From 2-d Polyakov Action to the 4-d Pseudo-Conformal Field Theory

C. N. Ragiadakos
email: ragiadak@gmail.com

ABSTRACT

The characteristic property of the 2-dimensional Polyakov action is its independence on the metric tensor, without being topological. A renormalizable 4-dimensional action is found satisfying this fundamental property. The fundamental quantity of this pseudo-conformal field theory (PCFT) is the lorentzian Cauchy-Riemann (LCR) structure. This action describes all current phenomenology: 1) The Poincaré group is determined. 2) Stable solitonic LCR-tetrads are found, which belong to representations of the Poincaré group and they are determined by the irreducible and reducible algebraic quadratic surfaces of CP3. 3) The static (irreducible) LCR-structure implies the Kerr-Newman manifold with g=2 gyromagnetic ratio and it is identified with the electron. The stationary (reducible) LCR-structure is identified with the neutrino. The antiparticles have conjugate LCR-structures. The Hawking-Penrose singularity theorems are bypassed in the electron LCR-manifold. 4) The LCR-tetrad defines Einstein’s metric and the U(2) electroweak connection. 5) An effective leptonic standard model action is derived using the Bogoliubov-Scharf recursive procedure. 6) The three generations of flavors are implied by the limited number (for curved spacetime) of permitted algebraic surfaces of CP3. 7) For every LCR-structure there exists a solitonic distributional gauge field configuration, identified with the corresponding quark, which explains the lepton-quark correspondence. It is explicitly computed for the static LCR-structure. 8) The derivation of a proper geometric SU(3) Cartan connection opens up the possibility to achieve Einstein’s goal to derive all interactions from the pure geometric LCR-structure.
Contents
1. INTRODUCTION
2. LORENTZIAN CR-STRUCTURE
3. THE COVARIANT ACTION OF PCFT
4. DERIVATION OF EINSTEIN’S GRAVITY
5. ELECTRON AND ELECTRODYNAMICS
   5.1 Microlocal analysis of the electron
   5.2 Derivation of quantum electrodynamics
   5.3 LCR-ray tracing in the electron LCR-manifold
6. NEUTRINO AND STANDARD MODEL
   6.1 The electroweak-U(2) gauge fields
   6.2 Derivation of standard model action
7. THE UP AND DOWN QUARKS
   7.1 A quark confining mechanism
8. A SU(3) CONNECTION FROM THE LCR-STRUCTURE
9. PERSPECTIVES
1 INTRODUCTION

The 2-dimensional Polyakov (string) action has the remarkable property to be metric independent without being topological (i.e. a pure surface integral). In fact this particular property is the essential origin of its mathematical beauty. The higher dimensional conformal field theories (the Weyl-transformation invariant covariant forms) are not metric independent, therefore they cannot be considered as the 4-dimensional versions of the Polyakov action. I found\[22\] and studied a 4-dimensional generally covariant action, which is metric independent without being topological. This action describes the current phenomenology without needing supersymmetry, which has not been observed. Besides gravity, it describes\[30\] the standard model as an effective field theory, with the only essential difference the hadronic sector. The strong interactions are described\[31\] from a 4-dimensional gauge field (gluon) which explicitly appears with a metric independent non-laplacian lagrangian (with first order derivatives) .

If we clarify the origin of the metric independence of the Polyakov action

$$I_S = \frac{1}{2} \int d\xi \sqrt{-g} \ g^{\alpha\beta} \partial_{\alpha}X^\mu \partial_{\beta}X^\nu \eta_{\mu\nu} \tag{1.1}$$

the invention of the 4-dimensional action is rather simple. Notice that the metric independence is caused by the general property of the 2-dimensional metrics to admit a coordinate system (the light-cone coordinates), which makes them off-diagonal, i.e.

$$ds^2 = 2g_{01}d\xi_+ d\xi_- \tag{1.2}$$

Apparently the 4-dimensional spacetime metrics cannot generally take an analogous off-diagonal form. Only metrics, which admit two geodetic and shear-free null congruences $\ell^\mu \partial_\mu, n^\mu \partial_\mu$ can take\[8\],\[9\] this form

$$ds^2 = 2g_{a\b}dz^\alpha dz^{\b}, \quad \alpha, \b = 0, 1 \tag{1.3}$$

where $z^b = (z^\alpha(x), z^{\b}(x))$ are generally complex coordinates. In this case we can write down the following metric independent Yang-Mills-like integral

$$I_G = \int dz \sqrt{-gg^{\alpha\b}g^{\b\g}} F_{\alpha\b}\tilde{F}_{\b\g} = \int d^4z \ F_{01}F_{\b0\g} \tag{1.4}$$

$$F_{j\a\b} = \partial_{\a}A_{j\b} - \partial_{\b}A_{j\a} - \gamma f_{jik}A_{i\a}A_{k\b}$$

which depends on the coordinates $(z^\alpha(x), z^{\b}(x))$, and it does not depend on the metric. This integral is apparently complex, because the structure coordinates are complex. Therefore the real spacetime action must be either its real or imaginary part. It will be clarified in section III, where the precise covariant action will be derived. The restriction on the metrics which admit two geodetic and shear free congruences, should not physically bother us, because the blackholes have this property. On the contrary, it is rather encouraging, because it
provides an argument why all the observed spacetimes are Schwarzschild type. The complete generally covariant action will be described in section III, after introducing the necessary mathematical notions.

Our first observation is on the fields necessary to achieve metric independence in two and four dimensions. The Polyakov action needs a scalar field $X^\mu(x)$, which string theory interprets as the embedding function of the Riemann surfaces in the 26-dimensional ordinary spacetime. My 4-dimensional metric independent action needs a peculiar gauge field, which is not of the laplacian type. The implied spherically symmetric equation and its static "potential" is

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2}\right)\Phi = [\text{source}]$$

$$\Phi = a + br$$

where in the second line, I write the spherically symmetric laplacian field equation and its corresponding static "potential". Notice the essential difference. The present 4-dimensional action gives a confining linear "potential" in the place of the $(\frac{1}{r})$ potential of the laplacian. Therefore the present gauge field will be identified with the gluon field. But the mathematical procedure will be quite sophisticated, because the symmetries of the present action do not permit the introduction of fermionic fields, as it happens in ordinary quantum field theory (QFT). That is the term $[\text{source}]$ cannot now be put in the action by hand, because it would destroy the renormalizability of the action. It must be derived. It is possible, because of the great advantage of PCFT to admit solitonic solutions, which are generalized functions (with distributional sources)[10]. The function, outside the compact (closed and bounded) source, is the "potential" and its "source" is the fermionic particle. A typical example is the defined electromagnetic field with its source been the electron. Or equivalently the electron in PCFT is a dressing electric field with its singularity. It is a fermion, because of the well known Carter observation[5] that the Kerr-Newman spacetime has a fermionic gyromagnetic ratio. This puzzle, that surprised general relativists[18], finds its "raison d’être" in the present theory. In the case of the gauge field, distributional configurations are found with their sources identified with the quarks. Therefore the 4-dimensional PCFT does not need supersymmetry to incorporate fermions. In fact my efforts to supersymmetrize it have also failed. Supersymmetry may not be compatible with the metric independence of PCFT.

The Polyakov action does not depend on the 2-dimensional metric, but it does depend on the more general notion of complex structure (after the usual Wick rotation). Recall that the string functional integral[20] is an integration over the 2-dimensional complex manifolds. My 4-dimensional action does not depend on the metric tensor, but it does depend on a special Cauchy-Riemann (CR) structure of the spacetime, the lorentzian CR-structure (LCR-structure), which is viewed as the existence of two geodetic and shear-free null congruences in the metric (riemannian) language of general relativity. The CR-structure was called pseudo-conformal by E. Cartan[4] and Tanaka, who first worked on real submanifolds of complex manifolds. This is the reason that I call the present
kind of 4-dimensional field theory pseudo-conformal field theory (PCFT), in order to stress its complete mathematical (but not physical) analogy with the 2-dimensional Polyakov action.

The present action is based on the lorentzian CR-structure\[28\], which is determined by two real and one complex independent 1-forms (\(\ell, m; n, \overline{m}\)) that satisfy the relations

\[
\begin{align*}
\ell & = Z_1 \wedge \ell + i \Phi_1 m \wedge \overline{m} \\
\overline{m} & = Z_2 \wedge n + i \Phi_2 m \wedge \overline{m} \\
m & = Z_3 \wedge m + \Phi_3 \ell \wedge n
\end{align*}
\]

(1.6)

\(\ell \wedge m \wedge n \wedge \overline{m} \neq 0\)

where the vector fields \(Z_1, Z_2\) are real, the vector field \(Z_3\) is complex, the scalar fields (called relative invariants) \(\Phi_1, \Phi_2\) are real and the scalar field \(\Phi_3\) is complex. One can easily check that these conditions are equivalent to the metric independent form of the geodetic and shear-free conditions\[6\]

\[
\begin{align*}
(\ell^\mu m^\nu - \ell^\nu m^\mu)(\partial_\mu \ell_\nu) & = 0 \quad , \quad (\ell^\mu m^\nu - \ell^\nu m^\mu)(\partial_\mu m_\nu) = 0 \\
(n^\mu \overline{m}^\nu - n^\nu \overline{m}^\mu)(\partial_\mu n_\nu) & = 0 \quad , \quad (n^\mu \overline{m}^\nu - n^\nu \overline{m}^\mu)(\partial_\mu \overline{m}_\nu) = 0
\end{align*}
\]

(1.7)

on the tetrad \((\ell, n, m, \overline{m})\). Notice that these conditions do not depend on a metric. It is a property of a basis of the tangent (and cotangent) space of a manifold.

The integrability conditions \(1.6\) of the LCR-structure are invariant under the transformations

\[
\begin{align*}
\ell'_\mu & = \Lambda \ell_\mu \\
n'_\mu & = N n_\mu \\
m'_\mu & = M m_\mu \\
Z'_1 & = Z_1 + d \ln \Lambda \\
Z'_2 & = Z_2 + d \ln N \\
Z'_3 & = Z_3 + d \ln M
\end{align*}
\]

(1.8)

which I will call tetrad-Weyl transformations. The fact that the tetrad-Weyl parameters \(\Lambda, N, M\) must not vanish, implies that the tetrad-Weyl transformation cannot annihilate the relative invariants. If they do not vanish, they may be fixed to take a constant numerical value, which is the reason of the used term "relative invariant". Notice that if all the relative invariants do not vanish, they may fix the tetrad-Weyl transformation.

In brief, the fundamental quantity of PCFT is not the metric (like general relativity) but the LCR-structure (like the Polyakov action). Starting from a LCR-structure i.e. a LCR-tetrad \(1.6\), we cannot define a unique symmetric tensor (an Einstein metric)

\[
g_{\mu\nu} = \ell_\mu n_\nu + \ell_\nu n_\mu - m_\mu \overline{m}_\nu - m_\nu \overline{m}_\mu
\]

(1.9)

Because of the tetrad-Weyl symmetry, we can only define a class of metrics \([g_{\mu\nu}]\), with equivalence relation the tetrad-Weyl transformations \(1.8\). Notice that for \(\Lambda N = M\overline{M}\), the tetrad-Weyl transformation becomes the ordinary Weyl
transformation. I will explicitly show how the charge conservation breaks the
tetrad-Weyl symmetry down to the ordinary Weyl symmetry and the energy-
momentum conservation breaks it farther.

The corresponding integrability conditions and transformations for the 2-
dimensional LCR-structure are
\[ d\ell = Z_1 \wedge \ell, \quad dn = Z_2 \wedge n \]
\[ \ell'_\mu = \Lambda \ell_\mu, \quad n'_\mu = N n_\mu \] (1.10)
which are satisfied for all the 2-dimensional independent 1-forms \((\ell, n)\). But
there is an essential difference. In two dimensions the LCR-structure is always
degenerate and the transformation coincides with the ordinary Weyl transforma-
tion, while in four dimensions the relative invariants \(\Phi_j\) make the LCR-structure
non-degenerate and they generate gravity, electromagnetism and all the leptonic
sector.

I want to point out that the Wick rotation in four dimensions "destroys" the
LCR-structure, because simply the Minkowski metric spacetime does not admit
a (real tensor) hermitian structure\[8\], \[9\]. In fact even in two dimensions, we do
not need the Wick rotation to show the dependence of the Polyakov action on
the algebraic curves of \(CP^2\). The 2-dimensional LCR-manifold may be viewed
as the product of two real submanifolds of a complex manifold (identified with
an algebraic curve).

In section II, I will describe the fundamental properties of the LCR-structure,
which will permit us to write down the generally covariant form of the action
of PCFT in section III. The holomorphic Frobenius theorem conducts to two
intersections of the lines with the hypersurfaces of \(CP^3\), the 4-dimensional real
submanifolds (spacetimes) of the grassmannian manifold \(G_{4,2}\) and finally the
Poincaré group, which will be identified with the (observed) conserved group in
nature. This identification will permit us to unfold and describe the vacuum,
the static LCR-manifold, which is identified with the electron and the stationary
LCR-structure, which is identified with the neutrino. In section IV Gravity is
derived\[25\]. Electromagnetism is derived in section V, and the same Bogoliubov
method is extended to the standard model derivation\[30\] in section VI, where
the three particle generations are implied as a restriction of LCR-structure inte-
grability conditions. In section VII, stable static solitonic solutions of the gauge
field equations are found, which have distributional sources, identified with the
quarks\[31\]. A quark-antiquark system, described in section VIII, could be the
origin of quark confinement.

The reader will see that the fundamental mathematical framework of PCFT\[29\]
is essentially analogous to that of string theory. We simply pass from the alge-
braic curves to the algebraic surfaces. I will stick to this analogy as long as it is
permitted, in order to facilitate string theory researchers to understand PCFT.
2 LORENTZIAN CR-STRUCTURE

The 4-dimensional pseudo-conformal field theory (PCFT) is based on the LCR-structure of the spacetime, like the Einstein relativity is based on the metric structure of the spacetime. Recall that after the success of general relativity, Einstein tried (and failed) to extend its fundamental structure from the lorentzian metric first to a 5-dimensional Kaluza-Klein theory and after to a metric with torsion. PCFT is an attempt in this direction providing all the leptonic interactions.

The LCR-structure definition (1.6) is an integrability condition that permits the application of Frobenius theorem. But here there is a subtlety that is essential in four dimensions. The existence of the complex tangent 1-form $m_\mu dx^\mu$ makes necessary first to complexify spacetime and after apply the holomorphic Frobenius theorem.

The complexification locally makes the spacetime a real surface of $C^4$ and we have to be restricted to real analytic functions. But this real-analyticity is not necessary to be on the entire spacetime. It must be valid on a large connected region of spacetime so that the two (imaginary) sides of spacetime communicate through the analytic continuation. Hence there may exist isolated local compact regions, which will be the singular regions of the considered generalized functions[10]. That is, this complexification permits the consideration of generalized functions and especially in the picture of the Sato’s hyperfunctions[12] [15]. The found solitonic configurations are generalized functions (Schwartz distributions). The distributional sources of these generalized functions will be identified with the leptons and the quarks and the regular part of the distributional configurations will be identified with the gravitational, electromagnetic and gluonic “dressings” of the particles. This will become clear in section VI, where a microlocal analysis of the electron-configuration will be presented.

The application of the holomorphic Frobenius theorem implies the existence of four complex functions $(z^\alpha, z^{\tilde{\alpha}})$, $\alpha = 0, 1$, such that

\begin{equation}
\begin{align*}
dz^\alpha &= f_\alpha \ell_\mu dx^\mu + h_\alpha m_\mu dx^\mu, \\
dz^{\tilde{\alpha}} &= f_{\tilde{\alpha}} n_\mu dx^\mu + h_{\tilde{\alpha}} \overline{m_\mu} dx^\mu
\end{align*}
\end{equation}

(2.1)

This LCR-structure[28] is called realizable or embedable and the complex functions are called LCR-structure coordinates. Notice that the corresponding result for the 2-dimensional LCR-structure is the existence of two structure coordinates $(z^0, \tilde{z}^0)$, such that

\begin{equation}
\begin{align*}
dz^0 &= f_0 \ell_\mu dx^\mu, \\
dz^{\tilde{0}} &= f_{\tilde{0}} n_\mu dx^\mu
\end{align*}
\end{equation}

(2.2)
We will assume them generally complex in order to keep the analogy between
the 2-dimensional and the 4-dimensionnal LCR-structures.

The tangent 1-forms \( \ell^\mu dx^\mu \) and \( n^\mu dx^\mu \) are real, and the 1-forms \( m^\mu dx^\mu \) and \( \overline{m^\mu} dx^\mu \) are complex conjugate. These (reality) relations imply

\[
\begin{align*}
dz^0 \wedge dz^1 & \wedge d\overline{z^0} \wedge d\overline{z^1} = 0 \\
dz^\tilde{0} \wedge dz^\tilde{1} & \wedge d\overline{z^\tilde{0}} \wedge d\overline{z^\tilde{1}} = 0 \\
dz^\tilde{0} \wedge dz^1 & \wedge d\overline{z^\tilde{0}} \wedge d\overline{z^1} = 0 \\
dz^0 \wedge dz^1 & \wedge dz^\tilde{0} \wedge dz^\tilde{1} \neq 0
\end{align*}
\]

(2.3)

where the last one is implied by the linear independence of the LCR-tetrad. It
is convenient to write the first three conditions as

\[
\rho_{11}(z^\alpha, z^{\tilde{\alpha}}) = 0, \quad \rho_{12}(z^\alpha, z^{\tilde{\alpha}}) = 0, \quad \rho_{22}(z^{\tilde{\alpha}}, z^{\tilde{\alpha}}) = 0
\]

\[
(2.4)
\]

where the two functions \( \rho_{11}, \rho_{22} \) are real and \( \rho_{12} \) is a complex function (i.e.
two real functions). Notice the particular dependence of these functions on the
structure coordinates. The two real conditions determine two ordinary hyper-
surface type CR-structures, which are connected through the complex condition.
The LCR-structure is essentially a special totally real CR-structure[11]. But un-
lke the ordinary totally real CR-structures, which are invariant under a general
holomorphic transformation \( z^b = f^b(z^c) \), the LCR-structure is invariant (and
considered to be equivalent) under the special holomorphic transformations

\[
z^\beta = f^\beta(z^\alpha) \quad , \quad z^{\tilde{\beta}} = f^{\tilde{\beta}}(z^{\tilde{\alpha}})
\]

(2.5)

where the transformations of the tilded and untilded structure coordinates are
independent.

In the case of the 2-dimensional LCR-structure (on which the Polyakov ac-
tion is based) the corresponding defining functions and the LCR-transformations
are

\[
\rho_1(z^0, z^0) = 0 \quad , \quad \rho_2(z^{\tilde{0}}, z^{\tilde{0}}) = 0 \quad ,
\]

\[
\frac{\partial \rho_j}{\partial z^b} \neq 0 \neq \frac{\partial \rho_j}{\partial z^b}
\]

(2.6)

I want to point out that I have not yet introduced any riemannian metric.
The CR-structure does not need the metric structure. The Einstein metric will
be defined in section IV, where we will study its limitations and really amazing
consequences. The vector and their dual 1-form tetrads are related by the
following inversion relations
\[ e_0^\mu dx^\mu \equiv \ell_\mu dx^\mu , \ e_1^\mu dx^\mu \equiv m_\mu dx^\mu , \ e_0^\mu \partial_\mu \equiv \eta^\mu \partial_\mu , \ e_1^\mu \partial_\mu \equiv \nu^\mu \partial_\mu , \ e_0^\mu \partial_\mu \equiv -\mu^\mu \partial_\mu , \ e_1^\mu \partial_\mu \equiv -\mu^\mu \partial_\mu \]

\[ e_a^\mu e_\mu^b = \delta_a^b , \ e_a^\mu e_\nu^a = \delta_\nu^\mu \]  

No lowering and raising index mechanism has been defined yet, because we have not defined the metric.

In order to clarify the relative essential differences between the diffeomorphic CR-transformations and the LCR-transformations, I will recall the historical discovery of 2-dimensional CR-structure by Poincaré. I think it is well known that any real submanifold \( \rho(x^\mu) = 0 \) can take the one coordinate form \( y = 0 \) after a diffeomorphic transformation (using the implicit function theorem). A real submanifold (curve) \( \rho(z,z) = 0 \) of the complex plane \( \mathbb{C} \) can take the real axis form \( z - \bar{z} = 0 \) after a holomorphic transformation. But Poincaré showed that this is not possible for real subsurfaces of \( \mathbb{C}^2 \). In higher dimensional complex manifolds the holomorphic transformations cannot transform a real surface to any other real surface of the same dimension.

In the case of the LCR-structure transformations we have an analogous restriction. The 2-dimensional LCR-transformations can give the (real analytic at the neighborhood of a point) defining functions (2.6) the simple (trivial) form

\[ z_0 - \bar{z}_0 = 0 , \ z_0 - \bar{z}_0 = 0 \]

But in four dimensions there is a restriction. A LCR-transformation can simplify a real analytic structure (2.4) to the non-trivial form

\[ \text{Im } z^0 = \phi_{11}(z_1, \bar{z}_1, \text{Re } z^0) , \ \text{Im } \bar{z}_0 = \phi_{22}(\bar{z}_1, \bar{z}_1, \text{Re } \bar{z}_0) \]

\[ z^{\bar{1}} - \bar{z} = \phi_{12}(z^1, \bar{z}) \]

\[ \phi_{11}(0) = \phi_{22}(0) = \phi_{12}(0) = 0 , \ d\phi_{11}(0) = d\phi_{22}(0) = d\phi_{12}(0) = 0 \]

and the corresponding coordinates are called regular LCR-coordinates. The LCR-transformations cannot completely remove (annihilate) the real analytic functions \( \phi_{ij} \). But a general holomorphic transformation \( z^b = f^b(z^c) \) can remove these functions. That is, a general holomorphic transformation makes a real analytic LCR-structure equivalent to the degenerate totally real CR-structure[1], which cannot be generally done with a LCR-transformation.

The 2-dimensional LCR-structure has two disconnected structure coordinates \( (z^0, \bar{z}_0) \), which in string theory are directly related to the two chiral sectors. In the degenerate 4-dimensional LCR-structure the tilded and untilded chiral regular coordinates are connected with the relation \( z^{\bar{1}} - \bar{z} = 0 \), which are the two chiral representations of the Lorentz group. This indicates the pathway to reveal the Poincaré group, which will be identified with the corresponding observed symmetry group of nature. I postpone this derivation for section IV,
where the Einstein metric is defined and the flat geodetic and shear-free null congruence conditions are solved through the Kerr theorem and its homogeneous holomorphic function $K(Z^m)$.

In order to understand the present work and applied mathematical techniques, the reader must be aware of the two mathematical approaches to study the geometry of a complex manifold. In the present case, it is the ambient complex space of the LCR-manifold. The first is the algebraic approach, through the possible embedding of the complex manifold into a projective space (external approach). The second (internal) approach is the sheaf cohomology. The interplay between these two approaches is regulated (for compact complex manifolds) by the Kodaira theorem[13]. Below I will describe the algebraic approach, which provides a clear-cut definition of the Poincaré group, which is essential to the understanding of the implied quantum field theory. After I will describe a "physical" view of the internal approach, which is better suited for general relativists.

The defining relations (2.4) of the quite general class of LCR-manifolds[28] take the following form of real surfaces of the grassmannian manifold $G_{4,2}$

$$\rho_{11}(X^{m_1}, X^{n_1}) = 0 \quad , \quad \rho_{22}(X^{m_2}, X^{n_2}) = 0 \quad , \quad \rho_{12}(X^{m_1}, X^{n_2}) = 0$$

$$K(X^{m_j}) = 0$$

(2.10)

where all the functions are homogeneous relative to the coordinates $X^{n_1}$ and $X^{n_2}$ independently, which must be roots of the homogeneous holomorphic Kerr polynomial $K(X^{n_i})$. The charts of its typical non-homogeneous (projective) coordinates are determined by the invertible pairs of rows. If the first two rows constitute an invertible matrix, the chart is determined by $\det \lambda^{A_j} \neq 0$ and the corresponding projective (affine space coordinates) $r_{A'A}$ are defined by

$$X = \begin{pmatrix} X^{01} & X^{02} \\ X^{11} & X^{12} \\ X^{21} & X^{22} \\ X^{31} & X^{32} \end{pmatrix} \equiv \begin{pmatrix} \lambda^{A_j} \\ -ir_{A'A} \lambda^{A_j} \end{pmatrix}$$

(2.11)

$$r_{A'A} = \eta_{ab}r^{a'b}\sigma_{A'A}^{b}$$

The matrix $\eta_{ab}$ is the ordinary Minkowski metric and $\sigma_{A'A}^{b}$ are the identity and the three hermitian Pauli matrices

$$\sigma_{A'B}^{0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \sigma_{A'B}^{1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_{A'B}^{2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_{A'B}^{3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(2.12)
and the spinor indices are lowered and raised with the antisymmetric matrix

\[ \lambda^A = \epsilon^{AB} \lambda_B, \quad \lambda_C = \lambda^B \epsilon_{BC} \]

\[ \lambda^A \xi_A = \lambda^A \xi^B \epsilon_{BA} = -\lambda_B \xi^B, \quad \lambda^A \lambda_A = 0 \]  

\[ \epsilon^{AB} = \epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon^B \epsilon_{AC} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]  

I point out that this notation is not exactly that used in the classical book of Penrose and Rindler\[19\].

The grassmannian manifold \( G_{4,2} \) is the projective space of the lines of \( CP^3 \). The homogeneous coordinates \( X^{mi} \) matrix of \( G_{4,2} \) are two points \( X^{mi} \) of the hypersurface of \( CP^3 \) determined by the irreducible or reducible Kerr polynomial \( K(Z^n) \). From the above LCR-structure conditions (2.10) we see that the untilded structure coordinates \( z^a \) determine the point \( X^{m1} \) and the tilded structure coordinates \( \tilde{z}^\alpha \) determine the point \( X^{m2} \). I point out that the embedding of the ambient complex manifold of the LCR-manifold into the grassmannian manifold is the essential step to relate PCFT with the particle physics. Recall that particles are representations of the Poincaré group, and in this review I simply explain my efforts to find the stable LCR-manifolds which are irreducible representations of the Poincaré algebra.

We must be careful with the passage to the "physical" Poincaré group from the general projective \( SL(4, \mathbb{C}) \) symmetry of the \( G_{4,2} \) geometry. The general complex Poincaré group is an affine subgroup of \( SL(4, \mathbb{C}) \). It is directly related with an affine chart of \( G_{4,2} \). In the present work our "physical" Poincaré group will be that imposed by the condition \( \det \lambda_{Aj} \neq 0 \).

The parameterization\[13\] of the algebraic manifolds is a very useful tool to study algebraic surfaces. The Newman generally complex trajectory\[17\] is a physically intuitive parameterization, where the Kerr holomorphic function \( K(Z^n) \) is replaced\[28\] by a trajectory \( \xi^b(\tau) \) and the following form of the homogeneous coordinates

\[ X = \begin{pmatrix} \lambda^{A1} \\ -i \xi_{A'A} \lambda^{A2} \end{pmatrix} = \begin{pmatrix} \lambda^{A1} \\ -i \xi_{A'A}(\tau_1) \lambda^{A2} \end{pmatrix} = \begin{pmatrix} \lambda^{A2} \\ -i \xi_{A'A}(\tau_2) \lambda^{A1} \end{pmatrix} \]  

\[ (r_{A':A} - \xi_{A'A}(\tau)) \lambda^A = 0 \quad \rightarrow \quad (r^a - \xi^a(\tau))(r^b - \xi^b(\tau)) \eta_{ab} = 0 \]  

The last condition assures the existence of a non-vanishing solution of \( \lambda^{Ai} \) and permits the computation of \( \tau \) as a function of \( r^a \). Notice that this procedure of one trajectory must provide at least two solutions \( \tau_1 \) and \( \tau_2 \) which are used to determine the structure coordinates of the two columns. Apparently we may take two independent complex trajectories. A general complex linear trajectory
corresponds to the following quadratic polynomial

$$\xi^a(\tau) = v^a \tau + c^a, \quad v^a v^b \eta_{ab} = 1$$

$$(v_{10} Z^0 + v_{11} Z^1)(iZ^2 - c_{10} Z^0 + c_{01} Z^1) - (v_{00} Z^0 + v_{01} Z^1)(iZ^3 - c_{00} Z^0 + c_{11} Z^1) = 0$$

(2.15)

In fact, the linear trajectory is the rational parameterization of this precise (Poincaré invariant) parametrized quadric. In this case we usually assume ($z^0 = \tau_1$, $z^\eta = \tau_2$). The Newman complex trajectory is mathematically implied by considering ruled surfaces of $CP^3$. They are surfaces which contain a straight line

$$Z^m(\tau, s) = (1 - s)Z^m_1(\tau) + sZ^m_2(\tau) = Z^m_1(\tau) + sT^m(\tau)$$

(2.16)

where $Z^m_1(\tau)$ is a curve of $CP^3$ and $T^m(\tau)$ indicates the direction of the generating line which meets $Z^m_1(\tau)$ (the generatrix) at $\tau$. This line is a point of $G_{4,2}$, determined by

$$\xi(\tau) =: iX_2 X_1^{-1} =: \begin{pmatrix} \xi^0 - \xi^3 \\ - (\xi^1 + i\xi^2) \\ \xi^0 + \xi^3 \end{pmatrix}$$

$$X_1 =: \begin{pmatrix} Z_1^0 \\ Z_1^1 \\ Z_1^2 \end{pmatrix}, \quad X_2 =: \begin{pmatrix} Z_2^0 \\ Z_2^1 \\ Z_2^2 \end{pmatrix}$$

(2.17)

The curve is called non-degenerate if the following determinant does not identically vanish

$$\det[Z^m_i, Z^m_j, \frac{dZ^m_i}{d\tau}, \frac{dZ^m_j}{d\tau}] = \det \begin{pmatrix} X_1 & \dot{X}_1 \\ -i\xi X_1 & -i(\xi X_1 + \dot{X}_1) \end{pmatrix} = -\det(\xi)(\det X_1)^2$$

(2.18)

This happens if and only if $\xi^a \xi^b \eta_{ab} \neq 0$. This condition will differentiate the massive from the massless partner (neutrino) of a leptonic generation. The complex trajectory is related to the ordinary classical trajectory of the particle viewed as a soliton. If they are real, they are identified with the well known trajectories of the Lienard-Wiechert potential. The degenerate trajectory occurs for $T^m(\tau) = \frac{dZ^m}{d\tau}$, which are called developable surfaces. Hence a non-degenerate ruled surface ($\xi^a \xi^b \eta_{ab} \neq 0$) corresponds to the massive particle (say electron) and the developable surface ($\xi^a \xi^b \eta_{ab} = 0$) corresponds to the massless particle (neutrino) of the leptonic generation (family).

Let us now turn into the internal "physical" approach, noticing that the LCR-tetrad defines and is defined by the following classes of symmetric and
antisymmetric tensors

\[
\begin{align*}
[g_{\mu\nu}] &= \ell_\mu n_\nu + \ell_\nu n_\mu - m_\mu m_\nu - m_\nu m_\mu \\
[J_{\mu\nu}] &= \ell_\mu n_\nu - \ell_\nu n_\mu - m_\mu m_\nu + m_\nu m_\mu
\end{align*}
\]

\[\det(g_{\mu\nu}) \neq 0 \neq \det(J_{\mu\nu})\]

The class is defined relative to the regular tetrad-Weyl transformations [13], which have non-vanishing factors for every coordinate patch, with the appropriate fitting relations in the intersections of the patches (sheaf requirement). The equivalent properties are that the metric \(g_{\mu\nu}\) admits 2-geodetic and shear-free congruences (\(\ell^\mu\) and \(n^\mu\)) and that \(J^\mu_{\nu}\) satisfies [8, 9] the equivalent Nijenhuis condition. That is Einstein’s gravity with 2-geodetic and shear-free congruences determines back the LCR-structure.

A typical example of LCR-structure is

\[
\begin{align*}
\ell_\mu dx^\mu &= dt - \frac{\mathbf{x}^2}{\Delta} dr - a \sin^2 \theta d\varphi \\
n_\mu dx^\mu &= \frac{1}{2\rho}(dt + \frac{\mathbf{x}^2}{\Delta} dr - a \sin^2 \theta d\varphi) \\
m_\mu dx^\mu &= \frac{1}{\eta\sqrt{2}}(ia \sin \theta dt - \rho^2 d\theta - i(r^2 + a^2) \sin \theta d\varphi)
\end{align*}
\]

\[\eta \equiv r + ia \cos \theta, \quad \rho^2 \equiv \eta\bar{\eta}, \quad \sqrt{-g} = \rho^2 \sin \theta \]

\[\Delta \equiv r^2 - 2Mr + a^2 + q^2\]

which corresponds to the static trajectory \(\xi^a = (r, 0, 0, ia)\). Because of the tetrad-Weyl symmetry, the tetrad does not need multiplicative factors in order to fix an LCR-structure. The precise above form is the geodetic and shear-free null tetrad of the Kerr-Newman spacetime [6]. Its contravariant components are

\[
\begin{align*}
\ell^\mu \partial_\mu &= \frac{1}{\Delta}((r^2 + a^2)\partial_\tau + \Delta \partial_r + a \partial_\varphi) \\
n^\mu \partial_\mu &= \frac{1}{2\rho}((r^2 + a^2)\partial_\tau - \Delta \partial_r + a \partial_\varphi) \\
m^\mu \partial_\mu &= \frac{1}{\eta\sqrt{2}}(ia \sin \theta \partial_\tau + \partial_\theta + \frac{1}{\sin \theta} \partial_\varphi)
\end{align*}
\]

and its Newman-Penrose (NP) spin coefficients are

\[
\begin{align*}
\varepsilon &= 0, \quad \beta = \frac{\cos \theta}{\sin \theta \eta\sqrt{2}}, \quad \pi = \frac{ia \sin \theta}{(\eta\sqrt{2})} \\
\tau &= -\frac{ia \sin \theta}{\rho\sqrt{2}}, \quad \rho = -\frac{1}{\eta}, \quad \mu = -\frac{1}{\eta\sqrt{2}} \\
\gamma &= -\frac{\Delta}{2\rho^2 \eta} + \frac{r-M}{2\rho^2}, \quad \alpha = \pi - \beta = -\frac{ia \sin \theta}{(\eta\sqrt{2})} - \frac{\cos \theta}{\sin \theta \eta\sqrt{2}}
\end{align*}
\]

The reader should not confuse the symbol \(\rho^2 \equiv \eta\bar{\eta}\) with the spin-coefficient \(\rho\).

The tetrad-Weyl gauge fields 1-forms \(Z_{\mu\nu} dx^\nu\) and the relative invariants \(\Phi_i\) are found using the standard relations

\[
\begin{align*}
dl &= [(\varepsilon + \tau)n - (\alpha + \beta - \pi)m - (\pi + \beta - \tau)m] \wedge \ell + (\rho - \bar{\rho})m \wedge \bar{m} \\
dn &= [-(\gamma + \tau)\ell + (\alpha + \beta - \pi)m + (\pi + \beta - \tau)m] \wedge n + (\mu - \bar{\mu})m \wedge \bar{m} \\
dm &= [(\gamma - \tau + \bar{\mu})\ell + (\varepsilon - \tau - \rho)n - (\beta + \tau)m] \wedge m - (\tau + \bar{\pi})\ell \wedge n
\end{align*}
\]

\[\text{(2.23)}\]
where I have assumed that the tetrad is geodetic and shear-free \( \kappa = \sigma = 0 = \lambda = \nu \). Notice that this LCR-structure has non-vanishing relative invariants

\[
\begin{align*}
\Phi_1 &= \frac{\partial}{\partial z} = -\frac{2a \cos \theta}{\eta} \\
\Phi_2 &= \frac{\partial}{\partial \bar{z}} = -\frac{2a \cos \theta}{\eta} \\
\Phi_3 &= -(\tau + \bar{\tau}) = \frac{2a \cos \theta}{\eta r}
\end{align*}
\] (2.24)

The structure coordinates are

\[
\begin{align*}
z^0 &= t - f_0(r) + ia \cos \theta - ia, \quad z^1 = e^{i\varphi} e^{-iaf_1(r)} \tan \frac{\theta}{2} \\
\bar{z}^0 &= t + f_0(r) - ia \cos \theta + ia, \quad \bar{z}^1 = e^{-i\varphi} e^{-iaf_1(r)} \tan \frac{\theta}{2}
\end{align*}
\] (2.25)

After straightforward calculations I find the following relations between the structure coordinates and the LCR-tetrad

\[
\begin{align*}
dz^0 &= dt - \frac{\sqrt{2}a^2}{\Delta} dr - ia \sin \theta d\theta \\
dln z^1 &= \frac{ia}{\Delta} dr + \frac{1}{\sin \theta} d\theta + id\varphi = \frac{ia}{\Delta} \ell_\mu dx^\mu - \frac{1}{\eta \sin \theta} m_\mu dx^\mu \\
\ell_\mu dx^\mu &= dz^0 + ia \sin^2 \theta d\ln z^1 \\
\eta_\mu dx^\mu &= \frac{ia \sin \theta}{\eta \sqrt{2}} dz^0 - \frac{(r^2 + a^2) \sin \theta}{\sqrt{2} \eta} d\ln z^1
\end{align*}
\] (2.26)

and

\[
\begin{align*}
d\bar{z}^0 &= dt + \frac{\sqrt{2}a^2}{\Delta} dr + ia \sin \theta d\theta = \frac{2\sqrt{2}a^2}{\Delta} n_\mu dx^\mu - \frac{ia \sqrt{2} \sin \theta}{\eta} m_\mu dx^\mu \\
d\ln \bar{z}^1 &= \frac{-ia}{\Delta} dr + \frac{1}{\sin \theta} d\theta - id\varphi = -\frac{2ia}{\Delta} n_\mu dx^\mu - \frac{1}{\eta \sin \theta} m_\mu dx^\mu \\
n_\mu dx^\mu &= \frac{\Delta}{2a^2 \eta} dz^0 - \frac{ia \sin^2 \theta}{2a^2} d\ln \bar{z}^1 \\
\bar{m}_\mu dx^\mu &= -\frac{ia \sin \theta}{\eta \sqrt{2}} dz^0 - \frac{(r^2 + a^2) \sin \theta}{\sqrt{2} \eta} d\ln \bar{z}^1
\end{align*}
\] (2.27)

This static LCR-manifold is a stable soliton. In the context of PCFT the term soliton should not be confused with that in QFT without gravity. The fact that gravity is contained in PCFT, the energy-momentum and angular momentum of the configuration is derived from the source integrals of linearized Einstein general relativity. On the other hand, besides the topological invariants, the LCR-manifolds have the relative invariants, which take discrete values and act as stabilizers. We have already found that all the relative invariants of the present LCR-structure do not vanish, which is not the case of the neutrino LCR-manifold, as I will show in section VI.

I want to point out that the gravitation of the particle is a generalized function (distribution). The singular support of the gravitational and electromagnetic fields are the "locations" of the electron, and the regular supports are the gravitational and electromagnetic "dressings" of the electron.

In order to stress the physical significance of the LCR-structure, let me mention that it is exactly this common property, that implies the observed
correspondence between the leptonic and hadronic sectors. That is, the up and down quarks have the same LCR-structures with the neutrino and the electron, with additional solitonic solutions (with distributional sources) of the non-abelian gauge field. This will be extensively described in section VII.

3 THE COVARIANT ACTION OF PCFT

The integral \((1.4)\) is complex and not generally covariant. It is written in the LCR-structure (chiral) coordinates (where the metric independence appears) in order to clarify how the metric independence of the Polyakov action triggered the search, discovery and study of the dynamical content of the 4-dimensional PCFT.

The fact that the structure coordinates are generally complex implies that the original metric independent form \((1.4)\) is complex, while the final action must be real. In order to make things clear, I will start from the LCR compatible gauge connection and its curvature

\[
\begin{align*}
(D_{\alpha})_{ij} & = \partial_{\alpha} \delta_{ij} - \gamma j_{ikj} A_{k\alpha}, \\
(D_{\tilde{\alpha}})_{ij} & = \partial_{\tilde{\alpha}} \delta_{ij} - \gamma j_{ikj} A_{k\tilde{\alpha}}, \\
F_{\alpha\beta} & = \partial_{\alpha} A_{\beta} - \partial_{\beta} A_{\alpha} - \gamma f_{ikj} A_{j\alpha} A_{k\beta}, \\
F_{i\tilde{\alpha}\tilde{\beta}} & = \partial_{i} A_{\tilde{\alpha}} - \partial_{\tilde{\alpha}} A_{i} - \gamma f_{ikj} A_{j\tilde{\alpha}} A_{k\tilde{\beta}}.
\end{align*}
\]

in structure coordinates. The gauge invariant and metric independent 4-form is

\[
F \wedge \tilde{F} = \left( \frac{1}{2} F_{\alpha\beta} dz^\alpha \wedge dz^\beta \right) \wedge \left( \frac{1}{2} F_{i\tilde{\alpha}\tilde{\beta}} dz^\tilde{\alpha} \wedge dz^\tilde{\beta} \right) = F_{i01} F_{\tilde{i}\tilde{0}\tilde{1}} dz^0 \wedge dz^1 \wedge dz^\tilde{0} \wedge dz^\tilde{1}
\]

Using the identity

\[
\begin{align*}
\delta^\mu_\nu & = \ell^\mu n_\nu + n^\mu \ell_\nu - m^\mu \tilde{m}_\nu - \tilde{m}^\mu m_\nu, \\
\delta^\alpha_\beta & = n^\alpha \ell_\beta - \tilde{m}^\alpha m_\beta, \\
\delta^\tilde{\alpha}_{\tilde{\beta}} & = \ell^\tilde{\alpha} n_{\tilde{\beta}} - \tilde{m}^\tilde{\alpha} \tilde{m}_{\tilde{\beta}}
\end{align*}
\]

in structure coordinates, the complexified 4-form becomes

\[
F \wedge \tilde{F} = \ell \wedge m \wedge n \wedge \tilde{m} (\ell^\mu m^\nu F_{\mu\nu}) (n^\rho \tilde{m}^\sigma F_{\rho\sigma})
\]

When we return back to the real spacetime, it becomes the complex 4-form

\[
(F \wedge \tilde{F})|_S = \ell \wedge m \wedge n \wedge \tilde{m} (\ell^\mu m^\nu F_{\mu\nu}) (n^\rho \tilde{m}^\sigma F_{\rho\sigma}) d^4x \sqrt{-g}
\]

\[
g = \det(g_{\mu\nu}) = \det(\eta_{ab}) \det(\epsilon_{\mu\nu}^a)^2 = [\det(\epsilon_{\mu\nu}^a)]^2
\]

\[
\eta_{ab} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}
\]

15
Hence we may assume as gauge field action either its real or its imaginary part

\[ I_R = \int d^4x \sqrt{-g}(\ell^\mu m^\nu F_{i\mu\nu}(n^\rho m^\sigma F_{i\rho\sigma}) - (\ell^\mu m^\nu F_{i\mu\nu})(n^\rho m^\sigma F_{i\rho\sigma})) \]

\[ I_I = \int d^4x \sqrt{-g}(\ell^\mu m^\nu F_{i\mu\nu}(n^\rho m^\sigma F_{i\rho\sigma}) + (\ell^\mu m^\nu F_{i\mu\nu})(n^\rho m^\sigma F_{i\rho\sigma})) \] (3.6)

\[ F_{j\mu\nu} = \partial_\mu A_{j\nu} - \partial_\nu A_{j\mu} - \gamma f_{jik} A_{i\mu} A_{k\nu} \]

Both actions are apparently invariant under the tetrad-Weyl transformation. Notice that only the null self-dual 2-forms appear in the actions. The non-null self-dual component does not appear in the action, because simply it is not multiplicatively transformed relative to the tetrad-Weyl transformation.

In fact these two actions are strongly related. The appearing gauge tensors \( F_{i\mu\nu} \) are each other duals, because \( \ell^{[\mu} m^{\nu]} \) and \( n^{[\rho} m^{\sigma]} \) are self-duals (relative to their corresponding metric). One of these two actions will be the starting point for the emergence of chromodynamics in the context of PCFT. In the hadronic sector, we will see how the \( I_R \) action implies field equations, which admit distributional solitons, which could be identified with the quarks [31].

We saw that the existence of a globally defined LCR-structure is the new (fundamental) mathematical notion, which corresponds to the metric structure of general relativity. In two dimensions all the smooth manifolds are LCR-manifolds, therefore in Polyakov functional integral we simply integrate over all 2-dimensional manifolds. But in four dimensions we have to consider only the LCR-manifolds. The simple way to impose this restriction is to use the Lagrange multiplier technique to add the following action term with the integrability conditions (1.7) on the tetrad

\[ I_C = \int d^4x \sqrt{-g}\{\phi_0(\ell^\mu m^\nu - \ell^\nu m^\mu)(\partial_\mu \ell_\nu) + \phi_1(\ell^\mu m^\nu - \ell^\nu m^\mu)(\partial_\mu m_\nu) + \phi_0(n^\mu m^\rho - n^\rho m^\mu)(\partial_\mu n_\nu) + \phi_1(n^\mu m^\rho - n^\rho m^\mu)(\partial_\mu m_\nu) + \text{c.conj.} \} \] (3.7)

These Lagrange multipliers make the complete action \( I = I_R + I_C \) self-consistent and the usual quantization techniques may be applied [24]. The action is formally renormalizable [24], because it is dimensionless and metric independent. Recall that even the (ordinary Weyl symmetric) conformal action is renormalizable, with the problem being that it contains non-removable negative-norm states, because of its higher order derivatives. The path-integral quantization of PCFT is also formulated [29] as functional summation of open and closed 4-dimensional LCR-manifolds in complete analogy to the summation of 2-dimensional surfaces in string theory [20]. These transition amplitudes of a quantum theory of LCR-manifolds provide (in principle) the self-consistent algorithms for the computation of the physical quantities.

The LCR-manifolds are defined with the existence of a tetrad (\( \ell, m; n, \overline{m} \)), which satisfies the integrability conditions. But if they are realizable [1], i.e. they admit structure coordinates (\( z^\alpha(x); \overline{z}^\beta(x) \)), which satisfy the conditions (2.3), they may be considered as real submanifolds of complex manifolds. In this case the structure coordinates may replace the tetrad as dynamical variables.
Then the LCR-transformation may be viewed as a proper vector bundle on a LCR-manifold, which will permit us better understand the gauge field solitonic solutions, which will be identified with the quarks.

The ambient complex manifold of the LCR-manifold (implied by the holomorphic Frobenius theorem) has two commuting complex structures. The trivial one defined by the complexification of the real spacetime coordinates and the second one defined by the structure coordinates \((z^\alpha(x); \bar{z}^\alpha(x))\). The holomorphic (relative to the trivial complex structure) transformation between these two complex structures is \((z^\alpha(r^b); \bar{z}^\alpha(r^b))\). This permit us to separate the total \((d)\) and partial \((\partial, \partial)\) exterior derivatives into the LCR-exterior derivative \((\partial', \partial'')\) as follows

\[
d = \partial + \partial = (\partial' + \partial') + (\partial + \partial')
\]

\[
\partial' f = \frac{\partial f}{\partial z^\alpha} dz^\alpha, \quad \partial'' f = \frac{\partial f}{\partial \bar{z}^\alpha} d\bar{z}^\alpha
\]  

\[A_\mu dr^b = A'_\alpha dz^\alpha + A''_\alpha d\bar{z}^\alpha
\]

In the last line I separate the 1-forms into marked 1-LCR-forms. In order to familiarize the reader with this new formalism we make the transcription \(A \rightarrow A' + A''\) in details

\[
A_\mu dr^\mu = A_\mu \delta_\mu^\nu dr^\nu = A_\mu (\ell^\mu n_\nu + n^\mu \ell_\nu - \bar{m}m_\nu - m^\mu \bar{m}_\nu) dr^\nu =
\]

\[
= [(n^\mu A_\mu)\ell_\alpha - (\bar{m}m^\mu A_\mu) m_\alpha] dz^\alpha + [(\ell^\mu A_\mu) n_\alpha - (m^\mu A_\mu) \bar{m}_\alpha] d\bar{z}^\alpha =
\]

\[
= A'_\alpha dz^\alpha + A''_\alpha d\bar{z}^\alpha
\]

The reader should be careful with the "complex bar" on \(m\). After the complexification of \(x\), I had to replace it with a tilde, but I hope it will be understood from the general content. Then the connections of the LCR-bundle take the form

\[
D' = \partial' + A', \quad D'' = \partial'' + A''
\]  

(3.10)

where the connection belongs to the Lie algebra of the gauge group.

If the ambient complex manifold is considered as a submanifold of the grassmannian space \(G_{4,2}\), the connection is essentially identified with the connection on the hypersurface of \(CP^3\) determined by the Kerr polynomial. The connections \(A'\) and \(A''\) correspond to the two branches of the hypersurface, which are necessary to define the LCR-structure. That is, to the left and right columns of the homogeneous coordinates of \(G_{4,2}\). This point of view and the chirality of gauge field solitonic solutions may explain why the pions are pseudoscalars. This will be explained in section VII.

Using the LCR-connection, the action takes the following compact form

\[
I_G = \int_M \left( \partial' A' + \gamma f_{jik} A'_i \wedge A'_k \right) \wedge \left( \partial'' A'' + \gamma f_{jim} A''_j \wedge A''_m \right) + \text{c.c.}
\]

\[
I_C = \int_M \left\{ \phi_0 dz^0 \wedge d\bar{z}^0 \wedge \bar{d}z^0 \wedge \bar{d}z^0 + \phi_0 dz^0 \wedge d\bar{z}^0 \wedge d\bar{z}^0 \wedge \bar{d}z^0 \right. \\
+ \phi dz^0 \wedge d\bar{z}^0 \wedge d\bar{z}^0 \wedge \bar{d}z^0 \right\}
\]  

(3.11)
The indication of the LCR-manifold $M$ in the integral sign is not necessary here, only to the Lagrange multipliers assure that $M$ admits a LCR-structure. I want only to stress the natural emergence of integral geometry, which will be very helpful to define the LCR-structure measure for the functional integration.

Using the relations (2.1), the action $I_G$ takes the better manageable form

$$I_G = \int d^4 x \left[ \det(\partial_z a^\alpha) \right] \left\{ (\partial_\mu x^\alpha)(\partial_\nu x^\alpha) F_{\mu\nu} \right\} \left\{ (\partial_\xi x^\alpha)(\partial_\chi x^\alpha) \right\} + c. c.]$$

where the $4 \times 4$ matrix $(\partial_\mu x^\alpha)$ is the inverse of $(\partial_\mu z^\beta)$. This form of the action permits the direct use of LCR-transformations to define conserved currents applying Noether’s theorem. Energy-momentum and angular momentum are defined as charges of such currents.

The action is invariant under the following two infinitesimal pseudo-conformal (LCR-structure preserving) transformations

$$\delta z^\beta \simeq \varepsilon \phi^\beta(z^\gamma) \, , \, \delta z^\tilde{\beta} \simeq \tilde{\varepsilon} \tilde{\phi}^\tilde{\beta}(\tilde{z}^\tilde{\gamma})$$

where

$$\delta \phi_0 = -\phi_0[(\partial_\xi x^\alpha)\varepsilon + (\partial_\xi \psi^\alpha)\tilde{\varepsilon}]$$

$$\delta \tilde{\phi}_0 = -\tilde{\phi}_0[(\partial_\chi x^\alpha)\varepsilon + (\partial_\chi \psi^\alpha)\tilde{\varepsilon}]$$

$$\delta \phi = -\phi[(\partial_\xi x^\alpha)\varepsilon + (\partial_\xi \psi^\alpha)\tilde{\varepsilon}]$$

Notice that the transformations of the "left" and "right" structure coordinates are independent, like the conformal transformations in the ordinary 2-dimensional conformal field theory (the Polyakov action).

Using such a general transformation we derive the conservation of the following "left" and "right" LCR-currents

$$J^\lambda \equiv -\det(\partial_z a^\alpha) \, F_{j_01} \psi^\gamma F_{j_0\gamma} e^{\tilde{\alpha} \tilde{\beta}}(\partial_\gamma x^\lambda) - \epsilon_{\alpha\beta} \psi^\gamma \epsilon^{\lambda r s}(\partial_\gamma z^\beta)[\phi_0(\partial_\mu z^\gamma)(\partial_\sigma z^\alpha) + \phi(\partial_\mu z^\gamma)(\partial_\sigma z^\alpha)]$$

$$\tilde{J}^\lambda \equiv -\det(\partial_z a^\alpha) \, \psi^\gamma \epsilon^{\lambda r s}(\partial_\gamma x^\lambda) F_{j_01} - \epsilon_{\alpha\beta} \psi^\gamma \epsilon^{\lambda r s}(\partial_\gamma z^\beta)[\phi_0(\partial_\mu z^\gamma)(\partial_\sigma z^\alpha) + \phi(\partial_\mu z^\gamma)(\partial_\sigma z^\alpha)]$$

An appropriate definition of the structure coordinates and their relation to the Poincaré group permit us to find the energy-momentum and angular momentum conserving currents. Notice that the explicit contribution of the gluon field indicate that we could in principle find a way to calculate the mass differences between leptons and hadrons.

The canonical and BRST quantization of the PCFT action is straightforward and I will not repeat it here. The path-integral quantization is analogous to that of the Polyakov action, where the measures are geometric. Here we
sum over the 4-dimensional LCR-structures instead of the 2-dimensional complex structures, because the Wick rotation destroys the LCR-structure. On the other hand, as we will see below, here the elementary particles are solitonic (distributional) configurations. The computation of path-integrals for soliton-soliton scattering processes looks quite formidable. Therefore I will use more intuitive solitonic technics to describe the experimental consequences of PCFT.

4 DERIVATION OF EINSTEIN’S GRAVITY

In this section I will properly define gravity from the LCR-structure defining conditions (2.10). As I mentioned in the introduction, because of its tetrad-Weyl symmetry a LCR-structure does not uniquely define a tetrad \((\ell, m; n, \overline{m})\). Hence

\[
[g_{\mu\nu}] = \ell_\mu n_\nu + \ell_\nu n_\mu - m_\mu \overline{m}_\nu - m_\nu \overline{m}_\mu
\]  

(4.1)

defines a class of symmetric tensors. In this form the LCR-tetrad is the null tetrad of the Newman-Penrose (NP) formalism\(^6\), for all the metrics of the class. Recall that the NP formalism is essentially the Cartan formalism adapted to the null tetrad. The LCR-conditions (1.7) are the geodetic and shear-free conditions of the null tetrad, which in the NP formalism coincide with the annihilation of the spin coefficients \(\kappa = \sigma = \lambda = \nu = 0\). This imposes the restriction to the Einstein metric to admit a geodetic and shear-free null tetrad (two geodetic and shear-free congruences). This restriction is experimentally in favor to the PCFT, because all the observed spacetimes have this property.

In the context of riemannian geometry, where the metric is the fundamental structure and the tetrad is derived, we have the local \(\text{SO}(1, 3)\) symmetry of the tetrad. But in PCFT, where the LCR-structure is the dynamical variable, there is no local \(\text{SO}(1, 3)\) symmetry. Instead we have the tetrad-Weyl symmetry (1.8).

Let us now consider the class of metrics \([\eta_{\mu\nu}]\), which are compatible with the Minkowski spacetime. The Penrose form\(^19\) of the Kerr solution for two geodetic and shear-free flat congruences is

\[
X^{mi}E_{mn}X^{nj} = 0 \quad , \quad K(X^{mi}) = 0
\]

\[
E_{mn} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]  

(4.2)

where \(X^{mi}\) are the homogeneous coordinates of \(G_{4,2}\) already defined in (2.11). The first relation implies that the projective coordinates \(r_{A'}_A\) are hermitian, which means that the Shilov boundary of the \(SU(2, 2)\) symmetric classical domain is the "real axis" of \(\mathbb{C}^4\), identified with the Minkowski spacetime. Besides, these are exactly the four conditions (2.10), which determine the LCR-structure. The relation \(K(X^{mi}) = 0\) is the Kerr holomorphic function, which determines the hypersurface of \(CP^3\). Notice that for the Minkowski spacetime, the conditions required for two null congruences to be geodetic and shear-free are only
the (corresponding two or one common) Kerr holomorphic functions. In the context of PCFT, the important point is that the flat spacetime is defined by the algebraic conditions of the LCR-structure without direct reference to the metric.

I want to point out that PCFT is intended to describe the elementary particles and not the macroscopic bodies, which must be viewed as made up of elementary particles. The macroscopic spherically symmetric metrics are simple approximations of macroscopic bodies. In the context of quantum field theory the crucial property is the Poincaré Lie algebra. We saw that in the context of LCR-structures, the identification of a Poincaré Lie algebra is achieved through a local embedding (immersion) of the LCR-structure ambient complex manifold into an affine subspace of $G_{4,2}$, viewed as a pair of points of $CP^3$. The LCR-manifolds, which admit a $G_{4,2}$ immersion, will be called ”particles”, and the rest ”unparticles”. Notice that the class of metrics $g_{\mu\nu}$ can be defined for unparticles, but a Poincaré algebra cannot be properly defined.

Let us also clarify the real analyticity problems implied by the projection of ambient complex manifold down to its LCR-manifold. This is understood as a limit of the complex structure coordinates in the ambient complex manifold down to the LCR-subsurface. In higher dimensional surfaces this can be done in an infinite number of cones with edge the limit point of the LCR-surface. In a real analytic LCR-structure, all these limits coincide. But it is not generally necessary. If it does not happen, the structure coordinates $z^b(x)$ become generalized functions. Notice that this point of view is essentially the Sato’s approach to generalized functions [15].

Let us now consider the quadratic surfaces of $CP^3$ implied by the linear trajectory (2.15) with all the constants real. After a Poincaré transformation, $v^a = (1, 0, 0, 0)$ and $c^a = 0$. The LCR-structure is

$$z^0 = t - r, \quad z^1 = e^{i\varphi} \tan \frac{\theta}{2} = \frac{x + iy}{r + z},$$
$$z^{\bar{0}} = t + r, \quad z^{\bar{1}} = e^{-i\varphi} \tan \frac{\theta}{2} = \bar{z}$$

$$\ell = dz^0 = \frac{1}{r(r+z)} (r dt - \vec{r} \, d\vec{r}) \quad n = dz^{\bar{0}} = \frac{1}{r(r+z)} (r dt + \vec{r} \, d\vec{r})$$
$$m = dz^1 = \frac{1}{r(r+z)^2} \left( (r^2 + rz + x^2) + ixy \right) dx +$$
$$+ (xy + (r^2 + rz + y^2) dy + [x(r + z) + iy(r + z)] dz$$

$$\ell \wedge n \wedge m \wedge \bar{m} = \frac{4 \pi i}{r(r+z)} dt \wedge dx \wedge dy \wedge dz \quad (4.3)$$

which is degenerate ($\equiv$ with vanishing relative invariants). It has singularities at $r = 0 = r + z$ and possibly at the projective infinity, where the affine coordinates are not valid. It is known that the grassmannian manifold $G_{4,2}$ admits an affine space where the infinite LCR-submanifold $G_{4,2}$ is compactified. It is apparently interesting to see how our infinite physical ”flat spacetime” globally appears in the precise ”unphysical” affine chart, where the Poincaré transformation is no longer quasi-linear. For that we have to find the Cayley transformation between
the corresponding projective coordinates. Consider the transformation

\[
Y = \left( \begin{array}{c} \mu \\ \tilde{z}\mu \end{array} \right) = B \left( \begin{array}{cc} \lambda & 0 \\ 0 & -i\bar{\lambda} \end{array} \right), \quad B = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} I & I \\ I & -I \end{array} \right)
\]

\[E' = BEB^\dagger = \left( \begin{array}{cc} I & 0 \\ 0 & -I \end{array} \right)\]

(4.4)

which gives the Cayley transformation between the bounded \(E'\) and the unbounded realization \(E\) of the domain

\[
\tilde{r} = i(I - \tilde{z})(I + \tilde{z})^{-1} = i(I + \tilde{z})^{-1}(I - \tilde{z}) , \quad \tilde{r} = 0
\]

\[
\tilde{z} = (iI - \tilde{r})(iI + \tilde{r})^{-1} = (iI + \tilde{r})^{-1}(iI - \tilde{r}) , \quad I - \tilde{z}^\dagger\tilde{z} = 0
\]

(4.5)

They are boundaries of the bounded and unbounded realizations of the \(SU(2,2)\) classical domain. It transforms the unbounded real surface (4.2) \(R^4\) to the bounded \(U(2)\) boundary of the classical domain. It is a \(1 \leftrightarrow 2\) mapping. Using the parameterization

\[
U = e^{i\tau} \left( \begin{array}{cc} \cos \rho + i \sin \rho \cos \sigma & -i \sin \rho \sin \sigma e^{-i\chi} \\ -i \sin \rho \sin \sigma e^{i\chi} & \cos \rho - i \sin \rho \cos \sigma \end{array} \right)
\]

\[
\tau \in (-\pi, \pi) , \quad \rho \in [0, 2\pi) , \quad \sigma \in [0, \pi) , \quad \chi \in (0, 2\pi)
\]

(4.6)

we see that the one \(R^4\) patch is

\[
x_+^0 = \frac{\sin \tau}{\cos \tau + \cos \rho}
\]

\[
x_+^1 + ix_+^2 = \frac{\sin \rho}{\cos \tau + \cos \rho} \sin \sigma e^{i\chi}
\]

\[
x_+^3 = \frac{\sin \rho}{\cos \tau + \cos \rho} \cos \sigma
\]

\[
\tau \in (-\pi, \pi) , \quad \rho \in [0, \pi) , \quad \sigma \in [0, \pi) , \quad \chi \in (0, 2\pi)
\]

\[
s := \frac{\sin \rho}{\cos \tau + \cos \rho} > 0 \quad \leftrightarrow \quad \cos \tau + \cos \rho > 0
\]

(4.7)

and the second \(R^4\) patch is

\[
x_-^0 = \frac{\sin \tau}{\cos \tau + \cos \rho}
\]

\[
x_-^1 + ix_-^2 = -\frac{\sin \rho}{\cos \tau + \cos \rho} \sin \sigma e^{i\chi}
\]

\[
x_-^3 = -\frac{\sin \rho}{\cos \tau + \cos \rho} \cos \sigma
\]

\[
\tau \in (-\pi, \pi) , \quad \rho \in [0, \pi) , \quad \sigma \in [0, \pi) , \quad \chi \in (0, 2\pi)
\]

\[
s := \frac{\sin \rho}{\cos \tau + \cos \rho} < 0 \quad \leftrightarrow \quad \cos \tau + \cos \rho < 0
\]

(4.8)

in order to cover the rest of \(U(2)\) universe. Apparently these two patches do not overlap, therefore they cannot form an atlas.

Let us now look for a quadratic polynomial, which is invariant under the left and right massless Poincaré transformations \([E = \pm p^3]\). From the detailed
analysis (presented in the neutrino section), the unique (in the chosen Poincaré group) degenerate quadratic surface of $CP^3$ is

$$K(Z^m) = Z^0Z^1 = 0$$  \hspace{1cm} (4.9)

The structure coordinates are

$$z^0 = t - z , \quad z^1 = x + iy$$  \hspace{1cm} (4.10)

which are regular in $\mathbb{R}^4$. In compacted coordinates they have the form

$$z^0 = \frac{\sin \tau - \sin \rho \cos \theta}{\cos \tau + \cos \rho} \sin \theta e^{i\phi}$$

$$z^{-0} = \frac{\sin \tau + \sin \rho \cos \theta}{\cos \tau + \cos \rho} \sin \theta e^{i\phi}$$

(4.11)

with an apparent singularity at $\pm \text{scri}$ ($\cos \tau + \cos \rho = 0$). It seems to be a quite reasonable flat vacuum. Notice that the following relation

$$dz^0 = \frac{x^i}{2r}dz^i - \frac{x^j}{2r}dz^{i\bar{j}} - \frac{x^i}{2r}dz^{i\bar{j}}$$

(4.12)

indicates that the LCR-structures (4.3) and (4.9) are not equivalent, despite the fact that both are degenerate.

The algebraic definition of the "flat" class of metrics $[\eta_{\mu\nu}]$ indicates the algebraic definition of gravity. We simply replace the algebraic LCR-structure conditions (4.2) with

$$X^\dagger E X = \begin{pmatrix} G_{11}(X^{m1}, X^{m1}) & G_{12}(X^{m1}, X^{m2}) \cr G_{12}(X^{m1}, X^{m2}) & G_{22}(X^{m2}, X^{m2}) \end{pmatrix}$$

$$K(X^{mj}) = 0$$

(4.13)

where $G_{ij} = G_{ij}(X^{mj}, X^{mj})$ are homogeneous functions with this precise dependence on the two points of the algebraic variety determined by the Kerr polynomial. The non-vanishing of these terms implies that the complex component of $r^b = x^b + iy^b$ does not vanish and gravity emerges. In order to compute this gravity $g^b(x^a)$ generated by LCR-structure, it is convenient to use the projective coordinates of the grassmannian space $G_{4,2}$ in a precise coordinate patch (affine variety) with the following spinorial form

$$X^{mj} = \begin{pmatrix} \lambda_{A_{\bar{B}}} \\ -i\epsilon_{A_{\bar{B}}B} \lambda_{Bj} \end{pmatrix}$$

(4.14)

of the rank-2 matrix $X^{mj}$, and define the tetrad

$$L^a = \frac{1}{\sqrt{2}} \lambda^{A1} \lambda^{B1} \sigma^a_{AB} , \quad N^a = \frac{1}{\sqrt{2}} \lambda^{A2} \lambda^{B2} \sigma^a_{AB} , \quad M^a = \frac{1}{\sqrt{2}} \lambda^{A1} \lambda^{B1} \sigma^a_{AB}$$

$$\epsilon_{AB} \lambda^{A1} \lambda^{B2} = 1$$

(4.15)
which is null relative to the Minkowski metric $\eta_{ab}$. Then the above relations (4.13) take the form

$$2\sqrt{2}y^a L_a = G_{11}(\overline{Y^{m1}}, Y^{n1})$$

$$2\sqrt{2}y^a M_a = G_{12}(\overline{Y^{m1}}, Y^{n2})$$

$$2\sqrt{2}y^a N_a = G_{22}(\overline{Y^{m2}}, Y^{n2})$$

Recall that $y^a$ is the imaginary part of the projective coordinates $r^a = x^a + iy^a$ defined by the relation $r_{A'B'} = r^a \sigma_a A'B'$ and $\sigma_a A'B'$ being the identity and the three Pauli matrices (2.11-2.13). The normalization of the spinors is permitted, because of the homogeneity of the functions. These conditions are formally "solved" by

$$y^a = \frac{1}{2\sqrt{2}}\left[G_{22} N_a + G_{11} L_a - G_{12} M_a - G_{12} M_a\right]$$

which combined with the computation of $\lambda^A_i$ as functions of $r^a$, using the Kerr conditions $K_i(X^{m}) = 0$, permit us to perturbatively compute $y^a$ as functions of the real part of $r^a$. From the physical point of view, we may say that this procedure gives the gravitational "dressing" of the soliton (particle) in the form $y^a = y^a(x)$ of the (totally real) lorentzian CR-submanifold expressed in the projective coordinates of $G_{4,2}$. The explicit form of $y^a(x)$ is implied by the precise dependence of $G_{ij}(X^{m1}, X^{m2})$, considered real analytic, and their expansion into a series relative to $y^a$. This is just a simple application of the implicit function theorem.

The definition of the Einstein metric permit us to define energy-momentum and angular momentum as conserved quantities in the linearized Einstein gravity approximation. We find the following linearized gravity relations in the limit

$$g_{\mu\nu} = \eta_{\mu\nu} + kh_{\mu\nu} + O(k^2)$$

$$\hat{R}_{\mu\rho\sigma\tau} = \lim_{k\to0}(k^{-1}R_{\mu\rho\sigma\tau}) = 2\partial_{\mu}\partial_{[\sigma} h_{\tau]} + \frac{1}{2}\delta^{\nu}_{\mu}\hat{R}$$

for the curvature tensor. The second Bianchi identities imply the conservation condition of the Einstein tensor

$$\partial_{\mu}\hat{E}^\mu_{\nu} = \partial_{\mu}[\hat{R}^\mu_{\nu} - \frac{1}{2}\delta^\mu_{\nu}\hat{R}] = 0$$

$$\hat{E}_{\rho\tau} = \hat{R}_{\rho\tau} - \frac{1}{2}\eta_{\rho\tau}\hat{R} = \frac{1}{2}[\partial^2 h_{\rho\tau} + \partial_{\rho}\partial_{\tau} h_{\nu\nu} - \partial_{\rho}(\partial_{\nu} h_{\tau}) - \partial_{\tau}(\partial_{\nu} h_{\rho})]$$

This means that the linearized Einstein tensor defines the energy-momentum density as a preserved tensor distribution.

The standard model does not explain the existence of only three generations of leptons and quarks. In the context of PCFT context the three generations of flavors is imposed by gravity, despite the fact that the standard model does not contain gravity. It is well known in general relativity, that in a geodetic and
shear-free null tetrad the first $\Psi_0$ and last $\Psi_4$ components of the Weyl tensor in the Newman-Penrose formalism vanish, i.e.

$$
\Psi_0 = \Psi_{ABCD}o^Ao^Bo^Co^D = 0 , \quad \Psi_4 = \Psi_{ABCD}^A_1B^C_1D = 0
$$

(4.20)

where $\Psi_{ABCD}$ is the conformal tensor in spinorial coordinates and $o^A, i^A$ is the geodetic and shear-free spinor dyad. In the zero gravity approximation we have $o^A = \lambda^A_1$ and $i^A = \lambda^A_2$, the two spinors which appear in the homogeneous coordinates (2.11) of a flat LCR-structure. Hence in the linearized Einstein gravity approximation we have the relations

$$
\Psi_{ABCD}o^Ao^Bo^Co^D \simeq k\hat{\Psi}_{ABCD}\lambda^A_1\lambda^B_1\lambda^C_1\lambda^D_1 + O(k^2) = 0
$$

(4.21)

That is, at every point of spacetime a gravitating (with non vanishing conformal tensor) LCR-manifold is implied by at least a quadratic hypersurface of $CP^3$ (already known) and at most to a quartic branched hypersurface of $CP^3$. This restriction imposes the existence of three generations of solitonic LCR-manifolds, which are identified with leptons. We will see below that they are the Petrov type D (the generation of the electron), the Petrov type II (the muon generation) and the Petrov type I (the tau generation). The Petrov type III spacetimes (LCR-manifolds) may not be realizable as elementary particles.

Generalizing the above simple examples of scalar LCR-structures we may consider the general real Newman trajectories $\xi^b(\tau)$, which may generally viewed as ”interacting” LCR-structures. The $CP^3$ embedding (2.14) imply

$$(r^a - A^A(\tau))\lambda^A = 0
$$

(4.22)

which completely determines the holomorphic solutions $\tau(r^a)$ and projectively $\lambda^A(r^a)$. The term ”real” here means that all the parameters of the function $\xi^b(\cdot)$ are real or the mathematically correct condition that the coefficients of a $r^a$-local Taylor expansion are real numbers, but the solutions $\tau(r^a), \lambda^A(r^a)$ may be complex. The relations (projections) $r^a(x^b)$ down to the real plane $\mathbb{R}^4$ is implied by (4.17). The number of solutions $\tau(r^a), \lambda^A(r^a)$ depends on the spacetime points and essentially characterize the multiplicity of the generally forth degree polynomial of the conformal tensor $\Psi_{ABCD}$. Hence we may have the following possibilities: 1) At the points $r^a(x^b)$ where we have one solution $\tau(r^a(x))$ and $\lambda^A(r^a(x))$, the corresponding LCR-manifold needs a second trajectory to be defined. Using as second trajectory that of the vacuum, then we interpret it as an object with only one chirality. 2) At the points with two solutions $\tau_i(r^a), \lambda^A_i(r^a)$ with $i = 1, 2$, is the most common form of the LCR-manifold, because it provides the classical notion of a generally interacting particle. This is completely clarified in the newtonian approximation. I first normalize the parameter $\tau$ as $\xi^b = ct$, where the velocity of light $c$ is explicitly written. Then, the condition for the existence of non-vanishing solutions $\lambda^A(r^a)$ from (4.22),
takes the form
\[
(c r^0 - c r)^2 - (r^j - \xi^j(\tau))^2 = 0 \\
z^0 = \tau_1 = r^0 - \frac{1}{2} \sqrt{(r^j - \xi^j(\tau_1))^2} \approx t - \frac{1}{2} \sqrt{(r^j - \xi^j(t))^2} + O(1/c) \\
z^0 = \tau_2 = r^0 + \frac{1}{2} \sqrt{(r^j - \xi^j(\tau_2))^2} \approx t + \frac{1}{2} \sqrt{(r^j - \xi^j(t))^2} + O(1/c)
\]  
(4.23)

We see that the structure coordinates \(z^0, \tilde{z}^0\) are retarded and advanced “wave” functions indicating that physical causality comes from the LCR-structure. Besides, these “waves” come from the same point \(r^j - \xi^j(t) = 0\), interpreted as the trajectory of the LCR-manifold (particle). 3) At the points with three solutions \(\tau_i, i = 1, 2, 3\), we may generally have the three LCR-structures \((\tau_1, \tau_2), (\tau_1, \tau_3), (\tau_2, \tau_3)\) and their complex conjugates. Recall that integral curves bifurcate. Hence we may have a LCR-structure (two roots \((\tau'_1, \tau'_2)\), one particle) in which and at a given point the integral curve \(\ell^\mu \partial_\mu\) bifurcates \(\tau'_1 \to (\tau_1, \tau_2)\). Then the natural implied LCR-structure (particle) evolution is
\[
(\tau'_1, \tau'_2) \to (\tau_1, \tau'_2) + (\tau_2, \tau'_2)
\]  
(4.24)

which is a typical particle disintegration. 4) Using the linearized gravity approximation, I have already showed that the LCR-manifold with four solutions is the largest Einstein spacetime (with gravity). The implied bifurcations are more complicated. The natural particle picture is when two decoupled stable particles, one LCR-manifold with two decoupled pairs \((\tau_1, \tau_2), (\tau_3, \tau_4)\) of solutions through the formation of LCR-structures with up to four \(\lambda^A(r^a)\) solutions, which finally disintegrate into stable particles, considered embedded into a flat spacetime (which is compatible with any number of geodetic and shear free congruences).

5 ELECTRON AND ELECTRODYNAMICS

Like the 2-dimensional Polyakov action (and its supersymmetric evolution), the present 4-dimensional PCFT does not explicitly contain the observed particles as independent fields. Therefore they have to be found as stable configurations. In string theory the guiding clue was the Poincaré group of the 26-dimensional Minkowski space, emerging after the identification of the \(X^\mu(x)\) field with the embedding function of the string in to the 26-dimensional Minkowski space. It is well known that string theory tried to identify the observed elementary particles with the lowest string modes. In the context of PCFT the gluon field is identified with the gauge field (LCR vector bundle), which explicitly appears in the action, and the observed elementary particles are identified with precise (distributional) solitons.

If we identify the 4-dimensional flat spacetime with the boundary of the \(SU(2, 2)\) classical domain, the linear subgroup of \(SU(2, 2)\), which fixes the projective “infinity” (the scri in the Penrose terminology), becomes the physical Poincaré×Dilation group. The particles will emerge as stable solitonic
(configurations) generalized functions (Schwartz distributions or Sato’s hyperfunctions) viewed as potentials of their distributional sources identified with the fermionic flavors (leptons and quarks). The stable particles (electron, neutrino, and up and down quarks) admit the automorphisms of time translation and z-axis rotation, which make them eigenstates of the corresponding generators of a Poincaré representation. The unstable (decaying) elementary particles admit only the z-rotation automorphism. That is we only consider that they can have only exact spin.

The Poincaré × Dilation transformation in the Siegel (chiral) realization is

\[
\begin{pmatrix}
\lambda' \\
-ir'\lambda'
\end{pmatrix} = \begin{pmatrix} B & 0 \\ -iT & (B^\dagger)^{-1} \end{pmatrix} \begin{pmatrix}
\lambda \\
-ir\lambda
\end{pmatrix} \tag{5.1}
\]

\[r' = (B^{-1})^\dagger r (B^{-1}) + T \quad T^\dagger = T\]

where for \(\det B = 1\) is the Poincaré transformation. This is an automorphism of the degenerate LCR-structure \(4.3\). The proof \(29\) uses the Newman replacement of the quadratic Kerr polynomial \(?\) with a trajectory \(\xi^a = (\tau, 0, 0, 0)\).

Under a Poincaré × Dilation transformation this trajectory becomes a real linear trajectory \(\xi^a = v^a\tau + c^a (2.15)\) with \(v^a v^b \eta_{ab} = 1\). The two structure coordinates \(z_0 = \tau_1\) and \(\tilde{z}_0 = \tau_2\) are determined from

\[(x^a - \xi^a(\tau_j))^2 = 0 \tag{5.2}\]

for each column of the homogeneous coordinates \(X^{mi}\), and found to be invariant, because the above defining form is invariant. The spinors \(\lambda^A_j\) are the dyad, which determine the null vectors \(\Delta^a_{(j)} = (x^a - \xi^a(\tau_j))\), that is \(19\)

\[(x^a - \xi^a(\tau_j)) = \lambda^j \sigma^a \lambda^j \tag{5.3}\]

with the same trivial trajectory for both \(j = 1, 2\) before and after the Poincaré transformation. The (real) translation does not affect the spinors. The Lorentz transformation changes the trajectory, but its form remains the same for \(j = 1, 2\). We precisely find

\[
\begin{pmatrix}
\Delta^0_{(j)} \\
\Delta_{(j)} \\
\Delta^1_{(j)} \\
\Delta^2_{(j)}
\end{pmatrix} = \begin{pmatrix}
\Delta & \Delta & \Delta & \Delta \\
\Delta & \Delta & \Delta & \Delta \\
\Delta & \Delta & \Delta & \Delta \\
\Delta & \Delta & \Delta & \Delta
\end{pmatrix} = \begin{pmatrix}
\lambda^0_j \lambda^0_j + \lambda^1_j \lambda^1_j \\
\lambda^1_j \lambda^0_j + \lambda^0_j \lambda^1_j \\
i\lambda^1_j \lambda^0_j - i\lambda^0_j \lambda^1_j \\
i\lambda^0_j \lambda^0_j - \lambda^1_j \lambda^1_j
\end{pmatrix} \tag{5.4}
\]

\[\Delta_{(1)} = \Delta \quad \Delta_{(2)} = -\Delta\]

Before the Lorentz transformation with zero velocity the LCR-structure condition \(X^{m1}E_{mn}X^{n2} = 0\) implies \(z^1 = \bar{z}^1\). After the transformation \(\Delta, \theta, \varphi\) change, but the relation of the structure coordinates remains the same, \(z^i = \bar{z}^i\), i.e.

\[z^1 = \frac{\lambda^1}{\chi^{n1}} = \frac{\Delta}{\chi^2} \sin \theta e^{i\varphi} \quad , \quad \bar{z}^1 = -\frac{\lambda^2}{\chi^{n2}} = \frac{\Delta}{\chi^2} \sin \theta e^{-i\varphi} \tag{5.5}\]

26
From their definition the two spinors have opposite chiralities. That is, even the vacuum configurations "see" the two chiralities, on which the standard model is built up.

We will now look for solitonic LCR-structures. The knowledge of the "physical" Poincaré group permit us to look for static and axially symmetric LCR-manifolds. That is massive LCR-structures, which admit time translation and z-rotations as automorphisms. These stable solitons are states of the Hilbert space and hence eigenstates of the translation and the z-rotation generators.

For a LCR-manifold embeddable in $G_{4,2}$, I consider the following structure coordinates and LCR-conditions

$$z^0 = iX^{21}, \quad z^1 = X^{11}, \quad z^0 = iX^{32}, \quad z^1 = -X^{03}$$

$$z^0 - U\left(\frac{z^0 + z^1}{2}, z^0, z^1\right) = 0, \quad z^1 - Z(z^0, z^0, z^1) = 0, \quad z^0 - z^1 = 0 \quad (5.6)$$

Then the infinitesimal time-translation and z-rotation are

$$\delta X^{0i} = 0, \quad \delta X^{1i} = 0, \quad \delta X^{2i} = -i\epsilon^0 X^{0i}, \quad \delta X^{3i} = -i\epsilon^0 X^{1i}$$
$$\delta z^0 = \epsilon^0, \quad \delta z^1 = 0, \quad \delta z^0 = \epsilon^0, \quad \delta z^1 = 0$$

$$\delta X^{0i} = -i\epsilon^{12} X^{0i}, \quad \delta X^{1i} = i\epsilon^{12} X^{1i}, \quad \delta X^{2i} = -i\epsilon^{12} X^{2i}, \quad \delta X^{3i} = i\epsilon^{12} X^{3i}$$
$$\delta z^0 = 0, \quad \delta z^1 = i\epsilon^{12} z^1, \quad \delta z^0 = 0, \quad \delta z^1 = -i\epsilon^{12} z^1$$

and the LCR-structure conditions become

$$\frac{z^0}{2i} - U(z^0 z^1) = 0, \quad z^1 - z^0 W(z^0 z^0) = 0, \quad \frac{z^1}{2i} - V(z^1 z^1) = 0 \quad (5.7)$$

I also found that only the quadratic Kerr polynomial

$$K(X^m) = X^1 X^2 - X^0 X^3 + 2a X^0 X^1 = 0 \quad (5.9)$$

admits these automorphisms among all the polynomials of maximal degree four.

Notice that if we try to impose the dilation as an additional automorphism, we find $a = 0$, which is the "spherical" degenerate LCR-structure (??). The quite general LCR-tetrad (2.19) (with $\Delta(r)$ arbitrary) satisfies these conditions, the additional condition of asymptotic flatness at null infinity

$$X^{m1} E_{mn} X^{n2} = 0 = X^{m2} E_{mn} X^{n2}$$

$$\frac{z^0}{2i} + 2a \frac{z^1 z^0}{1 + z^1 z^1} = 0, \quad z^1 - z^0 W(z^0 z^0) = 0, \quad \frac{z^1}{2i} - 2a \frac{z^1 z^1}{1 + z^1 z^1} = 0 \quad (5.10)$$

and a symmetry between the left and right chiral columns $z^1 z^1 = z^1 z^1$. Its
embedding in $G_{4,2}$ is

$$
X^{mi} = \begin{pmatrix}
1 & -z^3 \\
z^1 & 1 \\
-iz^0 & iz^1(z^0 - 2ia) \\
-iz^1(z^0 + 2ia) & -iz^0
\end{pmatrix}
$$

(5.11)

$$
z^0 = t - f_0(r) - 2ia \sin^2 \frac{\theta}{2}, \quad z^1 = e^{i\varphi}e^{-iaf_1(r)} \tan \frac{\theta}{2}
$$

$$
z^\bar{0} = t + f_0(r) + 2ia \sin \frac{\theta}{2}, \quad z^\bar{1} = e^{-i\varphi}e^{-iaf_1(r)} \tan \frac{\theta}{2}
$$

$$
f_0(r) = \int \frac{e^{2a^2} \Delta dr}{\Delta}, \quad f_1(r) = \int \frac{1}{\Delta} dr
$$

The gravitational dressing of the electron can be easily computed. It is stable relative to the vacuum, because it has non-vanishing all its relative-invariants $\Phi_j$.

A different way to find a static and axially symmetric LCR-structure is first to solve the problem for flat compatible LCR-structures which satisfy the Kerr polynomial (5.9). After we apply the well known Kerr-Schild ansatz to find the corresponding curved LCR-structure. The final result [23] is the same LCR-manifold (2.19).

The general quadratic form, which is invariant (but not automorphic) under a Poincaré transformation is

$$
A_{mn} Z^m Z^n = 0
$$

(5.12)

$$
P = \begin{pmatrix}
\omega & P^+ \\
P^- & 0
\end{pmatrix}, \quad P = \begin{pmatrix}-(p^1 - ip^2) & -p^0 + p^3 \\
p^0 + p^3 & (p^1 + ip^2)\end{pmatrix} = -p \epsilon
$$

The variables $p^\mu$ are the momentum (boost) parameters and $\omega$ is the spin. If we first make a boost transformation, we can annihilate the momenta. After we make a general complex translation

$$
\begin{pmatrix}
X'_1 \\
X'_2
\end{pmatrix} = \begin{pmatrix}
I & 0 \\
C & I
\end{pmatrix} \begin{pmatrix}
X_1 \\
X_2
\end{pmatrix}
$$

(5.13)

$$
r' = r + iC, \quad C^\dagger \neq \pm C
$$

Then the spin matrix transforms as follows

$$
\omega' = \omega + 2m \begin{pmatrix}-(C^1 - iC^2) & C^3 \\
C^3 & (C^1 + iC^2)\end{pmatrix}
$$

(5.14)

We see that a real translation ($C = -iT$) cannot remove the spin matrix. But a complex translation can do it. This means that the spin can be considered as a complex space translation in $G_{4,2}$. That is, the spin can be considered as an imaginary space translation in $G_{4,2}$, which explains why the Newman "magic" complex translation [18] of the Schwartzschild metric implies the Kerr metric. Besides notice that a complex time translation does not affect the quadric.
5.1 Microlocal analysis of the electron

We saw that the LCR-tetrad defines the class of symmetric tensors $[g_{\mu\nu}]$, which appear as the gravitation field. Besides the LCR-tetrad defines the class of antisymmetric tensors

$$[J_{\mu\nu}] = \ell_{\mu}n_{\nu} - \ell_{\nu}n_{\mu} - m_{\mu}\overline{m}_{\nu} + m_{\nu}\overline{m}_{\mu}$$  \hspace{1cm} (5.15)

Flaherty observed\[8\] that the metric $g_{\mu\nu}$ defines an integrable pseudo-complex structure (pseudo, because it is not a real tensor)

$$J_{\mu\nu} = \ell_{\mu}n_{\nu} - n_{\mu}\ell_{\nu} - m_{\mu}\overline{m}_{\nu} + \overline{m}_{\mu}m_{\nu}$$  \hspace{1cm} (5.16)

Its Nijenhuis integrability conditions coincide with the LCR-structure conditions. Notice that this tensor is invariant under the tetrad-Weyl transformation and that the LCR-tetrad are eigenvectors of this tensor. That is $g_{\mu\nu}$ and $J_{\mu\nu}$ determine the LCR-structure.

In the special case of the stable LCR-structure (2.19), the self-dual 2-form admits a multiplicative function, which makes it ”closed” up to a singular source.

$$G^+ = \frac{2C}{(r + i\alpha \cos \theta)^2} (\ell \wedge n - m \wedge \overline{m}) = G - i \ast G$$  \hspace{1cm} (5.17)

where $C$ is an arbitrary complex constant. That is, it defines an electromagnetic field $G$ determined by the self-dual 2-form $G^+$. It is closed outside a distributional singularity concentrated at the ring-singularity of the LCR-manifold, which provides a generally complex (electric plus magnetic) charge. Hence for an arbitrary complex constant $C$, this complex 2-form defines a real 2-form $G$ such that

$$dG = - \ast j_m, \quad d * G = - \ast j_e$$  \hspace{1cm} (5.18)

where $j_e$ and $j_m$ are the ”electric” and ”magnetic” currents. These are apparently analogous to the symmetric Maxwell equations (with both electric and magnetic monopoles), which were used by Dirac to prove the quantization\[7\] of the electric charge. It implies that the general electric charge is quantized\[25\]. But the apparent symmetry under the duality rotation absorbs the magnetic charge (or electric charge) leaving detectable only one kind of monopoles, as observed in nature

$$dG = 0, \quad d * G = - \ast j_e$$  \hspace{1cm} (5.19)

That is, here we have a ”self-quantization” of the electric charge. But once fixed, the conserved electric charge reduces the general tetrad-Weyl symmetry\[1,8\] down to the ordinary Weyl symmetry of the electromagnetic field. The precise tetrad-Weyl factors used in (2.21) give a metric, which coincides with the linearized gravity approximation, and hence define the Poincaré conserved quantities. This fact fixes the remaining ordinary Weyl transformation. That is the precise tetrad-Weyl factors, which provide the conserved charge, momentum and angular momentum of the electron, fix (break) the tetrad-Weyl symmetry.

Now it is trivial to show that the positron is the conjugate LCR-structure ($z^\alpha, z^\beta$), which corresponds to the tetrad $(\ell, \overline{m}; n, m)$. From the definition of the
electromagnetic form (5.17) we easily see that its electric charge has opposite sign from that of the electron LCR-manifold. Hence we have to identify the conjugate LCR-structure with the antiparticle as long as these two conjugate structures are not equivalent.

In order to avoid any confusion, I want to point out that the derivation of the electromagnetic equations (5.19) must be interpreted that the static solitonic LCR-manifold (2.19) admits a distributional potential implied by the closed self-dual 2-form (5.17). Other solitonic LCR-manifolds, having this precise 2-form closed, will be considered to have an electromagnetic charge. No more generalizations are permitted. The other important point is to realize the meaning of the ring-singularity, which essentially determines the electron. In the context of the Einstein gravity (based on riemannian geometry), the ring-singularity is an essential singularity. That is, it cannot be removed by a real coordinate transformation, in contrast to the (soft) horizon singularities, which are coordinate singularities. In PCFT the ring-singularity comes from the branch curve of the regular quadratic hypersurface of $CP^3$, which is a coordinate singularity. It is implied by the projection of the two sheets (branches) of the surface into a $CP^2$ subspace of $CP^3$.

Recall that electromagnetism (either classical or quantum) and gravity start imposing the sources as independent "objects". But here the solitonic electron comes with the metric and the distributional closed self-dual 2-form, which contains both its gravitation and electromagnetic field "dressing" with their sources. Using the generalized function terminology, we state that the electron is the singular support, and electromagnetism (and gravity) is the regular support of the soliton configuration, being a generalized function. The Kerr-Newman manifold has been extensively studied, but I describe here its electromagnetic field in oblate spheroidal coordinates, in order to compare its singular part with the corresponding singular part of the gluonic field of the quark soliton. The self-dual 2-form is

$$x = \sqrt{r^2 + a^2 \cos \varphi \sin \theta} \quad , \quad y = \sqrt{r^2 + a^2 \sin \varphi \sin \theta} \quad , \quad z = r \cos \theta$$

$$\cos \theta = \frac{\tilde{z}}{r} \quad , \quad \sin^2 \vartheta = \frac{x^2 + y^2}{r^2 + a^2} \quad , \quad \frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1$$

and cartesian coordinates, in order to compare its singular part with the corresponding singular part of the gluonic field of the quark soliton. The self-dual 2-form is

$$G^+ = G - i \ast G = \frac{e}{4\pi (r^2 + ia \cos \theta)} (\ell \wedge n - m \wedge \overline{m}) =$$

$$= \frac{e}{4\pi (r^2 + ia \cos \theta)} [dt \wedge dr - ia \sin \theta dt \wedge d\theta + a \sin^2 \theta dr \wedge d\varphi - i(r^2 + a^2) \sin \theta d\theta \wedge d\varphi]$$

in oblate spheroidal coordinates. In cartesian coordinates its electric $E$ and magnetic $B$ fields have the form

$$E^1 = -\frac{exr^3}{4\pi (r^2 + a^2 z^2)^2} \quad , \quad E^2 = \frac{-eyr^3}{4\pi (r^2 + a^2 z^2)^2} \quad , \quad E^3 = \frac{-exr^3(s^2 - a^2)}{4\pi (r^2 + a^2 z^2)^2}$$

$$B^1 = \frac{exr^3}{4\pi (r^2 + a^2 z^2)^2} \quad , \quad B^2 = \frac{eyr^3}{4\pi (r^2 + a^2 z^2)^2} \quad , \quad B^3 = \frac{exr^3(s^2 + z^2)}{4\pi (r^2 + a^2 z^2)^2}$$

(5.22)
The singularities occur at the ring \((r, z) = (0, 0)\). After a Poincaré transformation, this local singularity moves with a constant velocity, as expected from the solitonic origin of the configuration.

The LCR-structure relations imply the eikonal relations

\[ g^{\mu\nu}(\partial_\mu z^\alpha)(\partial_\nu z^\beta) = 0, \quad J^{\mu\nu}(\partial_\mu z^\alpha)(\partial_\nu z^\beta) = 0 \]

which indicate wavefront singularities. These are singularities of a generalized function determined by the position and the direction of its Fourier transform \((x; k)\). A singular point \(x\), in all the directions \(k\), will be called localizing singularity. Typical examples of such singularities are the delta (Dirac) functions. The singular points \(x\) with precise cones of "bad" directions will be called quantum singularities. These are essentially singularities implied by the characteristics of wave equations on their solutions. The electromagnetic (and gravitational) "dressing potential" singularities are localizing singularities, and we will treat them as classical solutions determining the "particle". The characteristics of the differential operators of the free photon and electron are the quantum modes of photon and electron. The naïf way to consider the general solution of equation (5.18) is

\[ A_\mu = A^C_\mu + A^Q_\mu, \quad j^\mu = j^C_\mu + e\bar{\psi}\gamma^\mu \psi \]

where the first term is the classical solitonic solution with the localization singularity and the second is the quantum solution with the wavefront singularity. The last fermionic equation is a general self-consistent condition imposed by the current conservation implied by (5.19). Notice that the electromagnetic "dressing" introduces a repulsive potential of order \(e^2\). The gravitational "dressing" may also enter with the classical electron tetrad, but I will ignore it below.

From the mathematical point of view the classical \(A^C_\mu\) contains the localizing singular and regular part of the distributional solution and the quantum part \(A^Q_\mu\) contains the other wavefront singularities implied by the principal symbol of the pseudo-differential operator (it represents the wave-particle duality). The quantum part \(A^Q_\mu\) of the electromagnetic field interacts with the quantum part (wavefront propagation) of the electron. The classical part \(A^C_\mu\) (electromagnetic dressing of the electron) intervenes through the quantum electron propagator. The Schwartz proper definition of generalized functions lead to rigged Hilbert space and the Hormander formulation of the wavefront singularities. Recall that the positive and negative energy solutions of the free photon with two polarizations \(i = 1, 2\),

\[ A^\pm_\mu = \varepsilon_\mu(i, k)e^{\pm ikx} \]

\[ k^0 = \frac{1}{|k|} \]
are generalized functions with wave front $WF(A^\pm_\mu) = [(t, \vec{x}); (\pm |\vec{k}|, \vec{k} \neq 0)]$. Hence quantum creation, propagation and annihilation may be understood as the creation, propagation and annihilation of the wavefront singularities. The Bogoliubov\cite{2} reformulation of the Wightman axioms, using his microcausality relation, clarifies quantum field theory. The S-matrix is properly defined as a series with coefficients recursively computed, turning the renormalization problem to an appropriate definition of the product of the implied distributions\cite{33}. This formalism is used in the next subsection to derive an effective quantum electrodynamics, where the local singularity of the electron is incorporated besides the conventional quantum singularities of the photon and electron.

5.2 Derivation of quantum electrodynamics

I will apply the Bogoliubov-Medvedev-Polivanov\cite{2} axiomatic formulation of a quantum field theory, viewed as a method for the construction of renormalizable effective quantum field theories. This method has been extensively described in the Bogoliubov-Shirkov book\cite{3}. It approaches the axiomatic formulation of a quantum field theory starting from the S-matrix and the introduction of a "switching on and off" function $c(x) \in [0, 1]$ and assuming the following expansion of the S-matrix

$$S = 1 + \sum_{n \geq 1} \frac{1}{n!} \int S_n(x_1, x_2, ..., x_n)c(x_1)c(x_2)...c(x_n)[dx]$$

(5.26)

where $S_n(x_1, x_2, ..., x_n)$ are generalized functions, which depend on the complete free field functions (the local Poincaré representations of the particles) and not its separate "positive" and "negative" frequency parts. That is, the S-matrix is an operator valued functional in the Fock space of free relativistic particles. Apparently this perturbative expansion needs the existence of a small coupling constant. The imposed axioms are

\begin{align*}
&\text{Poincaré covariance} : \quad U_p S_n(x_1, x_2, ..., x_n) U_p^\dagger = S_n(Px_1, Px_2, ..., Px_n) \\
&\text{Unitarity} : \quad SS^\dagger = S^\dagger S = 1 \\
&\text{Microcausality} : \quad \frac{\delta}{\delta c(x)} \frac{\delta S(c)}{\delta c(x)} S^\dagger(c) = 0 \quad \text{for} \quad x \preceq y \\
&\text{Correspondence principle} : \quad S_1(x) = iL_{int}[\phi(x)]
\end{align*}

(5.27)

where $\phi(x)$ denotes the free particle fields and $x \preceq y$ means $x^0 < y^0$ or $(x-y)^2 < 0$. A general solution of these conditions is

$$S = T[\exp(iL[\phi(x); c(x)])]$$

$$L[\phi(x); c(x)] = L_{int}[\phi(x)]c(x) + \sum_{n \geq 1} \frac{1}{n!} \int \Lambda_{n+1}(x, x_1, ..., x_n)c(x)c(x_1)...c(x_n)[dx]$$

(5.28)

where $\Lambda_n(x, x_1, ..., x_n)$ are quasilocal quantities (arbitrary add-ons of generalized functions\cite{10}), which permit the renormalization process. This order by order construction of a finite S-matrix (with possibly infinite hamiltonian and
lagrangian) provides a well-established algorithm to distinguish renormalizable with non-renormalizable interaction lagrangians. The mathematical origin of renormalization is a non-permitted multiplication of time step functions with other distributions which appear in the initial form of the action. Epstein-Glaser showed\textsuperscript{33} that the recursive procedure does not essentially need these non-well-defined multiplications.

The formalism is based on the well-defined rigged Hilbert-Fock space of the free quantum field representations of the Poincaré group. The advantage of the Bogoliubov procedure is that it can be used in the opposite sense. Knowing the (free) Poincaré representations, they are identified with "free particles" with precise mass and spin. Then they are described with the corresponding free fields, which are used to write down an effective interaction lagrangian, suggested by the fundamental dynamics. In the present case, the fundamental dynamics is the PCFT and the particles are the solitonic solutions and their corresponding potentials which satisfy the wave equations. The suggested interaction takes the place of the "correspondence principle" in the Bogoliubov procedure. In the present case of effective electrodynamics, the suggested interaction is

\[ L_{EM} = e \bar{\psi} \gamma^\mu \psi A_\mu \] (5.29)

where \( \psi \) is the Dirac field and \( A_\mu \) is the quantum electromagnetic field with its propagator implied by (5.24). Notice that in the derived quantum electrodynamics the electron field is not exactly the free Bogoliubov field. It continues to be a representation of the Poincaré group, but incorporates the electromagnetic dressing of the electron. The order by order computation introduces counterterms to the action (with up to first order derivatives). If the number of the forms of the counterterms is finite, the action is renormalizable and the model is considered compatible with quantum mechanics, otherwise the whole construction is rejected as inapplicable. The great value of this constructive procedure will appear in its application for the construction of the effective action of the standard model. The perturbative dependence of the S-matrix on the tempered distributions of the free fields permits the application of nilpotent Q-charge of Scharf and collaborators,\textsuperscript{34} which assures the elimination of negative norm states and the renormalization of the action.

In the Bogoliubov procedure we do not need all the interactions from the beginning. The order by order (perturbative) calculation of the S-matrix, permits the incorporation of all the "needed" additional lagrangian interactions imposed by the emerging counterterms. The restriction is that the final implied order-by-order lagrangian must have a finite number of terms without higher order derivatives, which are the conditions of renormalizability and compatibility with quantum mechanics. The effective quantum electrodynamics, derived from the classical photon-electron current interaction (correspondence principle), does not need additional terms. But in its extension with gravitational and (some) weak interaction terms, additional terms and conditions between the masses and the coupling constants will be needed for the interaction lagrangian to become self-consistent (renormalizable).
The perturbative approach permits the definition of general dynamical variables through the generating functional introduced considering the formal existence of a "classical" current \( J(x) \) for every field \( \phi(x) \) of the action. The generating functional \( Z_0(J) \) and the connected generating functional are

\[
Z_0(J) = \langle 0 | \exp \{ i \int L_I(x) + \phi(x)J(x) \} dx | 0 \rangle
\]

\[
Z_c(J) = -i \ln[Z_0(J)]
\]

Any field \( \phi(x) \) defines a generating field \( \Phi(x; J) \) and the Legendre transformation

\[
\Phi(x; J) = \frac{\delta Z_c(J)}{\delta J(x)}
\]

\[
Z_c(J) \rightarrow W(\Phi) = Z_c(J) - \int \Phi(x; J)J(x)dx
\]

In the context of the Bogoliubov-Shirkov notation\[3\]

\[
\Phi(x; g) = -\frac{\delta H(x; g)}{\delta J(x)} = -i\frac{\delta S}{\delta J(x)}S|_{J=0}
\]

\[
H(x; g) = i\frac{\delta S(g)}{\delta g(x)}S(g)
\]

where \( H(x; g) \) is the "quantum" hamiltonian of the system. The expected relation of a "dressing" potential of the elementary particles in PCFT and the above formalism is

\[
A_1(x; 1) = -\frac{\delta E(J)}{\delta J(x)}|_{J=0} = -i\frac{\delta S}{\delta J(x)}S|_{J=0}
\]

\[
\Phi_1 = (2\pi)^\frac{3}{2}a_1^-(-\frac{\vec{k}}{\hbar})\Phi_0
\]

where \( \Phi_1 \) is the one-electron state. Notice that the elementary particle has the same initial and final energies and their creation and annihilation operators are outside the time ordering. The physical intuition is that we use the classical current \( J(x) \) as a sensor of the potential generated by a particle. The relativistic field equations are also derived from the causal perturbative approach and all the experimental results are properly computed. Hence (5.33) is going to provide precise self-consistency conditions between PCFT and current quantum field theories, which we will describe below.

The first term of the effective electron potential in conventional quantum electrodynamics is

\[
A_{1\mu}(x; 1) \simeq -\frac{i}{2}\Phi_1 \frac{\delta S_1(J)}{\delta \phi^\mu(x)}\Phi_1 |_{J=0}
\]

\[
S_2(J) = \int T((L_I(x_1) + A_\nu(x_1)J^\nu(x_1))(L_I(x_2) + A_\nu(x_2)J^\nu(x_2))dx
\]

which becomes
\[ A_\mu(x) \simeq -e \int D_0(x - y) \Phi_1^\mu : \bar{\psi}_e(y) \gamma^\mu \psi_e(y) : \Phi_1 d^2 y \]  

\[ \Phi_1 = (2\pi)^3 a_\mu \ell_\mu \Phi_0 \]  

The electromagnetic dressing of the electron LCR-manifold in cartesian coordinates

\[ A = \frac{qr^3}{4\pi(r^4 + a^2(x^3)^2)}(dx^0 - \frac{r^2 - ax^2}{r^4 + a^2} dx^1 - \frac{r^2 + ax^1}{r^4 + a^2} dx^2 - \frac{r^2}{r^4 + a^2} dx^3) = \]

\[ \ell_\mu dx^\mu \]

is proportional to \( \ell_\mu \) and has the proper asymptotic charge \( e \) and magnetic moment \( ea \), already computed by Carter without any reference to quantum electrodynamics. Besides, all its components are locally integrable functions determining through derivations the "ladder" of the generalized functions (electric field, magnetic field, etc). Hence it strongly suggests to relate this form with the sum of all the orders of the effective potential of quantum electrodynamics. But \( (5.36) \) is singular at the ring with radius \( a \), while the perturbative terms \( (5.35) \) are singular at the point \( \ell = 0 \), which emerge after an expansion of \( (5.36) \) and the definition of \( r \) in powers of \( a = \frac{\hbar}{2m} \). The emergence of the Planck constant \( \hbar \) strongly indicates that \( (5.36) \) includes the contributions of loop diagrams.

Recall that the Kerr-Newman metric (with the electromagnetic potential) satisfy the Einstein field equations with

\[ h_{\mu\nu} = 2f(x)\ell_\mu \ell_\nu \]

\[ g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \]

Therefore causal perturbative approach has to start with gravity too with initial interaction

\[ L_I = \frac{1}{2} h^{\mu\nu} : (\bar{\psi}_e \gamma_\mu \partial_\nu \psi_e - (\partial_\nu \bar{\psi}_e) \gamma_\mu \psi_e) : \]

and possibly the first order gravity-gravity interaction, which has already computed by Scharf and his collaborators \[34\], using the nilpotent \( Q \) gauge charge method. Their observation that the computed first terms coincide with the expansion of the Einstein-Hilbert action should be expected, because the other gravitational scalars contain second order derivatives, which imply negative norm particle states, removed by their method.

The causal perturbative approach of quantum field theory provides the transition amplitudes between the free elementary particles (the stable asymptotic LCR-manifolds), but it is practically impossible to sum up all the terms. That is, quantum field theory cannot compute the geometric ring singularity of the elementary LCR-manifolds, which determines the particles themselves and the
geometry of the background \( \mathbb{R} \times S^3 \) universe. That is, the expansion of \( A \) and \( r \) of (5.36) hides the ring singularity of the global geometric solution. This singularity permits us to bypass the Hawking-Penrose singularity theorems for lorentzian riemannian manifolds, as described in the following subsection.

### 5.3 LCR-ray tracing in the electron LCR-manifold

The electron mass \( M_e \), charge \( e^2 \) and spin parameter \( a \) have the values

\[
M_e = 4.2 \times 10^{-23} \\
e^2 = \frac{a^2}{4\pi\varepsilon_0\hbar c} = \frac{1}{137} \\
a = \frac{\hbar}{2M_e} = 2.1 \times 10^{23}
\]  

(5.39)

in dimensionless units \( c = G = \hbar = 1 \). Hence \( a^2 + e^2 - M_e^2 > 0 \), which implies that its Kerr-Newman metric has a naked essential singularity. Because of this singularity the Kerr-Newman metric cannot be related with the electron despite the extraordinary fact of fermionic gyromagnetic ratio \( g = 2 \). The purpose of this subsection is to show that the LCR-manifold is well defined, permitting its identification with the electron.

The static electron is identified with the static axially symmetric LCR-structure determined with the linear trajectory \( \xi^a = (r, 0, 0, ia) \). That is, we have

\[
X^m = \begin{pmatrix}
1 & -z^1 & \bar{z}^1 \\
z^1 & 1 & i(z^0 - ia)z^1 \\
-i(z^0 - ia) & i(z^0 - ia)z^1 & -i(z^0 + ia) \\
-i(z^0 + ia) & -i(z^0 + ia) & 1
\end{pmatrix}
\]  

(5.40)

where \((z^a; \bar{z}^\alpha)\) are now the structure coordinates. Here I will first derive the "flat" LCR-structure (defined by \( X^m = 0 \)) and after I will make a "Kerr-Schild" ansatz adapted to the LCR-tetrad to finally refine the axially symmetric LCR-structure, which is identified with the electron. I think this approach will make general relativists more confident to the final picture of the electron as a gaussian beam (in the optics terminology) in \( U(2) \) spacetime.

This procedure implies first the "flat" LCR-structure coordinates

\[
\begin{align*}
 z^0 &= t - r + ia \cos \theta , & z^1 &= e^{i\varphi} \tan \frac{\theta}{2} \\
 \bar{z}^0 &= t + r - ia \cos \theta , & \bar{z}^1 &= \frac{r + ia}{r - ia} e^{-i\varphi} \tan \frac{\theta}{2}
\end{align*}
\]  

(5.41)

from which we find the tetrad compatible with the Minkowski metric

\[
\begin{align*}
 L_m dx^m &= \Lambda [dt - dr - a \sin^2 \theta d\varphi] \\
 N_m dx^m &= N [dt + \frac{r^2 + 2a^2 \cos^2 \theta - a^2}{r^2 + a^2 \cos^2 \theta - a^2} dr - a \sin^2 \theta d\varphi] \\
 M_m dx^m &= M [-a \sin \theta (dt - dr) + (r^2 + a^2 \cos^2 \theta) d\theta + i \sin \theta (r^2 + a^2) d\varphi]
\end{align*}
\]  

(5.42)

36
where the tetrad-Weyl factors are not determined as expected. They are determined by simply imposing that the tetrad gives the Minkowski metric. But for that, we have to find first the relation of the cartesian coordinates with the present convenient "asymmetric" coordinates \((t, r, \theta, \varphi)\), which are not the same with the "symmetric" ones.

The general relation between the projective coordinates and the homogeneous coordinates of \(G_{4,2}\) is found by simply inverting their definition formula \((2.11)\). We finally find

\[
\begin{align*}
\rho &= \frac{(X^{01}X^{32} - X^{02}X^{31}) + (X^{01}X^{12} - X^{02}X^{11})}{2(X^{01}X^{12} - X^{02}X^{11})}, \\
\rho &= \frac{(X^{11}X^{32} - X^{12}X^{31}) + (X^{11}X^{22} - X^{12}X^{21})}{2(X^{11}X^{22} - X^{12}X^{21})}, \\
\rho &= \frac{(X^{11}X^{32} - X^{12}X^{31}) - (X^{11}X^{22} - X^{12}X^{21})}{2(X^{11}X^{22} - X^{12}X^{21})}.
\end{align*}
\]

We already know that the imaginary part of \(r^b = x^b + iy^b\) determines the gravitational "dressing", because the algebraic "flatness" condition implies \(y^b = 0\). The Minkowski coordinates \(x^b\) are related with the "asymmetric" \((t, r, \theta, \varphi)\) via the relation

\[
\begin{align*}
x^0 &= t, \\
x^1 + ix^2 &= (r - ia) \sin \theta e^{i\varphi}, \\
x^3 &= r \cos \theta, \\
r^4 - [(x^1)^2 + (x^2)^2 + (x^3)^2 - a^2]^2 - a^2(x^3)^2 &= 0,
\end{align*}
\]

with the following diffeomorphic relations

\[
\begin{align*}
dx^0 &= dt, \\
dx^1 &= \sin \theta \cos \varphi dr + \cos \theta (r \sin \varphi + a \sin \varphi) d\theta - \sin \theta (r \sin \varphi - a \cos \varphi) d\varphi, \\
dx^2 &= \sin \theta \sin \varphi dr + \cos \theta (r \sin \varphi - a \cos \varphi) d\theta + \sin \theta (r \cos \varphi + a \sin \varphi) d\varphi, \\
dx^3 &= \cos \theta dr - r \sin \theta d\theta.
\end{align*}
\]

Their inversion implies

\[
\begin{align*}
dt &= dx^0, \\
dr &= \frac{r x^1 - a x^2}{r^2 + a^2} dx^1 + \frac{a x^1 + r x^2}{r^2 + a^2} dx^2 + \frac{x^3}{r} dx^3, \\
d\theta &= \frac{x^1(r x^1 - a x^2)}{r^2(\sqrt{r^2 + a^2})((x^1)^2 + (x^2)^2)} dx^1 + \frac{x^3(a x^1 + r x^2)}{r^2(\sqrt{r^2 + a^2})((x^1)^2 + (x^2)^2)} dx^2 - \frac{\sqrt{(x^1)^2 + (x^2)^2}}{r \sqrt{r^2 + a^2}} dx^3, \\
d\varphi &= -\frac{a x^1 + r x^2}{r((x^1)^2 + (x^2)^2)} dx^1 + \frac{r x^1 - a x^2}{r((x^1)^2 + (x^2)^2)} dx^2.
\end{align*}
\]

Hence, we finally find that the conventional tetrad corresponding to the
Minkowski metric is
\[ L \mu dx^\mu = [dt - dr - a \sin^2 \theta d\varphi] \]
\[ N \mu dx^\mu = \frac{2 r^2 a^2}{(r^2 + a^2 \cos \theta)} [dt + \frac{r^2 + 2 a^2 \cos^2 \theta - a^2}{r^2 + a^2} dr - a \sin^2 \theta d\varphi] \]
\[ M \mu dx^\mu = \frac{-1}{\sqrt{2(r + ia \cos \theta)}} [-ia \sin \theta (dt - dr) + (r^2 + a^2 \cos^2 \theta) d\theta + i \sin \theta (r^2 + a^2) d\varphi] \] (5.47)

The general tetrad is found with the "Kerr-Schild" ansatz adapted to the LCR-structure formalism
\[ \ell_\mu = L_\mu , \quad m_\mu = M_\mu , \quad n_\mu = N_\mu + \frac{h(r)}{2(r + a \cos \theta)} L_\mu \] (5.48)
I want to point out that we find the same static LCR-structure looking for LCR-structures admitting time translation and axisymmetric symmetries.

With the above definition of the coordinates \((t, r, \theta, \varphi)\), the structure coordinates have the form
\[ z^0 = t - r + ia \cos \theta \quad , \quad z^1 = e^{i \varphi} \tan \frac{\theta}{2} \]
\[ \tilde{z} = t + r - ia \cos \theta - 2f_1 \quad , \quad \tilde{\varphi} = \frac{r + ia}{r - ia} e^{2ia} e^{-i \varphi} \tan \frac{\theta}{2} \] (5.49)
where the two new functions are
\[ f_1(r) = \int \frac{h}{r^2 + a^2 + h} \, dr \quad , \quad f_2(r) = \int \frac{h}{(r^2 + a^2 + h)(r^2 + a^2)} \, dr \] (5.50)
The Newman-Penrose spin coefficients are found to be
\[ \alpha = \frac{ia(1+ \sin^2 \theta) - r \cos \theta}{2 \sqrt{2} \sin \theta \, (r - ia \cos \theta)^2} , \quad \beta = \frac{\cos \theta}{2 \sqrt{2} \sin \theta \, (r + ia \cos \theta)} \]
\[ \gamma = \frac{-a^2 + ia r \cos \theta + h}{2 r^2 (r - ia \cos \theta)} + \frac{h}{4 r^2} , \quad \varepsilon = 0 \]
\[ \delta = \frac{r^2 + a^2 + h}{2 r^2 (r - ia \cos \theta)} , \quad \pi = \frac{ia \sin \theta}{\sqrt{2} (r - ia \cos \theta)^2} \]
\[ r = \frac{r - ia \cos \theta}{\sqrt{2} (r - ia \cos \theta)^2} , \quad \nu = \frac{ia \sin \theta}{\sqrt{2} (r - ia \cos \theta)^2} \]
\[ \kappa = 0 , \quad \sigma = 0 , \quad \nu = 0 , \quad \lambda = 0 \] (5.51)
which will be useful for our computations. Recall that the Kerr-Newman spacetime has \(h(r) = -2Mr + e^2\). In this case the integrals are
\[ f_1(r) = \int \frac{-2Mr + e^2}{r^2 + a^2 + 2Mr + e^2} \, dr = -M \ln \left[ \frac{\Delta_1}{\Delta} + \frac{2M^2 - e^2}{\Theta} \right] + \frac{2M^2 - e^2}{\Theta} \arctan \left[ \frac{\Theta}{r - M} \right] \]
\[ f_2(r) = \int \frac{-2Mr + e^2}{(r^2 + a^2 - 2Mr + e^2)(r^2 + a^2)} \, dr = \frac{1}{2ia} \ln \left[ \frac{r - ia}{r + ia} \right] + \frac{2M^2 - e^2}{\Theta} \arctan \left[ \frac{\Theta}{r - M} \right] \] (5.52)
\[ \Delta := r^2 + a^2 - 2Mr + e^2 , \quad \Theta := \sqrt{a^2 + e^2 - M^2} \]
where \(r_1\) and \(r_2\) are normalization constants, and the structure coordinates of the "Kerr-Newman" LCR-manifold are
\[ z^0 = t - r + ia \cos \theta \quad , \quad z^1 = e^{i \varphi} \tan \frac{\theta}{2} \]
\[ \tilde{z} = t + r - ia \cos \theta + 2M \ln \left[ \frac{\Delta_1}{\Delta} + \frac{2M^2 - e^2}{\Theta} \right] \arctan \left[ \frac{\Theta}{r - M} \right] \]
\[ z^1 = r_2 \left[ \frac{r - M + i \Theta}{r - M - i \Theta} \right] e^{-i \varphi} \tan \frac{\theta}{2} \] (5.53)
Notice the singularities in the ambient complex manifold at the two complex values of $r = M \pm i\Theta$. It is well known to general relativists that this choice of tetrad-Weyl factors preserve the electromagnetic current and the energy-momentum and angular momentum currents. Hence, fixing the factors of the LCR-tetrad (to achieve conservation of the currents) implies a breaking of the tetrad-Weyl symmetry.

Notice that the electron LCR-structure coordinates (5.53) of the embedding of the LCR-manifold in the ambient complex manifold may be viewed as an anti-meromorphic deformation

$$z^\beta = f^\beta(z^\alpha; r)$$

where the deformation parameter is the real variable $r$.

The static axially symmetric LCR-structure (identified with the electron) is stable, because all its relative invariants

$$\Phi_1 = \bar{\nu} - \frac{\nu}{r^2} = \frac{2a\cos \theta}{r^2 + a^2 \cos^2 \theta}$$
$$\Phi_2 = \frac{\mu - \pi}{r} = \frac{(r^2 + a^2 + h)\cos \theta}{(r^2 + a^2 \cos^2 \theta)^2}$$
$$\Phi_3 = -(\tau + \pi) = \frac{\sqrt{2}r^2\cos \theta}{(r + ia \cos \theta)(r - ia \cos \theta)}$$

do not vanish.

The $L^\mu \partial_\mu z^\alpha = 0$ annihilation implies that the outgoing integral curves (rays) are determined by the surfaces

$$s_1 := t - r , \ s_2 := \theta , \ s_3 := \phi$$

We use the caustic coordinates $(r, s_1, s_2, s_3)$, which have the property $(0, s_1, \pi, s_3)$ to be on the caustic. In this caustic coordinate system the LCR-rays are traced by the relation

$$x_L^0(r) = s_1 + r$$
$$x_L^1(r) = (r \cos \phi + a \sin \phi) \sin \theta$$
$$x_L^2(r) = (r \sin \phi - a \cos \phi) \sin \theta$$
$$x_L^3(r) = r \cos \theta$$

**Jacobian**

$$[r^2 + a^2 \cos^2 \theta] \sin \theta$$

The source of the LCR-rays are at $r = 0$, i.e. the disk

$$x_L^0(0) = s_1$$
$$x_L^1(0) = a \sin \phi \sin \theta$$
$$x_L^2(0) = -a \cos \phi \sin \theta$$
$$x_L^3(0) = 0$$

The $N^\mu \partial_\mu \tilde{z}^\alpha = 0$ annihilation implies that its incoming rays are determined by the surfaces

$$s'_1 := t + r , \ s'_2 := \theta , \ s'_3 := \phi + \arctan \frac{2ar}{b^2 - r^2}$$
Then we find the congruence
\[ x_0^0(r) = s_1^1 - r \]
\[ x_1^0(r) = [r \cos s_3^1 - a \sin s_3^1] \sin \theta \]
\[ x_2^0(r) = [r \sin s_3^1 + a \cos s_3^1] \sin \theta \]
\[ x_3^0(r) = r \cos \theta \]

(5.60)

Jacobian = \([r^2 + a^2 \cos^2 \theta] \sin \theta\)

As expected the velocities \(\dot{x}_L^i(t)\) and \(\dot{x}_N^i(t)\) have asymptotically opposite radial directions.

We will now show that the origin of the essential singularity of the Kerr manifold is the intersection of the two sheets of the static electron regular quadric (in the unbounded Siegel realization)

\[ X^1 X^2 - X^0 X^3 + 2a X^0 X^1 = 0 \]

(5.61)
of \(CP^3\). In the flatprint case we have

\[
\begin{align*}
X^0 &= 1 \quad , \quad X^1 = \lambda \quad , \quad X^2 = -i[(x^0 - x^3) - (x^1 + ix^2)\lambda] \\
X^3 &= -i[−(x^1 + ix^2) + (x^0 + x^3)\lambda]
\end{align*}
\]

(5.62)

and the Kerr polynomial and its two solutions are

\[
(x^1 - ix^2)\lambda^2 + 2(x^3 - ia)\lambda - (x^1 + ix^2) = 0
\]

\[
\lambda_{1,2} = \frac{-(x^3 - ia) \pm \sqrt{\Delta}}{x - iy}, \quad \Delta = (x^1)^2 + (x^2)^2 + (x^3)^2 - a^2 - 2iax^3
\]

(5.63)

where \(\lambda_{1,2}\) are the two values of \(\lambda\) on the two sheets of the quadric. The intersection curve of these two sheets is

\[
\Delta = (x^1)^2 + (x^2)^2 + (x^3)^2 - a^2 - 2iax^3 = 0
\]

(5.64)

\[
x^3 = 0 \quad , \quad (x^1)^2 + (x^2)^2 = a^2
\]

which, after the LCR projection to \(R^4\), becomes the singularity ring of the electron (Kerr-Newman) manifold. Notice that the quadratic surface is regular and the intersection of the two branches is implied by the projection. The points of the algebraic intersection curve (the branch curve) of the (regular) quadric of \(CP^3\) are regular points like any other point of the quadric.

We have already pointed out that the entire LCR-manifold (universe) is the spinorial \(U(2)\) manifold which needs more than two \(R^3\) charts. Recall that the one \(R^4\) chart of the \(U(2) \to R^4\) Cayley 2 \(\to 1\) transformation is \((47)\) \(x_+^i\) for \(s > 0\) and the second (non-intersecting chart) is \((48)\) \(x_-^i\) for \(s < 0\), where the affine parameter is related with

\[
r = \begin{cases} 
\left\{ \frac{s^2-a^2}{2} + \sqrt{\frac{(s^2-a^2)^2}{2} + a^2(x^3)^2} \right\}^{\frac{1}{2}} & \text{for } s > 0 \\
\left\{ \frac{s^2-a^2}{2} + \sqrt{\frac{(s^2-a^2)^2}{2} + a^2(x^3)^2} \right\}^{\frac{1}{2}} & \text{for } s < 0
\end{cases}
\]

(5.65)
Notice that in the identified region (the disc for both charts) \( r = 0 \) in both charts. That is, \( r = 0 \) occurs at \( x^3 = 0 \) and \( s^2 \leq a^2 \) for both charts \( s \geq 0 \).

The two LCR-congruences \( L^\mu = \frac{dx^\mu}{dr} \) and \( N^\mu = \frac{dx^\mu}{ds} \) of the flatprint electron LCR-manifold can be easily implied from my calculations of the previous section. The starting idea is that the structure coordinates \( z^\alpha(x) \) provide the three invariants \((s_1, s_2, s_3)\) along the \( L^\mu \)-ray \( x^\mu_L(r) \), and the structure coordinates \( z^\beta(x) \) provide the invariants \((s'_1, s'_2, s'_3)\), which label the \( N^\mu \)-ray \( x^\mu_N(r) \). Hence we simply have the same forms, but we let \( r \) have the same forms, but we let \( r \) pass to the second \( x^\mu_L(r), x^\mu_N(r) \in \mathbb{R}^4 \) sheet.

The second way is tracing the rays \( w_{L,N}(r; s_1, s_2, s_3) \in U(2) \) in the complete bounded universe \( U(2) \) taking \( r \in (-\infty, +\infty) \) as the parameter indicating the ray points. From the relation

\[
Y^0 = \frac{1}{\sqrt{2}}(X^0 + X^2) \quad , \quad Y^1 = \frac{1}{\sqrt{2}}(X^1 + X^3) \\
Y^2 = \frac{1}{\sqrt{2}}(X^0 - X^2) \quad , \quad Y^3 = \frac{1}{\sqrt{2}}(X^1 - X^3)
\]

between the bounded \( Y^{mi} \) and unbounded \( X^{mi} \) homogeneous coordinates and \((5.40)\) we find

\[
Y^{mi} = \frac{i}{\sqrt{2}} \begin{pmatrix}
1 - i(z^0 - ia) & (-1 + i(z^0 - ia))z^1 \\
(1 - i(z^0 + ia))z^1 & 1 - i(z^0 + ia) \\
1 + i(z^0 - ia) & -(1 + i(z^0 - ia))z^1 \\
(1 + i(z^0 + ia))z^1 & 1 + i(z^0 + ia)
\end{pmatrix}
\]

Like previously, we use the relations \((5.41)\) to find the labels \((5.56)\) of \( L^\mu \) rays \((r; s_1, s_2, s_3)\), assuming the \( r \) parameter to indicate the points of one ray. The coordinates \( z^\alpha \) do not depend on \( r \), remain invariant along the rays, therefore I keep them unchanged. Then we express only \( z^{\tilde{\alpha}} \) as functions of the proper \( L^\mu \)-ray coordinates \((r; s_1, s_2, s_3)\)

\[
z^0 = s_1 + 2r - ia \cos s_2 \quad , \quad z^{\tilde{1}} = \frac{r + ia}{r - ia} e^{-i3} \tan \frac{s_3}{2}
\]

and we find the rays in homogeneous coordinates \( Y^{mi}(r; s_1, s_2, s_3) \).

In the context of the quadratic \( CP^3 \) hypersurface, along the \( L^\mu \) integral curves, the one intersection point with the line is preserved constant and changes the second. For the \( N^\mu \) integral curves the role of the intersection points are interchanged. Now using the relation

\[
w_{11} = \frac{y^{12}y^{13}y^{14}y^{22} + y^{12}y^{13}y^{14}y^{22}}{y^{12}y^{13}y^{14}y^{22}} \quad , \quad w_{12} = \frac{y^{11}y^{22}y^{13}y^{14}y^{22}}{y^{11}y^{12}y^{13}y^{14}y^{22}}
\]

\[
w_{21} = \frac{y^{12}y^{13}y^{14}y^{22}}{y^{12}y^{13}y^{14}y^{22}} \quad , \quad w_{22} = \frac{y^{11}y^{22}y^{13}y^{14}y^{22}}{y^{11}y^{12}y^{13}y^{14}y^{22}}
\]

between the bounded projective \( w \in U(2) \) and homogeneous \( Y^{mi} \) coordinates, we finally find the rays \( w_L(r; s_1, s_2, s_3) \in U(2) \) in the complete bounded universe \( U(2) \).
The intersection of the two $\mathbb{R}^4$ charts in $U(2)$ coordinates can be computed by simply making the Cayley transformation of the cartesian form of the ring singularity. Then we find that in $(\tau, \rho, \sigma, \chi)$ coordinates the ring singularity (the caustic of the congruence) and its "tube" connecting the two charts is

$$
\sigma = \frac{\pi}{2}, \quad R_0^2 \frac{\sin^2 \rho}{(\cos \tau + \cos \rho)^2} \leq a^2 \\
-\pi < \rho < \pi, \quad -\pi < \tau < \pi
$$

(5.70)

which apparently contains both rings of the two $\mathbb{R}^4$ copies.

6 NEUTRON AND STANDARD MODEL

The search for the electron-soliton started from the quite general assumptions to be massive and automorphic relative to time translation and $z$-rotation. That is in quantum theory terminology, looking for massive eigenstates of the Hamiltonian and $z$-component angular momentum. The found stable LCR-manifold is quite restrictive without any indication for the existence of other connected massive configuration. On the other hand the trajectory (5.11) of the Poincaré group in the set of quadratic algebraic surfaces of $CP^3$ provides two possibilities. The massive irreducible regular (rank-4) quadratic surface (5.11) with $\det p \neq 0$, which is identified with the free electron, and the massless reducible surface $\det p = 0$, which is apparently singular. Therefore, I focus my search for a massless stable LCR-structure, described by this reducible quadratic surface of $CP^3$.

It is computationally easier to first look for a LCR-structure compatible with a minkowskian class $[\eta_{\mu \nu}]$ of metrics (a flatprint in the terminology of general relativity) and after applying a Kerr-Schild ansatz to find a curved candidate. So we look for a Kerr polynomial (5.11) with $\det p = 0$, which is automorphic relative to the $z$-rotation (5.7). No rank-3 quadratic surface of the form (5.11) survives this condition. For every helicity $[E = \pm p^3]$ of the neutrino LCR-structure, I only find the rank-2 union of the following two planes

$$
[E = -p^3]: \quad X^3 - aX^1 = 0, \quad X^0 = 0 \\
[E = +p^3]: \quad X^1 = 0, \quad X^2 + bX^0 = 0
$$

(6.1)

in the frame with $p^1 + ip^2 = 0$.

The union of two planes is singular at their intersection line, if they are embedded in $CP^3$. It is well known in algebraic geometry that this kind of singularities are resolved with the blowing-up procedure [13]. That is, this singularity is essentially fictitious implied by the embending of both hyperplanes in $CP^3$. It disappears if they are embedded in larger projective spaces. The analogous simple example is the union of two lines embedded in $RP^2$, which are singular at their intersection point. But the union of two non-intersecting lines embedded in $RP^3$ are generally nowhere singular.
The intersection (complex) line of the first two hyperplanes of \( CP^3 \) of (6.1) is at the infinity of the \( X^0 = 1 \) affine space, while the intersection line of the next two hyperplanes of (6.1) is at the infinity of the \( X^1 = 1 \) affine space. Therefore no neutrino trajectory (singular line) is seen in the affine space \( X_{ni} = \left( \lambda A^i - i x_{A'} A^i \right) \) (6.2) of the grassmannian \( G_{4,2} \), where the trajectory [5.11] of the Poincaré group has been considered. This fact should be interpreted that the two stationary chiral neutrinos do not have a classical trajectory in spacetime. These stationary neutrinos have \( \Phi_2 = \Phi_3 = 0 \) vanishing relative invariants. Besides, unlike the electron LCR-structure, the corresponding 2-form [5.17] does not admit electromagnetic sources, hence they are chargeless.

It is very instructive to consider the following linear Kerr polynomials

\[
X^{31} - aX^{11} = 0 \quad , \quad X^{22} + bX^{02} = 0
\] (6.3)

for the left and right columns of the homogeneous coordinates. It is not stationary, but it has a singular trajectory

\[
-(x^1 + ix^2)\lambda^{01} + (x^0 + x^3 - ia)\lambda^{11} = 0 \\
(x^0 - x^3 + ib)\lambda^{02} - (x^1 - ix^2)\lambda^{12} = 0
\]

\[
\lambda^{01} : \lambda^{11} \sim \lambda^{02} : \lambda^{12} \quad \Rightarrow \\
(b + a)x^3 + (b - a)x^0 = 0 \quad , \quad (x^1)^2 + (x^2)^2 = ab[(2x^0)^2 + 1]
\] (6.4)

For \( ab > 0 \) it is not realistic, because the singularity ring "explodes". But if \( b = 0 \) it has a massless line trajectory. For \( a \neq 0 \neq b \) the LCR-structure conditions are

\[
X^{mi} E_{mn} X^{nj} = 0 \\
\frac{x^0 - x^\rho}{2i} + az^\rho z^\bar{\rho} = 0 \quad , \quad z^i z^\bar{\rho} + (z^0 + ia)z^\bar{\rho} = 0 \quad , \quad \frac{z^0 - x^\rho}{2i} - bz^i z^\bar{\rho} = 0
\] (6.5)

This is like a twisted natural \( U(2) \) LCR-structure[28], because its third relative invariant vanishes, \( \Phi_3 = 0 \). The LCR-tetrad can be directly found.

For \( b = 0 \), the structure coordinates are
The convenient structure coordinates are

\[ z^0 \equiv i \frac{X^{21}}{X^{11}}, \quad z^1 \equiv \frac{X^{22}}{X^{11}}, \quad z^\vartheta \equiv i \frac{X^{32}}{X^{12}}, \quad z^{\tilde{1}} \equiv -\frac{X_{02}}{X_{12}} \]

\[ z^0 = x^0 - x^3 - \frac{(x^1)^2 + (x^2)^2}{x^1 + x^2 - i \theta}, \quad z^1 = \frac{x^1 + i x^2}{x^1 + x^2 - i \theta}, \quad z^{\vartheta} = \frac{x^1 - i x^2}{x^1 + x^2 - i \theta} \quad \text{(6.6)} \]

\[ u = x^0 - x^3 - \frac{(x^1)^2 + (x^2)^2}{x^1 + x^2 - i \theta} (x^0 + x^3) \]
\[ v = x^0 + x^3 - \frac{(x^1)^2 + (x^2)^2}{x^1 + x^2 - i \theta} \]

The convenient structure coordinates are

\[ z^0 = -\frac{1}{z^\vartheta} = u' - i az^{\vartheta}, \quad z^1 = -\frac{1}{z^{\vartheta}} = \zeta' \]
\[ z^\vartheta = \tilde{z}^0 = v', \quad z^{\tilde{1}} = \frac{z^1}{z^0 + i \theta} = \tilde{z}^{\tilde{1}} \quad \text{(6.7)} \]

In these coordinates the following tetrad can be easily computed

\[ L = du' + ia \zeta' d\zeta' - ia \zeta' d\zeta', \quad N = dv', \quad M = d\zeta' \]
\[ dL = -2iaM \wedge \overline{M}, \quad dN = 0, \quad dM = 0 \quad \text{(6.8)} \]

\[ L^\mu \partial_\mu = \partial_\nu, \quad N^\mu \partial_\mu = \partial_\nu, \quad M^\mu \partial_\mu = -ia \zeta' \partial_\nu - \partial_{\zeta'} \]

Note that this flat LCR-structure has vanishing \( \Phi_2 = \Phi_3 = 0 \) relative invariants.

The Kerr-Schild ansatz may be applied either on \( L \) or on \( N \). I will consider the later case, in order to show an interesting effect of gravity. Let

\[ \ell = L, \quad m = M, \quad n = N + fL \quad \text{(6.9)} \]

The LCR-structure condition fixes the form of \( f(x) \) because

\[ dn = df \wedge L + f dL = Z_2 \wedge n + i \Phi_2 m \wedge \overline{m} \]
\[ M^\mu \partial_\mu f = 0 \quad \implies \quad \partial_\nu f = 0, \quad \partial_{\zeta'} f = 0 \quad \text{(6.10)} \]

Hence \( f = f(v') \) depends only on \( v' \). Notice that the structure relations now take the form

\[ df = -2ia m \wedge \overline{m}, \quad dm = 0 \]
\[ df \wedge L + f dL = Z_2 \wedge n + i \Phi_2 m \wedge \overline{m} \quad \text{(6.11)} \]

with non-vanishing relative invariants \( \Phi_1 \neq 0 \neq \Phi_2 = -2ia f \). That is gravity may generate a right chirality and the neutrino flatprint LCR-structure may not be a smooth deformation of the curved one, as it happens for the electron LCR-structure.

The next question we have to answer is whether the neutrino admits an electromagnetic potential. By analogy to the electron LCR-structure, the neutrino electromagnetic field” should be defined by the self-dual 2-form.
\[ F^+ = Ce^{2ia'v'}(\ell \wedge n - m \wedge \bar{m}) = Ce^{2ia'v'}(L \wedge N - M \wedge \bar{M}) \quad (6.12) \]

where \( C \) is an arbitrary complex constant. It is closed and exact, because a straightforward application of Stokes' theorem on the \( t \) and \( r \) constant sphere implies no sources. One can see it by simply observing that it is an exact form, because the imaginary term \( ia \) does not permit any singularity. Hence, I conclude that the "charge" of the neutrino (6.3) vanishes.

We have already pointed out that the apparent trajectories of the elementary leptonic particles strongly indicate that they are ruled surfaces (l-12c) of \( CP^3 \). The radiating electron is naturally described by their Newman complex trajectory. The relation (l-12e) proves that a ruled surface is characterized by a massive and a massless trajectory. Hence, the emergence of massive-massless pairs is a characteristic property of the ruled surfaces. In this context the three leptonic generations (electronic, muonic and tauonic) could correspond to quadric (already studied above), cubic and quartic curves \( Z^n(r) \) of the ruled surface.

### 6.1 The electroweak-U(2) gauge fields

In conventional field theory the interactions have to be imposed as connections. In the computation of the electron and its neutrino distributional LCR-structures, these fields appear as gravitational and electromagnetic dressing distributions with precise compact singular support. The purpose of the present subsection is to provide the algorithmic derivation of the weak connection[32]. This generalizes the surprising identification of the electron electromagnetic dressing \( A_\mu(x) \) with the vector \( \ell_\mu \) of the LCR-tetrad (and the induced gravitational dressing).

The general solution of a realizable LCR-structure is a special totally real submanifold of \( \mathbb{C}^4 \), determined by the conditions (??). Its characteristic local coframe of the surface contains the normal bundle \( dp_{ij} \) and the tangent 1-forms

\[ \ell = i(\partial - \bar{\partial})\rho_{11}, \quad n = i(\partial - \bar{\partial})\rho_{22}, \quad m = i(\partial - \bar{\partial})\rho_{12} \]

\[
\begin{pmatrix}
\ell & n
\end{pmatrix}
= i(\partial - \bar{\partial})
\begin{pmatrix}
\rho_{11} & \rho_{12} \\
\rho_{21} & \rho_{22}
\end{pmatrix}
\]

arranged to a 2×2 hermitian matrix, and considered as a function on the algebra of the Lie group \( U(2) \). Hence, it may be considered as an electroweak Cartan connection.

The "natural U(2)" LCR-structure is

\[ e = -iw^{-1}dw =: \begin{pmatrix} \ell & m \\ m & n \end{pmatrix}, \quad de - ie \wedge e = 0 \]

\[ dl = im \wedge \overline{m}, \quad dn = -im \wedge \overline{m}, \quad dm = i(\ell - n) \wedge m \]

\[ (6.14) \]
This form strongly suggests to osculate the LCR-structure with the $U(2)$ group. The first step of that is to cast a LCR-tetrad into the hermitian matrix

$$e' := \begin{pmatrix} \ell' & m' \\ m' & n' \end{pmatrix} = i(\partial - \overline{\partial}) \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{12} & \rho_{22} \end{pmatrix}$$  \hspace{1cm} (6.15)$$

Hence the electroweak gauge fields and the corresponding curvature are

$$B = B_{I\mu} dx^\mu t_I = \begin{pmatrix} \ell' & m' \\ m' & n' \end{pmatrix}, \quad [t_I, t_J] = iC_{IJK} t_K$$

$$F = dB - iB \wedge B \rightarrow DF := dF + iB \wedge F - iF \wedge B = 0 \hspace{1cm} (6.16)$$

where $t_I$ are generators of $U(2)$. Apparently a gauge transformation breaks the tetrad-Weyl symmetry, because the implied tetrad is an LCR-tetrad. Therefore we chose the LCR-tetrad $e'$ such that $\Phi'_1 = 1 = -\Phi'_2$. That is, we partly fix the tetrad-Weyl symmetry for non-trivial LCR-structures with $\Phi_1 \neq 0 \neq \Phi_2$.

Recall the general tetrad-Weyl transformation

$$\ell' = \Lambda \ell, \quad n' = N n, \quad m' = M m$$

$$Z'_1 = Z_1 + d(\ln \Lambda), \quad \Phi'_1 = \frac{\Lambda}{M} \Phi_1$$

$$Z'_2 = Z_2 + d(\ln N), \quad \Phi'_2 = \frac{\Lambda}{N} \Phi_2$$

$$Z'_3 = Z_3 + d(\ln M), \quad \Phi'_3 = \frac{\Lambda}{N} \Phi_1$$

(6.17)

In the case of the following generators

$$t_0 = I, \quad t_k = \frac{2a_k}{r} \rightarrow C_{ijk} = \epsilon_{ijk}$$

we have

$$B_{0\mu} + \frac{1}{2} B_{3\mu} = \ell'_\mu, \quad B_{0\mu} - \frac{1}{2} B_{3\mu} = n'_\mu, \quad \frac{1}{2}(B_{1\mu} + iB_{2\mu}) = m'_\mu$$

$$F_{0\mu\nu} = \partial_\mu B_{0\nu} - \partial_\nu B_{0\mu}, \quad F_{i\mu\nu} = \partial_\mu B_{i\nu} - \partial_\nu B_{i\mu} - \epsilon_{ijk} B_{j\mu} B_{k\nu}$$

(6.19)

Notice the direct relation of the gravitational tetrad with the electroweak potentials of the standard model.

In the case of the electron LCR-tetrad

$$\Phi_1 = \frac{-2a \cos \theta}{\sqrt{r^2 + a^2 \cos^2 \theta}}, \quad \Phi_2 = \frac{-2a \cos \theta}{\sqrt{(r^2 + a^2 \cos^2 \theta)^2}}$$

$$\Phi_3 = \frac{2a \sin \theta}{\sqrt{2(r + ia \cos \theta)^2(r - ia \cos \theta)}}$$

(6.20)

we make first the tetrad-Weyl transformation to reach the condition $\Phi'_1 = 1 = -\Phi'_2$. We find

$$N = -\frac{2a \cos \theta}{\sqrt{r^2 + a^2 \cos^2 \theta}} \Lambda$$

$$M \overline{M} = -\frac{2a \cos \theta}{\sqrt{r^2 + a^2 \cos^2 \theta}} \Lambda$$

(6.21)
The electromagnetic dressing is found with $\Lambda = \frac{qr}{r^2 + a^2 \cos^2 \theta}$. Then the connection $B$ is found with

$$
\begin{align*}
\Lambda &= \frac{qr}{r^2 + a^2 \cos^2 \theta} \\
N &= -\frac{qr}{4\pi(r^2 + a^2 \cos^2 \theta)} \\
M/M &= -\frac{qr a \cos \theta}{2\pi(r^2 + a^2 \cos^2 \theta)^2}
\end{align*}
$$

(6.22)

up to $M$ phase tetrad-Weyl transformation. We see that the $U(2)$ gauge field is directly related to the LCR-tetrad and the Higgs field is related with the $\Phi_i$ factors, which determine the relative invariants of the LCR-structure.

We will now prove that the above definition of the electroweak connection permits the emergence of distributional singularities in the embedded LCR-manifold. The starting point is to write the surface $\rho_{ij} = 0$, using the regular coordinates

\[
\begin{align*}
\text{Im} z^0 &= \phi_{11}(z^1, z^1, \text{Re} z^0), \quad \text{Im} \bar{z}^0 = \phi_{22}(z^1, \bar{z}^1, \text{Re} \bar{z}^0), \quad z^1 - \bar{z}^1 = \phi_{12}(z^\beta, \bar{z}^\beta) \\
\phi_{11}(p) &= \phi_{22}(p) = \phi_{12}(p) = 0, \quad d\phi_{11}(p) = d\phi_{22}(p) = d\phi_{12}(p) = 0
\end{align*}
\]

(6.23)

in a neighborhood of a point $p$. But the LCR-structure is a special totally real CR-structure, which at a real analytic neighborhood admits a general analytic transformation $r^b = f^b(z^\alpha)$, which makes it trivial

\[
\frac{r^a - \bar{r}^a}{2i} = 0
\]

(6.24)

This last analytic transformation is not generally an LCR-transformation. Hence, it breaks the LCR-structure, but there is no reason to worry for that now, because we are going to look for a connection, which has already broken the LCR-symmetry. The essential point here is the neighborhood of $p$ in the ambient complex manifold, where the analytic transformation can be extended. The case of the distributional electron (and neutrino) indicates that the analytic transformation cannot be extended around their location. Besides the entire region of the LCR-manifold can be described by a distribution with a representative (locally integrable function), which at the regular point $p$ appears as a regular potential with each source at the location of the electron. The location of the electron is not a real analytic region of the LCR-manifold, because it does not admit analytic extensions in both sides of the real surface $\rho_{ij} = 0$ in the ambient complex manifold. Recall that it is the Sato’s definition of generalized functions.

### 6.2 Derivation of standard model action

The successful application of the Bogoliubov recursive procedure to build up an effective quantum electrodynamics and its extraordinary experimental verification, suggest us to extend it including the massless neutrino soliton as a left-hand field $\frac{1}{2\pi} \gamma^\alpha \psi_\nu$, and all the permitted charged and neutral currents. No neutrino electromagnetic interaction should be introduced or permit it to appear through the Bogoliubov recursive procedure. It has already been shown.
that assuming the existence of all the standard model particles (for every generation separately) the implied standard model lagrangian is a consequence of the renormalizability condition. The appearance of the Poincaré group and the Schwartz distributions imposes the use of the basis of the rigged Hilbert space of the tempered distributions. A free field is an operator valued distribution in the appropriate Lorentz group representation corresponding to every elementary particle and all its dressings. Let us now enumerate the fields and the interactions we will consider in the beginning (correspondence principle) of the Bogoliubov procedure, indicating their existence in the context of PCFT:

1) The massive Dirac electron field $\psi_e(x)$, which satisfies the free Dirac equation and hence it implies a free massive Dirac propagator in the time ordering term $\langle 5.28 \rangle$. This fermionic solitonic LCR-manifold (with $g = 2$ gyromagnetic ratio) has been extensively studied.

2) The left-hand part of the massless Dirac neutrino field $\frac{1-e^{-\gamma}}{2} \psi_\nu$, which satisfies the free Dirac equation and hence it implies a free massless Dirac propagator in the time ordering term. All the considered currents will contain only the left-hand part of the neutrino field. This is the massless developable surface which corresponds to the massive electron ruled surface.

3) The massless electromagnetic field $A_\mu(x)$, which implies the corresponding massless propagator. It is the potential of the real part of the closed self-dual 2-form $\langle 5.17 \rangle$.

4) The electromagnetic interaction between the electron and the photon as indicated by the electromagnetic dressing of the electron LCR-structure.

5) The $U(2)$ gauge fields $\langle 6.19 \rangle$ viewed as an extension of electromagnetism. These fields are properly coupled with the charged and neutral currents of the electron neutrino pair. The most convenient method seems to be the Scharf $\langle 34 \rangle$ Q-charge operator with the $U(2)$ breaking will be implied by the difference of the electron and neutrino masses.

6) The derivation of the Cartan $U(2)$ connection strongly suggests that the Higgs scalar field is related to the relative invariants $\Phi_i$ of the LCR-structure. The real $\Phi_1$ and $\Phi_2$ are fixed with $\langle 5.22 \rangle$, and the phase of the complex $\Phi_3$ can be absorbed by $m_\mu dx^\mu$. Hence the radial of $\Phi_3$ should be identified with the scalar Higgs field of the standard model.

Including all these assumptions in the initial action through the ”correspondence principle”, the Bogoliubov procedure implies a closed lagrangian form only if the well known relations between the coupling constants and the masses of particles are valid $\langle 34 \rangle$. This means that the ”internal symmetry” $U(2)$ breaking mechanism, is a consequence of the initial mass difference between the electron and the neutrino and the renormalizability condition of the Bogoliubov procedure, viewed as an effective action generating mechanism! There is no initial internal group in PCFT. The ad hoc assumption of a fundamental $U(2)$ internal symmetry misled the scientific research to grand unified theories and their supersymmetric extensions, which have not been observed.

Let us now turn to the origin $\langle 25 \rangle$ of the three elementary particle generations (families). The three particle generations are a consequence of the gravity potentials of these solitons which emerge through the Einstein metric $g_{\mu\nu} \langle 1.1 \rangle$. 48
It is well known that the Einstein metric $g_{\mu\nu} = \eta_{ab} e_{\mu}^{a} e_{\nu}^{b}$, where $e_{\mu}^{a}$ are the four Cartan moving frames. They are defined up to a local $SO(1,3)$ transformation $e_{\mu}^{a} = S_{a}^{b} e_{\mu}^{b}$ which generates and relates the Cartan connection with the ordinary metric $g_{\mu\nu}$ connection. Newman and Penrose have noticed that assuming a null tetrad, the Cartan formalism acquires very useful properties easily applied to the radiation problems. In this formalism the LCR-structure coincides with the existence of two geodetic and shear-free null congruences, which have the simple form $\kappa = \sigma = \lambda = \nu = 0$. Besides, the use of the spinor dyad $(\varphi^{A}, i^{B})$ through the relations (4.15) imply the spinorial formulation of general relativity. I have already pointed out that a metric does not always admit two geodetic and shear-free congruences. In this case of metrics, using an arbitrary non geodetic and shear-free null tetrad, the spinor form of the conformal tensor $\Psi_{ABCD}$ can always be defined, and it admits two spinors $(\lambda^{A1}, \lambda^{B2})$, which satisfy the relations (4.20). In the linearized gravity approximation they become the spinors of the first two rows of the homogeneous coordinates of $G_{4,2}$. Hence locally, a non-conformally flat metric compatible with an LCR-structure has at most four geodetic and shear-free null congruences, i.e. at most four branches (sheets). Every two of them determine a LCR-structure. From the Petrov classification, we have the types of spacetime with four (type I), three (type II), two double (type D) and a triple (type III) principal null directions. Apparently the electron and the neutrino solitons correspond to type D spacetimes. The fact that I have not found static LCR-structures for cubic and quartic Kerr polynomials, suggests us to correspond the decaying muon and tau generations to the two Petrov types II and I respectively. The soliton stability is assured by the different degrees of the quadric (electron and its neutrino), cubic (muon and its neutrino) and quartic (tau and its neutrino) algebraic surfaces. The internal stability of each flavor is assured by the different relative invariants.

7 THE UP AND DOWN QUARKS

Concerning the electromagnetic and weak interactions, the hadronic sector of the elementary particles is (about) a copy of the leptonic sector. Quarks simply have the additional strong interaction, which should provide a confining mechanism. The standard model does not explain the general copy-picture, while the artificial add-on of the $SU(3)$ gauge group gives some answers to some phenomena, but it fails to imply (in the continuum) confinement and chirality breaking, which are the characteristic properties of strong interactions.

PCFT is mathematically a vector bundle (with a gauge field) over a lorentzian CR-manifold. The gluon field is identified with the gauge field of the action and the LCR-structure describes (contains) gravity, electromagnetic and weak interactions as outlined in the previous section, where we have assumed that the found distributional solitons have vanishing gluon field configuration. In this section I will explicitly find stable gluonic configurations for the electron and the neutrino LCR-manifolds, which I will identify with down and up quarks.
That is, the origin of the observed general copy-picture between the leptons and quarks is simply their common LCR-structure (which contains gravitational, electromagnetic and weak interactions), they differ to the non-vanishing "gluonic" dressing of the quarks.

Variation of the actions (3.6) relative to the gauge field implies the field equations

\[
\Gamma^{\mu \nu \rho \sigma} = \frac{1}{2} (\ell^{\mu} m^{\nu} - \ell^{\nu} m^{\mu}) (n^{\rho} m^{\sigma} - n^{\sigma} m^{\rho}) + (n^{\rho} m^{\mu} - n^{\mu} m^{\rho}) (\ell^{\sigma} m^{\nu} - \ell^{\nu} m^{\sigma})
\]

\[
(D_{\mu})_{ij} = \frac{1}{\sqrt{-g}} (\frac{\sqrt{-g}}{g} (\Gamma^{\mu \nu \rho \sigma} - \Gamma^{\mu \nu \rho \sigma}) F_{j \rho \sigma}) = 0
\]

\[
(D_{\mu})_{ij} = \frac{1}{\sqrt{-g}} (\frac{\sqrt{-g}}{g} (\Gamma^{\mu \nu \rho \sigma} + \Gamma^{\mu \nu \rho \sigma}) F_{j \rho \sigma}) = 0
\]

Recall that the derivation of quantum electrodynamics (as an affective field theory) was triggered by the existence of a source in the closed self-dual anti-symmetric tensor of the massive static soliton. But the above (7.1) both field equations are exact. We cannot replace (ad hoc) the zero of the second part of the equation with a source, because the symmetries of the action will be destroyed, and subsequently the renormalizability of the action will be destroyed too. The solution [31] to this obstruction comes after a close look at the form of the field equations (7.1). Notice that they are the sum or difference of two complex conjugate terms. This does not permit us to apply the complexification (necessary for the application of the Frobenius theorem) and use the convenient form that the LCR-structure tetrad takes in the ambient complex manifold through the structure coordinates \(z^a(x)\).

Therefore I find convenient to give the PDEs (7.1) the following equivalent forms

\[
I_R \rightarrow \frac{1}{\sqrt{-g}} (D_{\mu})_{ij} \{ \frac{\sqrt{-g}}{g} (\Gamma^{\mu \nu \rho \sigma} - \Gamma^{\mu \nu \rho \sigma}) F_{j \rho \sigma}) + (n^{\rho} m^{\mu} - n^{\mu} m^{\rho}) (\ell^{\sigma} m^{\nu} - \ell^{\nu} m^{\sigma}) \} = -k^\nu_i
\]

\[
I_I \rightarrow \frac{1}{\sqrt{-g}} (D_{\mu})_{ij} \{ \frac{\sqrt{-g}}{g} (\Gamma^{\mu \nu \rho \sigma} + \Gamma^{\mu \nu \rho \sigma}) F_{j \rho \sigma}) + (n^{\rho} m^{\mu} - n^{\mu} m^{\rho}) (\ell^{\sigma} m^{\nu} - \ell^{\nu} m^{\sigma}) \} = -ik^\nu_i
\]

where \(k^\nu_i(x)\) is a real vector field. The PDEs look like the equations of a gauge field with a color-electric and color-magnetic source respectively. Notice the natural emergence of the sources. I will solve these partial differential equations in the static (electron) LCR-structure (2.19). This is possible, because the LCR-structure defining equations completely decouple from the gauge field equations. The LCR-structure is first fixed (via the Lagrange multipliers) and after we proceed to the solution of the field equations, which involve the gauge field. This property of PCFT is essentially behind the physical observation of the lepton-quark correspondence! That is a quark has the same LCR-structure with the corresponding lepton. But the quark has in addition a stable non-vanishing distributional gauge field configuration (from which it gets its color), while the lepton has vanishing gauge field.
Recall that a distribution has two parts. The singular part and the regular part. A classical solution of the gauge field with a singular compact source will be interpreted as a colored soliton (the quark) with its gluon potential ("dressing") being the regular part of the generalized function. If we apply again with the gauge covariant derivative \((D_\nu)_{ij}\) and use the commutation relation

\[
[(D_\mu), (D_\nu)]_{ik} = -\gamma f_{ijk} F_{\mu\nu}
\]

(7.3)

we find that the current must be gauge covariantly conserved \((D_\nu)_{ij} k^\nu = 0\) for a classical solution to exist. We will look for fundamental distributional solutions which have compact singular sources, which may be interpreted as localized “particles”. I will work out the derivation of a (null) distributional solution for the first PDE (action \(I_R\)), where such a solution can exist, and I will simply indicate why the second PDE (action \(I_I\)) does not admit a corresponding color-magnetic solution.

In the case of gravity and electromagnetism we found distributional (fundamental) solutions, where the singular part is compact and located at the ring singularity. It is identified with the electron, while its gravitational and electromagnetic fields are the regular part of the distribution (the gravitational and electromagnetic dressings) located outside the singular support of the source (the electron). Now we will apply the same point of view for the computation of magnetic fields are the regular part of the distribution (the gravitational and electromagnetic) solutions, where the singular part is compact and located at the ring.

In these complex coordinates, the metric takes the off-diagonal form

\[
(z^\alpha(x), z^0(x)), \text{ and use their following powerful properties}
\]

\[
d z^\alpha = f^\alpha_0 \ell^\mu_0 dx^\mu + f^\alpha_1 m_\mu dx^\mu, \quad d z^\hat{0} = f^\hat{0}_0 n_\mu dx^\mu + f^\hat{0}_1 m_\mu dx^\mu
\]

(7.4)

In these complex coordinates, the metric takes the off-diagonal form

\[
\sqrt{-g} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = -i \ell \wedge m \wedge n \wedge \hat{m} = -i \hat{g} dz^0 \wedge dz^1 \wedge dz^\hat{0} \wedge dz^\hat{1}
\]

\[
g_{ab} = \begin{pmatrix} 0 & \hat{g}_{\alpha\beta} \\ \hat{g}_{\beta\alpha} & 0 \end{pmatrix}, \quad g^{ab} = \begin{pmatrix} 0 & \hat{g}^{\alpha\beta} \\ \hat{g}^{\beta\alpha} & 0 \end{pmatrix}
\]

(7.5)

Hence after the complexification we have to replace \(\sqrt{-g} \rightarrow -i \hat{g}\). Notice, that now we deal with a complex metric (pseudo-metric), and we must not take complex conjugations before returning back to real \(x\). Then (7.2) takes the
we may have the following solutions apparently coincide with the \((\text{abelian})\) equations

\[
\text{For } b = 0 \quad \partial_k F_{001} - \gamma f_{ikj} A_{k1} F_{j01} = (D_1)_{ij} F_{j01} = -\tilde{g}k^0_i
\]

\[
\text{For } b = 1 \quad \partial_k F_{010} - \gamma f_{ikj} A_{k0} F_{j01} = (D_0)_{ij} F_{j01} = \tilde{g}k^1_i
\]

\[
\text{For } b = 0 \quad \partial_k F_{010} - \gamma f_{ikj} A_{k1} F_{j01} = (D_1)_{ij} F_{j01} = -\tilde{g}k^0_i
\]

\[
\text{For } b = 1 \quad \partial_k F_{010} - \gamma f_{ikj} A_{k0} F_{j01} = (D_0)_{ij} F_{j01} = \tilde{g}k^1_i
\]

\(7.6\)

written separately for every structure coordinate in order to help a non-familiar reader to understand the subsequent mathematical operations. The integrability conditions imply

\[
[(D_0), (D_1)]_{ik} F_{k01} = -\gamma f_{ikj} F_{j01} - (D_n)_{ij} (\tilde{g}k^n_i)
\]

\[
[(D_0), (D_1)]_{ik} F_{k01} = -\gamma f_{ikj} F_{j01} - (D_\tilde{a})_{ij} (\tilde{g}k^\tilde{a}_i)
\]

\(7.7\)

They vanish outside the compact singular gluonic source.

As expected, the written in LCR-structure coordinates equations do not contain the complexified "metric" \(g_{\alpha\tilde{a}}\), and contain only the self-dual left-hand component \(F_{j01}\) and right-hand component \(F_{\tilde{a}01}\) of the gauge field strength, because the present gauge field action has been constructed to be metric independent.

It is evident that if

\[
f_{ijk} F_{j01} F_{k01} = -\frac{1}{2} f_{ijk} (n^\mu \tilde{m}^\nu F_{j\mu\nu}) (\ell^\mu m^\nu F_{k\mu\nu}) \neq 0
\]

\(7.8\)

does not vanish outside the sources, the differential equations \(7.6\) do not accept (fundamental) solutions with compact sources. Hence my conclusion is that, outside the singular compact part (the quark) of the generalized function, we may have solutions only if \(F_{j01}\) or \(F_{\tilde{a}01}\) vanish for non vanishing \(f_{ijk}\). That is, we may have the following solutions

\[
A_\alpha = \frac{1}{\gamma} (\partial_\alpha U)U^{-1} \quad , \quad (\ell^\mu m^\nu F_{k\mu\nu}) = (\tilde{e}^\mu m^\nu - \tilde{e}^\nu m^\mu) F_{k01} \neq 0
\]

\[
(n^\mu \tilde{m}^\nu F_{k\mu\nu}) = (n^0 \tilde{m}^1 - n^1 \tilde{m}^0) F_{k01} \neq 0 \quad , \quad A_\tilde{a} = \frac{1}{\gamma} (\partial_\alpha U)U^{-1}
\]

\(7.9\)

where \(U\) and \(U'\) are arbitrary elements of the gauge group in a prescribed gauge group representation.

Hence, the two gauge field equations become abelian

\[
\partial_\alpha F_{01} - \gamma [A_\alpha, F_{01}] = 0 \quad \Rightarrow \quad \partial_\alpha F'_{01} = 0 \quad , \quad F_{01} = U' F'_{01} U'^{-1}
\]

\[
\partial_\alpha F_{01} - \gamma [A_\alpha, F_{01}] = 0 \quad \Rightarrow \quad \partial_\alpha F'_{01} = 0 \quad , \quad F_{01} = U' F'_{01} U'^{-1}
\]

\(7.10\)

Now returning back in the real LCR-manifold these two partial differential equations apparently coincide with the (abelian) equations

\[
\frac{1}{\sqrt{-g}} \partial_\mu \left\{ \sqrt{-g} (\ell^\mu m^\nu - \ell^\nu m^\mu) (n^\rho m^\sigma F_{\rho\sigma}) \right\} = -k^\nu_j \quad , \quad \ell^\mu m^\nu F_{j\mu\nu} = 0
\]

\[
n^\rho m^\sigma F_{j\rho\sigma} = 0 \quad , \quad \frac{1}{\sqrt{-g}} \partial_\mu \left\{ \sqrt{-g} (n^\rho m^\sigma - n^\sigma m^\rho) (\ell^\rho m^\sigma F_{j\rho\sigma}) \right\} = -k^\nu_i
\]

\(7.11\)
Notice that the essential non-vanishing term in both solutions is null, therefore we will look for completely null solutions, i.e. $(\ell^\nu n^\nu - m^\nu m^\nu)F_{j\rho\sigma} = 0$. Hence we will look for null abelian solutions which satisfy the equations

$$d\{\ell \wedge m(n^\nu m^\nu F_{j\rho\sigma})\} = i \ast k_j, \quad \ell^\mu m^\nu F_{j\mu \nu} = 0, \quad (\ell^\nu n^\nu - m^\nu m^\nu)F_{j\rho\sigma} = 0$$

$$n^\nu m^\nu F_{j\mu \nu} = 0, \quad d\{(n \wedge m(n^\nu m^\nu F_{j\rho\sigma})\} = i \ast k_i', \quad (\ell^\nu n^\nu - m^\nu m^\nu)F_{j\rho\sigma} = 0 \quad (7.12)$$

The LCR-structure coordinates \((4.3)\) determine the two characteristic 2-forms of the static quadratic surface of \(CP^3\).

$$dz^0 \wedge dz^1 = (f_0^0 f_1^0 - f_0^1 f_0^1)\ell \wedge m, \quad dz^\delta \wedge dz^\gamma = (f_0^0 f_1^0 - f_0^1 f_0^1)n \wedge m \quad (7.13)$$

These two surfaces are one-side extensions of the corresponding searched solutions on \(\mathbb{R}^4\) boundary of the classical domain. Hence they can be solved.

The non-vanishing closed 2-forms (with sources) are found to be

$$d\{\frac{C_i'}{\sin \theta (r - ia \cos \theta)}\ell \wedge m\} = i \ast k'_j \quad (7.14)$$

$$d\{\frac{C''_i}{(r^{2 + a^2}) \sin \theta} \ell \wedge m\} = i \ast k''_j$$

where \(C'_i\) and \(C''_i\) are arbitrary complex constants, which are fixed using Stokes’ theorem. In the oblate spheroidal coordinates the solutions have the explicit forms

$$\frac{C'_i}{\sin \theta (r - ia \cos \theta)}\ell \wedge m = \frac{C'_i}{\sqrt{2} \sqrt{r^2 + a^2}} \left\{ \frac{ia}{r^2 + a^2} \left[ \frac{1}{\sin \theta} dt \wedge d\theta - \frac{1}{\sin \theta} dt \wedge d\varphi - idt \wedge d\varphi + \right. \right.$$

$$\left. \frac{1}{\sin \theta} - idr \wedge d\varphi - a \sin \theta \phi d\theta \wedge d\varphi \right) \right\}$$

$$\frac{C''_i}{(r^{2 + a^2}) \sin \theta} n \wedge m = \frac{C''_i}{2 \sqrt{2} \sqrt{r^2 + a^2}} \left\{ \frac{ia}{r^2 + a^2} \left[ \frac{1}{\sin \theta} dt \wedge d\theta + idt \wedge d\varphi - \right. \right.$$

$$\left. \frac{1}{\sin \theta} - idr \wedge d\varphi - a \sin \theta \phi d\theta \wedge d\varphi \right) \right\} \quad (7.15)$$

After a straightforward calculation I find

$$\int_{t, r = const} \frac{C'_i}{\sin \theta (r - ia \cos \theta)}\ell \wedge m = -2\sqrt{2} \pi C'_j a \equiv \gamma'_j \quad (7.16)$$

$$\int_{t, r = const} \frac{C''_i}{(r^{2 + a^2}) \sin \theta} n \wedge m = -\sqrt{2} \pi C''_j a \equiv \gamma''_j$$

which implies that the constants must be real for the sources to be real and the original field equations to be satisfied. Notice that they are proportional to the coefficient \(a\) (the spin of the soliton) implying that the scalar LCR-structures \([4.3]\) do not define colored configurations with sources. The physical meaning of this remark is that PCFT does not permit glueballs.

We finally find the solutions...
The two solutions with sources are expected to have the forms
\[
F_j' = \frac{-\gamma_i}{\pi a \alpha^2} \left[ \frac{a}{r^2 + a^2} dt \wedge dr - d(t - r) \wedge d\varphi \right] = \\
= d\left[ \frac{-\gamma_i}{\pi a \alpha^2} (t - r) \left( \frac{a}{r^2 + a^2} dr - d\varphi \right) \right]
\]
\[
F_j'' = \frac{-\gamma_i}{\pi a} \left[ \frac{a}{r^2 + a^2} dt \wedge dr + d(t + r) \wedge d\varphi \right] = \\
= d\left[ \frac{-\gamma_i}{\pi a} (t + r) \left( \frac{a}{r^2 + a^2} dr + d\varphi \right) \right]
\]
(7.17)
with the corresponding potentials been apparent. Notice that the parameter \(a\) (the radius of the singularity ring) appears in the denominator. This seems to be the origin of the confining potential asymptotic solution (??) of the gluonic equations in contradiction to the electromagnetic solution.

The second PDE of (7.2), which is implied by the action \(I_I\), may be written as
\[
I_I \rightarrow \sqrt{-g} (D_{\mu})_{ij} \left\{ \sqrt{-g} \left( \ell^{\mu} m^{\nu} - \ell^{\nu} m^{\mu} \right) \right\} + (n^{\nu} m^{\mu} - n^{\mu} m^{\nu} \left( \ell^{\mu} m^{\nu} * F_{j\rho\sigma} \right) + - i k'_3 \right\} = - i k'_3
\]
(7.18)
because \(\ell^{[\rho} m^{\sigma]}\) and \(n^{[\nu} m^{\mu]}\) are self-dual. This has exactly the form of the first PDE, with the gauge field tensor replaced by its dual. Hence the solutions of the second PDE will be \(- * F_j'\) and \(- * F_j''\), which is impossible, because they have sources, i.e. \(d * F_j' \neq 0 \neq d * F_j''\).

The left \(F_{j01}\) and right \(F_{j10}\) solutions may coexist in the same region if they do not vanish for \(i\) and \(j\) in the abelian subalgebra. In the physically interesting case of the \(su(3)\) Lie algebra can happen if \(i\) and \(j\) take the values 3 and 8. But in this case the final classical solution will not be null. It is a non-null solution with precise gluonic charges. The final form of the non-null gluonic solution is
\[
A_j^{(g)} = \frac{-\gamma_i}{\pi a} (\tan^{-1} \frac{x}{r} dt + rd\varphi) = \\
= \frac{-\gamma_i}{4\pi a} (\tan^{-1} \frac{1}{a} dx^0 - \frac{ax^1 + r x^2}{(x^1)^2 + (x^2)^2} dx^1 + \frac{rx^1 - ax^2}{(x^1)^2 + (x^2)^2} dx^2)
\]
\[
A_j^{(e)} = \frac{aq^3}{4\pi (r^2 + a^2)^2} (dx^0 - \frac{r x^1 - ax^2}{r^2 + a^2} dx^1 - \frac{rx^1 + ax^2}{r^2 + a^2} dx^2 - \frac{x^3}{r} dx^3)
\]
(7.19)
where the last formula is the electromagnetic dressing. The gluonic monopole potential is singular at \(a = 0\) of the spin parameter and and its magnetic part is linear in \(r\). These characteristics do not appear in the electromagnetic potential and may provide confinement.

The second quark of the massless LCR-structure can be found using the same procedure. Let us consider the first (\([E = -p^3]\)) LCR-structure of (6.1) with the corresponding structure coordinates and tetrad
\[
[E = -p^3]: \quad X^3 - aX^1 = 0 \quad , \quad X^0 = 0
\]
\[
z^0 = x^0 - ia - x^3 - \frac{(x^1)^2 + (x^2)^2}{2(x^1)^2 + (x^2)^2} ia \quad , \quad z^1 = \frac{x^1 - ix^2}{x^2 - x^1}
\]
\[
z^0 = x^0 + x^3 - \frac{(x^1)^2 + (x^2)^2}{2x^2 - x^1} \quad , \quad z^1 = \frac{x^1 + ix^2}{x^2 - x^1}
\]
(7.20)
The two solutions with sources are expected to have the forms
\[ F' = f_j(z^0, z^1)dz^0 \wedge dz^1 \rightarrow \ dF' = - * k'_j \]  
\[ F'' = f_j(\tilde{z}^0, \tilde{z}^1)d\tilde{z}^0 \wedge d\tilde{z}^1 \rightarrow \ dF'' = - * k''_j \]  
(7.21)

But the naive Stokes' theorem does not apply. This problem has to be treated in the "unphysical" grassmannian chart (4.6). Apparently, we may bypass this difficulty by assuming a mass term and repeat the preceding calculations, in order to experimentally check PCFT.

In analogy to electrodynamics we may introducing the quark field as the source. The implied self-consistent equations become

\[ (D_\mu)_{ij} H^\mu_\nu = \gamma \tau^\nu \tau_i q \quad \text{and} \quad (D_\mu)_{ij} * H^\mu_\nu = 0 \quad \text{and} \quad \gamma_\nu (i \partial_\mu - \gamma A_\mu \tau_i) q - m q = 0 \]

\[ H^\mu_\nu \equiv \frac{1}{2} \left( (\ell^\mu m^\nu - \ell^\nu m^\mu)(n^\sigma m^\rho F_{j,\sigma \rho}) + (\ell^\mu m^\nu - \ell^\nu m^\mu)(n^\sigma m^\rho F_{j,\sigma \rho}) + \right. 
\left. + (n^\sigma m^\mu - n^\nu m^\mu)(\ell^\rho m^\sigma F_{j,\rho \sigma}) + (n^\sigma m^\mu - n^\nu m^\mu)(\ell^\rho m^\sigma F_{j,\rho \sigma}) + \right. 
\left. + (n^\rho m^\mu - n^\nu m^\mu)(\ell^\sigma m^\rho F_{j,\sigma \rho}) - (n^\rho m^\mu - n^\nu m^\mu)(\ell^\sigma m^\rho F_{j,\sigma \rho}) \right) \]

*H^\mu_\nu \equiv \frac{1}{2} \left[ (\ell^\mu m^\nu - \ell^\nu m^\mu)(n^\sigma m^\rho F_{j,\sigma \rho}) - (\ell^\mu m^\nu - \ell^\nu m^\mu)(n^\sigma m^\rho F_{j,\sigma \rho}) + \right. 
\left. + (n^\rho m^\mu - n^\nu m^\mu)(\ell^\sigma m^\rho F_{j,\sigma \rho}) - (n^\rho m^\mu - n^\nu m^\mu)(\ell^\sigma m^\rho F_{j,\sigma \rho}) \right] \]

(7.22)

It is not clear to me what this set of PDEs represent.

### 7.1 A quark confining mechanism

In order to understand the implied confinement, we have to understand the mathematical framework of PCFT. Therefore I think it will be helpful to the reader, if I briefly recapitulate it, using now the Sato's hyperfunction point of view [15] for the distributions (generalized functions).

The LCR-manifold is a special totally real 4-dimensional submanifold of a complex 4-dimensional manifold satisfying the relations

\[ \rho = \begin{pmatrix} \rho_{11}(x^0, z^0) & \rho_{12}(x^0, z^0) \\ \rho_{12}(x^0, z^0) & \rho_{22}(x^0, z^0) \end{pmatrix} = 0 \]  
(7.23)

in a neighborhood of \( z^0(p) \). The CR-submanifolds are usually considered as boundaries of domains of holomorphy. The LCR-manifold may be considered as the boundary of the domain

\[ \rho = \begin{pmatrix} \rho_{11}(x^0, z^0) & \rho_{12}(x^0, z^0) \\ \rho_{12}(x^0, z^0) & \rho_{22}(x^0, z^0) \end{pmatrix} \succ 0 \]  
(7.24)

\[ \det \rho > 0 \quad \text{and} \quad \text{trace}(\rho) > 0 \]

where the symbol \( \succ \) means that the matrix \( \rho \) is positive definite, which is equivalent to the last two conditions. In the zero gravity approximation, the present ambient complex manifold is the \( SU(2, 2) \) classical (Cartan) and its characteristic (Shilov) boundary is \( S^1 \times S^3 \), which is a double cover of \( \mathbb{R}^4 \), as boundaries of the corresponding Siegel domains.
If the ambient complex manifold is a projective variety, precise special dependence of the defining functions from the structure coordinates \((z^\alpha, \tilde{z}^\beta)\) suggest us to consider it to be the lines of \(CP^3\) (points of the Grassmannian manifold \(G_{4,2}\), which intersect two sheets (branches) of a hypersurface \(K(Z^m) = 0\). The structure coordinates \(z^\alpha\) determine the one intersection point at the one branch and \(\tilde{z}^\beta\) determine the other intersection point at another branch. The two branches intersect at a branch curve of \(CP^3\), which corresponds to the branch points of the Riemann surfaces (algebraic curves) of \(CP^2\). Recall that in \(CP^2\), analyticity is restored by using a branch cut that joins two branch points. In the present case of \(CP^3\), the cut is done at a surface, which has the branch curve as boundary. The LCR-structure essentially projects this picture down to the LCR-manifold (the boundary of the domain). Then the holomorphic functions on the domains of holomorphy become generalized functions (Sato’s hyperfunctions) on the LCR-manifold (the real spacetime) with real analytic forms in some regions and distributional sources into others. Below I will perform these calculations in the case of the static solitonic LCR-manifold (with zero gravity) and the colored solutions of the gauge field. These calculations may be elementary for the mathematicians, but we (particle physicists) are not familiar with these techniques.

The precise subset of quadratic polynomials, which is closed relative to the Poincaré transformations have the form

\[ A_{mn}Z^mZ^n = 0 \]

\[ A_{mn} = \begin{pmatrix} \omega & P \\ P^\top & 0 \end{pmatrix}, \quad P = \begin{pmatrix} -(p^1 + ip^2) & -p^0 + p^3 \\ p^0 + p^3 & (p^1 + ip^2) \end{pmatrix} = -p\epsilon \tag{7.25} \]

\[ p = \begin{pmatrix} p^0 - p^3 \\ -p^1 + ip^2 \\ 0 \\ p^0 + p^3 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

where \(p^\mu\) is the (real) 4-momentum and \(\omega\) the \(2 \times 2\) spin-matrix of the solitonic LCR-structure. The projection (from an external point) of the quadric to a \(CP^2 \subset CP^3\) is a double cover (it has two sheets). Let us consider the following projection of the quadric

\[ Z = X + \tau Y \quad , \quad X = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad Y = \begin{pmatrix} Y^1 \\ Y^1 \\ Y^2 \\ Y^3 \end{pmatrix} \tag{7.26} \]

\[ X^mX^nA_{mn} + 2\tau X^mY^nA_{mn} + \tau^2Y^mY^nA_{mn} = 0 \]

The two roots of \(\tau\) determine the two sheets of the quadric. In our case, it is more convenient to consider the two intersection points between the two branches and the line determined by \(r \in G_{4,2}\) as the two roots of the (projective) equation

\[ Z^m = \begin{pmatrix} \lambda \\ -ir\lambda \end{pmatrix} \]

\[ A_{mn}Z^mZ^n = \lambda^T(\omega - iP\tau - ir^TP^\top)\lambda = 0 \tag{7.27} \]

56
The branch curve is given by the double root, i.e. it is

$$\det(\omega - iPr - iv^TP^T) = 0 \quad (7.28)$$

It intersects the zero gravity LCR-submanifold of $G_{4,2}$, when $r = x$ is hermitian matrix. Recall that in zero gravity the domain becomes the $SU(2, 2)$ symmetric classical domain and its boundary (in the chiral Siegel realization) is now the real $\mathbb{R}^4$ submanifold.

We already have found that in one from the two branches the gauge field strength $F_{\mu\nu}$ must vanish. Otherwise the gauge field does not admit null solution with localized distributional source. Let us start with the first left-hand solution of (7.17), which at the one side of the boundary it is

$$\text{Then the flatprint of the LCR-tetrad takes the form}
\begin{align*}
F_{j\ell}(z)dz^\ell \wedge dz^j &= \frac{-\gamma_j}{2\sqrt{2 \pi a \sin \theta (r - ia \cos \theta)}} \ell \wedge m \\
d\{\frac{-\gamma_j}{2\sqrt{2 \pi a \sin \theta (r - ia \cos \theta)}} \ell \wedge m\} &= i * k_j
\end{align*} \quad (7.29)$$

and at the other side it vanishes.

It is more convenient to use cartesian coordinates (5.20), where the differential forms are

$$\begin{align*}
dx &= \frac{r \cos \varphi \sin \theta \theta}{\sqrt{r^2 + a^2}} dx + \sqrt{r^2 + a^2} \cos \varphi \cos \theta \theta \sin \vartheta \sin \varphi \sin \theta \theta \varphi \\
dy &= \frac{r \sin \varphi \sin \theta \theta}{\sqrt{r^2 + a^2}} dy + \sqrt{r^2 + a^2} \cos \varphi \cos \theta \theta \sin \varphi \sin \theta \theta \sin \theta \theta \varphi \\
dz &= \cos \theta \theta \theta \theta \varphi
\end{align*} \quad (7.30)$$

which are inverted to

$$\begin{align*}
d\varphi &= \frac{1}{\sqrt{r^2 + a^2} \sin \theta} (\cos \varphi dy - \sin \varphi dx) \\
d\theta &= \frac{1}{\sqrt{r^2 + a^2} \cos \theta \cos \varphi} (dx + \sqrt{r^2 + a^2} \cos \theta \sin \varphi \sin \varphi \sin \theta \theta \varphi) \\
dr &= \frac{r \cos \varphi \sin \theta \theta}{\sqrt{r^2 + a^2}} dx + \frac{r \sin \varphi \sin \theta \theta}{\sqrt{r^2 + a^2}} dy \sin \varphi \sin \theta \theta \sin \theta \theta \varphi + \frac{r \sin \theta \theta \varphi}{(r^2 + a^2) \cos \theta \cos \varphi} dz
\end{align*} \quad (7.31)$$

Then the flatprint of the LCR-tetrad takes the form

$$\begin{align*}
\ell_\mu dx^\mu &= dt + \frac{b_\mu - a_\mu}{r^2 + a^2} dx - \frac{a_\mu + b_\mu}{r^2 + a^2} dy - \frac{r_\mu - a_\mu}{r^2 + a^2} dz \\
n_\mu dx^\mu &= \frac{r^2 (r^2 + a^2)}{2(r^2 + a^2)} [dt + \frac{a_\mu + b_\mu}{r^2 + a^2} dx + \frac{a_\mu - b_\mu}{r^2 + a^2} dy + \frac{r_\mu - a_\mu}{r^2 + a^2} dz] \\
\frac{1}{\sqrt{2 \pi a \sin \theta (r - ia \cos \theta)}} m_\mu dx^\mu &= \frac{r^2}{(r^2 + a^2)^2} [i dt + \frac{a_\mu y + b_\mu x}{r^2 + a^2} (- \frac{z z}{r} + iy) dx \\
&\quad - \frac{r_\mu y + a_\mu x}{r^2 + a^2} (\frac{\varphi}{r} + i x) dy + rdz]
\end{align*} \quad (7.32)$$

from which I will compute the self-dual 2-form

$$\begin{align*}
G_j - i * G_j &= \frac{-\gamma_j}{2\sqrt{2 \pi a \sin \theta (r - ia \cos \theta)}} \ell \wedge m = -(E_j^1 + iB_j^1) dt \wedge dx - \\
-(E_j^2 + iB_j^2) dt \wedge dy - (E_j^3 + iB_j^3) dt \wedge dz - i(E_j^3 + iB_j^3) dx \wedge dy + \\
i(E_j^3 + iB_j^3) dx \wedge dz - i(E_j^1 + iB_j^1) dy \wedge dz
\end{align*} \quad (7.33)$$

57
Hence it defines an effective real 2-form $G_j$, where $\vec{E}_j^I$ is its color electric component and $\vec{B}_j^I$ is its color magnetic component. The color electric and magnetic fields of the first solution are

$$\vec{E}_j^I = \frac{\gamma_i r^2}{(r^4 + a^2 z^2) \pi a \sqrt{2}} \begin{pmatrix} \frac{a(y_i - rx)}{r^2 + a^2} - \frac{y_i (r^2 + a^2)}{x^2 + y^2} & \frac{y_i (r^2 + a^2)}{x^2 + y^2} - \frac{az}{r^2 + a^2} \\ \frac{x^4 + a^2}{x^4 + y^2} & \frac{x^2 + a^2}{x^4 + y^2} \end{pmatrix}$$

(7.34)

and

$$\vec{B}_j^I = \frac{\gamma_i r}{(r^4 + a^2 z^2) \pi a \sqrt{2}} \begin{pmatrix} \frac{(r^2 + a^2)xz}{(r^2 + a^2)yz} & \frac{(r^2 + a^2)yz}{(r^2 + a^2)xz} \\ -1 \end{pmatrix}$$

(7.35)

respectively.

Recall that the variable $r$ is an oblate spheroidal coordinate and satisfies the relation

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1$$

(7.36)

Now we are ready to reveal the singularities of the solution, which occur at

$$(r, z) = (0, 0) \quad \text{and} \quad x^2 + y^2 = a^2$$

and

$$x^2 + y^2 = 0 \quad \Rightarrow \quad z = \pm r$$

(7.37)

This is the ring circle and the z-axis. An additional singularity (discontinuity) exists on the branch cut ($x^2 + y^2 < 0$), which is not seen by the field outside the branch cut. At the one side of the branch cut the color gauge field does not vanish, while at the other side it does vanish, because crossing the disk branch cut we pass to the other branch with vanishing gauge field. Recall that a point of the grassmanian space corresponds to two points of $\mathbb{C}P^3$, which belong to different branches, branched at the branch cut. In brief the quark is located at the ring singularity and the positive (or negative) z-axis, where it has non-vanishing gluon field. Compare the singularities of the present gluonic field with the corresponding singularities of the electromagnetic field (5.22). The gluonic field has the additional singularity at the z-axis which obstructs its free existence. Hence, unlike the electron, the quark cannot be free.

In the case of a quark and antiquark system, located at the two end rings of a finite tube and having non-vanishing gluon field inside the tube and vanishing outside the tube, form a ”fat” Nielsen-Olesen string [7], which implies confinement. Recall that the non-vanishing component of the gluonic field inside the string will be either the left or the right one, because there is no solution with both being non-zero. Hence the scalar meson bound state (pion) will have a precise chirality and it will be a pseudoscalar.

The second (right-hand) solution of (7.17), which at the one side of the boundary it is
\[
F_{\mu\nu}(z^a)dz^a \wedge dz^\bar{a} = \frac{-\gamma''_j (r - i a \cos \theta)}{\sqrt{2\pi a (r^2 + a^2) \sin \theta}} n \wedge \bar{m}
\]
\[
G''_j - i * G''_j = \frac{-\gamma_j (r - i a \cos \theta)}{\sqrt{2\pi a (r^2 + a^2) \sin \theta}} n \wedge \bar{m}
\]
(7.38)

where \(\gamma''_j\) is the effective color-electric charge.

The color electric and magnetic fields of the second solution are

\[
\bar{E}'_j = \frac{\gamma''_j r^2 \sqrt{2}}{(r^2 + a^2 - x^2) \pi a} \left( \begin{array}{c} y(r^2 + a^2) - \frac{a(x + y)}{r^2 + a^2} - \frac{a}{r} \\ z \frac{a(x + y)}{r^2 + a^2} - \frac{x(r^2 + a^2)}{r^2 + a^2} + 1 \end{array} \right)
\]
(7.39)

and

\[
\bar{B}'_j = \frac{\gamma''_j r \sqrt{2}}{(r^2 + a^2 - x^2) \pi a} \left( \begin{array}{c} (x^2 + a^2) yz \\ (x^2 + a^2) zy \\ (x^2 + a^2) xz \end{array} \right)
\]
(7.40)

respectively. This non-vanishing right-hand solution is slightly different than the left-hand one, but it has the same singularities.

Let us now see that it does not seem possible to incorporate this gluonic solution to the standard model action, using the Bogoliubov recursive procedure. One can check that the color electric and magnetic vectors satisfy the following null relations

\[
(E'_j)^2 - (B'_j)^2 = 0 \quad , \quad E'_j \cdot B'_j = 0 \quad , \quad \text{No} \ j \ \text{summation}
\]
(7.41)

and the effective field equations

\[
dG'_j = 0 \quad , \quad d * G'_j = i * k'_j
\]
\[
\nabla \cdot \bar{B}'_j = 0 \quad , \quad \partial_0 \bar{B}'_j + \nabla \times \bar{E}'_j = 0
\]
\[
\nabla \cdot \bar{E}'_j = k'^0_j \quad , \quad \partial_0 \bar{E}'_j - \nabla \times \bar{B}'_j = -k'_j
\]
(7.42)

where the Minkowski metric is assumed. The non-vanishing right-hand solution satisfies the same equations.

8 A SU(3) CONNECTION FROM THE LCR-STRUTURE

Recall that Einstein was looking for a geometric structure, which could replace the lorentzian metric, and produce all the interactions. His higher dimensional Kaluza-Klein model did not succeed in describing electromagnetism. The same fate had the metric with torsion suggested by Cartan. We already showed that
the LCR-structure implies the metric structure and the electroweak $U(2)$
connection, which are manifestations of the LCR-tetrad. In the previous
sections, we found distributional solutions of the gauge field related to the static
LCR-structure. If we also succeed to derive the $SU(3)$ connection from the LCR-
structure, Einstein could be justified. Everything (gravity, electromagnetism,
weak and strong interactions and the fermionic particles) are manifestations
of the pure geometric LCR-structure, without any additional gauge field. The
suggestion of this section is that a $SU(3)$ Cartan connection exists, which could
provide the distributional solutions found above.

A realizable LCR-structure is based on hypersurfaces of $CP^3$, which are
covariant relative to $SL(4, \mathbb{C})$ transformation. That is $SL(4, \mathbb{C})$ preserves LCR-
structure. Following the Griffiths-Harris\cite{14} moving frame approach, we will
look for a possible emergence of the gluonic connection.

Let a frame $\{A_0, A_1, A_2, A_3\}$ of $CP^3$ determined by the corresponding four
vectors of $\mathbb{C}^4$, where $A_0$ determines the point where the $CP^3$
defining lines of $\mathbb{C}^4$ pass through. Assuming it to be a unitary basis, the Cartan moving frame
relations are

$$dA_m = \omega_{mn} A_n \quad , \quad d\omega_{mn} = \omega_{ml} \wedge \omega_{ln} \quad , \quad \omega_{mn} = \overline{\omega_{nm}} \quad (8.1)$$

$\omega_{ml}$ are 1-forms, which take values in the Lie algebra of $SU(4)$. $CP^3$ is deter-
mined by the annihilation of the 1-forms $\omega_{0i}$, (which is a basis of the cotangent
space $T^*(CP^3)$) because of the Frobenius relation

$$d\omega_{0i} = \omega_{00} \wedge \omega_{0i} + \omega_{0j} \wedge \omega_{ji} \quad (8.2)$$

Hence the projection

$$A_0 : U(4) \rightarrow CP^3 \quad (8.3)$$

gives a principal $U(1) \times U(3)$ fibration with corresponding vector bundles, the
line bundle $L_{A_0} = O_{A_0}$ and the universal quotient bundle $Q_{A_0} = \mathbb{C}^4/L_{A_0}$. In
our case, the Kerr function (2.10) generates a Darboux unitary frame

$$\hat{Z}_0(\tau, s), \hat{Z}_1(\tau, s), \hat{Z}_2(\tau, s), \hat{Z}_3(\tau, s)$$
$$d\hat{Z}_0 = \theta_{00} \hat{Z}_0 + \theta_{01} \hat{Z}_1$$
$$d\theta_{0i} - iA_{ij} \wedge \theta_{0j} = 0 \quad , \quad \hat{A}_{ij} = A_{ji} \quad (8.4)$$

$$A = (A_{ij}) = \sum_{I=1}^{8} A_{I\beta} dz^\beta(t_I)_{ij} \quad , \quad [t_I, t_J] = i\delta_{IJK}t_K$$

$$F = \partial A - iA \wedge A \longrightarrow DF := \partial F + iA \wedge F - iF \wedge A = 0$$

where the general form of antihermition connection has been replaced with the
usual hermitian gauge field $A = (A_{ij})$, $t_I$ are the generators of $SU(3)$, and
$\partial(A_{I\beta} dz^\beta) = \frac{\partial A_{I\beta}}{\partial z^\alpha} dz^\alpha \wedge dz^\alpha$. The explicit form of the curvature is

$$F_{I\alpha\beta} = \partial_\alpha A_{I\beta} - \partial_\beta A_{I\alpha} - f_{IJK} A_{J\alpha} A_{K\beta}$$
$$F_{I01} = \partial_0 A_{I1} - \partial_1 A_{I0} - f_{IJK} A_{J0} A_{K1} \quad (8.5)$$
Notice that the Bianchi identity is identically satisfied, because of the (complex) dimension-2 of the analytic hypersurface. In this context the leptonic particles correspond to \( F_{I\alpha\beta} = 0 \) and the quarks (hadrons) to \( F_{I\alpha\beta} \neq 0 \).

Recall that a complex point \((z^\alpha, \tilde{z}^{\tilde{\beta}})\) in the 4-dimensional ambient Kaehler manifold is determined by the two complex points \( z^\alpha \) and \( \tilde{z}^{\tilde{\beta}} \) of the hypersurface of \( CP^3 \). Hence the above holomorphic connection adapted to the analytic surface implies the following section

\[
A_J = A_{J\alpha}(z^\beta)dz^\alpha + A_{J\tilde{\alpha}}(\tilde{z}^{\tilde{\beta}})d\tilde{z}^{\tilde{\alpha}}
\]

\[
F_{I\alpha\beta}(z^\beta) = \partial_\alpha A_{I\beta} - \partial_\beta A_{I\alpha} - f_{IJ\kappa} A_{J\kappa} A_{I\beta} \\
F_{I\tilde{\alpha}\tilde{\beta}}(\tilde{z}^{\tilde{\beta}}) = \partial_{\tilde{\alpha}} A_{I\tilde{\beta}} - \partial_{\tilde{\beta}} A_{I\tilde{\alpha}} - f_{IJ\kappa} A_{J\kappa} A_{I\tilde{\beta}} \\
G_{I\alpha\tilde{\beta}} = 0
\]

which is reduced down to the (real) 4-dimensional LCR-manifold

\[
A_J = A_{J\alpha}(z^\beta(x))\frac{\partial z^\alpha}{\partial x^\mu}dx^\mu + A_{J\tilde{\alpha}}(\tilde{z}^{\tilde{\beta}}(x))\frac{\partial \tilde{z}^{\tilde{\alpha}}}{\partial x^\mu}dx^\mu
\]

Achieving such a framework could provide Einstein’s objective to show that all the interactions observed in nature have a geometric origin. Besides the color group is fixed to the observed in nature \( SU(3) \) group.

### 9 PERSPECTIVES

The recent experimental results of the LHC experiments at CERN show that supersymmetric particles do not exist and subsequently, quantum string theory does not describe nature. Hence, the 4-dimensional PCFT remains the only known model, compatible with quantum theory, which provides the general experimentally observed framework, without needing supersymmetry to introduce fermions and internal symmetries.

Paraphrasing Euclid (of Alexandria) we may say that ”there is no royal road to .... the theory of everything too”. We already realized the background algebraic geometric structure in string theory based on the algebraic curves. PCFT has essentially the same mathematical basis as the Polyakov action, it simply transfers the mathematics to the algebraic surfaces of \( CP^3 \) and their intersection with the lines, which constitute the Grassmannian space \( G_{4,2} \). The Lorentzian CR-structure projection to a real 4-dimensional submanifold complicates further the mathematical problems, but it clarifies the general physical picture. No moving strings and hidden dimensions are needed. The particles and their potentials (dressing) are distributional solitons. The singular part of the distribution (the source) is the fermionic particle and the regular part is its ”potential”. The observed spacetime is just the ”superposition” of these elementary solitons. Mathematically they are real 4-dimensional manifolds (boundaries of domains of holomorphy) \([1,13]\). The Minkowski spacetime is the Shilov boundary of this \( SU(2,2) \) classical domain in its unbounded realization and precisely its universal cover \( \mathbb{R} \times S^3 \).
The mathematical structure is simple and beautiful, because the action is fully geometric and apparently renormalizable by simple conventional power counting. It is just the vector bundle on a 4-dimensional lorentzian CR-structure. The riemannian metric of general relativity is replaced by the LCR-structure as the fundamental quantity in PCFT. Around the notion of the LCR-structure we have to build up again the appropriate computational techniques to derive the observed phenomena. Up to now, I used the conventional solitonic techniques, which seem to be quite limited. Only the static (electron) and stationary (neutrino) solitons, and the corresponding quarks have been revealed. The three particle generations and the correspondence between leptons and quarks have been easily derived, but I do not actually see the way to compute "numbers". The computation of the masses and coupling constants of the effective standard model should be the ultimate goal of PCFT. They should emerge from the intimate relation between the electroweak gauge fields and the geodetic and shear free null tangent vectors of Einstein's gravity. The apparent difference between the conventional quantum chromodynamics with the present derived gluonic interaction could be used to provide precise experimental tests of PCFT in HL-LHC experiments.

The Bogoliubov axiomatic formulation of quantum field theory is essentially based on the rigged Hilbert spaces (the Gelfand triplet) of the representations of the Poincaré group. The Epstein-Glaser observation makes the Bogoliubov mathematical formulation intimately related with the Schwartz distributions. In the context of PCFT, the elementary particles are distributional solitons. They are essentially the wavefront singularities of the LCR-manifolds (the leptons) and the compatible gauge field (the quarks). Notice that the "free" particles emerge with their corresponding gravitational, electromagnetic and gluonic (for the quarks) dressings. In conventional QFT, it is clear that the generalized functions is the adequate framework to formulate quantum field theory. PCFT seems to invert the reasoning. That is, the particle-wave duality (quantum mechanics) may also be a consequence of the distributional solitonic nature of the structures of PCFT. I think that we should start investigating this provoking possibility too.

References

[1] M. S. Baouendi, P. Ebenfelt and L. Rothschild, "Real submanifolds in complex space and their mappings", Princeton University Press, Princeton, (1999).

[2] N. N. Bogoliubov, A.A. Logunov and I.T. Todorov, "Introduction to Axiomatic Quantum Field Theory", W.A. Benjamin Publishing Company, Inc. USA (1975).

[3] N. N. Bogoliubov and D. V. Shirkov, "Introduction to the Theory of Quantized Fields", John Wiley and sons, Inc. USA, (1980).
[4] E. Cartan, Ann. Math. Pure Appl. (4) 11 (1932), 17.

[5] B. Carter, Phys. Rev. 174 (1968), 1559.

[6] S. Chandrasekhar, “The Mathematical Theory of Black Holes”, Clarendon, Oxford, (1983).

[7] Felsager B., “Geometry, Particles and Fields”, Odense Univ. Press, (1981).

[8] E. J. Jr Flaherty, Phys. Lett. A46, (1974) 391.

[9] E. J. Jr Flaherty, “Hermitian and Kählerian geometry in Relativity”, Lecture Notes in Physics 46, Springer, Berlin, (1976).

[10] I. M. Gel’fand and G. E. Shilov, “Generalized Functions”, vol. 1, Academic Press Inc., New York, (1964).

[11] I. M. Gel’fand and N. Ya. Vilenkin, “Generalized Functions”, vol. 4, Academic Press Inc., New York, (1964).

[12] U. Graf, “Introduction to Hyperfunctions and Their Integral Transforms”, The Birkhauser/Springer Basel AG, (2010).

[13] P. Griffiths and J. Harris, ”Principles of Algebraic Geometry”, John Willey and sons, Inc. New York, (1978).

[14] P. Griffiths and J. Harris, ”Algebraic Geometry and Local Differential Geometry”, Ann. scient. Ec. Norm. Sup. vol. 12 (1979), 355.

[15] M. Morimoto, ”An Introduction to Sato’s Hyperfunctions”, Translation of AMS, (1993).

[16] C. N. Misner, K. S. Thorn and J. A. Wheeler, ”GRAVITATION”, W. H. Freeman and Co, (1973).

[17] E. T. Newman, J. Math. Phys. 14, (1973), 102.

[18] E. T. Newman, ”Assymptotically flat space-time and its hidden recesses: An enigma from GR”, arXiv:[gr-qc]/1602.07218v1.

[19] Penrose R. and Rindler W., “Spinors and space-time”, vol. I and II, Cambridge Univ. Press, Cambridge, (1984).

[20] J. Polchinski, ”STRING THEORY”, vol. I, Cambridge Univ. Press, Cambridge, (2005).

[21] I. I. Pyatetskii-Shapiro, “Automorphic functions and the geometry of classical domains”, Gordon and Breach, New York, (1969).

[22] C. N. Ragiadakos, ”A Four Dimensional Extended Conformal Model”, Phys. Lett. B251, (1990), 94.
[23] C. N. Ragiadakos, "Solitons in a Four Dimensional Generally Covariant Conformal Model" Phys. Lett. B269, (1991), 325.

[24] C. N. Ragiadakos, "Quantization of a Four Dimensional Generally Covariant Conformal Model", J. Math. Phys. 33, (1992), 122.

[25] C. N. Ragiadakos, "Geometrodynamic solitons", Int. J. Math. Phys. A14, (1999), 2607.

[26] C. N. Ragiadakos (2008), “Renormalizability of a modified generally covariant Yang-Mills action”, arXiv:hep-th/0802.3966v2.

[27] C. N. Ragiadakos (2008), “A modified Y-M action with three families of fermionic solitons and perturbative confinement”, arXiv:hep-th/0804.3183v1.

[28] C. N. Ragiadakos (2013), "Lorentzian CR structures", arXiv:hep-th/1310.7252.

[29] C. N. Ragiadakos (2017), "Pseudo-conformal Field Theory", arXiv:hep-th/1704.00321.

[30] C. N. Ragiadakos (2018), "Standard Model Derivation from a 4-d Pseudo-conformal Field Theory", arXiv:hep-th/1805.11966.

[31] C. N. Ragiadakos (2018), "Hadronic Sector in 4-d Pseudo-conformal Field Theory", arXiv:hep-th/1811.04428.

[32] C. N. Ragiadakos, Research eBook of "Pseudo-Conformal Field Theory" in my personal page www.pcft.gr.

[33] G. Scharf, “Finite Quantum Electrodynamics: The causal approach”, Springer-Verlag, Berlin, (1995).

[34] G. Scharf, “Quantum Gauge Theorie: A true ghost story”, John Wiley & Sons, Inc. USA, (2001).

[35] R. E. Strichartz, “A Guide to Distribution Theory and Fourier Transforms”, CRC Press Inc., Florida, (1994).