The field theory limit of multiloop string amplitudes

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Abstract

We report on recent progress in the use of string techniques for the computation of field theory amplitudes. We show how one-loop renormalization constants in Yang–Mills theory can be computed using the open spinning string, we review the calculation of two-loop scalar amplitudes with the bosonic string, and we briefly indicate how the technique can be applied to the two-loop vacuum bubbles of Yang–Mills theory.

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1 Introduction

All string theories contain a parameter, the string tension $T = 1/(2\pi\alpha')$, having the dimension of a mass squared. In the infinite tension limit, $\alpha' \to 0$, the heavy string states become infinitely massive and decouple, while the light states survive. Thus, when $\alpha' \to 0$, a string theory simply reduces to a quantum field theory of pointlike objects. Moreover, the parameter $1/\alpha'$ acts in string theory as an ultraviolet regulator in the integrals over loop momenta, making the string free from ultraviolet divergences. For these reasons, string theory may be useful not only for the construction of unified theories but also as an efficient tool to understand the structure of perturbative field theories, and to compute regularized amplitudes.

There are several features that make string theories very useful for this purpose. First of all, at each order of string perturbation theory, one does not get the large number of diagrams that one encounters in field theories. For example, with closed strings there is only one diagram at each perturbative order. Secondly, there exist compact and explicit expressions for string scattering amplitudes that are valid at any perturbative order \[1\]. This is to be contrasted with what happens in field theory where no such all–loop formulas exist. Finally, string amplitudes are always written in a way that takes maximal advantage of all symmetries, and in particular of gauge invariance. All these properties have been known since the very early days of string theory, but it is only in more recent times that they have been exploited for explicit calculations in Yang–Mills theory \[2\].

The aim of this talk is to review and extend some of the results of Refs. \[3\] and \[4\], where the limit $\alpha' \to 0$ of bosonic string theory was performed in order to analyze the structure of ultraviolet divergences in Yang–Mills theory at one loop, and to compute scattering amplitudes of a $\Phi^3$ theory at two loops.

Our analysis clearly shows that only some corners of the string moduli space contribute to the field theory limit, namely those corners where the integrand of the string amplitude exhibits a singular behaviour when $\alpha' \to 0$. For each such corner of moduli space, the string gives directly a Schwinger parameter integral, with momentum integrations, Lorentz and gauge algebra already performed. As a consequence, different regularizations are easily implemented: we work mainly with dimensional regularization, but it would be possible, for example, to use a proper–time cutoff on the Schwinger parameters. In particular, string theory gives correctly also the contributions of the diagrams (such as tadpoles or vacuum bubbles) that vanish in dimensional regularization.

It is also worth pointing out that actually there are several ways to take the infinite tension limit in a string amplitude. One of these, perhaps the most obvious and natural one, is to let $\alpha' \to 0$ in such a way that only the massless states survive. Performing this limit in the bosonic string, for example, one recovers a non–abelian Yang–Mills theory from the open sector, unified with an extended version of gravity from the closed sector, containing besides the graviton also an antisymmetric tensor and a dilaton. However, there are other consistent possibilities. For example, in
the open bosonic string one can take the infinite tension limit and retain only the massive scalar state instead of the massless vector ones. This can be achieved by giving an arbitrary mass \( m \) to the scalars, such that \( \alpha' m^2 = -1 \) when \( \alpha' \to 0 \), and by suitably modifying the measure of integration in the string amplitudes to avoid the inconsistencies associated with tachyon propagation. Thus, in this case, the infinite tension limit of the open bosonic string yields a \( \Phi^3 \) field theory [5].

The talk is organized as follows: in Sect. 2 we show how to compute renormalization constants in Yang–Mills theory at one loop, starting from the two–gluon scattering amplitude of the open spinning string. This is an extension of the results of Ref. [3], where similar calculations were performed in the open bosonic string and the inconsistencies due to tachyon exchanges were removed by hand. One sees that the results are independent of the string model one starts with. In Sect. 3 we show how to obtain the correctly normalized two–loop vacuum bubbles of the \( \Phi^3 \) theory starting from the two–loop tachyon amplitudes of the open bosonic string. Finally, we conclude by briefly indicating how the same calculation can be generalized to the vacuum bubbles of Yang–Mills theory.

### 2 Yang–Mills Renormalization Constants from the Spinning String

Given two gluons with momenta \( p_1 \) and \( p_2 \), polarizations \( \varepsilon_1 \) and \( \varepsilon_2 \), and \( SU(N) \) color indices \( a_1 \) and \( a_2 \), the corresponding scattering amplitude at one loop in the open spinning string is

\[
A_2^{(1)} = N \text{Tr} (\lambda^{a_1} \lambda^{a_2}) \varepsilon_1 \cdot \varepsilon_2 p_1 \cdot p_2 \frac{g_d^2}{(4\pi)^{d/2}} (2\alpha')^{2-d/2} R(p_1 \cdot p_2),
\]

where, with the same normalizations of Ref. [3], \( g_d \) is the (dimensionful) Yang–Mills coupling constant in \( d \) dimensions, and \( R(s) \) is defined by

\[
R(s) = \sum_a \int_0^1 \frac{dk}{k^{3/2}} \left( -\frac{1}{2} \log k \right)^{-d/2} \prod_{n=1}^\infty (1 - k^n)^{2-d} \text{Z}_F^{[n]}(k) \\
\times \int_1^k \frac{dz}{z^2} \left[ (\partial_2 G_B(1, z_2))^2 - (G_F^{[a]}(1, z_2))^2 \right] e^{2\alpha's} G_B(1, z_2).
\]

In this equation, \( \sum_a \) denotes the sum over the even spin structures, \( k \) is the modular parameter of the annulus, \( \text{Z}_F^{[n]}(k) \) is the fermionic partition function of the \( a \)-th spin structure, and \( G_B \) and \( G_F^{[a]} \) are the bosonic and fermionic Green functions, respectively. Notice that if the gluons are on mass shell, the two–gluon amplitude in Eq. 2.1 becomes ill–defined, because the kinematical prefactor vanishes, while the integral \( R \) diverges. As explained in Ref. [3], this problem can be consistently cured by keeping the gluons off–shell.
If we are interested in pure Yang–Mills theory, the sum in Eq. 2.2 must be restricted to the two even spin structures \( (a = 0, 1) \) of the Neveu–Schwarz sector, which describe bosonic string states circulating in the loop. The corresponding fermionic partition functions are

\[
Z_F^{[a]}(k) = \frac{(-1)^a}{2} \prod_{n=1}^{\infty} \left( 1 + (-1)^a k^{n-1/2} \right)^{2-d},
\]

while the fermionic Green functions are

\[
G_F^{[a]}(z, w) = \sum_{n=-\infty}^{\infty} (-1)^{a+n} \frac{k^{n/2}}{k^n z - w},
\]

for \( a = 0, 1 \). The bosonic Green function \( G_B \) is obviously independent of the spin structure and is given by

\[
G_B(z, w) = \frac{1}{2 \log k} \left( \log \frac{z}{w} \right)^2 + \log \left[ \frac{z - w}{\sqrt{z w}} \prod_{n=1}^{\infty} \frac{1 - k^n z}{1 - k^n w} \right].
\]

To discuss the field theory limit of Eq. 2.1 it is convenient to introduce the variables \( \tau = -\log k/2 \) and \( \nu = -\log z_2/2 \), which turn out to be related directly to the Schwinger proper times \( t \) and \( t_1 \) of the Feynman diagrams contributing to the two-point function. In particular, \( t \sim \alpha' \tau \) and \( t_1 \sim \alpha' \nu \), where \( t_1 \) is the proper time associated with one of the two internal gluon propagators, while \( t \) is the total proper time around the loop. In the field theory limit these proper times have to remain finite, and thus the limit \( \alpha' \to 0 \) must be accompanied by the limit \( \{ \tau, \nu \} \to \infty \) in the integrand of Eq. 2.2. Thus the field theory limit of a string amplitude is determined by the asymptotic behavior of the Green functions \( G_B \) and \( G_F^{[a]} \) for large \( \tau \), which is given by

\[
G_B(\nu, \tau) \sim \nu - \frac{\nu^2}{\tau} - e^{-2\nu} - e^{-2\tau+2\nu} + 2e^{-2\tau},
\]

and

\[
G_F^{[a]}(\nu, \tau) \sim 1 + (-1)^a e^{-\tau+2\nu} + (-1)^{a+1} e^{-\tau}.\]

We now substitute these results in Eq. 2.2 and keep only those terms that remain finite when \( k = e^{-2\tau} \to 0 \). While in the bosonic string tachyon exchanges produce divergent terms which must be discarded by hand, here our procedure leads directly to the desired contributions due to gluon exchanges, since in the spinning string tachyons are projected out by the sum over spin structures. Indeed, by defining \( \tilde{\nu} \equiv \nu/\tau \) and performing simple manipulations, we get

\[
R(s) = \int_{0}^{\infty} d\tau \int_{0}^{1} d\tilde{\nu} \, \tau^{1-d/2} e^{2\alpha' s (\tilde{\nu} - \tilde{\nu}^2) \tau} \left[ (1 - 2\tilde{\nu})^2 (d - 2) - 8 \right].
\]

\(^1\)The third even spin structure corresponds to fermionic states of the Ramond sector propagating in the loop, and should be considered if one were interested in supersymmetric Yang–Mills theories.
which is precisely the same result that emerges from the bosonic string \[8] after
discarding tachyons. In this case, the term proportional to \((d - 2)\) in the square
bracket is generated by the bosonic Green function of Eq. 2.2, while the factor \(-8\)
is produced by the fermionic Green functions. The integrals in Eq. 2.8 are now
elementary and yield

\[
\mathcal{R}(s) = -\Gamma\left(2 - \frac{d}{2}\right) (-2\alpha' s)^{d/2-2} \frac{6 - 7d}{1-d} B\left(\frac{d}{2} - 1, \frac{d}{2} - 1\right),
\]

(2.9)

where \(B\) is the Euler beta function.

If we substitute Eq. 2.9 into Eq. 2.1, we see that the \(\alpha'\) dependence cancels,
as it must. The ultraviolet finite string amplitude in Eq. 2.1 has been replaced by
a field theory amplitude which diverges in four space–time dimensions, because of
the pole in the \(\Gamma\) function in Eq. 2.9. The result coincides exactly with the one–
loop gluon vacuum polarization of the \(SU(N)\) gauge field theory that one computes
with the background field method, in the Feynman gauge and using dimensional
regularization.

Setting \(d = 4 - 2\epsilon\) and defining as usual a dimensionless coupling constant
\(g = g_d \mu^{-\epsilon}\), with \(\mu\) an arbitrary mass scale, the divergent part of the two–point
amplitude is

\[
A_2^{(1)} = -N \frac{g^2}{(4\pi)^2} \frac{11}{3} \frac{1}{\epsilon} \delta^{ab} \varepsilon_1 \cdot \varepsilon_2 \ p_1 \cdot p_2 + O(\epsilon^0) \quad .
\]

(2.10)

From this result it is immediate to extract the minimal subtraction wave function
renormalization constant at one loop,

\[
Z_A = 1 + N \frac{g^2}{(4\pi)^2} \frac{11}{3} \frac{1}{\epsilon} \quad .
\]

(2.11)

With similar techniques, one can compute the three and four–point vertex renor-
malization constants, \(Z_3\) and \(Z_4\), at one loop and verify that they satisfy the correct
Ward identity \(Z_3 = Z_4 = Z_A\) appropriate to the background field method [3].

### 3 Two–Loop Diagrams in \(\Phi^3\) Theory

We now consider the open bosonic string and outline the procedure to obtain the
two–loop Feynman diagrams of the \(\Phi^3\) field theory from the two–loop string am-
plitudes in the limit \(\alpha' \rightarrow 0\). This is a preliminary but important step towards the
more interesting case of multiloop amplitudes in Yang–Mills theories. By selecting
scalar particles instead of gluons, one can avoid the computational difficulties re-
lated to the fact that the gluons are not the lowest states of the spectrum, and one
can focus on the precise identification of the corners of moduli space contributing
to the field theory limit.
Our starting point is the planar two–loop scattering amplitude of \( M \) scalar particles of momenta \( p_1, \ldots, p_M \), and \( U(N) \) color indices \( a_1, \ldots, a_M \), which is

\[
A^{(2)}_M = N^2 \text{Tr}(\lambda^{a_1} \cdots \lambda^{a_M}) \frac{1}{(4\pi)^d} \frac{g^{2+M}}{2^{8+3M}} (2\alpha')^{3-d+M} \times \int \frac{dk_1}{k_1^2} \int \frac{dk_2}{k_2^2} \int \frac{d\eta_1}{(1-\eta_1)^2} \prod_{i=1}^M \left[ \frac{dz_i}{V_i'(0)} \right] \times \left[ \frac{1}{4} \left( \log k_1 \log k_2 - \log^2 \eta_1 \right) \right]^{-d/2} \prod_{i<j} \left[ \frac{\exp \left( G^{(2)}(z_i, z_j) \right)}{\sqrt{V_i'(0) V_j'(0)}} \right]^{2\alpha' p_i \cdot p_j}.
\]

In this formula \( G^{(2)}(z_i, z_j) \) is the two–loop bosonic Green function, while \( V_i(z) \) parametrizes the local coordinates around the puncture \( z_i \). Notice that in Eq. 3.12 we have used the two–loop integration measure of the open bosonic string in the limit of small \( k_1 \) and \( k_2 \). This is the obvious two–loop generalization of the limit \( k = e^{-2\tau} \to 0 \) considered in the one loop case.

Eq. 3.12 serves as a ‘master formula’ for all two–loop amplitudes of the theory defined by the Lagrangian

\[
L = \text{Tr} \left[ \partial_\mu \Phi \partial^\mu \Phi + m^2 \Phi^2 - \frac{g}{3!} \Phi^3 \right],
\]

where \( \Phi = \Phi^a \lambda^a \) is a scalar field in the adjoint representation of \( U(N) \). All amplitudes are determined by the single function \( G^{(2)}(z_i, z_j) \), which must be evaluated in different corners of moduli space. The integrations in Eq. 3.12 correspond to a sum over all the different positions of the vertex operators representing external legs, as well as over all the different shapes of the world–sheet surface. In our case the surface is represented in the Schottky parametrization, as shown in Fig. 1.

![Fig. 1: In the Schottky parametrization, the two–annulus corresponds to the part of the upper–half plane which is inside the big circle passing through \( A' \) and \( B' \), and which is outside the circles \( \mathcal{K}_1, \mathcal{K}_1' \) and \( \mathcal{K}_2 \).](image)

The width of the two–annulus around the hole represented by the two circles \( \mathcal{K}_1 \sim \mathcal{K}_1' \) is proportional to \( k_1 \), while the width around the other hole is proportional to \( k_2 \); on the other hand \( \eta_1 \) can be seen as the “distance” between the two loops.
The points $A$, $B$, $C$ and $D$ have to be identified with $A'$, $B'$, $C'$ and $D'$ respectively, so that it is easy to realize that the two segments $(AA')$ and $(DD')$ represent the two inner boundaries of the two–annulus, while the union of $(BC)$ and $(C'B')$ represents the external boundary.

In the limit $\alpha' \to 0$ the integrals in Eq. 3.12 are dominated by the thin two–annulus with $k_1, k_2 \to 0$; moreover if $\eta_1 \to 0$, the distance between the two holes vanishes and the loops glue together giving 1PI diagrams, while if $\eta_1 \to 1$, one recovers the reducible diagrams in which the loops are separated by a propagator. Unlike what we did in the previous section, we want now to select the scalar state; the easiest way to do this is to relax the condition $1 + \alpha' m^2 = 0$ for the mass of the scalar state and change it into $a + \alpha' m^2 = 0$. We can now transform the tachyon into a normal scalar particle with arbitrary mass $m$ by rewriting each double pole in the measure in Eq. 3.12 according to

$$x^{-2} \to x^{-1-a} = x^{-1} \exp \left(-a \log x \right) = x^{-1} \exp \left[ m^2 \alpha' \log x \right] . \quad (3.14)$$

As an easy example of the procedure just outlined, we will calculate the vacuum bubble diagrams. In this simple case there is no dependence on the Green function and we only have to integrate over the Schottky parameters. However it is more convenient to introduce new variables

$$q_1 = \frac{k_2}{\eta_1} , \quad q_2 = \frac{k_1}{\eta_1} , \quad q_3 = \eta_1 , \quad (3.15)$$

so that Eq. 3.12 becomes

$$A_0^{(2)} = \frac{N^3}{(4\pi)^d} \frac{g^2}{2^{8}} \frac{1}{(2\alpha')^{3-d}} \int_{0}^{1} \frac{dq_3}{(1 - q_3)^{1+a}} \int_{0}^{q_3} \frac{dq_2}{q_2^{1+a}} \int_{0}^{q_2} \frac{dq_1}{q_1^{1+a}}$$

$$\times \left[ \frac{1}{4} \left( \log q_1 \log q_2 + \log q_1 \log q_3 + \log q_2 \log q_3 \right) \right]^{-d/2} . \quad (3.16)$$

The region of integration we used for the $q_i$ can be derived from the geometrical interpretation described above [3]. We now consider the reducible and irreducible diagrams separately. In the first case ($q_3 \to 1$), Eq. 3.16 becomes

$$A_0^{(2)} \bigg|_{\text{red}} = \frac{N^3}{(4\pi)^d} \frac{g^2}{2^5} \int_{0}^{\infty} dt_3 \int_{0}^{\infty} dt_2 \int_{0}^{t_2} dt_1 \ e^{-m^2(t_1 + t_2 + t_3)} \ (t_1 t_2)^{-d/2} , \quad (3.17)$$

where we introduced the mass $m$ as explained in Eq. 3.14, and the Schwinger proper times $t_i$ according to

$$t_1 = -\alpha' \log q_1 , \quad t_2 = -\alpha' \log q_2 , \quad t_3 = -\alpha' \log (1 - q_3) ; \quad (3.18)$$

in the second case ($q_3 \to 0$), Eq. 3.16 becomes

$$A_0^{(2)} \bigg|_{\text{irr}} = \frac{N^3}{(4\pi)^d} \frac{g^2}{2^5} \int_{0}^{\infty} dt_3 \int_{0}^{t_3} dt_2 \int_{0}^{t_2} dt_1 \ e^{-m^2(t_1 + t_2 + t_3)}$$

$$\times \ (t_1 t_2 + t_1 t_3 + t_2 t_3)^{-d/2} ; \quad (3.19)$$

$$6$$
where \( t_3 \) is now defined by \( t_3 = -\alpha' \log q_3 \).

One can check that the results above coincide with the Schwinger parametrization of the same diagrams in field theory. We have explicitly verified that this method gives the correct results also in the presence of external states, even if they are off shell; in this latter case the result depends on the particular choice of the local coordinates \( V_i \). It turns out that the correct field theory amplitudes are obtained if one defines \( V_i \) to satisfy

\[
(V'_i(0))^{-1} = \left| \frac{1}{z_i - \rho_a} - \frac{1}{z_i - \rho_b} \right|,
\]

(3.20)

where \( \rho_a \) and \( \rho_b \) are the two fixed points on the left and on the right of \( z_i \) (this definition of \( V_i \) corresponds to the one used in Ref. [8]).

### 4 Conclusions

We would like to conclude by noting that the procedure described in Section 3 can be readily generalized to gluon propagation, and in fact we have already completed the calculation of the vacuum bubble diagrams in Yang–Mills theory at two loops, obtaining the correct result. Lack of space prevents us from describing the calculation in this contribution. However we note the main features: it is clearly necessary to expand the various terms in the string–theoretic measure to next–to–leading order in the multipliers \( k_i \). Then tachyon poles are discarded as was done at one loop. The main new feature is the diagram with the four gluon vertex, which can be traced to a contact term left over from the propagation of a tachyon in the corresponding channel. However it is interesting to notice that there does not appear to be a natural mapping between individual diagrams and points in moduli space. Only when the different contributions are recombined the correct answer is recovered in the form of a standard Schwinger parameter integral [9].

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