Associativity of fusion products of $C_1$-cofinite $\mathbb{N}$-gradable modules of vertex operator algebra

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Abstract

We prove an associative law of the fusion products $\boxtimes$ of $C_1$-cofinite $\mathbb{N}$-gradable modules for a vertex operator algebra $V$. To be more precise, for $C_1$-cofinite $\mathbb{N}$-gradable $V$-modules $A, B, C$ and their fusion products $(A \boxtimes B, \mathcal{Y}^{AB})$, $(A \boxtimes B) \boxtimes C, \mathcal{Y}^{AB}C, (B \boxtimes C, \mathcal{Y}^{BC}), (A \boxtimes (B \boxtimes C), \mathcal{Y}^{ABC})$ with logarithmic intertwining operators $\mathcal{Y}^{AB}, \ldots, \mathcal{Y}^{ABC}$ satisfying the universal properties for $\mathbb{N}$-gradable modules, we prove that four-point correlation functions $(\theta, \mathcal{Y}^{AB}(v, x)\mathcal{Y}^{BC}(u, y)w)$ and $(\theta', \mathcal{Y}^{AB}(v, x-y)u, y)w)$ are locally normally convergent over $\{(x, y) \in \mathbb{C}^2 | 0 < |x-y| < |y| < |x|\}$.

We then take their respective principal branches

$$\tilde{F}(\theta, \mathcal{Y}^{AB}(v, x)\mathcal{Y}^{BC}(u, y)w)$$

on $D^2 = \{(x, y) \in \mathbb{C}^2 | 0 < |x-y| < |y| < |x|, x, y, x-y \notin \mathbb{R} \leq 0\}$ and then show that there is an isomorphism $\phi^{*}_{[AB]C} : (A \boxtimes B) \boxtimes C \rightarrow A \boxtimes (B \boxtimes C)$ such that

$$\tilde{F}(\theta, \mathcal{Y}^{AB}(v, x)\mathcal{Y}^{BC}(u, y)w)) = \tilde{F}(\phi^{*}_{[AB]C}(\theta), \mathcal{Y}^{AB}(v, x-y)u, y)w))$$

on $D^2$ for $\theta \in (A \boxtimes (B \boxtimes C))^\vee$, $v \in A$, $u \in B$, and $w \in C$, where $W^\vee$ denotes the contragredient module of $W$ and $\phi^{*}_{[AB]C}$ denotes the dual of $\phi_{[AB]C}$. We also prove the pentagon identity.

1 Introduction

As a counterpart of a tensor product of two modules in the VOA theory, Huang and Lepowsky[4] have introduced a concept of fusion products $\boxtimes_{p(x)}$. Huang [3] has also shown that four-point correlation functions for three $C_1$-cofinite $\mathbb{N}$-gradable modules satisfy differential equations with regular singularity and then as an application, he has proved an associative law of fusion products under some conditions, where a $V$-module $W$ is called to be ”$C_1$-cofinite” if $\dim V/C_1(W) < \infty$ for $C_1(W) := \text{span}_{\mathbb{C}}\{\alpha_{-1}w \in W | w \in W, \alpha \in V, \text{wt}(\alpha) \geq 1\}$ and a $V$-module $W$ is called $\mathbb{N}$-gradable if $W$ has a decomposition $W = \oplus_{m \in \mathbb{N}} W_{(m)}$ satisfying $\alpha_{k}W_{(m)} \subseteq W_{(m+\text{wt}(\alpha)-k-1)}$ for $\alpha \in V$ and $k \in \mathbb{Z}$. From now on, $\mathcal{N}_1$ denotes the set of all $C_1$-cofinite $\mathbb{N}$-gradable $V$-modules. Before we explain our results, let us show basic knowledge for $C_1$-cofinite $\mathbb{N}$-gradable modules. If

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*Partially supported by the Grants-in-Aids for Scientific Research, No.21K03195, The Ministry of Education, Science and Culture, Japan
We write $\text{gr}(w)$ homogeneous and denoted by $\text{wt}(w)$. We always choose this grading for indecomposable modules $\alpha$.

For $A, B \in \mathcal{NC}_1$, as the author has shown in [3], a pair $(A \boxtimes B, \mathcal{Y}^{AB})$ of a $V$-module $A \boxtimes B$ and a logarithmic intertwining operator $\mathcal{Y}^{AB} \in I_{AB}^{\otimes V}$ satisfying the universal property for $\mathbb{N}$-graded modules (see §2) is always uniquely well-defined up to isomorphism and $A \boxtimes B$ is also in $\mathcal{NC}_1$ and $\mathcal{Y}^{AB}$ is also a logarithmic intertwining operator. Let $A, B, C \in \mathcal{NC}_1$ and assume to be $\mathbb{N}$-graded. Since $A \boxtimes B, B \boxtimes C \in \mathcal{NC}_1$, we also have $((A \boxtimes B) \boxtimes C, \mathcal{Y}^{(AB)C})$ and $(A \boxtimes (B \boxtimes C), \mathcal{Y}^{A(BC)})$ and we denote these logarithmic intertwining operators by

$$\mathcal{Y}^{A(BC)}(v, x) = \sum_{h=0}^{K_1} \mathcal{Y}_h^{A(BC)}(v, x) \log^h(x),$$

$$\mathcal{Y}^{BC}(u, y) = \sum_{k=0}^{K_2} \mathcal{Y}_k^{BC}(u, y) \log^k(y),$$

$$\mathcal{Y}^{AB}(v, x-y) = \sum_{k=0}^{K_3} \mathcal{Y}_k^{AB}(v, x-y) \log^k(x-y),$$

$$\mathcal{Y}^{(AB)C}((\delta, y)) = \sum_{k=0}^{K_4} \mathcal{Y}_k^{(AB)C}((\delta, y) \log^k(y),$$

for $v \in A, u \in B, \delta \in A \boxtimes B$ with formal $\mathbb{C}$-power series $\mathcal{Y}_h^{A(BC)}(v, x)$ of $x$ with coefficients in $\text{Hom}(B \boxtimes C, A \boxtimes (B \boxtimes C))$, etc. To simplify the notation, we use notation $K^2 = \{(h, k) \in \mathbb{N}^2 \mid h \leq K_1, k \leq K_2\}, K^2 = \{(h, k) \in \mathbb{N}^2 \mid h \leq K_3, k \leq K_4\}$ and we often denote $A \boxtimes (B \boxtimes C)$ and $(A \boxtimes B) \boxtimes C$ by $A(BC)$ and $(AB)C$, respectively. Then for $\theta \in (A(BC))^\vee, \theta' \in ((AB)C)^\vee$, we define two four-point correlation functions and the coefficients of $\log^h(x)$ and $\log^k(y)$ and $\log^h(x-y) \log^k(y)$ in them as follows:

$$F^{A(BC)}(\theta, v, u, w; x, y) := \langle \theta, \mathcal{Y}_h^{A(BC)}(v, x) \mathcal{Y}_k^{BC}(u, y)w \rangle,$$

$$F^{(AB)C}(\theta', v, u, w; x-y, y) := \langle \theta', \mathcal{Y}_h^{(AB)C}(v, x-y) \mathcal{Y}_k^{AB}(u, y)w \rangle,$$

$$F_{h,k}^{(AB)C}(\theta, v, u, w; x, y) := \langle \theta, \mathcal{Y}_h^{A(BC)}(v, x) \mathcal{Y}_k^{BC}(u, y)w \rangle,$$

$$F_{h,k}^{A(BC)}(\theta', v, u, w; x-y, y) := \langle \theta', \mathcal{Y}_h^{(AB)C}(v, x-y) \mathcal{Y}_k^{AB}(u, y)w \rangle.$$

Set $\Omega = (A(BC))^\vee \times A \times (BC)$ and $\Omega' = ((AB)C)^\vee \times A \times (BC)$. We use $\vec{\xi}$ and $\vec{\xi}'$ to denote a quadruple $(\theta, v, u, w) \in \Omega$ and $(\theta', v, u, w) \in \Omega'$ so that $F_{h,k}^{A(BC)}(\vec{\xi}; x, y)$ denotes $F_{h,k}^{A(BC)}(\theta, v, u, w; x, y)$ and so on. For $\alpha \in V$, we use notation $\alpha_i^n$ to denote the action of $\alpha_n$ on the $i$-th coordinate of $\Omega$, e.g. $\alpha^{[2]}_n \vec{\xi} = (\theta, \alpha_n v, u, w)$. For $\vec{\xi} = (\theta, v, u, w) \in \Omega$, define $\text{gr}(\vec{\xi}) = \text{gr}(\theta) + \text{gr}^{234}(\vec{\xi})$ and the total grade $\text{gr}(\vec{\xi}) = \text{gr}(\theta) + \text{gr}^{234}(\vec{\xi})$.

The main purpose in this paper is to show the associative law of fusion products (Theorem 4.1) for $A, B, C \in \mathcal{NC}_1$. We may assume that $A, B, C$ are indecomposable. As a proof, we will construct a surjective homomorphism $\phi^{A(BC)}_A : (A \boxtimes B) \boxtimes C \rightarrow A \boxtimes (B \boxtimes C)$ by starting from $F^{A(BC)}(\vec{\xi}; x, y)$. The reverse homomorphism is given by starting from $F^{(AB)C}(\vec{\xi}^2; x, y)$ with the similar argument. Our proofs are based on Huang’s ideas in [3]. The differences from [3] are that we will treat formal $\mathbb{C}$-power series $F_{h,k}^{A(BC)}(\vec{\xi}; x, y)$ andshow that their modified versions satisfy two Fuchsian systems on discs by fixing one variable $y = y_0 \not\in \mathbb{R}^{\leq 0}$ or $x = x_0 \not\in \mathbb{R}^{\leq 0}$. Furthermore, in order to expand $F^{A(BC)}(\vec{\xi}; x, y_0)$ in a $\mathbb{C}$-power series of $(x-y_0)$ with logarithm functions $\log^h(x-y_0)$ as a component.
of a solution of Fuchsian system, we restrict the variable $x$ into a domain $D^2_{x,y}(\zeta,\xi) = \{x \in \mathbb{C} \mid 0 < |x-y|_0 < |y|_0 < |x|, \text{ and } x,y, x-y \notin \mathbb{R}^\leq 0\}$ and take a principal branch $F^{A(BC)}(\zeta) \in F^{A(BC)}(\xi;x,y)$. A key result we get from our Fuchsian system is that there is a finite set $\Delta \subseteq \mathbb{C}$ such that the powers of $(x-y)_0$ in these expansions are all in $\Delta - \text{gr}(v) - \text{gr}(u) + \mathbb{N}$ for any $\zeta$. This supports the existence of an $\mathbb{N}$-gradable $V$-module isomorphic to $A \otimes B$.

In order to get Fuchsian sets, we consider finite spaces. Let $P_{W}$ denote a complement of $C_1(W)$ in $W$ and let $\tilde{P}_W$ be a finite dimensional subspace of $W$ containing $P_W$ and spanned by homogeneous elements. For $N \in \mathbb{N}$, we define $W_N = \oplus_{m=0}^{\infty} \text{Hom}(W_m, \mathbb{C})$ and set $\Omega_N = \{ (\theta, u, v, w) \in (A(BC))_N \times A \times B \}$.

Choose bases $j_{P_A} = \{ i \mid i \in \mathcal{P}_A \}$, $j_{P_B} = \{ j \mid j \in \mathcal{P}_B \}$, $j_{P_C} = \{ q \mid q \in \mathcal{P}_C \}$. Furthermore, for the residue classes of coefficients modulo $\mathbb{N}$, the coefficients in $\mathbb{C}$ and take a principal branch.

Then we will obtain the following reduction theorem.

**Theorem 3.1 for $G^{A(BC)}$** For $\zeta = (\theta, u, v, w) \in \Omega_N$, $\alpha \in V$, and $x_0 \neq y_0$, we have:

1. $G^{A(BC)}_{h,k}(\zeta; x_0, y_0)$ is a linear combination of $\{ G^{A(BC)}_{h,k}(\zeta; x_0, y_0) | \bar{\mu} \in \mathcal{J}_N \}$ with coefficients in $\mathbb{C}[x_0, y_0]([-x_0, y_0])^{-1} \subseteq \mathbb{C}[x_0, y_0]$ and
2. $G^{A(BC)}_{h,k}(\zeta; x, y)$ is a linear combination of $\{ G^{A(BC)}_{h,k}(\zeta; x, y) | \bar{\mu} \in \mathcal{J}_N \}$ with coefficients in $\mathbb{C}[x_0, y_0]([-x_0, y_0])^{-1} \subseteq \mathbb{C}[x_0, y_0]$.

Furthermore, for the residue classes of coefficients modulo $\mathbb{C}[x_0, y_0]([-x_0, y_0])^{-1}$, we have:

- $G^{A(BC)}_{h,k}(\zeta; x_0, y_0) \equiv 0 \mod \mathbb{C}[x_0, y_0]([-x_0, y_0])^{-1}$
- $G^{A(BC)}_{h,k}(\zeta; x, y) \equiv 0 \mod \mathbb{C}[x_0, y_0]([-x_0, y_0])^{-1}$

Using $L(-1)$-derivative properties: $F^{A(BC)}(L(-1)^{[3]} \zeta; x, y)$ and $F^{A(BC)}(L(-1)^{[2]} \zeta; x, y)$, we will show that there are $\lambda^{34}_{\bar{\mu}, p, q}(x_0, y_0) \in \mathbb{C}[x_0, y_0]([-x_0, y_0])^{-1} \subseteq \mathbb{C}[x_0, y_0]([-x_0, y_0])^{-1}$ such that

- $\frac{\partial}{\partial y} G^{A(BC)}_{h,k}(\zeta; x_0, y_0) = \frac{1}{y} \sum_{\bar{\mu} \in \mathcal{J}_N} \sum_{(p, q) \in K} \lambda^{34}_{\bar{\mu}, p, q}(x_0, y_0) G_{p,q}^{A(BC)}(\bar{\mu}; x_0, y_0)$
- $\frac{\partial}{\partial x} G^{A(BC)}_{h,k}(\zeta; x, y_0) = \frac{1}{x-x_0} \sum_{\bar{\mu} \in \mathcal{J}_N} \sum_{(p, q) \in K} \lambda^{34}_{\bar{\mu}, p, q}(x, y_0) G_{p,q}^{A(BC)}(\bar{\mu}; x, y_0)$

as formal $\mathbb{C}$-power series. Set $r = |\mathcal{J}_N| K^2$ and define matrix-valued functions

- $\Lambda^{34}(x_0, y_0) = (\lambda^{34}_{\bar{\mu}, p, q}(x_0, y_0))_{(\bar{\mu}, p, q) \in \mathcal{J}_N \times K^2} \in M_{r \times r}(\mathcal{O}(\mathbb{D}|_{y_0}(y_0)))$
- $\Lambda^{34}(x, y_0) = (\lambda^{34}_{\bar{\mu}, p, q}(x, y_0))_{(\bar{\mu}, p, q) \in \mathcal{J}_N \times K^2} \in M_{r \times r}(\mathcal{O}(\mathbb{D}|_{x_0}(0)))$.

Here and hereafter
\[ \mathbb{D}_R(z_0) = \{ z \in \mathbb{C} \mid |z - z_0| < R \} \] and \( \mathcal{O}(U) \) denotes the set of holomorphic functions on \( U \).

We note \( \lambda_{\mu,\nu}^{ABC,}\xi,0 = 0 = \lambda_{\mu,\nu}^{ABC,}\xi,0 \) if \( \mu \notin \mathcal{J}_N^0 \). Then we get:

**Theorem 3.2 [Differential systems]** Fix \( x_0 \neq 0 \) and \( N \in \mathbb{N} \). Then a vector valued function \( G^y(x_0, y) := (G^{ABC}_{h,k}(\xi, x_0, y))_{\xi \in \mathcal{J}_N(h,k) \subseteq K^2} \) satisfies a Fuchsian system

\[
\frac{d}{dy} G^y(x_0, y) = \frac{\Lambda_{AB}(x_0, y)}{y} G^y(x_0, y)
\]

over \( \mathbb{D}_{|x_0|}(0) \) with polar locus \( \{0\} \). Similarly, for a fixed \( 0 \neq \xi_0 \in \mathbb{C} \) and \( N \in \mathbb{N} \), a vector valued function \( G^{x-y}(x, \xi_0) := (G^{ABC}_{h,k}(x-y(\xi_0, x, y)))_{\xi \in \mathcal{J}_N(h,k) \subseteq K^2} \) satisfies a Fuchsian system

\[
\frac{d}{dx} G^{x-y}(x, \xi_0) = \frac{\Lambda_{AB}(x, y)}{x-y} G^{x-y}(x, \xi_0)
\]

over \( \mathbb{D}_{|\xi_0|}(0) \) with polar locus \( \{\xi_0\} \). Furthermore, the sets of nonzero eigenvalues (without multiplicities) of the constant matrices \( \Lambda_{AB}(x_0, 0) \) of (D1) and \( \Lambda_{AB}(y_0, 0) \) of (D2) are determined by only the choices of the bases \( J_{PA} \) and \( J_{PB} \) of \( P_A \) and \( P_B \).

As a corollary of (D1) for \( G^{ABC}_{h,k} \) and (D2) for \( G^{ABC}_{h,k} \), we will obtain the following:

**Corollary 3.3** \( F^{ABC}_{h,k}(\xi, x, y) \) and \( F^{ABC}_{h,k}(\xi, x-y, y) \) are locally normally convergent on \( \{(x, y) \in \mathbb{C}^2 \mid 0 < |x| < |x| \} \) and \( \{(x, y) \in \mathbb{C}^2 \mid 0 < |x-y| < |y| \} \), respectively.

Set \( \mathcal{D}^2 = \{(x, y) \in \mathbb{C}^2 \mid 0 < |x-y| < |y| < |x| \} \) and \( x, y, x-y \notin \mathbb{R} \). Let \( y_0 \notin \mathbb{R} \) and we take a principal branch \( G_{h,k}^{ABC}(x-y(\xi_0, x, y)) \) of \( G_{h,k}^{ABC}(x-y(\xi_0, x, y)) \) on a small domain \( \mathcal{D}^2_{(x,y_0)} = \{ x \in \mathbb{C} \mid (x, y_0) \in \mathcal{D}^2 \} \). As we will explain in Proposition 2.9 in \( \S 2 \), there is a finite subset \( \Delta' = \{ d_1, ..., d_p \} \subseteq \mathbb{C} \) which depends only on the choice of \( J_{PA} \) and \( J_{PB} \) such that all components of solutions of the Fuchsian system (D2) with polar locus \( \{y_0\} \) are written as

\[
\sum_{d=\Delta'} \sum_{m=\in\mathbb{N}} \sum_{t=0}^{K} r_{d+m,t}(x-y_0)^{d+m} \log^t(x-y_0)
\]

with \( r_{d+m,t} \in \mathbb{C} \) at \( \mathbb{D}_{|y_0|}(0) = \{ x \in \mathbb{C} \mid 0 < |x-y_0| < |y_0| \} \). Since this is true for every \( y = y_0 \notin \mathbb{R} \) and \( \xi \), there is \( r_{d+m,t}^{h,k}(\xi, y) \in \mathbb{C} \) and \( K(\xi) \in \mathbb{N} \) such that

\[
\tilde{G}_{h,k}^{ABC}(x-y(\xi, x, y)) = \sum_{\xi \in \mathcal{J}_N(h,k)} \sum_{m=\in\mathbb{N}} \sum_{t=0}^{K(\xi)} r_{d+m,t}(\xi, y)(x-y)^{d+m} \log^t(x-y_0)
\]

Multiplying it by \( (x-y)^{-\varepsilon} \log^h(x) \log^k(y) \) for \( h, k \) and taking a sum of them and replacing \( \log(x) \) by \( \sum_{j=0}^{\infty} \frac{(-1)^j}{j+1} (x-y)^{j+1} + \log(y) \), we obtain an expansion of a principal branch.
A singularity of such a meromorphic system is a point in $U$ (at least one of the coefficients of $F$ is not analytic at that point).

For technical reasons, one usually fixes a discrete subset $\Sigma$ of poles of $\Phi$, and speaks of meromorphic systems $\frac{d}{dz}Y = AY$ over $U$ with polar locus $\Sigma$ if the "true polar locus" (i.e., the set of poles of $A$) is contained in $\Sigma$.

A meromorphic system of linear differential equations is a differential equation of the form $\frac{d}{dz}Y = AY$, where $A \in M_{r \times r}(\mathcal{M}(U))$ is a square matrix whose coefficients are meromorphic functions over some complex domain $U \subseteq \mathbb{C}$ (open, nonempty).

A singularity of such a meromorphic system is a point in $U$ corresponding to a pole of (at least one of the coefficients of) $A$.

For technical reasons, one usually fixes a discrete subset $\Sigma \subseteq U$, and speaks of meromorphic systems $\frac{d}{dz}Y = AY$ over $U$ with polar locus $\Sigma$ if the "true polar locus" (i.e., the set of poles of $A$) is contained in $\Sigma$.

A meromorphic system $\frac{d}{dz}Y = \frac{A}{z-z_0}Y$ with a holomorphic matrix function $A$ of size $r \times r$ defined on $U$ is called a Fuchsian system and its singularity at $z = z_0$ is called a Fuchsian singularity.

Let $\frac{d}{dz}Y = AY$ be a meromorphic system of rank $r$ over $U \subseteq \mathbb{C}$. Let $U' \subseteq U$ be open. A fundamental solution of $\frac{d}{dz}Y = AY$ over $U'$ is a matrix-valued meromorphic function $\Phi \in M_{r \times r}(\mathcal{M}(U'))$ such that

$$\frac{d}{dz}\Phi(z) = A(z)\Phi(z) \quad \forall z \in U',$$
and moreover
\[ \det(\Phi)(z) \neq 0 \quad \forall z \in U'. \]

From now on, let \( z_0 \in U \) and let \( A = \sum_{k \geq 0} (z - z_0)^k A_k \) with \( A_k \in M_{r \times r}(\mathbb{C}) \) and we will consider
\[
\frac{d}{dz} Y = \frac{A}{z - z_0} Y, 
\]
which we will call Fuchsian system. Set \( D_R(z_0) = \{ z \in \mathbb{C} \mid |z - z_0| < R \} \). If \( A = A_0 \in M_{r \times r}(\mathbb{C}) \) and \( z_0 \in U \), then
\[
\frac{d}{dz} Y = \frac{A_0}{z - z_0} Y, 
\]
is called Euler system. In this case, there is \( G \in GL_r(\mathbb{C}) \) such that \( GA_0G^{-1} \) has a Jordan standard form and \( \frac{d}{dz}(GY) = G \frac{d}{dz} Y = \frac{GA_0G^{-1}}{z - z_0} (GY) \).

**Lemma 2.3** If an Euler system \((2.2)\) has a Jordan cell \( A_0 = J_r(a) \), that is, \( Y = \{ (f_1, \ldots, f_r) \} \) satisfies \((z - z_0) \frac{d}{dz} f_i = a f_i + f_{i+1} \) for \( i = 1, \ldots, r \) with \( f_{r+1} = 0 \), then \( \Phi = \{ (z - z_0)^s \times (\log(z - z_0)^t), 1, 0, \ldots, 0 \mid s = 0, \ldots, r - 1 \} \) is a fundamental solution, where \( (\cdot)^t \) denotes the transpose of \((\cdot)\).

Therefore, in generally, all components of solutions of Euler system \((2.2)\) are written as
\[
f(z) = \sum_{i=1}^{r} \sum_{t=0}^{r} r_{i,t}(z - z_0)^{d_i} \log^{t}(z - z_0) 
\]
with \( r_{i,t} \in \mathbb{C} \) and eigenvalues \( d_i \) of \( A_0 \). We also note that if \( \frac{d}{dz} Y = \frac{A}{z - z_0} Y \), then
\[
\frac{d}{dz} ((z - z_0)^d Y) = \frac{(A + dE_r)}{z - z_0} ((z - z_0)^d Y), 
\]
where \( E_r \) is the identity matrix of size \( r \).

**Definition 2.4** Let \( \frac{d}{dz} Y = AY \) be a meromorphic system over \( U \) with polar locus \( \Sigma \).

- A holomorphic gauge transformation over \( U' \subseteq U \) is the change of variable \( Y = \Delta Z \), yielding a meromorphic system \( \frac{d}{dz} Z = BZ \) over \( U' \) with polar locus \( \Sigma \cap U' \), where \( \Delta \in GL_r(\mathcal{O}(U')) \), i.e. \( \Delta \) is a holomorphic matrix function over \( U' \) with non-vanishing determinant.

- A meromorphic gauge transformation over \( U' \subseteq U \) is the change of variable \( Y = \Delta Z \), yielding a meromorphic system \( \frac{d}{dz} Z = BZ \) over \( U' \) with polar locus \( Z \cap U' \), where \( \Delta \in GL_r(\mathcal{M}(U')) \), i.e. \( \Delta \) is a meromorphic matrix function over \( U' \) whose determinant is not identically zero on any connected component of \( U' \), such that moreover, \( \Delta|_{U' \setminus \Sigma} \in M_{r \times r}(\mathcal{O}(U' \setminus \Sigma)) \) and \( \Delta^{-1}|_{U' \setminus \Sigma} \in M_{r \times r}(\mathcal{O}(U' \setminus \Sigma)) \).

- As usual, two meromorphic systems over \( U \) with polar locus \( \Sigma \) are said to be holomorphically (resp. meromorphically) gauge equivalent, if they are related via a global holomorphic (resp. meromorphic) gauge transformation.
Theorem 2.7 (Theorem 3.8 in [2]) Let $A(z) = \frac{\text{d}z}{z} Y = \frac{\text{d}z}{z}$ of rank $r$ over $\mathbb{D}$, with polar locus $\{0\}$, is said to be of Levelt normal form if $B = \sum_{k \geq 0} B_k z^k$ such that for any $k \in \mathbb{N}$, the matrix $B_k \in M_{r,r} (\mathbb{C})$ satisfies $\text{ad}(B_{0,s})(B_k) = k B_k$, where $B_0 = B_{0,s} + B_{0,n}$ is the Dunford decomposition of $B_0$ into semisimple part $B_{0,s}$ and nilpotent part $B_{0,n}$ with $B_{0,s} B_{0,n} = B_{0,n} B_{0,s}$.

Lemma 2.6 The eigenvalues of $B(1) = \sum_{k=0}^{\infty} B_k$ are equal to the eigenvalues of $B_0$.

[Proof] We may assume that $B_{0,s}$ is a diagonal matrix $(\lambda_1, \ldots, \lambda_r)$ and $B_{0,n}$ is an upper triangular matrix. Furthermore, we may assume $\Re(\lambda_1) \geq \cdots \geq \Re(\lambda_r)$. Then since $\text{ad}(B_{0,s}) E_{i,j} = (\lambda_i - \lambda_j) E_{i,j}$, $\text{ad}(B_{0,s}) B_k = k B_k$ means that $B_k$ are strictly upper triangular and $B(1) = \sum_{k=0}^{\infty} B_k$ is an upper triangular matrix with diagonal entries $\{\lambda_1, \ldots, \lambda_r\}$. □

Theorem 2.7 (Theorem 3.8 in [2]) Let $A(z) = \frac{\text{d}z}{z} Y = \frac{\text{d}z}{z}$ be a Fuchsian system over $\mathbb{D}$. This system is holomorphically gauge equivalent to some Fuchsian system $\frac{\text{d}z}{z} \hat{Y} = \frac{\text{d}z}{z} \hat{Y}$ of Levelt normal form over $\mathbb{D}$.

Remark 2.1 Under the setting of Theorem 2.7, the set of eigenvalues of $B(0)$ coincides with the set of eigenvalues of $A(0)$ counted with multiplicity.

Theorem 2.8 (Lemma 3.7 in [2]) Let $B_{0,s} \in M_{r,s} (\mathbb{C})$ be semisimple. Then there exists the unique semisimple matrix $L$ such that $L$ commutes with $B_{0,s}$ and has only integer eigenvalues and any eigenvalues $\mu$ of $B_{0,s} - L$ satisfies $\Re(\mu) \in [0, 1]$. Moreover, if $\frac{\text{d}z}{z} Y = \frac{\text{d}z}{z} Y$ is a Fuchsian system of Levelt normal form with $B(0) = B_{0,s} + B_{0,n}$ as above, then $B = z^t B(1) z^{-L}$.

Remark 2.2 As mentioned in the proof of Corollary 3.9 in [2], under the assumption in Theorem 2.8, $\hat{Y} := z^{-L} Y$ satisfies an Euler system $\frac{\text{d}z}{z} \hat{Y} = \frac{\text{d}z}{z} \hat{Y}$.

Using the these results, we get the following result.

Proposition 2.9 Let $A(z) = \frac{\text{d}z}{z} Z$ be a Fuchsian system with $B \in M_{r,s} (\mathbb{C}(U))$ and let $\{d_1, \ldots, d_r\}$ be the set of eigenvalues of the constant matrix $B_0$. Choose $t_i \in \mathbb{Z}$ so that $0 \leq d_i - t_i < 1$. Then all components of solutions of $\frac{\text{d}z}{z} Z = \frac{\text{d}z}{z} Z$ are written as

$$\sum_{j=1}^{r} \sum_{i=1}^{r} \sum_{m \in \mathbb{N}} \sum_{\ell=0}^{r} r_{i,m,t} z^{d_j - t_j + \ell_i + m} \log^\ell (z)$$

(2.5)

with $r_{i,m,t} \in \mathbb{C}$.

[Proof] If $\frac{\text{d}z}{z} Z = \frac{\text{d}z}{z} Z$ and $\frac{\text{d}z}{z} Y = \frac{\text{d}z}{z} Y$ are holomorphically equivalent, then all components of solutions of one are linear combinations of components of solutions of the other with coefficients in $\mathbb{C}[[z]]$ and so if all components of solutions of $\frac{\text{d}z}{z} Z = \frac{\text{d}z}{z} Z$ satisfy (2.5), then those of solutions of $\frac{\text{d}z}{z} Y = \frac{\text{d}z}{z} Y$ also satisfy (2.5). Therefore, we may assume that $\frac{\text{d}z}{z} Z = \frac{\text{d}z}{z} Z$ is a Fuchsian system of Levelt normal form over $\mathbb{D}$ by Theorem 2.7. Furthermore we may assume that $B_{0,s} = \text{diag}(d_1, \ldots, d_r)$ satisfying $\Re(d_1) \geq \cdots \geq \Re(d_r)$ and $B_{0,n}$ is an upper triangular matrix. In this case, $L$ in Theorem 2.8
is a diagonal matrix diag(\ell_1, ..., \ell_r). Then by Theorem 2.8 and Remark 2.1, we have \(B = z^L B(1) z^{-L}\) and \(\tilde{Y} := z^{-L} Y\) satisfies an Euler system \(\frac{d}{dz} \tilde{Y} = \frac{B(1) - L}{z} \tilde{Y}\). By Lemma 2.6, \(B(1)\) is an upper triangular matrix with diagonal entries \(\{d_1, ..., d_r\}\) and the eigenvalues of \(B(1) - L\) are \(d_1 - \ell_1, ..., d_r - \ell_r\). Hence all components of a solution \(\tilde{Y}\) are written as \(\sum_{j=1}^r \sum_{t \in \mathbb{N}} r_j^j z^{d_j - \ell_j + t} \log^t(z)\) by Lemma 2.3. In this case, since \(Y = z^L \tilde{Y}\), all components of solutions of \(\frac{d}{dz} Y = \frac{B}{z} Y\) are written as \(\sum_{i,j=1}^r \sum_{t \in \mathbb{N}} r_i^j d_i - \ell_i + t \log^t(z)\), as we desired. □

Regarding the radius of convergence of such a solution, we quote the following theorem.

**Theorem 2.10 (Theorem 1.6)** Let \(A = \sum_{k \in \mathbb{N}} A_k (z - z_0)^k\) with \(A_k \in M_{r \times r}(\mathbb{C})\) be a matrix power series with radius of convergence \(R > 0\). Let \(Y = \sum_{k=0}^\infty Y_k (z - z_0)^k\) with \(Y_k \in M_{r \times r}(\mathbb{C})\) be a formal power series satisfying \(\frac{d}{dz} Y = \frac{A}{z - z_0} Y\). Then the radius of convergence of \(Y\) is at least \(R\).

### 2.2 Intertwining operators among \(C_1\)-cofinite modules

Let \(A, B, C\) be \(V\)-modules. Let \(I(\mathcal{C})\) denote the set of logarithmic intertwining operators of type \((\mathcal{A}, \mathcal{B})\). We introduce a generalized concept of (logarithmic) intertwining operators. For a nonempty domain \(U \subseteq \mathbb{C}\) \(\setminus\{0\}\), we define a local intertwining operator \(Y\) of type \((\mathcal{A}, \mathcal{B})\) on \(U\) as a linear map \(A \otimes B \to C \otimes \mathcal{O}(U)\), we denote it by \(\langle \theta, Y(v, z)u \rangle\) for \(v \in A, u \in B, \theta \in C^\wedge\) and \(z \in U\), satisfying

(I 1) [commutativity] \(\langle \theta, \alpha_m Y(v, z)u - Y(v, z)\alpha_m u \rangle = \sum_{j=0}^\infty \binom{m}{j} \langle \theta, Y(\alpha_j v, z)u \rangle z^{m-j},\)

(I 2) [associativity] \(\langle \theta, Y(\alpha_m v, z)u \rangle = \sum_{j=0}^\infty \binom{m}{j} \langle \theta, \alpha_{m-j} z^j Y(v, z)u - (-1)^m Y(v, z)\alpha_j z^{m-j} u \rangle\), and

(I 3) \([L(-1)]\)-derivative property \(\langle \theta, Y(L(-1)v, z)u \rangle = \frac{d}{dz} \langle \theta, Y(v, z)u \rangle\)

for \(\alpha \in V\) and \(m \in \mathbb{Z}\), where \(\langle \theta, \alpha_m Y(v, z)u \rangle \) denotes \(\langle (\alpha_m)^* \theta, Y(v, z)u \rangle\) with a contragredient operator \((\alpha_m)^*\) of \(\alpha_m\).

An aim in this subsection is to show that every local intertwining operator for \(A, B, C\) in \(\mathcal{C}_1\) on a domain \(U\) is a branch of a logarithmic intertwining operator on \(U\).

Let \(W \in \mathcal{C}_1\). Since \(C_1(W) = \text{Span}_\mathbb{C}(\{(\alpha_1 - (P_W + C_1(W))| \alpha \in V_{\geq 1}\})\), we have:

**Lemma 2.11** \(W = \text{Span}_\mathbb{C}(\alpha_1 \cdot \alpha_k w | w \in P_W, \alpha^j \in V_{\geq 1}, k \geq 0\). In particular, \(W\) is a finitely generated \(V\)-module.

Therefore, there are finitely many \(d_1, ..., d_r \in \mathbb{C}\) such that \(W = \bigoplus_{j=1}^{d_1} \bigoplus_{n \in \mathbb{N}} W_{d_1 + n}\) and \(\dim W_s < \infty\) for \(s \in \mathbb{C}\), where \(W_s\) denotes a generalized eigenspace in \(W\) of \(L(0)\) with eigenvalue \(s\). In particular, if \(W\) is indecomposable, then \(W = \bigoplus_{n \in \mathbb{N}} W_{d_1 + n}\) for some \(d \in \mathbb{C}\) and \(W_d \neq 0\). We use the notation \(\text{wt}(w) = s\) for \(w \in W_s\) and \(\text{wt}(\alpha) = \text{wt}(\alpha) - n - 1\) for \(\alpha \in V\). Define \(\text{wt}(1) \in \text{End}_\mathbb{C}(W)\) by \(\text{wt}(w) := \text{wt}(w)\) for \(w \in W\). Since \(W_{(m)} \leq \text{Span}_\mathbb{C}(\alpha_1 \cdot \alpha_k w | w \in P_W, \alpha^j \in V_{\geq 1}, \sum_{j=1}^k \text{wt}(\alpha^j) \leq m)\), we obtain:

**Lemma 2.12** For \(m \in \mathbb{N}\), there is \(\mu(m) \in \mathbb{N}\) such that \(\dim W_{(m)} \leq \mu(m) \times \dim P_W\). Moreover, the restricted dual \(W^\vee = \bigoplus_{n \in \mathbb{N}} \text{Hom}(W_{(m)}, \mathbb{C})\) of an \(\mathbb{N}\)-graded module \(W = \bigoplus_{s \in \mathbb{C}} W_s\) in \(\mathcal{C}_1\) is isomorphic to the contragredient module \(\bigoplus_{s \in \mathbb{C}} \text{Hom}(W_s, \mathbb{C})\) of \(W = \bigoplus_{s \in \mathbb{C}} W_s\). In particular, \(W^\vee\) does not depend on the choice of an \(\mathbb{N}\)-grading on \(W\).
Since \( \text{wt}(\alpha_n w) = \text{wt}(w) + \text{wt}(\alpha) - n - 1 \) and \( L(0) \alpha_n w = (\text{wt}(\alpha) - n - 1) \alpha_n w + \alpha_n L(0) w = (\text{wt}(\alpha_n w) \alpha_n w + \alpha_n (L(0) - \text{wt}(w)) w \), we get:

**Lemma 2.13** \( L(0) - \text{wt}(1) \) is a \( V \)-homomorphism. If \( W \in \mathcal{NC}_1 \), then there is \( K \in \mathbb{N} \) such that \( (L(0) - \text{wt}(1))^K W = 0 \) and \( (L(0) - \text{wt}(1))^K W^\vee = 0 \).

**[Proof]** As we have just proved, \( L(0) - \text{wt}(1) \) is a \( V \)-homomorphism. Since \( \dim P_W < \infty \), there is \( n \in \mathbb{N} \) such that \( P_W \subseteq \oplus_{i=0}^n W(i) \), which is \( L(0) - \text{wt}(1) \)-invariant and \( \dim \oplus_{i=0}^n W(i) < \infty \). Hence \( (L(0) - \text{wt}(1))^K (\oplus_{i=0}^n W(i)) = 0 \) for some \( K \). Since \( W \) is generated from \( P_W \) by the actions of \( V \), we have \( (L(0) - \text{wt}(1))^K W = 0 \). From \( (L(0) - \text{wt}(1))^K W^\vee, W = (W^\vee, (L(0) - \text{wt}(1))^K W = 0, \) we also obtain \( (L(0) - \text{wt}(1))^K W^\vee = 0 \). \( \square \)

**Lemma 2.14** Let \( A, B, C \in \mathcal{NC}_1 \) and let \( \mathcal{Y} \) be a local intertwining operator of type \( ^C A_B \) on a domain \( U \) with a form \( \langle \theta, \mathcal{Y}(v, z) u \rangle = \sum_{h=0}^{t_{h,v,u}} \langle \theta, Y_h(v, z) u \rangle \log^h(z) \) with \( \mathbb{C} \)-formal power series \( Y_h(v, z) u \) and \( K(\theta, v, u) \in \mathbb{N} \). Then there is \( K \in \mathbb{N} \) such that \( K(\theta, v, u) \leq K \).

**[Proof]** Since

\[
\sum_{h=0}^{K(\theta, v, u)} \langle \theta, Y_h(L(-1)v, z) u \rangle \log^h(z) = \frac{\partial}{\partial z} \left\{ \sum_{h=0}^{K(\theta, v, u)} \langle \theta, Y_h(v, z) u \rangle \log^h(z) \right\},
\]

we have \( Y_{k+1}(v, z) u = Y_k(L(-1)v, z) u - \frac{\partial}{\partial z} (Y_k(v, z) u) \times \frac{1}{k+1} \). We also get

\[
\langle \theta, (L(0) - \text{wt}(1))(Y_k(v, x) u) \rangle = \langle \theta, (L(0) - \text{wt}(1))(\sum m v_{m,k} u z^{-m-1}) \rangle
\]

\[
= (\text{gr}(v) + \text{gr}(u)) \langle \theta, (L(0) - \text{wt}(1))(\sum m v_{m,k} u z^{-m-1}) \rangle
\]

\[
= (\text{gr}(v) + \text{gr}(u)) \langle \theta, Y_k(v, x) u \rangle + \frac{\partial}{\partial z} \langle \theta, Y_k(v, z) u \rangle z.
\]

Using the notation \( \Xi = L(0) - \text{wt}(1) \) and \( \{X + Y + Z\} = \sum_{i,j,p} X^i Y^j Z^p \), we obtain

\[
\langle \theta, \{(X + Y + Z)^\ell \} \rangle = \sum_{i,j,p} X^i Y^j Z^p,
\]

so that

\[
\langle \theta, Y_k(v, x) u \rangle = \sum_{i,j,p} \langle \theta, Y_k(v, x) \Xi^i \rangle \langle \theta, Y_k(v, x) \Xi^p \rangle.
\]

Since there is \( K \in \mathbb{N} \) such that \( (\Xi)^K A = (\Xi)^K B = (\Xi)^K C = 0 \) by Lemma 2.13, we have \( \langle \theta, Y_k(v, z) u \rangle = 0 \) for \( k \geq 3K \), that is, we can take \( K(\theta, v, u) \leq 3K \) for any \( \theta, v, u \). \( \square \)

We will use the following theorem after the proof of Theorem 4.1.

**Theorem 2.15** Let \( A, B, C \in \mathcal{NC}_1 \) and \( N \in \mathbb{N} \). For a local intertwining operator \( \mathcal{Y} \) of type \( ^C A_B \) over \( U \) satisfying \( (11) \sim (13) \), there is a finite subset \( \Delta_N \) of \( C \) and \( K \in \mathbb{N} \) such that \( \langle \theta, \mathcal{Y}(v, z) u \rangle \) has the following expression:

\[
\langle \theta, \mathcal{Y}(v, z) u \rangle = \sum_{t=0}^{K} \sum_{s=-1 \in \Delta_N - \text{gr}(v) - \text{gr}(u) + \text{gr}(\theta)} \langle \theta, v_{s,t} u \rangle z^{-s-1} \log^t(z)
\]

for any \( \theta \in C_{\Delta_N}, v \in A, u \in B \), where \( v_{s,t} \in \text{Hom}(B, C) \) for \( v \in V \) and we take a principal branch of \( z^{-s} \) and \( \log(z) \). Furthermore, \( \tilde{\mathcal{Y}}(v, z) = \sum_{t=0}^{K} \sum_{s \in \mathbb{C}} v_{s,t} z^{-s} \log^t(z) \in I_{(C)} \). Namely, \( \mathcal{Y} \) is a branch of a logarithmic intertwining operator \( \tilde{\mathcal{Y}} \) on \( U \).
Set $S(\theta, v, u; z) = \langle \theta, \mathcal{Y}(v, z)u \rangle$. Choose $N \in \mathbb{N}$ so that $\text{gr}(\theta) \leq N$. Let $Q_N = \{ \theta^p : p \in \mathcal{Q}_N \}$, $J_A = \{ v^i \mid i \in \mathcal{P}_A \}$, $J_B = \{ w^j \mid j \in \mathcal{P}_B \}$ be bases of $C^{(\leq N)}$, $P_A$, $P_B$, respectively. From (I 1) and (I 2), for $\alpha \in V_{\geq 1}$, we have

$$
S(\theta, \alpha_{-1}v, u; z) = \sum_{j=0}^{\infty} S((\alpha_{-j})^*\theta, v, u; z) z^j + \sum_{j=0}^{\infty} S(\theta, v, \alpha_j u; z) z^{-j-1} + \sum_{j=0}^{\infty} (1) S(\theta, v, u; z) z^{-j-1}.
$$

Define $\text{gr}^- (\theta, v, u) = \text{gr}(\theta) + \text{gr}(v) + \text{gr}(u)$ and the total grade $\text{gr}^+ (\theta, v, u) = \text{gr}(\theta) + \text{gr}(v) + \text{gr}(u)$ and set $T(\theta, v, u; z) = \mathcal{F}(\theta, v, u; z) z^{\text{gr}^- (\theta, v, u)}$. Then we obtain

$$
T(\theta, \alpha_{-1}v, u; z) = \sum_{j=0}^{\infty} T((\alpha_{-j})^*\theta, v, u; z) + \sum_{j=0}^{\infty} T(\theta, v, \alpha_j u; z) z^{-j-1}.
$$

Therefore, for a vector valued function $Z = (T(\theta^p, v^i, u^j; z))_{(p, i, j) \in \mathcal{Q}_N \times \mathcal{P}_A \times \mathcal{P}_B}$ of size $s$, there is a matrix $A_0 \in M_{s \times s}(\mathbb{C})$ such that

$$
\frac{d}{dz} Z = A_0 Z,
$$

where $s = |\mathcal{Q}_N| \times |\mathcal{P}_A| \times |\mathcal{P}_B|$, that is, $Z$ satisfies an Euler system (2.7) and $T(\theta, v, u; z)$ is a component of a solution of (2.7) for $(\theta, v, u) \in \mathcal{Q}_N \times J_A \times J_B$. By Lemma 2.3, all components of solutions $T(\theta, v, u; z)$ of (2.7) on $\mathbb{C} \setminus \{0\}$ are linear combinations of $\{ z^s \log^t(z) \mid s, t \in \mathbb{N} \}$. Namely, for each $N, v, u$ with $N \geq \text{gr}(\theta)$, there is a finite set $\Delta_N \subseteq \mathbb{C}$ and $K(\theta, v, u) \in \mathbb{N}$ such that

$$
\langle \theta, \mathcal{Y}(v, z)u \rangle = \sum_{s_j \in \Delta_N - \text{gr}(v) - \text{gr}(u) + \text{gr}(\theta)} \sum_{t=0}^{K(\theta, v, u)} \langle \theta, v_{-s_j-1}u \rangle z^{s_j} \log^t(z)
$$

with $v_{-s_j-1} \in \text{Hom}(B, C)$. Since $A$ and $B$ are generated from $P_A$ and $P_B$ as $V$-modules, respectively, we know that Eq.(2.8) holds for any $\theta \in C^\vee, v \in A, u \in B$ and $z \in U$ by using (2.6). Although our argument depends on the choice of $N \geq \text{gr}(\theta)$, we note that there is $K$ such that $K(\theta, v, u) \leq K$ by Lemma 2.4. Therefore it is easy to check that $\hat{\mathcal{Y}}(v, z) = \sum_{t=0}^{K} \sum_{s \in \mathbb{C}} v_{s,t} z^{-s-1} \log^t(z)$ is a logarithmic intertwining operator of type $( \overset{\circ}{C}_{AB} )$ and $\mathcal{Y}$ is a branch of it on $U$.

[**Fusion product**] In this paper, for $N$-gradable $V$-modules $A$ and $B$, a fusion product $(A \boxtimes B, \mathcal{Y}^{AB})$ is defined to be a pair of an $N$-gradable $V$-module $A \boxtimes B$ and $\mathcal{Y}^{AB} \in I(\overset{\circ}{A}_{AB})$ satisfying the universal property for $N$-gradable modules, that is, for any $N$-gradable $V$-module $C$ and $\mathcal{Y} \in I(\overset{\circ}{A}_{B})$, there is the unique isomorphism $\phi : A \boxtimes B \to C$ such that $\phi(\mathcal{Y}^{AB}(v, z)u) = \mathcal{Y}(v, z)u$ for $v \in A$ and $u \in B$.  

\[ \square \]

10
2.3 Borcherds Identities for four-point correlation functions

From the commutativity (I 1) and associativity (I 2), we have the following identities:

**Lemma 2.16 (Borcherds identities)** Let $\mathcal{Y}^1 \in I(A^{(ABC)}_B), \mathcal{Y}^2 \in I(B^{(A)}_C), \mathcal{Y}^3 \in I(A^{(AB)}_B), \mathcal{Y}^4 \in I(A^{(ABC)}_B)$. For $\theta \in (A^{(BC)})^\vee, \theta' \in (A^{(AB)}C)^\vee, v \in A, u \in B$, and $w \in C$ we set $F^{12}(\xi; x, y) = \langle \theta, \mathcal{Y}^1(v, x) \mathcal{Y}^2(u, y)w \rangle$ and $F^{34}(\xi; x, y) = \langle \theta', \mathcal{Y}^3(v, x - y)u, y)w \rangle$. Then for $\alpha \in V$ and $n \in \mathbb{Z}$, as formal $\mathbb{C}$-power series with logarithm functions, we have:

\[
\begin{align*}
(1A) & \quad F^{12}(\alpha_n^*; x, y) = \sum_{j=0}^{\infty} \binom{n}{j} F^{12}(\alpha_j^*; x, y)x^{n-j} \\
& \quad + \sum_{j=0}^{\infty} \binom{n}{j} F^{12}(\alpha_j^*; x, y)y^{n-j} + F^{12}(\alpha_n^*; x, y).
\end{align*}
\]

\[
\begin{align*}
(1B) & \quad F^{34}(\alpha_n^*; x, y) = \sum_{j=0}^{\infty} \binom{n}{j} F^{34}(\alpha_j^*; x, y)t_{y, x-y}\{x^{n-j}\} \\
& \quad + \sum_{j=0}^{\infty} \binom{n}{j} F^{34}(\alpha_j^*; x, y)t_{y, x-y}\{x^{n-j}\} + F^{34}(\alpha_n^*; x, y).
\end{align*}
\]

\[
\begin{align*}
(2A) & \quad F^{12}(\alpha_n^*; x, y) = \sum_{j=0}^{\infty} \binom{n}{j} F^{12}(\alpha_{n-j}^*; x, y)(-x)^j \\
& \quad - \sum_{j=0}^{\infty} \binom{n}{j} F^{12}(\alpha_j^*; x, y)x_{y, x}\{(y - x - y)^{n-j}\} - \sum_{j=0}^{\infty} \binom{n}{j} F^{12}(\alpha_j^*; x, y)(-x)^{n-j}.
\end{align*}
\]

\[
\begin{align*}
(2B) & \quad F^{34}(\alpha_n^*; x, y) = \sum_{j=0}^{\infty} \binom{n}{j} F^{34}(\alpha_{n-j}^*; x, y)t_{y, x-y}\{(y - x - y)^{n-j}\} \\
& \quad - \sum_{j=0}^{\infty} \binom{n}{j} F^{34}(\alpha_j^*; x, y)x_{y, x-y}\{(y - x - y)^{n-j}\}.
\end{align*}
\]

\[
\begin{align*}
(3A) & \quad F^{12}(\alpha_n^*; x, y) = \sum_{j=0}^{\infty} \binom{n}{j} F^{12}(\alpha_{n-j}^*; x, y) y^j \\
& \quad - \sum_{j=0}^{\infty} \binom{n}{j} F^{12}(\alpha_j^*; x, y)x_{y, x}\{(y - x - y)^{n-j}\} - \sum_{j=0}^{\infty} \binom{n}{j} F^{12}(\alpha_j^*; x, y)(y)^{n-j}.
\end{align*}
\]

\[
\begin{align*}
(3B) & \quad F^{34}(\alpha_n^*; x, y) = \sum_{j=0}^{\infty} \binom{n}{j} F^{34}(\alpha_{n-j}^*; x, y) t_{y, x-y}\{(y - x - y)^{n-j}\} \\
& \quad - \sum_{j=0}^{\infty} \binom{n}{j} F^{34}(\alpha_j^*; x, y)x_{y, x-y}\{(y - x - y)^{n-j}\}.
\end{align*}
\]

\[
\begin{align*}
(4A) & \quad F^{12}(\alpha_n^*; x, y) = F^{12}(\alpha_n^*; x, y) - \sum_{j=0}^{\infty} \binom{n}{j} F^{12}(\alpha_j^*; x, y)x^{n-j} \\
& \quad - \sum_{j=0}^{\infty} \binom{n}{j} F^{12}(\alpha_j^*; x, y)y^{n-j}.
\end{align*}
\]

\[
\begin{align*}
(4B) & \quad F^{34}(\alpha_n^*; x, y) = F^{34}(\alpha_n^*; x, y) - \sum_{j=0}^{\infty} \binom{n}{j} F^{34}(\alpha_j^*; x, y)t_{y, x-y}\{x^{n-j}\} \\
& \quad - \sum_{j=0}^{\infty} \binom{n}{j} F^{34}(\alpha_j^*; x, y)y^{n-j}.
\end{align*}
\]

Here $(\alpha_n^*)^*$ denotes its adjoint operator of $\alpha_n$ and $t_{x, y}\{(x - y)^s\} := \sum_{j=0}^{\infty} \binom{n}{j} x^{s-j}(y)^j$ and $t_{y, x-y}\{x^s\} := t_{y, x-y}\{(y + (x - y))^s\} = \sum_{j=0}^{\infty} \binom{n}{j} y^{s-j}(x - y)^j$.

**Comment 2** (1) The coefficients are in $\mathbb{C}[x^\pm 1, y^\pm 1, \tau_{x, y}\{(x - y)^\pm 1\}, \tau_{y, x-y}\{x^\pm 1\}]$. Hence the same equations also hold for coefficients of logarithm functions.

(2) Then for $n \in \mathbb{Z}, m, \ell = 1, 2, 3, 4,$ and $j \geq 0$, the coefficients of $F^{12}(\alpha_n^*; x, y)$ in RHS of $(mA)$ for $F^{12}(\alpha_n^*; x, y)$ are equal to the corresponding coefficients of $F^{34}(\alpha_n^*; x, y)$ in RHS of $(mB)$ for $F^{34}(\alpha_n^*; x, y)$ excepting the expansions by $t_{x, y}$ or $t_{y, x-y}$. Hence the corresponding coefficients in $(mA)$ and $(mB)$ are same on a domain $\{(x, y) \in \mathbb{C}^2 | 0 < |x - y| < |y| < |x|\}$.

**Proof** First, we prepare the following equations:

\[
\begin{align*}
\sum_{\ell=0}^{\infty} \sum_{i=0}^{\infty} \binom{n}{i} \binom{i}{\ell} x^{n-j}y^{\ell}\xi \mathcal{Y} = \sum_{\ell=0}^{\infty} \binom{n}{i} \xi \mathcal{Y} (x + y)^{n-\ell} Z^\ell \\
\sum_{i, j \in \mathbb{N}} \binom{n}{i} \binom{i}{j} (-1)^{i+j}(x - y)^j y^j Z^{i+j} = \sum_{\ell=0}^{\infty} \binom{n}{i} (-1)^{i+j} t_{y, x-y}\{x^\ell\} Z^\ell
\end{align*}
\]

and we will use the equations between the coefficients of $Z^\ell$ of LHS and RHS.
By commutativity (I 1) and associativity (I 2), we have:

(1A) \( \langle \langle (\alpha_n)^{\theta}, \mathcal{Y}^1(v, x) \rangle \mathcal{Y}^2(u, y) \rangle w = \langle \langle \alpha_n, \mathcal{Y}^1(v, x) \mathcal{Y}^2(u, y) \rangle \theta \rangle w \)

\[ = \theta, \mathcal{Y}^1(v, x) \mathcal{Y}^2(u, y) (\alpha_n) \rangle w + \sum_{j=0}^{\infty} \langle \langle \alpha_n, \mathcal{Y}^1(v, x) \mathcal{Y}^2(u, y) \rangle \theta \rangle w x^{n-j} \]
\[ + \sum_{j=0}^{\infty} \langle \langle \alpha_n, \mathcal{Y}^1(v, x) \mathcal{Y}^2(u, y) \rangle \theta \rangle w y^{n-j}. \]

(1B) \( \langle \alpha_n^{\theta}, \mathcal{Y}^3(v, x - y) u, y) \rangle w = \langle \langle \alpha_n, \mathcal{Y}^3(v, x - y) u, y) \rangle \theta \rangle w \)

\[ = \theta, \mathcal{Y}^3(v, x - y) u, y) (\alpha_n) \rangle w + \sum_{j=0}^{\infty} \langle \langle \alpha_n, \mathcal{Y}^3(v, x - y) u, y) \rangle \theta \rangle w \]
\[ + \sum_{j=0}^{\infty} \langle \langle \alpha_n, \mathcal{Y}^3(v, x - y) u, y) \rangle \theta \rangle w y^{n-j}. \]

(2A) \( \langle \theta, \mathcal{Y}^1(\alpha_n v, x) \mathcal{Y}^2(u, y) \rangle w = \sum_{j=0}^{\infty} \langle \langle \theta, \mathcal{Y}^1(\alpha_n v, x) \mathcal{Y}^2(u, y) \rangle \theta \rangle w x^{n-j} \]
\[ - \sum_{j=0}^{\infty} \langle \langle \theta, \mathcal{Y}^1(\alpha_n v, x) \mathcal{Y}^2(u, y) \rangle \theta \rangle w y^{n-j}. \]

(2B) \( \langle \theta, \mathcal{Y}^3(\alpha_n v, x) \mathcal{Y}^3(u, y) \rangle w = \sum_{j=0}^{\infty} \langle \langle \theta, \mathcal{Y}^3(\alpha_n v, x) \mathcal{Y}^3(u, y) \rangle \theta \rangle w x^{n-j} \]
\[ - \sum_{j=0}^{\infty} \langle \langle \theta, \mathcal{Y}^3(\alpha_n v, x) \mathcal{Y}^3(u, y) \rangle \theta \rangle w y^{n-j}. \]

(3A) \( \langle \theta, \mathcal{Y}^1(v, x) \mathcal{Y}^2(u, y) \rangle w = \sum_{j=0}^{\infty} \langle \langle \theta, \mathcal{Y}^1(v, x) \mathcal{Y}^2(u, y) \rangle \theta \rangle w x^{n-j} \]
\[ + \sum_{j=0}^{\infty} \langle \langle \theta, \mathcal{Y}^1(v, x) \mathcal{Y}^2(u, y) \rangle \theta \rangle w y^{n-j}. \]

(3B) \( \langle \theta, \mathcal{Y}^3(v, x - y) (\alpha_n u, y) \rangle w = \langle \langle \theta, \mathcal{Y}^3(v, x - y) (\alpha_n u, y) \rangle \theta \rangle w \)
\[ - \sum_{j=0}^{\infty} \langle \langle \theta, \mathcal{Y}^3(v, x - y) (\alpha_n u, y) \rangle \theta \rangle w x^{n-j} \]
\[ - \sum_{j=0}^{\infty} \langle \langle \theta, \mathcal{Y}^3(v, x - y) (\alpha_n u, y) \rangle \theta \rangle w y^{n-j}. \]
Comment 3 Since $\text{gr}((\alpha_{-1})^\ast \theta) < \text{gr}(\theta)$, we have $(\alpha_{-1})^\ast \theta \in (A(BC))'_\leq N$ for $\alpha \in V_{\geq 1}$ and $j \in \mathbb{N}$. Since $\text{gr}(\alpha_j \omega) = \text{gr}(\alpha_{-1} \omega) - j - 1$ for $j \in \mathbb{N}$, the total grades of quadruples of each term in RHS are less than the total grade of quadruples in LHS, that is, $\text{gr}(((\alpha_{-1})^\ast [i] \xi), \text{gr}(\alpha_k \xi) < \text{gr}(\alpha_{-1} \xi)$. Therefore by iterating these reductions as long as we have an element of $C_1(A), C_1(B)$ or $C_1(C)$ in the second, third or fourth coordinates of quadruples, we finally get an expression of $F_{h,k}^{A(BC)}(\xi'; x, y)$ as a linear combination of

\[
\{ F_{h,k}^{A(BC)}(\vec{\eta}; x, y) : \vec{\eta} \in J_0^N \}
\]

with coefficients in $\mathbb{C}[x, x^{-1}, y, y^{-1}, t_{x,y} \{ (x-y)^{-1} \}]$ for any $\xi \in \Omega_N$.

Comment 4 Since Borcherds identities for $F_{h,k}^{A(BC)}$ and $F_{h,k}^{(AB)C}$ have the same coefficients except the way of expansions, we can get Borcherds identities (2B) $\sim$ (4B) by viewing $F(\xi)$ as $F_{h,k}^{(AB)C}(\xi'; x, y)$ and expanding all coefficients by $t_{x,y}$.
Furthermore, for the residue classes of coefficients modulo $C$ coefficients in $G$, Theorem 3.1 Here the first columns are Borcherds identities for $n$ and $x, y$. We hence consider

$$
\begin{align*}
\mathcal{G}_{h,k}^{(BC)}(\vec{\xi}, x, y) & := F_{h,k}^{(BC)}(\vec{\xi}; x, y) y^{gr_{234}(\vec{\xi})}, \\
\mathcal{G}_{h,k}^{(BC):x-y}(\vec{\xi}, x, y) & := F_{h,k}^{(BC)}(\vec{\xi}; x, y) (x - y)^{gr_{234}(\vec{\xi})}, \\
\mathcal{G}_{h,k}^{(ABC)(C)}(\vec{\xi}, w; x, y) & := F_{h,k}^{(ABC)(C)}(\vec{\xi}; x, y) y^{gr_{234}(\vec{\xi})}, \quad \text{and} \\
\mathcal{G}_{h,k}^{(ABC):x-y}(\vec{\xi}, x, y) & := F_{h,k}^{(ABC):x-y}(\vec{\xi}; x, y) (x - y)^{gr_{234}(\vec{\xi})}.
\end{align*}
$$

(3.1)

Then we have:

**Theorem 3.1** For $\vec{\xi} = (\theta, \nu, u, w)$ with $\theta \in J_N$, $\alpha \in \mathbb{Z}$, and $x_0 \neq 0 = y_0$, (1) $G_{h,k}^{(BC)}(\vec{\xi}; x_0, y)$ is a linear combination of $\{G_{h,k}^{(ABC)}(\vec{\mu}; x_0, y) \mid \vec{\mu} \in J_N^0\}$ with coefficients in $\mathbb{C}[x_0, y, x_0 - y]$, with $\vec{\mu} \in J_N^0$ and

$$
\begin{align*}
\mathcal{G}_{h,k}^{(BC):x-y}(\vec{\xi}, x, y) & := \sum_{j=0}^{\infty} \mathcal{G}_{h,k}^{(BC):x-y}(\vec{\xi}; x, y) y^{j} (\mod \mathbb{C}[x - y_0]) (x - y_0), \\
\mathcal{G}_{h,k}^{(ABC):x-y}(\vec{\xi}, x, y) & := \sum_{j=0}^{\infty} (-1)^{j+1} \mathcal{G}_{h,k}^{(ABC):x-y}(\vec{\xi}; x, y) y^{j} (\mod \mathbb{C}[x - y_0]) (x - y_0), \\
\mathcal{G}_{h,k}^{(ABC):y}(\vec{\xi}, x, y) & := \sum_{j=0}^{\infty} \mathcal{G}_{h,k}^{(ABC):y}(\vec{\xi}; x, y) y^{j} (\mod \mathbb{C}[y]), \\
\mathcal{G}_{h,k}^{(ABC):y}(\vec{\xi}, x, y) & := \sum_{j=0}^{\infty} (-1)^{j+1} \mathcal{G}_{h,k}^{(ABC):y}(\vec{\xi}; x, y) y^{j} (\mod \mathbb{C}[y]).
\end{align*}
$$

We also obtain the same results for $G_{h,k}^{(ABC):x-y}(\vec{\xi}; x_0, y_0)$ and $G_{h,k}^{(ABC):y}(\vec{\xi}; x_0, y_0)$ by replacing $A(BC), \vec{\xi}, \vec{\mu}, J_N, J_N^0, \iota_{x,y}$ by $(AB)(C), \vec{\xi}, \vec{\mu}, J_N, J_N^0, \iota_{x,y}$, respectively.

| Borcherds identities | D:coefficient | q | $D \times (x-y)^{q}$ and $D \times x^{q}$ |
|----------------------|---------------|---|----------------------------------|
| (2A) $F(\vec{\xi})$ | $\vec{\xi}$ | 1 | 0 |
| (2B) $F(\vec{\mu})$ | $\vec{\mu}$ | 2 | 0 |
| (2C) $F(\vec{\nu})$ | $\vec{\nu}$ | 3 | 0 |
| (2D) $F(\vec{\omega})$ | $\vec{\omega}$ | 4 | 0 |
| (2E) $F(\vec{\eta})$ | $\vec{\eta}$ | 5 | 0 |
| (2F) $F(\vec{\zeta})$ | $\vec{\zeta}$ | 6 | 0 |

Table 1: Borcherds identity

Comment 5 In the case of $D \times (x-y)^{q}$, negative powers appear only on $x$ or $y$. In the case of $D \times x^{q}$, negative powers occur only on $(x-y)$ or $x$. We hence consider

$$
\begin{align*}
\mathcal{G}_{h,k}^{(BC)}(\vec{\xi}; x_0, y) & := F_{h,k}^{(BC)}(\vec{\xi}; x_0, y) y^{gr_{234}(\vec{\xi})}, \\
\mathcal{G}_{h,k}^{(BC):x-y}(\vec{\xi}; x_0, y) & := F_{h,k}^{(BC)}(\vec{\xi}; x, y_0) (x - y_0)^{gr_{234}(\vec{\xi})}, \\
\mathcal{G}_{h,k}^{(ABC)(C)}(\vec{\xi}, w; x, y) & := F_{h,k}^{(ABC)(C)}(\vec{\xi}; x, y) y^{gr_{234}(\vec{\xi})}, \quad \text{and} \\
\mathcal{G}_{h,k}^{(ABC):x-y}(\vec{\xi}; x, y) & := F_{h,k}^{(ABC):x-y}(\vec{\xi}; x, y) (x - y)^{gr_{234}(\vec{\xi})}.
\end{align*}
$$

Then we have:
[Proof] We will show the proofs for $G_{h,k}^{A(BC):\ast}(\xi', x, y)$. To simplify the notation, we denote $G_{h,k}^{A(BC):\ast}(\xi'; x, y)$ by $G^\ast(\xi'; x, y)$ for $* = y, x - y$. From Table 1, we have

$$
G^y(\alpha_{-1}^2, x, y) = F(\alpha_{-1}^2, x, y)y^{p+\omega(a)} = \sum_{j=0}^\infty G^y((\alpha_{-1-j})^y \xi_0, x, y)x_0^jy^{\omega(a)} + \sum_{j=0}^\infty G^y(\alpha_j, x, y)x_0^j(x_0 - y)^{j-1}y^j + \sum_{j=0}^\infty G^y(\alpha_j, x, y)x_0^jy^j + \sum_{j=0}^\infty G^y(\alpha_j, x, y)x_0^jy^j + \sum_{j=0}^\infty G^y(\alpha_j, x, y),
$$

For $y = y_0 \neq 0$ and a variable $x$, we have:

$$
G^{x-y}(\alpha_{-1}^2, x, y_0) = \sum_{j=0}^\infty G(((\alpha_{-1-j})^y \xi_0, x, y_0)x^j(x - y_0)^{\omega(a)} + \sum_{j=0}^\infty G(\alpha_j, x, y_0)x_0^j(x - y_0)^{j-1}y^j + \sum_{j=0}^\infty G(\alpha_j, x, y_0)x_0^jy^j + \sum_{j=0}^\infty G(\alpha_j, x, y_0),
$$

In particular, for the residue classes of coefficients modulo $\mathbb{C}[[x-y]][x-y]$, it is easy to check the statements. Since these processes don't depend on the choice of $G^{A(BC)}$ nor $G^{(AB)C}$, we get the same expressions for $G^{(AB)C:y}(\xi'; x, y)$ and $G^{(AB)C:x-y}(\xi'; x, y)$ except the notation and expansions by $\xi'$. This completes the proof of Theorem 3.1.

3.2 Proof of Theorem 3.2 (Differential systems)

We will prove the Fuchsian systems (D1) and (D2) for $G_{h,k}^{A(BC):y}$ and $G_{h,k}^{A(BC):x-y}$. The proofs for the other cases are similar. From the $(L^{-1})$-derivative property of intertwining operators, as $\mathbb{C}$-formal power series with logarithmic terms, for $\xi = (\theta, v, u, w) \in \Omega$ and $\xi' = (\theta', v, u, w) \in \Omega'$, we have

$$
\frac{\partial}{\partial y} F^{A(BC)}(\xi', x, y) = F^{A(BC)}(L(-1)^{[3]}\xi'; x, y),
$$

$$
\frac{\partial}{\partial x} F^{A(BC)}(\xi'; x, y) = F^{A(BC)}(L(-1)^{[2]}\xi'; x, y),
$$

and

$$
\frac{\partial}{\partial (x-y)} F^{(AB)C}(\xi'; x, y) = F^{(AB)C}(L(-1)^{[2]}\xi'; x, y).
$$

From (2.11) in Remark 2.3, we get

$$
\frac{\partial}{\partial (x-y)} \left\{G_{h,k}^{A(BC):y}(\xi'; x, y) \right\} = \frac{\partial}{\partial (x-y)} \left\{F_{h,k}^{A(BC)}(\xi'; x, y)y^{p+34}\xi + 1 \right\}
$$

$$
= \frac{\partial}{\partial (x-y)} \left\{F_{h,k}^{A(BC)}(\xi'; x, y)y^{p+34}\xi + 1 \right\} + (y^{p+34}(\xi'; x, y)y^{p+34}(\xi))
$$

$$
= G_{h,k}^{A(BC):y}(L(-1)^{[3]}\xi'; x, y) - (k+1)G_{h,k}^{A(BC):y}(\xi'; x, y) + (y^{p+34}(\xi'; x, y)y^{p+34}(\xi))
$$

$$
= \sum_{\xi \in J_0} \xi'(x, y)G_{h,k}^{A(BC):y}(\xi'; x, y),
$$

15
where \( s_\xi(x, y) \in \mathbb{C}[x, x^{-1}, \xi, y, \{x - y\}^{-1}] \). On the other hand, we obtain:

\[
\frac{\partial}{\partial y} \left( G_{h,k}^{ABC:y} \left( \xi, x, y \right) \right) = \frac{\partial}{\partial y} \left( G_{h,k}^{ABC:y} \left( \xi, x, y \right) \right) + G_{h,k}^{ABC:y} \left( \xi, x, y \right).
\]

Combining it with the above, and then using Theorem 3.1, we get

\[
\frac{d}{dy} \left( G_{h,k}^{ABC:y} \left( \xi, x, y \right) \right) = y^{-1} \frac{d}{dy} \left( G_{h,k}^{ABC:y} \left( \xi, x, y \right) \right) - y^{-1} G_{h,k}^{ABC:y} \left( \xi, x, y \right) = \frac{1}{y} \sum_{\mu \in J_N} \sum_{(p, q) \subset K^2} \lambda_{\mu, p, q}^{34,ponents, the set of nonzero eigenvalues of the constant matrix \( \Lambda \)

Therefore, there are \( \lambda_{\mu, p, q}^{34,ponents, the set of nonzero eigenvalues of the constant matrix \( \Lambda \)

\[
\left( x - y_0 \right) \frac{\partial}{\partial (x - y_0)} G_{h,k}^{ABC:x-y} \left( \xi, x, y_0 \right) = G_{h,k}^{ABC:x-y} \left( L(-1)^{2} \xi^{2}, x, y_0 \right) - (h + 1) G_{h+k}^{ABC:x-y} \left( \xi, x, y_0 \right) x^{-1} (x - y_0) + \text{gr}^{234} \left( \xi \right) G_{h,k}^{ABC:x-y} \left( \xi, x, y_0 \right).
\]

\[
\frac{\partial}{\partial x} G_{h,k}^{ABC:x-y} \left( \xi, x, y_0 \right) = \frac{1}{x - y_0} \sum_{\mu \in J_N} \sum_{(p, q) \subset K^2} \lambda_{\mu, p, q}^{23,ponents, the set of nonzero eigenvalues of the constant matrix \( \Lambda \)

\[
\frac{d}{dx} G_{h,k}^{ABC:x-y} \left( x, y_0 \right) = \frac{\Lambda_{\mu, p, q}^{23,ponents, the set of nonzero eigenvalues of the constant matrix \( \Lambda \)

We next investigate the set of eigenvalues of constant matrices. Since \( \lambda_{\mu, p, q}^{23,ponents, the set of nonzero eigenvalues of the constant matrix \( \Lambda \)

\[
(x - y_0) \frac{\partial}{\partial (x - y_0)} G_{h,k}^{ABC:x-y} \left( \xi, x, y_0 \right) \equiv G_{h,k}^{ABC:x-y} \left( L(-1)^{2} \xi^{2}, x, y_0 \right) + \text{gr}^{234} \left( \xi \right) G_{h,k}^{ABC:x-y} \left( \xi, x, y_0 \right) \mod \mathbb{C}[x - y_0] (x - y_0).
\]

Furthermore, on each step of the reduction of \( G_{h,k}^{ABC:x-y} \left( L(-1)^{2} \xi^{2}, x, y_0 \right) \) into a linear combination of \( \{G_{h,k}^{ABC:x-y}(\mu; x, y) \mid \mu \in J_N\} \), the coefficients at the first term with \( (\alpha_j)^{[1]} \xi \) and at the 4th term with \( \alpha_j^{[4]} \xi \) are contained in \( \mathbb{C}[x - y_0] (x - y_0) \) by Theorem 3.1. In other words, modulo \( \mathbb{C}[x - y_0] (x - y_0) \), the process of these reductions are independent from the choice of \( (\theta, w) \) and we don’t change the subscripts \( h, k \). Therefore, for \( \xi = (\theta, v, u, w), \mu = (\theta, \tilde{v}, \tilde{u}, \tilde{w}) \in J_N \), there are \( \lambda_{(\tilde{v}, \tilde{w})}^{(\theta, v, u, w), h,k}(x, y_0) \equiv \delta_{h,p} \delta_{k,q} \delta_{\theta, \theta} \delta_{v, v} \lambda_{(\tilde{v}, \tilde{w})}^{(v,u), h,k}(x, y_0) \mod \mathbb{C}[x - y_0] (x - y_0) \), (3.3)

which means that the set of eigenvalues (without counting multiplicity) of the constant matrix \( \Lambda^{23,ponents, the set of nonzero eigenvalues of the constant matrix \( \Lambda \)

16
3.3 Corollary 3.3 (Local normal convergence)

Let \( S^1 \) be an indecomposable direct summand of \( B \boxtimes C \) and \( T^1 \) an indecomposable direct summand of \( A \boxtimes S^1 \). We note that \( A \boxtimes (B \boxtimes C) \) is a direct sum of such \( T^1 \)'s. We may choose \( \theta \in (T^1)^\vee \). Set \( p_1 = d(T^1) - d(A) - d(S^1) - 1 \) and \( p_2 = d(S^1) - d(B) - d(C) + 1 \). Then there are \( \lambda_{m,h,k}(\xi) \in \mathbb{C} \) such that

\[
G_{h,k}^{\theta}(\xi;x,y) = \langle \theta, \mathcal{Y}_{h}^{\theta}(v,x)\mathcal{Y}_{k}^{\theta}(u,y)w \rangle y^{gr_{234}(\xi)} = \sum_{m=0}^{\infty} \lambda_{m,h,k}(\xi)x^{p_1}y^{p_2}(y/x)^m,
\]

where \( \mathcal{Y}_{h}^{\theta}(v,z) = \pi_{T^1}(\mathcal{Y}_{h,k}^{\theta}(v,z)\delta) \) and \( \mathcal{Y}_{k}^{\theta}(u,z)w = \pi_{S^1}(\mathcal{Y}_{k}^{BC}(u,z)w) \) for \( \delta \in S^1 \) and \( \pi_{P} \) denotes the projection to \( P \).

Therefore, in order to prove its convergence on a domain \( \{(x,y) \in \mathbb{C}^2 \mid 0 < |y| < |x|\} \), it is enough to check the case where \( x = 1 \) and \( 0 < |y| < 1 \). In other words, it is enough to treat a one-variable valued function \( Z = (G_{h,k}^{\theta}(\xi;y,1))_{\xi \in \mathcal{J}_N,(h,k) \in K^2} \) satisfying a Fuchsian system \( \frac{dZ}{dy} = (\xi_{n,h,k}(1,y)) \) with polar locus \( \{0\} \), where \( \xi_{n,h,k}(1,y) \in \mathbb{C}[(1-y)^{-1},y] \) by Theorem 3.1. Since \((1-y)^{-1} \) and \( y \) have radius of convergence at least 1, so does \( \xi_{n,h,k}(1,y) - \delta_{(\xi,h,k),(\mu,p,q)} \). Therefore, from Theorem 2.10 the radius of convergence of \( G_{h,k}^{\theta}(\xi,y,1) \) (and also of \( F_{h,k}^{\theta}(\xi,y,1) \)) is at least 1 for \( \xi \in \Omega \).

In a similar way, by using (D2) for \( F^{\theta}(\xi,v,x,0) \), we can show that \( F^{\theta}(\xi,v,x,0) \) is locally convergent on a domain \( \{(x,y) \in \mathbb{C}^2 \mid 0 < |y-x| < |y|\} \).

This completes the proof of Corollary 3.3.

4 Associativity

Set \( \mathcal{D}^2 = \{(x,y) \in \mathbb{C}^2 \mid 0 < |x-y| < |y| < |x| \} \), and \( x,y,x-y \notin \mathbb{R}^{\leq 0} \). In this section, we will prove the following main theorem.

**Theorem 4.1** On \( \mathcal{D}^2 \), we choose a principal branch \( \tilde{F}(\langle \theta, \mathcal{Y}_{(A,B)}^{(v,x)}\mathcal{Y}_{(u,y)}^{BC}(u,y)w \rangle) \) of \( \langle \theta, \mathcal{Y}_{(A,B)}^{(v,x)}\mathcal{Y}_{(u,y)}^{BC}(u,y)w \rangle \) by taking the values of \( \log(x), \log(y) \) which satisfy \( -\pi < \mathfrak{R}(\log(x)), \mathfrak{I}(\log(y)) < \pi \) and viewing \( x^d_1, y^d_2 \) as \( e^{d_1\log(x)+d_2\log(y)} \). Similarly \( \tilde{F}(\langle \theta', \mathcal{Y}_{(A,B)}^{(v,x-y)}\mathcal{Y}_{(u,y)}^{BC}(u,y)w \rangle) \) is a branch of \( \langle \theta', \mathcal{Y}_{(A,B)}^{(v,x-y)}\mathcal{Y}_{(u,y)}^{BC}(u,y)w \rangle \) by taking the values of \( \log(y), \log(x-y) \) which satisfy \( -\pi < \mathfrak{R}(\log(y)), \mathfrak{R}(\log(x-y)) < \pi \) and considering \( y^d_1(x-y)^d_2 = e^{d_1\log(y)+d_2\log(x-y)} \). Then there is an isomorphism \( \phi_{(A,B)}^{(B,C)} : (A \boxtimes B) \boxtimes C \rightarrow (A \boxtimes (B \boxtimes C)) \) such that

\[
\tilde{F}(\langle \theta, \mathcal{Y}_{(A,B)}^{(v,x)}\mathcal{Y}_{(u,y)}^{BC}(u,y)w \rangle) = \tilde{F}(\langle \phi_{(A,B)}^{(B,C)}(\theta), \mathcal{Y}_{(A,B)}^{(v,x-y)}\mathcal{Y}_{(u,y)}^{BC}(u,y)w \rangle)
\]

on \( \mathcal{D}^2 \) for any \( \theta \in ((A \boxtimes B) \boxtimes C)^\vee, v \in A, u \in B, w \in C \), where \( \phi_{(A,B)}^{(B,C)} \) denotes the dual of \( \phi_{(A,B)}^{(B,C)} \).

4.1 Equation (4.9)

We may still assume \( \theta \in (T^1)^\vee \). Hence there are \( p_1, p_2 \in \mathbb{C} \) such that

\[
G_{h,k}^{\theta}(\xi;x,y) = \langle \theta, \mathcal{Y}_{h,k}^{(v,x)}\mathcal{Y}_{k}^{BC}(u,y)w \rangle (x-y)^{gr_{234}(\xi)} \in \mathbb{C}[y/x]x^{p_1}y^{p_2}
\]
is absolutely convergent on \( \{(x, y) \in \mathbb{C}^2 \mid 0 < |y| < |x|\} \). We also recall the definition \( \mathcal{J}_N \) which is the set of quadruples of basis of \((A(BC))_N^\prime, P_A, \hat{P}_B, \) and \( \hat{P}_C \). We choose a branch \( \tilde{G}_{h,k}^{A(BC):x-y}(\xi; x, y) \) of \( G_{h,k}^{A(BC):x-y}(\xi; x, y) \) on \( \mathcal{D}^2 \) by taking values of \( \log(x), \log(y) \) which satisfy \(-\pi < \Im(\log(x)), \Im(\log(y)) < \pi \) and viewing \( x^{p_1+m_y}y^{p_2+n} \) as \( e^{(p_1+m_y)\log(x)+(p_2+n)\log(y)} = e^{p_1\log(x)+p_2\log(y)}x^my^n \) for \( m, n \in \mathbb{Z} \). Then by Theorem 3.2, for \( y_0 \not\in \mathbb{R}^{-}\), a vector valued function

\[
G_{h,k}^{A(BC):x-y}(\xi; x, y_0) = \left( \tilde{G}_{h,k}^{A(BC):x-y}(\xi; x, y_0) \right)_{\xi \in \mathcal{J}_N, (h,k) \in \mathbb{C}^2}
\]

satisfies a Fuchsian system \((D2)\) for \( \xi \in \mathcal{J}_N \). Furthermore, the set of nonzero eigenvalues of the constant matrix \( \Lambda_{a,b}(y_0, y_0) \) depends only on the choice of bases \( J_{P_A} \) and \( J_{P_B} \). Therefore, by Proposition 2.9, there is a finite subset \( \Delta' \subseteq \mathbb{C} \) such that all components of solutions of \((D2)\) are written as

\[
\sum_{t=0}^{K} \sum_{d \in \Delta'} \sum_{m \in \mathbb{N}} r_{d,m,t}(x - y_0)^{d+m} \log^t(x - y_0)
\]

with \( r_{d,m,t} \in \mathbb{C} \). We note that we have also taken a branch of \( (x - y_0)^{d+m} \log^t(x - y_0) \) on \( \mathcal{D}^2(x,y_0) \) by the same way. Since this holds for \( \tilde{G}_{h,k}^{A(BC):x-y}(\xi; x, y_0) \) with \( \xi \in \mathcal{J}_N \) and \( y = y_0 \not\in \mathbb{R}^{-}\), there are \( h,k,\tilde{\xi} \) \( \in \mathbb{C} \) and \( K(\tilde{\xi}) \in \mathbb{N} \) such that we can write

\[
\tilde{G}_{h,k}^{A(BC):x-y}(\xi; x, y) = \sum_{s \in \Delta' + \mathbb{N}} \sum_{t=0}^{K(\tilde{\xi})} r_{h,k}^{s,t}(\xi; y)(x - y)^s \log^t(x - y) \quad (4.1)
\]

on \( \mathcal{D}^2 \). Multiplying (4.1) by \( \log^h(x) \log^k(y)(x - y)^{-a_{234}(\tilde{\xi})} \) for all \( (h,k) \in \mathbb{C}^2 \) and taking the sum of all of them, we obtain an equality:

\[
\tilde{F}_{A(BC)}(\xi; x, y) = \sum_{h,k} \sum_{s \in \Delta' + \mathbb{N}} \sum_{j=0}^{K(\tilde{\xi})} r_{h,k}^{s,j}(\xi; y) \log^k(y) \log^h(x)(x - y)^s \log^j(x - y) \quad (4.2)
\]

for a branch \( \tilde{F}_{A(BC)}(\xi; x, y) \) of \( F^{A(BC)}(\xi; x, y) \) on \( \mathcal{D}^2 \). About the derivations, we use \( \frac{\partial x}{\partial (x - y)} = 1 \) and \( \frac{\partial y}{\partial (x - y)} = 0 \). Since \( \frac{\partial \log(x)}{\partial x} = \frac{1}{x} = \frac{1}{y} \frac{\partial \log(x)}{\partial y} = \sum_{j=0}^{\infty} (-1)^j \left( \frac{x - y}{y} \right)^j \) and \( \frac{\partial \log(x)}{\partial y} = 0 \), we define

\[
p(x, y) := \sum_{j=0}^{\infty} \frac{(-1)^j}{j + 1} \left( \frac{x - y}{y} \right)^{j + 1} + \log(y), \quad (4.3)
\]

which satisfies the same property with \( \log(x) \) in a neighborhood of \( y \not\in \mathbb{R}^{-}\). We replace \( \log(x) \) in (4.2) by \( p(x, y) \). Since the powers of \( x - y \) in \( p(x, y) \) are non-negative, we can write

\[
\tilde{F}_{A(BC)}(\xi; x, y) = \sum_{l=0}^{K(\tilde{\xi})} \sum_{s \in \text{gr}(v(\xi) - \Delta' - \mathbb{N})} g_{s,t}(\xi; y)(x - y)^{-s - 1} \log^t(x - y) \quad (4.4)
\]

with \( g_{s,t}(\xi; y) \in \mathbb{C} \) for \( \xi \in \mathcal{J}_N \). Since \( \text{dim} \ P_C < \infty \), using \( \Delta = \{ d - \max\{\text{gr}(w) : w \in P_C \} \mid d \in \Delta' \} \), we can rewrite (4.4) into

\[
\tilde{F}_{A(BC)}(\xi; x, y) = \sum_{l=0}^{K(\tilde{\xi})} \sum_{s \in \text{gr}(v(\xi) + \text{gr}(u)) - \Delta - \mathbb{N}} g_{s,t}(\xi; y)(x - y)^{-s - 1} \log^t(x - y) \quad (4.5)
\]
for $\xi = (\theta, v, u, w) \in \Omega$ with $w \in P_C$.

**Lemma 4.2** (4.5) holds for all $\xi \in \Omega$.

**[Proof]** Suppose false and let $\xi = (\theta, v, u, w')$ be a counterexample. Subject to counterexample, we choose $w'$ with the smallest grade. Since (4.5) holds for elements with $w$ in $P_C$, we may assume $w' = \alpha - 1 w$ with $\alpha \in V_{\geq 0}$ and $g_{s,t}(\theta, v, u, \alpha - 1 w; y) \neq 0$ for some $s \not\in wt(v) + wt(u) - 1 - \Delta - N$. In this case, by Borcherds identities, we obtain

$$
\tilde{F}^{A(BC)}(\alpha^{-1}[4] \xi; x, y) = \tilde{F}^{A(BC)}((\alpha^{-1})^{[1]} \xi; x, y) - \sum_{j=0}^{\infty} (-1)^j \tilde{F}^{A(BC)}(\alpha_j^{[2]} \xi; x, y)x^{-j-1}
- \sum_{j=0}^{\infty} (-1)^j \tilde{F}^{A(BC)}((\alpha_j^{[1]} \xi; x, y)y^j
- \sum_{i \in \mathbb{N}} (i) \tilde{F}^{A(BC)}((\alpha_i^{[1]} \xi; x, y)(x - y)^{-1}
- \sum_{j=0}^{\infty} (-1)^i j \sum_{i \in \mathbb{N}} (i) (-1)^{i+j+1} \tilde{F}^{A(BC)}((\alpha_j^{[2]} \xi; x, y)y^j
+ \sum_{j=0}^{\infty} \sum_{i \in \mathbb{N}} (i) (-1)^{i+j+1} \tilde{F}^{A(BC)}((\alpha_i^{[1]} \xi; x, y)y^j,
$$

by (2.9). However, since $\text{gr}(\alpha_j w) < \text{gr}(\alpha - 1 w) = \text{gr}(w')$ for $j \geq 0$ and the 2nd and 3rd components of $(\alpha_j^{[1]} \xi)$ in RHS of the last equation are $v$ and $u$, none of the terms in RHS of the last equation have nonzero coefficients of $(x - y)^{-s-1} \log'(x - y)$ by the minimality of $\text{gr}(w')$, which is a contradiction.

Since $g_{s,t}(\xi; y)$ is multi-linear on $\xi \in \Omega_N = (A(BC))^\vee \times A \times B \times C$, we split $\xi = (\theta, v, u, w)$ into $\{v, u\}$ and $\{\theta, u\}$. In other words, using a formal operator $\tilde{Y}^3_{s,t} \in \text{Hom}(A \otimes B, \text{Hom}(C, A(BC) \otimes O(\mathbb{C} \setminus \mathbb{R}^2)))$, we can write

$$
\tilde{F}^{A(BC)}(\xi; x, y) = \sum_{s,s,t}^{K(\xi)} \sum_{s \in \text{gr}(v) + \text{gr}(u) - \Delta - N - 1} \langle \theta, \tilde{Y}^3_{s,t}(v, u; y)w \rangle (x - y)^{-s-1} \log'(x - y) \tag{4.6}
$$

for each $(x, y) \in D^2$. Namely, $\langle \theta, \tilde{Y}^3_{s,t}(v, u; y)w \rangle = \text{Cl}_{s,t}(\tilde{F}^{A(BC)}(\xi; x, y))$, where

$$
\text{Cl}_{s,t}(f(x, y)) \text{ denotes a coefficient } \lambda_{s,t}(f(x, y)) \text{ of } (x - y)^{-s-1} \log'(x - y) \text{ in } f(x, y) = \sum_{t \in \mathbb{N}} s \text{, with } \lambda_{s,t}(y)(x - y)^{-s-1} \log'(x - y).
$$

We introduce a formal vector space $A \otimes_{s,t} B$, which is isomorphic to $A \otimes B$ as a vector space and its isomorphism is given by $v \otimes v \mapsto v \otimes_{s,t} u$ for $s, t$. Set $A \otimes_{s,t} B = \oplus_{(s,t) \in \mathbb{N}^2} (A \otimes_{s,t} B)$ and define a formal operator $\tilde{Y}^3$ of type $(A \otimes_{s,t} B)$ by $\tilde{Y}^3((A \otimes B) \otimes O(D^2))$ by $\langle \theta, \tilde{Y}^3(v \otimes_{s,t} u, y)w \rangle = \langle \theta, \tilde{Y}^3(v, u; y)w \rangle (x - y)^{-s-1} \log'(x - y)$. Then we can rewrite Eq. (4.6) into

$$
\tilde{F}^{A(BC)}(\xi; x, y) = \langle \theta, \tilde{Y}^3(\tilde{Y}^4(v, x - y)u; y)w \rangle \tag{4.7}
$$

As it is well-known, $V[z, z^{-1}]$ has a Lie algebra structure by

$$
[a z^m, b z^n] = \sum_{j \in \mathbb{N}} \binom{m}{j} (a_j b) z^{n+m-j}.
$$

19
Lemma 4.3 Define the action of $V[z, z^{-1}]$ on $A \otimes_\infty B$ by $(\alpha z^m)(v \otimes_{s,t} u) = v \otimes_{s,t} \alpha_m u + \sum_{j=0}^\infty (\alpha_j v \otimes_{s+m-j,t} u)$ for $\alpha z^m \in V[z, z^{-1}]$. Then $A \otimes_\infty B$ is a $V[z, z^{-1}]$-module.

[Proof] To simplify the notation, we will omit the index $t$ from $v \otimes_{s,t} u$. We also denote the action of $\alpha z^m$ by $\alpha|_m$. Then for $\alpha, \beta \in V$, we obtain

$$\alpha|_m \beta|_n (v \otimes s u) = \alpha|_m (\sum_{j=0}^\infty (\beta_j v) \otimes_{s+n-j} u) + \alpha|_m (v \otimes s (\beta_n u))$$

We hence get:

$$\alpha|_m \beta|_n (v \otimes s u) = \sum_{j=0}^\infty \sum_{i=0}^\infty (\alpha_i \beta_j v) \otimes_{s+m+n-i-j} u + \sum_{j=0}^\infty (\beta_j v) \otimes_{s+n-j} (\alpha_m u)$$

On the other hand, we have:

$$\sum_{p=0}^\infty (\alpha_p \beta)_{m+n-p} (v \otimes s u) = v \otimes s \sum_{p=0}^\infty (\alpha_p \beta)_{m+n-p} u + \sum_{p=0}^\infty (\alpha_p \beta)_{m+n-p} q = \sum_{p=0}^\infty (\alpha_p \beta)_{m+n-i-j} u + v \otimes s ([\alpha_m, \beta_n] u)$$

which coincides with the last terms in (4.8). Therefore $A \otimes_\infty B$ is a $V[z, z^{-1}]$-module. □

Definition 4.4 Let $\Gamma$ be the kernel of $\widetilde{\mathcal{Y}}^3$, that is,

$$\Gamma = \{ \gamma \in A \otimes_\infty B \mid \langle \theta, \widetilde{\mathcal{Y}}^3(\gamma, y)w \rangle = 0 \text{ for all } \theta \in (A(BC))^\vee, w \in C \text{ and } y \notin \mathbb{R}^{\leq 0} \}.$$ 

In other words, $\sum_{n=1}^\infty v^t \otimes_{s,t} u^t \in \Gamma$ if and only if $\sum_{n=1}^\infty \text{Cf}_{s,t}(\tilde{F}(BC)(\theta, v^t, w; x, y)w) = 0$ for any $\theta \in (A(BC))^\vee$ and $w \in C$. Clearly, $\Gamma$ is a subspace of $A \otimes_\infty B$. Define $A \otimes_\infty B = (A \otimes_\infty B)/\Gamma$. We denote $v \otimes s,t u + \Gamma$ in $A \otimes_\infty B$ by $[v \otimes s,t u]$ or $[v_{s,t} u]$. 

Lemma 4.5 $\widetilde{\mathcal{Y}}^3$ satisfies associativity (12) and $\Gamma$ is $V[z, z^{-1}]$-invariant.

[Proof] By the definition of the action of $V[z, z^{-1}]$, we get:

$$\langle \theta, \widetilde{\mathcal{Y}}^3(\alpha|_m (v \otimes s,t u), y)w \rangle = \langle \theta, \widetilde{\mathcal{Y}}^3(v \otimes s,t \alpha_m u), y)w \rangle + \sum_{i=0}^\infty \langle \theta, \widetilde{\mathcal{Y}}^3(\alpha_i v \otimes_{s+n-i,t} u, y)w \rangle$$

$$= \text{Cf}_{s,t}(\tilde{F}(BC)(\theta, v, \alpha_n u, w; x, y)w) + \sum_{i=0}^\infty \text{Cf}_{s+n-i,t}(\tilde{F}(BC)(\theta, \alpha v, u; x, y)w).$$

20
Since $\widetilde{F}^{A(BC)}$ satisfies Borcherds identity (3B), we also have:

$$
\text{Cl}_{s,t}(\tilde{F}^{A(BC)}(\theta, v, \alpha_n u, w; x, y); y) = \text{Cl}_{s,t} \left( \left( \sum_{j=0}^{\infty} (-1)^j \tilde{F}^{A(BC)}((\alpha_{n-j})^{[1]} \xi^{[4]}, \epsilon^{[4]}; x, y)y^j \right) \right)
$$

Therefore, we have:

$$
\langle \theta, \tilde{\mathcal{Y}}^3(\alpha_n \gamma, y)w \rangle = \sum_{j=0}^{\infty} \langle \alpha_n \gamma, y \rangle \langle \tilde{\mathcal{Y}}^3, \psi_1, u \rangle w^y + \sum_{j=0}^{\infty} \langle \alpha_n \gamma, y \rangle \langle \tilde{\mathcal{Y}}^3, \psi_1, u \rangle w^{y^j}
$$

We next show that $\Gamma$ is $\mathbb{C}[z, z^{-1}]-$invariant. If $\gamma = \sum_{i=1}^r \theta_i^{\otimes s_i,t_i}$ then we have

$$
\langle \theta, \tilde{\mathcal{Y}}^3(\alpha_n \gamma, y)w \rangle = \sum_{j=0}^{\infty} \langle \alpha_n \gamma, y \rangle \langle \tilde{\mathcal{Y}}^3, \psi_1, u \rangle w^y + \sum_{j=0}^{\infty} \langle \alpha_n \gamma, y \rangle \langle \tilde{\mathcal{Y}}^3, \psi_1, u \rangle w^{y^j} = 0
$$

for any $\theta$ and $w$, which means $\alpha_n \gamma \in \Gamma$ for any $\alpha z^n = \alpha_n$. Hence $\Gamma$ is $\mathbb{C}[z, z^{-1}]-$invariant.

Clearly, we can define formal operators $\mathcal{Y}^3$ of type $(A^{(BC)}_{A^{B}}} B_C$ by $\mathcal{Y}^3(u, v) = [\mathcal{Y}^3(u, v)] = \sum_{s,t} [v \otimes s, t] u \xi^{-1} \text{log}(z)$ and $\langle \theta, \mathcal{Y}^3([v, s, t], u)w \rangle = \langle \theta, \mathcal{Y}^3(v \otimes s, t, u)w \rangle$ for $\xi \in \Omega$. Then we have

$$
\bar{F}^{A(BC)}(x, y; \xi) = \langle \theta, \mathcal{Y}^3(\mathcal{Y}^3(u, x-y)u, y)w \rangle
$$

for $\xi \in \Omega$. We use $\mathcal{Y}^3(x, y)$ to denote $\langle \theta, \mathcal{Y}^3(\mathcal{Y}^3(u, x-y)u, y)w \rangle$.

**Theorem 4.6** $\langle \theta, \mathcal{Y}^3(\mathcal{Y}^3(u, x-y)u, y)w \rangle$ satisfies the Borcherds identities (1B) $\sim$ (4B).

**Proof** As we mentioned in Remark 2, the coefficients of $F^{A(BC)}(\mu; x, y)$ in the expansion of $F^{A(BC)}(\alpha_n^{[1]} \xi^{[4]}, x, y)$ by Borcherds identities are all $C[x, x+y, \epsilon^{-1}], (x-y)^\pm 1]$, that is, integer powers of $x, y$ and $x-y$. Hence branches $\bar{F}^{A(BC)}(x, y)$ of them on $D^2$ also satisfy the Borcherds identities (1A) $\sim$ (4A). By Eq. (4.9), $\bar{F}^{A(BC)}(x, y)$ also satisfies the Borcherds identities (1A) $\sim$ (4A) on $D^2$. Since the corresponding coefficients in (1A) $\sim$ (4A) and (1B) $\sim$ (4B) are same on $D^2$, we have the desired identities.

**Proposition 4.7** $\mathcal{Y}^3$ and $\mathcal{Y}^4$ satisfy $L(-1)$-derivative properties.

**Proof** We note that we are considering the derivations satisfying $\frac{d}{d(x-y)}y = 0 = \frac{d}{d(y)}y = 1$, and $\frac{d}{d(x-y)}(x-y)$. By the choice of $p(x, y)$ in (4.4), we get

$$
\langle \theta, \mathcal{Y}^3(\mathcal{Y}^3(L(-1)u, x-y)u, y)w \rangle = \bar{F}^{A(BC)}(L(-1)\xi^{[4]}, x, y) = \frac{d}{d(x-y)}E^{A(BC)}(\xi, x, y)
$$

and

$$
\langle \theta, \mathcal{Y}^3(\mathcal{Y}^3(u, x-y)u, y)w \rangle = \frac{d}{d(x-y)}(\theta, \mathcal{Y}^3(\mathcal{Y}^3(u, x-y)u, y)w).
$$
We fix \( v \in A \). Since the above holds for every \( \theta \in (A(BC))^\gamma, u \in B, w \in C \), the above implies \( L(-1) \)-derivative property of \( \mathcal{Y}^4 \): \( \mathcal{Y}^4(L(-1)v, x - y) = \frac{d}{dx-y} \mathcal{Y}^4(v, x - y) \). Using \( F^{34}(L(-1)^3)\tilde{\xi}; x, y) = \frac{\partial}{\partial y} F^{34}(\tilde{\xi}; x, y) \), we obtain

\[
\langle \theta, \mathcal{Y}^4(L(-1)\mathcal{Y}^4(v, x - y)u, y)w \rangle = \langle \theta, \mathcal{Y}^3(\mathcal{Y}^4(L(-1)v, x - y)u, y)w \rangle + \langle \theta, \mathcal{Y}^3(\mathcal{Y}^4(v, x - y)L(-1)u, y)w \rangle = \frac{\partial}{\partial x} F^{A(BC)}(\tilde{\xi}; x, y) + \frac{\partial}{\partial y} F^{A(BC)}(\tilde{\xi}; x, y) = (\frac{\partial}{\partial x} + \frac{\partial}{\partial y}) F^{A(BC)}(\tilde{\xi}; x, y).
\]

Since \( (\frac{\partial}{\partial x} + \frac{\partial}{\partial y})(x - y) = 0 \) and \( (\frac{\partial}{\partial y} + \frac{\partial}{\partial x})y = 1 \), we also get

\[
\langle \theta, \mathcal{Y}^3(L(-1)\mathcal{Y}^4(v, x - y)u, y)w \rangle = \frac{\partial}{\partial y} \mathcal{Y}^3(\mathcal{Y}^4(v, x - y)u, y)w
\]

under the assumption \( \frac{\partial(x-y)}{\partial y} = 0 \), which proves \( L(-1) \)-derivative property of \( \mathcal{Y}^3 \). \( \square \)

As a corollary of the above and the existence of a finite set \( \Delta \) in (4.5), we have:

**Corollary 4.8** \( L(-1)[v_{s,t}u] = -s[v_{s-1,t}u] + (t + 1)[v_{s-1,t+1}u] \) and \( wt([v_{s,t}u]) = wt(v) + wt(u) - s - 1 \). The weights of elements in \( A \bar{\otimes} B \) are contained in \( \Delta + N \) and so \( A \bar{\otimes} B \) is \( N \)-gradable. In particular, \( \alpha_n[v_{s,t}u] = 0 \) for \( n \gg 0 \) for \( \alpha \in V \).

Although Borchers identities are given from (I 1) and (I 2), we will show the reverse.

**Proposition 4.9** \( \mathcal{Y}^3 \) and \( \mathcal{Y}^4 \) satisfy (I 1) and (I 2).

**Proof** We have already proved (I 2) for \( \mathcal{Y}^3 \). By the action of \( V \) on \( A \otimes \infty B \), we have:

\[
\langle \theta, \mathcal{Y}^3(\alpha n)\mathcal{Y}^4(v, x - y)u, y)w \rangle = \langle \theta, \tilde{\mathcal{Y}}^3(\alpha n)\tilde{\mathcal{Y}}^4(v, x - y)u, y)w \rangle = \langle \theta, \mathcal{Y}^3(\mathcal{Y}^4(v, x - y)\alpha n u, y)w \rangle + \sum_{j=0}^{\infty} \binom{n}{j} \langle \theta, \mathcal{Y}^3(\mathcal{Y}^4(v, x - y)\alpha j u, y)w \rangle (x - y)^{n-j}
\]

for all \( \theta \) and \( w \), which implies (I 1) for \( \mathcal{Y}^3 \). From Borchers identity (2B), we obtain:

\[
\langle \theta, \mathcal{Y}^3(\mathcal{Y}^4(v, x - y)\alpha n u, y)w \rangle = \sum_{j=0}^{\infty} \binom{n}{j} \langle \theta, \mathcal{Y}^3(\mathcal{Y}^4(v, x - y)\alpha j u, y)w \rangle (x - y)^{n-j}.
\]

On the other hand, from the associativity (I 2) for \( \mathcal{Y}^3 \), we have

\[
\sum_{j=0}^{\infty} \binom{n}{j} (-1)^j \langle \theta, \mathcal{Y}^3(\mathcal{Y}^4(v, x - y)\alpha j u, y)w \rangle (x - y)^{n-j}.
\]

Since \( \sum_{j=0}^{\infty} \binom{n}{j} (-1)^j (x-y)^j y^{n-j} = \binom{n}{0} x^n \) for \( \ell \in N \) and \( \sum_{j=0}^{\infty} \binom{n}{j} (-1)^j (x-y)^j y^{n-j} = \binom{n}{0} (-1)^{n-j}x^n \) by (2.9), we obtain (I 2) for \( \mathcal{Y}^4 \). Commutativity (I 1) for \( \mathcal{Y}^3 \) is a direct consequence of Borchers identity (B1) and the definition of action of \( V \) on \( A \bar{\otimes} B \). \( \square \)
4.2 Recovery of $V$-modules and intertwining operators

**Theorem 4.10** $A\boxtimes B$ is an $N$-gradable $V$-module.

[Proof] We have proved that $A\boxtimes B$ is an $N$-gradable $V[z, z^{-1}]$-module in Corollary 4.8. In particular, the action of $V$ on $A\boxtimes B$ satisfies commutativity. Therefore, the remaining thing is to prove associativity of the action of $V$. In particular, it is enough to show

$$
\langle \theta, \mathcal{Y}^3(\alpha_{[-1]}\beta_{[-1]}\gamma, y) w \rangle = \langle \theta, \mathcal{Y}^3(\alpha_{[-1]}\beta_{[-1]}\gamma, y) w \rangle + \sum_{j=0}^{\infty} \langle \theta, (\alpha_{[-2-j]}\beta_{[j]} + \beta_{[-2-j]}\alpha_{[j]})\gamma, y) w \rangle
$$

(4.10)

for $\gamma \in A\boxtimes B, w \in C, \theta \in (A(BC))^\vee$. We will develop the action of $V$ into a triple $(A(BC))^\vee \times A\boxtimes B \times C$ by using (I 1) and (I 2). The rule is simple, that is, we shift $\alpha_{<0}$ and $\beta_{<0}$ to the left and $\alpha_{\geq 0}$ and $\beta_{\geq 0}$ to the right. Finally, the actions of $V$ on $A\boxtimes B$ and $C$ are given by $\alpha_{\geq 0}$ or $\beta_{\geq 0}$. We use notation $P(X(\alpha, \beta))$ to denote $X(\alpha, \beta) + X(\beta, \alpha)$ for any form $X$ of $\alpha$ and $\beta$. For example, $P(\langle \theta, \mathcal{Y}^3(\alpha_{[-2-j]}\beta_{[j]}\gamma, y) w \rangle)$ denotes $\langle \theta, \mathcal{Y}^3(\alpha_{[-2-j]}\beta_{[j]} + \beta_{[-2-j]}\alpha_{[j]})\gamma, y) w \rangle$. Then by (I 2) and associativity of the action of $V$ on $A(BC)$ and $C$, we have:

**LHS of (4.10)**

$$
\sum_{i,j \in \mathbb{N}} \langle \theta, \mathcal{Y}^3(\alpha_{[-1]}\beta_{[-1]}\gamma, y) w \rangle y^j + \sum_{i,j \in \mathbb{N}} \langle \theta, \mathcal{Y}^3(\gamma, y)(\alpha_{[-1]}\beta_{[-1]}w) y^{j-1}j \rangle
$$

$$
= \sum_{i,j \in \mathbb{N}} \langle \theta, \mathcal{Y}^3(\gamma, y)w \rangle y^j + \sum_{i,j \in \mathbb{N}} \langle \theta, \mathcal{Y}^3(\gamma, y)w \rangle y^{j-1}j
$$

Since there are many subscripts, in order to count the coefficients of terms in summations, we use monomials. For example, we denote $\langle \theta, \beta_{0}\mathcal{Y}^3([v,a], y)\alpha_{[b]}w \rangle y^c$ by $X^a Y^b Z^c$. Then since $i, j \in \mathbb{N}$, keeping the non-negative powers of $Y$, the following equation

$$
\sum_{i,j \in \mathbb{N}} X^{-2-i-j}Y^iz^j + \sum_{i,j \in \mathbb{N}} X^{-1-i-j}Y^iz^j = \sum_{i,j \in \mathbb{N}} X^{-1-i-j}Y^iz^j
$$

implies

$$
\sum_{i,j \in \mathbb{N}} \langle \theta, \beta_{-2-i-j}\mathcal{Y}^3(\gamma, y)\alpha_{i}w \rangle y^j + \sum_{i,j \in \mathbb{N}} \langle \theta, \beta_{-1-i-j}\mathcal{Y}^3(\gamma, y)\alpha_{j}w \rangle y^{j-1}j
$$

Furthermore, from

$$
\langle \theta, \mathcal{Y}^3(\alpha_{[-1]}\beta_{[-1]}[v,a], y) w \rangle
$$

$$
= \sum_{i,j \in \mathbb{N}} \langle \theta, \mathcal{Y}^3(\alpha_{[-1]}\beta_{[-1]}[v,a], y) w \rangle y^{j+i+j} + \sum_{i,j \in \mathbb{N}} \langle \theta, \mathcal{Y}^3(\gamma, y)\beta_{i}w \rangle y^{j-i-j}
$$

$$
+ \sum_{i,j \in \mathbb{N}} \langle \theta, \mathcal{Y}^3(\gamma, y)\beta_{i}w \rangle y^{j-i-j}
$$

$$
+ \sum_{i,j \in \mathbb{N}} \langle \theta, \mathcal{Y}^3(\gamma, y)\beta_{i}w \rangle y^{j-i-j},
$$

23
we get

LHS of (4.10) \(-\langle \theta, \mathcal{Y}^3(\alpha_{-1}, \beta_{-1})[v, u], y \rangle w \)
= \(P(\sum_{i,j,k \in \mathbb{N}} \langle \theta, \beta_{-2-i-j}[\alpha_i, \beta_{j-i}]\mathcal{V}^3(\gamma, y) \rangle y^{j-i-k}) - P(\sum_{i,j,k \in \mathbb{N}} \langle \theta, \beta_{-2-i-j}[\alpha_i, \beta_{j-i}]\mathcal{V}^3(\gamma, y) \rangle y^{j-i-k}) \)

On the other hand, we obtain

RHS of (4.10) \(-\langle \theta, \mathcal{Y}^3(\alpha_{-1}, \beta_{-1})[v, u], y \rangle w \)
= \(P(\sum_{i,j,k \in \mathbb{N}} (-2i)\langle \theta, \beta_{-2-i-j}[\alpha_i, \beta_{j-i}]\mathcal{V}^3(\gamma, y) \rangle y^{j-i-k}) \)

which coincides with the above, since we obtain

\[
P(\sum_{i,j,k \in \mathbb{N}} (-2i)\langle \theta, \beta_{-2-i-j}[\alpha_i, \beta_{j-i}]\mathcal{V}^3(\gamma, y) \rangle y^{j-i-k}) = P(\sum_{i,j,k \in \mathbb{N}} \langle \theta, \beta_{-2-i-j}[\alpha_i, \beta_{j-i}]\mathcal{V}^3(\gamma, y) \rangle y^{j-i-k})
\]

from

\[
\sum_{i,j,k \in \mathbb{N}} \langle \theta, \beta_{-2-i-j}[\alpha_i, \beta_{j-i}]\mathcal{V}^3(\gamma, y) \rangle y^{j-i-k} = \sum_{i,j,k \in \mathbb{N}} X^{j-i-k}Z^j(Z + Y)^i
\]

where we denote \(\langle \theta, \alpha_a \mathcal{Y}^3(\beta[\gamma], y) w \rangle y^c \) by \(X^a Y^b Z^c \) and \(a, b \in \mathbb{N} \) and we also have

\[
P(\sum_{i,j,k \in \mathbb{N}} (-2i)\langle \theta, \beta_{-2-i-j}[\alpha_i, \beta_{j-i}]\mathcal{V}^3(\gamma, y) \rangle y^{j-i-k}) = P(\sum_{i,j,k \in \mathbb{N}} (-2i)\langle \theta, \beta_{-2-i-j}[\alpha_i, \beta_{j-i}]\mathcal{V}^3(\gamma, y) \rangle y^{j-i-k})
\]

where we replace \(\langle \theta, \mathcal{Y}^3(\beta[\gamma], y) \alpha_a \rangle w \) with \(X^a Y^b Z^c \) and \(a, b \in \mathbb{N} \).

This completes the proof of Theorem 4.10.

We now start the proof of Theorem 4.11. Since \(A \boxtimes B \) is an \(\mathbb{N}\)-graded modules, by Theorem 2.15 \(\mathcal{Y}^3 \) and \(\mathcal{Y}^4 \) are branches of some logarithmic intertwining operators of type \((\mathcal{A}^{(BC)}) \) and \((\mathcal{A}^{AB}) \) on \(\mathcal{D}^2 \). By the construction of \(\mathcal{Y}^3, \mathcal{Y}^4 \), for any \(\theta \in (\mathcal{A}^{BC}) \), there are \(v \in A, u \in B, w \in C \) such that \(\langle \theta, \mathcal{Y}^3(v, x - y)u, y \rangle w \) \(\neq 0 \), that is, the set of coefficients of \((x - y)^ry^s \log^a(x - y) \log^b(y) \) in \(\{\mathcal{Y}^3(\theta, v, x - y)u, y \rangle w \mid v \in A, u \in B, w \in C \} \) spans \(A^{(BC)} \). From the universal properties of fusion products (\(A \boxtimes B \boxtimes C, \mathcal{X}^{(A^{BC})}\)) and \((A \boxtimes B \boxtimes C, \mathcal{Y}^{(A^{BC})}) \), there is a surjective \(V\)-homomorphism \(\phi_{\mathcal{A}^{BC}} : (A \boxtimes B) \boxtimes C \to A \boxtimes (B \boxtimes C) \) such that

\[
\langle \theta, \phi_{\mathcal{A}^{BC}}(\mathcal{Y}^{(A^{BC})})(\mathcal{Y}AB(v, x - y)u, y)w \rangle = \langle \theta, \mathcal{Y}^3(\mathcal{Y}^{(A^{BC})}(v, x - y)u, y)w \rangle = \langle \theta, \mathcal{Y}^{(A^{BC})}(v, x) \mathcal{Y}^{(A^{BC})}(u, y)w \rangle
\]
on $D^2$ for any $v \in A, u \in B, w \in C$ and $\theta \in (A(BC))^\vee$. Similarly, by starting from $\langle \theta', Y^{AB(C)}(v, x-y)u, y, w \rangle$, we also have a surjective homomorphism $\phi_{A|BC|} : (A \boxtimes B) \boxtimes C \rightarrow A \boxtimes (B \boxtimes C)$ satisfying

$$\langle \theta', \phi_{A|BC|}(Y^{AB(C)}(v, x))Y^{BC}(u, y)w \rangle = \langle \theta', Y^{AB}(v, x-y)u, y, w \rangle$$
on $D^2$ for all $v, u, w, \theta'$. From the construction, it is easy to see that $\phi_{A|BC|} \circ \phi_{AB|C} = 1$ and $\phi_{A|BC|} \circ \phi_{A|BC|} = 1$.

This completes the proof of the associativity law of fusion products. \hfill $\square$

**Corollary 4.11** If $V^\vee \cong V$ and $A$ and $A^\vee$ are both $C_1$-cofinite $N$-gradable modules, then for any non-zero $C_1$-cofinite $N$-gradable $V$-module $B$, $A \boxtimes B$ and $B \boxtimes A$ are not zero.

**[Proof]** There is a surjective intertwining operator of type $(V^\vee_Q^{A})$ which comes from the $V$-module structure on $A$ by skew-symmetry and duality. Hence $B \cong B \boxtimes V$ is a homomorphism image of $B \boxtimes (A \boxtimes A^\vee) \cong (B \boxtimes A) \boxtimes A^\vee$. Therefore, $(B \boxtimes A) \boxtimes A^\vee \neq 0$ and $B \boxtimes A \neq 0$. Similarly, we get $A \boxtimes B \neq 0$.

## 5 Pentagon axiom

**Theorem 5.1 (Pentagon axiom)** For $C_1$-cofinite $N$-gradable modules $A, B, C, D$, by using the above isomorphisms $\phi_{[a]}$, we have the following commutative diagram.

$$
\begin{array}{ccc}
(A \boxtimes B) \boxtimes C & \xrightarrow{\phi_{A|BC|} \times 1_D} & (A \boxtimes (B \boxtimes C)) \boxtimes D \\
\downarrow \phi_{A|BC|} & & \downarrow \phi_{A|BC|} \\
(A \boxtimes B) \boxtimes (C \boxtimes D) & \xrightarrow{1_A \times \phi_{A|BC|} D} & A \boxtimes (B \boxtimes (C \boxtimes D))
\end{array}
$$

**[Proof]** We consider a domain $D^3 = \{(x, y, z) \in \mathbb{C}^3 \mid x > |y| > |z| > |x-z| > |y-z| > |x-y| > 0, \text{ and } x, y, z, x-y, y-z, x-z \notin \mathbb{R}^{<0}\}$. We note $(7, 6, 4) \in D^3 \neq \emptyset$. Simplify the notation, we omit the notation $\boxtimes$. Let $\theta \in (A(B(CD)))^\vee, v \in A, u \in B, w \in C, d \in D$ and let $\xi$ denote a quadruple $(v, u, w, d) \in A \times B \times C \times D$. We fix all intertwining operators for fusion products, say $Y^{AB}, Y^{BC}, Y^{CD}, Y^{(BC)D}, Y^{A(B(CD))}, \ldots,$, and consider their five-point correlation functions:

$$
\begin{align*}
F^{A(B(CD))}(\theta^1, \xi; x, y, z) &= \langle \theta^1, Y^{A(B(CD))}(v, x)Y^{BC}(u, y)Y^{CD}(w, z) \rangle \\
F^{A((BC)D)}(\theta^2, \xi; x, y-z, z) &= \langle \theta^2, Y^{A((BC)D)}(v, x)Y^{BC}(u, y-z)c, z) \rangle \\
F^{A(BC)D}(\theta^3, \xi; x-z, y-z, z) &= \langle \theta^3, Y^{A(BC)D}(v, x-z)Y^{BC}(u, y-z)w, z) \rangle \\
F^{A((AB)C)D}(\theta^4, \xi; x-y, y-z, z) &= \langle \theta^4, Y^{A((AB)C)D}(v, x-y)u, y-z)w, z) \rangle \\
F^{A((AB)CD)}(\theta^5, \xi; x-y, y, z) &= \langle \theta^5, Y^{A((AB)CD)}(v, x-y)u, y)Y^{CD}(w, z) \rangle
\end{align*}
$$

(5.1)

for $\theta^1 \in (A(B(CD)))^\vee, \theta^2 \in (A((BC)D))^\vee, \theta^3 \in ((A(BC))D)^\vee, \theta^4 \in (((AB)C)D)^\vee, \theta^5 \in ((AB)(CD))^\vee$. By the same way as we did for four-point correlation functions, it is not difficult to see that these five-point correlation functions are all locally normal convergent
on \( \{ (x, y, z) \in \mathbb{C}^3 \mid |x| > |y| > |z| > |x - z| > |y - z| > |x - y| > 0 \} \). We next choose their principle branches

\[
\tilde{F}^{A(B(CD))}(\theta^1, \tilde{\xi}; x, y, z), \tilde{F}^{A((BC)D)}(\theta^2, \tilde{\xi}; x - y, z, \ldots), \tilde{F}^{(AB)(CD)}(\theta^5, \tilde{\xi}; x - y, y, z)
\]

on \( D^3 \) by taking the values of \( \log(x), \log(y), \log(z), \log(x - y), \log(x - z), \log(y - z) \) which satisfy \(-\pi < \Im(\log(x)), \Im(\log(y)), \Im(\log(z)), \Im(\log(x - y)), \Im(\log(x - z)), \Im(\log(y - z)) < \pi \) and viewing \( x^a, y^b, z^c, (x - y)^d, (x - z)^e, (y - z)^f \) as \( e^{a\log(x)}, e^{b\log(y)}, e^{c\log(z)}, e^{d\log(x - y)}, e^{e\log(x - z)}, e^{f\log(y - z)} \), respectively.

From our construction of isomorphisms \( \phi_{[*],*} \), by Theorem 4.1, we have equations:

\[
\tilde{F}^{A((BC)D)}(\theta^1, \tilde{\xi}; x, y, z) = \tilde{F}^{A((BC)D)}((1_A \times \phi_{[BC]D}^*)(\theta^1), \tilde{\xi}; x - y, z, z),
\]

\[
\tilde{F}^{A((BC)D)}(\theta^2, \tilde{\xi}; x - y, z, z) = \tilde{F}^{A((BC)D)}(\phi_{[A(BC)]D}^*(\theta^2), \tilde{\xi}; x - z, y - z, z),
\]

\[
\tilde{F}^{((AB)C)D)}(\theta^3, \tilde{\xi}; x - y, y - z, z) = \tilde{F}^{((AB)C)D)}((\phi_{[AB]C}^* \times 1_D)(\theta^3), \tilde{\xi}; x - y, y - z, z),
\]

\[
\tilde{F}^{((AB)CD)}(\theta^4, \tilde{\xi}; x - y, y, z) = \tilde{F}^{((AB)CD)}(\phi_{([AB]CD)}^*(\theta^4), \tilde{\xi}; x - y, y, z),
\]

\[
\tilde{F}^{((AB)CD)}(\theta^5, \tilde{\xi}; x - y, y, z) = \tilde{F}^{((AB)CD)}(\phi_{([AB]CD)}^*(\theta^5), \tilde{\xi}; x - y, y, z),
\]

on \( D^3 \) at least, where \( \phi^* : U^\vee \to W^\vee \) denotes the dual of \( \phi : W \to U \). We hence have

\[
\tilde{F}^{((AB)CD)}((\phi_{[AB]C}^* \otimes 1_D)(\phi_{[A(BC)]D}^*(1_A \otimes \phi_{BC}^*)(\theta^1)), \tilde{\xi}; x - y, y - z, z)
\]

\[
= \tilde{F}^{A((BC)D)}(\theta^1, \tilde{\xi}; x, y, z) = \tilde{F}^{A((BC)D)}(\phi_{[A(BC)]D}^*(\phi_{[AB]C}^* \otimes 1_D)(\theta^1)), \tilde{\xi}; x - y, y - z, z)
\]

for all \( \theta^1 \in (A(B(CD)))^\vee \) and \( \tilde{\xi} \). As a consequence, we obtain \( 1_A \otimes \phi_{[BC]D}^* \circ \phi_{[A(BC)]D}^* \circ \phi_{[AB]C}^* \otimes 1_D = \phi_{[AB]CD}^* \circ \phi_{[AB]CD}^* \), which implies the commutativity of the diagram.

This completes the proof of Theorem 5.1.

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