Bohmian mechanics and Fisher information for $q$-deformed Schrödinger equation

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Abstract

We discuss the Bohmian mechanics by means of the deformed Schrödinger equation for position dependent mass, in the context of a $q$-algebra inspired by nonextensive statistics. A derivation of the Bohmian quantum formalism is performed by means of a deformed Fisher information functional, from which a deformed Cramér-Rao bound is derived. Lagrangian and Hamiltonian formulations, inherited by the $q$-algebra, are also developed. Then, we illustrate the results with a particle confined in an infinite square potential well. The preservation of the deformed Cramér-Rao bound for the stationary states plays the role played by the $q$-algebraic structure.

Keywords: $q$-deformed Schrödinger equation, Bohmian mechanics, Fisher information, Cramér-Rao bound

PACS: 03.65.Ca, 03.67.-a, 05.90.+m

1. Introduction

Along several decades it has been shown that fundamental disciplines can be treated as theories of inference, where the available information about the system allows one to derive the dynamics from making use of probability theory. Between the most important methods of inference the maximum entropy one is found, where a rule is given (typically, the maximization of a functional $S(\rho)$) for obtaining the distribution $\rho$ that represents the best knowledge of the system constrained by the available information [1]. When $S(\rho)$ is chosen to be the Shannon-Gibbs entropy then the so-called MaxEnt method results. In particular, a functional of interest is the Fisher information (FI) $I_F(\rho)$, which measures the information of an observable variable of the estimated parameters. The FI can be used to derive the quantum and relativistic mechanics by means of variational principles, where the constraints contain the physics [2].

In this context, an interesting application of MaxEnt and FI is the deduction of the Bohmian quantum formalism [3,4], which was introduced by Bohm [5] as an alternative interpretation of the quantum mechanics using the idea of the de Broglie pilot wave [6]. Plastino et al. at [7] studied Hamiltonians with a position-dependent effective mass, which are widely used in many areas, both experimentally and theoretically: semiconductors [8], quantum dots [9], many body theory [10], superintegrable systems [12], quantum liquids [13], inversion potential for NH$_3$ [14], astrophysics [15], nonlinear optics [16], relativistic quantum mechanics [17], nuclear physics [18], etc.

In the mathematical description of quantum systems with position-dependent effective mass, the mass operator $m(\vec{x})$ and the linear momentum $\vec{p}$ are not commutating. A general form for the Hermitian kinetic energy operator has been suggested by O. von Roos [8] which characterizes the most of those used in the literature. The ordering problem of the kinetic energy operator has been investigated by BenDaniel and Duke [19], Gora and Williams [20], Zhu and Kroemer [21], Li and Kuhn [22]. Recently, a $q$-deformed Schrödinger equation inspired in nonextensive statistics, in this paper we discuss a $q$-deformed Bohmian quantum theory associated with the $q$-deformed Schrödinger equation, along with the corresponding Lagrangian and Hamiltonian formulations. Also, we derive a Cramér-Rao bound associated with the FI proposed.

The paper is organized as follows. In Section 2 we review the deformed Schrödinger equation for position dependent effective mass. Section 3 is devoted to a deformed Bohmian quantum theory based on the deformed Schrödinger equation and using the de Broglie wave-pilot interpretation. Next, in Section 4 we present a deformed Fisher functional for a position-dependent mass $m(\vec{x})$ and applying the variational principle to a $q$-deformed version of the FI inspired in nonextensive statistics, in this paper we discuss a $q$-deformed Bohmian quantum theory associated with the $q$-deformed Schrödinger equation, along with the corresponding Lagrangian and Hamiltonian formulations. Also, we derive a Cramér-Rao bound associated with the FI proposed.

The paper is organized as follows. In Section 2 we review the deformed Schrödinger equation for position dependent effective mass. Section 3 is devoted to a deformed Bohmian quantum theory based on the deformed Schrödinger equation and using the de Broglie wave-pilot interpretation. Next, in Section 4 we present a deformed Fisher functional for a position-dependent mass system. Here we deduce a Cramér-Rao bound associated with the deformed FI. Then, in Section 5 we illustrate the results with a particle confined in an infinite square potential well. For comparing, we calculate the deformed Cramér-Rao bound and the standard one for some stationary states. Finally, in Section 6 we draw some conclusions and future directions are outlined.

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Preprint submitted to Elsevier

September 3, 2018
2. Review of the \(q\)-deformed Schrödinger equation for position-dependent mass

Nonextensive statistical mechanics constitutes a formalism of wide applicability in several areas of physics \([28,29]\). The mathematical background of this approach is based on the generalized functions: the \(q\)-exponential, \(\exp_q(a)\equiv [1 + (1 - q)u]^\frac{1}{q-1}\), with \([A]\_q = \max(A,0)\), where the parameter \(q\) (the so-called entropic index) has been found to have several physical interpretations \([29]\). Considering the \(q\)-addition and \(q\)-subtraction operators defined respectively by \(a \circ_q b = a + b + (1 - q)ab\) and \(a \circ_q b = \frac{a^{1-q}b + b^{1-q}a}{1-q}\) (\(b \neq \frac{1}{q-1}\)), the \(q\)-exponential satisfies \(\exp_q(a)\exp_q(b) = \exp_q(a \circ_q b)\) and \(\exp_q(a)/\exp_q(b) = \exp_q(a/b)\). Moreover, as proposed in \([27]\), from the definition of the \(q\)-deformed infinitesimal element

\[
d_u = \lim_{u' \to u} u' \Theta_u = \frac{du}{1 + (1 - q)u},
\]

one can define the \(q\)-deformed derivative operator

\[
D_q f(u) = \lim_{u' \to u} \frac{f(u') - f(u)}{u' \Theta_u} = [1 + (1 - q)u] \frac{df}{du},
\]

and the \(q\)-deformed integral

\[
\int f(u) d_q u = \int f(u) \frac{du}{1 + (1 - q)u}.
\]

These operators satisfy the properties \(D_q \exp_q(a) = \exp_q(a)\) and \(\int \exp_q(a) d_q u = \exp_q(a)\) + constant.

Recently, Costa Filho et al. \([23,24,25,26]\) have introduced a generalized translation operator which produces nonadditive spatial displacements, i.e.,

\[
\hat{T}_\gamma(x) = |x + e + \gamma xe|
\]

where \(e\) is an infinitesimal displacement and \(\gamma\) is a parameter with dimension of inverse length whose physical role is as follows. If \(L_0\) is the characteristic volume of the system, then by defining \(\gamma_{L_0} \equiv (1 - q)/\xi\), where \(\xi\) is a characteristic length such that \(\gamma_{L_0} L_0 \sim 1\) (i.e., \(1 - q \sim \xi/L_0\)) then \(1 - q\) can be interpreted as a coupling measurement between \(\xi\) and \(L_0\). The lower the ratio \(\xi/L_0\), the closer the parameter \(\gamma\) should be to 1. Thus, the right hand side of \([4]\) can be identified as the \(q\)-addition \(\xi(x/\xi) \delta_q(x/\xi)\). The usual case is recovered for \(q \to 1\) (\(\gamma \to 0\)).

The operator \([4]\) leads to a generator operator of spatial translations corresponding to a position-dependent linear momentum, and consequently it represents a particle with position-dependent mass. More generally, an Hermitian generator operator of spatial translations was obtained in \([24,25]\), given by

\[
\hat{P}_q = \frac{\hat{\xi}(1 + \gamma_{\hat{q}} \hat{\xi})}{2} + \frac{\hat{\rho}(1 + \gamma_{\hat{q}} \hat{\xi})}{2} = \frac{(1 + \gamma_{\hat{q}} \hat{\xi})^{1/2}}{\gamma_{\hat{q}}} \hat{\rho}(1 + \gamma_{\hat{q}} \hat{\xi})^{1/2}.
\]

A canonically conjugated space operator for the deformed linear momentum operator is defined by

\[
\hat{\xi}_q = \frac{\ln(1 + \gamma_{\hat{q}} \hat{\xi})}{\gamma_{\hat{q}}} = \xi \ln(\exp_q(\xi/\xi)).
\]

Hence, \((\hat{\xi}_q, \hat{P}_q) \to (\hat{\xi}, \hat{P})\) constitutes a point canonical transformation (PCT) which maps a particle with constant mass \(m_0\) into another one with position-dependent mass. In fact, the Hamiltonian operator \(\hat{H}(\hat{\xi}_q, \hat{P}_q) = \frac{\hat{P}_q^2}{2m_q} + \hat{V}(\hat{\xi}_q)\) is mapped into \(\hat{\tilde{H}}(\hat{\xi}, \hat{\tilde{P}}) = \hat{T} + \hat{V}(\hat{\xi})\) whose the kinetic energy operator is

\[
\hat{T} = \frac{1}{2} [\hat{m}(\hat{\xi})]^{1/2} \hat{\rho} [\hat{m}(\hat{\xi})]^{-1/2} \hat{\rho} [\hat{m}(\hat{\xi})]^{-1/4},
\]

with

\[
m(x) = \frac{m_0}{(1 + \gamma_q x)^2}
\]

the effective mass, according to \([8]\). In consequence, the time-dependent \(\hat{q}\)-deformed Schrödinger equation for position-dependent mass in terms of wave function \(\Psi(x,t)\) is

\[
\left(\frac{i}{\hbar} \frac{d\Psi(x,t)}{dt}\right) - \frac{\hbar^2(1 + \gamma_q x)^2}{2m_0} \frac{\partial^2 \Psi(x,t)}{\partial x^2} - \frac{\hbar^2 \gamma_q (1 + \gamma_q x)^4}{8m_0} \frac{\partial \Psi(x,t)}{\partial x} + \frac{\hbar}{2m_0} \frac{\partial^2 \Psi(x,t)}{\partial x^2} + V(x)\Psi(x,t),
\]

(9)

The \(\Psi(x,t)\) can be adequately rewritten by means of a field \(\Phi_q(x,t)\) related to \(\Psi(x,t)\) by (see, for instance, \([30]\))

\[
\Psi(x,t) = \sqrt{\frac{m(x)}{m_0}} \Phi_q(x,t) = \frac{\Phi_q(x,t)}{\sqrt{1 + \gamma_q x}}.
\]

(10)

Thus, one obtains a \(q\)-deformed Schrödinger equation \([23,24]\):

\[
\left(\frac{i}{\hbar} \frac{d\Phi_q(x,t)}{dt}\right) - \frac{\hbar^2}{2m_0} \frac{\partial^2 \Phi_q(x,t)}{\partial x_q^2} + V(x)\Phi_q(x,t),
\]

(11)

where \(D_q \equiv (1 + \gamma_q x)\partial_x\) is a deformed derivative and

\[
\hat{H}^q = \frac{\hbar^2}{2m_0} \frac{\partial^2}{\partial x_q^2} + V(x).
\]

(12)

is a non-Hermitian operator.

Some remarks that deserve to be mentioned are the following. First, one can see that the Eq.’s \([9]\) and \([11]\) represent systems having the same energy spectrum (isospectral systems). That is, although operator \(\hat{H}^q\) is non-Hermitian, it has real energy eigenvalues (see \([31]\) for some details). Second, from Eq. \([10]\) the probability densities \(\rho(x,t) = |\Psi(x,t)|^2\) and the \(q\)-deformed one \(\rho_q(x,t) = |\Phi_q(x,t)|^2\) satisfy

\[
\rho(x,t) = \frac{\rho_q(x,t)}{1 + \gamma_q x}
\]

and \(\int_{-\infty}^{\infty} \rho(x,t) dx = \int_{-\infty}^{\infty} \rho_q(x,t) d_q x = 1\). That is, while the distribution probability is normalized by a standard integral, the \(q\)-deformed one is normalized by a \(q\)-deformed one.

By last, we emphasize that there is an equivalence between a Hermitian Hamiltonian system for position-dependent mass and a deformed non-Hermitian one in terms of the \(q\)-derivative, that results by replacing the field \(\Psi(x,t)\) by the deformed one \(\Phi_q(x,t)\). This may be understood as the effect of the position-dependent mass \([5]\) being imitated by a deformed derivative operator in the Schrödinger equation \([9]\).
3. Bohmian quantum theory for q-deformed Schrödinger equation

We present a deformed Bohmian theory for a position-dependent mass system, and we obtain the dynamics in the classical limit. Next, we explore a classical approach in the context of the Hamilton-Jacobi theory.

3.1. q-Deformed Bohmian quantum theory

In order to obtain a de Broglie–Bohm theory for the system with position-dependent mass, we use the field $\Phi_q(x,t)$. The same results can be obtained from the field $\Psi(x,t)$. Consider the field of the q-deformed Schrödinger equation expressed as a pilot wave, that is $\Phi_q(x,t) = \sqrt{\rho(x,t)}e^{iS_q(x,t)/\hbar}$, where $S_q(x,t)$ is a real phase that will be physically interpreted in the following. The q-deformed Hamilton-Jacobi equation in the quantum formalism:

$$\frac{1}{2\hbar^2}[D_q\psi(x,t)]^2 + V(x) = \frac{h^2}{2\hbar^2}D_x^2 \rho + \frac{\partial S_q}{\partial t} = 0,$$  

(13)

From the imaginary part of Eq. (14), we obtain a q-deformed continuity equation

$$\frac{\partial \rho_q(x,t)}{\partial t} + D_{\gamma_q}\mathcal{J}_q(x,t) = 0,$$  

(15)

where the deformed current density $\mathcal{J}_q(x,t)$ is defined by

$$\mathcal{J}_q(x,t) = \frac{\partial \rho_q(x,t)}{\partial t} + D_{\gamma_q}\mathcal{J}_q(x,t) = \frac{\partial \rho_q(x,t)}{\partial t} + D_{\gamma_q}\left[\frac{\partial \rho_q(x,t)}{\partial t} + \frac{\partial S_q}{\partial t}\right].$$  

(16)

Equivalently, from Eq. (14) one can obtain

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial J_q(x,t)}{\partial x} = 0,$$  

(17)

with $J_q(x,t) = J_t(x,t)$ expressed in terms of $\rho(x,t)$ or $\Psi(x,t)$ as

$$J_q(x,t) = \frac{\rho(x,t)}{m(x)} \frac{\partial S_q}{m(x)} = \frac{\rho(x,t)}{m(x)} \frac{\partial \rho(x,t)}{\partial \Psi(x,t)} = \frac{\rho(x,t)}{m(x)} \frac{\partial \rho(x,t)}{\partial \Psi(x,t)}.$$

(18)

As in the case of the classical theory, the spatial variation in the phase of the function varies in relation to the flow of probability: the more the phase changes, the more intense is the probability flow.

From the real part of Eq. (14), we obtain the following q-deformed Hamilton-Jacobi equation in the quantum formalism:

$$\frac{1}{2\hbar^2}[D_x \psi(x,t)]^2 + V(x) = \frac{h^2}{2\hbar^2}D_x^2 \rho + \frac{\partial S_q}{\partial t} = 0,$$  

(19)

where in terms of the mass function, we have

$$\frac{1}{2m(x)} \left( \frac{\partial S_q(x,t)}{\partial t} \right)^2 + V(x) = \frac{h^2}{2m(x)} \frac{\partial S_q}{\partial m(x)} + \frac{\partial S_q}{\partial t} = 0,$$  

(20)

with $Q_q(x,t)$ a deformed de Broglie-Bohm quantum potential given by

$$Q_q(x,t) = \frac{\hbar^2}{2m(x)} \frac{1}{D_x \psi(x,t)} \frac{\partial \psi(x,t)}{\partial D_x \psi(x,t)} = -\frac{\hbar^2}{2m(x)} \frac{1}{D_x \psi(x,t)} \frac{\partial \psi(x,t)}{\partial D_x \psi(x,t)} = \frac{h^2}{2m(x)} \left( \frac{1}{D_x \psi(x,t)} \right)^2 - \frac{1}{2m(x)} \frac{\partial^2 \psi(x,t)}{\partial D_x \psi(x,t)}.$$

(21)

From Eq. (13), the de Broglie-Bohm quantum potential in terms of the density probability $\rho(x,t)$ is

$$Q_q(x,t) = Q^{(1)}_q(x,t) + Q^{(2)}_q(x,t) + Q^{(3)}_q(x,t),$$  

(22)

in accordance to the Bohmian formulation proposed in [7]. The contributions $Q^{(1)}_q$ and $Q^{(2)}_q$ depend on the probability density $\rho$, whereas $Q^{(3)}_q$ remains independent.

3.2. Classical dynamics for system with position-dependent mass

Taking the classical limit $\hbar \to 0$ in (19), one obtains

$$\frac{1}{2m(x)} \left[ D_x \psi(x,t) \right]^2 + V(x) = \frac{\partial S_q(x,t)}{\partial t} = 0,$$  

(24)

corresponding to a q-deformed Hamilton-Jacobi equation in classical mechanics. Considering the separating of variables method, where $S_q(x,t) = W_q(x) - Et$, and $W_q(x)$ is a q-deformed Hamilton’s characteristic function, we have

$$W_q(x) = \pm \int^x \sqrt{2m(x')} [E - V(x')] dx'$$  

(25)

whose classical linear momentum is given by

$$p = \frac{\partial S_q(x,t)}{\partial x} = \frac{dW_q(x)}{dx} = \sqrt{2m(x)[E - V(x)]}.$$  

(26)

It follows that the deformed action $S_q(x,t)$ can be written as

$$S_q(x,t) = \pm \int^x \sqrt{2m(x')} [E - V(x')] dx' - Et$$  

(27)

Then,

$$\frac{dS_q(x,t)}{dt} = \frac{\partial S_q}{\partial x} \dot{x} + \frac{\partial S_q}{\partial t} = px - H = L,$$  

(28)
where we used (26) and \( H(x, \partial S/\partial x) + \partial S_q/\partial t = 0 \). It should be noted that \( S_q(x, t) \) coincides with the classical action. Thus, we have \( S_q = \int L(x, \dot{x}) dt \), with the Lagrangian function given by

\[
L(x, \dot{x}) = \frac{1}{2}m(x)\dot{x}^2 - V(x).
\]

Therefore, in the limit \( h \to 0 \) the classical mechanics for position-dependent mass system is recovered.

4. Fisher information for \( q \)-deformed Schrödinger equation

Considering the \( q \)-deformed Schrödinger equation, we apply the principle of minimum action to a deformed FI, from which we obtain the complete Bohm quantun potential. Kinetic energy operator for stationary states in the context of the Thomas-Fermi-Dirac theory, along with Lagrangian and Hamiltonian formulations are developed. Then, a deformed version of the Cramér-Rao bound is presented.

4.1. \( q \)-Deformed Fisher information and Bohm quantum potential

In a previous work [3], Reginatto derived the Bohmian quantum theory for systems with constant mass by mean of the principle of minimum Fisher information. Plastino et al. [2] extended the result for systems with position-dependent mass whose kinetic energy operator is \( \hat{\mathcal{T}} = \frac{\hat{p}^2}{2m(x)} \hat{\rho} \), which is different from the one given by Eq. (7). The authors obtained directly the quantum potential \( \mathcal{Q}_q(x, t) \) from the Fisher functional given by

\[
\mathcal{F}_q = \int_{\mathcal{H}} \mathcal{L}_q[\rho_q] d\mathcal{H},
\]

where the ordinary derivative and integral operators are replaced by the \( q \)-derivative and the \( q \)-integral respectively. From the equivalence between the Schrödinger equation for position-dependent mass and the \( q \)-deformed Schrödinger equation, one can relate the corresponding Fisher functionals (the proposed by Plastino and the deformed one). Using the Eq.’s (13) and (12), we have that the \( q \)-deformed FI can be written as

\[
\mathcal{F}_q = \int_{\mathcal{H}} \mathcal{L}_q[\rho_q] d\mathcal{H}.
\]

Thus, using Eq. (30) and applying an integration by parts in the second term, this leads to

\[
\mathcal{F}_q = \mathcal{F}_q - \gamma_q^2.
\]

If we apply the functional derivative

\[
\delta \mathcal{F}_q / \delta \rho_q = \frac{\partial \mathcal{F}_q}{\partial \rho_q} - \mathcal{D}_\rho \left( \frac{\partial \mathcal{F}_q}{\partial (\mathcal{D}_\rho \rho_q)} \right),
\]

where \( \mathcal{D}_\rho = \rho(x, t)[\mathcal{D}_\rho \ln \rho_q(x, t)]^2 \) is a \( q \)-deformed FI density, we obtain the complete deformed de Broglie-Bohm quantum potential expressed by the functional derivative [compare with Eq. (31)]

\[
\mathcal{Q}_q(x, t) = \hbar^2 \frac{\delta \mathcal{F}_q [\rho_q]}{\delta \rho_q}.
\]

From Eq.’s (22), (23), (31), (34) and (35), we have

\[
\mathcal{F}_q = \mathcal{F}_q - \gamma_q^2.
\]

i.e., an invariant under transformation \( \rho_q \leftrightarrow \rho \).

4.2. Kinetic energy operator for stationary states

In terms of \( \Psi(x, t) \), the Fisher functional (30) can be written as

\[
\mathcal{F}[\rho] = \int_{\mathcal{H}} \mathcal{J}_q[\rho] d\mathcal{H},
\]

where \( \mathcal{J}_q[\rho] = \int_{\mathcal{H}} \mathcal{L}_q[\rho_q] d\mathcal{H} \). Thus, the classical linear momentum obeys the relation \( \hat{p} = \hbar \frac{\partial \mathcal{F}}{\partial \Psi} \). Therefore, it follows that the Fisher functional (38) can be interpreted as a measure of nonclassicality between the quantum kinetic term and the classic one given by

\[
\mathcal{F}[\rho] = \int_{\mathcal{H}} \mathcal{J}_q[\rho] d\mathcal{H}.
\]

It should be noted that (39) has been also studied for a constant mass in [3].

For stationary states we have a constant current density \( J \), so using the Eq. (18) and integrating by parts, the classical contribution to \( \mathcal{F}[\rho] \) results

\[
\left( \frac{\rho^2}{m(x)} \right)_{\text{classic}} = \int_{\mathcal{H}} \mathcal{S}_q[\rho_q] d\mathcal{H},
\]

Accordingly, the FI for the stationary states \( \psi_n(x) \) is

\[
\mathcal{F}[\rho_n] = \int_{\mathcal{H}} \mathcal{J}_q[\rho_n] d\mathcal{H}.
\]
Considering the $q$-deformed FI, we have for the stationary states

$$I_q[\rho_{q,t}] = 4 \int_{x_i}^{x_f} \left[ \mathcal{D}_y \Phi_q(x,t) \right]^2 dx = \frac{8m_0}{\hbar^2} \left( \hat{F} \right),$$

in accordance to the von Weizsäcker’s kinetic energy functional operator \[33] in Thomas-Fermi-Dirac theory.

Some researchers \[34, 35, 36] have considered the information theory for systems with position-dependent mass by means using the standard FI

$$I_F[\rho] = 4 \int_{x_i}^{x_f} \left( \frac{d\rho_{q,t}}{dx} \right)^2 dx = \frac{(\hbar^2/2)^2}{(\hbar/2)^2}.$$  

Section 5 uses Eq. (43) for completeness.

4.3. $q$-Deformed Lagrangian and Hamiltonian formulations from variational principle

Consider the Lagrangian formulation by defining the deformed Lagrangian density as

$$\mathcal{L}_q(x,t) = \left\{ \frac{\partial S_q(x,t)}{\partial t} + \frac{1}{2m_0} \frac{(\mathcal{D}_y S_q(x,t))^2}{q} + V(x) \right\} \rho_q(x,t)$$

and its corresponding action as

$$A = \int_{x_i}^{x_f} \int_{t_i}^{t_f} \mathcal{L}_q(x,t) d\rho_q dxt,$$

then it follows that by applying the variational principle $\delta A = 0$ we get the equations of motion \[15] and \[19] for the fields $S_q(x,t)$ and $\rho_q(x,t)$ related to the Bohmian quantum formalism.

Alternatively, the equations of motion \[17] and \[20], associated to the Schrödinger equation for systems with position-dependent mass \[69] emerge from the variational principle applied to the standard Lagrangian density

$$\mathcal{L}(x,t) = \left\{ \frac{\partial S_q(x,t)}{\partial t} + \frac{1}{2m(x)} \frac{(\partial S_q(x,t))^2}{\partial x} + V(x) \right\} \rho(x,t)$$

whose corresponding action is

$$A = \int_{x_i}^{x_f} \int_{t_i}^{t_f} \mathcal{L}(x,t) \rho(x,t) dxt,$$

A Hamiltonian formulation also can be developed. For the Hamiltonian \[12] we have that the energy of the system is given by the $q$-integral

$$E = \int_{x_i}^{x_f} \Phi_q(x,t) \hat{H} \Phi_q(x,t) d\rho_q dxt$$

and, in terms of $S_q(x,t)$ and $\rho_q(x,t)$, we can write

$$E = \int_{x_i}^{x_f} \left( \frac{\hbar^2}{2m_0} \mathcal{D}_y S_q(x,t) + V(x) \right) \rho_q d\rho_q dxt + \frac{\hbar^2}{8m_0} \rho_{q,t}. \]$$

Similarly, there is an equivalent Hamiltonian formulation corresponding to the Schrödinger equation for system position-dependent mass. In this case, the energy of the system is given by

$$E = \int_{x_i}^{x_f} \Psi(x,t) \hat{H} \Psi(x,t) dx$$

which in terms of $\rho(x,t)$ and $S_q(x,t)$ can be recasted as

$$E = \int_{x_i}^{x_f} \mathcal{H} \rho d\rho dxt$$

where $\mathcal{H}$ is the Hamiltonian density for the position-dependent mass system.
Again, from the Hamilton’s equations for the fields $\rho(x,t)$ and $S_q(x,t)$ we recover the equations of motion

\[
\frac{\partial \rho}{\partial t} = \frac{\partial H}{\partial S_q} = \frac{\partial H}{\partial S_q} - \frac{\partial}{\partial x} \left( \frac{\partial H}{\partial (\partial_x S_q)} \right),
\]

and

\[
\frac{\partial S_q}{\partial t} = \frac{\partial H}{\partial \rho} = \frac{\partial H}{\partial \rho} + \frac{\partial}{\partial x} \left( \frac{\partial H}{\partial (\partial_x \rho)} \right) = -\frac{1}{2m(x)} \left( \frac{\partial S_q}{\partial x} \right)^2 - V - Q_q.
\]

### 4.4. q-Deformed Cramér-Rao bound

The Cramér-Rao bound states an uncertainty principle for probability distributions in terms of the FI and the variance. If $\rho(x)$ is a probability distribution and $\langle \Delta x \rangle^2 = \int \rho(x)(x - \langle x \rangle)^2 dx$ is the variance, then this is given by $I_q[\rho][\Delta x]^2 \geq 1$. It can be shown that the family of distributions that minimize the Cramér-Rao bound are the Gaussian ones.

Following a way similar proposed by Furuichi [37], now we develop a Cramér-Rao bound associated with the proposed q-deformed FI [26]. Consider the expected value in terms of the q-deformed density probability for stationary states $\langle f(\hat{q}) \rangle = \int_0^{\gamma_q} f(x)q(x)dx = \int_0^{\gamma_q} f(x)\rho(x)dx$. In particular, we have that $I_q[\rho] = \langle \Omega_q \rangle$, where $\Omega_q = D_q \ln \rho$ is a q-deformed score function. Thus, one has

\[
\langle (x - \langle \hat{x} \rangle) \Omega_q(x) \rangle = \int_0^{\gamma_q} (x - \langle \hat{x} \rangle)q(x)\Omega_q(x)dx
\]

\[
= -\int_0^{\gamma_q} q(x)dx = -(1 + \gamma_q \langle \hat{x} \rangle)
\]

Therefore, it follows that

\[
0 \leq \left( \Gamma_q \right) = I_q[\rho] + \frac{2}{(\Delta x)^2} \langle (x - \langle x \rangle) \rangle \Omega_q(x) + \frac{(x - \langle \hat{x} \rangle)^2}{(\Delta x)^2}
\]

\[
= I_q[\rho] - \frac{2(1 + \gamma_q \langle \hat{x} \rangle)}{(\Delta x)^2} + \frac{1}{(\Delta x)^2}
\]

from which one obtains

\[
I_q[\rho] \geq \frac{1}{(\Delta x)^2} \geq 1 + 2\gamma_q \langle \hat{x} \rangle.
\]

that constitutes a q-deformed version of the Cramér-Rao bound. Note that since $I_q[\rho] \rightarrow I_F[\rho]$ when $\gamma_q \rightarrow 0$, then the standard one is recovered in the limit $\gamma_q \rightarrow 0$.

### 5. Application: particle in an infinite square potential well

Consider a particle with position-dependent mass $m(x)$ given by Eq. (8) in an infinite one-dimensional square potential well of width $L$. The eigenfunctions for this problem are given by

\[
\psi_n(x) = \frac{A_q}{\sqrt{1 + \gamma_q x}} \sin \left[ \frac{k_{q,n} x}{\gamma_q} \ln(1 + \gamma_q x) \right],
\]

for $0 \leq x \leq L$ and $\psi_n(x) = 0$ otherwise, where $A_q^2 = 2/L_q$, $k_{q,n} = n\pi/L_q$ ($n$ is a non integer), and $L_q = \gamma_q^{-1} \ln(1 + \gamma_q L)$ is the length of the box at the deformed space $\langle \delta_{q} \rangle$ obtained by the transformation $[6]$. The corresponding solutions using the q-deformed Schrödinger equation are

\[
\varphi_q(x) = A_q \sin \left[ \frac{k_{q,n} x}{\gamma_q} \ln(1 + \gamma_q x) \right],
\]

for $0 \leq x \leq L$ and $\varphi_q(x) = 0$ otherwise. In this case, the expected value of the kinetic energy operator coincides with the energy of the eigenstates, given by

\[
E_n = \langle \hat{T} \rangle = \frac{h^2}{2m_0} \ln^2(1 + \gamma_q L).
\]

The expected values for $\langle \hat{x} \rangle$, $\langle \hat{x}^2 \rangle$, $\langle \hat{p} \rangle$ and $\langle \hat{p}^2 \rangle$ are [26]

\[
\langle \hat{x} \rangle = \gamma_q L - \ln(1 + \gamma_q L) - \frac{L \ln(1 + \gamma_q L)}{\gamma_q \ln(1 + \gamma_q L) + (2\pi)n^2},
\]

\[
\langle \hat{x}^2 \rangle = 2\gamma_q L^2 - 2\gamma_q L + 2 \ln(1 + \gamma_q L) - 2 \frac{\gamma_q^2 L^2}{\gamma_q \ln(1 + \gamma_q L) + (2\pi)n^2}
\]

\[
\langle \hat{p} \rangle = 0,
\]

\[
\langle \hat{p}^2 \rangle = \frac{h^2 k_{q,n}^2 (1 + \gamma_q L^2 - 1)}{2(1 + \gamma_q L)^2 \ln(1 + \gamma_q L)} \left[ 1 + \frac{\gamma_q^2}{4k_{q,n}^2 + \gamma_q^2} \right],
\]

which satisfy the uncertainty principle $\Delta x \Delta p \geq h/2$ [26].

Now, in order to analyze this result we use the FI. The q-deformed FI for the stationary states [59] is

\[
I_q[\rho] = 4A_q^2 k_{q,n}^2 \int_0^L \frac{\gamma_q^2 \ln(1 + \gamma_q x)}{1 + \gamma_q x} dx
\]

\[
= 4k_{q,n}^2 \gamma_q^2.
\]

in accordance with the Eq. (42) for the kinetic energy [60]. Note that Eq. (59) has the same form than the constant mass case, used in [59]. From Eq’s (54) and (62), the FI proposed by Plastino et al. results $I_F[\rho] = 4k_{q,n}^2 + \gamma_q^2$. We can also calculate the standard FI disregarding the effect of mass locality on functional. From Eq. (43) and (61), we get

\[
I_F[\rho] = \frac{2k_{q,n}^2 (1 + \gamma_q L^2 - 1)}{(1 + \gamma_q L)^2 \ln(1 + \gamma_q L)} \left[ 1 + \frac{\gamma_q^2}{4(\gamma_q + k_{q,n}^2)} \right].
\]
Figure 1 (a) shows the relation $I_0(\Delta x)^2$ as function of $\gamma_q L$. Note that $I_0(\Delta x)^2 < 1$ as $\gamma_q L$ approaches to $-1$. Figure 1 (b) shows that the $q$-deformed Cramér-Rao inequality (57) is satisfied for different values $\gamma_q L$. Figure 1 (c) shows the standard Cramér-Rao inequality $I_0(\Delta x)^2$. One can see that the Cramér-Rao inequality is only satisfied for their standard version and the $q$-deformed one, thus showing the consistence of the $q$-deformed structure. In all cases, the Cramér-Rao inequality for particle with constant mass is recovered in the limit $\gamma_q L \to 0$.

6. Conclusions

We have proposed an alternative way for obtaining the Bohmian quantum formalism using the deformed Schrödinger equation and the de Broglie wave pilot interpretation, in the context of a $q$-algebra structure. Specifically, a deformed derivative was used to represent a particle with a position-dependent mass with the advantage of controlling the deformation by means of the parameter $\gamma_q$. Additionally, we have proposed a deformed Fisher functional that allows one to derive a deformed Hamilton-Jacobi equation which emerges from the Bohmian formalism for a system with a position-dependent effective mass. The Lagrangian and Hamiltonian formulations have been also established.

Then, we have formulated a deformed Cramér-Rao bound associated with the deformed Fisher functional proposed. We have illustrated with a particle confined in a infinite square potential well. We found that the deformed Cramér-Rao bound is satisfied by the stationary states, while the product of the FI for position-dependent mass (previously defined in [23]) with the variance of the position violates the inequality, i.e., it results lower than one. This can be interpreted as an evidence that the $q$-structure is preserved, at least in the example studied. By the same way, another potentials could be analyzed, e.g., the harmonic oscillator, Morse and Coulomb potentials, etc. Furthermore, one can see that the scenario with position-dependent mass can be treated, equivalently, by means of a $q$-deformed one where the mass remains constant at the expense of deforming the space. An scheme is shown in Fig. 2.

Finally, we mention that the deformed FI and the Cramér-Rao bound presented in this work are inspired in the $q$-deformed Schrödinger equation proposed in [23]. Thus, they are not exhaustive and further generalizations based on other deformed derivatives could be formulated in future researches.

Acknowledgments

This work was partially supported by National Institute of Science and Technology for Complex Systems (INCT-SC) and Capes.

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