Abstract

The string-inspired technique is used for a first calculation of the one-loop axial-vector vacuum polarisation in a general constant electromagnetic field. A compact result is reached for the difference between this tensor and the corresponding vector vacuum polarisation. This result is confirmed by a Feynman diagram calculation. Its physical relevance is briefly discussed.
1 Introduction: Standard Model Processes in Constant Electromagnetic Fields

Following the calculation of the one-loop vector–vector and vector–axialvector vacuum polarisation tensors in a general constant electromagnetic field, presented in parts I [1] and II [2] of this series, in the present third part we consider the axialvector–axialvector case. As in the previous cases, we will use the “string-inspired” worldline path integral formalism [3]–[16] to arrive at a compact integral representation of this quantity. As a check we will also perform a Feynman diagrammatic calculation of it. With both methods it will turn out to be considerably simpler not to compute the axialvector vacuum polarisation itself, but rather the difference between this and the known [17, 18, 19, 2] vector vacuum polarisation in a constant field.

As in the case of the vector–axialvector amplitude, considered in part II, our main physical interest in this quantity stems from its relevance for low-energy neutrino processes. In particular, we refer to processes where the external momentum flux through heavy gauge boson propagators remains small compared to $m_W$, neutrino energies $E_\nu \ll m_W^3/eF$ and field strengths $eF \ll m_W^2$, so that the local limit of the standard model interaction, the Fermi theory, is applicable ($m_W$ denotes the heavy gauge boson mass). Vacuum polarisation, or phrased differently, the virtual existence of the neutrinos as charged lepton pairs, transfers electromagnetic properties to the neutrinos without requiring additional non-standard parameters (magnetic moments etc.). These loop-induced properties allow for neutrino–photon interactions or interactions of neutrinos with external electromagnetic fields.

The amplitude considered here occurs, for instance, in scattering processes involving 4 neutrinos and an arbitrary number of soft photons, and in decay processes involving 2 neutrinos and a lepton pair in an external field. For magnetic fields, the latter have been studied intensively in [20]–[25], and it is believed that processes of this type can contribute significantly to neutrino energy loss in astrophysical processes involving extreme conditions. Therefore, neutrino heating and cooling processes can be partly governed by those neutrino interactions enhanced by electromagnetic fields. The present work allows for a generalization of such results for magnetic fields to the case of a general electromagnetic field; this provides for new dimensions in parameter space involving electromagnetic invariants also with electric components. Although $E \ll B$ in most realistic scenarios, the invariant $\mathcal{G} = E \cdot B$ can have a sizeable value and, moreover, owing to its pseudoscalar nature, allow for processes that are forbidden in a purely magnetic field (see, e.g., [26]).

2 Worldline Calculation of the Axialvector Vacuum Polarisation Tensor in a Constant Field

According to the formalism developed in [14, 15, 3] the one-loop axialvector vacuum polarisation tensor in a constant field can be represented as the following integral of a worldline correlator of two axialvector vertex operators $\mathbf{1}$:

1 We work initially in the Euclidean spacetime with a positive definite metric $g_{\mu\nu} = \text{diag}(++,++)$. The Euclidean field strength tensor is defined by $F^{ij} = \varepsilon_{ijk}B_k$, $i,j = 1,2,3$, $F^{4i} = -iE_i$, its dual by $F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ with $\varepsilon^{1234} = 1$. The corresponding Minkowski space amplitudes are obtained by rotating $g_{\mu\nu} \rightarrow \eta_{\mu\nu} = \text{diag}(-+++)$, $k^4 \rightarrow -ik^0$, $T \rightarrow is$, $\varepsilon^{1234} \rightarrow i\varepsilon^{1230}$, $\varepsilon^{0123} = 1$, $F^{4i} \rightarrow F^{0i} = E_i$, $\tilde{F}^{\mu\nu} \rightarrow -i\tilde{F}^{\mu\nu}$. 

\[1\]
\[ \Pi_{55}^{\mu\nu}(k) = \frac{e^2}{\sin T} A_5^{\mu}(k) A_5^{\nu}(-k), \]

\[ \langle A_5^{\mu}(k_1) A_5^{\nu}(k_2) \rangle = 2 \int_0^\infty \frac{dT}{T} e^{-m^2 T (4\pi T)^{-\frac{D}{2}}} \det \left[ \tan \left( \frac{\mathcal{Z}}{\Xi} \right) \right] \int_0^T d\tau_1 \int_0^T d\tau_2 \times \left( i k_{1\mu} + 2\psi^{\mu}(\tau_1) \hat{x}(\tau_1) \cdot \psi(\tau_1) + \sqrt{D - 2} z^{\mu}(\tau_1) \right) e^{ik_{1\cdot x}(\tau_1)} \right) \times \left( i k_{2\nu} + 2\psi^{\nu}(\tau_2) \hat{x}(\tau_2) \cdot \psi(\tau_2) + \sqrt{D - 2} z^{\nu}(\tau_2) \right) e^{ik_{2\cdot x}(\tau_2)} \right). \]

(2.1)

Here \( T \) denotes the global Schwinger proper-time variable for the loop fermion, and \( Z_{\mu\nu} = eF_{\mu\nu} T \) with \( F_{\mu\nu} \) the constant field strength tensor. The spacetime dimension \( D \) enters through dimensional regularisation. On the right-hand side the angular brackets denote Wick contraction using the basic field-dependent worldline correlators:

\[ \langle y^{\mu}(\tau_1) y^{\nu}(\tau_2) \rangle = -G_B^{\mu\nu}(\tau_1, \tau_2) = \frac{T}{2} \left[ \frac{1}{(\mathcal{Z})^2} \left( \frac{\mathcal{Z}}{\sin(\mathcal{Z})} e^{-iZ\hat{G}_{B12}} + iZ\hat{G}_{B12} - 1 \right) \right]^{\mu\nu}, \]

\[ \langle \dot{y}^{\mu}(\tau_1) \dot{y}^{\nu}(\tau_2) \rangle = -\dot{G}_B^{\mu\nu}(\tau_1, \tau_2) = \left[ \frac{i}{\mathcal{Z}} \left( \frac{\mathcal{Z}}{\sin(\mathcal{Z})} e^{-iZ\hat{G}_{B12}} - 1 \right) \right]^{\mu\nu}, \]

\[ \langle \ddot{y}^{\mu}(\tau_1) \ddot{y}^{\nu}(\tau_2) \rangle = \ddot{G}_B^{\mu\nu}(\tau_1, \tau_2) = 2\delta(\tau_1 - \tau_2) g^{\mu\nu} - \frac{2}{T} \left[ \frac{\mathcal{Z}}{\sin(\mathcal{Z})} e^{-iZ\hat{G}_{B12}} \right]^{\mu\nu}, \]

\[ \langle \psi^{\mu}(\tau_1) \psi^{\nu}(\tau_2) \rangle = \frac{1}{2} G_F^{\mu\nu}(\tau_1, \tau_2) = \frac{1}{2} G_{F12} \left[ \frac{e^{-iZ\hat{G}_{B12}}}{\cos(\mathcal{Z})} \right]^{\mu\nu}, \quad (2.2) \]

where

\[ \dot{G}_{B12} = \text{sign}(\tau_1 - \tau_2) - 2\left( \frac{\tau_1 - \tau_2}{T} \right), \]

\[ G_{F12} = \text{sign}(\tau_1 - \tau_2), \quad (2.3) \]

and the trigonometric expressions should be understood as power series in the Lorentz matrix \( \mathcal{Z} \). The field \( z \) is auxiliary and has a trivial correlator:

\[ \langle z^{\mu}(\tau_1) z^{\nu}(\tau_2) \rangle = 2\delta(\tau_1 - \tau_2) g^{\mu\nu}. \]

(2.4)

After explicit Wick contraction, the expression in angular brackets becomes

\[ \langle \cdots \rangle_{A_5 A_5} = e^{-k \hat{G}_{B12}} \left\{ k^{\mu} k^{\nu} + k^{\mu} \left[ G_{F22}(\hat{G}_{B21} - \hat{G}_{B22})k \right]^{\nu} + k^{\nu} \left[ G_{F11}(\hat{G}_{B12} - \hat{G}_{B11})k \right]^{\mu} \right. \]

\[ + \left. \left( G_{F11}^{\mu\rho} G_{F22}^{\rho\sigma} - G_{F12}^{\mu\rho} G_{F12}^{\rho\sigma} \right) \left[ \hat{G}_{B12}^{\rho\sigma} - \left[ (\hat{G}_{B11} - \hat{G}_{B12})k \right]^{\rho} \left[ (\hat{G}_{B21} - \hat{G}_{B22})k \right]^{\sigma} \right] \right) \]

\[ + 2(D - 2)\delta_{12} g^{\mu\nu} \right\}, \quad (2.5) \]
where \( k = k_1 = -k_2 \) and

\[
\hat{G}_{B12} = G_B(\tau_1, \tau_2) - G_B(\tau, \tau) = \frac{T}{2Z} \left( e^{-i\hat{G}_{B12}Z} - \cos(Z) + i\hat{G}_{B12} \right).
\]

We write out the integrand explicitly using (2.2) \[\text{[2.2]}\]. Some terms involve products of Lorentz matrices and can be simplified, for instance

\[
G_{\sigma\tau}^{\rho\sigma}[\hat{G}_{B12} - \hat{G}_{B11}]^\rho(\hat{G}_{B21} - \hat{G}_{B22})^\sigma = G_{F12} k \cdot \left[ \frac{1 + \cos^2(Z)}{\cos(Z)\sin^2(Z)} \cos(\hat{G}_{B12}Z) - \frac{1}{\sin^2(Z)} \right] \cdot k.
\]

The result reads

\[
\langle \cdots \rangle_{A5A5} = e^{-k\hat{G}_{B12}k} \left\{ -\left[ \left( \frac{\mathcal{E}_{12}}{\sin(Z)} - \frac{1}{\sin(Z)\cos(Z)} k \right)^\mu \left( \frac{\mathcal{E}_{21}}{\sin(Z)} - \frac{1}{\sin(Z)\cos(Z)} k \right)^\nu \right] + \left( \frac{\mathcal{E}_{12}}{\cos(Z)} k \right)^\mu \left( \frac{\mathcal{E}_{21}}{\cos(Z)} k \right)^\nu \cdot \left[ \frac{1 + \cos^2(Z)}{\cos(Z)\sin^2(Z)} \cos(\hat{G}_{B12}Z) - \frac{1}{\sin^2(Z)} \right] \cdot k \\
+ \frac{2}{T} \left[ \left( \frac{\mathcal{E}_{12}}{\cos(Z)} \right)^\mu \right] \left( \frac{Z}{\sin(Z)\cos(Z)} - \left[ \frac{1 + \sin^2(Z)Z\mathcal{E}_{12}}{\sin(Z)\cos^2(Z)} \right] \right) \cdot k \right\},
\]

where we abbreviated \( e^{-i\hat{G}_{B12}Z} = \mathcal{E}_{12} \). Since from the axial Ward identity we know that, in the massless case, the axialvector–axialvector amplitude must coincide with the vector–vector amplitude, we subtract from this the corresponding integrand for the vector–vector case, given in section 4 of part I:

\[
\langle \cdots \rangle_{AA} = \langle \hat{x}^\mu(\tau_1) + 2i\psi^\mu(\tau_1)k_1 \cdot \psi(\tau_1) \rangle e^{ik_1 \cdot x_1} \left( \hat{x}^\nu(\tau_2) + 2i\psi^\nu(\tau_2)k_2 \cdot \psi(\tau_2) \right) e^{ik_2 \cdot x_2} \]

\[
= e^{-k\hat{G}_{B12}k} \left\{ \hat{G}_{B12} - G_{F12}^\mu k \cdot G_{F12}^\nu k \\
- \left[ \left( \hat{G}_{B11} - G_{F11}^\mu \right)^\mu \left( \hat{G}_{B21} - G_{F21}^\mu \right)^\nu + G_{F12}^\mu G_{F21}^\nu \right] k^\kappa k^\lambda \right\}.
\]

This indeed leads to some simplification,

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2We remark that care must be taken in the determination of coincidence limits, owing to the sign–function appearing in \( \hat{G}_B \). For example, \( \lim_{\tau_2 \to \tau_1} G_{F12}^\mu = -i(\tan(Z))^\mu, \) but \( \lim_{\tau_2 \to \tau_1} G_{F12}^\mu G_{F12}^\nu = g^{\mu\nu} g^{\rho\sigma} - (\tan(Z))^{\mu\nu}(\tan(Z))^{\rho\sigma} \neq G_{F11}^\mu G_{F11}^\nu. \)
\begin{align}
\langle \cdots \rangle_{A_A^5} - \langle \cdots \rangle_{A_A^5} = e^{-k\tilde{G}_{B12}k} \left\{ -4\delta_{12}g^{\mu\nu} + 2\left( \frac{E_{12}}{\cos(Z)} \right)^{\mu\nu} k \cdot U_{12} \cdot k + \frac{2}{T} \left[ \left( \frac{E_{12}}{\cos(Z)} \right)^{\mu\nu} \text{tr} \left( \frac{Z}{\sin(Z)\cos(Z)} \right) - 2\left( \frac{\sin(Z)Z\text{E}_{12}}{\cos^2(Z)} \right)^{\mu\nu} \right] \right\},
\end{align}

where

\begin{align}
U_{12} \equiv \frac{1 - \cos(Z\tilde{G}_{B12})}{\sin^2(Z)}
\end{align}

was introduced in part II, eq. (4.9). The difference should vanish in the massless case, i.e. the integrand should turn into a total derivative. And indeed, if one adds to the above the following two total derivative terms [3]

\begin{align}
0 &= \frac{8}{(4\pi)^{\frac{D}{2}}} \int_0^\infty dT \frac{\partial}{\partial T} \left\{ \frac{e^{-m^2T}}{T^{1+\frac{D}{2}}} \text{det}^{-\frac{1}{2}} \left[ \frac{\tan(Z)}{Z} \right] \right\} \int_0^T d\tau_1 \int_0^\infty d\tau_2 e^{-k\tilde{G}_{B12}k} \left( \frac{E_{12}}{\cos(Z)} \right)^{\mu\nu} \\
&\quad - T^{-1} \frac{\partial}{\partial T} \left\{ \frac{4g^{\mu\nu} + T^2 - T^2 \frac{2}{T} m^2 + k^2}{6} \right\} \\
&\quad + \frac{4}{(4\pi)^{\frac{D}{2}}} \int_0^\infty \frac{dT}{T^{1+\frac{D}{2}}} e^{-m^2T} \text{det}^{-\frac{1}{2}} \left[ \frac{\tan(Z)}{Z} \right] \int_0^T d\tau_1 \int_0^\infty d\tau_2 \times \frac{\partial}{\partial \tau_1} \left\{ e^{-k\tilde{G}_{B12}k} \tilde{G}_{B12} \left( \frac{E_{12}}{\cos(Z)} \right)^{\mu\nu} \right\},
\end{align}

a cancellation of terms ensues, which is complete in the massless case [4]. In the massive case a single term survives, leading to

\begin{align}
\langle A_5^\mu(k)A_5^\nu(-k) \rangle - \langle A^\mu(k)A^\nu(-k) \rangle = \frac{-8m^2}{(4\pi)^{\frac{D}{2}}} \int_0^\infty \frac{dT}{T^{1+\frac{D}{2}}} e^{-m^2T} \text{det}^{-\frac{1}{2}} \left[ \frac{\tan(Z)}{Z} \right] \times \int_0^T d\tau_1 d\tau_2 \left[ \frac{E_{12}}{\cos(Z)} \right]^{\mu\nu} e^{-k\tilde{G}_{B12}k} \\
= \frac{-8m^2}{(4\pi)^{\frac{D}{2}}} \int_0^\infty \frac{dT}{T^{2-\frac{D}{2}}} e^{-m^2T} \text{det}^{-\frac{1}{2}} \left[ \frac{\tan(Z)}{\cos(Z)} \right] \int_0^1 du \left[ \frac{\cos(Z\tilde{G}_{B12}Z)}{\cos(Z)} \right]^{\mu\nu} e^{-Tk\cdot\Phi_{12}k}.
\end{align}

\footnote{At this point the reader should be warned that a naive application of the chain rule to expressions involving sign(\tau) can lead to errors. For example, for n odd one has $\frac{\partial}{\partial \tau_1} (\tilde{G}_{B12})^n = 2\delta_{12} - \frac{2}{T} (\tilde{G}_{B12})^{n-1} \neq n\tilde{G}_{B12}(\tilde{G}_{B12})^{n-1}$.

\footnote{Note that we do not need to put $D = 4$ for this cancellation mechanism to work. This confirms that, as stated in [3], the path integral construction given in [4] does not break the chiral symmetry for parity-even loops.}
Here as usual we have rescaled to the unit circle, \( \tau_{1,2} = T u_{1,2}, k \cdot \hat{G}_{B12} \cdot k \equiv T k \cdot \Phi_{12} \cdot k \), and set \( u_2 = 0 \). Only the cosine part of \( \mathcal{E}_{12} \) contributes to the integral.

This integral still contains a logarithmic ultraviolet divergence at \( T = 0 \), which becomes obvious if one sets the external field equal to zero:

\[
\langle A_\mu^\mu(k)A_\nu^\nu(-k) \rangle_{F=0} = -\frac{m^2}{2\pi^2} \int_0^\infty \frac{dT}{T^2} e^{-T(m^2 + \frac{1}{4}(1-\hat{G}_{B12}^2)k^2)} g^{\mu\nu}.
\]  

(2.12)

Similar to the renormalisation of the vector vacuum polarisation tensor in part I, we remove this divergence by subtracting the same expression at vanishing field and momentum; the meaning of this subtraction will be discussed below. In this way we obtain for the renormalised axialvector vacuum polarisation tensor

\[
\bar{\Pi}_{55}^{\mu\nu}(k) = e^2 \bar{\Pi}_{\text{spin}}^{\mu\nu}(k) + \bar{N}_{55}^{\mu\nu}(k),
\]

\[
\bar{N}_{55}^{\mu\nu}(k) = \frac{e^2 m^2}{2\pi^2} \int_0^\infty \frac{dT}{T} e^{-m^2 T} \left\{ \det \frac{1}{2} \left[ \tan(\frac{Z}{2}) \right] \int_0^1 dv \right\} \int_0^1 du \left[ \frac{\cos(\hat{G}_{B12} Z)}{\cos(\frac{Z}{2})} \right] e^{-T \cdot \Phi_{12} \cdot k - g^{\mu\nu}},
\]

(2.13)

with \( \bar{\Pi}_{\text{spin}}^{\mu\nu} \) as given in part I, Eqs. (4.9) and (4.11); here, the overbar characterises renormalised quantities. Using the decomposition formulas from section 3.2 in part I, and continuing to Minkowski space, we obtain our final result for this amplitude:

\[
\bar{\Pi}_{55}^{\mu\nu}(k) = \frac{e^2}{e^2} \bar{\Pi}_{\text{spin}}^{\mu\nu} + \frac{e^2 m^2}{2\pi^2} \int_0^\infty \frac{ds}{s} e^{-s m^2} \int_0^1 dv \left\{ \frac{z^+ z^-}{\tanh(z^+) \tanh(z^-)} \right\} \exp \left[ -is \sum_{\alpha=+,-} \frac{\cosh(z_{\alpha} v) - \cosh(z_{\alpha})}{2z_{\alpha} \sinh(z_{\alpha})} k \cdot \hat{Z}_{\alpha}^2 \cdot k \right] \sum_{\alpha=+,-} \frac{\cosh(z_{\alpha} v)}{\cosh(z_{\alpha})} (\hat{Z}_{\alpha}^2)^{\mu\nu} + \eta^{\mu\nu},
\]

(2.14)

where \( v = \hat{G}_{B12} = 1 - 2u_1 \),

\[
z^+ = ise_a,
\]

\[
z^- = -esb,
\]

\[
(\hat{Z}_{\alpha}^2)^{\mu\nu} = \frac{(\hat{F}^2)^{\mu\nu} - b^2 \eta^{\mu\nu}}{a^2 + b^2},
\]

(2.15)

\[
(\hat{Z}_{\alpha}^2)^{\mu\nu} = -\frac{(\hat{F}^2)^{\mu\nu} + a^2 \eta^{\mu\nu}}{a^2 + b^2},
\]

and \( a, b \) are the secular invariants:

\[
a = \sqrt{\sqrt{\hat{F}^2 + \hat{G}^2} + \hat{F}},
\]

\[
b = \sqrt{\sqrt{\hat{F}^2 + \hat{G}^2} - \hat{F}},
\]

(2.16)
Figure 1: $W$-boson (left-hand side) and $Z$-boson (right-hand side) exchange contributing to the axialvector amplitude in the context of Standard-Model neutrino interactions.

with

\[ \mathcal{F} = \frac{1}{2} (B^2 - E^2), \]
\[ \mathcal{G} = \mathbf{E} \cdot \mathbf{B}. \]  

(2.17)

We have verified that this integral representation agrees for $b \to 0$ with the result for the magnetic special case given in [21]. The field-free case has, of course, been known for a long time (see, e.g., [27]). Note that the new contribution is not transversal, i.e. that $k_\mu \tilde{N}_{\mu\nu}^{55} \neq 0$; this is not astonishing, since there is no Ward identity that could protect the transversality of $\Pi_{\mu\nu}^{55}$.

It remains to elucidate the significance of the counterterm introduced above. Its meaning depends on the context. Since it is proportional to $A_5^2$, it obviously corresponds to the introduction of a mass term for the axial gauge field, indicating that the “axial QED” consisting only of a massive fermion coupled to an axial gauge field is not renormalisable.

In the context of Fermi theory, on the other hand, the axialvector-field coupling is associated with the coupling of a left-handed neutrino current to the electron loop, and $e_5$ is proportional to $g_A G_F$, i.e. the axial coupling to the electron times Fermi’s constant. Subtracting the divergence mentioned above thereby corresponds to a “renormalisation” of Fermi’s constant (disregarding the fact that Fermi’s theory is generally non-renormalisable).

Finally, in the context of the renormalisable microscopic theory, the Standard Model, the present divergence has two origins; in order to recognize this, consider the diagrams in Fig. 1, which, among others, contribute to our amplitude. The left-hand side depicts a $W$-boson exchange diagram that is finite by simple power-counting owing to the $1/p^2$ momentum dependence of the $W$-boson propagator for large loop momentum $p^2 \gg m_W^2$. But upon taking the local limit by approximating the propagator by $1/m_W^2$, in order to arrive at the effective Fermi theory, artificial divergences are introduced. By contrast, the right-hand side of Fig. 1 depicts a $Z$-boson exchange with a self-energy correction to the boson. This diagram is, of course, divergent before taking the local limit, and this divergence is absorbed in a wave function and coupling constant renormalisation of the $Z$-boson. The subtraction described above takes both types of divergences into account and is normalised by requiring that vacuum polarization has no physical effect on the axialvector field at zero momentum if there are no external fields.
3 Feynman Diagram Calculation of the Axialvector Vacuum Polarisation Tensor in a Constant Field

As a check of this result, we will now perform a standard Feynman diagram calculation of this quantity (in Minkowski space).

Decomposition Formulas

We will use the same approach as before to the decomposition of the field strength tensor, although in the slightly different conventions of [28, 29, 30]. Defining

\[ C_{\mu\nu} = \frac{1}{a^2 + b^2} (-b F_{\mu\nu} + a \tilde{F}_{\mu\nu}), \quad B_{\mu\nu} = \frac{1}{a^2 + b^2} (a F_{\mu\nu} + b \tilde{F}_{\mu\nu}), \]  \hspace{1cm} (3.1)

we have the relations

\[ C^2_{\mu\nu} = \frac{1}{a^2 + b^2} (F^2_{\mu\nu} + a^2 \eta_{\mu\nu}), \quad B^2_{\mu\nu} = \frac{1}{a^2 + b^2} (F^2_{\mu\nu} - b^2 \eta_{\mu\nu}). \]  \hspace{1cm} (3.2)

The inverse relations are easily found:

\[ F_{\mu\nu} = -b C_{\mu\nu} + a B_{\mu\nu}, \quad \tilde{F}_{\mu\nu} = a C_{\mu\nu} + b B_{\mu\nu}, \]
\[ F^2_{\mu\nu} = b^2 C^2_{\mu\nu} + a^2 B^2_{\mu\nu}, \quad \eta_{\mu\nu} = C^2_{\mu\nu} - B^2_{\mu\nu}. \]  \hspace{1cm} (3.3)

Most importantly, we get the decomposition relations by a straightforward calculation

\[ (CB)_{\mu\nu} = 0 = (BC)_{\mu\nu}, \quad C^3_{\mu\nu} = C_{\mu\nu}, \quad B^3_{\mu\nu} = -B_{\mu\nu}, \]
\[ C^{2\mu}_{\mu} \equiv -C_{\mu\nu} C^{\nu\mu} = -2, \quad B^{2\mu}_{\mu} \equiv -B_{\mu\nu} B^{\mu\nu} = 2, \]  \hspace{1cm} (3.4)

where it becomes obvious that the decomposition of the field strength tensor into $C$ and $B$ corresponds to a decomposition into orthogonal magnetic and electric subspaces.

Employing this representation of the field strength tensor, we can decompose any function of $F_{\mu\nu}$, regular at $F = 0$, into

\[ f(F)_{\mu\nu} = f(-bC)_{\mu\nu} + f(aB)_{\mu\nu} = f_{\text{odd}}(-b) C_{\mu\nu} + f_{\text{even}}(-b) C^2_{\mu\nu} - i f_{\text{odd}}(ia) B_{\mu\nu} - f_{\text{even}}(ia) B^2_{\mu\nu}, \]  \hspace{1cm} (3.5)

where $f_{\text{even,odd}}$ denotes the even/odd part of $f$.

Decomposition of $\Pi_{55}$

The axialvector–axialvector amplitude in an arbitrary external field is defined by

\[ \Pi_{55}^{\mu\nu}(k) = \frac{i e^2}{2} \text{tr} \gamma \int p \gamma^\mu \gamma_5 g(p) \gamma^\nu \gamma_5 g(p - k), \]  \hspace{1cm} (3.6)

\[ ^5\text{Our Dirac matrix conventions are } \{ \gamma_\mu, \gamma_\nu \} = -2 \eta_{\mu\nu} \mathbb{1}, \gamma_5^2 = 1. \]
where \( g(p) \) denotes the Fourier transform of the Dirac Green’s function, and \( \int_p = \int \frac{d^4p}{(2\pi)^4} \). In constant external fields, this object is given by

\[
g(p) = i \int_0^{\infty} ds e^{-im^2 s} e^{-Y(is)} \left[ m - \gamma_\alpha \left( p - i eF Xp \right)^\alpha \right] e^{-pXp} e^{ie\frac{1}{2} s \sigma F},
\]

(3.7)

where

\[
Y(s) = \frac{1}{2} \text{tr} \ln[\cos(eFs)], \quad X(s) = \frac{\tan(eFs)}{eF},
\]

(3.8)

and we employed matrix notation, \( \sigma F = \sigma_{\mu\nu} F^{\mu\nu} \), \( \sigma_{\mu\nu} = (i/2)[\gamma_\mu, \gamma_\nu] \). Using

\[
[e^{ieF s \sigma F}, \gamma_5] = 0,
\]

(3.9)

it can be shown that

\[
g(p) |_{m \to -m} = -\gamma_5 g(p) \big|_{m \to -m} \left( = \gamma_5 g(-p) \right).
\]

(3.10)

Adding and subtracting the mass term with the correct sign, we get

\[
g(p) |_{m \to -m} = g(p) - 2i m \int_0^{\infty} ds e^{-im^2 s} e^{-Y(is)} e^{-pXp} e^{ie\frac{1}{2} s \sigma F}.
\]

(3.11)

Inserting Eqs. (3.11) and (3.10) into Eq. (3.6), we again find a decomposition

\[
\Pi_{\mu\nu}^{\mu\nu}(k) = \frac{e^2}{e^2} \Pi_{\text{spin}}^{\mu\nu}(k) + N_{55}^{\mu\nu}(k),
\]

(3.12)

where \( \Pi_{\text{spin}}^{\mu\nu} \) denotes the vector polarisation tensor, and the additional term arising from the second term of Eq. (3.11) is defined by

\[
N_{55}^{\mu\nu}(k) = 2ie^2 m^2 \int_{p}^{\infty} dsds' e^{-im^2(s+s')} e^{-(Y'+Y)} e^{-p(X+X')p} e^{2pX'k} e^{-kX'k} \times \text{tr} \left[ \gamma^\mu e^{ie\frac{1}{2} s \sigma F} \gamma^\nu e^{ie\frac{1}{2} s' \sigma F} \right].
\]

(3.13)

**Calculation of \( N_{55}^{\mu\nu} \)**

The \( p \) integral is Gaussian and gives (in Minkowski space)

\[
\int_{p} e^{-p(X+X')p} e^{2pX'k} e^{-kX'k} = \frac{i}{(4\pi)^2} \left( \text{det}(X + X') \right)^{-1/2} e^{-k \frac{X' + X}{2X'k}}.
\]

(3.14)

Inserting Eq. (3.14) into Eq. (3.13), we encounter the combination

\[
e^{-(Y'+Y)} \left( \text{det}(X + X') \right)^{-1/2} = \sqrt{\frac{e^F}{\sin eFs}}, \quad \text{where } \tilde{s} := s + s'.
\]

(3.15)

\[^6\text{Our definition of } \Pi_{\text{spin}}^{\mu\nu} \text{ here differs by a sign from the one used in } [19] \text{ (it agrees with } [31]).\]
Putting it all together, the desired quantity reads
\[ N_{\mu\nu}^{\mu\nu}(k) = -\frac{2e_0^2 m^2}{(4\pi)^2} \int_0^\infty ds ds' e^{-im^2 s} \sqrt{\det \frac{eF}{\sin eF}} e^{-k \frac{x' x'}{x + x'}} k \text{tr} \gamma \left[ \gamma_\mu e^{i\frac{e}{2} \sigma} F \gamma_\nu e^{i\frac{e}{2} \sigma'} F \right]. \]

Employing the field strength tensor decomposition described above, it is straightforward to show that
\[ \sqrt{\det \frac{eF}{\sin eF}} = -\frac{ea \ e b}{\sinh e b \ sin e a s}, \]
and that the exponent in Eq. (3.16) yields
\[ k \frac{XX'}{X + X'} = -i \left[ \cosh e b v s - \cosh e b s k C^2 k + \frac{\cos e a v s - \cos e a s}{2 e a \ sin e a s} k B^2 k \right], \]
where \( v = 1 - \frac{2 e'}{s} \). Read together with the mass term \( e^{-im^2 s} \) in Eq. (3.16), we find the usual phase factor of the polarisation tensor or the axialvector–vector amplitude
\[ e^{-im^2 s} e^{-k \frac{x' x'}{x + x'}} k = e^{-i\phi_0}, \]
where
\[ \phi_0 = m^2 - \frac{k C^2 k \cosh e b s - \cosh v e b s}{2 e b \ sinh e b s} - \frac{k B^2 k \cos v e a s - \cos e a s}{2 e a \ sin e a s}. \]

The final task is to find an appropriate representation of the Dirac trace in Eq. (3.16). The trace has been evaluated, for instance, in App. C of [19]; in terms of the \( C \) and \( B \) decomposition the result reads
\[ \text{tr} \gamma \left[ \gamma_\mu e^{i\frac{e}{2} \sigma} F \gamma_\nu e^{i\frac{e}{2} \sigma'} F \right] = 4 \left\{ -\cos e a s \cosh e b v s C^2_{\mu\nu} + \cos e a v s \cosh e b s B^2_{\mu\nu} \right. \]
\[ \left. + \cos e a s \sinh e b v s C_{\mu\nu} - \sin e a v s \cosh e b s B_{\mu\nu} \right\}. \]

Upon substitution of \( \int_0^\infty ds \int_{-1}^1 \frac{dv}{2} \) for \( \int_0^\infty ds ds' \), we observe that the \( v \) integration is even, the phase factor is even in \( v \), but the terms \( \sim C_{\mu\nu}, B_{\mu\nu} \) are odd and thus drop out. The final result before renormalisation then reads \( \tilde{s} \to s \)
\[ N_{\mu\nu}^{\mu\nu}(k) = \frac{e_0^2}{2 \pi^2} m^2 \int_0^\infty ds \int_{-1}^1 \frac{dv}{2} e^{-is\phi_0} \frac{e a s \ sinh e a s}{\sin e a s \ sinh e b s} \]
\[ \times \left[ -\cos e a s \cosh e b v s C^2_{\mu\nu} + \cos e a v s \cosh e b s B^2_{\mu\nu} \right], \]
in agreement with (2.14) since \( B^2_{\mu\nu} = (\tilde{\gamma}^2_+)_{\mu\nu}, C^2_{\mu\nu} = -(\tilde{\gamma}^2_-)_{\mu\nu} \).
4 Discussion

We have used both the worldline formalism and standard Feynman diagrams for a first calculation of the axialvector vacuum polarisation tensor in an arbitrary constant electromagnetic field. With both methods, we obtained the same compact integral representation for the difference between this tensor and the well-known vector vacuum polarisation tensor in a constant field. For the special case of a purely magnetic field, our result agrees with the one of [21]; the case of a general constant field has, as far as is known to the authors, not been treated before.

As in the vector and vector–axialvector cases, the performance of suitable partial integrations has turned out to be essential for reaching a maximally compact integral representation in the worldline formalism. However, while in those cases the appropriate partial integrations involved only the loop variables, and could be easily found following the Bern-Kosower prescription [3] of removing all second derivatives of worldline Green’s functions, in the present case a judiciously chosen combination of partial integrations in both the loop variables and the global proper-time was found necessary for this purpose. It would be clearly useful to investigate the systematics of this partial integration procedure for the general vector–axialvector amplitudes, starting from the (vacuum) master formulas given in [15].

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