Probability Distribution of Curvatures of Isosurfaces in Gaussian Random Fields

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Abstract

An expression for the joint probability distribution of the principal curvatures at an arbitrary point in the ensemble of isosurfaces defined on isotropic Gaussian random fields on $\mathbb{R}^n$ is derived. The result is obtained by deriving symmetry properties of the ensemble of second derivative matrices of isotropic Gaussian random fields akin to those of the Gaussian orthogonal ensemble.

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I. INTRODUCTION

We closely follow the notation in [1] and [2]. Let $T$ be a tensor with rank $a$ and dimensions $(d_1, \ldots, d_a)$. The bijective linear map vec associates $T$ to the vector $\text{vec}(T)$ in $\mathbb{R}^{N}$, $N = \prod_{i=1}^{a} d_i$, with entry $(\text{vec}(T))_k$, $k \in \{1, \ldots, \prod_{i=1}^{a} d_i\}$, given by $(\text{vec}(T))_k = T_{i_1, \ldots, i_a}$ with $i_j \in \{1, \ldots, d_j\}$ uniquely defined by $k = 1 + \sum_{j=1}^{a} (i_j - 1) \prod_{i=1}^{j-1} d_i$. Let $d_i = n$, $i = 1, \ldots, a$. The linear operator diag maps $T$ to the vector $\text{diag}(T)$ in $\mathbb{R}^n$ with entry $(\text{diag}(T))_k$, $k \in \{1, \ldots, n\}$, given by $T_{i,\ldots,i}$ uniquely defined by $k = (j-1)n - j(j-1)/2 + i$. This operator maps a matrix $T$ to a vector containing the entries of $T$ read along its columns, but ignoring the elements above the main diagonal.

Following the notation in [1], the $(n, n)$ identity matrix is denoted by $I_n$, $C_n$ denotes the $(n^2, n^2)$ commutation matrix, i.e., the unique $(n^2, n^2)$ matrix such that $\text{vec}(T^T) = C_n \text{vec}(T)$ for all $(n, n)$ matrices $T$. The matrix $D_n$ denotes the $(n^2, (n(n+1))/2)$ duplication matrix, i.e., the unique $(n^2, (n(n+1))/2)$ matrix such that $\text{vec}(T) = D_n \text{uni}(T)$ for all $(n, n)$ symmetric matrices $T$. From this definition, we have $\text{uni}(T) = D_n^+ \text{vec}(T)$, where $A^+$ indicates the Moore-Penrose inverse of the matrix $A$. The duplication matrix and the commutation matrix are related through the identity $D_n D_n^+ = (1/2)(I_{n^2} + C_n)$. Finally, $I_{n,m}$ denotes an $(n, m)$ matrix of ones.

A random scalar field in the set $X$ is a stochastic process, i.e., an indexed set $\mathcal{F}_X = \{\mathcal{F}_x, x \in X\}$ of random variables $\mathcal{F}_x$ defined over the same probability space $(\Omega, \sigma_\Omega, P)$. A random tensor fields is defined analogously, with $\mathcal{F}_x$ a vector-valued function.

Let $\mathcal{F}_x^1$ and $\mathcal{F}_y^2$ be two random scalar fields as above, and assume that $\mathcal{F}_x^1$ is zero mean, which, for the purposes of this work, implies no loss of generality. If the expectation $E\{\mathcal{F}_x^1 \mathcal{F}_y^2\}$ taken over the joint distribution of $\mathcal{F}_x^1$ and $\mathcal{F}_y^2$ is defined for all $x$ and $y$ in $X$, the function $R_{\mathcal{F}_x^1, \mathcal{F}_y^2}(x,y) = E\{\mathcal{F}_x^1 \mathcal{F}_y^2\}$ defines the cross-covariance function of the random fields. If $\mathcal{F}_x^1 = \mathcal{F}_x^2 = \mathcal{F}_x$, the notation is simplified to $R_{\mathcal{F}} \triangleq R_{\mathcal{F}, \mathcal{F}}$, and the function $R_{\mathcal{F}}$ is referred to as the autocovariance function of $\mathcal{F}_x$. The conditional autocovariance function of $\mathcal{F}_x^1$ given $\mathcal{F}_y^2$, $R_{\mathcal{F}_x^1|\mathcal{F}_y^2}$, is defined by taking the expectation of $\mathcal{F}_x^1$ over the conditional distribution of $\mathcal{F}_x^1$ and $\mathcal{F}_y^1$ given $\mathcal{F}_y^2$. In the case of a random tensor field the definitions are analogous, with the product $\mathcal{F}_x^1 \mathcal{F}_y^2$ replaced by a tensor product $\text{vec}(\mathcal{F}_x^1) \otimes \text{vec}(\mathcal{F}_y^2)^T$ and the expectation taken over each entry of the tensor. Henceforth, the term “random field” will be used in reference to both scalar and tensorial random fields.
Let \( X \) be a vector space. Zero-mean random fields \( \mathcal{F}^1_X \) and \( \mathcal{F}^2_X \) on \( X \) are wide-sense stationary if their cross-covariance function \( R_{\mathcal{F}^1_X,\mathcal{F}^2_X}(x,y) \) satisfies \( R_{\mathcal{F}^1_X,\mathcal{F}^2_X}(x,y) = R_{\mathcal{F}_X}(s) \) with \( s = x - y \) for all \( x \) and \( y \) in \( X \). Henceforth we will assume that \( X = \mathbb{R}^n \), and therefore the subscript \( X \) in \( \mathcal{F}_X \) can then be safely omitted by defining \( \mathcal{F} \equiv \mathcal{F}_{\mathbb{R}^n} \). A stationary random field on \( \mathbb{R}^n \) is isotropic if its autocovariance function \( R_F(s) \) satisfies \( R_F(s) = \sigma^2 \rho(||s||) \), where \( \rho \) is a correlation function [2] and \( ||\cdot|| \) is the standard Euclidean norm in \( \mathbb{R}^n \).

II. DERIVATIVES OF ISOTROPIC RANDOM FIELDS

Great simplification is achieved in the derivations that follow if \( \rho(||s||) \) can be rewritten as \( \rho(||s||) = r(||s||)/\sigma^2 \Leftrightarrow \rho(\sqrt{||s||}) = r(||s||)/\sigma^2 \). For \( ||s|| > 0 \) the smoothness of \( r(||s||) \) is contingent upon that of \( \rho(||s||) \). However, for \( ||s|| = 0 \) the non-differentiability of \( \sqrt{||s||} \) could be a problem. This is not the case, as shown in appendix. The symbols \( \rho^{(i)}_0 \) and \( r^{(i)}_0 \) denotes the \( i \)-th derivative of \( \rho(||s||) \) and \( r(||s||) \) with respect to \( s = ||s|| \) at \( s = 0 \).

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a scalar function. The symbol \( \frac{\partial^{a+b} f}{\partial x^a \partial x^b} \) is used to describe the matrix of dimensions \((n^a, n^b)\) of partial derivatives of \( f \), i.e.,

\[
\left( \frac{\partial^{a+b} f}{\partial x^{a_1} \partial x^{b_1}} \right)_{A,B} \triangleq \frac{\partial^{a+b} f}{\partial x_{a_1} \ldots \partial x_{a_i} \partial x_{b_1} \ldots \partial x_{b_j}},
\]

where \( a_i \in \{1, 2, \ldots, n\} \) and \( b_j \in \{1, 2, \ldots, n\} \) are uniquely defined by \( A = 1 + \sum_{i=1}^n (a_i - 1)n^{i-1} \) and \( B = 1 + \sum_{j=1}^n (b_j - 1)n^{j-1} \). Second differentiability of the autocovariance function \( R_F(x,y) \) of a random field at the pair \((x, x)\) implies mean square differentiability of the random field itself, as demonstrated in theorem 2.4 of [2]. If a mean-square differentiable random field is stationary, its derivatives will also be stationary. For \( \mathcal{F}^1 \) and \( \mathcal{F}^2 \) jointly stationary with cross-covariance \( R_{\mathcal{F}^1,\mathcal{F}^2}(s) \), we define \( R^{(0)}_{\mathcal{F}^1,\mathcal{F}^2} \equiv R_{\mathcal{F}^1,\mathcal{F}^2}(0) \).

The theorem that follows is central to this work:

**Theorem 1.** Let \( \mathcal{F} \) be an isotropic random field on \( \mathbb{R}^n \) with autocovariance function \( R_F(s) = \sigma^2 \rho(||s||) \) where \( \rho(s) \) is four-differentiable. Let \( \partial \mathcal{F} \) and \( \partial^2 \mathcal{F} \) be the tensor fields defined as

\[
\partial \mathcal{F} \triangleq \frac{\partial \mathcal{F}}{\partial x^i} \quad \text{and} \quad \partial^2 \mathcal{F} \triangleq \frac{\partial^2 \mathcal{F}}{\partial x^i \partial x^j}.
\]
Then

\[ R_{\varphi}^0 = -\sigma^2 \rho_0^{(2)} I_n, \]  
(1a)

\[ R_{\varphi, \varphi^2}^0 = O_{n, n^2}, \]  
(1b)

\[ R_{\varphi^2}^0 = \sigma^2 \rho_0^{(4)} (I_n^2 + C_n + \text{vec } I_n \text{ vec }^T I_n). \]  
(1c)

**Proof.** Using lemma 4 in appendix we write \( \rho(||s||) = r(||s||^2) / \sigma^2 \). From the four-differentiability of \( r \) and theorem 2.4 in [2] we have

\[
R_{\varphi}(s) = -\frac{\partial^2 r(||s||^2)}{\partial s^T \partial s}, \tag{2a}
\]

\[
R_{\varphi, \varphi^2}(s) = -\frac{\partial^3 r(||s||^2)}{\partial s^T \partial s^2}, \tag{2b}
\]

\[
R_{\varphi^2}(s) = \frac{\partial^4 r(||s||^2)}{\partial s^2 \partial s^2}. \tag{2c}
\]

The chain rule can be used to expand \( (2a) \)–\( (2c) \), and substituting the identities \( \frac{\partial x}{\partial x} = \frac{\partial x^T}{\partial x} = I_n \), \( \frac{\partial (x^T \otimes I_n)}{\partial x} = I_n \otimes I_n \), and \( \frac{\partial (I_n \otimes x^T)}{\partial x} = C_n \) in the result produces

\[
R_{\varphi}(s) = -4r^{(2)}(||s||^2) s \otimes s^T - 2r^{(1)}(||s||^2) I_n,
\]

\[
R_{\varphi, \varphi^2}(s) = -4\{2r^{(3)}(||s||^2) s \otimes s^T \otimes s^T + r^{(2)}(||s||^2)(s^T \otimes I_n + I_n \otimes s^T + s^T \otimes C_n + C_n \otimes s^T I_n)\},
\]

\[
R_{\varphi^2}(s) = 4\{8r^{(4)}(||s||^2) s \otimes s \otimes s^T \otimes s^T + 8r^{(3)}(||s||^2)(\text{vec } I_n \otimes s^T \otimes s^T + 2s \otimes (I_n \otimes s^T + s^T \otimes I_n) + s \otimes (s \otimes \text{vec } I_n^T) + r^{(2)}(||s||^2) (I_n \otimes I_n + C_n + \text{vec } I_n \otimes \text{vec }^T I_n)\}.
\]

Making \( s = 0 \) completes the proof. \( \square \)

### III. CURVATURES OF GAUSSIAN RANDOM FIELDS

Let \( f(x) \) be a second-differentiable scalar function on \( \mathbb{R}^n \). For each \( x \in \mathbb{R}^n \) for which \( \partial f / \partial x \neq 0 \) we define the set \( F_x \) as \( F_x = \{ y \in \mathbb{R}^n \text{ such that } f(y) = f(x) \text{ and } \partial f / \partial y \neq 0 \} \). If \( F_x \neq \emptyset \), \( F_x \) is a \((n - 1)\)-hypersurface in \( \mathbb{R}^n \) [3]. Let \( \partial f \equiv \partial f / \partial x^T \) and \( \partial^2 f \equiv \partial^2 f / \partial x^T \partial x \). The principal
curvatures of the hypersurface $F_x$ at $x$ are given by the set of eigenvalues $\kappa$ obtained by solving the eigenproblem

$$
- \left( I_n - \frac{\partial f \otimes (\partial f)^T}{\|\partial f\|^2} \right) \frac{\partial^2 f}{\|\partial f\|_x} v = \kappa v,
$$

with $v \in \mathbb{R}^n$, $\|v\| = 1$ and $v^T \partial f = 0$ [4, pg. 138]. Let $(n_i, i = 1, \ldots, n - 1)$ be an orthonormal basis for the null-space of $\partial f$, i.e., $n_i^T \partial f = 0$ and $n_i^T n_j = \delta_{ij}$, and let $N$ be the matrix $N = [n_1 \ldots n_{n-1}]$.

The eigenproblem in (4) can be rewritten as

$$
- \frac{N^T (\partial^2 f) N}{\|\partial f\|} u = \kappa u,
$$

with $u \in \mathbb{R}^{n-1}$, $\|u\| = 1$. Equation (5) is still valid if the function $f$ is the realization $F(\omega)$, $\omega \in \Omega$, of a mean-square second-differentiable scalar random field $F$ on $\mathbb{R}^n$. Therefore the random tensor field $\mathcal{K}$ of curvatures of isotropic Gaussian random fields is implicitly defined at $x$ such that $\partial F_x(\omega) \neq 0$ by the solutions of the equation

$$
- \frac{N^T (\partial^2 F) N}{\|\partial F\|} U = \mathcal{K} U,
$$

where $N$ is a random tensor field satisfying $N^T_x \partial F_x = 0$, and $N^T_x N_x = I_{n-1}$, and $U$ is the tensor field such that $U_x(\omega)$ are the eigenvectors associated to the eigenvalues $\mathcal{K}_x(\omega)$.

Henceforth we assume that $F$ is an isotropic, second-differentiable Gaussian random field, which is defined simply as an isotropic random field for which the joint distribution of any finite set of random variables $\{F_A\}$, $A \in \mathbb{R}^n$, is Gaussian. This assumption implies that the zero-mean random tensors $\partial F$ and $\partial^2 F$ are also Gaussian, and therefore $\partial F_x$ and $\partial^2 F_x$ are fully characterized by their covariance matrices, given by (1a) and (1c) in theorem 1. However, because $\partial^2 F$ is symmetric, its probability density must be handled with care, since $R^0_{\partial^2 F}$ is not invertible.

The following lemma is a trivial corollary of the theorems in [1, sec. 7].

**Lemma 1.** Let $A_n$ be a $(n,n)$ invertible matrix. Then

$$
(A_n \otimes A_n) R^0_{\partial^2 F} (A_n^{-1} \otimes A_n^{-1}) = R^0_{\partial^2 F}.
$$

Let $\partial^2 F$ be as in theorem 1, and let $R$ be a $(n,m)$, $n \geq m$, orthonormal tensor field independent of $\partial^2 F$, i.e., a random tensor field such that for all $x \in \mathbb{R}^n$ any realization $R(\omega)$ of $R$ satisfies $R^T_x(\omega) R_x(\omega) = I_m$ and $R_x$ is independent of $\partial^2 F_x$. Define $\partial^2 F' \triangleq R^T \partial^2 F R = \{R^T_x \partial^2 F_x R_x, x \in \mathbb{R}^n\}$. We prove the following lemma:
Lemma 2. \( \partial^2 \mathcal{F}' \) is a Gaussian random tensor field with autocovariance function \( R_{\partial^2 \mathcal{F}'}(s) \) such that

\[
R_{\partial^2 \mathcal{F}'}^0 = \sigma^2 \rho_0^{(4)}(I_m + C_m + \text{vec} I_m \text{vec}^T I_m).
\]  

(8)

Proof. Let \( P_{\partial^2 \mathcal{F}'|\mathcal{R}_x} \) be the conditional probability of the random tensor \( \partial^2 \mathcal{F}' \) given \( \mathcal{R}_x \). Since \( \partial^2 \mathcal{F}_x \) is independent of \( \mathcal{R}_x \), \( \partial^2 \mathcal{F}'_x \) given \( \mathcal{R}_x \) is a linear function of \( \partial^2 \mathcal{F}_x \), and therefore it is zero-mean Gaussian. Using the identity \( \text{vec} (ABC) = (CT \otimes A) \text{vec} B \) and properties of commutator matrices [1] we can write \( \text{vec} \partial^2 \mathcal{F}'_x = (\mathcal{R}_x^T \otimes \mathcal{R}_x^T) \text{vec} \partial^2 \mathcal{F}_x \), and therefore

\[
R_{\partial^2 \mathcal{F}'|\mathcal{R}_x}^0 = (\mathcal{R}_x^T \otimes \mathcal{R}_x^T)R_{\partial^2 \mathcal{F}}^0(\mathcal{R}_x \otimes \mathcal{R}_x)
\]

(9)

\[
= \sigma^2 \rho_0^{(4)}(I_m + C_m + \text{vec} I_m \text{vec}^T I_m),
\]

(10)

using lemma 1. Since \( \partial^2 \mathcal{F}'_x \) given \( \mathcal{R}_x \) is zero-mean Gaussian and \( R_{\partial^2 \mathcal{F}'|\mathcal{R}_x}^0 \) does not depend on \( \mathcal{R}_x \) for fixed \( m \) and \( n \), we have \( P_{\partial^2 \mathcal{F}'|\mathcal{R}_x} = P_{\partial^2 \mathcal{F}'_x} \). Therefore \( P_{\partial^2 \mathcal{F}'_x} \) is zero-mean Gaussian with autocovariance function satisfying (8).

Lemma 2 justifies the notation \( \partial^2 \mathcal{F}^m \doteq \mathcal{R}^T \partial^2 \mathcal{F} \mathcal{R} \), for \( \mathcal{R} (n,m) \). Let \( \delta^2 \mathcal{F}^n \) be the \((n(n + 1)/2, 1)\) vector defined as \( \delta^2 \mathcal{F}^n = D_n^x \text{vec} \partial^2 \mathcal{F}^n \). Its covariance matrix \( \Sigma_n \) is invertible and given by

\[
\Sigma_n = D_n^x R_{\partial^2 \mathcal{F}^n}^0 D_n^x \text{vec}^T.
\]

(11)

Therefore the probability density \( p_{\partial^2 \mathcal{F}^n} \) of \( \delta^2 \mathcal{F}^n \) is standard:

\[
p_{\partial^2 \mathcal{F}^n}(h) = \frac{1}{\sqrt{2\pi \Sigma_n}} \exp \left( -\frac{h^T \Sigma_n^{-1} h}{2} \right).
\]

(12)

We now define the random fields \((\mathcal{R}, \mathcal{L}) \doteq \{(\mathcal{R}_x, \mathcal{L}_x) \in SO(n) \times \mathbb{R}^n, x \in \mathbb{R}^n | \mathcal{R}_x^T \mathcal{R}_x = I_n \) and \( \mathcal{R}_x^T \text{diag}^{-1} \mathcal{L}_x \mathcal{R}_x^T = \partial^2 \mathcal{F}_x^n \}, \) where \( SO(n) \) is the special orthogonal group of \((n,n)\) matrices. Let \( \text{eig}^{-1}_n \) be the mapping given by

\[
\text{eig}^{-1}_n : SO(n) \times \mathbb{R}^n \rightarrow \mathbb{S}(n)
\]

(13)

\[
(R, \Lambda) \mapsto S = \text{eig}^{-1}_n(R, \Lambda),
\]

where \( \mathbb{S}(n) \) is the set of \((n,n)\) symmetric matrices. This mapping is differentiable and onto, and therefore the joint probability density \( p_{\mathcal{R}_x, \mathcal{L}_x} \) of \( \mathcal{R}_x \) and \( \mathcal{L}_x \) is given by

\[
p_{\mathcal{R}_x, \mathcal{L}_x}(R, \Lambda) = p_{\partial^2 \mathcal{F}^n}(\text{uni}(R^T \text{diag}^{-1} \Lambda R)) |J(R, \Lambda)|,
\]

(14)
\(|J(R, \lambda)|\) is the absolute value of the Jacobian determinant of \(\text{eig}^{-1}\).

Theorem 3.3.1 in [5] provides a “closed-form” expression of the probability density of the eigenvalues of random matrices in the Gaussian orthogonal ensemble (GOE\(_n\)). This is the ensemble of \((n, n)\) real symmetric matrices \(M\) with probability density invariant with respect to similarity transformations \(M \to R^T M R\) for any given \((n, n)\) orthonormal \(R\) and such that the probability distribution of distinct entries are independent from each other. Even though each realization of \(\partial^2 \mathcal{F}^n\) is real, symmetric, and, by applying lemma 2, invariant to the same similarity transformations, \(\partial^2 \mathcal{F}^n\) is different from GOE\(_n\), because the distinct entries of \(\partial F^x\) are not independent. However, the assumption of independency used in [5] is important only to derive an expression for the joint probability density of the entries of random matrices in GOE\(_n\), and we already have that for the matrices in \(\delta^2 \mathcal{F}^n\). Once such an expression is available the result in [5] derives from the observation that, in an expression analogous to (14), the term \(R\) appeared only on \(|J(R, \lambda)|\), and therefore the probability density of the eigenvalues of matrices in GOE\(_n\) is obtained through the integration of \(|J(R, \lambda)|\) over \(SO(n)\). The next lemma shows that this is also the case for the probability density \(p_{\mathcal{L}^n}\) of the eigenvalues of \(\delta^2 \mathcal{F}^n\):

**Lemma 3.** Let \(\lambda \in \mathbb{R}^n\), \(R \in SO(n)\), and \(\Sigma_n\) as in (11). Then

\[
p_{\partial^2 \mathcal{F}_x}(\text{uni}(R^T \text{diag}^{-1} \lambda R)) = \frac{1}{\sqrt{2\pi \Sigma_n}} \exp \left( -\frac{\lambda^T \Sigma_n^{-1} \lambda}{2} \right),
\]

where \(\tilde{\Sigma}_n = \sigma^2 \rho_0^{(4)} (2I_n + 1_n \otimes 1_n)\).

**Proof.** The following identity can be easily verified:

\[
(R_{\partial^2 \mathcal{F}^n})^+ = \frac{1}{4\sigma^2 \rho_0^{(4)}} (I_n^2 + C_n - \frac{2}{2+n} \text{vec} I_n \text{vec}^T I_n).
\]

Let \(\Lambda = \text{diag}^{-1} \lambda\) and \(u_R \equiv \text{uni}(R^T AR)\). Therefore,

\[
u_R^T \Sigma_n^{-1} u_R = [\text{uni}(R^T AR)]^T \Sigma_n^{-1} [\text{uni}(R^T AR)] = [D_n^+ \text{vec}(R^T \Lambda R)]^T \Sigma_n^{-1} [D_n^+ \text{vec}(R^T \Lambda R)]
\]

\[
= (\text{vec} \Lambda)^T (R \otimes R) D_n^+ \Sigma_n^{-1} D_n^+ (R^T \otimes R^T) \text{vec} \Lambda,
\]

and, using \([(R \otimes R)D_n]^+ = D_n^+(R^T \otimes R^T)\),

\[
u_R^T \Sigma_n^{-1} u_R = (\text{vec} \Lambda)^T [(R \otimes R)D_n \Sigma_n D_n^T (R^T \otimes R^T)]^+ \text{vec} \Lambda,
\]
which, using $D_n \Sigma_n D_n^T = R_n^{(0)}$, yields
\[
\begin{align*}
\mathbf{u}_R^T \Sigma_n^{-1} \mathbf{u}_R & = (\text{vec} \Lambda)^T [(\mathbf{R} \otimes \mathbf{R}) R_n^{(0)} (\mathbf{R}^T \otimes \mathbf{R}^T)]^+ \text{vec} \Lambda \\
& = (\text{vec} \Lambda)^T (R_n^{(0)})^+ \text{vec} \Lambda,
\end{align*}
\]
\[= \lambda^T \Sigma_n^{-1} \lambda
\]  \hfill (17)
using lemma 1 and (16). \hfill □

The integration of $|J(\mathbf{R}, \lambda)|$ over $SO(n)$, carried out in [5], gives
\[
\int_{SO(n)} |J(\mathbf{R}, \lambda)| d\mathbf{R} = \left(\frac{\pi^{(n+1)/4}}{2}\right)^n \prod_{i=1}^{n-1} \prod_{j=i+1}^n \frac{\Gamma(\lambda_j - \lambda_i)}{\Gamma(1 + i/2)},
\]
and it can be shown that the determinant of $\Sigma_n$ is given by
\[|\Sigma_n| = 2^{n-1} (2 + n) (\sigma^2 \rho_n^{(0)})^n (n+1)/2. \]  \hfill (19)
Together with lemma 3, these results demonstrate the following theorem:

**Theorem 2.** The probability distribution $p_{L^k}$ of the eigenvalues of $\partial^2 F_x^n$ is

\[
p_{L^k}(\lambda) = \frac{2^{(2 - 7n - n^2)/4}}{\sqrt{2 + n(\sigma^2 \rho_n^{(0)})^n (n+1)/4}} \prod_{i=1}^{n-1} \prod_{j=i+1}^n \frac{\Gamma(\lambda_j - \lambda_i)}{\Gamma(1 + i/2)} \prod_{i=1}^{n-1} \prod_{j=i+1}^n \frac{\Gamma(\lambda_j - \lambda_i)}{\Gamma(1 + i/2)} \exp \left( -\lambda^T \Sigma_n^{-1} \lambda \right).
\]  \hfill (20)

Since $N_x$ in (6) is a function of $\partial^2 F_x^n$, theorem 1(1b) implies that $\partial^2 F_x^n$ is independent of $N_x$ for all $x$. Therefore $\partial^2 F_x^{-1} = N_x \partial^2 F_x^n N_x$ according to lemma 2. Theorem 2 can then be applied to obtain an expression for the probability density of the eigenvalues of the numerator of (6), $p_{L^{-1}}$. Using theorem 1(1a), we can show that the denominator of (6), $||\partial^2 F_x^n||$, is distributed according to $\sigma(-\rho_0^{(0)})^{1/2} \chi(n)$, where $\chi(n)$ follows a $\chi$-distribution with $n$ degrees of freedom, and therefore its probability density $p_{||\partial^2 F_x^n||}$ is given by
\[
p_{||\partial^2 F_x^n||}(u) = \frac{2u^{n-1} \exp[u^2/(2\sigma^2 \rho_n^{(2)})]}{(-2\sigma^2 \rho_n^{(2)})^{n/2} \Gamma(n/2)},
\]  \hfill (21)
We can now prove our main result:

**Theorem 3.** Let $\mathcal{K}$ be as in (6). Then
\[
p_{\mathcal{K}}(\kappa) = \frac{2^{(n^2 - 7n + 8)/4} \Gamma[n(n + 1)/4]}{\sqrt{1 + n \Gamma(n/2) \prod_{i=1}^{n-1} \Gamma(1 + i/2)}} \frac{\alpha^{n(n-1)/4} \prod_{i=1}^{n-2} \prod_{j=i+1}^{n-1} \kappa_j - \kappa_i}{\{\alpha[\Sigma_{i=1}^{n-1} \kappa_i^2 - \frac{1}{n+1}(\sum_{i=1}^{n-1} \kappa_i^2)] + 1\}} \frac{\exp[\alpha^2 \sum_{i=1}^{n-1} \kappa_i^2]}{\Gamma(n/4)},
\]  \hfill (22)
where $\alpha = -\rho_0^{(2)}/(2\rho_0^{(4)})$. 8
Proof. Since $\partial F^n_x$ and $\partial^2 F^{n-1}_x$ are independent, so will be $\|\partial F^n_x\|$ and $L^{n-1}_x$. Using (6), we have.

$$p_{K_x}(k) = \int_0^\infty u^{n-1} p_{F^{n-1}_x}(ku)p_{\|\partial F^n_x\|}(u) \, du.$$  \hspace{1cm} (23)

Substituting (20) and (21) in (23), we obtain (22). \hfill \square

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APPENDIX: DIFFERENTIABILITY OF THE AUTOCORRELATION FUNCTION

Correlation functions are characterized by the Wiener-Khintchine theorem, a simplified version of which, shown below, is quoted verbatim from [2]:

**Theorem 4.** A real function $\rho(||s||)$ on $\mathbb{R}^n$ is a correlation function if and only if it can be represented in the form

$$\rho(||s||) = 2^{(d-2)/2}\Gamma(d/2) \int_0^\infty \frac{J_{(d-2)/2}(ks)}{(ks)^{(d-2)/2}} \, d\Phi(k),$$  \hspace{1cm} (A.1)

where the function $\Phi(k)$ on $\mathbb{R}$ has the properties of a distribution function and $J_v$ is a Bessel functions of the first kind and order $v$.

**Lemma 4.** Let the $i$-th moment of the distribution $\Phi(k)$ in theorem 4 be defined. Then, the $i$-th derivative of $r(||s||)$, $r^{(i)}(||s||)$, exists and is given by

$$r^{(i)}(||s||) = 2^{(d-2)/2}\Gamma(d/2) \int_0^\infty k^i \frac{J_{(d-2)/2}(ks)}{(ks)^{(d-2)/2}} \, d\Phi(k).$$  \hspace{1cm} (A.2)

**Proof.** Define the operator $D_i$ acting on a function $f(u)$ as

$$D_i[f(u)] = \left(\frac{1}{u \, du} \right)^i [f(u)]$$  \hspace{1cm} (A.3)

where the term in the right-hand side is recursively defined as

$$\left(\frac{1}{u \, du} \right) [f(u)] = \frac{1}{u} \frac{df(u)}{du}$$  \hspace{1cm} (A.4)

and

$$\left(\frac{1}{u \, du} \right)^i [f(u)] = \left(\frac{1}{u \, du} \right) \left[\left(\frac{1}{u \, du} \right)^{i-1} [f(u)] \right].$$  \hspace{1cm} (A.5)
It can be shown by induction that
\[ r^{(i)}(||s||) = \left( \frac{1}{u} \frac{d}{du} \right) \rho(u) \bigg|_{u = \sqrt{||s||}}. \]  
(A.6)

The operator \( D_i \) and the integral in theorem 4 can be interchanged, since the functions and the measure \( d\Phi(k) \) involved satisfy the conditions of Lebesgue’s dominated convergence theorem. The identity
\[ D_i \left[ \frac{J_\nu(u)}{u^\nu} \right] = (-1)^i \frac{J_{\nu+i}(u)}{u^{\nu+i}}, \]  
(A.7)
found in [6], completes the proof. □

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