The exclusivity principle singles out the quantum violation of the Bell inequality

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We show that the exclusivity principle exactly singles out the Tsirelson bound of the Clauser-Horne-Shimony-Holt Bell inequality. The proof is surprisingly simple and does not require an infinite universe.

Introduction.—Quantum theory is arguably the most accurate scientific theory of all times. Nevertheless, despite its mathematical simplicity, its fundamental principles are still unknown. In the quest for these principles, a key question is why quantum theory is exactly as contextual and nonlocal as it is. The principles of information causality [1] and macroscopic locality [2] single out the maximum quantum violation (i.e., the Tsirelson bound [3]) of the Clauser-Horne-Shimony-Holt (CHSH) Bell inequality [4, 5], but cannot exclude superquantum correlations in tripartite scenarios [6, 7] or explain the quantum violation of noncontextuality inequalities [8]. In contrast, the exclusivity principle [9–14] singles out the maximum quantum violation of the simplest noncontextuality inequality [9], excludes extremal nonlocal boxes in tripartite scenarios [12, 14], explains some complete sets of quantum correlations [13] and there is strong evidence that singles out scenarios [12, 14], explains some complete sets of quantum correlations [13]. Here we will prove that the E principle exactly singles out the Tsirelson bound of the CHSH Bell inequality.

The Tsirelson bound.—The discovery that quantum theory (QT) violates the CHSH Bell inequality, a condition that any local realistic (LR) theory must satisfy, is one of the most celebrated results of science [4]. The CHSH Bell inequality [5] is defined in a scenario in which there are two distant observers, Alice and Bob, and in each run each of them measures an observable randomly chosen between two. Alice chooses between $A_0$ and $A_1$, and Bob between $B_0$ and $B_1$. Each measurement has two possible results: $+1$ and $-1$. Alice’s (Bob’s) choice is spacelike separated from Bob’s (Alice’s) measurement result. This implies that one observer’s result cannot be influenced by the other observer’s choice, assuming that influences do not propagate faster than the speed of light in vacuum.

The CHSH Bell inequality states that, for any LR theory,

$$S \leq 3,$$

where

$$S = P(A_0+,B_0+) + P(A_0-,B_0-) + P(A_0+,B_1+)$$

$$+ P(A_0-,B_1-) + P(A_1+,B_0+) + P(A_1-,B_0-$$

$$+ P(A_1+,B_1-) + P(A_1-,B_1+),$$

(2)

where, e.g., $P(A_1+,B_1-)$ is the joint probability of the event $(A_1+,B_1-)$, defined as “Alice obtains $+1$ when she measures $A_1$, and Bob obtains $-1$ when he measures $B_1$”. However, according to QT,

$$S_{\text{opt}} \leq 2 + \sqrt{2} \approx 3.414.$$

(3)

This upper bound is the Tsirelson bound [3]. Assuming that QT is correct, this means that the universe is “nonlocal” (i.e., cannot be explained with LR theories), but only up to a certain limit. Which is the principle that enforces this limit? [15] It is not relativistic causality [15] or triviality of communication complexity [16]. Information causality [1] and macroscopic locality [2] single out the Tsirelson bound, but cannot explain neither why superquantum correlations in tripartite scenarios are impossible [6, 7] nor the quantum bound of noncontextuality inequalities [8].

The E principle.—A physical theory satisfies the E principle when any set of pairwise mutually exclusive events is jointly exclusive. Therefore, from Kolmogorov’s axioms of probability, the sum of the probabilities of any set of pairwise mutually exclusive events cannot be higher than 1. Events are mutually exclusive if there exists a set of co-measurable observables that distinguish between them. For example, $(A_0+,B_0-)$ and $(A_0+,B_0+)$ are mutually exclusive since $B_0$ distinguishes between them. Observables $A$ and $B$ are co-measurable if there exists an observable $M_{A,B}$ whose outcome set is the Cartesian product of the outcome sets of $A$ and $B$ and such that, for all states, the outcome probability distributions of $A$ and $B$ are recovered as marginals of the outcome probability distribution of $M_{A,B}$ [17]. If some observables are co-measurable then there exists a joint probability distribution for them. If the same observable $A$ belongs to two different sets of co-measurable observables, the sum of the probabilities of both sets is larger than the probability of $A$. This implies that one observer’s result cannot be influenced by the other observer’s choice, assuming that influences do not propagate faster than the speed of light in vacuum.

Proof.—In the CHSH Bell scenario, given the 8 joint probabilities in (2), the normalization of the probabilities and the condition of no-signalling determines the values of the 8 complementary probabilities $P(A_0+,B_0-), \ldots, P(A_1-,B_1-)$.
and the 8 marginal probabilities \( P(A_0+) \ldots P(B_1-) \). For simplicity, we start by assuming that the maximum of \( S \) is attained when each of the 8 joint probabilities in (2) takes the same value \( p \). We will remove this assumption later. In that case, the value of each of the 8 complementary probabilities is \( \frac{1}{2} - p \) and the value of each of the 8 marginal probabilities is \( \frac{1}{2} \). To determine the maximum value of \( p \) allowed by the E principle we proceed in three steps.

(I) Consider two independent CHSH Bell experiments, one performed in one city, e.g., Vienna, on pairs of particles prepared in a certain state and another one in another city, e.g., Stockholm, on different pairs of particles also prepared in the same state. In Vienna, Alice randomly measures \( A_0 \) or \( A_1 \), and Bob randomly measures \( B_0 \) or \( B_1 \). In Stockholm, Alice' randomly measures \( A'_0 \) or \( A'_1 \) (which are the same ones Alice is measuring in Vienna), and Bob' randomly measures \( B'_0 \) or \( B'_1 \) (which are the same ones Bob is measuring in Vienna). Since the Vienna and Stockholm experiments are independent, the joint probability of an event involving Vienna’s and Stockholm’s events is the product of the probabilities of the respective events. That is,

\[
P(A_1, B_j, A'_k, B'_l) = P(A_1, B_j) P(A'_k) P(B'_l).
\]

(4)

Therefore, e.g., \( P(A_0+, B_0+, A'_0+, B'_1+) = p^2 \) and \( P(A_0+, B_0-, A'_0+, B'_1-) = \left( \frac{1}{2} - p \right)^2 \).

(II) Now recall that the E principle can be applied to scenar-

ios involving measurements that might be performed in addition to those needed for the CHSH Bell experiment [9]. Let us assume that the dichotomic observables \( A_0, A'_0 \) and \( A_1, A'_1 \) defined as follows exist: (i) \( A_i A'_i \), with \( i = 0, 1 \), is co-

measureable with \( A_i \) and \( A'_i \) and gives the result \( A_i A'_i = 1 \) if \( A_i = 1 \) and \( A'_i = 1 \) or if \( A_i = 0 \) and \( A'_i = 0 \), and gives \( A_i A'_i + 1 \) if \( A_i = 1 \) and \( A'_i = 0 \) or if \( A_i = 0 \) and \( A'_i = 1 \). (ii) \( A_0 A'_0 \) and \( A_1 A'_1 \) are co-

measureable. Then, after Alice has measured both Alice' has measured \( A'_0 \) in a CHSH Bell experiment, Alice and Alice' can measure \( A_0 A'_0 \); see Fig. 1 (a). By definition of \( A_0 A'_0 \), the probability of \( (A_0+, B_0+, A'_0+, B'_1+, A_0 A'_0+) \) is equal to the probability of \( (A_0+, B_0+, A'_0+, B'_1+) \).

In addition, consider an alternative experiment consists of measuring \( A_0 A'_0 \) and \( A_1 A'_1 \) on the particles of Alice and Alice'; see Fig. 1 (b). Using the values of the marginals and the independence of the Vienna and Stockholm experiments, it is easy to see that \( P(A_0+, A'_0+) = P(A_0+, A'_0-) = P(A_0-, A'_0+) = P(A_0-, A'_0-). \) This implies that \( P(A_0 A'_0) = P(A_0 A'_0) = \frac{1}{2} \) and, therefore, that, e.g., \( P(A_0 A'_0+) = P(A_0 A'_0-) = \frac{1}{2} \).

(III) Now consider the events given in Table I. Since events \( e_i \), with \( i = 1, \ldots, 9 \), are pairwise mutually exclusive and events \( f_i \) are also pairwise mutually exclusive, the E principle enforces

\[
\sum_{i=1}^{9} P(e_i) \leq 1, \quad \sum_{i=1}^{9} P(f_i) \leq 1.
\]

(5)

Summing both inequalities, we obtain

\[
\sum_{i=1}^{9} P(e_i) + P(f_i) \leq 2.
\]

(6)

Taking into account the probabilities given in Table I, we obtain

\[
p \leq \frac{2 + \sqrt{2}}{8}.
\]

(7)

Therefore,

\[
S \leq 2 + \sqrt{2}.
\]

(8)

The fact that QT attains this maximum [5] shows that this is a tight bound enforced by the E principle.

If the 8 joint probabilities in (2) take arbitrary values, notice that there are only two sets of 9 mutually exclusive events containing \( (A_0 A'_0-, A_1 A'_1-) \) like the one in Table I, i.e., with 8 events \( (A_i, B_j, A'_k, B'_l) \) such that in 4 of them \( (A_i, B_j, A'_k, B'_l) \) and \( (A'_k, B'_l, A_i, B_j) \) are in (2) and in the other 4 are in the complementary set. Consider the two sets corresponding to each of \( (A_0 A'_0+, A_1 A'_1+) \), \( (A_0 A'_0+, A_1 A'_1-) \), \( (A_0 A'_0-, A_1 A'_1-) \), \( (A_0 A'_0-, A_1 A'_1+) \). For each set, the E principle enforces a restriction like the ones in (5). Summing all of them, we obtain

\[
S^2 + (4 - S)^2 + 4 \leq 16.
\]

(9)
and we obtain, again, inequality (8). This finishes the proof.

The proof makes no reference to QT at all. It assumes that experiments can be independent, that the laws of physics are the same for independent experiments, that observables with properties (i) and (ii) exist (or, with more generality, that any structure of relationships of exclusivity is feasible), that the laws of physics are consistent with additional experiments that may be made, and the E principle. In QT, $A_0 A_0'$ and $A_1 A_1'$ with properties (i) and (ii) exist for the choices needed to attain the Tsirelson bound (even though $A_i$ and $A_i'$ are not co-measurable). Similarly for $A_0 A_1'$ and $A_1 A_0'$.

**Conclusions.**—Until now, the E principle failed to explain a fundamental result in QT: the maximum violation of the CHSH Bell inequality. Here we have presented such a proof. Regarding the proof itself, it has the virtue that, unlike other ones [1, 2], is simple and does not require an infinite number of copies of the target experiment (the CHSH Bell experiment) and, consequently, an infinite universe [20]. It just needs two copies and additional measurements. On the other hand, the fact that the maximum violation of the CHSH Bell inequality is singled out by the E principle implies (by Result 2 in Ref. [13]) that the maximum quantum violation of another noncontextuality inequality is exactly the maximum value allowed by the E principle [21], showing that this result may be key to explain quantum correlations in other scenarios.

However, the most interesting feature of our proof is that it shows that, as conjectured in Ref. [9], the full power of the E principle emerges when is applied to all possible extensions of the target experiment (e.g., in our proof, to two copies of the target experiment plus additional measurements). It is therefore incorrect to identify the set of correlations singled out by the E principle with the set of correlations obtained by applying the E principle solely to copies of the target experiment or to a particular extension of the target experiment [22].

The problem of whether or not the E principle determines quantum correlations (defined as the correlations between the outcomes of co-measurable quantum observables [23]) for any possible scenario is still open. However, the result presented here, which indicates that the set of correlations singled out by the E principle is actually the set singled out when any conceivable extension of the target experiment is taken into account, and the recently introduced multigraph approach to quantum correlations [24] (extending Refs. [25, 26]), which shows how to encode the relationships of exclusivity of any possible noncontextuality and Bell inequality and how to represent any conceivable structure of relationships of exclusivity, allows us to define with precision the set of correlations singled out by the E principle.

In any case, the fact that the E principle determines so many quantum correlations, including the maximum quantum violations of the most fundamental noncontextuality and Bell inequalities, puts the E principle as the most solid candidate for being the fundamental principle of quantum correlations and, possibly, of QT. If this were the case, the fact that the E principle looks like an axiom of probability would reinforce the thesis according to which QT is an evolution from Kolmogorov’s probability theory rather than from Newtonian and Maxwellian physics [27]. The main conclusion would be that QT is a probability theory in which the E principle is a fundamental axiom needed because “unperformed experiments have no results” [28] and not all observables are co-measurable.

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**Table I:** Events used for the proof and their probabilities. The events in each column are pairwise mutually exclusive.

| Event | Probability |
|-------|-------------|
| $e_1 = (A_0^+, B_0^+, A_0'^+, B_1'^+, A_0 A_0')$ | $p^2$ |
| $e_2 = (A_0^-, B_0^-, A_0'^-, B_1'^-, A_0 A_0')$ | $p^2$ |
| $e_3 = (A_0^+, B_0^+, A_0'^+, B_1'^+, A_0 A_0')$ | $(\frac{1}{2} - p)^2$ |
| $e_4 = (A_0^-, B_0^-, A_0'^-, B_1'^-, A_0 A_0')$ | $(\frac{1}{2} - p)^2$ |
| $e_5 = (A_1^+, B_0^+, A_1'^+, B_1'^-, A_1 A_1')$ | $p^2$ |
| $e_6 = (A_1^-, B_0^-, A_1'^-, B_1'^-, A_1 A_1')$ | $p^2$ |
| $e_7 = (A_1^+, B_0^-, A_1'^+, B_1'^+, A_1 A_1')$ | $(\frac{1}{2} - p)^2$ |
| $e_8 = (A_1^-, B_0^+, A_1'^-, B_1'^-, A_1 A_1')$ | $(\frac{1}{2} - p)^2$ |
| $e_9 = (A_0 A_0' - A_1 A_1')$ | $P(e_9)$ |
| $f_1 = (A_0^+, B_0^+, A_0'^+, B_0'^+, A_0 A_0')$ | $p^2$ |
| $f_2 = (A_0^-, B_0^-, A_0'^-, B_0'^-, A_0 A_0')$ | $p^2$ |
| $f_3 = (A_0^+, B_0^+, B_0'^-, A_0 A_0')$ | $(\frac{1}{2} - p)^2$ |
| $f_4 = (A_0^-, B_0^-, B_0'^+, A_0 A_0')$ | $(\frac{1}{2} - p)^2$ |
| $f_5 = (A_1^+, B_0^+, B_1'^+, A_1 A_1')$ | $p^2$ |
| $f_6 = (A_1^-, B_0^-, B_1'^-, A_1 A_1')$ | $p^2$ |
| $f_7 = (A_1^+, B_0^-, B_1'^+, A_1 A_1')$ | $(\frac{1}{2} - p)^2$ |
| $f_8 = (A_1^-, B_0^+, B_1'^-, A_1 A_1')$ | $(\frac{1}{2} - p)^2$ |
| $f_9 = (A_0 A_0' - A_1 A_1')$ | $\frac{1}{2} - P(e_9)$ |

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