given by a set of eigenvalues. It turns out one can give
expressions of the inverse operator, see [11], which leads to spectral information.

In the context of random processes with random environment an important model is the
so-called Brox diffusion, see [1], amply studied in the literature. This process can be worked
out as a Markov process, and informally speaking the generator has the form

\[ Lf(t) = \frac{1}{2}(-f''(t) + W'(t)f'(t)), \quad t \in \mathbb{R}. \]

It turns out that one can analyze \( L \) by finding its inverse, as done in the companion paper [5],
where a bounded version of the Brox diffusion is studied. Moreover, there is a remarkable
similarity with an operator arising in the theory of random matrices, see [12]. Loosely
speaking, such operator plays the role of the infinite random matrix, and the spectrum helps
to characterize the limiting eigenvalues of a random matrix.

As it is traditionally thought, knowing spectral information of the inverse helps to analyze
the differential operator. As demonstrated in [5], the inverse of \( L \) helps to obtain spectral
information which eventually leads to information of the probability density function. From
a more theoretical point of view, one can see that is possible to deal with the inverse in fairly
friendly way, without making use of machinery such as the theory of distributions. This is
so from well-known tools in the Sturm-Liouville theory and the stochastic calculus.

In this paper the two operators that we consider are given informally by the expressions:

\[ (Lf)(t) := f''(t) - W(t)f'(t) - W'(t)f(t), \]

and

\[ (Lf)(t) := \frac{f''(t)}{2} - \frac{W'(t)f'(t)}{2}. \]
In order to make sense of the term $W'$, we will define these operators in a weak sense using the inner product. In that way we can make sense of the term $\int_a^b W'(t) h(t) dt$ by rewriting it as
\[
\int_a^b h(t) dW(t).
\] (1)

After specifying the domains, our goal is to find the inverse of these two operators defined in weak sense. This inverse operator is called the Green operator. In the classical Sturm-Liouville theory, to tackle this problem one should consider the solutions of the homogeneous problem $Lf = 0$. Here we will also consider the solutions of the homogeneous equation but in a weak sense, again using the inner product. It turns out that the homogeneous solutions are explicit functions of the Brownian motions.

We start in the coming Section 2 with some preliminaries, where we present the concept of a weak operator. In Section 2.1 we also mention some ideas on strong operators associated to bilinear forms. Then in Section 3.2 we deal with the first weak operator and find explicitly the solutions of $Lf = 0$. These solutions will help to construct the green operator associated. In Section 3.1 we mention how to find the strong operator associated to the weak random operator. In a similar fashion, in Section 4 we work with the second weak operator, and we also find explicit solutions of the homogeneous equation using approximations of the Brownian motion.

2 Preliminaries

We will work with two weak random operators whose domain are functions defined on an interval $[a, b]$. More precisely, the domain is the set of functions $f \in L_2[a, b]$ absolutely continuous that satisfies the Dirichlet conditions $f(a) = 0 = f(b)$. Our first goal is to give the proper definitions of the operators that we work using bilinear forms. Next we find solutions for the homogeneous equation which eventually will lead to the inverse operator.

The first operator that we consider has the following formal expression:
\[
(Lf)(t) = f''(t) - W(t)f'(t) - W'(t)f(t).
\]
where $W := \{W(t) : t \in [a, b]\}$ is a Brownian motion, and $W'$ denotes its derivative, sometimes called the white noise.

The second operator that we consider can be expressed as follows:
\[(Lf)(t) = \frac{f''(t)}{2} - \frac{W'(t)f'(t)}{2}.\]

A natural space to work with these operators is the Hilbert space \(L_2[a, b]\) with its inner product
\[\langle f, g \rangle = \int_a^b f(x)g(x)dx.\]

To define properly the domain of our operators we need to introduce the following Sobolev space:
\[H_1 := \{h \in L_2[a, b] : h \text{ is absolutely continuous, } h(a) = h(b) = 0\}.

Note that \(H_1\) is Hilbert space with the norm
\[\|f\|_1^2 := \int_a^b [f(x)]^2dx + \int_a^b [f'(x)]^2dx,\] (2)
and the corresponding inner product.

The idea to define weak operators is to think of an operator \(L\) by describing its effect through the inner product, thus we will propose a bilinear form. More specifically:
\[\langle Lf, h \rangle = \int_a^b Lf(t)h(t)dt, \text{ for all } f, h \in H_1.\] (3)

We take this point of view from the work of Anatolii Vladimirovich Skorohod, see [14].

**Definition 1** Consider the mapping \(\varepsilon(f, g)\) defined on a Hilbert space with the following conditions:

1. \(\varepsilon(\alpha_1f_1 + \alpha_2f_2, \beta_1g_1 + \beta_2g_2) = \sum_{i,j=1}^2 \alpha_i \beta_j \varepsilon(f_i, g_j),\)

2. \(\varepsilon(f_n, g_n)\) converges to \(\varepsilon(f, g)\) in probability as \(f_n \to f\) and \(g_n \to g\).

We say that \(\varepsilon\) defines a weak random operator \(L\), through the expression \(<Lf, g> := \varepsilon(f, g)\).

On the other hand, as we mentioned in the Introduction, we need to find the solutions of the homogeneous equation \(Lf = 0\). So, if \(L\) is a weak random operator we have the following definition of solving \(Lf = 0\).
Definition 2 We say that a stochastic process \( \{u(t) : t \in [a, b]\} \) is a solution of the equation \( Lf = 0 \), if for all \( h \in H_1 \),
\[
(Lu, h) = \varepsilon(u, h) = 0 \quad \text{almost surely.}
\] (4)

It turns out that it is possible to find solutions of this problem for the operators we consider.

2.1 On strong operators

In some cases it is possible to find an operator in strong sense associated to the bilinear form. Generally speaking, such situation occurs if \( \varepsilon \) is what it is called a symmetric closed lower semibounded bilinear form on a Hilbert space with inner product \( \langle \cdot, \cdot \rangle \). The reader can see [13] as a general reference, in particular Chapter 10. The two examples that we will study are not symmetric, however the two bilinear forms that we consider can be written as \( \varepsilon = \varepsilon_1 + \varepsilon_2 \), where \( \varepsilon_1 \) is symmetric and \( \varepsilon_2 \) is coercive. We can use this decomposition to find a strong operator associated to \( \varepsilon \).

More precisely, on a linear subspace \( D \) of the Hilbert space \( H \) with norm \( \| \cdot \| \), a symmetric bilinear form \( \varepsilon_1 \) is lower semibounded if there exists a constant \( C \) such that \( \varepsilon_1(f, f) \geq C\|f\|^2 \) for all \( f \in D \). It is also said that \( \varepsilon_1 \) is closed if \( D \) is complete with the norm
\[
\|f\|_{\varepsilon_1} := \left[ \varepsilon_1(f, f) + (1 - C)\|f\|^2 \right]^{\frac{1}{2}}.
\] (5)

Then, we will be able to appeal to the Corollary 10.8 in [13] to show that \( \varepsilon_1 \) have associated a self-adjoint operator, i.e. there exists an operator \( L_1 \) such that \( \varepsilon_1(f, g) = \langle L_1 f, g \rangle \).

On the other hand, for the bilinear form \( \varepsilon_2 \) we will use the Lax-Milgram theorem. To use this theorem we need to show that \( \varepsilon_2 \) is bounded and coercive, i.e. if there are two constants \( C > 0 \) and \( c > 0 \) such that \( |\varepsilon_2(f, f)| \leq C\|f\|^2 \) and \( |\varepsilon_2(f, f)| \geq c\|f\|^2 \), respectively. If a bilinear form satisfies the previous properties on the Hilbert space \( H \) then there exists a operator \( L_2 \) such that \( \varepsilon_2(f, g) = \langle L_2 f, g \rangle \), where \( \langle \cdot, \cdot \rangle \) is the inner product on \( H \).

Then we obtain that
\[
\varepsilon(f, g) = \varepsilon_1(f, g) + \varepsilon_2(f, g) = \langle L_1 f, g \rangle + \langle L_2 f, g \rangle = \langle (L_1 + L_2) f, g \rangle.
\]

The previous equality shows that the bilinear form \( \varepsilon \) has associated the operator \( L_1 + L_2 \).

Let us stress out that although it becomes feasible to give this association of a strong operator, in this paper our main goal is to study the weak random operator. This comes from the interest to carry out calculations relying on the bilinear forms alone.
3 With random potential and random coefficient

In this section we consider the operator with the following formal expression:

\[(Lf)(t) = f''(t) - W(t)f'(t) - W'(t)f(t).\] (6)

We can consider (6) in the following weak sense, using the inner product

\[
\langle Lf, h \rangle := \int_a^b f''(t)h(t)dt - \int_a^b f'(t)W(t)h(t)dt - \int_a^b f(t)h(t)dW(t).
\] (7)

Now, we use integration by parts in the first term of (7) and the Itô’s formula in the third term of (7) to obtain the following definition.

**Definition 3** For any pair \(f, h \in H_1\), we define the bilinear form \(\varepsilon\) as

\[
\varepsilon(f, h) := -\int_a^b f'(t)h'(t)dt + \int_a^b f(t)h'(t)W(t)dt.
\] (8)

As we mentioned in the previous section, we consider this bilinear form as a weak random operator \(L\) through the expression \(\langle Lf, g \rangle := \varepsilon(f, g)\). We do not go into details, but it is possible to show that \(L\), i.e. \(\varepsilon\), fits into Definition 1.

Before we study the inverse operator of \(L\), let us mention how we can find an operator associated to the bilinear.

### 3.1 A strong operator

In order to find a strong operator associated to the bilinear form \(\varepsilon\) we carry on the following decomposition.

Notice that \(\varepsilon = \varepsilon_1 + \varepsilon_2\) where

\[
\varepsilon_1(f, h) := -\frac{1}{2} \int_a^b f'(t)h'(t)dt + \frac{1}{2} \int_a^b f(t)h'(t)W(t)dt + \frac{1}{2} \int_a^b f'(t)h(t)W(t)dt,
\]

and

\[
\varepsilon_2(f, h) := -\frac{1}{2} \int_a^b f'(t)h'(t)dt + \frac{1}{2} \int_a^b f(t)h'(t)W(t)dt - \frac{1}{2} \int_a^b f'(t)h(t)W(t)dt.
\]

One can see that \(\varepsilon_1\) is symmetric form on \(H_1\) but \(\varepsilon_2\) is not symmetric on \(H_1\).
Let us see that $\varepsilon_1$ is lower semibounded and closed bilinear form. Let $M := \max_{a \leq s \leq b} |W(s)|$. Using $|ab| \leq \frac{a^2 + b^2}{2}$, then we have

$$
\varepsilon_1(f, f) = -\frac{1}{2} \int_a^b [f'(t)]^2 dt + \int_a^b f(t)f'(t)W(t)dt \\
\geq -\frac{1}{2} \int_a^b [f'(t)]^2 dt - M \int_a^b |f(t)f'(t)| dt \\
\geq -\frac{1}{2} \int_a^b [f'(t)]^2 dt - \frac{M}{2} \int_a^b [f(t)]^2 dt - \frac{M}{2} \int_a^b [f'(t)]^2 dt \\
\geq C\|f\|_1,
$$

where $C$ is a constant that depends on $W$, and $\| \cdot \|_1$ is defined in (2).

Then $\varepsilon_1$ is a semibounded form on $H_1$. Let us now see that $\varepsilon_1$ is closed, this happens if the Sobolev space $H_1$ is complete with the norm $\| \cdot \|_{\varepsilon_1}$. Indeed, this is the case because $\| \cdot \|_{\varepsilon_1}$ is equivalent to the norm $\| \cdot \|_1$ of $H_1$. This implies that $\varepsilon_1$ is a closed form on $H_1$. Therefore, using the Corollary 10.8 from [13], there exists an operator $L_1$ in strong sense with domain $H_1$ associated with the bilinear form $\varepsilon_1$, i.e. $\varepsilon_1(f, g) = \langle L_1 f, g \rangle_1$, where $\langle \cdot, \cdot \rangle_1$ is the inner product associated with the norm $\| \cdot \|_1$.

On the other hand, we use the Lax-Milgram theorem to show that there exists an operator $L_2$ such that $\varepsilon_2(f, g) = \langle L_2 f, g \rangle_1$. To do that, we show that $\varepsilon_2$ is bounded and coercive. We have

$$
|\varepsilon_2(f, f)| = \frac{1}{2} \int_a^b [f'(t)]^2 dt \leq C\|f\|_1^2.
$$

The previous inequality shows that $\varepsilon_2$ is bounded. Let us see why it is coercive. To do that, we use the Poincaré inequality: $\|f\| \leq K\|f'\|$ for some constant $K > 0$ and for all $f \in H_1$. Then

$$
|\varepsilon_2(f, f)| = \frac{1}{2} \int_a^b [f'(t)]^2 dt \\
= \frac{1}{4}\|f''\|^2 + \frac{1}{4}\|f'\|^2 \\
\geq \frac{1}{4}\|f''\|^2 + \frac{1}{4\cdot K}\|f\|^2 \\
\geq c\|f\|_1^2.
$$
Thus, $\varepsilon_2$ is coercive. Therefore, using the Lax-Milgram theorem there exists an operator $L_2$ such that $\varepsilon_2(f, g) = \langle L_2 f, g \rangle_1$.

Then our bilinear form $\varepsilon$ is associated with the operator $L_1 + L_2$ with respect to the inner product $\langle \cdot, \cdot \rangle_1$.

### 3.2 The Green operator

Now, we want to construct the Green operator associated to the weak random operator $L$ from the Definition (8). To this end, we need to find two solutions linearly independent of the homogeneous equation. Intuitively we have

$$f''(t) - W(t)f'(t) - W'(t)f(t) = 0.$$  

This equation can be rewritten as

$$f''(t) = [W(t)f(t)]'.$$

Moreover, integrating both side we arrive at

$$f'(t) = W(t)f(t) + C,$$  

where $C$ is a constant.

This equation is easy to solve, and we exhibit the solutions in the following theorem. However, we rigourously verify that the solutions satisfies the equation $Lf = 0$.

**Theorem 4** Two linearly independent solutions of the problem $Lf = 0$ are the following

$$u(t) := \frac{\int_a^t e^{\int_a^s W(r)dr} ds}{\int_a^b e^{\int_a^r W(r)dr} dr},$$  \hspace{1cm} (9)

$$v(t) := \frac{\int_t^b e^{\int_s^b W(r)dr} ds}{\int_a^b e^{\int_a^r W(r)dr} dr},$$  \hspace{1cm} (10)

Furthermore, they satisfy $u(a) = 0$, $u(b) = 1$, $v(a) = 1$ and $v(b) = 0$. 

8
Proof. Let us verify that $u$ is solution. For $v$ is similar. According to the Definition 3 we need to show that $\langle Lu, h \rangle = 0$ for all $h \in H_1$, i.e.

$$\int_a^b u'(t)h'(t)dt + \int_a^b u(t)h'(t)W(t)dt = 0. \quad (11)$$

Note that

$$u'(t) = \frac{W(t)e^{\int_a^t W(s)ds} \int_a^t e^{-\int_a^r W(r)dr}ds + 1}{e^{\int_a^b W(s)ds} \int_a^b e^{-\int_a^r W(r)dr}dr}. \quad (12)$$

Substituting (12) and the definition of $u$ in (11), we end up with $\langle Lu, h \rangle = 0$. □

Using previous two solutions, we construct the Green operator. The following theorem shows the construction.

**Theorem 5** Let $u, v$ two solution of $Lf = 0$, such that $u(a) = 0$ and $u(b) = 1$ always, and $v(a) = 1$ and $v(b) = 0$ always. The stochastic Green operator associated to $L$ is given by

$$(Tf)(t) := \int_a^b G(t, s)f(s)ds, \quad (13)$$

where

$$G(t, s) := \begin{cases} u(t)v(s), & a \leq s \leq t \leq b; \\ \frac{u(s)v(t)}{\alpha(s)}, & a \leq t \leq s \leq b. \end{cases}$$

and

$$\alpha(t) := u'(t)v(t) - v'(t)u(t).$$

This operator $T$ is the right inverse of the operator $L$ in the sense that for all $h \in H_1$

$$\varepsilon(Tf, h) = \langle LTf, h \rangle = \langle f, h \rangle \text{ almost surely.}$$

**Proof.** We want to proof that $\langle L(Tf), h \rangle = \langle f, h \rangle$. First note that

$$(Tf)(t) = u(t) \int_a^t \frac{v(s)f(s)}{\alpha(s)}ds + v(t) \int_t^b \frac{u(s)f(s)}{\alpha(s)}ds. \quad (14)$$
Calculating the derivative of \((T f)\) and simplifying yield

\[
\frac{d((T f)(t))}{dt} = u'(t) \int_a^t \frac{v(s)f(s)}{\alpha(s)} ds + v'(t) \int_t^b \frac{u(s)f(s)}{\alpha(s)} ds.
\] (15)

From the Definition 8 we have

\[
\langle L(T f), h \rangle = - \int_a^b (T f)'(t)h'(t)dt + \int_a^b (T f)(t)h'(t)W(t)dt.
\] (16)

After plugging (15) into (16) one arrives at

\[
\langle L(T f), h \rangle = - \int_a^b u'(t) \left[ \int_a^t \frac{v(s)f(s)}{\alpha(s)} ds \right] h'(t)dt + \int_a^b u(t) \left[ \int_t^b \frac{v(s)f(s)}{\alpha(s)} ds \right] h'(t)W(t)dt
- \int_a^b v'(t) \left[ \int_t^b \frac{u(s)f(s)}{\alpha(s)} ds \right] h'(t)dt + \int_a^b v(t) \left[ \int_t^b \frac{u(s)f(s)}{\alpha(s)} ds \right] h'(t)W(t)dt.
\] (17)

Now, we add and subtract in (17) the following three terms:

\[
\int_a^b \frac{u'(t)v(t)f(t)h(t)}{\alpha(t)} dt,
\] (18)

\[
\int_a^b \frac{u(t)v'(t)f(t)h(t)}{\alpha(t)} dt,
\] (19)

\[
\int_a^b \frac{u(t)v(t)f(t)h(t)W(t)}{\alpha(t)} dt.
\] (20)

Hence, after calculations,

\[
\langle L(T f), h \rangle = - \int_a^b u'(t) \left[ h(t) \int_a^t \frac{v(s)f(s)}{\alpha(s)} ds \right]' dt + \int_a^b u(t) \left[ h(t) \int_t^b \frac{v(s)f(s)}{\alpha(s)} ds \right]' W(t)dt
- \int_a^b v'(t) \left[ h(t) \int_t^b \frac{u(s)f(s)}{\alpha(s)} ds \right] dt + \int_a^b v(t) \left[ h(t) \int_t^b \frac{u(s)f(s)}{\alpha(s)} ds \right]' W(t)dt
+ \int_a^b \frac{u'(t)v(t)f(t)h(t)}{\alpha(t)} dt - \int_a^b \frac{v'(t)u(t)f(t)h(t)}{\alpha(t)} dt.
\] (21)
Using the fact that \( Lu = 0 \) and \( Lv = 0 \), we obtain the result. ■

One can see that almost surely \( T \) is a compact operator, thus it has a discrete spectrum. It means that the relation \( Te = \lambda e \) holds for some eigenvalue \( \lambda \) and eigenfunction \( e \). After taking \( \langle LTe, h \rangle \) we arrive to the equation

\[
\langle Le, h \rangle = \langle e/\lambda, h \rangle.
\]

Therefore

**Corollary 6** Almost surely, the weak operator \( L \) has a discrete spectrum in the sense that for all \( h \in H_1 \), the relation

\[
\langle Le, h \rangle = \langle \lambda e, h \rangle
\]

holds for a countable number of \( \lambda \) and \( e \in H_1 \).

### 4 With random potential

Informally speaking, we consider the following stochastic operator

\[
(Lf)(t) = \frac{f''(t)}{2} - \frac{W'(t)f'(t)}{2}.
\]

Taking into account equation (1) and (3), we define (22) in the following weak sense

\[
\langle Lf, h \rangle := \int_a^b \frac{f''(t)h(t)}{2}dt - \int_a^b \frac{f'(t)h(t)}{2}dW(t). \tag{23}
\]

We go an step further and instead of (23), we use integration by parts to obtain the following definition.

**Definition 7** For any pair \( f, h \in H_1 \), we define the bilinear form \( \varepsilon \) as

\[
\varepsilon(f, h) := -\int_a^b \frac{f'(t)h(t)}{2}dt - \int_a^b \frac{f'(t)h(t)}{2}dW(t), \tag{24}
\]

and \( L \) through \( \langle Lf, g \rangle = \varepsilon(f, g) \).

As we mentioned in previous section, one can check that \( L \) satisfies the properties in Definition 1.
4.1 A strong operator

To talk about the strong operator associated to $\varepsilon$, in this case we consider the Sobolev space

$$W^{2,2} := \{ h \in L_2[a, b] : h', h'' \in L_2[a, b], h(a) = h(b) = 0 \},$$

with the norm

$$\| f \|_2^2 := \int_a^b |f(x)|^2 dx + \int_a^b |f'(x)|^2 dx + \int_a^b |f''(x)|^2 dx. \quad (25)$$

We want to prove the existence of an associated operator. Indeed, using the Itô's formula, we obtain for $f \in W^{2,2}$,

$$\varepsilon(f, h) = -\frac{1}{2} \int_a^b f'(t)h'(t)dt + \int_a^b \frac{1}{2} [f''(t)h(t) + f'(t)h'(t)] W(t)dt.$$

Notice that $\varepsilon = \varepsilon_1 + \varepsilon_2$, where

$$\varepsilon_1(f, h) := -\int_a^b \frac{f'(t)h'(t)}{4} dt + \int_a^b \frac{f''(t)h(t)W(t)}{4} dt + \int_a^b \frac{f(t)h''(t)W(t)}{4} dt + \int_a^b \frac{f'(t)h'(t)}{2} W(t)dt,$$

and

$$\varepsilon_2(f, h) := -\int_a^b \frac{f'(t)h'(t)}{4} dt + \frac{1}{4} \int_a^b f''(t)h(t)W(t)dt - \frac{1}{4} \int_a^b f(t)h''(t)W(t)dt.$$

Let us see that $\varepsilon_1$ is symmetric lower semibounded and closed bilinear form on $W^{2,2}$. Take $f \in W^{2,2}$, and let $M := \max_{a \leq s \leq b} |W(s)|$, then

$$\varepsilon_1(f, f) = -\int_a^b \frac{1}{4} [f'(t)]^2 dt + \int_a^b \frac{1}{2} [f''(t)f(t) + [f'(t)]^2] W(t)dt \geq -\int_a^b \frac{1}{4} [f'(t)]^2 dt - \frac{M}{2} \left[ \int_a^b |f''(t)f(t)| dt + \int_a^b [f'(t)]^2 dt \right] \geq -\frac{1}{2} \int_a^b [f'(t)]^2 dt - \frac{M}{2} \left[ \frac{1}{2} \int_a^b [f''(t)]^2 dt + \frac{1}{2} \int_a^b [f(t)]^2 dt + \int_a^b [f'(t)]^2 dt \right] \geq C \left[ \int_a^b |f(t)|^2 dt + \int_a^b [f'(t)]^2 dt + \int_a^b [f''(t)]^2 dt \right],$$

where $C$ is a constant depending $W$. Then we have that the bilinear form $\varepsilon_1$ satisfies

$$\varepsilon_1(f, f) \geq C \|f\|_2^2,$$
which concludes that $\varepsilon_1$ is a semibounded form on the Sobolev space $W^{2,2}$.

Now we point out why $\varepsilon_1$ is closed. This is the case because the norm $\| \cdot \|_2$, which makes $W^{2,2}$ complete, is actually equivalent to the norm $\| \cdot \|_{\varepsilon_1}$, as one can check it. This implies that $\varepsilon_1$ is a closed form on $W^{2,2}$, and using the Corollary 10.8 from [13], there exists an operator $L_1$ associated with the bilinear form $\varepsilon_1$, that is, such that $\varepsilon_1(f, g) = \langle L_1 f, g \rangle$, where $\langle \cdot, \cdot \rangle_2$ is the inner product associated with the norm $\| \cdot \|_2$.

For $\varepsilon_2$ we apply the Lax-Milgram theorem. As in Section 3.1, one can see that $\varepsilon_2$ is bounded and coercive. Then we obtain that there exists an operator $L_2$ such that $\varepsilon_2(f, h) = \langle L_2 f, h \rangle$. Then the bilinear form $\varepsilon$ is associated with the operator $L_1 + L_2$ using the inner product of $W^{2,2}$.

4.2 The Green operator

Our aim is to construct the so-called Green operator associated to the weak random operator $L$ from the Definition (24). To do this task, we notice that we need to find two linearly independent solutions of the problem $L f = 0$.

It happens that the two linearly independent solutions always exist; we will prove this fact later on. For the moment, let us suppose that we already have the two solutions $u$ and $v$ of the homogeneous equation. With these functions we are going to construct an operator $T$, called the Green operator, which will be the inverse operator of the weak random operator $L$.

The following theorem shows how to use the two solutions of the homogeneous problem to construct $T$. We take the idea of this constructions from the Sturm-Liouville theory.

**Theorem 8** Let $u, v$ be solutions of $L f = 0$, such that $u(a) = 0$ and $u(b) = 1$ a.s., and $v(a) = 1$ and $v(b) = 0$ a.s. The stochastic Green operator associated to the weak random operator $L$ is given by

$$\langle T f \rangle(t) := \int_a^b G(t, s) f(s) ds,$$

where

$$G(t, s) := \begin{cases} 
2u(t)v(s), & a \leq s \leq t \leq b; \\
\frac{2u(t)v(s)}{\alpha(s)}, & a \leq t \leq s \leq b.
\end{cases}$$
and
\[ \alpha(t) := u'(t)v(t) - v'(t)u(t). \]

The operator \( T \) in (26) is the right inverse of \( L \) in the sense that for all \( h \in H_1 \)
\[ \varepsilon(Tf, g) = \langle LTf, h \rangle = \langle f, h \rangle \text{ almost surely.} \]

**Proof.** Let \( u, v \) be solutions of \( Lf = 0 \), such that \( u(a) = 0 \) and \( u(b) = 1 \) always, and that \( v(a) = 1 \) and \( v(b) = 0 \) always as well.

Note that
\[ (Tf)(t) = 2u(t) \int_a^t \frac{v(s)f(s)}{\alpha(s)} ds + 2v(t) \int_t^b \frac{u(s)f(s)}{\alpha(s)} ds. \]  

(27)

On calculating the derivative of (27) we obtain
\[ \frac{d[(Tf)(t)]}{dt} = 2u'(t) \int_a^t \frac{v(s)f(s)}{\alpha(s)} ds + 2u(t)v(t)f(t) + 2v'(t) \int_t^b \frac{u(s)f(s)}{\alpha(s)} ds - \frac{2u(t)v(t)f(t)}{\alpha(t)}. \]  

(28)

Note that in the above expression the first and last term are canceled. Now, by using Definition 7
\[ \langle L(Tf), h \rangle = \frac{-1}{2} \left[ \int_a^b (Tf(t))'h'(t)dt + \int_a^b (Tf(t))'h(t)dW(t) \right]. \]

(29)

Inserting (28) into (29) we arrive at
\[ \langle L(Tf), h \rangle = - \int_a^b u'(t) \left[ \int_a^t \frac{v(s)f(s)}{\alpha(s)} ds \right] h'(t)dt - \int_a^b u'(t) \left[ \int_t^b \frac{u(s)f(s)}{\alpha(s)} ds \right] h(t)dW(t) \]
\[ - \int_a^b v'(t) \left[ \int_t^b \frac{u(s)f(s)}{\alpha(s)} ds \right] h'(t)dt - \int_a^b v'(t) \left[ \int_t^b \frac{u(s)f(s)}{\alpha(s)} ds \right] h(t)dW(t) \]

(30)

Therefore, if we add and subtract in (30) the following two terms
\[ \int_a^b u'(t)v(t)f(t)h(t) dt, \quad \text{and} \quad \int_a^b u(t)v'(t)f(t)h(t) dt, \]

14
and we use the fact that
\[
\left[ h(t) \int_a^t \frac{v(s)f(s)}{\alpha(s)} \, ds \right]' = h'(t) \int_a^t \frac{v(s)f(s)}{\alpha(s)} \, ds + h(t) \frac{v(t)f(t)}{\alpha(t)},
\]
(31)
we arrive at
\[
\langle L(T f), h \rangle = - \int_a^b u'(t) \left[ h(t) \int_a^t \frac{v(s)f(s)}{\alpha(s)} \, ds \right]' \, dt - \int_a^b u'(t) \left[ h(t) \int_a^t \frac{v(s)f(s)}{\alpha(s)} \, ds \right] dW(t) - \int_a^b v'(t) \left[ h(t) \int_t^b \frac{u(s)f(s)}{\alpha(s)} \, ds \right]' \, dt - \int_a^b v'(t) \left[ h(t) \int_t^b \frac{u(s)f(s)}{\alpha(s)} \, ds \right] dW(t),
\]
(32)
Now, using the fact that \( u \) and \( v \) are solutions of \( Lf = 0 \) in the sense of Definition 2, we see that only the last two terms in (32) survive. Thus we finally arrive at
\[
\langle L(T f), h \rangle = \int_a^b \frac{u'(t)v(t)f(t)h(t)}{\alpha(t)} \, dt - \int_a^b \frac{v'(t)u(t)f(t)h(t)}{\alpha(t)} \, dt.
\]
where we have substitute the very definition of \( \alpha \). This concludes the proof. ■

As previous section, since \( T \) is compact, we have that

**Corollary 9** The operator \( L \) has a discrete spectrum in a weak sense.

Now, in order to use previous theorem, we need to find the two solutions of \( Lf = 0 \). We do so by using approximations of Brownian motion.

First, to obtain intuitively such so solutions we consider the followings approximations of \( W \),
\[
W_n(t) := n \left[ \left( \frac{j + 1}{n} - t \right) W \left( \frac{j}{n} \right) + \left( t - \frac{j}{n} \right) W \left( \frac{j + 1}{n} \right) \right]
\]
where \( t \in \left[ \frac{j}{n}, \frac{j+1}{n} \right] \), and \( j = 0, \pm 1, \pm 2 \ldots \) Therefore, the random function \( W_n \) is almost everywhere differentiable.
Then the following equation is valid for almost every \( t \in [a, b] \)
\[
U''_n(t) = W''_n(t)U''_n(t).
\]
We want to use \( U_n(t) \) to find heuristically a solution of
\[
U''(t) = W'(t)U'(t).
\]
We consider the change of variable \( Z_n(t) := U'_n(t) \). Then we obtain the new equation
\[
Z''_n(t) = W'_n(t)Z_n(t). \tag{33}
\]
From the Corollary of Theorem 7.3 of [7], we have that there exist a sequence \( Z_n(t) \) of solutions of (33) such that, with probability one
\[
Z_n(t) \rightarrow Z(t), \quad \text{as} \quad n \rightarrow \infty, \tag{34}
\]
where \( Z(t) \) is solution of the stochastic differential equation
\[
dZ(t) = Z(t)dW(t). \tag{35}
\]
Then we obtain that with probability one
\[
U'_n(t) \rightarrow Z(t), \quad \text{as} \quad n \rightarrow \infty. \tag{36}
\]
On the other hand, the equation (35) has unique solution, and this solution is
\[
Z(t) = e^{W(t)-\frac{1}{2}t}. \tag{37}
\]
Hence
\[
U'_n(t) \rightarrow e^{W(t)-\frac{1}{2}t}, \quad \text{as} \quad n \rightarrow \infty. \tag{38}
\]
This implies that
\[
U_n(t) \rightarrow \int_a^t e^{W(s)-\frac{1}{2}s}ds, \quad \text{as} \quad n \rightarrow \infty. \tag{39}
\]
In the following theorem we verify rigourously that \( u(t) := C \cdot \int_a^t e^{W(s)-\frac{1}{2}s}ds \) satisfies \( Lu = 0 \), where \( C \) is an appropriate constant. We also consider other solution \( v \) that we need to construct the Green operator.
Theorem 10. Two linearly independent solutions of the problem $Lf = 0$ are the following integrals of Geometric Brownian motion

$$u(t) := \frac{\int_t^b e^{W(s)-\frac{s}{2}} ds}{\int_a^b e^{W(s)-\frac{s}{2}} ds},$$  \hspace{1cm} (40)$$

$$v(t) := \frac{\int_t^b e^{W(s)-\frac{s}{2}} ds}{\int_a^b e^{W(s)-\frac{s}{2}} ds},$$  \hspace{1cm} (41)$$

Furthermore, they satisfy $u(a) = 0$, $u(b) = 1$, $v(a) = 1$ and $v(b) = 0$.

Proof. We verify that $u$ is solution of $Lf = 0$. For $v$ is similar. To do that, according to Definition 7 we want to show that $\langle Lu, h \rangle = 0$ for all $h \in H_1$, i.e.

$$\int_a^b \frac{u'(t)h'(t)}{2} dt + \int_a^b \frac{u'(t)h(t)}{2} dW(t) = 0.$$  \hspace{1cm} (42)$$

From the definition of $u$ in (40), we have

$$u'(t) = \frac{e^{W(t)-\frac{t}{2}}}{\int_a^b e^{W(s)-\frac{s}{2}} ds}. \hspace{1cm} (43)$$

Let $\beta := \left[ \int_a^b e^{W(s)-\frac{s}{2}} ds \right]^{-1}$, then

$$\langle Lu, h \rangle = -\frac{\beta}{2} \left[ \int_a^b e^{W(t)-\frac{t}{2}} h'(t) dt + \int_a^b e^{W(t)-\frac{t}{2}} h(t) dW(t) \right]. \hspace{1cm} (44)$$

On the other hand, applying the Itô’s formula we obtain

$$\int_a^b e^{W(s)-\frac{s}{2}} h(s) dW(s) = h(b)e^{W(b)-\frac{b}{2}} - h(a)e^{W(a)-\frac{a}{2}} - \int_a^b e^{W(s)-\frac{s}{2}} h'(s) ds.$$  \hspace{1cm} (45)$$
Substituting (45) in (44), and recalling that \( h \in H_1 \), we arrive at

\[
\langle Lu, h \rangle = \frac{-\beta}{2} \left[ \int_a^b e^{W(t) - \frac{t}{2}} h'(t) dt + h(b)e^{W(b) - \frac{b}{2}} - h(a)e^{W(a) - \frac{a}{2}} - \int_a^b e^{W(t) - \frac{t}{2}} h'(t) dt \right] = 0.
\]

\[(46)\]

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