Inertial drag-out problem : sheets and films on a rotating drum

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Abstract

The so-called Landau-Levich-Derjagin problem treats the coating flow dynamics of thin viscous liquid films that form on freely-driven solid surfaces. Such flows are not only relevant to film coating and liquid entrainment in industrial processes but are also a starting block for lubrication theory. In this context, we use a simple experimental set-up consisting of a partially immersed rotating drum in a water tank to study the role of inertia, and also curvature, on the liquid entrainment phenomena. Using water and UCON™ mixtures, we point out a rich phenomenology in the presence of strong inertia. Instead of a 2D, or axisymmetric, dynamic meniscus as seen in the classical problem, the inertial effects bring about one or more thin liquid sheets from which a liquid film develops on the drum’s front end. In addition, this film then undergoes atomisation at the rear end of the drum due to the centrifugal acceleration. Both viscous and surface tension forces play a key role in deciding the film thickness as in the classical Landau-Levich problem until a critical Weber number based on Landau-Levich dynamic meniscus. Thereafter, strong inertial effects influence the film flow rate over the drum via lateral entrainment and via a modified inertial dynamic meniscus at large radius-to-immersion-depth ratio.

Keywords : film coating problem; inertial Landau-Levich film; liquid entrainment.

1 Introduction

Be it a fibre that is dragged out of a polymeric solution, or a horizontal roll that pulls out a thin liquid film from a reservoir where the liquid is at rest, or a viscous liquid that coats the inside of a narrow tube, the entrainment of a liquid by a wall is a common configuration in many industrial applications. They may include transmission of liquid lubricants, performance of roller crystallizers for continuous metal casting, Fourdrinier machines in the paper industries and a wide range of coating processes such as photographic films, microelectronics, adhesives on films or tapes, laminated composites, food processing, etc. From the point of view of fundamental fluid mechanics, the liquid entrainment problem is not only relevant for the lubrication theory to understand thin film flows and their stability when the film Reynolds number is sufficiently small but also to investigate the resulting liquid sheets, filaments and drops in the inertial regime.

The simplest entrainment flow concerns the characteristics of a liquid film entrained by a long vertical plate (figure 1a), or a thin vertical fiber, as it is withdrawn at a steady speed $U$ from
a large reservoir of a viscous liquid of density $\rho$ and viscosity $\mu$. It is the so-called Landau-Levich-Derjagin dip-coating flow (Deryagin, 1943, Goucher and Ward, 1922, Levich and Landau, 1942, Morey, 1940, Van Rossum, 1958). The liquid surface tension $\sigma$ plays a subtle role on the coating flow via the Laplace pressure at the dynamic meniscus between the film and the reservoir. However, far away from the dynamic meniscus, the flat entrained film is at the ambient pressure. And so, a pressure gradient which opposes liquid entrainment should exist along the vertical film. This lead Deryagin (1943), Levich and Landau (1942) to obtain

$$\delta_f^{(LLD)} \simeq l_c \left[ 0.946 Ca^{2/3} - 0.107 Ca + O \left( Ca^{4/3} \right) \right],$$

when the surface tension forces dominate viscous entrainment, i.e. at sufficiently small capillary numbers $Ca = \mu U / \sigma$. Note that $l_c = \sqrt{\sigma / \rho g}$ is the capillary length. This pioneering result correlates well with numerous experimental investigations for small capillary numbers up to $O(10^{-2})$ (Groenveld, 1970b, Maleki et al., 2011, Morey, 1940, Snoeijer et al., 2008, Van Rossum, 1958) and also, for moderate values of $Ca$ (Groenveld, 1970a, Kizito et al., 1999, Spiers et al., 1974, White and Tallmadge, 1965).

However, when both inertia and surface tension are negligible and, if the film is flat far away from the liquid bath, it can also be deduced that the film thickness $\delta_f$ results from a simple balance between the weight of the liquid film and the viscous drag of the plate. Thus, in this viscosity-gravity driven regime

$$\delta_f^{(g)} \sim l_c Ca^{1/2} = \sqrt{\frac{\mu U}{\rho g}},$$

where $g$ is the acceleration due to gravity. In fact, this regime was also observed experimentally above some critical capillary number (Groenveld, 1970a, Jin et al. 2005).

These results also apply for the case of fiber dip-coating as long as the film thickness is small compared to the fiber radius and the coating flow Reynolds number is lesser than or equal to $O(1)$ (see Quéré (1999) and references therein for a review on the subject). Within the context of the lubrication approximation, subsequent works on the liquid entrainment by a moving flat plate, or a fiber, considered the role of non-Newtonian rheology (de Ryck and Quéré, 1998a), inertia on thin films (de Ryck and Quéré, 1998b, Jin et al., 2005, Kizito et al., 1999), wettability (Snoeijer et al., 2006), surfactants (Campana et al., 2010, Krechentikov and Homsy, 2005), adsorbed particles on fluid interfaces (Campana et al., 2011, Duxit and Homsy, 2013, Gans et al., 2019, Jung and Ahn, 2013, Palma and Lhuissier, 2019), textured flat plates (Nasto et al., 2018, Seiwert et al., 2011) and also, the existence of non-unique solutions for the LLD problem (Benilov et al., 2010).

1Note that the higher-order terms in capillary number are first obtained by Wilson (1982) using a matched-asymptotic development.
Rotary entrainment occurs in many common roll coating processes which consist of multiple rolls and meniscus of liquid between the rollers (Ruschak, 1985). Rotating disks are also used to improve mass and heat transfer in gas-liquid chemical reactors, pharmaceutical industries, etc. In this context, Rubashkin (1967) & Middleman (1978) investigated the pertinence of the LLD scaling for the case of a partially immersed rotating drum. Using horizontal cylinders of radius \( R = 2.5 - 4 \text{ cm} \) and about 10 to 20 \text{ cm} long in a viscous oil bath, they observed that the film thickness is about that of the \( \text{viscosity-gravity} \) entrained by a rotating disk at speeds greater than the commonly studied limit of 1 \( \text{ms}^{-1} \) in the inertial limit, when the flow is not 2D anymore: for example when a less viscous fluid, like water, is via three-dimensionality and unsteadiness of the free-surface flows. Reynolds number based on coating thickness to about 20. Thereafter, principal difficulties arise and applications, both experimental and computational investigations often restricted the working range of small capillary numbers. They also noted that, by including inertial effects in the \( \text{flat plate LLD law} \) to improve mass and heat transfer in gas-liquid chemical reactors, pharmaceutical industries, etc. With the advent of numerical techniques, earlier theoretical efforts (such as Soroka and Tallmadge (1971) & Tharmalingam and Wilkinson (1978)) on the lubrication equations wherein inertial effects could be included via 1D film flow modeling and \( Oseen-like \) corrections were later complemented by fully non-linear and 2D film flow simulations (Campanella and Cerro, 1984, Cerro and Scriven, 1980, Hasan and Naser, 2009, Nigam and Esmail, 1980). Along with Tharmalingam and Wilkinson (1978), these authors reported that the LLD scaling for an inclined plate, as in \( \text{figure 1b-c} \), if \( \alpha \) is the angle between the inclined plate which is tangent to the partially immersed cylindrical drum at its line of contact with the liquid bath at rest and the horizontal line, they showed that the non-dimensional film thickness \( T_0 = \delta f \sqrt{\frac{\mu U}{\rho g}} \) just after the \( \text{dynamic meniscus} \) between the cylinder and the liquid reservoir is given by

\[
\frac{T_0}{(1 - T_0^2)^{2/3}} \sim 0.94Ca^{1/6}\left(\frac{\sin \alpha}{1 - \cos \alpha}\right),
\]

(3)

With the advent of numerical techniques, earlier theoretical efforts (such as Soroka and Tallmadge (1971) & Tharmalingam and Wilkinson (1978)) on the lubrication equations wherein inertial effects could be included via 1D film flow modeling and \( Oseen-like \) corrections were later complemented by fully non-linear and 2D film flow simulations (Campanella and Cerro, 1984, Cerro and Scriven, 1980, Hasan and Naser, 2009, Nigam and Esmail, 1980). Along with Tharmalingam and Wilkinson (1978), these authors reported that the LLD scaling for an inclined plate, as in \( \text{figure 3} \) holds for a good range of small capillary numbers. They also noted that, by including inertial effects in the \( \text{inclined-plate LLD law} \), it gave a better match with experiments when the film Reynolds number is of \( O(1) \). However, similar to the flat plate case, above a certain critical capillary number that depends on fluid properties, their results suggest that the film thickness no longer follows the LLD law.

Related to this rotary entrainment problem is also the steady and unsteady (instabilities) motion of a thin liquid film on the outside or the inside a hollow rotating cylinder. The former is the classical \( \text{Moffatt-Pukhnachev-Yih (MPY) flow} \) (Moffatt, 1977, Pukhnachev, 1977, Yih, 1960) and the latter is the so-called rimming flow. As first illustrated by these authors, such flows develop axial instabilities on the free-surface of the film leading to \( \text{liquid rings} \) (see Evans et al. 2005 and references therein for further reading).

Thus, for almost 80 years now, coating flow studies gave special attention to the role of the fluid interfacial properties and viscosity, be it Newtonian or non-Newtonian rheology. Nonetheless, recent works by De Ryck and Quéré (1996) and Kizito et al. (1999) strongly suggest that the role of inertia in flat plate and fiber LLD flows, respectively, results in a transition from the LLD law to a viscosity-gravity scaling at a critical Weber number based on the \( \text{LLD meniscus} \) length \( \lambda \simeq 0.65\sqrt{\mu U/\rho g} \). A good number of works (Campanella and Cerro, 1984, Evans et al., 2005, Jin et al., 2005, Kizito et al., 1999, Middleman, 1978) also suggest formation of \( \text{cusped menisci} \) and \( \text{wavy free-surface structures} \). But within the context of lubrication approximations and applications, both experimental and computational investigations often restricted the working Reynolds number based on coating thickness to about 20. Thereafter, principal difficulties arise via three-dimensionality and unsteadiness of the free surface flows.

In so far, it is not known if the above results hold for rotary entrainment flows in the strongly inertial limit, when the flow is not 2D anymore: for example when a less viscous fluid, like water, is entrained by a rotating disk at speeds greater than the commonly studied limit of 1 \( \text{ms}^{-1} \), similar
Figure 2: Entrainment of water by a partially immersed rotating disk (radius $R = 21$ cm, water depth $h = 1$ cm and linear velocity $U = 5.2$ m/s). Three dominant liquid flow structures appear: liquid sheet, film, ligaments and drops.

Table 1: Properties of liquids used in the present work. UCON™ Lubricant 75-H-90,000 was used for all mixtures used here.

| Liquid                  | Density ($\rho$) $kg m^{-3}$ | Viscosity ($\mu$) $\times 10^{-3} Pa s$ | Surface tension ($\sigma$) $\times 10^{-3} N m$ |
|------------------------|-------------------------------|-----------------------------------------|-----------------------------------------------|
| Water                  | 1000                          | 0.9                                     | 72                                            |
| Water-UCON™ 1 (WU1)    | 1015                          | 5.5                                     | 58                                            |
| Water-UCON™ 2 (WU2)    | 1022                          | 11                                      | 58                                            |
| Water-UCON™ 3 (WU3)    | 1044                          | 82                                      | 58                                            |

The latter is then simply ejected by centrifugal forces. What are the characteristic length scales of the liquid sheet as a function of immersion depth $h$ and the disk speed $U = R\Omega$? What is influence of the fluid viscosity $\mu$? How does the resulting liquid flow rate on the rotating drum influence the LLD scaling? Motivated by these questions, the present work is interested in this inertial limit of the rotary entrainment problem wherein a large cylinder rotates relatively fast in a large liquid bath.

2 Experimental set-up

As illustrated in figure 3 the experimental set-up consists of a large plexiglass liquid tank $40 \times 41.5 \times 100$ cm$^3$. An asynchronous motor is used along with an AC Inverter Drive (Parker AC10) to drive rigid PVC disks of radii $R = 10$ cm, 13.5 cm and 21 cm and thickness 4.5 cm. Experimental data in the following sections concern these disks of thickness 4.5 cm and qualitative results for a thicker disk are also presented in the discussion section. The driving frequency varied between 20 and 120 $rpm$ leading to a maximum azimuthal speed ($U = R\Omega$) of about 4 $m s^{-1}$ for all wheel
radius. The disk is partly submerged in the liquid bath and the liquid level \( h \), as illustrated in figure 1, is properly verified before each run. Various immersion depths \( h \) in the range of 0.05 – 0.8 times the disk radius \( R \) are used. Table 1 presents physical properties of the different liquids used in the experiments. Liquid density was measured using hydrometers whereas viscosity was obtained using a falling-sphere viscometer and by properly taking into account the Reynolds numbers corrections for the viscous drag on the spheres (Brown and Lawler, 2003). Surface tension was obtained by noting the liquid contact angle of a drop placed on a plexiglass plate. It is also ensured that the working liquid did not contain any surfactants.

A typical series of experiments at a fixed water level \( h \) for a given liquid (water or water-UCON\textsuperscript{TM} mixtures) is started by running the disk at a constant angular velocity \( \Omega \) as indicated by a tachometer. After a few seconds, the resulting liquid sheet along with the film flow rate are measured. Then the angular velocity is increased to repeat the measurements. The following sections describe in detail the results obtained using this experimental set-up along with the measurement techniques.

### 3 Experimental results

#### 3.1 Inertial menisci – a liquid sheet

A sequence of images in figure 4 depicts the evolution of the emerging liquid sheet as a function of azimuthal velocity \( U \). At the lowest of the speeds shown here (\( U = 0.88 \) ms\(^{-1}\)), a small quasi-static dynamic meniscus appears at the contact between the wheel and the stagnant water far away. Even at this stage the flow on the disk’s rim is completely distinct from the classical LLD flow. Firstly, we can observe a small dimple at the location where the meniscus emerges from the liquid bath. Secondly, it is no longer a smooth two-dimensional (or axisymmetric) structure, as is the case for LLD films on flat plates (or fibres), but a closer inspection shows spatial variations in the sheet thickness and the presence of one or two liquid rims. A further increase in wheel speed leads to the formation of a liquid sheet with a single liquid rim. At higher speeds, the rim presents fluctuations, holes with apparently random rim ruptures resulting in liquid drops occur on the sheet (see figure 5 and supplementary material) and even multiple rims are present when the wheel thickness is large (not presented here).

The same phenomenology, with a few exception, is observed when a more viscous liquid is used (figure 6). Here, the liquid used is a Water-UCON\textsuperscript{TM} oil mixture \( WU3 \) (see table 1 for details).
Figure 4: Instantaneous side view of the inertia-driven liquid menisci on a partially-immersed rotating disk in water ($R = 21$ cm, $H/R = 0.2$). See supplementary videos for more information.
Figure 5: Visualization of the liquid sheet entrained by the wheel, for $R = 21$ cm, $h = 1$ cm and $U = 5.2$ m/s: holes are formed in the water sheet (for a closer look, see supplementary material).

Already at $U = 0.42$ ms$^{-1}$, a three dimensional quasi-static meniscus appears as in the case with pure water. In addition, pendant drops are also seen to descend along the rim of the rotating disk (see supplementary material). This film flow is similar to that observed by Evans et al. (2004, 2005) in their numerical simulations of the classical Moffatt-Pukhnachev-Yih (MPY) flow at low speeds. No trace of descending pendant drops is seen at larger speeds but a quasi-steady liquid sheet emerging from the liquid bath is clearly visible. Even in this highly viscous liquid, a lubrication approximation for the related flow might become irrelevant since the related Reynolds number is sufficiently greater than unity. At higher speeds up to 1.85 ms$^{-1}$, a fine stable liquid sheet which terminates with a thick rim flow is observed. From these images of Water-UCON oil mixtures (WU3), it is evident that the liquid rim is a result of a capillary recession, as in a Savart sheet (Savart, 1833, Villermaux et al., 2013). This recession is known to occur at the Taylor-Culick velocity of $v_c = \sqrt{\sigma/\rho \delta_s}$ where $\delta_s$ is the local liquid sheet thickness (Culick, 1960, Savva and Bush, 2009, Taylor, 1959). As the speed is further increased, the rim of the sheet shows strong corrugations. For the case of the most viscous liquid (WU3), figures 4 & 6 also illustrate that the sheet height $H_m$ with respect to the free surface on liquid bath is about 3 to 5 times larger than the water case at the same disk speed $U$.

As indicated in figure 4, the liquid sheet height $H_m$ is measured from the surface of the pool. Figure 7 presents these measurements as a function of the rotation speed $U$, for three disk radii, and several depths $h$ when the entrained liquid is water. The sheet height $H_m$ increases monotonically though non-linearly with velocity. In figure 7, each symbol corresponds to a different depth of immersion $h$ of the rotating wheel. This depth seems to have little influence on the height of the liquid sheet, except for the case $h/R = 0.5$ (denoted by $\diamond$) at larger velocities. These particular cases correspond to situations where the top of the liquid sheet stretches significantly beyond the top of the rotating disk. The disk radius has a non negligible impact on the height, especially for velocities larger than 3 ms$^{-1}$. Again this corresponds to cases where the height of the liquid sheet is comparable to the wheel radius. For these particular cases, it is expected that a smaller wheel (larger curvature) will lead to a smaller sheet for a given linear velocity $U$. Similar trend is observed in figure 7, wherein liquid sheet height $H_m$ for different mixtures of water and UCON oil of varying viscosity ($\mu = 5.5, 11$ and $82 \times 10^{-3}$ Pa.s) is given when the disk radius $R = 21$ cm. By comparing with the case of pure water (open symbols), it can be concluded that a strong increase in sheet height $H_m$ occurs when viscosity is increased. Besides, for the highest viscosity liquid, the sheet height seems to increase linearly with the wheel speed.

A simple picture to describe the formation of the liquid sheet can be obtained by considering the motion of fluid particles as **ballistic** when they are dragged out of the bath by the rotating drum.
Figure 6: Instantaneous side view of the inertia-driven liquid menisci on a partially-immersed rotating disk ($R = 21\,\text{cm}$, $H/R = 0.2$) in a viscous liquid ($\mu = 82\,\text{mPa s}$), a mixture of UCON™ oil and water. For videos see supplementary material.

Figure 7: Liquid sheet height $H_m$ (in meters) as a function of the rotational speed $U = R\omega$ for various disk radii $R = 10$ (green), 13.5 (red), 21 (black) cm and immersion depth to radius ratio $h/R$. Data correspond to the experiments with (a) pure water (open symbols) (b) various UCON™/Water mixtures (filled symbols) when $R = 21\,\text{cm}$.
Figure 8: Viscosity increases the sheet height $H_m$ (in meters) as seen here when data for a constant disk radius ($R = 21$ cm) are compared between various UCON™ oil and water mixtures (filled symbols) and the case with pure water (open symbols) at different immersion depth to radius ratio $H/R$. Dashed line represents $H_m = U^2/2g$ (or simply, $H_m/R = Fr^2$).

at a velocity proportional to the drum speed $U = \Omega R$. The fluid particles then follow a *free-flight* trajectory in the liquid sheet to attain a height $H_m \propto U^2$. Figure 8 illustrates that $H_m$ is indeed proportional to $U^2/2g$ when all data from figure 7a (water) are non-dimensionalized in terms of $H_m/R$ versus Froude number $Fr = U/\sqrt{2gR}$ based on the disk radius. This trend is also verified for UCON™/water mixtures corresponding to $\mu = 5.5 & 11 \times 10^{-3}$ Pa.s. Therefore, this suggests that the simple *ballistic* model captures the general trend and provides a first approximation for the inertial sheet height.

Furthermore, the same data from figure 7a-b are renormalized as $2gH_m/U^2$ in figure 8c to explore more closely the departures from this global ballistic trend. Here, it is clear that $H_m$ is significantly smaller than this ballistic height as $2gH_m/U^2 \sim 0.3$, for the case of water. In addition, this ratio decreases at larger speeds, down to around 15% of the ballistic height for the smaller wheel. While at small Froude numbers $Fr = U/\sqrt{2gR}$, the sheet height shows a linear trend with the wheel speed $H_m \propto U$ for the more viscous water-UCON™ mixtures, all data for rescaled height $2gH_m/U^2$ lie close to unity when $Fr \geq 1$. The reason could be two-fold. Firstly, the fluid elements which quit the rim before ending up in the sheet do not all have the same momentum and so, the corresponding fluid element may only reach a smaller fraction of the ballistic height $U^2/2g$. Secondly, a delayed capillary recession in a viscous fluid can retard the truncation of the liquid sheet as it stretches and thins out.

In order to better understand the effect of viscosity on the retardation of the Taylor-Culick cut-off and thereby, explain the larger heights observed when viscosity is increased, further information on the inertial liquid sheet thickness $\delta_s$ is obtained experimentally. It is well-known that the light attenuation across a homogeneous material follows Beer-Lambert law. So, the liquid sheet thickness is then proportional to the light intensity, if it is properly exposed to a uniform back light. A standard technique is to mix the working liquid with a small amount of fine titanium oxide particles (TiO$_2$) whereby the liquid opacity is increased. By properly calibrating the relationship between liquid thickness and the light intensity, it is then possible to obtain the local thickness $\delta_s$ for various wheel immersion depths and rotational speeds. These results are given in figures 9 & 10.

For both $h/R = 0.2$ & $h/R = 0.4$, the contour plots in figure 9 illustrate that the sheet thickness $\delta_s$ is not constant. It is as thick as the wheel’s rim at its base where fluid elements are ejected with a kinetic energy proportional to $U^2/2$. And it rapidly thins out to attain a critical thickness at which capillary recession occurs. This then leads to a thick liquid rim. As mentioned previously, this recession leads to sheet truncation at a location where the local fluid velocity is smaller than the Taylor-Culick velocity of $v_c = \sqrt{\sigma/\rho\delta_s}$ (Culick 1960, Savva and Bush 2009, Taylor 1959). Thus, for a given water depth, the critical sheet thickness where the sheet forms a thick rim should decrease with increasing rotational speed. These observations are all the more
Figure 9: Liquid sheet thickness ($\delta_s$) at different immersion depth to radius ratio $H/R$ and rotational speeds for water. All data correspond to a wheel radius of 21cm.

Figure 10: Sheet thickness $\delta_s$ as a function of the distance $s$ traveled along a ballistic trajectory by a hypothetical fluid element that is ejected with an initial velocity equal to $U = R\Omega$ at the emerging side of the wheel ($R = 21$cm).
clear in figure 10(a). In addition, figure 10(b) indicates that the water depth only weakly influences this critical sheet thickness where capillary recession leads to truncation. Figure 10(c) compares the measured thickness for both Water and UCON\textsuperscript{TM}/Water mixture at a $h/R = 0.2$ and a wheel speed of 115 rpm. In comparison, the liquid sheet width decreases in a very similar trend for both the cases until the point where a liquid rim is formed for the case of water at about $s = 7.5$ cm and $\delta_s \approx 1.5$ cm whereas the formation of a liquid rim occurs much later for the UCON\textsuperscript{TM}/Water mixture. In the latter case, the critical sheet thickness $\delta_s$ where recession truncates the sheet is about 0.3 cm. This suggests that, compared to the case of water, capillary recession is more delayed for the more viscous UCON\textsuperscript{TM}/Water mixture.

Finally, it is pointed out that, in all experimental results given here, the upper limit of the measurable thickness is as large as the wheel width (4.5 cm) and it occurs at the base of the emerging side of the wheel for all cases. Furthermore, the contour plots in figure 9 for the case of $h/R = 0.8$ present a distinct phenomenon. They show that the thickness is almost constant throughout the sheet and that it is remarkably close to, or larger, than the wheel width, for all velocities presented here. This trend is also visible in figure 10(b) where the liquid sheet thickness (blue squares) remains constant for a good distance from the wheel and then the sheet suddenly disintegrates without a rim.

### 3.2 Entrained liquid film

As mentioned in the introductory section, the inertial entrainment produced by a rotating wheel does not only drag-out a thin liquid sheet but also a liquid film on the wheel rim. And it is this liquid film that emanates from the top of the liquid sheet is considered in the following. While it is conventional in the LLD approach to characterize the film thickness $\delta_f$, a global flow rate measurement is privileged in this study over the thickness measurements as this method is much easier and at the same time, it provides a good estimate of the efficiency of the overall inertial entrainment processes. Most of the film flow rate measurements were done using a simple scraping technique which consists in applying the sharp edge of a flexible, transparent plastic sheet on the declining side of the wheel rim and thereby scraping the film flow out of the rim. The former is then collected into a large receptacle, including droplets ejected from the film itself for larger rotation speeds. Since disk speed $U$ can show large variations when one scrapes-off the lubrication film, special care was taken to avoid such variations by systematically monitoring the tachometer over the time interval during which the liquid is allowed to flow in the receptacle. This technique showed very good repeatability. It is also possible to use a local measurement technique which measures the film thickness at a given location on the cylinder, as is common at low Reynolds number entrainment flows: due to the large Reynolds numbers at the scale of the liquid film, a series of local film thickness measurement using Chromatic Confocal Imaging showed strong spatio-temporal fluctuations of the film thickness at any given point on the rim\footnote{For the sake of brevity, these measurements are not reported here. An interested reader is directed towards the supplementary material where an instantaneous visualization of sheet and entrained film thickness is shown.}. In fact, the scraping technique is found to be more robust compared to the local measurement technique as it directly provides a spatio-temporal average of the film flow rate over the disk circumference, including the length and breadth of the rim, during a fixed time interval.

Figure 11 depicts the variations of the time-averaged film-flow rate as a function of velocity, measured for three disks of radius $R = 10$ cm, 13.5 cm and 21 cm over nine different immersion heights in water. In all cases, the entrained flow rate increases monotonically with the disk speed $U = R \Omega$ and also, with the height of the disk that is immersed. Note that all data seems to be relatively independent of the water height $h/R$ up to some critical azimuthal speed $U$. After this limit, figure 11 clearly indicates that the entrained water flow rate is strongly influenced by the immersion height $h$. For example, the film-flow rate at $h/R = 0.8$ is three times larger than that at $h/R = 0.05$ for disks $R = 10$ cm & 13.5 cm. And it gets as large as eight-folds when $R = 21$ cm.

The role of the working liquid is explored using various water-UCON\textsuperscript{TM} mixtures (see table 1 for details). Figure 12 illustrates that, as expected, increasing the liquid viscosity increases the
flow rate. Thus, when the liquid viscosity is increased about 100-fold, an increase in the liquid entrainment by an order of magnitude is observed. The effect of liquid depth observed for water is also observed for these liquids and, more importantly, figure 12 suggests that the corresponding critical velocity depends on the working liquid.

It is conventional in film coating flows to study the entrained flow by monitoring the film thickness $\delta_f$ as a function of capillary number $Ca = \mu U/\sigma$. In the creeping flow regime ($Re \ll 1$), the entrained mass flow rate is then given by

$$\dot{Q}_f = \rho U \delta_f w \left[ 1 - \frac{1}{3} \left( \frac{\delta_f}{\delta_f^g} \right)^2 \right],$$

(4)

where $w$ is the rim thickness. Note that the film flow thickness $\delta_f$, at sufficiently small $Ca$, should correspond to the expression $[1]$ associated with the classical Landau-Levich-Derjaguin dip-coating flow. However, when $Ca \gg 1$, previous observations [Kizito et al., 1999; Ruschak, 1985] report that the observed mass flow rate in the entrained film is obtained from the above expression if the film thickness $\delta_f$ is taken as in the expression $[2]$. This corresponds to the viscosity-gravity driven dip-coating flow and in this case, $\dot{Q}_f = 2/3 \times \rho U w \delta_f^g$. More recently, Jin et al. (2005) suggested, via numerical simulations of creeping flow ($Re \ll 1$) at sufficiently large capillary numbers, that the right expression for the flow rate with $\delta_f = \delta_f^g$ should contain a different pre-
factor 0.58 instead of $\frac{2}{3}$. In order to compare our experimental data with these results, we render the experimental flow rates of figure 12 dimensionless with the respective expressions for the flow rate. The resulting data is then presented in figure 13 wherein $\dot{Q}_{LLD}$ and $\dot{Q}_G$ are obtained from the expression \eqref{eq:4} when the film thickness is taken to be $\delta_f^{LLD} \eqref{eq:1}$ and $\delta_f^G \eqref{eq:2}$, respectively.

Figure 13 indicates that both $\dot{Q}_{LLD}$ and $\dot{Q}_G$ capture the correct order of magnitude of the entrained mass flow rate, which, as inferred from figure 12, varies over more than two orders of magnitude. The main dispersion observed in the rescaled experimental data is due to depth-to-radius ratio $h/R$. In addition, figures 13 suggest that this dispersion kicks-in at a critical capillary number which depends on the working liquid. Finally, below this critical capillary number, the ratio $\dot{Q}_f/\dot{Q}_{LLD}$ is approximately equal to unity for all liquids while the ratio $\dot{Q}_f/\dot{Q}_G$ significantly departs from unity for experiments in water. This suggests that a better adequacy of the LLD prediction \eqref{eq:1} compared to the visco-gravity film flow rate prediction \eqref{eq:2}. Therefore, in the subsequent discussions only the LLD framework is considered.

### 3.2.1 Transition to inertial regime in rotary entrainment flows

In fact, it is already known in coating flows (see De Ryck and Quéré \cite{1996} for thin fiber coating and Kizito et al. \cite{1999} for vertical plate coating), that a transition from the LLD law to an inertia-dominated regime happens when the capillary pressure based on the quasi-static LLD meniscus length $\lambda \simeq 0.65 l_c Ca^{1/3}$ is no longer large compared to the wall-imposed dynamic pressure $\rho U^2$. This is the case as the Weber number based on the LLD meniscus length i.e., $We_\lambda$, becomes sufficiently large. Hereby, in figure 14 we further investigate how the liquid depth modifies the film thickness by plotting all the above data as a function of this LLD Weber number $We_\lambda$. It is remarkable that all data collapse about a single horizontal line given by $\dot{Q}_f = \dot{Q}_{LLD}$ up to $We_\lambda \approx 10$. To the authors’ knowledge, this is the first ever demonstration of the existence of such a critical Weber number wherein inertial transition from LLD scaling in rotary film entrainment problem occurs. Thereafter, depending on the water depth $h/R$, the film thickness is either underestimated or over-estimated by the LLD dip coating law.

Note that the rotary flow entrainment problem in our case is particularly distinct from that of the classical coating flow processes. Here, instead of a dynamic LLD menisci at the base of the liquid film flow, a corrugated liquid sheet is present due to inertial ejection of the liquid at the emerging side of the disk. As already studied in section 3.1, the latter presents a three dimensional flow structure and its height is proportional to the square of the Froude number.

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**Figure 13:** Rescaled data from figure 12. Here, $\dot{Q}_G$ and $\dot{Q}_{LLD}$ are obtained from the expression \eqref{eq:4} when the film thickness $\delta_f$ is taken as $\delta_f^{LLD} \eqref{eq:1}$ and $\delta_f^G \eqref{eq:2}$, respectively.
Moreover, the range of Reynolds numbers \( Re_f = \rho U_f \delta_f / \mu \) in the film flow varies between \( \mathcal{O}(1) \) to \( \mathcal{O}(10^3) \). Therefore, it is quite unexpected that a good agreement with 2D creeping flow analysis of Landau-Levich-Deryagin is retrieved here for the entrained film on a rotating disk, as long as \( We_\lambda < 10 \). In addition, this model provides a satisfactory order of magnitude for the film flow rate even when \( We_\lambda > 10 \).

### 3.2.2 Impact of depth on entrained flow rate

Figures 11(a)–(c) & 12 suggest that beyond a critical speed the role of depth must be accounted for. Tallmadge (1971), Tharmalingam and Wilkinson (1978) suggested a suitably modified version of the dip-coating flow (see schematic 1c). In fact, via lubrication approximation along with LLD-type asymptotic analysis, these authors previously predicted that the average mass flow rate should be

\[
\dot{Q}_{TT} = \rho U \delta_f w \left[ 1 - \frac{1}{3} \left( \frac{\delta_f}{\delta_f^*} \right)^2 \sin \alpha \right],
\]

where \( \delta_f \) is computed from expression (3) and \( \alpha \) is the angle between the tangent to the partially immersed cylindrical drum at its line of contact with the liquid bath at rest and the horizontal line. When the measured data is compared with the estimates from this model (continuous and dashed lines in figures 11), a reasonable qualitative agreement is observed, showing an increase in the flow rate with increasing depth. Nevertheless the model largely underestimates the entrained flow rate. This arises from the fact that the model accounts for the squeezing of the meniscus occurring at large \( \alpha \), or low depths, and therefore predicts a smaller flow rate than that of equation (1) corresponding to the drag-out problem of a vertical plate (\( \alpha = 90^\circ \)).

A detailed observation of the flow around the rotating wheel shows that a supplementary liquid film is dragged out of the liquid bath by the disk’s lateral walls. This film, in turn, contributes to the film flow on the disk’s rim provided that centrifugal forces overcome gravity, which occurs for \( Fr > 1 \). In order to better quantify this contribution, two wiper blades were used to scrap-off the lateral film: the wiper blades were placed right next where the disk’s lateral walls and the liquid bath meet. The results are shown in figure 15 for \( R = 21 \text{ cm} \) at various water depth \( h/R \) (filled symbols) where they are also compared with data from figure 11(c). Here, it is evident that the water flow on the lateral wall modifies the entrained flow on the rim. Clearly, data from
Figure 15: Direct evidence for the contribution from liquid entrained via lateral walls (Water, and wheel radius $R = 21$ cm). Open symbols represent the cases without lateral scrapping while closed symbols represent data from scrapped cases. Remarkably, data corresponding to the depth to radius ratio $h/R = 0.4$ (black, filled squares) follows the LLD-scaling when lateral entrainment is scrapped off. However, for $h/R = 0.6$ and 0.8, lateral entrainment cannot completely explain the increase in mass flow rate.

Experiments with lateral scrapping for the case of $h/R = 0.2$ and $h/R = 0.4$ closely follow a unique curve given by $\dot{Q}_{LLD}$. Nonetheless, at water depths corresponding to $h/R = 0.6$ and 0.8, the supplementary entrainment via lateral walls accounts only for about 10–20% increase in film flow rate on the disk’s rim. Furthermore, experimental data in figure 15 (right) strongly suggests that, for both $h/R = 0.6$ & $h/R = 0.8$, the entrained flow rate in the absence of contribution from the film flow along lateral walls increases as a power-law of $We_\lambda$, independent of $h/R$.

A possible explanation can be found from a closer inspection of the dynamics of the liquid sheet on the emerging side of the wheel. As previously noted at the end of section 3.1, when the water depth is sufficiently large, the whole liquid sheet is as thick as the wheel rim (see for example, figure 9 when $h/R = 0.8$). This implies that at its top, the liquid sheet wets the entire thickness of rotating wheel: liquid is entrained from this liquid rim onto the wheel. Taking this action of the liquid sheet into account, it is possible to extend the classical Landau-Levich scaling to explain the liquid entrainment without lateral wall contributions.

In the classical entrainment problem, as illustrated by Maleki et al. (2011), Landau-Levich asserted that the viscous driving force per unit volume $\mu U/\delta_f^2$ on a fluid element is balanced out by the restoring force per unit volume $(\sigma/l_c)/\lambda$ which is taken solely due to the pressure gradient arising from the Laplace pressure across the dynamic menisci of length $\lambda$. Here, $\delta_f$ is the entrained film thickness and $l_c = \sqrt{\sigma/\rho g}$ is the capillary length. The missing length scale $\lambda$ is then obtained by matching the curvature of the dynamic menisci $\delta_f/\lambda^2$ to the static curvature $1/l_c$. Thereby, it is possible to obtain $\delta_f^{(LLD)} \sim l_c Ca^{2/3}$ which is exactly the first-order approximation to the Wilson formula in equation [1]. However, the fact that the wheel rim is entirely covered by a thick liquid sheet from which the thin liquid film emanates is indeed very distinct from the Landau-Levich case wherein the liquid film emerges from a quasi-static liquid bath. Therefore, it is no longer appropriate to match the curvature of dynamic menisci $\delta_f/\lambda^2$ with that of a static menisci $1/l_c$. Since the sheet height $H_{sm}$ depends only weakly on the wheel radius $R$, a simplest dimensional analysis points out that the appropriate curvature before the dynamic menisci should be $g/U^2 \times f(h/R)$. By matching the curvature of dynamic menisci $\delta_f/\lambda^2$ with this inertial menisci
curvature, the film thickness $\delta_f^I$ is obtained as

$$\delta_f^I \sim I_c Ca^{2/3} \left( \frac{U^2}{g l_c} \right)^{1/3} \zeta \left( \frac{h}{R} \right),$$

(6)

where $\zeta(h/R)$ is an arbitrary function of only the water depth to radius ratio. Thus, the corresponding entrained film flow rate is given by

$$\dot{Q}_f^I \sim \dot{Q}_{f,\text{LLD}}^{1/3} C a^{-1/9} \zeta \left( \frac{h}{R} \right).$$

(7)

where only the first-order contribution of the thickness in the expression (4) is taken. The modified LLD scaling (7) can be compared with the experimental data for the case when the liquid sheet occupies the entire rim i.e., when $h/R \geq 0.6$. As seen in figure 15 (right), despite many simplifying assumptions, the above scaling argument matches relatively well with the available data when the contribution from lateral entrainment is scrapped-off for both $h/R = 0.6$ & $h/R = 0.8$. Therefore, up to a first estimate, the observed entrainment in the inertial regime of a rotating wheel should arise from the presence of a thick liquid sheet.

### 4 Conclusion

In summary, the classical *Landau-Levich-Derjagin* entrainment problem is revisited for the case of a partially immersed rotating drum of radius $R$ and immersion depth $h$. Experiments show that, when the rotational speed is sufficiently large ($U \gtrsim 0.8$ ms$^{-1}$), a thin liquid sheet appears on the emerging side of the rotating drum which in turn leaves out a thin liquid film on the rim of the rotating drum.

Firstly, the liquid sheet height $H_m$ is shown to be proportional to the maximum ballistic height $U^2/2g$ attained by a fluid particle. And it is independent of the immersion depth $h/R$ except for the case of $h/R = 0.8$ wherein combined immersion depth and the sheet height become larger than the wheel diameter $h + H_m \gtrsim 2R$. In particular, when the immersion depth is small compared to the radius ($h/R < 0.5$), the sheet thickness measurements indicated that the sheet rapidly thins out until capillary recession inhibits further decrease. Since the capillary recession is delayed in viscous liquids, the sheet height in viscous UCON$^\text{TM}$/Water mixtures is found to be much larger than those observed in water at the same velocity. For large immersion ratio, no capillary recession is observed. Instead a thick liquid sheet as large as the rotating wheel’s rim appears.

Despite the large entrainment velocities in the present experiments with both water and UCON$^\text{TM}$/Water mixtures, the liquid film entrainment follows remarkably the *Landau-Levich-Derjagin* scaling (with appropriate higher-order corrections by Wilson (1982)) until a critical capillary number $Ca = \mu U/\sigma$. The transition to inertia-dominated entrainment regime occurs when the capillary pressure based on the quasi-static LLD meniscus length $\lambda \simeq 0.65 l_c Ca^{1/3}$ becomes lesser than the wall-imposed dynamic pressure $\rho U^2$. Experimental data in the present study strongly demonstrates that such a transition happens when $W e_\chi = \rho U^2 \lambda/\sigma$ is about 10. Beyond this limit, two major contributions are identified for inertial entrainment: (1) lateral wall entrainment and (2) entrainment from thick liquid sheet. The former is present when the Froude number based on the wheel radius is larger than unity and at this stage, the liquid film entrained on the lateral wall is centrifuged towards the wheel rim. This seems to be the dominant mechanism when the water depth is small compared to the wheel radius. The latter modifies the curvature of the quasi-static liquid in front of the entrained film and thereby leads to an enhanced film flow rate. This contribution is predominant when the liquid sheet thickness is comparable to that of the wheel’s rim ($h/R > 0.5$).

Finally, the authors hope that the present work motivates numerical simulations on inertia-dominated liquid film flows and also, the more challenging fully 3D two-phase flow problem of rotary entrainment at large Reynolds numbers.
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