Abstract. We present a new, completely three-dimensional proof of the fact, due to Gabai-Eliashberg-Thurston, that every closed, oriented, irreducible 3-manifold with nonzero second homology carries a universally tight contact structure.

The main result of this paper is a new three-dimensional proof of the following:

**Theorem 0.1** (Gabai-Eliashberg-Thurston). Let $M$ be a closed, oriented, connected, irreducible 3-manifold with $H_2(M, \mathbb{Z}) \neq 0$. Then $M$ carries a universally tight contact structure.

In [14], the authors proved the following variant of the above theorem for manifolds with boundary:

**Theorem 0.2.** Let $(M, \gamma)$ be an oriented, compact, connected, irreducible, sutured 3-manifold which has nonempty boundary, is taut, and has annular sutures. Then $(M, \gamma)$ carries a universally tight contact structure.

Our interest in reproving these theorems is twofold. First, if the starting point is a sutured manifold decomposition and the goal is to build a universally tight contact structure, it should not be necessary (indeed it is not) to construct a taut foliation, perturb it into a contact structure, and argue using symplectic filling techniques that the resulting contact structure is universally tight. Our other motivation is to use these theorems as guidelines in the development of a cut-and-paste theory of contact topology. This theory contrasts with foliation theory right from the start. Given a tight contact structure, it is very easy to produce useful decompositions of the space (so-called convex decompositions, see [14]). On the other hand, given two pieces of a manifold, each with a universally tight contact structure, it is surprisingly difficult to find gluing theorems which allow one to conclude that the contact structure on their union is universally tight. In the theory of taut foliations, the relative difficulty levels of the two appear to be switched.

We have used the search for cut-and-paste proofs of the Gabai-Eliashberg-Thurston theorem as a method for finding new gluing theorems. Theorem 0.2 of [14] was proved using gluing techniques (pioneered by Colin [3, 4]). The key gluing theorem used in the proof was:
Theorem 0.3 (Colin [4]). Let \((M, \xi)\) be an oriented, compact, connected, irreducible, contact 3-manifold and \(S \subset M\) an incompressible convex surface with nonempty Legendrian boundary and \(\partial\)-parallel dividing set \(\Gamma_S\). If \((M \setminus S, \xi|_{M \setminus S})\) is universally tight, then \((M, \xi)\) is universally tight.

The condition \(\partial S \neq \emptyset\) is an important condition in the proof of Theorem 0.3. Therefore, Theorem 0.3 is not applicable when \(M\) is a closed 3-manifold and the first cut in the sutured manifold decomposition is along a closed surface. Instead, we will make use of the main technical result of this paper, Theorem 2.1, which is a gluing theorem along certain closed surfaces.

A predecessor to this gluing theorem is the gluing theorem along incompressible tori [14], where we rephrased and gave a slightly different proof of a gluing result of Colin [4]. Using it, we presented, among other things, a foliation-theory-free proof of the existence theorem for tight contact structures in the case of a closed, irreducible, toroidal manifold (without the assumption \(H_2(M, \mathbb{Z}) \neq 0\)). The two key ingredients of Theorem 0.1 in the toroidal case are the above-mentioned variant of the Gluing Theorem (Theorem 2.1) along incompressible tori and a good understanding of universally tight contact structures on \(T^2 \times [0, 1]\).

It suffices, for the purposes of this paper, to prove a gluing theorem for atoroidal manifolds, along closed convex surfaces \(\Sigma\) of genus \(g > 1\) that satisfy the extremal condition
\[
\langle e(\xi), \Sigma \rangle = \pm (2g - 2),
\]
where the left-hand side is the Euler class of \(\xi\) evaluated on \(\Sigma\). (Note that the condition is trivially satisfied for genus one surfaces.) Tight contact structures \(\xi\) on \(M\) satisfying this condition are said to be extremal along \(\Sigma\) for the following reason. The Bennequin inequality [2, 5] states that:
\[
-(2g - 2) \leq \langle e(\xi), \Sigma \rangle \leq 2g - 2.
\]

One of the main results of [15] is the classification of extremal tight contact structures on \(\Sigma \times [0, 1]\) in the case of two dividing curves on each boundary component. This result, and its implications for covering spaces of \(\Sigma \times [0, 1]\), are enough to construct contact structures satisfying the hypotheses of the Gluing Theorem.

In Section 1, we introduce the notion of straddling or marking and reformulate the classification theorem of [15] in a language more suitable for the application of the Gluing Theorem. Section 2 contains a proof of the Gluing Theorem. In Section 3, Gabai’s well-groomed sutured manifold decomposition theory is used to construct a universally tight contact structure on the cut-open manifold, and then the Gluing Theorem completes the proof of Theorem 0.1.

We adopt the following conventions:
1. The ambient manifold \(M\) is an oriented, compact 3-manifold.
2. \(\xi = \) positive contact structure which is co-oriented by a global 1-form \(\alpha\).
3. A convex surface \(\Sigma\) is either closed or compact with Legendrian boundary.
4. \(\Gamma_\Sigma = \) dividing multicurve of a convex surface \(\Sigma\).
5. \(\#\Gamma_\Sigma = \) number of connected components of \(\Gamma_\Sigma\).
6. \( \Sigma \setminus \Gamma = \Sigma_+ \cup \Sigma_- \), where \( \Sigma_+ \) (resp. \( \Sigma_- \)) is the region where the normal orientation of \( \Sigma \) is the same as (resp. opposite to) the normal orientation for \( \xi \).

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1. Straddling

The following proposition will be used repeatedly throughout the paper.

**Proposition 1.1.** Let \((M, \xi)\) be a tight contact 3-manifold with convex boundary, \(\Sigma\) a component of \(\partial M\), and \(\gamma, \gamma'\) a pair of parallel disjoint curves in \(\Gamma_\Sigma\). Suppose there is a bypass \(B_\alpha \subset M\) attached along an arc \(\alpha \subset \Sigma\) that starts on \(\gamma'\), crosses \(\gamma\), and then ends on a third curve in \(\Gamma_\Sigma - (\gamma \cup \gamma')\). Let \(\beta\) be a Legendrian arc in \(\Sigma\) that starts on \(\gamma\), crosses \(\gamma'\), ends on a point of \(\Gamma_\Sigma - (\gamma \cup \gamma')\), and does not intersect \(\alpha\) or any other points of \(\Gamma_\Sigma\). Then attaching a bypass \(B_\beta\) to \(M\) along \(\beta\) produces a manifold contact isomorphic to the manifold obtained by removing a convex neighborhood of \(B_\alpha\).

**Proof.** Let \(B \subset \Sigma\) be a regular neighborhood of the union of \(\alpha, \beta\), and the annulus in \(\Sigma\) bounded by \(\gamma\) and \(\gamma'\) — we assume \(B\) is convex with Legendrian boundary. Let \(V\) be a small neighborhood of the union of \(B, B_\alpha,\) and \(B_\beta\). See Figure 1. Topologically, \(V\) is the product of an annulus and an interval, i.e., a solid torus. After the necessary edge-rounding, we see that the dividing set of \(\partial V\) has two components, each of which intersects a compressing disk in a single point. There is a unique tight contact structure on \(V\) with this boundary condition, as can be seen by splitting \(V\) along the compressing disk and appealing to Eliashberg’s uniqueness theorem on a ball \([3]\). It remains to verify that the contact structure on \(V\) is indeed tight — for this we simply remark that an explicit model can be found inside the unique (product) tight contact structure on \(V\). Since the contact structure on \(V\) is a product, it follows that adding a bypass along \(\beta\) is equivalent to removing a bypass along \(\alpha\). \(\square\)

![Figure 1. Neighborhood of B](image)

Let \((M, \xi)\) be a tight contact 3-manifold with convex boundary and let \(\Sigma\) be a connected component of \(\partial M\) of genus \(g\) which satisfies the extremal condition \(\langle e(\xi), \Sigma \rangle = -(2g - 2)\).
It follows that $\Sigma_-$ has zero Euler characteristic and is a disjoint union of annuli $A_i$, for $i = 1, \ldots, n$. Denote $\partial A_i = \gamma_i^0 \cup \gamma_i^1$, and call $\gamma_i^0$ and $\gamma_i^1$ a parallel pair of dividing curves.

We say a dividing curve $\gamma_i^j$ is straddled if there is a bypass in $M$ with an attaching arc $\alpha_i^j$ which starts on $\gamma_i^{1-j}$, crosses $\gamma_i^j$, and ends on (i) a point of $\Gamma_{\Sigma} - (\gamma_i^0 \cup \gamma_i^1)$ or on (ii) $\gamma_i^0$ or $\gamma_i^1$, but only after first going around a nontrivial loop on $\Sigma$ which is not parallel to $\gamma_i^j$. Our bypasses may be degenerate, i.e., the two endpoints of the arc of attachment are allowed to coincide. The attaching arcs of these bypasses are called straddling arcs.

A closed convex surface $\Sigma$ of genus $g > 1$ which is a connected component of $\partial M$ admits a full marking if the following hold:

1. $\xi$ is extremal along $\Sigma$ and satisfies $\langle e(\xi), \Sigma \rangle = -(2g - 2)$.
2. The components of $\Sigma_-$ are pairwise nonparallel annuli.
3. There is a collection $S = \{\alpha_i\}$ of straddling arcs (called a straddling set) and a corresponding collection of bypasses $B = \{B_{\alpha_i}\}$ in $M$ such that:
   (a) at least one curve in each parallel pair of $\Gamma_{\Sigma}$ is straddled by a bypass in $B$,
   (b) every curve in $\Gamma_{\Sigma}$ is straddled by at most one bypass in $B$,
   (c) if $i \neq j$, then either $B_{\alpha_i}$ and $B_{\alpha_j}$ are disjoint or intersect only at the endpoints of their corresponding arcs of attachment, and
   (d) $S$ is an essential family, i.e., $\Sigma_+ - (\bigcup_{i=1}^{n} \alpha_i)$ has no disk components. (Equivalently, a thickening of $\Sigma_- \cup (\bigcup_{i=1}^{n} \alpha_i)$ is an incompressible subsurface of $\Sigma$.)

**Proposition 1.2.** Let $(M, \xi)$ be a tight contact manifold with convex boundary, and let $(\tilde{M}, \tilde{\xi})$ be a finite cover. If $\Sigma$ is a boundary component of $(M, \xi)$ which admits a full marking, then the preimage $\tilde{\Sigma}$ of $\Sigma$ admits a full marking in $(\tilde{M}, \tilde{\xi})$.

**Proof.** Let $S$ be the straddling set for $\Sigma$ and $B$ the corresponding set of bypasses. Their preimages $\tilde{S}$ and $\tilde{B}$ satisfy all of the axioms necessary for $\tilde{\Sigma}$ to admit a full marking, except that there may be several arcs in $\tilde{S}$ which straddle the same dividing curve of $\tilde{\Sigma}$. Removing extra components of $\tilde{S}$ decreases the Euler characteristics of the complementary regions. Thus no disk components are produced, and $\tilde{\Sigma}$ is fully straddled. \qed

**Proposition 1.3.** Let $(M, \xi)$ be a tight contact manifold with convex boundary, $\Sigma$ a boundary component of $M$ which admits a full marking, and $S$ a corresponding straddling set. Let $\alpha$ be an arc of $S$ which straddles the dividing curve $\gamma$. Suppose there exists an arc $\alpha' \subset \Sigma$ such that:

1. $S \cup \{\alpha'\}$ is an embedded, essential family of arcs, and
2. $\alpha'$ starts on $\gamma$, crosses the dividing curve $\gamma'$ parallel to $\gamma$, and ends on a point of $\Gamma_{\Sigma} - (\gamma \cup \gamma')$, while intersecting no other points of $\Gamma_{\Sigma}$.

Then the manifold $(M', \xi')$, obtained by attaching a bypass along $\alpha'$ to $(M, \xi)$, also admits a full marking along the boundary component $\Sigma'$ corresponding to $\Sigma$. If $(M, \xi)$ is universally tight, then so is $(M', \xi')$. 

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Proof. Attaching a bypass along $\alpha'$ decreases the number of dividing curves on $\Sigma$ by two. The new collection of straddling arcs $S'$ is defined to be $S - \{\alpha\}$. To see that $S'$ is an essential family of arcs, it is necessary to relate the complementary region $\Omega' = \Sigma_+ - (\cup_{\beta \in S'} \beta)$ to the corresponding complementary region $\Omega = \Sigma_+ - (\cup_{\beta \in S} \beta)$. Performing the bypass along $\alpha'$ is equivalent to the following two steps:

- First cut $\Omega$ along a subarc of $\alpha'$. This creates no disks since $S \cup \{\alpha\}$ is an essential family.
- Next attach a band between two disjoint intervals on the boundary of the modified $\Omega$. This can only decrease the Euler characteristic of the complementary regions.

Finally, deleting $\alpha$ from $S$ decreases the Euler characteristic of the complementary regions, and hence $S'$ is an essential family. The universal tightness of $(M', \xi')$ follows directly from Proposition 1.1.

Proposition 1.4. Let $M$ be a closed hyperbolic manifold, $\xi$ a universally tight contact structure on $M$, and $\Sigma$ an incompressible, closed, convex surface in $M$ which satisfies the extremal condition. Let $\Sigma_0$ be a boundary component of $M \setminus \Sigma$ which admits a full marking. If $\gamma$ and $\gamma'$ form a parallel pair of $\Gamma_{\Sigma_0}$ and $\gamma$ is straddled, then $\gamma'$ is unstraddled, i.e., there is no bypass in $(M \setminus \Sigma, \xi)$ straddling $\gamma'$.

Proof. We argue by contradiction. Let $S$ be the straddling set for $\Sigma_0$, $B$ be the corresponding set of bypasses, and $\alpha$ be the arc in $S$ which straddles $\gamma$. Suppose there exists a straddling arc $\beta$ for $\gamma'$. We claim there exists a finite cover $\pi: (\tilde{M}, \tilde{\xi}) \rightarrow (M, \xi)$ in which components $\tilde{\alpha}$ and $\tilde{\beta}$ of preimages of $\alpha$ and $\beta$ straddle a parallel pair of dividing curves $\tilde{\gamma}$ and $\tilde{\gamma'}$, and are disjoint. To prove the claim we use the following theorem of Allman-Hamilton [1].

Theorem 1.5 (Abelian subgroup separability). Let $M$ be a hyperbolic 3-manifold. Then abelian subgroups $H$ of $\pi_1(M, \ast)$ are separable, i.e., for any $g \in \pi_1(M, \ast) - H$ there exists a finite index subgroup $K \supset H$ which does not contain $g$.

Let $\sigma$ be the core of the annulus $A$ bounded by $\gamma$ and $\gamma'$. If $\alpha$ intersects only two dividing curves, let $\pi$ be a closed loop formed by the union of $\alpha$ and a subarc of $\sigma$. From the definition of a straddling arc, $\pi$ is not in the subgroup $\langle \sigma \rangle$ generated by $\sigma$. According to Theorem 1.3, after passing to a finite cover, we may assume that $\alpha$ intersects 3 distinct dividing curves. Similarly, we may assume that $\beta$ intersects 3 distinct dividing curves. Next, by applying Theorem 1.3 to the trivial subgroup $H = \{e\}$ (or by using residual finiteness), there exists a finite cover $\tilde{M}$ which does not contain $\sigma^k$. Let $\tilde{\alpha}$ be a component of the preimage of $\alpha$ which intersects a component $\tilde{A}$ of the preimage of $A$ in an arc. If $k$ is large enough, there is a component $\tilde{\beta}$ of the preimage of $\beta$ which straddles the other boundary component of $\tilde{A}$ and is disjoint from $\tilde{\alpha}$. This proves the claim.

Now, Proposition 1.1 shows that attaching a bypass along $\tilde{\beta}$ from the exterior of $\tilde{M} \setminus \tilde{\Sigma}$ produces a contact structure isomorphic to the one obtained by digging out the bypass in $M \setminus \Sigma$ attached along $\alpha$, and, in particular, this new contact structure must be universally
tight. On the other hand, $\tilde{\beta}$ is assumed to be the attaching arc for a bypass in $\tilde{M} \setminus \tilde{\Sigma}$, and it follows that attaching a bypass along $\tilde{\beta}$ to the outside of $\tilde{M} \setminus \tilde{\Sigma}$ produces an overtwisted disk. This is a contradiction.

If $\xi$ is a universally tight contact structure and $\Sigma$ is a component of $\partial M$ which admits a full marking, then let $\Gamma(S)$ be the union of the dividing curves of $\Sigma$ which are straddled by the (full) straddling set $S$. In view of Proposition \[\text{Proposition 1.4},\] $\Gamma(S)$ is an invariant of $\xi$.

We now consider the case where $M = \Sigma \times [0,1]$ has convex boundary which consists of $\Sigma_i = \Sigma \times \{i\}, i = 0, 1$, and where each $\Gamma_{\Sigma_i}$ is just a single pair of parallel dividing curves. We proved the following classification theorem in [15].

**Theorem 1.6.** Let $\Sigma$ be a closed oriented surface of genus $g \geq 2$ and $M = \Sigma \times [0,1]$. Suppose that $\Gamma_{\Sigma_i}, i = 0, 1$, is a pair $\gamma_i \cup \gamma'_i$ of parallel nonseparating curves which cobound an annulus $A_i \subset \Sigma_i$ and that $\Gamma_{\Sigma_0} \neq \Gamma_{\Sigma_1}$. Choose a characteristic foliation $F$ on $\partial M$ which is adapted to $\Gamma_{\Sigma_0} \cup \Gamma_{\Sigma_1}$. Then there exist, up to isotopy rel boundary, exactly 4 tight contact structures which satisfy the boundary condition $F$, and all of them are universally tight. Moreover:

1. For each of the 4 tight contact structures, both $\Sigma_0$ and $\Sigma_1$ are fully straddled, and the 4 cases exactly correspond to the 4 possible choices of $\Gamma(S)$, consisting of pairs of straddled curves, one from each $\Gamma_{\Sigma_i}$.
2. Let $\delta_i$ be a closed Legendrian curve on $\Sigma_i$ which has geometric intersection $|\delta_i \cap \gamma_i| = 1$. Then there exists a (degenerate) bypass in $\Sigma \times I$ along $\delta_i$ which straddles the straddled curve of $\Gamma_{\Sigma_i}$.

**Remark.** The focus of this paper is on surfaces of genus greater than one. However, the definitions of straddled, unstraddled, and full marking make sense on a torus with only slight modification to allow for the necessity of more than two parallel dividing curves. The statements and proofs above apply to tori, with one exception: the uniqueness statement for universally tight contact structures on $T^2 \times I$ with a given full marking. The classification theorem in this setting requires an additional nonnegative integer invariant called the *torsion* (see [9], [4], [14]).

2. **Gluing along surfaces which admit full markings**

The atoroidal hypothesis in the following Gluing Theorem allows us to assume $M$ is hyperbolic and apply Proposition \[\text{Proposition 1.4}.\] A substantially similar argument can be made without the atoroidal hypothesis by using results of Long-Niblo [16] instead of Allman-Hamilton [1].

**Theorem 2.1** (Gluing Theorem). Let $(M, \xi)$ be an atoroidal contact manifold which is extremal along a closed, convex, incompressible surface $\Sigma$, and let $\Sigma^0$ and $\Sigma^1$ be the boundary components of $M \setminus \Sigma$ corresponding to $\Sigma$. Suppose that, for each $i = 0, 1$, $\Sigma^i$ admits a full marking in $M \setminus \Sigma$ with a straddling set $S^i$, and the dividing curves of $\Sigma^i$ are straddled if and only if they are unstraddled in $\Sigma^{1-i}$. Suppose further that $S^0 \cup S^1$ is an embedded,
essential family of arcs in $\Sigma$. Then $(M, \xi)$ is universally tight if and only if $(M \setminus \Sigma, \xi|_{M \setminus \Sigma})$ is universally tight.

Proof. The proof that $(M, \xi)$ is tight also applies to finite covers of $(M, \xi)$, in light of Proposition 1.2. Atoroidal Haken 3-manifolds are hyperbolic, and hence have residually finite fundamental groups. Therefore, any overtwisted disk that exists in the universal cover also exists in some finite cover, and the proof below will also imply that $(M, \xi)$ is universally tight.

The general strategy for proving tightness is explained in detail in \cite{14} and \cite{12}, so we will only provide a brief summary. Arguing by contradiction, we assume $(M, \xi)$ contains an overtwisted disk $D$. There exists a sequence $\Sigma = \Sigma_0, \Sigma_1, \ldots, \Sigma_n$ of isotopic surfaces, where each step is a single bypass attachment and $\Sigma_n$ is disjoint from $D$. Since we can extricate $D$ from (isotopic copies of) $\Sigma$ in stages, if we show that each $\xi|_{M \setminus \Sigma_i}$ is tight, this implies the tightness of $\xi|_M$. Now, for universal tightness, we pass to a large finite cover $\tilde{M}$ of $M$ and extricate some lift $\tilde{D}$ of $D$ from the preimage $\tilde{\Sigma}$ of $\Sigma$. Lifting to a cover has the following advantages:

1. A bypass which is $\#\Gamma$-increasing can be made trivial by using the residual finiteness of $M$.
2. A bypass whose attaching arc intersects only two distinct curves but is not trivial or $\#\Gamma$-increasing can be made to intersect three distinct curves by Theorem 1.5.

Therefore, by lifting as necessary so that each bypass satisfies Conditions 1 and 2 above, we have a sequence

$$(\tilde{M}_0 = M, \tilde{\Sigma}_0 = \Sigma, \tilde{\xi}_0 = \xi), (\tilde{M}_1, \tilde{\Sigma}_1, \tilde{\xi}_1), \ldots, (\tilde{M}_n, \tilde{\Sigma}_n, \tilde{\xi}_n),$$

where $\tilde{\Sigma}_i \subset \tilde{M}_i, \tilde{M}_{i+1}$ is a finite cover of $\tilde{M}_i$, $\tilde{\xi}_i$ is the pullback of $\xi$ to $\tilde{M}_i$, $\tilde{\Sigma}_{i+1}$ is obtained from the preimage of $\tilde{\Sigma}_i$ by attaching a bypass along an arc which straddles three distinct curves, and a lift $\tilde{D}$ of $D$ in $\tilde{M}_n$ is disjoint from $\tilde{\Sigma}_n$. If we can show that $\tilde{\xi}_i|_{\tilde{M}_i \setminus \tilde{\Sigma}_i}$ is universally tight for all $i$, we are done. This involves making sure that at each step the contact structure admits a full marking and satisfies the conditions of the theorem — therefore we need to exercise extra care when choosing the covers.

Consider the attachment of the first bypass. We start with $\Sigma$ and $M$, and take a finite cover which satisfies Conditions 1 and 2. (To simplify notation, we will still write $M$ for the finite cover of $M$ and $\Sigma$ for the preimage of $\Sigma$.) Then the bypass $B_\alpha$ has an attaching arc $\alpha$ which intersects three distinct dividing curves of $\Sigma$. Denote the copies of $\Sigma$ in $M \setminus \Sigma$ by $\Sigma^i$, $i = 0, 1$, and let $S^i$ be the straddling set for $\Sigma^i$. By Proposition 1.2, $\Sigma$ admits a full marking on both sides (i.e., on $\Sigma^0$ and on $\Sigma^1$), and it is clear that the union $S^0 \cup S^1$ is still an embedded essential family of arcs. After passing to a larger finite cover and removing duplicate straddling arcs which straddle the same dividing curve as in Proposition 1.2.
3. The arc of attachment of a bypass can be made disjoint from the straddling sets $S^i$, $i = 0, 1$, by using residual finiteness as in the proof of Proposition 1.3.

Suppose we dig out $B_\alpha$ from the $\Sigma^0$ side and reattach $B_\alpha$ along $\alpha$ to the $\Sigma^1$ side. This gives us $M \setminus \Sigma'$, where $\Sigma'$ is a surface parallel to and disjoint from $\Sigma$. By Proposition 1.4, the curve $\gamma$ straddled by $\alpha$ must be in $\Gamma(S^0)$. Since $\gamma$ is straddled by an element $\beta_0$ of $S^0$ if and only if $\gamma$ is unstraddled by an element of $S^1$, it follows that the parallel curve $\gamma'$ is straddled by an arc $\beta_1 \in S^1$.

It is clear that digging $B_\alpha$ preserves universal tightness. Since $\alpha$ and $\beta_1$ are disjoint (by Condition 3), Proposition 1.4 tells us that, on $M \setminus \Sigma$, digging $B_\alpha$ out is equivalent to attaching a bypass $B'_{\beta_1}$ along $\beta_1$ onto the $\Sigma^0$ side, i.e., $M \setminus (\Sigma \cup B_\alpha)$ is contactomorphic to $M' = (M \setminus \Sigma) \cup N(B'_{\beta_1})$, where $N(F)$ is a small neighborhood of $F$. We will write $\partial M' = (\Sigma')^0 \cup \Sigma^1$. By Proposition 1.3, since $S^0 \cup \{\beta_1\}$ is essential, $(\Sigma')^0 \subset M'$ admits a full marking obtained by dropping $\beta_0$ from $S^0$. On the other hand, by Proposition 1.4 again, attaching a bypass $B'_{\alpha}$ to $\Sigma^1 \subset M'$ along $\alpha$ is equivalent to digging a bypass $B_{\beta_1}$ along $\beta_1$. In other words, $M' \cup N(B_{\beta_1})$ is contactomorphic to $M' \setminus B_{\beta_1}$ and also to $M \setminus \Sigma'$. Therefore, $\xi|M_{\Sigma'}$ is universally tight and admits full markings $S^0 - \{\beta_0\}$ on the $(\Sigma')^0$ side and $S^1 - \{\beta_1\}$ on the $(\Sigma')^1$ side (by applying Proposition 1.3 again). The union of the two straddling sets is an embedded essential family of arcs.

We can now inductively construct $(\tilde{M}_i, \tilde{\Sigma}_i, \tilde{\xi}_i)$ that admit full markings on both sides which satisfy the conditions of the theorem, by choosing finite covers where Conditions 1, 2, and 3 are met. This ensures universal tightness of each step and finishes the proof.

3. The Gabai-Eliashberg-Thurston Theorem

Proof of Theorem 1.2. According to Gabai [7], there is a well-groomed sutured manifold decomposition of $M$,

$$M \overset{\Sigma}{\sim} (M_1, \gamma_1) \overset{S_1}{\sim} \ldots \overset{S_{n-1}}{\sim} (M_n, \gamma_n) = \cup (B^3, S^1 \times I),$$

where $\Sigma$ is a nonseparating surface. Since $M$ and $\Sigma$ are closed, $\gamma_1 = \emptyset$. For $i \geq 1$, $S_i$ may be chosen to have nonempty boundary (see [14], Theorem 1.3 for a statement of this version of Gabai’s theorem). It follows that for $i \geq 2$, $(M_i, \gamma_i)$ has annular sutures, that is, all sutures are annuli and every component of $\partial M_i$ contains at least one suture.

By the results of [14], the sutured manifold decomposition

$$((M_2, \gamma_2) \overset{S_2}{\sim} (M_3, \gamma_3) \overset{S_4}{\sim} \ldots \overset{S_{n-1}}{\sim} (M_n, \gamma_n) = \cup (B^3, S^1 \times I),$$

gives rise to a convex decomposition

$$((M_2, \Gamma_2) \overset{(S'_2, \sigma_2)}{\sim} (M_3, \Gamma_3) \overset{(S'_3, \sigma_3)}{\sim} \ldots \overset{(S'_{n-1}, \sigma_{n-1})}{\sim} (M_n, \Gamma_n) = \cup (B^3, S^1),$$

and then (and this requires that each of the $(M_i, \Gamma_i)$ above have annular sutures) $(M_2, \Gamma_2)$ carries a universally tight contact structure by Theorem 6.1 of [13]. Note that the proof of
Theorem 6.1 in [13] uses the perturbation of a taut foliation into a universally tight contact structure; we give a foliation-theory-free proof of the same fact in [14].

Gabai’s construction gives \((M_1, \gamma_1 = \emptyset) \supseteq (M_2, \gamma_2)\), so to apply Theorem 6.1 of [13], it is necessary to produce a convex structure \((M_1, \Gamma_1)\) with annular sutures (in particular, \(\Gamma_1 \neq \emptyset\)) and a splitting surface \((S'_1, \sigma_1)\) such that \((M_1, \Gamma_1) \supseteq (M_2, \Gamma_2)\). Let \(\Sigma^0\) and \(\Sigma^1\) be the components of \(\partial M_1\) corresponding to the original splitting surface \(\Sigma\). Since \(S_1\) is well-groomed, the components of \(S_1 \cap \Sigma^i\), \(i = 0, 1\), are parallel oriented nonseparating curves in the isotopy class \(s_i\). Let \(A_i\) be an annular neighborhood of a curve dual to \(s_i\), and denote \(\partial A_i = \delta_i \cup \delta'_i\). Define \(\Gamma_1 = \delta_0 \cup \delta'_0 \cup \delta_1 \cup \delta'_1\). The convex structure \((M_1, \Gamma_1)\) is defined by decreeing that \(\Sigma^i \setminus A_i \subset (\Sigma^i)_+\) and \(A_i \subset (\Sigma^i)_-\) if and only if the orientation induced from \(\Sigma \setminus A\) agrees with the outward pointing normal orientation on \(\Sigma^i \setminus A_i\).

Now, \((S'_1, \sigma_1)\) is defined so that \(S'_1 = S_1\) and \(\sigma_1\) is the unique dividing set which is \(\partial\)-parallel and gives rise to \((M_2, \Gamma_2)\) after the splitting. The bypasses corresponding to the \(\partial\)-parallel dividing curves straddle curves of \(\Gamma_1\). Due to the well-grooming of \(S_1\), there is a unique choice of straddled curve for each pair \(\delta_i \cup \delta'_i\). It now follows that there is a universally tight contact structure on \((M_1, \Gamma_1)\) and that \(\partial M_1\) admits a full marking with \(\Gamma(S) = \delta_0 \cup \delta_1\), for example.

Next consider \(M = M_1 \cup (\Sigma \times [0, 1])\), where we identify \(\Sigma'\) with \(\Sigma \times \{i\}, i = 0, 1\). By Theorem 1.1, there is a (unique) universally tight contact structure on \(\Sigma \times I\) with \(\Gamma_{\theta(\Sigma \times I)} = \delta_0 \cup \delta'_0 \cup \delta_1 \cup \delta'_1\) and \(\Gamma(S) = \delta'_0 \cup \delta'_1\). We apply Theorem 2.1 twice to obtain a universally tight contact structure on \(M\), as follows. First consider the gluing of \(\Sigma^0\) to \(\Sigma \times \{0\}\). Denote the straddling set for \(\Sigma^0\) by \(S^0\) and the straddling set for \(\Sigma \times \{0\}\) by \(S^1\). Then \(S^0\) consists of one closed curve isotopic to \(s_0\) (note that the corresponding bypass is degenerate). Choose \(S^1\) so that it consists of one closed curve dual to \(\delta_0\) and \(S^0 \cup S^1\) is an embedded, essential family on \(\Sigma^0 = \Sigma \times \{0\}\). This is easily arranged since the genus of \(\Sigma^0\) is \(\geq 2\). Finally, by (2) of Theorem 1.1, there exists a (degenerate) bypass in \(\Sigma \times I\) along the unique closed curve in \(S^1\) which straddles \(\delta'_0\). Now we can apply Theorem 2.1. The second gluing is identical. \qed

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