ON SOME CLASSES OF RINGS AND THEIR LINKS

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Abstract

The paper deals with Armendariz rings, their relationships with some well known rings. Then we treat generalizations of Armendariz rings, such as McCoy ring, abelian ring and their links. We also consider a skew version of some classes of rings, with respect to a ring endomorphism \( \alpha \).

Keywords: Armendariz ring, skew polynomial ring, reversible, symmetric, semi-commutative ring.

MSC: 16S36; 16W20; 16S99

1. Introduction

This paper investigates a class of rings called Armendariz rings, which generalizes fields and integral domains. These rings are associative with identity and they have been introduced by Rege and Chhawchharia in [19]. A ring \( R \) is called Armendariz if whenever the product of any two polynomials in \( R[x] \) is zero, then so is the product of any pair of coefficients from the two polynomials.

In [2] E.P.Armendariz proved that if the product of two polynomials, whose coefficients belong to a ring without nonzero nilpotent elements, equals zero then all possible pair wise products of coefficients of these polynomials equal zero.

Let \( R \) be a ring and \( \alpha : R \rightarrow R \) be an endomorphism. Then \( \alpha \)-derivation \( \delta \) of \( R \) is an additive map such that \( \delta(ab) = \delta(a)b + \alpha(a)\delta(b) \), for all \( a, b \in R \). The Ore extension \( R[x; \alpha, \delta] \) of \( R \) is the ring with the new multiplication \( xr = \alpha(r)x + \delta(r) \) in the polynomial ring over \( R \), where \( r \in R \). If \( \delta = 0 \), we write \( R[x; \alpha] \) and it is said to be a skew polynomial ring (also The Ore extension of endomorphism type.)

Some properties of skew polynomial rings have been studied in [6], [7], [9], [17] and [18]. According to Krempa [12], an endomorphism \( \alpha \) of a ring \( R \) is called rigid, if for \( r \in R \) the condition \( r\alpha(r) = 0 \) implies \( r = 0 \). In [8], a ring \( R \) has been called \( \alpha \)-rigid if there exists a rigid endomorphism \( \alpha \) of \( R \). In the same paper it has been shown also that any rigid endomorphism of a ring is a monomorphism and \( \alpha \)-rigid rings are reduced.

Hong et al. [9], introduced the concept of \( \alpha \)-Armendariz ring, which is a generalization of \( \alpha \)-rigid ring and Armendariz ring. A ring \( R \) is called \( \alpha \)-Armendariz ring, if
whenever the product of any two polynomials in $R[x; \alpha]$ is zero, then so is the product of any pair of coefficients from the two polynomials.

The organization of the paper is as follows. First, we consider the relationship between Armendariz rings and some other classes of rings (Section 1), then we treat generalizations of Armendariz rings (Section 2). The skew version of some classes of rings are considered in Section 3. Through the paper $\alpha$ stands for an endomorphism of ring $R$.

2. Armendariz rings and other rings

In this section we explore relationships between several classes of rings. Recall that a ring $R$ is said to be von Neumann regular, if $a \in aRa$ for any element $a$ of $R$. Every Boolean ring is von Neumann regular. Reduced rings are Armendariz, but the converse does not hold. Anderson and Camillo (see [1]) proved that a von Neumann regular ring is Armendariz, if and only if it is reduced.

**Proposition 1.** A commutative von Neumann regular ring is reduced.

**Proof.** Let $R$ be a commutative von Neumann regular ring and $a$ be an element of $R$. Suppose that $a^2 = 0$. By the hypothesis, $a = aba = a^2b = 0$. Hence $R$ is reduced. □

By Kaplansky [10], a ring $R$ is called a right $p.p$-ring, if the right annihilator $Ann_r(a)$ of each element $a$ of $R$ is generated by an idempotent. A ring $R$ is called Baer, if the right annihilator of every nonempty subset of $R$ is generated by an idempotent. Clearly Baer ring is right $p.p$-ring. Any Baer ring has nonzero central nilpotent element, then a commutative Baer ring is Armendariz.

Reduced rings can be included in the class of Armendariz rings and the class of semicommutative rings. The last two are abelian. It is natural to explore the relationships between them. A ring is said to be semicommutative, if it satisfies the following condition:

whenever elements $a, b \in R$ satisfy $ab = 0$, then $aRb = 0$.

Semicommutative rings are abelian, but the converse does not hold, which has been showed by Kim and Lee in [11].

Another class of rings is the class of Guassian rings, which has been treated by Anderson and Camillo [1]. The content $c(f)$ of a polynomial $f(x) \in R[x]$ is the ideal of $R$ generated by the coefficients of $f(x)$. A commutative ring $R$ with identity is Guassian, if $c(fg) = c(f)c(g)$ for all $f(x), g(x) \in R[x]$. The Guassian rings are Armendariz, but the converse is not true. Any integral domain is Armendariz, but it is not necessarily Guassian. A field is Guassian, thus it is Armendariz.

Recall that, a ring $R$ is called symmetric, if $abc = 0$ implies $acb = 0$ for $a, b$ and $c$ in $R$. A ring $R$ reversible provided $ab = 0$ implies $ba = 0$ for $a, b \in R$. Semicommutative ring is a generalization of reversible ring. A ring is said to be abelian if any its idempotent is central. Further we make use the notation “ $\implies$ ” to denote for one class of rings to be a subclass of another class.
**Theorem 1.** The following implications hold true:

reduced $\implies$ symmetric $\implies$ reversible $\implies$ semi-commutative $\implies$ abelian.

**Proof.** 1. “reduced $\implies$ symmetric :” Let $R$ be a reduced ring and $abc = 0$ for $a, b, c \in R$. Then $c(abc)ab = 0$ and $(cab)^2 = 0$. Since $R$ is reduced, we get $(cab) = 0$. Hence $aba(cab)ac = (abac)^2 = 0$ and $abac = 0$ (by reducibility). Thus $bacb(abac)ba = (baca)^2 = 0$ then $bacb = 0$. Multiply the last from the right hand side by $c$ we obtain $(bac)^2 = 0$. By using the reducibility of $R$ we have $bac = 0$.

2. “symmetric $\implies$ reversible :” Let $R$ be a symmetric ring and $ab = 0$ for $a, b \in R$. Since $R$ is a ring with identity, we have $a \cdot b \cdot 1 = 0$ and $b \cdot a \cdot 1 = 0$. Therefore $R$ is reversible ring.

3. “reversible $\implies$ semicommutative :” Let $R$ be a reversible ring and $ab = 0$. We claim that $aRb = 0$. By the hypothesis, $ba = 0$. Let $c$ be an arbitrary element of $R$. Hence $c(ba) = 0$, and $(cb)a = 0$. By using the reversibility of $R$ we have $acb = 0$. Thus $aRb = 0$. Therefore $R$ is semicommutative ring.

4. “semi-commutative $\implies$ abelian :” Let $e$ be an idempotent element of a semicommutative ring $R$. Then $e^2 - e = 0$. Since $ea(e - 1) = 0$ for each element $a$ of $R$, we get $ea = eae$. Since $(1 - e)$ is idempotent, then $(1 - e)^2 - 1 + e = 0$. Hence $(e - 1)e = 0$. By using the semicommutativity of $R$ we obtain $(e - 1)ae = 0$ for each $a \in R$. Then $eae = ae$. Thus $e$ is central. Therefore $R$ is abelian.

Here is an example of ring that is commutative, Boolean, von Neumann regular, $p.p.$-ring, reduced and Armendariz, but is not Baer (see [15]).

**Example 1.** (Dorroh extension) We refer the example of [15]. Let $S_0 = \mathbb{Z}_2$, $S_1 = \mathbb{Z}_2 * \mathbb{Z}_2$, $S_3 = S_2 * \mathbb{Z}_2$, ..., $S_n = S_{n-1} * \mathbb{Z}_2$, ..., where the operation on $S_n$ is defined as follows: for $(a, \bar{b}), (c, \bar{d}) \in S_n$ with $a, c \in S_{n-1}$

$$(a, \bar{b}) + (c, \bar{d}) = (a + c, \bar{b} + \bar{d})$$

and

$$(a, \bar{b})(c, \bar{d}) = (ac + bc + da, bd)$$

where $n = 1, 2, \ldots$. It is clear that there is the ring-monomorphism $f : S_{n-1} \longrightarrow S_n$ defined by $f(x) = (x, 0)$. Now construct the direct product $\prod_{n=1}^{\infty} S_n$ with $S_1 \subset S_2 \subset \ldots$ and consider

$R = \left( \bigoplus_{n=1}^{\infty} S_n, 1_s \right)$. Clearly, $R$ is a $\mathbb{Z}_2$-subalgebra of $\prod_{n=1}^{\infty} S_n$, generated by $\bigoplus_{n=1}^{\infty} S_n$ and $1_s$,

where $S = \prod_{n=1}^{\infty} S_n$. Every $S_n$ is Boolean and also von Neumann regular. Therefore $R$ is commutative von Neumann regular ring, we get $R$ is Armendariz. On the other hand, every $S_n$ is Boolean, then $R$ is Boolean. This implies that $R$ is a $p.p.$-ring.
3. Generalizations of Amendariz rings

Abelian rings are generalization of Armendariz rings. This result due to Kim and Lee [11]. The following theorem specifies a subclass of the class of abelian rings which is Armendariz.

**Proposition 2.** An abelian right p.p.–ring is Armendariz.

**Proof.** Let \( r \) be a nilpotent and \( e \) be an idempotent elements of \( R \). Suppose that \( r^2 = 0 \). Since \( r \in \text{Ann}_R(r) = eR \), there exists \( r' \in R \) such that \( r = er' \) and \( er = e^2 r' = er' \). Hence \( re = re = 0 \). Which means that \( R \) is reduced, therefore it is Armendariz. □

McCoy rings are another generalization of Armendariz rings. Recall that a ring \( R \) is a left McCoy, if whenever \( g(x) \) is a right zero-divisor in \( R[x] \) there exists a non-zero element \( c \in R \) such that \( cg(x) = 0 \). Right McCoy ring is defined dually. A ring is said to be McCoy ring, if it is both left and right McCoy. Armendariz rings are McCoy (see [19]), the converse does not hold. Commutative rings are McCoy (Scott [20]), but there are examples of commutative non-Armendariz rings.

Here is an example of a noncommutative McCoy ring that is not Armendariz.

**Example 2.**

\[
\begin{pmatrix}
  a & a_{12} & a_{13} & \ldots & a_{1n} \\
  0 & a & a_{23} & \ldots & a_{2n} \\
  0 & 0 & a & \ldots & a_{3n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \ldots & a
\end{pmatrix} : a, a_{ij} \in R
\]

Let \( R \) be a reduced ring and \( R_n = \left\{ \left( \begin{array}{cccc}
  a & a_{12} & a_{13} & \ldots & a_{1n} \\
  0 & a & a_{23} & \ldots & a_{2n} \\
  0 & 0 & a & \ldots & a_{3n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \ldots & a
\end{array} \right) : a, a_{ij} \in R \right\} \)

Then \( R_n \) is McCoy for any \( n \geq 1 \) [14], but it is not Armendariz for \( n \geq 4 \) [11].

All the previous results are included in Figure 1 below.

4. Skew version of rings

In this section, we consider a skew version of some classes of rings, with respect to a ring endomorphism \( \alpha \). When \( \alpha \) is the identity endomorphism, this coincides with the notion of ring.

Kwak [13], called an endomorphism \( \alpha \) of a ring \( R \), right (respectively, left) symmetric if whenever \( abc = 0 \) implies \( \alpha c \alpha b = 0 \) (respectively, \( \alpha b \alpha c = 0 \)) for \( a, b, c \in R \). A ring \( R \) is called right (respectively, left) \( \alpha \)-symmetric if there exists a right (respectively, left) symmetric endomorphism \( \alpha \) of \( R \). The ring \( R \) is \( \alpha \)-symmetric if it is right and left \( \alpha \)-symmetric. Obviously, domains are \( \alpha \)-symmetric for any endomorphism \( \alpha \).

Başer et al. [4], called a ring \( R \) right (respectively, left) \( \alpha \)-reversible if whenever \( ab = 0 \) for \( a, b \in R \) then \( b \alpha c = 0 \) (respectively, \( \alpha b a = 0 \)). The ring \( R \) is called \( \alpha \)-reversible if it is both right and left \( \alpha \)-reversible.
Proposition 3. An \( \alpha \)-symmetric ring is \( \alpha \)-reversible.

Proof. Let \( R \) be an \( \alpha \)-symmetric ring. Suppose that \( ab = 0 \) for \( a, b \in R \). Obviously, \( 1 \cdot a \cdot b = 0 \). Since \( R \) is right \( \alpha \)-symmetric, then \( ba(a) = 0 \). Hence \( R \) is right \( \alpha \)-reversible. It can be easily shown that \( R \) is left \( \alpha \)-reversible by the same way as above. Therefore \( R \) is \( \alpha \)-reversible. \( \square \)

Baser et al. (see [3]), defined the notion of an \( \alpha \)-semincommutative ring with the endomorphism \( \alpha \) as a generalization of \( \alpha \)-rigid ring and an extension of semicommutative ring.

An endomorphism \( \alpha \) of a ring \( R \) is called \textit{semincommutative} if \( ab = 0 \) implies \( aRa(b) = 0 \) for \( a, b \in R \). A ring \( R \) is called \( \alpha \)-semicommutative if there exists a semicommutative endomorphism \( \alpha \) of \( R \).

Proposition 4. An \( \alpha \)-symmetric ring is \( \alpha \)-semincommutative.
Proposition 5. A reduced $\alpha$-reversible ring is $\alpha$-semicommutative.

Proof. Suppose that $ab = 0$ for $a, b \in R$. Let $c$ be an arbitrary element of $R$. Then $abc = 0$ and $aca(b) = 0$. Hence $aRa(b) = 0$. Therefore $R$ is $\alpha$-semicommutative.

Proposition 6. An $\alpha$-reversible ring that satisfies the condition $(C_\alpha)$ is $\alpha$-semicommutative.

Proof. Suppose that $R$ is an $\alpha$-reversible ring with $(C_\alpha)$ condition and $ab = 0$ for $a, b \in R$. Let $c$ be an arbitrary element of $R$. Hence $\alpha(b)a = 0$ (by $\alpha$-reversibility) and $\alpha(b)ac = 0$. Then $aca(b) = 0$, due to $\alpha$-reversibility of $R$. Since $R$ satisfies the condition $(C_\alpha)$, we get $aca(b) = 0$. Therefore $R$ is $\alpha$-semicommutative.

Corollary 1. An $\alpha$-reversible $\alpha$-compatible ring is $\alpha$-semicommutative.

Proof. It is obvious.

Başer et al [3], proved that for $\alpha$-semicommutative ring $R$, $\alpha(1) = 1$ if and only if $\alpha(e) = e$, where 1 is the identity and $e$ is the idempotent element of $R$.

Proposition 7. An $\alpha$-semicommutative ring $R$ with $\alpha(1) = 1$ is abelian.

Proof. Let $e$ be an idempotent element of $R$. Then $e(1-e) = 0$ and $eR\alpha(1-e) = 0$. On the other hand, $(1-e)e = 0$ and $(1-e)R\alpha(e) = 0$. Hence $er(1-e) = (1-e)re = 0$ for all $r \in R$, This implies that $er = re$ for all $r \in R$. Therefore $R$ is abelian.

The following example shows that the condition “$\alpha(1) = 1$” cannot be dropped.

Example 3. Let $R = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \middle| a, b, c \in \mathbb{Z} \right\}$, and $\alpha$ be an endomorphism of $R$ defined by $\alpha\left( \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$, ($\alpha(1) \neq 1$).

For $A = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$, and $B = \begin{pmatrix} a_2 & 0 \\ b_2 & c_2 \end{pmatrix} \in R$ if $AB = 0$, we obtain $c_1c_2 = 0$. Let $C = \begin{pmatrix} a_3 \\ b_3 \end{pmatrix}$ be an arbitrary element of $R$. Then $AC\alpha(B) = \begin{pmatrix} 0 & 0 \\ 0 & c_1c_2c_3 \end{pmatrix} = 0$. Hence $A\alpha(B) = 0$. Therefore $R$ is $\alpha$-semicommutative ring.
Two idempotents of $R$ (i.e., $\begin{pmatrix} 0 & 0 \\ b & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix}$) aren’t central. Therefore $R$ is not abelian.

In the next theorem we show the relationship between semicommutative and $\alpha$-semicommutative rings.

**Theorem 2.** Let $R$ be an $\alpha$-compatible ring. Then the following hold.

1. $R$ is symmetric if and only if $R$ is $\alpha$-symmetric ring.
2. $R$ is reversible if and only if $R$ is $\alpha$-reversible ring.
3. $R$ is semicommutative if and only if $R$ is $\alpha$-semicommutative.

**Proof.** 1. Let $R$ be a symmetric ring and $abc = 0$, for $a, b, c \in R$. Then $acb = 0$ (by symmetric property) and $acb = 0$ (by $\alpha$-compatibility). Hence $R$ is right $\alpha$-symmetric. Since $R$ is symmetric, then it is reversible and $\alpha(b)ac = 0$. Thus $R$ is left $\alpha$-symmetric. Therefore $R$ is $\alpha$-symmetric ring.

   Conversely, let $R$ be an $\alpha$-symmetric ring and $abc = 0$ for $a, b, c \in R$. Then $acb = 0$ and $acb = 0$ (by $\alpha$-compatibility). Therefore $R$ is symmetric ring.

2. Let $R$ be a reversible ring and $ab=0$, for $a, b \in R$. Then $ba = 0$. Hence $b\alpha(a) = 0$ (by $\alpha$-compatibility). Therefore $R$ is right $\alpha$-reversible.

   On the other hand, $ab = 0$ we have $aa(b) = 0$. Hence $\alpha(b)a = 0$ (by reversibility). Thus $R$ is left $\alpha$-reversible. Therefore $R$ is $\alpha$-reversible.

   Conversely, let $ab = 0$ for $a, b \in R$. Then $b\alpha(a) = 0$ (by right $\alpha$-reversibility) and $ba = 0$ (by $\alpha$-compatibility). Therefore $R$ is reversible.

3. Let $R$ be a semicommutative ring and $ab = 0$ for $a, b \in R$. Hence $aRb = 0$. Since $R$ is $\alpha$-compatible, it implies that $\alpha(a)b = 0$. Therefore $R$ is $\alpha$-semicommutative ring. The “only if” part is obvious.

   □

**Proposition 8.** Let $R$ be an $\alpha$-semicommutative ring with $(C_\alpha)$ condition then $R$ is semicommutative.

**Proof.** Let $R$ be an $\alpha$-semicommutative ring and $ab = 0$ for $a, b \in R$. Then $a\alpha(b) = 0$. Since $R$ satisfies the condition $(C_\alpha)$, we get $aRb = 0$. Therefore $R$ is semicommutative ring.

   □

All the previous results are summarized in Figure 2.
Figure 2. Links between skew version of rings.

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