On the unitarity and low energy expansion of the Coon amplitude

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The Coon amplitude is a deformation of the Veneziano amplitude with logarithmic Regge trajectories and an accumulation point in the spectrum, which interpolates between string theory and field theory. With string theory, it is the only other solution to duality constraints explicitly known and it constitutes an important data point in the modern S-matrix bootstrap. Yet, its basic properties are essentially unknown. In this paper we fill this gap and derive the conditions of positivity and the low energy expansion of the amplitude. On the positivity side, we discover that the amplitude switches from a regime where it is positive in all dimensions to a regime with critical dimensions, that connects to the known $d = 26, 10$ when the deformation is removed. En passant, we find that the Veneziano amplitude can be extended to massive scalars of masses up to $m^2 = 1/3$, where it has critical dimension $6.3$. On the low-energy side, we compute the first few couplings of the theory in terms of $q$-deformed analogues of the standard Riemann zeta values of the string expansion. We locate their location in the EFT-hedron, and find agreement with a recent conjecture that theories with accumulation points populate this space. We also discuss their relation to low spin dominance.

The Coon amplitude [1–3] is, together with the Veneziano amplitude, the only explicitly known four-point tree-level amplitude that describes an infinite exchange of higher-spin resonances which solves the duality constraints. It was discovered as a deformation of the Veneziano amplitude to non-linear Regge trajectories. The deformation is given in terms of a parameter $q$ ($0 \leq q \leq 1$), which characterises a family of amplitudes defined by (in units $\alpha' = 1$)

$$A_q(s, t) = (q - 1)q^{-\log(\tau) + \log(r)} \prod_{n=0}^{\infty} \frac{(\sigma - q^n)(1 - q^{a+1})}{(\sigma - q^n)(\tau - q^n)} \tag{1}$$

with

$$\sigma = 1 + (s - m^2)(q - 1), \quad \tau = 1 + (t - m^2)(q - 1) \tag{2}$$

where $s, t$ are Mandelstam variables (c.f. appendix). This amplitude describes the scattering of four identical scalars of mass $m^2$. At $q = 0$, it reduces to a scalar theory, and at $q = 1$ it gives back the Veneziano model:

$$\lim_{q \to 0} A_q(s, t) = \frac{1}{s - m^2} + \frac{1}{t - m^2} + 1 \tag{3}$$

$$\lim_{q \to 1} A_q(s, t) = A^V(s, t) = - \frac{\Gamma(-s + m^2)\Gamma(-t + m^2)}{\Gamma(-s - t + 2m^2)} \tag{4}$$

Unlike for the Veneziano model, no worldsheet theory was found for the Coon amplitude, and to this day, its physical origin remains mysterious. In addition, and what concerns us in this paper, its basic properties; unitarity conditions and low-energy expansion, are essentially unknown.

In more recent times, the Coon amplitude was brought forward in the bootstrap analysis as an exception to the universality of linear Regge trajectories in [4], coming from the existence of an accumulation point in its spectrum, similar to that of the hydrogen atom, which allows the theory to evade the theorem of [4]. Related bootstrap constraints applied to the Wilson coefficients of effective field theories (EFTs) coming from unitarity, crossing and analyticity are known to impose bounds [5] that carve theory islands [6–10], and it appears that they are bigger than what is required to describe the basic theories of the world around us [11–14]. Even more interestingly, [14] recently conjectured that the space of gravitational EFTs is actually populated generically of theories with an accumulation point. Since the Coon amplitude has an accumulation point and connects continuously string theory and field theory, it provides an extremely interesting testing ground to investigate some aspects of these questions.

The main results of our analysis are as follows. Firstly, we map the positivity region of the amplitude in the $(q, m^2)$-space, see fig. 2: for each point $(q, m^2)$ we determine the maximal dimension in which no ghosts are exchanged as intermediate states. This generalises the known $d = (10)/26$ critical dimensions of (super)string theory for $m^2 = 0$ [15]. We discover a surprising regime of the amplitude where we can prove that it is ghost-free in all dimensions [15]. This goes against standard intuition that in high enough dimensions, string-like theories eventually cease to be unitary. In the other regime, assisted by analytical arguments, we determine numerically the positivity surface, which interpolates from infinite critical dimensions to the standard critical dimensions of string theory. Along the way, we also realised that the Coon and a fortiori Veneziano amplitudes can be extended to positive $m^2 \to 1/3$, with corresponding critical dimension $d \simeq 6.3$ for the Veneziano amplitude [16].

Secondly, we compute some low energy couplings of the Coon amplitude in terms of $q$-polylogarithm values, which generalise the known zeta-values of the string low-energy expansion. We compute explicitly the first coefficients $g_2(q), g_3(q), g_4(q)$ and map their location in the space of couplings, comparing to [6]. We also comment on the connection to the notion of low-spin dominance.
of \( \frac{1}{2} \mathfrak{s} \).

\[ A_q(s, t) \sim f(t)s^j(t), \quad j(t) = \frac{\log((t-m^2)(q-1)+1)}{\log(q)} \]

\[ A_q(s, t) \sim e^{\log(s)\log(-t)/\log(q)} \sim s^{\log(s \cos(\theta))/\log(q)}. \]
the critical dimensions of the Coon amplitude. The upper bound is imposed on us by unitarity: for $m^2 > 0$ and $m^2 = 1$, the Coon amplitude has ghosts in all dimensions, and unitarity demands that these should be positive, as a negative coefficient implies that the corresponding exchanged state has negative norm.

In the case of string theory, the no-ghost theorem guarantees that such states decouple from all scattering amplitudes in for $d \leq 26$ or 10. At the level of the four-point function (the Veneziano amplitude), a recent paper showed that residues should all be positively expandable on Gegenbauer polynomials in $d \leq 6$, (see also for the states on the leading Regge trajectory in $d = 4$). It would still be desirable to be able to bridge the gap to $d = 10$ or 26 and maybe the $q \to 1$ limit of the Coon amplitude could open an avenue to be combined with the techniques of these papers.

As regards Coon, an early study did investigate the presence of ghosts in the amplitude in four dimensions. Some partial results were obtained, showing that some regions in the $q, M^2$ parameter space are ghost-free. While their (numerical) method finds ghosts in four dimensions, we do not, for any values of $q$. This is because we look at a different set-up: for them, the mass of the external particles $M^2$ was different from $m^2$, the lowest mass of the amplitude, while for us, $M^2 = m^2$.

The more recent reference which also studied the problem (which was unaware of) is largely inconclusive and before the present work, nothing was known on the critical dimensions of the Coon amplitude.

Our results are summarized in fig. They show the existence of two regimes for $q$, distinguished critical value $q \lessapprox q_\infty(m^2)$, where

$$q_\infty(m^2) = \frac{m^2 - 3 + \sqrt{9 + 2m^2 + m^4}}{2m^2}. \quad (12)$$

We find that, for $q < q_\infty(m^2)$ the amplitude is ghost-free in all dimensions, which we demonstrate analytically. Then, for $q > q_\infty(m^2)$, are critical dimensions exist and we numerically determined them, backed by an estimate of the envelope of the critical dimensions near $q_\infty(m^2)$. Let us now give some details on this analysis.

**Mass range: $-1 \leq m^2 \leq 1/3$**

Before describing the behaviour in $q$, we describe the range of $m^2$. For masses $m^2 < -1$, the Veneziano amplitude is known to not be unitary (corresponding to intercepts greater than 1). For the Coon amplitude, a related statement holds: for $m^2 < -1$, there is a ghost at scalar mass-level 2 for $q > \frac{m^2}{m^2-1}$. For $q < \frac{m^2}{m^2-1}$, this state becomes a positive-norm state. We gathered solid numerical evidence that there exists a unitarity range similar to the one we describe below, for all masses $m^2 \to -\infty$. However, to keep a smooth $q \to 1$ limit to string theory we restricted ourselves to $m^2 \geq -1$, but the $m^2 < -1$ regime might contain some physics worth studying.

The upper bound is imposed on us by unitarity: for $m^2 > \frac{1}{3}$, the Coon amplitude has ghosts in all dimensions and for all values of $q$. One can check this explicitly by looking for example at the coefficient $c_{1,1}$, which is given
Critical dimensions for \( q > q_\infty(m^2) \)

In this range, ghosts are not excluded by our previous argument, and therefore we performed an extensive numerical study of the sign of the residues. We computed the Gegenbauer coefficients for the first 50 resonances and all spins for a grid of values of \( m^2 \) between \(-1\) and \( \frac{1}{2} \) and varied \( q \) and \( d \) by small increments to obtain the critical dimension \( d(m^2, q) \) for which all the coefficients become positive for each \( (m^2, q) \), with accuracy \( 1/25 = 4\% \). This allowed us to map the surface in parameter space that separates the unitary and non-unitary regions, given by the green spline in fig. 3. In the \( q \rightarrow 1 \) limit, the critical dimensions of the \( m^2 = -1 \) and \( m^2 = 0 \) models match the known values for the Veneziano and Neveu-Schwarz amplitudes \((d = 26 \) and \( d = 10 \) respectively). Moreover, our results point towards a possible extremal unitary amplitude with \( m^2 = \frac{1}{3} \) and critical dimension \( d \approx 6.3 \). Those three curves are plotted specifically in fig. 3. We provide some more details on the numerics in the appendix.

One interesting observation from that study is that the scalar ghost sector seems define the unitarity surface. Below, we give a proof of this fact in the large \( d \) limit and an estimate of the critical dimensions for \( q \) near the critical line which matches the numerics, see fig. 3. Surprisingly, while our arguments fail when \( q \) goes closer to 1, we observe numerically that the scalar ghost criterion continues to hold. Proving this fact fully, maybe using the methods of [28], would allow to prove exactly the unitarity of Coon (up to finite numerical accuracy in \( d \)) and of Veneziano amplitude as a function of \( m^2 \).

The argument about the scalar sector and the critical dimensions goes as follows. Consider the pole at \( \sigma = q^m \).

We want to relate the coefficients \( c_{n,L} \) in the Gegenbauer decomposition of \( P_n \) to the coefficients \( p_{n,k} \) appearing in \((15)\), which are given in terms of the roots \( x_{j,n} \) by

\[
p_{n,k} = (-1)^k \sum_{i_1 < i_2 < \ldots < i_k} x_{i_1,n} \ldots x_{i_k,n}.
\]  

(17)

At fixed \( n \), the \( c_{n,L} \) coefficient only receives contributions from the \( p_{n,k} \) coefficients with \( k \geq L \) and with the same parity. Explicitly, for the scalar sector we have

\[
c_{n,0} = p_{n,0} + \frac{p_{n,2}}{-1 + d} + \frac{3p_{n,4}}{(-1 + d)(1 + d)} + \ldots
\]  

(18)

From the form of the roots \( x_{j,n} \) one can see that the \( p_{n,L} \) coefficients follow a well-defined pattern: when decreasing \( q \) starting from values where the amplitude is not unitary, \( p_{n,L} \) become positive in decreasing order of spin. In particular, at some point, all the \( p_{n,k} \) coefficients will have become positive except \( p_{n,0} \).

The important point is that right before \( p_{n,0} \) turns positive (when \( x_{n-1,n} \) becomes negative), \( c_{n,0} \) will become
positive by continuity, as only the first term in \([18]\) remains negative. Thus, at large \(d\), \(c_{n,0}\) becomes positive when
\[
d = \frac{p_{n,2}}{-p_{n,0}} \tag{19}
\]
Since \(p_{n,0}\) is small and negative, \(d\) is large as expected. As all the other \(c_{n,L}\) coefficients do not receive contributions from \(p_{n,0}\), we see that the coefficient \(c_{n,0}\) is the last one to become positive and thus the scalar ghosts are the ones defining the transition to the unitary regime in the large \(d\) limit.

Finally, similar arguments show that this ratio decreases as \(n \to \infty\) and therefore the limit curves envelops the region of unitarity of the amplitude. It turns out that the limit can be explicitly evaluated by summing the infinite \(n\) limit of the double sum \(-p_{n,2}/p_{n,0} = -\sum_{j_1 \neq j_2} \frac{1}{x_{n,j_1} x_{n,j_2}}\). The resummed function is given in appendix, and is plotted in 3, and match nicely the numerical results at large \(d\).

**LOW ENERGY EXPANSION AND EFT-HEDRON**

Recent times have witnessed a renewal of activity revolving around implications of dispersion relations, crossing symmetry and unitarity. Following the ideas of [5], various studies explored how the Wilson coefficients of weakly coupled EFTs are constrained [6, 10, 33] and live in some (sometimes small) regions of some positive region dubbed “EFThedron”.

In this section, we compute the first few low energy couplings of the Coon amplitude. Because the Coon amplitude is well behaved at infinity, and respects analyticity and crossing, it admits dispersion relations and must fall in those positivity regions. We will see that the couplings indeed draw 1-dimensional varieties within those regions, parametrized by the value of \(q\).

The first amplitude we consider is an \((s, t, u)\) symmetric version of Coon, for external massless scalars \((m^2 = 0)\) with no color indices:
\[
M_q(s, t, u) = A_q(s, t) + A_q(t, u) + A_q(u, s) \tag{20}
\]
The \((s, t, u)\) symmetry and momentum conservation \(s + t + u = 0\) allow to expand at small \(s, t, u\) this function in terms of \(\sigma_2 = s^2 + t^2 + u^2\) and \(\sigma_3 = stu\), so that
\[
M_q(s, t, u) = \frac{1}{s} + \frac{1}{t} + \frac{1}{u} + g_0(q) + g_2(q)\sigma_2 + g_3(q)\sigma_3 + g_4(q)(\sigma_2)^2 + \ldots \tag{21}
\]
where the coefficients of this expansion are classically interpreted as low energy Wilson coefficients.

A lengthy but straightforward explicit calculation gave us the first few coefficients, up to \(g_4(q)\). Trivially, \(g_0(q) = 1 - q\). The next ones are given by functions related to \(q\)-zeta values, for instance \(g_2(q)\) reads
\[
g_2(q) = \frac{1}{2}(q - 1)^3 \left(3h_1(q) + 5h_2(q) + 2h_3(q)\right) - \frac{(q - 1)^3}{\log(q)} \tag{22}
\]
where
\[
h_m(q) = \sum_{n=1}^{\infty} \frac{q^{nm}}{(1 - q^n)^m} = \text{Li}_m(q^m; q) \tag{23}
\]
can be written in terms of \(q\)-deformed polylogarithms as defined for instance in [34], and whose \(q\)-zeta values are classically defined as \(q\)-values of those functions. Note that, compared to string theory, different orders of \(q\)-transcendentality appear to be mixed. The other couplings \(g_3(q)\) and \(g_4(q)\) are given in eqs. [37], [38]. We also verified that when \(q \to 1\), they descend to the values given by the symmetrized sum \(A^V(s, t) + A^V(t, u) + A^V(u, s)\):
\[
g_2(1) = -\zeta_3, \quad g_3(1) = 9/4 \, \zeta_4, \quad g_4(1) = -\zeta_5/2 \tag{24}
\]
For generic EFTs, the allowed range of coefficients \(g_2, g_3, g_4\) was determined in [6], fig. 8, in terms of dimensionless ratios \(g_3 = g_3 M^2/g_2\) and \(g_4 = g_4 M^4/g_2\) with \(M^2\) given by the scale of the first massive mode, which in our conventions is \(M^2 = [1] = 1\). We show in fig. 4 the value of those ratios. They fall neatly in the domain determined in [6], albeit approaching tangentially the boundary at intermediate values of \(q\).

One can also couple a massless scalar to a massive Coon amplitude, since \(0 \leq m^2 \leq 1/3\) are allowed. These amplitudes reduce to the extreme scalar case of [6] of a coupling the massless scalar to a single massive scalar of mass \(M^2 = m^2\) and therefore accumulate to the upper right corner of their fig. 8. It is not surprising, and the same happens when coupling by a massless scalar to an amplitude made of massive Veneziano blocks.

**Low spin dominance.** The Coon amplitude \(A_q(s,t)\), together with its Veneziano limit, exhibit a form of low-spin dominance, albeit weaker than that of [11]. It is a
low spin dominance were not only the scalar state dominates the partial waves, but also the spin 1 state \( a_{4,0} \). Let us explain how this comes about.

Following the conventions of [11], we Taylor expand the amplitude as \( A_J(t, -s - t) = \frac{1}{2} + \frac{1}{2} \sum_{p < k} a_{k,p} s^{k-p} p^p \). In order to match to [11], we look at the coefficients at level \( k = 2, 4, 6 \), which we compare to that of an amplitude given by a sum of

\[
A^{(J)}(t, u) = (-1)^J \left( \frac{\mathcal{P}_J(1 + 2s/M^2)}{t - M^2} + \frac{\mathcal{P}_J(1 + 2t/M^2)}{u - M^2} \right)
\]

with \( J = 0, 1 \). Denoting by \( a_{k,q}^{(J)} \) the low energy coefficients of expanding \( A^{(J)}(t, -s - t) \) in powers of \( s, t \), the model mentioned above states that \( a_{k,0} \sim a_{k,0}^{(0)} + q a_{k,0}^{(1)} \). This corresponds to the straight yellow line in fig. 5 and can be seen to match very well the dots near \( q = 1 \) (top-right corner), where the amplitude matches pure low-spin-dominance, where only \( J = 0 \) contributes. Within this spin 0-1 model, it is immediate to verify for instance that

\[
\begin{align*}
a_{2,1} &= \frac{2 - 2q}{1 + q} \\
a_{2,0} &= \frac{2}{1 + q} \\
a_{4,1} &= \frac{2(q + 2)}{q + 1}, \quad a_{4,2} = \frac{6}{q + 1}
\end{align*}
\]

For \( q \in [0, 1] \), this implies in particular that \( 0 \leq \frac{a_{2,1}}{a_{2,0}} \leq 1 \) and \( 3 \leq \frac{a_{4,1}}{a_{4,0}} \leq 4 \) and \( 3 \leq \frac{a_{4,2}}{a_{4,0}} \leq 6 \). The upper bound correspond to pure low-spin dominance, while the lower bound is pure spin-0 + spin-1 model. While for the coefficients \( a_{2,1}/a_{2,0} \), the bounds are exactly satisfied, at \( k = 4 \) it can be seen that string theory lies a bit away from that, at \( \frac{a_{4,1}}{a_{4,0}}, \frac{a_{4,2}}{a_{4,0}} \approx (2.9, 2.9) \). The relative accuracy of the model is explained by the fact that spin \( J \) exchanges come with \( q^{J(J+1)/2} \), for which the linear approximation (spin 0 and 1) is a good approximation away from \( q = 1 \).

**PERSPECTIVES**

This study opens many perspectives, already mentioned in the text.

Firstly, it resonates very neatly with a conjecture of [14] that amplitudes with accumulation points populate the EFT-hedron of gravitational theories away from the small portion where usual theories seem to live. It would be very important to study this problem in more details, in relation with the Coon amplitude.

Secondly, it opens a way to attack the question of the positivity of the Veneziano amplitude recently studied in [28] thanks to its extra \( q \)-dependence. In particular, if one could prove our empirical observation that only the scalar ghost determines the critical dimension for values of \( q \) arbitrarily close to one, maybe using the techniques of [28], one could use the smoothness of the limit to prove positivity of the string theory four-point amplitude.

More generally, it would be of course essential to extend these results to the \( N \)-point function. Indeed, to make a statement about the unitarity of a theory (rather than the amplitude), in absence of a no-ghost theorem, one should indeed prove that ghosts decouple in all exchanges of all amplitudes. \( N \)-point functions were proposed as infinite sums in [22, 30] and factorization was proven in [37, 58].

It would also be interesting to study the Coon version of the Lovelace-Shapiro amplitude [39, 40]. This amplitude, only recently understood from string theory [41], constitutes an interesting example where the \( N \)-point function shows defaults of unitarity.

Finally, the relation to \( q \)-zeta values might open an interesting avenue to produce worldsheet models for the Coon amplitude, and relate to various integral representation proposed in the literature [42, 43].

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**Acknowledgements** We would like to thank Eduardo Casali for collaboration at initial stages of a related project. We would like to thank Simon Caron-Huot for some useful discussions, Sasha Zhiboedov for many very stimulating discussions and comments, Pierre Vanhove for comments on FF’s internship thesis related to this paper, and Massimo Bianchi, Paolo Di Vecchia and Oliver Schlotterer for useful discussions and detailed comments on the paper.

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APPENDIX

Conventions

Kinematics. Here we define our conventions for the article. In the centre-of-mass frame, we have

$$s = -(p_1 + p_2)^2 = 4E^2 = 4(p^2 + m^2),$$

$$t = -(p_1 + p_4)^2 = -2p^2(1 - \cos(\theta)),$n

$$u = -(p_1 + p_3)^2 = -2p^2(1 + \cos(\theta)),$$

where $E$ is the center of mass energy, $\mathbf{p}$ the momentum transfer and $\theta$ the scattering angle. Note that the $t \leftrightarrow u$ crossing simply sends $\theta \to \pi - \theta$ in the case of scattering of identical particles.

The standard relations for the Mandelstam invariants read

$$s + t + u = 4m^2,$$

and

$$\cos(\theta) = 1 + \frac{2t}{s - 4m^2} = \frac{u - t}{u + t}.$$
are defined in terms of hypergeometric functions by
\[
\mathcal{P}_j^{(d)}(z) = 2F_1 \left( -J, J + d - 3, \frac{d - 2}{2}, \frac{1 - z}{2} \right) = \frac{\Gamma(1 + J)\Gamma(d - 3)}{\Gamma(J + d - 3)} G_j^{(\pm \frac{d}{2})}(z),
\] (33)
where \( G_j^{(\pm \frac{d}{2})}(z) \) are known as Gegenbauer polynomials. They satisfy the following orthogonality relation:
\[
\frac{1}{2} \int_{-1}^{1} dz \, (1 - z^2)^{\frac{d-2}{2}} \mathcal{P}_j^{(d)}(z) \mathcal{P}_j^{(d)}(z) = \frac{\delta_{jj}}{N_{d, j}^{(d)}},
\] (34)
with normalization factors defined by
\[
N_{d} = \frac{(16\pi)^{\frac{d-2}{d}}}{\Gamma\left(\frac{d-2}{2}\right)}, \quad n_{j}^{(d)} = \frac{(4\pi)^{\frac{d}{2}}(d + 2J - 3)\Gamma(d + J - 3)}{\pi\Gamma\left(\frac{d-2}{2}\right)\Gamma(J + 1)}.
\] (35)

Positivity of a given monomial \( x^n \) follows explicit expression [38]:
\[
\int_{-1}^{1} x^{n+2\rho}(1 - x^2)^{\frac{d-2}{2}} \mathcal{P}_n^{(d)}(x) dx =
2^{-n+1}n!\Gamma(1 + n)\Gamma(2\rho + 1)\Gamma(n + \rho + \frac{d-2}{2}).
\] (36)
for \( \rho \) positive integer. When \( n \) and \( J \) do not have the same parity modulo 2, \( \int_{-1}^{1} x^n G_j^{(\nu)}(1 - x^2)^{\nu-1/2} \) vanishes. This condition on \( \rho \) and \( n \), together with a factor of 2 are missing in [38].

**Regge trajectories**

For the interested reader, we show below how the Regge trajectories \( j(t) \)

**Numerical study**

We mapped the surface that separates the unitary and non-unitary regions in parameter space by a combination of two methods:

To obtain the vertical part corresponding to the high \( d \) behaviour we started from a value \( q > q_\infty(m^2) \) where ghosts are present and decreased progressively \( q \), while keeping \( m^2 \) and dimension \( d \) fixed until all the coefficients became positive. This allowed us to identify a \( d \)-dependent critical \( q \), which we call \( q_c(m^2, d) \), that determines the boundary of the region of 3-dimensional parameter space \( q, m^2, d \) in which the amplitude is unitary and approaches \( q_\infty(m^2) \) in the large \( d \) limit.

For the rest of the surface we used the fact that for each \( m^2 \) there exist a critical dimension below which the amplitude is ghost-free for all values of \( q \). Fixing \( q \) and \( m^2 \) and starting from \( d \) lower than this critical dimension, we progressively increased \( d \) until we found the presence of ghosts, thus determining the critical surface.

In both cases we computed all the Gegenbauer coefficients for the first 50 resonances at each value of \( q \) with a numerical uncertainty on the critical dimension of 4%.

**q-values and envelope curve**

We further find
\[
g_3(q) = -\frac{3}{2} (q - 1)^4 (h_1(q))^2 + 2 (h_2(q) + 1) h_1(q) + h_2(q) (h_2(q) + 3) - h_4(q) +
\frac{(q - 1)^4 (12 (h_1(q) + h_2(q)) + 7)}{2 \log^2(q)} - \frac{3(q - 1)^4}{4 \log(q)}
\] (37)
and
\[
g_4(q) = \frac{1}{8} (q - 1)^5 (h_1(q))^2 + 2 (h_2(q) + 3) h_1(q) + h_2(q) (h_2(q) + 21) + 28 h_3(q) + 17 h_4(q) + 4 h_5(q) +
\frac{(q - 1)^5 (-12 (h_1(q) + h_2(q)) - 11)}{48 \log^2(q)} + \frac{(q - 1)^5}{8 \log^2(q)}
\] (38)

A related type of computation gave us the envelope of the unitarity curve at large \( d \), plotted in fig. 3 which we found to be:
\[
\lim_{n \to \infty} \frac{p_{n,2}}{p_{n,0}} = \frac{1}{8} f^2 \left(1 + 3m^2(q-1)\right)^2 \left(\frac{3}{2} - \log(q) \left(\psi^{(0)}(\frac{3}{2}) \log \left(\frac{1}{q}\right) - \psi^{(0)}(\frac{1}{2}) + \log \left(\frac{q}{f}\right) + \log(1-q) \right) - \text{Li}_2 \left(1; \frac{f}{q}\right)\right),
\]
with \( f = \frac{2}{3 + m^2(q-1)} \).

**Relation to Veneziano**

Here we review briefly how the Coon amplitude relates to the Veneziano amplitude by allowing non-linearities and solving an ansatz, following the original reference [1].

The Veneziano amplitude [49] can be seen as the unique answer to the following question: Given a theoretical model with spectrum of resonances \( m_n^2 \) and assuming that its tree-level 4-point amplitude can be written in a product representation in terms of its poles \( m_n^2 \) and zeros \( \lambda_n \) of the form

\[
A(s,t) = \prod_{n=0}^{\infty} \frac{\alpha_n s + \beta_n t - \lambda_n}{\alpha'(s - m_n^2)(\alpha'(t - m_n^2))},
\]

what are the conditions on the \( m_n^2 \) and \( \lambda_n \) for the amplitude to have polynomial residues?

It is immediate to realize that with this ansatz, polynomiality of the residues implies the linear spectrum of the Veneziano amplitude \( m_n^2 = 4(n - \alpha_0)/\alpha' \) and that the zeroes \( \lambda_n \) must be located at \( s + t = m_n^2 \) [1][2].

The Coon amplitude was born out as a way to generalize the Veneziano amplitude by allowing an extra \( st \) term in the numerator, such that one is now looking for an amplitude which takes the form

\[
A(s,t) = \prod_{n=0}^{\infty} \frac{\alpha_n s + \beta_n ct + \gamma_n st - \lambda_n}{(s - m_n^2)(t - m_n^2)}
\]

Demanding that the amplitude reduces to the Veneziano model in some limit where the deformation parameters go to zero is enough to obtain the Coon amplitude.

In conclusion, the Coon amplitude is the unique solution to the duality constraints with non-linear trajectories.

**Explicit expression for the coefficients up to level 3**

The Gegenbauer coefficients of the first three levels together with their expansion around \( q = 1 \) are given by:

\[
c_{0,0} = 1, \quad c_{1,0} = \frac{(m^2 - 1) q}{2} + 1 = \frac{(m^2 + 1)}{2} + \frac{(m^2 - 1)}{2}(q-1),
\]

\[
c_{1,1} = \frac{(1 - 3m^2)}{2(d-3)} q = \frac{1}{2} - \frac{3m^2}{2(d-3)} + \frac{1}{2} - \frac{3m^2}{2(d-3)}(q-1),
\]

\[
c_{2,0} = \frac{(d-1)(q(1-m^2) + q^2 - 2)}{4(d-1)(q+1)} \left(\frac{(4m^4 - 10dm^2 - 12d + 40m^4 - 62m^2 + 40)}{16(d-1)} + O((q-1)^2)\right),
\]

\[
c_{2,1} = \frac{q(-3m^2 + q + 1)}{2(d-3)(q+1)} \left(\frac{(-15m^4 + 27m^2 - 8)}{8(d-3)} + O((q-1)^2)\right),
\]

\[
c_{2,2} = \frac{q^3(-3m^2 + q + 1)^2}{2(d-3)(d-1)(q+1)} \left(\frac{(2 - 3m^2)^2}{4(d-3)(d-1)} + \frac{(45m^4 - 72m^2 + 28)}{8(d-3)(d-1)} + O((q-1)^2)\right).
\]

It can be checked that the limit \( q \to 1 \) reproduces the coefficients of the Veneziano amplitude and allow to recover the critical dimensions of (super)string theory for \( m^2 = -1, 0 \). For instance, for \( m^2 = -1 \) and \( q = 1 \) one has

\[
c_{2,0} = -\frac{d - 26}{8(d-1)},
\]

which shows that the Veneziano amplitude is non-unitary for \( d > 26 \).