Rationality of conformally invariant local correlation functions on compactified Minkowski space

Nikolay M. Nikolov* and Ivan T. Todorov†

Institute for Nuclear Research and Nuclear Energy
72 Tsarigradsko Chaussee, BG-1784 Sofia, Bulgaria

October 23, 2000

Abstract

Rationality of the Wightman functions is proven to follow from energy positivity, locality and a natural condition of global conformal invariance (GCI) in any number $D$ of space-time dimensions. The GCI condition allows to treat correlation functions as generalized sections of a vector bundle over the compactification $\mathcal{M}$ of Minkowski space $M$ and yields a strong form of locality valid for all non-isotropic intervals if assumed true for space-like separations.

Introduction

The study of conformal (quantum) field theory (CFT) in four (or, in fact, any number of) space time dimensions (see, e.g., [3] [23] [14] [10] [11] [6] [7] [8] [13] as well as the reviews [20] [17] [18] and references therein) preceded the continuing excitement with 2-dimensional (2D) CFT (for a modern textbook and references to original work - see [2]). Interest in higher dimensional CFT was revived (starting in late 1997) by the discovery of the AdS-CFT correspondence in the context of string theory and supergravity (recent advances in this crowded field can be traced back from [1]). The present paper is more conservative in scope: we try to revitalize the old program of combining conformal invariance and operator product expansion with the general principles of quantum field theory using some insight gained in the study of 2D CFT models.

The main result of the paper (Theorems 3.1 and 4.1) can be formulated (omitting technicalities) as follows.

We say that a QFT (obeying Wightman axioms [16], [5]) satisfies global conformal invariance (GCI) if for any conformal transformation $g$ and for any set of points $(x_1, ..., x_n)$ in Minkowski space $M$ such that their images $(g x_1, ..., g x_n)$ also lie in $M$ the Wightman function $W(x_1, ..., x_n)$ stays invariant under $g$. We note that this requirement (stated more precisely in Sec.2) is stronger than the one used (under the same name) in [1]. An important tool in using GCI is the fact that mutually non-isotropic pairs of points form a single conformal orbit on

*e-mail: mitov@inrne.bas.bg
†e-mail: todorov@inrne.bas.bg
compactified Minkowski space (Proposition 1.1). Together with local commutativity for space like separations this implies the vanishing of the commutator \([\phi_1(x), \phi_2(x)]\) whenever the difference \(x_{12} = x_1 - x_2\) is non-isotropic (it follows from Lemma 3.2). We deduce from this strong locality property combined with the energy positivity condition that the Wightman functions are rational in \(x_{ij}\) (Theorem 3.1). Hilbert space positivity gives strong restrictions on the degrees of the poles of the resulting rational functions (Theorems 4.1, 4.2 and Proposition 4.3).

This result severely limits the class of local QFT satisfying GCI and makes feasible the construction of general conformal invariant correlation functions in such theories. That is illustrated on a simple example in the concluding Sec. 5. The interesting problem of exploiting the presence of conserved currents and the stress energy tensor and classifying their correlation functions is the subject of a separate study carried out currently in collaboration with Yassen Stanev.

**1 Conformally compactified Minkowski space \(\overline{M}\) and its complexification. The orbit of non-isotropic pairs of points in \(\overline{M} \times \overline{M}\)**

Conformally compactified Minkowski space \(\overline{M}\) of dimension \(D\) is a homogeneous space of the connected conformal group \(C_0\) (for even \(D\), \(C_0\) is isomorphic to \(SO_0(D,2)/\mathbb{Z}_2\); for odd \(D\), \(C_0 \cong SO_0(D,2)\)). Unless otherwise stated we shall suppose that the dimension is an arbitrary integer \(D \geq 1\). We shall use also in the exceptional cases \(D = 1, 2\) the group \(SO_0(D,2)\) of Möbius transformations (as the infinite dimensional conformal group present in those cases is not an invariance group of the vacuum state). The Minkowski space \(M\) is embedded as a dense open subset in \(\overline{M}\) in such a way that the isotropy relation in \(M \times M\) extends to a conformally invariant "isotropy relation" in \(\overline{M} \times \overline{M}\). More generally, for every element \(g \in SO_0(D,2)\) there exists a quadratic polynomial \(\omega(x,g)\) in the coordinates \(x\) in \(M \cong \mathbb{R}^{D-1,1}\) such that the pseudo-euclidean interval transforms as:

\[
(gx - gy)^2 = \frac{(x - y)^2}{\omega(x,g) \omega(y,g)},
\]

where \(x \mapsto gx\) is the nonlinear (coordinate) conformal action of \(SO_0(D,2)\) on \(M\) (with singularities, see Appendix A). The complement \(K_\infty := \overline{M} \setminus M\), the set of points at infinity, is an isotropic \((D - 1)\)-cone: there is a unique point \(p_\infty \in K_\infty\), the tip of the cone, such that \(K_\infty\) is the set of all points \(p\) in \(\overline{M}\) isotropic to \(p_\infty\). Thus the stabilizer \(C_\infty\) of \(p_\infty\) in \(C_0\) leaves \(M\) (and \(K_\infty\) ) invariant; it is the Poincaré group with dilations of \(M\) (also called the Weyl group).

**Proposition 1.1** Any pair \((p_0,p_1)\) of mutually non-isotropic points of \(\overline{M}\) can be mapped into any other such pair \((p'_0,p'_1)\) by a conformal transformation.

**Proof.** Due to the transitivity of the action of \(C_0\) there are elements \(g_0\) and \(g_0'\) of \(C_0\) which carry \(p_0\) and \(p'_0\) into the point \(p_\infty: g_0p_0 = g_0'p_0' = p_\infty\). Then the images \(g_0p_1\) and \(g_0'p_1'\) of the two other points will both belong to \(M\) (because the original pairs are mutually non isotropic) and can hence be moved to one another by a translation \(t\) (in \(C_\infty\)) which leaves \(p_\infty\) invariant: 
\[
g_0p_1 = tg_0p_1 .
\]
So the element \(g \in C_0\) which transforms the pair \((p_0,p_1)\) into \((p'_0,p'_1)\) is given by 
\[
g = (g_0')^{-1}tg_0 .
\]

\[\square\]
Remark 1.1 It is important to note that we are just dealing in Proposition 1.1 with pairs of points: a continuous time-like world line cannot be mapped into a space-like one by a conformal transformation. In fact, the connected conformal group preserves causal ordering on such lines—see [1] and [14]. □

We shall be primarily interested in this paper in the space-time bundle of (Fermi and) Bose fields of (half)integer dimension transforming under a representation of the finite covering \( C = \text{Spin}_0(D, 2) \) of the conformal group \( C_0 \) of \( \mathcal{M} \). (For even \( D \), \( C \) is a 4-fold covering of \( C_0 = \text{SO}_0(D, 2) / \mathbb{Z}_2 \); for odd \( D \) it is a 2-fold covering of \( C_0 = \text{SO}_0(D, 2) \).) The group \( C \) acts (transitively) on \( \mathcal{M} \) through its canonical projection on \( C_0 \) (its centre acting trivially).

We shall use the following atlas on \( \mathcal{M} \). Let \( x : M \cong \mathbb{R}^{D-1,1} \) be the standard Minkowski chart in \( \mathcal{M} \); for every \( g \in C \) we set \( M_{(g)} := g^{-1}M \) and introduce the coordinatization map \( x_{(g)} := x \circ g : M_{(g)} \cong \mathbb{R}^{D-1,1} \) thus obtaining an atlas \( \{ x_{(g)} ; g \in C \} \) over \( \mathcal{M} \). The transformation from the coordinates \( x_{(g)} \) to \( x_{(g')} \) is \( x_{(g')} = g' g^{-1} x_{(g)} \). (One can in fact show that \( \mathcal{M} \) can be covered by just 3 charts of this type.) We need also an atlas on \( \mathcal{M}^{\times n} \) in the study of \( n \)-point correlation functions. To this end we note that such an atlas is given by the diagonal subsystem of the Cartesian power of the above atlas:

\[
\left\{ (x_{1(\bar{g})}, \ldots, x_{n(\bar{g})}) : M_{(g)}^{\times n} \cong \mathbb{R}^{Dn} ; \; g \in C \right\}
\]

(to simplify notation we write here and in what follows \( \mathbb{R}^{D} \) instead of \( \mathbb{R}^{D-1,1} \)). Indeed, \( \bigcup_{g \in C} M_{(g)}^{\times n} = \mathcal{M}^{\times n} \), which is equivalent to the statement that for every set of points \( p_1, \ldots, p_n \in \mathcal{M} \) there exists \( g \in C \) such that \( gp_1, \ldots, gp_n \in \mathcal{M} \). The last statement can be proven by induction in \( n \) (see also the argument in Appendix C and the proof of Lemma 3.2).

Finally, let us introduce the complexification \( \mathcal{M}_c \) of \( \mathcal{M} \). It is needed because of the condition of energy positivity in QFT, which implies that the vector valued distribution \( F(x) = \phi(x)|0\rangle \) where \( \phi(x) \) is an arbitrary (local) Wightman field [12, 14] can be viewed as the boundary value of an analytic function \( F(x + iy) \) holomorphic in the (forward) tube domain \( T^+ \) where

\[
T^\pm = \{ x \pm iy ; \; x \in M , \; y \in V_+ \} , \quad V_\pm = \{ y \in M ; \; \pm y^0 > |y| \} .
\]

Clearly, \( T^\pm \subset \mathcal{M}_C \) and each of them is a homogeneous space of the (real) conformal group \( C \) [22], the stabilizer of a point being conjugate to the maximal compact subgroup \( \text{Spin} (D) \times \text{Spin} (2) \) of \( C \).

2 Global conformal invariance of Wightman functions

The Wightman functions, the vacuum expectation values of products \( \phi_1(x_1) \ldots \phi_n(x_n) \) of a multicomponent field \( \phi(x) \), are defined as tensor valued tempered distributions on \( M^{\times n} \cong \mathbb{R}^{Dn} \):

\[
\mathcal{W}_{1\ldots n}(x_1, \ldots, x_n) = \langle 0 | \phi_1(x_1) \phi_2(x_2) \ldots \phi_n(x_n) | 0 \rangle \in S'(M^{\times n}, F^{\otimes n}) .
\] (2.1)

where \( F \) is a finite dimensional (complex) vector space. In Eq.(2.1) and further we are using Faddeev’s shorthand for tensor products \( (\phi_1 \ldots \phi_n = \phi(x_1) \otimes \ldots \otimes \phi(x_n)) \).

We shall assume that \( C \) acts (locally) on \( M \times F \):

\[
M \times F \ni (x, v) \longmapsto (gx, \pi_x(g)v) \in M \times F \; \text{ iff } \; gx \in M
\]

where \( \pi_x(g) \in \text{Aut} (F) \), \( \pi_x(g_1 g_2) = \pi_{g_2 x}(g_1) \pi_x(g_2) \)

\[ (2.2) \]
\(\pi_x(g)\) being a \textit{single-valued} real analytic function defined for all pairs \(x \in M, g \in C\) for which \(gx \in M\) (thus the multiplicity of \(\pi_x\) in \((2.2)\) holds under the provision that both \(g_2x\) and \(g_1g_2x\) belong to \(M\)). Moreover, we will assume that if \(g\) is a translation, \(t_a x = x + a\), then \(\pi_x\) acts trivially on \(F\), i.e., \(\pi_x(t_a) = id\). Note that these conditions are satisfied by the usually considered local induced representations of \(C\) (see \[3\] or Chapter 2 of \[14\]).

\textbf{Example 2.1} Let \(\phi\) be a vector \textit{current} \(j^\mu(x)\); then \(\text{dim } F = D\). For \(x^2 \neq 0\) the action of the \textit{Weyl reflection} \(w \in C_0\) is given by:

\[
w: (x^\mu, j^\mu) \mapsto \left(\frac{x^0}{x^2}, \frac{-x}{x^2}; -\frac{r^\mu_\nu(x)}{(x^2)^d} j^\nu\right), \quad r^\mu_\nu(x) = \delta^\mu_\nu - 2 \frac{x^\mu x^\nu}{x^2}, \tag{2.3}\]

where \(d\) is the conformal dimension of the current \(j\). The result differs by a space reflection from the \textit{conformal inversion}, \(I_r: (x^\mu, j^\mu) \mapsto \left(\frac{x^0}{x^2}, \frac{r^\mu_\nu(x)}{(x^2)^d} j^\nu\right)\) (which does not belong to \(C_0\)). It is easy to verify that \(I_r\), and hence \(w\), leave invariant the current conservation law iff \(d = D - 1\); indeed, this follows from the identity

\[
\partial_\mu \left(\frac{r^\mu_\nu(x)}{(x^2)^d} j^\nu \left(\frac{x}{x^2}\right)\right) = \frac{1}{(x^2)^{d+1}} \partial_\nu j^\nu \left(\frac{x}{x^2}\right) + 2 \frac{d + 1 - D}{(x^2)^{d+1}} x_\nu j^\nu \left(\frac{x}{x^2}\right). \tag{2.4}\]

Note that the function \(\pi_x(w)\) (as well as \(\pi_x(I_r)\)) is well defined and single-valued (for all \(x\) such that \(x^2 \neq 0\)), iff \(d\) is an integer. \(\Box\)

\textbf{Remark 2.1} The condition for a trivial action of the translations in \(M\) onto \(F\) - i.e. \(\pi_x(t_a) = id\) - is not restrictive. If it is not satisfied we can pass to an equivalent action:

\[
\pi'_x(g) := \pi_{gx}(t_{gx}^{-1}) \pi_x(g) \pi_0(t_x) = \pi_0(t_{gx}^{-1}gt_x) \tag{2.5}\]

for which \(\pi'_x(t_a) = id\) does hold. It also follows from \((2.5)\) that the growth of \(\pi_x(g)\) for \(x \to \infty\) is not more than polynomial. \(\Box\)

The fact that the map \((2.2)\) is single valued outside its singularities allows to treat \(\pi_x(g)\) as a cocycle on a fibre bundle over \(\overline{M}\) with a standard fibre \(F\). More generally, for every \(n = 1, 2, 3, \ldots\) and the atlas \((2.2)\) we have a cocycle:

\[
(x(g), \mathcal{W}) \mapsto (x(g') = g'g^{-1}x(g), \pi_{(n)}(x(g); g'g^{-1}) \mathcal{W}) \in M^{\times n} \times F^{\otimes n}
\]

where:

\[
gx := (gx_1, \ldots, gx_n) ; \quad \pi_{(n)}(x; g) := \pi_{x_1}(g) \otimes \ldots \otimes \pi_{x_n}(g)
\]

for \(x = (x_1, \ldots, x_n)\) and \(g \in C\) \tag{2.6}\]

which gives a fibre bundle \(E_n\) over \(\overline{M}^{\times n}\) with a standard fibre \(F^{\otimes n}\). It allows to consider the Wightman functions \(\mathcal{W}_{1\ldots n}\) as generalized sections \(\mathcal{W}\) of \(E_n\) which admit for each chart \(M^{\times n}_{(g)}\) of the atlas \((2.2)\) a coordinate expression \(\mathcal{W}_{(g)}(x_1, \ldots, x_n) \in \mathcal{D}'(\mathbb{R}^n, F^{\otimes n})\) \((\mathcal{D}' being the space of distributions over test functions of compact support)\). These coordinate expressions should satisfy the consistency condition

\[
\mathcal{W}_{(g')}(g'g^{-1}x_1, \ldots, g'g^{-1}x_n) = \pi_{(n)}(x; g'g^{-1}) \mathcal{W}_{(g)}(x_1, \ldots, x_n) \tag{2.7}\]

\textbf{4}
be viewed as a multiplicator (regular in the neighbourhood of $(x ; g^{-1})$). Because of the consistency condition (2.7) this is equivalent to the following requirement.

\[ \pi(n) (x; g^{-1}) \] should be interpreted locally: for a fixed $g \in C$, \[ \pi(n) (x; g^{-1}) \] is a multiplicator. We can further define the space $D_n$ of test sections for a generalized section $W$, which is actually the space of all smooth sections of the fibre bundle over $\mathcal{M}^{\times n}$ with a standard fibre $F \otimes^\infty n$ and the cocycle:

\[
\left( x(g) ; f \right) \mapsto \left( x(g') = g'g^{-1}x(g) , \pi(n) (x(g); g'g^{-1}) f \right) \in M^{\times n} \times F_n
\]

\[
\pi(n) (x; g) := \pi(n) (x ; g)^{-1} J (x ; g)^{-1}
\]

and $J$ is the Jacobian of the transformation $x \mapsto gx$. (For the concepts of test functions and distributions on a manifold see, also, [21].)

**Proposition 2.1** (a) Let the distributions $W(g)_i$ be defined for any $g \in C$ and satisfy the consistency condition (2.7). Then there exists a unique linear functional $W$ on $D_n$ that is a generalized section of the vector bundle $E_n$ with coordinate expressions $W(g)_i$.

(b) Each $W(g)_i$ actually belongs to the subspace of tempered distributions $S' (\mathbb{R}^n, F_n)$ of $D' (\mathbb{R}^n, F_n)$.

**Sketch of proof.** (a) Since $\mathcal{M}^{\times n}$ is compact there exists a finite partition of unity $1 = \sum_{i=1}^k a_i$, where $a_i$ is a smooth function of compact support in some charts $\mathcal{M}^{\times n}$ ($i = 1, ..., k$). The linear functional $W$ is then uniquely defined by

\[
\langle W, f \rangle := \sum_{i=1}^k \langle W(g)_i ; (a_i f) (g_i)(x) \rangle,
\]

where $(a_i f) (g_i)(x)$ is the coordinate expression of the test section $a_i f$.

(b) Getting a test section $f$ with a coordinate expression $f(g)_i (x) \in S (\mathbb{R}^n, F_n)$, we obtain this statement by Eq. (2.9) because of the continuity of the linear maps:

\[
S (\mathbb{R}^n, F_n) \ni f(g)_i (x) \mapsto (a_i f) (g_i)(x) \in D (\mathbb{R}^n, F_n).
\]

\[ \square \]

The action (2.2) not only defines a cocycle on $E_n$: it also gives rise to a natural action of the conformal group $C$ on $E_n$ that is linear on the fibres. A generalized section $W$ of $E_n$ is called *conformally invariant* if it does not change under the action of $C$. This is equivalent to requiring that $W$ has the same coordinate expression in every chart:

\[ W(g) (x_1, ..., x_n) = W(x_1, ..., x_n) \] .

Because of the consistency condition (2.7) this is equivalent to the following requirement.

**GCI** *Global conformal invariance*

\[
\pi_x(g)^{-1} \otimes \ldots \otimes \pi_x(g)^{-1} W(gx_1, ..., gx_n) = W(x_1, ..., x_n) \text{ for } gx_1, ..., gx_n \in M.
\]

This implies, in particular, (unrestricted) Poincaré and dilation invariance. We recall that Eq. (2.12) should be interpreted locally: for a fixed $g \in C$, $\pi_x(g)^{-1} \otimes \ldots \otimes \pi_x(g)^{-1}$ should be viewed as a multiplicator (regular in the neighbourhood of $(x_1, ..., x_n) \in M^{\times n}$) while $g$

\( (x_1, \ldots, x_n) \mapsto (gx_1, \ldots, gx_n) \) is a local diffeomorphism. (That is, both sides of (2.12) are to be smeared with test functions with support in the above neighbourhood.)

Thus we have an one-to-one correspondence between the tempered distributions
\( W(x_1, \ldots, x_n) \in \mathcal{S}'(\mathbb{R}^{Dn}, \mathbb{C}^{\otimes n}) \) satisfying the condition (GCI) and the conformal invariant generalized sections of the bundle \( E_n \).

Remark 2.2 Local conformal invariance requires the existence of a continuous curve \( g(\tau) \in C \), such that \( g(0) = 1 \), \( g(1) = g \) and \( (g(\tau)x_1, \ldots, g(\tau)x_n) \in M^{\times n} \) for all \( \tau \in [0,1] \); it is equivalent to invariance under infinitesimal conformal transformations. The (GCI) condition (2.12) does not demand the existence of such a curve; moreover, if several curves of this type exist it says that \( \pi_x(g) \) is independent of the path \( g(\tau) \) connecting \( g \) with the group unit. The 2-point function of a hermitean scalar field of dimension \( d \),

\[
W_d(x_{12}) = \frac{\Gamma(d)}{(4\pi)^{D/2}} \left( \frac{4}{x_{12}^2 + i0x_{12}^0} \right)^d = \frac{2\pi}{\Gamma(d+1-\frac{D}{2})} \int \theta(p^0) \frac{\theta(-p^2) e^{ipx}}{(-p^2)^{D/2-d}} d^Dp \quad (2.13)
\]

(\( x_{12} = x_1 - x_2 \), satisfies all Wightman axioms (including positivity for any \( d \geq \frac{D}{2} - 1 \), \( D > 2 \)) as well as local conformal invariance, however, it only obeys (GCI) for positive integer \( d \), because of the requirement of singlevaluedness of \( \pi_x(g) \) inherent in (GCI). Thus (GCI) is indeed stronger than local conformal invariance. \( \square \)

The invariance of the \( i0x_1^0 \) prescription in (2.13) is implied by the following more general statement.

**Proposition 2.2** Any product of 2-point like functions of type (2.13) satisfies the invariance condition (GCI):

\[
[\omega(x_1, g)]^{-\mu_1} \cdots [\omega(x_n, g)]^{-\mu_n} \prod_{1 \leq j < k \leq n} \left( (gx_j - gx_k)^2 + i0 (gx_j^0 - gx_k^0) \right)^{-\mu_{jk}} = \\
= \prod_{1 \leq j < k \leq n} (x_{jk}^2 + i0x_{jk}^0)^{-\mu_{jk}}, \quad (2.14)
\]

where \( \mu_{jk} \in \mathbb{Z} \), \( \mu_k = \sum_{j=1}^{k-1} \mu_{jk} + \sum_{l=k+1}^{n} \mu_{kl} \), and the factors \( \omega(x, g) \) are introduced in (1.1).

Away from the singularities, for \( x_{ij}^2 \neq 0 \), the statement follows from Eq. (1.1). Only establishing conformal invariance of the \( "+i0x_{ij}^0" \) prescription poses a problem. The somewhat technical proof of this statement is relegated to Appendix B. It is based on the fact that \( (x_{jk}^2 + i0x_{jk}^0)^{-\mu} \) is a limit of a holomorphic function of \( x_j - \zeta_k \) for \( T_+ \ni \zeta_k \to x_k \) and of the conformal invariance of the forward tube (in the transformation of \( \zeta_k \) in \( x_j - \zeta_k \mapsto gx_j - g\zeta_k \)). \( \square \)

**Example 2.2** Let the points \( x_1, x_2 \) of \( M \) satisfy \( x_1^2 > 0 \) and \( x_1^2 x_2^2 < 0 \). Then \( wx_1, wx_2 \in M \) (where \( wx \) is defined in Example 2.1) but there is no continuous curve \( g(\tau) \in C \) that relates \( (x_1, x_2) \) with \( (wx_1, wx_2) \) in \( M \times M \). Indeed, \( (wx_1 - wx_2)^2 = \frac{x_1^2 - x_2^2}{x_1^2} < 0 \); if there were \( x_i(\tau) = g(\tau)x_i \in M \), \( i = 1, 2 \) which connect continuously \( x_i \) with \( wx_i \) then there would have been a point \( 0 < \tau_0 < 1 \) such that \( (x_1(\tau_0) - x_2(\tau_0))^2 = 0 \) which is impossible (cf. Remark 1.1). \( \square \)
3 Wightman axioms and global conformal invariance imply rationality

We will recall first some general properties of Wightman functions (2.1) (see for more detail [1] or [3]). Thus \( W \) is a tempered distribution with values in the \( n \)-fold tensor power \( F^\otimes n \) of a finite dimensional (complex) vector space \( F \). For complex fields the vector \( \phi \in F \) is assumed to contain along with each component \( \phi_n \) also its conjugate \( \phi_n^* \). The splitting of the fields into bosonic and fermionic ones amounts to defining a \(*\)-invariant \( \mathbb{Z}_2 \)-grading \( F = F_0 \oplus F_1 \).

Then the automorphisms \( \pi_x (g) \) of \( F \) in the conformal action (2.2) will preserve the conjugation and the \( \mathbb{Z}_2 \)-grading of \( F \). The implications of translation invariance and energy positivity and of locality can be summed up in the following conditions for \( W(x_1, ..., x_n) \).

\((TS)\) Translation invariance and spectral condition. The Fourier transform of

\[
W(x_1, ..., x_n) = W(x_1 - x_2, ..., x_{n-1} - x_n)
\]

(omitting the indices \( 1...n \) in both sides),

\[
\hat{W}(q_1, ..., q_{n-1}) = \int_{M^{x(n-1)}} W(y_1, ..., y_{n-1}) e^{-i(q_1 y_1 + ... + q_{n-1} y_{n-1})} d^n y_1 ... d^n y_{n-1},
\]

has support in the product of (closed) future light cones in \( M \):

\[
supp \hat{W} \subseteq \left( \nabla^+ \right)^{x(n-1)}, \quad \nabla^\pm = \{ q \in M ; \pm q^0 \geq |q| \}.
\]

This is the relativistic (Lorentz invariant) form of energy positivity.

\((L)\) Locality

\[
W_1 ... i i_{i+1} ... n(x_1, ..., x_i, x_{i+1}, ..., x_n) = \epsilon_i i_{i+1} W_1 ... i_{i+1} i ... n(x_1, ..., x_{i+1}, x_i, ..., x_n)
\]

for \( x_i i_{i+1}^2 > 0 \)

(3.4)

\(x_i i_{i+1} = x_i - x_{i+1} \); where \( \epsilon_{ij} (i \neq j) \) is a sign factor, \( \epsilon_{ij} = -1 \) if both \( i \) and \( j \) refer to Fermi fields (elements of \( F_1 \)) and \( \epsilon_{ij} = 1 \) otherwise. In other words Fermi fields anticommute among themselves while, Bose fields commute with both Bose and Fermi fields for space-like separations.

Permutation of field indices accompanied by the sign factors \( \epsilon_{ij} \) give rise to an action of the symmetric group \( S_n \) on \( F^\otimes n \). An \( n \)-point function \( F : M^{x n} \rightarrow F^{\otimes n} \) is said to be \( \mathbb{Z}_2 \)-symmetric if it is invariant under this action combined with the corresponding permutations of the coordinates.

\textbf{Remark 3.1} The restriction (2.1) on the class of distributions is not essential for theories satisfying \((GCI)\). Indeed, if we assume \( W \in \mathcal{D}' \) (or \( \hat{W} \in \mathcal{D}' \)) then dilation invariance (which has a similar form in coordinate and in momentum space) implies \( W \in \mathcal{S}'(2.1) \). \( \square \)

\textbf{Theorem 3.1} The tempered distribution \( W_{1...n}(x_1, ..., x_n) \) satisfies conditions \((TS)\), \((L)\) and \((GCI)\) iff it can be expressed in terms of a rational function of the type

\[
W_{1...n}(x_1, ..., x_n) = P_{1...n}(x_1, ..., x_n) \prod_{1 \leq j < k \leq n} (x_{jk}^2 + i0x_{jk}^0)^{-r_{jk}^a},
\]

\[
(x_{jk} = x_j - x_k).
\]

(3.5)
Appendix C. To complete the proof of Lemma 3.2 it remains to choose \( h \in C \) such that \( g \) is constant coefficients; hence \( \text{supp} g \subseteq M \) which is possible since \( M \) is an open set in \( \overline{M} \) and \( C \) acts continuously on \( \overline{M} \). Hence, \( g = hg' \) satisfies the conclusion of Lemma 3.2.

We continue with the proof of Theorem 3.1. Assume first that \( W \) satisfies \((TS), (L) \) and \((GCI)\). Lemma 3.2 and \((GCI)\) imply the locality property \((L)\) whenever \( x_i x_{i+1} \neq 0 \). Then \( W \) will be \( \mathbb{Z}_2\)-symmetric in the domain \( U \subset M^{\times n} \) of all \((x_1, ..., x_n)\) which are mutually non-isotropic. Since \( W_{1...n} \) is a (tempered) distribution its singularities have a finite order. Therefore, there are integers \( \mu_{ij}^n \) such that

\[
P_{1...n}(x_1, ..., x_n) = \left( \prod_{1 \leq i < j \leq n} (x_{ij}^2)^{\mu_{ij}^n} \right) W_{1...n}(x_1, ..., x_n)
\]

is a translation invariant distribution that is \( \mathbb{Z}_2\)-symmetric in the entire Cartesian product space \( M^{\times n} \). The Fourier transform of \( P_{1...n}(x_1, ..., x_{n-1}) = P_{1...n}(x_1, ..., x_n) \)

\[
\hat{P}_{1...n}(q_1, ..., q_{n-1}) = \int_{M^{\times (n-1)}} P_{1...n}(y_1, ..., y_{n-1}) e^{-i(q_1 y_1 + ... + q_{n-1} y_{n-1})} d^n y_1 ... d^n y_{n-1}
\]

is obtained from \( \hat{W}_{1...n}(q_1, ..., q_{n-1}) \) by the action of a differential operator in \( q_1, ..., q_{n-1} \) with constant coefficients; hence \( \text{supp} \hat{P}_{1...n}(q_1, ..., q_{n-1}) \subseteq \text{supp} \hat{W} \subseteq \overline{(V^+)^{\times (n-1)}} \). On the other hand, the total \( \mathbb{Z}_2\)-symmetry of \( P \) implies

\[
P_{1...n}(x_1, ..., x_n) = P_{n...1}(x_n, ..., x_1) \Rightarrow P_{1...n}(y_1, ..., y_{n-1}) = \epsilon P_{n...1}(-y_{n-1}, ..., -y_1)
\]

where \( \epsilon = \prod_i \epsilon_{i+1} \). Hence \( \hat{P}_{1...n}(q_1, ..., q_{n-1}) = \epsilon \hat{P}_{n...1}(-q_{n-1}, ..., -q_1) \) implying \( \text{supp} \hat{P}_{n...1} \subseteq \overline{(V^-)^{\times (n-1)}} \). Since the order of field labels is arbitrary and \( V^+ \cap V^- = \{0\} \) we conclude that \( \hat{P} \) is a polynomial and the same is true for \( P \).

If we combine this result with the fact that \( W \) admits an analytic continuation \( W(\zeta_1, ..., \zeta_{n-1}) \) in the backward tube \( (T_-)^{\times (n-1)} \) as a consequence of energy positivity then we end up with Eq. (3.3). Actually, both sides of (3.3) are equal in the domain \( U \) introduced above (in accord
with (3.3)). On the other hand, they have analytic continuations in the tube domain, which thus must be equal, too. Clearly, the rational function \( R_{1...n}(x_1, ..., x_n) \) so obtained is fully \( \mathbb{Z}_2 \)-symmetric and conformally invariant.

Conversely, if \( \mathcal{W} \) is given by (3.3) for a completely \( \mathbb{Z}_2 \)-symmetric and conformally invariant \( R_{1...n}(x_1, ..., x_n) \), then conditions (TS) and (L) follow as \( \mathcal{W} \) is the boundary value of a symmetric analytic function in the tube domain. Condition (GCI) is a corollary of Proposition 2.2 since \( \mathcal{P}_{1...n} \) is a regular multiplicator. □

Remark 3.2 By the proof of Theorem 3.1 it is clear that the (GCI) condition is just needed for extending the locality property (3.4) for all non-isotropic separations. The rationality of the Wightman functions (of the type (3.3) without conformal invariance) is then equivalent to this strong locality property and energy positivity (TS). □

4 Constraints on pole degrees coming from Wightman positivity

Up to this point we did not use Hilbert space (Wightman) positivity. Taking it into account allows to deduce that the order of poles in correlation functions are uniformly bounded with respect to the number \( n \) of points.

Theorem 4.1 Let \( \phi(x) = \phi_a(x) \) be a (multicomponent) field satisfying Wightman axioms as well as the condition (GCI) for Wightman functions. Then the orders of the poles of the rational functions \( \mathcal{W}(x_1, ..., x_n) \) are uniformly bounded, i.e., the integers \( \mu^a_{jk} \) in (3.5) can be chosen independent of \( n \).

Proof. Consider the vector valued distributions

\[
\Phi_{1...k}(x_1, ..., x_k) = \phi_1(x_1) ... \phi_k(x_k) |0\rangle
\]

It defines a continuous (finite order) map of the test function space \( \mathcal{S}(M^{\times k}) \) into the Hilbert space \( \mathcal{H} \) (16 Sec.3.3 or 3 Sec.3.3B). Moreover, \( \Phi \) is the boundary value of an analytic vector valued function \( \Phi_{1...k}(x_1 + iy_1, ..., x_k + iy_k) \) holomorphic for \( y_k \in V^+, y_{i+1} \in V^-, i = 1, ..., k - 1 \). For \( k = 2 \) it follows from Theorem 3.1 and from Reeh-Schlieder theorem (16, Theorem 4-2) that \( \Phi_{12}(x_1, x_2) = \langle 0 | \Phi_{12}(x_1, x_2) \rangle \) for any pair of mutually non-isotropic \((x_1, x_2)\). Taking into account the finite order of the distribution \( \Phi_{21} \) (in \( \mathcal{S}' \)) we deduce that there is a positive integer \( \mu \) such that the distributions

\[
f_{12}(x_1, x_2) = \langle x_2^{\mu} \psi | \Phi_{12}(x_1, x_2) \rangle = f_{21}(x_2, x_1) \quad \text{for all} \quad \psi \in \mathcal{H}, \quad (x_1, x_2) \in M^{\times 2}
\]

are \( \mathbb{Z}_2 \)-symmetric on \( M \times M \). Choosing, in particular, \( \langle \psi | = \langle 0 | \psi_1(\zeta_1), ..., \psi_k(\zeta_k) \rangle \) with \( \{\zeta_j = x_j + iy_j ; \ 1 \leq j \leq k\} \) in the above tube domain of analyticity and substituting \( \Phi_{12}(x_1, x_2) \) by \( \Phi_{k+1 \ k+2}(x_{k+1}, x_{k+2}) \) we deduce that the order of the pole of \( \mathcal{W}_{1...k+2}(x_1, ..., x_{k+2}) \) in \( x_{k+1 \ k+2} \) does not exceed \( \mu \) and is hence independent of \( k \). Locality then implies that this holds for any pair of arguments. □

We proceed now to estimate (give a realistic upper bound) of the value of \( \mu \) in (14.2) in the important special case of 4 dimensional space time, in which the (4-fold covering of the) conformal group \( C \) coincides with a group of \( 4 \times 4 \) pseudounitary matrices:

\[
C = Spin_0(4, 2) = SU(2, 2) = \{u \in SL(4, \mathbb{C}) \ , \ u\beta u^* = \beta \}
\]
where $\beta := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

We shall assume that our fields $\phi$ transform under elementary induced representations of $C$ (see [8] or Sec.2A of [17]). This means, in particular, that $\pi_x (g)$ of Eq. (4.2) provides an $x$-independent finite dimensional irreducible representation (IR) of the quantum mechanical Lorentz group $SL(2, \mathbb{C})$ with dilations:

$$\mathbb{R}^+ \times SL(2, \mathbb{C}) = \left\{ \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \in SU(2, 2) \ , \ \rho = \det A > 0 \right\} \ .$$

(For $x = x^0 \mathbf{1} + x^\sigma$, where $\sigma_1$, $\sigma_2$, $\sigma_3$ are the Pauli matrices we have $\rho A x = A x A^*$ where $(Ax)^\mu = A^\nu x^\nu$ is a proper Lorentz transformation.) It follows that the representation $\pi (g_\rho)$ of the dilation subgroup $g_\rho : x \mapsto \rho x$ ($\rho > 0$) is scalar:

$$g_\rho : \phi(x) \mapsto \pi_x (g_\rho)^{-1} \phi(\rho x) = \rho^d \phi(\rho x) \ , \ \ \ d > 0 \ .$$

The exponent $d$ is called the (conformal) dimension of $\phi$. Labeling, as customary, the IR of $SL(2, \mathbb{C})$ by a pair of (non-negative) half integers $(j_1, j_2)$ $(2j_1$ and $2j_2$ giving the numbers of undotted and dotted indices of the spin-tensor $\phi$) one proves that for singlevalued $\pi_x (g)$ the (positive) number $d + j_1 + j_2$ should be an integer. (In other words, $d$ should be an integer for Bose fields, for which $j_1 + j_2 \in \mathbb{Z}_+$, and half odd integer for Fermi fields.)

Let us note some implications for a Wightman QFT coming from the additional condition (GCI) (in 4 dimension). First, the Wightman functions have an analytic continuation in the domain of all mutually-nonisotropic points of $M_C$. They are rational and conformal invariant, and in particular their euclidean restrictions, the Euclidean Green functions, satisfy the condition of weak conformal invariance studied by Lüscher and Mack in [3]. Therefore ([3], Prop.1) there exists a unitary representation in the Hilbert space $\mathcal{H}$ of physical states, of the quantum-mechanical conformal group $\tilde{C}$, the universal covering of $C$. Thus (following [4]) $\mathcal{H}$ can be decomposed in a direct sum or integral of irreducible representation spaces of $\tilde{C}$.

It was shown by Mack in [3] that all irreducible unitary representations of $\tilde{C}$ with positive energy are field representations acting on

$$\mathcal{H}_{(j_1, j_2; d)} = \text{Span} \left\{ \phi^{(j_1, j_2; d)} (x) |0\rangle \right\} \ ,$$

where $\phi^{(j_1, j_2; d)} (x)$ is a field which transforms under the elementary induced representation of $\tilde{C}$ of weight $(j_1, j_2; d)$. In our case, because of the rationality and the conformal invariance on $M$ for the Wightman functions, it follows that in the decomposition of $\mathcal{H}$ take part only such $\mathcal{H}_{(j_1, j_2; d)}$ for which $d + j_1 + j_2$ is an integer. Thus the decomposition of $\mathcal{H}$ will be a direct sum:

$$\mathcal{H} = \mathbb{C} |0\rangle \oplus \bigoplus_{d + j_1 + j_2 \in \mathbb{N}} \left( \mathcal{N}_{(j_1, j_2; d)} \otimes \mathcal{H}_{(j_1, j_2; d)} \right) \ ,$$

where the Hilbert space $\mathcal{N}_{(j_1, j_2; d)}$ gives the multiplicity of $\mathcal{H}_{(j_1, j_2; d)}$ in $\mathcal{H}$.

Energy positivity and unitarity restrict each dimension $d$ according to the following result:
Theorem 4.2 ([3]; also see Theorem 3.18 of [17]). If $\phi$ is an elementary conformal field of weight $(j_1, j_2; d)$ then requirement (TS) and Wightman positivity imply

$$d \geq j_1 + j_2 + 1 \quad \text{for} \quad j_1j_2 = 0 ; \quad d \geq j_1 + j_2 + 2 \quad \text{for} \quad j_1j_2 > 0 .$$

(4.8)

The proof is based, in part, on an analysis of the positivity properties of the conformally invariant 2-point function

$$\langle 0 | \phi^{*}_{j_2j_1}(x) \phi_{j_1j_2}(y) | 0 \rangle = \frac{H_{2j_1+2j_2}(x-y)}{(x-y)^2 + i0 (x^0 - y^0))^{d+j_1+j_2}}$$

(4.9)

where $H_n(x)$ is a tensor-valued homogeneous harmonic polynomial of degree $n$ that is determined (up to a normalization constant) from conformal invariance. □

Example 4.1 If $J^\mu(x)$ is a conserved current in $D$ dimension then the 2-point function

$$\langle 0 | J^\mu(x_1) J_\nu(x_2) | 0 \rangle$$

is proportional to $(x_{12}^2 + i0 x_{12}^0)^{-D} (x_{12}^2 \delta^\mu_\nu - 2x_{12}^\mu(x_{12})_\nu)$. The harmonic polynomial $H_{2l}^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_l}(x)$ appearing in the 2-point function of a rank $l$ symmetric traceless tensor is written as a symmetrized product of factors of the type $x_{12}^\mu \delta^\nu_\mu - 2x_{12}^\mu(x_{12})_\nu$ with subtracted traces. □

Proposition 4.3 Let a system of fields (in dimension $D = 4$) satisfy the conditions of Theorem 4.1. Let $\phi(x)$ and $\psi(x)$ be two fields in this system which are transforming under elementary induced representations of $C$ of weights $(j_1'; j_2'; d')$ and $(j_1'', j_2''; d'')$ respectively. Then the pole degree $\mu$ of $((x-y)^2 + i0 (x^0 - y^0))^{-\mu}$ in any Wightman function $\langle 0 | \phi(x) \psi(y) \rho(y) | 0 \rangle$ has the upper limit:

$$\mu \leq \left[ \frac{d' + j_1' + j_2' + d'' + j_1'' + j_2''}{2} - \frac{1 - \delta_{j_1'j_1''} \delta_{j_2'j_2''} \delta_{x\nu}}{2} \right] ,$$

(4.10)

where $[a]$ stands for the integer part of the real number $a$ (i.e. the maximal $n \in \mathbb{Z}$ for which $n \leq a$).

Proof. As it was pointed out in the proof of Theorem 4.1, $\mu$ is the order of the vector-valued distribution $\phi(x) \psi(y) | 0 \rangle$. Then because of the decomposition ([17]) of the physical Hilbert space $\mathcal{H}$, $\mu$ does not exceed the order in $x - y$ of the two-point function $\langle 0 | \phi(x) \psi(y) | 0 \rangle$, or the maximal order of possible (non-zero) three-point conformal invariant Wightman functions $\langle 0 | \phi(x) \psi(y) \phi^{(j_2', j_1' ; d')} (z) | 0 \rangle$. The two-point function of $\phi$ and $\psi$ may be non-zero only for $(j_1', j_2') = (j_2'', j_1'')$, $d' = d''$ and then it saturates the upper limit ([10]), according to (4.3). It remains to verify (4.10) for the orders $\mu_{(j_1', j_2'; d)}$ of the pole in $x - y$ of the three-point functions $\langle 0 | \phi(x) \psi(y) \phi^{(j_2', j_1' ; d')} (z) | 0 \rangle$. We will use the results in [17] (Lemma 10) for the general form of such three-point functions. Thus we obtain:

$$\mu_{(j_1', j_2'; d)} \leq \frac{1}{2} \left( d' + d'' - d + L_{j_1, j_2} \right) ,$$

(4.11)

where $L_{j_1, j_2}$ is the maximal of the integers $l$ for which the $SL(2, \mathbb{C})$ representation $(\frac{l}{2}, \frac{l}{2})$ occurs in the triple tensor product

$$(j_1', j_2') \otimes (j_1'', j_2'') \otimes (j_1, j_2)$$

(4.12)
(when such \( l \) does not exist then the three-point function must be zero). Since \( (j'_1 + j''_1 + j_1, j'_2 + j''_2 + j_2) \) is the maximal weight occurring in the product (4.12), then when \( L_{j_1j_2} \) exists it will be equal to:

\[
L_{j_1j_2} = 2\min \{ j'_1 + j''_1 + j_1, j'_2 + j''_2 + j_2 \}.
\]

(4.13)

This will certainly happen for when \( j'_1 + j''_1 + j_1 = j'_2 + j''_2 + j_2 \). Assume, for the sake of definiteness that

\[
\frac{1}{2} L_{j_1j_2} = j'_1 + j''_1 + j_1 \leq j'_2 + j''_2 + j_2 ;
\]

(4.14)

then from Theorem 4.2 (Eq. (4.8)) and Eqs. (4.11), (4.14) we obtain:

\[
\mu_{(j_1,j_2;d)} \leq \frac{1}{2} (d' + d'' + j'_1 + j''_1 + j'_2 + j''_2) - \frac{1 + \theta_{j_1j_2}}{2} ,
\]

(4.15)

where \( \theta_0 = 0 \), \( \theta_{ij} = 1 \) for \( ij > 0 \). We further need to maximize \(-\frac{1 + \theta_{j_1j_2}}{2}\) or to minimize \(\theta_{j_1j_2}\). If \( j'_1 + j''_1 + j_1 = j'_2 + j''_2 + j_2 \) then we can choose \( (j_1, j_2) = (j, 0) \) or \((0, j)\) when \( \theta_{j_1j_2} = 0 \). So we obtain the upper limit (4.10) using the condition \( \mu \in \mathbb{Z} \) (owing to the rationality of Wightman functions).

\[\square\]

**Corollary 4.4** Under the assumptions of Proposition 4.3 for \( \phi = \psi^* \) each truncated Wightman function \( \langle 0 | \ldots | \psi^*(x) \ldots | \psi(y) \ldots | 0 \rangle^T \) will have a strictly smaller power \( \mu \) of the pole in \( (x - y)^2 + i0 (x^0 - y^0) \) than the 2-point function \( \langle 0 | \psi^*(x) \psi(y) | 0 \rangle \). If the weight of \( \psi \) is \( (j_1, j_2 ; d) \) then

\[
\mu \leq d + j_1 + j_2 - 1 .
\]

(4.16)

**Proof.** It follows from the definition of truncated functions (5 Sec.3.5C) that the order of the pole in \( (x - y)^2 \) does not exceed the order of the vector valued distribution \( (1_H - |0\rangle \langle 0|) \psi^*(x) \psi(y) | 0 \rangle \). Hence, we only need to take into account the non-vacuum contributions in the decomposition (4.7). These are determined by the 3-point functions. The estimate (4.16) then follows from Eq. (4.13).

\[\square\]

The following example illustrates the case of \( \phi \neq \psi^* \) (when the 2-point function of the fields in the product vanishes).

**Example 4.2** Let \( \psi \) be the free massless Dirac field transforming under the reducible representation \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\) of \( SL(2, \mathbb{C}) \) (it becomes irreducible if we extend the Lorentz group by space reflections). Its (normalized) 2-point function is given by (4.9) with \( H_1 (x) = \frac{1}{2 \pi^2} \gamma^\mu \gamma_\mu x^\nu \). The leading term of the operator product expansion of \( \psi \) with the conserved current \( J^\nu (x) = : \psi(x) \gamma^\nu \psi(x) : \) saturates the bound (4.10):

\[
\psi(x) J^\nu(y) |0\rangle \simeq \langle \psi(x) \bar{\psi}(y) \rangle (\psi(y) |0\rangle + O \left( (x - y)^2 \right)) = \frac{\gamma (x - y) \psi(y) + O \left( (x - y)^2 \right)}{2\pi^2 \left[ (x - y)^2 + i0 (x^0 - y^0) \right]} |0\rangle .
\]

(4.17)

We note that Eq. (4.11) gives a better estimate for the degree of the pole corresponding to the contribution of the field \( (j_1, j_2 ; d) \) to the operator product expansion.

\[\square\]
This example is typical in that the leading term in the small distance expansion of the product of a charge carrying conformal field with a conserved current (or the stress energy tensor) is the term involving the same charged field. This property is a consequence of the Ward(-Takahashi) identity.

5 Cluster property. Discussion

The conformal hamiltonian $H$ is the Hermitean generator of the conformal Lie algebra corresponding to $J_{-1,0}$ (i.e., the generator of the rotations in the euclidean $(-1,0)$-plane). The significance of both $H$ and the associated conformal time variable $\hat{\tau}$ has been repeatedly stressed by I. Segal (see, e.g. [14]) who has pointed out, in particular, that energy positivity with respect to the usual relativistic energy operator $P^0$ implies positivity of $H$ because of the relation

$$H = P^0 + U_w P^0 U_w^{-1}$$

(5.1)

where $U_w$ is the representation of the Weyl group element $w$ (of Example 2.1). Wightman axioms (including the uniqueness of the vacuum state) together with the GCI postulate imply that the spectrum of $H$ is contained in $\mathbb{Z}_+$ (because of the rationality of Wightman functions) and that there is a unique state in the Hilbert space $\mathcal{H}$ (the state space of our QFT), the vacuum, corresponding to eigenvalue 0 of $H$. In fact, these properties of the conformal Hamiltonian in the vacuum supeselection sector, conversely, implies rationality of correlation functions of local observable fields. In terms of the contraction semigroup $\{e^{-tH}, t \geq 0\}$ exploited in [6] the cluster decomposition property can be formulated as follows (cf. [6] Sec. 5).

**Proposition 5.1** For any pair of vectors

$$\langle \Phi | = \langle 0 | \phi_1 (x_1) ... \phi_n (x_n) \ , \ | \Phi' \rangle = \phi'_1 (y_1) ... \phi'_l (y_l) | 0 \rangle$$

(5.2)

the following limiting factorization property holds

$$\lim_{t \to \infty} \langle \Phi | e^{-tH} | \Phi' \rangle = \langle \Phi | 0 \rangle \langle 0 | \Phi' \rangle .$$

(5.3)

$\square$

Eq. (5.3) can be rewritten in terms of Wightman functions as

$$1 \otimes ... \otimes 1 \otimes \pi_{y_1} (e^{-tH}) ... \otimes \pi_{y_l} (e^{-tH}) \ W_{k+l} (x_1, ..., x_k, e^{-tH} y_1, ..., e^{-tH} y_l) \longrightarrow_{t \to \infty} W_k (x_1, ..., x_k) \ W_l (y_1, ..., y_l) .$$

(5.4)

The limit in Eq. (5.4) may be given two different (valid) interpretations: first, as a limit for every fixed set of points $(x_1, ..., x_k)$ and $(y_1, ..., y_l)$ in the corresponding tube domains; second, as a limit of rational functions (because of the rationality of both sides of (5.4)). Note that $\{e^{-tH}, t \geq 0\}$ is a subsemigroup of the complex conformal group $C_C$. Theorem 3.1 implies that in a QFT satisfying GCI the correlation functions viewed as rational functions would satisfy (5.4) for any semigroup $h e^{-tH} h^{-1}$ conjugate to $e^{-tH}$ in $C_C$. This is true, in particular, for the two opposite dilation semigroups:

$$U_\rho \phi_i^* (y_i) U_\rho^{-1} = \rho^{d_i} \phi_i^* (\rho y_i) \ , \ i = 1, ..., l \ , \ \text{for} \ \rho \geq 1 \ \text{or} \ \rho \leq 1 .$$

(5.5)
As a corollary of Proposition 5.1 we have:

\[ \rho^{d_1 + \ldots + d_l} W_{k+l} (x_1, \ldots, x_k, \rho y_1, \ldots, \rho y_l) \xrightarrow{\rho^{x_1 \to 0}} W_k (x_1, \ldots, x_k) W_l (y_1, \ldots, y_l) \quad (5.6) \]

Combined with locality this gives (Sec.3.5C)

**Proposition 5.2** For any splitting of the arguments \((x_1, \ldots, x_n)\) of a truncated Wightman function \(W_n^T\) into two disjoint subsets \(x_{i_1}, \ldots, x_{i_k}\) and \(x_{j_1}, \ldots, x_{j_l}\), we have

\[ \rho^{d_{i_1} + \ldots + d_{i_k}} W_n^T (x_{i_1}, \ldots, x_{i_k}, \rho x_{j_1}, \ldots, \rho x_{j_l}) \to 0 \quad \text{for} \quad \rho^{x_1} \to 0 \quad (5.7) \]

The general principles of QFT together with GCI are so restrictive that they allow, in principle, the computation of conformally invariant correlation functions. We shall illustrate this fact by writing down the general 4-point function of a neutral scalar field \(\phi\) of low conformal dimension.

**Proposition 5.3** The general 4-point function of a neutral scalar field \(\phi\) of (an integer) dimension \(d\) satisfying (TS) (L) (GCI) and the implications of positivity contained in Proposition 4.3 has the form

\[ W(x_1, x_2, x_3, x_4) = D_d (x_1, x_2, x_3, x_4) \mathcal{P} (\eta_1, \eta_2) \]

\[ D_d (x_1, x_2, x_3, x_4) = \left( \frac{x_{13}^2 x_{34}^2}{(x_{13}^2 + i0 x_{13}^0)(x_{34}^2 + i0 x_{34}^0)} \right)^d \quad (5.8) \]

where \(\eta_{1,2}\) are the cross-ratios

\[ \eta_1 = \frac{x_{12}^2 x_{34}^2}{(x_{13}^2 + i0 x_{13}^0)(x_{24}^2 + i0 x_{24}^0)} \quad \eta_2 = \frac{x_{14}^2 x_{24}^2}{(x_{13}^2 + i0 x_{13}^0)(x_{24}^2 + i0 x_{24}^0)} \quad (5.9) \]

\(\mathcal{P}\) is a polynomial in \(\eta_1, \eta_2\) of overall degree \(2d\),

\[ \mathcal{P} (\eta_1, \eta_2) = \sum_{\mu \geq 0, \nu \geq 0, \mu + \nu \leq 2d} C_{\mu\nu} \eta_1^\mu \eta_2^\nu \quad (5.10) \]

Locality implies invariance of \(\mathcal{P}\) under the 6 element dihedral group \(D_3 \cong S_3\):

\[ s_{12} : \quad \eta_2^{2d} \mathcal{P} \left( \frac{\eta_1}{\eta_2}, \frac{1}{\eta_2} \right) = \mathcal{P} (\eta_1, \eta_2) \]

\[ s_{12} : \quad \eta_1^{2d} \mathcal{P} \left( \frac{1}{\eta_1}, \frac{\eta_2}{\eta_1} \right) = \mathcal{P} (\eta_1, \eta_2) \]

\[ s_{13} = s_{12} s_{23}, s_{12} = s_{23} s_{12} s_{23} \quad : \quad \mathcal{P} (\eta_2, \eta_1) = \mathcal{P} (\eta_1, \eta_2) \quad (5.11) \]

(s\(_{ij}\) standing for the permutation of the arguments \(i, j\) of the rational function \(W\).) The normalization of \(\mathcal{P}\) is related to the normalization of the 2-point function of \(\phi\):

\[ W(x_1, x_2) = \frac{N_d}{(x_{12}^2 + i0 x_{12}^0)^d} \quad \Rightarrow \quad \mathcal{P} (1, 0) = N_d^2 = \mathcal{P} (0, 1) \quad (5.12) \]

The truncated 4-point function admits a similar representation,

\[ W_T (x_1, x_2, x_3, x_4) = D_d (x_1, x_2, x_3, x_4) \mathcal{P}_T (\eta_1, \eta_2) \quad (5.13) \]
where the polynomial $P_T$ has the properties (5.10) and (5.11) of $P$ while instead of (5.12) it satisfies

$$P_T(1,0) = P_T(0,1) = 0 .$$

(5.14)

Proof. The general form (5.8) is implied by the fact that the cross ratios (5.9) form a basis of (rational) invariants of 4 points. The summation limits in (5.10) follow from Proposition 4.3. The $i\eta^0_{jk}$ prescription in (5.12) reflects energy positivity. Eqs. (5.12) and (5.14) are consequences of Eqs. (5.6) and (5.7), respectively. $\square$

The simplest special cases, $d = 1, 2$, show that the range of summation in (5.10) can be further restricted by combining the operator product expansion implied by (5.8) with positivity. Indeed, the most general polynomials $P$ satisfying (5.11) and (5.12) for the above values of $d$ are

$$P_1(\eta_1, \eta_2) = C_1 [(1 - \eta_1 - \eta_2)^2 - 4 \eta_1 \eta_2] + N_1^2 (\eta_1 + \eta_2 + \eta_1 \eta_2)$$

$$+ C_{21} [\eta_1 (1 - \eta_1)^2 + \eta_2 (1 - \eta_2)^2 + \eta_1 \eta_2 (\eta_1 - \eta_2)^2] +$$

$$+ N_2^2 (\eta_1^2 + \eta_2^2 + \eta_1 \eta_2^2) + C_2 \eta_1 \eta_2 (1 + \eta_1 + \eta_2) .$$

(5.15)

On the other hand, for $d = 1$ the 2-point Wightman function $W(x_1, x_2) = W_1(x_{12})$ satisfies the d’Alembert equation $\Box W_1(x) = 0$. It then follows from the above cited Reeh-Schlieder theorem and from Wightman positivity that $\Box \phi(x) = 0$. Consequently, $C_1 = 0$ in (5.15) and we end up with a free field theory. More generally, the vanishing of $C_1$ - as well as that of $C_{20}$ and $C_{21}$ - follows from Corollary 4.4. (We owe this remark to Yassen Stanev.)

A systematic study of the implications of operator product expansions combined with positivity is relegated to the sequel of this paper (announced in the Introduction).

Acknowledgements

We thank Yassen Stanev for his inquisitive questioning and for helpful discussions. The authors acknowledge partial support by the Bulgarian National Council for Scientific Research under contract F−828.

Appendix A  The Klein-Dirac quadric

Viewed as a homogeneous space of the group $SO_0(D, 2)$, compactified $D$-dimensional Minkowski space $\overline{M}$ can be defined as a projective isotropic cone (or quadric) of signature $(D, 2)$ (see [3]). Points in $\overline{M}$ are identified with isotropic rays in $\mathbb{R}^{D,2}$, two collinear (isotropic) vectors, $\xi$ and $\lambda \xi$ ($\lambda \neq 0$) representing the same point in $\overline{M}$:

$$\overline{M} = Q/\mathbb{R}^* , \quad Q := \left\{ \xi \in \mathbb{R}^{D,2} ; \xi \neq 0, \xi^2 := \xi_0^2 + \xi_D^2 - \xi_0^2 - \xi_{-1}^2 = 0 \right\} .$$

(A.1)

Here $\mathbb{R}^*$ is the multiplicative group of non-zero reals, $\xi^2 := \xi_1^2 + \ldots + \xi_{D-1}^2$. In this representation of $\overline{M}$, the embedding of Minkowski space $M$ in $\overline{M}$ is given by the Klein-Dirac compactification
formulae:

\[ M \ni x \mapsto \left\{ \lambda \vec{\xi}_x \right\} \in \overline{M} \] , where:

\[ \vec{\xi}_x = x^\mu \vec{e}_\mu + \frac{1 + x^2}{2} \vec{e}_{-1} + \frac{1 - x^2}{2} \vec{e}_D \]  \hfill (A.2)

and \( \vec{e}_1, \ldots, \vec{e}_D, \vec{e}_{-1}, \vec{e}_0 \) is the standard basis in \( \mathbb{R}^{D,2} \). Then we have the following expression for the pseudo-euclidean interval:

\[ (x - y)^2 = -2 \vec{\xi}_x \cdot \vec{\xi}_y = \left( \vec{\xi}_x - \vec{\xi}_y \right)^2 \] . \hfill (A.3)

Eq. (A.3) shows that two points \( p_1, p_2 \in \overline{M} \) are (mutually) isotropic if the inner product \( \vec{\xi}_1, \vec{\xi}_2 \) of (any of) their representatives \( \vec{\xi}_1, \vec{\xi}_2 \) is zero. Thus it is obviously a \( SO_0(D,2) \)-invariant relation in \( \overline{M} \times \overline{M} \). The point \( p_\infty \in \overline{M} \) can be defined as \( p_\infty = \left\{ \lambda \vec{\xi}_\infty \right\} \), where \( \vec{\xi}_\infty = \vec{e}_D - \vec{e}_{-1} \).

Then the representatives \( \vec{\xi}_x \) of the points \( x \) of Minkowski space \( M \) are characterized by the "normalization" \( \vec{\xi}_x \cdot \vec{\xi}_\infty = 1 \) so that the set \( K_\infty \) of points at infinity is indeed

\[ (K_{p_\infty} \equiv \ K_\infty) = \left\{ p \in \overline{M} \mid (p, p_\infty) = 0, \ i.e., \ \vec{\xi}_x \cdot \vec{\xi}_\infty = 0 \ \text{for} \ p = \left\{ \lambda \vec{\xi}_x \right\}, \ p_\infty = \left\{ \lambda \vec{\xi}_\infty \right\} \right\} \] . \hfill (A.4)

As an illustration of the transitivity property used in the proof of Proposition 1.1 of Sec. 1 we note that the rotation on \( \pi \) in the \( (\xi^{-1}, \xi^0) \)-plane interchanges the origin in \( M \) with \( p_\infty \) (this is the Weyl reflection used in the Examples 2.1 and 2.2).

Equation (A.3) also allows to compute the conformal factor \( \omega (x, g) \omega (y, g) \) multiplying the interval \((x - y)^2\) under the action of \( g \in SO_0(D,2) \). Indeed, \( \omega (x, g) \) can be defined by:

\[ g \vec{\xi}_x = \omega (x, g) \vec{\xi}_{gx} \ \text{for} \ \vec{\xi}_x \cdot \vec{\xi}_\infty = 1 = \vec{\xi}_{gx} \cdot \vec{\xi}_\infty \ \Rightarrow \ \omega (x, g) = g \vec{\xi}_x \cdot \vec{\xi}_\infty \] \hfill (A.5)

and is, hence, a second degree polynomial in \( x \). Here \( \vec{\xi} \mapsto g \vec{\xi} \) is the linear action of the group \( SO_0(D,2) \) on \( \mathbb{R}^{D,2} \), while \( x \mapsto gx \) is the nonlinear action of \( SO_0(D,2) \) on \( M \) which can be computed from (A.3). We then find (cf. (1.1)):

\[ (gx - gy)^2 = -2 \vec{\xi}_{gx} \cdot \vec{\xi}_{gy} = -2 \frac{g \vec{\xi}_x \cdot g \vec{\xi}_y}{\omega (x, g) \omega (y, g)} = \frac{(x - y)^2}{\omega (x, g) \omega (y, g)} \] . \hfill (A.6)

So, the elements of the group \( SO_0(D,2) \) act indeed conformally on \( M \) outside their singularities. Because the dimension of \( SO_0(D,2) \) is equal to that of the conformal Lie algebra coming from Liouville theorem (see, e.g., [17] Appendix A) then \( SO_0(D,2) \) must be locally isomorphic to the conformal group \( C_0 \).

The double cover \( \overline{M} \) of \( \overline{M} \) (A.1) is diffeomorphic to the product of the \((D - 1)\)-sphere, \( S^{D - 1} \), with the unit circle \( S^1 \); \( \overline{M} \) is obtained from it by identifying opposite points:

\[ \overline{M} = S^{D - 1} \times S^1 \] ,

\[ S^{D - 1} = \left\{ (\xi, \xi_D) \mid \xi^2 + \xi_D^2 = 1 \right\} \] , \[ S^1 = \left\{ (\xi_{-1}, \xi_0) \mid \xi_{-1}^2 + \xi_0^2 = 1 \right\} \] ,

\[ \overline{M} = \overline{M} / \mathbb{Z}_2 = \overline{M} / \left( \vec{\xi} \cong -\vec{\xi} \right) \left( \vec{\xi} \in \overline{M} \right) \] . \hfill (A.7)
Observation: It follows from \([\text{A.1}]\) that compactified Minkowski space \(\overline{M}\) is orientable for even \(D\) and non-orientable for odd \(D\). □

Eq. \([\text{A.1}]\) admits a complex version, \(\overline{M}_C = \mathbb{Q}_C / \mathbb{C}^*\) where \(\mathbb{Q}_C\) is the complexification of the quadric \(Q\) and \(\mathbb{C}^*\) is the multiplicative group of (non-zero) complex numbers. \(\overline{M}_C\) is a homogeneous space of the complexified conformal group \(C\). \(\overline{M}_C\) and \(C\) admit a unique antianalytic involution * whose fixed points belong to \(\overline{M}\) and \(C\), respectively. In the above realization we have \(p^* = \left\{ \lambda \xi^* \right\}\), \((gp)^* = g^*p^*\) where \(\xi^*\) is the (componentwise) complex conjugate of \(\xi\) and \(g^*\) is the complex conjugate of the matrix \(g \in SO_0 (D, 2; \mathbb{C})\).

**Appendix B  Proof of Proposition 2.2**

We shall prove the statement by induction in the number of points. For \(n = 2\) the 2-point vacuum correlator is a boundary value of the analytic function \([|x - \zeta|^2]^{-\mu}\) holomorphic for \((x, \zeta) \in M \times T_+\) which satisfies \((\text{GCI})\) as a consequence of \([\text{I.1}]\) and of the conformal invariance of the forward tube. (In fact, this case can also be covered by our induction inference if we interpret the \((\text{GCI})\) property for \(n = 1\) as the identity \(1 = 1\).) Assume now that the statement is established for all \((n - 1)\)-point functions for some \(n > 1\). To prove it for the \(n\)-point functions we fix a \(g \in C\) and \(n\) open sets \(U_1, ..., U_n\) in \(M\) whose closures \(\overline{U}_1, ..., \overline{U}_n\) are compact and such that the mapping \(g : x \mapsto gx\) has no singularity for \(x \in \overline{U}_i\). So we will establish that Eq. \((2.14)\) is locally valid for \((x_1, ..., x_n) \in U_1 \times ... \times U_n\). The distribution in the right hand side of \((2.14)\) is then the boundary value for \(\text{Im} (\zeta) \to 0\), \(\zeta \in T_+\) of the analytic function

\[
\mathcal{W}_n (x_1, ..., x_{n-1}, \zeta) = \mathcal{W}_{n-1} (x_1, ..., x_{n-1}) \prod_{j=1}^{n-1} \left( (x_j - \zeta)^2 \right)^{-\mu_{jn}} \quad (\zeta \in T_+) \quad (B.1)
\]

(with \(\mathcal{W}_{n-1}\) again given by the right hand side of \((2.14)\) for \(n\) substituted by \(n - 1\)). Applying the induction assumption to \(\mathcal{W}_{n-1}\) and Eq. \((\text{I.1})\) to the other factors in the product in the right hand side of \((B.1)\) (treating them as a multiplicator) we deduce

\[
\mathcal{W}_n (x_1, ..., x_{n-1}, x_n + iy) = \omega (x_1, g)^{-\mu_1} ... \omega (x_{n-1}, g)^{-\mu_{n-1}} \omega (x_n + iy, g)^{-\mu_n} \mathcal{W}_{n-1} (gx_1, ..., gx_{n-1}) \prod_{j=1}^{n-1} \left( (gx_j - gx_n - \zeta_y (gx_n))^2 \right)^{-\mu_{jn}}, \quad (B.2)
\]

where \(\zeta_y (gx_n) := g (x_n + iy) - gx_n \in T_+\) because of the conformal invariance of \(T_+\). We also have the limit:

\[
\lim_{\nu \ni y \to 0} \mathcal{W}_{n-1} (x'_1, ..., x'_{n-1}) \prod_{j=1}^{n-1} \left( (x'_j - x'_n - \zeta_y (x'_n))^2 \right)^{-\mu_{jn}} = \mathcal{W}_{n-1} (x'_1, ..., x'_{n-1}) \prod_{j=1}^{n-1} \left( (x'_j - x'_n)^2 + i0 \left( x'_j - x'_0 \right) \right)^{-\mu_{jn}}, \quad (B.3)
\]

which combined with \((B.2)\) gives \((2.14)\) (due to the continuity of multiplication of a distribution with a multiplicator). □
Appendix C  Completion of the proof of Lemma 3.2

It remains to prove the following statement:
If the points \( y_1', y_2' \in M \) are non-isotropic, \( p \in K_\infty \) and \( C_{y_1', y_2'} \) is the stabilizer of the pair \( y_1', y_2' \) in \( SO_0(D, 2) \), then in any neighbourhood of unity in \( C_{y_1', y_2'} \) there exists an element \( h \) such that \( hp \notin K_\infty \).
In the case \( D = 1 \), \( K_\infty \) consists of a single point, \( p_\infty \), and it is not stable for \( C_{y_1', y_2'} \), so one does not need the argument below. To prove the above statement for \( D \geq 2 \) we shall use the representatives \( \xi_1, \xi_2 \) and \( \xi \) of the points \( y_1', y_2' \) and \( p \) (cf. (A.2)) in the quadric (A.1). We have \( \xi_1, \xi_2 \neq 0 \neq \xi_\infty \xi_a \) for \( a = 1, 2 \); \( \xi_\infty \xi_\infty = 0 \) (\( \Leftrightarrow \xi \in K_\infty \)). The metric in \( \text{Span} \{ \xi_1, \xi_2 \} \) being non-degenerate (since \( \xi_1, \xi_2 \neq 0 \)) while \( \xi_1^2 = \xi_2^2 = 0 \) there is a pseudo-orthogonal decomposition
\[
\mathbb{R}^{D,2} = \text{Span} \{ \xi_1, \xi_2 \} \oplus \{ \xi_\infty \xi_a \}^\perp.
\]
Let \( \xi_\infty = \xi_\infty', \xi = \xi' + \xi'' \left( \xi_\infty' \in \text{Span} \{ \xi_1, \xi_2 \} \right) \) be the corresponding decompositions of \( \xi_\infty \) and \( \xi \). Since \( \xi_\infty', \xi_\infty, \xi_1, \xi_2 \neq 0 \) and \( \xi_1^2 = 0 \) then \( \xi_\infty' \neq 0 \) and \( \xi_\infty'' \neq 0 \). It follows also that \( \xi'' \neq 0 \) (i.e. \( \xi \) does not belong to \( \text{Span} \{ \xi_1, \xi_2 \} \) since \( \xi^2 = 0 \) and \( \xi_\infty \xi_\infty = 0 \)). Let \( G \) be the connected component of the orthogonal group of the subspace \( \{ \xi_1, \xi_2 \}^\perp \) (\( G \cong SO_0(D - 1, 1) \)). Clearly, \( G \) is a subgroup of the stabilizer of the points \( y_i' = \{ \lambda \xi_i' \} \), \( i = 1, 2 \):
\[
G \subset C_{y_1', y_2'} = C_{\text{Span} \{ \xi_1, \xi_2 \}} := SO_0 \left( \text{Span} \{ \xi_1, \xi_2 \} \right) \times SO_0 \left( \{ \xi_1, \xi_2 \}^\perp \right).
\]
For \( h \) varying in \( G \), \( h\xi'' \) moves on a hyperboloid in \( \{ \xi_1, \xi_2 \}^\perp \). Consequently, the (real valued) function \( f(h) = h\xi_\infty \xi_\infty = \xi' \xi_\infty + \xi'' h^{-1} \xi_\infty'' \) cannot be a constant. On the other hand it is real analytic (in fact, algebraic) and \( f(1) = 0 \). Hence, in any (arbitrarily small) neighbourhood of the group unit in \( G \) (and also in \( C_{y_1', y_2'} \)) there exist elements \( h \) such that \( f(h) \neq 0 \), i.e. \( hp \notin K_\infty \). \( \square \)

References

[1] Aharony, O., Gubser, S.S., Maldacena, J., Ooguri, H., Oz, Y.: Large N field theories, string theory and gravity. hep-th/9905111, Phys. Reports 323, 183 (2000)
[2] Di Francesco, P., Mathieu, P., Senechal, D.: Conformal field theories. Berlin et al.: Springer, 1996
[3] Dirac, P.A.M.: Wave equations in conformal space. Ann. Math. 37, 429–442 (1936)
[4] Flato, M., Sternheimer, D.: Remarques sur les automorphismes causals de l’espace temps. Comptes Rendus Acad. Sci., Paris 263 A, 935–936 (1966)
[5] Jost, R.: The general theory of quantized fields. Providence, R.I.: Amer. Math. Soc. Publ., 1965
[6] Lüscher, M., Mack, G.: Global conformal invariance in quantum field theory. Commun. Math. Phys. 41, 203–234 (1975)

[7] Mack, G.: Convergence of operator product expansions on the vacuum in conformal invariant quantum field theory. Commun. Math. Phys. 53, 155–184 (1977)

[8] Mack, G.: All unitary representations of the conformal group $SU(2,2)$ with positive energy. Commun. Math. Phys. 55, 1–28 (1977)

[9] Mack, G., Symanzik, K.: Currents, stress tensor and generalized unitarity in conformal invariant quantum field theory. Commun. Math. Phys. 27, 247–281 (1972)

[10] Mack, G., Todorov, I.T.: Conformal invariant Green functions without ultra-violet divergences. Phys. Rev. D8, 1764–1787 (1973)

[11] Polyakov, A.M.: Conformal symmetry of crucial fluctuations. Zh. ETF Pis. Red. 12, 538 (1970) (transl.: JEPT Lett. 12, 381 (1970))

[12] Polyakov, A.M.: Nonhamiltonian approach in the conformal invariant quantum field theory. Zh. Eksp. Teor. Fiz. 66, 23 (1974)

[13] Ruhl, W., Yunn, B.C.: The transformation behaviour of fields in conformally covariant quantum field theory. Fortschritte d.Physik 25, 83–99 (1977)

[14] Segal, I.E.: Causally oriented manifolds and groups. Bull. Amer. Math. Soc. 77, 958–959 (1971)

[15] Stanev, Ya.S.: Stress-energy tensor and $U(1)$-current operator product expansion in conformal QFT. Bulg. J. Phys. 15, 93–107 (1988)

[16] Streater, R.F., Wightman, A.S.: PCT, spin and statistics and all that. Second Printing. Reading, MA: Benjamin/Cummings, 1978

[17] Todorov, I.T.: Local field representations of the conformal group and their applications. In: Streit, L. (ed.) Mathematics + Physics, Lectures on recent results. Vol. 1, pp. 195–338. Singapore: World Scientific, 1985

[18] Todorov, I.T.: Infinite dimensional Lie algebras in conformal QFT models. In: Barut, A.O. and Doebner, H.-D. (eds.) Conformal groups and related symmetries. Physical results and mathematical background, Lecture Notes in Physics 261, pp. 387–443. Berlin et al.: Springer, 1986

[19] Todorov, I.T.: Conformal description of spinning particles. Trieste Notes in Physics. Berlin et al.: Springer, 1986

[20] Todorov, I.T., Mintchev, M.C., Petkova, V.B.: Conformal invariance in quantum field theory. Scuola Normale Superiore, pp. 1–274. Pisa: 1978

[21] Treves, F.: Topological vector spaces, distributions and kernels. N.Y.: Acad. Press, 1967; for a brief review, see Treves, F.: Introduction to pseudodifferential operators and Fourier integral operators. Vol. 1, Sec. I.7. N.Y., London: Plenum Press, 1980
[22] Uhlmann, A.: Remarks on the future tube. Acta Phys, Pol. 24, 193 (1963); The closure of Minkowski space. ibid. pp. 295–296; Some properties of the future tube. preprint KMU-HEP 7209 (Leipzig, 1972)

[23] Wess, J.: The conformal invariance in quantum field theory. Nuovo Cimento 18, 1086 (1960)