Modified $U(1)$ lattice gauge theory: towards realistic lattice QED

V.G. Bornyakov $^1$, V.K. Mitrjushkin $^2$, M. Müller-Preussker $^2$.

$^1$ Institute for High Energy Physics
142284 Protvino (Moscow Region), Russia
$^2$ Humboldt-Universität zu Berlin, Fachbereich Physik
Institut für Elementarteilchenphysik
Invalidenstr. 110, D-1040 Berlin, Germany

Abstract

We study properties of the compact 4D $U(1)$ lattice gauge theory with monopoles removed. Employing Monte Carlo simulations we calculate correlators of scalar, vector and tensor operators at zero and nonzero momenta $\vec{p}$. We confirm that the theory without monopoles has no phase transition, at least, in the interval $0 < \beta \leq 2$. There the photon becomes massless and fits the lattice free field theory dispersion relation very well. The energies of the $0^{++}$, $1^{+-}$ and $2^{++}$ states show a rather weak dependence on the coupling in the interval of $\beta$ investigated, and their ratios are practically constant. We show also a further modification of the theory suppressing the negative plaquettes to improve drastically the overlap with the lowest states (at least, for $J = 1$).

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‡Permanent adress: Joint Institute for Nuclear Research, Dubna, Russia
1 Introduction

Despite the big efforts invested last years into the nonperturbative study of QED (for reviews see, for instance, [1, 2]) we still have no clear understanding of this theory. For the standard compact formulation there is a strong indication for a first order phase transition, thus leaving no room for the continuum limit. The noncompact formulation seemed to be more promising because the (chiral) phase transition turned out to be of second order. However, the noncompactness is realized only in the pure gauge part of the action, while in the fermionic part a compact $U(1)$ phase remains. This part of the action will eventually induce the compact term back into the action, thus living us with the question of selfconsistency of the (so-called) noncompact theory. Moreover, the compact part of the action is responsible for the appearance of monopoles, which are lattice artifacts and are supposed to be at least in part the driving force of the chiral phase transition. If the latter is correct then the chiral phase transition in the noncompact QED might not have a continuum analogue [3].

The problem of lattice artifacts should be taken very seriously in lattice calculations. Within the Monte Carlo approach, in order to calculate the average of any field operator one generates a sequence of (equilibrium) gauge field configurations: \( \{ U_{\text{link}}^{(1)} \}, \{ U_{\text{link}}^{(2)} \}, \ldots \) with the statistical weight \( P(U_i) \sim \exp \left[ -S(U_i) \right] \), where \( S(U_i) \) is the Euclidean lattice discretized action, and averages over all of them. The crucial question in this approach is, whether in a certain range of coupling(s) these field configurations can provide an adequate representation of continuum physics. It is a well-known fact that artifacts can change the structure of a theory drastically. One example of the influence of lattice artifacts is provided by the study of topological properties of gauge theories. Small–scale fluctuations, i.e., the fluctuations living on the scale size of, say, one lattice spacing, and carrying a non-trivial topological charge (dislocations) can lead to the divergence of the topological susceptibility in the continuum limit (at least for the geometric definition of the topological charge). A special source of troubles are field configurations containing negative plaquettes: \( \frac{1}{2} \text{Tr} U_P \simeq -1 \) (for \( SU(2) \)) [1, 2, 3]. A similar situation takes place for the \( CP^{n-1} \) model with \( n \leq 3 \) [4]. The fact that singular configurations may be important in quantum field theory was demonstrated also for the three–dimensional non–linear \( O(3) \) model [5] and for the compact Sine-Gordon model [6]. It was speculated that point–like singularities (topological defects) are the sources of a topological anomaly, and a new coupling constant controls their behaviour. This ‘hidden’ coupling constant without making important contributions to the short–distance behaviour becomes crucial in understanding the infrared structure (long–distance behaviour) of the theory. Rough fluctuations can also exhibit themselves in the form of \( Z_N \)– strings and \( Z_N \)– monopoles [6–15] (\( Z_N \)– artifacts). In the paper [16] the influence of small–size \( Z_2 \)–artifacts (negative plaquettes) on ‘physical’ values was studied for pure gauge \( SU(2) \) theory. It was shown that these \( Z_2 \)–artifacts can strongly influence large–scale objects, e.g.,
Wilson loops and their ratios. In this connection it could be instructive to note the following. The number of artifacts (e.g., negative plaquettes) $N_-$ decreases exponentially with increasing $\beta$. But this in itself does not mean that they become unimportant at large $\beta$. They can still strongly influence any lattice average $\langle O(U) \rangle(\beta)$ (say, Wilson loops) decaying faster then $N_-(\beta)$ with increasing $\beta$, and, therefore, even a small admixture of these rapidly varying fields can become competitive.

The question of contamination from lattice artifacts should be addressed also to the pure gauge $U(1)$ lattice theory which is expected to be the free photon’s theory in the continuum limit. During last years this theory was rather well studied (for a noncomplete list of references see, for instance, [17-28]). On the lattice the pure gauge $U(1)$ theory in four dimensions with Wilson action has two phases separated by a first order phase transition. In the strong coupling phase monopoles (i.e., artifacts) are supposed to form the condensate which causes the confinement of electric charges. At the phase transition point $\beta_c$ this condensate dissolves, and the transition to the Coulomb phase occurs. This theory has a nontrivial spectrum which differs within these phases. Its $0^{++}$, $1^{-+}$ and $2^{++}$ states were studied in [20].

To recover the continuum limit of this theory it is important to study the role of the underlying singular configurations to be separated from the ‘real’ physics. One can modify the lattice action in such a way that the unphysical short–range fluctuations (monopoles, . . .) become suppressed leaving the ‘physical’ fields untouched. The question of interest is then whether this modified theory can serve as a better lattice approximation to that of noninteracting photons in the continuum. To study this question in detail is the goal of the present work.

The paper is organized as follows. First we discuss the lattice monopoles and the modification of the action. Section 3 is devoted to the discussion of correlators and ‘glueball’ wave functions. Section 4 presents the results of Monte Carlo calculations and their analysis. The last section contains a discussion of the results as well as some speculations.

## 2 Monopoles and modified lattice action

The standard Wilson action (WA) for the pure gauge $U(1)$ theory is

$$S_W(U_l) = \beta \cdot \sum_{x;\mu\nu} \left(1 - \cos \theta_{x;\mu\nu}\right), \quad (1)$$

where $\beta = 1/g_{\text{bare}}^2 \equiv 1/g^2$, and $U_l \equiv U_{x\mu} = \exp(i\theta_{x;\mu}), \quad \theta_{x;\mu} \in (-\pi, \pi]$ are the field variables defined on the links $l = (x, \mu)$. Plaquette angles $\theta_p = \theta_{x;\mu\nu}$ are given by

$$\theta_{x;\mu\nu} = \theta_{x;\mu} + \theta_{x+\mu;\nu} - \theta_{x+\nu;\mu} - \theta_{x;\nu}. \quad (2)$$
To extract magnetic monopoles the plaquette angle $\theta_P$ is split up

$$\theta_P = \bar{\theta}_P + 2\pi n_P, \quad -\pi < \bar{\theta}_P < \pi, \quad n_P = 0, \pm 1, \pm 2,$$

where $\bar{\theta}_P$ describes the electromagnetic flux through the plaquette and $n_P$ is the number of Dirac strings passing through it. The net number of Dirac strings going out of an elementary 3D cube determines the monopole charge within this cube as follows. The monopole current out of the cube $c_{n,\mu}$, labeled by the dual link $(n, \mu)$, is defined to be

$$K_{n,\mu} = \frac{1}{4\pi} \varepsilon_{\mu\nu\rho\sigma} \Delta_\nu \bar{\theta}_{n,\rho\sigma},$$

where the lattice derivative $\Delta_\nu$ is defined by $\Delta_\nu f_n = f_{n+\nu} - f_n$. Then we have

$$2\pi K_{n,\mu} = \sum_P \bar{\theta}_P = \sum_P (\theta_P - 2\pi n_P) = -2\pi \sum_P n_P,$$

where the sums are over oriented plaquettes $P$ in the surface of the cube $c_{n,\mu}$.

$$\sum_\mu \Delta_\mu K_{n,\mu} = 0,$$

which means that the monopole currents form closed loops on the dual lattice.

To suppress monopoles one can use the modified action (MA)

$$S_{MA} = S_W + \lambda_K \sum_c |K_c|,$$

where the parameter $\lambda_K(\beta)$ plays the role of a chemical potential for monopoles. In our calculations we have chosen $\lambda_K = \infty$. The partition function is

$$Z_{MA} = \int \prod_{\text{links}} dU_l \prod_c \delta_{K_c,0} \cdot \exp[-S_W(U_l)]$$

with periodic boundary conditions, and the average of any field functional $O(U)$ is defined as

$$\langle O(U) \rangle_{MA} = Z_{MA}^{-1} \cdot \int \prod_{\text{links}} dU_l O(U) \cdot \prod_c \delta_{K_c,0} \cdot \exp[-S_W(U_l)].$$

The action $S_{MA}$ was used in [22, 23] in order to investigate the behaviour of the average plaquette $\langle P \rangle$.

For some $\beta$-values we employed a further modification of the action (MA1) suppressing all lattice artifacts characterized by negative plaquette values

$$S_{MA1} = S_{MA} + \lambda_P \sum_P \left(1 - \text{sign}(\cos \theta_P)\right), \quad \lambda_P = \infty.$$

Note that both modifications (MA and MA1) do not influence the formal continuum limit and do not change any perturbative aspects.
3 Correlators and boson wave functions

Energies of the excited states (‘glueballs’) can be calculated by measuring the asymptotic behaviour of the correlation functions of operators $\Phi$ with the proper $J^{PC}$ quantum numbers and momenta $\vec{p}$.

The corresponding correlation functions $\Gamma(\tau)$ on a $L_t \cdot L_s^3$ lattice are defined as follows

$$\Gamma(\tau) = \langle \Phi^*(t+\tau) \cdot \Phi(t) \rangle^c$$
$$\equiv \frac{1}{n-1} \sum_{i=1}^{n} \frac{1}{L_t} \sum_{t=0}^{L_t-1} \left[ \Phi^*_i(t+\tau) \cdot \Phi_i(t) - \Phi^*_i(t+\tau) \cdot \Phi(t) \right], \quad (11)$$

where $n$ is the number of measurements, $t \oplus \tau = (t + \tau) \bmod(L_t)$ and

$$\Phi(t) = \frac{1}{n} \sum_{i=1}^{n} \Phi_i(t). \quad (12)$$

This definition provides us with unbiased estimators for $\Gamma(\tau)$. The correlators defined above are the superpositions of the exponents corresponding to different energy levels, e.g.,

$$\frac{\Gamma(\tau)}{\Gamma(0)} = A \cdot \left[ e^{-E \cdot \tau} + e^{-E \cdot (N_t-\tau)} \right] + \ldots, \quad (13)$$

where dots correspond to higher state contributions and

$$A = \frac{1}{\Gamma(0)} \cdot |\langle J^{PC} | \Phi^{J^{PC}}(0) | 0 \rangle|^2 \leq 1. \quad (14)$$

In principle the lowest state energy can be defined from the ratio of correlators at large enough $\tau$ (but $\tau \ll L_t/2$):

$$E_{\text{eff}}(\tau) \equiv -\ln \frac{\Gamma(\tau)}{\Gamma(\tau-1)}, \quad (15)$$

and $E = E_{\text{eff}}$ if $E \cdot (\tau - 1) \gg 1$. In practice only for comparatively small values of energy ($E \sim 1 - 2$) we can use the separation distance up to $\tau = 5$. For larger values of energy the correlators decay very quickly, the signal–to–noise ratio becomes too bad, and at $E > 4$ it is possible to use only $\tau = 0$ and $\tau = 1$ values with reasonable statistical errors. In this case the effective energy $E_{\text{eff}}(\tau)$ in eq.(15) is nothing but an estimate of the true energy $E$. The question how close $E_{\text{eff}}(\tau)$ is to the true value $E$ is the question of contamination from higher energy states. The larger the overlap of our ‘glueball’ wave function with the lowest lying ‘glueball’ state, the closer $A$ is to 1, the less the contamination from higher energy states. The common way to suppress the higher states contribution is to

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consider 'smeared' wave functions which are linear combinations of large numbers of Wilson loops and have a large overlap with low lying states \[30, 31, 32\].

In our work we used the simplest plaquette wave functions which are known to have comparatively small overlaps with the lowest glueball states (at least for the standard Wilson action). For us this is not the point of great importance because our main goal is the comparison of predictions for the different actions. Also the plaquette wave functions are proved to work very well in the case of comparatively small energies (where we can use the larger values of \( \tau \)). We shall come back to this question later on, when discussing the influence of \(Z_2\)-artifacts.

To construct the one-plaquette operators we followed the standard recipe described elsewhere (for a review see, for instance, \[29\]). The first step is to construct local operators \(\Phi(t; \vec{r})\) whose \(\vec{p}=0\)-parts have the desired \(J^PC\) properties. Real parts of Wilson loop operators have \(C\)-parity \(C=+1\) and imaginary parts have \(C\)-parity \(C=-1\). For the \(0^{++}\) state we first sum up the real parts of all six plaquettes belonging to the smallest space–like cube

\[
\Phi(t; \vec{r}) = \frac{1}{6} \sum_{k=1}^{6} \cos(\theta_{P_k}) , \tag{16}
\]

where \(\vec{r}\) is the centre of the cube. This construction is invariant under \(\pi/2\) and \(\pi\) rotations about all three axes in the 3D space passing through its centre. It is easy to show that the zero momentum operator

\[
\Phi(t) = \sum_{\vec{r}} \Phi(t; \vec{r}) \tag{17}
\]

corresponds to \(J^{PC} = 0^{++}, 4^{++}, \ldots\) excitations. We assume that the state with \(J = 0\) has the minimal mass, so, we shall refer to it as the \(0^{++}\) state.

The Fourier transform with momentum \(\vec{p}\) gives us non-zero momentum operators:

\[
\Phi(t; \vec{p}) = \sum_{\vec{r}} e^{-i\vec{r}\cdot\vec{p}} \cdot \Phi(t; \vec{r}) \tag{18}
\]

For constructing a tensor operator one has to take plaquettes parallel to the quantization axis (say, \(OY\)) and subtract from it the same but rotated by \(\pi/2\) about \(OY\). After summing through the three–dimensional space we obtain the operator corresponding to \(J^{PC} = 2^{++}, 4^{++}, \ldots\) excitations. As far as it is commonly assumed that the state with \(J = 2\) has the minimal mass it is referred as the \(2^{++}\) state with spin projection \(\pm 2\) onto the quantization axis. In this case the momentum \(\vec{p}\) in eq.(18) must be chosen parallel to the quantization axis to avoid the admixture of a scalar state.

Finally, summing up the imaginary parts of plaquette operators parallel to the quantization axis results in \(1^{+}, 3^{+}, \ldots\) states (or just \(1^{++}\) state). Comparing with the photon it has the wrong parity at zero momentum. But at \(\vec{p} \neq 0\) it has a
non-zero overlap with the $1^{--}$ (photon) state \[20\]. This fact can be easily checked in the naive continuum limit.

\section{Monte Carlo calculations}

Our calculations were made on a $L_t \cdot L_s^3 = 12 \cdot 6^3$ lattice at $0.1 \leq \beta \leq 2$. For each $\beta$ value our statistics reaches from 40,000 to 50,000 sweeps, and 2000 sweeps without measurements were done to reach equilibrium. For calculating the mean square errors and biases of our estimators we used the jackknife method \[35, 36\].

We calculated autocorrelation functions $C_{\text{auto}}(n_{\text{sweep}})$ for all our observables (plaquettes $P_{s,t}$, correlators $\Gamma(\tau; \vec{p})$, etc.) and defined the decorrelation length $n_{\text{dec.}}$ as a minimal number of sweeps such that

$$C_{\text{auto}}(n_{\text{dec.}}) < \frac{1}{5} \cdot C_{\text{auto}}(0).$$

For the standard Wilson action $n_{\text{dec.}}$ gets a sharp peak at $\beta = \beta_c \sim 1$ with $n_{\text{dec.}} \gtrsim 100$, while for the modified actions this value is comparatively small ($n_{\text{dec.}} \sim 4 - 6$) and has a very weak dependence on $\beta$ in the chosen interval.

In Fig.1 we show the dependence of the average plaquette $\langle P \rangle$ on $\beta$ for WA and MA. For the modified action this dependence is very smooth and shows no trace of crossover or phase transition in agreement with the calculations of \[22, 23\].

Another important parameter which we care about is the average number of negative plaquettes $\langle N_- \rangle$, i.e., plaquettes $P_{x,\mu\nu}$ with

$$\cos(\theta_{x,\mu\nu}) < 0.$$ \hspace{1cm} (20)

In Fig.2 we show the dependence of its density $\rho_{\text{neg.}} \equiv \langle N_- \rangle / N_P$ on $\beta$ for WA and MA. It is interesting to note that the suppression of monopoles decreases the density of negative plaquettes at $\beta \gtrsim 1.1$, while at larger $\beta$ values $\rho_{\text{neg.}}$ is approximately the same for both the actions.

As an example in Fig.3 we show the $\tau$ dependence of the vector state correlator $\Gamma(\tau)$ at nonzero momentum $\vec{p} = (0, 1, 0) \cdot \pi/3a$ at $\beta = 0.3$ and $\beta = 1.0$ for WA and MA. For the modified action the correlator does not depend on the coupling giving an indication for the decoupling of the photon. Because the energy is comparatively small these correlators have reasonable errors up to $\tau = 5$. On the contrary, for the Wilson action at $\beta = 0.3$ we recover the expected strong fall–off of the correlator.

In order to extract masses from the effective energy $E_{\text{eff}}(\vec{p}; \tau)$ we need a dispersion relation (DR). In the free field boson case the propagator on the lattice provides us with the DR
\[
\sinh^2 \frac{aE}{2} = \sinh^2 \frac{am}{2} + \sum_{i=1}^{3} \sin^2 \frac{ap_i}{2},
\]
which at small enough spacing and momenta gives us the continuum DR:

\[
E^2 = m^2 + \vec{p}^2.
\]

(22)

In general the connection between the momentum and the energy can be very nontrivial on lattice, and a DR which we can expect is of the form (see, e.g., [33])

\[
\sinh^2 \frac{aE}{2} = z_1(\beta; J^P C) \cdot \sinh^2 \frac{am}{2} + z_2(\beta; J^P C) \cdot \sum_{i=1}^{3} \sin^2 \frac{ap_i}{2} + \ldots
\]

(23)

We believe that in the continuum limit

\[
z_i(\beta; J^P C) \rightarrow 1 \quad \text{at} \quad i = 1, 2;
\]

\[
z_i(\beta; J^P C) \rightarrow 0 \quad \text{at} \quad i \geq 3.
\]

(24)

The second line in eq.(24) provides the necessary condition for the restoration of rotational invariance (at small enough \( \vec{p} \)), and the first one yields the necessary condition for the restoration of Lorentz invariance.

In Figs.4 we show the effective energy of the vector boson \( E_{\text{eff}}(\vec{p}; \tau) \) as a function of \( \beta \) for Wilson action (Fig.4a) and for the modified action (Fig.4b). Different symbols correspond to different momenta \( \vec{p} \) as indicated in the figures. Broken lines correspond to the free field theory dispersion relation eq.(21) with zero mass and different momenta \( \vec{p} \). In the theory with the MA a massless photon exists in the whole \( \beta \) region we have chosen, and the free field theory dispersion relation (21) works astonishingly well. Therefore, it provides a very convincing evidence that in the theory without monopoles a massless photon exists even at very small \( \beta \). In the theory with the WA a massless photon exists in the Coulomb phase as already observed in [20] (see also [28]) while below the critical point the photon acquires a nonzero mass.

Another interesting observation is that the DR (21) works very well for both the actions at \( \beta \gtrsim 1 \) even for \( \tau = 1 \). This fact shows that the overlap coefficient \( A(\beta) \) is close enough to unity in this phase, and the contamination from higher states becomes negligible with increasing \( \beta \).

Analogously to Figs.4 in Figs.5 we show the effective energy \( E_{\text{eff}}(\vec{p}; \tau = 1) \) of the \( 0^{++} \) boson as a function of \( \beta \) for the Wilson action (Fig.5a) and modified action (Fig.5b). For the MA the dependence on the coupling is very smooth, and at increasing \( \beta \) the energies for both actions tend to be rather close to each other. It
is difficult to check the validity of the free field theory DR with our data because the energies are comparatively big, and their dependence on the momentum is weak.

Similar observations can be made for the tensor states. In Figs.6 we show the effective energy $E_{\text{eff}}(\vec{p}; \tau = 1)$ of the $2^{++}$ boson as a function of $\beta$ for the Wilson action (Fig.6a) and the modified action (Fig.6b).

Masses $m^\gamma$ we calculated are in units of the (unknown) spacing $a(\beta)$:

$$m^\gamma = m^\gamma_{\text{ph.}} \cdot a(\beta); \quad \gamma = 0^{++}, \ldots; \quad (25)$$

but their ratio

$$\frac{m^{2^{++}}}{m^{0^{++}}} = \frac{m^{2^{++}}_{\text{ph.}}}{m^{0^{++}}_{\text{ph.}}} \quad (26)$$

must not depend on the coupling in the continuum limit. Comparing the data for scalar and tensor states it is easy to see that this ratio has a very weak $\beta$ dependence in the whole interval and is rather close to unity.

As it was shown above the simple one-plaquette wave functions work very well for $\tau \geq 2$ (at least, for the vector state). This means that the overlap coefficient $A(\beta)$ which can be defined as

$$A(\beta) = \frac{\Gamma^2(\tau = 1)}{\Gamma(\tau = 2)}. \quad (27)$$

is not too far from unity. Fig.7a shows its dependence on $\beta$ for the vector state at $\vec{p} = (0; 1; 0) \cdot \pi/3a$. For MA (open circles) $A \sim 0.65$ at small $\beta$, and at $\beta \sim 1$ it becomes close to 1. It is tempting to assume that there is some correlation between this dependence and the $\beta$-dependence of the number of negative plaquettes (compare with Fig.2). To check this assumption we used another modified action (MA1) with the full suppression of negative plaquettes [10]. Calculations were made at $\beta = 0.3$ and $\beta = 0.5$, and the corresponding values of $A(\beta)$ are shown in Fig.7a with full circles. In Fig.7b we show the effective energy $E^\beta_{\text{eff}}(\vec{p}; \tau = 1)$ at $\beta = 0.3$ as a function of $\hat{p}^2 = \sum_{i=1}^3 4 \sin^2(a p_i/2)$. Open circles correspond to MA, and full circles correspond to MA1. The solid line corresponds to the dispersion relation (21). Therefore, after suppressing negative plaquettes the agreement between the $\tau = 1$ effective energy and the dispersion relation (21) becomes practically perfect. So, we conclude that the $Z_2$-artifacts are, at least partly, responsible for not good enough overlap with the higher states. It is worthwhile to note here that the suppression of negative plaquettes does not influence the energy of the photon while the energies of the massive scalar, vector and tensor states tend to increase.

5 Conclusions

We have studied the properties of the compact four-dimensional $U(1)$ lattice gauge theory without monopoles. Employing Monte Carlo simulations we calculated correlators of scalar, vector and tensor operators at zero and nonzero momenta $\vec{p}$ on
a $12 \cdot 6^3$ lattice at $0.1 \leq \beta \leq 2$. We confirm that the theory without monopoles has no phase transition, at least, in the interval of $\beta$ studied. In the modified theory a massless photon exists in the whole interval of $\beta$, fitting the lattice free field theory dispersion relation very well. The energies of $0^{++}$, $1^{+-}$ and $2^{++}$ states show a rather weak dependence on the coupling in the chosen interval of $\beta$, and their ratios are practically constant. We show also that other lattice artifacts – negative plaquettes – are, at least partly, responsible for the overlap in the strong coupling region. The further modification of the theory by suppressing the negative plaquettes drastically improves the overlap with the lowest states (at least, for $J = 1$).

We believe that the problem of artifacts in lattice calculations deserves much more detailed and thorough study, and still our understanding of pure gauge $U(1)$ theory is far from being complete. Even after suppressing monopoles (and negative plaquettes) there is still some residual interaction which occurs in the appearance of massive states ($0^{++}, \ldots$). One can speculate that there are other kinds of lattice artifacts which are responsible for this interaction, and Dirac strings are among the possible candidates for a further study.

We conclude that the modified theory without lattice artifacts (monopoles, negative plaquettes, etc.) should be a more reliable candidate for the QED calculations with fermions in the strong coupling area.

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Figure captions

**Fig.1** The plaquette value $\langle P \rangle$ for WA (squares) and MA (circles) as a function of $\beta$.

**Fig.2** The density of negative plaquettes $\rho_{neg.}$ for WA (squares) and MA (circles) vs. $\beta$.

**Fig.3** Correlator $\Gamma(\tau; \vec{p})$ for the vector state at $\beta = 0.3$ and $\beta = 1.0$ for WA and MA. $\vec{p} \equiv \vec{k} \cdot 2\pi/aL_s$.

**Fig.4** The effective energy of the vector state $E_{eff}(\beta; \tau)$ at different $\vec{p} \equiv \vec{k} \cdot 2\pi/aL_s$ as a function of $\beta$ for WA (a) and for MA (b). Different symbols correspond to different momenta $\vec{p}$ shown in the picture. Broken lines correspond to the dispersion relation (21) with zero mass.

**Fig.5** The effective energies $E_{eff}(\vec{p}; \tau = 1)$ for the scalar state as a function of $\beta$ for WA (a) and for MA (b).

**Fig.6** The effective energies $E_{eff}(\vec{p}; \tau = 1)$ of the tensor state as a function of $\beta$ for WA (a) and for MA (b).

**Fig.7** $A(\beta) = \Gamma^2(\tau = 1)/\Gamma(\tau = 2)$ vs. $\beta$ for the vector state for MA and MA1 (a); The effective energy $E_{eff}(\vec{p}; \tau = 1)$ of the vector state at $\beta = 0.3$ as a function of $\vec{p}^2 = \sum_{i=1}^{3} 4 \sin^2 (ap_i/2)$ for MA and MA1. The momenta are the same as in Fig.4. The solid line corresponds to the dispersion relation (21) (b).