Kinetic limit of \( N \)-body description
of wave-particle self-consistent interaction

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A system of \( N \) particles \[\xi^N = (x_1, v_1, ..., x_N, v_N)\] interacting self-consistently with \( M \) waves \[Z_n = A_n \exp(i\phi_n)\] is considered. Given initial data \((Z^N(0), \xi^N(0))\), it evolves according to hamiltonian dynamics to \((Z^N(t), \xi^N(t))\). In the limit \( N \to \infty \), this generates a Vlasov-like kinetic equation for the distribution function \( f(x, v, t) \), abbreviated as \( f(t) \), coupled to envelope equations for the \( Z_n \): initial data \((Z(0), f(0))\) evolve to \((Z(t), f(t))\).

The solution \((Z, f)\) exists and is unique for any initial data with finite energy. Moreover, for any time \( T > 0 \), given a sequence of initial data with \( N \) particles distributed so that the particle distribution \( f^N(0) \to f(0) \) weakly and with \( Z^N(0) \to Z(0) \) as \( N \to \infty \), the states generated by the hamiltonian dynamics at all times \( 0 \leq t \leq T \) are such that \((Z^N(t), f^N(t))\) converges weakly to \((Z(t), f(t))\).

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I. INTRODUCTION

Recent work on the dynamics of wave-particle interaction has led to extensive use of $N$-body hamiltonian models in parallel with the more traditional kinetic approach. The present paper aims at discussing to what extent the two approaches agree in the limit $N \to \infty$, where $N$-body dynamics formally reduces to kinetic theory. This is a classical problem of statistical physics, notoriously unsolved for particles interacting through short-range forces, where it amounts to deriving the Boltzmann equation from the Liouville equation: systematic rigorous derivations of the kinetic equation from BBGKY hierarchy are still lacking—notwithstanding the pioneering work of Lanford and King, limited to short timescales \[19,20\], and recent advances \[7,27\]. However, for long-range forces, and more precisely for smooth enough mean-field interactions, the formal limit $N \to \infty$ commutes with the dynamics \[22,26\]. We show in this paper how the mean-field methods apply also to wave-particle interactions.

A physical motivation for this work is that wave-particle interacting systems are typical of plasmas and common to many physical phenomena. The paradigm of such interactions is provided by the self-consistent hamiltonian $H_{sc}^{N,M}$ describing the evolution of $N$ particles and $M$ Langmuir waves \[1–3,5,8–10,16,21,23,24,27,28\]. In particular this hamiltonian enables a mechanical approach of classical plasma problems like Landau damping and beam-plasma instability, by treating Langmuir waves as $M$ harmonic oscillators self-consistently coupled to $N$ quasiresonant beam particles. For simplicity we present our results in one space dimension with periodic boundary conditions, which conforms to the physical conditions considered in models of plasmas \[4,12,13\].

The basic characteristic of such models is that particles do not interact directly with each other: they only interact with the modes; symmetrically, the modes do not interact directly with each other: they only interact directly with the particles. Inasmuch the modes are spatial Fourier components of some fields, these components are not localized spatially: this invites to describe the many-body limit $N \to \infty$ as a mean-field limit and enables us
to apply the techniques which succeed in the case of particle-particle mean-field coupling.

The present work takes advantage of this observation to show that the kinetic limit $N \to \infty$ and the time evolution over any time interval $[0, T]$ commute. Our result implies that numerical simulations with increasing number of particles behave ever closer to the predictions of kinetic theory (if one uses $N$ ‘large enough’...).

A preliminary form of our result was announced in reference [14]. In Sec. II we describe the model and its evolution equations. The main results are stated in Sec. III. Sec. IV is devoted mainly to a finite $N$ estimate and a technical remark, preparing the proof presented in Sec. V. The final section is devoted to the conclusion.

II. SELF-CONSISTENT HAMILTONIAN AND KINETIC LIMIT

We consider a system of $N$ particles with respectively mass $m_r$, charge $q_r$, position $x_r$ and momentum $p_r$, interacting with $M$ waves with respectively natural frequency $\omega_{j0}$, phase $\theta_j$ and intensity $I_j$. The evolution of this system is described by the hamiltonian

$$H_{sc}^{N,M} = \sum_{r=1}^{N} \frac{p_r^2}{2m_r} + \sum_{j=1}^{M} \omega_{j0} I_j - \varepsilon \sum_{r=1}^{N} \sum_{j=1}^{M} q_r k_j^{-1} \beta_j \sqrt{2I_j} \cos(k_j x_r - \theta_j)$$

where the first term corresponds to free particles, the second term to free waves (harmonic oscillators) and the third term to their coupling. The coupling constants are expressed in such a way to ease the kinetic limit $N \to \infty$ : we shall keep the ‘wave susceptibilities’ $\beta_j$ constant in this limit. A simple change of variables enables one to ensure that all coefficients $\beta_j > 0$, which is assumed in the following. The overall coupling factor $\varepsilon$ in the interaction term of (1) emphasizes our interest in the weak-coupling regime ($\varepsilon \ll 1$) [2,11].

Assuming periodic boundary conditions, the particles move on $(\mathbb{R}/L) = S_L$ and the wavenumbers are quantized ($k_j = n_j 2\pi / L$ for some integer $n_j$). The phase space of this system is thus $(S_L \times \mathbb{R})^N \times \mathcal{Z}^M$ where $\mathcal{Z} = S_{2\pi} \times \mathbb{R}^+$ for each mode.

The natural scaling of our model in the limit $N \to \infty$ is easily deduced from its equilibrium (Gibbs) thermodynamics [13]. Then the energy $E = H_{sc}$ and the wave intensities $I_j$ are
extensive (i.e. $O(N)$), and the coupling constant scales as $\varepsilon = O(N^{-1/2})$. The extensivity of wave intensities can easily be interpreted as, in the physical regime of the model, we expect particles to be mostly resonant with the waves, each such particle contributing then to wave intensities by evolving in their potential well. This prompts us to introduce intensive wave variables

$$z_j = N^{-1/2} Z_j = N^{-1/2} \sqrt{2 I_j} e^{-i \theta_j} = |z_j| e^{-i \theta_j}$$

for which $Z$ reduces to $C$, and renormalized coupling constants $\beta'_j = N^{1/2} \varepsilon \beta_j$.

The evolution equations of the hamiltonian (1) read

$$\dot{x}_r = p_r / m_r$$
$$\dot{p}_r = i q_r \sum_{j=1}^{M} \beta'_j (z_j e^{ik_j x_r} - z_j^* e^{-ik_j x_r})$$
$$\dot{z}_j = -i \omega_{j0} z_j + \frac{i}{N} \beta'_j \kappa_{j}^{-1} \sum_{r=1}^{N} q_r e^{-ik_j x_r}$$

To simplify calculations, we introduce non-canonical variables, namely particle velocities

$$v_r = p_r / m_r$$

and mode envelopes

$$a_j = z_j e^{i \omega_{j0} t}$$

bringing (3-4-5) to the form

$$\dot{x}_r = v_r$$
$$\dot{v}_r = \frac{i q_r}{2 m_r} \sum_{j=1}^{M} \beta'_j (a_j e^{ik_j x_r} - i \omega_{j0} v_r - \bar{a}_j^* e^{-ik_j x_r} + i \omega_{j0} v_r)$$
$$\dot{a}_j = \frac{i}{N} \beta'_j \kappa_{j}^{-1} \sum_{r=1}^{N} q_r e^{-ik_j x_r} + i \omega_{j0} v_r$$

The usual space of kinetic theory is Boltzmann’s $\mu$-space $\Lambda = S_L \times \mathbb{R}$. The positions and velocities of the $N$ particles determine a distribution $f$ on $\Lambda$.
\[ f(x,v,t) = \frac{1}{N} \sum_{r=1}^{N} \delta(x-x_r(t))\delta(v-v_r(t)) \]  

(11)

which is normalized to unity \((\int_{\Lambda} f(x,v,t) dxdv = 1)\) irrespective of the number \(N\) of particles (for simplicity, we assume a single species: all \(q_r = q > 0, m_r = m\)). The kinematic limit, formally \(N \to \infty\), corresponds to considering a sequence of \(N\)-particle distributions \(f^N\) converging to a distribution \(f^\infty\) in the weak sense for a natural space of observables \(D\).

Denote by \(F\) the space of positive normalized distributions on \(\Lambda\) with finite momentum and kinetic energy, i.e. 
\[ \mathcal{F} \equiv \{ f \in L^1(\Lambda, dxdv) : f \geq 0, \int f dxdv = 1, \int v^2 f dxdv < \infty \} \]  

and define on \(F\) the bounded-Lipshitz distance

\[ d_{bl}(f,f') \equiv \sup_{\phi \in D} \left| \int_{\Lambda} \phi f dxdv - \int_{\Lambda} \phi f' dxdv \right| \]  

(12)

with the set of bounded, Lipschitz-continuous normalized observables

\[ D \equiv \{ \phi : \Lambda \to [0,1], |\phi(x,v) - \phi(x',v')| \leq \|(x,v) - (x',v')\| \forall (x,v),(x',v') \in \Lambda \} \]  

(13)

Here \(\Lambda\) is equipped with the distance \(\|(x,v) - (x',v')\| \equiv \alpha(||(x-x')\text{mod}L| + \tau|v-v'||)\), where \(\alpha^{-1}\) and \(\tau\) are respectively convenient length and time scales to be chosen below.

Then we consider the distance on \(Z^M\),

\[ \|a - a'| = \sum_{j=1}^{M} w_j |a_j - a'_j| \]  

(14)

where real positive coefficients \(w_j\) will be chosen below in (31), and \(|a_j|\) is the modulus of the complex number \(a_j\). Our distance on \(\mathcal{F} \times Z^M\) is just

\[ \|(f,a) - (f',a')\| = d_{bl}(f,f') + \|a - a'|| \]  

(15)

The kinetic evolution system of equations, dual to (8-9-10), is the system

\[ \partial_t f + v \partial_x f + \frac{iq}{2m} \sum_{j=1}^{M} \beta'_j (a_j e^{ik_j x - i\omega_j t} - a_j^* e^{-ik_j x + i\omega_j t}) \partial_v f = 0 \]  

(16)

\[ \dot{a}_j = iq \beta'_j k_j^{-1} \int_{\Lambda} f(x,v,t) e^{-ik_j x + i\omega_j t} dxdv \]  

(17)

This dynamics leaves \(\mathcal{F} \times Z^M\) invariant.
III. MAIN RESULTS

The self-consistent dynamics \((3-4-5)\) preserves two constants of the motion, namely total energy \(H\) and total momentum \(P = \sum_r p_r + \sum_j k_j I_j\). In the kinetic limit, we consider the normalized constants \(h = H/N\) and \(p = P/N\):

\[
h(f, a) = \int_{\Lambda} \frac{mv^2}{2} f dx dv + \sum_j \omega_j \frac{|a_j|^2}{2} - \int_{\Lambda} \sum_j q k_j^{-1} \beta_j' \Re(a_j e^{ik_j x - i\omega_j t}) f dx dv \tag{18}
\]

\[
p(f, a) = \int_{\Lambda} mv f dx dv + \sum_j k_j \frac{|a_j|^2}{2} \tag{19}
\]

where \(\Re\) denotes the real part. For any finite \(N\) and \(h\), the energy surface \(H_{sc}^{N,M} = Nh\) in \(\Lambda^N \times \mathcal{Z}^M\) is compact, and the vector field \((3-4-5)\) is continuous and bounded on it. This ensures that the dynamics generates a group for all initial conditions.

Moreover, the first variation of the dynamics \((8-9-10)\) generates a linear operator \(\mathcal{M} = \partial(x_r, v_r, a_j)/\partial(x_r, v_r, a_j)\), depending continuously on \((x_r, v_r, a_j)\). As the energy surface is compact for any given \(N\), \(\mathcal{M}\) is bounded. With the specific form of \(H_{sc}\), we show that, with appropriate choice of the constants \(w_j\):

\[
\|\mathcal{M}\| \leq \tau^{-1} + \gamma[a(t)] \tag{20}
\]

where

\[
\tau = \left(\frac{q^2}{m} \sum_j \beta_j'^2\right)^{-1/3}, \tag{21}
\]

\[
\gamma[a(s)] = \sum_j \frac{qT}{m} \beta_j' k_j |a_j(s)| \tag{22}
\]

The positive function \(\gamma[a(s)]\) is continuous on the energy surface, on which it has an upper bound uniform with respect to \(N\).

The kinetic limit, \(N \to \infty\), admits a similar bound, ensuring the existence and uniqueness

**Theorem**: Given initial data \((f_0, a(0)), (f'_0, a'(0)) \in \mathcal{F} \times \mathcal{Z}^M\), with \(h_0 = h(f_0, a(0))\) and \(h'_0 = h(f'_0, a'(0))\), the kinetic evolution equations \((16-17)\) generate for all times \(t \geq 0\) states \((f_t, a(t))\) and \((f'_t, a'(t))\) respectively from these data. Moreover,
\[ dbL(f_t, f'_t) + \|a(t) - a'(t)\| \leq e^{Ct} (dbL(f_0, f'_0) + \|a(0) - a'(0)\|) \]  

(23)

for some \( C = C(h_0, h'_0) < \infty \).

This theorem implies the

**Corollary**: Given a distribution \( f_0^\infty \in \mathcal{F} \) and a sequence of finite-\( N \) Dirac distributions \( f_N^0 \in \mathcal{F} \) for particle initial data, such that \( \lim_{N \to \infty} dbL(f_N^0, f_0^\infty) = 0 \), given initial waves \( a(0) \in \mathcal{Z}^M \), and given any time \( T > 0 \), consider for all \( 0 \leq t \leq T \) the resulting distributions \( f_t^N \) and waves \( a^N(t) \) generated by \( H_{sc}^{N,M} \) and the kinetic solution \( (f_t^\infty, a^\infty(t)) \). Then \( \lim_{N \to \infty} dbL(f_t^N, f_t^\infty) = 0 \) and \( \lim_{N \to \infty} a^N(t) = a^\infty(t) \), uniformly on \( [0; T] \).

In other words, the following diagram commutes for all \( t > 0 \):

\[
\begin{array}{ccc}
(f_0^N, a(0)) & \xrightarrow{(8-13)} & (f_N^0, a^N(t)) \\
\downarrow N \to \infty & & \downarrow N \to \infty \\
(f_0^\infty, a(0)) & \xrightarrow{(16-17)} & (f_t^\infty, a(t))
\end{array}
\]  

(24)

**IV. PRELIMINARY REMARKS**

For given \( N \) and finite energy \( H \), the first variation \( \mathcal{M} \) of the dynamics \((8)-(9)-(10)\) has bounded norm (with the \( L_1 \) distance):

\[
\| (\delta\dot{x}, \delta\dot{v}, \delta\dot{a}) \|_1 \equiv N^{-1} \sum_{r=1}^{N} \alpha (|\delta\dot{x}_r| + \tau |\delta\dot{v}_r|) + \sum_{j=1}^{M} w_j |\delta\dot{a}_j| 
\]

(25)

\[
= N^{-1} \sum_{r=1}^{N} \| (\delta\dot{x}_r, \delta\dot{v}_r) \| + \sum_{j=1}^{M} w_j |\delta\dot{a}_j|
\]

We readily find

\[
|\delta\dot{x}_r| = |\delta\dot{v}_r|,
\]

(26)

\[
|\delta\dot{a}_j| = N^{-1} \left| \sum_{r=1}^{N} \beta'_j q e^{ik_jx_r - i\omega_j t} \delta x_r \right| \leq N^{-1} \beta'_j q \sum_{r=1}^{N} |\delta x_r|
\]

(27)

and
\[
|\delta \dot{v}_r| = \left| \sum_{j=1}^{M} \frac{q_j \beta_j'}{m} \Re(e^{i k_j x_r - i \omega_j t} (a_j k_j \delta x_r - i \delta a_j)) \right| 
\]  \tag{28}

\[
\leq \sum_{j=1}^{M} \frac{q_j \beta_j'}{m} |k_j| a_j | \cdot |\delta x_r| + \sum_{j=1}^{M} \frac{q_j \beta_j'}{m} \cdot |\delta a_j| 
\]  \tag{29}

so that

\[
\|(\delta \dot{x}, \delta \dot{v}, \delta \dot{a})\|_1 \leq N^{-1} \sum_{r=1}^{N} \alpha \tau^{-1}(\tau \gamma[a(t)] \cdot |\delta x_r| + \tau |\delta v_r|) 
\]

\[
+ \sum_{j=1}^{M} w_j N^{-1} \beta_j' q \sum_{r=1}^{N} |\delta x_r| + \alpha \tau \sum_{j=1}^{M} \frac{q_j \beta_j'}{m} \cdot |\delta a_j| 
\]  \tag{30}

with \(\gamma[a(s)]\) defined by (22). The four causes for the divergence of trajectories in \(\Lambda^N \times \mathcal{Z}^M\) are saddle points (in \((x, v)\) plane) associated with maxima of the modes’ potentials (the \(|a_j|\) contribution to \(\gamma[a(t)]\)), velocity shear (the velocity term), the dependence of the modes source on the particle positions, and the dependence of the saddle points themselves on the mode envelopes.

An appropriate choice of constants \(\alpha, \tau, w_j\) keeps the estimates as small as possible. Thus let

\[
w_j = \alpha w_0 \beta_j' , \tag{31}
\]

and solve

\[
\tau^{-1} = w_0 \sum_{j} q_j \beta_j'^2 = \frac{q \tau}{m w_0} \tag{32}
\]

This leads to the expression of \(\tau\) announced in (21) and to

\[
w_0 = (qm)^{-\frac{1}{4}} \left( \sum_{j} \beta_j'^2 \right)^{-\frac{2}{3}} , \tag{33}
\]

so that (30) reduces to

\[
\|(\delta \dot{x}, \delta \dot{v}, \delta \dot{a})\| \leq \tau^{-1} \|(\delta x, \delta v, \delta a)\| + \gamma[a(t)] N^{-1} \sum_{r} \alpha |\delta x_r| \tag{34}
\]

which implies (20). Constant \(\alpha\) remains arbitrary, as it only determines the scale of the distances in \(\Lambda\) and \(\mathcal{Z}\), and (20) is homogeneous (degree 1). Considering only the restricted dynamics on \(\Lambda\), with \(\delta a = 0\), (30) straightforwardly leads to the continuity equation

9
\[ \| (\delta \dot{x}, \delta \dot{v}) \| \leq \gamma'[a(t)] \| (\delta x, \delta v) \| \]  

(35)

with

\[ \gamma'[a(t)] = \max(\tau^{-1}, \gamma[a(t)]). \]  

(36)

Note that \( \gamma'[a(t)] \) is bounded uniformly in time, as the positive function \( \gamma[a] \) is bounded above on the energy surface by a function which does not grow faster than \( h^{1/2} \) in the large energy limit. More precisely, let \( \lambda > 0 \) solve

\[ \lambda^2 \sum_j \omega_j^{-1} (q\beta'_j k_j/m)^2 = 2h + \sum_j \omega_j^{-1} (q\beta'_j/k_j)^2. \]

Then \( \sum_j |\beta'_j k_j a_j q/m| \leq (q^2/m) \sum_j \omega_j^{-1} \beta'^2_j (1 + k^2_j \lambda/m) \). It should be noted once more that (35) reflects that the divergence rate in \((x, v)\) is controlled by velocity shear and by saddle points of the pendulum-like potential depending on wave amplitudes. The latter situation typically corresponds to a trapping regime for large enough wave intensities.

Finally, note the following

**Proposition 1**: Let \( Y : \Lambda \to \Lambda \) be a Lipschitz mapping with constant \( L \geq 1 \) on \( \Lambda \), and \( \mu, \nu \in \mathcal{F} \). Then :

\[ \sup_{\phi \in \mathcal{D}} \left| \int_{\Lambda} \phi \circ Y d(\mu - \nu) \right| \leq L d_{\mathcal{B}}(\mu, \nu) \]  

(37)

**Proof**: Clearly \( L^{-1} \phi \circ Y \in \mathcal{D} \) for any \( \phi \in \mathcal{D} \). Hence \( \sup_{\phi \in \mathcal{D}} \left| \int_{\Lambda} L^{-1} \phi \circ Y d(\mu - \nu) \right| \leq d_{\mathcal{B}}(\mu, \nu) \).

V. PROOF OF THE MAIN RESULT

The proof of theorem 1 uses the fact that the two types of degrees of freedom have no ‘self’-interaction. Indeed the motion of particle \( r \) is completely determined by its initial position and velocity and by the modes history, i.e. the data of the modes \( a_j(\cdot) \) over a time interval \([s, t]\) defines the vector field \( G \) so that:

\[ \frac{d}{dt}(x_r(t), v_r(t)) = G[a(t)](x_r(t), v_r(t)) \]  

(38)

This vector field is Lipschitz-continuous on \( \Lambda \) according to (35) and subsequent remarks. Thus Cauchy-Lipschitz theorem ensures the existence and unicity of the flow \( T \):
\begin{equation}
(x_r(t), v_r(t)) = T_{t,s}[a(.)](x_r(s), v_r(s)) \tag{39}
\end{equation}

By duality the measure \( \mu_s \) on \( \Lambda \) is transported by the flow to

\begin{equation}
\mu_t = \mu_s \circ T_{s,t}[a(.)] \tag{40}
\end{equation}

Similarly, the evolution of mode \( j \) is also completely determined by its initial data \( a_j(s) \) and by the history of the measure on particle phase space \( \Lambda \) in the right hand side of (17) which defines a flow \( S \) by

\begin{equation}
a_j(t) = S_{t,s}[\mu.]a_j(s) \tag{41}
\end{equation}

Solving kinetic equations (16)-(17) with initial data \(( \mu_0, a(0) )\) amounts to finding a fixed point of the coupled system (40)-(41) in the space \(( F \times Z_t^M ) \mathbb{R}^r \). Our strategy now follows that of Neunzert [22] and Spohn [26], who considered direct particle-particle interaction of mean-field type.

Thus consider two solutions \(( f, a(.) )\) and \(( f', a'(.) )\) of (16)-(17). To shorten notations, write \( b = a' \) and denote by \( d\mu = f dxdv \) and \( d\nu = f' dxdv \) the corresponding measures. Their distance at time \( t \) satisfies

\begin{equation}
\|(\mu_t, a(t)) - (\nu_t, b(t))\| = d_{bL}(\mu_0 \circ T_{0,t}[a(.)], \nu_0 \circ T_{0,t}[b(.)]) + \| S_{t,0}[\mu.]a(0) - S_{t,0}[\nu.]b(0) \| \tag{42}
\end{equation}

where

\begin{equation}
\| S_{t,0}[\mu.]a(0) - S_{t,0}[\nu.]b(0) \| \leq d_1(t) + d_2(t) \tag{43}
\end{equation}

\begin{equation}
d_{bL}(\mu_0 \circ T_{0,t}[a(.)], \nu_0 \circ T_{0,t}[b(.)]) \leq d_3(t) + d_4(t) \tag{44}
\end{equation}

and

\begin{equation}
d_1(t) = \| S_{t,0}[\mu.]a(0) - S_{t,0}[\mu.]b(0) \| \tag{45}
\end{equation}

\begin{equation}
d_2(t) = \| S_{t,0}[\mu.]b(0) - S_{t,0}[\mu.]b(0) \| \tag{46}
\end{equation}
\[ d_3(t) = d_{bL}(\mu_0 \circ T_{0,t}[a(.)], \nu_0 \circ T_{0,t}[a(.)]) \] (47)

\[ d_4(t) = d_{bL}(\nu_0 \circ T_{0,t}[a(.)], \nu_0 \circ T_{0,t}[b(.)]) \] (48)

Straightforward integration of (17) shows that

\[ d_1(t) = d_1(0) = \|a(0) - b(0)\| \] (49)

because the flow \( S[\mu] \) is just a translation in \( \mathcal{Z}^M \).

To estimate \( d_2 \) we integrate (17) with the right hand sides given by \( \mu \) and \( \nu \) :

\[ d_2(t) = \sum_{j=1}^{M} w_j q_j \beta_j^{-1} \int_0^t \int_{\Lambda} e^{-ik_j x + i\omega j_s} d(\mu_s - \nu_s) ds \] (50)

\[ = 2 \sum_{j=1}^{M} w_j q_j \beta_j^{-1} \int_0^t \int_{\Lambda} \frac{1 + i + e^{-ik_j x + i\omega j_s}}{2} d(\mu_s - \nu_s) ds \] (51)

\[ \leq \sqrt{2} \tau^{-1} \int_0^t d_{bL}(\mu_s, \nu_s) ds \] (52)

In (52) the inequality uses the fact that \( \alpha(1+\cos(k_j x - \theta))/k_j \in \mathcal{D} \) and \( \alpha(1+\sin(k_j x - \theta))/k_j \in \mathcal{D} \) for any real \( \theta \), provided that \( 2\alpha \leq \min k_j \).

Estimating \( d_3 \) is also straightforward, as proposition 1 implies

\[ d_3(t) \leq d_{31}(t) d_{bL}(\mu_0, \nu_0) \] (53)

provided that \( d_{31}(t) \) is a Lipschitz constant for \( T_{t,0}[a(.)] \). Now, \( \forall (x, v), (x', v') \in \Lambda \),

\[ \|T_{t,0}[a(.)](x, v) - T_{t,0}[a(.)](x', v')\| \]

\[ \leq \|(x, v) - (x', v')\| + \int_0^t \|G[a(s)]T_{s,0}[a](x, v) - G[a(s)]T_{s,0}[a](x', v')\| ds \] (54)

\[ \leq \|(x, v) - (x', v')\| + \int_0^t \gamma'(a(s))\|T_{s,0}[a](x, v) - T_{s,0}[a](x', v')\| ds \] (55)

Hence, \( d_{31}(t) \leq 1 + \int_0^t \gamma'(a(s)) d_{31}(s) ds \), which implies

\[ d_{31}(t) \leq \exp \int_0^t \gamma'[a(s)] ds \] (56)

by Gronwall’s lemma.
Finally,

\[ d_4(t) = \sup_{\phi \in \mathcal{D}} \left| \int_{t}^{\Lambda} (\phi \circ T_{t,0}[a(.)] - \phi \circ T_{t,0}[b(.)])d\nu_0 \right| \leq d_{40}(t) \]  

(57)

where

\[ d_{40}(t) := \sup \|T_{t,0}[a(.)](x, v) - T_{t,0}[b(.)](x, v)\| \leq d_{41}(t) + d_{42}(t) \]  

(58)

with

\[ d_{41}(t) := \sup \| \int_{0}^{t} (G[a(s)]T_{s,0}[a(.)](x, v) - G[a(s)]T_{s,0}[b(.)](x, v))ds \| \leq \int_{0}^{t} \gamma'[a(s)]d_{40}(s)ds \]  

(59)

\[ d_{42}(t) := \sup \| \int_{0}^{t} (G[a(s)]T_{s,0}[b(.)](x, v) - G[b(s)]T_{s,0}[b(.)](x, v))ds \| \leq \int_{0}^{t} \sup \|G[a(s)] - G[b(s)]\|ds \]  

(60)

Definition (38) shows that

\[ \| G[a(s)](x, v) - G[b(s)](x, v)\| = \left| \frac{\alpha q \tau}{m} \sum_j \beta'_j(a_j(s) - b_j(s))e^{ik_jx} \right| \leq \tau^{-1}\|a(s) - b(s)\| \]  

(63)

Now define \( \varphi(t) \), a majorant of the sum \( d_1(t) + d_3(t) \), and \( d_5(t) \), a majorant of the sum \( d_2(t) + d_4(t) \) as

\[ \varphi(t) = \|a(0) - b(0)\| + e^{\int_{0}^{t} \gamma'[a(s)]ds}d_{bL}(\mu_0, \nu_0) \]  

(64)

\[ d_5(t) = d_2(t) + d_{41}(t) + d_{42}(t) \]  

(65)

Then previous inequalities for the \( d_i \) lead to

\[ d_5(t) \leq \sqrt{2}\tau^{-1} \int_{0}^{t} d_{bL}(\mu_s, \nu_s)ds + \int_{0}^{t} \gamma'[a(s)]d_5(s)ds + \tau^{-1} \int_{0}^{t} d_1(s)ds \]  

(66)

and \( d_{bL}(\mu_s, \nu_s) + d_1(s) \leq d_1(s) + d_3(s) + d_4(s) \leq \varphi(s) + d_5(s) \) so that

\[ d_5(t) \leq \int_{0}^{t} \sqrt{2}\tau^{-1}\varphi(s)ds + \int_{0}^{t} (\sqrt{2}\tau^{-1} + \gamma'[a(s)])d_5(s)ds \]  

(67)
which Gronwall’s inequality readily estimates by
\[
d_5(t) \leq \int_0^t \sqrt{2\tau^{-1}} \varphi(s)e^{\int_s^t (\sqrt{2\tau^{-1}} + \gamma'[a(u)])da} ds
\] (68)

The resulting complete estimate
\[
\|(\mu_t, a(t)) - (\nu_t, b(t))\| \leq \varphi(t) + d_5(t) \] (69)
depends on two functions \(\gamma'[a(s)]\) and \(\varphi(s)\). Note that \(\varphi(0) = \|(\mu_0, a(0)) - (\nu_0, b(0))\| \) and \(d_5(0) = 0\). Estimate (69) does not grow faster than exponentially, with upper bound on its growth rate
\[
C = \sqrt{2\tau^{-1}} + 2 \sup_{0 \leq s \leq t} \gamma'[a(s)]
\] (70)
which is bounded by a function of \(h_0\) as discussed in Section IV. This completes the proof of the theorem.

The corollary follows in a standard way.

Remark: our estimate for the growth rate \(C\) in the kinetic case is larger than the finite-\(N\) estimate for \(|M|\) in phase space. This is due to the fact that the distance \(d_{bL}\) makes no distinction between \(x\)-components and \(v\)-components, while estimates of Sec. IV relied on treating these components of the phase space points separately to obtain (69).

VI. CONCLUSION

This work supports theoretically the use of full \(N\)-body dynamical schemes \([3,11,16,18]\) to study the wave-particle interactions, as an alternative to kinetic-theory based models. However the regularity of the limit \(N \to \infty\) is tempered by the rapid growth of the right hand side in the upper bound (23).

It also identifies the fundamental cause of phase space mixing and approach to equilibrium in this many-body system: particles passing near the instantaneous saddle points associated with the modes undergo exponential dichotomy, with a divergence rate controlled
by amplitudes $|z_j| = |a_j|$. This implies that the phase space regions where discrepancies between the kinetic description and the finite-$N$ description show up most rapidly correspond to the neighbourhood of the ‘separatrices’ associated with the envelopes in the particles’ $\mu$-space $\Lambda$, as was observed in numerical simulations for $M = 1$ by Guyomarc’h [17,18].

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