TRANSFER AND CHARACTERISTIC IDEMPOTENTS FOR SATURATED FUSION SYSTEMS

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Abstract. We construct a transfer map to the \( p \)-local Burnside ring of a saturated fusion system \( F \) from the \( p \)-local Burnside ring of the underlying \( p \)-group \( S \). Using such transfer maps, we give a new explicit construction of the characteristic idempotent of \( F \) – the unique idempotent in the \( p \)-local double Burnside ring of \( S \) satisfying properties of Linckelmann and Webb. We describe this idempotent both in terms of fixed points and as a linear combination of transitive bisets. Additionally, using fixed points we determine the map for Burnside rings given by multiplication with the characteristic idempotent, and show that this is the transfer map previously constructed. Applying these results, we show that for every saturated fusion system the ring generated by all (non-idempotent) characteristic elements in the \( p \)-local double Burnside ring is isomorphic to the \( p \)-local “single” Burnside ring of the fusion system, and we disprove a conjecture by Park-Ragnarsson-Stancu on the composition product of fusion systems.

1. Introduction

Saturated fusion systems are abstract models for the \( p \)-local structure of finite groups. The canonical example comes from a finite group \( G \) with Sylow \( p \)-subgroup \( S \). The fusion system \( F_S(G) \) associated to \( G \) (and \( S \)) is a category whose objects are the subgroups of \( S \) and where the morphisms between subgroups are the homomorphisms induced by conjugation by elements of \( G \). As shown by Ragnarsson-Stancu in [12,13], there is a one-to-one correspondence between the saturated fusion systems on a finite \( p \)-group \( S \) and their associated characteristic idempotents in \( A(S,S)_{(p)} \), the \( p \)-localized double Burnside ring of \( S \).

In this paper we introduce a transfer map \( \pi: A(S)_{(p)} \to A(F)_{(p)} \) between Burnside rings for a saturated fusion system \( F \) and its underlying \( p \)-group \( S \). By using this transfer map we give a new explicit construction of the characteristic idempotent \( \omega_F \) for a saturated fusion system \( F \). This enables us to calculate the fixed points and coefficients of \( \omega_F \) and give a precise description of the products \( \omega_F \circ X \) and \( X \circ \omega_F \) for any element \( X \) of the double Burnside ring. We give an application of these results to a conjecture by Park-Ragnarsson-Stancu on the composition product of saturated fusion systems.

In more detail, we first consider the transfer map for Burnside rings of fusion systems: The Burnside ring \( A(S) \) for a finite \( p \)-group \( S \) is the Grothendieck group formed from the monoid of isomorphism classes of finite \( S \)-sets, with disjoint union as addition and cartesian product as multiplication. Let

\[
\Phi: A(S) \to \prod_{Q \leq S \text{ up to } S\text{-conj.}} \mathbb{Z}
\]

be the homomorphism of marks, i.e., the injective ring homomorphism whose \( Q \)-coordinate \( \Phi_Q(X) \) counts the number of fixed points \( |X^Q| \) when \( X \) is an \( S \)-set. Given a fusion system \( F \) on \( S \), we say that a finite \( S \)-set \( X \), or a general element of \( A(S) \), is \( F \)-stable if the action

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on $X$ is invariant under conjugation in $\mathcal{F}$ – see section 3.1. The $\mathcal{F}$-stable elements form a subring of $A(S)$ which we call the Burnside ring of $\mathcal{F}$ and denote by $A(\mathcal{F})$.

**Theorem A.** Let $\mathcal{F}$ be a saturated fusion system on a finite $p$-group $S$. We let $A(\mathcal{F})_{(p)}$ denote the $p$-localized Burnside ring of $\mathcal{F}$ as a subring of the $p$-localized Burnside ring $A(S)_{(p)}$ for $S$. Then there is a transfer map $\pi: A(S)_{(p)} \to A(\mathcal{F})_{(p)}$, which is a homomorphism of $A(\mathcal{F})_{(p)}$-modules and which restricts to the identity on $A(\mathcal{F})_{(p)}$. In terms of the fixed point homomorphisms the transfer map $\pi$ satisfies

$$\Phi_Q(\pi(X)) = \frac{1}{[Q]_{\mathcal{F}}} \sum_{Q' \in [Q]_{\mathcal{F}}} \Phi_{Q'}(X),$$

where $[Q]_{\mathcal{F}}$ is the conjugacy class of $Q$ in $\mathcal{F}$.

If we apply the $\pi$ to the transitive $S$-sets $S/P$ for $P \leq S$, we get elements $\beta_P := \pi(S/P)$, which form a $\mathbb{Z}_{(p)}$-basis for the $p$-localized Burnside ring $A(\mathcal{F})_{(p)}$ by proposition 4.5, and where $\beta_P = \beta_Q$ if and only if $P$ and $Q$ are conjugate in $\mathcal{F}$. In proposition 4.7, we show that when $\mathcal{F}$ arises from a finite group $G$ with Sylow $p$-subgroup $S$, then the basis elements $\beta_P$ are closely related to the transitive $G$-sets $G/P$ for $P \leq S$, and the $p$-localized Burnside ring $A(\mathcal{F})_{(p)}$ is isomorphic to the part of $A(G)_{(p)}$ where all stabilizers are $p$-subgroups.

The (double) Burnside module $A(S,T)$ is defined for a pair of $p$-groups similarly to the Burnside ring of a group, except that we consider isomorphism classes of $(S,T)$-biset, which are sets equipped with both a right $S$-action and a left $T$-action that commute with each other. The Burnside module $A(S,T)$ is then the Grothendieck group of the monoid formed by isomorphism classes of finite $(S,T)$-biset with disjoint union as addition. The $(S,T)$-biset correspond to sets with a left $(T \times S)$-action, and the transitive biset correspond to transitive sets $(T \times S)/D$ for subgroups $D \leq T \times S$. Note that we do not make the usual requirement that the bisets have a free left action, and the results below hold for non-free biset as well.

For every triple of $p$-groups $S, T, U$ we have a composition map $\circ: A(T,U) \times A(S,T) \to A(S,U)$ given on biset by $Y \circ X := Y \times_T X = Y \times X/\sim$ where $(yt,x) \sim (y,tx)$ for all $y \in Y$, $x \in X$, and $t \in T$. For each $D \leq T \times S$ we have a fixed point homomorphism $\Phi_D: A(S,T) \to \mathbb{Z}$, but it is only a homomorphism of abelian groups. An element $X \in A(S,T)$ is still fully determined by the number of fixed points $\Phi_D(X)$ for $D \leq T \times S$. Subgroups in $T \times S$ of particular interest are the graphs of homomorphisms $\varphi: P \to T$ for $P \leq S$, where the graph of $\varphi: P \to T$ is the subgroup $\Delta(P,\varphi) := \{(\varphi(g),g) \mid g \in P\}$.

The transitive $(T \times S)$-set $(T \times S)/\Delta(P,\varphi)$ corresponds to a transitive $(S,T)$-biset whose isomorphism class we denote by $[P,\varphi]^T_S$.

Given a saturated fusion system $\mathcal{F}$, a particularly nice class of elements in the $p$-localized double Burnside ring $A(S,S)_{(p)}$ are the $\mathcal{F}$-characteristic elements, which satisfy the following properties put down by Linckelmann-Webb: An element $X \in A(S,S)_{(p)}$ is $\mathcal{F}$-characteristic if it is

- **$\mathcal{F}$-generated**: $X$ is a linear combination of the $(S,S)$-biset $[P,\varphi]^S_S$ where $\varphi: P \to S$ is a morphism of $\mathcal{F}$.
- **Right $\mathcal{F}$-stable**: For all $P \leq S$ and $\varphi \in \mathcal{F}(P,S)$ we have $X \circ [P,\varphi]^S_P = X \circ [P,\text{id}]^S_P$ as elements of $A(P,S)_{(p)}$.
- **Left $\mathcal{F}$-stable**: For all $P \leq S$ and $\varphi \in \mathcal{F}(P,S)$ we have $[\varphi P,\varphi^{-1}]^S_P \circ X = [P,\text{id}]^S_P \circ X$ as elements of $A(P,S)_{(p)}$.

and an additional technical condition to ensure that $X$ is not degenerate.
In this paper we give a new proof that every saturated fusion system $\mathcal{F}$ on $S$ has an associated element $\omega_{\mathcal{F}} \in A(S,S)_{(p)}$ that is both $\mathcal{F}$-characteristic and idempotent. To construct $\omega_{\mathcal{F}}$ we consider the product fusion system $\mathcal{F} \times \mathcal{F}$ on $S \times S$ and apply the transfer map of theorem [A] to $S \times S/\Delta(S, id)$. The resulting element $\beta_{\Delta(S, id)}$ then turns out to be both $\mathcal{F}$-characteristic and idempotent when considered as an element of $A(S,S)_{(p)}$. The new construction of the characteristic idempotent for instance enables us to calculate the fixed points of $\omega_{\mathcal{F}}$.

**Theorem B.** Let $\mathcal{F}$ be a saturated fusion system on a finite $p$-group $S$. Then there exists a unique $\mathcal{F}$-characteristic idempotent $\omega_{\mathcal{F}} \in A(S,S)_{(p)}$, and it satisfies:

For all graphs $\Delta(P, \varphi) \leq S \times S$ with $\varphi \in \mathcal{F}(P,S)$, we have

$$\Phi_{\Delta(P, \varphi)}(\omega_{\mathcal{F}}) = \frac{|S|}{|\mathcal{F}(P, S)|};$$

and $\Phi_D(\omega_{\mathcal{F}}) = 0$ for all other subgroups $D \leq S \times S$. Consequently, if we write $\omega_{\mathcal{F}}$ in the basis of $A(S,S)_{(p)}$, we get the expression

$$\omega_{\mathcal{F}} = \sum_{\Delta(P, \varphi) \leq S \times S \atop \text{with } \varphi \in \mathcal{F}(P,S)} \frac{|S|}{\Phi_{\Delta(P, \varphi)}([P, \varphi]_S)} \left( \sum_{P \leq Q \leq S} \frac{|\{\psi \in \mathcal{F}(Q, S) \mid \psi|_P = \varphi\}|}{|\mathcal{F}(Q, S)|} \cdot \mu(P, Q) \right) [P, \varphi]_S,$$

where the outer sum is taken over $(S \times S)$-conjugacy classes of subgroups, and where $\mu$ is the Möbius function for the poset of subgroups in $S$.

A closer look on the way theorem [A] is applied to construct $\omega_{\mathcal{F}}$ reveals an even closer relationship between the transfer map and the characteristic idempotent, and we get a precise description of what happens when other elements are multiplied by $\omega_{\mathcal{F}}$.

**Theorem C.** Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be saturated fusion systems on finite $p$-groups $S_1$ and $S_2$ respectively, and let $\omega_1 \in A(S_1, S_1)_{(p)}$ and $\omega_2 \in A(S_2, S_2)_{(p)}$ be the characteristic idempotents.

For every element of the Burnside module $X \in A(S_1, S_2)_{(p)}$, the product $\omega_2 \circ X \circ \omega_1$ is right $\mathcal{F}_1$-stable and left $\mathcal{F}_2$-stable, and satisfies

$$\Phi_D(\omega_2 \circ X \circ \omega_1) = \frac{1}{|[D]_{\mathcal{F}_2 \times \mathcal{F}_1}|} \sum_{D' \in [D]_{\mathcal{F}_2 \times \mathcal{F}_1}} \Phi_D(X),$$

for all subgroups $D \leq S_2 \times S_1$, where $[D]_{\mathcal{F}_2 \times \mathcal{F}_1}$ is the isomorphism class of $D$ in the product fusion system $\mathcal{F}_2 \times \mathcal{F}_1$ on $S_2 \times S_1$.

In particular, corollary [5.8] recovers the transfer map of theorem [A] for a saturated fusion system $\mathcal{F}$ as multiplication by $\omega_{\mathcal{F}}$ on the Burnside module $A(1, S)_{(p)}$ generated by finite left $S$-sets. Let $A(\mathcal{F}_1, \mathcal{F}_2)_{(p)}$ denote the right $\mathcal{F}_1$-stable and left $\mathcal{F}_2$-stable elements of $A(S_1, S_2)_{(p)}$. Then the characteristic idempotents $\omega_1$ and $\omega_2$ act trivially on $A(\mathcal{F}_1, \mathcal{F}_2)_{(p)}$, and theorem [C] gives a transfer homomorphism of modules over the double Burnside rings $A(\mathcal{F}_1, \mathcal{F}_1)_{(p)}$ and $A(\mathcal{F}_2, \mathcal{F}_2)_{(p)}$ as described in proposition [5.10].

For a saturated fusion system $\mathcal{F}$ on $S$, the double Burnside ring $A(\mathcal{F}, \mathcal{F})_{(p)}$ is the subring of $A(S, S)_{(p)}$ consisting of all the elements that are both left and right $\mathcal{F}$-stable. An even smaller subring is the collection of all the elements that are $\mathcal{F}$-generated as well as $\mathcal{F}$-stable. We denote this subring $A_{\text{char}}(\mathcal{F})_{(p)}$ since a generic $\mathcal{F}$-generated, $\mathcal{F}$-stable element is actually $\mathcal{F}$-characteristic. Hence we have a sequence of inclusions of subrings

$$A_{\text{char}}(\mathcal{F})_{(p)} \subseteq A(\mathcal{F}, \mathcal{F})_{(p)} \subseteq A(S, S)_{(p)}.$$
The last inclusion is not unital since $\omega_F$ is the multiplicative identity of the first two rings, and $[S, \text{id}]_S^S$ is the identity of $A(S,S)(p)$. According to proposition 6.3, $A^{\text{char}}(\mathcal{F})(p)$ has a $\mathbb{Z}(p)$-basis consisting of elements $\beta_{\Delta(p,\text{id})}$, which only depend on $F \leq S$ up to $\mathcal{F}$-conjugation, and each element of $A^{\text{char}}(\mathcal{F})(p)$, written

$$X = \sum_{p \leq S} \sum_{\text{up to } \mathcal{F}\text{-conj.}} c_{\Delta(p,\text{id})} \beta_{\Delta(p,\text{id})},$$

is $\mathcal{F}$-characteristic if and only if $c_{\Delta(S,\text{id})}$ is invertible in $\mathbb{Z}(p)$.

For every $(S,S)$-biset $X$, we can quotient out the right $S$-action in order to get $X/S$ as a left $S$-set. Quotienting out the right $S$-action preserves disjoint union and extends to a collapse map $q: A(S,S)(p) \to A(S)(p)$, and by restriction to subrings we get maps

$$A^{\text{char}}(\mathcal{F})(p) \subseteq A(\mathcal{F}, \mathcal{F})(p) \subseteq A(S,S)(p)$$

$$\xrightarrow{\text{collapse}} A(\mathcal{F})(p) \subseteq A(S)(p)$$

where $\mathcal{F}$-stable bisets are collapsed to $\mathcal{F}$-stable sets. In general the collapse map does not respect the multiplication of the double Burnside ring, but combining the techniques of theorems $A$ and $C$ we show that on $A^{\text{char}}(\mathcal{F})(p)$ the collapse map is not only a ring homomorphism but actually an isomorphism of rings!

**Theorem D.** Let $\mathcal{F}$ be a saturated fusion system on a finite $p$-group $S$.

Then the collapse map $q: A^{\text{char}}(\mathcal{F})(p) \to A(\mathcal{F})(p)$, which quotients out the right $S$-action, is an isomorphism of rings, and it sends the basis element $\beta_{\Delta(p,\text{id})}$ of $A(\mathcal{F}, \mathcal{F})(p)$ to the basis element $\beta_P$ of $A(\mathcal{F})(p)$.

This generalizes a similar result for groups where the Burnside ring $A(S)$ embeds in the double Burnside ring $A(S,S)$ with the transitive $S$-set $S/P$ corresponding to the transitive biset $[P, \text{id}]_S^S$. As an immediate consequence of theorem $D$ we get an alternative proof that the characteristic idempotent $\omega_F$ is unique: Corollary 6.6 shows that $\beta_{\Delta(S,\text{id})} = \omega_F$ is the only non-zero idempotent of $A^{\text{char}}(\mathcal{F})(p)$ by proving that $0$ and $S/S$ are the only idempotents of $A(\mathcal{F})(p)$.

The final section of this paper applies theorem $C$ to disprove a conjecture by Park-Ragnarsson-Stancu, $11$, on the composition product of fusion systems. Let $\mathcal{F}$ be a saturated fusion system on a $p$-group $S$, and let $\mathcal{H}, \mathcal{K}$ be saturated fusion subsystems on subgroups $R, T \leq S$ respectively. In the terminology of Park-Ragnarsson-Stancu, we then say that $\mathcal{F}$ is the composition product of $\mathcal{H}$ and $\mathcal{K}$, written $\mathcal{F} = \mathcal{H} \mathcal{K}$, if $S = RT$ and for all subgroups $P \leq T$ it holds that every morphism $\varphi \in \mathcal{F}(P,R)$ can be written as a composition $\varphi = \psi \rho$ where $\psi$ is a morphism of $\mathcal{H}$ and $\rho$ is a morphism of $\mathcal{K}$.

Park-Ragnarsson-Stancu conjectured that $\mathcal{F} = \mathcal{H} \mathcal{K}$ is equivalent to the following equation of characteristic idempotents:

$$[R, \text{id}]_S^S \circ \omega_\mathcal{F} \circ [T, \text{id}]_T^T = \omega_\mathcal{H} \circ [R \cap T, \text{id}]_T^R \circ \omega_\mathcal{K}. \tag{1.1}$$

A special case of the conjecture was proven in $11$, in the case where $R = S$ and $K$ is weakly normal in $\mathcal{F}$, and the general conjecture was inspired by the group case, where $H, K \leq G$ satisfy $G = HK$ if and only if there is an isomorphism of $(K, H)$-bisets $G \cong H \times_{H \cap K} K$.

By direct calculation via theorem $C$ we can now characterize all cases where (1.1) holds:
Theorem E. Let $F$ be a saturated fusion system on a $p$-group $S$, and suppose that $H, K$ are saturated fusion subsystems of $F$ on subgroups $R, T \leq S$ respectively.

Then the characteristic idempotents satisfy
\begin{equation}
[R, id]^F_R \circ \omega_F \circ [T, id]^F_T = \omega_H \circ [R \cap T, id]^F_T \circ \omega_K
\end{equation}
if and only if $F = HK$ and for all $Q \leq R \cap T$ we have
\begin{equation}
|F(Q, S)| = \frac{|H(Q, R)| \cdot |K(Q, T)|}{|\text{Hom}_{K\cap H}(Q, R \cap T)|}.
\end{equation}

In particular (1.2) always implies $F = HK$, but the converse is not true in general. In example 7.1, the alternating group $A_6$ gives rise to a composition product $F = HK$ where (1.3) fails – hence we get a counter-example to the general conjecture of Park-Ragnarsson-Stancu.

At the same time, proposition 7.2 proves a special case of the conjecture where $K$ is weakly normal in $F$, which is a generalization of the case proved by Park-Ragnarsson-Stancu.

Earlier results on Burnside rings for fusion systems. An earlier definition for the Burnside ring of a fusion system, was given by Diaz-Libman in [5]. The advantage of the Diaz-Libman definition of the Burnside ring is that it is constructed in close relation to a nice orbit category for the centric subgroups in a saturated fusion system $F$. However, by construction the Burnside ring of Diaz-Libman doesn’t see the non-centric subgroup at all, in contrast to the definition of $A(F)$ used in this paper where we have basis elements corresponding to all the subgroups. In proposition 4.8, we compare the two definitions and show that if we quotient out the non-centric part of $A(F)(p)$ we recover the centric Burnside ring of Diaz-Libman, and we relate the basis elements given by Diaz-Libman to the basis elements $\beta_P$ used in this paper.

Theorem A and the construction of characteristic idempotents in this paper is strongly inspired by an algorithm by Broto-Levi-Oliver. Originally, in [4], Broto-Levi-Oliver gave a procedure for constructing a characteristic biset $\Omega$ from a saturated fusion system $F$, and using such a biset, they then constructed a classifying spectrum for $F$. In [12] Ragnarsson took a characteristic biset as constructed by Broto-Levi-Oliver, and proceeded to refine this biset to get an idempotent. This proof used a Cauchy sequence argument in the $p$-completion $A(S, S)^\wedge_p$ of the double Burnside ring in order to show that a characteristic idempotent exists. A later part of [12] showed that $\omega_F$ is unique and that in fact $\omega_F$ lies in the $p$-localization $A(S, S)_{(p)}$ as a subring of the $p$-completion. The new construction of $\omega_F$ given in this paper takes the original procedure by Broto-Levi-Oliver and refines it in order to construct $\omega_F$ directly as an element of $A(S, S)_{(p)}$ – without needing to work in the $p$-completion. Furthermore, this refined procedure generalizes in order to give us the transfer map of theorem A.

Finally, we note that the formula for the fixed points of $\omega_F$ given in theorem B coincides with the work done independently by Boltje-Danz in [3]. The calculations by Boltje-Danz are done by working in their ghost ring for the double Burnside ring and applying the steps of Ragnarsson’s proof for the uniqueness of $\omega_F$. This way they are able to calculate what the fixed points of $\omega_F$ have to be, assuming that $\omega_F$ exists. In this paper, the fixed points follow as an immediate consequence of the way we construct $\omega_F$.

Outline. Section 2 recalls the definition and basic properties of saturated fusion systems, and establishes the related notation used throughout the rest of the paper. Section 3 gives
a similar treatment to the Burnside ring of a finite group as well as the Burnside ring for a saturated fusion system. Section 4 is the first main section of the paper, where we consider the structure of the \( p \)-localization \( A(\mathcal{F})_p \) of the Burnside ring for a saturated fusion system \( \mathcal{F} \) on a finite \( p \)-group \( S \). In particular, we construct a stabilization map that sends every finite \( S \)-set to an \( \mathcal{F} \)-stable element in a canonical way, and we prove theorem \( \text{A} \). The other main section, section 5, is subdivided in three parts. In 5.1 we recall the double Burnside ring of a group. In 5.2 we apply the stabilization map above for the fusion system \( \mathcal{F} \times \mathcal{F} \) in order to construct the characteristic idempotent for \( \mathcal{F} \) and prove theorem \( \text{B} \). In 5.3 we prove theorem \( \text{C} \) and study the strong relation between the stabilization homomorphism of theorem \( \text{A} \) and multiplying with the characteristic idempotent. In section 6 we prove theorem \( \text{D} \) relating the \( \mathcal{F} \)-characteristic elements to the Burnside ring of \( \mathcal{F} \). Finally, section 7 concerns the composition product of fusion systems and theorem \( \text{E} \).

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2. Fusion systems

The next few pages contain a very short introduction to fusion systems, which were originally introduced by Puig under the name “full Frobenius systems.” The aim is to introduce the terminology from the theory of fusion systems that will be used in the paper, and to establish the relevant notation. For a proper introduction to fusion systems see, for instance, Part I of “Fusion Systems in Algebra and Topology” by Aschbacher, Kessar and Oliver, [2].

**Definition 2.1.** A fusion system \( \mathcal{F} \) on a \( p \)-group \( S \), is a category where the objects are the subgroups of \( S \), and for all \( P, Q \leq S \) the morphisms must satisfy:

(i) Every morphism \( \varphi \in \text{Mor}_\mathcal{F}(P, Q) \) is an injective group homomorphism, and the composition of morphisms in \( \mathcal{F} \) is just composition of group homomorphisms.

(ii) \( \text{Hom}_S(P, Q) \subseteq \text{Mor}_\mathcal{F}(P, Q) \), where

\[
\text{Hom}_S(P, Q) = \{c_s \mid s \in N_S(P, Q)\}
\]

is the set of group homomorphisms \( P \to Q \) induced by \( S \)-conjugation.

(iii) For every morphism \( \varphi \in \text{Mor}_\mathcal{F}(P, Q) \), the group isomorphisms \( \varphi: P \to \varphi P \) and \( \varphi^{-1}: \varphi P \to P \) are elements of \( \text{Mor}_\mathcal{F}(P, \varphi P) \) and \( \text{Mor}_\mathcal{F}(\varphi P, P) \) respectively.

We also write \( \text{Hom}_\mathcal{F}(P, Q) \) or just \( \mathcal{F}(P, Q) \) for the morphism set \( \text{Mor}_\mathcal{F}(P, Q) \); and the group \( \mathcal{F}(P, P) \) of automorphisms is denoted by \( \text{Aut}_\mathcal{F}(P) \).

The canonical example of a fusion system comes from a finite group \( G \) with a given \( p \)-subgroup \( S \). The fusion system of \( G \) on \( S \), denoted \( \mathcal{F}_S(G) \), is the fusion system on \( S \) where the morphisms from \( P \leq S \) to \( Q \leq S \) are the homomorphisms induced by \( G \)-conjugation:

\[
\text{Hom}_{\mathcal{F}_S(G)}(P, Q) := \text{Hom}_G(P, Q) = \{c_g \mid g \in N_G(P, Q)\}.
\]

A particular case is the fusion system \( \mathcal{F}_S(S) \) consisting only of the homomorphisms induced by \( S \)-conjugation.
Let $\mathcal{F}$ be an abstract fusion system on $S$. We say that two subgroup $P, Q \leq S$ are $\mathcal{F}$-conjugate, written $P \sim_{\mathcal{F}} Q$, if they are isomorphic in $\mathcal{F}$, i.e., there exists a group isomorphism $\varphi \in \mathcal{F}(P, Q)$. $\mathcal{F}$-conjugation is an equivalence relation, and the set of $\mathcal{F}$-conjugates to $P$ is denoted by $[P]_{\mathcal{F}}$. The set of all $\mathcal{F}$-conjugacy classes of subgroups in $S$ is denoted by $\text{Cl}(\mathcal{F})$. Similarly, we write $P \sim_S Q$ if $P$ and $Q$ are $S$-conjugate, the $S$-conjugacy class of $P$ is written $[P]_S$ or just $[P]$, and we write $\text{Cl}(S)$ for the set of $S$-conjugacy classes of subgroups in $S$. Since all $S$-conjugation maps are in $\mathcal{F}$, any $\mathcal{F}$-conjugacy class $[P]_{\mathcal{F}}$ can be partitioned into disjoint $S$-conjugacy classes of subgroups $Q \in [P]_{\mathcal{F}}$.

We say that $Q$ is $\mathcal{F}$- or $S$-subconjugate to $P$ if $Q$ is respectively $\mathcal{F}$- or $S$-conjugate to a subgroup of $P$, and we denote this by $Q \lesssim_{\mathcal{F}} P$ or $Q \lesssim_S P$ respectively. In the case where $\mathcal{F} = \mathcal{F}_S(G)$, we have $Q \lesssim_{\mathcal{F}} P$ if and only if $Q$ is $G$-conjugate to a subgroup of $P$; in this case the $\mathcal{F}$-conjugates of $P$ are just those $G$-conjugates of $P$ that are contained in $S$.

A subgroup $P \leq S$ is said to be fully $\mathcal{F}$-normalized if $|N_S P| \geq |N_S Q|$ for all $Q \in [P]_{\mathcal{F}}$; similarly $P$ is fully $\mathcal{F}$-centralized if $|C_S P| \geq |C_S Q|$ for all $Q \in [P]_{\mathcal{F}}$.

**Definition 2.2.** A fusion system $\mathcal{F}$ on $S$ is said to be saturated if the following properties are satisfied for all $P \leq S$:

(i) If $P$ is fully $\mathcal{F}$-normalized, then $P$ is fully $\mathcal{F}$-centralized, and $\text{Aut}_S(P)$ is a Sylow $p$-subgroup of $\text{Aut}_{\mathcal{F}}(P)$.

(ii) Every homomorphism $\varphi \in \mathcal{F}(P, S)$ with $\varphi(P)$ fully $\mathcal{F}$-centralized extends to a homomorphism $\varphi \in \mathcal{F}(N_{\varphi}, S)$, where

$$N_{\varphi} := \{ x \in N_S(P) \mid \exists y \in S: \varphi \circ c_x = c_y \circ \varphi \}. $$

The saturation axioms are a way of emulating the Sylow theorems for finite groups; in particular, whenever $S$ is a Sylow $p$-subgroup of $G$, then the Sylow theorems imply that the induced fusion system $\mathcal{F}_S(G)$ is saturated (see e.g. [2, Theorem 2.3]).

In this paper, we shall rarely use the defining properties of saturated fusion systems directly. We shall instead mainly use the following lifting property, which saturated fusion systems satisfy:

**Lemma 2.3 ([15]).** Let $\mathcal{F}$ be saturated. If $P \leq S$ is fully normalized, then for each $Q \in [P]_{\mathcal{F}}$ there exists a homomorphism $\varphi \in \mathcal{F}(N_S Q, N_S P)$ with $\varphi(Q) = P$.

For the proof, see lemma 4.5 of [15] or lemma 2.6(c) of [2].

3. Burnside rings for groups and fusion systems

In this section we recall the Burnside ring of a finite group $S$ and how to describe its structure in terms of the homomorphism of marks, which embeds the Burnside ring into a suitable ghost ring. We also recall the Burnside ring of a saturated fusion system $\mathcal{F}$, in the sense of [14], which has a similar mark homomorphism and ghost ring.

Let $S$ be a finite group, not necessarily a $p$-group. Then the isomorphism classes of finite $S$-sets form a semiring with disjoint union as addition and cartesian product as multiplication. The Burnside ring of $S$, denoted $A(S)$, is then defined as the additive Grothendieck group of the semiring, and $A(S)$ inherits the multiplication as well. Given a finite $S$-set $X$, we let $[X]$ denote the isomorphism class of $X$ as an element of $A(S)$. The isomorphism classes $[S/P]$ of transitive $S$-sets form an additive basis for $A(S)$, and two transitive sets $S/P$ and $S/Q$ are isomorphic if and only if the subgroups $P$ and $Q$ are conjugate in $S$. 
For each element \( X \in A(S) \) we define \( c_P(X) \), with \( P \leq S \), to be the coefficients when we write \( X \) as a linear combination of the basis elements \([S/P]\) in \( A(S)\), i.e.

\[
X = \sum_{[P] \in Cl(S)} c_P(X) \cdot [S/P],
\]

where \( Cl(S) \) denotes the set of \( S \)-conjugacy classes of subgroup in \( S \). The resulting maps \( c_P : A(S) \rightarrow \mathbb{Z} \) are group homomorphisms, but they are not ring homomorphisms.

To describe the multiplication of \( A(S) \), it is enough to know the products of basis elements \([S/P]\) and \([S/Q]\). By taking the cartesian product \((S/P) \times (S/Q)\) and considering how it breaks into orbits, one reaches the following double coset formula for the multiplication in \( A(S)\):

\[
[S/P] \cdot [S/Q] = \sum_{\pi \in P\backslash S/Q} [S/(P \cap \pi Q)],
\]

where \( P\backslash S/Q \) is the set of double cosets \( PsQ \) with \( s \in S\).

Instead of counting orbits, an alternative way of characterising a finite \( S \)-set is counting the fixed points for each subgroup \( P \leq S\). For every \( P \leq S \) and \( S \)-set \( X \), we denote the number of \( P \)-fixed points by \( \Phi_P(X) := |X^P| \). This number only depends on \( P \) up to \( S \)-conjugation. Since we have

\[
|(X \cup Y)^P| = |X^P| + |Y^P| \quad \text{and} \quad |(X \times Y)^P| = |X^P| \cdot |Y^P|
\]

for all \( S \)-sets \( X \) and \( Y \), the fixed point map \( \Phi_P \) for \( S \)-sets extends to a ring homomorphism \( \Phi_P : A(S) \rightarrow \mathbb{Z} \). On the basis elements \([S/P]\), the number of fixed points is given by

\[
\Phi_Q([S/P]) = |(S/P)^Q| = \frac{|N_S(Q,P)|}{|P|},
\]

where \( N_S(Q,P) = \{ s \in S \mid \pi Q \leq P \} \) is the transporter in \( S \) from \( Q \) to \( P \). In particular, \( \Phi_Q([S/P]) \neq 0 \) if and only if \( Q \leq_S P \) (\( Q \) is subconjugate to \( P \)).

We have one fixed point homomorphism \( \Phi_P \) per conjugacy class of subgroups in \( S \), and we combine them into the homomorphism of marks \( \Phi = \Phi^S : A(S) \rightarrow \prod_{[P] \in Cl(S)} \mathbb{Z} \). This ring homomorphism maps \( A(S) \) into the product ring \( \tilde{\Omega}(S) := \prod_{[P] \in Cl(S)} \mathbb{Z} \), the so-called ghost ring for the Burnside ring \( A(S) \).

Results by tom Dieck and others show that the mark homomorphism is injective, and that the cokernel of \( \Phi \) is the obstruction group \( \text{Obs}(S) := \prod_{[P] \in Cl(S)} (\mathbb{Z}/|W_S P|\mathbb{Z}) \), where \( W_S P := N_S P / P \). These statements are combined in the following proposition, the proof of which can be found in [7, Chapter 1], [8], and [16].

**Proposition 3.1.** Let \( \Psi = \Psi^S : \tilde{\Omega}(S) \rightarrow \text{Obs}(S) \) be given by the \([P]\)-coordinate functions

\[
\Psi_P(\xi) := \sum_{\pi \in W_S P} \xi_{(s)P} \pmod{|W_S P|}.
\]

Here \( \xi_{(s)P} \) denotes the \([s]P\)-coordinate of an element \( \xi \in \tilde{\Omega}(S) = \prod_{[P] \in Cl(S)} \mathbb{Z} \).

The following sequence of abelian groups is then exact:

\[
0 \rightarrow A(S) \xrightarrow{\Phi} \tilde{\Omega}(S) \xrightarrow{\Psi} \text{Obs}(S) \rightarrow 0.
\]

\( \Phi \) is a ring homomorphism, but \( \Psi \) is just a group homomorphism.
The homomorphism of marks enables us to perform calculations for the Burnside ring $A(S)$ inside the much nicer product ring $\Omega(S)$, where we identify each element $X \in A(S)$ with its fixed point vector $(\Phi_Q(X))_{Q \in \text{Cl}(S)}$.

3.1. **The Burnside ring of a saturated fusion system.** Let $S$ be a finite $p$-group, and suppose that $\mathcal{F}$ is a saturated fusion system on $S$. We say that a finite $S$-set is $\mathcal{F}$-stable if the action is unchanged up to isomorphism whenever we act through morphisms of $\mathcal{F}$.

More precisely, if $P \leq S$ is a subgroup and $\varphi: P \to S$ is a homomorphism in $\mathcal{F}$, we can turn $X$ into a $P$-set by using $\varphi$ to define the action $g.x := \varphi(g)x$ for $g \in P$. We denote the resulting $P$-set by $p_\varphi X$. In particular when $\text{incl}: P \to S$ is the inclusion map, $p_{\text{incl}} X$ has the usual restriction of the $S$-action to $P$. Restricting the action of $S$-sets along $\varphi$ extends to a ring homomorphism $r_\varphi: A(S) \to A(P)$, and we let $p_\varphi X$ denote the image $r_\varphi(X)$ for all elements $X \in A(S)$.

We then say that an element $X \in A(S)$ is $\mathcal{F}$-stable if it satisfies

\[(3.3) \quad p_\varphi X = p_{\text{incl}} X \quad \text{inside} \quad A(P), \quad \text{for all} \quad P \leq S \quad \text{and homomorphisms} \quad \varphi: P \to S \quad \text{in} \quad \mathcal{F}.
\]

Alternatively, one can characterize $\mathcal{F}$-stability in terms of fixed points and the mark homomorphism, and the following three properties are equivalent for all $X \in A(S)$:

1. $X$ is $\mathcal{F}$-stable.
2. $\Phi_P(X) = \Phi_{\varphi P}(X)$ for all $\varphi \in \mathcal{F}(P, S)$ and $P \leq S$.
3. $\Phi_P(X) = \Phi_Q(X)$ for all pairs $P, Q \leq S$ with $P \sim_{\mathcal{F}} Q$.

A proof of this claim can be found in [9, Proposition 3.2.3] or [14]. We shall primarily use (ii) and (iii) to characterize $\mathcal{F}$-stability.

It follows from property (iii) that the $\mathcal{F}$-stable elements form a subring of $A(S)$. We define the **Burnside ring of $\mathcal{F}$** to be the subring $A(\mathcal{F}) \subseteq A(S)$ consisting of all the $\mathcal{F}$-stable elements. Equivalently, we can consider the actual $S$-sets that are $\mathcal{F}$-stable: The $\mathcal{F}$-stable sets form a semiring, and we define $A(\mathcal{F})$ to be the Grothendieck group hereof. These two constructions give rise to the same ring $A(\mathcal{F})$ – see [14]. As is the case for the Burnside ring of a group, $A(\mathcal{F})$ has an additive basis, where the basis elements are in one-to-one correspondence with the $\mathcal{F}$-conjugacy classes of subgroups in $S$.

For each $X \in A(\mathcal{F})$ the fixed point map $\Phi_P(X)$ only depends on $P$ up to $\mathcal{F}$-conjugation. The homomorphism of marks for $A(S)$ therefore restricts to the subring $A(\mathcal{F})$ as an injective ring homomorphism

$$\Phi_\mathcal{F}: A(S) \xrightarrow{\prod_{P \in \mathcal{F}} \Phi_P} \prod_{[P]} \mathbb{Z},$$

where $\text{Cl}(\mathcal{F})$ denotes the set of $\mathcal{F}$-conjugacy classes of subgroups in $S$. We call this map the **homomorphism of marks for $A(\mathcal{F})$**, and the ring $\Omega(\mathcal{F}) := \prod_{[P]} \mathbb{Z}$ is the **ghost ring** for $A(\mathcal{F})$.

As for the Burnside ring of a group, we also have an explicit description of the cokernel of $\Phi_\mathcal{F}$ as the group

$$\text{Obs}(\mathcal{F}) := \prod_{[P]} (\mathbb{Z}/|W_S P|\mathbb{Z}),$$

where $P$ is taken to be a fully normalized representative for each $\mathcal{F}$-conjugacy class of subgroups. According to [14], we have a short-exact sequence similar to proposition 3.1.
Proposition 3.2. Let $\Psi = \Psi^F : \tilde{\Omega}(F) \to \text{Obs}(F)$ be given by the \([P]_F\)-coordinate functions

$$\Psi_P(\xi) := \sum_{\pi \in W_S P} \xi_{(s)P} \pmod{|W_S P|},$$

when $P$ is fully $F$-normalized, and $\xi_{(s)P}$ denotes the $[(s)P]_F$-coordinate of an element $\xi \in \tilde{\Omega}(F) = \prod_{[P] \in \mathcal{C}(F)} \mathbb{Z}$.

The following sequence of abelian groups is then exact:

$$0 \to A(F) \xrightarrow{\Phi} \tilde{\Omega}(F) \xrightarrow{\Psi} \text{Obs}(F) \to 0.$$

$\Phi$ is a ring homomorphism, but $\Psi$ is just a group homomorphism.

4. The $p$-localized Burnside ring

Let $F$ be a saturated fusion system on a $p$-group $S$. In this section we show that there is a well-defined stabilization map $A(S)_{(p)} \to A(F)_{(p)}$ between $p$-localized Burnside rings. This map is shown to be a homomorphism of $A(F)_{(p)}$-modules, and it has an simple expression in terms of the mark homomorphism for $A(S)_{(p)}$. Using the stabilization homomorphism, we give a new basis for $A(F)_{(p)}$. It was shown in [14] that the irreducible $F$-stable sets form a basis for $A(F)$, but very little is known about their actual structure. The new basis for $A(F)_{(p)}$, though it only exists after $p$-localization, is easily described in terms of the stabilization homomorphism of marks. We use this basis in section 5 for the product fusion system $F \times F$ on $S \times S$, to give a new construction of the so-called characteristic idempotent for the saturated fusion system $F$. In section 4.1 we compare $A(F)_{(p)}$, including its basis, with the centric Burnside ring of $F$ defined by Diaz and Libman in [6]. When $F$ is realized by a group $G$, we also relate $A(F)_{(p)}$ to the $p$-subgroup part of $A(G)_{(p)}$.

It is useful to have a procedure for constructing an $F$-stable set from a general $S$-set. Such a procedure was used by Broto, Levi and Oliver in [1] to show that every saturated fusion system has at least one “characteristic biset,” a set with left and right $S$-actions satisfying properties suggested by Linckelmann and Webb. A similar procedure was used in [14], to construct all irreducible $F$-stable $S$-sets. Both constructions follow the same general idea: To begin with, we are given a finite $S$-set $X$ (or in general an element of the Burnside ring). We then consider each $F$-conjugacy class of subgroups in $S$ in decreasing order and add further $S$-orbits to $X$ until the set becomes $F$-stable. To construct the irreducible $F$-stable sets, we start with a transitive $S$-set $[S/P]$; to construct a characteristic biset, we start with $S$ itself considered as an $(S,S)$-biset.

The construction changes the number of elements and orbits in the set $X$ that we stabilize, and the number of added orbits depends heavily on the set that we start with – if $X$ is already $F$-stable we need not add anything at all. Because of this, we expect the stabilized sets to behave quite differently from the sets we start with, for instance, the stabilization procedure does not even preserve addition.

In this section we adjust the construction of [4, 14] such that instead of just adding orbits to stabilize a set, we subtract orbits as well, in a way such that all changes cancel “up to $F$-conjugation.” This results in a nicely behaved stabilization procedure that works for all $S$-sets, with one disadvantage: we must work in the $p$-localization $A(S)_{(p)}$ instead of $A(S)$.

The following lemmas [4, 14] are needed to show that the later calculations work in $A(S)_{(p)}$, i.e., that we never divide by $p$. Lemma [4, 14] is also interesting in itself since it
shows that for any fully normalized subgroup $P \leq S$, the number of $\mathcal{F}$-conjugates of $P$ is the same as the number of $S$-conjugates up to a $p'$-factor.

**Lemma 4.1.** Let $\mathcal{F}$ be a saturated fusion system on $S$, and let $P \leq S$ be fully $\mathcal{F}$-normalized. Then the number of $\mathcal{F}$-conjugates of $P$ is equal to $|\mathcal{F}(P, S)| = \frac{|S|}{|\mathcal{F}(P)|} \cdot k$, where $p \nmid k$.

Equivalently, $|\mathcal{F}(P, S)| = \frac{|S|}{|\mathcal{F}(P)|} \cdot k'$, with $p \nmid k'$.

**Proof.** Recall that $[P]_{\mathcal{F}}$ denotes the set of subgroups in $S$ that are $\mathcal{F}$-conjugate to $P$. We then have $|\mathcal{F}(P, S)| = |\text{Aut}_S(P)| \cdot |[P]_{\mathcal{F}}|$ for all $P \leq S$. When $P$ is fully $\mathcal{F}$-normalized, we furthermore get

$$|\text{Aut}_S(P)| = |\text{Aut}_S(P)| \cdot k'' = \frac{|N_S P|}{[\mathcal{F}(P)]} \cdot k''$$

where $p \nmid k''$ since $\mathcal{F}$ is saturated. It follows that the two statements in the lemma are equivalent for $P \leq S$ fully normalized.

We proceed by induction on the index $|S : P|$. If $P = S$, then $|[S]_{\mathcal{F}}| = 1 = \frac{|S|}{|\mathcal{F}(S)|}$.

Assume therefore that $P < S$ is fully normalized; since $P \neq S$, we then have $P < N_S P$. The $\mathcal{F}$-conjugacy class $[P]_{\mathcal{F}}$ is a disjoint union of the $S$-conjugacy classes $[Q]_S$ where $Q \sim_{\mathcal{F}} P$. The $S$-conjugacy class $[Q]_S$ has $|S|/|N_S Q|$ elements, and $\frac{|S|}{|\mathcal{F}(P)|}$ is divisible by $|N_S Q|$ since $P$ is fully normalized. In particular, $\frac{|S|}{|\mathcal{F}(P)|}$ divides $|[P]_{\mathcal{F}}|$.

Furthermore, we have $|[Q]_S| \cdot \frac{|N_S P|}{|S|} = \frac{|N_S P|}{|S|} = 0 \pmod{p}$ whenever $Q \sim_{\mathcal{F}} P$ isn't fully normalized. It follows that

$$|[P]_{\mathcal{F}}| \cdot \frac{|N_S P|}{|S|} = \sum_{[Q]_S \subseteq [P]_{\mathcal{F}}} |[Q]_S| \cdot \frac{|N_S P|}{|S|}$$

$$= \sum_{[Q]_S \subseteq [P]_{\mathcal{F}}} |[Q]_S| \cdot \frac{|N_S P|}{|S|} = \frac{|[P]_{\mathcal{F}}|}{\frac{|N_S P|}{|S|}} \cdot \frac{|N_S P|}{|S|} \pmod{p},$$

where “f.n.” is short for “fully normalized,” and $[P]_{\mathcal{F}}$ is the set of $Q \sim_{\mathcal{F}} P$ that are fully normalized. We conclude that $|[P]_{\mathcal{F}}| = \frac{|S|}{|\mathcal{F}(P)|} \cdot k$, with $p \nmid k$, if and only if $|[P]_{\mathcal{F}}| = \frac{|S|}{|\mathcal{F}(P)|} \cdot k'$, with $p \nmid k'$.

Suppose that $Q \sim_{\mathcal{F}} P$ is fully normalized. Since $P$ is fully normalized, we have a homomorphism $\varphi \in \mathcal{F}(N_S Q, N_S P)$ with $\varphi(Q) = P$ by lemma 2.3 and since $Q$ is fully normalized, $\varphi$ is an isomorphism. It follows that every $Q \in [P]_{\mathcal{F}}$ is a normal subgroup of exactly one element of $[N_S P]_{\mathcal{F}}$, namely $N_S Q \in [N_S P]_{\mathcal{F}}$.

Let $K \sim_{\mathcal{F}} N_S P$. We let $[P]^{\leq K}_{\mathcal{F}}$ denote the set of $Q \sim_{\mathcal{F}} P$ such that $Q < K$. Any such $Q < K$ is in particular fully normalized since $[K] = [N_S P]$. Any $\mathcal{F}$-isomorphism $N_S P \sim K$ gives a bijection $[P]^{\leq K}_{\mathcal{F}} \sim [P]^{\leq K}_{\mathcal{F}}$.

The set $[P]_{\mathcal{F}}$ is thus seen to be the disjoint union of the sets $[P]^{\leq K}_{\mathcal{F}}$ where $K \sim_{\mathcal{F}} N_S P$, and these sets all have the same number of elements as $[P]^{\leq K}_{\mathcal{F}}$:

$$|[P]_{\mathcal{F}}| = \sum_{K \in [N_S P]_{\mathcal{F}}} |[P]^{\leq K}_{\mathcal{F}}| = |[N_S P]_{\mathcal{F}}| \cdot |[P]^{\leq N_S P}_{\mathcal{F}}|.$$

Let $K \sim_{\mathcal{F}} N_S P$ be fully normalized, then there is some $Q \in [P]^{\leq K}_{\mathcal{F}}$. We have $Q \sim_{\mathcal{F}} P$, and $Q$ is fully normalized with $N_S Q = K$ that is itself fully normalized. By letting $Q$ take the place of $P$, we can therefore assume that $N_S P$ is fully normalized.
Any two elements $Q, R \in [P]_{\mathcal{F}}^{\leq N_S P}$ are mapped $Q \simeq R$ by some $\mathcal{F}$-automorphism of $N_S P$ (since $N_S P$ is the normalizer of both $Q$ and $R$); hence $\text{Aut}_\mathcal{F}(N_S P)$ acts transitively on $[P]_{\mathcal{F}}^{\leq N_S P}$. Let $X \leq \text{Aut}_\mathcal{F}(N_S P)$ be the subgroup stabilizing $P$ under this action; then

$$|[P]_{\mathcal{F}}^{\leq N_S P}| = |\text{Aut}_\mathcal{F}(N_S P) : X|.$$ 

The number of elements in $[P]_{\mathcal{F}}^{\leq N_S P}$ is therefore equal to

$$|[P]_{\mathcal{F}}^{\leq N_S P}| = |[N_S P]_{\mathcal{F}}| \cdot |\text{Aut}_\mathcal{F}(N_S P) : X|.$$ 

We know that $\frac{|S|}{[N_S(N_S P)]}$ divides $|[P]_{\mathcal{F}}^{\leq N_S P}|$; and by the induction assumption we have $|[N_S P]_{\mathcal{F}}| = \frac{|S|}{[N_S(N_S P)]} \cdot k$, where $p \nmid k$, since $N_S P$ is fully normalized. We can therefore conclude that $\frac{|S|}{[N_S(N_S P)]}$ divides the index $|\text{Aut}_\mathcal{F}(N_S P) : X|$.

We now consider the following diagram of subgroups of $\text{Aut}_\mathcal{F}(N_S P)$:

$$\begin{array}{ccc}
\text{Aut}_\mathcal{F}(N_S P) & & \text{Aut}_\mathcal{S}(N_S P)\\
\downarrow & & \downarrow \\
X & & X \cap \text{Aut}_\mathcal{S}(N_S P)
\end{array}$$

The index $|\text{Aut}_\mathcal{F}(N_S P) : \text{Aut}_\mathcal{S}(N_S P)|$ is coprime to $p$ since $N_S P$ is fully normalized and $\mathcal{F}$ is saturated. We have $C_\mathcal{S}(N_S P) \leq C_P \leq N_S P$, which tells us that $C_\mathcal{S}(N_S P) = Z(N_S P)$; and consequently

$$\text{Aut}_\mathcal{S}(N_S P) \cong N_S(N_S P)/Z(N_S P).$$

From the definition of $X$, we get that

$$X \cap \text{Aut}_\mathcal{S}(N_S P) = \{ \varphi \in \text{Aut}_\mathcal{F}(N_S P) \mid \varphi P = P \} \cap \{ c_s \in \text{Aut}_\mathcal{F}(N_S P) \mid s \in N_S(N_S P) \} = \{ c_s \in \text{Aut}_\mathcal{F}(N_S P) \mid s \in N_S P \} = \text{Inn}(N_S P) \cong N_S P/Z(N_S P).$$

The index $|\text{Aut}_\mathcal{F}(N_S P) : X \cap \text{Aut}_\mathcal{S}(N_S P)|$ is therefore equal to $\frac{|N_S(N_S P)|}{[N_S P]}$.

The right side of the subgroup diagram now shows that the highest power of $p$ dividing the index $|\text{Aut}_\mathcal{F}(N_S P) : X \cap \text{Aut}_\mathcal{S}(N_S P)|$ is $\frac{|N_S(N_S P)|}{[N_S P]}$. The highest power of $p$ dividing $|\text{Aut}_\mathcal{F}(N_S P) : X|$ is thus at most $\frac{|N_S(N_S P)|}{[N_S P]}$ – and we already know that this power of $p$ divides $|\text{Aut}_\mathcal{F}(N_S P) : X|$. We conclude that $|\text{Aut}_\mathcal{F}(N_S P) : X| = \frac{|N_S(N_S P)|}{[N_S P]} \cdot k'$ for some $k'$ coprime to $p$, and we finally have

$$|[P]_{\mathcal{F}}^{\leq N_S P}| = |[N_S P]_{\mathcal{F}}| \cdot |\text{Aut}_\mathcal{F}(N_S P) : X|$$

$$= \frac{|S|}{[N_S(N_S P)]} \cdot \frac{|N_S(N_S P)|}{[N_S P]} \cdot k' = \frac{|S|}{[N_S P]} \cdot kk';$$

and $p \nmid kk'$.
Lemma 4.2. Let $P, Q \leq S$, then $|[Q]_F|$ divides $|[Q']_S|$ in $\mathbb{Z}_{(p)}$ for all $Q' \sim_F Q$; and furthermore

$$\frac{1}{|[Q]_F|} \sum_{Q' \leq [Q]_F} \Phi_{Q'}([S/P]) = \sum_{[Q']_S \leq [Q]_F} \frac{|[Q']_S|}{|[Q]_F|} \Phi_{Q'}([S/P]) = \frac{|\mathcal{F}(Q, P)| \cdot |S|}{|P| \cdot |\mathcal{F}(Q, S)|} \in \mathbb{Z}_{(p)}.$$

Proof. By lemma 4.1 we can express the number of $\mathcal{F}$-conjugates as $|[Q]_F| = \frac{|S|}{|N_{S}Q_0|} \cdot k$, with $p \nmid k$, where $Q_0 \sim_F Q$ is fully normalized. At the same time, the number of $S$-conjugates of $Q'$ is given by $|[Q']_S| = \frac{|S|}{|N_{S}Q'|}$. Since $|N_{S}Q'| \leq |N_{S}Q_0|$, it then follows that $|[Q]_F|$ divides $|[Q']_S|$ in $\mathbb{Z}_{(p)}$.

We try to simplify the sum in the lemma:

$$\sum_{[Q']_S \leq [Q]_F} \frac{|[Q']_S|}{|[Q]_F|} \Phi_{Q'}([S/P]) = \frac{1}{|[Q]_F|} \sum_{[Q']_S \leq [Q]_F} \frac{|S|}{|N_{S}(Q')|} \cdot \frac{|N_{S}(Q', P)|}{|P|}$$

$$= \frac{|S|}{|P| \cdot |[Q]_F|} \sum_{[Q']_S \leq [Q]_F} \frac{|N_{S}(Q', P)|}{|N_{S}(Q')|}$$

$$= \frac{|S|}{|P| \cdot |[Q]_F|} \sum_{[Q']_S \leq [Q]_F} |\{ R \in [Q']_S \mid R \leq P \}|$$

$$= \frac{|\mathcal{F}(Q, P)| \cdot |S|}{|P| \cdot |\mathcal{F}(Q, S)|}.$$ 

The last equality follows from multiplying with $|\text{Aut}_F(Q)|$ in both the numerator and the denominator. 

Given any element $X$ in the $p$-localized Burnside ring $A(S)_{(p)}$, we stabilize $X$ according to the following idea: We run through the subgroups $Q \leq S$ in decreasing order and subtract/add orbits to $X$ such that it becomes $\mathcal{F}$-stable at the conjugacy class of $Q$ in $\mathcal{F}$, i.e., such that $\Phi_{Q'}(X) = \Phi_{Q}(X)$ for all $Q' \sim_F Q$. Here we take care to “add as many orbits as we remove” at each step. The actual work of the stabilization procedure is handled in the following technical lemma 4.3, which is then applied in theorem 4.3 to construct the stabilization map $A(S)_{(p)} \rightarrow A(\mathcal{F})_{(p)}$.

Recall that $c_P(X)$ denotes the coefficient of $[S/P]$ when $X$ is written in the standard basis of $A(S)_{(p)}$, and $\Phi_P: A(S)_{(p)} \rightarrow \mathbb{Z}_{(p)}$ for $P \leq S$ denote the fixed point homomorphisms.

Lemma 4.3. Let $\mathcal{F}$ be a saturated fusion system on a $p$-group $S$, and let $\mathcal{H}$ be a collection of subgroups of $S$ such that $\mathcal{H}$ is closed under taking $\mathcal{F}$-subconjugates, i.e., if $P \in \mathcal{H}$, then $Q \in \mathcal{H}$ for all $Q \leq_F P$. Assume that $X \in A(S)_{(p)}$ has the property that $\Phi_P(X) = \Phi_P'(X)$ for all pairs $P \sim_F P'$, with $P, P' \notin \mathcal{H}$.

Then there exists a uniquely determined element $\pi X \in A(\mathcal{F})_{(p)} \leq A(S)_{(p)}$ satisfying the following three properties:

(i) $\Phi_P(\pi X) = \Phi_P(X)$ and $c_P(\pi X) = c_P(X)$ for all $P \notin \mathcal{H}$, $P \leq S$.

(ii) For all $P \leq S$ we have

$$\sum_{[P^s]_S \leq [P]_F} c_{P^s}(\pi X) = \sum_{[P^s]_S \leq [P]_F} c_{P^s}(X).$$
Proof. We proceed by induction on the size of $H$ as well as fully normalized. \( \overline{H} \neq \emptyset \). We therefore assume that for each conjugacy class $F$ to the mark homomorphism when we stabilize: We simply take the mean of the fixed another orbit $[Z]_0$. Similarly, we have $\Psi_{S/P} \simeq S/P$ since $P$ is $S$-stable by assumption $\Phi_{N_S P'}(\Phi(X)) = 0$. In particular, the $P$-coordinate function satisfies $\Phi_{N_S P'}(\Phi(X)) = 0$, that is,

\[
\Phi_P(\pi X) = \frac{1}{|P|} \sum_{P' \in |P|} \Phi_{P'}(X) = \frac{1}{|P|} \sum_{P' \in |P|} \Phi_{P'}(X).
\]

Here $[P]_F$ denotes the set of $F$-conjugates of $P$. In the sums we pick one representative $P'$ for each $P$-conjugacy class $[P']_S$ contained in $[P]_F$, and by lemma \ref{lem:fixed_elements} the fractions $\frac{|[P']_S|}{|P|}$ make sense in $Z(p)$. Property \ref{item:iii} ensures that we do not destroy the part of $X$ that has already been stabilized. Property \ref{item:iii} is the requirement that the total number of orbits is constant for each $F$-conjugacy class of subgroups. We are only allowed to “replace” an orbit $[S/P]$ by another orbit $[S/P']$ where $P' \sim F P$. Finally, property \ref{item:iii} tells us exactly what happens to the mark homomorphism when we stabilize: We simply take the mean of the fixed points for each conjugacy class in $F$. Property \ref{item:iii} also implies that the resulting $F$-stable element $\pi X$ is independent of the choice of collection $\mathcal{H}$, as long as the chosen collection $\mathcal{H}$ satisfies the assumptions of the lemma.

Let $L \sim F P$. Then there is a homomorphism $\varphi \in F(N_S P', N_S P)$ with $\varphi(P') = P$ by lemma \ref{lem:fixed_elements} since $F$ is saturated. The restriction of $S$-actions to the subgroup $\varphi(N_S P')$ gives a ring homomorphism $A(S)_{(p)} \to A(\varphi(N_S P'))_{(p)}$ that preserves the fixed-point homomorphisms $\Phi_Q$ for $Q \leq \varphi(N_S P') \leq N_S P$.

If we consider $X$ as an element of $A(\varphi(N_S P'))$, we can apply the short exact sequence of proposition \ref{prop:short_exact} to get $\Psi_{\varphi(N_S P')}(\Phi(X)) = 0$. In particular, the $P$-coordinate function satisfies $\Phi_{\varphi(N_S P')}(\Phi(X)) = 0$, that is,

\[
\Phi_{\varphi N_S P'}(\Phi(X)) \equiv 0 \pmod{|\varphi(N_S P')/P'|}.
\]

Similarly, we have $\Psi_{\varphi(N_S P')}(\Phi(X)) = 0$, where the $P'$-coordinate $\Psi_{\varphi N_S P'}(\Phi(X)) = 0$ gives us

\[
\Phi_{\varphi N_S P'}(\Phi(X)) \equiv 0 \pmod{|N_S P'/P'|}.
\]

Since $P$ is maximal in $\mathcal{H}$, we have by assumption $\Phi_Q(\Phi(X)) = \Phi_{Q'}(\Phi(X))$ for all $Q \sim F Q'$ where $P$ is $F$-conjugate to a proper subgroup of $Q$. Specifically, we have

\[
\Phi_{\varphi N_S P'}(\Phi(X)) = \Phi_{\varphi N_S P'}(\Phi(X)) = \Phi_{\varphi N_S P'}(\Phi(X))
\]

for all $s \in N_S P'$ with $s \notin P'$. It then follows that

\[
\Phi_P(\pi X) - \Phi_{P'}(\pi X) = \sum_{\pi \in \varphi(N_S P')/P} \Phi_{\varphi N_S P'}(\Phi(X)) - \sum_{\pi \in N_S P'/P'} \Phi_{\varphi N_S P'}(\Phi(X)) \equiv 0 - 0 \pmod{|W_{P'}|}.
\]
We can therefore define \( \lambda_P^r := (\Phi P (X) - \Phi P^r(X))/|W_S P'| \in \mathbb{Z}(p) \). We now recall from lemma 4.1 that \(|[P]'| = [S]/|N_S P'| \cdot k \) where \( p \nmid k \), and since \( k \) is invertible in \( \mathbb{Z}(p) \), we can define

\[
c := \left( \sum_{[P]'|s \subseteq [P]'_F} \lambda_P^r \right) / k \in \mathbb{Z}(p),
\]

as well as \( \mu_P^r := \lambda_P^r - [W_S P]/|W_S P'| \cdot c \in \mathbb{Z}(p) \). We use the \( \mu_P^r \) as coefficients to construct a new element

\[
X' := X + \sum_{[P]'|s \subseteq [P]'_F} \mu_P^r \cdot [S/P'] \in A(S)(p).
\]

We then at least have \( c_Q(X') = c_Q(X) \) for all \( Q \not\in F \). The definition of \( c \) ensures that

\[
\sum_{[P]'|s \subseteq [P]'_F} [W_S P]/|W_S P'| \cdot c = c \cdot \sum_{[P]'|s \subseteq [P]'_F} [N_S P]/|N_S P'| \cdot \sum_{[P]'|s \subseteq [P]'_F} |[P]'_S| = c \cdot |N_S P|/|S| \cdot |[P]'_F| = c \cdot k = \sum_{[P]'|s \subseteq [P]'_F} \lambda_P^r;
\]

which in turn gives us

\[
\sum_{[P]'|s \subseteq [P]'_F} c_{P'}(X') - \sum_{[P]'|s \subseteq [P]'_F} c_P(X) = \sum_{[P]'|s \subseteq [P]'_F} \mu_P^r - \sum_{[P]'|s \subseteq [P]'_F} [W_S P]/|W_S P'| \cdot c = 0.
\]

Next we recall that \( \Phi_Q([S/P']) = 0 \) unless \( Q \subseteq S \cdot P' \), which implies that \( \Phi_Q(X') = \Phi_Q(X) \) for every \( Q \not\in H \). We then calculate \( \Phi_P^r(X') \) for each \( P' \sim F \): \( P \):

\[
\Phi_P^r(X') = \Phi_P^r(X) + \sum_{[P]'|s \subseteq [P]'_F} \mu_P^r \cdot \Phi_P^r([S/P']) \]

\[
= \Phi_P^r(X) + \mu_P^r \cdot \Phi_P^r([S/P']) = \Phi_P^r(X) + \mu_P^r [W_S P']
\]

\[
= \Phi_P^r(X) + \lambda_P^r [W_S P'] - [W_S P]/|W_S P'| \cdot c \cdot |W_S P'|
\]

\[
= \Phi_P^r(X) - [W_S P]|c;
\]

which is independent of the choice of \( P' \in [P]'_F \).

We define \( H' := H \setminus [P]'_F \) as \( H \) with the \( F \)-conjugates of \( P \) removed. Because \( P \) is maximal in \( H \), the subcollection \( H' \) again contains all \( F \)-subconjugates of any \( H \in H' \).

From \([4.2]\) we get that \( \Phi_Q(X) = \Phi_{Q'}(X) \) for all \( Q \sim F \) and \( Q, Q' \not\in H' \). By induction we can therefore apply lemma \([4.3]\) to \( X' \) and the smaller collection \( H' \). We get an element \( \pi X' \in A(F)(p) \) satisfying

(i) \( \Phi_Q(\pi X') = \Phi_Q(X') \) and \( c_Q(\pi X') = c_Q(X') \) for all \( Q \not\in H' \).

(ii) For all \( Q \leq S \) we have

\[
\sum_{[Q]'|s \subseteq [Q]'_F} c_Q(\pi X') = \sum_{[Q]'|s \subseteq [Q]'_F} c_Q(X').
\]
(iii) For every $Q \leq S$:

$$
\Phi_Q(\pi X') = \frac{1}{|Q|_F} \sum_{Q' \in [Q]_F} \Phi_{Q'}(X').
$$

We claim that $\pi X := \pi X'$ satisfies the properties of the lemma for $X$ and $H$ as well.

We immediately have that $\Phi_Q(\pi X') = \Phi_Q(X') = \Phi_Q(X)$ and $c_Q(\pi X') = c_Q(X') = c_Q(X)$ for all $Q \notin H$, so property \([\text{i}]\) is satisfied. Since $c_Q(X') = c_Q(X)$ when $Q \not\sim_F P$, we get for all $Q \in H'$ that

$$
\sum_{|Q'| \leq |Q|_F} c_Q(\pi X') = \sum_{|Q'| \leq |Q|_F} c_Q(X') = \sum_{|Q'| \leq |Q|_F} c_Q(X).
$$

Furthermore, since $P \notin H'$, we have $c_{P'}(\pi X') = c_{P'}(X')$ for $P' \sim_F P$. Using (4.1) we then get

$$
\sum_{|P'| \leq |P|_F} c_{P'}(\pi X') = \sum_{|P'| \leq |P|_F} c_{P'}(X') = \sum_{|P'| \leq |P|_F} c_{P'}(X).
$$

This proves that \([\text{ii}]\) is satisfied. Since $c_Q(X') = c_Q(X)$ when $Q \not\sim_F P$, we have $\Phi_Q(X') = \Phi_Q(X)$ for all $Q$ that are not $F$-subconjugate to $P$. Consequently we have

$$
\Phi_Q(\pi X') = \frac{1}{|Q|_F} \sum_{Q' \in [Q]_F} \Phi_{Q'}(X')
$$

$$
= \frac{1}{|Q|_F} \sum_{Q' \in [Q]_F} \Phi_{Q'}(X),
$$

when $Q$ is not $F$-subconjugate to $P$. We need the small lemma 4.2 below to show that every $P' \sim_F P$ satisfies

$$
\frac{1}{|Q|_F} \sum_{Q' \in [Q]_F} \Phi_{Q'}([S/P']) = \frac{|\mathcal{F}(Q, P)| \cdot |S|}{|P| \cdot |\mathcal{F}(Q, S)|} \in \mathbb{Z}_p
$$

for all $Q \leq S$. In the case where $Q$ is subconjugate to $P$ in $\mathcal{F}$, we can then use both (4.1) and (4.2) to show that

$$
\Phi_Q(\pi X') = \frac{1}{|Q|_F} \sum_{Q' \in [Q]_F} \Phi_{Q'}(X')
$$

$$
= \frac{1}{|Q|_F} \sum_{Q' \in [Q]_F} \Phi_{Q'}(X) + \sum_{|P'| \leq |P|_F} \mu_{P'} \left( \frac{1}{|Q|_F} \sum_{Q' \in [Q]_F} \Phi_{Q'}([S/P']) \right)
$$

$$
= \frac{1}{|Q|_F} \sum_{Q' \in [Q]_F} \Phi_{Q'}(X) + \sum_{|P'| \leq |P|_F} \mu_{P'} \cdot \frac{|\mathcal{F}(Q, P)| \cdot |S|}{|P| \cdot |\mathcal{F}(Q, S)|}
$$

$$
= \frac{1}{|Q|_F} \sum_{Q' \in [Q]_F} \Phi_{Q'}(X) + 0;
$$

which proves that $\pi X'$ satisfies \([\text{iii}]\). □
**Theorem A.** Let $F$ be a saturated fusion system on a finite $p$-group $S$. We let $A(F)_{(p)}$ denote the $p$-localized Burnside ring of $F$ as a subring of the $p$-localized Burnside ring $A(S)_{(p)}$ for $S$. Then there is a transfer map $\pi : A(S)_{(p)} \to A(F)_{(p)}$, which is a homomorphism of $A(F)_{(p)}$-modules and which restricts to the identity on $A(A)_{(p)}$. In terms of the fixed point homomorphisms the transfer map $\pi$ satisfies

$$\Phi_Q(\pi(X)) = \frac{1}{|[Q]_F|} \sum_{Q' \in [Q]_F} \Phi_{Q'}(X),$$

where $[Q]_F$ is the conjugacy class of $Q$ in $F$.

**Proof.** To construct $\pi(X)$ we apply lemma 4.3 to $X$ and the collection $H$ of all subgroups in $S$. This results in a stable element $\pi(X) \in A(F)_{(p)}$ satisfying

$$\Phi_Q(\pi(X)) = \frac{1}{|[Q]_F|} \sum_{Q' \in [Q]_F} \Phi_{Q'}(X)$$

as wanted. If we apply $\pi$ to an element $X$ that is already $F$-stable, then

$$\Phi_Q(\pi X) = \frac{1}{|[Q]_F|} \sum_{Q' \in [Q]_F} \Phi_{Q'}(X) = \frac{1}{|[Q]_F|} \sum_{Q' \in [Q]_F} \Phi_{Q'}(X) \Phi_Q(Y)$$

so $\pi(X) = X$. Hence $\pi$ is the identity map when restricted to $A(F)_{(p)}$.

If $X \in A(F)_{(p)}$ and $Y \in A(S)_{(p)}$, then since the fixed point homomorphisms preserve products, we have

$$\Phi_Q(\pi(XY)) = \frac{1}{|[Q]_F|} \sum_{Q' \in [Q]_F} \Phi_{Q'}(XY) = \frac{1}{|[Q]_F|} \sum_{Q' \in [Q]_F} \Phi_{Q'}(X) \Phi_{Q'}(Y)$$

$$= \Phi_Q(X) \frac{1}{|[Q]_F|} \sum_{Q' \in [Q]_F} \Phi_{Q'}(Y) = \Phi_Q(X) \Phi_Q(\pi(Y)).$$

This shows that $\pi(XY) = X \cdot \pi(Y)$, and by a similar argument, $\pi$ preserves addition. Hence $\pi$ is a homomorphism of $A(F)_{(p)}$-modules.

**Remark 4.4.** As stated in lemma 4.3 the stabilization homomorphism $\pi : A(F)_{(p)} \to A(S)_{(p)}$ also satisfies

$$\sum_{[P'] \leq [P]_F} c_{P'}(\pi(X)) = \sum_{[P'] \leq [P]_F} c_{P'}(X).$$

Hence $\pi$ replaces orbits of $X$ within each $F$-conjugation class, but doesn’t otherwise add or remove orbits from $X$. This fact will be important for describing the action of the characteristic idempotent on bises in theorem 5 of section 5.

We know that the transitive $S$ sets $[S/P]$ form a basis for $A(S)_{(p)}$. We now apply the projection $\pi : A(S)_{(p)} \to A(F)_{(p)}$ to this basis, and we get a new basis for the $p$-localized Burnside ring $A(F)_{(p)}$.

**Proposition 4.5.** Let $\beta_P \in A(F)_{(p)}$ be defined by $\beta_P := \pi([S/P])$. In terms of the homomorphism of marks, $\beta_P$ is then given by

$$\Phi_Q(\beta_P) = \frac{|\mathcal{F}(Q, P)| \cdot |S|}{|P| \cdot |\mathcal{F}(Q, S)|} \in \mathbb{Z}_{(p)},$$

hence $\beta_P$ only depends on $P$ up to $F$-conjugation.

The elements $\beta_P$ defined this way form a $\mathbb{Z}_{(p)}$-basis for $A(F)_{(p)}$. 


Proposition 4.6. For each \( P \leq S \), the element \( \beta_P \in A(S)_{(p)} \) is given by the following expression when written as a \( \mathbb{Z}_p \)-linear combination of \( S \)-sets:

\[
\beta_P = \sum_{[R]_S} \frac{1}{|R|} \left( \sum_{R \leq Q \leq S} \Phi_Q(\beta_P) \cdot \mu(R, Q) \right) \left[ \frac{S}{R} \right]
\]

where the last expression, and thus \( \beta_P \), only depends on \( P \) up to conjugation in \( \mathcal{F} \).

Because the transitive \( S \)-sets \( [S/P] \) for \( P \leq S \) generate \( A(S)_{(p)} \), and since \( \pi \) is surjective, the elements \( \beta_P \) must generate all of \( A(\mathcal{F})_{(p)} \).

We now order the \( \mathcal{F} \)-conjugacy classes \( [P]_\mathcal{F} \) according to decreasing order of \( P \), and the mark homomorphism \( \Phi: \text{Span} \{ \beta_P \} \to \tilde{\Omega}(\mathcal{F})_{(p)} \) is then represented by a matrix \( M \) with entries

\[
M_{Q,P} = \frac{|\mathcal{F}(Q, P)| \cdot |S|}{|P| \cdot |\mathcal{F}(Q, S)|}.
\]

If \( Q \) is not \( \mathcal{F} \)-sub conjugate to \( P \), then \( M_{Q,P} = 0 \); so \( M \) is a lower triangular matrix with diagonal entries

\[
M_{P,P} = \frac{|\mathcal{F}(P, P)| \cdot |S|}{|P| \cdot |\mathcal{F}(P, S)|} \neq 0.
\]

Since all diagonal entries are non-zero, we conclude that the \( \beta_P \) are linearly independent over \( \mathbb{Z}_p \).

The mark homomorphism \( \Phi: A(S)_{(p)} \to \tilde{\Omega}(S)_{(p)} \) embeds the Burnside ring of \( S \) into its ghost ring, and since we know the value of \( \Phi_Q(\beta_P) \) from proposition 4.5, we know the image of \( \beta_P \) inside \( \tilde{\Omega}(S)_{(p)} \). We might then wonder whether we can pull back our knowledge from \( \tilde{\Omega}(S)_{(p)} \) to \( A(S)_{(p)} \) and write \( \beta_P \) explicitly as a linear combination of transitive \( S \)-sets.

In [10], David Gluck gives a method on how to do exactly this. Because \( A(S)_{(p)} \) embeds in the ghost ring \( \tilde{\Omega}(S)_{(p)} \) as a subring of finite index, if we take the tensor product with \( Q \), we get an isomorphism \( \Phi: A(S) \otimes Q \cong \tilde{\Omega}(S) \otimes Q \). What [10] then contains is an expression for the inverse isomorphism. Let \( \epsilon_Q := (0, \ldots, 0, 1, 0, \ldots, 0) \) be the standard basis element of \( \tilde{\Omega}(S) \otimes Q \) corresponding to the subgroup \( Q \leq S \).

The inverse \( \Phi^{-1}: \tilde{\Omega}(S) \otimes Q \to A(S) \otimes Q \) is then given by

\[
\Phi^{-1}(\epsilon_Q) = \frac{1}{|N_S Q|} \sum_{R \leq Q} \mu(R, Q) \cdot |R| \cdot |S/R|,
\]

where \( \mu \) is the Möbius-function for the poset of subgroups in \( S \).

Since we know the image \( \Phi(\beta_P) \), we can apply the isomorphism above to get an expression for \( \beta_P \) inside \( A(S) \otimes Q \); and because \( A(S)_{(p)} \) is embedded in \( A(S) \otimes Q \), the expression holds in \( A(S)_{(p)} \) as well.

Theorem 4.6. For each \( P \leq S \), the element \( \beta_P \in A(\mathcal{F})_{(p)} \) is given by the following expression when written as a \( \mathbb{Z}_p \)-linear combination of \( S \)-sets:

\[
\beta_P = \sum_{[R]_S} \frac{1}{|R|} \left( \sum_{R \leq Q \leq S} \Phi_Q(\beta_P) \cdot \mu(R, Q) \right) \left[ \frac{S}{R} \right]
\]

where the last expression, and thus \( \beta_P \), only depends on \( P \) up to conjugation in \( \mathcal{F} \).
Proposition 4.7. Suppose that the bases of the rings correspond as well.

Libman in [6] as example 3.9 and theorem A, respectively. New in this section is the fact to each other in a suitable way. Both of these isomorphisms are originally due to Diaz-

is isomorphic to the “centric part” of defined using only the centric subgroups of the Burnside ring the ring is realized by a group G: We see that A(F)_p is isomorphic to the ring A(G:p)_p generated by G-sets [G/P] where P ≤ G is a p-group, and the basis element β_p almost corresponds to the transitive G-set [G/P]. After that, we consider the Burnside ring A^cent(F) introduced by Antonio Diaz and Assaf Libman in [5], which is defined using only the centric subgroups of F: We show that after p-localization A^cent(F)_p is isomorphic to the “centric part” of A(F)_p, again with the basis elements corresponding to each other in a suitable way. Both of these isomorphisms are originally due to Diaz-

Libman in [6] as example 3.9 and theorem A, respectively. New in this section is the fact that the bases of the rings correspond as well.

Proposition 4.7. Suppose that S is a Sylow p-subgroup of G, and let F := F_S(G). Define A(G;p) to be the subring of A(G) where all isotropy subgroups are p-groups.

Then the transitive G-set [G/S] is invertible in A(G;p)_p, and we get an isomorphism of rings A(F)_p ≅ A(G;p)_p by

\[ β_P \mapsto \frac{[G/P]}{[G/S]} \]

This isomorphism is in a way the best we could hope for, since the basis element β_P only depends on the fusion data in F_S(G), while the transitive G-set [G/P] depends on the actual group G. If we replace G with a product G' = G × H where H is a p'-group, then the fusion system F_S(G') and β_P are the same for G' as for G, but the transitive set [G'/P] has increased in size by a factor |H|. However, as we see in the proof below, the quotient \frac{[G/P]}{[G/S]} depends only on the fusion system and not on G.

Proof. We first show that [G/S] is invertible in A(G)_p. For every Q ≤ S that is fully F-normalized, we have

\[ Φ_Q([G/S]) = \frac{|N_G(Q,S)|}{|S|} = \frac{|N_GQ|}{|S|} \cdot |\{Q' ≤ S|Q' ∼_F Q\}|. \]
By lemma 4.1, we have \(|Q' \leq S|Q' \sim_\mathcal{F} Q\rangle = \frac{|S|}{|N_S Q|} \cdot k\) with \(p \nmid k\). We thus get

\[ \Phi_Q([G/S]) = \frac{|N_G Q|}{|N_S Q|} \cdot k, \]

which is invertible in \(\mathbb{Z}(p)\) since \(Q\) is fully \(\mathcal{F}\)-normalized.

If \(H \leq G\) is not a \(p\)-group, then \(\Phi_H(X) = 0\) for all \(X \in A(G;p)\). We also know that every \(p\)-subgroup of \(G\) is conjugate to a subgroup of \(S\) by Sylow’s theorems, and therefore the mark homomorphism for \(A(G;p)\) restricts to an inclusion

\[ \Phi: A(G;p) \to \prod_{[Q,F]} \mathbb{Z}(p) = \tilde{\Omega}(\mathcal{F})_{(p)}, \]

and \(A(G;p)\) has finite index in \(\tilde{\Omega}(\mathcal{F})_{(p)}\) for rank reasons.

Because \(\Phi_Q([G/S])\) is invertible in \(\mathbb{Z}(p)\), \([G/S]\) is invertible in the ghost ring \(\tilde{\Omega}(\mathcal{F})_{(p)}\).

It follows that multiplication with \([G/S]\) is a bijection \(\tilde{\Omega}(\mathcal{F})_{(p)} \to \tilde{\Omega}(\mathcal{F})_{(p)}\), which sends \(A(G;p)\) into itself. Since \(A(G;p)\) has finite index in \(\tilde{\Omega}(\mathcal{F})_{(p)}\), multiplication with \([G/S]\) must then also be a bijection of \(A(G;p)\) to itself, hence \([G/S]\) is invertible in \(A(G;p)\).

It thus makes sense to consider the elements \(\frac{[G/P]}{[G/S]}\) for \(P \leq S\), and we calculate

\[ \Phi_Q \left( \frac{[G/P]}{[G/S]} \right) = \frac{|N_G(Q,P)| \cdot |S|}{|P| \cdot |N_G(Q,S)|} = \frac{|F(Q,P)| \cdot |S|}{|P| \cdot |F(Q,S)|} = \Phi_Q(\beta_P). \]

It follows that \(\frac{[G/P]}{[G/S]} = \beta_P\) as elements of \(\tilde{\Omega}(\mathcal{F})_{(p)}\), giving the isomorphism \(A(G;p) \cong A(\mathcal{F})_{(p)}\).

The Burnside ring defined by Diaz-Libman in [3] for a saturated fusion system \(\mathcal{F}\), is constructed in terms of an orbit category over the \(\mathcal{F}\)-centric subgroups of \(S\). A subgroup \(P \leq S\) is \(\mathcal{F}\)-centric if all \(\mathcal{F}\)-conjugates \(P' \sim_\mathcal{F} P\) are self-centralizing, i.e., \(C_S(P') \leq P'\).

We denote the Diaz-Libman Burnside ring by \(A^\text{cent}(\mathcal{F})\), and it comes equipped with an additive basis \(\xi_P\) indexed by the \(\mathcal{F}\)-conjugacy classes of \(\mathcal{F}\)-centric subgroups. As shown in [3] there is also an injective homomorphism of marks

\[ \Phi^\text{cent}: A^\text{cent}(\mathcal{F}) \to \prod_{P \text{ is } \mathcal{F}\text{-centric}} \mathbb{Z}, \]

with finite cokernel, and on basis elements \(\Phi^\text{cent}\) is given by

\[ \Phi_Q^\text{cent}(\xi_P) = \frac{|Z(Q)| \cdot |F(Q,P)|}{|P|}. \]

**Proposition 4.8.** Let \(\mathcal{F}\) be a saturated fusion system on a \(p\)-group \(S\), and write \(N \leq A(\mathcal{F})_{(p)}\) for the \(\mathbb{Z}(p)\)-submodule generated by \(\beta_P\) for non-\(\mathcal{F}\)-centric \(P\). Then \(N\) is an ideal in the Burnside ring \(A(\mathcal{F})_{(p)}\), and there is a ring isomorphism \(A(\mathcal{F})_{(p)}/N \cong A^\text{cent}(\mathcal{F})_{(p)}\) with the Burnside ring of Diaz-Libman. The basis element \(\xi_S\) is invertible in \(A^\text{cent}(\mathcal{F})_{(p)}\), and the isomorphism is given by

\[ \beta_P \mapsto \frac{\xi_P}{\xi_S} \]

for \(\mathcal{F}\)-centric \(P \leq S\).
Proof. If $P$ is $\mathcal{F}$-centric, then any subgroup containing $P$ is $\mathcal{F}$-centric as well, hence the collection of non-$\mathcal{F}$-centric subgroup is closed under $\mathcal{F}$-conjugation and taking subgroups. By the double coset formula [1.1] for $A(S)(p)$, the $\mathbb{Z}(p)$-submodule generated by the elements $[S/P]$ with $P$ non-$\mathcal{F}$-centric is an ideal in $A(S)(p)$. Let us denote this ideal $M \leq A(S)(p)$.

The stabilization map $\pi: A(S)(p) \rightarrow A(\mathcal{F})(p)$ is a homomorphism of $A(\mathcal{F})(p)$-modules, so the image $N := \pi(M)$ is an ideal of $A(\mathcal{F})(p)$, and at the same time $N$ is the $\mathbb{Z}(p)$-submodule generated by the elements $\pi([S/P]) = \beta_P$ where $P$ is non-$\mathcal{F}$-centric. By proposition 4.5, we have $\Phi_Q(\beta_P) = 0$ whenever $Q$ is $\mathcal{F}$-centric and $P$ is not. Hence the homomorphism

$$A(\mathcal{F})(p) \xrightarrow{\Phi} \prod_{\mathcal{F}} \mathbb{Z}(p) \rightarrow \prod_{\mathcal{F}} \mathbb{Z}(p)$$

send $N$ to 0, and therefore induces a ring homomorphism

$$\Phi: A(\mathcal{F})(p)/N \rightarrow \prod_{\mathcal{F}} \mathbb{Z}(p),$$

Let $\bar{\beta}_P$ denote the equivalence class of $\beta_P$ in $A(S)(p)/N$ when $P$ is $\mathcal{F}$-centric. The quotient ring $A(\mathcal{F})(p)/N$ then has a basis consisting of $\bar{\beta}_P$ for each $\mathcal{F}$-centric $P$ up to $\mathcal{F}$-conjugation.

The rest of this proof follows the same lines as the proof of proposition 4.7. For the basis element $\xi_S$ of $A^{\text{cent}}(\mathcal{F})(p)$ the image under the mark homomorphism has the form $\Phi_Q^{\text{cent}}(\xi_S) = \frac{|Z(Q)|}{|S|} \cdot |\mathcal{F}(Q, S)|$, which by lemma 4.1 is invertible in $\mathbb{Z}(p)$. Hence $\xi_S$ is invertible in the ghost ring

$$\prod_{\mathcal{F}} \mathbb{Z}(p),$$

and since $\Phi^{\text{cent}}$ has finite cokernel, it follows that $\xi_S$ is invertible in $A^{\text{cent}}(\mathcal{F})(p)$ as well. It therefore makes sense to form the fractions $\frac{\beta_P}{\xi_S}$. Applying the fixed point homomorphisms to these fractions, we then get

$$\Phi_Q^{\text{cent}} \left( \frac{\xi_S}{\xi_S} \right) = \frac{|Z(Q)| \cdot |\mathcal{F}(Q, P)| \cdot |S|}{|P| \cdot |S|} \cdot |\mathcal{F}(Q, S)| \cdot |Z(Q)| = \Phi_Q(\beta_P)$$

for all $\mathcal{F}$-centric subgroup $Q, P \leq S$. This shows that the ring homomorphism

$$\Phi: A(\mathcal{F})(p)/N \rightarrow \prod_{\mathcal{F}} \mathbb{Z}(p),$$

sends $\bar{\beta}_P$ to $\Phi^{\text{cent}}(\frac{\beta_P}{\xi_S})$, which proves that $\Phi$ is injective on $A(\mathcal{F})(p)$ and that $\beta_P \mapsto \frac{\beta_P}{\xi_S}$ gives a ring isomorphism $A(\mathcal{F})(p)/N \cong A^{\text{cent}}(\mathcal{F})(p)$. □

5. The characteristic idempotent

In this section we make use of the stabilization homomorphism of theorem A to give new results on the characteristic idempotent for a saturated fusion system. These idempotents were shown by Ragnarsson and Stancu to classify the saturated fusion systems on a given $p$-group. In section 5.1 we recall the structures of the double Burnside rings and modules, and the category that they form. In section 5.2 we give a new construction of the characteristic idempotent $\omega_{\mathcal{F}}$ for a saturated fusion system $\mathcal{F}$ on $S$ by stabilizing the diagonal subgroup $\Delta(S) \leq S \times S$ with respect to the fusion system $\mathcal{F} \times \mathcal{F}$. As a consequence we discover the value of the mark homomorphism on the idempotent, and
this is the content of theorem [B]. In section [3.3] we discuss multiplication \( X \to \omega_F \circ X \) with the characteristic idempotent – both for elements \( X \) of the double Burnside ring of \( S \), but also more generally when \( X \) is just some finite set with an \( S \)-action. Theorem [C] describes the action of \( \omega_F \) in terms of the homomorphism of marks.

5.1. **The category of Burnside modules.** For finite groups \( G \) and \( H \), a \((G,H)\)-biset is a set with both a left \( H \)-action and a right \( G \)-action, and such that the two actions commute. A \((G,H)\)-biset \( X \) gives rise to a \((H \times G)\)-set by defining \((h,g).x := hxg^{-1} \), and vice versa. The transitive \((G,H)\)-bisets have the form \([H \times G]/D\) for subgroups \( D \leq H \times G \). The isomorphism classes of finite \((G,H)\)-bisets form a monoid, and the Grothendieck group \( A(G,H) \) is called the **Burnside module** of \( G \) and \( H \). Additively \( A(G,H) \) is isomorphic to \( A(H \times G) \) and we have a basis consisting of the transitive bisets \([H \times G]/D\) where \( D \leq H \times G \) is determined up to \((H \times G)\)-conjugation.

The multiplication for the Burnside modules is different from the non-biset Burnside rings. We have multiplication/composition maps \( \circ : A(H,K) \times A(G,H) \to A(G,K) \), defined for every \((G,H)\)-biset \( X \) and \((H,K)\)-biset \( Y \) as

\[
Y \circ X := Y \times_H X = Y \times X / \sim
\]

where \((yh,x) \sim (y,hx)\) for all \( y \in Y, x \in X \) and \( h \in H \). With this composition, the Burnside modules form the Hom-sets of a category with finite groups as objects. The ring of endomorphisms \( A(G,G) \) of \( G \) is the **double Burnside ring** of \( G \). The identity element of \( A(G,G) \) is the group \( G \) considered as a \((G,G)\)-biset. On transitive bisets, the composition is given by a double coset formula

\[
[(K \times H)/D] \circ [(H \times G)/C] = \sum_{\pi \in \pi_2 D \Delta H/C} [(K \times G)/D \ast (x,1)C]
\]

where the subgroup \( B \ast A \) is defined as \( \{(k,g) \in K \times G \mid \exists h \in H : (h,k,g) \in B, (h,g) \in A\} \) for subgroups \( B \leq K \times H \) and \( A \leq H \times G \).

Given a homomorphism \( \varphi : U \to H \) with \( U \leq G \), the graph \( \Delta(U,\varphi) = \{(\varphi u,u) \mid u \in U \} \) is a subgroup of \( H \times G \). We introduce the notation \([U,\varphi]_H^U\) as a shorthand for the biset \([H \times G]/\Delta(U,\varphi)\], and if the groups \( G,H \) are clear from context, we just write \([U,\varphi]\).

The bisets \([U,\varphi]\) generate the \((G,H)\)-bisets that have a free left \( H \)-action. For these basis elements, (5.1) takes the form

\[
[T,\psi]_H^U \circ [U,\varphi]_H^U = \sum_{\pi \in \pi T \cap H/\varphi U} [\varphi^{-1}(T \pi) \cap U,\psi \varphi]_H^U.
\]

From the isomorphism \( A(G,H) \cong A(H \times G) \) of additive groups, the Burnside modules inherit fixed point homomorphisms \( \Phi_C : A(G,H) \to \mathbb{Z} \) for each \((H \times G)\)-conjugacy class of subgroups \( C \leq H \times G \). Note however that the fixed point homomorphisms for \( A(G,G) \) are not ring homomorphisms – they are only homomorphisms of abelian groups.

Given any \((G,H)\)-biset \( X \), we can swap the actions to get an \((H,G)\)-biset \( X^{op} \) with \( g \cdot x^{op} h := h^{-1}.x.g^{-1} \), which extends to a group isomorphism \((-)^{op} : A(G,H) \to A(H,G) \). We clearly have \( ([H \times G]/D)^{op} = [(G \times H)/D^{op}] \) and \( \Phi_C(X^{op}) = \Phi_{C^{op}}(X) \), where \( C^{op},D^{op} \) are the subgroups \( C,D \) with the coordinates swapped. Any element of the double Burnside ring \( X \in A(G,G) \) that satisfies \( X^{op} = X \) is called symmetric.

5.2. **A new construction of the characteristic idempotent.** Let \( F \) be a fusion system on a \( p \)-group \( S \). We then say that an element of the \( p \)-localized double Burnside ring \( A(S,S)_{(p)} \) is \( F \)-characteristic if it satisfies the Linckelmann-Webb properties: The element
is $\mathcal{F}$-generated (see 5.1), it is $\mathcal{F}$-stable (see 5.2), and finally there is a $p'$-condition for the number of elements (see 5.4).

K. Ragnarsson showed in [12] that for every saturated fusion system $\mathcal{F}$ on a $p$-group $S$, there is a unique idempotent $\omega_{\mathcal{F}} \in A(S, S)_{(p)}$ that is $\mathcal{F}$-characteristic, and [13] shows how $\mathcal{F}$ can be reconstructed from $\omega_{\mathcal{F}}$ (or any $\mathcal{F}$-characteristic element). To construct the characteristic idempotent of a saturated fusion system, Ragnarsson used a Cauchy-sequence argument in the $p$-completion $A(S, S)_{(p)}$ to construct $\omega_{\mathcal{F}}$ as an element of $A(S, S)_{(p)}$. Later arguments then showed that $\omega_{\mathcal{F}}$ lives already in the $p$-localization $A(S, S)_{(p)}$, and that it is unique.

In this section we give a new construction of $\omega_{\mathcal{F}}$ inside $A(S, S)_{(p)}$ directly; in fact $\omega_{\mathcal{F}}$ turns out to be the basis element $\beta_{\Delta(S)}$ of proposition 4.5 with respect to the fusion system $\mathcal{F} \times \mathcal{F}$ on $S \times S$. As a consequence we learn the value of the fixed point homomorphisms on $\omega_{\mathcal{F}}$ as stated in theorem B and we also gain a (complicated) decomposition of $\omega_{\mathcal{F}}$ into $(S, S)$-orbits.

**Definition 5.1.** Let $\mathcal{F}$ be a fusion system on a $p$-group $S$. An element $X \in A(S, S)$ is then said to be $\mathcal{F}$-generated if $X$ is expressed solely in terms of basis elements $[P, \varphi]$ where $\varphi : P \to S$ is a morphism of $\mathcal{F}$. The $\mathcal{F}$-generated elements form a subring $A_{\mathcal{F}}(S, S)$ of the double Burnside ring, and since $[P, \varphi]^\text{op} = [\varphi P, \varphi^{-1}]$ for all $\varphi \in \mathcal{F}(P, S)$, the ring $A_{\mathcal{F}}(S, S)$ of $\mathcal{F}$-generated elements is stable with respect to the reflection $(-)^\text{op}$.

Any subgroup of a graph $\Delta(P, \varphi)$ with $\varphi \in \mathcal{F}(P, S)$ has the form $\Delta(R, \varphi|_R)$ for some subgroup $R \leq P$. By [5.2] we thus have $\Phi_D([P, \varphi]) = 0$ unless $D$ is the graph of a morphism in $\mathcal{F}$. An element $X \in A(S, S)_{(p)}$ is therefore $\mathcal{F}$-generated if and only if $\Phi_D([P, \varphi]) = 0$ for all subgroup $D \leq S \times S$ that are not graphs from $\mathcal{F}$.

**Definition 5.2.** For the Burnside ring of a group $A(S)$ we defined by (3.3) what it means for an $S$-set to be $\mathcal{F}$-stable. With bisets we now have both a left and a right actions, hence we get two notions of stability:

Let $\mathcal{F}_1, \mathcal{F}_2$ be fusion systems on $p$-groups $S_1, S_2$ respectively. Any $X \in A(S_1, S_2)_{(p)}$ is said to be right $\mathcal{F}_1$-stable if it satisfies

$$ (5.3) \quad X \circ [P, \varphi]^{S_1}_{P} = X \circ [P, \text{id}]^{S_1}_{P} \quad \text{inside } A(P, S_2)_{(p)}, \text{ for all } P \leq S_1 \text{ and } \varphi : P \to S_1 \text{ in } \mathcal{F}_1. $$

Similarly $X \in A(S_1, S_2)_{(p)}$ is left $\mathcal{F}_2$-stable if it satisfies

$$ (5.4) \quad [\varphi P, \varphi^{-1}]^{S_2}_{P} \circ X = [P, \text{id}]^{S_2}_{P} \circ X \quad \text{inside } A(S_1, P)_{(p)}, \text{ for all } P \leq S_2 \text{ and } \varphi : P \to S_2 \text{ in } \mathcal{F}_2. $$

Because $([P, \varphi]^\text{op} = [\varphi P, \varphi^{-1}]$ when $\varphi$ is injective, we clearly have that $X$ is right $\mathcal{F}$-stable if and only if $X^\text{op}$ is left $\mathcal{F}$-stable. For the double Burnside ring $A(S, S)_{(p)}$, any element that is both left and right $\mathcal{F}$-stable is said to be fully $\mathcal{F}$-stable or just $\mathcal{F}$-stable.

As with $\mathcal{F}$-stability in $A(S)_{(p)}$, we can characterize left and right stability in terms of the homomorphism of marks for the double Burnside ring.

**Lemma 5.3.** Let $\mathcal{F}_1, \mathcal{F}_2$ be fusion systems on $p$-groups $S_1, S_2$ respectively. The following are then equivalent for all $X \in A(S_1, S_2)_{(p)}$:

(i) $X$ is both right $\mathcal{F}_1$-stable and left $\mathcal{F}_2$-stable.

(ii) $X$ considered as an element of $A(S_2 \times S_1)_{(p)}$ is $(\mathcal{F}_2 \times \mathcal{F}_1)$-stable.

(iii) $\Phi_D(X) = \Phi_{D'}(X)$ for all subgroups $D, D' \leq S_2 \times S_1$ that are $(\mathcal{F}_2 \times \mathcal{F}_1)$-conjugate.

The analogue statements for right and left stability follow if we let $\mathcal{F}_1$ or $\mathcal{F}_2$ be trivial fusion systems.
For the purposes of this paper it would be sufficient to state lemma 5.3 and later results only for bisets where both actions are free, in which case the proof of lemma 5.3 would be easier. However, all the later proofs are nearly identical in the bifree and non-free cases, so for completeness sake we include the general statements – though the following proof becomes harder.

**Proof.** The equivalence of (ii) and (iii) is just the characterization of stability in Burnside rings (see page 9).

Suppose that \( X \in A(S_1, S_2)_{(p)} \) is both right \( F_1 \)-stable and left \( F_2 \)-stable. Let the map \( \varphi \in \text{Hom}_{F_2 \times F_1}(D, S_2 \times S_1) \) be any homomorphism in the product fusion system, and let the ring homomorphism \( \varphi^* : A(S_2 \times S_1)_{(p)} \to A(D)_{(p)} \) be the restriction along \( \varphi \). For subgroups \( D \leq C \leq S_2 \times S_1 \) we also let \( \text{incl}^C_D \) denote the inclusion of \( D \) in \( C \). We then wish to show that \( \varphi^*(X) = (\text{incl}^{S_2 \times S_1}_D)^*(X) \). Define \( D_i \) to be the projection of \( D \) to the group \( S_i \), then by definition of the product fusion system \( \varphi \) has the form \( (\varphi_2 \times \varphi_1)|_D \) for suitable morphisms \( \varphi_i \in F_i(D_i, S_i) \). The restriction homomorphism \( \varphi^* \) thus decomposes as

\[
\varphi^* : A(S_2 \times S_1)_{(p)} \xrightarrow{(\varphi_2 \times \varphi_1)^*} A(D_2 \times D_1)_{(p)} \xrightarrow{(\text{incl}^{D_2 \times D_1}_D)^*} A(D)_{(p)}.
\]

On \((S_1, S_2)\)-bisets the composition

\[
[\varphi_2 D_2, \varphi_2^{-1} |_{S_2} D_2] \circ X \circ [D_1, \varphi_1]_{S_1}^{S_1}
\]

is exactly the same as the restriction \((\varphi_2 \times \varphi_1)^*\) of \((S_2 \times S_1)\)-sets, and by the assumed stability of \( X \) we therefore get

\[
(\varphi_2 \times \varphi_1)^*(X) = [\varphi_2 D_2, \varphi_2^{-1} |_{S_2} D_2] \circ X \circ [D_1, \varphi_1]_{S_1}^{S_1}
= [D_2, id |_{D_2} D_2 \circ X \circ [D_1, id]_{D_1}^{S_1} = (\text{incl}^{S_2 \times S_1}_D)^*(X).
\]

Restricting further to \( D \), we then have \( \varphi^*(X) = (\text{incl}^{S_2 \times S_1}_D)^*(X) \) as claimed.

Suppose conversely that \( X \) is \( F_2 \times F_1 \)-stable. Then in particular we assume that \((id \times \varphi)^*(X) = (\text{incl}^{S_2 \times S_1}_D)^*(X)\) for all maps \( \varphi \in F_1(P, S_1) \), hence we have

\[
X \circ [P, \varphi]_{S_1}^{S_1} = (id \times \varphi)^*(X) = (\text{incl}^{S_2 \times S_1}_D)^*(X) = X \circ [P, id]_{S_1}^{S_1}
\]

so \( X \) is right \( F_1 \)-stable. Similarly we get that \( X \) is left \( F_2 \)-stable as well. \( \square \)

Let \( A^<(S, S)_{(p)} \) be the subring of the double Burnside ring generated by left-free bisets, i.e., the subring with basis elements \([P, \varphi]\) where \( \varphi : P \to S \) is any group homomorphism. We then define an augmentation map \( \varepsilon(X) := |X|/|S| \) for any biset \( X \). Since \( \varepsilon(X \circ Y) = |X \times S Y|/|S|^2 = |X||Y|/|S|^2 = \varepsilon(X)\varepsilon(Y) \), we get a ring homomorphism \( \varepsilon : A^<(S, S)_{(p)} \to \mathbb{Z}_{(p)} \).

**Definition 5.4.** Let \( F \) be a fusion system on a \( p \)-group \( S \). An element \( X \in A(S, S)_{(p)} \) is said to be right/left/fully \( F \)-characteristic if:

(i) \( X \) is \( F \)-generated.

(ii) \( X \) is right/left/fully \( F \)-stable respectively.

(iii) \( \varepsilon(X) \) is invertible in \( \mathbb{Z}_{(p)} \).

A fully \( F \)-characteristic element is also just called \( F \)-characteristic.
Remark 5.5. We will now give a new proof that every saturated fusion system has a fully $F$-characteristic idempotent.

To see that the characteristic idempotent for $F$ is unique, one can use the uniqueness part of Ragnarsson’s proof in [12]. Alternatively, corollary 6.6 establishes the uniqueness of $\omega_F$. Until that corollary is proved, we let $\omega_F$ denote only the particular $F$-characteristic idempotent constructed below.

Theorem B. Let $F$ be a saturated fusion system on a finite $p$-group $S$. Then there exists a (unique) fully $F$-characteristic idempotent $\omega_F \in A(S,S)(p)$, and it satisfies:

For all graphs $\Delta(P, \varphi) \leq S \times S$ with $\varphi \in F(P, S)$, we have

$$\Phi_{\Delta(P, \varphi)}(\omega_F) = \frac{|S|}{|F(P, S)|};$$

and $\Phi_D(\omega_F) = 0$ for all other subgroups $D \leq S \times S$. Consequently, if we write $\omega_F$ in the basis of $A(S, S)(p)$, we get the expression

$$\omega_F = \sum_{[\Delta(P, \varphi)] \times S \text{ with } \varphi \in F(P, S)} \frac{|S|}{\Phi_{\Delta(P, \varphi)}([P, \varphi]^S)} \left( \sum_{P \leq Q \leq S} \frac{|\{ \psi \in F(Q, S) \mid \psi|_P = \varphi \}|}{|F(Q, S)|} \cdot \mu(P, Q) \right) [P, \varphi]^S,$$

where the outer sum is taken over $(S \times S)$-conjugacy classes of subgroups, and where $\mu$ is the Möbius function for the poset of subgroups in $S$.

The general strategy of the construction is as follows: We consider the saturated fusion system $F \times F_S$ on $S \times S$, where $F_S := F_S(S)$ is the trivial fusion system on $S$. For this product fusion system we then apply the stabilization map of theorem A to $[S, id]$ and get $\beta_{\Delta(S)} \in A(F \times F_S)(p)$. By construction $\beta_{\Delta(S)}$ is only left $F$-stable, but fixed point calculations will show that $\beta_{\Delta(S)}$ is right stable as well. Finally, using lemma 4.3, we will show that $\beta_{\Delta(S)}$ is idempotent.

Alternatively, we could in theory stabilize $[S, id]$ with respect to $F \times F$, in order to immediately get a fully $F$-stable element. The fixed point formulas imply that this would give us exactly the same element $\beta_{\Delta(S)}$ as before. However, by stabilizing with respect to a larger fusion system, lemma 4.3 yields less information about the orbits of the stabilized element, hence it would be harder to show that $\beta_{\Delta(S)}$ is idempotent. This is why we use the first, asymmetric approach to the construction.

Proof. Let $F_S := F_S(S)$ denote the trivial fusion system on $S$, then $F \times F_S$ is a product of saturated fusion systems and is therefore a saturated fusion system on $S \times S$.

Next we remark that the $(F \times F_S)$-conjugates of a graph $\Delta(P, \varphi)$ with $\varphi \in F(P, S)$ are all the other graphs $\Delta(P', \psi)$ with $P' \sim_S P$ and $\psi \in F(P', S)$. Furthermore, the subgroups of the diagonal $\Delta(S) := \Delta(S, id)$ in $S \times S$ are the graphs $\Delta(P, id)$ for $P \leq S$, and consequently the subgroups of $S \times S$ that are $(F \times F_S)$-subconjugate to $\Delta(S)$ are exactly the graphs $\Delta(P, \varphi)$ with $\varphi \in F(P, S)$.

Recall that the basis element $\beta_{\Delta(S)} \in A(F \times F_S)(p)$ of proposition 4.5 is constructed by applying lemma 4.3 to the $(S, S)$-biset $[S \times S/\Delta(S)] = [S, id]$. For subgroups $D \leq S \times S$ we have $\Phi_D([S, id]) = 0$ unless $D$ is $(S \times S)$-subconjugate to $\Delta(S)$.

When applying lemma 4.3 to $[S, id]$, we can therefore use the collection of subgroups $H$ consisting of the graphs $\Delta(P, \varphi)$ with $\varphi \in F(P, S)$, since $\Phi_D([S, id]) = 0$ for all other subgroups $D \leq S \times S$. As remarked right after lemma 4.3, the stable element that the lemma constructs does not depend on the collection $H$ used. Hence we still get $\beta_{\Delta(S)}$ even
though we use a smaller collection \( \mathcal{H} \) than in section 3 (where \( \mathcal{H} \) contained all subgroups). By lemma 4.3, \( \beta_{\Delta(S)} \) then satisfies

(i) \( \Phi_D(\beta_{\Delta(S)}) = \Phi_D([S, id]) = 0 \) and \( c_D(\beta_{\Delta(S)}) = c_D([S, id]) = 0 \) for all \( D \leq S \times S \), not on the form \( \Delta(P, \varphi) \) with \( \varphi \in \mathcal{F}(P, S) \).

(ii) For all \( \Delta(P, \varphi) \) with \( \varphi \in \mathcal{F}(P, S) \), we have

\[
\sum_{[\Delta(P, \varphi)]_{S \times S} \subseteq [\Delta(P, \varphi)]_{\mathcal{F} \times \mathcal{F}}} c_{\Delta(P, \varphi)}(\beta_{\Delta(S)}) = \sum_{[\Delta(P, \varphi)]_{S \times S} \subseteq [\Delta(P, \varphi)]_{\mathcal{F} \times \mathcal{F}}} c_{\Delta(P, \varphi)}([S, id]).
\]

By proposition 4.5, the element \( \beta_{\Delta(S)} \) also satisfies

\[
\Phi_{\Delta(P, \varphi)}(\beta_{\Delta(S)}) = \frac{|\text{Hom}_{\mathcal{F} \times \mathcal{F}}(\Delta(P, \varphi), \Delta(S, id))| \cdot |S \times S|}{|\Delta(S, id)| \cdot |\text{Hom}_{\mathcal{F} \times \mathcal{F}}(\Delta(P, \varphi), S \times S)|}
\]

for all \( P \leq S \) and \( \varphi \in \mathcal{F}(P, S) \).

Property (ii) shows that \( \beta_{\Delta(S)} \in A(S, S)_{(p)} \) is a linear combination of basis elements \( [P, \varphi] \) with \( \varphi \in \mathcal{F}(P, S) \). Hence \( \beta_{\Delta(S)} \) is \( \mathcal{F} \)-generated. As a consequence of (ii), the value of the augmentation map on \( \beta_{\Delta(S)} \) is

\[
\varepsilon(\beta_{\Delta(S)}) = \sum_{[\Delta(P, \varphi)]_{S \times S} \subseteq [\Delta(P, \varphi)]_{\mathcal{F} \times \mathcal{F}}} c_{\Delta(P, \varphi)}(\beta_{\Delta(S)}) \cdot \varepsilon([P, \varphi])
\]

\[
= \sum_{[\Delta(P, \varphi)]_{S \times S} \subseteq [\Delta(P, \varphi)]_{\mathcal{F} \times \mathcal{F}}} c_{\Delta(P, \varphi)}(\beta_{\Delta(S)}) \cdot \frac{|S|}{|P|}
\]

\[
= \sum_{[P]_{S} \subseteq [\Delta(P, \varphi)]_{\mathcal{F} \times \mathcal{F}}} \left( \sum_{[\Delta(P, \varphi)]_{S \times S} \subseteq [\Delta(P, \varphi)]_{\mathcal{F} \times \mathcal{F}}} c_{\Delta(P, \varphi)}([S, id]) \right) \cdot \frac{|S|}{|P|}
\]

\[
= c_{\Delta(S, id)}([S, id]) \cdot \frac{|S|}{|S|} = 1.
\]

By construction, \( \beta_{\Delta(S)} \) is stable as an \( (S \times S) \)-set with respect to the fusion system \( \mathcal{F} \times \mathcal{F}_S \). Therefore, by lemma 5.3, \( \beta_{\Delta(S)} \) is left \( \mathcal{F} \)-stable as an element of \( A(S, S)_{(p)} \). We have thus proved that \( \beta_{\Delta(S)} \in A(S, S)_{(p)} \) is a left characteristic element for \( \mathcal{F} \).

We now consider the value of \( \Phi_{\Delta(P, \varphi)}(\beta_{\Delta(S)}) \) with \( \varphi \in \mathcal{F}(P, S) \) in more detail. First we remark that \( \Phi_{\Delta(P, \varphi)}(\beta_{\Delta(S)}) = \Phi_{\Delta(P, id)}(\beta_{\Delta(S)}) \) since \( \beta_{\Delta(S)} \) is left \( \mathcal{F} \)-stable. Then (5.5) gives us

\[
\Phi_{\Delta(P, id)}(\beta_{\Delta(S)}) = \frac{|\text{Hom}_{\mathcal{F} \times \mathcal{F}_S}(\Delta(P, id), \Delta(S, id))| \cdot |S \times S|}{|\Delta(S, id)| \cdot |\text{Hom}_{\mathcal{F} \times \mathcal{F}_S}(\Delta(P, id), S \times S)|}.
\]

The morphisms of \( \text{Hom}_{\mathcal{F} \times \mathcal{F}_S}(\Delta(P, id), S \times S) \) are the pairs \( (\varphi, c_s) \) where \( \varphi \in \mathcal{F}(P, S) \) and \( c_s \in \mathcal{F}_S(P, S) \), hence

\[
|\text{Hom}_{\mathcal{F} \times \mathcal{F}_S}(\Delta(P, id), S \times S)| = |\mathcal{F}_S(P, S)| \cdot |\mathcal{F}(P, S)|.
\]
The image of $\Delta(P, id)$ under a morphism $(\varphi, c_s) \in \text{Hom}_{\mathcal{F} \times \mathcal{F}_S}(\Delta(P, id), S \times S)$ is

$$(\varphi, c_s)(\Delta(P, id)) = \{(\varphi(g), c_s(g)) \mid g \in P\} = \Delta(P, \varphi \circ (c_s)^{-1}).$$

This image lies in $\Delta(S, id)$ if and only if $\varphi \circ (c_s)^{-1} = id$, i.e., if $\varphi = c_s$. The number of morphisms in $\text{Hom}_{\mathcal{F} \times \mathcal{F}_S}(\Delta(P, id), \Delta(S, id))$ is therefore simply $|\mathcal{F}_S(P, S)|$.

Returning to the expression for $\Phi_{P, id}(\beta_{\Delta(S)})$ we then have

$$\Phi_{\Delta(P, id)}(\beta_{\Delta(S)}) = \frac{|\mathcal{F}_S(P, S)| \cdot |S \times S|}{|\Delta(S, id)| \cdot (|\mathcal{F}_S(P, S)| \cdot |\mathcal{F}(P, S)|)} = \frac{|S|}{|\mathcal{F}(P, S)|},$$

which only depends on the $\mathcal{F}$-conjugacy class of $P$. We conclude that for all $(P, \varphi)$ with $\varphi \in \mathcal{F}(P, S)$, and $(Q, \psi)$ with $\psi \in \mathcal{F}(Q, S)$, and such that $P \sim \mathcal{F} Q$, we have

$$\Phi_{\Delta(P, \varphi)}(\beta_{\Delta(S)}) = \Phi_{\Delta(Q, \psi)}(\beta_{\Delta(S)}).$$

Recalling that $\Phi_D(\beta_{\Delta(S)}) = 0$ when $D \leq S \times S$ is not a graph $\Delta(Q, \psi)$ with $\psi \in \mathcal{F}(Q, S)$, lemma 5.3 says that $\beta_{\Delta(S)}$ is fully $\mathcal{F}$-stable and not just left $\mathcal{F}$-stable.

We have proven that $\beta_{\Delta(S)}$ is fully $\mathcal{F}$-characteristic, so we now need to show that $\beta_{\Delta(S)}$ is actually idempotent. Since $\beta_{\Delta(S)}$ is right $\mathcal{F}$-stable, we have $\beta_{\Delta(S)} \circ [P, \varphi] = \beta_{\Delta(S)} \circ [P, id]$ for all $\varphi \in \mathcal{F}(P, S)$. We can therefore calculate

$$\beta_{\Delta(S)} \circ \beta_{\Delta(S)} = \beta_{\Delta(S)} \circ \left( \sum_{[\Delta(P, \varphi)]_{S \times S} \text{ with } \varphi \in \mathcal{F}(P, S)} c_{\Delta(P, \varphi)}(\beta_{\Delta(S)}) \cdot [P, \varphi] \right)$$

$$= \sum_{[\Delta(P, \varphi)]_{S \times S} \text{ with } \varphi \in \mathcal{F}(P, S)} c_{\Delta(P, \varphi)}(\beta_{\Delta(S)}) \cdot (\beta_{\Delta(S)} \circ [P, id])$$

$$= \sum_{[P, S]} \left( \sum_{[\Delta(P, \varphi)]_{S \times S} \subseteq [\Delta(P, id)]_{\mathcal{F} \times \mathcal{F}_S}} c_{\Delta(P, \varphi)}(\beta_{\Delta(S)}) \right) \cdot (\beta_{\Delta(S)} \circ [P, id])$$

$$= \sum_{[P, S]} \left( \sum_{[\Delta(Q, \psi)]_{S \times S} \subseteq [\Delta(P, id)]_{\mathcal{F} \times \mathcal{F}_S}} c_{\Delta(P, \varphi)}([S, id]) \right) \cdot (\beta_{\Delta(S)} \circ [P, id])$$

$$= c_{\Delta(S, id)}([S, id]) \cdot (\beta_{\Delta(S)} \circ [S, id]) = \beta_{\Delta(S)};$$

so $\beta_{\Delta(S)}$ is a characteristic idempotent for $\mathcal{F}$.

Finally, proposition 4.6 gives the coefficients of $\beta_{\Delta(S)}$ in terms of the basis in $A(S, S)_{(p)}$:

$$c_{\Delta(P, \varphi)}(\beta_{\Delta(S)}) = \frac{1}{\Phi_{\Delta(P, \varphi)}([P, \varphi])] \left( \sum_{D \geq \Delta(P, \varphi)} \Phi_D(\omega_{\mathcal{F}}) \cdot \mu(\Delta(P, \varphi), D) \right)$$

$$= \frac{1}{\Phi_{\Delta(P, \varphi)}([P, \varphi])] \left( \sum_{\Delta(Q, \psi) \geq \Delta(P, \varphi)} \frac{|S|}{|\mathcal{F}(Q, S)|} \cdot \mu(\Delta(P, \varphi), \Delta(Q, \psi)) \right)$$

$$= \frac{|S|}{\Phi_{\Delta(P, \varphi)}([P, \varphi])} \left( \sum_{Q \geq P} \frac{|\left\{ \psi \in \mathcal{F}(Q, S) \mid \psi|_P = \varphi \right\}|}{|\mathcal{F}(Q, S)|} \cdot \mu(P, Q) \right),$$

which expresses the characteristic idempotent $\beta_{\Delta(S)}$ as the linear combination in the theorem. \qed
Corollary 5.6. Let $F$ be a saturated fusion system on a $p$-group $S$, and let $\omega_F$ be the characteristic idempotent constructed in theorem [B].

If $X \in A(S,T)_{(p)}$ is right $F$-stable, then $X \circ \omega_F = X$. Similarly if $X \in A(T,S)_{(p)}$ is left $F$-stable, then $\omega_F \circ X = X$.

Proof. To calculate the product $X \circ \omega_F$ when $X$ is right $F$-stable, we apply the same technique used in theorem [B] to show that $\beta_{\Delta(S)}$ is idempotent:

$$X \circ \omega_F = \sum_{\Delta(P,\varphi) | S \times S \text{ with } \varphi \in F(P,S)} c_{\Delta(P,\varphi)}(\omega_F) \cdot (X \circ [P, \varphi])$$

$$= \sum_{[P]_S} \left( \sum_{\Delta(P,\varphi) | S \times S \subseteq [\Delta(P,id), F \times F_S]} c_{\Delta(P,\varphi)}(\omega_F) \right) \cdot (X \circ [P, id])$$

$$= \sum_{[P]_S} \left( \sum_{\Delta(P,\varphi) | S \times S \subseteq [\Delta(P,id), F \times F_S]} c_{\Delta(P,\varphi)}([S, id]) \right) \cdot (X \circ [S, id]) = c_{\Delta(S, id)}([S, id]) \cdot (X \circ [S, id]) = X.$$  

From theorem [B] we have

$$\Phi_{\Delta(P,\varphi)}(\omega_F) = \frac{|S|}{|F(P,S)|} = \Phi_{\Delta(P,\varphi^{-1})}(\omega_F) = \Phi_{\Delta(P,\varphi)}(\omega_F^\text{op}),$$

which implies that $\omega_F^\text{op} = \omega_F$. Hence, if $X \in A(T,S)_{(p)}$ is left $F$-stable, then equivalently $X^\text{op} \in A(S,T)_{(p)}$ is right $F$-stable and

$$\omega_F \circ X = (X^\text{op} \circ \omega_F^\text{op})^\text{op} = (X^\text{op})^\text{op} = X$$

follows by the right $F$-stable case above. \qed

5.3. The action of the characteristic idempotent. In this section we explore how a characteristic idempotent $\omega_F$ acts by multiplication on elements of the double Burnside ring and other Burnside modules. Theorem [C] gives a precise description of the action of $\omega_F$ in terms of the fixed point maps, and in this way we recover the stabilization homomorphism of theorem [A]. The Burnside ring $A(S)_{(p)}$ is isomorphic the the Burnside module $A(1,S)_{(p)}$, and through this isomorphism the stabilization homomorphism of theorem [A] is given by multiplication with $\omega_F$ from the left.

We warm up with a result about basis elements for Burnside modules $A(S_1,S_2)_{(p)}$, where $S_1$ and $S_2$ are $p$-groups. We already know that a transitive $(S_1,S_2)$-set $(S_2 \times S_1)/D$ only depends on $D$ up to $(S_2 \times S_1)$-conjugation, and now we show that when we multiply $(S_2 \times S_1)/D$ by characteristic idempotents the result only depends on the subgroup $D$ up to conjugation in the corresponding saturated fusion systems.

Lemma 5.7. Let $F_1$ and $F_2$ be saturated fusion systems on the $p$-groups $S_1$ and $S_2$ respectively, and let $\omega_1 \in A(S_1,S_1)_{(p)}$ and $\omega_2 \in A(S_2,S_2)_{(p)}$ be their respective characteristic idempotents.

Then for all subgroups $D,C \leq S_2 \times S_1$, if $D$ and $C$ are conjugate in $F_2 \times F_1$, we have

$$\omega_2 \circ [(S_2 \times S_1)/D] \circ \omega_1 = \omega_2 \circ [(S_2 \times S_1)/C] \circ \omega_1$$

in $A(S_1,S_2)_{(p)}$. 

Proof. Suppose that the subgroups $D, C \leq S_2 \times S_1$ are conjugate in $F_2 \times F_1$, and let $\varphi \in \text{Hom}_{F_2 \times F_1}(D, C)$ be an isomorphism.

By definition of $F_2 \times F_1$, the homomorphism $\varphi$ extends to $(\varphi_2 \times \varphi_1): D_2 \times D_1 \to C_2 \times C_1$ where $D_1$ is the projection of $D$ onto $S_1$, similarly for $C_1$, and where $\varphi_i \in F_i(D_i, C_i)$. By assumption, $\varphi$ is invertible in $F_2 \times F_1$, hence the inverse $\varphi^{-1}$ also extends to a homomorphism $C_2 \times C_1 \to D_2 \times D_1$, which shows that $\varphi_1$ and $\varphi_2$ are invertible in $F_1$ and $F_2$ respectively. With this we have

$$[(S_2 \times S_1)/D] = [D_2, id_{D_2}^S \circ [(D_2 \times D_1)/D] \circ [D_1, id_{D_1}^S]$$

$$= [D_2, id_{D_2}^S \circ [C_2, \varphi^{-1}_{D_2}]_{C_2} \circ [(C_2 \times C_1)/C] \circ [D_1, \varphi_1]^C_{D_1} \circ [D_1, id_{D_1}^S]$$

$$= [C_2, \varphi^{-1}_{D_2}]_{C_2} \circ [(C_2 \times C_1)/C] \circ [D_1, \varphi_1]^C_{D_1}. $$

Since $\omega_2$ is right $F_2$-stable, and $\omega_1$ is left $F_1$-stable, it follows that

$$\omega_2 \circ [(S_2 \times S_1)/D] \circ \omega_1 = \omega_2 \circ [C_2, \varphi^{-1}_{D_2}]_{C_2} \circ [(C_2 \times C_1)/C] \circ [D_1, \varphi_1]^C_{D_1} \circ \omega_1$$

$$= \omega_2 \circ [C_2, id_{C_2}^S \circ [(C_2 \times C_1)/C] \circ [C_1, id_{C_1}^S] \circ \omega_1$$

$$= \omega_2 \circ [(S_2 \times S_1)/C] \circ \omega_1. $$

\[Theorem\ C.\] Let $F_1$ and $F_2$ be saturated fusion systems on finite $p$-groups $S_1$ and $S_2$ respectively, and let $\omega_1 \in A(S_1, S_1)(p)$ and $\omega_2 \in A(S_2, S_2)(p)$ be their characteristic idempotents as constructed earlier.

For every element of the Burnside module $X \in A(S_1, S_2)(p)$, the product $\omega_2 \circ X \circ \omega_1$ is right $F_1$-stable and left $F_2$-stable, and satisfies

$$\Phi_D(\omega_2 \circ X \circ \omega_1) = \frac{1}{|[D]_{F_2 \times F_1}|} \sum_{D' \in [D]_{F_2 \times F_1}} \Phi_{D'}(X),$$

for all subgroups $D \leq S_2 \times S_1$, where $[D]_{F_2 \times F_1}$ is the isomorphism class of $D$ in the product fusion system $F_2 \times F_1$ on $S_2 \times S_1$.

Note that by the fixed point formula, theorem [C] states that the map $X \mapsto \omega_{F_2} \circ X \circ \omega_{F_1}$ coincides with the stabilization map $A(S_2 \times S_1)(p) \to A(F_2 \times F_1)(p)$ of theorem [A].

Proof. Any product $\omega_2 \circ X \circ \omega_1$ is right $F_1$-stable by definition since $\omega_1$ is right $F_1$-stable, and similarly we see that $\omega_2 \circ X \circ \omega_1$ is left $F_2$-stable.

Consider the element $X \in A(S_1, S_2)(p)$ as an element of $A(S_2 \times S_1)(p)$. The fusion system $F_2 \times F_1$ on $S_2 \times S_1$ is saturated by [4, Lemma 1.5], and we apply theorem [A] to get an $(F_2 \times F_1)$-stable element $\pi X$ satisfying

$$\Phi_D(\pi X) := \frac{1}{|[D]_{F_2 \times F_1}|} \sum_{D' \in [D]_{F_2 \times F_1}} \Phi_{D'}(X)$$

for all $D \leq S_2 \times S_1$. By lemma [5.3], $\pi X$ is left $F_2$-stable and right $F_1$-stable when considered as an element $X \in A(S_1, S_2)(p)$.

Furthermore, by remark [4.4], $\pi X$ also satisfies

$$\sum_{[D']_{s_2 \times s_1} \subseteq [D]_{F_2 \times F_1}} c_{D'}(\pi X) = \sum_{[D']_{s_2 \times s_1} \subseteq [D]_{F_2 \times F_1}} c_{D'}(X),$$

or equivalently

$$\sum_{[D']_{s_2 \times s_1} \subseteq [D]_{F_2 \times F_1}} c_{D'}(\pi X - X) = 0.$$
Using lemma 5.7 we then have
\[ \omega_2 \circ (\pi X - X) \circ \omega_1 = \sum_{|D|F_2 \times F_1} \left( \sum_{|D'|S_2 \times S_1 \subseteq |D|F_2 \times F_1} c_{D'}(\pi X - X) \cdot (\omega_2 \circ (S_2 \times S_1/D') \circ \omega_1) \right) \]
\[ = \sum_{|D|F_2 \times F_1} \left( \sum_{|D'|S_2 \times S_1 \subseteq |D|F_2 \times F_1} c_{D'}(\pi X - X) \right) \cdot (\omega_2 \circ (S_2 \times S_1/D) \circ \omega_1) \]
\[ = \sum_{|D|F_2 \times F_1} 0 \cdot (\omega_2 \circ (S_2 \times S_1/D) \circ \omega_1) = 0. \]

From which we conclude
\[ \omega_2 \circ X \circ \omega_1 = \omega_2 \circ \pi X \circ \omega_1 = \pi X, \]
where the last equality holds by corollary 5.6 since \( \pi X \) is left \( F_2 \)-stable and right \( F_1 \)-stable.

\[ \square \]

**Corollary 5.8.** Let \( F \) be a saturated fusion system on a \( p \)-group \( S \). A set with a left action of \( S \) is the same as a \((1, S)\)-biset, so the Burnside module \( A(1, S)_{(p)} \) is isomorphic to the Burnside ring \( A(S)_{(p)} \). Through this isomorphism left multiplication with \( \omega_F \) in \( A(1, S)_{(p)} \) coincides with the stabilization homomorphism \( \pi: A(S)_{(p)} \rightarrow A(F)_{(p)} \) of theorem A.

**Proof.** The subgroups of \( S \times 1 \) are all on the form \( Q \times 1 \) for som \( Q \leq S \), and the characteristic idempotent for the unique fusion system on the trivial group is just \([1, id]_1 = [pt]_1\).

By theorem C we then have
\[ \Phi_{Q \times 1}(\omega_F \circ X) = \frac{1}{|Q|F} \sum_{Q' \in [Q]_F} \Phi_{Q' \times 1}(X) = \Phi_Q(\pi(X)) \]
for all \( X \in A(1, S)_{(p)} \), so \( \omega_F \circ X = \pi(X) \) as claimed.

\[ \square \]

**Definition 5.9.** For saturated fusion systems \( F_1, F_2 \) on \( p \)-groups \( S_1, S_2 \), we define the **Burnside module** \( A(F_1, F_2)_{(p)} \) as the \( \mathbb{Z}(p)\)-submodule of \( A(S_1, S_2)_{(p)} \) consisting of the elements that are right \( F_1 \)-stable and left \( F_2 \)-stable.

The elements \( \omega_{F_2} \circ [(S_2 \times S_1)/D] \circ \omega_{F_1} \) generate \( A(F_1, F_2)_{(p)} \) over \( \mathbb{Z}(p) \). By the fixed point calculation of theorem C the element \( \omega_{F_2} \circ [(S_2 \times S_1)/D] \circ \omega_{F_1} \) actually corresponds to the element \( \beta_D \in A(F_2 \times F_1)_{(p)} \), so it follows that the elements \( \{ \omega_{F_2} \circ [(S_2 \times S_1)/D] \circ \omega_{F_1} \mid D \leq S_2 \times S_1 \} \) form a \( \mathbb{Z}(p) \)-basis for the Burnside module \( A(F_1, F_2)_{(p)} \). Two subgroups \( C \) and \( D \) give the same basis element if and only if \( C \) and \( D \) are conjugate in \( F_2 \times F_1 \). The existence of such basis elements nicely generalizes the basis we have for the Burnside modules of groups.

As for groups, the Burnside modules \( A(F_1, F_2)_{(p)} \) form the Hom-sets of a category where the objects are all saturated fusion systems on \( p \)-groups. We define the \((p\text{-localized})\) **double Burnside ring** of a saturated fusion system \( F \) to be the ring \( A(F, F)_{(p)} \). The double Burnside ring \( A(F, F)_{(p)} \) has a 1-element: It is simply the characteristic idempotent \( \omega_F = \omega_F \circ [S, id] \circ \omega_F \), which is also one of the \( \mathbb{Z}(p) \)-basis elements.
Multiplication with characteristic idempotents $\omega_1$ and $\omega_2$ defines a map $A(S_1,S_2)_{(p)} \to A(\mathcal{F}_1,\mathcal{F}_2)_{(p)}$. In the spirit of theorem \ref{thm:transfer_and_characteristic}, we proceed to show that this map is a homomorphism of modules.

**Proposition 5.10.** Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be saturated fusion systems on $p$-groups $S_1$ and $S_2$ respectively, and let $\omega_1 \in A(S_1,S_1)_{(p)}$ and $\omega_2 \in A(S_2,S_2)_{(p)}$ be their characteristic idempotents as constructed earlier.

Then the map $\pi: A(S_1,S_2)_{(p)} \to A(\mathcal{F}_1,\mathcal{F}_2)_{(p)}$ given by $\pi(X) := \omega_2 \circ X \circ \omega_1$ is a homomorphism of left $A(\mathcal{F}_2,\mathcal{F}_2)_{(p)}$-modules and right $A(\mathcal{F}_1,\mathcal{F}_1)_{(p)}$-modules.

**Proof.** We only show that $\pi$ is a homomorphism of right $A(\mathcal{F}_1,\mathcal{F}_1)_{(p)}$-modules, since the other case is similar.

Let $X \in A(S_1,S_2)_{(p)}$ be given, and let $Z \in A(\mathcal{F}_1,\mathcal{F}_1)_{(p)}$ be a fully $\mathcal{F}_1$-stable element of $A(S_1,S_1)_{(p)}$. Then $\mathcal{F}_1$-stability ensures that $\omega_1 \circ Z = Z \circ \omega_1 = Z$ by corollary \ref{cor:transfer_and_characteristic}, hence we get

$$\pi(X \circ Z) = \omega_2 \circ X \circ Z \circ \omega_1 = \omega_2 \circ X \circ \omega_1 \circ Z = \pi(X) \circ Z.$$ 

\qed

6. **The Burnside ring embeds in the double Burnside ring**

In this section we show that the “one-sided” Burnside ring $A(\mathcal{F})_{(p)}$ of sections \ref{sec:burnside} and \ref{sec:transfer_and_characteristic} always embeds in the double Burnside ring $A(\mathcal{F},\mathcal{F})_{(p)}$ for $\mathcal{F}$ defined above. In fact, theorem \ref{thm:transfer_and_characteristic} states that $A(\mathcal{F})_{(p)}$ is isomorphic to the subring generated by all $\mathcal{F}$-characteristic elements. Through this isomorphism we can describe the structure of the $\mathcal{F}$-characteristic elements, and in particular we prove that there is only one $\mathcal{F}$-characteristic idempotent.

The isomorphism between the “one-sided” Burnside ring and a subring of the double Burnside ring is inspired by a similar result for finite groups, where the Burnside ring $A(G)$ embeds in the double Burnside ring $A(G,G)$. Let us therefore first analyze the situation for Burnside rings of $p$-groups and see what might be generalized to fusion systems:

**Example 6.1.** Let $S$ be a finite $p$-group. Recall the double coset formula for the multiplication of basis elements in $A(S)$:

$$[S/P] \cdot [S/Q] = \sum_{\pi \in P\backslash S/Q} [S/(P \cap \pi Q)].$$

If we then consider the bisets $[P, id]_S$ and $[Q, id]_S$ for subgroup $P, Q \leq S$ in the double Burnside ring for $S$, then the double coset formula \ref{eq:double_coset_formula} for $A(S,S)$ shows us that

$$[P, id]_S \circ [Q, id]_S = \sum_{\pi \in P \cap S/Q} [P^\pi \cap Q, c_\pi]_S = \sum_{\pi \in P \cap S/Q} [P \cap \pi Q, id]_S.$$

If we compare the two formulas, we discover that the basis elements $[S/P]$ in $A(S)$ and the basis elements $[P, id]_S$ in $A(S,S)$ satisfy exactly the same multiplication formula. Hence we get an injective ring homomorphism $\iota: A(S) \to A(S,S)_{(p)}$ mapping $[S/P] \mapsto [P, id]$, which embeds $A(S)$ as the subring of $A(S,S)$ generated by $[P, id]$ for $P \leq S$.

The basis elements $[P, id]$ are precisely the basis elements $[(S \times S)/D]$ for $D = \Delta(P, c_P)$ the graph of an $S$-conjugation map — recall that the subgroup $\Delta(P, c_P)$ is only determined up to $(S \times S)$-conjugation, so $[P, c_P] = [P, id]$. The subring generated by $[P, id]$ for $P \leq S$, is therefore the ring $A_{\mathcal{F}_S}(S,S)$ of all $\mathcal{F}_S$-generated elements (see definition \ref{def:fas}), where $\mathcal{F}_S$ is the trivial fusion system on $S$. This suggests that we should consider the $\mathcal{F}$-generated elements for general fusion systems.
Finally the inverse of $\iota$ is the map $q: A_{\mathcal{F}}(S, S) \to A(S)$ given on bisets by $X \mapsto X/S$. Here we eliminate the right $S$-action by quotienting out, equivalently this can be expressed by the multiplication $X \mapsto X \circ \left( (S \times 1)/(S \times 1) \right)$ from $A_{\mathcal{F}}(S, S)$ to $A(1, S)$. To see that this map is the inverse of $\iota$ we simply examine the basis elements and note that

$$q([P, id]_S^S) = q([S \times P S]) = [(S \times P S)/S] = [S/P].$$

It is not clear that $q$ preserves the multiplication, but this must be true since $q = \iota^{-1}$.

A similar situation occurs in Theorem D: We state the theorem for the nice map $\iota$ as the subring formed by all elements that are $S$-stable and a product of $F$-generated elements (hence $\iota$ as ring homomorphism from the start).

**Definition 6.2.** For a saturated fusion system $\mathcal{F}$ on $S$, the double Burnside ring $A(\mathcal{F}, \mathcal{F})_{(p)}$ was defined to be the subring of $A(S, S)_{(p)}$ consisting of the elements that are both left and right $\mathcal{F}$-stable. Example 6.1 suggests that we should look at those elements of $A(\mathcal{F}, \mathcal{F})_{(p)}$ that are in addition $\mathcal{F}$-generated. We therefore define

$$A_{\text{char}}(\mathcal{F})_{(p)} := A(\mathcal{F}, \mathcal{F})_{(p)} \cap A_S(S, S)_{(p)}$$

as the subring formed by all elements that are $\mathcal{F}$-stable as well as $\mathcal{F}$-generated. Hence we have a sequence of inclusions of subrings

$$A_{\text{char}}(\mathcal{F})_{(p)} \subseteq A(\mathcal{F}, \mathcal{F})_{(p)} \subseteq A(S, S)_{(p)}.$$

The last inclusion is not unital since $\omega_{\mathcal{F}}$ is the multiplicative identity of the first two rings, and $[S, id]_S^S$ is the identity of $A(S, S)_{(p)}$.

We use the notation $A_{\text{char}}(\mathcal{F})_{(p)}$ for this particular subring because the following proposition shows that $A_{\text{char}}(\mathcal{F})_{(p)}$ is generated, over $\mathbb{Z}_{(p)}$, by all the $\mathcal{F}$-characteristic elements in $A(S, S)_{(p)}$. Note that not all elements of $A_{\text{char}}(\mathcal{F})_{(p)}$ are $\mathcal{F}$-characteristic, but the non-characteristic elements of $A_{\text{char}}(\mathcal{F})_{(p)}$ are few, and they form a proper $\mathbb{Z}_{(p)}$-submodule.

**Proposition 6.3.** Let $\mathcal{F}$ be a saturated fusion systems on a $p$-group $S$, and let $A_{\text{char}}(\mathcal{F})_{(p)}$ be defined as above. Then $A_{\text{char}}(\mathcal{F})_{(p)}$ is also the subring of $A(S, S)_{(p)}$ generated by the $\mathcal{F}$-characteristic elements, and it has a $\mathbb{Z}_{(p)}$-basis consisting of the elements $\beta_{\Delta(P, id)} = \omega_{\mathcal{F}} \circ [P, id] \circ \omega_{\mathcal{F}}$, which are in one-to-one correspondence with the $\mathcal{F}$-conjugacy classes of subgroup $P \leq S$.

The characteristic elements of $\mathcal{F}$ are those elements $X \in A_{\text{char}}(\mathcal{F})_{(p)}$ where the coefficient of $X$ at the basis element $\beta_{\Delta(S, id)} = \omega_{\mathcal{F}}$ is invertible in $\mathbb{Z}_{(p)}$.

**Proof.** We first claim that $A_{\text{char}}(\mathcal{F})_{(p)} = \omega_{\mathcal{F}} \circ A_{\mathcal{F}}(S, S)_{(p)} \circ \omega_{\mathcal{F}}$, where $A_{\mathcal{F}}(S, S)_{(p)}$ is the subring of $\mathcal{F}$-generated elements in $A(S, S)_{(p)}$. Each element in $\omega_{\mathcal{F}} \circ A_{\mathcal{F}}(S, S)_{(p)} \circ \omega_{\mathcal{F}}$ is $\mathcal{F}$-stable and a product of $\mathcal{F}$-generated elements (hence $\mathcal{F}$-generated as well), so it is contained in $A_{\text{char}}(\mathcal{F})_{(p)}$.

Conversely, suppose $X \in A_{\text{char}}(\mathcal{F})_{(p)}$. Because $X$ is $\mathcal{F}$-stable, we have $X = \omega_{\mathcal{F}} \circ X \circ \omega_{\mathcal{F}}$ by corollary 5.6, so $X$ lies in the product $\omega_{\mathcal{F}} \circ A_{\mathcal{F}}(S, S)_{(p)} \circ \omega_{\mathcal{F}}$. We conclude that we have $A_{\text{char}}(\mathcal{F})_{(p)} = \omega_{\mathcal{F}} \circ A_{\mathcal{F}}(S, S)_{(p)} \circ \omega_{\mathcal{F}}$ as claimed.

We know that $A_{\mathcal{F}}(S, S)_{(p)}$ is generated by the sets $[P, \varphi]$ with $\varphi \in \mathcal{P}(P, S)$ by definition. Hence $A_{\text{char}}(\mathcal{F})_{(p)}$ is generated by the elements $\omega_{\mathcal{F}} \circ [P, \varphi] \circ \omega_{\mathcal{F}}$ with $\varphi \in \mathcal{P}(P, S)$, and by lemma 5.7 we have $\omega_{\mathcal{F}} \circ [P, \varphi] \circ \omega_{\mathcal{F}} = \omega_{\mathcal{F}} \circ [P, id] \circ \omega_{\mathcal{F}} = \beta_{\Delta(P, id)}$ as an element of $A(\mathcal{F} \times \mathcal{F})_{(p)}$. So the elements $\beta_{\Delta(P, id)}$ generate $A_{\text{char}}(\mathcal{F})_{(p)}$ and are linearly independent over $\mathbb{Z}_{(p)}$ since they are already part of a basis for the double Burnside ring $A(\mathcal{F}, \mathcal{F})_{(p)}$. 

Two basis elements \( \beta_{\Delta(P,id)} \) and \( \beta_{\Delta(Q,id)} \) are equal exactly when \( \Delta(P,id) \) and \( \Delta(Q,id) \) are \((\mathcal{F} \times \mathcal{F})\)-conjugate, which happens if and only if \( P \) and \( Q \) are \( \mathcal{F} \)-conjugate.

The elements \( X \in A^{\text{char}}(\mathcal{F})_{(p)} \) are already \( \mathcal{F} \)-stable and \( \mathcal{F} \)-generated, so the only extra condition that \( \mathcal{F} \)-characteristic elements must satisfy is that \( \varepsilon(X) \) is invertible in \( \mathbb{Z}_{(p)} \), i.e., \( \varepsilon(X) \not\equiv 0 \text{ (mod } p) \) in \( \mathbb{Z}_{(p)} \). Any basis element of \( A^{\text{char}}(\mathcal{F})_{(p)} \) other than \( \beta_{\Delta(S,id)} \) is of the form \( \omega_{\mathcal{F}} \circ [P, \text{incl}] \circ \omega_{\mathcal{F}} \) with \( P < S \). Because \( \varepsilon(\omega_{\mathcal{F}}) = 1 \), by the proof of theorem B we therefore have
\[
\varepsilon(\omega_{\mathcal{F}} \circ [P, \text{incl}] \circ \omega_{\mathcal{F}}) = 1 \cdot \varepsilon([P, \text{incl}]) \cdot 1 = \frac{|S|}{|P|} \equiv 0 \text{ (mod } p)\]
for all \( P < S \). So whether \( \varepsilon(X) \not\equiv 0 \text{ (mod } p) \) depends only on the coefficient of \( X \) at the basis element \( \beta_{\Delta(S,id)} = \omega_{\mathcal{F}} \).

**Lemma 6.4.** Let \( \iota^S : A(S)_{(p)} \to A(S,S)_{(p)} \) be the injective ring homomorphism of example 6.1 given by \([S/P] \mapsto [P, id]\). For every \( X \in A(S)_{(p)} \) and subgroup \( D \leq S \times S \), we have
\[
\Phi_D(\iota^S(X)) = 0 \text{ unless } D \text{ is } (S \times S)\text{-subconjugate to } \Delta(Q,id) \text{ for some } Q \leq S. \text{ In that case }
\[
\Phi_{\Delta(Q,id)}(\iota^S(X)) = \Phi_Q([S/P]) \cdot |C_S(Q)|.
\]
Furthermore, \( \iota^S(X) \) is symmetric for all \( X \in A(S)_{(p)} \), i.e., \( \iota^S(X)^{op} = \iota^S(X) \).

**Proof.** By linearity in \( X \in A(S)_{(p)} \), it is enough to prove the lemma for basis elements \([S/P] \in A(S)_{(p)}\), where \( P \leq S \). The symmetry is obvious since \( \iota^S([S/P]) = [P, id] \), which is symmetric.

Since \( \iota^S([S/P]) = [P, id] \), we apply the formula [32] for the fixed-point homomorphisms on basis elements: For \( D \leq S \times S \) we have \( \Phi_D([P, id]) = 0 \) unless \( D \) is \((S \times S)\)-subconjugate to \( \Delta(P, id) \). The subgroups of \( \Delta(P, id) \) are \( \Delta(R, id) \) for \( R \leq P \), hence \( D \) has to be of the form \( \Delta(Q, c_s) \) for \( Q \leq S \) and \( s \in S \), which is \((S \times S)\)-conjugate to \( \Delta(Q, id) \). For the graph \( \Delta(Q, id) \) we then have
\[
\Phi_{\Delta(Q,id)}(\iota^S([S/P])) = \frac{|N_{S \times S}(\Delta(Q,id), \Delta(P,id))|}{|\Delta(P,id)|} = \frac{|\{(s,t) \mid s,t \in N_S(Q,P) \text{ and } c_s = c_t \in \text{Hom}_S(Q,P)\}|}{|P|} = \frac{|N_S(Q,P)|}{|P|} \cdot |C_S(Q)| = \Phi_Q([S/P]) \cdot |C_S(Q)|. \]

**Lemma 6.5.** Let \( \mathcal{F} \) be a saturated fusion system on a \( p \)-group \( S \). For all basis elements \( \beta_P \in A(\mathcal{F})_{(p)} \) it holds that
\[
\omega_{\mathcal{F}} \circ \iota^S(\beta_P) \circ \omega_{\mathcal{F}} = \omega_{\mathcal{F}} \circ \iota^S(\beta_P) = \iota^S(\beta_P) \circ \omega_{\mathcal{F}} = \beta_{\Delta(P,id)}.
\]

By linearity, we get for all \( X \in A(\mathcal{F})_{(p)} \) that \( \omega_{\mathcal{F}} \circ \iota^S(X) \circ \omega_{\mathcal{F}} = \omega_{\mathcal{F}} \circ \iota^S(X) = \iota^S(X) \circ \omega_{\mathcal{F}} \).

**Proof.** Because the basis element \( \beta_{\Delta(P,id)} \in A^{\text{char}}(\mathcal{F})_{(p)} \) is \( \mathcal{F} \)-generated, we have that \( \Phi_D(\beta_{\Delta(P,id)}) = 0 \) unless \( D \) has the form \( \Delta(Q, \psi) \) with \( \psi \in \mathcal{F}(Q,S) \), and because \( \beta_{\Delta(P,id)} \) is \( \mathcal{F} \)-stable, we have \( \Phi_{\Delta(Q,\psi)}(\beta_{\Delta(P,id)}) = \Phi_{\Delta(Q,id)}(\beta_{\Delta(P,id)}) \) when \( \psi \in \mathcal{F}(Q,S) \). Considered as an element of \( A(\mathcal{F} \times \mathcal{F})_{(p)} \) we know these fixed point values from proposition 4.5.
For the product \( \omega_F \circ \iota^S(\beta_p) \) we apply theorem C to give us

\[
\Phi_{\Delta(Q, \psi)}(\omega_F \circ \iota^S(\beta_p)) = \sum_{\Delta(Q, \psi') \in \Delta(Q, \psi)} \Phi_{\Delta(Q, \psi')}(\iota^S(\beta_p)),
\]

where \( F_S \) is the trivial fusion system on \( S \). By lemma 6.4, \( \Phi_{\Delta(Q, \psi')}(\iota^S(\beta_p)) = 0 \) unless \( \Delta(Q', \psi') \) is \((S \times S)\)-conjugate to \( \Delta(Q', id) \). Since \( Q' \sim_S Q \) for all subgroups \( \Delta(Q', \psi') \in \Delta(Q, \psi) \), we conclude that all summands are zero unless \( \Delta(Q, id) \in \Delta(Q, \psi) \). Hence \( \Delta(Q, \psi) \) should be conjugate to \( \Delta(Q, id) \) inside \( F \times F_S \), i.e., \( \psi \) must lie in \( F \).

In this case, by left \( F \)-stability of \( \omega_F \circ \iota^S(\beta_p) \), we have

\[
\Phi_{\Delta(Q, \psi)}(\omega_F \circ \iota^S(\beta_p)) = \Phi_{\Delta(Q, id)}(\omega_F \circ \iota^S(\beta_p)).
\]

We still get \( \Phi_{\Delta(Q', \psi')}(\iota^S(\beta_p)) = 0 \) unless \( \Delta(Q', \psi') \) is actually \((S \times S)\)-conjugate to \( \Delta(Q', id) \) and \( \Delta(Q, id) \). In the calculation of \( \Phi_{\Delta(Q, id)}(\omega_F \circ \iota^S(\beta_p)) \) we can therefore omit all the summands that are zero, and we get

\[
\Phi_{\Delta(Q, \psi)}(\omega_F \circ \iota^S(\beta_p)) = \Phi_{\Delta(Q, id)}(\omega_F \circ \iota^S(\beta_p))
\]

\[
= \frac{1}{|\Delta(Q, id)|_{F \times F_S}} \sum_{\Delta(Q', \psi') \in \Delta(Q, id)} \Phi_{\Delta(Q', \psi')}(\iota^S(\beta_p))
\]

\[
= \frac{|\Delta(Q, id)|_{S \times S}}{|\Delta(Q, id)|_{F \times F_S}} \cdot \Phi_{\Delta(Q, id)}(\iota^S(\beta_p))
\]

\[
= \frac{|\text{Hom}_{S \times S}(\Delta(Q, id), S \times S)| \cdot |\text{Aut}_{F \times F_S}(\Delta(Q, id))|}{|\text{Aut}_{S \times S}(\Delta(Q, id))| \cdot |\text{Hom}_{F \times F_S}(\Delta(Q, id), S \times S)|} \cdot \Phi_Q(\beta_P) \cdot |C_S(Q)|
\]

\[
= \frac{|\text{Hom}_{S \times S}(\Delta(Q, id), S \times S)|}{|\text{Aut}_{S \times S}(\Delta(Q, id))|} \cdot \Phi_Q(\beta_P) \cdot \frac{|S|}{|F(Q, S)|} \cdot \frac{|C_S(Q)|}{|C_S(Q)|}
\]

\[
= \Phi_Q(\beta_P) \cdot \frac{|S|}{|F(Q, S)|} = \Phi_{\Delta(Q, \psi)}(\beta_{\Delta(p, id)}).
\]

This shows that \( \omega_F \circ \iota^S(\beta_p) = \beta_{\Delta(p, id)} \); and by symmetry we have

\[
\beta_{\Delta(p, id)} = (\beta_{\Delta(p, id)})^{op} = (\omega_F \circ \iota^S(\beta_p))^{op} = \iota^S(\beta_P)^{op} \circ \omega_F^{op} = \iota^S(\beta_P) \circ \omega_F.
\]

Finally, \( \omega_F \circ (\iota^S(\beta_P) \circ \omega_F) = \omega_F \circ (\omega_F \circ \iota^S(\beta_P)) = \omega_F \circ \iota^S(\beta_P) \).

\[\square\]

**Theorem D.** Let \( F \) be a saturated fusion system on a finite \( p \)-group \( S \).

Then the collapse map \( q \colon A^{\text{char}}(F)_{(p)} \to A(F)_{(p)} \), which quotients out the right \( S \)-action, is an isomorphism of rings, and it sends the basis element \( \beta_{\Delta(p, id)} \) of \( A(F, F)_{(p)} \) to the basis element \( \beta_P \) of \( A(F)_{(p)} \).

**Proof.** For a biset \( X \) the quotient \( X/S \) is the same as the product \( X \times_S pt \), so the collapse map \( q \colon A(S, S)_{(p)} \to A(S)_{(p)} \) is alternatively given as right-multiplication with the one-point \((1, S)\)-biset \([pt]_1^S \). The one-point biset has \( \Phi_D([pt]_1^S) = 1 \) for all \( D \leq S \times 1 \), and by theorem C we then also have \( \Phi_D(\omega_F \circ [pt]_1^S) = 1 \) for all \( D \leq S \times 1 \), so \( \omega_F \circ [pt]_1^S = [pt]_1^S \).


If we apply the collapse map \( q \) to the basis elements \( \beta_{\Delta(p, id)} = \omega_F \circ [P, id] \circ \omega_F \) of \( A^{\text{char}}(F)_{(p)} \) we therefore get

\[
q(\beta_{\Delta(p, id)}) = \omega_F \circ [P, id]^S_F \circ \omega_F \circ [pt]^S_F = \omega_F \circ [P/id]^S_F = \omega_F \circ [S/P]^S_F.
\]

By corollary 6.5 multiplication with \( \omega_F \) in \( A(1, S)_{(p)} \) is the same as the stabilization map of theorem A, so \( q(\beta_{\Delta(p, id)}) = \omega_F \circ [S/P]^S_F = \beta_p \) as elements of \( A(S)_{(p)} \).

Now we define a \( \mathbb{Z}(p) \)-homomorphism \( v^F : A(F)_{(p)} \to A^{\text{char}}(F)_{(p)} \) by

\[
v^F(X) = \omega_F \circ v^S(X) \circ \omega_F,
\]

and by lemma 6.5 we then have \( v^F(\beta_p) = \beta_{\Delta(p, id)} \). Because \( q \) sends \( \beta_{\Delta(p, id)} \in A^{\text{char}}(F)_{(p)} \) to \( \beta_p \in A(F)_{(p)} \), and \( v^F \) sends it back again, the two maps \( q \) and \( v^F \) are inverse isomorphisms of \( \mathbb{Z}(p) \)-modules \( A(F)_{(p)} \) and \( A^{\text{char}}(F)_{(p)} \).

Finally, we recall that \( v^S \) is a ring homomorphism, and apply lemma 6.5 to show that all elements \( X, Y \in A(F)_{(p)} \) satisfy

\[
(\omega_F \circ v^S(X) \circ \omega_F) \circ (\omega_F \circ v^S(Y) \circ \omega_F) = \omega_F \circ v^S(X) \circ v^S(Y) \circ \omega_F = \omega_F \circ v^S(XY) \circ \omega_F.
\]

Hence \( v^F \) preserves multiplication, and consequently the inverse \( q : A^{\text{char}}(F)_{(p)} \to A(F)_{(p)} \) does as well.

We now apply the ring isomorphism \( A^{\text{char}}(F)_{(p)} \cong A(F)_{(p)} \) to determine all idempotents of \( A^{\text{char}}(F)_{(p)} \). In particular, this finally completes the proof that a saturated fusion system \( F \) has exactly one characteristic idempotent.

**Corollary 6.6.** Let \( F \) be a saturated fusion system on a finite \( p \)-group \( S \). The only idempotents of \( A(F)_{(p)} \) are 0 and the 1-element \([S/S]\). Hence it follows that \( A^{\text{char}}(F)_{(p)} \) has exactly one non-zero idempotent, hence the characteristic idempotent \( \omega_F \) is unique.

**Proof.** By proposition 3.2 the Burnside ring \( A(F)_{(p)} \) fits in a short-exact sequence of \( \mathbb{Z}(p) \)-modules

\[
0 \to A(F)_{(p)} \xrightarrow{\Phi} \tilde{\Omega}(F)_{(p)} \xrightarrow{\Psi} Obs(F)_{(p)} \to 0.
\]

Here \( \Phi \) is the mark homomorphism, \( Obs(F)_{(p)} \) is the group

\[
Obs(F)_{(p)} = \prod_{\substack{P \in \mathcal{F}(F) \setminus \mathcal{F}(p) \\ P \text{ 1-norm.}}} (\mathbb{Z}/|W_S P|\mathbb{Z}),
\]

and \( \Psi \) is given by the \([P]_{\mathcal{F}} \)-coordinate functions

\[
\Psi_P(\xi) := \sum_{\pi \in W_S P} \xi_{(s)P} \pmod{|W_S P|},
\]

when \( P \) is fully \( \mathcal{F} \)-normalized, and \( \xi_{(s)P} \) denotes the \([<s>P]_{\mathcal{F}} \)-coordinate of an element \( \xi \in \tilde{\Omega}(F)_{(p)} = \prod_{P \in \mathcal{F}(F)} \mathbb{Z}(p) \).

Let \( \omega \) be an idempotent in \( A(F)_{(p)} \), then since \( \Phi \) is a ring homomorphism, the fixed point vector \( \Phi(\omega) \) must be idempotent in the product ring \( \tilde{\Omega}(F)_{(p)} \). Since \( \Phi(\omega) \) is an element of a product ring, it is idempotent if and only if each coordinate \( \Phi_Q(\omega) \) is idempotent in \( \mathbb{Z}(p) \). The only idempotents of \( \mathbb{Z}(p) \) are 0 and 1, so \( \omega \in A(F)_{(p)} \) is idempotent if and only if we have \( \Phi_Q(\omega) \in \{0, 1\} \) for all \( Q \leq S \).
Let the top coordinate $\Phi_S(\omega)$ be fixed as either 0 or 1, then we will prove by induction on the index of $Q \leq S$ that there is at most one possibility for the coordinate $\Phi_Q(\omega)$. Suppose that $Q < S$, and that $\Phi_R(\omega)$ is determined for all $R$ with $|R| > |Q|$. Then because $\Psi \Phi = 0$, the fixed points must satisfy

$$\sum_{\pi \in W_S Q} \Phi_{(s)} Q(\omega) \equiv 0 \pmod{|W_S Q|},$$

or if we isolate $\Phi_Q(\omega)$:

$$\Phi_Q(\omega) \equiv -\sum_{\pi \in W_S Q \pi \neq 1} \Phi_{(s)} Q(\omega) \pmod{|W_S Q|}.$$

We have $|\langle s \rangle| > |Q|$ for all $s \in N_S Q$ with $s \notin Q$, so all the numbers $\Phi_{(s)} Q(\omega)$ are already determined by induction. In addition $Q < S$ implies $Q < N_S Q$, so $|W_S Q| \geq 2$, and thus $\Phi_Q(\omega) = 0$ and $\Phi_Q(\omega) = 1$ cannot both satisfy the congruence relation.

We conclude that once $\Phi_S(\omega)$ is fixed, there is at most one possibility for $\omega$. The empty set $0 = [\emptyset]$ is idempotent and satisfies $\Psi_S(0) = 0$, and the one point set $[S/S]$ is idempotent and satisfies $\Phi_S([S/S]) = 1$, so both possibilities exist.

\[ \square \]

7. On the composition product of saturated fusion systems

In this final section we apply the earlier theorems \[B\] and \[C\] about characteristic idempotents to a conjecture of Park-Ragnarsson-Stancu in \[11\] concerning composition products of fusion systems and how to characterize them in terms of characteristic idempotents.

Let $\mathcal{F}$ be a fusion system on a $p$-group $S$, and let $\mathcal{H}, \mathcal{K}$ be fusion subsystems on subgroups $R,T \leq S$ respectively. In the terminology of Park-Ragnarsson-Stancu, we then say that $\mathcal{F}$ is the composition product of $\mathcal{H}$ and $\mathcal{K}$, written $\mathcal{F} = \mathcal{H} \mathcal{K}$, if $S = RT$ and for all subgroups $P \leq T$ it holds that every morphism $\varphi \in \mathcal{F}(P,R)$ can be written as a composition $\varphi = \psi \rho$ such that $\psi$ is a morphism of $\mathcal{H}$ and $\rho$ is a morphism of $\mathcal{K}$.

For a finite group $G$ with subgroups $H,K \leq G$, we can ask whether $G = HK$, i.e., if every $g \in G$ can be written as $g = hk$ with $h \in H$ and $k \in K$. It turns out that the answer to this question is detected by the structure of $G$ as an $(K,H)$-biset. With a little thought one can show that $G = HK$ if and only if the $(K,H)$-biset $G$ is isomorphic to the transitive biset $H \times_{H\cap K} K$. This result for groups inspired Park-Ragnarsson-Stancu to conjecture that $\mathcal{F} = \mathcal{H} \mathcal{K}$ is equivalent to a similar relation between the characteristic idempotents:

\[(7.1) \quad [R,id]^R_S \circ \omega_T \circ [T,id]^T_S = \omega_H \circ [R \cap T,id]^R_T \circ \omega_K.\]

Thanks to theorem \[C\] we can now directly calculate under which circumstances \[7.1\] holds, which results in the following theorem.

**Theorem E.** Let $\mathcal{F}$ be a saturated fusion system on a $p$-group $S$, and suppose that $\mathcal{H}, \mathcal{K}$ are saturated fusion subsystems of $\mathcal{F}$ on subgroups $R,T \leq S$ respectively.

Then the characteristic idempotents satisfy

\[(7.2) \quad [R,id]^R_S \circ \omega_T \circ [T,id]^T_S = \omega_H \circ [R \cap T,id]^R_T \circ \omega_K,\]

if and only if $\mathcal{F} = \mathcal{H} \mathcal{K}$ and for all $Q \leq R \cap T$ we have

\[(7.3) \quad |\mathcal{F}(Q,S)| = \frac{|\mathcal{H}(Q,R)| \cdot |\mathcal{K}(Q,T)|}{|\text{Hom}_{H \cap K}(Q,R \cap T)|}.\]
Proof. The element l.h.s. := \([R, id]^S_T \circ \omega_F \circ [T, id]^S_T\) is the characteristic idempotent for \(F\) restricted to \(A(T, R)_{(p)}\). For subgroups \(D \leq R \times T\) we therefore have \(\Phi_D([R, id]^S_T \circ \omega_F \circ [T, id]^S_T) = 0\) unless \(D\) has the form \(\Delta(P, \varphi)\) with \(P \leq T\) and \(\varphi \in F(P, R)\), and for such \(P\) and \(\varphi\) we get

\[
\Phi_{\Delta(P, \varphi)}([R, id]^S_T \circ \omega_F \circ [T, id]^S_T) = \frac{|S|}{|F(P, S)|}.
\]

For the right hand side we know from theorem \(\Box\) that r.h.s. := \(\omega_H \circ [R \cap T, id]^R_T \circ \omega_K\) is equal to the basis element \(\beta_{\Delta(R \cap T, id)}\) in \(A(H \times K)_{(p)}\). Hence we have \(\Phi_D(\omega_H \circ [R \cap T, id]^R_T \circ \omega_K) = 0\) unless \(D\) is \((H \times K)\)-conjugate to \(\Delta(P, id)\) for some \(P \leq R \cap T\), i.e., \(D\) has the form \(\Delta(p^{-1}Q, \psi\rho)\) with \(Q \leq T \cap R\), \(p^{-1} \in K(Q, T)\), and \(\psi \in H(Q, R)\); and if \(D\) has this form, then we get

\[
\Phi_{\Delta(p^{-1}Q, \psi\rho)}(r.h.s.) = \Phi_{\Delta(p^{-1}Q, \psi\rho)}(\omega_H \circ [R \cap T, id]^R_T \circ \omega_K) = \Phi_{\Delta(Q, id)}(\beta_{\Delta(R \cap T, id)})
\]

\[
= \frac{|\text{Hom}_{H \times K}(\Delta(Q, id), \Delta(R \cap T, id)) \cdot |R \times T|}{|\Delta(R \cap T, id)| \cdot |\text{Hom}_{H \times K}(\Delta(Q, id), R \times T)|}
\]

\[
\frac{\beta_{\Delta(R \cap T, id)}}{\Delta(R \cap T, id)} = \frac{|\text{Hom}_{H \times K}(Q, R \cap T)| \cdot |R| \cdot |T|}{|R \cap T| \cdot |H(Q, R)| \cdot |K(Q, T)|}
\]

\[
= |RT| \cdot \frac{|\text{Hom}_{H \times K}(Q, R \cap T)|}{|H(Q, R)| \cdot |K(Q, T)|}.
\]

Suppose that \(\Box\) is true. Comparing \(\Phi_{\Delta(1, id)}(l.h.s.) = |S|\) and \(\Phi_{\Delta(1, id)}(r.h.s.) = |RT|\), we see that we must have \(|S| = |RT|\), and consequently \(S = RT\). Furthermore we know that \(\Phi_{\Delta(P, \varphi)}(l.h.s.) \neq 0\) if \(P \leq T\) and \(\varphi \in F(P, R)\), it is therefore a requirement for \(\Box\) that \(\Phi_{\Delta(P, \varphi)}(r.h.s.) \neq 0\) as well, which is the case exactly when \(\Delta(P, \varphi)\) has the form \(\Delta(p^{-1}Q, \psi\rho)\) with \(p \in K\) and \(\psi \in H\), hence \(\varphi = \psi\rho \in H \times K\), so we must have \(F = H \times K\). Because \(S = RT\), the equality \(\Phi_{\Delta(Q, id)}(l.h.s.) = \Phi_{\Delta(Q, id)}(r.h.s.)\) gives us \(\Box\).

If we conversely suppose that \(F = H \times K\), then \(\Phi_{\Delta(P, \varphi)}(l.h.s.)\) and \(\Phi_{\Delta(P, \varphi)}(r.h.s.)\) are non-zero for the same indices, and because \(S = RT\), the only obstacle for equality of fixed points \(\Phi_{\Delta(p^{-1}Q, \psi\rho)}(l.h.s.) = \Phi_{\Delta(p^{-1}Q, \psi\rho)}(r.h.s.)\) is whether it holds that

\[
\frac{1}{|F(Q, S)|} = \frac{|\text{Hom}_{H \times K}(Q, R \cap T)|}{|H(Q, R)| \cdot |K(Q, T)|}
\]

for all \(Q \leq R \cap T\), which is \(\Box\).

\[\Box\]

Example 7.1. The following example shows that the conjecture of Park-Ragnarsson-Stancu fails in general. We consider the alternating group \(A_6\), and identify one of its Sylow 2-subgroups with the dihedral group \(D_8\). The associated fusion system \(F := F_{D_8}(A_6)\) is the saturated fusion system on \(D_8\) wherein all five subgroups of order 2 are conjugate. Let \(R, T \leq D_8\) be the two Klein four-groups inside \(D_8\), and let \(H = F_{R}(R \rtimes \mathbb{Z}/3), K = F_{T}(T \rtimes \mathbb{Z}/3)\) be fusion subsystems of \(F\) on \(R\) and \(T\) respectively, with \(\mathbb{Z}/3\) acting nontrivially on \(R \cong T \cong \mathbb{Z}/2 \times \mathbb{Z}/2\). Then \(H\) and \(K\) both contain the order 3 automorphisms of the Klein four-group, and both are saturated.

We claim that \(F = H \times K\). First of all \(D_8 = RT\) is clear. Next, there is no isomorphism between \(R\) and \(T\) in \(F\), so the only subgroups of \(T\) that map to \(R\) in \(F\), are the subgroups of order 2 and the trivial group. Suppose \(A \leq T\) has order 2. Then every morphism \(\varphi \in F(A, R)\) factors through \(Z(D_8) = R \cap T\), and can therefore be factored as \(\varphi = \rho \psi\) with \(\psi \in K(A, Z(D_8))\) and \(\rho \in H(Z(D_8), R)\). Hence we have \(F = H \times K\).
Furthermore, the Q(Q) \in H(R) \in \mathbb{R}$.  

\[ \text{Proof. By theorem } \text{E} \text{ it is sufficient to prove that } \text{the same map as } \eta \text{ and for each } \phi \in \mathbb{R} \text{ we can factor } Q \leq K \text{ weakly normal in } F, \text{ i.e., } K \text{ is saturated and } F \text{-invariant in the sense of } [1]. \]

Then \( F = HK \) if and only if the characteristic idempotents satisfy 
\[ [R, \text{id}]_S^R \circ \omega_F \circ [T, \text{id}]_T^R = \omega_H \circ [R \cap T, \text{id}]_T^R \circ \omega_K. \]

\[ \text{Proof. By theorem } \text{E} \text{ it is sufficient to prove that } \text{F} = HK \text{ implies } (7.3), \text{ so suppose } \text{F} = HK. \text{ The subsystem } K \text{ being } F \text{-invariant means that } T \text{ is strongly closed in } F, \text{ and whenever we have } Q, R \leq P \leq T \text{ and } \varphi \in F(P, T), \text{ conjugation by } \varphi \text{ induces a bijection } K(Q, R) \xrightarrow{\varphi(-)\varphi^{-1}} K(\varphi Q, \varphi R). \]

According to [1, Lemma 3.6], the intersection \( H \cap K \) is an \( H \)-invariant fusion system on \( R \cap T \). Suppose we have subgroups \( Q \leq R \cap T \) and \( Q' \sim_H Q \), and choose an isomorphism \( \varphi \in \mathcal{H}(Q, Q') \). Because \( T \) is strongly closed in \( F \), \( R \cap T \) is strongly closed in \( H \), hence \( Q' \leq R \cap T \). By the Frattini property of \( H \)-invariant subsystems, [1, Section 3], \( \varphi \) can be factored as \( \varphi = \eta \kappa \) with \( \kappa \in (H \cap K)(Q, R \cap T) \) and \( \eta \in \text{Aut}_H(R \cap T) \). If we let \( Q'' := \kappa(Q) \), we then have \( |K(Q, T)| = |K(Q'', T)| \) and \( |(H \cap K)(Q, T)| = |(H \cap K)(Q'', T)| \). Furthermore, the \( H \)-isomorphism \( \eta: Q'' \to Q' \) is defined on all of \( R \cap T \), so the \( F \)-stability of \( K \) and \( H \)-stability of \( H \cap K \) implies that \( \eta \) induces bijections \( K(Q'', T) \cong K(Q', T) \) and \( (H \cap K)(Q'', T) \cong (H \cap K)(Q', T) \).

We will now prove (7.3), and because \( T \) is strongly closed in \( F \), we must show 
\[ |F(Q, T)| = \frac{|H(Q, R \cap T)| \cdot |K(Q, T)|}{|(H \cap K)(Q, R \cap T)|} \]
for all \( Q \leq R \cap T \). Let therefore \( Q \leq R \cap T \) be given. For every homomorphism \( \varphi \in F(Q, T) \), we can factor \( \varphi^{-1}: \varphi Q \to Q \) as \( \varphi^{-1} = \eta^{-1} \kappa^{-1} \) with \( \eta^{-1} \in H \) and \( \kappa^{-1} \in K \), or equivalently \( \varphi = \kappa \eta \). We will enumerate \( F(Q, T) \) by counting the number of pairs of isomorphisms \( (\kappa, \eta) \) with \( \eta: Q \to Q' \in H \) and \( \kappa: Q' \to Q'' \in K \). The number of choices for \( \eta \) is \( |H(Q, R \cap T)| \), and for each \( \eta: Q \to Q' \) the number of choices for \( \kappa \) is \( |K(Q', T)| \). Because \( Q' \) is isomorphic to \( Q \) in \( H \), the arguments above imply that \( |K(Q', T)| = |K(Q, T)| \), which is independent of the chosen \( \eta \in H(Q, R \cap T) \). The total number of composable pairs \( (\kappa, \eta) \) is therefore
\[ |H(Q, T)| \cdot |K(Q, T)|. \]

Given a pair \( (\kappa, \eta) \) of composable isomorphisms \( Q \xrightarrow{\eta} Q' \xrightarrow{\kappa} Q'' \), we then count the number of other pairs \( Q \xrightarrow{\eta'} Q' \xrightarrow{\kappa'} Q'' \) that represent the same isomorphism in \( F \). If \( (\kappa, \eta) \) and \( (\kappa', \eta') \) give the same isomorphism \( Q \to Q'' \) in \( F \), then we have \( \kappa \eta = \kappa' \eta' \) or equivalently \( (\kappa')^{-1} \kappa = \eta' \eta^{-1} \in (H \cap K)(Q', R \cap T) \). Conversely, given any \( \rho \in (H \cap K)(Q', R \cap T) \), the pair \( (\kappa \rho^{-1}, \rho \eta) \) defines the same \( F \)-homomorphism as \( (\kappa, \eta) \). The number of pairs representing the same map as \( (\kappa, \eta) \) is therefore \( |(H \cap K)(Q', R \cap T)| = |(H \cap K)(Q, R \cap T)| \), which is
independent of the chosen pair \((\kappa, \eta)\). Hence there are \(|(H \cap K)(Q, R \cap T)|\) pairs representing each homomorphism \(\varphi \in \mathcal{F}(Q, T)\)

\[
|\mathcal{F}(Q, T)| = \frac{|H(Q, R \cap T)| \cdot |K(Q, T)|}{|(H \cap K)(Q, R \cap T)|}
\]

as we wanted. \(\square\)

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