REMOVABILITY RESULTS FOR SUBHARMONIC FUNCTIONS, FOR SEPARATELY SUBHARMONIC FUNCTIONS, FOR HARMONIC FUNCTIONS AND FOR HOLOMORPHIC FUNCTIONS

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ABSTRACT. Blanchet has shown that a $C^2$ subharmonic function can be extended through a $C^1$ hypersurface provided the function is continuous throughout and satisfies certain $C^1$-type continuity conditions on the exceptional hypersurface. Later we improved Blanchet’s result, at least in a certain sense, measuring the exceptional set with the aid of Hausdorff measure. Our result was flexible, and with the aid of it, we gave certain extension results for harmonic and for holomorphic functions, related to Besicovitch’s and Shiffman’s well-known extension results, at least in some sense. Now we return to this subject. First, we refine our subharmonic function extension result slightly still more, improving also our previous proof. Though our result might be considered a little bit technical and even complicated, it is, nevertheless, flexible. As an example of its flexibility, we give a new and concise extension result for subharmonic functions. Second, we slightly refine our previous corollaries for harmonic and for holomorphic functions. In addition, and as a new application, we give related extension results for separately subharmonic functions. We also recall a slightly related extension result for holomorphic functions.

1. INTRODUCTION

1.1. An outline. We will consider extension problems for subharmonic, separately subharmonic, harmonic and holomorphic functions. Our results are based on an extension result for subharmonic functions, see Theorem 1 in Section 2 below. The starting point for this result is a result of Blanchet [3], Theorem 3.1, p. 312. As a matter of fact, Blanchet has shown that hypersurfaces of class $C^1$ are removable singularities for subharmonic functions, provided the considered subharmonic functions satisfy certain extra assumptions. Previously we have shown that, in certain cases, it is sufficient that the exceptional sets are of finite (n-1)-dimensional Hausdorff measure, see [28], Theorem, p. 568, [29], Theorem 3, p. 51, and [30], Theorem 1, p. 154, and these results we now improve. Moreover, we give related results for separately subharmonic functions, see Theorem 2 in Section 3 below.

In Sections 4 and 5 we will apply our subharmonic function result Theorem 1 below to get extension results both for harmonic and for holomorphic functions. In addition, in subsection 5.4, in Theorem 5 we give a related result for holomorphic functions.

1.2. Notation. Our notation is more or less standard, see [24, 27, 28, 29, 30]. However, for the convenience of the reader we recall here the following. We use the common convention $0 \cdot \pm \infty = 0$. For each $n \geq 1$ we identify $C^n$ with $\mathbb{R}^{2n}$. In integrals we will write $dx$ or $dm_n$ for the Lebesgue measure in $\mathbb{R}^n$, $n \in \mathbb{N}$. Let $0 \leq \alpha \leq n$ and $A \subset \mathbb{R}^n$, $n \geq 1$. Then we write $\mathcal{H}^\alpha(A)$ for the $\alpha$-dimensional Hausdorff (outer) measure of $A$. Recall that $\mathcal{H}^0(A)$ is the number of points

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of $A$. If $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $n \geq 2$, and $j \in \mathbb{N}$, $1 \leq j \leq n$, then we write $x = (x_j, X_j)$, where $X_j = (x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$. Moreover, if $A \subset \mathbb{R}^n$, $1 \leq j \leq n$, and $x_j^0 \in \mathbb{R}$, $X_j^0 \in \mathbb{R}^{n-1}$, we write

$$A(x_j^0) = \{ X_j \in \mathbb{R}^{n-1} : x = (x_j^0, X_j) \in A \} \quad A(X_j^0) = \{ x_j \in \mathbb{R} : x = (x_j, X_j^0) \in A \}.$$ 

If $\Omega \subset \mathbb{R}^n$ and $p > 0$, then $L^p_{\text{loc}}(\Omega)$, $p > 0$, is the space of functions $u$ in $\Omega$ for which $|u|^p$ is locally integrable on $\Omega$.

For the definition and properties of harmonic and subharmonic functions, see e.g. [5, 14, 15]. For the definition and properties of holomorphic functions see e.g. [5, 14, 15].

2. Extension results for subharmonic functions

2.1. A result of Federer. The following important result of Federer on geometric measure theory will be used repeatedly.

**Lemma.** ([7], Theorem 2.10.25, p. 188, and [31], Corollary 4, Lemma 2, p. 114) Suppose that $E \subset \mathbb{R}^n$, $n \geq 2$. Let $\alpha \geq 0$ and let $\pi_k : \mathbb{R}^n \rightarrow \mathbb{R}^k$ denote the projection onto the first $k$ coordinates.

1. If $\mathcal{H}^{k+\alpha}(E) = 0$, then $\mathcal{H}^\alpha(E \cap \pi_k^{-1}(x)) = 0$ for $\mathcal{H}^k$-almost all $x \in \mathbb{R}^k$.

2. If $\mathcal{H}^{k+\alpha}(E) < +\infty$, then $\mathcal{H}^\alpha(E \cap \pi_k^{-1}(x)) < +\infty$ for $\mathcal{H}^k$-almost all $x \in \mathbb{R}^k$.

2.2. A result of Blanchet. Blanchet has given the following result:

**Blanchet’s theorem.** ([3], Theorems 3.1, 3.2 and 3.3, pp. 312-313) Let $\Omega$ be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $S$ be a hypersurface of class $C^1$ which divides $\Omega$ into two subdomains $\Omega_1$ and $\Omega_2$. Let $u \in C^0(\Omega) \cap C^2(\Omega_1 \cup \Omega_2)$ be subharmonic (respectively convex or respectively plurisubharmonic provided $\Omega$ is then a domain in $\mathbb{C}^n$, $n \geq 1$) in $\Omega_1$ and $\Omega_2$. If $u_i = u|_{\Omega_i} \in C^1(\Omega_i \cup S)$, $i = 1, 2$, and

$$\frac{\partial u_i}{\partial n^k} \geq \frac{\partial u_k}{\partial n^k}$$

on $S$ with $i, k = 1, 2$, then $u$ is subharmonic (respectively convex or respectively plurisubharmonic) in $\Omega$.

Above $\vec{n}^k = (\vec{n}_1^k, \vec{n}_2^k, \ldots, \vec{n}_n^k)$ is the unit normal exterior to $\Omega_k$, and $u_k \in C^1(\Omega_k \cup S)$, $k = 1, 2$, means that there exist $n$ functions $v_j^i$, $j = 1, 2, \ldots, n$, continuous on $\Omega_k \cup S$, such that

$$v_j^i(x) = \frac{\partial u_k}{\partial x_j}(x)$$

for all $x \in \Omega_k$, $k = 1, 2$ and $j = 1, 2, \ldots, n$. 

The following example shows that one cannot drop the above condition (1) in Blanchet’s theorem.

Example 1. The function \( u : \mathbb{R}^2 \to \mathbb{R} \),
\[
u(z) = u(x + iy) = u(x, y) := \begin{cases} 1 + x, & \text{when } x < 0, \\ 1 - x, & \text{when } x \geq 0, \end{cases}
\]
is continuous in \( \mathbb{R}^2 \) and subharmonic, even harmonic in \( \mathbb{R}^2 \setminus \{0\} \times \mathbb{R} \). It is easy to see that \( u \) does not satisfy the condition (1) on \( S = \{0\} \times \mathbb{R} \) and that \( u \) is not subharmonic in \( \mathbb{R}^2 \).

Remark 1. For related results, previous and later, see Khabibullin’s results [16], Lemma 2.2, p. 201, Fundamental Theorem 2.1, pp. 200-201, and [17], Lemma 4.1, p. 503, Theorem 2.1, p. 498, Theorems 3.1 and 3.2, pp. 500-501. In this connection, see also [10], 1.4.3, pp. 21-22.

2.3. An improvement to the result of Blanchet. Already in [24], Theorem 4, pp. 181-182, we have given partial improvements to the cited subharmonic removability results of Blanchet. For more recent improvements, see [28], Theorem, p. 568, and [30], Theorem 1, p. 154. Now we improve these recent results slightly still more, see Theorem 1 below. Instead of hypersurfaces of class \( C^1 \), we consider again arbitrary sets of finite \((n-1)\)-dimensional Hausdorff measure as exceptional sets. Then, however, the condition (1) is replaced by another, related condition, the condition (iv) below, which is now, at least seemingly, less stringent than before.

Theorem 1. Suppose that \( \Omega \) is a domain in \( \mathbb{R}^n \), \( n \geq 2 \). Let \( E \subset \Omega \) be closed in \( \Omega \) and let \( H^{n-1}(E) < +\infty \). Let \( u : \Omega \setminus E \to \mathbb{R} \) be subharmonic and such that the following conditions are satisfied:

(i) \( u \in L^1_{\text{loc}}(\Omega) \).
(ii) \( u \in C^2(\Omega \setminus E) \).
(iii) For each \( j, 1 \leq j \leq n \), \( \frac{\partial^2 u}{\partial x_j^2} \in L^1_{\text{loc}}(\Omega) \).
(iv) For each \( j, 1 \leq j \leq n \), and for \( H^{n-1} \)-almost all \( x_j \in \mathbb{R}^{n-1} \) such that \( E(x_j) \) is finite, the following condition holds:
For each \( x_j^0 \in E(x_j) \) there exist sequences \( x_{j,l}^0, x_{j,l}^0 \in (\Omega \setminus E)(X_j), l = 1, 2, \ldots \), such that \( x_{j,l}^0 \to x_j^0 \) as \( l \to +\infty \), and
(iv(a)) \( \lim_{l \to +\infty} u(x_{j,l}^0, x_j) = \lim_{l \to +\infty} u(x_{j,l}^0, x_j) \in \mathbb{R} \),
(iv(b)) \( -\infty < \lim inf_{l \to +\infty} \frac{\partial u}{\partial x_j}(x_{j,l}^0, x_j) \leq \lim sup_{l \to +\infty} \frac{\partial u}{\partial x_j}(x_{j,l}^0, x_j) < +\infty \).

Then \( u \) has a subharmonic extension to \( \Omega \).
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Proof. Observe first that using suitable subsequences one can replace the assumption (iv) by the following, (only) seemingly stronger condition:

(iv*) For each $x_j^0 \in E(X_j)$ there exist sequences $x_{j,l}^{0,1}, x_{j,l}^{0,2} \in (\Omega \setminus E)(X_j)$, $l = 1, 2, \ldots$, such that $x_{j,l}^{0,1} \rightarrow x_j^0$, $x_{j,l}^{0,2} \rightarrow x_j^0$ as $l \rightarrow +\infty$, and

\[ \text{lim}_{l \rightarrow +\infty} u(x_{j,l}^{0,1}, X_j) = \text{lim}_{l \rightarrow +\infty} u(x_{j,l}^{0,2}, X_j) \in \mathbb{R}, \]

\[ -\infty < \text{lim}_{l \rightarrow +\infty} \frac{\partial u}{\partial x_j}(x_{j,l}^{0,1}, X_j) \leq \text{lim}_{l \rightarrow +\infty} \frac{\partial u}{\partial x_j}(x_{j,l}^{0,2}, X_j) < +\infty. \]

It is sufficient to show that

\[ \int u(x) \Delta \varphi(x) \, dx \geq 0 \]

for all nonnegative testfunctions $\varphi \in \mathcal{D}(\Omega)$, see e.g. [12], Corollary 1, p. 13.

Take $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geq 0$, arbitrarily. Let $K = \text{spt}\, \varphi$. Choose a domain $\Omega_1$ such that $K \subset \Omega_1 \subset \overline{\Omega}_1 \subset \Omega$ and $\overline{\Omega}_1$ is compact. Since $u \in C^2(\Omega \setminus E)$ and $u$ is subharmonic in $\Omega \setminus E$, $\Delta u(x) \geq 0$ for all $x \in \Omega \setminus E$. Thus the claim follows if we show that

\[ \int u(x) \Delta \varphi(x) \, dx \geq \int \Delta u(x) \varphi(x) \, dx. \]

For this purpose fix $j$, $1 \leq j \leq n$, arbitrarily for a while. By Fubini’s theorem, see e.g. [6], Theorem 1, pp. 22-23,

\[ \int u(x) \frac{\partial^2 \varphi}{\partial x_j^2}(x) \, dx = \int \left[ \int u(x_j, X_j) \frac{\partial^2 \varphi}{\partial x_j^2}(x_j, X_j) \, dx_j \right] \, dX_j. \]

Using the above Lemma, assumptions (i), (ii) and (iii), and Fubini’s theorem, we see that for $\mathcal{H}^{n-1}$-almost all $X_j \in \mathbb{R}^{n-1},$

\[ \begin{cases} 
\begin{aligned} 
u(x, X_j) & \in L^1_{\text{loc}}(\Omega(X_j)), \\
\frac{\partial^2 u}{\partial x_j^2}(x, X_j) & \in L^1_{\text{loc}}(\Omega(X_j)), \\
E(X_j) & \text{is finite, thus there exists} \ M = M(X_j) \in \mathbb{N}_0 \text{ such that} \\
E(X_j) & = \{x_j^1, x_j^2, \ldots, x_j^M \} \text{ where} \ x_j^k < x_j^{k+1}, k = 1, 2, \ldots, M - 1. 
\end{aligned} 
\end{cases} \]

Let $X_j \in \mathbb{R}^{n-1}$ be arbitrary as above in (2). We may suppose that $\Omega(X_j)$ is a finite interval. Choose for each $k = 1, 2, \ldots, M$ numbers $a_k, b_k \in (\Omega \setminus E)(X_j)$ such that $a_k < x_j^k < b_k$, $k = 1, 2, \ldots, M$, $a_{k+1} = b_k$, $k = 1, 2, \ldots, M - 1$, and that $a_1, b_M \in (\Omega \setminus \overline{\Omega}_1)(X_j)$.
With the aid of (iv*) we find for each \( x^k_j \in E(X_j) \) sequences \( x^{k,1}_{j,l}, x^{k,2}_{j,l} \in (\Omega \setminus E)(X_j), l = 1, 2, \ldots, x^{k,1}_{j,l} \nearrow x^k_j, x^{k,2}_{j,l} \searrow x^k_j \) as \( l \to +\infty \), such that

\[
\lim_{l \to +\infty} u(x^{k,1}_{j,l}, X_j) = \lim_{l \to +\infty} u(x^{k,2}_{j,l}, X_j) \in \mathbb{R},
\]

and

\[
-\infty < \lim_{l \to +\infty} \frac{\partial u}{\partial x_j}(x^{k,1}_{j,l}, X_j) \leq \lim_{l \to +\infty} \frac{\partial u}{\partial x_j}(x^{k,2}_{j,l}, X_j) < +\infty.
\]

Take \( k, 1 \leq k \leq M \), arbitrarily and consider the interval \((a_k, b_k)\), where \( a_k < x^k_j < b_k \). To simplify the notation, write \( a := a_k, b := b_k \) and \( x^{0,1}_j := x^0_j \). Then

\[
a < x^{0,1}_{j,l} \nearrow x^0_j, \ b > x^{0,2}_{j,l} \searrow x^0_j \text{ as } l \to +\infty.
\]

Using partial integration we get:

\[
\int_a^b u(x_j, X_j) \frac{\partial^2 \varphi}{\partial x_j^2}(x_j, X_j) \, dx_j = \int_a^{x^0_j} u(x_j, X_j) \frac{\partial^2 \varphi}{\partial x_j^2}(x_j, X_j) \, dx_j + \int_{x^0_j}^b u(x_j, X_j) \frac{\partial^2 \varphi}{\partial x_j^2}(x_j, X_j) \, dx_j
\]

\[
+ \int_a^{x^0_j} u(x_j, X_j) \frac{\partial \varphi}{\partial x_j}(x_j, X_j) \, dx_j - \int_{x^0_j}^b u(x_j, X_j) \frac{\partial \varphi}{\partial x_j}(x_j, X_j) \, dx_j
\]

\[
= \lim_{l \to +\infty} \int_a^{x^0_{j,l}} u(x_j, X_j) \frac{\partial \varphi}{\partial x_j}(x_j, X_j) \, dx_j - \lim_{l \to +\infty} \int_{x^0_{j,l}}^b u(x_j, X_j) \frac{\partial \varphi}{\partial x_j}(x_j, X_j) \, dx_j
\]

\[
+ \lim_{l \to +\infty} \left[ u(b, X_j) \frac{\partial \varphi}{\partial x_j}(b, X_j) - u(a, X_j) \frac{\partial \varphi}{\partial x_j}(a, X_j) \right] +
\]

\[
+ \lim_{l \to +\infty} \left[ u(x^{0,1}_{j,l}, X_j) \frac{\partial \varphi}{\partial x_j}(x^{0,1}_{j,l}, X_j) - \int_a^{x^{0,1}_{j,l}} u(x_j, X_j) \frac{\partial \varphi}{\partial x_j}(x_j, X_j) \, dx_j \right] +
\]

\[
- \lim_{l \to +\infty} \left[ u(x^{0,2}_{j,l}, X_j) \frac{\partial \varphi}{\partial x_j}(x^{0,2}_{j,l}, X_j) + \int_{x^{0,2}_{j,l}}^b u(x_j, X_j) \frac{\partial \varphi}{\partial x_j}(x_j, X_j) \, dx_j \right].
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\[
\begin{aligned}
&= \left[ u(b, X_j) \frac{\partial \phi}{\partial x_j}(b, X_j) - u(a, X_j) \frac{\partial \phi}{\partial x_j}(a, X_j) \right] + \\
&- \lim_{l \to +\infty} \int_a^{x_{j,l}} \frac{\partial u}{\partial x_j}(x_j, X_j) \frac{\partial \phi}{\partial x_j}(x_j, X_j) \, dx_j + \\
&- \lim_{l \to +\infty} \int_{x_{j,l}}^{b} \frac{\partial u}{\partial x_j}(x_j, X_j) \frac{\partial \phi}{\partial x_j}(x_j, X_j) \, dx_j \\
&= \left[ u(b, X_j) \frac{\partial \phi}{\partial x_j}(b, X_j) - u(a, X_j) \frac{\partial \phi}{\partial x_j}(a, X_j) \right] + \\
&- \lim_{l \to +\infty} \left[ \int_a^{x_{j,l}} \frac{\partial u}{\partial x_j}(x_j, X_j) \Phi(x_j, X_j) - \int_a^{x_{j,l}} \frac{\partial^2 u}{\partial x_j^2}(x_j, X_j) \Phi(x_j, X_j) \, dx_j \right] + \\
&- \lim_{l \to +\infty} \left[ \int_{x_{j,l}}^{b} \frac{\partial u}{\partial x_j}(x_j, X_j) \Phi(x_j, X_j) - \int_{x_{j,l}}^{b} \frac{\partial^2 u}{\partial x_j^2}(x_j, X_j) \Phi(x_j, X_j) \, dx_j \right] \\
&= \left[ u(b, X_j) \frac{\partial \phi}{\partial x_j}(b, X_j) - u(a, X_j) \frac{\partial \phi}{\partial x_j}(a, X_j) \right] + \\
&+ \left[ \frac{\partial u}{\partial x_j}(a, X_j) \Phi(a, X_j) - \frac{\partial u}{\partial x_j}(b, X_j) \Phi(b, X_j) \right] + \\
&- \lim_{l \to +\infty} \left[ \frac{\partial u}{\partial x_j}(x_{j,l}, X_j) \Phi(x_{j,l}, X_j) \right] + \\
&+ \lim_{l \to +\infty} \left[ \frac{\partial u}{\partial x_j}(x_{j,l}, X_j) \Phi(x_{j,l}, X_j) \right] + \\
&+ \int_a^{b} \frac{\partial^2 u}{\partial x_j^2}(x_j, X_j) \Phi(x_j, X_j) \, dx_j \\
&= \left[ u(b, X_j) \frac{\partial \phi}{\partial x_j}(b, X_j) - u(a, X_j) \frac{\partial \phi}{\partial x_j}(a, X_j) \right] + \\
&+ \left[ \frac{\partial u}{\partial x_j}(a, X_j) \Phi(a, X_j) - \frac{\partial u}{\partial x_j}(b, X_j) \Phi(b, X_j) \right] + \\
&+ \left[ \lim_{l \to +\infty} \frac{\partial u}{\partial x_j}(x_{j,l}, X_j) - \lim_{l \to +\infty} \frac{\partial u}{\partial x_j}(x_{j,l}, X_j) \right] \Phi(x_{j,l}, X_j) + \\
&+ \int_a^{b} \frac{\partial^2 u}{\partial x_j^2}(x_j, X_j) \Phi(x_j, X_j) \, dx_j
\end{aligned}
\]
\[ \geq u(b, X_j) \frac{\partial \phi}{\partial x_j}(b, X_j) - u(a, X_j) \frac{\partial \phi}{\partial x_j}(a, X_j) + \]
\[ + \left[ \frac{\partial u}{\partial x_j}(a, X_j) \varphi(a, X_j) - \frac{\partial u}{\partial x_j}(b, X_j) \varphi(b, X_j) \right] + \int_a^b \frac{\partial^2 u}{\partial x_j^2}(x_j, X_j) \varphi(x_j, X_j) \, dx_j. \]

Above we have used just standard properties of limits and our assumption (iv* (b)).

To return to our original notation, we have thus obtained for each \( k = 1, 2, \ldots, M \),
\[ \int_{a_k}^{b_k} u(x_j, X_j) \frac{\partial^2 \varphi}{\partial x_j^2}(x_j, X_j) \, dx_j \geq \left[ u(b_k, X_j) \frac{\partial \phi}{\partial x_j}(b_k, X_j) - u(a_k, X_j) \frac{\partial \phi}{\partial x_j}(a_k, X_j) \right] + \]
\[ + \left[ \frac{\partial u}{\partial x_j}(a_k, X_j) \varphi(a_k, X_j) - \frac{\partial u}{\partial x_j}(b_k, X_j) \varphi(b_k, X_j) \right] + \int_{a_k}^{b_k} \frac{\partial^2 u}{\partial x_j^2}(x_j, X_j) \varphi(x_j, X_j) \, dx_j. \]

Then just sum over \( k = 1, 2, \ldots, M \):
\[ \int u(x_j, X_j) \frac{\partial^2 \varphi}{\partial x_j^2}(x_j, X_j) \, dx_j = \int_{a_1}^{b_M} u(x_j, X_j) \frac{\partial^2 \varphi}{\partial x_j^2}(x_j, X_j) \, dx_j = \]
\[ = \sum_{k=1}^{M} \int_{a_k}^{b_k} u(x_j, X_j) \frac{\partial^2 \varphi}{\partial x_j^2}(x_j, X_j) \, dx_j \geq \]
\[ \geq \sum_{k=1}^{M} \left[ u(b_k, X_j) \frac{\partial \phi}{\partial x_j}(b_k, X_j) - u(a_k, X_j) \frac{\partial \phi}{\partial x_j}(a_k, X_j) \right] + \]
\[ + \sum_{k=1}^{M} \left[ \frac{\partial u}{\partial x_j}(a_k, X_j) \varphi(a_k, X_j) - \frac{\partial u}{\partial x_j}(b_k, X_j) \varphi(b_k, X_j) \right] + \]
\[ + \sum_{k=1}^{M} \int_{a_k}^{b_k} \frac{\partial^2 u}{\partial x_j^2}(x_j, X_j) \varphi(x_j, X_j) \, dx_j = \]
\[ = \int_{a_1}^{b_M} \frac{\partial^2 u}{\partial x_j^2}(x_j, X_j) \varphi(x_j, X_j) \, dx_j = \int \frac{\partial^2 u}{\partial x_j^2}(x_j, X_j) \varphi(x_j, X_j) \, dx_j. \]

Above we have used the choice of the numbers \( a_k, b_k, k = 1, 2, \ldots, M \), and the fact that \( a_1, b_M \in (\Omega \setminus \Omega_1)(X_j) \).

Integrate then with respect to \( X_j \) and use again Fubini’s theorem:
\[ \int u(x) \frac{\partial^2 \varphi}{\partial x_j^2}(x) \, dx = \int \left[ \int u(x_j, X_j) \frac{\partial^2 \varphi}{\partial x_j^2}(x_j, X_j) \, dx_j \right] \, dX_j \geq \]
\[ \geq \int \left[ \int \frac{\partial^2 u}{\partial x_j^2}(x_j, X_j) \varphi(x_j, X_j) \, dx_j \right] \, dX_j = \int \frac{\partial^2 u}{\partial x_j^2}(x) \varphi(x) \, dx. \]
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Summing over $j = 1, 2, \ldots, n$ gives the desired inequality

$$\int u(x)\Delta\varphi(x)\,dx = \int u(x)\sum_{j=1}^{n} \frac{\partial^2 \varphi}{\partial x_j^2}(x)\,dx \geq \int \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}(x)\varphi(x)\,dx = \int \Delta u(x)\varphi(x)\,dx \geq 0,$$

concluding the proof. \qed

Example 2. The function $u : \mathbb{R}^2 \to \mathbb{R}$ given already in Example 1 is continuous in $\mathbb{R}^2$ and subharmonic, even harmonic in $\mathbb{R}^2 \setminus \{0\} \times \mathbb{R}$, but not subharmonic in $\mathbb{R}^2$. Observe that $u$ satisfies the above conditions (i), (ii), (iii) and (iv(a)) in $\mathbb{R}^2 \setminus \{0\} \times \mathbb{R}$. However, $u|\mathbb{R}^2 \setminus \{0\} \times \mathbb{R}$ does not satisfy the condition (iv(b)). Thus one cannot drop the condition (iv(b)) in Theorem 1.

2.4. The below stated five corollaries reflect the strength of Theorem 1. As a matter of fact, one of these, Corollary 3, which previously has not been explicitly stated, gives a concise extension result for subharmonic functions, which might be of interest in itself.

Corollary 1. ([28], Theorem, p. 568, and [30], Theorem 1, p. 154) Suppose that $\Omega$ is a domain in $\mathbb{R}^n$, $n \geq 2$. Let $E \subset \Omega$ be closed in $\Omega$ and let $\mathcal{H}^{n-1}(E) < +\infty$. Let $u : \Omega \setminus E \to \mathbb{R}$ be subharmonic and such that the following conditions hold:

(i) $u \in L^1_{\text{loc}}(\Omega)$.

(ii) $u \in C^2(\Omega \setminus E)$.

(iii) For each $j$, $1 \leq j \leq n$, $\frac{\partial^2 u}{\partial x_j^2} \in L^1_{\text{loc}}(\Omega)$.

(iv) For each $j$, $1 \leq j \leq n$, and for $\mathcal{H}^{n-1}$-almost all $X_j \in \mathbb{R}^{n-1}$ such that $E(X_j)$ is finite, the following condition holds:

For each $x_j^0 \in E(X_j)$ there exist sequences $x_{j,l}^{0,1}, x_{j,l}^{0,2} \in (\Omega \setminus E)(X_j), l = 1, 2, \ldots$, such that $x_{j,l}^{0,1} \to x_j^0, x_{j,l}^{0,2} \to x_j^0$ as $l \to +\infty$, and

(iv(a)) $\lim_{l \to +\infty} u(x_{j,l}^{0,1}, X_j) = \lim_{l \to +\infty} u(x_{j,l}^{0,2}, X_j) \in \mathbb{R}$,

(iv(b)) $-\infty < \lim_{l \to +\infty} \frac{\partial u}{\partial x_j}(x_{j,l}^{0,1}, X_j) \leq \lim_{l \to +\infty} \frac{\partial u}{\partial x_j}(x_{j,l}^{0,2}, X_j) < +\infty$.

Then $u$ has a subharmonic extension to $\Omega$.

Corollary 2. ([24], Corollary 4A, pp. 185-186) Suppose that $\Omega$ is a domain in $\mathbb{R}^n$, $n \geq 2$. Let $E \subset \Omega$ be closed in $\Omega$ and let $\mathcal{H}^{n-1}(E) < +\infty$. Let $u : \Omega \to \mathbb{R}$ be such that

(i) $u \in C^1(\Omega)$,

(ii) $u \in C^2(\Omega \setminus E)$,
(iii) for each \( j, 1 \leq j \leq n \), \( \frac{\partial^2 u}{\partial x_j^2} \in L^1_{\text{loc}}(\Omega) \),

(iv) \( u \) is subharmonic in \( \Omega \setminus E \).

Then \( u \) is subharmonic.

**Corollary 3.** Suppose that \( \Omega \) is a domain in \( \mathbb{R}^n, n \geq 2 \). Let \( E \subset \Omega \) be closed in \( \Omega \) and let \( \mathcal{H}^{n-1}(E) = 0 \). Let \( u : \Omega \setminus E \to \mathbb{R} \) be subharmonic and such that the following conditions hold:

(i) \( u \in L^1_{\text{loc}}(\Omega) \),

(ii) \( u \in C^2(\Omega \setminus E) \),

(iii) for each \( j, 1 \leq j \leq n \), \( \frac{\partial^2 u}{\partial x_j^2} \in L^1_{\text{loc}}(\Omega) \).

Then \( u \) has a subharmonic extension to \( \Omega \).

**Proof.** Follows directly from Theorem 1 and from the above Lemma of Federer. \( \square \)

3. Extension Results for Separately Subharmonic Functions

3.1. Next we will give an extension result for separately subharmonic functions. Our proof will be based on Theorem 1 and on the following nice result. Observe here that the below in Proposition 1 used hypoharmonic functions are in our terminology just subharmonic functions.

**Proposition 1.** ([12], Proposition 1, p. 33) Suppose that \( \Omega \) is a domain in \( \mathbb{R}^{p+q}, p,q \geq 2 \). Let \( w : \Omega \to [-\infty, +\infty) \) be nearly subharmonic. Let \( w^* : \Omega \to [-\infty, +\infty) \) be the regularized function of \( w \), which is then subharmonic. Then the following properties are equivalent.

1. The distribution \( \Delta x w = \Delta x w^* = (\text{sum of the square second order derivatives of } w \text{ or } w^* \text{ with respect to the } p \text{ coordinates of } x) \) is positive.
2. For all \( y \in \mathbb{R}^q \) the function \( \Omega(y) \ni x \mapsto w^*(x,y) \in [-\infty, +\infty) \) is hypoharmonic.
3. For almost every \( y \in \mathbb{R}^q \) the function \( \Omega(y) \ni x \mapsto w^*(x,y) \in [-\infty, +\infty) \) is subharmonic.
4. For almost every \( y \in \mathbb{R}^q \) the function \( \Omega(y) \ni x \mapsto w^*(x,y) \in [-\infty, +\infty) \) is nearly subharmonic.
3.2. **An extension result for separately subharmonic functions.** Then our result:

**Theorem 2.** Suppose that $\Omega$ is a domain in $\mathbb{R}^{p+q}$, $p, q \geq 2$. Let $E \subset \Omega$ be closed in $\Omega$ and let $\mathcal{H}^{p+q-1}(E) < +\infty$. Let $w : \Omega \setminus E \to \mathbb{R}$ be separately subharmonic, that is,

$$(\Omega \setminus E)(y) \ni x \mapsto w(x, y) \in \mathbb{R} \text{ is subharmonic for all } y \in \mathbb{R}^q$$

and

$$(\Omega \setminus E)(x) \ni y \mapsto w(x, y) \in \mathbb{R} \text{ is subharmonic for all } x \in \mathbb{R}^p,$$

and such that the following conditions are satisfied:

(i) $w \in L^1_{\text{loc}}(\Omega)$.

(ii) $w \in C^2(\Omega \setminus E)$.

(iii) For each $j$, $1 \leq j \leq p$, $\frac{\partial^2 w}{\partial x_j^2} \in L^1_{\text{loc}}(\Omega)$, and for each $k$, $1 \leq k \leq q$, $\frac{\partial^2 w}{\partial y_k^2} \in L^1_{\text{loc}}(\Omega)$.

(iv) For each $j$, $1 \leq j \leq p$, and for $\mathcal{H}^{p-1+q}$-almost all $(X_j, y) \in \mathbb{R}^{p-1+q}$ such that $E(X_j, y)$ is finite, the following condition holds:

For each $X_j \in E(X_j, y)$ there exist sequences $x_{j,l}^{0,1} , x_{j,l}^{0,2} \in (\Omega \setminus E)(X_j, y)$, $l = 1, 2, \ldots$, such that $x_{j,l}^{0,1} \nearrow x_j^0, x_{j,l}^{0,2} \searrow x_j^0$ as $l \to +\infty$, and

(iv(a)) $\lim_{l \to +\infty} w(x_{j,l}^{0,1}, X_j, y) = \lim_{l \to +\infty} w(x_{j,l}^{0,2}, X_j, y) \in \mathbb{R}$,

(iv(b)) $-\infty < \liminf_{l \to +\infty} \frac{\partial w}{\partial x_j}(x_{j,l}^{0,1}, X_j, y) \leq \limsup_{l \to +\infty} \frac{\partial w}{\partial x_j}(x_{j,l}^{0,2}, X_j, y) < +\infty$.

(v) For each $k$, $1 \leq k \leq q$, and for $\mathcal{H}^{p+q-1}$-almost all $(x, Y_k) \in \mathbb{R}^{p+q-1}$ such that $E(x, Y_k)$ is finite, the following condition holds:

For each $Y_k \in E(x, Y_k)$ there exist sequences $y_{k,l}^{0,1} , y_{k,l}^{0,2} \in (\Omega \setminus E)(x, Y_k)$, $l = 1, 2, \ldots$, such that $y_{k,l}^{0,1} \nearrow y_k^0, y_{k,l}^{0,2} \searrow y_k^0$ as $l \to +\infty$, and

(v(a)) $\lim_{l \to +\infty} w(x, y_{k,l}^{0,1}, Y_k) = \lim_{l \to +\infty} w(x, y_{k,l}^{0,2}, Y_k) \in \mathbb{R}$,

(v(b)) $-\infty < \liminf_{l \to +\infty} \frac{\partial w}{\partial y_k}(x, y_{k,l}^{0,1}, Y_k) \leq \limsup_{l \to +\infty} \frac{\partial w}{\partial y_k}(x, y_{k,l}^{0,2}, Y_k) < +\infty$.

Then $w$ has a separately subharmonic extension to $\Omega$.

**Proof.** Using Theorem 1 one sees at once that $w : \Omega \setminus E \to \mathbb{R}$ has a subharmonic extension $w^* : \Omega \to [-\infty, +\infty]$.

Next we show that for $\mathcal{H}^q$-almost all $y \in \mathbb{R}^q$ the subharmonic function

$$(\Omega \setminus E)(y) \ni x \mapsto w(x, y) \in \mathbb{R}$$

has a subharmonic extension

$$\Omega(y) \ni x \mapsto w^*(x, y) \in [-\infty, +\infty].$$
For this purpose fix \( j, 1 \leq j \leq p \), arbitrarily for a while. Let

\[
A := \{ (X_j, y) \in \mathbb{R}^{p-1+q} : E(y)(X_j) \text{ is finite} \}.
\]

Using then the assumption that \( \mathcal{H}^{p-1+q}(E) < +\infty \) and the above Lemma of Federer, we see that for \( \mathcal{H}^{p-1+q} \)-almost all \( (X_j, y) \in \mathbb{R}^{p-1+q} \) the set \( E(y)(X_j) \) is finite.

Thus

\[
\mathcal{H}^{p-1+q}(A^c) = 0 \iff m_{p-1+q}(A^c) = 0 \iff \int_{\mathbb{R}^{p-1+q}} \chi_{A^c}(X_j, y) \, dm_{p-1+q}(X_j, y) = 0,
\]

where \( \chi_{A^c}(\cdot, \cdot) \) is the characteristic function of the set \( A^c \), the complement taken in \( \mathbb{R}^{p-1+q} \).

Next use Fubini’s theorem:

\[
0 = \int_{\mathbb{R}^{p-1+q}} \chi_{A^c}(X_j, y) \, dm_{p-1+q}(X_j, y) = \int_{\mathbb{R}^q} \left[ \int_{\mathbb{R}^{p-1}} \chi_{A^c}(X_j, y) \, dm_{p-1}(X_j) \right] \, dm_q(y).
\]

Since

\[
\int_{\mathbb{R}^{p-1}} \chi_{A^c}(X_j, y) \, dm_{p-1}(X_j) \geq 0,
\]

we see that in fact

\[
\int_{\mathbb{R}^{p-1}} \chi_{A^c}(X_j, y) \, dm_{p-1}(X_j) = 0
\]

for \( \mathcal{H}^{q} \)-almost all \( y \in \mathbb{R}^q \). Consider then one such \( y \in \mathbb{R}^q \). Since

\[
\int_{\mathbb{R}^{p-1}} \chi_{A^c}(X_j, y) \, dm_{p-1}(X_j) = 0,
\]

we see that

\[
\chi_{A^c}(X_j, y) = 0
\]

for almost all \( X_j \in \mathbb{R}^{p-1} \) or equivalently

\[
\chi_A(X_j, y) = 1
\]

for almost all \( X_j \in \mathbb{R}^{p-1} \). But this means that \( (X_j, y) \in A \) for \( \mathcal{H}^{p-1} \)-almost all \( X_j \in \mathbb{R}^{p-1} \).

For almost all \( y \in \mathbb{R}^q \) we can then apply Theorem 1. Observe that its assumptions are indeed satisfied:

- \( w(\cdot, y) \in L^1_{\text{loc}}(\Omega(y)) \) by Fubini’s theorem.
- \( w(\cdot, y) \in C^2((\Omega \setminus E)(y)) \), since \( w \in C^2(\Omega \setminus E) \).
- For each \( j, 1 \leq j \leq p \), \( \frac{\partial^2 w}{\partial x_j^2}(\cdot, y) \in L^1_{\text{loc}}(\Omega(y)) \) by Fubini’s theorem.
- \( \mathcal{H}^{p-1}(E(y)) < +\infty \), this with the aid of Federer’s Lemma.
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• Moreover the following: Take $j$, $1 \leq j \leq p$, arbitrary. Then for $\mathcal{H}^{p-1}$-almost all $X_j \in \mathbb{R}^{p-1}$ we have $(X_j, y) \in A$, and thus for almost all $X_j \in \mathbb{R}^{p-1}$ the set $E(y)(X_j)$ is finite, thus the following condition holds:

For each $x_j^0 \in E(y)(X_j)$ there exist sequences $x_{j,l}^{0,1}, x_{j,l}^{0,2} \in (\Omega \setminus E)(y)(X_j)$, $l = 1, 2, \ldots,$ such that $x_{j,l}^{0,1} \rightarrow x_j^0, x_{j,l}^{0,2} \searrow x_j^0$ as $l \rightarrow +\infty$, and

$$\lim_{l \rightarrow +\infty} w(x_{j,l}^{0,1}, X_j, y) = \lim_{l \rightarrow +\infty} w(x_{j,l}^{0,2}, X_j, y) \in \mathbb{R},$$

$$-\infty < \liminf_{l \rightarrow +\infty} \frac{\partial w}{\partial x_j}(x_{j,l}^{0,1}, X_j, y) \leq \limsup_{l \rightarrow +\infty} \frac{\partial w}{\partial x_j}(x_{j,l}^{0,2}, X_j, y) < +\infty.$$ Using then Theorem 1 we see that for almost all $y \in \mathbb{R}^q$ the function

$$(\Omega \setminus E)(y) \ni x \mapsto w(x, y) \in \mathbb{R}$$

has a subharmonic extension

$$\Omega(y) \ni x \mapsto w^{**}(x, y) \in [-\infty, +\infty).$$

Similarly we see that for almost all $x \in \mathbb{R}^p$ the function

$$(\Omega \setminus E)(x) \ni y \mapsto w(x, y) \in \mathbb{R}$$

has a subharmonic extension

$$\Omega(x) \ni y \mapsto w^{**}(x, y) \in [-\infty, +\infty).$$

Since $w^* : \Omega \rightarrow [-\infty, +\infty)$ is subharmonic, it follows from Proposition 1 that for all $y \in \mathbb{R}^q$ the function

$$(\Omega \setminus E)(y) \ni x \mapsto w(x, y) \in \mathbb{R}$$

has a subharmonic extension

$$(\Omega)(y) \ni x \mapsto w^{**}(x, y) \in [-\infty, +\infty).$$

Similarly we see that for all $x \in \mathbb{R}^p$ the function

$$(\Omega \setminus E)(x) \ni y \mapsto w(x, y) \in \mathbb{R}$$

has a subharmonic extension

$$\Omega(x) \ni y \mapsto w^{**}(x, y) \in [-\infty, +\infty).$$

Thus the proof is completed.

**Example 3.** The function $u : \mathbb{R}^4 \rightarrow \mathbb{R}$,

$$u(z_1, z_2) = u(x_1 + iy_1, x_2 + iy_2) = u(x_1, y_1, x_2, y_2) := \begin{cases} 1 + x_1, & \text{when } x_1 < 0, \\ 1 - x_1, & \text{when } x_1 \geq 0, \end{cases}$$

is continuous in $\mathbb{R}^4$ and separately subharmonic, even separately harmonic in $\mathbb{R}^4 \setminus \{0\} \times \mathbb{R}^3$, but not separately subharmonic in $\mathbb{R}^4$. Observe that $u$ satisfies
the above conditions (i), (ii), (iii), (iv(a)) and (v(a)) in \( \mathbb{R}^4 \setminus (\{0\} \times \mathbb{R}^3) \). However, \( u|_{\mathbb{R}^4 \setminus (\{0\} \times \mathbb{R}^3)} \) does not satisfy the conditions (iv(b)) and (v(b)). Thus these conditions cannot be dropped in Theorem 2.

**Corollary 4.** Suppose that \( \Omega \) is a domain in \( \mathbb{R}^{p+q} \), \( p, q \geq 2 \). Let \( E \subset \Omega \) be closed in \( \Omega \) and let \( \mathcal{H}^{p+q-1}(E) = 0 \). Let \( w : \Omega \setminus E \to \mathbb{R} \) be separately subharmonic, that is,

\[
(\Omega \setminus E)(y) \ni x \mapsto w(x, y) \in \mathbb{R} \text{ is subharmonic for all } y \in \mathbb{R}^q
\]

and

\[
(\Omega \setminus E)(x) \ni y \mapsto w(x, y) \in \mathbb{R} \text{ is subharmonic for all } x \in \mathbb{R}^p,
\]

and such that the following conditions are satisfied:

(i) \( w \in L^1_{\text{loc}}(\Omega) \),

(ii) \( w \in C^2(\Omega \setminus E) \),

(iii) for each \( j, 1 \leq j \leq p \), \( \frac{\partial^2 w}{\partial x_j^2} \in L^1_{\text{loc}}(\Omega) \) and for each \( k, 1 \leq k \leq q \), \( \frac{\partial^2 w}{\partial y_k^2} \in L^1_{\text{loc}}(\Omega) \).

Then \( w \) has a separately subharmonic extension to \( \Omega \).

**Proof.** Follows directly from Theorem 2 and from the above Lemma of Federer. \( \square \)

### 4. Extension results for harmonic functions

4.1. For removability results for harmonic functions see, among others, [8, 9, 19, 32] and the references therein, say.

Now, using our Theorem 1, we give the following extension result for harmonic functions:

**Theorem 3.** Suppose that \( \Omega \) is a domain in \( \mathbb{R}^n \), \( n \geq 2 \). Let \( E \subset \Omega \) be closed in \( \Omega \) and let \( \mathcal{H}^{n-1}(E) < +\infty \). Let \( u : \Omega \setminus E \to \mathbb{R} \) be harmonic and such that the following conditions are satisfied:

(i) \( u \in L^1_{\text{loc}}(\Omega) \).

(ii) For each \( j, 1 \leq j \leq n \), \( \frac{\partial^2 u}{\partial x_j^2} \in L^1_{\text{loc}}(\Omega) \).

(iii) For each \( j, 1 \leq j \leq n \), and for \( \mathcal{H}^{n-1} \)-almost all \( X_j \in \mathbb{R}^{n-1} \) such that \( E(X_j) \) is finite, the following condition holds:

For each \( x_j^0 \in E(X_j) \) there exist sequences \( x_{j,l}^{0,1}, x_{j,l}^{0,2} \in (\Omega \setminus E)(X_j) \), \( l = 1, 2, \ldots \), such that \( x_{j,l}^{0,1} \nearrow x_j^0 \), \( x_{j,l}^{0,2} \searrow x_j^0 \) as \( l \to +\infty \), and

(iii(a)) \( \lim_{l \to +\infty} u(x_{j,l}^{0,1}, X_j) = \lim_{l \to +\infty} u(x_{j,l}^{0,2}, X_j) \in \mathbb{R} \),
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\[(iii(b)) \quad - \infty < \liminf_{l \to +\infty} \frac{\partial u}{\partial x_j}(x_{j,l}, X_j) = \limsup_{l \to +\infty} \frac{\partial u}{\partial x_j}(x_{j,l}, X_j) < +\infty.\]

Then $u$ has a unique harmonic extension to $\Omega$.

Proof. Since the assumptions of Theorem 1 do hold for the subharmonic function $u$, $u$ has a subharmonic extension $u^*$ to $\Omega$. On the other hand, the assumptions of Theorem 1 hold also for the subharmonic function $v = -u$. Thus $v = -u$ has a subharmonic extension $v^* = (-u)^*$ to $\Omega$. As above in the proof of Theorem 1, we may suppose that the limits
\[
\lim_{l \to +\infty} \left[ \frac{\partial u}{\partial x_j}(x_{j,l}, X_j) \right] \quad \text{and} \quad \lim_{l \to +\infty} \left[ \frac{\partial u}{\partial x_j}(x_{j,l}, X_j) \right]
\]
indeed exist.

Since $-v^* = u^*$, the extension $u^*$ of $u$ is both subharmonic and superharmonic, thus harmonic and the claim follows. \qed

4.2. Then a concise special case to Theorem 3:

**Corollary 5.** Suppose that $\Omega$ is a domain in $\mathbb{R}^n$, $n \geq 2$. Let $E \subset \Omega$ be closed in $\Omega$ and let $H^{n-1}(E) = 0$. Let $u : \Omega \setminus E \to \mathbb{R}$ be harmonic and such that the following conditions are satisfied:

(i) $u \in L^1_{\text{loc}}(\Omega)$,

(ii) for each $j$, $1 \leq j \leq n$, $\frac{\partial^2 u}{\partial x_j^2} \in L^1_{\text{loc}}(\Omega)$.

Then $u$ has a unique harmonic extension to $\Omega$.

Proof. With the aid of the above Lemma one sees easily that the assumptions of Theorem 3 are satisfied. \qed

5. Extension results for holomorphic functions

5.1. Below we give certain counterparts to two of Shiffman’s well-known extension results for holomorphic functions. For these results of Shiffman, see, among others, [31, 8, 9, 19].

5.2. First a counterpart to the following result:

**Shiffman’s theorem.** ([31], Lemma 3, p. 115, and [9], Theorem 1.1 (b), p. 703)

Let $\Omega$ be a domain in $\mathbb{C}^n$, $n \geq 1$. Let $E \subset \Omega$ be closed in $\Omega$ and let $H^{2n-1}(E) < +\infty$. If $f : \Omega \to \mathbb{C}$ is continuous and $f|\Omega \setminus E$ is holomorphic, then $f$ is holomorphic in $\Omega$.

Shiffman’s proof was based on coordinate rotation, on the use of Cauchy integral formula and on the cited result of Federer, the above Lemma.
For slightly more general versions of Shiffman’s result with different proofs, see [21], Theorem 3.1, p. 49, Theorem 3.5, p. 52, Corollary 3.7, p. 54, and [22], Theorem 3.1, p. 333, Corollary 3.3, p. 336.

Using again our above Theorem 1, or more directly Theorem 3, we get the following counterpart to Shiffman’s above result. See also our preliminary result [30], Theorem 3, p. 156.

**Theorem 4.** Suppose that $\Omega$ is a domain in $\mathbb{C}^n$, $n \geq 1$. Let $E \subset \Omega$ be closed in $\Omega$ and let $\mathcal{H}^{2n-1}(E) < +\infty$. Let $f = u + iv : \Omega \setminus E \to \mathbb{C}$ be holomorphic and such that the following conditions are satisfied:

(i) $f \in \mathcal{L}^1_{\text{loc}}(\Omega)$.

(ii) For each $j$, $1 \leq j \leq 2n$, $\frac{\partial^2 u}{\partial x_j^2} \in \mathcal{L}^1_{\text{loc}}(\Omega)$ and $\frac{\partial^2 v}{\partial x_j^2} \in \mathcal{L}^1_{\text{loc}}(\Omega)$.

(iii) For each $j$, $1 \leq j \leq 2n$, and for $\mathcal{H}^{2n-1}$-almost all $X_j \in \mathbb{R}^{2n-1}$ such that $E(X_j)$ is finite, the following condition holds:

For each $x^0_j \in E(X_j)$ there exist sequences $x^{0,1}_j, x^{0,2}_j \in (\Omega \setminus E)(X_j)$, $l = 1, 2, \ldots$, such that $x^{0,1}_j \to x^0_j, x^{0,2}_j \to x^0_j$ as $l \to +\infty$, and

(iii(a)) $\lim_{l \to +\infty} f(x^{0,1}_j, X_j) = \lim_{l \to +\infty} f(x^{0,2}_j, X_j) \in \mathbb{C},$

(iii(b)) $-\infty < \liminf_{l \to +\infty} \frac{\partial u}{\partial x_j}(x^{0,1}_j, X_j) = \limsup_{l \to +\infty} \frac{\partial u}{\partial x_j}(x^{0,2}_j, X_j) < +\infty$ and $-\infty < \liminf_{l \to +\infty} \frac{\partial v}{\partial x_j}(x^{0,1}_j, X_j) = \limsup_{l \to +\infty} \frac{\partial v}{\partial x_j}(x^{0,2}_j, X_j) < +\infty.$

Then $f$ has a unique holomorphic extension to $\Omega$.

**Proof.** It is sufficient to show that $u$ and $v$ have harmonic extensions $u^*$ and $v^*$ to $\Omega$. As a matter of fact, then $f^* = u^* + iv^* : \Omega \to \mathbb{C}$ is $C^\infty$ and thus a continuous function. Therefore the claim follows from Shiffman’s theorem or also from [21, 22].

Another possibility for the proof is just to observe that the in $\Omega \setminus E$ harmonic functions $u$ and $v$ have by Theorem 3 harmonic extensions $u^*$ and $v^*$ to $\Omega$. Since $u^*$ and $v^*$ are thus $C^\infty$ functions, the holomorphy of the extension $f^* = u^* + iv^*$ in $\Omega$ follows easily. $\square$

5.3. As a corollary we get a counterpart to another result of Shiffman, at least in some sense, namely the following concise result:

**Corollary 6.** ([29], Theorem 3, p. 51, [30], Theorem 4, p. 157) Suppose that $\Omega$ is a domain in $\mathbb{C}^n$, $n \geq 1$. Let $E \subset \Omega$ be closed in $\Omega$ and let $\mathcal{H}^{2n-1}(E) = 0$. Let $f : \Omega \setminus E \to \mathbb{C}$ be holomorphic and such that the following conditions are satisfied:

(i) $f \in \mathcal{L}^1_{\text{loc}}(\Omega)$,
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(ii) For each $j$, $1 \leq j \leq 2n$, $\frac{\partial^2 f}{\partial x_j^2} \in L^1_{\text{loc}}(\Omega)$.

Then $f$ has a unique holomorphic extension to $\Omega$.

The here referred result of Shiffman is the following:

**Another theorem of Shiffman.** ([31], Lemma 3, p. 115, and [9], Theorem 1.1 (c), p. 703) Let $\Omega$ be a domain in $\mathbb{C}^n$, $n \geq 1$. Let $E \subset \Omega$ be closed in $\Omega$ and let $\mathcal{H}^{2n-1}(E) = 0$. If $f : \Omega \setminus E \to \mathbb{C}$ is holomorphic and bounded, then $f$ has a unique holomorphic extension to $\Omega$.

Shiffman’s proof was also here based on coordinate rotation, on the use of Cauchy integral formula, on the already stated important result of Federer, the Lemma above, and on the following classical result of Besicovitch:

**Besicovitch’s theorem.** ([2], Theorem 1, p. 2) Let $D$ be a domain in $\mathbb{C}$. Let $E \subset D$ be closed in $D$ and let $\mathcal{H}^1(E) = 0$. If $f : D \setminus E \to \mathbb{C}$ is holomorphic and bounded, then $f$ has a unique holomorphic extension to $D$.

For slightly more general versions of Shiffman’s result with different proofs, see again [21], Corollary 3.2, p. 52, and [22], Corollary 3.3, p. 336.

5.4. **A previous, slightly related result.** Observe that, in addition to Corollary 6, also the following result holds:

**Theorem 5.** Suppose that $\Omega$ is a domain in $\mathbb{C}^n$, $n \geq 1$. Let $E \subset \Omega$ be closed in $\Omega$ and let $\mathcal{H}^{2n-1}(E) = 0$. Let $f : \Omega \setminus E \to \mathbb{C}$ be holomorphic. If for each $j$, $1 \leq j \leq 2n$, $\frac{\partial f}{\partial x_j} \in L^2_{\text{loc}}(\Omega)$, then $f$ has a holomorphic extension to $\Omega$.

The proof follows at once from the following, rather old result:

**Proposition 2.** ([13], Corollary 3.6, p. 301) Suppose that $\Omega$ is a domain in $\mathbb{C}^n$, $n \geq 1$. Let $E \subset \Omega$ be closed in $\Omega$ and let $\mathcal{H}^{2n-1}(E) = 0$. Let $f : \Omega \setminus E \to \mathbb{C}$ be holomorphic. If for some $p \in \mathbb{R}$,

$$\int_{\Omega \setminus E} |f(z)|^p \frac{1}{2} \sum_{j=1}^{n} \left| \frac{\partial f}{\partial z_j}(z) \right|^2 dm_{2n}(z) < +\infty,$$

then $f$ has a meromorphic extension $f^*$ to $\Omega$. If $p \geq 0$, then $f^*$ is holomorphic.

For related, partly previous and partly more general results, see [4], Theorem, p. 284, [13], Theorem 3.5, pp. 300-301, and [23], Theorem 3.1, pp. 925-926.
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