THE PROBLEM OF HARMONIC ANALYSIS ON
THE INFINITE–DIMENSIONAL UNITARY GROUP

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ABSTRACT. The goal of harmonic analysis on a (noncommutative) group is to decompose the most “natural” unitary representations of this group (like the regular representation) on irreducible ones. The infinite–dimensional unitary group $U(\infty)$ is one of the basic examples of “big” groups whose irreducible representations depend on infinitely many parameters. Our aim is to explain what the harmonic analysis on $U(\infty)$ consists of.

We deal with unitary representations of a reasonable class, which are in 1–1 correspondence with characters (central, positive definite, normalized functions on $U(\infty)$). The decomposition of any representation of this class is described by a probability measure (called spectral measure) on the space of indecomposable characters. The indecomposable characters were found by Dan Voiculescu in 1976.

The main result of the present paper consists in explicitly constructing a 4–parameter family of “natural” representations and computing their characters. We view these representations as a substitute of the nonexisting regular representation of $U(\infty)$. We state the problem of harmonic analysis on $U(\infty)$ as the problem of computing the spectral measures for these “natural” representations. A solution to this problem is given in the next paper [BO4], joint with Alexei Borodin.

We also prove a few auxiliary general results. In particular, it is proved that the spectral measure of any character of $U(\infty)$ can be approximated by a sequence of (discrete) spectral measures for the restrictions of the character to the compact unitary groups $U(N)$. This fact is a starting point for computing spectral measures.

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Introduction

(a) Preface. The problem of noncommutative harmonic analysis consists in decomposing “natural” unitary representations of a given group into irreducible ones.

For instance, if $K$ is a compact group, then the decomposition of the (bi)regular representation in the space $L^2(K)$ is described by the classical Peter–Weyl theorem.

Another well–known example is the decomposition of the quasiregular representation of a noncompact simple Lie group $G$ acting in the $L^2$ space on the Riemannian symmetric space $G/K$.

Note that in the Peter–Weyl theorem, the spectrum of the decomposition is discrete, while for $L^2(G/K)$ the spectrum is continuous. The decomposition of $L^2(G/K)$ is described by a measure living on a region of dimension equal to the rank of the space $G/K$.

The problems of decomposing the $L^2$ space on $G$ and on a pseudo–Riemannian symmetric space $G/H$ belong to the next levels of difficulty.

In the present paper and the next one (joint with Alexei Borodin, [BO4]), we deal with the problem of harmonic analysis in a totally different situation. The novelty is that the group is no longer compact or locally compact, its dual space has infinite dimension, and the decomposition into irreducibles is governed by a measure with infinite–dimensional support.

Our main result (established in [BO4]) is an explicit description of measures which arise in this way (we call them spectral measures). The description is given in the language of stochastic point processes. These probabilistic objects have never emerged in classical representation theory. However, they happen to be an adequate tool for groups with infinite–dimensional dual.

The present paper contains results of two kinds:

First, we construct a family of representations which we consider as “natural” ones. In our situation, when the group is not locally compact, the conventional definition of a regular representation (or a quasi–regular representation associated with a homogeneous space) is not applicable directly. This forces us to choose another, more sophisticated, way to produce representations.

Second, we prove necessary general theorems concerning the spectral measures with infinite–dimensional support.

This finally allows us to convert the problem of harmonic analysis to an asymptotic problem of the form which is typical for random matrix theory or asymptotic combinatorics. We are lead, however, to a new model, which was not previously examined.

We proceed now to a more detailed description of the contents of the present paper.

(b) The group. Consider the chain of the compact classical groups $U(N)$, $N = 1, 2, \ldots$, which are embedded one into another in a natural way, and let $U(\infty)$ be their union. Equivalently, elements of $U(\infty)$ are infinite unitary matrices $U = [U_{ij}]$, where the indices $i, j$ take values $1, 2, \ldots$, and we assume that $U_{ij} = \delta_{ij}$ for $i + j$ large enough. The group $U(\infty)$ is one of the fundamental examples of inductive limit groups (another such example is $S(\infty)$, the union of the finite symmetric groups).

Following the philosophy of [Ol1], [Ol3], we form a $(G, K)$–pair, where $G$ is the group $U(\infty) \times U(\infty)$ and $K$ is the diagonal subgroup in $G$, isomorphic to $U(\infty)$. This is a Gelfand pair in the sense of [Ol3].
(c) **Representations and characters.** We are dealing with unitary representations $T$ of the group $G$ possessing a distinguished cyclic $K$–invariant vector $\xi$. Such representations are called **spherical representations** of the pair $(G, K)$. They are completely determined by the corresponding matrix coefficients $\psi(\cdot) = (T(\cdot)\xi, \xi)$. The $\psi$’s are called the **spherical functions**. These are certain $K$–biinvariant functions on $G$, which can be converted (via restriction to the subgroup $U(\infty) \times \{e\} \subset G$) to certain central functions $\chi$ on $U(\infty)$. The functions $\chi$ thus obtained are called the **characters** of the group $U(\infty)$. The correspondence $T \leftrightarrow \chi$ makes it possible to employ both languages, that of spherical representations and that of characters, each of which has its own merits. To **irreducible** representations $T$ correspond **extreme** characters $\chi$ (i.e., extreme points in the convex set of all characters). Irreducible spherical representations of $(G, K)$ and extreme characters of $U(\infty)$ admit a complete description. They depend on countably many continuous parameters.

(d) **How to get “natural” representations.** Set $G(N) = U(N) \times U(N)$ and let $K(N)$ be the diagonal in $G(N), N = 1, 2, \ldots$. The homogeneous space $G(N)/K(N)$ can be identified with the group space $U(N)$, and then the action of $G(N)$ on $G(N)/K(N)$ turns into the two–sided action of $U(N)$ on itself. Let $\text{Reg}_N$ denote the quasi–regular representation of $G(N)$ in $L^2(G(N)/K(N))$. This is nothing else than the biregular representation of the compact group $U(N)$ whose decomposition is determined by the Peter–Weyl theorem.

We seek for a counterpart $\text{Reg}_\infty$ of the representations $\text{Reg}_N$ for the pair $(G, K)$. As was mentioned above, there is no good measure on $G/K$, hence no space $L^2(G/K)$. To overcome this difficulty, two general recipes are known. First, to embed representations $\text{Reg}_N$ in each other and then to define $\text{Reg}_\infty$ as the inductive limit representation $\varprojlim_{N \to \infty} \text{Reg}_N$. Second, to embed $G/K$ into an appropriate $G$–space $\overline{G/K}$ possessing an invariant (or quasiinvariant) measure $m$ and then to realize $\text{Reg}_\infty$ in $L^2(\overline{G/K}, m)$.

A realization of any of these recipes is by no means an automatic exercise:

A subtle point of the inductive limit construction is that there are many different embeddings $\iota_N : \text{Reg}_N \to \text{Reg}_{N+1}$ for each $N$, and the limit representation $\text{Reg}_\infty$ highly depends of the chain $\{\iota_N\}$ chosen. Trying all the possible chains $\{\iota_N\}$ one gets too many limit representations, so that a fine selection rule must be imposed.

As for the second way, we have to guess what the ambient space $\overline{G/K}$ should be. With $\overline{G/K}$ specified, we next have to find good measures $m$ (and moreover, to select some 1–cocycles that are also involved in the construction).

We employ both methods and finally get a family $\{T_{zw}\}$ of representations depending on two complex parameters $z, w$ such that $\Re(z + w) > -\frac{1}{2}$. We believe that the representations $T_{zw}$ are “natural” objects of harmonic analysis.

The space $\overline{G/K}$ is constructed as follows. We define certain projections of the group spaces $U(N) \to U(N-1)$, where $N = 2, 3, \ldots$, and then take the projective limit space $\mathfrak{U} = \varprojlim U(N)$. This is our $\overline{G/K}$.

(e) **Gelfand–Tsetlin graph and coherent systems.** The Gelfand–Tsetlin graph is a convenient tool for writing characters of $U(\infty)$. The vertices of the graph symbolize the irreducible representations of various groups $U(N)$ while the edges code the inclusion relations between irreducible representations of $U(N)$ and $U(N+1)$. By $\mathfrak{GT}_N$ we denote the subset of vertices corresponding to irreducibles of $U(N)$; elements of $\mathfrak{GT}_N$ are identified with dominant weights for $U(N)$, i.e., these are
Given a character $\chi$ of $U(\infty)$, we can expand its restriction to the subgroup $U(N)$ into a convex combination of the functions $\chi^\lambda(\cdot)/\chi(\cdot)$, where $\chi^\lambda$ stands for the irreducible character (in the conventional sense) of $U(N)$, indexed by $\lambda \in \mathcal{G}T_N$. The coefficients $P_N(\lambda)$ of this expansion determine a probability distribution $P_N$ on the discrete set $\mathcal{G}T_N$.

In this way, we get a bijection $\chi \leftrightarrow \{P_N\}_{N=1,2,...}$ between characters $\chi$ and certain sequences $\{P_N\}$ of probability distributions. These sequences are called \textit{coherent systems}, because, for any $N$, the distributions $P_N$ and $P_{N+1}$ are connected by a certain “coherency relation”.

(f) The characters $\chi_{zw}$ and their analytic continuation. Having constructed the representations $T_{zw}$, we proceed to the corresponding characters $\chi_{zw}$. We evaluate them in terms of the associated coherent systems $\{P_N\}$. The expressions that we get for $P_N$ make sense, via analytic continuation, for a larger set of parameters.

Specifically, let $z, z', w, w'$ be the coordinates in $\mathbb{C}^4$, and let $\mathcal{D}$ be the open half-space in $\mathbb{C}^4$ determined by the inequality $\Re(z + w + z' + w') > -1$. We exhibit an “admissible subset” $\mathcal{D}_{adm} \subset \mathcal{D}$ of real dimension 4, containing as a proper subset all the quadruples $(z, z, w, w') \in \mathcal{D}$ with $z' = \bar{z}$, $w' = \bar{w}$, and such that for any quadruple $(z, z, w, w') \in \mathcal{D}_{adm}$, the following formulas provide a coherent system:

$$P_N(\lambda \mid z, z, w, w') = (S_N(z, z, w, w'))^{-1} \times \prod_{1 \leq i < j \leq N} \frac{(\lambda_i - \lambda_j + j - i)^2}{(j - i)^2} \times \prod_{i=1}^{N} \frac{1}{\Gamma(z - \lambda_i + i)\Gamma(z' - \lambda_i + i)\Gamma(w + N + 1 + \lambda_i - i)\Gamma(w' + N + 1 + \lambda_i - i)},$$

where $N = 1, 2, \ldots$, $\lambda$ ranges over $\mathcal{G}T_N$, and $S_N(z, z, w, w')$ is a normalization constant:

$$S_N(z, z, w, w') = \prod_{i=1}^{N} \frac{\Gamma(z + z' + w + w' + i)}{\Gamma(z + w + i)\Gamma(z + w' + i)\Gamma(z' + w + i)\Gamma(z' + w' + i)\Gamma(i)}.$$

Thus, to any $(z, z, w, w') \in \mathcal{D}_{adm}$, we may assign a character $\chi_{z, z, w, w'}$. The initial characters $\chi_{zw} = \chi_{z, z, z, w}$ form the “principal series”, and the remaining characters belong to its analytic continuation. The whole picture resembles the conventional principal, complementary and degenerate series for semi–simple Lie groups, so that we even employ this terminology. However, in our context, these concepts refer not to (generically) irreducible representations, as in the conventional context, but to highly reducible ones.

Note that the construction of the coherent systems implies a curious summation formula,

$$\sum_{\lambda_1 \geq \cdots \geq \lambda_n \in \mathbb{Z}} P_N(\lambda \mid z, z, w, w') = 1.$$

In the simplest case $N = 1$ it looks as

$$\sum_{k \in \mathbb{Z}} \frac{1}{\Gamma(z - k + 1)\Gamma(z' - k + 1)\Gamma(w + k + 1)\Gamma(w' + k + 1)} = \frac{\Gamma(z + z' + w + w' + 1)}{\Gamma(z + w + 1)\Gamma(z + w' + 1)\Gamma(z' + w + 1)\Gamma(z' + w' + 1)};$$
which is equivalent to a classical identity due to Dougall, see [AAR, Chapter 2, Theorem 2.8.2 and Exercise 42(b)], [Er, §1.4].

**(g) Abstract theorems on spectral measures.** Let us return now to extreme characters. There is a bijective correspondence $\chi(\omega) \leftrightarrow \omega$ between extreme characters and points $\omega$ of an infinite-dimensional “region”

$$\Omega \subset \mathbb{R}^{4\infty+2} = \mathbb{R}^{\infty} \times \mathbb{R}^{\infty} \times \mathbb{R}^{\infty} \times \mathbb{R} \times \mathbb{R}.$$  

For the functions $\chi(\omega)(U)$, where $U \in U(\infty)$, there is a remarkable explicit formula due to Voiculescu (see (1.2) below).

Our first “abstract” theorem says that for any character $\chi$, there exists a unique probability measure $P$ on $\Omega$ such that

$$\chi(U) = \int_\Omega \chi(\omega)(U) P(d\omega), \quad U \in U(\infty).$$

We call $P$ the *spectral measure* of $\chi$. Conversely, any probability measure $P$ on $\Omega$ is a spectral measure for a certain $\chi$. This result is a refinement of a theorem due to Voiculescu [Vo]. It provides a nice general description of the whole set of characters.

The next result shows that spectral measures can be, in principle, computed. Specifically, we define, for any $N = 1, 2, \ldots$, an embedding $\mathbb{G}T_N \hookrightarrow \Omega$ such that the image of $\mathbb{G}T_N$ looks as a discrete approximation of $\Omega$, which becomes more and more exact as $N \to \infty$.

Now let $\chi$ be an arbitrary character, $\{P_N\}$ be the corresponding coherent system, and $P$ be the spectral measure of $\chi$. Then, according to the second “abstract” theorem, we have $P = \lim_{N \to \infty} P_N$, where we identify $P_N$ with its pushforward under the embedding $\mathbb{G}T_N \hookrightarrow \Omega$.

**(h) The problem of harmonic analysis on $U(\infty)$.** Now we are in a position to state this problem explicitly:

Let $\chi = \chi_{z,z,w,w'}$, where $(z, z, w, w') \in \mathcal{D}_{\text{adm}}$, and let $\{P_N\}$ be the corresponding coherent system. Recall that $P_N = P_N(\cdot \mid z, z, w, w')$ is a probability measure on $\mathbb{G}T_N$, given by the explicit formula above. Let us carry over $P_N$ to the space $\Omega$. Then the problem consists in evaluating the limit of the measures $P_N$ in the ambient space $\Omega$ as $N \to \infty$.

(By virtue of the “abstract” theorems above, the limit always exists and coincides with the spectral measure.)

The solution to the problem is presented in the next paper, [BO4].

**(i) Connections with infinite random matrices [BO3].** Let $\mathfrak{H}$ be the space of all infinite Hermitian matrices and let $U(\infty)$ act on $\mathfrak{H}$ by conjugations. There is a parallelism:

Characters $\chi$ of $U(\infty)$.

Extreme characters $\chi^{(\omega)}$, indexed by points $\omega$ of a region $\Omega \subset \mathbb{R}^{4\infty+2}$.

Decomposition $\chi = \int_\Omega \chi^{(\omega)} P(d\omega)$, where $P$ is a probability measure

Invariant probability measures $M$ on $\mathfrak{H}$.

Ergodic invariant measures $M^{(\omega)}$, indexed by points $\omega$ of a region $\Omega \subset \mathbb{R}^{2\infty+2}$.

Decomposition $M = \int_\Omega M^{(\omega)} P(d\omega)$, where $P$ is a probability measure.
on $\Omega \subset \mathbb{R}^{4\infty+2}$.

Distinguished characters $\chi = \chi_{zw}$, where $z, w \in \mathbb{C}$, $\Re(z + w) > -\frac{1}{2}$.

The problem of computing the spectral measures $P$ for distinguished characters $\chi$.

And so on.

In the paper [BO3], we constructed the distinguished measures $m^{(s)}$ and computed their spectral measures. The whole theory of $U(\infty)$–invariant measures can be viewed as a simplified version of the theory of characters.

On the other hand, the results of [BO3] are directly used in the present paper: the space $\mathfrak{U}$ mentioned above in subsection (d) can be identified (within a negligible subset) with the space $\mathfrak{H}$, and the measures $m$ involved in the construction of the representations $T_{zw}$ are nothing else than the measures $m^{(s)}$.

(j) **Characters of $S(\infty)$**. As was mentioned above, the group $S(\infty)$, the union of finite symmetric groups $S(n)$, is another fundamental example of an inductive limit group. The representation theories of the both groups, $U(\infty)$ and $S(\infty)$, reveal deep analogies and links. Of course, in the classical representation theory, it is well known that representations of the groups $S(n)$ and $U(N)$ are connected by the Schur–Weyl duality. But in the case of infinite dimension the connections between the symmetric and unitary groups turn out to be much closer.

The problem of harmonic analysis on $S(\infty)$ was stated in [KOV] and then further developed in a cycle of papers [P.I-V], [Bor1-2], [BO1-2]. The construction [KOV] of the “generalized regular representations” $T_z$ served as a guiding example for the construction of the representations $T_{zw}$ in the present paper. Further, the experience of our work on the spectral measures for $S(\infty)$ helped us very much in the work [BO4].

(k) **Generalization to other $(G,K)$–pairs. Works of Pickrell [Pi] and Neretin [Ner3]**. The $(G,K)$–pair $(U(\infty) \times U(\infty), \text{diagonal } U(\infty))$ is a representative of the family of ten $(G,K)$–pairs, which come from the ten classical series of compact Riemannian symmetric spaces. This family is a natural framework for developing representation theory (see [O11], [OL3], [Ner1]) and, in particular, harmonic analysis. In the pioneer work [Pi], Pickrell considered the pair $G = \lim_{\to} U(2N)$, $K = U(N) \times U(N)$, which corresponds to the series of complex Grassmanians $U(2N)/U(N) \times U(N)$. He constructed the ambient space $G/K$ as a projective limit of Grassmanians and defined a family of measures $m$ which give rise to “natural” representations. Pickrell’s construction was the starting point of [KOV] and of the present paper. In the recent paper by Neretin, [Ner3], Pickrell’s construction is carried over to all ten pairs.

Note that, as compared with Pickrell’s results, our construction of the representations $T_{zw}$ incorporates a few new observations.

First, following [KOV], we introduce a complex parameter instead of a real one (the idea is to employ a wider family of cocycles for an action of the group $G$). It is worth noting that the same generalization makes sense for all ten pairs $(G,K)$ mentioned above.

Second, for our pair $(G,K)$, it is actually possible to introduce two complex parameters $z, w$. This (nonevident) fact was prompted by Neretin’s results on
Hua–type integrals, see [Ner2], [Ner3].

Third, we observe that the construction of the representations $T_{z,w}$ makes sense for all $z,w \in \mathbb{C}$. But for $\Re z + \Re w \leq -1/2$ we get representations without a distinguished $K$–invariant vector. It would be interesting to study these representations.

(1) Acknowledgment. This paper is part of a joint project with Alexei Borodin. I am very grateful to him and also to Sergei Kerov, Yurii Neretin, and Anatoly Vershik for numerous discussions.

1. Characters

Definition 1.1. Let $K$ be a topological group. By a character of $K$ we mean any continuous complex–valued function $\chi$ on $K$ satisfying the following three conditions:

(i) $\chi$ is central, i.e., constant on conjugacy classes;
(ii) $\chi$ is positive definite, i.e., for any finite collection $g_1, \ldots, g_n$ of elements of $K$, the $n \times n$ matrix $[\chi(g_j^{-1} g_i)]_{1 \leq i,j \leq n}$ is Hermitian and nonnegative;
(iii) $\chi$ is normalized at the unit element, i.e., $\chi(e) = 1$.

Let $\mathcal{X}(K)$ denote the set of the characters of $K$. Evidently, $\mathcal{X}(K)$ is a convex set. Its extreme points are called extreme (or indecomposable) characters.\(^1\)

Example 1.2. Let $K$ be a finite group or, more generally, a compact separable group, and let $\hat{K}$ be its dual space, i.e., the set of equivalence classes of irreducible representations. Then $\hat{K}$ is a finite or a countably infinite set. Let $\lambda$ range over $\hat{K}$ and $\chi^\lambda$ denote the irreducible character corresponding to $\lambda$ (i.e., the trace of an irreducible representation in the class $\lambda$). The extreme characters of $K$ in the sense of Definition 1.1 are exactly the normalized irreducible characters

$$\tilde{\chi}^\lambda(g) = \frac{\chi^\lambda}{\chi^\lambda(e)}, \quad \lambda \in \hat{K}, \quad (1.1)$$

while general characters $\chi \in \mathcal{X}(K)$ are convex combinations of the extreme ones,

$$\chi = \sum_{\lambda \in \hat{K}} P(\lambda) \tilde{\chi}^\lambda, \quad P(\lambda) \geq 0, \quad \sum_{\lambda \in \hat{K}} P(\lambda) = 1.$$

Thus, the set $\mathcal{X}(K)$ is a simplex with vertices indexed by $\lambda \in \hat{K}$.

In the example above the set $\mathcal{X}(K)$ is large enough to separate conjugacy classes. If $K$ is not compact it may well happen that $\mathcal{X}(K)$ is very small (e.g., is exhausted by the function $\chi \equiv 1$). Actually, the class of groups $K$ with $\mathcal{X}(K)$ large enough is rather restricted. However, it includes important examples leading to a rich theory.

In the present paper we take as $K$ the infinite–dimensional unitary group, which is defined as

$$U(\infty) = \bigcup_{N \geq 1} U(N),$$

where $U(N)$ is the group of $N \times N$ unitary matrices. The embedding $U(N) \hookrightarrow U(N + 1)$ is defined as follows: we identify $U(N)$ with the subgroup in $U(N + 1)$

\(^1\)This terminology differs from that used in [Th1], [Th2], [VK1], [VK2], [Vo]. In those papers, only extreme characters were considered, and they were called simply characters.
fixing the \((N+1)\)st basis vector. Equivalently, \(U(\infty)\) is the group of infinite unitary matrices \(U = [U_{ij}], i, j = 1, 2, \ldots\), with finitely many matrix entries \(U_{ij}\) distinct from \(\delta_{ij}\).

We equip \(U(\infty)\) with the inductive limit topology (i.e., a function on \(U(\infty)\) is continuous if its restriction to each subgroup \(U(N)\) is continuous). Note that \(U(\infty)\) is not locally compact.

First, let us describe the conjugacy classes in \(U(\infty)\). Recall that the conjugacy classes in \(U(N)\) are parameterized by spectra of unitary matrices, that is, by unordered \(N\)-tuples \(u_1, \ldots, u_N\) of complex numbers with modulus 1. In this notation, the embedding \(U(N) \hookrightarrow U(N+1)\) is described as \((u_1, \ldots, u_N) \mapsto (u_1, \ldots, u_N, 1)\). Thus, the conjugacy classes in \(U(\infty)\) can be parameterized by countable collections of complex numbers \((u_1, u_2, \ldots)\) such that \(|u_i| = 1\) and only finitely many of \(u_i\)'s are different from 1; the ordering of \(u_i\)'s is unessential. As a representative of a conjugacy class indexed by \((u_1, u_2, \ldots)\) one can take the diagonal matrix \(\text{diag}(u_1, u_2, \ldots)\).

To describe the extreme characters of \(U(\infty)\) we need some notation.

Let \(\mathbb{R}^\infty\) denote the product of countably many copies of \(\mathbb{R}\), and set \(\mathbb{R}^{4\infty+2} = \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R} \times \mathbb{R}\).

Let \(\Omega \subset \mathbb{R}^{4\infty+2}\) be the subset of sextuples \(\omega = (\alpha^+, \beta^+; \alpha^-, \beta^-; \delta^+, \delta^-)\) such that

\[
\alpha^\pm = (\alpha_1^\pm \geq \alpha_2^\pm \geq \cdots \geq 0) \in \mathbb{R}^\infty, \quad \beta^\pm = (\beta_1^\pm \geq \beta_2^\pm \geq \cdots \geq 0) \in \mathbb{R}^\infty,
\]

\[
\sum_{i=1}^\infty (\alpha_i^\pm + \beta_i^\pm) \leq \delta^\pm, \quad \beta_1^+ + \beta_1^- \leq 1.
\]

Set

\[
\gamma^\pm = \delta^\pm - \sum_{i=1}^\infty (\alpha_i^\pm + \beta_i^\pm)
\]

and note that \(\gamma^+, \gamma^-\) are nonnegative.

To any \(\omega \in \Omega\) we assign a function \(\chi(\omega)\) on \(U(\infty)\):

\[
\chi(\omega)(U) = \prod_{u \in \text{Spectrum}(U)} \left\{ e^{\gamma^+(u-1)+\gamma^-(u^{-1}-1)} \prod_{i=1}^\infty \frac{1 + \beta_i^+(u - 1)}{1 - \alpha_i^+(u - 1)} \frac{1 + \beta_i^-(u^{-1} - 1)}{1 - \alpha_i^-(u^{-1} - 1)} \right\}.
\]

Here \(U\) is a matrix from \(U(\infty)\) and \(u\) ranges over the set of its eigenvalues. All but finitely many \(u\)'s equal 1, so that the product over \(u\) is actually finite. The product over \(i\) is convergent, because the sum of the parameters is finite.

**Theorem 1.3.** The functions \(\chi(\omega)\), where \(\omega\) ranges over \(\Omega\), are exactly the extreme characters of the group \(U(\infty)\).

The coordinates \(\alpha_i^\pm, \beta_i^\pm\), and \(\gamma^\pm\) (or \(\delta^\pm\)) are called the **Voiculescu parameters** of the extreme character \(\chi(\omega)\). Theorem 1.3 is similar to Thoma’s theorem which
describes the extreme characters of the infinite symmetric group, see [Th1], [VK1], [Wa], [KOO].

The study of extreme characters of the group $U(\infty)$ was initiated by Voiculescu [Vo]. He proved (among other things) that all functions (1.2) are actually extreme characters. The fact that Voiculescu’s list is exhaustive was established later by Boyer [Boy1] and by Vershik–Kerov [VK2]. They independently pointed out that Theorem 1.3 is implied by old Edrei’s result [Ed] about two–sided totally positive sequences.

On the other hand, Vershik and Kerov outlined in [VK2] another approach to Theorem 1.3, based on the idea of approximating extreme characters by normalized irreducible characters of the compact groups $U(N)$. The same idea was employed in their previous work [VK1] on the infinite symmetric group $S(\infty)$.

A detailed proof of Theorem 1.3 by Vershik–Kerov’s asymptotic method was given much later by Okounkov–Olshanski [OkOl] (this paper also contains more general results). In a special case, a detailed proof was earlier given by Boyer [Boy2].

Here are some comments to Voiculescu’s formula (1.2).

**Remark 1.4.** The simplest extreme characters are of the form $\chi(U) = \det^k(U)$, where $k \in \mathbb{Z}$. The corresponding parameters are as follows: all of them are equal to zero except the first $|k|$ coordinates in $\beta^+$ (if $k > 0$) or in $\beta^-$ (if $k < 0$), which are equal to 1.

**Remark 1.5.** Given a character $\chi$, define the function $\chi \otimes \det^k(\cdot)$ as the pointwise product $\chi(U) \det^k(U))$. Then $\chi \otimes \det^k(\cdot)$ is a character, too. If $\chi$ is extreme then $\chi \otimes \det^k(\cdot)$ is extreme. In terms of Voiculescu’s parameters, tensoring with $\det(\cdot)$ reduces to

$$\beta^+ \mapsto (1 - \beta_1^-, \beta_1^+, \beta_2^+, \ldots), \quad \beta^- \mapsto (\beta_{2}^-, \beta_3^-, \ldots).$$

**Remark 1.6.** Here is a comment to the condition $\beta_1^+ + \beta_1^- \leq 1$. Even if this condition is dropped, Voiculescu’s formula still defines an extreme character. However, using the identity

$$[1 + b^+(u - 1)][1 + b^-(u^{-1} - 1)] = [1 + (1 - b^-)(u - 1)][1 + (1 - b^+)(u^{-1} - 1)],$$

we can always modify the beta parameters so that the condition $\beta_1^+ + \beta_1^- \leq 1$ will be satisfied. With this condition imposed, no freedom to change the parameters remains: if $\omega_1 \neq \omega_2$ then $\chi^{(\omega_1)} \neq \chi^{(\omega_2)}$, see [OkOl, §5, Step 3].

**Remark 1.7.** Let $SU(\infty)$ denote the subgroup of the matrices $U \in U(\infty)$ with determinant 1. I.e., $SU(\infty)$ is the union of the groups $SU(N)$. Restricting any character to $SU(\infty)$ one gets a character of this group, and extreme characters remain extreme. However, it may well happen that $\chi^{(\omega_1)} |_{SU(\infty)}$ equals $\chi^{(\omega_2)} |_{SU(\infty)}$ for certain $\omega_1 \neq \omega_2$. As parameters for the characters $\chi^{(\omega)} |_{SU(\infty)}$ one may take $\alpha^\pm, \gamma^\pm$, and a multiset $B$ in $[-\frac{1}{2}, \frac{1}{2}]$, which is obtained by mixing together $\beta^+$ and $\beta^-$. Specifically,

$$B = \{\beta_i^+ - \frac{1}{2}\}_{i=1,2,\ldots} \cup \{-\beta_j^- + \frac{1}{2}\}_{j=1,2,\ldots}.$$

Given a point $b \in B$, it is no longer possible to decide whether it comes from a coordinate of $\beta^+$ or of $\beta^-$. 
Remark 1.8. On the set of all characters, there is a natural operation: pointwise conjugation. For extreme characters, it reduces to the transposition \((\alpha^+, \beta^+, \delta^+) \leftrightarrow (\alpha^-, \beta^-, \delta^-)\).

Remark 1.9. One can check that the functions (1.2) separate conjugacy classes in \(U(\infty)\).

Remark 1.10. The reader might notice that each extreme character \(\chi^{(\omega)}\) is a multiplicative function with respect to a natural product on the set of conjugacy classes of \(U(\infty)\): taking disjoint union of two collections of eigenvalues. Such a multiplicativity property holds for many other groups and can be established independently of the classification of extreme characters, see the survey [Ol4].

2. Spherical and admissible representations

There are two ways to establish a connection between characters and unitary representations:

First, extreme characters of a group \(K\) parameterize its finite factor representations, i.e., unitary representations generating finite von Neumann factors. See, e.g., [Th2].

Second, extreme characters of \(K\) parameterize irreducible spherical representations of the pair \((G, \text{diag} K)\), where \(G = K \times K\) and \(\text{diag} K\) is the diagonal subgroup in \(G\). See [Ol1] and [Ol3, Theorem 24.5]. In these papers, it is also explained how to establish the relationship between both kinds of representations directly.

In the present paper we are dealing with spherical representations. Below we review basic facts concerning the correspondence between characters and spherical representations, specialized to the particular case \(K = U(\infty)\). For proofs of the claims stated below we refer to [Ol3].

- Set \(G = U(\infty) \times U(\infty)\) and \(K = \text{diag} U(\infty)\). Let \(\Psi\) denote the set of continuous functions \(\psi\) on the group \(G\) that are \(K\)-biinvariant, positive definite, and normalized at the unit element. Such functions will be called spherical functions. The set \(\Psi\) is convex.
- There exists a natural bijective correspondence \(\chi \leftrightarrow \psi\) between characters \(\chi \in \mathcal{X}(U(\infty))\) and spherical functions \(\psi \in \Psi\), which is defined as follows

\[
\chi(U) = \psi(U, 1), \quad \psi(U_1, U_2) = \chi(U_1 U_2^{-1}), \quad U, U_1, U_2 \in U(\infty).
\]

- The correspondence \(\chi \leftrightarrow \psi\) is an isomorphism of convex sets, so that extreme characters exactly correspond to extreme spherical functions.
- Any extreme function \(\psi \in \Psi\) is also extreme in a wider convex set, which is formed by all (not necessarily \(K\)-biinvariant) positive definite normalized functions on \(G\).
- By a spherical representation of \((G, K)\) we mean any pair \((T, \xi)\), where \(T\) is a unitary representation of \(G\) and \(\xi\) is a fixed cyclic unit \(K\)-invariant vector in the Hilbert space of \(T\). The vector \(\xi\) is called the spherical vector.
- There is a bijective correspondence \(\psi \leftrightarrow (T, \xi)\) between spherical functions and (equivalence classes of) spherical representations, defined by

\[
\psi(g) = (T(g)\xi, \xi), \quad g = (U_1, U_2) \in G.
\]
Equivalently, there is a bijection $\chi \leftrightarrow (T, \xi)$ between characters and (equivalence classes of) spherical representations, defined by
$$\chi(U) = (T(U, 1)\xi, \xi), \quad U \in U(\infty).$$

- Under this bijection, extreme spherical functions (or extreme characters) exactly correspond to irreducible spherical representations.
- In an irreducible spherical representation, the subspace of $K$-invariant vectors has dimension 1.

Combining these claims with Theorem 1.3 we get the following

**Corollary 2.1.** Irreducible spherical representations of the pair $(G, K)$, where $G = U(\infty) \times U(\infty)$ and $K = \text{diag} \, U(\infty)$, are parameterized by the points $\omega \in \Omega$, where the region $\Omega \subset \mathbb{R}^{1\infty+2}$ is defined in §1.

An explicit realization of the irreducible spherical representations is given in [Ol3], [Ol2].

The class of spherical representations is part of a wider class of unitary representations, which is defined as follows.

**Definition 2.2.** Let $G, K$ be as above and let $K_n$ denote the subgroup of $K \simeq U(\infty)$ constituted by matrices of the form $\begin{bmatrix} 1_n & 0 \\ 0 & * \end{bmatrix}$. A unitary representation $T$ of the group $G$ is called an admissible representation of $(G, K)$ if the subspace of $K_n$-invariant vectors (with varying $n$) is dense in the Hilbert space of $T$.

It is readily proved that any spherical representation is admissible. A (conjecturally, exhaustive) list of irreducible admissible representations, together with their explicit realization, can be found in [Ol3], [Ol2]. Each representation from this list is specified by a point $\omega \in \Omega$ and some additional discrete data. It is worth noting that, although $G$ is the product of two copies of $U(\infty)$, generic irreducible admissible representations of $(G, K)$ are not tensor products of two irreducible representations of $U(\infty)$.

3. The representations $T_{z,w}$

In this section we construct a family of representations of the group $G = U(\infty) \times U(\infty)$ depending on 2 complex parameters $z, w$.

Let us abbreviate
$$G(N) = U(N) \times U(N) \subset G, \quad K(N) = \text{diag} \, U(N) \subset K, \quad N = 1, 2, \ldots.\$$

We consider $U(N)$ as the homogeneous space $K(N) \setminus G(N)$ with the following right action of $G(N)$:
$$(U, (U_1, U_2)) \mapsto U_2^{-1}UU_1, \quad U \in U(N), \quad (U_1, U_2) \in G(N).$$

Let $N \geq 2$. Given a unitary matrix $U \in U(N)$, write it in the block form corresponding to the partition $N = (N-1) + 1$:
$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$
so that $D = U_{NN}$ is a complex number while $A$ is a $(N-1) \times (N-1)$ matrix, and set
$$p_N(U) = \begin{cases} A - B(1 + D)^{-1}C, & D \neq -1, \\ A, & D = -1. \end{cases}$$
Lemma 3.1. The map $p_N$ defines a projection $U(N) \to U(N-1)$, which commutes with the action of $G(N-1)$, that is, with left and right shifts by elements of $U(N-1)$.

We will call $p_N$ the canonical projection. The claim of the lemma is by no means new (see, e.g., [Ner3]). For the reader’s convenience we give a detailed proof below.

Proof. The fact that $p_N$ commutes with left and right shifts by elements of $U(N-1)$ is evident. Next, if $D = -1$ then $B = 0$, $C = 0$, and $A$ is a unitary matrix; therefore, in this case $p_N(U) \in U(N-1)$. It remains to prove that if $D \neq -1$ then the matrix $V = A - B(1 + D)^{-1}C$ is unitary.

Let $\Gamma \subset \mathbb{C}^N \oplus \mathbb{C}^N$ be the graph of $\tilde{U} = \begin{bmatrix} A & B \\ -C & -D \end{bmatrix}$, that is

$$\Gamma = \{ x \oplus y \mid x \in \mathbb{C}^N, y \in \mathbb{C}^N, y = \tilde{U}x \}.$$ 

Write

$$x = x_1 \oplus x_2, \quad y = y_1 \oplus y_2, \quad x_1, y_1 \in \mathbb{C}^{N-1}, \quad x_2, y_2 \in \mathbb{C}^1.$$ 

Intersect $\Gamma$ with the hyperplane $x_2 = y_2$ and denote by $\Gamma_1$ the image of this intersection under the projection

$$\mathbb{C}^N \oplus \mathbb{C}^N \to \mathbb{C}^{N-1} \oplus \mathbb{C}^{N-1}, \quad x \oplus y \mapsto x_1 \oplus y_1.$$ 

Clearly, $\Gamma_1$ is described by the linear equations

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} A & B \\ -C & -D \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad y_2 = x_2.$$ 

Or, equivalently,

$$y_1 = Ax_1 + Bx_2, \quad y_2 = -Cx_1 - Dx_2, \quad y_2 = x_2.$$ 

Excluding $x_2$ and $y_2$ from this system we conclude that $\Gamma_1$ coincides with the graph of $V$.

Since $\tilde{U}$ is unitary, we have $(x, x) = (y, y)$ for any $x \oplus y \in \Gamma$, or, equivalently, $(x_1, x_1) + |x_2|^2 = (y_1, y_1) + |y_2|^2$. Therefore, $x_2 = y_2$ implies $(x_1, x_1) = (y_1, y_1)$, which means that $V$ is unitary. □

Remark 3.2. The above interpretation of the projection $p_N$ in terms of graphs of operators also works in the exceptional case $D = -1$. More generally, it can be used to describe the projection $p_{M,N} = p_{M-1} \circ \cdots \circ p_N$ from $U(N)$ to $U(M)$ for any $M < N$. To do this we have to write the matrices $U \in U(N)$ in the block form corresponding to the partition $N = M + (N - M)$ and then repeat the same argument.

Remark 3.3. The projection $p_N$ is continuous on the open subset of $U(N)$ consisting of matrices $U$ with $U_{NN} \neq -1$ but not on the whole group $U(N)$. Indeed, the first claim is evident. To demonstrate discontinuity, take $N = 2$ and remark that

$$p_2 \begin{bmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{bmatrix} = \begin{cases} 1, & \cos \varphi \neq -1 \\ -1, & \cos \varphi = -1. \end{cases}$$
Remark 3.4. The projection $p_{M,N} : U(N) \to U(M)$ mentioned in Remark 3.2 is connected with characteristic functions in the sense of Livšic et al. Write $U \in U(N)$ in the block form corresponding to the partition $N = M + (N - M)$:

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $A$ is of the size $M \times M$. The matrix–valued function

$$\phi_U(\zeta) = A + \zeta B(1 - \zeta D)^{-1}C$$

is called the characteristic function of $U$: it is an important invariant of the action of the subgroup

$$\left\{ g \in U(N) \mid g = \begin{bmatrix} 1_M & 0 \\ 0 & \ast \end{bmatrix} \right\} \simeq U(N - M)$$

on $U(N)$ by conjugations. The function $\phi_U(\zeta)$ is an inner function: $\|\phi_U(\zeta)\| < 1$ in the open disk $|\zeta| < 1$ while $\phi_U(\zeta) \in U(M)$ on the boundary $|\zeta| = 1$. See [Ner1, Appendix E] for further information and references to original papers. We have $p_{M,N}(U) = \phi_U(-1)$.

Although $p_N$ is not continuous on the whole $U(N)$, it is clearly a Borel map. Hence, given a Borel measure on $U(N)$, we may project it to $U(N - 1)$ by means of $p_N$.

In particular, let $\mu_N$ denote the normalized Haar measure on $U(N)$. Then Lemma 3.1 implies that $p_N$ takes $\mu_N$ to an invariant probability measure on $U(N - 1)$, that is, to $\mu_{N-1}$.

Let $\mathcal{D} = \{ D \in \mathbb{C} \mid |D| \leq 1 \}$ denote the closed unit disk. Introduce a projection

$$\varepsilon_N : U(N) \to \mathcal{D}, \quad U \mapsto D = U_{NN}.$$ 

This map is constant on double cosets modulo $U(N - 1)$. Moreover, one can prove that $\varepsilon_N$ separates these cosets, i.e., if two matrices have the same $(N,N)$–entry then they belong to the same double coset modulo $U(N - 1)$.

Denote by $\nu_N$ the image of $\mu_N$ under $\varepsilon_N$; this is a probability measure on the disk $\mathcal{D}$.

Lemma 3.5.

$$\nu_N(dD) = \text{const } (1 - |D|^2)^{N-2} \ell(dD),$$

where $\ell$ is the Lebesgue measure on $\mathcal{D}$.

Proof. This claim is well known. It can be deduced from the fact that $\nu_N$ coincides with the image of the uniform measure on the $(2N - 1)$–dimensional sphere $\{ \zeta \in \mathbb{C}^N \mid |\zeta_1|^2 + \ldots + |\zeta_N|^2 = 1 \}$ under the projection $\zeta \mapsto \zeta_N \in \mathcal{D}$. □

Following [Ner3, §1.2], we combine $p_N$ and $\varepsilon_N$ into a single projection

$$\widetilde{p}_N : U(N) \to U(N - 1) \times \mathcal{D}.$$
Lemma 3.6. The image of $\mu_N$ under $\tilde{p}_N$ is $\mu_{N-1} \times \nu_N$.

Proof. Indeed, if $U \in U(N)$ and $\tilde{p}_N(U) = (V, D)$ then, for any $U_1, U_2 \in U(N - 1)$, we have $\tilde{p}_N(U_1^{-1}UU_1) = (U_2^{-1}VU_1, D)$. It follows that the pushforward of $\mu_N$ under $\tilde{p}_N$ splits into the direct product of $\mu_{N-1}$ with a certain probability measure on $\mathcal{D}$. Then the latter measure must coincide with $\nu_N$. \[\square\]

Set $H_N = L^2(U(N), \mu_N)$ and denote by $\text{Reg}_N$ the (bi)regular representation of $G(N)$ in $H_N$:

$$(\text{Reg}_N(g)f)(U) = f(U_2^{-1}UU_1), \quad f \in H_N, \quad U \in U(N), \quad g = (U_1, U_2) \in G(N).$$

Next, introduce the subspace $H^0_N \subset H_N$ of functions depending only on $\tilde{p}_N(\cdot)$.

Lemma 3.7. $H^0_N$ is a $G(N-1)$-invariant subspace, and we have a natural isometry

$$H^0_N \simeq H_{N-1} \otimes V_N, \quad V_N = L^2(\mathcal{D}, \nu_N).$$

Proof. Indeed, this follows from the definitions and Lemma 3.6. \[\square\]

By Lemma 3.7, any unit vector $v \in V_N$ determines an embedding $\text{Reg}_{N-1} \rightarrow \text{Reg}_N$,

$$H_{N-1} \ni f \mapsto f \otimes v \in H^0_N \subset H_N.$$  

Fix two complex numbers $z, w$. Our next aim is to assign a meaning to the expression

$$f_{z,w} \mid_N(U) = \det((1 + U)^z(1 + U^{-1})^w), \quad U \in U(N).$$

To do this we need a preparation which concludes in Definition 3.10.

Let $\text{Mat}(N, \mathbb{C})$ denote the space of complex $N \times N$ matrices. For $X \in \text{Mat}(N, \mathbb{C})$ let $\Re X = \frac{1}{2}(X + X^*)$. We set

$$\text{Mat}(N, \mathbb{C})_+ = \{X \in \text{Mat}(N, \mathbb{C}) \mid \Re X > 0\},$$

$$\text{Mat}(N, \mathbb{C})_{>-1} = \{X \in \text{Mat}(N, \mathbb{C}) \mid \Re X > -1\} = -1 + \text{Mat}(N, \mathbb{C})_+.$$  

These are open domains in $\text{Mat}(N, \mathbb{C})$, isomorphic to a matrix wedge.

Lemma 3.8. For any fixed $z \in \mathbb{C}$, the expression $(1 + X)^z$ makes sense for $X \in \text{Mat}(N, \mathbb{C})_{>-1}$ and defines a holomorphic map $\text{Mat}(N, \mathbb{C})_{>-1} \rightarrow \text{GL}(N, \mathbb{C})$. Moreover,

$$(1 + X)^z \cdot (1 + X)^{z'} = (1 + X)^{z + z'}, \quad X \in \text{Mat}(N, \mathbb{C})_{>-1}, \quad z, z' \in \mathbb{C}.$$  

Proof. Setting $1 + X = Y$ we must prove that $Y \mapsto Y^z$ is a correctly defined holomorphic map $\text{Mat}(N, \mathbb{C})_+ \rightarrow \text{GL}(N, \mathbb{C})$.

First, remark that the spectrum of any matrix $Y \in \text{Mat}(N, \mathbb{C})_+$ lies in the open half–plane $\mathbb{C}_+ = \{\zeta \mid \Re \zeta > 0\}$. Indeed, let $\zeta$ be an eigenvalue of $Y$ and $v \neq 0$ be any eigenvector corresponding to $\zeta$. By the definition of $\text{Mat}(N, \mathbb{C})_+$,

$$0 < ((Y + Y^*)v, v) = \zeta + \overline{\zeta},$$

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which means that $\zeta \in \mathbb{C}_+$. Now we can define $Y^z$ via the functional calculus:

$$Y^z = \int_C \zeta^z (\zeta 1 - Y)^{-1} d\zeta,$$

where $\zeta^z$ is correctly defined in $\mathbb{C}_+$ (we choose the branch taking value 1 at $\zeta = 1 \in \mathbb{C}_+$) and $C$ is any simple contour in $\mathbb{C}_+$ containing the spectrum of $Y$.

Clearly, $Y^z$ is nonsingular together with $Y$.

Now, to check the second claim it suffices to remark that it obviously holds for matrices $X$ close to 0. □

**Corollary 3.9.** For any fixed $z \in \mathbb{C}$, the function $X \mapsto \text{det}((1 + X)^z)$ is correctly defined and holomorphic in $\text{Mat}(N, \mathbb{C})_{>-1}$. Moreover,

$$\text{det}((1 + X)^z) \text{det}((1 + X)^{z'}) = \text{det}((1 + X)^{z + z'}), \quad X \in \text{Mat}(N, \mathbb{C})_{>-1}, \quad z, z' \in \mathbb{C}.$$ □

Note that, denoting by $x_1, \ldots, x_N$ the eigenvalues of $X$, we have

$$\text{det}((1 + X)^z) = \prod_{k=1}^{N} (1 + x_k)^z.$$ The right–hand side makes sense, because $\Re x_k > -1$. On the contrary, the expression $(\text{det}(1 + X))^z = (\prod (1 + x_k))^z$ is ambiguous.

**Definition 3.10.** Let $U(N)' \subset U(N)$ denote the set of unitary matrices which do not have $-1$ as an eigenvalue. Note that $U(N)' = U(N) \cap \text{Mat}(N, \mathbb{C})_{>-1}$. By Corollary 3.9, the function

$$f_{z,w|N}(U) = \text{det}((1 + U)^z(1 + U^{-1})^w)$$

is well defined on $U(N)'$.

Equivalently, denoting by $u_1, \ldots, u_N$ the eigenvalues of $U$ and assuming that none of them equals $-1$, we have

$$f_{z,w|N}(U) = \prod_{k=1}^{N} (1 + u_k)^z(1 + \bar{u}_k)^w.$$

On the complement of $U(N)'$, which is a negligible set with respect to the Haar measure, we agree to continue the function by 0. □

Thus, we have defined the function $f_{z,w|N}$ for any complex $z, w$ and each $N = 1, 2, \ldots$. In a few lemmas below we describe some special properties of these functions which are used in the construction of the representations.

**Lemma 3.11.** Write arbitrary $N \times N$ matrices in block form according to the partition $N = (N - 1) + 1$. The map

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \rightarrow X_1 = A - B(1 + D)^{-1}C$$
is correctly defined in the domain \( \text{Mat}(N, \mathbb{C})_{>-1} \) and projects it onto the domain \( \text{Mat}(N-1, \mathbb{C})_{>-1} \).

Consequently, \( p_N \) maps \( U(N)' \) onto \( U(N-1)' \).

**Proof.** First of all, note that \( X \in \text{Mat}(N, \mathbb{C})_{>-1} \) implies \( \Re D > -1 \), so that \( 1 + D \neq 0 \) and the matrix \( X_1 \) is well defined.

Next, we use the argument and notation of Lemma 3.1. Consider the graph \( \Gamma \) of the operator \( \begin{bmatrix} A & B \\ -C & -D \end{bmatrix} \). In terms of \( \Gamma \), the condition \( \Re X > -1 \) means

\[
(x_1, x_1) + |x_2|^2 + \Re((x_1, y_1) - x_2 \bar{y}_2) > 0, \quad x \oplus y \in \Gamma.
\]

When \( x_2 = y_2 \), this condition turns into

\[
(x_1, x_1) + \Re(x_1, y_1) > 0, \quad x_1 \oplus y_1 \in \Gamma_1,
\]

which means that \( X_1 \in \text{Mat}(N-1, \mathbb{C})_{>-1} \).

Together with Lemma 3.1 this implies that \( p_N \) maps \( U(N)' \) to \( U(N-1)' \). The surjectivity is evident. \( \square \)

**Lemma 3.12.** For any \( z \in \mathbb{C} \) and \( X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{Mat}(N, \mathbb{C})_{>-1} \) we have

\[
\det((1 + X)^z) = \det((1 + p_N(X))^z) \cdot (1 + D)^z.
\]

**Proof.** First of all, note that all the terms of this formula make sense. Indeed, \( (1 + X)^z \) is correctly defined by Lemma 3.8, \( (1 + p_N(X))^z \) is correctly defined by virtue of Lemma 3.8 and Lemma 3.11, and, finally, \( (1 + D)^z \) is well defined, because \( \Re D > -1 \).

Next, we remark that these three terms are holomorphic functions in \( X \in \text{Mat}(N, \mathbb{C})_{>-1} \). Hence, it suffices to prove the formula when \( X \) is near zero. Then we may interchange the symbol of determinant and exponentiation, thus reducing the problem to the identity

\[
\det(1 + X) = \det(1 + p_N(X)) \cdot (1 + D),
\]

which is equivalent to

\[
\det \begin{bmatrix} 1 + A \\ C \\ 1 + D \end{bmatrix} = \det((1 + A) - B(1 + D)^{-1}C) \cdot (1 + D).
\]

This is a special case of the well-known identity for the determinant of a block matrix (of an arbitrary format),

\[
\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det(a - bd^{-1}c) \det d.
\]

\( \square \)

As a corollary we get the following formula:

\[
f_{z, w|N}(U) = f_{z, w|N-1}(p_N(U)) \cdot (1 + D)^z(1 + \overline{D})^w,
\]

(3.1)

where

\[
U \in U(N)', \quad D = U_{NN} = \varepsilon_N(U) \in \mathcal{D} \setminus \{-1\}.
\]
Lemma 3.13. The function $D \mapsto (1+D)^{(1+D)^w}$ is square integrable with respect to the measure $\nu_N$ on the unit disk $\mathcal{D}$ provided that $2Rz + 2Rw + N > 0$.

Assume that this condition is satisfied and let $v_{z,w|N}$ denote the above function viewed as a vector of the Hilbert space $V_N = L^2(\mathcal{D}, \nu_N)$. Then we have

$$\|v_{z,w|N}\|^2 = \int_\mathcal{D} \|(1+D)^{(1+D)^w}\|^2 \nu_N(dD) = \frac{\Gamma(N)\Gamma(N + z + \bar{z} + w + \bar{w})}{\Gamma(N + z + \bar{w})\Gamma(N + \bar{z} + w)}.$$

Proof. The measure $\nu_N$ is given by the formula of Lemma 3.5. Using it we get

$$\int_\mathcal{D} \|(1+D)^{(1+D)^w}\|^2 \nu_N(dD) = \text{const} \int_\mathcal{D} (1+D)^{z+w}(1+D)^{\bar{z}+w}(1-|D|^2)^N - 2 \ell(dD),$$

where $\ell$ denotes the Lebesgue measure. A way of calculating the latter integral is given in [Ner3, §1.8]. The constant is fixed by the requirement that the whole expression equals 1 for $z = w = 0$. □

Lemma 3.14. Assume that $Rz + Rw > -\frac{1}{2}$. Then the function $f_{z,w|N}$ belongs to the Hilbert space $H_N = L^2(U(N), \mu_N)$, and

$$\|f_{z,w|N}\|^2 = \prod_{k=1}^N \frac{\Gamma(k)\Gamma(k + z + \bar{z} + w + \bar{w})}{\Gamma(k + z + \bar{w})\Gamma(k + \bar{z} + w)}.$$

Proof. Using the identification of two spaces indicated in Lemma 3.7 we can rewrite formula (3.1) as

$$f_{z,w|N} = f_{z,w|N-1} \otimes v_{z,w|N}, \quad N \geq 2.$$

Note that our assumption on the parameters implies that $2Rz + 2Rw + N > 0$ for any $N = 2, 3, \ldots$ (even for $N = 1$). Consequently, $v_{z,w|N}$ is well defined as a vector of $V_N$. Applying Lemma 3.13 we get that $f_{z,w|N}$ is square integrable provided that $f_{z,w|N-1}$ is square integrable, and we have by recurrence

$$\|f_{z,w|N}\| = \|f_{z,w|1}\| \cdot \|v_{z,w|2}\| \cdots \|v_{z,w|N}\|.$$

It remains to check that the function $f_{z,w|1}$ is square integrable on $U(1)$ (the unit circle) provided that $Rz + Rw > -\frac{1}{2}$, and

$$\|f_{z,w|1}\|^2 = \frac{\Gamma(1)\Gamma(1 + z + \bar{z} + w + \bar{w})}{\Gamma(1 + z + \bar{w})\Gamma(1 + \bar{z} + w)}.$$

That is, denoting by $|du|$ the Lebesgue measure on the unit circle $|u| = 1$,

$$\int_{|u|=1} (1 + u)^{z+w}(1 + \bar{u})^{\bar{z}+w} |du| = \frac{\Gamma(1)\Gamma(1 + z + \bar{z} + w + \bar{w})}{\Gamma(1 + z + \bar{w})\Gamma(1 + \bar{z} + w)}.$$

This can be proved by the same argument as in [Ner3, §1.8].

Note also that the value of the above integral coincides with the value of the expression of Lemma 3.13 formally specialized at $N = 1$. This coincidence has an explanation. Indeed, consider the expression for the measure $\nu_N$ on the disk $\mathcal{D}$ given in Lemma 3.5, and assume that the parameter $N$ is a real number. Then, as $N$ tends to 1 from above, the measure degenerates to the uniform measure concentrated on the unit circle. □

The above results make it possible to give the following definition.
Definition 3.15.

- Let us fix arbitrary \( z, w \in \mathbb{C} \). If \( N \) is large enough \( (N > -2\Re z - 2\Re w) \) then the vector \( v_{z,w|N} \in V_N \) is well defined, and we can normalize it by setting

\[
v'_{z,w|N} = \frac{v_{z,w|N}}{\|v_{z,w|N}\|},
\]

where the norm is given by Lemma 3.13.
- Define an isometric embedding \( L_{z,w|N} : H_{N-1} \to H_N \) using the identification of Lemma 3.7:

\[
H_{N-1} \ni f \mapsto f \otimes v'_{z,w|N} \in H_{N-1} \otimes V_N \simeq H_N^0 \subset H_N.
\] (3.2)

It commutes with the action of \( G(N-1) \). Consequently, we can form the inductive limit Hilbert space \( H = \lim \downarrow H_N \) and the natural unitary representation \( \lim \downarrow \text{Reg}_N \) in \( H \). We denote it by \( T_{z,w} \).
- If \( \Re z + \Re w > -\frac{1}{2} \) then the functions \( f_{z,w|N} \) are square integrable and we can normalize them:

\[
f'_{z,w|N} = \frac{f_{z,w|N}}{\|f_{z,w|N}\|}.
\]

Then (3.2) implies existence of the inductive limit vector

\[
\xi_{z,w} = \lim \downarrow f'_{z,w|N} \in H.
\]

It is \( K \)-invariant, because all the functions \( f_{z,w|N} \) are constant on conjugacy classes.
- It is worth noting that the definition of this distinguished vector makes sense only when \( \Re z + \Re w > -\frac{1}{2} \).

Remark 3.16. Removing the normalization of the function \( v_{z,w|N} \) we get a map \( \tilde{L}_{z,w|N} \), which differs from \( L_{z,w|N} \) by a scalar multiple and has the form

\[
(\tilde{L}_{z,w|N} f)(U) = f(p_N(U)) v_{z,w|N}(U_{NN}), \quad U \in U(N).
\]

It is worth noting that \( \tilde{L}_{z,w|N} \) makes sense for any \( N \), irrespective of the values of the parameters \( z, w \), and can be applied to any function \( f \) on \( U(N-1) \). By virtue of (3.1), we have

\[
\tilde{L}_{z,w|N} : f_{z,w|N-1} \mapsto f_{z,w|N}.
\]

\( \square \)

The formula (3.2) describes the embedding \( L_{z,w|N} : H_{N-1} \to H_N \). More generally, for \( M < N \) (where \( M \) is large enough), we shall now describe the embedding

\[
L_{z,w|M,N} : H_M \to H_N, \quad L_{z,w|M,N} = L_{z,w|N} \circ \cdots \circ L_{z,w|M+1}.
\] (3.3)

Write any \( U \in U(N) \) in block form \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) according to the partition \( N = M + (N-M) \). Then \( D \) lies in the matrix ball \( \mathcal{D}_{N-M} \), the set of complex matrices.
of size \((N - M) \times (N - M)\), with norm \(\leq 1\). Generalizing the definitions of \(p_N\) and \(\varepsilon_N\), consider the maps

\[
\varepsilon_{M,N} : U(N) \to \mathfrak{D}_{N-M}, \quad U \mapsto D,
\]

\[
\overline{\varepsilon}_{M,N} : U(N) \to U(M) \times \mathfrak{D}_{N-M}, \quad \overline{\varepsilon}_{M,N} = p_{M,N} \times \varepsilon_{M,N},
\]

where \(p_{M,N}\) was introduced in Remark 3.2 (if \(U \in U(N)\) then \(p_{M,N}(U) = A - B(1 + D)^{-1}C\)). Let \(\nu_{M,N}\) be the image of the Haar measure \(\mu_N\) under the map \(\varepsilon_{M,N} : U(N) \to \mathfrak{D}_{N-M}\), and let \(V_{M,N}\) denote the Hilbert space \(L^2(\mathfrak{D}_{N-M}, \nu_{M,N})\).

Next, let \(H^0_{M,N} \subset H_N = L^2(U(N), \mu_N)\) be the subspace of functions depending on \(\overline{\varepsilon}_{M,N}(\cdot)\) only. As in Lemma 3.7, we have a natural isometry between \(H^0_{M,N}\) and \(H_M \otimes V_{M,N}\); we use it to identify these two spaces.

Set \(\mathfrak{D}'_{N-M} = \mathfrak{D}_{N-M} \cap \text{Mat}(N - M)_{> -1}\); this subset of \(\mathfrak{D}_{N-M}\) contains the interior of the matrix ball. Generalizing the definition of the vector \(v_{z,w|N}\) (see Lemma 3.13), we introduce a function \(v_{z,w|M,N}\) on \(\mathfrak{D}_{N-M}\) as follows:

\[
v_{z,w|M,N}(D) = \begin{cases} 
\det((1 + D)^2) \det((1 + \overline{D})^w), & D \in \mathfrak{D}'_{N-M}; \\
0, & D \in \mathfrak{D}_{N-M} \setminus \mathfrak{D}'_{N-M}.
\end{cases}
\]

**Proposition 3.17.** Let \(M\) be so large that the embeddings \(L_{z,w|N}\) are well defined for all \(N > M\). Fix \(N > M\).

(i) The function \(v_{z,w|M,N}\) defined in (3.4) lies in the Hilbert space \(V_{M,N} = L^2(\mathfrak{D}_{M,N}, \nu_{M,N})\).

(ii) The embedding \(L_{z,w|M,N}\) defined in (3.3) maps \(H_M\) into the subspace \(H^0_{M,N}\) of \(H_N\). Under the identification \(H^0_{M,N} = H_M \otimes V_{M,N}\), this embedding has the form

\[
L_{z,w|N} : f \mapsto f \otimes v'_{z,w|M,N}, \quad v'_{z,w|M,N} = \frac{v_{z,w|M,N}}{\|v_{z,w|M,N}\|}.
\]

**Proof.** From the definition (3.3) it follows that

\[
(L_{z,w|N} f)(U) = f(p_{M,N}(U))g(U),
\]

where \(g\) is a certain function not depending on \(f\). We shall prove that \(g(U)\) is proportional to \(v_{z,w|M,N}(\varepsilon_{M,N}(U))\). It will follow that the image of \(L_{z,w|N}\) lies in \(H^0_{z,w|M,N}\) and then all the claims of the proposition will become clear.

It is convenient to pass to the maps defined in Remark 3.16. Then we get, similarly to (3.3), a map \(\tilde{L}_{z,w|M,N}\), which differs from \(L_{z,w|M,N}\) by a scalar multiple. Similarly to (3.5), we have

\[
(\tilde{L}_{z,w|M,N} f)(U) = f(p_{M,N}(U))\tilde{g}(U),
\]

where \(\tilde{g}\) is proportional to \(g\). According to Remark 3.16, we may apply formula (3.6) to any function \(f\), not necessarily a square integrable one. So, we may take \(f = f_{z,w|M}\). By the last claim of Remark 3.16, \(\tilde{L}_{z,w|N}\) takes \(f_{z,w|M}\) to \(f_{z,w|N}\), which implies

\[
\tilde{g}(U) = \frac{f_{z,w|N}(U)}{f_{z,w|M}(p_{M,N}(U))}.
\]

On the other hand, the same argument as in the proof of Lemma 3.12 shows that the right-hand side is equal to \(v_{z,w|M,N}(D)\), where \(D = \varepsilon_{M,N}(U) \in \mathfrak{D}_{N-M}\).

Thus, \(\tilde{g}(U) = v_{z,w|M,N}(D)\). Since \(\tilde{g}\) is proportional to \(g\), this concludes the proof. □
**Theorem 3.18.** All representations constructed in Definition 3.15 are admissible in the sense of Definition 2.2.

**Proof.** Assume as above that $M$ is a sufficiently large natural number, so that the embedding $H_M \subset H$ is well defined. By the very definition, the subspace $\cup_M H_M$ is dense in $H$.

On the other hand, denote by $H'_M$ the subspace in $H$ formed by all $K_M$-invariant vectors (recall that $K_M \subset K \simeq U(\infty)$ is the subgroup of matrices of the form

$\begin{bmatrix} 1 & 0 \\ 0 & * \end{bmatrix}$, see Definition 2.2). The subspaces $H'_M$ form an ascending chain (because the subgroups $K_M$ form a descending chain). We must prove that $\cup_M H'_M$ is dense in $H$. To do this we shall prove that $H'_M \supset H_M$.

For any $N > M$, let $K_M(N)$ denote the subgroup in $K \simeq U(\infty)$ that is the intersection of $K_M$ with $U(N)$. Clearly, $K_M(N)$ is isomorphic to $U(N-M)$ and we have $K_M = \cup_{N>M} K_M(N)$. Note that the function $v_{z,w}|_{M,N}$ on the matrix ball $D_{N-M}$ is invariant with respect to conjugation by unitary matrices from $U(N-M)$.

Note also that, for any $U \in U(N)$, the matrix $p_{M,N}(U)$ does not change when we conjugate $U$ by a matrix from $K_M(N) \subset U(N)$. Combining this with the description of the embedding $H_M \to H_N$ we conclude that the vectors from $H_M$ are invariant with respect to $K_M(N)$. Since this holds for any $N$, it follows that all these vectors are $K_M$-invariant, i.e., $H_M \subset H'_M$. □

4. **The space $\mathfrak{U}$ of virtual unitary matrices and another description of the representations $T_{z,w}$**

The construction of the representations $T_{z,w}$ described in §3 is quite similar to the second construction of the “generalized regular representations” $T_z$ in [KOV]. In this section we present the counterpart of the first construction from [KOV].

**Definition 4.1.** Let $\mathfrak{U} = \varprojlim U(N)$ be the projective limit of the spaces $U(N)$, as $N \to \infty$, taken with respect to the projections $p_N : U(N) \to U(N-1)$. By analogy with the space of virtual permutations from [KOV] we call $\mathfrak{U}$ the space of virtual unitary matrices.

By definition, a point of $\mathfrak{U}$ is an arbitrary sequence $x = (x_N)$, where $x_N \in U(N)$ for any $N = 1, 2, \ldots$, and $p_N(x_N) = x_{N-1}$ for any $N \geq 2$. We denote by $\pi_N$ the natural projection map $\mathfrak{U} \to U(N)$ sending $x$ to $x_N$.

There is a natural embedding $U(\infty) \hookrightarrow \mathfrak{U}$ assigning to a matrix $U \in U(\infty)$ a sequence $x = (x_N)$ such that $x_N = U$ provided that $N$ is so large that $U \in U(N) \subset U(\infty)$. Thus, the image of $U(\infty)$ in $\mathfrak{U}$ consists of the stabilizing sequences $x = (x_N)$.

However, general elements of the space $\mathfrak{U}$ cannot be interpreted as unitary matrices.

Recall that the map $p_N$ is continuous on the open subset $U(N)' \subset U(N)$ but discontinuous on the whole space $U(N)$ (Remark 3.3). This is an obstacle to equipping the space $\mathfrak{U}$ with a natural topology. However, $p_N$ certainly are Borel maps, so that $\mathfrak{U}$ has a natural Borel structure. We keep in mind this structure while speaking of measures on $\mathfrak{U}$.

**Definition 4.2.** By Lemma 3.11, $p_N$ maps $U(N)' \to U(N-1)'$, which makes it possible to define the following subset in $\mathfrak{U}$:

$\mathfrak{U}' = \varprojlim U(N)' \subset \mathfrak{U}$. 

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Definition 4.3. A subset of \( \mathcal{U} \) will be called *negligible* if for any \( N \), its image under \( \pi_N : \mathcal{U} \to U(N) \) is a null set with respect to the Haar measure on \( U(N) \).

For instance, \( \mathcal{U} \setminus \mathcal{U}’ \) is a negligible set. Let us agree to identify functions on \( \mathcal{U} \) that coincide outside a negligible set. Let also agree that a function is well defined on \( \mathcal{U} \) if it is defined outside a negligible set.

Definition 4.4. We define a right action \( (x,g) \mapsto x.g \) of the group \( G = U(\infty) \times U(\infty) \) on the space \( \mathcal{U} \) by

\[
\pi_N(x.g) = U_2^{-1} \pi_N(x) U_1, \quad x \in \mathcal{U}, \quad g \in (U_1, U_2) \in G,
\]

where \( N \) is so large that \( g \in G(N) = U(N) \times U(N) \). The correctness of this definition follows from the basic equivariance property of the canonical projection \( p_N \), see Lemma 3.1.

In other words, this action arises from the right actions of the groups \( G(N) \) on the spaces \( U(N) \) defined in the beginning of §3. On \( U(\infty) \subset \mathcal{U} \), this action coincides with \( (U, (U_1, U_2)) \mapsto U_2^{-1} U U_1 \). Note also that the action of the subgroup \( K = \text{diag} U(\infty) \) arises from the actions of the groups \( U(N) \) on themselves by conjugations.

Note that the shift of a negligible set by an element of \( G \) is negligible, too.

Definition 4.5. A function on \( \mathcal{U} \) is called *cylindrical* if it has the form \( F(x) = F_N(\pi_N(x)) \) for a certain \( N \) and a certain function \( F_N \) on \( U(N) \). More generally, we extend (in an evident way) this definition to functions defined outside a negligible set.

Lemma 4.6. Let \( z, w \in \mathbb{C} \), \( g \in G \), and \( x \in \mathcal{U}' \cap (\mathcal{U}', g^{-1}) \). If \( N \) is large enough then the expression

\[
\frac{f_{z, w|N}(\pi_N(x.g))}{f_{z, w|N}(\pi_N(x))}
\]

does not depend on \( N \).

Proof. Indeed, assume \( g \in G(N - 1) \), and let us prove that the expression above does not change when \( N \) is replaced by \( N - 1 \). Recall that the function \( f_{z, w|N} \) is the product of two determinants (which correspond to the particular cases \( w = 0 \) and \( z = 0 \), respectively). We shall check the above claim for the first determinant; for the second determinant it is proved similarly, so that the desired claim will follow.

Let \( X = \pi_N(x) \) and \( g = (U_1, U_2) \); then \( \pi_N(x.g) = U_2^{-1} X U_1 \). Since \( g \in G(N - 1) \), the matrices \( U_1, U_2 \) lie in \( U(N - 1) \). It follows that \( p_N(U_2^{-1} X U_1) = U_2^{-1} p_N(X) U_1 \) (Lemma 3.1) and that the matrix entry \( D = X_{NN} \) does not change when \( X \) is replaced by \( U_2^{-1} X U_1 \).

Then, applying Lemma 3.12, we get that the ratio

\[
\frac{\det((1 + U_2^{-1} X U_1)^z)}{\det((1 + X)^z)}
\]

does not change when \( X \) is replaced by \( p_N(X) \), which concludes the proof. \( \square \)

As a corollary we get
Proposition 4.7. Fix arbitrary $z, w \in \mathbb{C}$. For $g \in G$ and $x \in \Omega$, set

$$C_{z,w}(x,g) = \text{the stable value of } \frac{f_{z,w|N}(\pi_N(x).g))}{f_{z,w|N}(\pi_N(x))} \text{ as } N \to \infty.$$ 

For any $g$, this is a correctly defined cylindrical function in $x$ in the sense of Definition 4.5. Furthermore, $C_{z,w}(x,g)$ possesses the multiplier property

$$C_{z,w}(x,g_1)C_{z,w}(x.g_1,g_2) = C_{z,w}(x,g_1g_2), \quad x \in \Omega, \quad g_1,g_2 \in G.$$ 

Finally,

$$C_{z,w}(\cdot,g) \equiv 1 \quad \text{for } g \in K.$$ 

Proof. The first claim follows from Lemma 4.6. The second claim is evident. The third claim follows from the fact that the function $f_{z,w|N}$ on $U(N)$ is central (constant on conjugacy classes). □

Fix an arbitrary $s \in \mathbb{C}$ with $\Re s > -\frac{1}{2}$. For each $N = 1, 2, \ldots$ we consider a measure $\mu^{(s)}_N$ on $U(N)$ such that

$$\mu^{(s)}_N(dU) = \left( \prod_{k=1}^N \frac{\Gamma(k)\Gamma(k+s+\bar{s})}{\Gamma(k+s)\Gamma(k+\bar{s})} \right)^{-1} |\det((1+U)^s)|^2 \mu_N(dU),$$

where, as above, $\mu_N$ is the normalized Haar measure on $U(N)$. When $s = 0$, this measure reduces to $\mu_N$. Further, using the formula of Corollary 3.9 and Lemma 3.14, we get that

$$\frac{\mu^{(s)}_N(dU)}{\mu_N(dU)} = \frac{|f_{z,w|N}(U)|^2}{\|f_{z,w|N}\|^2} \quad \text{for any } z, w \text{ such that } z + \bar{w} = s.$$ 

This implies that $\mu^{(s)}_N$ is a probability measure. Since $f_{z,w|N}$ is central, $\mu^{(s)}_N$ is invariant under the action of $U(N)$ on itself by conjugations.

Lemma 4.8. The family $\{\mu^{(s)}_N\}_{N \geq 1}$ is consistent with the projections $p_N$, i.e., the pushforward of $\mu^{(s)}_N$ under $p_N$ is $\mu^{(s)}_{N-1}$.

Proof. Same reasoning, based on Lemma 3.12, as in the proof of Lemma 3.13. □

This makes it possible to give the following

Definition 4.9. For any $s \in \mathbb{C}$, $\Re s > -1/2$, we denote by $\mu^{(s)}$ the projective limit of the family $\{\mu^{(s)}_N\}$ as $N \to \infty$. This is a probability Borel measure on $\Omega$.

Note the following properties of the measures $\mu^{(s)}$:

- $\mu^{(s)}$ is invariant with respect to the action of $K$ on $\Omega$;
- the measure $\mu^{0}$ is the projective limit of the Haar measures $\mu_N$; consequently, it is $G$–invariant;
- negligible sets are null sets with respect to any measure $\mu^{(s)}$. 

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Proposition 4.10. For any $s \in \mathbb{C}$ with $\Re s > -\frac{1}{2}$, the measure $\mu^{(s)}$ is $G$-quasiinvariant. Its Radon–Nikodym derivative
\[ \tau_s(x, g) = \frac{\mu^{(s)}(d(x,g))}{\mu^{(s)}(dx)}, \quad x \in \mathcal{U}, \ g \in G, \]
is a cylindrical function in $x$. Moreover, for any complex $z, w$ such that $z + \bar{w} = s$ we have
\[ \tau_s(x, g) = |C_{z,w}(x, g)|^2, \quad x \in \mathcal{U}, \ g \in G, \]
where $C_{z,w}(x, g)$ is the multiplier introduced in Proposition 4.7.

Proof. Indeed, this follows from the definition of the measures $\mu^{(s)}_{\mathcal{U}}$ and Proposition 4.7. $\square$

Theorem 4.11. Let $s, z, w \in \mathbb{C}$ be such that $\Re s > -\frac{1}{2}$, $z + \bar{w} = s$, and let $C_{z,w}(x, g)$ be the multiplier introduced in Proposition 4.7. The following formula defines a unitary representation $T_{z,w}$ of the group $G = U(\infty) \times U(\infty)$ in the Hilbert space $H = L^2(\mathcal{U}, \mu^{(s)})$:
\[ (T_{z,w}(g)f)(x) = f(x,g)C_{z,w}(x, g), \quad g \in G, \ f \in H, \ x \in \mathcal{U}. \]

Further, the constant function $1 \in L^2(\mathcal{U}, \mu^{(s)})$ is a unit $K$–invariant vector in $H$.

We call $1$ the distinguished vector of the representation $T_{z,w}$.

Proof. The correctness of the definition follows from Proposition 4.10, and the second claim follows from the $K$–invariance property of the multiplier, see the last claim of Proposition 4.7. $\square$

Proposition 4.12. Let $z, w \in \mathbb{C}$ and $s = z + \bar{w}$. Assume that $\Re s > -\frac{1}{2}$. There exists a Hilbert space isometry between the space $H$ of the representation $T_{z,w}$ and the space $H$ of the representation $T_{z,w}$ constructed in section 3; this isometry intertwines the representations $T_{z,w}$ and $T_{z,w}$ and takes the distinguished vector $1 \in H$ to the distinguished vector $\xi_{z,w} \in H$ introduced in Definition 3.15.

Proof. Recall that $H = \lim_{N} H_{N}$, where $H_{N} = L^2(\overline{U}(N), \mu_{N})$. On the other hand, since $\mu^{(s)} = \lim_{N} \mu^{(s)}_{\mathcal{U}}$, we have $H = \lim_{N} H_{N}$, where $H_{N} = L^2(\overline{U}(N), \mu^{(s)}_{\mathcal{U}})$. For each $N = 1, 2, \ldots$, the operator of multiplication by the function
\[ f'_{z,w|N}(\cdot) = \frac{f_{z,w|N}(\cdot)}{\|f_{z,w|N}\|} \]
defines an isometry $H_{N} \to H_{N}$ taking the constant function $1$ to the vector $f'_{z,w|N}$.

By the very construction, these isometries are consistent with the embeddings $H_{N} \to H_{N+1}$ and $H_{N} \to H_{N+1}$ and, consequently, define an isometry $H \to H$. Clearly, the isometry $H \to H$ takes $1$ to $\xi_{z,w}$. Moreover, again by the construction of Theorem 4.11, $H \to H$ is an intertwining operator between the representations $T_{z,w}$ and $T_{z,w}$. $\square$

Remark 4.13. The above results can be extended, with appropriate modifications, to the case when $z, w$ are arbitrary complex numbers (and $s = z + \bar{w}$, as usual). When $\Re s \leq -\frac{1}{2}$, the measure $\mu^{(s)}$ is infinite and the constant function $1$ is not a vector of $H$. Recall that the definition of the distinguished vector $\xi_{z,w}$ also fails in this case. The equivalence of $T_{z,w}$ and $T_{z,w}$ remains valid for all values of the parameters.
5. Measures on the space of Hermitian matrices

Here we give another description of the measures \( \mu^{(s)} \) which were introduced in §4.

Let \( \text{Herm}(N) \) denote the space of \( N \times N \) complex Hermitian matrices.

**Definition 5.1.** We introduce a bijection \( U \leftrightarrow X \) between \( U(N)' \subset U(N) \) and \( \text{Herm}(N) \) as follows:

\[
U = \frac{i - X}{i + X}, \quad X = \frac{1 - U}{1 + U},
\]

where \( i = \sqrt{-1} \) is identified with the scalar matrix \( i \cdot 1_N \). We call \( U(N)' \rightarrow \text{Herm}(N) \) the *Cayley transform* and \( \text{Herm}(N) \rightarrow U(N)' \) the inverse Cayley transform.

We define the projection \( p'_N : \text{Herm}(N) \rightarrow \text{Herm}(N - 1) \) as the operation of removing the \( N \)th row and the \( N \)th column from a \( N \times N \) matrix.

**Lemma 5.2.** The following diagram is commutative

\[
\begin{array}{ccc}
U(N)' & \xrightarrow{\text{Cayley}} & \text{Herm}(N) \\
\downarrow p_N & & \downarrow p'_N \\
U(N - 1)' & \xrightarrow{\text{Cayley}} & \text{Herm}(N - 1)
\end{array}
\]

*Proof.* Indeed, this is verified by a direct calculation. The easiest way is to use the interpretation of \( p_N \) in terms of graphs of operators, see the proof of Lemma 3.2. \( \square \)

**Proposition 5.3.** The Cayley transform takes the Haar measure \( \mu_N \) on \( U(N)' \) to the measure

\[
m_N(dX) = \text{const} \det(1 + XX^*)^{-N} \ell(dX)
\]
on \( \text{Herm}(N) \). Here \( \ell \) denotes the Lebesgue measure.

More generally, for any \( s \in \mathbb{C} \) with \( \Re s > -\frac{1}{2} \), the measure \( \mu_N^{(s)} \) is transformed to the measure

\[
m_N^{(s)}(dX) = \text{const} \det((1 - iX)^{-s}) \det((1 + iX)^{-\bar{s}}) \det(1 + XX^*)^{-N} \ell(dX).
\]

*Proof.* Direct computation. \( \square \)

Note that for real values of \( s \) the latter expression can be simplified:

\[
m_N^{(s)}(dX) = \text{const} \det(1 + X^2)^{-s-N} \ell(dX), \quad s \in \mathbb{R}.
\]

**Corollary 5.4.** Fix \( s \in \mathbb{C}, \Re s > -\frac{1}{2} \). The measures \( m_N^{(s)} \) with varying \( N \) are consistent with the projections \( p'_N \).

*Proof.* Indeed, this follows from Lemma 5.2, Proposition 5.3, and the similar claim for the measures \( \mu_N^{(s)} \). \( \square \)

Of course, a direct verification of Corollary 5.4 is also possible. For real values of \( s \) it was first carried out by Hua, see the proof of Theorem 2.1.5 in his remarkable book [Hua].

The considerations above lead to the following
**Definition 5.5.** Let $\mathfrak{H}$ denote the space of all infinite Hermitian matrices. We may view it as a projective limit space: $\mathfrak{H} = \varprojlim \text{Herm}(N)$. For any $s \in \mathbb{C}$ with $\Re s > -\frac{1}{2}$, the family $\{m^{(s)}_N\}_{N=1,2,\ldots}$ defines a probability measure on the space $\mathfrak{H}$, which will be denoted by $m^{(s)}$ and called the *Hua–Pickrell measure* (with parameter $s$).

The Hua–Pickrell measures are studied in detail in [BO3]. The measures with real parameter $s$ are exact counterparts of measures introduced by Pickrell [Pi] (instead of $\mathfrak{H}$, he considered the space of all infinite complex matrices).

The group $U(\infty)$ operates on the space $\mathfrak{H}$ by conjugations, and all the measures $m^{(s)}$ are clearly invariant with respect to this action.

The Cayley transforms $U(N)' \to \text{Herm}(N)$ (Definition 5.1) define a bijection $\mathfrak{H}' \to \mathfrak{H}$, which is an isomorphism of measures spaces $(\mathfrak{H}', \mu^{(s)}) \to (\mathfrak{H}, m^{(s)})$ for any $s$. Since $\mathfrak{H} \setminus \mathfrak{H}'$ is a negligible set, this can also be viewed as an isomorphism $(\mathfrak{H}, \mu^{(s)}) \to (\mathfrak{H}, m^{(s)})$.

**Remark 5.6.** The bijection $\mathfrak{H}' \to \mathfrak{H}$ takes the action of $G$ on $\mathfrak{H}$ to an action by fractional–linear transformations on $\mathfrak{H}$. The transformation of a matrix $X \in \mathfrak{H}$ by an element $g = (U_1, U_2) \in G$ has the form

$$X \mapsto (Xb + d)^{-1}(Xa + c), \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{U_1 + U_2}{2} & -i\frac{U_1 - U_2}{2} \\ i\frac{U_1 - U_2}{2} & \frac{U_1 + U_2}{2} \end{bmatrix}.$$ 

Given $g \in G$, this transformation is defined almost everywhere.

**Remark 5.7.** The above results can be extended to the case of an arbitrary complex parameter $s$, cf. Remark 4.13. However, when $\Re s \leq -1/2$, the measures become infinite.

6. The characters $\chi_{z,w}$

Let $\chi$ be a character of $U(\infty)$. Then, for each $N = 1, 2, \ldots$, the restriction of $\chi$ to $U(N) \subset U(\infty)$ is a character of $U(N)$ (in the sense of Definition 1.1). According to Example 1.2, we have

$$\chi |_{U(N)} = \sum_{\lambda \in \hat{U}(N)} P_N(\lambda) \hat{\chi}^\lambda,$$

(6.1)

where $P_N(\lambda)$ are nonnegative coefficients whose sum is equal to 1, and $\hat{\chi}^\lambda$ are the normalized irreducible characters of $U(N)$, see (1.1). Note that the series converges uniformly on $U(N)$, because $|\hat{\chi}^\lambda(\cdot)| \leq 1$.

Any character is uniquely determined by the collection of its coefficients $P_N(\lambda)$. Indeed, the coefficients with fixed $N$ determine the restriction of the character to $U(N)$, and if we know these restrictions for all $N$ then we know the character itself.

**Definition 6.1.** Following the general scheme described in §2, we introduce a family of characters attached to the representations $T_{z,w}$, as follows. Let $z, w \in \mathbb{C}$ satisfy the condition $\Re z + \Re w > -\frac{1}{2}$ ensuring the existence of the distinguished $K$-invariant vector $\xi_{z,w}$ in the representation $T_{z,w}$, see Definition 3.15. We consider the matrix coefficient determined by this vector and then pass to the corresponding character, which we denote by $\chi_{z,w}$:

$$\chi_{z,w}(U) = (T_{z,w}(U, 1)\xi_{z,w}, \xi_{z,w}), \quad U \in U(\infty).$$
Lemma 6.2. Assume $U \in U(N) \subset U(\infty)$. Then
\[
\chi_{z,w}(U) = \frac{1}{\|f_{z,w|N}\|^2} \int_{U(N)} f_{z,w|N}(VU) \overline{f_{z,w|N}(V)} \mu_N(dV).
\]

Proof. By the very construction of the representations $T_{z,w}$, if $U \in U(N)$ then
\[
\chi_{z,w}(U) = (\text{Reg}_N(U,1)f'_{z,w|N},f'_{z,w|N}) = \int_{U(N)} f'_{z,w|N}(VU) \overline{f'_{z,w|N}(V)} \mu_N(dV),
\]
which is equivalent to the desired formula. □

Our aim is to describe explicitly the expansion (6.1) of the characters $\chi_{z,w}$.

We shall interpret the labels $\lambda \in U(N)^\wedge$ of irreducible characters of $U(N)$ as signatures of length $N$, i.e., as ordered $N$-tuples of integers:
\[
\lambda = (\lambda_1 \geq \cdots \geq \lambda_N), \quad \lambda_i \in \mathbb{Z}.
\]

Theorem 6.3. Assume $\Re z + \Re w > -\frac{1}{2}$. Denote by $P_{z,w|N}(\lambda)$ the coefficients in the expansion (6.1) of the characters $\chi_{z,w}$. We have
\[
P_N(\lambda | z, w) = (S_N(z, w))^{-1} \prod_{i=1}^{N} \left| \frac{1}{\Gamma(z - \lambda_i + i) \Gamma(w + N + 1 + \lambda_i - i)} \right|^2 \cdot \text{Dim}_N^2(\lambda),
\]
where
\[
S_N(z, w) = \prod_{i=1}^{N} \frac{\Gamma(z + w + \bar{w} + i)}{\Gamma(z + w + i) \Gamma(z + \bar{w} + i) \Gamma(z + \bar{w} + i) \Gamma(z + w + i)},
\]
and $\text{Dim}_N(\lambda)$ is the dimension of the irreducible character $\chi^\lambda$, given by Weyl’s formula:
\[
\text{Dim}_N(\lambda) = \prod_{1 \leq i < j \leq N} \frac{\lambda_i - \lambda_j + j - i}{j - i}.
\]

The proof of Theorem 6.3 is partitioned into a few lemmas.

Note that the irreducible characters $\chi^\lambda$, where $\lambda$ ranges over $U(N)^\wedge$, form an orthonormal basis in the subspace of $H_N = L^2(U(N), \mu_N)$ constituted by central functions. Therefore, there exists an expansion
\[
f_{z,w|N} = \sum_{\lambda \in U(N)^\wedge} c_{z,w|N} \chi^\lambda
\]
with certain coefficients $c_{z,w|N}$.

Lemma 6.4. We have
\[
P_N(\lambda | z, w) = \frac{|c_{z,w|N}|^2}{\|f_{z,w|N}\|^2}.
\]

Proof. This follows from Lemma 6.2 and the formula
\[
\int_{U(N)} \chi^\lambda(VU) \overline{\chi^\mu(V)} \mu_N(dV) = \delta_{\mu \nu} \widehat{\chi^\lambda}(U), \quad \lambda, \mu \in GT_N,
\]
which in turn follows from Schur’s orthogonality relations. □

Since the norm $\|f_{z,w|N}\|$ is known (Lemma 3.14), our problem is entirely reduced to evaluating the coefficients $c_{z,w|N}$. 

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Lemma 6.5. Assume that

\[ f_1(u) = \sum_{l=-\infty}^{\infty} c(l)u^l, \quad c(l) \in \mathbb{C}, \]

is a function on the unit circle \(|u| = 1\). For any \(N = 1, 2, \ldots\) consider the following function on \(U(N)\), which is constant on conjugacy classes:

\[ f_N(U) = f_1(u_1) \cdots f_1(u_N), \]

where \(U\) ranges over \(U(N)\) and \((u_1, \ldots, u_N)\) denotes the spectrum of \(U\). Expand \(f_N\) in the “Fourier series” on the irreducible characters,

\[ f_N = \sum_{\lambda \in U(N)^\vee} c(\lambda)\chi^\lambda. \]

Then the “Fourier coefficients” \(c(\lambda)\) are given by determinants of order \(N\),

\[ c(\lambda) = \det[c(\lambda_i - i + j)]_{1 \leq i, j \leq N}. \]

Proof. This is a well-known combinatorial fact, based on Weyl’s character formula. See, e.g., [Vo, Lemme 2] or [Hua, Theorem 1.2.1]. □

Observe that, by Definition 3.10,

\[ f_{z,w|N}(\text{diag}(u_1, \ldots, u_N)) = f_{z,w|1}(u_1) \cdots f_{z,w|1}(u_N), \]

as in Lemma 6.5. By virtue of this lemma, the problem of evaluating the “Fourier coefficients” of \(f_{z,w|N}\) is split into two parts: first, calculate the Fourier coefficients of \(f_{z,w|1}\) and, second, calculate the above determinants.

Lemma 6.6. The coefficients of the Fourier expansion

\[ f_{z,w|1}(u) = \sum_{l=-\infty}^{\infty} c_{z,w|1}(l)u^l \]

have the form

\[ c_{z,w|1}(l) = \frac{\Gamma(1 + z + w)}{\Gamma(1 + z - l)\Gamma(1 + w + l)}. \]

Proof. Setting \(u = e^{i\theta}\) we have

\[ c_{z,w|1}(l) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 + u)^z(1 + \bar{u})^wu^{-l}d\theta. \]

To evaluate this integral one can, e.g., reduce it to [Er, 1.5 (30)]. See also [Ner3]. □
Lemma 6.7. The determinants of Lemma 6.5 corresponding to the Fourier coefficients of Lemma 6.6 can be explicitly calculated:

\[
c_{z,w|N}(\lambda) = \prod_{i=1}^{N} \frac{\Gamma(z + w + i) \Gamma(i)}{\Gamma(z - \lambda_i + i) \Gamma(w + N + 1 + \lambda_i - i)} \cdot \text{Dim}_N(\lambda).
\]  

(6.4)

Proof. We will reduce the determinant in question to a known one, given in [Kra, Lemma 3]:

\[
det \begin{bmatrix} x_i + a_k \cdot x_j + b_k \end{bmatrix}^{N}_{j-1} = \prod_{i,j=1}^{N} (x_i - x_j) \cdot \prod_{1 \leq i \leq j \leq N-1} (a_i - b_j).
\]  

(6.5)

To do this we abbreviate \( l_i = \lambda_i - i \) and we transform our determinant as follows:

\[
c_{z,w|N}(\lambda) = \frac{(\Gamma(1 + z + w))^N}{\prod_{i=1}^{N} \Gamma(z - \lambda_i + i) \Gamma(w + N + 1 + \lambda_i - i)} \cdot \text{det}[A(i,j)],
\]

where

\[
A(i,j) = \frac{\Gamma(z - l_i) \Gamma(w + N + 1 + l_i)}{\Gamma(1 + z - l_i - j) \Gamma(w + 1 + l_i + j)} = \frac{(z - l_i - 1) \cdots (z - l_i - j + 1)}{N-j} \cdot \frac{(w + N + l_i) \cdots (w + l_i + j + 1)}{N-j}.
\]

Then the reduction to (6.5) is carried out by setting

\[
x_i = l_i, \quad a_j = -z + j, \quad b_j = w + 1 + j.
\]

Then, after simple transformations, we get (6.4). □

Proof of Theorem 6.3. Follows from the lemmas above and the formula for the norm in Lemma 3.14. □

Remark 6.8. Note some symmetry properties of the characters \( \chi_{z,w} \).

- Conjugation:

\[
\overline{\chi_{z,w}(U)} = \chi_{\overline{w},\overline{z}}(U), \quad U \in U(\infty).
\]

Indeed, this follows from Lemma 6.2 and the fact that

\[
\overline{f_{z,w|N}(U)} = f_{\overline{w},\overline{z}|N}(U), \quad U \in U(N).
\]

- The character \( \chi_{z,w} \) is invariant under the symmetries of the parameters generated by \( z \to \overline{z} \) and \( w \to \overline{w} \). Indeed, this follows at once from the corresponding symmetry of the coefficients \( P_N(\lambda \mid z, w) \). Note that this fact is not obvious from the construction of the characters \( \chi_{z,w} \).

- Let \( \lambda^* = (-\lambda_N, \ldots, -\lambda_1) \) denote the dual signature to \( \lambda \): this is the label of the conjugate irreducible character \( \overline{\chi^\lambda} \). Another symmetry property of the coefficients \( P_N(\lambda \mid z, w) \) is as follows:

\[
P_N(\lambda \mid z, w) = P_N(\lambda^* \mid w, z).
\]

Theorem 6.3 is an analog of Proposition 4.8 in [Pi]. This theorem leads to an important consequence for the representations \( T_{z,w} \), cf. Lemma 4.5 in [Pi].
Corollary 6.9. Let \( z, w \in \mathbb{C} \) be such that \( \Re z + \Re w > -1/2 \) and \( z, w \notin \mathbb{Z} \). Then the distinguished vector \( \xi_{z,w} \) (see Definition 3.15) is cyclic.

Proof. Indeed, for nonintegral values of \( z, w \), the coefficients \( P_N(\lambda \mid z,w) \) are non-vanishing for all \( N \) and all \( \lambda \in U(N)^\sim \). This implies that the vector \( f_{z,w}|_N \in H_N \) is a cyclic vector of the representation \( \text{Reg}_N \) for any \( N \). This concludes the proof. \( \Box \)

Remark 6.10. Another possible way of evaluating the coefficients \( c_{z,w}|_N \) is to use Neretin’s results, see [Ner4].

7. Other series of characters

For two signatures \( \nu \) and \( \lambda \), of length \( N-1 \) and \( N \), respectively, write \( \nu \prec \lambda \) if
\[
\lambda_1 \geq \nu_1 \geq \lambda_2 \geq \nu_2 \geq \cdots \geq \nu_{N-1} \geq \lambda_N.
\]
The relation \( \nu \prec \lambda \) appears in the Gelfand–Tsetlin branching rule for the irreducible characters of the unitary groups, see, e.g., [Zh]:
\[
\chi^\lambda|_{U(N-1)} = \sum_{\nu \prec \lambda} \chi^\nu.
\]

Definition 7.1. The Gelfand–Tsetlin graph \( \mathbb{G}T \) is a \( \mathbb{Z}_+ \)-graded graph whose \( N \)th level \( \mathbb{G}T_N \) consists of signatures of length \( N \). Two vertices \( \nu \in \mathbb{G}T_{N-1} \) and \( \lambda \in \mathbb{G}T_N \) are connected by an edge if \( \nu \prec \lambda \). We agree that \( \mathbb{G}T_0 \) consists of a single element denoted as \( \emptyset \); it is connected to all \( \lambda \in \mathbb{G}T_1 \). \( \Box \)

Definition 7.2. For \( \nu \in \mathbb{G}T_{N-1} \) and \( \lambda \in \mathbb{G}T_N \), where \( N = 1, 2, \ldots \), set
\[
q(\nu, \lambda) = \begin{cases} 
\frac{\text{Dim}_{N-1} \nu}{\text{Dim}_N \lambda}, & \nu \prec \lambda, \\
0, & \nu \nprec \lambda.
\end{cases}
\]
This is the cotransition probability function of the Gelfand–Tsetlin graph; it satisfies the relation
\[
\sum_{\nu \in \mathbb{G}T_{N-1}} q(\nu, \lambda) = 1, \quad \forall \lambda \in \mathbb{G}T_N.
\]
We agree that \( q(\emptyset, \lambda) = 1 \) for all \( \lambda \in \mathbb{G}T_1 \). \( \Box \)

One can imagine that each edge \((\nu, \mu)\) of the graph \( \mathbb{G}T \) is equipped with a label, which is the positive number \( q(\nu, \lambda) \). Note that the Gelfand–Tsetlin branching rule is equivalent to the relation
\[
\tilde{\chi}^\lambda|_{U(N-1)} = \sum_{\nu \in \mathbb{G}T_{N-1}} q(\nu, \lambda) \tilde{\chi}^\nu. \tag{7.1}
\]

Definition 7.3. Assume that for each \( N = 0, 1, \ldots \) we are given a probability measure \( P_N \) on the discrete set \( \mathbb{G}T_N \). The family \( \{P_N\} \) is called a coherent system if
\[
P_{N-1}(\nu) = \sum_{\lambda \in \mathbb{G}T_N} q(\nu, \lambda) P_N(\lambda), \quad N = 1, 2, \ldots, \quad \nu \in \mathbb{G}T_{N-1}. \tag{7.2}
\]
\( \Box \)

Note that if \( P_N \) is an arbitrary probability measure on \( \mathbb{G}T_N \) then (7.2) defines a probability measure on \( \mathbb{G}T_{N-1} \) (indeed, this follows at once from the above relation for \( q(\nu, \lambda) \)). Thus, in a coherent system \( \{P_N\} \), the \( N \)th term is a refinement of the \((N-1)\)th one.
Proposition 7.4. There exists a natural bijective correspondence \( \chi \leftrightarrow \{P_N\} \) between characters of the group \( U(\infty) \) and coherent systems on the Gelfand–Tsetlin graph. The correspondence is defined by the relations

\[
\chi_N = \sum_{\lambda \in \Gamma_T^N} P_N(\lambda) \tilde{\chi}^\lambda, \quad \chi_N := \chi \mid_{U(N)}, \quad N = 1, 2, \ldots.
\] (7.3)

Proof. Let \( \chi \) be a character of \( U(\infty) \). We repeat the argument at the beginning of §6. Since \( \chi_N \) is a character of \( U(N) \) in the sense of Definition 1.1 \( (N = 1, 2, \ldots) \), we get, according to Example 1.2, the expansion (7.3), where \( P_N(\cdot) \) is a probability measure on \( \Gamma_T^N \). Then the relation (7.2) follows from (7.1) and the evident relation \( \chi_N \mid_{U(N-1)} = \chi_{N-1} \).

Conversely, let \( \{P_N\} \) be a coherent system. We define by means of (7.3) a sequence \( \{\chi_N\} \) of characters of the groups \( U(N) \). The coherency property (7.2) ensures that \( \chi_N \mid_{U(N-1)} = \chi_{N-1} \) for any \( N = 2, 3, \ldots \), so that there exists a function \( \chi \) on \( U(\infty) \) such that \( \chi \mid_{U(N)} = \chi_N \) for all \( N = 1, 2, \ldots \). Obviously, \( \chi \) is a character of \( U(\infty) \). □

Using Proposition 7.4 we will construct new series of characters by analytic continuation of the formulas (6.2)–(6.3).

Let \( z, z', w, w' \) be complex parameters. For any \( N = 1, 2, \ldots \) and any \( \lambda \in \Gamma_T^N \) set

\[
P'_N(\lambda \mid z, z', w, w') = \text{Dim}_N^2(\lambda)
\times \prod_{i=1}^{N} \frac{1}{\Gamma(z - \lambda_i + i) \Gamma(z' - \lambda_i + i) \Gamma(w + N + 1 + \lambda_i - i) \Gamma(w' + N + 1 + \lambda_i - i)},
\]

where \( \text{Dim}_N^2(\lambda) \) was defined in §6. Clearly, for any fixed \( N \) and \( \lambda \), \( P'_N(\lambda \mid z, z', w, w') \) is an entire function on \( \mathbb{C}^4 \). This expression was obtained by analytic continuation from the expression given in (6.2).

Set

\[
\mathcal{D} = \{(z, z', w, w') \in \mathbb{C}^4 \mid \Re(z + z' + w + w') > -1\}.
\]

This is a domain (a half–space) in \( \mathbb{C}^4 \).

Proposition 7.5. Fix an arbitrary \( N = 1, 2, \ldots \). The series of entire functions

\[
\sum_{\lambda \in \Gamma_T^N} P'_N(\lambda \mid z, z', w, w')
\]

converges in the domain \( \mathcal{D} \), uniformly on compact sets. Its sum is equal to

\[
S_N(z, z', w, w') = \prod_{i=1}^{N} \frac{\Gamma(z + z' + w + w' + i)}{\Gamma(z + w + i) \Gamma(z + w' + i) \Gamma(z' + w + i) \Gamma(z' + w' + i) \Gamma(i)}
\] (7.5)

Proof. Let us prove that for any \( \lambda \in \Gamma_T^N \)

\[
|P'_N(\lambda \mid z, z', w, w')| \leq \text{const} \prod_{i=1}^{N} (1 + |\lambda_i|)^{-\Re(z + z' + w + w' + 2)},
\] (7.6)
uniformly on compact sets in \( \mathcal{D} \).

Indeed, using the formula \( (\Gamma(z)\Gamma(1-z))^{-1} = \sin(\pi z)/\pi \) we get

\[
\frac{1}{\Gamma(z - \lambda_i + i)\Gamma(z' - \lambda_i + i)\Gamma(w + N + 1 + \lambda_i - i)\Gamma(w' + N + 1 + \lambda_i - i)}
\]

\[
= \frac{\sin(\pi z)\sin(\pi z')}{\pi^2} \frac{\Gamma(-z + 1 + \lambda_i - i)\Gamma(-z' + 1 + \lambda_i - i)}{\Gamma(w + N + 1 + \lambda_i - i)\Gamma(w' + N + 1 + \lambda_i - i)}
\]

\[
= \frac{\sin(\pi w)\sin(\pi w')}{\pi^2} \frac{\Gamma(-w - N - \lambda_i + i)\Gamma(-w' - N - \lambda_i + i)}{\Gamma(z - \lambda_i + i)\Gamma(z' - \lambda_i + i)}.
\]

Suppose \(|\lambda_i| \to \infty\). Then, applying the first or the second equality above (depending on whether \(\lambda_i \gg 0\) or \(\lambda_i \ll 0\)) and the asymptotic formula

\[
\lim_{x \to +\infty} \frac{\Gamma(a + x)}{\Gamma(b + x)} = x^{a-b}, \quad a, b \in \mathbb{C},
\]

we get

\[
\frac{1}{|\Gamma(z - \lambda_i + i)\Gamma(z' - \lambda_i + i)\Gamma(w + N + 1 + \lambda_i - i)\Gamma(w' + N + 1 + \lambda_i - i)|}
\]

\[
\leq \text{const} \,(1 + |\lambda_i|)^{-\Re(z + z' + w + w' + 2N)}, \quad (7.7)
\]

where the estimate is uniform on compact sets in \( \mathcal{D} \).

Next, Weyl’s formula implies that \( \text{Dim}^2 \lambda \) is a polynomial in \( \lambda_1, \ldots, \lambda_N \) of degree \( 2N - 2 \) with respect to each variable. Combining this fact with (7.7) we get (7.6).

The bound (7.6) ensures the convergence of the series (7.4), uniformly on compact sets in \( \mathcal{D} \), to a holomorphic function in \( \mathcal{D} \). It remains to show that this function coincides with (7.5). To do this we remark that (7.5) is a holomorphic function coinciding with the function \( S_N(z, w) \) (see (6.3)) on the subset \( \mathcal{D} \). For this subset, the claim holds by virtue of Theorem 6.3. Clearly, this subset is a set of uniqueness for holomorphic functions in \( \mathcal{D} \). This implies that the claim holds on the whole domain \( \mathcal{D} \). \( \square \)

Note that in the special case \( N = 1 \), the set \( \mathcal{GT}_1 \) is simply \( \mathbb{Z} \) and the identity

\[
\sum_{\lambda \in \mathcal{GT}_1} P'_1(\lambda \mid z, z, w, w') = S_1(z, z, w, w')
\]

is equivalent to the well-known Dougall’s formula, see [AAR, Chapter 2, Theorem 2.8.2 and Exercise 42(b)], [Er, §1.4]).

**Definition 7.6.** The set of **admissible values** of the parameters \( z, z, w, w' \) is the subset \( \mathcal{D}_{adm} \) of the quadruples \( (z, z, w, w') \in \mathcal{D} \) such that:

First, \( P'_N(\lambda \mid z, z, w, w') \geq 0 \) for any \( N \) and any \( \lambda \in \mathcal{GT}_N \).

Second, for any \( N \), the above inequality is strict for at least two different \( \lambda \)'s.

A quadruple \( (z, z, w, w') \) will be called **admissible** if it belongs to \( \mathcal{D}_{adm} \).

Consider the subdomain

\[
\mathcal{D}_0 = \{(z, z', w, w') \in \mathcal{D} \mid z + w, z + w', z' + w, z' + w' \neq -1, -2, \ldots\}
\]

\[
= \{(z, z, w, w') \in \mathcal{D} \mid S_N(z, z', w, w') \neq 0\}.
\]
For any \((z, z, w, w') \in \mathcal{D}_0\) we set
\[
P_N(\lambda \mid z, z, w, w') = \frac{P'_N(\lambda \mid z, z, w, w')}{S_N(z, z, w, w')}, \quad N = 1, 2, \ldots, \quad \lambda \in \mathbb{G} \mathbb{T}_N. \quad (7.8)
\]

Note that \(\mathcal{D}_{adm} \subset \mathcal{D}_0\) (indeed, if \((z, z, w, w') \in \mathcal{D}_{adm}\) then \(S_N(z, z, w, w')\) is strictly positive), so that formula \((7.8)\) makes sense for any admissible \((z, z, w, w')\).

**Proposition 7.7.** For any \((z, z, w, w') \in \mathcal{D}_0\), the expressions \((7.8)\) satisfy the coherency relation \((7.2)\).

**Proof.** Plug in \(P_{N-1}(\nu) = P_{N-1}(\nu \mid z, z, w, w')\) and \(P_N(\lambda) = P_N(\lambda \mid z, z, w, w')\) to \((7.2)\). First of all, since \(0 \leq q(\nu, \lambda) \leq 1\), the series in the right-hand side of \((7.2)\) converges, uniformly on compact sets in \(\mathcal{D}_0\), by virtue of Proposition 7.5. Then we apply the same argument as in the proof of Proposition 7.5. Namely, we remark that the relation holds provided that \(z' = \bar{z}, w' = \bar{w}\), and then conclude that it must hold for any \((z, z, w, w') \in \mathcal{D}_0\) by analytic continuation. \(\square\)

**Corollary 7.8.** For any \((z, z, w, w') \in \mathcal{D}_{adm}\), the expressions \((7.8)\) form a coherent system. Hence, by Proposition 7.4, there exists a character \(\chi_{z, z, w, w'}\) corresponding to this coherent system.

Our aim is to describe the set \(\mathcal{D}_{adm}\) explicitly.

Define the subset \(\mathcal{Z} \subset \mathbb{C}^2\) as follows:
\[
\mathcal{Z} = \mathcal{Z}_{princ} \sqcup \mathcal{Z}_{compl} \sqcup \mathcal{Z}_{degen},
\]
\[
\mathcal{Z}_{princ} = \{(z, z') \in \mathbb{C}^2 \setminus \mathbb{R}^2 \mid z' = \bar{z}\},
\]
\[
\mathcal{Z}_{compl} = \{(z, z') \in \mathbb{R}^2 \mid \exists m \in \mathbb{Z}, m < z, z < m + 1\},
\]
\[
\mathcal{Z}_{degen} = \bigcup_{m \in \mathbb{Z}} \mathcal{Z}_{degen, m},
\]
\[
\mathcal{Z}_{degen, m} = \{(z, z') \in \mathbb{R}^2 \mid z = m, z' > m - 1, \quad \text{or} \quad z' = m, z > m - 1\},
\]

where “princ”, “compl”, and “degen” are abbreviations for “principal”, “complementary”, and “degenerate”, respectively. The terminology is justified by the following lemma.

**Lemma 7.9.** Let \((z, z') \in \mathbb{C}^2\).

(i) The expression \((\Gamma(z - k + 1)\Gamma(z' - k + 1))^{-1}\) is nonnegative for all \(k \in \mathbb{Z}\) if and only if \((z, z') \in \mathcal{Z}\).

(ii) If \((z, z') \in \mathcal{Z}_{princ} \cup \mathcal{Z}_{compl}\) then this expression is strictly positive for all \(k \in \mathbb{Z}\).

(iii) If \((z, z') \in \mathcal{Z}_{degen, m}\) then this expression vanishes for \(k = m + 1, m + 2, \ldots\) and is strictly positive for \(k = m, m - 1, \ldots\).

**Proof.** Assume first that both \(z\) and \(z'\) are nonintegral. Then the expression \((\Gamma(z - k + 1)\Gamma(z' - k + 1))^{-1}\) does not vanish for any \(k \subset \mathbb{Z}\). Clearly, it is strictly positive for all \(k \in \mathbb{Z}\) whenever \((z, z')\) is in \(\mathcal{Z}_{princ}\) or in \(\mathcal{Z}_{compl}\). Let us check the inverse claim. Dividing \(\Gamma(z - k + 1)\Gamma(z' - k + 1)\) by \(\Gamma(z - k)\Gamma(z' - k)\) we see that \((z - k)(z' - k)\) must be strictly positive for all \(k \in \mathbb{Z}\). But this implies that \((z, z')\) belongs either to \(\mathcal{Z}_{princ}\) or to \(\mathcal{Z}_{compl}\).

Thus, we have verified all the claims in the case when both \(z\) and \(z'\) are nonintegral. Now we shall do the same when at least one of them is integral. By virtue of the
symmetry $z \leftrightarrow z'$, we may assume that $z = m \in \mathbb{Z}$ and $z' \neq m - 1, m - 2, \ldots$. Then
the expression $(\Gamma(z-k+1)\Gamma(z'-k+1))^{-1}$ vanishes for $k = m+1, m+2, \ldots$ and does not
vanish for $k = m, m-1, \ldots$. If $z'$ is real and strictly greater than $m-1$ then
the expression is strictly positive for $k = m, m-1, \ldots$, because then both $\Gamma(z-k+1)$
and $\Gamma(z'-k+1)$ are strictly positive. Conversely, let $\Gamma(z-k+1)\Gamma(z'-k+1)$ be
strictly positive for $k = m, m-1, \ldots$. As $\Gamma(z-k+1) = \Gamma(m-k+1)$ is strictly
positive, $\Gamma(z'-k+1)$ must be strictly positive, too $(k = m, m-1, \ldots)$. Hence
the same holds for the ratio $\Gamma(z' - k + 1)/\Gamma(z' - k + 2)$. Therefore, $z' - k + 1 > 0$ for
all $k = m, m-1, \ldots$, which implies $z' > m - 1$. This concludes the proof. □

**Proposition 7.10.** The set $\mathcal{D}_{\text{adm}}$ introduced in Definition 7.6 consists of the
quadruples $(z, z, w, w') \in \mathcal{D}$ satisfying the following two conditions:

First, both $(z, z')$ and $(w, w')$ belong to $\mathcal{Z}$.

Second, in the particular case when both $(z, z')$ and $(w, w')$ are in $\mathcal{Z}_{\text{degen}}$, an extra
condition is added: let $k, l$ be such that $(z, z') \in \mathcal{Z}_{\text{degen}, k}$ and $(w, w') \in \mathcal{Z}_{\text{degen}, l}$; then
we require $k + l \geq 1$.

**Proof.** Let us abbreviate $P'_N(\lambda) = P'_N(\lambda | z, z, w, w')$.

Assume that the above two conditions on $(z, z, w, w') \in \mathcal{D}$ are satisfied and
prove that $(z, z, w, w') \in \mathcal{D}_{\text{adm}}$. Indeed, examine in detail the possible cases. If
none of the parameters is integral then claim (ii) of Lemma 7.9 shows that $P'_N(\lambda)$
is always strictly positive. If $(z, z') \in \mathcal{Z}_{\text{degen}, m}$ while $w, w'$ are nonintegral then
claim (iii) of Lemma 7.9 (applied to the couple $(z, z')$) together with claim (ii)
(applied to the couple $(w, w')$) show that $P'_N(\lambda)$ is strictly positive if $\lambda_1 \leq m$
and vanishes otherwise. Likewise, if $(w, w') \in \mathcal{Z}_{\text{degen}, m}$ (for some $m \in \mathbb{Z}$) and $z, z'$
are nonintegral then $P'_N(\lambda)$ is strictly positive when $\lambda_N \geq -m$ and vanishes otherwise.
Finally, if $(z, z') \in \mathcal{Z}_{\text{degen}, k}$ and $(w, w') \in \mathcal{Z}_{\text{degen}, l}$ (with some $k, l \in \mathbb{Z}$) then $P'_N(\lambda)$
is strictly positive if $\lambda$ satisfies the inequalities $k \geq \lambda_1 \geq \cdots \geq \lambda_N \geq -l$
and vanishes otherwise. Since $k > -l$ by the second condition, these inequalities are satisfied
by $\geq 2$ different $\lambda$’s. Hence we conclude that in all cases $(z, z, w, w') \in \mathcal{D}_{\text{adm}}$, as
required.

Conversely, assume that $(z, z, w, w') \in \mathcal{D}_{\text{adm}}$ and prove that the above two conditions
hold. We will verify that $(z, z') \in \mathcal{Z}$. Then the similar claim concerning
$(w, w')$ will follow by virtue of the symmetry property

$$P'_N(\lambda_1, \ldots, \lambda_N | z, z, w, w') = P'_N(-\lambda_N, \ldots, -\lambda_1 | w, w', z, z').$$

As for the second condition, $k + l \geq 1$, it follows from the argument above.

We need two simple lemmas.

**Lemma 7.11.** Assume that for a given $N$ and a certain $\lambda \in \mathcal{G}_T_N$ the following
two conditions hold:

- $\lambda^\downarrow := (\lambda_1 - 1, \lambda_2, \ldots, \lambda_N) \in \mathcal{G}_T_N$, i.e., $\lambda_1 > \lambda_2$ if $N \geq 2$,
- $P'_N(\lambda) > 0$ and $P'(\lambda^\downarrow) > 0$.

Then

$$\frac{(z - \lambda_1 + 1)(z' - \lambda_1 + 1)}{(w + N + \lambda_1 - 1)(w' + N + \lambda_1 - 1)} > 0.$$ 

**Proof.** Indeed, the above expression coincides with the ratio $P'_N(\lambda)/P'_N(\lambda^\downarrow)$. □
Lemma 7.12. Assume that for a given $N$ and a certain $\lambda \in \mathbb{GT}_N$ the following two conditions hold:

- $\lambda^\uparrow := (\lambda_1, \ldots, \lambda_{N-1}, \lambda_N + 1) \in \mathbb{GT}_N$, i.e., $\lambda_{N-1} > \lambda_N$ if $N \geq 2$,
- $P_N'(\lambda) > 0$ and $P'(\lambda^\uparrow) > 0$.

Then

$$\frac{(w + \lambda_N + 1)(w' + \lambda_N + 1)}{(z - \lambda_N + N - 1)(z' - \lambda_N + N - 1)} > 0.$$ 

Proof. Indeed, the above expression coincides with the ratio $P_N'(\lambda)/P_N'(\lambda^\uparrow)$.

Now we resume the proof of Proposition 7.10. To prove that $(z, z') \in \mathcal{Z}$ we will examine in succession four possible cases.

Case 1: all parameters are nonintegral. Then $P_N'(\lambda)$ is always nonzero, hence we have $P_N(\lambda) > 0$ for all $N$ and all $\lambda \in \mathbb{GT}_N$. Given $N = 1, 2, \ldots$ and $k \in \mathbb{Z}$, choose $\lambda \in \mathbb{GT}_N$ such that $\lambda_1 = k$ and (if $N \geq 2$) $\lambda_2 > \lambda$. Then, applying Lemma 7.11 we get

$$\frac{(z - k + 1)(z' - k + 1)}{(w + N + k - 1)(w' + N + k - 1)} > 0.$$ 

Fix $k$ and let $N \to \infty$. Then

$$\frac{(z - k + 1)(z' - k + 1)}{(w + N + k - 1)(w' + N + k - 1)} \sim \frac{(z - k + 1)(z' - k + 1)}{N^2} > 0.$$ 

This implies $(z - k + 1)(z' - k + 1) > 0$. Since this inequality holds for any $k \in \mathbb{Z}$ we conclude that $(z, z')$ belongs either to $\mathcal{Z}_{\text{princ}}$ or to $\mathcal{Z}_{\text{compl}}$.

Case 2: at least one of the parameters $z, z'$ is integral and at least one of the parameters $w, w'$ is integral, too. Then, using the symmetries $z \leftrightarrow z'$ and $w \leftrightarrow w'$, we may assume, without loss of generality, that

$$z = k \in \mathbb{Z}, \quad z' \neq k - 1, k - 2, \ldots; \quad w = l \in \mathbb{Z}, \quad w' \neq l - 1, l - 2, \ldots.$$ 

Let $N$ be arbitrary. We have: $P_N'(\lambda) = 0$ whenever $\lambda_1 > k$ or $\lambda_N < -l$. Next, if $k \geq \lambda_1 \geq \cdots \geq \lambda_N \geq -l$ then $P_N'(\lambda) > 0$. It follows that $k \geq -l$ and even more, $k > -l$, because two different $\lambda$'s with $P_N'(\lambda) > 0$ must exist.

Now we apply Lemma 7.11 to $\lambda = (k, k - 1, \ldots, k - 1)$. The assumptions of the lemma are satisfied, and we get the same inequality as in Case 1 above. Moreover, as $z - k + 1$ reduces to 1, the same argument as above shows that $z' - k + 1 > 0$, i.e., $z' > k - 1$. Hence, $(z, z') \in \mathcal{Z}_{\text{degen}, k}$.

Case 3: at least one of the parameters $z, z'$ is integral while both $w$ and $w'$ are nonintegral. We may assume that $z = k \in \mathbb{Z}$ and $z' \neq k - 1, k - 2, \ldots$. In this case, $P_N'(\lambda)$ does not vanish (and hence is strictly positive) whenever $\lambda_1 \leq k$. Then we apply the argument of Case 2 and get exactly the same conclusion.

Case 4: both $z, z'$ are nonintegral while at least one of the parameters $w, w'$ is integral. We may assume that $w = l \in \mathbb{Z}$ and $w' \neq l, l - 1, \ldots$. Then we have $P_N'(\lambda) > 0$ whenever $\lambda_N \geq -l$. We apply first Lemma 7.11, where we take any $\lambda$ such that $\lambda_1 = -l + i$ with $i = 1, 2, \ldots$, and $\lambda_2 < \lambda_1$ (if $N \geq 2$). The same argument as above then gives the inequalities

$$ (z + l - i + 1)(z' + l - i + 1) > 0, \quad i = 1, 2, \ldots.$$
Next, we apply Lemma 7.12, where we take any \( \lambda \) such that \( \lambda_N = -l \) and \( \lambda_{N-1} > \lambda_N \) (if \( N \geq 2 \)). This leads to the inequality

\[
\frac{(w - l + 1)(w' - l + 1)}{(z + l + N - 1)(z' + l + N - 1)} > 0, \quad N = 1, 2, \ldots .
\]

Applying the same trick as above \( (N \to \infty) \) we see that the numerator must be strictly positive. But then the denominator must be strictly positive for any \( N \). This results in the inequalities

\[(z + l + j)(z' + l + j) > 0, \quad j = N - 1 = 0, 1, \ldots .\]

Combining these two families of inequalities we get that \((z + k)(z' + k) > 0\) for any \( k \in \mathbb{Z} \), which means that \((z, z')\) belongs either to \( \mathcal{Z}_{\text{princ}} \) or to \( \mathcal{Z}_{\text{compl}} \).

This completes the proof of Proposition 7.10.

8. Topology on the space of extreme characters

Recall that there is a 1–1 correspondence between extreme characters of \( U(\infty) \) and points of the set \( \Omega \subset \mathbb{R}^{4\infty+2} \), see Theorem 1.3 and Remark 1.6. The aim of this section is to prove that this correspondence is a homeomorphism with respect to natural topologies on both spaces, the space of extreme characters and the space \( \Omega \). This result is used in the proof of Theorem 9.1 below, it is also of independent interest.

First, we have to define the topologies in question. Let us start with the space \( \Omega \). We equip it with the product topology of the ambient space \( \mathbb{R}^{4\infty+2} \). Since \( \mathbb{R}^{4\infty+2} \) is a separable metrizable space, so is \( \Omega \). From the definition of \( \Omega \) it follows that this is a locally compact space. Furthermore, for any positive constant \( c \), the subset of the form \( \{ \omega \in \Omega \mid \delta^+ + \delta^- \leq c \} \) is compact.

Now let us turn to the space of extreme characters. This is a subset of the space \( \mathcal{X}(U(\infty)) \) of all characters. We equip \( \mathcal{X}(U(\infty)) \) with the topology of uniform convergence on each of the compact subgroups \( U(N) \subset U(\infty), N = 1, 2, \ldots . \) Then we restrict this topology to the subspace of extreme characters. It is readily seen that the topological space thus obtained is separable and metrizable.

**Theorem 8.1.** The correspondence \( \omega \mapsto \chi(\omega) \) between the points \( \omega \in \Omega \) and the extreme characters of the group \( U(\infty) \) is a homeomorphism with respect to the topologies defined above.

**Proof.** Recall that any character \( \chi \in \mathcal{X}(U(\infty)) \) is uniquely determined by the coefficients \( P_N(\lambda) \) of the expansions (7.3), where \( N = 1, 2, \ldots \) and \( \lambda \) ranges over \( \mathbb{GT}_N \). It is readily seen that the topology of \( \mathcal{X}(U(\infty)) \) just defined coincides with the topology of simple convergence of these “Fourier coefficients”. Indeed, the crucial point here is that the coefficients are nonnegative and each sum of the form \( \sum P_N(\lambda) \), where \( \lambda \) ranges over \( \mathbb{GT}_N \), equals 1.

By virtue of the multiplicativity property of formula (1.2), the topology on extreme characters is defined by convergence of the coefficients \( P_1(\lambda) \), where \( \lambda \in \mathbb{GT}_1 = \mathbb{Z} \). These are simply the ordinary Fourier coefficients of the functions on the unit circle in \( \mathbb{C} \), which are given by the expression in curved brackets in (1.2), i.e.,

\[
F(\omega)(u) = e^{\gamma^+(u-1)+\gamma^-(u^{-1}-1)} \prod_{i=1}^{\infty} \frac{1 + \beta_i^+(u - 1)}{1 - \alpha_i^+(u - 1)} \frac{1 + \beta_i^-(u^{-1} - 1)}{1 - \alpha_i^-(u^{-1} - 1)}, \quad u \in \mathbb{C}, \quad |u| = 1.
\]

(8.1)
According to the remarks above, the claim of the theorem is equivalent to the following one: in the set of functions of the form (8.1), the uniform convergence on the unit circle (or, which is the same, the simple convergence of the Fourier coefficients) is equivalent to the convergence of the labels \( \omega \) in the space \( \Omega \). Let us notice once again that the reformulation in terms of the convergence of the Fourier coefficients is possible because the functions (8.1) are positive definite and normalized at \( u = 1 \).

We proceed in a few steps.

**Step 1.** Let us prove that the map \( \omega \mapsto F^{(\omega)} \) is continuous. Rewrite (8.1) in the form

\[
e^{\delta^+(u-1)+\delta^-(u^{-1}-1)} \prod_{i=1}^{\infty} \frac{(1 + \beta_i^+(u-1))e^{-\beta_i^+(u-1)}}{(1 - \alpha_i^+(u-1))e^{\alpha_i^+(u-1)}} \frac{(1 + \beta_i^-(u^{-1}-1))e^{-\beta_i^-(u^{-1}-1)}}{(1 - \alpha_i^-(u^{-1}-1))e^{\alpha_i^-(u^{-1}-1)}},
\]

where we used the fact that \( \gamma^\pm = \delta^\pm - \sum (\alpha_i^\pm + \beta_i^\pm) \). It suffices to show that the infinite product in (8.2) converges uniformly on \( u \) and \( (\alpha^+, \beta^+, \alpha^-, \beta^-) \), where \( u \) ranges over the unit circle and the sum of the parameters \( \alpha_i^\pm, \beta_i^\pm \) is bounded by a constant. This is reduced to the following elementary fact: the infinite product \( \prod (1 + x_i) \exp(-x_i) \) converges uniformly on any subset of \( \mathbb{C}^\infty \) of the form \( \{(x_i) \in \mathbb{C}^\infty \mid \sum |x_i| \leq c \} \), where \( c \) is an arbitrary positive constant.

Note that without exponentials, the convergence is nonuniform. For this reason, we cannot take \( \gamma^\pm \) instead of \( \delta^\pm \). Note also that \( \gamma^+ \) and \( \gamma^- \) are not continuous functions of \( \omega \).

**Step 2.** For \( \omega \in \Omega \), set \( ||\omega|| = \delta^+ + \delta^- \). We claim that the inverse map \( F^{(\omega)} \rightarrow \omega \) is continuous provided that \( \omega \) is subject to the restriction \( ||\omega|| \leq c \), where \( c \) is an arbitrary positive constant. Indeed, this follows from the result of step 1, because any subset of the form \( ||\omega|| \leq c \) is compact. Here we use the fact that a bijective continuous map of a compact space on a Hausdorff space is a homeomorphism.

**Step 3.** For \( \omega \in \Omega \), set

\[
||\omega||' = \gamma^+ + \gamma^- + \sum \alpha_i^+(1+\alpha_i^+) + \sum \beta_i^+(1-\beta_i^+) + \sum \alpha_i^-(1+\alpha_i^-) + \sum \beta_i^-(1-\beta_i^-).
\]

Note that all summands are nonnegative and the sums are finite. Let \( (\omega_n) \) be a sequence of points of \( \Omega \) such that the corresponding sequence of the functions \( F^{(\omega_n)} \) is convergent. We claim that then \( ||\omega_n||' \) remains bounded.

Indeed, recall that any function of the form (8.1) is positive definite on the unit circle and normalized at 1. Hence, it is the characteristic function (i.e., Fourier transform) of a probability measure on \( \mathbb{Z} \). Since (8.1) is a real analytic function, the corresponding measure possesses finite moments of any order. Note also that the uniform convergence of characteristic functions on the unit circle is equivalent to the weak convergence of the corresponding probability measures.

We will need the following

**Lemma [OkOl, Lemma 5.2].** Let \( (M_n) \) be a sequence of probability measures on \( \mathbb{Z} \) (or, even more generally, on \( \mathbb{R} \)) such that each \( M_n \) has finite moment of order 4 and the sequence \( (M_n) \) weakly converges to a probability measure. Assume that the second moments of \( M_n \)’s tend to infinity. Then the fourth moments grow faster than the squares of the second moments.

Actually, we will use a corollary of the lemma.
Corollary. Let \((M_n)\) be a sequence of probability measures on \(\mathbb{Z}\) with finite 4-th moments, weakly convergent to a probability measure. Assume that for any \(n\), the 4-th moment of \(M_n\) is bounded by the square of the 2-nd moment times a constant which does not depend on \(n\). Then the 2-nd moments are uniformly bounded.

Consider the functions \(G_n(u) = F(\omega_n)(u) F(\omega_n)(u)\). These are also positive definite normalized functions on the unit circle, and the sequence \((G_n)\) is uniformly convergent by the assumption on the initial functions. Let \(M_n\) be the probability measures corresponding to the functions \(G_n\). Then the measures \(M_n\) weakly converge to a probability measure. We will prove that these measures obey the assumption of the corollary, which will imply the uniform boundedness of their second moments. This, in turn, will imply that \(\|\omega_n\|\) remains bounded, as required.\(^2\)

Let us realize this plan. Set \(u = e^{i\theta}\). If \(M\) is a probability measure on \(\mathbb{Z}\) and \(G(u) = G(e^{i\theta})\) is its characteristic function, then the 2-nd and 4-th moments of \(M\) are equal, within number factors, to the 2-nd and 4-th coefficients of the Taylor expansion of the function \(G(e^{i\theta})\), respectively.

Given \(\omega \in \Omega\), introduce new variables as follows

\[
\{a_i\} = \{a_j^+(1+a_j^-)\} \cup \{a_k^-(1+a_k^+)\}, \quad \{b_i\} = \{b_j^-(1-\beta_j^+)\} \cup \{\beta_k^- (1-\beta_k^-)\}, \quad c = \gamma^+ + \gamma^-
\]

Here the ordering of the variables \(a_i, b_i\) is unessential. In this notation we get, after a simple computation,

\[
F(\omega)(e^{i\theta}) F(\omega)(e^{i\theta}) = e^{2c + (\text{cos } \theta - 1) \sum a_i + (\text{cos } \theta - 1) \sum b_i} \prod_{i=1}^\infty \frac{1 + 2b_i (\text{cos } \theta - 1)}{1 - 2a_i (\text{cos } \theta - 1)}
\]

\[
= e^{-c\theta^2 + \frac{1}{12} c^2 \theta^4 + ...} \prod_{i=1}^\infty \frac{1 - b_i \theta^2 + \frac{1}{12} b_i \theta^4 + ...}{1 + a_i \theta^2 - \frac{1}{12} a_i \theta^4 + ...}.
\]

In this expression, the coefficient of \(\theta^2\) equals

\[
-(c + \sum a_i + \sum b_i) = -\|\omega\|'
\]  \hspace{1cm} (8.3)

and the coefficient of \(\theta^4\) equals

\[
\frac{1}{12} (c + \sum a_i + \sum b_i) + \frac{1}{2} c^2 + \sum a_i^2 + e_2(c, a_1, a_2, ..., \theta, b_1, b_2, ...), \hspace{1cm} (8.4)
\]

where \(e_2\) stands for the 2-nd elementary symmetric function. It is readily seen that the expression (8.4) is bounded from above by the square of the expression (8.3) times a constant. This shows that in our situation, the 4-th moment is always bounded by a constant times the square of the 2-nd moment. As was explained above, this implies that the second moments remain bounded. By virtue of (8.3) this exactly means that \(\|\omega_n\|'\) remains bounded, as was required.

Step 4. Let a sequence \((\omega_n)\) be such that \((F(\omega_n))\) is convergent. Here we will show that \(\|\omega_n\|\) remains bounded. By virtue of step 2 this will imply that \((\omega_n)\) converges in \(\omega\). As we noticed in the very beginning of the section, our topological spaces are

\(^2\)We have replaced the initial functions by the squares of their moduli, because this simplifies the estimation of the moments.
separable and metrizable. Therefore, it will follow that the inverse map \( F(\omega) \mapsto \omega \) is continuous, which will complete the proof of the theorem.

By virtue of step 3, \( \|\omega_n\|' \) remains bounded. We would like to deduce from this that \( \|\omega_n\| \) remains bounded, too.

Let us compare \( \|\omega\| \) and \( \|\omega\|' \). We have

\[
\alpha_i^\pm \leq \alpha_i^\pm (1 + \alpha_i^\pm)
\]

and

\[
\beta_i^\pm \leq 2\beta_i^\pm (1 - \beta_i^\pm), \quad \text{provided that } \beta_i^\pm \in [0, \frac{1}{2}].
\]

In general, we only know that the coordinates \( \beta_i^\pm \) are in \([0, 1]\), so that there is no universal estimate of the form \( \|\omega\| \leq \text{const} \cdot \|\omega\|' \). We will bypass this difficulty as follows.

For any \( \omega \in \Omega \), only finitely many coordinates \( \beta_i^+ \) or \( \beta_i^- \) are strictly greater than \( \frac{1}{2} \). Let us call these coordinates “bad”, and let \( K(\omega) \) denote their number. Then we have an estimate of the form \( \|\omega\| \leq C \cdot \|\omega\|' \), where \( C \) depends only on \( K(\omega) \).

Now it suffices to show that \( K(\omega_n) \) remains bounded, which will imply that \( \|\omega_n\| \) remains bounded.

Recall that for any \( \omega \in \Omega \), \( \beta_i^+ + \beta_i^- \leq 1 \). Hence, both \( \beta_i^+ \) and \( \beta_i^- \) cannot be “bad”. Therefore, if \( K(\omega) > 0 \), then the “bad” coordinates are either the first \( K(\omega) \) coordinates of \( \beta^+ \) or the first \( K(\omega) \) coordinates of \( \beta^- \). Define a new element \( \tilde{\omega} \in \Omega \) as follows. If the “bad” coordinates were in \( \beta^+ \) then we remove them, then add to \( \beta^- \) new coordinates \( 1 - \beta_i^+, \ldots, 1 - \beta_i^+ \omega_{k(\Omega)} \), and finally rearrange all coordinates in \( \beta^- \) in the descending order. If the “bad” coordinates were in \( \beta^- \) then we do the same operation with \( \beta^+ \) and \( \beta^- \) interchanged. Then we have

\[
K(\tilde{\omega}) = 0, \quad \|\tilde{\omega}\|' = \|\omega\|'. \tag{8.5}
\]

On the other hand (see Remark 1.5),

\[
F(\tilde{\omega})(u) = F(\omega)(u)u^+K(\omega). \tag{8.6}
\]

Let us assume now that in our sequence \( (\omega_n) \), the quantity \( K(\omega_n) \) is unbounded. Taking a subsequence, we may assume that \( K(\omega_n) \to \infty \). Consider the corresponding sequence \( (\tilde{\omega}_n) \). Since \( \|\omega_n\|' \) remains bounded and \( \|\tilde{\omega}_n\|' = \|\omega_n\|' \) (see (8.5)), we conclude that \( \|\tilde{\omega}_n\|' \) is bounded, too. Since \( \tilde{\omega}_n \) does not have “bad” coordinates (see (8.5)), the argument above shows that \( \|\tilde{\omega}_n\| \) remains bounded. It follows that the corresponding functions on the unit circle form a relatively compact set in the topology of uniform convergence. Taking a subsequence again, we may assume that the functions \( F(\tilde{\omega}_n) \) converge. On the other hand, by the initial assumption, the functions \( F(\omega_n) \) converge, too. Now taking account of (8.6) we obtain a contradiction. Indeed, we get two sequences of continuous functions on the unit circle, normalized at 1, say \( F_n(u) \) and \( \tilde{F}_n(u) \), such that

\[
\tilde{F}_n(u) = F_n(u)u^+K_n, \quad \text{where } K_n \to \infty,
\]

and such that both \( (F_n) \) and \( (\tilde{F}_n) \) converge in the uniform metric. These conditions imply that the ratios

\[
\frac{\tilde{F}_n(u)}{F_n(u)} = u^+K_n
\]

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uniformly converge in a neighborhood of \( u = 1 \), which is impossible, because \( K_n \to \infty \).

This contradiction shows that actually \( K(\omega_n) \) must be bounded, which completes the proof. \( \square \)

9. Existence and uniqueness of spectral decomposition

The aim of this section is to rederive (and slightly refine) a result due to Voiculescu:

**Theorem 9.1 (Cf. [Vo, Théorème 2]).** For any character \( \chi \) of the group \( U(\infty) \) there exists a probability measure \( P \) on the topological space \( \Omega \) such that

\[
\chi(U) = \int_{\Omega} \chi(\omega)(U) P(d\omega), \quad U \in U(\infty).
\]

Moreover, such a measure is unique.

We call \( P \) the spectral measure of \( \chi \).

**Comments.**
1. Recall (see the beginning of §8) that \( \Omega \) is a good topological space (locally compact, separable, metrizable), so that there is no problem in defining measures on it. Specifically, we take Borel measures with respect to the natural Borel structure of \( \Omega \).
2. The integral above makes sense, because \( \chi(\omega)(U) \) is a continuous function in \( \omega \) (see Theorem 8.1) and \( |\chi(\omega)(U)| \leq 1 \).
3. It is readily seen that for any probability Borel measure \( P \) on \( \Omega \), the above formula defines a character, so that the correspondence \( \chi \mapsto P \) is a bijection between the space \( \mathcal{X}(U(\infty)) \) of characters and the space of probability Borel measures on \( \Omega \).
4. In Remark 9.4 below we compare our approach with that of Voiculescu.

We shall derive Theorem 9.1 from a more general claim, see Theorem 9.2 below.

Assume we are given a sequence \( \Gamma_0, \Gamma_1, \Gamma_2, \ldots \) of nonempty sets, where \( \Gamma_0 \) consists of a single point denoted by the symbol \( \emptyset \) while each \( \Gamma_N \) with \( N \geq 1 \) is a finite or countable set. Let \( \Delta_N \) denote the space of formal convex combinations of the points of \( \Gamma_N \); this is a simplex whose vertices are points of \( \Gamma_N \). Further, assume we are given a function \( q(\nu, \lambda) \) defined on couples \((\nu, \lambda) \in \Gamma_{N-1} \times \Gamma_N\), where \( N = 1, 2, \ldots \), such that \( 0 \leq q(\nu, \lambda) \leq 1 \) and, for any \( \lambda \in \Gamma_N \), \( \sum_{\nu} q(\nu, \lambda) = 1 \). In the particular case \( N = 1 \) this means that \( q(\emptyset, \lambda) = 1 \) for all \( \lambda \in \Gamma_1 \).

For each \( N = 1, 2, \ldots \), there exists a unique affine map \( \Delta_N \to \Delta_{N-1} \) taking any \( \lambda \in \Gamma_N \) to the convex combination \( \sum_{\nu} q(\nu, \lambda) \nu \in \Delta_{N-1} \). Let \( \Delta = \lim \Delta_N \) denote the projective limit taken with respect to these affine maps.

Let \( \Gamma = \Gamma_0 \sqcup \Gamma_1 \sqcup \Gamma_2 \sqcup \ldots \) be the disjoint union of the sets \( \Gamma_N \); this is a countable set. Denote by \( \mathcal{F} \) the vector space of all real functions on \( \Gamma \) and equip it with the topology of pointwise convergence on \( \Gamma \). Note that \( \mathcal{F} \) is a locally convex vector space. It is metrizable, because \( \Gamma \) is countable.

We identify \( \Delta \) with the subset of \( \mathcal{F} \) formed by nonnegative functions \( f \) such that \( f(\emptyset) = 1 \) and \( f(\nu) = \sum_\lambda q(\nu, \lambda) f(\lambda) \) for any \( \nu \in \Gamma_{N-1}, \quad N = 1, 2, \ldots \), where the summation is taken over \( \lambda \in \Gamma_N \) (note that these conditions imply that \( \sum_\lambda f(\lambda) = 1 \), summed over \( \lambda \in \Gamma_N \)).
Clearly, $\Delta$ is a convex subset of $\mathcal{F}$. As a subset of $\mathcal{F}$, $\Delta$ inherits its topology. We also consider the Borel structure on $\Delta$ generated by this topology. One can prove that this structure is standard.

Let the symbol $\text{Ex}(\cdot)$ denote the subset of extreme points of a convex set.

**Theorem 9.2.** Let $\Delta$ be the convex set $\lim \Delta_N$ defined above. The subset $\text{Ex}(\Delta)$ of extreme points of $\Delta$ is a Borel subset and each point $f \in \Delta$ is uniquely represented by a probability Borel measure $P$ on $\text{Ex}(\Delta)$: $f = \int_{\text{Ex}(\Delta)} gP(df)$. That is, $f(\lambda) = \int_{\text{Ex}(\Delta)} g(\lambda)P(dg)$ for any $\lambda \in \Gamma$.

**Derivation of Theorem 9.1 from Theorem 9.2.** Take $\Gamma_N = \mathcal{G}T_N$, $N = 1, 2, \ldots$ and take as $q(\nu, \lambda)$ the cotransition probability function (Definition 7.2). Then $\Delta_N$ turns into the set of probability measures on $\mathcal{G}T_N$ and the set $\Delta$ becomes the set of coherent families of measures on $\mathcal{G}T$ (Definition 7.3). By virtue of Proposition 7.4 we get a bijection between $\Delta$ and $\mathcal{X}(U(\infty))$, which is an isomorphism of convex sets. Moreover, this is a homeomorphism of topological spaces (see the discussion of the topology on $\mathcal{X}(U(\infty))$ in the beginning of §8). Applying Theorem 1.3, which provides an explicit parameterization of extreme characters, we get a bijective correspondence between $\text{Ex}(\Delta)$ and $\Omega$. By Theorem 8.1, this correspondence is a homeomorphism of topological spaces. Hence, it preserves the Borel structures. This turns Theorem 9.1 into a special case of Theorem 9.2.

The proof of Theorem 9.2 is based on the following lemma.

**Lemma 9.3.** The convex set $\Delta$ is a Choquet simplex, i.e., the cone generated by $\Delta$ is a lattice.

**Proof.** Let $C$ denote this cone. It coincides with the subset of $\mathcal{F}$ that is described similarly to $\Delta \subset \mathcal{F}$: the only difference is that the condition $f(\emptyset) = 1$ is dropped.

We extend the function $q(\nu, \lambda)$ to any couples $\nu \in \Gamma_M$, $\lambda \in \Gamma_N$ with $M < N$ using the following recurrence relation:

$$q(\nu, \lambda) = \sum_{\mu \in \Gamma_{N-1}} q(\nu, \mu)q(\mu, \lambda).$$

Given $f_1, f_2 \in C$ we construct their lowest upper bound as follows. Define a function $f$ on $\Gamma$ by

$$f(\nu) = \lim_{N \to \infty} \sum_{\lambda \in \Gamma_N} q(\nu, \lambda) \max(f_1(\lambda), f_2(\lambda)).$$

The limit exists, because, for any fixed $\nu$, the $N$th sum monotonically increases as $N \to \infty$ and remains bounded from above by $f_1(\nu) + f_2(\nu)$. It is readily verified that $f$ belongs to the cone and is the lowest upper bound for $f_1$ and $f_2$.

The existence of the greatest lower bound is verified similarly: it suffices to substitute “min” for “max”.

**Proof of Theorem 9.2.** If all sets $\Gamma_N$ are finite, then $\Delta$ is compact. Since $\Delta$ is metrizable (as $\mathcal{F}$ is metrizable), the claims of Theorem 9.2 immediately follow from Lemma 9.3 and Choquet’s theorem, see [Ph]. When the sets $\Gamma_N$ are allowed to be countable, the space $\Delta$ may be noncompact, so that Choquet’s theorem is not applicable.
directly applicable. To overcome this difficulty we embed $\Delta$ into a bigger set $\tilde{\Delta} \subset \mathcal{F}$ which is compact.

Specifically, let $\tilde{\Delta}$ be the set of nonnegative functions $f$ on $\Gamma$ such that $f(\emptyset) = 1$ and $f(\nu) \geq \sum_{\lambda} q(\nu, \lambda) f(\lambda)$ for any $\nu \in \Gamma_{N-1}$, $N = 1, 2, \ldots$, where $\lambda$ ranges over $\Gamma_N$ (i.e., in the definition of $\Delta$, we have replaced the equality by an inequality). We note that $\tilde{\Delta}$ is a compact convex set containing $\Delta$.

Next, for any $M = 0, 1, \ldots$, let $\tilde{\Delta}_M$ be the set of nonnegative functions on $\Gamma$ such that $f(\emptyset) = 1$, $f(\nu) = \sum_{\lambda} q(\nu, \lambda) f(\lambda)$ for all $\nu \in \Gamma_{N-1}$ with $N \leq M$, and $f(\cdot) \equiv 0$ on $\Gamma_{M+1} \cup \Gamma_{M+2} \cup \ldots$. Note that $\tilde{\Delta}_M$ is a convex subset of $\tilde{\Delta}$, isomorphic to $\Delta_M$.

Finally, dropping the condition $f(\emptyset) = 1$ in the definitions above we get the cones spanned by the sets $\tilde{\Delta}$ and $\tilde{\Delta}_M$; we denote them by $\tilde{C}$ and $\tilde{C}_M$, respectively.

Now the crucial remark is that any element $f \in \tilde{C}$ is uniquely represented as a sum $f = f_0 + f_1 + \cdots + f_\infty$, where $f_M \in C_M$ and $f_\infty \in C$. Conversely, any such sum represents an element of $\tilde{C}$ provided that $\sum_M f_M(\emptyset) < +\infty$, cf. [Ol3, §22].

This implies a number of consequences. First, the cone $\tilde{C}$ is a lattice, so that we may apply Choquet’s theorem to $\tilde{\Delta}$. Next, $\text{Ex}(\tilde{\Delta})$ is the disjoint union of the sets $\text{Ex}(\tilde{\Delta}_M) \simeq \Gamma_M$, where $M = 0, 1, \ldots$, and the set $\text{Ex}(\Delta)$. Thus, $\text{Ex}(\tilde{\Delta})$ is the difference of $\text{Ex}(\tilde{\Delta})$, which is a Borel set, and a countable set; whence, $\text{Ex}(\Delta)$ is a Borel set. Finally, if $f$ belongs to $\Delta$ then its representing measure on $\text{Ex}(\tilde{\Delta})$ is concentrated on $\text{Ex}(\Delta) \subset \text{Ex}(\Delta)$. This concludes the proof. □

**Remark 9.4.** Here is a comment to Voiculescu’s original result [Vo, Théorème 2]. Its formulation says that any $\chi \in \mathcal{X}(U(\infty))$ is uniquely represented by a probability measure $P$ on the space $\text{Ex}(\mathcal{X}(U(\infty)))$, equipped with the Borel structure inherited from the ambient space $\mathcal{X}(U(\infty))$. To prove the existence of $P$, Voiculescu also embedded $\mathcal{X}(U(\infty))$ into a compact space. But he used a special property of the cotransition function $q(\nu, \lambda)$ of the graph $G \mathcal{T}$ (specifically, the fact that $q(\nu, \lambda)$ tends to zero when $\nu \in G \mathcal{T}_{N-1}$ is fixed and $\lambda \in G \mathcal{T}_N$ goes to infinity). This allowed him to take, instead of our space $\tilde{\Delta}$, a smaller set. Voiculescu’s proof of the uniqueness statement is quite different from ours: it substantially relies on the multiplicativity property of extreme characters. The argument presented above seems to be more general and direct.

Voiculescu’s original formulation did not involve the space $\Omega$, because at that time it was not yet clear whether the characters $\chi^{(\omega)}$ exhaust the whole set $\text{Ex}(\mathcal{X}(U(\infty)))$ of extreme characters. Our (modest) supplement consists in checking that the bijection between the spaces $\text{Ex}(\mathcal{X}(U(\infty)))$ and $\Omega$ preserves the Borel structures, so that $P$ can be carried over from the “abstract” space $\text{Ex}(\mathcal{X}(U(\infty)))$ to the “concrete” space $\Omega$. Of course, this is a pure technical claim whose validity seems to be beyond doubt. However, it is not completely trivial.

## 10. Approximation of Spectral Measures

The aim of this section is to establish a relationship between the spectral measure $P$ of an arbitrary character $\chi$ and the coherent system $\{P_N\}$ corresponding to $\chi$. Recall that $P_N$ is a probability measure on the discrete set $G \mathcal{T}_N \subset \mathbb{Z}^N$ ($N = 1, 2, \ldots$) while $P$ is a probability Borel measure on the region $\Omega \subset \mathbb{R}^{4\infty+2}$ (see Proposition 7.4 and Theorem 9.1). We will prove that, as $N \to \infty$, the measures
$P_N$ approximate the measure $P$ in a certain sense. To precisely state this claim we need a few definitions.

Given a signature $\lambda \in \mathbb{GT}_N$, we denote by $\lambda^+$ and $\lambda^-$ its positive and negative parts. These are two Young diagrams such that $\ell(\lambda^+) + \ell(\lambda^-) \leq N$, where $\ell(\cdot)$ is the number of nonzero rows of a Young diagram. That is,

$$\lambda = (\lambda^+, \ldots, \lambda^+_k, 0, \ldots, 0, -\lambda^-_1, \ldots, -\lambda^-_l), \quad k = \ell(\lambda^+), \quad l = \ell(\lambda^-).$$

Next, given a Young diagram $\nu$, we denote by $d(\nu)$ the number of diagonal boxes in $\nu$, and we introduce the Frobenius coordinates of $\nu$:

$$p_i(\nu) = \nu_i - i, \quad q_i(\nu) = \nu'_i - i, \quad i = 1, \ldots, d(\nu),$$

where $\nu'$ stands for the transposed (conjugate) diagram.

Then, following Vershik–Kerov [VK1], we introduce the modified Frobenius coordinates of $\nu$ as follows

$$\tilde{p}_i(\nu) = p_i(\nu) + \frac{1}{2} = \nu_i - i + \frac{1}{2}, \quad \tilde{q}_i(\nu) = q_i(\nu) + \frac{1}{2} = \nu'_i - i + \frac{1}{2}, \quad i = 1, \ldots, d(\nu).$$

Note that

$$\tilde{p}_1(\nu) > \cdots > \tilde{p}_{d(\nu)}(\nu) > 0, \quad \tilde{q}_1(\nu) > \cdots > \tilde{q}_{d(\nu)}(\nu) > 0,$$

$$\sum_{i=1}^{d(\nu)} (\tilde{p}_i(\nu) + \tilde{q}_i(\nu)) = |\nu|,$$

where $|\nu|$ denotes the total number of boxes in $\nu$. We also agree to set

$$\tilde{p}_i(\nu) = \tilde{q}_i(\nu) = 0, \quad i = d(\nu) + 1, d(\nu) + 2, \ldots,$$

so that the coordinates $\tilde{p}_i(\nu)$ and $\tilde{q}_i(\nu)$ are now defined for any $i = 1, 2, \ldots$.

**Definition 10.1.** For any $N = 1, 2, \ldots$, we embed the set $\mathbb{GT}_N$ into $\Omega$ as follows

$$\mathbb{GT}_N \ni \lambda \mapsto \omega = (a^+, b^+, a^-, b^-, c^+, c^-) \in \Omega,$$

$$a^+_i = \frac{\tilde{p}_i(\lambda^+_N)}{N}, \quad b^+_i = \frac{\tilde{q}_i(\lambda^+_N)}{N}, \quad (i = 1, 2, \ldots), \quad c^\pm = \frac{|\lambda^\pm_N|}{N}.$$

**Theorem 10.2.** Let $\chi$ be an arbitrary character of $U(\infty)$, $P$ be its spectral measure, and $\{P_N\}$ be the coherent system corresponding to $\chi$. Further, for any $N$, convert $P_N$ to a probability measure $P_N$ on $\Omega$, which is the pushforward of $P_N$ under the embedding $\mathbb{GT}_N \hookrightarrow \Omega$ introduced in Definition 10.1.

Then, as $N \to \infty$, the measures $P_N$ weakly converge to the spectral measure $P$. I.e., for any bounded continuous function $F$ on $\Omega$,

$$\lim_{N \to \infty} \int_{\Omega} F(\omega) P_N(d\omega) = \int_{\Omega} F(\omega) P(d\omega).$$

The proof is quite similar to that of [BO3, Theorem 5.3]. We will see that Theorem 10.2 is a corollary of another general result, Theorem 10.7.
A path in the Gelfand–Tsetlin graph $\mathcal{G}$ is an infinite sequence $t = (t_1, t_2, \ldots)$ such that $t_N \in \mathcal{G}T$ and $t_N \prec t_{N+1}$ for any $N = 1, 2, \ldots$. The set of the paths will be denoted by $\mathcal{T}$.

We also need finite paths. A finite path of length $N$ is a sequence $\tau = (\tau_1, \ldots, \tau_N)$, where $\tau_1 \in \mathcal{G}T_1$, $\ldots$, $\tau_N \in \mathcal{G}T_N$ and $\tau_1 \prec \cdots \prec \tau_N$. The set of finite paths of length $N$ will be denoted by $\mathcal{T}_N$. One can identify $\mathcal{T}$ with the projective limit space $\varprojlim \mathcal{T}_N$, where the projection $\mathcal{T}_N \to \mathcal{T}_{N-1}$ is the operation of removing the last vertex $\tau_N$.

Consider the natural embedding $\mathcal{T} \subset \prod_N \mathcal{G}T_N$. We equip $\prod_N \mathcal{G}T_N$ with the product topology (the sets $\mathcal{G}T_N$ are viewed as discrete spaces). The set $\mathcal{T}$ is closed in this product space. We equip $\mathcal{T}$ with the induced topology. Equivalently, the topology is that of the projective limit space $\varprojlim \mathcal{T}_N$. Then $\mathcal{T}$ turns into a totally disconnected topological space.

Given a finite path $\tau = (\tau_1, \ldots, \tau_N) \in \mathcal{T}_N$, define the cylinder set $C_\tau \subset \mathcal{T}$ as the inverse image of $\{\tau\}$ under the projection $\mathcal{T} \to \mathcal{T}_N$, 

$$C_\tau = \{ t \in \mathcal{T} \mid t_1 = \tau_1, \ldots, t_N = \tau_N \}.$$ 

The cylinder sets form a base of topology in $\mathcal{T}$.

Consider an arbitrary signature $\lambda \in \mathcal{G}T_N$. The set of finite paths $\tau = (\tau_1 \prec \cdots \prec \tau_N)$ ending at $\lambda$ has cardinality equal to $\text{Dim}_N \lambda = \chi^\lambda(e)$. The cylinder sets $C_\tau$ corresponding to these finite paths $\tau$ are pairwise disjoint, and their union coincides with the set of infinite paths $t$ passing through $\lambda$.

A central measure is any probability Borel measure on $\mathcal{T}$ such that the mass of any cylinder set $C_\tau$ depends only on its endpoint $\lambda$. Clearly, central measures form a convex set.

These definitions are inspired by [VK1].

**Proposition 10.3.** There exists a natural bijective correspondence $M \leftrightarrow \{P_N\}$ between central measures $M$ and coherent systems $\{P_N\}$, defined by the relations

$$\text{Dim}_N \lambda \cdot M(C_\tau) = P_N(\lambda),$$

where $N = 1, 2, \ldots$, $\lambda \in \mathcal{G}T_N$, and $\tau$ is an arbitrary finite path ending at $\lambda$.

In other words, the relations mean that for any $N$, the pushforward of $M$ under the natural projection

$$\prod_{N=1}^\infty \mathcal{G}T_N \supset \mathcal{T} \to \mathcal{G}T_N$$

coincides with $P_N$.

**Proof.** Let $\{P_N\}$ be a coherent system. For any $N$, we define a measure $M_N$ on the discrete space $\mathcal{T}_N$ as follows. Given $\tau \in \mathcal{T}_N$, we set

$$M_N(\{\tau\}) = \frac{1}{\text{Dim}_N \lambda} P_N(\lambda),$$

where $\lambda \in \mathcal{G}T_N$ is the end of $\tau$. This is a probability measure. Its pushforward under the projection $\mathcal{T}_N \to \mathcal{T}_{N-1}$ coincides with $M_{N-1}$: indeed, this is exactly a reformulation of the coherency property (7.2). Hence the family $\{M_N\}$ determines
a probability measure $M$ on the projective limit space $\mathcal{T}$. Clearly, $M$ is a central measure, and we have the relations (10.1).

Conversely, let $M$ be a central measure. For any $N$, define a probability measure $P_N$ on $\mathbb{GT}_N$ as the pushforward of $M$ under the projection (10.2). The fact that $M$ is central then implies that the family $\{P_N\}$ satisfies the coherency property. □

**Corollary 10.4.** There is a natural bijective correspondence $\chi \longleftrightarrow M$ between characters and central measures. This correspondence is an isomorphism of convex sets.

**Proof.** The bijection is established by means of the bijections $\chi \longleftrightarrow \{P_N\}$ (Proposition 7.4) and $\{P_N\} \longleftrightarrow M$ (Proposition 10.3). From the proofs of these propositions it is clear that this is an isomorphism of convex sets. □

**Definition 10.5.** Let $t = (t_N) \in \mathcal{T}$ be a path. We say that $t$ is regular if the images of the $t_N$’s under the embeddings $\mathbb{GT}_N \hookrightarrow \Omega$ (see Definition 10.1) converge to a point $\omega \in \Omega$. Then $\omega$ is called the end of $t$. Equivalently, the condition means that there exist limits

$$
\alpha_+^\pm = \lim_{N \to \infty} \frac{\tilde{p}_i(t_N^\pm)}{N}, \quad \beta_+^\pm = \lim_{N \to \infty} \frac{\tilde{q}_i(t_N^\pm)}{N} \quad (i = 1, 2, \ldots), \quad \delta_+^\pm = \lim_{N \to \infty} \frac{|t_N^\pm|}{N},
$$

where $t_N^+$ and $t_N^−$ denote the positive and negative parts of $t_N \in \mathbb{GT}_N$. The set of regular paths will be denoted by $\mathcal{T}_{reg}$.

Introduce the map

$$
\pi : \mathcal{T}_{reg} \to \Omega, \quad \mathcal{T}_{reg} \ni t \longmapsto \{\text{the end of } t\} \in \Omega. \tag{10.3}
$$

Further, for any $N = 1, 2, \ldots$, we introduce the map $\pi_N : \mathcal{T} \to \Omega$ as the composition

$$
\pi_N : \mathcal{T} \xrightarrow{t \mapsto t_N} \mathbb{GT}_N \xrightarrow{\text{Definition 10.1}} \Omega. \tag{10.4}
$$

For any $t \in \mathcal{T}_{reg}$, $\pi_N(t)$ converges (as $N \to \infty$) to $\pi(t)$ in the topology of the space $\Omega$. Indeed, this holds by the very definition of regular paths, see Definition 10.5.

**Lemma 10.6.** $\mathcal{T}_{reg} \subset \mathcal{T}$ is a Borel set, and the maps $\pi$, $\pi_N$ are Borel maps from $\mathcal{T}_{reg}$ to $\Omega$.

**Proof.** Indeed, for any $N$, the functions

$$
t \mapsto \frac{\tilde{p}_i(t_N^\pm)}{N}, \quad t \mapsto \frac{\tilde{q}_i(t_N^\pm)}{N} \quad (i = 1, 2, \ldots), \quad t \mapsto \frac{|t_N^\pm|}{N}
$$

are continuous functions on the space $\mathcal{T}$. This implies that the regularity condition determines a set of type $F_{\sigma\delta}$, hence a Borel set.

Each $\pi_N$, being a cylindrical map, is continuous on $\mathcal{T}$. Its restriction to $\mathcal{T}_{reg}$ is also continuous, hence Borel. Finally, $\pi$ is the pointwise limit of the $\pi_N$’s, hence it is a Borel map, too.
Theorem 10.7. Any central measure $M$ is concentrated on the set $\mathcal{T}_{\text{reg}} \subset \mathcal{T}$. The pushforward $\pi(M)$ under the projection (10.3) coincides with the spectral measure $P$ of the character $\chi$ that corresponds to $M$.

This claim makes sense, because $\mathcal{T}_{\text{reg}}$ is a Borel set.

**Derivation of Theorem 10.2 from Theorem 10.7.** Let $M$ be the central measure corresponding to $\chi$. By Proposition 10.3, we have $\pi_N(M) = P_N$. Now we will interpret $M$ as a probability measure on $\mathcal{T}_{\text{reg}}$, which is possible by virtue of Theorem 10.7. Also, $\pi_N$ will be viewed as a projection $\mathcal{T}_{\text{reg}} \to \Omega$. Then the equality $\pi_N(M) = P_N$ remains true. On the other hand, $\pi(M) = P$ (again by Theorem 10.7). Thus, both $P_N$ and $P$ can be viewed as pushforwards of one and the same probability measure $M$.

Now let $F$ be a bounded continuous function on $\Omega$. Since $\pi_N(t) \to \pi(t)$ for any $t \in \mathcal{T}_{\text{reg}}$, $F(\pi_N(t)) \to F(\pi(t))$, $t \in \mathcal{T}_{\text{reg}}$. Therefore, we get a sequence $\{F(\pi_N(\cdot))\}$ of uniformly bounded functions on $\mathcal{T}_{\text{reg}}$ converging pointwise to the function $F(\pi(\cdot))$. All these functions are Borel functions, because $\pi_N$ and $\pi$ are Borel maps (Lemma 10.6). Consequently, as $N \to \infty$,

$$\int_{\mathcal{T}_{\text{reg}}} F(\pi_N(t)) M(dt) \to \int_{\mathcal{T}_{\text{reg}}} F(\pi(t)) M(dt).$$

Since $\pi_N(M) = P_N$ and $\pi(M) = P$, we can convert these integrals to integrals over $\Omega$,

$$\int_{\mathcal{T}_{\text{reg}}} F(\pi_N(t)) M(dt) = \int_{\Omega} F(\omega) P_N(d\omega), \quad \int_{\mathcal{T}_{\text{reg}}} F(\pi(t)) M(dt) = \int_{\Omega} F(\omega) P(d\omega),$$

which concludes the proof. □

The rest of the section is devoted to the proof of Theorem 10.7.

Given two signatures, $\nu \in \mathcal{GT}_n$ and $\lambda \in \mathcal{GT}_N$, where $n < N$, denote by $\dim_{nN}(\nu, \lambda)$ the number of paths starting at $\nu$ and ending at $\lambda$. I.e., the number of chains

$$\tau = (\tau_n < \tau_{n+1} < \cdots < \tau_{N-1} < \tau_N), \quad \tau_n = \nu, \quad \tau_N = \lambda.$$

We extend the definition of the cotransition function by setting (cf. the proof of Lemma 9.3)

$$q(\nu, \lambda) = \frac{\dim_n(\nu) \dim_{nN}(\nu, \lambda)}{\dim_N \lambda}, \quad \nu \in \mathcal{GT}_n, \quad \lambda \in \mathcal{GT}_N, \quad n < N.$$

Note that

$$q(\nu, \lambda) = \sum_\tau \prod_{i=n+1}^N q(\tau_{i-1}, \tau_i),$$

summed over all $\tau$’s as above.

For any coherent system $\{P_N\}$, we have

$$P_n(\nu) = \sum_{\lambda \in \mathcal{GT}_N} q(\nu, \lambda) P_N(\lambda), \quad \nu \in \mathcal{GT}_n, \quad n < N.$$

Indeed, this relation is obtained by iterating the coherency relation (7.2).
Proposition 10.8. Let \( \{P_N\} \) be a coherent system and \( M \) be the corresponding central measure. Assume that \( M \) is extreme. Then for \( M \)-almost all paths \( t = (t_N) \in \mathcal{T} \)

\[
\lim_{N \to \infty} q(\nu, t_N) = P_n(\nu), \quad n = 1, 2, \ldots, \ \nu \in \mathbb{GT}_n. \tag{10.5}
\]

**Comment.** This result actually holds for general “branching graphs” in the sense of Vershik–Kerov [VK3]. It was stated in their paper [VK1] for the Young graph, associated with the infinite symmetric group \( S(\infty) \). A detailed proof (an adaptation of the proof of the Birkhoff–Khinchine ergodic theorem) is contained in an unpublished work by Kerov. We present below a similar argument but use a limit theorem for reversed martingales (as was suggested in [VK1]). Note also a very close earlier result due to Vershik, see [Ve, Theorem 1] and [OV, Theorem 3.2 and Remark 3.6].

**Proof. Step 1.** Let us say that two paths \( t = (t_m), t' = (t'_m) \) are \( N \)-equivalent if \( t_m = t'_m \) for \( m \geq N \). Denote by \( \xi_N(t) \) the \( N \)-equivalence class containing \( t \). Then

\[
\{t\} = \xi_1(t) \subset \xi_2(t) \subset \ldots.
\]

We say that \( t \) and \( t' \) are \( \infty \)-equivalent if they are \( N \)-equivalent for \( N \) large enough. The \( \infty \)-equivalence class of \( t \) is denoted by \( \xi_\infty(t) \). Clearly, \( \xi_\infty(t) = \cup \xi_N(t) \).

Let \( \mathcal{B}_-N \) denote the \( \sigma \)-algebra of those Borel sets in \( \mathcal{T} \) that are saturated with respect to the \( N \)-equivalence relation, where \( N = 1, 2, \ldots \). We define \( \mathcal{B}_-\infty \) likewise. We have

\[
\mathcal{B}_-\infty \subset \cdots \subset \mathcal{B}_-2 \subset \mathcal{B}_-1 = \{\text{All Borel sets}\}, \quad \mathcal{B}_-\infty = \bigcap_{N=1}^\infty \mathcal{B}_-N.
\]

Fix an arbitrary bounded Borel function \( \psi \) on \( \mathcal{T} \). We will view \( \psi \) as a random variable defined on the probability space \( (\mathcal{T}, M) \). Let \( \psi_N = \mathbb{E}(\psi \mid \mathcal{B}_-N) \) and \( \psi_\infty = \mathbb{E}(\psi \mid \mathcal{B}_-\infty) \) be the conditional expectations of \( \psi \) with respect to the \( \sigma \)-algebras \( \mathcal{B}_-N \) and \( \mathcal{B}_-\infty \), respectively.

By the theorem on convergence of (reversed) martingales, we have

\[
\lim_{N \to \infty} \psi_N = \psi_\infty \quad \text{almost everywhere with respect to } M,
\]

see, e.g., Doob [Do, Chapter VII, Theorem 4.2].

**Step 2.** So far we did not use the assumption that \( M \) is central. Now we remark that this assumption makes it possible to describe each \( \psi_N \) explicitly as the averaging over the \( N \)-equivalence classes:

\[
\psi_N(t) = \frac{1}{\text{Dim}_N(t_N)} \sum_{t' \in \xi_N(t)} \psi(t'). \tag{10.6}
\]

Next, we use the assumption that \( M \) is an extreme central measure to conclude that each set \( A \in \mathcal{B}_-\infty \) has mass 0 or 1. Indeed, assume \( 0 < M(A) < 1 \), and let \( 1_A \) stand for the characteristic function of \( A \). Then the measure \( M \) can be written as a nontrivial convex combination of two probability measures,

\[
M = M(A) \cdot \frac{1_A M}{M(A)} + (1 - M(A)) \frac{1_{T \setminus A} M}{1 - M(A)}.
\]
Since $A$ is saturated with respect to the $\infty$–equivalence relation, these two measures are central, which contradicts the extremality assumption.

It follows that $\psi_\infty$ is constant almost everywhere. Since the means of all $\psi_N$’s are the same and equal to the mean of $\psi$, the constant is the mean of $\psi$.

Therefore, we conclude: let $\psi$ be any bounded Borel function on $T$ and let $\psi_N(t)$ be defined by (10.6); then

$$
\lim_{N\to\infty} \psi_N(t) = \int_T \psi(t')M(dt') \quad \text{for } M\text{–almost all } t \in T. 
$$

(10.7)

**Step 3.** Fix $n$ and $\nu \in GT_n$, and set

$$
\psi(t) = \begin{cases} 1, & \text{if } t_n = \nu, \\ 0, & \text{otherwise}. \end{cases}
$$

For any $N \geq n$, we have

$$
\psi_N(t) = \frac{\operatorname{Dim}_n(\nu) \operatorname{Dim}_{nN}(\nu, t_N)}{\operatorname{Dim}_N t_N} = q(\nu, t_N).
$$

Indeed, the set $\xi_N(t)$ contains exactly $\operatorname{Dim}_n(\nu) \operatorname{Dim}_{nN}(\nu, t_N)$ paths $t'$ passing through $\nu$. Finally, the mean of $\psi$ equals $P_n(\nu)$.

Now we apply the result of Step 2 and get from (10.7) the required claim. □

Assume that for any $N = 1, 2, \ldots$ we are given a character (in the sense of Definition 1.1) of the group $U(N)$, denoted as $\chi^{(N)}$. Let us say that the sequence $\{\chi^{(N)}\}$ converges to a function $\chi^{(\infty)}$ defined on the group $U(\infty)$ if

$$
\text{as } N \to \infty, \quad \chi^{(N)}|_{U(n)} \xrightarrow{\text{uniformly}} \chi^{(\infty)}|_{U(n)}, \quad \text{for any fixed } n = 1, 2, \ldots 
$$

(10.8)

Clearly, the limit function $\chi^{(\infty)}$ is a character of $U(\infty)$. Expand $\chi^{(N)}$ on irreducible normalized characters of $U(N)$:

$$
\chi^{(N)} = \sum_{\lambda \in GT_N} P^{(N)}_\lambda \overline{\chi}^\lambda.
$$

Then, for any $n < N$,

$$
\chi^{(N)}|_{U(n)} = \sum_{\nu \in GT_n} \left( \sum_{\lambda \in GT_N} q(\nu, \lambda)P^{(N)}_\lambda(\lambda) \right) \overline{\chi}^{\nu}.
$$

It follows that (10.8) is equivalent to the following condition: for any $n$ and any $\nu \in GT_n$,

$$
\lim_{N\to\infty} \sum_{\lambda \in GT_N} q(\nu, \lambda)P^{(N)}_\lambda(\lambda) = P_n(\nu), \quad (10.9)
$$

where $\{P_N\}$ is the coherent system corresponding to $\chi^{(\infty)}$. 47
Proposition 10.9. Let \( \{\lambda(N) \in \mathbb{G} T_N\}_{N=1,2,...} \) be a sequence of signatures and let \( \chi^{(N)} = \tilde{\chi}^{\lambda(N)} \) be the corresponding normalized irreducible characters.

The sequence \( \{\chi^{(N)}\} \) converges to a function \( \chi^{(\infty)} \) if and only if the images of the \( \lambda(N) \)'s under the embeddings of Definition 10.1 converge to a point \( \omega \in \Omega \), and then the limit function \( \chi^{(\infty)} \) coincides with the extreme character \( \chi(\omega) \).

Proof. This result is due to Vershik–Kerov [VK2]. For a detailed proof, see [OkOl]. □

Proposition 10.10. Let \( \omega \in \Omega \) and \( M^{(\omega)} \) be the extreme central measure corresponding to the extreme character \( \chi^{(\omega)} \). The measure \( M^{(\omega)} \) is concentrated on the subset \( \pi^{-1}(\omega) \subset T_{\text{reg}} \) of regular paths ending at \( \omega \).

Proof. By Proposition 10.8, the measure \( M^{(\omega)} \) is concentrated on the set of paths \( t = (t_N) \in T \) satisfying the condition (10.5), where \( \{P_N\} \) is the coherent system corresponding to the character \( \chi^{(\omega)} \). Let us show that this set coincides with \( \pi^{-1}(\omega) \).

Indeed, the condition (10.5) coincides with the condition (10.9) for the characters \( \chi^{(N)} = \tilde{\chi}^{t_N} \) (here \( P(N) \) is reduced to the delta measure at \( t_N \)). As explained above, the latter condition is equivalent to the convergence of the characters \( \chi^{(N)} \) to the character \( \chi^{(\omega)} \). By Proposition 10.9, this exactly means that \( t \) is a regular path ending at \( \omega \). □

Proof of Theorem 10.7. Translating Theorem 9.1 into the language of central measures we get the decomposition

\[ M = \int_{\Omega} M^{(\omega)} P(d\omega). \]

By Proposition 10.10, any extreme measure \( M^{(\omega)} \) is concentrated on the subset \( \pi^{-1}(\omega) \subset T_{\text{reg}} \). Hence, \( M \) is concentrated on \( T_{\text{reg}} \). This proves the first claim of Theorem 10.7. The second claim also follows from the above decomposition and Proposition 10.10. □

11. Conclusion: the problem of harmonic analysis

Now we are in a position to state the problem of harmonic analysis on the group \( U(\infty) \):

Let \( (z, z, w, w') \in D_{\text{adm}} \) be an arbitrary admissible quadruple of parameters (Proposition 7.10), let \( P_N = \{P_N(\cdot \mid z, z, w, w')\}_{N=1,2,...} \) be the coherent system as defined in (7.8), let \( \chi_{z,z,w,w'} \) be the corresponding character, and let \( P \) be the spectral measure of \( \chi_{z,z,w,w'} \).

Describe explicitly the measure \( P \).

Theorem 10.2 suggests the idea to evaluate \( P \) by means of the limit transition from the measures \( P_N \). This idea is realized in the subsequent paper [BO4].

Let \( \chi \) be an arbitrary character of \( U(\infty) \) and \( T_\chi \) be the corresponding spherical representation of \( (G, K) \), see §2. One can prove that the spectral measure of \( \chi \) also determines the decomposition of \( T_\chi \) into a (multiplicity free) direct integral of irreducible spherical representations. Recall that in §3, we constructed a family \( \{T_{zw}\} \) of unitary representations. If \( \Re(z + w) > -\frac{1}{2} \) then \( T_{zw} \) possesses a distinguished vector, and if, moreover, both \( z \) and \( w \) are nonintegral, then this vector is
cyclic (Proposition 6.9), so that $T_{zw}$ coincides with the representation $T_{\chi}$ with the character $\chi = \chi_{zw}$. Thus, in this case, the spectral measure of the character $\chi_{zw}$ also governs the decomposition of $T_{zw}$.

If $\Re(z + w) > -\frac{1}{2}$ but at least one of the parameters $z, w$ is integral, then the spectral measure of the character $\chi_{zw}$ refers to a proper subrepresentation of $T_{zw}$. To decompose the whole representation $T_{zw}$ we need additional tools. Cf. [KOV].

Finally, recall that the construction of the representations $T_{zw}$ makes sense even when $\Re(z + w) \leq -\frac{1}{2}$, although then the characters $\chi_{zw}$ disappear. It would be interesting to study the decomposition problem in this case as well. Cf. a similar problem concerning infinite measures $m(s)$ stated in [BO3, §8].

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