On a successive property of strongly starlikeness for multivalent functions

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Abstract

For $f$ analytic in the unit disk $\mathbb{D}$, of the form $f(z) = z^p + \cdots$, we consider some consequences of strongly starlikeness of $f^{(p-1)}(z)/p!$.

Keywords Starlike · Strongly starlike · Multivalent

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1 Introduction

We denote by $\mathcal{H}$ the class of functions $f(z)$ which are holomorphic in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Denote by $\mathcal{A}_p$, $p \in \mathbb{N} = \{1, 2, \ldots\}$, the class of functions $f(z) \in \mathcal{H}$ given by

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (z \in \mathbb{D}). \quad (1.1)$$

Lemma 1.1 [2, Theorem 5] If $f(z) \in \mathcal{A}_p$, then for all $z \in \mathbb{D}$, we have

$$\Re \left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} > 0 \quad \Rightarrow \quad \forall k \in \{1, \ldots, p-1\} : \quad \Re \left\{ \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} \right\} > 0. \quad (1.2)$$

In this paper we consider a generalization of the above result. In Lemma 1.1 we have assumed that $zf^{(p)}(z)/f^{(p-1)}(z)$ lies in the right half-plane while in this paper we work with a sector. The problem we solve here is: for what values of $\alpha$, $\beta$ does an analytic function of the form (1.1) satisfy...
\[ \arg \left\{ \frac{zf((p-1)(z))}{f(p-1)(z)} \right\} < \frac{\pi \alpha}{2} \Rightarrow \forall k \in \{1, \ldots, p-1\} : \arg \left\{ \frac{zf((k)(z))}{f(k-1)(z)} \right\} < \frac{\pi \beta}{2} \]

Recall that if \( f(z) \in \mathcal{A}_p \) and

\[ \Re \left\{ \frac{zf(p)(z)}{f(p-1)(z)} \right\} > 0 \quad (z \in \mathbb{D}), \]

then \( f^{(p-1)}(z)/p! \in \mathcal{A}_1 \) is univalent in \( \mathbb{D} \) and \( f^{(p-1)}(z)/p! \) is called a starlike function. If \( f(z) \in \mathcal{A}_p, \gamma \in (0, 1) \), and

\[ \arg \left\{ \frac{zf(p)(z)}{f(p-1)(z)} \right\} < \frac{\pi \gamma}{2}, \quad z \in \mathbb{D}, \quad (1.3) \]

then \( f^{(p-1)}(z)/p! \) is called a strongly starlike function of order \( \gamma \) and such functions we consider in the paper. This class for the case \( p = 1 \) was introduced by Brannan and Kirwan [1]. Also, if \( f(z) \in \mathcal{A}_p \) satisfies (1.3), then \( f(z) \) is called \( p \)-valently strongly starlike function of order \( \gamma \). For the proof of main result we need the following lemma.

**Lemma 1.2** [3] Let \( q(z) = 1 + \sum_{n \geq m} c_n z^n, c_m \neq 0 \) be analytic function in \( |z| < 1 \) with \( q(0) = 1, q(z) \neq 0 \). If there exists a point \( z_0, |z_0| < 1 \), such that

\[ |\arg \{q(z_0)\}| < \frac{\pi \beta}{2}, \quad z \in \mathbb{D}, \]

then \( q(z_0)/z \) is analytic in \( |z| < |z_0| \), and

\[ |\arg \{q(z_0)\}| = \frac{\pi \beta}{2} \]

for some \( \beta > 0 \), then we have

\[ \frac{zq'(z_0)}{q(z_0)} = \frac{2ik \arg \{q(z_0)\}}{\pi}, \]

for some \( k \geq m(a + a^{-1})/2 \geq m \), where

\( \{q(z_0)\}^{1/\beta} = \pm ia, \quad \text{and} \quad a > 0. \)

**2 Main results**

For given \( 0 < \beta_{s-1} \leq 1 \) let us consider the number

\[ \beta_s = \beta_{s-1} + \frac{2}{\pi} \tan^{-1} \frac{\beta_{s-1} n(\beta_{s-1}) \sin[\pi(1 - \beta_{s-1})/2]}{sm(\beta_{s-1}) + \beta_{s-1} n(\beta_{s-1}) \cos[\pi(1 - \beta_{s-1})/2]}, \quad s = 2, 3, \ldots, p, \]  

(2.1)

where

\[ m(\beta_{s-1}) = (1 + \beta_{s-1})^{(1 + \beta_{s-1})/2}, \quad \text{and} \quad n(\beta_{s-1}) = (1 - \beta_{s-1})^{(1 - \beta_{s-1})/2}. \]  

(2.2)

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Notice that if \( 0 < \beta_{s-1} \leq 1 \), then \( 0 < \beta_s \leq 1 \) too because from (2.1), (2.2), we have
\[
\beta_s = \beta_{s-1} + \frac{2}{\pi} \tan^{-1} \frac{\beta_{s-1} n(\beta_{s-1}) \sin[\pi (1 - \beta_{s-1})/2]}{\beta_{s-1} n(\beta_{s-1}) + \beta_{s-1} n(\beta_{s-1}) \cos[\pi (1 - \beta_{s-1})/2]},
\]
\[
\leq \beta_{s-1} + \frac{2}{\pi} \tan^{-1} \frac{\beta_{s-1} n(\beta_{s-1}) \sin[\pi (1 - \beta_{s-1})/2]}{\beta_{s-1} n(\beta_{s-1}) \cos[\pi (1 - \beta_{s-1})/2]},
\]
\[
= 1.
\]

Therefore, if we have a number \( \beta_1 \in (0, 1) \), then from (2.1), we can find a sequence \( \beta_p, \beta_{p-1}, \ldots, \beta_2, \beta_1 \), such that
\[
0 < \beta_1 \leq \beta_2 \leq \cdots \leq \beta_{p-1} \leq \beta_p \leq 1. \tag{2.3}
\]

**Theorem 2.1** Let \( f(z) \in A_p, p \geq 2 \). For given \( \beta_{p-1} \in (0, 1) \) there exists \( \beta_p \in (0, 1) \) of the form (2.1) such that for all \( z \in \mathbb{D} \), we have
\[
\left| \arg \left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} \right| < \frac{\pi \beta_p}{2} \Rightarrow \left| \arg \left\{ \frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)} \right\} \right| < \frac{\pi \beta_{p-1}}{2}. \tag{2.4}
\]

**Proof** Let us put
\[
q_1(z) = \frac{zf^{(p-1)}(z)}{2f^{(p-2)}(z)}, \quad q_1(0) = 1.
\]
Then it follows that
\[
\frac{zq_1'(z)}{q_1(z)} = 1 + \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} - \frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}
\]
and
\[
2q_1(z) + \frac{zq_1'(z)}{q_1(z)} = 1 + \frac{zf^{(p)}(z)}{f^{(p-1)}(z)}
\]
and so
\[
\arg \{q_1(z)\} + \arg \left\{ 2 + \frac{zq_1'(z)}{q_1(z)} \right\} = \arg \left\{ 1 + \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\}.
\]

If there exists a point \( z_0 \in \mathbb{D} \) such that
\[
|\arg\{q_1(z)\}| < \pi \beta_{p-1}/2 \quad \text{for} \quad |z| < |z_0|, \quad |\arg\{q_1(z_0)\}| = \pi \beta_{p-1}/2,
\]
\[
\{q_1(z_0)\}^{1/\beta_{p-1}} = \pm ia, \quad \text{and} \quad a > 0,
\]
then from Lemma 1.2, we have
\[
\frac{z_0q_1'(z_0)}{q_1(z_0)} = \frac{2ik \arg\{q_1(z_0)\}}{\pi} \tag{2.5}
\]
for some real \( k \) with \( k \geq (a + a^{-1})/2 \geq 1 \). For the case \( \arg\{q_1(z_0)\} = \pi \beta_{p-1}/2 \), we have
\[
\arg\left\{ \frac{z_0f^{(p)}(z_0)}{f^{(p-1)}(z_0)} \right\} \geq \arg\left\{ 1 + \frac{z_0f^{(p)}(z_0)}{f^{(p-1)}(z_0)} \right\}
\]
\[
= \arg\{q_1(z_0)\} + \arg\left\{ 2 + \frac{z_0q_1'(z_0)}{q_1(z_0)} \frac{1}{q_1(z_0)} \right\}
\]
\[
\geq \frac{\pi \beta_{p-1}}{2} + \arg\left\{ 2 + e^{i\pi(1-\beta_{p-1})/2} \frac{1}{(ia)^{\beta_{p-1}}} \right\}.
\]
where \( q_1(z_0) \) is a positive real number. Applying Lemma 1.2 we obtain
\[
\arg \left\{ \frac{z_0f^{(p)}(z_0)}{f^{(p-1)}(z_0)} \right\} \geq \frac{\pi \beta_{p-1}}{2} + \arg \left\{ e^{i\pi(1-\beta_{p-1})/2} \left( \frac{1 + \beta_{p-1}}{1 - \beta_{p-1}} \right)^{(1-\beta_{p-1})/2} \right\} + 2
\]
\[
= \frac{\pi \beta_{p-1}}{2} + \tan^{-1} \frac{\beta_{p-1}}{2 + \beta_{p-1}} \left( \frac{1 - \beta_{p-1}}{1 + \beta_{p-1}} \right)^{(1-\beta_{p-1})/2} \sin \frac{\pi(1-\beta_{p-1})}{2} \cos \frac{\pi(1-\beta_{p-1})}{2}
\]
\[
= \frac{\pi \beta_{p-1}}{2} + \tan^{-1} \frac{\beta_{p-1}n(\beta_{p-1})}{2m(\beta_{p-1}) + \beta_{p-1}n(\beta_{p-1})} \sin \frac{\pi(1-\beta_{p-1})}{2} \frac{\pi(1-\beta_{p-1})}{2}.
\]

From (2.1), we can see that
\[
\arg \left\{ \frac{z_0f^{(p)}(z_0)}{f^{(p-1)}(z_0)} \right\} \geq \frac{\pi \beta_{p}}{2}. \tag{2.6}
\]

This contradicts hypothesis in (2.4).

For the case \( \arg q_1(z_0) = -\pi \beta_{p-1}/2 \), applying the same method as the above, gives
\[
\arg \left\{ \frac{z_0f^{(p)}(z_0)}{f^{(p-1)}(z_0)} \right\} \leq -\frac{\pi \beta_{p}}{2}. \tag{2.7}
\]

This also contradicts hypothesis in (2.4) and therefore, we have
\[
|\arg q_1(z)| < \pi \beta_{p-1}/2 \quad \text{for} \quad |z| < |1|.
\]

This completes the proof. \(\square\)

Let us go to next step and define the function
\[
q_2(z) = \frac{zf^{(p-2)}(z)}{3f^{(p-3)}(z)}, \quad q_2(0) = 1
\]

and applying the same method as the above, we have the following theorem.

**Theorem 2.2** Let \( f(z) \in A_p, \ p \geq 2, \ 0 < \beta_2 \leq 1 \) and suppose that
\[
\left| \arg \left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} \right| < \frac{\pi \beta_{p}}{2}, \quad z \in \mathbb{D}. \tag{2.8}
\]

Then we have
\[
\left| \arg \left\{ \frac{zf^{(p-2)}(z)}{f^{(p-3)}(z)} \right\} \right| < \frac{\pi \beta_{p-2}}{2}, \quad z \in \mathbb{D}. \tag{2.9}
\]

where \( \beta_{p-2} \) we obtain from \( \beta_{p-1} \) using formula (2.1). Furthermore,
and where $\beta_{p-1}$ we obtain from $\beta_p$ using formula (2.1) too.

Applying the same step as the above and under the hypothesis of Theorem 2.1, we have the following theorem

**Theorem 2.3** Let $f(z) \in A_p$, $p \geq 2$. For given $\beta_1 \in (0, 1]$ there exist $\beta_k \in (0, 1]$, $k = 2, \ldots, p$, of the form (2.1) such that for all $z \in \mathbb{D}$, we have

$$\left| \arg \left\{ \frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)} \right\} \right| < \frac{\pi \beta_p}{2}, \quad z \in \mathbb{D}, \quad (2.10)$$

Furthermore

$$0 < \beta_1 \leq \beta_2 \leq \cdots \leq \beta_{p-1} \leq \beta_p \leq 1.$$  

It is easy to see that Theorem 2.3 holds for the case $\beta_p = \beta_{p-1} = \cdots = \beta_1 = 1$ and then Theorem 2.3 becomes Lemma 1.1 and in this sense Theorem 2.3 improves Lemma 1.1.

**Corollary 2.4** Let $f(z) \in A_p$, $p \geq 2$. If $\beta_1 \in (0, 1]$ and $\beta_k \in (0, 1]$, $k = 2, \ldots, p$ are of the form (2.1), then for all $k = 1, \ldots, p - 1$ and for all $z \in \mathbb{D}$, we have

$$\left| \arg \left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} \right| < \frac{\pi \beta_k}{2} \quad \Rightarrow \quad \left| \arg \left\{ \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} \right\} \right| < \frac{\pi \beta_k}{2}.$$  

**Proof** For given $\beta_1 \in (0, 1]$ there exist $\beta_k \in (0, 1]$, $k = 2, \ldots, p$, of the form (2.1) such that

$$\left| \arg \left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} \right| < \frac{\pi \beta_p}{2} \quad \Rightarrow \quad \left| \arg \left\{ \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} \right\} \right| < \frac{\pi \beta_k}{2},$$

where

$$0 < \beta_1 \leq \beta_2 \leq \cdots \leq \beta_{p-1} \leq \beta_p \leq 1. \quad (2.11)$$

Therefore, from (2.11), we have

$$\left| \arg \left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} \right| < \frac{\pi \beta_p}{2} \quad \Rightarrow \quad \left| \arg \left\{ \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} \right\} \right| < \frac{\pi \beta_k}{2}.$$  

**Corollary 2.5** If $f(z) \in A_p$, $p \geq 2$, then for all $\gamma \in (0, 1]$ and for all $k \in \{1, \ldots, p\}$ and for all $s \in \{k, \ldots, p - 1\}$, and for all $z \in \mathbb{D}$, we have

$$\left| \arg \left\{ \frac{zf^{(s)}(z)}{f^{(s-1)}(z)} \right\} \right| < \frac{\pi \gamma}{2} \quad \Rightarrow \quad \left| \arg \left\{ \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} \right\} \right| < \frac{\pi \gamma}{2}.$$
Corollary 2.6 Let \( f(z) \in A_p, p \geq 2, 0 < \beta_p \leq 1 \) and suppose that
\[
\left| \arg \left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} \right| < \frac{\pi \beta_p}{2}, \ z \in \mathbb{D}.
\]

Then we have
\[
\sup_{z \in \mathbb{D}} \left| \arg \left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} \right| > \sup_{z \in \mathbb{D}} \left| \arg \left\{ \frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)} \right\} \right| > \cdots > \sup_{z \in \mathbb{D}} \left| \arg \left\{ \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} \right\} \right|.
\]

Theorem 2.7 Let \( \beta = \alpha + \left( \frac{2}{\pi} \right) \tan^{-1} \alpha \) and \( f(z) \in A_p, p \geq 2 \). Suppose also that
\[
\left| \arg \left\{ f^{(p)}(z) \right\} \right| < \frac{\pi \beta_p}{2}, \ z \in \mathbb{D}, \quad (2.12)
\]

Then we have
\[
\left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi \beta_1}{2}, \ z \in \mathbb{D}, \quad (2.13)
\]

where \( \beta_1 \) is described in (2.1) with \( \beta_p = \alpha + \beta \).

Proof If
\[
\left| \arg \left\{ \frac{f^{(p-1)}(z)}{z} \right\} \right| < \frac{\pi \alpha}{2} \quad (2.14)
\]

in \( |z| < |z_0| \) and
\[
\arg \left\{ \frac{f^{(p-1)}(z_0)}{z_0} \right\} = \frac{\pi \alpha}{2} \quad \text{or} \quad \arg \left\{ \frac{f^{(p-1)}(z_0)}{z_0} \right\} = -\frac{\pi \alpha}{2}, \quad (2.15)
\]

then for the first case in (2.15), from Lemma 1.2, we have
\[
\frac{z_0 f^{(p)}(z_0)}{f^{(p-1)}(z_0)} - 1 = ik\alpha
\]

for some \( k \geq 1 \). This gives
\[
\arg \left\{ f^{(p)}(z_0) \right\} = \arg \left\{ \frac{f^{(p-1)}(z_0)}{z_0} (ik\alpha + 1) \right\}
\]
\[
= \arg \left\{ \frac{f^{(p-1)}(z_0)}{z_0} \right\} + \arg \{ik\alpha + 1\}
\]
\[
\geq \frac{\pi}{2} \left\{ \alpha + \left( \frac{2}{\pi} \right) \tan^{-1} \alpha \right\} = -\frac{\pi \beta}{2}.
\]

This contradicts hypothesis (2.12). In the second case in (2.15), applying the same method as in the first case, we obtain
\[
\arg \left\{ f^{(p)}(z_0) \right\} \leq -\frac{\pi}{2} \left\{ \alpha + \left( \frac{2}{\pi} \right) \tan^{-1} \alpha \right\} = \frac{\pi \beta}{2}.
\]
This also contradicts hypothesis (2.12). So (2.14) holds in the whole unit disc $\mathbb{D}$. From (2.12) and (2.14), we have

$$\left| \arg \left\{ \frac{zf'(p)(z)}{f'(p-1)(z)} \right\} \right| = \left| \arg \left\{ \frac{f'(p)(z)z}{f'(p-1)(z)} \right\} \right|$$

$$\leq \left| \arg \left\{ f'(p)(z) \right\} \right| + \left| \arg \left\{ \frac{f'(p-1)(z)}{z} \right\} \right|$$

$$< \frac{\pi(\alpha + \beta)}{2}, \quad z \in \mathbb{D}.$$ 

Applying Theorem 2.3, we obtain (2.13). $\square$

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