Hidden Symmetries in Second Class Constrained Systems - Are New Fields Necessary?

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Abstract

For many systems with second class constraints, the question posed in the title is answered in the negative. We prove this for a range of systems with two second class constraints. After looking at two examples, we consider a fairly general proof. It is shown that, to unravel gauge invariances in second class constrained systems, it is sufficient to work in the original phase space itself. Extension of the phase space by introducing new variables or fields is not required.

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1 Introduction

The conversion of systems with second class constraints into those with first class ones has been of interest in recent times. Since first class constraints are generators of gauge transformations such conversions are useful in having a better and more illuminating view of second class constrained systems. The unearthing of inherent gauge symmetries, implied by the modification into first class constraints, allows a broader study of the system, in contrast to the limited view offered by the original second class constraints.

The basic premise behind such a conversion is that the second class constrained system is considered to be a gauge fixed version of a gauge theory; the latter goes back to the former under a certain set of gauge fixing conditions. The advantage in having a gauge theory lies in the fact that other gauges can also be considered, sometimes more profitably. Further such conversions into gauge theories can result in more than one gauge theory for the same second class system, with some gauge theories being more relevant than the others. This also raises the interesting possibility of (many) inequivalent gauge - fixed versions for the same gauge theory.

The motivation for this conversion into gauge theories came originally from anomalous gauge theories. In these theories the classical gauge invariance is lost upon quantisation. In terms of constraints, this means the classical first class constraints become second class upon quantisation. In this context, the conversion to gauge theories would mean recovering the lost gauge invariance.

There are basically two ideas proposed to convert second class constrained systems into gauge theories. One idea, proposed by Faddeev and Shatashvili [1], uses an enlarged phase space; the other, given in [2], is confined to the original phase space itself. Both ideas are based on the possibility that a system with second class constraints can be considered to be a gauge - fixed version of some gauge theory.

Based on these two ideas, methods have been developed and applied to realise hidden symmetries in various systems. While the Batalin - Fradkin method [3] follows the Faddeev - Shatashvili idea, the Gauge Unfixing method of [2, 4] uses the original phase space. There are other related methods too; while the one given by Wotzasek [5] uses an extended phase space, the method of Bizdadea and Saliu [6] is developed in the original phase space, with BRST quantisation.
Even though the Batalin - Fradkin and the Gauge Unfixing formulations appear quite different, when applied to various systems they give essentially the same results! The first class constraints may look different, but relevant observables obtained by demanding their gauge invariance in both the methods are essentially the same. In this context we refer to [7], which compares results of the two methods applied to the chiral Schwinger model, the Proca model and abelian Chern-Simons theory. For these theories it was found that, in both classical and path integral context, the gauge invariant Hamiltonians and the actions obtained using both methods are the same! Hence, as far as these systems were concerned, an enlarged phase space was found to be not really necessary to obtain the hidden gauge invariances.

In this paper, we pursue this matter further and compare the two methods in a more general context. As a first step towards demonstrating this formal equivalence, we consider two examples and apply and compare the two methods. For the general case, to simplify matters, we consider only two second class constraints. We will see that even in a fairly general context, the two methods when compared on their respective first class constrained surfaces give equivalent results. Hence we show that, contrary to widely accepted belief, extra fields are not really necessary for inducing gauge symmetries in second class constrained systems.

In Section 2 we review the two methods for the case of two second class constraints. In Section 3 we look at two specific systems, the chiral Schwinger model and the non-linear sigma model. In Section 4 we present a fairly general proof, and conclude in Section 5.

2 The Formalisms

We consider a finite dimensional system [5] with phase space co-ordinates $q^i$ and conjugate momenta $p_i$ ($i = 1, 2, \ldots N$). The system has two second class constraints,

$$Q_1(q, p) \approx 0, \quad Q_2(q, p) \approx 0,$$

(1)

defining a constraint surface $\Sigma_2$. Due to their second class nature, the $2 \times 2$ antisymmetric matrix $\mathcal{E}$ whose elements $\mathcal{E}_{ab}$ are Poisson brackets among the $Q$’s,

$$\mathcal{E}_{ab}(q, p) = \{Q_a, Q_b\} \quad a, b = 1, 2,$$

(2)
is invertible everywhere, even on the surface $\Sigma_2$. The canonical Hamiltonian is $H_c$ and the total Hamiltonian is

$$H = H_c + \mu_1 Q_1 + \mu_2 Q_2,$$

where the multipliers $\mu_1, \mu_2$ are determined by demanding the consistency conditions $\{Q_a, H\} = 0$, $a = 1, 2$ on the surface $\Sigma_2$. Other relevant physical quantities must also have similar properties with respect to the $Q_a$. These considerations can also be extended to field theories.

### 2.1 Batalin - Fradkin (BF) method

As mentioned in the Introduction, this method [3] is formulated in an enlarged phase space, the extent of enlargement depending on the number of second class constraints. Here since this number is two, the phase space is enlarged by introducing two new variables $\Phi^a (a = 1, 2)$. The enlarged phase space $(q, p, \Phi)$ has the basic Poisson brackets

$$\{q^i, p_j\} = \delta^i_j,$$

$$\{\Phi^a, \Phi^b\} = \omega^{ab},$$

with all other Poisson brackets zero. The antisymmetric $2 \times 2$ matrix $\omega^{ab}$ is a constant matrix, unspecified for the present.

The first class constraints are obtained as functions in this extended phase space. Since we had initially two second class constraints, there will now be two first class constraints, given in general by

$$\tilde{Q}_a (q^i, p_i, \Phi^a) = Q_a + \sum_{m=1}^{\infty} Q_a^{(m)},$$

$$\tilde{Q}_a (q^i, p_i, 0) = Q_a,$$

where the second line gives the boundary condition. The terms of various orders in the expansion for $\tilde{Q}_a$ are obtained by demanding that the $\tilde{Q}_a$ are strongly first class,

$$\{\tilde{Q}_a, \tilde{Q}_b\} = 0, \quad a, b = 1, 2.$$

For instance for the lowest order this requirement gives

$$\mathcal{E}_{ab} = -X_{ac}(q, p)\omega^{cd}X_{db},$$

where $X_{ac}$ are determined by demanding that $\tilde{Q}_a$ are strongly first class.
where the matrix $X$ is also unspecified for the present. Using (2) and (4), eqn. (7) can be satisfied, if we write and substitute

$$Q^{(1)}_a = X_{ab}(q,p)\Phi^b,$$

in (5) and consider terms at lowest order in (6). Taking $\omega_{ab}$ and $X^{ab}$ to be inverses to $\omega^{ab}$ and $X_{ab}$ respectively, the higher order terms are given by

$$Q^{(n+1)}_a = -\frac{1}{(n+2)} \Phi^b \omega_{bc} X^{cd} B^{(n)}_{da}, \quad n \geq 1,$$

$$B^{(1)}_{ab} = \{Q_{[a}, Q_{b]}^{(1)}\}_{(q,p)}$$

$$B^{(n)}_{ab} = \frac{1}{2} B^{(n)}_{[ab]} = \sum_{m=0}^{n} \{Q^{(n-m)}_a, Q^{(m)}_b\}_{(p,q)} + \sum_{m=0}^{n-2} \{Q^{(n-m)}_a, Q^{(m+2)}_b\}_{(\Phi)} \quad n \geq 2,$$

where the square brackets in the subscript imply antisymmetrization. In the last two lines in (9) the subscript $(q,p)$ implies evaluation of the corresponding Poisson bracket with respect to only the $(q^i, p_i)$, while the subscript $(\Phi)$ implies evaluation with respect to only the $\Phi$. Further, in the above equations the matrix $X$ along with the matrix $\omega$ (and hence the $\Phi^a$) are chosen according to convenience. This implies an inherent *arbitrariness* in our choice of a convenient gauge theory.

It is important to note that eqn. (7) can always be written so for the case of 2 constraints. For more than two second class constraints, this has to be taken as an assumption, which however may not hold in a very general context. In a sense, the matrix $X$ can be called the "square root" of the matrix $E$. We will come back to this issue later.

To get gauge invariant observables, we note that in general relevant quantities of the original second class system cannot be used here directly, since they are not invariant (i.e., do not have zero Poisson brackets) with respect to the new first class constraints. They are made gauge invariant by modifying them in the extended phase space. For a function $A(q,p)$ on the original phase space, the corresponding gauge invariant variables are,

$$\tilde{A}(q^i, p_i, \Phi) = A + \sum_{m=1}^{\infty} A^{(m)} \quad A^{(m)} \sim (\Phi^a)^m,$$

with the terms of various orders obtained by demanding that

$$\{\tilde{A}, Q_a\} = 0 \quad a = 1, 2.$$
The terms $A^{(m)}$ in the expansion (10) are
\[ A^{(m+1)} = -\frac{1}{(m+1)} \Phi^a \omega_{ab} X^{bc} G_c^{(m)} \quad m \geq 0, \]
\[
G_a^{(0)} = \{ Q_a, A \} \\
G_c^{(1)} = \{ Q_c^{(1)}, A \} + \{ Q_c, A^{(1)} \} + \{ Q_c^{(2)}, A^{(1)} \} (\Phi) \\
G_c^{(m)} = \sum_{n=0}^{m} \{ Q_c^{(m-n)}, A^{(n)} \}_{(q,p)} + \sum_{n=0}^{m-2} \{ Q_c^{(m-n)}, A^{(n+2)} \}_{(\Phi)} + \{ Q_c^{(m+1)}, A^{(1)} \}_{(\Phi)} \quad m \geq 2,
\]
where in the last line, the subscripts $(q, p)$ and $(\Phi)$ stand for evaluation of corresponding Poisson brackets with respect to $(q^i, p_i)$ and $\Phi$ respectively. Thus in this method, the first class constraints $\tilde{Q}_a \approx 0$ and the various gauge invariant observables $\tilde{A}$ describe the new gauge theory.

### 2.2 The Gauge Unfixing (GU) method

This method \cite{4}, in stark contrast to the BF method, makes no enlargement of the phase space while extracting a gauge theory from a second class constrained system. Rather, since the number of second class constraints is even (we consider here only bosonic constraints), this method attempts to treat half these constraints to form a first class subset, and the other half as the corresponding gauge fixing subset. This latter subset is discarded, retaining only the first class subset, and so we have a gauge theory.

In a general system, getting a first class subset is a non-trivial issue \cite{4}; this might be possible only under certain conditions. However in the case of only two second class constraints, the first class constraint can always be chosen.

For instance, we can choose $Q_1$ as our first class constraint, and $Q_2$ as its gauge fixing constraint. We redefine, using (2),
\[ Q_1 \rightarrow \chi = \mathcal{E}_{12}^{-1} Q_1, \quad Q_2 \rightarrow \psi, \]
and discard the $\psi$ as a constraint (i.e., no longer consider $\psi = 0$). To obtain the gauge invariant Hamiltonian and other physical quantities we construct a projection operator $\mathcal{P}$ by defining its operation on any phase space function $A$ as
\[
\mathcal{P}(A) = \tilde{A} \equiv : e^{-\psi \hat{\chi}} : A = A - \psi \{ \chi, A \} + \frac{1}{2!} \psi^2 \{ \chi, \{ \chi, A \} \} - \frac{1}{3!} \psi^3 \{ \chi, \{ \chi, \{ \chi, A \} \} \} + \ldots - \ldots
\]
where it may noted that the ψ is always outside the Poisson brackets on the right hand side. The gauge invariant quantities are the \( \mathcal{P}(A) = \tilde{A} \), since they satisfy the gauge invariance condition \( \{ \chi, \tilde{A} \} = 0 \). These and the first class constraint \( \chi = 0 \) describe the new gauge theory.

It must be noted that even in this method, there is an inherent arbitrariness; of the two second class constraints the first class constraint can be chosen in two ways. The two choices define two different projection operators, and the gauge theories so constructed will in general be different. This arbitrariness can be exploited to advantage.

### 3 Examples

#### 3.1 The chiral Schwinger model

This well known anomalous gauge theory \[8\] involves chiral fermions coupled to a \( U(1) \) gauge field in \((1 + 1)\) dimensions. Classically the theory has gauge invariance, but this is lost upon quantisation. We look at its bosonised version, the advantage being that the corresponding classical theory itself has no gauge invariance. We have

\[
\mathcal{L} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} (\partial_\mu \phi)^2 + e (g^{\mu \nu} - \epsilon^{\mu \nu}) (\partial_\mu \phi) A_\nu + \frac{1}{2} e^2 \alpha A_\mu^2. \tag{15}
\]

where \( g^{\mu \nu} = \text{diag}(1, -1) \), \( \epsilon^{01} = - \epsilon^{10} = 1 \) and \( \alpha \) is the regularisation parameter. The Lagrangian is gauge non-invariant for all values of \( \alpha \). We consider the case \( \alpha > 1 \).

The canonical Hamiltonian density is

\[
\mathcal{H}_c = \frac{1}{2} \pi_1^2 + \frac{1}{2} \pi_\phi^2 + \frac{1}{2} (\partial_1 \phi)^2 + e (\partial_1 \phi + \pi_\phi) A_1 + \frac{1}{2} e^2 (\alpha + 1) A_1^2 - A_0 \left[ -\partial_1 \pi_1 + \frac{1}{2} e^2 (\alpha - 1) A_0 + e (\partial_1 \phi + \pi_\phi) + e^2 A_1 \right] \tag{16}
\]

where \( \pi_1 = F^{01} = \partial^0 A^1 - \partial^1 A^0 \) and \( \pi_\phi = \partial_0 \phi + e (A_0 - A_1) \) are the momenta conjugate to \( A_1 \) and \( \phi \) respectively. The constraints are

\[
Q_1 = \pi_0 \approx 0 \quad Q_2 = - \partial_1 \pi_1 + e^2 (\alpha - 1) A_0 + e (\partial_1 \phi + \pi_\phi) + e^2 A_1 \approx 0 \tag{17}
\]

defining a constraint surface \( \Sigma_2 \). These are of the second class,

\[
\mathcal{E}_{12} = \{ Q_1(x), Q_2(y) \} = -e^2 (\alpha - 1) \delta(x - y). \tag{18}
\]
Following the BF method [10], the phase space is extended by introducing two fields $\Phi^1, \Phi^2$, with Poisson bracket relations of the form (4). The new first class constraints have the general form (5), with the first order term as given in (8). As mentioned earlier, there is a natural arbitrariness in choosing the matrices $\omega^{ab}$ and $X_{ab}$. The choice
\[
\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \delta(x - y) \quad \text{and} \quad X(x, y) = e^{\sqrt{\alpha - 1}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta(x - y)
\]  
(19)
allows the two new fields to form a canonically conjugate pair. The higher order terms beyond the first in the expansion (5) are all zero. Then the first class constraints are
\[
\tilde{Q}_a = Q_a + e^{\sqrt{\alpha - 1}} \Phi^a, \quad a = 1, 2,
\]  
(20)
which, using (4), (18) and (19), can be verified to be strongly first class.

Using the general expressions in (10) and (12) the gauge invariant Hamiltonian for the choice (19) is
\[
\tilde{H}_{BF} = H_c + \int dx \left[ -\frac{(\pi_1 + e(\alpha - 1)\partial_1 A_1)}{\sqrt{\alpha - 1}} \Phi^1 + \frac{e^2}{2(\alpha - 1)} (\Phi^1)^2 + \frac{1}{2} (\partial_1 \Phi^1)^2 + \frac{1}{2} (\Phi^2)^2 - \frac{\tilde{Q}_2 \Phi^2}{e^{\sqrt{\alpha - 1}}} \right],
\]  
(21)
with $H_c$ given by (16). This $\tilde{H}_{BF}$ has zero PBs with the constraints in (20).

Coming to the Gauge Unfixing (GU) method [11], we reiterate that no new field need be introduced. The first class constraint is taken to be just one of the two existing constraints. We choose, after a rescaling
\[
\chi = \frac{1}{e^{2(\alpha - 1)}} Q_2,
\]  
(22)
so that the relevant constraint surface $\Sigma_1$ is defined by $\chi \cong 0$. The gauge fixing-like constraint is $\psi = 0$, and is discarded (that is unfixed). The gauge invariant Hamiltonian is obtained by constructing a projection operator $\Pi$ of the form (14) and using it on the $H_c$. We get $\Pi(H_c) = \tilde{H}_{GU}$,
\[
\tilde{H}_{GU} = H_c + \int dx \left[ \frac{(\pi_1 + (\alpha - 1)\partial_1 A_1)}{\alpha - 1} Q_1 + \frac{(\partial_1 Q_1)^2}{2e^2(\alpha - 1)} + \frac{Q_1^2}{2(\alpha - 1)^2} \right],
\]  
(23)
which satisfies $\{\chi, \tilde{H}_{GU}\} = 0$. 8
It can be seen that, if we make the identification \( \Phi^1 = -\frac{Q_1}{e\sqrt{\alpha} - 1} \), the \( \tilde{H}_{BF} \) in (21) and the \( \tilde{H}_{GU} \) in (23) are almost the same. The difference between these two Hamiltonians are the extra terms \( \int dx \left( \frac{(\Phi^2)^2}{2} - \frac{\Phi^2}{e\sqrt{\alpha} - 1} \tilde{Q}_2 \right) \), appearing in (21). The second of these is zero due to (20). The first term, when rewritten using (20), is proportional to \( \tilde{Q}_2 \) and the constraint \( \chi \) in (22).

We emphasise the two rather different paths used to get these Hamiltonians. One requires the introduction of an extra (canonical) pair of fields, while the other doesn’t need this. In both cases extra terms are needed to make the original Hamiltonian gauge invariant. For the \( \tilde{H}_{BF} \) these terms had to be written down using the extra fields, whereas in the \( \tilde{H}_{GU} \) these terms involve a variable already present in the original theory.

We look at the path integral quantisation for these two gauge invariant Hamiltonians. For the Batalin-Fradkin Hamiltonian \( \tilde{H}_{BF} \), we first redefine

\[
\Phi^1 \rightarrow \theta \quad \Phi^2 \rightarrow \pi_\theta,
\]

and the partition function is

\[
Z_{BF} = \int D(\pi_\mu, A^\mu, \pi_\phi, \phi, \theta, \pi_\theta, \lambda_1, \lambda_2) \; e^{iS_{BF}}
\]

\[
S_{BF} = \int dt \left[ \pi_0 \dot{A}^0 + \pi_1 \dot{A}^1 + \pi_\phi \dot{\phi} + \pi_\theta \dot{\theta} - \tilde{H}_{BF} - \lambda_1 \tilde{Q}_1 - \lambda_2 \tilde{Q}_2 \right].
\]

Here \( \lambda_1, \lambda_2 \) are undetermined Lagrange multipliers corresponding to the first class constraints \( \tilde{Q}_1, \tilde{Q}_2 \) respectively. The integration over the \( \pi_0 \) gives the delta function \( \delta(\dot{A}_0 - \lambda_1) \), which can be used while integrating over the \( \lambda_1 \). We next make the transformations

\[
A_0 \rightarrow A_0' = A_0 - \lambda_2 + \frac{\pi_\theta}{e\sqrt{\alpha} - 1}, \quad \pi_1 \rightarrow \pi_1' = \pi_1 + \partial_0 A_1 - \partial_1 A_0' - \frac{\pi_\theta}{\sqrt{\alpha} - 1},
\]

\[
\pi_\phi \rightarrow \pi_\phi' = \pi_\phi - \dot{\phi} - e(A_0' - A_1), \quad \lambda_2 \rightarrow \lambda_2' = e\sqrt{\alpha} - 1 \lambda_2 - \dot{\theta}
\]

and after rearranging terms, we get the action to be

\[
S_{BF} = \int dt \left[ -\frac{1}{2}(\pi_1')^2 - \frac{1}{2}(\pi_\phi')^2 - \frac{1}{2}(\lambda_2')^2 + \frac{1}{2}(\partial_0 A_1 - \partial_1 A_0')^2 + \frac{1}{2}(\partial_\mu \phi)^2 
+ e(\dot{\phi} A_0' - \partial_1 A_1) - e(\dot{\phi} A_1 - A_0' \partial_1 \phi) + \frac{e^2 \alpha}{2}[(A_0')^2 - A_1^2]
+ \frac{1}{2}(\partial_\mu \theta)^2 - e \theta \sqrt{\alpha} - 1 (A_0' - \partial_1 A_1) - \frac{e \theta}{\sqrt{\alpha} - 1}(A_1 - \partial_1 A_0') \right].
\]
Putting this in the path integral, the \( \pi_1', \pi_\phi', \lambda_2' \), are integrated over. We redefine \( \theta' = \frac{\theta}{\sqrt{\alpha - 1}} \) and dropping the primes on \( \theta' \) and \( A_0' \), we get

\[
Z_{BF} = \int \mathcal{D}(A^\mu, \phi, \theta) \ e^{iS_{BF}}
\]

\[
S_{BF} = \int dt \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{e^2 \alpha}{2} A_\mu A^\mu + e(\eta^{\mu\nu} - \epsilon^{\mu\nu})(\partial_\mu \phi) A_\nu \right.
\]

\[
+ \left. \frac{1}{2} (\partial_\mu \phi)^2 + \frac{\alpha - 1}{2} (\partial_\mu \theta)^2 - e\theta [(\alpha - 1) \eta^{\mu\nu} + \epsilon^{\mu\nu}] (\partial_\mu A_\nu) \right) .
\]

The action \( S_{BF} \) above is just the gauge invariant version of the chiral Schwinger model. As is well known, this action was obtained earlier by adding the (Wess-Zumino) terms \([12]\) in the variable \( \theta \) to the original bosonised action \((15)\). Other arguments have also been used to get the same result \([1, 13]\). In the Batalin-Fradkin approach, these Wess Zumino terms and \( \theta \) come up due to the enlargement of the phase space.

In the Gauge Unfixing method, the path integral is

\[
Z_{GU} = \int \mathcal{D}(A^\mu, \pi_\mu, \phi, \pi_\phi, \mu) \exp \left( i \int dt \left[ \pi_0 \dot{A}^0 + \pi_1 A^1 + \pi_\phi \dot{\phi} - \tilde{H}_{GU} - \mu \chi \right] \right),
\]

with \( \tilde{H}_{GU} \) given by \((23)\). Here \( \mu \) is the arbitrary Lagrange multiplier. We make the transformations

\[
A_0 \to A'_0 = A_0 - \frac{\mu}{e^2 (\alpha - 1)}, \quad \pi_1 \to \pi'_1 = \pi_1 + \partial_0 A_1 - \partial_1 A'_0 + \frac{\pi_0}{\alpha - 1},
\]

\[
\pi_\phi \to \pi'_\phi = \pi_\phi - \dot{\phi} + e A_1 - e A'_0, \quad \mu \to \mu' = \mu + \partial_0 \pi_0.
\]

Dropping the prime on the \( A'_0 \) and integrating over \( \pi_1', \pi_\phi' \) and \( \mu' \) we get

\[
Z_{GU} = \int \mathcal{D}(A^\mu, \phi, \pi_0) \ e^{iS_{GU}}
\]

\[
S_{GU} = \int dt \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{e^2 \alpha}{2} A_\mu A^\mu + e(\eta^{\mu\nu} - \epsilon^{\mu\nu})(\partial_\mu \phi) A_\nu + \frac{1}{2} (\partial_\mu \phi)^2 \right.
\]

\[
+ \left. \frac{1}{2 e^2 (\alpha - 1)} \left[ (\alpha - 1) \eta^{\mu\nu} + \epsilon^{\mu\nu} \right] (\partial_\mu A_\nu) \right).
\]

On making the replacement \( \pi_0 = -e (\alpha - 1) \theta \) in \((29)\), we get the same path integral and action as in the Batalin-Fradkin case \((27)\). Here this is achieved without introducing extra fields. The extra field \( \theta \) of the BF method is found here within the original phase space.

Further the Wess Zumino terms are the same in both cases. It may also be noted that, on comparing the gauge invariant Hamiltonians in \((21)\) and \((23)\), the extra terms in the \( \pi_\theta \) in \((21)\) have been integrated away, and so these do not appear in \((27)\).
3.2 O(N) Invariant Nonlinear Sigma Model

In the earlier example, gauge invariant observables like the Hamiltonian had finite number of terms, either in the new variables (BF method) or in the discarded constraint of the GU method. The general formalisms of Section 2 showed that these observables in general have infinite number of terms. The nonlinear sigma model presents an example where the gauge invariant observables have infinite number of terms. But these can be rewritten in closed form. Even here the two methods give the same results.

The model consists of a multiplet of $N$ real scalar fields $n^a$, $a = 1, 2, \ldots, N$ and is described by the Lagrangian density

$$
\mathcal{L} = \frac{1}{4}(\partial_\mu n^a)(\partial^\mu n^a) - \lambda(n^a n_a - 1),
$$

where $\lambda$ is a Lagrange multiplier. The canonical Hamiltonian density is

$$
\mathcal{H}_c = \pi^a \pi_a + \frac{1}{4}(\partial_1 n^a)(\partial_1 n_a) + \lambda(n^a n_a - 1),
$$

with $\pi_a = \dot{n}_a^2$, the conjugate momenta. The constraints are of the second class,

$$
Q_1 = (n^a n_a - 1) \approx 0, \quad Q_2 = n^a \pi_a \approx 0,
$$

$$
\{Q_1(x), Q_2(y)\} = 2|n|^2 \delta(x - y) = 2(Q_1 + 1) \approx 2.
$$

The form of $\lambda$ can be fixed by demanding time independence of $Q_1, Q_2$. We then get

$$
H_T = \int dx \left[ \pi_a \pi_a |n|^2 + \frac{n^a \partial_1^2 n_a}{4} (|n|^2 - 2) \right].
$$

This total Hamiltonian ensures time independence of the constraints (32) on the constrained surface defined by both these constraints.

We first apply the Gauge Unfixing method. Using (33) we rescale $Q_2$ and rewrite as

$$
\chi = -\frac{Q_2}{2|n|^2}, \quad \psi = Q_1 = |n|^2 - 1.
$$

Choosing $\chi \approx 0$ as our first class constraint, we disregard $\psi = 0$ as a constraint. Since the original Hamiltonian $H_T$ is not invariant with respect to $\chi$ on the new surface defined by $\chi \approx 0$, we construct and use a projection operator of the form (14). Here we do not
apply this directly on $H_T$ to get the gauge invariant Hamiltonian; instead we first apply the operator on the fields $n^a, \pi_a$ to get their *gauge invariant* analogs. We find that an infinite series of the form (14) is required here. For the $n^a, \pi_a$, these series can be rewritten in closed form. The results are,

$$\tilde{n}^a_{(GU)} = n^a \left( 1 - \frac{\psi}{|n|^2} \right)^{1/2} \quad \text{and} \quad \tilde{\pi}_a_{(GU)} = (\pi_a + 2n_a\chi) \left( 1 - \frac{\psi}{|n|^2} \right)^{-1/2}. \quad (36)$$

These satisfy $\{\chi, \tilde{n}^a_{(GU)}\} = 0$ and $\{\chi, \tilde{\pi}_a_{(GU)}\} = 0$. Using a property [4] of such projected fields, we substitute these gauge invariant fields in $H_T$, and get our gauge invariant Hamiltonian,

$$\tilde{H}_{T(GU)} = \int dx \left[ (\pi_a + 2n_a\chi)^2|n|^2 + \frac{\tilde{n}_{(GU)}^a}{4} \partial^2 \tilde{n}_{(GU)}^a (|n|^2 - \psi - 2) \right]$$

$$= \int dx \left[ (\pi_a + 2n_a\chi)^2|n|^2 - \frac{\tilde{n}_{(GU)}^a}{4} \partial^2 \tilde{n}_{(GU)}^a \right], \quad (37)$$

where we have used (35). It can be verified that $\tilde{H}_{T(GU)}$ satisfies $\{\chi, \tilde{H}_{T(GU)}\} = 0$. This $\tilde{H}_{T(GU)}$ together with the $\chi = 0$ describes a gauge theory here.

We now apply the Batalin - Fradkin method to this model. We first make the choice

$$\omega = 2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \delta(x - y) \quad X(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & -|n|^2 \end{pmatrix} \delta(x - y) \quad (38)$$

so that the new (first class) constraints are

$$\tilde{Q}_1(x) = Q_1 + \Phi^1 \approx 0 \quad \text{and} \quad \tilde{Q}_2(x) = Q_2 - |n|^2\Phi^2 \approx 0, \quad (39)$$

with $\Phi^1$ and $\Phi^2$ being the new variables introduced to enlarge the phase space (they are not exact canonical conjugates).

With respect to these first class constraints, the gauge invariant Hamiltonian is obtained by resorting to the general series (10). Even here we do not directly construct this Hamiltonian; we look for gauge invariant analogs of the $n^a, \pi_a$. Using an infinite series of the form (10) we get closed form expressions,

$$\tilde{n}^a_{(BF)} = n^a \left( 1 + \frac{\Phi^1}{|n|^2} \right)^{1/2} \quad \tilde{\pi}_a_{(BF)} = (\pi_a - n_a\Phi^2) \left( 1 + \frac{\Phi^1}{|n|^2} \right)^{-1/2}. \quad (40)$$
Replacing the $n^a$ and the $\pi_a$ in $H_T$ by the $\bar{n}^a_{(BF)}$ and the $\bar{\pi}_a_{(BF)}$ we get the gauge invariant Hamiltonian

$$\tilde{H}_{T(BF)} = \int dx \left[ |\bar{\pi}_{(BF)}|^2 |\bar{n}_{(BF)}|^2 + \bar{n}_{(BF)}^a \frac{\partial^2 \bar{n}_a_{(BF)}}{4} (|\bar{n}_{(BF)}|^2 - 2) \right]$$

$$= \int dx \left[ (\pi_a - n_a \Phi^2)^2 |n|^2 - \frac{\bar{n}_{(BF)}^a}{4} \frac{\partial^2 \bar{n}_a_{(BF)}}{4} (\bar{Q}_1 - 1) \right],$$

which maintains the time consistency of the two first class constraints in (39). These constraints together with the $\tilde{H}_{T(BF)}$ describe a gauge theory in the BF method.

On comparing the gauge invariant observables in (36) of the GU method and the gauge invariant observables in (40) of the BF method, we see that they are the same if we make the identification $\Phi^1 = -\psi$ and $\Phi^2 = -2\chi$. Obviously due to this identification, the gauge invariant Hamiltonians in (37) and (41) are also the same, apart from the term in $\bar{Q}_1$ in (41). Thus even here extra variables are not required to get gauge symmetries. What comes out as an extra variable in the BF method can actually be found in the original phase space in the GU method.

We look at path integral quantisation for the gauge theories obtained in these two methods. For the $\tilde{H}_{T(GU)}$, the partition function is

$$Z_{GU} = \int \mathcal{D}(n^a, \pi_a, \mu) \exp \left( i \int dx dt \left[ \pi_a \dot{n}^a - \tilde{H}_{T(GU)} - \mu \chi \right] \right),$$

with $\mu$ being an arbitrary Lagrange multiplier. We make the transformations

$$\mu \to \mu' = \left( \frac{\mu}{2 |n|^2} + (n^a \pi_a) \right)$$

and

$$\pi_a \to \pi'_a = \left( \pi_a - \frac{\dot{n}_a}{2 |n|^2} - \frac{\mu' n_a}{2 |n|^2} \right)$$

and then integrate over the $\pi'$. We get

$$Z_{GU} = \int \mathcal{D}(n^a, \mu') (\det |n^a n_a|)^{1/2} \exp \left( i \int dx dt \left[ -\frac{(\partial_1 \bar{n}^a_{(GU)}) (\partial_1 \bar{n}_a_{(GU)})}{4} + |n|^2 (\mu'')^2 \right] \right),$$

where the $\mu''$ is the (once again) redefined arbitrary multiplier $\mu'' = \left( \frac{\mu' n^a}{2 |n|^2} + \frac{\dot{n}_a}{2 |n|^2} \right)$. It may be noted from (36) that $\psi$ is contained within the $\bar{n}^a_{GU}$.

For the Hamiltonian $\tilde{H}_{T(BF)}$, the partition function is

$$Z_{BF} = \int \mathcal{D}(\pi_a, n^a, \Phi^1, \Phi^2, \lambda_1, \lambda_2) e^{iS_{BF}}$$

$$S_{BF} = \int dx dt \left[ \pi_a \dot{n}^a + \frac{1}{2} \Phi^2 \dot{\Phi}^1 - \tilde{H}_{T(BF)} - \lambda_1 \bar{Q}_1 - \lambda \bar{Q}_2 \right],$$
with \( \lambda_1, \lambda_2 \) being undetermined Lagrange multipliers. We make the transformations

\[
\pi_a \rightarrow \pi'_a = \left( \pi_a - n_a \Phi^2 + \frac{\lambda_2 n^a}{2|n|^2} - \frac{n^a}{2|n|^2} \right)
\]

\[
\lambda_1 \rightarrow \lambda'_1 = \lambda_1 + \frac{\tilde{\eta}_a BF \partial_1 \tilde{n}_{a(BF)}}{4}
\]

\[
\lambda_2 \rightarrow \lambda'_2 = \left( \frac{\lambda_2 n^a}{2|n|^2} - \frac{n^a}{2|n|^2} \right)
\]

and integrate over \( \pi'_a, \lambda'_1 \). The latter integration gives a delta function \( \delta(\Phi^1 + |n|^2 - 1) \).

Integration over the \( \Phi^1 \) will replace \( \Phi^1 \) everywhere by \(-|n|^2 + 1 = -\psi \). We then get

\[
Z_{BF} = \mathcal{D}(n^a, \lambda_2^{'}) \left( \text{det} \left| -n^2 \right| \right)^{1/2} \exp \left( i \int dx dt \left[ |n|^2 (\lambda'_2)^2 - \frac{\partial_1 \tilde{n}_a^{BF} (\partial_1 \tilde{n}_{a(BF)})}{4} \right] \right) \quad \text{(45)}
\]

Due to the delta function \( \delta(\Phi^1 + |n|^2 - 1) \), the \( \Phi^1 \) in the expression (40) for \( \tilde{n}_a^{BF} \) is now replaced by \(-\psi = (-|n|^2 + 1)\), so that from (36) we now have \( \tilde{n}_a^{BF} = \tilde{n}_a^{GU} \). Using this, and taking note of the arbitrary nature of the multipliers \( \mu'' \) in (43) and the \( \lambda' \) in (45), we see that we get the \textit{same results} from both the BF and the GU methods!

### 4 General Proof for Two Second Class Constraints

Having considered the examples in the earlier section, we now arrive at a general proof of the equivalence between the Batalin-Fradkin and Gauge Unfixing methods. We consider the case of two second class constraints.

In the \textbf{gauge unfixing} method, we redefine the two constraints as

\[
\chi = \frac{1}{E} Q_1, \quad \psi = Q_2, \quad \text{ (46)}
\]

where \( E(q, p) = \{Q_1, Q_2\} \). We retain the \( \chi \) as the first class constraint, and discard the \( \psi \) (other choices are also possible). The construction and application of the corresponding \textbf{projection operator} \( IP \) on a phase space function \( A \) gives the gauge invariant function

\[
\tilde{A}_{GU} = : e^{-\psi \hat{\chi}} A = A - \psi \{\chi, A\} + \frac{\psi^2}{2!} \{\chi, \{\chi, A\}\} - \frac{\psi^3}{3!} \{\chi, \{\chi, \{\chi, A\}\}\} + \ldots \quad \text{(14)}
\]

with an infinite number of terms. Of these, apart from the \( A \), we give below terms up to the fourth order. Using (46) and \( E = \{Q_1, Q_2\} \), these terms are

\[
- Q_2 \{\chi, A\} = - \frac{Q_2}{E} \{Q_1, A\} + \ldots \ldots\ldots\ldots
\]
\[
+ \frac{Q^2}{2!} \{\chi, \{\chi, A\}\} = \frac{1}{2!} \frac{Q^2}{E^2} \left[\{Q_1, \{Q_1, A\}\} + E \left\{Q_1, \frac{1}{E}\right\} \{Q_1, A\}\right] + \ldots \ldots \ldots \ldots (47)
\]
\[
- \frac{Q^2}{3!} \{\chi, \{\chi, \{\chi, A\}\}\} = - \frac{1}{3!} \frac{Q^2}{\bar{E}^3} \left[\{Q_1, \{Q_1, \{Q_1, A\}\}\} + 3E \left\{Q_1, \frac{1}{E}\right\} \{Q_1, \{Q_1, A\}\} + E^2 \left\{Q_1, \frac{1}{E}\right\} \{Q_1, A\} + E \left\{Q_1, \left\{Q_1, \frac{1}{E}\right\} \{Q_1, A\}\right\} \right] + \ldots \ldots \ldots \ldots
\]
\[
+ \frac{Q^4}{4!} \{\chi, \{\chi, \{\chi, \{\chi, A\}\}\}\} = \frac{1}{4!} \frac{Q^4}{E^4} \left(\{Q_1, \{Q_1, \{Q_1, \{Q_1, A\}\}\}\} + 6E \left\{Q_1, \frac{1}{E}\right\} \{Q_1, \{Q_1, \{Q_1, A\}\}\} + 7E^2 \left\{Q_1, \frac{1}{E}\right\} \{Q_1, A\} + E^3 \left\{Q_1, \frac{1}{E}\right\} \{Q_1, A\} + 4E \left\{Q_1, \left\{Q_1, \frac{1}{E}\right\}\right\} \left[\{Q_1, \{Q_1, A\}\} + E \left\{Q_1, \frac{1}{E}\right\} \{Q_1, A\}\right] + E \left\{Q_1, \left\{Q_1, \left\{Q_1, \frac{1}{E}\right\}\right\} \{Q_1, A\}\right\} \right] + \ldots \ldots \ldots \ldots
\]

In the right hand sides of each equation in (47) above we have explicitly given only those terms which are proportional to only the \(\psi \) (= \(Q_2\)). There are other terms, which are proportional to the first class constraint \(\chi\). These terms can however be put to zero, by using \(\chi = 0\).

In the BF method, the (modified) first class constraints have the general form (5), an infinite series in the new variables. We will consider the case where this series is truncated after the second term. Since the choice of the matrices \(X_{ab}\) and \(\omega_{ab}\) of eqn. (7) reflects the arbitrariness in the new gauge theory, we make here a specific choice,

\[
\omega_{ab} = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}, \quad X_{ab} = \begin{pmatrix}
X & 0 \\
0 & 1
\end{pmatrix}
\quad (48)
\]
\[
X = - E, \quad (49)
\]
with \(E(q, p) = \{Q_1, Q_2\}\). Here (49) is obtained by substituting (48) in the first order equation (7). Using (48), (49) and (9), and by equating the second order term from (9) to zero, we get the condition for truncating the series (5) after the second term as

\[
\{Q_2, E\} = 0. \quad (50)
\]
Higher order terms are also zero. We will assume (50) from now on. We thus get the new first class constraints,

\[ \tilde{\mathcal{Q}}_1 = Q_1 - E \Phi^1, \quad \tilde{\mathcal{Q}}_2 = Q_2 + \Phi^2, \]  

(51)

For the choice (48) we now look at the various terms in the series (10) for a general gauge invariant variable. If for example we consider the first and second order terms,

\[ A^{(1)} = (\tilde{Q}_a - Q_a) (\mathcal{E}^{-1})^{ab} \{Q_b, A\}, \]

\[ A^{(2)} = (\tilde{Q}_a - Q_a) (\mathcal{E}^{-1})^{ab} \left( \frac{1}{2} \tilde{Q}_c - Q_c \right) (\mathcal{E}^{-1})^{cd} \left[ \{Q_b, \{Q_d, A\}\} + X_{de} \{Q_b, X^{ef}\} \{Q_f, A\} \right] \]

we see that terms proportional to both \((\tilde{Q}_1 - Q_1)\) and \((\tilde{Q}_2 - Q_2)\) are present. The higher order terms will also have terms proportional to the \((\tilde{Q}_1 - Q_1)\) and \((\tilde{Q}_2 - Q_2)\). Since both \(\tilde{Q}_1\) and \(\tilde{Q}_2\) are first class constraints in the BF construction, we can ignore the terms proportional to \(\tilde{Q}_1\) and \(\tilde{Q}_2\). We are then left with terms separately proportional to the \(Q_1\) and \(Q_2\), and terms containing the product \(Q_1 Q_2\). We then have, up to the fourth order,

\[ A^{(1)} = - \frac{Q_2}{E} \{Q_1, A\} + \ldots \]

\[ A^{(2)} = \frac{1}{2!} \left( \frac{Q_2}{E} \right)^2 \left[ \{Q_1, \{Q_1, A\}\} + E \left( Q_1, \frac{1}{E} \right) \{Q_1, A\} \right] + \ldots \]

\[ A^{(3)} = - \frac{1}{3!} \left( \frac{Q_2}{E} \right)^3 \left[ \{Q_1, \{Q_1, \{Q_1, A\}\}\} + 3E \left( Q_1, \frac{1}{E} \right) \{Q_1, \{Q_1, A\}\} \right] \]

\[ + E \left( Q_1, \left\{ Q_1, \frac{1}{E} \right\} \right) \left\{ Q_1, A \right\} + E^2 \left( Q_1, \frac{1}{E} \right)^2 \{Q_1, A\} \]

\[ + \ldots \]

\[ A^{(4)} = \frac{1}{4!} \left( \frac{Q_2}{E} \right)^4 \left[ \{Q_1, \{Q_1, \{Q_1, \{Q_1, A\}\}\}\} + 6E \left( Q_1, \frac{1}{E} \right) \{Q_1, \{Q_1, \{Q_1, A\}\}\} \right] \]

\[ + 7E^2 \left( Q_1, \frac{1}{E} \right)^2 \{Q_1, \{Q_1, A\}\} + E^3 \left( Q_1, \frac{1}{E} \right)^3 \{Q_1, A\} \]

\[ + 4E \left( Q_1, \left\{ Q_1, \frac{1}{E} \right\} \right) \left( \{Q_1, \{Q_1, A\}\} + E \left( Q_1, \frac{1}{E} \right) \{Q_1, A\} \right) \]

\[ + E \left( Q_1, \left\{ Q_1, \left\{ Q_1, \frac{1}{E} \right\} \right\} \right) \{Q_1, A\} \]

\[ + \ldots \]
where we have explicitly given terms proportional to only the $Q_2$. The terms proportional to the $Q_1$ will be, as explained below, ignored.

To compare the gauge invariant $\tilde{A}_{BF}$ and $\tilde{A}_{GU}$ of the two methods, we look at the terms of different orders in (47) and (52). It can be seen that, for each of the orders considered, the terms proportional to the $\psi$ ($= Q_2$) in (47) are the same as those proportional to the $Q_2$ in (52). Even though this is shown here for terms upto the fourth order, it can be verified to be true for higher terms also. Both (47) and (52) will also have terms proportional to the $Q_1$ (or $\chi$), though these need not be the same. We thus conclude that

$$\tilde{A}_{BF} = \tilde{A}_{GU} + \sum_{m=1}^{\infty} (\ ) Q_1^m. \quad (53)$$

Since the second term (a series) on the RHS is proportional to the first class constraint $\chi$, it goes to zero on the constraint surface, and so it can be ignored. Thus the gauge invariant observables from the two methods are equivalent upto terms involving the first class constraint of the gauge unfixing method.

5 Conclusions

We conclude by going back to the question posed in the title of this paper: Are new variables necessary to extract hidden symmetries in second class constrained systems? We find that, for a fairly general class of systems, this question is answered in the negative. Extra variables are not necessary; rather the hidden gauge symmetry can be found within the original system itself.

The above conclusion has been demonstrated by first looking at two theories as examples and then by presenting a proof for a fairly general second class constrained system. We have shown that in all these, the Batalin-Fradkin and Gauge Unfixing methods, even though widely different in construction, give the same results. A similar conclusion was also made in an earlier paper [7].

In the past gauge invariances have been induced in some systems (like anomalous gauge theories) by sometimes introducing what are known as compensating fields. These correspond to the gauge degrees of freedom. The extra variables of the BF method can be identified with these compensating fields. Using the GU method we can then say that these compensating fields can be found within the original phase space.
The general proof given in Section 4 considers a case where the first class constraints in the BFT method have a particular form, with terms beyond the first order in the extra variables being zero. It is to be seen if a similar proof holds for a more general form of the first class constraints.

The general proof of Section 4 involved only two second class constraints. One has to see if a similar proof of equivalence can be obtained for more than two second class constraints. In this context, before looking for gauge invariant observables, one must see whether first class constraints can always be obtained in both methods. It may be recalled that mention was made of the $X$ matrix of the BFT method being a “square root” of the $E$ matrix of (2). For more than two second class constraints, getting this $X$ matrix may become a non-trivial issue in the general case. Similarly, in the GU method, the classification into first class and gauge fixing-like constraints in a global manner may be a nontrivial issue. Looking for equivalence of the two methods is to be done only after these issues are resolved.

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