ON THE UNIVERSAL FUNCTION FOR WEIGHTED SPACES
$L^p_\mu[0,1], \ p \geq 1$

MARTIN GRIGORYAN$^1$, TIGRAN GRIGORYAN$^1$ and ARTSRUN SARGSYAN$^2$

Abstract. In the paper it is shown that there exist a function $g \in L^1[0,1]$ and a weight function $0 < \mu(x) \leq 1$, so that $g$ is universal for each classes $L^p_\mu[0,1], \ p \geq 1$ with respect to signs–subseries of its Fourier–Walsh series.

1. Introduction and preliminaries

Let $|E|$ be the Lebesgue measure of a measurable set $E \subseteq [0,1]$, $\chi_E(x)$ – its characteristic function, $L^p(E)$ ($p > 0$) – the class of all those measurable functions on $E$ that satisfy the condition $\int_E |f(x)|^p dx < +\infty$, $L^p_\mu[0,1]$ (weighted space) – the class of all those measurable functions on $[0,1]$ that satisfy the condition $\int_0^1 |f(x)|^p \mu(x) dx < +\infty$, where $0 < \mu(x) \leq 1$ is a weight function, and $\{\varphi_k\}$ – a complete orthonormal system in $L^2[0,1]$.

Definition 1.1. Let $0 < \mu(x) \leq 1$, be a measurable on $[0,1]$ function. We say that a function $g \in L^1[0,1]$ is universal for a class $L^p_\mu[0,1]$ with respect to signs–subseries of its Fourier series by the system $\{\varphi_k\}$, if for each function $f \in L^p_\mu[0,1]$ one can choose numbers $\delta_k = \pm 1, 0$ so that the series

$$\sum_{k=0}^{\infty} \delta_k c_k(g) \varphi_k(x), \quad \text{with} \quad c_k(g) = \int_0^1 g(x) \varphi_k(x) dx,$$

converges to $f$ in $L^p_\mu[0,1]$ metric, i.e.

$$\lim_{m \to \infty} \int_0^1 \left| \sum_{k=0}^{m} \delta_k c_k(g) \varphi_k(x) - f(x) \right|^p \mu(x) dx = 0.$$

Let us recall the definition of the Walsh orthonormal system $\{W_n(x)\}_{n=0}^\infty$. Functions of the Walsh system are defined by means of Rademacher’s functions

$$R_n(x) = \text{sign}(\sin 2^n \pi x), \quad x \in [0,1], \quad n = 1, 2, \ldots,$$

in the following way (see [1]): $W_0(x) \equiv 1$ and for $n \geq 1$

$$W_n(x) = \prod_{i=1}^{p} R_{k_i+1}(x),$$

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*Corresponding author.
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where \( n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_p} \) \((k_1 > k_2 > \cdots > k_p)\).

In the present paper the following theorem is proved for the Walsh system:

**Theorem 1.2.** There exist a function \( g \in L^1[0,1] \) and a weight function \( 0 < \mu(x) \leq 1 \), so that \( g \) is universal for each class \( L^p_\mu[0,1], \ p \geq 1 \) with respect to signs–subseries of its Fourier–Walsh series.

Moreover, it will be shown that the measure of the set on which \( \mu(x) = 1 \) can be made arbitrarily close to 1, and the function \( g \in L^1[0,1] \) can be chosen to have strictly decreasing Fourier–Walsh coefficients and converging to it by \( L^1[0,1] \) norm Fourier–Walsh series.

**Remark 1.3.** In the proved theorem the weight function \( \mu(x) \) cannot be made equal to 1 everywhere in \([0,1]\). Moreover, there does not exist a universal function \( g \in L^1[0,1] \) (defined above) for any class \( L^p[0,1], \ p \geq 1 \).

It can be easily shown that the assumption of existence of such universal function simply leads to contradiction. Indeed, if that assumption was true, then for the function \( k_0c_{k_0}(g)W_{k_0}(x) \), where \( k_0 > 1 \) is any natural number with condition \( c_{k_0}(g) \neq 0 \), one could find numbers \( \delta_k = \pm 1, 0 \) so that

\[
\lim_{m \to \infty} \int_0^1 \left| \sum_{k=0}^m \delta_k c_k(g)W_k(x) - k_0c_{k_0}(g)W_{k_0}(x) \right|^p dx = 0.
\]

Hence, we would simply get a contradiction: \( \delta_{k_0} = k_0 > 1 \).

Existences of functions, which are universal in different senses, were considered by mathematicians since the beginning of the 20-th century. The first type of universal function was considered by G. Birkhoff [2] in 1929. He proved, that there exists an entire function \( g(z) \), which is universal with respect to translations, i.e. for every entire function \( f(z) \) and for each number \( r > 0 \) there exists a growing sequence of natural numbers \( \{n_k\}_{k=1}^\infty \), so that the sequence \( \{g(z+n_k)\}_{k=1}^\infty \) uniformly converges to \( f(z) \) on \(|z| \leq r \). In 1952 G. MacLane [3] proved a similar result for another type of universality, namely, there exists an entire function \( g(z) \), which is universal with respect to derivatives, i.e. for every entire function \( f(z) \) and for each number \( r > 0 \) there exists a growing sequence of natural numbers \( \{n_k\}_{k=1}^\infty \), so that the sequence \( \{g^{(n_k)}(z)\}_{k=1}^\infty \) uniformly converges to \( f(z) \) on \(|z| \leq r \). Further, in 1975 S. Voronin [4] proved the universality theorem for the Riemann zeta function \( \zeta(s) \), which states that any nonvanishing analytic function can be approximated uniformly by certain purely imaginary shifts of the zeta function in the critical strip, namely, if \( 0 < r < \frac{1}{3} \) and \( g(s) \) is a nonvanishing continuous function on the disk \(|s| \leq r \), that is analytic in the interior, then for any \( \varepsilon > 0 \), there exists such a positive real number \( \tau \) that

\[
\max_{|s| \leq r} \left| g(s) - \zeta(s + 3/4 + i\tau) \right| < \varepsilon.
\]

In 1987 K. Grosse–Erdman [5] proved the existence of infinitely differentiable function with universal Taylor expansion, namely, there exists a function \( g \in C^\infty(R) \) with \( g(0) = 0 \), such that for every function \( f \in C(R) \) with \( f(0) = 0 \)
and for each number \( r > 0 \) there exists a growing sequence of natural numbers \( \{n_k\}_{k=1}^{\infty} \), so that the sequence

\[
S_{n_k}(g, 0) = \sum_{m=1}^{n_k} \frac{g^{(m)}(0)}{m!} x^m
\]

uniformly converges to \( f(x) \) on \( |x| \leq r \).

In papers \([6]\) and \([7]\) authors studied existances of universal functions for classes \( L^p[0, 1] \), \( p \in (0, 1) \) with respect to signs-subseries of Fourier–Walsh series and signs of Fourier–Walsh coefficients, respectively. In particular, it was shown in \([6]\) that for each number \( p \in (0, 1) \) one can construct a function from \( L^1[0, 1] \) with convergent in \( L^1[0, 1] \) Fourier–Walsh series having decreasing coefficients, which is universal for the class \( L^p[0, 1] \) with respect to signs-subseries of Fourier–Walsh series.

Note that the definition of function universality which we gave above could be done in terms of Fourier series universality in corresponding sense. The topic of universal series existence (in the common sense, with respect to rearrangements, partial series, signs of coefficients and etc.) in various classical orthogonal systems was also investigated intensively. The most general results were obtained by D. Menshov \([8]\), A. Talalyan \([9]\), P. Ulyanov \([10]\) and their disciples (see \([11]\)–\([22]\)).

Regarding to the result of the present paper the following questions arise, the answer to which is unknown yet:

**Question 1.4.** Is the theorem 1.2 true for other orthonormal systems (trigonometric system, Franklin system and etc.)?

**Question 1.5.** Is it possible to acheive universality with respect to signs of Fourier–Walsh coefficients (i.e. exclude 0 values from the sequence \( \delta_k \)) in theorem 1.2?

2. MAIN LEMMAS

Let us start from known properties of the Walsh system, which will be used during the proofs. It is known (see \([1]\)) that for each natural number \( m \)

\[
\sum_{k=0}^{2^m-1} W_k(x) = \begin{cases} 2^m, & \text{when } x \in [0, 2^{-m}), \\ 0, & \text{when } x \in (2^{-m}, 1], \end{cases} \quad (2.1)
\]

and, consequently,

\[
\sum_{k=2^m}^{2^{m+1}-1} W_k(x) = \begin{cases} 2^m, & \text{when } x \in [0, 2^{-m-1}), \\ -2^m, & \text{when } x \in (2^{-m-1}, 2^{-m}), \\ 0, & \text{when } x \in (2^{-m}, 1], \end{cases}
\]

thus, for each \( p > 0 \) we have

\[
\int_0^1 \left| \sum_{k=2^m}^{2^{m+1}-1} W_k(x) \right|^p dx = 2^{m(p-1)}. \quad (2.2)
\]
Let
\[ \| \cdot \|_{L^p(E)} = \left( \int_E | \cdot |^p \, dx \right)^{\frac{1}{p}} \] and
\[ \| \cdot \|_{L^p_{\mu}[0,1]} = \left( \int_0^1 | \cdot |^p \mu(x) \, dx \right)^{\frac{1}{p}}, \]
where \( p \geq 1, E \subseteq [0,1] \) and \( 0 < \mu(x) \leq 1 \), be the norms of spaces \( L^p(E) \) and \( L^p_{\mu}[0,1] \), respectively. Obviously, for any natural number \( M \in [2^m, 2^{m+1}) \) and numbers \( \{a_k\}_{k=2^m}^{2^{m+1}-1} \)
\[ \left\| \sum_{k=2^m}^M a_k W_k \right\|_{L^1[0,1]} \leq \left\| \sum_{k=2^m}^{2^{m+1}-1} a_k W_k \right\|_{L^2[0,1]} . \tag{2.3} \]

Note also that the basicity of the Walsh system in spaces \( L^p[0,1], p > 1 \) provides the existence of a constant \( C_p > 0 \), so that for each function \( f \in L^p[0,1] \) the following inequality holds:
\[ \| S_k(f) \|_{L^p[0,1]} \leq C_p \| f \|_{L^p[0,1]}, \quad \forall k \in \mathbb{N}, \tag{2.4} \]
where \( \{S_k(f)\} \) are partial sums of its expansion by the Walsh system \([1]\).

In the paper we use the following lemma, which was proved in \([23]\):

**Lemma 2.1.** For each dyadic interval \( \Delta = \left[ \frac{i}{2^K}, \frac{i+1}{2^K} \right], 0 \leq i < 2^K, \) and for every natural number \( M > K, \) such that \( \frac{M-K}{2} \) is a whole number, there exists a polynomial in the Walsh system
\[ H(x) = \sum_{k=2^M}^{2^{M+1}-1} a_k W_k(x), \]
so that
1) \( |a_k| = 2^{\frac{M-K}{2}}, \) when \( 2^M \leq k < 2^{M+1}, \)
2) \( H(x) = -1, \) if \( x \in E_1, |E_1| = \frac{1}{2} |\Delta|, \)
3) \( H(x) = 1, \) if \( x \in E_2, |E_2| = \frac{1}{2} |\Delta|, \)
4) \( H(x) = 0, \) if \( x \notin \Delta, \)
where \( E_1 \) and \( E_2 \) are finite unions of dyadic intervals.

One of the main building blocks in the proof of the theorem 1.2 is Lemma 2.3 which is proved by the help of Lemma 2.2.

**Lemma 2.2.** Let \( p > 1, n_0 \) be some natural number and \( \Delta \subseteq [0,1] \) be a dyadic interval, then for any numbers \( 0 < \varepsilon < 1, l \neq 0 \) and natural number \( q \) there exist a measurable set \( E_q \subset \Delta \) with measure \( |E_q| = (1 - 2^{-q}) |\Delta| \) and polynomials
\[ P_q(x) = \sum_{k=2^{n_0}}^{2^{n_0}} a_k W_k(x) \] and \( H_q(x) = \sum_{k=2^{n_0}}^{2^{n_0}} \delta_k a_k W_k(x), \) \( \delta_k = \pm 1, 0, \)
in the Walsh system, so that \( H_q(x) = 0 \) outside \( \Delta, \)
1) \( 0 < a_{k+1} \leq a_k < \varepsilon \) when \( k \in [2^{n_0}, 2^{n_0} - 1], \)
2) \( \| I_{\chi_{\Delta}} - H_q \|_{L^p(E_q)} = 0, \)
3) \[ \max_{2^{n_0} \leq M < 2^{n_q}} \left\| \sum_{k=2^{n_0}}^{M} \delta_k a_k W_k \right\|_{L^p[0,1]} < 2^i C |l| |\Delta|^{\frac{1}{p}}, \]

where \( C \) is a constant defined by the space \( L^p[0,1] \), and

4) \[ \max_{2^{n_0} \leq M < 2^{n_q}} \left\| \sum_{k=2^{n_0}}^{M} a_k W_k \right\|_{L^1[0,1]} < \varepsilon. \]

**Proof.** The proof is performed using mathematical induction with respect to the number \( q \). Let \( \Delta = \left[ \frac{i}{2^K}, \frac{i+1}{2^K} \right] \subset [0,1] \). Choosing a natural number \( K_1 > K \) such that

\[ |l| 2^{-\frac{K_1+1}{2}} < \frac{\varepsilon}{2}, \quad (2.5) \]

we present the interval \( \Delta \) in the form of union of disjoint dyadic intervals

\[ \Delta = \bigcup_{i=1}^{N_1} \Delta_i^{(1)} \]

with measure \( |\Delta_i^{(1)}| = 2^{-K_1-1} \), \( i = 1, N_1 \). Obviously, \( N_1 = 2^{K_1-K+1} \).

By denoting \( K_0^{(1)} \equiv n_0 - 1 \), for each natural number \( i \in [1, N_1] \) we choose a natural number \( K_i^{(1)} > K_{i-1}^{(1)} \) \( (K_i^{(1)} > K_1) \) such that the following conditions take place:

a) \( \frac{K_i^{(1)} - K_1 - 1}{2} \) is a whole number,

b) \( (K_i^{(1)} - K_{i-1}^{(1)}) |l| 2^{-\frac{K_i^{(1)}+K_{i-1}^{(1)}}{2}} < \frac{\varepsilon}{4N_1} \),

c) \( 2 |l| 2^{-\frac{K_i^{(1)}+1}{2}} < \frac{\varepsilon}{2} \).

It immediately follows from (2.5) that

\[ |l| 2^{-\frac{K_i^{(1)}+K_{i-1}^{(1)}+1}{2}} < \varepsilon. \quad (2.6) \]

By successively applying lemma 2.1 for each interval \( \Delta_i^{(1)} \) \( (i = 1, N_1) \) and corresponding number \( K_i^{(1)} \), we can find polynomials in the Walsh system

\[ H_i^{(1)}(x) = \sum_{k=2^{K_i^{(1)}}}^{2^{K_i^{(1)}+1}-1} a_k W_k(x), \quad i = 1, N_1 \]

such that

\[ |a_k| = |l| 2^{-\frac{K_i^{(1)}+K_{i-1}^{(1)}+1}{2}}, \quad \text{when} \quad k \in [2^{K_i^{(1)}}, 2^{K_i^{(1)}+1}) \], \quad (2.8) \]

\[ H_i^{(1)}(x) = \begin{cases} -l, & \text{for} \quad x \in \tilde{E}_i^{(1)} \subset \Delta_i^{(1)}, \quad |\tilde{E}_i^{(1)}| = \frac{1}{2} |\Delta_i^{(1)}|, \\ l, & \text{for} \quad x \in \tilde{E}_i^{(1)} \subset \Delta_i^{(1)}, \quad |\tilde{E}_i^{(1)}| = \frac{1}{2} |\Delta_i^{(1)}|, \\ 0, & \text{for} \quad x \notin \Delta_i^{(1)}. \end{cases} \quad (2.9) \]
Hence, by denoting
\[
H_1(x) = \sum_{i=1}^{N_1} \overline{H}_i^{(1)}(x),
\]
we get
\[
H_1(x) = \begin{cases} 
-l, & \text{for } x \in \tilde{E}_1 \subset \Delta, \quad |\tilde{E}_1| = \frac{|\Delta|}{2}, \\
0, & \text{for } x \notin \Delta.
\end{cases}
\]

As the polynomial \( \overline{H}_i^{(1)}(x) \) is a linear combination of Walsh functions from \( K_1^{(1)} \) group, it is clear, that the set \( \tilde{E}_1 \) can be presented as a union of certain \( N_2 \) number of disjoint dyadic intervals
\[
\tilde{E}_1 = \bigcup_{i=1}^{N_2} \Delta_i^{(2)}
\]
with measure \(|\Delta_i^{(2)}| = 2^{-K_{N_1}^{(1)}}, \ i = \overline{1, N_2} \).

By defining
\[
E_1 = \Delta \setminus \tilde{E}_1
\]
and
\[
\begin{cases} 
\bar{a}_k = |l|2^{-K_{i+1}^{(1)}}, & \text{when } k \in \left[2^{K_{i-1}^{(1)}}, 2^{K_{i}^{(1)}}\right), \ i \in [1, N_1], \\
\bar{\delta}_k = \begin{cases} 0, & \text{when } k \in \left[2^{K_{i-1}^{(1)}}, 2^{K_{i}^{(1)}}\right), \ i \in [1, N_1], \\
1, & \text{when } k \in \left[2^{K_{i}^{(1)}}, 2^{K_{i+1}^{(1)}}\right), \end{cases} \\
\bar{a}_k = |\bar{a}_k|, \ \bar{\delta}_k = \bar{\delta}_k \cdot \frac{\bar{a}_k}{|\bar{a}_k|}, & \text{when } k \in \left[2^\bar{n}, 2^{K_{i+1}^{(1)}}\right],
\end{cases}
\]
let us verify that the set \( E_1 \) and polynomials
\[
P_1(x) = \sum_{k=2^\bar{n}}^{2^{K_{i+1}^{(1)}-1}} a_k W_k(x) \quad \text{and} \quad H_1(x) = \sum_{k=2^\bar{n}}^{2^{K_{i+1}^{(1)}-1}} \delta_k a_k W_k(x), \ \delta_k = \pm 1, 0
\]
satisfy all statements of lemma 2.2 for \( q = 1 \). Indeed, by using (2.11) and (2.12) we obtain \(|E_1| = (1 - 2^{-1})|\Delta|\). The statement 1) follows from (2.6), (2.8), (2.13) and from monotonicity of numbers \( K_{i}^{(1)} (i = 1, N_1) \). The statement 2) immediately follows from (2.11) and (2.12). To prove statements 3) and 4) we present the natural number \( M \in \left[2^{\bar{n}}, 2^{K_{N_1}^{(1)}+1}\right] \) in the form \( M = 2^{\bar{n}} + s, \ s \in [0, 2^{\bar{n}}], \) where \( \bar{n} \in \left(K_{m-1}^{(1)}, K_m^{(1)}\right) \) for some \( m \in [1, N_1] \). Since intervals \( \Delta_i^{(1)} (i = 1, N_1) \) are disjoint, by using (2.4), (2.7), (2.9)–(2.13) we have
\[
\left\| \sum_{k=2^\bar{n}}^{M} \delta_k a_k W_k \right\|_{L^p[0,1]} \leq \left\| \sum_{i=1}^{m-1} \overline{H}_i^{(1)} \right\|_{L^p[0,1]} + \left\| \sum_{k=2^\bar{n}}^{2^\bar{n}+s} \delta_k a_k W_k \right\|_{L^p[0,1]} \leq \left\| H_1 \right\|_{L^p[0,1]} + C_p \left\| \overline{H}_m^{(1)} \right\|_{L^p[0,1]} \leq
\]
\[
\left( |l|^p |E_1| + |l|^p |\tilde{E}_1| \right)^{\frac{1}{p}} + C_p |l| |\Delta_m^{(1)}|^{\frac{1}{p}} < 2C |l| |\Delta|^{\frac{1}{p}},
\]
where \( C = C_p + 1 \).

Further, for each natural number \( n \in [n_0, K_{N_1}^{(1)}] \) we denote \( b_n = a_k, \ k \in [2^n, 2^{n+1}) \) (coefficients \( a_k \) of Walsh functions from \( n \)-th group are equal in \( H_1(x) \)). Taking into account (2.2), (2.3), (2.5), (2.8), (2.13) and b) condition for numbers \( K_{i}^{(1)} \ (i = 1, N_1) \), we get

\[
\sum_{n=n_0}^{K_{N_1}^{(1)}} b_n = \sum_{n=K_{i-1}^{(1)}+1}^{N_1} b_n = \sum_{i=1}^{N_1}(K_i^{(1)} - K_{i-1}^{(1)}) |l|2^{-\frac{k^{(i)}+K_{i+1}^{(1)}}{2}} < \frac{\varepsilon}{4},
\]

\[
\left\| \sum_{k=2^n}^{M} a_k W_k \right\|_{L^1[0,1]} \leq \sum_{n=n_0}^{\bar{n}-1} b_n + \left\| \sum_{k=2^n}^{2^n+1} a_k W_k \right\|_{L^1[0,1]} \leq \varepsilon + |l|2^{-\frac{K_{i}^{(1)}+K_{i+1}^{(1)}}{2}} \frac{\sigma}{2} < \varepsilon,
\]

which proves the statement 4) of lemma 2.2.

Assume, that for \( q > 1 \) natural numbers

\[ K_1^{(q)} < \ldots < K_{N_1}^{(q)} < \ldots < K_1^{(q-1)} < \ldots < K_{N_{q-1}^{(q-1)}}^{(q-1)}, \]

sets

\( \tilde{E}_{q-1} \subset \Delta \) and \( E_{q-1} = \Delta \setminus \tilde{E}_{q-1} \)

and polynomials

\[
P_{q-1}(x) = \sum_{k=2^n}^{2^{k^{(q-1)}+1}-1} a_k W_k(x),
\]

\[
H_{q-1}(x) = \sum_{k=2^n}^{2^{k^{(q-1)}+1}-1} \delta_k a_k W_k(x), \ \delta_k = \pm 1, 0
\]

are already chosen to satisfy the conditions

\[
a') \ K_{i-1}^{(q-1)} \frac{K_{N_{q-1}^{(q-1)}}^{(q-1)} - K_{i-1}^{(q-1)} - 1}{2} \text{ is a whole number } (K_{N_0}^{(0)} \equiv K_1),
\]

\[
b') \ (K_i^{(q)} - K_{i-1}^{(q)}) 2^{(q-1)} |l|2^{-\frac{K_{i}^{(q)}+K_{i+1}^{(q-1)}+1}{2}} < \frac{\varepsilon}{2^{q+1} N_{q'}},
\]

\[
c') \ 2^{q'} |l|2^{-\frac{K_{i}^{(q)}+K_{i+1}^{(q-1)}+1}{2}} < \frac{\varepsilon}{2},
\]

\[
a_k = 2^{q-1} |l|2^{-\frac{K_{i}^{(q)}+K_{i+1}^{(q-1)}+1}{2}} \text{ for } k \in \left[ 2^{K_{i}^{(q-1)}+1}, 2^{K_{i}^{(q-1)}+1} \right), \ (2.14)
\]

\[
K_{0}^{(q)} = \begin{cases} K_{N_{q-1}^{(q-1)}}^{(q-1)}, & \text{if } \nu > 1, \\ n_0 - 1, & \text{if } \nu = 1, \end{cases}
\]
for any natural numbers \( i \in [1, N_\nu] \) and \( \nu \in [1, q - 1] \). Besides,

\[
\sum_{n=n_0}^{K_{q-1}} b_n < \sum_{k=1}^{q-1} \varepsilon \frac{q}{2k+1}, \quad \text{where} \quad b_n \equiv a_k, \quad k \in [2^n, 2^{n+1}),
\]

(2.15)

\[
H_{q-1}(x) = \begin{cases} 
-(2^{q-1} - 1)l, & \text{for} \ x \in \tilde{E}_{q-1}, \\
l, & \text{for} \ x \in E_{q-1}, \\
0, & \text{for} \ x \notin \Delta,
\end{cases}
\]

(2.16)

\[
|\tilde{E}_{q-1}| = 2^{-q+1} \Delta \quad \text{and} \quad |E_{q-1}| = (1 - 2^{-q+1}) \Delta
\]

(2.17)

and the set \( \tilde{E}_{q-1} \) can be presented as a union of certain \( N_q \) number of disjoint dyadic intervals

\[
\tilde{E}_{q-1} = \bigcup_{i=1}^{N_q} \Delta_i^{(q)}
\]

(2.18)

with measure \( |\Delta_i^{(q)}| = 2^{-K_{q-1}^{(q)-1}}, \ i = 1, N_q. \)

For each natural number \( i \in [1, N_q] \) we choose a natural number \( K_i^{(q)} > K_{i-1}^{(q)} \) \((K_0^{(q)} = K_{N_q-1}^{(q)} = 0)\) such that the following conditions hold:

\[
a'^{''} \ \frac{K_i^{(q)} - K_{i-1}^{(q)-1}}{2} \ \text{is a whole number},
\]

\[
b'^{''} \ (K_i^{(q)} - K_{i-1}^{(q)}) 2^{(q-1)} |l| 2^{-\frac{K_i^{(q)}+K_{i-1}^{(q)-1}+1}{2}} < \frac{\varepsilon}{2^{q+1} N_q},
\]

\[
ce'^{''} 2^q |l| 2^{-\frac{K_i^{(q)+1}}{2}} < \frac{\varepsilon}{2}.
\]

By successively applying lemma 2.1 for each interval \( \Delta_i^{(q)} \subset \tilde{E}_{q-1} \ (i = 1, N_q) \) and corresponding number \( K_i^{(q)} \), we can find polynomials in the Walsh system

\[
\overline{H}_i^{(q)}(x) = \sum_{k=2^{K_i^{(q)}}}^{2^{K_i^{(q)+1}-1}} \bar{a}_k W_k(x), \quad i = 1, N_q,
\]

(2.19)

such that

\[
|\bar{a}_k| = 2^{q-1} |l| 2^{-\frac{K_i^{(q)}+K_{i-1}^{(q)-1}+1}{2}}, \quad \text{when} \ k \in [2^{K_i^{(q)}}, 2^{K_i^{(q)+1}}),
\]

(2.20)

\[
\overline{H}_i^{(q)}(x) = \begin{cases} 
-2^{q-1}l, & \text{for} \ x \in \tilde{E}_i^{(q)} \subset \Delta_i^{(q)}, \quad |\tilde{E}_i^{(q)}| = \frac{1}{2} |\Delta_i^{(q)}|, \\
2^{q-1}l, & \text{for} \ x \in \tilde{E}_i^{(q)} \subset \Delta_i^{(q)}, \quad |\tilde{E}_i^{(q)}| = \frac{1}{2} |\Delta_i^{(q)}|, \\
0, & \text{for} \ x \notin \Delta_i^{(q)}.
\end{cases}
\]

(2.21)

Hence, by denoting

\[
H_q(x) = H_{q-1}(x) + \sum_{i=1}^{N_q} \overline{H}_i^{(q)}(x)
\]

(2.22)
and taking into account (2.16) and (2.18), we obtain
\[
H_q(x) = \begin{cases} 
-(2^q - 1)l, & \text{for } x \in \tilde{E}_q \subset \tilde{E}_{q-1}, \\
l, & \text{for } x \in \Delta \setminus \tilde{E}_q, \\
0, & \text{for } x \notin \Delta.
\end{cases}
\] (2.23)

Now, after defining
\[
E_q = \Delta \setminus \tilde{E}_q
\] (2.24) and
\[
\begin{align*}
\bar{a}_k &= 2^{q-1}|l|2^{-\frac{K^{(q)}_i + K^{(q-1)}_i + 1}{2}}, & \text{when } k \in [2K^{(q)}_{i-1} + 1, 2K^{(q)}_{i}], \\
\bar{\delta}_k &= \begin{cases} 
0, & \text{when } k \in [2K^{(q)}_{i-1} + 1, 2K^{(q)}_{i}), \\
1, & \text{when } k \in [2K^{(q)}_{i}, 2K^{(q)}_{i+1}), \\
\end{cases} \\
\delta_k &= \bar{\delta}_k \cdot \frac{\bar{a}_k}{|\bar{a}_k|}, & \text{when } k \in [2^n, 2K^{(q)}_{n+1}],
\end{align*}
\] (2.25)

let us verify that the set \(E_q\) and polynomials
\[
P_q(x) = \sum_{k=2^n}^{2^{n+1}-1} a_k W_k(x),
\]
\[
H_q(x) = \sum_{k=2^n}^{2^{n+1}-1} \delta_k a_k W_k(x),
\]
where \(n_q \equiv K^{(q)}_{n_q} + 1\), satisfy all statements of lemma 2.2. Indeed, from (2.23) and (2.24) it follows that \(|E_q| = (1 - 2^{-q})|\Delta|\). The statement 1) follows from (2.6), (2.14), (2.20), (2.25) and from monotonicity of numbers \(K^{(\nu)}_i\), \(i \in [1, N_q]\), \(\nu \in [1, q]\). The statement 2) immediately follows from (2.23) and (2.24). To prove statements 3) and 4) we present the natural number \(M \in [2^n, 2^{n+1})\) in the form \(M = 2^n + s, \ s \in [0, 2^n]\). Let us consider only the case when \(\bar{n} \in (K^{(q)}_{n_q-1}, K^{(q)}_{n_q})\), since all other cases are under consideration in previous steps of induction. Let \(\bar{n} \in (K^{(q)}_{m-1}, K^{(q)}_{m})\) for some \(m \in [1, N_q]\). From (2.4), (2.16)–(2.19), (2.21) and (2.25) we have
\[
\left\| \sum_{k=2^n}^{M} \delta_k a_k W_k \right\|_{L^p[0,1]} \leq \left\| H_{q-1} + \sum_{i=1}^{m-1} \overline{H}^{(q)}_i \right\|_{L^p[0,1]} + \\
\left\| \sum_{k=2^n}^{2^{n+1}-s} \delta_k a_k W_k \right\|_{L^p[0,1]} \leq \left\| H_q \right\|_{L^p[0,1]} + C_p \left\| \overline{H}^{(q)}_m \right\|_{L^p[0,1]} < \\
< \left( |l| \left| E_q \right| + 2^{pq}|l|^p \left| \tilde{E}_q \right| \right)^{\frac{1}{p}} + C_p 2^{q-1} |l| \left| \Delta^{(q)}_m \right|^{\frac{1}{p}} < 2^q C |l| |\Delta|^{\frac{1}{p}}
\]
\((C = C_p + 1)\), which proves the statement 3).

Further, for each natural number \(n \in [n_0, K^{(q)}_{n_q}]\) we denote
\[
b_n \equiv a_k, \ \text{when } k \in [2^n, 2^{n+1}).
Taking into account (2.2), (2.3), (2.15), (2.20), (2.25), (c') condition for number \( K_{N_q-1}^{(q-1)} \) and \( b' \) condition for numbers \( K_i^{(q)} (i = 1, N_q) \) we get

\[
\left\| \sum_{k=2^{n_0}}^{M} a_k W_k \right\|_{L^1[0,1]} \leq \sum_{n=n_0}^{\tilde{n}-1} b_n + \left\| \sum_{k=2^n}^{2^{n+s}} a_k W_k \right\|_{L^1[0,1]} \leq \sum_{n=n_0}^{K_{N_q-1}^{(q-1)}} b_n + \sum_{n=K_{i-1}^{(q)}+1}^{K_i^{(q)}} b_n + \left\| \sum_{k=2^n}^{2^{n+1}-1} b_n W_k \right\|_{L^2[0,1]} \leq \sum_{k=1}^{q-1} \frac{\varepsilon}{2k+1} + \sum_{i=1}^{N_q} \left( K_i^{(q)} - K_{i-1}^{(q)} \right) 2^{q-1} |l| 2^{\frac{K_i^{(q)} + K_{i-1}^{(q-1)} + 1}{2}} + 2^{q-1} |l| 2^{\frac{K_{i}^{(q)} + K_{i-1}^{(q-1)} + 1}{2}} < \varepsilon,
\]

which proves the statement 4).

Lemma 2.2 is proved. \[\square\]

**Lemma 2.3.** Let numbers \( p_0 > 1, \ n_0 \in \mathbb{N}, \ 0 < \varepsilon < 1 \) and polynomial \( f(x) \neq 0 \) in the Walsh system be given. Then one can find a measurable set \( E_\varepsilon \) with measure \( |E_\varepsilon| > 1 - \varepsilon \) and polynomials

\[
P(x) = \sum_{k=2^{n_0}}^{2^n-1} a_k W_k(x) \quad \text{and} \quad H(x) = \sum_{k=2^{n_0}}^{2^n-1} \delta_k a_k W_k(x), \quad \delta_k = 0, \pm 1,
\]
in the Walsh system, which satisfies the following conditions:

1) \( 0 < a_{k+1} < a_k < \varepsilon, \quad k \in [2^{n_0}, 2^n - 1], \)

2) \( \| f(x) - H(x) \|_{L^{p_0}(E_\varepsilon)} < \varepsilon, \)

3) \( \max_{2^{n_0} \leq M < 2^n} \left\| \sum_{k=2^{n_0}}^{M} \delta_k a_k W_k(x) \right\|_{L^p(e)} < \| f(x) \|_{L^p(e)} + \varepsilon \)

for any measurable set \( e \subseteq E_\varepsilon \) and \( p \in [1, p_0], \)

4) \( \max_{2^{n_0} \leq M < 2^n} \left\| \sum_{k=2^{n_0}}^{M} a_k W_k(x) \right\|_{L^1[0,1]} < \varepsilon. \)

**Proof.** We choose a natural number \( q \), so that

\[
2^{-q} < \varepsilon, \quad (2.26)
\]

and present the function \( f(x) \) in the form

\[
f(x) = \sum_{j=1}^{i_0} l_j \chi_{\Delta_j}(x),
\]
where \( l_j \neq 0 \), \( j = 1, \nu_0 \), and \( \{ \Delta_j \}_{j=1}^{\nu_0} \) are disjoint dyadic subintervals of the section \([0, 1]\). Without loss of generality we can assume that all these intervals have the same length and are small enough to provide the condition
\[
\max_{1 \leq j \leq \nu_0} \left\{ 2^q C |l_j| \left| \Delta_j \right|^1 \right\} < \frac{\varepsilon}{2}, \tag{2.27}
\]

By successively applying lemma 2.2 for each interval \( \Delta_j \), \( j = 1, \nu_0 \), and taking into account (2.26) and (2.27), we can find sets \( E_q^{(j)} \subset \Delta_j \) with measure
\[
\left| E_q^{(j)} \right| = (1 - 2^{-q}) \left| \Delta_j \right| > (1 - \varepsilon) \left| \Delta_j \right| \tag{2.28}
\]
and polynomials
\[
\bar{P}_q^{(j)}(x) = \sum_{k=2^{n_j-1}}^{2^{n_j}} \bar{a}_k^{(j)} W_k(x),
\]
\[
\bar{H}_q^{(j)}(x) = \sum_{k=2^{n_j-1}}^{2^{n_j}} \delta_k^{(j)} \bar{a}_k^{(j)} W_k(x), \quad \delta_k^{(j)} = \pm 1, 0
\]
in the Walsh system, so that \( \bar{H}_q^{(j)}(x) = 0 \) outside \( \Delta_j \),
\[
\left\{ \begin{array}{ll}
0 < \bar{a}_k^{(j)} & \leq \bar{a}_k^{(1)} < \frac{\varepsilon}{2}, \\
0 < \bar{a}_k^{(j)} & \leq \bar{a}_k^{(j-1)} < \bar{a}_k^{(j-1)-1},
\end{array} \right. \quad \text{for all } k \in [2^{n_0}, 2^{n_j-1} - 1), \tag{2.29}
\]
\[
\left\| l_j \chi_{\Delta_j} - \bar{H}_q^{(j)} \right\|_{L^p[0,1]} = 0, \tag{2.30}
\]
\[
\max_{2^{n_j-1} \leq M < 2^{n_j}} \left\| \sum_{k=2^{n_j-1}}^{M} \delta_k^{(j)} \bar{a}_k^{(j)} W_k \right\|_{L^p[0,1]} < 2^q C |l_j| \left| \Delta_j \right|^\frac{1}{p_0} < \frac{\varepsilon}{2}, \tag{2.31}
\]
\[
\max_{2^{n_j-1} \leq M < 2^{n_j}} \left\| \sum_{k=2^{n_j-1}}^{M} \bar{a}_k^{(j)} W_k \right\|_{L^1[0,1]} < \frac{\varepsilon}{2^{j+1}}. \tag{2.32}
\]

We define a set
\[
E_\varepsilon = \bigcup_{j=1}^{\nu_0} E_q^{(j)} \cup ([0, 1] \setminus \bigcup_{j=1}^{\nu_0} \Delta_j) \tag{2.33}
\]
and polynomials
\[
\bar{P}(x) = \sum_{j=1}^{\nu_0} \bar{P}_q^{(j)}(x) = \sum_{k=2^{n_0}}^{2^{n_0 \nu_0 - 1}} \bar{a}_k W_k(x),
\]
\[
\bar{H}(x) = \sum_{j=1}^{\nu_0} \bar{H}_q^{(j)}(x) = \sum_{k=2^{n_0}}^{2^{n_0 \nu_0 - 1}} \delta_k \bar{a}_k W_k(x),
\]
where \( \bar{a}_k = \bar{a}_k^{(j)} \) and \( \delta_k = \delta_k^{(j)} \), when \( k \in [2^{n_j-1}, 2^{n_j}) \). Note that \( \bar{H}_q^{(j)} = 0 \) on the set \([0, 1] \setminus \bigcup_{j=1}^{\nu_0} \Delta_j \) (in case it is not empty) for any \( j \in [1, \nu_0] \).

From (2.28)–(2.30) and (2.33) it follows that
\[
\left| E_\varepsilon \right| > 1 - \varepsilon,
\]
\[0 < \bar{a}_{k+1} \leq \bar{a}_k < \frac{\varepsilon}{2}, \quad \text{when} \quad k \in \left[2^{n_0}, 2^{n_0} - 1\right),\]  
(2.34)

\[\| f - \bar{H} \|_{L^p(E_\varepsilon)} \leq \sum_{j=1}^{\nu_0} \| l_j \chi_{\Delta_j} - \bar{H}_q^{(j)} \|_{L^p(E_q^{(j)})} = 0.\]  
(2.35)

Further, let \( M \) be a natural number from \( \left[2^{n_0}, 2^{n_0}\right) \). Then \( M \in \left[2^{n_0-1}, 2^{n_0}\right) \) for some \( m \in [1, \nu_0] \). Taking into account (2.30), (2.31) and (2.33), for any measurable set \( e \subseteq E_\varepsilon \) and \( p \in [1, p_0] \) we have

\[\left\| \sum_{k=2^{n_0}}^{M} \delta_k \bar{a}_k W_k \right\|_{L^p(e)} \leq \]  
(2.36)

\[\left\| \sum_{j=1}^{m-1} \bar{H}_q^{(j)} \right\|_{L^p(e)} + \left\| \sum_{k=2^{n_0-1}}^{M} \delta_k^{(m)} a_k^{(m)} W_k \right\|_{L^p(e)} \leq \]

\[\leq \sum_{j=1}^{m-1} \| l_j \chi_{\Delta_j} - \bar{H}_q^{(j)} \|_{L^p(e)} + \sum_{j=1}^{m-1} \| l_j \chi_{\Delta_j} \|_{L^p(e)} + \]

\[+ \left\| \sum_{k=2^{n_0-1}}^{M} \delta_k^{(m)} a_k^{(m)} W_k \right\|_{L^p[0,1]} < \| f \|_{L^p(e)} + \frac{\varepsilon}{2}\]

and, by using (2.32), we obtain

\[\left\| \sum_{k=2^{n_0}}^{M} \bar{a}_k W_k \right\|_{L^1[0,1]} \leq \sum_{j=1}^{\nu_0} \max_{2^{n_0-j-1} \leq N < 2^{n_0}} \left\| \sum_{k=2^{n_0-j-1}}^{N} \delta_k^{(j)} W_k \right\|_{L^1[0,1]} < \frac{\varepsilon}{2}.\]  
(2.37)

Hence, polynomials \( \bar{P}(x) \) and \( \bar{H}(x) \) satisfy all statements of lemma 3 except for 1). To have strict inequalities between coefficients we choose such a natural number \( N_0 \) that

\[2^{-N_0} < \frac{\varepsilon}{2}\]  
(2.38)

and define polynomials

\[P(x) = \sum_{k=2^{n_0}}^{2^{n_0}-1} a_k W_k(x) \quad \text{and} \quad H(x) = \sum_{k=2^{n_0}}^{2^{n_0}-1} \delta_k a_k W_k(x),\]

where

\[a_k = \bar{a}_k + 2^{-(N_0+k)}\]  
(2.39)

It is not hard to verify that polynomials \( P(x) \) and \( H(x) \) satisfy all statements of lemma 2.3. Indeed, the statement 1) immediately follows from (2.34), (2.38) and (2.39). Further, considering (2.35)–(2.39) for each natural number \( M \in \left[2^{n_0}, 2^{n_0}\right) \), measurable set \( e \subseteq E_\varepsilon \) and \( p \in [1, p_0] \) we get

\[\| f - H \|_{L^p(E_\varepsilon)} \leq \| f - \bar{H} \|_{L^p(E_\varepsilon)} + \| \sum_{k=2^{n_0}}^{2^{n_0}-1} \delta_k 2^{-(N_0+k)} W_k \|_{L^p[0,1]} \leq\]
is proved.

Lemma 2.4. For any \( \delta \in (0, 1) \) there exist a weight function \( 0 < \mu(x) \leq 1 \), with \( |\{ x \in [0, 1]; \, \mu(x) = 1 \}| > 1 - \delta \), so that for any numbers \( p_0 > 1, \, n_0 \in \mathbb{N}, \, \varepsilon \in (0, 1) \) and polynomial \( f(x) \neq 0 \) in the Walsh system, one can find polynomials in the Walsh system

\[
P(x) = \sum_{k=2^{n_0}}^{2^{n_1} - 1} a_k W_k(x) \quad \text{and} \quad H(x) = \sum_{k=2^{n_0}}^{2^{n_1} - 1} \delta_k a_k W_k(x), \quad \delta_k = \pm 1, 0
\]

satisfying the following conditions:

1) \( 0 < a_{k+1} < a_k < \varepsilon, \quad k \in [2^{n_0}, 2^n - 1) \),

2) \( \| f - H \|_{L^p[0,1]} < \varepsilon \),

3) \( \max_{2^{n_0} \leq M < 2^n} \left\| \sum_{k=2^{n_0}}^{M} \delta_k a_k W_k \right\|_{L^p[0,1]} < 2 \| f \|_{L^p[0,1]} + \varepsilon, \quad \forall p \in [1, p_0] \),

4) \( \max_{2^{n_0} \leq M < 2^n} \left\| \sum_{k=2^{n_0}}^{M} a_k W_k \right\|_{L^1[0,1]} < \varepsilon \).

Proof. Let \( p_m \to +\infty, \, \delta \in (0, 1), \, N_0 = 1 \) and \( \{ f_m(x) \}_{m=1}^\infty \), \( x \in [0, 1] \), be a sequence of all polynomials in the Walsh system with rational coefficients. By
successively applying lemma 2.3, one can find sets \( E_m \subset [0, 1] \) and polynomials in the Walsh system of the form

\[
P_m(x) = \sum_{k=2^{N_m-1}}^{2^{N_m-1}} a_k^{(m)} W_k(x),
\]

\[
H_m(x) = \sum_{k=2^{N_m-1}}^{2^{N_m-1}} \delta_k^{(m)} a_k^{(m)} W_k(x), \quad \delta_k^{(m)} = \pm 1, 0,
\]

which satisfy the following conditions for any natural number \( m \):

\[
|E_m| > 1 - \frac{1}{2^{m+1}},
\]

\[
0 < a_{k+1}^{(m)} < a_k^{(m)} < \frac{1}{4^{N_m-1}}, \quad k \in [2^{N_m-1}, 2^{N_m} - 1),
\]

\[
\|f - H_m\|_{L^p(E_m)} < \frac{1}{2^{m+2}},
\]

\[
\max_{2^{N_m-1} \leq M < 2^{N_m}} \left\| \sum_{k=2^{N_m-1}}^{M} \delta_k^{(m)} a_k^{(m)} W_k \right\|_{L^p(e)} < \|f\|_{L^p(e)} + \frac{1}{2^{m+2}},
\]

for any measurable set \( e \subseteq E_m \) and \( p \in [1, p_m] \), and

\[
\max_{2^{N_m-1} \leq M < 2^{N_m}} \left\| \sum_{k=2^{N_m-1}}^{M} a_k^{(m)} W_k \right\|_{L^1[0,1]} < \frac{1}{2^{m+2}}.
\]

We set

\[
\begin{align*}
\Omega_n &= \bigcap_{m=n}^{+\infty} E_m, \quad n \in \mathbb{N}, \\
E &= \Omega_{\tilde{n}} = \bigcap_{m=\tilde{n}}^{+\infty} E_m, \quad \tilde{n} = \lfloor \log_{1/2} \delta \rfloor + 1, \\
B &= \Omega_{\tilde{n}} \cup \left( \bigcup_{n=\tilde{n}+1}^{+\infty} \Omega_n \setminus \Omega_{n-1} \right).
\end{align*}
\]

It is clear (see (2.42) and (2.47)) that

\[
|B| = 1, \quad |E| > 1 - \delta.
\]

We define a function \( \mu(x) \) in the following way:

\[
\mu(x) = \begin{cases} 
1, & x \in E \cup ([0, 1] \setminus B), \\
\mu_n, & x \in \Omega_n \setminus \Omega_{n-1}, \ n \geq \tilde{n} + 1,
\end{cases}
\]

where

\[
\mu_n = \frac{1}{2p_n(n+2)} \cdot \left[ \prod_{m=1}^{n} h_m \right]^{-1},
\]
\[ h_m = \max_{1 \leq p \leq \tilde{n}} \left( 1 + \int_0^1 |f_m(x)|^p dx + \max_{2^{N-1} \leq M < 2^N} \int_0^1 \left| \sum_{k=2^{N-1}}^M \delta_k^{(m)} a_k^{(m)} W_k(x) \right|^p dx \right). \]

It follows from (2.47)–(2.49) that for all \( m \geq \tilde{n} \)
\[
\int_{[0,1] \setminus \Omega_m} |H_m(x)|^{p_m} \mu(x) dx = \sum_{n=m+1}^{+\infty} \left( \int_{\Omega_n \setminus \Omega_{n-1}} |H_m(x)|^{p_m} \mu_n dx \right) < (2.50)
\]
\[
< \sum_{n=m+1}^{+\infty} \frac{1}{2^p(n+2) h_m} \left( \int_0^1 |H_m(x)|^{p_m} dx \right) < \frac{1}{2^p(m+2)}.
\]

In a similar way for all \( m \geq \tilde{n} \), \( M \in [2^{N-1}, 2^N) \) and \( p \in [1, p_m] \) we have
\[
\int_{[0,1] \setminus \Omega_m} |f_m(x)|^{p_m} \mu(x) dx < \frac{1}{2^p(m+2)} \quad (2.51)
\]
and
\[
\int_{[0,1] \setminus \Omega_m} \left| \sum_{k=2^{N-1}}^M \delta_k^{(m)} a_k^{(m)} W_k(x) \right|^p \mu(x) dx < \frac{1}{2^p(m+2)} \quad (2.52)
\]

Since \( \Omega_m \subset E_m \), by using conditions (2.44), (2.47)–(2.51) and Jensen’s inequality, for all \( m \geq \tilde{n} \) we obtain
\[
\int_0^1 |f_m(x) - H_m(x)|^{p_m} \mu(x) dx = \int_{\Omega_m} |f_m(x) - H_m(x)|^{p_m} \mu(x) dx +
\]
\[
+ \int_{[0,1] \setminus \Omega_m} |f_m(x) - H_m(x)|^{p_m} \mu(x) dx < \frac{1}{2^p(m+2)} + 2^{p_m} \frac{1}{2^p(m+2)} < \frac{1}{2^p(m-1)},
\]
or
\[
\|f_m - H_m\|_{L^p_{\mu}[0,1]} < \frac{1}{2^{m-1}} \quad (2.53)
\]

Further, taking relations (2.45), (2.47)–(2.49), (2.52) and Jensen’s inequality into account for all \( M \in [2^{N-1}, 2^N) \), \( p \in [1, p_m] \) and \( m \geq \tilde{n} + 1 \) we get
\[
\int_0^1 \left| \sum_{k=2^{N-1}}^M \delta_k^{(m)} a_k^{(m)} W_k(x) \right|^p \mu(x) dx = \int_{\Omega_m} \left| \sum_{k=2^{N-1}}^M \delta_k^{(m)} a_k^{(m)} W_k(x) \right|^p \mu(x) dx +
\]
\[
+ \int_{[0,1] \setminus \Omega_m} \left| \sum_{k=2^{N-1}}^M \delta_k^{(m)} a_k^{(m)} W_k(x) \right|^p \mu(x) dx <
\]
\[
< \int_{\Omega_{\tilde{n}}} \left| \sum_{k=2^{N-1}}^M \delta_k^{(m)} a_k^{(m)} W_k(x) \right|^p \mu(x) dx +
\]
\[
+ \sum_{n=\tilde{n}+1}^m \mu_n \cdot \int_{\Omega_n \setminus \Omega_{n-1}} \left| \sum_{k=2^{N-1}}^M \delta_k^{(m)} a_k^{(m)} W_k(x) \right|^p dx + \frac{1}{2^p(m+2)} <
\]
\[
\left( \| f_m \|_{L^p(\Omega_n)} + \frac{1}{2^{m+2}} \right)^p + \sum_{n=n+1}^m \mu_n \left( \| f_m \|_{L^p(\Omega_n \setminus \Omega_{n-1})} + \frac{1}{2^{m+2}} \right)^p + \frac{1}{2^{p(m+2)}} \leq
\]
\[
\leq 2^p \left( \int_{\Omega_n} |f_m(x)|^p \, dx + \sum_{n=n+1}^m \int_{\Omega_n \setminus \Omega_{n-1}} |f_m(x)|^p \cdot \mu_n \, dx \right) +
\]
\[
+ \frac{1}{2^{p(m+2)}} \left( 2^p + 2^p \cdot \sum_{n=n+1}^m \mu_n + 1 \right) < 2^p \| f_m \|_{L^p_n[0,1]}^p + \frac{1}{2^{p(m+2)}}
\]
or
\[
\left\| \sum_{k=2^{N_{m-1}}}^M \delta_k^{(m)} a_k^{(m)} W_k \right\|_{L^p_n[0,1]}^p < 2 \| f_m \|_{L^p_n[0,1]} + \frac{1}{2^{m-1}}. \quad (2.54)
\]

Let \( n_0 \in \mathbb{N} \) and \( \varepsilon \in (0, 1) \) be arbitrarily given. From the sequence \( \{ f_m(x) \}_{m=1}^{\infty} \) we choose such a function \( f_{m_0}(x) \) that
\[
m_0 > \max \left\{ n, \log_2 \frac{8}{\varepsilon} \right\}, \quad p_{m_0} > p_0, \quad 2^{N_{m_0} - 1} > 2^{n_0}, \quad (2.55)
\]
and for \( k \in [2^{n_0}, 2^{N_{m_0}}) \) set
\[
a_k = \begin{cases} 
\delta_k^{(m_0)} + \frac{1}{2^{n_0}}, & \text{when } k \in [2^{n_0}, 2^{N_{m_0} - 1}) \\
\delta_k^{(m_0)}, & \text{when } k \in [2^{N_{m_0} - 1}, 2^{N_{m_0}}) \\
0, & \text{when } k \in [2^{n_0}, 2^{N_{m_0} - 1}) 
\end{cases} \quad (2.57)
\]
\[
\delta_k = \begin{cases} 
\pm 1, 0, & \text{when } k \in [2^{N_{m_0} - 1}, 2^{N_{m_0}}) 
\end{cases} \quad (2.58)
\]
and
\[
P(x) = \sum_{k=2^{n_0}}^{2^{N_{m_0} - 1}} a_k W_k(x) = \sum_{k=2^{n_0}}^{2^{N_{m_0} - 1}} a_k W_k(x) + P_{m_0}(x),
\]
\[
H(x) = \sum_{k=2^{n_0}}^{2^{N_{m_0} - 1}} \delta_k a_k W_k(x) = H_{m_0}(x).
\]

Now it is not hard to verify that the function \( \mu(x) \) and polynomials \( P(x) \) and \( H(x) \) satisfy all requirements of lemma 2.4. Indeed, statements 1)–3) immediately follow from (2.43), (2.53)–(2.58). Further, by using (2.46), (2.55)–(2.57) we obtain
\[
\max_{2^{n_0} \leq M < 2^{N_{m_0}}} \left\| \sum_{k=2^{n_0}}^{M} a_k W_k \right\|_{L^1[0,1]} \leq \max_{2^{n_0} \leq M_1 < 2^{N_{m_0} - 1}} \left\| \sum_{k=2^{n_0}}^{M_1} a_k W_k \right\|_{L^1[0,1]} +
\]
\[
+ \max_{2^{N_{m_0} - 1} \leq M_2 < 2^{N_{m_0}}} \left\| \sum_{k=2^{N_{m_0} - 1}}^{M_2} a_k^{(m_0)} W_k \right\|_{L^1[0,1]} < \max_{2^{n_0} \leq M_1 < 2^{N_{m_0} - 1}} \left\| \sum_{k=2^{n_0}}^{M_1} a_k W_k \right\|_{L^1[0,1]} + \frac{\varepsilon}{2}. \]
Let $M_1$ be an arbitrary natural number from $[2^{n_0}, 2^{n_0 - 1})$. Then $M_1 \in [2^{n_1}, 2^{n_1 + 1})$ for some $n_1 \in [n_0, N_{m_0 - 1})$ and, considering (2.1), we have

$$\left\| \sum_{k=2^{n_0}}^{M_1} a_k W_k \right\|_{L^1[0,1]} < a_2^{(n_0)} 2^{n_{m_0} - 1} \cdot \left\| \sum_{k=2^{n_0}}^{2^{n_1} - 1} W_k \right\|_{L^1[0,1]} + a_2^{(n_0)} \cdot 2^{n_1 + 1} + \sum_{k=2^{n_0}}^{M_1} \frac{1}{2^{k+m_0}} < \frac{\varepsilon}{2},$$

which proves the statement 4).

Lemma 2.4 is proved.

3. Proof of theorem 1.2

Let $\delta \in (0, 1), p_m \nearrow \infty$ and $\{f_m(x)\}_{m=1}^{\infty}, x \in [0,1]$, be a sequence of all polynomials in the Walsh system with rational coefficients. By applying Lemma 2.4, we obtain a weight function $0 < \mu(x) \leq 1$ with $|\{x \in [0,1], \mu(x) = 1\}| > 1 - \delta$ and polynomials in the Walsh systems

$$P_m(x) = \sum_{k=N_{m-1}}^{N_{m-1}} a_k^{(m)} W_k(x), \quad (3.1)$$

$$H_m(x) = \sum_{k=N_{m-1}}^{N_{m-1}} \delta_k^{(m)} a_k^{(m)} W_k(x), \quad \delta_k^{(m)} = \pm 1, 0, \quad (3.2)$$

which satisfy the following conditions for any natural number $m$:

$$\begin{cases}
0 < a_{k+1}^{(1)} < a_k^{(1)}, \\
0 < a_k^{(m)} < a_k^{(m)} < \min\{2^{-m}, a_{N_{m-1} - 1}^{(m-1)}\} \quad \text{for} \quad m > 1,
\end{cases} 
\quad (3.3)$$

when $k \in [N_{m-1}, N_m - 1)$,

$$\left\| f_m - H_m \right\|_{L^p_m[0,1]} < 2^{-m-1}, \quad (3.4)$$

$$\max_{N_{m-1} \leq M < N_m} \left\| \sum_{k=N_{m-1}}^{M} \delta_k^{(m)} a_k^{(m)} W_k \right\|_{L^p_m[0,1]} < 2\|f_m\|_{L^p_m[0,1]} + 2^{-m}, \quad (3.5)$$

for any $p \in [1, p_m]$, and

$$\max_{N_{m-1} \leq M < N_m} \left\| \sum_{k=N_{m-1}}^{M} a_k^{(m)} W_k \right\|_{L^1[0,1]} < 2^{-m-1}. \quad (3.6)$$

From (3.1) and (3.6) it immediately follows that

$$\left\| \sum_{m=1}^{\infty} P_m \right\|_{L^1[0,1]} \leq \sum_{m=1}^{\infty} \|P_m\|_{L^1[0,1]} < +\infty. \quad (3.7)$$
By denoting
\[ P_0(x) = \sum_{k=0}^{N_0-1} a_k W_k(x), \] (3.8)
where coefficients \( a_k, \ k \in [0, N_0) \), are arbitrary monotonically decreasing positive numbers with \( a_{N_0} = a_{N_0}^{(1)} \), we define a function \( g(x) \) and a series \( \sum_{k=0}^{\infty} a_k W_k(x) \) as follows:
\[ g(x) = \sum_{m=0}^{\infty} P_m(x), \] (3.9)
\[ a_k = a_k^{(m)}, \text{ when } k \in [N_{m-1}, N_m), \ m \in \mathbb{N}, \] (3.10)
and \( a_k \) are coefficients in \( P_0(x) \) (see (3.8)), when \( k \in [0, N_0) \). By using (3.3), (3.6)–(3.10) we conclude that the series \( \sum_{k=0}^{\infty} a_k W_k(x) \) converges to \( g \in L^1[0,1] \) in \( L^1[0,1] \) metric, and \( a_k = \int_0^1 g(t) W_k(t) dt \sim 0. \)

Let \( p \geq 1 \) and let \( f \in L_p^p(0,1) \). We choose such a polynomial \( f_{\nu_1}(x) \) from the sequence \( \{f_m(x)\}_{m=1}^{\infty} \) that
\[ \|f - f_{\nu_1}\|_{L_p^p[0,1]} < 2^{-2} \quad \text{and} \quad p_{\nu_1} > p. \] (3.11)

By denoting
\[ \delta_k = \begin{cases} \delta_k^{(\nu_1)} = \pm 1, 0, & \text{when } k \in [N_{\nu_1-1}, N_{\nu_1}), \\ 0, & \text{when } k \in [0, N_{\nu_1-1}), \end{cases} \] and taking into account (3.2), (3.4), (3.5) and (3.11), we have
\[ \left\| f - \sum_{k=0}^{N_{\nu_1}-1} \delta_k a_k W_k \right\|_{L_p^p[0,1]} \leq \|f - f_{\nu_1}\|_{L_p^p[0,1]} + \|f_{\nu_1} - H_{\nu_1}\|_{L_p^p[0,1]} < 2^{-2} + 2^{-\nu_1-1} < 2^{-1}, \]
and
\[ \max_{N_{\nu_1-1} \leq M < N_{\nu_1}} \left\| \sum_{k=N_{\nu_1-1}}^{M} \delta_k a_k W_k \right\|_{L_p^p[0,1]} < 2\|f_{\nu_1}\|_{L_p^p[0,1]} + 2^{-\nu_1}. \]

Assume that for \( q > 1 \) numbers \( \nu_1 < \nu_2 < \cdots < \nu_{q-1} \) and \( \{\delta_k = \pm 1, 0\}_{k=0}^{N_{\nu_{q-1}}-1} \) are already chosen, so that for each natural number \( j \in [1, q-1] \) the following conditions hold:
\[ \delta_k = \begin{cases} \delta_k^{(\nu_j)} = \pm 1, 0, & \text{when } k \in [N_{\nu_j-1}, N_{\nu_j}), \\ 0, & \text{when } k \notin \bigcup_{j=1}^{q-1} [N_{\nu_j-1}, N_{\nu_j}), \end{cases} \]
\[ \left\| f - \sum_{k=0}^{N_{\nu_j}-1} \delta_k a_k W_k \right\|_{L_p^p[0,1]} < 2^{-j}, \] (3.12)
\[ \max_{N_{\nu_j-1} \leq M < N_{\nu_j}} \left\| \sum_{k=N_{\nu_j-1}}^{M} \delta_k a_k W_k \right\|_{L_p^p[0,1]} < 2\|f_{\nu_j}\|_{L_p^p[0,1]} + 2^{-\nu_j}. \]
We choose a function \( f_{\nu q}(x) \) from the sequence \( \{ f_m(x) \}_{m=1}^{\infty} \) with \( \nu_q > \nu_{q-1} \) so that
\[
\left\| f - \sum_{k=0}^{N_{\nu_q-1}-1} \delta_k a_k W_k(x) - f_{\nu q} \right\|_{L_p^p[0,1]} < 2^{-q-1}, \tag{3.13}
\]
and define
\[
\delta_k = \begin{cases} 
\delta_k^{(\nu_q)} = \pm 1, 0, & \text{when } k \in [N_{\nu_q-1}, N_{\nu_q}),
0, & \text{when } k \notin \bigcup_{j=1}^{q} [N_{\nu_q-1}, N_{\nu_q}).
\end{cases} \tag{3.14}
\]
Taking into account (3.2), (3.4), (3.13) and (3.14), we get
\[
\left\| f - \sum_{k=0}^{N_{\nu_q-1}-1} \delta_k a_k W_k \right\|_{L_p^p[0,1]} \leq \left\| f - \sum_{k=0}^{N_{\nu_q-1}-1} \delta_k a_k W_k - f_{\nu q} \right\|_{L_p^p[0,1]} + \left\| f_{\nu q} - H_{\nu q} \right\|_{L_p^p[0,1]} < 2^{-q-1} + 2^{-\nu_q - 1} < 2^{-q}.
\]
Further, from (3.12) and (3.13) we have
\[
\left\| f_{\nu q} \right\|_{L_p^p[0,1]} < \left\| f - \sum_{k=0}^{N_{\nu_q-1}-1} \delta_k a_k W_k - f_{\nu q} \right\|_{L_p^p[0,1]} + \left\| f - \sum_{k=0}^{N_{\nu_q-1}-1} \delta_k a_k W_k \right\|_{L_p^p[0,1]} < 2^{-q-1} + 2^{-q+1} < 2^{-q+2}.
\]
Thus, from (3.5) and (3.14) it follows that for each natural number \( M \in [N_{\nu_q-1}, N_{\nu_q}) \)
\[
\left\| \sum_{k=N_{\nu_q-1}}^{M} \delta_k a_k W_k \right\|_{L_p^p[0,1]} < 2 \left\| f_{\nu q} \right\|_{L_p^p[0,1]} + 2^{-\nu_q} < 2^{-q+4}. \tag{3.16}
\]
Clearly, by using induction one can determine growing sequence of indexes \( \{ \nu_q \}_{q=1}^{+\infty} \) and numbers \( \{ \delta_k = \pm 1, 0 \}_{k=0}^{+\infty} \) so that conditions (3.14)–(3.16) hold for any \( q \in \mathbb{N} \). Hence, we obtain a series
\[
\sum_{k=0}^{+\infty} \delta_k a_k W_k(x), \quad \delta_k = \pm 1, 0, \tag{3.17}
\]
which converges to \( f \) in \( L_p^p[0,1] \) metric. Indeed, from (3.15) it follows that the subsequence \( \{ S_{N_{\nu q}}(x) \}_{q=1}^{+\infty} \) of its partial sums
\[
S_N(x) \equiv \sum_{k=0}^{N-1} \delta_k a_k W_k(x), \quad N = 1, 2, \ldots,
\]
converges to $f$ in $L^p_{\mu}[0, 1]$ metric, and (3.16) provides the convergence of the whole sequence $S_N(x)$.

The theorem 1.2 is proved.

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1Department of Physics, Yerevan State University, A. Manoogian 1, 0025 Yerevan, Armenia.
E-mail address: gmarting@ysu.am; t.grigoryan@ysu.am

2CANDLE SRI, Acharyan 31, 0040 Yerevan, Armenia; Russian-Armenian (Slavonic) University, H. Emin 123, 0051 Yerevan, Armenia.
E-mail address: asargsyan@ysu.am