Abstract. Moisil in 1941, while constructing the algebraic models of \( n \)-valued Lukasiewicz logic defined the set \( B^{[n]} \), where \( B \) is a Boolean algebra and ‘\( n \)’ being a natural number. Further it was proved by Moisil himself the representations of \( n \)-valued Lukasiewicz Moisil algebra in terms of \( B^{[n]} \). In this article, structural representation results for Stone, dual Stone and double Stone algebras are proved similar to Moisil’s work by showing that elements of these algebras can be looked upon as monotone ordered tuple of sets. 3-valued semantics of logic for Stone algebras, dual Stone algebras and 4-valued semantics of logic for double Stone algebras are proposed and established soundness and completeness results.

Key words: Stone algebras, double Stone algebras, 3-valued logic, 4-valued logic, Rough sets.

1 Introduction

The Stone’s \([1]\) representation theorem for Boolean algebra identifies an element of a given Boolean algebra as a set, and has vital role in algebra and logic. It shows that algebraic, set theoretic and True-False semantics of classical propositional logic are equivalent. On the other hand, there are some works where an element of a given algebra is identified by a pair of sets. Some well known examples are:

- (Moisil (cf. \([2]\))) Moisil represented each element of a given 3-valued LM algebra \( \mathcal{A} \) can be looked upon as a monotone ordered pair of sets.
- Rasiowa \([3]\) represented De Morgan algebras as set-based De Morgan algebras, where De Morgan negation is defined by an involution function.
- In Dunn’s \([4,5]\) representation, each element of a De Morgan algebra can be identified with an ordered pair of sets, where De Morgan negation is defined as reversing the order in the pair.

Note that all these representation results have their logical insight, e.g., Dunn’s representation leads to his well known 4-valued semantics of De Morgan logic. In \([6]\), author provided a representation result for Kleene algebras which is very
similar to Moisil’s representation of 3-valued LM algebra, and studied 3-valued aspect of corresponding logic (for Kleene algebras).

In this article, we prove structural theorems for Stone and dual Stone algebras similar to the Moisil representation of 3-valued LM algebra. Let us define the algebras.

**Definition 1.** An algebra $S := (S, ∨, ∧, ∼, 0, 1)$ is a Stone algebra if

1. $(S, ∨, ∧, ∼, 0, 1)$ is a bounded distributive pseudo complemented lattice, i.e.,
   \[ ∀a ∈ S, \sim a = \max\{c ∈ S : a ∧ c = 0\} \text{ exists.} \]
2. $\sim a ∨ \sim ∼ a = 1$, for all $a ∈ S$.

The axiom $\sim a ∨ \sim ∼ a = 1$ first appeared in Problem 70 - What is the most general pseudo-complemented distributive lattice in which $a^* ∨ a^{**} = I$ identically? (attributed to M.H.Stone) of Birkhoff’s book [7]. The formal definition of Stone algebra (lattice) appeared in Gratzer and Schmidt [8] and have been extensively studied in literature (cf. [9]). The dual notion of a given Stone algebra is known as dual Stone algebra. For the self explanatory of this article, let us explicitly define the dual Stone algebra.

**Definition 2.** An algebra $DS := (DS, ∨, ∧, ∼, 0, 1)$ is a dual Stone algebra if

1. $(DS, ∨, ∧, ∼, 0, 1)$ is a bounded distributive dual pseudo complemented lattice, i.e.,
   \[ ∀a ∈ DS, \sim a = \min\{c ∈ DS : a ∨ c = 1\} \text{ exists.} \]
2. $\sim a ∧ \sim ∼ a = 0$, for all $a ∈ DS$ (dual Stone property).

In this article, we also present a representation result for double Stone algebra, where each element of a given double Stone algebra is identified as a monotone ordered 3-tuple of sets. Double Stone algebra is a bounded distributive lattice which is both Stone and dual Stone algebra.

**Definition 3.** An algebra $A := (A, ∨, ∧, ∼, ¬, 0, 1)$ is a double Stone algebra if

1. $(A, ∨, ∧, ∼, 0, 1)$ is a bounded distributive lattice,
2. $(A, ∨, ∧, ∼, 0, 1)$ is a Stone algebra,
3. $(A, ∨, ∧, ¬, 0, 1)$ is dual Stone algebra,

It is well known that 3-valued LM algebras are algebraic models of 3-valued Lukasiewicz logic, and Kleene algebras, Stone algebras, dual Stone algebras and double Stone algebras appear as reduct algebras [10] of 3-valued LM algebras. So, it is natural to ask ‘can we provide multi valued semantics of logics corresponding these algebras?’. In [3], author provided a 3-valued semantics of logic corresponding to Kleene algebras. In this paper we provide 3-valued semantics of logic ($L_S$) for Stone, logic ($L_{DS}$) for dual Stone algebras and a 4-valued semantics of logic ($L_{DBS}$) for double Stone algebras.

In other aspect of this paper, we make explicit connections between rough sets, Stone and dual Stone algebras by providing rough set representations of these algebras. Rough set theory, introduced by Pawlak [11] in 1982, also provides a way to look elements of various algebras as monotone ordered pair of sets. There are various algebraic representations in terms of rough sets. We mention here some well known representation results. For a good expositions of various algebraic representation results in terms of rough sets, we refer to [13].
1. (Comer [14]) Every regular double Stone algebra is isomorphic to an algebra of rough sets in a Pawlak approximation space.

2. (Pagliani, [15]) Any finite semi simple Nelson algebra is isomorphic to a Nelson algebra formed by rough sets for some appropriate approximation space.

3. (Järvinen, Radeleczki [16]) Every Nelson algebra defined over an algebraic lattice is isomorphic to an approximation space based on a quasi order.

4. (Kumar, Banerjee [6]) Every Kleene algebra is isomorphic to an algebra of rough sets in a Pawlak approximation space.

Since Pawlak introduced the rough set theory, the algebraic and logical study of rough set theory evolved simultaneously. In fact, there are some works where the appearance of logics are motivated by the algebraic representations of rough sets. It is noteworthy to mention here some logics which arose in context of rough set theory.

1. (Banerjee and Chakraborty [17]) The emergence of Pre-rough logic was motivated by algebraic representation of Pre-rough algebra in terms of rough sets. It has been shown (in [17]) that Pre-rough logic is sound and complete in class of all Pre-rough algebra formed by rough sets, for all approximation spaces.

2. (Järvinen, Radeleczki [16], Järvinen, Pagliani and Radeleczki [18]) Constructive logic with strong negation (CLSN) is sound and complete in class of all finite Nelson algebras formed by rough sets, for all approximation spaces.

3. (Kumar and Banerjee [6]) The \( L_{K} \) emerged as a result of rough set representation of Kleene algebras. Moreover, the logic \( L_{K} \) is sound and complete in class of all Kleene algebras formed rough sets, for all approximation spaces.

In this work also, our algebraic and logical study of rough set theory evolved simultaneously. Rough set representations of Stone and dual Stone algebra which further leads to equivalency between rough set semantics, 3-valued semantics and algebraic semantics of the logics \( L_{S} \) and \( L_{DS} \).

The study of logics in this article is completely based on DLL. Let us present the logic. The language consists of

- Propositional variables: \( p, q, r, \ldots \).
- Logical connectives: \( \lor, \land \).

The well-formed formulas of the logic are then given by the scheme:

\[ p \mid \alpha \lor \beta \mid \alpha \land \beta . \]

**Notation 1** Denote the set of propositional variables by \( \mathcal{P} \), and that of well-formed formulas by \( \mathcal{F} \).

Let \( \alpha \) and \( \beta \) be two formulas. The pair \( (\alpha, \beta) \) is called a consequence pair. The rules and postulates of the logic DLL are presented in terms of consequence pairs. Intuitively, the consequence pair \( (\alpha, \beta) \) reflects that \( \beta \) is a consequence of \( \alpha \). In
the representation of a logic, a consequence pair \((\alpha, \beta)\) is denoted by \(\alpha \vdash \beta\) (called a consequent). The logic is now given through the following postulates and rules, taken from Dunn’s [5] and [19]. These define reflexivity and transitivity of \(\vdash\), introduction, elimination principles and the distributive law for the connectives \(\land\) and \(\lor\).

**Definition 4.** \((\text{DLL- postulates})\)

1. \(\alpha \vdash \alpha\) (Reflexivity).
2. \(\alpha \vdash \beta, \beta \vdash \gamma \vdash \gamma\) (Transitivity).
3. \(\alpha \land \beta \vdash \alpha, \alpha \land \beta \vdash \beta\) (Conjunction Elimination)
4. \(\alpha \vdash \beta, \alpha \vdash \beta \vdash \gamma \vdash \gamma\) (Conjunction Introduction)
5. \(\alpha \lor \beta, \beta \vdash \alpha \lor \beta\) (Disjunction Introduction)
6. \(\alpha \lor \gamma, \beta \vdash \gamma \lor \beta \vdash \gamma\) (Disjunction Elimination)
7. \(\alpha \land (\beta \lor \gamma) \vdash (\alpha \land \beta) \lor (\alpha \land \gamma)\) (Distributivity)

Further Dunn in [20] extended the language of \(\text{DLL}\) by adding,

- Propositional constants: \(\top, \bot\).

Then, he added the following postulate to extend \(\text{DLL}\) to give a logic \(\text{BDLL}\), whose algebraic models are bounded distributive lattices.

- \(\alpha \vdash \top\) (Top); \(\bot \vdash \alpha\) (Bottom).

In this paper, semantics of a logic is defined via valuations. Let \(\mathcal{A} = (A, \lor, \land, f_1, f_2, 0, 1)\) be a lattice based algebra, and \(\mathcal{F}_{f_1, f_2}\) be extension of \(\mathcal{F}\) by adding unary connectives \(f_1\) and \(f_2\). A map \(v : \mathcal{F}_{f_1, f_2} \rightarrow A\) is called a valuation on \(A\) if

1. \(v(\alpha \land \beta) = v(\alpha) \land v(\beta)\),
2. \(v(\alpha \lor \beta) = v(\alpha) \lor v(\beta)\).
3. \(v(f_1(\alpha)) = f_1(v(\alpha)), v(f_2(\alpha)) = f_2(v(\alpha))\).
4. \(v(\bot) = 0, v(\top) = 1\).

A consequent \(\alpha \vdash \beta\) is **valid in \(A\)** **under the valuation** \(v\), if \(v(\alpha) \leq v(\beta)\). If the consequent is valid under all valuations on \(A\), then it is **valid in \(A\)**, and denote it as \(\alpha \Vdash \beta\). Let \(\mathcal{A}\) be a class of algebras. If the consequent \(\alpha \vdash \beta\) is valid in each algebra of \(\mathcal{A}\), then we say \(\alpha \vdash \beta\) is **valid in \(\mathcal{A}\)**, and denote it as \(\alpha \Vdash \mathcal{A} \beta\).

This paper is organized as follows. In Section 2 we provide structural representations of Stone and dual Stone algebras in which elements of these algebra are defined by pairs of Boolean elements and rough sets. Negations in corresponding algebras are defined using Boolean complements. Moreover, we also obtained the 3-valued and rough set semantics of the proposed logics \(\mathcal{L}_S\) and \(\mathcal{L}_{DS}\). In Section 3 we show that each element of a double Stone algebra can be represented as ordered monotone 3-tuple of Boolean elements, and hence 3-tuple of sets. This leads to a 4-valued semantics of the proposed logic \(\mathcal{L}_{DBS}\). Finally, we conclude this article in Section 4.

The lattice theoretic results used in this article are taken from [21]. We use the convention of representing a set \(\{x, y, z, \ldots\}\) by \(xyz\ldots\).
2 Boolean representations of Stone and dual Stone algebras and corresponding logics

It has been a general trend in algebra to construct a new type of algebra from a given class of algebras. Some well known examples of such constructions are:

- Nelson algebra from a given Heyting algebra (Vakarelov [22], Fidel [23]).
- Kleene algebras from distributive lattices (Kalman [24]).
- 3-valued Łukasiewicz-Moisil (LM) algebra from a given Boolean algebra (Moisil, cf. [2]).
- Regular double Stone algebra from a Boolean algebra (Katriňák [25], cf. [10]).

More importantly, the afore mentioned constructions can be reversed in the sense of representations of these new type of algebras in terms of the given class of algebras. In the same lines, our work in this section, is based on Moisil’s construction of a 3-valued LM algebra. As we mentioned in the Introduction, De Morgan, Kleene, Stone and double Stone algebras appear as reduct of 3-valued LM algebra (c.f. [10]). Exploiting this fact, we prove structural representation theorems for Stone and dual Stone algebras and provide 3-valued semantics for logics of Stone and dual Stone algebras.

2.1 Boolean representations of Stone and dual Stone algebras

Let $B := (B, \lor, \land, \circ, 0, 1)$ be a Boolean algebra, the set $B^2 := \{(a, b) : a \leq b, a, b \in B\}$ was first studied by Moisil, while constructing the algebraic models for 3-valued Łukasiewicz logic. In fact, Moisil showed that the structure $(B^2, \lor, \land', \Delta, (0, 0), (1, 1))$ is a 3-valued Łukasiewicz-Moisil (LM) algebra, where $\lor, \land$ are component wise operations, and for $(a, b) \in B^2$ \((a, b)' := (b', a')\) and $\Delta(a, b) = (a, a)$. Moreover, the following structural representation result was proved by Moisil.

**Theorem 1.** (cf. [2]) Let $LM = (LM, \lor, \land', \Delta, 0, 1)$ be a 3-valued Łukasiewicz-Moisil (LM) algebra, then there exists a Boolean algebra $B$ such that $LM$ is embeddable into $(B^2, \lor, \land', \Delta, (0, 0), (1, 1))$.

It is well known that 3-valued Łukasiewicz-Moisil algebra, regular double Stone algebra and semi simple Nelson algebra are equivalent in the sense that one can be obtained from the other by providing appropriate transformations. Hence a 3-valued Łukasiewicz-Moisil (LM) algebra is also an Stone and a double Stone algebra.

**Proposition 1.** [10]

1. $E^2 := (B^2, \lor, \land, \sim, (0, 0), (1, 1))$ is a Stone algebra, where, for $(a, b) \in B^2$, $\sim(a, b) := (b', b')$.
2. $B^2 := (B^2, \lor, \land, \sim, (0, 0), (1, 1))$ is a dual Stone algebra, where, for $(a, b) \in B^2$, $\sim(a, b) := (a', a')$. 
Let us demonstrate the proof of 1, proof of 2 follows similarly.

**Proof.** \((a, b) \land (c, d) = (a \land c, b \land d) = (0, 0)\). So, \(c \leq a^c\) and \(d \leq b^c\). Hence \((c, d) \leq (b^c, b^c)\). Clearly, we have \((a, b) \land (b^c, b^c) = (0, 0)\). Hence \(\sim (a, b) = (b^c, b^c)\).

□

It is well known that, with pseudo negation \(\sim\) and dual pseudo negation \(\neg\), 1, 2 and 3 are the only subdirect irreducible Stone and dual Stone algebras (Fig. 1). Hence using Birkhoff [26] well known representation result, we have the followings. To distinguish the Stone and dual Stone algebra based on lattice 3, we use 3\(\sim\) and 3\(\neg\) respectively.

**Theorem 2.** [9]

1. Let \(\mathcal{S} = (S, \lor, \land, \sim, 0, 1)\) be an Stone algebra. There exists a (index) set \(I\) such that \(\mathcal{S}\) can be embedded into Stone algebra \(3^I_{\sim}\).

2. Let \(\mathcal{DS} = (DS, \lor, \land, \neg, 0, 1)\) be a dual Stone algebra. There exists a (index) set \(I\) such that \(\mathcal{S}\) can be embedded into dual Stone algebra \(3^I_{\neg}\).

So, if \(B\) is a Boolean algebra, then the Stone algebra \(B^2[2]\) and dual Stone algebra \(B^2[2]\) are embeddable into \(3^I_{\sim}\) and \(3^I_{\neg}\) respectively, for appropriate index sets \(I\) and \(J\).

Atoms play fundamental role in the study of Boolean algebras. Completely join irreducible elements of lattices are counterpart of atoms in Boolean algebras, and play fundamental role in establishing isomorphism between certain classes of lattice based algebra. An example of such can be seen in rough set representation [16] of Nelson algebras.

**Definition 5.** [21] Let \(\mathcal{L} := (L, \lor, \land, 0, 1)\) be a complete lattice.

(i) An element \(a \in L\) is said to be completely join irreducible, if \(a = \lor \ S\) implies that \(a \in S\), for every subset \(S\) of \(L\).
**Notation 2** Let $J_L$ denote the set of all completely join irreducible elements of $L$, and $J(x) := \{a \in J_L : a \leq x\}$, for any $x \in L$.

(ii) A set $S$ is said to be join dense in $L$, provided for every element $a \in L$, there is a subset $S'$ of $S$ such that $a = \bigvee S'$.

The illustration of importance of completely join irreducible elements can be seen by a result of Birkhoff.

**Lemma 1.** [27] Let $L$ and $K$ be two completely distributive lattices. Further, let $J_L$ and $J_K$ be join dense in $L$ and $K$ respectively. Let $\phi : J_L \rightarrow J_K$ be an order isomorphism. Then the extension map $\Phi : L \rightarrow K$ given by

$$\Phi(x) := \bigvee (\phi(J(x))) \quad (\text{where } J(x) := \{a \in J_L : a \leq x\}), \quad x \in L,$$

is a lattice isomorphism.

In [6] we characterized the completely join irreducible elements of lattices $3^I$ and $B[2]$, where $B$ is a complete atomic Boolean algebra. Let $i, k \in I$. Denote by $f^x_i, \quad x \in \{a, 1\}$, the following element in $3^I$.

$$f^x_i(k) := \begin{cases} x & \text{if } k = i \\ 0 & \text{otherwise} \end{cases}$$

**Proposition 2.** [6]

1. The set of completely join irreducible elements of $3^I$ is given by:

$$J_{3^I} = \{f^a_i, f^1_i : i \in I\}.$$

Moreover, $J_{3^I}$ is join dense in $3^I$.

2. Let $B$ be a complete atomic Boolean algebra. The set of completely join irreducible elements of $B[2]$ is given by

$$J_{B[2]} = \{(0, a_i), (a_i, a) : a \in J_B\}.$$

Moreover, $J_{B[2]}$ is join dense in $B[2]$.

Figure 2 shows the Hasse diagrams of $J_{3^I}$ and $J_{B[2]}$. We also established the following isomorphism.

![Fig. 2. Hasse diagram of $J_{3^I}$](image)
Theorem 3. The sets of completely join irreducible elements of \(3^I\) and \((2^I)^2\) are order isomorphic.

Now, we know that the pseudo and dual pseudo negations (if exist) are defined via the order of the given partially ordered sets. Moreover Stone and dual Stone algebras are equational algebras, hence using Lemma 1, we can deduce the following Theorem.

Theorem 4. 1. The algebras \(3^I\) and \((2^I)^2\) are lattice isomorphic.
2. The Stone (dual Stone) algebras \(3^I\) and \((2^I)^2\) are isomorphic.
3. Let \(S\) be an Stone algebra. There exists a (index) set \(I\) such that \(S\) can be embedded into Stone algebra \((2^I)^2\).
4. Let \(DS\) be a dual Stone algebra. There exists a (index) set \(I\) such that \(DS\) can be embedded into dual Stone algebra \((2^I)^2\).

2.2 3-valued semantics of logic for Stone and dual Stone algebras

As mentioned earlier, Moisil in 1941 (cf. [2]) proved that \(B]\) forms a 3-valued LM algebra. So, while discussing the logic corresponding to the structures \(B]\), one is naturally led to 3-valued Lukasiewicz logic. Our focus in this section is to study the logic corresponding to the classes of Stone and dual Stone algebras and the structures \(B]\) and \(B]\). Our approach to the study is motivated by Dunn’s 4-valued semantics of the De Morgan consequence system \(D\) : \(\vdash_{0,1}\) (or \(\vdash_0\) or \(\vdash_1\)), wherein valuations are defined in the De Morgan algebra \(4\) (Figure 3).

![Fig. 3. De Morgan lattice 4](image_url)

The 4-valued semantics arises from the fact that each element of a De Morgan algebra can be looked upon as a pair of sets. In this connection, we exploit Theorem 4 to provide a 3-valued semantics of logic for Stone and dual Stone algebras. However, an easy consequence of Stone’s representation theorem and Theorem 4 we have:
**Theorem 5.** 1. Given a Stone algebra \( S = (S, \lor, \land, \sim, 0, 1) \), there exist a set \( U \) such that \( S \) can be embeddable into Stone algebra formed by \((\mathcal{P}(U))^{[2]}\).

2. Given a dual Stone algebra \( DS = (DS, \lor, \land, \sim, 0, 1) \), there exist a set \( U \) such that \( DS \) can be embeddable into dual Stone algebra formed by \((\mathcal{P}(U))^{[2]}\).

The 2-valued property of classical set theory arises from the fact that given a set \( U \) and \( A \subseteq U \), then \( \{ A, A^c \} \) is a partition of the set \( U \). Hence if \( v \) is a valuation from classical propositional sentences to \( \mathcal{P}(U) \) for some set \( U \), then \( v \) determine a class of 2-valued valuations \( \{ v_x : x \in U \} \) on classical propositional sentences, where where \( v_x(\gamma) = 1 \) if \( x \in v(\gamma) \) and \( v_x(\gamma) = 0 \) if \( x \notin v(\gamma) \). This shows the 2-valued semantics and set theoretic semantics of classical propositional are equivalent. Now, the Stone’s representation theorem for Boolean algebras guarantee that algebraic (Boolean algebras) semantics, 2-valued semantics and set theoretic semantics of propositional logic are equivalent.

In this section, we follow the same approach to establish the completeness results for \( L_S, L_{DS} \) and \( L_{DBS} \) (defined below).

Let \( \alpha, \beta \in F_\sim \) and \( \alpha, \beta \in F_\neg \) be extensions of \( F \) by adding unary connectives \( \sim \) and \( \neg \) respectively. Let \( \alpha, \beta \in F_\sim \) and \( L_S \) denote the logic \( BDLL \) along with following rules and postulates.

1. \( \alpha \vdash \beta/ \sim \beta \vdash \sim \alpha \) (Contraposition)
2. \( \sim \alpha \land \sim \beta \vdash \sim (\alpha \lor \beta) \) (\lor-linearity).
3. \( \top \vdash \bot \) (Nor).
4. \( \alpha \land \sim \beta \vdash \gamma \land \sim \gamma \vdash \beta \)
5. \( \alpha \land \sim \alpha \vdash \bot \)
6. \( \top \vdash \alpha \lor \sim \alpha \)

For \( \alpha, \beta \in F_\neg \), let \( L_{DS} \) denote the logic \( BDLL \) along with following rules and postulates.

1. \( \alpha \vdash \beta/ \neg \beta \vdash \neg \alpha \) (Contraposition)
2. \( \neg (\alpha \land \beta) \vdash \neg \alpha \lor \neg \beta \) (\land-linearity).
3. \( \neg \top \vdash \bot \)
4. \( \gamma \vdash \alpha \lor \neg \beta \vdash \alpha \lor \neg \gamma \)
5. \( \top \vdash \alpha \lor \neg \alpha \)
6. \( \neg \alpha \land \neg \alpha \vdash \bot \)

Let \( A_S \) denote the class of all Stone algebras, \( SB^{[2]} \) denote the class of all Stone algebras formed by the set \( B^{[2]} \) for all Boolean algebras \( B \), \( S(\mathcal{P}(U))^{[2]} \) denote the class of all Stone algebras formed by the collection \( \mathcal{P}(U) \) for all sets \( U \). Now, utilizing Theorem 4 and Corollary 5 in classical manner we have.

**Theorem 6.** For \( \alpha, \beta \in F_\sim \), the followings are equivalent:

1. \( \alpha \vdash_{L_S} \beta \)
2. \( \alpha \models_{A_S} \beta \)
3. \( \alpha \models_{SB^{[2]}} \beta \)
4. \( \alpha \models_{S(\mathcal{P}(U))^{[2]}} \beta \)
In similar manner utilizing Theorem 4 and Theorem 5, we have results for the
logic $L_{DS}$ and dual Stone algebras. Let $A_{DS}$ denote the class of all dual Stone
algebras, $DSB^{[2]}$ denote the class of all dual Stone algebras formed by the set
$B^{[2]}$ for all Boolean algebras $B$, $DS(P(U)^[2]}$ denote the class of all dual Stone
algebras formed by the collection $P(U)^[2]}$ for all sets $U$.

**Theorem 7.** For $\alpha, \beta \in F_\sim$, the followings are equivalent:

1. $\alpha \vdash_{DS} \beta$.
2. $\alpha \vDash_{DS} \beta$.
3. $\alpha \vDash_{DSB^{[2]}} \beta$.
4. $\alpha \vDash_{DS(P(U)^[2]}} \beta$.

Now, let us define the following semantic consequence relations.

**Definition 6.**

1. Let $\alpha, \beta \in F_\sim$.
   
   (i) $\alpha \vdash_1 \beta$ if and only if, for all valuations $v$ in $3_\sim$ if $v(\alpha) = 1$ then
   $v(\beta) = 1$ (Truth preservation).
   (ii) $\alpha \vdash_0 \beta$ if and only if, for all valuations $v$ in $3_\sim$ if $v(\beta) = 0$ then
   $v(\alpha) = 0$ (Falsity preservation).
   (iii) $\alpha \vdash_{1,0} \beta$ if and only if, $\alpha \vdash \beta$ and $\alpha \vdash 0 \beta$.

2. Let $\alpha, \beta \in F_\sim$.
   
   (i) $\alpha \vdash_{DS} \beta$ if and only if, for all valuations $v$ in $3_\sim$ if $v(\alpha) = 1$ then
   $v(\beta) = 1$ (Truth preservation).
   (ii) $\alpha \vdash_{DS0} \beta$ if and only if, for all valuations $v$ in $3_\sim$ if $v(\beta) = 0$ then
   $v(\alpha) = 0$ (Falsity preservation).
   (iii) $\alpha \vdash_{DS1,0} \beta$ if and only if, $\alpha \vdash_{DS} \beta$ and $\alpha \vdash_{DS0} \beta$.

**Proposition 3.** [10] Let $S = (S, \lor, \cdot, \sim, 0, 1)$ and $DS = (DS, \lor, \cdot, \sim, 0, 1)$ be
Stone and dual Stone algebra respectively, then for $a, b \in S$ and $x, y \in DS$

(i) $\sim \sim (a \lor b) = \sim \sim a \lor \sim \sim b$ and $\sim \sim (a \land b) = \sim \sim a \land \sim \sim b$.

(ii) $\neg \neg (x \land y) = \neg \neg x \land \neg \neg y$ and $\neg \neg (x \lor y) = \neg \neg x \lor \neg \neg y$.

**Lemma 2.** 1. For $\alpha, \beta \in F_\sim$, if $\alpha \vdash_1 \beta$ then $\alpha \vdash S \beta$.

2. For $\alpha, \beta \in F_\sim$, if $\alpha \vdash_{DS} \beta$ then $\alpha \vdash_{DS} \beta$.

**Proof.** 1. Let $\alpha \vdash_1 \beta$, and $v$ be a valuation in $3_\sim$ such that $v(\beta) = 0$. As
$\alpha \vdash_1 \beta$, so $v(\alpha) \neq 1$. If $v(\alpha) = 0$, then our work is done. So, assume that
$v(\alpha) = a$. Define a map $v^* : F_\sim \to 3_\sim$ as:
$v^*(x) = \sim \sim v(x)$.

Let us show that $v^*$ is indeed a valuation $3_\sim$. For this, we have to show that
$v^*(\gamma \land \delta) = v^*(\gamma) \land v^*(\delta)$, $v^*(\gamma \lor \delta) = v^*(\gamma) \lor v^*(\delta)$, $v^*(\sim \gamma) = \sim v^*(\gamma)$,
$v^*(\bot) = 0$ and $v^*(\top) = 1$, but this follows immediately from Proposition 5.
Hence $v^*$ is a valuation and $v^*(\alpha) = 1$ and $v^*(\beta) = 0$ but this contradicts to the
fact that $\alpha \vdash_1 \beta$. So, $\alpha \vdash_1 \beta$ implies $\alpha \vdash S \beta$. 


2. Now, let \( \alpha \vdash_{\bar{0}}^{DS} \beta \), and \( v \) be a valuation in \( 3. \) such that \( v(\alpha) = 1. \) As \( \alpha \vdash_{\bar{0}}^{DS} \beta \), so \( v(\beta) \neq 0. \) If \( v(\beta) = 1 \), then our work is done. So, assume that \( v(\beta) = a. \) In a similar fashion as in previous case, define a map \( v^* : \mathcal{F}^* \to 3. \) as:

\[
v^*(\gamma) = \neg\neg v(\gamma).
\]

Similar to the previous case, using Proposition \([3]\) we can easily established that \( v^* \) is indeed a valuation \( 3. \). This arises a contradiction to \( \alpha \vdash_{\bar{0}}^{DS} \beta. \)

Note that converse of the above statements are not true, for example \( \sim \sim \alpha \vdash_{\bar{0}}^{S} \alpha \) but \( \sim \sim \alpha \not\vdash_{\bar{0}}^{S} \beta \) and \( \beta \vdash_{\bar{0}}^{S} \sim \sim \beta \) but \( \beta \vdash_{\bar{1}}^{S} \sim \sim \beta. \) This is in contrary to the Dunn’s consequence relations \( \vdash_{0}, \vdash_{1} \) and \( \vdash_{0,1} \) where all these three turn out be equivalent.

**Theorem 8.** 1. \( \alpha \vdash_{SP(U)\bar{1}}^{S} \beta \) if and only if \( \alpha \vdash_{\bar{1}}^{S} \beta, \) for any \( \alpha, \beta \in \mathcal{F}^*. \)
2. \( \alpha \vdash_{SP(U)\bar{2}}^{S} \beta \) if and only if \( \alpha \vdash_{\bar{0}}^{DS} \beta, \) for any \( \alpha, \beta \in \mathcal{F}^*. \)

**Proof.** 1. Let \( \alpha \vdash_{SP(U)\bar{2}}^{S} \beta \) and \( v : \mathcal{F} \to 3. \) be a valuation. By Theorem \([5]\) \( 3. \) is embeddable to a Stone algebra of \( \mathcal{P}(U)^{[2]} \) for some set \( U. \) If this embedding is denoted by \( \phi, \phi \circ v \) is a valuation in \( \mathcal{P}(U)^{[2]} \). Then \( (\phi \circ v)(\alpha) \leq (\phi \circ v)(\beta) \) implies \( v(\alpha) \leq v(\beta). \) Thus if \( v(\alpha) = 1, \) we have \( v(\beta) = 1. \)

Now, let \( \alpha \vdash_{\bar{2}}^{S} \beta. \) Let \( U \) be a set, and \( \mathcal{P}(U)^{[2]} \) be the corresponding Stone algebra. Let \( v \) be a valuation on \( \mathcal{P}(U)^{[2]} \) – we need to show \( v(\alpha) \leq v(\beta). \) For any \( \gamma \in \mathcal{F} \) with \( v(\gamma) := (A, B), \ A, B \subseteq U, \) and for each \( x \in U, \) define a map \( v_x : \mathcal{F} \to 3. \) as

\[
v_x(\gamma) := \begin{cases} 1 & \text{if } x \in A \\ a & \text{if } x \in B \setminus A \\ 0 & \text{if } x \notin B. \end{cases}
\]

Consider any \( \gamma, \delta \in \mathcal{F}, \) with \( v(\gamma) := (A, B) \) and \( v(\delta) := (C, D), \) \( A, B, C, D \subseteq U. \) It is easy to show that (for a complete proof, we refer to \([6]\)), \( v_x(\gamma \wedge \delta) = v_x(\gamma) \wedge v_x(\delta), \) \( v_x(\gamma \vee \delta) = v_x(\gamma) \vee v_x(\delta). \) Let us show the following: \( v_x(\sim \gamma) = \sim v_x(\gamma). \)

Note that \( v(\sim \gamma) = (B^c, B^c). \)

**Case 1** \( v_x(\gamma) = 1: \) then \( x \in A, \) i.e. \( x \notin A^c \) and so \( x \notin B^c. \) Hence \( v_x(\sim \gamma) = 0 = \sim v_x(\gamma). \)

**Case 2** \( v_x(\gamma) = a: \) \( x \notin A \) but \( x \in B, \) so \( x \notin B^c. \) Hence \( v_x(\sim \gamma) = 0 = \sim v_x(\gamma). \)

**Case 3** \( v_x(\gamma) = 0: \) \( x \notin B, \) i.e. \( x \in B^c. \) So \( v_x(\sim \gamma) = 1 = \sim v_x(\gamma). \)

Hence \( v_x \) is a valuation in \( 3. \). Now let us show that \( v(\alpha) \leq v(\beta). \) Let \( v(\alpha) := (A', B'), \ v(\beta) := (C', D'), \) and \( x \in A'. \) Then \( v_x(\alpha) = 1, \) and as \( \alpha \vdash_{\bar{1}}^{S} \beta, \) by definition, \( v_x(\beta) = 1. \) This implies \( x \in C', \) whence \( A' \subseteq C'. \)

On the other hand, if \( x \notin D', \ v_x(\beta) = 0. \) Then using Lemma \([2]\) we have \( v_x(\alpha) = 0, \) so that \( x \notin B', \) giving \( B' \subseteq D'. \)

2. We prove second part only. For this let \( \alpha \vdash_{\bar{0}}^{DS} \beta. \) Let \( U \) be a set, and \( \mathcal{P}(U)^{[2]} \) be the corresponding dual Stone algebra. Let \( v \) be a valuation on \( \mathcal{P}(U)^{[2]}, \) we show that \( v(\alpha) \leq v(\beta). \) Very similar to previous case, for any \( \gamma \in \mathcal{F} \) with
\( v(\gamma) := (A, B), \ A, B \subseteq U, \) and for each \( x \in U, \) define a map \( v_x : F \rightarrow 3_\omega, \) as
\[
\begin{align*}
v_x(\gamma) & := \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{if } x \notin B.
\end{cases}
\end{align*}
\]
Consider any \( \gamma, \delta \in \mathcal{F}, \) with \( v(\gamma) := (A, B) \) and \( v(\delta) := (C, D), \ A, B, C, D \subseteq U. \) Let us show the following: \( v_x(\neg \gamma) = \neg v_x(\gamma). \)

\begin{itemize}
\item \textbf{Case 1.} \( v_x(\gamma) = 1: \) Then \( x \in A, \) i.e. \( x \notin A^c. \) Hence \( v_x(\neg \gamma) = 0 = \neg v_x(\gamma). \)
\item \textbf{Case 2.} \( v_x(\gamma) = a: x \notin A \) but \( x \in B, \) so \( x \in A^c. \) Hence \( v_x(\neg \gamma) = 0 = \neg v_x(\gamma). \)
\item \textbf{Case 3.} \( v_x(\gamma) = 0: x \notin B, \) and so \( x \notin A, \) i.e. \( x \in A^c. \) So \( v_x(\neg \gamma) = 1 = \neg v_x(\gamma). \)
\end{itemize}

Hence \( v_x \) is a valuation in \( 3_\omega. \) To complete the proof let us show that \( v(\alpha) \leq v(\beta). \) Let \( v(\alpha) := (A', B'), v(\beta) := (C', D'), \) and \( x \in A'. \) Then \( v_x(\alpha) = 1, \) and as \( \alpha \vdash_1 DS \beta, \) by Lemma 2 \( v_x(\beta) = 1. \) This implies \( x \in C', \) whence \( A' \subseteq C'. \)

On the other hand, if \( x \notin D', \) \( v_x(\beta) = 0. \) Then by our assumption \( \alpha \vdash_0 DS \beta, \) we have \( v_x(\alpha) = 0, \) so that \( x \notin B', \) giving \( B' \subseteq D'. \)

\[ \square \]

Note that in proof of previous Theorem, while defining the valuations \( v_x \) we have utilized the fact that if \( U \) is a set and \( A, B \subseteq U \) with \( A \subseteq B, \) then the collection \( \{A, B \setminus A, B^c\} \) is a partition of \( U. \) So, for \( x \in U, \) either \( x \in A \) or \( x \in B \setminus A \) or \( x \in B^c. \) Nevertheless, finally we have the following Theorem.

\textbf{Theorem 9. (3-valued semantics)}

1. \( \alpha \vdash_L \beta \) if and only if \( \alpha \vdash_1 \beta. \)
2. \( \alpha \vdash_{LDS} \beta \) if and only if \( \alpha \vdash_1 DS \beta. \)

\begin{itemize}
\item \textbf{2.3 Rough set models for 3-valued logics} \end{itemize}

Rough set theory, introduced by Pawlak in 1982, deals with a domain \( U \) that is the set of objects, and an equivalence (indiscernibility) relation \( R \) on \( U. \) The pair \((U, R)\) is called an (Pawlak) approximation space. For any \( A \subseteq U, \) one defines the lower and upper approximations of \( A \) in the approximation space \((U, R), \) denoted \( LA \) and \( UA \) respectively, as follows.

\[
LA := \bigcup\{[x] \in X \mid x \in X\}, \quad UA := \bigcup\{[x] \in X \mid x \cap X \neq \emptyset\}. \tag{4}
\]

As the information about the objects of the domain is available modulo the equivalence classes in \( U, \) the description of any concept, represented extensionally as the subset \( A \) of \( U, \) is inexact. One then ‘approximates’ the description from within and outside, through the lower and upper approximations respectively. Unions of equivalence classes are termed as definable sets, signifying exactly describable concepts in the context of the given information. In particular, sets of the form \( LA, UA \) are definable sets.

\textbf{Definition 7.} Let \((U, R)\) be an approximation space. For each \( A \subseteq U, \) the ordered pair \((LA, UA)\) is called a rough set in \((U, R)\).
Notation 3 $\mathcal{RS} := \{ (LA, UA) : A \subseteq U \}$.

The ordered pair $(D_1, D_2)$, where $D_1 \subseteq D_2$ and $D_1, D_2$ are definable sets, is called a generalized rough set in $(U, R)$.

Notation 4 $D$ denotes the collection of definable sets and $R$ that of the generalized rough sets in $(U, R)$.

In rough set theory, if $(U, R)$ is an approximation space and $A \subseteq U$ then \{LA, UA \ LA, (UA)^c\} is a partition of $U$. So for any $x \in U$, one of the following is true: either $x \in LA$ or $x \in UA \ LA$ or $x \in (UA)^c$. This fact leads to the following interpretations in rough set theory.

1. $x$ certainly belongs to $A$, if $x \in LA$, i.e. all objects which are indiscernible to $x$ are in $A$.
2. $x$ certainly does not belong to $A$, if $x /\in UA$, i.e. all objects which are indiscernible to $x$ are not in $A$.
3. Belongingness of $x$ to $A$ is not certain, but possible, if $x \in UA$ but $x \not\in LA$. In rough set terminology, this is the case when $x$ is in the boundary of $A$: some objects indiscernible to $x$ are in $A$, while some others, also indiscernible to $x$, are in $A^c$.

The phrase ‘certainly’ in the above interpretations derived because of indiscernible behavior of the equivalence relation $R$. For any set $A \subseteq U$, LA, (UA)^c are certain regions and the set UA \ LA is uncertain region.

Nevertheless, these interpretations have led to much work in the study of connections between 3-valued algebras or logics and rough sets, see for instance [28,29,30,31,32,33]. In [6] author came up with a logic $L_K$, which explicitly capture these interpretations via a 3-valued semantics. It also worth mention here the work of Avron and Konikowska, in [32], they have obtained a non-deterministic logical matrix and studied the 3-valued logic generated by this matrix. A simple predicate language is used, with no quantifiers or connectives, to express membership in rough sets. Connections, in special cases, with 3-valued Kleene, Lukasiewicz and two paraconsistent logics are established.

Now, let us provide rough set representations of Stone and dual Stone algebras and rough set semantics for $\mathcal{L}_S$ and $\mathcal{L}_{DS}$. Let $\mathcal{RS}_s$ and $\mathcal{RS}_d$ denote respectively the Stone algebra and dual Stone algebra formed by $\mathcal{RS}$ for an approximation space $(U, R)$.

Theorem 10. 1. (a) Given an Stone algebra $\mathcal{S} = (S, \lor, \land, \neg, 0, 1)$, then there exists an approximation space $(U, R)$ such that $\mathcal{S}$ can be embedded into Stone algebra $\mathcal{RS}_s$.

(b) Let $A_{\mathcal{RS}_s}$ denote the class of all Stone algebras formed by $\mathcal{RS}$. Then we have for $\alpha, \beta \in F_s$:

$$\alpha \models_{\mathcal{L}_s} \text{ if and only if } \alpha \models_{A_{\mathcal{RS}_s}} \beta.$$

2. (a) Given a dual Stone algebra $\mathcal{DS} = (DS, \lor, \land, \neg, 0, 1)$, then there exists an approximation space $(U, R)$ such that $\mathcal{DS}$ can be embedded into dual Stone algebra $\mathcal{RS}_d$. 


Let $\mathcal{A}_{DSRS}$ denote the class of all dual Stone algebras formed by $\mathcal{RS}$. Then we have for $\alpha, \beta \in F$:

$$\alpha \vdash_{\mathcal{L}_{DS}} \text{iff } \alpha \models_{\mathcal{A}_{DSRS}} \beta.$$ 

**Proof.** In [6], we proved that given a set $U'$, there exists an approximation space $(U, R)$ such that the partially ordered set $(\mathcal{P}(U')^2, \leq)$ is order isomorphic to collection of rough sets $\mathcal{RS}$ corresponding to $(U, R)$. This order isomorphism can easily be extended to an Stone or dual Stone isomorphism. This proves 1(a) and 2(a). 1(b) and 2(b) follows from 1(a), 2(a), Theorem 6 and Theorem 7.

Hence, in particular, Stone algebra $3_{\sim}$ and dual Stone algebra $3_{\neg}$ are isomorphic to $\mathcal{RS}$ for some approximation space. Figure 4 and Figure 5 depict the isomorphisms, where $U = \{x\}$ and $\mathcal{RS}$ is collection of rough sets corresponding to approximation $(U' = \{x, x'\}, R' = \{(x, x), (x, x'), (x', x), (x', x')\})$.

$$1 = \sim 0 \quad (U, U) = \sim (\emptyset, \emptyset) \quad (L U, L U) = \sim (L \emptyset, U \emptyset)$$

$$3_{\sim} := a \quad \mathcal{P}(U)_{\sim} := (\emptyset, U) \quad \mathcal{RS}_{\sim} := (L x, L x) = (L x', L x')$$

$$0 = \sim 1 = \sim a \quad (\emptyset, \emptyset) = \sim (\emptyset, U) = \sim (U, U) \quad (L \emptyset, U \emptyset) = \sim (L U, L U)$$

$$\Rightarrow (L x, L x) = (L x', L x')$$

**Fig. 4.** $3_{\sim} \cong \mathcal{P}(U)_{\sim} \cong \mathcal{RS}_{\sim}$

$$1 = \neg 0 \quad (U, U) = \neg (\emptyset, \emptyset) \quad = \neg (L x, L x) = \neg (L x', L x') \quad (L U, L U) = \neg (L \emptyset, U \emptyset)$$

$$3_{\neg} := u \quad \mathcal{P}(U)_{\neg} := (\emptyset, U) \quad \mathcal{RS}_{\neg} := (L x, L x) = (L x', L x')$$

$$0 = \sim 1 = \sim u \quad (\emptyset, \emptyset) = \neg (\emptyset, U) = \neg (U, U) \quad (L \emptyset, U \emptyset) = \neg (L U, L U)$$

**Fig. 5.** $3_{\neg} \cong \mathcal{P}(U)_{\neg} \cong \mathcal{RS}_{\neg}$

Now, we would explicate the relationship between rough sets and the 3-valued semantics of the logic $\mathcal{L}_S$ and $\mathcal{L}_{DS}$ indicated in Theorem 10 and Theorem 9.
Let $\alpha$ be a formula in $\mathcal{L}_3$ and $\gamma$ be a formula in $\mathcal{L}_{DS}$. Let $v$ be a valuation in $\mathcal{RS}_\approx$ for some approximation space $(U, R)$ such that $v(\alpha) := (LA, UA), A \subseteq U$, and $v'$ be a valuation in $\mathcal{RS}'_\approx$ for some approximation space $(U', R')$ such that $v(\gamma) := (LA', UA'), A' \subseteq U'$. Let $x \in U$ and $y \in U'$. We define the following semantic consequence relations.

1. $v, x \vDash_{\mathcal{RS}_\approx} \alpha$ if and only if $x \in LA$.
2. $v, x \vDash_{\mathcal{RS}_\approx} \alpha$ if and only if $x \notin UA$.
3. $v, x \vDash_{\mathcal{RS}_\approx} \alpha$ if and only if $x \notin LA, x \in UA$.

Next, let us define the following relations. Let $\alpha, \beta \in \mathcal{F}_\approx$ and $\gamma, \delta \in \mathcal{F}_\approx$,

1. $\Gamma \vDash_{1, \mathcal{RS}_\approx} \beta$ if and only if $v, \Gamma \vDash_{1, \mathcal{RS}_\approx} \alpha$ implies $v, \Gamma \vDash_{1, \mathcal{RS}_\approx} \beta$, for all valuations $v$ in $\mathcal{RS}_\approx$ and $x \in U$.
2. $\Gamma \vDash_{0, \mathcal{RS}_\approx} \beta$ if and only if $v, x \vDash_{0, \mathcal{RS}_\approx} \beta$ implies $v, \Gamma \vDash_{0, \mathcal{RS}_\approx} \alpha$, for all valuations $v$ in $\mathcal{RS}_\approx$ and $x \in U$.
3. $\Gamma \vDash_{1, \mathcal{RS}_\approx} \beta$ if and only if $\alpha \vDash_{1, \mathcal{RS}_\approx} \beta$ and $\Gamma \vDash_{0, \mathcal{RS}_\approx} \beta$.

Now we link the syntax and semantics.

**Definition 8.**
1. $\alpha \vdash \beta$ is valid in an approximation space $(U, R)$, if and only if $\alpha \vDash_{1, \mathcal{RS}_\approx} \beta$.
2. $\alpha \vdash \beta$ is valid in a class $\mathcal{F}$ of approximation spaces if and only if $\alpha \vdash \beta$ is valid in all approximation spaces $(U, R) \in \mathcal{F}$.

**Theorem 11.** Let $\alpha, \beta \in \mathcal{F}_\approx$ and $\gamma, \delta \in \mathcal{F}_\approx$, then

1. $\alpha \vDash_{A_{\mathcal{RS}_\approx}} \beta$ if and only if $\alpha \vdash \beta$ is valid in the class of all approximation spaces.
2. $\gamma \vDash_{A_{\mathcal{AD}_{\mathcal{RS}_\approx}}} \delta$ if and only if $\gamma \vdash \delta$ is valid in the class of all approximation spaces.
Proof. 1. Let $\alpha \models_{\mathcal{A}_{RS}} \beta$. Let $(U, R)$ be an approximation space, and $v$ be a valuation in $\mathcal{R}_S$, with $v(\alpha) := (L_A, U_A)$ and $v(\beta) := (L_B, U_B)$. $A, B \subseteq U$. By the assumption, $L_A \subseteq L_B$ and $U_A \subseteq U_B$. Now, let us show that $\alpha \models_{\mathcal{R}_S} \beta$. So, let $x \in U$ and $v, x \models_{\mathcal{R}_S} \alpha$, i.e., $x \in L_A$. But we have $L_A \subseteq L_B$, hence $v, x \models_{\mathcal{R}_S} \beta$.

Now, suppose $\alpha \vdash \beta$ is valid in the class of all approximation spaces. We want to show that $\alpha \models_{\mathcal{A}_{RS}} \beta$. Let $v$ be a valuation in $\mathcal{R}_S$ as taken above. We have to show that $L_A \subseteq L_B$ and $U_A \subseteq U_B$. Let $x \in L_A$, i.e., $v, x \models_{\mathcal{R}_S} \alpha$. Hence by our assumption, $v, x \models_{\mathcal{R}_S} \beta$, i.e., $x \in L_B$. So $L_A \subseteq L_B$. Now, let $y \notin U_B$, using Lemma 2 we have $v, y \not\models_{\mathcal{R}_S} \beta$. By our assumption, $v, y \not\models_{\mathcal{R}_S} \alpha$, i.e., $y \notin U_A$.

2. The proof of this part is very similar to that of part 1 which uses lemma 2.

\[ \square \]

3 Boolean representation of double Stone algebra and corresponding logic

In this section, we follow the same idea as in above section. We provide a structural representation of double Stone algebra in which negations present in double Stone algebra are described using Boolean complement.

3.1 Boolean representation of double Stone algebra

Moisil in 1941 (cf. [10]) generalizes the construction of $B^{[\mathfrak{a}]}$. He defined for any natural number $n$, the set

$$B^n = \{(b_1, b_2, ..., b_n) \in B^n : b_1 \leq b_2 \leq ... \leq b_n \},$$

as examples of n-valued Łukasiewicz Moisil algebras. Our interest in this section is to consider $B^{[\mathfrak{a}]}$ as double Stone algebra.

Proposition 4. [10] $(B^{[\mathfrak{a}]}, \lor, \land, \sim, \gamma, (0, 0, 0), (1, 1, 1))$ is a double Stone algebra, where, for $(a, b, c), (d, e, f) \in B^{[\mathfrak{a}]}$,

$$(a, b, c) \lor (d, e, f) := (a \lor d, b \lor e, c \lor f), (a, b, c) \land (d, e, f) := (a \land d, b \land e, c \land f), \sim (a, b, c) := (c', c', c'), \gamma (a, b, c) := (a^*, a^*, a^*),$$

It is well known that 1, 2, 3 and 4 are the only subdirect irreducible (Figure 6) double Stone algebras.

Hence, again using Birkhoff’s representation theorem we have the following.

Theorem 12. [10]

Let $\mathcal{DS} = (DS, \lor, \land, \sim, \gamma, 0, 1)$ be a double Stone algebra. There exists a (index) set $I$ such that $\mathcal{DS}$ can be embedded into $4^I$.

Hence in particular given a Boolean algebra $B$, the double Stone algebra $B^{[\mathfrak{a}]}$ is embeddable into $4^I$ for index set $I$. Now, following similar idea as above, let us characterize completely join irreducible elements of $B^{[\mathfrak{a}]}$ and $4^I$. Let us denote by

$$f^*_I(k) := \begin{cases} x & \text{if } k = i \\ 0 & \text{otherwise} \end{cases}$$
Proposition 5. 1. The set of completely join irreducible elements of the algebra $4^I$ is given by

$$J_{4^I} = \{f^a_i, f^b_i, f^1_i : i \in I\}.$$  
Moreover, $J_{4^I}$ is join dense in $4^I$.

2. Let $B$ be a complete atomic Boolean algebra. The set of completely join irreducible elements of $B^{[3]}$ is given by

$$J_{B^{[3]}} = \{(0,0,a), (0,a,a), (a,a,a) : a \in J_B\}.$$  
Moreover, $J_{B^{[3]}}$ is join dense in $B^{[3]}$.

Proof. Proof of this proposition is very similar to the Proposition 5 and can be seen [6].

The order structure of $J_{4^I}$ and $J_{(2^I)^{[3]}}$ can be visualized in the Figure 7.

Fig. 6. Subdirectly irreducible double Stone algebras

Fig. 7. Hasse diagram of $J_{4^I}$
Theorem 13. (i) The sets of completely join irreducible elements of $4^I$ and $(2^I)^{[3]}$ are order isomorphic. The algebras $4^I$ and $(2^I)^{[3]}$ are lattice isomorphic.

(ii) The double Stone algebras $4^I$ and $(2^I)^{[3]}$ are isomorphic.

(iii) Let $DBS$ be a double Stone algebra. There exists a (index) set $I$ such that $DBS$ can be embedded into Stone algebra $(2^I)^{[3]}$.

Proof. We define the map $\phi : J_{4^I} \to J_{(2^I)^{[3]}}$ as follows. For $i \in I$,

$\phi(f_{2i}^{1}) := (0, 0, g_{1}^{1})$,

$\phi(f_{1i}^{1}) := (0, g_{1}^{1}, g_{1}^{1})$,

$\phi(f_{1i}^{1}) := (g_{1}^{1}, g_{1}^{1}, g_{1}^{1})$.

Here $g_{1}^{1}$ is an atom of the Boolean algebra $2^I$, defined as follows:

$g_{1}^{1}(k) := \begin{cases} 1 & \text{if } k = i \\ 0 & \text{otherwise} \end{cases}$

It can be easily seen that $\phi$ is an order isomorphism. Hence using Lemma [1], $\phi$ can be extended to $4^I$ as $\Phi$, then $\Phi$ is a lattice isomorphism from $4^I$ to $(2^I)^{[3]}$.

Now, as, double Stone algebras are equational algebras, so the extended map of $\Phi$ is also isomorphic as double Stone algebras. This proves (i) and (ii). (iii) follows from (ii) and [12].

Now, let illustrate this Theorem through an example.

Example 1. Let us consider a lattice whose Hasse diagram is given in the Figure 8. Note that $\sim 1 = \sim a = \sim b = \sim c = \sim d = 0 = \sim 0 = \sim a = \sim b = \sim c = \sim d = 1 = \sim 1$. It can be easily verified that the given lattice is also a double Stone algebra. The given double Stone algebra is isomorphic (Figure 8) to a subalgebra double Stone algebra $(2^2)^{[2]}$. Here $2^2$ is the 4-element Boolean algebra $\{0, x, y, 1\}$.

We end this section by providing an easy consequence of Stone’s representation and Theorem 13.

Theorem 14. Given a double Stone algebra $DBS = (DBS, \lor, \land, \sim, \neg, 0, 1)$, then there exists a set $U$ such that $DBS$ can be embedded into double Stone algebra formed by the set $(P(U))^{[3]}$.

3.2 4-valued semantics of logic for double Stone algebras

The idea of arriving at 4-valued semantics of $L_{DBS}$ is to represent an element of a given double Stone algebra as a tuple $(A, B, C)$, where $A \subseteq B \subseteq C$ are subsets of some set $U$. This proposition is established by Theorem 14. So for any $x \in U$, there are 4 possibilities.

$x \in A$, $x \in B \setminus A$, $x \in C \setminus B$, and $x \notin C$.

These 4 possibilities leads to the 4-valued semantics of the logic $L_{DBS}$. Now, let us formally present the concerned logic $L_{DBS}$. Let $F_{\sim, \neg}$ denote the set of formulae by extending the syntax of $BDLL$ by adding unary connectives $\sim, \neg$. The logic $L_{DBS}$ denote the logic $BDLL$ along with following rules and postulates.
1. $\sim \alpha \land \sim \beta \vdash \sim (\alpha \lor \beta)$, $\neg (\alpha \land \beta) \vdash \neg \alpha \lor \neg \beta$.
2. $\top \vdash \bot$, $\neg \top \vdash \bot$.
3. $\alpha \vdash \beta$, $\alpha \vdash \beta$, $\neg \alpha \vdash \neg \alpha$.
4. $\sim \alpha \land \sim \beta \vdash \sim (\alpha \lor \beta)$, $\sim (\alpha \land \beta) \vdash \sim \alpha \lor \sim \beta$.
5. $\alpha \land \sim \alpha \vdash \bot$, $\alpha \lor \neg \alpha \vdash \bot$.
6. $\alpha \land \beta \vdash \gamma / \alpha \land \sim \gamma \leq \sim \beta$, $\gamma \vdash \alpha \lor \beta / \neg \beta \vdash \alpha \lor \neg \gamma$.
7. $\top \vdash \alpha \land \sim \alpha$, $\neg \alpha \land \neg \alpha \vdash \bot$.

With usual constructions of Lindenbaum-Tarski algebra and an easy consequence of Theorem 14 is.

**Theorem 15.** The followings are equivalent:
1. $\alpha \vdash_{L_{\text{DBS}}} \beta$.
2. $\alpha \models_{A_{\text{DBS}}} \beta$.
3. $\alpha \models_{\text{DBS}^p \beta}$. 
4. $\alpha \models_{\text{DBS}^p(U)[3]} \beta$.

**Theorem 16.** $\alpha \models_{\text{DBS}^p(U)[3]} \beta$ if and only if $\alpha \models_4 \beta$, for any $\alpha, \beta \in \mathcal{F}$.

**Proof.** Let $\alpha \models_{\text{DBS}^p(U)[3]} \beta$, and $v : \mathcal{F} \to 4$ be a valuation. By Theorem 14, $4$ is isomorphic to a double Stone algebra of $\mathcal{P}(U)^3$ for some set $U$. If this isomorphism is denoted by $\phi$, $\phi \circ v$ is a valuation in $\mathcal{P}(U)^3$. Then $(\phi \circ v)(\alpha) \leq (\phi \circ v)(\beta)$ implies $v(\alpha) \leq v(\beta)$.

Now, let $\alpha \models_4 \beta$. Let $U$ be a set, and $\mathcal{P}(U)^3$ be the corresponding double Stone algebra. Let $v$ be a valuation on $\mathcal{P}(U)^4$ – we need to show $v(\alpha) \leq v(\beta)$. For any $\gamma \in \mathcal{F}$ with $v(\gamma) := (A, B, C)$, $A, B, C \subseteq U$, and for each $x \in U$, define a
map \( v_x : F \to 4 \) as
\[
v_x(\gamma) := \begin{cases} 
  t & \text{if } x \in A \\
  u_1 & \text{if } x \in B \setminus A \\
  u_2 & \text{if } x \in C \setminus B \\
  f & \text{if } x \notin C. 
\end{cases}
\]

Consider any \( \gamma, \delta \in F \), with \( v(\gamma) := (A, C, B) \) and \( v(\delta) := (D, E, F) \), \( A, B, C, D, E, F \subseteq U \). It is easy to show that: \( v_x(\gamma \land \delta) = v_x(\gamma) \land v_x(\delta) \), \( v_x(\gamma \lor \delta) = v_x(\gamma) \lor v_x(\delta) \).

Let us show the followings.

\[ - \quad v_x(\neg \gamma) = \neg v_x(\gamma). \]

Note that \( v(\neg \gamma) = (C, C, A^c) \).

**Case 1** \( v_x(\gamma) = t \): Then \( x \in A \), i.e. \( x \notin A^c \) and so \( x \notin C^c \). Hence \( v_x(\neg \gamma) = f = \neg v_x(\gamma) \).

**Case 2** \( v_x(\gamma) = u_1 \): \( x \notin A \) but \( x \in B \), i.e. \( x \notin B^c \) so, \( x \notin C \). Hence \( v_x(\neg \gamma) = f = \neg v_x(\gamma) \).

**Case 3** \( v_x(\gamma) = u_2 \): \( x \notin B \) but \( x \in C \), i.e. \( x \notin C^c \). Hence \( v_x(\neg \gamma) = f = \neg v_x(\gamma) \).

**Case 4** \( v_x(\gamma) = f \): \( x \notin C \), i.e. \( x \in C^c \). So \( v_x(\neg \gamma) = t = \neg v_x(\gamma) \).

Hence \( v_x \) is a valuation in \( 4 \). Now let us show that \( v(\alpha) \leq v(\beta) \). Let \( v(\alpha) := (A', B', C') \), \( v(\beta) := (D', E', F') \).

Let \( x \in A' \) then \( v_x(\alpha) = t \). As \( \alpha \vdash_4 \beta \) hence \( v_x(\beta) = t \). This implies \( x \in D' \), whence \( A' \subseteq D' \).

Let \( x \in B' \setminus A' \) then \( v_x(\alpha) = u_1 \). As \( \alpha \vdash_4 \beta \) hence \( v_x(\beta) = t \) or \( u_1 \). This implies \( x \in E' \), whence \( B' \subseteq E' \).

On the other hand, if \( x \notin F' \), \( v_x(\beta) = f \). By assumption, we have \( v_x(\alpha) = f \), so that \( x \notin C' \), giving \( C' \subseteq F' \).

Finally, we have the following 4-valued semantics of the logic \( L_{DDBS} \).

**Theorem 17.** (4-valued semantics) \( \alpha \vdash_{L_{DDBS}} \beta \) if and only if \( \alpha \vdash_4 \beta \).

Now, a natural question arises here, what happens if we follow the Dunn’s semantic consequence relation of De Morgan logic?

**Definition 9.** Let \( \alpha, \beta \in F_{DDBS} \).

(i) \( \alpha \vdash_1 \beta \) if and only if, if \( v(\alpha) = 1 \) then \( v(\beta) = 1 \) (Truth preservation).

(ii) \( \alpha \vdash_0 \beta \) if and only if, if \( v(\beta) = 0 \) then \( v(\alpha) = 0 \) (Falsity preservation).

(iii) \( \alpha \vdash_{0,1} \beta \) if and only if, \( \alpha \vdash_1 \beta \) and \( \alpha \vdash_0 \beta \).
Lemma 3. $\alpha \models_1 \beta$ if and only if $\alpha \models_0 \beta$.

Proof. We prove ‘if’ part, proof of ‘only if’ part follows similarly.

Assume $\alpha \models_1 \beta$ and let $v : \mathcal{F} \to 4$ such that $v(\beta) = 0$. As, $\alpha \models_1 \beta$ so $v(\alpha) \neq 1$. Assume $v(\alpha) = u_1$ and define $v_1 : \mathcal{F} \to 4$ as $v_1(\gamma) = \neg \neg v(\gamma)$. Then $v_1$ is a valuation in $4$ and $v_1(\alpha) = 1$. So, $v_1(\beta) = 1$ but $v_1(\beta) = v(\beta) = 0$ which is absurd. Hence $v(\alpha) \neq u_1$. Similarly $v(\alpha) \neq u_2$. □

Proposition 6. If $\alpha \models_4 \beta$ then $\alpha \models_{1,0} \beta$.

Let us show through an example below that converse of the above proposition may not be true.

Example 2. Consider the sequent $\alpha \land \neg \alpha \models \beta \lor \neg \beta$. Let us show that $\alpha \land \neg \alpha \models_{1,0} \beta \lor \neg \beta$. So, assume that $v$ is a valuation in $4$ such that $v(\alpha \land \neg \alpha) = 1$. So, $v(\alpha) = 1$ and $\neg v(\alpha) = 1$. But there is no $x \in 4$ such that $x \land \neg x = 1$. Hence $\alpha \land \neg \alpha \models_1 \beta \lor \neg \beta$ is vacuously true. Similarly, there is no $x \in 4$ such that $x \lor \neg x = 0$. Hence $\alpha \land \neg \alpha \models_{0} \beta \lor \neg \beta$ is vacuous.

So, $\alpha \land \neg \alpha \models_{1,0} \beta \lor \neg \beta$.

Now let us show that $\alpha \land \neg \alpha \not\models_4 \beta \lor \neg \beta$. Define a valuation $v$ in $4$ such that $v(\alpha) = u_1$ and $v(\beta) = u_2$. Then $v(\alpha \land \neg \alpha) = v(\alpha) \land \neg v(\alpha) = u_1 \land 1 = u_1$ and $v(\beta \lor \neg \beta) = u_2$. But $u_1 \not\leq u_2$, hence $\alpha \land \neg \alpha \not\models_4 \beta \lor \neg \beta$.

4 Conclusions

The lattices $B^{[2]}$ and $B^{[3]}$ can be extended to form various algebraic structures. In this article, we have studied $B^{[2]}$ as (dual) Stone algebra and $B^{[3]}$ as double Stone algebra. Moisil obtained representation of 3-valued and 4-valued Lukasiewicz algebra in terms of $B^{[2]}$ and $B^{[3]}$ respectively. Similar to his works we have obtained the representations of (dual) Stone and double Stone algebras in terms of $B^{[2]}$ and $B^{[3]}$ respectively.

The 2 element Boolean algebra play a fundamental role in classical propositional logic (via True-False) semantics and Boolean algebra (via Stone’s representation theorem). We have established the same for:

1. The Stone algebra $3_-$, the logic $\mathcal{L}_S$ and the class of all Stone algebras.
2. The Stone algebra $3_+$, the logic $\mathcal{L}_{DS}$ and the class of all dual Stone algebras.
3. The double Stone algebra $4$, the logic $\mathcal{L}_{DBS}$ and the class of all double Stone algebras.

Moreover, due to rough set representation result of (dual) Stone algebra, we have been able to provide rough set semantics of the logic ($\mathcal{L}_{DS}$) $\mathcal{L}_S$ which thus leads to the equivalence of 3-valued, algebraic and rough set semantics of the logic ($\mathcal{L}_{DS}$) $\mathcal{L}_S$. 

References

1. Stone, M.: The theory of representation for Boolean algeb ras. Transaction of American Mathematical Society 40(1) (1936) 37–111
2. Cignoli, R.: The algebras of Łukasiewicz many-valued logic: A historical overview. In Aguzzoli, S., Ciabattoni, A., Gerla, B., Manara, C., Marra, V., eds.: Algebraic and Proof-theoretic Aspects of Non-classical Logics, LNAI 4460. Springer-Verlag, Berlin Heidelberg (2007) 69–83
3. Rasiowa, H.: An Algebraic Approach to Non-classical Logics. North-Holland (1974)
4. Dunn, J.: The algebra of intensional logic. Doctoral dissertation, University of Pittsburg h (1966)
5. Dunn, J.: A comparative study of various model-theoretic treatments of negation: A history of formal negations. In Gabbay, D., Wansing, H., eds.: What is Negation? Kluwer Academic Publishers, Netherlands (1999) 23–51
6. Kumar, A., Banerjee, M.: Kleene algebras and logic: Boolean and rough set representations, 3-valued, rough set and perp semantics. Studia Logica (2016) doi:10.1007/s11225-016-9696-6
7. Birkhoff, G.: Lattice Theory. Colloquium Publications, Vol. XXV, 2nd edn., American Mathematical Society, Providence (1948)
8. Gratzer, G., Schmidt, E.T.: On a problem of M.H. stone. Acta Math Acad. Sci. Hungar 8 (1957) 455–460
9. Balbes, R., Dwinger, P.: Distributive Lattices. University of Missouri Press, Columbia (1974)
10. Boicescu, V., Filipoiu, A., Georgescu, G., Rudeanu, S.: Łukasiewicz-Moisil Algebras. North-Holland, Amsterdam (1991)
11. Pawlak, Z.: Rough sets. International Journal of Computer and Information Sciences 11 (1982) 341–356
12. Pawlak, Z.: Rough Sets: Theoretical Aspects of Reasoning About Data. Kluwer Academic Publishers (1991)
13. Banerjee, M., Chakraborty, M.K.: Algebras from rough sets. In Pal, S.K., Polkowski, L., Skowron, A., eds.: Rough-neuro Computing: Techniques for Computing with Words. Springer-Verlag, Berlin (2004) 157–184
14. Comer, S.: Perfect extensions of regular double Stone algebras. Algebra Universalis 34(1) (1995) 96–109
15. Pagliani, P.: Rough sets and Nelson algebras. Fundamenta Informaticae 27(2-3) (1996) 205–219
16. Järvinen, J., Radeleczki, S.: Representation of Nelson algebra by rough sets determined by quasiorder. Algebra Universalis 66 (2011) 163–179
17. Banerjee, M., Chakraborty, M.K.: Rough sets through algebraic logic. Fundamenta Informaticae 28(3-4) (1996) 211–221
18. Järvinen, J., Pagliani, P., Radeleczki, S.: Information completeness in Nelson algebras of rough sets induced by quasiorders. Studia Logica 101(5) (2013) 1073–1092
19. Dunn, J.: Negation in the context of gaggle theory. Studia Logica 80 (2005) 235–264
20. Dunn, J.: Positive modal logic. Studia Logica 55 (1995) 301–317
21. Davey, B.A., Priestley, H.A.: Introduction to Lattices and Order. Cambridge University Press (2002)
22. Vakarelov, D.: Notes on N-lattices and constructive logic with strong negation. Studia Logica 36 (1977) 109–125
23. Fidel, M.: An algebraic study of a propositional system of Nelson. In Arruda, A.I., da Costa, N.C.A., Chuaqui, R., eds.: Mathematical Logic: Proceedings of First Brazilian Conference. Lecture Notes in Pure and Applied Mathematics Vol. 39, M.Dekker Inc., New York (1978) 99–117
24. Kalman, J.: Lattices with involution. Transactions of American Mathematical Society 87 (1958) 485–491
25. Katriňák, T.: Construction of regular double p-algebras. Bull. Soc. Roy. Sci. Liege 43 (1974) 238–246
26. Birkhoff, G.: Subdirect unions in universal algebra. Bull. Amer. Math. Society 56 (1944) 764–768
27. Birkhoff, G.: Lattice Theory. Colloquium Publications, Vol. XXV, 3rd edn., American Mathematical Society, Providence (1995)
28. Banerjee, M.: Rough sets and 3-valued Lukasiewicz logic. Fundamenta Informaticae 31 (1997) 213–220
29. Düntsch, I.: A logic for rough sets. Theoretical Computer Science 179 (1997) 427–436
30. Pagliani, P.: Rough set theory and logic-algebraic structures. In Orłowska, E., ed.: Incomplete Information: Rough Set Analysis, volume 3 of Studies in Fuzziness and Soft Computing. Springer Physica-Verlag (1998) 109–190
31. Iturrioz, L.: Rough sets and three-valued structures. In Orłowska, E., ed.: Logic at Work: Essays Dedicated to the Memory of Helena Rasiowa, volume 24 of Studies in Fuzziness and Soft Computing. Springer Physica-Verlag (1999) 596–603
32. Avron, A., Konikowska, B.: Rough sets and 3-valued logics. Studia Logica 90 (2008) 69–92
33. Ciucci, D., Dubois, D.: Three-valued logics, uncertainty management and rough sets. Transactions on Rough Sets XVII, LNCS 8375, Springer Berlin Heidelberg (2014) 1–32