Countering quantum noise with supplementary classical information

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We consider situations in which i) Alice wishes to send quantum information to Bob via a noisy quantum channel, ii) Alice has a classical description of the states she wishes to send and iii) Alice can make use of a finite amount of noiseless classical information. After setting up the problem in general, we focus attention on one specific scenario in which Alice sends a known qubit down a depolarizing channel along with a noiseless cbit. We describe a protocol which we conjecture is optimal and calculate the average fidelity obtained. A surprising amount of structure is revealed even for this simple case which suggests that relationships between quantum and classical information could in general be very intricate.

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I. INTRODUCTION

In the study of quantum information theory it is often assumed that classical information is effectively noiseless, free and unlimited. In this context many problems become trivial. For example, consider a situation in which Alice wants to ‘teleport’ a quantum state, whose identity is known to her, to Bob (this has come to be known as ‘remote state preparation’). If classical information is considered to be free, then no teleportation-type procedure is actually needed. Alice can simply call Bob on the telephone and tell him what the state is. If they don’t care how long the call lasts then Bob can construct a state arbitrarily close to Alice’s original.

Remote state preparation becomes non-trivial if we wish to restrict the amount of classical information that Alice can send to Bob. Of course, if Alice and Bob share a perfect singlet (or one ebit), then they can achieve perfect teleportation with the transmission of only two classical bits (cbits). But in general it is not possible to distill a perfect singlet from a finite number of mixed states, so the resulting states are still noisy. (ii) some states only admit distillation if collective operations are allowed, that is operations on more than one pair at once, (for example this is true of Werner states) and this may be impractical in a given situation and most importantly (iii) distillation itself involves the sending of cbits and if this is expensive, distillation may not be the best option.

It may rarely be the case that the sending of (relatively noiseless) cbits is expensive compared with the sending of (potentially noisy) qubits. Even if so, by assuming always that classical information is effectively free and thereby not bothering to count it, we may miss out on interesting theoretical relations between quantum and classical information.

II. THE PROBLEM

With the above in mind, we consider the following problem. Alice and Bob are separated by a noisy quantum channel. Alice sends into the channel some quantum state, drawn from an ensemble \( \{ p_j, |\psi_j\rangle \} \) (where the state \( |\psi_j\rangle \) is drawn with probability \( p_j \) and Alice and Bob both know the ensemble). Alice is given a classical description of which state went into the channel and can generate any state she wants to be sent into the channel. The difference lies in the fact that Alice may, in
this alternative case, generate and send a state which is
different from the one which she wants Bob ultimately to
end up with. Here we do not investigate this possibility
but concentrate only on the scenario in which the state
entering the channel is always identical with the state
Alice wishes Bob to prepare. This would be the case if
Alice has no control over what goes into the channel but
is simply given a classical description. Or if Alice sends
the state she wants Bob to have into the channel expect-
it to be noiseless and only finds out about the noise
later, at which point she decides to send some additional
classical information.

We start by describing how the most general possible
scheme will work. It is well known that the most general
evolution which a quantum state can undergo (assuming
that if measurements are performed, their results are to
be averaged over) corresponds to a completely positive
trace-preserving map $[10]$. In the Kraus representation,
this can be written as:

$$\rho \rightarrow \rho' = \sum_i M_i \rho M_i^\dagger,$$  \hspace{1cm} (1)

where

$$\sum_i M_i^\dagger M_i = I \hspace{1cm} (2)$$

and $I$ is the identity. We refer to such a map as a quan-
tum operation. Both the noise experienced by the state
as it passes down the channel and Bob's operation take
this form. Now consider that the experiment described
above is repeated many times - each time, Alice sends a
quantum state down the noisy channel, and $n$ classical
bits, and Bob performs some quantum operation. Fo-
cus attention on those runs of the experiment in which
the classical bit-string has a certain value, say $k$. We
may as well regard Bob as performing the same quantum
operation on each of these runs. We use the fact that
a probabilistic mixture of quantum operations is itself
a quantum operation. So we can stipulate without loss of
generality that which quantum operation Bob applies
depends deterministically on the values of the $n$ cbits.

We can also stipulate without loss of generality that
the values of the $n$ cbits sent by Alice depend determi-

nistically on which quantum state she is sending. Suppose,
to the contrary, that a particular quantum state deter-

mines only probabilistically the values of the cbits. Then,
instead of regarding Alice and Bob as using a probabili-
stic scheme, one might regard them as using one from
several deterministic schemes, with certain probabilities.
But then the average fidelity obtained will be the average
over that obtained for each of the deterministic schemes
and we would do better simply to use whichever of these
is the best.

It follows from the above that we lose no generality if
we restrict ourselves to schemes which work as follows.
The ensemble is divided up into $2^n$ sub-ensembles. Alice
uses the $n$ classical bits to tell Bob which sub-ensemble
the state she is sending lies in. Bob has a choice of $2^n$
possible quantum operations to perform. Which one he
performs is determined by the values of the $n$ classical
bits. The problem is to find the scheme which leads to
the maximum value for $F$.

We can split this problem into two. The first part is
to determine, for a general ensemble of quantum states,
$\{p_i, |\psi_i\rangle\}$ which undergo some noise process of the form
$\rho \rightarrow S(\rho)$, where $S$ is a quantum operation: What is the
best operation to perform in order to undo this noise as
well as possible? In other words, we wish to find an op-
eration $T$ such that $\sum_i p_i \langle \psi_i | T(S(|\psi_i\rangle \langle \psi_i|)) \psi_i \rangle$ is
maximized. The second part is to determine the best way for
Alice to divide the initial ensemble into $2^n$ sub-ensembles,
given that an answer to the first part will determine for
Bob an operation to perform on each sub-ensemble.

Unfortunately, even the first of these appears to be a
difficult problem in itself. Some progress is made by
Barnum and Knill in $[11]$, but they are concerned with
maximizing entanglement fidelity and their results are
only valid for ensembles of commuting density operators
and so are not immediately useful for our problem.

For the rest of this paper, we are less ambitious. We
consider only a very simple instance of the problem in
which Alice sends just one cbit and the pure states she
sends are qubit states drawn from a distribution which
is uniform over the Bloch sphere. The noisy quantum
channel is a depolarizing channel, which acts as:

$$\rho \rightarrow \alpha \rho + (1 - \alpha) I/2, \hspace{1cm} (3)$$

where $0 \leq \alpha \leq 1$. We will see that the solution even
to this seemingly trivial problem involves a surprising
amount of structure - suggesting that relationships be-
tween classical and quantum information in general may
well be very intricate.

III. ONE QUBIT AND ONE CBIT

Consider the scenario in which Alice sends to Bob a
pure state drawn from a uniform distribution over the
Bloch sphere, which gets depolarized on the way, and a
single noiseless classical bit. From the above, we know that
Alice must divide the surface of the Bloch sphere
into two subsets, $S_0$ and $S_1$, which correspond to the cbit
taking the value ‘0’ or ‘1’. We must then find, in each
case, the optimal quantum operation for Bob to perform,
given that the depolarized qubit lies in that particular
subset.

We begin by assuming that Alice divides up the Bloch
sphere in the following fashion:

**Assumption 1** For a general state, $|\psi\rangle$, we have that
$|\psi\rangle \in S_0$ iff $|\langle \psi | 0 \rangle|^2 \geq \cos^2(\beta/2)$, where $|0\rangle$ is some
fixed basis state corresponding to the point $(0,0,1)$, or
the north pole, on the Bloch sphere and $0 \leq \beta \leq \pi/2$.
Otherwise $|\psi\rangle \in S_1$. 

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We conjecture that this assumption leads to an optimal scheme (it seems very likely, for example, that in the optimal scheme the sets $S_0$ and $S_1$ will be simply connected and unlikely that the optimal scheme will be less symmetric than the one presented). In the rest of this section, we derive the optimal quantum operations for Bob to perform, in the cases that $|\psi\rangle \in S_0$ and $|\psi\rangle \in S_1$.

It is helpful to write quantum operations in a different way. Suppose that a general qubit density matrix is written

$$\rho = \frac{1}{2}(I + \vec{r}, \vec{s}),$$  \hspace{1cm} (4)

where $\vec{r}$ is a real 3-vector and $|\vec{r}| \leq 1$. Then, from the fact that a quantum operation is linear, we can write it in the form $[10]$:

$$\vec{r} \rightarrow \vec{r}' = A\vec{r} + \vec{b},$$  \hspace{1cm} (5)

where $A$ is a real $3 \times 3$ matrix and $\vec{b}$ is a real 3-vector. We have also automatically included the conditions that a quantum operation must be trace-preserving and positive. The condition of complete positivity imposes further constraints on $A$ and $\vec{b}$. Of course we must also have that $|\vec{r}'| \leq 1 \forall \vec{r}'$. Further, we can write $A$ in the form $U.S$, where $U$ is orthogonal (i.e. a rotation) and $S$ is symmetric. So we can view a quantum operation as a deformation of the Bloch sphere along principal axes determined by $S$, followed by a rotation, followed by a translation.

Suppose now that $|\psi\rangle \in S_0$. Bob performs an operation characterized by $A$ and $\vec{b}$. From the symmetry of the problem, it follows that the fidelity obtained (averaged over all $|\psi\rangle$ such that $|\psi\rangle \in S_0$) is unchanged if Bob performs a different operation, characterized by $A'$ and $\vec{b}'$, where $A' = O(\theta).A.O(\theta)^T$ and $\vec{b}' = O(\theta)\vec{b}$ and where $O(\theta)$ is a rotation of angle $\theta$ about the z-axis. It follows from this that the fidelity is also unchanged if Bob performs an operation characterized by $A''$ and $\vec{b}''$, where:

$$A'' = \frac{1}{2\pi} \int_{0}^{2\pi} d\theta O(\theta).A.O(\theta)^T$$  \hspace{1cm} (6)

and

$$\vec{b}'' = \frac{1}{2\pi} \int_{0}^{2\pi} d\theta O(\theta)\vec{b}.$$  \hspace{1cm} (7)

This means that without loss of generality, we can restrict Bob to actions of the form $\vec{r} \rightarrow V.A\vec{r} + \vec{b}$, where $A = \text{Diag}(\gamma, \gamma, \delta)$, $\vec{b} = (0,0,k)$ and $V$ is a fixed rotation about the z-axis. From the condition that $|\vec{r}'| \leq 1 \forall \vec{r}'$, we get $|\gamma|, |\delta|, |k| \leq 1$. Quantum operations are contractions on the Bloch sphere.

Recall that the qubit which Bob receives has been depolarized. We can write its density matrix in the form $\rho = \frac{1}{2}(I + \alpha \vec{r}, \vec{s})$, where $|\vec{r}| = 1$. Ideally, Bob would like an operation which takes $\alpha \vec{r} \rightarrow \vec{r}'$, at least for those states belonging to $S_0$, but this is not allowed (such an operation is not a contraction). Bob's operation will in fact consist of a translation in the z-direction and contractions parametrized by $\gamma$ and $\delta$. It is clear geometrically that in the optimum scheme, $V = I$, where $I$ is the identity.

Our aim is now, for fixed $\alpha$, $\beta$ and $k$, to find the optimum values of $\gamma$ and $\delta$, consistently with their describing a genuine quantum operation (which, recall, must correspond to a completely positive map on the set of density matrices). In fact, one can show that complete positivity implies that:

$$0 \leq k \leq 1,$$  \hspace{1cm} (8)

$$0 \leq \delta \leq 1 - k,$$  \hspace{1cm} (9)

and

$$0 \leq \gamma \leq \sqrt{1-k}.$$  \hspace{1cm} (10)

These conditions are necessary but not sufficient. The actual derivation of these conditions is unenlightening, so we do not reproduce it here.

It is easy to see now that Bob's best operation will be characterized by setting $\gamma = \sqrt{1-k}$ and $\delta = 1-k$. This gives:

$$A = \text{Diag}(\sqrt{1-k}, \sqrt{1-k}, 1-k)$$  \hspace{1cm} (11)

and

$$\vec{b} = (0,0,k).$$  \hspace{1cm} (12)

In fact, remarkably, this corresponds to an already well known quantum operation, usually described as an 'amplitude damping channel'. Amplitude damping is usually studied for its physical relevance - it corresponds to many natural physical processes. For example, it may describe an atom coupled to a single mode of electromagnetic radiation undergoing spontaneous emission, or a single photon mode from which a photon may be scattered by a beam splitter. This suggests that our scheme should be easily implementable experimentally.

One can run through similar arguments for the case $|\psi\rangle \in S_1$. Again, it turns out that Bob's optimal operation is essentially an amplitude damping operation, except that in this case, the vector $\vec{b}$ will point in the opposite direction i.e. Bob's operation will involve a translation of the Bloch sphere downwards, towards the south pole, as well as some contraction. For the rest of this paper we calculate the optimum fidelity that Bob can achieve for a given $\alpha$ and $\beta$. One can optimize over the value of $k$ separately for the cases $|\psi\rangle \in S_0$ and $|\psi\rangle \in S_1$ (the optimum value of $k$ may sometimes be zero implying that Bob's best operation is to do nothing). One can then optimize over $\beta$. 

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IV. ACHIEVABLE FIDELITY

After the action of the depolarizing channel and Bob’s quantum operation, we have that:

\[
\vec{r}' = \begin{pmatrix}
\sin \theta \cos \phi \\
\sin \theta \sin \phi \\
\cos \theta
\end{pmatrix}
\]

\[\rightarrow \vec{r}' = \begin{pmatrix}
\sqrt{1 - k} \alpha \sin \theta \cos \phi \\
\sqrt{1 - k} \alpha \sin \theta \sin \phi \\
(1 - k) \alpha \cos \theta + k
\end{pmatrix}.
\]

The average fidelity is given by:

\[
\bar{F} = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\beta d\theta \sin \theta \\
\left[\frac{1}{2} \left(1 + \alpha \sqrt{1 - k} \sin^2 \theta + \alpha (1 - k) \cos^2 \theta + k \cos \theta\right)\right] \\
+ \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_\beta^\pi d\theta \sin \theta \\
\left[\frac{1}{2} \left(1 + \alpha \sqrt{1 - k'} \sin^2 \theta + \alpha (1 - k') \cos^2 \theta - k' \cos \theta\right)\right],
\]

where \(k'\) is Bob’s quantum operation parameter in the case that \(|\psi\rangle \in S_1\) and is defined so that \(A = \text{Diag}(\sqrt{1 - k'}, \sqrt{1 - k'}, 1 - k')\) and \(\vec{b} = (0, 0, -k')\).

Optimizing over \(k, k'\) and \(\beta\) numerically leads to the graph shown in Figure 1 which shows the achievable fidelity for a depolarizing channel parametrized by \(\alpha\). Also shown on the graph is the fidelity obtained in the case that Alice sends no cbit and Bob performs no quantum operation.

![Figure 1](image1.png)

We finish this section by noting some features of the graph.

1. As we might expect, our scheme always yields an advantage when compared with doing nothing.

2. If Alice can send to Bob one cbit but cannot use a quantum channel, then the best obtainable fidelity is \(3/4\) (Alice tells Bob ‘upper’ or ‘lower’ hemisphere and Bob prepares a state which is spin up or spin down accordingly). With our scheme we have \(\bar{F} > 3/4\) if \(\alpha > 0\). Thus the quantum channel is some use for any \(\alpha > 0\).

3. There is a kink in the graph at \(\alpha \approx 0.54\). Further numerical investigations reveal why this is the case. Denote the optimum value of \(\beta\) (the angle which describes how Alice is dividing up the Bloch sphere) by \(\beta_{\text{opt}}\). Below this value of \(\alpha\), we have \(\beta_{\text{opt}} = \pi/2\). At \(\alpha \approx 0.54\), \(\beta_{\text{opt}}\) suddenly jumps to \(\approx 1.1\) and then decreases as \(\alpha\) increases. This is shown in Figure 2.

![Figure 2](image2.png)

4. In the \(\alpha < \sim 0.54\) region, where \(\beta_{\text{opt}} = \pi/2\), we can calculate \(\bar{F}\) analytically yielding \(\bar{F} = \frac{3}{4} + \frac{1}{4} \alpha^2\).

5. As \(\alpha \to 0\), Bob’s operation tends towards a simple ‘swap’ operation which maps all points in the Bloch sphere to one of the poles depending on which hemisphere the qubit lies in.

6. If \(\alpha < \sim 0.72\), then \(\bar{F} < \sim 0.872\) and Alice and Bob, if they can, would do better to use a protocol due to Gisin in which Alice sends two noiseless cbits and no quantum information.

V. CONCLUSION

We have considered situations in which Alice and Bob wish to use noiseless classical information to offset quantum noise - a kind of error correction. An important feature is that Alice possesses a classical description of the quantum states she wishes to send. After considering these situations in generality, we turned to consider a very specific scenario in which Alice sends one qubit...
which passes through a depolarizing channel accompanied by a noiseless classical bit. We described a scheme which we conjecture is optimal which involves Alice dividing up the Bloch sphere as in assumption 1 above and Bob performing ‘amplitude damping’ operations. Our results for this scheme were obtained by brute force. Clearly a more principled approach is desirable. One idea might be to regard the depolarization as actually coming about through the actions of an eavesdropper, Eve. Eve gains some information about the quantum state passing through and must therefore gain some information about its identity. It follows that even after Bob’s recovery operation, some disturbance to the state is inevitable [13]. This way, one might be able to derive an upper bound on Bob’s achievable fidelity for more general scenarios than the one considered here.

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