ON TAMENESS OF ZONOIDS

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Abstract. We prove that in a tame family of convex bodies the set of zonoids is tame. Here “tame” means “definable in the o-minimal structure generated by globally subanalytic sets and the graph of the exponential function”.

1. Introduction

A zonoid in $\mathbb{R}^n$ is a convex body that can be approximated, in the Hausdorff metric, by finite sums of line segments. Zonoids appear in several different contexts of modern mathematics: functional analysis, measure theory, stochastic geometry, geometric convexity, enumerative geometry [AL16, Bol71, BLM89, BL18, GW92, GW93, LM20, NRZ08, Vit91, Wit73]. Despite the simple definition, the problem of deciding whether a given convex body is a zonoid is considerably hard, and it is referred to as the zonoid problem.

In $\mathbb{R}^2$ every centrally symmetric convex body is a zonoid, but for $n \geq 3$ no simple geometric characterization exists for zonoids in $\mathbb{R}^n$. For example, an interesting non-trivial characterization says that a convex body is a zonoid if and only if its support function is conditionally positive definite\(^1\): this characterization involves nice inequalities on the values of the support function, but an infinite number of them.

In this paper we show that, as soon as we restrict to a “tame” family of convex bodies, the set of zonoids in this family can be described using only a finite number of “tame” conditions.

Here tame means definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$, i.e. the o-minimal structure generated by globally subanalytic sets and the graph of the exponential function. With this notation, our main result is the following theorem.

Theorem 1. In a tame family of convex bodies, the set of zonoids is tame.

Remark 2. Our study was motivated by the following question, suggested by B. Sturmfels: given a semialgebraic convex body, is there an algorithm to decide whether it is a zonoid or not? For example, let $p \in \mathbb{R}[x_1, \ldots, x_n]_d$ be a polynomial of degree $d$ in $n$ variables and let $K_p := \{x \in \mathbb{R}^n \mid p(x) \geq 0\}$. We define $P := \{p \in \mathbb{R}[x_0, \ldots, x_n] \mid K_p \text{ is a convex body}\}$. This defines a tame (semi-algebraic) family of convex bodies. Is there an algorithm that, given the polynomial $p$, decides whether $K_p$ is a zonoid or not? Consider now the set:

$$\mathcal{Z}(P) := \{p \in P \mid K_p \text{ is a zonoid}\}.$$  

If $\mathcal{Z}(P)$ is semialgebraic, such an algorithm (in the sense of [BPR06]) exists. In general, for $n \geq 3$, it is not known if $\mathcal{Z}(P)$ is semi-algebraic. Our Theorem 1 above proves that it is at least definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$, and therefore it cannot be “wild” (in particular it shares with semialgebraic sets many finiteness properties). Our proof of Theorem 1 is constructive but it is not clear to us whether this can be called an algorithm.

\(^1\)A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be conditionally positive definite, see [Sch14, Notes for Section 3.5, pag. 204], if for all $k \in \mathbb{N}$ and for all $x_1, \ldots, x_k \in \mathbb{R}^n$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ such that $\alpha_1 + \cdots + \alpha_k = 0$ we have $\sum_{i,j=1}^{k} \alpha_i \alpha_j f(x_i - x_j) \geq 0$. 

Remark 3. The main idea of the proof is to reduce the problem to the study of a “tame” family of distributions on the unit interval. More precisely, our first step consists in associating to our family of convex bodies a new tame family of rotational invariant convex bodies. The support functions of these convex bodies depend on one variable only and zonoids correspond to members of the family such that an appropriate integral transform of their support function is a non-negative measure. Using the language of distributions, we show that the condition of being a non-negative measure can be expressed in a tame way using a second primitive of the integral transform.

If we start with a semialgebraic family, both the process of moving to rotational invariant bodies and of taking the second primitive take us out of the semialgebraic world, but keeps us inside the tame one. (If in addition the support function – and not just the body – is polynomial, then it is easy to see that the condition of being a zonoid is semialgebraic in the coefficients of the polynomial.)

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2. Preliminaries

2.1. Zonoids. Let us recall some definitions from convex geometry. For a more detailed treatment we refer to [Sch14]. Our main object of study are convex bodies, i.e. non-empty, compact and convex subsets of Euclidean space. Recall that the support function of a convex body $K \subset \mathbb{R}^{n+1}$ is the function on $h_K : S^n \to \mathbb{R}$ given by:

$$h_K(u) = \sup \{ \langle u, x \rangle \mid x \in K \}.$$ 

A convex body $K$ is called a zonoid if there exists an even, non-negative measure $\mu$ on $S^n$ such that the support function of $K$ can be written in the following form

$$h_K(u) = \frac{1}{2} \int_{S^n} |\langle u, x \rangle| d\mu(x).$$

We will use the following notion introduced by Lonke in [Lon].

Definition 4. Let $f : S^n \to \mathbb{R}$ be a measurable function and $e \in S^n$. We denote by

$$O(e) := \text{Stab}_{O(n+1)}(e) \simeq O(n) \subset O(n+1)$$

the stabilizer of $e$ in $O(n+1)$, endowed with the normalized Haar measure $dg$. We then define the measurable function $S_e f : S^n \to \mathbb{R}$ by

$$S_e f(u) := \int_{O(e)} f(g(u)) dg.$$ 

If $K \subset \mathbb{R}^{n+1}$ is a centrally symmetric convex body whose support function is $h_K$ we define $S_e K$ to be the convex body whose support function is given by:

$$h_{S_e K} := S_e h_K.$$ 

The fact that $S_e h_K$ is the support function of a convex body follows from the characterization of support functions as positively homogeneous sublinear functions [Sch14, Section 1.7].

Using Definition 4 one can give the following alternative characterization of zonoids, see [Lon, Theorem 1.1].

Lemma 5. A convex body $K \subset \mathbb{R}^{n+1}$ is a zonoid if and only if for all $e \in S^n$, $S_e K$ is a zonoid.
2.2. O-minimal structures and tameness. We denote by \( \mathbb{R}_{an,exp} \) the o-minimal structure generated by the globally subanalytic sets together with the graph of the exponential function, see \([vdD99]\). We say that a set or a function is tame if it is definable in \( \mathbb{R}_{an,exp} \).

We will use the following crucial result \([CM11, \text{Theorem 1.3}]\).

**Theorem 6.** Let \( F: P \times \mathbb{R}^m \to \mathbb{R} \) be a tame function, with \( P \) subanalytic. Suppose that for all \( p \in P \) the function \( F(p,\cdot) : \mathbb{R}^m \to \mathbb{R} \) is integrable. Then the the function \( I(F) : P \to \mathbb{R} \) defined by \( I(F)(p) := \int_{\mathbb{R}^m} F(p,x)dx \) is tame.

**Remark 7.** In the case \( F : P \times \mathbb{R}^m \to \mathbb{R}^m \) is semialgebraic, then the parametrized integral function \( I(F) \) is definable in a structure strictly smaller than \( \mathbb{R}_{an,exp} \), see \([Kai13]\).

**Corollary 8.** If \( h : S^n \to \mathbb{R} \) is tame, the function \( (e,u) \mapsto S_nh(u) \) is also tame.

**Proof.** Consider a tame function \( F : S^n \times O(n) \to O(n+1) \) such that for almost all \( e \in S^n \) the function \( F(e,\cdot) \) is a tame isomorphism between \( O(n) \) and \( O(e) \). (Since we are only requiring that \( F \) is tame, such function can also be defined piecewise.) Then we can write:

\[
S_eh_p(u) = \int_{O(n)} h_p(F(e,\tilde{g})(u)) |\det JF(e,\cdot)|d\tilde{g},
\]

where \( d\tilde{g} \) is the normalized Haar measure on \( O(n) \). Since the integrand is tame, the result follows by applying Theorem 6 after noticing that there is a tame diffeomorphism between an open dense of subset of \( O(n) \) and \( \mathbb{R}^m \), \( m = n(n-1)/2 \) (for instance one can take the restriction of the Riemannian exponential map at the identity on an appropriate tame domain). \( \square \)

**Definition 9.** Let \( P \) be a tame set. A tame family of convex bodies in \( \mathbb{R}^{n+1} \) is a tame set \( T \subset P \times \mathbb{R}^{n+1} \) such that for every \( p \in P \) the set

\[
K_p := \left\{ x \in \mathbb{R}^{n+1} \mid (p,x) \in T \right\}
\]

is a convex body. For \( p \in P \), we will denote by \( h_p \) the support function of \( K_p \) (instead of \( h_{K_p} \)).

Notice that if \( T \subset P \times \mathbb{R}^{n+1} \) is a tame family of convex bodies, the function \( H : P \times S^n \to \mathbb{R} \), given by \( H(p,u) = h_{K_p}(u) \), is tame. Moreover, if \( P \) is a tame set and \( T \subset P \times \mathbb{R}^{n+1} \) is tame, denoting by \( T_p := \left\{ x \in \mathbb{R}^{n+1} \mid (p,x) \in T \right\} \), it is immediate to see that the following sets are tame:

(i) \( \mathcal{K}(P) := \left\{ p \in P \mid T_p \text{ is a convex body} \right\} \);

(ii) \( \mathcal{K}_0(P) := \left\{ p \in P \mid T_p \text{ is a centrally symmetric convex body, centered at the origin} \right\} \).

The following lemma will also be useful for us.

**Lemma 10.** Let \( F : P \times [-1,1] \to \mathbb{R} \) be a tame function. For every \( p \in P \) and for \( k \in \mathbb{N} \) consider the set \( \Sigma_k(p) \subset [-1,1] \) defined by \( \Sigma_k(p) := \left\{ z \in [-1,1] \mid F(p,\cdot) \text{ is not } C^k \text{ at } z \right\} \). Then the set \( \Sigma_k := \left\{ (p,z) \in P \times [-1,1] \mid z \in \Sigma_k(p) \right\} \) is tame.

**Proof.** Consider the case \( k = 1 \). The function \( F(p,\cdot) \) is \( C^1 \) at \( z \) if and only if the limit \( \lim_{h \to 0} \frac{F(p,z+h)-F(p,z)}{h} \) exists and is finite. This is a first order formula involving only tame objects. This proves that the complement of \( \Sigma_1 \) is tame and thus \( \Sigma_1 \) itself is tame. The general case is proved similarly. \( \square \)
2.3. Distributions. Let $X$ be a smooth manifold. We denote by $C_c^\infty(X)$ the space of compactly supported, smooth functions on $X$, endowed with the $C^\infty$ topology (see [Hir94]). The space of distributions on $X$ is denoted by $\mathcal{D}'(X)$: it is the dual of $C_c^\infty(X)$, endowed with the weak-* topology.

**Remark 11.** If $X$ is compact $C_c^\infty(X) = C^\infty(X)$.

**Remark 12.** Let $G$ be a group. Any group action $G \curvearrowright X$ gives rise to a group action on $C_c^\infty(X)$ by $g \cdot \varphi(x) := \varphi(g^{-1} \cdot x)$ for all $g \in G$, $\varphi \in C_c^\infty(X)$ and $x \in X$. This induces a group action on $\mathcal{D}'(X)$ defined by $(g \cdot \varphi)(x) := \langle \rho, g^{-1} \cdot \varphi \rangle$ for all $g \in G$, $\varphi \in C_c^\infty(X)$ and $\rho \in \mathcal{D}'(X)$. We denote by $C_c^\infty(X)^G$ (respectively $\mathcal{D}'(X)^G$) the functions (respectively distributions) which are invariant under these actions. Note that $(C_c^\infty(X)^G)^* = \mathcal{D}'(X)^G$.

In our case $X$ will be $S^n$, $[-1, 1]$ or $(-1, 1)$ an $G$ will be $O(e)$ for some $e \in S^n$.

**Definition 13.** We denote by $C_c^\infty(S^n)^O(e)$ (respectively $\mathcal{D}'(S^n)^O(e)$) the functions (respectively distributions) on $S^n$ invariant by the group generated by $O(e)$ and the antipodal map $u \mapsto -u$. Similarly we denote by $C_c^\infty([-1, 1])$ (respectively $\mathcal{D}'([-1, 1])$) the even functions (respectively distributions) on $[-1, 1]$.

**Remark 14.** Recall that there is a dense embedding $C_c^\infty(S^n) \hookrightarrow \mathcal{D}'(S^n)$, $f \mapsto \rho_f$ given, for all $g \in C_c^\infty(S^n)$, by:

\[
\langle \rho_f, g \rangle := \int_{S^n} f(x) g(x) dx,
\]

where $dx$ denotes the integration with respect to the standard volume form on $S^n$. There is a similar dense embedding for $[-1, 1]$.

**Lemma 15.** Consider the map $\sigma : [-1, 1] \to [0, 1]$ given by $\sigma(z) := \sqrt{1 - z^2}$. Then the operator $\mathcal{P}$ defined for all $\varphi \in C_c^\infty([-1, 1])$ by $\mathcal{P} \varphi := \varphi \circ \sigma$ is a continuous automorphism of $C_c^\infty([-1, 1])$.

**Proof.** It is enough to know that for every $\varphi \in C_c^\infty([-1, 1])$ there exists $\psi \in C^\infty([0, 1])$ such that $\varphi(z) = \psi(z^2)$ and that the map $\varphi \mapsto \psi$ is continuous in the $C^\infty$ topology. This is done in [Whi43]. \(\square\)

**Proposition 16.** For every $e \in S^n$ there is a linear homeomorphism

\[
\zeta_n : C_c^\infty(S^n)^O(e) \to C_c^\infty([-1, 1]).
\]

Its inverse is given by $(\zeta_n^{-1}) \varphi(u) := \varphi((u, e))$. By density this gives a linear homeomorphism

\[
\zeta_n : \mathcal{D}'(S^n)^O(e) \to \mathcal{D}'([-1, 1]).
\]

**Proof.** Let $f \in C_c^\infty(S^n)^O(e)$ and let us consider an orthonormal basis $v_1, \ldots, v_n, e$. In this basis

\[
(\zeta_n f)(z) = f(0, \ldots, 0, \sqrt{1 - z^2}, z).
\]

Denoting by $\gamma : [-1, 1] \to S^n$ the curve $\gamma(z) := (0, \ldots, 0, \sqrt{1 - z^2}, z)$, we see that $(\zeta_n f) = f \circ \gamma$.

This function is clearly even and $C^\infty$ away from $z = \pm 1$. We show now that it is also smooth near these points. Consider the local coordinate chart $\pi$ around the north pole $(0, \ldots, 0, 1)$ that is the projection on the first $n$ coordinates. Then near the north pole, $\zeta_n f = \bar{f} \circ \pi^{-1}$, the function $\bar{f}$ is smooth because it is a composition of smooth functions. Consider now first the case $n = 1$, for which we have

\[
(\zeta_1 f)(z) = \bar{f}(\sqrt{1 - z^2}) = (\mathcal{P} \bar{f})(z).
\]
In this case it is enough to apply the previous lemma. For the general case we note that $\zeta_n$ is the composition of $\zeta_1$ with the restriction to any 2-plane containing $e$. This proves that $\zeta_n f \in C^\infty_{\text{even}}([-1,1])$.

The map $\zeta_n$ is linear and, by (2.1) and Lemma 15, it is also continuous. Its inverse, given by $(\zeta_n^{-1} \varphi)(u) := \varphi((u,e))$, is also clearly continuous in the $C^\infty$ topology. \hfill \Box

The isomorphism $\zeta_n = \zeta_{n,e}$ defined above depends on $e \in S^n$. However this dependence is tame in the following sense.

**Proposition 17.** Let $F : S^n \times S^n \to \mathbb{R}$ be a tame function such that for all $e \in S^n$ the function $f_e := F(e, \cdot)$ is $O(e)$-invariant. Then the function $(e, z) \mapsto (\zeta_n f_e)(z)$ is a tame function on $S^n \times [-1,1]$.

**Proof.** Take a tame function $v_0 : S^n \to S^n$ such that for all $e \in S^n$, $(e, v_0(e)) = 0$. Then we can write $(\zeta_n f_e)(z) = F(e, z + \sqrt{1 - z^2} \ v_0(e))$, and this is tame because it is a composition of tame functions. \hfill \Box

**Definition 18.** Let $\varphi$ be a function on $[-1,1]$ or $(-1,1)$. Suppose $\varphi$ is locally integrable and thus defines a distribution. We will denote by $D^k \varphi$ its $k$-th distributional derivative and by $\varphi^{(k)}$ the classical derivative whenever this is defined and locally integrable (we will also use $\frac{d^k \varphi}{dx^k}$ when we need to specify the variable of differentiation), i.e.:

$$\langle D^k \varphi, \psi \rangle = (-1)^k \int_I \varphi \psi^{(k)} \quad \text{and} \quad \langle \varphi^{(k)}, \psi \rangle = \int_I \varphi^{(k)} \psi.$$

Let $\rho \in \mathcal{D}'(X)$. We say that $\rho$ is non negative and write $\rho \geq 0$ if for all $\varphi \in C^\infty_c(X)$ such that $\varphi \geq 0$ we have $\langle \rho, \varphi \rangle \geq 0$.

Let us recall some properties of distributions. For proof and details, see [Sch66]. For $x \in X$ we denote by $\delta_x$ the Dirac delta measure at $x$.

**Proposition 19.** Distributions satisfy the following properties:

(i) let $\rho \in \mathcal{D}'(X)$. Then $\rho \geq 0$ if and only if $\rho$ is a positive (finite) Borel measure on $X$;

(ii) let $x \in [-1,1]$ and let $\rho \in \mathcal{D}'([-1,1])$ such that supp($\rho$) = $\{x\}$. Then there exist $N \in \mathbb{N}$ and $r_0, \ldots, r_N$ such that $\rho = \sum_{i=0}^N r_i D^i \delta_x$; if moreover $\rho \geq 0$, then $r_0 \geq 0$ and $r_i = 0$ for all $i > 0$;

(iii) let $\rho \in \mathcal{D}'((-1,1))$. Then $\rho \geq 0$ iff there is a convex function $\gamma : (-1,1) \to \mathbb{R}$ such that $D^2 \gamma = \rho$.

**Remark 20.** Let $\rho \in \mathcal{D}'(S^n)^{O(e)}$. Then $\rho \geq 0$ in $\mathcal{D}'(S^n)^{O(e)}$ if and only if $\zeta_n \rho \geq 0$ in $\mathcal{D}'_{\text{even}}([-1,1])$. This is because $f \geq 0$ iff $\zeta_n f \geq 0$.

A straightforward computation gives also the following lemma.

**Lemma 21.** Let $\psi$ be $k$ times differentiable on $[a,b]$ with $-1 \leq a < b \leq 1$. Then the following properties are true:

(i) If $\psi^{(k)} \in L^1([a,b])$, then

$$D^k \psi = \psi^{(k)} + \sum_{i=0}^{k-1} \psi^{(i)}(a) D^{k-1-i} \delta_a - \psi^{(i)}(b) D^{k-1-i} \delta_b;$$

(ii) for every $x \in [a,b]$ we have

$$\psi \cdot D^k \delta_x = (-1)^k \sum_{i=0}^k (-1)^i \binom{k}{i} \psi^{(k-i)}(x) D^i \delta_x.$$
Next lemma characterizes non–negative measures on \([-1, 1]\).

**Lemma 22.** Let \(\Lambda \in \mathcal{D}'([-1, 1])\) and let \(W^{2,1}([-1, 1])\) be the Sobolev space of (equivalence classes of) functions on \([-1, 1]\) with integrable second derivative. Then \(\Lambda \geq 0\) if and only if the following two conditions are both satisfied:

(i) There is \(\gamma : [-1, 1] \to \mathbb{R}\) in \(W^{2,1}([-1, 1])\) convex such that \(D^2\gamma|_{(-1,1)} = \Lambda|_{(-1,1)}\); 

(ii) There exist \(\lambda_{-1}, \lambda_1 \geq 0\) such that:

\[
\Lambda - \gamma^{(2)} = \lambda_{-1}\delta_{-1} + \lambda_1\delta_1.
\]

**Proof.** Assume that (i) and (ii) are satisfied. Then \(\gamma^{(2)} \geq 0\) and therefore (2.2) defines a positive measure.

Now suppose \(\Lambda \geq 0\). Then \(\Lambda|_{(-1,1)} \geq 0\) and by point (iii) of Proposition 19 there is \(\gamma : (-1, 1) \to \mathbb{R}\) convex such that \(D^2\gamma|_{(-1,1)} = \Lambda|_{(-1,1)}\). Since \(\Lambda\) is finite on the closed interval \([-1, 1]\), then \(\gamma^{(2)}\) is integrable near \(\{\pm 1\}\), i.e. \(\gamma\) is in \(W^{2,1}([-1, 1])\) (in particular \(\gamma\) is also defined in \(\{\pm 1\}\)). This shows that (i) is satisfied.

Now since \(\gamma^{(2)} = D^2\gamma|_{(-1,1)} = \Lambda|_{(-1,1)}\), then \(\Lambda - \gamma^{(2)}\) has support on \(\{\pm 1\}\) and by point (ii) of Proposition 19 it is a sum of deltas and its derivatives. Since both \(\Lambda\) and \(\gamma^{(2)}\) are measures, then \(\Lambda - \gamma^{(2)}\) has no derivatives of deltas. Thus there exist \(\lambda_1, \lambda_{-1} \in \mathbb{R}\) such that \(\Lambda = \gamma^{(2)} + \lambda_{-1}\delta_{-1} + \lambda_1\delta_1\). Since \(\Lambda \geq 0\) point (ii) follows. \(\square\)

2.4. Integral Transforms. We recall now some classical construction of integral transforms. We refer the reader to [GW92] for more details.

The *Cosine Transform* \(T_n\) and the *Radon Transform* \(R_n\) are the endomorphisms of \(C^\infty_{\text{even}}(S^n)\) given for all \(f \in C^\infty_{\text{even}}(S^n)\) by

\[
T_nf(u) := \frac{1}{2} \int_{S^n} \langle u, x\rangle f(x) \, dx \quad \text{and} \quad R_nf(u) := \int_{S(u^\perp)} f(x) \, dx,
\]

where \(S(u^\perp) = S^n \cap u^\perp\) is the unit sphere on \(u^\perp\) and \(dx\) denotes the integration with respect to the standard volume forms of the corresponding spheres.

**Remark 23.** The two operators \(T_n\) and \(R_n\) are linear continuous bijections from the space \(C^\infty_{\text{even}}(S^n)\) to itself. One can define the transpose of these operators on the dual space \(\mathcal{D}'_{\text{even}}(S^n)\) in the usual way. More precisely, for every \(\rho \in \mathcal{D}'_{\text{even}}(S^n)\) and for every \(f \in C^\infty_{\text{even}}(S^n)\), we set:

\[
\langle (T_n)^t \rho, f \rangle := \langle \rho, T_n f \rangle,
\]

and similarly we do for \((R_n)^t\).

With this definition it turns out that

\[
(T_n)^t|_{C^\infty_{\text{even}}(S^n)} = T_n \quad \text{and} \quad (R_n)^t|_{C^\infty_{\text{even}}(S^n)} = R_n,
\]

when the space of smooth functions is embedded into the space of distribution in the usual way (see Remark 14). Thus \((T_n)^t\) and \((R_n)^t\) can be seen as extensions of the original operators to the space of distributions, and we will still denote these extension by \(T_n\) and \(R_n\), see [GW92].

Observe that a convex body \(K\) in \(\mathbb{R}^{n+1}\) is a zonoid if and only if there is a non negative measure \(\mu\) on \(S^n\) such that:

\[
(2.3) \quad h_K = T_n \mu.
\]

Following [GW92], we define the following differential operator on \(C^\infty_{\text{even}}(S^n)\):

\[
\square := \Delta_{S^n} + n,
\]
where $\Delta_{S^n}$ is the Spherical Laplacian on $S^n$. The various operators defined so far satisfy the following intertwining properties [GW92, Proposition 2.1]:

$$\Box T_n = R_n \quad \text{and} \quad T_n^{-1} = \Box R_n^{-1}. \tag{2.4}$$

We use now Proposition 16 to turn these operators into operators on the space of distributions on the interval, as follows.

**Definition 24.** We define the operators $r_n, t_n$ on $\mathcal{D}'([-1, 1])$ by:

$$r_n := \zeta_n R_n \zeta_n^{-1} \quad \text{and} \quad t_n := \zeta_n T_n \zeta_n^{-1}. \tag{2.5}$$

**Remark 25.** Since $\Box$ commutes with the action of $O(n+1)$, it preserves the space $C_{\text{even}}^\infty(S^n)^{O(e)}$. Using the isomorphism $\zeta_n$ we consider the operator $\zeta_n \Box \zeta_n^{-1}$ that we still denote by $\Box$. This is a differential operator on $C_{\text{even}}^\infty([-1, 1])$. For any $\varphi \in C_{\text{even}}^\infty([-1, 1])$, it is given by

$$\Box \varphi(z) = (1 - z^2) \varphi^{(2)}(z) - nz\varphi^{(1)}(z) + n\varphi(z). \tag{2.6}$$

In fact, let $\theta = \theta(u)$ be the angle between $e$ and $u$. By Proposition 16, $f := \zeta_n^{-1} \varphi$ as a function of $\theta$ is given by $f(\theta) = \varphi(\cos(\theta))$. Equivalently $\varphi(z) = f(\arccos(z))$. It is then enough to consider the spherical Laplacian of a function that depends only on $\theta$ which is given by $\Delta_{S^n} f(\theta) = (\sin \theta)^{1-n} \frac{\partial}{\partial \theta} \left( (\sin \theta)^{n-1} \frac{\partial f}{\partial \theta} \right)$, and apply the change of variables $z = \cos \theta$ to obtain (2.5).

The following proposition gives an integral expression for $r_n$.

**Proposition 26.** Let $\varphi \in C_{\text{even}}^\infty([-1, 1])$ and let $z' := \sqrt{1 - z^2}$. We have:

$$\begin{align*}
(r_n \varphi)(z) &= c_n \frac{1}{(z')^{n-2}} \int_0^{z'} \varphi(w) \left( (z')^2 - w^2 \right)^{\frac{n-3}{2}} dw
\end{align*} \tag{2.6}$$

where $c_n = 2 \text{vol}_{n-2}(S^{n-2})$.

**Proof.** Let $f := \zeta_n^{-1} \varphi$ in such a way that $r_n \varphi = \zeta_n R_n f$. Consider an orthonormal basis $v_1, \ldots, v_n, e$ and let $u(z) := z e + z' v_n$. The unit sphere in the space $u(z)^\perp$ can be parametrized by $z' \cos \phi e + z \cos \phi v_n + \sin \phi v$ where $\phi \in [0, \pi]$ and $v$ belongs to the unit sphere of $\text{Span} \{v_1, \ldots, v_{n-1}\}$. Then, using the $O(e)$ invariance and the fact that $f$ is even, we get

$$\zeta_n R_n f(z) = R_n f(u(z)) = 2 \int_0^{\frac{\pi}{2}} f(z' \cos \phi e + z \cos \phi v_n) (\sin \phi)^{n-2} \text{vol}_{n-2}(S^{n-2}) d\phi.$$

We then note that $f(z' \cos \phi e + z \cos \phi v_n) = \varphi(z' \cos \phi)$ and apply the change of variable $w = z' \cos \phi$ to get (2.6).

We notice the particular cases:

$$\begin{align*}
r_2 \varphi(z) &= c_2 \int_0^{z'} \frac{\varphi(w)}{\sqrt{(z')^2 - w^2}} dw \quad \text{and} \quad r_3 \varphi(z) = \frac{c_3}{z} \int_0^{z'} \varphi(w) dw.
\end{align*} \tag{2.7}$$

The equation on the left in (2.7) is known as Abel Equation and the second one can be inverted as follows. Recall that the operator $\mathcal{P}$ from Lemma 15 is defined by $\mathcal{P} \varphi(z) = \varphi(z')$, with $z' = \sqrt{1 - z^2}$.

**Corollary 27.** The inverse of $r_3$ is given by $r_3^{-1} = \frac{1}{c_3} \left( 1 + \frac{1}{d_3} z \right) \circ \mathcal{P}$.

**Proof.** We differentiate equation (2.7) to obtain $\frac{d}{dz} r_3 \varphi(z) = -\frac{1}{2} r_3 \varphi(z) + \frac{1}{d_3} \varphi(z')$. Then write $\varphi = r_3^{-1} \psi$ and change $z'$ for $z$. \qed

The following can be deduced from (2.6) with a straightforward computation.
**Proposition 28.** Let \( \varphi \in C^\infty_{\text{even}}([-1, 1]) \). For any positive integer \( k \leq n/2 - 1 \)
\[
\left( \frac{1}{z'} \frac{d}{dz'} \right)^k (z')^{n-2} r_n \varphi(z) = c_{n,k} (z')^{n-2-2k} r_{n-2k} \varphi(z),
\]
where \( c_{n,k} = \frac{(n-3)!}{(n-2k-1)!} c_{n-2k} c_n \).

Together with Corollary 27 this implies the following.

**Corollary 29.** For odd \( n := 2m + 1 \), \( m \geq 1 \) and for \( \psi \in C^\infty_{\text{even}}([-1, 1]) \) the inverse of \( r_n \) is given by
\[
(r_{2m+1})^{-1} \psi(z) = \frac{1}{c_{2m+1,m-1} c_3} z \left( \frac{1}{z} \frac{d}{dz} \right)^m (z^{2m-1} \mathcal{P} \psi(z)).
\]

**Proof.** Apply Proposition 28 with \( k = m - 1 \) to obtain
\[
\varphi(z) = \frac{1}{c_{2m+1,m-1} c_3} r_3^{-1} \left( \frac{1}{z'} \frac{d}{dz'} \right)^{m-1} (z')^{2m-1} r_{2m+1} \varphi(z).
\]

Then write \( \psi := r_{2m+1} \varphi \) and apply Corollary 27. It is then enough to note that, as operators,
\[
\left( \frac{d}{dz} + 1 \right) \frac{1}{z} = \frac{d}{dz}.
\]

**Remark 30.** Note that in (2.8) the coefficients of the differential operator on the right hand side are polynomial functions of \( z \). Indeed the term of lowest degree is \( z \left( \frac{1}{z} \frac{d}{dz} \right)^m (z^{2m-1}) \) which is of degree zero. In other words, there exist polynomials \( Q_{k,m} \) such that
\[
(r_{2m+1})^{-1} \psi(z) = \sum_{k=0}^m Q_{k,m}(z) \left( \frac{d}{dz} \right)^k \mathcal{P} \psi(z).
\]

Thus we can still define the operator \( \sum_{k=0}^m Q_{k,m}(z) D^k \mathcal{P} \) on distributions. By density this is also equal to the inverse of \( r_{2m+1} \).

Remark 30 together with (2.4) and Remark 25 proves the following.

**Lemma 31.** Let \( f \in C^\infty([-1, 1]) \). There are polynomials \( P_{k,m} \) such that
\[
(t_{2m+1})^{-1} \psi(z) = \sum_{k=0}^{m+2} P_{k,m}(z) \left( \frac{d}{dz} \right)^k \mathcal{P} \psi(z).
\]

This allows us to define the operator \( (t_{2m+1})^{-1} := \sum_{k=0}^{m+2} P_{k,m} D^k \mathcal{P} \) on \( \mathcal{D}'([-1, 1]) \).

3. Proof of Theorem 1.

In this section we prove Theorem 1, which states that in a tame family of convex bodies the set of zonoids is tame. We reformulate this as follows.

**Theorem 32.** Let \( P \) be a tame set and let \( \{K_p \mid p \in P\} \) be a tame family of convex bodies in \( \mathbb{R}^{n+1} \). Then the set \( \mathcal{Z}(P) := \{p \in P \mid K_p \text{ is a zonoid}\} \) is tame.

**Proof.** Let us consider \( n = 2m + 1 \) with \( m \geq 1 \). We will prove that the set
\[
S \mathcal{Z} := \{(p, e) \in P \times S^n \mid S_e K_p \text{ is a zonoid}\}
\]
is tame. By Lemma 5 we have that \( \mathcal{Z}(P) = \{p \in P \mid \forall e \in S^n, (p, e) \in S \mathcal{Z}\} \). Thus if \( S \mathcal{Z} \) is tame then \( \mathcal{Z}(P) \) is also tame.

For \( p \in P \) let \( h_p \) be the support function of \( K_p \) and let \( e \in S^n \). Then by (2.3) \( S_e K_p \) is a zonoid if and only if \( (T_{2m+1})^{-1} S_e h_p \geq 0 \).
Let \( \eta = \eta_{p,e} := \zeta_n S_e h_p \in \mathcal{D}'_{\text{even}}([-1, 1]) \) and let \( \Lambda = \Lambda_{p,e} := t_{2m+1}^{-1} \eta \in \mathcal{D}'_{\text{even}}([-1, 1]) \). Then, by Remark 20, \( S_e K_p \) is a zonoid if and only if \( \Lambda \geq 0 \). By Lemma 22 this is equivalent to having the following two conditions satisfied:

1. there is \( \gamma : [-1, 1] \to \mathbb{R} \) in \( W^{2,1}([-1, 1]) \) convex such that \( D^2 \gamma |_{[-1, 1]} = \Lambda |_{[-1, 1]} \);
2. there exist \( \lambda_{-1}, \lambda_1 \geq 0 \) such that \( \Lambda - \gamma(2) = \lambda_{-1} \delta_{-1} + \lambda_1 \delta_1 \).

The function \( S_e h_p \) is tame (Corollary 8) thus, by Proposition 17, \( \eta \) is tame (as a function of \( e, p \) and \( z \)) and thus piecewise \( C^{m+2} \) as a function of \( z \). Let \(-1 = a_1 < a_2 < \cdots < a_\nu = 1\) be the points of \([-1, 1]\) where \( \eta \) is not \( C^{m+2} \) and the boundary points. Note that \( \nu \) and the points \( \{a_i\} \) depend on \((p, e)\) in a definable way, but we omit this dependence in the notation when not necessary. By tameness (Lemma 10) \( \nu = \nu(p, e) \) is bounded, i.e. there exists \( N \in \mathbb{N} \) such that \( \forall (p, e) \in P \times S^n \) we have \( \nu \leq N \).

For all \( 1 \leq i \leq N \) we define \( I_i := [a_i, a_{i+1}] \) if \( i = 1, \ldots, \nu \) and \( I_i := \emptyset \) if \( \nu < i \leq N \). For \( 1 \leq i \leq \nu \) we define \( a_i := (a_i + a_{i+1})/2 \) in \( I_i \). Let \( \eta_i := \eta |_{I_i} \). By Lemma 31 \((t_{2m+1})^{-1} \eta_i \) is a continuous tame function on \( I_i \) and, as a distribution, it equals \( (\Lambda_{p,e}) |_{I_i} \). For \( i = 1, \ldots, \nu \), we define the following function on \( I_i \):

\[
\tilde{\psi}_{p,e,i}(z) := \int_{a_i}^z \int_{a_i}^s \bigg( (t_{2m+1})^{-1} \eta_i \bigg) (w) \, dw \, ds.
\]

The integrand is tame and Theorem 6 implies that \( \tilde{\psi}_{p,e,i} \) is tame (as a function of \( p, e \) and \( z \)).

For \( \alpha, \beta \in \mathbb{R}^N \) we let \( \psi_{p,e}^{\alpha,\beta} \) be the function on \([-1, 1] \times \{a_i\}_{i=1}^{\nu} \) given by

\[
\psi_{p,e}^{\alpha,\beta}(z) := \sum_{i=1}^\nu \left( \tilde{\psi}_{p,e,i}(z) + \alpha_i z + \beta_i \right) \mathbb{1}(a_i, a_{i+1})(z).
\]

Condition (i) above if satisfied if and only if there exist \( \alpha, \beta \in \mathbb{R}^N \) such that \( \psi_{p,e}^{\alpha,\beta} \) extends to a convex function on \([-1, 1]\); note that if such \( \alpha, \beta \) exist, the convex function \( \psi_{p,e}^{\alpha,\beta} \) is in \( W^{2,1}([-1, 1]) \) because its second derivative equals \( \Lambda = t_{2m+1}^{-1} \eta \) (which is a distribution, thus locally, and therefore globally on \([-1, 1]\), integrable).

Since \( \tilde{\psi}_{p,e,i} \) is tame, and the characteristic functions of the intervals \((a_i, a_{i+1})\) are tame, we can express this condition as a first order formula involving only tame objects, therefore (i) is a tame condition.

Suppose (i) is satisfied, then we can write \( \Lambda_{p,e} - (\psi_{p,e}^{\alpha,\beta})^{(2)} \) as

\[
\Lambda_{p,e} - (\psi_{p,e}^{\alpha,\beta})^{(2)} = \sum_{i=0}^{m+2} \left( \lambda_{i-1}^{(i)} D^i \delta_{-1} + \lambda_1^{(i)} D^i \delta_1 \right),
\]

where the order of the distribution is bounded by \( m+2 \) because of Lemma 31. Moreover the numbers \( \lambda_{i}^{(i)} \) are tame functions of \((p, e)\). Indeed by Lemma 21, their expression only involves the value of \( P_{k,m} \) from (2.9), \( \eta_{p,e} \) and their derivatives at the points \( \pm 1 \). Thus condition (ii) is tame as well:

\[
\Leftrightarrow \lambda_{i}^{(0)} \geq 0 \quad \text{and} \quad \lambda_{i}^{(1)} = 0 \quad \forall i = 1, \ldots, m + 2.
\]

This proves that the set \( \mathcal{S}_{\mathcal{F}} := \{(p, e) \in P \times S^{2m+1} \mid \Lambda_{p,e} \geq 0\} \) is tame, as claimed.

Now if \( n = 2m \) is even, \( m \geq 1 \), we consider the natural embedding

\[
\iota : \mathbb{R}^{2m+1} \hookrightarrow \mathbb{R}^{2m+2} \quad \quad (x_1, \ldots, x_{2m+1}) \mapsto (x_1, \ldots, x_{2m+1}, 0),
\]

which induces an embedding \( \iota_\ast \) at the level of convex bodies. A convex body \( K \subset \mathbb{R}^{2m+1} \) is a zonoid if and only if \( \iota_\ast K \subset \mathbb{R}^{2m+2} \) is a zonoid. Moreover we have \( h_{\iota_\ast K}(u) = h_K(\pi(u)) \) where
$\pi : \mathbb{R}^{2m+2} \to \mathbb{R}^{2m+1}$ is the projection on the first $2m+1$ coordinates. This implies that the family $\{\iota_* K_p \mid p \in P\}$ is a tame family of convex bodies in $\mathbb{R}^{2m+2}$. In particular

$$\mathcal{Z}(P) = \{ p \in P \mid K_p \text{ is a zonoid} \} = \{ p \in P \mid \iota_* K_p \text{ is a zonoid} \}$$

is tame by the previous part of the proof.

\[\blacksquare\]

References

[AL16] Guillaume Aubrun and Cécilia Lancien. Zonoids and sparsification of quantum measurements. Positivity, 20(1):1–23, 2016.

[BL18] Peter Bürgisser and Antonio Lerario. Probabilistic schubert calculus. Crelle’s journal, 2018. to appear.

[BLM89] J. Bourgain, J. Lindenstrauss, and V. Milman. Approximation of zonoids by zonotopes. Acta Math., 162(1-2):73–141, 1989.

[Bol71] E. D. Bolker. Research Problems: The Zonoid Problem. Amer. Math. Monthly, 78(5):529–531, 1971.

[BSR06] Saugata Basu, Richard Pollack, and Marie-Françoise Roy. Algorithms in real algebraic geometry, volume 10 of Algorithms and Computation in Mathematics. Springer-Verlag, Berlin, second edition, 2006.

[CM11] Raf Cluckers and Daniel J. Miller. Stability under integration of sums of products of real globally subanalytic functions and their logarithms. Duke Math. J., 156(2):311–348, 2011.

[GW92] Paul Goodey and Wolfgang Weil. Centrally symmetric convex bodies and the spherical Radon transform. J. Differential Geom., 35(3):675–688, 1992.

[GW93] Paul Goodey and Wolfgang Weil. Zonoids and generalisations. In Handbook of convex geometry, Vol. A, B, pages 1297–1326. North-Holland, Amsterdam, 1993.

[Hir94] Morris W. Hirsch. Differential topology, volume 33 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1994. Corrected reprint of the 1976 original.

[Kai13] Tobias Kaiser. Integration of semialgebraic functions and integrated Nash functions. Math. Z., 275(1-2):349–366, 2013.

[LM20] Antonio Lerario and Léo Mathis. Probabilistic schubert calculus: Asymptotics. Arnold Mathematical Journal, 2020.

[Lou] Yossi Lonke. A characterization of zonoids. preprint.

[NRZ08] Fedor Nazarov, Dmitry Ryabogin, and Artem Zvavitch. On the local equatorial characterization of zonoids and intersection bodies. Adv. Math., 217(3):1368–1380, 2008.

[Sch66] Laurent Schwartz. Théorie des distributions. Publications de l’Institut de Mathématique de l’Université de Strasbourg, No. IX-X. Nouvelle édition, entièrement corrigée, refondue et augmentée. Hermann, Paris, 1966.

[Sch14] Rolf Schneider. Convex bodies: the Brunn-Minkowski theory, volume 151 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, expanded edition, 2014.

[vD99] Lou van den Dries. o-minimal structures and real analytic geometry. In Current developments in mathematics, 1998 (Cambridge, MA), pages 105–152. Int. Press, Somerville, MA, 1999.

[Vit91] R. A. Vitale. Expected absolute random determinants and zonoids. Ann. Appl. Probab., 1(2):293–300, 1991.

[Whi43] Hassler Whitney. Differentiable even functions. Duke Math. J., 10(1):159–160, 03 1943.

[Wit73] H. S. Witsenhausen. Metric inequalities and the zonoid problem. Proc. Amer. Math. Soc., 40:517–520, 1973.