Angular Ordering in Gluon Radiation

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Abstract

The assumption of angular ordering in gluon radiation is essential to obtain quantitative results concerning gluonic behaviors. In order to prove the validity of this assumption, we have applied our momentum space flux-tube formalism to check out the angular dependences of gluon radiation. We have calculated the probability amplitudes to get new gluon, and have found that the new gluon is generally expected to have the maximum amplitude when it is produced between the momentum directions of the last two partons.
The gluonic behaviors in non-perturbative regions are not so much understood that many phenomenological models are introduced to account for the relevant processes such as small-x physics\[1\], fragmentations into hadrons\[2\], and the confining aspects in bound states\[3\]. In order to provide for a systematic approach to these problems\[4\], one of us has developed flux-tube formalism\[5\] in which gluonic flux-tubes are classified and related to form topological spaces. General measures on the defined topological spaces can be used to predict the gluonic structures of hadrons\[6\] and, when applied in momentum space, the particle multiplicity distributions in jets can be analyzed systematically\[7\]. In this paper, we will try to apply the developed flux-tube formalism to the prediction of gluon radiation and to prove the validity of angular ordering assumption which is extensively used in predicting particle fragmentations.

It is well-known that the gluonic behaviors in perturbative region can be described by CCFM equation\[8\], which becomes A-P equation\[9\] for large x and BFKL equation\[10\] for small x. The probability to have a new gluon is usually given by

\[ dp = \Delta_s \tilde{P} dz dq_T^2 \Theta(\theta - \theta'), \]  

(1)

where \( \Delta_s \) is the Sudakov form factor\[12\], and \( \tilde{P} \) is the gluon-gluon splitting function\[13\]. The factor \( \Theta(\theta - \theta') \) represents the condition that the radiated gluons are ordered in angles. However, the final direction of radiation cannot be determined by this short range condition. In order to prove the validity of this assumption in long range scale\[14\], we now turn to a brief introduction of our flux-tube formalism.

The starting point for a systematic description of flux-tubes rests on the classification of flux-tubes, which are taken to start from quark boundaries and to end at antiquark boundaries. It is sufficient to count the number of quarks and antiquarks to classify flux-tubes, and we can represent the set of flux-tubes with \( a \) quarks and \( b \) antiquarks sitting at boundaries as \( F_{a,b} \). Omitting the number 0, except for \( F_0 \) representing glueballs, mesonic flux-tubes are represented by \( F_{1,1} \), and baryonic and antibaryonic flux-tubes by \( F_3 \) and \( F_\bar{3} \). For the classified flux-tube sets, we can consider relationships between them which are generated by quark pair creations and annihilations. The division of a flux-tube is generated by a quark pair creation, and the union of two flux-tubes by a pair annihilation. These relationships can be
used to construct topological spaces of flux-tubes, which are necessary to define physical amplitudes. The assumptions for the construction of topological spaces are

1. Open sets are stable flux-tubes.
2. The union of stable flux-tubes becomes a stable flux-tube.
3. The intersection between a connected stable flux-tube and disconnected stable flux-tubes is the reverse operation of the union.

With these assumptions, we can follow the flux-tube sets that can be produced from a given flux-tube by repeating union and intersection operations. If the produced sets are closed under these operations, we can classify the constructed topological spaces and this classification procedure can be done by counting the numbers of incoming and outgoing 3-junctions in a given closed set. When we include the excited flux-tube set $F_0$, two 3-junctions can be created making it impossible to assign fixed numbers of 3-junctions to a given set, and therefore, we omit this possibility in this paper. Then the simplest non-trivial topological space is

$$T_0 = \{ \phi, F_{1,1}, F_{1,1}^2, \ldots, F_{1,1}^n, \ldots \}$$

where $F_{1,1}^n$ represents $n$ quarkonium meson states. Since $F_{1,1}$ can be multiplied repeatedly without violating the law of baryon number conservation, we may reduce the notation into $T_0 \equiv \{ \phi, F_{1,1} \}$. In this notation, the topological space for baryon-meson system can be represented as

$$T_1 = \{ \phi, F_3 \},$$

and the baryon-meson-baryon space becomes

$$T_2 = \{ \phi, F_3^2 \}.$$  

When outgoing 3-junctions exist, we need another index to represent the topological space. For example, the space with two incoming 3-junctions and one outgoing 3-junction is represented as

$$T_{2,1} = \{ \phi, F_3^2 F_3, F_3 F_2, F_4 \}$$

In general, we can write down the spaces as

$$T_{i,j} = \{ \phi, F_3^i F_3^j, F_3^{i-1} F_3^{j-1} F_2, \ldots \}$$
where \( i \) is the number of incoming 3-junctions and \( j \) that of outgoing 3-junctions.

Now let’s try to define physical amplitude related to the measures on flux-tubes. It is physically natural to define the amplitudes \( A \) for a quark to be connected to another quark or antiquark through given flux-tube open set. In order to quantify \( A \), we can assume the existence of a measure \( M \) of \( A \) satisfying the conditions

1. \( M(A) \) decreases as \( A \) increases,
2. \( M(A_1) + M(A_2) = M(A_1A_2) \) when \( A_1 \) and \( A_2 \) are independent.

From these two conditions, the measure \( M \) of \( A \) can be solved as functions of \( A 
\[
M(A) = -k \ln \frac{A}{A_0},
\]
where \( A_0 \) is a normalization constant and \( k \) is an appropriate parameter. One reasonable method to convert the amplitude \( A \) into a concrete form is to consider the measure \( M \) as a metric function defined on the flux-tube. A general form of distance function between the two boundary points \( x \) and \( y \) can be written down as \( |x - y|^\nu \) with \( \nu \) being an arbitrary number. This distance function can be made metric for the points \( z \) satisfying

\[
|x - z|^\nu + |z - y|^\nu \geq |x - y|^\nu.
\]

The set of points \( z \) not satisfying this triangle inequality can be taken as forming the inner part of the flux-tube where it is impossible to define a metric from boundary points with given \( \nu \). If we take \( |x - y|^\nu \) as an appropriate measure for \( A \), we need to sum over contributions from different \( \nu \)'s. For a small increment \( d\nu \), the product of the two probability amplitudes for \( |x - y|^\nu \) and \( |x - y|^{\nu + d\nu} \) to satisfy the metric conditions can be accepted as the probability amplitude for the increased region to be added to the inner connected region which is out of the metric condition. Considering all possibilities, the full connection amplitude becomes

\[
A = A_0\exp\{-\frac{1}{k} \int_1^{\alpha} F(\nu)r^\nu d\nu\},
\]

where the lower limit of \( \nu \) is fixed to 1 because there exists no point \( z \) satisfying the triangle inequality with \( \nu < 1 \), and the upper limit \( \alpha \) is arbitrary.
The weight factor $F(\nu)$ has been introduced in order to account for possible different contributions from different $\nu'$s, and the variable $r$ is

$$r = \frac{1}{l}|x - y|$$

with $l$ being a scale parameter. When we take the case of $\alpha = 2$, which corresponds to a spherical shape flux-tube, and the case of equal weight $F(\nu) = 1$, we get

$$A = A_0 \exp\left\{-\frac{1}{k} \frac{r^2 - r}{\ln r}\right\}.$$  \hfill (11)

We will use this form of connection amplitude.

For scattering states, we need to formulate the connection amplitude in momentum space. By applying the same arguments, we can write down the connection amplitude for two boundary points with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$

$$A = A_0 \exp\left\{-\frac{1}{\tau} \int_1^\alpha G(\nu)|\mathbf{p}_1 - \mathbf{p}_2|^\nu d\nu\right\},$$

where $G(\nu)$ and $\alpha$ are weight factor and the upper limit of $\nu$. In case of $G(\nu) = 1$ and $\alpha = 2$, we get

$$A(p) = A_0 \exp\left\{-\frac{1}{\tau} \frac{p^2 - p}{\ln p}\right\}$$

with $p = |\mathbf{p}_1 - \mathbf{p}_2|$.  \hfill (13)

Now let’s consider the two partons with momenta $p_1$ and $p_2$ subtending an angle $\theta$. We may take the parton 1 as a quark and the second one as a radiated gluon, or may take both particles as quarks or gluons. In any case, we want to calculate the probability amplitude to have the third gluon with given momentum in some direction. In our momentum space flux-tube model, the probability amplitude to have the third gluon is taken to be proportional to the connection amplitude representing the connections of third gluon with the other two particles. Of course, these connections are formulated in non-perturbative region, that is, in long range region. Our calculations have been carried out with the form of $A$ given in Eq.(13), varying the magnitude and angle of the momentum $\mathbf{p}_3$ of the third gluon. The amplitude is

$$A = A_0^2 \exp\left\{-\frac{1}{\tau} \frac{|\mathbf{p}_1 - \mathbf{p}_3|^2 - |\mathbf{p}_1 - \mathbf{p}_3|}{\ln |\mathbf{p}_1 - \mathbf{p}_3|} - \frac{1}{\tau} \frac{|\mathbf{p}_3 - \mathbf{p}_2|^2 - |\mathbf{p}_3 - \mathbf{p}_2|}{\ln |\mathbf{p}_3 - \mathbf{p}_2|}\right\}.$$  \hfill (14)
In Fig.1, the probability amplitudes to have the third gluon in the same plane of $p_1$ and $p_2$ are shown as functions of the magnitude of gluon momentum. We have fixed some parameters as $A_0 = 1$, $\tau = 0.7$ and $p_1 = 1.0$, and we have shown the case of $p_2 = 0.3$ and $\theta = \pi/6$ in Fig.1. The different curves correspond to different angles which have been selected by dividing the given angle $\theta$ into 9 equally spaced angles. The lower curve corresponds to smaller angle, and so we can see that the maximum probability occurs at higher values of momentum as the value of angle increases. In any case, the maximum probability results from a momentum value between the two given values $p_1 = 1.0$ and $p_2 = 0.3$. We have also changed the values of $p_2$ and $\theta$, and obtained similar results. The general shape is closely related to the momentum or energy distributions of particles in a jet\cite{15}.

For angular variations, we have checked the amplitude dependence on angles by changing the values of $p_2$ and $\theta$. For a fixed value of $p_2$, the shapes of the curves are not so much changed as we vary the values of subtending angle $\theta$. However, if we vary the values of $p_2$ for fixed $\theta$ as in Fig.2, we can see that the probability amplitude becomes peaked around the direction of larger momentum. In Fig.2, we have fixed $\theta$ to $\pi/4$ and the angle is measured from the axis of larger momentum $p_1$. We have obtained quite similar results for other choices of $\theta$. For larger values of $p_2$ comparable with $p_1$, the peak values of $A$ appear near $\frac{\theta}{2}$, and move toward the direction of $p_1$ as the value of $p_2$ decreases. This result can be accepted as consistent with the angular ordering condition in Eq(1). Moreover, we can predict the maximal direction in which a new gluon will be radiated. Since the variations of the probability amplitude are continuous, it will be possible to introduce smooth function instead of step function $\Theta$ to predict gluon radiation. In this case, the probability to violate the strict angular ordering represented by $\Theta(\theta - \theta')$ is not zero as can be seen in Fig.2.

In summary, we have calculated the probability amplitude of gluon radiation by using momentum space flux-tube model. We have found that the direction of a radiated gluon depends on the magnitude and directions of momenta of the two precedent partons. The new gluon will be most likely radiated in the direction between the two initial partons, which corresponds to the usual process of fragmentation\cite{16}. This result is consistent with the angular ordering assumption appearing in the equations such as CCFM.
equation. However, there exists the possibility in our formalism that the new gluon is to be radiated in a direction outside the angle subtended by the two initial partons. We need further work to replace the simple step function representing angular ordering with some smooth function which can be manageable in solving the equations describing gluonic behaviors.

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Figure Caption

Fig. 1  The probability amplitude to have new gluon as functions of gluon momentum. We have fixed $p_1 = 1.0, p_2 = 0.3$ with the angle $\theta = \frac{\pi}{6}$. The different curves correspond to different directions.

Fig. 2  Angular variation of the probability amplitude. We fixed the angle $\theta$ as $\frac{\pi}{3}$ and $p_1 = 1.0$. The different curves correspond to different values of $p_2$. 
 gamma vs momentum
Fig. 2

gamma

angle (rad)