Differential calculus on a novel cross-product quantum algebra

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Abstract. We investigate the algebro-geometric structure of a novel two-parameter quantum deformation which exhibits the nature of a semidirect or cross-product algebra built upon \( GL(2) \otimes GL(1) \), and is related to several other known examples of quantum groups. Following the R-matrix framework, we construct the \( L^\pm \) functionals and address the problem of duality for this quantum group. This naturally leads to the construction of a bicovariant differential calculus that depends only on one deformation parameter, respects the cross-product structure and has interesting applications. The corresponding Jordanian and hybrid deformation is also explored.

1. The quantum algebra \( A_{r,s} \)

The biparametric \( q \)-deformation \( A_{r,s} \) is defined [1] to be the semidirector cross-product
\[ GL_r(2) \rtimes \mathbb{C}[f,f^{-1}] \] built on the vector space \( GL_r(2) \otimes \mathbb{C}[f,f^{-1}] \) where \( GL_r(2) = \mathbb{C}[a,b,c,d] \) modulo the relations
\[
\begin{align*}
ab &= r^{-1}ba, & bd &= r^{-1}db \\
ac &= r^{-1}ca, & cd &= r^{-1}dc \\
bc &= cb, & [a,d] &= (r^{-1} - r)bc
\end{align*}
\]
and \( \mathbb{C}[f,f^{-1}] \) has the cross relations
\[
\begin{align*}
af &= fa, & cf &= sf c \\
f &\triangleright b = sb, & df &= fd
\end{align*}
\]
\( A_{r,s} \) can also be interpreted as a skew Laurent polynomial ring \( GL_r[f,f^{-1};\sigma] \) where \( \sigma \) is the automorphism given by the action of element \( f \) on \( GL_r(2) \). If we let \( A = GL_r(2) \) and \( H = \mathbb{C}[f,f^{-1}] \), then \( A \) is a left \( H \)-module algebra and the action of \( f \) is given by
\[
\begin{align*}
 f \triangleright a &= a, & f \triangleright b &= sb, & f \triangleright c &= s^{-1}c, & f \triangleright d &= d
\end{align*}
\]

2. The dual algebra \( U_{r,s} \)

Knowing properties of cross-product algebras [2, 3], we already know that the algebra dual to \( A_{r,s} \) would be the cross-coproduct coalgebra \( U_{r,s} = U_r(gl(2)) \rtimes \mathbb{C}[[\phi]] \) with \( \phi \) as an element dual to \( f \). As a vector space, the dual is \( U_{r,s} = U_r(gl(2)) \otimes U(u(1)) \). Now, the duality relation between \( \langle GL_r(2), U_r(gl(2)) \rangle \) is already well-known [4], while that between \( \langle \mathbb{C}[f,f^{-1}], U(u(1)) \rangle \) is given by \( \langle f, \phi \rangle = 1 \), i.e., \( U(u(1)) = \mathbb{C}[[\phi]] \). More precisely, we work
algebraically with \( \mathbb{C}[s^\phi, s^{-\phi}] \) where \( \langle f, s^\phi \rangle = s \). This induces duality on the vector space tensor products, the left action dualises to the left coaction, and this results in the dual algebra being a cross-coproduct \( \mathcal{U}_{r,s} = U_r(gl(2)) \times \mathbb{C}[[\phi]] \). Let us recall that \( U_r(gl(2)) \), the algebra dual to \( GL_r(2) \), is isomorphic to the tensor product \( U_r(sl(2)) \otimes \hat{U}(u(1)) \) where \( U_r(sl(2)) \) has the usual generators \( \{H, X_\pm\} \) and \( \hat{U}(u(1)) = \mathbb{C}[\xi] = \mathbb{C}[r^\xi, r^{-\xi}] \) with \( \xi \) central. Therefore, \( \mathcal{U}_{r,s} \) is nothing but \( U_r(sl(2)) \) and two central generators \( \xi \) and \( \phi \), where \( \xi \) is the generating element of \( \hat{U}(u(1)) \) and \( \phi \) is the generating element of \( U(u(1)) \). Also note that \( s^\phi \) is dually paired with the element \( f \) of \( \mathcal{A}_{r,s} \).

3. R-matrix relations

In the quantum group language, \( \mathcal{A}_{r,s} \) is understood as a novel Hopf algebra \([5, 6]\) generated by \( \{a, b, c, d, f\} \) arranged in the matrix form

\[
T = \begin{pmatrix} f & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}
\] (4)

with the labelling 0, 1, 2, and \( \{r, s\} \) are the two deformation parameters. The \( R \)-matrix

\[
R = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & S^{-1} & 0 & 0 \\ 0 & \Lambda & S & 0 \\ 0 & 0 & 0 & R_r \end{pmatrix}
\] (5)

is in block form, i.e., in the order \([00], [01], [02], [10], [20], [11], [12], [21], [22] \) (which is chosen in conjunction with the block form of the \( T \)-matrix) where

\[
R_r = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & r \end{pmatrix}; \quad S = \begin{pmatrix} s & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \quad \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}; \quad \lambda = r - r^{-1}
\]

The \( RTT \) relations \( RT_1T_2 = T_2T_1R \) (where \( T_1 = T \otimes \mathbf{1} \) and \( T_2 = \mathbf{1} \otimes T \)) then yield the commutation relations (1) and (2) between the generators. The Hopf algebra structure underlying \( \mathcal{A}_{r,s} \) is \( \Delta(T) = T \otimes T, \varepsilon(T) = 1 \). The Casimir operator \( \delta = ad - r^{-1}bc \) is invertible and the antipode is

\[
S(f) = f^{-1}, \quad S(a) = \delta^{-1}d, \quad S(b) = -\delta^{-1}rb, \quad S(c) = -\delta^{-1}rc, \quad S(d) = \delta^{-1}a \quad (6)
\]

The quantum determinant \( \mathcal{D} = \delta f \) is group-like but not central. There are several interesting features of this deformation \([5, 6, 1]\) which cannot be mentioned here. In the \( R \)-matrix formulation of matrix quantum groups, a basic step is to construct functionals (matrices) \( L^+ \) and \( L^- \) which are dual to the matrix of generators (4) in the fundamental representation. These functions are defined by their value on the matrix of generators \( T \)

\[
\langle (L^\pm)_b, T_d \rangle = (R^\pm)_{bd}^{ac} \quad (7)
\]

where

\[
(R^+)^{ac}_{bd} = c^+(R^-)^{ca}_{db} \quad (8)
\]

\[
(R^-)^{ac}_{bd} = c^-(R^-)^{ac}_{bd} \quad (9)
\]
and $c^+, c^-$ are free parameters. For $A_{r,s}$ we make the following ansatz for the $L^\pm$ matrices:

\[
L^+ = \begin{pmatrix} J & 0 & 0 \\ 0 & M & P \\ 0 & 0 & N \end{pmatrix} \quad \text{and} \quad L^- = \begin{pmatrix} J' & 0 & 0 \\ 0 & M' & 0 \\ 0 & Q & N' \end{pmatrix}
\]

where

\[
J = s^{\frac{1}{2}(F-A+D-1)}r^{\frac{1}{2}(F-A-D+1)}, \quad J' = s^{\frac{1}{2}(F-A+D-1)}r^{\frac{1}{2}(F-A-D+1)}
\]

\[
M = s^{\frac{1}{2}(F-A-D+1)}r^{\frac{1}{2}(-F+A-D+1)}, \quad M' = s^{\frac{1}{2}(F-A-D+1)}r^{\frac{1}{2}(-F+A-D+1)}
\]

\[
N = s^{\frac{1}{2}(F+A+D-1)}r^{\frac{1}{2}(-F-A+D+1)}, \quad N' = s^{\frac{1}{2}(F+A+D-1)}r^{\frac{1}{2}(-F-A+D+1)}
\]

\[
P = \lambda C,
\]

and \(\{A, B, C, D, F\}\) is the set of generating elements of the dual algebra \(U_{r,s}\). This is consistent with the action on the generators of \(A_{r,s}\) and gives the correct duality pairings. The commutation algebra is given by the \(RLL\) relations \(R_{12}L^+_2 L^+_1 = L^+_1 L^+_2 R_{12}, R_{12}L^-_2 L^-_1 = R_{12}L^-_1 L^-_2 L^-_2, \) where \(L^+_1 = L^+ \otimes 1, L^+_2 = 1 \otimes L^+,\) and \(R_{12}\) is the same as (5). Finally, we obtain a single-parameter deformation of \(U(gl(2)) \otimes U(u(1))\) as an algebra. Including the coproduct, we again obtain \(U_{r,s}\) as a semidirect product \(U_r(gl(2)) \rtimes U_u(u(1))\).

4. Differential calculus on \(A_{r,s}\)

The \(R\)-matrix procedure [7] is known to provide a natural framework to construct differential calculus on matrix quantum groups. We note here that the \(A_{r,s}\) deformation is not a full matrix quantum group, but an appropriate quotient of one (of multiparameter \(g\)-deformed \(GL(3)\), to be precise). Nevertheless, it turns out that the constructive differential calculus methods [8] work equally well for such quotients. The bimodule \(\Gamma\) (space of quantum one-forms \(\omega\)) is characterised by the commutation relations between \(\omega\) and \(a \in \mathcal{A}(= A_{r,s})\)

\[
\omega a = (1 \otimes g) \Delta(a) \omega
\]

and the linear functional \(g \in \mathcal{A}'(= \text{Hom}(\mathcal{A}, \mathbb{C}))\) is defined in terms of the \(L^\pm\) matrices

\[
g = S(L^+)L^-
\]

Thus, in terms of components we have

\[
\omega_{ij}a = [(1 \otimes S(l^+_i l^-_j)) \Delta(a)] \omega_{kl}
\]

using \(L^\pm = l^\pm_{ij}\) and \(\omega = \omega_{ij}\) where \(i, j = 1..3\). From these relations, one can obtain the commutation relations of all the left-invariant one-forms with the generating elements of \(\mathcal{A}\).

The left-invariant vector fields \(\chi_{ij}\) on \(\mathcal{A}\) are given by the expression

\[
\chi_{ij} = S(l^+_i l^-_j) - \delta_{ij} \varepsilon
\]

The vector fields act on the generating elements as

\[
\chi_{ij}a = (S(l^+_i l^-_j) - \delta_{ij} \varepsilon)a
\]
Furthermore, using the formula $da = \sum_i (\chi_i \ast a)\omega^i$, we obtain the action of the exterior derivatives ($d : A \rightarrow \Gamma$):

\[
d a = (r^-2 - 1)a \omega^1 - \lambda b \omega^+ \\
d b = \lambda^2 b \omega^1 - \lambda a \omega^- + (r^-2 - 1)b \omega^2 \\
d c = (r^-2 - 1)c \omega^- - \lambda c \omega^+ \\
d d = \lambda^2 d \omega^1 - \lambda c \omega^- + (r^-2 - 1)d \omega^2 \\
d f = (r^-2 - 1)f \omega^0
\]

(16) (17) (18) (19) (20)

where $\omega^0 = \omega_{11}, \omega^1 = \omega_{22}, \omega^+ = \omega_{23}, \omega^- = \omega_{32}, \omega^2 = \omega_{33}$. $dA$ generates $\Gamma$ as a left $A$-module, and this defines a first-order differential calculus $(\Gamma, d)$ on $A_{r,s}$. The calculus is bicovariant due to the coexistence of the left ($\Delta_L : \Gamma \rightarrow A \otimes \Gamma$) and the right ($\Delta_R : \Gamma \rightarrow \Gamma \otimes A$) actions. Curiosly, using the Leibniz rule it can be checked that

\[
d (af - fa) = 0, \quad d(cf - sf c) = 0, \quad d(bf - s^{-1} fb) = 0, \quad d(df - fd) = 0, \quad (21)
\]

which is consistent with cross relations (2), and so the differential calculus also respects the cross-product structure of $A_{r,s}$.

5. The Jordanian deformation $A_{m,k}$

$A_{r,s}$ can be contracted [6] (by means of singular limit of similarity transformations) to obtain a nonstandard or Jordanian analogue, say $A_{m,k}$, with deformation parameters $\{m, k\}$ and the associated $R$-matrix is triangular. In analogy with $A_{r,s}$, $A_{m,k}$ can also be considered as the semidirect or cross-product $GL_m(2) \ltimes C[f, f^{-1}]$ built upon the vector space $GL_m(2) \otimes C[f, f^{-1}]$, where $GL_m(2)$ is itself a Jordanian deformation of $GL(2)$. Thus, $A_{m,k}$ can also be interpreted as a skew Laurent polynomial ring $GL_m[f, f^{-1}; \sigma]$ where $\sigma$ is the automorphism given by the action of element $f$ on $GL_m(2)$.

6. Conclusions

The $A_{r,s}$ and $A_{m,k}$ deformations provide interesting new examples of cross-product quantum algebras, both of which have $GL(2) \otimes GL(1)$ as their classical limits. The differential calculus on $A_{r,s}$ also has an inherent cross-product structure, embeds the calculus on $GL_q(2)$ and is also related to the calculus on $GL_{p,q}(2)$. It would be interesting to investigate the calculus on the Jordanian $A_{m,k}$, and on the hybrid/intermediate [9] deformation obtained during the course of the contraction of $A_{r,s}$ to $A_{m,k}$.

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