ON 3–PARAMETER FAMILIES OF PIECEWISE SMOOTH VECTOR FIELDS IN THE PLANE.

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Abstract. This paper is concerned with the local bifurcation analysis around typical singularities of piecewise smooth planar dynamical systems. Three–parameter families of a class of non–smooth vector fields are studied and the tridimensional bifurcation diagrams are exhibited. Our main results describe the unfolding of the so called fold–cusp singularity by means of the variation of 3 parameters.

1. Introduction

NSDS’s have become certainly one of the common frontiers between Mathematics and Physics or Engineering. Problems involving impact or friction are piecewise–smooth, as are many control systems with thresholds. Many authors have contributed to the study of Filippov systems (see for instance [7] and [10]). One of the starting points for a systematic approach to the geometric and qualitative analysis of non–smooth dynamical systems (NSDS’s, for short) is [13], on smooth systems in 2–dimensional manifolds with boundary. The generic singularities that appear in NSDS’s, to the best of our knowledge, were first studied in [15]. Bifurcations and related problems involving or not sliding regions were studied in papers like [6, 8, 2, 3]. The classification of codimension–1 local and some global bifurcations for planar systems was given in [11]. In [9] codimension–2 singularities were discussed and it was shown how to construct the homeomorphisms which lead to topological equivalences between two NSDS’s when the discontinuity set is a planar smooth curve. See [16] or [4] for a survey on NSDS’s and references there in.

The specific topic addressed in this paper is the qualitative analysis of fold–cusp singularities of NSDS’s, where a fold and a cusp coincide. Moreover, the bifurcation diagrams are exhibited.

Specifically, we distinguish the following cases (see Figure 1):

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• Unfolding of an invisible fold–cusp singularity:

\begin{equation}
Z_{\lambda, \beta, \mu} = \begin{cases} 
X_\lambda = \left( \frac{1}{-x + \lambda} \right) & \text{if } y \geq 0, \\
Y_\beta = \left( -x^2 + \beta - \frac{\partial B}{\partial x}(x, \beta, \mu) \right) & \text{if } y \leq 0,
\end{cases}
\end{equation}

where \((\lambda, \beta) \in (-\lambda_0, \lambda_0) \times (-\beta_0, \beta_0)\), with \(\lambda_0 > 0\) and \(\beta_0 > 0\) sufficiently small and \(B\) is a \(C^2\)-bump function such that \(B(x, \beta, \mu) = 0\) if \(\beta \leq 0\) and

\begin{equation}
B(x, \beta, \mu) = \begin{cases} 
0, & \text{if } x < -\sqrt{\beta} \text{ or } x > 4\sqrt{\beta}; \\
B_1(x, \beta) + f(\beta, \mu), & \text{if } -\sqrt{\beta} \leq x \leq \sqrt{\beta}; \\
B_2(x, \beta) + f(\beta, \mu), & \text{if } \sqrt{\beta} < x \leq 4\sqrt{\beta}.
\end{cases}
\end{equation}

if \(\beta > 0\), where

\[
B_1(x, \beta) = \frac{-3}{128\beta} \left( \frac{x^2(208 + 3\beta) - 4x\sqrt{\beta}(176 + 15\beta) + \beta(688 + 93\beta)}{\beta(176 + 15\beta)} \right),
\]

\[
B_2(x, \beta) = \frac{-1}{48\beta} \left( \frac{(x - 4\sqrt{\beta})(x^2 + \beta)(-16 + 9\beta) - 2x\sqrt{\beta}(16 + 15\beta)}{\beta(16 + 15\beta)} \right)
\]

and

\[
f(\beta, \mu) = \frac{\mu}{48} \left( \frac{-8\beta(128 + 3\beta)\mu\sqrt{\beta}(256 + 63\beta)\mu - (-64 + 45\beta)\mu^2}{\beta^{-1/2}(80 + 3\beta)\mu^3 + \beta^{-1}(-16 + 9\beta)\mu^4} \right).
\]

• Unfolding of a visible fold–cusp singularity:

\begin{equation}
Z_{\lambda, \beta} = \begin{cases} 
X_\lambda = \left( \frac{1}{x - \lambda} \right) & \text{if } y \geq 0, \\
Y_\beta = \left( \frac{1}{-x^2 + \beta} \right) & \text{if } y \leq 0,
\end{cases}
\end{equation}

where \((\lambda, \beta) \in (-\lambda_0, \lambda_0) \times (-\beta_0, \beta_0)\), with \(\lambda_0 > 0\) and \(\beta_0 > 0\) sufficiently small.

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**Figure 1.** Fold–cusp singularities. Following the notation of [9], a point \(p_0 \in \Sigma\) is a **fold–cusp singularity** of \(Z = (X, Y)\) if it is a \(\Sigma\)-fold point of \(X\) and a \(\Sigma\)-cusp point of \(Y\) (see precise definitions in Section 2).
1.1. **Setting the problem.** Denote both, $Z_{\beta,\lambda,\mu}$ in (1) and $Z_{\beta,\lambda}$ in (3), by $Z = (X,Y)$. In short our goal is to study the local dynamics of $Z$ consisting of two smooth vector fields $X$ and $Y$ in $\mathbb{R}^2$ such that on one side of a smooth surface $\Sigma = \{ y = 0 \}$ we take $Z = X$ and on the other side $Z = Y$.

In [9] the analysis of the bifurcation diagram of the 2–parameter family

$$W_{\mu,\epsilon} = \begin{cases} 
X_{\mu} = \begin{pmatrix} 1 \\ x - \mu \end{pmatrix} & \text{if } y \geq 0, \\
Y_{\epsilon} = \begin{pmatrix} -1 \\ -x^2 + \epsilon \end{pmatrix} & \text{if } y \leq 0.
\end{cases}$$

of NSDS’s presenting an invisible fold–cusp singularity is performed. A challenging problem is to extend the analysis of [9] in answering the following question: Can we find families of NSDS’s presenting fold–cusp singularities whose dynamics is richer than the family exhibited in [9] ? In this paper such an extension has been carried out. By means of the positive answer to the previous question, we are able to say that two parameters is not enough to explain the birth of some new topological types around $Z_{0,0,0}$.

In fact, ours results cover the study done in [9] and we can obtain the bifurcation diagram presented in [9] assuming $\beta = \mu^2$ and $\mu \leq 0$ in (1). For example, the configuration in Figure 2 is not observed in [9] and is present at the bifurcation diagram of (1).

![Figure 2](image)

**Figure 2.** Configuration nearby $Z_{0,0,0}$ not observed in [9].

![Figure 3](image)

**Figure 3.** The local and the global bifurcation observed in (1) when $\beta > 0$, $\lambda = \sqrt{3}$ and $\mu = 0$.

We mention two particular situations illustrated in Figure 3 that occur in (1) when $\beta > 0$. In this *resonant* configuration we note, simultaneously, a two–fold singularity (which is a local phenomenon) and a loop passing through the visible $\Sigma$–fold of $Y$ (which is a global phenomenon).

1.2. **Statement of the Main Results.** Theorems 1, 2 and 3 pave the way for the proof of Theorem A. Theorem B is self contained.

**Theorem 1.** If $\mu = 0$ in Equation (1) then its bifurcation diagram in the $(\lambda, \beta)$–plane contains essentially 17 distinct phase portraits (see Figure 23).
It is easy to see that the cases covered by Theorem 1 do not represent the full unfolding of the invisible fold—cusp singularity. Because of this, the next two theorems are necessary.

**Theorem 2.** If $0 < \mu < \mu_0$ in Equation (1) then its bifurcation diagram in the $(\lambda, \beta)-$plane contains essentially 19 distinct phase portraits (see Figure 25).

**Theorem 3.** If $-\mu_0 < \mu < 0$ in Equation (1) then its bifurcation diagram in the $(\lambda, \beta)-$plane contains essentially 19 distinct phase portraits (see Figure 25).

Finally, we are in position to state the main results of the paper.

**Theorem A.** The bifurcation diagram of Equation (1) exhibits 55 distinct cases representing 23 distinct phase portraits (see Figure 27).

**Theorem B.** The bifurcation diagram of Equation (3) exhibits 11 distinct phase portraits (see Figure 33).

The paper is organized as follows. In Section 2 we present some basic elements on the theory of NSDS’s. In Sections 3, 4 and 5 we pave the way for the proofs of the main results of the paper (Theorems A and B). Section 6 is devoted to prove Theorem A and exhibit the Bifurcation Diagram of (1). In Section 7 the proof of Theorem B and the Bifurcation Diagram of (3) are presented and in Section 8 some concluding remarks are discussed. In our paper we follow basically the terminology and the approach of [11] or [9] and no one sophisticated tool is needed.

## 2. Preliminaries

Let $K \subseteq \mathbb{R}^2$ be a compact set such that $\partial K$ is a smooth 1—manifold and $\Sigma \subseteq K$ given by $\Sigma = f^{-1}(0)$, where $f : K \to \mathbb{R}$ is a smooth function having $0 \in \mathbb{R}$ as a regular value (i.e. $\nabla f(p) \neq 0$, for any $p \in f^{-1}(0)$) such that $\partial K \cap \Sigma = \emptyset$ or $\partial K \cap \Sigma$. Clearly the switching manifold $\Sigma$ is the separating boundary of the regions $\Sigma^+ = \{q \in K | f(q) \geq 0\}$ and $\Sigma^- = \{q \in K | f(q) \leq 0\}$.

Designate by $\chi$ the space of $C^1$—vector fields on $K$ endowed with the $C^1$—topology. Call $\Omega = \Omega(K, f)$ the space of vector fields $Z : K \to \mathbb{R}^2$ such that

\begin{equation}
Z(x, y) = \begin{cases} 
X(x, y), & \text{for } (x, y) \in \Sigma^+,
Y(x, y), & \text{for } (x, y) \in \Sigma^-,
\end{cases}
\end{equation}
where \( X = (f_1, g_1) \), \( Y = (f_2, g_2) \) are in \( \chi \). We write \( Z = (X, Y) \), which we will accept to be multivalued in points of \( \Sigma \). We endow \( \Omega \) with the product \( C^1 \)-topology. The trajectories of \( Z \) are solutions of \( \dot{q} = Z(q) \), which has, in general, discontinuous righthand side. The basic results of differential equations, in this context, were stated by Filippov in [7].

**Definition 1.** A \( k \)-parameter family of elements in \( \Omega \) is a \( C^1 \)-mapping, with \( r > 1 \),

\[
\zeta: S^k \quad \rightarrow \quad \Omega
\]

where \( S^k = [-\epsilon_1, \epsilon_1] \times [-\epsilon_2, \epsilon_2] \times \ldots \times [-\epsilon_k, \epsilon_k] \) with \( \epsilon_i > 0 \), \( i = 1, 2, \ldots, k \), sufficiently small.

**Definition 2.** We say that \( W, \tilde{W} \in \chi \) defined in open sets \( U \) and \( \tilde{U} \), respectively, are \( C^0 \)-orbitally equivalent if there exists an orientation preserving homeomorphism \( h: U \rightarrow \tilde{U} \) that sends orbits of \( W \) to orbits of \( \tilde{W} \). Here, orbit of \( W \) means the image of a solution of \( \dot{x} = W(x) \). We say that \( h \) to be multivalued in points of \( \Sigma \) equivalent if there exists an orientation preserving homeomorphism \( h: \Sigma \rightarrow \Sigma \) with \( \sigma > 0 \) and \( \Sigma \subseteq U \cap \Sigma^+ \) in the orbits of \( \tilde{X} \) restrict to \( \tilde{U} \cap \Sigma^+ \), and the orbits of \( Y \) restrict to \( U \cap \Sigma^- \) in the orbits of \( \tilde{Y} \) restrict to \( \tilde{U} \cap \Sigma^- \).

Consider the notation

\[
X.f(p) = \langle \nabla f(p), X(p) \rangle \quad \text{and} \quad X^i.f(p) = \langle \nabla X^{i-1}.f(p), X(p) \rangle, i \geq 2
\]

where \( \langle ., . \rangle \) is the usual inner product in \( \mathbb{R}^2 \).

**Remark 1.** The vertical dotted lines present in almost all figures of this paper represent the points \( p \in K \subseteq \mathbb{R}^2 \) where \( X.f(p) = 0 \) or \( Y.f(p) = 0 \).

We distinguish the following regions on the discontinuity set \( \Sigma \):

(i) \( \Sigma^e \subseteq \Sigma \) is the sewing region if \( (X.f)(Y.f) > 0 \) on \( \Sigma^e \).

(ii) \( \Sigma^e \subseteq \Sigma \) is the escaping region if \( (X.f) > 0 \) and \( (Y.f) < 0 \) on \( \Sigma^e \).

(iii) \( \Sigma^s \subseteq \Sigma \) is the sliding region if \( (X.f) < 0 \) and \( (Y.f) > 0 \) on \( \Sigma^s \).

Consider \( Z \in \Omega \). The sliding vector field associated to \( Z \) is the vector field \( Z^s \) tangent to \( \Sigma^s \) and defined at \( q \in \Sigma^s \) by \( Z^s(q) = m - q \) with \( m \) being the point of the segment joining \( q + X(q) \) and \( q + Y(q) \) such that \( m - q \) is tangent to \( \Sigma^s \) (see Figure 4). It is clear that if \( q \in \Sigma^s \) then \( q \in \Sigma^e \) for \( -Z \) and then we can define the escaping vector field on \( \Sigma^e \) associated to \( Z \) by \( Z^e = (-Z)^s \). In what follows we use the notation \( Z^\Sigma \) for both cases.

We say that \( q \in \Sigma \) is a \( \Sigma \)-regular point if

(i) \( (X.f(q))(Y.f(q)) > 0 \) or
(ii) \((X.f(q))(Y.f(q)) < 0\) and \(Z^\Sigma(q) \neq 0\) (that is \(q \in \Sigma^e \cup \Sigma^s\) and it is not an equilibrium point of \(Z^\Sigma\)).

The points of \(\Sigma\) which are not \(\Sigma\)-regular are called \(\Sigma\)-singular. We distinguish two subsets in the set of \(\Sigma\)-singular points: \(\Sigma^t\) and \(\Sigma^p\). Any \(q \in \Sigma^p\) is called a pseudo equilibrium of \(Z\) and it is characterized by \(Z^\Sigma(q) = 0\). Any \(q \in \Sigma^t\) is called a tangential singularity and is characterized by \(Z^\Sigma(q) \neq 0\) and \((X.f(q))(Y.f(q)) = 0\) (\(q\) is a contact point).

We say that a point \(p_0 \in \Sigma\) is a \(\Sigma\)-fold point of \(X\) if \(X.f(p_0) = 0\) but \(X^2.f(p_0) \neq 0\). Moreover, \(p_0 \in \Sigma\) is a visible (respectively invisible) \(\Sigma\)-fold point of \(X\) if \(X.f(p_0) = 0\) and \(X^2.f(p_0) > 0\) (respectively \(X^2.f(p_0) < 0\)). We say that a point \(q_0 \in \Sigma\) is a \(\Sigma\)-cusp point of \(Y\) if \(Y.f(q_0) = Y^2.f(q_0) = 0\) and \(Y^3.f(q_0) \neq 0\). Moreover, a \(\Sigma\)-cusp point \(q_0\) of \(Y\) is of kind 1 (respectively kind 2) if \(Y^3.f(q_0) > 0\) (respectively \(Y^3.f(q_0) < 0\)). In particular, \(\Sigma\)-fold and \(\Sigma\)-cusp points are tangential singularities.

A pseudo equilibrium \(q \in \Sigma^p\) is a \(\Sigma\)-saddle provided that one of the following conditions is satisfied: (i) \(q \in \Sigma^e\) and \(q\) is an attractor for \(Z^ \Sigma\) or (ii) \(q \in \Sigma^s\) and \(q\) is a repeller for \(Z^ \Sigma\). A pseudo equilibrium \(q \in \Sigma^p\) is a \(\Sigma\)-repeller (resp. \(\Sigma\)-attractor) provided \(q \in \Sigma^c\) (respectively \(q \in \Sigma^s\)) and \(q\) is a repeller (respectively, attractor) equilibrium point for \(Z^ \Sigma\).

Given a point \(q \in \Sigma^c\), we denote by \(r(q)\) the straight line through \(q + X(q)\) and \(q + Y(q)\).

**Definition 4.** The \(\Sigma\)-regular points \(q \in \Sigma^c\) such that either \(\{X(q), Y(q)\}\) is a linearly dependent set or \(r(q) \cap \Sigma = \emptyset\) are called virtual pseudo equilibria.
Let us consider a smooth autonomous vector field $W$ defined in an open set $U$. Then we denote its flow by $\phi_W(t,p)$. In this way,
\[
\begin{aligned}
\frac{d}{dt}\phi_W(t,p) &= W(\phi_W(t,p)), \\
\phi_W(0,p) &= p,
\end{aligned}
\]
where $t \in I = I(p,W) \subset \mathbb{R}$, an interval depending on $p \in U$ and $W$.

The following definition was stated in [9], pg 1971.

**Definition 5.** The **local trajectory** of a NSDS given by $(\Omega)$ is defined as follows:

- For $p \in \Sigma^+$ and $p \in \Sigma^-$ the trajectory is given by $\phi_Z(t,p) = \phi_X(t,p)$ and $\phi_Z(t,p) = \phi_Y(t,p)$ respectively, where $t \in I$.
- For $p \in \Sigma^c$ such that $X.f(p) > 0$, $Y.f(p) > 0$ and taking the origin of time at $p$, the trajectory is defined as $\phi_Z(t,p) = \phi_Y(t,p)$ for $t \in I \cap \{t \leq 0\}$ and $\phi_Z(t,p) = \phi_X(t,p)$ for $t \in I \cap \{t \geq 0\}$. For the case $X.f(p) < 0$ and $Y.f(p) < 0$ the definition is the same reversing time.
- For $p \in \Sigma^c \cup \Sigma^s$ such that $Z^\Sigma(p) \neq 0$ we define $\phi_Z(t,p) = \phi_Z^s(t,p)$ for $t \in I$.
- For $p \in \partial \Sigma^c \cup \partial \Sigma^c \cup \partial \Sigma^s$ such that the definitions of trajectories for points in a full neighborhood of $p$ in $\Sigma$ can be extended to $p$ and coincide, the trajectory through $p$ is this trajectory.
- For any other point $\phi_Z(t,p) = p$ for all $t \in \mathbb{R}$. This is the case of points in $\partial \Sigma^c \cup \partial \Sigma^c \cup \partial \Sigma^s$ which are not regular tangential singularities and the equilibrium points of $X$ in $\Sigma^+$, of $Y$ in $\Sigma^-$ and of $Z^\Sigma$ in $\Sigma^s \cup \Sigma^e$.

**Definition 6.** The **local orbit—arc** of the vector field $W$ passing through a point $p \in U$ is the set $\gamma_W(p) = \{\phi_W(t,p) : t \in I\}$.

Since we are dealing with autonomous systems, from now on we will use trajectory and orbit—arc indistinctly when there is no danger of confusion.

**Definition 7.** Consider $Z = (X,Y) \in \Omega$.

1. A **canard cycle** is a closed curve $\Gamma = \bigcup_{i=1}^{n} \sigma_i$ composed by the union of orbit—arcs $\sigma_i$, $i = 1, \ldots, n$, of $X|_{\Sigma^+}$, $Y|_{\Sigma^-}$ and $Z^\Sigma$ such that:
   - Either there exists $i_0 \in \{1, \ldots, n\}$ with $\sigma_{i_0} \subset \gamma_X$ (respectively $\sigma_{i_0} \subset \gamma_Y$) and then there exists $j \neq i_0$ with $\sigma_j \subset \gamma_Y \cup \gamma_{Z^\Sigma}$ (respectively $\sigma_j \subset \gamma_X \cup \gamma_{Z^\Sigma}$), or $\Gamma$ is composed by a single arc $\sigma_i$ of $Z^\Sigma$;
   - the transition between arcs of $X$ and arcs of $Y$ occurs in sewing points;
   - the transition between arcs of $X$ (or $Y$) and arcs of $Z^\Sigma$ occurs through $\Sigma$—fold points or regular points in the escaping or sliding arc, respecting the orientation. Moreover if $\Gamma \neq \Sigma$ then
there exists at least one visible $\Sigma$–fold point on each connected component of $\Gamma \cap \Sigma$.

(2) A canard cycle $\Gamma$ of $Z$ is of:

- **Kind I** if $\Gamma$ meets $\Sigma$ just in sewing points;
- **Kind II** if $\Gamma = \Sigma$;
- **Kind III** if $\Gamma$ contains at least one visible $\Sigma$–fold point of $Z$.

In Figures 5, 6 and 7 arise canard cycles of kind I, II and III respectively.

(3) A canard cycle $\Gamma$ of $Z$ is hyperbolic if one of the following conditions are satisfied:

(i) $\Gamma$ is of kind I and $\eta'(p) \neq 1$, where $\eta$ is the first return map defined on a segment $T$ with $p \in T \cap \gamma$;

(ii) $\Gamma$ is of kind II;

(iii) $\Gamma$ is of kind III, $\Sigma^e \cap \Sigma^s \cap \Gamma = \emptyset$ and either $\Gamma \cap \Sigma \subseteq \Sigma^e \cup \Sigma^s \cup \Sigma^t$ or $\Gamma \cap \Sigma \subseteq \Sigma^e \cup \Sigma^s \cup \Sigma^t$.

#### Figure 5. Canard cycle of kind I.

#### Figure 6. Canard cycle of kind II.

#### Figure 7. Canard cycle of kind III.

Following Theorem 2 of [17], locally it is possible to consider $f(x, y) = y$ and conclude that any $X \in \Sigma^+$ presenting a $\Sigma$–fold point is $C^0$–orbitally equivalent to the normal form $X_0(x, y) = (\rho_1, \rho_2 x)$ with $\rho_1 = \pm 1$ and $\rho_2 = \pm 1$.

Following [13], we can take $f(x, y) = y$ and derive that any $Y \in \Sigma^-$ presenting a $\Sigma$–cusp point is $C^0$–orbitally equivalent to the normal form $Y_0(x, y) = (\rho_3, \rho_4 x^2)$ with $\rho_3 = \pm 1$ and $\rho_4 = \pm 1$.

**Lemma 8.** Let $Z = (X, Y) \in \Omega$ presenting a fold–cusp singularity, then $Z$ is $\Sigma$–equivalent to the standard form $Z_0 = Z_{\rho_1, \rho_2, \rho_3, \rho_4}$ given by

\[
Z_\rho = Z_{\rho_1, \rho_2, \rho_3, \rho_4} = \begin{cases} 
X_{\rho_1, \rho_2} = \begin{pmatrix} \rho_1 \\ \rho_2 x \end{pmatrix} & \text{if } y \geq 0, \\
Y_{\rho_3, \rho_4} = \begin{pmatrix} \rho_3 \\ \rho_4 x^2 \end{pmatrix} & \text{if } y \leq 0 
\end{cases}
\]

where $\rho_1, \rho_2, \rho_3, \rho_4 = \pm 1$. 
Observe that the values of $\rho_i$, $i = 1, 2, 3, 4$, in Lemma 8 depend on the orientation of $X$ and $Y$. In Subsection 2.3 we prove Lemma 8, i.e., we exhibit the homeomorphism that characterizes the equivalence between any fold–cusp singularity and the standard form given by (5).

Consider $Z_{ivb,k1}^{ivb,k1}, Z_{vis,k2}^{vis,k2} \in \Omega$ written in the following standard forms (similar forms were stated in Section 12 of [9]):

\[
Z_{ivb,k1}^{ivb,k1} = \begin{cases}
X_{0}^{ivb} = \begin{pmatrix} 1 \\ -x \end{pmatrix} & \text{if } y \geq 0, \\
Y_{0}^{k1} = \begin{pmatrix} -1 \\ -x^2 \end{pmatrix} & \text{if } y \leq 0, \text{ and}
\end{cases}
\]

\[
Z_{vis,k2}^{vis,k2} = \begin{cases}
X_{0}^{vis} = \begin{pmatrix} 1 \\ x \end{pmatrix} & \text{if } y \geq 0, \\
Y_{0}^{k2} = \begin{pmatrix} 1 \\ -x^2 \end{pmatrix} & \text{if } y \leq 0.
\end{cases}
\]

Note that $X_{0}^{ivb}$ presents an invisible $\Sigma$–fold point on its phase portrait, $X_{0}^{vis}$ presents a visible $\Sigma$–fold point, $Y_{0}^{k1}$ presents a $\Sigma$–cusp point of kind 1 and $Y_{0}^{k2}$ presents a $\Sigma$–cusp point of kind 2. Moreover, in (6) we made $\rho_1 = 1$ and $\rho_2 = \rho_3 = \rho_4 = -1$ in (5) and in (7) we made $\rho_1 = \rho_2 = \rho_3 = 1$ and $\rho_4 = -1$ in (5). For simplicity we restrict our study to the normal forms given above, i.e., (6) and (7). All the other choices on the values of $\rho_i$, $i = 1, 2, 3, 4$ in (5) are treated similarly.

The main problem is to exhibit the bifurcation diagram of $Z_{0}^{\tau,\rho}$ where $\tau = ivb$ or $vis$ and $\rho = k1$ or $k2$.

In order to detect a larger range of topological behaviors near an invisible fold–cusp singularity we have to refine the analysis done in [9]. This refinement can be obtained adding a bump function on the expression of the NSDS.

Denote

\[F(x) = \int g_2(x, \beta, \mu)dx = \frac{x^3}{3} - \beta x + B(x, \beta, \mu) + c_0,\]

where $c_0 = -2\beta \sqrt{3}/3$ and $g_2$ is the second coordinate of $Y_{\beta,\mu}$ in (11). The $C^1$–bump function $B$ satisfies the following properties when $\beta > 0$:

- It has exactly one point of local minimum in the interval $(-\sqrt{3}, 4\sqrt{3})$. This point is located at $x_0 = \sqrt{3}$.
- $F(3\sqrt{3} + \mu) = 0$ (see Figure 8). By means of this last property the orbit–arc of $Y_{\beta,\mu}$ that has a quadratic contact to $\Sigma$ at $q_0 = (-\sqrt{3}, 0)$ turns to collide with $\Sigma$ at the point $q_1 = (3\sqrt{3} + \mu, 0)$. So, the first coordinate of $q_1$ is bigger (respectively, smaller) than $3\sqrt{3}$ as $\mu$ is bigger (respectively, smaller) than $0$. 


Remark 2. It is worth saying that the parameter $\mu$ breaks the strong proportionality between the roots of $g_2(x, \beta, 0)$. At the limit value $\mu_0 = 0$, $Z_{\lambda, \beta, \mu}$ presents distinct topological behaviors for $\mu < \mu_0$ or $\mu > \mu_0$.

Remark 3. Note that in Equations (1) and (3) the perturbations considered depend only on the variable $x$. The local geometry of a NSDS presenting a cusp–fold singularity becomes rather different if perturbations involving the variables $x$ and $y$ are admitted.

2.1. Global Bifurcation. As said before, the configuration illustrated in Figure 8 plays a very important role in our analysis. The configuration of this figure is reached from (1), by taking $\beta > 0$, $\lambda = \sqrt{\beta}$ and $\mu = 0$. In this section we deal with this global phenomenon.

Emphasizing, let $Z_0 = (X_0, Y_0) \in \Omega$ having the following properties:

- The discontinuity set $\Sigma$ is represented by $f(x, y) = y$.
- Consider $X_0 = (f_0^1, g_0^1)$ and $Y_0 = (f_0^2, g_0^2)$ and assume that $f_0^1(p) > 0$ if $p \in \Sigma^+$ and $f_0^2(p) < 0$ if $p \in \Sigma^-$.
- $q_0 \in \Sigma$ is a visible $\Sigma$–fold point of $Y_0$ and $X.f(q_0) > 0$.
- The orbit $\gamma_{\lambda_0}(q_0)$ of $X_0$ through $q_0$ meets transversally $\Sigma$ at a point $q_1$.
- The orbit $\gamma_{\lambda_0}(q_1)$ of $Y_0$ through $q_1$ meets tangentially $\Sigma$ at $q_0$. Call $\Gamma$ the degenerate canard cycle composed by $\gamma_{\lambda_0}(q_0)$ and $\gamma_{\lambda_0}(q_1)$. Let $M$ be the compact region in the plane bounded by $\Gamma$.

2.1.1. Transition Fold Map. As $q_0 \in \Sigma$ is a visible $\Sigma$–fold point of $Y_0$, we may assume (see [17]) coordinates around $q_0$ such that the system is represented by $(\dot{x}, \dot{y}) = (-1, x)$ with $q_0 = (0, 0)$. The solutions of this differential equation are given by:

$$\phi_{a,b}(t) = (-t + a, -(t^2/2) + at + b).$$

The orbit–arc $\phi_0$ through $(0, 0)$ is represented by $\phi_0(t) = (-t, -t^2/2)$.

Let $\delta$ be a very small positive number. We construct the Transition Map $\xi : L_1 \to L_0$, from $L_1 = \{(x, y), \ y = -\delta, \ x \geq \sqrt{2\delta}\}$ to $L_0 = \{(x, 0), \ x \geq 0\}$, following the orbits of $Y_0$ (see Figure 9). The curve $L_1$ is transverse to
$Y_0$ at $p_0 = (\sqrt{2\delta}, -\delta)$. Since the solutions $\phi_\delta$ through $((x, -\delta) \in L_1$ meet $\Sigma = \{y = 0\}$ at time $t = x + \sqrt{x^2 - 2\delta}$ we obtain that $\xi(x) = \sqrt{x^2 - 2\delta}$ and $\xi$ is an homeomorphism. Moreover, $\xi^{-1}(x) = \sqrt{x^2 + 2\delta}$, $\xi^{-1}$ is differentiable at 0 and $(\xi^{-1})'(0) = 0$.

2.1.2. First Return Map associated to $\Gamma$. Let $g_X$ be the transition map from $L_0$ to $L_2 \subset \Sigma$ via $X_0$--trajectories and $g_Y$ be the transition map from $L_2$ to $L_1$ via $Y_0$--trajectories (see Figure 9). Observe that the linear part of the composition $g_Y \circ g_X$ is nonzero due to the transversality conditions of the problem. For simplicity, let $J_\epsilon = [0, \epsilon) \times \{0\}$ be a small semi-open interval of $\Sigma$.

So the First Return Map of $Z_0$ at $q_0$ is $\kappa(x) = (\xi \circ g_Y \circ g_X)(x)$ for $x \in J_\epsilon$. Its inverse $\kappa^{-1}$ is a differentiable map at 0 and satisfies $(\kappa^{-1})'(0) = 0$. So, $\Gamma$ locally repeals the orbits of $Z_0$ closed to $\Gamma$ and in the interior of $\Gamma$.

In conclusion, if $Z$ is very close to $Z_0$ in $\Omega$ in such a way that it possesses a canard cycle nearby $\Gamma$ then it is a hyperbolic repeller canard cycle. Under some other conditions on $Z_0$ (reversing the directions of $X_0$ and $Y_0$) we can derive that such canard cycle is an attractor.

2.1.3. Analysis around the two--fold singularity. In Equation (1), for $\beta > 0$, it is possible to define a First Return Map $\psi_\lambda^\mu : (\sqrt{\beta}, 3\sqrt{\beta} + \mu) \to (\sqrt{\beta}, 3\sqrt{\beta})$, associated to $Z_{\lambda,\beta,\mu}$, given by

$$\psi_\lambda^\mu(x) = (g_{X_\lambda} \circ g_{Y_{\mu,\beta}})(x)$$

where $g_{Y_{\mu,\beta}}(x)$ is the first return to $\Sigma$ of the orbit--arc of $Y_{\mu,\beta}$ that passes through $p = (x, 0)$ and $g_{X_\lambda}(\bar{x})$ is the first return to $\Sigma$ of the orbit--arc of $X_\lambda$ that passes through $p = (\bar{x}, 0)$.

Lemma 9. If $\beta > 0$, $\lambda = \sqrt{\beta}$ and $\mu = 0$ in (1) then (see Figure 9) the First Return Map $\psi_\lambda^\mu(x)$ satisfies

(i) $\psi_\lambda^\mu(x) < x$, $\forall x \in (\sqrt{\beta}, 3\sqrt{\beta})$ and

(ii) $|\psi_\lambda^\mu'(\sqrt{\beta})| \neq 1$.

Proof. Consider Figure 9. Given a point $p \in L_2$, the positive $Y$--orbit by $p$ reaches $L_3$ at the point $q = (q_1, q_2)$ and the negative $X$--orbit by $p$ reaches...
$L_0$ at the point $\tilde{p} = (\tilde{p}_1, \tilde{p}_2)$. The negative $Y$-orbit by $\tilde{p}$ reaches $L_3$ at the point $\tilde{q} = (\tilde{q}_1, \tilde{q}_2)$. Since
\[ q_2 - \tilde{q}_2 = \frac{(p_1 - 3\sqrt{\beta})(p_1 - \sqrt{\beta})(p_1 (1744 + 99\sqrt{\beta}) - \sqrt{\beta}(6256 + 321\beta))}{384\beta} \]
and $\sqrt{\beta} < p_1 < 3\sqrt{\beta}$ we conclude that $q_2 - \tilde{q}_2 > 0$ and item (i) is proved. Item (ii) follows from Section 2.1.2. \hfill \square

Note that Lemma 9 implies that $Z_{\sqrt{\beta}, \beta, 0}$ does not have closed orbits in the interior of the closed curve of $Z$ passing through the visible $\Sigma$-fold point of $Y_{0, \beta}$. Moreover, when $\mu < 0$ (see Figure 10), Lemma 9 guarantees that $\psi_\mu$ has a unique fixed point $x$ where $x < 3\sqrt{\beta} + \mu$. And, in this case, $|((\psi_\mu)'(x))| \neq 1$, i.e., $x$ is a hyperbolic fixed point for $\psi_\mu$ that corresponds to a hyperbolic canard cycle of $Z_{\lambda, \beta, \mu}$. When $\mu > 0$ (see Figure 10), $\psi_\mu(x) < x$ for all $x \in (\sqrt{\beta}, 3\sqrt{\beta} + \mu)$ and closed orbits of $Z_{\lambda, \beta, \mu}$ do not arise.

![Figure 10. Graph of the First Return Map $\psi_\mu$.](image)

Given $Z = (X, Y)$, we describe some properties of both $X = X_\lambda$ and either $Y = Y_{\beta, \mu}$ or $Y = Y_\beta$.

The parameter $\lambda$ measures how the $\Sigma$-fold point $d = (\lambda, 0)$ of $X$ is translated away from the origin. More specifically, if $\lambda < 0$ then $d$ is translated to the left hand side and if $\lambda > 0$ then $d$ is translated to the right hand side.

The parameter $\beta$ distinguishes the contact order between a trajectory of $Y$ and $\Sigma$. In this way, it occurs one, and only one, of the following situations:

- **$Y^+$**: In this case $\beta > 0$. So $Y$ has two $\Sigma$-fold points in such a way that one of them invisible and the other one visible. These points are expressed by $a = a_\beta = (-\sqrt{\beta}, 0)$ and $b = b_\beta = (\sqrt{\beta}, 0)$. Moreover, a third point $c = c_{\beta, \mu} = (3\sqrt{\beta} + \mu, 0)$ plays an important role at the analysis of $Z_{\lambda, \beta, \mu}$. This point is the locus where the orbit-arc $\gamma_Y(a)$ intersects transversally $\Sigma$ for negative time (see Figure 11). Using the bump function $B$ the distance between $c$ and $b$ is bigger or smaller than the distance between $a$ and $b$ according to the value of the parameter $\mu$. This fact will be important to change from Theorem 1 to Theorems 2 and 3.
• $Y^0$: In this case $\beta = 0$. So $Y$ has a $\Sigma$–cusp point $e = (0, 0)$ (see Figure 1).
• $Y^-$: In this case $\beta < 0$. So $Y$ does not have $\Sigma$–fold points. In this way, $Y.f \neq 0$ and $Y$ is transversal to $\Sigma$ (see Figure 12).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure11}
\caption{Case $Y^+$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure12}
\caption{Case $Y^-$.}
\end{figure}

2.2. The Direction Function. The next function will be very useful in the sequel.

On $\Sigma$, consider the point $C = (C_1, C_2)$, the vectors $X(C) = (D_1, D_2)$ and $Y(C) = (E_1, E_2)$ (as illustrated in Figure 13). Observe that the straight line $r(C)$ by $q + X(q)$ and $q + Y(q)$, generically, meets $\Sigma$ in a point $p(C)$. We define the $C^r$–map

$$p : \Sigma \rightarrow \Sigma, \quad z \mapsto p(z).$$

We choose local coordinates such that $\Sigma$ is the $x$–axis; so $C = (C_1, 0)$ and $p(C) \in \mathbb{R} \times \{0\}$ can be identified with points in $\mathbb{R}$. According with this identification, the direction function on $\Sigma$ is defined by

$$H : \mathbb{R} \rightarrow \mathbb{R}, \quad z \mapsto p(z) - z.$$

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure13}
\caption{Direction function.}
\end{figure}

Remark 4. We obtain that $H$ is a $C^r$–map. When $C \in \Sigma^e \cup \Sigma^s$ the following holds:

• if $H(C) < 0$ then the orientation of $Z^\Sigma$ in a small neighborhood of $C$ is from $B$ to $A$;
• if $H(C) = 0$ then $C \in \Sigma^p$;
• if $H(C) > 0$ then the orientation of $Z^\Sigma$ in a small neighborhood of $C$ is from $A$ to $B$. 
Simple calculations show that \( p(C_1) = \frac{E_2(D_1 + C_1) - D_2(E_1 + C_1)}{E_2 - D_2} \) and consequently,
\[
H(C_1) = \frac{E_2D_1 - D_2E_1}{E_2 - D_2}.
\]

\( \text{Remark 5.} \) If \( X.f(p) = 0 \) and \( Y.f(p) \neq 0 \) then, in a neighborhood \( V_p \) of \( p \) in \( \Sigma \), holds \( H(V_p)D_1 > 0 \), where \( X(p) = (D_1, D_2) \). In fact, \( X.f(p) = 0 \) and \( Y.f(p) \neq 0 \) are equivalent to say that \( D_2 = 0 \) and \( E_2 \neq 0 \) in \( \Sigma \). So,
\[
\lim_{(D_2, E_2) \to (0, k_0)} H(p_1) = D_1, \text{ where } k_0 \neq 0 \text{ and } p = (p_1, p_2).
\]

Considering the previous notation and identifying \( \Sigma \) with the \( x \)-axis, we have that \( r(C) \cap \Sigma = \emptyset \) when \( E_2 = D_2 \). In such a case, \( H \) is not defined at \( C \). The following property is immediate.

\( \text{Proposition 10.} \) If \( n_1 \) is the number of pseudo equilibria and \( n_2 \) is the number of virtual pseudo equilibria then \( n_1 + n_2 = v_1 + v_2 \) where \( v_1 \) is the number of zeros of \( H \) and \( v_2 \) is the number of points \( q \) of \( \Sigma \) such that \( r(q) \cap \Sigma = \emptyset \).

\( \text{Proof.} \) Straightforward according to Remark 4, Equation (8) and Definition 4.

\( \square \)

\( \text{Remark 6.} \) Given \( Z_{\lambda, \beta, \mu} \), we list some properties of the function \( H \). According to (8) we have that the expression of \( H \) is
\[
H(x, \lambda, \beta, \mu) = \frac{H_1(x, \lambda, \beta, \mu)}{H_2(x, \lambda, \beta, \mu)}
\]
where \( H_1(x, \lambda, \beta, \mu) = -x^2 - x + \lambda + \beta - \frac{\partial B}{\partial x}(x, \beta, \mu) \) and \( H_2(x, \lambda, \beta, \mu) = -x^2 + x - \lambda + \beta - \frac{\partial B}{\partial x}(x, \beta, \mu) \). So,

(i) When \( x = \lambda \) we get \( H_1(\lambda, \lambda, \beta, \mu) = H_2(\lambda, \lambda, \beta, \mu) \).

(ii) For the parameter values satisfying \( \beta = \lambda^2 + \frac{\partial B}{\partial x}(\lambda, \beta, \mu) > 0 \) we have \( H_1(\lambda, \lambda, \beta, \mu) = H_2(\lambda, \lambda, \beta, \mu) = 0 \).

(iii) Since \( H_1(0, 0, 0, 0) = 0 \) (respectively \( H_2(0, 0, 0, 0) = 0 \)) and \( \frac{\partial H_1}{\partial x}(0, 0, 0, 0) = -1 \) (respectively \( \frac{\partial H_2}{\partial x}(0, 0, 0, 0) = 1 \)), by the Implicit Function Theorem there is a unique \( x = x_{H_1}(\lambda, \beta, \mu) \) such that \( H_1(x_{H_1}(\lambda, \beta, \mu), \lambda, \beta, \mu) = 0 \) (respectively \( H_2(x_{H_2}(\lambda, \beta, \mu), \lambda, \beta, \mu) = 0 \)). Therefore, there is only one zero of \( H_1 \) and only one zero of \( H_2 \) in a sufficiently small neighborhood of \( x = 0 \). These points are called \( p_1 \) and \( r_1 \), respectively, in Figure 14. The pseudo equilibrium \( p_1 \) and the virtual pseudo equilibrium \( r_1 \) are the unique roots of \( H_1 \) and \( H_2 \), respectively, that are relevant to our analysis. In fact, the other roots are far from the origin.
2.3. Proof of Lemma

Now we prove Lemma

Proof of Lemma Here we construct a $\Sigma-$preserving homeomorphism $h$ that sends orbits of $Z = (X,Y)$ to orbits of $\tilde{Z} = (\tilde{X},\tilde{Y})$, where $\tilde{Z} = Z_\rho$ is given by (5) with $\rho_1 = 1$ and $\rho_i = -1$, $i = 2, 3, 4$. The other choices on parameters $\rho_i$, $i = 1, 2, 3, 4$, are treated in a similar way. Let $p$ (respectively, $\tilde{p}$) be the fold–cusp singularity of $Z$ (respectively, $\tilde{Z}$) (see Figure 15). Identify $p$ with $\tilde{p}$, i.e., $h(p) = \tilde{p}$. Consider a point $q \in \gamma$ (respectively, $\tilde{q} \in \tilde{\gamma}$),
where $\gamma$ (respectively, $\tilde{\gamma}$) is the orbit–arc of $Y$ (respectively, $\tilde{Y}$) starting at $p$ (respectively, $\tilde{p}$). Identify $\gamma$ with $\tilde{\gamma}$ (i.e., $h(\gamma) = \tilde{\gamma}$) from a reparametrization by arc–length. Let $T$ (respectively, $\tilde{T}$) be transversal sections to $Y$ (respectively, $\tilde{Y}$) passing through $q$ (resp., $\tilde{q}$) with small amplitude. Identify $T$ with $\tilde{T}$ (i.e., $h(T) = \tilde{T}$) by arc–length. Let $q^1_s \in T$ be a point on the left of $q$. Using the Implicit Function Theorem (abbreviated by IFT), there exists a time $t'_s < 0$, depending on $q^1_s$, such that $\phi_Y(q^1_s, t'_s) := p^1_s \in \Sigma$. Since $h(t' = \tilde{T}$, there exists $\tilde{q}^1_s \in \tilde{T}$ such that $h(q^1_s) = \tilde{q}^1_s$. Using IFT, there exists a time $\tilde{t}'_s < 0$, depending on $\tilde{q}^1_s$, such that $\phi_{\tilde{Y}}(\tilde{q}^1_s, \tilde{t}'_s) := \tilde{p}^1_s \in \tilde{\Sigma}$. Identify the orbit–arc $\sigma^p_{q^1_s}(Y)$ of $Y$ joining $p^1_s$ to $q^1_s$ with the orbit–arc $\sigma^p_{\tilde{q}^1_s}(\tilde{Y})$ of $\tilde{Y}$ joining $\tilde{p}^1_s$ to $\tilde{q}^1_s$ (i.e., $h(\sigma^p_{q^1_s}(Y)) = \sigma^p_{\tilde{q}^1_s}(\tilde{Y})$) by arc–length. Fix the notation for the orbit–arcs of a given vector field joining two points. Since $p$ (respectively, $\tilde{p}$) is a $\Sigma$–fold point of $X$ (respectively, $\tilde{X}$), using the IFT, there exists a time $t^2_s > 0$ (respectively, $\tilde{t}^2_s > 0$), depending on $p^1_s$ (respectively, $\tilde{p}^1_s$), such that $\phi_X(p^1_s, t^2_s) := p^2_s \in \Sigma$ (respectively, $\phi_{\tilde{X}}(\tilde{p}^1_s, \tilde{t}^2_s) := \tilde{p}^2_s \in \tilde{\Sigma}$). Identify $\sigma^p_{p^1_s}(X)$ with $\sigma^p_{\tilde{p}^1_s}(\tilde{X})$ (i.e., $h(\sigma^p_{p^1_s}(X)) = \sigma^p_{\tilde{p}^1_s}(\tilde{X})$) by arc–length. Using the IFT, there exists a time $t'_s > 0$ (respectively, $\tilde{t}'_s > 0$), depending on $p^2_s$ (respectively, $\tilde{p}^2_s$), such that $\phi_{\tilde{X}}(\tilde{p}^2_s, t'_s) := \tilde{q}^2_s \in \tilde{T}$ (resp., $\phi_X(p^2_s, t'_s) := q^2_s \in T$). Identify $\sigma^p_{\tilde{p}^2_s}(\tilde{Y})$ with $\sigma^p_{\tilde{p}^2_s}(\tilde{Y})$ (i.e., $h(\sigma^p_{\tilde{p}^2_s}(\tilde{Y})) = \sigma^p_{\tilde{p}^2_s}(\tilde{Y})$) by arc–length.

So, the homeomorphism $h$ sends $\Sigma$ to $\tilde{\Sigma}$ and sends orbits of $Z$ to orbits of $\tilde{Z}$.

\[ \square \]

3. Proof of Theorem 1

**Proof of Theorem 1.** In Case 1 we assume that $Y$ presents the behavior $Y^-$ where $\beta < 0$. In Cases 2, 3, and 4 we assume that $Y$ presents the behavior $Y^0$ where $\beta = 0$. In these cases canard cycles do not arise (for a proof, see \[5\]).

- **Case 1.** $\beta < 0$: The points of $\Sigma$ on the left of $d$ belong to $\Sigma^c$ and the points on the right of $d$ belong to $\Sigma^c$. See Figure 1\[6\]. Since $\beta < 0$, the graph of $H$ is illustrated in H-3 of Figure 1\[4\]. We get that $p_1 = (-1 + \sqrt{1 + 4\beta + 4\lambda^2}, 0) \in \Sigma^c$ is a $\Sigma$–repeller and $r_1 = (1 - \sqrt{1 + 4\beta - 4\lambda^2}, 0) \in \Sigma^c$.

- **Case 2.** $\lambda < 0$, **Case 3.** $\lambda = 0$ and **Case 4.** $\lambda > 0$: The configuration of the connected components of $\Sigma$ is the same as Case 1. Since $\beta = 0$, the graph of $H$, when $\lambda \neq 0$, is given by H-3 of Figure 1\[4\]. When $\lambda = 0$ (Case 3), the graph of $H$ is given by H-2 of Figure 1\[4\] and $p_1 = r_1$. These cases are illustrated in Figure 1\[7\].

In Cases 5 - 17 we assume that $Y$ presents the behavior $Y^+$ where $\beta > 0$.
\( d < e \)

\( d = e \)

\( d > e \)

\( \lambda < -\sqrt{\beta} \): The points of \( \Sigma \) on the left of \( d \) belong to \( \Sigma^e \), the points inside the interval \((a, b)\) belong to \( \Sigma^s \) and the points on \((d, a)\) and on the right of \( b \) belong to \( \Sigma^c \). The graph of \( H \) is like H-3 of Figure 14. We can prove that \( p_1 \) is a \( \Sigma \)-repeller situated on the left of \( d \) and \( r_1 \in (d, a) \). Canard cycles do not arise. See Figure 18.

\( \lambda = -\sqrt{\beta} \): In this case the points on the right of \( b \) belong to \( \Sigma^c \), the points on \((a = d, b)\) belong to \( \Sigma^s \) and the points on the left of \( a = d \) belong to \( \Sigma^c \). Since \( \beta = \lambda^2 \), \( H \) is like H-2 of Figure 14 and \( p_1 = r_1 \). There exists a non hyperbolic canard cycle \( \Gamma \) of kind III passing through \( a \) and \( e \). See Figure 19.

\( \lambda = 0 \) and \( 0 < \lambda < \sqrt{\beta} \): The configuration of the connected components of \( \Sigma \) is like Case 5 replacing \( a \) by \( d \) and vice-versa. The graph of \( H \) is like H-1 of Figure 14. We observe
that \( p_1 \in (d, b) \) is a \( \Sigma \)-attractor and \( r_1 \in (a, d) \). There exists a hyperbolic repeller canard cycle \( \Gamma \) of kind III passing through \( a \) and \( c \). See Figure 20.

\[ b < d < 0 \quad d = 0 \quad 0 < d < c \]

\[ 7 \quad 8 \quad 9 \]

Figure 20. Cases 7-9.

\( \diamond \) Case 10. \( \lambda = \sqrt{\beta} \): In this case the points on the left of \( a \) belong to \( \Sigma^e \) and the points on the right of \( a \) belong to \( \Sigma^e \), except by \( Q = (b, 0) \in \Sigma \). Since \( \beta = \lambda^2 \), \( H \) is like H-2 of Figure 14 and \( p_1 = r_1 \). Since \( \mu = 0 \) and \( d = b \), by the construction of the bump function \( B \) it is straightforward to show that the point \( Q \) behaves itself like a weak attractor for \( Z \) and there exists a non hyperbolic canard cycle of kind III passing through \( a \) and \( c \). See Figure 3. This case has already been discussed previously in Subsection 2.1. Note that in [9] the authors avoid this case.

\( \diamond \) Case 11. \( \sqrt{\beta} < \lambda < L_1 \): The meaning of the value \( L_1 \) will be given below in this case. The points of \( \Sigma \) on the left of \( a \) and on \( (b, d) \) belong to \( \Sigma^e \). The points on \( (a, b) \) and on the right of \( d \) belong to \( \Sigma^e \). The graph of \( H \) is like H-3 of Figure 14. We can prove that \( p_1 \in (b, d) \) is a \( \Sigma \)-repeller and \( r_1 \) is on the right of \( d \). Since the point \( Q \) of the previous case is a weak attractor, in a neighborhood of \( d \) occurs a Like Hopf Bifurcation. Moreover, according to Lemma 9 there is a unique canard cycle \( \Gamma_1 \) in a neighborhood of \( d \) and a unique canard cycle \( \Gamma_2 \) in a neighborhood of \( c \). Observe that both are of kind I, \( \Gamma_1 \) is attractor, \( \Gamma_2 \) is repeller and \( \Gamma_1 \) is located within the region bounded by \( \Gamma_2 \). See Figure 21. Note that, as \( \lambda \) increases, \( \Gamma_1 \) becomes bigger and \( \Gamma_2 \) becomes smaller. When \( \lambda \) assumes the limit value \( L_1 \), one of them collides with the other.

\( \diamond \) Case 12. \( \lambda = L_1 \): The distribution of the connected components of \( \Sigma \) and the behavior of \( H \) are the same as Case 11. Since \( \lambda = L_1 \), as described in the previous case, there exists a non hyperbolic canard cycle \( \Gamma \) of kind I which is an attractor for the trajectories inside it and is a repeller for the trajectories outside it. See Figure 21.

\( \diamond \) Case 13. \( L_1 < \lambda < 2\sqrt{\beta} \), Case 14. \( \lambda = 2\sqrt{\beta} \), Case 15. \( 2\sqrt{\beta} < \lambda < 3\sqrt{\beta} \), Case 16. \( \lambda = 3\sqrt{\beta} \) and Case 17. \( \lambda > 3\sqrt{\beta} \): The distribution of the connected components of \( \Sigma \) and the behavior of \( H \) are the same as Case 11. Canard cycles do not arise. See Figure 22.

The Bifurcation Diagram is illustrated in Figure 23. □
Remark 7. In Cases 9₁ and 11₁ the ST−bifurcations (as described in [9], Subsections 11.2 and 12.2) arise. In fact, note that the trajectory passing through $a$, in Case 9₁, and $c$, in Case 11₁, can make more and more turns.
around $p_1$. This fact characterizes a global bifurcation also reached in other cases.

4. Proof of Theorem 2

Proof of Theorem 2. In Case 1 we assume that $Y$ presents the behavior $Y^-$. In Cases 2, 3 and 4 we assume that $Y$ presents the behavior $Y^0$. In Cases 5 - 19 we assume that $Y$ presents the behavior $Y^+$.

- **Case 1.** $\beta < 0$, Case 2. $\lambda < 0$, Case 3. $\lambda = 0$, Case 4. $\lambda > 0$. **Case 5.** $\lambda < -\sqrt{\beta}$. **Case 6.** $\lambda = -\sqrt{\beta}$. **Case 7.** $-\sqrt{\beta} < \lambda < 0$ and Case 8. $\lambda = 0$.

By the choice of the bump function $B$, these cases are analogous to Cases 1, 2, 3, 4, 5, 6, 7 and 8.

- **Case 9.** $0 < \lambda < \sqrt{\beta}$. The analysis of this case is done in a similar way as the Case 9. In this case and in Cases 7 and 8 there exists a hyperbolic repeller canard cycle $\Gamma$ of kind III passing through $a$ and $c$.

- **Case 10.** $\lambda = \sqrt{\beta} - \mu/2$: The points of $\Sigma$ on the left of $a$ belong to $\Sigma^e$ and the points on $(d,b)$ belong to $\Sigma^s$. The points on $(a,d)$ and on the right of $b$ belong to $\Sigma^c$. The graph of $H$ is like H-3 of Figure 14. Observe that $p_1 \in (d,b)$ is a $\Sigma$-attractor and $r_1$ is on the right of $b$. In this case the arc $\gamma_X(a)$ of $X$ passing through $a$ returns to $\Sigma$ at the point $c$. So, in this case there arises a non hyperbolic canard cycle $\Gamma = \gamma_X(a) \cup \gamma_Y(c)$. By the discussion on subsection 2.1.2, we have that $\Gamma$ is a repeller and we do not have other canard cycles inside $\Gamma$. See Figure 24.

![Figure 24. Cases 10 - 12.](image)

- **Case 11.** $\sqrt{\beta} - \mu/2 < \lambda < \sqrt{\beta}$: The configuration on $\Sigma$ and the graph of $H$ are the same as Case 10. Since $g_X^{-1}(c) \in (a,d)$ there exists a point $Q \in (g_X^{-1}(c), g_X^{-1}(b))$ such that $\eta(Q) = 1$. So there exists a hyperbolic repeller canard cycle $\Gamma$, of kind I, passing through $Q$. See Figure 24. Moreover, by Lemma 9 this canard cycle is unique. In Figure 11 we introduce the point $x$ which plays the same role of $Q$.

- **Case 12.** $\lambda = \sqrt{\beta}$: The points of $\Sigma$ on the left of $a$ belong to $\Sigma^e$ and the points on the right of $a$ belong to $\Sigma^c$, except by the tangential singularity $c = d$. The graph of $H$ is like H-2 of Figure 14. The repeller canard cycle $\Gamma$ presented in the previous case is persistent. Recall that this canard cycle is born from the bifurcation of Case 10. So, the radius of $\Gamma$ does not tend
to zero when $\lambda$ tends to $\sqrt{\beta}$. Moreover, the tangential singularity $b = d$ behaves itself like a weak attractor. See Figure 24.

- Case 13. $\sqrt{\beta} < \lambda < L_1$, Case 14. $\lambda = L_1$, Case 15. $L_1 < \lambda < 2\sqrt{\beta} + \mu/2$, Case 16. $\lambda = 2\sqrt{\beta} + \mu/2$, Case 17. $2\sqrt{\beta} + \mu/2 < \lambda < 3\sqrt{\beta} + \mu$, Case 18. $\lambda = 3\sqrt{\beta} + \mu$ and Case 19. $\lambda > 3\sqrt{\beta} + \mu$: The analysis of these cases is done in a similar way as Cases 11, 12, 13, 14, 15, 16 and 17, respectively.

The bifurcation diagram is illustrated in Figure 25.

The bifurcation diagram is illustrated in Figure 25. □

5. PROOF OF THEOREM 3

Proof of Theorem 3. In Case 1 we assume that $Y$ presents the behavior $Y^-$. In Cases 2, 3 and 4 we assume that $Y$ presents the behavior $Y^0$. In Cases 5 - 9 we assume that $Y$ presents the behavior $Y^+$. 

- Case 1. $\beta < 0$, Case 2. $\lambda < 0$, Case 3. $\lambda = 0$, Case 4. $\lambda > 0$, Case 5. $\lambda < -\sqrt{\beta}$, Case 6. $\lambda = -\sqrt{\beta}$, Case 7. $-\sqrt{\beta} < \lambda < 0$, Case 8. $\lambda = 0$ and Case 9. $0 < \lambda < \sqrt{\beta}$: By the choice of the bump function $B$, these cases are analogous to Cases 1, 2, 3, 4, 5, 6, 7, 8 and 9.

- Case 10. $\lambda = \sqrt{\beta}$: The distribution of the connected components of $\Sigma$ and the behavior of $H$ are the same as Case 12. This case differs from Case 12 because, as observed in Subsection 2.1.3 when $\lambda = \sqrt{\beta}$ and $\mu > 0$ canard cycles of $Z$ do not arise (see Figure 10) bifurcating from the non hyperbolic canard cycle $\Gamma$ of Case 12 below. Moreover, the tangential singularity $d = b$ behaves itself like a weak attractor. See Figure 26. There exists a hyperbolic repeller canard cycle $\Gamma$ of kind III passing through $a$ and $c$.

- Case 11. $\sqrt{\beta} < \lambda < \sqrt{\beta} - \mu/2$: The points of $\Sigma$ on the left of $a$ and on $(b, d)$ belong to $\Sigma^c$. The points on $(a, b)$ and on the right of $d$ belong to $\Sigma^c$. The graph of $H$ is like H-3 of Figure 14. We can prove that $p_1 \in (b, d)$.
Figure 26. Cases $10_3 - 12_3$.

is a $\Sigma$-repeller and $r_1$ is on the right of $d$. Since $g_Y(g_X(a)) \in (a, b)$ there exists a point $Q \in (g_Y(g_X(a)), g_Y(d))$ such that $\eta'(Q) = 1$. So there exists a hyperbolic attractor canard cycle $\Gamma$, of kind I, passing through $Q$. See Figure 26. By Lemma 8 in this Hopf Bifurcation a unique canard cycle arises. Moreover, there exists a hyperbolic repeller canard cycle $\Gamma$ of kind III passing through $a$ and $c$.

$\diamond$ Case $12_3. \lambda = \sqrt{\beta} - \mu/2$: The configuration on $\Sigma$ and the graph of $H$ are the same as Case $11_3$. The attractor canard cycle $\Gamma$ presented in the previous case is persistent. Recall that this canard cycle is born from the bifurcation of Case $10_3$. So, the radius of $\Gamma$ does not tend to zero when $\lambda$ tends to $\sqrt{\beta} + \mu/2$. Moreover, it appears a non hyperbolic canard cycle passing through $a$ and $c$. See Figure 24.

$\diamond$ Case $13_3. \sqrt{\beta} - \mu/2 < \lambda < L_1$, Case $14_3. \lambda = L_1$, Case $15_3. L_1 < \lambda < 2\sqrt{\beta} - \mu/2$, Case $16_3. \lambda = 2\sqrt{\beta} - \mu/2$, Case $17_3. 2\sqrt{\beta} - \mu/2 < \lambda < 3\sqrt{\beta} - \mu$, Case $18_3. \lambda = 3\sqrt{\beta} - \mu$ and Case $19_3. \lambda > 3\sqrt{\beta} - \mu$: The analysis of these cases is done in a similar way as Cases $11_1, 12_1, 13_1, 14_1, 15_1, 16_1$ and $17_1$, respectively.

The bifurcation diagram is illustrated in Figure 25 replacing the number 2 subscript by the number 3. $\square$

6. Proof of Theorem A

Proof of Theorem A. Since in Equation (1) we can take $\mu \in (-\mu_0, \mu_0)$, from Theorems 1, 2 and 3 we derive that it bifurcation diagram contains all the 55 cases described in Theorems 1, 2 and 3. But some of them are $\Sigma$-equivalent and the number of distinct topological behaviors is 23. Moreover, each topological behavior can be represented respectively by the Cases $1_1, 2_1, 3_1, 4_1, 5_1, 6_1, 7_1, 8_1, 9_1, 10_1, 11_1, 12_1, 13_1, 14_1, 15_1, 16_1, 17_1, 10_2, 11_2, 12_2, 10_3, 11_3$ and $12_3$.

The full behavior of the three-parameter family of NSDS’s expressed by Equation (1) is illustrated in Figure 27 where we consider a sphere around the point $(\lambda, \beta, \mu) = (0, 0, 0)$ with a small ray and so we make a stereographic projection defined on the entire sphere, except the south pole. Still in relation to this figure, the numbers pictured correspond to the occurrence of the cases described in the previous theorems. As expected, the cases $3_1$
and $3_2$ are not represented in this figure because they are, respectively, the center and the south pole of the sphere.

Figure 27. Bifurcation diagram of the invisible fold–cusp singularity.

7. Proof of Theorem B

When we consider Equation (3) the function $H$, given by (8), is constant and equal to 1 independently of the value of $\mu$. Moreover, distinct values of the bump function $\tilde{B}$ (where $\tilde{B} \neq B$) do not produce any topological change in the bifurcation diagram of the singularity. In another words, two parameters are enough to describe the full behavior of this singularity. Observe that, by Proposition 10, we have $\Sigma^f = \emptyset$ and it does not have virtual pseudo equilibria.
Proof of Theorem B. Since $X$ has a unique $\Sigma$-fold point which is visible we conclude that canard cycles do not arise. In Case $1_B$ we assume that $Y$ presents the behavior $-Y^\circ$. In Cases $2_B$, $3_B$ and $4_B$ we assume that $Y$ presents the behavior $Y^0$. In Cases $5_B - 11_B$ we assume that $Y$ presents the behavior $Y^+$. 

- **Case 1** $\beta < 0$: The points of $\Sigma$ on the left of $d$ belong to $\Sigma^c$ and the points on the right of $d$ belong to $\Sigma^e$. See Figure 28.

- **Case 2** $\lambda < 0$, **Case 3** $\lambda = 0$ and **Case 4** $\lambda > 0$: The configuration of the connected components of $\Sigma$ is the same as Case $1_B$. Note that, when $\lambda < 0$ (Case $2_B$), it appears a tangential singularity $P = (\lambda, 0) \in \Sigma^e$ but $Z_\Sigma^\circ$ is always oriented from the left to the right. These cases are illustrated in Figure 29.

- **Case 5** $\lambda < -2\sqrt{\beta}$, **Case 6** $\lambda = -2\sqrt{\beta}$ and **Case 7** $-2\sqrt{\beta} < \lambda < -\sqrt{\beta}$: The points of $\Sigma$ on the right of $b$ and inside the interval $(d, a)$ belong to $\Sigma^c$. The points on $(a, b)$ and on the left of $d$ belong to $\Sigma^e$. See Figure 30.

- **Case 8** $\lambda = -\sqrt{\beta}$: In this case $a = d$ and the configuration of the connected components of $\Sigma$ is illustrated in Figure 31.

- **Case 9** $-\sqrt{\beta} < \lambda < \sqrt{\beta}$: The points of $\Sigma$ on the right side of $b$ belong to $\Sigma^c$ and the points inside the interval $(a, d)$ belong to $\Sigma^s$. The points on $(d, b)$ and on the left of $a$ belong to $\Sigma^c$. See Figure 32.

- **Case 10** $\lambda = \sqrt{\beta}$: In this case $d = b$ and the configuration of the connected components of $\Sigma$ is illustrated in Figure 32.
\[ d = c \]

**Figure 30.** Cases $5_B - 7_B$.

\[ \diamond \text{Case 8}_B. \lambda > \sqrt{\beta}: \text{The points of } \Sigma \text{ on the right of } d \text{ belong to } \Sigma^e \text{ and the points inside the interval } (a,b) \text{ belong to } \Sigma^s. \text{ The points on } (b,d) \text{ and on the left of } a \text{ belong to } \Sigma^c. \text{ See Figure 32} \]

\[ d = b \]

**Figure 31.** Case $8_B$.

\[ \diamond \text{Case 11}_B. \lambda > \sqrt{\beta}: \text{The points of } \Sigma \text{ on the right of } d \text{ belong to } \Sigma^e \text{ and the points inside the interval } (a,b) \text{ belong to } \Sigma^s. \text{ The points on } (b,d) \text{ and on the left of } a \text{ belong to } \Sigma^c. \text{ See Figure 32} \]

\[ d = b \]

**Figure 32.** Cases $9_B - 11_B$.

The bifurcation diagram is illustrated in Figure 33.

\[ \square \]

8. **Concluding Remarks**

The results in Section 12 of [9] were revisited and extended in this paper. The bifurcation diagram of a three-parameter family of NSDS’s presenting a fold–cusp singularity is exhibited. In particular it is shown the existence of some new interesting global bifurcations around the standard fold–cusp singularity expressed by (5). Moreover, the simultaneous occurrence of such local and global bifurcations indicates how complex is the behavior of this singularity.
\begin{equation}
\lambda = -\sqrt{\beta} \\
\lambda = -2\sqrt{\beta} \\
\lambda = \sqrt{\beta}
\end{equation}

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{bifurcation_diagram.png}
\caption{Bifurcation Diagram of Theorem B.}
\end{figure}

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