The control chart with practical importance for linear profile

Yuqiong Liao, Xuemin Zi
Tianjin University of Technology and Education, Tianjin, 300222, China
E-mail: 404377736@qq.com

Abstract. In recent years, linear profile monitoring has become one of the popular research directions in SPC. Although the linear profile model is simple and widely applicable, it is too sensitive to small parameter changes, leading to an increase in false alarm rates. This paper presents a new control chart with the practical importance for linear profile. The control chart can be more tolerant for the small shifts comparing the conventional control chart with considering the practical importance, so as to ensure that the really important changes are detected. The simulation study shows that the new control chart can be used to detect the change of intercept and slope efficiently. Based on control chart provided by Kim (2003), we obtain the threshold for the control chart with tolerance under null hypothesis with nominal in control run length ($ARL_0$) and also yield run length for the situations of out of control ($ARL_c$).

1. Introduction
In many production processes, the product quality we have learned is no longer simply represented by the mean value or variance (or distribution) of one or several indicators, but characterized by the relationship between a response variable and some independent variables [1]. This kind of relationship can be described by profile. Therefore, more and more attention is focusing on monitor linear profile monitoring, which has very important applications in many fields. For example, Lawless et al. (1999) [2] gave an example of automotive engineering. Kang and Albin (2000) [3] introduced semiconductor manufacturing applications. Salimi et al. (2005) [4] described a production process of food mayonnaise with profile. For these different applications, we need to use a variety of appropriate control charts to monitor the parameters. Kang and Albin (2000) [3] proposed two schemes control charts to monitor product quality linear process. Kim (2003) [5] discussed the method of combining three univariate control charts, which can be used to simultaneously monitor intercept, slope, and standard deviation. The list of research on such issues can be referred to Woodall et al. (2004) [6] and Woodall (2007) [7], which summarizes the existing methods and future research directions of linear profile. The method suggested above are dependent on the classical hypothesis testing, but in practice, it does not need to detect the small deviation of these parameters as soon as possible. As Box et al. (2003-2004) [8] pointed out, the idea that “the monitored system (factory) is either “good” or “bad” is too simplistic.” In this case, Dett (2016) [9] studies the test problem with tolerance, which overcomes the deficiency brought by the test of standard hypothesis. In this paper, a linear profile control chart with tolerance is proposed to monitor the changes of practical importance. Through the simulation study, the monitoring performance of the proposed method and Kim (2003) method is compared, and the effectiveness of the proposed method is further verified.
2. Control chart with tolerance

Suppose \( \{(x_i, y_{ij}), i = 1, 2, ..., n\} \) is a random sample of the profile collected at the \( j \)th time. When the process is in-control, the general model is

\[
y_{ij} = A_0 + A_1 x_i + \varepsilon_{ij}
\]

where \( \varepsilon_{ij} \) is an independent, identically distributed (i.i.d.) normal random variable with mean 0 and variance \( \sigma^2 \). In the consideration of Phase II, the in-control parameters \( A_0, A_1 \) and \( \sigma \) of equation (2.1) are known. By centralizing the covariant \( x_i \), Kim (2003) converted (2.1) to an equivalent model

\[
y_{ij} = B_0 + B_1 x_i^* + \varepsilon_{ij}
\]

where \( B_0 = A_0 + A_1 x_i, B_1 = A_1, x_i^* = x_i - \bar{x}, \bar{x} = 1/n \sum_{i=1}^{n} x_i \).

The models introduced above are based on classical hypothesis testing \( H_0: \theta = \theta_0 \leftrightarrow H_1: \theta \neq \theta_0 \) (3) where \( \theta \) and \( \theta_0 \) represent the parameters before and after the change in the process. As for the linear profile model, the commonly used method is to use the parameter model to describe the profile in the in-control state, and then monitor the changes of the model parameters. In the above method, for the in-control model parameters \( B_0, B_1 \) and \( \sigma \), when the model parameters \( B_0, B_1, \sigma \) are known, the process is considered being stable.

However, this assumption is overly sensitive to small changes in parameters, does not have practical importance, and is often prone to false alarms. If there are too many false alarms, the alarms are often ignored and monitoring will become invalid. Hence, based on Kim's (2003) method, we consider the hypothesis with tolerance in the case of phase II. In this situation, the profile is no longer accurately represented by a line, and our linear profiles may be approximate. This means that we add an tolerate range to intercept, slope and other parameters in the linear model. When the difference between parameters is in this range, we think it is in-control, that is, considering the assumption of tolerance

\[
H_0: ||\theta - \theta_0|| \leq \Delta \leftrightarrow H_1: ||\theta - \theta_0|| > \Delta
\]

where \( || \cdot || \) represents the norm on the parameter space. In simple linear models, we usually use absolute values to measure the distance between parameters. \( \Delta (> 0) \) is a preset constant representing the "maximum" change that can be accepted in practice. Based on the above assumptions, we construct a linear profile model with tolerance

\[
y_{ij} = (B_0 + \Delta_0) + (B_1 + \Delta_1) x_i^* + \varepsilon_{ij}
\]

where \( \Delta_0 \) and \( \Delta_1 \) represent the tolerance values of \( B_0 \) and \( B_1 \) respectively. In this model, when \( \Delta_0 = 0 \), it means that only the slope with tolerance is considered. In the same way, when \( \Delta_1 = 0 \), this means that only the slope with tolerance is considered. For the \( j \)-th sample, as Kim (2003) proposed, the least squares estimates of \( B_0, B_1, \sigma \) are given as follows

\[
b_{o0} = \bar{y}_j, b_{o1} = S_{xy(j)} / S_{xx}, MSE_j = 1/(n - 2) \sum_{i=1}^{n} (y_{ij} - b_{o1} x_i^* - b_{o0})^2
\]

where \( \bar{y}_j = 1/n \sum_{i=1}^{n} y_{ij}, S_{xx} = \sum_{i=1}^{n} (x_i - \bar{x})^2, S_{xy(j)} = \sum_{i=1}^{n} (x_i - \bar{x}) y_{ij} \). Note that these three estimators are independent. Therefore, the model (2.5) can be effectively monitored by three EWMA control charts. The \( EWMA_1 \) control chart of monitoring intercept \( (B_0) \) is defined

\[
EWMA_1(j) = \lambda b_{o0} + (1 - \lambda) EWMA_1(j - 1)
\]

where \( \lambda \) is the smooth parameter of \( EWMA_1 \), and \( EWMA_1(0) = B_0 \), if \( EWMA_1(j) < LCL \) or \( EWMA_1(j) > UCL \), this control chart alarms

\[
LCL = B_0 - L_1 \sigma \sqrt{\lambda / ((2 - \lambda)n)}, UCL = B_0 + L_1 \sigma \sqrt{\lambda / ((2 - \lambda)n)}
\]

where \( L_1 (> 0) \) is the control limit chosen to satisfy a given in-control average run length (ARL). Similarly, we define a \( EWMA_S \) control chart that monitors the slope \( (B_1) \)

\[
EWMA_S(j) = \lambda b_{o1} + (1 - \lambda) EWMA_S(j - 1)
\]

where \( \lambda \) is the smooth parameter of \( EWMA_S \), and \( EWMA_S(0) = B_1 \), if \( EWMA_S(j) < LCL \) or \( EWMA_S(j) > UCL \), the control chart gives out of control signals

\[
LCL = B_1 - L_S \sigma \sqrt{\lambda / ((2 - \lambda) S_{xx})}, UCL = B_1 + L_S \sigma \sqrt{\lambda / ((2 - \lambda) S_{xx})}
\]
where $L_C (> 0)$ is the control limit chosen to satisfy a given in-control average run length ($ARL_0$). For the control chart of monitoring standard deviation ($\sigma$), we use Crowder (1992) method and value to obtain statistics

$$EWM_{\lambda}(j) = \max(\lambda \ln(MSE_j) + (1 - \lambda)EWM_{\lambda}(j-1), \ln(\sigma^2))$$

where $\lambda$ is the smooth parameter of $EWM_{\lambda}$, and $EWM_{\lambda}(0) = \ln(\sigma^2)$. The difference between the above two control charts is that the $EWM_{\lambda}$ control chart only considers oneside. When $EWM_{\lambda}(j)$ exceeds the upper control limit given in the following formula, the process will give a signal

$$UCL = L_C \sqrt{\frac{\lambda}{2 - \lambda} \text{Var}[\ln MSE_j]}$$

where $\text{Var}[\ln MSE_j] \approx \frac{2}{(n-2)} + \frac{2}{(n-2)^2} + \frac{4}{3(n-2)^3} - \frac{16}{15(n-2)^5}$.

Use the above three control charts in combination. If any control chart generates an alarm, the monitoring process stops.

### 3. Simulation and performance

In this section, we use $ARL$ to measure the monitoring performance of the proposed linear profile scheme with tolerance. In order to make the control chart comparable, for each $EWM_{\lambda}$ control chart considered separately, we set its in-control $ARL$ to 584, so that the overall in-control $ARL$ is approximately 200. Moreover, choosing different smoothing parameters may lead to different $ARL$ for a given shift size. Generally, smaller smoothing parameters can detect small shift faster. Based on Lucas and Saccucci (1990), we set the smoothing parameter to 0.2, which is a classical value. In our simulation study, we repeat the simulation 10000 times to obtain the control line with tolerance, and give the out-of-control average run length ($OC ARL$) under different shifts, and analyze the monitoring performance of its $EWM_{\lambda}$ control chart.

Consider using the same example as Kim (2003) in the simulation study. Its simple linear model is $y_{ij} = 13 + 2x_i^j + \varepsilon_{ij}$, where $\varepsilon_{ij}$ is i.i.d normal random variable with mean 0 and variance 1. Fixed $x_i$ values were used in the simulation study, respectively 3, -1, 1, 3. Next, we consider three different types of shift with practical importance: intercept with tolerance, slope with tolerance, intercept and slope with tolerance.

#### Table 1. intercept with tolerance.

| $\Delta_0$ | $L$ | 0.05 | 0.10 | 0.15 | 0.20 | 0.25 | 0.30 | 0.35 | 0.40 | 0.45 | 0.50 |
|------------|-----|------|------|------|------|------|------|------|------|------|------|
| 0.00       | 3.1056 | 180.0 | 133.7 | 90.2 | 59.2 | 40.4 | 28.3 | 21.1 | 16.3 | 13.1 | 10.8 |
| 0.05       | 3.1482 | 158.2 | 109.9 | 71.6 | 48.0 | 33.6 | 24.1 | 18.3 | 14.6 | 11.9 | 10.0 |
| 0.10       | 3.3919 | 150.6 | 98.7  | 66.1 | 44.2 | 31.8 | 23.0 | 17.9 | 14.4 | 11.8 | 9.9  |
| 0.15       | 3.6515 | 142.2 | 94.7  | 63.4 | 43.4 | 30.9 | 22.7 | 17.7 | 14.1 | 11.8 | 10.1 |
| 0.20       | 3.9650 | 143.9 | 98.2  | 65.0 | 44.4 | 30.9 | 22.7 | 17.7 | 14.1 | 11.8 | 10.5 |
| 0.25       | 4.2468 | 142.8 | 96.2  | 62.8 | 43.7 | 31.1 | 23.7 | 18.2 | 15.0 | 12.4 | 10.6 |
| 0.30       | 4.5244 | 139.9 | 94.3  | 63.1 | 43.0 | 31.1 | 23.3 | 18.3 | 14.9 | 12.5 | 10.6 |
| 0.35       | 4.8719 | 146.8 | 101.4 | 66.4 | 46.5 | 32.7 | 24.7 | 19.1 | 15.5 | 13.2 | 11.4 |
| 0.40       | 5.1813 | 151.0 | 99.9  | 68.5 | 46.8 | 33.7 | 25.1 | 19.8 | 16.2 | 13.5 | 11.6 |
| 0.45       | 5.4492 | 143.2 | 98.6  | 66.1 | 45.6 | 32.4 | 24.6 | 19.6 | 15.9 | 13.4 | 11.7 |
| 0.50       | 5.7605 | 143.2 | 100.4 | 66.6 | 45.9 | 33.3 | 25.0 | 19.9 | 16.3 | 13.7 | 11.9 |

In Table 1, we discuss the change of intercept with tolerance from $B_0$ in in-control state to $B_0 + \alpha \sigma$ in out-of-control state, where $\alpha$ is 0(0.1)1 and compare it with Kim (2003) control chart without tolerance. It can be seen that the performance of the proposed method is much better than that of Kim (2003) control chart in detecting medium and small shift. However, in case of large shift, the monitoring effect of the control chart with tolerance ($\Delta > 0$) is slightly worse than that of the control chart without tolerance ($\Delta = 0$). This is because when we increase the range of monitoring parameters,
the corresponding control lines will also increase, resulting in less alarming. In Figure 1, we select three cases of different tolerance ($\Delta = 0.1, 0.25, 0.5$) in Table 1 for drawing. In the case of $\Delta = 0.5$, when $B_0$ has small and medium shift of $\alpha \in [0, 0.4]$, its $OC$ ARL with tolerance is lower than that without tolerance. When the shift of $\alpha$ is 0.45 and 0.50, the corresponding $OC$ ARL with tolerance is 13.7 and 11.9, which is slightly higher than the $OC$ ARL without tolerance.

![Figure 1: $B_0$ of ARL Values](image1)

![Figure 2: $B_1$ of ARL Values](image2)

In Table 2 and Figure 2, we discuss the change of slope with tolerance from $B_1$ in in-control state to $B_1 + \beta \sigma$ in out-of-control state. We take $\beta$ from 0 (0.025) 0.250. By comparison, we can get a similar conclusion with table 1, that is, we observed that when the $B_1$ occurred in small and medium shift, the control chart with tolerance can detect the change of shift more quickly. When $B_1$ has a large shift, the detection of the control chart with tolerance is slightly slower than that of the control chart without tolerance. In addition, we find that the tolerance increased to a certain extent for the same shift and its $OC$ ARL did not change much.

| $\Delta$ | $B_1$ |
|----------|-------|
| 0.000    | 3.0109|
| 0.025    | 3.1656|
| 0.050    | 3.4441|
| 0.075    | 3.7678|
| 0.100    | 4.1125|
| 0.125    | 4.4641|
| 0.150    | 4.7500|
| 0.175    | 5.1081|
| 0.200    | 5.4520|
| 0.225    | 5.7552|
| 0.250    | 6.1000|

In Table 3, we study the model with tolerances in both the intercept and slope. The $EWMA_3$ represents a case without tolerance, and the $EWMA_{imp}$ represents a control chart with tolerance. In order to facilitate comparison, the tolerance and the shift of the intercept are taken as 0 (0.05) and 0.50. Similarly, the tolerance and the shift of the slope are taken as 0 (0.025) and 0.250. For example, in the $EWMA_3$ control chart, we consider that the intercept shifts by 0.05 and the slope shifts by 0.025, but...
when neither of them has tolerances, the $OC\ ARL$ is 157.6. In $EWMA_{imp}$, the intercept changes by 0.05 and the slope changes by 0.025. When both the intercept and the slope are considered with tolerance equal to their shift, the resulting $OC\ ARL$ is 122.5. In general, the monitoring performance with tolerance is better than that without tolerance, which verifies the effectiveness of $EWMA$ control chart with practical importance. It is worth noting that in the case of intercept or slope only considering large shift, we do not recommend using a control chart with tolerance. Taking intercept and tolerance $\Delta_0(\alpha)$ as 0.45, we find that their $OC\ ARL$ (bold in black in Table 3) are slightly higher than those without tolerance.

Table 3. intercept and slope with tolerance.

| $EWMA_3$ | $EWMA_{imp}$ | 0.025 | 0.050 | 0.075 | 0.100 | 0.125 | 0.150 | 0.175 | 0.200 | 0.225 | 0.250 |
|----------|--------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.05     | 157.6        | 114.7 | 74.8  | 48.3  | 32.2  | 22.5  | 16.9  | 13.2  | 10.7  | 8.9   |       |
| 0.10     | 122.1        | 94.6  | 66.4  | 44.9  | 30.7  | 21.9  | 16.6  | 13.1  | 10.6  | 8.9   |       |
| 0.15     | 88.6         | 64.0  | 45.6  | 33.2  | 25.6  | 19.2  | 15.8  | 13.5  | 11.6  | 10.4  |       |
| 0.20     | 58.5         | 47.5  | 37.7  | 29.7  | 23.7  | 18.2  | 15.5  | 13.2  | 11.4  | 10.3  |       |
| $\Delta_0(\beta)$ |       | 0.25  |       |       |       |       |       |       |       |       |       |
| 0.25     | 39.5         | 36.5  | 32.3  | 27.1  | 22.0  | 17.8  | 14.4  | 11.9  | 10.0  | 8.5   |       |
| 0.30     | 28.2         | 26.9  | 24.7  | 22.0  | 18.8  | 15.7  | 13.2  | 11.2  | 9.6   | 8.3   |       |
| 0.35     | 20.9         | 20.2  | 19.1  | 17.6  | 15.8  | 13.9  | 12.1  | 10.5  | 9.1   | 8.0   |       |
| 0.40     | 18.9         | 18.4  | 17.5  | 16.4  | 15.2  | 13.4  | 12.4  | 11.4  | 10.3  | 9.5   |       |
| 0.45     | 16.2         | 15.9  | 15.3  | 14.5  | 13.5  | 12.1  | 10.9  | 9.7   | 8.6   | 7.6   |       |
| 0.50     | 16.0         | 15.5  | 15.0  | 14.3  | 13.4  | 12.4  | 11.6  | 10.8  | 9.91  | 9.2   |       |
| 13.5     | 13.3         | 12.9  | 12.4  | 12.1  | 11.3  | 10.8  | 10.1  | 9.5   | 9.0   | 8.6   |       |
| 11.8     | 11.7         | 11.5  | 11.4  | 11.1  | 10.6  | 10.1  | 9.6   | 9.1   | 8.6   |       | 4. Conclusion

In this paper, the practical importance of data is combined with the linear profile model, which overcomes the shortcomings of traditional control chart that is too sensitive to shift. It is found that the monitoring effect of control chart with tolerance is better than that without tolerance for medium and small shift. However, in the case of large shift, this advantage will be relatively weakened. The new linear profile control chart can not only effectively reduce the occurrence of false alarms, but also quickly detect important changes in the process. In the future, the research on practical importance will be extended to the control chart of multivariate exponentially weighted moving average ($MEWMA$).

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