Controlling Exceptional Points with Light

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We propose and show that application of light leads to an intriguing platform for controlling exceptional points in non-Hermitian topological systems. We demonstrate our proposal using two different systems – nodal line semimetals and semi-Dirac semimetals – and show that using illumination with light one can engineer the dynamics and stability of exceptional points. We illustrate the topological properties and map out the light-driven topological phase transitions.

Introduction– In the last few years, the study of topological phases in non-Hermitian systems has attracted wide interest. This rapidly burgeoning field encompasses both theoretical and experimental advancements in atomic physics, quantum optics, microwave cavities, and topological laser systems, using controlled dissipation with incorporation of gain and loss terms in open systems. Considering the interplay between non-Hermiticity and topology, several varieties of non-Hermitian topological semimetals have been proposed, including knot, nodal line, nodal ring, Hopf link, Dirac, Weyl, and semi-Dirac, to name just a few. Some of these have been experimentally realized recently.

Non-Hermitian topological systems show a distinct class of spectral degeneracies, known as exceptional points (EPs), where not only eigenvalues but also eigenvectors coalesce and result in the Hamiltonian becoming non-diagonalizable. Enclosing such an EP yields a quantized topological invariant revealing its underlying topological nature.

On the other hand, there has been a growing interest in creating new topological phases of matter using light – a field dubbed Floquet engineering. Application of light to low-dimensional systems, including graphene and silicene, has led to various topological phases, creating non-trivial gaps in their spectra. Photon dressing and Floquet dynamics of various topological band systems including Weyl semimetals, nodal semimetals, Chern insulators, Dirac and semi-Dirac semimetals have been well studied and their rich topological properties have been intensively explored.

Very recently, non-Hermitian Floquet topological phases have also been studied with great interest. The study of periodically driven non-Hermitian systems opens up an avenue to explore new features of non-Hermitian topological phases. We highlight among them the study of topological edge state properties, their dynamical characterization, and their transport properties.

Motivated by these rapid developments, in this work, we combine the two ideas and propose to use Floquet engineering to create and control tunable non-Hermitian phases. Generally, experimental setups used to study the Floquet topological phases are dissipative systems subject to gain and loss. Therefore, it becomes particularly important to study non-Hermitian Floquet topological phases in search of new exciting phenomena. In this contribution, we show that using light one can control the dynamics and stability of EPs. We demonstrate our proposal using two examples in different dimensions, namely non-Hermitian nodal line semimetal in three dimensions and non-Hermitian semi-Dirac semimetal in two dimensions. We show that one can tune the position of the EPs and annihilate them by using light. We use analytical as well as numerical calculations to illustrate the topological properties and map out the topological phase transitions arising from the application of light. Our hope is that these findings will motivate future theoretical and experimental investigations to engineer non-Hermitian Floquet topological phases.

Nodal line semimetal in three dimensions– The continuum Hamiltonian describing a two-band spinless non-Hermitian nodal-line semimetal is

$$H_0(q) = e_0(q)I + (m - Bq^2)\sigma_x + (v_z q_z + i\gamma)\sigma_z,$$  \hspace{1cm} (1)

where total squared momentum $q^2 = q_x^2 + q_y^2 + q_z^2$, $\sigma_i$ ($i = x, y, z$) is the triad of Pauli matrices and $I$ is the identity matrix. Here $v_z$ denotes the Fermi velocity along the $z$ direction, and $m$ and $B$ are parameters with the dimensions of energy and inverse energy, respectively. Note that the Hamiltonian is rotationally symmetric in the $q_x - q_y$ plane. Here $i\gamma_0$ is the non-Hermitian term associated with particle gain and loss between the two orbitals, with strength $\gamma$. We obtain the energy eigenvalues as

$$E = e_0(q) \pm \sqrt{(m - Bq^2)^2 + v_z^2 q_z^2 - \gamma^2 + 2iv_z q_z \gamma},$$  \hspace{1cm} (2)

where $\pm$ denote the conduction and valence band respectively. The energy eigenvalues are in general complex. Without loss of generality, we choose $e_0$ to be zero. In the absence of the non-Hermitian term, the conduction and valence bands touch each other and form a nodal ring in the $q_z = 0$ plane for $mB > 0$. The relative sign of $mB$ controls the transition between a trivial insulator phase and a nodal semimetal phase. In the presence of the non-Hermitian term, $i\gamma\sigma_z$, the original nodal ring splits into two exceptional rings (ERs) for $\gamma < m$. The band diagrams clearly demonstrating these features are presented in Fig. 1. We note that the energy is real both
inside the inner ER and outside the outer ER, while being imaginary between the two ERs.

The non-Hermitian term creates four exceptional points (EPs) along the nodal line in the $q_z = 0$ plane, where the valence and conduction band touch each other. The location of the four EPs are given by $q_{EP} = \pm \sqrt{(m-s\gamma)/B}$, where $s = \pm 1$ in the $q_x - q_y$ plane. At the critical value of $\gamma$, i.e. $\gamma = m$, the inner EPs coincide with each other, and the inner ER becomes a point.

We now introduce an off-resonant laser beam of frequency $\omega$ polarized in the $yz$ plane on the non-Hermitian nodal semimetal. We choose the vector potential to be $A(t) = a_x \eta \cos \omega t + a_z \cos (\omega t + \phi) \hat{z}$, where angle $\phi$ controls the polarization of light. Here $\eta = \pm 1$ represents the left or right handedness of the incident light beam. We obtain the full time-dependent Hamiltonian considering the minimal coupling $q \rightarrow q + eA(t)$. Using Floquet formalism, the time-dependent Hamiltonian can be approximated as

$$H_{eff} = H + \frac{[H_{-1}, H_{+1}]}{\omega} + O(1/\omega^2), \quad (3)$$

where $H_{\pm 1} = \frac{\omega}{2\pi} \int_0^T H(t) e^{\pm i\omega t} dt$ are Fourier coefficients of the time-dependent Hamiltonian, and $T = 2\pi/\omega$ is the time period of the incident light. We note that such an approximation is valid in the high frequency domain for small laser power. The effective Hamiltonian in the presence of light is

$$H_{eff} = (m - Bq^2)\sigma_x + (v_zq_z + i\gamma)\sigma_z + \frac{2e^2Bv_z a_y a_z q_y \sin \phi}{\omega} \sigma_y, \quad (4)$$

Notice that in the effective Hamiltonian, we get the photon-dressed term linear in $q_y$, explicitly breaking the rotational symmetry in the $q_x - q_y$ plane. The $\sin \phi$ dependence indicates that the linearly polarized light does not affect the band structure. The light-induced band structure for the non-Hermitian nodal line semimetal is shown in Fig. 1 at different values of light intensity for circularly polarized light ($\phi = \pi/2$). The calculated band diagram shows that in this case the light illumination does not create a gap in the spectrum, but it affects the rotational symmetry of ERs along $q_y$. As a consequence of the competition between linear and quadratic terms in $q_y$ the energy bands start losing the nodal ring symmetry along the $q_x = 0$ line. With increasing light intensity, the linear term in $q_y$ dominates over the quadratic term. Eventually beyond a critical value of light intensity, the bands change topology, turning into two arcs in the $q_x - q_y$ plane. Instead of $yz$ plane, if we had started with the laser beam polarized in the $xz$ plane, the light-induced term would have been of the form $\frac{2e^2Bv_z a_x a_z q_z \sin \phi}{\omega} \sigma_x$. Such a term would affect the rotational symmetry along $q_x = 0$. This means that with increasing light intensity, we would have obtained two arcs along $q_y = 0$ line, i.e., in a direction orthogonal to the case when the laser beam is polarized in the $yz$ plane. This observation suggests that we can control the dynamics of the EPs along any direction in the $q_x - q_y$ plane by simply rotating the laser beam in the plane perpendicular to the plane originally containing the nodal line.

The light-induced term allows control over the position of the EPs. In order to characterize the topological properties, we adopt cylindrical polar coordinates and
Hermitian systems. When we encircle an EP, the as-
phase transitions using topological invariants for non-
across these critical values due to the tuning by light.

\[ \theta = \tan^{-1} \frac{q_y}{q_x} \]

We see that rotational symmetry reduces one degree of
freedom in the system. Upon choosing \( a_x = a_y = a \)
and \( \phi = \pi/2 \), the dispersion relation now reads
\[ E = \pm \sqrt{(m - Bq_\gamma)^2 + v_\gamma^2 q_\gamma^2 - \gamma^2 + 2iv_\gamma q_\gamma \gamma + (2e^2 B q_\gamma a^2 q/\omega)^2} \]

We discover that our system now has four EPs, sym-
metrically situated about the nodal line, located at
\[ q_{EP} = \pm \sqrt{\frac{(2mb - \zeta^2) \pm (\zeta^2 - 2mb^2) - 4B^2(m^2 - \gamma^2)}{2B^2}}, \]
where \( \zeta = 2e^2 B v_\gamma a^2/\omega \). Locations of the EPs are shown for different values of \( a \) in Fig. 2. We find that at a critical value of \( a \), the four EPs annihilate each other and their positions become imaginary. From Eq. 5, we obtain two conditions for the EPs to disappear (i) \((\zeta^2 - 2mb^2)^2 \leq 4B^2(m^2 - \gamma^2)\) and (ii) \( m \leq \gamma \). Since, we only consider \( m \geq \gamma \), we focus on condition (i), which leads to the critical values
\[ \zeta_c = \sqrt{2mb \pm \sqrt{4B^2(m^2 - \gamma^2)}}, \]
\[ a_c = (\omega^2/4e^4B^2v_\gamma^2)^{1/4}[2mb \pm \sqrt{4B^2(m^2 - \gamma^2)}]^{1/4}. \]

We expect non-Hermitian topological phase transitions across these critical values due to the tuning by light. Next, we characterize these light induced topological phase transitions using topological invariants for non-Hermitian systems. When we encircle an EP, the associated bands get swapped in the parameter space of complex energy due to the Riemann sheet geometry of the energy bands while returning to initial states. One can define the vorticity, \( \nu_{mn} \), directly associated with the complex energy dispersion for any pair of bands as
\[ \nu_{mn}(\Gamma) = -\frac{1}{2\pi} \int_{\Gamma} \nabla_k \arg[E_m(k) - E_n(k)] \, dk, \]
where \( \Gamma \) is a closed loop encircling the EP in the momentum space. The energy eigenvalue for a single complex band of non-Hermitian Hamiltonian in general can be written as \( E(k) = |E(k)|e^{i\theta(k)} \), where \( \theta = \tan^{-1}(\text{Im}E/\text{Re}E) \). The complex energy bands evolve in a periodic cycle \( \theta(k) \rightarrow \theta(k) + 2\pi \nu \) (\( \nu \) being an integer).

Since an EP is a defective point in the spectrum, encircling an EP leads to a quantized vorticity. For our photon dressed non-Hermitian nodal line semimetal case, we illustrate the topological transitions by calculating the evolution of the two bands encircling an EP at different values of light intensity. In Fig. 2 we present the vorticity of one of the EPs for zero light intensity and clearly see the swapping of two bands, indicating band degeneracy as a signature of nontrivial topological phase. Gradually increasing the light intensity leads to the deformation of evolution of the two complex bands. Eventually at a critical value of light intensity (see Eq. 6), when the four EP locations become imaginary we see the two energy bands get separated and topological properties of the system are lost, as there is no swapping between energy bands in the complex energy plane. Overall, our analysis shows that one can control the stability of EPs by tuning the light amplitude and eventually also annihilate them.

Semi-Dirac semimetal in two dimensions– As a second

![Image](a1)

![Image](a2)

![Image](b1)

![Image](b2)

![Image](c1)

![Image](c2)

![Image](d1)

![Image](d2)
FIG. 3. Exceptional points in semi-Dirac semimetals under illumination with varying light amplitude. The trajectory of the two complex eigenvalues, when the contour parameterized by $\theta_L \in [0, 2\pi)$ enclosing the EP with light intensity (a2) $a = 0.0$, (b2) $a = 0.45$, (c2) $a = 0.452$, (d2) $a = 0.46$. The position of EPs in (a1), (b1), (c1), and (d1) with same light intensity as in the vorticity diagrams in the lower panels. At the critical laser intensity [(c1) and (c2)], the two EPs annihilate each other and the two complex energy bands yield a trivial vorticity. Beyond this critical amplitude [(d1) and (d2)], the system does not exhibit EPs. Here we set $\gamma = 0.2$ and $\delta_0 = 0.5$.

Example of our proposal, we consider a non-Hermitian semi-Dirac semimetal in two dimensions described by the Hamiltonian

$$H(q) = \left(\frac{q^2}{2m} - \delta_0\right) \sigma_x + v_f q_y \sigma_z + i\gamma \sigma_z. \quad (8)$$

Here also $\gamma$ is the gain and loss coefficient between the two orbitals. We now introduce circularly polarized off-resonant light, with a vector potential $A(t) = a(\eta \sin \omega t \hat{x} + \cos \omega t \hat{y})$. Using Floquet formalism discussed previously, we obtain the effective Hamiltonian in the presence of light as

$$H_{\text{eff}} = \left(\frac{q^2}{2m} - \delta_0\right) \sigma_x + (v_z q_z + i\gamma) \sigma_z - \frac{\eta e^2 a^2 v_f q_x}{m\omega} \sigma_y. \quad (9)$$

We have evaluated the effect of light on the EPs in this two-dimensional system (see Supplemental Material for detailed calculations). The modified positions of the EPs, due to light, along the nodal line $q_y = 0$ become

$$q_{\text{EP}} = \pm \sqrt{\frac{m}{2}} \sqrt{(2\delta_0 - 2m\xi^2) \pm \sqrt{(2\delta_0 - 2m\xi^2)^2 - 4(\delta_0^2 - \gamma^2)}}. \quad (10)$$

where $\xi = \frac{\eta e^2 a^2 v_f q_x}{m\omega}$. We present the locations of EPs for different values of $a$ in upper panels of Fig. 3. We discover that one can indeed tune the positions of EPs by changing the laser intensity. We find a critical laser intensity, at which the four EPs annihilate each other pairwise, rendering their positions imaginary, as

$$\xi_c = \sqrt{\frac{2\delta_0 \pm 2 \sqrt{(\delta_0^2 - \gamma^2)}}{2m}}, \quad a_c = \left(\frac{2\delta_0 \pm 2 \sqrt{(\delta_0^2 - \gamma^2)}}{2m}\right)^{1/4}. \quad (11)$$

Next, we characterize our system under illumination using the vorticity. We illustrate the topological changes of the system in Fig. 3 for increasing values of light intensity. We find a change in evolution of two bands as $a$ is increased. Two pairs of the EPs approach each other and mutually annihilate. So, we have found that the laser intensity can be used in topological tuning and the accompanying topological phase transitions in two-dimensional semi-Dirac semimetals.

Possible experimental realization—Over the last few years, there have been several exciting experiments suggested, as well as realized, to explore rich topological phases by tuning the gain and loss terms in a controllable manner, mainly in dissipative waveguide systems and cold atomic gases [53,54]. Furthermore many ingenious experiments have been devoted to study the topological signatures of exceptional points [55,56]. Very recently, Cerjan et al. demonstrated Weyl exceptional rings in a three-dimensional photonic lattice consisting of evanescently coupled single mode helical waveguides [57]. Incorporating circularly polarized light in their setup could be a promising potential platform to test our proposal and explore the possibility of controlling the dynamics of
induced topological properties of EPs. Additionally, topological laser systems can be potential hosts to study the light induced topological properties of EPs by tuning the laser intensity. Considering the recent experimental advancements in non-Hermitian topological phases, we are confident that our proposal can be interesting playground for realizing light induced dynamics of non-Hermitian topological phases.

Summary and conclusions– In summary, we have proposed a new platform for controlling exceptional points in non-Hermitian topological systems using light. Using different examples – nodal line semimetals and semi-Dirac semimetals – we have shown that light can tune the dynamics and stability of exceptional points. Furthermore, one can generalize our formalism beyond these two systems to other non-Hermitian systems which can be tuned using light. We hope that these findings will motivate exploration of using light to engineer exceptional points in non-Hermitian systems.

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I. SUPPLEMENTAL MATERIAL

A. Floquet engineering of non-Hermitian semi-Dirac semimetals

In this section we present further calculation details of Floquet analysis of a semi-Dirac semimetal. The model Hamiltonian describing low-energy electronic bands of a two-dimensional non-Hermitian semi-Dirac semi-metal incorporating a gain and loss term between two orbitals is

$$H(q) = \left( \frac{q^2}{2m} - \delta_0 \right) \sigma_x + v_f q_y \sigma_z + i \gamma \sigma_z,$$

(1)

where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices in the pseudospin space and $\gamma$ can be thought as the gain and loss coefficient between the two orbitals.

We get the energy eigenvalues as

$$E = \pm \sqrt{\left( \frac{q^2}{2m} - \delta_0 \right)^2 + v_f^2 q_y^2 - \gamma^2 + 2iv_f q_y \gamma}.$$

(2)

As a consequence of non-Hermitian band degeneracy, we find a nodal line along $q_y = 0$ consisting of four EPs with symmetrically displaced locations given by

$$q_{EP}^x = \pm \sqrt{2m(\delta_0 \pm \gamma)}.$$

(3)

We now study the effect of circularly polarized off-resonant light of frequency $\omega$, which generates a vector potential $A(t) = a(\eta \sin \omega t \hat{x} + \cos \omega t \hat{y})$, on non-Hermitian semi-Dirac semimetal. Here $\eta = \pm 1$ for right and left circularly polarized beams, respectively. Using Floquet formalism discussed in the main text, we get light induced term for the Hamiltonian in Eq. (1) as

$$[H_{-1}, H_{+1}] = -\frac{\eta e^2 a^2 v_f q_x}{m} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

(4)

such that the effective Hamiltonian in the presence of light becomes

$$H_{\text{eff}} = \left( \frac{q^2}{2m} - \delta_0 \right) \sigma_x + (v_z q_z + i \gamma) \sigma_z - \frac{\eta e^2 a^2 v_f q_x}{m \omega} \sigma_y.$$

(5)

Note that similar to the nodal line semimetal, the perturbing term in the effective Hamiltonian is proportional to momentum, i.e. it is linear in $q_x$. We see in the band diagram presented in Fig. 1(a) that without the light induced term there are four EPs along the nodal line $q_y = 0$. With increasing light intensity Fig. 1(b), (c) and (d), the positions of the EPs shift and eventually they annihilate each other resulting in a gap in the spectrum.

The dispersion relation for the effective Hamiltonian becomes...
FIG. 1. Band diagrams of photon-dressed non-Hermitian semi-Dirac semimetals. The real and the imaginary part of the energy dispersion (a) and (e) in the absence of light \((a = 0)\), (b) and (f) with moderate laser intensity \((a = 0.45)\), and (c) and (g) at critical light intensity \((a = 0.452)\). At the critical value the nodal line symmetry of the spectrum is lost along the \(q_y = 0\) line. Beyond this critical light intensity, (d) the real and (h) imaginary part of the spectrum are shown. The real part of the spectrum become gapped. We choose the following values for the other parameters: \(B = v_f = 1.0, e = \omega = 1.0, m = 1, \delta = 0.5\) and \(\gamma = 0.2\), keeping them unchanged unless otherwise specified.

\[
E = \pm \sqrt{\left(\frac{q_x^2}{2m} - \delta_0\right)^2 + v_f^2 q_y^2 - \gamma^2 + 2i v_f q_y \gamma + \left(\frac{\eta e^2 a^2 v_f q_x}{m \omega}\right)^2}
\]  

(6)

The positions of the EPs along the nodal line \(q_y = 0\) change to

\[
q_{\text{EP}} = \pm \sqrt{m \left(\sqrt{2 \delta_0 - 2m \xi^2} \pm \sqrt{(2 \delta_0 - 2m \xi^2)^2 - 4(\delta_0^2 - \gamma^2)}\right)}
\]  

(7)

where \(\xi = \frac{\eta e^2 a^2 v_f q_x}{m \omega}\). In the main text, we have presented the locations of EPs for different values of \(a\) and we have shown that we can tune the positions of EPs by changing the laser intensity. We find the critical laser intensity, at which the four EPs annihilate each other pairwise, rendering their positions imaginary as

\[
\xi_c = \sqrt{\frac{2 \delta_0 \pm 2 \sqrt{(\delta_0^2 - \gamma^2)}}{2m}}
\]

\[
a_c = \left(\frac{2 \delta_0 \pm 2 \sqrt{(\delta_0^2 - \gamma^2)}}{2m}\right)^{1/4}
\]  

(8)

So, the laser intensity can be used in topological tuning accompanying topological phase transitions. This result is confirmed by our vorticity analysis presented in the main text.