GLOBAL EXISTENCE FOR THE DERIVATIVE NONLINEAR
SCHRÖDINGER EQUATION WITH ARBITRARY SPECTRAL
SINGULARITIES

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ABSTRACT. We show that the derivative nonlinear Schrödinger (DNLS) equation is globally well-posed in the weighted Sobolev space $H^{2,2}(\mathbb{R})$. Our result exploits the complete integrability of DNLS and removes certain spectral conditions on the initial data required by our previous work [5], thanks to Zhou’s analysis on spectral singularities in the context of inverse scattering [20].

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1. Introduction

In this paper, we prove global well-posedness of the Cauchy problem for the Derivative Nonlinear Schrödinger equation (DNLS)

\[
\begin{aligned}
&i u_t + u_{xx} - i\varepsilon(|u|^2u)_x = 0, \quad \varepsilon = \pm 1 \\
&u(x, t = 0) = u_0(x)
\end{aligned}
\]

with initial condition \(u_0\) in the weighted Sobolev space

\[
H^{2,2}(\mathbb{R}) = \{u \in L^2(\mathbb{R}) : u''(x), x^2u(x) \in L^2(\mathbb{R})\}
\]

In contrast to previous work using PDE methods [4, 18, 3], we impose no upper bound on the \(L^2\)-norm of the initial data (although we require more smoothness and decay than these authors), and in contrast to previous work using completely integrable methods [5, 6, 11, 12, 14], we make no spectral restrictions to “generic initial data” that rule out singularities of the spectral data associated to the initial condition. We use the complete integrability of DNLS discovered by Kaup and Newell [8] and we also draw on exploit Zhou’s approach to inverse scattering with arbitrary spectral singularities [20, 22].

To describe our results more precisely, we recall that the invertible gauge transformation

\[
G(u)(x) = u(x) \exp \left(i \int_x^\infty |u(y)|^2 \, dy \right)
\]

maps solutions of (1.1) to solutions of

\[
\begin{aligned}
&i\dot{q} + q_{xx} + i\varepsilon q^2\dot{q} + \frac{1}{2}|q|^4q = 0, \quad \varepsilon = \pm 1 \\
&q(x, t = 0) = q_0(x).
\end{aligned}
\]

Equation (1.2) is more directly amenable to inverse scattering. Global wellposedness in \(H^{2,2}(\mathbb{R})\) for equations (1.1) and (1.2) are equivalent since \(G\) is a Lipschitz continuous map from \(H^{2,2}(\mathbb{R})\) to itself.

The main result of the paper is the following theorem.

**Theorem 1.1.** Suppose that \(q_0 \in H^{2,2}(\mathbb{R})\). There exists a unique solution \(q(x, t)\) of (1.2) with \(q(x, t = 0) = q_0\) and \(t \mapsto q(\cdot, t) \in C([-T, T], H^{2,2}(\mathbb{R}))\) for every \(T > 0\). Moreover, the map \(q_0 \mapsto q\) is Lipschitz continuous from \(H^{2,2}(\mathbb{R})\) to \(C([-T, T], H^{2,2}(\mathbb{R}))\) for every \(T > 0\).

Global well-posedness of DNLS was first considered by Hayashi and Ozawa [4] who proved that for any initial condition \(u_0 \in H^1(\mathbb{R})\) such that \(\|u_0\|_{L^2} < \sqrt{2\pi}\), global wellposedness holds in \(H^1(\mathbb{R})\). More recently, Wu [18] proved global well-posedness for initial data in \(H^{1/2}(\mathbb{R})\) under the same smallness assumption on the \(L^2\)-norm of the initial data. Finally, Fukaya, Hayashi, and Inui [3] improved the \(L^2\)-smallness condition to \(\|u_0\|_{L^2} < \sqrt{4\pi}\) (or \(\|u_0\|_{L^2} = \sqrt{4\pi}\) with negative momentum). In this paper, we will use the fact, due to Kaup and Newell [8], that the DNLS is completely integrable, to remove the smallness condition at the expense of requiring greater regularity and decay of the initial data. On the other hand, in contrast to previous approaches to the problem by inverse scattering methods, we make no spectral assumptions on the initial data. As we will explain, an essential ingredient of our work is Xin Zhou’s approach to inverse scattering with arbitrary
It should be noted that Zhou’s methods are quite general and are likely applicable to wellposedness questions for other integrable PDE’s in one space dimension.

The present paper also builds on previous work of the co-authors which proved global wellposedness of DNLS for initial conditions in weighted Sobolev spaces was proved under some additional conditions, namely excluding the so-called spectral singularities. This restriction is related to the fact that, in the study of the direct and inverse scattering transform, a hypothesis on the character of the spectrum of the linear operator was made, namely that no spectral singularity is present. In this context, we proved global well-posedness for data in an open and dense set of $H^{2,2}(\mathbb{R})$ which allows finitely many resonances but no spectral singularities, and also establishes the long-time behavior of solution in the form of the soliton resolution. We will discuss precisely in Section 2 the meaning of spectral singularities. In the present paper, we remove all spectral assumptions on the initial data and obtain global wellposedness of the DNLS equation for general initial condition in $H^{2,2}(\mathbb{R})$.

Our approach here is inspired by the work of Zhou, who, in a series of papers, developed new tools to construct direct and inverse scattering maps that are insensitive to singularities of the spectral data. We emphasize that spectral singularities may affect the long-time behavior of solutions, in the same way that resonances affect the long-time behavior of solutions through soliton resolution (see where the soliton resolution conjecture is proved for generic initial data). In the case of the focusing cubic nonlinear Schrödinger equation, Kamvissis studied the effect of a single spectral singularity on the large-time behavior of solutions. He showed that the latter is limited to the region of the $(x,t)$-plane in which the spectral singularity is close to the point of stationary phase, and there, slightly modifies the rate of decay. In a future paper, we will investigate how spectral singularities affect the long-time behavior of DNLS solutions.

To explain our methods, we will sketch the completely integrable method for (1.2) as discovered by Kaup and Newell in two steps. First, we describe how the method works when the initial data does not support solitons or spectral singularities. Next, we describe how Zhou’s method can be extended to the DNLS equation to construct global solutions in the presence of solitons and spectral singularities.

1.1. The Inverse Scattering Method: No Singularities. Kaup and Newell showed that the flow determined by (1.2) may be linearized by spectral data associated to the linear problem

$$\frac{d}{dx}\Psi(x,\zeta) = -\zeta^2 \sigma \Psi + \zeta Q(x)\Psi + P(x)\Psi$$

where $\Psi(x,\zeta)$ is a $2 \times 2$ matrix-valued function of $x$ and

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$Q(x) = \begin{pmatrix} 0 & q(x) \\ -q(x) & 0 \end{pmatrix}, \quad P(x) = \frac{i}{2} \begin{pmatrix} |q(x)|^2 & 0 \\ 0 & -|q(x)|^2 \end{pmatrix}.$$
Later, it will be convenient to set \( \Psi(x, \zeta) = m(x, \zeta)e^{ix\zeta^2/2} \), so that \( m \) solves the equation

\[
\frac{d}{dx} m(x, \zeta) = -i\zeta^2 \text{ad}(\sigma) m + \zeta Q(x)m + P(x)m
\]

where

\[
\text{ad}(\sigma)A = \sigma A - A\sigma.
\]

Equation (1.3) admits bounded solutions provided \( q \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) and \( \zeta \in \mathbb{R} \cup i\mathbb{R} \). There exist unique solutions \( \Psi^\pm(x, \zeta) \) of (1.3) satisfying the respective boundary conditions

\[
\lim_{x \to \pm\infty} \Psi^\pm(x, \zeta)e^{i\zeta^2/2} = I, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

These \textit{Jost solutions} have determinant 1 and define action-angle variables \( a \) and \( b \) for the flow (1.2) through the relation

\[
\Psi^+(x, \zeta) = \Psi^-(x, \zeta) \begin{pmatrix} a(\zeta) & b(\zeta) \\ \tilde{b}(\zeta) & \tilde{a}(\zeta) \end{pmatrix}.
\]

That is, if \( q(x, t) \) solves (1.2), and \( a(\zeta, t) \) and \( b(\zeta, t) \) are the corresponding scattering data for \( q(t, \cdot) \), then

\[
\dot{a}(\zeta, t) = 0, \quad \dot{b} = -4i\zeta^2b(\zeta, t).
\]

Thus, if the map \( q \mapsto (a, b) \) can be inverted, one can hope to solve (1.2) via a composition of the direct scattering map \( q \mapsto (a, b) \), the flow map defined by (1.7), and the inverse map \( (a, b) \mapsto q \).

The functions \( a \) and \( \tilde{a} \) have analytic extensions to the respective regions \( \Omega^- = \{ \text{Im} \ z^2 < 0 \} \) and \( \Omega^+ = \{ \text{Im} \ z^2 > 0 \} \) (see Figure 1.1). Zeros of \( a \) (resp. \( \tilde{a} \)) in \( \Omega^- \) (resp. \( \Omega^+ \)) are associated to soliton solutions of (1.2), while zeros of \( a \) or \( \tilde{a} \) on \( \mathbb{R} \cup i\mathbb{R} \) are called spectral singularities. For the moment, we assume that \( a \) and \( \tilde{a} \) are zero-free in their respective regions of definition. This allows us to define the reflection coefficients

\[
r(\zeta) = \frac{\tilde{b}(\zeta)}{a(\zeta)}, \quad \tilde{r}(\zeta) = \frac{b(\zeta)}{\tilde{a}(\zeta)}.
\]

The map \( q \mapsto r \) is the \textit{direct scattering map}. One can recover \( \tilde{r} \) from \( r \) by solving a scalar Riemann-Hilbert problem.

In his thesis, J.-H. Lee [10] formulated the inverse scattering map as a Riemann-Hilbert problem (RHP) in which \( r \) and \( \tilde{r} \) enter as jump data for a piecewise analytic function. To describe it, denote by \( \mathbb{R} \cup i\mathbb{R} \) the oriented contour, shown in Figure 1.1a, that bounds \( \Omega^\pm \) with \( \Omega^+ \) to the left and \( \Omega^- \) to the right. An oriented contour that divides \( \mathbb{C} \) into two such regions \( \Omega^+ \) and \( \Omega^- \), is called a \textit{complete contour}.

Denote by \( m^\pm \) the \textit{renormalized Jost solutions} \( m^\pm = \Psi^\pm e^{-ix\zeta^2/2} \). Let \( m^\pm_{(1)} \) and \( m^\pm_{(2)} \) denote the first and second columns of \( m^\pm \), with a similar notation \( m^\pm_{(1)} \), \( m^\pm_{(2)} \) for the columns of \( m^- \). From the integral equations (2.3)–(2.6), it is easy to see that \( m^-_{(1)}(x, \zeta) \) and \( m^-_{(2)}(x, \zeta) \) extend to analytic functions of \( z \in \Omega^+ \), while \( m^+_{(1)}(x, \zeta) \) and \( m^+_{(2)}(x, \zeta) \) extend to analytic functions of \( z \in \Omega^- \). From these columns, one can construct left and right \textit{Beals-Coifman solutions} \( M(x, z) \) of (1.5) which are piecewise analytic for \( z \in \mathbb{C}\setminus(\mathbb{R} \cup i\mathbb{R}) \) and normalized so that \( \lim_{x \to \infty} M(x, z) = I \).
(right-normalized, (2.13)) or \( \lim_{z \to -\infty} M(x, z) = I \) (left-normalized, (2.14)). Enforcing these normalizations involves division by \( a \) and \( \bar{a} \) so any zeros of \( a \) and \( \bar{a} \) would create new singularities.

The Beals-Coifman solution solves a Riemann-Hilbert problem (RHP) in the \( z \) variable. Thus \( x \) plays the role of a parameter and, for each \( x \), the function \( M(x, z) \) is piecewise analytic in \( z \) with prescribed asymptotics as \( z \to \infty \) and prescribed multiplicative jumps along the contour \( \mathbb{R} \cup i\mathbb{R} \).

The Riemann-Hilbert Problem 1.2. For each \( x \in \mathbb{R} \), find an analytic\(^1 \) function \( M(x, \cdot) : \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}) \to SL(2, \mathbb{C}) \) with:

(i) \( \lim_{z \to \infty} M(x, z) = I \),

(ii) \( M \) has continuous boundary values \( M_\pm \) as \( \pm \text{Im} \xi \downarrow 0 \), and

(iii) \( M_\pm \) obey the jump relation

\[
M_+(x, \zeta) = M_-(x, \zeta)e^{-ix\xi^2}\text{ad} \sigma v(\zeta)
\]

where

\[
v(\zeta) = \begin{pmatrix} 1 + |r(\zeta)|^2 & r(\zeta) \\ \bar{r}(\zeta) & 1 \end{pmatrix}.
\]

The matrix \( v \) is called the jump matrix for the RHP 1.2. We recover \( q(x) \) through the asymptotic formula

\[
q(x) = 2i \lim_{z \to \infty} zM_{12}(x, z).
\]

RHP 1.2 and the reconstruction formula 1.9 define the inverse scattering map.

\(^1\)If \( a \) has zeros, \( m \) is meromorphic and discrete data for each pole must be added to close the problem.
1.2. The Inverse Scattering Method: Singularities. So far, we have assumed that $a$ and $\bar{a}$ are zero-free; however, zeros of $a$ and $\bar{a}$ do occur for data of physical interest. By the symmetries

$$a(\zeta) = \bar{a}(\zeta), \quad a(-\zeta) = a(\zeta),$$

zeros of $a$ and $\bar{a}$ in $\mathbb{C}\setminus(\mathbb{R} \cup i\mathbb{R})$ occur in “quartets” as shown in Figure 1.1b. These quartets correspond to soliton solutions of (1.2). The further symmetry

$$b(-\zeta) = -b(\zeta), \quad \bar{b}(\zeta) = -\bar{b}(\zeta)$$

and the determinant condition

$$a(\zeta)\bar{a}(\zeta) - b(\zeta)\bar{b}(\zeta) = 1$$

imply that

$$|a(it)|^2 - |b(it)|^2 = 1$$

for all real $t$, so $a$ has no zeros on the imaginary axis. However, zeros of $a$ on the real axis may occur and correspond to spectral singularities. RHP 1.2 is no longer solvable since the jump matrix $v$ now has singularities on the contour $\mathbb{R} \cup i\mathbb{R}$; moreover, any zeros of $a$ and $\bar{a}$ in their domains of analyticity will make the Beals-Coifman solutions meromorphic rather than analytic. We exhibit data $q_0 \in H^{2,2}(\mathbb{R})$ with spectral singularities in Appendix C.

On the other hand, any zeros of $a$ and $\bar{a}$ lie in the disc

$$B(0, R) = \{z : |z| < R\},$$

where $R$ is determined by $\|q_0\|_{H^{2,2}}$ (see, for example, [13, Proposition 3.2.5]). Moreover, for $\|q_0\|_{H^{2,2}}$ sufficiently small, $a$ and $\bar{a}$ are zero-free on their respective domains. We will say that such a potential has zero-free scattering data.

Zhou’s insight in [20, 22] is that RHP 1.2 can be modified in the following way. First, choose $R$ so large that $a$ and $\bar{a}$ have no zeros in $\mathbb{C}\setminus B(0, R)$, and denote by $\Sigma_R$ the circle of radius $R$ centered at 0. Next, choose $x_0 > 0$ sufficiently large so that the potential

$$q_{x_0}(x) = \begin{cases} 0, & x \leq x_0 \\ q(x), & x > x_0 \end{cases}$$

has zero-free scattering data; a sufficient condition is that

$$\sup_{|z| \leq R} \|zQ + P\|_{L^1(x > x_0)} < 1/2$$

(see §2, (2.17) and the discussion that follows). Note that both $x_0$ and $R$ may be chosen uniform for $q$ in a bounded subset of $H^{2,2}(\mathbb{R})$. Next, let $M^{(0)}(x, z)$ denote the solution to RHP 1.2 for $q_{x_0}$. The function $M^{(0)}$ is analytic in $\mathbb{C}\setminus(\mathbb{R} \cup i\mathbb{R})$ with continuous boundary values $M^{(0)}_\pm$ on $\mathbb{R} \cup i\mathbb{R}$. Indeed, eigenvalues and spectral singularities for $q_{x_0}$ are ruled out by the analytic Fredholm theory. We define

$$M^{(1)}(x, \zeta) = I + \int_{x_0}^x e^{i(y-x)\zeta^2} \text{ad} \sigma \left( \zeta Q(y) M^{(\pm)}(y, \zeta) + P(x) M^{(\pm)}(y, \zeta) \right) dy$$

and

$$M^{(2)}(x, \zeta) = M^{(1)}(x, \zeta) e^{-i(x-x_0)\zeta^2} \text{ad} \sigma M^{(0)}(x_0, \zeta).$$
One can now define a new piecewise analytic function $M(x, z)$ on the domain shown in Figure 1.2a by

$$
M(x, z) = \begin{cases} 
M(x, z), & z \in \tilde{\Omega}_1 \cap \tilde{\Omega}_2 \cap \tilde{\Omega}_3 \cap \tilde{\Omega}_4 \\
M^{(2)}(x, z), & z \in \tilde{\Omega}_5 \cap \tilde{\Omega}_6 \cap \tilde{\Omega}_7 \cap \tilde{\Omega}_8,
\end{cases}
$$

where $\Sigma_{\infty}$ is the circle of radius $R$ centered at 0. The function $M(x, z)$ is piecewise analytic on $\mathbb{C}\setminus (B(0, R) \cup (\mathbb{R} \cup i\mathbb{R}))$ because $a$ and $\tilde{a}$ are zero-free for $|z| > R$. By construction, the function $M^{(2)}$ is piecewise analytic in $B(0, R) \setminus (\mathbb{R} \cup i\mathbb{R})$. The new unknown $M(x, z)$ obeys Riemann-Hilbert problem 3.7 described in section 3.

The jump matrix of the Riemann-Hilbert problem for $M(x, z)$ is unchanged outside the circle $\Sigma_{\infty}$ but is replaced inside by new jump data that may be explicitly computed from $q_0$ and $q_{x0}$; see §2 for full discussion. Since $M(x, \zeta) = M(x, \zeta)$ in a neighborhood of infinity, we can still recover $q$ from the reconstruction formula (1.9). To carry out the analysis, we change variables from $\zeta$ to $\lambda = \zeta^2$ and actually analyze Riemann-Hilbert problem 3.1.

(a) The augmented contour $\Sigma = \mathbb{R} \cup i\mathbb{R} \cup \Sigma_{\infty}$ in the $\zeta$-plane

(b) The augmented contour $\Gamma = \mathbb{R} \cup \Gamma_{\infty}$ in the $\lambda$-plane

Figure 1.2. Augmented Contours for the Modified Riemann-Hilbert Problem

To analyze the direct map (from the given potential $q_0$ to the jump matrix for the augmented contour $\Sigma$) and the inverse map (from the jump matrix to the recovered potential) it is helpful to exploit the symmetry reduction of the spectral problem (1.3) to the spectral variable $\lambda = \zeta^2$. Under the map $\zeta \mapsto \zeta^2$, the contour $\mathbb{R} \cup i\mathbb{R}$ maps to $\mathbb{R}$ with its usual orientation and $\Omega^\pm$ map to $\mathbb{C}^\pm$. The augmented contour $\Sigma$ shown in Figure 1.2a maps to the the contour $\Gamma$ shown in Figure 1.2b, and $\Sigma_{\infty}$ maps to the circle of radius $S_{\infty} = R^2$. In what follows, we will set

$$
R_{\infty} = \mathbb{R}\setminus [-S_{\infty}, S_{\infty}],
$$

the part of the contour $\mathbb{R}$ outside the circle $\Sigma_{\infty}$. One can compute the jump data for the Riemann-Hilbert problem on the contour $\Gamma$ explicitly in terms of scattering...
data for $q$, scattering data for $q_{x_0}$, and normalized Jost solutions for $q$ (see Figure 2.1 and Proposition 2.2; it is then easy to show that the direct spectral map from $q \in H^{2,2}(\mathbb{R})$ to these scattering data is continuous in a natural topology on the jump data (see Theorem 2.7 for a precise statement).

It remains to show that the scattering data can be time-evolved continuously and that Riemann-Hilbert problem with scattering data as described in Theorem 2.7 can be uniquely solved and used to recover the potential $q$. To do so, much as in [5] and [12], we show that the Riemann-Hilbert problem in the $\lambda$ variables is equivalent to a Riemann-Hilbert problem in the $\zeta$-variable which is uniquely solvable. We then apply Zhou’s uniqueness theorem (see Proposition 2.1 and [19]) to obtain unique solvability. We also need to show that the recovered potential is continuous in the scattering data; this will actually follow from results of Zhou [22] and our previous results on the scattering transform in [12].

Finally, we sketch the contents of the paper. Section 2 is devoted to the direct scattering map.

In Section 2.1, we recall the basic properties of the scattering problem and Beals-Coifman solutions in the $\zeta$ variables. In Sections 2.2 and 2.3, we construct the scattering data in the $\zeta$ and $\lambda$ variables. For this purpose, we implement Zhou’s method to deal with spectral singularities. The goal is to choose the scattering data so the inverse scattering problem will allow a reconstruction formula for the potential. The idea is to augment the contour $\mathbb{R} \cup i\mathbb{R}$ with a large circle centered at $\zeta = 0$ that contains all resonances and spectral singularities. Figures 1.2a and 1.2b show the augmented contours in the $\zeta$ and $\lambda$ variables. In this setting, the usual Beals-Coifman solutions are changed to piecewise analytic functions according to (1.14). We give explicit formulas for the corresponding jump matrices along the augmented contours. We use Zhou’s approach [22] (see also Trogdon-Olver [17]) to address the matching conditions at the intersection points of the contours and give a full description of the jump matrices and their factorization.

In Section 2.4, we establish the time evolution of the scattering data. Finally, as in the absence of spectral singularities where right and left RHPs are needed for the reconstruction of the potential for $x \in [-a, \infty)$ and $x \in (-\infty, a]$, there is a left and right RHP and their corresponding jump matrices that are related through an auxiliary matrix (see (2.43)).

In Section 3, we show that the RHP with the augmented contour and the jump matrices, as derived in Sections 2.2 and 2.3 has a unique solution. The proof follows the lines of the proof given in [13]: Suppose the RHP in $\lambda$ has a null vector $\nu$. This null vector induces a null matrix $\mu$ for the RHP in the $\zeta$ variables. Apply the vanishing lemma from [19] to show that the null matrix $\mu$ is in fact the zero vector. By explicit formulas relating $\mu$ and $\nu$, we conclude that the original null vector is zero, so by Fredholm theory, the RHP in $\lambda$ with the augmented contour is uniquely solvable. As is [5], we establish the existence and uniqueness of solutions to the RHP for scattering data in a larger space $Y$ (see Definition 3.3) in order to obtain uniform resolvent estimates for scattering data in bounded sets of a smaller space.

In Section 4, we establish the mapping properties of the inverse scattering map and estimate the potential obtained from the reconstruction formula in the $\lambda$-variables. This analysis requires another technical step taken from Zhou’s method [22]. As seen in Fig. 1.2b, the orientation of the piece of the contour $(S_{x_-}^-, S_{x_-}^+)$

\footnote{In [22] the non-zero off-diagonal entries are not calculated explicitly.}
goes from right to left. A second augmentation shown in Figure 4.1 allows the new contour to have the usual orientation thus allowing standard estimates of the Cauchy projectors on \( \mathbb{R} \) to be used to obtain decay estimates on the potential. The Lipschitz continuity follows from the second resolvent identity.

To analyze Riemann-Hilbert problems with self-intersecting contours, we make use of certain Sobolev spaces of functions that obey continuity conditions at self-intersection points. For the reader’s convenience, we briefly describe these Sobolev spaces in Appendix A. In Appendix B, we present the necessary abstract functional analysis tools used to prove uniform resolvent estimates needed for the Lipschitz continuity of the inverse scattering map in Section 4. In Appendix C, we analyze a particular family of potentials of the form \( q(x) = A \text{sech}(x) e^{i\varphi(x)} \) (see (C.1)) for which one can compute explicitly the scattering data, thus illustrating various characterizations of the spectral map \([2, 16]\). In [5], we used this family to exhibit initial conditions of arbitrary large \( L^2 \)-norm for which global wellposedness holds. We show that it is possible to find potentials of this form for which the associated spectral problem has either no discrete spectrum, or exactly \( n \) resonances and no spectral singularities, or \( n \) resonances and one spectral singularity.

2. The Direct Scattering Map

2.1. The scattering problem in the \( \zeta \) variable. The system (1.5) can be written in the form of the integral equation for the \( 2 \times 2 \) matrix \( m(x, \zeta) \)

\[
(2.1) \quad m(x, \zeta) = I + \int_{\delta}^{x} e^{i(y-x)\zeta^2/2} \left( \zeta Q(y) m(y, \zeta) + P(x)m(y, \zeta) \right) dy,
\]

where the lower limit \( \delta \) can be different for various choices of normalization. We will use several solutions of (2.1). The standard AKNS method starts with the following two Volterra integral equations as special cases of (2.1) for \( \text{Im} \zeta^2 = 0 \):

\[
(2.2) \quad m^{(\pm)}(x, \zeta) = I + \int_{\pm \infty}^{x} e^{i(y-x)\zeta^2/2} \left( \zeta Q(y)m^{(\pm)}(y, \zeta) + P(x)m^{(\pm)}(y, \zeta) \right) dy,
\]

which are expressed in componentwise form as

\[
(2.3) \quad \begin{pmatrix} m_{11}^{+}(x, \zeta) \\ m_{21}^{+}(x, \zeta) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \int_{x}^{\infty} \begin{pmatrix} \zeta q m_{21}^{+} + p_{1} m_{11}^{+} \\ -\zeta m_{11}^{+} + p_{2} m_{21}^{+} \end{pmatrix} dy,
\]

\[
(2.4) \quad \begin{pmatrix} m_{12}^{+}(x, \zeta) \\ m_{22}^{+}(x, \zeta) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \int_{x}^{\infty} \begin{pmatrix} \zeta q m_{22}^{+} + p_{1} m_{12}^{+} \\ -\zeta m_{12}^{+} + p_{2} m_{22}^{+} \end{pmatrix} dy,
\]

\[
(2.5) \quad \begin{pmatrix} m_{11}^{-}(x, \zeta) \\ m_{21}^{-}(x, \zeta) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \int_{-\infty}^{x} \begin{pmatrix} \zeta q m_{21}^{-} + p_{1} m_{11}^{-} \\ -\zeta m_{11}^{-} + p_{2} m_{21}^{-} \end{pmatrix} dy,
\]

\[
(2.6) \quad \begin{pmatrix} m_{12}^{-}(x, \zeta) \\ m_{22}^{-}(x, \zeta) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \int_{-\infty}^{x} \begin{pmatrix} \zeta q m_{22}^{-} + p_{1} m_{12}^{-} \\ -\zeta m_{12}^{-} + p_{2} m_{22}^{-} \end{pmatrix} dy.
\]

By uniqueness theory for ODEs, any two solutions of (1.5) with nonvanishing determinant are related by a matrix \( A(\zeta) \) with \( \text{det} A(\zeta) = 1 \), so that

\[
(2.7) \quad m^{(+)}(x, \zeta) = m^{(-)}(x, \zeta) e^{-i\zeta^2 x/2} A(\zeta), \quad A(\zeta) = \begin{pmatrix} a & b \\ b & \bar{a} \end{pmatrix}.
\]
The matrix-valued function $A(\zeta)$ is expressed in terms of $m(\zeta)$ as
\begin{align}
(2.8) \quad a(\zeta) &= 1 - \int_{-\infty}^{\infty} \left( \zeta m_{21}^+ + p_1 m_{11}^+ \right) dy = 1 + \int_{-\infty}^{\infty} \left( -\zeta m_{12}^- + p_2 m_{22}^- \right) dy,
(2.9) \quad \tilde{a}(\zeta) &= 1 - \int_{-\infty}^{\infty} \left( -\zeta m_{12}^+ + p_2 m_{22}^+ \right) dy = 1 + \int_{-\infty}^{\infty} \left( \zeta m_{21}^- + p_1 m_{11}^- \right) dy,
(2.10) \quad b(\zeta) &= \int_{-\infty}^{\infty} e^{-2i\kappa^2 y} \left( \zeta m_{11}^+ - p_2 m_{21}^+ \right) dy \\
&= \int_{-\infty}^{\infty} e^{-2i\kappa^2 y} \left( \zeta m_{11}^- - p_2 m_{21}^- \right) dy,
(2.11) \quad \tilde{b}(\zeta) &= -\int_{-\infty}^{\infty} e^{2i\kappa^2 y} \left( \zeta m_{22}^+ + p_1 m_{12}^+ \right) dy \\
&= -\int_{-\infty}^{\infty} e^{2i\kappa^2 y} \left( \zeta m_{22}^- + p_1 m_{12}^- \right) dy.
\end{align}

We now construct the Beals-Coifman solutions needed for the RHP in the form of piecewise analytic matrix functions. An obvious choice is
\begin{align}
(2.12) \quad M(x, z) &= \begin{cases} 
(m_1^{(-)}(x, z), m_2^{(+)}(x, z)), & \text{Im } z^2 > 0 \\
(m_1^{(+)}(x, z), m_2^{(-)}(x, z)), & \text{Im } z^2 < 0.
\end{cases}
\end{align}

The left and right Beals-Coifman solutions are obtained from the normalized Jost solutions as follows:
\begin{align}
(2.13) \quad M_R(x, z) &= \begin{cases} 
\left[ \frac{m_1^{(-)}(x, z), m_2^{(+)}(x, z)}{\tilde{a}(z)} \right], & \text{Im } z^2 > 0 \\
\left[ m_1^{(+)}(x, z), \frac{m_2^{(-)}(x, z)}{a(z)} \right], & \text{Im } z^2 < 0
\end{cases}
\end{align}
\begin{align}
(2.14) \quad M_L(x, z) &= \begin{cases} 
\left[ m_1^{(-)}(x, z), \frac{m_2^{(+)}(x, z)}{\tilde{a}(z)} \right], & \text{Im } z^2 > 0 \\
\left[ \frac{m_1^{(+)}(x, z), m_2^{(-)}(x, z)}{a(z)} \right], & \text{Im } z^2 < 0.
\end{cases}
\end{align}

The Beals-Coifman solutions, if they exist, are piecewise analytic with continuous boundary values as \( \pm \text{Im } z^2 \downarrow 0 \) denoted \( M_{L, \pm} \) and \( M_{R, \pm} \). The Beals-Coifman solution corresponding to the potential \( q_{x_0} \) are constructed similarly. From here onward, we will analyze the right Beals-Coifman solution (2.13) and drop the subscripts \( R \) and \( L \).

2.2. Construction of the scattering data in the \( \zeta \) variables. We first construct \( M^{(2)}(x, z) \) defined in (1.14). Combining (2.4) and (2.5), we obtain
\begin{align}
(2.15) \quad \left[ m_1^{(-)}, m_2^{(+)} \right] = I + \int_{\delta}^{\infty} e^{i(y-x)\zeta} \sigma \left( \zeta Q(y) + P(y) \right) \left( m_1^{(-)}(y), m_2^{(+)}(y) \right) dy
\end{align}
where \( \delta \) is chosen differently for the various entries of the matrix, namely \( \delta = -\infty \) for the (1-1) and (2-1) entries and \( \delta = +\infty \) for the (1-2) and (2-2) entries. Using (2.9), we rewrite (2.15) as
\[
\begin{pmatrix} m_1(-), m_2^+ \end{pmatrix} = \begin{pmatrix} \hat{a} & 0 \\ 0 & 1 \end{pmatrix} + \int_{\delta}^{x} e^{i(y-x)\zeta^2} \text{ad} \sigma \left( (\zeta Q(y) + P(y)) \begin{pmatrix} m_1(-)(y), m_2^+(y) \end{pmatrix} \right) dy
\]
where \( \delta = -\infty \) for the (2-1) entry and \( \delta = +\infty \) for the (1-1), (1-2) and (2-2) entries. Notice that if the inverse of \( \begin{pmatrix} \hat{a} & 0 \\ 0 & 1 \end{pmatrix} \) exists, we obtain a Fredholm equation for \( M_R \) defined in (2.13):
\[
(2.16) \quad \begin{pmatrix} m_1(-)/\hat{a}, m_2^+ \end{pmatrix} = I + \int_{\delta}^{x} e^{i(y-x)\zeta^2} \text{ad} \sigma \left( (\zeta Q(y) + P(y)) \begin{pmatrix} m_1(-)/\hat{a}, m_2^+ \end{pmatrix} \right) dy
\]
It is easy to see that \( \begin{pmatrix} m_1(-)/\hat{a}, m_2^+ \end{pmatrix} \) solves (2.16) iff \( \hat{a}(\zeta) \neq 0 \).

Let \( x_0 \in \mathbb{R} \) be such that \( q_{x_0} = q_{\lambda(x_0, \infty)} \) satisfies the smallness condition
\[
(2.17) \quad \|\zeta Q_{x_0} + P_{x_0}\|_{L^1} < 1/2.
\]
for all \( \zeta, |\zeta| \leq R \) where \( R \) is the radius of a large circle containing all zeros of \( a \) (see (1.14)). Let \( M^{(0)} \) be a bounded Beals-Coifman solution of the form (2.13) associated to the potential \( q_{x_0} \) and normalized at \( x \to \infty \). It is shown in [21] by analytic Fredholm theory that a solution to
\[
(2.18) \quad m(x, \zeta) = I + \int_{\delta}^{x} e^{i(y-x)\zeta^2} \text{ad} \sigma \left( (\zeta Q_{x_0}(y)m(y, \zeta) + P_{x_0}(x)m(y, \zeta)) \right) dy
\]
where \( \delta = -\infty \) for the (2-1) entry and \( \delta = +\infty \) for the (1-1), (1-2) and (2-2) entry exists under the smallness condition (2.17). This precisely means \( \tilde{a}_0(z) \) associated to \( q_{x_0} \) is nonzero for \( \text{Im} \, z^2 \geq 0 \). We prove this statement by contradiction. Suppose that there exists \( \zeta_0 \) such that (2.17) holds and \( m(x, \zeta_0) \) solves (2.18), but \( \hat{a}(\zeta_0) = 0 \). By uniqueness, we have
\[
m(x, \zeta_0) = \begin{pmatrix} m_1(-), m_2^+ \end{pmatrix} e^{-ix\zeta_0^2} \text{ad} \sigma B(\zeta_0).
\]
Here \( m_1(-), m_2^+ \) are the Jost functions in the form of (2.4) and (2.5) associated to the potential \( q_{x_0} \). Letting \( x \to +\infty \) and using (2.9), we obtain
\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} e^{-i\zeta_0^2 \text{ad} \sigma} B(\zeta_0),
\]
which leads to a contradiction. Thus the cutoff potential \( q_{x_0} \) does not support resonances and spectral singularities. We now define
\[
M^{(1)}(x, \zeta) = I + \int_{x_0}^{x} e^{i(y-x)\zeta^2} \text{ad} \sigma \left( (\zeta Q(y)M^{(1)}(y, \zeta) + P(x)M^{(1)}(y, \zeta)) \right) dy
\]
and
\[
M^{(2)}(x, \zeta) = M^{(1)}(x, \zeta)e^{-i(x-x_0)\zeta^2 \text{ad} \sigma} M^{(0)}(x_0, \zeta).
\]
Since \( a \) and \( \hat{a} \) approach 1 as \( \zeta \to \infty \), they do not have any zero near \( \zeta = \infty \).

Let \( \Sigma_{\infty} \) be a circle centered at \( \zeta = 0 \) such that \( M \) does not have any singularities outside the circle (see Figure 1.2a). Let
\[
\tilde{\Omega}_+ = \tilde{\Omega}_1 \cup \tilde{\Omega}_3 \cup \tilde{\Omega}_6 \cup \tilde{\Omega}_8, \quad \tilde{\Omega}_- = \tilde{\Omega}_2 \cup \tilde{\Omega}_4 \cup \tilde{\Omega}_5 \cup \tilde{\Omega}_7.
\]
From $M$ and $M_2$, we construct a new matrix-valued function $M$ that is piecewise analytic in $\mathbb{C}$ with jump along the augmented contour $\Sigma$ in the following way. Let

\begin{equation}
M(x, z) = \begin{cases} 
M(x, z), & z \in \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4, \\
M^{(2)}(x, z), & z \in \Omega_5 \cup \Omega_6 \cup \Omega_7 \cup \Omega_8.
\end{cases}
\end{equation}

We can compute the jump matrix

\[ v(\zeta) = e^{ix\zeta^2 \sigma M(x, \zeta)^{-1}M(x, \zeta)} \]

explicitly across the various parts of the augmented contour $\Sigma$. Along the contour $\mathbb{R} \cup i\mathbb{R}$, outside of the circle,

\begin{equation}
v(\zeta) = \begin{pmatrix} 1 - r(\zeta)\tilde{r}(\zeta) & r(\zeta) \\
-\tilde{r}(\zeta) & 1 \end{pmatrix}
\end{equation}

Along the contour $\mathbb{R} \cup i\mathbb{R}$ inside of the circle,

\begin{equation}
v(\zeta) = \begin{pmatrix} 1 & -r_0(\zeta) \\
r_0(\zeta) & 1 - r_0(\zeta)\tilde{r}_0(\zeta) \end{pmatrix}
\end{equation}

Here, the subscript "0" denotes the scattering data generated by $q_{x_0}$.

Since both $M$ and $M^{(2)}$ are solutions of (1.5) with non-vanishing determinant, we have

\[ v(\zeta) = e^{ix\zeta^2 \sigma M^{(2)}(x, \zeta)^{-1}M(x, \zeta)} \]

along the circle $\Sigma_{x_0}$. In particular, setting $x = x_0$, we get $v(\zeta)$ in terms of Jost functions. Across the arc in the first and third quadrant we have:

\begin{equation}
e^{-ix\zeta^2 \sigma} v(\zeta) = M^{(2)}(x_0, \zeta)^{-1}M(x_0, \zeta) = \begin{pmatrix} 1 & 0 \\
\frac{m_{\overline{a}}(x_0, \zeta)}{a(\zeta)a_0(\zeta)} & 1 \end{pmatrix}
\end{equation}

Across the arc in the second and fourth quadrant we have:

\begin{equation}
e^{-ix\zeta^2 \sigma} v(\zeta) = M^{(2)}(x_0, \zeta)^{-1}M(x_0, \zeta) = \begin{pmatrix} 1 & -\frac{m_{\overline{a}}(x_0, \zeta)}{a(\zeta)a_0(\zeta)} \\
0 & 1 \end{pmatrix}
\end{equation}

Denote by $A^\dagger$ the hermitian conjugate of the matrix $A$. The following property of $v$ will be used later to prove the unique solvability of the RHP (see Proposition 3.9).

**Proposition 2.1.** The jump matrix $v$ on $\Sigma$ defined in (2.20)-(2.23) satisfies:

(i) $v(\zeta) + v(\zeta)^\dagger$ is positive definite for $\zeta \in \mathbb{R}$.

(ii) $v(\overline{\zeta}) = v(\zeta)^\dagger$ for $\zeta \in \Sigma \setminus \mathbb{R}$.

**Proof.** An immediate consequence of the symmetries (1.10)–(1.11), the formulas (1.8), and the explicit formulas (2.20)–(2.23).
2.3. Construction of the scattering data in the $\lambda$ variables. In the absence of resonances and spectral singularities, we reduced the scattering problem (1.5) of $\zeta \in \mathbb{R} \cup i\mathbb{R}$ to scattering problem for $\lambda = \zeta^2 \in \mathbb{R}$, and identified a single scattering datum $\rho(\lambda)$ defining the direct scattering map [12]; we can carry out a similar reduction here. Let $m(x, \zeta)$ be a solution to (1.5). We then set
\[
m^\xi(x, \zeta) = \begin{pmatrix} m_{11}(x, \zeta) & \zeta^{-1}m_{12}(x, \zeta) \\ \zeta m_{21}(x, \zeta) & m_{22}(x, \zeta) \end{pmatrix},
\]
$m^\xi$ is an even function of $\zeta$. Defining $\lambda = \zeta^2$, $n(x, \lambda) = m^\xi(x, \zeta)$, the map
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} a & \zeta^{-1}b \\ \zeta c & d \end{pmatrix}
\]
is an automorphism of $2 \times 2$ matrices and commutes with differentiation in $x$. It follows that the functions $n^\pm$ obtained from $m^\pm$ by this map obey
\[
\begin{align*}
\frac{dn^\pm}{dx} &= -i\lambda \text{ad} \sigma(n^+) + \begin{pmatrix} 0 & q \\ -\lambda^2 & 0 \end{pmatrix} n^\pm + Pn^\pm \\
\lim_{x \to \pm \infty} n^+(x, \lambda) &= 1
\end{align*}
\]
and satisfy
\[
n^\pm(x, \lambda) = n^-(x, \lambda)e^{-i\lambda x \text{ad} \sigma} \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) \\ \lambda \beta(\lambda) & \alpha(\lambda) \end{pmatrix}
\]
where $\alpha(\zeta) = a(\zeta)$, $\beta(\zeta) = \zeta^{-1}b(\zeta)$ and the relation $|\alpha(\zeta)|^2 + |\beta(\zeta)|^2 = 1$ holds.

In the presence of arbitrary spectral singularities, we perform the change of variable $\zeta \rightarrow \lambda$ in the same fashion as in [12] and obtain the corresponding row vector-valued Beals-Coifman solutions $N^{(0)}$, $N^{(2)}$ and $N$. More explicitly,
\[
N^{(0)} = \text{first row of } (\zeta^{-1/2} \ 0 \ \ 0 \ \ \zeta^{1/2}) M^{(0)} (\zeta^{1/2} \ 0 \ \ 0 \ \ \zeta^{-1/2}),
\]
\[
N^{(2)} = \text{first row of } (\zeta^{-1/2} \ 0 \ \ 0 \ \ \zeta^{1/2}) M^{(2)} (\zeta^{1/2} \ 0 \ \ 0 \ \ \zeta^{-1/2}),
\]
\[
N = \text{first row of } (\zeta^{-1/2} \ 0 \ \ 0 \ \ \zeta^{1/2}) M (\zeta^{1/2} \ 0 \ \ 0 \ \ \zeta^{-1/2}).
\]
The contour $\Gamma$ for the new RHP, shown in Figure 1.2b, is the image of the contour $\Sigma$ in Figure 1.2a under the change of variable $\lambda = \zeta^2$.

As shown in Figures A.1a and A.1b, the contour $\Gamma$ can be viewed simultaneously as the boundary of $\Omega_+$ and $\Omega_-$. Referring to Figure 1.2b, we see that $\Omega_+ = \partial \Omega_1 \cup \Omega_4$ and $\Omega_- = \partial \Omega_2 \cup \Omega_3$. Also $\Gamma_+ = \partial \Omega_1 \cup \partial \Omega_4$ and $\Gamma_- = \partial \Omega_2 \cup \partial \Omega_3$. Define the piecewise analytic function $N$ (compare (2.19)) as
\[
N(x, z) = \begin{cases} N(x, z), & z \in \Omega_1 \cup \Omega_3, \\
N^{(2)}(x, z), & z \in \Omega_3 \cup \Omega_4. \end{cases}
\]
By setting

\[
\begin{align*}
\alpha(\lambda) &= a(\zeta), & \tilde{\alpha}(\lambda) &= \tilde{a}(\zeta), \\
\rho(\lambda) &= \zeta^{-1}r(\zeta), & \rho_0(\lambda) &= \zeta^{-1}r_0(\zeta), \\
n_{21}^{-}(x, \lambda) &= \zeta m_{21}^{-}(x, \zeta), & n_{12}^{-}(x, \lambda) &= \zeta^{-1}m_{12}^{-}(x, \zeta),
\end{align*}
\]

we obtain from (2.20) – (2.23) the jump matrices \( J(\lambda) \) for the piecewise analytic matrix \( N \).

**Proposition 2.2.** The jump matrices for \( N \) along the various parts of the contour \( \Gamma \) are given as follows:

(i) across the part of the real line outside the circle:

\[
J(\lambda) = \begin{pmatrix}
1 + \lambda|\rho(\lambda)|^2 & \rho(\lambda) \\
\lambda\rho(\lambda) & 1
\end{pmatrix}
\]

(ii) across the part of the real line inside the circle:

\[
J(\lambda) = \begin{pmatrix}
1 & -\rho(\lambda) \\
-\lambda\rho(\lambda) & 1 + \lambda|\rho(\lambda)|^2
\end{pmatrix}
\]

(iii) across the arc in \( \mathbb{C}^+ \):

\[
J(\lambda) = \begin{pmatrix}
1 & 0 \\
e^{-2ix_0\lambda} \frac{n_{21}^{-}(x_0, \lambda)}{\tilde{a}(\lambda)\tilde{a}_0(\lambda)} & 1
\end{pmatrix}
\]

(iv) across the arc in \( \mathbb{C}^- \):

\[
J(\lambda) = \begin{pmatrix}
1 & -e^{2ix_0\lambda} \frac{n_{12}^{-}(x_0, \lambda)}{\alpha(\lambda)\alpha_0(\lambda)} \\
0 & 1
\end{pmatrix}
\]
Remark 2.3. We define the scattering data associated to the potential $q$ as the entries of the various jump matrices along $\Gamma$, as listed in Proposition 2.2 and shown in Figure 2.1. Unlike the problem without spectral singularities, we are not able to explicitly factorize the entries of the matrix $J(\lambda)$ along the arcs of the circle (see (2.29)-(2.30)). However, we show these factorizations exist and obtain estimates in appropriate Sobolev spaces. The choice of scattering data is motivated by the inverse problem. Indeed, from these spectral data, we will, in the next section, define an inverse map and a reconstruction of the potential. Note that the scattering data depend on the choice of $x_0$ as well as the choice of the large circle $\Sigma_{\infty}$. Indeed in [20], scattering data are seen as an equivalence class. In the study of the inverse map, we will need the fact that the reconstruction formula does not depend on $x_0$ and $\Sigma_{\infty}$. This is because the reconstruction formula involves a limit as $\lambda$ tends to infinity of the entry $p_{1,2}$ of the solution of a RHP, and will not be affected by the exact position of the cut-off point or the circle $\Sigma$, although the RHP itself depends on it. For more details, we refer the reader to [20, Theorem 3.3.15].

To give a full characterization of the scattering data we will use the spaces $H^k(\Gamma)$ and $H^k(\Sigma)$ described in Appendix A to define the notion of $k$-regularity [17, Definition 2.54] of a given jump matrix along an admissible contour. Note that all contours in this paper are admissible in the sense of [17, Definition 2.40].

**Definition 2.4.** A jump matrix $J$ defined on an admissible contour $\Gamma$ is $k$-regular if $\Gamma$ is complete and $J$ has a factorization

$$J(s) = J^{-1}_-(s)J_+(s)$$

where $J_\pm(s) - I$ and $J^{-1}_\pm(s) - I \in H^k(\Gamma)$. 

**Figure 2.1.** Scattering data for $q$

$$
\begin{pmatrix}
1 & 0 \\
\frac{1}{e^{2ix_0}\frac{\alpha_{\Gamma}(x_0, \lambda)}{\alpha_{\Sigma}(x_0, \lambda)}} & 1
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 + \lambda|\rho|^2 & \rho \\
\lambda \rho & 1
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 & -\rho \\
-\lambda \rho & 1 + \lambda|\rho|^2
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 + \lambda|\rho|^2 & \rho \\
\lambda \rho & 1
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 & 0 \\
\frac{-e^{-2ix_0}\frac{\alpha_{\Gamma}(x_0, \lambda)}{\alpha_{\Sigma}(x_0, \lambda)}} & 1
\end{pmatrix}
$$
Definition 2.5. Assume $a \in \gamma_0$, the set of self-intersections of $\Gamma$. Let $\Gamma_1, \ldots, \Gamma_m$ be a counter-clockwise ordering of sub-components of $\Gamma$ which contain $z = a$ as an endpoint. For $J \in H^k(\Gamma)$ we define $\hat{J}_i$ by $J|_{\Gamma_i}$, if $\Gamma_i$ is oriented outwards and by $(J|_{\Gamma_i})^{-1}$ otherwise. We say that $J$ satisfies the $(k-1)$th-order product condition if, using the $(k-1)$th-order Taylor expansion of each $J_i$, we have

$$
\prod_{i=1}^{m} \hat{J}_i = I + \mathcal{O}(|z-a|^k) \quad \forall a \in \gamma_0.
$$

The following theorem is due to Zhou [23]; see also Trogdon-Olver [17, Theorem 2.56].

Theorem 2.6. The two following statements are equivalent:

(i) $J - I$ and $J^{-1} - I \in H^k(\Gamma)$ away from points of self intersection and $J$ satisfies the $(k-1)$th-order product condition;

(ii) $J$ is $k$-regular.

In the next theorem, we check that the jump matrix $J(\lambda)$ satisfies the condition (i) of Theorem 2.6 and characterize the large-$\lambda$ decay of scattering data in terms of weighted Sobolev spaces. In what follows,

$$
H^{2,2}(\partial \Omega_2) = \{ f \in H^2(\partial \Omega_2) : f|_{\mathbb{R}_x} \in H^{2,2}(\mathbb{R}_x) \},
$$

$$
H^{1,1}(\partial \Omega_1) = \{ f \in H^1(\partial \Omega_1) : f|_{\mathbb{R}_x} \in H^{1,1}(\mathbb{R}_x) \}.
$$

Theorem 2.7 should be compared to (C2.28) of [21], where the scattering matrix is characterized as belonging to $H^k$ for any $k \geq 1$ given initial data $q_0$ is in Schwartz class. Theorem 2.7 shows that the direct scattering transform maps initial data in the weighted Sobolev space $H^{2,2}(\mathbb{R})$ into scattering data belonging to appropriate weighted Sobolev spaces.

Theorem 2.7. The matrix $J(\lambda)$ admits a triangular factorization

$$
J(\lambda) = J_{-}^{-1}(\lambda)J_{+}(\lambda)
$$

where:

(i) $J_{+}(\lambda) - I \in H^{2,2}(\partial \Omega_2)$, $J_{-}(\lambda) - I \in H^2(\partial \Omega_3)$, $J_{+}(\lambda) - I \in H^2(\partial \Omega_4)$ and $J_{-}(\lambda) - I \in H^{1,1}(\partial \Omega_1)$, and

(ii) $J_+|_{\partial \Omega_2} - I$ and $J_-|_{\partial \Omega_2} - I$ are strictly lower triangular while $J_+|_{\partial \Omega_3} - I$ and $J_-|_{\partial \Omega_3} - I$ are strictly upper triangular.

Moreover,

(iii) The matrix $J(\lambda)$ satisfies the first-order product condition at the singular points $\pm S_x$ on the real $\lambda$-axis.

Proof. We denote by $J_i$ the restriction of $J$ to the contour $\Gamma_i$, $1 \leq i \leq 5$, where the contours $\Gamma_i$ are shown in Figure A.2. Recall that $\mathbb{R}_x = \mathbb{R}\setminus[-S_x,S_x]$. Note that, on $\mathbb{R}_x$, the scattering matrix $J_1$ admits the factorization:

$$
J_1(\lambda) = J_{1,-}(\lambda)^{-1}J_{1,+}(\lambda) = \begin{pmatrix} 1 & \rho(\lambda) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \lambda \rho(\lambda) & 1 \end{pmatrix},
$$

with the same decomposition for $J_5(\lambda)$, while on $(-S_x,S_x)$ the scattering matrix $J_3$ admits the factorization:

$$
J_3(\lambda) = J_{3,-}(\lambda)^{-1}J_{3,+}(\lambda) = \begin{pmatrix} 1 & 0 \\ \lambda \rho(\lambda) & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\rho(\lambda) & 1 \end{pmatrix}.
$$
(i) The methods of section 3 of [12] can be used to show that \( \rho \in H^{2,2}(\mathbb{R}_x) \). It follows from this fact and the explicit factorization (2.32) that

\[
[J_1,-(\lambda) - I]|_{\mathbb{R}_x} \in H^{2,2}(\mathbb{R}_x)
\]

and

\[
[J_1,+(\lambda) - I]|_{\mathbb{R}_x} \in H^{1,1}(\mathbb{R}_x).
\]

This same fact and (2.33) show that the restrictions of \( J_\pm(\lambda) - I \) to \((-S_\infty, S_\infty)\) belong to \( H^2(-S_\infty, S_\infty) \). The remaining Sobolev estimates all involve the bounded semicircular contours \( \Gamma_2 \). From (2.29) and (2.30), since \( H^2(\Gamma) \) is an algebra for any semicircular contour \( \Gamma \), it suffices to show that the functions \( n_{12}(x_0, \lambda), 1/\alpha(\lambda), \) and \( 1/\alpha_0(\lambda) \) belong to \( H^2(\Gamma^+) \) and that the functions \( n_{21}(x, \lambda), 1/\alpha(\lambda), \) and \( 1/\alpha_0(\lambda) \) belong to \( H^2(\Gamma^-) \). This is easily deduced from the Volterra integral equations corresponding to (2.24) and the integral representations for \( \alpha \) and \( \bar{\alpha} \) that can be deduced from (2.25).

(ii) The assertions about triangularity follow from the factorizations (2.32) and (2.33) together with the formulas (2.29) and (2.30).

(iii) Using the relation (2.25), the scattering matrices \( J_2 \) and \( J_4 \) are given at the self-intersection point \( S_\infty \) respectively by

\[
J_2(S_\infty) = \begin{pmatrix}
1 & 0 \\
e^{-2ix_0 S_\infty} \frac{n_{21}(x_0, S_\infty)}{\bar{\alpha}(S_\infty)} & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & 0 \\
e^{-2ix_0 S_\infty} \frac{n_{21}(S_\infty)}{\bar{\alpha}(S_\infty)} & 1
\end{pmatrix} \times \begin{pmatrix}
1 & 0 \\
S_\infty S_\alpha(S_\infty) & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & 0 \\
-S_\infty \rho_0(S_\infty) & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
S_\infty \rho(S_\infty) & 1
\end{pmatrix}
\]

and

\[
J_4(S_\infty) = \begin{pmatrix}
1 & -e^{2ix_0 S_\infty} \frac{n_{12}(x_0, S_\infty)}{\alpha(S_\infty)\alpha_0(S_\infty)} \\
0 & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & -e^{2ix_0 S_\infty} \frac{n_{12}(S_\infty)}{\alpha(S_\infty)\alpha_0(S_\infty)} \\
0 & 1
\end{pmatrix} \times \begin{pmatrix}
1 & 0 \\
S_\infty S_\alpha(S_\infty) & 1
\end{pmatrix}
\]
The Volterra integral equations that define these normalized Jost solutions (see (2.36))

\[ \begin{pmatrix}
1 & e^{2i x_0 S_x} \frac{\beta(S_x) e^{-2i x_0 S_x} \nu_{11}^+(S_x)}{\alpha(S_x) \alpha_0(S_x)} \\
0 & 1 \\
\end{pmatrix} = 
\begin{pmatrix}
1 & \rho(S_x) \\
0 & 1 \\
\end{pmatrix} \begin{pmatrix}
1 & -\rho_0(S_x) \\
0 & 1 \\
\end{pmatrix}. \]

We want to establish (2.31) for \( k = 2 \), that is,

\[ \prod_{i=1}^4 J_i = I + O \left( |z - S_x|^2 \right). \]

Denoting by \( \tilde{J}_i \) the Taylor polynomial of \( J_i \) at \( S_x \), \( i = 1, \ldots, 4 \), proving (2.34) is equivalent to proving that

\[ \tilde{J}_1 \tilde{J}_2^{-1} = \tilde{J}_4 \tilde{J}_3^{-1} + O \left( |z - S_x|^2 \right). \]

It is clear that \( J_1(S_x) J_2(S_x)^{-1} = J_4(S_x) J_3(S_x)^{-1} \). We can also check that

\[ (J_1 J_2^{-1})_{\lambda}(S_x) = (J_4 J_3^{-1})_{\lambda}(S_x). \]

By direct calculation of

\[ \lim_{\lambda \to S_x} \frac{d}{d\lambda} e^{2i x_0 \lambda} \nu_{21}^-(x_0, \lambda) \]

we can check the \( (k-1) \)th order product condition for \( k = 2 \). By Theorem 2.6, we conclude that \( J_+(\lambda) \) satisfies the matching condition (A.1) and \( J_-(\lambda) \) satisfies the matching condition (A.2) at the non-smooth point \((S_x, 0)\). A similar proof shows that the same conclusion holds for \((-S_x, 0)\).

\[ \square \]

The following Lipschitz continuity result can be deduced by the methods of [12, 13].

**Proposition 2.8.** The maps

(2.35) \( q \mapsto \rho \big|_{S_x} \in H^{2, 2}(\mathbb{R}_x) \), \( q_{x_0} \mapsto \rho_0 \in H^{2, 2}(\mathbb{R}) \),

(2.36) \( q \mapsto \nu_{21}^-(x_0, \cdot) \in H^2(\Gamma_+^x) \), \( q \mapsto \nu_{12}^-(x_0, \cdot) \in H^2(\Gamma_-^x) \),

(2.37) \( q \mapsto \frac{1}{\alpha} \in H^2(\Gamma_+^x) \), \( q \mapsto \frac{1}{\alpha_0} \in H^2(\Gamma_-^x) \),

(2.38) \( q \mapsto 1/\alpha \in H^2(\Gamma_+^x) \), \( q \mapsto 1/\alpha_0 \in H^2(\Gamma_-^x) \)

are locally Lipschitz continuous from \( H^{2, 2}(\mathbb{R}) \) into the respective ranges.

**Proof.** Continuity of the maps (2.35) follows from [12, Propositions 3.2–3.3] together with the symmetry (1.22) and the identities (3.2)–(3.3) there. The exclusion of \( |\lambda| < R \) means that \(|\alpha(\lambda)|\) is strictly positive so division by \( \alpha \) does not spoil the estimates.

Continuity of the maps (2.36) follows from the proof of [12, Proposition 3.2] and the symmetry (1.22) in [12] together with the observation that the analysis given there is unchanged for \( \lambda \) in the appropriate half-plane owing to the form of the Volterra integral equations that define these normalized Jost solutions (see [12, (3.4a), (3.4b)]).
Continuity of the maps (2.37) and (2.38) follows from a simple modification of [12, Proposition 3.2] together with the symmetry (1.22) and the identity (3.2) there. The analysis still applies on the complex contours $\Gamma^{\pm}_x$ for the same reason as stated above.

Proposition 2.8, formulas (2.27) – (2.30), and the factorizations (2.32) and (2.33) immediately imply the following Lipschitz continuity of the scattering data.

**Proposition 2.9.** The maps
\[ q \to J^+_1(\lambda) - I \in H^{2,2}(\partial \Omega_2) \]
\[ q \to J^+_2(\lambda) - I \in H^{2}(\partial \Omega_3) \]
\[ q \to J^+_3(\lambda) - I \in H^{1,1}(\partial \Omega_4) \]
\[ q \to J^+_4(\lambda) - I \in H^{2}(\partial \Omega_4) \]
are locally Lipschitz mappings from $H^{2,2}(\mathbb{R})$ into their respective ranges.

### 2.4. Time evolution of the scattering data.

A key property of the inverse scattering method is the simple time evolution of its scattering data. In [13], we calculated the time evolution of the scattering data where they reduce to a reflection coefficient and discrete data. We need to complement the analysis by examining the time evolution of the scattering matrix on the additional section of the contour $\Gamma_x$ (see Figure 1.2b). As before, we work in the $\zeta$ variable and carry out the change of variable $\zeta \to \lambda$. Given $M^+ (x, t; \zeta) = M^- (x, t; \zeta) v_x (t; \zeta)$ we compute the time derivative
\[ (2.39) \quad M^+ (x, t; \zeta)_t = M^- (x, t; \zeta)_t v_x (t; \zeta) + M^- (x, t; \zeta) v_x (t; \zeta)_t. \]

We recall that $M^\pm$ are fundamental solutions for the Lax equations
\[ (2.40a) \quad \frac{\partial M}{\partial x}(x, t, \zeta) = -i \zeta^2 \text{ad} \sigma(M) + \zeta Q(x, t) M + P(x, t) M, \]
\[ (2.40b) \quad \frac{\partial M}{\partial t}(x, t, \zeta) = -2i \zeta^4 \text{ad} \sigma(M) + A(x, t, \zeta) M. \]

where $\sigma, P, Q$ are given in terms of $q = q(x, t)$ by (1.4) and
\[ (2.41) \quad A(x, t, \zeta) = 2 \zeta^3 \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix} + i \zeta^2 \begin{pmatrix} |q|^2 & 0 \\ 0 & -|q|^2 \end{pmatrix} + i \zeta \begin{pmatrix} 0 & q_x \\ \overline{q_x} & 0 \end{pmatrix} \]
\[ + \frac{i}{4} \begin{pmatrix} |q|^4 & 0 \\ 0 & -|q|^4 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} q_x \overline{q} - q \overline{q}_x & 0 \\ 0 & -q_x \overline{q} + q \overline{q}_x \end{pmatrix}. \]

Taking the limit $x \to +\infty$ in (2.39), using the normalization of $M^\pm$ at $+\infty$, and using the fact that $\text{ad} \sigma$ is a derivation, we obtain
\[ v_x (t, \zeta)_t = -2i \zeta^4 \text{ad} \sigma v_x (t, \zeta). \]

Integrating we obtain
\[ v_x (t, \zeta) = e^{-2i \zeta^4 t \text{ad} \sigma} v_x (\zeta) \]
or equivalently
\[ (2.42) \quad J_x (t, \lambda) = e^{-2i \lambda^2 t \text{ad} \sigma} J_x (\lambda). \]

The map $(f, t) \to e^{-2i \lambda^2 t} f$ is a bounded continuous map from $X \times [-T, T]$ to $X$ for $X = H^{2,2}(\Omega_2), H^{1,1}(\Omega_1), H^2(\Omega_3)$ and $H^2(\Omega_4)$. This map is also Lipschitz
continuous in $X$ uniformly for $f$ in a bounded subset of $X$ and $t \in [-T, T]$, for a fixed $T > 0$.

From Proposition 2.9 and these facts, we deduce the following continuity result.

**Proposition 2.10.** Suppose that $q_0 \in H^{2,2}(\mathbb{R})$ and that $J(\lambda)$ is the scattering data associated to $q_0$. Denote by $J \pm (\lambda, t)$ the matrices $e^{i\lambda t \mathbf{ad} \sigma} J_{\pm}(\lambda)$ where $J_{\pm}(\lambda)$ are the factors given in Theorem 2.7. For any $T > 0$, the maps

\[
H^{2,2}(\mathbb{R}) \times [-T, T] \ni (q_0, t) \mapsto J_-(\lambda, t) - I \in H^{2,2}(\partial \Omega_2)
\]

\[
H^{2,2}(\mathbb{R}) \times [-T, T] \ni (q_0, t) \mapsto J_-(\lambda, t) - I \in H^2(\partial \Omega_3)
\]

\[
H^{2,2}(\mathbb{R}) \times [-T, T] \ni (q_0, t) \mapsto J_+(\lambda, t) - I \in H^{2,2}(\partial \Omega_4)
\]

are all continuous, and uniformly Lipschitz in $q_0$ for $t \in [-T, T]$ and $q_0$ in a bounded subset of $H^{2,2}(\mathbb{R})$.

2.5. **Auxiliary scattering matrix.** In Section 2.2, we have chosen $x_0 \in \mathbb{R}$ such that the cut-off potential $q_{x_0} = q_\chi(x_0, x)$ satisfies the smallness condition (2.17).

Without loss of generality, we assume $q_{x_0} = q_\chi(-x_0, -x_0)$ also satisfies (2.17). Let $\tilde{N}$ be constructed in the same way as $N$ (see (2.26)) but with potential $q_0$ changed to $\tilde{q}_0$ with normalization at $x \to -\infty$. We define the auxiliary matrix $s$:

\[
s(\lambda) = e^{iz\lambda \mathbf{ad} \sigma} \tilde{N}^{-1}(x, \lambda) N(x, \lambda)
\]

For instance, for $z \in \Omega_1 \cup \Omega_2$

\[
s(\lambda) = \begin{pmatrix}
\delta(z)^{-1} & 0 \\
0 & \delta(z)
\end{pmatrix}
\]

with $\delta(z) = \begin{pmatrix}
\bar{\alpha}(z) \\
\alpha(z)
\end{pmatrix}$, $\text{Im} z > 0$.

The jump matrix $\tilde{J}$ for $\tilde{N}$ is obtained from $J$ by conjugation, as $\tilde{J} = s^{-1}_- J_s$. In analogy with Theorem 2.7, we have:

**Theorem 2.11.** The matrix $\tilde{J} = s^{-1}_- J_s$ admits a triangular factorization $\tilde{J}(\lambda) = \tilde{J}_-^{-1}(\lambda) \tilde{J}_+(\lambda)$ where:

(i) $\tilde{J}_+ = I \in H^{2,2}(\partial \Omega_1)$, $\tilde{J}_- = I \in H^{1,1}(\partial \Omega_2)$, $\tilde{J}_- = I \in H^2(\partial \Omega_3)$ and $\tilde{J}_+ = I \in H^{2}(\partial \Omega_4)$.

(ii) $\tilde{J}_+ \upharpoonright \partial \Omega_1 = I$ and $\tilde{J}_- \upharpoonright \partial \Omega_2 = I$ are strictly upper triangular while $\tilde{J}_- \upharpoonright \partial \Omega_3 = I$ and $\tilde{J}_+ \upharpoonright \partial \Omega_4 = I$ are strictly lower triangular.

3. **Unique Solvability of the RHP.**

In this section, we prove the unique solvability of the following Riemann-Hilbert problem on the contour $\Gamma = \mathbb{R} \cup \Gamma_x$ shown in Figure 1.2b.

For a $2 \times 2$ matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we denote

\[
A_x = e^{-i\lambda x \mathbf{ad} \sigma} A = \begin{pmatrix} a & e^{-2i\lambda x} b \\ e^{2i\lambda x} c & d \end{pmatrix}.
\]

**Riemann-Hilbert Problem 3.1.** Fix $x \in \mathbb{R}$. Find a row vector-valued function $N(x, \cdot)$ on $\mathbb{C} \setminus \Gamma$ with the following properties:

(i) (Analyticity) $N(x, z)$ is an analytic function of $z$ for $z \in \mathbb{C} \setminus \Gamma$,
(ii) (Normalization) \( \mathbf{N}(x,z) = (1,0) + \mathcal{O}(z^{-1}) \) as \( z \to \infty \), and

(iii) (Jump condition) For each \( \lambda \in \Gamma \), \( \mathbf{N} \) has continuous boundary values \( \mathbf{N}_\pm(\lambda) \) as \( z \to \lambda \) from \( \Omega_\pm \). Moreover, the jump relation

\[
\mathbf{N}_+(x, \lambda) = \mathbf{N}_-(x, \lambda) J_x(\lambda)
\]

holds, where

\[
J_x(\lambda) = e^{-i\lambda x \ad \sigma}
\]

\[
\begin{pmatrix}
1 + \lambda |\rho(\lambda)|^2 & \rho(\lambda) \\
\lambda \rho(\lambda) & 1
\end{pmatrix}, \quad \lambda \in \mathbb{R}
\]

\[
\begin{pmatrix}
1 & -\rho_0(\lambda) \\
\lambda \rho_0(\lambda) & 1 + \lambda |\rho_0(\lambda)|^2
\end{pmatrix}, \quad \lambda \in (-S_\infty, S_\infty)
\]

\[
\begin{pmatrix}
1 & 0 \\
e^{-2ix_0 \lambda} \frac{n_{2\lambda}(x_0, \lambda)}{\alpha(\lambda) \alpha_0(\lambda)} & 1 \\
1 & -e^{2ix_0 \lambda} \frac{n_{2\lambda}(x_0, \lambda)}{\alpha(\lambda) \alpha_0(\lambda)}
\end{pmatrix}, \quad \lambda \in \Gamma^+_\infty,
\]

\[
\begin{pmatrix}
1 & 1 \\
e^{-2ix_0 \lambda} \frac{n_{2\lambda}(x_0, \lambda)}{\alpha(\lambda) \alpha_0(\lambda)} & 1
\end{pmatrix}, \quad \lambda \in \Gamma^-_\infty.
\]

**Definition 3.2.** We say that row vector-valued function \( \mathbf{N}(x,z) \) is a null vector for RHP 3.1 if \( \mathbf{N}(x,z) \) satisfies (i) and (iii) above but \( \mathbf{N}(x,z) = \mathcal{O}(z^{-1}) \) as \( z \to \infty \).

The scattering data that determine the jump matrix \( J \) are the functions

\[
SD = (\rho, \rho_0, \alpha, \alpha_0, \tilde{\alpha}_0, n_{2\lambda}(x_0, \cdot), n_{2\lambda}(x_0, \cdot))
\]

Although these functions are not independent, for the purpose of proving existence and uniqueness of solutions to RHP 3.1 we may consider them so. Recalling (1.15), we seek a Banach space \( Y_0 \) with the following properties:

- There is an injection \( i : H^{2,2}(\mathbb{R}) \to Y_0 \) that maps bounded subsets of \( H^{2,2}(\mathbb{R}) \) to precompact subsets of \( Y_0 \)
- For each \( \rho \in Y_0, (1 + |\cdot|)\rho(\cdot) \in L^2(\mathbb{R}) \cap L^2(\mathbb{R}_\infty) \)
- Each \( \rho \in Y_0 \) is a continuous function with \( \lim_{\lambda \to \infty} \lambda \rho(\lambda) = 0 \). This will allow uniform rational approximation of \( (\cdot)\rho(\cdot) \) in \( L^\infty \).

Consider the scale of spaces

\[
H^{\alpha, \beta}(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : \langle \xi \rangle^\alpha \hat{f}(\xi), \langle x \rangle^\beta \hat{f} \in L^2(\mathbb{R}) \right\}
\]

and recall that for any \( \varepsilon > 0 \), \( H^{1/2+\varepsilon,0}(\mathbb{R}) \subset C_0(\mathbb{R}) \), where \( C_0(\mathbb{R}) \) denotes the continuous functions vanishing at infinity. Also, recall that the embedding \( i : H^{\alpha, \beta}(\mathbb{R}) \hookrightarrow H^{\alpha', \beta'}(\mathbb{R}) \) is compact for \( \alpha > \alpha' \) and \( \beta > \beta' \). From the estimates

\[
\| \langle \cdot \rangle \rho(\cdot) \|_{L^2} \leq \| \rho \|_{H^{0,1}(\mathbb{R})}, \quad \| \langle \cdot \rangle \rho \|_{H^{1/2,0}(\mathbb{R})} \leq \| \rho \|_{H^{1/2,2}(\mathbb{R})}
\]

it follows by interpolation that for any \( \varepsilon > 0 \),

\[
\| \langle \cdot \rangle \rho \|_{H^{1/2+\varepsilon,0}(\mathbb{R})} \leq \| \rho \|_{H^{1+2\varepsilon,3/2+2\varepsilon}}.
\]
If we take $Y_0 = H^{1+2\varepsilon,3/2+\varepsilon}(\mathbb{R}_x)$, the restriction of the fractional Sobolev space $H^{1+2\varepsilon,3/2+\varepsilon}(\mathbb{R})$ on $\mathbb{R}_x$, it follows that any $\rho \in Y_0$ belongs to $C_0(\mathbb{R}_x)$ and it is easy to check that the remaining properties also hold.

**Definition 3.3.** We denote by $Y$ the Banach space of scattering data $SD$ with $\rho \in Y_0$ and all other data in $H^1$.

**Remark 3.4.** Note that, for $SD \in Y$, the entries of $J$ all belong to $L^2 \cap L^\infty$.

Let $Z_0 = H^{2,2}(\mathbb{R}_x)$. By Proposition 2.8, the range of the direct scattering map actually lies in the following stronger space:

**Definition 3.5.** We denote by $Z$ the set of scattering data $SD$ with $\rho \in Z_0$ and all other data in $H^2$.

We choose to consider $SD$ in the larger space in order to obtain uniform resolvent estimates for scattering data in bounded subsets of $Z$ later by a continuity-compactness argument (see Appendix B). We will exploit the fact that, under the natural continuous embedding of $Z$ in $Y$, bounded subsets of $Z$ are identified with precompact subsets of $Y$. We will prove:

**Theorem 3.6.** Suppose that the scattering data $J(\lambda)$ are given by (2.27)–(2.30) with $SD \in Y$. Then RHP 3.1 has a unique solution for each $x_0 \in \mathbb{R}$.

Following the pattern of the uniqueness result in [5, 13], we will prove the existence and uniqueness of solutions in the following way. First, we show that RHP 3.1 is equivalent to a Fredholm integral equation (the Beals-Coifman integral equation, (3.2)), for an unknown function $\nu(x, \cdot)$ on $\Gamma$. By the Fredholm alternative, it suffices to show that the corresponding homogeneous equation, (3.3), has only the trivial solution. In order to do so, in Subsection 3.2, we derive similar integral equations associated to an equivalent Riemann-Hilbert Problem, RHP 3.7, on the contour $\Sigma$. These integral equations involve an unknown function $\mu$; the inhomogeneous equation is (3.8) and the homogeneous equation is (3.9). We can use Zhou’s uniqueness theorem to show that RHP 3.7 is uniquely solvable, or, equivalently, (3.9) has only the trivial solution. Finally, we show that any solution $\nu$ of the homogeneous equation (3.2) induces a solution of (3.9), It then follows from explicit formulae connecting $\nu$ and $\mu$ that $\nu = 0$, establishing the Fredholm alternative for the original Beals-Coifman equation (3.2).

### 3.1. RHPs and singular integral equations

We now derive the Beals-Coifman integral equation for RHP 3.1. The unique solvability of RHP 3.1 is equivalent to the unique solvability of its associated integral equation. We define the nilpotent matrices $W_x^+$ and $W_x^-$ in the various parts of the contour as

$$J_x(\lambda) = (J_{x-})^{-1} J_{x+} = (I - W_x^-)^{-1} (I + W_x^+)$$

and the Beals-Coifman solution

$$\nu = N^+ (I + W_x^+)^{-1} = N^- (I - W_x^-)^{-1}$$

(3.1)
where

\[
(W_x^+, W_x^-) = \begin{cases}
\left( \begin{pmatrix} 0 & 0 \\ \lambda \rho(\lambda)e^{2i\lambda x} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \rho(\lambda)e^{-2i\lambda x} \\ 0 & 0 \end{pmatrix} \right), & \lambda \in \mathbb{R}_\times, \\
\left( \begin{pmatrix} 0 & -\rho_0(\lambda)e^{-2i\lambda x} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -\lambda \rho_0(\lambda)e^{2i\lambda x} & 0 \end{pmatrix} \right), & \lambda \in (-S_x,S_x), \\
\left( \begin{pmatrix} 0 & 0 \\ e^{2i\lambda x}S_1(\lambda) & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ e^{2i\lambda x}S_2(\lambda) & 0 \end{pmatrix} \right), & \lambda \in \Gamma_\times^+, \\
\left( \begin{pmatrix} 0 & -e^{2i\lambda x}S_3(\lambda) \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & e^{2i\lambda x}S_4(\lambda) \\ 0 & 0 \end{pmatrix} \right), & \lambda \in \Gamma_\times^-.
\end{cases}
\]

The coefficients $S_i(\lambda)$, $i = 1, \cdots, 4$, are not explicitly determined. Only the sums $S_1(\lambda) + S_2(\lambda)$ and $S_3(\lambda) + S_4(\lambda)$ identify to the entries $(2,1)$ and $(1,2)$ of the jump matrix $J_x(\lambda)$ respectively, in the corresponding part of the contour.

Note that if $SD \in Y$, then $W_x^\pm$ in $L^\infty \cap L^2$, while if $SD \in Z$, $W_x^\pm \in H^1$.

We can write the Beals-Coifman solution $\nu(x, \lambda)$ explicitly in terms of the Jost functions. In each case we give two equivalent formulas.

\[
\nu(x, \lambda) = \begin{cases}
\left( \begin{pmatrix} n_{11}(x, \lambda) \\ \alpha(\lambda) \end{pmatrix}, n_{12}(x, \lambda) \right) \begin{pmatrix} 1 \\ -e^{2i\lambda x} \lambda \rho(\lambda) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & \lambda \in \mathbb{R}_\times, \\
\left( \begin{pmatrix} n_{11}(x, \lambda) \\ \alpha(\lambda) \end{pmatrix}, n_{12}(x, \lambda) \right) \begin{pmatrix} 1 \\ \rho(\lambda)e^{-2i\lambda x} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & \lambda \in \Gamma_\times^+, \\
\left( \begin{pmatrix} N_{11+}(x, \lambda) \\ N_{12+}(x, \lambda) \end{pmatrix}, \begin{pmatrix} 1 \\ e^{2i\lambda x}S_1(\lambda) \end{pmatrix} \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & \lambda \in \Gamma_\times^- \\
\left( \begin{pmatrix} N_{11+}(x, \lambda) \\ N_{12+}(x, \lambda) \end{pmatrix}, \begin{pmatrix} \rho_0(\lambda)e^{-2i\lambda x} \\ 1 \end{pmatrix} \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & \lambda \in (-S_x,S_x) \\
\left( \begin{pmatrix} N_{11-}(x, \lambda) \\ N_{12-}(x, \lambda) \end{pmatrix}, \begin{pmatrix} 1 \\ -\lambda \rho_0(\lambda)e^{2i\lambda x} \end{pmatrix} \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & \lambda \in \Gamma_\times^-.
\end{cases}
\]
The goal is an existence and uniqueness result for solution to Problem (3.6) \( \nu \) is given by Figure 1.2a.

\[
\nu(x, \lambda) = \begin{cases} 
\left( \frac{N_{11}^{(2)}(x, \lambda)}{N_{12}^{(2)}(x, \lambda)} \right) \begin{pmatrix} 1 & -e^{-2i\lambda x} S_3(\lambda) \\ 0 & 1 \end{pmatrix} & \lambda \in \Gamma^- \\
\left( \frac{n_{11}^+(x, \lambda)}{n_{12}^-(x, \lambda)} \right) \begin{pmatrix} 1 & e^{-2i\lambda x} S_4(\lambda) \\ 0 & 1 \end{pmatrix} & \end{cases}
\]

From (3.1), we have
\[
N^+ - N^- = \nu \left( W_x^+ + W_x^- \right).
\]

The Plemelj formula and the normalization condition (ii) give the Beals-Coifman integral equation:

(3.2) \[ \nu(x, \lambda) = (1, 0) + (C_W, \nu)(\lambda) \]
= \[ (1, 0) + C^+_\Gamma (\nu W_x^-)(\lambda) + C^-_\Gamma (\nu W_x^+)(\lambda). \]

RHP 3.1 is equivalent to the integral equation (3.2) [19, Proposition 3.3]. Similarly, if \( N \) is a null vector for RHP 3.1 in the sense of Definition 3.2 and \( \nu \) is defined by (3.1), we have

(3.3) \[ \nu(x, \lambda) = C_W, \nu(\lambda) = C^+_\Gamma (\nu W_x^-)(\lambda) + C^-_\Gamma (\nu W_x^+)(\lambda). \]

Note that if \( SD \in Y \), we consider (3.2) (resp. (3.3)) as integral equations for \( \nu - 1 \in L^2(\Gamma) \) (resp. \( \nu \in L^2(\Gamma) \)), while if \( SD \in Z \) we consider (3.2) (resp. (3.3)) as equations for \( \nu - 1 \in H^1(\Gamma) \) (resp. \( \nu \in H^1(\Gamma) \)).

For \( \lambda \in \mathbb{R}_x \), (3.2) reads:

(3.4) \[ \nu_{11}(x, \lambda) = 1 + \int_{\mathbb{R}_x} \frac{\nu_{12}(x, s)\rho(s)e^{2i\pi x}}{s - \lambda - i0} \frac{ds}{2\pi i} \]

\[ - \int_{-S_x}^{S_x} \frac{\nu_{12}(x, s)\rho_0(s)e^{2i\pi x}}{s - \lambda} \frac{ds}{2\pi i} \]

\[ + \int_{\Gamma^+_x} \frac{\nu_{12}(x, s)\left(S_1(s) + S_2(s)\right)e^{2i\pi x}}{s - \lambda} \frac{ds}{2\pi i} \]

(3.5) \[ \nu_{12}(x, \lambda) = \int_{\mathbb{R}_x} \frac{\nu_{11}(x, s)\rho(s)e^{-2i\pi x}}{s - \lambda - i0} \frac{ds}{2\pi i}, \]

\[ - \int_{-S_x}^{S_x} \frac{\nu_{11}(x, s)\rho_0(s)e^{-2i\pi x}}{s - \lambda} \frac{ds}{2\pi i} \]

\[ + \int_{\Gamma^+_x} \frac{\nu_{11}(x, s)\left(S_3(\lambda) + S_4(\lambda)\right)e^{-2i\pi x}}{s - \lambda} \frac{ds}{2\pi i} \]

The integral equations for \( \lambda \in (-S_x, S_x) \) and \( \lambda \in \Gamma^+_x \) are obtained analogously.

Note that the solution to Problem 3.1 is given, in terms of \( \nu = (\nu_{11}, \nu_{12}) \) by

(3.6) \[ N(x, z) = (1, 0) + \frac{1}{2\pi i} \int_{\Gamma} \frac{\nu(x, s)(W_x^+(s) + W_x^-(s))}{s - z} \frac{ds}{s - z}. \]

The goal is an existence and uniqueness result for solution to Problem 3.1. To make use of the symmetry relations of the jump conditions and Zhou’s vanishing lemma, we need to consider the equivalent RHP in the \( \zeta \) variables with jump contour \( \mathbb{R} \cup i\mathbb{R} \) given by Figure 1.2a.
**Riemann-Hilbert Problem 3.7.** Fix $x \in \mathbb{R}$. Find a matrix-valued function $M(x, \cdot)$ with the following properties:

(i) (Analyticity) $M(x, z)$ is a $2 \times 2$ matrix-valued analytic function of $z$ for $z \in \mathbb{C}\setminus\Sigma$ where the contour $\Sigma$ is given by Figure 1.2a.

(ii) (Normalization) $M(x, z)$ is the $2 \times 2$ matrix-valued analytic function of $z$ for $z \in \mathbb{C}\setminus\Sigma$ where the contour $\Sigma$ is given by Figure 1.2a.

(iii) (Jump condition) For each $\zeta \in \Sigma$, $M$ has continuous boundary values $M_{\pm}(\lambda)$ as $z \to \zeta$ from $\Omega_{\pm}$. Moreover, the jump relation

$$M_+(x, \zeta) = M_-(x, \zeta)e^{-i\pi \zeta^2 \sigma v(\zeta)}$$

holds, where $v(\zeta)$ is given by (2.20)-(2.23).

**Definition 3.8.** We say that a matrix-valued function $M(x, z)$ is a null vector for RHP 3.7 if $M(x, z)$ satisfies (i) and (iii) above but $M(x, z) = O(z^{-1})$ as $|z| \to \infty$.

Observe that, given scattering data $SD$ for RHP 3.1 in the space $Y$ from Definition 3.3, the induced scattering data for RHP 3.7 consists of bounded continuous functions, square-integrable on the unbounded contours. Thus RHP 3.7 makes sense with the $O(z^{-1})$ conditions replaced by an $L^2$-condition on $M_{\pm} - I$ for the RHP, and the condition $M_{\pm} \in L^2$ for a null vector.

**Proposition 3.9.** The only $L^2$ null vector for Problem 3.7 with scattering data induced from $SD \in Y$ is the zero vector.

**Proof.** A direct consequence of Proposition 2.1 and Theorem 9.3 of [19].

It will be most useful to formulate the result of Proposition 3.9 in terms of the homogeneous Beals-Coifman equation for RHP 3.7, which we now derive.

The jump matrix $v(\zeta)$ admits the following factorization

$$v_x = (1 - w_x)\mu^{-1} + w_x^+.$$

We set

$$\mu = M^+(1 + w_x^+)^{-1} = M^-(1 - w_x^-)^{-1}.$$  \hspace{1cm} (3.7)

In analogy with Problem 3.1, we deduce the Beals-Coifman integral equation for Problem 3.7:

$$\mu = I + C_{w_x} \mu = I + C^+_{\Sigma}(\mu w_x^-) + C^-_{\Sigma}(\mu w_x^+).$$  \hspace{1cm} (3.8)

where $I$ is the $2 \times 2$ identity matrix. If $M$ is a null vector in the sense of Definition 3.8 and $\mu$ is defined by (3.7), then

$$\mu = C_{w_x} \mu = C^+_{\Sigma}(\mu w_x^-) + C^-_{\Sigma}(\mu w_x^+).$$  \hspace{1cm} (3.9)

Thus, we can reformulate Proposition 3.9 as follows:

**Proposition 3.10.** Suppose that $w_x^\pm$ are derived from scattering data $SD \in Y$. Then, the only solution to (3.9) in $L^2(\Sigma)$ is the zero vector.
3.2. A Mapping Between Null Spaces. To complete the proof of Theorem 3.6, we show that any solution \( \nu \) of (3.3) induces a solution \( \mu \) of (3.9) and that, if \( \mu = 0 \), then \( \nu = 0 \). For notational brevity we suppress the dependence of \( \mu \) and \( \nu \) on \( x \), which remains fixed throughout the discussion.

**Lemma 3.11.** Suppose that \( W^\pm_x \) are generated from scattering data \( SD \in Y \). For \( \nu = (\nu_1, \nu_2) \) a solution of the homogeneous Beals-Coifman equation (3.3) in \( L^2(\Gamma) \), define

\[
\mu(x, \zeta) = \begin{pmatrix}
\mu_{11}(x, \zeta) & \mu_{12}(x, \zeta) \\
\mu_{21}(x, \zeta) & \mu_{22}(x, \zeta)
\end{pmatrix} = \begin{pmatrix}
\nu_{11}(x, \zeta^2) & \zeta \nu_{12}(x, \zeta^2) \\
-\zeta \nu_{12}(x, \zeta^2) & \nu_{11}(x, \zeta^2)
\end{pmatrix}.
\]

Then \( \mu \in L^2(\Sigma) \) solves Equation (3.9).

**Remark 3.12.** We can invert (3.10) to recover \( \nu \) via the formulas

\[
\nu_{11}(x, \zeta^2) = \mu_{11}(x, \zeta), \quad \nu_{12}(x, \zeta^2) = \mu_{12}(x, \zeta)/\zeta.
\]

In particular, if \( \mu = 0 \), then \( \nu = 0 \).

**Proof.** Define a matrix-valued function \( \mu \) by (3.11) for a given solution \( \nu \) of (3.3). It is easy to see that

\[
\mu_{11}(x, -\zeta) = \mu_{11}(x, \zeta), \quad \mu_{12}(x, -\zeta) = -\mu_{12}(x, \zeta).
\]

In [13, Lemma 5.2.2] we have shown that for \( \nu \in L^2(\Gamma) \) and \( \rho \in Y_0 \),

\[
\mu_{11}(x, \zeta)r(\zeta) = \nu_{11}(x, \zeta^2)\zeta\rho(\zeta^2), \quad \mu_{12}(x, \zeta)\bar{r}(\zeta) = \zeta\nu_{12}(x, \zeta^2)\bar{\rho}(\zeta^2)
\]

are both square-integrable on the part of the \( \Sigma \) contour outside the circle \( \Sigma_x \).

Thus \( \mu w^\pm_x \) is an \( L^2 \) function on \( \Sigma \). Once we establish the transition from Equation (3.3) to Equation (3.9), \( \mu \in L^2(\Sigma) \) follows from the boundedness of Cauchy projection on \( L^2 \) functions.

In [13, Chapter 5], we established the transition from Equation (3.3) to Equation (3.9) when \( \Gamma = \mathbb{R} \) and \( \Sigma = \mathbb{R} \cup i\mathbb{R} \). Thus we only deal with the two contour integrals:

\[
I^+ := \int_{\Gamma^+_x} \frac{\nu_{12}(x, s)(S_1(s) + S_2(s))e^{2isx}}{s - \lambda} \, ds \quad 2\pi i
\]

and

\[
I^- := \int_{\Gamma^-_x} \frac{\nu_{11}(x, s)(S_3(\lambda) + S_4(\lambda))e^{-2isx}}{s - \lambda} \, ds \quad 2\pi i
\]

Let \( \lambda = \zeta^2 \) and fix the branch \([0, 2\pi)\).

\[
I^+ = \int_{\Gamma^+_x} \frac{\nu_{12}(x, \lambda)n_{21}(x, \lambda)e^{2i\lambda(x-x_0)}}{(\lambda - \lambda_0)\tilde{a}(\lambda)\tilde{a}_0(\lambda)} \, d\lambda \quad 2\pi i
\]

\[
= \int_C \left( \frac{\mu_{12}(x, \zeta)\mu_{21}(x, 0, \zeta)e^{2i\zeta^2(x-x_0)}}{(\zeta - \zeta_0)\tilde{a}(\zeta)\tilde{a}_0(\zeta)} - \frac{\mu_{12}(x, \zeta)\mu_{21}(x, 0, \zeta)e^{2i\zeta^2(x-x_0)}}{(-\zeta - \zeta_0)\tilde{a}(\zeta)\tilde{a}_0(\zeta)} \right) \frac{d\zeta^2}{2\zeta^2} \frac{1}{2\pi i}
\]

\[
= \int_C \left( \frac{\mu_{12}(x, \zeta)m_{21}(x, 0, \zeta)e^{2i\zeta^2(x-x_0)}}{(\zeta - \zeta_0)\tilde{a}(\zeta)\tilde{a}_0(\zeta)} - \frac{\mu_{12}(x, \zeta)m_{21}(x, 0, \zeta)e^{2i\zeta^2(x-x_0)}}{(-\zeta - \zeta_0)\tilde{a}(\zeta)\tilde{a}_0(\zeta)} \right) \frac{d\zeta^2}{2\pi i}
\]

that, by Lemma 3.12, equivalent to solvability of \( RHP \).

We write Proposition 3.13. The resolvent \( (I - C_{W,x})^{-1} \) exists for all \( x \in \mathbb{R} \) and all \( SD \in Y \).
4. Mapping Properties of the Inverse Scattering Map

We begin with the reconstruction formula for the potential $q$ with $J_\pm$ characterized in Theorem 2.7:

$$q(x) = \left(- \frac{1}{\pi} \int_{\Gamma} \nu(x, \lambda) \left(W^+_x(\lambda) + W^-_x(\lambda)\right) d\lambda \right)_{12}$$

$$= \left(- \frac{1}{\pi} \int_{\Gamma} \nu(x, \lambda)e^{-i\lambda x \sigma}(J_+(\lambda) - J_-(\lambda)) d\lambda \right)_{12}$$

where the 12 subscript denotes the second entry of the row vector. We show the estimate for $x \geq 0$ first. Following a reduction technique from [22], we construct functions $\omega_+ \in A(\Omega_\pm)$ such that, for $k = 2$:

1. $\omega_+ \in R(\partial \Omega_\pm)$ and $\omega_+ - I = O(z^{-2})$ as $z \to \infty$.
2. $\omega_+$ has the same triangularity as $J_\pm$, and
3. $\omega_+(z) = J_\pm(z) + o((z - a)^{k-1})$ for $a \in \mathbb{R} \cap \Gamma$.

Here $A(\Omega_\pm)$ denotes the space of analytic functions in the $\Omega_\pm$ region of the complex plane. $R(\partial \Omega_\pm)$ denotes the space of functions whose restrictions on $\partial \Omega_\pm$ are rational functions.

The construction of $\omega_+$ is given in [20, Appendix I]. For example, consider the approximation of $J_{-1}(\partial \Omega_2)$. Since $(J_- - I)_{|\partial \Omega_2}$ is in $H^2$, we construct a rational function $\omega_-$ such that $(\omega_- - J_-)_{|\partial \Omega_2}$ vanishes at $\pm S_\infty$ to order 1. Explicitly, we need

$$\omega_-(\pm S_\infty) - I = \rho(\pm S_\infty), \quad \omega_-'(\pm S_\infty) = \rho'(\pm S_\infty)$$

This is performed by the following steps:

(i). Choose $z_0 \notin \overline{\Omega_2}$ and denote $p_\pm$ the Taylor polynomial of degree 1 of $(z - z_0)^{-n}p(z)$ at $z = \pm S_\infty$. We choose $n \geq 6$.

(ii). By [20, Lemma A1.2], there is $p(z)$ a polynomial of degree at most 3 such that

$$p(z) - p_\pm(z) = O((z \mp S_\infty)^{n-2})$$

(iii). Set $\omega_-(z) = (z - z_0)^{-n}p(z)$. Clearly, $\omega_-(z) - \rho(z)$ vanishes at $\pm S_\infty$ to order 1. Since $n \geq 6$, $\omega_- \in H^{2,2}(\partial \Omega_2)$ and $\omega$ is analytic in $\Omega_2$.

By construction,

$$J = \omega_-^{-1}(J_-\omega_-^{-1})^{-1}(J_+\omega_+^{-1})\omega_+ \equiv \omega_-^{-1}J_-^{-1}J_+\omega_+ \equiv \omega_-^{-1}J\omega_+$$

The advantage of working with $J$ is that $J_\pm$ vanishes at $\pm S_\infty$ to order 1:

$$J_\pm(\lambda) = I + o((\lambda - a)^{1}), \quad a = \pm S_\infty.$$  

Notice that $J_\pm$ are characterized as $J_\pm$ in Theorem 2.7 and they will be used when establishing estimates such as (4.10) and (4.11). For $x \geq 0$, $J_\pm$ is the jump condition for the RHP

$$N_+(x, \lambda) = N_-(x, \lambda)J_\pm(\lambda) \quad \lambda \in \Gamma$$

if and only if $J_\pm$ is the jump condition for the RHP 3.1. Here $N = Ne^{-i\lambda x \sigma}$. $\omega \in AL^2(\mathbb{C} \setminus \Gamma)$, and $AL^2(\mathbb{C} \setminus \Gamma)$ is guaranteed by construction. Note that $N$ and $\omega$ give rise to the same $\nu$, solution of the associated Beals-Coifman solutions. The potential is given by

$$q(x) = \left(- \frac{1}{\pi} \int_{\Gamma} \nu(x, \lambda)e^{-i\lambda x \sigma}(J_+(\lambda) - J_-(\lambda)) d\lambda \right)_{12},$$
which shows that $\omega$ gives no contribution to the reconstruction of $q$ for $x \geq 0$. We may thus as well work with $J$.

The next step consists of augmenting the contour as in figure 4.1 below. The advantage of this new contour is that it reverses the orientation of the segment $(S^-, S^+)$ and thus allows to prove usual estimates of the Cauchy projections when the contour is restricted to $\mathbb{R}$. The added ellipse has no effect of the RHP since the jump matrices there are chosen to be the identity.

![Figure 4.1. The newly modified contour $\Gamma_n$](image)

We redefine $J_{\pm}$ as follows:
1. $J_{\pm} = I$ on the added ellipse,
2. $J_{\pm}(\lambda)$ is the lower/upper triangular factor in the lower/upper triangular factorization of $J$ ($J^{-1}$) on $\mathbb{R}$ for $|\lambda| > |S_{\pm}|$, $|\lambda| < |S_{\pm}|$) and
3. for $\lambda \in \Gamma_\pm$, $J_{\pm}(\lambda) = I$ for $\text{Im} \lambda \leq 0$ and $J_{\pm}(\lambda) = J(\lambda)$ for $\text{Im} \lambda \geq 0$.

The newly defined $J_{\pm}$ satisfy all properties characterized by Theorem 2.7.

4.1. Decay property of the reconstructed potential.

**Theorem 4.1.** If $J$ is characterized by Theorem 2.7, then $q \in H^{0,2}(\mathbb{R})$ where

$$H^{0,2}(\mathbb{R}) = \{ q \in L^2(\mathbb{R}) : x^2 q(x) \in L^2(\mathbb{R}) \}.$$

Moreover, the map from data $J = J_+^{-1} J_-$ defined as in (4.2) and obeying the hypothesis of Theorem 3.6 to $q \in H^{2,2}(\mathbb{R})$ is Lipschitz continuous.

**Proposition 4.2.** Define operator $C_{J_{\pm}}$ by

$$C_{J_{\pm}} \phi = C^{+} \phi(J_{x+} - I) + C^{-} \phi(I - J_{x-}).$$

For any $x \in [a, \infty)$, $a \in \mathbb{R}$ and $J = \omega J_{\pm}^{-1}$ where $J$ has the form of (2.27)-(2.30) and is constructed from $SD \in B(Z)$ which is a bounded subset of $Z$ (see Definition 3.5) and $J$ admits an algebraic factorization $J = J_+^{-1} J_-$ with $J_{\pm}$ having the same triangularities characterized in Theorem 2.7, then the norm of the resolvent operator

$$\| (I - C_{J_{\pm}})^{-1} \|_{L^2}$$

is uniformly bounded.

**Proof.** The proof consists of the following steps:

(i) Fix $J$. Let $\tilde{J}_{\pm} = (J_{\pm})^{-1}$ and construct the operator

$$T_{J_x} = C^{+} (C^{-} \phi(J_{x+} - J_{x-})) (I - \tilde{J}_{x-}) + C^{-} (C^{+} \phi(J_{x+} - J_{x-})) (\tilde{J}_{x+} - I).$$
We have
\[(4.4)\quad I - T_{J_x} = (I - C_{J_x^+})(I - C_{J_x^-}).\]

(ii) ([19, Theorem 6.1]) Approximate \(\tilde{J}^\pm\) by rational functions \(u^\pm\). Then
\[
C^+ (C^- \phi(J_x^+ - J_x^-)) (I - \tilde{J}_x^-)
\]
is approximated in \(L^2\) by \(C^+ (C^- \phi(J_x^+ - J_x^-)) (I - u_x^-)\) which is equal to
\[
\sum_{\lambda_i \in \Omega^-} (C_{\lambda_i} \phi(J_x^+ - J_x^-)) (I - u_x^-) \lambda_i.
\]

Here \((I - u_x^-) \lambda_i\) refers to the residue of \((I - u_x^-) \lambda_i\) at \(\lambda_i\). Using the triangularity of \(u_x^\pm\), we conclude that \(\|T_{J_x}\|_{L^2}\) approaches 0 in \(L^2\) as \(x \to +\infty\). Fix \(b \gg 0\) such that \(\|T_{J_x}\|_{L^2} \leq 1/2\) for \(x > b\). Using Neumann series, we see that
\[
(I - C_{J_x^\pm})^{-1} = \sum_{k=0}^{\infty} (-1)^k T_{J_x}^k (I - C_{J_x^\pm})
\]
which implies that
\[
\sup_{x \in [b, \infty)} \|(I - C_{J_x^\pm})^{-1}\|_{L^2} \leq K
\]
for all \(x > b\).

(iii) Since \(x \mapsto (I - C_{J_x^\pm})^{-1}\) is continuous in \(x\), we have that, for fixed scattering data \(J^\pm \in B(Z)\),
\[
\sup_{x \in [a, \infty)} \|(I - C_{J_x^\pm})^{-1}\|_{L^2} < \infty.
\]

(iv) Using the fact that the space \(Z\) is precompact \(Y\) (Definition 3.3) and Proposition B.1 of Appendix B, we have
\[(4.5)\quad \sup_{J} \left( \sup_{x \in [a, \infty)} \|(I - C_{J_x^\pm})^{-1}\|_{L^2} \right) < \infty.\]

\(\square\)

**Definition 4.3.** Define the subsets of the contour \(\Gamma\):
\[(4.6)\quad \Gamma_\pm := \mathbb{R} \cup (\{\text{Im } \lambda \geq 0\} \cap \Gamma),\]
\[(4.7)\quad \Gamma' := \Gamma_{\pm}^\pm = \text{either } \Gamma_{\infty} \cap \{\text{Im } \lambda \geq 0\} \text{ or } \mathbb{R}.\]

**Lemma 4.4.** (See [22, Lemma 2.9]) For \(x \geq 0\),
\[(4.8)\quad \|C_{\Gamma_-^\pm} (I - J_x^-)\|_{L^2} \leq \frac{c}{1 + x^2} \|J_x^- - I\|_{H^2},\]
\[(4.9)\quad \|C_{\Gamma_+^\pm} (I - J_x^+)\|_{L^2} \leq \frac{c}{(1 + x^2)^{1/2}} \|J_x^+ - I\|_{H^1},\]
\[(4.10)\quad \|J_x^+ - I\|_{L^2(\Gamma_+) \cap H^1} \leq \frac{c}{(1 + x^2)^{1/2}} \|J_x^+ - I\|_{H^1},\]
\[(4.11)\quad \|J_x^- - I\|_{L^2(\Gamma_-) \cap H^1} \leq \frac{c}{1 + x^2} \|J_x^- - I\|_{H^2},\]
\[(4.12)\quad \left( \|(C_{J_x^\pm}^\pm) I\|_{L^2(\Gamma')} \right)_{11} \leq \frac{c}{1 + x^2} \|J_x^+ - I\|_{H^1} \|J_x^- - I\|_{H^2},\]
\[(4.13)\quad \left( \|(C_{J_x^\pm}^\pm) I\|_{L^2(\Gamma')} \right)_{22} \leq \frac{c}{(1 + x^2)^{1/2}} \|J_x^+ - I\|_{H^1} \|J_x^- - I\|_{H^2}.\]
Proof of Theorem 4.1. The two previous lemmas provide the tools for estimating the decay of the potential \( q \), recalling that \( \nu \) appearing in (4.1) is equal to \((I - C_{J^±})^{-1}(1, 0)\). We will work with the following integral

\[
\int \left( \left( I - C_{J^±} \right)^{-1} I \right) e^{-i\lambda x} d\mu (J_+ - J_-) d\lambda = \int_1 + \int_2 + \int_3 + \int_4
\]

(4.14)

(where the right-hand integrals are defined in (4.14)–(4.17)) and extract information of \( q(x) \) from the (1-2) entry. Here and thereafter the integral sign without subscripts refers to taking integral on the entire contour given by Figure 4.1. We write

\[
\int_1 := \int_{\mathbb{R}} (J_{x+} - J_{x-}) + \int_{\Gamma_{x}^{+}} (J_{x+} - I) + \int_{\Gamma_{x}^{-}} (I - J_{x-}).
\]

Notice that \( J_{x}^{-} - I \) is strictly upper triangular on \( \Gamma \) and \( J_{x}^{-} - I \) is in \( H^2 \) so we conclude that the (1-2) entry of the integral above is in \( H^{0, 2} \) by Fourier transform and (4.11). The second integral

\[
\int_2 := \int (C_{J_{x}^±} I) (J_{x+} - J_{x-})
\]

is zero on the (1-2) entry, it thus has no contribution to the reconstruction of \( q \).

For the third integral,

\[
\int_3 := \int ((C_{J_{x}^±})^2 I) (J_{x+} - J_{x-}) = \int (C^+ (C^- (J_{x+} - I)) (I - J_{x-})) (J_{x+} - I) + \int (C^- (C^+ (I - J_{x-})) (J_{x+} - I)) (I - J_{x-}),
\]

we notice that the (1-2) entry is given by

\[
\int_{\Gamma_{x}^{+}} \left( C_{t_{x}^{-}}^{-1} C_{t_{x}^{+}}^{+} (I - J_{x-}) \right) (J_{x+} - I) (I - J_{x-}) + \int_{\mathbb{R}} \left( C_{t_{x}^{+}}^{+} C_{t_{x}^{-}}^{-} (I - J_{x-}) \right) (J_{x+} - I) C_{\mathbb{R}}^{+} (I - J_{x-}),
\]

and from (4.8) and (4.11), we conclude that

\[
\left| \left( \int_3 \right)_{12} \right| \leq \frac{c}{(1 + x^2)^{2}}.
\]

Finally we set

\[
g = (1 - C_{J_{x}^±})^{-1} ((C_{J_{x}^±})^2 I)
\]

and write

\[
\left( \int_4 \right) := \left[ \int [(C^+ g(I - J_{x-})) (J_{x+} - I) + (C^- g(J_{x+} - I)) (I - J_{x-})] \right].
\]

(4.17)

Again, the 1-2 entry is given by

\[
\int_{\Gamma_{x}^{+}} (C^- g(J_{x+} - I)) (I - J_{x-}) + \int_{\mathbb{R}} (C_{t_{x}^{+}}^{+} g(J_{x+} - I)) C_{\mathbb{R}}^{+} (I - J_{x-})
\]

and from (4.8), (4.11), (4.12), and Lemma 4.2 we conclude that

\[
\left| \left( \int_4 \right)_{12} \right| \leq \frac{c}{(1 + x^2)^{2}}.
\]
The estimate on \((-\infty, a)\) can be obtained by considering the RHP with jump condition described in Theorem 2.11. This concludes the proof of Theorem 4.1. Lipschitz continuity of the map follows from the uniform boundedness of the resolvent operator given by (4.5) and the second resolvent identity. □

4.2. Smoothness property of the reconstructed potential.

**Theorem 4.5.** If the jump matrix \(J\) is characterized by Theorem 2.7, then \(q \in H^{2,0}(\mathbb{R})\). Moreover, the map from data \(J\) defined as in (4.2) and obeying the hypothesis of Theorem 3.6 to \(q \in H^{2,2}(\mathbb{R})\) is Lipschitz continuous.

In order to study smoothness properties of the reconstructed potential, we first show that the functions \(M_{\pm}\) solving RHP 3.7 solve a differential equation in the \(x\) variable. It follows that the same is true of the solution \(\mu\) of (3.8) since \(\mu\) is obtained from either \(M_{+}\) or \(M_{-}\) through postmultiplication by a matrix of the form \(e^{ix\zeta^2 \text{ad} \sigma A(\zeta)}\). We can then change variables to find a differential equation in \(x\) obeyed by the (matrix-valued) solution \(\nu\) of RHP 3.1.

**Proposition 4.6.** The functions \(M_{\pm}\) obey the differential equation

\[
M(x, \zeta) = I + \int_{\mathbb{R} \cup i\mathbb{R}} \frac{\mu(x, s) \left( w^+_{x}(s) + w^-_{x}(s) \right)}{s - \zeta} \, ds \quad 2\pi \, i \tag{4.18}
\]

\[
Q(x) = -\frac{1}{2\pi} \text{ad} \sigma \left( \int_{\mathbb{R} \cup i\mathbb{R}} \mu(x, \zeta) \left( w^+_{x}(\zeta) + w^-_{x}(\zeta) \right) \, d\zeta \right) \tag{4.19}
\]

\[
P(x) = Q(x)i(\text{ad} \sigma)^{-1}Q(x). \tag{4.20}
\]

The proof of the above proposition is a slight modification of the proof of Proposition 5.3.1 in [13]. We then notice that \(\mu\) given by (3.7) solves the linear problem (1.5):

\[
\frac{d}{dx} \mu = (-i\zeta^2 \text{ad} \sigma + \zeta Q(x) + P(x)) \mu. \tag{4.21}
\]

We now use the change of variable \(\zeta \to \lambda\) to obtain

\[
\frac{d}{dx} \nu = \left( -i\lambda \text{ad} \sigma + \begin{pmatrix} 0 & q \\ -\lambda q & 0 \end{pmatrix} + P \right) \nu. \tag{4.22}
\]

We further write

\[
\frac{d}{dx} \left( \nu e^{-i\lambda x \text{ad} \sigma} (J_+ - J_-) \right) = \\
\left( -i\lambda \text{ad} \sigma + \begin{pmatrix} 0 & q \\ -\lambda q & 0 \end{pmatrix} + P \right) \left( \nu e^{-i\lambda x \text{ad} \sigma} (J_+ - J_-) \right). \tag{4.23}
\]

**Remark 4.7.** Unlike RHP 3.1 in which \(\nu\) appears as a row vector, here \(\nu\) is a 2 \times 2 matrix:

\[
\nu = \begin{pmatrix} \nu_{11}(x, \lambda) & \nu_{12}(x, \lambda) \\ -\lambda \nu_{12}(x, \lambda) & \nu_{11}(x, \lambda) \end{pmatrix}
\]

and its first row \((\nu_{11}, \nu_{12})\) is the solution to RHP 3.1.
Proof of Theorem 4.5. We integrate both sides of (4.23) along the contour shown in Figure 4.1
\[
\frac{d}{dx} \int (\nu e^{-i\lambda x} \text{ad } \sigma (J_+ - J_-)) = \int \left( -i\lambda \text{ad } \sigma + \begin{pmatrix} 0 & q \\ -\lambda \sigma & 0 \end{pmatrix} + P \right) (\nu e^{-i\lambda x} \text{ad } \sigma (J_+ - J_-)).
\]
The potential \(q\) is given by the (1-2) entry of this matrix form integral. Using that \(J_\sigma \in H^{2,2}(\Gamma)\), the (1-2) entry of
\[
\int -i\lambda \text{ad } \sigma (\nu e^{-i\lambda x} \text{ad } \sigma (J_+ - J_-))
\]
is an \(L^2\) function of \(x\) following the same argument as in the proof of Theorem 4.1.
To show that the (1-2) entry of \(\hat{z} = \begin{pmatrix} \hat{z} & \lambda q \\ 0 & \hat{z} \end{pmatrix} \sigma \hat{z} \)
is an \(L^2\)-function of \(x\), we use that \(q \in L^2 \cap L^\infty\) which comes from the fact that \(|q| \leq c/(1 + x^2)^2\) shown in Theorem 4.1. This proves that \(q_x \in L^2\). To estimate \(q_{xx}\), we differentialte (4.23) with respect to \(x\). Explicitly, we have that
\[
\frac{d^2}{dx^2} \int (\nu e^{-i\lambda x} \text{ad } \sigma (J_+ - J_-)) = \int_1 + \int_2
\]
where
\[
\int_1 := \int \left( -i\lambda \text{ad } \sigma + \begin{pmatrix} 0 & q \\ -\lambda \sigma & 0 \end{pmatrix} + P \right)^2 (\nu e^{-i\lambda x} \text{ad } \sigma (J_+ - J_-)).
\]
and
\[
\int_2 := \int \left( -i\lambda \text{ad } \sigma + \begin{pmatrix} 0 & q \\ -\lambda \sigma & 0 \end{pmatrix} \right) (\nu e^{-i\lambda x} \text{ad } \sigma (J_+ - J_-)).
\]
Again following the previous argument and using that \(J_\sigma \in H^{2,2}\) and \(q \in H^1\), we conclude that \(q_{xx} \in L^2(-\infty, a)\). A similar argument using scattering data given by Theorem 2.11 and solving the corresponding RHP shows that \(q_{xx} \in L^2(-\infty, a)\).
Lipschitz continuity of the map follows from the uniform boundedness of the resolvent operator given by (4.5) and the second resolvent identity. This completes the proof of Theorem 4.5. \(\square\)

Finally, the proof of Theorem 1.1 uses Theorems 4.1 and 4.5 and the following proposition (see [13, Proposition 7.0.4]):

**Proposition 4.8.** Suppose that \(M_\pm\) solve the RHP (3.7). Let
\[
Q(x, t) = -\frac{1}{2\pi} \text{ad } \sigma \left( \int_{\Sigma} \mu (w_{x,t}^+ + w_{x,t}^-) \right)
\]
\[
P(x, t) = iQ(x, t)(\text{ad } \sigma)^{-1}Q(x, t) = \frac{i}{2} \begin{pmatrix} |q|^2 & 0 \\ 0 & -|q|^2 \end{pmatrix},
\]
and \(A(x, t)\) given as in (2.41). Then \(M_\pm\) are fundamental solutions of the Lax equations (2.40).
Proof of Theorem 1.1. Given initial data $q_0 \in H^{2,2}(\mathbb{R})$, the direct scattering map has the continuity properties asserted in Proposition 2.9, so that the time-evolved scattering data has the continuity properties asserted in Proposition 2.10. By Theorem 3.6, RHP 3.1 is uniquely solvable for each $x, t$, and by Theorems 4.1 and 4.5, the map from scattering data $J(\cdot, t)$ to reconstructed potential $q(\cdot, t)$ is Lipschitz continuous into $H^{2,2}(\mathbb{R})$. The map $(q_0, t) \mapsto q(x, t)$ defined by the composition of the direct scattering map, the flow map, and the inverse scattering map is jointly continuous in $(q_0, t)$ and locally Lipschitz continuous in $q_0$. □

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Appendix A. Sobolev Spaces on Self-Intersecting Contours

In this appendix we define and discuss the Sobolev spaces $H^k_{\tilde{\Gamma}}(\Gamma)$ and $H^k_{\Gamma}(\Gamma)$ needed for the analysis of RHP 3.1. These Sobolev spaces were all introduced by Zhou [19]; see [17, §2.6–2.7] for a very readable account of these spaces and their role in the analysis of Beals-Coifman integral equations associated to RHP’s.

Recall that if $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_n$ and the $\Gamma_i$ are either half-lines, line segments, or arcs, the space $H^k(\Gamma)$ consists of those functions $f$ on $\Gamma$ with the property that $f|_{\Gamma_i} \in H^k(\Gamma_i)$. The space $H^k(\Gamma_i)$ has an obvious meaning since each $\Gamma_i$ can be parameterized by arc length and functions on $\Gamma_i$ viewed as functions on a subset of $\mathbb{R}$. Any function $f \in H^k(\Gamma_i)$ has a representative which is continuous together with its derivatives $f^{(j)}$ up to order $k - 1$. Limits of $f^{(j)}$ at the endpoints of each $\Gamma_i$ are well-defined for $0 \leq j \leq k - 1$. The space $H^k_{\tilde{\Gamma}}(\Gamma)$ (resp. $H^k_{\Gamma}(\Gamma)$) consists of those functions in $H^k(\Gamma)$ which are continuous together with their derivatives up to order $k - 1$ along the solid and dashed components shown in Figure A.1a (resp. Figure A.1b).

(a) Boundary components of $\Omega^+$

(b) Boundary components of $\Omega^-$

![Figure A.1. Boundary Components of $\Omega^\pm$](image)

To describe the continuity conditions, let

$$(f_i^j)^\pm = \lim_{z \to S_{z,\pm}^\Gamma} f^{(j)}(z)$$

where the contours $\Gamma_i$ are as shown in Figure A.2 below. A function $f \in H^k_{\tilde{\Gamma}}(\Gamma)$ obeys the conditions

(A.1) $(f_1^j)^- = (f_1^j)^+, \quad (f_2^j)^- = (f_3^j)^-, \quad (f_3^j)^+ = (f_3^j)^+, \quad (f_4^j)^+ = (f_4^j)^+$

for $0 \leq j \leq k - 1$, where in each case the first condition comes from continuity across the solid contour, and the second from continuity across the dashed contour.
Similarly, a function \( f \in H^k(\Gamma) \) obeys the conditions
\[
(A.2) \quad (f^1_j)_- = (f^2_j)_- = (f^3_j)_-, \quad (f^3_j)_+ = (f^3_j)_+, \quad (f^4_j)_+ = (f^4_j)_+ \\
for 0 \leq j \leq k - 1.
\]
The space \( H^k_\pm(\Gamma) \) consists of those functions in \( H^k(\Gamma) \) which obey the following zero-sum conditions at the two intersection points \( \pm S_\infty \).
\[
(A.3) \quad (f^1_j)_+ + (f^4_j)_+ - (f^3_j)_+ - (f^3_j)_+ = 0 \\
(f^3_j)_+ + (f^4_j)_+ - (f^3_j)_+ - (f^4_j)_+ = 0
\]
where the \( \pm \) signs are determined by the orientation of the contour (see Figures A.1a and A.1b).

**Figure A.2. Zero-Sum Conditions**

\[
\begin{align*}
(a) \quad -S_\infty & \quad \Gamma_2 \\
\Gamma_1 & \quad \Gamma_4 \\
\Gamma_3 & \quad S_\infty \\
(b) \quad S_\infty & \quad \Gamma_5 \\
\Gamma_4 & \quad \Gamma_3
\end{align*}
\]

It is easy to see from \((A.1), (A.2), \) and \((A.3)\) that \( H^k_\pm(\Gamma) \subset H^k(\Gamma) \). In [17, Lemma 2.51] it is shown that if \( f \in H^k_\pm(\Gamma) \), then \( C^\pm f \in H^k_\pm(\Gamma) \). This mapping property is very natural since \( C^+ f \) (resp. \( C^- f \)) is the boundary value of a function analytic in \( \Omega_+ \) (resp. \( \Omega^- \)). From these facts, it follows that that
\[
H^k_\pm(\Gamma) = H^k_+ (\Gamma) + H^k_- (\Gamma)
\]
with the decomposition given by the Cauchy projectors: \( f = C^+_\Gamma (f) + C^-_\Gamma (f) \).

**Appendix B. The Continuity-Compactness Argument**

In this appendix we give the abstract functional-analytic argument needed to prove uniform resolvent estimates required in section 4 for the Lipschitz continuity of the inverse scattering map. Proposition B.1 can also be used to simplify proofs of analogous uniform estimates in [5, 12]. In what follows \( \mathcal{B}(X) \) denotes the Banach space of bounded operators on the Banach space \( X \).

**Proposition B.1.** Let \( X, Y, \) and \( Z \) be Banach spaces and suppose that there is a continuous embedding \( i : Z \to Y \) with the property that bounded subsets of \( Z \) map to precompact subsets of \( Y \). Suppose that \( C_{1,x} \) is a family of compact operators on a Banach space \( X \) indexed by \( J \in Y \) and \( x \in \mathbb{R} \). Finally, suppose that:
(i) The map \((J, x) \mapsto C_{J, x}\) is continuous as a map from \(Y \times \mathbb{R}\) into \(\mathcal{B}(X)\), and the estimate
\[
\sup_{x \in \mathbb{R}} \|C_{J, x} - C_{J', x}\|_{\mathcal{B}(X)} \lesssim \|J - J'\|_Y
\]
holds,
(ii) The resolvent \((I - C_{J, x})^{-1}\) exists for each \(x \in \mathbb{R}\) and \(J \in Y\), and
(iii) For each \(J \in Y\), the estimate
\[
\sup_{x \in [a, \infty)} \|(I - C_{J, x})^{-1}\|_{\mathcal{B}(X)} < \infty
\]
holds.

Then for any bounded subset \(B\) of \(Z\),
\[
\sup_{J \in B} \left( \sup_{x \in [a, \infty)} \|(I - C_{J, x})^{-1}\|_{\mathcal{B}(X)} \right) < \infty
\]
and the map
\[
J \mapsto \{x \mapsto (I - C_{J, x})^{-1}\}
\]
is locally Lipschitz continuous as a map from \(Z\) into \(C([a, \infty); \mathcal{B}(X))\).

Remark B.2. 1. In applications, (i) is easy to prove from the explicit form of the Beals-Coifman integral operators, (ii) follows from Fredholm theory and a vanishing theorem for the RHP, and (iii) follows from the continuity of the map
\[
x \mapsto (I - C_{J, x})^{-1}
\]
and the fact that, in the limit \(x \to \infty\), the integral kernel of the operator is highly oscillatory. 2. In applications, the bound in hypothesis (iii) is typically only true for half-lines. One can replace \([a, \infty)\) by \((-\infty, a]\) and obtain the same result.

Proof. Denote by \(C([a, \infty), \mathcal{B}(X))\) the Banach space of continuous \(\mathcal{B}(X)\)-valued functions of \(x\) equipped with the norm
\[
\|f\|_{C([a, \infty), \mathcal{B}(X))} = \sup_{x \in \mathbb{R}} \|f(x)\|_{\mathcal{B}(X)}.
\]

Consider the map
\[
Y \ni J \mapsto \{x \mapsto (I - C_{J, x})^{-1}\} \in C([a, \infty), \mathcal{B}(X)).
\]
Assumptions (i), (ii), (iii) and the second resolvent formula show that this map is well-defined and continuous. Using the injection \(i\) we can identify bounded subsets of \(Z\) with precompact subsets of \(Y\). We can then use the continuity of the map \((B.1)\) to conclude that the image of any bounded subset of \(Z\) has compact closure in \(C([a, \infty), \mathcal{B}(X))\) and hence is bounded. The local Lipschitz continuity now follows from the second resolvent formula.

**Appendix C. Spectral Singularities**

We establish the existence of a family of potentials \(q \in H^{2,2}(\mathbb{R})\) with arbitrary \(L^2\)-norm for which the forward spectral map generates any number of resonances and show explicitly how spectral singularities naturally emerge. The main result of the section is the following.
Proposition C.1. The scattering map can be explicitly computed for the family of potentials

\begin{equation}
q(x) = \nu \text{sech}(x)^{1-2i\mu} e^{i(S_0+\nu^2 \tanh(x)-2\delta x)}, \quad \|q\|_{L^2(\mathbb{R})}^2 = 2\nu^2,
\end{equation}

where \(\nu > 0\), and \(\mu, \delta, S_0 \in \mathbb{R}\). Let

\begin{equation}
\text{Re} \lambda \text{ is principally branched and maps (C.3)} \quad \text{where (C.4)}
\end{equation}

\begin{equation}
\alpha \lambda \text{ is principally branched and maps (C.6)} \quad \text{and (C.7)}
\end{equation}

\begin{equation}
\text{then for any integer } n \in \mathbb{N} \text{ if}
\end{equation}

- \(E(\delta, \mu, \nu) > -\frac{1}{4}\) then the discrete spectrum is empty.
- \(E(\delta, \mu, \nu) \in \left(-\left(n + \frac{1}{2}\right)^2, -\left(n - \frac{1}{2}\right)^2\right)\), then the discrete spectrum has exactly \(n\) resonances in \(\mathbb{C}^+\) and no spectral singularities.
- \(E(\delta, \mu, \nu) = -(n + \frac{1}{2})^2\) then the system has \(n\) resonances in \(\mathbb{C}^+\) and a single spectral singularity at \(\lambda = \delta = -\left(\frac{\mu}{\nu}\right)^2 - \left(n + \frac{1}{2}\right)^2\).

In our previous paper \([5]\) we showed that for any potential \(q\) in the family (C.1), the linear system (2.24) can be solved explicitly in terms of hypergeometric functions using ideas of Tovbis-Venakides \([16]\) and diFranco-Miller \([2]\). The result of that calculation gives

\begin{equation}
\bar{\alpha}(\lambda) = e^{\nu^2} \frac{\Gamma(c)(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}
\end{equation}

where

\begin{equation}
a = i\mu + i\nu\sqrt{-\varepsilon} R(\lambda), \quad b = i\mu - i\nu\sqrt{-\varepsilon} R(\lambda),
\end{equation}

\begin{equation}
c = -i\lambda + i(\mu + \delta) + \frac{1}{2}.
\end{equation}

and

\begin{equation}
R(\lambda) = \sqrt{\lambda + \left(\frac{\mu}{\nu}\right)^2}
\end{equation}

is principally branched and maps \(\mathbb{C}^+\) onto \(\mathbb{C}^{++}\).

From the formulas \(c = -i\lambda + i(\mu + \delta) + \frac{1}{2}\) and \(c-a-b = -i\lambda - i(\mu - \delta) + \frac{1}{2}\), it is easy to see that if \(\lambda \in \mathbb{R}\) then \(\text{Re}(c) = \text{Re}(c-a-b) = 1/2\), so that \(\alpha(\lambda)\) has no poles along the real line. Moreover, \(\text{Re}(c) > 1/2\) and \(\text{Re}(c-a-b) > 1/2\) for \(\lambda \in \mathbb{C}^+\) so that \(\bar{\alpha}(\lambda)\) is holomorphic in \(\mathbb{C}^+\) as required.

Proof of Proposition C.1. Recall that resonances are the zeros of \(\bar{\alpha}(\lambda)\) in \(\mathbb{C}^+\), and that spectral singularities are real zeros of \(\bar{\alpha}(\lambda)\). \(\bar{\alpha}(\lambda) = 0\) whenever \(c-a = 1-n\) or \(c-b = 1-n\). We have

\begin{equation}
c-a = -i\lambda + i\delta - i\nu R(\lambda) + \frac{1}{2}
\end{equation}

\begin{equation}
c-b = -i\lambda + i\delta + i\nu R(\lambda) + \frac{1}{2}.
\end{equation}

Notice that \(\text{Re} c-a = \text{Im} \lambda + \nu \text{Im}(R(\lambda)) > 0\) as \(R(\lambda) \in \mathbb{C}^{++}\) when \(\lambda \in \mathbb{C}^+\). Thus zeros of \(\bar{\alpha}(\lambda)\) must satisfy \(c-b = 1-n\) or equivalently

\begin{equation}
\lambda_n - \nu R(\lambda_n) = \delta - i(n-1/2), \quad \lambda_n \in \mathbb{C}^+ \quad n = 1, 2, 3, \ldots.
\end{equation}

The left hand side of (C.6), \(w = \lambda + R(\lambda)\), is a conformal map of \(\mathbb{C}^+\) onto the set

\begin{equation}
\mathcal{W} = \mathbb{C}^+ \cup \left\{ w \in \mathbb{C}^- : \text{Re}(w) < -\left(\frac{\mu}{\nu}\right)^2 - \text{Im}(w)^2 \right\}.
\end{equation}
It is clear from (C.6) that if $\delta > - (\mu / \nu)^2$ then $\tilde{\alpha}(\lambda)$ is zero-free. If $\delta \leq - (\mu / \nu)^2$, then for the right hand side of (C.6) to lie in the range of $\lambda + R(\lambda)$ requires that
\[
\delta < - \left( \frac{\mu}{\nu} \right)^2 - \left( n - \frac{1}{2} \right)^2
\]
which is equivalent to $E(\delta, \mu, \nu) < - (n - \frac{1}{2})^2$. For a fixed set of parameters $(\delta, \mu, \nu)$ there is a last integers $n$ for which the inequality holds and (C.6) can be inverted to find a unique resonance corresponding to each integer $k \leq n$. If it happens that $E(\delta, \mu, \nu) = - (n + \frac{1}{2})^2$ then in addition to the $n$ resonances, a spectral singularity is generated at the pre-image of $\delta - i(n + \frac{1}{2})$ on the real line at $\lambda = \delta$.

\[\square\]

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