Littlest modular seesaw

Ivo de Medeiros Varzielas, a Steve F. King b and Miguel Levy a

a CFTP, Departamento de Física, Instituto Superior Técnico, Universidade de Lisboa, Avenida Rovisco Pais 1, 1049 Lisboa, Portugal
b School of Physics and Astronomy, University of Southampton, Southampton SO17 1BJ, United Kingdom

E-mail: ivo.de@udo.edu, s.f.king@soton.ac.uk, miguelplevy@ist.utl.pt

ABSTRACT: We present the first complete model of the Littlest Modular Seesaw, based on two right-handed neutrinos, within the framework of multiple modular symmetries, justifying the use of multiple moduli fields which take their values at 3 specific stabilizers of $\Gamma_4 \simeq S_4$, including a new phenomenological possibility. Using a semi-analytical approach, we perform a $\chi^2$ analysis of each case and show that good agreement with neutrino oscillation data is obtained, including predictive relations between the leptonic mixing angles and the ratio of light neutrino masses, which non-trivially agree with the experimental values. It is noteworthy that in this very predictive setup, the models fit the global fits of the experimental data remarkably well, both with and without the Super-Kamiokande atmospheric data, for both models presented here. By extending the model to include a weighton and the double cover group $\Gamma_4' \simeq S_4'$, we are able to also account for the hierarchy of the charged leptons using modular symmetries, without altering the neutrino predictions.

KEYWORDS: Discrete Symmetries, Flavour Symmetries, Theories of Flavour

ArXiv ePrint: 2211.00654

Dedicated to Graham G. Ross
1 Introduction

The mysterious threefold replication of the fermion generations is one important open issue of the Standard Model (SM) at the heart of the flavour problem. The most promising solution are symmetries that relate the generations, known as family symmetries or flavour symmetries. Recent reviews include [1–3].

The idea of modular invariance [4, 5] has been suggested as key ingredient in solutions to the flavour problem [6]. In these promising scenarios, a modular symmetry associated with transformations of a modulus field can lead to very predictive models of flavour. We will focus on the modular $S_4$ group and its double cover, which have been used in interesting flavour models [7–9]. Nevertheless, in order to apply the methodology of residual flavour symmetries, it is relevant to consider all their fixed points or stabilizers [10, 11]: special values for the modulus field where part of the modular transformations are preserved.\footnote{The issue of moduli stabilisation which brings the moduli to these special values is an open problem, and beyond the scope of the present work. We note that this topic has been discussed recently in [12].}

Furthermore, if multiple residual symmetries are desired, multiple modular symmetries, each with its respective modulus, can be employed — as proposed in [13] and expanded upon in [14–16].
Modular symmetries can further be exploited to explain the mass hierarchy of the fermions by use of an extra field referred to as a weighton [17]. While similar to the Froggatt-Nielsen mechanism, the weighton explicitly relies on modular invariance and does not require extra symmetry.

Another mass hierarchy that is puzzling is the lightness of neutrino masses. Although the type I seesaw mechanism can qualitatively explain the smallness of neutrino masses through the heavy right-handed neutrinos (RHNs), if one doesn’t make other assumptions, it contains too many parameters to make any particular predictions for neutrino mass and mixing. The sequential dominance (SD) [18, 19] of right-handed neutrinos proposes that the mass spectrum of heavy Majorana neutrinos is strongly hierarchical, i.e. $M_{\text{atm}} \ll M_{\text{sol}} \ll M_{\text{dec}}$, where the lightest RHN with mass $M_{\text{atm}}$ is responsible for the atmospheric neutrino mass, that with mass $M_{\text{sol}}$ gives the solar neutrino mass, and a third largely decoupled RHN gives a suppressed lightest neutrino mass. It leads to an effective two right-handed neutrino (2RHN) model [20, 21] with a natural explanation for the physical neutrino mass hierarchy, with normal ordering and the lightest neutrino being approximately massless, $m_1 = 0$.

A very predictive minimal seesaw model with two right-handed neutrinos and one texture zero is the so-called constrained sequential dominance (CSD) model [22–31]. The CSD($n$) scheme assumes that the two columns of the Dirac neutrino mass matrix are proportional to $(0, 1, -1)$ and $(1, n, 2 - n)$ respectively in the RHN diagonal basis, where the parameter $n$ was initially assumed to be a positive integer, but in general may be a real number. For example the CSD(3) (also called Littlest Seesaw model) [24–28], CSD(4) models [29, 30] and CSD($-1/2$) [32] can give rise to phenomenologically viable predictions for lepton mixing parameters and the two neutrino mass squared differences $\Delta m_{21}^2$ and $\Delta m_{31}^2$, corresponding to special constrained cases of TM1 lepton mixing. As was observed, modular symmetry remarkably suggests CSD($1 - \sqrt{6}$) $\approx$ CSD($-1.45$) [10, 33], although such a model would require multiple moduli and so far there is no complete model of this kind in the literature.

In this paper, we construct the first complete model of the Littlest Modular Seesaw (LMS), based on CSD($1 - \sqrt{6}$) $\approx$ CSD($-1.45$), within a consistent framework based on multiple modular symmetries. We also propose a new related possibility based on CSD($1 + \sqrt{6}$) $\approx$ CSD($3.45$), intermediate between CSD($3$) and CSD($4$). In each case, three $S_4$ modular symmetries are introduced, each with their respective modulus field at a distinct stabilizer, leading to three separate residual subgroups, thus dispensing with vacuum alignment mechanisms. The result, in the symmetry basis, is a diagonal charged lepton mass matrix and a LMS scenario of a particular kind. In order to account for the hierarchy of the charged lepton masses, we subsequently introduce a weighton field, where this model is implemented by upgrading the modular symmetries to the respective double covers $S'_4$. Using a semi-analytical approach, we perform a $\chi^2$ analysis of each case and show that good agreement with neutrino oscillation data is obtained, for both possible octants of atmospheric angle, including predictive relations between the leptonic mixing angles and the ratio of light neutrino masses, which non-trivially agree with the experimental values. It is noteworthy that in this very predictive setup, all the models fit the experimental data...
remarkably well, depending on the choice of stabilizers and data set, in one case to within approximately $1\sigma$.

In section 2.1 we present the model with the respective fields and their assignments under the multiple modular symmetries. The charged-lepton structure is shown in section 2.2, and the neutrino seesaw matrix is shown in section 2.3. Analytical results for the leptonic mixing angles and the neutrino masses are given in section 2.4 and a numerical analysis is done in section 2.5. We conclude in section 3. Appendix C gives two alternative models where the charged-lepton hierarchies are naturally explained by including a weighton.

## 2 The model

### 2.1 Symmetries and stabilizers

The model we are building features three commuting $S_4$ modular symmetries, which we label as $S^A_4$, $S^B_4$, $S^C_4$. At low energies, due to the VEVs of fields $\Phi_{AC}$ and $\Phi_{BC}$, they are broken down to the diagonal subgroup, as described in [13]. Table 1 contains the transformation properties (representations and modular weights) under the modular symmetries of the fields and of the relevant modular forms, where we also take usual $SU(2)$ doublets $H_{u,d}$ to transform trivially under all flavour symmetries, and so we omit them from table 1. These assignments are very similar to those used in [13].

Our goal is to achieve a CSD(3.45) [10] structure from the multiple modular symmetries. To that end, the desired directions of the modular forms are obtained for these representations and weights at specific stabilizers [10, 11, 13]. Namely, following the basis of [13], we compute the modular forms:

$$\tau_A = \frac{1}{2} + \frac{i}{2} ; \quad Y_{3'}^4(\tau_A) = (0, -1, 1) , \quad (2.1)$$

### Table 1. Transformation properties of fields and modular forms (Yuk/Mass) under the modular symmetries.

| Field | $S^A_4$ | $S^B_4$ | $S^C_4$ | $2k_A$ | $2k_B$ | $2k_C$ |
|-------|--------|--------|--------|--------|--------|--------|
| $L$   | 1      | 1      | 3      | 0      | 0      | 0      |
| $e^c$ | 1      | 1      | 1'     | 0      | 0      | $-6$   |
| $\mu^c$ | 1    | 1      | 1'     | 0      | 0      | $-4$   |
| $\tau^c$ | 1    | 1      | 1'     | 0      | 0      | $-2$   |
| $N^A_\ell$ | 1'   | 1      | 1      | $-4$   | 0      | 0      |
| $N^B_\ell$ | 1    | 1'     | 1      | 0      | $-2$   | 0      |
| $\Phi_{AC}$ | 3    | 1      | 3      | 0      | 0      | 0      |
| $\Phi_{BC}$ | 1    | 3      | 3      | 0      | 0      | 0      |

| Yuk/Mass | $S^A_4$ | $S^B_4$ | $S^C_4$ | $2k_A$ | $2k_B$ | $2k_C$ |
|----------|--------|--------|--------|--------|--------|--------|
| $Y_4(\tau_C)$ | 1  | 1      | 3'     | 0      | 0      | 6      |
| $Y_\mu(\tau_C)$ | 1  | 1      | 3'     | 0      | 0      | 4      |
| $Y_\tau(\tau_C)$ | 1  | 1      | 3'     | 0      | 0      | 2      |
| $Y_{A/\tau}(\tau_A)$ | $3'$ | 1      | 1      | 4      | 0      | 0      |
| $Y_{B/\tau}(\tau_B)$ | 1  | 3'     | 1      | 0      | 2      | 0      |
| $M_{A/\tau}(\tau_A)$ | 1  | 1      | 1      | 8      | 0      | 0      |
| $M_{B/\tau}(\tau_B)$ | 1  | 1      | 1      | 0      | 4      | 0      |

$^2$We note there is a typo in [13] where RH leptons and the respective modular forms should have primes, as the modular form $Y_\tau(\tau_C)$ (weight 2) only exists as a $3'$.

$^3$This choice is not unique, and $\tau'_A = (-3 + i)/2$ also gives the same modular form.
for one of the Dirac mass matrix columns, and
\[
\tau_B = \frac{3}{2} + \frac{i}{2} : \quad Y^{(2)}_B(\tau_B) = (1, 1 - \sqrt{6}, 1 + \sqrt{6}), \tag{2.2}
\]
or
\[
\tau_B = -\frac{1}{2} + \frac{i}{2} : \quad Y^{(2)}_B(\tau_B) = (1, 1 + \sqrt{6}, 1 - \sqrt{6}), \tag{2.3}
\]
for the other. These specific modular forms lead to the desired CSD structure. In the same basis, we want to enforce a diagonal structure for the charged-lepton Yukawa coupling matrices. This can be easily achieved through the weights 2, 4, and 6 modular forms transforming as $3'$, for $\tau_C = \omega \equiv e^{2\pi i/3}$:
\[
Y^{(2)}_3(\tau_C) = (0, 1, 0) \tag{2.4a}
\]
\[
Y^{(4)}_3(\tau_C) = (0, 0, 1) \tag{2.4b}
\]
\[
Y^{(6)}_3(\tau_C) = (1, 0, 0) \tag{2.4c}
\]

A subtlety should be noted here. Indeed, for weight 6, there are two independent $3'$ modular forms, which could spoil the diagonal arrangement of the charged-leptons. Nevertheless, for $\tau = \omega$, one of them vanishes, introducing no further parameters.

In appendix B it is shown that $\tau_A$ and $\tau'_A$ are stabilisers of $U$, and that $\tau_B$ (either version) is a stabiliser of $SU$ in our chosen basis. It is also shown that the respective modular forms we are using are eigenvectors of the $3'$ representation matrices.

For clarity, we note that the basis in which the modular forms are computed in the present work follows reference [13], which is different from [10]. To be precise, although the $S_4$ basis used here and [13] is the same as that in [10], the basis of modular generators is different, and hence the modular forms differ also. However the physics should be and is basis independent, and indeed the Yukawa alignments shown above can be achieved for different values of the modulus field in the two different bases. It useful to present the different stabilisers for both cases which lead to the desired modular forms, which is shown in table 2.4

2.2 Charged leptons

With the fields and assignments of the previous subsection, we write the respective lepton sector superpotential as
\[
w_L = \frac{1}{\Lambda} \left[ L\Phi_{AC}Y_A(\tau_A)N_A^c + L\Phi_{BC}Y_B(\tau_B)N_B^c \right] H_u
+ \left[ LY_e(\tau_C)e^c + LY_\mu(\tau_C)\mu^c + LY_\tau(\tau_C)\tau^c \right] H_d
+ \frac{1}{2} M_A(\tau_A)N_A^c N_A^c + \frac{1}{2} M_B(\tau_B)N_B^c N_B^c + M_{AB}(\tau_A, \tau_B)N_A^c N_B^c . \tag{2.5}
\]

4Note that with multiple moduli, transforming under a diagonal $S_4$ subgroup, it is meaningful to have fixed points outside the fundamental domain. This can be understood for a case with two moduli, one inside and one outside the fundamental domain, the relative difference in residual subgroups is relevant — the transformation of the diagonal $S_4$ subgroup that brings one inside takes the other one outside.
\[
Y_{\Delta'}^{(4)}(\tau) = \begin{pmatrix} 0 & -1 & 1 \end{pmatrix} \quad Y_{\Delta'}^{(2)}(\tau) = \begin{pmatrix} 1 & 1 - \sqrt{6} & 1 + \sqrt{6} \end{pmatrix} \quad Y_{\Delta'}^{(2)}(\tau) = \begin{pmatrix} 1 & 1 + \sqrt{6} & 1 - \sqrt{6} \end{pmatrix}
\]

| Basis | \[\tau = \frac{1+i}{2}, \quad \tau = \frac{-1+i}{2}\] | \[\tau = \frac{1+i}{2}\] | \[\tau = \frac{-1+i}{2}\] |
|-------|---------------------------------|-----------------|-----------------|
| Basis 2 | \[\tau = 2 + i, \quad \tau = \frac{-2+i}{3}\] | \[\tau = \frac{-2+i}{3}\] | \[\tau = i\] |

Table 2. Relevant stabilisers to obtain the desired modular forms to achieve either a CSD(3.45) or a CSD(-1.45) model, both for basis 1 (used throughout this paper), and basis 2 used in [10]. For both cases, the charged leptons are at the left cusp: \(\tau_C = \omega\). Note that the convention of 3 and 3' is exchanged.

Expanding the superpotential of eq. (2.5), we can find the mass matrices for the fields after the electroweak symmetry breaking, where we are assuming the minimal form of the \(S_4\) tensor. Due to the nature of the \(S_4\) tensor products in our chosen basis, and the particular structure chosen for the bi-triplets VEVs, the \(3 \otimes 3\) tensor products are non-diagonal:

\[
\begin{align*}
(a \otimes b)_1 &= a_1 b_1 + a_2 b_3 + a_3 b_2, \\
(a \otimes \langle \Phi \rangle \otimes b)_1 &\propto a_1 b_1 + a_2 b_3 + a_3 b_2.
\end{align*}
\]

Hence, the charged-lepton mass matrix is simply given by

\[
M_l = v_d \begin{pmatrix}
(Y_e)_1 & (Y_\mu)_1 & (Y_\tau)_1 \\
(Y_e)_3 & (Y_\mu)_3 & (Y_\tau)_3 \\
(Y_e)_2 & (Y_\mu)_2 & (Y_\tau)_2
\end{pmatrix},
\]

where we omit the \(\tau_c\) dependency for clarity, and \(v_d\) stands for \(\langle H_d \rangle\). Plugging in the specific shapes of the modular forms given in eqs. (2.4a)–(2.4c) we arrive at a diagonal charged-lepton mass matrix when \(\tau_C = \omega\):

\[
M_l = v_d \begin{pmatrix}
y_e & 0 & 0 \\
0 & y_\mu & 0 \\
0 & 0 & y_\tau
\end{pmatrix},
\]

where \(v_d y_{e,\mu,\tau}\) are the electron, muon, and tau masses respectively. In this model, the hierarchical masses of the charged-leptons are not addressed. In order to naturally deal with this issue, we present two modifications of this model in the appendix C, where a weighton is responsible for the hierarchy of the masses, without affecting the remaining predictions of the model. Other mechanisms to address the hierarchies rely on small displacements from the fixed points [35–38]. However, our set-up relies on residual symmetries that are preserved in the fixed point to make the model predictive.

---

5 The choice of the minimal Kähler potential is common in modular flavour constructions, as a generic Kähler potential compatible with modular invariance would reduce the predictive power of the model [34].
2.3 Neutrinos

We now turn to the Majorana mass terms for the neutrinos, $N^c_A$ and $N^c_B$. From table 1, we see that $N^c_A N^c_A$ as well as $N^c_B N^c_B$ are $S^4_A \times S^4_B \times S^4_C$ singlets. As such, we just need to cancel out the weight with a singlet Yukawa form. From [8, 39] we see that the Yukawa modular forms of weight 4 do have a singlet representation, needed for the $M_A(\tau_A)$ term. Due to the properties of the modular terms, this implies that there is also a singlet modular form of weight 8, required for $M_B(\tau_B)$. Conversely, as $N^c_A N^c_B$ transforms non-trivially under both $S^4_A$ and under $S^4_B$, there are no one-dimensional modular forms of weight 2 and the respective term is forbidden by the symmetries, and the RH neutrino mass matrix is diagonal:

$$M_R = \begin{pmatrix} M_A(\tau_A) & 0 \\ 0 & M_B(\tau_B) \end{pmatrix}. \quad (2.10)$$

Finally, we need to check the shape of the Dirac mass matrices. Given the VEVs for the bi-triplets $\Phi_{AC}, \Phi_{BC}$, the tensor products after SSB will mimic those of the usual $S_4$ (the diagonal $S_4$ preserved by the bi-triplets symmetry breaking), as explained in [13–16]. This feature is preserved also in the weighton versions of the model, that are using $S'_4$. The Dirac mass matrix is then given by:

$$M_D = v_u \begin{pmatrix} (Y_A)_1 (Y_B)_1 \\ (Y_A)_3 (Y_B)_3 \\ (Y_A)_2 (Y_B)_2 \end{pmatrix}, \quad (2.11)$$

where, as usual, $v_u$ denotes the $H_u$ VEV, and the $3 \times 2$ structure comes from the CSD with just two RH neutrinos. Choosing specific stabilisers for the two remaining moduli fields, we can achieve a new CSD(3.45) structure with $n = 1 + \sqrt{6}$:

$$M_D = v_u \begin{pmatrix} 0 & b \\ a & b (1 + \sqrt{6}) \\ -a & b (1 - \sqrt{6}) \end{pmatrix}, \quad \tau_A = -\frac{3}{2} + \frac{i}{2}, \quad \tau_B = \frac{3}{2} + \frac{i}{2}. \quad (2.12)$$

We can similarly achieve the case CSD(−1.45) with $n = 1 - \sqrt{6}$ already discussed in [10]:

$$M_D = v_u \begin{pmatrix} 0 & b \\ a & b (1 - \sqrt{6}) \\ -a & b (1 + \sqrt{6}) \end{pmatrix}, \quad \tau_A = -\frac{3}{2} + \frac{i}{2}, \quad \tau_B = -\frac{1}{2} + \frac{i}{2}. \quad (2.13)$$

---

6Although we use a different basis, the assignments of the representations are identical, as can be seen by the weight 2 modular forms. Furthermore, we have explicitly checked that the tensor product of $(Y^{(2)}_A \otimes Y^{(2)}_B)_1$ does not vanish for the relevant $\tau_A$ nor any of $\tau_B$. This ensures a non-zero $M_A$ and $M_B$. 

- 6 -
The type-I seesaw mechanism will lead to an effective mass matrix for the light neutrinos:

\[
m_\nu = M_D \cdot M_R^{-1} \cdot M_D^T = v_u^2 \begin{pmatrix}
\frac{b^2}{M_B} & \frac{b^2 n}{M_B} & \frac{b^2 (2 - n)}{M_B} \\
\frac{a^2}{M_A} + \frac{b^2 n^2}{M_B} & -\frac{a^2}{M_A} + \frac{b^2 n(2 - n)}{M_B} & \frac{a^2}{M_A} + \frac{b^2 (2 - n)^2}{M_B} \\
. & . & .
\end{pmatrix},
\tag{2.14}
\]

where \( n = 1 + \sqrt{6} \approx 3.45 \) or \( n = 1 - \sqrt{6} \approx -1.45 \).

### 2.4 Analytic results

The effective mass matrix for the light neutrinos can be split into two contributions,

\[
m_\nu = \frac{v_u^2}{M_A} |a|^2 \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & -1 \\
0 & -1 & 1
\end{pmatrix} + \frac{v_u^2}{M_B} |b|^{2i\beta} \begin{pmatrix}
1 & n & 2 - n \\
n & n^2 & n(2 - n) \\
2 - n & n(2 - n) & (2 - n)^2
\end{pmatrix}.
\tag{2.15}
\]

It is worth noting that the above neutrino mass matrix in the diagonal charged lepton mass basis is determined effectively by two real parameters, \( m_a = \frac{v_u^2 |a|^2}{M_A} \), \( m_b = \frac{v_u^2 |b|^2}{M_B} \), one phase \( \beta \) and a discrete choice of \( n = 1 \pm \sqrt{6} \). For a given choice of \( n \), the remaining three real parameters determine all the parameters in the neutrino sector, namely all the neutrino masses and the entire PMNS matrix.

These two terms above can be simultaneously block-diagonalized by the following Tribimaximal mixing matrix,

\[
U_{TBM} = \begin{pmatrix}
-\frac{\sqrt{2}}{3} & \sqrt{\frac{1}{3}} & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{\sqrt{1}}{2} \\
\frac{1}{\sqrt{2}} & -\frac{\sqrt{1}}{2} & 0
\end{pmatrix},
\tag{2.16}
\]

leading to

\[
m_\nu' = U_{TBM}^T \cdot m_\nu \cdot U_{TBM} = \frac{v_u^2 |a|^2}{M_A} \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 2
\end{pmatrix} + \frac{v_u^2}{M_B} |b|^{2i\beta} \begin{pmatrix}
1 & 0 & 0 \\
0 & 3 & \sqrt{6}(n - 1) \\
0 & \sqrt{6}(n - 1) & 2(n - 1)^2
\end{pmatrix}.
\tag{2.17}
\]

We diagonalize the remaining \((2,2)\) block through the matrix

\[
U_\alpha = \begin{pmatrix}
1 & 0 & 0 \\
0 & c_\alpha & e^{i\gamma} s_\alpha \\
0 & -e^{-i\gamma} s_\alpha & c_\alpha
\end{pmatrix},
\tag{2.18}
\]

such that

\[
U_\alpha^T \cdot m_\nu' \cdot U_\alpha = \text{diag}(0, m_1, m_2).
\tag{2.19}
\]
To ensure that \( m_1, m_2 \) are real and positive, we use the phase matrix, \( P_\nu \), such that:

\[
U_\nu^T \cdot m_\nu \cdot U_\nu = \text{diag}(0, |m_1|, |m_2|),
\]

where

\[
U_\nu \equiv (U_{\text{TBM}} U_\alpha P_\nu) = \begin{pmatrix}
-\sqrt{\frac{2}{3}} \\
\sqrt{\frac{1}{6}} c_\alpha - e^{-i\gamma} s_\alpha \\
\sqrt{\frac{1}{6}} c_\alpha + e^{-i\gamma} s_\alpha
\end{pmatrix} \begin{pmatrix}
c_\gamma s_\alpha \\
e^{i\gamma} s_\alpha \\
\sqrt{\frac{2}{3}}
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & e^{i\phi_2} & 0 \\
0 & 0 & e^{i\phi_3}
\end{pmatrix}. \tag{2.21}
\]

As this is effectively a 2 \times 2 diagonalization, it is possible to find analytical relations for \( \alpha \). Namely, by requiring a vanishing \((U_\nu^T m_\nu U_\alpha)_{23}\) element we find [25]:

\[
t \equiv \tan 2\alpha = \frac{2y}{z \cos(\varphi - \gamma) - x \cos \gamma},
\]

\[
\tan \gamma = \frac{z \sin \varphi}{x + z \cos \varphi}, \quad \text{with} \quad \varphi = \phi_z - \beta, \tag{2.23}
\]

where we defined

\[
m_\nu' = \begin{pmatrix}
0 & 0 & 0 \\
x e^{i\delta} & y e^{i\beta} \\
y e^{i\beta} & z e^{i\phi_3}
\end{pmatrix}, \tag{2.24}
\]

with

\[
x = 3m_b, \quad y = \sqrt{6}(n-1)m_b, \quad z = |2(m_a + e^{i\beta}(n-1)^2m_b)|, \quad m_a = v_u^2 |a|^2, \quad m_b = v_d^2 |b|^2. \tag{2.25}
\]

To relate this to the PMNS matrix in its standard parametrization, we must also take into account the charged-lepton rotation. In our specific realisation, the modular representations of the charged-leptons were chosen in such a way that its mass matrix is already diagonal. As such, the LH rotation is, in general, a diagonal phase matrix

\[
U_L = \begin{pmatrix}
e^{i\delta_x} & 0 & 0 \\
0 & e^{i\delta_\mu} & 0 \\
0 & 0 & e^{i\delta_\tau}
\end{pmatrix}, \tag{2.26}
\]

which can be used to match the standard parametrization [40]:

\[
U_{\text{PMNS}} = \begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
-s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\
s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23}
\end{pmatrix} \begin{pmatrix}
e^{i\eta_1} & 0 & 0 \\
0 & e^{i\eta_2} & 0 \\
0 & 0 & 1
\end{pmatrix}, \tag{2.27}
\]

which has the measured mixing angles and CP-violating phase, and \( s_{ij} (c_{ij}) \) denotes \( \sin (\theta_{ij}) (\cos (\theta_{ij})) \).

\footnote{Indeed, the RH fields rotate away the possible phases of \( M_l \) and, as such, when we write down \( m_\nu \) we are already in a basis where \( M_l \) is diagonal and positive. The LH rotation was used to enforce the reality of \( a \). In general, this won’t be the basis where the light neutrino masses are real. \( U_L \) is then required to rotate into the standard parametrization basis.}
Now, we can relate our Unitary matrix $U_{\nu}$ to $U_{\text{PMNS}}$ and find out the relations between the measured neutrino data, and our model’s parameters. The resulting relations are

$$\sin \theta_{13} = \frac{\sin \alpha}{\sqrt{3}} = \frac{1}{\sqrt{6}} \sqrt{1 - \sqrt{1 + t^2}},$$

(2.28)

$$\tan \theta_{12} = \frac{\cos \alpha}{\sqrt{2}} = \frac{1}{\sqrt{2}} \sqrt{1 - 3 \sin^2 \theta_{13}},$$

(2.29)

$$\tan \theta_{23} = \frac{|1 + \epsilon_\alpha|}{|1 - \epsilon_\alpha|},$$

(2.30)

where

$$\epsilon_\alpha = \sqrt{\frac{2}{3}} e^{i\gamma} \tan \alpha = \sqrt{\frac{2}{3}} e^{i\gamma} \sqrt{1 + t^2} - 1.$$  

(2.31)

Note that the mixing angles depend only on two parameters, with $\theta_{13}$ and $\theta_{12}$ depending only on $t$. Since the mixing is unaffected by an overall factor, we can factorise $m_b$ in eq. (2.24), leading to

$$m'_{\nu} = m_b \begin{pmatrix} 0 & 0 & 0 \\ 0 & x'e^{i\beta} & y'e^{i\beta} \\ 0 & y'e^{i\beta} & z'e^{i\phi_z} \end{pmatrix},$$

(2.32)

where

$$x' = 3, \quad y' = \sqrt{6}(n - 1), \quad z' = 2 \left( \frac{1}{r} + e^{i\beta}(n - 1)^2 \right),$$

(2.33)

$$\phi_z = \arg \left( \frac{1}{r} + e^{i\beta}(n - 1)^2 \right), \quad r = \frac{m_b}{m_a},$$

(2.34)

where we note how $\phi_z$ and $z'$ depend on $r$ and $\beta$. For fixed $n$, the mixing angles themselves will depend solely on $r$ and $\beta$.

To obtain the neutrinos masses, we proceed as in [25] by taking the trace and determinant of the hermitian combination $H'_\nu = m'_{\nu}^\dagger m'_{\nu}$, and equating it to the sum and product of the squared masses, respectively. Given that the LS paradigm forcibly leads to a massless light neutrino and thus, to Normal Ordering, the obtained masses can be readily equated to the $\Delta m^2_{21}$ and $\Delta m^2_{31}$ observables. Defining the combinations of parameters, that depend on those of eqs. (2.23) and (2.33)–(2.34),

$$\Sigma = \frac{m^2_b}{2} \left( x'^2 + 2y'^2 + z'^2 \right),$$

(2.35)

$$\delta M = \frac{m^2_b}{2} \sqrt{x'^2(4y'^2 - 2z'^2) + x'^4 + 8x'y'^2z' \cos \varphi + 4y'^2z'^2 + z'^4},$$

(2.36)

then

$$\Delta m^2_{21} = m^2_1 = \Sigma - \delta M,$$

(2.37)

$$\Delta m^2_{31} = m^2_3 = \Sigma + \delta M,$$

(2.38)

which are functions of $r$ and $\beta$, and with the overall factor given by $m_b$, which cancels out in the ratio. As such, $\Delta m^2_{21}/\Delta m^2_{31}$, the 3 mixing angles, and the CP-phase are all functions of just two effective parameters.
with SK atmospheric data

| Normal Ordering | without SK atmospheric data | with SK atmospheric data |
|-----------------|----------------------------|-------------------------|
|                 | bfp ±1σ | 3σ range | bfp ±1σ | 3σ range |
| sin²θ_{12}      | 0.304±0.013 | 0.269 → 0.343 | 0.304±0.012 | 0.269 → 0.343 |
| θ_{12}/°        | 33.44±0.77  | 31.27 → 35.86 | 33.45±0.77  | 31.27 → 35.87 |
| sin²θ_{23}      | 0.573±0.018 | 0.405 → 0.620 | 0.450±0.019 | 0.408 → 0.603 |
| θ_{23}/°        | 49.2±1.0   | 39.5 → 52.0   | 42.1±1.1    | 39.7 → 50.9   |
| sin²θ_{13}      | 0.02220±0.00068 | 0.02034 → 0.02430 | 0.02240±0.00062 | 0.02060 → 0.02435 |
| θ_{13}/°        | 8.57±0.13  | 8.20 → 8.97   | 8.62±0.12   | 8.25 → 8.98   |
| δ/°             | 194±52     | 105 → 405     | 230±36      | 144 → 350     |
| Δm_{21}^{2}/10^{-5} eV² | 7.42±0.21 | 6.82 → 8.04   | 7.42±0.21   | 6.82 → 8.04   |
| Δm_{23}^{2}/10^{-3} eV² | +2.515±0.028 | +2.431 → +2.599 | +2.510±0.027 | +2.430 → +2.593 |

Table 3. Normal Ordering NuFit 5.1 values [42, 43] for the neutrino observables.

The CP-phase of the PMNS matrix, as well as the physical Majorana phase (since there is one massless neutrino, only η₂ of eq. (2.27) is physical⁸) can be extracted through careful combinations of the elements [41], and lead to

\[
\delta = -\arg \left( \text{sign}(t) e^{i\beta} \left( 4 \sqrt{t^2 + 1} - 1 \right) + (-2 + 3e^{2i\gamma}) t^2 \right), \quad (2.39)
\]

\[
\eta_2 = (-\gamma - \delta - (\phi_3 - \phi_2)). \quad (2.40)
\]

2.5 Numerical analysis

Using the analytical expressions, we plot the allowed experimental ranges for the lepton mixing parameters in the (r, β) plane. We present both the case where τ_B = (3 + i)/2 and τ_B = (-1 + i)/2, corresponding to the modular forms of Eqs (2.2) and (2.3). The results shown correspond to the NuFit 5.1 values [42, 43] without SK atmospheric data in figure 1 and with SK atmospheric data in figure 2. We reproduce the ranges used in table 3. In both figures, the top row displays the 3σ ranges, the bottom row the 1σ ranges, the left column the n = 1 + √6 case and the right column the n = 1 − √6 case.⁹

We note the significant differences between the two possibilities n = 1 + √6 and n = 1 − √6. This corresponds to a change of sign in the effective parameter t, which does not affect the predictions for r, θ_{13}, θ_{12}, but does affect the prediction for θ_{23} and δ. This can be understood as the change of sign corresponds to changing from the tangent to a cotangent in the θ_{23} expression (2.30), and for δ (2.39) to adding π.

---

⁸This is made clear when computing m_{ee}. Alternatively, we can always rotate ν₁ to absorb η₁, but this will not influence the second and third columns.

⁹The results for n = 1 − √6 match the results of [10], as expected.
Figure 1. Allowed $3\sigma$ (top) and $1\sigma$ (bottom) experimental ranges in the $(r, \beta)$ plane using NuFit 5.1 values without SK atmospheric data for the $n = 1 + \sqrt{6}$ case (left) and for the $n = 1 - \sqrt{6}$ case (right). The red circle indicates the best fit region.

While qualitatively both possibilities are similarly successful in reproducing the experimental data at $3\sigma$, it is visible from the plots how the $1\sigma$ range clearly favours different cases. It is worth emphasising how the new case we are considering is able to fit all observables at $1\sigma$, with the exception of $\theta_{12}$, for which the $1\sigma$ contour is just slightly above the intersection of all other observables, which include the very narrow contours from $\theta_{13}$ and from the mass ratio. To better quantify this we define

$$\chi^2 = \sum_i \left( \frac{x_{i,\text{pred}} - x_{i,\text{exp}}}{\sigma_i} \right)^2$$

and list the respective $\chi^2$ values in table 4. For the $n = 1 + \sqrt{6}$ case, $\chi^2 = 1.87$ can be obtained. Table 4 also gives the predictions for $m_{ee}$ for the best-fit point in each case, where [40]:

$$m_{ee} = \left| \sum_i U_{ei}^2 m_i \right|,$$
which, in our case (since we are working in a basis where the charged-leptons are already diagonal, positive, and ordered) can be extracted simply from

$$m_{ee} = \left| (m_\nu)_{1,1} \right|. \quad (2.43)$$

From eq. (2.14), we can see that this is identically $m_b$.

3 Conclusion

In this paper, we have constructed the first complete model of the Littlest Modular Seesaw (LMS), based on CSD$(1 - \sqrt{6}) \approx$ CSD$(-1.45)$, within a consistent framework based on multiple modular symmetries. We also proposed a new related possibility based on CSD$(1 + \sqrt{6}) \approx$ CSD$(3.45)$. In each case, three $S_4$ modular symmetries are introduced, each with their respective modulus field at a distinct stabilizer, leading to three separate residual subgroups, thus dispensing with vacuum alignment mechanisms. Of the three moduli, two are responsible implementing the viable Littlest Seesaw leading to Trimaximal 1 mixing, which correlates non-trivially with the observed ratio of neutrino masses.
The remaining modulus guarantees the charged lepton mass matrix is diagonal in the same basis, preserving the predictive power of the model. The result, in the symmetry basis, is a diagonal charged lepton mass matrix and a LMS scenario of a particular kind.

Using a semi-analytical approach, we performed a $\chi^2$ analysis of each case and showed that good agreement with neutrino oscillation data is obtained, for both possible octants of atmospheric angle, including predictive relations between the leptonic mixing angles and the ratio of light neutrino masses, which non-trivially agree with the experimental values. It is noteworthy that in this very predictive setup, all the models fit the experimental data very well, depending on the choice of stabilizers and data set, in one case to within approximately $1\sigma$. This is a remarkable achievement, given that the neutrino mass matrix in the diagonal charged lepton mass basis is determined effectively by two real parameters, $m_a$, $m_b$ and one phase $\beta$ together with a discrete choice of $n = 1 \pm \sqrt{6}$. For a given choice of $n$, the remaining three real parameters determine all the parameters in the neutrino sector, namely all the neutrino masses and the entire PMNS matrix.

By extending the model to include a weighton and the double cover group $\Gamma'_4 \simeq S'_4$, we are able to also account for the hierarchy of the charged leptons using modular symmetries, without altering the neutrino predictions.

In summary, we have presented an extremely economical model of leptonic masses and mixing, by combining multiple modular symmetries with the littlest seesaw, and optionally adding a weighton. The latter accounts elegantly for the observed hierarchy of the lepton masses without the need for additional Froggatt-Nielsen style symmetries.

We argue that this is a minimal model of leptonic mixing, as we do not count the moduli as free continuous parameters given that we take them as stabilizers. As such, we have 3 real parameters in the charged lepton sector to fit the 3 masses, 1 real parameter that governs the overall neutrino mass scale, and just 2 effective parameters (the ratio $r = m_b/m_a$ and the phase $\beta$) which fit the remaining observables: the neutrino mass ratio, the 3 PMNS mixing angles, the Dirac CP phase and a Majorana phase. The lightest neutrino mass is predicted to be zero and the PMNS phases are predicted in terms of the other observables. Within this predictive setup we are able to fit all the neutrino oscillation data to within approximately $1\sigma$. 

### Table 4

Our $\chi^2$ values for the different cases $n = 1 + \sqrt{6}$ and $n = 1 - \sqrt{6}$. Note that from eq. (2.14) and the definition eq. (2.25), the output parameter $m_{ee}$ is directly equal to the input parameter $m_b$. The neutrino squared-masses $m_2^2$ and $m_3^2$ are given in eV$^2$.

| $n$ | $\chi^2$ | $r$ | $\beta/\pi$ | $m_b/10^{-3}$ | $m_2^2/10^{-5}$ | $m_3^2/10^{-3}$ | $\theta_{12}$ | $\theta_{23}$ | $\theta_{13}$ | $\delta$ |
|-----|----------|-----|-------------|----------------|----------------|----------------|------------|------------|------------|--------|
| $1 + \sqrt{6}$ | 29.47 | 0.076 | 1.26 | 2.33 | 7.19 | 2.53 | 34.29 | 43.06 | 8.78 | 262 |
| $1 - \sqrt{6}$ | 4.96 | 0.073 | 0.76 | 2.23 | 7.45 | 2.51 | 34.34 | 48.26 | 8.55 | 284 |
| $1 + \sqrt{6}$ | 1.87 | 0.074 | 1.24 | 2.30 | 7.42 | 2.51 | 34.33 | 42.03 | 8.62 | 257 |
| $1 - \sqrt{6}$ | 25.79 | 0.077 | 0.74 | 2.33 | 7.15 | 2.52 | 34.28 | 46.76 | 8.82 | 277 |

The final $\chi^2$ values for the different cases $n = 1 + \sqrt{6}$ and $n = 1 - \sqrt{6}$. Note that from eq. (2.14) and the definition eq. (2.25), the output parameter $m_{ee}$ is directly equal to the input parameter $m_b$. The neutrino squared-masses $m_2^2$ and $m_3^2$ are given in eV$^2$. 

The remaining modulus guarantees the charged lepton mass matrix is diagonal in the same basis, preserving the predictive power of the model. The result, in the symmetry basis, is a diagonal charged lepton mass matrix and a LMS scenario of a particular kind.

Using a semi-analytical approach, we performed a $\chi^2$ analysis of each case and showed that good agreement with neutrino oscillation data is obtained, for both possible octants of atmospheric angle, including predictive relations between the leptonic mixing angles and the ratio of light neutrino masses, which non-trivially agree with the experimental values. It is noteworthy that in this very predictive setup, all the models fit the experimental data very well, depending on the choice of stabilizers and data set, in one case to within approximately $1\sigma$. This is a remarkable achievement, given that the neutrino mass matrix in the diagonal charged lepton mass basis is determined effectively by two real parameters, $m_a$, $m_b$ and one phase $\beta$ together with a discrete choice of $n = 1 \pm \sqrt{6}$. For a given choice of $n$, the remaining three real parameters determine all the parameters in the neutrino sector, namely all the neutrino masses and the entire PMNS matrix.

By extending the model to include a weighton and the double cover group $\Gamma'_4 \simeq S'_4$, we are able to also account for the hierarchy of the charged leptons using modular symmetries, without altering the neutrino predictions.

In summary, we have presented an extremely economical model of leptonic masses and mixing, by combining multiple modular symmetries with the littlest seesaw, and optionally adding a weighton. The latter accounts elegantly for the observed hierarchy of the lepton masses without the need for additional Froggatt-Nielsen style symmetries.

We argue that this is a minimal model of leptonic mixing, as we do not count the moduli as free continuous parameters given that we take them as stabilizers. As such, we have 3 real parameters in the charged lepton sector to fit the 3 masses, 1 real parameter that governs the overall neutrino mass scale, and just 2 effective parameters (the ratio $r = m_b/m_a$ and the phase $\beta$) which fit the remaining observables: the neutrino mass ratio, the 3 PMNS mixing angles, the Dirac CP phase and a Majorana phase. The lightest neutrino mass is predicted to be zero and the PMNS phases are predicted in terms of the other observables. Within this predictive setup we are able to fit all the neutrino oscillation data to within approximately $1\sigma$. 

---

- 13 –
Table 5. In the basis used, the representation matrices for $T$, $S$ and $U$, with $\omega = e^{2\pi i/3}$.

### Acknowledgments

IdMV acknowledges funding from Fundação para a Ciência e a Tecnologia (FCT) through the contract UID/FIS/00777/2020 and was supported in part by FCT through projects CFTP-FCT Unit 777 (UID/FIS/00777/2019), PTDC/FIS-PAR/29436/2017, CERN/FIS-PAR/0004/2019 and CERN/FIS-PAR/0008/2019 which are partially funded through POCTI (FEDER), COMPETE, QREN and EU. The work of ML is funded by FCT Grant No.PD/BD/150488/2019, in the framework of the Doctoral Programme IDPASC-PT. SFK acknowledges the STFC Consolidated Grant ST/L000296/1 and the European Union’s Horizon 2020 Research and Innovation programme under Marie Sklodowska-Curie grant agreement HIDDeN European ITN project (H2020-MSCA-ITN-2019//860881-HIDDeN).

### A Group theory of $S_4$

In this appendix we summarize some relevant group theoretical details of $S_4$ (see [13] and references therein). The products of irreps follow:

$$
\begin{align*}
1' \otimes 1' &= 1, \\
1' \otimes 2 &= 2, \\
1' \otimes 3 &= 3', \\
1' \otimes 3' &= 3, \\
2 \otimes 2 &= 1 \oplus 1' \oplus 2, \\
2 \otimes 3 &= 2 \otimes 3' = 3 \oplus 3', \\
3 \otimes 3 &= 3' \otimes 3' = 1 \oplus 2 \oplus 3 \oplus 3', \\
3 \otimes 3' &= 1' \oplus 2 \oplus 3 \oplus 3' .
\end{align*}
$$

(A.1)

In the basis we are using, the representation matrices for $T$, $S$ and $U$ are shown in table 5.

In this basis, the product of 3 dimensional irreps $a$ and $b$:

$$
\begin{align*}
(ab)_{1} &= a_{1}b_{1} + a_{2}b_{3} + a_{3}b_{2}, \\
(ab)_{2} &= (a_{2}b_{2} + a_{1}b_{3} + a_{3}b_{1}, a_{3}b_{3} + a_{1}b_{2} + a_{2}b_{1})^{T}, \\
(ab)_{3} &= (2a_{1}b_{1} - a_{2}b_{3} - a_{3}b_{2}, 2a_{3}b_{3} - a_{1}b_{2} - a_{2}b_{1}, 2a_{2}b_{2} - a_{3}b_{1} - a_{1}b_{3})^{T}, \\
(ab)_{3'} &= (a_{2}b_{3} - a_{3}b_{2}, a_{1}b_{2} - a_{2}b_{1}, a_{3}b_{1} - a_{1}b_{3})^{T}.
\end{align*}
$$

(A.2)
for

\begin{align}
1_i &= 1, \quad 3_i = 3, \quad 3_j = 3' \quad \text{for } a \sim b \sim 3, \ 3', \\
1_i &= 1', \quad 3_i = 3', \quad 3_j = 3 \quad \text{for } a \sim 3 \quad \text{and } b \sim 3'.
\end{align}
(A.3)

The expressions for the product of 2 dimensional irreps \( a = (a_1, a_2)^T \) and \( b = (b_1, b_2)^T \) are:

\begin{align}
(ab)_1 &= a_1 b_2 + a_2 b_1, \quad (ab)_1' = a_1 b_2 - a_2 b_1, \quad (ab)_2 = (a_2 b_2, a_1 b_1)^T.
\end{align}
(A.4)

B Stabilizers and residual symmetry

In the basis we work in, we can make the following mapping of modular generators [13]:

\begin{align}
S &= T_\tau^2, \quad T = S_\tau T_\tau, \quad U = T_\tau S_\tau T_\tau^2 S_\tau,
\end{align}
(B.1)

where \( S_\tau \) and \( T_\tau \) are the usual modular generators of the full modular group \( \Gamma \):

\begin{align}
S_\tau &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T_\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\end{align}
(B.2)

which act on the modulus field as

\begin{align}
\gamma \tau &= \frac{a \tau + b}{c \tau + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\end{align}
(B.3)

With the requirement that \( \tau = \tau + 4 \), which must hold true for \( \Gamma_4 \), we can compute the corresponding \( \gamma \) for \( U \) and \( SU \) [13]:

\begin{align}
\gamma(U) &= \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, \quad \gamma(SU) = \begin{pmatrix} 5 & -3 \\ 2 & -1 \end{pmatrix}.
\end{align}
(B.4)

Now, due to \( T_\tau^4 = 1 \), the choice of \( \gamma(g) \) is not unique. Indeed, any element of \( S_4 \), \( \gamma(g) \):

\begin{align}
\gamma(g) &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{Z},
\end{align}
(B.5)

is equivalent to

\begin{align}
\gamma'(g) &= (\pm 1) \begin{pmatrix} 4k_a + a & 4k_b + b \\ 4k_c + c & 4k_d + d \end{pmatrix}, \quad 4k_a k_d + ak_d + dk_a = 4k_b k_c + bk_c + ck_d, \quad k_x \in \mathbb{Z}
\end{align}
(B.6)

where the constraint comes from requiring that \( \gamma'(g) \) also satisfies \( ad - bc = 1 \).
By choosing the following sets of integers, we arrive at equivalent representations of the $\gamma(U)$ and $\gamma(SU)$ matrices:

\[
\begin{align*}
\gamma_1(U) &= \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \equiv \gamma(U), \\
\gamma_2(U) &= \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}, \quad k_a = -1 \ k_b = -1 \ k_c = 0 \ k_d = 1, \\
\gamma_1(SU) &= \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}, \quad k_a = -1 \ k_b = 1 \ k_c = -1 \ k_d = 0, \\
\gamma_2(SU) &= \begin{pmatrix} -3 & 5 \\ -2 & 3 \end{pmatrix}, \quad k_a = -2 \ k_b = 2 \ k_c = -1 \ k_d = 1.
\end{align*}
\]

(B.7) (B.8) (B.9) (B.10)

Using these matrices, it is straightforward to show that

\[
\begin{align*}
\gamma_1(U) \tau_A &= \tau_A, \quad \tau_A = \frac{1 + i}{2}, \\
\gamma_2(U) \tau'_A &= \tau'_A, \quad \tau'_A = \frac{-3 + i}{2}, \\
\gamma_1(SU) \tau_B &= \tau_B, \quad \tau_B = \frac{-1 + i}{2}, \\
\gamma_2(SU) \tau_B &= \tau_B, \quad \tau_B = \frac{3 + i}{2}.
\end{align*}
\]

(B.11) (B.12) (B.13) (B.14)

In other words, $\tau_A$ and $\tau'_A$ are stabilisers of the modular generator $U$, and that $\tau_B$ (either version) is a stabiliser of the modular generator $SU$ in our chosen basis.

To further corroborate that the stabilisers are leaving an unbroken subgroup, we can check that the respective modular forms are eigenvectors of the appropriate representation matrices. From appendix A, we have

\[
\rho_3'(S) = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad \rho_3'(U) = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho_3'(SU) = \frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & -2 & 1 \\ -2 & 1 & -2 \end{pmatrix},
\]

(B.15)

from which is straightforward to arrive at

\[
\begin{align*}
\rho_3'(U) \cdot \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} &= (+1) \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \rho_3'(SU) \cdot \begin{pmatrix} 1 \\ 1 \pm \sqrt{6} \\ 1 \mp \sqrt{6} \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ 1 \pm \sqrt{6} \\ 1 \mp \sqrt{6} \end{pmatrix},
\end{align*}
\]

(B.16)

agreeing with the expected results. We note that both modular forms $\left(1 \ 1 \pm \sqrt{6} \ 1 \mp \sqrt{6}\right)$ have an eigenvalue $-1$, which is a consequence of [13]

\[(c\tau + d)^{-2k} = (2\tau_{SU} + 1)^{-2k} = (-1)^k,\]

(B.17)

where $k = 1$ for $Y_3^{(2)}$. As such, we preserve a residual flavour symmetry $U$ by the modular form of $\tau_A$ (eigenvalue +1), whereas the modular forms of $\tau_B$ (eigenvalue −1) do not preserve the residual flavour symmetry $SU$, but do preserve the corresponding residual modular symmetry, taking into account the automorphy factor.

\[\text{– 16 –}\]
In this subsection we provide an alternative weighton model, that does not require assigning large modular weights to the charged lepton fields.

### C.1 A minimal weighton model

We now modify the model presented in the main text to include a weighton field $\phi$. In order to preserve the features of the previous model (particularly the diagonal charged-lepton mass matrix) we employ $S^I_4$ modular symmetries \cite{8} instead of $S_4$.

The assignments of the fields under the symmetries are listed in table 6. Notice that this implementation of the weighton is distinguished from the standard one as the weighton is assigned to non-trivial representations of $S^A_4$, $S^B_4$, and $S^C_4$. Due to this and the representations of the charged leptons, the invariant terms have the desired modular forms $Y_\tau$, $Y_\mu$, and $Y_e$ respectively for the field combinations $L\tau^c$, $L\mu^c\phi$ and $Le^c\phi^3$. This is shown (in green) in table 7, where other possibilities are not invariant.

Since there are no charged leptons with weights under $S^A_4$, $S^B_4$, the charged-leptons Yukawa modular forms must be singlets under $S^A_4$, $S^B_4$ with weight 0 under these symmetries.

By having chosen the weighton to have a positive weight under $S^C_4$, there are no additional contributions beyond the leading order ones, as the Yukawa modular forms also have positive weight. An alternative solution, where the weighton has a negative weight under $S^C_4$, is presented in appendix C.2.

### C.2 An alternative weighton model

In this subsection we provide an alternative weighton model, that does not require assigning large modular weights to the charged lepton fields.
### Table 8. Assignments of fields for the alternative weighton version of the model.

| Field | $S_A^4$ | $S_B^4$ | $S_C^4$ | $2k_A$ | $2k_B$ | $2k_C$ |
|-------|---------|---------|---------|--------|--------|--------|
| $L_e$ | 1       | 1       | 3       | 0      | 0      | 0      |
| $\mu_e$ | $\hat{1}$ | $\hat{1}$ | 1'     | 0      | 0      | 0      |
| $\tau_e$ | $\hat{1}'$ | $\hat{1}'$ | 1'     | 0      | 0      | -2     |
| $N^c_A$ | 1' | 1' | 1      | -4     | 0      | 0      |
| $N^c_B$ | 1' | 1' | 1      | 0      | 0      | -2     |
| $\Phi_{AC}$ | 3 | 1 | 3      | 0      | 0      | 0      |
| $\Phi_{BC}$ | 1 | 3 | 3      | 0      | 0      | 0      |
| $\phi$ | $\hat{1}$ | $\hat{1}$ | $\hat{1}$ | 0      | 0      | -2     |

This allows fields (in particular charged lepton fields) to be assigned as distinct non-trivial singlets of the underlying modular symmetries, as shown in Table 8. The invariant combinations are highlighted in green.

### Table 9. Irreps of the leptonic tensor products with different powers of the weighton following the new assignments. The invariant combinations are highlighted in green.

| $\phi^0$ | $\phi^1$ | $\phi^2$ | $\phi^3$ | $\phi^4$ |
|----------|----------|----------|----------|----------|
| $L_e$    | $(\hat{1}, \hat{1}, \hat{3})$ | $(1', 1', 3, -2)$ | $(\hat{1}', \hat{1}', \hat{3}', -4)$ | $(1, 1, 3', -6)$ | $(\hat{1}, \hat{1}, \hat{3}', -8)$ |
| $L_\mu$  | $(\hat{1}', \hat{1}', \hat{3})$ | $(1, 1, 3')$ | $(\hat{1}', \hat{1}', \hat{3}')$ | $(1, 1, 3')$ | $(\hat{1}, \hat{1}, \hat{3}')$ |
| $L_\tau$ | $(1, 1, 3')$ | $(\hat{1}, \hat{1}, \hat{3})$ | $(1', 1', 3')$ | $(1, 1, 3')$ | $(\hat{1}, \hat{1}, \hat{3})$ |
| $N^c_A N^c_A$ | $(1, 1, 3, -2)$ | $(\hat{1}, \hat{1}, \hat{3})$ | $(1', 1', 3')$ | $(1, 1, 3')$ | $(\hat{1}, \hat{1}, \hat{3})$ |
| $N^c_B N^c_B$ | $(1', 1', 3')$ | $(\hat{1}, \hat{1}, \hat{3})$ | $(1', 1', 3')$ | $(1, 1, 3')$ | $(\hat{1}, \hat{1}, \hat{3})$ |
| $N^A_{\Phi AC} N^A_{\Phi AC}$ | $(1, 1, 3)$ | $(\hat{1}, \hat{1}, \hat{3})$ | $(1', 1', 3')$ | $(1, 1, 3')$ | $(\hat{1}, \hat{1}, \hat{3})$ |
| $N^B_{\Phi BC} N^B_{\Phi BC}$ | $(1', 1', 3')$ | $(\hat{1}, \hat{1}, \hat{3})$ | $(1', 1', 3')$ | $(1, 1, 3')$ | $(\hat{1}, \hat{1}, \hat{3})$ |

This is because the model is charged under $S_A^4$, and the 1D irreps have at most $\tau^4 = 1$, there will always be corrections to the leading terms with 4 more weighton insertions. This is avoided by taking the weighton model of appendix C.1.

10 We estimate this suppression factor should to be around $10^{-5}$ by assuming $\mathcal{O}(1)$ couplings for the charged leptons.$^{11}$

11 Namely, we take $\langle \phi \rangle / M = \epsilon = 6.5 \times 10^{-2}$, to have $m_\mu \sim 0.92 \epsilon m_\tau$ and $m_e \sim 1.08 \epsilon^2 m_\tau$.
Open Access. This article is distributed under the terms of the Creative Commons Attribution License (CC-BY 4.0), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited. SCOAP3 supports the goals of the International Year of Basic Sciences for Sustainable Development.

References

[1] S.F. King, *Unified Models of Neutrinos, Flavour and CP Violation*, *Prog. Part. Nucl. Phys.* **94** (2017) 217 [arXiv:1701.04413] [SPIRE].

[2] Z.-z. Xing, *Flavor structures of charged fermions and massive neutrinos*, *Phys. Rept.* **854** (2020) 1 [arXiv:1909.09610] [SPIRE].

[3] F. Feruglio and A. Romanino, *Lepton flavor symmetries*, *Rev. Mod. Phys.* **93** (2021) 015007 [arXiv:1912.06028] [SPIRE].

[4] S. Ferrara, D. Lust, A.D. Shapere and S. Theisen, *Modular Invariance in Supersymmetric Field Theories*, *Phys. Lett. B* **225** (1989) 363 [SPIRE].

[5] S. Ferrara, D. Lust and S. Theisen, *Target Space Modular Invariance and Low-Energy Couplings in Orbifold Compactifications*, *Phys. Lett. B* **233** (1989) 147 [SPIRE].

[6] F. Feruglio, *Are neutrino masses modular forms?,* in A. Levy, S. Forte and G. Ridolfi eds., *From My Vast Repertoire ...: Guido Altarelli’s Legacy*, (2019), pp. 227–266, DOI: 10.1142/9789813238053_0012 [arXiv:1706.08749] [SPIRE].

[7] J.T. Penedo and S.T. Petcov, *Lepton Masses and Mixing from Modular $S_4$ Symmetry*, *Nucl. Phys. B* **939** (2019) 292 [arXiv:1806.11040] [SPIRE].

[8] P.P. Novichkov, J.T. Penedo and S.T. Petcov, *Double cover of modular $S_4$ for flavour model building*, *Nucl. Phys. B* **963** (2021) 115301 [arXiv:2006.03058] [SPIRE].

[9] X.-G. Liu, C.-Y. Yao and G.-J. Ding, *Modular invariant quark and lepton models in double covering of $S_4$ modular group*, *Phys. Rev. D* **103** (2021) 056013 [arXiv:2006.10722] [SPIRE].

[10] G.-J. Ding, S.F. King, X.-G. Liu and J.-N. Lu, *Modular $S_4$ and $A_4$ symmetries and their fixed points: new predictive examples of lepton mixing*, *JHEP* **12** (2019) 030 [arXiv:1910.03460] [SPIRE].

[11] I. de Medeiros Varzielas, M. Levy and Y.-L. Zhou, *Symmetries and stabilisers in modular invariant flavour models*, *JHEP* **11** (2020) 085 [arXiv:2008.05329] [SPIRE].

[12] P.P. Novichkov, J.T. Penedo and S.T. Petcov, *Modular flavour symmetries and modulus stabilisation*, *JHEP* **03** (2022) 149 [arXiv:2201.02020] [SPIRE].

[13] I. de Medeiros Varzielas, S.F. King and Y.-L. Zhou, *Multiple modular symmetries as the origin of flavor*, *Phys. Rev. D* **101** (2020) 055033 [arXiv:1906.02208] [SPIRE].

[14] S.F. King and Y.-L. Zhou, *Trimaximal TM1 mixing with two modular $S_4$ groups*, *Phys. Rev. D* **101** (2020) 015001 [arXiv:1908.02770] [SPIRE].

[15] I. de Medeiros Varzielas and J.a. Lourenço, *Two $A_4$ modular symmetries for Tri-Maximal 2 mixing*, *Nucl. Phys. B* **979** (2022) 115793 [arXiv:2107.04042] [SPIRE].

[16] I. de Medeiros Varzielas and J.a. Lourenço, *Two $A_5$ modular symmetries for Golden Ratio 2 mixing*, *Nucl. Phys. B* **984** (2022) 115974 [arXiv:2206.14869] [SPIRE].
[17] S.J.D. King and S.F. King, Fermion mass hierarchies from modular symmetry, *JHEP* **09** (2020) 043 [arXiv:2002.00969] [insPIRE].

[18] S.F. King, Atmospheric and solar neutrinos with a heavy singlet, *Phys. Lett. B* **439** (1998) 350 [hep-ph/9806440] [insPIRE].

[19] S.F. King, Atmospheric and solar neutrinos from single right-handed neutrino dominance and $U(1)$ family symmetry, *Nucl. Phys. B* **562** (1999) 57 [hep-ph/9904210] [insPIRE].

[20] S.F. King, Large mixing angle MSW and atmospheric neutrinos from single right-handed neutrino dominance and $U(1)$ family symmetry, *Nucl. Phys. B* **576** (2000) 85 [hep-ph/9912492] [insPIRE].

[21] P.H. Frampton, S.L. Glashow and T. Yanagida, Cosmological sign of neutrino CP violation, *Phys. Lett. B* **548** (2002) 119 [hep-ph/0208157] [insPIRE].

[22] S.F. King, Predicting neutrino parameters from $SO(3)$ family symmetry and quark-lepton unification, *JHEP* **08** (2005) 105 [hep-ph/0506297] [insPIRE].

[23] S. Antusch, S.F. King, C. Luhn and M. Spinrath, Trimaximal mixing with predicted $\theta_{13}$ from a new type of constrained sequential dominance, *Nucl. Phys. B* **856** (2012) 328 [arXiv:1108.4278] [insPIRE].

[24] S.F. King, Minimal predictive see-saw model with normal neutrino mass hierarchy, *JHEP* **07** (2013) 137 [arXiv:1304.6264] [insPIRE].

[25] S.F. King, Littlest Seesaw, *JHEP* **02** (2016) 085 [arXiv:1512.07531] [insPIRE].

[26] S.F. King and C. Luhn, Littlest Seesaw model from $S_4 \times U(1)$, *JHEP* **09** (2016) 023 [arXiv:1607.06276] [insPIRE].

[27] P. Ballett, S.F. King, S. Pascoli, N.W. Prouse and T. Wang, Precision neutrino experiments vs the Littlest Seesaw, *JHEP* **03** (2017) 110 [arXiv:1612.01999] [insPIRE].

[28] S.F. King, S. Molina Sedgwick and S.J. Rowley, Fitting high-energy Littlest Seesaw parameters using low-energy neutrino data and leptogenesis, *JHEP* **10** (2018) 184 [arXiv:1808.01005] [insPIRE].

[29] S.F. King, Minimal see-saw model predicting best fit lepton mixing angles, *Phys. Lett. B* **724** (2013) 92 [arXiv:1305.4846] [insPIRE].

[30] S.F. King, A model of quark and lepton mixing, *JHEP* **01** (2014) 119 [arXiv:1311.3295] [insPIRE].

[31] F. Björkeroth and S.F. King, Testing constrained sequential dominance models of neutrinos, *J. Phys. G* **42** (2015) 125002 [arXiv:1412.6996] [insPIRE].

[32] P.-T. Chen, G.-J. Ding, S.F. King and C.-C. Li, A New Littlest Seesaw Model, *J. Phys. G* **47** (2020) 065001 [arXiv:1906.11414] [insPIRE].

[33] G.-J. Ding, S.F. King and C.-Y. Yao, Modular $S_4 \times SU(5)$ GUT, *Phys. Rev. D* **104** (2021) 055034 [arXiv:2103.16311] [insPIRE].

[34] M.-C. Chen, S. Ramos-Sánchez and M. Ratz, A note on the predictions of models with modular flavor symmetries, *Phys. Lett. B* **801** (2020) 135153 [arXiv:1909.06910] [insPIRE].

[35] H. Okada and M. Tanimoto, Modular invariant flavor model of $A_4$ and hierarchical structures at nearby fixed points, *Phys. Rev. D* **103** (2021) 015005 [arXiv:2009.14242] [insPIRE].
[36] F. Feruglio, V. Gherardi, A. Romanino and A. Titov, *Modular invariant dynamics and fermion mass hierarchies around $\tau = i$, JHEP 05 (2021) 242 [arXiv:2101.08718] [inSPIRE].*

[37] P.P. Novichkov, J.T. Penedo and S.T. Petcov, *Fermion mass hierarchies, large lepton mixing and residual modular symmetries, JHEP 04 (2021) 206 [arXiv:2102.07488] [inSPIRE].*

[38] S.T. Petcov and M. Tanimoto, *$A_4$ Modular Flavour Model of Quark Mass Hierarchies close to the Fixed Point $\tau = \omega$, arXiv:2212.13336 [inSPIRE].*

[39] P.P. Novichkov, J.T. Penedo, S.T. Petcov and A.V. Titov, *Modular $S_4$ models of lepton masses and mixing, JHEP 04 (2019) 005 [arXiv:1811.04933] [inSPIRE].*

[40] Particle Data Group collaboration, *Review of Particle Physics, PTEP 2022 (2022) 083C01 [inSPIRE].*

[41] S. Antusch, J. Kersten, M. Lindner and M. Ratz, *Running neutrino masses, mixings and CP phases: Analytical results and phenomenological consequences, Nucl. Phys. B 674 (2003) 401 [hep-ph/0305273] [inSPIRE].*

[42] I. Esteban, M.C. Gonzalez-Garcia, M. Maltoni, T. Schwetz and A. Zhou, *The fate of hints: updated global analysis of three-flavor neutrino oscillations, JHEP 09 (2020) 178 [arXiv:2007.14792] [inSPIRE].*

[43] NuFit webpage, http://www.nu-fit.org.