Practically efficient methods for performing bit-reversed permutation in C++11 on the x86-64 architecture

Christian Knauth
Freie Universität Berlin
Institut für Informatik

Boran Adas
Freie Universität Berlin
Institut für Informatik

Daniel Whitfield
Freie Universität Berlin
Institut für Informatik

Xuesong Wang
Freie Universität Berlin
Institut für Informatik

Lydia Ickler
Freie Universität Berlin
Institut für Informatik

Tim Conrad
Freie Universität Berlin
Institut für Informatik

Oliver Serang*
University of Montana
Department of Computer Science

August 8, 2017

Abstract

The bit-reversed permutation is a famous task in signal processing and is key to efficient implementation of the fast Fourier transform. This paper presents optimized C++11 implementations of five extant methods for computing the bit-reversed permutation: Stockham auto-sort, naive bitwise swapping, swapping via a table of reversed bytes, local pairwise swapping of bits, and swapping via a cache-localized matrix buffer. Three new strategies for performing the bit-reversed permutation in C++11 are proposed: an inductive method using the bitwise XOR operation, a template-recursive closed form, and a cache-oblivious template-recursive approach, which reduces the bit-reversed permutation to smaller bit-reversed permutations and a square matrix transposition. These new methods are compared to the extant approaches in terms of theoretical runtime, empirical compile time, and empirical runtime. The template-recursive cache-oblivious method is shown to be competitive with the fastest known method; however, we demonstrate that the cache-oblivious method can more readily benefit from parallelization on multiple cores and on the GPU.

Introduction

The classic Cooley-Tukey fast Fourier transform (FFT) works by recursively reducing an FFT to two FFTs of half the size (in the case of decimation in time these smaller FFTs are evaluated on the even indices and odd indices of the original array) [1]. Cooley-Tukey is extremely important for its many uses in scientific computing: for example, the FFT permits numeric computation of the convolution of two arrays of length \( n \) in \( O(n \log(n)) \) steps rather than the \( O(n^2) \) required by the naive convolution algorithm [2]. The simplicity, efficiency, and broad utility of Cooley and Tukey’s work has led it to be considered one of the most influential computer science methods of the 20th century [3].

The simplest way to compute the Cooley-Tukey FFT is to compute it out-of-place and using a buffer of size \( n \). Then, the values from the original array can be copied into different indices of the buffer so that the even indices of the array are copied to indices \( 0, 1, \ldots, \frac{n}{2} - 1 \) of the buffer and so

*To whom correspondence should be addressed
that the odd indices of the array are copied to indices \( \frac{n}{2}, \frac{n}{2} + 1, \ldots n - 1 \) of the buffer. This buffered approach is commonly referred to as “Stockham auto-sort”\(^4\).

However, computing an in-place FFT (i.e., computing an FFT by overwriting the array evaluated, without requiring a result buffer) is more challenging. The even-odd permutation performed by the Stockham FFT is equivalent to an \( \frac{n}{2} \times 2 \) matrix transposition. For example, if \( n = 8 \), then the indices \([0, 1, 2, 3, 4, 5, 6, 7]\) can be thought of as a \( 4 \times 2 \) matrix, whose transposition corresponds to the desired even-odd permutation:

\[
\begin{bmatrix}
0 & 1 \\
2 & 3 \\
4 & 5 \\
6 & 7
\end{bmatrix}^T =
\begin{bmatrix}
0 & 2 & 4 & 6 \\
1 & 3 & 5 & 7
\end{bmatrix},
\]

where the even-index values are now in the first row and the odd-index values are now in the second row. This transposition is the inverse of the “faro shuffle” (also called “perfect shuffle”)\(^5\), wherein a deck of cards is cut in half and then the two halves are interleaved. In-place FFT requires these matrix transpositions to be performed in place (i.e., transposing the matrices while using substantially less than \( n \) additional memory); however, in-place transposition of non-square matrices is challenging because the destination index being written to does not necessarily swap with the source value (as would be the case with a square matrix). To transpose an \( M \times N \) matrix using at most \( O(M + N) \) space, existing algorithms require runtime \( \in \Omega(MN \log(MN)) \)\(^6\). Furthermore, even if a fast \( O(n) \) algorithm were available to perform the even-odd permutation without significant allocating of memory (perhaps exploiting the fact that this is a special case of matrix transposition, where the numbers of rows and columns are both powers of two), performing this permutation separates the FFT of length \( n \) from the recursive FFTs of length \( \frac{n}{2} \); this can have a negative influence and prevent the compiler from recognizing shared code in the recursive calls, such as reused trigonometric constants. FFT code that can be optimized by the compiler is highly valued, and can produce a substantial speedup\(^7\).

One approach is to perform the even-odd permutations for all of the recursive calls at the start of the FFT. Each even-odd permutation can be thought of as making the least significant bit into the most significant bit, and thus performing all of them sequentially is equivalent to permuting by swapping the indices with their bitwise reversed indices. The “bit-reversed permutation” of \([0, 1, 2, 3, 4, 5, 6, 7]\) (or \([000, 001, 010, 011, 100, 101, 110, 111]\) in binary) would be \([0, 4, 2, 6, 1, 5, 3, 7]\) (or \([000, 100, 010, 101, 001, 101, 011, 111]\) in binary). Given a function \( \text{rev} \) that reverses integer indices bitwise, the bit-reversed permutation is fairly straightforward to perform (Listing 1). Note that the bit-reversed permutation can be performed in-place by simply swapping index and reversed index pairs; this is less complicated than in-place transposition of a non-square matrix, because we are guaranteed that \( \text{rev}(\text{rev}(i)) = i \) (in a manner similar to square matrix transposition). After applying a bit-reversed permutation, the FFT can be performed by simply invoking a slightly rewritten version of the FFTs, which can now assume that at every level of recursion, the even-odd permutation has already been performed.

Listing 1: Performing the bit-reversed permutation with the help of an external rev function. Let \( \text{LOG}_N \) be the problem size, which is a constexpr (i.e., it is known a constant known at compile time). Note that the function \( \text{rev} \) is templated to take the word size used for reversal.

```cpp
void naive_bitwise_permutation(T* __ restrict const v) {
    constexpr unsigned long int N = 1ul << LOG_N;
    for (unsigned long index=1; index<(N-1); ++index) {
        unsigned long reversed = rev<LOG_N>(index);
        // Comparison ensures swap is performed only once per unique pair (otherwise, 
        // every index will be swapped and then swapped back to re-create the original 
        // array):
        if (index<reversed)
            std::swap(v[index], v[reversed]);
    }
}
```
Although some digital signal processors (DSPs) provide native bit reversal operations, many modern desktop CPUs (including the x64 CPUs at time of publication), do not have an opcode to perform the rev function; therefore, efficient functions to compute rev are necessary. Bit reversal can of course be performed bitwise in \( \Theta(b) \), where \( n = 2^b \) and where \( b \) corresponds to \( \log_2 N \) in the C++ code. Performing the full bit-reversed permutation using this naive bit reversal method thus requires \( \Theta(nb) = \Theta(n \log(n)) \) steps.

**Bytewise bit reversal:** Because the bit reversal function rev is called at every index, optimizing the rev function can be quite beneficial for performance. One way to perform bit reversal significantly faster than the naive bitwise approach is to swap with byte blocks instead of bits. This is efficiently accomplished by hard-coding an array of unsigned char types, which contains 256 entries, one for each possible byte. For any byte \( B \), the accessing the table at index \( B \) returns the bit-reversed value of that byte[8, 9]. With this reversed byte table, the same approach as the naive bitwise method can be used, requiring \( b \) steps instead of the \( b \) steps required by the naive bitwise method. This bytewise method is much faster in practice, even though its asymptotic runtime, like the naive bitwise method, is still \( \in \Theta(n \log(n)) \).

However, even if the bit reversal function rev were natively supported and ran in \( O(1) \) CPU clock cycles, the non-sequential memory accesses performed by the bit-reversal permutation do not cache effectively using the standard hierarchical cache model (which loads contiguous chunks into the cache every time a non-cached value is fetched from the next layer of memory hierarchy), which is used by x64 computers. This contrasts with the other FFT code, which accesses memory in an almost perfectly sequential manner (and thus caches very effectively). Thus as a result, the cost of computing a large FFT can actually spend a significant percent of its time performing the bit-reversed permutation.

**Reversing bitwise using pairs of bits:** One method for achieving better cache performance proceeds bitwise but working 2 bits at a time (exchanging the most significant and least significant bits). Then, instead of computing the full bit-reversed index and then swapping if \( \text{index} < \text{rev(index)} \), this paired bitwise method performs multiple swaps, one after each pair of bits are exchanged[10]. Thus a greater number of swaps on the array are performed, but the swaps achieve better spatial locality. Even though the blocks of memory accessed are not sequential, they are more contiguous than in the naive bitwise and bytewise approaches. Despite the greater number of swap operations performed on the array, the asymptotic runtime of this approach is no different from the naive bit reversal approach, and is thus \( \in \Theta(n \log(n)) \).

**Bit reversal using a matrix buffer:** Carter & Gatlin proposed a cache-optimized method which uses a matrix buffer. The principal idea is that the buffer is small enough to fit completely within the L1 or L2 cache, and thus the buffer can be accessed in either row-major or column-major order and still achieve good cache performance. Given \( \text{index} = x y z \) where the bit strings \( x \) and \( z \) have the same size \( \log(\sqrt{t}) \), their method uses a matrix buffer of size \( \sqrt{t} \times \sqrt{t} = t \)[11].

Because \( \text{rev(index)} = \text{rev(z)} \text{rev(y)} \text{rev(x)} \), an out-of-place method for performing the bit-reversed permutation would copy \( \forall x, \forall y, \forall z, \text{dest}[\text{rev(z)} \text{rev(y)} \text{rev(x)}] \leftarrow \text{source}[x y z] \). The method (denoted COBRA in[11]) proceeds in two steps:

\[
\forall y,
\forall x, \forall z, \\text{buff}[\text{rev(x)} \text{rev(z)}] \leftarrow \text{source}[x y z]
\]

\[
\forall x, \forall z, \\text{dest}[z \text{rev(y)} x] \leftarrow \text{buff}[x z].
\]

This can be modified slightly:

\[
\forall y,
\forall x, \forall z, \\text{buff}[\text{rev(x)} z] \leftarrow \text{source}[x y z]
\]
\[ \forall x, \forall z, \text{dest}[\text{rev}(z) \text{ rev}(y) \ x] \leftarrow \text{buff}[x \ z]. \]

Although Carter & Gatlin describe an out-of-place implementation (meaning that it writes to dest rather than modifying source), it is possible to adapt the method to an in-place version by swapping data with buff rather than simply copying to or from buff. This requires a third looping step to propagate the changes made to the buffer back into the array. Note that even this “in-place” variant of COBRA still requires a buffer, and so it is not truly an in-place method.

Assuming a \( \Theta(b) \) \text{rev} method, the asymptotic runtime of COBRA can be found as follows: The \( \forall y \) loop requires \( \frac{n}{\tau} \) steps, and the \( \forall x \) and \( \forall z \) loops each require \( \sqrt{t} \) steps. By caching bit-reversed values (e.g., \text{rev}(y)) as soon as they can be computed, the runtime is

\[
\frac{n}{\tau} \left( \frac{\log(n)}{t} \right) + \frac{2\tau}{\sqrt{t}} \left( \frac{\log(\sqrt{t})}{\tau} + \sqrt{t} \right).
\]

Asymptotically, this runtime is \( \in \Theta \left( n + \frac{n}{\tau} \log(\frac{n}{\tau}) \right) \). Note that using a large buffer (i.e., \( t \gg 1 \)), the runtime approaches \( \Theta(n) \); however, this is achieved by sacrificing the improved cache performance that can be achieved when \( t \) is small enough to fit into the L1 or L2 cache. Conversely, when \( n \gg t \), then the runtime will be \( \in \Theta(n \log(n)) \). Unlike cache-oblivious methods, which perform well for any hierarchical caches, for any problem size \( n \), the COBRA algorithm needs to optimize the parameter \( t \) for a particular architecture.

COBRA has previously been demonstrated to outperform a method from Karp\[11\]; before that, Karp’s method had been shown to outperform 30 other methods\[12\].

In this paper we introduce three additional methods for performing the bit-reversed permutation. We then compare the theoretical and practical performance to existing methods by benchmarking fast implementations of all methods in C++11.

**Methods**

**Inductive XOR method for generating bit-reversed indices:** We first propose a method that avoids calling the \text{rev} function altogether. This is accomplished by inductively generating bit-reversed indices without the use of bitwise reversing or bytewise reversing with a reversed byte table.

Begin with the first index and its reversed value: \text{index} = \text{rev(index)} = 0. The next index can be trivially computed as \text{index+1}, but the next reversed index is found via \text{rev(index+1)}, which will not necessarily equal \text{rev(index)+1}. Rather than explicitly call a \text{rev} function (whether it be bitwise or bytewise), we avoid this by making use of the bitwise XOR function. \text{index} XOR \text{index+1} reveals only the bits that have changed by incrementing. Reversing both \text{index} and \text{index+1} and computing \text{rev(index)} \text{ XOR } \text{rev(index+1)} will be equivalent to computing \text{rev(index XOR index+1)}, because the bitwise XOR operation does not exchange any information between bits. Furthermore, \text{index} XOR \text{index+1} will reflect the fact that all differing bits reflect a carrying operation in binary addition; therefore, \text{index} XOR \text{index+1} must be of the form 000...0111...1. A bit string of this form can be reversed efficiently by simply shifting left by the number of leading zeros, thereby producing a bit string of the form 1...1110...000.

The number of leading zeros in a bit string of \( b \) bits can be computed trivially in \( O(b) \) steps, but it can also be computed in \( O(\log(b)) \) steps. An \( O(\log(b)) \) runtime is accomplished by bitmasking with a bit string with the most significant \( \frac{b}{2} \) equal to 1 and the least significant \( \frac{b}{2} \) bits equal to 0, thereby uncovering which half contains a 1 bit. The first half with a 1 can be iteratively subdivided to find the most significant 1 in \( O(\log(b)) \) steps\[9\].

The number of leading zeros can also be computed in \( O(1) \) via the integer \( \log_2 \): this can be performed by casting to a float and bitmasking to retrieve the exponent; however, this approach
First, index+1 is computed. Second, the bits that are different between index and index+1 are computed via XOR; the resulting bit string will be of the form 000...0111...1. Third, because the differing bits will be of the form 000...0111...1, they can be reversed by bit shifting left by the number of leading zeros. Fourth, rev(index+1) can be computed by flipping only the differing bits (via XOR).

Figure 1: Illustration of the inductive XOR method for bit reversal. Beginning with index and its bit-reverse value rev(index), the next values (index+1 and rev(index+1)) are computed. First, index+1 is computed. Second, the bits that are different between index and index+1 are computed via XOR; the resulting bit string will be of the form 000...0111...1. Third, because the differing bits will be of the form 000...0111...1, they can be reversed by bit shifting left by the number of leading zeros. Fourth, rev(index+1) can be computed by flipping only the differing bits (via XOR).

only works for $b \leq 24$\[9\]. For large problems with $b > 24$, that shortcoming can be solved by casting to double, although casting to double would be less efficient. Fortunately, even though the x64 architecture does not have an opcode for performing rev in $O(1)$, it does feature an opcode for computing the number of leading zeros. In g++ and clang this can be accessed via the ___builtin_clz name (which counts the leading zeros in an unsigned long int).

As a result, rev(index) XOR rev(index+1) can be reversed in very few operations by computing the bits differing between rev(index) and rev(index+1). The next reversed index, rev(index+1), is then computed by flipping only those bits differing from rev(index) by using XOR (Figure 1). The XOR method performs a constant number of XOR operations as well as a single count leading zeros operation to compute the next index and reversed index. Count leading zeros will require $\in \Theta(\log(b)) = \Theta(\log(\log(n)))$ operations in the general case, and can be performed in $O(1)$ when $b \leq 24$ or when hardware support a built-in operation. Thus the overall runtime of the full bit-reversed permutation will be $\in \Theta(n \log(\log(n)))$ in the general case and $\in \Theta(n)$ when counting leading zeros can be performed in $O(1)$.

The XOR method is very efficient at computing the next index and next reversed index; however, even if that computation were instantaneous, the algorithm achieves only moderate performance for two reasons: The first reason is that the bit-reversed indices are not accessed in a contiguous fashion (and thus are not cache-optimized). The second reason is the if statement, which is also found in Listing 1. This if statement prevents loop unrolling from being fully effective, because the compiler will not yet know the effects of swapping in the previous iteration (limiting the ability to swap in parallel, even if hardware would support it). It should also be noted that the frequency with which index < rev(index) will change during the loop, diminishing the benefits of branch prediction.

Unrolling (template-recursive closed form): Eliminating the if (index < rev(index)) comparison is difficult, because it requires computing in advance the indices on which this will be true and then only visiting those indices; correctly computing the pattern of indices where this is true is strikingly similar to performing bit reversal. However, for problems of a fixed size (e.g., $b = 10$ bits or equivalently $n = 1024$), it is possible to simply compute all indices that should be swapped over
the bit-reversed permutation. This offers two benefits: First, the overhead of looping, performing the
bit reversal on indices, and checking whether \( \text{index} < \text{rev(index)} \) are all eliminated. Second, the
compiler could (in theory) rearrange the swap operations to be more sequentially contiguous, thereby
improving cache performance.

This could implemented via a hard-coded function `unrolled_permutation_10`, but alternatively
it could also be implemented by generating that code at compile time via template recursion. For any
bit string of the form \( \text{index} = z \times y \) (where \( z \times y \) is the concatenation of bit strings \( z \), \( x \), and \( y \)),
the reverse will be \( \text{rev(index)} = \text{rev(y)} \cdot \text{rev(x)} \cdot \text{rev(z)} \). When the bit strings \( z \) and \( y \) consist of a
single bit, then their reversal can be ignored: \( \text{rev(index)} = y \cdot \text{rev(x)} \cdot z \). Thus, it is possible to start
from both ends (the most significant bit remaining and least significant bit remaining) and proceed
inward to recursively generate problems of the same form (these recursive calls are implemented via
template recursion to unroll them at compile time). Some of the template recursions can be aborted in
a branch-and-bound style (Figure 2): Bit strings of the form \( \text{index} = 1 \times 0 \) will never be less than
their reverse, regardless of the value of the bit string \( x \); therefore, further recursion can be aborted.
Likewise, bit strings of the form \( \text{index} = 0 \times 1 \) will always be less than their reverse, and therefore
swap operations should be performed for every possible \( x \). Lastly, bit strings of the form \( 0 \times 0 \) and
\( 1 \times 1 \) will be less than their reverse when \( x < \text{rev}(x) \). This produces a problem of the same form as
finding whether \( \text{index} < \text{rev(index)} \), but two bits smaller, and thus it can be performed recursively.

As a result, the runtime of this method is defined by the recurrence \( r(b) = \frac{2^{b-2}}{\text{green}} + \frac{2 \cdot r(b-2)}{\text{yellow}} + \frac{0}{\text{red}} \)
(where the colors labeling the three terms correspond to the coloring in Figure 2). Using \( r(1) = 0 \)
(because there are 0 swaps necessary with a single bit) and \( r(2) = 1 \) (there is one swap necessary
using 2 bits), then this recurrence has closed form \( r(b) = 2^\frac{b}{2} \times \left((-1)^b \frac{\sqrt{2} - 1}{2} + \frac{1 + \sqrt{2}}{4}\right) + 2^{b-1} \). Hence,
regardless of whether \( b \) is even or odd, \( r(b) = 2^{b-1} - c \cdot 2^\frac{b}{2} \) (c = -1 when \( b \) is even and \( c = -1 \sqrt{2} \) when
\( b \) is odd). Regardless, the asymptotic runtime is dominated by the \( 2^{b-1} \) term; therefore, asymptotic
runtime is a linear function of \( n \) (because \( n = 2^b \)).

For small problems, this template-recursive “unrolled” method is very efficient. However, a dis-
advantage of the method is that on large problems, a great deal of code will be generated, which can
result in large compilation times and also can mean that the order of the swapping operations will
not be optimized effectively by the compiler. In fact, there may be no cache-efficient order to visit the
indices being swapped because bit reversed indices jump around. Thus, unless the compiler possesses
mathematical insight about the swaps being performed (enabling the compiler to transform the code
into one of the other algorithms listed), the unrolled method’s mediocre efficiency on large problems
will not justify its substantial compilation times.

| index   | rev(index) |
|---------|------------|
| 1 x 0   | 0 rev(x) 1 |
| 0 x 0   | 0 rev(x) 0 |
| 1 x 1   | 1 rev(x) 1 |
| 0 x 1   | 1 rev(x) 0 |

∀ index, index > rev(index). Return.

∀ index < rev(index) when x < rev(x).

∀ x : x < rev(x), swap index and rev(index) by recursing on a problem 2 bits smaller.

∀ index, index < rev(index). Swap ∀ index.

Figure 2: Illustration of the unrolled method for bit reversal. All possible bit strings where
\( \text{index} < \text{rev(index)} \) are found at compile time by beginning with the most significant bit remaining
and least significant bit remaining and progressing inward recursively. Bit strings of the form \( 1 \times 0 \) will never be less than their reverse (highlighted in red). Bit strings of the form \( 0 \times 1 \) will always be
less than their reverse (highlighted in green). Bit strings of the form \( 0 \times 0 \) and \( 1 \times 1 \) will only be
less than their reverse when \( x < \text{rev}(x) \) (highlighted in yellow).
∀ x, recursively perform a smaller bit-reversed permutation.  

\[
\begin{array}{cccc}
  x & y & x & y \\
  0 & 1 & 2 & 3 \\
  4 & 5 & 6 & 7 \\
  8 & 9 & 10 & 11 \\
  12 & 13 & 14 & 15 \\
\end{array}
\]

∀ y, recursively perform a smaller bit-reversed permutation.  

\[
\begin{array}{cccc}
  y & x & y & x \\
  0 & 1 & 2 & 3 \\
  4 & 5 & 6 & 7 \\
  8 & 9 & 10 & 11 \\
  12 & 13 & 14 & 15 \\
\end{array}
\]

Figure 3: Illustration of recursive bit-reversed permutation. A bit-reversed permutation on \( b \) bits is performed as several smaller bit-reversed permutations on \( \frac{b}{2} \) bits, an in-place transposition of a square matrix and another batch of smaller bit-reversed permutations on \( \frac{b}{2} \) bits. The spatial locality of each of these steps yields a cache-performant algorithm.

**Cache-oblivious recursive bit-reversed permutation:** When the number of bits \( b \) is even, an index bit string can be partitioned into two equally sized bit strings of \( \frac{b}{2} \) bits each: \( \text{index} = x \ y \). The reversed index would be \( \text{rev(index)} = \text{rev(y)} \ \text{rev(x)} \). This can be computed in three steps: First, reversing the least significant bits (which form \( y \)) to produce \( x \ \text{rev}(y) \). Second, swapping the most significant bits forming \( x \) with the least significant bits forming \( \text{rev}(y) \), thus producing \( \text{rev}(y) \ x \). Third, repeating the first step and reversing the least significant bits (which at this point contain \( x \)) to produce \( \text{rev}(y) \ \text{rev}(x) = \text{rev}(\text{index}) \).

This recursive technique is not only useful for computing a reversed index: it can also be used to perform the entire bit-reversed permutation in a recursive manner with localized operations that do not need to be tuned for the cache size. The first and third steps are performed in a manner identical to one another: for all possible most significant bit strings, perform a smaller, local bit-reversed permutation of size \( \frac{b}{2} \). These recursive bit-reversed permutations will all be applied to much smaller and contiguous blocks of memory, thereby improving the cache performance. The second step, wherein the most significant and least significant bit strings are reversed corresponds to a matrix transposition, with the most significant bits corresponding to the rows of the matrix and the least significant bits corresponding to the columns (using C-style row-order array organization). Since \( x \) and \( y \) use equal numbers of bits, this swap corresponds to a transposition of a square matrix, which can be performed in place. Furthermore, an optimal cache-oblivious algorithm (meaning that it is close to the optimal possible runtime for any hierarchical cache organization) for matrix transposition is known, which performs the transposition in further and further subdivisions\[13\]. Thus performing a bit-reversed permutation can be reduced to smaller bit reversed permutations on contiguous blocks of memory and a square matrix transposition (Figure 3).

When the number of bits \( b \) is odd, then a single even-odd permutation can be performed first:
Reversing index = x y z where z consists of a single bit yields \texttt{rev(index)} = z \texttt{rev(y)} \texttt{rev(x)}.
An even-odd permutation produces z x y. Applying a \( b - 1 \) bit-reversed permutation when z=0 and another \( b - 1 \) bit-reversed permutation z=1 will yield z \texttt{rev(y)} \texttt{rev(x)}. This even-odd permutation for preprocessing when \( b \) is odd will slightly increase the runtime in that case (this even-odd permutation is also performed out of place by using a buffer of size \( \frac{n}{2} \)).

The runtime of the recursive method is defined by the recurrence \( r(b) = 2 \cdot 2^b \cdot r(\frac{b}{2}) + 2^b \). This recurrence has closed form \( r(b) = 2^{b-3} \cdot b \cdot c + 2^b \cdot (b - 1) \), where \( c \) is a constant. Thus \( r(b) \in \Theta(2^b \cdot b) = \Theta(n \log(n)) \).

In spite of the flaws in the unrolled closed form when \( b \gg 1 \), it is very efficient for small to moderately-sized problems (e.g., \( b \leq 14 \), which corresponds to \( n \leq 16384 \)); therefore, it is an ideal base case for the recursive method. The recursive calls can be made using template recursion, thereby enabling the compiler to optimize code shared by these recursive calls. This recursive method generalizes to a semi-recursive method, which not only calls the unrolled implementation when the number of bits is below a threshold, it also calls the unrolled implementation when the recursion depth passes beyond a threshold. This can allow the compiler greater optimizations, because all of the template-recursive bit-reversed permutations can be inlined (for larger recursion depths, compilers do not always inline all of the code, which can be seen by the much higher compilation times when decorating the recursive bit-reversed permutation function with \texttt{attribute (\_always_inline\_)}), which forces inlining in \texttt{g++ and clang}). Note that a semi-recursive method that only allows a single recursion will have runtime \( r(b) = 2 \cdot 2^b \cdot 2^b + 2^b \), because the recursive calls \( r(\frac{b}{2}) \) will be replaced with the unrolled method runtime, \( 2^b \). Therefore, the runtime of that semi-recursive method becomes \( 2 \cdot 2^b + 2^b \in \Theta(2^b) = \Theta(n) \).

An advantage of the recursive method over COBRA is that (at least when the number of bits \( b \) is even, \( \frac{b}{2} \) is even, \( \ldots \) until the base case size or recursion limit are reached) it can be performed completely in place and without a buffer. Because the recursive method reduced bit-reversed permutation to smaller bit-reversed permutations (each of which are performed independently and in an in-place manner) and to a square matrix transposition, the method is well suited to parallelization via SIMD, multiple cores, or by broadcasting over GPUs; parallelization of COBRA would require duplicate buffers for each parallelization in order to prevent race conditions.

The asymptotic runtimes for all methods are shown in Table 1.

| Runtime (asymptotic) | Stockham | Bitwise | Bytewise | Pair bitwise | COBRA | Unrolled | XOR | Recursive |
|----------------------|----------|---------|----------|--------------|-------|----------|-----|----------|
| \( n \log(n) \)     | \( n \log(n) \) | \( n \log(n) \) | \( n \log(n) \) | \( n \log(n) \) | \( n + \frac{n}{2} \log(\frac{n}{2}) \) | \( n \) | \( n \) or \( n \log(\log(n)) \) | \( n \) or \( n \log(n) \) |

Table 1: Theoretical runtimes. The asymptotic runtimes for each algorithm are given. Note that the algorithms with lowest theoretical runtime are not necessarily superior in practice: For example, the bitwise and bytewise methods are both \( \in \Theta(n \log(n)) \), but the bytewise method has a superior runtime constant because it uses a table to reverse words using 8 bits at a time. Likewise, the pair bitwise, COBRA, and recursive methods all access memory in more contiguous, local fashions and therefore have superior cache locality. For the COBRA method, \( t \) is the buffer size used. The runtime of the inductive XOR method will be either \( \in \Theta(n) \) (when count leading zeros is \( \Theta(1) \)) or \( \Theta(n \log(\log(n))) \) (when count leading zeros cannot be performed in hardware, and thus requires \( \in \Theta(\log(\log(n))) \) steps). The recursive method will require \( \in \Theta(n \log(n)) \) steps in general, but can be sped up to \( \in \Theta(n) \) steps when using a semi-recursive approach, which limits the recursion depth.

Results

All methods were implemented in \texttt{C++11}, making use of template recursion, along with \texttt{constexpr} variables and functions. Runtimes were compared, benchmarking all methods by compiling with both
clang++ 3.8.0 and g++ 6.2.0. On both compilers, compilation was optimized with flags \texttt{-Ofast -march=native -mtune=native}. In order not to risk pointer aliasing or other interference between different benchmarks, each algorithm and input size was benchmarked in a separate \texttt{main.cpp} file. Compile times with \texttt{clang++} are plotted in \textbf{Figure 4}, compile times with \texttt{g++} in \textbf{Figure 5}. All methods were applied to arrays of type \texttt{std::complex<double>}, since our primary motivation of performing fast bit-reversed permutation lies in evaluating the methods' suitability for FFT implementation. Note that individual performances of the methods can vary slightly when other data types with different sizes (e.g., \texttt{int} or \texttt{float}) are used.

For both the in-place and out-of-place COBRA methods, the buffer size was optimized for each problem size. The cache-oblivious recursive method used a base cases of size $b \leq 9$, at which point it computed the unrolled closed form of bit-reversed permutation of that size. The runtime was not very sensitive to this choice of base case size; this is reminiscent of the base case size used in cache-oblivious matrix transposition, where it is merely used to amortize out the cost of recursions by ensuring the computational cost of the base case is non-trivial.

Benchmarked problem sizes ranged from from $n = 2^8$ (which requires $\approx 4$KB to store) to $n = 2^{30}$ (which requires $\approx 16.4$GB to store). All measurements were averaged over 100 runs, and runtimes are reported in seconds per element (i.e., the total elapsed time for the bit-reversed permutation divided by $n$). The CPU specifications of the computer used for benchmarking are listed in \textbf{Table 2}. The runtimes for all tests, compiled with \texttt{clang++} and \texttt{g++} are shown in \textbf{Figure 6} and \textbf{Figure 7}, respectively.

In order to measure its inherently parallelizable nature, parallel versions of the semi-recursive method were also implemented and compared. The first of these multiple cores via OpenMP and the \texttt{-fopenmp} compiler option. This OpenMP implementation uses \texttt{#pragma omp parallel for} to allow the recursive calls (which will be implemented via the unrolled closed form since only one recursion is permitted) to be run in parallel. The second parallel version adapts the semi-recursive method to use the GPU via the CUDA toolkit. This GPU implementation permits two recursions and is hard-coded for $b = 24$ (it requires the indices swapped to be stored in an array, which is broadcast over the GPU). $b = 24$ was chosen because it was the largest problem that would wholly fit on the GPU such that $b$ is divisible by 4 (problems divisible by 4 can be expanded by two recursions without performing an even-odd permutation as preprocessing). This CUDA implementation also performs the in-place transposition on the GPU[14], and therefore parallelizes both the recursive calls and the transposition (this has an added benefit of only needing to move data to the GPU once). Both the OpenMP and the GPU parallel versions were compiled with \texttt{g++} (and the GPU version used \texttt{nvcc}), because of limited support to-date for both OpenMP and CUDA in \texttt{clang++}. These parallel versions are compared to the best-performing single-threaded methods in \textbf{Figure 8}.

| L1d | L1i | L2   | L3   | MAX_SPEED | RAM  |
|-----|-----|------|------|-----------|------|
| 32K | 32K | 256K | 15360K | 3.8Ghz    | 65GB |

\textbf{Table 2: CPU specification used for benchmarking.} The size of the L1 data cache, the L1 instruction cache, the L2 cache, the L3 cache, and the clockspeed as well as the RAM-size of the computer used for benchmarking are shown. To relate this to the benchmark-results we have shown, note that 32K can hold an array of $n = 2^{11}$ elements of type \texttt{std::complex<double>}, 256K can hold $n = 2^{14}$ elements, and 15360K can hold $n = 2^{20}$ elements. The 65GB of RAM can hold $n = 2^{31}$ elements.

\textbf{Discussion}

With both \texttt{g++} and \texttt{clang++}, the Stockham and naive bitwise shuffling methods have roughly similar performance to one another and are the least efficient methods investigated here. The Stockham
The Stockham method is an out-of-place procedure (thus increasing the burden on the cache) and performs more swap operations than the bitwise shuffle; however, the Stockham method accesses data by even and odd elements and thus access the data in a roughly contiguous fashion, while the bitwise method accesses the data in a less contiguous fashion because it accesses index \( \text{rev}(i) \) at every iteration.

The bytewise table method and the XOR method achieve similar performance to one another. Both methods scale considerably better than the Stockham and bitwise methods. Both the bytewise table method and the XOR method achieve faster bit reversal, but neither achieves a contiguous memory access pattern; therefore, neither is very cache performant. The proposed XOR method may still have use in areas where memory is at a premium (e.g., embedded systems), since it achieves similar performance to the byte table without storing or accessing a table of 256B.

For small problems, the unrolled approach achieves essentially optimal performance, seeing as the entire array can be fit into the L1 cache and there is no overhead for looping, bit reversal, or branch statements. Furthermore, the operations can be compiled to use immediate mode addressing in the assembly code, meaning that the indices swapped are natively encoded into the assembly instructions themselves and need not be encoded in a separate array. The runtime benefit of this unrolled method does not extend to larger problems (because the array doesn’t fit into the L1 or L2 caches) and the compile time of the unrolled method becomes very expensive (roughly 100 seconds when \( b = 16 \) for both \texttt{g++} and \texttt{clang++}).

The most performant algorithms on large problems are those designed with the cache in mind: this includes the pair bitwise method, the COBRA method (both in-place and out-of-place), and the
recursive method and its semi-recursive variant. The recursive and semi-recursive methods benefit when applied to problems with an even number of bits, because no even-odd permutation need be performed as a preprocessing step (this results in a sawtooth pattern in the runtimes of large problems). The semi-recursive method achieves slightly greater performance than the recursive method, but at the cost of an increased compilation time (between 10 to 13 seconds when $b = 28$ using both $g++$ and $clang++$). For smaller problems, the out-of-place COBRA method is less efficient than its in-place variant, because of the greater burden on the cache from using twice as much memory. However, for larger problems, a greater performance is achieved by the out-of-place COBRA method, because the out-of-place method copies values via a buffer, while the in-place method swaps values via the buffer; therefore the in-place method must copy to the buffer, then swap between the buffer and the array, then swap those resulting changes to the buffer back into the array.

The pair bitwise method performs more swap operations, but with greater locality and without the use of a result buffer (as is required by out-of-place algorithms like Stockham). The pair bitwise method, the COBRA methods, and the recursive and semi-recursive methods perform roughly similarly for large problems; the out-of-place COBRA method performs slightly better for larger problems, but its performance gains are muted when the matrix buffer size is not optimized for the CPU. There are small increases in the runtime per element at cache boundaries, most notably at the L3 cache boundary encountered at $b = 18$ for in-place methods and at $b = 17$ for out-of-place methods.

Unlike the COBRA methods, which achieve their performance via a matrix buffer, the recursive and semi-recursive methods are inherently cache efficient and use no such buffer; therefore, they are
Figure 6: Runtimes per element with g++. For each method, bit-reversed permutations of various sizes are performed. For each method and on each problem size $b$, 100 replicate runtimes (in seconds) were measured, and the average runtime per element (i.e., elapsed time $t$ divided by $n = 2^b$) are plotted. The shaded areas around each series depict the minimum and the maximum runtimes in all of the 100 replicates. The left panel shows all methods from $b = 8$ to $b = 30$ and the right panel shows only the highest performance series on larger problems $b = 20$ to $b = 30$. The methods with poor performance on large problem sizes (Stockham, bitwise, bytewise and XOR) have been excluded from benchmarking for $b > 26$.

well suited to parallelism. Figure 8 shows the strong performance benefit that can be achieved by either adding coarse-grain parallelism via OpenMP or by adding fine-grain parallelism with the GPU. The recursions do not need to access any information from other elements, so are perfectly suitable for concurrency (including the fact that modern architectures sometimes include separate L1 caches for each hardware core). This inherent parallelizable quality to the recursive and semi-recursive methods could likely also be exploited to greater effect by further optimizing the GPU code.

At $b = 24$ in Figure 8 the runtime of the fastest non-parallelized method requires just under $8 \times 10^{-9}$ seconds per element, or roughly 0.13 seconds total. In comparison, the OpenMP variant of the semi-recursive method requires roughly 0.1 seconds, and the GPU variant of the semi-recursive method requires 0.08 seconds. As a point of reference, the numpy FFT (which uses the FFTPACK library) takes roughly 2 seconds for problems of that size. The naive bitwise approach requires roughly 1 second, indicating that around 50% of the total FFT runtime would be spent in the bit-reversed permutation, meaning that the speedup from using these faster bit-reversed permutation methods is non-trivial; although the FFT butterfly code performs more sophisticated manipulations of complex numbers, it does so in a fully contiguous fashion, making the bit-reversed permutation more important than it may seem. Furthermore, using this semi-recursive method to perform the full FFT on the GPU would not only benefit from performing small FFT butterfly operations in parallel, it would also substantially reduce the runtime of the bit-reversed permutation by avoiding the cost of copying data to and from the GPU.

Aside from the ability to parallelize the recursive methods, their benefit comes from the fact that they achieve high performance without tuning for a specific cache architecture (which we have found substantially influences the efficiency of the COBRA methods). Specifically, the recursive method is cache-oblivious. When paired with an optimal cache-oblivious method for matrix transposition,
Figure 7: Runtimes per element with g++. For each method, bit-reversed permutations of various sizes are performed. For each method and on each problem size, 100 replicate runtimes (in seconds) were measured, and the average runtime per element (i.e., elapsed time t divided by $n = 2^b$) are plotted. The shaded areas around each series depict the minimum and the maximum runtimes in all of the 100 replicates. The left panel shows all methods from $b = 8$ to $b = 30$ and the right panel shows only the highest performance series on larger problems $b = 20$ to $b = 30$. The methods with poor performance on large problem sizes (Stockham, bitwise, bytewise and XOR) have been excluded from benchmarking for $b > 26$.

it guarantees fairly contiguous memory accesses without any knowledge of the cache architecture. Like the recursive method, the semi-recursive strategy also does not use a buffer with size tuned for cache performance; however, the semi-recursive is not cache-oblivious, because large problems may be reduced to smaller bit-reversed permutation problems that still do not fit in the cache. The recursive method is not only interesting for future research that would investigate the performance benefit from parallelizing it, the fact that it achieves high performance without architecture-specific cache tuning suggests that it is a promising foothold in the search for an optimal cache-oblivious bit-reversed permutation algorithm.

Availability

All C++11 source code accompanying this paper, preliminary GPU implementation of the recursive algorithm, as well as benchmarking scripts, and the LaTeX code for the paper itself are freely available (Creative Commons license) at https://bitbucket.org/orserang/bit-reversal-methods.

Acknowledgements

We are grateful to Thimo Wellner and Guy Ling for their contributions.

This paper grew out of the masters course in Scientific Computing taught by Oliver Serang. Translations of this paper are also available in the mother tongues of every student: German, Chinese,
Figure 8: **Performance gains from parallelization.** Benchmarks were repeated from the right panel of Figure 7 but with the inclusion of a parallelized versions of the semi-recursive method. Because of the semi-recursive methods inherently parallel nature, OpenMP easily results in faster performance. Likewise, broadcasting operations over the GPU (via CUDA) achieves even greater parallelism.

Author Contributions

The new algorithms and non-parallel implementations in C++11 were created by and the project was supervised by O.S. The Introduction section was written by X.W. The Methods section and runtime analysis were performed by B.A., X.W., and D.W. The OpenMP implementation was made by O.S. and the GPU variant was made by B.A. The benchmarks, figures, and Results sections were created by C.K. The Discussion section was created by X.W. and C.K. Testing and optimization were performed by B.A. and D.W. Final benchmarks were done by C.K., L.I., T.C., and O.S.

Author order was chosen by a secret ballot among the students.
References

[1] J. W. Cooley and J. W. Tukey. An algorithm for the machine calculation of complex Fourier series. *Mathematics of computation*, 19(90):297–301, 1965.

[2] J. G. Proakis and D. G. Manolakis. *Introduction to digital signal processing*. Prentice Hall Professional Technical Reference, 1988.

[3] B. A. Cipra. The best of the 20th century: editors name top 10 algorithms. *SIAM News*, 33(4):1–2, 2000.

[4] W. T. Cochran, J. W. Cooley, D. L. Favin, H. D. Helms, R. A. Kaenel, W. W. Lang, G. C. Maling, D. E. Nelson, C. M. Rader, and P. D. Welch. What is the fast Fourier transform? *Proceedings of the IEEE*, 55(10):1664–1674, 1967.

[5] R. Sedgewick. *Algorithms*. Addison-Wesley, Reading, MA, 1983.

[6] F. E. Fich, J. I. Munro, and P. V. Poblete. Permuting in place. *SIAM Journal on Computing*, 24(2):266–278, 1995.

[7] V. Myrnyy. A simple and efficient FFT implementation in C++: Part I, 2007.

[8] Matt J. [http://stackoverflow.com/questions/746171/best-algorithm-for-bit-reversal-from-msb-lsb-to-lsb-msb-in-c](http://stackoverflow.com/questions/746171/best-algorithm-for-bit-reversal-from-msb-lsb-to-lsb-msb-in-c) Accessed: 20 Jan. 2017.

[9] S. E. Anderson. Bit twiddling hacks. [graphics.stanford.edu/~seander/bithacks.html](http://graphics.stanford.edu/~seander/bithacks.html) Accessed: 20 Jan. 2017.

[10] J. M. P´ez-Jord. In-place self-sorting fast Fourier transform algorithm with local memory references. *Computer physics communications*, 108(1):1–8, 1998.

[11] L. Carter and K. S. Gatlin. Towards an optimal bit-reversal permutation program. In *Foundations of Computer Science, 1998. Proceedings. 39th Annual Symposium on*, pages 544–553. IEEE, 1998.

[12] A. H. Karp. Bit reversal on uniprocessors. *SIAM review*, 38(1):1–26, 1996.

[13] H. Prokop. *Cache-oblivious algorithms*. PhD thesis, Massachusetts Institute of Technology, 1999.

[14] M. Harris. An efficient matrix tranpose in CUDA C/C++. 2013.