AN IMMERSED $S^n$ $\lambda$-HYPERSURFACE

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Abstract. In this paper, we construct an immersed, non-embedded $S^n$ $\lambda$-hypersurface in Euclidean spaces $\mathbb{R}^{n+1}$.

1. Introduction

In the study of mean curvature flow, the important class of solutions are those in which the hypersurface evolves under self-similar shrinking. Indeed, under general mean curvature flow, singularities often develop which can be modeled using self-shrinking solutions. Such solutions can be identified with a single time-slice of the flow, which gives us a hypersurface called a self-shrinker. Let $X : M^n \to \mathbb{R}^{n+1}$ be a smooth $n$-dimensional immersed hypersurface in the $(n + 1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. An $n$-dimensional hypersurface $M^n \subset \mathbb{R}^{n+1}$ is called a self-shrinker if it satisfies the equation

$$H + \langle X, \vec{N} \rangle = 0,$$

(1.1)

where $H$ is the mean curvature of the hypersurface $M^n$, $X$ is the position vector and $\vec{N}$ is the unit normal vector of the hypersurface.

In [3], Cheng and Wei introduced the notation of $\lambda$-hypersurfaces by studying the weighted volume-preserving mean curvature flow, which is defined as the following: a family $X(\cdot, t)$ of smooth immersions $X(\cdot, t) : M^n \to \mathbb{R}^{n+1}$ with $X(\cdot, 0) = X(\cdot)$ is called a weighted volume-preserving mean curvature flow if

$$\frac{\partial X(t)}{\partial t} = -\alpha(t)\vec{N}(t) + \vec{H}(t)$$

holds, where

$$\alpha(t) = \frac{\int_M H(t)\langle \vec{N}(t), \vec{N} \rangle e^{-\frac{|X|^2}{2}} d\mu}{\int_M \langle \vec{N}(t), \vec{N} \rangle e^{-\frac{|X|^2}{2}} d\mu},$$

$$\vec{H}(t) = \vec{H}(\cdot, t)$$

and $\vec{N}(t)$ denote the mean curvature vector and the unit normal vector of hypersurface $M_t = X(M^n, t)$ at point $X(\cdot, t)$, respectively and $\vec{N}$ is the unit normal vector of $X : M^n \to \mathbb{R}^{n+1}$. One can

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prove that the flow preserves the weighted volume $V(t)$ defined by $V(t) = \int_M \langle X(t), \vec{N} \rangle e^{-\frac{|X|^2}{2}} dm$. The weighted area functional $A : (-\epsilon, \epsilon) \rightarrow \mathbb{R}$ is defined by $A(t) = \int_M e^{-\frac{|X|^2}{2}} dm$, where $dm$ is the area element of $M$ in the metric induced by $X(t)$. Let $X(t) : M \rightarrow \mathbb{R}^{n+1}$ with $X(0) = X$ be a variation of $X$. If $V(t)$ is constant for any $t$, we call $X(t) : M \rightarrow \mathbb{R}^{n+1}$ is a weighted volume-preserving variation of $X$. Cheng and Wei [3] have proved that $X : M \rightarrow \mathbb{R}^{n+1}$ is a critical point of the weighted area functional $A(t)$ for all weighted volume-preserving variations if and only if there exists constant $\lambda$ such that the hypersurfaces satisfy the equation

$$H + \langle X, \vec{N} \rangle = \lambda.$$  

An immersed hypersurface $X(t) : M^n \rightarrow \mathbb{R}^{n+1}$ is called a $\lambda$-hypersurfaces if the equation (1.2) is satisfied. Moreover, they defined a $F$-functional of $\lambda$-hypersurfaces and studied $F$-stability, which extended a result of Colding-Minicozzi [8].

From the definition, we know that if $\lambda = 0$, $X : M^n \rightarrow \mathbb{R}^{n+1}$ is a self-shrinker of mean curvature flow, that is, $\lambda$-hypersurfaces is the generalization of self-shrinkers. $\lambda$-hypersurfaces were also studied by McGonagle and Ross in [15], where they investigate the following isoperimetric type problem in a Gaussian weighted Euclidean space. It turns out that critical points of this variational problem are $\lambda$-hypersurfaces and the only smooth stable ones are hyperplanes. More information on $\lambda$-hypersurfaces can be found in [5], [13], [18] and [19].

The simplest examples of self-shrinkers in $\mathbb{R}^{n+1}$ are the sphere of radius $\sqrt{n}$ centered at the origin, cylinders with an axis through the origin and radius $\sqrt{k}$, $1 \leq k \leq n - 1$, and planes through the origin. In 1992, Angenstein [1] constructed an embedded self-shrinker which is diffeomorphic to $S^1 \times S^{n-1}$. In 1994, Chopp [7] described a numerical algorithm to compute surfaces that are approximately self-similar under mean curvature flow. Kapouleas, Kleene and Møller [14] given the first rigorous construction of complete, embedded self-shrinking hypersurfaces under mean curvature flow. Besides, Drugan and Kleene [10] presented a new family of non-compact properly embedded, self-shrinking, asymptotically conical, positive mean curvature ends that are hypersurfaces of revolution with circular boundaries and proved the important classification result. In [16], Møller proved that there is a rigorous construction of closed, embedded, smooth mean curvature self-shrinkers with high genus, embedded in Euclidean space $\mathbb{R}^3$. Drugan [9] constructed an immersed and non-embedded $S^2$ self-shrinker in $\mathbb{R}^3$. In 2015, Cheng and Wei [4] constructed an embedded $\lambda$-torus. More recently, Drugan and Nguyen [12] used variational methods and a modified curvature flow to give an alternative proof of the existence of a self-shrinking torus under mean curvature flow. Drugan, Lee and Nguyen [11] surveyed known results on closed self-shrinkers for mean curvature ow and discuss techniques used in
recent constructions of closed self-shrinkers with classical rotational symmetry. Ross [17] showed that there exists a closed, embedded \(\lambda\)-hypersurface which is diffeomorphic to \(S^{n-1} \times S^{n-1} \times S^1\) in \(\mathbb{R}^{2n}\). In this paper, inspired by Drugan’s paper [9], we construct an immersed, non-embedded \(S^n\) \(\lambda\)-hypersurfaces in \(\mathbb{R}^{n+1}\).

**Theorem 1.1.** For small \(\lambda < 0\), there exists an immersion \(\lambda\)-hypersurface \(X : S^n \to \mathbb{R}^{n+1}\) which is not embedding.

The basic idea of the proof of Theorem [14] is to construct a curve \(\Gamma(s)\) to the geodesic shooting problem with self-intersections whose rotation about the \(x\)-axis is an topologically \(S^n\). In fact, using the symmetry of the curve equation with respect to reflections across the \(z\)-axis, it is sufficient to find a curve \(\Gamma\) that intersects the \(z\)-axis perpendicularly. Using comparison arguments, we give a detailed description of the first two branches of the curve \(\Gamma\) when the initial height is small. Following the approach of Angenent in [1], we use the continuity argument to find an initial condition that corresponds to a solution whose rotation about the \(x\)-axis is an immersed and non-embedded \(S^n\) \(\lambda\)-hypersurface.

2. **Equations of Rotational \(\lambda\)-Hypersurfaces in \(\mathbb{R}^{n+1}\)**

Let \(\Gamma(s) = (x(s), z(s))\), \(s \in (a_1, a_2)\) be a curve with \(z > 0\) in the upper half plane \(\mathbb{H} = \{x + iz \mid z > 0, x \in \mathbb{R}, i = \sqrt{-1}\}\), where \(s\) is arc length parameter of \(\Gamma(s)\). We consider a rotational hypersurface \(X : (a_1, a_2) \times S^{n-1}(1) \to \mathbb{R}^{n+1}\) in \(\mathbb{R}^{n+1}\) defined by

\[
X : (a_1, a_2) \times S^{n-1}(1) \to \mathbb{R}^{n+1}, \quad X(s, \alpha) = (x(s), z(s)\alpha) \in \mathbb{R}^{n+1},
\]

where \(S^{n-1}(1)\) is the \((n-1)\)-dimensional unit sphere.

By a direct calculation, we get the unit normal vector and the mean curvature as follows:

\[
\vec{N} = \left(-\frac{dz(s)}{ds}, \frac{dx(s)}{ds}\right), \quad H = -\frac{d^2 x(s)}{ds^2} \frac{dz(s)}{ds} + \frac{dx(s)}{ds} \frac{d^2 z(s)}{ds^2} - \frac{n-1}{z(s)} \frac{dx(s)}{ds}.
\]

And then,

\[
\langle X, \vec{N} \rangle = -x(s) \frac{dz(s)}{ds} + z(s) \frac{dx(s)}{ds}.
\]

Since \(s\) is arc length parameter of the profile curve \(\Gamma(s) = (x(s), z(s))\), we have \(\left(\frac{dx(s)}{ds}\right)^2 + \left(\frac{dz(s)}{ds}\right)^2 = 1\). Thus, it follows that \(\frac{d^2 x(s)}{ds^2} = \frac{dx(s)}{ds} \frac{d^2 z(s)}{ds^2} + \frac{dz(s)}{ds} \frac{d^2 x(s)}{ds^2} = 0\).

Therefore, it follows that \(X : (a_1, a_2) \times S^{n-1}(1) \to \mathbb{R}^{n+1}\) is a \(\lambda\)-hypersurface in \(\mathbb{R}^{n+1}\) if and only if

\[
-\frac{d^2 x(s)}{ds^2} \frac{dz(s)}{ds} + \frac{dx(s)}{ds} \frac{d^2 z(s)}{ds^2} + \left(z(s) - \frac{n-1}{z(s)}\right) \frac{dx(s)}{ds} - x(s) \frac{dz(s)}{ds} = \lambda.
\]

(2.1)
That is,
\[
\frac{d^2x(s)}{dz(s)} = \left( z(s) - \frac{n-1}{z(s)} \right) \frac{dx(s)}{dz(s)} - x(s) \frac{dz(s)}{ds} - \lambda, \tag{2.2}
\]
where \(\frac{dx}{dz} + \frac{dz}{ds} = 0\).

Taking the third derivative of (2.2), we have the following equation,
\[
\frac{d^3x(s)}{dz^3} = - \frac{dx(s)}{dz(s)} \left( \frac{d^2x(s)}{dz^2} \right)^2 + \left( z(s) - \frac{n-1}{z(s)} \right) \frac{d^2x(s)}{dz^2} + \frac{n-1}{z(s)^2} \frac{dx(s)}{dz(s)} \frac{dz(s)}{ds} + \frac{x(s) \frac{dz(s)}{ds}}{\frac{dz(s)}{ds}}, \tag{2.3}
\]
where \(\frac{dx}{dz} + \frac{dz}{ds} = 0\).

Obviously, there are several special solutions of (2.1):

1. \((x, z) = (0, 0)\) is a solution. This curve corresponds to the hyperplane through \((0, 0)\) and \(\lambda = 0\).

2. \((x, z) = (a \cos \frac{s}{2}, a \sin \frac{s}{2})\) is a solution, where \(a = -\lambda + \sqrt{\lambda^2 + 4n}\). This circle \(x^2 + z^2 = a^2\) corresponds to a sphere \(S^n(a)\) with radius \(a\).

3. \((x, z) = (-s, a)\) is a solution, where \(a = -\lambda + \sqrt{\lambda^2 + 4(n-1)}\). This straight line corresponds to a cylinder \(S^{n-1}(a) \times \mathbb{R}\).

3. THE FIRST PART OF THE PROFILE CURVES

In this section, for \(b > -(4n+1)\lambda\), we will study the solutions of (2.2) with \((x(0), z(0)) = (b, 0)\) and \((\frac{dx(0)}{ds}, \frac{dz(0)}{ds}) = (0, 1)\). In [9], Drugan have obtained the existence of solutions of self-shrinker near \(z = 0\) by discussing the existence, uniqueness and continuous dependence of solutions on initial height at \(z\) near 0. That’s also true to \(\lambda\)-hypersurface. After this, we can use some different comparison estimates to describe the basic behavior of the profile curve \(\Gamma(s)\). In the end, we complete the section with a detailed description of \(\Gamma(s)\) when the initial height \(b > -(4n+1)\lambda\) is small.

3.1. Basic shape of the first branch of the curve \(\Gamma(s)\). For \(b > -(4n+1)\lambda\), let \(\Gamma(s)\) be the solution of (2.2) with \((x(0), z(0)) = (b, 0)\) and \((\frac{dx(0)}{ds}, \frac{dz(0)}{ds}) = (0, 1)\). Then \(\frac{d^2x(0)}{ds^2} = -\frac{1}{n}(b + \lambda) < 0\) so that \(\Gamma\) starts out concave to the left. We will describe the basic shape of the curve \(\Gamma(s)\) through following several lemmas.

Lemma 3.1. \(\frac{d^2x(s)}{ds^2} < 0\).

Proof. Since \(\frac{d^2x(0)}{ds^2} = -\frac{1}{n}(b + \lambda)\), we have that \(\frac{d^2x(s)}{ds^2} < 0\) at \(s\) near 0. Assuming \(\frac{d^2x(s)}{ds^2} = 0\) for some \(s > 0\). Choosing \(\tilde{s}\) so that \(\frac{d^2x(\tilde{s})}{ds^2} = 0\) and
\[ \frac{d^2x(s)}{ds^2} < 0 \text{ for } s \in [0, \tilde{s}], \text{ then } \frac{d^2x(s)}{ds^2} \geq 0. \text{ Also because } \frac{dx(0)}{ds} = 0, \text{ we have } \frac{d^2x(s)}{ds} < 0. \text{ Using equation (2.3), we get} \]

\[ 0 \leq \frac{d^3x(\tilde{s})}{ds^3} = \frac{n-1}{z(\tilde{s})^2} \left( \frac{dz(\tilde{s})}{ds} \right)^2 \frac{dx(\tilde{s})}{ds} < 0, \]

it’s contradictory. So we have \( \frac{d^2x(s)}{ds^2} < 0. \) \( \square \)

Let \( s_1 = s(b) > 0 \) be the arc length of the first time at which the unit tangent vector is \((-1,0)(\frac{dz(s)}{ds}) = 0). \) In \([6], \) Cheng and Wei prove that an entirely graphic \( \lambda \)-hypersurface in Euclidean space is a hyperplane, then \( s_1 \) must exist and \( z(s) \) can’t take the value at infinity. By Lemma 3.1 and the definition of \( s_1, \) we have that \( \frac{dx(s)}{ds} < 0 \text{ and } \frac{dz(s)}{ds} > 0 \text{ in } (0,s_1). \) Hence, the curve \( \Gamma(s) \) can be written as a graph of \( x = \gamma(z), \) where \( 0 < z < z(s_1). \) If \( \gamma'(z) = \frac{dz}{ds} = -\infty, \) the profile curve \( \Gamma \) has a horizontal tangent point, in a sense where \( \gamma(z) \) blow up. It follows from (2.2) that

\[ \frac{\gamma''(z)}{1 + \gamma'(z)^2} = \left( z - \frac{n-1}{z} \right) \gamma'(z) - \gamma(z) - \lambda \sqrt{1 + \gamma'(z)^2}, \tag{3.1} \]

where \( \gamma''(z) = \frac{d^2x(z)}{dx^2} \), \( \gamma'(z) = \frac{dx(z)}{ds} < 0 \) and \( \frac{dz(s)}{ds} = \frac{1}{\sqrt{1 + \gamma'(z)^2}} > 0. \) By Lemma 3.1, we have that \( \gamma''(z) < 0 \) and \( \gamma'(z) \leq 0 \) on \([0,z(s_1)]\). Taking the third and fourth derivatives of (3.1), we have the following equations,

\[ \frac{\gamma'''}{1 + \gamma'^2} = \frac{2\gamma'(\gamma'')^2}{1 + \gamma'^2} + \left( z - \frac{n-1}{z} \right) \gamma''' + \frac{n-1}{z^2} \gamma' - \frac{\lambda \gamma \gamma''}{\sqrt{1 + \gamma'^2}}, \tag{3.2} \]

and

\[ \frac{\gamma^{(4)}}{1 + \gamma'^2} = 6\gamma'\gamma''\gamma''' + 2(\gamma'')^3 - \frac{8(\gamma'')^2(\gamma'')^3}{(1 + \gamma'^2)^3} + \left( z - \frac{n-1}{z} \right) \gamma''' - \frac{2(n-1)}{z^3} \gamma' + \left( 1 + \frac{2(n-1)}{z^2} \right) \gamma''' + \frac{\lambda(\gamma ' \gamma '')^2}{(1 + \gamma'^2)^{3/2} - \frac{\lambda(\gamma ' \gamma '')^2}{\sqrt{1 + \gamma'^2}}. \tag{3.3} \]

**Lemma 3.2.** \( z(s_1) > -\frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2}. \)

**Proof.** Since \( \left( \frac{dx(s)}{ds} \right)^2 + \left( \frac{dz(s)}{ds} \right)^2 = 1 \) and \( \frac{dz(s)}{ds} = 0, \) we have \( \frac{dx(s)}{ds} = -1. \) Supposing \( z(s_1) = -\frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2}, \) there exists a special solution of (2.1) which \( (x, z) = (-s, a) \) with \( a = -\frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2}, \) by the existence and uniqueness of solutions for the differential equation, it’s contradictory. Supposing \( z(s_1) < -\frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2}, \) since \( \frac{dz(s_1)}{ds} = 0, \) \( \frac{dx(s_1)}{ds} = -1 \) and \( \frac{dx(s_1)}{ds} = 0. \)
\[ \frac{dz}{ds} \frac{d^2z}{ds^2} = 0, \] using equation (3.1), we have
\[ \frac{d^2z(s)}{ds^2} = \frac{n - 1}{z(s)} - z(s) - \lambda > 0. \]

Then, there exists \( \delta > 0 \) so that \( \frac{d^2z(s)}{ds^2} > 0 \) for \( s \in (s_1 - \delta, s_1) \), and then \( \frac{dz(s)}{ds} < 0 \) for \( s \in (s_1 - \delta, s_1) \), it’s contradictory. We can get
\[ z(s_1) > -\frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2}. \]

\[ \square \]

**Lemma 3.3.** \( \lim_{s \to s_1} x(s) = -\infty. \)

**Proof.** When the profile curve \( \Gamma \) can be written in the form \((\gamma(z), z)\), \( \gamma \) be a solution of (3.1) with \( \gamma(0) = b > -(4n + 1)\lambda, \gamma'(0) = 0 \). By Lemma 3.2, we have that \( z(s_1) > -\frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2} \) and \( \lim_{s \to z(s_1)} \gamma'(z) = -\infty \). Fixing \( 0 < \delta < 1 \) so that \( z(s_1) - \delta > -\frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2} \) and letting \( M \) so that \( z - \frac{n-1}{2} + \lambda > M \) for \( z \in (z(s_1) - \delta, z(s_1)) \). Choosing \( M > 0 \) so that \( M \geq \frac{3}{2M} \) and \( M \geq -\gamma'(z(s_1) - \delta) \).

For \( \delta > \varepsilon > 0 \), \( g_\varepsilon(z) \) is defined as
\[ g_\varepsilon(z) = \frac{M}{\sqrt{(z(s_1) - \varepsilon) - z}}, \quad z \in (z(s_1) - \delta, z(s_1) - \varepsilon). \]

Then we have
\[ g_\varepsilon(z) > M, \quad g_\varepsilon'(z) = \frac{M}{2((z(s_1) - \varepsilon) - z)^{1/2}} > 0 \]
and
\[ g_\varepsilon''(z) = \frac{3}{2M}g_\varepsilon(z)^2g_\varepsilon'(z) \leq \left( z - \frac{n-1}{z} + \lambda \right) g_\varepsilon(z)^2g_\varepsilon'(z), \]
for \( z \in (z(s_1) - \delta, z(s_1) - \varepsilon) \). We use the function \( g_\varepsilon \) to prove that \(-\gamma'\) blow-up no faster than \( \frac{M}{\sqrt{z(s_1) - z}} \).

Letting \( f(z) = -\gamma'(z) \), then we have \( f(z) \geq 0 \) and \( f'(z) > 0 \). Using equation (3.2), we get
\[ f''(z) = \frac{2f(z)f'(z)^2}{1 + f(z)^2} + \left( z - \frac{n-1}{z} \right) f'(z)(1 + f(z)^2) \]
\[ + \frac{(n-1)f(z)}{z^2}(1 + f(z)^2) + \lambda f'(z)f(z)\sqrt{1 + f(z)^2} \]
\[ \geq \left( z - \frac{n-1}{z} \right) f'(z)(1 + f(z)^2) + \lambda f'(z)f(z)\sqrt{1 + f(z)^2} \]
\[ \geq \left( z - \frac{n-1}{z} + \lambda \right) f'(z)(1 + f(z)^2) \]
\[ \geq \left( z - \frac{n-1}{z} + \lambda \right) f'(z)f(z)^2, \]
where $\lambda < 0$, $z \geq -\frac{\lambda + \sqrt{\lambda^2 + 4(n - 1)}}{2}$.

Next, the purpose is to prove $f \leq g_\varepsilon$. It is known that

$$f(z(s_1) - \delta) \leq M < g_\varepsilon(z(s_1) - \delta)$$

and

$$f(z(s_1) - \varepsilon) < \lim_{z \to (z(s_1) - \varepsilon)} g_\varepsilon(z).$$

Therefore, if there exists some points on $(z(s_1) - \delta, z(s_1) - \varepsilon)$ so that $f > g_\varepsilon$, then $f - g_\varepsilon$ achieves a positive maximum at point $\tilde{z} \in (z(s_1) - \delta, z(s_1) - \varepsilon)$. This leads to $(f - g_\varepsilon)'(\tilde{z}) = 0$ and $(f - g_\varepsilon)''(\tilde{z}) \leq 0$. We have

$$0 \geq (f - g_\varepsilon)''(\tilde{z}) \geq \left(\tilde{z} - \frac{n - 1}{\tilde{z}} + \lambda\right) f'(\tilde{z}) (f(\tilde{z})^2 - g_\varepsilon(\tilde{z})^2) > 0,$$

it’s contradictory. It follows that $f \leq g_\varepsilon$ on $(z(s_1) - \delta, z(s_1) - \varepsilon)$. Taking $\varepsilon \to 0$, we have that

$$\gamma'(z) \geq \frac{-M}{\sqrt{z(s_1) - z}},$$

for $x \in (z(s_1) - \delta, z(s_1))$. Integrating the inequality from $z(s_1) - \delta$ to $z(s_1)$, we have

$$\lim_{z \to z(s_1)} \gamma(z) \geq 2M\sqrt{\delta} + \gamma(z(s_1) - \delta) > -\infty.$$ 

Then we get $\lim_{s \to s_1} x(s) > -\infty$. \hfill $\Box$

### 3.2. Estimates for small initial height.

From the above lemmas, we have the basic description of the $\Gamma$ curves: For $b > -(4n + 1)\lambda$, when the profile curve $\Gamma$ can be written in the form $(\gamma_b(z), z)$, $\gamma_b$ be a solution of (3.1) with $\gamma_b(0) = b$, $\gamma_b'(0) = 0$. Then $\gamma_b$ is decreasing, concave down and there exists a point $z(s_1^b) \in (\frac{-\lambda + \sqrt{\lambda^2 + 4(n - 1)}}{2}, \infty)$ so that $\gamma_b$ is defined on $[0, z(s_1^b))$ and $\lim_{z \to z(s_1^b)} \gamma_b'(z) = -\infty$ (in a sense $\gamma_b$ blow up at $z(s_1^b)$). There also exists a point $x(s_1^b) \in (-\infty, b)$ so that $\gamma_b(z(s_1^b)) = x(s_1^b)$. Next, we show estimates for $x(s_1^b)$ and $z(s_1^b)$ when the initial height $b > -(4n + 1)\lambda$ is small.

**Proposition 3.4.** For $b > -(4n + 1)\lambda$, let $\gamma_b$ denote the solution of (3.1) with $\gamma_b(0) = b$ and $\gamma_b'(0) = 0$. Let $z(s_1^b)$ denote the point where $\gamma_b$ blow-up and $x(s_1^b) = \gamma_b(z(s_1^b))$. There exists $\bar{b} > 0$ so that if $b \in (0, \bar{b})$, then

$$z(s_1^b) \geq \sqrt{\ln \frac{1}{2\sqrt{\pi} (b + \lambda)}},$$

$$-\frac{4(3n + 1)}{4\ln \frac{1}{2\sqrt{\pi} (b + \lambda)} + 8\lambda} \leq x(s_1^b) < 0,$$

and there exists a point $z_0^b \in [-\frac{\lambda + \sqrt{\lambda^2 + 4n}}{2}, \sqrt{\frac{2nb}{b + \lambda}}]$ so that $\gamma_b(z_0^b) = 0$. 

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\[ \text{where } \lambda < 0, \quad z \geq -\frac{\lambda + \sqrt{\lambda^2 + 4(n - 1)}}{2}. \]
Lemma 3.5. Suppose $-(4n+1)\lambda < b < \sqrt{\frac{1}{16\pi \varepsilon^{2n}}} - \lambda$, then $z(s_1) > 3\sqrt{2n}$ and $|\gamma'(z)| \leq \frac{1}{2}$ for $z \in [0,3\sqrt{2n}]$.

Proof. Since $\gamma'(0) = 0$, $\gamma'' < 0$ and $\lim_{z \to z(s_1)} \gamma'(z) = -\infty$, we have that there exists $\tilde{z} \in (0,z(s_1))$ so that $\gamma'(') = -\frac{1}{2}$. For $z \in (0,\tilde{z})$, we get

$$\frac{d}{dz} \left( e^{-z^2} \gamma'(z) \right) = e^{-z^2} \gamma''(z) - 2ze^{-z^2} \gamma'(z)$$

$$\geq \frac{2}{1 + \gamma'(z)^2} e^{-z^2} \gamma''(z) - 2ze^{-z^2} \gamma'(z)$$

$$= 2e^{-z^2} \left[ \left( z - \frac{n - 1}{z} \right) \gamma' - \gamma(z) - 2\lambda e^{-z^2} \sqrt{1 + (\gamma')^2} \right] - 2ze^{-z^2} \gamma'(z)$$

$$\geq - e^{-z^2} (2\gamma(z) + 2\lambda).$$

Integrating both sides of this inequality from 0 to $\tilde{z}$,

$$-\frac{1}{2} e^{-(\tilde{z})^2} \geq - \int_0^{\tilde{z}} e^{-z^2} (2\gamma(z) + 2\lambda)dz \geq -2(b+\lambda) \int_0^{\tilde{z}} e^{-z^2} dx \geq -2(b+\lambda) \sqrt{\pi},$$

we get $\tilde{z} \geq \sqrt{\ln \frac{1}{4\pi (b+\lambda)}}$. When $b < \sqrt{\frac{1}{16\pi \varepsilon^{2n}}} - \lambda$, we have $e^{-(\tilde{z})^2} < e^{-18n}$, and then $z(s_1) > \tilde{z} > 3\sqrt{2n}$. \hfill $\square$

Lemma 3.6. If $|\gamma'(z)| \leq \frac{1}{2}$ for $z \in [0,3\sqrt{2n}]$ and $-(4n+1)\lambda < b < \frac{1}{3\sqrt{2n}}$, then $\frac{z\gamma'(z) - \gamma(z)}{1 + \gamma'(z)^2}$ is non-increasing on $[0,3\sqrt{2n}]$.

Proof. Taking the derivative of $\frac{z\gamma'(z) - \gamma(z)}{1 + \gamma'(z)^2}$, we have

$$\frac{d}{dz} \left( \frac{z\gamma'(z) - \gamma(z)}{1 + \gamma'(z)^2} \right) = \gamma''(z) \frac{z + \gamma(z)\gamma'(z)}{(1 + \gamma'(z)^2)^{3/2}}.$$

Next, the purpose is to prove that $z + \gamma(z)\gamma'(z) \geq 0$. Since $z + \gamma(z)\gamma'(z) = 0$ at $z = 0$, we only need to prove $1 + \gamma(z)\gamma''(z) + \gamma'(z)^2 \geq 0$ on $(0,3\sqrt{2n}]$. For
z \in (0, 3\sqrt{2n}]$, according to $|\gamma'(z)| \leq \frac{1}{2}$ and $-(4n + 1)\lambda < b < \frac{1}{3\sqrt{2n}}$, we get

$$\gamma''(z) = (1 + \gamma'(z)^2) \left[ \left( z - \frac{n - 1}{z} \right) \gamma'(z) - \gamma(z) - \lambda \sqrt{1 + \gamma'(z)^2} \right]$$

$$\geq (1 + \gamma'(z)^2) \left[ z\gamma'(z) - \gamma(z) - \lambda \right]$$

$$\geq \frac{5}{4} \left[ 3\sqrt{2n}(-\frac{1}{2}) - b \right]$$

$$\geq \frac{5}{4} \left[ (-\frac{3}{2})\sqrt{2n} - \frac{1}{3\sqrt{2n}} \right] \geq -3\sqrt{2n}.$$  

It follows that $1 + \gamma(z)\gamma''(z) + \gamma'(z)^2 \geq 0$. \hfill \Box

**Lemma 3.7.** If $|\gamma'(z)| \leq \frac{1}{2}$ for $z \in [0, 3\sqrt{2n}]$ and $-(4n + 1)\lambda < b < \frac{1}{3\sqrt{2n}}$, then $\gamma'''(z) < 0$ for $z \in (0, z(s_1))$.

**Proof.** Using equations (3.2) and (3.3) at $z = 0$, we have that $\gamma'''(0) = 0$ and

$$\gamma^{(iv)}(0) = \frac{3}{n + 2} \gamma''(0) \left[ 2\gamma''(0)^2 - \lambda\gamma''(0) + 1 \right].$$

And for $\gamma''(0) < 0$ and $-(4n + 1)\lambda < b < \frac{1}{3\sqrt{2n}}$, we have that $\gamma^{(iv)}(0) < 0$ and $\frac{-\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2} < 3\sqrt{2n}$. Therefore, $\gamma'''(z) < 0$ when $z > 0$ is near 0. Besides, using equation (3.2), we get

$$\frac{\gamma''}{1 + (\gamma')^2} = \frac{2\gamma'\gamma''}{1 + (\gamma')^2} + \left( z - \frac{n - 1}{z} \right) \gamma'' + \frac{n - 1}{z^2} \gamma' - \lambda \gamma'' \frac{\gamma'}{1 + (\gamma')^2}$$

$$< \frac{2\gamma'\gamma''}{1 + (\gamma')^2} + \left( z - \frac{n - 1}{z} + \lambda \right) \gamma'' + \frac{n - 1}{z^2} \gamma',$$

where $0 < \frac{-\gamma'}{\sqrt{1 + (\gamma')^2}} < 1$. It follows that $\gamma'''(z) < 0$ when $z \geq \frac{-\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2}$. If there are some points on $(0, z(s_1))$ so that $\gamma'''(z) = 0$, there must exist $\tilde{z} \in (0, \frac{-\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2})$ so that $\gamma'''(\tilde{z}) = 0$ and $\gamma'''(z) < 0$ for $z \in (0, \tilde{z})$. It follows that $\gamma^{(iv)}(\tilde{z}) \geq 0$. It is obvious that $z\gamma''(z) - \gamma'(z)$ is decreasing and negative on $(0, \tilde{z})$. By Lemma 3.6, we have $\gamma'(\tilde{z}) \geq -3\sqrt{2n}$. Then, using
equation (3.3) and $|\gamma'(\bar{z})| \leq \frac{1}{2}$, we get

$$
\frac{\gamma''(\bar{z})}{1 + \gamma'(\bar{z})^2} = 2(\gamma''(\bar{z}))^3 \frac{1 - 3\gamma'(\bar{z})^2}{(1 + \gamma'(\bar{z})^2)^3} + \gamma''(\bar{z}) + 2(n - 1) \frac{\bar{z}\gamma''(\bar{z}) - \gamma'(\bar{z})}{(\bar{z})^3}
$$

\[
+ \frac{\lambda(\gamma'(\bar{z})\gamma''(\bar{z}))^2}{(1 + \gamma'(\bar{z})^2)^{3/2}} - \frac{\lambda(\gamma''(\bar{z}))^2}{\sqrt{1 + \gamma'(\bar{z})^2}}
\]

\[
= 2(\gamma''(\bar{z}))^3 \frac{1 - 3\gamma'(\bar{z})^2}{(1 + \gamma'(\bar{z})^2)^3} + \gamma''(\bar{z}) + 2(n - 1) \frac{\bar{z}\gamma''(\bar{z}) - \gamma'(\bar{z})}{(\bar{z})^3}
\]

\[
- \frac{\lambda(\gamma''(\bar{z}))^2}{(1 + \gamma'(\bar{z})^2)^{3/2}}
\]

\[
\leq 2(\gamma''(\bar{z}))^3 \frac{1 - 3\gamma'(\bar{z})^2}{(1 + \gamma'(\bar{z})^2)^3} + \gamma''(\bar{z})(1 - \lambda\gamma''(\bar{z})) \frac{(1 + \gamma'(\bar{z})^2)^{3/2}}{(1 + \gamma'(\bar{z})^2)^{3/2}} + 2(n - 1) \frac{\bar{z}\gamma''(\bar{z}) - \gamma'(\bar{z})}{(\bar{z})^3}
\]

\[
\leq 2(\gamma''(\bar{z}))^3 \frac{1 - 3\gamma'(\bar{z})^2}{(1 + \gamma'(\bar{z})^2)^3} + \gamma''(\bar{z})(1 + 3\sqrt{2n\lambda}) \frac{(1 + \gamma'(\bar{z})^2)^{3/2}}{(1 + \gamma'(\bar{z})^2)^{3/2}} + 2(n - 1) \frac{\bar{z}\gamma''(\bar{z}) - \gamma'(\bar{z})}{(\bar{z})^3}
\]

\[
< 0.
\]

where $\gamma''(\bar{z}) < 0$, $|\gamma'(\bar{z})| \leq \frac{1}{2}$ and $-(4n + 1)\lambda < b < \frac{1}{3\sqrt{2n}}$, it’s contradictory. So we have $\gamma''(z) < 0$ for $z \in (0, z(s_1))$. $\square$

**Lemma 3.8.** Suppose $-(4n + 1)\lambda < b < \sqrt{\frac{1}{16\gamma - \text{sec}^4 \lambda}} - \lambda$, then there exists a point $z_0 \in \left[-\frac{\lambda + \sqrt{\lambda^2 + 4n}}{2}, \frac{2nb}{b + \lambda}\right]$ so that $\gamma(z_0) = 0$.

**Proof.** We assume $-(4n + 1)\lambda < b < \sqrt{\frac{1}{16\gamma - \text{sec}^4 \lambda}} - \lambda(b < \frac{1}{3\sqrt{2n}})$, by Lemma 3.5, we have that $z(s_1) > 3\sqrt{2n}$ and $|\gamma'(z)| \leq \frac{1}{2}$ for $x \in [0, 3\sqrt{2n}]$. Therefore, by Lemma 3.7, we have $\gamma'' < 0$ on $(0, z(s_1))$. Integrating this inequality from 0 to $z$ repeatedly, we have

$$
\gamma(z) < b - \frac{1}{2n}(b + \lambda)z^2.
$$

If $b - \frac{1}{2n}(b + \lambda)z^2 \leq 0$, we have $\gamma(z) < 0$ for $z \geq \sqrt{\frac{2nb}{b + \lambda}}$. Since $b > -(4n + 1)\lambda$, we have $\sqrt{\frac{2nb}{b + \lambda}} < 2\sqrt{n}$. Besides, $z(s_1) > 3\sqrt{2n}$, there exists $z_0 \in (0, \sqrt{\frac{2nb}{b + \lambda}})$ so that $\gamma(z_0) = 0$.

Next, we will evaluate the lower bound of $z_0$. We write equation (3.1) as the form

$$
\frac{d}{dz} \left( \frac{z^{n-1}\gamma'(z)}{\sqrt{1 + \gamma'(z)^2}} \right) = z^{n-1} \cdot \frac{z\gamma'(z) - \gamma(z)}{\sqrt{1 + \gamma'(z)^2}} - \lambda z^{n-1}.
$$

(3.4)
From Lemma 3.6, we have that \( \frac{z' \gamma'(z) - \gamma(z)}{\sqrt{1 + \gamma'(z)^2}} \geq \frac{-z_0 \gamma'(z_0)}{\sqrt{1 + \gamma'(z_0)^2}} \) for \( z \in [0, z_0] \).

Integrating (3.3) from 0 to \( z_0 \), we get
\[
\frac{z_0^{n-1} \gamma'(z_0)}{\sqrt{1 + \gamma'(z_0)^2}} = \int_0^{z_0} z^{n-1} \cdot \frac{z' \gamma'(z) - \gamma(z)}{\sqrt{1 + \gamma'(z)^2}} dz - \int_0^{z_0} \frac{\lambda z^{n-1}}{n} dz
\]
\[
\geq \frac{z_0 \gamma'(z_0)}{\sqrt{1 + \gamma'(z_0)^2}} \int_0^{z_0} z^{n-1} dz - \frac{\lambda}{n} (z_0)^n
\]
\[
= \frac{1}{n} (z_0)^n \frac{z_0 \gamma'(z_0)}{\sqrt{1 + \gamma'(z_0)^2}} - \frac{\lambda}{n} (z_0)^n
\]
\[
\geq \frac{1}{n} (z_0)^n \frac{z_0 \gamma'(z_0)}{\sqrt{1 + \gamma'(z_0)^2}} + \frac{\lambda}{n} (z_0)^n \cdot \gamma'(z_0).
\]

Therefore, we have that \( 1 \leq \left( \frac{z_0}{n} \right)^2 + \frac{\lambda}{n} z_0 \), then \( z_0 \geq \frac{-\lambda + \sqrt{\lambda^2 + 4n}}{2} \). This proves the last statement of the Lemma. \( \square \)

According to Lemma 2.8, if \( b - \frac{1}{2n} (b + \lambda) z^2 \leq \lambda \), we have \( \gamma(z) \leq \lambda \) for \( z \geq \sqrt{\frac{2n(b - \lambda)}{b + \lambda}} \). If \( b - \frac{1}{2n} (b + \lambda) z^2 \geq 8n \lambda \), we have \( \gamma(z) \leq 8n \lambda \) for \( z \geq \sqrt{\frac{2n(b - 8n \lambda)}{b + \lambda}} \).

At this time, \( \sqrt{\frac{2n(b - \lambda)}{b + \lambda}} < 2 \sqrt{n} \) and \( \sqrt{\frac{2n(b - 8n \lambda)}{b + \lambda}} < 4 \sqrt{n} \) for \( b > -(4n + 1) \).

**Lemma 2.9.** Suppose \(-4n + 1) \lambda < b < \sqrt{\frac{1}{16 \pi e^{6n} - \lambda}}, z(s_1) > 3 \sqrt{2n} \) and there exists a point \( z_0 \in [-\frac{-\lambda + \sqrt{\lambda^2 + 4n}}{2}, \sqrt{\frac{2nb}{b + \lambda}}] \) so that \( \gamma(z_0) = 0 \). Then, for \( z \in [z_0, z(s_1)] \),
\[
\gamma(z) > \frac{12n}{z} \gamma'(z).
\]

**Proof.** Letting \( \Phi(z) = \frac{1}{12n} z \gamma(z) - \gamma'(z) \). To prove \( \gamma(z) > \frac{12n}{z} \gamma'(z) \) for \( z \in [z_0, z(s_1)] \), it only need to be satisfied \( \Phi(z) > 0 \). Since \( \Phi(z_0) = -\gamma'(z_0) > 0 \) and
\[
\frac{1}{12n} z \gamma(z) = \frac{1}{12n} \int_{z_0}^{z} \gamma'(\xi)d\xi
\]
\[
> \frac{1}{12n} z(z - z_0) \gamma'(z),
\]
we have \( \Phi(z) > \frac{1}{12n} \gamma'(z)(z^2 - z_0 z - 12n) \). Besides, for \( z_0 \geq \frac{-\lambda + \sqrt{\lambda^2 + 14n}}{2} \), we have that \( \Phi(z) > 0 \) when \( z \leq \frac{-\lambda + \sqrt{\lambda^2 + 4n} + \sqrt{(-\lambda + \sqrt{\lambda^2 + 4n})^2 + 48n}}{2} \).

Letting \( \frac{-\lambda + \sqrt{\lambda^2 + 14n}}{2} + \sqrt{(\frac{-\lambda + \sqrt{\lambda^2 + 14n}}{2})^2 + 48n} \), it is obvious that \( z > 4 \sqrt{n} \). If there are some points \( z \in [z_0, z(s_1)] \) so that \( \Phi(z) = 0 \), then there exists a point \( \bar{z} \in (z, z(s_1)) \) so that \( \Phi(\bar{z}) = 0 \) and \( \Phi(z) > 0 \) for \( z \in [z_0, \bar{z}] \). This
shows that $\Phi'(\hat{z}) \leq 0$ and $\frac{1}{12n}\hat{z}\gamma(\hat{z}) = \gamma'(\hat{z})$. Since $\gamma(\hat{z}) < 0$ and $\gamma'(\hat{z}) < 0$, we get

$$0 \geq \Phi'(\hat{z})$$

$$= \frac{1}{12n}\gamma(\hat{z}) + \frac{1}{12n}\hat{z}\gamma'(\hat{z}) - \gamma''(\hat{z})$$

$$\geq \frac{1}{12n}\gamma(\hat{z}) + \frac{1}{12n}\hat{z}\gamma'(\hat{z}) - \frac{\gamma''(\hat{z})}{1 + \gamma'(\hat{z})^2}$$

$$= \frac{1}{12n}\gamma(\hat{z}) + \frac{1}{(12n)^2}(\hat{z})^2\gamma(\hat{z}) - \left[ \frac{1}{12n}\hat{z}\gamma(\hat{z}) \left( \hat{z} - \frac{n - 1}{\hat{z}} \right) - \gamma(\hat{z}) - \lambda \sqrt{1 + \gamma'(\hat{z})^2} \right]$$

$$\geq \frac{13}{12}\gamma(\hat{z}) - \frac{12n - 1}{(12n)^2}(\hat{z})^2\gamma(\hat{z}) + \lambda(1 - \gamma'(\hat{z}))$$

$$= \gamma(\hat{z}) \left( \frac{13}{12} - \frac{\lambda}{12n} - \frac{12n - 1}{(12n)^2}(\hat{z})^2 \right) + \lambda.$$

Letting $f(z) = \frac{13}{12} - \frac{\lambda}{12n}z - \frac{12n - 1}{(12n)^2}z^2$ for $z \geq 4\sqrt{n}$. Since $-(4n + 1)\lambda < b < \sqrt{\frac{1}{16\pi e^{15n}}} - \lambda$, we have $-\lambda < \frac{1}{16n}\sqrt{\frac{1}{16\pi e^{15n}}}$. Through the monotonicity of the quadratic function, we have $f(z) < 0$ for $z \geq 4\sqrt{n}$. Since $\hat{z} > 4\sqrt{n}$, we have $\gamma(\hat{z}) < 8n\lambda < 0$, and then

$$\gamma(\hat{z}) \left( \frac{13}{12} - \frac{\lambda}{12n} \hat{z} - \frac{12n - 1}{(12n)^2}(\hat{z})^2 \right) + \lambda$$

$$> \lambda \left( \frac{26n}{3} + 1 \right) - \frac{2\lambda}{3}(\hat{z} - \frac{12n - 1}{18n}(\hat{z})^2)$$

$$> \lambda \left( \frac{26n}{3} + 1 \right) - \frac{2\lambda}{3} \left( 4\sqrt{n} - \frac{12n - 1}{18n}(4\sqrt{n})^2 \right)$$

$$= \lambda \left( -2n + \frac{17}{9} - \frac{8\sqrt{n}\lambda}{3} \right) > 0,$$

where $-\lambda < \frac{1}{16n}\sqrt{\frac{1}{16\pi e^{15n}}}$ and $n \geq 2$. It’s contradictory. So we have that $\gamma(z) > \frac{12n}{z}\gamma'(z)$ for $z \in [z_0, z(s_1))$. \hfill \Box

**Proof of Proposition 3.4.** Let $\gamma$ be the solution of (3.1) with $\gamma(0) = b > 0$ and $\gamma'(0) = 0$. By Lemma 3.2 and Lemma 3.3, there exists a point $z(s_1) \in (-\lambda + \frac{\sqrt{1 + 4(n-1)}}{2}, \infty)$ and a point $x(s_1) \in (-\infty, b)$ so that $\lim_{z \to z(s_1)} \gamma'(z) = -\infty$ and $\gamma(z(s_1)) = x(s_1)$. We want to refine the estimate from Lemma 3.5 to establish a lower bound for $z(s_1)$ in terms of $b$. Let $z_1 \in (0, z(s_1))$ be the point where $\gamma'(z_1) = -1$. Using the same method we used in the proof of Lemma 3.5, integrating both sides of this inequality

$$\frac{d}{dz} \left( e^{-z^2} \gamma'(z) \right) \geq -2(\gamma(z) + \lambda)e^{-z^2}$$
from 0 to $z_1$. We have $-e^{-(z_1)^2} \geq -2(b + \lambda)\sqrt{n}$, then

$$z_1 \geq \sqrt{-\ln \frac{1}{2\sqrt{n}(b + \lambda)}}.$$  

Assuming $-(4n + 1)\lambda \leq b \leq \sqrt{\frac{1}{4\pi e^{64n}}} - \lambda$, we have $z_1 \geq 4\sqrt{2n} > \frac{2n\lambda}{b + \lambda}$, and then $\frac{13n-1}{z} < \frac{3n}{4}$ for $z > z_1$. By Lemma 3.8, there exists $z_0 \in [-\lambda + \sqrt{\lambda^2 + 4n}, \sqrt{\frac{2n\lambda}{b + \lambda}}]$ so that $\gamma(z_0) = 0$, then we have from Lemma 3.9 that $\gamma(z) > \frac{12n}{z}\gamma'(z)$ for $z \in [z_0, z(s_1))$. In particular, at $z_1$, we get

$$\gamma(z_1) > -\frac{12n}{z_1}.$$  

To extend this estimate of $\gamma(z_1)$ to $\gamma(z(s_1)) = x(s_1)$. For $z \geq z_1$, we get

$$\gamma''(z) \leq \gamma'(z)^2 \frac{\gamma''(z)}{1 + \gamma'(z)^2}$$

$$= \gamma'(z)^2 \left[ \left( z - \frac{n - 1}{z} \right) \gamma'(z) - \gamma(z) - \lambda \sqrt{1 + \gamma'(z)^2} \right]$$

$$< (z - \frac{13n-1}{z} + 2\lambda) \gamma'(x)^3$$

$$\leq (\frac{1}{4}z + 2\lambda) \gamma'(z)^3,$$

where we have used that $\gamma'(z) \leq -1$ and $\gamma(z) > \frac{12n}{z}\gamma'(z)$ for $z \geq z_1$. Integrating both sides of this inequality from $z$ to $z(s_1)$, we get

$$\gamma'(z)^2 \leq \frac{4}{(z(s_1) - z)(z(s_1) + z + 16\lambda)},$$

for $z \geq z_1$. Since $\gamma'(z) < 0$ and $-(4n + 1)\lambda < b \leq \sqrt{\frac{1}{4\pi e^{64n}}} - \lambda$, we have that

$$-\frac{1}{4n} \sqrt{\frac{1}{4\pi e^{64n}}} < \lambda < 0, \quad z_1 + 8\lambda > 0.$$  

Then

$$\gamma'(z) \geq -\frac{2}{\sqrt{z(s_1) - z}\sqrt{z(s_1) + z + 16\lambda}}$$

$$\geq -\frac{2}{\sqrt{z(s_1) - z} \cdot \sqrt{z(s_1) + z_1 + 16\lambda}},$$

for $z \in [z_1, z(s_1))$. Taking value at $z_1$, we get

$$-\frac{\sqrt{z(s_1) - z_1}}{\sqrt{z(s_1) + z_1 + 16\lambda}} \geq -\frac{2}{z(s_1) + z_1 + 16\lambda}.$$  

Integrating (3.5) from $z_1$ to $z(s_1)$, we have

$$\gamma(z(s_1)) - \gamma(z_1) \geq -\frac{4}{\sqrt{z(s_1) + z_1 + 16\lambda}} \cdot \sqrt{z(s_1) - z_1},$$
then
\[
\gamma(z(s_1)) \geq \gamma(z_1) - \frac{4}{\sqrt{z(s_1) + z_1 + 16\lambda}} \cdot \sqrt{z(s_1) - z_1} \\
\geq -\frac{12n}{z_1} - \frac{8}{z(s_1) + z_1 + 16\lambda} \\
\geq -\frac{12n}{z_1 + 8\lambda} - \frac{8}{z(s_1) + z_1 + 16\lambda} \\
\geq -\frac{4(3n + 1)}{z_1 + 8\lambda} \\
\geq -\frac{4(3n + 1)}{\sqrt{\ln \frac{1}{2\sqrt{8\lambda}} + 8\lambda}}.
\]

Finally, we complete the proof of the Proposition 3.4 when \( \bar{b} = \sqrt{\frac{1}{4\pi \lambda e^{4\pi\lambda}}} - \lambda \).

\[\square\]

4. The second part of the profile curves

The basic shape of the curve \( \Gamma(s) \) is described in the previous section, we have that \( \frac{dx(s)}{ds} < 0 \) and \( \frac{d^2x(s)}{ds^2} < 0 \) when \( s \in (0, s_1) \). Then the first branch of the curve \( \Gamma(s) \) can be written as the curve \( (\gamma(z), z) \) where \( \gamma \) is the solution to \( (3.1) \) with \( \gamma(0) = b > -(4n+1)\lambda \) and \( \gamma'(0) = 0 \). At \( s = s_1 \), we have \( \frac{dx(s_1)}{ds} = -1 \) and \( \frac{dz(s_1)}{ds} = 0 \). According to the equation \( (2.2) \) and \( \frac{dz}{ds} \frac{d^2z}{ds^2} + \frac{d^2z}{ds^2} = 0 \), we have \( \frac{d^2z(s_1)}{ds^2} = \frac{n-1}{z(s_1)} - z(s_1) - \lambda < 0 \), where \( z(s_1) > -\frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2} \). This shows that the curve \( \Gamma \) is concave down at \( (x(s_1), z(s_1)) \) and heads back towards the \( x \)-axis as \( s \) increases. Let \( s_2 = s(b) > 0 \) be the arc length of the second time, if any, at which either \( \frac{dz}{ds} = 0 \) or \( z(s_2) = 0 \).

In this section, we study the curves \( \Gamma(s) \) as they travel from \( (x(s_1), z(s_1)) \) toward the \( x \)-axis. From this definition of \( s_2 \), we have \( \frac{dz}{ds} < 0 \) for \( s \in (s_1, s_2) \). This curve can be written as a graph of \( x = \beta(z) \), where \( z(s_2) < z < z(s_1) \). It follows from \( (2.2) \) that

\[
\frac{\beta''(z)}{1 + \beta'(z)^2} = \left( z - \frac{n-1}{z} \right) \beta'(z) - \beta(z) + \lambda \sqrt{1 + \beta'(z)^2}, \tag{4.1}
\]

where \( \beta''(z) = \frac{d^2x(z)}{(dx/ds)^4} \) and \( \frac{dz}{ds} = \frac{-1}{\sqrt{1 + \beta'(z)^2}} < 0 \). Then, \( \beta \) is the unique solution to \( (4.1) \) with \( \beta(z(s_1)) = x(s_1) \), \( \beta(z(s_2)) = x(s_2) \) and \( \lim_{z \to z(s_1)} \beta'(z) = \infty \). Taking the third derivative of \( (4.1) \), we have the following equation,

\[
\frac{\beta'''}{1 + (\beta')^2} = \frac{2\beta'(\beta'')^2}{(1 + (\beta')^2)^2} + \left( z - \frac{n-1}{z} \right) \beta'' + \frac{n-1}{z^2} \beta' + \frac{\lambda \beta' \beta''}{\sqrt{1 + (\beta')^2}} \tag{4.2}
\]
Since $\frac{dx(s)}{ds} = -1$ near $s = s_1$, the curve near the point $(x(s_1), z(s_1))$ can be written as a graph of $z = \alpha(x)$. It follows from (2.1) that
\[
\frac{\alpha''(x)}{1 + \alpha'(x)^2} = \left( \frac{n-1}{\alpha} - \alpha \right) + x\alpha'(x) - \lambda \sqrt{1 + \alpha'(x)^2},
\]
where $\alpha''(x) = \frac{d^2z(s)}{(dx(s))^2}$ and $\frac{dx(s)}{ds} = \frac{1}{\sqrt{1+\alpha'(x)^2}}$. Taking the third derivative of (4.3), we have the following equation,
\[
\frac{\alpha'''}{1 + (\alpha')^2} = \frac{2\alpha'(\alpha'')^2}{(1 + (\alpha')^2)^2} - \frac{n-1}{\alpha^2} \alpha' + x\alpha'' - \frac{\lambda\alpha'\alpha''}{\sqrt{1 + (\alpha')^2}}.
\]

Next, we will show that for small $b > (4n+1)\lambda$, $x(s)\text{ achieves a negative minimum at a point} s_m \in (s_1, s_2)$, the curve $\Gamma(s)$ is concave to the right, $z(s_2) > 0$ and $0 < x(s_2) < \infty$.

### 4.1. Basic shape of the second branch of the curve $\Gamma(s)$

In the first, we give some properties of the second branch of the curve $\Gamma(s)$ that are consequences of equations (2.2) and (4.1).

**Lemma 4.1.** There exists at most one point $s_m \in (s_1, s_2)$ so that $\frac{dx(s_m)}{ds} = 0$. If the point $s_m$ exists, then $\frac{d^2x(s)}{ds^2} > 0$ on $(s_1, s_2)$ and $x(s_m) < \lambda$.

**Proof.** According to the equation (2.2) and $\frac{dx(s)}{ds} = \frac{dz(s)}{ds} + \frac{dx(s)}{ds} = 0$, we have $\frac{d^2z(s_1)}{ds^2} = \frac{n-1}{\alpha(s_1)} - z(s_1) - \lambda < 0$, where $z(s_1) > -\frac{\lambda + 1 \sqrt{\lambda^2 + 4(n-1)}}{2}$, $\frac{dx(s_1)}{ds} = -1$ and $\frac{dx(s_1)}{ds} = 0$. There exists $\delta > 0$ so that for $s \in (s_1, s_1 + \delta)$, $\frac{dx(s)}{ds} < 0$, $\frac{dz(s)}{ds} < 0$ and $\frac{d^2z(s)}{ds^2} < 0$. Using the equation $\frac{dx(s)}{ds} + \frac{dx(s)}{ds} = 0$, then we have $\frac{d^2x(s)}{ds^2} > 0$ for $s \in (s_1, s_1 + \delta)$.

Next, we will prove that there exists at most one point $s_m \in (s_1, s_2)$ so that $\frac{dx(s_m)}{ds} = 0$. Supposing that there are two adjacent zeros $s_m$ and $s_n(s_m < s_n)$ to $\frac{dx(s_m)}{ds} = 0$, then there exists $s \in (s_m, s_n)$ so that $\frac{d^2x(s)}{ds^2} = 0$. We will discuss this in three cases. In the first case, if $\frac{dx(s)}{ds} \equiv 0$ for $s \in (s_m, s_n)$, we have $\frac{d^2x(s)}{ds^2} = 0$ and $\frac{dx(s)}{ds} = -1$ for $s \in (s_m, s_n)$, and then $x(s) = \lambda$ for $s \in (s_m, s_n)$. By the uniqueness of solutions for the differential equations, $x$ must be the constant function $x(s) \equiv \lambda$ on $(s_1, s_2)$, it’s contradictory. In the second case, if $\frac{d^2x(s)}{ds^2} > 0$ on $(s_m, s)$ and $\frac{d^2x(s)}{ds^2} < 0$ on $(s, s_n)$, then we have $\frac{dx(s)}{ds} > 0$ and $\frac{d^2x(s)}{ds^3} \leq 0$. Using equation (2.3), we get
\[
0 > \frac{d^3x(s)}{ds^3} = \frac{n-1}{z(s)} \frac{dz(s)}{ds}^2 \left( \frac{dx(s)}{ds} \right)^2 > 0,
\]

it’s contradictory. In the third case, if $\frac{d^2x(s)}{ds^2} < 0$ on $(s_m, s)$ and $\frac{d^2x(s)}{ds^2} > 0$ on $(s, s_n)$, then we have $\frac{dx(s)}{ds} < 0$ and $\frac{d^3x(s)}{ds^3} \geq 0$. Using equation (2.3) again,
Lemma 4.2. Suppose there exists $s \in (s_1, s_2)$. Since there exists $(2.3)$, we get
\[ s_{\bar{}} \]
it’s contradictory. The same method, supposing $s_{\bar{}} < \lambda$. Arguing as in the same proof of lemma 3.1, supposing $s_{\bar{}} = 0$ for some points $s \in (s_1, s_m)$. Choosing $s_{\bar{}} \in (s_1, s_m)$ so that $s_{\bar{}} = 0$ and $s_{\bar{}} > 0$ for $s \in (s_{\bar{}}, s_m)$, then we have that $\frac{dx_{\bar{}}}{ds}_{\bar{}} < 0$ and $\frac{dx_{\bar{}}}{ds}_{\bar{}} > 0$. Using equation $(2.3)$, we get
\[ 0 \leq \frac{d^3x(s_{\bar{}})}{ds^3} = \frac{n - 1}{z(s_{\bar{}})^2} \frac{dz(s_{\bar{}})}{ds} \frac{dx(s_{\bar{}})}{ds} < 0, \]
it’s contradictory. The same method, supposing $\frac{d^3x(s_{\bar{}})}{ds^3} = 0$ for some points $s \in (s_m, s_2)$. Choosing $s_{\bar{}} \in (s_m, s_2)$ so that $\frac{d^3x(s_{\bar{}})}{ds^3} = 0$ and $\frac{d^3x(s_{\bar{}})}{ds^3} > 0$ for $s \in (s_{\bar{}}, s_{\bar{}})$, then we have that $\frac{dx_{\bar{}}}{ds}_{\bar{}} > 0$ and $\frac{dx_{\bar{}}}{ds}_{\bar{}} < 0$. Using equation $(2.3)$, we get
\[ 0 \geq \frac{d^3x(s_{\bar{}})}{ds^3} = \frac{n - 1}{z(s_{\bar{}})^2} \frac{dz(s_{\bar{}})}{ds} \frac{dx(s_{\bar{}})}{ds} > 0, \]
it’s contradictory. Then, if there exists one point $s_m \in (s_1, s_2)$ so that $\frac{dx_{s_m}}{ds} = 0$, we have that $\frac{dx_{s_m}}{ds} > 0$ for $s \in (x_1, x_2)$. □

**Lemma 4.2.** Suppose there exists $s_m \in (s_1, s_2)$ so that $\frac{dx_{s_m}}{ds} = 0$, if $s \in (s_m, s_2)$ and $z(s) \geq \frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2}$, then $x(s) < \lambda$; if $s \in (s_m, s_2)$ and $z(s) \geq \frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2}$, then $x(s) < 0$.

**Proof.** Since there exists $s_m \in (s_1, s_2)$ so that $\frac{dx_{s_m}}{ds} = 0$. By Lemma 4.1, we have $\frac{dx_{s_m}}{ds} > 0$ for $s \in (s_1, s_2)$ and $\frac{dx_{s_2}}{ds} > 0$ for $s \in (s_m, s_2)$. If the second branch of the curve $\Gamma(s)$ may be written as $(\beta(z), z)$, then $\beta''(z) = \frac{d^2x_{s_2}}{ds^2} \frac{dx_{s_2}}{ds} > 0$ for $z \in (z(s_2), z(s_1))$ and $\beta'(z) = \frac{dx_{s_2}}{ds} < 0$ for $z \in (z(s_2), z(s_m))$.  


It follows from (4.1) that for $z \in (z(s_2), z(s_m))$,

$$0 < \frac{\beta''(z)}{1 + \beta'(z)^2} = \left( z - \frac{n - 1}{z} \right) \beta'(z) - \beta(z) + \lambda \sqrt{1 + \beta'(z)^2}$$

$$\leq \left( z - \frac{n - 1}{z} \right) \beta'(z) - \beta(z) + \lambda(1 + \beta'(z))$$

$$= \left( z - \frac{n - 1}{z} + \lambda \right) \beta'(z) - (\beta(z) - \lambda),$$

where $\beta'(z) < 0$.

When $z \geq -\frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2}$, we have $\beta(z) < \lambda$. For $z \in (z(s_2), z(s_m))$, according to the equation (4.1), again, we have that

$$0 < \frac{\beta''(z)}{1 + (\beta'(z))^2} = \left( z - \frac{n - 1}{z} \right) \beta'(z) - \beta(z) + \lambda \sqrt{1 + \beta'(z)^2}$$

$$\leq \left( z - \frac{n - 1}{z} \right) \beta'(z) - \beta(z) - \lambda \beta'(z)$$

$$= \left( z - \frac{n - 1}{z} - \lambda \right) \beta'(z) - \beta(z),$$

where $\beta'(z) < 0$. When $z \geq \frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2}$, we have $\beta(z) < 0$. 

\[\square\]

**Lemma 4.3.** Suppose there exists a point $s_m \in (s_1, s_2)$ so that $\frac{dx(s_m)}{ds} = 0$, then $z(s_2) < \frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2}$.

**Proof.** There exists $s_m \in (s_1, s_2)$ so that $\frac{dx(s_m)}{ds} = 0$. By Lemma 4.1, we have $\frac{d^2x(s)}{ds^2} > 0$ for $s \in (s_1, s_2)$ and $\frac{dx(s)}{ds} > 0$ for $s \in (s_m, s_2)$. By Lemma 4.2, we have that $x(s) < 0$ when $s \in (s_m, s_2)$ and $z(s) \geq \frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2}$. Letting $M = -x(s_m)$, then we have $x(s) \geq -M$ for $s \in (s_1, s_2)$. If $z(s_m) \leq \frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2}$, then $z(s_2) < z(s_m) \leq \frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2}$. If $z(s_m) > \frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2}$, the second branch of the curve $\Gamma(s)$ may be written as $(\beta(z), z)$, for any small $\varepsilon > 0$, there exists a constant $m_{\varepsilon} > 0$ so that $z - \frac{n - 1}{z} - \lambda > m_{\varepsilon}$ for $z \in \left[\frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2} + \varepsilon, z(s_m)\right)$. Using inequality (4.6),

$$\beta'(z) > \frac{\beta(z)}{z - \frac{n - 1}{z} - \lambda} \geq -\frac{M}{m_{\varepsilon}},$$

for $z \in \left[\frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2} + \varepsilon, z(s_m)\right)$. Therefore, $|\beta(z)|$ and $|\beta'(z)|$ are uniformly bounded for $z \in \left[\frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2} + \varepsilon, z(s_m)\right)$. By the existence theory for the differential equations, we have $z(s_2) < \frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2} + \varepsilon$. In fact, if
\[ z(s_2) \geq \frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2} + \varepsilon, \] then \( \lim_{z \to z(s_2)} \beta'(z) = -\infty \) is contradictory with uniform boundedness of \(|\beta'(z)|\) on \( \left[ \frac{\lambda + \sqrt{\lambda^2 + 2(n-1)}}{2} + \varepsilon, z(s_m) \right) \). Taking \( \varepsilon \to 0 \), we have
\[ z(s_2) \leq \frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2}. \]

Next, we will show that \( z(s_2) < \frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2} \). Supposing \( z(s_2) = \frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2} \), then \( x(s_2) \in [-M, 0] \). Since \( \frac{dx(s_2)}{ds} = 0 \) and \( \frac{dx(s_2)}{ds} = 1 \), by the existence and uniqueness of solutions for the differential equations, we deduce that \( z(s) \) is the constant function \( z(s) = \frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2} \), it’s contradictory.

**Lemma 4.4.** \( \lim_{s \to s_2} x(s) < \infty \).

**Proof.** Supposing \( \lim_{s \to s_2} x(s) = \infty \). Since \( \frac{dx(s)}{ds} < 0 \) for \( s \) close to \( s_1 \), there exists a point \( s_m \in (s_1, s_2) \) so that \( \frac{dx(s_m)}{ds} = 0 \). By Lemma 4.1, we have that \( \frac{d^2x(s)}{ds^2} > 0 \) for \( s \in (s_1, s_2) \) and \( x(s_m) < \lambda \). It follows that there exists a point \( s_0 \in (s_m, s_2) \) so that \( x(s_0) = 0 \). By Lemma 4.2, we have \( z(s_0) < \frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2} \). By Lemma 4.3, we have \( z(s_2) < \frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2} \).

In particular, when \( z \in (z(s_2), z(s_0)) \), there exists some \( M > 0 \) so that \( z(s) - \frac{n-1}{z(s)} - \lambda \leq -\frac{1}{2M} \).

When the second branch of the curve \( \Gamma(s) \) may be written as \( (\beta(z), z) \), there exists \( k > 0 \) so that \( \beta'(z(s_0)) = -k \). Letting \( f = -\beta' \), we have that \( f(z) > 0 \) for \( z \in (z(s_2), z(s_0)) \) and \( f'(z) < 0 \). Using equation (1.2), we get
\[
\begin{align*}
f''(z) &= \frac{2f(z)f'(z)^2}{1+f(z)^2} + \left( z - \frac{n-1}{z} \right) f'(z)(1 + f(z)^2) \\
&\quad + \frac{(n-1)f(z)}{z^2}(1 + f(z)^2) - \lambda f'(z)f(z)\sqrt{1 + f(z)^2} \\
&\geq \left( z - \frac{n-1}{z} \right) f'(z)(1 + f(z)^2) - \lambda f'(z)f(z)\sqrt{1 + f(z)^2} \\
&\geq \left( z - \frac{n-1}{z} - \lambda \right) f'(z)(1 + f(z)^2) \\
&\geq \left( z - \frac{n-1}{z} - \lambda \right) f'(z)f(z)^2 \\
&\geq -\frac{1}{2M} f'(z)f(z)^2,
\end{align*}
\]

Where \( z \in (z(s_2), z(s_0)) \). For small \( \varepsilon > 0 \), letting
\[
g_\varepsilon(z) = k\sqrt{z(s_0) - (z(s_2) + \varepsilon)} + \sqrt{3M} \sqrt{z - (z(s_2) + \varepsilon)},
\]
for $z \in (z(s_2) + \varepsilon, z(s_0))$, then we get

$$g'_\varepsilon = -\frac{1}{2} k \sqrt{z(s_0) - (z(s_2) + \varepsilon) + \sqrt{3M}} \frac{\varepsilon}{(z - (z(s_2) + \varepsilon))^{3/2}} < 0$$

and

$$g''_\varepsilon = -\frac{3}{2} \frac{1}{k \sqrt{z(s_0) - (z(s_2) + \varepsilon) + \sqrt{3M}}} g'_\varepsilon \cdot g'\varepsilon \leq -\frac{1}{2M} g'_\varepsilon \cdot g'_\varepsilon.$$

Next, the purpose is to prove $f \leq g_\varepsilon$. It is known that

$$k + \frac{\sqrt{3M}}{\sqrt{z(s_0) - (z(s_2) + \varepsilon)}} = g_\varepsilon(z(s_0)) > f(z(s_0)) = k$$

and

$$\infty = \lim_{z \to z(s_2) + \varepsilon} g_\varepsilon(z) > f(z(s_2) + \varepsilon).$$

Therefore, if there are some points on $(z(s_2) + \varepsilon, z(s_0))$ so that $f > g_\varepsilon$, then $f - g_\varepsilon$ achieves a positive maximum at point $\tilde{z} \in (z(s_2) + \varepsilon, z(s_0))$. This leads to $(f - g_\varepsilon)'(\tilde{z}) = 0$ and $(f - g_\varepsilon)''(\tilde{z}) \leq 0$. Then we have

$$0 \geq (f - g_\varepsilon)''(\tilde{z}) \geq -\frac{1}{2M} f'(\tilde{z}) \left( f(\tilde{z})^2 - g_\varepsilon(\tilde{z})^2 \right) > 0,$$

it’s contradictory. Therefore, we have $f \leq g_\varepsilon$. Taking $\varepsilon \to 0$, we have the estimate

$$f(z) \leq k \sqrt{z(s_0) - z(s_2) + \sqrt{3M}} \sqrt{z - z(s_2)},$$

for $z \in (z(s_2), z(s_0))$. Integrating this inequality from $z$ to $z(s_0)$,

$$\beta(z) - \beta(z(s_0)) \leq 2 \left( k \sqrt{z(s_0) - z(s_2) + \sqrt{3M}} \right) \left( \sqrt{z(s_0) - z(s_2)} - \sqrt{z - z(s_2)} \right).$$

Since $\beta(z(s_0)) = 0$, it follows that

$$\lim_{z \to z(s_2)} \beta(z) \leq 2 \left( \sqrt{z(s_0) - z(s_2)} \right) \left( k \sqrt{z(s_0) - z(s_2) + \sqrt{3M}} \right).$$

Then we have $\lim_{z \to z(s_2)} \beta(z) < \infty$.

According to Lemma 4.4, there exists a point $s_m \in (s_1, s_2)$ so that $\frac{dx(s_m)}{ds} = 0$, then $z(s_2) > 0$. At this time, $\frac{dx(s_2)}{ds} = 0$ and $\frac{dx(s_2)}{ds} = 1$.

**Lemma 4.5.** Suppose there exists a point $s_m \in (s_1, s_2)$ so that $\frac{dx(s_m)}{ds} = 0$, then $z(s_2) > 0$.

**Proof.** If there exists a point $s_m \in (s_1, s_2)$ so that $\frac{dx(s_m)}{ds} = 0$, then we have $\frac{d^2x(s)}{ds^2} > 0$. When the second branch of the curve $\Gamma(s)$ may be written as $(\beta(z), z)$, we have $-1 < \frac{\beta'(z)}{\sqrt{1 + \beta(z)^2}} < 0$ for $z \in (z(s_2), z(s_m))$, and then there
Lemma 4.4, there exists \( s \in (z(s_2), z(s_m)) \) so that \( \frac{\beta'(z_\varepsilon)}{\sqrt{1+\beta'(z_\varepsilon)^2}} = -\varepsilon \). Besides, by Lemma 4.6, there exists \( M \geq 0 \) so that \( \beta(z) < M \) for \( z \in (z(s_2), z(s_1)) \). Letting \( \theta(z) = \arctan(\beta'(z)) \), for \( z \in (z(s_2), z(s_1)) \), we get that
\[
\frac{d}{dz} \left( \ln \frac{\sin \theta(z)}{\sin \theta(\varepsilon)} \right) = 1 - \frac{\beta''(z)}{\beta'(z)} = \frac{\beta''(z)}{\beta'(z)} \left( 1 - \frac{n-1}{z} \right)
\]
where \( \beta'(z) < 0 \). Integrating this inequality from \( z \to z_\varepsilon \), we get
\[
\ln \left( \frac{\sin \theta(z_\varepsilon)}{\sin \theta(\varepsilon)} \right) \leq \frac{1}{2} (z_\varepsilon)^2 (n-1) \ln \left( \frac{z}{z_\varepsilon} \right) - \lambda z_\varepsilon + \frac{(M - \lambda)z_\varepsilon}{(-\beta'(z_\varepsilon))}.
\] (4.7)
Since \( \sin \theta(z_\varepsilon) = \frac{\beta'(z_\varepsilon)}{\sqrt{1+\beta'(z_\varepsilon)^2}} = -\varepsilon \), taking \( z \to z(s_2) \) for (4.7), we get
\[
0 < \varepsilon \leq (-\sin \theta(z(s_2))) \cdot \left( \frac{z(s_2)}{z_\varepsilon} \right)^{n-1} e^{\frac{1}{2}(z_\varepsilon)^2 - \lambda z_\varepsilon + \frac{(M - \lambda)z_\varepsilon}{(-\beta'(z_\varepsilon))}}.
\]
Then \( z(s_2) > 0 \).
\[\square\]

4.2. **Behavior of the second branch of the curve \( \Gamma(s) \) for small \( b \) \(- (4n + 1)\lambda \).** From the above lemmas, we can get the basic description of the second branch of the curve \( \Gamma(s) \): If there exists a point \( s_m \in (s_1, s_2) \) so that \( \frac{dx(s_m)}{ds} = 0 \), then we have \( \frac{dx^2(s)}{ds^2} > 0 \) on \( (s_1, s_2) \) and \( 0 < z(s_2) < \frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2} \). Next, we will study the dependence of the second branch of the curve \( \Gamma(s) \) and \( (x(s_2), z(s_2)) \) on the initial height.

Fixing \( b \in (0, b^\dagger) \), where \( b \) is defined in Proposition 3.4. For \( b > -(4n + 1)\lambda \), let \( \gamma_b \) denote the solution of (3.1) with \( \gamma_b(0) = b \) and \( \gamma'_b(0) = 0 \). Let \( z(s_1^b) \) denote the point where \( \gamma_b \) blow-up and \( x(s_1^b) = \gamma_b(z(s_1^b)) \). Let \( \beta_b \) be the unique solution to (4.1) with \( \beta(z(s_1^b)) = x(s_1^b) \) and \( \lim_{z \to z(s_1^b)} \beta'(z) = \infty \).

There exists \( s_2^b > s_1^b \) so that \( \beta_b \) is defined on \( (z(s_2^b), z(s_1^b)) \) and \( \beta_b \) blow-up as \( z \to z(s_1^b) \) or \( z(s_2^b) = 0 \).

**Lemma 4.6.** Supposing \( z(s_1) \geq \frac{-\lambda + 2\sqrt{2} + 2\sqrt{\lambda^2 + 4n}}{2} \), then there exists \( s_m \in (s_1, s_2) \) so that \( z(s_m) \in [z(s_1) - \sqrt{2}, z(s_1)] \) and \( \beta'(z(s_m)) = 0 \).

**Proof.** Supposing \( \beta'(z) > 0 \) for \( z \in [z(s_1) - \sqrt{2}, z(s_1)] \). By Lemma 3.8, for \( s \in (0, s_1) \), if \( z(s) \geq \sqrt{\frac{2n(b - \lambda)}{b + \lambda}} \), we have \( x(s) = \beta(z(s)) < \lambda \). Since
b > -(4n+1)\lambda, \text{ then } \sqrt{\frac{2n(b-\lambda)}{b+\lambda}} < 2\sqrt{\alpha}. \text{ When } z(s_1) \geq -\frac{\lambda+2\sqrt{\lambda^2+4(\lambda-1)}}{2}, \text{ we have that } z(s_1) - \sqrt{\alpha} \geq -\frac{\lambda+2\sqrt{\lambda^2+4(\lambda-1)}}{2} \quad \text{ and } \beta(z(s_1)) < \lambda. 

Using the equation (4.1), for \( z > \frac{1}{b} \), we have that \( \in \quad \text{ and } \beta(z(s_1)) < \lambda. 

Using the equation (4.1), for \( z \in [z(s_1) - \sqrt{\alpha}, z(s_1)) \), we have

\[
\frac{\beta''(z)}{1 + \beta'(z)^2} = \left( z - \frac{n - 1}{z} \right) \beta'(z) - \beta(z) + \lambda \sqrt{1 + \beta'(z)^2} \\
\geq \left( z - \frac{n - 1}{z} \right) \beta'(z) - \beta(z) + \lambda(1 + \beta'(z)) \\
\geq \left( z - \frac{n - 1}{z} + \lambda \right) \beta'(z) - (\beta(z) - \lambda),
\]

then \( \beta''(z) > 0 \) on \([z(s_1) - \sqrt{\alpha}, z(s_1))\) and \( \beta''(z(s_1) - \sqrt{\alpha}) \geq -(\beta(z(s_1) - \sqrt{\alpha}) - \lambda) \). Therefore, for \( z \in [z(s_1) - \sqrt{\alpha}, z(s_1)) \), using the equation (4.2), we have

\[
\frac{\beta''''(z)}{1 + \beta'(z)^2} = \frac{2\beta'(z)(\beta''(z))^2}{(1 + \beta'(z)^2)^2} + \left( z - \frac{n - 1}{z} \right) \beta''(z) + \frac{n - 1}{z^2} \beta'(z) + \frac{\lambda \beta'(z) \beta''(z)}{\sqrt{1 + \beta'(z)^2}} \\
\geq (z - \frac{n - 1}{z}) \beta''(z) + \frac{\lambda \beta'(z) \beta''(z)}{\sqrt{1 + \beta'(z)^2}} \\
\geq (z - \frac{n - 1}{z} + \lambda) \beta''(z),
\]

then \( \beta''''(z) > 0 \) on \([z(s_1) - \sqrt{\alpha}, z(s_1))\). For \( z \in [z(s_1) - \sqrt{\alpha}, z(s_1)) \), integrating this inequality repeatedly from \( z(s_1) - \sqrt{\alpha} \) to \( z \), we get

\[
\beta(z) \geq \beta(z(s_1) - \sqrt{\alpha}) - \frac{1}{2} (\beta(z(s_1) - \sqrt{\alpha}) - \lambda) \left( z - (z(s_1) - \sqrt{\alpha}) \right)^2,
\]

where we use \( \beta''''(z(s_1) - \sqrt{\alpha}) \geq -(\beta(z(s_1) - \sqrt{\alpha}) - \lambda) \). This shows us that \( \beta(z(s_1)) \geq \lambda \), it’s contradictory. \( \square \)

Let \( \bar{b} > 0 \) be the value given in the conclusion of Proposition 3.4 so that if \( b \in (0, \bar{b}] \), \( \gamma(z) \leq 0 \) for \( z \in [\sqrt{\frac{2nb}{\lambda+\lambda} + 1}, z(s_1)) \). We also assume that \( \bar{b} \) is chosen so small that \( z(s_m) > 2\sqrt{2n} \) and \( x(s) \geq -\frac{1}{s\sqrt{2n}} > -\frac{7n+1}{4\sqrt{2n}} \) when \( b \in (0, \bar{b}] \). So we will get the following conclusion.

Lemma 4.7. If \( b \in (0, \bar{b}] \), then \( 2x(s) \leq \beta(z) < 0 \) for \( z \in [2\sqrt{2n}, z(s_1)] \).

\textbf{Proof.} Let \( \alpha(x) \) denote the curve that connects \( \gamma \) and \( \beta \) near the point \( (x(s_1), z(s_1)) \), then \( \alpha \) is a solution of (4.3) with \( \alpha'(x(s_1)) = z(s_1) \) and \( \alpha''(x(s_1)) = 0 \). Using equation (4.4), we have \( \alpha''(x(s_1)) = x(s_1)\alpha''(x(s_1)) \). Therefore, we get Taylor expansion of \( \alpha(x), \)

\[
\alpha(x) = z(s_1) + \frac{1}{2} \alpha''(x(s_1))(x - x(s_1))^2 \\
+ \frac{1}{6} x(s_1) \alpha''(x(s_1))(x - x(s_1))^2 + O(|x - x(s_1)|^4), \quad (4.8)
\]
as \( x \to x(s_1) \).

For \( z < z(s_1) \) and near \( z(s_1) \), the curve \( \Gamma \) is concave down and heads back towards the \( x \)-axis. We can find \( s, t > 0 \) so that

\[
\alpha(x(s_1) + t) = z = \alpha(x(s_1) - s).
\]

Using (4.8) and \( x(s_1) < 0 \), we can get \( t > s \).

To prove the lemma, we consider the function \( \delta(z) = \gamma(z) + \beta(z) \). Since \( \gamma(z) \leq 0 \) and \( \beta(z) < 0 \) for \( z \in [2\sqrt{2n}, z(s_1)) \), we have \( \delta(z) < 0 \). By the previous discussion, then

\[
\delta(z) = (x(s_1) + t) + (x(s_1) - s) > 2x(s_1) = \delta(z(s_1)),
\]

for \( z < z(s_1) \) and near \( z(s_1) \).

Now, we will prove that \( \delta(z) > 2x(s_1) \) on \( [2\sqrt{2n}, z(s_1)) \). Supposing \( \delta(z) = 2x(s_1) \) for some \( z \in [2\sqrt{2n}, z(s_1)) \). Choosing \( \tilde{z} \in [2\sqrt{2n}, z(s_1)) \) so that \( \delta(\tilde{z}) = 2x(s_1) \). It follows from the previous discussion that there exists a point \( \tilde{z} \in (\tilde{z}, z(s_1)) \) so that \( \delta \) achieves a negative maximum. This leads to \( \delta'(\tilde{z}) = 0 \) and \( \delta''(\tilde{z}) \leq 0 \). Therefore, we get

\[
0 \geq \frac{\delta''(\tilde{z})}{1 + \gamma'((\tilde{z})^2)} = -\delta(\tilde{z}) > 0,
\]

it’s contradictory. So we have that \( \delta > 2x(s_1) \) on \( [2\sqrt{2n}, z(s_1)) \). Since \( \gamma(z) \leq 0 \) on \( [2\sqrt{2n}, z(s_1)) \), we have that \( 2x(s_1) \leq \gamma(z) + \beta(z) \leq \beta(z) < 0 \) on \( [2\sqrt{2n}, z(s_1)) \).

\[\square\]

**Lemma 4.8.** Letting \( b \in (0, \tilde{b}] \). If \( \beta < 0 \) on \( (z(s_2), z(s_1)) \), then

\[
z(s_2) \leq \frac{8(n - 1)}{(\pi - 2) + \frac{16n - 1}{\sqrt{2n}}(-\lambda)}(-x(s_1)).
\]

**Proof.** According to our previous assumptions on \( \tilde{b} \), we have that \( z(s_m) > 2\sqrt{2n} \), \( \beta'' > 0 \) and \( \beta(2\sqrt{2n}) \geq 2x(s_1) \). By Lemma 4.2, we have \( 2x(s_1) \leq \beta(2\sqrt{2n}) \leq \lambda \). For \( z \in (z(s_2), z(s_m)) \), using the inequality (4.5) and choosing \( z = 2\sqrt{2n} \), we have

\[
\beta'(2\sqrt{2n}) > \frac{\beta(2\sqrt{2n}) - \lambda}{2\sqrt{2n} - \frac{n-1}{2\sqrt{2n}} + \lambda} \geq \frac{2x(s_1) - \lambda}{2\sqrt{2n} + \lambda} > -1.
\]

For \( z \in (z(s_2), 2\sqrt{2n}) \), we have that

\[
\frac{d}{dz}(\arctan \beta'(z)) = \left(z - \frac{n-1}{z}\right) \beta'(z) - \beta(z) + \lambda \sqrt{1 + \beta'(z)^2}
\]

\[
\leq - \frac{n-1}{z} \beta'(z) - \beta(z) + \lambda(1 + \beta'(z))
\]

\[
\leq (-\frac{n-1}{z(s_2) + \lambda}) \beta'(z) + (\lambda - \beta(2\sqrt{2n})),
\]
where $\beta'(z) < 0$. Integrating this inequality from $z(s_2)$ to $2\sqrt{2n}$, then

$$\frac{\pi}{4} \leq \arctan \beta'(2\sqrt{2n}) + \frac{\pi}{2} \leq (-\frac{n-1}{x(s_2)} + \lambda) \beta(2\sqrt{2n}) + 2\sqrt{2n}(\lambda - \beta(2\sqrt{2n})) = (-\beta(2\sqrt{2n})(\frac{n-1}{z(s_2)} - \lambda) + 2\sqrt{2n}) + 2\sqrt{2n}\lambda \leq (-2x(s_1)(\frac{n-1}{z(s_2)} - \lambda) + 2\sqrt{2n}) + 2\sqrt{2n}\lambda,$$

where $\beta(z(s_2)) < 0$, $2x(s_1) \leq \beta(2\sqrt{2n}) < \lambda$ and $\beta'(2\sqrt{2n}) > -1$. Rearranging this inequality to estimate $z(s_2)$ and using $x(s_1) \geq -\frac{1}{8\sqrt{2n}}$, we get

$$z(s_2) \leq \frac{8(n-1)}{(\pi - 2) + \frac{16n-1}{\sqrt{2n}}(-\lambda)}(-x(s_1)).$$

Lemma 4.9. There exists $\tilde{b} > 0$ so that for $b \in (0,\tilde{b}]$, there is a point $z(s_m^b) \in (z(s_2^b), z(s_1^b))$ so that $\beta'_b(z(s_m^b)) = 0$ and $0 < \beta_b(z(s_2^b)) < \infty$.

Proof. Fixing $b \in (0,\tilde{b}]$ and letting $\beta = \beta_b$. By Lemma 4.6, we have that there exists $s_m \in (s_1, s_2)$ so that $z(s_m) > 2\sqrt{2n}$ and $\beta'(z(s_m)) = 0$. Therefore, we have $\beta'' > 0$ on $(z(s_2), z(s_1))$ and $\beta' < 0$ on $(z(s_2), z(s_m))$. Besides, by Lemma 4.3, we get $\beta(z(s_2)) < \infty$.

Supposing $\beta(z(s_2)) \leq 0$ so that $\beta < 0$ on $(z(s_2), z(s_m))$. By Lemma 4.8, there exists $0 < \epsilon < \frac{\lambda + \sqrt{\lambda^2 + (n-1)}}{2}$ so that $\beta'(\epsilon) = -1$ for sufficiently small $b$. Therefore, using equation (1.2), we get

$$\frac{\beta''(z)}{1 + \beta'(z)^2} = \frac{2\beta'(z)(\beta''(z))^2}{(1 + \beta'(z)^2)^2} + \left(z - \frac{n-1}{z}\right)\beta''(z) + \frac{n-1}{z^2}\beta'(z) + \frac{\lambda \beta'(z)\beta''(z)}{\sqrt{1 + \beta'(z)^2}} \leq (z - \frac{n-1}{z} - \lambda)\beta''(z),$$
then $\beta''(z) < 0$ for $z \in (z(s_2), \frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2})$. Using the equation (4.1), then
\[
\beta''(z) \geq \frac{\beta''(z)}{1 + \beta'(z)^2} = \left( z - \frac{n-1}{z} \right) \beta'(z) - \beta(z) + \lambda \sqrt{1 + \beta'(z)^2} 
\geq \left( z - \frac{n-1}{z} \right) \beta'(z) - \beta(z) + \lambda (1 - \beta'(z)) 
= \left( z - \frac{n-1}{z} - \lambda \right) \beta'(z) - (\beta(z) - \lambda),
\]
Taking $z = \frac{\lambda + \sqrt{\lambda^2 + (n-1)}}{2}$, we have $\beta'' \left( \frac{\lambda + \sqrt{\lambda^2 + (n-1)}}{2} \right) \geq - (\beta \left( \frac{\lambda + \sqrt{\lambda^2 + (n-1)}}{2} \right) - \lambda)$, and then $\beta''(z) \geq -(\beta \left( \frac{\lambda + \sqrt{\lambda^2 + (n-1)}}{2} \right) - \lambda)$ for $z \in (z(s_2), \frac{\lambda + \sqrt{\lambda^2 + (n-1)}}{2})$.
Integrating this inequality from $\epsilon$ to $\frac{\lambda + \sqrt{\lambda^2 + (n-1)}}{2}$. In addition, $\beta'(z) < 0$ for $z \in (\epsilon, \frac{\lambda + \sqrt{\lambda^2 + (n-1)}}{2})$, we have
\[
\beta'(\epsilon) \leq \left( \frac{\lambda + \sqrt{\lambda^2 + (n-1)}}{2} \right) - \epsilon (\beta(\epsilon) - \lambda).
\]
Therefore,
\[
\beta(\epsilon) \geq - \frac{1}{\frac{\lambda + \sqrt{\lambda^2 + (n-1)}}{2} - \epsilon} + \lambda.
\]
Choosing $z \in (z(s_2), \epsilon)$, we have
\[
\frac{\beta''(z)}{\beta'(z)} = \left( z - \frac{n-1}{z} \right) (1 + \beta'(z)^2) - \frac{\beta(z)}{\beta'(z)} (1 + \beta'(z)^2) + \frac{\lambda(1 + \beta'(z)^2)^{\frac{1}{2}}}{\beta'(z)} 
\leq \left( z - \frac{n-1}{z} + \lambda \frac{1}{\beta'(z)} (1 + \beta'(z)^2)^{\frac{1}{2}} \right) (1 + \beta'(z)^2) 
\leq \left( z - \frac{n-1}{z} - 2\lambda \right) (1 + \beta'(z)^2) 
\leq z - \frac{n-1}{z} - 2\lambda,
\]
where $\beta'(z) < \beta'(\epsilon) = -1$. Integrating (4.9) repeatedly from $z$ to $\epsilon$, we get
\[
\beta(z) \geq \beta(\epsilon) - \beta'(\epsilon)e^{-\frac{(z}{t}^2 - 2\lambda \epsilon} \int_z^\epsilon \frac{t}{t}^{n-1} dt = \beta(\epsilon) + e^{-\frac{(z}{t}^2 - 2\lambda \epsilon} \int_z^\epsilon \frac{t}{t}^{n-1} dt.
\]
Combining this with $\beta(\epsilon) \geq - \frac{1}{\frac{\lambda + \sqrt{\lambda^2 + (n-1)}}{2} - \epsilon} + \lambda$, we have that
\[
0 > \beta(z) \geq e^{-\frac{(z}{t}^2 - 2\lambda \epsilon} \int_z^\epsilon \frac{t}{t}^{n-1} dt - \frac{1}{\frac{\lambda + \sqrt{\lambda^2 + (n-1)}}{2} - \epsilon} + \lambda.
\]
By Proposition 3.4 and Lemma 4.8, we have that \(z(s_2)\) is sufficiently close to 0 when \(b > -(4n + 1)\alpha\) is sufficiently small, and then \(\beta(z) \geq 0\) for \(z \to z(s_2)\), it’s contradictory. Therefore, \(0 < \beta_b(z(s_2^b)) < \infty\) for sufficiently small \(b > -(4n + 1)\alpha\).

\[\square\]

5. An Immerged \(S^n\) \(\lambda\)-Hypersurface

In this section, we will complete the proof of Theorem 1.1. We consider the set

\[T = \{\tilde{b} : \forall b \in (0, \tilde{b}], \exists s_m^b \in (s_2^b, s_1^b) \text{ so that } \frac{dx(s_m^b)}{ds} = 0 \text{ and } x(s_2^b) > 0\}.\]

By Lemma 4.9, we have that this set is not empty. Following Ang enent’s argument in [1], let \(b_0\) be the supremum of this set:

\[b_0 = \sup_b T.\]

In [4], we know that the curve \(\Gamma(s) = (b \cos \frac{s}{b}, b \sin \frac{s}{b})\) is a circle, where \(b = \frac{-\lambda + \sqrt{\lambda^2 + 4n}}{2}\), which is special solution of (2.2). It’s obvious that \(b_0 \leq \frac{-\lambda + \sqrt{\lambda^2 + 4n}}{2}\). We want to show \(\Gamma_{b_0}\) intersects the \(x\)-axis perpendicularly at \(z(s_2^{b_0}) (x(s_2^{b_0}) = 0)\).

**Lemma 5.1.** \(b_0 < \frac{-\lambda + \sqrt{\lambda^2 + 4n}}{2}\).

**Proof.** Suppose \(\frac{dx(s)}{ds} < 0\) and \(\frac{d^2x(s)}{ds^2} \geq 0\) on \((s_1^{b_0}, s_2^{b_0})\). It writes the second branch of the curve \(\Gamma_{b_0}(s)\) as a form of \((\beta_{b_0}(z), z)\), we have that \(\beta'_{b_0} > 0\) and \(\beta''_{b_0} \geq 0\) on \((z(s_2^{b_0}), z(s_1^{b_0}))\). Then, the curve \(\beta_{b_0}\) must intersect with the \(x\)-axis \((z(s_2^{b_0}) = 0)\) and there exists \(m > 0\) so that \(\beta_{b_0}(z) \leq -m\) for \(z \in (0, \frac{1}{2}]\). Let \(\{b_n\}\) be an increasing sequence that converges to \(b_0\). Let \(\beta_n\) be the solution of (1.1) corresponding to the initial height \(b_n\). Fixing \(0 < \varepsilon < \frac{1}{2}\), by continuity of the curve \(\Gamma_b(s)\), there exists \(N = N(\varepsilon) > 0\) so that for \(n > N\), we have that \(|z(s_2^n) - z(s_2^{b_0})| < \varepsilon\) and \(|\beta_n(\varepsilon) - \beta_{b_0}(\varepsilon)| < \varepsilon\). Therefore, \(z(s_2^n) < \varepsilon\) and \(\beta_n(\varepsilon) < -m\), it follows that \(\beta_n\) intersects the \(z\)-axis at point \(z(s_0^n) < \varepsilon\). Besides, the curve \((\beta_n(z), z)\) can be written as the curve \((x, \alpha_n(x))\) for \(x \in [-m/2, 0]\), it follows from (2.2) that

\[\frac{\alpha_n''(x)}{1 + \alpha_n'(x)^2} = \left(\frac{n - \frac{m}{2}}{\alpha_n} - \alpha_n\right) + x\alpha_n'(x) + \lambda \sqrt{1 + \alpha_n'(x)^2}, \quad (5.1)\]

where \(\alpha_n''(x) = \frac{d^2x(s)}{ds^2} \left(\frac{dx(s)}{ds}\right)^2\) and \(\frac{dx(s)}{ds} = \frac{1}{\sqrt{1 + \alpha_n'(x)^2}}\).

In fact, since \(\beta_n(\varepsilon) \leq -m < -\frac{m}{2}\), \(\alpha_n'(x) = \frac{1}{\beta_n'(z)} < 0\) and \(\alpha_n''(x) = -\frac{\beta_n''(z)}{\beta_n'(z)^2}\), we have the following estimates

\[0 < \alpha_n(x) < \alpha_n(\frac{-m}{2}) < \varepsilon, \quad -\frac{2\varepsilon}{m} \leq \alpha_n'(x) < 0, \quad \alpha_n''(x) \geq 0, \quad x \in [-m/2, 0].\]
Using equation (5.1), for \( x \in [-m/2, 0] \), then
\[
\alpha_n''(x) \geq \frac{\alpha_n''(x)}{1 + \alpha_n'(x)^2}
= \left( \frac{n - 1}{\alpha_n} - \alpha_n \right) + x\alpha_n'(x) + \lambda \sqrt{1 + \alpha_n'(x)^2}
\geq \left( \frac{n - 1}{\alpha_n} - \alpha_n \right) + x\alpha_n'(x) + \lambda(1 - \alpha_n'(x))
\geq \frac{n - 1}{\alpha_n} - \alpha_n + \lambda(1 + \frac{2\varepsilon}{m})
\geq -\varepsilon + \lambda(1 + \frac{2\varepsilon}{m}),
\]
where \( 0 < \alpha_n(x) < \varepsilon \) and \( -\frac{2\varepsilon}{m} \leq \alpha_n'(x) < 0 \). For small \( \varepsilon > 0 \), we have
\[
\alpha_n''(x) \geq \frac{n - 1}{2\varepsilon} + \lambda(1 + \frac{2\varepsilon}{m}).
\]
Integrating this inequality repeatedly from \( x \) to \( 0 \), then
\[
\alpha_n(x) \geq \alpha_n(0) + \alpha_n'(0)x + \frac{1}{2} \left( n - 1 + \lambda \left( 1 + \frac{2\varepsilon}{m} \right) \right) x^2 \geq \frac{1}{2} \left( n - 1 + \lambda \left( 1 + \frac{2\varepsilon}{m} \right) \right) x^2.
\]
Taking \( x = \frac{-m}{4} \), we have \( \alpha_n\left( \frac{-m}{4} \right) \geq \frac{m^2}{32} \left( \frac{n - 1}{2\varepsilon} + \lambda(1 + \frac{2\varepsilon}{m}) \right) \). Since there exists the point \( \bar{z} \in (z(s^0), \varepsilon) \) so that \( \beta_\alpha(\bar{z}) = -\frac{m}{4} \), we have that \( \frac{m^2}{32} \left( \frac{n - 1}{2\varepsilon} + \lambda(1 + \frac{2\varepsilon}{m}) \right) \leq \bar{z} < \varepsilon \). This contradicts the fact that if \( \varepsilon > 0 \) is sufficiently small, where \( \lambda \) is bounded negative. Then, there dose not exist \( \frac{dx(s)}{ds} < 0 \) and \( \frac{d^2x(s)}{ds^2} \geq 0 \) on \((s^{b_0}_1, s^{b_0}_2)\). When \( b_0 = \frac{-\lambda + \sqrt{\lambda^2 + 4m}}{2} \), we have that \( x(s) = b_0 \cos \frac{s}{b_0} < 0 \) and \( z(s) = b_0 \sin \frac{s}{b_0} > 0 \) for \( s \in (s^{b_0}_1, s^{b_0}_2) \), then \( \frac{dx(s)}{ds} = -\sin \frac{s}{b_0} < 0 \) and \( \frac{d^2x(s)}{ds^2} = -\frac{1}{b_0} \cos \frac{s}{b_0} > 0 \) for \( s \in (s^{b_0}_1, s^{b_0}_2) \). So we get \( b_0 < \frac{-\lambda + \sqrt{\lambda^2 + 4m}}{2} \).

**Proof of Theorem 1.1.** The proof of the theorem 1.1 is divided into two parts. Part one, we will show that there exists \( s^m_n \in (s^1_n, s^2_n) \) so that \( \frac{dx(s^m_n)}{ds} = 0 \). Supposing that \( \frac{dx(s)}{ds} < 0 \) on \((s^{b_0}_1, s^{b_0}_2)\). By Lemma 5.1, we have that it is impossible to have \( \frac{dx(s)}{ds} < 0 \) and \( \frac{d^2x(s)}{ds^2} \geq 0 \) on \((s^{b_0}_1, s^{b_0}_2)\). Therefore, there exists \( \frac{dx(s)}{ds} < 0 \) for near \( s^{b_0}_2 \). Let \( \{b_n\} \) be an increasing sequence that converges to \( b_0 \). Let \( \Gamma_\alpha(s) \) be the solution of (2.2) corresponding to the initial height \( b_0 \). According to the definition of \( b_0 \), we have that for sufficiently large \( n \), \( \frac{dx(s)}{ds} > 0 \) on \((s^{b_n}_1, s^{b_n}_2)\), which contradicts the fact that \( \frac{d^2x(s)}{ds^2} < 0 \) for near \( s^{b_n}_2 \).

Part two, we will show that \( x(s^{b_0}_2) = 0 \). According to the discussion of the previous paragraph, we have that \( z(s^{b_0}_2) > 0 \) and \( |x(s^{b_0}_2)| < \infty \). Therefore, by the continuous dependence of the curve \( \Gamma_\alpha(s) \) on the initial height, there is a \( \delta > 0 \) so that when \( |b - b_0| < \delta \) and \( b > 0 \) there exists \( s^b_m \in (s^1_b, s^2_b) \) so that
\[ \frac{dx(s^b_{m})}{ds} = 0. \] To complete the proof, we will discuss three cases: \( x(s^b_{0}) > 0, \) \( x(s^b_{2}) < 0 \) and \( x(s^b_{2}) = 0. \) Our purpose is to prove that the first and the second case cannot occur. If \( x(s^b_{2}) > 0, \) it follows from the continuity of the curve \( \Gamma_b(s) \) that we can find \( \delta_1 \in (0, \bar{\delta}) \) so that when \( |b - b_0| < \delta_1 \) and \( b > 0 \) the curve \( \Gamma_b \) has a vertical tangent point \( (x(s^b_{2}), z(s^b_{2})) \) with \( x(s^b_{2}) > 0. \) Then \( b_0 + \delta_1/2 \in T, \) this contradicts the definition of \( b_0. \) If \( x(s^b_{2}) < 0, \) it follows from the continuity of the curve \( \Gamma_b(s) \) that there exists \( \delta_2 \in (0, \bar{\delta}) \) so that when \( |b - b_0| < \delta_2 \) and \( b > 0 \) the curve \( \Gamma_b \) has a vertical tangent point \( (x(s^b_{2}), z(s^b_{2})) \) with \( x(s^b_{2}) < 0. \) However, \( b_0 - \delta_2/2 \in T, \) this also contradicts the definition of \( T. \)

From the above we have that the curve \( \Gamma_{b_0}(s) \) intersects the \( z \)-axis perpendicularly at the point \((0, z(s^b_{2}))\) where \( z(s^b_{2}) \in (0, \frac{\lambda + \sqrt{\lambda^2 + 4(n-1)}}{2}). \) Let \( \Gamma_{b_0} \) be the solution to (2.2) obtained by shooting perpendicularly to the \( x \)-axis at the point \((b_0, 0)\) in the right-half plane. We show that \( \Gamma_{b_0} \) follows the curve \( (\gamma_{b_0}(z), z) \) from the \( x \)-axis, across the \( z \)-axis, to the point \((x(s^b_{1}), z(s^b_{1}))\), and then it follows the curve \( (\beta_{b_0}(z), z) \) until it intersects the \( z \)-axis perpendicularly at \((0, z(s^b_{2}))\). Since the equation is symmetric with respect to reflections across the \( z \)-axis, we see that \( \Gamma_{b_0} \) continues along the reflected curves \( (-\beta_{b_0}(z), z) \) and \( (-\gamma_{b_0}(z), z) \) until it exits the right-half plane, where it intersects the \( x \)-axis perpendicularly at the point \((-b_0, 0)\). That is, \( \Gamma_{b_0} = \gamma_{b_0} \cup \beta_{b_0} \cup -\beta_{b_0} \cup -\gamma_{b_0}. \) Since the rotation of \( \Gamma_{b_0} \) about the \( x \)-axis is smooth in a neighborhood of these symmetric points and \( T \), is smooth in the vertical tangent points \( z(s^b_{1}) \) where the \( \gamma \) and \( \beta \) curves meet, then the rotation of \( \Gamma_{b_0} \) about the \( x \)-axis is a smooth manifold. Notice the facts that \( (\gamma_{b_0}(z), z) \) and \( (-\beta_{b_0}(z), z) \) intersect transversally and the shapes of \( \gamma \) and \( \beta \) are convex, we have that the curve \( \Gamma_{b_0} \) has two self-intersections. Therefore, its rotation about the \( x \)-axis is an immersed, non-embedded \( S^n \) \( \lambda \)-hypersurface in \( \mathbb{R}^{n+1}. \) \( \square \)

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