A FIRST LOOK AT THE BOSONIC MASTER-FIELD EQUATION OF THE IIB MATRIX MODEL

F.R. KLINKHAMER

Institute for Theoretical Physics, Karlsruhe Institute of Technology (KIT), 76128 Karlsruhe, Germany
frans.klinkhamer@kit.edu

The bosonic large-$N$ master field of the IIB matrix model can, in principle, give rise to an emergent classical spacetime. The task is then to calculate this master field as a solution of the bosonic master-field equation. We consider a simplified version of the algebraic bosonic master-field equation and take dimensionality $D = 2$ and matrix size $N = 6$. For an explicit realization of the pseudorandom constants entering this simplified algebraic equation, we establish the existence of a solution and find, after diagonalization of one of the two obtained matrices, a band-diagonal structure of the other matrix.

Keywords: Strings and branes; M theory; quantum gravity – lattice and discrete methods.

1. Introduction

The definitive formulation of nonperturbative superstring theory, also known as $M$-theory, is still outstanding. One suggestion is the IKKT matrix model. That matrix model reproduces the basic structure of the light-cone string field theory of type-IIB superstrings and the model is also called the IIB matrix model. It is, therefore, important to investigate the IIB matrix model.

First results on the partition function of the IIB matrix model were reported in Refs. and numerical simulations of the Lorentzian version of the model were presented in Refs. (related numerical results for the Euclidean model were discussed in Ref.). Still, the conceptual question of how classical spacetime emerges from the IIB matrix model was essentially left unanswered.

We have recently suggested, in the context of the IIB matrix model, to consider Witten’s large-$N$ master field as a possible source of the emerging classical spacetime. Assuming the matrices of the bosonic master field to be known and assuming these matrices to have a band-diagonal structure, we have shown that, in principle, classical spacetime points can be extracted and an emerging spacetime metric calculated. The technical details have been presented in our original paper, further work on the emerging cosmological metric has been discussed in a follow-up paper, and the heuristics of the spacetime extraction from the master field has been explained in the recent review.

But all these discussions assume the bosonic master field to exist and to have
matrices with a band-diagonal structure (first hints of a band-diagonal structure were obtained in numerical results for the Lorentzian model). The goal of our present paper is to embark upon a preliminary analysis of the bosonic master-field equation and to check for the possible appearance of a band-diagonal structure. As the full bosonic master-field equation from the IIB-matrix model is extremely complicated, we first consider a simplified equation and, then, are able to obtain some nontrivial results. Let us, however, emphasize that the present paper is purely exploratory. As such, the paper has a limited scope, which implies, in particular, that all discussion of statistical issues is postponed to future work.

2. IIB Matrix Model

2.1. Action

The IIB matrix model has a finite number of $N \times N$ traceless Hermitian matrices: ten bosonic matrices $A^\mu$ and essentially eight fermionic (Majorana–Weyl) matrices $\Psi_\alpha$. The partition function $Z$ of the IIB matrix model is defined by the following “path” integral:

$$Z = \int dA d\Psi e^{-S[A, \Psi]/\ell^4} = \int dA e^{-S_{\text{eff}}[A]/\ell^4}, \quad (1a)$$

$$S[A, \Psi] = S_{\text{bos}}[A] + S_{\text{term}}[A, \Psi] = -\text{Tr} \left( \frac{1}{4} [A^\mu, A^\nu] [A^\rho, A^\sigma] \tilde{\delta}_{\mu\rho} \tilde{\delta}_{\nu\sigma} + \frac{1}{2} \Psi_\beta \tilde{\Gamma}_{\beta\alpha} \tilde{\delta}_{\mu\nu} [A^\nu, \Psi_\alpha] \right), \quad (1b)$$

$$\tilde{\delta}_{\mu\nu} = \left[ \text{diag}(1, 1, \ldots, 1) \right]_{\mu\nu}, \quad \text{for } \mu, \nu \in \{1, 2, \ldots, 10\}. \quad (1c)$$

In addition to the partition function $Z$, there are expectation values of observables, which will be discussed in Sec. 2.2.

The fermions appear quadratically in the action (1b). The fermionic integrals in the first part of (1a) are then Gaussian and can be performed analytically. In this way, the following effective action is obtained:

$$S_{\text{eff}}[A] = S_{\text{bos}}[A] + S_{\text{ind}}[A], \quad (2)$$

where the induced term $S_{\text{ind}}$ may, for example, contain a high-order term with commutators and anticommutators of the bosonic matrices.

We have two technical remarks. First, the model shown in (1) is the original model with “Euclidean” coupling constants $\tilde{\delta}_{\mu\nu}$, but it is also possible to consider a “Lorentzian” version with a complex Feynman phase factor $\exp \left( i S/\ell^4 \right)$ in the path integral and coupling constants $\eta_{\mu\nu} = \left[ \text{diag}(-1, 1, \ldots, 1) \right]_{\mu\nu}$, for $\mu, \nu \in \{0, 1, \ldots, 9\}$, in the action.

Second, a model length scale $\ell$ has been introduced in (1), so that $A^\mu$ has the dimension of length and $\Psi_\alpha$ the dimension of $(\text{length})^{3/2}$. From now on, we set

$$\ell = 1, \quad (3)$$
so that the model contains only dimensionless variables.

Now, the IIB matrix model (1) just gives numbers, $Z$ and further expectation values to be discussed later, while the (dimensionless) matrices $A^\mu$ and $\Psi_\alpha$ in (1a) are merely integration variables. Moreover, there is no obvious small dimensionless parameter to motivate a saddle-point approximation. Hence, the following conceptual question arises: where is the classical spacetime?

For a possible origin of classical spacetime in the context of the IIB matrix model, we have suggested 16 to revisit an old idea, the large-$N$ master field of Witten 11 (see Ref. 12 for a review and Refs. 13, 14, 15 for a selection of subsequent research papers). In the next two subsections, we briefly recall the meaning of this mysterious master field (a name coined by Coleman 12) and then discuss its “field” equation.

2.2. Large-$N$ bosonic master field

Consider the following bosonic observable:

$$w^{\mu_1 \ldots \mu_m} \equiv \text{Tr} (A^{\mu_1} \ldots A^{\mu_m}) .$$

(4)

There are also fermionic observables (for example, $\text{Tr} \overline{\Psi} \Psi$), but here we focus on bosonic observables of the type (4). Then, arbitrary strings of these $w$ observables have expectation values

$$\langle w^{\mu_1 \ldots \mu_m} w^{\nu_1 \ldots \nu_n} \ldots w^{\omega_1 \ldots \omega_z} \rangle = \frac{1}{Z} \int dA (w^{\mu_1 \ldots \mu_m} w^{\nu_1 \ldots \nu_n} \ldots w^{\omega_1 \ldots \omega_z}) e^{-S_{\text{eff}}},$$

(5)

with normalization $\langle 1 \rangle = 1$. For large values of $N$, these observables display a remarkable factorization property:

$$\langle w^{\mu_1 \ldots \mu_m} w^{\nu_1 \ldots \nu_n} \ldots w^{\omega_1 \ldots \omega_z} \rangle \approx \langle w^{\mu_1 \ldots \mu_m} \rangle \langle w^{\nu_1 \ldots \nu_n} \rangle \ldots \langle w^{\omega_1 \ldots \omega_z} \rangle ,$$

(6)

where the equality holds to leading order in $N$.

According to Witten 11, the factorization (6) implies that the path integrals (5) are saturated by a single configuration, the so-called master field $\hat{A}^\mu$. To leading order in $N$, the expectation values are then given by

$$\langle w^{\mu_1 \ldots \mu_m} w^{\nu_1 \ldots \nu_n} \ldots w^{\omega_1 \ldots \omega_z} \rangle \approx \hat{w}^{\mu_1 \ldots \mu_m} \hat{w}^{\nu_1 \ldots \nu_n} \ldots \hat{w}^{\omega_1 \ldots \omega_z},$$

(7a)

$$\hat{w}^{\mu_1 \ldots \mu_m} \equiv \text{Tr} (\hat{A}^{\mu_1} \ldots \hat{A}^{\mu_m}) .$$

(7b)

Hence, we do not have to perform the path integrals on the right-hand side of (5): we just need ten traceless Hermitian matrices $\hat{A}^\mu$ to get all these expectation values from the simple procedure of replacing each $A^\mu$ in the observables by the corresponding $\hat{A}^\mu$. Most likely, there is more than one master field, all these master fields being equivalent [giving, in the large-$N$ limit, exactly the same results for all possible observables of the type (4)]; see, e.g., Ref. 13 for a discussion of this point. But, for definiteness, we will talk, in the following, only about a single master field.
Now, the meaning of the suggestion at the end of Sec. 2.1 is clear: classical spacetime may reside in the bosonic master-field matrices $\hat{A}^\mu$ of the IIB matrix model. The heuristics of this idea has been discussed in Sec. 4.4 of a recent review paper. Next, assume that the matrices $\hat{A}^\mu$ of the IIB-matrix-model bosonic master field are known and that they are approximately band-diagonal. If, for simplicity, we consider $N = Kn$ with positive integers $K$ and $n$, then it is possible to extract from these matrices $\hat{A}^\mu$ a discrete set of spacetime points $\{\hat{x}^\mu_k\}$ with an index $k \in \{1, \ldots, K\}$. These discrete spacetime points sample a smooth manifold with continuous spacetime coordinates $x^\mu$ and an emergent inverse metric $g^{\mu\nu}(x)$, for which there is an explicit expression in terms of the density distribution and correlation functions of the extracted spacetime points. The metric $g_{\mu\nu}(x)$ is obtained as matrix inverse of $g^{\mu\nu}(x)$. The emerging metric may have a Lorentzian signature, even if the original matrix model is Euclidean; see Appendix B of Ref. 16 and Appendix D of Ref. 18 for further discussion.

The task is to really calculate the bosonic master-field matrices $\hat{A}^\mu$ of the IIB matrix model and, if possible, to establish a band-diagonal structure. For this calculation, we need the “field” equation for these master matrices.

### 2.3. Bosonic master-field equation

Building on previous work by Greensite and Halpern, we have obtained the IIB-matrix-model bosonic master field in the following “quenched” form:

$$\hat{A}^\rho_{kl} = e^{i(\hat{p}_k - \hat{p}_l)\tau_{eq}} \hat{a}^\rho_{kl}, \quad (8a)$$

where the matrix indices $k$ and $l$ take values from $\{1, \ldots, N\}$ and the directional index $\rho$ runs over $\{1, 2, \ldots, D\}$. The dimensionless time $\tau_{eq}$ in (8a) must have a sufficiently large value in order to represent an equilibrium situation ($\tau$ is the fictitious Langevin time of the stochastic-quantization procedure). The $\tau$-independent matrix $\hat{a}^\rho$ on the right-hand side of (8a) solves the following algebraic equation:

$$i(\hat{p}_k - \hat{p}_l) \hat{a}^\rho_{kl} = -\left. \frac{\delta S_{\text{eff}}}{\delta \hat{A}^\rho_{lk}} \right|_{\hat{A} = \hat{a}} + \hat{\eta}^\rho_{kl}, \quad (8b)$$

in terms of the master momenta $\hat{p}_k$ (uniform random numbers) and the master-noise matrices $\hat{\eta}^\rho_{kl}$ (Gaussian random numbers); further details and references can be found in Ref. 13.

The matrices $\hat{a}^\rho$ are $N \times N$ traceless Hermitian matrices and the number of real bosonic degrees of freedom is

$$N_{\text{d.o.f.}} = D (N^2 - 1). \quad (9)$$

These degrees of freedom are determined by the algebraic equation (8b) for fixed random constants $\hat{p}_k$ and $\hat{\eta}^\rho_{kl}$. It remains to solve this algebraic equation, which is not quite trivial, as there is a complicated high-order term in $S_{\text{eff}}$ from fermion
induction effects. In this paper, we will take a first step by considering a simplified version of (8b).

3. Simplified Algebraic Equation

3.1. General case

For $N \times N$ traceless Hermitian matrices $\hat{a}^\rho$ with index $\rho$ running over \{1, 2, \ldots, D\}, we will consider the following simplified algebraic equation:

$$i (\hat{p}_k - \hat{p}_l) \hat{a}_{kl}^\rho = \tilde{g}_{\mu\nu} \left[ \hat{a}_\mu^\rho, [\hat{a}^\nu, \hat{a}^\rho]\right]_{kl} + \hat{\eta}^\rho_{kl},$$  (10a)

$$\tilde{g}_{\mu\nu} = \left[ \text{diag} (\tilde{g}_{11}, \tilde{g}_{22}, \ldots, \tilde{g}_{DD}) \right]_{\mu\nu} = \left[ \text{diag} (\tilde{s}, 1, \ldots, 1) \right]_{\mu\nu},$$  (10b)

$$\tilde{s} = 1,$$  (10c)

where $k$ and $l$ are matrix indices running over \{1, \ldots, N\} and where we omit matrix indices inside the double commutator on the right-hand side of (10a). The choice $\tilde{s} = 1$ corresponds to “Euclidean” coupling constants $\tilde{g}_{\mu\nu}$ from (10b). The pseudorandom numbers $\hat{p}_k$ and $\hat{\eta}^\rho_{kl}$ will be specified in Sec. 3.2 for a special case which can be easily generalized.

The crucial simplification of (10), compared to the full algebraic equation (8b), is that the effects of the fermions are neglected, which are contained in the $S_{\text{ind}}$ contribution to the effective action (2). Remark also that our simplified algebraic equation (10) resembles the “classical equation” studied in Ref. 9, as given by Eq. (2.6) in that reference with implicit “Lorentzian” coupling constants.

3.2. Special case: $D = 2$ and $N = 4$

In order to be specific, let us first consider the case

$$\{D, N\} = \{2, 4\}.$$  (11)

The discussion of this subsection trivially extends to larger values of $D$ and $N$. For the case (11), we parameterize the matrices $\hat{a}^1$ and $\hat{a}^2$ as follows:

$$\hat{a}^1 = \begin{pmatrix}
a_{11} & a_{12} + i A_{12} & a_{13} + i A_{13} & a_{14} + i A_{14} 
a_{12} - i A_{12} & a_{22} & a_{23} + i A_{23} & a_{24} + i A_{24} 
a_{13} - i A_{13} & a_{23} - i A_{23} & a_{33} & a_{34} + i A_{34} 
a_{14} - i A_{14} & a_{24} - i A_{24} & a_{34} - i A_{34} & -a_{11} - a_{22} - a_{33}
\end{pmatrix},$$  (12a)

$$\hat{a}^2 = \begin{pmatrix}
b_{11} & b_{12} + i B_{12} & b_{13} + i B_{13} & b_{14} + i B_{14} 
b_{12} - i B_{12} & b_{22} & b_{23} + i B_{23} & b_{24} + i B_{24} 
b_{13} - i B_{13} & b_{23} - i B_{23} & b_{33} & b_{34} + i B_{34} 
b_{14} - i B_{14} & b_{24} - i B_{24} & b_{34} - i B_{34} & -b_{11} - b_{22} - b_{33}
\end{pmatrix},$$  (12b)

in terms of the 15 real variables $\{a_{11}, a_{12}, a_{13}, \ldots, A_{34}\}$ and the 15 real variables $\{b_{11}, b_{12}, b_{22}, \ldots, B_{34}\}$. Hence, the number of unknowns is 30.
Similarly, the master coupling constants are parameterized as follows:

\[ \hat{p} = \{ \hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4 \}, \]  

(13a)

\[ \hat{\eta}^1 = \begin{pmatrix} e_{11} & e_{12} + iE_{12} & e_{13} + iE_{13} & e_{14} + iE_{14} \\ e_{12} - iE_{12} & e_{22} & e_{23} + iE_{23} & e_{24} + iE_{24} \\ e_{13} - iE_{13} & e_{23} - iE_{23} & e_{33} & e_{34} + iE_{34} \\ e_{14} - iE_{14} & e_{24} - iE_{24} & e_{34} - iE_{34} & -e_{11} - e_{22} - e_{33} \end{pmatrix}, \]  

(13b)

\[ \hat{\eta}^2 = \begin{pmatrix} f_{11} & f_{12} + iF_{12} & f_{13} + iF_{13} & f_{14} + iF_{14} \\ f_{12} - iF_{12} & f_{22} & f_{23} + iF_{23} & f_{24} + iF_{24} \\ f_{13} - iF_{13} & f_{23} - iF_{23} & f_{33} & f_{34} + iF_{34} \\ f_{14} - iF_{14} & f_{24} - iF_{24} & f_{34} - iF_{34} & -f_{11} - f_{22} - f_{33} \end{pmatrix}. \]  

(13c)

The entries in (13) are given by pseudorandom rational numbers with ranges \([-1/2, 1/2]\) for the master momenta and \([-1, 1]\) for the master noise:

\[ \hat{p}_k = \left( \frac{\text{randominteger}[-500, +500]}{1000} \right)_k, \]  

(14a)

\[ e_{kl} = \left( \frac{\text{randominteger}[-1000, +1000]}{1000} \right)_{kl}, \]  

(14b)

\[ E_{kl} = \left( \frac{\text{randominteger}[-1000, +1000]}{1000} \right)_{kl}, \]  

(14c)

\[ f_{kl} = \left( \frac{\text{randominteger}[-1000, +1000]}{1000} \right)_{kl}, \]  

(14d)

\[ F_{kl} = \left( \frac{\text{randominteger}[-1000, +1000]}{1000} \right)_{kl}, \]  

(14e)

where the pseudorandom integers on the right-hand sides are taken from uniform distributions with ranges as indicated. The practical reason for taking rational numbers is that we can then easily write down their exact values, whereas real numbers would require an infinite number of digits (or an implicit defining relation, as for the irrational number \( \sqrt{2} \)).

Strictly speaking, the pseudorandom master-noise numbers \( e_{kl}, E_{kl}, f_{kl}, \) and \( F_{kl} \) must be taken from a Gaussian (normal) distribution, but here we have used, for simplicity, a uniform distribution with a finite range. For later reference, we can mention that an improved procedure would use a truncated Gaussian distribution,

\[ P_{\text{trunc-Gauss}}(x) = \begin{cases} \nu e^{-\frac{1}{2} x^2/\sigma^2}, & \text{for } |x| \leq x_{\text{trunc}}, \\ 0, & \text{for } |x| > x_{\text{trunc}}, \end{cases} \]  

(15)

with spread \( \sigma > 0 \), cut-off value \( x_{\text{trunc}} > 0 \), and normalization factor \( \nu = \nu(\sigma, x_{\text{trunc}}) \). In principle, we should have \( x_{\text{trunc}} \gg \sigma \), so that \( \sigma \) approaches the
standard deviation. As we are primarily interested in establishing the existence of a nontrivial solution and giving a qualitative discussion of its properties, we have used a simple and explicit procedure in (14b)–(14e) with rational numbers \(n/1000\), for integer \(n \in [-1000, +1000]\), taken from a uniform distribution [which corresponds to \(x_{\text{trunc}} = 1\) and \(\sigma \gg 1\) in (15)]. An in-depth discussion of the statistical aspects of the obtained solutions requires significantly larger values of \(N\) and also a larger dimensionality, \(D = 10\).

From (10) and (11), we have 30 coupled algebraic equations for the 30 real unknowns \(\{a_{11}, \ldots, B_{34}\}\). It appears impossible to obtain a general analytic solution in terms of the 34 constants \(\{\hat{p}_k, \ldots, F_{34}\}\). (Remark that an exact solution was obtained in Sec. 5 of Ref. 13 for a single-matrix model at \(N = 2\).) In our case with two matrices and \(N = 4\), we will consider the 30 coupled algebraic equations with an explicit choice for the 34 pseudorandom constants. The reader who is allergic to seeing too many numbers may skip ahead to Sec. 5.

4. Numerical Solutions

4.1. \(D = 2\) and \(N = 4\)

We present, here, a representative numerical solution of the 30 coupled algebraic equations mentioned at the end of Sec. 3.2. Take, for example, the following pseudorandom constants:

\[
\hat{p}_{\text{num}} = \left\{ \begin{array}{c}
\frac{159}{1000} - \frac{73}{250} - \frac{141}{500} \frac{209}{500} \end{array} \right\}, \quad (16a)
\]

\[
\hat{\eta}_{\text{num}}^1 = \left( \begin{array}{cccccccc}
-\frac{97}{200} & \frac{209}{500} & -\frac{111}{1000} & -\frac{229}{1000} & \frac{221}{500} & \frac{23}{50} & \frac{23}{40} \\
\frac{209}{500} & \frac{111}{1000} & -\frac{1}{125} & \frac{923}{1000} & \frac{333}{500} & \frac{77}{500} & \frac{169}{500} \\
\frac{229}{1000} & \frac{221}{500} & \frac{923}{1000} & \frac{333}{500} & \frac{273}{500} & \frac{471}{500} & \frac{681}{500} \\
\frac{23}{50} & \frac{23}{40} & \frac{77}{500} & \frac{169}{500} & \frac{471}{500} & \frac{681}{500} & \frac{1039}{1000} \\
\end{array} \right), \quad (16b)
\]

\[
\hat{\eta}_{\text{num}}^2 = \left( \begin{array}{cccccccc}
-\frac{701}{1000} & \frac{543}{1000} & \frac{463}{500} & -\frac{419}{1000} & -\frac{453}{1000} & \frac{307}{1000} & \frac{339}{1000} \\
\frac{543}{1000} & -\frac{1}{2} & -\frac{559}{1000} & -\frac{299}{1000} & \frac{249}{1000} & \frac{301}{1000} \\
\frac{419}{1000} & \frac{453}{1000} & \frac{559}{1000} & \frac{299}{1000} & \frac{191}{1000} & \frac{171}{1000} & \frac{439}{1000} \\
\frac{307}{1000} & \frac{339}{1000} & \frac{249}{1000} & \frac{301}{1000} & \frac{171}{1000} & \frac{437}{1000} & \frac{1039}{1000} \\
\end{array} \right). \quad (16c)
\]
Then, a numerical solution of the simplified algebraic equation (11), for the parameters (11), is given by

\[
\hat{\alpha}^1_{\text{num-sol}} = \begin{pmatrix}
-0.674 & -0.024 - 0.432 i & 0.763 + 0.788 i & 0.071 + 0.578 i \\
-0.024 + 0.432 i & 0.382 & 0.791 + 0.512 i & 0.67 - 1.57 i \\
0.763 - 0.788 i & 0.791 - 0.512 i & 0.565 & 0.566 + 0.698 i \\
0.071 - 0.578 i & 0.67 + 1.57 i & 0.566 - 0.698 i & -0.274
\end{pmatrix}, \quad (17a)
\]

\[
\hat{\alpha}^2_{\text{num-sol}} = \begin{pmatrix}
0.690 & 0.128 - 0.292 i & 0.217 + 0.356 i & 0.536 - 0.437 i \\
0.128 + 0.292 i & -0.652 & 0.593 + 0.518 i & -0.314 + 0.776 i \\
0.217 - 0.356 i & 0.593 - 0.518 i & 0.631 & -0.041 + 0.201 i \\
0.536 + 0.437 i & -0.314 - 0.776 i & -0.041 - 0.201 i & -0.669
\end{pmatrix}, \quad (17b)
\]

where only two or three significant digits are shown (typically, we have a 24-digit working precision). Incidentally, the solution (17) is not unique, as there is, at least, one other solution.

Considering the absolute values of the entries in the matrices (17), we have

\[
\text{Abs} [\hat{\alpha}^1_{\text{num-sol}}] = \begin{pmatrix}
0.674 & 0.433 & 1.10 & 0.582 \\
0.433 & 0.382 & 0.942 & 1.71 \\
1.10 & 0.942 & 0.565 & 0.899 \\
0.582 & 1.71 & 0.899 & 0.274
\end{pmatrix}, \quad (18a)
\]

\[
\text{Abs} [\hat{\alpha}^2_{\text{num-sol}}] = \begin{pmatrix}
0.690 & 0.319 & 0.417 & 0.691 \\
0.319 & 0.652 & 0.788 & 0.837 \\
0.417 & 0.788 & 0.631 & 0.206 \\
0.691 & 0.837 & 0.206 & 0.669
\end{pmatrix}, \quad (18b)
\]

and we see no obvious band-diagonal structure. (Recall that the diagonal is singled out by the Hermiticity condition on the matrix \(\hat{\alpha}^1\), making the entries on its diagonal real, and similarly for the matrix \(\hat{\alpha}^2\).) The presence or absence of a band-diagonal structure can be quantified by the following averages and ratios:

\[
\left\{ \langle \text{all}, \langle \text{band-diag}, \langle \text{off-band-diag}, \right. \rangle \text{ ratio}\right\}^{(N=4)}_{\text{Abs}[\hat{\alpha}^1_{\text{num-sol}}]} = \{0.827, 0.644, 1.13, 0.570\} , \quad (19a)
\]

\[
\left\{ \langle \text{all}, \langle \text{band-diag}, \langle \text{off-band-diag}, \right. \rangle \text{ ratio}\right\}^{(N=4)}_{\text{Abs}[\hat{\alpha}^2_{\text{num-sol}}]} = \{0.572, 0.527, 0.649, 0.812\}, \quad (19b)
\]

which give, respectively, the average absolute value of all (16) matrix entries, the average absolute value of the band-diagonal (3+4+3) matrix entries, the average absolute value of the off-band-diagonal (3+3) matrix entries, and the ratio of the
A first look at the bosonic master-field equation of the IIB matrix model

band-diagonal value over the off-band-diagonal value. The ratios shown as the last entries of (19) are of order unity.

Next, we diagonalize one of the matrices, while ordering the eigenvalues, and look at the other matrix to see if it has a band-diagonal structure (even for the very small value of \( N \) we are considering). Recall that the diagonalization is achieved by use of a similarity transformation,

\[
\hat{a}^{\rho \text{ (new)}} = S \cdot \hat{a}^{\rho} \cdot S^{-1},
\]

where an appropriate choice of the matrix \( S \) can make one of the matrices diagonal (see, e.g., Sec. 11.0 of Ref. [19] for a clear discussion).

If we diagonalize and order \( \hat{a}^{1}_{\text{num-sol}} \) (the new matrices are denoted by a prime), we get

\[
\hat{a}^{1}_{\text{num-sol}} = S_{1} \cdot \hat{a}^{1}_{\text{num-sol}} \cdot S_{1}^{-1} = \text{diag}(-2.40, -1.19, 1.60, 1.99),
\]

\[
\hat{a}^{2}_{\text{num-sol}} = S_{1} \cdot \hat{a}^{2}_{\text{num-sol}} \cdot S_{1}^{-1} = \begin{pmatrix}
-0.507 & 0.249 + 0.415 i & 0.0355 - 0.0451 i & -0.0316 + 0.0235 i \\
0.249 - 0.415 i & 0.484 & 0.181 + 0.031 i & -0.0756 - 0.0105 i \\
0.0355 + 0.0451 i & 0.181 - 0.031 i & 0.432 & 1.330 + 0.258 i \\
-0.0316 - 0.0235 i & -0.0756 + 0.0105 i & 1.330 - 0.258 i & -0.772
\end{pmatrix},
\]

where \( S_{1} \) is a short-hand notation for \( S_{1,\text{num}} \) and where, again, only two or three significant digits are shown.

For the record, we give the absolute values of the entries of the matrix (21b),

\[
\text{Abs} \left[ \hat{a}^{2}_{\text{num-sol}} \right] = \begin{pmatrix}
0.507 & 0.484 & 0.0574 & 0.0394 \\
0.484 & 0.847 & 0.183 & 0.0763 \\
0.0574 & 0.183 & 0.432 & 1.35 \\
0.0394 & 0.0763 & 1.35 & 0.772
\end{pmatrix},
\]

where we see that the far-off-diagonal elements (e.g., at positions 13, 14, and 24) are rather small in comparison to those close to the diagonal, which is not the case for the original matrix (18b). Again, this can be quantified by the following averages and ratio:

\[
\left\{ \langle \text{all} \rangle, \langle \text{band-diag} \rangle, \langle \text{off-band-diag} \rangle, \text{ratio} \right\}_{(N=4)}^{(N=4)} = \text{Abs}[\hat{a}^{2}_{\text{num-sol}}] = \{0.434, 0.660, 0.0577, 11.4\},
\]

where the meaning of the quantities has been explained on the lines below (19). The ratio of the band-diagonal value over the off-band-diagonal value from (23) is of order 10.

Similar results are obtained if the other matrix \( \hat{a}^{2}_{\text{num-sol}} \) is diagonalized and ordered. The new matrices are denoted by a double prime and are obtained by a
similarity transformation (20) with an appropriate matrix $S_2$. The corresponding averages and ratio are

$$\{\langle \text{all} \rangle, \langle \text{band-diag} \rangle, \langle \text{off-band-diag} \rangle, \text{ratio}\}^{(N=4)}_{\text{Abs}[\hat{a}''_{\text{num-sol}}]} = \{0.655, 0.977, 0.120, 8.17\},$$

where the ratio of the band-diagonal value over the off-band-diagonal value is of order 10.

For both choices of the bases $S = S_1$ or $S = S_2$ in the similarity transformation (20), we see a more or less band-diagonal structure if one of the matrices is diagonalized. With different realizations of the pseudorandom constants, there is significant scatter in the values of the ratios mentioned in (23) and (24), but all values are apparently above unity. As mentioned before, a proper analysis of statistical issues must wait for solutions at larger values of $N$ and $D$.

These results were obtained from the simplified algebraic equation (10a) with “Euclidean” coupling constants $\tilde{g}_{\mu\nu}$ from (10b) having $\tilde{s} = 1$. Similar results are obtained with “Lorentzian” coupling constants $\tilde{g}_{\mu\nu}$ having $\tilde{s} = -1$.

### 4.2. $D = 2$ and $N = 6$

The numerical solution of Sec. 4.1 has suggested that a band-diagonal structure appears if one of the matrices $\hat{a}^p$ is diagonalized. But the matrix size considered ($N = 4$) was rather small, with the number of off-band-diagonal elements ($N_{\text{off-band-diag}} = 6$) being smaller than the number of band-diagonal elements ($N_{\text{band-diag}} = 10$). In principle, we would like to have $N_{\text{off-band-diag}} = O(N^2) \gg N_{\text{band-diag}} = O(N)$, for very large $N$ and fixed width $\Delta N$ of the band diagonal. We take a modest step in the right direction by using $N = 6$ and $N_{\text{off-band-diag}} = 20 > N_{\text{band-diag}} = 16$.

We present a representative numerical solution for the case

$$\{D, N\} = \{2, 6\}$$

and a particular choice of pseudorandom constants $\{\hat{p}_{\text{num}}, \hat{\eta}^1_{\text{num}}, \hat{\eta}^2_{\text{num}}\}$, which are shown explicitly in Appendix A. The simplified algebraic equation (10) with parameters (25a) then gives a numerical solution $\hat{a}^1_{\text{num-sol}}$ and $\hat{a}^2_{\text{num-sol}}$, which is also shown explicitly in Appendix A.

Here, we only display the absolute values of the entries of these last matrices,

$$\text{Abs } [\hat{a}^1_{\text{num-sol}}] = \begin{pmatrix}
0.234 & 0.416 & 0.636 & 1.12 & 0.831 & 0.232 \\
0.416 & 0.721 & 0.949 & 0.566 & 0.611 & 0.366 \\
0.636 & 0.949 & 0.445 & 0.754 & 1.20 & 0.623 \\
1.12 & 0.566 & 0.754 & 0.0504 & 0.114 & 0.620 \\
0.831 & 0.611 & 1.20 & 0.114 & 0.256 & 0.631 \\
0.232 & 0.366 & 0.623 & 0.620 & 0.631 & 0.248
\end{pmatrix},$$

(26a)
A first look at the bosonic master-field equation of the IIB matrix model

\[
\text{Abs} \left[ \hat{\alpha}_{\text{num-sol}}^{2} \right] = \begin{pmatrix}
0.560 & 0.926 & 0.560 & 0.400 & 0.565 & 0.620 \\
0.926 & 0.866 & 0.701 & 0.813 & 0.624 & 1.11 \\
0.560 & 0.701 & 0.623 & 1.04 & 0.0857 & 0.624 \\
0.400 & 0.813 & 1.04 & 0.459 & 0.469 & 0.911 \\
0.565 & 0.624 & 0.0857 & 0.469 & 0.400 & 0.485 \\
0.620 & 1.11 & 0.624 & 0.911 & 0.485 & 1.19
\end{pmatrix}, \quad (26b)
\]

where we do not observe any obvious band-diagonal structure. In fact, from the values given in (26), we calculate the following averages and ratios:

\[
\{ \langle \text{all}, \langle \text{band-diag}, \langle \text{off-band-diag}, \text{ratio}\rangle \rangle \}^{(N=6)}_{\text{Abs}[\hat{\alpha}_{\text{num-sol}}^{1}]} = \{0.592, 0.480, 0.681, 0.705\},
\]

\[
\{ \langle \text{all}, \langle \text{band-diag}, \langle \text{off-band-diag}, \text{ratio}\rangle \rangle \rangle \}^{(N=6)}_{\text{Abs}[\hat{\alpha}_{\text{num-sol}}^{2}]} = \{0.666, 0.709, 0.631, 1.12\},
\]

which give, respectively, the average absolute value of all (36) matrix entries, the average absolute value of the band-diagonal (5+6+5) matrix entries, the average absolute value of the off-band-diagonal (10+10) matrix entries, and the ratio of the band-diagonal value over the off-band-diagonal value. The ratios shown as the last entries of (27) are of order unity.

Diagonalizing and ordering the matrix \(\hat{\alpha}_{\text{num-sol}}^{1}\), we get new matrices (denoted by a prime), which are given explicitly in Appendix A. For the record, we give here the absolute values of the matrix entries of \(\hat{\alpha}_{\text{num-sol}}^{1}\) from (A.3a) and the absolute values of the matrix entries of \(\hat{\alpha}_{\text{num-sol}}^{2}\) from (A.3b) and (A.3c).

\[
\text{Abs} \left[ \hat{\alpha}_{\text{num-sol}}^{1} \right] = \text{diag} \left( 2.03, 1.69, 0.653, 0.569, 1.21, 2.59 \right), \quad (28a)
\]

\[
\text{Abs} \left[ \hat{\alpha}_{\text{num-sol}}^{2} \right] = \begin{pmatrix}
0.0175 & 1.08 & 0.209 & 0.0608 & 0.0651 & 0.0610 \\
1.08 & 2.38 & 0.140 & 0.140 & 0.0458 & 0.0541 \\
0.209 & 0.140 & 1.46 & 0.804 & 0.115 & 0.0341 \\
0.0608 & 0.140 & 0.804 & 0.459 & 1.25 & 0.201 \\
0.0651 & 0.0458 & 0.115 & 1.25 & 1.45 & 0.371 \\
0.0610 & 0.0541 & 0.0341 & 0.201 & 0.371 & 0.965
\end{pmatrix}, \quad (28b)
\]

where we see in the last matrix that the far-off-diagonal elements (e.g., at positions 15, 16, and 26) are rather small in comparison to those close to the diagonal, which is not the case for the original matrix (26b). This can be quantified by the following averages and ratio:

\[
\{ \langle \text{all}, \langle \text{band-diag}, \langle \text{off-band-diag}, \text{ratio}\rangle \rangle \}^{(N=6)}_{\text{Abs}[\hat{\alpha}_{\text{num-sol}}^{2}]} = \{0.444, 0.877, 0.0986, 8.89\},
\]

(29)
where the meaning of the quantities has been explained on the lines below (27). The ratio of the band-diagonal value over the off-band-diagonal value from (29) is of order 10, just as what was found in Sec. 4.1 for \( N = 4 \).

Similar results are obtained if the other matrix \( \hat{a}_{num-sol}^2 \) is diagonalized and ordered (the new matrices are denoted by a double prime and are given explicitly in Appendix A), and the corresponding numbers are

\[
\left\{ \langle \text{all} \rangle, \langle \text{band-diag} \rangle, \langle \text{off-band-diag} \rangle, \text{ratio} \right\}_{\text{Abs}[\hat{a}_{num-sol}^{{''}1}]}^{(N=6)} = \\
\{0.411, 0.774, 0.121, 6.40\}.
\]  

(30)

For both choices of the bases, we see a more or less band-diagonal structure appearing if one of the matrices is diagonalized.

In closing, we comment briefly on the results \( \hat{a}_{1,2}^1 \) for \( D = 2 \) and \( N = 6 \). The entries of these matrices (70 real numbers in total) were obtained by the numerical minimization routine \texttt{FindMinimum} of MATHEMATICA 12.1 (cf. Ref. 20). This minimization operates on an auxiliary function, which consists of a sum of 70 squares, each square containing the real or imaginary part of one of the components of the simplified algebraic matrix equation (10a). (The auxiliary function has a size of about 7 MB, as simplifications are difficult to obtain.) The accuracy of the obtained 70 numbers can, in principle, be increased arbitrarily. Hence, given the exact (pseudorandom) constants (A.1), the obtained matrices (A.2) may be called “quasi-exact.”

### 4.3. Corresponding bosonic master-field matrices

Up till now, we have focused on the bosonic matrices \( \hat{a}^\rho \). The corresponding master-field matrices \( \hat{A}^\rho \) follow from (5a). That expression can be rewritten as follows:

\[
\hat{A}^\rho = D \cdot \hat{a}^\rho \cdot D^{-1},
\]  

(31a)

\[
D_{kl} \equiv \left[ \text{diag} \left( e^{i \hat{p}_1 \tau_{eq}}, \ldots, e^{i \hat{p}_N \tau_{eq}} \right) \right]_{kl},
\]  

(31b)

where we suppress the dependence of \( \tau_{eq} \) in \( D \) and \( \hat{A}^\rho \).

Explicit matrices \( \hat{a}_{num-sol}^{1,2} \) were obtained in Secs. 4.1 and 4.2 from the simplified algebraic equation (10) for \( D = 2 \) and with a particular realization of the pseudorandom constants \( \hat{p} \) and \( \hat{\eta}^\rho \). The corresponding master-field matrices are

\[
\hat{A}_{num-sol}^\rho = D_{num} \cdot \hat{a}_{num-sol}^\rho \cdot D_{num}^{-1},
\]  

(32a)

\[
\left[ D_{num} \right]_{kl} = \left[ \text{diag} \left( e^{i \hat{p}_1 \tau_{eq, num}}, \ldots, e^{i \hat{p}_N \tau_{eq, num}} \right) \right]_{kl},
\]  

(32b)

where \( \tau_{eq, num} \) is an appropriate numerical value for \( \tau_{eq} \). [We conjecture that the value of \( \tau_{eq, num} \) must be so large that the diagonal entries in (32b), for given values of \( \hat{p}_k, \tau_{eq, num} \), cover the unit circle in the complex plane more or less uniformly in the limit \( N \to \infty \).]
With the similarity transformation (20) on $\hat{a}^\rho_{\text{num-sol}}$ to diagonalize the $\rho = 1$ matrix (which requires $S = S_1$), we get from (32)

$$\hat{A}^\rho_{\text{num-sol}} = T_1 \cdot \hat{a}^\rho_{\text{num-sol}} \cdot T_1^{-1},$$

(33a)

and similarly for the other diagonalization $\hat{a}''^\rho_{\text{num-sol}}$ from $S = S_2$.

From the expression (33a), we conclude that the diagonal/band-diagonal structure discovered for $\hat{a}^\rho_{\text{num-sol}}$ in (21) and (A.3) directly carries over to the master-field matrices $\hat{A}^\rho_{\text{num-sol}}$. But we prefer to focus the discussion of this paper on the $\tau$-independent matrices $\hat{a}^\rho$, as a proper value for $\tau_{\text{eq, num}}$ is not required for that discussion.

5. Conclusion

The large-$N$ bosonic master field (8a) for bosonic observables in the IIB matrix model is essentially determined as a solution of the algebraic equation (8b). To solve that equation is a formidable task and we have, instead, considered the simplified algebraic equation (10).

For low dimensionality ($D = 2$) and small matrix sizes ($N = 4$ and 6), we have established the existence of one or more nontrivial solutions of that simplified algebraic equation for an explicit realization of the pseudorandom constants; see Secs. 4.1 and 4.2. The obtained matrices do not show an obvious band-diagonal structure. But if one of these two matrices is diagonalized (with ordered eigenvalues), then the other matrix does get a band-diagonal structure. For $N=6$, this has been quantified by the averages and ratios given in (29) and (30).

There are, however, many open questions and let us mention two. First, how does the appropriately defined width $\Delta N$ of the band diagonal in $\hat{a}''^2$ (or in $\hat{a}''^1$) depend on $N$ and how do the off-band-diagonal entries scale with $N$? Second, is there indeed no essential difference for the appearance of a diagonal/band-diagonal structure between the Euclidean ($\tilde{s} = 1$) and the Lorentzian ($\tilde{s} = -1$) models? For the first question, we need to consider larger and larger values of $N$. For the second question, we need to go to larger dimensionality, for example, $D = 4$ or 10. Insight into both extensions (larger $N$ and larger $D$) may perhaps come from approximative results; cf. the earlier work reported in Ref. 15.

Also missing is a convincing explanation for why the matrix solution of the simplified algebraic equation (10) has a hidden diagonal/band-diagonal structure (some preliminary considerations are presented in Appendix B). Still, the diagonal/band-diagonal structure found in the “quasi-exact” solutions from Secs. 4.1 and 4.2 admitted only for low dimensionality and relatively small values of the matrix size, provides clear evidence for one of the assumption in our previous discussion of spacetime extraction from the bosonic IIB-matrix-model master field.
Acknowledgments

It is a pleasure to thank J. Nishimura for informative discussions over the last years and the referee of the present paper for constructive remarks.

Note Added in Proof

We have been informed by T. Fischbacher\cite{Fischbacher21} that he and his colleagues at Google Research, Zürich, have obtained numerical solutions of the algebraic equation (10) for $(D, N) = (10, 50)$ and that these solutions display a diagonal/band-diagonal structure.

Appendix A. Matrices for $D = 2$ and $N = 6$

The matrices from Sec. 4.2 are rather big and it is better to show their real and imaginary parts separately.

Taking the following pseudorandom constants:

\[
\hat{p}_{\text{num}} = \left\{ \frac{53}{500}, \frac{9}{100}, \frac{441}{1000}, \frac{217}{1000}, \frac{371}{1000}, \frac{19}{40} \right\}, \quad (A.1a)
\]

\[
\text{Re} \left[ \hat{\eta}_{\text{num}}^1 \right] = \begin{pmatrix}
81 & 71 & 151 & 371 & 83 & 491 \\
125 & 1000 & 500 & 500 & 200 & 1000 \\
71 & 279 & 259 & 13 & 493 & 449 \\
1000 & 1000 & 500 & 1000 & 500 & 1000 \\
151 & 259 & 413 & 911 & 203 & 299 \\
0 & 500 & 1000 & 1000 & 250 & 1000 \\
371 & 13 & 911 & 671 & 417 & 913 \\
500 & 1000 & 1000 & 1000 & 500 & 1000 \\
83 & 493 & 203 & 417 & 51 & 181 \\
200 & 500 & 250 & 500 & 125 & 250 \\
491 & 449 & 299 & 913 & 181 & 261 \\
1000 & 1000 & 1000 & 1000 & 250 & 1000
\end{pmatrix}, \quad (A.1b)
\]

\[
\text{Im} \left[ \hat{\eta}_{\text{num}}^1 \right] = \begin{pmatrix}
0 & 441 & 17 & 87 & -127 & 199 \\
1000 & 0 & 177 & 783 & -303 & 969 \\
17 & 177 & 0 & 259 & -14 & 711 \\
250 & 250 & 0 & 25 & 1000 & -1000 \\
87 & 783 & 14 & 0 & 43 & 1 \\
1000 & 1000 & 25 & 0 & 250 & 125 \\
127 & 303 & 259 & 43 & 0 & 491 \\
200 & 500 & 1000 & 250 & 0 & 1000 \\
199 & 969 & 711 & 1 & 491 & 0 \\
500 & 1000 & 1000 & 125 & 1000 & 0
\end{pmatrix}, \quad (A.1c)
\]
we obtain from the simplified algebraic equation (10), for following numerical solution:

\[
\text{Re} \left[ \tilde{\eta}^2_{\text{num}} \right] = \begin{bmatrix}
41 & 53 & 241 & 621 & 3 & -51 \\
200 & 1000 & 250 & 1000 & 20 & 200 \\
53 & 139 & 23 & -557 & 7 & 137 \\
1000 & -500 & 200 & 1000 & 100 & 200 \\
241 & 23 & 31 & -22 & 14 & 31 \\
250 & 200 & 500 & 125 & 25 & 100 \\
621 & 557 & 22 & 289 & 227 & 103 \\
1000 & 1000 & 125 & 1000 & 1000 & 200 \\
3 & 7 & 14 & -227 & 17 & 369 \\
20 & 100 & 25 & 1000 & 1000 & 1000 \\
51 & 137 & 31 & 103 & 369 & 171 \\
-200 & -200 & 100 & -200 & 1000 & 1000 \\
\end{bmatrix}, \quad (A.1d)
\]

\[
\text{Im} \left[ \tilde{\eta}^2_{\text{num}} \right] = \begin{bmatrix}
0 & 449 & -31 & 233 & -413 & -807 \\
500 & 250 & 1000 & 500 & 1000 \\
449 & 500 & -56 & 7 & 77 & 23 \\
500 & 0 & 125 & 50 & 200 & 500 \\
31 & 56 & 0 & 409 & 57 & 689 \\
250 & 125 & 500 & 250 & 1000 \\
233 & 7 & -409 & 0 & 189 & 953 \\
1000 & 50 & 500 & 500 & 1000 \\
413 & 77 & -57 & 189 & 0 & 47 \\
500 & 200 & 250 & 500 & 200 \\
807 & 23 & 689 & 953 & 47 & 0 \\
1000 & 500 & 1000 & 1000 & 200 \\
\end{bmatrix}, \quad (A.1e)
\]

we obtain from the simplified algebraic equation (10), for \( D = 2 \) and \( N = 6 \), the following numerical solution:

\[
\text{Re} \left[ \tilde{a}^1_{\text{num-sol}} \right] = \begin{bmatrix}
0.234 & 0.346 & 0.578 & 0.328 & 0.336 & -0.152 \\
0.346 & -0.721 & -0.698 & 0.0277 & 0.483 & -0.344 \\
0.578 & -0.698 & 0.445 & -0.208 & -0.466 & 0.613 \\
0.328 & 0.0277 & -0.208 & 0.0504 & 0.0989 & 0.543 \\
0.336 & 0.483 & -0.466 & 0.0989 & -0.256 & -0.319 \\
-0.152 & -0.344 & 0.613 & 0.543 & -0.319 & 0.248 \\
\end{bmatrix}, \quad (A.2a)
\]

\[
\text{Im} \left[ \tilde{a}^1_{\text{num-sol}} \right] = \begin{bmatrix}
0 & 0.231 & 0.267 & 1.08 & 0.760 & -0.176 \\
-0.231 & 0 & 0.643 & -0.566 & -0.374 & -0.125 \\
-0.267 & -0.643 & 0 & 0.724 & 1.11 & 0.108 \\
-1.08 & 0.566 & -0.724 & 0 & -0.0565 & -0.301 \\
-0.760 & 0.374 & -1.11 & 0.0565 & 0 & 0.545 \\
0.176 & 0.125 & -0.108 & 0.301 & -0.545 & 0 \\
\end{bmatrix}, \quad (A.2b)
\]
\[
\text{Re} \left[ \tilde{\alpha}^2_{\text{num-sol}} \right] = \begin{pmatrix}
-0.560 & 0.820 & 0.493 & -0.348 & 0.377 & -0.126 \\
0.820 & -0.866 & -0.430 & -0.714 & 0.489 & 1.10 \\
0.493 & -0.430 & -0.623 & -0.0737 & 0.0523 & 0.451 \\
-0.348 & -0.714 & -0.0737 & 0.459 & -0.279 & 0.763 \\
0.377 & 0.489 & 0.0523 & -0.279 & 0.400 & -0.195 \\
-0.126 & 1.10 & 0.451 & 0.195 & -0.195 & 1.19
\end{pmatrix}, \quad (A.2c)
\]

\[
\text{Im} \left[ \tilde{\alpha}^2_{\text{num-sol}} \right] = \begin{pmatrix}
0 & 0.429 & 0.266 & 0.198 & -0.421 & 0.607 \\
-0.429 & 0 & 0.554 & -0.388 & 0.388 & -0.139 \\
-0.266 & -0.554 & 0 & -1.04 & -0.0679 & -0.431 \\
-0.198 & 0.388 & 1.04 & 0 & -0.378 & 0.498 \\
0.421 & -0.388 & 0.0679 & 0.378 & 0 & 0.444 \\
-0.607 & 0.139 & 0.431 & -0.498 & -0.444 & 0
\end{pmatrix}, \quad (A.2d)
\]

where only three significant digits are shown.

Diagonalizing and ordering the matrix \( \tilde{\alpha}_{\text{num-sol}}^1 \) gives (the new matrices are denoted by a prime)

\[
\tilde{\alpha}_{\text{num-sol}}^1 = \text{diag}( -2.03, -1.69, -0.653, 0.569, 1.21, 2.59 ), \quad (A.3a)
\]

\[
\text{Re} \left[ \tilde{\alpha}_{\text{num-sol}}^{1/2} \right] = \begin{pmatrix}
-0.0175 & 0.491 & -0.195 & 0.0240 & -0.0384 & -0.0596 \\
0.491 & -2.38 & -0.140 & 0.108 & -0.0356 & -0.0540 \\
-0.195 & -0.140 & 1.46 & 0.642 & 0.0403 & 0.00702 \\
0.0240 & 0.108 & 0.642 & 0.459 & 1.10 & -0.0952 \\
-0.0384 & -0.0356 & 0.0403 & 1.10 & 1.45 & -0.229 \\
-0.0596 & -0.0540 & 0.00702 & -0.0952 & -0.229 & -0.965
\end{pmatrix}, \quad (A.3b)
\]

\[
\text{Im} \left[ \tilde{\alpha}_{\text{num-sol}}^{1/2} \right] = \begin{pmatrix}
0 & -0.962 & -0.0748 & -0.0559 & 0.0525 & 0.0129 \\
0.962 & 0 & -0.00653 & 0.0886 & -0.0287 & 0.00331 \\
0.0748 & 0.00653 & 0 & -0.485 & -0.108 & -0.0334 \\
0.0559 & -0.0886 & 0.485 & 0 & 0.604 & 0.177 \\
-0.0525 & 0.0287 & 0.108 & -0.604 & 0 & -0.293 \\
-0.0129 & -0.00331 & 0.0334 & -0.177 & 0.293 & 0
\end{pmatrix}, \quad (A.3c)
\]

where, again, only three significant digits are shown. Similarly, diagonalizing and ordering the matrix \( \tilde{\alpha}_{\text{num-sol}}^2 \) gives (the new matrices are denoted by a double prime)

\[
\text{Re} \left[ \tilde{\alpha}_{\text{num-sol}}^{1/1} \right] = \begin{pmatrix}
-1.73 & -0.00903 & 0.115 & 0.0429 & 0.0344 & -0.0108 \\
-0.00903 & 2.08 & 0.736 & -0.151 & 0.0896 & -0.00741 \\
0.115 & 0.736 & 0.967 & 0.00290 & 0.265 & -0.0423 \\
0.0429 & -0.151 & 0.00290 & -1.81 & 0.256 & -0.0382 \\
0.0344 & 0.0896 & 0.265 & 0.256 & -0.111 & -0.385 \\
-0.0108 & -0.00741 & -0.0423 & -0.0382 & -0.385 & 0.608
\end{pmatrix}, \quad (A.4a)
\]
A first look at the bosonic master-field equation of the IIB matrix model

\[ \text{Im} \left[ \tilde{a}_{\text{num-sol}}^{\prime \prime} \right] = \begin{pmatrix}
0 & 0.0384 & -0.0693 & 0.0689 & -0.0424 & -0.0215 \\
0.0384 & 0 & -0.463 & 0.135 & 0.108 & 0.0611 \\
0.0693 & 0.463 & 0 & -0.579 & -0.337 & 0.0693 \\
-0.0689 & -0.135 & 0.579 & 0 & 0.211 & -0.0408 \\
0.0424 & -0.108 & 0.337 & -0.211 & 0 & 0.606 \\
0.0215 & -0.00611 & -0.0669 & 0.0408 & -0.606 & 0 \\
\end{pmatrix}, \quad (A.4b) \]

\[ \tilde{a}_{\text{num-sol}}^{\prime \prime 2} = \text{diag}(-2.81, -1.19, -0.429, 0.413, 1.44, 2.58), \quad (A.4c) \]

with three significant digits shown. Further discussion of these results appears in Sec. 4.2.

Appendix B. Possible Large-\(N\) Behavior

We have found, in Sec. 4, a clear hint of a diagonal/band-diagonal structure in the solutions of the simplified algebraic equation (10), albeit at low dimensionality (\(D = 2\)) and small matrix size (\(N = 4\) or 6). Even while keeping \(D = 2\), we do not really know if this structure survives in the large-\(N\) limit. In this appendix, we present some preliminary ideas.

The simplified algebraic equation (10), for fixed values of \(\tilde{p}_k\) and \(\tilde{\eta}_{\rho}^{\nu} \tilde{a}_{kl}^{\rho}\), is surprisingly difficult to solve and understand. Some minor progress can be made if we consider a judicial approximation.

Take the absolute values of the master-noise matrix entries \(\tilde{\eta}_{\rho}^{\nu} \tilde{a}_{kl}^{\rho}\) to be of order unity and consider the absolute values of the master momenta \(\tilde{p}_k\) to be very much smaller than unity [in principle, these small values can be compensated by considering very much larger values of \(\tau_{eq}\) in (31)]. Then, it makes sense to look at the following approximate algebraic equation:

\[ 0 = \left[ \tilde{a}_\nu, \left[ \tilde{a}_\nu, \tilde{a}_\rho \right] \right] + \tilde{\eta}_\rho, \quad (B.1) \]

where the matrix indices have been suppressed altogether and where an index \(\nu\) has been lowered with the Euclidean metric \(\tilde{g}_{\mu\nu}\) from (10b) and (10c).

If we make a similarity transformation (20) with a matrix \(S = \tilde{S}_1\), then (B.1) keeps the same form if the master-noise matrix is transformed accordingly,

\[ 0 = \left[ \tilde{a}_\nu', \left[ \tilde{a}_\nu', \tilde{a}_\rho' \right] \right] + \tilde{\eta}'_\rho, \quad (B.2a) \]

\[ \tilde{\eta}'_\rho = \tilde{S}_1 \cdot \tilde{\eta}_\rho \cdot \tilde{S}_1^{-1}, \quad (B.2b) \]

where \(\tilde{a}_\nu'\) is now assumed to be diagonal, hence the notation with the single prime.

With given master-noise matrices \(\tilde{\eta}_\rho\), the goal is to solve equation (B.1) for the unknown matrices \(\tilde{a}_\rho\). Let us also assume that we have

\[ D = 2, \quad (B.3) \]

so that there are only two unknown matrices in the problem.
This problem is still difficult. So, let us, instead, consider the inverse problem: make certain Ansätze $\hat{a}_{\text{Ansatz}}^1$ and $\hat{a}_{\text{Ansatz}}^2$, calculate the corresponding matrices $\hat{\eta}_{\text{result}}^{\rho \nu}$ from (B.1), and then ask if these calculated matrices are more or less noise-like (all entries pseudorandom and with an absolute value of order unity).

In fact, we will consider the approximate algebraic equation (B.2) with diagonal $\hat{a}^1$. Making appropriate Ansätze $\hat{a}_{\text{Ansatz}}^1$ and $\hat{a}_{\text{Ansatz}}^2$, we calculate the resulting matrices

$$\hat{\eta}_{\text{result}}^{\rho \nu} = -\left[\hat{a}_{\text{Ansatz}}^{\nu}, \left[\hat{a}_{\text{Ansatz}}^{\nu}, \hat{a}_{\text{Ansatz}}^{\rho}\right]^{\dagger}\right].$$

For $\hat{a}_{\text{Ansatz}}^1$, we take a diagonal matrix with ordered eigenvalues and, for $\hat{a}_{\text{Ansatz}}^2$, we take the sum of a real band-diagonal matrix (with width $\Delta N = 3$, in order to be specific) and a purely random matrix in the bulk. Both of these Ansatz matrices are traceless and Hermitian. We will try to scale the entries of the matrices $\hat{a}_{\text{Ansatz}}^1$ by appropriate powers of $N$, so that the entries of the resulting matrices (B.4) are more or less random and have average absolute values which are approximately constant as $N$ becomes very large.

Specifically, we use the following construction for the matrices $\hat{a}_{\text{Ansatz}}^1$. Assume, for simplicity, $N$ to be even. With the matrix indices $k, l$ running over $\{1, \ldots N\}$ and the directional index $\rho$ running over $\{1, \ldots, D\}$ for $D = 2$, we then define

$$\hat{a}_{\text{tmp}}^1_{kl} = \Xi^2 N^{-2/3} \left[\text{diag}(-N/2, \ldots, -2, -1, 1, 2, \ldots, N/2)\right]_{kl},$$

$$\hat{a}_{\text{tmp}}^2_{kl} = \left[\hat{a}_{\text{tmp-rand}}^2\right]_{kl} + \left[\hat{a}_{\text{tmp-band}}^2\right]_{kl} = \Xi^{-1} N^{-1/3} \left(\text{randominteger}[1, 1] + i \text{randominteger}[1, 1]\right),$$

$$\hat{a}_{\text{tmp-band}}^2_{kl} = \begin{cases} \Xi^{-4} N^{-2/3} \text{randominteger}[-N/2, +N/2], & \text{for } k > l + 1 \wedge l < k + 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$N = 2 K, \text{ for } K \in \mathbb{N}^+, \quad (B.5a)$$

$$\Xi \equiv 126/100, \quad (B.5b)$$

and make all matrices traceless ($\hat{a}_{\text{tmp}}^1$ is already traceless),

$$\hat{a}_{\text{tmp-traceless}}^{\rho} = \hat{a}_{\text{tmp}}^{\rho} - \frac{1}{N} \text{Tr} \left(\hat{a}_{\text{tmp}}^{\rho}\right) \mathbb{1}_N, \quad (B.5c)$$

and Hermitian,

$$\hat{a}_{\text{Ansatz}}^{\rho} = \frac{1}{2} \left[\hat{a}_{\text{tmp-traceless}}^{\rho} + \left(\hat{a}_{\text{tmp-traceless}}^{\rho}\right)^{\dagger}\right]. \quad (B.5d)$$

For $\hat{a}_{\text{Ansatz}}^2$, we take the sum of a real band-diagonal matrix (with width $\Delta N = 3$, in order to be specific) and a purely random matrix in the bulk. Both of these Ansatz matrices are traceless and Hermitian. We will try to scale the entries of the matrices $\hat{a}_{\text{Ansatz}}^2$ by appropriate powers of $N$, so that the entries of the resulting matrices (B.4) are more or less random and have average absolute values which are approximately constant as $N$ becomes very large.

Specifically, we use the following construction for the matrices $\hat{a}_{\text{Ansatz}}^2$. Assume, for simplicity, $N$ to be even. With the matrix indices $k, l$ running over $\{1, \ldots N\}$ and the directional index $\rho$ running over $\{1, \ldots, D\}$ for $D = 2$, we then define

$$\hat{a}_{\text{tmp}}^1_{kl} = \Xi^2 N^{-2/3} \left[\text{diag}(-N/2, \ldots, -2, -1, 1, 2, \ldots, N/2)\right]_{kl},$$

$$\hat{a}_{\text{tmp}}^2_{kl} = \left[\hat{a}_{\text{tmp-rand}}^2\right]_{kl} + \left[\hat{a}_{\text{tmp-band}}^2\right]_{kl} = \Xi^{-1} N^{-1/3} \left(\text{randominteger}[1, 1] + i \text{randominteger}[1, 1]\right),$$

$$\hat{a}_{\text{tmp-band}}^2_{kl} = \begin{cases} \Xi^{-4} N^{-2/3} \text{randominteger}[-N/2, +N/2], & \text{for } k > l + 1 \wedge l < k + 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$N = 2 K, \text{ for } K \in \mathbb{N}^+, \quad (B.5a)$$

$$\Xi \equiv 126/100, \quad (B.5b)$$

and make all matrices traceless ($\hat{a}_{\text{tmp}}^1$ is already traceless),

$$\hat{a}_{\text{tmp-traceless}}^{\rho} = \hat{a}_{\text{tmp}}^{\rho} - \frac{1}{N} \text{Tr} \left(\hat{a}_{\text{tmp}}^{\rho}\right) \mathbb{1}_N, \quad (B.5c)$$

and Hermitian,

$$\hat{a}_{\text{Ansatz}}^{\rho} = \frac{1}{2} \left[\hat{a}_{\text{tmp-traceless}}^{\rho} + \left(\hat{a}_{\text{tmp-traceless}}^{\rho}\right)^{\dagger}\right]. \quad (B.5d)$$
Observe that the diagonal entries of $\hat{a}_1^{\text{Ansatz}}$ and the band-diagonal entries of $\hat{a}_2^{\text{Ansatz}}$ grow as $O\left(\frac{N}{3}\right)$, while the off-band-diagonal entries of $\hat{a}_2^{\text{Ansatz}}$ drop as $O\left(\frac{N-1}{3}\right)$. Incidentally, the Ansatz (B.5) is strictly rational if we take $N = 2^3p$, for positive integer $p$.

With the Ansatz matrices $\hat{a}_\rho^{\text{Ansatz}}$ from (B.4), we calculate the matrices $\hat{\eta}_\rho^{\text{result}}$ from (B.5). For $N = 8, 64, \text{and } 512$, we have some representative results which show that the matrices $\hat{\eta}_\rho^{\text{result}}$ are more or less noise-like, except that $\hat{\eta}_1^{\text{result}}$ has somewhat large entries on the diagonal and that $\hat{\eta}_2^{\text{result}}$ has zeros on the diagonal and relatively small values in a wide band around the diagonal. Still, the matrices relevant to the original problem (B.1) are obtained by a similarity transformation,

$$\hat{\eta}_\rho^{\text{result}} = S^{-1}_1 \cdot \hat{\eta}_\rho^{\text{result}} \cdot S_1,$$

which will change the diagonal values of, in particular, $\hat{\eta}_2^{\text{result}}$. More important is that the calculated matrices $\hat{\eta}_\rho^{\text{result}}$ have entries that do not seem to grow drastically with $N$. Indeed, the arithmetic mean (average) of the absolute values of the entries of the calculated matrices $\hat{\eta}_\rho^{\text{result}}$ stays more or less constant at unity for values of $N$ up to 512; see Table 1. Admittedly, the results from Table 1 are not perfect (the maximum values grow with $N$, as do the mean values to a lesser extent), but perhaps the Ansatz can be improved, as will be discussed in the next paragraph.

Based on the results obtained above, we conjecture that the solution matrices $\hat{a}_\rho$ of the algebraic equation (10) may display, after diagonalization of one of them, a diagonal/band-diagonal structure, having off-band-diagonal entries that drop in magnitude with $N$ (perhaps as $\frac{N}{3}$) and diagonal entries that grow with $N$ (perhaps as $N^{1/3}$). Here, the assumption is that the left-hand side of (10) can, in first approximation, be neglected. Two of our further assumptions are perhaps less adequate. First, the assumption of a constant band-diagonal width $\Delta N = 3$ in $\hat{a}_2^{\text{Ansatz}}$ may be invalid and perhaps $\Delta N$ increases with a small positive power of $N$. Second, the assumption of a constant range for the entries on the band diagonal in $\hat{a}_2^{\text{result}}$ may also be invalid and an improvement would have an over-all structure matching the ordered eigenvalues of $\hat{a}_1^{\text{result}}$. Still, it appears not at all impossible that some type of diagonal/band-diagonal structure in the matrices $\hat{a}_\rho^{\text{result}}$ remains in the large-$N$ limit.

Table 1. Matrices $\hat{\eta}_\rho^{\text{result}}$ calculated from (B.5) with Ansätze (B.5). Shown are representative results for the minimal value, the maximal value, and the mean value of the absolute values of the matrix entries (the matrix size is $N$).

| $N$  | $\text{Abs} \left[\hat{\eta}_1^{\text{result}}\right]$ | $\text{Abs} \left[\hat{\eta}_2^{\text{result}}\right]$ |
|------|--------------------------------------------------|--------------------------------------------------|
| 8    | $\{0.158, 3.92, 1.14\}$                          | $\{0, 4.33, 0.851\}$                             |
| 64   | $\{0.0227, 12.1, 1.04\}$                         | $\{0, 9.62, 1.05\}$                             |
| 512  | $\{0.00206, 45.4, 1.38\}$                        | $\{0, 21.8, 1.94\}$                             |
References

1. E. Witten, “String theory dynamics in various dimensions,” Nucl. Phys. B 443, 85 (1995), arXiv:hep-th/9503124.
2. P. Horava and E. Witten, “Heterotic and type I string dynamics from eleven dimensions,” Nucl. Phys. B 460, 506 (1996), arXiv:hep-th/9510209.
3. N. Ishibashi, H. Kawai, Y. Kitazawa, and A. Tsuchiya, “A large-N reduced model as superstring,” Nucl. Phys. B 498, 467 (1997), arXiv:hep-th/9612115.
4. H. Aoki, S. Iso, H. Kawai, Y. Kitazawa, A. Tsuchiya, and T. Tada, “IIB matrix model,” Prog. Theor. Phys. Suppl. 134, 47 (1999), arXiv:hep-th/9908038.
5. W. Krauth, H. Nicolai, and M. Staudacher, “Monte Carlo approach to M-theory,” Phys. Lett. B 431, 31 (1998), arXiv:hep-th/9803117.
6. P. Austing and J.F. Wheater, “Convergent Yang–Mills matrix theories,” JHEP 04, 019 (2001), arXiv:hep-th/0103159.
7. S.W. Kim, J. Nishimura, and A. Tsuchiya, “Expanding (3+1)-dimensional universe from a Lorentzian matrix model for superstring theory in (9+1)-dimensions,” Phys. Rev. Lett. 108, 011601 (2012), arXiv:1108.1540.
8. J. Nishimura and A. Tsuchiya, “Complex Langevin analysis of the space-time structure in the Lorentzian type IIB matrix model,” JHEP 1906, 077 (2019), arXiv:1904.05019.
9. K. Hatakeyama, A. Matsumoto, J. Nishimura, A. Tsuchiya, and A. Yosprakob, “The emergence of expanding space-time and intersecting D-branes from classical solutions in the Lorentzian type IIB matrix model,” Prog. Theor. Exp. Phys. 2020, 043B10 (2020), arXiv:1911.08132.
10. K.N. Anagnostopoulos, T. Azuma, Y. Ito, J. Nishimura, T. Okubo, and S. K. Papadodis, “Complex Langevin analysis of the spontaneous breaking of 10D rotational symmetry in the Euclidean IKKT matrix model,” JHEP 06, 069 (2020), arXiv:2002.07410.
11. E. Witten, “The 1/N expansion in atomic and particle physics,” in G. ’t Hooft et al. (eds.), Recent Developments in Gauge Theories, Cargese 1979 (Plenum Press, New York, USA, 1980).
12. S. Coleman, “1/N (Erice 1979),” in: Aspects of Symmetry: Selected Erice Lectures (Cambridge University Press, Cambridge, UK, 1985), Chap. 8.
13. J. Greensite and M.B. Halpern, “Quenched master fields,” Nucl. Phys. B 211, 343 (1983).
14. J. Carlson, J. Greensite, M.B. Halpern, and T. Sterling, “Detection of master fields near factorization,” Nucl. Phys. B 217, 461 (1983).
15. J.M. Alberty and J. Greensite, “Approximation techniques for the quenched master field equations,” Nucl. Phys. B 238, 39 (1984).
16. F.R. Klinkhamer, “IIB matrix model: Emergent spacetime from the master field,” Prog. Theor. Exp. Phys. 2021, 013B04 (2021), arXiv:2007.08485.
17. F.R. Klinkhamer, “IIB matrix model and regularized big bang,” Prog. Theor. Exp. Phys. 2021, 063B05 (2021), arXiv:2009.06525.
18. F.R. Klinkhamer, “M-theory and the birth of the Universe,” Acta Phys. Pol. B 52, 1007 (2021), arXiv:2102.11202.
19. W.H. Press, S.A. Teukolsky, W.T. Vetterling, and B.P. Flannery, Numerical Recipes in FORTRAN: The Art of Scientific Computing (Cambridge University Press, Cambridge, UK, 1986).
20. S. Wolfram, Mathematica: A System for Doing Mathematics by Computer, Second Edition (Addison–Wesley, Redwood City CA, USA, 1991).
21. T. Fischbacher, private communication, May 18, 2021.