The relations of Choquet Integral and G-Expectation

Ju Hong Kim

Department of Mathematics, Sungshin Women’s University, Seoul 02844, Republic of Korea

Abstract

In incomplete financial markets, there exists a set of equivalent martingale measures (or risk-neutral probabilities) in an arbitrage-free pricing of the contingent claims. Minimax expectation is closely related to the g-expectation which is the solution of a certain stochastic differential equation. We show that Choquet expectation and minimax expectation are equal in pricing European type options, whose payoff is a monotone function of the terminal stock price $S_T$.

Keywords: Choquet expectation, G-expectation, Minimax expectation, Submodular Capacity, Comonotonicity

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1. Introduction

Nonlinear expectations such as Choquet expectation, minimax expectation and g-expectation are applied to many areas like statistics, economics and finance. Choquet expectation [3] has a difficulty in defining a conditional expectation. Wang [11] introduces the concept of conditional Choquet expectation which is the conditional expectation with respect to a submodular capacity.

Choquet expectation [1, 5] is equivalent to the convex(or coherent) risk measure if given capacity is submodular. G-expectation (see papers [4, 6, 7, 8, 9, 10, 13] for the related topics) is the solution of the following nonlinear backward stochastic differential equation (BSDE),

$$y_t = \xi + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_s dB_s, \quad 0 \leq t \leq T.$$  \hspace{1cm} (1.1)

G-expectation very much depends on the generator $g$ in the BSDE (1.1). If $g$ is sublinear with respect to $z$, then $g$-expectation is represented as

$$y_0 = \sup_{Q \in \mathcal{P}} E_Q[\xi] \quad \forall \xi \in L^2(\Omega, \mathcal{F}_T, P)$$

where $y_t$ is the solution of the BSDE (1.1), $E_Q$ represents the expectation with respect to $Q$ and $\mathcal{P}$ is a set of risk-neutral probability measures. Minimax expectation [12] is the expectation taken supremum or infimum over a set of probability measures. Minimax expectation is very much related to $g$-expectation.
In this paper, we will show that Choquet and minimax expectations are equal in pricing European type options, whose payoff is a monotone function of the terminal stock price $S_T$. First, it is shown that the Choquet and minimax expectations are equal on the space of real-valued, bounded, $\mathcal{F}_T$-measurable functions, $\mathcal{B}(\Omega, \mathcal{F}_T, P)$. Second, the function space of $\mathcal{B}(\Omega, \mathcal{F}_T, P)$ is extended to a monotone subset of $L^2(\Omega, \mathcal{F}_T, P)$.

2. $G$-expectation and Choquet expectation

In this section, we define the upper and the lower Choquet expectations, and also find the specific solution of the BSDE (1.1) when the generator $g$ is sublinear with respect to $z$. Minimax pricing rules are closely related to $g$-expectation, the solution of the BSDE (1.1).

Let $(\Omega, \mathcal{F}, P)$ be a given completed probability space. Let $(\Omega, (\mathcal{F}_t)_{t \in [0,T]}, P)$ be the given filtered probability space. The filtration $\mathcal{F}_t = \sigma\{B_s : s \leq t\}$ is generated by $(B_t)_{t \in [0,T]}$, a one-dimensional standard Brownian motion. Let $\mathcal{B}(\Omega, \mathcal{F}_T, P)$ be the space of real-valued, bounded, $\mathcal{F}_T$-measurable functions, and let $V : \mathcal{B}(\Omega, \mathcal{F}_T, P) \to \mathbb{R}$ be a functional.

Definition 2.1. A set function $c : \mathcal{F}_T \to [0,1]$ is called monotone if
\[ c(A) \leq c(B) \quad \text{for } A \subset B, \text{ and } A, B \in \mathcal{F}_T \]
and normalized if
\[ c(\emptyset) = 0 \quad \text{and} \quad c(\Omega) = 1. \]

The monotone and normalized set function is called a capacity. A monotone set function is called submodular or 2-alternating if
\[ c(A \cup B) + c(A \cap B) \leq c(A) + c(B) \quad A, B \in \mathcal{F}_T. \]

The risk of an asset position $X + Y$ will be lower than the sum of each risk, because of the diversification effects. The property of comonotonicity is that if there is no way for $X$ to serve as a hedge for $Y$, then it is simply adding up the risks.

Two real functions $X, Y \in \mathcal{B}(\Omega, \mathcal{F}_T, P)$ are called comonotonic if
\[ [X(\omega_1) - X(\omega_2)][Y(\omega_1) - Y(\omega_2)] \geq 0, \quad \omega_1, \omega_2 \in \Omega. \]

The functional $V$ is said to be comonotonic additive if
\[ X, Y \text{ are comonotonic} \implies V(X + Y) = V(X) + V(Y). \]

Definition 2.2. Let $c : \mathcal{F}_T \to [0,1]$ be a capacity. The Choquet expectation with respect to $c$ is defined as
\[ \int_{\Omega} X dc := \int_{-\infty}^{0} (c(X > x) - 1) \, dx + \int_{0}^{\infty} c(X > x) \, dx, \quad X \in L^2(\Omega, \mathcal{F}_T, P). \]
The following theorem of Schmeidler [15] tells us that there exists a capacity that the normalized, monotone, and comonotonic additive functional is equal to Choquet expectation on $\mathcal{B}(\Omega, \mathcal{F}_T, P)$.

**Theorem 2.3** ([15]). Let $V$ be a functional from $\mathcal{B}(\Omega, \mathcal{F}_T, P)$ to $\mathbb{R}$. The following statements are equivalent.

1. $V$ is normalized, monotone, and comonotonic additive.
2. There exists a unique capacity $c : \mathcal{F}_T \to [0, 1]$ such that
   
   $$V(X) = \int_{-\infty}^{0} (c(X > x) - 1) \, dx + \int_{0}^{\infty} c(X > x) \, dx \quad \forall X \in \mathcal{B}(\Omega, \mathcal{F}_T, P). \quad (2.1)$$

Let $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a function that $(y, z) \mapsto g(t, y, z)$ is measurable for each $(y, z) \in \mathbb{R} \times \mathbb{R}^n$ and satisfy the following conditions

$$|g(t, y, z) - g(t, \bar{y}, \bar{z})| \leq K(|y - \bar{y}| + |z - \bar{z}|) \quad \forall t \in [0, T], \forall (y, z), (\bar{y}, \bar{z}) \in \mathbb{R} \times \mathbb{R}^n, \text{ for some } K > 0,$$

$$\int_{0}^{T} |g(t, 0, 0)|^2 \, dt < \infty,$$

$$g(t, y, 0) = 0 \text{ for each } (t, y) \in [0, T] \times \mathbb{R}. \quad (2.2c)$$

The space $L^2(\Omega, \mathcal{F}_T, P)$ is defined as

$$L^2(\Omega, \mathcal{F}_T, P) := \{ \xi \mid \xi \text{ is } \mathcal{F}_T\text{-measurable random variable and } E[|\xi|^2] < \infty \}.$$

**Theorem 2.4** ([14]). For every terminal condition $\xi \in L^2(\mathcal{F}_T) := L^2(\Omega, \mathcal{F}_T, P)$ the following backward stochastic differential equation

$$-dy_t = g(t, y_t, z_t) \, dt - z_t dB_t, \quad 0 \leq t \leq T,$$

$$y_T = \xi \quad (2.3a)$$

has a unique solution

$$(y_t, z_t)_{t \in [0, T]} \in L^2_T([0, T]; \mathbb{R}) \times L^2_T([0, T]; \mathbb{R}^n).$$

**Definition 2.5.** For each $\xi \in L^2(\mathcal{F}_T)$ and for each $t \in [0, T]$ $g$–expectation of $\xi$ and the conditional $g$–expectation of $\xi$ under $\mathcal{F}_t$ is respectively defined by

$$\mathcal{E}_g[\xi] := y_0, \quad \mathcal{E}_g[\xi|\mathcal{F}_t] := y_t,$$

where $y_t$ is the solution of the BSDE (2.3).

Let $\{S_t\}$ be the stock price evolving as a stochastic differential equation

$$\frac{dS_t}{S_t} = \mu_t \, dt + \sigma_t dB_t$$

where $\{\mu_t\}$ is a market return rate, and $\{\sigma_t\}$ is a market volatility.
In a Black-Scholes world, there exists a unique risk-neutral probability measure $Q$ defined as
\[
d\frac{dQ}{dP} = e^{-\frac{1}{2} \int_0^T (\frac{\mu_s}{\sigma_s})^2 ds + \int_0^T \frac{\mu_s}{\sigma_s} dB_s},
\]
where $r$ is a riskless interest rate. In a real world, the parameters $\mu_t$ and $\sigma_t$ are not known exactly. We assume that $\mu_t$ belong to some interval, i.e. $\mu_t \in [r - k\sigma_t, r + k\sigma_t]$ for a constant $k > 0$. Then the risk-neutral probability measures belong to
\[
\mathcal{P} = \left\{ Q^\nu : \frac{dQ^\nu}{dP} = e^{-\frac{1}{2} \int_0^T |\nu_s|^2 ds + \int_0^T \nu_s dB_s}, \sup_{t \in [0, T]} |\nu_t| \leq k \right\}
\]
where $\nu_t := (\mu_t - r)/\sigma_t$. There are two pricing methods of a contingent claim $\xi$, i.e. minimax pricing rules which are
\[
\mathcal{E}[\xi] := \inf_{Q \in \mathcal{P}} E_Q[\xi], \quad \mathcal{E}^\nu[\xi] := \sup_{Q \in \mathcal{P}} E_Q[\xi].
\]

Let $\xi \in L^2(\Omega, \mathcal{F}_T, P)$. The conditional $g$-expectations $\mathcal{E}[\xi|\mathcal{F}_t]$ and $\mathcal{E}^\nu[\xi|\mathcal{F}_t]$ are given as
\[
\mathcal{E}[\xi|\mathcal{F}_t] = \text{ess sup}_{Q \in \mathcal{P}} E_Q[\xi|\mathcal{F}_t], \quad \mathcal{E}^\nu[\xi|\mathcal{F}_t] = \text{ess inf}_{Q \in \mathcal{P}} E_Q[\xi|\mathcal{F}_t],
\]
which are the solutions of BSDE (2.3) when the generators are $g(t, y_t, z_t) = k|z_t|$ and $g(t, y_t, z_t) = -k|z_t|$ respectively. The equations (2.4) will be proved in Lemma 2.1.

It is clear that
\[
\mathcal{E}[\xi|\mathcal{F}_0] = \mathcal{E}[\xi], \quad \mathcal{E}^\nu[\xi] = \mathcal{E}^\nu[\xi] := \inf_{Q \in \mathcal{P}} E_Q[\xi].
\]

The upper and the lower Choquet integrals(or expectations) are respectively defined as
\[
\bar{V}(\xi) := \int_{-\infty}^0 (\bar{c}(\xi > x) - 1) \, dx + \int_0^\infty \bar{c}(\xi > x) \, dx,
\]
\[
\underline{V}(\xi) := \int_{-\infty}^0 (\underline{c}(\xi > x) - 1) \, dx + \int_0^\infty \underline{c}(\xi > x) \, dx,
\]
where $\bar{c}$ and $\underline{c}$ are defined as
\[
\bar{c}(A) = \sup_{Q \in \mathcal{P}} Q(A) \quad \text{and} \quad \underline{c}(A) = \inf_{Q \in \mathcal{P}} Q(A) \quad \text{for} \ A \in \mathcal{F}_T.
\]

We will use the notation of $\bar{V}(\xi) := \int \xi \, d\bar{c}$ and $\underline{V}(\xi) := \int \xi \, d\underline{c}$ or sometimes integration notation just for the convenience of proof.
It can be easily seen that

\[ V(\xi) \leq E[\xi] \leq \bar{E}[\xi] \leq \bar{V}(\xi). \]

In the complete market where \( P \) has a single element, we can see that

\[ V(\xi) = E[\xi] = \bar{E}[\xi] = \bar{V}(\xi). \]

**Theorem 2.6** ([1]). Suppose that \( g \) satisfies the condition (2.2a)-(2.2c). Then there exists a Choquet integral whose restriction to \( L^2(\Omega, \mathcal{F}_T, P) \) is equal to a \( g \)-expectation if and only if \( g \) does not depend on \( y \) and is linear in \( z \), that is, there exists a continuous function \( \nu_t \) such that

\[ g(y, z, t) = \nu_t z. \]

The Theorem 2.6 implies that the generator \( g \) in (2.3) should be linear function for both Choquet integral and \( g \)-expectation to be equal. We will show that \( \bar{E}[\xi|\mathcal{F}_t] \) and \( E[\xi|\mathcal{F}_t] \) are the solutions of the BSDEs (2.5) in the following Lemma 2.1.

**Lemma 2.1.** For \( \xi \in L^2(\Omega, \mathcal{F}_T, P) \), let \((Y_t, z_t)\) and \((y_t, z_t)\) be the unique solution of the following BSDEs

\begin{align*}
    Y_t &= \xi + \int_t^T k|z_s| \, ds - \int_t^T z_s \, dB_s, \quad t \in [0, T], \quad (2.5a) \\
    y_t &= \xi - \int_t^T k|z_s| \, ds - \int_t^T z_s \, dB_s, \quad t \in [0, T] \quad (2.5b)
\end{align*}

respectively. Then \( Y_t \) and \( y_t \) are respectively represented as

\begin{align*}
    Y_t &= \text{ess sup}_{Q \in \mathcal{P}} E_Q[\xi|\mathcal{F}_t] = \bar{E}[\xi|\mathcal{F}_t], \quad (2.6a) \\
    y_t &= \text{ess inf}_{Q \in \mathcal{P}} E_Q[\xi|\mathcal{F}_t] = E[\xi|\mathcal{F}_t]. \quad (2.6b)
\end{align*}

**Proof.** First, we show (2.6a). Let \( \nu_t = k \text{ sgn}(z_t) \). Then \( \sup_{t \in [0, T]} |\nu_t| \leq k \). If we define \( z'_t \) as

\[ z'_t = \exp \left( -\frac{1}{2} \int_0^t |\nu_s|^2 \, ds + \int_0^t \nu_s dB_s \right), \quad 0 \leq t \leq T, \]

then \((z'_t)_{0 \leq t \leq T}\) is a \( P \)-martingale since \( dz'_t / z'_t = \nu_t \cdot dB_t \). Also \( z'_T \) is a \( P \)-density on \( \mathcal{F}_T \) since \( 1 = z'_0 = E[z'_T] \).

Define an equivalent martingale probability measure \( Q' \) and a Brownian motion \( \bar{B}_t \) as

\[ \frac{dQ'}{dP} = e^{-\frac{1}{2} \int_0^T |\nu_s|^2 \, ds + \int_0^T \nu_s dB_s}, \quad \bar{B}_t = B_t - \int_0^t \nu_s \, ds. \]
Then $Q^\nu \in \mathcal{P}$, and Girsanov’s theorem implies that $\{\bar{B}_t\}$ is a $Q^\nu$-Brownian motion.

The BSDE (2.5a) is expressed as

$$ Y_t = \xi - \int_t^T z_s^\theta \, dB_s. $$

So we get

$$ Y_t = E_{Q^\nu} [\xi \mid \mathcal{F}_t] \leq \text{ess sup}_{Q \in \mathcal{P}} E_Q [\xi | \mathcal{F}_t]. \quad (2.7) $$

Let $\{\theta_t\}$ be a adapted process satisfying

$$ \sup_{t \in [0,T]} |\theta_t| \leq k. $$

Consider the following BSDE

$$ Y^\theta_t = \xi + \int_t^T \theta_s z_s^\theta \, ds - \int_t^T z_s^\theta \, dB_s, \quad t \in [0,T]. \quad (2.8) $$

Define an equivalent martingale probability measure $Q^\theta$ and a Brownian motion $\bar{B}^\theta_t$ as

$$ \frac{dQ^\theta}{dP} = e^{-\frac{1}{2} \int_0^T |\theta_s|^2 ds + \int_0^T \theta_s \, dB_s}, \quad \bar{B}^\theta_t = B_t - \int_0^t \theta_s \, ds. $$

Then $Q^\theta \in \mathcal{P}$, and Girsanov’s theorem implies that $\{\bar{B}^\theta_t\}$ is a $Q^\theta$-Brownian motion. The BSDE (2.8) is expressed as

$$ Y^\theta_t = \xi - \int_t^T z_s \, d\bar{B}^\theta_s. $$

So we get

$$ Y^\theta_t = E_{Q^\theta} [\xi | \mathcal{F}_t]. $$

Since $\theta_t z_t \leq k|z_t|$ for all $(z_t, t) \in \mathbb{R} \times [0,T]$, the Comparison Theorem applied to (2.5a) and (2.8), implies that

$$ E_{Q^\theta} [\xi | \mathcal{F}_t] = Y^\theta_t \leq Y_t \quad \forall t \in [0,T] $$

Hence we obtain

$$ \text{ess sup}_{Q \in \mathcal{P}} E_Q [\xi | \mathcal{F}_t] \leq Y_t. \quad (2.9) $$

The inequalities (2.7) and (2.9) implies that

$$ \bar{E}[\xi | \mathcal{F}_t] := \text{ess sup}_{Q \in \mathcal{P}} E_Q [\xi | \mathcal{F}_t] $$
is the solution of (2.5a).

In the same fashion, we can show that

$$E[\xi | F_t] := \inf_{Q \in P} E_Q[\xi | F_t]$$

is the solution of (2.5b) by setting $\nu_t = -k \text{sgn}(z_t)$.

3. Choquet expectation and minimax expectation

In this section, we show that Choquet expectation and minimax expectation are equal in pricing European type options, whose payoff is a monotone function of the terminal stock price $S_T$. We also prove that the minimax expectation attains a maximum or a minimum on the set of equivalent martingale probability measures which is weakly compact.

At the expiration date $T$, let the stock price $S_T \in L^2(\Omega, \mathcal{F}_T, P)$ be a unique solution of the following SDE

$$dS_t = \mu_t S_t \, dt + \sigma_t S_t \, dB_t, \quad t \in [0, T].$$

Let $\Phi$ be a monotone function such that $\Phi(S_T) \in L^2(\Omega, \mathcal{F}_T, P)$. Let $(Y_t, z_t)$ and $(y_t, z_t)$ be the unique solution of the following BSDE

$$Y_t = \Phi(S_T) + \int_t^T \mu_s |z_s| \, ds - \int_0^T z_s \, dB_s,$n$$

$$y_t = \Phi(S_T) - \int_t^T \mu_s |z_s| \, ds - \int_0^T z_s \, dB_s,$n$$

respectively.

In Lemma 2.1 we have shown that

$$Y_t = \bar{E}^\Phi(S_T)[F_t], \quad y_t = E^\Phi(S_T)[F_t].$$

For example, in the option pricing, the monotone functions $\Phi(x) = (x - K)^+$ or $\Phi(x) = (K - x)^+$ is the payoff function of European call or put option, respectively. Here $K$ is an exercise price of the option. We want to show that

$$\bar{E}^\Phi(S_T) = \bar{V}^\Phi(S_T), \quad E^\Phi(S_T) = V^\Phi(S_T),$$

where $\bar{V}$ and $V$ are the upper and lower Choquet expectations, respectively.

Since $\bar{E}[\xi]$ is defined as

$$\bar{E}[\xi] := \sup_{Q \in P} E_Q[\xi],$$

it is obvious that $\bar{E}$ is normalized and monotone.

For each $i = 1, 2$, let the random variables $\xi'_i$ be comonotonic functions.
\( \bar{E} \) is comonotonic additive since 
\[
\bar{E}[\xi_1 + \xi_2] = \sup_{Q \in P} E_Q[\xi_1 + \xi_2] = \sup_{Q \in P} E_Q[\xi_1] + \sup_{Q \in P} E_Q[\xi_2] = \bar{E}[\xi_1] + \bar{E}[\xi_2].
\]

So Theorem 2.3 says that there exists a unique capacity \( c : \mathcal{F}_T \to [0, 1] \)
satisfying 
\[
\bar{E}[X] = \int_{-\infty}^{0} (c(X > x) - 1) \, dx + \int_{0}^{\infty} c(X > x) \, dx \quad \forall X \in \mathcal{B}(\Omega, \mathcal{F}_T, P). \tag{3.1}
\]

If we take \( X = I_A \) for \( A \in \mathcal{F}_T \), then (3.1) becomes 
\[
\bar{E}[I_A] = \int_{-\infty}^{0} (c(I_A > x) - 1) \, dx + \int_{0}^{\infty} c(I_A > x) \, dx. \tag{3.2}
\]

Thus we have 
\[
c(A) = \sup_{Q \in P} Q[A] := \bar{c}(A). \tag{3.3}
\]

So we have \( c = \bar{c} \).

Therefore, the equation (3.3) becomes 
\[
\bar{E}[X] = \int_{-\infty}^{0} (\bar{c}(X > x) - 1) \, dx + \int_{0}^{\infty} \bar{c}(X > x) \, dx := \tilde{V}(X) \quad \forall X \in \mathcal{B}(\Omega, \mathcal{F}_T, P).
\]

From now on, we will show that the equation (3.3) can be extended from \( \mathcal{B}(\Omega, \mathcal{F}_T, P) \) to a set of the monotone functions which is a subset of \( L^2(\Omega, \mathcal{F}_T, P) \).

**Lemma 3.1.** The capacity \( \bar{c} \) in (3.3) is submodular.

**Proof.** It’s easily shown that \( \bar{c} \) is monotone and normalized. Since \( I_{A \cap B} \) and \( I_{A \cup B} \) are a pair of comonotone functions for all \( A, B \in \mathcal{F}_T \), the comonotonicity of \( \bar{E} \) implies 
\[
\bar{c}(A \cap B) + \bar{c}(A \cup B) = \bar{E}[I_{A \cap B}] + \bar{E}[I_{A \cup B}] = \bar{E}[I_{A \cap B} + I_{A \cup B}] = \bar{E}[I_A + I_B] \leq \bar{E}[I_A] + \bar{E}[I_B] = \bar{c}(A) + \bar{c}(B).
\]

So the proof is done. \( \square \)

**Lemma 3.2.** \( \bar{E}[\xi] := \sup_{Q \in P} E_Q[\xi] \) is \( L^2 \)-continuous for comonotonic functions \( \xi \in L^2(\Omega, \mathcal{F}_T, P) \).

**Proof.** Let \( \xi_1 \) and \( \xi_2 \) be comonotonic functions. Since \( \bar{E} \) is comonotonic additive, 
\[
|\bar{E}[\xi_2] - \bar{E}[\xi_1]| = |\bar{E}[\xi_2 - \xi_1]| = \sup_{Q \in P} E_Q[|\xi_2 - \xi_1|] \leq \sup_{Q \in P} E_Q[|\xi_1 - \xi_2|] = \bar{E}[|\xi_2 - \xi_1|].
\]
Now we’ll show that $\tilde{E}$ is $L^2$-bounded. Let an adapted process $\{\theta_t\}$ bounded by $k$ be such that

$$
\frac{dQ^\theta}{dP} = e^{-\frac{1}{2}\int_0^T |\theta_s|^2 ds + \int_0^T \theta_s dB_s}.
$$

By the Hölder’s inequality, we have

$$
E_Q(\xi) = E\left( E\left[ \frac{dQ^\theta}{dP} \right] \right)^\frac{1}{2} \leq \left( E\left[ |\xi|^2 \right] \right)^\frac{1}{2} \left( E\left[ e^{-\frac{1}{2}\int_0^T |\theta_s|^2 ds + \int_0^T \theta_s dB_s} \right] \right)^\frac{1}{2} \leq \left( E\left[ |\xi|^2 \right] \right)^\frac{1}{2} e^{\frac{1}{2}k^2T}.
$$

So we get

$$
\tilde{E}[\xi] = \sup_{Q \in \mathcal{P}} E_Q[\xi] \leq (E[|\xi|^2])^{\frac{1}{2}} e^{\frac{1}{2}k^2T}. \quad (3.4)
$$

Thus we have

$$
|\tilde{E}[\xi_2] - \tilde{E}[\xi_1]| \leq (E[|\xi_2 - \xi_1|^2])^{\frac{1}{2}} e^{\frac{1}{2}k^2T}.
$$

Therefore, $\tilde{E}$ is $L^2$-continuous for the comonotonic random variables.

On $L^2(\Omega, \mathcal{F}_T, P)$, denote Choquet integral as

$$
\int_{\Omega} X \, dc := \int_{-\infty}^{0} (c(X > x) - 1) \, dx + \int_{0}^{\infty} c(X > x) \, dx \quad \forall X \in L^2(\Omega, \mathcal{F}_T, P),
$$

just for the convenience of proof.

**Theorem 3.1.** Let $X, Y$ be real-valued measurable functions defined on $\Omega$. If a capacity $c$ is submodular and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$
\int_{\Omega} |XY| \, dc \leq \left( \int_{\Omega} |X|^p \, dc \right)^{\frac{1}{p}} \left( \int_{\Omega} |Y|^q \, dc \right)^{\frac{1}{q}}.
$$

The following is the main theorem.

**Theorem 3.2.** Let $X \in L^2(\Omega, \mathcal{F}_T, P)$ be a monotone function. Then we have

$$
\tilde{E}[X] = \int_{\Omega} X \, d\tilde{c}, \quad \mathcal{E}[X] = \int_{\Omega} X \, dc.
$$
Proof. Since \( \bar{\mathbb{E}}[X] = -\bar{\mathbb{E}}[-X] \), we only prove that \( \bar{\mathbb{E}}[X] = \int_{\Omega} X \, d\bar{c} \). Let \( X \in L^2(\Omega, \mathcal{F}_T, P) \) be a monotone random variable. Let \( f \) be a simple function. Let \( \epsilon > 0 \) be given.

\[
|\bar{\mathbb{E}}[X] - \int_{\Omega} X \, d\bar{c}| \leq |\bar{\mathbb{E}}[X] - \bar{\mathbb{E}}[f]| + \left| \bar{\mathbb{E}}[f] - \int_{\Omega} f \, d\bar{c} \right| + \left| \int_{\Omega} f \, d\bar{c} - \int_{\Omega} X \, d\bar{c} \right|. \tag{3.5}
\]

Since simple functions are dense in \( L^2(\Omega, \mathcal{F}_T, P) \), there exists an increasing simple function \( f \uparrow X \) satisfying both

\[
\|X - f\|_{L^2} < e^{-\frac{1}{2}k^2T} \cdot \frac{\epsilon}{3} \quad \text{and} \quad \left( \int_{\Omega} |f - X|^2 \, d\bar{c} \right)^{\frac{1}{2}} = \left( \int_{0}^{\infty} \bar{c}(|f - X|^2 > x) \, dx \right)^{\frac{1}{2}} < \frac{\epsilon}{3}.
\]

Since the \( \bar{\mathbb{E}} \) is \( L^2 \)-continuous for the comonotonic random variables \( X \) and \( f \) by Lemma 3.2, the first term of the right hand side of (3.5) is less than \( \epsilon/3 \). The equation (3.1) implies that the second term of the right hand side of (3.5) is zero.

The capacity \( \bar{c} \) is submodular by Lemma 3.1 and so Theorem 3.1 implies that the third term of the right hand side of (3.5) becomes

\[
\left| \int_{\Omega} f \, d\bar{c} - \int_{\Omega} X \, d\bar{c} \right| \leq \int_{\Omega} |f - X| \, d\bar{c} \leq \left( \int_{\Omega} |f - X|^2 \, d\bar{c} \right)^{\frac{1}{2}} \left( \int_{\Omega} 1_{\Omega}^2 \, d\bar{c} \right)^{\frac{1}{2}} \leq \left( \int_{\Omega} |f - X|^2 \, d\bar{c} \right)^{\frac{1}{2}} < \frac{\epsilon}{3}.
\]

So we obtain

\[
|\bar{\mathbb{E}}[X] - \int_{\Omega} X \, d\bar{c}| < \epsilon.
\]

Therefore, the proof is done. \( \square \)

We will show that there exists \( Q \in \mathcal{P} \) such that the minimax expectation takes a maximum or minimum.

Lemma 3.3. The set of densities

\[
\mathcal{D} := \left\{ \frac{dQ}{dP} \mid Q \in \mathcal{P} \right\}
\]

is weakly compact in \( L^2(\Omega, \mathcal{F}_T, P) \).
Proof. As in the proof of Lemma 3.2, we can prove
\[ E \left[ \left( \frac{dQ}{dP} \right)^2 \right] \leq e^{\frac{1}{2} k^2 T}. \]

So we have \( \frac{dQ}{dP} \in L^2(\Omega, \mathcal{F}_T, P) \). Thus we have \( D \subset L^2(\Omega, \mathcal{F}_T, P) \)

We want to show that \( D \) is weakly closed in \( L^2(\Omega, \mathcal{F}_T, P) \). Suppose that the sequence \( (Z_n) \) in \( D \) converges weakly to \( Z \). I.e., \( f(Z_n) \to f(Z) \) for all \( f \in (L^2)^* \), where \( (L^2)^* \) is the set of continuous dual functionals of \( L^2 \).

We want to show \( Z \in D \).

For \( X \in L^2(\Omega, \mathcal{F}, P) \), define the linear functional \( J_X \) as
\[ J_X(Z) := E[XZ] \quad \forall Z \in D. \quad (3.6) \]

By the Hölder’s inequality, we have
\[ |J_X(Z)| \leq E[|XZ|^2]^{1/2} \cdot (\int |Z|^2 dP)^{1/2} < +\infty. \]

So \( J_X \) is bounded and thus continuous on \( L^2 \).

By the assumption, we have \( J_X(Z_n) \to J_X(Z) \) as \( n \to \infty \).

That is,
\[ \lim_{n \to \infty} \int X dQ_n = \lim_{n \to \infty} E[XZ_n] = E[XZ] = \int X dQ. \]

Since \( Z_n \in D \), there exist \( \theta^{(n)}_t \) and \( Q^{(n)}_t \in \mathcal{P} \) satisfying
\[ Z_n = \frac{dQ^{(n)}_t}{dP} = \exp \left( -\frac{1}{2} \int_0^T |\theta^{(n)}_s|^2 ds + \int_0^T \theta^{(n)}_s dW_s \right). \]

Let \( \lim_{n \to \infty} \theta^{(n)}_t = \theta_t \). Then we have
\[ Z' = \lim_{n \to \infty} Z_n = \exp \left( -\frac{1}{2} \int_0^T |\theta_s|^2 ds + \int_0^T \theta_s dB_s \right). \]

So we have
\[ \int_0^T XZ dP = \int_0^T XZ' dP \quad \forall X \in L^2(\Omega, \mathcal{F}_T, P). \]

Therefore, it becomes \( Z = Z' \) a.e. and thus \( Z \in D \). It is proven that \( D \) is a weakly compact set. \( \square \)
Theorem 3.3 (James’ Theorem). A weakly closed subset $D$ of a Banach space $L^2(\Omega, \mathcal{F}_T, P)$ is weakly compact if and only if each continuous linear functional on $L^2(\Omega, \mathcal{F}_T, P)$ attains a maximum or a minimum on $D$.

By James’ Theorem, the linear functional $J_X$ as in (3.6) attains a maximum on $D$. That is, there exists $Q^* \in \mathcal{P}$ such that

$$\sup_{Q \in \mathcal{P}} E_Q[\xi] = E_{Q^*}[\xi], \quad \xi \in L^2(\Omega, \mathcal{F}_T, P).$$

To specify $Q^* \in \mathcal{P}$, we need Lemma 3.4 which gives the restriction to the generator $g$ of BSDE (3.8), in addition to Theorem 2.6. Let $\{S_t\}$ be the solution of the following stochastic differential equation,

$$S_t = S_0 + \int_0^t \eta(t, S_t) dt + \int_0^t \sigma(t, S_t) dB_t, \quad t \in [0, T], \quad (3.7)$$

where $\eta, \sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ are continuous in $(t, S)$ and Lipschitz continuous in $S$.

Lemma 3.4 (1). Let $\{S_t\}$ be the solution of (3.7). Let $\Phi$ be the monotone function such that $\Phi(S_T) \in L^2(\Omega, \mathcal{F}_T, P)$. Let $(y_t, z_t)$ be the solution of the following BSDE

$$y_t = \Phi(S_T) + \int_t^T \theta_s |z_s| \, ds - \int_t^T z_s dB_s, \quad (3.8)$$

Then the followings hold,

1. $z_t \sigma(t, S_t) \geq 0$, a.e. $t \in [0, T)$, if $\Phi$ is an increasing function
2. $z_t \sigma(t, S_t) \leq 0$, a.e. $t \in [0, T)$, if $\Phi$ is a decreasing function.

Suppose that $\Phi$ is an increasing function. Then for $|\theta_s| \leq k$, by Theorem 3.4, the solution $(y^{(k)}_t, z^{(k)}_t)$ of (3.8) becomes the unique solution of the form of BSDE

$$y_t^{(k)} = \Phi(S_T) + \int_t^T \theta_s z^{(k)}_s \, ds - \int_t^T z^{(k)}_s dB_s,$$

where $\tilde{B}^\theta_t = B_t - \int_0^t \theta_s \, ds$.

Let $(y^{(k)}_t, z^{(k)}_t)$ be the unique solution of the following BSDE

$$y_t^{(k)} = \Phi(S_T) + \int_t^T k \, z^{(k)}_s \, ds - \int_t^T z^{(k)}_s dB_s. \quad (3.10)$$

As we did at the end of Section 2, we have $y^{(k)}_t \geq y^{(\theta)}_t$ for all $t \in [0, T]$ by applying the Comparison Theorem for BSDEs to (3.9) and (3.10). Therefore, we get

$$y^{(k)}_0 = E_{Q_k}[\Phi(S_T)] \geq y^{(\theta)}_0 = E_{Q_\theta}[\Phi(S_T)],$$
where $Q_k$ and $Q_\theta$ are respectively defined as
\[
\frac{dQ_k}{dP} = e^{-\frac{1}{2} \int_0^T k^2 ds + \int_0^T k dB_s}, \quad \frac{dQ_\theta}{dP} = e^{-\frac{1}{2} \int_0^T \theta^2 ds + \int_0^T \theta dB_s}.
\]

Thus we have
\[
E_{Q_k}[\Phi(S_T)] = \sup_{Q_k \in \mathcal{P}} E_{Q_k}[\Phi(S_T)] := \bar{\mathcal{E}}[\Phi(S_T)],
\]
since $Q_k \in \mathcal{P}$ and $|\theta| \leq k$. In the similar fashion, we can also show that
\[
E_{Q_{-k}}[\Phi(S_T)] = \inf_{Q \in \mathcal{P}} E_Q[\Phi(S_T)] := \underline{\mathcal{E}}[\Phi(S_T)],
\]
where $Q_{-k}$ is defined as
\[
\frac{dQ_{-k}}{dP} = e^{-\frac{1}{2} k^2 T - kB_T}.
\]

Now suppose that $\Phi$ is a decreasing function. Then $-\Phi$ is an increasing function. So we have
\[
\bar{\mathcal{E}}[\Phi(S_T)] = -\underline{\mathcal{E}}[-\Phi(S_T)] = -E_{Q_{-k}}[-\Phi(S_T)] = E_{Q_{-k}}[\Phi(S_T)],
\]
\[
\underline{\mathcal{E}}[\Phi(S_T)] = -\bar{\mathcal{E}}[-\Phi(S_T)] = -E_{Q_k}[-\Phi(S_T)] = E_{Q_k}[\Phi(S_T)].
\]

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