THE ARITHMETIC VOLUME OF HYPERSURFACES IN TORIC VARIETIES AND MAHLER MEASURES

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ABSTRACT. In this paper we determine the canonical arithmetic volume of hypersurfaces in smooth projective toric varieties. As a consequence, we prove a generalized Hodge index theorem on hypersurfaces in smooth projective toric varieties.

Keywords: Arithmetic volume; Height; Mahler measure.

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1. INTRODUCTION

Let $Z$ be an arithmetic variety over $\text{Spec}(\mathbb{Z})$, that is a projective, integral and flat scheme over $\mathbb{Z}$. Let $d + 1$ be the absolute dimension of $Z$. Let $\mathcal{L} = (\mathcal{L}, \| \cdot \|_\phi)$ be a Hermitian line bundle on $Z$, such that the norm $\| \cdot \|_\phi$ is defined by a continuous weight $\phi$. For any $k \in \mathbb{N}_{\geq 1}$, $k\mathcal{L}$ denotes $\mathcal{L}^{\otimes k}$.

The arithmetic volume $\hat{\text{vol}}(\mathcal{L})$ is defined by

$$
\hat{\text{vol}}(\mathcal{L}) = \lim_{k \to \infty} \sup \frac{\hat{h}^0(H^0(Z, k\mathcal{L})_{(\sup, k\phi)})}{k^{d+1}/(d+1)!},
$$

where $\hat{h}^0(H^0(Z, k\mathcal{L})_{(\sup, k\phi)}) := \log \# \{ s \in H^0(Z, k\mathcal{L}) \mid \| s \|_{\sup, k\phi} \leq 1 \}$. This arithmetic invariant was introduced by Moriwaki in [14].
One of the main results of [14] can be stated as follows. Let \( \mathcal{L} \) be a nef \( C^\infty \) Hermitian line bundle on \( Z \) (see Section 3). Then, the height of \( Z \) with respect to \( \mathcal{L} \) equals to the arithmetic volume of \( Z \) with respect to \( \mathcal{L} \). Namely
\[
\hat{\text{vol}}(\mathcal{L}) = h_\mathcal{L}(Z),
\]
(see [9] for the definition of the heights). The proof of this result is difficult and relies on the property of continuity of arithmetic volume function proved in [14].

The explicit determination of the arithmetic volume is a very difficult problem. When \( Z \) is a toric variety, and \( \mathcal{L} \) is toric in the sense of Burgos-Philippon-Sombra [5], the arithmetic volume of \( Z \) with respect to \( \mathcal{L} \) possesses a nice integral representation. Namely the following equation.
\[
\hat{\text{vol}}(\mathcal{L}) = (d+1)! \int_{\Delta_{\mathcal{L}}} \max(0, \vartheta_{\mathcal{L}}) d\text{vol}_M,
\]
where \( M \) is a free \( \mathbb{Z} \)-module of rank \( d \), and \( \Delta_{\mathcal{L}} \) is a rational polytope in \( M \otimes_{\mathbb{Z}} \mathbb{R} \) attached to \( \mathcal{L} \), \( \vartheta_{\mathcal{L}} \) a concave function defined in terms of the metric of \( \mathcal{L} \), and \( d\text{vol}_M \) is a normalized Lebesgue measure on \( M \otimes_{\mathbb{Z}} \mathbb{R} \), see [4, 10, 15]. We can prove (2) using three ingredients. Namely, the following equation
\[
\hat{\text{vol}}(\mathcal{L}) = \limsup_{k \to \infty} \log \frac{\# \{ s \in H^0(Z, kL) \mid \|s\|_{\mu,k\phi} \leq 1 \}}{k^{d+1}/(d+1)!},
\]
where \( \|\cdot\|_{\mu,k\phi} \) is a Euclidean norm (see Section 2), the fact that the Euclidean lattice \( \overline{H^0(Y, kL)}_{(\mu,k\phi)} \) possesses a natural orthonormal basis with respect to \( \|\cdot\|_{\mu,k\phi} \) and some classical results from the geometry of numbers [8]. Note that in [4], the authors bypass the use of the \( L^2 \)-norm with the fact that the basis of toric sections is orthogonal with respect to the sup-norm for all places, both Archimedean or ultrametric.

Burgos, Moriwaki, Philippon and Sombra [4] gave a combinatorial proof of (1) in the toric setting.

Let \( M \) be a free \( \mathbb{Z} \)-module of rank \( d \) and \( N \) its dual. We consider a fan \( \Sigma \) on \( M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} \) and we denote by \( Y \) the associated toric variety over \( \mathbb{Z} \), see for instance [16] or [12, Paragraph 2.2]. In the sequel, we assume that \( Y \) is smooth and projective (this is equivalent to the fact that \( \Sigma \) is nonsingular and the support of \( \Sigma \) is \( M_{\mathbb{R}} \), see [16, Theorems 1.10 and 1.11]). We set \( T_M := \text{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) \simeq (\mathbb{C}^*)^d \) and we denote by \( S_N \simeq (S^1)^d \) its compact torus. We have an open dense immersion \( T_M \hookrightarrow X \) with an action of \( T_M \) on \( Y \) which extends the action of \( T_M \) on itself by translations. There exists a canonical volume form on \( S_N \) (see [12, Remarque 6.1.2, and p. 97]) which is denoted by \( d\mu_{\infty} \).
Let $s$ be a nonzero rational function on $Y$. Mahler measure of $s$ is defined as follows.

$$m(s) = \int_{\mathbb{S}^n} \log |s| d\mu_\infty.$$ 

Let $L$ be an equivariant line bundle on $Y$. It is well known that $L$ possesses a natural continuous Hermitian metric which we denote by $\| \cdot \|_{\infty}$ and which is given in terms of the combinatorial structure of $Y$ [12, Paragraph 3.4]. This metric is called the canonical metric of $L$. We denote this Hermitian line bundle by $L_{\infty}$. Let $P^N$ be the projective space of dimension $N$. The canonical metric of the standard line bundle $\mathcal{O}(1)$ on $P^N$ is given as follows.

$$\| \cdot \|_{\phi_{\infty}} = \max(|x_0|, \ldots, |x_N|),$$

where $x_0, x_1, \ldots, x_N$ are the standard homogeneous coordinates in $Y$. Let $L$ be an equivariant line bundle generated by its global sections on $Y$. Then $L$ defines an equivariant morphism $\psi_L$ on $Y$ with image in $P^{h^0(Y, L)-1}$. We can show that $L_{\phi_{\infty}} = \psi_L^* \mathcal{O}(1)_{\phi_{\infty}}$, see [12, Paragraph 3.3.3]. It is worth noting that, up to a positive multiplicative constant, $\| \cdot \|_{\phi_{\infty}}$ is the unique toric metric on $L$ which has Bernstein-Markov property (see Section 2) with respect to $\mu_\infty$, see Theorem 2.2.

Let $X$ be a hypersurface in $Y$ which is defined by a nonzero rational section $s$ of an equivariant line bundle $\mathcal{E}$ on $Y$. Let $D$ be an equivariant divisor on $Y$ such that $\mathcal{E} \simeq \mathcal{O}(D)$. Let $s_D$ be the rational function of $Y$ which corresponds to $s$ by this isomorphism. The canonical height of the hypersurface $X$ with respect to $L_{\phi_{\infty}}$ is given in terms of Mahler measure of $s_D$. Namely

$$h_{\mathcal{L}_{\phi_{\infty}}} (X) = \deg(\mathcal{L}_Q) m(s_D),$$

see [12, Proposition 7.2.1]. In the sequel, $\mathcal{E}_{\psi_{\infty}}$ denotes the line bundle $\mathcal{E}$ endowed with its canonical metric $\| \cdot \|_{\psi_{\infty}}$.

Our first goal is to determine the canonical arithmetic volume of $X$ with respect to $L_{\phi_{\infty}}$. Our first result is the following equation.

$$\widetilde{\text{vol}}_X(\mathcal{L}_{\phi_{\infty}}) = \text{vol}(\mathcal{L}_Q) m(s_D),$$

where $\widetilde{\text{vol}}_X(\mathcal{L}_{\phi_{\infty}})$ is the arithmetic volume of $X$ with respect to $\mathcal{L}_{\phi_{\infty}}$ and $\text{vol}(\mathcal{L}_Q)$ is the geometric volume of $\mathcal{L}_Q$, see Theorem 4.4. As a first application, we deduce that

$$\widetilde{\text{vol}}_X(\mathcal{L}_{\phi_{\infty}}) = h_{\mathcal{L}_{\phi_{\infty}}} (X).$$
In [10, Proposition 1.5], we proved that $\overline{L}_{\phi_\infty}$ is arithmetically nef but not big, and hence not arithmetically ample on $Y$ (see Section 3 for the definitions of arithmetically nef, big and ample Hermitian line bundles).

Classically, (1) can be used to prove (4). Let us outline the proof of this. Let us first recall that $\overline{L}_{\phi_\infty}$ can be approximated by a sequence of nef $C^\infty$ Hermitian line bundles [19, Paragraph 2.1]. Then use the fact that the arithmetic volume function and the height are continuous with respect to the variation of the metrics [2, Proposition 3.2.2] and [14, Proposition 4.2].

Our method for the proof of (4) is different and more direct. As a consequence of this study, we show that the equation

$$\hat{\text{vol}}_X(\overline{L}_\phi) = h_{\overline{L}_\phi}(X),$$

holds for every Hermitian line $\overline{L}_\phi$ generated by its small sections on $Y$, see Theorem 5.2. Thus we partially recover (1).

We proved a generalized Hodge index theorem on toric varieties, see [11, Theorem 5.5]. In this paper, we show that the methods of [11] can be used to prove a generalized Hodge index theorem on hypersurfaces in $Y$. Let $\phi$ be a semipositive weight on $L$. We shall prove that

$$\hat{\text{vol}}_X(\overline{L}_\phi) \geq h_{\overline{L}_\phi}(X),$$

see Theorem 5.3.

The approach proposed here for the study of these problems relies on the combinatorial structure of the toric variety $Y$. The space $Y(\mathbb{C})$ possesses a canonical measure which is denoted by $\mu_\infty$. Its restriction to the compact torus of $Y(\mathbb{C})$ is a Haar measure. To $\mu_\infty$, $\overline{L}_{\phi_\infty}$ and $\overline{E}_{\psi_\infty}$ we attach an Euclidean lattice $\overline{H}^0(Y, kL + E)_{(\mu_\infty, k\phi_\infty + \psi_\infty)}$ for every $k \in \mathbb{N}$ (see Section 2). We call $\overline{H}^0(Y, kL + E)_{(\mu_\infty, k\phi_\infty + \psi_\infty)}$ the canonical Euclidean lattice associated with $\mu_\infty$, $k\overline{L}_{\phi_\infty}$ and $\overline{E}_{\psi_\infty}$. This lattice plays a central role in this paper.

The Euclidean lattice $\overline{H}^0(Y, kL + E)_{(\mu_\infty, k\phi_\infty + \psi_\infty)}$ induces a structure of Euclidean lattice on $H^0(X, (kL + E)|_X)$, which we denote by $\overline{H}^0(X, (kL + E)|_X)_{\text{sq}(\mu_\infty, k\phi_\infty + \psi_\infty)}$ (see Section 2 for more details on the construction). We are naturally led to study the following limits

$$\limsup_{k \to \infty} \frac{\overline{H}^0(X, (kL + E)|_X)_{\text{sq}(\mu_\infty, k\phi_\infty + \psi_\infty)}}{k^d/d!},$$
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and

$$\limsup_{k \to \infty} \frac{\deg \left( H^0(X, (kL + E)|_X)^{sq} \right)_{q(\mu_{\infty, k\phi_{\infty} + \psi_{\infty})}}}{k^d / d!}.$$ 

In Proposition 4.3 we prove that

$$\limsup_{k \to \infty} \frac{\hat{h}^0 \left( H^0(X, (kL + E)|_X)^{sq} \right)_{q(\mu_{\infty, k\phi_{\infty} + \psi_{\infty})}}}{k^d / d!} = \limsup_{k \to \infty} \frac{\hat{h}^0 \left( H^0(X, kL^q)|_X \right)_{q(\mu_{\infty, k\phi_{\infty})}}}{k^d / d!}.$$ 

An important point to note here is that the $\chi$-arithmetic volume of canonical Euclidean lattices is zero. We use the additivity of the $\chi$-arithmetic degree on admissible metrized sequences, and a theorem due to Szegö and generalized by Deninger, to deduce the following inequality

$$\hat{\text{vol}}_X(\mathcal{L}_{\phi_{\infty}}) \geq \text{vol}(\mathcal{L}_Q)m(s_D).$$

Let $(\| \cdot \|_{\phi_p})_{p=1,2,...}$ be a sequence of smooth Hermitian metrics on $\mathcal{L}$ converging uniformly to $\| \cdot \|_{\phi_{\infty}}$.

We shall show that

$$\lim_{p \to \infty} \limsup_{k \to \infty} \frac{\hat{h}^0 \left( H^0(X, kL^q)|_X \right)_{q(\mu_{\infty, k\phi_{\infty})}}}{k^d / d!} = \limsup_{k \to \infty} \frac{\hat{h}^0 \left( H^0(X, kL^q)|_X \right)_{q(\mu_{\infty, k\phi_{\infty})}}}{k^d / d!},$$

where $\mu$ is any smooth probability measure on $Y$ (see Proposition 4.1). Using Bernstein-Markov’s property (see Section 2) we shall deduce that

$$\lim_{p \to \infty} \hat{\text{vol}}_X(\mathcal{L}_{\phi_p}) = \limsup_{k \to \infty} \frac{\hat{h}^0 \left( H^0(X, kL^q)|_X \right)_{q(\mu_{\infty, k\phi_{\infty})}}}{k^d / d!}.$$ 

It is not difficult to prove that

$$\lim_{p \to \infty} \hat{\text{vol}}_X(\mathcal{L}_{\phi_p}) = \hat{\text{vol}}_X(\mathcal{L}_{\phi_{\infty}}).$$

Using the technical lemma 3.2 we should deduce the following

$$\lim_{k \to \infty} \frac{\deg \left( H^0(X, (kL + E)|_X)^{sq} \right)_{q(\mu_{\infty, k\phi_{\infty} + \psi_{\infty})}}}{k^d / d!} = \limsup_{k \to \infty} \frac{\hat{h}^0 \left( H^0(X, (kL + E)|_X)^{sq} \right)_{q(\mu_{\infty, k\phi_{\infty} + \psi_{\infty})}}}{k^d / d!}.$$ 

Gathering all these computations, we shall conclude the proof of (3).

2. Preliminaries

A normed $\mathbb{Z}$-module $\mathcal{M} = (M, \| \cdot \|)$ is a $\mathbb{Z}$-module of finite type endowed with a norm $\| \cdot \|$ on the $\mathbb{C}$-vector space $M_\mathbb{C} = M \otimes \mathbb{Z} \mathbb{C}$. Let $M_\text{tors}$ denote the torsion-module.
of \( M \), \( M_{\text{free}} = M/M_{\text{tors}} \), and \( M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R} \). We let \( B = \{ m \in M_\mathbb{R} : \|m\| \leq 1 \} \).

There exists a unique Haar measure on \( M_\mathbb{R} \) such that the volume of \( B \) is 1. We let \( \hat{\chi}(M, \| \cdot \|) = \log \# M_{\text{tors}} - \log \text{vol}(M_\mathbb{R}/(M/M_{\text{tors}})) \).

Equivalently, we have
\[
\hat{\chi}(M, \| \cdot \|) = \log \# M_{\text{tors}} - \log \left( \frac{\text{vol}(M_\mathbb{R}/(M/M_{\text{tor}}))}{\text{vol}(B(M, \| \cdot \|))} \right),
\]
for any choice of a Haar measure of \( M_\mathbb{R} \).

The arithmetic degree of \((M, \| \cdot \|)\) is defined as follows
\[
\hat{\text{deg}}(M, \| \cdot \|) = \hat{\text{deg}}\overline{M} = \hat{\chi}(M) - \hat{\chi}(\overline{\mathbb{Z}}),
\]
where \( \hat{\chi}(\overline{\mathbb{Z}}) = -\log (\Gamma(\frac{r}{2} + 1)\pi^{-\frac{r}{2}}) \), with \( r \) is the rank of \( M \otimes_\mathbb{Z} \mathbb{Q} \).

When the norm \( \| \cdot \| \) is induced by a Hermitian product \((\cdot, \cdot)\), we say that \( \overline{M} \) is an Euclidean lattice. In this situation, we have
\[
\hat{\text{deg}}(M) = \log \# M/(s_1, \ldots, s_r) - \log \sqrt{\det((s_i, s_j))_{1 \leq i, j \leq r}},
\]
where \( s_1, \ldots, s_r \) are elements of \( M \) such that their images in \( M_\mathbb{Q} \) form a basis.

We define \( \hat{H}^0(\overline{M}) \) and \( \hat{h}^0(\overline{M}) \) to be
\[
\hat{H}^0(\overline{M}) = \{ m \in M : \|m\| \leq 1 \} \quad \text{and} \quad \hat{h}^0(\overline{M}) = \log \# \hat{H}^0(\overline{M}).
\]
We let
\[
\hat{H}^1(\overline{M}) := \hat{H}^0(\overline{M}^\vee) \quad \text{and} \quad \hat{h}^1(\overline{M}) := \hat{h}^0(\overline{M}^\vee),
\]
where \( \overline{M}^\vee \) is the \( \mathbb{Z} \)-module \( \overline{M}^\vee = \text{Hom}_\mathbb{Z}(M, \mathbb{Z}) \) endowed with the dual norm \( \| \cdot \|^\vee \) defined as follows
\[
\|f\|^\vee = \sup_{x \in M_\mathbb{R}\setminus\{0\}} \frac{|f(x)|}{\|x\|}, \quad \forall f \in M^\vee.
\]

Gillet and Soulé [8] proved the following
\[
-\log(6) \text{ rank } M \leq \hat{h}^0(\overline{M}) - \hat{\text{deg}}(\overline{M}) - \hat{h}^1(\overline{M}) \leq \log(\frac{3}{2}) \text{ rank } M + 2 \log((\text{rank } M)!),
\]
see also [14, Proposition 2.1].

A short exact sequence of Euclidean lattices
\[
0 \rightarrow \overline{N} \overset{i}{\rightarrow} \overline{M} \overset{\pi}{\rightarrow} \overline{Q} \rightarrow 0,
\]
is said to be admissible if \( i_\mathbb{R} \) and the transpose of \( \pi_\mathbb{R} \) are isometries with respect to the Euclidean norms on \( N_\mathbb{R}, M_\mathbb{R} \) and \( Q_\mathbb{R} \) defining the Euclidean lattices \( \overline{N}, \overline{M} \) and \( \overline{Q} \) (for more details see \([3]\)).

We denote by \( \| \cdot \|_{sq} \) the norm on \( Q \) induced by \( \overline{M} \). It is given by
\[
\| v \|_{sq} := \inf_{m \in M_\mathbb{R}, \pi_\mathbb{R}(m) = v} \| m \|, \quad \forall v \in Q_\mathbb{R}.
\]

Let \( Y \) be an arithmetic variety over \( \text{Spec}(\mathbb{Z}) \) of absolute dimension \( d + 1 \). We assume that \( Y_\mathbb{Q} \) is smooth. Let \( L \) be a line bundle on \( Y \).

A weight \( \phi \) on \( L(\mathbb{C}) \) is a locally integrable function on the complement of the zero-section in the total space of the dual line bundle \( L^{-1}(\mathbb{C}) \) satisfying the log-homogeneity property
\[
\phi(\lambda v) = \log |\lambda| + \phi(v)
\]
for all non-zero \( v \in L^{-1}(\mathbb{C}) \) and \( \lambda \in \mathbb{C} \). Let \( \phi \) be a weight function on \( L \). \( \phi \) defines a Hermitian metric on \( L \), which we denote by \( \| \cdot \|_\phi \). We denote by \( L_\phi \) the line bundle \( L \) endowed with the metric \( \| \cdot \|_\phi \).

Let \( \mu \) be a probability measure on \( Y(\mathbb{C}) \). Let \( \phi \) (resp. \( \psi \)) be a continuous weight function on \( L \) (resp. \( E \)). Let \( k \) be a positive integer. We endow the space of global sections \( H^0(Y,kL+E) \otimes_\mathbb{Z} \mathbb{C} \) with the \( L^2 \)-norm given as follows
\[
\| s \|_{(\mu,k\phi+\psi)} := \left( \int_{Y(\mathbb{C})} \| s(x) \|_{k\phi+\psi,\mu}^2 \right)^{\frac{1}{2}} \quad \forall s \in H^0(Y,kL+E) \otimes_\mathbb{Z} \mathbb{C}.
\]

Let \( (\cdot,\cdot)_{(\mu,k\phi+\psi)} \) denote the associated inner product. Also we consider the sup-norm defined by
\[
\| s \|_{\sup,k\phi+\psi} := \sup_{x \in Y(\mathbb{C})} \| s(x) \|_{k\phi+\psi} \quad \forall s \in H^0(Y,kL+E) \otimes_\mathbb{Z} \mathbb{C}
\]

Let \( X \) be a subvariety of \( Y \). We let
\[
\| s \|_{\sup,(k\phi+\psi),|X|} := \sup_{x \in X(\mathbb{C})} \| s(x) \|_{(k\phi+\psi),|X|} \quad \forall s \in H^0(X,(kL+E)|_X) \otimes_\mathbb{Z} \mathbb{C}
\]
where \( (k\phi+\psi)|_X \) denotes the weight of the restriction of \( \| \cdot \|_{k\phi+\psi} \) to \( (kL+E)|_X \).

The Bergman distortion function \( \rho(\mu,\overline{L}) \) is by definition the function given at a point \( x \in X \) by
\[
\rho(\mu,\phi)(x) := \sup_{s \in H^0(X,L)\setminus\{0\}} \frac{\| s(x) \|_{\phi}^2}{\| s \|_{(\mu,\phi)}^2}.
\]
If \( \{s_1, \ldots, s_N\} \) is a \((\mu, \phi)\)-orthonormal basis of \( H^0(X, \mathcal{L})_\mathbb{C} \), where \( N = \dim_{\mathbb{C}} H^0(X, \mathcal{L})_\mathbb{C} \), then it is well known that

\[
\rho(\mu, \phi)(x) = \sum_{j=1}^{N} \|s_j(x)\|_{\phi}^2 \quad \forall x \in X,
\]

see [1, p. 357].

We say that \( \mu \) has the Bernstein-Markov property with respect to \( \| \cdot \|_{\phi} \) if for all \( \varepsilon > 0 \) we have

\[
\sup_{X} \rho(\mu, k\phi)^{\frac{1}{2}} = O(e^{k\varepsilon}).
\]

**Remark 2.1.** If \( \mu \) is a smooth positive volume form and \( \| \cdot \|_{\phi} \) is a continuous metric on \( \mathcal{L} \) then \( \mu \) has the Bernstein-Markov property with respect to \( \| \cdot \|_{\phi} \) (see [1, Lemma 3.2]).

The following result provides a new characterization of canonical metrics on equivariant line bundles on toric varieties.

**Theorem 2.2.** Let \( Y_\mathbb{C} \) be nonsingular complex projective variety. Let \( L \) be an equivariant line bundle on \( Y_\mathbb{C} \) generated by its global sections. Let \( \| \cdot \|_{\phi} \) be a toric Hermitian metric on \( L \). Then

(i) \( \| \cdot \|_{\phi_\infty} \) has Bernstein-Markov property with respect to \( \mu_\infty \).

(ii) \( \| \cdot \|_{\phi} \) has Bernstein-Markov property with respect to \( \mu_\infty \) if and only if \( \| \cdot \|_{\phi} = \lambda \| \cdot \|_{\phi_\infty} \) where \( \lambda \) is a positive constant.

**Proof.**

(i) Let \( k \in \mathbb{N}_{\geq 1} \). It is clear that \( \|\chi_m\|_{\sup, k\phi_\infty} = 1 \) and \( \|\chi_m\|_{\mu_\infty, k\phi_\infty} = 1 \) for every \( m \in k\Delta_L \cap M \). Using this, it is not difficult to see that

\[
\|s\|_{\sup, k\phi_\infty} \leq \sum_{m \in k\Delta_L \cap M} |a_m| \leq \sqrt{\#(k\Delta_L \cap M)} \|s\|_{\mu, \phi} \quad \forall s \in H^0(Y, kL)_\mathbb{C}
\]

where the complex coefficients \( a_m \) are such that \( s = \sum_{m \in k\Delta_L \cap M} a_m \chi_m \). So we have proved (i).

(ii) Let us assume that for every \( \varepsilon > 0 \) we have

\[
\|s\|_{\sup, k\phi} \leq Ce^{k\varepsilon} \|s\|_{\mu_\infty, k\phi} \quad \forall k \in \mathbb{N}
\]

for every \( s \in H^0(Z, kL) \), where \( C \) is a positive constant.

For \( s = \chi^m \) with \( m \in k\Delta_L \cap M \), we get

\[
e^{-k\bar{g}(\frac{m}{k})} \leq Ce^{k\varepsilon} e^{k\bar{g}(0)}.
\]
where \( g(u) := \log \|s_L\|(e^{-u}) \) for every \( u \in \mathbb{R} \) with \( s_L \) is an equivariant rational section of \( L \) and \( \hat{g} \) is the Legendre-Fenchel transform of \( g \), see [5] for the definition of Legendre-Fenchel transform. We deduce from (7) the following
\[
\hat{g}(x) \geq -g(0) - \varepsilon \quad \forall x \in \Delta_L.
\]
Observe that \( \hat{g}(x) \leq -g(0) \). It follows that
\[
\hat{g}(x) = -g(0) \quad \forall x \in \Delta_L.
\]
By [18, Corollary 12.2.1], we infer that \( g = g_{\infty} - g(0) \) where \( g_{\infty}(u) = \log \|s_L\|_{\phi_{\infty}}(e^{-u}) \).

3. Arithmetic volume of hypersurfaces in projective toric varieties

For \( X \) an irreducible hypersurface of \( Y \), the arithmetic volume of \( X \) with respect to \( \mathcal{L}_{|X} \) is denoted by \( \hat{\text{vol}}_X(\mathcal{L}_{|X}) \) or \( \hat{\text{vol}}_X(\mathcal{L}, \| \cdot \|_{\phi}) \). In other words,
\[
\hat{\text{vol}}_X(\mathcal{L}_{\phi}) := \limsup_{k \to \infty} \frac{1}{k^d/\text{dim}} \hat{\text{h}}^0(\mathcal{H}^0(X, k\mathcal{L}_{|X})_{(\sup, k\phi,X)}).
\]

Unless otherwise stated we assume that \( Y \) is a smooth projective toric variety, and \( \mathcal{L} \) is an equivariant line bundle generated by its global sections on \( Y \). Let \( \mathcal{E} \) be a line bundle on \( Y \) such that the defining equation of \( X \) is given by a global section \( s \) of \( \mathcal{E} \).

Note that the following sequence is exact.
\[
0 \to \mathcal{H}^0(\mathcal{Y}, k\mathcal{L}) \xrightarrow{i} \mathcal{H}^0(\mathcal{Y}, k\mathcal{L} + \mathcal{E}) \xrightarrow{\pi} \mathcal{H}^0(X, (k\mathcal{L} + \mathcal{E})_{|X}) \to 0 \quad \text{for } k = 1, 2, \ldots
\]
where \( i \) is the multiplication map by \( s \).

Let \( k \geq 1 \). We consider the following admissible exact sequences.

(8)
\[
0 \to \mathcal{H}^0(\mathcal{Y}, k\mathcal{L})_{(\mu, k\phi,s)} \xrightarrow{i} \mathcal{H}^0(\mathcal{Y}, k\mathcal{L} + \mathcal{E})_{(\mu, k\phi)} \xrightarrow{\pi} \mathcal{H}^0(X, (k\mathcal{L} + \mathcal{E})_{|X})_{\text{sq},(\mu,k\phi)} \to 0,
\]
and

(9)
\[
0 \to \mathcal{H}^0(\mathcal{Y}, k\mathcal{L})_{(\sup, k\phi,s)} \xrightarrow{i} \mathcal{H}^0(\mathcal{Y}, k\mathcal{L} + \mathcal{E})_{(\sup, k\phi)} \xrightarrow{\pi} \mathcal{H}^0(X, (k\mathcal{L} + \mathcal{E})_{|X})_{\text{sq},(\sup,k\phi)} \to 0,
\]
where the metrics of \( \mathcal{H}^0(\mathcal{Y}, k\mathcal{L})_{(\sup, k\phi,s)} \) and \( \mathcal{H}^0(X, (k\mathcal{L} + \mathcal{E})_{|X})_{\text{sq},(\sup,k\phi,s)} \) (resp. \( \mathcal{H}^0(\mathcal{Y}, k\mathcal{L})_{(\mu, k\phi,s)} \) and \( \mathcal{H}^0(X, (k\mathcal{L} + \mathcal{E})_{|X})_{\text{sq},(\mu,k\phi,s)} \)) are induced by the norm considered on \( \mathcal{H}^0(\mathcal{Y}, k\mathcal{L} + \mathcal{E})_{(\sup, k\phi)} \) (resp. \( \mathcal{H}^0(\mathcal{Y}, k\mathcal{L} + \mathcal{E})_{(\mu, k\phi)} \)).
**Theorem 3.1.** Let $\phi$ be a weight on $\mathcal{L}$. We assume that $\| \cdot \|_\phi$ is smooth and positive.

Let $\mu$ be a smooth probability measure on $Y(\mathbb{C})$. We have

(i) \[ \limsup_{k \to \infty} \frac{\hat{h}^0 \left( H^0(X, (kL + E)|_X)_{\text{sq}}, (\mu, k\phi) \right)}{k^d/d!} = \limsup_{k \to \infty} \frac{\hat{h}^0 \left( H^0(X, (kL + E)|_X)_{\text{sq}}, (\mu, k\phi) \right)}{k^d/d!}. \]

(ii) \[ \limsup_{k \to \infty} \frac{\hat{h}^0 \left( H^0(X, kL|_X)_{\text{sq}}, (\mu, k\phi) \right)}{k^d/d!} = \hat{\text{vol}}_X (\mathcal{L}_\phi). \]

**Proof.** Let $\phi$ be a weight on $\mathcal{L}$ such that the metric $\| \cdot \|_\phi$ is smooth and positive. By [17, Theorem B, (2.7.3)], we know that for every $\varepsilon > 0$ and $k = 1, 2, \ldots$

\[ \|s\|_{\text{sq}, (\mu, k\phi + \psi)} \leq Ce^{k\varepsilon} \|s\|_{\sup, (\mu, k\phi + \psi)} \quad \forall s \in H^0(Y, (kL|_X)_{\text{sup}}) \otimes \mathbb{C}, \]

where $C$ is a positive constant depending only on $\phi$ and $\mu$.

It is clear that

\[ \|s\|_{\sup, \chi} \leq \|s\|_{\text{sq}, (\mu, \phi)} \quad \forall s \in H^0(Y, (kL + E)|_X) \otimes \mathbb{C} \]

Combining (12) and (13), and following the proof [15, Lemma 2.1]), it follows immediately that

\[ \limsup_{k \to \infty} \frac{\hat{h}^0 \left( H^0(Y, (kL + E)|_X)_{\text{sup}, (k\phi + \psi)} \right)}{k^d/d!} = \limsup_{k \to \infty} \frac{\hat{h}^0 \left( H^0(Y, (kL + E)|_X)_{\text{sq}, (k\phi + \psi)} \right)}{k^d/d!}. \]

By Gromov’s inequality, there exists a constant $C'$ such that for every $\varepsilon > 0$ and $k \in \mathbb{N}$,

\[ \|s\|_{(\mu, k\phi + \psi)} \leq \|s\|_{\text{sq}, (\mu, k\phi + \psi)} \leq C'e^{k\varepsilon} \|s\|_{(\mu, k\phi + \psi)} \quad \forall s \in H^0(Y, kL + E) \otimes \mathbb{C}. \]

Hence

\[ \|s\|_{\text{sq}, (\mu, k\phi + \psi)} \leq \|s\|_{\text{sq}, (\mu, k\phi + \psi)} \leq C'e^{k\varepsilon} \|s\|_{\text{sq}, (\mu, k\phi + \psi)} \quad \forall s \in H^0(Y, (kL + E)|_X). \]

So we deduce (i). The proof (ii) follows from (i). \[ \square \]

Let $Z$ be an arithmetic variety over $\text{Spec}(\mathbb{Z})$ of dimension $N + 1$. According to [13], there are three kinds of positivity of $\mathcal{L} = (\mathcal{L}, \| \cdot \|)$ a Hermitian line bundle on $Z$. 
ample : \( \mathcal{L} \) is ample if \( L \) is ample on \( Z \), the first Chern form \( c_1(\mathcal{L}) \) is positive on \( Z(\mathbb{C}) \) and, for a sufficiently large integer \( k \), \( H^0(Z, k\mathcal{L}) \) is generated by the set
\[
\{ s \in H^0(Z, k\mathcal{L}) \mid \|s\|_{\text{sup}} < 1 \},
\]
as a \( \mathbb{Z} \)-module.

nef : \( \mathcal{L} \) is nef if the first Chern form \( c_1(\mathcal{L}) \) is semipositive and \( \hat{\deg}(\mathcal{L}|_\Gamma) \geq 0 \) for any 1-dimensional closed subscheme \( \Gamma \) in \( Z \).

big : \( \mathcal{L} \) is big if \( \mathcal{L} \) is big on \( Z, Q \) and there is a positive integer \( k \) and a non-zero section \( s \) of \( H^0(Z, k\mathcal{L}) \) with \( \|s\|_{\text{sup}} < 1 \).

In the notation of \([13]\) we have
\[
(14) \quad \hat{h}^1(H^0(Z, k\mathcal{L}), \| \cdot \|_{\text{sup}}^{k\mathcal{L}}) = o(k^{N+1}), \quad (k \to \infty),
\]
for every ample Hermitian line bundle \( \mathcal{L} \) on \( Z \), see \([14, p. 428]\).

The following lemma can be regarded as a slight generalization of (14).

**Lemma 3.2.** Let \( \mu \) be a smooth positive volume form on \( Y \). Let \( \mathcal{L} \) be an equivariant line bundle generated by its global sections on \( Y \). Let \( \| \cdot \|_{\phi} \) be a continuous Hermitian metric on \( \mathcal{L} \) such that \( \| \chi^m \|_{\text{sup}, \phi} \leq 1 \) for every \( m \in \Delta_L \cap M \).

Let \( X \) be an irreducible hypersurface of \( Y \). With the notations of the previous section, we have
\[
(15) \quad |\gamma(\pi(e_m))| \leq \|\pi(e_m)\|_{\text{sq}} \leq \|e_m\| \leq 1, \quad \forall m \in (k\Delta_L + \Delta E) \cap M.
\]

**Remark 3.3.** Lemma 3.2 can be applied to \( \mathcal{L}_{\phi, \infty} \), endowed with its canonical metric.

**Proof of Lemma 3.2.** To shorten notation, we write \( \| \cdot \| \) and \( \| \cdot \|_{\text{sq}} \) instead of \( \| \cdot \|_{\text{sup}} \) and \( \| \cdot \|_{\text{sq}, \text{sup}, \phi + \psi_{\infty}} \) respectively. The proof of the second assertion can be deduced from the first one by using Bernstein-Markov’s property.

We let \( e_m := \chi^m \) for every \( m \in \Delta_{kL+E} \).

Let \( k \geq 1 \). Let \( \gamma \in \hat{H}^1(H^0(X, (kL + E)|_X)_{\text{sq}}) \). We have
\[
|\gamma(\pi(e_m))| \leq \|\pi(e_m)\|_{\text{sq}} \leq \|e_m\| \leq 1, \quad \forall m \in (k\Delta_L + \Delta E) \cap M.
\]

Note that
\[
\pi^* : H^0(X, (kL + E)|_X)^{\vee} \longrightarrow H^0(Y, kL + E)^{\vee}
\]
is injective. We consider $\pi^*(\gamma) \in H^0(Y, k\mathcal{L} + \mathcal{E})^\vee$. There exists a sequence of integers $(a_m)_{m \in (k\Delta_L + \Delta_E) \cap M}$ such that

$$\gamma \circ \pi = \sum_{m \in \mathbb{N}^{k+1}} a_m e_m^\vee,$$

where $\{e_m^\vee\}_{m \in k\Delta_L + \Delta_E \cap M}$ denote the dual basis of $\{e_m\}_{m \in k\Delta_L + \Delta_E \cap M}$. From (15) we see that

$$a_m \in \{-1, 0, 1\}, \quad \forall m \in k\Delta_L + \Delta_E \cap M.$$

Let $f$ be a rational function which defines $X$. We have

$$(\gamma \circ \pi)(f \chi^\mu) = 0 \quad \forall \mu \in k\Delta_L \cap M.$$

So

$$0 = \sum_{\nu \in k\Delta_L + \Delta_E \cap M} a_{\nu} e_{\nu}^\vee(f \chi^\mu)$$

$$= \sum_{\nu \in k\Delta_L + \Delta_E \cap M} a_{\nu} \sum_{m \in \Delta_E} b_m e_{\nu}^\vee(\chi^m \chi^\mu)$$

$$= \sum_{\nu \in k\Delta_L + \Delta_E \cap M} a_{\nu} \sum_{m \in \Delta_E} b_m e_{\nu}^\vee(e_{m+\mu}).$$

Hence

$$0 = \sum_{\nu \in k\Delta_L + \Delta_E \cap M} a_{\nu} b_{\nu - \mu}, \quad \forall \mu \in k\Delta_L \cap M,$$

where we have made the convention that $b_{\nu - \mu} = 0$ whenever $\nu - \mu \notin k\Delta_L \cap M$.

Let us consider the matrix

$$C_k = (c_{\mu, m})_{\mu \in k\Delta_L \cap M, m \in k\Delta_L + \Delta_E \cap M},$$

where $c_{\mu, m} = b_{m-\mu}$ for any $\mu \in k\Delta_L \cap M$ and $m \in k\Delta_L + \Delta_E \cap M$. So $C_k$ is a $h^0(Y, k\mathcal{L}) \times h^0(Y, k\mathcal{L} + \mathcal{E})$-matrix, where its $\mu$-row is given in terms of the coefficients of $f \chi^\mu$.

We claim that the rank of $C_k$ is $h^0(Y, k\mathcal{L})$. Indeed, let $y = (y_m)_{m \in k\Delta_L \cap M} \in \mathbb{R}^{h^0(Y, k\mathcal{L})}$. By basic linear algebra, we observe that $C_k^t y = 0$ (where $C_k^t$ is the transpose of $C_k$) if and only if $f \sum_{m \in k\Delta_L \cap M} y_m \chi^m = 0$.

It follows that

$$\dim \ker C_k = h^0(Y, k\mathcal{L} + \mathcal{E}) - h^0(Y, k\mathcal{L}) = o(k^d),$$
as \( k \to \infty \).

Note that \((a_m)_{m \in \Delta_{k,L} + E \cap M} \in \ker C_k \) and recall that \( a_m \in \{-1, 0, 1\} \), so we can conclude that

\[
\# \hat{H}^1(\overline{H^0(X, (kL + E)|_X)}_{\text{sq}}) = 3^{o(kd)}.
\]

\[\square\]

4. Canonical arithmetic volume of hypersurfaces

Assume that \( \mathcal{L} \) is generated by its global sections on \( Y \). Let \( (\phi_p)_{p=1,2,...} \) be the sequence of continuous weights on \( \mathcal{L} \) given as follows

\[
\| s(x) \|_{\phi_p} = \frac{|s(x)|}{\left(\sum_{v \in \Delta_L \cap M} |\chi^v(x)|^p\right)^{1/p}}, \quad p = 1, 2, \ldots
\]

for every local section \( s \) of \( \mathcal{L} \).

It is well-known that the sequence \( (\| \cdot \|_{\phi_p})_{p=1,2,...} \) converges uniformly to \( \| \cdot \|_{\phi_{\infty}} \).

From now on, we assume moreover that the probability measure \( \mu \) is invariant under the action of the compact torus of \( Y(\mathbb{C}) \).

**Proposition 4.1.** Let \( \mathcal{L} \) be an equivariant line bundle generated by its global sections. Under the above notations and assumptions, we have

\[
\lim_{p \to \infty} \limsup_{k \to \infty} \frac{\hat{h}^0(\overline{H^0(X, k\mathcal{L} + E)|_X}_{\text{sq}, (\mu, k\phi_p)})}{k^d/d!} = \limsup_{k \to \infty} \frac{\hat{h}^0(\overline{H^0(X, k\mathcal{L} + E)|_X}_{\text{sq}, (\mu_{\infty}, k\phi_{\infty})})}{k^d/d!}.
\]

**Proof.** There exists an equivariant map

\[
\psi_{\mathcal{L}} : Y \longrightarrow \mathbb{P}^{r_{\mathcal{L}}}, \quad x \mapsto \psi_{\mathcal{L}}(x) = (\chi^m(x))_{m \in \Delta_{\mathcal{L} \cap M}
\]

where \( r_{\mathcal{L}} := \#(\Delta_{\mathcal{L} \cap M}) - 1 \). We have

\[\| \cdot \|_{\mathcal{L}_{\phi_{\infty}}} = \psi_{\mathcal{L}}^* \| \cdot \|_{\mathbb{P}(1)_{\phi_{\infty}}}.
\]

Let \( \delta \in [0, 1] \). Let \( v_0 \in \Delta_{\mathcal{L} \cap M} \). For every \( v \in \Delta_{\mathcal{L} \cap M} \), we let

\[
E_{v,\delta} := \left\{ x \in Y(\mathbb{C}) : \chi^{v_0}(x) \neq 0, \quad \delta \frac{|\chi^v(x)|}{|\chi^{v_0}(x)|} \leq \frac{|\chi^{v'}(x)|}{|\chi^{v_0}(x)|} \leq \frac{|\chi^v(x)|}{|\chi^{v_0}(x)|} \text{ for every } v' \in (\Delta_{\mathcal{L} \cap M})\right\}.
\]

It is clear that

\[
Y(\mathbb{C}) \setminus \text{div}(\chi^{v_0}) = \bigcup_{v \in \Delta_{\mathcal{L} \cap M}} E_{v,0}.
\]
For $0 < \delta < 1$, and for every $p = 1, 2, \ldots, k = 1, 2, \ldots$, and $m \in k\Delta_L \cap M$,

$$(\chi^m, \chi^m)(\mu, k\phi_p) \geq \sum_{v \in \Delta_L \cap M} \int_{E_v, \delta} \frac{|\chi^m(x)|^2}{(\sum_{v \in \Delta_L \cap M} |\chi^m(x)|^p)^{\frac{2}{p}}} \mu \geq \frac{\delta^{2k}}{(r_L + 1)^{\frac{2k}{p}}} I_{\delta},$$

where we have put $I_{\delta} := \sum_{v \in \Delta_L \cap M} \int_{E_v, \delta} \mu$.

That is

$$(17) \quad (\chi^m, \chi^m)(\mu, k\phi_p) \geq \frac{\delta^{2k}}{(r_L + 1)^{\frac{2k}{p}}} I_{\delta}.$$

On one hand, by noticing that the metrics are invariant under the action of the compact group $S$, it is easy to check that (17) gives the following

$$(s, s)_{(\mu, k\phi_p)} \geq \frac{\delta^{2k} I_{\delta}}{(r_L + 1)^{\frac{2k}{p}}} (s, s)_{(\mu, k\phi_\infty)}, \quad \forall \ s \in H^0(Y, k\mathcal{L}) \otimes \mathbb{Z} \mathbb{C}.$$ (18)

On the other hand, we have

$$(19) \quad (s, s)_{(\mu, k\phi_p)} \leq (s, s)_{(\mu, k\phi_\infty)}, \quad \forall \ s \in H^0(Y, k\mathcal{L}) \otimes \mathbb{Z} \mathbb{C}.$$

In order to see this, let $s = \sum_{m \in k\Delta_L \cap M} c_m \chi^m$ be an element of $H^0(Y, k\mathcal{L}) \otimes \mathbb{Z} \mathbb{C}$. By the invariance of the metrics, we obtain that

$$(s, s)_{(\mu, k\phi_p)} = \sum_{m \in k\Delta_L \cap M} |c_m|^2 \int_{\mathbb{C}} \|\chi^m\|^2_{k\phi_p} \mu$$

$$\leq \sum_{m \in k\Delta_L \cap M} |c_m|^2$$

$$= (s, s)_{(\mu, k\phi_\infty)},$$

where we have used the fact that $\|\chi^m\|_{k\phi_p} \leq \|\chi^m\|_{k\phi_\infty} \leq 1$.

So, we have proved the following.

$$(s, s)_{(\mu, k\phi_\infty)} \leq (s, s)_{(\mu, k\phi_p)} \leq (s, s)_{(\mu, k\phi_\infty)} \quad \forall \ s \in H^0(Y, k\mathcal{L}) \otimes \mathbb{Z} \mathbb{C}.$$ (18)

That is

$$(s, s)_{(\mu, k\phi_\infty)} \leq \|s\|_{\mu, k\phi_\infty} \quad \forall \ k \in \mathbb{N}.$$ (19)
From these inequalities, we infer that
\[
\limsup_{k \to \infty} \frac{\hat{h}^0(H^0(X, kL|_X))_{sq,(\mu, k, \phi)}}{k^{d/d!}} \leq \limsup_{k \to \infty} \frac{\hat{h}^0(H^0(X, kL|_X))_{sq,(\mu, k, \phi)}}{k^{d/d!}} \leq \limsup_{k \to \infty} \frac{\hat{h}^0(H^0(X, kL|_X))_{sq,(\mu, k, \phi)}}{k^{d/d!}} - \log \frac{\delta^2}{(rL+1)^{\frac{d}{2}}}.
\]

By letting \( \delta \to 1^- \), we obtain
\[
(20) \quad \left| \limsup_{k \to \infty} \frac{\hat{h}^0(H^0(X, kL|_X))_{sq,(\mu, k, \phi)}}{k^{d/d!}} - \limsup_{k \to \infty} \frac{\hat{h}^0(H^0(X, kL|_X))_{sq,(\mu, k, \phi)}}{k^{d/d!}} \right| \leq \frac{2}{p} \log(rL+1).
\]

\[\square\]

Lemma 4.2.
\[
\lim_{k \to \infty} \frac{1}{k^d} \left( \chi(\mathbb{Z}(\#(\Delta_{kL+E}\cap M)) - \chi(\mathbb{Z}(\#(kL\cap M))) \right) = 0.
\]

Proof. This is a consequence of Stirling’s asymptotic formula. \[\square\]

Proposition 4.3.
\[
(21) \quad \limsup_{k \to \infty} \frac{\hat{h}^0(H^0(X, (kL+E)|_X)_{sq,(\mu, k, \phi, + \psi)}}{k^{d/d!}} \leq \limsup_{k \to \infty} \frac{\hat{h}^0(H^0(X, kL|_X))_{sq,(\mu, k, \phi)}}{k^{d/d!}}.
\]

Proof. Let \( v \) be a global section of \( E \) which does not vanish on the compact torus \( S \) of \( Y \). We denote by \( V \) the hypersurface defined by \( v \). We can show that the following sequence is exact.
\[
0 \to H^0(X, kL|_X) \xrightarrow{i_v} H^0(X, (kL+E)|_X) \xrightarrow{\pi_v} H^0(V, (kL+E)|_V) \to 0,
\]
where \( i_v \) is the multiplication map by \( v \) and \( \pi_v \) is the natural projection map.

Let us consider the following admissible metrized exact sequence
\[
0 \to H^0(X, kL|_X) \xrightarrow{i_v} H^0(X, (kL+E)|_X) \xrightarrow{\pi_v} H^0(V, (kL+E)|_V) \to 0,
\]

(22)
On one hand, there exists a positive constant $c$ such that for every $t \in H^0(Y,k\mathcal{L}) \otimes_{\mathbb{Z}} \mathbb{C}$ we have

$$\|vt\|_{\mu,\kappa,\phi}^2 = \int_S \|v(x)\|_{\psi}^2 \|t(x)\|_{\kappa,\phi}^2 \mu \geq c \|t\|_{\mu,\kappa,\phi}^2.$$ 

On the other hand,

$$\|vt\|_{\mu,\kappa,\phi}^2 \leq \|v\|_{\sup,\psi}^2 \|t\|_{\mu,\kappa,\phi}^2,$$

for every $t \in H^0(Y,k\mathcal{L})_{\mathbb{C}}$.

This leads to the following

$$c \|t\|_{\mu,\kappa,\phi}^2 \leq \|vt\|_{\mu,\kappa,\phi}^2 \leq \|v\|_{\sup,\psi}^2 \|t\|_{\mu,\kappa,\phi}^2 \forall t \in H^0(Y,k\mathcal{L}) \otimes_{\mathbb{Z}} \mathbb{C}.$$ 

An easy adaptation of the proof of [11, Theorem 4.1] can be used to deduce that

$$\limsup_{k \to \infty} \frac{\hat{h}_0^0(H^0(Y,k\mathcal{L}|_X)_{\mu,\kappa,\phi})}{k^d/d!} = \limsup_{k \to \infty} \frac{\hat{h}_0^0(H^0(X,k\mathcal{L}|_X)_{\mu,\kappa,\phi})}{k^d/d!}.$$ 

From (22) and using [3, (3.3.2), (3.3.3) p. 59] and by [11, Theorem 4.1], we infer that

$$\limsup_{k \to \infty} \frac{\hat{h}_0^0(H^0(X,(k\mathcal{L}+\mathcal{E})|_X)_{\mu,\kappa,\phi})}{k^d/d!} = \limsup_{k \to \infty} \frac{\hat{h}_0^0(H^0(X,(k\mathcal{L}+\mathcal{E})|_X)_{\mu,\kappa,\phi})}{k^d/d!}.$$ 

This concludes the proof of the proposition. $\square$

**Theorem 4.4.** We have

i) $$\lim_{k \to \infty} \frac{\hat{h}_0^0(H^0(X,(k\mathcal{L}+\mathcal{E})|_X)_{\mu,\kappa,\phi})}{k^d/d!} = h_{\overline{\mathbb{T}}_{\phi}}(X),$$

ii) $$\hat{\text{vol}}_X(\overline{\mathcal{T}}_{\phi}) = h_{\overline{\mathbb{T}}_{\phi}}(X).$$

**Remark 4.5.** Chen [6] proved that the limsup in the definition of arithmetic volume is in fact a limit.

**Proof.** From Theorem 3.1, we get for every $p = 1,2,\ldots$

$$\limsup_{k \to \infty} \frac{\hat{h}_0^0(H^0(X,(k\mathcal{L}+\mathcal{E})|_X)_{\mu,\kappa,\phi})}{k^d/d!} = \limsup_{k \to \infty} \frac{\hat{h}_0^0(H^0(X,(k\mathcal{L}+\mathcal{E})|_X)_{\mu,\kappa,\phi})}{k^d/d!} = \hat{\text{vol}}_X(\overline{\mathcal{T}}_{\phi}).$$
We have
\[
\lim_{p \to \infty} \hat{\text{vol}}(\mathcal{L}_{\phi_p}) = \lim_{p \to \infty} \limsup_{k \to \infty} \frac{\hat{h}^0(H^0(X, k\mathcal{L}|_X)_{\text{sq}},(\mu, k\phi_p))}{k^d/d!} \quad \text{(by (11))}
\]
\[
= \limsup_{k \to \infty} \frac{\hat{h}^0(H^0(X, k\mathcal{L}|_X)_{\text{sq}}(\mu, k\phi_\infty))}{k^d/d!} \quad \text{(by (16))}
\]
\[
= \limsup_{k \to \infty} \frac{\hat{h}^0\left(H^0(X, (k\mathcal{L} + \mathcal{E})|_X)_{\text{sq}}(\mu, k\phi_\infty + \psi_\infty)\right)}{k^d/d!} \quad \text{(by (21))}
\]
Hence
\[
\lim_{k \to \infty} \frac{\hat{h}^0\left(H^0(X, (k\mathcal{L} + \mathcal{E})|_X)_{\text{sq}}(\mu, k\phi_\infty + \psi_\infty)\right)}{k^d/d!} = \hat{\text{vol}}(\mathcal{L}_{\phi_\infty}),
\]
where we have used that \(\lim_{p \to \infty} \hat{\text{vol}}(\mathcal{L}_{\phi_p}) = \hat{\text{vol}}(\mathcal{L}_{\phi_\infty})\).

Note that
\[
\hat{\chi}(H^0(Y, k\mathcal{L})(\mu_\infty, k\phi_\infty, s)) = \det\left(\left<s\chi^m, s\chi^{m'}\right>(\mu_\infty, k\phi_\infty)\right)_{m, m' \in k\Delta_L \cap M},
\]
(we recall that \(\langle \cdot, \cdot \rangle(\mu_\infty, k\phi_\infty)\) is the scalar product associated with \(\| \cdot \|_{(\mu_\infty, k\phi_\infty)}\)).

We have
\[
\lim_{k \to \infty} \log \left(\det\left(\int_{S_N} \chi^m \chi^{m'} |s|^2 d\mu_\infty\right)_{m, m' \in (k\Delta_L) \cap M}\right)^{1/|k\Delta_L \cap M|} = \text{vol}(\mathcal{L}_Q) \int_S \log |s|^2 d\mu_\infty,
\]
see [7, Theorem 4, p.49].

We obtain that
\[
\lim_{k \to \infty} \frac{\hat{\chi}(H^0(Y, k\mathcal{L})(\mu_\infty, k\phi_\infty, s))}{k^d/d!} = \text{vol}(\mathcal{L}_Q) \int_S \log |s|^2 d\mu_\infty.
\]
Applying [17, p. 81] to (8), we obtain that
\[
\hat{\deg}(H^0(X, (k\mathcal{L} + \mathcal{E})|_X))_{\text{sq}}(\mu_\infty, k\phi_\infty + \psi_\infty) = \hat{\deg}(H^0(Y, (k\mathcal{L} + \mathcal{E})))(\mu_\infty, k\phi_\infty + \psi_\infty)
\]
holds for every \(k \in \mathbb{N}\).
An easy computation shows that
\[ \hat{\chi}(H^0(Y, (kL + E))|_{\mu_{\infty}, k \phi_{\infty} + \psi_{\infty}}) = 0 \quad \forall \; k \in \mathbb{N}. \]

Hence, for all \( k \in \mathbb{N}, \)
\[
\deg(H^0(X, (kL + E)|_X))_{sq, (\mu_{\infty}, k \phi_{\infty} + \psi_{\infty})} = \hat{\chi}(H^0(Y, kL + E)|_{\mu_{\infty}, k \phi_{\infty} + \psi_{\infty}}) - \hat{\chi}(H^0(Y, kL)|_{\mu_{\infty}, k \phi_{\infty} + \psi_{\infty}}) \\
- \hat{\chi}(\mathbb{Z}^d(Y, (kL+E))) + \hat{\chi}(\mathbb{Z}^d(Y, kL)) \\
= \det \left( \left\langle s \chi^m, s \chi^{m'} \right\rangle_{(\mu_{\infty}, k \phi_{\infty})} \right)_{m, m' \in k \Delta \mathcal{E} \cap M} \\
- \hat{\chi}(\mathbb{Z}^d((\Delta \mathcal{E} + \epsilon \cap M)) + \hat{\chi}(\mathbb{Z}^d(k \Delta \mathcal{E} \cap M)).
\]

Since \( \dim H^0(Y, kL + E) \otimes_{\mathbb{Z}} \mathbb{Q} = O(k^d) \) as \( k \to \infty, \) we can use Lemma 4.2 to conclude that
\[
\lim_{k \to \infty} \frac{\deg(H^0(X, (kL + E)|_X))_{sq, (\mu_{\infty}, k \phi_{\infty} + \psi_{\infty})}}{k^d/d!} = \text{vol}(\mathcal{L}_\mathbb{Q}) \int_S \log |s|^2 d\mu_{\infty}.
\]

It is clear that the metric \( \| \cdot \|_{\phi_{\infty}} \) satisfies the conditions of Lemma 3.2. Using (5), we get
\[
\lim_{k \to \infty} \frac{\deg(H^0(X, (kL + E)|_X))_{sq, (\mu_{\infty}, k \phi_{\infty} + \psi_{\infty})}}{k^d/d!} = \limsup_{k \to \infty} \frac{h^0(H^0(X, (kL + E)|_X))_{sq, (\mu_{\infty}, k \phi_{\infty} + \psi_{\infty})}}{k^d/d!}.
\]

So, by (23) and (26),
\[
\text{vol}(\mathcal{L}_\mathbb{Q}) \int_S \log |s|^2 d\mu_{\infty} = \text{vol}_X(\mathcal{Z}_{\phi_{\infty}}).
\]

We conclude that
\[ \text{vol}_X(\mathcal{Z}_{\phi_{\infty}}) = h^{\mathbb{Z}_{\phi_{\infty}}}(X), \]
where we have used the fact that
\[ h^{\mathbb{Z}_{\phi_{\infty}}}(X) = \text{vol}(\mathcal{L}_\mathbb{Q}) \int_S \log |s|^2 d\mu_{\infty}, \]
see [12, Proposition 7.2.1]. \qed

5. A GENERALIZED HODGE INDEX THEOREM ON HYPERSURFACES IN TORIC VARIETIES

We introduced the theory of arithmetic theta invariants associated with Hermitian line bundles on arithmetic varieties, see [11, Section 4]. One of the main result of [11] is a generalized Hodge index theorem on toric varieties. In this section we show that
the methods of [11] can be generalized to prove a generalized Hodge index theorem on hypersurfaces in toric varieties.

**Theorem 5.1.** Let \( Z \) be an arithmetic variety of dimension \( n + 1 \) and with smooth generic fibre. Let \((\mathcal{L}, \| \cdot \|_\phi)\) and \((\mathcal{L}, \| \cdot \|_\psi)\) be two \( \mathcal{C}^\infty \) semipositive Hermitian line bundles on \( Z \). We assume \( \psi \leq \phi \) and \((\mathcal{L}, \| \cdot \|_\psi)\) is generated by small sections. We have

\[
\hat{\text{vol}}(\mathcal{L}, \| \cdot \|_\phi) - \hat{\deg}(\hat{c}_1(\mathcal{L}, \| \cdot \|_\phi)^{n+1}) = \hat{\text{vol}}(\mathcal{L}, \| \cdot \|_\psi) - \hat{\deg}(\hat{c}_1(\mathcal{L}, \| \cdot \|_\psi)^{n+1}).
\]

**Proof.** See [11, Theorem 4.11]. \(\square\)

**Theorem 5.2.** Let \( Y \) be a smooth toric variety over \( \mathbb{Z} \) of dimension \( d + 1 \). Let \( \mathcal{L} \) be an equivariant line bundle on \( Y \). We assume that it admits a semipositive weight \( \phi \) and that \( \mathcal{L} \phi \) is generated by small sections. Let \( X \) be a hypersurface in \( Y \). Then

\[
\hat{\text{vol}}_X(\mathcal{L}_\phi) = h_{\mathcal{L}_\phi}(X).
\]

**Proof.** The arithmetic volume function is continuous with respect to the variation of the metric, see [15, p. 513]. Then we can assume that \( \mathcal{L}_\phi \) is generated by strictly small sections. We claim that the proof can be obtained by the same method as in the proof of [11, Theorem 5.4]. Indeed, there exists \( 0 < \alpha \leq 1 \) such that

\[
\alpha \| \cdot \|_\phi \leq \| \cdot \|_{\phi^\infty}.
\]

So by Theorem 5.1 and a continuity argument, we can show that

\[
\hat{\text{vol}}_X(\mathcal{L}, \alpha \| \cdot \|_\phi) - \hat{\deg}(\hat{c}_1(\mathcal{L}|_X, \alpha \| \cdot \|_\phi)^d) = \hat{\text{vol}}_X(\mathcal{L}, \| \cdot \|_{\phi^\infty}) - \hat{\deg}(\hat{c}_1(\mathcal{L}|_X, \| \cdot \|_{\phi^\infty})^d),
\]

and

\[
\hat{\text{vol}}_X(\mathcal{L}, \alpha \| \cdot \|_\phi) - \hat{\deg}(\hat{c}_1(\mathcal{L}|_X, \alpha \| \cdot \|_\phi)^d) = \hat{\text{vol}}_X(\mathcal{L}, \| \cdot \|_\phi) - \hat{\deg}(\hat{c}_1(\mathcal{L}|_X, \| \cdot \|_\phi)^d).
\]

Since

\[
\hat{\text{vol}}(\mathcal{L}_{\phi^\infty}) = h_{\mathcal{L}_{\phi^\infty}}(X),
\]

(see (ii) of Theorem 4.4) we deduce that

\[
\hat{\text{vol}}_X(\mathcal{L}, \| \cdot \|_\phi) = h_{\mathcal{L}_\phi}(X).
\]

\(\square\)

**Theorem 5.3.** [A generalized Hodge index theorem] Let \( Y \) be a smooth toric variety over \( \mathbb{Z} \) of dimension \( d + 1 \). Let \( \mathcal{L} \) be an equivariant line bundle on \( Y \). Let \( \phi \) be a semipositive weight on \( \mathcal{L} \). Let \( X \) be a hypersurface in \( Y \). We have

\[
\hat{\text{vol}}_X(\mathcal{L}_\phi) \geq h_{\mathcal{L}_\phi}(X).
\]
Proof. The proof is similar to the proof of [11, Theorem 5.5]. The $\mathbb{Z}$-algebra $\bigoplus_{k \geq 0} H^0(X, k\mathcal{L}|_X)$ is generated by $\chi^m$ for $m \in k\Delta_\mathcal{L} \cap M$.

Let $\alpha$ be a positive real number such that

$$0 < \alpha < 1 \quad \text{and} \quad \alpha \|\chi^m\|_{\sup, \phi|_X} < 1 \quad \text{for} \quad m \in k\Delta_\mathcal{L} \cap M.$$ 

It follows that $(\mathcal{L}, \alpha \| \cdot \|_\phi)$ is ample.

From Theorem 5.2, we get

$$\widehat{\text{vol}}_X(\mathcal{L}, \alpha \| \cdot \|_\phi) = \widehat{\text{deg}}(\hat{c}_1(\mathcal{L}, \alpha \| \cdot \|_{\phi|_X})^d).$$

By [11, Proposition 4.6 and Theorem 4.7] we see that

$$\widehat{\text{vol}}_X(\mathcal{L}, \| \cdot \|_\phi) - \widehat{\text{deg}}(\hat{c}_1(\mathcal{L}, \| \cdot \|_{\phi|_X})^d) \geq \widehat{\text{vol}}(\mathcal{L}, \alpha \| \cdot \|_{\phi|_X}) - \widehat{\text{deg}}(\hat{c}_1(\mathcal{L}, \alpha \| \cdot \|_{\phi|_X})^d).$$

Therefore

$$\widehat{\text{vol}}_X(\mathcal{L}, \| \cdot \|_\phi) \geq \widehat{\text{deg}}(\hat{c}_1(\mathcal{L}, \| \cdot \|_{\phi|_X})^d).$$

\[ \square \]

References

[1] Robert Berman and Sébastien Boucksom. Growth of balls of holomorphic sections and energy at equilibrium. *Invent. Math.*, 181(2):337–394, 2010.

[2] J.-B. Bost, H. Gillet, and C. Soulé. Heights of projective varieties and positive Green forms. *J. Amer. Math. Soc.*, 7(4):903–1027, 1994.

[3] Jean-Benoît Bost. *Theta invariants of Euclidean lattices and infinite-dimensional Hermitian vector bundles over arithmetic curves*, volume 334 of *Progress in Mathematics*. Birkhäuser/Springer, Cham, [2020] ©2020.

[4] José Ignacio Burgos Gil, Atsushi Moriwaki, Patrice Philippon, and Martín Sombra. Arithmetic positivity on toric varieties. *J. Algebraic Geom.*, 25(2):201–272, 2016.

[5] José Ignacio Burgos Gil, Patrice Philippon, and Martín Sombra. Arithmetic geometry of toric varieties. Metrics, measures and heights. *Astérisque*, (360):vi+222, 2014.

[6] Huayi Chen. Arithmetic Fujita approximation. *Ann. Sci. Éc. Norm. Supér. (4)*, 43(4):555–578, 2010.

[7] Christopher Deninger. Mahler measures and Fuglede-Kadison determinants. *Münster J. Math.*, 2:45–63, 2009.

[8] H. Gillet and C. Soulé. On the number of lattice points in convex symmetric bodies and their duals. *Israel J. Math.*, 74(2-3):347–357, 1991.

[9] Henri Gillet and Christophe Soulé. Characteristic classes for algebraic vector bundles with Hermitian metric. I. *Ann. of Math. (2)*, 131(1):163–203, 1990.

[10] Mounir Hajli. Sur le volume arithmétique sur les schémas torique lisses. *Kyoto J. Math.*, 54(4):819–840, 2014.

[11] Mounir Hajli. The theta invariants and the volume function on arithmetic varieties. *Trans. Amer. Math. Soc.*, 376(3):2237–2256, 2023.
[12] Vincent Maillot. Géométrie d’Arakelov des variétés toriques et fibrés en droites intégrables. *Mém. Soc. Math. Fr. (N.S.)*, 80:vi+129, 2000.

[13] Atsushi Moriwaki. Arithmetic height functions over finitely generated fields. *Invent. Math.*, 140(1):101–142, 2000.

[14] Atsushi Moriwaki. Continuity of volumes on arithmetic varieties. *J. Algebraic Geom.*, 18(3):407–457, 2009.

[15] Atsushi Moriwaki. Big arithmetic divisors on the projective spaces over Z. *Kyoto J. Math.*, 51(3):503–534, 2011.

[16] Tadao Oda. Convex bodies and algebraic geometry—toric varieties and applications. I. In *Algebraic Geometry Seminar (Singapore, 1987)*, pages 89–94. World Sci. Publishing, Singapore, 1988.

[17] Hugues Randriambololona. Métriques de sous-quotient et théorème de Hilbert-Samuel arithmétique pour les faisceaux cohérents. *J. Reine Angew. Math.*, 590:67–88, 2006.

[18] R. Tyrrell Rockafellar. *Convex analysis*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1997. Reprint of the 1970 original, Princeton Paperbacks.

[19] Shouwu Zhang. Small points and adelic metrics. *J. Algebraic Geom.*, 4(2):281–300, 1995.

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