Positivity for the curvature of the diffeomorphism group corresponding to the incompressible Euler equation with Coriolis force

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We investigate the geometry of the central extension \(\hat{D}_\mu(S^2)\) of the group of volume-preserving diffeomorphisms of the 2-sphere equipped with an \(L^2\)-metric, for which geodesics correspond to solutions of the incompressible Euler equation with Coriolis force. In particular, we calculate the Misio\'lak curvature of this group. This value is related to the existence of a conjugate point and its positivity directly implies the positivity of the sectional curvature.

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1. Introduction

An example of a stable multiple zonal jet flow can be observed on Jupiter’s surface. Despite attracting many researchers over the years, its mechanism has not yet been well understood. The incompressible 2D-Navier–Stokes equations on a rotating sphere form one of the simplest models of this flow. Williams [24] was the first to find that turbulent flow evolves into multiple jet flows for such models. However, he assumed a high degree of symmetry for the flow field. Later, Yoden and Yamada [26] and Nozawa and Yoden [14] made further progress. In particular, Obuse, Takehiro, and Yamada [15] calculated non-forced 2D-Navier–Stokes flow (without symmetry assumptions) on a rotating sphere, observing multiple zonal jet flows merging with each other, and finally, obtaining only two or three broad zonal jets remaining in the overall flow (see [16–18] for further progress). Therefore, it seems that we need to develop totally different ideas to clarify the existence of stable multiple zonal jet flow on the sphere. For recent developments in this field, see the work of Sasaki, Takehiro, and Yamada [19, 20], who apply a spectral method to the linearized fluid equations. For a mathematical analysis on the resonant interaction of Rossby waves on a beta plane (approximating the curved surface as a 2D plane), see for example [7, 25].

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However, as far as the authors are aware, few mathematical studies have attempted to investigate the effect of the Coriolis force on a sphere (see [3] for one of the few), and hence our motivation for the present mathematical study becomes clarifying the mechanism underlying the Coriolis force on various manifolds. Let us explain more precisely. The Euler equations for incompressible flow on a two-dimensional sphere $S^2$ are expressed as follows:

$$\begin{align*}
\frac{\partial u}{\partial t} + \nabla u \cdot u &= -\text{grad} \ p, \\
\text{div} \ u &= 0, \\
u|_{t=0} &= u_0.
\end{align*} \tag{1}$$

We drop the viscosity term because we only focus on the stability of the largest-scale zonal flow with large-scale perturbation. This is considered the simplest model for Jupiter (if we also add the Coriolis force). However, rigorously, Jupiter is not a sphere; a perceptible bulge appears around its equatorial middle and the poles are flattened (see [4] and [21, Table 4]).

As a first step, we studied equation (1) (without the Coriolis force) not only on $S^2$ but also on a two-dimensional manifold $M = M_s$, which is defined by

$$M_s := \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = s^2(1-z^2) \} \tag{2}$$

for $s \geq 1 ([22])$. Note that $M_s$ is an ellipsoid if $s > 1$ and $S^2$ if $s = 1$. As for the sequence of study, in this paper, we investigate the effect of the Coriolis force using the same method in [22], namely, the Arnol’d method, which we outline below.

Let $G$ be a (infinite-dimensional) Lie group and $\mathfrak{g}$ its Lie algebra with an inner product $\langle , \rangle$. By right translation, we extend $\langle , \rangle$ to a right-invariant metric on $G$. Then, the geodesic equation on $G$ is equivalent to the differential equation on $\mathfrak{g}$ called the Euler–Arnol’d equation [23] (see Appendix A):

$$u_t = [u, u]^*, \tag{3}$$

where $u$ denotes a curve on $\mathfrak{g}$, $[ , ]$ the Lie bracket of $\mathfrak{g}$, and $[ , ]^*$ is defined by

$$\langle [X, Y], Z \rangle = \langle Y, [X, Z]^* \rangle \quad \text{for } X, Y, Z \in \mathfrak{g} \tag{4}$$

if it exists. For certain $G$, it is known that this differential equation becomes a physically and mathematically important differential equation. For example, if $G$ is the group of volume preserving diffeomorphisms $D_\mu(M)$ of a compact Riemannian manifold $M$, (3) is the incompressible Euler equation (1), which was discovered by Arnol’d [1]. Moreover, (3) is the KdV equation if $G$ is the Bott–Virasoro group $\widehat{D}(S^1)$, which is the central extension of the diffeomorphism group $D(S^1)$ of the unit circle (e.g., see [13, 23]).

Therefore, the geometry of a group with a right-invariant metric has attracted much attention. In particular, its sectional curvature is important because the structure of geodesics is closely related to it. For example, Misiołek showed that the non-positivity of the sectional curvature of $G$ implies the Lagrangian instability of solutions in [11, Lem. 4.2]. The explicit calculation of the curvature appears in [1, 9, 13, 27] where $G$ is $D_\mu(M)$ for some $M$.

Given the direction of this study, the existence of a conjugate point on $G$ also has attracted some attention because in a Lagrangian sense it can be thought of as the stability of the corresponding solution of the Euler–Arnol’d equation. Indeed, the existence of a conjugate
point \( p \in G \) on a geodesic \( \gamma(t) \) means that a family of geodesics that despite initially diverging have almost the same initial value of \( \gamma \) converge to \( p \) at some later moment.

**Definition 1.1** (Conjugate points). Let \( D \) be a Riemannian manifold and \( \eta(t) := \exp_p(tV) \) a geodesic for some \( V \in T_p D \), for which \( \exp_p : T_p D \to D \) is the exponential map at \( p \in D \). Then, we say that \( \eta(1) \) is a **conjugate point** or **conjugate** to \( p \) along \( \eta \) if the differential \( T_V \exp_p : T_V(T_p D) \to T_{\eta(1)} D \) of the exponential map at \( p \) is not bijective. (For \( \dim D = \infty \), there are two reasons for a point to be conjugate to another; see [6].)

We note that, in principle, greater positivity of the sectional curvature on \( G \) creates a conjugate point, and greater negativity implies the absence of a conjugate point. For a study of the existence of conjugate points, see for example [5, 11–13].

In [12], Misiołek calculated the second variation of a geodesic corresponding to a certain stationary solution \( X \) of (3) and showed the existence of a conjugate point along it if \( G \) is the diffeomorphism group of the flat torus \( T^2 \). Moreover, he also revealed the importance of the value

\[
MC_{X,Y}^g := -||[X,Y]||^2 - \langle X, [[X,Y], Y] \rangle
\]

where \( Y \in g \). Specifically, he essentially proved Fact 2.1, which states that the \( MC_{X,Y}^g > 0 \) ensures the existence of a conjugate point on \( G \). We call this important value \( MC_{X,Y}^g \) the **Misiołek curvature** and study when it is positive or otherwise.

In this article, we calculate the Misiołek curvature for the incompressible Euler equation with the Coriolis force \( az (a > 0) \) on \( M_s \):

\[
\frac{\partial u}{\partial t} + \nabla_u u = az \ast (u) - \text{grad} p, \\
\text{div} u = 0, \\
u|_{t=0} = u_0,
\]

where \( \ast \) denotes the Hodge operator. In this case, the solutions correspond to geodesics on the central extension \( \hat{D}_\mu(M_s) \) of the group of volume-preserving diffeomorphisms of \( M_s \) (see Section 3), for which the Lie algebra is identified with \( g \oplus \mathbb{R} \). Our main result is the positivity of the Misiołek curvature \( MC_{X,Y}^{g\oplus\mathbb{R}} \) with respect to a west-facing zonal flow:

**Definition 1.2.** We call a vector field \( Z \) on \( M_s \) a **zonal flow** if \( Z \) has the form

\[
Z = F(z)(x\partial_y - y\partial_x)
\]

for some function \( F \). Moreover, if \( F \leq 0 \), we call \( Z \) a **west-facing** zonal flow.

Note that the definition of the west-facing zonal flow is just for simplicity. See Corollary 4.3. Actually, we just need the positivity of the second term in the right-hand side of (47). This means that we can generalize Theorem 1.3 to cover oscillatory situations. Here, we say that a zonal flow \( Z = F(r)\partial_\theta \) is oscillating if the sets \( \{ r \mid F(r) > 0 \} \) and \( \{ r \mid F(r) < 0 \} \) are nonempty. For example, Theorem 1.3 is still true for any oscillating zonal flow \( Z = F(r)\partial_\theta \) if we only consider perturbations \((Y,b)\) that are sufficiently large on the set \( \{ r \mid F(r) < 0 \} \) and \( |Y(r)| \ll 1 \) on \( \{ r \mid F(r) > 0 \} \). We also note that any zonal flow is a stationary solution of (1) and (6). Then, our main results are the following:
Theorem 1.3. Let $Z$ be a nonzero west-facing zonal flow and $a \in \mathbb{R}_{>0}$. Then we have

$$MC_{(Z,a),(Y,b)} > MC_{Z,Y}$$

for any $(Y,b) \in \mathfrak{g} \oplus \mathbb{R}$.

Corollary 1.4. Suppose $s > 1$. Let $Z$ be a nonzero west-facing zonal flow for which support is contained in $M_s \setminus \{(0,0,\pm1)\}$ and $a \in \mathbb{R}_{>0}$. Then, there exists $Y \in \mathfrak{g}$ satisfying $MC_{(Z,a),(Y,b)} > 0$ for any $b \in \mathbb{R}$.

The theorem states that for any west-facing zonal flow $Z$, the Misiolek curvature of $Z$ regarded as a solution of (6) is greater than the Misiolek curvature regarded as a solution of (1), which can be understood as the stability of $Z$ under Coriolis force. We note that the positivity of the Misiolek curvature directly implies the positivity of the sectional curvature on the corresponding group (see Definition B.4 and Lemma B.6 in Appendix). Therefore, the corollary implies the positivity of the sectional curvature on $\hat{D}_\mu(M_s)$ under some support condition.

2. Misiolek curvature

We next define the Misiolek curvature and explain its importance. We refer to [12, 22].

Let $G$ be an (infinite-dimensional) Lie group with right-invariant metric $\langle \cdot, \cdot \rangle$, and $\mathfrak{g}$ the Lie algebra of $G$. Then, we define the Misiolek curvature $MC_{X,Y} := MC_{X,Y}^\theta$ by

$$MC_{X,Y} := -||[X,Y]||^2 - \langle X,[[X,Y],Y]\rangle.$$  \hfill (9)

The initial importance of this value is that the positivity of $MC$ directly implies that of the curvature (see Definition B.4 and Lemma B.6 in Appendix). Note that this formula for $MC$ seems to be simpler than the general formula of the curvature on the group with right-invariant metric (see Lemma B.2).

The main importance of $MC$ is Fact 2.1 given below. In [12], this fact is proved for $G$ being the group $D^s_\mu(T^2)$ of volume-preserving $H^s$-diffeomorphisms of the 2-dimensional flat torus $T^2$. (For $D^s_\mu(M)$, where $M$ is a compact $n$-dimensional Riemannian manifold, see also [22].) The essential point of the proof in [12] is that the inverse function theorem holds for the Riemannian exponential map $\exp : T_e D^s_\mu(M) \to D^s_\mu(M)$. Here, we say that the inverse function theorem holds for $\exp$ if $\exp$ is an isometry near $X \in T_e D^s_\mu(M)$ whenever the differential of $\exp$ is an isomorphism at $X$. Therefore, we obtain the following.

Fact 2.1. Suppose that there exists the (Riemannian) exponential map $\exp : \mathfrak{g} \to G$ and the inverse function theorem holds for $\exp$. Let $X \in \mathfrak{g}$ be a stationary solution of the Euler–Arnol’d equation. Suppose that there exists $Y \in \mathfrak{g}$ satisfying $MC_{X,Y} > 0$. Then, there exists a point conjugate to the identity element $e \in G$ along the geodesic corresponding to $X$ on $G$.

Fact 2.1 states that the positivity of the Misiolek curvature ensures the existence of a conjugate point.

3. Central extension of volume-preserving diffeomorphism group

We now briefly recall basics regarding the central extension of the volume-preserving diffeomorphism group by a Lichnerowicz cocycle. Our main references are [11, 23].
Let \((M, g)\) be a compact \(n\)-dimensional Riemannian manifold and \(\mathcal{D}_\mu(M)\) the group of volume-preserving \(C^\infty\)-diffeomorphisms with \(L^2\)-metric
\[
\langle X, Y \rangle := \int_M g(X,Y)\mu,
\]
where \(\mu\) is the volume form. We write \(g\) for the space of divergence-free vector fields on \(M\), which is identified with the tangent space of \(\mathcal{D}_\mu(M)\) at the identity element.

For a closed 2-form \(\eta\), we define a Lichnerowicz 2-cocycle \(\Omega\) on \(g\) by
\[
\Omega(X,Y) := \int_M \eta(X,Y).
\]
If \(\eta \in H^2(M, \mathbb{Z}) \subset H^2(M, \mathbb{R})\), this cocycle integrates the group \(\mathcal{D}^{ex}_\mu(M)\) of exact volume preserving diffeomorphism. This group coincides with the identity component of \(\mathcal{D}_\mu(M)\) if \(H^{n-1}(M, \mathbb{R}) = 0\). Thus, in this case, there exists a central extension \(\hat{\mathcal{D}}_\mu(M)\) of the identity component of \(\mathcal{D}_\mu(M)\), the tangent space at the identity being \(g \oplus \mathbb{R}\) and its Lie bracket and inner product are given by
\[
\langle (X,a), (Y,b) \rangle = \langle [X,Y], \Omega(X,Y) \rangle + ab,
\]
Consider a \((n-2)\)-form \(B\) satisfying \(\eta = \iota_B(\mu)\) or, equivalently,
\[
\Omega(X,Y) = \int_M \eta(X,Y)\mu = \int_M \mu(B,X,Y) = \int_M g(B \times X,Y),
\]
where \(B \times X := \ast(B \wedge X)\). Note that this can be rewritten as
\[
\Omega(X,Y) = \langle P(B \times X), Y \rangle,
\]
where \(P : \mathfrak{X}(M) \to g\) is the projection to the divergence-free part. Then, the Euler–Arnol’d equation of \(\hat{\mathcal{D}}_\mu(M)\) is
\[
\frac{\partial u}{\partial t} = -\nabla_u u + au \times B - \text{grad} p.
\]

Remark 3.1. This formula is slightly different from [23] because the sign convention differs. To clarify, we present our conventions in Appendix.

We state the content of this section setting \(\dim M = 2\).

**Proposition 3.2.** Suppose that \(\dim M = 2\), \(\eta \in H^2(M, \mathbb{Z})\) and \(H^1(M, \mathbb{R}) = 0\). Then, there exists a central extension group \(\hat{\mathcal{D}}_\mu(M)\) of the identity component of \(\mathcal{D}_\mu(M)\), the Euler–Arnol’d equation for which is
\[
\frac{\partial u}{\partial t} = -\nabla_u u + au \times B - \text{grad} p.
\]
Moreover, the Misiołek curvature \(MC_{(X,a),(Y,b)} := MC_{g \oplus \mathbb{R}}^{g \oplus R}(X,a),(Y,b)\) is given by
\[
MC_{(X,a),(Y,b)} = -||[X,Y]||^2 - \langle X, [[X,Y],Y] \rangle - \Omega(X,Y)^2 - a\Omega([X,Y],Y)
\]
\[= MC_{X,Y} - \Omega(X,Y)^2 - a\Omega([X,Y],Y)
\]
for \((X,a), (Y,b) \in g \oplus \mathbb{R}\).

**Proof.** Note that \(u \times B = u \ast B\) if \(\dim M = 2\). Moreover, the assertion of the Misiołek curvature follows from definition, (12) and (13).
4. \( M = M_s \) case

We apply the results of Section 3 with \( M = M_s \). Recall

\[
M_s := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = s^2(1 - z^2)\}.
\]

**Proposition 4.1.** Let \( B := z \) and \( \eta := z\mu \). Then, there exists a group \( \hat{D}_\mu(M_r) \) for which the Euler–Arnold equation is (17).

**Proof.** Note that \( \eta = 0 \in H^2(M_s, \mathbb{Z}) \) because

\[
\int_{M_s} z\mu = 0.
\]

Thus, the proposition follows from Proposition 3.2.\( \blacksquare \)

Consider a “spherical coordinate” of \( M_s \):

\[
\phi := \phi_s : (-d, d) \times (-\pi, \pi) \rightarrow M_s
\]

\[
(r, \theta) \mapsto (c_1(r) \cos \theta, c_1(r) \sin \theta, c_2(r))
\]

in such a way that \( c_2(0) = 0, c_1(r) > 0, \dot{c}_2(r) > 0 \), and that \( \dot{c}_1^2 + \dot{c}_2^2 = 1 \). Note that \( (c_1, c_2, d) = (\cos(r), \sin(r), \pi/2) \) for \( s = 1 \) (\( M_1 = S^2 \)). Then, we obtain

\[
g(\partial_r, \partial_r) = 1, \quad g(\partial_r, \partial_\theta) = 0, \quad g(\partial_\theta, \partial_\theta) = c_1^2
\]

and

\[
\mu = c_1(r)d\theta \wedge dr.
\]

This implies

\[
\star \partial_r = -\frac{\partial_\theta}{c_1}, \quad \star dr = -c_1 d\theta, \quad \star \partial_\theta = c_1 \partial_r, \quad \star d\theta = \frac{dr}{c_1},
\]

and

\[
\langle X, Y \rangle = \int_{-d}^{d} \int_{-\pi}^{\pi} (X_1 Y_1 + X_2 Y_2 c_1^2) c_1^2 d\theta dr
\]

for \( X = X_1 \partial_r + X_2 \partial_\theta \) and \( Y = Y_1 \partial_r + Y_2 \partial_\theta \), which are elements of \( \mathfrak{g} \). Moreover, we have

\[
\text{grad } f = \partial_r f \partial_r + c_1^{-2} \partial_\theta f \partial_\theta,
\]

\[
\text{div } u = (\partial_r + c_1^{-1} \partial_r c_1) u_1 + \partial_\theta u_2
\]

for a function \( f \) on \( M \) and \( u = u_1 \partial_r + u_2 \partial_\theta \). Recall that we call a vector field \( Z \) on \( M_s \) a zonal flow if \( Z \) has the form

\[
Z = F(r) \partial_\theta,
\]

for some function depending only on variable \( r \). Moreover, if \( F \leq 0 \), we call \( Z \) a west-facing zonal flow.

**Lemma 4.2.** Let \( Z = F(r) \partial_\theta \) be a zonal flow. Then, we have

\[
\Omega(Z, Y) = 0
\]

\[
\Omega(Y, [Y, Z]) = \int_{-d}^{d} \int_{-\pi}^{\pi} c_1^2 Y_1^2 F \partial_r c_2 dr d\theta.
\]

for \( Y = Y_1 \partial_r + Y_2 \partial_\theta \in \mathfrak{g} \).
Proof. Recall that $B = z = c_2(r)$ and that
\[ \Omega(Z, Y) = \langle P(B \times Z), Y \rangle = \langle P(B \ast Z), Y \rangle. \tag{31} \]
The last equality follows with \(\text{dim } M = 2\). In contrast,
\[ B \ast Z = c_1(r)c_2(r)F(r)\partial_r. \tag{32} \]
This expression implies the existence of function \(f\) satisfying \(\text{grad } f = B \ast Z\). Therefore, we have \(P(B \ast Z) = 0\), which implies the first equality.

For the second equality, we have
\[ [Y, Z] = -F\partial_y Y_1\partial_r + (Y_1\partial_r F - F\partial_y Y_2)\partial_\theta \tag{33} \]
\[ \ast BY = B\left(c_1 Y_2\partial_r - \frac{Y_1}{c_1}\partial_\theta\right). \tag{34} \]
Moreover,
\[ \Omega(Y, [Y, Z]) \]
\[ = \langle P(\ast BY), [Y, Z] \rangle \]
\[ = \langle \ast BY, [Y, Z] \rangle \]
\[ = \int_{-\pi}^{\pi} \int_{-d}^{d} B(-Fc_1 Y_2\partial_y Y_1 - c_1 Y_1(Y_1\partial_r F - F\partial_y Y_2)) c_1 drd\theta \tag{38} \]
\[ = -\int_{-\pi}^{\pi} \int_{-d}^{d} B\left(c_1^2 Y_2^2\partial_r F - 2c_1^3 Y_1 F \partial_y Y_2 \right) drd\theta. \tag{39} \]
This is equal to
\[ = -\int_{-\pi}^{\pi} \int_{-d}^{d} BFC_1^2 \int_{-\pi}^{\pi} \partial_y(Y_1 Y_2) d\theta dr \tag{40} \]
\[ -\int_{-\pi}^{\pi} \int_{-d}^{d} B\left(c_1^2 Y_2^2\partial_r F - 2c_1^3 Y_1 F \partial_y Y_2 \right) drd\theta \tag{41} \]
\[ = -\int_{-\pi}^{\pi} \int_{-d}^{d} B\left(c_1^2 Y_2^2\partial_r F - 2c_1^3 Y_1 F \partial_y Y_2 \right) drd\theta. \tag{42} \]
Recall that
\[ \text{div } Y = \partial_r Y_1 + \frac{\partial_c c_1}{c_1} Y_1 + \partial_y Y_2. \]
Thus, a divergence-free \(Y\) implies
\[ \Omega(Y, [Y, Z]) \]
\[ = \int_{-\pi}^{\pi} \int_{-d}^{d} B\left(c_1^2 Y_2^2\partial_r F + 2c_1^3 Y_1 F \left(\partial_r Y_1 + \frac{\partial_c c_1}{c_1} Y_1\right)\right) drd\theta \tag{43} \]
\[ = \int_{-\pi}^{\pi} \int_{-d}^{d} B\left(c_1^2 Y_2^2\partial_r F + c_1^2 \partial_c Y_1^2 F + \partial_r (c_1^2 Y_1 F)\right) drd\theta. \tag{44} \]
By Stokes’ theorem, this is equal to
\[ = \int_{-\pi}^{\pi} \int_{-d}^{d} \partial_r B c_1^2 Y_2^2 F drd\theta. \tag{46} \]
This completes the proof.
**Corollary 4.3.** Let $Z = F(r)\partial_\theta$ be a zonal flow. Then, we have

$$MC(Z,a) = YF_{\partial_\theta} - a \int_{-d}^{d} \int_{-\pi}^{\pi} c_1^2 Y_1^2 F \sqrt{1 - c_1^2} dr d\theta$$

for $Y = Y_1 \partial_r + Y_2 \partial_\theta \in \mathfrak{g}$.

**Proof.** This is a consequence of (18), Lemma 4.2, and the definition of $c_1, c_2$. Note that $\Omega([Z,Y], Y) = \Omega(Y, [Y,Z])$ because $\Omega$ is a 2-cocycle. ■

**Proof of Theorem 1.3.** Corollary 4.3 and (18) imply the theorem. ■

**Proof of Corollary 1.4.** This follows from Theorem 1.3 and Fact 4.4. ■

**Fact 4.4 ([22, Thm. 1.2]).** Let $s > 1$. Then, for any zonal flow $Z$ on $M_s$ for which support is contained in $M_s \setminus \{(0,0,\pm 1)\}$, there exists $Y \in \mathfrak{g}$ satisfying $MC_{Z,Y} > 0$.

### 5. Final remark

Note that we do not know whether $\hat{D}_\mu(M_s)$—the existence of which is guaranteed by Proposition 4.1—satisfies the assumption of Fact 2.1. Therefore, we cannot conclude the existence of a conjugate point on $\hat{D}_\mu(M_s)$. This problem remains open.

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A. Appendix

A.1. Sign conventions

To clarify our sign conventions, we briefly derive formula (16). All content in this section is well known. We refer to [8, Section 46] or [23, Section 2].

A.1.1. Right-invariant Maurer–Cartan form. Let $G$ be a (possibly infinite-dimensional) Lie group and $\mathfrak{g}$ its Lie algebra.

Definition A.1. The right-invariant Maurer–Cartan form $\omega$ is the $\mathfrak{g}$-valued 1-form on $G$ defined by

$$\omega_g(X) := r_g^{-1}X \in \mathfrak{g},$$

(A1)

where $r_g^{-1}$ denotes the differential of the right translation map $R_{g^{-1}}(h) := h g^{-1}$.

For $X \in \mathfrak{g}$, we write $X^R$ for the right-invariant vector fields on $G$ with $X^R(e) = X$. Note that

$$[X^R, Y^R] = -[X, Y]^R.$$  

(A2)

Lemma A.2. For $X, Y \in \mathfrak{g}$, we have

$$\omega([X^R, Y^R]) = [\omega(X^R), \omega(Y^R)].$$  

(A3)

Proof. By (A2), we have

$$\omega([X^R, Y^R]) = \omega(-[X, Y]^R) = -[X, Y]^R = [X^R, Y^R] = [\omega(X^R), \omega(Y^R)],$$

(A4)

which completes the proof. ■

Lemma A.3. For smooth vector fields $U, V$ on $G$, we have

$$d\omega(U, V) = -[\omega(U), \omega(V)].$$  

(A5)
Proof. Recall that
\[ d\omega(U, V) = U(\omega(V)) - V(\omega(U)) - \omega([U, V]). \] (A6)
With \( U = X^R \) and \( V = Y^R \), we have
\[ d\omega(X^R, Y^R) = -\omega([X^R, Y^R]) \] (A7)
\[ = -[\omega(X^R), \omega(Y^R)]. \] (A8)
This equation describes \( \mathfrak{g} \)-valued 2-forms and, therefore, holds for any \( U \) and \( V \).

**Corollary A.4.** For smooth vector fields \( U, V \) on \( G \), we have
\[ [\omega(U), \omega(V)] = -U(\omega(V)) + V(\omega(U)). \] (A9)

Proof. This is obvious from the preceding lemma.

**A.1.2. Euler–Arnol'd equation.** Let \( G \) be a (possibly infinite-dimensional) Lie group with right-invariant metric \( \langle \cdot, \cdot \rangle \) and \( \mathfrak{g} = T_eG \) the Lie algebra. Define \( [\cdot, \cdot]^* \) by
\[ \langle [X, Y]^*, Z \rangle = \langle Y, [X, Z] \rangle, \] (A10)
if it exists. Hereafter, we always assume the existence of \( [\cdot, \cdot]^* \).

**Lemma A.5.** Let \( \eta \) be a curve on \( G \). Define a curve \( c: [0, t_0] \to \mathfrak{g} \) by \( c(t) := \omega(\dot{\eta}) - \dot{\eta} \). Then, \( \eta \) is a geodesic if and only if \( c \) satisfies
\[ \partial_t c = [c, c]^* \] (A11)
Moreover, for a curve \( c: [0, t_0] \to \mathfrak{g} \) satisfying (A11), there exists a geodesic \( \eta \) on \( G \) satisfying \( c(t) = r_{\eta^{-1}}(\dot{\eta}) \) if \( G \) is regular in the sense of [10, Def. 7.6].

Proof. Consider the energy function of a curve \( \eta \) on \( G \):
\[ E(\eta) = \frac{1}{2} \int_0^t \langle \dot{\eta}, \dot{\eta} \rangle dt \] (A12)
\[ = \frac{1}{2} \int_0^t \|r_{\eta^{-1}}(\dot{\eta})\| dt. \] (A13)
For a proper variation \( \eta_s \) of \( \eta \), define \( c_s(t) := r_{\eta^{-1}}(\dot{\eta}) = \omega(\dot{\eta}) \in \mathfrak{g} \). \( X_s(t) := \partial_s c_s(t) \), and \( x_s(t) := \omega(X_s) \) where \( \omega \) is the right-invariant Maurer–Cartan form. Then, the first variation is
\[ \partial_s E(\eta_s) = \int_0^t \langle c_s, \partial_s c_s \rangle dt. \] (A14)
Corollary A.4 implies
\[ \partial_s c_s = X_s(\omega(\dot{\eta})) \] (A15)
\[ = [\omega(\dot{\eta}), \omega(X_s)] + \dot{\eta}(\omega(X_s)) \] (A16)
\[ = [c_s, x_s] + \partial_t x_s. \] (A17)
Thus,
\[ \partial_s E(\eta_s) = \int_0^t \langle c_s, [c_s, x_s] + \partial_t x_s \rangle dt \] (A18)
\[ = \int_0^t \langle [c_s, c_s]^* - \partial_t c_s, x_s \rangle dt. \] (A19)
This completes the proof.
Definition A.6. We define the Euler–Arnol’d equation of $G$ by
\[ u_t = [u, u]^* . \] (A20)
for $u : [0, t_0] \rightarrow g$. In other words,
\[ u_t = - \text{ad}^*_u u , \] (A21)
where $\text{ad}_u v = - [u, v]$ and $\langle \text{ad}_u v, w \rangle = \langle v, \text{ad}^*_u w \rangle$.

A.1.3. Euler–Arnol’d equation of $\mathcal{D}_\mu(M)$. Let $\mathcal{D}_\mu(M)$ be the group of volume-preserving $C^\infty$-diffeomorphisms of a $n$-dimensional compact Riemannian manifold $(M, g)$ with right-invariant Riemannian metric
\[ \langle X, Y \rangle := \int_M g(X, Y)\mu, \] (A22)
where $X, Y \in g := T_e \mathcal{D}_\mu(M)$. Let $P : \mathfrak{X}(M) \rightarrow g$ be the projection onto the divergence-free part, where $\mathfrak{X}(M)$ is the space of vector fields.

Lemma A.7. For $X, Y \in g$, we have
\[ \nabla_X^M Y^R = -(P(\mathcal{N}^M_X Y))^R , \] (A23)
where $\nabla^M$ is the Levi-Civita connection on $M$.

Proof. The Koszul formula and the right-invariance imply
\[ 2\langle \nabla_X^M Y^R, Z^R \rangle \] (A24)
\[ = \langle [X^R, Y^R], Z^R \rangle - \langle [X^R, Z^R], Y^R \rangle - \langle [Y^R, Z^R], X^R \rangle \] (A25)
\[ = -\langle [X, Y]^R, Z^R \rangle + \langle [X, Z]^R, Y^R \rangle + \langle [Y, Z]^R, X^R \rangle \] (A26)
\[ = -\langle [X, Y], Z \rangle + \langle [X, Z], Y \rangle + \langle [Y, Z], X \rangle \] (A27)
This completes the proof using the Koszul formula on $M$ and the right-invariance of $\langle , \rangle$. ■

Lemma A.8. For $X \in g$, we have
\[ [X, X]^* = - P(\mathcal{N}^M_X X) . \] (A28)

Proof. By the Koszul formula, we have
\[ 2\langle \nabla_X^M X, Z \rangle \] (A29)
\[ = -2\langle \nabla_X X^R, Z^R \rangle \] (A30)
\[ = -\langle [X^R, X^R], Z^R \rangle + \langle [X^R, Z^R], X^R \rangle + \langle [X^R, Z^R], X^R \rangle \] (A31)
\[ = 2\langle X^R, [X^R, Z^R] \rangle \] (A32)
\[ = -2\langle X^R, [X, Z] \rangle \] (A33)
\[ = -2\langle X, [X, Z] \rangle . \] (A34)
This completes the proof. ■

Corollary A.9. The Euler–Arnol’d equation of $\mathcal{D}_\mu(M)$ is
\[ u_t = - \nabla^M_u u - \text{grad } p . \] (A35)
A.1.4. Lichnerowicz 2-cocycle. Let $D_\mu(M)$ be the groups of volume-preserving diffeomorphisms of a compact $n$-dimensional Riemannian manifold $M$, and $\mathfrak{g} := T_e D_\mu(M)$, which is identified with the space of divergence-free vector fields on $M$. For a closed 2-form $\eta$, define a skew-symmetric bilinear form $\Omega$ on the Lie algebra $\mathfrak{g}$ by

$$\Omega(X,Y) := \int_M \eta(X,Y). \tag{A36}$$

**Lemma A.10.** The form $\Omega$ defines a 2-cocycle on $\mathfrak{g}$. Specifically, it satisfies the "Jacobi identity"

$$\Omega([[X,Y],Z]) + \Omega([Z,X],Y) + \Omega([Y,Z],X) = 0 \tag{A37}$$

for any $X,Y,Z \in \mathfrak{g}$.

**Proof.** Because $\eta$ is closed, there exists a one-form $\alpha$ on $M$ such that $\eta = d\alpha$. Recall the formula for the exterior derivative of a 1-form,

$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]). \tag{A38}$$

This implies

$$\Omega([X,Y],Z) = \int_M ((X,Y)(\alpha(Z)) - Z(\alpha([X,Y])) - \alpha([[X,Y],Z])) \mu \tag{A39}$$

and

$$= \int_M \alpha([[X,Y],Z]) \mu. \tag{A40}$$

Note that the second equality follows from

$$\int_M X(f) \mu = \int_M g(\text{grad } f, X) \mu = 0 \tag{A42}$$

for any $f \in C^\infty(M)$ and a divergence-free vector field $X$. Thus, we have

$$\Omega([X,Y],Z) + \Omega([Z,X],Y) + \Omega([Y,Z],X) = \int_M (\alpha([[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X])) \mu \tag{A43}$$

and

$$= 0 \tag{A45}$$

by the Jacobi identity of $[,]$. This completes the proof. $\blacksquare$

This Lemma allows us to endow $\widehat{\mathfrak{g}} := \mathfrak{g} \oplus \mathbb{R}$ with a Lie algebra structure defined by

$$[[X,a),(Y,b)] := ([X,Y],\Omega(X,Y)). \tag{A46}$$

Take $B \in C^\infty(\wedge^{n-2}TM)$ satisfying $\iota_B \mu = \eta$. Then, we have

$$\Omega(X,Y) = \int_M \iota_B \mu(X,Y) = \int_M \mu(B,X,Y) = \int_M g(B \times X,Y) = \langle P(B \times X), Y \rangle, \tag{A47}$$

where $B \times X = *(B \wedge X)$ (see (A65) and Lemma A.18) and $P : \mathfrak{X}(M) \to \mathfrak{g}$ is the projection onto the divergence-free part.

**Lemma A.11.** For $(X,a), (Y,b) \in \widehat{\mathfrak{g}}$, we have

$$[(X,a),(Y,b)]^* = ([X,Y]^* + bP(B \times X), 0). \tag{A48}$$
Proof.

\[
\langle ([X, a], (Y, b))^*, (Z, c) \rangle = \langle (Y, b), ([X, a], (Z, c)) \rangle
\]

\[
= \langle (Y, b), ([X, Z], \Omega(X, Z)) \rangle
\]

\[
= \langle Y, [X, Z] \rangle + b\Omega(X, Z)
\]

\[
= \langle [X, Y]^*, Z \rangle + \langle bP(B \times X), Z \rangle.
\]

This completes the proof. \[\square\]

Thus, we have

**Theorem A.12.** The Euler–Arnol’d equation of \( \hat{g} \) is

\[
u_t = [u, u]^* + aP(B \times u)
\]

or, equivalently,

\[
u_t = -\nabla u + a(B \times u) - \text{grad} \ p.
\]

**A.1.5. Formulae on Riemannian manifold.** For the convenience of readers, we briefly review the formulae concerning Riemannian manifolds. Let \((M, g)\) be a \(n\)-dimensional Riemannian manifold and \(\mu\) the volume form. Write \(X^1(M)\) for the space of \(1\)-vector fields on \(M\), and \(E^1(M)\) for the space of \(1\)-forms on \(M\).

**Definition A.13.** Define \(\♭: X^1(M) \to E^1(M)\) and \(\♯: E^1(M) \to X^1(M)\) by

\[
V^\♭ := g(V, \cdot) \in E^1(M),
\]

\[
g(\alpha^\sharp, \cdot) = \alpha \in E^1(M)
\]

for \(V \in X^1(M)\) and \(\alpha \in E^1(M)\). We extend these isomorphisms to \(\♭: X^p(M) \to E^q(M)\) and \(\♯: E^p(M) \to X^q(M)\) for any \(p \in \mathbb{Z}\).

**Definition A.14.** Define \(\langle , \rangle_X: X^p(M) \otimes_{C^\infty(M)} X^p(M) \to C^\infty(M)\) by

\[
\langle V, W \rangle_X := \iota_V(W^\♭),
\]

where \(\iota\) is the interior derivative. Similarly, define \(\langle , \rangle_E: E^p(M) \otimes_{C^\infty(M)} E^p(M) \to C^\infty(M)\) by

\[
\langle \alpha, \beta \rangle_E := \iota_{\alpha^\sharp}(\beta).
\]

**Lemma A.15.** Let \(V, W \in X^1(M)\). Then, we have

\[
\langle V, W \rangle_X = g(V, W).
\]

**Proof.** By Definition A.13 and A.14, we have

\[
\langle V, W \rangle_X = \iota_V(W^\♭) = W^\♭(V) = g(W, V).
\]

This completes the proof. \[\square\]

**Definition A.16.** We define the Hodge star operator \(\ast: X^p(M) \to X^{n-p}(M)\) and \(\ast: E^p(M) \to E^{n-p}(M)\) by

\[
V \wedge \ast W = \langle V, W \rangle_X \mu^2 \quad \text{for any } V \in X^p(M)
\]

\[
\alpha \wedge (\ast \beta) = \langle \alpha, \beta \rangle_E \mu \quad \text{for any } \alpha \in E^p(M).
\]
Note that
\[ \star^2 \alpha = (-1)^{n-1} \alpha \quad \text{for } \alpha \in \mathcal{E}^1(M). \quad (A63) \]
Moreover, applying $\mu$ to (A61), we have
\[ \mu(V \wedge \star W) = \langle V, W \rangle_X. \quad (A64) \]

**Definition A.17.** For $X \in \mathcal{X}^1(M)$ and $B \in \mathcal{X}^{n-2}(M)$, define
\[ B \times X := \star (B \wedge X) \in \mathcal{X}^1(M). \quad (A65) \]

**Lemma A.18.** For $X, Y \in \mathcal{X}^1(M)$ and $B \in \mathcal{X}^{n-2}(M)$, we have
\[ \int_M \iota_B(\mu)(X, Y) \mu = \int_M g(B \times X, Y) \mu. \quad (A66) \]

**Proof.** By definition, we have
\[ \int_M \iota_B(\mu)(X, Y) \mu = \int_M \mu(B, X, Y) \mu. \quad (A67) \]
Alternatively,
\[ \int_M g(B \times X, Y) \mu = \int_M \langle Y, B \times X \rangle \mu \]
\[ = \int_M \langle Y, \star (B \wedge X) \rangle \mu \]
\[ = \int_M \mu(Y \wedge \star^2 (B \wedge X)) \mu \]
\[ = (-1)^{n-1} \int_M \mu(Y \wedge B \wedge X) \mu \]
\[ = \int_M \mu(B, X, Y) \mu. \quad (A72) \]
The third and fifth equalities follow from (A63) and (A64), respectively. \[\blacksquare\]

**B. Appendix II**

**B.1. Misiolek curvature**

We briefly recall properties of the Misiolek curvature. We refer the reader to [12, 22].

**B.1.1. Curvature for a group with right-invariant metric.** We note certain formulae concerning a group with a right-invariant metric. All content in this section is known.

Let $G$ be a (possibly infinite-dimensional) Lie group with a right-invariant metric $\langle \cdot, \cdot \rangle$, and $\mathfrak{g}$ its Lie algebra. Define
\[ \langle [X, Y], Z \rangle = \langle Y, [X, Z]^* \rangle. \quad (B1) \]

**Lemma B.1.** For $X, Y \in \mathfrak{g}$, we have
\[ 2 \nabla_{X^R} Y^R = (-[X, Y] + [X, Y]^* + [Y, X]^*)^R \]
where $\nabla$ is the right-invariant Levi-Civita connection on $G$. In particular,
\[ \nabla_{X^R} X^R = [X, X]^* R. \quad (B3) \]
Proof. The Koszul formula and right-invariance imply
\[
2\langle \nabla_X Y^R, Z^R \rangle
\]
\[
= \langle [X^R, Y^R], Z^R \rangle - \langle [X^R, Z^R], Y^R \rangle - \langle [Y^R, Z^R], X^R \rangle
\]
\[
= -\langle [X, Y]^R, Z^R \rangle + \langle [X, Z]^R, Y^R \rangle + \langle [Y, Z]^R, X^R \rangle
\]
\[
= -\langle [X, Y], Z \rangle + \langle [X, Z], Y \rangle + \langle [Y, Z], X \rangle
\]
\[
= \langle -[X, Y] + [X, Y]^* + [Y, X]^*, Z \rangle.
\]
This completes the proof. ■

For simplicity, put
\[
A^\pm(X, Y) = [X, Y]^* \pm [Y, X]^*.
\]
Define
\[
R(X^R, Y^R) = \nabla_X \nabla_{Y^R} - \nabla_{Y^R} \nabla_X - \nabla_{[X,Y]^R}.
\]

Lemma B.2 ([2, Thm. 2.1 in IV. §2]). For \( X, Y \in \mathfrak{g} \), we have
\[
4\langle R(X^R, Y^R)Y^R, X^R \rangle
\]
\[
= -4\langle [Y, Y]^*, [X, X]^* \rangle + ||A^+(X, Y)||^2 - 3||[X, Y]||^2 - 2\langle [X, Y], A^-(X, Y) \rangle.
\]

Proof. By the properties of the Levi-Civita connection and right-invariance, we have
\[
4\langle \nabla_X \nabla_{Y^R}Y^R, X^R \rangle
\]
\[
= 4\langle \nabla_X \nabla_{Y^R}Y^R, X^R \rangle - 4\langle \nabla_{Y^R}Y^R, \nabla_X X^R \rangle
\]
\[
= -4\langle [Y, Y]^R, [X, X]^*R \rangle.
\]

Similarly,
\[
-4\langle \nabla_{Y^R} \nabla_X Y^R, X^R \rangle
\]
\[
= -4\langle \nabla_{Y^R} \nabla_X Y^R, X^R \rangle + 4\langle \nabla_X Y^R, \nabla_{Y^R} X^R \rangle
\]
\[
= \langle (-[X, Y] + A^+(X, Y))^R, (-[Y, X] + A^+(X, Y))^R \rangle
\]
\[
= ||A^+(X, Y)||^2 - ||[X, Y]||^2.
\]

Moreover,
\[
-4\langle \nabla_{[X,Y]^R}Y^R, X^R \rangle
\]
\[
= 4\langle \nabla_{[X,Y]^R}Y^R, X^R \rangle
\]
\[
= 2\langle ([X, Y], Y^*) + [X, Y]^* + [Y, [X, Y]^*], X \rangle
\]
\[
= 2\langle [X, Y], [Y, X]^* \rangle + 2\langle [Y, [X, Y]], X \rangle + 2\langle [X, Y], [Y, X] \rangle
\]
\[
= 2\langle [X, Y], [Y, X]^* \rangle - [X, Y]^* - 2\langle [X, Y] \rangle^2
\]
\[
= -2\langle [X, Y], A^-(X, Y) \rangle - 2\langle [X, Y] \rangle^2.
\]
This completes the proof. ■
B.1.2. Definition of Misiolek curvature. Let $G$ be a (possibly infinite-dimensional) Lie group with a right-invariant metric $\langle \cdot, \cdot \rangle$ and Lie algebra $\mathfrak{g}$. Define
\[
\langle [X, Y], Z \rangle = \langle Y, [X, Z]^* \rangle, 
\]
\[
R^{X^R}(Y^R) = \nabla_{X^R}Y^R - \nabla_{Y^R}X^R - \nabla_{[X^R, Y^R]}^R, 
\]
\[
A^\pm(X, Y) = [X, Y]^* \pm [Y, X]^*. 
\]
Then, we have
\[
2\nabla_{X^R}Y^R = -[X, Y]^R + A^+(X, Y)^R. 
\]

Note that $[X, Y]^* = 0$ implies that $X$ is a stationary solution of the Euler–Arnol’d equation of $G$.

Lemma B.3. Let $X \in \mathfrak{g}$ satisfying $[X, X]^* = 0$ and $\eta$ a geodesic corresponding to $X \in T_eG$. For $Y \in \mathfrak{g}$ and $f \in C^\infty(G)$ with $f(e) = f(t_0) = 0$ for some $t_0 > 0$, define a vector field $\tilde{Y}$ along $\eta$ by $\tilde{Y}(\eta(t)) = f(t) \cdot Y^R(\eta(t))$. Then, the second variation $E''(\eta)$ of the energy function of $\eta$ is
\[
E''(\eta)(\tilde{Y}, \tilde{Y}) = \int_0^{t_0} \left( \dot{f}^2 \|Y\|^2 - f^2 \langle R(X^R, Y^R)Y^R, X^R \rangle - \|\nabla_{X^R}Y^R\|^2 \right) dt 
\]
where $\dot{f} := X^R f$.

Proof. Note that, by assumption, $\dot{\eta} = X^R$. The general formula for the second variation of the energy function implies
\[
E''(\eta)(\tilde{Y}, \tilde{Y}) = \int_0^{t_0} \|\nabla_{X^R}Y\|^2 dt - \int_0^{t_0} \langle R(X^R, \tilde{Y})\tilde{Y}, X^R \rangle dt. 
\]
For the first term, we have
\[
\nabla_{X^R}\tilde{Y} = \dot{f}Y^R + f \nabla_{X^R}Y^R 
\]
Thus,
\[
\|\nabla_{X^R}\tilde{Y}\|^2 = \dot{f}^2 \|Y\|^2 + 2f \dot{f} \langle Y, \nabla_{X^R}Y^R \rangle + f^2 \|\nabla_{X^R}Y^R\|^2. 
\]
Then, $\langle Y^R, \nabla_{X^R}Y^R \rangle = -\langle \nabla_{X^R}Y^R, Y^R \rangle$ implies
\[
\|\nabla_{X^R}\tilde{Y}\|^2 = \dot{f}^2 \|Y\|^2 + f^2 \|\nabla_{X^R}Y^R\|^2. 
\]
In addition,
\[
\langle R(X^R, \tilde{Y})\tilde{Y}, X^R \rangle = f^2 \langle R(X^R, Y^R)Y^R, X^R \rangle. 
\]
This completes the proof.

Definition B.4. Let $X \in \mathfrak{g}$ satisfying $[X, X]^* = 0$. Define the Misiolek curvature $MC_{X,Y} := MC_{X,Y}^G$ by
\[
MC_{X,Y} = \langle R(X^R, Y^R)Y^R, X^R \rangle - \|\nabla_{X^R}Y^R\|^2 
\]
for $Y \in \mathfrak{g}$.
Theorem B.5. Let $X \in \mathfrak{g}$ satisfying $[X, X]^* = 0$ and $\eta$ a geodesic corresponding to $X \in T_e G$. Suppose $Y \in \mathfrak{g}$ satisfies $MC_{X,Y} > 0$. For $s > 0$, define

$$t_s := \pi ||Y|| \sqrt{\frac{s}{MC_{X,Y}}} \in \mathbb{R}_{>0} \quad (B36)$$

$$f_s(t) := \sin \left( \frac{t}{||Y||} \sqrt{\frac{MC_{X,Y}}{s}} \right) \in C^\infty(\mathbb{R}_{\geq 0}) \quad (B37)$$

and a vector field $\tilde{Y}$ along $\eta$ by $\tilde{Y}(\eta(t)) = f_s(t) Y(\eta(t))$. Then, the second variation $E''$ of the energy function of $\eta$ is

$$E''(\eta)(\tilde{Y}, \tilde{Y}) = \frac{\pi}{2} (1 - s) ||Y|| \sqrt{\frac{MC_{X,Y}}{s}}. \quad (B38)$$

In particular, $E''(\eta)(\tilde{Y}, \tilde{Y}) < 0$ if $0 < s < 1$.

Proof. For simplicity, set $M := MC_{X,Y}$. By Lemma B.3, we have

$$E''(\eta)(\tilde{Y}, \tilde{Y}) = \int_0^{t_e} \left( f_s^2 ||Y||^2 - f_s^2 M \right) dt \quad (B39)$$

$$= \int_0^{t_e} \left( \frac{M}{s} \cos^2 \left( \frac{t}{||Y||} \sqrt{\frac{M}{s}} \right) - M \sin^2 \left( \frac{t}{||Y||} \sqrt{\frac{M}{s}} \right) \right) dt \quad (B40)$$

$$= M \int_0^\pi \left( \frac{1}{s} \cos^2(x) - \sin^2(x) \right) ||Y|| \sqrt{\frac{s}{M}} dx \quad (B41)$$

$$= \frac{\pi}{2} (1 - s) ||Y|| \sqrt{\frac{M}{s}}. \quad (B42)$$

This completes the proof. ■

Lemma B.6. Let $X \in \mathfrak{g}$ satisfying $[X, X]^* = 0$ and $Y \in \mathfrak{g}$. Then, we have

$$MC_{X,Y} = -||[X,Y]||^2 - \langle [[X,Y],Y],X \rangle. \quad (B43)$$

Proof. By Lemma B.2, we have

$$4(R(X^R,Y^R)Y^R,X^R) = ||A^+(X,Y)||^2 - 3||[X,Y]||^2 - 2\langle [X,Y], A^+(X,Y) \rangle. \quad (B44)$$

Alternatively,

$$-4||\nabla_X^e Y^R||^2 \quad (B45)$$

$$= -|| - [X,Y] + A^+(X,Y)|| \quad (B46)$$

$$= -||[X,Y]||^2 + 2\langle [X,Y], A^+(X,Y) \rangle - ||A^+(X,Y)||^2. \quad (B47)$$

This completes the proof. ■
B.1.3. Misiołek curvature of $\mathcal{D}_\mu(M)$. Let $\mathcal{D}_\mu(M)$ be the group of volume-preserving $C^\infty$-diffeomorphisms of a compact $n$-dimensional manifold $M$ with the $L^2$ right-invariant metric:

$$
\langle X, Y \rangle := \int_M g(X, Y) \mu.
$$

(B49)

Here $X, Y \in \mathfrak{g} = T_eG$, which is identified with the space of divergence-free vector fields.

**Lemma B.7.** Let $X \in \mathfrak{g}$ satisfying $[X, X]^* = 0$ and $Y \in \mathfrak{g}$. Then,

$$
MC_{X,Y} = \langle \nabla^M_X [X, Y] + \nabla^M_{[X,Y]}X, Y \rangle,
$$

(B50)

where $\nabla^M$ is the Levi-Civita connection on $M$.

**Proof.** By the Koszul formula, we have

$$
2\langle \nabla^M_X [X, Y], Y \rangle = \langle [[X, X], Y], Y \rangle - \langle [X, Y], [X, Y] \rangle - \langle [[X, Y], Y], X \rangle,
$$

(B51)

$$
2\langle \nabla^M_{[X,Y]}X, Y \rangle = \langle [[X, Y], X], Y \rangle - \langle [[X, Y], Y], X \rangle - \langle [X, Y], [X, Y] \rangle.
$$

(B52)

Thus, Lemma B.6 implies the lemma.  

\[\square\]