Improved Parallel Algorithms for Generalized Baumslag Groups

Caroline Mattes¹ and Armin Weiß¹⋆

Universität Stuttgart, Institut für Formale Methoden der Informatik, Germany

Abstract. The Baumslag group had been a candidate for a group with an extremely difficult word problem until Myasnikov, Ushakov, and Won succeeded to show that its word problem can be solved in polynomial time. Their result used the newly developed data structure of power circuits allowing for a non-elementary compression of integers. Later this was extended in two directions: Laun showed that the same applies to generalized Baumslag groups $G_{1,q}$ for $q \geq 2$ and we established that the word problem of the Baumslag group $G_{1,2}$ can be solved in $TC^1$. In this work we further improve upon both previous results by showing that the word problem of all the generalized Baumslag groups $G_{1,q}$ can be solved in $TC^1$ – even for negative $q$. Our result is based on using refined operations on reduced power circuits. Moreover, we prove that the conjugacy problem in $G_{1,q}$ is strongly generically in $TC^1$ (meaning that for “most” inputs it is in $TC^1$). Finally, for every fixed $g \in G_{1,q}$ conjugacy to $g$ can be decided in $TC^1$ for all inputs.

Keywords: Algorithmic group theory · power circuit · $TC^1$ · word problem · conjugacy problem · Baumslag group · parallel complexity.

1 Introduction

In the early 20th century, Dehn [7] introduced the word problem as one of the basic algorithmic problems in group theory: given a word over the generators of a group $G$, the question is whether this word represents the identity of $G$. Already in the 1950s, Novikov and Boone constructed finitely presented groups with an undecidable word problem [5,28]. Still, many natural classes of groups have an (efficiently) decidable word problem – most prominently, the class of linear groups (groups embeddable into a matrix group over some field): their word problem is in \textsc{Logspace} [18,31] – in particular, in \textsc{NC}, i.e., decidable by Boolean circuits of polynomial size and polylogarithmic depth (or, equivalently decidable in polylogarithmic time using polynomially many processors). There are several other results on word problems of groups in small complexity classes defined by circuits, for example for solvable linear groups in $TC^0$ (constant depth with threshold gates) [16], for Baumslag-Solitar groups in \textsc{Logspace} [33], and for hyperbolic groups in $\text{SAC}^1 \subseteq \text{NC}^1$ [19]. Nevertheless, there are also finitely presented groups with decidable, yet arbitrarily hard, word problems [30].

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A one-relator group is a group that can be written as a free group modulo a normal subgroup generated by a single element (relator). A famous algorithm called the Magnus breakdown procedure [21] shows that one-relator groups have decidable word problems (see also [20,22]). Its complexity remains an open problem: while it is not even clear whether the word problems of one-relator groups are solvable in elementary time, [3] asks for polynomial-time algorithms.

In 1969 Gilbert Baumslag defined the group $G_{1,2} = \langle a, b \mid bab^{-1}a = a^2bab^{-1} \rangle$ as an example of a one-relator group enjoying certain remarkable properties. It is infinite and non-abelian, but all its finite quotients are cyclic and, thus, it is not residually finite [4]. Moreover, Gersten showed that the Dehn function of $G_{1,2}$ is non-elementary [10] meaning that it cannot be bounded by any fixed tower of exponentials (see also [29]). This made the Baumslag group a candidate for a group with a very difficult word problem.

Indeed, when applying the Magnus breakdown procedure to an input word of length $n$, one obtains as intermediate results words of the form $v^i_1 \cdots v^m_n$ where $v_i \in \{a, b, bab^{-1}\}$, $x_i \in \mathbb{Z}$, and $m \leq n$. The issue is that the $x_i$ might grow up to $\tau_2(\log n)$ (with $\tau_2(0) = 1$ and $\tau_2(i + 1) = 2^{\tau_2(i)}$ for $i \geq 0$ – the tower function). However, Myasnikov, Ushakov and Won succeeded to show that the word problem of $G_{1,2}$ is, indeed, decidable in polynomial time [26]. Their crucial contribution were so-called power circuits in [27] for compressing the $x_i$ in the above description.

Roughly speaking, a (base-2) power circuit is a directed acyclic graph with edges labelled by numbers from $\{-1, 0, 1\}$. One defines an evaluation of a vertex $P$ as two raised to the power of the (weighted) sum of the successors of $P$. Hence, the value $\tau_2(n)$ can be represented by an $n + 1$-vertex power circuit – thus, power circuits allow for a non-elementary compression. The crucial feature for the application to the Baumslag group is that they not only efficiently support the operations $+$, $-$, and $(x, y) \mapsto x \cdot 2^y$, but also the test whether $x = y$ or $x < y$ for two integers represented by power circuits can be done in polynomial time. The main technical part of the comparison algorithm is to compute a so-called reduced power circuit.

Based on these striking results, Diekert, Laun and Ushakov [8] improved the running time for the word problem of the Baumslag group from $O(n^2)$ down to $O(n^3)$ and described a polynomial-time algorithm for the word problem of the Higman group $H_4$ ([13]. Subsequently, more applications of power circuits to similar groups emerged: In [17] Laun gave a polynomial-time solution for the word problem of generalized Baumslag groups $G_{1,q} = \langle a, b \mid bab^{-1}a = a^qbab^{-1} \rangle$ for $q \geq 1$ and also for generalized Higman groups. In order to do so, he generalized power circuits to arbitrary bases $q \geq 2$ and adapted the corresponding algorithms from [27,8]. Of particular interest here is the computation of so-called compact markings, which allow for a unique representation of integers; for arbitrary bases it is considerably more involved than for base two.

In [9] the conjugacy problem of the Baumslag group is shown to be strongly generically in $P$ and in [2] the same is done for the conjugacy problem of the Higman group. Here “generically” roughly means that the algorithm works for most inputs – for a precise definition, see Section 1.2 below. The idea is that often the “generic-case behavior” of an algorithm is more relevant than its average-case or worst-case behavior. We refer to [14,15] where the foundations of this theory were developed and to [25] for applications in cryptography.

Finally, in [23], we studied the word problem of the Baumslag group $G_{1,2}$ from the point of view of parallel complexity. We showed that it can be solved in the circuit class $TC^2$. The proof consists of two main steps: first, to show that for a power circuit of logarithmic depth a corresponding reduced power circuit can be computed in $TC^1$. 

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(in contrast to the general case where computing reduced power circuits is P-complete [23, Theorem C]) and, second, to show that the Magnus breakdown procedure can be performed in a tree-shape manner leading to a logarithmic number of rounds with each individual round doable in $TC^1$.

**Contribution.** In this work we combine the results of [17] and [23] by considering the parallel complexity of the word problem of generalized Baumslag groups $G_{1,q} = \langle a, b \mid bab^{-1}a = a^qb^{-1} \rangle$. As a first step, we show how to compute compact base-$q$ signed-digit representations in $AC^0$ (see Theorem 7). Moreover, we not only unify [17] and [23] but also prove improved complexity bounds:

**Theorem A** For every $q \in \mathbb{Z}$ with $|q| \geq 2$ the word problem of the generalized Baumslag group $G_{1,q}$ is in $TC^1$.

Note that for the first time we allow $q$ to be negative. We do not consider the case $q = \pm 1$ since then the word problem can be solved even in $TC^0$ using a different approach; we refer to future work. The main ingredient to the improvement of the complexity from $TC^2$ in [23]) to $TC^1$ (here) is that we succeed to perform all operations directly on reduced base-$|q|$ power circuits. For this we allow operations in a SIMD (single instruction multiple data) fashion: many operations of the same type are performed on the same power circuit in parallel in $TC^0$. Furthermore, we improve the algorithm to get power circuits of quasi-linear size—thus, close to the optimal size as in the sequential algorithms [8, 17].

In the last part of our paper, we consider the conjugacy problem for $G_{1,q}$. We use our results for the word problem to improve the complexity of the strongly generic algorithm from [9] by showing:

**Theorem B** For every $q \in \mathbb{Z}$ with $|q| \geq 2$ the conjugacy problem of the generalized Baumslag group $G_{1,q}$ is strongly generically in $TC^1$.

Moreover, for every fixed $g \in G_{1,q}$, the problem to decide whether some input word $w$ is conjugate to $g$ is in $TC^1$.

Note that for the second part of Theorem B not even a polynomial-time algorithm has been described before. Also, it seems to stand in strong contrast to the hypothesis that the conjugacy problem of $G_{1,2}$ cannot be solved in elementary time (see e.g. [9, Corollary 2]). The crucial point here is that $g \in G_{1,q}$ is fixed.

### 1.1 Notation and Preliminaries

The logarithm $\log$ is with respect to base two, while $\log_q$ denotes the base-$q$ logarithm. Let $q \in \mathbb{N}$. Then the base-$q$ tower function $\tau_q : \mathbb{N} \to \mathbb{N}$ is defined by $\tau_q(0) = 1$ and $\tau_q(i + 1) = q^{\tau_q(i)}$ for $i \geq 0$. It is primitive recursive, but already $\tau_q(0)$ written in binary cannot be stored in the memory of any conceivable real-world computer. We denote the support of a function $f : X \to \mathbb{R}$ by $\sigma(f) = \{ x \in X \mid f(x) \neq 0 \}$. Furthermore, the interval of integers $\{ i, \ldots, j \} \subseteq \mathbb{Z}$ is denoted by $[i..j]$. For $q, x \in \mathbb{Z}$, we write $q \mid x$ if $q$ does not divide $x$. Moreover, $\text{sgn}(x)$ denotes the sign of $x \in \mathbb{Z}$. We write $\mathbb{Z}[1/q] = \{ m/q^n \in \mathbb{Q} \mid m, k \in \mathbb{Z} \}$ for the set of fractions with powers of $q$ as denominators.

Let $\Sigma$ be a set. The set of words over $\Sigma$ is denoted by $\Sigma^* = \bigcup_{n \in \mathbb{N}} \Sigma^n$. The length of $w \in \Sigma^*$ is denoted by $|w|$. A dag is a directed acyclic graph. For a dag $\Gamma$ we write $\text{depth}(\Gamma)$ for the length (number of edges) of a longest path in $\Gamma$. 

1.2 Complexity

We assume the reader to be familiar with the complexity classes LOGSPACE and $P$ (polynomial time); see e.g. [1] for details. Most of the time, however, we use circuit complexity within $NC$.

Throughout, we assume that languages $L$ (resp. inputs to functions $f$) are encoded over the binary alphabet $\{0, 1\}$. Let $k \in \mathbb{N}$. A language $L$ (resp. function $f$) is in $AC^k$ if there is a family of polynomial-size Boolean circuits of depth $O(log^k n)$ (where $n$ is the input length) deciding $L$ (resp. computing $f$). More precisely, a Boolean circuit is a dag (directed acyclic graph) where the vertices are either input gates $x_1, \ldots, x_n$, or NOT, AND, or OR gates. There are one or more designated output gates (for computing functions there is more than one output gate – in this case they are numbered from 1 to $m$). All gates may have unbounded fan-in (i.e., there is no bound on the number of incoming wires). A language $L \subseteq \{0, 1\}^*$ belongs to $AC^k$ if there exist a family $(C_n)_{n \in \mathbb{N}}$ of Boolean circuits such that $x \in L \cap \{0, 1\}^n$ if and only if the output gate of $C_n$ evaluates to 1 when assigning $x = x_1 \cdots x_n$ to the input gates. Moreover, $C_n$ may contain at most $n^{O(1)}$ gates and have depth $O(log^k n)$. Likewise $AC^k$-computable functions are defined.

The class $TC^k$ is defined analogously also allowing MAJORITY gates (which output 1 if the input contains more 1s than 0s). Moreover, $NC = \bigcup_{k \geq 0} TC^k = \bigcup_{k \geq 0} AC^k$. For more details on circuits we refer to [32]. We use two basic building blocks, iterated addition and sorting, which can be done in $TC^0$.

Example 1. Base-q iterated addition is as follows: on input of $n$ base-$q$ numbers $A_1, \ldots, A_n$ each having $n$ digits, compute $\sum_{i=1}^{n} A_i$ (in base-$q$ representation). For binary numbers this is well-known to be in $TC^0$ (see e.g. [32, Theorem 1.37]). The standard proof can be translated for other bases. The result also can be easily derived from other existing results: as iterated multiplication and division are in $TC^0$ by [11,12], one can convert a base-$q$ integer to a binary integer, do the addition in binary, and convert the number back.

The class $NC$ is contained in $P$ if we consider uniform circuits. Roughly speaking, a circuit family is called uniform if the $n$-input circuit can be computed efficiently from the string $1^n$. In order not to overload the presentation, throughout, we state all our results in the non-uniform case – all uniformity considerations are left to the reader.

**Generic case complexity.** A set $I \subseteq \Sigma^*$ is called strongly generic if the probability to find a random string outside $I$ converges exponentially fast to zero – more precisely, if $|\Sigma^n \setminus I|/|\Sigma^n| \in 2^{-O(n)}$. Let $C$ be some complexity class. A problem $L \subseteq \Sigma^*$ is called strongly generically in $C$ if there is a strongly generic set $I \subseteq \Sigma^*$ and a (partial) algorithm (or circuit family) $A$ running within the bounds of $C$ such that $A$ computes the correct answer for every $w \in I$; outside of $I$ it provides either the correct answer or none (or outputs “unknown”).

2 Compact representations

Based on the concept of compact sums and power circuits with base 2 (as introduced in [27]), Laun [17] described so-called power sums: A power sum to base $q \geq 2$ is a sum $\sum_{i \geq 0} a_i q^i$ with $a_i \in [-q+1..q-1]$ and only finitely many $a_i$ are non-zero. We are interested in compact representations of such power sums. In [17, Proposition 2.18] it
is shown that each power sum has a unique compact representation, which is obtained using a confluent rewriting system. Using Boolean formulas for this construction we show that it is in AC$^0$. This will be an important ingredient for our power circuit operations to be in TC$^0$. Observe that in [23, Theorem 11] we gave a proof for base $q = 2$. Here, we fix $q \geq 2$.

**Definition 2.** Let $A = (a_0, \ldots, a_{m-1})$ be a sequence with $a_i \in [-q + 1 .. q - 1]$.
- We define $\text{val}_q(A) = \sum_{i=0}^{m-1} a_i \cdot q^i$.
- We call $A$ a (base-$q$) signed-digit representation (short sdr) of $\text{val}_q(A)$.
- We call $A$ compact if the following conditions hold for all $i \in [0 .. m - 2]$:
  1. if $|a_i| = q - 1$, then $|a_{i+1}| < q - 1$,
  2. if $|a_i| \neq 0$, then $a_{i+1} = 0$ or $\text{sgn}(a_i) = \text{sgn}(a_{i+1})$.

We set $a_i = 0$ for $i \geq m$. Note that, if $a_i \in [0 .. q - 1]$, we have a usual base-$q$ representation of an integer. Allowing negative digits gives more flexibility when working with power circuits. The nice thing about compact base-$q$ signed-digit representations (for short base-$q$ csdr) is that they can be compared easily.

**Lemma 3 ([17, Proposition 2.18]).** For every $x \in \mathbb{Z}$ there is a unique compact base-$q$ signed-digit representation $A = (a_0, \ldots, a_{m-1})$ with $\text{val}_q(A) = x$.

Moreover, two base-$q$ csdrs $A = (a_0, \ldots, a_{m-1})$ and $B = (b_0, \ldots, b_{m-1})$ can be compared using the lexicographical order – more precisely, $\text{val}_q(A) < \text{val}_q(B)$ if and only if $a_i < b_i$ for $i = \max \{i \in [0 .. m - 1] \mid a_i \neq b_i\}$.

If $A$ is a base-$q$ sdr, in the following we will write $\text{CR}_q(A)$ for its compact base-$q$ signed-digit representation.

**Lemma 4.** If $A = (a_0, \ldots, a_{m-1})$ is a base-$q$ csdr, then $\text{val}_q(A) \leq \left\lfloor \frac{m+1}{q^2-1} \right\rfloor$.

**Proof.** It is clear that $\text{val}_q(A)$ becomes maximal if $a_{m-1} = q - 1$ and then (going from right to left) $q - 2$ and $q - 1$ alternate. Thus, the maximal value can be computed as

$$\sum_{i=0}^{m-1} (q - 2)q^i + \sum_{j=0}^{\frac{(m-1)}{2}} q^{m-1-2j} = \frac{q^{m+1}}{q^2-1}.$$  

Next, we construct the base-$q$ csdr of a given sdr $A = (a_0, \ldots, a_{m-1})$. We start by restricting $A$ to only non-negative digits (i.e., a usual base-$q$ representation). In a first step we need the following formula for $i \geq 0$:

$$e_i = \bigvee_{j \in [1..i]} \left( (a_j = q - 1) \land (a_{j-1} = q - 1) \right) \land \bigwedge_{k \in [i+1..m-1]} ((a_k = q - 1) \lor (a_{k+1} = q - 1) \land (a_k \geq q - 2))$$

**Lemma 5.** For every signed-digit representation $A = (a_0, \ldots, a_{m-1})$ with $a_i \in [0 .. q - 1]$ the sequence $B = (b_0, \ldots, b_m)$ defined by $b_i = a_i - q \cdot e_{i+1} + e_i$ satisfies:
- $\text{val}_q(A) = \text{val}_q(B)$
− $b_i \in [-1..q-1]$
− $b_i = -1 \implies b_{i+1} = 0$
− $b_i = q - 1 \implies b_{i+1} < q - 1$

Proof. First observe that because $e_{m+1} = e_0 = 0$, $\text{val}_q(A) = \text{val}_q(B)$ follows directly from the definition of the $b_i$.

Next we show that $b_i \in [-1..q-1]$. First assume that $e_{i+1} = 1$. Then $a_i \geq q - 2$. If $a_i = q - 2$, then $e_i = 1$ as otherwise $e_{i+1} = 1$ is not possible. So if $e_{i+1} = 1$, then $b_i \in [-1..q-1]$. Further, if $e_i = q - 1$ and $e_i = 1$, then $e_{i+1} = 1$. So $b_i \in [-1..q-1]$ if $e_i = 1$. If $e_i = e_{i+1} = 0$, then $b_i = a_i \in [0..q-1]$.

To show that $b_i = -1$ implies that $b_{i+1} = 0$ we first assume for a contradiction that $b_{i+1} = -1$ meaning that

$$b_i = a_i - q \cdot e_{i+1} + e_i = -1 \quad \text{and} \quad b_{i+1} = a_{i+1} - q \cdot e_{i+2} + e_{i+1} = -1.$$ (1)

If $e_j = e_{j+1} = 0$, then $b_j = a_j$. But $a_j \geq 0$ for all $j$, so this is not possible if $b_j = -1$. Now we construct a table as follows: We consider the remaining possibilities for $e_i, e_{i+1}, e_{i+2}$ and calculate $a_i$ and $a_{i+1}$ assuming (1). In each row of the table, the blue entries lead to a contradiction which is described in the last column.

| $e_i$ | $e_{i+1}$ | $e_{i+2}$ | $a_i$ | $a_{i+1}$ | $b_i = -1 \implies b_{i+1} \neq -1$ |
|------|-----------|-----------|------|-----------|----------------------------------|
| 0    | 1         | 0         | $q-1$| $-2$      | $a_i \geq 0$                     |
| 0    | 1         | 1         | $q-1$| $q-2$     | $e_{i+1} = 1$                    |
| 1    | 0         | 1         | $-2$ | $q-1$     | $a_i \geq 0$                     |
| 1    | 1         | 0         | $q-2$| $-2$      | $a_i \geq 0$                     |
| 1    | 1         | 1         | $q-2$| $q-2$     | $e_{i+1} = 1$                    |

Hence, we have shown that $b_i = -1$ implies $b_{i+1} \neq -1$. We now assume that

$$b_i = a_i - q \cdot e_{i+1} + e_i = -1$$
$$b_{i+1} = a_{i+1} - q \cdot e_{i+2} + e_{i+1} > 0$$

From the above we already know that the cases $e_{i+1} = e_i = 0$, and $e_i = 1, e_{i+1} = 0$ lead to a contradiction. Moreover, $e_i = 0$ and $e_{i+1} = 1$, is only possible if $a_i = a_{i+1} = q - 1$. If $e_{i+1} = 1$ and $a_i = q - 2$, then $a_{i+1} = q - 1$. For the remaining cases we construct a table in the same way as we did above. Again, the blue entries lead to the contradiction in the last column.

| $e_i$ | $e_{i+1}$ | $e_{i+2}$ | $a_i$ | $a_{i+1}$ | $e_{i+2} = 0$ |
|------|-----------|-----------|------|-----------|---------------|
| 0    | 1         | 0         | $q-1$| $q-1$     | $a_i < q - 1$ |
| 0    | 1         | 1         | $q-1$| $q-1$     | $a_i < q - 1$ |
| 1    | 0         | 0         | $q-2$| $q-1$     | $e_{i+2} = 0$ |
| 1    | 1         | 1         | $q-2$| $q-1$     | $a_i < q - 1$ |

So we showed that, if $b_i = -1$, then $b_{i+1} = 0$. It remains to show that $b_i = q - 1$ implies that $b_{i+1} < q - 1$. We assume that

$$b_i = a_i - q \cdot e_{i+1} + e_i = q - 1$$
$$b_{i+1} = a_{i+1} - q \cdot e_{i+2} + e_{i+1} = q - 1$$
First observe that if \( e_{j+1} = 1 \), then \( a_j > q - 1 \) for \( j = i, i + 1 \). If \( e_i = e_{i+1} = e_{i+2} = 0 \), then \( a_i = a_{i+1} = q - 1 \), which is a contradiction to \( e_{i+1} = 0 \). It remains the case \( e_i = 1 \), \( e_{i+1} = e_{i+2} = 0 \). But then \( a_i = q - 2 \) and \( a_{i+1} = q - 1 \). With \( e_{i+1} = 1 \) we obtain a contradiction to \( e_{i+1} = 0 \). This shows the lemma.

For the second step - to make a signed-digit representation \( B = (b_0, \ldots, b_m) \) as in Lemma 5 compact – we need the following formula:

\[
f_i = \bigvee_{j \in [i,m]} (b_j = -1) \land \bigwedge_{\ell \in [i-1,j-1]} (b_\ell > 0)
\]

**Lemma 6.** For every signed-digit representation \( B = (b_0, \ldots, b_m) \) satisfying the conditions of the output of Lemma 5 (i.e., \( b_i \in [-1..q - 1] \), \( b_i = q - 1 \) implies \( b_{i+1} < q - 1 \) and \( b_i = -1 \) implies \( b_{i+1} = 0 \) the sequence \( C = (c_0, \ldots, c_m) \) defined by \( c_i = b_i - q \cdot f_{i+1} + f_i \) satisfies:

- \( \text{val}_q(B) = \text{val}_q(C) \)
- \( c_i \in [-q + 1..q - 1] \)
- \( C \) is compact.

**Proof.** As in the proof of the previous lemma, because of \( f_{m+1} = f_0 = 0 \), \( \text{val}_q(B) = \text{val}_q(C) \) follows directly from the definition of the \( c_i \). Before proving the other conditions, we first make the following observations:

(I) \( b_i = 0 \implies f_i = f_{i+1} = 0; \) therefore, \( b_i = 0 \implies c_i = 0. \)

(II) \( b_i = -1 \) implies that \( b_{i+1} = 0 \), and thus also \( f_{i+1} = 0. \)

(III) If \( f_i = 1 \) and \( f_{i+1} = 0 \), then \( b_i = -1. \)

It is clear that, if \( 0 < b_i < q - 1, \) then \( c_i \in [-q + 1..q - 1]. \) If \( b_i = 0, \) then \( c_i = 0 \) by (I). If \( b_i = -1, \) then \( c_i \in \{-1,0\} \) by (II). Now assume that \( b_i = q - 1. \) If \( f_i = f_{i+1} = 0, \) then \( c_i = q - 1. \) If \( f_{i+1} = 1, \) then \( c_i \in \{-1,0\}. \) Because of (III), \( f_i = 1 \) and \( f_{i+1} = 0 \) is not possible. So we showed that \( c_i \in [-q + 1..q - 1]. \)

Now we will prove that \( C \) is compact. First, we show that \( |c_i| = |c_{i+1}| = q - 1 \) is not possible. Observe that \( |c_i| = q - 1 \) is only possible if \( b_i \in \{0,1,q-2,q-1\} \). Because of (I) only the cases \( b_i, b_{i+1} \in \{1,q-2,q-1\} \) remain.

If \( f_i = f_{i+1} = f_{i+2} = 0, \) then \( c_i = b_i \) and \( c_{i+1} = b_{i+1} \) and we are done as already in \( B \) there are no two \( q - 1 \)'s next to each other. Moreover, if \( b_j \in \{1,q-2,q-1\}, \) then \( b_j - q - 1 < q - 1. \) So there is nothing more to show in case \( f_{i+1} = f_{i+2} = 1. \) If \( f_{i+2} = 1 \) and \( b_i, b_{i+1} \in \{1,q-2,q-1\}, \) then \( f_{i+1} = 1 \) so the previous case applies. The remaining cases \( (f_{i+2} = 0 \) and either \( f_{i+1} = 1 \) or \( f_i = 1) \) are ruled out by (III).

We still have to show that if \( c_i \neq 0, \) then \( c_{i+1} = 0 \) or \( \text{sgn}(c_i) = \text{sgn}(c_{i+1}). \) First assume that \( c_i > 0. \) Then \( f_{i+1} = 0 \) and by (I) we have \( b_i > 0. \) If \( c_i < 0 \) and \( f_{i+2} = 0, \) then \( b_{i+1} = -1. \) Because \( b_i > 0 \) this implies that \( f_{i+1} = 1, \) which is a contradiction. So if \( c_{i+1} < 0, \) then \( f_{i+2} = 1 \) and so \( b_{i+1} > 0. \) Together with \( b_i > 0 \) this implies that \( f_{i+1} = 1. \) So, \( c_i > 0 \) and \( c_{i+1} < 0 \) is not possible.

Now assume that \( c_i < 0. \) If \( f_{i+1} = 0, \) then \( b_i = -1. \) So by assumption, \( b_{i+1} = 0. \) Thus, \( c_{i+1} = 0 \) by (I). So assume that \( f_{i+1} = 1 \) and \( c_{i+1} > 0. \) It follows that \( f_{i+2} = 0. \) So (III) implies that \( b_{i+1} = -1. \) Thus, \( c_{i+1} = 0. \) So we considered all possible cases and showed that \( C \) is compact. This proves the lemma. 

\[ \square \]
Theorem 7. The following is in \( \text{AC}^0 \):

Input: A base-\( q \) signed-digit representation \( A = (a_0, \ldots, a_{m-1}) \).
Output: A base-\( q \) csdr \( B = (b_0, \ldots, b_m) \) such that \( \text{val}_q(A) = \text{val}_q(B) \).

Proof. Observe that there exist signed-digit representations \( C = (c_0, \ldots, c_{m-1}) \) and \( D = (d_0, \ldots, d_{m-1}) \) such that \( c_i, d_i \in [0, q - 1] \) and such that \( \text{val}_q(A) = \text{val}_q(C) - \text{val}_q(D) \) (we just collect the negative digits of \( A \) into \( D \) and the positive ones into \( C \)). Now, we compute \( |\text{support of } \Lambda| \):

Define power circuits with respect to an arbitrary base \( q \). Here, following \[17\], we extend \( \text{AC}^0 \)— see e.g. \[32\, \text{Theorem 1.15} \] for base 2; the general case follows the same way) and make it compact by first applying Lemma 5 and then Lemma 6. If \( \text{val}_q(C) - \text{val}_q(D) < 0 \), we invert this number digit by digit.

## Power circuits

The original definition \[27\] is for power circuits with base 2. Here, following \[17\], we define power circuits with respect to an arbitrary base \( q \) — hence, from now on we fix \( q \geq 2 \).

Consider a pair \( (\Gamma, \delta) \) where \( \Gamma \) is a set of \( n \) vertices and \( \delta \) is a mapping \( \delta : \Gamma \times \Gamma \to [-q + 1 \ldots q - 1] \). The support of \( \delta \) is the subset \( \sigma(\delta) \subseteq \Gamma \times \Gamma \) consisting of those \( (P, Q) \) with \( \delta(P, Q) \neq 0 \). We write 0 for the all-zero marking. Thus, \( (\Gamma, \sigma(\delta)) \) is a directed graph without multi-edges. Throughout we require that \( (\Gamma, \sigma(\delta)) \) is acyclic — i.e., it is a dag. In particular, \( \delta(P, P) = 0 \) for all vertices \( P \). A marking is a mapping \( M : \Gamma \to [-q + 1 \ldots q - 1] \). Each node \( P \in \Gamma \) is associated in a natural way with a marking \( A_P : \Gamma \to [-q + 1 \ldots q - 1], Q \mapsto \delta(P, Q) \) called its successor marking. The support of \( A_P \) consists of the target nodes of outgoing edges from \( P \). We define the \textit{evaluation} \( \varepsilon(P) \) of a node \( (\varepsilon(M) \) of a marking resp.) bottom-up in the dag by induction:

\[
\begin{align*}
\varepsilon(\emptyset) &= 0, \\
\varepsilon(P) &= q^{\varepsilon(A_P)} \quad \text{for a node } P, \\
\varepsilon(M) &= \sum_P M(P)\varepsilon(P) \quad \text{for a marking } M.
\end{align*}
\]

We have \( \varepsilon(A_P) = \log_q(\varepsilon(P)) \), i.e., the marking \( A_P \) plays the role of a logarithm. Note that nodes of out-degree zero (sinks) evaluate to 1 and every node evaluates to a positive real number. However, we are only interested in the case that all nodes evaluate to integers:

Definition 8. A (base-\( q \)) power circuit is a pair \( (\Gamma, \delta) \) with \( \delta : \Gamma \times \Gamma \to [-q + 1 \ldots q - 1] \) such that \( (\Gamma, \sigma(\delta)) \) is a dag and all nodes evaluate to an integer in \( q^n \).

The size of a power circuit is the number of nodes \( |\Gamma| \). By abuse of language, we also simply call \( \Gamma \) a power circuit and suppress \( \delta \) whenever it is clear. If \( M \) is a marking on \( \Gamma \) and \( S \subseteq \Gamma \), we write \( M|_S \) for the restriction of \( M \) to \( S \). Let \( (\Gamma', \delta') \) be a power circuit, \( \Gamma' \subseteq \Gamma' \), \( \delta = \delta'|_{\Gamma \times \Gamma} \), and \( \delta'|_{\Gamma \times (\Gamma' \setminus \Gamma)} = 0 \). Then \( (\Gamma, \delta) \) itself is a power circuit. We call it a \textit{sub-power circuit} and denote this by \( (\Gamma, \delta) \leq (\Gamma', \delta') \). If \( M \) is a marking on \( S \subseteq \Gamma \), we extend \( M \) to \( \Gamma \) by setting \( M(P) = 0 \) for \( P \in \Gamma \setminus S \). With this convention, every marking on \( \Gamma \) also can be seen as a marking on \( \Gamma' \) if \( (\Gamma, \delta) \leq (\Gamma', \delta') \). If \( M \) is a marking, we write \( -M \) for the marking defined by \( -M(P) = -(M(P)) \), which clearly evaluates to \( -\varepsilon(M) \). For a list of markings \( \vec{M} = (M_1, \ldots, M_n) \) we define \( S(\vec{M}) = \sum_{i=1}^n |\sigma(M_i)| \) (and \( S(M) = |\sigma(M)| \) for a single marking).
Example 9. A power circuit of size $n + 1$ to base $q$ can realize $\tau_q(n)$ since a directed path of $n + 1$ nodes represents $\tau_q(n)$ as the evaluation of the last node. The following power circuit to base 2 realizes $\tau_2(5)$ using 6 nodes:

![Power Circuit Diagram]

Example 10. We can represent every integer in the range $[-q^n + 1, q^n - 1]$ by some marking on a base $q$ power circuit with nodes $\{P_0, \ldots, P_{n-1}\}$ with $\varepsilon(P_i) = q^i$ for $i \in [0..n-1]$. Thus, we can convert the $q$-ary notation of an $n$-digit integer into a power circuit with $n$ vertices, $O(n \log_q n)$ edges (each successor marking requires at most $\lceil \log_q n \rceil + 1$ edges) and depth at most $\log_q n$. For an example of a marking representing the integer 187 to base 3, see Fig. 1.

![Marking Diagram]

Fig. 1. Each integer $z \in [-242..242]$ can be represented by a marking on the following power circuit. The marking given in blue is representing the number 187.

Definition 11. We call a marking $M$ compact if for all $P, Q \in \sigma(M)$ with $P \neq Q$ we have $\varepsilon(P) \neq \varepsilon(Q)$ and, if $|M(P)| = |M(Q)| = q - 1$ or $\text{sgn}(M(P)) \neq \text{sgn}(M(Q))$, then $|\varepsilon(P) - \varepsilon(Q)| \geq 2$. A reduced power circuit of size $n$ is a power circuit $(\Gamma, \delta)$ with $\Gamma$ given as a sorted list $\Gamma = (P_0, \ldots, P_{n-1})$ such that all successor markings are compact and $\varepsilon(P_i) < \varepsilon(P_j)$ whenever $i < j$. In particular, all nodes have pairwise distinct evaluations.

Note that by [23, Theorem 37] it is crucial that the nodes in $\Gamma$ are sorted by their values. Still, sometimes it is convenient to treat $\Gamma$ as a set – we write $P \in \Gamma$ or $S \subseteq \Gamma$ with the obvious meaning.

Also note some slight differences compared to other literature: In [8,17], the definition of a reduced power circuit also contains a bit-vector indicating which nodes have successor markings differing by one – we compute this information on-the-fly whenever needed. Moreover, in [17] the (successor) markings of a reduced power circuit do not have to be compact. Working only with compact markings helps us to compare them in $\mathbf{AC^0}$.

Remark 12. If $(\Gamma, \delta)$ is a reduced power circuit with $\Gamma = (P_0, \ldots, P_{n-1})$, we have $\delta(P_i, P_j) = 0$ for $j \geq i$. Thus, the order on $\Gamma$ by evaluations is also a topological order on the dag $(\Gamma, \sigma(\delta))$. 
Definition 13. Let $(\Gamma, \delta)$ be a reduced power circuit with $\Gamma = (P_0, \ldots, P_n)$. 

(i) A chain $C$ of length $\ell = |C|$ in $\Gamma$ is a sequence $(P_i, \ldots, P_{i+\ell-1})$ such that $\varepsilon(P_{i+1}) = q \cdot \varepsilon(P_i)$ for all $i \in [0..\ell-2]$.

(ii) We call a chain $C$ maximal if it cannot be extended in either direction. We denote the set of all maximal chains by $\mathcal{C}_\Gamma$.

(iii) There is a unique maximal chain $C_0$ containing the node $P_0$ of value 1. We call $C_0$ the initial maximal chain of $\Gamma$ and denote it by $C_0 = C_0(\Gamma)$.

(iv) Let $\mu$ be a marking on $(\Gamma, \delta)$ and $C = (P_i, \ldots, P_{i+\ell-1}) \in \mathcal{C}_\Gamma$. For $a_j = \mu(P_{i+j})$ for $j \in [0..\ell-1]$ we write digit$_\Gamma (\mu) = (a_0, \ldots, a_{\ell-1})$.

Note that a marking $\mu$ is maximal if and only if digit$_\Gamma (\mu)$ is compact (in the sense of Section 2) for all $\mu \in \mathcal{C}_\Gamma$.

3.1 Operations on reduced power circuits

We continue with fixed $q \geq 2$ and assume that all power circuits are with respect to base $q$. Following [23, Proposition 14], we can also compare compact markings on reduced base-$q$ power circuits in AC$^0$. The proof is a straightforward application of Lemma 3.

Lemma 14. The following problem is in AC$^0$:

Input: A reduced power circuit $(\Gamma, \delta)$ with compact markings $L, M$.

Question: Is $\varepsilon(L) \leq \varepsilon(M)$?

The next lemma turns out to be quite versatile and of interest on its own. In particular, it allows to compare a marking on a reduced power circuit in TC$^0$ with some integer given in binary. Moreover, we use it to get rid of the technical condition $\mu \leq \frac{2^{C_0(\Gamma)}+1}{3}$ of [23, Lemma 20] leading to Lemma 17 below.

Lemma 15. The following problem is in TC$^0$:

Input: A reduced power circuit $(\Gamma, \delta)$ and $\mu \in \mathbb{N}$ given in unary.

Output: A reduced power circuit $(\Gamma', \delta')$ such that $|C_0(\Gamma')| \geq \mu$ and $(\Gamma, \delta) \leq (\Gamma', \delta')$ (and $|\Gamma'| \leq |\Gamma| + \mu$).

Proof. Writing $\nu = \lceil \log_q \mu \rceil$, we know that every node to be added for the desired chain $C_0(\Gamma')$ has a successor marking using only the first $\nu$ nodes of $C_0(\Gamma')$. Thus, in a first (most difficult) step, we will extend $C_0(\Gamma)$ to have length at least $\nu$.

We start by creating a new reduced power circuit $\Delta = (Q_0, \ldots, Q_\nu)$ such that $\varepsilon(Q_i) = q^i$ (we just need to write every number $i \in [0..\nu]$ as its compact base-$q$ signed-digit representation and then create a node with the respective successor marking as outlined in the proof of [24, Theorem 32]).

Next, we wish to merge $\Delta$ with our input power circuit $\Gamma = (P_0, \ldots, P_n)$. In order to do so, we “guess” a subset $X \subseteq [0..\nu]$ (note that there are at most $2^{\nu+1}$ guesses, so they can be checked all in parallel). For $i \in X$ we write $\lambda(i) = j - 1$ if $i$ is the $j$-th element in the sorted order of $X$ (i.e., $\lambda(i) = \{|j \in X \mid j < i\}$).

For the guessed set $X$ we want to check whether $\varepsilon(Q_i) = \varepsilon(P_{\lambda(i)})$ for all $i \in X$ holds and whether $X$ is maximal with this property, i.e., the intuition behind $X$ is that it comprises all nodes of $\Delta$ which are also present in $\Gamma$.

To verify whether $\varepsilon(Q_i) = \varepsilon(P_{\lambda(i)})$ holds for all $i \in X$, ideally, we would check whether $A_{Q_i} = A_{P_{\lambda(i)}}$. However, this is not possible as the successor markings are on different power circuits. Instead, for all $i \in X$ we check whether...
Lemma 16 (\textbf{UpdateNodes}, [23, Lemma 19]). The following is in $\text{TC}^0$:

\begin{itemize}
  \item Input: A power circuit $(\Gamma \cup \Xi, \delta)$ as above.
  \item Output: A reduced power circuit $(\Gamma', \delta')$ such that
    \begin{itemize}
      \item for each $Q \in \Xi$ there is a node $P \in \Gamma'$ with $\varepsilon(P) = \varepsilon(Q),$
      \item $(\Gamma, \delta_{|\Gamma' \cup \Gamma}) \leq (\Gamma', \delta'),$
      \item $|\Gamma'| \leq |\Gamma| + |\Xi|$, and
      \item $|\Gamma'| \leq |\Gamma| + |\Xi|.$
    \end{itemize}
\end{itemize}
Lemma 17 (ExtendChains, [23, Lemma 20]). The following is in TC\(^0\):

Input: A reduced power circuit \((\Gamma, \delta)\) and \(\mu \in \mathbb{N}\) given in unary.

Output: A reduced power circuit \((\Gamma', \delta')\) such that

- for each \(P \in \Gamma\) and each \(i \in [0..\mu]\) there is a node \(Q \in \Gamma'\) with 
  \(\varepsilon(A_Q) = \varepsilon(A_P) + i\),
- \((\Gamma, \delta) \leq (\Gamma', \delta')\),
- \(|\Gamma'| \leq |\Gamma| + |\mathcal{C}_\Gamma| \cdot \mu\), and
- \(|\mathcal{C}_{\Gamma'}| \leq |\mathcal{C}_\Gamma|\).

Since we changed the statement slightly, we indicate the differences in the proof. Note that in [23, Lemma 20] we still had the technical condition that \(\mu \leq \left\lfloor \frac{2^{\varepsilon(C_0(i)) + 1}}{k} \right\rfloor\) (compare to Lemma 4 for \(q = 2\)). By using Lemma 15 we do not need this condition anymore.

Proof. We use Lemma 15 to prolongate \(C_0\) to a new chain \(\tilde{C}_0\) such that the last \(\mu\) nodes of \(\tilde{C}_0\) are not already present in \(\Gamma\) (this can be done e.g. by applying Lemma 15 to prolongate \(C_0\) to length \(i\) for all \(i \in [1..(\mu + 1) \cdot |\Gamma|]\) and checking the number of newly introduced nodes at the end of the chain). This replaces Step 1 in the proof of [23, Lemma 20]. Now, we can proceed exactly as Step 2 in the proof of [23, Lemma 20] – except that we do not prolongate the chain \(\tilde{C}_0\) any more.

For compact markings \(L, M\) on \(\Gamma\), it should be clear that \(\text{CR}_j(\varepsilon(M|_{C_0}) + \mu)\) can be represented as a compact marking on \(\tilde{C}_0\); thus, we can check whether \(\varepsilon(L) \leq \varepsilon(M) + \mu\) in AC\(^0\) like in [23, Proposition 14] and create new nodes with successor markings of value \(\varepsilon(M) + i\) for \(i \in [1..\mu]\) if necessary.

On input of a red-PC rep. for \(\ell, k\) we want to construct a red-PC rep. for \(m = k + \ell\) or \(m = k \cdot q^\ell\). If we proceed as in [23, Lemma 15], the power circuit representing \(m\) will not be (almost) reduced in general, even if we start with a reduced power circuit. We adapt these operations such that we obtain (almost) reduced power circuits as intermediate results. Thus the construction of a red-PC rep. for \(m\) is in TC\(^0\).

Lemma 18 (Addition). The following is possible in TC\(^0\):

Input: A reduced power circuit \((\Gamma, \delta)\) with compact markings \(L_j^{(i)}\) on \(\Gamma\) for \(i \in [1..\ell], j \in [1..k]\).

Output: A reduced power circuit \((\Gamma', \delta')\) and compact markings \(M^{(i)}\) on \(\Gamma'\) with \(\varepsilon(M^{(i)}) = \varepsilon(L_1^{(i)}) + \cdots + \varepsilon(L_k^{(i)})\) for \(i \in [1..\ell]\) and

- \((\Gamma', \delta') \leq (\Gamma', \delta')\),
- \(|\Gamma'| \leq |\Gamma| + \lceil \log_q(k) \rceil \cdot |\mathcal{C}_\Gamma|\),
- \(|\mathcal{C}_{\Gamma'}| \leq |\mathcal{C}_{\Gamma'}|\),
- \(|\sigma(M^{(i)})| \leq \sum_{j=1}^k |\sigma(L_j^{(i)})|\) for each \(i \in [1..\ell]\).
Lemma 19. The following is possible in $\text{TC}^0$:

1. **MultByIdPower:**
   
   **Input:** A reduced power circuit $(\Gamma, \delta)$ with compact markings $K^{(i)}, L^{(i)}$ on $\Gamma$ for $i \in [1..\ell]$ such that $\varepsilon(K^{(i)}) \cdot q^{\varepsilon(L^{(i)})} \in \mathbb{Z}$.
   
   **Output:** A reduced power circuit $(\Gamma', \delta')$ with compact markings $M^{(i)}$ such that $\varepsilon(M^{(i)}) = \varepsilon(K^{(i)}) \cdot q^{\varepsilon(L^{(i)})}$ for $i \in [1..\ell]$.

2. **MakeFloatingPoint:**
   
   **Input:** A reduced power circuit $(\Gamma, \delta)$ with compact markings $K^{(i)}$ for $i \in [1..\ell]$.
   
   **Output:** A reduced power circuit $(\Gamma', \delta')$ with compact markings $U^{(i)}, E^{(i)}$ such that $\varepsilon(K^{(i)}) = \varepsilon(U^{(i)}) \cdot q^{\varepsilon(E^{(i)})}$ with $\varepsilon(U^{(i)}) = 0$ or $q \nmid \varepsilon(U^{(i)})$ for $i \in [1..\ell]$.

In both cases we have $(\Gamma, \delta) \leq (\Gamma', \delta')$ and

\[
|\Gamma'| \leq |\Gamma| + |C_{\Gamma}| + \sum_{i=1}^\ell |\sigma(K^{(i)})| \leq |C_{\Gamma'}| + \sum_{i=1}^\ell |\sigma(K^{(i)})| + |\sigma(M^{(i)})| = |\sigma(K^{(i)})| \quad (\text{resp. } |\sigma(U^{(i)})| = |\sigma(K^{(i)})|) \text{ for all } i \in [1..\ell].
\]

Notice that the size of $\Gamma'$ and $|\sigma(M^{(i)})|$ does not depend on $L^{(i)}$. 

\[\text{Lemma 4, we have} \quad \prod_{j=1}^k \text{digit}_{C'}(M^{(i)}) = \text{CR}_{q} \left( \sum_{j=1}^k \text{digit}_{C'}(L_j^{(i)}) \right). \tag{2}\]

Note that the last $\lceil \log_q(k) \rceil$ nodes of $C'$ are not marked by any $L_j^{(i)}$. Therefore, by Lemma 4, we have

\[
\sum_{j=1}^k \text{val}_q(\text{digit}_{C'}(L_j^{(i)})) \leq k \cdot \left\lfloor \frac{|C'| - \lceil \log_q(k) \rceil + 1}{q^2 - 1} \right\rfloor \leq \left\lfloor \frac{|C'| + 1}{q^2 - 1} \right\rfloor.
\]

Therefore, the right side of (2), indeed, is a compact representation using only $|C'|$ digits.

Finally, if we define $M^{(i)}$ like this on all maximal chains, it is clear that $M^{(i)}$ is a compact marking and that $\varepsilon(M^{(i)}) = \varepsilon(L_1^{(i)}) + \cdots + \varepsilon(L_k^{(i)})$. It is also clear that $|\sigma(M^{(i)})| = \sum_{j=1}^k |\sigma(L_j^{(i)})|$. Lemma 17 shows that $(\Gamma, \delta) \leq (\Gamma', \delta')$ as well as the size conditions of Lemma 18. Because base-$q$ iteration is in $\text{TC}^0$ (see Example 1), with Theorem 7 it follows that the right side in (2) can be constructed in $\text{TC}^0$. This shows the lemma. \(\square\)
Proof. **MULTIPower:** We start by applying EXTENDCHAINS(1). We denote the resulting reduced power circuit by \((\Gamma', \delta_1)\). Observe that \((\Gamma, \delta) \leq (\Gamma_1, \delta_1)\). Next, we apply a construction similar to the one in [23, Lemma 15]. For each \(i \in [1..\ell]\) and each node \(P \in \sigma(K^{(i)})\) we construct a node \(R^{(i)}\) as follows: Let \(C\) be a maximal chain in \(I_1\). As the last node of \(C\) is neither marked by \(A_P\) nor by \(L^{(i)}\), by Lemma 4, $\text{CR}_q\left(\text{digit}_C(A_P) + \text{digit}_C(L^{(i)})\right)$ uses at most \(|C|\) digits. Thus,

$$\text{digit}_C(A^{(i)}_P) = \text{CR}_q\left(\text{digit}_C(A_P) + \text{digit}_C(L^{(i)})\right)$$

is well-defined and can be computed in \(\mathsf{TC}^0\) by Theorem 7. We take the marking \(A^{(i)}_P\) defined like that as the successor marking of \(R^{(i)}\). Then

$$\varepsilon(R^{(i)}_P) = q^\varepsilon(A^{(i)}_P) = q^{\varepsilon(A_P) + \varepsilon(L^{(i)})} = \varepsilon(P) \cdot q^{\varepsilon(L^{(i)})}.$$

We obtain a (not necessarily reduced) power circuit \((\Gamma_1 \cup \Xi, \delta)\), containing all the newly constructed nodes, with

$$\Xi = \left\{R^{(i)}_P \mid i \in [1..\ell], P \in \sigma(K^{(i)})\right\}.$$ 

All markings \(A^{(i)}_P\) are on \(\Gamma_1\) and compact by construction. Observe that \(|\Xi| = \sum_{i=1}^\ell |\sigma(K^{(i)})|\). We apply \textsc{UpdateNodes} with input of \((\Gamma_1 \cup \Xi, \delta)\). By Lemma 17 and Lemma 16 the construction of the resulting reduced power circuit \((\Gamma', \delta')\) is possible in \(\mathsf{TC}^0\) such that \((\Gamma, \delta) \leq (\Gamma', \delta')\) and such that the size conditions on \(|\Gamma'|\) and \(|C_{\Gamma'}|\) in the lemma are satisfied. While applying \textsc{UpdateNodes}, for each node \(R^{(i)}_P\) we remember which node in \(\Gamma'\) has the same evaluation.

Now we need to define the markings \(M^{(i)}\) on \(\Gamma'\). Let \(Q \in \Gamma'\). If there exists a node \(P \in \sigma(K^{(i)})\) such that \(\varepsilon(Q) = \varepsilon(R^{(i)}_P)\) (i.e., \(A^*_Q = A^{(i)}_P\)), then we set \(M^{(i)}(Q) = K^{(i)}(P)\). Otherwise, we set \(M^{(i)}(Q) = 0\).

As \(\varepsilon(R^{(i)}_P) = \varepsilon(P) \cdot q^{\varepsilon(L^{(i)})}\) for all \(P \in \sigma(K^{(i)})\), the marking \(M^{(i)}\) is just a “shift” of \(K^{(i)}\); thus, it is well-defined and compact. Observe that \(|\sigma(M^{(i)})| = |\sigma(K^{(i)})|\). We obtain that

$$\varepsilon(M^{(i)}) = \sum_{Q \in \Gamma'} M^{(i)}(Q) \cdot \varepsilon(Q) = \sum_{P \in \sigma(K^{(i)})} K^{(i)}(P) \cdot \varepsilon(P) \cdot q^{\varepsilon(L^{(i)})} = \varepsilon(K^{(i)}) \cdot q^{\varepsilon(L^{(i)})}.$$

This proves part 1.

**MAKEFloatingpoint:** Let \(\sigma(K^{(i)}) = \{Q_1, \ldots, Q_k\}\). If \(k = 0\), then \(\varepsilon(K^{(i)}) = 0\), so we set \(\varepsilon(U^{(i)}) = \varepsilon(E^{(i)}) = 0\). Now let \(k \geq 1\). Because \((\Gamma, \delta)\) is reduced, we know that \(\varepsilon(Q_j) < \varepsilon(Q_j)\) for all \(j \in [2..k]\). Therefore, \(u = \varepsilon(K^{(i)}) \cdot q^{-\varepsilon(A_{Q_1})}\) is integral but not divisible by \(q\). We set \(E^{(i)} = A_{Q_1}\) and use \textsc{MULTIPower} with input \(K^{(i)}\) and \(E^{(i)}\) to compute a marking \(U^{(i)}\) with \(\varepsilon(U^{(i)}) = u\). By the first part of the lemma, we can do this for all \(i \in [1..\ell]\) in parallel. \qed

We want to represent elements \(r \in \mathbb{Z}[1/q]\) as floating point numbers using compact markings in reduced power circuit:
Definition 20. Let \((\Gamma, \delta)\) be a reduced power circuit and \(r \in \mathbb{Z}[1/q]\). We call \(R = (U, E)\) a reduced power circuit representation (red-PC rep.) for \(r\) over \((\Gamma, \delta)\) if \(U\) and \(E\) are compact markings on \(\Gamma\) with \(r = \varepsilon(U) \cdot q^{(E)}\) and \(\varepsilon(U)\) is either zero or \(q \mid \varepsilon(U)\). We write \(\varepsilon(R) = \varepsilon(U) \cdot q^{(E)}\) and define \(S(R) = S(U)\) (recall that \(S(U) = |\sigma(U)|\), i.e., we only count nodes in the support of the mantissa).

Likewise, for \(m \in \mathbb{Z}\) we call a compact marking \(M\) on \(\Gamma\) with \(\varepsilon(M) = m\) a reduced power circuit representation (red-PC rep.) of \(m\) over \((\Gamma, \delta)\). Moreover, for \(\bar{R} = ((U_1, E_1), \ldots, (U_\ell, E_\ell))\) we write \(S(\bar{R}) = S((U_1, \ldots, U_\ell)) = \sum_{i=1}^\ell S((U_i, E_i))\).

The operations \textsc{Addition}, \textsc{MultByPower} and \textsc{MakeFloatingPoint} are our main ingredients for the Britton-reduction algorithm in \(G_{1,q}\). The next result combines these operations to work with floating point numbers (more precisely, their red-PC rep.s). For an analogous statement for floating point operations on non-reduced power circuits, see [23, Lemma 28]. Notice that, while [23, Lemma 28] deals only with a single operation, here we consider an unbounded number of operations of the same type on the same reduced power circuit. Moreover, the constructions are all in \(\text{TC}^0\), while in [23] the depth of the \(\text{TC}\) circuit depends on the depth of the input power circuit. This is because in [23] we need to reduce arbitrary power circuits which is not known to be in \(\text{TC}^0\) (except for power circuits of constant depth).

Corollary 21. The following constructions are in \(\text{TC}^0\):

\(\begin{align*}
\text{a) Input:} & \quad \text{A red-PC rep. } \bar{R} = (R^{(i)})_{i \in [1..\ell]} \text{ over } (\Gamma, \delta) \text{ for } r^{(i)} \in \mathbb{Z}[1/q] \text{ and compact markings } M^{(i)} \text{ on } \Gamma \text{ for } i \in [1..\ell]. \\
\text{Output:} & \quad \text{A red-PC rep. } \bar{S} = (S^{(i)})_{i \in [1..\ell]} \text{ for } r^{(i)} \cdot q^{(M^{(i)})} \text{ over a power circuit } (\Gamma', \delta') \text{ such that } S(S^{(i)}) = S(R^{(i)}).
\end{align*}\)

\(\begin{align*}
\text{b) Input:} & \quad \text{Red-PC rep.s } \bar{R} = (R^{(i)})_{i \in [1..\ell]} \text{ over } (\Gamma, \delta) \text{ for } s^{(i)} \in \mathbb{Z}[1/q]. \\
\text{Output:} & \quad \text{A reduced power circuit } (\Gamma', \delta') \text{ and for each } i \in [1..\ell] \text{ the answer whether } r^{(i)} \in \mathbb{Z} \text{ and, if yes, a compact marking } M^{(i)} \text{ such that } \varepsilon(M^{(i)}) = r^{(i)} \text{ and } S(M^{(i)}) = S(R^{(i)}).
\end{align*}\)

\(\begin{align*}
\text{c) Input:} & \quad \text{Red-PC rep.s } \bar{R} = (R^{(i)})_{i \in [1..\ell], j \in [1..k]} \text{ over } (\Gamma, \delta) \text{ for } r^{(i)} \in \mathbb{Z}[1/q]. \\
\text{Output:} & \quad \text{Red-PC rep.s } \bar{S} = (S^{(i)})_{i \in [1..\ell]} \text{ over a power circuit } (\Gamma', \delta') \text{ for } \sum_{j=1}^k r_j^{(i)} \text{ such that } S(S^{(i)}) \leq \sum_{j=1}^k S(R_j^{(i)}).
\end{align*}\)

In all cases we have \((\Gamma', \delta') \leq (\Gamma', \delta')\) and there is some constant \(c\) such that

\(- |\Gamma'| \leq |\Gamma| + c \cdot (|C| + S(\bar{R})) \text{ in (in cases a) and b)},
\(- |\Gamma'| \leq |\Gamma| + (c + \log_q k) \cdot (|C| + S(\bar{R})) \text{ (in case c)},
\(- |C_r'| \leq |C_r| + c \cdot S(\bar{R}).

Note that clearly, equality of red-PC rep. \((U_1, E_1)\) and \((U_2, E_2)\) can be tested in \(\text{AC}^0\) as it amounts to checking whether \(U_1 = U_2\) and \(E_1 = E_2\) or \(U_1 = 0 = U_2\).

\textbf{Proof.} We write \(R^{(i)} = (U^{(i)}, E^{(i)})\). For a), we just need to add the markings \(E^{(i)}\) and \(M^{(i)}\). This can be done by Lemma 18. The check whether \(r^{(i)} \in \mathbb{Z}\) in b) can be done by testing whether \(\varepsilon(E^{(i)}) \geq 0\), which is in \(\text{AC}^0\) by Lemma 14. If yes, we can apply Lemma 19.
In order to see c), let us write \( R_j^{(i)} = (U_j^{(i)}, E_j^{(i)}) \) and \( u_j^{(i)} = e(U_j^{(i)}) \) and \( e_j^{(i)} = e(E_j^{(i)}) \). Observe that

\[
\sum_{j=1}^{\ell} R_j^{(i)} = \sum_{j=1}^{\ell} q_j^{(i)} \cdot u_j^{(i)} = q_1^{(i)} \cdot \left( \sum_{j=1}^{\ell} q_j^{(i)} - e_j^{(i)} \cdot u_j^{(i)} \right)
\]

with \( e_j^{(i)} = \min_{1 \leq j \leq k} e_j^{(i)} \). Due to Lemma 19, we know that we can apply \textsc{MultByPower} and \textsc{MakeFloatingPoint} to several markings independently in parallel in \( \text{TC}^0 \). So we can calculate the red-PC rep. for \( \sum_{j=1}^{k} R_j^{(i)} \) using the following (constant number of) \( \text{TC}^0 \) steps (while in each step we apply the same kind of operation on all markings): We first reduce the \( q_j^{(i)} \cdot u_j^{(i)} \) to the common denominator \( e_j^{(i)} \) by applying \textsc{Addition}(2) (to compute \( e_j^{(i)} - e_j^{(i)} \)) followed by \textsc{MultByPower}. Now we apply \textsc{Addition}(k) to compute \( \sum_{j=1}^{k} q_j^{(i)} - e_j^{(i)} \cdot u_j^{(i)} \). Then we apply \textsc{MakeFloatingPoint} and, finally, multiply the result by \( q_1^{(i)} \) using part a) of this lemma (which uses one application of \textsc{Addition}(2)).

By Lemma 19, the marking representing \( q_1^{(i)} - e_1^{(i)} \cdot u_1^{(i)} \) has support of size \( |S(U_1^{(i)})| \). Moreover, each \( q_j^{(i)} - e_j^{(i)} \cdot u_j^{(i)} \) appears at most once in the sum in (3). By Lemma 18 it follows that \( S(S(i)) \leq \sum_{j=1}^{k} S(R_j^{(i)}) \).

Again by Lemma 18 and Lemma 19, each of the operations increases the number of maximal chains by at most \( S(\bar{R}) \). Therefore, \( |C_{\Gamma'}| \leq |C_{\Gamma}| + c \cdot S(\bar{R}) \) and the same bound applies to all intermediate power circuits. For the power circuit \( \Gamma' \) obtained after the operations \textsc{Addition}(2) and \textsc{MultByPower} we even have the stronger bound \( |C_{\Gamma'}| \leq |C_{\Gamma}| + S(\bar{R}) \). Thus, when applying \textsc{Addition}(k) we add at most \( |1 + \log_k(k) \cdot (|C_{\Gamma}| + S(\bar{R})) \) new nodes (Lemma 18) and for the other operations at most \( |C_{\Gamma}| + (c + 1) \cdot S(\bar{R}) \) new nodes (Lemma 19). Therefore, after making \( c \) sufficiently larger, the bound \( |\Gamma'_{\Gamma^*}| \leq |\Gamma| + (c + \log_k k) \cdot (|C_{\Gamma}| + S(\bar{R})) \) follows. \( \square \)

### 3.2 Modulo in power circuits

As usual for \( a, b \in \mathbb{Z} \), we write \( a \mod b \) for the unique \( x \in \mathbb{Z} \) with \( x \equiv a \mod b \) and \( x \in [0..b-1] \). In [9, Theorem 1] it is shown that to compute a marking for \( \varepsilon(M) \mod \varepsilon(L) \) on input of markings \( M \) and \( L \) on a power circuit can lead to a non-elementary blow-up. Thus, in general, calculating modulo is certainly not in \( \text{TC}^0 \). Nevertheless, this changes for a small modulus.

**Proposition 22.** For every fixed \( k \geq 2 \) the following problem is in \( \text{TC}^0 \):

- **Input:** A reduced power circuit \((\Gamma, \delta)\) and a compact marking \( M \) on \( \Gamma \).
- **Output:** \( \varepsilon(M) \mod k \).

Note that as \( k \) is a constant, the output can be given in any reasonable format.

**Proof.** In order to compute \( \varepsilon(M) \mod k \), it suffices to compute \( \varepsilon(P) \mod k \) for \( P \in \Gamma \) (then we can sum up these values using usual binary arithmetic and again compute modulo \( k \)). Let \( k = \ell r \) such that \( \ell \) is maximal with \( \gcd(\ell, q) = 1 \) (i.e., we collect all prime factors shared by \( k \) and \( q \) into \( r \)). Clearly, we have \( \gcd(\ell, r) = 1 \). We compute
\[ \varepsilon(P) \mod \ell \text{ and } \varepsilon(P) \mod r \text{ separately (and compose them in the end using the Chinese Remainder Theorem, as } r \text{ is a constant, we can hard-wire this into the circuit).} \]

First, let us explain how to compute \( \varepsilon(P) \mod r \). Let \( e \) be the maximum such that \( p^e \) divides \( r \) for some prime \( p \) (note that \( p \) divides \( q \)). Hence, we have \( q^i \equiv 0 \mod r \) as soon as \( i > e \). Thus, for computing \( \varepsilon(P) \mod r = q^{\varepsilon(A_P)} \mod r \), we first check whether \( \varepsilon(A_P) > e \) (using Lemma 14). If this is the case, we know that \( \varepsilon(P) \equiv 0 \mod r \); otherwise we can explicitly compute \( \varepsilon(P) = q^{\varepsilon(A_P)} \) as it is bounded by a constant (or hard-wire it into the circuit).

For computing \( \varepsilon(P) \mod \ell \), we proceed by induction: As \( \gcd(\ell,q) = 1 \), we know that \( q^{\varepsilon(\ell)} \equiv 1 \mod \ell \), where \( \varphi \) is Euler’s totient function. Therefore,

\[
\varepsilon(P) = q^{\varepsilon(A_P)} \equiv q^{\varepsilon(A_P) \mod \varphi(\ell)} \mod \ell.
\]

By induction, we know that \( \varepsilon(A_P) \mod \varphi(\ell) \) can be computed in \( \text{TC}^0 \) (note that \( \ell \) is a constant, so we only compose a constant number of \( \text{TC}^0 \) computations, which results again in a \( \text{TC}^0 \) computation).

**Remark 23.** From the proof of Proposition 22 it follows that actually when allowing \( k \) in unary as part of the input, the problem

- **Input:** A reduced power circuit \((\Gamma, \delta)\), a marking \(M\) on \(\Gamma\) and \(k \in \mathbb{N}\) given in unary.
- **Output:** \(\varepsilon(M) \mod k\).

is in \( \text{TC}^1 \). In order to see this, we apply the algorithm from Proposition 22. In order to compute \( \varphi(\ell) \), we factorize \( \ell \) (by trying all possible factors) and then compute it via the formula \( \varphi(mn) = \varphi(m)\varphi(n) \) for coprime \( m \) and \( n \). Notice that after a logarithmic number of applications of \( \varphi \), we arrive at \( \varphi(\ell) = 1 \). This is because at least in every second step \( \ell \) will be even – and if \( \ell \) is even we have \( \varphi(\ell) \leq \ell/2 \). As each round in the proof of Proposition 22 is in \( \text{TC}^0 \), the complete algorithm is in \( \text{TC}^1 \).

**Lemma 24.** Let \( r \in \mathbb{Z} \) be a constant. The following problem is in \( \text{TC}^0 \):

- **Input:** A reduced power circuit \((\Gamma, \delta)\) with compact markings \(K, L\).
- **Output:** A reduced power circuit \((\Gamma', \delta')\) with a compact marking \(M\) such that \(\varepsilon(M) = \varepsilon(L) \mod q^{\varepsilon(K)} \cdot r\).

**Proof.** W.l.o.g. \( q \) does not divide \( r \) (otherwise, we add some constant to \( \varepsilon(K) \), which can be done in \( \text{TC}^0 \)). By the Chinese Remainder Theorem, we may compute \( \varepsilon(L) \mod q^{\varepsilon(K)} \) and \( \varepsilon(L) \mod r \) separately and then compose them.

To compute \( \ell_K = \varepsilon(L) \mod q^{\varepsilon(K)} \), we check for all \( P \in \sigma(L) \) whether \( \varepsilon(A_P) \leq \varepsilon(K) \) and create a new marking \( L_K \) with \( \varepsilon(L_K) = \ell_K \) by setting \( L_K(P) = L(P) \) if \( \varepsilon(A_P) \leq \varepsilon(K) \) and \( L_K(P) = 0 \) otherwise. This can be done in \( \text{TC}^0 \) by Lemma 14 (for the comparison). The computation of \( \ell_r = \varepsilon(L) \mod r \) is in \( \text{TC}^0 \) by Proposition 22.

Now for \( \varepsilon(M) \) there remain the possibilities \( m_i = \ell_K + i \cdot q^{\varepsilon(K)} \) for \( i \in [0..r-1] \). For all these \( i \) we can compute \( m_i \mod r \) in \( \text{TC}^0 \) by Proposition 22 and choose \( i \) such that \( m_i \mod r = \ell_r \). By Lemma 18, we can compute the marking \( M \) with \( \varepsilon(M) = \ell_K + i \cdot q^{\varepsilon(K)}. \)
4 The word problem of $G_{1,q}$

4.1 Notations from group theory

First let us fix our notation from group theory.

**Group presentations.** A group $G$ is *finitely generated* if there is some finite set $\Sigma$ and a surjective monoid homomorphism $\eta: \Sigma^* \to G$ (called a presentation). Usually, we do not write the homomorphism $\eta$ and treat words over $\Sigma$ both as words and as their images under $\eta$. We write $v =_{G} w$ with the meaning that $\eta(v) = \eta(w)$. If $\Sigma = S \cup S^{-1}$ where $S^{-1}$ is some disjoint set of formal inverses and $R \subseteq \Sigma^* \times \Sigma^*$ is some set of relations, we write $(\Sigma \mid R)$ for the group $\Sigma^*/C(R)$ where $C(R)$ is the congruence generated by $R$ together with the relations $aa^{-1} = a^{-1}a = 1$ for $a \in \Sigma$. If $R$ is finite, $G$ is called *finitely presented*.

The word problem for a fixed group $G$ with presentation $\eta: \Sigma^* \to G$ is as follows:

**Input:** A word $w \in \Sigma^*$

**Question:** Is $w =_{G} 1$?

For $S \subseteq G$, we write $(S)$ for the subgroup generated by $S$. For further background on group theory, we refer to [20].

**The Baumslag-Solitar group.** Let $q \in \mathbb{Z}$ with $|q| \geq 2$. The Baumslag-Solitar group is defined by

$$BS_{1,q} = \langle a, t \mid tat^{-1} = a^q \rangle.$$  

We have $BS_{1,q} \cong \mathbb{Z}[1/q] \rtimes \mathbb{Z}$ via the isomorphism $a \mapsto (1,0)$ and $t \mapsto (0,1)$. Recall that $\mathbb{Z}[1/q] = \{ m/q^k \in \mathbb{Q} \mid m, k \in \mathbb{Z} \}$ is the set of fractions with powers of $q$ as denominators. It forms a groups with addition as group operation. The multiplication in $\mathbb{Z}[1/q] \rtimes \mathbb{Z}$ is defined by $(r,m) \cdot (s,n) = (r+q^m s, m+n)$. Inverses can be computed by the formula $(r,m)^{-1} = (-r, q^{-m}, -m)$. In the following we use $BS_{1,q}$ and $\mathbb{Z}[1/q] \rtimes \mathbb{Z}$ as synonyms.

**The Baumslag group.** The Baumslag group $G_{1,q}$ can be understood as an HNN extension (for a definition see [20] – this is how the Magnus breakdown procedure works) of the Baumslag-Solitar group:

$$G_{1,q} = \langle BS_{1,q}, b \mid bab^{-1} = t \rangle = \langle a, t, b \mid tat^{-1} = a^q, bab^{-1} = t \rangle.$$  

Observe that due to $bab^{-1} = t$, we can remove $t$ and we obtain exactly the presentation $(a, b \mid bab^{-1}a = a^q bab^{-1})$. Moreover, $BS_{1,q}$ is a subgroup of $G_{1,q}$ via the canonical embedding and we have $b(g,0)b^{-1} = (0, g)$, so a conjugation by $b$ “flips” the two components of the semi-direct product if possible. Henceforth, we will use the alphabet $\Sigma = \{1, a, a^{-1}, t, t^{-1}, b, b^{-1}\}$ to represent elements of $G_{1,q}$ (the letter 1 represents the group identity; it is there for padding reasons).
Britton reductions. Britton reductions are a standard way to solve the word problem in HNN extensions. Here we define them for the special case of $G_{1,q}$. We start with $G_{1,q}$. Let
\[ \Delta_q = BS_{1,q} \cup \{ b, b^{-1} \} \]
be an infinite alphabet (note that $\Sigma \subseteq \Delta_q$). A word $w \in \Delta_q^\ast$ is called Britton-reduced if it is of the form
\[ w = (s_0, n_0)\beta_1(s_1, n_1) \cdots \beta_k(s_k, n_k) \]
with $\beta_i \in \{ b, b^{-1} \}$ and $(s_i, n_i) \in BS_{1,q}$ for all $i$ (i.e., $w$ does not have two successive letters from $BS_{1,q}$ – note that here we identify $(0,0) \in BS_{1,q}$ with the empty word 1) and there is no factor of the form $b(g,0)b^{-1}$ or $b^{-1}(0,k)b$ with $g,k \in \mathbb{Z}$. If $w$ is not Britton-reduced, one can apply one of the rules
\[
\begin{align*}
(r,m)(s,n) & \to (r + q^m s, m + n) \\
bs{b(g,0)b^{-1}} & \to (0,g) \\
bs{b^{-1}(0,k)b} & \to (k,0)
\end{align*}
\]
in order to obtain a shorter word representing the same group element. The following lemma is well-known (see also [20, Section IV.2]).

**Lemma 25** (Britton’s Lemma for $G_{1,q}$ [6]). Let $w \in \Delta_q^\ast$ be Britton-reduced. Then $w \in BS_{1,q}$ as a group element if and only if $w$ does not contain any letter $b$ or $b^{-1}$. In particular, $w = g_{1,q}$ if and only if $w = (0,0)$ or $w = 1$ as a word.

**Example 26.** Let $q \geq 2$ and define words $w_0 = t$ and $w_{n+1} = bw_n w_n^{-1}$ for $n \geq 0$ with $w_n \in \Delta_q^\ast$ for all $n \geq 0$. Then we have $|w_n| = 2^n + 2 - 3$ but $w_n = g_{1,q} \cdot t_{\tau_q(n)}$. While the length of the word $w_n$ is only exponential in $n$, the length of its Britton-reduced form is $\tau_q(n)$.

### 4.2 Conditions for Britton-reductions in $G_{1,q}$

The following lemma was already used in [23] to find a maximal suffix of $u$ which cancels with a prefix of $v$ on input of two Britton-reduced words $u$ and $v$. The proof is exactly the same for arbitrary $q$ with $|q| \geq 2$, just replace 2 by $q$.

**Lemma 27** ([23, Lemma 31]). Let $w = \beta_1(r,m)\beta_2 x \beta_2^{-1}(s,n)\beta_1^{-1} \in \Delta_q^\ast$ with $\beta_1, \beta_2 \in \{ b, b^{-1} \}$ such that $\beta_1(r,m)\beta_2$ and $\beta_2^{-1}(s,n)\beta_1^{-1}$ both are Britton-reduced and $\beta_2 x \beta_2^{-1} = g_{1,q} (g,k) \in BS_{1,q}$ (in particular, $k = 0$ and $g \in \mathbb{Z}$, or $g = 0$).

Then $w \in BS_{1,q}$ if and only if the respective condition in the table below is satisfied. Moreover, if $w \in BS_{1,q}$, then $w \overset{\circ}{=}_{g_{1,q}} w$ according to the last column of the table.

| $\beta_1$ | $\beta_2$ | Condition | $\hat{w}$ |
|---|---|---|---|
| $b$ | $b$ | $r + q^{m+k}s \in \mathbb{Z}$, $m + n + k = 0$ | $(0, r + q^{-n}s)$ |
| $b$ | $b^{-1}$ | $r + q^m(g+s) \in \mathbb{Z}$, $m + n = 0$ | $(0, r + q^m(g+s))$ |
| $b^{-1}$ | $b$ | $r + q^{m+k}s = 0$ | $(m + n + k, 0)$ |
| $b^{-1}$ | $b^{-1}$ | $r + q^m(g+s) = 0$ | $(m + n, 0)$ |

Notice that in the case $\beta_1 = b^{-1}$ and $\beta_2 = b$, we have $r \neq 0$ and $s \neq 0$. 
Lemma 28. Let $\Delta_3$.

As this approach does not allow for good bounds on the size of the power circuits, we
in the third case of Lemma 27, we used the log-operation to compute the outcome.
use a different approach based on iterated addition as shown in the next lemma:

$$w \text{ a red-PC rep. for a Britton-reduced word}$$

Britton’s Lemma we obtain that, if $u$.

Note that, to simplify some notation, here we start with $\beta$.

have shown that $\beta_+$

Proof. Let $u, v \in G_{1,q}$ be Britton-reduced and denoted as in Eq. (5) with $i + 1 \leq \lceil \frac{1}{2} \rceil$.

Further assume that

$$uv[i, j] = \beta_{i+1}(r_i, m_i) \cdots \beta_1(r_0, m_0) (s_0, n_0) \beta_1 \cdots (s_j, n_j) \beta_{j+1}.$$

Note that, to simplify some notation, here we start with $\beta_+$ while in [23] we had a similar notation starting with $\beta_+$. Further notice that as an immediate consequence of Britton’s Lemma we obtain that, if $u$ and $v$ as in (5) are Britton-reduced and $uv[i, j] \in BS_{1,q}$ for some $i$, then also $uv[j, j] \in BS_{1,q}$ for all $j \leq i$. Moreover, $uv$ is Britton-reduced if and only if $\beta_1(r_0, m_0)(s_0, n_0) \beta_1 \notin BS_{1,q}$.

On input of red-PC rep.s of Britton-reduced words $u$ and $v$ we want to construct a red-PC rep. for a Britton-reduced word $w = u \in G_{1,q}$ $uv$ using Corollary 21. Note that in [23], in the third case of Lemma 27, we used the log-operation to compute the outcome. As this approach does not allow for good bounds on the size of the power circuits, we use a different approach based on iterated addition as shown in the next lemma:

**Lemma 28.** Let $u, v \in G_{1,q}$ be Britton-reduced and denoted as in Eq. (5) and $j \leq \lceil \frac{1}{2} \rceil$.

with $i + 1 \leq \min\{\lceil |u|_b \rceil, \lceil |v|_b \rceil\}$. Further assume that

$$uv[i, j] = b(r_i, m_i)b^{-1} \cdots b(r_{i-2j}, m_{i-2j})b^{-1} yb(s_{i-2j}, n_{i-2j}) \cdots b(s_i, n_i)b^{-1} \in BS_{1,q}$$

with $b^{-1}yb = (g, 0) \in BS_{1,q}$. Then, for $s_0 = \sum_{\zeta=0}^{\theta} m_{i-2\zeta}$ and $h_0 = i - (2\theta + 1)$ we have

$$uv[i, j] = \left(0, r_i + q^{s_j} \cdot (g + s_{h+1}) + \sum_{\theta=0}^{j-1} q^{s_{\theta}} \cdot (m_{h_{\theta}} + n_{h_{\theta}} + r_{h-1} + s_{h_{\theta}+1})\right).$$

**Proof.** For this proof all equal signs have to be read as equalities in $G_{1,q}$. We proceed by induction over $j$. First, let $j = 0$. Then,

$$b(r_0, m_0)b^{-1} yb(s_0, n_0)b^{-1} = (0, r_i + q^{s_0} \cdot (g + s_i))$$

with $b^{-1}yb = (g, 0)$. Because the empty sum evaluates to 0, we have

$$r_i + q^{s_0} \cdot (g + s_{h+1}) + \sum_{\theta=0}^{j-1} q^{s_{\theta}} \cdot (m_{h_{\theta}} + n_{h_{\theta}} + r_{h-1} + s_{h_{\theta}+1})$$

$$= r_i + q^{s_0} \cdot (g + s_i).$$

We now assume that for some $j \geq 1$ we know that $i - 2(j + 1) \geq 0$ and we already have shown that

$$b(r_i, m_i)b^{-1} \cdots b(r_{i-2j}, m_{i-2j})b^{-1} yb(s_{i-2j}, n_{i-2j}) \cdots b(s_i, n_i)b^{-1}$$

$$= \left(0, r_i + q^{s_j} \cdot (g + s_{h+1}) + \sum_{\theta=0}^{j-1} q^{s_{\theta}} \cdot (m_{h_{\theta}} + n_{h_{\theta}} + r_{h-1} + s_{h_{\theta}+1})\right).$$
with $b^{-1}gb = (g,k)$ and $\kappa_\theta$ and $h_\theta$ as above. As a preparation for the induction step we do the following calculation with $b^{-1}y'b = (g',k') = (g',0)$:

$$
\begin{align*}
&b^{-1}y'b = b^{-1}(r_{h_j}, m_{h_j})b(r_{h_j-1}, m_{h_j-1}) b^{-1}y' b (s_{h_j-1}, n_{h_j-1})b^{-1}(s_{h_j}, n_{h_j})b \\
&\quad= b^{-1}(r_{h_j}, m_{h_j}) \{ 0, r_{h_j-1} + q^{m_{h_j}-1} \cdot (g' + s_{h_j-1}) \} (s_{h_j}, n_{h_j})b \\
&\quad= (m_{h_j} + n_{h_j} + r_{h_j-1} + q^{m_{h_j}-1} \cdot (g' + s_{h_j-1}), 0).
\end{align*}
$$

Then,

$$
\begin{align*}
uv[i,i] &= b(r_i, m_i)b^{-1} \cdots b(r_{h_j+1}, m_{h_j+1}) b^{-1}(r_{h_j}, m_{h_j})b(r_{h_j-1}, m_{h_j-1}) b^{-1}y' b \\
&= b(r_i, m_i)b^{-1} \cdots b(r_{h_j+1}, m_{h_j+1}) \\
&\quad\sum_{\theta=0}^{j-1} q^{s_{\theta}} \cdot (m_{h_{\theta}} + n_{h_{\theta}} + r_{h_{\theta}-1} + s_{h_{\theta}+1}) \\
&= \left(0, r_i + q^{s_{j-1}} \cdot (m_{h_j} + n_{h_j} + r_{h_j-1} + q^{m_{h_j}-1} \cdot (g' + s_{h_j-1}) + s_{h_j+1})
\right)
\end{align*}
$$

This proves the lemma. □

### 4.3 The algorithm for $G_{1,q}$

We want to represent elements of the group $G_{1,q}$ using compact markings in reduced power circuits. We do this as follows:

**Definition 29.** Let $(\Gamma, \delta)$ be a reduced base-$|q|$ power circuit and $w = w_1 \cdots w_n \in \Delta^+ \Gamma$. A red-PC rep. of $w$ over $(\Gamma, \delta)$ is a list $W = ((B_i, U_i, E_i, M_i)_{i\in[1..n]} \text{ with } B_i \in \{b, b^{-1}, \$ \} \text{ and } U_i, E_i, M_i \text{ compact markings on } \Gamma \text{ such that for } i \in [1..n] \text{ we have}

- if $w_i \in \{b, b^{-1}\}$, then $B_i = w_i$ and $U_i, E_i, M_i$ are the zero marking,
- if $w_i = (r_i, m_i) \in \text{BS}_{1,q}$, then $B_i = \$ and $(U_i, E_i)$ is a red-PC rep. for $r_i$ and $M_i$ a red-PC rep. for $m_i$ (as in Definition 20).

We write $|W|_{\beta} = |w|_\beta$ and $S(W) = S(U_1, \ldots, U_n) + S(M_1, \ldots, M_n) = \sum_{i=1}^{n} (|\sigma(U_i)| + |\sigma(M_i)|)$ and call $n$ the length of $W$.

Note that we always use power circuits with a positive base – even when $q$ is negative! In the following, we do not specify the base of the power circuit – it is always $|q|$. Recall that we assume $|q| \geq 2$.

Be aware that for $S(W)$ we do not count the markings in $E$. In ADDITION or MULTByPOWER, the number of new nodes we insert in the worst case only depends on
the number of maximal chains and the markings \( M_i \) and \( U_i \), but not on the markings \( E_i \). Moreover, \( S(W) \) does not increase by any of our operations. This property is destroyed when also counting the markings \( E_i \) for \( S(W) \).

In the following lemma it is crucial that we use the same reduced power circuit for all \( w^{(i)} \). In [23], we used a separate (non-reduced) power circuit for each \( i \). Reducing an arbitrary power circuit is not known to be in \( TC^0 \) (unless it has constant depth).

**Lemma 30.** There is a constant \( c \) such that the following problem is in \( TC^0 \):

**Input:** Reduced-PC rep.s \( U^{(i)}, V^{(i)} \) for Britton-reduced words \( u^{(i)}, v^{(i)} \in \Delta_q^* \) over \( (\Gamma, \delta) \) for \( i \in [1..\nu] \).

**Output:** Reduced-PC rep.s \( W^{(i)} \) over \( (\Gamma', \delta') \) for Britton-reduced words \( u^{(i)} \in \Delta_q^* \) with \( w^{(i)} = \mathcal{G}_{1,q} u^{(i)} \) for \( i \in [1..\nu] \) and

\[
- \sum_{i=1}^{\nu} S(V^{(i)}) \leq S, \\
- |C_{\Gamma'}| \leq |C_\Gamma| + c \cdot S, \\
- |\Gamma'| \leq |\Gamma| + c \cdot \log(n) \cdot (|C_\Gamma| + S),
\]

where \( n = \max_{i \in [1..\nu]} \left| U^{(i)} \right|_\beta + \left| V^{(i)} \right|_\beta \) and \( S = \sum_{i=1}^{\nu} S(U^{(i)}) + S(V^{(i)}) \).

**Proof.** We describe the proof for the case that \( \nu = 1 \) writing \( u = u^{(1)}, v = v^{(1)} \) etc. The general case follows easily by the observation that Corollary 21, Lemma 18 and Lemma 19 allow the manipulation of several (bunches of) markings on the same reduced power circuit in parallel – we give some details at the end of the proof.

In order to find the Britton reduction for \( uv \), we need to find the maximal \( i \) such that \( uv[i, i]\in BS_{1,q} \) (where \( uv[i, i] \) is as in (6) in Section 4.2). So for each \( i < \min \{ |u|_\beta, |v|_\beta \} \) we proceed as follows: we compute a bit indicating whether \( uv[i, i] \in BS_{1,q} \) under the assumption that \( uv[i-1, i-1] \in BS_{1,q} \). Note that this is the same approach as in [23] – however, the way we compute this bit differs.

If \( \beta_j \neq \beta_j^{-1} \) for some \( j \leq i+1 \), we know that \( uv[i, i] \notin BS_{1,q} \). Otherwise, we assume that there exist \( g, k \in \mathbb{Z} \) such that \( uv[i-1, i-1] = \mathcal{G}_{1,q} (g, k) \in BS_{1,q} \). As we want to apply Lemma 27, we want to calculate the red-PC rep. for \( (g, k) \). To do the latter we also use Lemma 27: If \( \beta_i = \beta_{i-1} \), Lemma 27 tells us directly which operations we need to perform to compute \( (g, k) \) (like we did in [23]). If \( (\beta_i, \beta_{i-1}) = (b, b^{-1}) \), we proceed as follows: Let \( h_j = i - (2j + 2) \) be as in Lemma 28 (with \( i - 1 \) instead of \( i \)) and let \( j \) be minimal such that \( \beta_{h_{j+1}} = \beta_{h_j} \) or \( \beta_{h_j} = \beta_{h_{j-1}} \) or \( h_j \leq 1 \). We first compute \( uv[h_j, h_j] \) using the formula from Lemma 27 (note that it might happen that we need to apply Lemma 27 again to \( uv[h_j-1, h_j-1] \); however, after two steps we are in a case where the outcome can be computed without looking further inward). Now, Lemma 28 tells us how to compute \( (g, k) \).

Finally, if \( (\beta_i, \beta_{i-1}) = (b^{-1}, b) \), we can apply one of the already discussed cases to \( uv[i-2, i-2] \) (note also from the sum from Lemma 28) we only use operations described in Corollary 21, Lemma 18 and Lemma 19; so we can construct the red-PC rep. for \( (g, k) \) using constantly many \( TC^0 \) operations. Note that for \( q < 0 \) we need to take some extra care: computing \( q^r \) (for \( \kappa \) and \( r \) given as red-PC rep.s) is done in two steps: we compute \( |q^r| r \) as usual; then we compute \( z = \kappa \mod 2 \) using Proposition 22 and replace \( |q^r| r \) if \( z = 1 \).

Now we use the second column of the table in Lemma 27 to check whether \( uv[i, i] \in BS_{1,q} \) (given that \( uv[i-1, i-1] = \mathcal{G}_{1,q} (g, k) \)). Again we need only constantly many
$\text{TC}^0$-operations (note that also the checks for equality and $\in \mathbb{Z}$ are in $\text{TC}^0$ by Lemma 14 and Corollary 21 b)).

During this whole process we can use separate reduced power circuits for each $i$. We discard them after the checking is done. Hence, we do not need to consider the size of these circuits and this checking step is possible in $\text{TC}^0$.

This gives us Boolean values indicating whether $uv[i-1, i-1] \in \mathcal{B}_1$ implies $uv[i, i] \in \mathcal{B}_1$. Now, we only have to find the maximal $i_0$ such that for all $j \leq i_0$ this implication is true. Since $uv[i-1, -1] = 1 \in \mathcal{B}_1$, it follows inductively that $uv[i, i] \in \mathcal{B}_1$ for all $i \leq i_0$. Moreover, as the implication $uv[i_0, i_0] \in \mathcal{B}_1 \implies uv[i_0+1, i_0+1] \in \mathcal{B}_1$ fails, we have $uv[j, j] \notin \mathcal{B}_1$ for $j \geq i_0+1$.

Now let $uv[i_0, i_0] = a_i, (g, k) \in \mathcal{B}_1$. To obtain the power circuit representation of $(g, k)$, here called $\mathcal{R}$, we proceed as we did for $uv[i-1, i-1]$ during the checking step above (we apply Lemma 27 maybe followed by Lemma 28 maybe followed by Lemma 27). Observe that each application of one of the lemmas uses different $r_\theta, m_\theta, s_\theta, n_\theta$ and in the sum in Lemma 28 none of them appears twice — if we ignore the exponents $s_\theta$ (note that we do not consider these exponents for the estimation on $S(\mathcal{R})$).

We have seen that by Lemma 28 and Lemma 27 we need a constant number of $\text{TC}^0$ steps (from Section 3.1) to calculate the power circuit representation $\mathcal{R}$ of $(g, k)$. In the following capital letters refer to the power circuit representations of the respective lower case letters. By Corollary 21, Lemma 18 and Lemma 19 we have that

$$S(\mathcal{R}) \leq \sum_{\theta=0}^{i_0} (S(R_\theta) + S(M_\theta) + S(S_\theta) + S(N_\theta)). \tag{7}$$

Now, the final output is

$$(r_h, m_h)\tilde{\beta}_h \cdots (r_{i_0+2}, m_{i_0+2})\tilde{\beta}_{i_0+2} (s, m) \tilde{\beta}_{i_0+2}(s_{i_0+2}, n_{i_0+2}) \cdots \tilde{\beta}_k(s_k, n_k)$$

with $(s, m) = (r_{i_0+1} + q^{m_{i_0+1}} \cdot (g + q^k \cdot s_{i_0+1}), m_{i_0+1} + n_{i_0+1} + k)$. Let $\mathcal{W}$ denote the power circuit representation we obtain. By (7) it follows that

$$S(\mathcal{W}) \leq \sum_{\theta=0}^{\max(h, \ell)} (S(R_\theta) + S(M_\theta) + S(S_\theta) + S(N_\theta)) = S(\mathcal{U}) + S(\mathcal{V}) \tag{8}$$

with $r_\theta = m_\theta = 0$ for $\theta > h$ and $s_\theta = n_\theta = 0$ for $\theta > \ell$.

We have shown that the red-PC rep. $\mathcal{W}$ of a Britton-reduced word $w = a_i, g$ over a reduced power circuit $(G', \delta')$ can be constructed using constantly many operations described in Corollary 21 and the lemmas before in Section 3.1. As already mentioned above these operations can be applied independently in parallel to several (bunches of) markings. So we can apply the above process to the words $u^{(i)} v^{(i)}$ for $1 \leq i \leq \nu$ independently in parallel, while when applying an operation to several markings at the same time we make sure that it is the same operation for all markings.

We still need to prove the size conditions. By (8) we have $\sum_{i=1}^\nu S(\mathcal{W}^{(i)}) \leq S$. By Corollary 21, Lemma 18 and Lemma 19 we see that using one of these operations we add at most $(c + \log_{|q|} n) \cdot (|C_\ell| + S)$ new nodes and $c \cdot S$ new chains. Like in the proof of Corollary 21 we obtain the size conditions in Lemma 30, $\log_{|q|} n \leq \log n$. \hfill $\square$
Theorem 31. There exists a constant c such that the following is in $\text{TC}^1$:

Input: A red-PC rep. $W$ for a word $w \in \Delta^*$ over $(\Gamma, \delta)$ with $n = |w|$.
Output: A red-PC rep. $W'$ for a Britton-reduced word $w_{\text{red}} \in \Delta^*$ over $(\Gamma', \delta')$ such that $w_{\text{red}} = g_{1,q} w$ and $|\Gamma'| \leq |\Gamma| + c \cdot n \cdot \log(n)^3 \cdot |\Gamma|$.

Note that the size of the power circuits in Theorem 31 is close to the optimum $O(n)$ (for $|\Gamma| = 1$) for the sequential algorithm in [8] and much better than the rough polynomial bound in [23].

Recall that $\Sigma = \{1, a, a^{-1}, t, t^{-1}, b, b^{-1}\}$.

Corollary 32. There exists a constant c such that the following is in $\text{TC}^1$:

Input: A word $w \in \Sigma^*$.
Output: A red-PC rep. $W$ over $(\Gamma, \delta)$ for a Britton-reduced word $w_{\text{red}} \in \Delta^*$ such that $w_{\text{red}} = g_{1,q} w$ and $|\Gamma| \leq c \cdot |w| \cdot \log(|w|)^3$.

Proof. Let $|w| = n$ and $w = w_1 \cdots w_n$ with $w_i \in \Sigma$. Let $(\Gamma, \delta)$ be the reduced power circuit consisting of exactly one node. We construct a red-PC rep. $W = ((B_i, U_i, E_i, M_i))_{i \in [1..n]}$ of $w$ in a straightforward way: If $w_j = \beta$ with $\beta \in \{b, b^{-1}\}$, we set $B_i = \beta$ and $U_i = E_i = M_i = 0$. In the other cases, $B_i = \emptyset$ and $U_i, E_i, M_i$ are as follows: if $w_j = a^\alpha$ for $\alpha \in \{-1, 1\}$, then $\varepsilon(U_i) = \alpha$ and $\varepsilon(E_i) = \varepsilon(M_i) = 0$. If $w_j = t^\alpha$, then $\varepsilon(U_i) = \varepsilon(E_i) = 0$ and $\varepsilon(M_i) = \alpha$. The corollary follows using Theorem 31.

The following corollary also proves Theorem A from the introduction.

Corollary 33. (a) The word problem in $G_{1,q}$ and the subgroup membership problem for $BS_{1,q} \in G_{1,q}$ (given $w \in \Sigma^*$, decide whether $w$ presents some element in $BS_{1,q}$) are in $\text{TC}^1$.
(b) The word problem in $G_{1,q}$ and the subgroup membership problem for $BS_{1,q} \in G_{1,q}$ with the input word given as a red-PC rep. are in $\text{TC}^1$.

Proof. Theorem 31 and Corollary 32. For the subgroup membership problem note that by Britton’s Lemma $w$ represents an element of $BS_{1,q}$ if and only if the Britton reduction of $w$ consists of a single $(r, m) \in BS_{1,q}$. □

In [23, Corollary 39] we showed that the subgroup membership problem for $BS_{1,2}$ in $G_{1,2}$ with power circuit input is P-complete. The proof from [23] easily generalizes to $q \geq 2$. The main difference between Corollary 33 and [23, Corollary 39] is that here we only allow reduced power circuit representations as input while in [23] there is no restriction to the power circuits.

Proof (Proof of Theorem 31). The general idea of the proof is to proceed in a tree-shape manner: first we Britton-reduce all factors of length two, then all factors of length four of $w$ and so on using Lemma 30. Thus, after $\log n$ rounds we will end up with a red-PC rep. for a Britton-reduced word.

W.l.o.g. we assume that $n$ is a power of two. We can write $w = w_1 \cdots w_n$ with $w_i \in \{b, b^{-1}\}$ or $w_i = (r, m) \in \mathbb{Z}[1/q] \times \mathbb{Z}$ and $W = ((B_i, U_i, E_i, M_i))_{i \in [1..n]}$. For each $i$ we define a red-PC rep. $W_i^{(0)}$ of length one over $(\Gamma^{(0)}, \delta^{(0)}) = (\Gamma, \delta)$ by $W_i^{(0)} = ((B_i, U_i, E_i, M_i))$. Observe that $S(W) = \sum_{i=1}^n S(W_i^{(0)}) \leq 2n \cdot |\Gamma|$. 

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We now assume that we already have constructed red-PC rep. $W_j^{(k)}$ of Britton-reduced words $w_j^{(k)}$ for $j \in [1..n/2^k]$ all over $(\Gamma^{(k)}, \delta^{(k)})$ such that $\sum_{j=1}^{n/2^k} S(W_j^{(k)}) \leq S(W)$. By Lemma 30 we can construct in $\text{TC}^0$ red-PC rep. $W_j^{(k+1)}$ of Britton-reduced words $w_j^{(k+1)} = g_{1,q}^i w_j^{(k)}$ all over the same power circuit $(\Gamma^{(k+1)}, \delta^{(k+1)})$, with $\sum_{j=1}^{n/2^{k+1}} S(W_j^{(k+1)}) \leq \sum_{j=1}^{n/2^k} S(W_j^{(k)}) \leq S(W)$. Regarding the size constraints, by Lemma 30 there is a constant $c$ such that

$$|C_{\Gamma^{(k)}}| \leq |C_{\Gamma^{(k-1)}}| + c \cdot S(W) \leq \cdots \leq |C_{\Gamma^{(0)}}| + k \cdot c \cdot S(W)$$

and

$$|\Gamma^{(k)}| \leq |\Gamma^{(k-1)}| + c \cdot \log(n) \cdot (|C_{\Gamma^{(k-1)}}| + S(W)) \leq \cdots$$

where the last inequality is due to the fact that $S(W) \leq 2n \cdot |\Gamma|$. Because we only construct power circuits $(\Gamma^{(k)}, \delta^{(k)})$ for $k \leq \log(n)$, we know that for some large enough $c'$ we have

$$|\Gamma^{(k)}| \leq |\Gamma| + c' \cdot n \cdot \log(n)^3 \cdot |\Gamma|$$

for all occurring power circuits $(\Gamma^{(k)}, \delta^{(k)})$. This, firstly, shows the bound on $|\Gamma|$ stated in the theorem and, secondly, establishes that the inputs of all subsequent stages of the $\text{TC}^0$-circuit from Lemma 30 are of polynomial size. Therefore, as the whole algorithm consists of $\log(n)$ many $\text{TC}^0$-steps, it is in $\text{TC}^1$.

4.4 Conjugacy

Let $u \in \Delta_q^*$. A word $v \in \Delta_q^*$ is called a cyclic permutation of $u$ if we can write $u = xy$ and $v = yx$ for some $x, y \in \Delta_q^*$. A word $u \in \Delta_q^*$ is called cyclically Britton-reduced if all its cyclic permutations are Britton reduced (equivalently, if $u \in BS_{1,q}$ or $uu$ is Britton-reduced). For $g, h \in G_{1,q}$ and $A \subseteq G_{1,q}$ we write $g \sim_A h$ if they are conjugate by some element of $A$, i.e., if there exists some $z \in A$ with $g = g_{1,q}^z h z$.

**Lemma 34.** There exists a constant $c$ such that the following is in $\text{TC}^0$:

**Input:** A red-PC rep. $W$ of a Britton-reduced word $w \in \Delta_q^*$ over $(\Gamma, \delta)$

**Output:** A red-PC rep. $W'$ of a cyclically Britton-reduced word $w_{red}$ over $(\Gamma', \delta')$ with $w_{red} \sim_{G_{1,q}} w$ and $|\Gamma'| \leq |\Gamma| + c \cdot |w| \cdot \log(|w|) \cdot |\Gamma|$.

Note that using Theorem 31 and Lemma 34, we can compute on input of an arbitrary red-PC rep. $W$ a cyclically Britton-reduced word $w_{red}$ in $\text{TC}^1$.

**Proof.** It is a standard fact from group theory (see e.g. [33, Lemma 25]), that we can compute $w_{red}$ by performing one cyclic permutation to $w$ (cutting right through the middle) and applying Britton reductions. Thus, we write $w = w_0 w_1$ with $0 \leq |w_0| - |w_1| \leq 1$ and apply Lemma 30 to the corresponding red-PC rep. Thus, the lemma follows. $\square$
Lemma 35 (Collins' Lemma for $G_{1,q}$ [20, Theorem IV.2.5]). Let

$$u = \beta_1(r_1, m_1) \cdots \beta_h(r_h, m_h), \quad v = \tilde{\beta}_1(s_1, n_1) \cdots \tilde{\beta}_l(s_l, n_l)$$

be cyclically Britton-reduced with $h, l \geq 1$. Then $u \sim_{G_{1,q}} v$ if and only if $h = l$ and there is a cyclic permutation $v'$ of $v$ and $u \sim_{(t_i \cup s_i)} v'$.

Note that, by Britton's Lemma, $u \sim_{(t_i \cup s_i)} v$ implies that $\beta_i = \tilde{\beta}_i$ for all $i \in \{1, \ldots, \ell\}$. Thus, if $u \sim_{G_{1,q}} v$, then $\beta_1 \cdots \beta_h$ is a cyclic permutation of $\tilde{\beta}_1 \cdots \tilde{\beta}_l$.

Corollary 36. The following is in $\text{TC}^1$:

**Input:** Words $u, v \in \Sigma^*$.

**Output:** Is $u$ conjugate to some element in $\text{BS}_{1,q}$? If no, is $u \sim_{G_{1,q}} v$?

In particular, the conjugacy problem for $G_{1,q}$ is strongly generically in $\text{TC}^1$.

**Proof.** By Corollary 32, we can compute red-PC rep. representations for Britton-reduced $\hat{u}, \hat{v}$ with $\hat{u} = G_{1,q} u$ and $\hat{v} = G_{1,q} v$ in $\text{TC}^1$. Moreover, by Lemma 34 we can Britton-reduce them cyclically yielding $\hat{u}$ and $\hat{v}$. Now, by Collins' Lemma, $u$ is conjugate to some element of $\text{BS}_{1,q}$ if and only if $\hat{u}$ is a single letter from $\text{BS}_{1,q}$ — which can be easily tested in $\text{TC}^1$. Thus, in the following, we can assume that $u$ cannot be conjugated into $\text{BS}_{1,q}$.

Now, we follow [9, Theorem 3]. Its proof shows how to apply Collins' Lemma (note that the proof is only for $G_{1,2}$; however, the generalization to $G_{1,q}$ is a verbatim repetition with 2 replaced by $q$): Let $u, v$ be as in Lemma 35. After possibly replacing them by $u^{-1}$ and $v^{-1}$ or a cyclic permutation of either one, we may assume that the first letters of $u$ and $v$ are both $b^{-1}$ (otherwise, they are certainly not conjugate). Thus, we can write

$$u = \beta_1(r_1, m_1) \cdots \beta_h(r_h, m_h), \quad v = \tilde{\beta}_1(s_1, n_1) \cdots \tilde{\beta}_l(s_l, n_l) \quad (9)$$

with $\beta_1 = b^{-1}$ and we have

(I) if $l = 1$ (i.e., $u = b^{-1}(r, m)$ and $v = b^{-1}(s, n)$), then $u \sim_{G_{1,q}} v$ if and only if $q^n - m = q^s - q^n(m - m)$.

(II) If $l \geq 2$ and $\beta_2 = b$, let $e, f \in \mathbb{Z}$ such that $q^e r_1$ and $q^f s_1$ are integers not divisible by $q$ (note that $r_1 \neq 0 \neq s_1$ as otherwise $u$ or $v$ would not be Britton-reduced). Then $u \sim_{(t_i \cup s_i)} v$ if and only if $a^e u a^{-e} = G_{1,q} a^f v a^{-f}$.

(III) If $l \geq 2$ and $\beta_2 = b^{-1}$, then $u \sim_{(t_i \cup s_i)} v$ if and only if $a^{-m_1} u a^{m_1} = G_{1,q}$.

We treat all cyclic permutations in parallel. As outlined above, we can assume that $\hat{u}$ and $\hat{v}$ are written as in (9) and we need to check whether one of the conditions (I), (II) or (III) holds. Condition (I) can be checked in $\text{TC}^0$ by Corollary 21. The numbers $e, f$ in condition (II) can be directly read from the floating-point representation of $r$ and $s$. After that the condition reduces to one instance of the word problem (with power circuit representations as input), which is in $\text{TC}^1$ by Theorem 31. Likewise, (III) can be checked in $\text{TC}^1$. By [9, Theorem 4] the set of words $u \in \Delta_q^*$ representing elements of $G_{1,q} \setminus \text{BS}_{1,q}$ is strongly generic. Hence, the second part of the corollary follows. □

Proposition 37 ([9, Equation (5), Proposition 5 and 6]).

1. Let $m \geq 1$. Then $(r, m) \sim_{\text{BS}_{1,q}} (s, n)$ if and only if $m = n$ and there is some $k \in [0..m - 1]$ such that $r \cdot q^k \equiv s \mod (q^m - 1)$. 

2. Let $m \geq 1$. Then $(r, m) \sim_{\text{BS}_{1,q}} (s, n)$ if and only if $m = n$ and there is some $k \in [0..m - 1]$ such that $r \cdot q^k \equiv s \mod (q^m - 1)$.
2. Let \( r, m \in \mathbb{Z}, m \neq 0 \). If \((r, m) \not\sim_{BS_{1,q}} (0, m)\), then \((r, m) \sim_{G_{1,q}} (s, n)\) if and only if \((r, m) \sim_{BS_{1,q}} (s, n)\).

3. Let \( m, n \in \mathbb{Z}\). Then \((0, m) \sim_{G_{1,q}} (0, n)\) if and only if \((m, 0) \sim_{BS_{1,q}} (0, 0)\) if and only if \(\exists k \in \mathbb{Z}: m = q^k n\).

**Proposition 38.** For every fixed \( g \in G_{1,q}\) the following is in \( TC^1\):

Input: A word \( w \in \Sigma^*\).

Question: Is \( g \sim_{G_{1,q}} w\)?

**Proof (of Proposition 38).** By Corollary 36, we only need to consider the case that \( g = (r, m) \in BS_{1,q}\). By conjugating with some power of \((0, 1) \in BS_{1,q}\), we may assume that \( r \in \mathbb{Z}\) and \( q\) does not divide \( r\). Moreover, using Corollary 32 and Lemma 34 we can Britton-reduce the input word \( w \) leading to a red-PC rep. for \((s, n) \in BS_{1,q}\) if the cyclic reduction of \( w \) is not in \( BS_{1,q}\), we know that \( w \) is not conjugate to \( g\). Again by conjugating with some power of \((0, 1)\), we may assume that \( s \in \mathbb{Z}\) and \( q\) does not divide \( s\) (indeed, if \((U, E)\) is the power circuit representation of \( s\), we only need to set \( E = 0\)).

Now, first assume that \( m \neq 0 \) and \((r, m) \not\sim_{BS_{1,q}} (0, m)\). By replacing \( g \) and \((s, n)\) by their inverses if necessary, we may assume \( m \geq 1\). Hence, by [9, Proposition 5], we have \((r, m) \sim_{G_{1,q}} (s, n)\) if and only if \((r, m) \sim_{BS_{1,q}} (s, n)\). By [9, Equation (5)], we know that the latter holds if and only if \( m = n \) and there is some \( k \in [0..m - 1] \) such that \( r \cdot q^k \equiv s \mod (q^m - 1)\). Now, as \((q^m - 1)\) is a constant, we can compute \( r \mod (q^m - 1) \) and \( s \mod (q^m - 1)\) by Proposition 22. As there are only a constant number of possibilities for \( k\) we can check for all of them whether \( r \cdot q^k \equiv s \mod (q^m - 1)\).

If \((r, m) \sim_{G_{1,q}} (0, m)\), we also know that \((r, m) \sim_{G_{1,q}} (m, 0)\) (as \( b^{-1}(0, m)b = (m, 0)\)). Thus, it only remains to consider the case \( g = (r, 0)\) with \( r \in \mathbb{Z}\) (i.e., \( m = 0\)). In this case [9, Proposition 6] tells us that \((s, 0) \sim_{G_{1,q}} (r, 0) \iff \exists k \in \mathbb{Z}: s = q^k r\).

Clearly given a power circuit representation for \( s\), we can check this condition.

Thus, it remains to test whether either \( n = 0 \) or \((s, n) \not\sim_{G_{1,q}} (0, n)\) (as otherwise, it is straightforward to see that \((s, n) \not\sim_{G_{1,q}} (r, 0)\) for any \( r\)). Checking whether \( n = 0\) can be done by Lemma 14. For the latter test we proceed as follows:

After possibly replacing \((s, n)\) by \((s, n)^{-1}\) and \((r, m)\) by \((r, m)^{-1}\), we can assume that \( n > 0\). Now, because of [9, Proposition 6] we know that, if \( n \neq 0\) and \((s, n) \sim_{G_{1,q}} (r, 0)\), then \( n = q^k r\) for some \( k \in \mathbb{Z}\) (be aware that compared to the above notation we flipped the coordinates \( s \) and \( n\)); moreover, we can compute this \( k\) if it exists (Lemma 19).

By our assumption \( n = q^k r > 0\), we can use [9, Equation (5)] stating that \((s, q^k r) \sim_{G_{1,q}} (0, q^k r)\) if and only if there is some \( x \in [0..q^k r - 1]\) with \( s \cdot q^x \equiv 0 \mod q^{q^k r - 1}\). Recall that \( s \in \mathbb{Z}\) and \( q\) does not divide \( s\). As also \( q\) does not divide \( q^{q^k r - 1} \in \mathbb{Z}\), the only way for satisfying \( s \cdot q^x \equiv 0 \mod q^{q^k r - 1}\) is if it is already satisfied for \( x = 0\). Thus, we need to test whether \( s \equiv 0 \mod q^{q^k r - 1}\). As \( s\) is given as a compact marking \( U\) with \( \varepsilon(U) = s\), it is sufficient to compute \( \varepsilon(P) \mod q^{q^k r - 1}\) for all \( P \in \sigma(U)\).

Now, observe that \( q^k \mod q^{q^k r - 1} = q^k \mod q^{q^k r}\); thus, it suffices to compute \( \varepsilon(L) \mod q^{q^k r}\) for some arbitrary marking \( L\) on our power circuit. This can be done by Lemma 24. □
Conclusion. We have shown the word problem of generalized Baumslag groups $G_{1,q}$ to be in $TC^1$. The same complexity applies to the conjugacy problem for elements outside $BS_{1,q}$ or if one element is fixed. $TC^1$ seems to be the best possible using the current approach of tree-like Britton reductions. We conclude with some open questions: What is the “actual” complexity of the word problem of $G_{1,q}$? Are there any better lower bounds other than that it contains a non-abelian free subgroup? Can our methods be generalized to the Higman group $H_4$? This is closely related to the growth of its Dehn function, which, to the best of our knowledge, is not known to be in $\tau(O(\log n))$ like for the Baumslag group.

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