Entropy driven mechanism for noise induced patterns formation in reaction-diffusion systems

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Abstract

We have studied the entropy-driven mechanism leading to stationary patterns formation in stochastic systems with local dynamics and non-Fickian diffusion. We have shown that a multiplicative noise fulfilling a fluctuation-dissipation relation is able to induce and sustain stationary structures with its intensity growth. It was found that at small and large noise intensities the system is characterized by unstable homogeneous states. At intermediate values of the noise intensity three types of patterns are possible: nucleation, spinodal decomposition and stripes with liner defects (dislocations). Our analytical investigations are verified by computer simulations.

Key words: Stochastic systems; internal multiplicative noise; stationary structures.

PACS:

1 Introduction

It is well known that the interaction of noise and nonlinearity attracts a lot of interest in nonequilibrium systems theory [1]. Most of articles, concerning above noise induced phenomena, relate to problems of external noise influence: noise induced transitions in zero-dimensional systems [2]; noise-induced phase transitions [3, 4]; stochastic resonance [5]; noise sustained patterns [6–9], etc. Analytically, numerically and in natural experiments it was found that an external noise plays an organizing role only if its amplitude depends on the field.
variable — multiplicative noise (see, for example, Refs. [2, 4, 10, 11]). If more than one noise source is introduced into the system dynamics, then the noise cross-correlations start to play a crucial role in ordering processes: a kind of phase transitions principally changes [12, 13]. From a fundamental viewpoint such effects has a dynamical origin: in a short-time limit external fluctuations destabilize a disordered homogeneous state. An analytic description of extended systems with external fluctuations is provided with an approximately known stationary distribution function.

The possibility of noise induced patterns formation has been studied in last two decades (see, for example, Refs. [4, 6–9, 14–17]). One of the most interesting problem when the noise sustains patterns is widely discussed recently [17–21]. In the most of works, concerning this problem, external fluctuations influence on the dynamics of systems is considered. Recently a new mechanism for noise induced patterns formation was discovered. Its origin is in a relaxational dynamics governed by the field-dependent mobility/kinetic coefficient [22]. It was shown that corresponding self-organization processes are not related to a short-time instability of a homogeneous/disordered state. Here the principle role is played by entropy variations caused by the field-dependent mobility [23–28]. Thus, an ordered behaviour appears due to the balance between relaxing forces moving the system to the homogeneous state, and fluctuations dependent on the field variable pulling the system away from the disordered state. This mechanism belongs to a kind of entropy driven phase transitions that is an extension of noise induced unimodal-bimodal transitions in zero-dimensional systems [2]. In such systems a dissipation is related to fluctuations according the fluctuation dissipation relation. Hence, one gets an internal multiplicative noise with intensity reduced to the bath temperature. For such class of stochastic systems the corresponding distribution function, free energy and associated effective potential are known exactly. Therefore, coherent structures/patterns appeared in the course of self-organization processes can be analyzed without any dynamical reference.

In this paper we will study a behaviour of extended stochastic systems when the noise can induce or sustain spatial patterns according to entropy driven mechanism. To this end we consider a model of reaction-diffusion system of the field-dependent mobility with non-Fickian diffusion and internal fluctuations of the multiplicative character. Reaction-diffusion systems play an important role in the study of generic spatiotemporal behaviour of far from equilibrium systems. These models are usually applied to describe an extended systems when the physical space is divided in to the small cells and dynamics of each cell is described by a variable type of concentration. The model to be considered concerns the local dynamics of the variables inside each cell and the transport phenomena between cells. The local dynamics is determined by so-called chemical reactions inside each cell. The transport is caused by non-Fickian diffusion. We will study stationary spatial inhomogeneities with lengths bigger
than the linear dimensions of the cells in such system.

Formally, reaction-diffusion models are described by the equation for the field $x = x(r, t)$ in the form $\partial_t x = f(x) - \nabla J$, where $f(x)$ stands for the local dynamics, $J$ is the flux for transport phenomena, $\nabla = \partial_r$. Considering the non-Fickian diffusion, we exploit a gradient of molecular interactions potential $U(r)$ ($U(r) = -\int u(r - r')x(r')dr'$, where the spherically symmetric interaction potential $u(r)$ between molecules separated by a distance $|r|$ is introduced). The corresponding force given by the gradient of $U(r)$ governs the transport phenomena. In the case of the small interaction radius compared to the diffusion length the concentration field $x$ will not vary significantly within the interaction radius. It allows to approximate the integral by $\kappa x + \beta \nabla^2 x$, where $\kappa = \int u(r)dr$, $\beta = (1/2) \int |r|^2 u(r)dr$ ($\beta \simeq \kappa r_c^2/2$, where $r_c$ is a correlation radius); $\int r u(r)dr = 0$ due to symmetrical properties of the interaction potential [29]. Therefore, the obtained flux allows to describe phase separation processes with mutual (lateral) interactions; the combined model with local dynamics can be used to consider the spatial patterns induced by the fluctuations of the bath. With the help of such formalism one can find an explicit form of the stationary distribution functional and study spatial patterns using a variational principle. Analytical results we verify by computer simulations.

The paper is organized in the following manner. In the Section II we introduce a general model for stochastic reaction-diffusion systems and present the theoretical approach is useful in our consideration. In Section III we discuss the model and assumptions related to our study. Section IV deals with a theoretical investigation of stationary noise induced/sustained patterns. Finally we summarize our results in Section V.

2 Main equations

Let us start with the typical deterministic equation for the reaction-diffusion models in the form

$$\partial_t x = f[x(r, t)] - \nabla J[x(r, t)],$$

where $x$ stands collectively for the variables, $f[x]$ for the local dynamics inside each cell, $J[x]$ is the flux proportional to the conjugate thermodynamic force, arising from the spatial variation of the chemical potential $\mu[x(r, t)]$. One can exploit a definition $J[x(r, t)] = -D_{ef}[x(r, t)]\nabla \mu[x(r, t)]$, where the chemical potential $\mu = \delta F/\delta x$ is introduced through the free energy $F[x]$; $D_{ef}[x]$ is an effective diffusion coefficient related to the field-dependent mobility. In such a case, the deterministic evolution equation for the field $x$ reads

$$\partial_t x = f[x] + \nabla (D_{ef}[x] \nabla \mu[x]).$$
Formally, Eq.(2) can be written in a variational form as
\[ \partial_t x = -\frac{1}{D_{ef}[x]} \frac{\delta\mathcal{U}[x]}{\delta x}, \]  
(3)
where the potential functional \( \mathcal{U}[x] \) is represented through the definitions for \( f[x], D_{ef}[x] \) and \( \mathcal{F}[x] \). It plays a role of a Lyapunov functional for the deterministic dynamics. An explicit form for the functional \( \mathcal{U}[x] \) can be found only if \( \mu = x \). Indeed, here one has
\[ \mathcal{U}[x] = \int dr \left\{ -\int f[x']D_{ef}[x']dx' + (D_{ef}[x]\nabla x)^2/2 \right\}. \]

Generally, if the chemical potential \( \mu \) is a function of both \( x \) and \( \nabla x \), then one can obtain only its first variation,
\[ \delta\mathcal{U}[x] = -\int dr \delta x \left\{ f[x]D_{ef}[x] + D_{ef}[x]\nabla (D_{ef}[x]\nabla \mu[x]) \right\}, \]
(4)
which after substitution into Eq.(3) gives Eq.(2) immediately.

Considering the system under real conditions of thermal bath influence, one has to take into account corresponding fluctuations. In stochastic analysis we introduce a related multiplicative noise in an ad hoc form, following Ref. [17],
\[ \partial_t x = -\frac{1}{D_{ef}[x]} \frac{\delta\mathcal{U}[x]}{\delta x} + \frac{1}{\sqrt{D_{ef}[x]}} \xi(r,t). \]
(5)
where the fluctuation dissipation relation holds, i.e. \( \langle \xi(r,t)\xi(r',t') \rangle = 2\sigma^2 \delta(r-r')\delta(t-t'); \langle \xi(r,t) \rangle = 0; \sigma^2 \) is the intensity of a Gaussian noise \( \xi \).

Using the Stratonovich interpretation of the stochastic equation (5), the stationary solution of the corresponding Fokker-Planck equation can be written as follows [23]:
\[ \mathcal{P}_{st}[x] \propto \exp \left( -\frac{\mathcal{U}_{ef}[x]}{\sigma^2} \right), \]
(6)
where the effective potential functional \( \mathcal{U}_{ef}[x] \) is of the form
\[ \mathcal{U}_{ef}[x] = \mathcal{U}[x] - \frac{\sigma^2}{2} \int dr \ln D_{ef}[x]. \]
(7)
It follows, that the stationary distribution functional \( \mathcal{P}_{st} \) or the effective functional \( \mathcal{U}_{ef} \) are obtained exactly: the form of initial functional \( \mathcal{U} \) is supposed to be known, the second term in \( \mathcal{U}_{ef} \) can be calculated if needed. Let us note, if we assume that \( \mathcal{U} \) plays a role of an effective free energy functional, then rewriting the integral in Eq.(7) as \( \mathcal{S}_{ef} = \int dr \ln(D_{ef}[x])^{-1} \), the expression (7) can be transformed into the thermodynamic relation between free energy, internal energy and entropy functionals: \( \mathcal{U}_{ef} = \mathcal{U} + \sigma^2 \mathcal{S}_{ef} \). Therefore, according
to such relation the noise intensity $\sigma^2$ addresses to an effective temperature of the bath, whereas $S_{ef}$ plays a role of the effective entropy. Such situation is well known in the stochastic systems theory. It appears when the multiplicative fluctuations corresponded to the internal noise. The last one results to the entropy change that yields entropy driven phase transitions [18, 23, 26, 28]. In this paper we will not discuss above phase transitions, we will consider an ability of the noise to sustain or induce formation of spatial structures.

Stationary noise sustained structures $x_s(r)$ correspond to the extreme of the functional $U_{ef}[x]$. To find an associated equation for $x_s(r)$ we make the first variation of the effective functional with respect to $x$ and equal it to zero, that is

$$\delta U_{ef}[x] = -\int d\mathbf{r} \delta x \left\{ [f[x]D_{ef}[x] + D_{ef}[x]\nabla (D_{ef}[x]\nabla \mu[x])] + \frac{\sigma^2}{2D_{ef}[x]} \frac{\partial D_{ef}[x]}{\partial x} \right\} = 0.$$  

(8)

As a result, the corresponding equation takes the form

$$\nabla (D_{ef}[x]\nabla \mu[x]) + f[x] + \frac{\sigma^2}{2D_{ef}^2[x]} \frac{\partial D_{ef}[x]}{\partial x} = 0.$$  

(9)

Homogeneous states are defined as solutions of the reduced equation

$$fD_{ef}^2 = -\frac{\sigma^2}{2} \frac{\partial D_{ef}}{\partial x}.$$  

(10)

From the mathematical viewpoint solutions of Eq.(10) give extreme positions of the function $U_{ef}(x)$ obtained under supposition that $x(r) = \text{const}$.

3 The model

In order to investigate a possibility of patterns formation we use the well known assumptions for the diffusion coefficient $D_{ef}$: the diffusion coefficient decreases from the constant value with the field $x$ growth. Following Ref. [22], the generalized formula for $D_{ef}$ can be written in the form $D_{ef}(x) = (1-x^2)^\alpha$, where $\alpha > 0$. Under such an assumption the diffusion $D_{ef}$ decreases slightly in the vicinity of $x = 0$ and goes to zero with an increase in $x$ by absolute value. At small $\alpha$ one can use an approximation

$$D_{ef} = \frac{1}{1+\alpha x^2}, \quad \alpha \geq 0.$$  

(11)

Such a construction assumes that fluctuations can not disappear with an increase in $x$, moreover some analytic solutions of the problem can be obtained.
The reaction term $f(x)$ can be defined according to a chemical kinetics and generally is represented through a potential function, $V(x)$ in the standard way: $f(x) = -\partial V/\partial x$. In this work we study a case of nonlinear potential force of the form $f(x) = -\prod_i (x - x^0_i)$, where the set $\{x^0_i\}$ corresponds to zero values of the force and relates to stationary points of the deterministic system. In our approach this force associates with the potential

$$V(x) = \frac{x^4}{4} + \frac{\mu}{3} x^3 - \frac{\varepsilon}{2} x^2,$$

(12)

here $\mu$ and $\varepsilon$ are constants that control the system dynamics. The potential (12) has three extrema placed at $x^\pm_0 = -\mu/2 \pm \sqrt{\mu^2 + 4\varepsilon}$, and at $x_0 = 0$. A spinodal is given by equation $\varepsilon = -\mu^2/4$. A prototype model of chemical reactions is $A \rightleftharpoons B$ with transient reactions $A + 2X \rightleftharpoons 3X$, $X \rightleftharpoons B$, where first reaction occur with constant rates $k_1$, $k_2$, the second one is realized with $k_3$, $k_4$ [30]. Here $x \in [-1, 1]$ measures deviations of spices $X$ concentration from the constant fixed value controlled by parameters $\varepsilon$ and $\mu$, which relate to the constant rates $k_1$, $k_2$, $k_3$, $k_4$.

To make the quantitative analysis let us use the chemical potential in the standard form $\mu = \delta F/\delta x$, where the free energy functional is assumed in the Ginzburg-Landau form

$$\mathcal{F} = \int \text{d}r \left[ \phi(x) + \frac{\beta}{2} (\nabla x)^2 \right],$$

(13)

$\phi(x)$ the local potential; $\beta > 0$ is an inhomogeneity constant. In order to make a general description of patterns formation scenario let us consider the prototype model of phase separation where the local potential is of the form

$$\phi = -\frac{\kappa}{2} x^2 + \frac{1}{4} x^4,$$

(14)

where $\kappa$ is a control parameter. From the formal viewpoint one can consider two different models. Indeed, in the simplest case of lateral interactions one can use the local potential in the form $\phi(x) = -\frac{\kappa}{2} x^2$, $\kappa > 0$ [29]. A case of ordinary phase separation scenario corresponds to the $x^4$-construction, given by Eq.(14). Next, we derive a formalism for a general model and compare results for these two models of phase separation.
4 Results

4.1 Noise induced transitions

Let us note that the noise influence leads to a short-time instability in our model. Indeed, the linear stability analysis of the linearized Langevin equation (5) allows to find an evolution equation for the structure function \( S(k, t) = \langle x_k(t)x_{-k}(t) \rangle \), where \( x_k(t) = \int x(r, t)e^{ikr}dr \) is the Fourier transform of the concentration field, (...) represents an ensemble average over noise. Considering the state \( x = 0 \), a spherically averaged structure function \( S(k, t) = \int S(k, t)d\Omega_k \) (\( \Omega_k \) is the hyperspherical shell of radius \( k \)) evolves according to a linear equation

\[
\frac{1}{2} \frac{dS(k, t)}{dt} = -\omega(k)S(k, t) + \sigma^2,
\]

where \( \omega(k) = k^2(\beta k^2 - \kappa) - \varepsilon - \alpha \sigma^2 \). It follows that internal multiplicative noise leads to instability of the null state. The stationary value of the structure function is

\[
S_{st}(k) = \frac{\sigma^2}{k^2(\beta k^2 - \kappa) - \varepsilon - \alpha \sigma^2}.
\]

It is seen that at short-time scales with an increase in the noise intensity the peak position of the function \( S_{st}(k) \) is shifted toward large values of the wave vector.

To make an appropriate analysis of patterns formation scenario, let us consider a case of a homogeneous (zero-dimensional) system, assuming stochastic variable depending on the time only, i.e. \( x(r, t) = x(t) \). The main attention will be paid to stationary states investigation. In the case under consideration the stationary probability density function is of the form

\[ P_{st} \propto \exp\left(-\frac{U_{ef}(x)}{\sigma^2}\right), \]

where the functional is reduced to the function, \( U_{ef}[x] \rightarrow U_{ef}(x) \). Hence, the effective potential is \( U_{ef}(x) = -\int dx'f(x)D_{ef}(x) - (\sigma^2/2)\ln D_{ef}(x) \). Inserting definitions for \( f \) and \( D_{ef} \) into this construction, we obtain

\[
U_{ef}(x) = \frac{1}{2\alpha}x^2 + \frac{\mu}{\alpha}x - \frac{1}{2\alpha} \left( \varepsilon + \frac{1}{\alpha} \right) \ln(1+\alpha x^2) - \frac{\mu}{\alpha^{3/2}} \arctan(\alpha \sqrt{x}) + \frac{\sigma^2}{2} \ln(1+\alpha x^2).
\]

To consider a case \( x \in [-1, 1] \) let us assume values of both \( \varepsilon \) and \( \mu \) to locate a minimum \( U_{ef}(x_-) \) at \( x < 0 \), a minimum \( U_{ef}(x_+) \) we locate at \( x > 0 \). An appropriate choice is \( \varepsilon = 0.2, \mu = -0.5, \alpha > 0 \); the corresponding dependence \( U_{ef}(x, \sigma^2) \) is shown in Fig.1. To find most probable states we solve the problem \( dU_{ef}(x)/dx = 0 \). The corresponding equation takes the form (10). As it follows from our consideration the functional dependence of \( D_{ef} \) leads to bifurcations (a number of extrema of the effective potential changing). Hence, solutions of Eq.(10) allow to find bifurcation diagram illustrating extrema positions of the effective potential. In order to calculate critical points related
Fig. 1. The effective potential $U_{\text{ef}}(x, \sigma^2)$ at $\alpha = 0.2$, $\epsilon = 0.2$, $\mu = -0.5$.

to the bifurcations one needs to differentiate Eq.(10) with respect to $x$ and after solve both obtained equation and Eq.(10) simultaneously. Solutions of Eq.(10) show that a root $x_0 = 0$ exists always. Another two roots

$$x_\pm = -\frac{\mu}{2} \pm \frac{1}{2} \sqrt{\mu^2 + 4\epsilon - 4\sigma^2\alpha}$$  \hspace{1cm} (18)

are realized if $\sigma^2 < \sigma_c^2$ where

$$\sigma_c^2 = \frac{1}{\alpha} \left( \epsilon + \frac{\mu^2}{4} \right).$$  \hspace{1cm} (19)

At $\sigma^2 = \sigma_c^2$ solutions $x_-$ and $x_+$ degenerate, and at $\sigma^2 > \sigma_c^2$ only trivial one, $x_0 = 0$, remains. The corresponding dependencies $x_\pm(\sigma^2)$ are shown in Fig.2a. To understand transformations of the system states let us use the noise induced transitions formalism [2]. As it follows from naive consideration, the bimodal stationary distribution $P_{st}(x) \propto \exp(-U_{\text{ef}}(x)/\sigma^2)$ becomes unimodal with an increase in the noise intensity $\sigma^2$. In the case under consideration above transition occurs in the following manner. At $\sigma^2 = 0$ a form of the effective potential $U_{\text{ef}}$ is identical topologically to a form of the initial potential $V(x)$. With an increase in the noise intensity $\sigma^2$ a minimum of $U_{\text{ef}}(x_-)$ located at $x_-$ tends to zero, at $\sigma^2 = \sigma_s^2 = \epsilon/\alpha$ the effective potential has a double degenerated point, $x_0 = x_- = 0$. Therefore, the values $\sigma_s^2$ define a spinodal curve. At $\sigma_s^2 < \sigma^2 < \sigma_0^2$ the point $x_0$ relates to a minimum, whereas $x_-$ defines a maximum position of the function $U_{\text{ef}}$. These two minima differ in depth, i.e. $U_{\text{ef}}(0) > U_{\text{ef}}(x_+)$. At $\sigma^2 = \sigma_0^2$ one has $U_{\text{ef}}(0) = U_{\text{ef}}(x_+)$, therefore, $\sigma_0^2$ defines a coexistence line (binodal). With a further increase in $\sigma^2$ we get $U_{\text{ef}}(0) < U_{\text{ef}}(x_+)$. An equality $U_{\text{ef}}(x_-) = U_{\text{ef}}(x_+)$ is satisfied at
In plot (a): solid lines define stable states, dashed line corresponds to unstable solution. In plot (b): solid line corresponds to $\sigma^2_c$ values, dashed and dotted lines relate to $\sigma^2_s$ and $\sigma^2_0$, respectively.

$\sigma^2 = \sigma^2_c$, hence the bifurcation point, $\sigma^2_c$, define another spinodal. At $\sigma^2 > \sigma^2_c$ the effective potential has one well only. Therefore, in such a noise induced transition we have shift of the potential extreme, transformation of the global minimum into a local one, loss of its stability and, finally, change a number of extreme of the function $U_{ef}$. The corresponding diagram illustrating above situation is shown in Fig.2b.

4.2 Noise induced patterns

Next, let us consider a possibility of the system to manifest stationary structures formation. To this end we assume the spatial dependence of the stochastic field and solve the variation problem, $\delta U_{ef}[x]/\delta x = 0$. The corresponding equation takes the form of Eq.(9). Let us rewrite Eq.(9) in a more convenient form

$$D_{ef}\left[\beta \nabla^4 x - \frac{\partial^2 \phi}{\partial x^2} \nabla^2 x - \frac{\partial^3 \phi}{\partial x^3} (\nabla x)^2\right] - \frac{\partial D_{ef}}{\partial x} \frac{\partial^2 \phi}{\partial x^2} (\nabla x)^2 = f + \frac{\sigma^2}{2D_{ef}^2} \frac{\partial D_{ef}}{\partial x}.$$

As it follows from our consideration, the system stationary behaviour can be described in the 4-dimensional space $(x, y, z, u)$ where $y = \nabla x$, $z = \nabla^2 x$, $u = \nabla^3 x$. Using results form the homogeneous system analysis one can state that there are three fixed points describing homogeneous states with coordinates $(x_-, 0, 0, 0)$, $(0, 0, 0, 0)$ and $(x_+, 0, 0, 0)$ at $\sigma^2 < \sigma^2_c$ and the unique fixed point $(0, 0, 0, 0)$ at $\sigma^2 > \sigma^2_c$.

Firstly, let us investigate the stability of stationary solutions of Eq.(20) in the $r$–space. To this end one assumes $x(r)$ to be in the form $x(r) = x(0) + \delta \exp(\lambda r)$, $\delta \ll 1$ is a small perturbation in the vicinity of the corresponding
homogeneous solution. Inserting this assumption with corresponding expressions for all possible derivatives into Eq.(20), one gets the equation

$$\lambda^4 - \lambda^2 \left[ D_{ef} \frac{\partial^2 \phi}{\partial x^2} \right]_{x = \{x_0, x_-, x_+, x_-\}} - \left[ \frac{\partial}{\partial x} \left( f + \frac{\sigma^2}{2D_{ef}} \frac{\partial D_{ef}}{\partial x} \right) \right]_{x = \{x_0, x_-, x_+, x_-\}} = 0 \quad (21)$$

for eigenvalues $\vec{\lambda} = \vec{\gamma} \pm i \vec{k}$ of the Jacobi matrix, where $\vec{\gamma} = \{ \gamma_j \}_{j=1}^4$, $\vec{k} = \{ k_j \}_{j=1}^4$; $\gamma_j = \Re \lambda_j$ defines the local stability of the fixed point along $j$-th axis, $k_j = \Im \lambda_j$ gives the corresponding oscillations frequency that relates to the wave vector magnitude.

From the linear stability analysis it follows that relations for imaginary parts $k_1 = -k_2$, $k_3 = -k_4$ and real parts $\gamma_1 = -\gamma_2$, $\gamma_3 = -\gamma_4$ are hold. Solving Eq.(21) and using these symmetrical properties, we present only positive values of $\vec{k}$ and $\vec{\gamma}$ as functions of the noise intensity $\sigma^2$ in Fig.3. Obviously,

Fig. 3. Positive values of imaginary $\vec{k}$ (a) and real $\vec{\gamma}$ (b) parts of eigenvalues $\vec{\lambda}$ versus the noise intensity $\sigma^2$, other parameters are: $\varepsilon = 0.2$, $\mu = 0.5$, $\alpha = 0.2$, $\kappa = 1$. Solid, dashed and dotted lines correspond to stability of the fixed points $O(0, 0, 0, 0)$, $(x_+, 0, 0, 0)$ $(x_-, 0, 0, 0)$, respectively. Plots (a) correspond to lateral interaction model, plots (b) are related to $x^4$-model.

if one of the real part related to the fixed point of a homogeneous state is positive, then the such point is unstable in the r-space. In the case of zero values of all real parts for the fixed point with nonzero imaginary ones, the fixed point is a center of a manifold type of limit cycle (with a single non-trivial $k_j$) or torus (with more than one nonzero imaginary parts). To relate
$k_j$ to the profile $x(r)$ or the corresponding derivatives we have solved Eq.(20) in the one-dimensional case at the appropriate values for the system parameters. The corresponding description will be given below. As Fig.3 shows at small noise intensities with $\sigma^2 < \sigma^2_{T-}$, all fixed points are unstable, no patterns can be formed. If the threshold $\sigma^2_{T-}$ is crossed, then the stability of the point $(x_-,0,0,0)$ is changed. It becomes a center of a set of tori with two frequencies $k_1$ and $k_2$ displayed as dotted lines in Fig.3. When the threshold $\sigma^2_s$ is crossed, then the stability of the solution $(x_-,0,0,0)$ is changed again, it becomes an unstable one. Therefore, in the interval $\sigma^2_{T-} \leq \sigma^2 \leq \sigma^2_s$ the phase space is characterized by a single set of tori formed at the center of the point $(x_-,0,0,0)$. At $\sigma^2 > \sigma^2_s$ the null state, characterized by the point $(0,0,0,0)$ acquires a neutral stability, which is realized till $\sigma^2 = \sigma^2_{T0}$ (see solid lines in Fig.3), where

$$\sigma^2_{T0} = \sigma^2_s + \frac{\kappa^2}{4\alpha}.$$  \hspace{1cm} (22)

Moreover, at $\sigma^2 = \sigma^2_{T+}$ the fixed point $(x_+,0,0,0)$ becomes a center of an additional set of tori (dashed lines). Hence, in the interval of the noise intensity $\sigma^2_{T+} < \sigma^2 < \sigma^2_s$ two set of tori are observed. At $\sigma^2 = \sigma^2_s$ the stability of the point $(x_+,0,0,0)$ is changed, it becomes unstable due to the one of the corresponding real part is positive. Therefore, two sets of tori around the points $O(0,0,0,0)$ and $(x_+,0,0,0)$ are realized till the threshold $\sigma^2_s$. Due to the bifurcation related to the noise induced transition oscillations in the vicinity of the point $(x_+,0,0,0)$ disappears at $\sigma^2 = \sigma^2_c$ and as a result a single set of tori around the point $(0,0,0,0)$ can be observed.

The value $\sigma^2_{T0}$ as other thresholds except $\sigma^2_s$ and $\sigma^2_c$ are dependent on the parameter $\kappa$. Above, we have considered the case when $\sigma^2_c < \sigma^2_{T0}$. From our analysis it follows that at small noise intensities or without noise the homogeneous states are absolutely unstable. With an increase in the noise intensity two-period solutions of Eq.(20) are realized. At large noise intensity the system is characterized by the unstable homogeneous solution. Therefore, we get a reentrant picture of self-organization with the noise growth. The corresponding phase diagram illustrating patterns formation is shown in Fig.4. Here we plot the parameter $\alpha$ versus noise intensity $\sigma^2$ at fixed values for other control parameters. Unstable states are outside the filled domain. In the filled part we denote three sub-domains (a), (b) and (c). In the sub-domains (a) and (c) only one set of tori is formed around points $(x_-,0,0,0)$ and $(0,0,0,0)$, respectively. The domain (b) is characterized by two sets of tori in the vicinity of both points $(0,0,0,0)$ and $(x_+,0,0,0)$. All calculations are performed for two choices lateral interactions and $x^4$-model of phase separation. Comparing Fig.3a and Fig.3b, one can see that there are no qualitative changes in the dependencies of the eigenvalues of the Jacobi matrix versus the noise intensity. The situation is similar for these two models. The corresponding phase diagrams for two models of phase separations are topologically identical. In the $x^4$-model the size of the domain of patterns formation in the vicinity of
Fig. 4. Phase diagram of reentrant behaviour of self-organization process at \( \varepsilon = 0.2 \), \( \mu = -0.5 \), \( \beta = 1 \), \( \kappa = 1.0 \). Domains of tori formation are filed: (a) corresponds to a tori set around the point \((x_-,0,0,0)\); (c) is addressed to a tori set around the fixed point \((0,0,0,0)\); double filed domain (b) relates to two set of tori around points \((x_-,0,0,0)\) and \((x_+,0,0,0)\). Plot (a) correspond to lateral interaction model, (b) is related to \( x^4 \)-model.

Let us relate every root \( k_i \) to the corresponding solution of the stationary equation (20), namely to \( x(r), y = \nabla x(r), z = \nabla^2 x(r), u = \nabla^3 x(r) \). To make a more accurate analysis we use conditions that a ratio of two different wave vectors \( k_i/k_j \) gives a rational number and appropriate real values equal zero, \( \gamma_i = \gamma_j = 0 \), where \( i \neq j \). We plot all solutions in Fig.5 in a vicinity of every fixed point \((x_-,0,0,0)\), \((x_+,0,0,0)\) and \((0,0,0,0)\). It is seen that the profile \( x(r) \) has one large period related to small values of \( k \). Two-period solutions are related to profiles of spatial derivatives of the field \( x \). Therefore, the stationary noise sustained structures \( x_{st}(r) \) has one period defined by branches with small \( k \) values in Fig.3. The corresponding solutions for \( x^4 \)-model are of the same kind, no qualitatively changes are observed.

Solutions of the problem (20) with irrational ratio \( k_i/k_j \) are shown in Fig.6. Here we choose the noise intensity values to be about the well known irrational number with a good precision, \( k_i/k_j \approx \pi \). It follows that in the three-dimensional functional space solutions of the problem (20) is a three-dimensional line which lies on the torus. This line is an unclosed and the corresponding torus is dense filled (the corresponding fractal correlation di-

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\[\text{To identify that the ratio } k_i/k_j \text{ is about rational or irrational number we use an additional criteria. Due to in numerical calculations all numbers are understood as rational ones with given precision in our analysis we have used an algorithm allowing to determine a fractal correlation dimension } D_c \text{ of a manifold (torus) in the space } (x, \nabla x, \nabla^2 x, \nabla^3 x). \text{ In our calculations the value } D_c \approx 1 \text{ corresponds to the rational ratio } k_i/k_j \text{ (the solution of the problem (20) is a three-dimensional closed line (torus) in the space } (x, \nabla x, \nabla^2 x, \nabla^3 x), \text{ whereas } D_c \approx 2 \text{ relates to the irrational one (the corresponding solution is an unclosed line on the torus in the four–dimensional space).} \]
Fig. 5. Solutions of the stationary equation (20) spatial profiles for $x(r), y(r) = \nabla x$, $z(r) = \nabla y$, and $u(r) = \nabla z$: (a) $\sigma^2 = 0.75832$, $x(0) = -0.083$, $k_i/k_j = 4$; b) $\sigma^2 = 1.15896$, $x(0) = 0.425$, $k_i/k_j = 2$; c) $\sigma^2 = 1.45(45)$, $x(0) = 10^{-5}$, $k_i/k_j = 3$.

Other parameters are: $\alpha = 0.2$, $\varepsilon = 0.2$, $\mu = -0.5$, initial conditions for derivatives are $y(0) = z(0) = u(0) = 0$.

Fig. 6. Solutions of the stationary equation (20) (set of tori) in a space $(x, y = \nabla x, z = \nabla y)$ at ratio $k_i/k_j \simeq \pi$: a) around fixed point $x_-$ at $\sigma^2 = 0.635213$; b) around fixed point $x_+$ at $\sigma^2 = 1.2532123$; c) around fixed point $x_0$ at $\sigma^2 = 1.41767865$.

Other parameters are: $\alpha = 0.2$, $\varepsilon = 0.2$, $\mu = -0.5$, initial conditions for derivatives are: $y(0) = z(0) = u(0) = 0$.

$mension is D_c \simeq 2.0 \pm 0.02$).

4.3 Simulations

To investigate spatial structures numerically let us consider solutions of the Langevin equation (5) numerically. In order to perform numerical analysis,
we redefine the model considering a regular two-dimensional lattice with \( N^2 \) points and a lattice spacing \( \ell \). Then, the partial differential equation (5) is reduced to a set of usual differential equations written for an every cell \( i \) on a grid in the form

\[
\frac{dx_i}{dt} = f(x_i) + (\nabla_L)_{ij} D(x_j) (\nabla_R)_{jl} \frac{\partial F}{\partial x_l} + \frac{1}{\sqrt{D(x_i)}} \xi_i(t),
\]

where index \( i \) labels cells, \( i = 1, \ldots, N^2 \); the discrete left and right operators are introduced

\[
(\nabla_L)_{ij} = \frac{1}{\ell} (\delta_{i,j} - \delta_{i-1,j}), \quad (\nabla_R)_{ij} = \frac{1}{\ell} (\delta_{i+1,j} - \delta_{i,j}),
\]

\[
(\nabla_L)_{ij} = -(\nabla_R)_{ji}, \quad (\nabla_L)_{ij} (\nabla_R)_{jl} = \Delta_{ij} = \frac{1}{\ell^2} (\delta_{i,l+1} - 2\delta_{i,l} + \delta_{i,l-1}),
\]

all sums are taken according to the Einstein rule. A derivative of the free energy \( F \) is of the form

\[
\frac{\partial F}{\partial x_i} = \frac{d\phi(x_i)}{dx_i} - \beta \Delta_{ij} x_j.
\]

For stochastic sources the discrete correlator is of the form \( \langle \xi_i(t) \xi_j(t') \rangle = 2t^2 \sigma^2 \delta_{ij} \delta(t - t') \). In the following analysis we use the Stratonovich interpretation of the Langevin equations (5). In our integration procedure we choose \( \ell = 1 \). After discretization scheme described above, the resulting set of coupled stochastic ordinary differential equations has been integrated numerically. To simulate evolution of the system we have used a first-order Euler algorithm with a time step \( \delta t = 0.005 \). Initial conditions for the field were chosen from a random Gaussian distribution with fixed average \( \langle x(r,0) \rangle \) (in vicinity of the fixed point defined from analysis of the effective potential \( U_{ef}(x) \), see Fig.2) and small dispersion, \( \langle (\delta x(r,0))^2 \rangle = 0.003 \). Typical patterns appeared in a course of evolution at different values of the noise intensity \( \sigma^2 \) are shown in Fig.7. Evolution of the two first moments is shown in Fig.8. From numerical solutions it follows that at small noise intensity (see Fig.7a) the system evolves by a scenario when inside a matrix phase a new phase appears. The stationary value of the first statistical moment \( \langle x(r,t) \rangle \) goes from initial one \( \langle x(r,0) \rangle \approx x_- \) to a positive fixed value (see Fig.8a ). It means that at small noise intensity the effective potential \( U_{ef}(x) \) has a broken symmetry: \( U_{ef}(x_-) > U_{ef}(x_+) \). It leads to the effect when in a course of evolution the value \( \langle x \rangle \) goes to a positive constant. At \( \sigma^2 = \sigma_0^2 \) one has \( U_{ef}(0) = U_{ef}(x_+) \). Then, starting from the configuration \( \langle x(r,0) \rangle = x_- > 0 \) with small dispersion a picture kind of phase separation should be realized. Indeed, despite the picture shown in Fig.7b is noisy with domains of two phases and diffuse interfaces, the statistical average \( \langle x \rangle \) does not change its value, i.e. \( \langle x(r,0) \rangle = \langle x(r, t \to \infty) \rangle = x_- \) (see Fig.8b). Therefore, here we have a situation when dynamics is conserved: \( \int x(r,t) dr = const \), but there are no domain
Fig. 7. Spatial patterns appeared in a course of evolution at $t = 4000$: a) $\sigma^2 = 0.2$; b) $\sigma^2 = \sigma^2_0 = 1.2757$; c) $\sigma^2 = 2.0$. Other parameters are: $\varepsilon = 0.2$, $\mu = 0.5$, $\kappa = 1.0$, $\beta = 1.0$, $\alpha = 0.2$.

Fig. 8. Evolution of first and second statistical moments at $\sigma^2 = 0.2$ (a), $\sigma^2 = \sigma^2_0 = 1.2757$ (b) and $\sigma^2 = 2.0$ (c). Other parameters are: $\varepsilon = 0.2$, $\mu = 0.5$, $\kappa = 1.0$, $\beta = 1.0$, $\alpha = 0.2$.

Fig. 9. An averaged structure function at different values of the noise intensity at $t = 4000$ (the model of lateral interactions): triangles correspond to $\sigma^2 = 0.25$; circles correspond to $\sigma^2 = \sigma^2_0$; stars relate to $\sigma^2 = 1.5$. Other parameters are: $\varepsilon = 0.2$, $\mu = 0.5$, $\kappa = 1.0$, $\beta = 1.0$, $\alpha = 0.2$.

size growth law as it realizes in phase separation processes. At $\sigma^2 > \sigma^2_c$ (see Fig.7c) when the effective potential has only minimum placed at $x = 0$ the numerical solution (see Fig.7c) shows massive structures with $\langle x(r,t) \rangle \simeq 0.0$. Despite the system has trivial value for the first statistical moment, the second moment is not trivial (see Fig.8). We calculate the structure function at different values $\sigma^2$. The corresponding dependencies are shown in Fig.9. It is seen
that $S(k)$ has one peak located at values $k$ determined by the linear stability analysis (see Fig.3a) and depends on the noise intensity. With an increase in $\sigma^2$ the peak smears, it means that patterns have more diffuse interface.

The bifurcation values for $\sigma^2$ illustrating a qualitative change of the stationary distribution (see Fig.2a) can be calculated numerically. Due to we cannot investigate a form of the total stationary distribution functional $P_{st}[x]$ numerically, our analysis were performed for the quantity $\delta = \langle x \rangle - x_{mp}$, where $x_{mp}$ is the most probable value related to the position of the maximum for the distribution function $P_{st}$. In such a case $\delta$ plays a role of a criterion for the quantitative change of the stationary distribution form. Indeed, considering the interval $\sigma^2 \in [0, \sigma_s^2)$, one choose $x_{mp} = 0$ that yields $\delta = \langle x \rangle > 0$. It means that the stationary distribution has a broken symmetry with a large contribution related to values $x > 0$. At $\sigma^2 = \sigma_s^2$ the stationary distribution has a double degenerated point located at $x = x_0 = x_-$. It leads to the fact that a dependence $\delta(\sigma^2)$ breaks at $\sigma^2 = \sigma_s^2$ and decreases with $\sigma^2$ growth; one has to take into account the value $x_{mp} = x_- > 0$ in the parameter $\delta$ calculation at $\sigma^2 \in [\sigma_s^2, \sigma_c^2)$. At $\sigma^2 = \sigma_0^2$ one has $\langle x \rangle = x_-$ and $\delta = 0$. It means that the stationary distribution is of a symmetrical form with respect to the central extremum $x = x_-$. With further increase in the noise intensity the parameter $\delta$ becomes negative and changes its value abruptly at $\sigma^2 = \sigma_c^2$ where $x_{mp} = 0$. The corresponding dependencies (analytical and numerical) are shown in Fig.10. It is seen a good correspondence of analytical results with computer simulations.

Previously, we have studied stochastic dynamics when the system evolves into the stationary state, described by the functional $P_{st}[x] \propto \exp(-U_{ef}[x]/\sigma^2)$. Let us consider the case when the system relaxes according to the stationary distribution. To generate the field $x$ according to this distribution we can use the algorithm known variously as the “Langevin method” [31]. The principle idea lies in a supposition that the distribution is realized, already. Then, the stochastic field $x$ can be obtained as solution of an effective Langevin equation $\partial_t x = -\delta U_{ef}[x]/\delta x + \xi(r,t)$, where $\xi(r,t)$ is a white additive noise with intensity $\sigma^2$. Solutions of the deterministic equation $\partial_t x = -\delta U_{ef}[x]/\delta x$ at $t \to \infty$ allows to give the stationary patterns at fixed noise intensities. Here one assumes that the system is distributed according to an effective probability density functional $P_{ef} = \prod_i \langle \delta_i(\delta U_{ef}[x]/\delta x) \rangle$, where average is taken over initial conditions, $i$ relates to minima positions of the function $U_{ef}(x)$. The corresponding patterns at different $\sigma^2$ are shown in Fig.11. It is seen that at small noise intensity (Fig.11a) nuclei are formed, at intermediate values $(\sigma = \sigma_0^2)$ a picture type of spinodal decomposition is realized (see Fig.11b), at large $\sigma^2$ (see Fig.11c) linear defects of dislocations kind are realized. Obtained pictures correspond to profiles of the concentration $x_{st}$, shown in Fig.5: at small noise intensities the pointed profile is observed, whereas at intermediate and large $\sigma^2$ a symmetrical form of profiles are realized.
Fig. 10. The effective order parameter $\delta$ versus noise intensity: solid line corresponds to analytical investigation; triangles correspond to simulations on the lattice with linear size $N = 96$; circles are related to $N = 192$. Other parameters are: $\alpha = 0.2$, $\varepsilon = 0.2$, $\mu = -0.5$, $\beta = 1.0$, $\kappa = 1$.

![Graph of $\delta$ vs $\sigma^2$](image.png)

5 Conclusions

We have studied a simple deterministic model of reaction-diffusion systems which can qualitatively describe stationary noise patterns. In order to study noise induced mechanism for pattern formation we add a multiplicative noise in a way to satisfy a fluctuation-dissipation relation. We have found that the stationary distribution can be obtained exactly. Comparing noise induced transitions picture and pattern formation processes it was shown that the system follows the entropy driven mechanism by analogy with entropy driven phase transitions theory.

We have explored early stages of the system evolution and found that at small times the noise leads to instability of the null state. Considering the stationary case we solve the variation problem when the stationary distribution functional $P_{st}[x]$ is maximized. Solutions of the problem show that stationary patterns are formed in the vicinity of the maxima of the corresponding
stationary density function, \(P_{st}(x)\). In linear stability analysis in \(r\)-space in the stationary case we have found that at small noise intensities the system is characterized by unstable homogeneous solutions related to the maxima positions of the function \(P_{st}(x)\). With an increase in the noise strength the spatial structures \(x_{st}(r)\) are formed in the vicinity of the local maxima of \(P_{st}(x)\). Analytically and numerically we have obtained that at large noise intensity after a bifurcation point (the stationary probability density function \(P_{st}(x)\) is of one maximum) the noise can sustain stationary patterns. The process of the pattern formation is well described by the formalism of noise induced transitions in zero-dimensional systems, whereas the obtained effect is a some generalization of the noise induced transitions for extended systems. Our study shows that at small noise intensity the system manifest a nucleation regime, at fixed value of the noise strength a spinodal decomposition is realized, at large noise the system exhibits strip patterns with lineal defects. Strip structures exist in the fixed interval of the noise intensity — large fluctuations destroy the patterns.

Our results are in a good correspondence with experimental data obtained for patterns formation in polymer gels with chemical reactions (see Ref. [32]). It was shown that in such the reaction-diffusion system patterns can be formed only inside the fixed temperature interval \((10^6C < T < 60^6C)\). Author discusses diffusion-induced pattern formation mechanism of volume phase transition with pattern formation. He shows that straight strip patterns tend to appear in high temperature regime \((T \approx 57^6C)\), and random patterns appear at low temperature \((T \approx 30^6C)\). Moreover, the size of patterns decrease in a power law form with an increase in the temperature, \(L \propto T^{-\beta/2}\), \(\beta = 1.01 \pm 0.03\). In our calculations we have not used the corresponding chemical reactions, the local dynamics were governed by the Schlögl model [30], we have used the model of non-Fickian transport with the field dependent diffusion coefficient. The noise source satisfies the fluctuation-dissipation theorem and has intensity related to the bath temperature. To compare our results with above experimental investigations we need to stress that an actual noise intensity values are \(\sigma^2 \in (\sigma^2_s, \sigma^2_{T0})\). We have shown that random stationary patterns are realized if \(\sigma^2_s < \sigma^2 < \sigma^2_{c}\) (experimental photograph in Ref. [32] is the same as we show in Fig.11b). At \(\sigma^2 < \sigma^2_{T0}\) the strip patterns are realized (see Fig.11c), the same picture was obtained in Ref. [32]. Moreover, we have calculated the dependence \(L \propto (\sigma^2)^{-\beta/2}\), and obtained \(\beta = 3.2 \pm 0.02\). The qualitative correspondence in the wavelength dependence is observed. The difference in the exponent \(\beta\) values can be explained by the fact that we have used another kind of the local dynamics model, non-Fickian transport and white noise assumptions.

Obtained results can be applied to study patterns in adsorption/desorption processes in metals [29] deposition of a monolayer of molecules [33] and in processes of microstructure transformations of materials subject to intensive
irradiation. One can await that formation of point defect clusters and clusters of particles in the system kind of “vacanies+particles” where a phase decompositions and “chemical transformations” caused by irradiation can be observed.

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