Ladder operators and a second–order difference equation for general discrete Sobolev orthogonal polynomials

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Abstract

We consider a general discrete Sobolev inner product involving the Hahn difference operator, so this includes the well–known difference operators $\mathcal{D}_q$ and $\Delta$ and, as a limit case, the derivative operator. The objective is twofold. On the one hand, we construct the ladder operators for the corresponding nonstandard orthogonal polynomials and we obtain the second–order difference equation satisfied by these polynomials. On the other hand, we generalise some related results appeared in the literature as we are working in a more general framework. Moreover, we will show that all the functions involved in these constructions can be computed explicitly.

Keywords: Sobolev orthogonal polynomials · Ladders operators · Difference equations

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1 Introduction

The literature about discrete Sobolev orthogonal polynomials is very extensive, dating back to the early 90s. A general and historic vision is given in the survey \cite{32} where they rename these nonstandard orthogonal polynomials as Sobolev orthogonal polynomials of the second type, although the most frequently used names are Sobolev–type orthogonal polynomials and discrete Sobolev orthogonal polynomials. In some of those papers the discrete Sobolev inner product

$$(f, g)_S = \int f(x)g(x)\varrho(x)dx + Mf^{(j)}(c)g^{(j)}(c), \quad M > 0, \quad j \in \mathbb{N},$$

was considered, where $\varrho(x)$ is a continuous weight function on the real line and $c$ is located on the real axis. The authors of those papers studied extensively algebraic, differential and asymptotic properties of the corresponding Sobolev orthogonal polynomials (SOP),
including generalizations of the inner product or even the varying case where the constant
\( M \) is changed by a general sequence depending on \( n \) (see [30] for the varying case). Notice
that if we permit \( j = 0 \), then the inner product will be standard and therefore we have all the
advantages of the standard polynomials such as the three–term recurrence relation,
Christoffel–Darboux formula, etc. Thus, to use the word Sobolev properly we should take
\( j \) as a positive integer, i.e. \( j \in \mathbb{N} \).

On the other hand, there has been a growing interest in considering nonstandard inner
products with noncontinuous weights, i.e. weights supported on some lattices. According
to the type of weight, the differential operator is substituted either by the difference
operator \( \Delta \) or by the \( q \)–difference operator \( \mathcal{D}_q \). This leads to \( \Delta \)–SOP or \( q \)–SOP. In many
cases the approach to study these SOP was made by considering particular weights or
only one of these two difference operators (see, among others, [2, 6, 7, 8, 9, 10, 11, 18, 25,
27, 28, 33, 35]) although there are some papers where the three cases have been treated
in a unified way via the Hahn difference operator, for example, in [1, 3, 19].

In addition, ladder operators for orthogonal polynomials have been widely studied in
the literature, see among others [12]–[16], [26], [37]. One motivation for this, but not
the only one, is that they constitute a nice and useful tool to construct differential (or
difference) equations whose solutions are orthogonal polynomials.

Thus, the main goal of this paper is the study of the ladder operators for a wide
class of discrete Sobolev orthogonal polynomials using the Hahn difference operator as
a powerful tool and, as a consequence, providing the second–order differential/difference
equation satisfied by these nonstandard polynomials. Thus, we generalize some results of
other contributions to this topic where the authors considered particular cases such as in
(3, 20, 21, 22, 25, 34).

Getting to the point, we consider the discrete Sobolev inner product

\[
(f, g)_S = \int f(x)g(x)\varrho(x)dx + M\mathcal{D}_q^{(j)}f(c)\mathcal{D}_q^{(j)}g(c),
\]

where \( \varrho(x) \) is a weight function on the real line, \( c \) is located on the real axis, \( M > 0 \), \( j \) is
a non–negative integer and \( \mathcal{D}_q \) is the operator introduced by Hahn in [24, f. (1.3)] (see
also [29, f. (2.1.1)]) defined by

\[
\mathcal{D}_q f(x) = \begin{cases} 
\frac{f(qx + \omega) - f(x)}{(q-1)x + \omega}, & \text{if } x \neq \omega_0; \\
\frac{f'(\omega_0)}{q}, & \text{if } x = \omega_0,
\end{cases}
\]

where \( 0 < q < 1, \omega \geq 0, \omega_0 = \frac{\omega}{1-q} \), and further, following the notation given in [4], we
define

\[
\mathcal{D}_q^{(0)} f := f \quad \text{and} \quad \mathcal{D}_q^{(j)} f := \mathcal{D}_q \mathcal{D}_q^{(j-1)} f, \quad j \in \mathbb{N}.
\]

Thus, \( \mathcal{D}_q^{(j)} f(c) \) in (1) means

\[
\mathcal{D}_q^{(j)} f(c) = \left. \mathcal{D}_q^{(j)} f(x) \right|_{x=c}.
\]

To simplify the notation we will write \( \mathcal{D}_q^{(j)} f(c) \) instead of \( \mathcal{D}_q^{(j)} f(x) \left|_{x=c} \right. \) when no confusion
arises. Notice that when \( \varrho(x) \) is a discrete weight then the integral in (1) must be con-
sidered adequately, for example, according to the case, it could be the Jackson–Nörlund
integral (see [4]) or the Jackson integral.
It is well known that this class of operators given by (2) includes the $q$-difference operator $\mathcal{D}_q$ by Jackson when $\omega = 0$, the forward difference operator $\Delta$ when $q = 1$ and $\omega = 1$, and the derivative operator $\frac{d}{dx}$ as a limit case when $\omega = 0$ and $q \to 1$.

Notice that when $q = 1$ we have the difference operator $\mathcal{D}_\omega$ acting on general lattices, being $\Delta$ a particular case. However, we will only give the explicit details of our results in the Appendices A, B, C for the three operators: $\mathcal{D}_q$, $\Delta$, and $\frac{d}{dx}$. We will do this, and will not include an appendix for the operator $\mathcal{D}_\omega$, because the mentioned three operators are the most used ones in the literature in the framework of the Sobolev orthogonality (see the survey [32]), but the results hold for any $\omega \geq 0$.

We denote by $\{Q_n\}_{n \geq 0}$ the sequence of monic orthogonal polynomials with respect to the inner product (1). We also denote by $\{P_n\}_{n \geq 0}$ the sequence of monic polynomials orthogonal with respect to the inner product

$$(f, g)_\varrho = \int f(x)g(x)\varrho(x)dx,$$

thus, we have

$$(P_n, P_k)_\varrho = \int P_n(x)P_k(x)\varrho(x)dx = h_n\delta_{n,k}, \quad n, k \in \mathbb{N} \cup \{0\},$$

where $\delta_{n,k}$ denotes the Kronecker delta and $h_n$ is the square of the norm of these polynomials.

Since the inner product (4) is standard, i.e., the property $(xf, g)_\varrho = (f, xg)_\varrho$ holds, then it is known that the sequence of monic orthogonal polynomials $\{P_n\}_{n \geq 0}$ satisfies a three-term recurrence relation of the following form:

$$xp_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x), \quad n \geq 0,$$

with initial conditions $P_{-1}(x) = 0$ and $P_0(x) = 1$. In addition, we have $\beta_n = h_n/h_{n-1}$ for $n \geq 1$. When the weight $\varrho$ is symmetric, it is well known that $\alpha_n = 0$ (see, for example, [17]).

As we have mentioned previously, our objective is to obtain ladder operators and a linear second-order difference equation, which we will call holonomic equation, for the sequence of the monic Sobolev orthogonal polynomials $\{Q_n\}_{n \geq 0}$.

We assume that the sequence of the monic orthogonal polynomials $\{P_n\}_{n \geq 0}$ with respect to (4) satisfies the following relation:

$$A(x)\mathcal{D}_{q,\omega} P_n(x) = B_n(x)P_n(x) + C_n(x)P_{n-1}(x), \quad n \geq 1,$$

where $A(x)$ is a polynomial and $B_n(x)$ and $C_n(x)$ are certain functions. Notice that the relation (5) is very general. It holds for general standard orthogonal polynomials with respect to inner products involving any of the three operators $\mathcal{D}_q$, $\Delta$, and $\frac{d}{dx}$ (see [26]).

Then, we can define the lowering operator from (6) as

$$\Psi_n := A(x)\mathcal{D}_{q,\omega} - B_n(x), \quad n \geq 1,$$

hence

$$\Psi_n P_n(x) = C_n(x)P_{n-1}(x), \quad n \geq 1.$$
The raising operator is defined as follows:

\[ \hat{\Psi}_n := B_{n-1}(x) + \frac{C_{n-1}(x)(x - \alpha_{n-1})}{\beta_{n-1}} - A(x)\mathcal{D}_q, \quad n \geq 2, \]

hence, using (5), we deduce

\[ \hat{\Psi}_n P_{n-1}(x) = \frac{C_{n-1}(x)}{\beta_{n-1}} P_n(x), \quad n \geq 2. \]

Once we get the suitable properties of these ladder operators, we can deduce the holonomic equation for the polynomials \( Q_n(x) \).

This paper is structured as follows. In Section 2 we introduce some basic notation and provide a connection formula for the SOP, \( Q_n \), in terms of the standard ones, \( P_n \). Section 3 is devoted to finding several relationships between these two families of orthogonal polynomials. The use of the Hahn difference operator and its properties will be essential to deduce these relationships and will be the key to obtaining the main results of this paper. Thus, in Section 4 we obtain the expressions for the ladder operators and the second–order difference equation satisfied by the SOP, which constitute the goals of this work as we have mentioned previously. Finally, we provide one appendix for each one of the three operators: \( \mathcal{D}_q, \Delta, \) and \( \frac{d}{dx} \). In these appendices we give explicitly the expressions of the functions involved in the construction of the ladder operators as well as in the coefficients of the holonomic equation.

## 2 Connection formula

In this section we provide a connection formula for the orthogonal polynomials with respect to \( [1] \), \( \{Q_n\}_{n \geq 0} \), in terms of the orthogonal polynomials with respect to \( [2] \), \( \{P_n\}_{n \geq 0} \). In fact, the construction of this formula follows from a well–known standard technique but now using the Hahn operator.

We introduce some notation that will be used along the work. We define the kernel polynomials in the usual way

\[ K_n(x,y) := \sum_{i=0}^{n} \frac{P_i(x)P_i(y)}{h_i}. \quad (7) \]

It is well established that the kernel polynomials \( [17] \) satisfy the Christoffel–Darboux formula (see, for example, \( [17] \) or \( [36] \))

\[ K_n(x,y) = \frac{1}{h_n} \frac{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)}{x - y}. \quad (8) \]

As usual, we denote

\[ K_n^{(k,\ell)}(x,y) := \sum_{i=0}^{n} \frac{\mathcal{D}_q^{(k)}P_i(x) \mathcal{D}_q^{(\ell)}P_i(y)}{h_i}, \quad k, \ell \in \mathbb{N} \cup \{0\}. \quad (9) \]
In order to obtain the connection formula, we follow a similar technique to [30, Lemma 2] or [31, Sect. 2]. Since the sequence of orthogonal polynomials \( \{P_n\}_{n \geq 0} \) forms a basis of the linear space \( \mathbb{P}_n[x] \) of polynomials with real coefficients of degree at most \( n \), we can write
\[
Q_n(x) = \sum_{k=0}^{n} a_{n,k} P_k(x). \tag{10}
\]
The coefficient \( a_{n,n} = 1 \) because \( Q_n(x) \) and \( P_n(x) \) are monic polynomials. For \( 0 \leq k \leq n - 1 \), using the orthogonality of \( Q_n(x) \) and \( P_n(x) \), we get in a straightforward way
\[
a_{n,k} = -\frac{M \mathcal{D}^{(j)}_{q,\omega} Q_n(c) \mathcal{D}^{(j)}_{q,\omega} P_k(c)}{h_k}. \tag{11}
\]
So, applying (7), (10) and (11), we can write
\[
Q_n(x) = P_n(x) - M \mathcal{D}^{(j)}_{q,\omega} Q_n(c) K^{(0,j)}_{n-1}(x,c). \tag{12}
\]
To determine the value of \( \mathcal{D}^{(j)}_{q,\omega} Q_n(c) \), we apply appropriately the Hahn operator \( \mathcal{D}^{(j)}_{q,\omega} \) in (12) getting
\[
\mathcal{D}^{(j)}_{q,\omega} Q_n(c) = \mathcal{D}^{(j)}_{q,\omega} P_n(c) - M \mathcal{D}^{(j)}_{q,\omega} Q_n(c) K^{(j,j)}_{n-1}(c,c),
\]
so, we have
\[
\mathcal{D}^{(j)}_{q,\omega} Q_n(c) = \frac{\mathcal{D}^{(j)}_{q,\omega} P_n(c)}{1 + M K^{(j,j)}_{n-1}(c,c)}.
\]
Therefore, the connection formula between both families of orthogonal polynomials is given by
\[
Q_n(x) = P_n(x) - \frac{M \mathcal{D}^{(j)}_{q,\omega} P_n(c)}{1 + M K^{(j,j)}_{n-1}(c,c)} K^{(0,j)}_{n-1}(x,c). \tag{13}
\]

3 Hahn difference operator and technical relations between the families of orthogonal polynomials \( Q_n(x) \) and \( P_n(x) \)

In this section we are going to obtain different relations between \( P_n(x) \) and \( Q_n(x) \) with the objective of finding the ladder operators for the polynomials \( Q_n(x) \). First, we need some properties of the Hahn difference operator collected in the next statement.

**Proposition 1.** The Hahn difference operator has the following properties:

1. **Linearity:**
\[
\mathcal{D}_{q,\omega}(f + g)(x) = \mathcal{D}_{q,\omega}f(x) + \mathcal{D}_{q,\omega}g(x),
\]
\[
\mathcal{D}_{q,\omega}(af)(x) = a \mathcal{D}_{q,\omega}f(x), \quad a \in \mathbb{R}.
\]

2. **Product rule:**
\[
\mathcal{D}_{q,\omega}(fg)(x) = g(x) \mathcal{D}_{q,\omega}f(x) + f(qx + \omega) \mathcal{D}_{q,\omega}g(x). \tag{14}
\]
3. Quotient rule:

\[ D_{q,\omega}(f/g)(x) = \frac{g(x)D_{q,\omega}f(x) - f(x)D_{q,\omega}g(x)}{g(x)g(qx + \omega)} \] (15)

4. Leibniz formula:

\[ D^{(n)}_{q,\omega}(fg)(x) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] q^{k(n-k)} D^{(k)}_{q,\omega}g(x) D^{(n-k)}_{q,\omega}f(q^kx + \omega[k]_q), \quad n \geq 0, \] (16)

where the \(q\)-binomial coefficient is defined by (see, [29, f. (1.9.4)])

\[ \left[ \begin{array}{c} n \\ k \end{array} \right]_q := \frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}}, \] (17)

being \((a; q)_k\) a \(q\)-analogue of the Pochhammer symbol \((a)_k\) defined by (see, [29, f. (1.8.3)])

\[(a; q)_0 := 1 \quad \text{and} \quad (a; q)_k := \prod_{i=1}^{k} (1 - aq^{i-1}), \quad k \in \mathbb{N}. \] (18)

and for \(q \neq 0\) and \(q \neq 1\) we define (see, [29, f. (1.8.1)]) the basic \(q\)-number

\[ [\alpha]_q := \frac{1 - q^\alpha}{1 - q}, \quad \alpha \in \mathbb{R}. \] (19)

5. Let \(\alpha, \gamma \in \mathbb{R}\) be, we take \(f(x) = (\beta - \gamma x)^{-1}\), then

\[ D_{q,\omega}^{(n)}f(x) = \frac{\gamma^n(q; q)_n}{(1 - q)^n \prod_{i=0}^{n} (\beta - \gamma(q^ix + \omega[i]_q))}. \] (20)

Proof. The linearity follows directly from the definition. The product and quotient rules can be found, for example, in [4, f. (16-17)].

To establish (16) we need,

\[ D^{(n)}_{q,\omega}(fg)(x) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] q^{k(n-k)} D^{(k)}_{q,\omega}g(x) D^{(n-k)}_{q,\omega}f(q^kx + \omega[k]_q), \]

given in [4, Th. 3.1]. Then, it is enough to apply [4, f. (26)] repeatedly for obtaining the
result,

\[ \mathcal{D}_{q,\omega}^{(n)}(fg)(x) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{D}_{q,\omega}^{(k)}(g)(\mathcal{D}_{q,\omega}^{(n-k)}f)(q^k x + \omega[k]) \]

\[ = \sum_{k=0}^{n} \binom{n}{k} \mathcal{D}_{q,\omega}^{(k)}(g)(x) q^{-k} \mathcal{D}_{q,\omega}^{(n-k-1)}f(q^k x + \omega[k]) \]

\[ = \sum_{k=0}^{n} \binom{n}{k} \mathcal{D}_{q,\omega}^{(k)}(g)(x) q^{-k} \mathcal{D}_{q,\omega}^{(n-k)}f(q^k x + \omega[k]) \]

\[ = \sum_{k=0}^{n} \binom{n}{k} \mathcal{D}_{q,\omega}^{(k)}(g)(x) q^{-2k} \mathcal{D}_{q,\omega}^{(n-k-2)}f(q^k x + \omega[k]) \]

\[ = \cdots \]

\[ = \sum_{k=0}^{n} \binom{n}{k} \mathcal{D}_{q,\omega}^{(k)}(g)(x) q^{-k(n-k)} \mathcal{D}_{q,\omega}^{(n-k)}f(q^k x + \omega[k]) \]

\[ = \sum_{k=0}^{n} \binom{n}{k} q^{k(n-k)} \mathcal{D}_{q,\omega}^{(k)}(g)(x) \mathcal{D}_{q,\omega}^{(n-k)}f(q^k x + \omega[k]). \]

Finally, in order to establish (20), we use mathematical induction on \( n \). For \( f(x) = (\beta - \gamma x)^{-1} \) and \( n = 1 \),

\[ \mathcal{D}_{q,\omega}f(x) = \frac{(\beta - \gamma(qx + \omega))^{-1} - (\beta - \gamma x)^{-1}}{(qx + \omega) - x} \]

\[ = \frac{1}{(\beta - \gamma(qx + \omega))(q^{-1}x + \omega) - (\beta - \gamma x)((q^{-1}x + \omega))} \]

\[ = \frac{\gamma}{(\beta - \gamma x)(\beta - \gamma(qx + \omega))}. \]

Since \( (q; q)_1 = 1 - q \), (20) holds for \( n = 1 \).

We assume that the identity (20) holds for some \( n \geq 2 \), and we are going to prove it for \( n + 1 \). To do this, the following relations will be useful:

\[ \prod_{k=1}^{n} [k]_q = \prod_{k=0}^{n-1} [1 + k]_q = (1 - q)^{-n(q; q)_n}, \quad (21) \]

which can be deduced from (23 f. (1.2.45)) and (23 f. (1.2.47)). Then, by (21), establishing (20) is equivalent to proving

\[ \mathcal{D}_{q,\omega}^{(n)}f(x) = \frac{\gamma^n \prod_{i=1}^{n} [i]_q}{\prod_{i=0}^{n} (\beta - \gamma(q^i x + \omega[i]_q))}. \]

We denote

\[ f_{n}(x) := \frac{\gamma^n \prod_{i=1}^{n} [i]_q}{\prod_{i=0}^{n} (\beta - \gamma(q^i x + \omega[i]_q))}, \]
then, using the induction hypothesis, we get

\[ D^{(n+1)} q,\omega f(x) = D_q,\omega D^{(n)}_{q,\omega} f(x) = D_q,\omega f_n(x) = \frac{f_n(qx + \omega) - f_n(x)}{(q - 1)x + \omega} \]

\[ = \gamma_n \prod_{i=1}^{n} [i]_q \left( (\prod_{i=0}^{n-1}(\beta - \gamma(q^i(qx + \omega) + \omega[i])q))^{-1} - \prod_{i=0}^{n}(\beta - \gamma(q^i x + \omega[i]))^{-1} \right) \frac{(q - 1)x + \omega}{(q - 1)x + \omega} \]

\[ = \gamma_n \prod_{i=1}^{n} [i]_q \left( \prod_{i=0}^{n} (\beta - \gamma(q^i x + \omega[i])) \left( (q - 1)x + \omega \right) \right) \]

\[ = \gamma_n \prod_{i=1}^{n} [i]_q \left( \prod_{i=0}^{n} (\beta - \gamma(q^i x + \omega[i])) \right) \]

and this completes the induction.

The following five lemmas are essential to obtain the results in Section 4.

Lemma 1. Let \( \{Q_n(x)\}_{n \geq 0} \) and \( \{P_n(x)\}_{n \geq 0} \) be the sequences of monic orthogonal polynomials with respect to \( (1) \) and \( (4) \), respectively. Then,

\[ r_c(x)Q_n(x) = f_{1,c,n}(x)P_n(x) + g_{1,c,n}(x)P_{n-1}(x), \quad n \geq 1, \quad (22) \]

where

\[ r_c(x) = \prod_{k=0}^{j} (x - q^k c - \omega[k]), \quad (23) \]

\[ f_{1,c,n}(x) = r_c(x) - \frac{M \rho_{n,j,c}}{h_{n-1}} \left( \sum_{i=0}^{j} \frac{(q; q)_j Q^{(i)}(q) P_{n-1}(c) \prod_{k=0}^{i-1}(x - q^k c - \omega[k])}{(q; q)_i(1 - q)^{j-i}} \right), \quad (24) \]

\[ g_{1,c,n}(x) = \frac{M \rho_{n,j,c}}{h_{n-1}} \left( \sum_{i=0}^{j} \frac{(q; q)_j Q^{(i)}(q) P_{n}(c) \prod_{k=0}^{i-1}(x - q^k c - \omega[k])}{(q; q)_i(1 - q)^{j-i}} \right), \quad (25) \]

with \( \rho_{n,j,c} := Q^{(j)}(c)P_n(c) \frac{Q^{(i)}(q) P_{n}(c)}{1 + MK^{(j,j)}_{n-1}(c,c)}. \)
Proof. Using (5), (9), and applying the relationships (16,20), we deduce

\[ K^{(0,j)}_{n-1}(x,y) = \frac{1}{h_{n-1}} \left( P_n(x) \mathcal{D}_{q,\omega}^{(j)} \frac{P_{n-1}(y)}{x-y} - P_{n-1}(x) \mathcal{D}_{q,\omega}^{(j)} P_n(y) \right) \]

\[ = \frac{1}{h_{n-1}} \left( P_n(x) \sum_{i=0}^{j} \binom{j}{i} q^{(i-j)} \mathcal{D}_{q,\omega}^{(i)} P_{n-1}(y) \mathcal{D}_{q,\omega}^{(j-i)} (x - (q^iy + \omega [i]_q))^{-1} \right. \]
\[ - \left. P_{n-1}(x) \sum_{i=0}^{j} \binom{j}{i} q^{(i-j)} \mathcal{D}_{q,\omega}^{(i)} P_n(y) \mathcal{D}_{q,\omega}^{(j-i)} (x - (q^iy + \omega [i]_q))^{-1} \right) \]
\[ = \frac{1}{h_{n-1}} \left( P_n(x) \sum_{i=0}^{j} \frac{(q;q)_j q^{(i-j)}}{(q;q)_i (q;q)_{j-i}} \mathcal{D}_{q,\omega}^{(i)} P_{n-1}(y) \frac{q^{(j-i)}(q;q)_{j-i}}{(1-q)^{j-i} \prod_{k=0}^{j-1} (x - \omega [k]_q - q^{k+i}y - q^i\omega [k]_q)} \right. \]
\[ - \left. P_{n-1}(x) \sum_{i=0}^{j} \frac{(q;q)_j q^{(i-j)}}{(q;q)_i (q;q)_{j-i}} \mathcal{D}_{q,\omega}^{(i)} P_n(y) \frac{q^{(j-i)}(q;q)_{j-i}}{(1-q)^{j-i} \prod_{k=0}^{j-1} (x - \omega [k]_q - q^{k+i}y - q^i\omega [k]_q)} \right) \]
\[ = \frac{1}{h_{n-1}} \left( P_n(x) \sum_{i=0}^{j} \frac{(q;q)_j q^{(i-j)}}{(q;q)_i (q;q)_{j-i}} \mathcal{D}_{q,\omega}^{(i)} P_{n-1}(y) \frac{1}{(1-q)^{j-i} \prod_{k=0}^{j-1} (x - q^{k+i}y - \omega [k]_q)} \right. \]
\[ - \left. P_{n-1}(x) \sum_{i=0}^{j} \frac{(q;q)_j q^{(i-j)}}{(q;q)_i (q;q)_{j-i}} \mathcal{D}_{q,\omega}^{(i)} P_n(y) \frac{1}{(1-q)^{j-i} \prod_{k=0}^{j-1} (x - q^{k+i}y - \omega [k]_q)} \right) \]

So, evaluating the variable y at the point c, we have

\[ K^{(0,j)}_{n-1}(x,c) = \frac{1}{h_{n-1}} \left( P_n(x) \sum_{i=0}^{j} \frac{(q;q)_j q^{(i-j)}}{(q;q)_i (1-q)^{j-i} \prod_{k=0}^{j-1} (x - q^{k+i}c - \omega [k]_q)} \mathcal{D}_{q,\omega}^{(i)} P_{n-1}(c) \right. \]
\[ - \left. P_{n-1}(x) \sum_{i=0}^{j} \frac{(q;q)_j q^{(i-j)}}{(q;q)_i (1-q)^{j-i} \prod_{k=0}^{j-1} (x - q^{k+i}c - \omega [k]_q)} \mathcal{D}_{q,\omega}^{(i)} P_n(c) \right) \]

Now, substituting this expression into relation (13) we get

\[ Q_n(x) = P_n(x) - M \rho_{n,j,c} K^{(0,j)}_{n-1}(x,c) \]
\[ = P_n(x) - \frac{M \rho_{n,j,c}}{h_{n-1}} \left( P_n(x) \left( \sum_{i=0}^{j} \frac{(q;q)_j q^{(i-j)}}{(q;q)_i (1-q)^{j-i} \prod_{k=0}^{j-1} (x - q^{k+i}c - \omega [k]_q)} \mathcal{D}_{q,\omega}^{(i)} P_{n-1}(c) \right) \right. \]
\[ - \left. P_{n-1}(x) \left( \sum_{i=0}^{j} \frac{(q;q)_j q^{(i-j)}}{(q;q)_i (1-q)^{j-i} \prod_{k=0}^{j-1} (x - q^{k+i}c - \omega [k]_q)} \mathcal{D}_{q,\omega}^{(i)} P_n(c) \right) \right) \]
\[ = P_n(x) \left( 1 - \frac{M \rho_{n,j,c}}{h_{n-1}} \left( \sum_{i=0}^{j} \frac{(q;q)_j q^{(i-j)}}{(q;q)_i (1-q)^{j-i} \prod_{k=0}^{j-1} (x - q^{k+i}c - \omega [k]_q)} \mathcal{D}_{q,\omega}^{(i)} P_{n-1}(c) \right) \right) \]
\[ + P_{n-1}(x) \frac{M \rho_{n,j,c}}{h_{n-1}} \left( \sum_{i=0}^{j} \frac{(q;q)_j q^{(i-j)}}{(q;q)_i (1-q)^{j-i} \prod_{k=0}^{j-1} (x - q^{k+i}c - \omega [k]_q)} \mathcal{D}_{q,\omega}^{(i)} P_n(c) \right) \]
Finally, multiplying by \( r_c(x) = \prod_{k=0}^{j} (x - q^k c - \omega[k]q) \), we obtain the desired result:

\[
Q_n(x) \prod_{k=0}^{j} (x - q^k c - \omega[k]q) = P_n(x) \left( \prod_{k=0}^{j} (x - q^k c - \omega[k]q) \right) - \frac{M \rho_{n,j,c}}{h_{n-1}} \left( \sum_{i=0}^{j} \frac{(q; q)_i}{(1 - q)^{j-i}} \Pi_{k=0}^{i-1} (x - q^{k+i}c - \omega[k+i]q) \right)
\]

\[
+ P_{n-1}(x) \frac{M \rho_{n,j,c}}{h_{n-1}} \left( \sum_{i=0}^{j} \frac{(q; q)_i}{(1 - q)^{j-i}} \Pi_{k=0}^{j-i} (x - q^{k+i}c - \omega[k+i]q) \right).
\]

\[
\square
\]

**Lemma 2.** Let \( \{Q_n(x)\}_{n \geq 0} \) and \( \{P_n(x)\}_{n \geq 0} \) be the sequences of monic orthogonal polynomials with respect to (1) and (4), respectively. Then,

\[
r_c(x) \mathcal{D}_{q,\omega} Q_n(x) = f_{2,c,n}(x) P_n(x) + g_{2,c,n}(x) P_{n-1}(x), \quad n \geq 2,
\]

where

\[
f_{2,c,n}(x) = \frac{x - q^{j+1}c - \omega[j]q}{q^{j+1}(x - q^{-1}c - \omega[-1]q)} \left( \mathcal{D}_{q,\omega} f_{1,c,n}(x) + f_{1,c,n}(qx + \omega) \frac{B_n(x)}{A(x)} \right),
\]

\[
g_{1,c,n}(x) = -g_{1,c,n}(qx + \omega) \frac{C_{n-1}(x)}{\beta_{n-1} A(x)} - \frac{[j + 1]q}{x - q^{j+1}c - \omega[j+1]q} f_{1,c,n}(x), \quad \beta_{n-1} A(x) - [j + 1]q \mathcal{D}_{q,\omega} g_{1,c,n}(x) + f_{1,c,n}(qx + \omega) \frac{C_n(x)}{A(x)}
\]

\[
g_{2,c,n}(x) = \frac{x - q^{j+1}c - \omega[j]q}{q^{j+1}(x - q^{-1}c - \omega[-1]q)} \left( \mathcal{D}_{q,\omega} g_{1,c,n}(x) + f_{1,c,n}(qx + \omega) \frac{C_n(x)}{A(x)} \right)
\]

\[
+ g_{1,c,n}(qx + \omega) \left( \frac{B_{n-1}(x)}{A(x)} + \frac{x - \alpha_{n-1} C_{n-1}(x)}{A(x) \beta_{n-1}} \right) - \frac{[j + 1]q}{(x - q^{j+1}c - \omega[j]q)} g_{1,c,n}(x), \quad (28)
\]

with \( r_c(x) \), \( f_{1,c,n}(x) \) and \( g_{1,c,n}(x) \) the functions defined by (23) and (24).

**Proof.** When we apply the operator \( \mathcal{D}_{q,\omega} \) to (22) in Lemma 1, we get

\[
Q_n(x) \mathcal{D}_{q,\omega} r_c(x) + r_c(qx + \omega) \mathcal{D}_{q,\omega} Q_n(x) = P_n(x) \mathcal{D}_{q,\omega} f_{1,c,n}(x) + f_{1,c,n}(qx + \omega) \mathcal{D}_{q,\omega} P_n(x)
\]

\[
+ P_{n-1}(x) \mathcal{D}_{q,\omega} g_{1,c,n}(x) + g_{1,c,n}(qx + \omega) \mathcal{D}_{q,\omega} P_{n-1}(x),
\]

(29)

We are going to compute, and express appropriately, some terms in the previous
In the second place, we express $r_c(qx + \omega)$, then

\[
\mathcal{D}_{q,\omega} r_c(x) = \frac{r_c(qx + \omega) - r_c(x)}{(q - 1)x + \omega}
\]

\[
= \prod_{k=0}^{j} (qx - q^k c - \omega[k]_q) - \prod_{k=0}^{j} (x - q^k c - \omega[k]_q)
\]

\[
= \prod_{k=0}^{j+1} (x - q^{k-1} c - \omega[k-1]_q) - \prod_{k=0}^{j} (x - q^k c - \omega[k]_q)
\]

\[
= [j + 1]_q \prod_{k=0}^{j} (x - q^k c - \omega[k]_q)
\]

\[
= \frac{[j + 1]_q}{(x - q^1 c - \omega[j]_q)} \prod_{k=0}^{j} (x - q^k c - \omega[k]_q)
\]

\[
= \frac{[j + 1]_q}{(x - q^1 c - \omega[j]_q)} r_c(x).
\] (30)

In the second place, we express $r_c(qx + \omega) = \prod_{k=0}^{j} (qx - q^k c - \omega[k]_q)$ in terms of $r_c(x)$

\[
\prod_{k=0}^{j} (qx + \omega - q^k c - \omega[k]_q) = \prod_{k=0}^{j} (qx - q^k c - \omega(-1 + [k]_q)) = \prod_{k=0}^{j} q(x - q^{k-1} c - \omega[k-1]_q)
\]

\[
= q^{j+1} \prod_{k=0}^{j} (x - q^{k-1} c - \omega[k-1]_q)
\]

\[
= q^{j+1} \prod_{k=0}^{j} (x - q^k c - \omega[k]_q) \frac{x - q^{-1} c - \omega[-1]_q}{x - q^1 c - \omega[j]_q}
\]

\[
= q^{j+1} \frac{x - q^{-1} c - \omega[-1]_q}{x - q^1 c - \omega[j]_q} r_c(x).
\] (31)

So, using (30) and (31) the relation (29) can be rewritten as

\[
\frac{[j + 1]_q}{(x - q^1 c - \omega[j]_q)} r_c(x) Q_n(x) + q^{j+1} \frac{x - q^{-1} c - \omega[-1]_q}{x - q^1 c - \omega[j]_q} r_c(x) \mathcal{D}_{q,\omega} Q_n(x)
\]

\[
= P_n(x) \mathcal{D}_{q,\omega} f_{1,c,n}(x) + f_{1,c,n}(qx + \omega) \mathcal{D}_{q,\omega} P_n(x)
\]

\[
+ P_{n-1}(x) \mathcal{D}_{q,\omega} g_{1,c,n}(x) + g_{1,c,n}(qx + \omega) \mathcal{D}_{q,\omega} P_{n-1}(x).
\]
Then,
\[
\begin{aligned}
r_c(x) \mathcal{D}_{q,\omega} Q_n(x) &= \frac{x - q^j c - \omega[j]q}{q^{j+1} (x - q^{-1} c - \omega[-1]q)} P_n(x) \mathcal{D}_{q,\omega} f_{1,c,n}(x) \\
&+ \frac{x - q^j c - \omega[j]q}{q^{j+1} (x - q^{-1} c - \omega[-1]q)} f_{1,c,n}(qx + \omega) \mathcal{D}_{q,\omega} P_n(x) \\
&+ \frac{x - q^j c - \omega[j]q}{q^{j+1} (x - q^{-1} c - \omega[-1]q)} P_{n-1}(x) \mathcal{D}_{q,\omega} g_{1,c,n}(x) \\
&+ \frac{x - q^j c - \omega[j]q}{q^{j+1} (x - q^{-1} c - \omega[-1]q)} g_{1,c,n}(qx + \omega) \mathcal{D}_{q,\omega} P_{n-1}(x) \\
&- \frac{x - q^j c - \omega[j]q}{q^{j+1} (x - q^{-1} c - \omega[-1]q)} \frac{[j + 1]q}{(x - q^j c - \omega[j]q)} (f_{1,c,n}(x) P_n(x) + g_{1,c,n}(x) P_{n-1}(x)).
\end{aligned}
\]

Now, we can observe that from (5) we have
\[
P_{n-2}(x) = \frac{(x - \alpha_{n-1}) P_{n-1}(x) - P_n(x)}{\beta_{n-1}}, \quad n \geq 1.
\] (32)
Therefore, using (32) and after some algebraic manipulations, we get the result
\[
\begin{aligned}
r_c(x) \mathcal{D}_{q,\omega} Q_n(x) &= \frac{x - q^j c - \omega[j]q}{q^{j+1} (x - q^{-1} c - \omega[-1]q)} P_n(x) \mathcal{D}_{q,\omega} f_{1,c,n}(x) \\
&+ \frac{x - q^j c - \omega[j]q}{q^{j+1} (x - q^{-1} c - \omega[-1]q)} f_{1,c,n}(qx + \omega) \left( \frac{B_n(x)}{A(x)} P_n(x) + \frac{C_n(x)}{A(x)} P_{n-1}(x) \right) \\
&+ \frac{x - q^j c - \omega[j]q}{q^{j+1} (x - q^{-1} c - \omega[-1]q)} P_{n-1}(x) \mathcal{D}_{q,\omega} g_{1,c,n}(x) \\
&+ \frac{x - q^j c - \omega[j]q}{q^{j+1} (x - q^{-1} c - \omega[-1]q)} g_{1,c,n}(qx + \omega) \left( \frac{B_{n-1}(x)}{A(x)} P_{n-1}(x) \\
&+ \frac{C_{n-1}(x) (x - \alpha_{n-1}) P_{n-1}(x) - P_n(x)}{\beta_{n-1}} \right) \\
&- \frac{x - q^j c - \omega[j]q}{q^{j+1} (x - q^{-1} c - \omega[-1]q)} \frac{[j + 1]q}{(x - q^j c - \omega[j]q)} (f_{1,c,n}(x) P_n(x) + g_{1,c,n}(x) P_{n-1}(x)).
\end{aligned}
\]

Lemma 3. Let \( \{Q_n(x)\}_{n \geq 0} \) and \( \{P_n(x)\}_{n \geq 0} \) be the sequences of monic orthogonal polynomials with respect to (1) and (3), respectively. Then,
\[
r_c(x) Q_{n-1}(x) = f_{3,c,n}(x) P_n(x) + g_{3,c,n}(x) P_{n-1}(x), \quad n \geq 2,
\]
where
\[
\begin{aligned}
f_{3,c,n}(x) &= -\frac{g_{1,c,n-1}(x)}{\beta_{n-1}}, \\
g_{3,c,n}(x) &= f_{1,c,n-1}(x) + \frac{(x - \alpha_{n-1}) g_{1,c,n-1}(x)}{\beta_{n-1}}.
\end{aligned}
\]

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Proof. It is enough to use Lemma 1 and the relation (32) to obtain

\[
\begin{align*}
    r_c(x)Q_{n-1}(x) &= f_{1,c,n-1}(x)P_{n-1}(x) + g_{1,c,n-1}(x)P_{n-2}(x) \\
    &= f_{1,c,n-1}(x)P_{n-1}(x) + g_{1,c,n-1}(x)\left(\frac{(x - \alpha_{n-1})P_{n-1}(x) - P_{n}(x)}{\beta_{n-1}}\right) \\
    &= -\frac{g_{1,c,n-1}(x)}{\beta_{n-1}}P_{n}(x) + \left(f_{1,c,n-1}(x) + \frac{(x - \alpha_{n-1})g_{1,c,n-1}(x)}{\beta_{n-1}}\right)P_{n-1}(x),
\end{align*}
\]

where \(f_{1,c,n}(x)\) and \(g_{1,c,n}(x)\) are defined in (24) and (25), respectively. \(\square\)

Lemma 4. Let \(\{Q_{n}(x)\}\) and \(\{P_{n}(x)\}\) be the sequences of monic orthogonal polynomials with respect to (4) and (4), respectively. Then,

\[
r_c(x)Q_{n-1}(x) = f_{4,c,n}(x)P_{n}(x) + g_{4,c,n}(x)P_{n-1}(x), \quad n \geq 3,
\]

where

\[
\begin{align*}
    f_{4,c,n}(x) &= -\frac{g_{2,c,n-1}(x)}{\beta_{n-1}}, \\
    g_{4,c,n}(x) &= f_{2,c,n-1}(x) + \frac{(x - \alpha_{n-1})g_{2,c,n-1}(x)}{\beta_{n-1}}.
\end{align*}
\]

Proof. The proof is identical to the one made in Lemma 3 but now considering (26), (28).

\(\square\)

Lemma 5. For \(n \geq 3\), we have

\[
\begin{align*}
    P_{n}(x) &= r_c(x)\frac{g_{3,c,n}(x)Q_{n}(x) - g_{1,c,n}(x)Q_{n-1}(x)}{f_{1,c,n}(x)g_{3,c,n}(x) - g_{1,c,n}(x)f_{3,c,n}(x)}, \\
    P_{n-1}(x) &= r_c(x)\frac{-f_{3,c,n}(x)Q_{n}(x) + f_{1,c,n}(x)Q_{n-1}(x)}{f_{1,c,n}(x)g_{3,c,n}(x) - g_{1,c,n}(x)f_{3,c,n}(x)},
\end{align*}
\]

where the functions \(r_c(x)\), \(f_{1,c,n}(x)\), \(g_{1,c,n}(x)\), \(f_{3,c,n}(x)\) and \(g_{3,c,n}(x)\) are defined in Lemma 1 and Lemma 3.

Proof. From Lemma 1 and Lemma 3 we can write the following

\[
\begin{align*}
    \left\{ \begin{array}{l}
    f_{1,c,n}(x)P_{n}(x) + g_{1,c,n}(x)P_{n-1}(x) = r_c(x)Q_{n}(x), \\
    f_{3,c,n}(x)P_{n}(x) + g_{3,c,n}(x)P_{n-1}(x) = r_c(x)Q_{n-1}(x).
    \end{array} \right.
\]

It is enough to apply the very well–known Cramer rule to get the result. \(\square\)
4 Ladder operators and a second–order difference equation

We are ready to obtain a second–order linear difference equation satisfied by the orthogonal polynomials $Q_n(x)$ with respect to the inner product (1). The first step is to obtain the ladder operators for this family of polynomials. To do this, the key will be the Lemmas obtained in the previous section.

Theorem 1. (Ladder Operators) Let $\{Q_n(x)\}_{n \geq 0}$ be the sequence of monic orthogonal polynomials with respect to (1). Then, there exits a lowering difference operator $\Phi_n$ and a raising difference operator $\hat{\Phi}_n$, defined as

$$
\Phi_n := \varphi_{c,n}^{3,2}(x) + \varphi_{c,n}^{1,3}(x)\mathcal{R}_{q,\omega}, \quad n \geq 3,
$$

$$
\hat{\Phi}_n := \varphi_{c,n}^{1,4}(x) - \varphi_{c,n}^{1,3}(x)\mathcal{R}_{q,\omega}, \quad n \geq 3,
$$

satisfying

$$
\Phi_n Q_n(x) = \varphi_{c,n}^{1,2}(x)Q_{n-1}(x), \quad n \geq 3
$$

$$
\hat{\Phi}_n Q_{n-1}(x) = \varphi_{c,n}^{3,4}(x)Q_n(x), \quad n \geq 3
$$

with

$$
\varphi_{c,n}^{i,j}(x) := \begin{vmatrix}
    f_{i,c,n}(x) & f_{j,c,n}(x) \\
    g_{i,c,n}(x) & g_{j,c,n}(x)
\end{vmatrix}, \quad i, j \in \{1, 2, 3, 4\},
$$

where the functions $f_{i,c,n}(x)$ and $g_{i,c,n}(x)$, $i \in \{1, 2, 3, 4\}$, are defined in Lemmas 1-4.

Proof. To prove (35), we substitute relations (33) and (34) into (26) and simplify. Then, we get

$$
r_c(x)\mathcal{R}_{q,\omega} Q_n(x) = f_{2,c,n}(x)P_n(x) + g_{2,c,n}(x)P_{n-1}(x) \\
= f_{2,c,n}(x)\frac{r_c(x)}{\varphi_{c,n}^{1,3}(x)}\left(g_{3,c,n}(x)Q_n(x) - g_{1,c,n}(x)Q_{n-1}(x)\right) \\
+ g_{2,c,n}(x)\frac{r_c(x)}{\varphi_{c,n}^{1,3}(x)}\left(-f_{3,c,n}(x)Q_n(x) + f_{1,c,n}(x)Q_{n-1}(x)\right).
$$

Thus,

$$
\varphi_{c,n}^{1,3}(x)\mathcal{R}_{q,\omega} Q_n(x) = \left(f_{2,c,n}(x)g_{3,c,n}(x) - g_{2,c,n}(x)f_{3,c,n}(x)\right)Q_n(x) \\
+ \left(-f_{2,c,n}(x)g_{1,c,n}(x) + g_{2,c,n}(x)f_{1,c,n}(x)\right)Q_{n-1}(x) \\
= \varphi_{c,n}^{2,3}(x)Q_n(x) + \varphi_{c,n}^{1,2}(x)Q_{n-1}(x).
$$

Taking into account the obvious fact that $\varphi_{c,n}^{3,2}(x) = -\varphi_{c,n}^{2,3}(x)$ we deduce (35).

The proof of (36) is completely similar using the corresponding relations. Therefore, we omit it. □
Finally, we have all the ingredients to establish the following statement.

**Theorem 2. (Holonomic equation)** The monic orthogonal polynomials, $Q_n(x)$, satisfy the following second–order linear difference equation

$$
\sigma_{1,c,n}(x) D_2^{(2)} Q_n(x) + \sigma_{2,c,n}(x) Q_n(x) + \sigma_{3,c,n}(x) Q_n(x) = 0, \quad n \geq 3,
$$

where

$$
\begin{align*}
\sigma_{1,c,n}(x) &= \varphi_{c,n}^{1,3}(x) \varphi_{c,n}^{1,3}(q x + \omega) \varphi_{c,n}^{1,2}(x), \\
\sigma_{2,c,n}(x) &= \varphi_{c,n}^{1,3}(x) \left( \varphi_{c,n}^{1,2}(x) \left( \varphi_{c,n}^{3,2}(q x + \omega) + D_{q,\omega} \varphi_{c,n}^{3,3}(x) \right) - \varphi_{c,n}^{1,2}(q x + \omega) \varphi_{c,n}^{1,4}(x) - \varphi_{c,n}^{1,3}(x) D_{q,\omega} \varphi_{c,n}^{1,2}(x) \right), \\
\sigma_{3,c,n}(x) &= \varphi_{c,n}^{1,2}(q x + \omega) \left( \varphi_{c,n}^{1,2}(x) \varphi_{c,n}^{3,4}(x) - \varphi_{c,n}^{1,4}(x) \varphi_{c,n}^{3,2}(x) \right) + \varphi_{c,n}^{1,3}(x) \left( \varphi_{c,n}^{1,2}(x) D_{q,\omega} \varphi_{c,n}^{3,2}(x) - \varphi_{c,n}^{3,2}(x) D_{q,\omega} \varphi_{c,n}^{1,2}(x) \right).
\end{align*}
$$

**Proof.** Once we know the corresponding ladder operators given in Theorem 1 we can proceed as follows. Considering (36), we have

$$
\hat{\Phi}_n Q_{n-1}(x) = \varphi_{c,n}^{3,4}(x) Q_n(x).
$$

We know by (35)

$$
Q_{n-1}(x) = \frac{\Phi_n Q_n(x)}{\varphi_{c,n}^{1,2}(x)},
$$

so, we get

$$
\hat{\Phi}_n \frac{\Phi_n Q_n(x)}{\varphi_{c,n}^{1,2}(x)} = \varphi_{c,n}^{3,4}(x) Q_n(x). \tag{37}
$$

Now, we analyze the left–hand side in the above expression getting

$$
\hat{\Phi}_n \frac{\Phi_n Q_n(x)}{\varphi_{c,n}^{1,2}(x)} = \frac{\varphi_{c,n}^{1,4}(x)}{\varphi_{c,n}^{1,2}(x)} \Phi_n Q_n(x) - \varphi_{c,n}^{1,3}(x) D_{q,\omega} \frac{\Phi_n Q_n(x)}{\varphi_{c,n}^{1,2}(x)}
$$

$$
= \frac{\varphi_{c,n}^{1,4}(x)}{\varphi_{c,n}^{1,2}(x)} \left( \varphi_{c,n}^{3,2}(x) Q_n(x) + \varphi_{c,n}^{1,3}(x) D_{q,\omega} Q_n(x) \right)
$$

$$
- \varphi_{c,n}^{1,3}(x) \left( D_{q,\omega} \frac{\varphi_{c,n}^{3,2}(x) Q_n(x)}{\varphi_{c,n}^{1,2}(x)} + D_{q,\omega} \frac{\varphi_{c,n}^{1,3}(x) D_{q,\omega} Q_n(x)}{\varphi_{c,n}^{1,2}(x)} \right).
$$

To compute the above $D_{q,\omega}$-differences we use (14) and (15), obtaining

$$
D_{q,\omega} \frac{\varphi_{c,n}^{3,2}(x) Q_n(x)}{\varphi_{c,n}^{1,2}(x)}
$$

$$
= \frac{\varphi_{c,n}^{1,2}(x) D_{q,\omega} \varphi_{c,n}^{3,2}(x) - \varphi_{c,n}^{3,2}(x) D_{q,\omega} \varphi_{c,n}^{1,2}(x)}{\varphi_{c,n}^{1,2}(x) \varphi_{c,n}^{1,2}(q x + \omega)} Q_n(x) + \frac{\varphi_{c,n}^{3,2}(q x + \omega)}{\varphi_{c,n}^{1,2}(x) \varphi_{c,n}^{1,2}(q x + \omega)} D_{q,\omega} Q_n(x),
$$

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corresponding to the operators $\Delta$ be computed explicitly. In this way, we provide three appendices with all the details polynomials satisfy a second–order difference equation whose polynomial coefficients can
orthogonal polynomials with respect to an inner product involving the Hahn discrete
construct a computer program that can get automatically these ladder operators and
Finally, to get the second–order difference equation for the nonstandard polynomials $Q_n$, it only remains to multiply the previous expression by $\varphi_{c,n}^{1,2}(qx+\omega)$ and simplify.

In conclusion, we have obtained ladder operators for a wide family of discrete Sobolev orthogonal polynomials with respect to an inner product involving the Hahn discrete operator. Furthermore, we have proved that these families of nonstandard orthogonal polynomials satisfy a second–order difference equation whose polynomial coefficients can be computed explicitly. In this way, we provide three appendices with all the details corresponding to the operators $\mathcal{D}_q$, $\Delta$ and $\frac{d}{dx}$. As a future work, we are planning to construct a computer program that can get automatically these ladder operators and the corresponding second–order difference equation taking the data of the Sobolev inner product as inputs.

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Appendix A. $\mathcal{D}_q$ difference operator

The $\mathcal{D}_q$ difference operator is obtained taking $\omega = 0$ in the expression (2) of the Hahn difference operator $\mathcal{D}_{q,\omega}$. We provide explicit expressions for all the functions involved in the construction of the ladder operators (Theorem [1]) for the discrete Sobolev orthogonal polynomials $Q_n$ as well as for the coefficients of the second–order difference equation satisfied by $Q_n$. All these expressions can be computed using only the standard polynomials $P_n$, and their properties such as relation (3). This also occurs in the two following appendices.

$$\rho_{n,j,c} = \frac{\mathcal{D}_q^{(j)} P_n(c)}{1 + M \mathcal{D}_q^{(j)}(c, c)} ,$$

$$r_c(x) = \prod_{k=0}^{j} (x - q^k c),$$

$$f_{1,c,n}(x) = r_c(x) - M \rho_{n,j,c} \frac{\sum_{i=0}^{j} \frac{(q; q)_i}{(q; q)_j} \mathcal{D}_q^{(i)} P_{n-1}(c) \prod_{k=0}^{i-1} (x - q^k c)}{(q; q)_i (1 - q)^{j-i}} ,$$

$$g_{1,c,n}(x) = M \rho_{n,j,c} \frac{\sum_{i=0}^{j} \frac{(q; q)_i}{(q; q)_j} \mathcal{D}_q^{(i)} P_n(c) \prod_{k=0}^{i-1} (x - q^k c)}{(q; q)_i (1 - q)^{j-i}} ,$$

$$f_{2,c,n}(x) = \frac{x - q^j c}{q^{j+1} (x - q^{-1} c)} \left( \mathcal{D}_q f_{1,c,n}(x) + f_{1,c,n}(qx) \frac{B_n(x)}{A(x)} - g_{1,c,n}(qx) \frac{C_{n-1}(x)}{\beta_{n-1} A(x)} \right) - \frac{[j+1]_q}{q^{j+1} (x - q^{-1} c)} f_{1,c,n}(x) ,$$

$$g_{2,c,n}(x) = \frac{x - q^j c}{q^{j+1} (x - q^{-1} c)} \left( \mathcal{D}_q g_{1,c,n}(x) + f_{1,c,n}(qx) \frac{C_n(x)}{A(x)} + g_{1,c,n}(qx) \frac{B_{n-1}(x)}{A(x)} + \frac{(x - \alpha_{n-1}) C_{n-1}(x)}{A(x) \beta_{n-1}} \right) - \frac{[j+1]_q}{(x - q^j c)} g_{1,c,n}(x) ,$$

$$f_{3,c,n}(x) = - \frac{g_{1,c,n-1}(x)}{\beta_{n-1}} ,$$

$$g_{3,c,n}(x) = f_{1,c,n-1}(x) + \frac{(x - \alpha_{n-1}) g_{1,c,n-1}(x)}{\beta_{n-1}} ,$$

$$f_{4,c,n}(x) = - \frac{g_{2,c,n-1}(x)}{\beta_{n-1}} ,$$

$$g_{4,c,n}(x) = f_{2,c,n-1}(x) + \frac{(x - \alpha_{n-1}) g_{2,c,n-1}(x)}{\beta_{n-1}} ,$$

$$\sigma_{1,c,n}(x) = \varphi_{c,n}^{1,3}(x) \varphi_{c,n}^{1,3}(qx) \varphi_{c,n}^{1,2}(x) ,$$

$$\sigma_{2,c,n}(x) = \varphi_{c,n}^{1,3}(x) \left( \varphi_{c,n}^{1,2}(x) (\varphi_{c,n}^{3,2}(qx) + \mathcal{D}_q \varphi_{c,n}^{1,3}(x)) - \varphi_{c,n}^{1,2}(qx) \varphi_{c,n}^{1,4}(x) - \varphi_{c,n}^{1,3}(x) \mathcal{D}_q \varphi_{c,n}^{1,2}(x) \right) ,$$

$$\sigma_{3,c,n}(x) = \varphi_{c,n}^{1,2}(qx) \left( \varphi_{c,n}^{1,2}(x) \varphi_{c,n}^{3,4}(x) - \varphi_{c,n}^{1,4}(x) \varphi_{c,n}^{3,2}(x) \right) + \varphi_{c,n}^{1,3}(x) \left( \varphi_{c,n}^{1,2}(x) \mathcal{D}_q \varphi_{c,n}^{3,2}(x) - \varphi_{c,n}^{3,2}(x) \mathcal{D}_q \varphi_{c,n}^{1,2}(x) \right) ,$$

where

$$\varphi_{c,n}^{i,j}(x) = \begin{vmatrix} f_{i,c,n}(x) & f_{j,c,n}(x) \\ g_{i,c,n}(x) & g_{j,c,n}(x) \end{vmatrix} , \quad i, j \in \{1, 2, 3, 4\}.$$
Appendix B. Forward difference operator

The forward difference operator, $\Delta$, is obtained taking $\omega = 1$ and $q = 1$ in the expression (2) of the Hahn difference operator $\mathcal{D}_{q,\omega}$. Taking into account that (see [29])

$$\lim_{q \to 1} [a]_q = a, \quad \lim_{q \to 1} \frac{(q^n; q)_n}{(1 - q)^a} = (a)_n,$$

we get,

$$\rho_{n,j,c} = \frac{\Delta^{(j)} P_n(c)}{1 + M \mathcal{K}^{(j)}(c,c)},$$

$$r_c(x) = \prod_{k=0}^{j} (x - c - k) = (x - c - j)_{j+1},$$

$$f_{1,c,n}(x) = r_c(x) - \frac{M \rho_{n,j,c}}{h_{n-1}} \left( \sum_{i=0}^{j} \frac{j! \Delta^i P_{n-1}(c) \prod_{k=0}^{i-1} (x - c - k)}{i!} \right),$$

$$g_{1,c,n}(x) = \frac{M \rho_{n,j,c}}{h_{n-1}} \left( \sum_{i=0}^{j} \frac{j! \Delta^i P_{n-1}(c) \prod_{k=0}^{i-1} (x - c - k)}{i!} \right),$$

$$f_{2,c,n}(x) = \frac{x - c - j}{x - c + 1} \left( \Delta f_{1,c,n}(x) - g_{1,c,n}(x + 1) \frac{C_{n-1}(x)}{A(x) \beta_{n-1}} + B_{n}(x) A(x) f_{1,c,n}(x + 1) \right) - \frac{j + 1}{x - c + 1} f_{1,c,n}(x),$$

$$g_{2,c,n}(x) = \frac{x - c - j}{x - c + 1} \left( \Delta g_{1,c,n}(x) + f_{1,c,n}(x + 1) \frac{C_{n}(x)}{A(x)} + g_{1,c,n}(x + 1) \frac{B_{n}(x)}{A(x) \beta_{n-1}} + \frac{(x - \alpha_{n-1}) C_{n-1}(x)}{A(x) \beta_{n-1}} \right) - \frac{j + 1}{x - c + 1} g_{1,c,n}(x),$$

$$f_{3,c,n}(x) = -\frac{g_{1,c,n-1}(x)}{\beta_{n-1}},$$

$$g_{3,c,n}(x) = f_{1,c,n-1}(x) + \frac{(x - \alpha_{n-1}) g_{1,c,n-1}(x)}{\beta_{n-1}},$$

$$f_{4,c,n}(x) = -\frac{g_{2,c,n-1}(x)}{\beta_{n-1}},$$

$$g_{4,c,n}(x) = f_{2,c,n-1}(x) + \frac{(x - \alpha_{n-1}) g_{2,c,n-1}(x)}{\beta_{n-1}},$$

$$\sigma_{1,c,n}(x) = \varphi_{c,n}^{1,3}(x) \varphi_{c,n}^{1,3}(x + 1) \varphi_{c,n}^{1,2}(x),$$

$$\sigma_{2,c,n}(x) = \varphi_{c,n}^{1,3}(x) \left( \varphi_{c,n}^{1,2}(x) \left( \varphi_{c,n}^{3,2}(x + 1) + \Delta \varphi_{c,n}^{1,3}(x) \right) - \varphi_{c,n}^{1,2}(x + 1) \varphi_{c,n}^{1,4}(x) - \varphi_{c,n}^{1,3}(x) \Delta \varphi_{c,n}^{1,2}(x) \right),$$

$$\sigma_{3,c,n}(x) = \varphi_{c,n}^{1,2}(x + 1) \left( \varphi_{c,n}^{1,2}(x) \varphi_{c,n}^{3,4}(x) - \varphi_{c,n}^{1,4}(x) \varphi_{c,n}^{3,2}(x) \right) + \varphi_{c,n}^{1,3}(x) \left( \varphi_{c,n}^{1,2}(x) \Delta \varphi_{c,n}^{3,2}(x) - \varphi_{c,n}^{3,2}(x) \Delta \varphi_{c,n}^{1,2}(x) \right),$$

where

$$\varphi_{c,n}^{i,j}(x) = \begin{vmatrix} f_{i,c,n}(x) & f_{j,c,n}(x) \\ g_{i,c,n}(x) & g_{j,c,n}(x) \end{vmatrix}, \quad i,j \in \{1,2,3,4\}.$$
Appendix C. Derivative operator

The derivative operator is obtained as a limit case of the Hahn difference operator $\mathcal{D}_{q,\omega}$ when $\omega = 0$ and $q \to 1$. Then, the expressions of the functions are the following.

$$\rho_{n,j,c} = \frac{P^{(j)}_n(c)}{1 + MK^{(j)}_{n-1}(c,c)},$$

$$r_c(x) = (x - c)^{j+1},$$

$$f_{1,c,n}(x) = r_c(x) - \frac{M\rho_{n,j,c}}{h_{n-1}} \left( \sum_{i=0}^j \frac{j!P^{(i)}_n(c)(x-c)^i}{i!} \right),$$

$$g_{1,c,n}(x) = \frac{M\rho_{n,j,c}}{h_{n-1}} \left( \sum_{i=0}^j \frac{j!P^{(i)}_n(c)(x-c)^i}{i!} \right),$$

$$f_{2,c,n}(x) = f'_{1,c,n}(x) - g_{1,c,n}(x) \frac{C_{n-1}(x)}{A(x)\beta_{n-1}} + f_{1,c,n}(x) \left( \frac{B_n(x)}{A(x)} - \frac{j+1}{x-c} \right),$$

$$g_{2,c,n}(x) = g'_{1,c,n}(x) + f_{1,c,n}(x) \frac{C_n(x)}{A(x)} + g_{1,c,n}(x) \left( \frac{B_{n-1}(x)}{A(x)} + \frac{(x-\alpha_{n-1})C_{n-1}(x)}{A(x)\beta_{n-1}} - \frac{j+1}{x-c} \right),$$

$$f_{3,c,n}(x) = -\frac{g_{1,c,n-1}(x)}{\beta_{n-1}},$$

$$g_{3,c,n}(x) = f_{1,c,n-1}(x) + \frac{(x-\alpha_{n-1})g_{1,c,n-1}(x)}{\beta_{n-1}},$$

$$f_{4,c,n}(x) = -\frac{g_{2,c,n-1}(x)}{\beta_{n-1}},$$

$$g_{4,c,n}(x) = f_{2,c,n-1}(x) + \frac{(x-\alpha_{n-1})g_{2,c,n-1}(x)}{\beta_{n-1}},$$

$$\sigma_{1,c,n}(x) = (\varphi^{1,3}_{c,n}(x))^2 \varphi^{1,2}_{c,n}(x),$$

$$\sigma_{2,c,n}(x) = \varphi^{1,3}_{c,n}(x) \left( \varphi^{1,2}_{c,n}(x) \varphi^{3,2}_{c,n}(x) + \left( \varphi^{1,3}_{c,n}(x) \varphi^{1,4}_{c,n}(x) \right) - \left( \varphi^{1,2}_{c,n}(x) \varphi^{1,3}_{c,n}(x) \right) \right),$$

$$\sigma_{3,c,n}(x) = \varphi^{1,2}_{c,n}(x) \left( \varphi^{1,2}_{c,n}(x) \varphi^{3,4}_{c,n}(x) - \varphi^{1,4}_{c,n}(x) \varphi^{3,2}_{c,n}(x) \right) + \varphi^{1,3}_{c,n}(x) \left( \left( \varphi^{3,2}_{c,n}(x) \varphi^{1,2}_{c,n}(x) - \left( \varphi^{1,2}_{c,n}(x) \varphi^{1,3}_{c,n}(x) \right) \right) \right),$$

where

$$\varphi^{i,j}_{c,n}(x) = \begin{vmatrix} f_{i,c,n}(x) & f_{j,c,n}(x) \\ g_{i,c,n}(x) & g_{j,c,n}(x) \end{vmatrix}, \quad i, j \in \{1, 2, 3, 4\}.$$