Condensation and Lasing of Microcavity Polaritons: Comparison between two Models

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Abstract

Condensation of microcavity polaritons and the substantial influence of pair-breaking disorder and decoherence leading to a laser regime has been recently considered using two different models: a model for direct two band excitons in a disordered quantum well coupled to light and a model where the cavity mode couples instead to a medium of localised excitons, represented by two-level oscillators in the presence of dephasing processes. Even if complementary from the point of view of assumptions, the models share most of the main conclusions and show similar phase diagrams. The issue whether excitons are propagating or localised seems secondary for the polariton condensation and the way in which pair-breaking disorder and decoherence processes influence the condensation and drive the microcavity into a lasing regime is, within the approximations used in each model, generic. The reasons for the similarities between the two physical situations are analysed and explained.

Key words: A. Nanostructures; D. Phase transition
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1. Introduction

Since exciton-polaritons in semiconductor microcavities possess very light effective mass, there has been a large interest in recent years in realising polariton BEC [1]. So far, however, experiments concentrate on a very low density regime, in which a clear polariton splitting in a normal state is measured. Although the physical picture of polariton condensation is more complex at higher densities, recent works suggest that the fermionic structure of excitons does not prevent condensation and moreover, at higher densities, the condensate becomes more robust to pair-breaking decoherence and disorder, real antagonists of condensation [2,3]. In the absence of disorder, even at high excitation densities, a strongly coupled light-matter condensate can be realised and it is only a strong influence of the environment (all other processes other than dipole interaction) which leads to a weak light-matter coupling characteristic of a laser and governed by a Fermi Golden rule. The laser regime emerges from the polariton condensate at high densities when pair-breaking disorder...
or decoherence is large in a way similar to gapless superconductivity and it arises because these processes destroy the electronic polarisation leaving the photon component unchanged.

This conclusion and several other properties of polariton condensate has been shown to be independent on whether the excitons are propagating or localised. Two models have been recently considered, where the condensation of cavity polaritons and the crossover to a laser regime is not restricted by the assumption on the bosonic nature of polaritons [4,2,3], or, in other words, by the low density limit. In the first one [3], direct two band excitons in a disordered quantum well are affected by Coulomb interaction and are dipole coupled to the cavity light. In the second one, instead, excitons are localised, e.g. by disorder, and described as two-level oscillators coupled to light [4]. Here, decoherence effects can be introduced by coupling the system to external baths [2].

The main aim of this paper is to show and explain that similar conclusions can be drawn from both approaches. As far as some of the assumptions are concerned, the two models are in fact complementary (see the table 1). Firstly, while in the first one dispersion of electrons and holes is included, in the second one only the presence or absence of localised excitons is taken into account. As a result, Coulomb interaction and screening at high excitonic densities is self-consistently included in the band model, while in the two-level oscillators version only the on-site Coulomb interaction is taken into account. Secondly, because of technical restrictions, we limit the analysis of the band model to the high density regime, while in the two-level systems model all range of excitations are considered, e.g. by disorder, and described as two-level oscillators coupled to light [4]. Here, decoherence effects can be introduced by coupling the system to external baths [2].

The paper is organised as follows: The main results relative to each model are respectively delineated in the next two sections, while a comparison is described in the concluding one.

### 2. Band Model

The Hamiltonian for the coupled electron-hole/photon system can be separated in the following components:

$$
\hat{H} = \mu \hat{\mathcal{N}} + \hat{\mathcal{H}}_e + \hat{\mathcal{H}}_{\text{dis}} + \hat{\mathcal{H}}_{\text{ph}} + \hat{\mathcal{H}}_{\text{int}}.
$$

The Hamiltonian $\hat{\mathcal{H}}_e$ represents the interacting Hamiltonian for a direct-gap semiconducting quantum well:

$$
\hat{\mathcal{H}}_e = \sum_p \left( \frac{\hat{p}^2}{2m} - \frac{\mu - E_g}{2} \right) \left( \hat{b}_p \hat{b}_p^\dagger + \hat{a}_p \hat{a}_p^\dagger \right)
+ \frac{1}{2} \sum_{\mathbf{q} \neq 0} v(\mathbf{q}) \left( \rho_{\mathbf{q}} \rho_{-\mathbf{q}} - \sum_p \hat{b}_{p+\mathbf{q}} \hat{b}_p - \sum_p \hat{a}_p \hat{a}_{p+\mathbf{q}} \right),
$$

where the middle of the gap $E_g$ is taken as the energy reference and where $\rho_{\mathbf{q}} = \sum_p \langle \hat{b}_{p+\mathbf{q}} \hat{b}_p - \hat{a}_p \hat{a}_{p+\mathbf{q}} \rangle$ is the total electron density operator ($\hat{b}_p$ and $\hat{a}_p$ create an electron with momentum $p$ respectively in the conductance and valence band band). Without loss of generality, we assume electrons and holes to have the same mass $m$. In the high density regime of electrons and holes $\rho_{\mathbf{q}} \hat{a}_p^\dagger \gg 1 (\rho_{\mathbf{q}} \hat{a}_p^\dagger \hat{a}_p)$

| Band Model | Two-level systems |
|------------|-------------------|
| band dispersion | exciton localisation |
| Coulomb interaction and screening | on-site Coulomb interaction and phase space filling |
| dipole coupling to light | dipole coupling to light |
| non-pair-breaking disorder | inhomogeneous broadening of exciton energies |
| pair-breaking disorder | decoherence (coupling to external baths) |

Table 1
Principal components of the ‘band’ model and the ‘two-level systems’ one.
Fig. 1. Zero temperature mean-field phase diagram for the band model, for \( d = 200 \) and \( \tilde{g} = 1 \) (solid and dashed). The boundary between the condensed gapped and gapless region for \( \tilde{g} = 0.8 \) (dotted) is shown for comparison (in this case the regime where the BCS description loses validity starts at \( \rho_{\text{ex}} a_0^2 = 31 \)).

denotes the real density of electrons and holes and \( a_0 = 2 \epsilon_0/\epsilon^2 m^{-1} \) the exciton Bohr radius, Coulomb interaction, \( v(q) = 4 \pi \epsilon^2/[\epsilon_0(q^2 + \kappa^2)] \) is screened due to both electrons and holes and, in two dimensions, the screening length is set by the Bohr radius, \( 1/\kappa = a_0 \). Hence, in this limit, the Coulomb interaction can be replaced by a short range contact interaction \([6,7]\), where the coupling constant, \( g \), will be a decreasing function of the increasing total density of excitations, \( \rho_{\text{ex}} a_0^2 \) \( (5) \). Applying a self-consistent Hartree-Fock treatment, the Coulomb quartic interaction can be decoupled by means of the excitonic order parameter or polarisation field, \( \Sigma(r) = -g L^2 \langle g.s. | b^\dagger(r) a(r) | g.s. \rangle \). Note that ordinary Hartree-Fock pairings, while crucial in the low-density limit, effect only a small renormalisation of the single-particle energy. Therefore, the system shows a BCS-like instability around the Fermi surface, \( (\mu - E_g)/2 \), and the condensation in the exciton insulator state is signalled by the opening of a gap equal to \( \Sigma(r) \) \( [8] \). The second term in \( (1) \) describes the impurity potentials:

\[
\hat{H}_{\text{int}} = \sum_p \psi_p^\dagger \left( \omega_p - \mu \right) \psi_p
\]

\[
\hat{H}_{\text{int}} = g \int dr \left[ \psi(r) b^\dagger(r) a(r) + \text{h.c.} \right] ,
\]

where the dispersion \( \omega_p = \sqrt{\omega_e^2 + (\langle p \rangle)^2} \) is quantised in the direction perpendicular to the plane of the cavity mirrors. We notice that the decoupling of Coulomb interaction through the excitonic order parameter has place in the same channel as the coupling to photon. Therefore, at mean-field level, the condensate of the polariton system is characterised by an order parameter which is a combination of the polarisation and photon amplitudes, \( \Delta = \Sigma + g \psi \). Moreover, one can show the following constraint has to be verified \([3]\)

\[
g_c (\omega_c - \mu) |\psi| = g |\Sigma| ,
\]
which leads to the following renormalised pairing interaction, $\Delta = -g_{\text{eff}} L^2 \langle \rho \rangle$:

$$g_{\text{eff}} = g_{\text{c}} + \frac{g^2}{\omega_c - \mu}. \tag{4}$$

As a consequence, in the mean-field approximation and in absence of disorder, the condensation is described in BCS terms, where the interplay between electron-hole and photon excitations is invested in the effective coupling constant, $g_{\text{eff}}$, and in the chemical potential, $\mu$. Finally, we will suppose that the electron-hole/photon system is held in quasiequilibrium by tuning the chemical potential in (1) to fix the total number of excitations

$$N_{\text{ex}} = \sum_p \psi_p^\dagger \psi_p + \frac{1}{2} \sum_p \left( \hat{b}_p^\dagger \hat{b}_p + a_p a_p^\dagger \right), \tag{5}$$

where the density of excitations will be indicated with $\rho_{\text{ex}} = N_{\text{ex}} L^2$. However, how the system chooses to portion the excitations between the electron-hole and photon degrees of freedom depends sensitively on the properties of the condensate.

In particular, in the clean limit, it is easy to see that when the density $\rho_{\text{ex}} a_0^2$ is low, the majority of the excitations are invested in electrons and holes and therefore the density increases linearly with the chemical potential. Here, provided the density is large enough to keep the particles unbound, the condensed phase is reminiscent of the exciton insulator one, even if a small fraction of photons do contribute to the condensate. Therefore, in this regime the zero temperature phase diagram (see figure 1) mimics the behaviour of the symmetry broken exciton insulator: At a fixed value of disorder, the amplitude of the order parameter decreases increasing the density because of the Coulomb interaction screening. When the scattering rate $\tau_c$ is comparable to the value of the unperturbed order parameter, the system enters a gapless phase before the condensate is extinguished altogether. Further, adding more excitations into the system, eventually, as the chemical potential approaches the band edge, $\omega_c$, the photons are brought into resonance and the character of the condensate changes abruptly. Screening suppresses the excitonic coupling constant $g_{\text{c}}$, and at the same time the photon effective coupling constant $g^2/\omega_c - \mu$ grows in size. Here, the excitations become increasingly photon-like, with $\rho_{\text{ex}} a_0^2$ diverging exponentially when the chemical potential converges on $\omega_c$ [3]. As a consequence, the condensate becomes more and more robust against the disorder potential until, eventually, the complete quenching of coherence is inhibited. Interestingly, however, the residual effect of the disorder leaves open the possibility of a substantial region of the phase diagram where the system is gapless. Here, the condensate manifests the conventional properties of a semiconductor laser — i.e. a substantial coherent optical field, but a gapless spectrum of electron-hole pairs with negligible electronic polarisation. However, at sufficiently large densities, the photon-dominated order parameter becomes comparable with the Fermi energy and the BCS-type description loses its validity.

Measuring energies in units of the Rydberg ($Ry = e^2/2\epsilon_0 a_0$), the phase diagram in Fig. 1 is characterised by the total excitation density ($\rho_{\text{ex}} a_0^2$), the disorder strength $(1/\tau_c Ry)$, the dimensionless photon coupling strength $\tilde{g} = g(\nu L^2 / Ry)^{1/2}$ and the dimensionless Coulomb coupling strength, which, fixing the thickness of the quantum well of the order of the Bohr radius $a_0$, is a function of $x = (\omega_c - \mu) / Ry$, in particular $g_c \nu L^2 = 10[1 + 4(d - x)]^{-1/2}$. Here, the parameter $d = (\omega_c - E_g) / Ry$ characterises the crossover between the electron/hole and the photon dominated region, which is approximatively given by $d/4\pi$. This is the density of electronic excitations, which, in absence of photons, can be reached when the chemical potential is equal to $\omega_c$. In contrast, fixing the value of $d$, variations in the coupling $\tilde{g}$ bring small changes in the electron/hole dominated region, while, for example decreasing its value as shown in figure 1, it brings to a strong enhancement of the gapless region. Note as well that diminishing the value of $\tilde{g}$ pushes the strong coupling region to higher densities.

### 3. Excitons as Two-Level Oscillators

When the charged component of the disorder potential is negligible, while the neutral component...
is strong enough to localise the excitons, a simple model describing $N$ localised excitons can be rewritten as

$$\hat{H}_{ei} + \hat{H}_{\text{dis}} + \hat{H}_{\text{int}} = \sum_{j=1}^{N} \left( \epsilon_j - \frac{\mu}{2} \right) \left( b_j^\dagger b_j + a_j^\dagger a_j \right) + g \sum_{j=1}^{N} \sum_{p} \left( e^{-2\pi ip \cdot r_j} \psi_p^b b_j^\dagger a_j + \text{h.c.} \right), \quad (6)$$

while the coupling to photons, $\hat{H}_{\text{ph}}$, remains unchanged. As before, the system is described by two order parameters, the coherent photon field and the polarisation, with the difference that, since Coulomb interaction is contained in the on-site energies $\epsilon_j$, the polarisation does not have to be self-consistently determined, but it is locked to the photon field by the relation (3). Again the values of the order parameters depend on the density of excitations, $\rho_{\text{ex}} = N_{\text{ex}}/N$, or equivalently on the chemical potential, $\mu$. In addition, one can introduce dephasing processes, coupling the system (6) to external baths [2]:

$$\hat{H}_{\text{bath}} = \sum_{j,p} \Gamma_p^{(1)} \left( b_j^\dagger b_j - a_j^\dagger a_j \right) \left( c_{1,p}^\dagger + c_{1,p} \right) + \sum_{j,p} \Gamma_p^{(2)} \left( b_j^\dagger b_j + a_j^\dagger a_j \right) \left( c_{2,p}^\dagger + c_{2,p} \right) + \sum_p \Omega_p^{(1)} c_{1,p}^\dagger c_{1,p} + \sum_p \Omega_p^{(2)} c_{2,p}^\dagger c_{2,p}. \quad (7)$$

Here, our interest is restricted to processes which do not change the total number of excitations in the cavity, $N_{\text{ex}} = \sum_p \psi_p^b \psi_p^c + \frac{i}{2} \sum_{j} \left( b_j^\dagger b_j + a_j^\dagger a_j \right)$, as, for example, collisions with phonons. Analogously to disorder in the previous model (2), decoherence processes can be divided into non-pair-breaking ($\Gamma_p^{(1)}$) and pair-breaking ($\Gamma_p^{(2)}$) ones. The degrees of freedom of the bath, $c_{1,p}$ and $c_{2,p}$, can be integrated out, inducing a quartic interaction between different two-level systems, which, similar to how disorder has been treated in the previous model, can be analysed within a self-consistent Born approximation. The main difference lies in the fact that disorder is static, while the baths are characterised by a dynamics. However, for simplicity, one can assume a static limit, in which the bath is characterised by only one frequency, e.g. the lowest one. In this case the decoherence parameter for, say, the pair-breaking term, will be indicated with $2\gamma^2 = \sum_k \Gamma_p^{(2)}/\Omega_p^{(2)}$. In the opposite limit instead, one can assume each mode of the bath oscillating independently from the other ones. In this limit, called Markovian approximation, as for the previous one, the mean-field equations admit an analytical solution. Here we will refer to the first case.

In absence of decoherence, the minimum $\rho_{\text{ex}} = -0.5$ corresponds to the case where there are no photons and no electronic excitations. At low excitation densities, where $\rho_{\text{ex}} \gtrsim -0.5$, the usual bosonic polaritons limit is preserved, while away from the minimum the condensate gradually becomes more photon-like, the phase space filling effect of the fermionic states start to become relevant and the bosonic picture breaks down [4]. In the absence of dephasing, however, even at high excitation densities, the ground state of the system is a strongly coupled light-matter condensate with a large gap in the excitation spectrum, i.e. the incoherent luminescence, very different from a laser. The influence of the pair-breaking decoherence on the condensate in this model [2] is similar to the influence of the pair-breaking disorder in

![Fig. 2. Zero temperature mean-field phase diagram for the localised excitons model. The dashed lines show the phase boundaries between the gapped and the gapless condensate, while the solid line refer to the boundaries between the gapless condensate and the non-condensed system. The upper curves correspond to the resonance where $\omega_c - 2\epsilon = 0$, while the lower curves are for detuning $\omega_c - 2\epsilon = g$, where the photon field is above the exciton line by an energy equal to the dipole coupling.](image.png)
the band model described in the previous section. This is manifested in the phase diagram in Fig. 2. The dephasing processes lead to a suppression of the electronic polarisation and the gap in the excitation spectrum. At very low excitation densities and low decoherence there is a gapped condensate with photon and exciton components comparable in size. As the decoherence is increased, the system reaches a narrow gapless region and finally there is a complete suppression of the coherent fields. As the excitation density is increased, the gapless region becomes larger. For sufficiently large excitations, the coherent fields are present even in the presence of very large dephasing. In this regime the condensate is almost entirely photon-like with very small excitonic polarisation and the spectrum is gapless, reminiscent of the laser. In contrast, non-pair-breaking processes which give rise to inhomogeneous broadening of exciton energies have much weaker, only qualitative, influence on the coherent fields and do not cause any transitions.

4. Discussion

A comparison between the models described in the previous two sections can now be drawn. Despite the fact the two approaches start from different assumptions on the nature of the excitons, they share the main results and in particular exhibit a very similar phase diagram. First of all, the description of the condensate in both cases is done in terms of an order parameter to which both excitonic and photonic excitations contribute. In the first model the polarisation field derives from the self-consistent decoupling of Coulomb interaction, and, at mean-field level, is locked to the photon field by the relation (3), which similarly has place for the second model. Secondly, in both cases, increasing the density of excitations, the electron-hole dominated region precedes the one dominated by photons. While, due to a combination of screening of Coulomb interaction and a resonance mechanism when the chemical potential reaches the cavity frequency $\omega_c$ in the first case, in the second one this is due to the space filling effect of the fermionic degrees of freedom. Either way, this circumstance causes the pair-breaking disorder from one side and the pair-breaking (static) decoherence processes from the other one to act very similarly on the condensate: Complete quenching of the condensate and therefore loss of coherence in the system by pair-breaking mechanisms is possible only in the electron-hole dominated region, while, above a certain density threshold, coherence is preserved and strong pair-breaking effects drive the system into a gapless regime, where there is no coherence in the excitonic component of the order parameter but only in the photonic one. Here, the system exhibits the characteristic of a conventional laser. Note that, because of screening, in the band model the condensed region in the electron-hole dominated region is reduced by the increase of the density, while this effect is absent in the two-level systems' model.

There are several important issues not yet considered here. How would the phase diagram look like in the presence of realistic band-limited decoherence characteristic, for example, for acoustic phonons? Moreover, beyond the mean-field approximation, while the effect of fluctuations and the spectrum of collective excitations has been analysed within the band model [3] and in the two level systems one without decoherence [5], a similar study in presence of decoherence will be the subject of further investigations.

From the analysis above we can therefore conclude that, regardless whether excitons are propagating or localised, high densities do not preclude polariton condensation. On the contrary, since the gap in the density of state is proportional to the coherent field amplitude, the condensate becomes more robust at high densities. Real antagonists of condensation are pair-breaking disorder and decoherence processes.

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