Finite-dimensional analogs of string $s \leftrightarrow t$ duality and pentagon equation

Igor G. Korepanov* and Satoru Saito†

Department of Physics, Tokyo Metropolitan University
Minami-Ohsawa, Hachioji, Tokyo 192-0397, Japan

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Abstract

We put forward one of the forms of functional pentagon equation (FPE), known from the theory of integrable models, as an algebraic explanation to the phenomenon known in physics as $s \leftrightarrow t$ duality. We present two simple geometrical examples of FPE solutions, one of them yielding in a particular case the well-known Veneziano expression for 4-particle amplitude. Finally, we interpret our solutions of FPE in terms of relations in Lie groups.

1 Introduction

One of the most characteristic properties of scattering in string theory, as well as in some strong interaction processes, is the so-called $s \leftrightarrow t$ duality. Schematically, it can be represented as in Figure 1.

We assume here the point of view that the 4-point amplitudes $A(k,l,m,n)$, where $k,l,m,n$ are some ‘quantum numbers’, in both sides of Figure 1 are obtained out of two 3-point amplitudes $A(k,l,m)$ by integrating away the quantum number corresponding to the internal line, so that analytically the duality looks like

$$\int A(k,l,s)A(s,m,n)\,d\mu(s) = \int A(k,m,t)A(t,l,n)\,d\mu(t),$$

where $\mu$ is some measure. In (1) the 3-point amplitude is assumed to be symmetric in all its arguments. If it is not, the formula (1) must be written more carefully, as we will see below in section 4.

*Permanent address: South Ural State University, 76 Lenin ave., Chelyabinsk 454080, Russia. E-mail address: igor@prima.tu-chel.ac.ru
†E-mail address: saito@phys.metro-u.ac.jp
As is known, the $s \leftrightarrow t$ duality results in a dramatic reduction of the number of Feynman diagrams: any two diagrams with the same numbers of external lines and cycles are equivalent.

What mathematical structures form the basis of such duality? Of course, within the usual string theory it can be explained ‘geometrically’ by saying that to the two sides of Figure 1 corresponds in fact the same ‘string diagram’ (Figure 2). Suppose, however, that we want to have a general algebraic mechanism for $s \leftrightarrow t$ duality irrespective of such pictures and hopefully providing new possibilities for constructing string-like theories.

String theory has intimate connections with many fields of mathematics. We are mostly interested in its relations with integrable models. So, let us mention here
the string–soliton correspondence \[1, 2, 4\] and the fact that the string amplitudes satisfy the Yang–Baxter equation \[3\].

There exist, however, different kinds of fundamental equations responsible for integrability. The most important is believed to be the tetrahedron equation (TE), which deals with \(2 + 1\)-dimensional integrability: the quantum TE \[5\] for quantum models and the functional TE \[6, 8\] for both classical and quantum models. At the same time, the different equations are strongly connected with one another. In particular, the tetrahedron equation is connected with the pentagon equation \[9\].

In this paper we argue that the mathematical structure responsible for \(s \leftrightarrow t\) duality is the functional pentagon equation (FPE). We demonstrate it on simple examples. As is known, strings have close relations with infinite-dimensional groups. Nevertheless, we believe it is natural to start from FPE solutions related to finite-dimensional Lie groups. So, the modest aim of this paper is to demonstrate that there can exist some algebraic mechanism for \(s \leftrightarrow t\) duality based on FPE solutions, and if we can pass from finite-dimensional to infinite-dimensional groups (which looks very plausible), we will obtain wide range of new string-like theories.

Below, in section 2 we explain what is the FPE and how it arises naturally when dealing with \(s \leftrightarrow t\) duality. In sections 3 and 4 we present two simple geometric constructions for 3-point amplitudes that obey duality. The amplitude of section 4 generalizes the well-known Veneziano 4-particle amplitude. Finally, in section 5 we show that our constructions can be described algebraically in terms of relations in Lie groups, namely the group of movements of euclidean plane and the Heisenberg group.

## 2 Functional pentagon equation

Our idea of a duality mechanism is very simple. Suppose that, for any fixed quantum numbers on the external lines of both sides in Figure 1, there exists some correspondence law \(f: t \mapsto s\) such that

\[
 t = f(s) \Rightarrow A(k, l, s)A(s, m, n)\, d\mu(s) = A(k, m, t)A(t, l, n)\, d\mu(t). \tag{2}
\]
Here it is implied tacitly that function \( f \) also depends on the ‘outer’ variables \( k, l, m \) and \( n \).

It is clear that (2) is sufficient for (1) to hold. Consider now the scattering diagram (a) in Figure 3 and transform it into the diagram (b) in two ways, as shown in the Figure. Suppose that some quantum numbers have been attached to all lines of diagram (a) (including the internal ones). Using the function \( f \), we get quantum numbers for diagram (b) as well. It is very natural to require that the function \( f \) satisfy the compatibility condition: two sequences of transformations in Figure 3 must result in the same quantum numbers for diagram (b). And this compatibility condition is nothing but some version of functional pentagon equation.

To see where the pentagon is, let us draw a ‘Poincaré dual’ for Figure 3 as Figure 4. Here the vertices of Figure 3 are represented as triangles, and to a transform of the type of Figure 1 corresponds deleting of a diagonal of a quadrilateral and replacing it with the other diagonal.

Note that the variables (quantum numbers) are attached to the edges (sides and
diagonals) of the pentagon. There exist also other versions of FPE where variables belong e.g. to the triangles themselves, see [9, 7].

3 Geometric duality for edge lengths

The geometrical picture of Figure 4 suggests at once a possibility for choosing function $f$. Namely, let us draw, as Figure 5, the ‘Poincaré dual’ of Figure 1 on the euclidean plane and take the lengths of edges as ‘quantum numbers’. If the lengths $l_1, l_2, l_3, l_4$ and $l_5$ of edges in Figure 5 are given, then $l_6$ is determined from the equation

$$S_{ABD} + S_{BCD} = S_{ABC} + S_{ACD},$$

where $S_{...}$ is the area of the corresponding triangle expressed through the lengths of its sides. For example,

$$S_{ABD} = S(l_1, l_2, l_3) = \frac{1}{4}\sqrt{(l_1 + l_2 + l_3)(l_2 + l_3 - l_1)(l_3 + l_1 - l_2)(l_1 + l_2 - l_3)}.$$

The fact that such a transformation $f: l_3 \mapsto l_6$ obeys the pentagon equation is evident from geometrical argument (a pentagon in which the lengths of all sides and
two diagonals are given is a ‘rigid body’ where distance between any two points is fixed and does not depend on a chain of algebraic transformations we have used to calculate it).

Consider the obvious relation

\[ d(\alpha + \gamma) = -d(\beta + \delta) \quad (4) \]

for the angles in Figure 5, and let the sides of the quadrilateral \( ABCD \) be fixed and only its diagonals vary. Using formulae like

\[ \sin \alpha = \frac{2S_{ABD}}{l_1 l_2}, \quad \cos \alpha = \frac{l_1^2 + l_2^2 - l_3^2}{2l_1 l_2}, \]

we will find

\[ d\alpha = -\frac{d\cos \alpha}{\sin \alpha} = \frac{l_3 dl_3}{2S_{ABD}} \]

and similarly for \( \beta, \gamma \) and \( \delta \). Substituting these angle differentials into (4) and taking (3) into account, it is not hard to derive the relation

\[ \frac{l_3 dl_3}{S_{ABD} \cdot S_{BCD}} = -\frac{l_6 dl_6}{S_{ABC} \cdot S_{ACD}}. \quad (5) \]
The relation (3) together with (5) suggests the following form for 3-point amplitude \( A(l_1, l_2, l_3) \) and measure \( d\mu(l) \) satisfying the condition (2):

\[
A(l_1, l_2, l_3) = \frac{e^{\lambda S(l_1, l_2, l_3)}}{S(l_1, l_2, l_3)},
\]

where \( \lambda \) is an overall arbitrary constant, and

\[
d\mu(l) = l \, dl.
\]

As for the minus sign in (5), its role becomes clear when we choose the integration path in \( l_3 \) and/or \( l_6 \). Here some freedom seems to exist. If we regard the lengths as complex variables, the equation (3) determines, for fixed \( l_1, l_2, l_4 \) and \( l_5 \), some Riemann surface whose points are pairs \((l_3, l_6)\). So, probably, some cycles on that surface can be taken as integration contours. Here we will not go that far, but just naïvely assume \( ABCD \) to be a convex quadrilateral in a usual real euclidean plane, and let \( l_3 \) change from its minimal value compatible with this assumption (and with given \( l_1, l_2, l_4 \) and \( l_5 \)) to its maximal value. Then it is clear that \( l_6 \) changes from its maximal value to its minimal value.

If we reverse the direction of integration in \( l_6 \), the minus sign disappears, and finally the formula (1) acquires the form

\[
\int_{(l_3)_{\text{min}}}^{(l_3)_{\text{max}}} \frac{e^{\lambda S(l_1, l_2, l_3)}}{S(l_1, l_2, l_3)} \frac{e^{\lambda S(l_3, l_4, l_5)}}{S(l_3, l_4, l_5)} l_3 \, dl_3 = \int_{(l_6)_{\text{min}}}^{(l_6)_{\text{max}}} \frac{e^{\lambda S(l_1, l_5, l_6)}}{S(l_1, l_5, l_6)} \frac{e^{\lambda S(l_2, l_4, l_6)}}{S(l_2, l_4, l_6)} l_6 \, dl_6.
\]

4 Geometric duality for angular coefficients—a generalization of Veneziano amplitude

Figure 4 suggests in fact one more choice of a transformation satisfying the FPE. Let us take as a quantum number the \textit{angular coefficient} \( k \) of a given edge (if the ends of the edge are \((x_1, y_1)\) and \((x_2, y_2)\) in some fixed frame of reference, not necessarily orthogonal, then \( k = (y_2 - y_1)/(x_2 - x_1) \)). It is not hard to see that if values \((k_1, k_2, k_3, k_4)\) and \(k_5\) in Figure 6 are given, then \(k_6\) is determined uniquely. Let us write the formula for finding \(k_6\) in the following form:

\[
\frac{k_3 - k_2}{k_3 - k_1} \frac{k_3 - k_5}{k_3 - k_4} = \frac{k_6 - k_2}{k_6 - k_4} \frac{k_6 - k_5}{k_6 - k_1}.
\]
The structure of this relation is
\[ F(ABD) \cdot G(BCD) = H(ABC) \cdot K(ACD), \] (9)
b by which we mean that the l.h.s. is the product of two expressions corresponding
to triangles \( ABD \) and \( BCD \) respectively, while the r.h.s. corresponds in a similar
way to triangles \( ABC \) and \( ACD \). So, relation (8) is similar to (3), although it is
‘multiplicative’ rather than ‘additive’.

Note that (a) equation (8) has also the solution \( k_6 \equiv k_3 \) which we are not inter-
ested in. Let us agree that we have rejected that solution; (b) the transformation
\( f: k_3 \mapsto k_6 \), given \( k_1, k_2, k_4 \) and \( k_5 \), is an involution.

The fact that the transformation \( f \) obeys the pentagon equation follows from the
fact that a pentagon for which the angular coefficients of all sides and two diagonals
are given is determined uniquely up to a similarity and a shift.

Remarkably, there exist two more multiplicative relations yielding the same de-
pendence \( f: k_3 \mapsto k_6 \) and having the same structure (9):
\[ \frac{k_1 - k_2}{k_1 - k_3} \cdot \frac{k_4 - k_3}{k_4 - k_5} = \frac{k_4 - k_2}{k_4 - k_6} \cdot \frac{k_1 - k_6}{k_1 - k_5} \] (10)
and
\[ \frac{k_2 - k_1}{k_2 - k_3} \cdot \frac{k_5 - k_3}{k_5 - k_4} = \frac{k_2 - k_6}{k_2 - k_4} \cdot \frac{k_5 - k_1}{k_5 - k_6}. \] (11)

For the analog of formula (5) we can take
\[ \frac{dk_3}{(k_3 - k_2)(k_3 - k_5)} = -\frac{dk_6}{(k_6 - k_2)(k_6 - k_5)}. \] (12)

The reader can verify that (12) follows from (8), provided \( k_6 \not\equiv k_3 \).

An important feature of formulae (8–12) is that they remain valid after a Möbius (rational) transformation of all \( k \)'s:
\[ k_j \mapsto \frac{ak_j + b}{ck_j + d}, \quad j = 1, \ldots, 6. \]

Such transformations correspond just to another choice of coordinate axes for Figure 6.

The most general formula of type (1) that we can obtain from (8–12) results from raising all terms in (8) to some degree \( \alpha \), in (10) to some degree \( \beta \), in (11) to some degree \( \gamma \) and multiplying all together with corresponding terms of (12). Using the fact that
\[ k_3 = k_2 \iff k_6 = k_5 \quad \text{and} \quad k_3 = k_5 \iff k_6 = k_2, \]
we can choose e.g. a curve joining \( k_2 \) and \( k_5 \) as the integration path, and write the final formula as
\[
\int_{k_2}^{k_5} A(k_3, k_1, k_2| -\alpha, \beta, -\gamma) A(k_3, k_4, k_5| -\alpha, -\beta, \gamma) \, dk_3
\]
\[
= \int_{k_2}^{k_5} A(k_6, k_4, k_2| -\alpha, \beta, \gamma) A(k_6, k_1, k_5| -\alpha, -\beta, -\gamma) \, dk_6, \] (13)

where
\[ A(l, m, n|\lambda, \mu, \nu) = (l - m)^{\lambda - \mu}(m - n)^{\mu - \nu}(n - l)^{\nu - \lambda - 1}. \] (14)

Both sides of (13) generalize the well-known Veneziano expression [10, 15] for the 4-point amplitude. To see this, let us put
\[ k_1 = k_4 = \infty, \quad k_2 = 0, \quad k_5 = 1. \]
Then, for example, the l.h.s. of (13) yields, up to a constant multiplier,
\[ \int_0^1 k_3^{\alpha-\gamma-1}(k_3 - 1)^{\alpha+\gamma-1} \, dk_3, \]
which coincides with the Veneziano amplitude up to the obvious change of notations.

It will be certainly of big interest to compare the five-point and, more generally, \(N\)-point amplitudes that can be obtained in such way with those in classical papers [11, 12, 13, 14].

5 Discussion: a group-theoretical comment

Let us explain why we believe that our constructions are related to Lie groups. Consider two transformations acting on points \((x, y)\) of a euclidean plane: shifting by \(a\) along the \(x\) axis
\[ S(a): (x, y) \mapsto (x + a, y) \] (15)
and rotation through the angle \(\phi\)
\[ R(\phi): (x, y) \mapsto (x \cos \phi - y \sin \phi, x \sin \phi + y \cos \phi). \]
Then, the existence of a triangle with sides \(l, m, n\) and external angles \(\alpha, \beta, \gamma\) means the equality
\[ R(\alpha) \circ S(n) \circ R(\beta) \circ S(l) \circ R(\gamma) \circ S(m) = 1. \] (16)

Note that the group of movements of a euclidean plane is three-parametric, and that is why \(\alpha, \beta\) and \(\gamma\) can be determined from given \(l, m,\) and \(n\). Similar to (16) relations can be written also for quadrilaterals and pentagons, and all the geometric constructions of section 3 can be described in terms of such relations.

As for section 4, its constructions have nothing to do with euclidean distance, so, in our opinion, here more relevant is the three-parametric group generated by the transformations \(S(a)\) (13) and
\[ T(\kappa): (x, y) \mapsto (x, y + \kappa x). \]
Then the existence of a triangle whose sides have \(x\)-projections \(a, b, c\) (where \(a + b + c = 0\)) and angular coefficients \(l, m, n\) is described by the equality
\[ S(a) \circ T(l - m) \circ S(b) \circ T(m - n) \circ S(c) \circ T(n - l) = 1. \] (17)
From the abstract point of view, the group generated by $S(a)$ and $T(\kappa)$ is nothing but the Heisenberg group. One can readily see this from the relation
\[ T(\kappa) \circ S(a) \circ T(-\kappa) \circ S(-a) = C(\kappa a), \]
where
\[ C(a): (x, y) \mapsto (x, y + a) \]
is a central element for any $a$.

We believe that studying relations of the type (16,17) in greater groups is the algebraic clue for constructing wide generalizations of the string theory.

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