PURITY AND FLATNESS IN SYMMETRIC MONOIDAL CLOSED EXACT CATEGORIES

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Abstract. Let \((\mathcal{A}, \otimes)\) be a symmetric monoidal closed exact category. This category is a natural framework to define the notions of \(\otimes\)-purity and \(\otimes\)-flatness. We show that an object \(F\) in \(\mathcal{A}\) is \(\otimes\)-flat if and only if any conflation ending in \(F\) is \(\otimes\)-pure. Furthermore, we prove a generalization of the Lambek Theorem ([La64]) in \(\mathcal{A}\). In the case \(\mathcal{A}\) is a quasi-abelian category, we prove that \(\mathcal{A}\) has enough pure injective objects.

1. Introduction

Let \(\mathcal{A}\) be an additive category. A sequence \(X \xrightarrow{f} Y \xrightarrow{g} Z\) in \(\mathcal{A}\) is said to be a conflation if \(f\) is the kernel of \(g\) and \(g\) is the cokernel of \(f\). The map \(f\) is called an inflation and \(g\) is called a deflation ([Ke90, Appendix A]). For a given class \(\mathcal{E}\) of conflations in \(\mathcal{A}\), \((\mathcal{A}, \mathcal{E})\) is said to be an exact category if the following axioms hold.

(i) For any object \(A \in \mathcal{A}\), the identity morphism \(1_A\) is an inflation.
(ii) For any object \(A \in \mathcal{A}\), the identity morphism \(1_A\) is a deflation.
(iii) Deflations (resp. Inflations) are closed under composition.
(iv) The pullback (resp. pushout) of a deflation (resp. inflation) along an arbitrary morphism exists and yields a deflation (resp. inflation).

See [Ke90, Appendix A] for more details on exact categories. Assume that \((\mathcal{A}, \otimes)\) is a symmetric monoidal closed exact category. Then there is a bifunctor \(\text{Hom}_\mathcal{A}(-, -) : \mathcal{A}^{\text{op}} \times \mathcal{A} \to \mathcal{A}\) such that for any object \(G\) in \(\mathcal{A}\) the functor \(- \otimes G : \mathcal{A} \to \mathcal{A}\) has a right adjoint \(\text{Hom}_\mathcal{A}(G, -) : \mathcal{A} \to \mathcal{A}\), i.e. for any pair of objects \(F\) and \(K\) in \(\mathcal{A}\), we have an isomorphism

\[
\text{Hom}_\mathcal{A}(F \otimes G, K) \cong \text{Hom}_\mathcal{A}(F, \text{Hom}_\mathcal{A}(G, K))
\]

(1.1)

which is naturally in all three arguments (see [KM71] for more details). The bifunctor \(\text{Hom}_\mathcal{A}(-, -)\) is called the internal hom on \(A\).

In some situations, the category \(\mathcal{A}\) does not have enough projective objects ([Har97, Ex III.6.2]). This causes that, some of the most important theorems in homological algebra do not hold in general case. The Lazard-Govorov Theorem is one of them. It asserts that any flat module over a ring is a direct limit of finitely generated free modules ([La69] and [Go65]). This theorem has a significance role in the proofs of some important theorems in homological algebra, especially [Sten68, Theorem 3] and [EG98, Theorem 2.4]. In this work we prove a generalization of the main result of [La64], [Sten68, Theorem 3] and [EG98, Theorem 2.4] in \(\mathcal{A}\) regardless of whether the Lazard-Govorov Theorem holds in \(\mathcal{A}\) or not (see also [Em16]). To this end, we need to prove

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that for any object $F$ in $A$, $F^+ = \text{Hom}_A(F, \mathcal{J})$ is a pure injective object where $\mathcal{J}$ is an injective cogenerator for $A$. This approach is very beneficial. Because we can generalize it to the category of complexes in $A$ and deduce the same results for complexes.

**Setup:** In this work, all categories are symmetric monoidal closed unless otherwise specified.

### 2. Purity and flatness in $A$

This section is devoted to the relation between flatness and purity in $A$. A conflation $L : G' \longrightarrow G \longrightarrow G''$ in $A$ is said to be **pure** if for any object $G$ in $A$, $L \otimes G$ is also a conflation. An object $J$ in $A$ is called **injective** if for any conflation $L$ in $A$, $\text{Hom}_A(L, J)$ is a conflation in the ordinary abelian exact structure of abelian groups. Moreover $J$ is an injective cogenerator if for any sequence $L : G' \longrightarrow G \longrightarrow G''$ in $A$ where $\text{Hom}_A(L, J)$ is a conflation of abelian groups then $L$ is a conflation in $A$. Let $\mathcal{J}$ be an injective cogenerator for $A$ and $(-)^+ := \text{Hom}_A(-, \mathcal{J}) : A \longrightarrow A$. We use the contravariant functor $(-)^+$ and prove the following important interpretation of purity in $A$.

**Proposition 2.1.** A conflation $L : G' \longrightarrow G \longrightarrow G''$ in $A$ is pure if and only if $L^+$ splits.

**Proof.** If $L$ is pure then, $(G')^+ \otimes L$ is a conflation and so, $\text{Hom}_A((G')^+ \otimes L, \mathcal{J})$ is a conflation of abelian groups. By (1.1), we have an isomorphism

\[ \text{Hom}_A((G')^+ \otimes L, \mathcal{J}) \cong \text{Hom}_A((G')^+, \text{Hom}_A(L, \mathcal{J})) \]

of conflations of abelian groups. This implies that $L^+$ splits. Conversely, assume that $L^+$ splits. Then for any object $F$ in $A$, we have a conflation $\text{Hom}_A(F, \text{Hom}_A(L, \mathcal{J}))$ of abelian groups. By (1.1), there is an isomorphism

\[ \text{Hom}_A(F, \text{Hom}_A(L, \mathcal{J})) \cong \text{Hom}_A(F \otimes L, \mathcal{J}) \]

of conflations of abelian groups. Since $\mathcal{J}$ is an injective cogenerator then, $F \otimes L$ is a conflation and so we are done. \qed

In the next lemma we show the existence of pure injective objects in $A$. An object $\mathcal{E}$ in $A$ is called **pure injective** if it is injective with respect to pure confluations.

**Lemma 2.2.** For any object $\mathcal{X}$ in $A$, $\mathcal{X}^+$ is pure injective.

**Proof.** Let $\mathcal{L}$ be a pure conflation in $A$. By (1.1), we have the isomorphism,

\[ \text{Hom}_A(\mathcal{L}, \mathcal{X}^+) \cong \text{Hom}_A(\mathcal{L} \otimes \mathcal{X}, \mathcal{J}) \]

of conflations of abelian groups. It follows that, $\mathcal{X}^+$ is pure injective. \qed

An object $F$ in $A$ is called **flat** if $- \otimes F : A \longrightarrow A$ preserves confluations. By the proof of Lemma 2.2, for a given object $\mathcal{X}$ in $A$, $\mathcal{X}^+$ is pure injective. Now, the conditions for proving the main theorem of the article are available.

**Theorem 2.3.** The following conditions are equivalent.

(i) $F$ is a flat object.
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(ii) $F^+$ is injective.
(iii) Any conflation ending in $F$ is pure.

Proof. (i) ⇒ (ii) Let $F$ be a flat object and $L$ be a conflation in $A$. Then, $L \otimes F$ is also a conflation. Apply $\text{Hom}_A(-, J)$ and use the adjoint property of $- \otimes F$ and $\text{Hom}_A(F, -)$ to deduce the isomorphism

$$\text{Hom}_A(L \otimes F, J) \cong \text{Hom}_A(L, F^+)$$

of conflations of abelian groups. This shows the injectivity of $F^+$.

(ii) ⇒ (i) Let $F^+$ be an injective object. For a given conflation $L$ in $A$, $\text{Hom}_A(L, F^+)$ is a conflation of abelian groups and so, by the adjoint property of $- \otimes F$ and $\text{Hom}_A(F, -)$, $\text{Hom}_A(L \otimes F, J)$ is a conflation of abelian groups. Since $J$ is an injective cogenerator, then $L \otimes F$ is a conflation. This shows the flatness of $F$.

(i) ⇒ (iii). By (i) ⇔ (ii), $F^+$ is an injective object. So, for a given conflation $L : K \rightarrow G \rightarrow F$, $L$ splits and hence by Proposition 2.1, $L$ is pure.

(iii) ⇒ (i). By (i) ⇔ (ii), it is enough to show that $F^+$ is injective. Let

$$F^+ \xrightarrow{f} G \rightarrow K$$

be a conflation in $A$. By the axioms of a closed symmetric category, we have a morphism $d_F : F \rightarrow \text{Hom}_A(F^+, F \otimes F^+)$ and so by (1.1), there is a morphism $\lambda_F : F \rightarrow F^+$ in $A$ (see [KMT1, pp 97-99]). This implies that the composition $F^+ \rightarrow F^{+++} \rightarrow F^+$ is the identity $1_{F^+}$. By the axiom of an exact category, the top row of the following pullback diagram

$$\begin{array}{ccc}
K^+ & \longrightarrow & Q \\
\downarrow & & \downarrow t \\
K^+ & \longrightarrow & G^+ \xrightarrow{f^+} F^{+++}
\end{array}$$

is a pure conflation and hence, it is split ($K^+$ is pure injective). Then, we have a morphism $t' : F \rightarrow Q$ such that $tt' = 1_F$. If $g_1 = gt' : F \rightarrow G^+$, then $f^+g_1 = f^+gt' = itt' = i$. Consequently, in the following commutative diagram

$$\begin{array}{ccc}
F^+ & \xrightarrow{f} & G \\
\downarrow & & \downarrow k \\
F^{+++} & \rightarrow & G^{++} \rightarrow K^{++}
\end{array}$$

$g_1^+kf = g_1^+f^{++}j = i^+j = 1_{F^+}$. It follows that $F^+$ is an injective object.

□

2.1. Purity and flatness in the category of complexes in $A$. Recall that a complex in $A$ is a cochain

$$X : \cdots \rightarrow X^{n-1} \xrightarrow{\partial_X^{n-1}} X^n \xrightarrow{\partial_X^n} X^{n+1} \rightarrow \cdots$$

in $A$ such that for any $n \in \mathbb{Z}$, $\partial_X^n \partial_X^{n-1} = 0$. The category of all complexes in $A$ is denoted by $\mathcal{C}(A)$. A complex $X$ in $A$ is called acyclic if for any $n \in \mathbb{Z}$,

$$\text{Ker}\partial_X^n \rightarrow X^n \rightarrow \text{Im}\partial_X^n$$
is a conflation in $\mathcal{A}$ and it is called *pure acyclic* if for any object $G$ in $\mathcal{A}$, $X \otimes G$ is acyclic. An acyclic complex

\[
\mathbf{F} : \ldots \to F^{n-1} \to F^n \to F^{n+1} \to \ldots
\]

in $\mathcal{A}$ is said to be *flat* if for any $n \in \mathbb{Z}$, $\text{Ker} \partial^n_F$ is a flat object in $\mathcal{A}$. The exact structure on $\mathcal{A}$ induces an exact structure on $\mathcal{A}$ as follows. A sequence $X \xrightarrow{f} Y \xrightarrow{g} K$ in $\mathcal{C}(\mathcal{A})$ is a conflation if for any $n \in \mathbb{Z}$,

\[
X^n \xrightarrow{f^n} Y^n \xrightarrow{g^n} K^n
\]

is a conflation in $\mathcal{A}$. Proposition 2.1 will enable us to define a notion of purity in $\mathcal{C}(\mathcal{A})$ and prove a generalization of [EG98, Theorem 2.4] in $\mathcal{C}(\mathcal{A})$.

**Definition 2.1.1.** A conflation $L$ in $\mathcal{C}(\mathcal{A})$ is called pure if $L^+$ splits.

Notice that if $\mathcal{A}$ is locally finitely presented Grothendieck category with enough projective objects, then the following result and [EG98, Theorem 2.4] are equivalent (see also [Sten68]).

**Theorem 2.1.2.** The following conditions are equivalent.

(i) $\mathbf{F}$ is a flat complex in $\mathcal{A}$.

(ii) $\mathbf{F}^+$ is an injective complex in $\mathcal{A}$.

(iii) $\mathbf{F}$ is a pure acyclic complex of flat objects in $\mathcal{A}$.

(iv) Any conflation ending in $\mathbf{F}$ is pure.

**Proof.** The proof is straightforward. \hfill \Box

## 2.2. Pure injective objects

Pure injective objects are one of the most important generalizations of injective objects which has a significance role in homological algebra. For instance, they are essential tools in the Swan’s approach on Cup products, derived functors and Hochschild cohomology ([Sw99]).

Our motivation on this subsection is a question asked by Rosicky in [Ro09, Question 1] (see also [Sw99, Theorem 2.1]). This question concerning about the existence of enough $\lambda$-pure injective objects in a locally $\lambda$-presentable additive category ($\lambda$ is an infinite regular cardinal). It is known that there is another notion of purity which is different from the $\lambda$-purity. This purity is known as $\otimes$-purity and defined in monoidal categories. We are interested to ask [Ro09, Question 1] for this purity and find an answer for it. In this subsection, we show that any symmetric monoidal closed quasi-abelian category has enough $\otimes$-pure injective objects.

Assume that $\mathcal{A}$ is a pre-abelian category, that is, an additive category with kernels and cokernels (see [RW77] and [SW11] for more details). We know that $\mathcal{A}$ has a natural structure of an exact category where conflations are short exact sequences. A subobject $\mathcal{F}$ of an object $G$ in $\mathcal{A}$ is called *pure* if the canonical exact sequence

\[
0 \to \mathcal{F} \to G \to G/\mathcal{F} \to 0
\]

is pure in $\mathcal{A}$.

**Theorem 2.2.1.** The category $\mathcal{A}$ has enough pure injective objects.
Proof. By the axioms of a symmetric monoidal closed category, there is a morphism 
\( d_F : F \to \text{Hom}_A(F^+, F \otimes F^+) \) and so by (1.1), we have a morphism 
\( \lambda_F : F \to F^{++} \) in \( A \) (see [KM71, pp. 97-99]). We show that \( \lambda_F : F \to F^{++} \) is a pure monomorphism. Let \( K = \text{Ker} \lambda_F \). Then we have the following commutative diagram

\[
\begin{array}{ccc}
0 & \to & K \\
\downarrow \lambda_K & & \downarrow \lambda_F \\
0 & \to & F^{++}
\end{array}
\]

with exact rows. Since \( i^{++} \) is a monomorphism then \( \lambda_K = 0 \). This implies that \( K = 0 \) and so \( \lambda_F \) is a monomorphism. By Proposition 2.1, it is enough to show that the epimorphism \( (\lambda_F)^+ : (F^{++})^+ \to F^+ \to 0 \) admits a section. Since \( (F^{++})^+ \cong (F^+)^{++} \) then we have the following commutative diagram

\[
\begin{array}{ccc}
F^{++} & \xrightarrow{\lambda_F^+} & F^{+++} \\
\downarrow \text{id}_{F^+} & & \downarrow (\lambda_F)^+ \\
F^+ & \xrightarrow{(\lambda_F)^+} & F^+
\end{array}
\]

in \( A \) (see [KM71, pp. 100, diagram (1.3)]). So, by Proposition 2.1, \( F \to F^{++} \) is a pure monomorphism where \( F^{++} \) is pure injective by Lemma 2.2. □

This theorem gives another proof for [Sw99, Theorem 2.1] and [EEO16, Corollary 4.6, 4.8] and enable us to prove to prove the existence of pure injective preenvelope in \( A \).

Example 2.4. The category can be replaced by any symmetric monoidal closed Grothen-ddieck category. For example, the category of modules over an associative ring, the category of sheaves over an arbitrary topological space and the category of quasi–coherent sheaves over an arbitrary scheme (see [Har97] for the algebraic geometry background).

For more examples on non-abelian categories see the [Me18, Me12].

Theorem 2.2.2. The category \( C(A) \) has enough pure injective objects.

Proof. The proof is similar to the proof of Theorem 2.2.1. □

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