C¹ Positive Surface over Positive Scattered Data Sites

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Abstract

The aim of this paper is to develop a local positivity preserving scheme when the data amassed from different sources is positioned at sparse points. The proposed algorithm first triangulates the irregular data using Delaunay triangulation method, therewith interpolates each boundary and radial curve of the triangle by C¹ rational trigonometric cubic function. Half of the parameters in the description of the interpolant are constrained to keep up the positive shape of data while the remaining half are set free for users' requirement. Orthogonality of trigonometric function assures much smoother surface as compared to polynomial functions. The proposed scheme can be of great use in areas of surface reconstruction and deformation, signal processing, CAD/CAM design, solving differential equations, and image restoration.

1. Introduction

Data measured or amassed from many engineering and scientific fields, is often positioned at sparse points. For example, meteorological measurements at different weather stations [1], density measurements on different positions within the human body, heart potential measurements at random points in the diagnosis of various ailments of heart [2], 3D photography, aeronautical engineering and industrial design, structural graph networks [3], graph entropy [4], [5], [6]. A visual model is often required to get a clear understanding of underlying phenomena as colossal amount of data is difficult to analyse or communicate a message in raw form. Further, a meticulous visual representation obligates the interpolating function to affirm intrinsic attributes of data like positivity, monotonicity and convexity. Although, tensor product provides a robust medium for fitting surface to rectilinear data sites, it can not be used to fit a surface over sparse data points. This paper addresses the problem of retaining positivity over scattered data points.

Several approaches have been proposed in literature to address the problem of positivity preserving interpolating surfaces. Amidor [7] surveyed method to interpolate scattered data necessitating from electronic imaging system. The author mainly examined radial basis function method, tetrahedral interpolation, cubic triangular interpolation, triangle based blending interpolation, inverse distance method and neutral neighbourhood. The difference between
scattered data interpolation and scattered data fitting was also demonstrated in the survey. Cubic and quintic Hermite interpolants were used for preserving monotonicity, positivity and convexity of discrete data by [8]. Piah, Goodman, Unsworth [9] first triangulated the data points by Delaunay triangulation and constructed the interpolating surface consisting of “cubic Bezier triangular patches”. Positivity of data was achieved by imposing sufficient conditions on Bezier ordinates in each triangular patch. The proposed scheme was local and $C^1$ continuous. Hussain and Hussain [10] arranged the scattered data over a triangular grid to preserve the positivity and monotonicity. The authors used a cubic interpolant with one parameter to interpolate the boundary of each triangular patch while linear interpolant was used in Nielson side vertex method to obtain radial curves. Final surface patch was obtained by convex combination of interpolants. Positivity and monotonicity was retained by deriving data dependent constraints on free parameters. $C^1$ Quadratic splines and Powell-Sabin splines were used as interpolating function to tackle the problem of range restricted univariate and bivariate scattered data by Hermann et. al. [11]. The authors obtained a system of inequalities for the gradients and positivity was accomplished by deriving sufficient conditions on this system. A $C^1$ local rational cubic Bernstein Bezier interpolatory scheme was proposed by Hussain and Hussain [12] to retain positivity of scattered data. In each triangular patch, inner and boundary Bezier ordinates were confined for positivity. If in any triangular grid, Bezier ordinates failed to attain positive shape of data, then these were varied by the weights described in formation of rational cubic Bernstein Bezier interpolant. Sarfraz et. al. [1] established a local $C^1$ approach to keep up the positivity of scattered data positioned over a triangular domain. They employed $C^1$ rational cubic function with four parameters in Nielson side vertex technique to formulate the interpolating surface. Two of the four parameters were constrained for positivity.

Although several approaches have been proposed to retain the positivity of data, little attention has been paid towards the use of trigonometric basis function. This paper develops a positivity preserving scheme for scatter data by taking $C^1$ rational trigonometric function [13] into account. Delaunay triangulation method has been used to place scatter data as vertices of triangle. Nielson side vertex method [14] has been employed in each triangle to construct triangular patches. The $C^1$ rational trigonometric cubic function [13] with four parameter has been used for the interpolation along boundary and radial curve of the triangle. Positivity is attained by deriving data dependent condition on half of the parameters in the description of $C^1$ rational trigonometric cubic function [13].

The remainder of the paper is formulated as: Section 2 reviews the rational trigonometric cubic function [13]. Nielson side vertex method [14] to formulate triangular patches is detailed in Section 3. Positivity preserving algorithm is developed and explained in Section 4. Section 5 demonstrates the developed algorithm and presents graphical results. Section 5 summarizes this research and draws conclusion.

2. Rational Trigonometric Cubic Function

Let $\{(x_i, y_i), i = 0, 1, 2, \ldots, n-1\}$ be the given set of data points defined over the interval $[a, b]$ where $a = x_0 < x_1 < x_2 < \ldots < x_n = b$. A piecewise rational trigonometric cubic function is defined over each subinterval $I_j = [x_i, x_{i+1}]$ as

$$S_i(x) = \frac{p_i(\theta)}{q_i(\theta)}$$

(1)
The rational trigonometric cubic function (Eq 1) satisfies the following properties:

\[
p_i(\theta) = \alpha_i \sin^3 \theta + \left\{ \beta_i \sin \theta \left( 1 - \sin \theta \right)^2 \right\} \sin \theta \left( 1 - \sin \theta \right)^2 + \delta_i \cos \theta \left( 1 - \cos \theta \right)^3
\]

\[
q_i(\theta) = \alpha_i \sin^3 \theta + \beta_i \sin \theta \left( 1 - \sin \theta \right)^2 + \gamma_i \cos \theta \left( 1 - \cos \theta \right)^2 + \delta_i \left( 1 - \cos \theta \right)^3
\]

where

\[
\theta = \frac{\pi}{2} \left( \frac{x - x_i}{h_i} \right), \quad h_i = x_{i+1} - x_i, \quad i = 0, 1, 2, ..., n - 1
\]

The rational trigonometric cubic function (Eq 1) satisfy the following properties:

\[
S(x_i) = f_i, \quad S(x_{i+1}) = f_{i+1}, \quad S'(x_i) = d_i, \quad S'(x_{i+1}) = d_{i+1}.
\]

\(d_i\) and \(d_{i+1}\) are derivative at the endpoints of the interval \(I_i = [x_i, x_{i+1}]\). \(\alpha_i, \beta_i, \gamma_i\) and \(\delta_i\) are the free parameters. The following result has been proved in [13].

**Theorem 2.1** The \(C^1\) piecewise rational trigonometric cubic function preserve the positivity of positive data if in each subinterval \(I_i = [x_i, x_{i+1}]\), the parameters \(\beta_i\) and \(\gamma_i\) satisfy the following sufficient conditions

\[
\beta_i = u_i + \max \left\{ 0, \frac{-2h_i d_i}{\pi f_i} \right\}, \quad u_i > 0,
\]

\[
\gamma_i = v_i + \max \left\{ 0, \frac{2h_i d_{i+1}}{\pi f_{i+1}} \right\}, \quad v_i > 0.
\]

3. Nielson Side Vertex Method

Consider a triangle \(\triangle V_1 V_2 V_3\) with vertices \(V_1, V_2, V_3\) having edges \(e_1, e_2, e_3\) and \(u, v, w\) be the barycentric coordinates such that any point \(V\) on the triangle can be written as:

\[
V = uV_1 + vV_2 + wV_3,
\]

where

\(u + v + w = 1\) and \(u, v, w \geq 0\).

The interpolant defined by Nielson [14] to generate surface over each triangular patch is defined as the following convex combination:

\[
P(a, b, c) = \frac{v^2 w^2 Q_1 + u^2 w^2 Q_2 + u^2 v^2 Q_3}{v^2 w^2 + u^2 w^2 + u^2 v^2}.
\]

where \(Q_i\)s represent line segments joining vertices \(V_i\)s to points \(S_i\)s on the opposite boundary. Eq (4) interpolates data at the vertices as well as first order derivatives at the boundary. Since the barycentric coordinates at the vertices of triangle is simultaneously zero, the interpolant Eq (4) takes the following values:

\[
P(a, b, c) = Q_1 \quad \text{when} \quad v = w = 0,
\]

\[
P(a, b, c) = Q_2 \quad \text{when} \quad u = w = 0,
\]

\[
P(a, b, c) = Q_3 \quad \text{when} \quad v = u = 0
\]

where \(Q_i, i = 1, 2, 3\) are the ordinate values at the vertices \(V_i, i = 1, 2, 3\) of triangle.
4. Positive Scatter Data Interpolation

This section details the derivation of sufficient conditions for $C^1$ triangular patches to be positive. Let the given positive scattered data set arranged over a triangular domain be $\{(x_i, y_i, F_i), i = 1, 2, \ldots, n\}$. The resulting surface $S(x, y)$ described as

$$S(x_i, y_i) = F_i, i = 1, 2, \ldots, n,$$

is positive if

$$S(x, y) > 0, \forall (x, y) \in D. \quad (6)$$

4.1 Domain Triangulation

Triangulation of data is performed by Delaunay triangulation method such that data $F_i$ fall on vertices $\{V_i = (x_i, y_i), i = 1, 2, 3, \ldots, n\}$ of the triangles.

4.2 Estimation of Derivatives

Partial derivatives at the vertices $V_i, i = 1, 2, 3$ of each triangle are calculated by derivative estimation scheme suggested by Goodman et al. [15]

4.3 $C^1$ Positive triangular patch

Let $V_1 V_2 V_3$ be the given triangle with edges $e_i, i = 1, 2, 3$ opposite to the vertices $V_i, i = 1, 2, 3$ respectively and $S_i, i = 1, 2, 3$ be the points on the edges opposite to vertices $V_i, i = 1, 2, 3$. The radial curve $Q_1$ connecting vertex $V_1$ to the points $S_1$ on the opposite edges $e_1$ is defined as (Fig 1):

$$Q_1 = \frac{Q_{1n}}{Q_{1d}}, \quad (7)$$

where

$$Q_{1n} = (1 - \sin \hat{\lambda})^2 F_1 x_1 + \sin \hat{\lambda}(1 - \sin \hat{\lambda})^2 \left(\beta_1 F_1 + \frac{2R_1}{\pi} \delta_1 \right) + \cos \hat{\lambda}(1 - \cos \hat{\lambda})^2 \left(\gamma_1 F(S_1) - \frac{2\delta_1}{\pi} R_1 \right) + (1 - \cos \hat{\lambda})^3 \delta_1 F(S_1),$$

$$Q_{1d} = \beta_1 (1 - \sin \hat{\lambda})^2 + \beta_1 \sin \hat{\lambda}(1 - \sin \hat{\lambda})^2 + \gamma_1 \cos \hat{\lambda}(1 - \cos \hat{\lambda})^2 + \delta_1 (1 - \cos \hat{\lambda})^3.$$ 

such that

$$\hat{\lambda} = \frac{\pi}{2} (1 - u), \quad \hat{\lambda} = 1 - \lambda.$$ 

$R_1$ and $R_2$ are the directional derivatives at $V_1$ and $S_1$ (Fig 2) defined as

$$R_1 = (x_n - x_1) \frac{\partial f}{\partial x}(V_1) + (y_n - y_1) \frac{\partial f}{\partial y}(V_1),$$

$$R_2 = (x_n - x_1) \frac{\partial f}{\partial x}(S_1) + (y_n - y_1) \frac{\partial f}{\partial y}(S_1).$$

and $F(S_1)$ is the boundary curve along the edge $e_1$ evaluated from the following expression

$$F(S_1) = \frac{F_{1n}}{F_{1d}}.$$
Fig 1. Radial curve $Q_1$: connecting vertex $V_1$ to the point $S_1$.

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Fig 2. Directional derivatives along $S_1 V_1$.

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where

\[
F_{1n} = (1 - \sin r)^3 F_1 x_4 + \sin r (1 - \sin r)^2 \left( \beta_4 x_4 + \frac{2d_3 x_4}{\pi} \right) + \cos \hat{r} (1 - \cos \hat{r})^2 \left( \gamma_4 F_3 - \frac{2\delta_4 d_4}{\pi} \right) + (1 - \cos \hat{r})^3 \delta_4 F_4,
\]

\[
F_{1d} = x_4 (1 - \sin r)^3 + \beta_4 \sin r (1 - \sin r)^2 + \gamma_4 \cos \hat{r} (1 - \cos \hat{r})^2 + \delta_4 (1 - \cos \hat{r})^3.
\]

such that

\[
r = \frac{\pi(1 - v)}{2(v + w)}, \quad \hat{r} = \frac{\pi(1 - w)}{2(u + w)}.
\]

\(d_3\) and \(d_4\) are the directional derivatives along \(V_2 \vec{V}_3\) at \(V_2\) and \(V_3\) (Fig 3)

\[d_3 = (x_3 - x_2) \frac{\partial f}{\partial x}(V_2) + (y_3 - y_2) \frac{\partial f}{\partial y}(V_2),\]

\[d_4 = (x_3 - x_2) \frac{\partial f}{\partial x}(V_3) + (y_3 - y_2) \frac{\partial f}{\partial y}(V_3).\]
From Eq 7, \( Q_1 > 0 \) if

\[
Q_{1n} > 0 \text{ and } Q_{1d} > 0.
\]

Now, \( Q_{1n} > 0 \) if

\[
z_1 > 0, \; \delta_1 > 0, \\
\beta_1 > \frac{-2R_1z_1}{\pi F_1}, \\
\gamma_1 > \frac{2R_1\delta_1}{\pi F(S_1)}, \\
F(S_1) > 0.
\] (8)

From Theorem 2.1, \( F(S_1) > 0 \) if

\[
\beta_1 > \frac{-2z_1d_1}{\pi F_2}, \; \gamma_1 > \frac{2\delta_1d_1}{\pi F_3}.
\] (9)

Now, \( Q_{1d} > 0 \) if

\[
z_1 > 0, \beta_1 > 0, \gamma_1 > 0 \text{ and } \delta_1 > 0.
\]

Likewise, radial curve \( Q_2 \) connecting vertex \( V_2 \) to the points \( S_2 \) on the opposite edges \( e_2 \) is defined as

\[
Q_2 = \frac{Q_{2n}}{Q_{2d}}
\] (10)

where

\[
Q_{2n} = (1 - \sin \mu)^3F_2z_2 + \sin \mu(1 - \sin \mu)^2 \left( \beta_2F_2 + \frac{2R_1z_1}{\pi} \right) \\
+ \cos \tilde{\mu}(1 - \cos \tilde{\mu})^2 \left( \gamma_2F(S_2) - \frac{2\delta_1R_1}{\pi} \right) + (1 - \cos \tilde{\mu})^3 \delta_2F(S_2), \\
Q_{2d} = z_2(1 - \sin \mu)^3 + \beta_2\sin \mu(1 - \sin \mu)^2 + \gamma_2\cos \tilde{\mu}(1 - \cos \tilde{\mu})^2 + \delta_2(1 - \cos \tilde{\mu})^3.
\]

such that

\[
\mu = \frac{\pi}{2}(1 - v), \; \tilde{\mu} = 1 - \mu.
\] (11)

\( R_3 \) and \( R_4 \) are the directional derivatives at \( V_2 \) and \( S_2 \)

\[
R_3 = (x_2 - x_2) \frac{\partial f}{\partial x}(V_2) + (y_2 - y_2) \frac{\partial f}{\partial y}(V_2) \\
R_4 = (x_2 - x_2) \frac{\partial f}{\partial x}(S_2) + (y_2 - y_2) \frac{\partial f}{\partial y}(S_2)
\]

and \( F(S_2) \) is the boundary curve along the edge \( e_2 \) to be evaluated from the following expression

\[
F(S_2) = \frac{F_{2n}}{F_{2d}}.
\]
where

\[
F_{2n} = (1 - \sin s)^3 F_2 \alpha_2 + \sin r (1 - \sin r)^2 \left( \beta_2 F_3 + \frac{2d_3 \alpha_3}{\pi} \right)
\]

\[
+ \cos r (1 - \cos r)^2 \left( \gamma_2 F_1 - \frac{2\delta_2 d_2}{\pi} \right) + (1 - \cos r)^3 \delta_2 F_1,
\]

\[
F_{3d} = \alpha_3 (1 - \sin r)^3 + \beta_3 \sin r (1 - \sin r)^2 + \gamma_3 \cos r (1 - \cos r)^2 + \delta_3 (1 - \cos r)^3.
\]

d_5 and d_6 are the directional derivatives along \( V_1 V_3 \) at \( V_1 \) and \( V_3 \):

\[
d_5 = \frac{\partial \alpha_1}{\partial x}(V_1) + \frac{\partial \alpha_2}{\partial y}(V_1),
\]

\[
d_6 = \frac{\partial \alpha_3}{\partial x}(V_3) + \frac{\partial \alpha_2}{\partial y}(V_3).
\]

From Eq 10, \( Q_2 > 0 \) if

\[
Q_{2n} > 0 \text{ and } Q_{2d} > 0.
\]

Now, \( Q_{2n} > 0 \) if

\[
\alpha_2 > 0, \delta_2 > 0,
\]

\[
\beta_2 > \frac{-2R_2 x_2}{\pi F_2},
\]

\[
\gamma_2 > \frac{2R_2 \delta_2}{\pi F_1},
\]

\[
F(S_2) > 0.
\]

From Theorem 2.1, \( F(S_2) > 0 \) if

\[
\beta_5 > \frac{-2x_3 d_4}{\pi F_3}, \gamma_5 > \frac{2\delta_5 d_6}{\pi F_1}.\]

Now, \( Q_{2d} > 0 \) if

\[
\alpha_2 > 0, \beta_2 > 0, \gamma_2 > 0 \text{ and } \delta_2 > 0.
\]

and the radial curve \( Q_3 \) connecting vertex \( V_3 \) to the point \( S_3 \) on the opposite edge \( e_3 \) is defined as

\[
Q_3 = \frac{Q_{2n}}{Q_{2d}},
\]

where

\[
Q_{2n} = (1 - \sin v)^3 F_2 \alpha_3 + \sin v (1 - \sin v)^2 \left( \beta_3 F_3 + \frac{2R_3 x_3}{\pi} \right)
\]

\[
+ \cos v (1 - \cos v)^2 \left( \gamma_3 F(S_3) - \frac{2\delta_3 R_3}{\pi} \right) + (1 - \cos v)^3 \delta_3 F(S_3),
\]

\[
Q_{3d} = \alpha_3 (1 - \sin v)^3 + \beta_3 \sin v (1 - \sin v)^2 + \gamma_3 \cos v (1 - \cos v)^2 + \delta_3 (1 - \cos v)^3.
\]
where $R_5$ and $R_6$ are the directional derivatives at $V_3$ and $S_3$ defined as

$$R_5 = (x_{i_3} - x_{i_3}) \frac{\partial f}{\partial x}(V_3) + (y_{i_3} - y_{i_3}) \frac{\partial f}{\partial y}(V_3)$$

$$R_6 = (x_{i_3} - x_{i_3}) \frac{\partial f}{\partial x}(S_3) + (y_{i_3} - y_{i_3}) \frac{\partial f}{\partial y}(S_3)$$

and $F(S_3)$ is the boundary curve along the edge $e_3$ to be evaluated from the following expression

$$F(S_3) = \frac{F_{3n}}{F_{3d}}.$$

where

$$F_{3n} = (1 - \sin t)^3 F_1 x_6 + \sin t(1 - \sin t)^2 \left( \beta_6 F_1 + \frac{2d_6 z_6}{\pi} \right)$$

$$+ \cos t(1 - \cos t)^2 \left( \gamma_6 F_2 - \frac{2\delta_6 d_2}{\pi} \right) + (1 - \cos t)^3 \delta_6 F_2,$$

$$F_{3d} = x_6(1 - \sin t)^3 + \beta_6 \sin t(1 - \sin t)^2 + \gamma_6 \cos t(1 - \cos t)^2 + \delta_6 (1 - \cos t)^2.$$

$d_1$ and $d_2$ are the directional derivatives along $V_1 V_2$ at $V_1$ and $V_2$

$$d_1 = (x_2 - x_1) \frac{\partial f}{\partial x}(V_1) + (y_2 - y_1) \frac{\partial f}{\partial y}(V_1),$$

$$d_2 = (x_2 - x_1) \frac{\partial f}{\partial x}(V_2) + (y_2 - y_1) \frac{\partial f}{\partial y}(V_2).$$

From Eq 14, $Q_3 > 0$ if

$$Q_{3n} > 0 \text{ and } Q_{3d} > 0.$$

Now, $Q_{3n} > 0$ if

$$x_3 > 0, \delta_2 > 0,$$

$$\beta_3 > -\frac{2R_3 x_3}{\pi F_3},$$

$$\gamma_3 > \frac{2R_3 \delta_3}{\pi F(S_3)},$$

$$F(S_3) > 0.$$ (15)

From Theorem 2.1, $F(S_3) > 0$ if

$$\beta_6 > -\frac{2x_6 d_1}{\pi F_1}, \gamma_6 > \frac{2\delta_6 d_2}{\pi F_2}.$$ (16)

Now, $Q_{3d} > 0$ if

$$x_3 > 0, \beta_3 > 0, \gamma_3 > 0 \text{ and } \delta_3 > 0.$$

The above discussion leads to the following result:
Theorem 4.1 The $C^1$ triangular patch $P$ in Eq (4) is positive if the following conditions are attained.

\[
\begin{align*}
\alpha_1 > 0, \alpha_2 > 0, \alpha_3 > 0, \alpha_4 > 0, \alpha_5 > 0, \alpha_6 > 0, \\
\delta_1 > 0, \delta_2 > 0, \delta_3 > 0, \delta_4 > 0, \delta_5 > 0, \delta_6 > 0, \\
\beta_1 > \max\left\{0, \frac{-2R_1\alpha_1}{\pi F_1}\right\}, \gamma_1 > \max\left\{0, \frac{2R_1\delta_1}{\pi F(S_1)}\right\}, \\
\beta_2 > \max\left\{0, \frac{-2R_1\alpha_2}{\pi F_2}\right\}, \gamma_2 > \max\left\{0, \frac{2R_1\delta_2}{\pi F(S_2)}\right\}, \\
\beta_3 > \max\left\{0, \frac{-2R_1\alpha_3}{\pi F_3}\right\}, \gamma_3 > \max\left\{0, \frac{2R_1\delta_3}{\pi F(S_3)}\right\}, \\
\beta_4 > \max\left\{0, \frac{-2R_1\alpha_4}{\pi F_4}\right\}, \gamma_4 > \max\left\{0, \frac{2\delta_4 d_4}{\pi F_1}\right\}, \\
\beta_5 > \max\left\{0, \frac{-2R_1\alpha_5}{\pi F_5}\right\}, \gamma_5 > \max\left\{0, \frac{2\delta_5 d_5}{\pi F_1}\right\}, \\
\beta_6 > \max\left\{0, \frac{-2R_1\alpha_6}{\pi F_6}\right\}, \gamma_6 > \max\left\{0, \frac{2\delta_6 d_6}{\pi F_2}\right\}.
\end{align*}
\]

The above constraints can be rearranged as

\[
\begin{align*}
\beta_1 > l_1 + \max\left\{0, \frac{-2R_1\alpha_1}{\pi F_1}\right\}, \gamma_1 > m_1 + \max\left\{0, \frac{2R_1\delta_1}{\pi F(S_1)}\right\}, l_1, m_1 > 0, \\
\beta_2 > l_2 + \max\left\{0, \frac{-2R_1\alpha_2}{\pi F_2}\right\}, \gamma_2 > m_2 + \max\left\{0, \frac{2R_1\delta_2}{\pi F(S_2)}\right\}, l_2, m_2 > 0, \\
\beta_3 > l_3 + \max\left\{0, \frac{-2R_1\alpha_3}{\pi F_3}\right\}, \gamma_3 > m_3 + \max\left\{0, \frac{2R_1\delta_3}{\pi F(S_3)}\right\}, l_3, m_3 > 0, \\
\beta_4 > l_4 + \max\left\{0, \frac{-2R_1\alpha_4}{\pi F_4}\right\}, \gamma_4 > m_4 + \max\left\{0, \frac{2\delta_4 d_4}{\pi F_1}\right\}, l_4, m_4 > 0, \\
\beta_5 > l_5 + \max\left\{0, \frac{-2R_1\alpha_5}{\pi F_5}\right\}, \gamma_5 > m_5 + \max\left\{0, \frac{2\delta_5 d_5}{\pi F_1}\right\}, l_5, m_5 > 0, \\
\beta_6 > l_6 + \max\left\{0, \frac{-2R_1\alpha_6}{\pi F_6}\right\}, \gamma_6 > m_6 + \max\left\{0, \frac{2\delta_6 d_6}{\pi F_2}\right\}, l_6, m_6 > 0.
\end{align*}
\]

5. Numerical Examples

This section illustrates the positivity preserving scheme for scattered data devised in Section 4.3.

Example 5.1 Positive scattered data is taken in Table 1. Fig 4 represents corresponding delaunay triangulations. The data is interpolated first by Eq (4) for arbitrary values of free parameters, $\alpha_1 = 4.1, \alpha_2 = 3, \alpha_3 = 2.5, \alpha_4 = 1.6, \alpha_5 = 2.7, \alpha_6 = 2.8, \beta_1 = 3.8, \beta_2 = 2.4, \beta_3 = 4.2, \beta_4 = 2.5, \beta_5 = 1.5, \beta_6 = 4, \gamma_1 = 1, \gamma_2 = 6, \gamma_3 = 1, \gamma_4 = 2, \gamma_5 = 2, \gamma_6 = 3, \delta_1 = 1, \delta_2 = 3, \delta_3 = 3, \delta_4 = 1, \delta_5 = 2, \delta_6 = 1$. The resulting surface is displayed in Fig 5. It is clear from Fig 5 that the inherent shape feature
Table 1. A Positive scattered data set I.

| x       | y       | F       |
|---------|---------|---------|
| 0       | 0       | 0.7487  |
| 0       | 0.125   | 0.5779  |
| 0       | 0.25    | 0.4668  |
| 0       | 0.375   | 0.4042  |
| 0       | 0.625   | 0.4042  |
| 0       | 0.75    | 0.4668  |
| 0       | 0.875   | 0.5779  |
| 0       | 1       | 0.7487  |
| 0.125   | 0       | 0.5779  |
| 0.125   | 0.125   | 0.4248  |
| 0.125   | 0.5     | 0.251   |
| 0.125   | 0.625   | 0.2691  |
| 0.125   | 0.75    | 0.3252  |
| 0.125   | 1       | 0.5779  |
| 0.25    | 0       | 0.4668  |
| 0.25    | 0.125   | 0.3252  |
| 0.25    | 0.25    | 0.2331  |
| 0.25    | 0.375   | 0.1813  |
| 0.25    | 0.5     | 0.1645  |
| 0.25    | 0.875   | 0.3252  |
| 0.375   | 0.125   | 0.2691  |
| 0.375   | 0.25    | 0.1813  |
| 0.375   | 0.625   | 0.1317  |
| 0.375   | 0.75    | 0.1813  |
| 0.375   | 0.875   | 0.2691  |
| 0.375   | 1       | 0.4042  |
| 0.5     | 0       | 0.384   |
| 0.5     | 0.375   | 0.1157  |
| 0.5     | 0.625   | 0.1157  |
| 0.5     | 0.75    | 0.1645  |
| 0.5     | 0.875   | 0.251   |
| 0.5     | 1       | 0.384   |
| 0.625   | 0       | 0.4042  |
| 0.625   | 0.125   | 0.2691  |
| 0.625   | 0.375   | 0.1317  |
| 0.625   | 0.5     | 0.1157  |
| 0.625   | 0.625   | 0.1317  |
| 0.75    | 0       | 0.4668  |
| 0.75    | 0.125   | 0.3252  |
| 0.75    | 0.375   | 0.1813  |
| 0.75    | 0.75    | 0.2331  |
| 0.75    | 0.875   | 0.3252  |
| x       | y       | z       |
| 0.875   | 0       | 0.5779  |
| 0.875   | 0.125   | 0.4248  |
| 0.875   | 0.375   | 0.2691  |
| 0.875   | 0.625   | 0.2691  |

(Continued)
Table 1. (Continued)

| x    | y    | F     |
|------|------|-------|
| 0.875| 0.75 | 0.3252|
| 0.875| 0.875| 0.4248|
| 0.875| 1    | 0.5779|
| 1    | 0    | 0.7487|
| 1    | 0.125| 0.7779|
| 1    | 0.25 | 0.4668|
| 1    | 0.375| 0.5042|
| 1    | 0.5  | 0.384 |
| 1    | 0.625| 0.4042|
| 1    | 0.75 | 0.4668|
| 1    | 0.875| 0.5779|
| 1    | 1    | 0.7487|
| 0.75 | 1    | 0.4668|

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Fig 4. Delaunay triangulation of positive data in Table 1.
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of positivity of data could not be held in visual model. This detriment is removed in Figs 6, 7 and 8 by implementing positivity preserving conditions summarized in Theorem 4.1. It is worth mentioning here that parameters $\alpha_i$ and $\delta_i$ for $i = 1, 2, \ldots, 6$ are left free to refine the shape according to user's requirement. The effect of free parameters are shown in Figs 6, 7 and 8. Figs 6 and 7 are constructed against the parameter choice $\alpha_1 = 12, \alpha_2 = 0.4, \alpha_3 = 13, \alpha_4 = 0.22, \alpha_5 = 12, \alpha_6 = 0.33, \delta_1 = 13, \delta_2 = 0.3, \delta_3 = 14, \delta_4 = 0.3, \delta_5 = 15, \delta_6 = 12$ and $\alpha_1 = 0.1, \alpha_2 = 1.0, \alpha_3 = 0.5, \alpha_4 = 1.6, \alpha_5 = 0.7, \alpha_6 = 0.8, \delta_1 = 0.8, \delta_2 = 0.4, \delta_3 = 1.2, \delta_4 = 1.5, \delta_5 = 1.5, \delta_6 = 1.0$ respectively, which lacks smoothness. A smooth visibly pleasant representation is obtained in Fig 8 by setting $\alpha_1 = 1.0, \alpha_2 = 1.0, \alpha_3 = 0.5, \alpha_4 = 0.6, \alpha_5 = 0.7, \alpha_6 = 0.8, \delta_1 = 0.8, \delta_2 = 0.4, \delta_3 = 1.2, \delta_4 = 0.5, \delta_5 = 1.0, \delta_6 = 1.0$.

Example 5.2 A Positive scattered data set is displayed in Table 2. Delauny triangulation is illustrated in Fig 9 and the corresponding surface in Fig 10 is obtained by interpolating the data.
for arbitrary values of free parameters, $\alpha_1 = 4.1, \alpha_2 = 3, \alpha_3 = 2.0, \alpha_4 = 1.5, \alpha_5 = 2.7, \alpha_6 = 2.5, \beta_1 = 3.5, \beta_2 = 2.3, \beta_3 = 3.2, \beta_4 = 2.2, \beta_5 = 1.0, \beta_6 = 4.5, \gamma_1 = 1.4, \gamma_2 = 5.5, \gamma_3 = 1.5, \gamma_4 = 2.2, \gamma_5 = 1.5, \gamma_6 = 2.5, \delta_1 = 0.5, \delta_2 = 3.1, \delta_3 = 3.5, \delta_4 = 0.4, \delta_5 = 2, \delta_6 = 1.2$, in description of Eq (4). It is evident from Fig 10 that the positivity of data could not be conserved in visual model. This impediment is removed in Figs 11, 12 and Fig 13 by implementing positivity preserving constraints on parameters $\beta_i, \gamma_i$ for $i = 1, 2, \ldots, 6$, summarized in Theorem 4.1. Here, it is noteworthy that parameters $\alpha_i$ and $\delta_i$ for $i = 1, 2, \ldots, 6$ are set free to refine the shape as required by the user. The effect of free parameters are shown in Figs 11, 12 and 13. Figs 11 and 12 are constructed against the parameter choice $\alpha_1 = 2.0, \alpha_2 = 0.1, \alpha_3 = 0.5, \alpha_4 = 0.5, \alpha_5 = 1.0, \alpha_6 = 0.63, \delta_1 = 1.0, \delta_2 = 0.33, \delta_3 = 0.5, \delta_4 = 0.4, \delta_5 = 1.0, \delta_6 = 0.3$ and $\alpha_1 = 2.2, \alpha_2 = 1.1, \alpha_3 = 2.5, \alpha_4 = 1.5, \alpha_5 = 1.0, \alpha_6 = 1.0, \delta_1 = 1.0, \delta_2 = 1.0, \delta_3 = 1.5, \delta_4 = 1.4, \delta_5 = 1.0, \delta_6 = 0.3$ respectively, which lacks smoothness. A smooth visibly pleasant representation is obtained in Fig 13 by setting $\alpha_1 = 2.0, \alpha_2 = 0.4, \alpha_3 = 0.5, \alpha_4 = 0.5, \alpha_5 = 1.0, \alpha_6 = 0.63, \delta_1 = 0.3, \delta_2 = 0.33, \delta_3 = 0.5, \delta_4 = 0.3, \delta_5 = 0.5, \delta_6 = 0.2$.  

Fig 6. Positive surface generated from Theorem 4.1 of the positive data in Table 1. doi:10.1371/journal.pone.0120658.g006
Conclusion

In this study, positivity preserving algorithm for scattered data arranged over a triangular domain, is established. The rational trigonometric cubic function [13] with four free parameters is used for the interpolation along each boundary and radial curve. Nielson side vertex has been applied to construct the interpolating surface. Constraints on half of the parameters are obtained to guarantee the positive shape of data while half are set free for users modification. The proposed algorithm, surpasses many prevailing approaches in literature. In [10], authors
Table 2. A Positive scattered data set II.

| x    | y    | F     |
|------|------|-------|
| 0    | 0    | 0.4486|
| 0    | 0.125| 0.3616|
| 0    | 0.25 | 0.4692|
| 0    | 0.375| 0.6827|
| 0    | 0.5  | 0.786 |
| 0    | 0.625| 0.836 |
| 0    | 0.75 | 0.8765|
| 0    | 0.875| 0.9125|
| 0    | 1    | 0.9447|
| 0.125| 0    | 0.3369|
| 0.125| 0.125| 0.0001|
| 0.125| 0.375| 0.6256|
| 0.125| 0.625| 0.8621|

(Continued)
Table 2. (Continued)

| x     | y     | F     |
|-------|-------|-------|
| 0.125 | 0.875 | 0.9334 |
| 0.125 | 1     | 0.9634 |
| 0.25  | 0     | 0.4529 |
| 0.25  | 0.125 | 0.1767 |
| 0.25  | 0.25  | 0.3217 |
| 0.25  | 0.375 | 0.7005 |
| 0.25  | 0.5   | 0.8555 |
| 0.25  | 0.625 | 0.9327 |
| 0.25  | 0.75  | 0.9775 |
| 0.25  | 0.875 | 0.9686 |
| 0.25  | 1     | 0.9926 |
| 0.375 | 0     | 0.696  |
| 0.375 | 0.375 | 0.8363 |
| 0.375 | 0.625 | 1.2176 |
| 0.375 | 0.875 | 1.028  |
| 0.375 | 1     | 1.0284 |
| 0.5   | 0     | 0.8329 |
| 0.5   | 0.125 | 0.8315 |
| 0.5   | 0.25  | 0.821  |
| 0.5   | 0.375 | 0.8498 |
| 0.5   | 0.5   | 0.925  |
| 0.5   | 0.625 | 1.0925 |
| 0.5   | 0.75  | 1.1688 |
| 0.5   | 0.875 | 1.0568 |
| 0.5   | 1     | 1.0662 |
| 0.625 | 0     | 0.9049 |
| 0.625 | 0.125 | 0.8376 |
| 0.625 | 0.375 | 0.7163 |
| 0.625 | 0.5   | 0.8608 |
| 0.625 | 0.75  | 1.0671 |
| 0.625 | 0.875 | 1.0883 |
| 0.625 | 1     | 1.1023 |
| 0.75  | 0     | 0.9639 |
| 0.75  | 0.125 | 0.8326 |
| 0.75  | 0.25  | 0.6283 |
| 0.75  | 0.375 | 0.5976 |
| 0.75  | 0.5   | 0.8075 |
| 0.75  | 0.625 | 1.0136 |
| 0.75  | 0.75  | 1.0989 |
| 0.75  | 0.875 | 1.1231 |
| 0.75  | 1     | 1.134  |
| 0.875 | 0     | 1.0355 |
| 0.875 | 0.125 | 0.922  |
| 0.875 | 0.25  | 0.7477 |
| 0.875 | 0.375 | 0.7193 |
| 0.875 | 0.5   | 0.893  |
| 0.875 | 0.625 | 1.0638 |

(Continued)
### Table 2. (Continued)

| x     | y     | F   |
|-------|-------|-----|
| 0.875 | 0.75  | 1.1335 |
| 0.875 | 0.875 | 1.152 |
| 0.875 | 1     | 1.1597 |
| 1     | 0     | 1.1074 |
| 1     | 0.125 | 1.0598 |
| 1     | 0.25  | 0.9848 |
| 1     | 0.375 | 0.9745 |
| 1     | 0.5   | 1.054 |
| 1     | 0.625 | 1.1319 |
| 1     | 0.75  | 1.1646 |
| 1     | 0.875 | 1.1744 |
| 1     | 1     | 1.1791 |

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**Fig 9.** Delaunay triangulation of positive data in Table 2.

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utilized a cubic function with one free parameter to retain the positive shape of data. Positive surface was obtained by drawing data dependent constraints on this free parameter, and, hence the scheme did not offer refinement in the shape. The scheme suggested in this paper does not suffer this detriment. The developed algorithm is local and can be applied to data with or without derivatives. Moreover, shape preserving algorithms play an instrumental role in many areas of visualization such as geometric modelling, robot trajectories, evolution game theory, prisoner’s dilemma game [16], [17], [18], meshless method and inverse kinematics etc.
Fig 11. Positive surface generated from Theorem 4.1.

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Fig 12. Positive surface generated from Theorem 4.1.

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Author Contributions
Conceived and designed the experiments: FI MZH AB. Performed the experiments: FI MZH AB. Analyzed the data: FI MZH AB. Contributed reagents/materials/analysis tools: FI MZH AB. Wrote the paper: FI MZH AB.

References
1. Sarfraz M, Hussain MZ, Afan M. Positivity preserving scattered data interpolation scheme using the side vertex method. Applied Mathematics and Computation. 2012; 218(15):7898–7910. doi: 10.1016/j.amc.2012.01.072
2. Franke R, Nielson GM. Smooth interpolation of larger sets of scattered data. International Journal of Numerical Methods in Engineering. 1980; 15:1691–1704. doi: 10.1002/nme.1620151110
3. Kraus V, Dehmer M, Emmert-Streib F. Probabilistic inequalities for evaluating structural network measures. Information Sciences. 2014; 288:220–245. doi: 10.1016/j.ins.2014.07.018
4. Dehmer M, Emmert-Streib F, Shi Y. Interrelations of graph distance measures based on topological indices. PLoS ONE. 2014; 9.
5. Cao S, Dehmer M, Shi Y. Extremality of degree-based graph entropies. Information Sciences. 2014; 278:22–33. doi: 10.1016/j.ins.2014.03.133
6. Gupta MK, Niyogi R, Misra M. A 2D Graphical Representation of Protein Sequence and Their Similarity Analysis with Probabilistic Method. MATCH Communications in Mathematical and in Computer Chemistry. 2014; 72:519–532.
7. Amidor I. Scattered data interpolation methods for electronic imaging systems: survey. Journal of Electronic Imaging. 2002; 11(2):157–176. doi: 10.1117/1.1455013

8. Dougherty RL, Edelman Z, Hyman JM. Non-negativity, monotonicity or convexity preserving cubic and quintic Hermite interpolation. Mathematics of Computation. 1989; 52(186):471–494. doi: 10.1090/S0025-5718-1989-0962209-1

9. Piah ARM, Goodman TNT, Unsworth K. Positivity preserving scattered data interpolation. In: Martin R, Bez H, Sabin M, editors. Proceeding of Mathematics of Surfaces, Lecture Notes in Computer Science, 3604. New York: Springer-Verlag Berlin Heidelberg; 2005. p. 336–349.

10. Hussain MZ, Hussain M. Shape preserving scattered data interpolation. European Journal of Scientific Research. 2009; 25(1):151–164.

11. Hermann M, Mulansky B, Schmidt JW. Scattered data interpolation subject to piecewise quadratic range restriction. Journal of Computational and Applied Mathematics. 1996; 73:209–223. doi: 10.1016/0377-0427(96)00044-1

12. Hussain MZ, Hussain M. $C^1$ positivity preserving scattered data interpolation using Bernstein Bezier triangular patch. Journal of Applied Mathematics and Computing. 2011; 35:281–293. doi: 10.1007/s12190-009-0356-0

13. Ibraheem F, Hussain M, Hussain MZ, Bhatti AA. Data visualization using Trigonometric function. Journal of Applied Mathematics. 2011; 2012:1–19. doi: 10.1155/2012/247120

14. Nielson GM. The side-vertex method for interpolation in triangles. Journal of Approximation Theory. 1979; 25:316–336.

15. Goodman TNT, Said HB, Chang LHT. Local derivative estimation for scattered data interpolation. Applied Mathematics and Computation. 1995; 68:41–50. doi: 10.1016/0096-3003(94)00086-J

16. Davidson KR. The Evolution of Cooperation. Newyork: Basic Books; 2006.

17. Ma ZQ, Xia CY, Sun SW, Wang L, Wang HB, Wang J. Heterogeneous link weight promotes the cooperation in spatial prisoner’s dilemma. International Journal of Modern Physics C. 2011; 22:1257–1258. doi: 10.1142/S0129183111016877

18. Juai W, Yi XC, Yiling W, Shuai D, JunQing S. Spatial prisoner’s dilemma games with increasing size of the interaction neighborhood on regular lattices. Chinese Science Bulletin. 2012; 57:724–728. doi: 10.1007/s11434-011-4890-4