Coherence of quantum channels

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We investigate the coherence of quantum channels and establish a resource theory for quantifying the coherence of quantum channels via Choi matrix. To this aim, we define the incoherent channels and incoherent superchannels. This theory recovers the case of quantum states when we view quantum states as a special case of quantum channels and also, this theory allows some analytical expressions for coherence measures.

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I. INTRODUCTION

Coherence is a fundamental feature of quantum physics. In recent years, there have been many papers devoted to study the properties of coherence of quantum states, such as quantifications, interconversions, interpretations and applications (see reviews [1, 2] and references therein).

All these research for coherence of quantum states extensively enriched our understanding for quantum theory and leads to many applications. While the coherence of quantum states is still under active research, recently many researchers begin to consider the coherence of quantum channels [3–6]. The motivation to study the coherence of channels is somewhat obvious since most quantum information processings involve and depend on the properties of quantum channels. Among these research, a natural scheme is often adopted to define the coherence of a channel as the optimization of coherence over all output states from this channel. However, such kind of definitions are generally very hard to calculate even numerically.

In this work, we investigate the coherence of quantum channels from a new perspective. After some preliminaries about Choi matrices of channel and superchannel (Sec. II), we provide a definition of incoherent channels and then define and characterize some kinds of superchannels (Sec. III). Especially, we provide a framework for quantifying the coherence of channels via Choi matrix (Sec. IV). Our framework unifies the coherence quantification of channels and states in the sense that states can be viewed as special channels. A conclusion is given in Sec. V.

II. PRELIMINARIES

For clarity, we first give some prerequisites and notations, and also we postpone most proofs of this work to the Appendix part. Let \( H^A, H^B \) be two Hilbert spaces with dimensions \(|A|, |B|, \) and \( \{ |j \rangle \}_j = \{ |k \rangle \}_k \), \( \{ |\alpha \rangle \}_\alpha = \{ |\beta \rangle \}_\beta \) be orthonormal bases of \( H^A, H^B \), respectively. We always assume the orthonormal bases are fixed, i.e., base dependent, and adopt the tensor basis \( \{ |k \rangle |\alpha \rangle \} _{j\alpha} \) as fixed basis when considering the multipartite system \( H^{AB} = H^A \otimes H^B \). Let \( \mathcal{D}_A, \mathcal{D}_B \) be the set of all density operators on \( H^A \) and \( H^B \) respectively, and \( \mathcal{C}_{AB} \) denote the set of all channels from \( \mathcal{D}_A \) to \( \mathcal{D}_B \). A quantum channel \( \phi \in \mathcal{C}_{AB} \) can be represented by the Kraus operators \( \phi = \{ M_m \}_m \) with \( M_m^\dagger M_m = I^A \) the identity on \( H^A \), or by the Choi matrix

\[
J_\phi = \sum_{jk} |j\rangle \langle k| \otimes \phi(|j\rangle \langle k|) \tag{1}
\]

with \( \phi_{jk,\alpha\beta} = \langle \alpha | \phi(|j\rangle \langle k|) |\beta \rangle \).

It holds that \( J_\phi \geq 0 \), \( \sum_\alpha \phi_{jk,\alpha\alpha} = \delta_{jk} \), and

\[
\sum_n M_{n\alpha j} M_n^{\ast \beta k} = \phi_{jk,\alpha\beta} \tag{3}
\]

with \( M_m = \sum_{n\alpha j} M_{n\alpha j} |\alpha \rangle \langle j| \). Note that \( J_\phi \geq 0 \) means \( J_\phi \) is positive semidefinite, and \( \sum_\alpha \phi_{jk,\alpha\alpha} = \delta_{jk} \) is equivalent to \( \text{tr}_B J_\phi = I^A \) or \( \sum_m M_m^\dagger M_m = I^A \) or \( \text{tr}(\phi(|j\rangle \langle k|) = \delta_{jk} \).

We remark that expressions in Eqs. (2), (3) are used in [4].

The completely dephsing channel \( \Delta^A \in \mathcal{C}_{AA} \) is defined as

\[
\Delta^A(\rho^A) = \sum_j \langle j| \rho^A |j\rangle |j\rangle \langle j|, \rho^A \in \mathcal{D}_A. \tag{4}
\]

The notations \( \Delta^B \in \mathcal{C}_{BB}, I^A, I^{AB} = I^A \otimes I^B \), are similarly defined, and contractions \( |\alpha \rangle = |j\rangle |\alpha \rangle, |j\rangle |\alpha \rangle |\alpha' \rangle = |j\rangle |\alpha \rangle |j' \rangle |\alpha' \rangle \), complex number conjugate \( * \), matrix transpose \( \dagger \), Hermitian conjugate \( \dagger \) are used.

When we consider the other systems \( A', B' \), the similar symbols are defined, such as Hilbert spaces \( H^{A'}, H^{B'} \), dimensions \(|A'|, |B'|, \) fixed orthonormal bases \( \{ |j' \rangle \}_j \), \( \{ |\alpha' \rangle \}_\alpha \), \( \{ |\beta' \rangle \}_\beta \) of \( H^{A'}, H^{B'} \), the sets of density operators \( \mathcal{D}_{A'}, \mathcal{D}_{B'} \), the set of quantum channels \( \mathcal{C}_{A'B'}, \) the identity \( I^{A'} \), etc.

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A superchannel $\Theta \in SC_{ABA'B'}$, is a linear map from $C_{AB}$ to $C_{A'B'}$, we express its Choi matrix as

$$J_\Theta = \sum_{j,k,\alpha\beta} |j\alpha\rangle\langle k\beta| \otimes \Theta(|j\alpha\rangle\langle k\beta|)$$

(5)

and

$$J_\Theta = \sum_{j,k,\alpha\beta} \Theta_{jk,\alpha\beta,j'k',\alpha'\beta'} |j\alpha\rangle\langle j'\alpha'| \langle k\beta|\langle k'\beta'|$$

(6)

with $\Theta_{jk,\alpha\beta,j'k',\alpha'\beta'} = \langle j'\alpha'|\Theta(|j\alpha\rangle\langle k\beta|)|k'\beta'|$. The expression in Eq. (6) is just because of the linearity of superchannel. It is worth emphasizing that there is an analogy between Eq. (5) and Eq. (6), as well as between Eq. (2) and Eq. (6), when we regard the bipartite states $\{|j\alpha\rangle\}_{j\alpha}$ as single partite states.

A superchannel should satisfy some conditions [7, 8], when we express the Choi matrix $J_\Theta$ in terms of $\Theta_{jk,\alpha\beta,j'k',\alpha'\beta'}$ above, we have Proposition 1 below.

**Proposition 1.** $\Theta \in SC_{ABA'B'}$ iff (if and only if) $J_\Theta \geq 0$, and

$$\sum_{\alpha'\alpha} \Theta_{jk,\alpha\beta,j'k',\alpha'\alpha'} = \delta_{\alpha\beta} \rho_{j'k'},$$

(7)

$$\sum_{j} \rho_{j'k'} = \delta_{j'k'}.$$  

(8)

Similar to the case of channel, a superchannel $\Theta$ also has the expression of Kraus operators $\Theta = \{M_m\}_m \in SC_{ABA'B'}$, with

$$J_{\Theta(\phi)} = \sum_{m} M_m J_{\phi} M_{m}^\dagger, \forall \phi \in C_{AB},$$

(9)

$$\Theta_{jk,\alpha\beta,j'k',\alpha'\beta' \delta} = \sum_{m} M_{j'k',\alpha'\alpha} M^\ast_{jk,\alpha\beta} M_{mk'k',\beta'\beta}.$$  

(10)

where

$$M_m = \sum_{j'k',\alpha'\alpha} M_{m} j'k',\alpha'\alpha |j'\alpha'| \langle j\alpha|.$$  

(11)

Superchannel can be physically implemented as a simple quantum circuit [7, 8].

### III. INCOHERENT CHANNELS

We first specify the incoherent channels. Recall that for quantum states, a state $\sigma^A \in D_A$ is called incoherent if

$$\Delta^A(\sigma^A) = \sigma^A.$$  

(12)

$\Delta^A$ is a resource destroying map [9], that is to say, $\Delta^A(\sigma^A) = \sigma^A$ for any incoherent state $\sigma^A \in D_A$ and $\Delta^A(\rho^A)$ is incoherent for any $\rho^A \in D_A$. For channels, we give the following definition.

**Definition 1.** We call a channel $\phi \in C_{AB}$ an incoherent channel (IC) if

$$\Upsilon(\phi) = \phi,$$  

(13)

where

$$\Upsilon(\phi) = \Delta^B \phi \Delta^A, \forall \phi \in C_{AB}.$$  

(14)

We denote the set of all incoherent channels in $C_{AB}$ as $IC_{AB}$, and the set of all incoherent channels in $C_{A'B'}$ as $IC_{A'B'}$. The IC defined here is also called classical channel [10].

It is easy to check that $\Upsilon$ also is a resource destroying map [9], i.e., $\Upsilon(\phi) = \phi$ for any incoherent channel $\phi \in C_{AB}$ and $\Upsilon(\phi)$ is incoherent for any $\phi \in C_{AB}$. We call $\Upsilon$ the completely dephasing superchannel.

**Proposition 2.** $\phi \in IC_{AB}$, then $\phi \in IC_{A'B'}$ iff

$$\phi_{jk,\alpha\beta} = \delta_{jk} \delta_{\alpha\beta} \delta_{j'j,\alpha\beta}, \forall j, k, \alpha, \beta,$$  

(15)

that is, $J_\phi$ is diagonal.

We see that the intuitive explanation of incoherent channels is, for such channels quantum coherence of a state (off-diagonal entries) can be neither input into nor output from them.

It can be checked that the channel $\chi \in IC_{AB}$ admits the Kraus operators $\chi = \{M_{\alpha j}\}_{\alpha j}$ with $M_{\alpha j} = \sqrt{\chi_{\alpha j,\alpha\beta} \beta}|j|$. It is easy to see that $IC_{AB}$ is a convex set. Theorem 3 below characterizes the structure of $IC_{AB}$.

**Theorem 3.**

(1) $IC_{AB} = \text{conv}\{\phi \in C_{AB}|\phi_{jk,\alpha\beta} = \delta_{jk} \delta_{\alpha\beta} \delta_{j'j,\alpha\beta}\}$.  

(2) $IC_{AB} \subseteq PIO_{AB}$.  

(16)

$$\text{conv}$$ means convex hull, $f(j) \in \{\alpha\}_1^{|B|}, PIO_{AB}$ is the set of all PIOs (physically incoherent operations) [11, 12].

For the free operations of incoherent channels, we propose the definitions of MISC and ISC, they correspond to MIO (maximally incoherent operation) [13] and IO (incoherent operation) [14, 15] of quantum states.

**Definition 2.** A superchannel $\Theta$ is called a maximally incoherent superchannel (MISC) if $\Theta(\chi) \in IC_{A'B'}$ for $\forall \chi \in IC_{AB}$.

We denote the set of all MISCs by $MISC_{ABA'B'}$.  

**Definition 3.** A superchannel $\Theta \in SC_{ABA'B'}$ is called an incoherent superchannel (ISC) if it admits an expression of Kraus operators $\Theta = \{M_m\}_m$ such that for each $m$,

$$M_m = \sum_{j\alpha} M_{m} j\alpha |f(j\alpha)| \langle j\alpha|.$$  

(18)

where $f(j\alpha) = f(j, \alpha) \in \{j'\alpha'\}_{j'\alpha' = 1}^{|A'|}$. We call $\{M_m\}_m$ an incoherent expression for ISC $\Theta$ if for each $m$, $M_m$ has the form in Eq. (18).

We denote the set of all ISCs by $ISC_{ABA'B'}$.

It can be checked that $M_m J_m M_m^\dagger$ is diagonal for any $M_m$ in Eq. (18) and any $\chi \in IC_{AB}$.

It follows that

$$ISC_{ABA'B'} \subseteq MISC_{ABA'B'} \subseteq SC_{ABA'B'},$$  

(19)
and they are all convex sets.

Unitary channels are very fundamental in the sense of Stinespring dilation. For superchannels, we give the definition of preunitary superchannel.

**Definition 4.** For $|A| \geq |A'|$ and $|B| \leq |B'|$, a superchannel $\Theta = \{U\} \in SC_{AB'BA'}$ is called preunitary if it has an expression of only one Kraus operator $U$. When $|A| = |A'|$ and $|B| = |B'|$, a preunitary superchannel is called a unitary superchannel, for such case $U^d U = U^d = I^{AB}$.

Unitary incoherent channels are important for the discussion of PIO [11, 12]. For the case of superchannel, we have following theorem.

**Theorem 4.** For $|A| \geq |A'|$ and $|B| \leq |B'|$, a superchannel $\Theta = \{U\} \in SC_{AB'BA'}$ is preunitary, if it has the form

$$U = U \otimes V,$$

(20)

where $U$ is a $|A'| \times |A|$ coisometry, i.e., $UU^d = I^{A'}$, $V$ is a $|B'| \times |B|$ isometry, i.e., $V^d V = I^{B}$. Further, if $\Theta = \{U\}$ is preunitary and incoherent, then $U$ and $V$ all have at most one nonzero element in each column.

IV. A FRAMEWORK FOR QUANTIFYING COHERENCE OF QUANTUM CHANNELS

Quantum resource theory (QRT) provides a powerful tool for quantifying a certain quantum feature possessed in quantum systems or quantum processes (see [16] and references therein). In QRT, the specification of free operations is in principle not unique and different settings are motivated by various considerations. Recently a QRT for coherence of channels has been proposed in [4] where free channels are detection incoherent channels or creation incoherent channels. In this section, with $ISC$ as free channels and $ISC$ as superchannels, and inspired by the BCP framework for quantifying the coherence of quantum states in [14], we establish a framework for quantifying the coherence of quantum channels.

We propose the necessary conditions that any coherence measure $C$ for quantum channels should satisfy:

(C1). Faithfulness: $C(\phi) \geq 0$ for any $\phi \in C_{AB}$, and $C(\phi) = 0$ iff $\phi \in ISC_{AB}$.

(C2a). Nonincreasing under ISC: $C(\phi) \geq C(\Theta(\phi))$ for any $\Theta \in ISC_{AB'BA'}$.

(C2b). Nonincreasing under ISC on average: $C(\phi) \geq \sum_m p_m C(\phi_m)$ for any $\Theta \in ISC_{AB'BA'}$, with $\{M_m\}_{m}$ an incoherent expression of $\Theta$, $p_m = tr(M_m M_m^d)/|A'|$, $J_{\phi_m} = |A'|^{M_m J_{\phi_m} M_m^d}/tr(M_m M_m^d)$.

(C3). Convexity: $C(\sum_m p_m \phi_m) \leq \sum_m p_m C(\phi_m)$, for any $\{\phi_m\}_{m} \subset C_{AB}$ and probability $\{p_m\}_{m}$.

Notice that (C3) and (C2b) automatically imply (C2a).

Note that in (C2b), $J_{\phi_m} = |A'|^{M_m J_{\phi_m} M_m^d}/tr(M_m M_m^d)$ is not necessarily a Choi matrix for quantum channel, so this framework implies that any coherence measure $C$ should properly defined on such $J_{\phi_m}$.

From now on we discuss some properties of this framework. One advantage of this framework is that we can get some results for quantum channels by using the known results of quantum states.

**Theorem 5.** If $C$ is a coherence measure for quantum states in the BCP framework [14], then

$$C(\phi) = C(J_{\phi}/|A|), \phi \in C_{AB},$$

(21)

is a coherence measure for quantum channels.

The proof is straightforward by checking the four conditions above.

From Theorem 5 we can get many coherence measures for channels corresponding to the coherence measures for quantum states, for examples the coherence measure based on $t_1$ norm $C_{t_1}$ [14], coherence measure based on relative entropy $C_r$ [14], based on Tsallis relative entropy $C_{\alpha}$ [17, 18], based on robustness $C_R$ [19, 20], based on trace norm $C_{tr}$ [21], and the geometric coherence [22].

**Definition 5.** A channel is called a channel with maximal coherence, if it reaches the maximum for any coherence measure of channels.

**Theorem 6.** For $\phi \in C_{AB}$, if $J_{\phi}^{AB}$ is a quantum state with maximal coherence on the composite space $H^A \otimes H^B$, then $\phi$ is a channel with maximal coherence. As a result, for $|A| \leq |B|$, the isometry channel $U_{\text{max}} \in C_{AB}$,

$$U_{\text{max}}(|j\rangle) = \frac{1}{\sqrt{|B|}} \sum_{\alpha=1}^{|B|} e^{i\theta_{\alpha}^j} |\alpha\rangle,$$

(22)

and

$$\frac{1}{|B|} \sum_{\alpha=1}^{|B|} e^{i(\theta_{j\alpha}^j - \theta_{k\alpha})} = \delta_{jk},$$

(23)

is a channel with maximal coherence. Especially for $|A| \leq |B|$, the isometry channel

$$U_{\text{max}}^{(0)}(|j\rangle) = \frac{1}{\sqrt{|B|}} \sum_{\alpha=1}^{|B|} e^{2\pi i \theta_{j\alpha}^j} |\alpha\rangle,$$

(24)

is a channel with maximal coherence.

**Proof.** The first statement that $\phi$ is a channel with maximal coherence is a result of Theorem 5. From Eqs. (22) and (23) we can check that $J_{\phi_{\text{max}}}^{AB}$ is a quantum state with maximal coherence [23] and $U_{\text{max}}$ is an isometry channel.

Note that if $\rho^{AB}$ is a quantum state with maximal coherence on $H^A \otimes H^B$, then $|A| \rho^{AB}$ is not necessarily a Choi matrix for any channel. For example,

$$|\psi^{AB}\rangle = \frac{1}{\sqrt{|A||B|}} \sum_{j=1}^{|A|} \sum_{\alpha=1}^{|B|} |j\rangle \langle \alpha|,$$

(25)

is a quantum state with maximal coherence on $H^A \otimes H^B$ [14], but $|A| \langle \psi^{AB}| \psi^{AB}\rangle$ is not a Choi matrix for any channel since $tr_B(|A| \langle \psi^{AB}| \psi^{AB}\rangle) \neq I^A$ for $|A| \geq 2$. 

Another advantage of this framework is that we can regard it as a unified theory for quantifying coherence of both quantum states and quantum channels. A quantum state of system A can be viewed as a special quantum channel in $C_{AB}$ for $|A| = 1$. For $|A| = |A'| = 1$, we see that MISC degenerates to MIO, ISC degenerates to IO, and preunitary superchannel degenerates to isometry channel. With these observations we have proposition 7 below.

**Proposition 7.** A coherence measure for quantum channels degenerates to a coherence measure for quantum states in BCP framework [14] when $|A| = |A'| = 1$.

For the case $|B| = 1$, there is only one channel $\phi \in C_{AB}$ with Choi matrix $J_\phi = I^A$, which is the action of trace $\phi(\rho^A) = tr(\rho^A) = 1$. Note that this is an incoherent channel, this fact coincides with our intuition.

The third advantage of this framework is that it possesses the monotonicity under composition below.

**Proposition 8.** Monotonicity under composition.

1. For $\phi \in C_{AB}$, $\chi \in IC_{BB}$,
   
   $C(\chi \circ \phi) \leq C(\phi).$
   
   Especially, for $\phi \in C_{AB}$, $\Delta^B \in IC_{BB}$,
   
   $C(\Delta^B \circ \phi) \leq C(\phi).$

2. For $|A'| \leq |A|$, $\phi \in C_{AB}$, $\chi \in IC_{A'A}$, $\sum_{j,j'} \chi_{j} \chi'_{j'} \leq 1, \forall j$, $\phi \in C_{AB}$, $\chi \in IC_{BB}$,

   $C(\phi \circ \chi) \leq C(\phi).$

Finally we give an example to show the unity of quantifying coherence for both channels and states under Theorem 5.

**Example.** Consider the channel $\tilde{\phi} \in C_{AB}$,

$\tilde{\phi}(\rho^A) = p\phi(\rho^A) + (1-p)\rho^B, \forall \rho^A \in D_A,$

with $p \in [0,1]$, $\phi \in C_{AB}$, $\phi$ is a MIO, and fixed $\rho^B \in D_B$. Since $\phi$ is a MIO then

$\phi_{jj,\alpha\beta} = \delta_{\alpha\beta} \phi_{jj,\alpha\alpha}.$

The channel

$\phi_1(\rho^A) = \rho^B, \forall \rho^A \in D_A,$

has the Choi matrix

$J_{\phi_1} = I^A \otimes \rho^B.$

Hence we can equivalently express $\tilde{\phi}$ as

$\tilde{\phi}_{jk,\alpha\beta} = p\phi_{jk,\alpha\beta} + (1-p)\delta_{jk} \rho^B_{\alpha\beta},$

with $\rho^B_{\alpha\beta} = \langle \alpha | \rho^B | \beta \rangle$.

Using the coherence measure $C_{t_1}$,

$C_{t_1}(\phi) = \frac{1}{|A|} \sum_{jk,\alpha\beta} |\phi_{jk,\alpha\beta}| - 1, \forall \phi \in C_{AB},$

we get

$C_{t_1}(\tilde{\phi}) = pC_{t_1}(\phi) + (1-p)C_{t_1}(\rho^B).$

We see that under the coherence measure $C_{t_1}$, the total channel coherence $C_{t_1}(\tilde{\phi})$ includes two parts, $C_{t_1}(\phi)$ accounts for the contribution of channel $\phi$, and $C_{t_1}(\rho^B)$ accounts for the contribution of state $\rho^B$.

**V. CONCLUSION AND OUTLOOK**

We investigated the coherence of quantum channels, defined IC, MISC, ISC, preunitary superchannel, and under quantum resource theory we established a framework to quantify the coherence of quantum channels via Choi matrix. We showed that this framework has many advantages, such as getting some results from coherence theory of quantum states, allowing for a unified viewpoint combining coherence of channels and states, and satisfying a monotonicity of composition. We hope this work will open a new way to explore the coherence of quantum channels.

There are many questions for future research after this work. One is that whether or not there exist two or more coherence measures for channels they degenerate to the same coherence measure for quantum states when $|A| = |A'| = 1$. If it is not true, then coherence measures for both channels and states are the same under Theorem 5.

Another important question is the physical interpretations for the coherence measures of channels. Many works has been done for some coherence measures of quantum states such as [24, 25] which may provide inspiring evidences.

Also, Gaussian channels are very important in both theories and experiments, while it seems difficult to address the coherence since coherence is primarily defined on orthonormal bases but Gaussian states and Gaussian channels intrinsically defined in phase space.

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**Appendix**

In this appendix, we provide some necessary details and proofs for the results in the main text.
A. Choi matrices of Channels and superchannels

The Choi matrix \( J_\phi \) of a channel \( \phi \in C_{AB} \) is defined in Eq. (1). Choi theorem says \( \phi \) is completely positive definite iff \( J_\phi \geq 0 \). Since \( J_\phi \geq 0 \) then \( J_\phi \) has the decomposition of the form

\[
J_\phi = \sum_m |M_m\rangle \langle M_m|
\]

This can be equivalently written as

\[
[\phi(\rho^A)]_{\alpha\beta} = \sum_{jk} \rho^A_{jk} \delta_{jk}, \quad (A8)
\]

with \( \rho^A_{jk} = \langle j|\rho^A|k\rangle \) and \( [\phi(\rho^A)]_{\alpha\beta} = \langle \alpha|\phi(\rho^A)|\beta\rangle \). Since the channel \( \phi \) is trace preserving, then

\[
\sum_{\alpha} \phi_{jk,\alpha\alpha} = \delta_{jk}. \quad (A9)
\]

For superchannel \( \Theta \in SC_{ABA'B'} \) and channel \( \phi \in C_{AB} \), it holds that [7, 8, 26]

\[
J_{\Theta(\phi)} = tr_{AB}[J_\Theta(\phi)^t \otimes I^{A'B'}], \quad (A10)
\]

this can be equivalently expressed as

\[
[\Theta(\phi)]_{j'k',\alpha'\beta'} = \sum_{jk,\alpha\beta} \phi_{jk,\alpha\beta} \Theta_{jk,\alpha\beta,j'k',\alpha'\beta'}. \quad (A11)
\]

Taking Eq. (10) into Eq. (A11) we get Eq. (9).

For \( \phi \in C_{AB} \), \( \psi \in C_{BB'} \), the composition \( \psi \circ \phi \in C_{AB'} \), and

\[
(\psi \circ \phi)_{jk,\alpha'\beta'} = \sum_{\alpha\beta} \phi_{j',\alpha\beta} \psi_{\alpha\beta,\alpha'\beta'}. \quad (A12)
\]

this is equivalent to Eq. (3) in [27].

B. Proof of Proposition 1

Proposition 1 is equivalent to Eq. (19) in [8] and also the result in [7], here we give a proof for being self-contained and consistent with the notations used in this work.

From Choi theorem, \( \Theta \) is completely positive iff \( J_\Theta \geq 0 \). Suppose \( \phi \in C_{AB} \), then \( J_\phi \geq 0 \) (completely positive) and \( tr\phi(\langle j|\langle k\rangle|) = \delta_{jk} \) (trace preserving). Note that \( J_\phi \geq 0 \) implies \( J_{\phi}^\dagger = J_\phi \), \( J_\Theta \geq 0 \) implies \( J_{\Theta}^\dagger = J_{\Theta} \), that is

\[
\phi_{jk,\alpha\beta}^\dagger = \phi_{kj,\beta\alpha}, \quad (A13)
\]

\[
\Theta_{jk,\alpha\beta,j'k',\alpha'\beta'}^\dagger = \Theta_{kj,\beta\alpha,j'k',\alpha'\beta'}. \quad (A14)
\]

If \( \Theta(\phi) \) is trace-preserving, we have

\[
\sum_{jk,\alpha\beta,\alpha'} \phi_{jk,\alpha\beta} \Theta_{jk,\alpha\beta,j'k',\alpha'\beta'} = \delta_{j'k'}, \forall \phi \in C_{AB}. \quad (A15)
\]

For clarity, we set four steps to complete this proof.

Step 1. Let \( \phi_{jk,\alpha\beta} = \delta_{jk} \delta_{\alpha\beta} \phi_{jj,\alpha\alpha}, \phi_{jj,\alpha\alpha} \geq 0, \sum_{\alpha} \phi_{jj,\alpha\alpha} = 1 \), then

\[
\sum_{j\alpha\alpha'} \phi_{jj,\alpha\alpha} \Theta_{jj,\alpha\alpha,j'k',\alpha'\alpha'} = \delta_{j'k'}. \quad (A16)
\]

Varying \( \{\phi_{jj,\alpha\alpha}\}_{j\alpha} \) on the domain of \( \phi_{jj,\alpha\alpha} \in R : \phi_{jj,\alpha\alpha} \geq 0, \sum_{\alpha} \phi_{jj,\alpha\alpha} = 1 \), we get

\[
\sum_{\alpha'} \Theta_{jj,\alpha\alpha,j'k',\alpha'\alpha'} = \rho^{jj}_{j'k'}. \quad (A17)
\]
independent of \( \alpha \), and

\[
\sum_{j,\alpha'} \phi_{jj,\alpha\alpha} \Theta_{jj,\alpha\alpha,j'k',\alpha'\alpha'} = \sum_j (\sum_{\alpha} \phi_{jj,\alpha\alpha})(\sum_{\alpha'} \Theta_{jj,\alpha\alpha,j'k',\alpha'\alpha'}) = \sum_j \rho_{j'k'} = \delta_{j'k'}.
\]  

(A18)

This proves Eq. (8).

Step 2. Fix \( j_0, k_0 \in \{j\}_{j=1}^{|A|}, j_0 < k_0 \), let

\[
\phi_{jj,\alpha\alpha} = \frac{1}{|B|}, \forall j, \alpha,
\]

(A19)

\[
\phi_{j_0k_0,\alpha\alpha} = \delta_{j_0k_0}, \forall \alpha,
\]

(A20)

and others \( \phi_{jk,\alpha\beta} = 0 \). We also let \( |\phi_{j_0k_0,\alpha\alpha}| \leq \frac{1}{|B|} \) for \( \forall \alpha \) to ensure \( J_\phi \geq 0 \) according to the Gersgorin discs theorem.

For this case, subtract (A16) from (A15) we get

\[
\sum_{aa'} (\phi_{j_0k_0,\alpha\alpha} \Theta_{j_0k_0,\alpha\alpha,j'k',\alpha'\alpha'}) + \phi_{j_0k_0,\alpha\alpha} \Theta_{j_0k_0,\alpha\alpha,j'k',\alpha'\alpha'} = 0,
\]

(A21)

Notice that \( \sum_{\alpha} \phi_{j_0k_0,\alpha\alpha} = 0 \), hence varying \( \{\phi_{j_0k_0,\alpha\alpha}\}_\alpha \) in the domain \( \{\phi_{j_0k_0,\alpha\alpha} \in R : \sum_{\alpha} \phi_{j_0k_0,\alpha\alpha} = 0, |\phi_{j_0k_0,\alpha\alpha}| \leq \frac{1}{|B|}\} \), we get

\[
\sum_{\alpha'} (\Theta_{j_0k_0,\alpha\alpha,j'k',\alpha'\alpha'} + \Theta_{j_0k_0,\alpha\alpha,k'j',\alpha'\alpha'}) = 0,
\]

(A23)

is independent of \( \alpha \).

Again varying \( \{\phi_{j_0k_0,\alpha\alpha}\}_\alpha \) in the domain \( \{i\phi_{j_0k_0,\alpha\alpha} \in R : \sum_{\alpha} \phi_{j_0k_0,\alpha\alpha} = 0, |\phi_{j_0k_0,\alpha\alpha}| \leq \frac{1}{|B|}\} \), we get

\[
\sum_{\alpha'} (\Theta_{j_0k_0,\alpha\alpha,j'k',\alpha'\alpha'} - \Theta_{j_0k_0,\alpha\alpha,k'j',\alpha'\alpha'}) = 0,
\]

(A24)

is independent of \( \alpha \).

Thus

\[
\sum_{\alpha'} \Theta_{j_0k_0,\alpha\alpha,j'k',\alpha'\alpha'} = \rho_{j'k'}
\]

(A25)

is independent of \( \alpha \).

Step 3. Let \( \phi_{jj,\alpha\alpha} = \frac{1}{|B|}, \forall j, \alpha, \) and \( \phi_{j_0k_0,\alpha\alpha}\delta_{0} = \phi_{k_0j_0,\alpha\alpha\delta_{0}} \) for fixed \( j_0, k_0, \alpha, \alpha' \neq \beta_0 \) and others \( \phi_{jk,\alpha\beta} = 0 \).

We also let \( |\phi_{j_0k_0,\alpha\alpha}\delta_{0}| \leq \frac{1}{|B|} \) to ensure \( J_\phi \geq 0 \). For this case, subtract (A16) from (A15) we get

\[
\phi_{j_0k_0,\alpha\alpha}\delta_{0} \sum_{\alpha'} \Theta_{j_0k_0,\alpha\alpha\delta_{0},j'k',\alpha'\alpha'} = 0.
\]

(A26)

Varying \( \phi_{j_0k_0,\alpha\alpha}\delta_{0} \) in the domain \( |\phi_{j_0k_0,\alpha\alpha}\delta_{0}| \leq \frac{1}{|B|} \), we get

\[
\sum_{\alpha'} \Theta_{j_0k_0,\alpha\alpha\delta_{0},j'k',\alpha'\alpha'} = 0.
\]

(A27)

Combine Eqs. (A17), (A25), (A27), then we get Eq. (7).

Step 4. Conversely, if Eqs. (7), (8) hold, then for any \( \phi \in C_{AB} \),

\[
\sum_{\alpha'} (\Theta_{\alpha}'(\phi) j'k',\alpha'\alpha') = \sum_{j,\alpha,\beta} \phi_{jk,\alpha\beta} \Theta_{jk,\alpha\beta,j'k',\alpha'\alpha'}
\]

\[
= \sum_{j,\alpha,\beta} \phi_{jk,\alpha\beta} (\sum_{\alpha'} \Theta_{jk,\alpha\beta,j'k',\alpha'\alpha'}) = \sum_{\alpha'} (\sum_{j,\alpha,\beta} \phi_{jk,\alpha\beta}) \rho_{j'k'}(j^k)
\]

\[
= \sum_{j,\alpha} \delta_{jk}\rho_{j'k'}(j^k) = \sum_{j} \delta_{jk}\rho_{j'k'} = \delta_{j'k'}.
\]

(A28)

Then we complete this proof.

**Corollary A1.** For \( \Theta \in SC_{ABA'B'} \),

\[
trJ_\Theta = |B||A|^{-1}.
\]

(A29)

### C. Proof of Proposition 2

Note that \( \Delta^A \in C_{AA}, A^B \in C_{BB}, \) and

\[
\Delta_{jk,\alpha\beta} = \delta_{jk}\delta_{j\alpha}\delta_{j\beta}.
\]

(A30)

If \( \phi \in IC_{AB} \), then

\[
\phi(\rho^A) = \Delta^B \phi \Delta^A (\rho^A), \forall \rho^A \in D_A,
\]

(A31)

this yields

\[
\sum_{j} \rho_{jk}^A \phi_{jk,\alpha\beta} = \delta_{\alpha\beta} \sum_{j} \rho_{jj}^A \phi_{jj,\alpha\alpha}.
\]

(A32)

Let \( \rho^A = |j\rangle \langle j| \), we get

\[
\phi_{jj,\alpha\beta} = \delta_{\alpha\beta} \phi_{jj,\alpha\alpha}.
\]

(A33)

Let \( \rho^A = \frac{\rho^A_k}{|A|} \),

\[
\rho^A_k \rho^A_j \delta_{jk}\delta_{j\alpha}\delta_{j\beta} = \sum_{\alpha,\beta} \rho^A_{jk}\phi_{jk,\alpha\beta} = 0.
\]

(A34)

Let \( \rho^A_k \) be real numbers we get \( \phi_{jk,\alpha\beta} + \phi_{kj,\alpha\beta} = 0 \), let \( \rho^A_k \) be imaginary numbers we get \( \phi_{jk,\alpha\beta} - \phi_{kj,\alpha\beta} = 0 \), then we get

\[
\phi_{jk,\alpha\beta} = \delta_{jk}\phi_{jj,\alpha\beta}.
\]

(A35)

Combining Eqs. (A33) and (A35) we then get Eq. (15) and end this proof.
D. Proof of Theorem 3

Step 1. The first statement Eq. (16) is a result of Lemma 1 below.

**Lemma 1.** An $m \times n$ matrix $P = (P_{ja})$ is called a row stochastic matrix if it satisfies $P_{ja} \geq 0$ for any $j, \alpha$, and $\sum_{\alpha=1}^{N} P_{ja} = 1$ for any $j$. For any $m \times n$ row stochastic matrix $P$, there exists a probability $\{p_t\}_{t=1}^{N}$, and row stochastic matrices $\{D^{(i)}\}_{i=1}^{N}$ such that each row of $D^{(i)}$ has just one nonzero entry (that must be 1) for any $i$, $P = \sum_{i=1}^{N} p_t D^{(i)}$, and $N \leq m(n-1) + 1$.

Proof of Lemma 1. This proof is similar to the proofs of Theorem 2.1 and Theorem 2.2 in Ref. [28] where for the case of $n \times n$. For any $m \times n$ row stochastic matrix $P$, denote $P^{(0)} = P$, for $s = 0, 1, 2, \ldots$, define $m \times n$ matrices $Q^{(s)} = (Q^{(s)}_{ja})$ and $P^{(s)} = (P^{(s)}_{ja})$ as

$$Q^{(s)}_{ja} = \delta_{j,\alpha}, \quad (A36)$$

$$J_{s} = \max\{a \mid P_{ja} \geq P_{ja}, \forall \alpha\}, \quad (A37)$$

$$\kappa(P^{(s)}) = \min(\max(P^{(s)}_{ja})), \quad (A38)$$

$$P^{(s+1)} = P^{(s)} - \kappa(P^{(s)})Q^{(s)}. \quad (A39)$$

For each iteration of $s$, $P^{(s+1)}$ has at least one zero entry more than $P^{(s)}$. Hence after finite iterations we will attain $P^{(N)} = 0$. From Eq. (A39) and iterations we can reversely get an expression of $P$ which just has the form $P = \sum_{i=1}^{N} p_t D^{(i)}$.

Further since all $m \times n$ row stochastic matrices form a convex set with dimensions $mn - m$, then Carathéodory's theorem yields that $N \leq m(n-1) + 1$.

Step 2. We prove Eq. (17). Note that, for $\phi \in C_{AB}$ we can equivalently assume $|A| = |B|$ by adding extra dimensions to $A$ or $B$ system. As a result, we restate the Proposition 1 in Ref. [11] as follows.

**Lemma 2.** A channel $\phi \in C_{AB}$ is a PIO iff it can be expressed as a convex combination of channels each having Kraus operators $\{M_{ja}\}_{a}$ of the form

$$M_{ja} = U_{ja} P_{ja} = \sum_{j=1}^{|A|} e^{i\theta_j} |f_n(j)\rangle \langle j| P_{ja} \quad (A40)$$

where $\{P_{ja}\}_{a}$ form an orthogonal and complete set of incoherent projectors on system $A$ and $f_n(j) \in \{\alpha\}_{\alpha=1}^{|B|}$, $f_n(j) \neq f_n(k)$ for $j \neq k$.

Now, let $\phi = \{M_{ja}\}_{ja=1}^{\{a\}} \in C_{AB}$ as

$$P_{ja} = |\langle n|n\rangle|, \quad (A41)$$

then

$$M_{ja} = e^{i\theta_j} |f_n(n)\rangle \langle n|, \quad (A42)$$

$$|M_{ja}| = e^{i\theta_j} |f_n(n)|, \quad (A43)$$

$$|M_{ja}\rangle \langle M_{ja}| = |n\rangle \langle n| \otimes |f_n(n)\rangle \langle f_n(n)|, \quad (A44)$$

$$J_{\phi} = \sum_{n} |\langle n|n\rangle \otimes |f_n(n)\rangle \langle f_n(n)|, \quad (A45)$$

which just is an IC. Together with Eq. (16), we get $IC_{AB} \subseteq PIO_{AB}$.

To show $IC_{AB} \neq PIO_{AB}$, let channel $\phi = \{U\} \in C_{AB}$ as $U = \sum_{j=1}^{|A|} |f(j)\rangle \langle j|$ with $f(j) \in \{\alpha\}_{\alpha=1}^{|B|}$, $f(j) \neq f(k)$ for $j \neq k$, then

$$|U\rangle = \sum_{j=1}^{\{A\}} |j| \langle j|, \quad (A46)$$

$$J_{\phi} = |U\rangle \langle U| = \sum_{j,k} |j\rangle \langle k| \otimes |f(j)\rangle \langle f(k)|, \quad (A47)$$

which is evidently not an IC for $|A| \geq 2$.

E. Proof of Theorem 4

Step 1. Suppose $\Theta = \{U\} \in SC_{AB,A'B'}$ is preunitary, then for any $\phi \in C_{AB}$,

$$\Theta_{j,k,\alpha,\beta,j',k',\alpha',\beta'} = U_{j,k,\alpha,\alpha'} U_{k,j',\alpha',\beta}, \quad (A48)$$

$$\sum_{\alpha'} \Theta_{j,k,\alpha,\beta,j',k',\alpha',\beta'} = \sum_{\alpha'} U_{j,k,\alpha,\alpha'} U_{k,j',\alpha',\beta} = \delta_{\alpha,\beta} \delta_{j,k}', \quad (A49)$$

$$\sum_{\alpha} \rho_{\alpha} = \delta_{j',k}'. \quad (A50)$$

For fixed $j' = k' = j = k = 1$, $\{U_{1,\alpha,\alpha'}\}_{\alpha} \alpha'$ satisfies

$$\sum_{\alpha'} U_{1,\alpha,\alpha'} U_{1,\alpha,\alpha'} = \delta_{\alpha,\beta} \rho_{\alpha}', \quad (A51)$$

where $P_{11} \geq 0$, $V$ is a $|B'| \times |B|$ isometry, i.e. $V^\dagger V = I_{B'}$.

Note that for $P_{11} > 0$, $V^\dagger V = I_{B'}$ requires $|B| \leq |B'|$.

**Lemma 3.** If $\{\{\varphi_j\}_{j=1}^{n}\}$ and $\{\{\psi_j\}_{j=1}^{n}\}$ are two orthonormal bases of Hilbert space $H$, and for all $j, k$, $\langle \varphi_j | \psi_k \rangle = \delta_{jk} |\varphi_j \rangle |\psi_k \rangle$, $\langle \varphi_j | \psi_k \rangle = \langle \varphi_k | \psi_j \rangle = \delta_{jk} |\varphi_j \rangle |\psi_k \rangle$, then $|\langle \varphi_j | \psi \rangle \rangle = e^{i\theta} |\varphi_j \rangle |\psi \rangle \rangle$ for all with real number $\theta$.

Proof hint: expand $|\langle \varphi_j | \psi \rangle \rangle_{k=1}^{n}$ in $\{|\varphi_j \rangle \}_{k=1}^{n}$.

Using this lemma to $\{U_{j,j',\alpha,\alpha'}\}_{\alpha} \alpha'$ and $\{U_{1,\alpha,\alpha'}\}_{\alpha} \alpha'$, we get

$$\{U_{j,j',\alpha,\alpha'}\}_{\alpha} \alpha' = \sqrt{P_{j,j'}} e^{i\theta_{j,j'}} V \quad (A52)$$

with $P_{j,j} \geq 0$, $\theta_{j,j'} \in R$. That is

$$U = U \otimes V, \quad (A53)$$

with $U_{j,j'} = \sqrt{P_{j,j'}} e^{i\theta_{j,j'}}$. $\sum_{j,j'} \rho_{\alpha\alpha'}^{(j,j')} = \delta_{j,k}'$ yields that $U$ is a $|A'| \times |A|$ isometry, i.e. $UU^* = I_{A'}$. Again, $UU^* = I_{A'}$ requires $|A| \geq |A'|$.

We can write $U = U \otimes V$ as

$$U_{j,j',\alpha,\alpha'} = U_{j,j'} V_{\alpha,\alpha'} \quad (A54)$$

Step 2. Now suppose $\Theta = \{U\} \in ISC_{AB}$. For $\phi \in ISC_{AB}$ as

$$\phi_{j,j',j} f(j) = 1, \forall j, \quad (A55)$$
with \( f(j) \in \{|\alpha|^2|_\alpha=1| \), and other elements \( \phi_{jk,\alpha\beta} = 0 \). For this case
\[
[\Theta(\phi)]_{j'k',\alpha'\beta'} = \sum_j \phi_{j,\alpha}(f(j)) \Theta_{jj',f(j)} \Theta_{k,k',\alpha'\beta'} = \Theta_{jj',f(j)} \sum_j \Theta_{j,j}(f(j),j',k',\alpha'\beta') = \sum_j U_{j,j}^* V_{\alpha'f(j)} V_{\beta'f(j)} = \sum_j U_{j,j}^* U_{k,k}^* V_{\alpha'f(j)} V_{\beta'f(j)}.
\]
\[
(A56)
\]
Let \( j' = k' \), \( \alpha' \neq \beta' \), \( f(j) = \alpha \) for \( \forall j \),
\[
0 = [\Theta(\phi)]_{j'k',\alpha'\beta'} = (\sum_j U_{j,j}^* U_{j,j}^*) V_{\alpha'\alpha}, V_{\beta'\alpha} = V_{\alpha'\alpha} V_{\beta'\alpha}.
\]
\[
(A57)
\]
as a result, each column of \( V \) has at most one nonzero element. And since \( V^\dagger V = I^B \), then \( V \) has just one nonzero element with modulus 1 in each column and has at most one nonzero element in each row.

Let \( j' \neq k' \), suppose \( V_{\alpha'\alpha} \neq 0 \), then \( V_{\alpha'\beta} = 0 \) for \( \beta \neq \alpha \), let \( \alpha' = \beta' \), \( f(1) = \alpha \), \( f(j) \neq \alpha \) for \( j \neq 1 \),
\[
0 = [\Theta(\phi)]_{j'k',\alpha'\beta'} = \sum_j U_{j,j}^* V_{\alpha'f(j)} V_{\beta'f(j)} = U_{j,j}^* U_{k,k}^*,
\]
\[
(A58)
\]
this yields that the first column of \( U \) has at most one nonzero element. Similarly every column of \( U \) has at most one nonzero element.

\section{Proof of Proposition 8}

\textbf{Step 1.} We show that the superchannel \( \Theta(\phi) = \chi \circ \phi \) is an ISC. Let \( A = A' \), then \( \Theta \in ISC_{ABA'B'} \). From the convexity of coherence for channels in (C3) and the structure of \( \mathcal{I}C \) in Eq. (16), we only need to consider the case that \( \chi_{\alpha\beta,\alpha'\beta'} = \delta_{\alpha\beta} \delta_{\alpha'\beta'} \delta_{\alpha',f(\alpha)} \) with \( f(\alpha) \in \{\alpha\}^{|B'|} \). It follows that
\[
[\Theta(\phi)]_{j'k',\alpha'\beta'} = \sum_{j,k,\alpha\beta} \phi_{jk,\alpha\beta} \Theta_{jk,\alpha\beta,j'k',\alpha'\beta'} = [\chi \circ \phi]_{j'k',\alpha'\beta'} = \sum_{\alpha\beta} \phi_{j'k',\alpha'\beta'} \chi_{\alpha\beta,\alpha'\beta'} = \sum_{\alpha} \phi_{j'k',\alpha} \delta_{\alpha'\beta'} \delta_{\alpha',f(\alpha)}.
\]
\[
(A59)
\]
consequently,
\[
\Theta_{j,k,\alpha\beta,j'k',\alpha'\beta'} = \delta_{j,j'} \delta_{k,k'} \delta_{\alpha\beta} \delta_{\alpha'\beta'} \delta_{\alpha',f(\alpha)}.
\]
\[
(A60)
\]
which admits the decomposition of Eq. (10) with
\[
\mathcal{M}_{m,j,j',\alpha'\alpha} = \delta_{ma} \delta_{j,j'} \delta_{\alpha',f(\alpha)},
\]
\[
(A61)
\]
\[
\mathcal{M}_m = \sum_{jj'} \delta_{ma} \delta_{j,j'} \delta_{\alpha',f(\alpha)} |\alpha'\rangle \langle \alpha|.
\]
\[
(A62)
\]
\textbf{Compare to Eq. (18) we see that} \( \Theta = \{\mathcal{M}_m\}_m \in ISC_{ABA'B'} \).

\textbf{Step 2.} We show that \( \Theta(\phi) = \phi \circ \chi \) is an ISC. Let \( B = B' \), then \( \Theta \in ISC_{ABA'B'} \). From Eq. (16), for \( |A'| \leq |A| \), we have
\[
\{ \chi \in \mathcal{A}_{A'k} | \chi_{j'k',jk} = \delta_{j'k'} \delta_{jk} \phi_{j',j}, \sum_j \chi_{j'j',j} \leq 1, \forall j \}
\]
\[
= \text{conv}\{ \chi \in \mathcal{A}_{A'k} | \chi_{j'k',jk} = \delta_{j'k'} \delta_{jk} \delta_{j,j'}, f(j') \neq f(k') \text{ for } j' \neq k' \},
\]
\[
(A63)
\]
where \( f(j') \in \{j \}^{|A'|} \). Thus we only need to consider the case that \( \chi \in \mathcal{A}_{A'k} \), \( \chi_{j'k',jk} = \delta_{j'k'} \delta_{jk} \delta_{j,j'}, f(j') \neq f(k') \text{ for } j' \neq k' \). Similar to Eqs. (A60-A62) we get
\[
\Theta_{j,k,\alpha\beta,j'k',\alpha'\beta'} = \delta_{jk} \delta_{j'k'} \delta_{\alpha\alpha'} \delta_{\beta'\beta} \delta_{\beta,j,j'},
\]
\[
(A64)
\]
\[
\mathcal{M}_{m,j,j',\alpha'\alpha} = \delta_{m,j} \delta_{\alpha\alpha'} \delta_{\beta,j,j'},
\]
\[
(A65)
\]
\[
\mathcal{M}_m = \sum_{jj'} \delta_{m,j} \delta_{\alpha\alpha'} \delta_{\beta,j,j'} |\alpha'\rangle \langle \alpha|.
\]
\[
(A66)
\]
Comparing to Eq. (18) we see that \( \Theta = \{\mathcal{M}_m\}_m \in ISC_{ABA'B'} \).

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