The thermal history of the plasma and high-frequency gravitons

Massimo Giovannini

Department of Physics, Theory Division, CERN, 1211 Geneva 23, Switzerland and
INFN, Section of Milan-Bicocca, 20126 Milan, Italy

E-mail: massimo.giovannini@cern.ch

Received 15 September 2008, in final form 16 November 2008
Published 28 January 2009
Online at stacks.iop.org/CQG/26/045004

Abstract

Possible deviations from a radiation-dominated evolution, occurring prior to the synthesis of light nuclei, impacted on the spectral energy density of high-frequency gravitons. For a systematic scrutiny of this situation, the $\Lambda$CDM paradigm must be complemented by (at least two) physical parameters describing, respectively, a threshold frequency and a slope. The supplementary frequency scale sets the lower border of a high-frequency domain where the spectral energy grows with a slope which depends, predominantly, upon the total sound speed of the plasma right after inflation. While the infrared region of the graviton energy spectrum is nearly scale invariant, the expected signals for typical frequencies larger than $0.01 \text{ nHz}$ are hereby analyzed in a model-independent framework by requiring that the total sound speed of the post-inflationary plasma be smaller than the speed of light. Current (e.g., low-frequency) upper limits on the tensor power spectra (determined from the combined analysis of the three large-scale data sets) are shown to be compatible with a detectable signal in the frequency range of wideband interferometers. In the present context, the scrutiny of the early evolution of the sound speed of the plasma can then be mapped onto a reliable strategy of parameter extraction including not only the well-established cosmological observables but also the forthcoming data from wideband interferometers.

PACS numbers: 98.70.Vc, 98.80.Cq, 04.30.Db, 98.80.−k

(Some figures in this article are in colour only in the electronic version)

1. The general framework

Cosmological observations rely on three pivotal data sets, i.e. the cosmic microwave background (CMB) data, the determinations of the matter power spectrum from galaxy surveys...
and the supernova light curve observations. The large-scale measurements are inextricably bound to the model used to interpret the data. Consequently, the three aforementioned data sets are jointly analyzed in terms of a standard scenario which is often dubbed \( \Lambda \)CDM paradigm, where \( \Lambda \) qualifies the dark energy component and CDM denotes the cold dark matter component. While all the current observations are based, directly or indirectly, on the electromagnetic spectrum, there is the hope, in the future, that the electromagnetic observations could be complemented by the analysis of the spectrum of the relic gravitons which have been produced both in the context of the \( \Lambda \)CDM paradigm and in other related contexts. The problem is, therefore, two-fold: on the one hand reliable estimates of the spectrum of the relic gravitons arising in the \( \Lambda \)CDM paradigm are needed. On the other hand, it will be important to analyze other complementary scenarios. The purpose of the present paper is to address both issues in quantitative terms. The relic graviton background produced in the context of the \( \Lambda \)CDM paradigm is expected to be rather minute and undetectable by wideband interferometers \[1–4\] in one of their future realizations. There are, however, extensions of the \( \Lambda \)CDM scenario where the signal potentially detectable by wideband interferometers is much larger than in the current paradigm.

The recent WMAP 5 year data \[5–9\] set quite stringent bounds on the amplitude of the relic graviton spectral energy density for typical frequency scales\(^1\):

\[ \nu_p = \frac{k_p}{2\pi} = 3.092 \times 10^{-18} \text{ Hz} \equiv 3.092 \text{ aHz}, \]  

(1.1)

where, according to the prefixes of the international system of units, 1 aHz = \(10^{-18} \) Hz. The wave number \( k_p \) is also called sometimes pivot scale\(^2\) since it is customary, in the experimental analyses of CMB data \[8, 9\], to assign the amplitude of the scalar and tensor modes exactly at \( k_p \). In the same units of equation (1.1) the typical frequency interval potentially accessible to the observations of the wideband interferometers ranges between a few Hz and 10 kHz with a peak of sensitivity around 100 Hz.

The logic followed in the present investigation will be to compute as accurately as possible the relic graviton spectra in terms of the parameters of the putative \( \Lambda \)CDM paradigm. The CMB data will then be used to enforce the normalization of the spectra at \( \nu_p \). This will allow for the estimate of the spectral energy density at the frequency explored by wideband interferometers (i.e., approximately, 100 Hz) not only in the case of the \( \Lambda \)CDM paradigm but also in the context of its extensions. The latter extensions will be examined on the basis of their plausibility, e.g. the models which are already incompatible (or barely compatible) with CMB observations will not be analyzed and the attention will be focused on those scenarios which are not ruled out by (current) large-scale observations and which may lead to a potentially large signal at the wideband interferometer scale. In the present introductory section, after a general discussion of the typical frequencies of the graviton spectrum, the present status of CMB observations and wideband interferometers observations will be swiftly discussed. Specific attention will be paid to those quantitative aspects which are germane to our theme, i.e. the stochastic backgrounds of relic gravitons. Some of the concepts introduced here will also be more specifically addressed in the forthcoming sections. At the end of this introduction, the purposes of the present investigation will be more specifically outlined.

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\(^1\) Natural units \( \hbar = c = k_B = 1 \) will be consistently adopted all along the present investigation.

\(^2\) The pivot wave number \( k_p \) corresponds to an effective multipole \( \ell_{\text{eff}} \simeq 30. \)
1.1. Typical frequencies of the problem

To compare frequencies it is mandatory to specify the background and the appropriate conventions on the normalization of the scale factor. Consistently with the ΛCDM paradigm, the background geometry will be taken to be conformally flat, i.e.

\[ ds^2 = \overline{g}_{\mu \nu} \, dx^\mu \, dx^\nu \equiv a^2(\tau) [d\tau^2 - d\vec{x}^2], \quad \overline{g}_{\mu \nu} = a^2(\tau) \eta_{\mu \nu}, \]

where \( \eta_{\mu \nu} \) is the Minkowski metric with signature mostly minus, i.e. \((+, -, -, -)\). The scale factor at the present time will be normalized to unity, i.e. \( a_0 = 1 \). Within the latter convention, the comoving frequencies (or wavelengths) coincide, at the present time, with the physical frequencies (or wavelengths). The (conformal) time derivative of the logarithm of the scale factor will be used throughout the script and it is defined as

\[ \dot{\mathcal{H}} = \frac{a'}{a} = \frac{\text{d} \ln a}{\text{d} \tau}. \]

Note that, in equation (1.3), the prime denotes a derivation with respect to the conformal time coordinate \( \tau \); this notation will be consistently enforced in the whole investigation. The evolution of the background can be expressed in terms of \( \mathcal{H} \) and \( \dot{\mathcal{H}} \) and it is given by

\[ 3 \dot{\mathcal{H}}^2 = a^2 \mathcal{H}^2 \rho, \]

\[ 2(\mathcal{H}^2 - \dot{\mathcal{H}}^2) = a^2 \mathcal{H}^2 (\rho + p), \]

\[ \rho' + 3 \mathcal{H} (\rho + p) = 0, \]

where \( \rho \) and \( p \) denote, respectively, the total energy density and the total pressure of the plasma.

The frequency of equation (1.1) can be usefully compared with two other important frequencies, i.e. the frequency of matter-radiation equality (be it \( v_{\text{eq}} \)) and the frequency of neutrino decoupling (which also coincides, in loose terms, with the Hubble radius at the onset of big-bang nucleosynthesis). These two frequencies can then be written, respectively, as

\[ v_{\text{eq}} = \frac{k_{\text{eq}}}{2\pi} = 1.281 \times 10^{-17} \left( \frac{h^3_0 \Omega_{M0}}{0.1326} \right) \left( \frac{h^3_0 \Omega_{R0}}{4.15 \times 10^{-5}} \right)^{-1/2} \text{Hz}, \]

\[ v_{\text{bbn}} = 2.252 \times 10^{-11} \left( \frac{g^\rho}{10.75} \right)^{1/4} \left( \frac{T_{\text{bbn}}}{\text{MeV}} \right) \left( \frac{h^3_0 \Omega_{R0}}{4.15 \times 10^{-5}} \right)^{1/4} \text{Hz} \approx 0.01 \text{nHz}. \]

In equations (1.7) and (1.8) \( \Omega_{M0} \) and \( \Omega_{R0} \) denote, respectively, the present critical fraction of matter and radiation with typical values drawn from the best fit to the WMAP 5 year data alone and within the ΛCDM paradigm. In equation (1.8), \( g^\rho \) denotes the effective number of relativistic degrees of freedom entering the total energy density of the plasma. While \( v_{\text{eq}} \) is still close to the aHz, \( v_{\text{bbn}} \) is already in the nHz range.

The success of the CMB and BBN calculations implicitly demands that, after neutrino decoupling, the universe was already dominated by radiation. If we assume that the radiation dominates right at the end of inflation, then the maximal frequency of the graviton spectrum can be computed and it is given by

\[ v_{\text{max}} = 0.346 \left( \frac{\epsilon}{0.01} \right)^{1/4} \left( \frac{A_R}{2.41 \times 10^{-9}} \right)^{1/4} \left( \frac{h^3_0 \Omega_{R0}}{4.15 \times 10^{-5}} \right)^{1/4} \text{GHz}, \]

where \( A_R \) denotes the amplitude of the power spectrum of curvature perturbations evaluated at the pivot wave number \( k_p \). Between \( v_{\text{bbn}} \) and \( v_{\text{max}} \) there are roughly 20 orders of magnitude in frequency. In the ΛCDM scenario the relic graviton spectrum has, in this range, always the same slope.
1.2. CMB data and relic gravitons

As already mentioned, CMB experiments are sensitive to long wavelength gravitons with typical frequencies of the order of $\nu_p \sim aHz$ (see also equation (1.1)). The number of CMB parameters depends upon the specific model used to interpret (and fit) the data. The $\Lambda$CDM scenario probably contains the fewest number of parameters required to have a consistent fit of CMB data.

The $\Lambda$CDM parameters can be inferred from various experiments and, among them, a central role is played by WMAP [5–9] (see also [10–12] for first year data release and [13, 14] for the third year data release) as well as other experiments (see, for instance, [15] in connection with the 5 year WMAP data release). The TT, TE and partially EE angular power spectra have been measured by the WMAP experiment. Other (i.e., non-space-borne) experiments are now measuring polarization observables, in particular there are the 3 year Dasi release [16], the CAPMAP experiment [17], the recent QUAD data [19, 20], as well as various other experiments at different stages of development. The TT, TE and EE power spectra are customarily analyzed in the light of the minimal $\Lambda$CDM scenario but also other models are possible and they include, for instance, the addition of spatial curvature (i.e., the open-$\Lambda$CDM), more general parametrizations for the equation of state of dark energy and so on and so forth.

The combined analysis of the CMB data, of the large-scale structure data [22, 23] and of the supernova data [24, 25] can lead to quantitative upper limits on the possible contribution of the tensor modes to the initial conditions of the CMB temperature and polarization anisotropies. These upper limits can be phrased in terms of $r_T$, i.e. the ratio between the power spectrum of tensor fluctuations and the power spectrum of the scalar fluctuations evaluated at the pivot wave number $k_p = 0.002Mpc^{-1}$. In the minimal paradigm (i.e., the $\Lambda$CDM scenario) the tensor is not included in the fit.

If the inflationary phase is driven by a single scalar degree of freedom and if the radiation dominance kicks in almost suddenly after inflation, $r_T$ not only determines the tensor amplitude but also, thanks to the algebra obeyed by the slow-roll parameters, the slope of the tensor power spectrum, customarily denoted by $n_T$. To lowest order in the slow-roll expansion, therefore, the tensor spectral index is slightly red and it is related to $r_T$ (and to the slow-roll parameter) as $n_T \simeq -r_T/8 \simeq -2\epsilon$, where $\epsilon = -\dot{H}/H^2$ measures the rate of decrease of the Hubble parameter during the inflationary epoch. Within the established set of conventions the scalar spectral index $n_s$ is given by $n_s = (1 - 6\epsilon + 2\eta)$ and it depends not only upon $\epsilon$ but also upon the second slow-roll parameter $\eta = \mathcal{M}_P^2 \partial^2 V/\partial \phi^2$ (where $V$ is the inflation potential, $V,\phi\phi$ denotes the second derivative of the potential with respect to the inflation field and $\mathcal{M}_P = 1/\sqrt{8\pi G}$).

Depending upon the specific data used in the analysis, the upper limits on $r_T$ as well as the determination of the other cosmological parameters may change slightly. In table 1, the upper limits on $r_T$ are illustrated as they are determined from the combination of different data sets. For illustration the determined values of the scalar spectral index (i.e., $n_s$), of the dark energy and dark matter fractions (i.e., respectively, $\Omega_\Lambda$ and $\Omega_{\text{M0}}$), and of the typical wave number of equality $k_{eq}$ are also reported in the remaining columns. While different analyses can be performed, it is clear, by looking at table 1 that the typical upper bounds on $r_T$ range between

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3 Following the custom the TT correlations will simply denote the angular power spectra of the temperature autocorrelations. The TE and the EE power spectra denote, respectively, the cross power spectrum between temperature and polarization and the polarization autocorrelations.

4 The overdot will denote throughout the paper a derivation with respect to the cosmic time coordinate $t$ while the prime will denote a derivation with respect to the conformal time coordinate $\tau$. 

Table 1. The values of $r_T$ are reported as they have been estimated in the absence of any running of the (scalar) spectral index.

| Data                      | $r_T$     | $n_s$        | $\Omega_x$     | $\Omega_{\Lambda_1}$ | $k_{eq}\text{Mpc}$ |
|---------------------------|-----------|--------------|-----------------|------------------------|--------------------|
| WMAP5 alone               | $<0.43$   | 0.986 ± 0.22 | 0.770$^{+0.032}_{-0.032}$ | 0.236$^{+0.032}_{-0.032}$ | 0.009 36           |
| WMAP5 + Acbar             | $<0.40$   | 0.985$^{+0.019}_{-0.020}$ | 0.767 ± 0.032  | 0.233 ± 0.032          | 0.009 44           |
| WMAP5+ LSS + SN           | $<0.20$   | 0.968 ± 0.015 | 0.725 ± 0.015  | 0.275 ± 0.015          | 0.009 99           |
| WMAP5+ other CMB data     | $<0.36$   | 0.979 ± 0.020 | 0.775 ± 0.032  | 0.225 ± 0.032          | 0.009 22           |

say, 0.2 and 0.4. Slightly more stringent limits can also be obtained by adding supplementary assumptions. Within a conservative perspective, the tensor power spectra are, at least, ten times smaller than the power spectra of curvature perturbations. In the near future, the Planck explorer satellite [26] might be able to set more direct limits on $r_T$ by measuring (hopefully) the BB angular power spectra. The E-mode power spectra and the B-mode power spectra arise as two orthogonal combinations of the Stokes parameters which are frame-dependent (i.e., $Q$ and $U$). While the adiabatic mode leads naturally to the E-mode polarization, the only way of obtaining the B-mode (in the standard $\Lambda$CDM paradigm) is through the contribution of the tensor modes. Consequently, detection of the BB angular power spectra would be equivalent, in the $\Lambda$CDM framework, to a first determination of $r_T$. Having reviewed the essentials of CMB data and their connection with the relic graviton spectra, we can now move to higher frequencies and describe the status of the other devices which could shed light on the relic gravitons, i.e. the wideband interferometers.

1.3. Wideband interferometers

The wideband interferometers operate in a window ranging from a few Hz up to 10 kHz. The available interferometers are LIGO [1], VIRGO [2], TAMA [3] and GEO [4]. The sensitivity of a given pair of wideband detectors to a stochastic background of relic gravitons depends upon the relative orientation of the instruments. The wideness of the band (important for the correlation among different instruments) is not as large as 10 kHz but typically narrower and, in an optimistic perspective, it could range up to 100 Hz. The putative frequency of wideband detectors will therefore be indicated as $\nu_{LV}$, i.e. in loose terms, the LIGO/VIRGO frequency. There are daring projects of wideband detectors in space such as the Lisa [31], the BBO [32] and the Decigo [33] projects. The common denominator of these three projects is that they are all space-borne missions and sensitive to frequencies smaller than the mHz. While wideband interferometers are now operating and might even reach their advanced sensitivities along the incoming decade, the achievable sensitivities of space-borne interferometers are still on the edge of the achievable technologies. Since $\nu_{bbn} < \nu_{LV} < \nu_{\text{max}}$, the wideband interferometers are ideal instruments to investigate the relic graviton spectrum in the unknown territory where there are neither direct nor indirect tests on the thermal history of the plasma. The problem is that, as will carefully be shown, the spectral energy density of the relic gravitons produced within the $\Lambda$CDM model is quite minute and it is undetectable by interferometers even in their advanced version where the sensitivity is expected to improve by five or even six orders of magnitude in comparison with the present performances [34–36] (see also [37, 38]). This impasse, as previously stressed, stems from the assumption that, right after inflation, the radiation-dominated evolution kicks in almost suddenly. At the moment, there is no evidence

5 Forthcoming projects such as Clover [27], Brain [28], Quiet [29] and Spider [30] have polarization as a specific target.
either in favor of such a statement nor against it. The main theme of the present investigation will be to reverse this problem. It will be argued that wideband detectors, in their advanced version, will certainly be able to test definite deviations from a simplistic thermal history of the plasma, i.e. the one stipulating that, after inflation, the radiation was suddenly dominating the evolution.

1.4. Layout of the investigation

In the present investigation the spectral energy density of the relic gravitons will be calculated first at small frequencies (compatible with the CMB observations) and then at higher frequencies (compatible with the operational window of wideband interferometers). The latter calculation will be performed both in the case of the ΛCDM paradigm and in those extensions which may lead to a large spectral energy density at the scale of the wideband interferometers without violating any of the bounds stemming from CMB observations.

The spectral energy density of the relic gravitons will be introduced in section 2. Gravity is inherently a non-Abelian gauge theory, there are potential ambiguities in defining univocally an energy–momentum (pseudo)-tensor for the relic gravitons: it will be shown that different choices of the energy–momentum pseudo-tensor lead to the same spectral energy density of the relic gravitons. The punch line of section 2 will be that the spectral energy density can be more accurately performed with numerical methods rather than resorting, as often done, to semi-analytical estimates which amount to estimating first the power spectrum and then the spectral energy density.

In section 3, using the numerical techniques introduced in section 2, the spectral energy density of the relic gravitons will be computed in the case of the standard ΛCDM paradigm which leads to nearly scale-invariant spectra. The signal arising in the context of the ΛCDM paradigm will be confronted with the sensitivity of wideband interferometers. After showing the compatibility of the new methods with the results of the nearly scale-invariant spectra, possible scaling violations in the spectral energy density will be discussed in section 4. While in the ΛCDM paradigm the spectral energy density of the relic gravitons is nearly scale invariant, it is plausible to construct a class of models where the spectral energy density is fully compatible with the CMB and with the large-scale data at low frequencies while it is potentially detectable by wideband interferometers. When we say that it is plausible this simply means that it is not forbidden by any of the current observational data. The proposed extensions of the ΛCDM paradigm also have a physical interpretation (see section 4) since they naturally arise when the thermal history of the universe deviates, for sufficiently early times, from the usual assumptions of the ΛCDM scenario which stipulates that, right after inflation, the universe suddenly becomes dominated by radiation.

The minimal realization of the ideas pursued in section 4 is scrutinized in section 5 and it is dubbed TΛCDM scenario (for tensor-ΛCDM). The TΛCDM paradigm consists of two supplementary parameters, i.e., in broad terms, a new pivotal frequency and a new spectral slope. The new frequency marks the onset of the high-frequency branch of the spectral energy density of the relic gravitons. In section 5, the TΛCDM paradigm is compatible with the current bounds stemming not only from CMB and large-scale structure. It will also be required that the big-bang nucleosynthesis (BBN) as well as pulsar timing constraints be satisfied and this will allow us to spell out quantitatively the restrictions on the two supplementary parameters characterizing the TΛCDM scenario.
2. Basic equations

The basic technical tools required to pursue the present analysis will hereby be summarized. The first part of the present section (i.e., subsection 2.1) contains an introduction to the basic terminology and a swift derivation of the main equations. Subsections 2.2 and 2.3 contain the details of our numerical approach whose results will also be illustrated in various physically relevant examples (see subsection 2.4) and compared to the corresponding semi-analytical estimates (see subsection 2.5). Finally, the exponential damping of the relic graviton spectrum will be numerically discussed in subsection 2.6. As explained in the general layout of the investigation, all the considerations of the present section are bound to the $\Lambda$CDM paradigm so that the typical values of the cosmological parameters may be usefully drawn from table 1.

2.1. Generalities

In the $\Lambda$CDM paradigm the geometry is conformally flat (see equation (1.2)) and the corresponding tensor fluctuations are defined as

$$\delta^{(1)}_{\mu \nu} = -a^2(\tau) h_{\mu \nu}, \hspace{1cm} \delta^{(1)}_{\mu} = h^i_{\mu} a^2, \hspace{1cm} \delta^{(2)}_{\mu \nu} = -\frac{h'_i h^i_{\mu \nu}}{a^2},$$

(2.1)

where $h'_i = \partial_i h^i_j = 0$. The second-order action obeyed by $h_{\mu \nu}$ can be written as

$$S_{GW} = \frac{1}{8\ell^2_P} \int d^4x \sqrt{-g} g^{\mu \nu} \partial_{\mu} h_{\nu \rho} \partial_{\rho} h_{\nu \rho},$$

(2.2)

where $\ell^2_P$ is defined as

$$\ell^2_P = \frac{8\pi G}{M^2_P} = \frac{8\pi}{M^2_P}.$$

(2.3)

Equation (2.2) is effectively equivalent to the sum of the actions of two (scalar) degrees of freedom minimally coupled to the background geometry. To derive equation (2.2) the Einstein–Hilbert action must be perturbed to second order in the amplitude $h_{\mu \nu}$, i.e.

$$\delta^{(2)}_{\mu \nu} S = -\frac{1}{16\pi G} \int d^4x \left[ \delta^{(2)}_{\mu \nu} \sqrt{-g} R + \sqrt{-g} \delta_{\mu \nu}^{(2)} R + \delta_{\mu}^{(1)} \sqrt{-g} \delta_{\nu}^{(1)} R \right].$$

(2.4)

To evaluate equation (2.4) in explicit terms it is necessary to compute the Ricci tensors to both first and second orders, i.e.

$$\delta^{(1)}_{\mu \nu} R_{\mu \nu} = \frac{1}{2} \left[ h'''_{\mu \nu} + 27 h'_{\mu \nu} - \nabla^2 h_{\mu \nu} \right] + (\mathcal{H}' + 2\mathcal{H}^2) h_{\mu \nu},$$

(2.5)

$$\delta^{(2)}_{\mu \nu} R_{\mu \nu}^0 = \frac{1}{2} h'''_{\mu \nu} - \mathcal{H} h''_{\mu \nu} + \frac{1}{2} h''_{\mu \nu} \nabla^2 h_{\mu \nu},$$

(2.6)

$$\delta^{(2)}_{\mu \nu} R_{\mu \nu} = \frac{1}{2} h^{\mu \nu} \left[ \partial_{\mu} h_{\nu \lambda} - \partial_{\nu} h_{\mu \lambda} - \partial_{\lambda} h_{\mu \nu} \right]$$

$$- \frac{1}{2} \left[ \partial_{\mu} \left( h^{\mu \nu} h_{\lambda \nu} - \partial_{\nu} h_{\mu \lambda} - \partial_{\lambda} h_{\mu \nu} \right) \right] - \frac{\mathcal{H}}{2} h^{\mu \nu} h_{\rho \lambda} \delta_{\rho \lambda}$$

$$+ \frac{\mathcal{H}}{2} h^i_{\mu \nu} h_{i \lambda} - \frac{\mathcal{H}}{2} h^i_{\mu} h_{i \lambda} - \frac{\mathcal{H}}{2} h^i_{\nu} h_{i \lambda} - \frac{\mathcal{H}}{2} h^i_{\lambda} h_{i \mu} - \frac{\mathcal{H}}{2} h^i_{\lambda} h_{i \nu}$$

$$- \frac{1}{4} \left[ \partial^2_{\mu} h^i_{\nu} + \partial^2_{\nu} h^i_{\mu} - \partial^2_{\lambda} h^i_{\mu} \right] \left[ \partial_{\mu} h_{i \lambda} + \partial_{\nu} h_{i \lambda} - \partial_{\lambda} h_{i \mu} \right].$$

(2.7)

Some authors include a $\sqrt{8\pi}$ in the definition of the reduced Planck mass (which we call $M_P = \ell^{-1}_P$). This convention is per se harmless, however, it may be confusing in practice. In the present script the conventions expressed by equation (2.3) will always be carefully followed.
The Ricci scalar is zero to first order in the tensor fluctuations, i.e. $\delta^{(1)}_t R = 0$. This is due to the traceless nature of these fluctuations. To second order, however, $\delta^{(2)}_t R \neq 0$ and its form is

$$
\delta^{(2)}_t R = \frac{1}{a^2} \left\{ \frac{3}{4} h''_{\ell\ell} h^{\ell\ell} + \mathcal{H} h'_{\ell\ell} h^{\ell\ell} + \frac{1}{2} h^{i\ell}\nabla^2 h_{i\ell} - \frac{1}{4} \partial_i h^{i\ell} \partial^j h_{\ell j} \right\}
$$

$$
+ \frac{1}{a^2} \left\{ - \frac{1}{2} \partial_i \left[ h^{\ell\ell} \left( \partial_\ell h^i_{i\ell} - \partial^j h_{i\ell j} \right) \right]
$$

$$
- \frac{1}{4} \left[ \partial_i h_{\ell j} \partial^j h^i_{i\ell} - \partial^i h_{i\ell j} \partial^j h^i_{i\ell} + \partial_i h'^{i\ell} \partial^j h_{i\ell j} - \partial^i h_{i\ell j} \partial^j h^i_{i\ell} + \partial^j h^i_{i\ell} \partial^i h_{i\ell j} \right] \right\}. \tag{2.8}
$$

Using the results of equations (2.5)–(2.8) into equation (2.4) the second-order action for the tensor modes of equation (2.2) can be obtained by getting rid of a number of total derivatives.

According to equation (2.1), $h_{ij}$ carries two degrees of freedom associated with the two polarizations of the graviton in a Friedmann–Robertson–Walker (FRW) spacetime. Defining as $\hat{k} = \hat{k} / |\hat{k}|$ the direction along which a given tensor mode propagates, the two polarizations can be defined as

$$
\epsilon^{(0)}_{ij}(\hat{k}) = (\hat{m}_i \hat{n}_j - \hat{n}_i \hat{m}_j), \quad \epsilon^{(0)}_{ij}(\hat{k}) = (\hat{m}_i \hat{n}_j + \hat{n}_i \hat{m}_j), \tag{2.9}
$$

where $\hat{m}$ and $\hat{n}$ are two mutually orthogonal unit vectors which are also orthogonal to $\hat{k}$ (i.e., $\hat{m} \cdot \hat{n} = \hat{n} \cdot \hat{k} = \hat{m} \cdot \hat{k} = 0$). During the early stages of the ΛCDM model (i.e., during the inflationary phase) $h_{ij}(\vec{x}, \tau)$ can be expanded in terms of the appropriate creation and annihilation operators as

$$
\hat{h}_{ij}(\vec{x}, \tau) = \frac{\sqrt{2} \epsilon_{kl}}{(2\pi)^{1/2}} \sum_{\lambda} \int d^3k \epsilon^{(b)}_{ij}(\vec{k}) F_{k,\lambda}(\tau) a_{\lambda}^k e^{-i\vec{k} \cdot \vec{x}} + F^*_{k,\lambda}(\tau) \bar{a}_{\lambda}^k e^{i\vec{k} \cdot \vec{x}}. \tag{2.10}
$$

where the index $\lambda$ counts the two polarizations, i.e. $\lambda = \otimes, \oplus$; $k$ denotes the wave number and $F_{k,\lambda}(\tau)$ is the (complex) mode function obeying

$$
F^*_{k,\lambda} = G_{k,\lambda}, \tag{2.11}
$$

$$
G_{k,\lambda} = -2\mathcal{H} G_{k,\lambda} - k^2 F_{k,\lambda}. \tag{2.12}
$$

In equation (2.10) $[\hat{a}_{\lambda k}, \hat{a}_{\lambda' k}] = \delta^{(3)}(\vec{k} - \vec{p}) \delta_{\lambda \lambda'}$. The initial state $|0\rangle$ (annihilated by $\hat{a}_{\lambda k}$) minimizes the tensor Hamiltonian when all the wavelengths of the field are shorter than the event horizon at the onset of the inflationary evolution. The main observables which are used to characterize the relic graviton background are the two-point function evaluated at equal times and the spectral energy density in critical units. The two-point function is defined as

$$
\langle 0 | \hat{h}_{ij}(\vec{x}, \tau) \hat{h}_{ij}(\vec{y}, \tau) | 0 \rangle = \int^\infty_0 d \ln k P_T(k, \tau) \frac{\sin kr}{kr}, \quad r = |\vec{x} - \vec{y}|. \tag{2.13}
$$

$$
P_T(k, \tau) = \frac{4\epsilon_{ij}^2 k^3}{\pi^2} |F_k(\tau)|^2. \tag{2.14}
$$

The quantity $P_T(k, \tau)$ is, by definition, the tensor power spectrum. Equations (2.13) and (2.14) can be derived by recalling the following pair of relations:

$$
\epsilon^{(b)}_{ij} \epsilon^{(b)}_{ij} = 2\delta_{\lambda \lambda'}, \quad F_{k,\lambda}(\tau) = F_{k,\lambda}(\tau) \equiv F_k(\tau). \tag{2.15}
$$

Out of equations (2.13) and (2.14) it is sometimes practical to introduce the spectral amplitude $S_h(\nu, \tau)$, namely,

$$
\langle 0 | \hat{h}_{ij}(\vec{x}, \tau) \hat{h}_{ij}(\vec{x}, \tau) | 0 \rangle = \int^\infty_0 P_T(k, \tau) d \ln k = 4 \int^\infty_0 \nu S_h(\nu, \tau) d \ln \nu, \tag{2.16}
$$

8
where $k = 2\pi \nu$. By definition, $\rho_{GW}(\vec{x}, \tau) = \langle 0| T^{\mu\nu}_0(\vec{x}, \tau)|0 \rangle$ where $|0\rangle$ is, again, the state annihilated by $\hat{a}_{k, \lambda}$ (see also equation (2.10)) and where $T^{\mu\nu}_0$ is the energy–momentum (pseudo-)tensor of the relic gravitons. The spectral energy density of the relic gravitons in critical units can then be computed from the expectation value of $\rho_{GW} = \langle 0|T^{\mu\nu}_0|0 \rangle$,

$$\Omega_{GW}(k, \tau) = \frac{1}{\rho_{crit}} \frac{d\rho_{GW}}{dk}, \quad \rho_{GW} = \langle 0|T^{\mu\nu}_0|0 \rangle,$$

where $\rho_{crit} = \frac{3H^2}{\ell^2}$ is the critical energy density. In FRW spacetimes the energy–momentum pseudo-tensor of the relic gravitons can be assigned in manners which are conceptually different but physically complementary. This will be one of the topics discussed in subsections 2.2 and 2.3.

### 2.2. Transfer function for the amplitude

To connect the early moment of the normalization of the relic gravitons to the moment where the tensor modes of the geometry re-enter the Hubble radius and affect, in principle, terrestrial detectors, the customary approach is to solve for the tensor mode function and to define the so-called amplitude transfer function. From the amplitude transfer function the spectral energy density (see, e.g., equation (2.17)) can be computed. What we propose here is to do the opposite, i.e. to compute, numerically and in one shot, the transfer function for the spectral energy density. To show the equivalence (but also the inherent differences) of the two approaches, the present subsection will be concerned with the transfer function for the amplitude. In subsection 2.3 the transfer function of the spectral energy density will be more specifically discussed.

During the inflationary phase, the tensor power spectrum can easily be computed by solving equations (2.11) and (2.12) in the slow-roll approximation

$$F_k(\tau) = \frac{N}{a(\tau)\sqrt{2k}} \sqrt{-k\tau} H^{(1)}_\nu(-k\tau), \quad N = \frac{\pi}{2} e^{i\pi(\nu+1)/2}, \quad \nu = \frac{3 - \epsilon}{2(1 - \epsilon)},$$

where $H^{(1)}_\nu(z) = J\nu(z) + iY\nu(z)$ is the Hankel function of the first kind and where $\epsilon = -\dot{H}/H^2$. To obtain the result of equation (2.18) from equations (2.11) and (2.12) it is useful to bear in mind the following pair of identities:

$$H^2 + \mathcal{H}^2 = a^2 H^2(2 - \epsilon), \quad aH = -\frac{1}{\tau(1 - \epsilon)}.$$  

The second equality in equation (2.19) can be simply deduced (after integration by parts) from the relation between cosmic and conformal times, i.e. $a(\tau)\, d\tau = dt$. Physically equations (2.18) and (2.19) hold under the approximation that, at early times, the background geometry is of quasi-de Sitter type.

By substituting equation (2.18) into equation (2.14) the standard expression of the tensor power spectrum can be obtained. When the relevant modes exited the Hubble radius during inflation:

$$\mathcal{P}_T(k, \tau) = \ell_P^2 H^2 \frac{2^{2\nu}}{\pi^3} \Gamma^2(\nu)(1 - \epsilon)^{2\nu-1} \left(\frac{k}{aH}\right)^{3-2\nu}, \quad \nu = \frac{3}{2} + \epsilon + O(\epsilon^2),$$

where the small argument limit of the Hankel functions has been taken and where the slow-roll approximation has been enforced, i.e. in formulae:

$$x = k\tau \simeq \frac{k}{\mathcal{H}} = \frac{k}{aH} \ll 1, \quad \epsilon = -\frac{\dot{H}}{H^2} < 1.$$  

7 The natural logarithms will be denoted by ln, while the common logarithms will be denoted by log.
The two approximations introduced in equation (2.21) will be often employed and, therefore, it is appropriate to spell out clearly their physical meaning. The first relation of equation (2.21) implies that \( k \tau < 1 \) this means that the wave numbers are, in practice, smaller than the Hubble rate. Conversely, the corresponding wavelengths will be larger than the Hubble radius. The chain of equalities appearing in equation (2.21) can easily be understood since, by definition, \( H = aH \) and \( H \) is the Hubble rate. In this long wavelength limit, as we shall see, the evolution of the tensor modes can be derived in semi-analytical terms and it corresponds to a tensor mode function \( F_k(\tau) \) which is approximately constant. The latter statement holds if the geometry is of quasi-de Sitter type. The latter condition is verified if the second relation of equation (2.21), stipulating \( \epsilon < 1 \), holds. The latter conditions are also dubbed slow-roll approximation and it allows us to simplify the tensor power spectrum even further:

\[
P_T(k) \simeq \frac{2}{3\pi^2} \left( \frac{V}{M_p^2} \right) k \simeq \frac{128}{3} \left( \frac{V}{M_p^2} \right) k_{\text{H}}.
\]

(2.22)

The spectral index defined from equation (2.22) is nothing but

\[
n_T = \frac{\ln P_T}{\ln k} = -\frac{2}{1-\epsilon} = -2\epsilon + O(\epsilon^2).
\]

(2.23)

where the second equality can be derived with the standard rules of the slow-roll algebra. The spectral amplitude and slope are then parametrized, for practical purposes, as

\[
P_T(k) = A_T \left( \frac{k}{k_p} \right)^{n_T}, \quad k_p = 0.002 \text{ Mpc}^{-1},
\]

(2.24)

where, by definition, \( A_T \) is the amplitude of the tensor power spectrum evaluated at the pivot scale \( k_p \). The pivot wave number of equation (2.24) is simply related to the pivot frequency defined in equation (1.1) as \( \nu_p = k_p/(2\pi) \). Bearing in mind that the power spectrum of curvature perturbations is given, in single field inflationary models, as

\[
P_R(k) = \frac{8}{3} \left( \frac{V}{\epsilon M_p^2} \right) k_{\text{H}} \simeq A_R \left( \frac{k}{k_p} \right)^{n_s-1},
\]

(2.25)

the ratio between the tensor and the scalar power spectra is simply given by

\[
r_T = \frac{P_T(k)}{P_R(k)} = \frac{A_T}{A_R} = 16\epsilon.
\]

(2.26)

Equation (2.26) implies, recalling equation (2.23), that \( r_T = -8n_T \). In table 1, the values of \( r_T \) have been reported as they can be estimated in a few different analyses of the cosmological data sets.

Equation (2.24) correctly parametrizes the spectrum only when the relevant wavelengths are larger than the Hubble radius before matter-radiation equality. To transfer the spectrum inside the Hubble radius the procedure is to integrate numerically equations (2.11) and (2.12) (as well as equations (1.4)–(1.6)) across the relevant transitions of the background geometry. While the geometry passes from inflation to radiation equation (2.24) implies that the tensor mode function is constant while the relevant wavelengths are larger than the Hubble radius:

\[
F_k(\tau) = A_k + B_k \int \frac{d\tau'}{a^2(\tau')} , \quad \frac{k}{aH} \ll 1, \quad |A_k|^2 = \frac{\pi^2}{4\epsilon k^3} P_T(k).
\]

(2.27)

The term proportional to \( B_k \) in equation (2.27) leads to a decaying mode and \( F_k(\tau) \) is therefore determined, for \( |k\tau| \ll 1 \), by the first term whose squared modulus coincides with the spectrum computed in equation (2.22) and parametrized as in equation (2.24). The
evolution of the background (i.e., equations (1.4)–(1.6)) and of the tensor mode functions (i.e., equations (2.11)–(2.12)) should therefore be solved across the radiation matter transition and the usual approach is to compute the transfer function for the amplitude \[ \text{[41]} \]

\[ T_h(k) = \sqrt{\frac{\langle |F_k(\tau)|^2 \rangle}{\langle |f_k(\tau)|^2 \rangle}}. \]  

(2.28)

In equation (2.28), \( f_k(\tau) \) denotes the approximate form of the mode function (holding during the matter-dominated phase); \( F_k(\tau) \) denotes, instead, the solution obtained by fully numerical methods. As the wavelengths become shorter than the Hubble radius, \( F_k(\tau) \) oscillates. Consequently, to get \( T_h(k) \) the oscillations must be carefully averaged and this is the meaning of the averages appearing in equation (2.28). Hence, the calculation of \( T_h(k) \) requires careful matching over the phases between the numerical and the approximate (but analytical) solution. Consider, indeed, one of the most important applications of the previous results, i.e. the radiation-matter transition. After matter-radiation equality, the scale factor is going, approximately, as \( a(\tau) \approx \tau^2 \) and, therefore, the (approximate) solution of equations (2.11) and (2.12) is given by

\[ \bar{f}_k(\tau) = \frac{3j_i(k\tau)}{k\tau} A_k, \quad j_i(k\tau) = \frac{\sin k\tau}{(k\tau)^2} - \frac{\cos k\tau}{(k\tau)}, \]  

(2.29)

which is constant for \( k\tau < 1 \). In figure 1 the result of the numerical integration is reported in terms of \( T_h^2(k/k_{eq}) \). In figure 1 the fit to the numerical points is also reported and it can be parametrized as

\[ T_h(k/k_{eq}) = \sqrt{1 + c_1 \left( \frac{k}{k_{eq}} \right) + b_1 \left( \frac{k}{k_{eq}} \right)^2}. \]  

(2.30)

By applying the standard tools of the regression analysis \( c_1 \) and \( b_1 \) can be determined as \( c_1 = 1.260 \) and \( b_1 = 2.683 \). The latter result agrees with the findings of [41] who obtain...
\[ \tau_1 = 1.34 \text{ and } \vec{B}_1 = 2.50. \] The value of \( k_{\text{eq}} \) can be obtained directly from the experimental data (see, for instance, the last column of table 1 implying \( k_{\text{eq}} \approx O(0.0099 \text{ Mpc}^{-1}) \)). For instance, the WMAP 5 year data combined with the supernova data and with the large-scale structure data would give \( k_{\text{eq}} = 0.0099 \pm 0.00028 \text{ Mpc}^{-1} \). It turns out that a rather good analytical estimate of \( k_{\text{eq}} \) can be presented as

\[
k_{\text{eq}} = 0.0082879 \left( \frac{h_0^2 \Omega_{M0}}{0.1326} \right) \left( \frac{h_0^2 \Omega_{R0}}{4.15 \times 10^{-5}} \right)^{-1/2} \text{ Mpc}^{-1}, \tag{2.31}
\]

where the typical value selected for \( h_0^2 \Omega_{R0} \) is given by the sum of the photon component (i.e., \( h_0^2 \Omega_{\nu0} = 2.47 \times 10^{-5} \)) and of the neutrino component (i.e., \( h_0^2 \Omega_{\nu0} = 1.68 \times 10^{-5} \)): the neutrinos, consistently with the \( \Lambda \)CDM paradigm, are taken to be massless and their (present) kinetic temperature is just a factor \( (4/11)^{1/3} \) smaller than the (present) photon temperature.

From equation (2.31) it is straightforward to estimate the equality frequency of equation (1.7).

The analytical estimate stems from the observation that the exact solution of equations (1.4)–(1.6) for the matter-radiation transition can be given as \( a(\tau) = a_{\text{eq}}[\gamma^2 + 2y] \) where \( \gamma = \tau/\tau_1 \). The timescale \( \tau_1 = \tau_{\text{eq}}(\sqrt{2} + 1) \) is related to the equality time \( \tau_{\text{eq}} \) which can be estimated as

\[
\tau_{\text{eq}} = \frac{2(\sqrt{2} - 1) \sqrt{\Omega_{R0}}}{H_0} = 120.658 \left( \frac{h_0^2 \Omega_{M0}}{0.1326} \right)^{-1} \left( \frac{h_0^2 \Omega_{R0}}{4.15 \times 10^{-5}} \right)^{1/2} \text{ Mpc}. \tag{2.32}
\]

In the case of the WMAP 5 year data combined with the supernova and large-scale structure data \( h_0^2 \Omega_{M0} = 0.1368 \pm 0.0033 \). Consequently, equations (2.28)–(2.30) imply that the spectrum of the tensor modes is given, at the present time, as

\[
P_T(k, \tau_0) = \frac{9 j_1^2(k \tau_0)}{(k \tau_0)^2} T^2_T(k/ k_{\text{eq}}) \overline{P}_T(k). \tag{2.33}
\]

Within the standard approach, equation (2.33) is customarily connected to the spectral energy density of the relic gravitons. It will now be shown that the spectral energy density of the relic gravitons can directly be assessed, by numerical means, without resorting to equation (2.33).

### 2.3. Spectral energy density

Having presented the standard derivation of the amplitude transfer function we will now focus the attention on the transfer function of the spectral energy density. The latter approach leads more directly to the estimate of the present value of the spectral energy density. Before discussing the numerics in some detail, it is appropriate to recall the construction of an energy–momentum pseudo-tensor by following the same approach which has been proven to be successful in flat spacetime [42]. In a conformally flat geometry of the type introduced in equation (1.2), the energy–momentum pseudo-tensor can be derived by following two complementary strategies. The first one is to take the energy–momentum tensor associated with the action of equation (2.2). Since each polarization of the graviton in a FRW spacetime obeys the evolution equation of a minimally coupled scalar field, it is legitimate to establish that the energy–momentum pseudo-tensor is just given by the energy–momentum tensor of a pair of scalar degrees of freedom minimally coupled to the geometry. By formally taking the functional derivative of equation (2.2) with respect to \( \overline{\mathcal{E}}_{\mu\nu} \), \( T^\mu_\nu \) becomes

\[
T^\mu_\nu = \frac{1}{4 \ell_p^2} \left[ \partial_\mu h_{ij} \partial^\nu h^{ij} - \frac{1}{2} \delta^\mu_\nu \overline{\mathcal{E}}^{\alpha \beta} \partial_\alpha h_{ij} \partial_\beta h^{ij} \right] = \frac{1}{4 \ell_p^2} \sum_{\lambda} \left[ \partial_\mu h_{ij,(\lambda)} \partial^\nu h^{ij,(\lambda)} - \frac{1}{2} \delta^\mu_\nu \overline{\mathcal{E}}^{\alpha \beta} \partial_\alpha h_{ij,(\lambda)} \partial_\beta h_{ij,(\lambda)} \delta^\nu_\mu \right]. \tag{2.34}
\]
where the second equality follows from the first by using that \( h_{ij} = \sum \lambda h_{ij} \) and \( \epsilon_{ij}^{(2)} \epsilon^{(2)}_{ij} = 2 \delta_{ij} \). This perspective was adopted and developed, for the first time, in [43, 44] by Ford and Parker. A complementary approach is to use the energy–momentum pseudo-tensor defined from the second-order fluctuations of the Einstein tensor:

\[
T^*_\mu = -\frac{1}{\ell_p^2} \delta^{(2)}_{\mu} G^*_\mu, \quad G^*_\mu = R^*_\mu - \frac{1}{2} \delta^*_\mu R.
\]  

(2.35)

where \( \delta^{(2)}_\mu \) denotes the second-order tensor fluctuation of the corresponding quantity. The latter approach is more directly related to the well-known flat spacetime procedure [42]. The approach expressed by equation (2.35) has been described in [45, 46] and has been reprised, in a related context, by the authors of [47, 48], mainly in connection with conventional inflationary models where the universe is always expanding.

According to equation (2.34) the energy density is given by \( \rho^{(1)}_{GW} = \langle 0| T^0_0 |0 \rangle \) where \( |0\rangle \) is the state annihilated by the creation and destruction operators introduced in equation (2.10):

\[
\rho^{(1)}_{GW}(\tau) = \frac{1}{a^4} \int d\ln k \frac{k^3}{2\pi^2} \left[ |g_k(\tau)|^2 + (k^2 + H^2)|f_k(\tau)|^2 - 4\mathcal{H} f_k^*(\tau)g_k(\tau) + f_k(\tau)g_k^*(\tau) \right].
\]  

(2.36)

where \( f_k(\tau) = F_k(\tau) + i\pi \) and \( g_k(\tau) = f_k^*(\tau) \) have been introduced. The superscript appearing in equations (2.36) recalls that the energy density refers to the first choice of the energy–momentum tensor given in equation (2.34). According to equations (2.11) and (2.12) the tensor mode functions \( f_k \) and \( g_k \) obey

\[
f_k^* = g_k, \quad g_k^* = -[k^2 - (\mathcal{H}^2 + \mathcal{H}')] f_k.
\]  

(2.37)

By adopting the approach expressed by equation (2.35), the energy density of the relic gravitons \( \rho^{(2)}_{GW} = \langle 0| T^0_0 |0 \rangle \) become

\[
\rho^{(2)}_{GW}(\tau) = \int d\ln k \frac{k^3}{2\pi^2 a^4} \left[ |g_k(\tau)|^2 + (k^2 - 7\mathcal{H}^2)|f_k(\tau)|^2 + 3\mathcal{H} f_k^*(\tau)g_k(\tau) + f_k(\tau)g_k^*(\tau) \right].
\]  

(2.38)

To pass from equation (2.35) to equation (2.38) the simplest procedure is to obtain the second-order fluctuation of the Einstein tensor, i.e. \( \delta^{(2)}_\mu G^*_\mu \). This calculation can easily be carried on by using the results of equations (2.5)–(2.8). Furthermore, it should be appreciated that the two energy–momentum pseudo-tensors lead also to different pressures and this observation has an impact on the back-reaction problems [49]. From equations (2.36)–(2.38) the corresponding critical fractions are

\[
\Omega^{(1)}_{GW}(k, \tau) = \frac{1}{\rho_{crit}} \frac{d\rho^{(1)}_{GW}}{d\ln k}, \quad \Omega^{(2)}_{GW}(k, \tau) = \frac{1}{\rho_{crit}} \frac{d\rho^{(2)}_{GW}}{d\ln k}
\]  

(2.39)

If \( k/\mathcal{H} > 1 \), then \( f_k(\tau) \) will be, in the first approximation, plane waves and \( g_k(\tau) \approx \pm ik f_k(\tau) \) and the two versions of \( \Omega_{GW}(k, \tau) \) will be given by

\[
\Omega^{(1)}_{GW}(k, \tau) = \frac{k^5 \ell_p^2}{3\pi^2 a^2 \mathcal{H}^2} \left[ 1 + \frac{\mathcal{H}^2}{2k^2} \right] |f_k(\tau)|^2 = \left[ 1 + \frac{\mathcal{H}^2}{2k^2} \right] P_T(k, \tau),
\]  

(2.40)

\[
\Omega^{(2)}_{GW}(k, \tau) = \frac{k^5 \ell_p^2}{3\pi^2 a^2 \mathcal{H}^2} \left[ 1 - \frac{7\mathcal{H}^2}{2k^2} \right] |f_k(\tau)|^2 = \left[ 1 - \frac{7\mathcal{H}^2}{2k^2} \right] P_T(k, \tau),
\]  

(2.41)

where the second equality follows from the first by recalling that \( |f_k(\tau)|^2 = \pi^2 a^2 P_T(k, \tau)/(4\ell_p^6 k^3) \). Equations (2.40) and (2.41) coincide (up to corrections \( O(\mathcal{H}^2/k^4) \)).
This means, physically, that the energy density of the relic gravitons is effectively the same no matter what choice of the energy–momentum pseudo-tensor is adopted but provided the wavelengths of the gravitons are all inside (i.e., shorter than) the Hubble radius. When the given wavelengths are larger than the Hubble radius (i.e., \( k \tau \ll 1 \)), \( g_k = \mathcal{H} f_k \) and the corresponding expressions for the spectral energy densities can easily be obtained. In summary, for modes which are inside the Hubble radius the energy density of the relic gravitons can be expressed in terms of the power spectrum as

\[
\rho_{GW}(\tau) = \frac{2}{a^4} \int \frac{d\ln k}{2\pi^2} \frac{k^2 |f_k|^2}{H^2 a^2} = \frac{1}{4\pi^2 a^4} \int d\ln k k^2 P_T(k, \tau),
\]

and the critical fraction of relic gravitons at a given time as

\[
\Omega_{GW}(k, \tau) = \frac{1}{\rho_{\text{crit}}} \frac{d\rho_{GW}}{d\ln k} = \frac{k^2}{12H^2 a^2} \mathcal{T}(k, \tau).
\]

Specific examples of the numerical calculation of the spectral energy density will be given in subsection 2.4. The first example will be that of the radiation-matter transition, the second example will be that of the transition between a stiff phase and the radiation-dominated phase.

### 2.4. Transfer function for the spectral energy density: examples

The idea pursued in the present subsection is to use, as a pivot quantity for the numerical integration, not the power spectrum \( P(k, \tau) \) but rather the energy density itself. The evolution equations of the background geometry (i.e., equations (1.4)–(1.6)) and of the tensor mode functions (i.e., equation (2.37)) will be solved simultaneously and the energy density computed in one shot. This program will be illustrated in two simple examples, i.e. the radiation-matter transition and the case of a stiff background. It is useful to point out that equation (2.21) suggests that an appropriate variable for the numerical calculation is exactly \( x = k \tau \) whose definition we now repeat

\[
x = k \tau = \kappa \left( \frac{\tau}{\tau_{eq}} \right), \quad \kappa = \frac{k}{k_{eq}},
\]

(4.4)

It is both practical and physically sound to adopt \( x \) and \( \kappa \) as pivotal variables for the numerical integration around the radiation-matter transition. Indeed \( x \) is a smooth variable which interpolates between the sub-Hubble regime (where equations (2.40) and (2.41) are valid) and the super-Hubble regime where \( x > 1 \). The results of the numerical calculation are reported in figure 2 in terms of \( \Delta_{\rho}^{(1)}(x, \kappa) \) and in terms of the transfer function of the energy density (denoted by \( T_{\rho}(\kappa) \)). The quantities \( \Delta_{\rho}^{(1)}(x, \kappa) \) (and, analogously, \( \Delta_{\rho}^{(2)}(x, \kappa) \)) are nothing but

\[
\Delta_{\rho}^{(1)}(k, \tau) = [(g_k(\tau))^2 + (k^2 + \mathcal{H}^2)] |f_k(\tau)|^2 - \mathcal{H}[f_k^*(\tau)g_k(\tau) + f_k(\tau)g_k^*(\tau)],
\]

\[
\Delta_{\rho}^{(2)}(k, \tau) = [(g_k(\tau))^2 + (k^2 - 7\mathcal{H}^2)] |f_k(\tau)|^2 + 3\mathcal{H}[f_k^*(\tau)g_k(\tau) + f_k(\tau)g_k^*(\tau)].
\]

(4.5) and (4.6)

Equations (4.5) and (4.6) are simply related to the spectral energy densities in critical units, i.e.

\[
\Omega_{GW}^{(1)}(k, \tau) = \frac{k^3}{2\pi^2 a^4 \rho_{\text{crit}}} \Delta_{\rho}^{(1)}(k, \tau), \quad \Omega_{GW}^{(2)}(k, \tau) = \frac{k^3}{2\pi^2 a^4 \rho_{\text{crit}}} \Delta_{\rho}^{(2)}(k, \tau).
\]

(4.7)

As a function of \( x \) and \( k \Delta_{\rho}^{(1,2)}(x, \kappa) \) reaches a constant value when the relevant modes are evaluated deep inside the Hubble radius. The energy transfer function which is then defined as

\[
\lim_{x \to 1} \Delta_{\rho}^{(1,2)}(x, \kappa) \equiv T_{\rho}^{2}(\kappa) \Delta_{\rho}^{(1,2)}(x, x_1), \quad x_1 \ll 1.
\]

(4.8)
Figure 2. The functions given in equations (2.45) and (2.46) are numerically computed (plot on the left) for different values of $\kappa$ and in the case of the radiation-matter transition. In the plot on the right the transfer function for the energy density is illustrated.

Figure 3. The different definitions of energy–momentum pseudo-tensor (i.e., equations (2.45) and (2.46)) are compared in the determination of the asymptotic value of the energy transfer function.

The specific form of the energy–momentum tensor is immaterial for the determination of $T^2_\rho(\kappa)$: different forms of the energy–momentum tensor of the relic gravitons will lead to the same result. This occurrence can be appreciated from figure 3 where $\Delta^{(1,2)}(\kappa, x)$ has been reported for $\kappa = 10^{-2}$ (plot on the left) and for $\kappa = 10^{-4}$ (plot on the right). The dashed and the dot-dashed curves (in both plots) correspond, respectively, to $\Delta^{(1)}(\kappa, x)$ and to $\Delta^{(2)}(\kappa, x)$. The full line, in both plots, corresponds to the combination

$$k^2|f_k(\tau)|^2 + |g_k(\tau)|^2 = k(|c_+(k)|^2 + |c_-(k)|^2),$$

(2.49)

where $c_{\pm}(k)$ are the so-called mixing coefficients which parametrize, at a given time, the solution for the tensor mode functions when the relevant wavelengths are all inside the Hubble radius, i.e.

$$f_k(\tau) = \frac{1}{\sqrt{2k}}[c_+(k) e^{-i\kappa \tau} + c_-(k) e^{i\kappa \tau}], \quad g_k(\tau) = -i\sqrt{\frac{k}{2}}[c_+(k) e^{-i\kappa \tau} - c_-(k) e^{i\kappa \tau}],$$

(2.50)
where $\mathcal{F}_k(\tau)$ and $\bar{g}_k(\tau)$ are the solutions to leading order in the limit $k\tau \gg 1$. From equation (2.50), $c_{\pm}(k)$ are given by

$$
c_+(k) = \frac{e^{ik\tau}}{\sqrt{2k}}[k\mathcal{F}_k(\tau) + i\bar{g}_k(\tau)], \quad c_-(k) = \frac{e^{-ik\tau}}{\sqrt{2k}}[k\mathcal{F}_k(\tau) - i\bar{g}_k(\tau)].
$$

(2.51)

Using equations (2.50) and (2.51), equations (2.45) and (2.46) can be directly assessed in the limit $x = k\tau \gg 1$ with the result that

$$
\Delta_\rho^{(1)}(\kappa, x_f) = \Delta_\rho^{(2)}(\kappa, x_f) = \kappa(|c_+(\kappa)|^2 + |c_-(\kappa)|^2) + \mathcal{O}(\frac{1}{x_f}),
$$

(2.52)

which proves that the oscillating contributions are suppressed as $x_f^{-1}$ for $x_f \gg 1$.

To get to the results illustrated in figures 2 and 3 the evolution equations of the mode functions have been integrated by setting initial conditions deep outside the Hubble radius (i.e., $x = k\tau \ll 1$), by following the corresponding quantities through the Hubble crossing (i.e., $x \simeq 1$) and then, finally, deep inside the Hubble radius (i.e., $x \gg 1$). The initial value of the integration variable $x$ has been chosen to be $x_i = 10^{-3}$. The integration of the mode functions is most easily performed in terms of appropriately rescaled variables and since these rescalings are rather obvious, the relevant details will be omitted.

In the plot on the right of figure 2, the fit to the energy transfer function is reported with the full (thin) line on top of the diamonds defining the numerical points. The analytical form of the fit can then be written as

$$
T_{\rho}(k/k_{eq}) = \sqrt{1 + c_2 \left(\frac{k_{eq}}{k}\right)^2 + b_2 \left(\frac{k_{eq}}{k}\right)^4}, \quad c_2 = 0.5238, \quad b_2 = 0.3537.
$$

(2.53)

Equation (2.53) permits the accurate evaluation of the spectral energy density of relic gravitons, for instance, in the minimal version of the $\Lambda$CDM paradigm.

Yet another relevant physical situation for the present considerations is that where the background geometry, after inflation, transits from a stiff epoch to the ordinary radiation-dominated epoch. In the primeval plasma, stiff phases can arise: this idea goes back to the pioneering suggestions of Zeldovich [50] in connection with the entropy problem. The approach of Zeldovich was revisited in [51–54] by supposing that the stiff phase would take place after the inflationary phase with the main purpose of identifying a potential source of high-frequency gravitons which could even be interesting for the LIGO/VIRGO detectors in one of their advanced versions.

At the end of the inflation, in a model-independent approach, it is plausible to think that the onset of the radiation dominance could be delayed. This may happen, in concrete models, for various reasons. One possibility could be that the inflation field does not decay but rather changes its dynamical nature by acting as a quintessence field [59] (see also [60]). In these kinds of situations the geometry passes from a stiff phase where

$$w_1(\tau) = \frac{p_1}{\rho_1} > \frac{1}{3},
$$

(2.54)

$$c_a^2(\tau) = \frac{p_1'}{\rho_1'} = w_1 - \frac{w_1'}{3\mathcal{H}(w_1 + 1)} = w_1 - \frac{1}{3} \left(\frac{d}{a}\right)' > \frac{1}{3},
$$

(2.55)

to a radiation-dominated phase where $c_a = 1/\sqrt{3}$. According to equations (2.54) and (2.55), $c_a^2 = w_1$ if the (total) barotropic index is constant in time. In the limiting case $w_1 = 1 = c_a^2$ and the speed of sound coincides with the speed of light. As argued in [57], barotropic indices $w_1 > 1$ would not be compatible with causality (see, however, [58]). As in the case of the
matter-radiation transition the transfer function only depends upon $\kappa$ which is defined, this time, as $\kappa = k/k_s$, where $k_s = \tau_s^{-1}$ and $\tau_s$ parametrizes the transition time. A simple analytical form of the transition regime is given by

$$a(y) = a_s \sqrt{y^2 + 2y}, \quad y = \frac{\tau}{\tau_s}, \quad \tau_s = \frac{1}{a_i H_i} \sqrt{\frac{\rho_{si}}{\rho_{Ri}}},$$

where, by definition, $\rho_{si} = \rho_i(\tau_i)$ and $\rho_{Ri} = \rho_R(\tau_i)$. Equation (2.56) is a solution of equations (1.4)–(1.6) when the radiation is present together with a stiff component which has, in the case of equation (2.56), a sound speed which equals the speed of light. In the limit $y \to 0$ the scale factor expands as $a(y) \sim \sqrt{2y}$ while, in the opposite limit, $a(y) \sim y$. In figure 4 (plot on the left) $\Delta_\rho(\kappa, x)$ is illustrated for different values of $\kappa$. We shall not dwell here (again) on the possible different forms of the energy–momentum pseudo-tensor. The bottom line will always be that, provided the energy density is evaluated deep inside the Hubble radius, the different approaches to the energy density of the relic gravitons give the same result. From the numerical points reported in figure 4 (plot on the right) the semi-analytical form of the transfer function becomes, this time,

$$T^2_\rho(k/k_s) = 1 + 0.204 \left( \frac{k}{k_s} \right)^{1/4} - 0.980 \left( \frac{k}{k_s} \right)^{1/2} + 3.389 \left( \frac{k}{k_s} \right)^{1/2} - 0.067 \left( \frac{k}{k_s} \right)^{1/4} \ln^2 \left( \frac{k}{k_s} \right),$$

where $k_s = \tau_s^{-1}$. The value of $k_s$ can either be computed in an explicit model or it can be left as a free parameter. In section 3 both strategies will be explored by privileging, however, a model-independent approach. Taking into account that the energy density of the inflation will be exactly $\rho_{ei} \simeq H^2_i M^2_P$, the value of $k_s$ (as well as the duration of the stiff phase) will be determined, grossly speaking, by $H_i / M_P$.

### 2.5. Analytical estimates of the mixing coefficients

To obtain a fit of the transfer function for the spectral energy density it is useful to be aware of the analytical results which should always be reproduced by the numerical analysis when

---

**Figure 4.** The transition between the stiff phase and the radiation phase is illustrated. The energy transfer function increases with the frequency while the opposite is true for the radiation-matter transition (see figure 2).
\(\kappa\) is sufficiently larger than 1. This is the purpose of the present subsection where it will be shown that the semi-analytical results are consistent with the numerical evaluations which are, however, intrinsically more accurate.

Consider the transition from a generic accelerated phase to a decelerated stage of expansion. In this situation, by naming the transition point \(-\tau_1\), the continuous and differentiable forms of the scale factors can be written as

\[
a_i(\tau) = \left(-\frac{\tau}{\tau_1}\right)^{-\beta}, \quad \tau < -\tau_1, \tag{2.58}
\]

\[
a_i(\tau) = \left[\frac{\beta}{\alpha}\left(\frac{\tau}{\tau_1} + 1\right) + 1\right]^a, \quad \tau \geq -\tau_1, \tag{2.59}
\]

where the scale factors are continuous and differentiable at the transition point which has generically been indicated as \(\tau_1\). The pump fields of the tensor mode functions turn out to be

\[
a_i'' = \frac{\beta(\beta + 1)}{\tau^2}, \quad a_i'' = \frac{\alpha(\alpha - 1)}{\left[\tau + (\frac{\nu}{\beta} + 1)\tau_1\right]^2}. \tag{2.60}
\]

The solution of equation (2.37) can then be written as

\[
f_i(\tau) = \frac{N}{\sqrt{2k}} \sqrt{-1} H^{(1)}_\nu(-x), \quad \tau < -\tau_1, \quad x = k\tau, \tag{2.61}
\]

\[
f_i(\tau) = \frac{N}{\sqrt{2k}} \left[M c_+(k) H^{(2)}_\nu(y) + M c_-(k) H^{(2)}_\nu(y)\right], \quad \tau \geq -\tau_1,
\]

where \(y = k\tau + k\tau_1(1 + \frac{x}{\beta})\) and where

\[
N = \sqrt{\frac{\pi}{2}} e^{i(x+1/2)\pi/2}, \quad M = \sqrt{\frac{\pi}{2}} e^{-i(x+1/2)\pi/2}. \tag{2.62}
\]

The continuity of the tensor mode functions at the transition point (i.e., \(f_i(-\tau_1) = \tilde{f}_i(-\tau_1)\)) and \(g_i(-\tau_1) = \tilde{g}_i(-\tau_1)\) implies that the mixing coefficients are given by

\[
c_+(k) = \frac{i\tau}{8\sqrt{a}\beta} e^{i\tau(\nu+1)/2} \left\{[\beta(2\lambda + 1) + \alpha(2v + 1)]H^{(1)}_\nu(x_1)H^{(1)}_\nu(y_1)\right.
\]

\[
- 2\alpha x_1 \left[H^{(1)}_\nu(y_1)H^{(1)}_\nu(x_1) + H^{(1)}_\nu(x_1)H^{(1)}_\nu(y_1)\right] \right\},
\]

\[
c_-(k) = \frac{i\tau}{8\sqrt{a}\beta} e^{i\tau(\nu-1)/2} \left\{[\beta(2\lambda + 1) + \alpha(2v + 1)]H^{(1)}_\nu(x_1)H^{(1)}_\nu(y_1)\right.
\]

\[
- 2\alpha x_1 \left[H^{(2)}_\nu(y_1)H^{(2)}_\nu(x_1) + H^{(2)}_\nu(x_1)H^{(2)}_\nu(y_1)\right] \right\}, \tag{2.63}
\]

where, according to the notation previously established, \(y_1 = y(-\tau_1) = (\alpha/\beta)x_1\). The case \(\alpha = \beta = 1\) corresponds to a transition from the inflationary phase to a radiation-dominated phase. In this case we do know which are the mixing coefficients. The previous expressions give us

\[
c_-(k) = \frac{e^{2\pi x_1}}{2\chi^4}, \quad c_+(k) = \left(1 - \frac{1}{2\chi^4} + \frac{i}{\chi^4}\right), \tag{2.64}
\]

which clearly agree with previous results [62, 63]. In the case of equation (2.64) \(|c_+(k)|^2 - |c_-(k)|^2 = 1\) and \(k^4|c_-(k)|^2\) is exactly scale invariant. Another interesting situation is that of the transition from inflation to stiff, i.e. \(\beta = 1, \alpha = 1/2, y_1 = x_1/2\) which leads
coefficients can be written as

\[ c_-(k) = \frac{\sqrt{\pi}}{2} \frac{(i - 1)}{4x_1^{3/2}} e^{i\pi/4} \left\{ \sqrt{2} e^{-i\pi/2} [x_1^2 + 6ix_1 - 12] H_0^{(2)}(x_1/2) + (i + x_1) \left[ i\pi H_0^{(2)}(x_1/2) - 3i H_1^{(2)}(x_1/2) \right] \right\}, \tag{2.65} \]

\[ c_+(k) = \frac{\sqrt{\pi}}{2} \frac{(i + 1)}{4x_1^{3/2}} e^{i\pi/4} \left\{ x_1 H_0^{(1)}(x_1/2) + i(x_1 + 1) H_1^{(1)}(x_1/2) \right\}. \tag{2.66} \]

The above result can be expanded in for \( x_1 \ll 1 \) and the result is

\[ c_+(k) = -0.398(1 - i) x_1^{-1/2} + \sqrt{x_1} \left( 0.131 + 0.338i \right) - 0.149(1 - i) \ln x_1 + \mathcal{O}(x_1^{3/2}), \tag{2.67} \]

\[ c_-(k) = \frac{7.031 - 1.723i}{x_1^{1/2}} - 16.68(1 + i) \ln x_1 x_1^{-1/2} + \sqrt{x_1} \left( -0.621 + 0.265i \right) + 0.282(1 + i) \ln x_1 + \mathcal{O}(x_1^{3/2}). \tag{2.68} \]

The logarithms arising in equations (2.67) and (2.68) explain why, in equation (2.57), the transfer function of the spectral energy density contains logarithms. In spite of the fact that semi-analytical estimates can pin down the slope of the transfer functions in different intervals, they are insufficient for a faithful account of more realistic situations where the slow-roll corrections are relevant and when other dissipative effects (such as neutrino free streaming) are taken into account.

### 2.6. Exponential damping of the mixing coefficients

In a model-independent perspective, it can be argued that the relic gravitons are also characterized by a maximal frequency which is related to the modes which are maximally amplified. Let us consider, for instance, the case of the ΛCDM paradigm where the inflationary phase is almost suddenly followed by the radiation-dominated phase. By denoting the transition time as \( \tau_1 \), it is plausible to think that all the modes of the field such that \( k > a(t) H_i \simeq \tau_1^{-1} \) are exponentially suppressed \([64, 65]\). For the modes \( k \tau_1 > 1 \), the pumping action of the gravitational field is practically absent. The wave number \( k_{\text{max}} \) (which is related to the maximal frequency introduced in equation (1.9)) is the maximally amplified wave number which can be determined by requiring \( k \simeq \tau_1^{-1} \):

\[ k_{\text{max}} = 3.5661 \times 10^{22} \left( \frac{\epsilon}{0.01} \right)^{1/4} \left( \frac{A_s}{2.41 \times 10^{-9}} \right)^{1/4} \left( \frac{h_0^2 \Omega_{\text{CDM}}}{4.15 \times 10^{-5}} \right)^{1/4} \text{Mpc}^{-1}, \tag{2.69} \]

where the typical values of the slow-roll parameter have been derived by taking into account that, in the absence of running of the tensor spectral index, \( r_T = 16\epsilon \); since, according to the WMAP 5 year data alone, \( r_T < 0.43, \epsilon < 0.01 \). Note that \( v_{\text{max}} = 2\pi \epsilon k_{\text{max}} = 117.45 \times (\pi \epsilon A_s)^{1/4} \text{GHz} \) for the same typical values of the ΛCDM parameters. For phenomenological purposes it can be also interesting to know what kind of exponential suppression we can expect. From the analysis of various transitions it emerges that the mixing coefficients for \( k > k_{\text{max}} \) (or \( \nu > v_{\text{max}} \)) will satisfy

\[ |c_+(k)|^2 - |c_-|^2 = 1, \quad |c_+(k)|^2 + |c_-|^2 = e^{-2\beta k_{\text{max}}} + 1. \tag{2.70} \]
From equation (2.70) we can easily argue that, for $k > k_{\text{max}}$, $|c_+(k)| \to 1$ and $|c_-(k)| \simeq 2^{-1/2} \exp \left[-\beta k/k_{\text{max}}\right]$. The point is then to estimate the value of $\beta$ which depends on the nature of the transition regime. Typically, however, $\beta > 2$ for sufficiently smooth transitions. To justify this statement it is interesting to consider the following toy model where the scale factor interpolates between a quasi-de Sitter phase and a radiation-dominated phase:

$$a(\tau) = a_i \left[ \tau + \sqrt{\tau^2 + \tau_i^2} \right]. \quad (2.71)$$

For $\tau \to -\infty$ (i.e., $\tau \ll -\tau_i$), $a(\tau) \simeq -a_i/\tau$ and the quasi-de Sitter dynamics is recovered. In the opposite limit (i.e., $\tau \gg +\tau_i$), $a(\tau) \simeq a_i \tau$ and the radiation dominance is recovered. In figure 5 (plot on the left) the exponential damping of the mixing coefficients is numerically illustrated. The curve at the top (full line) illustrates the case $\kappa = 1$. The cases $\kappa = 2$ and $\kappa = 3$ are barely distinguishable at the bottom of the plot. Note, always in the right plot, the rather narrow range of times which are reported in a linear scale. In the plot on the right the asymptotic values of the mixing coefficients are reported for different values of $\kappa = k/k_{\text{max}}$. By fitting the numerical data with an equation of the form given in equation (2.70), the value of $\beta = 6.33$. Different examples can be presented on the same line of that discussed in figure 5. While it is clear, from the numerical data, that the decay is indeed exponential, the value of $\beta$ may well vary for different models of the transition. The latter observation is effectively equivalent to a rescaling of $k_{\text{max}}$ for different models of inflation–radiation transition. By positing, for instance that $k_{\text{max}} \to \tilde{k}_{\text{max}}/\beta$ we will have a new $\tilde{k}_{\text{max}}$ which differs slightly from $k_{\text{max}}$. It is clear that this indetermination on the maximal frequency of the relic graviton spectrum can only be solved by endorsing a given model (i.e., by theoretical prejudice) or by having direct measurements at those frequencies (which seems to be unlikely in the near future).

3. Nearly scale-invariant spectra

The transfer function of the spectral energy density has been numerically computed in the previous section and the numerical results have been corroborated by appropriate semi-analytical estimates. We are then ready for an explicit calculation of the spectral energy density in the $\Lambda$CDM scenario. In subsection 3.1, the spectral energy density will be computed in
terms of the amplitude transfer function and also directly in terms of the transfer function for the spectral energy density. Explicit calculations will show that the latter method is more accurate. Subsection 3.2 is devoted to various late time effects (e.g., neutrino anisotropic stress, late dominance of dark energy, progressive diminishment of the number of relativistic species) which are certainly present and which affect the amplitude of the spectral energy density. Finally, in subsection 3.3 current (and foreseen) sensitivities of wideband interferometers will be briefly compared to the (unfortunately minute) $\Lambda$CDM signal.

3.1. Spectral energy density in the $\Lambda$CDM paradigm

In figure 6, $h_0^2\Omega_{GW}(v, \tau_0)$ and the strain amplitude $S_h(v, \tau_0)$ are computed using the transfer function for the amplitude discussed (and rederived) in equations (2.28), (2.29) and (2.30). The strain amplitude appearing in the plot on the right of figure 6 is related to the spectral energy density as in equation (2.16) which also implies that $\Omega_{GW}(v, \tau_0)$ can be expressed in terms of $S_h(v, \tau_0)$. Indeed, according to equation (2.16), $P_T(k, \tau) = 4vS_h(v, \tau)$ and the spectral energy density becomes

$$\Omega_{GW}(v, \tau) = \frac{4\pi^2}{3H^2}v^3S_h(v, \tau), \quad (3.1)$$

where, in natural units, $k = 2\pi v$. By making more explicit the numerical factors and by inverting equation (3.1) in terms of $S_h(v, \tau_0)$ we obtain

$$S_h(v, \tau_0) = 7.981 \times 10^{-43} \left(\frac{100 \text{ Hz}}{v}\right)^3 h_0^2\Omega_{GW}(v, \tau_0) \text{ Hz}^{-1}, \quad (3.2)$$

where $H_0 = 3.24078 \times 10^{-18}h_0 \text{ Hz}$. The oscillations of figure 6 are related to the way the transfer function for the amplitude is derived. There are complementary forms of the strategy leading to the results of figure 6 (see, for instance, [66]). The plots of figure 6 have been obtained by using directly equations (2.28)–(2.30) inside equation (2.43). This is the procedure used, originally, in [41] (see also [67]). One could also define the transfer function as in equation (2.29) and then, at the level of the spectral energy density, compute

![Figure 6. The spectral energy density of relic gravitons in critical units (plot on the left). The strain amplitude is instead reported in the plot on the right. Note that while $h_0^2\Omega_{GW}(v, \tau_0)$ is dimensionless, $S_h(v, \tau_0)$ has dimensions of Hz$^{-1}$. The fiducial set of parameters used corresponds to the best fit to the WMAP 5 year data.](image-url)
\( g_\ell(\tau) = f'_k(\tau) \) (see appendix E of [66]). By working with the transfer function of the tensor amplitude the spectral energy density for frequencies \( \nu \gg \nu_{eq} \) is given by

\[
h_0^2 \Omega_{GW}(\nu, \tau_0) = N_h r_T \left( \frac{\nu}{v_p} \right)^{n_T} e^{-2\beta \ln \nu},
\]

\[
N_h = 7.992 \times 10^{-15} \left( \frac{h_0^2 \Omega_{M0}}{0.1326} \right)^{-2} \left( \frac{h_0^2 \Omega_{R0}}{4.15 \times 10^{-5}} \right) \left( \frac{d_A}{1.4115 \times 10^4 \text{Mpc}} \right)^{-4},
\]

where \( d_A(z_*) \) is the (comoving) angular diameter distance to decoupling. The dependence upon \( d_A(z_*) \) arises because we have to estimate \( \tau_0 \). In equations (3.3) and (3.4) (as well as in the program used for the numerical calculations) there are two complementary options: the first one is to give the angular diameter distance to decoupling (which is directly inferred from the CMB data). In the case of the 5 year WMAP data alone, \( d_A(z_*) = 14 115 \text{Mpc}^{\pm 188} \); this approach has been followed, for instance, in [67]. In a complementary perspective, it is also possible to take the best fit value of the total matter fraction \( \Omega_{M0} = 0.258 \) for the case of the WMAP 5 year data alone) and compute the comoving angular diameter distance according to the well-known expression for spatially flat universes:

\[
d_A(z_*) = \frac{1}{H_0} \int_{z_*}^{1} \frac{dz}{\sqrt{\Omega_{M0}(1+z)^3 + \Omega_{\Lambda} + \Omega_{R0}(1+z)^3}} = \frac{3.375}{H_0} = 14072 \text{Mpc},
\]

(3.5)

The latter strategy has been used, for instance, in [66]. The two strategies are compatible and, moreover, this explains why, in equation (3.4) the dependence upon \( \Omega_{M0} \) does not cancel. In equation (3.3) \( n_T \) denotes, as usual, the tensor spectral index which can also be written as

\[
n_T = -2\epsilon + \frac{\alpha_T}{2} \ln \left( \frac{k}{k_p} \right), \quad \alpha_T = \frac{r_T}{8} \left( (n_s - 1) + \frac{r_T}{8} \right),
\]

(3.6)

If \( \alpha_T = 0 \) equation (2.23) is recovered and the spectral index is independent on the frequency. In the case when \( \alpha_T \neq 0 \) and it is given by equation (3.6) the spectral index does depend upon the frequency: in the jargon this is often dubbed by saying that the spectral index runs. The frequency-dependent correction (i.e., \( \alpha_T \)) contains the scalar spectral index \( n_s \) and this is why the value of \( n_s \) is mentioned in the parameters of figure 6. The last remark concerning the result of equation (3.4) is that, in the limit, \( \nu \gg \nu_{eq} \), the oscillating terms have been appropriately averaged: this is done by setting the terms going as \( \cos^2 \left( 2\pi \nu \tau_0 \right) \) to \( 1/2 \). The latter procedure has been employed, for instance, in the analyses of [67, 70]. This procedure is justified in semi-analytical terms but rather odd in a fully numerical context. In what follows it will be argued that there is no need for these types of tricks if the spectral energy density is computed directly without passing through the transfer function of the tensor amplitude.

Along this perspective, the results of figure 6 should then be compared with figure 7 where the transfer function for the spectral energy density has been consistently employed. In figure 7 the spectral energy density of the relic gravitons as well as \( S_h(\nu, \tau_0) \) are reported for different values of \( r_T \) and for the same fiducial set of parameters used in figure 6. The first salient feature emerging from the comparison of figures 6 and 7 is that the oscillatory behavior disappears. The spectra of figure 7 have been obtained from the direct integration of the mode functions but can be parametrized, according to equation (2.53) as
By comparing equations (3.3) and (3.4) to equations (3.7) and (3.8), the amplitude for $\nu \gg \nu_{eq}$ differs by a factor which is roughly a factor 2. This occurrence is not surprising since equations (3.3) and (3.4) have been obtained by averaging over the oscillations (i.e., by replacing cosine squared with $1/2$) and by imposing that $|g_k|=k|f_k|$. In figure 7 the impact of the variation of $n_T$ is also illustrated. Recalling that the WMAP 5 year data alone suggest $r_T < 0.4$, the variation of the spectral energy density is more pronounced than the change of the strain power spectrum. This is because of the steepness of $S_h(\nu, \tau_0)$ in frequency.

### 3.2. Anisotropic stress and dark-energy contribution

The considerations of the previous subsection suggest that the results obtainable with the transfer function of the spectral energy density seem to be intrinsically more accurate. The obvious question is of course if we need this precision. There are two answers to these kinds of questions. The first one is that, of course, the accuracy in the estimate of the $\Lambda$CDM plateau is necessary for comparing the theoretical predictions with the data. Therefore it would be strange to treat very accurately the tensor contribution to CMB anisotropies but not to wideband detectors. The second issue is more theoretical. In the recent past the community investigated various late time effects which can modify the $\Lambda$CDM plateau for $\nu \gg \nu_{eq}$. All these effects compete with the accuracy which is inherent in the estimate of the transfer function. This will be the subject of the present subsection.

Let us therefore start by noting that, so far, the evolution of the tensor modes has been treated as if the anisotropic stress of the fluid was absent. After neutrino decoupling, the neutrinos free stream and the effective energy–momentum tensor acquires, to first order in the amplitude of the plasma fluctuations, an anisotropic stress, i.e.

$$\delta T^i_j = -\delta p \delta^i_j + \Pi^i_j,$$

$$\partial_i \Pi^i_j = 0.$$  

(3.9)
The presence of the anisotropic stress clearly affects the evolution of the tensor modes whose evolution is then dictated by

\[ h_{ij}'' + 2\dot{h}_{ij}' - \nabla^2 h_{ij} = -16\pi G a^2 \Pi_{ij}. \]  

Equation (3.10) reduces to an integro-differential equation which has been analyzed in [68] (see also [69–71]). The overall effect of collisionless particles is a reduction of the spectral energy density of the relic gravitons. Assuming that the only collisionless species in the thermal history of the universe are the neutrinos, the amount of suppression can be parametrized by the function

\[ F(R_\nu) = 1 - 0.539R_\nu + 0.134R^2_\nu, \]  

where \( R_\nu \) is the fraction of neutrinos in the radiation plasma, i.e.

\[ R_\nu = \frac{r}{r+1}, \quad r = 0.681 \left( \frac{N_\nu}{3} \right), \quad R_\gamma + R_\nu = 1. \]  

In equation (3.12) \( N_\nu \) represents the number of massless neutrino families. In the standard ΛCDM scenario the neutrinos are taken to be massless. In the case \( R_\nu = 0 \) (i.e., in the absence of collisionless particles) there is no suppression. If, in contrast, \( R_\nu \neq 0 \) the suppression can even reach one order of magnitude. In the case \( N_\nu = 3, R_\nu = 0.405 \) and the suppression of the spectral energy density is proportional to \( F^2(0.405) = 0.645 \). This suppression will be effective for relatively small frequencies which are larger than \( \nu_{\text{eq}} \) and smaller than the frequency corresponding to the Hubble radius at the time of big-bang nucleosynthesis, i.e. \( \nu_{\text{bbn}} \) of equation (1.8).

The effect of neutrino free streaming has been included in figure 8 together with the damping effect associated with the (present) dominance of the dark energy component. The redshift of Λ-dominance is defined as

\[ 1 + z_\Lambda = \left( \frac{a_0}{a_\Lambda} \right) = \left( \frac{\Omega_{\Lambda}}{\Omega_{M0}} \right)^{1/3}. \]  

Consider now the mode which will be denoted as \( k_\Lambda \), i.e. the mode re-entering the Hubble radius at \( \tau_\Lambda \). By definition \( k_\Lambda = H_\Lambda a_\Lambda \) must hold. But for \( \tau > \tau_\Lambda \) is constant, i.e. \( H_\Lambda = H_0 \)
where \( H_0 \) is the present value of the Hubble rate. Using now equation (3.13), it can easily be shown that \( k_\Lambda = (\Omega_M / \Omega_\Lambda)^{1/3} k_H \) where \( k_H = a_0 H_0 \). The frequency interval between \( \nu_H \) and \( \nu_\Lambda \) is rather tiny. Indeed, it turns out that \( \nu_\Lambda = k_\Lambda / (2 \pi) \) is given by

\[
\nu_\Lambda = 2.607 \times 10^{-19} \left( \frac{H_0}{0.719} \right) \left( \frac{\Omega_M M_0}{0.258} \right)^{1/3} \left( \frac{\Omega_\Lambda}{0.742} \right)^{1/3} \text{ Hz.} \tag{3.14}
\]

For the same choice of parameters of equation (3.14), \( \nu_H = H_0 / (2 \pi) = 3.708 \times 10^{-19} \text{ Hz} \) which is not so different than \( \nu_\Lambda = 2.607 \times 10^{-19} \text{ Hz} \). The adiabatic damping of the mode function across \( \tau_\Lambda \) reduces the amplitude of the spectral energy density by a factor \( (\Omega_M / \Omega_\Lambda)^2 \). For the typical choice of parameters of equation (3.14) we have that the suppression is of the order of 0.12. This class of effects has been repeatedly in a number of recent papers [72, 73]. The essence of the effect is captured by the following observation. Consider a mode \( k \) which re-enters before \( \tau_\Lambda \). The present value of the amplitude \( F_k(\tau) = f_k(\tau) / a(\tau) \) will be adiabatically suppressed since, as repeatedly stressed, in this regime \( f_k(\tau) \) will simply be plane waves. Consequently, defining as \( \tilde{f}_k \) the amplitude at \( k_\Lambda = H_0 a_\Lambda \) when the given mode crosses the Hubble radius, we will also have that

\[
F_k(\tau_0) = \left( \frac{a_k}{a_\Lambda} \right)_{\text{mat}} \left( \frac{a_\Lambda}{a_0} \right) \tilde{f}_k \equiv \left( \frac{k}{k_H} \right)^{-2} \left( \frac{\Omega_M M_0}{\Omega_\Lambda} \right) \tilde{f}_k, \tag{3.15}
\]

where the subscripts (in the first equality) denote the time range over which the corresponding redshift is computed, i.e. either matter-dominated or \( \Lambda \)-dominated stages. The second equality follows from the first one by appreciating that \( a(k_\Lambda) \simeq \tau_\Lambda^2 \simeq k^{-2} \) and by using equation (3.13). Equation (3.15) implies, immediately, that the spectral energy density of relic gravitons is corrected in two different fashions. For \( \nu < \nu_H \) the frequency dependence will be different and will be proportional to \( \Omega_{GW}(\nu, \tau_0) \propto (\nu/\nu_H)^{\nu_H-2} (\Omega_M M_0 / \Omega_\Lambda)^2 \). Conversely, in the range \( \nu > \nu_H \) the frequency dependence will be exactly that already computed but, overall, the amplitude will be smaller by a factor \( (\Omega_M M_0 / \Omega_\Lambda)^2 \). Two comments are in order. The modification of the frequency dependence is only effective between\(^9\) 0.36 aHz and 0.26 aHz: this effect is therefore unimportant and customarily ignored (see, for instance, [67, 72]) for phenomenological purposes. On the other hand, the overall suppression going as \( (\Omega_M M_0 / \Omega_\Lambda)^2 \) must be taken properly into account on the same footing of other sources of suppression of the spectral energy density.

There is, in principle, a third effect which may arise and it has to do with the variation of the effective number of relativistic species. The total energy density and the total entropy density of the plasma can be written as

\[
\rho_\Lambda = g_\rho(T) \frac{\pi^2}{30} T^4, \quad s_\Lambda = g_s(T) \frac{2\pi^2}{45} T^3. \tag{3.16}
\]

For temperatures much larger than the top quark mass, all the known species of the minimal standard model of particle interactions are in local thermal equilibrium, then \( g_\rho = g_s = 106.75 \). Below, \( T \simeq 175 \text{ GeV} \) the various species start decoupling, the notion of thermal equilibrium is replaced by the notion of kinetic equilibrium and the time evolution of the number of relativistic degrees of freedom effectively changes the evolution of the Hubble rate. In principle if a given mode \( k \) re-enters the Hubble radius at a temperature \( T_k \) the spectral energy density of the relic gravitons is (kinematically) suppressed by a factor which can be written as (see, for instance, [72])

\[
\left( \frac{g_\rho(T_k)}{g_\rho(0)} \right) \left( \frac{g_s(T_k)}{g_s(0)} \right)^{-4/3}. \tag{3.17}
\]

\(^9\) We are enforcing here the usual terminology stemming from the powers of 10: aHz (for atto Hz i.e. \( 10^{-18} \text{ Hz} \)), fHz (for femto Hz, i.e. \( 10^{-15} \text{ Hz} \)) and so on.
At the present time \( g_{\rho0} = 3.36 \) and \( g_{s0} = 3.90 \). In general terms the effect parametrized by equation (3.17) will cause a frequency-dependent suppression, i.e. a further modulation of the spectral energy density \( \Omega_{GW}(v, \tau_0) \). The maximal suppression one can expect can be obtained by inserting into equation (3.17) the highest possible number of degrees of freedom. So, in the case of the minimal standard model this would imply that the suppression (on \( \Omega_{GW}(v, \tau_0) \)) will be of the order of 0.38. In popular supersymmetric extensions of the minimal standard models \( g_{\rho} \) and \( g_{s} \) can be as high as, approximately, 230. This will bring down the figure given above to 0.29.

All the effects estimated in the last part of the present section (i.e., free streaming, dark energy, evolution of relativistic degrees of freedom) have common features. Both in the case of the neutrinos and in the case of the evolution of the relativistic degrees of freedom the potential impact of the effect could be larger. For instance, suppose that, in the early universe, the particle model has many more degrees of freedom and many more particles which can free stream, at some epoch. At the same time we can say that all the aforementioned effects decrease rather than increasing the spectral energy density. Taken singularly, each of the effects will decrease \( \Omega_{GW} \) by less than one order of magnitude. The net result of the combined effects will then be, roughly, a suppression of \( \Omega_{GW}(v, \tau_0) \) which is of the order of \( 3 \times 10^{-2} \) (for \( 10^{-16} \text{ Hz} < v < 10^{-11} \text{ Hz} \)) and of the order of \( 4 \times 10^{-2} \) for \( v > 10^{-11} \text{ Hz} \). The impact of the various damping effects is self-evident by looking at figure 8. The effects of the neutrinos are visible in the intermediate region where the spectrum exhibits a shallow depression.

### 3.3. Sensitivity of wideband interferometers to \( \Lambda CDM \) signal

It is now interesting to compare the ideas discussed in the present section with the sensitivity of wideband interferometers. In figure 9 the spectral density of relic gravitons is reported, at a specific frequency, as a function of \( r_T \). The specific frequency at which \( \Omega_{GW}(v, \tau_0) \) is computed is given, as indicated by \( v_{LV} = 100 \text{ Hz} \). The subscript \( LV \) is a shorthand notation for LIGO/VIRGO. In figure 9 (plot on the left) the tensor spectral index is frequency independent.
(i.e., $\sigma_T = 0$ in equation (2.23)). In the same figure (plot on the right) $n_T$ is allowed to run and $\sigma_T$ is given in terms of the scalar spectral index $n_T$ as in equation (3.6).

It is the moment of comparing the theoretical signal with the current sensitivity of wideband interferometers. This figure can be assessed, for instance, from [35] (see also [34, 36]) where the current limits on the presence of an isotropic background of relic gravitons have been illustrated. According to the LIGO Collaboration (see equation (19) of [35]) the spectral energy density of a putative (isotropic) background of relic gravitons can be parametrized as:

$$\Omega_{GW}(\nu, \tau_0) = \Omega_{GW, \beta} \left(\frac{\nu}{100 \text{ Hz}}\right)^{\beta+3}.$$  

(3.18)

It is worth mentioning that the parametrization of equation (3.18) fits very well with figure 9 where the pivot frequency $\nu_{LV} = 100 \text{ Hz}$ coincides with the pivot frequency appearing in the parametrization (3.18). For the scale-invariant case (i.e., $\beta = -3$ in equation (3.18)) the LIGO Collaboration sets a 90% upper limit of $1.20 \times 10^{-4}$ on the amplitude appearing in equation (3.18), i.e., $\Omega_{GW, -3}$. Using different sets of data (see [34, 36]) the LIGO Collaboration manages to improve the bound even by a factor 2 getting down to $6.5 \times 10^{-5}$. Keeping an eye on figure 9 shows that the current LIGO sensitivity is still too small.

As far as the $\Lambda$CDM model is concerned, direct detection looks equally hopeless also for the advanced interferometers. In the case of an exactly scale invariant spectrum the correlation of the two (coaligned) LIGO detectors with central corner stations in Livingston (Louisiana) and in Hanford (Washington) might reach a sensitivity to a flat spectrum which is [54–56]

$$h_0^2 \Omega_{GW}(\nu_{LV}, \tau_0) \simeq 6.5 \times 10^{-11} \left(\frac{1 \text{ yr}}{T}\right)^{1/2} \text{SNR}^2, \quad \nu_{LV} = 0.1 \text{ kHz},$$  

(3.19)

where $T$ denotes the observation time and SNR is the signal-to-noise ratio. Equation (3.19) is in close agreement with the sensitivity of the advanced LIGO apparatus [1] to an exactly scale-invariant spectral energy density [92–94]. Equation (3.19) together with the plots of figure 9 suggests that the relic graviton background predicted by the $\Lambda$CDM paradigm is not directly observable by wideband interferometers in their advanced version. The minuteness of $h_0^2 \Omega_{GW}(\nu_{LV}, \tau_0)$ stems directly from the assumption that the inflationary phase is suddenly followed by the radiation-dominated phase.

4. Scaling violations at high frequencies

According to the results of the previous section, even in the future, the sensitivity of wideband interferometers will be insufficient to reach into the parameter space of the $\Lambda$CDM scenario. The accurate techniques introduced in the present paper seem therefore a bit pleonastic. In this section the opposite will be argued insofar as the spectral energy density of relic gravitons may well be increasing (rather than decreasing) as a function of the frequency $\nu$.

The late and early time effects conspire, in the $\Lambda$CDM paradigm to make the spectral energy density slightly decreasing at high frequencies (see, e.g., figure 8). Different thermal histories allow for scaling violations which may also go in the opposite direction and make the spectral energy density increasing (rather than decreasing as in the $\Lambda$CDM case) for typical frequencies larger than a pivotal frequency $\nu_s$ which is related to the total duration of the stiff phase. If the stiff phase takes place before BBN, then $\nu_s > 10^{-2} \text{ nHz}$. If the stiff phase takes

10 To be completely faithful with the LIGO parametrization the variable $\beta$ will not be changed. It should be borne in mind, however, that $\beta$ is used, in the present paper, to quantify the theoretical error on the maximal frequency of the relic graviton spectrum (see, e.g., equation (2.70) and discussion therein).
place for equivalent temperatures larger than 100 GeV, then \( v_s \gg \mu \text{Hz} \). Finally, if the stiff phase takes place for \( T \gg 100 \text{ TeV} \), then \( v_s \gg \text{mHz} \).

In the early universe, the dominant energy condition might be violated and this observation will also produce scaling violations in the spectral energy density \([74]\). If we assume the validity of the \( \Lambda \)CDM paradigm, a violation of the dominant energy condition implies that, during an early stage of the life of the universe, the effective enthalpy density of the sources driving the geometry was negative and this may happen in the presence of bulk viscous stresses \([74]\) (see also \([75, 76]\) for interesting reprises of this idea). In what follows the focus will be on the more mundane possibility that the thermal history of the plasma includes a phase where the speed of sound was close to the speed of light. In equations \((2.54)\) and \((2.55)\) the stiff evolution has been parametrized in terms of an effective (i.e., fluid) description which can be realized in diverse models not necessarily related to a fluid behavior. If the energy–momentum tensor of the sources of the geometry is provided by a scalar degree of freedom (be it for instance \( \phi \)) the effective energy density, pressure and anisotropic stress of \( \phi \) will then be, respectively,

\[
\rho_{\phi} = \left( \frac{\dot{\phi}^2}{2} + V \right) + \frac{1}{2\alpha^2} (\partial_k \phi)^2, \quad p_{\phi} = \left( \frac{\dot{\phi}^2}{2} - V \right) - \frac{1}{6\alpha^2} (\partial_k \phi)^2, \quad \Pi^i_j(\phi) = -\frac{1}{a^2} \left[ \partial_i \phi \partial^j \phi - \frac{1}{3} (\partial_k \phi)^2 \delta^j_i \right].
\]

Equations \((4.1)\) imply that the effective barotropic index for the scalar system under discussion is simply given by

\[
w_\phi = \frac{p_{\phi}}{\rho_{\phi}} = \left( \frac{\dot{\phi}^2}{2} - V \right) - \frac{1}{2\alpha^2} (\partial_k \phi)^2.
\]

If \( \dot{\phi}^2 \gg V \) and \( \dot{\phi}^2 \gg (\partial_k \phi)^2/a^2 \), then \( w_\phi \equiv \rho_\phi \): in this regime the scalar field behaves as a stiff fluid. If \( V \gg \dot{\phi}^2 \gg (\partial_k \phi)^2/a^2 \), then \( w_\phi \approx -1 \): in this regime the scalar field is an inflation candidate. Finally if \( (\partial_k \phi)^2/a^2 \gg \dot{\phi}^2 \) and \( (\partial_k \phi)^2/a^2 \gg V \), then \( w_\phi \approx -1/3 \): in this regime the system is gradient dominated. Of course also intermediate situations are possible (or plausible).

From the purely phenomenological point of view it is not forbidden (by any phenomenological consideration) to have a sufficiently long stiff phase. This was the point of view invoked in \([51]\) (see also \([52, 53]\) where it was also suggested that the spectral energy density of relic gravitons may increase with frequency. The presence of a phase dominated in diverse models not necessarily related to a fluid behavior. If the energy–momentum tensor of the sources of the geometry is provided by a scalar degree of freedom (be it for instance \( \phi \)).

The maximal wave number of the spectrum will be given by

\[
k_{\text{max}} = M_P \left( \frac{H}{M_P} \right)^{1-a} \left( \frac{H_{\text{eq}}}{M_P} \right)^{a-1/2} \left( \frac{\rho_{\text{eq}}}{a_0} \right)^{1/2} \left( \dot{a}_0 \right)^{1/2}.
\]

where \( \alpha = 2/[3(w_t + 1)] \) is related to the specific kind of stiff dynamics (indeed, \( w_t > 1/3 \)). Equation \((4.4)\) can also be written as

\[
k_{\text{max}} = M_P \Sigma^{-1/2} \left( \frac{H_{\text{eq}}}{M_P} \right) \left( \frac{\dot{a}_0}{a_0} \right), \quad \nu_M = k_M/(2\pi),
\]

where

\[
\Sigma = \left( \frac{H}{M_P} \right)^{2-\alpha} \left( \frac{H_t}{M_P} \right)^{1-2/\alpha}.
\]
In the case $\Sigma = \mathcal{O}(1)$ (as it happens in the case $\alpha = 1/3$ if the initial radiation is in the form of quantum fluctuations) $v_M \simeq 100\text{ GHz}$, more precisely:

$$v_{\text{max}} = 1.177 \times 10^{11} \Sigma^{-1} \left( \frac{h_0^2 \Omega_{R0}}{4.15 \times 10^{-9}} \right)^{1/4} \text{ Hz.} \quad (4.7)$$

The strategy will now be to parametrize the violation of scale invariance in terms of the least possible number of parameters, i.e the frequency $v_s$ (defining the region of the spectrum at which the scaling violations take place) and the slope of the spectrum arising during the stiff phase. Of course the frequency $v_s$ can be dynamically related to the frequency of the maximum and, consequently, the first parameter can be traded for $\Sigma$. The slope of the spectrum during the stiff phase depends upon the total barotropic index and can therefore be traded for $w_t$. Assuming the presence of a single stiff (post-inflationary) phase we will have that

$$k_s = M_p \sqrt{\frac{h_0^2 \Omega_{R0}}{4.15 \times 10^{-9}}} \left( \frac{a_0}{a_0} \right)^{1/2} \sqrt{\frac{H_t}{M_p}} = M_p \left( \frac{H_t}{M_p} \right)^{1/2} \left( \frac{a_0}{a_0} \right)^{1/2} \Sigma^{1/(1-2\alpha)} \left( \frac{H}{M_p} \right)^{(\alpha-1)/(2\alpha-1)}, \quad (4.8)$$

where the second equality follows from the first by using the relation of $H_t$ to $\Sigma$ dictated by equation (4.8). From equation (4.8) the frequency turns out to be:

$$v_s = 1.173 \times 10^{11} \Sigma^{1/(1-2\alpha)} (\pi \epsilon A_R)^{\frac{\alpha-1}{2\alpha}} \left( \frac{h_0^2 \Omega_{R0}}{4.15 \times 10^{-9}} \right)^{1/4} \text{ Hz.} \quad (4.9)$$

The quantity $\Sigma$ is always smaller than 1 or, at most, of order 1. This is what happens within specific models. For instance, if the radiation present at the end of inflation comes from amplified quantum fluctuations (i.e., Gibbons–Hawking radiation), quite generically, at the end of inflation $\rho_\epsilon \simeq H^4$. More specifically

$$\rho_\epsilon = \frac{\pi^2}{30} N_{\text{eff}} T_H^4 = \frac{N_{\text{eff}} H^4}{480\pi^2}. \quad (4.10)$$

In equation (4.10) $N_{\text{eff}}$ is the number of species contributing to the quantum fluctuations during the quasi-de Sitter stage of expansion. In [61] (see also [52, 53, 59]) it has been argued that this quantity could be evaluated using a perturbative expansion valid in the limit of quasi-conformal coupling. It should be clear that $N_{\text{eff}}$ is conceptually different from the number of relativistic degrees of freedom $g_\nu$. Given $H$ and $N_{\text{eff}}$ the length of the stiff phase is fixed, in this case, by [59]

$$\lambda H^4 \left( \frac{a_0}{a_t} \right)^4 = H_t^2 M_p^2 \left( \frac{a_0}{a_t} \right)^{3(\alpha+1)} = H_t^2 M_p^2 \left( \frac{a_0}{a_t} \right)^{2/\alpha}, \quad (4.11)$$

where we used the fact that $\alpha = 2/[3(\epsilon + 1)]$ and where we defined $\lambda = N_{\text{eff}}/(480\pi^2)$. Equation (4.11) implies that

$$\left( \frac{a_t}{a_0} \right) = \lambda \frac{H}{M_p} \left( \frac{H}{M_p} \right)^{\frac{1}{\alpha}}, \quad \left( \frac{H_t}{M_p} \right) = \lambda \frac{1}{\alpha} \left( \frac{H}{M_p} \right)^{\frac{2}{\alpha}}. \quad (4.12)$$

Using the second relation in equations (4.12) and (4.6), it turns out that $\Sigma = \lambda^{1/4}$, which is always smaller than 1 and, at most, $\mathcal{O}(1)$. Instead of endorsing an explicit model by pretending to know the whole thermal history of the universe in reasonable detail, it is more productive to keep $\Sigma$ as a free parameter and to require that the scaling violations in the spectral energy density will take place before BBN. The variation of $\Sigma$, $w$ and $\tau_T$ can be simultaneously bounded. The essential constraint which must be enforced in any model of scaling violations implies that the frequency $v_s$ must necessarily exceed $v_{\text{bbn}}$ (see equation (1.8)). This requirement guarantees that the stiff dynamics will be over by the time
light nuclei start being formed. In a complementary approach one might also require that \( \nu_s > \nu_{\text{ew}} \) where \( \nu_{\text{ew}} \) corresponds to the value of the Hubble rate at the electroweak epoch, i.e.

\[
\nu_{\text{ew}} = 3.998 \times 10^{-6} \left( \frac{g_\rho}{106.75} \right)^{1/4} \left( \frac{T_s}{100 \text{ GeV}} \right)^{1/4} \left( \frac{h^2 \Omega_{R0}}{4.15 \times 10^{-5}} \right)^{1/4} \text{ Hz. (4.13)}
\]

Finally, yet a different requirement could be to impose that \( \nu > \nu_{\text{TeV}} \) where \( \nu_{\text{TeV}} \) is defined as

\[
\nu_{\text{TeV}} = 4.819 \times 10^{-3} \left( \frac{g_\rho}{228.75} \right)^{1/4} \left( \frac{T_s}{100 \text{ TeV}} \right)^{1/4} \left( \frac{h^2 \Omega_{R0}}{4.15 \times 10^{-5}} \right)^{1/4} \text{ Hz. (4.14)}
\]

The condition \( \nu > \nu_{\text{TeV}} \) (as opposed to \( \nu > \nu_{\text{ew}} \)) would imply that the stiff age did already finish by the time the universe had a temperature of the order of 100 TeV when, presumably, the number of relativistic degrees of freedom was much larger than in the minimal standard model\(^\text{11}\).

The constraints on \( \Sigma, w_t \), and \( r_T \) are summarized in figure 10. The value of \( \Sigma \) controls the position of the frequency at which the nearly scale-invariant slope of the spectrum will be violated. The barotropic index \( w_t \) is taken to be always larger than 1/3 (by definition of stiff fluid) and with a maximal value of 1. In figure 10, values of \( w_t \) as large as 2 have been allowed just for completeness since some authors like to speculate that models with \( w_t > 1 \) do not violate causality constraints. It is amusing to note that the cases \( w_t > 1 \) are, anyway, totally irrelevant from the phenomenological point of view. In these cases, in fact, the detectability prospects are forlorn (see section 5).

In figure 10 (plot on the right) the different curves denote, respectively, the cases \( \nu_s = \nu_{\text{bbn}} \) (full line), \( \nu_s = \nu_{\text{ew}} \) (dashed line) and \( \nu_s = \nu_{\text{TeV}} \) (dot-dashed line). To be compatible with the corresponding constraint we have to be above each curve and the shaded region are excluded. Of course since different curves are present, we decided to shade the region which might correspond, according to theoretical prejudice, to the most typical choice of parameters.

\(^{11}\) In equation (4.14), the typical value of \( g_\rho \) is that arising in the minimal supersymmetric extension of the standard model.
5. Relic gravitons and from the stiff age

5.1. Spectral energy density in the minimal $T\Lambda CDM$ scenario

The conclusion of the previous section has been that it is indeed plausible to parametrize the scaling violations (at high frequency) in terms of two parameters, i.e. the typical frequency $\nu_s$ at which scaling violations occur and the typical slope of the spectral energy density for $\nu > \nu_s$. The latter framework has been dubbed $T\Lambda CDM$ for tensor-$\Lambda CDM$ [77]. The two supplementary parameters physically depend upon the sound speed during the stiff phase (i.e., $c_s$) and the threshold frequency (i.e., $\nu_s$). Besides $c_s$ and $\nu_s$, there will also be $r_T$ which controls, at once, the normalization and the slope of the low-frequency branch of the spectral energy density. The remaining six parameters of the underlying $\Lambda CDM$ model will be fixed, just for illustration, to the best fit of the 5 year WMAP data alone [5–9]. In the numerical program used to compute the spectral energy density of the relic gravitons the putative values of the cosmological parameters can be changed at wish.

In spite of their present sensitivities [35] (see also equation (3.18) and discussion therein) terrestrial interferometers might be able, one day, to provide a prima facie evidence of relic gravitons. The present numerical approach will then be instrumental not only in setting upper limits but also in providing global fits of the cosmological observables within the $T\Lambda CDM$ model. According to this perspective, in the future the three (now available) cosmological data sets will be complemented by the observations of the wideband interferometers. Therefore, different choices of cosmological parameters (like, for instance, the various critical fractions of matter and dark energy) could slightly change the typical frequencies of the relic graviton spectrum as well as other features in the low-frequency region of the spectral energy density.

In the absence of any tensor contribution (i.e., $r_T = 0$) the 5 year WMAP data alone imply

$$\begin{pmatrix} \Omega_0, \Omega_{c0}, \Omega_\Lambda, h_0, n_s, \tau \end{pmatrix} = (0.0441, 0.214, 0.742, 0.719, 0.963, 0.087).$$  \hspace{1cm} (5.1)

If the tensors are included (i.e., $r_T \neq 0$) but without any running of the scalar spectral index the parameters (inferred from the 5 year best fit to the WMAP data alone) slightly change and become

$$\begin{pmatrix} \Omega_0, \Omega_{c0}, \Omega_\Lambda, h_0, n_s, \tau, r_T \end{pmatrix} = (0.0417, 0.188, 0.770, 0.751, 0.986, 0.090, <0.43).$$  \hspace{1cm} (5.2)

In equation (5.2) the last entry of the array contains $r_T$ and it is actually an upper limit (95% CL) corresponding to the first row appearing in table 1. Various other examples could be provided by considering, for instance, the combinations listed in table 1. In the numerical examples reported here, the $\Lambda CDM$ parameters will be fixed to their best fit values as they are reported in equation (5.1). In this situation the tensor contribution will be parametrized not only by $r_T$, but also by $c_s$ and $\nu_s$. The bounds on $r_T$ are spelled out in table 1.

In both plots of figure 11 the parameters are fixed to the values reported in equation (5.1). In figure 11 (plot on the left) the $\Lambda CDM$ scenario is complemented by a stiff phase with $w_0 = 1$ and for different values of $r_T$. Always in figure 11 the value of the barotropic index is slightly reduced from 1 to $w_i = 0.6$. In both plots of figure 11, $\alpha_T \neq 0$ and its value is given by equation (3.6). The effect associated with a slight frequency variation of the tensor spectral index is rather modest so that it can be hardly distinguished from $\alpha_T = 0$ except when $r_T$ is sufficiently large. A similar occurrence can be observed in the two plots reported in figure 9.

The infrared branch of the spectrum in both plots of figure 11 reproduces the results of figure 8.

12 If not otherwise specified, the value of the scalar spectral index used to compute $\alpha_T$ is consistent with the 5 year best fit to the WMAP data alone.
Figure 11. The spectral energy density of the relic gravitons coming from the stiff ages. In the plot on the left $w_t = 1$ while in the plot on the right $w_t = 0.6$. In both plots the value of $\Sigma$ has been fixed to 0.15.

As soon as the frequency increases from the aHz up to the nHz (and even larger) the spectral energy density increases sharply in comparison with the nearly scale-invariant case (see, e.g., figures 8 and 9) where the spectral energy density was, for $\nu > nHz$, at most $O(10^{-16})$. In the case of figure 11 the spectral energy density is clearly much larger. The accuracy in the determination of the infrared branch of the spectrum is a condition for the correctness of the estimate of the spectral energy density of the high-frequency branch. The plots of figure 11 demonstrate that the low-frequency bounds on $r_T$ do not forbid a larger signal at higher frequencies.

A decrease of $r_T$ implies a suppression of the nearly scale-invariant plateau in the region $\nu_{eq} < \nu < \nu_s$. At the same time the amplitude of the spectral energy density still increases for frequencies larger than the frequency of the elbow (i.e., $\nu_s$). The latter trend can be simply understood since, at high frequency, the transfer function for the spectral energy density grows faster than the power spectrum of inflationary origin. For instance, in the case $w_t = 1$ and neglecting logarithmic corrections, $\Omega_{GW}(\nu, \tau_0) \propto \nu^{n_T+1}$ for $\nu \gg \nu_s$. Now, recall that $n_T$ is given by equation (3.6). If $r_T \rightarrow 0$, the combination $(n_T + 1)$ will be much closer to 1 than in the case when, say, $r_T \simeq 0.3$. This aspect can be observed in both plots of figure 11 where different values of $r_T$ have been reported. By decreasing the $w_t$ from 1 to say, 0.6 the extension of the nearly flat plateau gets narrower. This is also a general effect which is particularly evident by comparing the two plots of figure 11. The slope of the high-frequency branch of the graviton energy spectrum can easily be deduced with analytic methods and it turns out to be

$$\frac{d \ln \Omega_{GW}}{d \ln \nu} = \frac{6w_t - 2}{3w_t + 1}, \quad \nu > \nu_s,$$

up to logarithmic corrections. The result of equation (5.3) stems from the simultaneous integration of the background evolution equations and of the tensor mode functions according to the techniques described in section 3. The semi-analytic estimate of the slope (see [51]) agrees with the results obtained by means of the transfer function of the spectral energy density. In figure 4 (plot on the left), for $\kappa = k/k_s > 1$ $T_s^r(\kappa) \simeq \kappa$ which is consistent with equation (5.3) in the case $w_t = 1$. The logarithmic corrections arising in the case $w_t = 1$ (see, for instance, equation (2.57)) have a simple analytic interpretation which is evident from the results reported in equations (2.65)–(2.68) for the mixing coefficients in the case $w_t = 1$. 
5.2. Phenomenological constraints

The spectra illustrated in figure 11 (as all the spectra stemming from the stiff ages) must be compatible not only with the CMB constraints (bounding, from above, the value of $r_T$) but also with other two classes of constraints, i.e. the pulsar timing constraints [78, 79] and the big-bang nucleosynthesis constraints [80–82]. The pulsar timing constraint demands

$$\Omega(\nu_{\text{pulsar}}, \tau_0) < 1.9 \times 10^{-8}$$

where $\nu_{\text{pulsar}}$ roughly corresponds to the inverse of the observation time along which the pulsars timing has been monitored. Assuming the maximal growth of the spectral energy density and the minimal value of $\nu_s$, i.e. $\nu_{\text{bhn}}$ we will have

$$h_0^2\Omega_{\text{GW}}(\nu, \tau_0) \propto \nu, \quad \nu \gg \nu_s \simeq \nu_{\text{bhn}}.$$  (5.5)

Since $\nu_{\text{pulsar}} \simeq 10^3\nu_{\text{bhn}}$, equation (5.5) implies that $h_0^2\Omega_{\text{GW}}(\nu_{\text{pulsar}}, \tau_0) \simeq 10^{-13}$ or even $10^{-14}$ depending upon $r_T$. But this value is always much smaller than the constraint stemming from pulsar timing measurements. If either $\nu_s \gg \nu_{\text{bhn}}$ or $c_s < 1$ the value of $h_0^2\Omega_{\text{GW}}(\nu_{\text{pulsar}}, \tau_0)$ will be even smaller. Consequently, even in the extreme cases when the frequency of the elbow is close to $\nu_{\text{bhn}}$, the spectral energy density is always much smaller than the requirement of equation (5.4). The conclusion is that the pulsar timing bound is not constraining for the TACDM model.

It is well known that the most significant constraint on the stiff spectra stems from BBN [49, 51]. Being massless, gravitons can increase the expansion rate at the BBN epoch. To avoid the overproduction of $^4$He, the number of relativistic species must be bounded from above. The BBN bound is customarily expressed in terms of (equivalent) extra fermionic species. According to equation (3.16), during the radiation-dominated era, the energy density of the plasma can be written as $\rho = g_{\nu}(\pi^2/30)T^4$ where $T$ denotes here the common (thermodynamic) temperature. An (ultra)relativistic fermion species with two internal degrees of freedom and in thermal equilibrium contributes $2\pi/8 = 7/4 = 1.75$ to $g_{\nu}$. Before neutrino decoupling the contributing relativistic particles are photons, electrons, positrons, and $N_\nu = 3$ species of neutrinos, giving $g_{\nu} = 10.75$.

The neutrinos have decoupled before electron–positron annihilation so that they do not contribute to the entropy released in the annihilation. While they are relativistic, the neutrinos still retain an equilibrium energy distribution, but after the annihilation their (kinetic) temperature is lower, $T_{\text{h}} = (4/11)^{1/3}T$. Thus $g_{\nu} = 3.36$ after electron–positron annihilation.

By now assuming that there are some additional relativistic degrees of freedom, which also have decoupled by the time of electron–positron annihilation, or just some additional component $\rho_s$ to the energy density with a radiation-like equation of state (i.e., $p_s = \rho_s/3$), the effect on the expansion rate will be the same as that of having some (perhaps a fractional number of) additional neutrino species. Thus its contribution can be represented by replacing $N_\nu$ with $N_\nu + \Delta N_\nu$ in the above. Before electron–positron annihilation we have $\rho_s = (7/8)\Delta N_\nu\rho_{\nu}$ and after electron–positron annihilation we have $\rho_s = (7/8)(4/11)^{4/3}\Delta N_\nu\rho_{\nu} \simeq 0.227\Delta N_\nu\rho_{\nu}$.

The critical fraction of CMB photons can be directly computed from the value of the CMB temperature and it is notoriously given by $h_0^2\Omega_s \equiv \rho_s/\rho_{\text{crit}} = 2.47 \times 10^{-5}$. If the extra energy density component has stayed radiation-like until today, its ratio to the critical density, $\Omega_s$, is given by

$$h_0^2\Omega_s \equiv h_0^2\frac{\rho_s}{\rho_c} = 5.61 \times 10^{-6}\Delta N_\nu\left(\frac{h_0^2\Omega_0\nu_{\text{bhn}}}{2.47 \times 10^{-5}}\right).$$  (5.6)

13This conclusion follows immediately from the hierarchy between $\nu_{\text{pulsar}}$ and $\nu_{\text{bhn}}$. If either $c_s < 1$ or $\nu_s \gg \nu_{\text{bhn}}, h_0^2\Omega_{\text{GW}}$ can only grow very little and certainly much less than required to violate the bound of equation (5.4).
If the additional species are relic gravitons, then [80–82]

\[ h_0^2 \int_{\nu_{\text{b}} \nu_{\text{max}}} \Omega_{GW}(\nu, \tau_0) \, d\ln \nu = 5.61 \times 10^{-6} \Delta N_{\nu} \left( \frac{h_0^2 \Omega_{\gamma 0}}{2.47 \times 10^{-5}} \right), \quad (5.7) \]

where \( \nu_{\text{b}} \) and \( \nu_{\text{max}} \) are given, respectively, by equations (1.8) and (4.7). Thus the constraint of equation (5.7) arises from the simple consideration that new massless particles could eventually increase the expansion rate at the epoch of BBN. The extra-relativistic species do not have to be, however, fermionic [81] and therefore the bounds on \( \Delta N_{\nu} \) can be translated into bounds on the energy density of the relic gravitons.

A review of the constraints on \( \Delta N_{\nu} \) can be found in [81]. Depending on the combined data sets (i.e., various light element abundances and different combinations of CMB observations), the standard BBN scenario implies that the bounds on \( \Delta N_{\nu} \) range from \( \Delta N_{\nu} \leq 0.2 \) to \( \Delta N_{\nu} \leq 1 \). Similar figures, depending on the priors of the analysis, have been obtained in a more recent analysis [82]. All the relativistic species present inside the Hubble radius at the BBN contribute to the potential increase in the expansion rate and this explains why the integral in equation (5.7) must be performed from \( \nu_{\text{b}} \) to \( \nu_{\text{max}} \) (see also [52] where this point was stressed in the framework of a specific model).

The existence of the exponential suppression for \( \nu > \nu_{\text{max}} \) (see figure 11) guarantees the convergence of the integral also in the case when the integration is performed up to \( \nu \to \infty \). The constraint of equation (5.7) can be relaxed in some non-standard nucleosynthesis scenarios [81], but, in what follows, the validity of equation (5.7) will be enforced by adopting \( \Delta N_{\nu} \approx 1 \) which implies, effectively

\[ h_0^2 \int_{\nu_{\text{b}}}^{\nu_{\text{max}}} \Omega_{GW}(\nu, \tau_0) \, d\ln \nu < 5.61 \times 10^{-6} \left( \frac{h_0^2 \Omega_{\gamma 0}}{2.47 \times 10^{-5}} \right). \quad (5.8) \]

The models illustrated in figure 11 are on the verge of saturating the bounds of equations (5.7) and (5.8). This conclusion stems directly from the form of spectral energy density: the broad spike dominates the (total) energy density of relic gravitons which are inside the Hubble radius at the time of big-bang nucleosynthesis. A practical way of enforcing the bounds of equations (5.7) and (5.8) is to integrate around the maximum of the curves depicted in figure 11. In figure 12, the energy density of the relic gravitons inside the Hubble radius...
5.3. Detectability prospects

The results presented in the previous subsection suggest that if $r_T$ is bounded from above by the cosmological data sets (see, e.g., table 1), a detectable signal is expected for $\nu \simeq 100$ Hz for $0.35 < w_t < 0.61$. In this case, following the parametrization of the LIGO Collaboration we could say that the expected signal can be parametrized as

$$\Omega_{GW}(\nu, \tau_0) = \Omega_{GW}(\nu, \tau_0) \nu^{n_T - 2},$$

which mirrors equation (19) of [35] where the upper limits on the amplitude $\Omega_{GW}$ have been set. This range turns out to be compatible with the bounds of equations (5.7) and (5.8). Relation (5.9) could be used by the experimenters to set bounds on $\Omega_{GW}$ in the same way as upper bounds are obtained in the case of nearly scale-invariant spectra (see equation (3.18) and discussion therein). Clearly, from the theoretical point of view, $\Omega_{GW}$ changes by varying the various $\Lambda$CDM parameters.

For instance, by lowering $w_t$, $h_0^2\Omega_{GW}(\nu, \tau_0)$ increases for $\nu = \nu_{LV} \simeq 0.1$ kHz. This trend can be inferred from figure 13 where the spectral energy density is evaluated exactly for $\nu = \nu_{LV}$. To be detectable by wideband interferometers the parameters of the $\Lambda$CDM must lie above the full lines. The region of low barotropic indices emerging neatly from figure 13, leads to spectral energy densities which are progressively flattening as $w_t$ diminishes toward
1/3. Low values of \( w_t \) bring the frequency of the elbow, i.e. \( \nu_s \) below \( 10^{-10} \) Hz which is unacceptable since it would mean that, during nucleosynthesis, the universe was dominated by the stiff fluid. In figure 10 (plot on the left) the region above the full line corresponds to a range of parameters for which \( \nu_s > \nu_{bbn} \); in such a range a decrease of \( w_t \) demands an increase of \( \Sigma_1 \).

The occurrence described in the previous paragraph is illustrated in figure 14 where, at the left, \( w_t = 0.5 \) and the values of \( \Sigma_1 \) are the same as those illustrated in figure 13. The full, dashed and dot-dashed curves illustrated in figure 14 (plot on the left) are incompatible with phenomenological considerations since the frequency of the elbow is systematically smaller than \( \nu_{bbn} \). Once more, this choice of parameters would contradict the bounds of figure 10 and would imply that the stiff phase is not yet finished at the BBN time. In the left plot of figure 14 the diamonds denote a model which is compatible with BBN considerations but whose signal at the frequency of interferometers is rather small (always three orders of magnitude larger than in the case of conventional inflationary models).

The compatibility with the phenomenological constraints demands that the parameters of the \( T/\Lambda_1 \)CDM paradigm must lie above the full lines of figure 10. The requirements of figure 10 suggest, therefore, that \( \Sigma_1 \) should be raised a bit. In this case the frequency of the elbow gets shifted to the right but, at the same time, the overall amplitude of the spike diminishes. The putative amplitude remains still much larger than the conventional inflationary signal reported in figure 9.

In figure 14 (plot on the right) \( \Sigma_1 = 0.2 \) and \( w_t = 0.6 \). The tensor spectral index is allowed to depend upon frequency according to equation (3.6) (i.e., \( \alpha_T \neq 0 \)). Two different values of \( n_s \) are reported. In the example of figure 14 the phenomenological bounds are all satisfied. In figure 15 the spectral energy density of the relic gravitons is illustrated as a function of \( r_T \) for a choice of parameters which is compatible with all the bounds applicable to the stochastic backgrounds of the relic gravitons. The three curves refer to three different frequencies, i.e. 0.1 kHz, 1 kHz and 10 kHz. Indeed, if the spectrum is nearly scale invariant (as in the case of figure 9) we can compare the potential signal with the central frequency of the window. If the signal increases with frequency it is interesting to plot the same curve for some significant frequencies inside the window of wideband interferometers. Even if the frequency window extends from a few Hz to 10 kHz the maximal sensitivity is in the central region and depends upon various important factors which will now be briefly discussed.
Figure 15. The graviton energy spectrum is illustrated, in the TACDM scenario, for \( \nu = \nu_{LV} \) and as a function of \( r_T \). As in figure 9 at the left \( \alpha_T = 0 \) while, at the right, \( \alpha_T \neq 0 \).

To illustrate more quantitatively this point we recall the expression of the signal-to-noise ratio in the context of optimal processing required for the detection of stochastic backgrounds:

\[
\text{SNR}^2 = \frac{3H_0^2}{2\sqrt{2\pi^2}} F \sqrt{T} \left\{ \int_0^{\infty} d\nu \gamma^2(\nu) \Omega_{GW}^2(\nu, \tau_0) \right\}^{1/2},
\]

\((5.10)\)

\((F\) depends upon the geometry of the two detectors and in the case of the correlation between two interferometers \( F = 2/5 \); \( T \) is the observation time). In equation \((5.10)\), \( S_k^{(i)}(f) \) is the (one-sided) noise power spectrum (NPS) of the \( k \)th \((k = 1, 2)\) detector. The NPS contains the important information concerning the noise sources (in broad terms seismic, thermal and shot noises) while \( \gamma(\nu) \) is the overlap reduction function which is determined by the relative locations and orientations of the two detectors. In \([54]\) equation \((5.10)\) has been used to assess the detectability prospects of gravitons coming from a specific model of stiff evolution with \( w_t = 1 \). At that time the various suppressions of the low-frequency amplitude as well as the free-streaming effects were not taken into account. Furthermore, the evaluation of the energy transfer function was obtained, in \([56]\), not numerically but by matching of the relevant solutions. We do know, by direct comparison, that such a procedure is justified but intrinsically less accurate than that proposed here. It would be interesting to apply equation \((5.10)\) for the (more accurate) assessment of the sensitivities of different instruments to a potential signal stemming from the stiff age\(^{14}\).

Equation \((5.10)\) assumes that the intrinsic noises of the detectors are stationary, Gaussian, uncorrelated, much larger in amplitude than the gravitational strain, and statistically independent on the strain itself \([92–94]\). The integral appearing in equation \((5.10)\) extends over all the frequencies. However, the noise power spectra of the detectors are defined in a frequency interval ranging from a few Hz to 10 kHz. In the latter window, for very small frequencies the seismic disturbances are the dominant sources of noise. For intermediate and

\(^{14}\) For intermediate frequencies the integral of equation \((5.10)\) is sensitive to the form of the overlap reduction function which depends upon the mutual position and relative orientations of the interferometers. The function \( \gamma(\nu) \) effectively cuts off the integral which defines the signal-to-noise ratio for a typical frequency \( \nu \approx 1/(2d) \), where \( d \) is the separation between the two detectors. Since \( \Omega_{GW} \) increases with frequency (at least in the case of relic gravitons from stiff ages) at most as \( \nu \) and since there is a \( \nu^{-6} \) in the denominator, the main contribution to the integral should occur for \( \nu < 0.1 \) kHz. This argument can be explicitly verified in the case of the calculations carried on in \([54]\) and it would be interesting to check it also in our improved framework.
The graviton energy spectrum is illustrated, in the T\(_\Lambda_1\)CDM scenario, for \(\nu = \nu_C\) and as a function of \(r_T\). As in figures 9 and 15 on the left \(\alpha_T = 0\) while, on the right, \(\alpha_T \neq 0\).

In the T\(_\Lambda_1\)CDM paradigm the maximal signal occurs in a frequency region between the MHz and the GHz. This intriguing aspect led to the suggestion [54, 55] that microwave cavities [83] can be used as GW detectors precisely in the mentioned frequency range. Prototypes of these detectors [84] have been described and the possibility of further improvements in their sensitivity received recently attention [85–90]. Different groups are now concerned with high-frequency gravitons. In [86] the ideas put forward in [83–85] have been developed by using electromagnetic cavities (i.e., static electromagnetic fields). In [87–89] dynamical electromagnetic fields (i.e., wave guides) have been studied always for the purpose of detecting relic gravitons. In [89] an interesting prototype detector was described with frequency of operation of the order of 100 MHz (see also [91]). In figure 16 the value of the spectral energy density is reported for \(\nu = \nu_C\) where \(\nu_C\) defines the frequency of operation of a given electromagnetic detector. In figure 16 \(\nu_C\) is taken in the MHz range. In both plots the horizontal lines denote the bounds of equations (5.7) and (5.8) for two typical values of \(\Delta N\), (i.e., more specifically, \(\Delta N = 1\) and \(\Delta N = 0.2\)). To be compatible with the bounds the values of the spectral energy density must be smaller than the horizontal lines. The region of large \(r_T\) (i.e.,
is already excluded from CMB upper limits: the plots have been extended also in that region for comparison with the analog plots illustrated in figure 9.

Absent direct tests on the thermal history of the plasma prior to neutrino decoupling, the current bounds on a tensor component affecting the initial conditions of the CMB anisotropies (and polarization) do not forbid a potentially detectable signal for typical frequencies compatible with the window of wideband interferometers. The numerical approach described in the present paper allows for a sufficiently accurate estimate of the spectral energy density of the relic gravitons. In the context of the class of models analyzed here it is plausible to imagine, in the years to come, a rather intriguing synergy between large-scale observations (e.g., CMB physics, measurements of the matter power spectrum and supernovae) and small-scale observations such as those conducted by wideband interferometers in the range between a few Hz and 10 kHz.

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