Astronomical redshifts and the expansion of space

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ABSTRACT
In homogeneous cosmological models, the wavelength $\lambda$ of a photon exchanged between two fundamental observers changes in proportion to expansion of the space $D$ between them, so $\Delta \log (\lambda/D) = 0$. This is exactly the same as for a pair of observers receding from each other in flat space–time where the effect is purely kinematic. The interpretation of this has been the subject of considerable debate, and it has been suggested that all redshifts are a relative velocity effect, raising the question of whether the wavelength always stretches in proportion to the emitter-receiver separation. Here, we show that, for low redshift at least, $\Delta \log (\lambda/D)$ vanishes for a photon exchanged between any two freely falling observers in a spatially constant tidal field, because such a field stretches wavelengths and the space between the observers identically. But in general there is a non-kinematic, and essentially gravitational, component of the redshift that is given by a weighted average of the gradient of the tidal field along the photon path. While the redshift can always be formally expressed using the Doppler formula, in situations where the gravitational redshift dominates, the ‘relative velocity’ is typically quite different from the rate of change of $D$ and it is misleading to think of the redshift as being a velocity or ‘kinematic’ effect.

Key words: galaxies: distances and redshifts – cosmology: theory.

1 INTRODUCTION

1.1 Overview, goals and outline
In spatially homogeneous and isotropic Friedmann, Robertson & Walker (FRW) cosmological models, the wavelength $\lambda$ of a photon exchanged between any two fundamental (or ‘comoving’) observers (FOs) changes in proportion to the change in the scale factor $a(t)$ or, equivalently, in proportion to the change in the proper separation $D$ of the observers, which for concreteness we take to be the integrated distance along the geodesic of the 3-space of constant proper time since the big bang that connects them, so

$$\Delta \log (\lambda/D) = 0.$$  \hspace{1cm} (1)

This ‘cosmic wavelength stretching relation’ also applies for FOs in homogeneous but anisotropic models. This is a well-known and familiar result. The stretching is sometimes described as being caused by the expansion of space, and can also be readily understood in terms of standing waves in an expanding cavity. But is also perhaps a little surprising since it has been known since Bondi (1947) that the redshift between FOs is, for low redshift at least, expressible as the product of a special relativistic (SR) Doppler shift and a gravitational redshift, these terms generating fractional wavelength shifts that are, respectively, linear and quadratic in distance (see Peacock 2008). Yet the stretching relation (1) is just the same as for a pair of observers receding from one another in flat space–time; the effect of gravity does not appear.

Subsequently, Synge (1960) showed that any redshift is expressible as a Doppler shift, with the relative velocity defined in terms of parallel transport of the emitter’s 4-velocity. This provides an elegant and unified way to view all redshifts. Significantly, Synge says that redshifts should not be considered to be a gravitational effect as the curvature tensor does not appear in the formulae.

This view has been revived by Narlikar (1994) and, more recently, by Bunn & Hogg (2009) who argue that all redshifts are Doppler, or ‘kinematic’, in nature, because, in their view, a gravitational redshift is just a Doppler shift viewed from an unnatural coordinate system.

Our interest in this was piqued by recent measurements of gravitational redshifts from clusters of galaxies (Wojtak, Hansen & Hjorth 2011). They measured the potential well depth difference between low specific energy brightest cluster galaxies (BCGs) and the general cluster population. Theoretical studies (e.g. Cappi 1995) considered the redshift $\Delta \lambda/\lambda = \Delta \phi/c^2$ that would be seen by static non-inertial observers, with $\Delta \phi$ the Newtonian gravitational potential difference. But the sources and observer are really in free fall. Does this make any difference? Obviously the main effect of this is to add a very large first-order Doppler effect from the relative motion, which has to be carefully averaged out to reveal the gravitational effect. Another complication is that one will see transverse Doppler and other kinematic effects that are generally of the same order as the static gravitational redshift (Kaiser 2013; Zhao, 2008), yet the stretching relation (1) is just the same as for a pair of observers receding from one another in flat space–time; the effect of gravity does not appear.

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Peacock & Li 2013). But imagine, if you will, a freely falling source and observer pair that both reside in the potential well of a relaxed static massive cluster and whose separation is known to have been the same at the times of emission and reception. What redshift would they see?

It would seem natural to interpret the kinematic view as saying that there would not be any redshift. After all this would be the case for a pair of FOs inside a small part of a closed FRW model at its phase of maximum expansion. Is this fundamentally any different from the observers in the cluster?

Another possibility would be to argue that they would not see any first-order potential difference $\Delta \phi \sim -D \cdot g$ from any spatially constant component of $g$ by virtue of the equivalence principle, but perhaps one might expect them to perceive any variation of the gravity, i.e. to lowest order at small separation the local tidal field. But there are some problems with this view. One is that there is a non-vanishing tide in the closed FRW example cited above, yet apparently this does not give rise to a redshift. Also, in this picture, any observer in a smooth cluster potential may consider itself to be at the bottom of a locally parabolic (though possibly tri-axial) potential well. This is certainly legitimate in the sense that, if that observer were to release a test particle at rest nearby, it would indeed see that particle accelerate slowly towards it. That might lead one to think that all observers would see an average blueshift for photons from neighbour particles that are, on average, not moving away or approaching the chosen observer.

But both of these might seem to conflict with the view – arguably the prevailing one in the field of cluster gravitational redshifts – that if the emitters are BCGs on low energy orbits close to the centre of the cluster and the receivers are galaxies on radial orbits much further out then they actually would see the full potential difference $\Delta \phi$ including any first order $\sim -D \cdot g$ effect as well as the higher order terms.

The goal of this paper is to try to see how the seemingly different, and perhaps conflicting, views described above can all be reconciled in a coherent way and to develop a consistent picture for how to think about redshifts in general, and particularly gravitational redshifts, in a way that avoids misconceptions or pitfalls.

The outline of the paper is as follows. In the rest of this Introduction, we review the history of how the homogeneous cosmology wavelength stretching has been interpreted and also review the extension to inhomogeneous situations. As our results are closely related to the proposal of Bunn & Hogg, we review their arguments in some detail. In Section 2, we perform a simple calculation of the domain of validity of the stretching relation by directly calculating the change in wavelength and proper separation of observers in arbitrary gravitation fields, though the calculation is limited to weak fields (i.e. low redshifts). We conclude with a summary and discussion, including an illustrative example.

1.2 Different interpretations of the redshift

In many popular accounts the relation $\Delta \log (\lambda / D) = 0$ is described as a causal effect of the expansion of space stretching the wavelength of light. Aside from giving the right answer, this finds support in the fact that Maxwell’s equations written in the comoving coordinates that are most natural in an expanding universe have an extra ‘damping’ or ‘friction’ term that gives a reduction of comoving energy density – consistent with occupation numbers being adiabatically conserved while wavelengths get stretched (reducing the energy per photon) – and that these equations admit e.g. standing waves where the separation of nodes grows with the expansion factor. In this picture, the redshifting of all radiation as $\lambda \propto a(t)$ appears inescapable and it is often stated as if self-evident that if the scale factor doubles the wavelength of light must double too (e.g. Harrison 2000; Lineweaver & Davis 2005).

These damping standing wave solutions are a good way to think about the cosmic microwave background, and this line of argument has the virtue of giving the right answer for photons exchanged between FOs also. But the ‘expanding-space’ paradigm has been criticized, mostly on the grounds that it obscures the central tenet of general relativity (GR), which is that space–time is locally flat, or Minkowskian (e.g. Whiting 2004; Chodorowski 2007, 2011; Peacock 2008; Bunn & Hogg 2009), so it does not define any local state of expansion. Writing Maxwell’s equations in expanding coordinates does not change the physics, or the solutions, and nothing in the physical laws says that wavelengths should increase. This problem is seen most starkly in matter-free FRW models such as in de Sitter space–time and the Milne model in Minkowski space–time which admit solutions for expanding (or contracting) families of FOs defining different expansion histories $a(t)$ – possibly overlapping in the same region of space–time – and also admit solutions to Maxwell’s equations that correspond to either redshifting or blueshifting radiation fields. In these solutions, the expansion rate is defined by the radiation itself; an observer who is in the zero momentum density frame at some point will see a Poynting flux that increases linearly with distance, and other observers who locally see zero momentum flux obey Hubble’s law. This is all ultimately determined by the initial condition for the field. Additionally, it may not be completely obvious how e.g. a classical wave packet being emitted and received by localized observers relates to an unbounded, perhaps infinite, standing wave.

The most satisfactory explanation of the wavelength–separation relation in FRW models is that of Peebles (1971). He argued that the overall wavelength ratio for a widely separated pair of FOs is the product of the ratios for a set of intervening exchanges between neighbouring FOs along the photon path, each of which are, to first order, Doppler effects because of the local flatness of space–time. This is clearly ‘GR compliant’ and leads to the differential equation $\delta \lambda / \lambda = da / a$ with solution $\lambda \propto a(t)$. In this picture, the change in wavelength can be thought of as an unchanging photon being seen by a succession of observers with different frames of reference.

While the interpretation of the cosmic wavelength stretching has been contentious, no one doubts that it is obeyed in homogeneous models. The question we shall ask here is whether this is of broader validity and whether it is valid in an inhomogeneous universe. It is not clear how the expanding space paradigm can be extended to inhomogeneous gravitating systems. One might try to make progress by modelling single-flow regions (if such exist) as being locally like anisotropic homogeneous models, but since galaxies and dark matter may, in general, have multiple streaming velocities at the same point in space it is obvious that one cannot even think about the local space in a bound system like a cluster of galaxies as having any state of expansion. But the concept of the space between any pair of observers or galaxies changing with time is still perfectly valid, and so one can ask, for instance, whether (1) still applies.

Relaxing the requirement of homogeneity, Bondi (1947) showed that, for small redshifts at least and in matter dominated spherically symmetric models, the redshift can be expressed as the product of a Doppler shift and a gravitational redshift (see Peacock 2008). This might be taken to suggest that there is something fundamentally different about inhomogeneous models. But this applies also to the homogeneous case, and is not inconsistent with the wavelength obeying the ‘stretching law’ (1). The resolution of the apparent
difference is that Bondi’s velocity is that of the receiver relative to the emitter at the time of reception, with the emitter being stationary at the centre of the expanding sphere of matter on whose surface the receiver resides (Peacock 2008). This is not what determines the change in the proper separation, which is some average of the observers relative velocity over the light travel time and it is not hard to show that allowing for the change of this velocity is equivalent to including the extra gravitational redshift factor.

Relaxing the requirement of spherical symmetry and matter domination, Synge (1960) has shown that the redshift is quite generally given by the usual SR Doppler formula alone, with the ‘relative velocity’ being defined in terms of parallel transport of the emitter 4-velocity along the null ray from the emission event to the reception event. The energy perceived by an observer (emitter or receiver) is the dot product of their 4-velocity $U$ with the photon 4-momentum $P$. These observed energies transform as scalars, and so can be directly compared, but this comparison can also be made by parallel transporting the emitter velocity and photon momentum from the emitter to the receiver (along the photon’s null path) and then comparing. But transporting the photon along its path does not change it, so the result, with $S$ and $O$ denoting source and observer, is

$$\lambda_0 = \frac{U_S \cdot P_S}{U_O \cdot P_O} = \frac{\tilde{U}_S \cdot P_O}{U_O \cdot P_O}$$

(2)

where $\tilde{U}_S$ is the parallel transported source 4-velocity. The last expression above is the usual SR Doppler effect (since all of the 4-vectors are defined at the point of reception and at this point, as at any other space–time is locally flat).

This is an elegant formalism that nicely unifies all redshifts, be they Doppler, gravitational or cosmological (Narlikar 1994). However, in discussing this, Synge says that the spectral shift is a velocity effect, and not a gravitational effect, because the Riemann curvature tensor does not appear in (2). This is surprising, since the parallel transport $U_S \rightarrow \tilde{U}_S$ depends on the connection, which, while not the same as the curvature is closely related to it.

Regarding the physical interpretation of Synge’s velocity, Chodorowski (2011) showed that, in FRW models, the difference between parallel transporting from the emitter to the receiver along the photon’s null-path and parallel transporting along the geodesic lying in the 3-space of constant proper time (at the time of emission, say) is just Bondi’s gravitational component of the redshift. This can also be understood in the weak-field and non-relativistic observer limit, which is applicable within a limited region of an FRW model. Parallel transport in time alone is then a rotation of the 4-velocity $U$ at an angular frequency $g/c$, where $g$ is the Newtonian gravity, so for non-relativistic observers: $dU/\,dr = \{0, g/c\}U^0$. In FRW models, the gravity $g$ increases linearly with distance from any reference position, so $g = -r \, d^2\phi/\,dr^2$. The photon time of flight is $\Delta t = r/c$, so $\Delta \nu = (r/c) \, dU/\,dt = -(d^2\phi/\,dr^2) r^2/c$, which is quadratic in separation and proportional to the constant tidal field.

More recently, Bunn & Hogg (2009), in their stimulating critique of the expanding space picture, discuss redshifts in the more general context of non-homogenous universes. Like Peebles, they consider the overall wavelength ratio to be the product of incremental shifts between neighbouring observers along the photon path and invoke local flatness. They argue that:

(i) An observed frequency shift in any space–time can be interpreted as either a kinematic (Doppler) shift or a gravitational shift by imagining a suitable family of observers along the photon’s path.

(ii) It is more natural to consider them to be Doppler or kinematic in nature.

(iii) In situations where any relative velocities are $\ll c$, and curvature is small (over the distance and time-scales travelled by the photon) one should have no hesitation in applying the Doppler formula.

The latter two statements resonate with Synge’s, but the way that are reached is actually different. The logic used here is that, while it might seem most natural to consider the imaginary intervening observers to be freely falling, one could also imagine them to be accelerated observers, with rocket motors strapped to their legs perhaps. If they are each comoving with one of another family of freely falling observers at the moment the photon passes, the same incremental wavelength shifts would apply since all that matters for the redshift is the tangent to the 4-velocity at the events of emission and reception, any curvature of the world lines before or after the event being irrelevant.

They then invoke the ‘parable of the speeding ticket’ in which a motorist caught speeding by a cop with a radar gun tries to avoid the fine by arguing that in a different coordinate frame the car and the cop were not moving relative to each other (though perhaps a more precise analogy would be a cop measuring the frequency of radiation from a source of known frequency mounted on the car). As illustrated in Fig. 1, one can indeed imagine a rod being uniformly accelerated (by rocket motors, say) passing by with two observers riding on it, one being present at the emission event and comoving with the car and the other present at the reception instantaneously comoving with the cop. In the coordinate system of the accelerated rod, it is claimed, the emitter and receiver are not in a state of relative motion; so there is no Doppler shift, but there is now a ‘gravitational’ redshift $\Delta \nu/\nu = al/c^2$ where $a$ is the acceleration and $l$ is the length of the rod. Thus, they argue, an enlightened cosmologist would never try to characterize a redshift as a velocity or gravitational effect, because the two interpretations arise from different choices of coordinates.

This is a fascinating and ingenious argument, but probably not one that Synge would have approved of. As he was at pains to emphasize in the preface to his book, despite its name, GR is an absolute theory since whether or not there is a gravitational field in some region of space is unambiguously measurable from geodesic deviation of freely falling test particles (though the values of the components of the curvature tensor are coordinate system dependent). The curvature, or tidal field, is unaffected by the presence of any observers (real or imaginary) who might be accelerated by rockets.1 If the curvature vanishes in the region of space–time containing the observers and the photon path then whatever happens there can hardly be said to be a gravitational redshift.

Similarly, while the velocity of an object depends on the frame from which it is observed, the relative velocity of two objects in their centre of velocity frame is another absolute quantity. Accelerated observers know that they are being accelerated. Once they allow for

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1 Rindler (1977) gives an interesting argument, which he attributes to Dennis Sciama, that the weight of objects sensed by an accelerated observer in a rocket can be thought of, in a Machian sense, as gravity arising from the relative acceleration of the rest of the Universe. That argument cannot be applied here, since the acceleration of the imaginary intervening observers is determined by the arbitrary choice of their velocities; this is generally varying along the photon path and the gradient of this is not equal to the real tidal field.
For example, while it is widely believed that the dipole anisotropy of the microwave background is the result of our being accelerated by large-scale structure, it is possible that some of the dipole is generated by a large-scale specific entropy gradient (Gunn 1988), but this indeterminacy of the local value of $g$ has no effect on local dynamics within the Milky Way or within the local supercluster say.
There are many ways, in principle at least, to directly measure the relative velocity of a pair of observers that are independent of the redshift, at least at low redshift. One can use rate of change of parallaxes or light-echo time delays. It has been suggested that one can directly measure velocities on cosmological scales by the shrinking of the angular size of bound objects with time (Darling 2013). And one could, in principle, use rate of change of luminosity of standard candles. It may not be practically achievable, but one can imagine a very large rigid lattice populated with observers with clocks and rulers who record the rate of motions of observers flying past. This, after all, is how we usually imagine measuring geodesic deviation in order to determine the gravitational field. The question here is not whether this can be done in practice; only whether it is possible in principle. As we describe below, we are free to construct the lattice such that the velocities of the emitter and receiver relative to the lattice are equal and opposite. All of these concepts converge, at low redshift, to an unambiguous operational definition of relative velocity in the centre of velocity frame.\(^3\) This, of course, is the physical quantity that one would normally associate with the phrase ‘relative-velocity’ (rather than the relatively abstract definition in terms of the mathematical operation of parallel transport).

In the weak-field limit, and for stationary non-inertial observers, Synge’s relative velocity is \(\Delta v = \int \mathbf{d}r \cdot \mathbf{g}/c\). As discussed, in the context of the Pound–Rebka experiment, this is the same as the constant relative velocity of a pair of particles launched so as to be tangent at the interaction events. But that is a flat-space (constant gravity) phenomenon. More interesting is how the velocities are related when curvature cannot be neglected. To this end, consider test particles in dynamic equilibrium in a static potential well. Imagine a dark matter halo that generates a smooth bowl-shaped potential well and imagine two families of test particles; a ‘hot’ high energy population and a ‘cold’ low energy population that have relatively negligible velocities and are confined to the very centre of the halo. The conventional view is that photons emitted from a cold particle and received by a randomly chosen hot particle at larger radius will on average suffer the usual static gravitational redshift \(\langle \Delta \lambda \rangle/\lambda \simeq \Delta \phi/c^2\), and that photons emitted by a randomly chosen hot particle and received by a cold particle will be blueshifted by the same amount. This is the redshift which would be seen by a static non-inertial observer being supported against falling by stress in its supporting structure, but the hot population is similarly being supported against falling by the stress of its kinetic pressure so it should be essentially identical. Though as mentioned above, the redshift is not exactly the same because there will also be a transverse Doppler and other second-order kinematic effects that give corrections to the naive prediction that, by virtue of the virial theorem, is guaranteed to be of the same order of magnitude (though will in general depend on the details of the structure and velocity dispersion of the cluster).

In this example, we see the important, and general, distinction between gravitational and Doppler redshifts; that gravitational redshifts are generally asymmetric and change sign if we reverse the direction of light propagation, while Doppler shifts are symmetric; both observers perceive a shift of the same sign. This means that

\[ g(\Delta t)^2/2 \]

\[ c\Delta t \]

**Figure 2.** Illustration of situation described in text where a photon is emitted by a ‘cold’ particles near the centre of a smooth potential well and is received by a randomly chosen ‘hot’ particle at large distance. A sample of orbits from the distribution of velocities – here a simple box-car – is shown as curves. The orbit for the average velocity particle is shown as the heavy curve. The average radial velocity is zero at the time of reception, but the average velocity over the time of flight of the photon is \(-g\Delta t/2\) and, for a parabolic potential, this velocity, in units of \(c\), is equal to the gravitational redshift. It is tantalizing to think that a generalization of this reconciles the GR kinematic view of redshifts with more conventional view of gravitational redshifts. As we will see, however, this result only obtains for this specific, rather special, form for the potential and does not apply in the general case.

Synge’s parallel transport velocity difference here is, in an average sense, positive in the first case (outgoing photons) and negative in the latter, whereas randomly chosen hot particles have, of course, zero real average relative velocity.

On the face of it this makes the kinematic picture seem a bit nonsensical; how can the relative velocity of the two populations depend on who was emitting and who was receiving? But a moment’s reflection reveals that in the case of outgoing (ingoing) photons, the hot halo particles that interacted with the photons actually did have a positive (negative) average radial velocity during the time of flight of the photon, as illustrated in Fig. 2. Consider, for simplicity, a parabolic potential well; \(\phi = \alpha r^2\). The average velocity, over the time of flight for an outgoing photon, is \(\langle v \rangle = -g\Delta t/2 = \alpha r^2/c\), so \(v/c = (\Delta \lambda)/\lambda\), just the usual gravitational redshift. So Synge’s velocity can here be interpreted as the pair-wise velocity averaged over the light travel time. Moreover, the change in separation during the photon trip is \(\langle \Delta D \rangle = g(\Delta \lambda)^2/2\), so, since \(\Delta t = D/c\), this means that \(\langle \Delta D \rangle/D = \Delta \phi/c^2\). Thus, in this situation, where curvature clearly plays an important role, the redshifts can still be considered to be ‘kinematic’ in nature in the sense that equation (1) applies. The question is whether this is a general phenomenon for freely falling observers. If this were the case it would nicely reconcile the elegant relativistic kinematic picture with the more pedestrian view of photons losing energy climbing out of potential wells as well as with the much maligned ‘expanding space’ picture.

But while attractive, there are reasons to suspect that equation (1) is not, in fact, universal. One is that it does not even apply exactly for all pairs of observers in Minkowski space; though it is a very good approximation for non-relativistic observers as the corrections are of the order of \((v/c)^3\). More importantly, the result above was for a very special form of the potential and does not prove that this works out for more general potentials. And indeed, for more realistic models for cluster haloes, the result would conflict with the conventional view that, aside from complications arising from the transverse Doppler effect etc., the redshift is given by the usual static gravitational redshift.

\[^3\text{It is hard to understand the reluctance to consider such velocities as legitimate; if there are any problems with the concept of relative velocity of two observers at the same time (i.e. in their centre of velocity frame) they seem to us to pale into insignificance compared to the more fundamental problem of defining the difference of velocities at different times in the face of the unknown absolute value of } g.\]
The question of whether redshifts should generally be considered to be kinematic, in the sense of (1) (perhaps to an approximation with a precision growing in some calculable way as the limit \( v/c \to 0 \) and/or small space–time curvature is approached), is readily answered by calculation. In the next section, we compute the change in wavelength and change in separation of a pair of observers who are freely falling in a weak, but otherwise arbitrary, gravitational field. We work to a precision sufficient to describe gravitational redshifts (second order in velocity and first order in gravitational potential). Consequently, the results are not applicable at high redshift, but should be adequate to decide whether the stretching law applies in the limit of low redshifts and small curvature.

We find that, to the stated level of precision equation (1) applies for any emitter/receiver pair that are freely falling in a spatially constant tidal field. This is of slightly broader applicability than just the redshift for a pair of FOs in homogeneous model, as it applies also to non-fundamental observers (i.e. observers who may be moving with respect to the local frame of rest defined by the matter). But in the presence of any inhomogeneity – which implies spatially varying curvature – the relationship does not hold beyond first order in the relative velocity. This is because the change in separation involves the gravity at the end points – which is the line integral of the tide – but the wavelength change involves the integral of the gravity. Only for a constant tide are these equal. If the change in the tide or curvature is small over the photon path length then (1) may be a very good approximation, but in general, applying this gives errors that can be as large as the gravitational redshift. Indeed, in astronomical situations in which an unsophisticated astronomer or physicist would describe the redshift as essentially gravitational in nature, the ‘velocity’ that, in the Doppler formula, gives the redshift tends to be very different from the real velocity.

To show this it is sufficient to use simple special relativity and gravitational redshifts as Einstein would predict from Newtonian gravity. This is adequate for systems of relative velocity substantially less than \( c \). We also work in terms of physical velocities and positions, which, we believe, helps clarify what is going on.

2 ANALYSIS

To calculate changes in the wavelength of light exchanged, or the distance, between a pair of freely falling particles (an emitter ‘1’ and a receiver ‘2’), we imagine a family of non-inertial observers who lie at the grid-points of a non-rotating (as determined by gyroscopes) and non-expanding rigid lattice armed with clocks and rulers to determine rates of motion of observers in their vicinity and weighing scales to determine their acceleration \( \mathbf{a} \), which is just the reflex of the Newtonian gravity \( g \).

Alternatively one could imagine a fleet of rocketeers who adjust their thrusters to maintain a rigid, and non-rotating, spatial relationship with their neighbours and who report the value of the acceleration of test particles they release; this would provide the Newtonian gravity \( g(r) \) relative to one of their number – the ‘reference observer’ – who is arbitrarily chosen and who does not activate his or her thrusters. The absolute value of the acceleration is somewhat arbitrary, but differentiating \( g(r) \) gives, unambiguously, the tidal field.

In either case, if a pair of these non-inertial observers exchange a photon, the redshift is just \( \Delta \lambda/\lambda = -\int \mathbf{a} \cdot \mathbf{g}/c^2 \).

First consider a pair of observers in empty space. Let us work in the centre of velocity frame – the ‘lab’ frame – and, without loss of generality, let the observers be moving in the \( x, y \) plane with velocities along the \( x \)-axis \( \mathbf{v} = \pm \beta \hat{x} \) with \( \beta = |v_x - v_1|/2c \) at some distance \( \pm \Delta y/2 \) from the \( x \)-axis. Let them exchange a photon that, in the lab-frame has 4-momentum \( p = \{p_0, \mathbf{p} = \{p_0, p_x, p_y, 0\} \) and define the angle \( \theta = \tan^{-1}(p_y/p_x) \) (see Fig. 3). Boosting this into the frames of the observers gives \( p_0' = p_0\gamma(1 \pm \mu \beta) \) where \( \mu = \cos(\theta) \) and, as usual, \( \gamma = (1 - \beta^2)^{-1/2} \). Defining \( \lambda \) and \( \Delta \lambda \) such that \( \lambda_{em} = \lambda - \Delta \lambda/2 \) and \( \lambda_{rec} = \lambda + \Delta \lambda/2 \), then \( \Delta \lambda/\lambda = (\lambda_{rec} - \lambda_{em})/\lambda_{em} \), there would be additional second-order terms. The definition used here leads to a cleaner result, and is legitimate as we will define \( \Delta D \) in the same way below.

If we now switch on gravity, then the emitter and receiver velocities will become time dependent, and their paths, as well as those of the photons, will become very slightly bent. The rocketeer’s servo-controlled thrusters will fire, and they will sense their acceleration and will also start to perceive that their clocks start to slowly drift out of synchronicity with their neighbours (by an amount that they can correct for if they wished using their measurable non-gravitational acceleration). The redshift is now the product of three terms; local Lorentz boosts to or from the emitter and receiver’s frame into the frame of the local rocketeer and then a static gravitational redshift as the photon propagates between the two rocketeers. The Lorentz boosts will now no longer be perfectly symmetric, but the difference between the Lorentz \( \gamma \) factors is \( \sim \delta \gamma \sim \delta v/c^2 \), but with \( \delta v \sim g\Delta t \sim \phi/c \) this is of third order in \( v/c \) and so we can ignore it. Similarly, any bending angles are of the order of the potential divided by \( c^2 \), so any change to the fractional wavelength shift which is already of first order in \( v/c \) is also third order and can be neglected, as can any corrections from the drifting of the clock synchronization. The only effects that appear at our stated precision goal are the change in the first-order Doppler effect and the static

Figure 3. Dashed lines are paths of emitter and receiver, in flat space–time and in the centre of velocity frame, who exchange a photon (heavy line) that travels in the \( x, y \) plane at angle \( \theta \) to the \( x \)-axis (which we can chose to be aligned with the velocity difference) from point \( E = r_1 \) to point \( R = r_2 \). Arrows indicate the velocity (\( \beta = v/c \)). The location of the emitter and receiver at the times of reception and emission, respectively, are \( E' \) and \( R' \). Distances between the observers in centre of velocity frame at times of emission (\( D_E = ER' \)) and reception (\( D_R = RE' \)) are indicated by thin lines and are given by \( \Delta y \sqrt{1 + \beta^2} \pm \mu \beta \sin(\theta) \), with \( \mu \equiv \cos(\theta) \).
gravitational redshift:

$$\frac{\Delta \lambda}{\lambda} = n \cdot (v_2(t_2) - v_1(t_1))/c - \int_{t_1}^{t_2} \frac{dr \cdot g(r)}{c^2}$$

$$= n \cdot (v_2 - v_1)_{t_1}/c + \int_{t_1}^{t_2} \frac{dr \cdot (g_2 - g(r))}{c^2}. \quad (3)$$

Thus, by working in a frame which is close to the centre-of-motion frame we end up with a simple Newtonian looking result.

If, as in the second line, we decompose the redshift into a first-order Doppler or kinematic component and a gravitational redshift then only changes in the gravity vector with position – to the lowest order tidal field – appear in the latter, consistent with the idea that one can ‘transform away’ any constant gravitational acceleration. The asymmetry in this formula – the fact that $g_2$ appears in the integral – is a consequence of the fact that we have chosen to use the first-order Doppler term at the initial time $t_1$. Had we used the final time then we would have $g_1$ in the integral and if we had used the relative velocity at the time the photon is half way along its path – arguably the most natural choice – we would have $(g_1 + g_2)/2$ in the integral.

However, had we tried to decompose $\Delta \lambda/\lambda$ into a first-order effect from the velocity difference at different times $v_2(t_2) - v_1(t_1)$ as in the first line of (3) then the gravitational effect is $-\int dr \cdot g/c^2$, which is a poorly defined concept in Newtonian gravity. It is not measurable from local geodesic deviation measurements; it may in principle have a contribution from structures at arbitrary large distances; as discussed, its absolute value is arbitrary. But there is no real physical problem here; the combination of these individually poorly defined terms does not depend on the arbitrary choice of zero-point of $g$. But it is salutary, nonetheless, that any difference of velocities at different times is poorly defined in Newtonian gravity; since GR contains Newtonian gravity as a limiting case one should also be wary about any discussion, or calculation, invoking difference of velocities at different times.

This result resolves the issue raised in the introduction which was the legitimacy of ‘transforming away’ the gravity in the vicinity of the receiver. This, in effect, is what we have done in the second line of (3) where the integrand is zero at the location of the receiver (particle 2) and consequently the gravitational term will, for a smooth potential, grow quadratically with distance from the receiver. That is fine, but only gives the correct redshift if it is combined with the velocity of the receiver determined at the time of emission. For randomly chosen observers in dynamical equilibrium, the average velocity at the emission time is not zero, and the net result is the usual static gravitational redshift (though there is still the complication of the traverse Doppler effect if one works in the rest-frame of the cluster).

Regardless of which velocity is chosen to define the ‘kinematic’ component, the gravitational component does involve the tidal field. The only distinction is whether the gravitational component is fully determined by the tide, as it is if the relative velocity is taken to be in the rest-frame, or whether it also involves the gravity, as is the case if the velocity difference is at different times. This is at odds with Synge’s statement that the curvature does not appear in the redshift.

What about the change in the separation during the light-propagational time? Letting the centre of velocity frame separations, in the absence of gravity, at reception and emission be $D_0$, $D_\Delta = (D + \Delta D)/2$, we see from the caption of Fig. 3 that $\Delta D/D$ is not precisely the same as the flat-space $\Delta \lambda/\lambda = 2\mu/\beta$. But the difference is of the order of $\beta^2$, so to our precision goal we can take them to be equal. Switching on gravity, the fractional change of the separation between the particles as measured by lattice based observers – i.e. the change in the proper separation in the centre of velocity frame – is

$$\Delta D/D = n \cdot (v_2 - v_1)_{t_1}/c + \Delta r \cdot (g_2 - g_1)/2c^2 \quad (4)$$

which is also a simple Newtonian looking result. Unsurprisingly, to first order in the relative velocity the fractional changes in wavelength and separation are identical. Both contain an additional gravitational term that is, to lowest order, a tidal effect. In general, these gravitational effects are not precisely equal – $g_2 - g_1$ is the integral of the tide while the wavelength shift involves the integral of the gravity – so the ‘cosmological relation’ that wavelengths vary precisely in proportion to the source-observer proper separation, does not hold in general.

But if the tide is spatially constant – i.e. the potential has no spatial derivatives higher than second – then the gravity varies linearly with position then we can write $g(r) = g_1 + (g_2 - g_1)|r - r_1|/|r_2 - r_1|$ and the gravitational terms are readily found to be identical. Thus, the relation seen in cosmology is of wider generality, and applies for an arbitrary pair of particles moving in a field that has a spatially constant tide. This includes, as a special case, a pair of particles with relative motion along their separation in a quadratic potential as in an FRW model containing matter and/or dark energy. Note that there is no need for the particles to be comoving with the matter density, though again this result does apply in that situation.

This is one of the two main results of this paper: a constant tide stretches wavelength of radiation just as it changes the separation of test-particles. Arguably this ‘explains’ the apparent stretching of wavelength of light by the expansion of space in FRW models.

To highlight the differences between separation and wavelength changes – what one might call the ‘non-kinematic’ component of the redshift – and to see how this depends on the tide (and its derivatives) we note the following:

First, we can write

$$\Delta D/D = n \cdot (v_2 - v_1)_{t_1}/c - \frac{\Delta r}{2c^2} \int dr \phi'(r), \quad (5)$$

where $\phi(r) = \phi(r\cdot n)$ is the gravitational potential and prime denotes the operator $\partial_\tau = n \cdot \nabla$, i.e. the spatial derivative along the photon path, so $\phi'(r) = n \cdot \nabla \phi(r) = -n \cdot g$. Thus, the gravitational contribution to $\Delta D/D$ is the average of the tide along the photon path times $(\Delta r/c)^2/2$.

Secondly, taking the difference of (3) and (4), we have

$$\Delta \log(\lambda/D) = \frac{1}{c^2} \left( d \cdot n \cdot (g_1 + g_2) - \int dr \cdot g \right), \quad (6)$$

where $d \equiv |r_2 - r_1|/2$. Taking the origin of coordinates to lie at $(r_1 + r_2)/2$ for simplicity, this is a weighted average of the gravity $-\phi'$:

$$\Delta \log(\lambda/D) = \frac{1}{c^2} \int dr W_0(r)\phi'(r) \quad (7)$$

with dimensionless weighting function $W_0(r) \equiv \theta_+(r)\theta_-(r) - d\delta(r - d) + \delta(r + d)$, where $\phi(r) \equiv \phi(r \cdot n)$; and where $\theta_+(r) \equiv \theta(\pm r - d)$ with $\theta$ and $\delta$ denoting the Heaviside function and the Dirac delta function, respectively. The weight function $W_0(r)$ is shown schematically as the upper plot in Fig. 4. The product of Heaviside functions is zero for $|r| > d$ so the range of integration is now unrestricted. The integral of $W_0(r)$ over all $r$ vanishes, so
difference is at most of the order of the square of the separation in units of the curvature radius \(\Delta D/D\) so any effect on \(\Delta D/D\) is of higher order. Similarly the asynchronicity between clocks carried by our non-inertial grid-based observers is negligible at our stated level of precision.

More fundamentally while matter tells space–time how to curve, it is only the curvature of time that tells non-relativistic matter how to move, and is also only the time–time part of the metric perturbation that is relevant for the calculation of the redshift. So at the specified level of precision, we can ignore the spatial part of the metric.

We have asked: what is the domain of validity of the relationship between wavelengths and emitter/receiver proper separation (1) that we see for FOs in homogeneous models? We have shown that a spatially constant time stretches wavelength in exactly the same way it affects the observers’ separation, but if the tide varies with position the relationship between wavelength and separation is modified.

From this perspective, the perfect correlation seen between changes in wavelengths and emitter/receiver proper separation (1) that we see for FOs in homogeneous models is of the order of \(1/\lambda D\) of light exchanged between FRW FOs and the change \(\Delta D/D\) in the space (between said FOs) is not a causal relationship, rather both the change in the wavelength and the change in the space between the observers are ‘caused’, or determined, by a combination of the observers’ initial velocities and the tidal field in which they and the photons propagate. Echoing Whiting (2004), the expansion rate defined by the matter content of the universe is irrelevant (which is a jolly good thing if the universe has a cosmological constant or a scalar field to realize dark energy since neither defines either a frame of motion or a state of expansion). Rather, on the parabolic potential generated by gravitating matter and dark energy one can have emitter/receiver pairs that recede from each other or pairs that approach each other, and what determines both the changes in the wavelengths and proper separations is a combination of initial conditions and the curvature, or tidal field.

The perfect correlation of \(\Delta \log (\lambda)\) and \(\Delta \log (D)\) in homogeneous models can be considered to be a reflection of the symmetry of the gravitational fields that are allowed in these models, in accord with the conjecture of Melia (2012).

In the Introduction, we asked what redshift would be seen by a pair of observers in a cluster who have the same separation at emission as at reception. Our analysis shows that they do not see the effect of any local tidal field and if they were residing in a constant density cluster core they would see no redshift. From the perspective of their centre of mass, the observers were moving apart at the moment of emission, but falling back together by the time of reception, so the Doppler shifts would cancel, and the net gravitational redshift also vanishes as any energy gained by the photon on the first half of its journey is cancelled by the redshift on the second half. The receiver could justifiably consider itself to be at rest at the centre of the parabolic potential well generated by the uniform matter density. From that perspective, the net redshift vanishes because the Doppler redshift at emission is cancelled by the gravitational blueshift as the photon rolls down the potential to the receiver.

In an inhomogeneous system such as the Solar system, or a galaxy, cluster or supercluster, the tidal field necessarily varies with position. There is then a non-kinematic component to the redshift that violates (1) and which is essentially gravitational in nature. Combining this with the kinematic redshift component, if any, one obtains complete consistency with the conventional view of the gravitational redshift in clusters of galaxies and other gravitating systems. Note that our description of components of the redshift is different from the terminology of Chodorowski (2011) who was

**Figure 4.** The upper plot shows, schematically, the dimensionless weight function \(W_0(r) = \theta_+(r)\theta_-(r) - d\delta(r - d) + \delta(r + d)/2\) that, when multiplied by the gravitational redshift \(\phi(r)\) gives the difference \(\Delta \lambda/\lambda - \Delta D/D\). The Dirac \(\delta\)-functions are shown as the narrow box-cars at \(r = \pm d\) and together have (minus) the same weight as the central box-car. As described in the text, this difference can also be computed as a weighted average of the tide \(\phi(r)\) using the weighting function \(W_1(r)\) shown in the centre plot, which is (minus) the integral of \(W_0(r)\), and also has zero net weight. The third way to compute the difference is averaging the gradient of the tide \(\phi''(r)\) with the weight function shown in the bottom plot.
considering the gravitational component of the redshift in FRW models that arises if the ‘kinematic’ component is taken to be the relative velocity at emission or reception rather than the average velocity. Here, we consider redshifts between observers in FRW models to be purely kinematic in the sense that (1) is obeyed.

For emitter/receiver pair separation that is small compared to the size of the gravitating system the difference between the fractional change in the wavelength and separation is of the order of the gravitational potential well depth times the cube of the separation in units of the overall system size. This is seen most easily from (9), and the fact that \( W(r) \sim d^2 \), which together imply \( \Delta \log(\lambda/D) \sim (d/R)^2 \phi/c^2 \) where \( R \) is the size of the system. So, just as the local gravity is invisible to freely falling observers, as far as the ratio of wavelength to separation is concerned, the local tide is also invisible. But if the path length has a similar size to the entire system the error is of the order of the gravitational potential.

To see better how this relates to Synge’s result that the redshift is always given by the Doppler formula, consider the case of an emitter at the centre of the potential for a small uniform spherical distribution of matter of mass \( M \) and radius \( r \) and a receiver outside at distance \( D \gg r \) who happens to be at rest at the moment of reception. In this situation, the redshift is the static gravitational redshift: \( \Delta \lambda/\lambda = \int dr/g/c^2 \sim GM/c^2r \). But the more distant the receiver, the smaller any fractional change in the emitter/receiver proper separation during the time of flight: \( \Delta D \lesssim (GM/D^2)\Delta \lambda^2/2 \) which implies \( \Delta D/D \lesssim GM/c^2D \ll \Delta \lambda/\lambda \). Evidently the kinematic relation does not apply here. But, following Peebles, we can still break the net wavelength ratio down into the product of ratios between a set of pairs of neighbouring particles. We can take these to be particles on a set of radial orbits such that each particle is at apogee at the time the photon passes (see Fig. 5). Thus, the \( n \)th particle has zero velocity as the photon passes it, as does the \((n+1)\)th particle. In the rest-frame of the pair of particles, however, at the time the photon passes the \((n+1)\)th particle the \( n \)th particle will have started to fall back towards the mass and will have picked up a velocity \( \delta v = \Delta \lambda \). The Doppler shift, evaluated using this small rest-frame velocity, is \( \delta \nu/c = \Delta \lambda/\lambda \), which is just the gravitational redshift for this element of the path, and this \( \delta \nu \) is also the same as the result of parallel transporting the 4-velocity of the \( n \)th particle along the null ray and subtracting it from the 4-velocity of the \((n+1)\)th particle. Either way, integrating these velocity increments gives the gravitational redshift, so there is no mathematical conflict with Synge. But this ‘velocity’ is not related in any sensible way to the rate of change of the emitter-receiver proper separation which is much smaller. It is therefore misleading to say that redshifts in the situation described here – that a non-sophisticated physicist would say are essentially gravitational – are kinematic in nature.

If instead we consider a similar pair of particles in a quadratic potential – where, unlike the Keplerian example above the tidal field is spatially constant – the redshift is again just the gravitational redshift, but in this case this is not inconsistent with the kinematic interpretation since at the time of emission the receiver was indeed closer to the source by an amount such that the fractional change in \( \lambda \) is indeed the same as the fractional change in separation.

We do not think that Synge would object strongly to our conclusions. While he did say that if one were to attribute a cause to the spectral shift one would have to say that it is caused by the relative velocity of the source and observer and is not a gravitational effect, this should not be taken out of context. He followed that immediately by emphasizing that this is true only given the specific definition of relative velocity in terms of parallel transport; that one is not obliged to accept that definition; and that arguments about

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{fig5}
\caption{Illustration of the example described in the text. Here, we have a potential with gravity \( g \sim r/(r^3 + r_0^3) \) which is like that for a uniform density sphere at \( r \ll r_0 \), and is Keplerian at \( r \gg r_0 \). The redshift between an emitter at \( r = 0 \) and a distant receiver at \( r \gg r_0 \) which happens to be turning around at the instant of reception is just the static gravitational redshift. But following Synge & Peebles, the wavelength ratio is also equal to the product of Doppler shifts between pairs of fictitious particles along the photon path which, for simplicity, can be taken to be all on radial orbits (dotted lines) that happen also to be turning around as the photon (dashed line) passes. The velocities are supposed all to be small compared to \( c \) but have been exaggerated here for clarity. Now the pairwise velocity differences have to be calculated in the rest-frame of each pair; i.e. the differences are between the space–time points connected by the horizontal lines. If the velocities were differenced along a continuous path – at a constant time say – the sum of the pairwise velocities and the relative velocity of the end points has to be equal. But when a null path is chopped up into a set of space-like intervals like this the connection between the ‘relative velocity’ obtained by summing pairwise differences and the true relative velocity is broken. For this type of potential, the resulting net ‘relative velocity’ is dominated by the transition region \( r \sim r_0 \) where there are only fictitious particles. For \( r \gg r_c \), the true rate of change of the separation of the only two real particles involved is much smaller than that calculated by summing these fictitious velocity differences.}
\end{figure}

whether a redshift is a gravitational or a velocity effect are futile without any attempt to analyse the meanings of the terms employed. Unfortunately he did not expand much on the physical interpretation of his relative velocity. That is what we have tried to do here. Later in his book he adopts a relatively conventional decomposition of spectral shifts into a product of velocity and gravitational effects, remarking that the earlier formalism should only be taken as ‘a matter of speaking’.

Synge’s reason for saying the redshift is not gravitational was that the Riemann tensor does not appear in the formulæ. This was emphasized by Narlikar (1994) and underlies Bunn & Hogg’s narrative. But gravity obviously does play a role. We believe that this has contributed to the confusion over the nature of redshifts. We
cannot know exactly what Synge had in mind, but perhaps he was referring to the fact that the parallel transport operation depends on the connection, which contains first derivatives of the metric, rather than the second derivatives that appear in the curvature or tide. As we have discussed, there is something a little disturbing about this since in the Newtonian limit this corresponds to the gravity, which is somewhat poorly defined, and that carries over into the more general theory. But to do anything useful with a redshift, one needs to compare it with something else – like the relative velocity, for instance – in which case the difference tells us something physically meaningful about the gravitational field and hence the mass distribution that generates it. As we have seen, provided the velocity differences are taken to be at the same time there is no ambiguity since the connection itself does not then appear.

In conclusion, we hope that the discussion and rather elementary analysis presented here makes clear that saying that redshifts are an effect of the relative velocity is either meaningless, as it is if there is no means at one’s disposal other than redshift to measure relative velocity, or, unless the tide happens to be constant, it is false. In many situations, of course, the kinematic relationship is an excellent approximation, because in most circumstances the first-order velocity effect dominates over any gravitational effects, which are generally of second order in the velocity. But as regards the latter, redshifts cannot, in general, be regarded as kinematic in nature.

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REFERENCES

Bondi H., 1947, MNRAS, 107, 410
Bunn E. F., Hogg D. W., 2009, Am. J. Phys., 77, 688
Cappi A., 1995, A&A, 301, 6
Chodorowski M. J., 2007, Old New Concepts Phys., 4, 15
Chodorowski M. J., 2011, MNRAS, 413, 585
Darling J., 2013, ApJ, 777, L21
Gunn J. E., 1988, in van den Bergh S., Pritchet C., eds, ASP Conf. Ser. Vol. 4, The Extragalactic Distance Scale. Astron. Soc. Pac., San Francisco, p. 344
Harrison E. R., 2000, Cosmology: The Science of the Universe. Cambridge Univ. Press, Cambridge
Kaiser N., 2013, MNRAS, 435, 1278
Lineweaver C. H., Davis T. M., 2005, Sci. Am., 292, 36
Melia F., 2012, MNRAS, 422, 1418
Narlikar J. V., 1994, Am. J. Phys., 62, 903
Peacock J. A., 2008, preprint (arXiv:0809.4573)
Peebles P. J. E., 1971, Physical Cosmology. Princeton Univ. Press, Princeton
Pound R. V., Rebka G. A., Jr, 1959, Phys. Rev. Lett., 3, 439
Rees M. J., Sciama D. W., 1968, Nature, 217, 511
Rindler W., 1977, Essential Relativity, 2nd edn. Springer-verlag, New York
Synge J. L., 1960, Relativity: the General Theory. North-Holland Publ., Amsterdam
Whiting A. B., 2004, The Observatory, 124, 174
Wojtak R., Hansen S. H., Hjorth J., 2011, Nature, 477, 567
Zhao H., Peacock J. A., Li B., 2013, Phys. Rev. D, 88, 043013

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