CLASSIFICATION OF AFFINE HOMOGENEOUS SPACES OF COMPLEXITY ONE

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Abstract. The complexity of an action of a reductive algebraic group $G$ on an algebraic variety $X$ is the codimension of a generic $B$-orbit in $X$, where $B \subset G$ is a Borel subgroup. We classify affine homogeneous spaces $G/H$ of complexity one. These results are the natural continuation of the classification of spherical affine homogeneous spaces, i.e., spaces of complexity zero.

1. Introduction

Let $G$ be a connected reductive algebraic group over an algebraically closed field $\mathbb{K}$ of characteristic zero and $H$ be an algebraic subgroup of $G$. The complexity $c(G/H)$ is an important integer-valued invariant of the homogeneous space $G/H$. The notion of complexity was introduced by Luna and Vust for homogeneous spaces [LV83] and by Vinberg for arbitrary $G$-varieties [Vi86]. It has been shown in further researches that the complexity plays the key role in the study of geometry of a homogeneous space, in the theory of its equivariant embeddings, in the theory of invariant Hamiltonian systems on the cotangent bundle $T^*(G/H)$, and in other fields of mathematics concerned with homogeneous spaces.

Let us recall the definition of complexity. Let a reductive group $G$ act on an irreducible algebraic variety $X$ and $B$ be a Borel subgroup of $G$. The complexity $c_G(X)$ of the $G$-variety $X$ is the minimal codimension of $B$-orbits in $X$ for the induced action $B : X$. (We shall write $c(X)$, if it does not lead to misunderstanding.) By Rosenlicht’s Theorem, $c_G(X)$ equals the transcendence degree of the field $\mathbb{K}(X)^B$ of rational $B$-invariant functions on $X$. In addition to complexity, there is one more important characteristic of the action. The rank $\text{rk}(X)$ of an irreducible $G$-variety $X$ is the rank of the weight lattice of $B$-semiinvariant rational functions on $X$.

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A normal $G$-variety $X$ is called spherical if $c(X) = 0$ or, equivalently, $\mathbb{K}(X)^H = \mathbb{K}$. A space $G/H$ and a subgroup $H \subseteq G$ are called spherical if $G/H$ is a spherical $G$-variety. The theory of spherical varieties is one of the most developed parts of the theory of algebraic transformation groups, see e.g. [Br97]. In particular, for spherical homogeneous spaces there exists a remarkable theory of embeddings [LV83], [Kn91], generalizing the theory of toric varieties. In [Ti97], D.A. Timashev obtained a similar (but much more difficult) description of embeddings for homogeneous spaces of complexity one.

Spaces of small complexity appear naturally in the study of the question ”whether the number of orbits in any embedding of a given homogeneous space is finite?”. For example, the number of orbits in any embedding of a spherical homogeneous space is finite. At the same time, any homogeneous space of complexity $\geq 1$ admits a projective embedding with infinitely many orbits [Ak85]. More generally, the maximal value of $G$-modality (that is the maximal number of parameters in a continuous family of $G$-orbits) over all embeddings of the space $G/H$ is equal to $c(G/H)$ [Vi86], [Ak88].

Suppose that the homogeneous space $G/H$ is affine and we consider only its affine embeddings. It is known that every affine embedding of the space $G/H$ of complexity $\geq 2$ contains finitely many $G$-orbits if and only if $G/H$ is affinely closed, that is any affine embedding of $G/H$ consists of only one orbit [AT01]. An affine homogeneous space of complexity one admits an affine embedding with infinitely many orbits if and only if there exists an extension of $H$ by a one-dimensional torus in $N_G(H)/H$ such that the complexity of the obtained homogeneous space is equal to one [AT01]. Among homogeneous spaces of simple groups all the spaces with this property correspond to item 3 of Table 2 of this paper. In the case of semisimple groups there are much more such spaces. They correspond to items 9, 10, 15, 17, and 26 of Table 4. For more information on relations between complexity and modality, see [Ar01].

Another reasons in favor of the investigation of homogeneous spaces of small complexity are problems of symplectic geometry and theory of integrability of invariant Hamiltonian systems. Let $G$ be a connected reductive real Lie group, and $K$ be its connected closed reductive subgroup. The cotangent bundle $T^*(G/K)$ has a natural structure of a symplectic variety, and the action $G : G/K$ induces the symplectic action $G : T^*(G/K)$. Denote by $P$ the momentum map $P : T^*(G/K) \to g^*$ corresponding to this action. The algebras of real analytic functions $C(T^*(G/K))$ and $C(g^*)$ have the canonical Poisson brackets. Note that for all $h_1, h_2 \in C(g^*)$ we have $\{h_1 \circ P, h_2 \circ P\} = \{h_1, h_2\} \circ P$. Functions on $T^*(G/K)$ having the form $h \circ P$ are called collective. By Noether’s Theorem, collective functions are integrals for every flow on $T^*(G/K)$ with $G$-invariant Hamiltonian. In other words, $\{h \circ P, f\} = 0$ for every
A $G$-invariant Hamiltonian system $\dot{x} = s\text{grad} f$ is said to be \textit{integrable in the class of Noether’s integrals} or \textit{collectively completely integrable}, if there exists a set \( \{ h_i \} \), \( i = 1, \ldots, \dim G/K \), of real analytical functions on \( g^* \) such that the functions \( h_i \circ P \) are functionally independent and every pair is in involution. It is known that any $G$-invariant Hamiltonian system on $T^*(G/K)$ is integrable in the class of Noether’s integrals if and only if the complexified homogeneous space $G^C/K^C$ is spherical \[\text{[My86]}\]. If the subgroup $K$ is compact, then these conditions are equivalent to the fact that the Poisson bracket vanishes on every pair of functions from $C(T^*(G/K))^G$ \[\text{[GS84]}, \text{see also \[\text{[Vi01, Prop. 9]}\].}\]

The recent works \[\text{[MS00], [My01]}\] are concerned with investigations of properties of Hamiltonian systems on cotangent bundles of homogeneous spaces of positive complexity. As shown in these papers, if $K$ is compact, then the number of independent $G$-invariant real analytical functions on $T^*(G/K)$, that are also independent from collective functions, is equal to $2c(G^C/K^C)$. (This number equals the corank of the symplectic action $G^C : T^*(G^C/K^C)$, see \[\text{[Vi01]}\].) On the other hand, the maximal number of independent collective functions in involution equals $\dim G/K - c(G^C/K^C)$. In the case of homogeneous spaces of complexity one \[1\], if we add to a maximal independent system of collective functions in involution an independent of them invariant function $F$, then we shall obtain a complete system of functions in involution. Choosing $F$ commuting with the Hamiltonian, one can show that any $G$-invariant Hamiltonian system on the cotangent bundle of a homogeneous space of complexity one is integrable \[\text{[My01, Prop. 12]}\].

We may conclude that homogeneous spaces of small complexity are natural and important objects. Hence the list of all such spaces $G/H$ might be useful. The case of a reductive subgroup $H$ is a natural beginning of the classification. (By Matsushima’s Criterion, this condition is equivalent to affinity of the space $G/H$.) Under this restriction, the list of homogeneous spaces of the complexity $\leq 1$ is not too long. This confirms the hypothesis stated at the end of \[\text{[Vi86]}, \text{where the problem of classification of homogeneous spaces of complexity one was posed for the first time.}\]

It is well known that the complexity of a homogeneous space $G/H$ is determined by the pair of the tangent algebras $(g, h)$ \[\text{(Proposition 1)}\]. This allows to reduce the classification to the enumeration of pairs $(g, h)$, where $g$ is a reductive Lie algebra, $h$ is its reductive subalgebra such that the generic codimension of the subspace $h + \text{Ad}(g)b$, $g \in G$ in the algebra $g$ equals one. Without loss of generality, it can be assumed that the algebra $g$ is semisimple \[\text{(Proposition 2)}\].

The aim of this paper is to classify all affine homogeneous spaces of complexity one for reductive groups in terms of their tangent algebras.\footnote{In \[\text{[My01]}\] they are called \textit{almost spherical spaces}.}
The results are represented in Table 4. The classification of spherical reductive subgroups in simple groups was obtained in [Kr79], the classification of spherical reductive subgroups in reductive groups was obtained in [My86] and, independently, in [Br87], and the classification of reductive subgroups of complexity one in simple groups is given in [Pa92], see also [MS00]. For convenience of the reader the results of these classifications are included in the text (see Table 1, Table 3 and Table 2 respectively). Note that the classification is made up to automorphisms of the ambient algebra \( g \). For example, there exist exterior automorphisms of \( \mathfrak{so}_8 \) that take the spherical pairs \((\mathfrak{so}_8, \mathfrak{spin}_7)\) and \((\mathfrak{so}_8, \mathfrak{sp}_4 \oplus \mathfrak{sl}_2)\) to the pairs corresponding to item 10 of Table 1 with \((n, m) = (7, 1) (5, 3)\) respectively, so these pairs are not represented in the table. In Tables 1 and 2 we also indicate \( \mathfrak{h}\)-module \( \mathfrak{m} \) corresponding to the isotropy representation. In Tables 3 and 4 we do not indicate the isotropy representation, but it can be found easily since the pairs of these tables are obtained by the operation of coupling in the pairs from Tables 1 and 2. (The definition of this operation will be given below.)

Suppose we make a coupling of a component \( \mathfrak{h} \); after this the isotropy representation will be the direct sum of the isotropy representations of the coupled pairs and one copy of the adjoint representation of \( \mathfrak{h} \). In Tables 3 and 4 we indicate the stationary subalgebras of general position for the isotropy representation and the rank of corresponding homogeneous spaces. (Note that the rank of a homogeneous space \( G/H \) is also determined by the pair \((\mathfrak{g}, \mathfrak{h})\), see [Pa90].)

The method used in this paper is similar to the method of I. V. Mykytyuk [My86]. In particular, in [My86] the operation of coupling was called "the extension of a pair". Let us remark that, using the operation of coupling of two pairs "algebra-subalgebra", we do not need to consider the depth of a subalgebra (M. Brion), and may use induction only on the number of simple components of the ambient algebra \( \mathfrak{g} \). The computation of complexity of a homogeneous space is based on the study of stationary subalgebras of general position (we use Elashvili’s Tables [El72]) and Panyushev’s formulæ (Theorems 1 and 2).

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2. SOME FACTS ABOUT THE COMPLEXITY AND THE RANK OF HOMOGENEOUS SPACES

Everywhere in the text Lie algebras of reductive groups are denoted by the corresponding Gothic lowercase letters to the exclusion of the exceptional simple Lie algebras. By tradition, they are denoted by the Latin capital letters. In this section we discuss some well-known facts about the complexity of homogeneous spaces.
Lemma 1. Let $F$ be a reductive subgroup of $G$, and $H$ be an algebraic subgroup of $F$. Then

1) $c_G(G/F) \leq c_G(G/H)$;
2) $c_F(F/H) \leq c_G(G/H)$.

Proof. By [Vi86], the complexity of a $G$-variety $X$ equals the modality of the induced action of $B$ on $X$. Let $X$ be the homogeneous space $G/H$. Then the complexity is the modality of the double-sided action of $B \times H$ on $G$ or of the right action of $H$ on $B \setminus G$. Hence, we have inequality 1). In the same way we prove inequality 2). Indeed, there exists a Borel subgroup $B$ of $G$ such that the intersection of $B$ and $F$ is a Borel subgroup $B_F$ of $F$. Then $B_F \setminus F$ is isomorphic to the $F$-orbit of the point $Be$ in $B \setminus G$. □

Proposition 1. The following conditions are equivalent:

1) $c(G/H) = m$;
2) there exists an open subset $W \subset G$ such that for any $g \in W$ we have $\text{codim}_{G}(h + \text{Ad}(g)b) = m$.

Proof. Condition 1) holds if and only if there exists an open subset $V \subset G$ such that for any $g \in V$ the codimension of $BgH$ (or of $g^{-1}BgH$ in $G$ is equal to $m$. The latter is equivalent to condition 2) for $W = V^{-1}$. □

We say that a pair $(g, h)$ has complexity $m$ if condition 2) holds. In the sequel by $c(g, h)$ denote the complexity of a pair $(g, h)$. Proposition 1 allows to reduce the classification of homogeneous spaces of complexity one to the classification of corresponding pairs $(g, h)$.

Proposition 2. Let $g = g^s \oplus \mathfrak{3}$ be the decomposition into the direct sum of the maximal semisimple ideal and the center. Then the complexity of the pair $(g, h)$ equals the complexity of the pair $(g^s, pr_1(h))$, where $pr_1 : g \rightarrow g^s$ is the projection along $\mathfrak{3}$.

Proof. Since $\mathfrak{3}$ is contained in $\text{Ad}(g)b$ for any $g \in G$, we have $h + \text{Ad}(g)b = pr_1(h) + \text{Ad}(g)b$. □

Proposition 2 allows to consider only semisimple algebras $g$. In the sequel the subgroup $H$ and the subalgebra $h$ are supposed to be reductive.

Consider the restriction of the adjoint representation $\text{Ad} : G \rightarrow GL(g)$ to the subgroup $H$. Since the subalgebra $h$ is an invariant subspace for this representation and $H$ is reductive, there exists an $H$-invariant subspace $m$ of $g$ such that $g = h \oplus m$. The $H$-module $m$ can be naturally identified with the tangent space to the affine homogeneous space $G/H$ at the point $eH$. The linear representation $\rho : H \rightarrow GL(m)$ is said to be the isotropy representation of the pair $(G, H)$. Its differential is uniquely defined by the pair $(g, h)$. By $s$ denote a stationary subalgebra of general position (s.s.g.p.) for the
It is known that $s$ is reductive [Pa90, Th.3]. Let $b$ be a Borel subalgebra of $g$, $b_s$ be a Borel subalgebra of $s$, and $N(g)$ be the number of positive roots in the root system corresponding to the algebra $g$. The next theorem shows that the rank of $G/H$ is uniquely determined by the pair $(g,h)$. This justifies the notation $\text{rk}(g,h)$.

**Theorem 1.** [Pa90, Th.3] The following formulae hold:
$$c(G/H) = c(g,h) = \dim b - \text{rk} g - \dim h + \dim b_s = N(g) - \dim h + N(s) + \text{rk} s,$$
$$\text{rk}(G/H) = \text{rk}(g,h) = \text{rk} g - \text{rk} s.$$

For our calculations we also need the following theorem.

**Theorem 2.** [Pa94, Th.1.2] Suppose that a subalgebra $h$ is contained in a Levi subalgebra $l$ of a parabolic subalgebra $p$ of the algebra $g$, and $S_1$ is a stabilizer of general position for the action of $H$ on $l/h$. Then
$$c(g,h) = c(l,h) + c_{S_1}(p^u),$$
where $p^u$ is the unipotent radical of the algebra $p$. In particular, if $c(l,h) = 0$, then
$$c(g,h) = \dim p^u - \dim b_{s_1} + \dim b_1,$$
where $b_1$ is a s.s.g.p. for the action of $b_{s_1}$ on $p^u$.

### 3. Reductive Subalgebras of Semisimple Lie Algebras

Let $g = g_1 \oplus \cdots \oplus g_s$ be a decomposition of a reductive Lie algebra $g$ into the direct sum of the simple ideals. We say that $g_i$ are the components of the algebra $g$. If $g$ is semisimple, then all $g_i$ are simple (noncommutative) Lie algebras. If $g$ is reductive, then some $g_i$ can be one-dimensional.

Let $h$ be a reductive subalgebra of $g$, and $h(i)$ be the projection of $h$ on $g_i$ along the other components. Then $h(i)$ is a reductive subalgebra of $g_i$. We say that the subalgebra $h$ is completely decomposable if $h = h(1) \oplus \cdots \oplus h(s)$. In the general case, $h$ is contained in the completely decomposable subalgebra $h^r = h(1) \oplus \cdots \oplus h(s)$, and $h^r$ is the minimal completely decomposable subalgebra containing $h$.

In order to describe reductive subalgebras of $g$ we have to do two steps. The first is the description of completely decomposable subalgebras (this is reduced to the description of reductive subalgebras of simple algebras) and the second is the description of reductive subalgebras such that the projection on any component of the algebra is surjective.

**Lemma 2.** Let $g = g_1 \oplus \cdots \oplus g_s$ be a semisimple Lie algebra, and $h$ be a reductive subalgebra of $g$. Suppose that the projection of $h$ on every component $g_i$ is surjective; then $h$ is semisimple, and any its component $h_j$ embeds in $g_1 \oplus \cdots \oplus g_s$ by the rule $x \rightarrow (pr_{j_1}(x), \ldots, pr_{j_s}(x))$, where
Proof. Since the projection $pr_i : h \to g_i$ is surjective, we see that ideals of $h$ are taken to ideals of $g_i$. As the algebra $g_i$ is simple, the image of any $h_j$ is either 0 or $g_i$. This implies that there do not exist commutative ideals of $h$, i.e., $h$ is semisimple. Moreover, in the second case $pr_{ji}$ is an isomorphism. Finally, the kernel of the projection $pr_i : h \to g_i$ is an ideal of $h$, therefore it is a sum of some components of the algebra $h$. Thus $pr_i$ determines an isomorphism between $g_i$ and the sum of some $h_j$ such that $h_j$ is not contained in the kernel of $pr_i$. This proves the last statement of the Lemma.

It is easy to generalize Lemma 2 to a reductive $g$. In this case the center of $h$ is the center of $g$.

Let $v = v_1 \oplus \cdots \oplus v_s$ be a reductive Lie algebra and $\phi_{i,j} : v_i \to v_j$ be an isomorphism for some $i \neq j$. We say that the subalgebra

$$h = \{(x_1, \ldots, x_i, \ldots, x_{j-1}, \phi_{ij}(x_i), x_{j+1}, \ldots, x_l) \mid x_i \in v_i\}$$

is obtained from the algebra $v$ by the operation of coupling. In general, $h$ is obtained from $v$ by a multiple coupling if it is obtained from $v$ by the successive application of a finite number of couplings.

Conclusion. Any reductive subalgebra $h$ of a reductive Lie algebra $g = g_1 \oplus \cdots \oplus g_s$ can be obtained from a completely decomposable subalgebra $w = w(1) \oplus \cdots \oplus w(s)$ (= $h'$), where $w(i)$ are reductive subalgebras of $g_i$, by a multiple coupling of some components of the algebra $w$.

4. The method of classification

We say that a pair $(g, h)$ is decomposable if there exists a non-trivial decomposition of $g$ into the direct sum of ideals $g^1$ and $g^2$ such that $h = h^1 \oplus h^2$, where $h^1 = g^i \cap h$. By Proposition 1 it follows that if $h = h^1 \oplus h^2$ is a decomposable subalgebra of $g = g^1 \oplus g^2$, then the complexity is additive: $c(g, h) = c(g^1, h^1) + c(g^2, h^2)$. Thus, it is sufficient to classify indecomposable pairs.

We shall make the classification by induction on the number $s$ of components of the algebra $g$. If $s = 1$, then the algebra $g$ is simple and pairs of complexity 0 (respectively, 1) are represented in Table 1 (respectively, in Table 2). Suppose that we have classified all pairs $(g, h)$ of complexity $\leq 1$ for all algebras $g$ such that $s = k$ (List $k$). Then we make a multiple coupling in the subalgebra $h^1 \oplus h^2$ of $g^1 \oplus g^2$, where $(g^1, h^1)$ is a pair from List $k$ and a pair $(g^2, h^2)$ is from Table 1 or 2. Among obtained pairs we have all required pairs $(g, h)$ with $s = k + 1$. Moreover, one of the pairs $(g^1, h^1)$ or $(g^2, h^2)$ should be spherical (by Lemma 1), it follows that $c(g, h) \geq c(g^1, h^1) + c(g^2, h^2))$. Note also
that it is sufficient to consider couplings of simple components lying in different summands.

A pair \((g, h)\) is called saturated if the center of the algebra \(h\) is a completely decomposable subalgebra of \(g\) (in [My86] it was called "a main pair"). This means that \(h\) is obtained from a completely decomposable subalgebra of the algebra \(g\) by couplings of noncommutative components. We classify saturated indecomposable pairs of complexity one. As a consequence, we obtain the classification of all indecomposable pairs of complexity one, see Remark 2 below.

Let \((g, h)\) be a pair of complexity \(\leq 1\) and \(h_i\) be a component of \(h\). We say that \(h_i\) is valent (respectively, 1-valent) if \(h_i\) is not one-dimensional and the subalgebra obtained by the coupling of the component \(h_i\) in the subalgebra \(h \oplus h_i\) is spherical in \(g \oplus h_i\) (respectively, of complexity 1). Such couplings are called elementary.

The notion of valency allows to reduce the number of couplings under consideration. Namely, suppose that we make a non-elementary coupling. Then to obtain all desired pairs we should make couplings of either two valent components or valent and 1-valent components and the multiple couplings in the obtained algebras. (Using Theorem 1, it is easy to show that the complexity of the direct sum of two elementary couplings \((g_1 \oplus h \oplus g_2, h_1 \oplus h \oplus h_2)\) is not greater than the complexity of the non-elementary coupling \((g_1 \oplus g_2, h_1 \oplus h \oplus h_2)\).)

Step 1. At first we determine all valent and 1-valent components of pairs \((g, h)\), where \(g\) is simple. By the definition of complexity, it follows that \(c(g, h) \geq N(g) - \dim h\). Consequently for the valency (respectively, the 1-valency) of the component \(h_i\) it is necessary that \(N(g) + N(h_i) - \dim h \leq 0\) (respectively, \(\leq 1\)). By means of this condition from pairs of Tables 1 and 2 we obtain the following list of candidates for the valency or the 1-valency (they are underlined):

\[
\begin{align*}
&\text{\underline{(\mathfrak{sl}_3, \mathfrak{so}_3)}}, \text{ \underline{\text{(\mathfrak{sl}_{n+m}, \mathfrak{sl}_m \oplus \mathfrak{sl}_n \oplus c)}}}^2, \text{ \underline{\text{(\mathfrak{sl}_{n+m}, \mathfrak{sl}_m \oplus \mathfrak{sl}_n)}}}^3, \text{ \underline{\text{(\mathfrak{so}_{n+m}, \mathfrak{so}_m \oplus \mathfrak{so}_n)}}}^4, \\
&(\mathfrak{so}_7, \mathfrak{G}_2), (\mathfrak{sp}_4, \mathfrak{sl}_2 \oplus c)^5, (\mathfrak{sp}_{2(n+m)}, \mathfrak{sp}_{2m} \oplus \mathfrak{sp}_{2n})^6, (\mathfrak{G}_2, \mathfrak{sl}_3), \\
&(\mathfrak{G}_2, \mathfrak{sl}_2 \oplus \mathfrak{sl}_2), (\mathfrak{F}_4, \mathfrak{sl}_2 \oplus \mathfrak{sp}_6), (\mathfrak{E}_6, \mathfrak{sl}_2 \oplus \mathfrak{sl}_6), (\mathfrak{E}_7, \mathfrak{sl}_2 \oplus \mathfrak{so}_{12}), \\
&(\mathfrak{E}_8, \mathfrak{sl}_2 \oplus \mathfrak{E}_7), (\mathfrak{sp}_{2n}, \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sp}_{2n-4}).
\end{align*}
\]

To know which of these components are valent or 1-valent we use Theorems 1 and 2.

\(^2\)under some conditions on \(m, n\)  
\(^3\)under some conditions on \(m, n\)  
\(^4\)under some conditions on \(m, n\)  
\(^5\)the embedding corresponds to item 17 of Table 1  
\(^6\)under some conditions on \(m, n\)
Remark 1. Consider the pairs $(\mathfrak{g}_1, \mathfrak{sl}_2 \oplus \mathfrak{h}_1)$ ($\mathfrak{s}_1$ is a s.s.g.p. for the isotropy representation, the complexity equals $c_1$), $(\mathfrak{g}_2, \mathfrak{sl}_2 \oplus \mathfrak{h}_2)$ ($\mathfrak{s}_2$ is a s.s.g.p., the complexity equals $c_2$) and the coupling of the component $\mathfrak{sl}_2$ in the subalgebra: $(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{sl}_2 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2)$ ($\mathfrak{s}$ is a s.s.g.p., the complexity equals $c$). By Theorem 1, it follows that $c = c_1 + c_2$ if and only if in one of the pairs the projection of $\mathfrak{s}_i$ on the component $\mathfrak{sl}_2$ is surjective and for another component this projection is non-zero. The condition $c = c_1 + c_2 + 1$ holds if and only if either one of the projections is surjective and another is zero or both projections are one-dimensional (their images are Cartan subalgebras of $\mathfrak{sl}_2$). In other cases $c \geq c_1 + c_2 + 2$.

The projection of the algebra $\mathfrak{s}$ on the component $\mathfrak{sl}_2$ is zero except for the case where the projections of $\mathfrak{s}_1$ and $\mathfrak{s}_2$ on the component $\mathfrak{sl}_2$ are surjective. In this case the projection of $\mathfrak{s}$ on the component $\mathfrak{sl}_2$ is one-dimensional.

By Remark 1, it follows that after an elementary coupling of components isomorphic to $\mathfrak{sl}_2$ the complexity increases by no more then 1 and an $\mathfrak{sl}_2$-component of a spherical pair is always valent or 1-valent (in the first case the pair is contained in Table 3). By the method of exception, we obtain 1-valent components from spherical pairs:

$$(\mathfrak{sl}_3, \mathfrak{so}_3), (G_2, \mathfrak{sl}_2 \oplus \mathfrak{sl}_2), (F_4, \mathfrak{sl}_2 \oplus \mathfrak{sp}_6),$$

$$(E_6, \mathfrak{sl}_2 \oplus \mathfrak{so}_6), (E_7, \mathfrak{sl}_2 \oplus \mathfrak{so}_{12}), (E_8, \mathfrak{sl}_2 \oplus E_7), (\mathfrak{sp}_4, \mathfrak{sl}_2 \oplus \mathfrak{c}),$$

$$(\mathfrak{sl}_3, \mathfrak{sl}_2), (\mathfrak{so}_{n+3}, \mathfrak{so}_3 \oplus \mathfrak{so}_n).$$

It is easy to show that we have the next 1-valent components from the candidates originated from the pairs of complexity one:

$$(\mathfrak{sp}_{2n}, \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sp}_{2n-4}), (\mathfrak{sl}_4, \mathfrak{sl}_2 \oplus \mathfrak{sl}_2).$$

Let us pass to the question about the valency of components different from $\mathfrak{sl}_2$. We apply Theorem 2 to the couplings $(\mathfrak{sl}_{n+m} \oplus \mathfrak{sl}_m, \mathfrak{sl}_m \oplus \mathfrak{sl}_n \oplus \mathfrak{c})$ and $(\mathfrak{sl}_{n+m} \oplus \mathfrak{sl}_m, \mathfrak{sl}_m \oplus \mathfrak{sl}_n)$. In the first case $\mathfrak{b}_1 = \mathfrak{c} \oplus \mathfrak{b}^{n-m}$ (here $\mathfrak{b}^{n-m}$ is a Borel subalgebra of $\mathfrak{sl}_{n-m}$) if $m < n$ and $\mathfrak{b}_1 = 0$ if $m \geq n$. We obtain the valent components:

$$(\mathfrak{sl}_{n+2}, \mathfrak{sl}_2 \oplus \mathfrak{sl}_n \oplus \mathfrak{c}), (\mathfrak{sl}_{n+1}, \mathfrak{sl}_m \oplus \mathfrak{c}), (\mathfrak{sl}_{n+2}, \mathfrak{sl}_2 \oplus \mathfrak{sl}_n, n \geq 3),$$

and the 1-valent components:

$$(\mathfrak{sl}_{n+3}, \mathfrak{sl}_2 \oplus \mathfrak{sl}_n \oplus \mathfrak{c}, n \geq 2), (\mathfrak{sl}_{n+3}, \mathfrak{sl}_2 \oplus \mathfrak{sl}_n, n \geq 4), (\mathfrak{sl}_{m+1}, \mathfrak{sl}_m), (\mathfrak{sl}_4, \mathfrak{sl}_2 \oplus \mathfrak{sl}_2).$$

Consider the pair $(\mathfrak{so}_{n+m} \oplus \mathfrak{so}_m, \mathfrak{so}_m \oplus \mathfrak{so}_n)$. Here the isotropy representation has the form $m = \pi_1 \otimes \pi_1' \oplus ad(\mathfrak{so}_m)$. Hence, $\mathfrak{s} = \mathfrak{so}_{n-m} \subset \mathfrak{so}_n$ if $n > m$ and $\mathfrak{s} = 0$ if $n \leq m$. Thus we obtain the valent component $(\mathfrak{so}_{m+1}, \mathfrak{so}_m)$ and the 1-valent component $(\mathfrak{so}_{n+3}, \mathfrak{so}_3 \oplus \mathfrak{so}_n)$.

Consider a pair $(\mathfrak{sp}_{2(n+m)} \oplus \mathfrak{sp}_{2m}, \mathfrak{sp}_{2m} \oplus \mathfrak{sp}_{2n})$. The isotropy representation has the form $m = \pi_1 \otimes \pi_1' \oplus ad(\mathfrak{sp}_{2m})$. Thus we have $\mathfrak{s} = \mathfrak{c} \oplus \mathfrak{sp}_{2n-2}$ if $m = 1$ and $\mathfrak{s} = \mathfrak{sp}_{2(n-m)} \subset \mathfrak{sp}_{2n}$ if $m \geq 2$ ($\mathfrak{s} = 0$ if $n \leq
Therefore we obtain the valent components \((\mathfrak{sp}_{2(n+2)}, \mathfrak{sp}_1 \oplus \mathfrak{sp}_{2n})\), 
\((\mathfrak{sp}_{2(n+1)}, \mathfrak{sp}_2 \oplus \mathfrak{sp}_{2n})\) and the 1-valent component \((\mathfrak{sp}_8, \mathfrak{sp}_6 \oplus \mathfrak{sp}_2)\).

The isotropy representation of the coupling \((\mathfrak{so}_7 \oplus G_2, G_2)\) has the form \(m = \pi_1 \oplus \text{ad}(G_2)\). Here \(s = 0\) and the component is 1-valent.

Consider the pair \((G_2 \oplus \mathfrak{sl}_3, \mathfrak{sl}_3)\). Here \(\mathfrak{sl}_3\) embeds in \(G_2\) 
"via the long roots." The short roots are linearly independent on the Cartan subalgebra of \(\mathfrak{sl}_3\), \(s = 0\), and the component is 1-valent.

Thus we obtain all valent and 1-valent simple components of pairs \((\mathfrak{g}, \mathfrak{h})\) of complexity \(\leq 1\), where \(\mathfrak{g}\) is simple.

**Step 2.** Now it is convenient to deviate from the standard induction on the parameter \(s\). We indicate all indecomposable saturated pairs \((\mathfrak{g}, \mathfrak{h})\) of complexity one such that \((\mathfrak{g}, \mathfrak{h})\) is obtained from pairs corresponding to simple algebras by couplings of components isomorphic to \(\mathfrak{sl}_2\).

By Remark 1 and the list of all valent and 1-valent components isomorphic to \(\mathfrak{sl}_2\) (or by an explicit form of the isotropy representation), it follows that if \((\mathfrak{g}, \mathfrak{h})\) is a pair from the list of candidates (Step 1) such that \(\mathfrak{h}\) has a component isomorphic to \(\mathfrak{sl}_2\), then the projection of the subalgebra \(\mathfrak{s}\) on this component is zero except for the pairs

1) \((\mathfrak{sl}_{n+2}, \mathfrak{sl}_2 \oplus \mathfrak{sl}_n \oplus \mathfrak{c}), n \geq 1\), \((\mathfrak{sl}_{n+2}, \mathfrak{sl}_2 \oplus \mathfrak{sl}_n), n \geq 2\), \((\mathfrak{sp}_{2n}, \mathfrak{sl}_2 \oplus \mathfrak{sp}_{2n-4}), n \geq 3\) — here the projection is one-dimensional;

2) \((\mathfrak{sp}_{2(n+1)}, \mathfrak{sl}_2 \oplus \mathfrak{sp}_{2n})\) — here the projection is surjective.

Remark 1 and the information about the projections allow to find all pairs of complexity 1 obtained by couplings of components \(\mathfrak{sl}_2\) from pairs \((\mathfrak{g}, \mathfrak{h})\), where \(\mathfrak{g}\) is simple. By this way, we obtain items 1 – 22 of Table 4.

**Step 3.** By direct calculations based on examination of the dimension and the isotropy representation, it is easy to prove that all non-elementary couplings of valent and 1-valent components different from \(\mathfrak{sl}_2\) lead to pairs of complexity \(> 1\).

Finally we put items 24 – 29 corresponding to obtained 1-valent components to Table 4.

**Step 4.** Now we enumerate valent and 1-valent components different from \(\mathfrak{sl}_2\) for pairs of complexity \(\leq 1\) with \(s = 2\). Consider a pair \((\mathfrak{g}, \mathfrak{h})\). Suppose that we obtain this pair by couplings in the pairs \((\mathfrak{g}_i, \mathfrak{h}_i)\), where \(\mathfrak{g}_i\) are simple; then we have to consider only components corresponding to the valent and the 1-valent components of \((\mathfrak{g}_i, \mathfrak{h}_i)\).

Start with item 9 of Table 3. The component \(\mathfrak{h}\) of this pair is 1-valent if and only if \(N(\mathfrak{h}) - \text{rk}\mathfrak{h} = 1\). This holds only if \(\mathfrak{h} = \mathfrak{sl}_3\) (this corresponds to item 30 of Table 4-4). By Lemma 1, we follow that if \(\mathfrak{h}_i\) is a component of a pair of complexity one, \(\mathfrak{h}_i \neq \mathfrak{sl}_2, \mathfrak{sl}_3\) then \(\mathfrak{h}_i\) can not be embedded in more than two simple components of the algebra \(\mathfrak{g}\).
It remains to check the valency or the 1-valency of the underlined components of the next pairs:

\[
(g_1 \oplus \mathfrak{sp}_{2n+4}, h_1 \oplus \mathfrak{sp}_{2n} \oplus \mathfrak{sp}_4), \quad (n = 1, 2), \\
(g_1 \oplus \mathfrak{sp}_8, h_1 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sp}_6), \\
(g_1 \oplus \mathfrak{sl}_{n+3}, h_1 \oplus \mathfrak{sl}_n \oplus \mathfrak{sl}_3), \quad (n = 2, 3), \\
(g_1 \oplus \mathfrak{sl}_{n+3}, h_1 \oplus \mathfrak{gl}_n \oplus \mathfrak{sl}_3), \quad (n = 2, 3), \\
(g_1 \oplus \mathfrak{g}_2, h_1 \oplus \mathfrak{sl}_3 \oplus \mathfrak{h}_2),
\]

where the first component of \( h \) is embedded only in \( g_1 \), the third one only in the second component of \( g \), and the second one in the both components of \( g \).

Theorem 1 shows that a necessary condition for underlined components to be valent or 1-valent is that the projection of the algebra \( s \) on this component for the indicated pair contains a Borel subalgebra of dimension no less than 3, 8, 2, 2 2 respectively. Using the explicit form of a subalgebra \( s \) computed for all pairs of Table 4 for \( s = 2 \) we have that this condition holds only for \((\mathfrak{sp}_{2n+2} \oplus \mathfrak{sp}_n, \mathfrak{sp}_{2n} \oplus \mathfrak{sl}_2 \oplus \mathfrak{sp}_4)\). It is easy to show that the component \( \mathfrak{sp}_4 \) is 1-valent here (Table 4-4, item 23).

**Step 5.** For \( s = 3 \) we have to consider a non-elementary coupling of the components different from \( \mathfrak{sl}_2 \) only for item 16 of Table 1 with \( m = 2 \) and item 3 of Table 3 with \( m = 2 \) (here the joint component is \( \mathfrak{sp}_4 \)). The complexity of this coupling is greater than 3.

**Step 6.** Similarly to Step 4 it can be checked that a pair with \( s = 3 \) has no valent and 1-valent components different from \( \mathfrak{sl}_2 \).

It remains to note that for any pair from Table 4 if there exist two different isomorphisms determining embeddings of a component of the subalgebra into a component of the algebra, then these isomorphisms give the subalgebras which are taken to each other by an automorphism of the ambient algebra. Thus we proved

**Theorem 3.** All indecomposable saturated pairs \((g, h)\) of complexity one, where \( g \) is a simple Lie algebra and \( h \) is its reductive subalgebra, are represented in Table 4 (up to automorphisms of the algebra \( g \)).

It is interesting to note that in Table 4 there are no pairs of even rank greater than 8 and there is only one series of pairs of odd rank greater than 7 (item 27).

**Remark 2.** We shall show how to describe all indecomposable pairs of complexity one having the description of all saturated indecomposable pairs of complexity one. The idea of this method was suggested by...
D. A. Timashev. For any indecomposable pair \((g, h)\) we have the saturated (maybe decomposable) pair \((g, \tilde{h})\), where the semisimple components of \(h\) and \(\tilde{h}\) are the same (we denote them by \(h^s\)) and the center of \(\tilde{h}\) (we denote it by \(\tilde{z} = z_1 \oplus \cdots \oplus z_s\)) is the minimal completely decomposable subalgebra containing the center of \(h\) (we denote the center of \(h\) by \(z\)). If \(c(g, h) = 1\), then \(c(g, \tilde{h}) \leq 1\). In the algebra \(\tilde{z}\) there exists the largest subalgebra \(p\) such that
\[
c(g, \tilde{h}^s) = c(g, h^s + p) = c(g, \tilde{h}) - (\dim \tilde{z} - \dim p).
\]

In fact, consider the maximal subtorus of the center of the group \(\tilde{H}\) such that the generic \(H^s\)-orbit on \(G/B\) is invariant under its action. The subalgebra \(p\) is the tangent algebra to this subtorus. Note that \(p\) can be decomposed into the direct sum of subalgebras corresponding to the decomposition of the pair \((g, \tilde{h})\) into indecomposable components. Thus it is sufficient to find the subalgebra \(p\) for every saturated indecomposable pair. Tables 1, 2, 3, 4 imply that the subalgebra \(p\) is zero in all cases except for the followings:

1) Table 1: it.2, \(n \neq m\); it.6; it.7, \(n\) is odd; it.25;

2) Table 2, it.2: here \(p\) is a one-dimensional subalgebra described in it.3;

3) Table 4, it.8, \(n = 1, 2, m \geq 3\) and \(m = 1, 2, n \geq 3\): here \(p\) is one-dimensional and corresponds to the bigger component.

It remains to note that the pair \((g, h)\) has the complexity 1 if and only if either \(c(g, \tilde{h}) = 1\) and the restriction of the projection \(\tilde{z} \to \tilde{z}/p\) to the subalgebra \(z\) is surjective, or \(c(g, \tilde{h}) = 0\) and the image of \(z\) under the projection is a hyperplane.

**Notations and conventions used in the Tables.** The indexes \(n, m\) in Tables 1 and 2 are assumed to be \(\geq 1\) and the indexes \(n, m, k, l\) in Tables 3 and 4 are assumed to be \(\geq 0\) (if the converse is not mentioned). If the index of a classical algebra has a non-positive value (for example, \(sp_{2n-2}\) with \(n = 0, 1\)), then the corresponding algebra is assumed to be zero. The symbol \(\delta^j_i\) equals 1 if \(i = j\) and 0 otherwise. In Tables 1 and 2 the column "\(h \subset g\)" determines the embedding of the subalgebra \(h\) in the algebra \(g\) by the restriction of the first fundamental representation of \(g\) to \(h\) (the nomenclature of the fundamental weights is taken from [OV88]). By \(\pi_i\) and \(\pi_i'\) we denote the fundamental weights of the first and the second simple components of the algebra \(h\) respectively. By \(\epsilon\) we denote the fundamental weight of the one-dimensional central subalgebra \(c\). We use multiplicative notation for weights (for example, \(\pi_2^?\) is the highest weight with the first mark 2 and other marks zero). By the sign "+" we denote the direct sum of representations.
By the symbol 1 we denote the one-dimensional trivial representation, by 2 — the two-dimensional trivial representation, and so on. In Tables 3 and 4 segments denote the embedding of the subalgebra in the algebra. If a segment does not determine the embedding uniquely, then the embedding is indicated on the segment as in Tables 1 and 2.
| $g$                          | $h$                          | $h \subset g$                                      | $m$                      |
|-----------------------------|------------------------------|---------------------------------------------------|--------------------------|
| $sl_n, n \geq 2$            | $so_n$                       | $\pi_1$                                           | $\pi_1^2 (n \neq 2, 4)$ |
| $sl_{n+m}, m \neq n$       | $sl_n \oplus sl_m \oplus c$ | $\pi_1 e^{m-n} + \pi'_1 e^{-n}$                   | $\pi_1 \otimes \pi'_1 e^{-n-n}$ |
| $sl_{n+m}$, $n \geq 2$     | $sl_n \oplus sl_m$          | $\pi_1 + \pi'_1$                                  | $\pi_1 \otimes \pi'_1 + \pi_{n-1} \otimes \pi'_1 + 1$ |
| $sl_{2n}, n \geq 2$        | $sp_{2n}$                    | $\pi_1$                                           | $\pi_2$                  |
| $so_{4n+2}$                 | $sl_{2n+1}$                  | $\pi_1 + \pi_{2n}$                                | $\pi_2 + \pi_{2n-1} + 1$ |
| $so_{2n+1}$                 | $sl_n \oplus c$              | $\pi_1 e + \pi_{n-1} e^{-1} + 1$                  | $\pi_1 e + \pi_{2n} e^{-2} + \pi_{n-1} e^{-1}$ |
| $so_n + m$                  | $so_n \oplus so_m$          | $\pi_1 + \pi'_1$                                  | $\pi_1 \otimes \pi'_1$ |
| $so_7$                      | $G_2$                        | $\pi_1$                                           | $\pi_1$                  |
| $so_8$                      | $G_2$                        | $\pi_1 + 1$                                       | $\pi_1^2$                |
| $so_{10}$                   | $so_7 \oplus c$              | $\pi_3 + \varepsilon + \varepsilon^{-1}$         | $\pi_1 + \varepsilon + \pi_3 e^{-1}$ |
| $sp_{2n}$                   | $sp_{2n} \oplus c$          | $\pi_1 e + \pi_{n-1} e^{-1}$                      | $\pi_1 e + \pi_{2n} e^{-2} + \pi_{n-1} e^{-1}$ |
| $sp_{2(n-1)} \oplus c$     | $sp_{2n} \oplus sp_{2m}$    | $\pi_1 + \pi'_1$                                  | $\pi_1 \otimes \pi'_1$ |
| $sp_{2n}$                   | $sp_{2(n-1)} \oplus c$      | $\pi_1 + \varepsilon + \varepsilon^{-1}$         | $\pi_1 e + \pi_{n-1} e^{-1} + \varepsilon + \varepsilon^{-2}$ |
| $G_2$                       | $sl_3$                       | $\pi_1 + \pi_2 + 1$                               | $\pi_1 + \pi_2$          |
| $F_4$                       | $so_9$                       | $\pi_1 + \pi_4 + 1$                               | $\pi_4$                  |
| $F_4$                       | $sl_2 \oplus sl_2$          | $\pi_2 + \pi_1 \otimes \pi'_1$                   | $\pi_3 \otimes \pi'_1$ |
| $E_6$                       | $sp_8$                       | $\pi_2$                                           | $\pi_4$                  |
| $E_6$                       | $F_4$                        | $\pi_1 + 1$                                       | $\pi_1$                  |
| $E_6$                       | $so_{10}$                    | $\pi_1 + \pi_5 + 1$                               | $\pi_4 + \pi_5 + 1$      |
| $E_6$                       | $so_{10} \oplus c$          | $\pi_1 e^2 + \pi_6 e^{-1} + e^{-1}$               | $\pi_4 e^6 + \pi_5 e^{-6}$ |
| $E_6$                       | $sl_6 \oplus sl_2$          | $\pi_4 + \pi_1 \otimes \pi'_1$                   | $\pi_3 \otimes \pi'_1$ |
| $E_7$                       | $E_6 \oplus c$               | $\pi_1 e + \pi_5 e^{-1} + \varepsilon + \varepsilon^{-1}$ | $\pi_1 e^4 + \pi_5 e^{-4}$ |
| $E_7$                       | $sl_8$                       | $\pi_2 + \pi_6$                                   | $\pi_4$                  |
| $E_7$                       | $sl_6 \oplus sl_2$          | $\pi_2 + \pi_1 \otimes \pi'_1$                   | $\pi_5 \otimes \pi'_1$ |
| $E_8$                       | $so_{16}$                    | $\pi_2 + \pi_8$                                   | $\pi_7$                  |
| $E_8$                       | $sl_2 \times E_7$           | $\pi_1 \otimes \pi'_1 + \pi_1 + \pi_6$          | $\pi_1 \otimes \pi'_1$ |
|    | g         | h         | h ⊂ g                                       | m                                      |
|----|-----------|-----------|--------------------------------------------|----------------------------------------|
| 1  | sl_{2n}   | sl_n ⊕ sl_n | \pi_1 + \pi_1'|                          | \pi_1 ⊗ \pi_1' + \pi_{n-1} ⊗ \pi_{n-1}' + 1 |
| 2  | sl_n      | sl_{n-2} ⊕ c ⊕ c | \pi_1 \varepsilon_1 + \varepsilon_1' + \varepsilon_2 - n | \pi_1 \varepsilon_1 + \pi_1 \varepsilon_1' + \pi_{n-1} \varepsilon_1' + \pi_{n-1} \varepsilon_2 - n + \pi_{n-3} \varepsilon_1 - 1 - n |
| 3  | sl_n      | sl_{n-2} ⊕ c | \pi_1 \varepsilon_1 + \varepsilon_1' + \varepsilon_2 - d_1 \neq d_2, d_1 + d_2 = 2 - n | \pi_1 \varepsilon_1 + \pi_1 \varepsilon_1' + \pi_{n-2} + \pi_{n-3} \varepsilon_2 - d_2 |
| 4  | sp_l      | sp_4 ⊕ sl_2 ⊕ c | \pi_1 \varepsilon_1 + \pi_1 \varepsilon_1' - 2 | \pi_1 \varepsilon_1 + \pi_1 \varepsilon_1' + \pi_2 + \pi_1 \varepsilon_1' - 1 |
| 5  | so_n      | so_{n-2}   | \pi_1 + 2                                   | \pi_1 + \pi_1 + 1                       |
| 6  | so_{2n+1} | sl_n       | \pi_1 + \pi_{n-1} + 1                       | \pi_1 + \pi_2 + \pi_{n-1} + \pi_{n-2} + 1 |
| 7  | so_{4n}   | sl_{2n}    | \pi_1 + \pi_{2n-1}                         | \pi_2 + \pi_{2n-2}                      |
| 8  | so_0      | G_2 ⊕ c    | \pi_1 + \varepsilon + \varepsilon^{-1}     | \pi_1 + \pi_1 \varepsilon + \pi_1 \varepsilon^{-1} |
| 9  | so_{11}   | sl_2 ⊕ so_7 | \pi_1' + \pi_3                           | \pi_1' ⊗ \pi_3' + \pi_1'              |
| 10 | so_{10}   | so_7       | \pi_3 + 2                                  | \pi_1 + \pi_3 + \pi_3 + 1             |
| 11 | sp_{2n}   | sl_n       | \pi_1 + \pi_{n-1}                         | \pi_1 + \pi_{n-1} - 1 + 1             |
| 12 | sp_{2n}   | sp_{2n-2}  | \pi_1 + 2                                  | \pi_1 + \pi_1 + 3                       |
| 13 | sp_{2n}   | sp_{2n-4} ⊕ sl_2 ⊕ sl_2 | \pi_1 + \pi_1' + \pi_1''                  | \pi_1 ⊗ \pi_1' + \pi_1 ⊗ \pi_1'' + \pi_1' ⊗ \pi_1'' |
| 14 | sp_8      | sl_2       | \pi_1'                                    | \pi_1'                                  |
| 15 | E_6       | so_9 ⊕ c   | \pi_1 \varepsilon_1 + \varepsilon_2 + \pi_4 \varepsilon_1^{-1} + \varepsilon_4^{-1} | \pi_1 + \pi_4 \varepsilon + \pi_4 \varepsilon^{-1} |
| 16 | E_7       | E_6        | \pi_1 + \pi_5 + 2                         | \pi_1 + \pi_5 + 1                       |
| 17 | F_4       | so_8       | \pi_1 + \pi_3 + \pi_4 + 2                 | \pi_1 + \pi_3 + \pi_4                 |
Table 3

|   | \(h \subset g\) | \(s\) | \(rk\) |
|---|-----------------|------|-------|
| 1 | \(\text{gl}_n\) \(\text{sl}_2\) \(\text{sp}_{2m}\) | \(n \geq 1\) | \(\text{gl}_{n-2} \oplus \text{sp}_{2m-2}\) | \(5 - \delta^0_m - \delta^0_n\) |
| 2 | \(\text{sl}_n\) \(\text{sl}_2\) \(\text{sp}_{2m}\) | \(n \geq 3\) | \(\text{sl}_{n-2} \oplus \text{sp}_{2m-2}\) | \(6 - \delta^0_m\) |
| 3 | \(\text{sp}_{2n+2}\) \(\text{sp}_{2m+2}\) | | \(\text{sp}_{2n-2} \oplus \text{sp}_{2m-2} \oplus \text{c}\) | \(3 - \delta^0_m - \delta^0_n\) |
| 4 | \(\text{sp}_{2n+2}\) \(\text{sp}_{2m+2}\) | \(\text{sp}_{2l+2}\) | \(\text{sp}_{2n-2} \oplus \text{sp}_{2m-2} \oplus \text{sp}_{2l-2}\) | \(6 - \delta^0_m - \delta^0_n - \delta^0_l\) |
| 5 | \(\text{sp}_{2n+2}\) \(\text{sp}_{4}\) \(\text{sp}_{2m+2}\) | \(\text{sp}_{2n}\) \(\text{sp}_{4}\) | \(\text{sp}_{2n-2} \oplus \text{sp}_{2m-2}\) | \(6 - \delta^0_m - \delta^0_n\) |
| 6 | \(\text{sp}_{2n+4}\) \(\text{sp}_{4}\) | | \(\text{sp}_{2n-4}\) | \(6 - \delta^1_n\) |
| 7 | \(\text{sl}_n\) \(\text{sl}_{n+1}\) | \(n \geq 2\) | | \(2n - 1\) |
| 8 | \(\text{so}_n\) \(\text{so}_{n+1}\) | \(n \geq 3\) | | \(n\) |
| 9 | \(h = \text{simple}\) | \(h\) | Cartan subalgebra | \(\text{rk}\ h\) |
| \( h \subset g \) | \( s \) | \( \text{rk} \) |
|---|---|---|
| \( \begin{array}{c}
\text{sp}_{2n+2} \\
\text{sp}_{2n} \\
\text{sl}_2
\end{array} \) | \( \text{sp}_{2n-2} \) | \( 4 - \delta_n^0 \) |
| \( \begin{array}{c}
\text{sp}_{2n+2} \\
\text{sp}_{2n} \\
\text{sl}_2 \\
\text{sl}_2
\end{array} \) | \( \text{sp}_{2n-2} \) | \( 4 - \delta_n^0 \) |
| \( \begin{array}{c}
\text{sp}_{2n+2} \\
\text{sp}_{2n} \\
\text{sl}_2 \\
\text{sp}_6
\end{array} \) | \( \text{sp}_{2n-2} \) | \( 6 - \delta_n^0 \) |
| \( \begin{array}{c}
\text{sp}_{2n+2} \\
\text{sp}_{2n} \\
\text{sl}_2 \\
\text{sl}_6
\end{array} \) | \( \text{sp}_{2n-2} \oplus \mathfrak{c} \oplus \mathfrak{c} \) | \( 6 - \delta_n^0 \) |
| \( \begin{array}{c}
\text{sp}_{2n+2} \\
\text{sp}_{2n} \\
\text{sl}_2 \\
\mathfrak{so}_{12}
\end{array} \) | \( \text{sp}_{2n-2} \oplus \text{sl}_2 \oplus \text{sl}_2 \oplus \text{sl}_2 \) | \( 6 - \delta_n^0 \) |
| \( \begin{array}{c}
\text{sp}_{2n+2} \\
\text{sp}_{2n} \\
\text{sl}_2 \\
\text{E}_7
\end{array} \) | \( \text{sp}_{2n-2} \oplus \mathfrak{so}_8 \) | \( 6 - \delta_n^0 \) |
| \( \begin{array}{c}
\text{sp}_{2n+2} \\
\text{sp}_{2n} \\
\text{sl}_2 \\
\mathfrak{so}_{k+3}
\end{array} \) | \( \text{sp}_{2n-2} \oplus \mathfrak{so}_{k-3} \) | \( 5 - \delta_n^0 - \delta_k^3 \) |
| \( \begin{array}{c}
\text{sl}_{n+2} \\
\text{sl}_2 \\
\text{gl}_m \\
\text{gl}_{n, m} \; n, m \geq 1
\end{array} \) | \( \text{gl}_{n-2} \oplus \text{gl}_{m-2} \) | \( 6 - \delta_n^1 - \delta_m^1 \) |
| $h \subset g$ | $s$ | $\text{rk}$ |
|---|---|---|
| $\mathfrak{gl}_n \oplus \mathfrak{sl}_m \ (m \geq 3, n \geq 1)$ | $\mathfrak{gl}_{n-2} \oplus \mathfrak{sl}_{m-2}$ | $7 - \delta_n^{1}$ |
| $\mathfrak{sl}_n \oplus \mathfrak{sl}_m \ (n, m \geq 3)$ | $\mathfrak{sl}_{n-2} \oplus \mathfrak{sl}_{m-2}$ | $8$ |
| $\mathfrak{sp}_{2n+2} \oplus \mathfrak{sl}_3 \oplus 1$ | $\mathfrak{sp}_{2n-2}$ | $4 - \delta_{n}^{0}$ |
| $\mathfrak{sp}_{2n+2} \oplus \mathfrak{sp}_{2n} \oplus \mathfrak{sp}_{2m} \oplus \mathfrak{sp}_{2m+2}$ | $\mathfrak{sp}_{2n-2} \oplus \mathfrak{sp}_{2m-2}$ | $7 - \delta_{n}^{0} - \delta_{m}^{0}$ |
| $\mathfrak{gl}_n \oplus \mathfrak{sp}_{2k+2} \oplus \mathfrak{sp}_{2m+2}$ | $\mathfrak{gl}_{n-2} \oplus \mathfrak{sp}_{2k-2} \oplus \mathfrak{sp}_{2m-2}$ | $7 - \delta_n^{1} - \delta_k^{0} - \delta_m^{0}$ |
| $\mathfrak{sl}_n \oplus \mathfrak{sp}_{2k+2} \oplus \mathfrak{sp}_{2m+2}$ | $\mathfrak{sl}_{n-2} \oplus \mathfrak{sp}_{2k-2} \oplus \mathfrak{sp}_{2m-2}$ | $8 - \delta_k^{0} - \delta_m^{0}$ |
| h ⊂ g | s | rk |
|-------|---|----|
| $\mathfrak{sl}_{n+2} \mathfrak{sp}_4 \mathfrak{sp}_{2m+2}$ | $\mathfrak{sl}_n \mathfrak{sl}_2 \mathfrak{sl}_2 \mathfrak{sp}_{2m}$ $n \geq 1$ | $\mathfrak{gl}_{n-2} \oplus \mathfrak{sp}_{2m-2}$ | $7 - \delta_n^1 - \delta_m^0$ |
| $\mathfrak{sl}_{n+2} \mathfrak{sp}_4 \mathfrak{sp}_{2m+2}$ | $\mathfrak{sl}_n \mathfrak{sl}_2 \mathfrak{sl}_2 \mathfrak{sp}_{2m}$ $n \geq 3$ | $\mathfrak{sl}_{n-2} \oplus \mathfrak{sp}_{2m-2}$ | $8 - \delta_m^0$ |
| $\mathfrak{sp}_{2k+4} \mathfrak{sp}_{2n+2}$ | $\mathfrak{sp}_{2k} \mathfrak{sl}_2 \mathfrak{sl}_2 \mathfrak{sp}_{2n}$ $k \geq 1$ | $\mathfrak{sp}_{2k-4} \oplus \mathfrak{sp}_{2n-2}$ | $6 - \delta_n^0 - \delta_k^1$ |
| $\mathfrak{sp}_{2n+2} \mathfrak{sp}_{2m+2} \mathfrak{sp}_{2l+2} \mathfrak{sp}_{2k+2}$ | $\mathfrak{sp}_{2n} \mathfrak{sp}_{2m} \mathfrak{sl}_2 \mathfrak{sp}_{2l} \mathfrak{sp}_{2k}$ | $\mathfrak{sp}_{2n-2} \oplus \mathfrak{sp}_{2m-2} \oplus \mathfrak{sp}_{2l-2} \oplus \mathfrak{sp}_{2k-2}$ | $8 - \delta_n^0 - \delta_m^0 - \delta_l^0 - \delta_k^0$ |
| $\mathfrak{sp}_{2n+2} \mathfrak{sp}_{2k} \mathfrak{sp}_4 \mathfrak{sp}_{2m+2}$ | $\mathfrak{sp}_{2n} \mathfrak{sp}_{2k} \mathfrak{sl}_2 \mathfrak{sl}_2 \mathfrak{sp}_{2m}$ | $\mathfrak{sp}_{2n-2} \oplus \mathfrak{sp}_{2m-2} \oplus \mathfrak{sp}_{2k-2}$ | $8 - \delta_n^0 - \delta_m^0 - \delta_k^0$ |
| $\mathfrak{sp}_{2n+2} \mathfrak{sp}_4 \mathfrak{sp}_{2m+2}$ | $\mathfrak{sp}_{2n} \mathfrak{sl}_2 \mathfrak{sl}_2 \mathfrak{sl}_2 \mathfrak{sp}_{2m}$ | $\mathfrak{sp}_{2n-2} \oplus \mathfrak{sp}_{2m-2}$ | $8 - \delta_n^0 - \delta_m^0$ |
| $\mathfrak{sp}_4 \mathfrak{sp}_{2n+2}$ | $\mathfrak{c} \mathfrak{sl}_2 \mathfrak{sp}_{2n}$ | $\mathfrak{sp}_{2n-2}$ | $4 - \delta_n^0$ |
Table 4-4

|   | $h \subset g$ | $s$ | rk |
|---|---------------|-----|----|
| 23 | $\text{spin}_4 \quad \text{spin}_6 \quad \text{spin}_{2n+2}$ | $\text{spin}_{2n-2}$ | $7 - \delta^0_n$ |
| 24 | $G_2$ \quad $\text{spin}_3$ | 0 | 4 |
| 25 | $\text{spin}_{n+3} \quad \text{spin}_3$ | $\text{spin}_3$ | $\geq 2$ | $\text{spin}_{n-3}$ | $7 - \delta^2_n$ |
| 26 | $\text{spin}_{n+3} \quad \text{spin}_3$ | $\text{spin}_3$ | $\geq 4$ | $\text{spin}_{n-3}$ | 8 |
| 27 | $\text{spin}_{n+1} \quad \text{spin}_n$ | $\text{spin}_n$ | $\geq 2$ | 0 | $2n - 1$ |
| 28 | $\text{spin}_8 \quad \text{spin}_6$ | $\text{spin}_2 \quad \text{spin}_6$ | 0 | 7 |
| 29 | $\text{spin}_7 \quad G_2$ | $G_2$ | 0 | 5 |
| 30 | $\text{spin}_3 \quad \text{spin}_3 \quad \text{spin}_3$ | $\text{spin}_3$ | 0 | 6 |
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