An Introduction to Harmonic Manifolds and the Lichnerowicz Conjecture

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Abstract

The title is self-explanatory. We aim to give an easy to read and self-contained introduction to the field of harmonic manifolds. Only basic knowledge of Riemannian geometry is required. After we gave the definition of harmonicity and derived some properties, we concentrate on Z. I. Szabó’s proof of Lichnerowicz’s conjecture in the class of compact simply connected manifolds.

1 Introduction

1.1 History of Lichnerowicz’s Conjecture

One attempt to find solutions of the Laplace equation \( \Delta f = 0 \) is to look for them only in special classes of functions. It is easy to find the solutions

\[
f_n : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}, \quad x \mapsto ||x||^{2-n}
\]

for \( n \neq 2 \) and

\[
f_2 : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}, \quad x \mapsto \log ||x||
\]

for \( n = 2 \) in the class of radially symmetric functions on \( \mathbb{R}^n \setminus \{0\} \).

In 1930 H. S. Ruse gave this ansatz a try for pointed open balls in general Riemannian manifolds and thought he had succeeded, cf. [Rus31]. Together with E. T. Copson he published the article [CR40] in which they described a mistake in Ruse’s proof. Consequently they defined, amongst other notions of harmonicity, completely harmonic space which are nowadays called locally harmonic spaces. A Riemannian manifold is said to be locally harmonic if it allows a non-constant
radially symmetric solution of the Laplace equation around every point in a small enough
neighbourhood. They also derived that this condition is equivalent to the constancy of the mean
curvature of small geodesic spheres. Furthermore they showed that locally harmonic spaces
are necessarily Einsteinian. Hence they have constant curvature in dimensions 2 and 3. See \cite{Pat76}
for a detailed description of H. S. Ruse’s work on locally harmonic manifolds. Interestingly there
are many more, fairly different, but equivalent formulations for harmonicity such as the validity of
the mean value theorem, which was proved by T. J. Willmore in \cite{Wil50}, or the radial symmetry
of the density function.

In 1944 A. Lichnerowicz conjectured that locally harmonic manifolds of dimension 4 are
necessarily locally symmetric spaces. He also gave some strong hints for a proof of his statement
and remarked that he did not know whether it holds in higher dimensions as well, cf. \cite{Lic44}, pp.
166-168. In \cite{Wal49, Theorem 1} A. G. Walker proved Lichnerowicz’s original conjecture. But
since the used arguments rely heavily on the dimension, there was no hope to generalise them.
The conjecture could be refined by A. J. Ledger since he showed that locally symmetric manifolds
are locally harmonic if and only if they are flat or have rank 1, \cite{Led57}. So what today is called
‘Lichnerowicz’s conjecture’ was born: ‘Every locally harmonic manifold is either flat or locally
symmetric of rank 1.’ A complete collection of the knowledge about locally harmonic manifolds at
its time was given in the book \cite{RWW61}.

An important result of global nature is due to A.-C. Allamigeon. He showed in \cite{All65}, p. 114
that complete simply connected locally harmonic manifolds are either Blaschkean or diffeomorphic
to $\mathbb{R}^n$. This established the connection with the generalised Blaschke conjecture, which is: ‘Every
Blaschke manifold is a compact Riemannian symmetric space of rank 1.’

Actually, there were several notions of harmonicity defined, which only coincide under additional
topological restrictions. Amongst others we have infinitesimal, local, global and strong harmonicity.
One uses ‘harmonic manifold’ as a collective term since it is clear from the context which type
of harmonicity is meant. In \cite{Mic76, Theorem 2} D. Michel used Brownian motion techniques to
show that compact simply connected globally harmonic manifolds are strongly harmonic. Later
on Z. I. Szabó gave a shorter and simpler proof, cf. \cite{Sza90, Theorem 1.1}. He also remarked that
the notions of infinitesimal, local and global harmonicity are equivalent in the class of complete
manifolds because of the Kazdan-DeTurck theorem, cf. \cite{DK81, Theorem 5.2}.

A. L. Besse found an embedding map for strongly harmonic manifolds into a Euclidean sphere
of suitable radius, cf. \cite{Bes78, Theorem 6.99}. The embedded manifold has unexpected additional
properties, e.g. it is minimal in the sphere and its geodesics are screw lines. Because of that it
was given the name ‘nice embedding’. The mentioned book also presented all of the at that time
known facts about harmonic manifolds and Blaschke manifolds.

The major breakthrough was made by Z. I. Szabó in 1990. He proved the Lichnerowicz conjec-
ture for the class of compact simply connected manifolds in his article \cite{Sza90}. In 2000 A. Ranjan
published a slightly changed version of Z. I. Szabó’s proof using a more careful analysis of a certain
ODE through perturbations. The interesting aspect about this is that it makes no use of the nice
embedding in one of the key steps of the proof, namely that the density function is a trigonometric
polynomial of a special form, cf. \cite{Ran00, Corollary 3.1}. A less technical argument can be found
in [Nik05, Theorem 2]. Furthermore, by using a result about the first eigenvalue of P-manifolds, cf. [RS97, Theorem 1], one can give an intrinsic proof without using an embedding.

Surprisingly, one of the more recent results is the following. There are globally harmonic manifolds in infinitely many dimensions greater or equal to 7 which are not locally symmetric, cf. [DR92, Corollary 1]. E. Damek and F. Ricci constructed one-dimensional extensions of Heisenberg-type groups which are simply connected and globally harmonic, but only symmetric if the used Heisenberg-type group has a centre of dimension 1, 3 or 7. This leaves the question what additional condition would be sufficient to force a harmonic manifold to be locally symmetric and whether there are counterexamples in every dimension greater or equal to 7.

In [Nik05, Theorem 1] Y. Nikolayevsky used the curvature conditions derived by A. J. Ledger, today called Ledger’s formulae, cf. [Wil93, pp. 231-232], to solve the conjecture in dimension 5, i.e. he showed that every locally harmonic manifold of dimension 5 has constant sectional curvature. Namely, after lengthy and tedious calculations he is able to compute the algebraic curvature tensors which satisfy the first two of Ledger’s formulae, yielding that they are parallel. Lichnerowicz’s conjecture remains unsolved in dimension 6.

A very recent result is due to J. Heber. In [Heb06, Corollary 1.2] he showed that a simply connected homogeneous globally harmonic space is either flat, symmetric of rank 1 or one of the non-symmetric spaces found by E. Damek and F. Ricci. This is achieved by carefully examining the structure of the group of isometries which is, endowed with a suitable metric, isometric to the manifold. First he showed that it is simply transitive and solvable and then that its commutator has codimension 1. Finally his calculation of the stable Jacobi tensors yields the claim.

There are many more related topics, results and open questions not mentioned yet. Here is a short list with some of the latest references: harmonicity in semi-Riemannian manifolds, k-harmonicity [NV06], infinitesimally harmonic at every point implies infinitesimally harmonic [Van81], non-compact strongly harmonic manifolds, commutative and D’Atri spaces [BTV95], Busemann functions in a harmonic manifold [RS03], asymptotical harmonicity [Heb06], etc.

1.2 Extended Abstract

This subsection contains a more detailed account of the structure of this article and its differences with and additions to Z. I. Szabó’s work.

The second section gives a concise introduction to the objects and notions needed to examine locally harmonic manifolds. Namely, it consists of the definitions of Jacobi tensors along geodesics, density function, geodesic involution, mean curvature of geodesic spheres, radial and averaged functions and screw lines as well as some of their properties. The approach to screw lines presented here is due to J. von Neumann and I. J. Schoenberg. Hence the detour over the notion of curvatures in [Sza90, Section 3] and [Sza90, Lemma 4.9] can be avoided, cf. Lemmata 2.6.2 and 8.5.2.

In Subsection 3.1 we present several, rather different, but equivalent definitions of local harmonicity, e.g. ‘geodesic spheres have constant mean curvature’, ‘every harmonic function satisfies the mean value property’ and ‘the radial derivative commutes with the average operator’, where the last one seems to be a new characterisation. For our considerations the local version of Z. I. Szabó’s so-called ‘basic commutativity’ is of greater interest. It states that local harmonicity
is equivalent to the commutating of the average operator with the Laplace operator. Its global version is used to find radial eigenfunctions of the Laplacian later on. We also prove that locally harmonic manifolds are Einsteinian. Hence they are analytic by the Kazdan-DeTurck theorem. Then we can show that the density function does not depend on the point.

Section 4 contains some basic facts about Blaschke manifolds and a proof of the (original) global version of the basic commutativity. We use a different argument to Z. I. Szabó’s one, cf. [Sza90, p. 5], since we only show that the radialised average is $C^2$ and not $C^\infty$, cf. Lemma 4.2.3.

The next aim is to understand the relation between the notions of locally, globally and strongly harmonic manifolds. Important for our argumentation is that they coincide under the hypothesis of a compact simply connected manifold and that we then get the Blaschke property.

Then we show that averaged eigenfunctions are solutions of a certain linear ODE involving the mean curvature by using the basic commutativity. This yields some findings on the structure of the spaces of (radial) eigenfunctions. Also contained in Section 6 is a characterisation of local harmonicity in Blaschke manifolds by means of the $L^2$-product.

In Section 7 we show that locally harmonic Blaschke manifolds which are not diffeomorphic to a sphere can be embedded into a Euclidean sphere of suitable radius, cf. Corollary 7.0.2. This is Z. I. Szabó’s new version of Besse’s so-called ‘nice embedding’ using a radial eigenfunction. In [Sza90, Theorem 3.1] it is stated with a weaker hypothesis, but without mentioning the exception of the sphere.

Finally, we are ready to prove the main result.

**Satz** (main result). Let $M$ be locally harmonic Blaschke manifold of dimension $n$ and diameter $\pi$. Then $M$, and therefore every compact simply connected locally harmonic manifold, is a Riemannian symmetric space of rank 1, i.e. isometric (up to scaling of the metric) to either $S^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$ or $\mathbb{O}P^2$.

Z. I. Szabó showed that the averaged eigenfunctions of the Laplacian can be written as polynomials in cosine by showing that the space spanned by their parallel displacements is finite-dimensional. The same is true for the square of the density function. Here he used the embedding theorem to be able to carry out calculations in a Euclidean space, cf. Lemma [Sza90, Lemma 4.3]. We present a slightly varied version of Y. Nikolayevsky’s proof of this statement which does not make use of an embedding, cf. Lemma 8.1.2. Then Z. I. Szabó derived restrictions to the possible roots of the mentioned polynomials. This rather technical part uses essentially the aforementioned linear ODE solved by the averaged eigenfunctions. Note that we give a new proof for [Sza90, Lemma 4.6], cf. Lemma 8.1.5. Consequently there is a strong restriction to the form of the density function and hence to the form of the mean curvature.

**Proposition.** There are $\alpha, \beta \geq 0$ such that

$$
\eta_p(q) = \frac{(\alpha + \beta) \cos d(p, q) + \beta}{\sin d(p, q)},
$$

where $\eta_p(q)$ is the mean curvature of the geodesic sphere of radius $0 < d(p, q) < \pi$ around $p \in M$ in the point $q \in M$. 

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Again by using the ODE this enabled Z. I. Szabó to find the spectrum and the radial eigenfunctions easily. This is the content of Subsection 8.2.

**Corollary.** The spectrum $(\lambda_k)_{k \in \mathbb{N}_0}$ of $M$ is given by $\lambda_k := k(k + \alpha + \beta)$. A radial eigenfunction to $\lambda_1$ around $p \in M$ is given by

$$M \ni q \mapsto \frac{\lambda_1}{n} \left( \cos d(p, q) + \frac{n - \lambda_1}{\lambda_1} \right).$$

In Subsection 8.3 we show that this radial eigenfunction to the first eigenvalue yields an especially nice embedding. More precisely, the geodesics are mapped into circles so that the geodesic symmetries are isometries. Hence the main result is established. Alternatively, one can use the Bott-Samelson theorem [Bes78, Theorem 7.23] and the statement of [RS97, Theorem 1] to give an intrinsic version of the proof, cf. Subsection 8.3.
1.3 Notations and Conventions

In this subsection we fix some notations and general hypotheses, which are valid for the whole article. This is meant to serve the reader as a guideline and to give them a feeling for the used notations.

Let \((M, g)\) be a connected Riemannian manifold of dimension \(n\) with metric \(g\). The Levi-Civita connection will be denoted by \(\nabla\). Denote by \(T_p M\) the tangent space in \(p \in M\) and by \(TM\) the tangent bundle of \(M\). Points in \(TM\) will be denoted by \((p, v)\) where \(p \in M\) and \(v \in T_p M\).

The geodesic distance between two points \(p, q \in M\) will be denoted by \(d(p, q)\). The metric sphere of radius \(R \geq 0\) around \(p \in M\) is then given by \(S^R_p (p) := \{q \in M \mid d(p, q) = R\}\).

We denote the cut locus of \(p \in M\) by \(C(p)\). We write injrad\((p)\) for the injectivity radius of \(M\) at \(p\) and injrad\((M)\) for the injectivity radius of \(M\). The diameter of \(M\) is denoted by diam\((M)\).

We also use the standard notation for the function spaces \(L^2(M), C^0(M), C^\infty(M)\), \(C^0([0, \infty[), C^\infty([0, \infty[), \ldots\) and the space \(\ell^2\) of square-integrable sequences.

For an eigenvalue \(\lambda \in \mathbb{R}\) of the Laplacian \(\Delta\) we have the space of eigenfunctions \(V^\lambda \subset C^\infty(M)\).

We abbreviate ‘Riemannian symmetric space of rank 1’ by ‘ROSS’. These are the Euclidean spheres \(S^n\), the projective spaces \(\mathbb{K}P^n\) and \(\mathbb{O}P^2\) and the hyperbolic spaces \(\mathbb{K}H^n\) and \(\mathbb{O}H^2\), where \(\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}\). Here \(m\) denotes the \(\mathbb{K}\)-dimension of the respective space, i.e. \(m \cdot \text{dim}_\mathbb{K}(\mathbb{K}) = n\).

We use \(\text{vol}(S^{n-1})\) for the volume of the sphere of radius 1 in \(\mathbb{R}^n\).

The open geodesic ball of radius \(0 < R \leq \text{injrad}(p)\) around \(p \in M\) is denoted by \(B_R(p)\). The related ball in \(T_p M\) is denoted by \(B_R(0_p)\). Furthermore, set \(\tilde{B}_R(p) := B_R(p) \setminus \{p\}\) and \(\tilde{B}_R(0_p) := B_R(0_p) \setminus \{0_p\}\) for the pointed balls. Similarly the geodesic sphere \(S_R(p)\) in \(M\) and the related sphere in \(S_R(0_p)\) in \(T_p M\) of radius \(0 < R < \text{injrad}(p)\) are defined.

Polar coordinates are used throughout this article, i.e. for a \(v \in \tilde{B}_R(0_p)\) where \(0 < R \leq \text{injrad}(p)\) we often write \(v = r\theta\) where \(r := \|v\|\) and \(\theta := \frac{v}{\|v\|} \in S_1(0_p)\).

For a smooth curve \(\gamma\) in \(M\) we denote by \(T^\perp\gamma\) the subbundle of \(\gamma^*TM\) normal to \(\gamma'\). Furthermore we define a section \(R_\gamma\) of \(\text{End}(T^\perp\gamma)\) by \(R_\gamma = R(\cdot, \gamma')\gamma'\) where \(R\) is the curvature tensor. For a section \(S \in \Gamma(\text{End}(T^\perp\gamma))\) of the endomorphism bundle we set \(S' := \nabla_\gamma S\) where \(\nabla\) is used for the induced connection on \(\text{End}(T^\perp\gamma)\).
2 Preliminaries

This section contains a big chunk of the necessary setup, as the definitions and some properties of the relevant objects in conjunction with locally harmonic manifolds are given. The most important results are the invariance of the density function under the geodesic involution (Lemma 2.3.2), an equation which relates mean curvature and the density (Lemma 2.4.2) and a formula for the Laplacian of radial functions (Lemma 2.5.6). In the last subsection we show that two screw lines are congruent if and only if they have got the same screw function (Lemma 2.6.2).

2.1 Jacobi Tensors

The concept of Jacobi tensors comes in handy later on because it reduces complexity of notation. A useful reference is [EO80, Section 2]. Let \( \gamma \) be a geodesic in \( M \) and assume that 0 is in its domain of definition.

**Definition 2.1.1** (Jacobi tensor). We call a section \( J \) of the endomorphism bundle \( \text{End}(T^\perp \gamma) \) which satisfies

\[
J'' + R_\gamma \circ J = 0
\]

a Jacobi tensor to \( \gamma \).

**Remark.** Set \( p := \gamma(0) \). Take a basis \((e_2, \ldots, e_n)\) of \( T^\perp p \gamma \) and denote by \((E_2, \ldots, E_n)\) its parallel translate along \( \gamma \). Choose Jacobi fields \( J_2, \ldots, J_n \) along \( \gamma \) with \( J_i(0), J'_i(0) \in T^\perp p \gamma \) where \( i = 2, \ldots, n \). We can define a Jacobi tensor \( J \) to \( \gamma \) by setting \( JE_i := J_i \) for \( i = 2, \ldots, n \). It is easy to see that every Jacobi tensor to \( \gamma \) can be written that way. If \((e_2, \ldots, e_n)\) is an orthonormal basis of \( T^\perp p \gamma \), we get

\[
JE_i = \sum_{j=2}^{n} g(J_i, E_j)E_j
\]

and if \((e_2, \ldots, e_n)\) is additionally positively oriented

\[
\det J = \det (g(J_i, E_j))_{i,j=2,\ldots,n}.
\]

**Definition 2.1.2** (associated Jacobi tensor). There is exactly one Jacobi tensor \( J \) to \( \gamma \) with \( J(0) = 0 \) and \( J'(0) = \text{id} \). We call it the Jacobi tensor associated to \( \gamma \).

2.2 Density Function

Local harmonicity is defined in terms of the density function, which will be examined in this subsection. From its definition it is not immediately clear why the density function is smooth and why it is called ‘density’. Therefore we give a formula for it in normal coordinates, which clarifies the situation. The results of this subsection can also be found in [Wil93, Section 6.6].

**Definition 2.2.1** (density function). Choose \( V \subset TM \) such that \( \exp : V \to M \) is defined. Let \((p, v) \in V \) with \( v \neq 0 \) and set \( \tilde{v} := \frac{v}{\|v\|} \). Let \( J_v \) be the Jacobi tensor associated to the normalised geodesic \( r \mapsto \exp_p r\tilde{v} \). The density function \( \omega \) is then defined by

\[
\omega : V \to \mathbb{R}, \ (p, v) \mapsto \|v\|^{1-n} \det \tau^\perp p \gamma (J_v(\|v\|))
\]
where we set \( \omega(p, 0_p) := 1 \).

**Remark.** The density function \( \omega \) is obviously continuous on \( V \) and \( \omega(p, v) = 0 \) if and only if \( p \) and \( \exp_p v \) are conjugate along \( r \mapsto \exp_p r \tilde{v} \). When fixing a point \( p \in M \) and choosing a normal coordinate neighbourhood \( U \) around \( p \), we will often write \( \omega_p(q) := \omega(p, \exp_p^{-1} q) \) for \( q \in U \). If \( M \) is complete, \( \omega \) is defined on the whole of \( TM \).

**Lemma 2.2.2** (density in normal coordinates). Let \( U \) be a normal neighbourhood around \( p \in M \). Take \( q \in U \) and let \((g_{q,ij})_{i,j=1...n}\) be the metric of \( T_q M \) expressed in the normal coordinates of \( U \). Then

\[
\omega_p(q) = \sqrt{\det(g_{q,ij})_{i,j=1...n}}.
\]

**Proof.** We have \( \omega_p(p) = 1 = \sqrt{\det(g_{p,ij})_{i,j=1...n}} \). So assume \( q \neq p \) and set \( v := \exp_p^{-1} q \) as well as \( e_1 := \frac{v}{\|v\|} \). Pick \( e_2, \ldots, e_n \in T_p M \) such that \((e_1, \ldots, e_n)\) is a positively oriented orthonormal basis of \( T_p M \). We identify this basis with the standard basis in \( \mathbb{R}^n \). Denote by \( J_1, \ldots, J_n \) the Jacobi fields along the geodesic \( r \mapsto \exp_p re_1 \) with initial conditions \( J_i(0) = 0_p \) and \( J'_i(0) = e_i \) where \( i = 1, \ldots, n \). We get

\[
g_{q,ij} = g_{\exp_p v,ij} = g_{\exp_p v} \left( (d\exp_p)_v(e_i), (d\exp_p)_v(e_j) \right) = \frac{1}{\|v\|^2} g_{\exp_p v} \left( J_i(\|v\|), J_j(\|v\|) \right).
\]

Taking the determinant yields

\[\det(g_{q,ij})_{i,j=1...n} = \omega_p(q)^2.\]

The claim follows since \( \omega_p \) is positive on \( U \).

**Remark.** This lemma shows that \( \omega \) is smooth in inner points of its domain. Additionally it explains why we call \( \omega \) the density function since the Riemannian volume is defined by integration of \( \omega_p \).

### 2.3 Geodesic Involution

In this subsection we show the invariance of the density under the geodesic involution. This result is important for the proof of Proposition 3.2.4. It is also contained in [Bes78, Section 6.B].

**Definition 2.3.1** ((canonical) geodesic involution). Let \( V \subset TM \) be the maximal subset of the tangent bundle such that \( \exp : V \to M \) is defined. The (canonical) geodesic involution \( i \) is then defined by

\[
i : V \to V, \quad (p, v) \mapsto \left( \exp_p(v), -(d\exp_p)_v(v) \right).
\]

**Remark.** Indeed, this is well-defined as \( i(V) \subset V \) and an involution as \( i(i(p,v)) = (p,v) \).

**Lemma 2.3.2** (density invariant under geodesic involution). Let \( V \subset TM \) be the maximal subset of the tangent bundle such that \( \exp : V \to M \) is defined. Then

\[
\forall (p,v) \in V : \quad \omega(p,v) = \omega(i(p,v)).
\]
Proof. For $v = 0_p$ the statement is true because $i(p, 0_p) = (p, 0_p)$.

So consider $(p, v) \in V$ with $v \neq 0_p$. Set $\tilde{v} := \frac{v}{\|v\|}$ and for $r \in [0, \|v\|]$ set $\gamma(r) := \exp_p (r\tilde{v})$. The density function in the point $(p, v)$ can be written as

$$\omega(p, v) = \|v\|^{1-n} \det_{\gamma(\|v\|)} T^\gamma(J(\|v\|))$$

where $J \in \Gamma(\End(T^{\perp} \gamma))$ is the Jacobi tensor associated to $\gamma$. By setting $\gamma(r) := \exp_p (v(-r(d\exp)_p v(\tilde{v})))$ for $r \in [0, \|v\|]$ we get

$$\omega(i(p, v)) = \|v\|^{1-n} \det_{\gamma(\|v\|)} \gamma(K(\|v\|))$$

where $\gamma \in \Gamma(\End(T^{\perp} \gamma))$ is the Jacobi tensor associated to $\gamma$. Define the section $K$ of $\End(T^{\perp} \gamma)$ by $K(r) := \gamma(||v|| - r)$ for $r \in [0, ||v||]$. We remark that $K'' + R_\gamma \circ K = 0$ holds because of $\nabla_{\gamma'} = \nabla_{\gamma'} = -\nabla_\gamma$ and $R_\gamma(||v|| - r) = R_\gamma(r)$ for $r \in [0, ||v||]$.

Then $\mathcal{J} := (J^T)' \circ K - J^T \circ K'$ is a section of $\End(T^{\perp} \gamma)$ where $(\cdot)^T$ means transposition of an endomorphism. We have

$$\mathcal{J}' = ((J^T)' \circ K - J^T \circ K')' = (J^T)'' \circ K + (J^T)' \circ K' - (J^T)' \circ K - J^T \circ K''$$

$$= (J^T)'' \circ K - J^T \circ K''$$

$$= -(R_\gamma \circ J)^T \circ K + J^T \circ (R_\gamma \circ K)$$

$$= -J^T \circ R_\gamma \circ K + J^T \circ R_\gamma \circ K$$

$$= -J^T \circ R_\gamma \circ K + J^T \circ R_\gamma \circ K$$

$$= 0.$$

Hence the section $\mathcal{J}$ is parallel along $\gamma$.

Because of

$$\mathcal{J}(0) = ((J^T)' \circ K)(0) - (J^T \circ K')(0) = K(0)$$

and

$$\mathcal{J}(\|v\|) = ((J^T)' \circ K)(\|v\|) - (J^T \circ K')(\|v\|) = (J^T)'(\|v\|) \circ \gamma(\|v\|) + J^T(\|v\|) \circ \gamma(\|v\|) = J^T(\|v\|)$$

we get that $J^T(\|v\|)$ is the parallel translate of $K(0)$ along $\gamma$. That means

$$\omega(p, v) = \|v\|^{1-n} \det_{\gamma(\|v\|)} T^\gamma(J(\|v\|)) = \|v\|^{1-n} \det_{\gamma(\|v\|)} T^\gamma(J(\|v\|))$$

$$= \|v\|^{1-n} \det_{\gamma(\|v\|)} (J^T(\|v\|))$$

$$= \|v\|^{1-n} \det_{\gamma(\|v\|)} (K(0))$$

$$= \|v\|^{1-n} \det_{\gamma(\|v\|)} (\gamma(\|v\|))$$

$$= \omega(i(p, v)).$$

$\square$
2.4 Mean Curvature

This subsection describes the relation between the mean curvature of geodesic spheres and the density function. Lemma 2.4.2 is central for the proof of various equivalences in the next section and the proof of Lichnerowicz’s conjecture. A useful reference is [EOS10, Section 2].

Definition 2.4.1 (mean curvature (of geodesic spheres)). Let \( q \in \tilde{B}_R(p) \) be a point in the pointed geodesic ball of radius \( 0 < R \leq \text{injrad}(p) \) around \( p \in M \). Set \( v := \exp_p^{-1} q \) and \( \tilde{v} := \frac{v}{\|v\|} \). Let \( J_v \) be the Jacobi tensor associated to the geodesic \( r \mapsto \exp_p \tilde{v} \). The mean curvature \( \eta_p(q) \) (of the geodesic sphere \( S_{\|v\|}(p) \)) in the point \( q \) is defined by

\[
\eta_p(q) := \text{tr}(J_v \circ J_v^{-1})(\|v\|).
\]

Remark. Define the section \( S_v \in \Gamma(\text{End}(T^1\gamma)) \) by \( \Gamma(T^1\gamma) \ni X \mapsto \nabla_X \gamma' \in \Gamma(T^1\gamma) \). Then \( S_v(\|v\|) \) is the shape operator of \( S_{\|v\|}(p) \) in the point \( q \). Because of

\[
\Gamma(T^1\gamma) \ni J'_vX - S_vJ_vX = \nabla_{\gamma'}J_vX - \nabla_{J_vX}\gamma' = [\gamma', J_vX] \perp \Gamma(T^1\gamma)
\]

we get \( J'_v = S_v \circ J_v \). Hence our definition of \( \eta_p \) coincides with the one usually given as the trace of the shape operator. We have

\[
\sum_{i=2}^n \nabla_{\tilde{E}_i}E_i = \sum_{i=2}^n g(\gamma', \nabla_{\tilde{E}_i}E_i)\gamma' = -\sum_{i=2}^n g(\nabla_{E_i}\gamma', E_i)\gamma' = -\eta_v\gamma'
\]

where \( E_2, \ldots, E_n \) are fields along \( \gamma \) such that \( (\gamma', E_2, \ldots, E_n) \) is orthonormal along \( \gamma \) and \( \nabla_{\perp} \) denotes the part of the connection tangent to \( \gamma' \), i.e. normal to the geodesic spheres.

Lemma 2.4.2 (mean curvature through density). For \( q \in \tilde{B}_R(p) \) as above set again \( v := \exp_p^{-1} q \), \( r := \|v\| \) and \( \tilde{v} := \frac{v}{r} \). Then

\[
\eta_p(q) = \frac{\partial_r \left( r^{n-1}\omega(p, rv) \right)}{r^{n-1}\omega(p, rv)} = \frac{n-1}{r} + \frac{\partial_r \omega(p, rv)}{\omega(p, rv)}.
\]

Proof. The first equality follows from the formula

\[
(\text{det } J_v)' = \text{tr}(J_v' \circ J_v^{-1}) \text{det}(J_v).
\]

Hence

\[
\eta_p(q) = \frac{\partial_r \left( r^{n-1}\omega(p, rv) \right)}{r^{n-1}\omega(p, rv)} = \frac{(n-1)r^{n-2}\omega(p, rv) + r^{n-1}\partial_r \omega(p, rv)}{r^{n-1}\omega(p, rv)} = \frac{n-1}{r} + \frac{\partial_r \omega(p, rv)}{\omega(p, rv)}.
\]

2.5 Radial and Averaged Functions

Note that we only consider functions on pointed geodesic balls in this subsection. More general considerations are given for the special case of a Blaschke manifold later on. Strictly speaking, there are no results in this subsection except of Lemma 2.5.6. We only define some notions for the following discussion. Fix a point \( p \in M \) and a number \( 0 < R \leq \text{injrad}(p) \).
Definition 2.5.1 (normal and outward vector field). Denote by \( E^p \) the normal and outward vector field of \( \hat{B}_R(p) \) which is given by \( (E^p)_q := (d \exp_p)_v \left( \frac{\nu}{\|\nu\|} \right) \) for \( q \in \hat{B}_R(p) \) with \( v := \exp_p^{-1} q \).

Remark. \( E^p \) is the unique unit vector field on \( \hat{B}_R(p) \) such that \( E^p \) is normal and outward along \( S_r(p) \) for all \( 0 < r < R \).

Definition 2.5.2 ((associated) radial function). For a smooth function \( F : ]0, R[ \to \mathbb{R} \) we define the (associated) radial function (around \( p \in M \)) on \( \hat{B}_R(p) \) by

\[
R_p F : \hat{B}_R(p) \to \mathbb{R}, \quad q \mapsto F(d(p, q)).
\]

We call \( R_p : C^\infty([0, R]) \to C^\infty(\hat{B}_R(p)) \) radial operator (around \( p \)). Functions \( f : \hat{B}_R(p) \to \mathbb{R} \) such that an \( F : ]0, R[ \to \mathbb{R} \) exists with \( f = R_p F \) are called radially symmetric functions (around \( p \)) or abbreviated radial functions (around \( p \)).

Remark. The radial operator is linear.

Definition 2.5.3 (average operator). Let \( f : \hat{B}_R(p) \to \mathbb{R} \) be smooth. The averaged function \( A_p f \) of \( f \) (around \( p \in M \)) is defined by

\[
A_p f : ]0, R[ \to \mathbb{R}, \quad r \mapsto (A_p f)(r) := \frac{1}{\operatorname{vol}(S_r(p))} \int_{S_r(p)} f \, dS_r(p).
\]

We call \( A_p : C^\infty(\hat{B}_R(p)) \to C^\infty([0, R[) \) average operator (around \( p \)).

Remark. The average operator is linear.

Definition 2.5.4 (radial derivative). Let \( E^p \) be the normal and outward vector field of \( \hat{B}_R(p) \). We define the radial derivative \( f' \) of \( f : \hat{B}_R(p) \to \mathbb{R} \) by

\[
f' : \hat{B}_R(p) \to \mathbb{R}, \quad q \mapsto f'(q) := (\nabla_{E^p} f)(q).
\]

Remark. In terms of polar coordinates and the exponential map we can write \( f'(\exp_p r \theta) = \partial_r f(\exp_p r \theta) \) where \( 0 < r < R \) and \( \theta \in S_1(0_p) \).

Lemma 2.5.5 (properties of the radial operator). Let \( h : \hat{B}_R(p) \to \mathbb{R} \) and \( F, G : ]0, R[ \to \mathbb{R} \). Then

1. \( A_p R_p F = F \)
2. \( R_p(FG) = R_p F R_p G \)
3. \( A_p(hR_p G) = GA_p h \)
4. \( (R_p F)' = R_p F' \)

Proof. The first three statements are clear.

Using the above remark we have for \( q \in \hat{B}_R(p) \) with \( q = \exp_p r \theta \)

\[
(R_p F)'(q) = \partial_r (R_p F)(\exp_p r \theta) = \partial_r F(r) = (R_p F')(q).
\]

\qed
Lemma 2.5.6 (Laplacian of radial functions). Let \( f : \hat{B}_r(p) \to \mathbb{R} \) be a radial function. Then

\[
\Delta f = -f'' - \eta_p f'.
\]

Proof. Fix \( 0 < r < R \) and let \( q \in S_r(p) \). Denote the connection on \( S_r(p) \) by \( \nabla \) and the associated Laplacian by \( \overline{\Delta} \). Since \( f \) is radial, \( f|_{S_r(p)} \) is constant and \( \overline{\Delta} f|_{S_r(p)} = 0 \). Take \( e_2, \ldots, e_n \in T_qM \) such that \( \langle (E^p)_q, e_2, \ldots, e_n \rangle \) is an orthonormal basis of \( T_qM \). In the point \( q \) we get

\[
(\Delta f)(q) = -\nabla^2_{(E^p)_q} f - \sum_{i=2}^n \nabla^2_{e_i,e_i} f
\]

\[
= -f''(q) - \sum_{i=2}^n (\nabla_{e_i} \nabla_{e_i} f - (\nabla_{e_i} e_i) f)
\]

\[
= -f''(q) - \sum_{i=2}^n (\nabla^2_{e_i,e_i} f - (\nabla_{e_i} e_i) f)
\]

\[
= -f''(q) - \sum_{i=2}^n (\nabla^2_{e_i,e_i} f - (\nabla_{e_i} e_i) f)
\]

\[
= -f''(q) - \eta_p(q) \nabla_{(E^p)_q} f
\]

\[
= -f''(q) - \eta_p(q) f'(q).
\]

\[\square\]

Remark. In particular it holds \( \Delta d(p, \cdot) = -\eta_p \) on \( \hat{B}_{majrad}(p) \).

2.6 Screw Lines

Let \( N \in \mathbb{N} \) and \( c : \mathbb{R} \to \mathbb{R}^N \) be a smooth curve which is parametrised by arc length. We will discuss some kind of generalisation of curves with constant curvatures called screw lines. This is needed when discussing the nice embedding. The following Lemma 2.6.2 is true for curves \( c : \mathbb{R} \to \ell^2 \) as well. The ideas can also be found in [vNS41, Part II].

Definition 2.6.1 (screw function and screw line). We define the screw function \( S_{s_0} \) in \( s_0 \in \mathbb{R} \) of \( c \) by

\[
S_{s_0} : \mathbb{R} \to \mathbb{R}, \quad s \mapsto \|c(s_0 + s) - c(s_0)\|^2.
\]

The curve \( c \) is called screw line if its screw functions are independent of the chosen points, i.e.

\[
\forall s_0 \in \mathbb{R} : \ S_{s_0} = S_0.
\]

Lemma 2.6.2. Let \( c \) and \( \varpi \) be screw lines which have the same screw function. Then they are congruent, i.e. there is an isometry \( I \in \text{Iso}(\mathbb{R}^N) \) with \( I(c(s)) = \varpi(s) \) for all \( s \in \mathbb{R} \).

Proof. Firstly, we remark that for all \( r, s, t \in \mathbb{R} \) holds

\[
\langle c(t) - c(r), c(s) - c(r) \rangle = \frac{1}{2} (S_0(t-r) + S_0(s-r) - S_0(t-r-s-r)) = \langle \varpi(t) - \varpi(r), \varpi(s) - \varpi(r) \rangle.
\]
Without loss of generality we may assume that $c(0) = 0 = \tau(0)$. We choose $t_1, \ldots, t_k \in \mathbb{R}$ such that $(c(t_1), \ldots, c(t_k))$ is a basis of the space span $\{c(t) \mid t \in \mathbb{R}\}$. By applying the Gram-Schmidt process to this basis we get an orthonormal basis $(e_1, \ldots, e_k)$. We denote by $a_{ij} \in \mathbb{R}$ the coefficients of the change of basis given by that process, i.e.

$$e_i = \sum_{j=1}^{i} a_{ij} c(t_j), \quad i = 1, \ldots, k.$$ 

We emphasise that the $a_{ij}$’s only depend on the scalar products

$$\langle c(t_\nu), c(t_\mu) \rangle, \quad \nu, \mu = 1, \ldots, k.$$ 

Furthermore we fix an $s \in \mathbb{R}$, write

$$c(s) = \sum_{i=1}^{k} b_i(s) e_i$$

and emphasise that the $b_i(s)$’s only depend on the scalar products

$$\langle c(t_\nu), c(t) \rangle, \quad \nu = 1, \ldots, k, \quad t \in \mathbb{R}.$$ 

Because of our first remark we get that $(\tau(t_1), \ldots, \tau(t_k))$ is a basis of the space span $\{\tau(t) \mid t \in \mathbb{R}\}$ and $(\tau_1, \ldots, \tau_k)$ with

$$\tau_i := \sum_{j=1}^{i} a_{ij} \tau(t_j), \quad i = 1, \ldots, k$$

is the orthonormal basis we get by applying the Gram-Schmidt process. Furthermore it holds

$$\tau(s) = \sum_{i=1}^{k} b_i(s) \tau_i.$$ 

Let $A \in O(N)$ be an orthonormal transformation mapping $e_i$ into $\tau_i$ for $i = 1, \ldots, k$. We get

$$Ac(s) = \sum_{i=1}^{k} b_i(s) A e_i = \sum_{i=1}^{k} b_i(s) \tau_i = \tau(s).$$

3 Local Harmonicity

A rough definition for $M$ being locally harmonic could be ‘locally the density function is radially symmetric’. The aim of this section is to state the definition more precisely and to give several characterisations of locally harmonic manifolds. Especially Parts (2.) and (6.) of Proposition 3.1.2 are important for our considerations. Furthermore we give examples and show that locally harmonic manifolds are Einsteinian (Proposition 3.2.1).
3.1 Definition and Equivalences

We give several equivalent definitions of a locally harmonic manifold. Note that we show with Corollary 3.2.3 that the following proposition is still true if we formulate it with \( \text{injrad}(p) \) instead of \( \varepsilon \). The basic commutativity (Proposition 3.1.2(6.)) can be found in [Sza90, Section 1]. The commuting of the averaging operator with the radial derivative (Proposition 3.1.2(3.)) seems to be nowhere mentioned. The rest of Proposition 3.1.2 can be found in [Bes78, Proposition 6.21].

Definition 3.1.1 (locally harmonic). The Riemannian manifold \( M \) is said to be locally harmonic at \( p \in M \) if there exists an \( \varepsilon > 0 \) such that \( \omega_p|_{\hat{B}_\varepsilon(p)} \) is radial. If \( M \) is locally harmonic at every point, we call it locally harmonic.

Remark. Equivalently, we could require the existence of an \( \Omega : [0, \varepsilon \rightarrow \mathbb{R} \) such that \( \forall v \in B_\varepsilon(0_p) : \omega(p, v) = \Omega(||v||) \).

Notice that the choice of \( \varepsilon \) and \( \Omega \) could depend on \( p \). Actually, it does not, as we will prove in Proposition 3.2.4. The property ‘locally harmonic’ is often abbreviated by ‘LH’. A manifold which is LH is often called LH-manifold.

Proposition 3.1.2 (equivalences). Let \( p \in M \). Then the following statements are equivalent:

1. \( M \) is locally harmonic at \( p \).

2. There is an \( \varepsilon > 0 \) and an \( H : ]0, \varepsilon[ \rightarrow \mathbb{R} \) with \( \eta_p = R_p H \), i.e. the mean curvature is radial.

3. There is an \( \varepsilon > 0 \) such that for every \( f : \hat{B}_\varepsilon(p) \rightarrow \mathbb{R} \) we have \( (A_p f)' = A_p f' \), i.e. the radial derivative commutes with the average operator.

4. There is an \( \varepsilon > 0 \) such that for every \( f \in C^\infty(\hat{B}_\varepsilon(p)) \) with \( \Delta f = 0 \) we have \( (A_p f)' = 0 \), i.e. every harmonic function satisfies the mean value property.

5. There is an \( \varepsilon > 0 \) and a non-constant \( F : ]0, \varepsilon[ \rightarrow \mathbb{R} \) with \( \Delta R_p F = 0 \), i.e. there is a non-constant radial solution of the Laplace equation.

6. There is an \( \varepsilon > 0 \) such that for every \( f : \hat{B}_\varepsilon(p) \rightarrow \mathbb{R} \) we have \( \Delta R_p A_p f = R_p A_p \Delta f \), i.e. the Laplace operator commutes with \( R_p \circ A_p \).

Proof. 1. \( \Rightarrow \) 2.: Choose an \( \varepsilon \) such that \( \omega_p : \hat{B}_\varepsilon(p) \rightarrow \mathbb{R} \) is radial. Then so is \( \omega'_p : \hat{B}_\varepsilon(p) \rightarrow \mathbb{R} \). By Lemma 2.4.2 the mean curvature is radial, too.

2. \( \Rightarrow \) 1.: Choose \( \varepsilon > 0 \) and \( H : ]0, \varepsilon[ \rightarrow \mathbb{R} \) such that \( \eta_p = R_p H \). Let \( \hat{H} \) be the antiderivative of \( H - \frac{n-1}{2} \) in \( ]0, \varepsilon[ \). Let \( \theta \in S_1(p) \). The solution of the ODE

\[
\frac{y'}{y} = H - \frac{n-1}{2}
\]

with initial condition

\[
y\left(\frac{\varepsilon}{2}\right) = \omega\left(p, \frac{\varepsilon\theta}{2}\right)
\]
is given by \( g(r) = C(\theta) \exp(\tilde{H}(r)) \) for \( r \in ]0, \varepsilon[ \) where \( C(\theta) \) is a constant depending on \( \theta \). Since \( r \mapsto \omega(p, r\theta) \) solves the ODE as well, we have \( \omega(p, r\theta) = C(\theta) \exp(\tilde{H}(r)) \) for \( r \in ]0, \varepsilon[ \). Because \( \omega \) is continuous in \((p, 0_p)\) with \( \omega(p, 0_p) = 1 \) we get that

\[
\lim_{r \to 0} C(\theta) \exp(\tilde{H}(r))
\]

exists and equals 1. Hence \( C(\theta) \) does not depend on \( \theta \) and \( \omega_p|_{\hat{B}_r(p)} \) is radial.

2. \( \Rightarrow \) 3.: Choose \( \varepsilon > 0 \), \( H : ]0, \varepsilon[ \to \mathbb{R} \) and \( \Omega : ]0, \varepsilon[ \to \mathbb{R} \) such that \( \eta_p = R_p H \) and \( \omega_p = R_p \Omega \) on \( \hat{B}_\varepsilon(p) \). Let \( 0 < r < \varepsilon \). By taking polar coordinates and Lemma 2.2.2 into account we have

\[
(A_p f)(r) = \int_{S_1(0_p)} \frac{1}{r^{n-1}} \omega(p, r\theta) \ d\theta \int_{S_1(0_p)} f(\exp_p(r\theta)) r^{n-1} \omega(p, r\theta) \ d\theta
\]

\[
= \frac{1}{\text{vol}(S^{n-1})} \int_{S_1(0_p)} f(\exp_p(r\theta)) \ d\theta.
\]

Taking the derivative yields the claim.

3. \( \Rightarrow \) 4.: Choose an \( \varepsilon > 0 \) such that for every \( f \in C^\infty(\hat{B}_\varepsilon(p)) \) we have \((A_p f)' = A_p f'\). Suppose \( \Delta f = 0 \). Hence for every \( 0 < r < \varepsilon \) we get by Green’s first identity

\[
(A_p f)'(r) = \frac{1}{\text{vol}(S_r(p))} \int_{S_r(p)} \nabla_E f \ dS_r(p)
\]

\[
= \frac{1}{\text{vol}(S_r(p))} \int_{S_r(p)} (\text{grad } f, E^p) \ dS_r(p)
\]

\[
= -\frac{1}{\text{vol}(S_r(p))} \int_{\hat{B}_r(p)} \Delta f \ d\hat{B}_r(p)
\]

\[
= 0.
\]

4. \( \Rightarrow \) 2.: Choose an \( \varepsilon > 0 \) such that for every \( f \in C^\infty(\hat{B}_\varepsilon(p)) \) with \( \Delta f = 0 \) we have \((A_p f)' = 0\). We set

\[
H(r) := \frac{\partial_r \text{vol}(S_r(p))}{\text{vol}(S_r(p))}
\]

and show that \( \eta_p = R_p H \). Take an \( 0 < r < \varepsilon \). By solving a Dirichlet problem we can find an \( f \in C^\infty(\hat{B}_r(p)) \) with \( \Delta f|_{\hat{B}_r(p)} = 0 \) and \( f|_{S_r(p)} = \eta_p - R_p H \). Because of

\[
0 = \text{vol}(S_r(p))(A_p f)'(r) = -\frac{\partial_r \text{vol}(S_r(p))}{\text{vol}(S_r(p))} \int_{S_r(p)} f \ dS_r(p) + \partial_r \int_{S_r(p)} f \ dS_r(p)
\]

\[
= -\int_{S_r(p)} f R_p H \ dS_r(p) + \partial_r \int_{S_1(0_p)} f(\exp_p(r\theta)) r^{n-1} \omega(p, r\theta) \ d\theta
\]

\[
= -\int_{S_r(p)} f R_p H \ dS_r(p) + \int_{S_r(p)} f' \ dS_r(p) + \int_{S_r(p)} f \eta_p \ dS_r(p)
\]

\[
= -\int_{S_r(p)} f R_p H \ dS_r(p) - \int_{\hat{B}_r(p)} \Delta f \ d\hat{B}_r(p) + \int_{S_r(p)} f \eta_p \ dS_r(p)
\]
\[
\int_{S_r(p)} (\eta_p - R_p H)^2 \, dS_r(p)
\]
the claim follows.

2. \(\Rightarrow\) 5.: Choose an \(\varepsilon > 0\) such that \(\eta_p : \hat{B}_\varepsilon(p) \to \mathbb{R}\) is radial and a function \(H : ]0, \varepsilon[ \to \mathbb{R}\) with \(R_p H = \eta_p\). Let \(F : ]0, \varepsilon[ \to \mathbb{R}\) be a non-constant solution of the ODE
\[
y'' - Hy' = 0.
\]

We have
\[
\Delta R_p F = -(R_p F)'' - \eta_p (R_p F)' = -R_p F'' - R_p H R_p F' = R_p (-F'' - HF') = 0.
\]

5. \(\Rightarrow\) 2.: Take an \(\varepsilon > 0\) and a non-constant \(F : ]0, \varepsilon[ \to \mathbb{R}\) with \(\Delta R_p F = 0\). Since
\[
0 = \Delta R_p F = -R_p F'' - \eta_p R_p F'
\]
we have
\[
\eta_p R_p F' = -R_p F''
\]
and
\[
0 = -A_p R_p F'' - A_p \eta_p A_p R_p F' = -F'' - A_p \eta_p F'.
\]

If \(F'\) had a zero \(0 < r_0 < \varepsilon\), \(F\) would be constant, since \(F\) would be a solution of the ODE
\[
y'' - A_p \eta_p y' = 0\]
with \(F'(r_0) = F''(r_0) = 0\). So \(\eta_p\) is radial with
\[
\eta_p = R_p \left( \frac{-F''}{F'} \right).
\]

2. \(\Rightarrow\) 6.: Choose \(\varepsilon > 0\) and \(H : ]0, \varepsilon[ \to \mathbb{R}\) such that \(\eta_p = R_p H\). For a fixed \(0 < r < \varepsilon\) denote the Laplacian on \(S_r(p)\) by \(\nabla\). As in the proof of Lemma 2.5.6 we get for a \(q \in S_r(p)\)
\[
(\Delta f)(q) = (\nabla f)(q) - f''(q) - \eta_p(q)f'(q).
\]

By Green’s first identity we have
\[
\int_{S_r(p)} \nabla f|_{S_r(p)} \, dS_r(p) = 0
\]
and therefore again in \(q \in S_r(p)\)
\[
(R_p A_p \Delta f)(q) = R_p A_p ((\nabla f)(q) - f''(q) - \eta_p(q)f'(q))
= -(R_p A_p f'')(q) - \eta_p(q)(R_p A_p f')(q)
= -(R_p (A_p f)')(q) - \eta_p(q)(R_p (A_p f))(q)
= (\Delta R_p A_p f)(q).
\]

6. \(\Rightarrow\) 2.: Choose an \(\varepsilon > 0\) such that for every \(f : \hat{B}_\varepsilon(p) \to \mathbb{R}\) we have \(\Delta R_p A_p f = R_p A_p \Delta f\). If we set \(f := d(p, \cdot)\), we get
\[
R_p A_p \Delta d(p, \cdot) = \Delta R_p A_p d(p, \cdot) = \Delta d(p, \cdot) = -\eta_p.
\]
This means that the mean curvature is a radial function.
3.2 Curvature Restrictions

The main result of this subsection is that LH-manifolds are Einsteinian and therefore analytic. The proof for this statement can be found in [Wil93, Section 6.8]. Furthermore, we can deduce that in an LH-manifold the density function $\omega(p,v)$ does not depend on the point $p$, cf. [Wil93, Proposition 6.7.3]. In this section we let $V \subset TM$ be the maximal subset of the tangent bundle such that $\exp : V \to M$ is defined.

**Proposition 3.2.1** (harmonic manifolds are Einsteinian). *Every LH-manifold is an Einstein manifold.*

**Proof.** Fix $p \in M$ and $\theta \in S_1(0_p)$. Choose an $\varepsilon > 0$ such that $\eta_p : \hat{B}_\varepsilon(p) \to \mathbb{R}$ is radial and a function $H : [0, \varepsilon] \to \mathbb{R}$ with $R_p H = \eta_p$. For $r \in [0, \varepsilon]$ set $\gamma(r) := \exp_p r\theta$. Denote by $J$ the Jacobi tensor associated to $\gamma$. The inverse tensor $J^{-1}$ has got a singularity of order $n-1$ in 0 because of $\lim_{r \to 0} r^{1-n} \det J(r) = \omega(p,0_p) = 1$. So the section $J := rJ' \circ J^{-1}$ of $\text{End}(T^\perp \gamma)$ is not singular in 0. We get

$$rJ' = rJ' \circ J^{-1} + r^2 J'' \circ J^{-1} - r^2 J' \circ J^{-1} \circ J' \circ J^{-1} = J - r^2 R_\gamma - J^2$$

since $(J^{-1})' = -J^{-1} \circ J' \circ J^{-1}$. Differentiating the equation $rJ' = J - r^2 R_\gamma - J^2$ yields

$$J' + rJ'' = J' - 2r R_\gamma - r^2 R'_\gamma - J' \circ J - J \circ J'$$

and differentiating once more yields

$$J'' + rJ''' = -2R_\gamma - 2r R'_\gamma - 2r R''_\gamma - J'' \circ J - J' \circ J' \circ J - J' \circ J - J \circ J'$$

Since $\lim_{r \to 0} \frac{J(r)}{r} = J'(0)$ we get from the definition of $J$ and the two equations above

$$J(0) = \text{id}, \quad J'(0) = 0 \quad \text{and} \quad J''(0) = -\frac{2}{3} R_\gamma(0).$$

Taking the trace in the last equation gives

$$-\frac{2}{3} \text{ric}_p(\theta,\theta) = \text{tr} J''(0) = (\text{tr} J')''(0) = (rH(r))''(0).$$

This shows that $\text{ric}_p(\theta,\theta)$ does not depend on the chosen $\theta$. Hence $M$ is Einsteinian.

**Remark.** In dimensions 2 and 3 this implies that $M$ has constant sectional curvature. Taking more and more derivatives of $rJ' = J - r^2 R_\gamma - J^2$ yields the so-called ‘Ledger’s formulae’, cf. [Wil93, Section 6.8]. With their help one can give an affirmative answer to Lichnerowicz’s conjecture in dimension 4, cf. [Bes78, Section 6.1].

**Theorem 3.2.2** (Kazdan-DeTurck, [DK81, Theorem 5.2]). *Let $(M,g)$ be an Einstein manifold. Then the representation of $g$ in normal coordinates is real analytic.*

**Remark.** This implies that normal coordinates define a real analytic atlas on $M$. So we see that the map $\exp : \text{int} V \to M$ is real analytic by using normal coordinates.
Corollary 3.2.3 (density function is analytic). Let $(M, g)$ be an LH-manifold. Then the density function $\omega : \text{int} V \to \mathbb{R}$ is real analytic.

Proof. The density is given by a composition of the real analytic functions $d\exp, \det$ and $g$. □

Remark. We emphasise that only now we know that the density $\omega_p$ of an LH-manifold is radial till the injectivity radius and that $\omega(p, v)$ only depends on $\|v\|$ for $(p, v) \in V$.

Proposition 3.2.4 (density independent of the point). Let $M$ be an LH-manifold. Then there is a function $\Omega : [0, \infty) \to \mathbb{R}$ such that

$$\forall (p, v) \in V : \omega(p, v) = \Omega(\|v\|).$$

Proof. Let $\sigma : [0, 1] \to M$ be a smooth curve in $M$. Set

$$\delta := \frac{1}{2} \min_{t \in [0, 1]} \text{injrad}(\sigma(t))$$

and

$$U := \bigcup_{t \in [0, 1]} B_{\delta}(\sigma(t)).$$

Then $U$ is open and connected. The density $\omega(p, r\theta)$ is defined for $p \in U$, $\theta \in S_1(0_p)$ and $0 \leq r < \delta$.

Pick an $0 \leq r < \delta$ and define $\varpi(r, \cdot) : U \to \mathbb{R}$ by $\varpi(r, p) := \omega(p, r\theta)$. This is well-defined, i.e. does not depend on $\theta \in S_1(0_p)$, because of the local harmonicity of $M$.

We will show that for every $p \in U$ with $\varpi(r, p) \neq 0$ the derivative, namely $(d\varpi(r, \cdot))_p : T_p M \to \mathbb{R}$, vanishes. This implies that $\varpi(r, \cdot)$ is constant on the components of $U \setminus \varpi(r, \cdot)^{-1}(\{0\})$. By the connectedness of $U$ and the continuity of $\varpi(r, \cdot)$ we get the following. In the case $\varpi(r, \cdot)^{-1}(\{0\}) = \emptyset$ we have a constant $\varpi(r, \cdot)$. In the case $\varpi(r, \cdot)^{-1}(\{0\}) \neq \emptyset$ we have $\varpi(r, \cdot) = 0$.

Let $u \in T_p M$. In order to show $(d\varpi(r, \cdot))_p(u) = 0$ we construct a curve through $p$ with initial velocity $u$. Take a normalised geodesic $\gamma : [0, r] \to M$ with $\gamma(r) = p$ and $g_p(\gamma'(r), u) = 0$. Set $q := \gamma(0)$. Because of $\omega(p, -r\gamma'(r)) = \varpi(r, p) \neq 0$ the points $p$ and $q$ are not conjugate along $\gamma$.

Choose an $\varepsilon > 0$ and a one-parameter family of geodesics $\gamma_s$ with $s \in [-\varepsilon, \varepsilon]$ such that $\gamma_s(0) = q$ for $s \in [-\varepsilon, \varepsilon]$ and

$$\frac{d}{ds} \bigg|_{s=0} \gamma_s(r) = u.$$  

By the invariance under the geodesic involution (Lemma 2.3.2) we have

$$\varpi(r, q) = \omega(q, \gamma_s'(0)) = \omega(\gamma_s(r), -r\gamma'(r)) = \varpi(r, \gamma_s(r)).$$

Hence

$$(d\varpi(r, \cdot))_p(u) = \frac{d}{ds} \bigg|_{s=0} \varpi(r, \gamma_s(r)) = \frac{d}{ds} \bigg|_{s=0} \varpi(r, q) = 0.$$  

We get that $\varpi(r, \cdot)$ is constant on $U$ and therefore

$$\omega(\sigma(0), r\theta) = \varpi(r, \sigma(0)) = \varpi(r, \sigma(1)) = \omega(\sigma(1), r\theta)$$

for $0 \leq r < \delta$. By the above Lemma 3.2.3 we get the claim. □
3.3 Examples

We compute the density functions of the ROSSs, cf. [Bes78, Section 3.E], and show that locally symmetric spaces of rank 1 are examples of LH-manifolds.

**Proposition 3.3.1** (density functions of the ROSSs). Let \( p \in M \) and \( \theta \in S_1(p) \). Set \( d(\mathbb{K}) = \text{dim}(\mathbb{K}) \) for \( \mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\} \) and denote by \( m \) the \( \mathbb{K} \)-dimension of the ROSSs. If we assume that the hyperbolic spaces have sectional curvature between \(-1\) and \(-\frac{1}{4}\) we get

\[
\begin{array}{|c|c|c|c|}
\hline
M & \text{RH}^m & \mathbb{K}H^m & \text{QH}^2 \\
\hline
r^{d(\mathbb{K})m-1} \omega(p, r\theta) & (\sinh r)^{m-1} & (\sinh r)^{d(\mathbb{K})-1}(2\sinh \frac{r}{2})^{d(\mathbb{K})}(m-1) & (\sinh r)^7(2\sinh \frac{r}{2})^8 \\
\hline
\end{array}
\]

for \( 0 \leq r < \infty \) and if we assume that the projective spaces have diameter \( \pi \) we get

\[
\begin{array}{|c|c|c|c|}
\hline
M & \text{S}^m & \mathbb{K}P^m & \text{Q}P^2 \\
\hline
r^{d(\mathbb{K})m-1} \omega(p, r\theta) & (\sin r)^{m-1} & 2^{d(\mathbb{K})}(m-1)(\sin r)^{d(\mathbb{K})-1}(1-\cos r)^{\frac{d(\mathbb{K})}{2}}(m-1) & 16(\sin r)^7(1-\cos r)^4 \\
\hline
\end{array}
\]

for \( 0 \leq r \leq \pi \).

**Proof.** We only consider \( M := \mathbb{C}P^m \) since the computations for the other spaces work similarly. Choose a geodesic \( \gamma \) with \( \gamma(0) = p \) and \( \gamma'(0) = \theta \). We denote the imaginary unit by \( i \). Choose \( e_3, \ldots, e_{2m} \in T_p^1 \gamma \) such that \( (\theta \cdot i, e_3, \ldots, e_{2m}) \) is a basis of \( T_p^1 \gamma \) in which \( R_\gamma(0) \) is diagonal. Denote by \( (E_2, E_3, \ldots, E_{2m}) \) the parallel translate of \( (\theta \cdot i, e_3, \ldots, e_{2m}) \) along \( \gamma \). Then \( R_\gamma \) is diagonal in the basis \( (E_2, E_3, \ldots, E_{2m}) \) since \( R \) is parallel.

In order to compute the Jacobi fields along \( \gamma \) we need the eigenvalues of \( R_\gamma(0) \). They are 1 and \( \frac{1}{4} \) because of

\[
g_p(R(\theta \cdot i, \theta), \theta \cdot i) = 1
\]

and

\[
g_p(R(e_j, \theta), e_j) = \frac{1}{4}, \quad j = 3, \ldots, 2m.
\]

So \( J_2(r) := (\sin r)E_2(r) \) and \( J_j(r) := (2\sin \frac{r}{2})E_j(r) \) are Jacobi fields along \( \gamma \) with the initial conditions \( J_2(0) = 0 \), \( J_2'(0) = \theta \cdot i \) and \( J_j(0) = 0 \), \( J_j'(0) = e_j \) where \( j = 3, \ldots, 2m \). Hence

\[
r^{2m-1} \omega(p, r\theta) = (\sin r)^2 \left( 2\sin \frac{r}{2} \right)^{2(m-1)} = 2^{m-1}(\sin r)^{1-\cos r}^{m-1}.
\]

**Corollary 3.3.2** (locally symmetric spaces and local harmonicity). Let \( M \) be a locally Riemannian symmetric space. Then \( M \) is LH if and only if it is of rank 1 or flat.

**Proof.** If \( M \) is LH and not of rank 1, it is flat, cf. [Esc80] or [Led57]. Since for every point in a locally symmetric space there is a neighbourhood which is isometric to a neighbourhood in a symmetric space, we are done by the above lemma.
4 Blaschke Manifolds

The aim of this section is to provide the definition and some properties of Blaschke manifolds, since we will show that compact simply connected LH-manifolds are of that type in the next section. Noteworthy are Propositions 4.1.4 and 4.1.5 and the (global) basic commutativity (Theorem 4.2.5).

4.1 Definition and Some Properties

We do not present any proofs in this subsection and refer to [Bes78, Sections 5.D and 5.E] for a detailed account.

Definition 4.1.1 (spherical cut locus). We say that $M$ has spherical cut locus at $p \in M$ if $S^d_{\text{injrad}}(p) = C(p)$.

Definition 4.1.2 (Blaschke manifold). We say that $M$ is a Blaschke manifold if $M$ is compact and has spherical cut locus at every $p \in M$.

Proposition 4.1.3 (metric spheres are submanifolds). In a Blaschke manifold every metric sphere is a submanifold.

Proposition 4.1.4 (injrad($M$) = diam($M$)). For a Blaschke manifold we have injrad($M$) = diam($M$) = $d(p, q) = \text{injrad}(p)$ where $p \in M$ and $q \in C(p)$.

Proposition 4.1.5 (simple and closed geodesics). In a Blaschke manifold every geodesic is simple and closed with length $2 \text{diam}(M)$.

Proposition 4.1.6 (special case: singleton cut locus). Let $M$ be a Blaschke manifold and $p \in M$. Assume that the cut locus consists of only one element, i.e. $C(p) = \{q_p\}$. Then the following statements hold.

1. $M$ is diffeomorphic to the sphere $S^n$.

2. The map $\sigma : M \to M, p \mapsto q_p$ is an involutive isometry.

3. The Riemannian quotient $\overline{M} := M/\sigma$ is Blaschkean and diffeomorphic to $\mathbb{R}P^n$.

4. The natural projection map $\pi : M \to \overline{M}$ is the universal Riemannian covering of $\overline{M}$.

Remark. Actually, $M$ is isometric to the sphere in this case, cf. [Bes78, Theorem D.1].

4.2 Radial and Averaged Functions

Let $M$ be a Blaschke manifold and set $D := \text{diam}(M)$. The pieces of notation we define in this subsection are used in the following argumentation in the context of a Blaschke manifold only. Note that the definitions given here coincide with the ones given earlier on pointed open geodesic balls. Anyway, the results provided here are mostly only true for Blaschke manifolds, cf. [Sza90, Section 1].
Definition 4.2.1 ((associated) radial function). For a smooth function \( F : [0, D] \to \mathbb{R} \) we define the (associated) radial function (around \( p \in M \)) by

\[
R_p F : M \to \mathbb{R}, \ q \mapsto F(d(p, q)).
\]

We call \( R_p : C^\infty([0, D]) \to C^0(M) \cap C^\infty(\hat{B}_D(p)) \) radial operator. Functions \( f \in C^\infty(M) \) such that an \( F : [0, D] \to \mathbb{R} \) exists with \( f = R_p F \) are called radially symmetric functions (around \( p \)) or abbreviated radial functions (around \( p \)).

Remark. The radial operator is linear. We emphasise that the function \( R_p F \) is not necessarily differentiable in \( p \) nor in points of \( \mathcal{C}(p) \).

Lemma 4.2.2 (criterium for smoothness). Let \( F : [0, D] \to \mathbb{R} \) be a smooth function. For every \( p \in M \) the following two statements are equivalent.

1. \( R_p F \) is of class \( C^{2m} \).
2. \( F^{(2i-1)}(0) = F^{(2i-1)}(D) = 0 \) holds for \( i = 1, \ldots, m \).

Proof. \( 1. \Rightarrow 2. \): This is clear.

2. \( \Rightarrow 1. \): Set \( k := \dim \mathcal{C}(p) \) and write \( \mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k \). The function \( R_p F \) is certainly of class \( C^{2m} \) in \( B_D(p) \). So pick a point \( q \in \mathcal{C}(p) \). Since \( \mathcal{C}(p) \) is a submanifold and geodesics emanating from \( p \) hit the cut locus \( \mathcal{C}(p) \) orthogonally, we can find a chart \( (\varphi, U) \) around \( q \) such that

1. \( \varphi(q) = 0 \),
2. \( \varphi : U \cap \mathcal{C}(p) \to \{0\} \times \mathbb{R}^k \) is a diffeomorphism,
3. \( \varphi : U \setminus \mathcal{C}(p) \to \mathbb{R}^{n-k} \times \{0\} \cap \varphi(U) \) is a diffeomorphism and
4. For every geodesic \( \hat{\gamma} : \mathbb{R} \to M \) through \( p \) and \( \overline{q} \in \mathcal{C}(p) \) the set \( \varphi(\gamma(\mathbb{R}) \cap U) \) is a line through \( \varphi(\overline{q}) \) which is orthogonal to \( \{0\} \times \mathbb{R}^k \).

The function \( R_p F \circ \varphi^{-1} \) is therefore of class \( C^{2m} \) since its partial derivatives of order \( 2m \) exist and are continuous. \( \square \)

Definition 4.2.3 (average operator). Let \( f : M \to \mathbb{R} \) be a smooth function. The averaged function \( A_p f \) of \( f \) (around \( p \in M \)) is defined by

\[
A_p f : [0, D] \to \mathbb{R}, \ r \mapsto (A_p f)(r) := \lim_{\varrho \to r} \left( A_p \left( f|_{B_D(p)} \right) \right)(\varrho).
\]

We call \( A_p : C^\infty(M) \to C^\infty([0, D]) \) average operator (around \( p \)).

Remark. The average operator is linear and we have \( (A_p f)(0) = f(p) \). If we lift \( f|_{\mathcal{C}(p)} \) to a function \( \tilde{f} := f \circ \exp_p \) on \( S_D(0)_p \), we easily see that the average of \( f \) taken over the cut locus of \( p \) equals the average of \( \tilde{f} \) taken over \( S_D(0)_p \). So the limit equals the actual average, i.e.

\[
(A_p f)(D) = \frac{1}{\text{vol}(\mathcal{C}(p))} \int_{\mathcal{C}(p)} f|_{\mathcal{C}(p)} \, d\mathcal{C}(p).
\]
Lemma 4.2.4 (properties of the radial operator). Let $h : M \to \mathbb{R}$ and $F,G : [0,D] \to \mathbb{R}$ be smooth and $p \in M$.

1. $A_p R_p F = F$
2. $R_p (FG) = R_p F R_p G$
3. $A_p (h R_p G) = GA_p h$

Lemma 4.2.5 ((global) basic commutativity). Let $M$ be a locally harmonic Blaschke manifold and $p \in M$. For every smooth function $f$ on $M$ the function $R_p A_p f : M \to \mathbb{R}$ is of class $C^2$ and it holds

$$\Delta R_p A_p f = R_p A_p \Delta f.$$ 

Proof. Since the equality holds on $\bar{B}_D(p)$, we only need to prove the first claim.

By Lemma 4.2.2 we only need to show that $(A_p f)'(0) = (A_p f)'(D) = 0$. Let $\Omega : [0,D] \to \mathbb{R}$ be the function with $R_p \Omega = \omega_p$. For $0 < r < D$ we have by Green’s first identity

$$(A_p f)'(r) = - \frac{1}{\operatorname{vol}(S_r(p))} \int_{B_r(p)} \Delta f \, dB_r(p)$$

$$= - \frac{1}{\int_{S_r(0_p)} r^{n-1} \Omega(r)} \int_0^r \int_{S_r(0_p)} (\Delta f)(\exp_p \varrho \theta) \varrho^{n-1} \Omega(\varrho) \, d\varrho d\theta.$$ 

Hence

$$|(A_p f)'(r)| \leq r \left| \frac{\max_{0 \leq \varrho \leq r} (\varrho^{n-1} \Omega(\varrho)) \max_{\theta \in S_r(0_p)} (\Delta f)(\exp_p \varrho \theta)}{r^{n-1} \Omega(r)} \right| \leq r \max_{q \in B_r(p)} (\Delta f)(q).$$ 

and

$$\lim_{r \to 0^+} (A_p f)'(r) = 0.$$ 

Because of

$$0 = \int_M \Delta f \, dM = \int_0^D \int_{S_1(0_p)} (\Delta f)(\exp_p \varrho \theta) \varrho^{n-1} \Omega(\varrho) \, d\varrho d\theta$$ 

we get for $0 < r < D$

$$|(A_p f)'(r)| = \left| \frac{1}{\int_{S_r(0_p)} r^{n-1} \Omega(r)} \int_M \Delta f \, dM + (A_p f)'(r) \right|$$

$$= \left| \frac{1}{\int_{S_r(0_p)} r^{n-1} \Omega(r)} \int_r^D \int_{S_r(0_p)} (\Delta f)(\exp_p \varrho \theta) \varrho^{n-1} \Omega(\varrho) \, d\varrho d\theta \right|$$

$$\leq (D - r) \max_{q \in M \setminus B_r(p)} (\Delta f)(q).$$ 

This proves the claim because

$$\lim_{r \to D^-} (A_p f)'(r) = 0.$$ 

Remark. $R_p A_p f$ is actually smooth, but this fact is not needed below.
5 Other Notions of Harmonicity

There are two more kinds of harmonicity which are of interest for our considerations. In this section we give the definitions for globally harmonic and strongly harmonic manifolds as well as topological conditions which force LH-manifolds to be globally respectively strongly harmonic. Noteworthy are Allamigeon’s theorem (Theorem 5.1.3) and Proposition 5.2.4.

5.1 Globally Harmonic Manifolds

The most important result of global nature for LH-manifolds is Allamigeon’s theorem, cf. [Bes78, Theorem 6.82], which allows us to use the statements of the previous section.

Definition 5.1.1 (globally harmonic). A complete Riemannian manifold $M$ is said to be globally harmonic if for every $p \in M$ there exists $\Omega : \mathbb{R}^\geq 0 \rightarrow \mathbb{R}$ such that

$$ \forall v \in T_p M : \omega(p, v) = \Omega(\|v\|) $$

Remark. Notice that the choice of $\Omega$ could depend on $p$. Actually, it does not by Proposition 3.2.4.

The property ‘globally harmonic’ is often abbreviated by ‘GH’. A manifold which is GH is often called GH-manifold.

Proposition 5.1.2 (LH-manifolds are GH). Every complete LH-manifold $M$ is GH.

Proof. Let $p \in M$. The density function $\omega(p, \cdot)$ is an analytic function $T_p M \rightarrow \mathbb{R}$. Since it is radially symmetric in a neighbourhood around $0_p$, it is radially symmetric on the whole of $T_p M$.

Theorem 5.1.3 (Allamigeon’s theorem). Every complete simply connected LH-manifold $M$ is either a Blaschke manifold or diffeomorphic to $\mathbb{R}^n$.

Proof. By the previous lemma we know that $M$ is GH. Let $p \in M$. For every $0 \neq v \in T_p M$ set $\gamma_v(r) := \exp_p \left( r \frac{v}{\|v\|} \right)$ for $r \in \mathbb{R}^\geq 0$. Suppose there is no conjugate point along $\gamma_v$ for all $0 \neq v \in T_p M$. Then $\exp_p : T_p M \rightarrow M$ is a covering map and, since $M$ is simply connected, a diffeomorphism.

So take a $0 \neq v_0 \in T_p M$ and an $r_0 \in \mathbb{R}^\geq 0$ such that the first conjugate point along $\gamma_{v_0}$ is $\gamma_{v_0}(r_0)$. Then the first conjugate point along $\gamma_v$ is $\gamma_v(r_0)$ for all $0 \neq v \in T_p M$, since $\omega(p, \cdot)$ is radial. Note that $r_0$ is the same for every point in $M$. This means that $M$ is a Blaschke manifold by the Allamigeon-Warner theorem, cf. [Bes78, Corollary 5.31].

5.2 Strongly Harmonic Manifolds

The interesting result of this subsection is Proposition 5.2.4, which can also be found in [Sza90, Theorem 1.1]. However, we do not need any of the following statements for our discussion.

Theorem 5.2.1 (heat kernel). Let $M$ be a compact Riemannian manifold. There exists a unique $k : \mathbb{R}^\geq 0 \times M \times M \rightarrow \mathbb{R}$ with the following properties.

1. $k$ is continuous, of class $C^1$ in the first variable and of class $C^2$ in the second.
2. \[ \forall t \in \mathbb{R}^>0 \ \forall q \in M : (\partial_t + \Delta)k(t, \cdot, q) = 0. \]

3. \[ \forall f \in C^\infty(M) \ \forall q \in M : \lim_{t \to 0} \int_M k(t, \cdot, q) f \ dM = f(q). \]

This \( k \) is actually smooth and \( k(t, p, q) = k(t, q, p) \) holds for every \( t \in \mathbb{R}^>0 \) and \( p, q \in M \).

Remark. A proof can be found in [BGM71, Section III.E].

Definition 5.2.2 (strongly harmonic). A compact Riemannian manifold \( M \) is said to be strongly harmonic if for every \( t \in \mathbb{R}^>0 \) there exists a \( K_t : \mathbb{R}^2 \to \mathbb{R} \) such that

\[ \forall t \in \mathbb{R}^>0 \ \forall p, q \in M : k(t, p, q) = K_t(d(p, q)). \]

Remark. The property ‘strongly harmonic’ is often abbreviated by ‘SH’. A manifold which is SH is often called SH-manifold.

Since a unique heat kernel also exists in the non-compact case, we could define a notion of strong harmonicity in this case as well, cf. [Str83, Theorem 3.5] and [Sza90, p. 7], but this is not needed in the following considerations.

Proposition 5.2.3 (SH-manifolds are GH). Every strongly harmonic manifold is globally harmonic.

Proof. It suffices to show that \( M \) is locally harmonic. For every \( t \in \mathbb{R}^>0 \) define \( K_t : [0, \text{injrad}(M)] \to \mathbb{R} \) such that \( k(t, \cdot, q) = R_q K_t \) for every \( q \in M \). Then we have

\[ R_q \partial_t K_t = \partial_t R_q K_t = \partial_t k(t, \cdot, q) = -\Delta k(t, \cdot, q) = -\Delta R_q K_t = R_q K''_t + \eta_q R_q K'_t. \]

In particular \( K_t : [0, \text{injrad}(M)] \to \mathbb{R} \) is a solution of a linear ODE of second order. Furthermore \( K'_t \) is non-zero in a dense subset of \([0, \text{injrad}(M)]\) since otherwise \( K_t \) would be constant and \( \partial_t K_t \) would be zero, which would contradict the third property of the heat kernel. Hence \( \eta_q \) is radial.

\[ \square \]

Theorem 5.2.4 (LH-manifolds are SH). Every compact simply connected LH-manifold is strongly harmonic.

Proof. We know that \( M \) is globally harmonic and a Blaschke manifold of diameter say \( D \). It suffices to show that \( \overline{K} : \mathbb{R}^>0 \times M \times M \to \mathbb{R}, \ (t, p, q) \mapsto \overline{K}(t, p, q) := (R_q A_q k(t, \cdot, q))(p) \) also satisfies the properties of the heat kernel \( k \), since it is unique. Pick \( t \in \mathbb{R}^>0 \) and \( p, q \in M \). The function \( \overline{K} \) is continuous, of class \( C^1 \) in the first variable and of class \( C^2 \) in the second. We have

\[ \partial_t \overline{K}(t, \cdot, q) = \partial_t R_q A_q k(t, \cdot, q) = R_q A_q \partial_t k(t, \cdot, q) = -R_q A_q \Delta k(t, \cdot, q) = -\Delta R_q A_q k(t, \cdot, q) = -\Delta \overline{K}(t, \cdot, q) \]
\[ \lim_{t \to 0} \int_M k(t, \cdot, q) f \, dM = \lim_{t \to 0} \int_M R_q A_q k(t, \cdot, q) f \, dM \]
\[ = \lim_{t \to 0} \int_0^D (A_q k(t, \cdot, q))(r) \int_{S_t(0)} f(\exp_q r\theta) r^{n-1} \omega(q, r\theta) \, d\theta \, dr \]
\[ = \lim_{t \to 0} \int_0^D (A_q f)(r) \int_{S_t(0)} k(t, \exp_q r\theta, q) r^{n-1} \omega(q, r\theta) \, d\theta \, dr \]
\[ = \lim_{t \to 0} \int_M k(t, \cdot, q) R_q A_q f \, dM \]
\[ = f(q) \]

where we use \( R_q A_q f \in C^\infty(M) \) in the last equality.

6 Radial Eigenfunctions

In this section we discuss some properties of radially symmetric eigenfunctions of the Laplacian in a locally harmonic Blaschke manifold \( M \). We fix an eigenvalue \( \lambda > 0 \) and write \( V_\lambda \) for the space of radial eigenfunctions around \( p \in M \). Set \( D := \operatorname{diam}(M) \) and denote by \( H : ]0, D[ \to \mathbb{R} \) the function with \( R_p H = \eta_p \). Since the linear ODE

\[ y'' + Hy' + \lambda y = 0 \]

is central to this section, we will refer to it as ‘the ODE’. The main results are summarised in Proposition 6.0.1 and Corollary 6.0.2. They can also be found in [Sza90, Section 2].

**Proposition 6.0.1.** The ODE has at exactly one solution \( y : ]0, D[ \to \mathbb{R} \) with the initial conditions

\[ \lim_{r \to 0} y(r) = 1 \quad \text{and} \quad \lim_{r \to 0} y'(r) = 0. \]

This solution can be extended to a smooth function \( \Phi_\lambda : ]0, D[ \to \mathbb{R} \). For every \( p \in M \) the function \( R_p \Phi_\lambda \) is smooth and for \( \varphi \in V_\lambda^p \) it holds \( \varphi = \varphi(p) R_p \Phi_\lambda \).

**Proof. Uniqueness:** Given two solutions \( y_1, y_2 : ]0, D[ \to \mathbb{R} \) with

\[ \lim_{r \to 0} y_i(r) = 1 \quad \text{and} \quad \lim_{r \to 0} y_i'(r) = 0, \quad i = 1, 2 \]

we get a solution \( \overline{y} := y_1 - y_2 \) with

\[ \lim_{r \to 0} \overline{y}(r) = 0 \quad \text{and} \quad \lim_{r \to 0} \overline{y}'(r) = 0. \]

We have to show that \( \overline{y} = 0 \). By multiplying the ODE with \( \overline{y} \) we get

\[ 0 = \overline{y}'' \overline{y} + H(\overline{y})^2 + \lambda \overline{y} \overline{y} = \frac{(\overline{y}')^2}{2} + H(\overline{y})^2 + \lambda \frac{(\overline{y}')^2}{2}. \]
By setting
\[ z := \frac{1}{2}((\overline{y}')^2 + \lambda y^2) \geq 0 \]
we get
\[ z' = \frac{1}{2}((\overline{y}')^2 + \lambda y^2)' = -H(\overline{y}')^2 \leq 0 \]
on \] with \( \varepsilon > 0 \) sufficiently small. Because of
\[ \lim_{r \to 0} z(r) = 0 \]
it follows that \( z|_{0,\varepsilon} = 0 \) and \( \overline{y}|_{0,\varepsilon} = 0 \). Then \( \overline{y} = 0 \) holds by the Picard-Lindelöf theorem.

Existence: Let be \( \varphi, \psi \in V^\lambda \) and \( p, q \in M \). The function \( R_p A_p \varphi \) is again an eigenfunction for the eigenvalue \( \lambda \) by the global basic commutativity (Theorem 4.2.5). In particular, \( R_p A_p \varphi \) is smooth. For \( p \) we pick \( \varphi \) such that \( \varphi(p) \neq 0 \) and set
\[ \Phi_\lambda := \frac{A_{q\varphi}}{\varphi(p)}. \]
This definition is independent of the choices since by Lemma 2.5.6 we get that \( A_{p\varphi}, A_{q\varphi} \) and \( A_{p\psi} \) solve the ODE. Hence the claim follows.

Remark. In the following, we will use the notation \( \Phi_\lambda : [0, D] \to \mathbb{R} \) for the unique extended solution of the ODE with the described initial conditions and call it ‘the solution’.

Corollary 6.0.2. The space of eigenfunctions is spanned by the radial eigenfunctions, i.e.
\[ V^\lambda = \text{span} \{ V_p \mid p \in M \} = \text{span} \{ R_p \Phi_\lambda \mid p \in M \}. \]

Proof. Assume there were a \( 0 \neq \varphi \in V^\lambda \) with \( \langle \varphi, R_p \Phi_\lambda \rangle_{L^2(M)} = 0 \) for all \( p \in M \). Hence
\[ 0 = \langle \varphi, R_p \Phi_\lambda \rangle_{L^2(M)} = \langle R_p A_p \varphi, R_p \Phi_\lambda \rangle_{L^2(M)} = \varphi(p) \langle R_p \Phi_\lambda, R_p \Phi_\lambda \rangle_{L^2(M)}. \]
So either \( \varphi = 0 \) or \( \| R_p \Phi_\lambda \|_{L^2(M)} = 0 \) for a \( p \in M \). Both possibilities contradict the assumptions.

Proposition 6.0.3 (harmonicity and \( L^2 \)-product). Let \( M \) be a locally harmonic Blaschke manifold. Then for every \( p \in M \) and smooth \( F, G : [0, D] \to \mathbb{R} \) the function
\[ M \to \mathbb{R}, \ q \mapsto \langle R_p F, R_q G \rangle_{L^2(M)} \]
is radial around \( p \), i.e. the \( L^2 \)-product of two radial functions is radial again.

Proof. Let \( q \in M \) and \( \Omega : [0, D] \to \mathbb{R} \) the function with \( R_q \Omega = \omega_q \). Denote by \( (\lambda_i)_{i \in \mathbb{N}_0} \) the spectrum of the Laplacian. Then \( (R_p \Phi_{\lambda_i})_{i \in \mathbb{N}} \) forms an orthogonal basis of the space of radial functions around \( p \). Let \( a_i \in \mathbb{R} \) be the coefficients of \( R_p F \) in this basis.

By Proposition 6.0.1, we get
\[ A_q R_p \Phi_{\lambda_i} = (A_q R_p \Phi_{\lambda_i})(0) \Phi_{\lambda_i} = (R_p \Phi_{\lambda_i})(q) \Phi_{\lambda_i}. \]
Hence

\[
(R_pF, R_qG)_{L^2(M)} = \int_M R_pF R_qG \, dM
= \sum_{i \in N_0} a_i \int_M R_p \Phi_\lambda, R_qG \, dM
= \sum_{i \in N_0} a_i \int_0^D \int_{S_1(0)} (R_p \Phi_\lambda)(\exp r \theta) G(r) r^{n-1} \Omega(r) \, d\theta dr
= \sum_{i \in N_0} a_i \text{vol}(S^{n-1}) \int_0^D (A_q R_p \Phi_\lambda)(r) G(r) r^{n-1} \Omega(r) \, dr
= \text{vol}(S^{n-1}) \left( \sum_{i \in N_0} \left( a_i \int_0^D \Phi_\lambda(r) G(r) r^{n-1} \Omega(r) \, dr \right) (R_p \Phi_\lambda)(q) \right).
\]

This implies the claim. \(\square\)

Remark. If we set \(F := \Phi_\lambda := G\) in the above computation, we get

\[
\langle R_p \Phi_\lambda, R_q \Phi_\lambda \rangle_{L^2(M)} = \left( \text{vol}(S^{n-1}) \int_0^D \Phi_\lambda(r)^2 r^{n-1} \Omega(r) \, dr \right) (R_p \Phi_\lambda)(q).
\]

The statement “if in a Blaschke manifold \(M\) the \(L^2\)-product of two radial functions is radial again, then \(M\) is locally harmonic” is also true, cf. [Sza90, Proposition 2.1].

7 The ‘Nice Embedding’ of Harmonic Manifolds

For this section let \(M\) be a locally harmonic Blaschke manifold. The density function \(\omega_p\) in \(p \in M\) is radial with \(\omega_p = R_p \Omega\) for a suitable \(\Omega : [0, D] \to \mathbb{R}\). For a smooth \(G : [0, D] \to \mathbb{R}\) we set

\[
\|G\|_{L^2_\Omega} := \sqrt{\int_0^D G(r)^2 r^{n-1} \Omega(r) \, dr}.
\]

Then it holds

\[
\|R_p G\|_{L^2(M)} = \sqrt{\text{vol}(S^{n-1}) \int_0^D G(r)^2 r^{n-1} \Omega(r) \, dr} = \sqrt{\text{vol}(S^{n-1}) \|G\|_{L^2_\Omega}}.
\]

The following results allow us to embed \(M\) in a Euclidean space such that the geodesics are mapped into congruent screw lines. Together with Lemma 8.3.2 this forms the key idea for the proof of Lichnerowicz’s conjecture. The finite-dimensional version can be found in [Bes78, Theorem 6.99], the infinite-dimensional in [Sza90, Theorem 3.1].

**Theorem 7.0.1 (embedding theorem).** For a non-constant \(G \in C^\infty([0, D])\) we define the map

\[
R^G : M \to L^2(M), \ p \mapsto R^G(p) := c_G R_p G
\]
with

\[ c_G := \frac{\sqrt{\lambda}}{\|G\|_{L^2(\mathbb{R})}^2 \sqrt{\text{vol}(S^{n-1})}}. \]

This map has the following properties.

1. \( R^G(M) \subset S_{C_G} \) where \( S_{C_G} \) is the sphere in \( L^2(M) \) of radius

\[ C_G := \frac{\|G\|_{L^2(\mathbb{R})}^2 \sqrt{\lambda}}{\|G\|_{L^2(\mathbb{R})}^2} \]

2. For a normalised geodesic \( \gamma \) of \( M \) the curve \( R^G \circ \gamma \) is a screw line of \( L^2(M) \). For two normalised geodesics \( \gamma \) and \( \sigma \) of \( M \) the screw lines \( R^G \circ \gamma \) and \( R^G \circ \sigma \) have the same screw function. They are therefore congruent.

3. \( R^G \) is an isometric immersion.

Proof. 1. For \( p \in M \) we have

\[ \left\| R^G(p) \right\|_{L^2(M)} = c_G \| R_pG \|_{L^2(M)} = c_G \sqrt{\text{vol}(S^{n-1})} \|G\|_{L^2(\mathbb{R})} = C_G. \]

This means \( R^G(M) \subset S_{C_G} \).

2. For \( p, q \in M \) we have

\[
\left\| R^G(p) - R^G(q) \right\|_{L^2(M)}^2 = \left\| R^G(p) \right\|_{L^2(M)}^2 + \left\| R^G(q) \right\|_{L^2(M)}^2 - 2 \left\langle R^G(p), R^G(q) \right\rangle_{L^2(M)}
\]

\[ = 2C_G^2 - 2 \left\langle R^G(p), R^G(q) \right\rangle_{L^2(M)} \]

\[ = 2C_G^2 - 2c_G^2 \left( R_pG, R_qG \right)_{L^2(M)}. \]

By Proposition 5.0.3 the function \( \left( R_pG, R_qG \right)_{L^2(M)} \) only depends on \( d(p, q) \). For \( s_0, s \in \mathbb{R} \) we set \( p := \gamma(s_0 + s) \) and \( q := \gamma(s_0) \) respectively \( p := \sigma(s_0 + s) \) and \( q := \sigma(s_0) \) to get the claim.

3. Pick \( p \in M \) and \( v \in T_pM \) with \( \|v\|_2 = 1 \). Let \( \gamma \) be a geodesic parametrised by arc length with \( \gamma(0) = p \) and \( \gamma'(0) = v \). We have

\[
\left\| (dR^G)_p(v) \right\|_{L^2(M)} = \left\| \left. \frac{d}{dt} \right|_{t=0} R^G(\gamma(t)) \right\|_{L^2(M)}
\]

\[ = c_G \left\| \left. \frac{d}{dt} \right|_{t=0} R_{\gamma(t)}G \right\|_{L^2(M)} \]

\[ = c_G \left\| \left. \frac{d}{dt} \right|_{t=0} G(d(\gamma(t), \cdot)) \right\|_{L^2(M)} \]

\[ = c_G \sqrt{\int_M \left( \left. \frac{d}{dt} \right|_{t=0} G(d(\gamma(t), \cdot)) \right)^2 \, dM}. \]
\[ c_G \sqrt{\int_0^D \int_{S_1(0,p)} \left( \frac{d}{dt} \bigg|_{t=0} G(d(\gamma(t), \exp_p r\theta)) \right)^2 r^{n-1} \Omega(r) \, d\theta \, dr} \]

\[ = c_G \sqrt{\int_0^D \int_{S_1(0,p)} G'(d(p, \exp_p r\theta))^2 \cos^2 (v, \theta) \, r^{n-1} \Omega(r) \, d\theta \, dr} \]

\[ = c_G \sqrt{\int_0^D G'(r)^2 r^{n-1} \Omega(r) \, dr} \sqrt{\int_{S_1(0,p)} \cos^2 (v, \theta) \, d\theta} \]

\[ = c_G \|G'\|_{L_0^2} \sqrt{\frac{\text{vol}(S^{n-1})}{n}} = 1. \]

This shows that \( R^G \) is an isometric immersion.

\[ \square \]

**Corollary 7.0.2** (Besse’s nice embedding: special case \( G = \Phi_\lambda \)). For an eigenvalue \( \lambda > 0 \) of the Laplacian denote by \( \Phi := \Phi_\lambda \) the solution of the ODE and set \( \overline{M} := R^\Phi(M) \).

1. Let \( \Phi(D) = 1 \) and \( M \) be diffeomorphic to the sphere \( S^n \). Then \( \overline{M} \) is diffeomorphic to \( \mathbb{RP}^n \) and a locally harmonic Blaschke manifold. The map \( R^\Phi : M \to \overline{M} \) is the universal Riemannian covering map.

2. Let \( \Phi(D) \neq 1 \) or \( M \) be not diffeomorphic to the sphere \( S^n \). Then the map \( R^\Phi : M \to V^\lambda \) is an injective isometric immersion, i.e. an embedding since \( M \) is compact. The manifold \( \overline{M} \) is a minimal submanifold of the sphere \( S_{C_n} \). For a unit speed geodesic \( \gamma \) of \( M \) set \( c := R^\Phi \circ \gamma \).

Then we have for every \( s_0, s \in \mathbb{R} \)

\[ (c(s_0), c(s))_{L_2(M)} = C_\Phi^2 \Phi(d(\gamma(s_0), \gamma(s))). \]

**Proof.** Let \( p, q \in M \) be points with \( R^\Phi(p) = R^\Phi(q) \). From the remark after Proposition 6.0.3 and the proof of the second statement of the embedding theorem we get

\[ 0 = \|R^\Phi(p) - R^\Phi(q)\|_{L_2(M)}^2 = 2C_\Phi^2 - 2C_\Phi^2 \langle R_p \Phi, R_q \Phi \rangle_{L_2(M)} \]

\[ = 2C_\Phi^2 - 2 \|\Phi'\|_{L_0^2}^2 \text{vol}(S^{n-1}) \|\Phi\|_{L_0^2}^2 \langle R_p \Phi, \Phi \rangle(q) \]

\[ = 2C_\Phi^2 - 2C_\Phi^2 \langle R_p \Phi, \Phi \rangle(q). \]

It follows \( 1 = \langle R_p \Phi(q) = \Phi(d(p, q)) \). This means that \( R^\Phi(p) = R^\Phi(q) \) for all \( q \in S_{d(p,q)}^d \). We recall that \( S_{d(p,q)}^d(p) \) is a submanifold of \( M \). But then it must be a single point since otherwise we had a contradiction to the fact that \( R^\Phi \) is an isometric immersion. The only case in which \( S_{d(p,q)}^d(p) \) is singleton occurs for \( M \) diffeomorphic to the sphere \( S^n \) and \( d(p, q) = D \), cf. Proposition 4.1.14.

Then \( \overline{M} \) is Blaschkean and diffeomorphic to \( \mathbb{RP}^n \). The map \( R^\Phi : M \to \overline{M} \) is the universal Riemannian covering map and therefore \( \overline{M} \) locally harmonic. This completes the first part.
We are left to show that the embedding is minimal. First we remark that for every $s$ and therefore $s$

screw lines, cf. [Sak82, Theorems 6.2 and 6.5].

through (minimal) embeddings into a sphere such that all geodesics are mapped into congruent

the canonical metric. Hence we need not consider the first case in the following considerations.

We have for every $p \in M$

$$\langle c(s_0), c(s) \rangle_{L^2(M)} = c_0^2 \langle R_{\gamma(s_0)} \Phi, R_{\gamma(s)} \Phi \rangle_{L^2(M)} = c_0^2 \text{vol}(S^{n-1}) \| \Phi \|_{L^2(M)}^2 \Phi(d(\gamma(s_0), \gamma(s)))$$

We are left to show that the embedding is minimal. First we remark that for every $p \in M$

$$\lambda = \frac{\langle \Delta R_p \Phi, R_p \Phi \rangle_{L^2(M)}}{\| R_p \Phi \|_{L^2(M)}^2} = \frac{1}{\| R_p \Phi \|_{L^2(M)}^2} \int_M (\text{grad} R_p \Phi, \text{grad} R_p \Phi)_{L^2(M)} \, dM$$

$$= \frac{1}{\| R_p \Phi \|_{L^2(M)}^2} \int_M \nabla_{E_p} R_p \Phi \nabla_{E_p} R_p \Phi \, dM = \frac{\| R_p \Phi' \|_{L^2(M)}^2}{\| R_p \Phi \|_{L^2(M)}^2} = \frac{\| \Phi' \|_{L^2(M)}^2}{\| \Phi \|_{L^2(M)}^2}$$

holds. Set $N := \text{dim} V^\lambda$ and choose an $L^2$-orthonormal basis $(\phi_1, \ldots, \phi_N)$ of $V^\lambda$. Coordinates $(x_1, \ldots, x_N)$ on $M$ are given by

$$x_i(R^\Phi(p)) := \langle \phi_i, R^\Phi(p) \rangle_{L^2(M)} = c_\Phi \int_M \phi_i R_p \Phi \, dM, \quad i = 1, \ldots, N.$$ 

The submanifold $M \subset S_{C_\Phi}$ is minimal if and only if every $x_i$ is an eigenfunction to the eigenvalue $\frac{n}{C_\Phi}$, cf. [KN96, Note 14, Example 3]. Because of $\Delta R_p \Phi = \lambda R_p \Phi$ this is equivalent to $\lambda = \frac{n}{C_\Phi}$.

\[\square\]

**Remark.** Since we show in the next section that a locally harmonic Blaschke manifold which is
diffeomorphic to $\mathbb{R}P^n$ carries the canonical metric, our $M$ in the first case is then the sphere with
the canonical metric. Hence we need not consider the first case in the following considerations.

Noteworthy is the characterisation of globally harmonic manifolds and Blaschke manifolds
through (minimal) embeddings into a sphere such that all geodesics are mapped into congruent
screw lines, cf. [Sak82, Theorems 6.2 and 6.5].

The embedding in the second case above is actually Besse’s nice embedding, cf. [Bes78, Theorem
6.99]. It is defined by

$$M \ni p \mapsto \sqrt{\frac{n \text{vol}(M)}{\lambda N}} (\phi_1(p), \ldots, \phi_N(p)) \in \mathbb{R}^N.$$ 

We have for every $p \in M$ and $i = 1, \ldots, N$

$$\langle R^\Phi(p), \phi_i \rangle_{L^2(M)} = c_\Phi \text{vol}(S^{n-1}) \| \Phi \|_{L^2(M)}^2 \phi_i(p)$$

and therefore

$$\sqrt{\frac{n \text{vol}(M)}{\lambda N}} = c_\Phi \text{vol}(S^{n-1}) \| \Phi \|_{L^2(M)}^2 = C_\Phi \| R_p \Phi \|_{L^2(M)}^2 = \sqrt{\frac{n}{\lambda}} \| R_p \Phi \|_{L^2(M)}^2$$

or

$$\text{vol}(M) = \frac{\text{vol}(M)}{\lambda N} \| R_p \Phi \|_{L^2(M)}^2.$$
8 Proof of Lichnerowicz’s Conjecture

In this section let $M$ be a locally harmonic Blaschke manifold and assume without loss of generality that $\text{diam}(M) = \pi$. By pinning down the possible density functions of $M$ (Lemma 8.1.7) we are able to find its first eigenvalue and to solve the ODE for it (Lemma 8.2.1). Then we present two variants of the proof of Lichnerowicz’s conjecture. The first one uses the nice embedding (Corollary 7.0.2) and Lemma 8.3.2. The second one is intrinsic, but more complex so that we only refer to the literature.

For the rest of the section we fix an eigenvalue $\lambda > 0$, a point $p \in M$ and the solution $\Phi := \Phi_\lambda$ of the ODE. From now on we consider the average $A_p f : [0, \pi] \to \mathbb{R}$ of a radial function $f : M \to \mathbb{R}$ around $p$ to be periodically extended to $\mathbb{R}$. That means we consider the function $f \circ \gamma : \mathbb{R} \to \mathbb{R}$, where $\gamma : \mathbb{R} \to M$ is a unit speed geodesic with $\gamma(0) = p$, instead of $A_p f : [0, \pi] \to \mathbb{R}$. This new function is $2\pi$-periodic and even. In particular, $\Phi : \mathbb{R} \to \mathbb{R}$ has these properties. Alternatively, we can set $A_p f : \mathbb{R} \to \mathbb{R}$, $r \mapsto A_p f(\pi - |\pi - r| \mod 2\pi)$ since $\forall r, t \in \mathbb{R} : d(\gamma(r), \gamma(t)) = \pi - |\pi - r - t| \mod 2\pi$ holds. Furthermore we set $\Omega := A_p \omega_p$ and $\hat{\Omega} : \mathbb{R} \to \mathbb{R}$, $r \mapsto r^{n-1}\Omega(r)$ so that in particular $\hat{\Omega}$ is odd, $\hat{\Omega}^2$ is even and

$$\Phi'' + \frac{\hat{\Omega}'}{\Omega} \Phi' + \lambda \Phi = 0.$$ 

holds on $\mathbb{R} \setminus \{k\pi \mid k \in \mathbb{Z}\}$.

8.1 Possible Density Functions

We present Szabó’s careful analysis of the possible forms of density functions for locally harmonic Blaschke manifolds. More precisely, our aim is it to show Lemma 8.1.7 which states that the function $\hat{\Omega}$ is the product of a power of sine and a power of cosine. We follow [Sza90, Section 4] with two exceptions. The proof of Lemma 8.1.2 is a slightly changed version of [Nik05, Theorem 2] and the proof of Lemma 8.1.5 is new.

First we show that $\Phi$ and $\hat{\Omega}^2$ are trigonometric polynomials of a special form.

Lemma 8.1.1. There is a polynomial $P : \mathbb{R} \to \mathbb{R}$ with real coefficients such that

$$\Phi = P \circ \cos.$$

Proof. Let $\gamma : \mathbb{R} \to M$ be a unit speed geodesic in $M$ with $\gamma(0) = p$. We have

$$\text{span} \{(R_q \Phi) \circ \gamma \mid q \in \gamma(\mathbb{R})\} = \text{span} \{\Phi(d(\gamma(\cdot), q)) \mid q \in \gamma(\mathbb{R})\} = \text{span} \{\Phi(\pi - |\pi - \cdot - t| \mod 2\pi) \mid t \in \mathbb{R}\} = \text{span} \{\Phi(\cdot - t) \mid t \in \mathbb{R}\}.$$
Since \( \text{span} \{ R_q \Phi \mid q \in \gamma(\mathbb{R}) \} \) is a subspace of the finite-dimensional \( V^\lambda \), it is finite-dimensional. 
Because precomposing with \( \gamma \) is linear, we have that \( \text{span} \{ \Phi(\cdot - t) \mid t \in \mathbb{R} \} \) is a finite-dimensional subspace of \( C^\infty(\mathbb{R}) \). Because \( \Phi \) is \( 2\pi \)-periodic and even, the claim follows from the Lemmata \[A.0.2\] and \[A.0.3\].

**Lemma 8.1.2.** There is a polynomial \( O : \mathbb{R} \to \mathbb{R} \) with real coefficients such that

\[
\hat{\Omega}^2 = O \circ \cos.
\]

**Proof.** Let \( \gamma : \mathbb{R} \to M \) be a unit speed geodesic in \( M \) with \( \gamma(0) = p \) and let \( (e_2, \ldots, e_n) \) be a positively oriented orthonormal basis of \( T_p^\perp \gamma \). Denote by \( (E_2, \ldots, E_n) \) its parallel translates along \( \gamma \). In this proof we will use the representation of Jacobi tensors in the basis \( (E_2, \ldots, E_n) \), i.e. they are considered to be maps \( \mathbb{R} \to \mathbb{R}^{(n-1) \times (n-1)} \).

Denote by \( J \) and \( K \) the Jacobi tensors along \( \gamma \) with initial conditions \( J(0) = 0, J'(0) = I \), \( K(0) = I \) and \( K'(0) = 0 \) where \( I \in \mathbb{R}^{(n-1) \times (n-1)} \) is the identity matrix. Let \( r \in \mathbb{R} \) and \( t \in \mathbb{R} \setminus \{ k\pi \mid k \in \mathbb{Z} \} \). We set

\[
L(t) := J^{-1}(t)K(t)
\]
and

\[
J(t) := J'(t)J^{-1}(t)K(t) - K'(t) = J'(t)L(t) - K'(t).
\]

Because of

\[
J^T(t)J'(t) - (J^T)'(t)J(t) = 0
\]
and

\[
J^T(t)K'(t) - (J^T)'(t)K(t) = -I
\]
we get

\[
J^T(t)J(t) = J^T(t)J'(t)J^{-1}(t)K(t) - J^T(t)K'(t) = (J^T)'(t)K(t) - J^T(t)K'(t) = I.
\]

Hence \( J(t) \) is invertible with \( \det J^{-1}(t) = \det J^T(t) = \det J(t) = \hat{\Omega}(t) \).

Set

\[
J_t(r) := (J(r)L(t) - K(r))J^{-1}(t).
\]

Because \( J_t \) is a Jacobi tensor along \( \gamma \) with

\[
J_t(t) = (J(t)L(t) - K(t))J^{-1}(t) = 0
\]
and

\[
J'_t(t) = (J'(t)L(t) - K'(t))J^{-1}(t) = I
\]
it holds \( \det J_t(r) = \hat{\Omega}(r - t) \). Hence

\[
\hat{\Omega}(r - t) = \det J_t(r) = \det (J(r)L(t) - K(r)) \det J^{-1}(t) = \det (J(r)L(t) - K(r)) \hat{\Omega}(t)
\]
and

\[
\hat{\Omega}^2(r - t) = \det (J(r)L(t) - K(r))^2 \hat{\Omega}^2(t).
\]
By expanding the determinant we see that \( \text{span} \{ \hat{\Omega}^2(\cdot - t) \mid t \in \mathbb{R} \setminus \{ k\pi \mid k \in \mathbb{Z} \} \} \) is finite-dimensional and therefore \( \text{span} \{ \hat{\Omega}^2(\cdot - t) \mid t \in \mathbb{R} \} \) as well. The Lemmata \( \Lambda.0.2 \) and \( \Lambda.0.3 \) yield the claim.

The next step is to examine \( P \) and \( O \) by finding restrictions to their possible roots.

**Lemma 8.1.3.** The numbers \(-1\) and \(1\) are roots of \( O \).

**Proof.** This follows from
\[
O(-1) = O(\cos \pi) = \hat{\Omega}(\pi)^2 = \pi^{2n-2}\Omega(\pi)^2 = 0
\]
and
\[
O(1) = O(\cos 0) = \hat{\Omega}(0)^2 = 0 \cdot \Omega(0)^2 = 0.
\]

**Lemma 8.1.4.** The following three statements hold.

1. All roots of \( P \) have multiplicity one.
2. All roots of \( P' \) have multiplicity one.
3. Except \(-1\) and \(1\), all roots of \( O \) are also roots of \( P' \).

**Proof.** In \( \mathbb{R} \setminus \{ k\pi \mid k \in \mathbb{Z} \} \) we have the equality
\[
\Phi'' + \frac{\hat{\Omega}'}{\Omega} \Phi' = -\lambda \Phi.
\]
In the first part of the proof we work in a compact interval of \( \mathbb{R} \setminus \{ k\pi \mid k \in \mathbb{Z} \} \) where \( \Phi' \) has no roots. By setting
\[
Q := O(P')^2(1 - \text{id}^2)
\]
we get
\[
Q \circ \cos = (O \circ \cos)(P' \circ \cos)^2(1 - \cos^2) = (O \circ \cos)(P' \circ \cos)^2 \sin^2 = \hat{\Omega}^2(\Phi')^2
\]
and
\[
(\log(Q \circ \cos))' = \left(\log \left( \frac{\hat{\Omega}^2(\Phi')^2}{\Omega^2(\Phi')^2} \right) \right)' = \frac{\left(\hat{\Omega}^2(\Phi')^2\right)'}{\hat{\Omega}^2(\Phi')^2} + \frac{\left((\Phi')^2\right)'}{\left(\Phi'\right)^2} = 2 \left( \frac{\hat{\Omega}'}{\Omega} + \frac{\Phi''}{\Phi'} \right) = -2\lambda \frac{\Phi'}{\Phi'}
\]
\[
= 2\lambda \frac{P \circ \cos}{(P' \circ \cos) \sin}.
\]
Hence
\[
\log(Q \circ \cos) = 2\lambda \int \frac{P \circ \cos}{(P' \circ \cos) \sin}
\]
and the substitution of \( \cos \) yields

\[
\log Q = -2\lambda \int \frac{P}{(1 - i\lambda^2)P'} \, dx.
\]

Let be \( x \in \mathbb{R} \) for the rest of the proof. Let \( \pi_1, \ldots, \pi_\nu \in \mathbb{C} \) be the (distinct) roots of \( P \) with multiplicities \( p_1, \ldots, p_\nu \). Denote by \( \varrho_1, \ldots, \varrho_\mu \in \mathbb{C} \) the (distinct) roots of \( P' \) which are not roots of \( P \) and by \( r_1, \ldots, r_\mu \) their multiplicities. Let the leading coefficients be \( A \) and \( B \) respectively. We can write

\[
P(x) = A(x - \pi_1)^{p_1} \cdots (x - \pi_\nu)^{p_\nu},
\]

\[
P'(x) = B(x - \pi_1)^{p_1 - 1} \cdots (x - \pi_\nu)^{p_\nu - 1}(x - \varrho_1)^{r_1} \cdots (x - \varrho_\mu)^{r_\mu}
\]

and

\[
\log Q(x) = -\frac{2\lambda A}{B} \int \frac{(x - \pi_1) \cdots (x - \pi_\nu)}{(1 - x)(1 + x)(x - \varrho_1)^{r_1} \cdots (x - \varrho_\mu)^{r_\mu}} \, dx.
\]

By the partial fraction expansion of the integrand we get that \( r_1 = \cdots = r_\mu = 1 \) and \( -1 \neq \varrho_i \neq 1 \) for \( i = 1, \ldots, \mu \) since otherwise \( Q \) would not be a polynomial. Moreover the partial fraction expansion gives us

\[
Q(x) = C(1 - x)^\sigma (1 + x)^\tau (x - \varrho_1)^{q_1} \cdots (x - \varrho_\mu)^{q_\mu}
\]

where \( \sigma, \tau, q_1, \ldots, q_\mu \in \mathbb{N}_0 \) and \( C \in \mathbb{R} \). By the definition of \( Q \) we even know \( \sigma, \tau \geq 1 \) and \( q_1, \ldots, q_\mu \geq 2 \).

Since \( O \) is a polynomial and

\[
O(x) = Q(x)(P')^{-2}(x)(1 - x^2)^{-1}
\]

\[
= C(1 - x)^\sigma (1 + x)^\tau (x - \varrho_1)^{q_1} \cdots (x - \varrho_\mu)^{q_\mu}
\]

\[
\cdot B^{-2}(x - \pi_1)^{-2(p_1 - 1)} \cdots (x - \pi_\nu)^{-2(p_\nu - 1)}(x - \varrho_1)^{-2} \cdots (x - \varrho_\mu)^{-2}
\]

\[
(1 - x^2)^{-1}
\]

\[
= CB^{-2}(1 - x)^{\sigma - 1}(1 + x)^{\tau - 1}(x - \varrho_1)^{q_1 - 2} \cdots (x - \varrho_\mu)^{q_\mu - 2}
\]

\[
(1 - x)(1 + x)(x - \varrho_1)^{r_1} \cdots (x - \varrho_\mu)^{r_\mu}
\]

holds, we get \( -2p_i + 2 \geq 0 \) for \( i = 1, \ldots, \nu \) and therefore \( p_1 = \cdots = p_\nu = 1 \).

We keep the notation of the above proof, i.e. denote by \( \pi_1, \ldots, \pi_\nu \) the roots of \( P \) and by \( \varrho_1, \ldots, \varrho_{\nu - 1} \) the roots of \( P' \). Then the roots of \( O \) are contained in \( \{-1, 1, \varrho_1, \ldots, \varrho_{\nu - 1}\} \).

**Lemma 8.1.5.** The roots of \( P \) and \( P' \) are real numbers and if we arrange them in ascending order, it holds

\[
-1 < \pi_1 < q_1 < \pi_2 < \cdots < \pi_{\nu - 1} < q_{\nu - 1} < \pi_{\nu} < 1.
\]

**Proof.** From the above proof we have

\[
(-\sin)(O(P')^2(1 - i\lambda^2))^\prime \cos = ((O(P')^2(1 - i\lambda^2))^\prime \cos)^\prime = \left(\widehat{\Pi}^2(\Phi')^2\right)^\prime = -2\lambda\widehat{\Pi}^2\Phi \Phi' = -2\lambda(-\sin)(OPP') \cos.
\]
Lemma [A.0.4] implies that the roots of $OPP'$ lie in the convex hull of the roots of $O(P')^2(1 - \text{id}^2)$, i.e.
\[
\{-1, 1, \pi_1, \ldots, \pi_\nu, \varrho_1, \ldots, \varrho_{\nu - 1}\} \subset \text{conv} \{-1, 1, \varrho_1, \ldots, \varrho_{\nu - 1}\}.
\]
From this we get
\[
\text{conv} \{-1, 1, \pi_1, \ldots, \pi_\nu\} \subset \text{conv} \{-1, 1, \varrho_1, \ldots, \varrho_{\nu - 1}\}.
\]
Because of
\[
\{\varrho_1, \ldots, \varrho_{\nu - 1}\} \subset \text{conv} \{\pi_1, \ldots, \pi_\nu\}
\]
we have
\[
\{-1, 1, \varrho_1, \ldots, \varrho_{\nu - 1}\} \subset \text{conv} \{-1, 1, \pi_1, \ldots, \pi_\nu\}
\]
and
\[
\text{conv} \{-1, 1, \varrho_1, \ldots, \varrho_{\nu - 1}\} \subset \text{conv} \{-1, 1, \pi_1, \ldots, \pi_\nu\}.
\]
Since
\[
\{\varrho_1, \ldots, \varrho_{\nu - 1}\} \cap \{\pi_1, \ldots, \pi_\nu\} = \emptyset
\]
we get
\[
\text{conv} \{-1, 1, \varrho_1, \ldots, \varrho_{\nu - 1}\} = \text{conv} \{-1, 1, \pi_1, \ldots, \pi_\nu\} = [-1, 1].
\]
From this the claim follows. 

**Lemma 8.1.6.** The polynomial $O$ has no roots other than $-1$ and $1$.

**Proof.** We prove the lemma by contradiction. Without loss of generality we may assume that $\varrho_1$ is a root of $O$. Since $-1 < \varrho_1 < 1$ by the last lemma, there is $0 < r_0 < \pi$ with $\cos r_0 = \varrho_1$. Then \(\hat{\Omega}^2(r_0) = O(\cos r_0) = O(\varrho_1) = 0\). This is a contradiction. 

We are now in the position to prove the result we were looking for.

**Proposition 8.1.7.** There are $\tilde{C}, \alpha, \beta \in \mathbb{R}$ such that
\[
\hat{\Omega} = \tilde{C}(1 - \cos)\beta \sin^\alpha.
\]

**Proof.** For all $x \in \mathbb{R}$ we can write
\[
O(x) = C(1 - x)^\sigma (1 + x)^\tau
\]
with suitable $\sigma, \tau \in \mathbb{N}$ and $C \in \mathbb{R}^>0$. Then for all $r \in \mathbb{R}$ holds
\[
\hat{\Omega}(r) = \sqrt{O(\cos r)} = \sqrt{C}(1 - \cos r)^{\tilde{\beta}} (1 + \cos r)^{\tilde{\tau}} = \sqrt{C}(1 - \cos r)^{\tilde{\beta} - \tilde{\tau}} \sin^{\tilde{\alpha}} r.
\]
Remark. We keep the notation and get for the mean curvature function
\[ H := \frac{\Omega'}{\Omega} = \frac{(1 - \cos)\beta \sin \alpha}{(1 - \cos)\beta \sin \alpha} = \frac{\beta(1 - \cos)\beta \sin \alpha + \alpha(1 - \cos)\beta \cos \sin \alpha - 1}{(1 - \cos) \sin} = \frac{(\alpha + \beta) \cos + \beta}{\sin}. \]

Using Proposition 3.2.1 and after some lengthy calculations we compute the Ricci curvature to be
\[ \alpha + \frac{1}{2} \beta. \]
Since \( \hat{\Omega} \) vanishes of order \( n - 1 \) in 0 we have \( \alpha + 2 \beta = n - 1 \). Because of \( \Omega(0) = 1 \) we can deduce \( \tilde{C} = 2^\beta \).

Actually we can say even more. By the Bott-Samelson theorem, cf. [Bes78, Theorem 7.23], we know that \( \hat{\Omega} \) vanishes of order \( n - 1, 0, 1, 3 \) or 7 in \( \pi \). Hence \( \alpha \) can only take the values \( n - 1, 0, 1, 3 \) or 7. Then \( \beta \) equals 0, \( \frac{n-1}{2} \), \( \frac{n-3}{2} \), or \( \frac{n-5}{2} \) respectively. If we set \( n = m, 2m, 4m \) or 16 respectively, we recover the density functions of the ROSSs (Proposition 3.3.1).

8.2 Spectrum and Radial Eigenfunctions

Because of Lemma 8.1.7 it is now easy to construct concrete eigenvalues and radial eigenfunctions of the Laplacian. We keep the notation of this lemma and additionally set \( \lambda_1 := \alpha + \beta + 1 \).

Lemma 8.2.1. The number \( \lambda_1 \) is an eigenvalue and
\[ \Phi : \mathbb{R} \to \mathbb{R}, \quad r \mapsto \frac{\lambda_1}{\lambda_1 + \beta} \left( \cos r + \frac{\beta}{\lambda_1} \right) \]
is the solution of the ODE, i.e. \( \Phi = \Phi_{\lambda_1} \).

Proof. The function \( R_p \Phi \) is obviously smooth for every \( p \in M \). We have
\[ \Phi' = -\frac{\lambda_1}{\lambda_1 + \beta} \sin \quad \text{and} \quad \Phi'' = -\frac{\lambda_1}{\lambda_1 + \beta} \cos. \]
The initial conditions \( \Phi(0) = 1 \) and \( \Phi'(0) = 0 \) are satisfied. Furthermore
\[ \Phi'' + \hat{\Omega}' \Phi' + \lambda_1 \Phi = \frac{\lambda_1}{\lambda_1 + \beta} (e^{-\cos - (\alpha + \beta) \cos \beta} + \lambda_1 \cos + \beta) = 0. \]
This implies the claim.

Lemma 8.2.2. Set \( \lambda_k := k(k + \alpha + \beta) \) for \( k \in \mathbb{N} \). Then \( \lambda_k \) is an eigenvalue and the solutions \( \Phi_{\lambda_k} \)
of the ODE is given by
\[ \Phi_{\lambda_k} : \mathbb{R} \to \mathbb{R}, \quad r \mapsto \sum_{i=0}^{k} a_i \cos^i r, \]
with certain \( a_i \in \mathbb{R} \). The spectrum of \( M \) is \( (\lambda_k)_{k \in \mathbb{N}_0} \).
Proof. Let \( k \in \mathbb{N} \). The function \( R_p \Phi_{\lambda k} \) is obviously smooth for every \( p \in M \). We have

\[
\Phi'_{\lambda k} = -\sin \sum_{i=0}^{k} ia_i \cos^{i-1},
\]

\[
\Phi''_{\lambda k} = -\sum_{i=0}^{k} ia_i \cos^i + (1 - \cos^2) \sum_{i=0}^{k} i(i-1)a_i \cos^{i-2} = - \sum_{i=0}^{k} i^2 a_i \cos^i + \sum_{i=-2}^{k-2} (i + 2)(i + 1)a_{i+2} \cos^i
\]

and

\[
\frac{\partial^2}{\partial \Omega^2} \Phi_{\lambda k} = - ((\alpha + \beta) \cos + \beta) \sum_{i=0}^{k} ia_i \cos^{i-1} = - (\alpha + \beta) \sum_{i=0}^{k} ia_i \cos^i - \beta \sum_{i=-1}^{k-1} (i + 1)a_{i+1} \cos^i.
\]

Hence

\[
0 = \Phi''_{\lambda k} + \frac{\partial^2}{\partial \Omega^2} \Phi_{\lambda k} + \lambda_k \Phi_{\lambda k}
\]

\[
= \sum_{i=0}^{k} ((k^2 - i^2 + (k - i)(\alpha + \beta))a_i + (-\beta i - \beta)a_{i+1} + (i^2 + 3i + 2)a_{i+2}) \cos^i
\]

where we set \( a_{k+2} := 0 =: a_{k+1} \). Since \( k^2 - i^2 + (k - i)(\alpha + \beta) \neq 0 \) for \( i \neq k \) we get a recursive formula for the \( a_i \) if we require \( \Phi_{\lambda k}(0) = 1 = \sum_{i=0}^{k} a_i \). Because \( (\Phi_{\lambda k})_{k \in \mathbb{N}} \) spans the space consisting of all polynomials in cosine, \( (\lambda_k)_{k \in \mathbb{N}_0} \) is the whole spectrum. \( \square \)

8.3 Two Variants of the Proof

We keep the definitions of \( \alpha, \beta, \lambda_1 \) and \( \Phi \) from the last section.

First Variant. So far we have not used the embedding at all. In order to be allowed to use the second part of Corollary 7.0.2 we only consider the case where \( M \) is not diffeomorphic to the sphere \( S^n \) in this first variant of the proof.

Lemma 8.3.1. All geodesics of \( R^2(M) \) are circles.

Proof. For a unit speed geodesic \( c \) in \( R^2(M) \) we have

\[
\langle c(0), c(s) \rangle = C_2^2 \frac{\lambda_1}{\lambda_1 + \beta} \left( \cos s + \frac{\beta}{\lambda_1} \right)
\]

for all \( s \in \mathbb{R} \) by the second part of Corollary 7.0.2. The screw function \( S_0 \) of \( c \) is therefore

\[
S_0(s) = 2C_2^2 - 2C_2^2 \frac{\lambda_1}{\lambda_1 + \beta} \left( \cos s + \frac{\beta}{\lambda_1} \right) = 2 \frac{\lambda_1}{\lambda_1 + \beta} C_2^2 - 2 \frac{\lambda_1}{\lambda_1 + \beta} C_2^2 \cos s.
\]

Because a circle of radius \( \sqrt{\frac{\lambda_1}{\lambda_1 + \beta}} C_2 \) has got the same screw function, \( c \) is a circle. \( \square \)

Remark. Taking the proof of Corollary 7.0.2 and the remark after Proposition 8.1.7 into account we get that \( C_2^2 = \frac{1}{\lambda_1} \) and \( \lambda_1 + \beta = n \) respectively. Hence the circles are of radius 1.
Lemma 8.3.2. Let $\overline{M}$ be the $n$-dimensional submanifold $R^\Phi(M)$ of $V^{\lambda_1}$. Then $\overline{M}$ is a ROSS.

Proof. Fix a point $p \in \overline{M}$. Denote by $T_p\overline{M}$ the normal space of $\overline{M}$ in $p$. Let $s_p : V^{\lambda_1} \to V^{\lambda_1}$ be the reflection at the affine subspace $T_p\overline{M}$. For a geodesic $c : \mathbb{R} \to \overline{M}$ of $\overline{M}$ with $c(0) = p$ we have $s_p(c(0)) = p$, $s_p(c'(0)) = -c'(0)$ and $s_p(c''(0)) = c''(0)$. Since a circle is determined by this data, we have $s_p(c(\mathbb{R})) = c(\mathbb{R})$. In particular, it holds $s_p(\overline{M}) = \overline{M}$. Since $s_p$ is an isometry of $V^{\lambda_1}$, it is one of $\overline{M}$. This shows that $\overline{M}$ is a Riemannian symmetric space. If it were not of rank 1, it would have non-closed geodesics in maximal flats.

Second Variant. The second variant is an intrinsic proof, which uses [RS97, Theorem 1]. Since the averaged eigenfunction $\Phi$ has got no saddle point, we only have to check that equality holds in Ros’s estimate for the first eigenvalue, cf. [Ros84, Theorem 4.2]. Equality holds because of

$$\lambda_1 = \alpha + \beta + 1 = n - 1 - 2\beta + \beta + 1 = n - \beta$$

and

$$\frac{1}{3}(2 \text{ric} + n + 2) = \frac{1}{3}(2\alpha + \beta + n + 2) = \frac{1}{3}(2n - 2 - 4\beta + \beta + n + 2) = n - \beta.$$
A Appendix

All the auxiliary results are collected here.

Lemma A.0.1. Let $F : \mathbb{R} \to \mathbb{R}$ be smooth. The following statements are equivalent.

1. The vector space
   $$V := \text{span} \{ F(\cdot-t) \mid t \in \mathbb{R} \} \subset C^\infty(\mathbb{R})$$
   is of finite dimension.

2. The function $F$ solves a linear ODE with constant coefficients.

3. There are $k \in \mathbb{N}, \alpha_i, \beta_i \in \mathbb{R}$ and polynomials $P_i, Q_i : \mathbb{R} \to \mathbb{R}$ with real coefficients such that
   $$\forall x \in \mathbb{R} : F(x) = k \sum_{i=1}^k (P_i(x) \sin \beta_i x + Q_i(x) \cos \beta_i x)e^{\alpha_i x}.$$

Proof. 1. $\Rightarrow$ 2.: For every $t \in \mathbb{R}$ the map
   $$B_t : V \to V, \ G \mapsto B_t G := G(\cdot-t)$$
is an endomorphism of $V$. Furthermore $(B_t)_{t \in \mathbb{R}}$ is a smooth one-parameter subgroup of $\text{End}(V)$. So there is $B \in \text{End}(V)$ with
   $$B_t = \exp(tB).$$
We have for all $x \in \mathbb{R}$
   $$F'(x) = \partial_x ((B_0F)(x)) = \partial_x ((B_xF)(0)) = \partial_x ((\exp(xB)F)(0))$$
   $$= (B(\exp(xB)F))(0) = (B(B_xF))(0) = (B(B_0F))(x)$$
   $$= (BF)(x).$$
This means that $F'$ is again in $V$. Because of $\dim V < \infty$ the functions $F, F', \ldots, F^{(\dim V)}$ are linearly dependent. Hence $F$ solves a linear ODE with constant coefficients.

2. $\Rightarrow$ 1.: The function $F$ solves a linear ODE with constant coefficients. For every $t \in \mathbb{R}$ this ODE is solved by $F(\cdot-t)$ as well. Since the space of solutions is finite-dimensional so is $\text{span} \{ F(\cdot-t) \mid t \in \mathbb{R} \}$.

2. $\Leftrightarrow$ 3.: This follows from standard linear ODE theory. \qed

Lemma A.0.2. Let $F : \mathbb{R} \to \mathbb{R}$ be smooth, $2\pi$-periodic and even. Assume that the vector space $\text{span} \{ F(\cdot-t) \mid t \in \mathbb{R} \}$ is of finite dimension. Then there are $k \in \mathbb{N}, Q_i \in \mathbb{R}$ and $\beta_i \in \mathbb{N}$ such that
   $$\forall x \in \mathbb{R} : F(x) = \sum_{i=1}^k Q_i \cos \beta_i x.$$
Proof. By Lemma A.0.1 and the fact that $F$ is $2\pi$-periodic and even we get $k \in \mathbb{N}$, $Q_i \in \mathbb{R}$ and $\beta_i \in \mathbb{R}$ with the desired property. We only need to show that $\beta_i \in \mathbb{N}$. We may assume that the $\beta_i$ are distinct and that $Q_i \neq 0$. Fix an $a \in \mathbb{R}$. Then $\cos \beta_1 x, \ldots, \cos \beta_k x$ and $\sin \beta_1 x, \ldots, \sin \beta_k x$ are linearly independent. Because of the $2\pi$-periodicity of $F$ we get

$$0 = F(x - 2\pi) - F(x + 2\pi) = \sum_{i=1}^{k} Q_i (\cos \beta_i x \cos 2\pi \beta_i + \sin \beta_i x \sin 2\pi \beta_i) - \sum_{i=1}^{k} Q_i (\cos \beta_i x \cos 2\pi \beta_i - \sin \beta_i x \sin 2\pi \beta_i)$$

$$= \sum_{i=1}^{k} 2Q_i \sin \beta_i x \sin 2\pi \beta_i,$$

This yields $\sin 2\pi \beta_i = 0$. Hence we get

$$0 = F(x - 2\pi) - F(x) = \sum_{i=1}^{k} Q_i \cos \beta_i x \cos 2\pi \beta_i - \sum_{i=1}^{k} Q_i \cos \beta_i x = \sum_{i=1}^{k} Q_i \cos \beta_i x (\cos 2\pi \beta_i - 1).$$

This yields $\cos 2\pi \beta_i = 1$ and hence the claim.

Lemma A.0.3. For every $m \in \mathbb{N}$ there are $a_{m,1}, \ldots, a_{m,m} \in \mathbb{R}$ such that

$$\forall x \in \mathbb{R} : \cos mx = \sum_{k=1}^{m} a_{m,k} \cos^k x.$$

Proof. We can prove the claim by induction on $m$. For $m = 1$ we have $a_{1,1} = 1$. If the claim is true for $1, \ldots, m$ then because of

$$\cos(m + 1)x + \cos(m - 1)x = \cos mx \cos x - \sin mx \sin x + \cos mx \cos x + \sin mx \sin x$$

$$= 2 \cos mx \cos x$$

we have for all $x \in \mathbb{R}$

$$\cos(m + 1)x = -\cos(m - 1)x + 2 \cos mx \cos x$$

$$= -\sum_{k=1}^{m-1} a_{m-1,k} \cos^k x + 2 \cos x \sum_{k=1}^{m} a_{m,k} \cos^k x$$

$$= -\sum_{k=1}^{m-1} a_{m-1,k} \cos^k x + 2 \sum_{k=1}^{m} a_{m,k} \cos^{k+1} x$$

$$= -a_{m-1,1} \cos x + \sum_{k=2}^{m} (2a_{m,k-1} - a_{m-1,k}) \cos^k x$$

$$+ 2a_{m,m-1} \cos^m x + 2a_{m,m} \cos^{m+1} x.$$  

Lemma A.0.4 (Gauß-Lucas' Theorem, [RS02, Theorem 2.1.1]). If $P : \mathbb{C} \to \mathbb{C}$ is a non-constant polynomial with complex coefficients, all roots of $P'$ belong to the convex hull of the set of roots of $P$.  

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Proof. Set \( m := \deg P \) and let \( \zeta_1, \ldots, \zeta_m \in \mathbb{C} \) be the (not necessarily distinct) roots of \( P \). We can write

\[
\forall z \in \mathbb{C} : \quad P(z) = A \prod_{i=1}^{m} (z - \zeta_i)
\]

where \( A \in \mathbb{C} \) is the leading coefficient of \( P \). First fix a \( w \in \mathbb{C} \) with \( P'(w) = 0 \) and \( P(w) \neq 0 \). We have

\[
0 = \frac{P'(w)}{P(w)} = \sum_{i=1}^{m} \frac{1}{w - \zeta_i} = \sum_{i=1}^{m} \frac{\overline{w} - \overline{\zeta_i}}{|w - \zeta_i|^2}.
\]

This implies

\[
\left( \sum_{i=1}^{m} \frac{1}{|w - \zeta_i|^2} \right) \overline{w} = \sum_{i=1}^{m} \frac{1}{|w - \zeta_i|^2} \overline{\zeta_i}
\]

and by taking conjugates

\[
w = \frac{1}{\sum_{i=1}^{m} \frac{1}{|w - \zeta_i|^2}} \sum_{i=1}^{m} \frac{1}{|w - \zeta_i|^2} \zeta_i.
\]

Hence we get \( w \in \text{conv} \{\zeta_1, \ldots, \zeta_m\} \). Now assume that \( P'(\zeta_j) = 0 \) for some \( 1 \leq j \leq m \). Since \( \zeta_j \in \text{conv} \{\zeta_1, \ldots, \zeta_m\} \) we are done. \( \square \)
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