On transverse triangulations

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Abstract. We show that every smooth manifold admits a smooth triangulation transverse to a given smooth map. This removes the properness assumption on the smooth map used in an essential way in Scharlemann’s construction [6].

1. Introduction

For $l \in \mathbb{Z}_{\geq 0}$, let $\Delta^l \subset \mathbb{R}^l$ denote the standard $l$-simplex. If $|K| \subset \mathbb{R}^N$ is a geometric realization of a simplicial complex $K$ in the sense of [5, Sec. 3], for each $l$-simplex $\sigma$ of $K$ there is an injective linear map $\iota_{\sigma}: \Delta^l \rightarrow |K|$ taking $\Delta^l$ to $|\sigma|$. If $X$ is a smooth manifold, a topological embedding $\mu: \Delta^l \rightarrow X$ is a smooth embedding if there exist an open neighborhood $\Delta^l_\mu$ of $\Delta^l$ in $\mathbb{R}^l$ and a smooth embedding $\bar{\mu}: \Delta^l_\mu \rightarrow X$ so that $\bar{\mu}|_{\Delta^l} = \mu$. A triangulation of a smooth manifold $X$ is a pair $T = (K, \eta)$ consisting of a simplicial complex and a homeomorphism $\eta: |K| \rightarrow X$ such that

$$\eta \circ \iota_{\sigma}: \Delta^l \rightarrow X$$

is a smooth embedding for every $l$-simplex $\sigma$ in $K$ and $l \in \mathbb{Z}_{\geq 0}$. If $T = (K, \eta)$ is a triangulation of $X$ and $\psi: X \rightarrow X$ is a diffeomorphism, then $\psi_* T = (K, \psi \circ \eta)$ is also a triangulation of $X$.

Theorem 1.1. If $X, Y$ are smooth manifolds and $h: Y \rightarrow X$ is a smooth map, there exists a triangulation $(K, \eta)$ of $X$ such that $h$ is transverse to $\eta|_{\text{Int } \sigma}$ for every simplex $\sigma \in K$.

This theorem is stated in [8] as Lemma 2.3 and described as an obvious fact. As pointed out to the author by Matthias Kreck, Scharlemann [6] proves Theorem 1.1 under the assumption that the smooth map $h$ is proper, and his argument makes use of this assumption in an essential way. For the purposes of [8], a transverse $C^1$-triangulation would suffice, and the existence of a such
triangulation is fairly evident from the point of view of the Sard-Smale Theorem [7, (1.3)]. On the other hand, according to Matthias Kreck, the existence of smooth transverse triangulations without the properness assumption is related to subtle issues arising from the topology of stratifolds [2]. In this note we give a detailed proof of Theorem 1.1 as stated above, using Sard’s theorem [3, Section 2].

2. Outline of the Proof of Theorem 1.1

If $K$ is a simplicial complex, we denote by $sd K$ the barycentric subdivision of $K$. For any nonnegative integer $l$, let $K_l$ be the $l$-th skeleton of $K$, i.e. the subcomplex of $K$ consisting of the simplices in $K$ of dimension at most $l$. If $\sigma$ is a simplex in a simplicial complex $K$ with geometric realization $|K|$, let $St(\sigma, K) = \bigcup_{\sigma \subset \sigma'} \mathsf{Int} \sigma'$ be the star of $\sigma$ in $K$, as in [5, Sec. 62], and $b_\sigma \in sd K$ the barycenter of $\sigma$. The main step in the proof of Theorem 1.1 is the following observation.

**Proposition 2.1.** Let $h : Y \rightarrow X$ be a smooth map between smooth manifolds. If $(K, \eta)$ is a triangulation of $X$ and $\sigma$ is an $l$-simplex in $K$, there exists a diffeomorphism $\psi_\sigma : X \rightarrow X$ restricting to the identity outside of $\eta(St(b_\sigma, sd K))$ so that $\psi_\sigma \circ \eta_{|\mathsf{Int} \sigma}$ is transverse to $h$.

If $\sigma$ and $\sigma'$ are two distinct simplices in $K$ of the same dimension $l$,

$$St(b_\sigma, sd K) \cap St(b_{\sigma'}, sd K) = \emptyset.$$  

Since $\psi_\sigma$ is the identity outside of $\eta(St(b_\sigma, sd K))$ and the collection $\{St(b_\sigma, sd K)\}$ is locally finite, the composition $\psi_l : X \rightarrow X$ of all diffeomorphisms $\psi_\sigma : X \rightarrow X$ taken over all $l$-simplices $\sigma$ in $K$ is a well-defined diffeomorphism\(^2\) of $X$. Since $\psi_l \circ \eta_{|\sigma} = \psi_\sigma \circ \eta_{|\sigma}$ for every $l$-simplex $\sigma$ in $K$, we obtain the following conclusion from Proposition 2.1.

**Corollary 2.2.** Let $h : Y \rightarrow X$ be a smooth map between smooth manifolds. If $(K, \eta)$ is a triangulation of $X$, for every $l = 0, 1, \ldots, \dim X$, there exists a diffeomorphism $\psi_l : X \rightarrow X$ restricting to the identity on $\eta(|K_{l-1}|)$ so that $\psi_l \circ \eta_{|\mathsf{Int} \sigma}$ is transverse to $h$ for every $l$-simplex $\sigma$ in $K$.

This corollary implies Theorem 1.1. By [4, Chap. II], $X$ admits a triangulation $(K, \eta_{-1})$. By induction and Corollary 2.2, for each $l = 0, 1, \ldots, \dim X - 1$ there exists a triangulation $(K, \eta_l) = (K, \psi_l \circ \eta_{l-1})$ of $X$ which is transverse to $h$ on every open simplex in $K$ of dimension at most $l$.

\(^2\)The locally finite property implies that the composition of these diffeomorphisms in any order is a diffeomorphism; by (1), these diffeomorphisms commute and so the composition is independent of the order.

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3. Proof of Proposition 2.1

Lemma 3.1. For every \( l \in \mathbb{Z}^+ \), there exists a smooth function \( \rho_l : \mathbb{R}^l \to \mathbb{R}^+ \) such that \( \rho_l^{-1}(\mathbb{R}^+) = \text{Int} \Delta^l \).

Proof. Let \( \rho : \mathbb{R} \to \mathbb{R} \) be the smooth function given by

\[
\rho(r) = \begin{cases} 
    e^{-1/r}, & \text{if } r > 0, \\
    0, & \text{if } r \leq 0.
\end{cases}
\]

The smooth function \( \rho_l : \mathbb{R}^l \to \mathbb{R} \) given by

\[
\rho_l(t_1, \ldots, t_l) = \rho \left( 1 - \sum_{i=1}^{i=l} t_i \right) \cdot \prod_{i=1}^{i=l} \rho(t_i)
\]
then has the desired property. \( \square \)

Lemma 3.2. Let \((K, \eta)\) be a triangulation of a smooth manifold \( X \) and \( \sigma \) an \( l \)-simplex in \( K \). If

\[
\tilde{\mu}_\sigma : \Delta^l_\sigma \times \mathbb{R}^{m-l} \to U_\sigma \subset X
\]
is a diffeomorphism onto an open neighborhood \( U_\sigma \) of \( \eta(|\sigma|) \) in \( X \) such that \( \tilde{\mu}_\sigma(t,0) = \eta(\iota_\sigma(t)) \) for all \( t \in \Delta_\sigma \), there exists \( c_\sigma \in \mathbb{R}^+ \) such that

\[
\{(t,v) \in (\text{Int} \Delta^l_\sigma) \times \mathbb{R}^{m-l} | |v| \leq c_\sigma \rho_l(t)\} \subset \tilde{\mu}_\sigma^{-1}(\eta(\text{St}(b_\sigma, \text{sd} K))).
\]

Proof. It is sufficient to show\(^3\) that there exists \( c_\sigma > 0 \) such that

\[
\{(t,v) \in (\text{Int} \Delta^l_\sigma) \times \mathbb{R}^{m-l} | |v| \leq c_\sigma \rho_l(t)\} \subset \tilde{\mu}_\sigma^{-1}(\eta(\text{St}(\sigma, K))).
\]

We assume that \( 0 < l < m \). Suppose \((t_p, v_p) \in (\text{Int} \Delta^l_\sigma) \times (\mathbb{R}^{m-l} - 0)\) is a sequence such that

\[
(t_p, v_p) \notin \tilde{\mu}_\sigma^{-1}(\eta(\text{St}(\sigma, K))), \quad |v_p| \leq \frac{1}{p} \rho_l(t_p).
\]

Since \( \eta(\text{St}(\sigma, K)) \) is an open neighborhood of \( \eta(\text{Int} \sigma) \) in \( X \), by shrinking \( v_p \) and passing to a subsequence we can assume that

\[
(t_p, v_p) \in \tilde{\mu}_\sigma^{-1}(\eta(|\tau'|)) \subset \tilde{\mu}_\sigma^{-1}(\eta(|\tau|))
\]
for an \( m \)-simplex \( \tau \) in \( K \) and a face \( \tau' \) of \( \tau \) so that \( \sigma \subset \tau' \), \( \tau' \subset \sigma \), and \( \sigma \subset \tau \).

Let \( \iota_\tau : \Delta^m \to |K| \) be an injective linear map taking \( \Delta^m \) to \( |\tau| \) so that

\[
\iota_\tau^{-1}(|\sigma|) = \Delta^m \cap \mathbb{R}^l \times 0 \subset \mathbb{R}^l \times \mathbb{R}^{m-l},
\]

\[
\iota_\tau^{-1}(|\tau'|) = \Delta^m \cap 0 \times \mathbb{R}^{m-1} \subset \mathbb{R}^1 \times \mathbb{R}^{m-1}.
\]

Choose a smooth embedding \( \mu_\tau : \Delta^m_\tau \to X \) from an open neighborhood of \( \Delta^m_\tau \) in \( \mathbb{R}^m \) such that \( \mu_\tau|\Delta^m_\tau = \eta \circ \iota_\tau \). Let \( \phi \) be the first component of the diffeomorphism

\[
\mu_\tau^{-1} \circ \tilde{\mu}_\sigma : \tilde{\mu}_\sigma^{-1}(\mu_\tau(\Delta^m_\tau)) \to \mu_\tau^{-1}(\tilde{\mu}_\sigma(\Delta^l_\sigma \times \mathbb{R}^{m-l})) \subset \mathbb{R}^1 \times \mathbb{R}^{m-1}.
\]

\(^3\)If \( K' \) is the subdivision of \( K \) obtained by adding the vertices \( b_\sigma' \) with \( \sigma' \supset \sigma \), then \( \text{St}(b_\sigma, \text{sd} K) = \text{St}(\sigma, K') \).
By (3), the second assumption in (4), the continuity of $d\phi$, and the compactness of $\Delta^l$,
\begin{equation}
|\phi(t_p, 0)| = |\phi(t_p, 0) - \phi(t_p, v_p)| \leq C|v_p|, \quad \forall \ p,
\end{equation}
for some $C > 0$. On the other hand, by the first assumption in (4), the vanishing of $\rho_t$ on $\text{Bd} \Delta^l$, the continuity of $d\rho_t$, and the compactness of $\Delta^l$,
\begin{equation}
|\rho_t(t_p)| \leq C|\phi(t_p, 0)|, \quad \forall \ p,
\end{equation}
for some $C > 0$. The second assumption in (2), (5), and (6) give a contradiction for $p > C^2$. \hfill \Box

**Lemma 3.3.** Let $h : Y \longrightarrow X$ be a smooth map between smooth manifolds, $(K, \eta)$ a triangulation of $X$, $\sigma$ an $l$-simplex in $K$, and
\[
\tilde{\mu}_\sigma : \Delta^l_\sigma \times \mathbb{R}^{m-l} \longrightarrow U_\sigma \subset X
\]
a diffeomorphism onto an open neighborhood $U_\sigma$ of $\eta(|\sigma|)$ in $X$ such that $\tilde{\mu}_\sigma(t, 0) = \eta(t(\sigma(t)))$ for all $t \in \Delta^l_\sigma$. For every $\epsilon > 0$, there exists $s_\sigma \in C^\infty(\text{Int} \Delta^l; \mathbb{R}^{m-l})$ so that the map
\begin{equation}
\tilde{\mu}_\sigma \circ (\text{id}, s_\sigma) : \text{Int} \Delta^l \longrightarrow X
\end{equation}
is transverse to $h$,
\begin{equation}
|s_\sigma(t)| < \epsilon^2 \rho_t(t) \quad \forall \ t \in \text{Int} \Delta^l,
\end{equation}
\[
\lim_{t \rightarrow \text{Bd} \Delta^l} \rho_t(t)^{-i} |\nabla^i s_\sigma(t)| = 0 \quad \forall \ i, j \in \mathbb{Z}^{\geq 0},
\]
where $\nabla^i s_\sigma$ is the multilinear functional determined by the $j$-th derivatives of $s_\sigma$.

**Proof.** The smooth map
\[
\phi : \text{Int} \Delta^l \times \mathbb{R}^{m-l} \longrightarrow X, \quad \phi(t, v) = \tilde{\mu}_\sigma(t, e^{-1/\rho(t)}v),
\]
is a diffeomorphism onto an open neighborhood $U'_\sigma$ of $\eta(\text{Int} \sigma)$ in $X$. The smooth map (7) with $s_\sigma = e^{-1/\rho(t)}v$ is transverse to $h$ if and only if $v \in \mathbb{R}^{m-l}$ is a regular value of the smooth map
\[
\pi_2 \circ \phi^{-1} \circ h : h^{-1}(U'_\sigma) \longrightarrow \mathbb{R}^{m-l},
\]
where $\pi_2 : \text{Int} \Delta^l \times \mathbb{R}^{m-l} \longrightarrow \mathbb{R}^{m-l}$ is the projection onto the second component. By Sard’s Theorem, the set of such regular values is dense in $\mathbb{R}^{m-l}$. Thus, the map (7) with $s_\sigma = e^{-1/\rho(t)}v$ is transverse to $h$ for some $v \in \mathbb{R}^{m-l}$ with $|v| < \epsilon^2$. The second statement in (8) follows from $\rho_t|_{\text{Bd} \Delta^l} = 0$. \hfill \Box

**Corollary 3.4.** Let $h : Y \longrightarrow X$ be a smooth map between smooth manifolds, $(K, \eta)$ a triangulation of $X$, $\sigma$ an $l$-simplex in $K$, and
\[
\tilde{\mu}_\sigma : \Delta^l_\sigma \times \mathbb{R}^{m-l} \longrightarrow U_\sigma \subset X
\]

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a diffeomorphism onto an open neighborhood $U_\sigma$ of $\eta(|\sigma|)$ in $X$ such that
\[ \bar{\mu}_\sigma(t, 0) = \eta(t_\sigma(t)) \text{ for all } t \in \Delta_\sigma. \]
For every $\epsilon > 0$, there exists a diffeomorphism $\psi'_\sigma$ of $\Delta^l_\sigma \times \mathbb{R}^{m-l}$ restricting to the identity outside of
\[ \{(t, v) \in (\text{Int} \Delta^l) \times \mathbb{R}^{m-l} : |v| \leq \epsilon \rho_l(t)\} \]
so that the map $\bar{\mu}_\sigma \circ \psi'_\sigma|_{\text{Int} \Delta^l \times 0}$ is transverse to $h$.

**Proof.** Choose $\beta \in C^\infty(\mathbb{R}; [0, 1])$ so that
\[ \beta(r) = \begin{cases} 
1, & \text{if } r \leq \frac{1}{2}; \\
0, & \text{if } r \geq 1.
\end{cases} \]
Let $C_\beta = \sup_{r \in \mathbb{R}} |\beta'(r)|$. With $s_\sigma$ as provided by Lemma 3.3, define
\[ \psi'_\sigma : \Delta^l_\sigma \times \mathbb{R}^{m-l} \rightarrow \Delta^l_\sigma \times \mathbb{R}^{m-l} \]
by
\[ \psi'_\sigma(t, v) = \begin{cases} 
(t, v + \beta \left( \frac{|v|}{\epsilon \rho_l(t)} \right) s_\sigma(t)), & \text{if } t \in \text{Int} \Delta^l, \\
(t, v), & \text{if } t \notin \text{Int} \Delta^l.
\end{cases} \]
The restriction of this map to $(\text{Int} \Delta^l) \times \mathbb{R}^{m-l}$ is smooth and its Jacobian is
\[ (J \psi'_\sigma)|_{(t, v)} = \left( \begin{array}{cc}
\mathbb{I}_t & 0 \\
(J \psi'_\sigma)|_{(t, v)} & \mathbb{I}_{m-l} + \beta' \left( \frac{|v|}{\epsilon \rho_l(t)} \right) \left( \frac{s_\sigma(t)}{\epsilon \rho_l(t)} v^r \right)
\end{array} \right), \]
where
\[ (J \psi'_\sigma)|_{(t, v)} = 0. \]
By the first property in (8), this matrix is non-singular if $\epsilon < 1/C_\beta$. If $W$ is any linear subspace of $\mathbb{R}^{m-l}$ containing $s_\sigma(t)$,
\[ \psi'_\sigma(t \times W) \subset t \times W, \quad \psi'_\sigma(t, v) = (t, v) \quad \forall \ v \in W \text{ such that } |v| \geq \epsilon \rho_l(t). \]
Thus, $\psi'_\sigma$ is a bijection on $t \times W$, a diffeomorphism on $(\text{Int} \Delta^l) \times \mathbb{R}^{m-l}$, and a bijection on $\Delta^l_\sigma \times \mathbb{R}^{m-l}$.

Since $\beta(r) = 0$ for $r \geq 1$, $\psi'_\sigma(t, v) = (t, v)$ unless $t \in \text{Int} \Delta^l$ and $|v| < \epsilon \rho_l(t)$. It remains to show that $\psi'_\sigma$ is smooth along
\[ \{(t, v) \in (\text{Int} \Delta^l) \times \mathbb{R}^{m-l} : |v| \leq \epsilon \rho_l(t)\} - (\text{Int} \Delta^l) \times \mathbb{R}^{m-l} = (\text{Bd} \Delta^l) \times 0. \]
Since $|s_\sigma(t)| \rightarrow 0$ as $t \rightarrow \text{Bd} \Delta^l$ by the first property in (8), $\psi'_\sigma$ is continuous at all $(t, 0) \in (\text{Bd} \Delta^l) \times 0$. By the first property in (8), $\psi'_\sigma$ is also differentiable at all $(t, 0) \in (\text{Bd} \Delta^l) \times 0$, with the Jacobian equal to $\mathbb{I}_m$. By (9) and the compactness of $\Delta^l$,
\[ |J \psi'_\sigma|_{(t, v)} - \mathbb{I}_m| \leq C (|\nabla s_\sigma(t)| + \rho(t)^{-1} |s_\sigma(t)|) \quad \forall \ (t, v) \in (\text{Int} \Delta^l) \times \mathbb{R}^{m-l} \]
for some $C > 0$. So $J \psi'_\sigma$ is continuous at $(t, 0)$ by the second statement in (8), as well as differentiable, with the differential of $J \psi'_\sigma$ at $(t, 0)$ equal
to 0. For \( i \geq 2 \), the \( i \)-th derivatives of the second component of \( \psi'_\sigma \) at \((t, v) \in (\text{Int } \Delta^l) \times \mathbb{R}^{m-l}\) are linear combinations of the terms

\[
\beta(i_1) \left( \frac{|v|}{\epsilon \rho_l(t)} \right) \left( \frac{|v|}{\epsilon \rho_l(t)} \right)^{i_1} \cdot \prod_{k=1}^{k=j} \left( \frac{\nabla^p \rho_l}{\rho_l(t)} \right) \cdot \frac{v_j}{|v|^{2j_2}} \cdot \nabla^{i_2} s_\sigma(t),
\]

where \( i_1, i_2, j_1, j_2 \in \mathbb{Z}_0^+ \) and \( p_1, \ldots, p_{j_1}, j_1 \in \mathbb{Z}_0^+ \) are such that

\[
i_1 + (p_1 + p_2 + \ldots + p_{j_1} - j_1) + i_2 = i, \quad j_1 + j_2 \leq i,
\]

and \( v_j \) is a \( j_2 \)-fold product of components of \( v \). Such a term is nonzero only if \( \epsilon \rho_l(t)/2 < |v| < \epsilon \rho_l(t) \) or \( i_1 = 0 \) and \( |v| < \epsilon \rho_l(t) \). Thus, the \( i \)-th derivatives of \( \psi'_\sigma \) at \((t, v) \in (\text{Int } \Delta^l) \times \mathbb{R}^{m-l}\) are bounded by

\[
C_i \sum_{i_1+i_2 \leq i} \rho_l(t)^{-i_1} |\nabla^{i_2} s_\sigma(t)|
\]

for some constant \( C_i > 0 \). By the second statement in (8), the last expression approaches 0 as \( t \to \text{Bd } \Delta \) and does so faster than \( \rho_l \). It follows that \( \psi'_\sigma \) is smooth at all \((t, 0) \in (\text{Bd } \Delta^l) \times 0\). \( \square \)

**Proof of Proposition 2.1.** Let \( \Delta^l_\sigma \) be a contractible open neighborhood of \( \Delta \) in \( \mathbb{R}^l \) and \( \mu_\sigma : \Delta^l_\sigma \to X \) a smooth embedding so that \( \mu_\sigma|_{\Delta^l} = \eta \circ t_\sigma \). By the Tubular Neighborhood Theorem [1, (12.11)], there exist an open neighborhood \( U_\sigma \) of \( \mu_\sigma(\Delta^l_\sigma) \) in \( X \) and a diffeomorphism\(^4\)

\[
\bar{\mu}_\sigma : \Delta^l_\sigma \times \mathbb{R}^{m-l} \to U_\sigma \quad \text{such that} \quad \bar{\mu}_\sigma(t, 0) = \mu_\sigma(t) \forall t \in \Delta^l_\sigma.
\]

Let \( c_\sigma > 0 \) be as in Lemma 3.2 and \( \psi'_\sigma \) as in Corollary 3.4 with \( \epsilon = c_\sigma \). The diffeomorphism

\[
\psi_\sigma = \bar{\mu}_\sigma \circ \psi'_\sigma \circ \bar{\mu}_\sigma^{-1} : U_\sigma \to U_\sigma
\]

is then the identity on \( U_\sigma - \text{St}(b_\sigma, sd K) \). Since \( \psi_\sigma \) is also the identity outside of a compact subset of \( U_\sigma \), it extends by identity to a diffeomorphism on all of \( X \). \( \square \)

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\(^4\)Since \( \Delta^l_\sigma \) is contractible, the normal bundle to the embedding \( \mu_\sigma \) is trivial.
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