A VIRIAL-MORAWETZ APPROACH TO SCATTERING FOR THE NON-RADIAL INHOMOGENEOUS NLS

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Abstract. Consider the focusing inhomogeneous nonlinear Schrödinger equation in $H^1(\mathbb{R}^N)$,

$$iu_t + \Delta u + |x|^{-b}|u|^{p-1}u = 0,$$

when $b > 0$ and $N \geq 3$ in the intercritical case $0 < s_c < 1$. In previous works, the second author, as well as Farah, Guzmán and Murphy, applied the concentration-compactness approach to prove scattering below the mass-energy threshold for radial and non-radial data. Recently, the first author adapted the Dodson-Murphy approach for radial data, followed by Murphy, who proved scattering for non-radial solutions in the 3d cubic case, for $b < 1/2$. This work generalizes the recent result of Murphy, allowing a broader range of values for the parameters $p$ and $b$, as well as allowing any dimension $N \geq 3$. It also gives a simpler proof for scattering nonradial, avoiding the Kenig-Merle road map. We exploit the decay of the nonlinearity, which, together with Virial-Morawetz-type estimates, allows us to drop the radial assumption.

1. Introduction

In this work, we consider the Cauchy problem for the focusing inhomogeneous nonlinear Schrödinger equation (INLS)

$$\begin{cases}
iu_t + \Delta u + |x|^{-b}|u|^{p-1}u = 0, \\
u(0) = u_0 \in H^1(\mathbb{R}^N),
\end{cases}$$

(1.1)

where $u : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$, $N \geq 3$, $0 < b < 2$, and

$$1 + \frac{4 - 2b}{N} < p < 1 + \frac{4 - 2b}{N - 2}.$$ (1.2)

These equations arise as a model in optics, to accounts for the inhomogeneity of the medium. For a physical point of view, we refer to Gill [15], Liu and Tripathi [22]. The INLS case appears naturally as a limiting case of potentials that decay as $|x|^{-b}$ at infinity (Genoud and Stuart [14]).

Moreover, this model is invariant under scaling. Indeed, if $u(x,t)$ is a solution to (1.1), then

$$u_\lambda(x,t) = \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t), \quad \lambda > 0,$$

is also a solution. Computing the homogeneous Sobolev norm, we obtain

$$\|u_\lambda(\cdot,0)\|_{H^s} = \lambda^{s - \frac{N}{2} - \frac{2}{p-1}} \|u_0\|_{H^s}.$$ 

The Sobolev index which leaves the scaling symmetry invariant is called the critical index and is defined as

$$s_c = \frac{N}{2} - \frac{2}{p} - \frac{b}{p-1}.$$ 

Note that the condition (1.2) is equivalent to $0 < s_c < 1$.

Solutions to the Cauchy problem (1.1) conserve mass $M[u]$ and energy $E[u]$, defined by

$$M[u(t)] = \int |u(t)|^2 dx = M[u_0],$$

$$E[u(t)] = \frac{1}{2} \int |\nabla u(t)|^2 dx - \frac{1}{p+1} \int |x|^{-b}|u(t)|^{p+1} dx = E[u_0].$$

2020 Mathematics Subject classification: 35Q55, 35P25, 35B40.
Keywords: Nonlinear Schrödinger-type equations, scattering, Morawetz estimates.
The homogeneous case \( b = 0 \) is known as the nonlinear Schrödinger (NLS) equation, which has been receiving attention over the past decades (see, for instance, the works of Bourgain [2], Cazenave [5], Linares-Ponce [21], and Tao [24]).

We briefly review the literature about (1.1). Genoud and Stuart [14] proved that (1.1) is locally well-posed in \( H^1(\mathbb{R}^N) \), \( N \geq 1 \) for \( 0 < b < \min\{2, N\} \). For other well-posedness results for this equation, we refer the reader to Guzmán [16] and Dinh [6]. Farah [11] proved global well-posedness for the INLS in \( H^1(\mathbb{R}^N) \) if

\[
M[u_0] \frac{1}{H^1} E[u_0] < M[\mathcal{Q}] \frac{1}{H^1} E[\mathcal{Q}]
\]

and

\[
\|u_0\|_{L^{2^*}} \|\nabla u_0\|_{L^2} < \|\mathcal{Q}\|_{L^{2^*}} \|\nabla \mathcal{Q}\|_{L^2},
\]

where \( \mathcal{Q} \) is the unique positive radial solution to the elliptic equation

\[
\Delta \mathcal{Q} - \mathcal{Q} + |x|^{-b} |\mathcal{Q}|^{p-1} \mathcal{Q} = 0,
\]

usually referred as the ground state associated to (1.1).

Scattering in \( H^1 \) under (1.3) and (1.4) was initially proved using the concentration-compactness-rigidity approach in the radial setting for \( N \geq 2 \) by Farah-Guzmán [10,12], by imposing some extra restrictions on \( p \) and \( b \). The first author [3] generalized the results for the whole intercritical setting in \( N \geq 3 \), extending the allowed range for \( p \) and \( b \), by adapting the ideas of Dodson-Murphy [7] to the radial NLS.

For the non-radial case, the lack of momentum conservation posed a technical difficulty. In the concentration-compactness-rigidity approach, a critical solution is constructed, whose orbit is compact under some symmetries, one of which is the translation parameter \( x(t) \). In the homogeneous case \((b = 0)\), the translation parameter associated to a zero-momentum (critical) solution under (1.3) and (1.4) which does not scatter satisfies \( x(t) = o(t) \). However, for the INLS, this control is not available through momentum arguments. Moreover, the non-radial interaction Morawetz approach by Dodson-Murphy [8] fails, due to the same lack of conservation.

However, it is possible to make use of the spatial decay of the nonlinearity in the INLS equation to extend to the non-radial case the proofs used in the radial case. This was shown in [4] by an adapted profile decomposition which eventually concluded that one could take \( x(t) = 0 \). As such, non-radial (critical) solutions to the INLS equation behaved similarly, in some sense, to the radial ones. Recently, Murphy [23] proved that a similar intuition works for the Virial-Morawetz approach, at least in the 3d cubic case, when \( 0 < b < 1/2 \). Here, inspired by [24], we show how to formalize this intuition in the non-radial case, for any \( N \geq 3 \), \( 0 < s_c < 1 \).

The key to main result is the scattering criterion, which was first proved for the 3d cubic NLS equation by Tao [25] (see also [11,3,24]).

**Theorem 1.1** (Scattering criterion). Let \( N \geq 3 \), \( 1 + \frac{4-2b}{N} < p < 1 + \frac{4-2b}{N-2} \) and \( 0 < b < \min\{N/2, 2\} \). Consider an \( H^1(\mathbb{R}^N) \)-solution \( u \) to (1.1) defined on \([0, +\infty)\) and assume the a priori bound

\[
\sup_{t \in [0, +\infty)} \|u(t)\|_{H^1} := E < +\infty.
\]

There exist constants \( R > 0 \) and \( \epsilon > 0 \) depending only on \( E, N, p \) and \( b \) (but never on \( u \) or \( t \)) such that if

\[
\liminf_{t \to +\infty} \int_{B(0, R)} |u(x, t)|^2 \, dx \leq \epsilon^2,
\]

then there exists a function \( u_+ \in H^1(\mathbb{R}^N) \) such that

\[
\lim_{t \to +\infty} \|u(t) - e^{it\Delta} u_+\|_{H^1(\mathbb{R}^N)} = 0,
\]

i.e., \( u \) scatters forward in time in \( H^1(\mathbb{R}^N) \).

**Remark 1.2.** The criterion above was proved for radial solutions in [3], and relied heavily on the so-called Strauss Lemma, which ensures spatial localization of radial \( H^1 \) functions. Here, we drop the radiality assumption, showing that the exact same criterion applies to non-radial solutions as well.
This shows that the decay of the nonlinearity implies in some kind of localization for solutions under the thresholds given by the ground state.

The localization effect caused by the decay of the nonlinearity can be expressed as the following proposition, which we show to hold for non-radial solutions.

**Proposition 1.3** (Virial-Morawetz estimate). For $N \geq 3$, $1 + \frac{4 - 2b}{N} < p < 1 + \frac{4 - 2b}{2}$ and $0 < b < \min\{N/2, 2\}$, let $u$ be a $H^1(\mathbb{R}^N)$-solution to (1.1) satisfying (1.3) and (1.4). Then, there exists $R > 0$ such that, for any $T > 0$,

$$\frac{1}{T} \int_0^T \int_{|x| \leq R} |x|^{-b}|u(x,t)|^p \, dx \, dt \lesssim u_{0,\delta} \frac{R}{T} + \frac{1}{R^b}. $$

The scattering criterion and the virial-Morawetz estimates allow us to prove the following theorem.

**Theorem 1.4.** Let $N \geq 3$, $1 + \frac{4 - 2b}{N} < p < 1 + \frac{4 - 2b}{2}$, $0 < b < \min\{N/2, 2\}$, and $u_0 \in H^1(\mathbb{R}^N)$ be such that

$$M[u_0]^{\frac{1}{1-\sigma}} E[u_0] < M[Q]^{\frac{1}{1-\sigma}} E[Q]$$

and

$$\|u_0\|_{L^2}^{\frac{1}{1-\sigma}} \|\nabla u_0\|_{L^2} \lesssim \|Q\|_{L^2}^{\frac{1}{1-\sigma}} \|\nabla Q\|_{L^2}. $$

Then the solution $u(t)$ to (1.1) exists globally in time and scatters in $H^1$ in both time directions.

**Remark 1.5.** The proofs in [4][5][12][23] use the so-called concentration-compactness-rigidity approach, pioneered by Kenig and Merle [20] in the context of the energy-critical ($s_c = 1$) NLS equation. More recently, Dodson and Murphy [7] developed a new approach, based on Tao’s scattering criterion in [25] and on Virial-Morawetz estimates. This approach was adapted to the INLS by [3], in the radial case, and by Murphy [24] in the 3d cubic, non-radial case. We develop here a modification of the approach in [24], closer to the one chosen in [3], replacing $L^1_t W^{1,3}_x$ estimates by smoother Strichartz estimates which, together with small data theory, make it possible to handle the inhomogeneity better, allowing for an optimal range of parameters in dimensions $N \geq 3$. The radial assumption is dropped vis-à-vis the $|x|^{-b}$ factor in the nonlinear term. In lower dimensions, this approach fails due to the slow decay on time of the Schrödinger operator $e^{it\Delta}$ and the slow decay in the Virial-Morawetz estimate due to the weaker non-radial decay.

This paper is organized as follows: in the next section, we introduce some notation and basic estimates. In Section 3, we prove the scattering criterion (Theorem 1.1). In Section 4, we apply this criterion, together with Morawetz/Virial estimates to prove Theorem 1.4.

2. Notation and basic estimates

We denote by $p'$ the Holder’s conjugate of $p \geq 1$. We use $X \lesssim Y$ to denote $X \leq CY$, where the constant $C$ only depends on the parameters (such as $N$, $p$, $b$, as well as $E$ in (1.5)) and exponents, but never on $u$ or on $t$. The notations $a^+$ and $a^-$ denote, respectively, $a + \eta$ and $a - \eta$, for a fixed $0 < \eta \ll 1$. We use $p^*$ to denote the critical exponent of the Sobolev embedding $H^1 \hookrightarrow L^{p^*}$, that is, $p^* = 2N/(N - 2)$, for $N \geq 3$.

**Definition 2.1.** If $N \geq 1$ and $s \in (-1,1)$, the pair $(q, r)$ is called $\dot{H}^s$-admissible if it satisfies the condition

$$\frac{2}{q} = \frac{N}{2} - \frac{N}{r} - s,$$

where

$$2 \leq q, r \leq \infty, \text{ and } (q, r, N) \neq (2, \infty, 2).$$

In particular, if $s = 0$, we say that the pair is $L^2$-admissible.

**Definition 2.2.** Given $N > 2$, consider the set

$$\mathcal{A}_0 = \left\{ (q, r) \text{ is } L^2 \text{-admissible} \mid 2 \leq r \leq \frac{2N}{N - 2} \right\}.$$
For $N > 2$ and $s \in (0, 1)$, consider also

$$A_s = \{(q, r) \text{ is } \dot{H}^s\text{-admissible} \mid \left( \frac{2N}{N-2s} \right)^+ \leq r \leq \left( \frac{2N}{N-2} \right)^- \}$$

and

$$A_{-s} = \{(q, r) \text{ is } \dot{H}^{-s}\text{-admissible} \mid \left( \frac{2N}{N-2s} \right)^+ \leq r \leq \left( \frac{2N}{N-2} \right)^- \}.$$

We define the following Strichartz norm

$$\|u\|_{S(H^s, I)} = \sup_{(q, r) \in A_s} \| u \|_{L^q_t L^r_x},$$

and the dual Strichartz norm

$$\|u\|_{S'(H^{-s}, I)} = \inf_{(q, r) \in A_{-s}} \| u \|_{L^{q'}_{t} L^{r'}_{x}}.$$

If $s = 0$, we shall write $S(\dot{H}^0, I) = S(L^2, I)$ and $S'(\dot{H}^0, I) = S'(L^2, I)$. If $I = \mathbb{R}$, we will omit $I$.

2.1. Strichartz Estimates. In this work, we use the following versions of the Strichartz estimates:

(i) The standard Strichartz estimates (Cazenave [5], Keel and Tao [19], Foschi [13])

$$\|e^{it\Delta} f\|_{S(L^2)} \lesssim \|f\|_{L^2}, \quad (2.1)$$

$$\|e^{it\Delta} f\|_{S(H^s)} \lesssim \|f\|_{H^s}, \quad (2.2)$$

$$\left\| \int_{\mathbb{R}} e^{it(-\tau)\Delta} g(\cdot, \tau) \, d\tau \right\|_{S(L^2, I)} + \left\| \int_0^t e^{it(-\tau)\Delta} g(\cdot, \tau) \, d\tau \right\|_{S(L^2, I)} \lesssim \|g\|_{S'(L^2, I)}. \quad (2.3)$$

(ii) The Kato-Strichartz estimate (Kato [18], Foschi [13])

$$\left\| \int_{\mathbb{R}} e^{it(-\tau)\Delta} g(\cdot, \tau) \, d\tau \right\|_{S(H^s, I)} + \left\| \int_0^t e^{it(-\tau)\Delta} g(\cdot, \tau) \, d\tau \right\|_{S(H^s, I)} \lesssim \|g\|_{S'(H^{-s}, I)}. \quad (2.4)$$

(iii) Local-in-time estimate

$$\left\| \int_a^b e^{it(-\tau)\Delta} g(\cdot, \tau) \, d\tau \right\|_{S(H^s, [a, b])} \lesssim \|g\|_{S(\dot{H}^{-s}, [a, b])}. \quad (2.5)$$

These relations are obtained from the decay of the linear operator (see, for instance, Linares and Ponce [21] Lemma 4.1))

$$\|e^{it\Delta} f\|_{L^p_t} \lesssim \frac{1}{|t|^\frac{N}{p}(\frac{2N}{N-2} - \frac{2}{p})} \|f\|_{L^{p'}_t}, \quad p \geq 2, \quad (2.6)$$

combined with Sobolev inequalities and interpolation. The inequalities (2.1) - (2.4) are standard in the theory [5]. The inequality (2.5) follows from (2.4) by noting that

$$\int_a^b e^{it(-\tau)\Delta} g(\tau) \, d\tau = \int_{\mathbb{R}} e^{it(-\tau)\Delta} \mathbb{1}_{[a, b]}(\tau) g(\tau) \, d\tau.$$
2.2. Other useful estimates. In what follows we also use the following standard estimates.

Lemma 2.3 (See \[3\] Section 2 and \[16\] Section 4). Let \( N \geq 3, u, v \in C_0^\infty(\mathbb{R}^{N+1}), 1 + \frac{4}{N-2} b < p < 1 + \frac{2}{N-2} \) and \( 0 < b < \min\{N/2, 2\} \). Then there exists \( 0 \leq \theta = \theta(N, p, b) \ll p^{-1} \) such that the following inequalities hold

\[
\| |x|^{-b}|u|^{p-1}v\|_{S(\mathcal{H}^{\frac{N}{2} - \epsilon}, t)} \lesssim \|u\|_{L^p_t H^s_x}^0 \|u\|_{S(\mathcal{H}^{\frac{N}{2} - \epsilon}, t)}^{p-\theta} \|v\|_{S(\mathcal{H}^{\frac{N}{2} - \epsilon}, t)},
\]

(2.7)

\[
\| |x|^{-b}|u|^{p-1}v\|_{S(\mathcal{H}^{2}, t)} \lesssim \|u\|_{L^p_t H^s_x}^0 \|u\|_{S(\mathcal{H}^{\frac{N}{2} - \epsilon}, t)}^{p-\theta} \|u\|_{S(\mathcal{H}^{2}, t)},
\]

(2.8)

\[
\| |x|^{-b}|u|^{p-1}v\|_{S(\mathcal{H}^{2}, t)} \lesssim \|u\|_{L^p_t H^s_x}^0 \|u\|_{S(\mathcal{H}^{\frac{N}{2} - \epsilon}, t)}^{p-\theta} \|\nabla u\|_{S(\mathcal{H}^{2}, t)},
\]

(2.9)

\[
\| |x|^{-b}|u|^{p-1}v\|_{L^p_t L^s_x} \lesssim \|u\|_{L^p_t H^s_x}^0.
\]

(2.10)

for \( \frac{2(N-b)}{N+4-2b} < r < \frac{2(N-b)}{N+2-2b} \).

Remark 2.4. Inequalities (2.7)–(2.9) were proved in \[16\] for \( 0 < b < b^* \) (\( b^* = \frac{N}{2} \), if \( N = 1, 2, 3 \) and \( b^* = 2 \), if \( N \geq 4 \)) and with the additional restriction \( p < 4 - 2b \) instead of \( p < 5 - 2b \) in the 3d case. The proof was extended to the full range in \[3\].

The next lemma was proved in \[16\] with the same restrictions mentioned in Remark 2.3. In view of the results in \[3\], the proof in \[16\] immediately extended to the new range of \( p \) and \( b \).

Lemma 2.5 (Small data theory, see \[3\] and \[16\]). Let \( N \geq 3, 1 + \frac{4}{N-2} b < p < 1 + \frac{2}{N-2} \) and \( 0 < b < \min\{N/2, 2\} \). Suppose \( \|u_0\|_{H^1} \leq E \). Then there exists \( \delta_{ad} = \delta_{ad}(E) > 0 \) such that if

\[
\|e^{it\Delta}u_0\|_{S(\mathcal{H}^{\frac{N}{2} - \epsilon}, [0, +\infty))} \leq \delta_{ad},
\]

then the solution \( u \) to (1.1) with initial condition \( u_0 \in H^1(\mathbb{R}^N) \) is globally defined on \([0, +\infty)\).

Moreover,

\[
\|u\|_{S(\mathcal{H}^{\frac{N}{2} - \epsilon}, [0, +\infty])} \leq 2 \|e^{it\Delta}u_0\|_{S(\mathcal{H}^{\frac{N}{2} - \epsilon}, [0, +\infty])},
\]

and

\[
\|u\|_{S(L^2([0, +\infty]))} + \|\nabla u\|_{S(L^2([0, +\infty]))} \lesssim \|u_0\|_{H^1}.
\]

Furthermore, \( u \) scatters forward in time in \( H^1 \), i.e., here exists \( u_+ \in H^1 \) such that

\[
\lim_{t \to +\infty} \|u(t) - e^{it\Delta}u_+\|_{H^1} = 0.
\]

3. Proof of the scattering criterion

The following result is the key to prove Theorem 1.1. It was proved initially for radial solutions to the INLS equation, in the intercritical setting, for \( N \geq 3, \) in \[3\]. Jason \[24\] extended the result for non-radial data in the 3d-cubic setting, for \( 0 < b < 1/2 \). Here, we prove the result for non-radial data the full intercritical range, for \( N \geq 3 \).

Lemma 3.1. Let \( N \geq 3, 1 + \frac{4}{N-2} b < p < 1 + \frac{2}{N-2} \), \( 0 < b < \min\{N/2, 2\} \) and \( u \) be a (possibly non-radial) \( H^1(\mathbb{R}^N) \)-solution to (1.1) satisfying (1.5). If \( u \) satisfies (1.6) for some \( 0 < \epsilon < 1 \), then there exist \( \gamma, T > 0 \) such that the following estimate is valid

\[
\|e^{i(-\gamma)\Delta}u(T)\|_{S(\mathcal{H}^{\frac{N}{2} - \epsilon}, [T, +\infty))} \lesssim \epsilon^\gamma.
\]

Proof. For \( N \geq 3 \), fix the parameters \( \alpha, \gamma > 0 \) (to be chosen later). From (2.22), there exists \( T_0 > \epsilon^{-\alpha} \) such that

\[
\|e^{it\Delta}u_0\|_{S(\mathcal{H}^{\frac{N}{2} - \epsilon}, [T_0, +\infty])} \leq \epsilon^\gamma.
\]

(3.1)

For \( T \geq T_0 \) to be chosen later, define \( I_1 := [T, T - \epsilon^{-\alpha}, T], I_2 := [0, T - \epsilon^{-\alpha}] \) and let \( \eta \) denote a smooth, spherically symmetric function which equals 1 on \( B(0, 1/2) \) and 0 outside \( B(0, 1) \). For any \( R > 0 \) use \( \eta_R \) to denote the rescaling \( \eta_R(x) := \eta(x/R) \).

From Duhamel’s formula

\[
u(T) = e^{iT\Delta}u_0 + i \int_0^T e^{i(T-s)\Delta}|x|^{-b}|u|^{p-1}u(s) \, ds,
\]
we obtain
\[e^{i(t-T)\Delta} u(T) = e^{it\Delta} u_0 + i F_1 + i F_2,\]
where, for \(i = 1, 2,\)
\[F_i = \int_I e^{i(t-s)\Delta} |x|^{-b} |u|^{p-1} u(s) \, ds.\]

We refer, as usual, to \(F_1\) as the “recent past”, and to \(F_2\) as the “distant past”. By (3.3), it remains to estimate \(F_1\) and \(F_2\).

**Step 1. Estimate on recent past.**

By hypothesis (1.6), we can fix \(T \geq T_0\) such that
\[\int \eta_R(x) |u(T, x)|^2 \, dx \lesssim \epsilon.\] (3.2)

Given the relation (obtained by multiplying (1.1) by \(\eta\), taking the imaginary part and integrating by parts, see Tao [25, Section 4] for details)
\[\partial_t \int \eta R |u|^2 \, dx = 2 \text{Im} \int \nabla \eta R \cdot \nabla \bar{u} \, u,\]
we have, from (1.5), for all times,
\[\left| \partial_t \int \eta R(x)|u(t, x)|^2 \, dx \right| \lesssim \frac{1}{R},\]
so that, by (3.2), for \(t \in I_1,\)
\[\int \eta_R(x) |u(t, x)|^2 \, dx \lesssim \epsilon^2 + \frac{\epsilon^{-\alpha}}{R}.\]

If \(R > \epsilon^{-(\alpha+2)},\) then we have \(\|\eta R u\|_{L^\infty_t L^2_x} \lesssim \epsilon.\)

Define \((\hat{q}, \hat{r}) \in A_{s_0}\) as
\[\hat{q} = \frac{4(p-1)(p+1-\theta)}{(p-1)(N(p-1)+2b)-\theta(N(p-1)+4+2b)}, \quad \hat{r} = \frac{N(p-1)(p+1-\theta)}{(p-1)(N-\theta)(2-\theta)}.

We have, by Hölder and Sobolev, for \(t \in I_1,\)
\[\|\eta R |x|^{-b} |u|^{p-1} u(t)\|_{L^\infty_t L^\infty_x} \lesssim \|u(t)\|_{H^\theta_1}^\theta \|u(t)\|_{L^p_x}^{p-\theta} \|\eta R u(t)\|_{L^q_x} \lesssim \|\eta R u(t)\|_{L^b_t}.\] (3.3)

Now, letting \(\hat{\theta}\) be the solution of \(\frac{1}{\hat{q}} = \frac{\hat{q}}{2} + \frac{1}{\frac{\hat{r}}{2}},\) we have,
\[\|\eta R u(t)\|_{L^\infty_t} \lesssim \|u(t)\|_{H^\theta_1}^{1-\hat{\theta}} \|\eta R u(t)\|_{L^b_t}^{\hat{\theta}} \lesssim \epsilon^{\hat{\theta}},\] (3.4)
uniformly on time in \(I_1.\) We now exploit the decay of the nonlinearity, instead of assuming radiality, to estimate, by Hölder and Sobolev, for \(R > 0\) large enough (depending on \(\epsilon\)) and \(t \in I_1,\)
\[\| (1 - \eta R) |x|^{-b} |u|^{p-1} u(t)\|_{L^\infty_x} \lesssim \| |x|^{-b} |u|^{p-1} u(t)\|_{L^\infty_x} \lesssim \| |x|^{-b} |u|^{p-1} u(t)\|_{L^\infty_x} \lesssim \| \frac{1}{R^{r_1-N}} \|u(t)\|_{H^\theta_1}^{\theta} \lesssim \epsilon^{\theta},\] (3.5)
where \(r_1\) and \(r_2\) are such that \(br_1 > N, \theta r_2 \in (2, N(p-1)/(2-b))\) and
\[\frac{1}{\hat{r}} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{p-\theta}{\theta}.\]

Using the local-in-time Strichartz estimate (2.5), together with estimates (3.3), (3.4) and (3.5), we bound
\[1.\] This is one of the crucial estimates which allow us to drop the radiality assumption.
\[ \left\| \int_I e^{it-s} \Delta |x|^{-b} |u|^{p-1} u(s) \, ds \right\|_{S(H^{s_c}(I_I, T, \infty))} \leq \left\| |x|^{-b} |u|^{p-1} u \right\|_{S(H^{s_c}, I_I)} \]
\[ \leq \left\| \eta R |x|^{-b} |u|^{p-1} u \right\|_{L_t^q L_x^{q'}} + \left\| (1 - \eta R) |x|^{-b} |u|^{p-1} u \right\|_{L_t^{q'} L_x^{q''}} \]
\[ \lesssim |I_I|^{1/q'} \varepsilon = \varepsilon^{\delta / 2}, \]
where we chose \( \alpha := q'/\delta / 2 \).

**Step 2. Estimate on distant past.**

The estimate for the distant past is the same as in [3], as radiality does not play a role in this part of the estimate. We provide the argument here for completeness. Let \((q, r) \in A_{s_c}\). Define, for small \( \delta > 0 \),
\[ \frac{1}{c} = \left( \frac{1}{1 - s_c} \right) \left[ \frac{1}{q} - \delta s_c \right] \]
and
\[ \frac{1}{d} = \left( \frac{1}{1 - s_c} \right) \left[ \frac{1}{r} - s_c \left( \frac{N - 2 - 4\delta}{2N} \right) \right]. \]
We see that \((c, d) \in A_0\) (see [3 Section 3]). By interpolation,
\[ \|F_2\|_{L_t^q \cap L_x^r} \leq \|F_2\|_{L_t^{q_c} L_x^{r_c}} \|F_2\|_{L_t^{q_c} L_x^{r_c}}^{2N} \lesssim \varepsilon^{\alpha \delta s_c}, \]

Using Duhamel’s principle, write
\[ F_2 = e^{it\Delta} \left[ e^{i(T + \epsilon^{-\alpha}) \Delta} u(T - \epsilon^{-\alpha}) - u(0) \right]. \]

Thus, by the Strichartz estimate (2.1),
\[ \|F_2\|_{L_t^q \cap L_x^r} \leq \left\| e^{it\Delta} \left[ e^{i(T + \epsilon^{-\alpha}) \Delta} u(T - \epsilon^{-\alpha}) - u(0) \right] \right\|_{L_t^{q_c} L_x^{r_c}}^{1-s_c} \|F_2\|_{L_t^{q_c} L_x^{r_c}}^{2N} \]
\[ \lesssim \left\| u \right\|_{L_t^q L_x^r}^{1-s_c} \|F_2\|_{L_t^{q_c} L_x^{r_c}}^{2N} \lesssim \varepsilon^{\alpha \delta s_c}, \]

since, by (2.6) and (2.10),
\[ \|F_2\|_{L_t^q L_x^r}^{2N} \lesssim \left\| \int_{I_I} \left| \cdot - s \right|^{-1+2\delta} \left\| |x|^{-b} |u|^{p-1} u(s) \right\|_{L_x^{2N/2+2}} \, ds \right\|_{L_t^q L_x^r}^{1/s_c} \]
\[ \lesssim \|u\|_{L_t^q H_x^s} \left\| (\cdot - T + \epsilon^{-\alpha})^{-2\delta} \right\|_{L_t^{q_c} L_x^{r_c}}^{1/s_c} \]
\[ \lesssim \varepsilon^{\alpha \delta}. \]
Therefore, defining \( \gamma := \min \{ \hat{\theta}/2, \alpha \delta s_c \} \) and recalling that
\[ e^{i(t-T)\Delta} u(T) = e^{it\Delta} u_0 + i F_1 + i F_2, \]
we have
\[ \left\| e^{i(T - T)\Delta} u(T) \right\|_{S(H^{s_c}, I_I, T, \infty)} \lesssim \varepsilon^\gamma. \]
Hence, Lemma 3.1 is proved. \( \square \)

**Proof of Theorem 1.7.** Choose \( \epsilon \) is small enough so that, by Lemma 3.1
\[ \left\| e^{i(T - T)\Delta} u(T) \right\|_{S(H^{s_c}, I_I, T, \infty)} \leq \left\| e^{i(T - T)\Delta} u(T) \right\|_{S(H^{s_c}, I_I, T, \infty)} \leq \epsilon \varepsilon^\gamma \leq \delta_{sd}, \]
where \( \delta_{sd} \) is given in Lemma 2.5. Thus, by small data theory, \( u \) scatters forward in time in \( H^1 \), as desired. \( \square \)
4. Proof of scattering

We now turn to Theorem 1.4. The main idea behind the proof is to combine the decay of the nonlinearity (instead of exploiting some form of radial decay) with a truncated Virial identity. By choosing a suitable weight, and employing coercivity on large balls around the origin, one can control a time-averaged weighed $L^p$ norm on these balls. Averaging is necessary due to the lack of uniform estimates in time, since we are not employing concentration-compactness as in Holmer-Roudenko [9,17].

We start with the following “trapping” lemmas, which can be found in [3] and in Farah and Guzmán [10].

Lemma 4.1 (Energy trapping). Let $N \geq 1$ and $0 < s_c < 1$. If

$$M[u_0]\frac{1}{\bar{R}} E[u_0] < (1 - \delta) M[u_0]\frac{1}{\bar{R}} E[u_0]$$

for some $\delta > 0$ and

$$\|u_0\|_{L^2} \|\nabla u_0\|_{L^2} \leq \|Q\|_{L^2} \|\nabla Q\|_{L^2},$$

then there exists $\delta' = \delta'(\delta) > 0$ such that

$$\|u_0\|_{L^2} \|\nabla u_0\|_{L^2} < (1 - \delta') \|Q\|_{L^2} \|\nabla Q\|_{L^2}.$$

for all $t \in I$, where $I \subset \mathbb{R}$ is the maximal interval of existence of the solution $u(t)$ to (1.1). Moreover, $I = \mathbb{R}$ and $u$ is uniformly bounded in $H^1$.

Lemma 4.2. Suppose, for $f \in H^1(\mathbb{R}^N)$, $N \geq 1$, that

$$\|f\|_{L^2} \|\nabla f\|_{L^2} < (1 - \delta) \|Q\|_{L^2} \|\nabla Q\|_{L^2}.$$

Then there exists $\delta' = \delta'(\delta) > 0$ so that

$$\int |\nabla f|^2 + \left(\frac{N - b}{p + 1} - \frac{N}{2}\right) \int |x|^{-b}|f|^{p+1} \geq \delta' \int |x|^{-b}|f|^{p+1}.$$

From now on, we consider $u$ to be a solution to (1.1) satisfying the conditions (1.3) and (1.4). In particular, by Lemma 4.1, $u$ is global and uniformly bounded in $H^1$. Moreover, there exists $\delta > 0$ such that

$$\sup_{t \in \mathbb{R}} \|u_0\|_{L^2} \|\nabla u(t)\|_{L^2} < (1 - 2\delta) \|Q\|_{L^2} \|\nabla Q\|_{L^2} \quad (4.1)$$

In the spirit of Dodson and Murphy [7], local coercivity was proved in [3]. They proved:

Lemma 4.3. For $N \geq 1$, let $\phi$ be a smooth cutoff to the set $\{|x| \leq \frac{1}{2}\}$ and define $\phi_R(x) = \phi\left(\frac{x}{R}\right)$. If $f \in H^1(\mathbb{R}^N)$, then

$$\int |\nabla (\phi_R f)|^2 = \int \phi_R^2 |\nabla f|^2 - \int \phi_R \Delta (\phi_R) |f|^2.$$

In particular,

$$\int |\nabla (\phi_R f)|^2 - \int \phi_R^2 |\nabla f|^2 \leq \frac{c}{R^2} \|f\|_{L^2}^2.$$

Lemma 4.4 (Local coercivity). For $N \geq 1$, let $u$ be a globally defined $H^1(\mathbb{R}^N)$-solution to (1.1) satisfying (4.1). There exists $\bar{R} = \bar{R}(\delta, M[u_0], Q, s_c) > 0$ such that, for any $R \geq \bar{R},$

$$\sup_{t \in \mathbb{R}} \|\phi_R u(t)\|_{L^2} \|\nabla (\phi_R u(t))\|_{L^2} \leq (1 - \delta) \|Q\|_{L^2} \|\nabla Q\|_{L^2}.$$

In particular, by Lemma 4.2 there exists $\delta' = \delta'(\delta) > 0$ such that

$$\int |\nabla (\phi_R u(t))|^2 + \left(\frac{N - b}{p + 1} - \frac{N}{2}\right) \int |x|^{-b}|\phi_R u(t)|^{p+1} \geq \delta' \int |x|^{-b}|\phi_R u(t)|^{p+1}.$$

We exploit the coercivity given by the previous lemma by making use of the Virial identity (see Dodson and Murphy [7, Lemma 3.3], Farah and Guzmán [10, Proposition 7.2]).
Lemma 4.5 (Virial identity). Let \( a : \mathbb{R}^N \to \mathbb{R} \) be a real-valued weight. If \(|\nabla a| \in L^\infty\), define
\[
Z(t) = 2 \text{Im} \int \bar{u} \nabla u \cdot \nabla a \, dx.
\]
Then, if \( u \) is a solution to (1.1), we have the following identity
\[
\frac{d}{dt} Z(t) = \left( \frac{4}{p+1} - 2 \right) \int |x|^{-b}|u|^{p+1} \, \Delta a - \frac{4b}{p+1} \int |x|^{-b-2}|u|^{p+1} x \cdot \nabla a
\]
\[- \int |u|^2 \Delta \Delta a + 4 \text{Re} \sum_{i,j} a_{ij} \bar{u}_i u_j.
\]

We now have all the basic tools needed to prove scattering. Let \( R \gg 1 \) to be determined below. We take \( a \) to be a smooth radial function satisfying
\[
a(x) = \begin{cases} |x|^2 & |x| \leq \frac{R}{2}, \\ 2R|x| & |x| > R. \end{cases}
\]
In the intermediate region \( \frac{R}{2} < |x| \leq R \), we impose that
\[
\partial_r a \geq 0, \quad \partial_r^2 a \geq 0, \quad |\partial^\alpha a(x)| \lesssim R |x|^{-|\alpha|+1} \text{ for } |\alpha| \geq 1.
\]
Here, \( \partial_r \) denotes the radial derivative, i.e., \( \partial_r a = \nabla a \cdot \frac{x}{|x|} \). Note that for \( |x| \leq \frac{R}{2} \), we have
\[
a_{ij} = 2 \delta_{ij}, \quad \Delta a = 2N, \quad \Delta \Delta a = 0,
\]
while, for \( |x| > R \), we have
\[
a_{ij} = \frac{2R}{|x|} \left[ \delta_{ij} - \frac{x_i x_j}{|x|^2} \right], \quad \Delta a = \frac{2(N-1)R}{|x|}, \quad |\Delta \Delta a(x)| \lesssim \frac{R}{|x|^2}.
\]

Proof of Proposition 1.3. We follow mostly [3], but highlighting the differences (extra terms appearing due to non-radiality and weaker decay) throughout the proof. Choose \( R \geq R(\delta, M[u_0], Q, s, x) \) as in Lemma 4.4. We define the weight \( a \) as above and define \( Z(t) \) as in Lemma 4.5. Using Cauchy-Schwarz inequality, and the definition of \( Z(t) \), we have
\[
\sup_{t \in \mathbb{R}} |Z(t)| \lesssim R. \tag{4.2}
\]
As in [3], we compute
\[
\frac{d}{dt} Z(t) = 8 \int |x| \leq \frac{R}{4} \nabla u|^2 + \left( \frac{N-b}{p+1} - \frac{N}{2} \right) \int |x| \leq \frac{R}{4} |x|^{-b}|u|^{p+1}
\]
\[+ \int |x| > \frac{R}{4} \left( \frac{4}{p+1} - 2 \right) (N-1) \Delta a - \frac{4b}{p+1} \frac{x \cdot \nabla a}{|x|^2} |x|^{-b}|u|^{p+1}
\]
\[+ 4 \int |x| > \frac{R}{4} \partial_r^2 a \partial_r u|^2 + 4 \int |x| > \frac{R}{4} \partial_r a \partial_r u^2 + \int |x| > \frac{R}{4} |\nabla u|^2 - \int |x| > \frac{R}{4} |u|^2 \Delta \Delta a,
\]
where we denote the angular derivative as \( \nabla u = \nabla u - \frac{x}{|x|^2} \nabla x \). Note that \( \nabla u \) is not necessarily zero, since we are not assuming radiality. Nevertheless, the first two terms in the last line can be dropped, by non-negativity.

As for the second line, one can bound \( \|u\|_{L^{p+1}} \) by \( E \), using Sobolev, so this term gives us only a decay of \( O(1/R^6) \). It is, of course, a weaker decay than that one in [3] (which used Strauss), but in dimensions \( N \geq 3 \), it is enough to close the argument. Therefore,
\[
\frac{d}{dt} Z(t) \geq 8 \int |x| \leq \frac{R}{4} \nabla u|^2 + \left( \frac{N-b}{p+1} - \frac{N}{2} \right) \int |x| \leq \frac{R}{4} |x|^{-b}|u|^{p+1}
\]
\[- \frac{cE^{p+1}}{R^6} - \frac{c}{R^2} M[u_0], \tag{4.3}
\]
Define $\phi^A$, $A > 0$, as a smooth cutoff to the set $\{|x| \leq \frac{1}{2}\}$ that vanishes outside the set $\{|x| \leq \frac{1}{2} + \frac{1}{R}\}$, and define $\phi^A_R(x) = \phi^A \left( \frac{R}{R} \right)$. In order to use Lemma 4.3, we introduce some smoothing in the first term of the last inequality (at the expense of an acceptable error, which decays with a power of $R$).

Using Lemma 4.3, we can write

$$I_A = \int |\nabla u|^2 + \left( \frac{N - b}{p + 1} - \frac{N}{2} \right) \int |x|^{-b} |u|^{p+1} = \int \left( \phi^A_R \nabla u \right)^2 + \left( \frac{N - b}{p + 1} - \frac{N}{2} \right) \int |x|^{-b} |\phi^A_R u|^{p+1}$$

$$- \int \int_{\frac{1}{4} < |x| \leq \frac{1}{4} + \frac{1}{4}} \left( \phi^A_R \nabla u \right)^2 + \left( \frac{N - b}{p + 1} - \frac{N}{2} \right) \int |x|^{-b} |\phi^A_R u|^{p+1}$$

Using Lemma 4.3, we can write

$$\int |\phi^A_R \nabla u|^2 + \left( \frac{N - b}{p + 1} - \frac{N}{2} \right) \int |x|^{-b} |\phi^A_R u|^{p+1} \geq \int |\nabla (\phi^A_R u)|^2 + \left( \frac{N - b}{p + 1} - \frac{N}{2} \right) \int |x|^{-b} |\phi^A_R u|^{p+1} - \frac{c}{R^2} M[u_0].$$

The inequalities (4.3), (4.4) and (4.5) can be rewritten as

$$\frac{d}{dt} \mathcal{Z}(t) \geq 8 \left[ \int |\nabla (\phi^A_R u)|^2 + \left( \frac{N - b}{p + 1} - \frac{N}{2} \right) \int |x|^{-b} |\phi^A_R u|^{p+1} \right]$$

$$- \frac{cE^{x+1}_{b+1}}{R^6} - \frac{c}{R^2} M[u_0] - 8I_A - 8II_A.$$

By Lemma 4.4 and recalling that $0 < b < 2$, we can write (4.6) as

$$\int |x|^{-b} |\phi^A_R u(t)|^{p+1} \leq \frac{d}{dt} \mathcal{Z}(t) + \frac{1}{R^6} + 8I_A + 8II_A.$$

We can now make $A \to +\infty$ to obtain $I_A + II_A \to 0$ by dominated convergence. Hence,

$$\int \mathcal{Z}(t) \leq \frac{d}{dt} \mathcal{Z}(t) + \frac{1}{R^6}.$$

We finish the proof integrating over time, and using (4.2). We have

$$\frac{1}{T} \int_0^T \int_{|x| \leq \frac{1}{4}} |x|^{-b} |u(t)|^{p+1} \leq \frac{1}{T} \sup_{t \in [0, T]} |Z(t)| + \frac{1}{R^6}$$

$$\leq \frac{R}{T} + \frac{1}{R^6}.$$

We are now able to prove some energy evacuation. Note that, unlike in [3], and inspired by [24] we keep the factor $|x|^{-b}$ in the integral, since otherwise it would jeopardize the decay.

**Proposition 4.6** (Energy evacuation). Under the hypotheses of Proposition 4.3, there exist a sequence of times $t_n \to +\infty$ and a sequence of radii $R_n \to +\infty$ such that

$$\lim_{n \to +\infty} \int_{|x| \leq R_n} |x|^{-b} |u(t_n)|^{p+1} = 0$$

(4.7)
Proof. Using Proposition 1.3 choose $T_n \to +\infty$ and $R_n = T_n^{-\frac{1}{2}}$, so that
\[
\frac{1}{T_n} \int_0^{T_n} \int_{|x| \leq R_n} |x|^{-b}|u(t)|^{p+1} \leq \frac{1}{T_n} \to 0 \text{ as } n \to +\infty.
\]
Therefore, by the Mean Value Theorem, there is a sequence $t_n \to +\infty$ such that (4.7) holds. The proof is complete. □

Using Proposition 1.6 we can prove Theorem 1.4. We prove only the case $t \to +\infty$, as the case $t \to -\infty$ is entirely analogous.

Proof of Theorem 1.4. Take $t_n \to +\infty$ and $R_n \to +\infty$ as in Proposition 1.6. Fix $\epsilon > 0$ and $R > 0$ as in Theorem 1.1. Choosing $n$ large enough, such that $R_n \geq R$, Hölder’s inequality yields
\[
\int_{|x| \leq R} |u(x, t_n)|^2 \leq R^{-\frac{2+bQ(N-1)}{p+1}} \left( \int_{|x| \leq R_n} |x|^{-b}|u(x, t_n)|^{p+1} \right)^{\frac{p}{p+1}} \to 0 \text{ as } n \to +\infty.
\]
Therefore, by Theorem 1.1, $u$ scatters forward in time. □

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