A construction of a $\frac{3}{2}$-tough plane triangulation with no 2-factor

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Abstract. In 1956, Tutte proved the celebrated theorem that every 4-connected planar graph is hamiltonian. This result implies that every more than $\frac{3}{2}$-tough planar graph on at least three vertices is hamiltonian and so has a 2-factor. Owens in 1999 constructed non-hamiltonian maximal planar graphs of toughness arbitrarily close to $\frac{3}{2}$ and asked whether there exists a maximal non-hamiltonian planar graph of toughness exactly $\frac{3}{2}$. In fact, the graphs Owens constructed do not even contain a 2-factor. Thus the toughness of exactly $\frac{3}{2}$ is the only case left in asking the existence of 2-factors in tough planar graphs. This question was also asked by Bauer, Broersma, and Schmeichel in a survey. In this paper, we close this gap by constructing a maximal $\frac{3}{2}$-tough plane graph with no 2-factor, answering the question asked by Owens as well as by Bauer, Broersma, and Schmeichel.

Keywords. 2-factor; plane triangulation; toughness

1 Introduction

We consider only simple graphs. Let $G$ be a graph. Denote by $V(G)$ and $E(G)$ the vertex set and edge set of $G$, respectively. We denote by $n(G)$ and $e(G)$ the sizes of $V(G)$ and $E(G)$, respectively, and by $f(G)$ the number of faces of $G$ if $G$ is embedded on a surface. Let $v \in V(G)$ and $S \subseteq V(G)$. Then $N_G(v)$ denotes the set of neighbors of $v$ in $G$ and $N_G(S) := (\bigcup_{x \in S} N_G(x)) \setminus S$. The subgraph of $G$ induced on $S$ and $V(G) \setminus S$ are denoted by $G[S]$ and $G - S$, respectively. For notational simplicity we write $G - x$ for $G - \{x\}$. Let $V_1, V_2 \subseteq V(G)$ be two disjoint vertex sets. Then $E_G(V_1, V_2)$ is the set of edges in $G$ with one end in $V_1$ and the other end in $V_2$ and $e_G(V_1, V_2) := |E_G(V_1, V_2)|$. We write $E_G(v, V_2)$ and $e_G(v, V_2)$ if $V_1 = \{v\}$ is a singleton. When $H \subseteq G$ and $S \subseteq V(G) \setminus V(H)$, we write $N_G(H)$, $E_G(H, S)$, and $e_G(H, S)$ respectively for $N_G(V(H))$, $E_G(V(H), S)$, and $e_G(V(H), S)$. For two integers $p$ and $q$, we let $[p, q] = \{i \in \mathbb{Z} : p \leq i \leq q\}$.
The number of components of $G$ is denoted by $c(G)$. Let $t \geq 0$ be a real number. The graph $G$ is said to be $t$-tough if $|S| \geq t \cdot c(G - S)$ for each $S \subseteq V(G)$ with $c(G - S) \geq 2$. The toughness $\tau(G)$ is the largest real number $t$ for which $G$ is $t$-tough, or is $\infty$ if $G$ is complete. This concept was introduced by Chvátal [4] in 1973. It is easy to see that if $G$ has a hamiltonian cycle then $G$ is 1-tough. Conversely, Chvátal [4] conjectured that there exists a constant $t_0$ such that every $t_0$-tough graph is hamiltonian. Bauer, Broersma and Veldman [1] have constructed $t$-tough graphs that are not hamiltonian for all $t < \frac{9}{4}$, so $t_0$ must be at least $\frac{9}{4}$ if Chvátal’s conjecture is true. The conjecture has been verified when restricted to a number of graph classes [2], including planar graphs, claw-free graphs, co-comparability graphs, and chordal graphs.

The study of cycle structures in planar graphs under a given toughness condition is particularly intensive and interesting, see, for examples [5, 6, 7, 9, 10]. Observe that any more than $\frac{3}{2}$-tough planar graph on at least 5 vertices is 4-connected. Thus the toughness conjecture of Chvátal holds for planar graphs with toughness greater than $\frac{3}{2}$ by the classic result of Tutte [13] that every 4-connected planar graph is hamiltonian. Furthermore, it is shown by Owens [9] that $t_0$ cannot be smaller than $\frac{3}{2}$. It is still unknown whether $t_0 = \frac{3}{2}$ is the sharp toughness bound for a planar graph to be hamiltonian. In fact, this question is even open for the existence of 2-factors in planar graphs.

A 2-factor in a graph $G$ is a spanning 2-regular subgraph. Thus a hamiltonian cycle of $G$ is a 2-factor with only one component. By the result of Tutte [13], we know that every more than $\frac{3}{2}$-tough planar graph on at least 3 vertices has a 2-factor. On the other hand, constructed by Owens [9], there are maximal planar graphs with toughness arbitrarily close to $\frac{3}{2}$ but with no 2-factor. Owens asked in the same paper whether there exists a non-hamiltonian maximal planar graph with toughness exactly $\frac{3}{2}$. Bauer, Broersma, and Schmeichel in the survey [2] commented that “one of the challenging open problems in this area is to determine whether every $\frac{3}{2}$-tough maximal planar graph has a 2-factor. If so, are they all hamiltonian? We also do not know if a $\frac{3}{2}$-tough planar graph has a 2-factor.” In this paper, we answer positively the question asked by Owens and negatively the questions raised by Bauer, Broersma, and Schmeichel.

**Theorem 1.** There exists a $\frac{3}{2}$-tough plane triangulation with no 2-factor.

The remainder of this paper is organized as follows: in Section 2, we introduce some notation and preliminary results, and in Section 3, we prove Theorem 1.

## 2 Preliminary results

Let $S$ and $T$ be disjoint sets of vertices of a graph $G$ and $D$ be a component of $G-(S\cup T)$. Then $D$ is said to be an odd component (resp. even component) if $e_G(D,T) \equiv 1 \pmod{2}$ (resp. $e_G(D,T) \equiv 0 \pmod{2}$). For each integer $k \geq 0$, we denote by $C_{2k+1}$ the set of all
odd components $D$ of $G - (S \cup T)$ such that $e_G(D, T) = 2k + 1$. Let $\mathcal{C} = \bigcup_{k \geq 0} \mathcal{C}_{2k+1}$ and let $c(S, T) = |\mathcal{C}|$.

Define $\delta(S, T) = 2|S| + \sum_{y \in T} d_{G-S}(y) - 2|T| - c(S, T)$. It is easy to see $\delta(S, T) \equiv 0 \pmod{2}$ for every $S, T \subseteq V(G)$ with $S \cap T = \emptyset$. We use the following criterion for the existence of a 2-factor, which is a special case of Tutte’s f-Factor Theorem.

**Theorem 2** (Tutte [12]). A graph $G$ has a 2-factor if and only if $\delta(S, T) \geq 0$ for every $S, T \subseteq V(G)$ with $S \cap T = \emptyset$.

An ordered pair $(S, T)$ consisting of disjoint sets of vertices $S$ and $T$ in a graph $G$ is called a barrier if $\delta_G(S, T) \leq -2$. By Theorem 2, if $G$ does not have a 2-factor, then $G$ has a barrier.

We need also the following result regarding the toughness of the square of a graph.

**Theorem 3** (V. Chvátal, [4, Theorem 1.7]). For any graph $G$, we have $\tau(G^2) \geq \kappa(G)$, where $G^2$ is obtained from $G$ by adding edges joining pairs of vertices of distance 2 in $G$, and $\kappa(G)$ is the connectivity of $G$.

### 3 Proof of Theorem 1

Suppose there exists a $\frac{3}{2}$-tough plane triangulation on $n$ vertices with no 2-factor. Let $(S, T)$ be a barrier of $G$, and $\mathcal{C}$ be the set of all odd components of $G - (S \cup T)$. Let $G_2$ be obtained from $G$ by deleting all the vertices in $S$; $G_1$ be obtained from $G_2$ by smoothing all the vertices of $T$ that have degree 2 in $G_2$ (for a degree 2 vertex $u$, smooth it amounts to deleting $u$ and joining an edge between its two neighbors) and deleting all the vertices of $T$ that have degree 1 in $G_2$; and $G_0$ be obtained from $G_1$ by contracting each graph $D$ in $\mathcal{C}$ into a single vertex (identify all the vertices of $D$ into a single vertex and join all the edges from the vertex to vertices of $N_{G_1}(D)$). We call $G_0$ the component graph of $G$. For any subgraph $G'$ of $G$, we also call the subgraph of $G_0$ obtained through the above process from $G'$ (replacing “contracting each graph in $\mathcal{C}$” by “contracting the restriction of each graph of $\mathcal{C}$ in $G'$”) the component graph of $G'$. Our construction of $G$ starts with its component graph $G_0$ and then reverse the process of getting $G_0$ from $G$ to obtain the graph $G$. In the construction, we will add vertices and edges to the existing graph in a way such that the resulting graph is still a plane graph. We will stick to this rule without mentioning it.

**Step 1**: The component graph $G_0$. 

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Figure 1: The component graph $G_0$ of $G$. 
Let \( G_0 \) be the plane graph drawn in Figure 1. The vertices of \( G_0 \) are depicted in 4 different colors: white vertices (those vertices form a color class of \( G_0 \)), light gray vertices, gray vertices, and black vertices, where all vertices of gray or black color have degree 2 in \( G_0 \). The filled vertices are labelled while the white vertices (except \( w \)) are not labelled due to the space limitation. However, we will denote the white vertices between two filled vertices \( v_i \) and \( v_{i+2} \) by \( v_{i+1} \) for each \( i \in [1, 85] \), where \( v_{87} := v_1 \). We denote by \( C_0 \) the cycle \( v_1 v_2 \ldots v_86 v_1 \). It is easy to see that the graph is bipartite with the bipartition as the set of filled vertices and the set of unfilled vertices in the drawing. As there are only 43 unfilled vertices in \( G_0 - w \), we have the following fact.

**Fact 1.** The size of a maximum matching in \( G_0 - w \) is 43.

Each face of \( G_0 \) that has exactly three degree 2 vertices of the same color on its boundary is called an \( S \)-triangle face associated with those degree 2 vertices. For example, the face with boundary \( wu_1 v_4 v_5 v_6 u_2 w \) is associated with \( u_1, u_2, v_5 \), and the face with boundary \( v_7 v_8 \ldots v_20 v_7 \) is associated with \( v_11, v_15, v_19 \).

The vertex \( w \) in \( G_0 \) is corresponding to the only component in \( C_{39} \), and all other vertices of \( G_0 \) are corresponding to graphs in \( C_3 \), which are all triangles. Thus, we can get to our second step of construction by replacing each vertex of \( G_0 \) with a graph.

**Step 2:** The construction of \( G_1 \).

The vertex \( w \) will be replaced with a 4-connected plane graph. Let a plane graph \( D \) be constructed as follows:

(i) Take four vertex-disjoint 39-cycles \( A_1, A_2, A_3, A_4 \) with vertex set \( V(A_i) = \{a_{i,1}, \ldots, a_{i,39}\} \) for \( i \in [1, 4] \) such that they are embedded in the plane with the faces bounded by \( A_1, A_2, A_3, A_3 \) and \( A_4 \) and \( A_4 \), respectively;

(ii) Add edges \( a_{i,j} a_{i+1,j} \) for each \( i \in [1, 3] \) and \( j \in [1, 39] \);

(iii) Add edges \( a_{i,j} a_{i+1,j-1} \) for each \( i \in [1, 3] \) and \( j \in [1, 39] \), where \( a_{i+1,0} := a_{i+1,30} \);

(iv) Add edges \( a_{4,1} a_{4,j} \) for each \( j \in [3, 38] \).

Note that \( D \) is a near triangulation with \( A_1 \) being the boundary of its non-triangle face. We prove that \( D \) is 2-tough. This fact will be used later on when we show that the finally constructed graph \( G \) is at least \( \frac{3}{2} \)-tough.

**Claim 1.** The graph \( D \) is 2-tough.

**Proof.** Let \( D_1 = D[V(A_1) \cup V(A_2)] \) and \( D_2 = D[V(A_3) \cup V(A_4)] \). We first show that each \( D_i \) for \( i \in [1, 2] \) is 2-tough. As \( D_2 \) contains a spanning subgraph isomorphic to \( D_1 \) by
the construction of $D$ and so $\tau(D_2) \geq \tau(D_1)$, we only need to show that $\tau(D_1) \geq 2$. By Theorem 3, we show that $D_1$ is the square of a 78-cycle. This is obvious as if we let

$$Q = a_1, a_2, a_3, \ldots, a_{i}, a_{i+1}, a_{1}, a_{i+1}, \ldots, a_{i}, a_{39}, a_{39}, a_{2}, a_{1}, a_{1}, a_{1},$$

be a 78-cycle, then we can easily check that any two vertices of distance 2 in $Q$ are adjacent in $D_1$, and the edges of $G_1$ consists of the edges of $Q$, and edges joining pairs of vertices that are distance 2 in $Q$. Thus $D_1 = Q^2$.

Now we have $\tau(D_2) \geq \tau(D_1) \geq 2$, and we show that $\tau(D) \geq 2$. Let $W$ be an arbitrary cutset of $D$. Let $W_i = W \cap V(D_i)$ for $i \in [1, 2]$. If $W_i$ is a cutset of $D_i$ for each $i \in [1, 2]$, then we have $|W| \geq 2c(D - W_i)$. As $c(D - W) \leq c(D_1 - W_1) + c(D_2 - W_2)$, it follows that $|W| \geq 2c(D - W)$.

Consider next that $W_i$ is not a cutset of $D_i$ for each $i \in [1, 2]$. Then $c(D_i - W_i) = 1$, and so $c(D - W) = 2$ as $W$ is a cutset of $D$. If $|W| \geq 4$, then we have $|W| \geq 2c(D - W)$ already. Thus we assume $|W| \leq 3$, and assume by symmetry, that $|W_1| \leq 1$. As $D_2 - W_2$ is a graph that contains at least $39 - 3 = 36$ vertices from $V(A_3)$, and $D_1 - W_1$ contains at least $39 - 1$ vertices from $V(A_2)$, it follows that $E_{D - W}(A_2 - W_1, A_3 - W_2) \neq \emptyset$ by the construction of $D$. Thus $E_{D - W}(D_1 - W_1, D_2 - W_2) \neq \emptyset$ and so $c(D - W) = 1$, a contradiction to the assumption that $W$ is a cutset of $D$.

Consider lastly, by symmetry, that $W_1$ is a cutset of $D_1$ but $W_2$ is not a cutset of $D_2$. Thus $c(D_2 - W_2) = 1$. We may further assume $|W_2| \leq 1$. For otherwise, we have $|W| = |W_1| + |W_2| \geq 2c(D_1 - W_1) + 1 \geq 2c(D - W)$ already. As each vertex from $V(A_2)$ has $D$ two neighbors from $V(A_3)$, if $D_1 - W_1$ has a component containing a vertex from $V(A_2)$, then we have $c(D - W) = c(D_1 - W_1)$ and so we get $|W| \geq 2c(D - W)$. Thus we assume that every component of $D_1 - W_1$ is disjoint with $A_2$. As a consequence, $V(A_2) \subseteq W$. As $A_1$ is a cycle and so is 1-tough, we know that $|W_1 \cap V(A_1)| \geq c(A_1 - (W_1 \cap V(A_1)))$. Thus $c(D - W) = c(A_1 - (W_1 \cap V(A_1))) + 1$. As $c(A_1 - (W_1 \cap V(A_1))) \geq 2$ by $W_1$ being a cutset of $D_1$, it follows that $|W_1 \cap V(A_1)| \leq 37$. Hence $|W| \geq 39 + |W_1 \cap V(A_1)| \geq 2c(A_1 - (W_1 \cap V(A_1))) + 1 = 2c(D - W)$, as desired.

\textbf{Step 2.1:} In $G_0$, we replace $w$ with $D$. That is, delete $w$, place $D$ in the position of $w$ such that $G_0 - w$ is embedded inside the non-triangle face of $D$, then add a perfect matching between $V(A)$ and $N_{G_0}(w)$ such that the resulting graph is still a plane graph.

\textbf{Step 2.2:} In the graph resulting from Step 2.1, we replace each vertex $v \in V(G_0) \setminus \{w\}$ with a triangle. The replacement distinguishes whether $d_{G_0}(v) = 2$ or $d_{G_0}(v) = 3$, and is illustrated below.

(i) If $d_{G_0}(v) = 3$, then we delete $v$, place a triangle in the position of $v$, and add a perfect matching between the three vertices of the triangle and the three neighbors of $v$ in the current graph;

(ii) If $d_{G_0}(v) = 2$, then $v$ is depicted in black or gray in Figure 1. We delete $v$, place a triangle $xyzx$ in the position of $v$, add a perfect matching between $\{x, y\}$ and the
two neighbors of $v$ in the current graph such that the vertex $z$ is embedded inside the $S$-triangle face of $G_0$ that is associated with $v$. See Figure 2(b) for an illustration of the resulting graph of the face boundary $wu_1v_1v_5v_6u_2w$ after this step.

The resulting graph from Steps 2.1-2.2 is called $G_1$. The faces of $G_1$ incident with a vertex of degree 2 of $G_1$ correspond to the $S$-triangle faces of $G_0$, and are still called $S$-triangle faces of $G_1$. The triangles used to replace the vertices of $G_0 - w$ in this step are called $C_3$-triangles.

![Figure 2: Constructing $G$ from $G_0$.](image)

**Step 3:** The construction of $G_2$.

We subdivide each edge of $G_1$ joining vertices from two distinct $C_3$-triangles, or with one from a $C_3$-triangle and the other from $D$, and let the set of those new vertices be $T_2$. For each vertex $v$ of degree 2 in $G_1$, those are the vertices like $z$ produced in Step 2.2 (ii), we add a new vertex $v'$, place $v'$ in the $S$-triangle face of $G_1$ that is incident with $v$, and add the edge $vv'$. The resulting graph is called $G_2$. Let the set of those new vertices $v'$ be $T_1$. See Figure 2(c) for an illustration. The faces of $G_2$ incident with a vertex of degree 1 of $G_2$ correspond to the $S$-triangle faces of $G_1$, and are still called the $S$-triangle faces of $G_2$. 
Let \[ T = T_1 \cup T_2. \] (1)

**Step 4:** The construction of \( G \).

For each face \( F \) of \( G_2 \), we do the following operations.

(i) If \( F \) is not an \( S \)-triangle face, we embed a new vertex inside the face and join an edge from the new vertex to all the vertices on the boundary of the face;

(ii) If \( F \) is an \( S \)-triangle face, let \( x_1, x_2, x_3 \) be the three vertices of degree 1 incident with the face. We embed a new triangle \( s_1s_2s_3s_1 \) inside the face, we first add the edges \( x_1s_1, x_1s_2, x_2s_2, x_2s_3, x_3s_3, x_3s_1 \), then in the current plane graph, we add edges joining \( s_i \) to all the non-adjacent vertices of \( s_i \) on the face boundary containing \( s_i \) for each \( i \in [1,3] \), where \( x_0 := x_3 \) (triangulate the face). Denote the set of all new vertices placed in the faces of \( G_2 \) by \( S \). See Figure 2(d) for an illustration.

The triangles such as \( s_1s_2s_3s_1 \) added in Step 4(ii) are called \( S \)-triangles. We also call \( \{s_1, s_2, s_3\} \), the vertex set of an \( S \)-triangle, an \( S \)-triangle just for notation simplicity. If one vertex of an \( S \)-triangle is adjacent in \( G \) to a vertex from \( V(D) \), we say that the \( S \)-triangle is associated with \( D \). Otherwise, the \( S \)-triangle is not associated with \( D \). The resulting graph from Step 4 is now defined to be \( G \). By the construction, \( G \) is a plane triangulation. Let \( p \) and \( q \) be respectively the number of vertices of degree 2 and 3 in \( G_0 \), and let \( f_s(G_0) \) be the number of \( S \)-triangle faces of \( G_0 \). By direct counting and calculations, we have

\[
\begin{align*}
p &= 17 \times 3 + 4 \times 3 = 63, \\
q &= 86 - (17 + 12) = 57, \\
n(G_0) &= p + q + 1 = 121, \\
e(G_0) &= \frac{1}{2}(39 + 2p + 3q) = 168, \\
f(G_0) &= 2 + e(G_0) - n(G_0) = 49, \\
f_s(G_0) &= 17 + 4 = 21, \\
|S| &= f(G_0) + 2 \times f_s(G_0) = 91, \\
|T| &= e(G_0) + 3f_s(G_0) = 231.
\end{align*}
\]

We define some notation before we proceed with the rest proofs. Let \( U = V(G) \setminus (S \cup T) \) and \( U_3 = U \setminus V(D) \). Vertices from \( S, T, \) and \( U \) are called \( S \)-vertices, \( T \)-vertices, and \( U \)-vertices, respectively.

For a vertex \( v \in V(G_0) \setminus \{w\} \), we let \( R(v) \) be the \( C_3 \)-triangle that was used to replace \( v \) in Step 2.2, and we write \( R(v) = v_{i,1}v_{i,2}v_{i,3}v_{i,1} \) if \( v = v_i \) for some \( i \in [1,86] \), and
$R(v) = u_{i,1}u_{i,2}u_{i,3}u_{i,1}$ if $v = u_i$ for some $i \in [1, 34]$. For each $i \in [1, 34]$, we assume that $u_{i,1}$ and a vertex from $D$ have in $G$ a common neighbor from $T$, and $u_{i,3}$ is embedded inside the $S$-triangle face of $G_1$ that is corresponding to the $S$-triangle face of $G_0$ associated with $u_i$. Furthermore, we assume that for each vertex $v_i$ with $i \in [1, 86]$, if $d_{G_0}(v_i) = 3$, then $v_{i,3}$ and a vertex $u_{j,2}$ for some $j \in [1, 34]$ or some vertex of $D$ have in $G$ a common neighbor from $T$, and if $d_{G_0}(v_i) = 2$, then $v_{i,3}$ is embedded inside the $S$-triangle face of $G_1$ that is corresponding to the $S$-triangle face of $G_0$ associated with $v_i$. We let the other two vertices of $R(v_i)$ for $i \in [1, 86]$ be labeled such that

$$C_1 = v_{1,1}v_{1,2}v_{1,2,1}v_{2,2} \ldots v_{86,1}v_{86,2}v_{1,1}$$

is a cycle in $G_1$. For an illustration, see Figure 3. The cycle in $G_2$ obtained from $C_1$ by subdividing each of its edges is denoted by $C$. The labels for the vertices $u_i$'s, vertices from $R(u_i)$, $v_j$'s, and vertices from $R(v_j)$ for $i \in [1, 34]$ and $j \in [1, 86]$ will be fixed throughout the paper.

![Figure 3: The labels of vertices from the $C_3$-triangles, where $w_1, w_2 \in V(D)$.](image)

A path $P$ in $G$ that can be denoted as $wt_1u_{i,1}u_{i,2}t_2v_{j,3}$ or $wt_1v_{j,3}$, where $w \in V(D)$, $t_1, t_2 \in T_2$, $i \in [1, 34]$, and $j \in [1, 86]$ is called a spoke of $G$. The former is a long spoke while the latter is a short spoke. For example, $w_1t_1x_{1,1}t_{2,2}x_{2,1}$ in Figure 5 is a long spoke.

Let $G^*$ be obtained from $G_2$ by deleting all the vertices from $T_1$, deleting $u_{i,3}$ for each $i \in [1, 34]$, and deleting $v_{i,3}$ if $d_{G_0}(v_i) = 2$. In other words, by our assumption of the labels of the vertices of $G$, $G^* = G - S - T_1 - (N_G(T_1) \cap U)$. For a face of $G^*$ with boundary $F$, we say that $F$ is the boundary of the set of $S$-vertices, say $S^*$, that are embedded in $G$ inside $F$. The subgraph $F \cap C$ is called the $C$-segment of $F$. If $S^*$ is associated with $D$, then the boundary of $S^*$ consists of two spokes and one $C$-segment. We let $S^*_1$ and $S^*_2$ respectively denote the two sets of $S$-vertices embedded inside the two faces of $G^*$ that share a spoke.
with \( F \) on its left and right, and let \( S^*_r \) be the set of \( S \)-vertices embedded inside the face of \( G^* \) that share the \( C \)-segment with \( F \). For example, using Figure 5 as an illustration, if we let

\[
F = w_1t_1x_1x_1x_2x_2x_2x_3x_3x_4x_4x_5x_5x_5x_6w_2w_1,
\]

and let the set of the three \( S \)-vertices of \( G \) embedded inside \( F \) be \( S^* \), then the set of the \( S \)-vertex adjacent in \( G \) to vertices \( w_2, t_6, x_5, 1, t_5 \) is \( S^*_r \), the set of the \( S \)-vertex adjacent in \( G \) to vertices \( w_1, t_1, x_1, 1, t_2 \) is \( S^*_l \), and the set of the \( S \)-vertices adjacent to \( x_2, 2, t_3, 3, 1, x_3, 2, t_4 \) is \( S^*_r \). An \( S \)-triangle \( S^* \) is internal if the boundary of each of \( S^*_l \) and \( S^*_r \) contains no short spoke.

For an \( S \)-triangle \( s_1s_2s_3s_1 \), the three vertices from \( T_1 \) that each have in \( G \) two neighbors from \( \{s_1, s_2, s_3\} \) are called the \( T \)-vertices associated with \( s_1s_2s_3s_1 \). The three \( C_3 \)-triangles that each have a vertex adjacent in \( G \) to a vertex from the \( T \)-vertices associated with \( s_1s_2s_3s_1 \) are called the \( C_3 \)-triangles associated with \( s_1s_2s_3s_1 \).

For a vertex \( y \in T \), a neighbor of \( y \) from \( U \) in \( G \) is called a \( U \)-neighbor of \( y \) in \( G \). For a vertex \( u \in U \) such that \( e_G(u, T) = 1 \), we let \( T(u) \) be the neighbor of \( u \) from \( T \).

We first show that \( G \) has no 2-factor.

**Claim 2.** The pair \((S, T)\) is a barrier of \( G \) with \( \delta(S, T) = -2 \). As a consequence, \( G \) does not have a 2-factor.

**Proof.** By the construction of \( G \), we know that \( T \) is an independent set in \( G \), and \( G - (S \cup T) \) has no even component. Thus \( \sum_{y \in T} d_{G-S}(y) = \sum_{k \geq 0} (2k + 1)|C_{2k+1}| \), and so

\[
\begin{align*}
\delta(S, T) &= 2|S| - 2|T| + \sum_{k \geq 0} (2k + 1)|C_{2k+1}| - \sum_{k \geq 0} |C_{2k+1}| \\
&= 2|S| - 2|T| + 2|C_3| + 38|C_{39}| \\
&= 2|S| - 2|T| + 2(p + q) + 38 = 182 - 462 + 240 + 38 = -2.
\end{align*}
\]

Thus by Theorem 2, \( G \) does not have a 2-factor. \( \Box \)

A vertex of a cutset \( W \) of \( G \) is said to be connected to a component of \( G - W \) if that vertex is adjacent in \( G \) to a vertex from the component. To finish proving Theorem 1, it remains to show that \( \tau(G) \geq \frac{3}{2} \). We will prove this by a contradiction. As \( \delta(G) = 3 \), we have \( \tau(G) \leq \frac{3}{2} \). Suppose to the contrary that \( \tau(G) < \frac{3}{2} \). We choose a cutset \( W \) of \( G \) such that

(a) \( h(W) := \frac{3}{2}c(G - W) - |W| \) is as large as possible; and
(b) subject to (a), \( |W| \) is as large as possible; and
(c) subject to (b), \( |W \cap T| \) is as small as possible.

Since we assumed \( \tau(G) < \frac{3}{2} \), there exists \( W \subseteq V(G) \) such that \( h(W) \geq \frac{1}{2} \). Also by the constraint (a) in the choice of \( W \), we have the following fact.
Fact 2. For any $W' \subseteq W$ such that $W \setminus W'$ is a cutset of $G$, vertices of $W'$ are connected in $G$ to at least $\frac{2}{3}|W'| + 1$ components of $G - W$. 

Our goal is to show that each component of $G - W$ is either a single vertex or an edge. This structure restriction of the components forces $W$ to be consisted of some special vertices. Based on that we will show that we actually have $h(W) < 0$, achieving a contradiction.

Claim 3. It holds that $W \cap (V(D) \setminus V(A_1)) = \emptyset$. As a consequence, $D - W$ is connected.

Proof. Recall that $D$ is the replacement graph for the vertex $w$ in $G_0$. Suppose to the contrary that $W \cap (V(D) \setminus V(A_1)) \neq \emptyset$. Let $W' = W \cap (V(D) \setminus V(A_1))$. As $W' \subseteq V(D) \setminus V(A_1)$ and $G$ is a plane graph, we know that for any component $Q$ of $G - W$ with $V(Q) \cap V(D) = \emptyset$, we have $E_G(W', Q) = \emptyset$. Hence the restriction of the components of $G - W$ to which vertices from $W'$ are connected are components of $D - W'$. Thus each vertex of $W'$ must be connected in $D$ to at least two components of $D - W'$ by Fact 2. Therefore $W'$ is a cutset of $D$, and so $c(D - W') \leq \frac{1}{2}|W'|$ as $D$ is 2-tough by Claim 1. As $h(W) > 0$ and $c(D - W') \leq \frac{1}{2}|W'|$, it follows that $|W| \geq |W'| + 1$ and $c(G - W) > \frac{2}{3}|W| > \frac{1}{2}|W'|$. Thus $c(G - W) \geq \lfloor \frac{1}{2}|W'| \rfloor + 1$ as $c(G - W)$ is an integer. Hence $c(G - (W \setminus W')) \geq c(G - W) - \lfloor \frac{1}{2}|W'| \rfloor + 2$, and so $W \setminus W'$ is a cutset of $G$. This gives a contradiction to Fact 2 since vertices of $W'$ are connected in $G$ to at most $\frac{1}{2}|W'|$ components of $G - W$. The consequence part of the statement is clear as every vertex of $V(A_1)$ has in $D$ a neighbor from $V(A_2)$ by the construction of $D$. \hfill \Box

Claim 4. Each component of $G - W$ has no cutvertex.

Proof. Suppose to the contrary that a component $Q$ of $G - W$ has a cutvertex, say $x$. Then $h(W \cup \{x\}) \geq h(W) + \frac{3}{2} > h(W)$, a contradiction to the choice of $W$. \hfill \Box

Claim 5. For any $y \in W \cap T$, we have $d_G(y) = 4$ and $N_G(y) \cap S \subseteq W$.

Proof. Since $y$ is connected to at least two components of $G - W$, it follows that $G[N_G(y)]$ is not a complete graph. Thus $d_G(y) = 4$ and so $G[N_G(y)]$ is a 4-cycle. By the construction of $G$, we have $|N_G(y) \cap S| = |N_G(y) \cap U| = 2$. Let $N_G(y) = \{u_1, u_2, s_1, s_2\}$ with $u_1, u_2 \in U$ and $s_1, s_2 \in S$. Then by Fact 2, we must have that either $u_1, u_2 \in W$ and $s_1$ and $s_2$ are separated in two different components of $G - W$, or $s_1, s_2 \in W$ and $u_1$ and $u_2$ are separated in two different components of $G - W$. Suppose to the contrary that $u_1, u_2 \in W$ and $s_1$ and $s_2$ are separated in two different components of $G - W$ is the case. For $i \in [1, 2]$, let $Q_i$ be the odd component of $G - (S \cup T)$ containing $u_i$. By the construction of $G$, each of $s_1$ and $s_2$ has two neighbors in $G$ from $V(Q_i)$, and $s_1$ and $s_2$ have in $G$ exactly one common neighbor from $V(Q_i)$. If $Q_i$ is a triangle, then $Q_i - W$ has at most one component; if $Q_i = D$, then again $Q_i - W$ is connected by Claim 3. As $s_1$ and $s_2$ are separated in two different components of $G - W$, we must have $|W \cap V(Q_i) \cap N_G(\{s_1, s_2\})| \geq 2$ for each $i \in [1, 2]$. Let $|(W \cap N_G(\{s_1, s_2\})) \cap V(Q_i)) \cup (W \cap N_G(\{s_1, s_2\}) \cap V(Q_j))| = j$, where $j \in [4, 6]$. As
implies that \( |N_G(s_1) \cap V(Q_i)| = |N_G(s_2) \cap V(Q_i)| = 2 \) and \( s_1 \) and \( s_2 \) have in \( G \) exactly one common neighbor from \( Q_i \), it follows that all vertices from \( W' := \{y\} \cup (W \cap N_G(\{s_1, s_2\}) \cap V(Q_1)) \cup (W \cap N_G(\{s_1, s_2\}) \cap V(Q_2)) \) are connected in \( G \) to at most \( 2 + (j - 4) = j - 2 \) components of \( G - W \): deleting two vertices from \( W \cap N_G(\{s_1, s_2\}) \cap V(Q_1) \) and two vertices from \( W \cap N_G(\{s_1, s_2\}) \cap V(Q_2) \) can create at most two components in \( G - W \), and each third deletion of a vertex from \( W \cap N_G(\{s_1, s_2\}) \cap V(Q_1) \) or \( W \cap N_G(\{s_1, s_2\}) \cap V(Q_2) \) can create at most one more component either by the adjacencies of vertices of a \( C_3 \)-triangle or by Claim 3 if one of \( Q_i \) is \( D \). As \( h(W) > 0 \) and \( |W| \geq j + 1 \), we have \( c(G - W) > \frac{2}{3}(j + 1) \). Since \( \frac{2}{3}(j + 1) - (j - 2) = \frac{2}{3} - \frac{1}{3} > 0 \), we know that \( W \setminus W' \) is a cutset of \( G \). This shows a contradiction to Fact 2 as \( j - 2 < \frac{2}{3}(j + 1) + 1 \) when \( j \in [4, 6] \). \( \square \)

**Claim 6.** For any \( y \in T \cap W \), \( y \) is connected in \( G \) to exactly two components of \( G - W \), and both of the components are trivial.

**Proof.** If \( y \in W \), then we have \( d_G(y) = 4 \) and \( N_G(y) \cap S \subseteq W \) by Claim 5. Thus \( y \) is connected in \( G \) to exactly two components of \( G - W \) by Fact 2. Furthermore, the two components of \( G - W \) connected to \( y \) in \( G \) respectively contain the two neighbors of \( y \) from \( U \). Let \( N_G(y) \cap U = \{u_1, u_2\} \) and assume that the component of \( G - W \) that contains \( u_1 \) is not trivial. Then \( (W \setminus \{y\}) \cup \{u_1\} \) is a cutset of \( G \) with the same size as \( W \) and \( h((W \setminus \{y\}) \cup \{u_1\}) = h(W) \). However \( (W \setminus \{y\}) \cup \{u_1\} \) contains less vertices from \( T \) than \( W \) does, a contradiction to the choice of \( W \). \( \square \)

**Claim 7.** For any \( y \in W \cap U \), we have \( N_G(y) \cap S \subseteq W \).

**Proof.** Let \( Q \) be the component of \( G - (S \cup T) \) containing \( y \). By the construction of \( G \) and Claim 3, we know that \( y \) is adjacent in \( G \) to a vertex from \( T \), two vertices from \( S \), and some vertices from \( Q \). As \( Q - W \) is connected, Fact 2 implies that \( y \) is connected in \( G \) to exactly two components of \( G - W \), where either the two components contain vertices of \( Q - W \) and \( N_G(y) \cap T \) respectively, or the two components respectively contain one distinct vertex from \( N_G(y) \cap S \). Let us suppose instead that \( N_G(y) \cap S \not\subseteq W \). Then we must have \( N_G(y) \cap T \subseteq W \). Let \( z \) be the vertex from \( N_G(y) \cap T \). As \( N_G(z) \cap S = N_G(y) \cap S \), Claim 5 implies \( N_G(y) \cap S \subseteq W \), a contradiction. \( \square \)

**Claim 8.** Let \( Q \) be a 2-connected component of \( G - W \), and \( F \) be the boundary of a face of \( Q \) such that \( E_G(F, W) \neq \emptyset \). Then \( V(F) \cap S = \emptyset \).

**Proof.** Suppose to the contrary that \( V(F) \cap S \neq \emptyset \). Let \( s_1 \in V(F) \cap S \). As \( s_1 \in V(F) \), \( E_G(V(F), W) \neq \emptyset \), and \( G \) is a plane triangulation, we have \( E_G(s_1, W) \neq \emptyset \). Then by Claims 5 and 7, we have \( N_G(s_1) \cap (T \cup U) \subseteq Q \). As a consequence, \( |V(Q)| \geq |N_G(s_1) \cap (T \cup U)| \geq 6 \). Since \( N_G(s_1) \cap (T \cup U) \subseteq Q \) and \( E_G(s_1, W) \neq \emptyset \), it follows that \( s_1 \) is one vertex from an \( S \)-triangle, say \( s_1s_2s_3s_4 \). Let \( t_1, t_2, t_3 \) be the three vertices from \( T \) such that \( t_is_{i+1} \in E(G) \) for each \( i \in [1, 3] \), where \( s_4 := s_1 \). Let \( u_1, u_2, u_3 \in U \) such that \( t_iu_i \in E(G) \) for \( i \in [1, 3] \).
If \(s_2, s_3 \in W\), then we have \(t_1, t_3 \in V(Q)\). Then \(c(Q - \{s_1, u_1, u_3\}) \geq 3\). Therefore, \(W \cup \{s_1, u_1, u_3\}\) is a cutset of \(G - W\) with \(h(W \cup \{s_1, u_1, u_3\}) \geq h(W)\), a contradiction to the choice of \(W\). Thus we assume that \(|W \cap \{s_2, s_3\}| \leq 1\). Since \(E_G(s_1, W) = E_G(s_1, W \cap \{s_2, s_3\}) \neq \emptyset\), we may assume, without loss of generality, that \(s_2 \in V(Q)\) and \(s_3 \in W\). Then \((N_G(s_1) \cup N_G(s_2)) \setminus \{s_3\} \subseteq V(Q)\) by Claims 5 and 7. This further implies that \(t_1, t_2, t_3 \in V(Q)\) by Claim 5. Let \(F'\) be the boundary of \(s_1s_2s_3s_1\), where recall that \(F'\) is defined as a face boundary of \(G^*\) before Claim 2. Then we know that one component of \(F' - W\) intersects with \(Q\), and the rest are paths.

Then we must have \(|V(F') \cap W \cap N_G(s_3)| \geq 5\) and \(c(F' - (W \cap N_G(s_3))) \geq 5\). For otherwise, let \(|V(F') \cap W| = j\) for \(j \in [2, 4]\). As \(F'\) is a cycle and so is 1-tough, we have \(c(F' - W) \leq j\). This implies that vertices of \(V(F') \cap W \cap N_G(s_3)\) are connected in \(G\) to at most \(j\) components of \(G - W\). However \(j < \frac{2}{5}(j + 1) + 1\), contradicting Fact 2. Thus \(|V(F') \cap W \cap N_G(s_3)| \geq 5\) and \(c(F' - (W \cap N_G(s_3))) \geq 5\).

When \(s_1s_2s_3s_1\) is associated with \(D\), by the construction of \(G\), we know that \(s_3\) is adjacent in \(G\) to at most 6 vertices from \(V(F')\) that are not contained in \(Q\). Thus it is impossible to get \(c(F' - (W \cap N_G(s_3))) \geq 5\) as \(F'\) is 1-tough. Therefore \(s_1s_2s_3s_1\) is not associated with \(D\). Since for any two \(S\)-triangles \(S_1\) and \(S_2\) that are not associated with \(D\), we have \(G[N_G(S_1)] \cong G[N_G(S_2)]\), we may assume that the component graph \(F'_0\) of \(F'\) is \(v_1v_8 \ldots v_{20}v_7\). We label vertices on \(F'\) as shown in Figure 4 for easily referring to them. For each \(i \in [7, 20]\), the triangle \(v_{i,1}v_{i,2}v_{i,3}v_{i,1}\) is the replace graph of \(v_i\) from \(F'_0\), \(\{a, b, c\} = \{s_1, s_2, s_3\}\), and \(\{t_1, t_2, t_3, t_4, t_5, t_6, \ldots, t_{14}\} \subseteq T\). Since \(s_1, s_2 \in V(Q)\) and \(G[N_G(b)] \cong G[N_G(c)]\), we may assume that \(s_1 = c\).

Consider first that \(s_2 = a\) and \(s_3 = b\). Let the \(S\)-vertex that is adjacent in \(G\) to each vertex from \(\{v_{20,3}, v_{20,1}, t_{14}, v_{19,2}, v_{19,1}, t_{13}, v_{18,2}, v_{18,3}\}\) be \(s^*\). If \(W \cup \{v_{18,2}, v_{19,1}\} \neq \emptyset\), then we must have \(s^* \in W\) by Claim 7. Then we have \(c(Q - \{a, c, v_{11,3}, v_{19,3}, v_{15,3}, t_{14}\}) \geq 5\). However, \(|W \cup \{a, c, v_{11,3}, v_{19,3}, v_{15,3}, t_{14}\}| > |W|\) and \(h(W \cup \{a, c, v_{11,3}, v_{19,3}, v_{15,3}, t_{14}\}) \geq h(W)\), a contradiction to the choice of \(W\). Therefore we assume that \(v_{18,2}, v_{19,1} \in V(Q)\) and so \(t_{13} \in V(Q)\) by Fact 2. Similarly, we have \(v_{15,2}, v_{10,1}, t_{10} \in V(Q)\). Thus the other components of \(F' - (W \cap N_G(s_3))\) not containing a vertex from \(V(Q)\) will possibly only contain vertices from \(\{t_{12}, v_{17,2}, v_{17,1}, t_{11}\}\). However, the maximum number of components we can have in \(F'[\{t_{12}, v_{17,2}, v_{17,1}, t_{11}\}]\) by deleting its vertices is 2. Thus \(c(F' - (W \cap N_G(s_3))) \leq 3\), a contradiction to the assumption that \(c(F' - (W \cap N_G(s_3))) \geq 5\).

Next we assume that \(s_2 = b\) and \(s_3 = a\). Suppose first that \(W \cap \{v_{19,2}, v_{20,1}\} \neq \emptyset\). Then the \(S\)-vertex adjacent in \(G\) to both \(v_{19,2}\) and \(v_{20,1}\) is contained in \(W\) by Claim 7. Then we have \(c(Q - \{b, c, v_{11,3}, v_{19,3}, v_{15,3}, t_{13}\}) \geq 5\). However, \(|W \cup \{b, c, v_{11,3}, v_{19,3}, v_{15,3}, t_{13}\}| > |W|\) and \(h(W \cup \{b, c, v_{11,3}, v_{19,3}, v_{15,3}, t_{13}\}) \geq h(W)\), a contradiction to the choice of \(W\). Thus \(W \cap \{v_{19,2}, v_{20,1}\} = \emptyset\). Thus \(v_{19,2} \in V(Q)\) and so \(t_{14} \in V(Q)\) by Claim 6. Hence \(v_{20,1} \in V(Q)\) as \(W \cap \{v_{19,2}, v_{20,1}\} = \emptyset\). Suppose then that \(W \cap \{v_{20,2}, v_{7,1}\} \neq \emptyset\). Then the \(S\)-vertex adjacent in \(G\) to both \(v_{20,2}\) and \(v_{7,1}\) is contained in \(W\) by Claim 7. Let \(s^*\) be the \(S\)-vertex adjacent to both \(t_{13}\) and \(t_{14}\). Then have \(c(Q - \{b, c, v_{11,3}, v_{19,3}, v_{15,3}, t_{14}, v_{19,1}, v_{18,2}, s^*\}) \geq 7\). However,
\[|W \cup \{b, c, v_{11,3}, v_{19,3}, v_{15,3}, t_{14}, v_{19,1}, v_{18,2}, s^*\}| > |W| \text{ and} \]
\[h(W \cup \{b, c, v_{11,3}, v_{19,3}, v_{15,3}, t_{14}, v_{19,1}, v_{18,2}, s^*\}) \geq h(W),\]
a contradiction to the choice of \(W\). Thus \(W \cap \{v_{20,2}, v_{7,1}\} = \emptyset\). Thus \(v_{20,2} \in V(Q)\) and so \(t_1 \in V(Q)\) by Claim 6. Hence \(v_{7,1} \in V(Q)\) as \(W \cap \{v_{20,2}, v_{7,1}\} = \emptyset\). Symmetrically, we also must have \(v_{11,1}, t_5, v_{10,2}, v_{10,1}, t_4, v_{9,2} \in V(Q)\). Thus the other components of \(F' - (W \cap N_G(s_3))\) not containing a vertex from \(V(Q)\) will possibly only contain vertices from \(\{t_2, v_{8,1}, v_{8,2}, t_3\}\). However, the maximum number of components we can have in \(F'[\{t_2, v_{8,1}, v_{8,2}, t_3\}]\) by deleting its vertices is 2. Thus \(c(F' - (W \cap N_G(s_3))) \leq 3\), a contradiction to the assumption that \(c(F' - (W \cap N_G(s_3))) \geq 5\).

Therefore \(V(F) \cap S = \emptyset\), proving the statement. \(\square\)

Figure 4: An \(S\)-triangle \(abca\) and the neighbors of the vertices \(a, b, c\). The edges joining \(a, b\) and \(c\) to other vertices for triangulating the three faces of length more than 3 are omitted in the drawing.

**Claim 9.** Let \(P\) be a spoke of \(G\) with endvertices as \(x\) and \(y\), where \(x \in V(D)\) and \(y \in V(C)\). If it holds that \(x \notin W\) or that \(y \notin W\) but \(y\) is not a component of \(G - W\), then \(P\) is contained in a component of \(G - W\).

**Proof.** Suppose first that \(x \notin W\). Then the vertex, say \(t_x\), from \(T\) adjacent to \(x\) is not contained in \(W\) by Claim 6. Let \(s_1, s_2 \in S\) be the two neighbors of \(t_x\) in \(G\). If \(\{s_1, s_2\} \notin W\),
say \( s_1 \not\in W \), then none of the \( S \)-vertices with the same boundary as \( s_1 \) is contained in \( W \) by Claim 8. As a consequence, we have \( V(P) \cap W = \emptyset \) by Claims 5 and 7. Thus we assume \( s_1, s_2 \in W \). As a consequence, all the \( S \)-vertices with the same boundary as \( s_1 \) or \( s_2 \) are contained in \( W \) by Claim 8. Then as every component of \( G - W \) has no cutvertex by Claim 4, we know that \( P \) is entirely contained in a component of \( G - W \). The argument for \( y \not\in W \) follows from the same idea.

Claim 10. It holds that \( G - W \) has at most one 2-connected component, which is the component containing a vertex of \( D \).

Proof. Otherwise, let \( Q \) be a 2-connected component of \( G - W \) that contains no vertex of \( D \), and let \( Q_0 \) be the component graph of \( Q \). Then \( Q_0 \) is also 2-connected by Claim 8: the boundary \( F \) of each face of \( Q \) for which \( E_G(F, W) \neq \emptyset \) is a cycle containing no vertex of \( S \) and so the component graph \( F_0 \) of \( F \) is a cycle in \( G_0 \). Since \( Q \) does not contain any vertex of \( D \), it follows that any graph from \( C \) that intersects \( Q \) is a \( C_3 \)-triangle. Let \( F \) be a face of \( Q \) with a vertex \( u \in V(F) \cap U \) satisfying \( e_G(u, W \cap U) \geq 1 \). Let \( u' \in W \cap U \) be a vertex for which \( e_G(u, u') = 1 \), and let \( t_{u'} \in T \) such that \( u't_{u'} \in E(G) \). Then as \( u' \in W \), we have \( N_G(u') \cap S \subseteq W \) by Claim 7 and \( t_{u'} \) being a component of \( G - W \) by Fact 2. Let \( u^* \) be the other vertex for which \( u, u' \) together form a \( C_3 \)-triangle. Then we must have \( u^* \in V(Q) \).

For otherwise, we have \( u^* \in W \) and so the \( S \)-vertex that is adjacent to both \( u^* \) and \( u \) in \( G \) is also contained in \( W \) by Claim 7. Thus \( u, u^* \in V(Q) \) and \( N_G(Q) \cap T \subseteq V(Q) \) by Claim 6. Hence if a \( C \)-triangle intersects both \( W \) and \( Q \), then the vertex of \( Q_0 \) that is corresponding to this \( C \)-triangle has degree 2 in \( Q_0 \).

Let \( b \) be the number of \( C_3 \)-triangles that intersects both \( W \) and \( Q \), and \( f_s \) be the number of \( S \)-triangles in \( Q \). We show that \( Q_0 \) has exactly \( 3f_s + b \) vertices of degree 2. Let \( v \in V(Q_0) \) such that \( d_{Q_0}(v) = 2 \), and let \( R \) be the \( C_3 \)-triangle corresponding to \( v \). If \( V(R) \cap W \neq \emptyset \) and \( V(R) \cap V(Q) \neq \emptyset \), then \( v \) corresponds to a \( C_3 \)-triangle that intersects both \( W \) and \( Q \) and so there are at most \( b \) such degree 2 vertices in \( Q \). Thus we assume \( R \subseteq Q \). Then \( v \) must also be a vertex of \( G_0 \) that is of degree 2 in \( G_0 \). By Claim 6, all the three vertices, say \( x, y, z \), from \( T \) that are adjacent in \( G \) to vertices from \( R \) are contained in \( Q \) as well. Since \( Q \) is 2-connected, some vertices from the \( S \)-triangle \( S^* \) for which some of its vertices are adjacent in \( G \) to \( x, y, z \) are contained in \( Q \) too. By Claim 8, the entire \( S \)-triangle \( S^* \) is contained in \( Q \). Let \( F \) be the boundary of the face of \( G_2 \) such that \( S^* \) is embedded inside \( F \). By Claims 5 and 7, we know that \( F \subseteq Q \). As \( F \) contains vertices from all the three \( C_3 \)-triangles associated with \( S^* \) where one of them is \( R \), it follows that the three \( C_3 \)-triangles in \( Q \) that are associated with \( S^* \) are all contained in \( Q \). Thus \( v \) is corresponding to one \( C_3 \)-triangle that is associated with \( S^* \) and so there are at most \( 3f_s \) such degree 2 vertices in \( Q \). The above argument shows that the number of vertices of degree 2 in \( Q_0 \) is at most \( b + 3f_s \).

On the other hand, by the argument from the first paragraph of this proof, we know
that every $C_3$-triangle $R$ that intersects both $W$ and $Q$ corresponds to a vertex of degree 2 in $Q_0$. Furthermore, for any $S$-triangle $S^*$ with $S^* \subseteq V(Q)$, let $F$ be the boundary of the face of $G_2$ such that $S^*$ is embedded inside $F$. Then we have $F \subseteq Q$ by Claims 5 and 7. As $F$ contains all the three $C_3$-triangles associated with $S^*$ together with the neighbors of the vertices of $S^*$ from $T$ in $G$, it follows that $S^*$ corresponds to three vertices of degree 2 of $Q_0$. Thus $Q_0$ has at least $b + 3f_s$ vertices of degree 2. Combining this with the assertion that the number of vertices of degree 2 in $Q_0$ is at most $b + 3f_s$, we know that $Q_0$ has exactly $3f_s + b$ vertices of degree 2. Thus we have $e(Q_0) = \frac{1}{2}(3n(Q_0) - 3f_s - b)$. Consequently by Euler’s formula, $f(Q_0) = e(Q_0) - n(Q_0) + 2 = 0.5n(Q_0) - 1.5f_s - 0.5b + 2$.

Let $S^*$ be the set of $S$-vertices that are embedded inside a face of $G_2$ with boundary $F$. By Claim 8, we have either $S^* \subseteq V(Q)$ or $S^* \n V(Q) = \emptyset$. If $S^* \subseteq V(Q)$, then we have $F \subseteq Q$ by Claims 5 and 7. Thus $Q_0$ has a face whose boundary is the component graph of $F$. Therefore we have $|V(Q) \cap S| \leq f(Q_0) - 1 + 2f_s$ as at least one face of $Q$ whose boundary has vertices adjacent in $G$ to vertices from $W$ and so there is no $S$-vertex embedded in $Q$ inside that face.

We will contract a cutset $W_Q$ of $Q$ for which $h(W \cup W_Q) \geq h(W)$ to get a contradiction to the choice of $W$. Let $M_0$ be a maximum matching in the component graph $Q_0$ of $Q$. For each $uv \in M_0$, there exists $x_u \in R(u)$ and $x_v \in R(v)$ such that $x_u$ and $x_v$ are both adjacent in $G$ to a vertex from $T$, where recall that $R(u)$ is the $C_3$-triangle corresponding to $u$. We call $x_u$ a representative vertex of $R(u)$. As $M_0$ is a matching, each component from $C_3$ either has no representative vertex or has a unique representative vertex.

Let $R \in C_3$ such that $V(R) \cap V(Q) \neq \emptyset$. By the argument in the first paragraph in the proof of Claim 10, we have that either $R \subseteq Q$ or $|V(R) \cap V(Q)| = 2$. If $R$ has a representative vertex, say $x$, let $W_R \subseteq (V(R) \cap V(Q)) \setminus \{x\}$ be the set of two vertices (if $R \subseteq Q$) or one vertex (if $|V(R) \cap V(Q)| = 2$) such that $e_G(R - W_R, T) = e_G(x, T)$. Otherwise, let $W_R \subseteq V(R)$ be a set of two vertices (if $R \subseteq Q$) or one vertex (if $|V(R) \cap V(Q)| = 2$) such that $e_G(R - W_R, T) = 1$.

Note that for any two representative vertices $x_1$ and $x_2$, $x_1$ and $x_2$ are adjacent in $G$ to the same vertex from $T$. We let $T^*$ be the set of all these vertices from $T$ that are adjacent in $G$ to a representative vertex of a $C_3$-triangle that intersects $Q$.

Let $W_Q$ be the set that consists of all vertices in $S \cap V(Q)$, $T^*$, and $W_R$ for all $R \in C_3$ such that $|V(R) \cap V(Q)| \geq 2$. Thus $|W_Q| \leq f(Q_0) - 1 + 2f_s + 2n(Q_0) - b + |T^*|$. Each component of $Q - W_Q$ is either a vertex from $T$, or an edge consisting of a vertex from $T$ and a vertex from $U$, or a vertex from $U$. For the last case, the vertex from $U$ corresponds to an endvertex of an edge of $M_0$. Since the three vertices from $T_1 \cap V(Q)$ that are adjacent in $G$ to vertices from every $S$-triangle of $Q$ are contained in $Q$ by Claim 5, we get $c(Q - W_Q) = e(Q_0) - |T^*| + 3f_s + 2|T^*|$.

Since $Q$ contains no vertex of $D$ and so contains no spoke of $G$, it follows that every $C_3$-triangle $R$ such that $V(R) \cap V(Q) \neq \emptyset$ and that $R$ contains a vertex, say $x$, of a spoke of
\(G\) satisfies the property that \(x \in W\) by Claim 9. As a consequence, all the \(S\)-triangles that are associated with \(D\) and are adjacent in \(G\) to a vertex from \(V(Q) \cap V(C)\) are all contained in \(W\) by Claims 7 and 8. Thus \(Q_0\) contains only vertices of \(C_0\). If a vertex \(t \in T_2 \cap V(C)\) is not contained in \(Q\), then the two \(S\)-vertices, say \(s_1\) and \(s_2\), adjacent to \(t\) in \(G\) must be contained in \(W\) by Claims 5 (if \(t \in W\)) or 6 (if \(t \notin W\)). Thus the two set of \(S\)-vertices with the same boundary as \(s_1\) or \(s_2\) are all contained in \(W\) by Claim 8. As \(Q_0\) is 2-connected, it then follows that \(Q_0 \cap C_0\) is connected. This, together with the fact that \(Q_0\) contains only vertices of \(C_0\) gives \(|T^*| = \lfloor \frac{1}{2} n(Q_0) \rfloor\). Letting \(\alpha = 0\) if \(n(G_0)\) is even and \(\alpha = 1\) if \(n(G_0)\) is odd, then we have

\[
\frac{|W_Q|}{e(Q - W_Q)} \leq \frac{f(Q_0) - 1 + 2f_s + 2n(Q_0) - b + \lfloor \frac{1}{2} n(Q_0) \rfloor}{e(Q_0) + 3f_s + \lfloor \frac{1}{2} n(Q_0) \rfloor} = \frac{3n(Q_0) + 0.5f_s + 1 - 1.5b - \alpha}{2n(Q_0) + 1.5f_s - 0.5b - \alpha}.
\]

We claim that \(b \geq 6\). We already argued that \(Q_0 \cap C_0\) is connected. Then as any cycle of \(G_0 - w\) has length at least 8, we know that \(Q_0\) contains at least 8 consecutive vertices of \(C_0\). As \(Q_0 \cap C_0\) is connected and \(Q_0\) is 2-connected, \(Q_0\) contains a cycle that contains \(Q_0 \cap C_0\) as a subgraph. As any cycle of \(G_0 - w\) that contains some consecutive vertices of \(C_0\) has at least 6 vertices that are adjacent in \(G_0\) to vertices from \(\{u_1, \ldots, u_{34}\}\), we get \(e_{G_0}(Q_0, G_0 - V(Q_0)) \geq 6\). For any \(u \in V(Q_0)\) such that \(e_{G_0}(u, G_0 - V(Q_0)) \geq 1\), we know that \(R(u)\) intersects both \(W\) and \(Q\) by the same argument as in the first paragraph of this proof. Thus \(e_{G_0}(Q_0, G_0 - V(Q_0)) \geq 6\) implies \(b \geq 6\). Hence \(1.5(c(Q - W_Q) - 1) - |W_Q| \geq 1.75f_s + 0.75b - 0.5\alpha - 2.5 \geq 0\). Then we have \(|W \cup W_Q| > |W|\) and \(h(W \cup W_Q) \geq h(W)\), giving a contradiction to the choice of \(W\).

By Claim 10, \(G - W\) has a unique 2-connected component that contains vertices of \(D\). We let \(Q\) be that component.

**Claim 11.** Let \(R\) be a subgraph of \(G - V(Q)\), and let \(R_0\) be the component graph of \(R\). Then the vertices of \(R\) can be separated in at most \(e(R_0) + |V(R) \cap T_1| + \alpha'(R_0)\) components of \(G - W\), where \(\alpha'(R_0)\) is the size of a maximum matching in \(R_0\).

**Proof.** Every component of \(R - W\) is either a single vertex or an edge by Claims 4 and 10. Since each vertex from \(S \cap V(R)\) has in \(G\) (and also in \(R\) by Claim 8) more than 4 neighbors from \(T \cup U\), Claims 5 and 7 imply that \(V(R) \cap S \subseteq W\). Hence each component of \(R - W\) is either a vertex from \(T \cap V(R)\), an edge consisting of one vertex from \(T \cap V(R)\) and one vertex from \(U_3 \cap V(R)\), or a vertex from \(U_3 \cap V(R)\). Note that it is impossible to have a component of \(R - W\) that is an edge consisting of two vertices say \(x\) and \(y\), from \(U_3\) by Claim 6, as if \(xy\) were a component of \(R - W\), then we must have \(T(x), T(y) \in W\) where recall that \(T(x)\) is the neighbor of \(x\) from \(T\) in \(G\), but that gives a contradiction to Claim 6. Also it is impossible to have a component in \(R - W\) containing a vertex of \(D\) as \(D - W\) is contained in \(Q\) by Claim 3.
Each component of $R - W$ that contains a vertex from $T$ either corresponds to an edge of $R_0$ (if the vertex is from $T_2$) or is a vertex from $V(R) \cap T_1$. For each component of $R - W$ that is a single vertex $u$ from $U_3$, we have $T(u) \subseteq W$. Thus $T(u) \in T_2$ by Claim 5. Let $v$ be the other $U$-neighbor of $T(u)$. Then both $u$ and $v$ are trivial components of $R - W$ by Claim 6. As $uT(u)v$ corresponds to an edge of $R_0$, the set of pairs of such components $u$ and $v$ of $R - W$ corresponds to a matching of $R_0$. Let $\alpha$ be the total number of pairs of such components $u$ and $v$. As the $\alpha$ vertices from $T$ that are adjacent in $G$ to such pairs of vertices $u$ and $v$ are contained in $W$, we know that $R - W$ has exactly $e(R_0) + |V(R) \cap T_1| - \alpha + 2\alpha$ components. The conclusion of the claim now follows since $\alpha \leq \alpha'(R_0)$.

**Claim 12.** Let $F$ be the boundary of an $S$-triangle $S^*$ with $S^* \subseteq W$. Then vertices of $F - W$ are separated in at least 6 distinct components of $G - W$ and so $c(F - W) \geq 6$.

**Proof.** We adopt the labeling of vertices from Figure 5 for convenient description. Let

$$F = w_1t_1x_{1,1}x_{1,2}t_2x_{1,2}t_3x_{3,3}x_{3,1}x_{3,2}t_4x_4t_5x_5t_6x_6w_2w_1.$$  

As $S^* \subseteq W$ and $Q$ is 2-connected, it follows that none of the three $T$-vertices respectively incident with $x_{1,3}, x_{3,3}, x_{5,3}$ is contained in $Q$. Furthermore, any of these $T$-vertices cannot be contained in $W$ by Claim 6. Therefore, these three $T$-vertices are three components of $G - W$. As a consequence, we must have $x_{1,3}, x_{3,3}, x_{5,3} \in W$.

Suppose otherwise that vertices of $F - W$ are separated in at most 5 distinct components of $G - W$. Let $|V(F) \cap W| = j$ for some integer $j \geq 0$. As $F$ is a cycle and so is 1-tough, we have $c(F - W) \leq j$, and so vertices of $F$ are separated in at most $\max\{j, 1\}$ distinct components of $G - W$ as well. Thus $j \leq 5$ and vertices from $V(F) \cap W \cup S^* \cup \{x_{1,3}, x_{3,3}, x_{5,3}\}$ are connected in $G$ to at most $\max\{1 + 3, j + 3\}$ components of $G - W$. We Claim that $W' := W \setminus (V(F) \cap W \cup S^* \cup \{x_{1,3}, x_{3,3}, x_{5,3}\})$ is still a cutset of $G$. It suffices to show that $G - W$ has a component that is connected to vertices of $W'$ but is not connected to vertices of $V(F) \cap W \cup S^* \cup \{x_{1,3}, x_{3,3}, x_{5,3}\}$. Suppose instead that vertices of $V(F) \cap W \cup S^* \cup \{x_{1,3}, x_{3,3}, x_{5,3}\}$ are connected to all the components of $G - W$. If $j = 0$, then we have $|W| \geq 6$ and $c(G - W) = 4$ as vertices of $V(F) \cap W \cup S^* \cup \{x_{1,3}, x_{3,3}, x_{5,3}\}$ are connected to exactly four components of $G - W$. However this gives $h(W) = 0$. Thus $j \geq 1$. By Claims 5 and 7, $W$ contains an $S$-vertex that is adjacent in $G$ to the vertices from $V(F) \cap W$ but is not contained in $S^*$. Thus $|W| \geq j + 7$, but we get $h(W) \leq \frac{3}{2}(j + 3) - (j + 7) \leq 0$ when $j \leq 5$. Therefore $G - W$ has a component that is connected to vertices of $W'$ but is not connected to vertices of $V(F) \cap W \cup S^* \cup \{x_{1,3}, x_{3,3}, x_{5,3}\}$, implying that $W'$ is a cutset of $G$. However, as vertices from $V(F) \cap W \cup S^* \cup \{x_{1,3}, x_{3,3}, x_{5,3}\}$ are connected in $G$ to at most $\max\{1 + 3, j + 3\}$ components of $G - W$ and $\max\{1 + 3, j + 3\} < \frac{3}{2}(j + 6) + 1$ when $j \leq 5$, we get a contradiction to Fact 2. Thus vertices of $F - W$ are separated in at least 6 distinct components of $G - W$ and so $c(F - W) \geq 6$.

**Claim 13.** Let $F$ be the boundary of an $S$-triangle $S^*$ that is associated with $D$. Suppose that $S^*$ is contained in $W$, then $S^*_1, S^*_r, S^*_c \subseteq W$. As a consequence, none of the spoke and the C-segment of $F$ is contained in $Q$.  

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Proof. We adopt the labels of vertices in Figure 5 for convenient description. Thus let 
\[ F = w_1t_1x_1,1x_1,2t_2x_2,1x_2,2t_3x_3,1x_3,2t_4x_4,1x_4,2t_5x_5,1x_5,2t_6w_2w_1. \]

By the same argument as in Claim 12, we have \( x_{1,3}, x_{3,3}, x_{5,3} \in W. \)

If \( S^*_c \not\subseteq W, \) then we have \( S^*_c \subseteq V(Q) \) by Claim 8. Then by the construction of of \( G_0 \) and Claim 11, we have \( c(F - W) \leq 5, \) showing a contradiction to Claim 12. Thus \( S^*_c \subseteq W. \)

We consider now, by symmetry, that \( S^*_c \subseteq V(Q) \) and \( S^*_t \subseteq W. \) Suppose \( w_2t_6x_5,2x_5,1t_5x_{4,2} \) is part of the boundary of \( S^*_c. \) By Claim 12, we need to have \( c(F - W) \geq 6. \) This in particular, implies that \( x_{4,1} \in W. \) Let \( W_Q = \{ w_2, x_5, 2, t_5, x_{4,3}, x \} \cup S^*_r, \) where \( x \) is the other \( U \)-neighbor of \( T(x_{4,3}). \) Then \( Q - W_Q \) has at least 5 components: the vertex \( t_6, \) the vertex \( x_{5,1}, \) the vertex \( x_{4,2}, \) the vertex \( T(x_{4,3}), \) and the component containing a vertex of \( D. \) Since \( |W_Q| = 6 \) and \( c(G - (W \cup W_Q)) \geq c(G - W) + 4, \) and so \( h(W \cup W_Q) \geq h(W), \) we get a contradiction to the choice of \( W. \) Therefore \( S^*_t, S^*_c, S^*_e \subseteq W. \)

We argue that it is impossible to have any of the two spokes contained in \( F \) to be contained in \( Q. \) Suppose, by symmetry, that \( w_2t_6x_5,2x_5,1t_5x_{4,2} \) is contained in \( Q. \) Then as \( c(F - W) \geq 6, \) we must have \( x_{4,1} \in W. \) Since \( Q \) is 2-connected and \( S^*_t, S^*_c, S^*_e \subseteq W, \) it follows that the \( U \)-neighbor, say \( x \) of \( T(x_{4,3}) \) is contained in \( Q. \) Let \( W_Q = \{ w_2, x_5, 2, t_5, x_{4,3}, x \}. \) Then again we get \( h(W \cup W_Q) \geq h(W), \) a contradiction to the choice of \( W. \) Lastly, the \( C \)-segment \( x_{2,1}x_{2,2}t_3x_{3,1}x_{3,2}t_4x_{4,1}x_{4,2} \) of \( F \) is not contained in \( Q \) since otherwise \( c(F - W) \leq 5 \) by Claim 11.

![Figure 5](image-url)

Figure 5: A subgraph of \( G \) containing four spokes, where each black vertex \( t_i \) with \( i \in [1, 14] \) is a \( T \)-vertex and \( w_1, w_2, w_3, w_4 \in V(D). \)

Claim 14. Let \( F \) be the boundary of an \( S \)-triangle \( S^* \) that is associated with \( D. \) Suppose \( S^* \subseteq V(Q), S^* \) is internal, and one of \( S^*_t \) or \( S^*_c \) is contained in \( W. \) Then \( S^*_e \subseteq V(Q). \)
Proof. We adopt the labels of vertices in Figure 5 for convenient description. Thus let

\[ F = w_1t_1x_1,1x_1,2t_2x_2,1x_2,2t_3x_3,1x_3,2t_4x_4,1x_4,2t_5x_5,1x_5,2t_6w_2w_1. \]

Since \( S^* \subseteq V(Q) \), we have \( F \subseteq Q \) and \( x_1,1, x_3,3, x_5,3, T(x_1,3), T(x_3,3), T(x_5,3) \in V(Q) \) by Claims 5 and 7.

Suppose, by symmetry, that \( S^*_r \subseteq W \) and \( w_2t_6x_5,2x_5,1t_5x_4,2 \) is part of the boundary of \( S^*_r \). Suppose to the contrary that \( S^*_r \subseteq W \). Consider first that \( x_2,3 \in W \). Let \( W_Q = \{x_1,3, x_3,3, x_5,3, w_1, x_1,1, t_2, x_2,2, x_3,1, x_4,1\} \cup S^* \). Then we know that \( |W_Q| = 12 \) and \( c(G - (W \cup W_Q)) \geq c(G - W) + 8 \) and so \( h(W \cup W_Q) = h(W) \), a contradiction to the choice of \( W \).

Thus we suppose \( x_2,3 \not\in W \). Let \( F_l \) be the boundary of \( S^*_l \). Then \( x_2,3 \in V(Q) \). As a consequence, we have \( t_7 \in V(Q) \) by Claim 6 and so \( z_1,2 \in V(Q) \) by Claim 7. We claim that \( z_1,1 \in W \). For otherwise, by the same logic as in the line above, we have \( t_8, y_4,2 \in V(Q) \). Let \( S' \) be the \( S \)-triangle with boundary \( F' = w_3t_9y_1,1y_1,2 \ldots y_5,1y_5,2t_4w_4w_3 \). Then we must have \( S' \in W \): otherwise \( F' \subseteq Q \), and so vertices of \( S^*_r \) are only connected in \( G \) to the component \( Q \) of \( G - W \), contradicting Fact 2. Now as \( S' \) is in \( W \), we have \( S'_l, S'_r, S'_c \in W \) by Claim 13, and none of the spokes of \( F' \) is contained in \( Q \) and the \( C \)-segment of \( F' \) is not contained in \( Q \) as well. Since none of the spokes of \( F' \) is contained in \( Q \), we have \( w_3, w_4 \in W \) by Claim 9. Since the \( C \)-segment of \( F' \) is not contained in \( Q \), some vertex of the \( C \)-segment of \( F' \) is contained in \( W \). Now as \( S'_l, S'_r, S'_c \in W \), it follows that the vertex \( z_1,1 \) is a cutvertex of \( Q \) (deleting \( z_1,1 \) separates \( t_8 \) with \( z_1,2 \)), a contradiction to \( Q \) being 2-connected.

Thus we have \( z_1,1 \in W \). By the same argument as above, we must have \( S' \in W \): otherwise \( F' \subseteq Q \), and so the maximum number of components from the boundary of \( S^*_l \) we can get is by deleting \( y_4,3 \) and \( z_1,1 \) to get \( t_8 \) and the rest as two components. As \( G \) is 3-connected and \( h(W) > 0 \), we know that \( c(G - W) \geq 3 \). As we deleted the vertex in \( S^*_l \) and at most two vertices \( y_4,3 \) and \( z_1,1 \) from \( F_l \) and these three vertices are connected in \( G \) to at most two distinct components of \( G - W \), we know that \( W \setminus (S^*_l \cup \{y_4,3, z_1,1\}) \) is a cutset of \( G \). However, \( 2 < \frac{2}{3} \times 3 + 1 \) contradicts Fact 2. Thus \( S' \) is in \( W \), and so \( S'_l, S'_r, S'_c \in W \) by Claim 13.

Let \( u \) be the other \( U \)-neighbor of \( T(z_1,3) \), and

\[ W_Q = \{z_1,2, u, x_2,2, x_2,3, x_3,1, x_3,3, x_4,1, t_2, x_1,1, x_1,3, x_5,3, w_1\} \cup S^*. \]

Then we created 10 extra components after deleting all the 10 vertices of \( W_Q \). Those components include single vertex components and components consisting of an edge:

\[ t_1, x_1,2, x_2,1, t_7, T(z_1,3)z_1,3, t_3, x_3,2t_4 \]

and the three vertices from \( T_1 \) that are associated with \( S^* \). However we get \( h(W \cup W_Q) \geq h(W) \), a contradiction to the choice of \( W \). □
Let $S^*$ be an $S$-triangle associated with $D$. If $S^* \subseteq V(Q)$, then the two spokes of $G$ that are contained in the boundary of $S^*$ are both contained in $Q$ by Claims 5 and 7. If $S^* \not\subseteq V(Q)$ then we have $S^* \subseteq W$ by Claim 8. By Claim 13, none of the two spokes contained in the boundary of $S^*$ is contained in $Q$, and the $C$-segment of the boundary of $S^*$ is also not contained in $Q$.

We say that two $S$-triangles $S^*$ and $S'$ associated with $D$ are in the same patch if $S^*_0 = S'_0$. A patch of $S$-triangles is the union of all the $S$-triangles that are from the same patch. Similarly, a patch of spokes is the union of the spokes contained in the boundaries of a patch of $S$-triangles. By Claim 14 and the argument immediately above, a patch of $S$-triangles are either all contained in $Q$ or all contained in $W$. In particular, when a patch of $S$-triangles are all contained in $Q$, then by Claims 5 and 7, all the spokes that are contained in the boundary of this patch of $S$-triangles are contained in $Q$ as well. When a patch of $S$-triangles are all contained in $W$, none of the spokes from the corresponding patch is contained in $Q$ by Claim 13.

Let $Q_0$ be the component graph of $Q$, $a = |V(Q) \cap V(A)|$, $b$ be the number of $C_3$-triangles that intersects both $W$ and $Q$, and $c$ be the number of components of $Q \cap C$. As $Q$ is 2-connected, Claim 8 implies that $Q_0$ is 2-connected. We first claim that $b \geq 2c$.

**Claim 15.** It holds that $b \geq 2c \geq 2$.

**Proof.** The conclusion is obvious if $Q_0 \cap C_0 \neq C_0$, as in this case we have $b = e_{G_0}(Q_0 - w, W \cap V(G_0)) \geq e_{G_0}(Q_0 \cap C_0, W \cap V(G_0)) \geq 2c$. Thus we may assume that $Q_0 \cap C_0 = C_0$ and $b \leq 1$. This implies that at most one spoke of $G$ is not contained in $Q$. By Claim 13, every $S$-triangle associated with $D$ is contained in $Q$, and the same holds for all the $S$-triangle not associated with $D$ as a patch of $S$-triangles are either all contained in $Q$ or all contained in $W$. Thus $T_1 \subseteq V(Q)$ by Claim 5. Then as $Q_0 \cap C_0 = C_0$, it follows that $G_0 - V(Q_0) - w$ consists of isolated vertices. This together with the fact that every component of $G - W - V(Q)$ that is a single vertex from $U_3$ corresponds to an edge of $G_0 - V(Q_0) - w$, it implies that no component of $G - W - V(Q)$ is a single vertex from $U_3$. Thus every component of $G - W - V(Q)$ is either a vertex from $T_2$ or an edge consisting of a vertex from $T_2$ and a vertex from $U_3$. Since $b \leq 1$ and $Q_0 \cap C_0 = C_0$, it follows that $Q_0$ contains all the edges that incident in $G_0$ with a vertex of $C_0$. Then as at most one spoke of $G$ is not contained in $Q$, it follows that $c(G - W) \leq 3$. As $G$ is 3-connected and $h(W) > 0$, it follows that $c(G - W) = 3$. Thus exactly one long spoke, say $P$, of $G$ is not contained in $Q$. Since we need to delete the two endvertices of $P$ for it to be not contained in $Q$ by Claim 9, at least three vertices of $P$ need be deleted to disconnect $P$ into two components. We also need to delete at least two $S$-vertices that have $P$ has part of their boundaries. Thus $|W| \geq 5$, showing a contradiction to $h(W) > 0$. □

In the next, we will show that we can construct a cutset $W_Q$ of $Q$ that contains $V(A) \cap V(Q)$ and that $1.5c(Q - W_Q) - 1 - |W_Q| \geq -0.75$. As $h(W) > 0$ implies $h(W) \geq 0.5$, it will then follow that $h(W \cup W_Q) \geq -0.25$. This will lead to a contradiction as $h(W \cup W_Q)$ should be at most $-0.5$. If $a = 0$, then we let $W_Q = \emptyset$, which certainly
satisfying $1.5c(Q - W_Q) - |W_Q| \geq 0.75$. Thus we assume $a \geq 1$ and construct the set $W_Q$. Let $f_s$ be the number of $S$-triangles contained in $Q$.

By the same argument as in the proof of Claim 10 and Euler’s formula, we have

\[ e(Q_0) = 0.5(3(n(Q_0) - 1) + a - 3f_s - b) = 1.5n(Q_0) + 0.5a - 1.5f_s - 0.5b - 1.5, \]
\[ f(Q_0) = e(Q_0) - n(Q_0) + 2 = 0.5n(Q_0) + 0.5a - 1.5f_s - 0.5b + 0.5. \]

Let $M_0$ be a maximum matching in $Q_0$, and let $a'$ be the number of short spokes that are contained in $Q$. As $e(Q_0 \cap C_0) = e(Q \cap C)$ and each component of $Q_0 \cap C_0$ is a path, we know that $|M_0| \geq 0.5(n(Q_0) - (a - a') - c)$. For each $uv, x \in R(u)$ and $x_v \in R(v)$ such that $x_u$ and $x_v$ are both adjacent in $G$ to a vertex $y \in T$. We call $x_u$ a representative vertex of $R(u)$. As $M_0$ is a matching, each graph from $C_3$ either has no representative vertex or has a unique representative vertex.

Let $R \in C_3$ such that $V(R) \cap V(Q) \neq \emptyset$. By the argument as in the first paragraph in the proof of Claim 10, we have that either $R \subseteq Q$ or $|V(R) \cap V(Q)| = 2$. If $R$ has a representative vertex, say $x$, let $W_R \subseteq (V(R) \cap V(Q)) \setminus \{x\}$ be the set of two vertices (if $R \subseteq Q$) or one vertex (if $|V(R) \cap V(Q)| = 2$) such that $e_G(R - W_R, T) = e_G(x, T)$. Otherwise, let $W_R \subseteq V(R)$ be a set of two vertices (if $R \subseteq Q$) or one vertex (if $|V(R) \cap V(Q)| = 2$) such that $e_G(R - W_R, T) = 1$. Note that for any two representative vertices $x_1$ and $x_2$, $x_1$ and $x_2$ are adjacent in $G$ to the same vertex from $T$. We let $T^*$ be the set of all these vertices from $T$ that are adjacent in $G$ to a representative vertex of components from $C_3$ that intersects $Q$.

Let $S^*$ be the set of $S$-vertices that are embedded inside a face with boundary, say $F$. By Claim 8, we have either $S^* \subseteq V(Q)$ or $S^* \cap V(Q) = \emptyset$. If $S^* \subseteq V(Q)$, then we have $F \subseteq Q$ by Claims 5 and 7. Thus $Q_0$ has a face whose boundary is the component graph of $F$. Therefore we have $|V(Q) \cap S| \leq f(Q_0) - 1 + 2f_s$, as at least one face of $Q$ whose boundary has vertices adjacent in $G$ to vertices from $W$ and so there is no $S$-vertex embedded in $Q$ inside that face.

Let $W_Q$ be the set of all vertices in $V(A) \cap Q, S \cap V(Q), T^*$, and $W_R$ for all $R \in C_3$ such that $|V(R) \cap V(Q)| \geq 2$. Then

\[ |W_Q| \leq (2(n(Q_0) - 1) - b) + a + f(Q_0) - 1 + 2f_s + |T^*| \]
\[ = 2n(Q_0) + a + f(Q_0) + 2f_s + |T^*| - b - 3 \]
\[ = 2.5n(Q_0) + 1.5a + 0.5f_s + |T^*| - 1.5b - 2.5. \]

In $Q - W_Q$, there are exactly $e(Q_0) - |T^*|$ components that contains a vertex of $T_2$, $3f_s$ components that contains a vertex of $T_1$, and $2|T^*|$ components that each is a single vertex from $U_3$, and $D - W - W_Q$ is also a component. Thus we have

\[ c(Q - W_Q) = (e(Q_0) - |T^*|) + 3f_s + 2|T^*| + 1 \]
\[ = 1.5n(Q_0) + 0.5a + 1.5f_s + |T^*| - 0.5b - 0.5. \]
Suppose that $Q$ contains precisely $h$ patches of $S$-triangles, where $h \in [0, 4]$. Then we have $a = 2(f_s - h) + a'$, as $Q$ contains $h$ $S$-triangles that are not associated with $D$. Then as $|T^*| \geq \frac{1}{2}(n(Q_0) + a' - a - c)$, we get

$$
1.5c(Q - W_Q) - |W_Q| = 1.5(1.5n(Q_0) + 0.5a + 1.5f_s + |T^*| - 0.5b - 0.5) - (2.5n(Q_0) + 1.5a + 0.5f_s + |T^*| - 1.5b - 2.5)
$$

$$
= -0.25n(Q_0) - 0.75a + 1.75f_s + 0.5|T^*| + 0.75b + 1.75
$$

$$
\geq -0.25n(Q_0) + 1.75f_s - 0.75a + 0.25(n(Q_0) + a' - a - c) + 0.75b + 1.75
$$

$$
= 1.75f_s + 0.25a' + 0.75b + 1.75 - a - 0.25c
$$

$$
\geq -0.25f_s - 0.75a' + 2h + 0.625b + 1.75 \quad (a = 2(f_s - h) + a' \text{ and } b \geq 2c).
$$

Note that $f_s \leq 21$, $b \geq 2c \geq 2$, and $a' \leq 5$. When $h = 4$, we have

$$
-0.25f_s - 0.75a' + 2h + 0.625b + 1.75 \geq -5.25 - 3.75 + 8 + 1.25 + 1.75 \geq 2.
$$

Thus $1.5(c(Q - W_Q) - 1) - |W_Q| > 0$ and so $h(W \cup W_Q) > h(W)$, a contradiction to the choice of $W$. Thus we have $h \leq 3$.

Suppose $h = 3$. Then we have $f_s \leq 17$ as there are at least four $S$-triangles in a patch. Thus

$$
-0.25f_s - 0.75a' + 2h + 0.625b + 1.75 \geq -4.25 - 3.75 + 6 + 1.25 + 1.75 \geq 1.
$$

Thus $1.5(c(Q - W_Q) - 1) - |W_Q| \geq -0.75$.

Next we suppose $h = 2$. Then we have $f_s \leq 13$. If $a' \leq 4$, then we get $1.5(c(Q - W_Q) - 1) - |W_Q| \geq -0.75$. Thus we suppose $a' = 5$. Since two patches of $S$-triangles are not contained in $Q$ but $Q$ contains all of the 5 short spokes, it follows that $c \geq 2$. Thus $b \geq 2c \geq 4$. Hence

$$
-0.25f_s - 0.75a' + 2h + 0.625b + 1.75 \geq -3.25 - 3.75 + 4 + 2.5 + 1.75 = 1.25.
$$

Thus again we get $1.5(c(Q - W_Q) - 1) - |W_Q| \geq -0.25 \geq -0.75$.

Then we suppose $h = 1$. We claim that $f_s \leq 5 + c$. A largest component of $Q_0 \cap C_0$ can contain the $C$-segments of boundaries of at most six $S$-triangles (the component that contain the $C$-segment of the boundary of an $S$-triangle from the unique patch that is contained in $Q$). Every other component of $Q_0 \cap C_0$ does not contain any $C$-segment of the boundary of any $S$-triangle from a patch and so it can contain the $C$-segment of the boundary of at most one $S$-triangle. Hence we have $f_s \leq 5 + c$. Since $c \leq 0.5b$, when $a' \leq 3$, we get

$$
-0.25f_s - 0.75a' + 2h + 0.625b + 1.75 \geq -1.25 - 0.75a' + 0.5b + 3.75 \geq 1.25.
$$

Thus $a' \geq 4$. This implies that $b \geq 2c \geq 4$. Then again, $-1.25 - 0.75a' + 0.5b + 3.75 \geq 0.75$. 

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Lastly, suppose \( h = 0 \). Then we have \( a \leq 2f_s + c + 1 \) and \( f_s \leq c + 1 \) by the construction of \( G_0 \). Then if \( a' \geq 1 \), as \( b \geq 2c \), we get

\[
1.5c(Q - W_Q) - |W_Q| \geq 1.75f_s + 0.25a' + 0.75b + 1.75 - a - 0.25c \\
\geq -0.25f_s - 1.25c + 0.25a' + 0.75b + 0.75 \\
\geq -1.5c + 0.25a' + 0.75b + 0.5 \geq 0.25a' + 0.5 \geq 0.75.
\]

Thus we suppose \( a' = 0 \). As a consequence, we have \( a = 2f_s \). Then

\[
1.5c(Q - W_Q) - |W_Q| \geq 1.75f_s + 0.25a' + 0.75b + 1.75 - a - 0.25c \\
\geq -0.25f_s + 0.25a' + 0.75b + 1.75 - 0.25c \\
\geq -0.25f_s + 0.625b + 1.75 \geq 1.75,
\]

as \( f_s \leq 5 \). Thus \( 1.5(c(Q - W_Q) - 1) - |W_Q| > 0 \) and so \( h(W \cup W_Q) > h(W) \), a contradiction to the choice of \( W \).

By the arguments above, we can always find \( W_Q \) with \( 1.5(c(Q - W_Q) - 1) - |W_Q| \geq -0.75 \). Let \( W^* = W \cup W_Q \). Then as \( h(W) > 0 \) implies \( h(W) \geq 0.5 \) by the definition of the function \( h \), it follows that \( h(W^*) \geq -0.25 \). Now we achieve a contradiction by showing that \( h(W^*) \leq -0.5 \).

Note that by the assumption that \( Q \) is the only 2-connected component of \( G - W \), we know that \( D - W^* \) is the only 2-connected component of \( G - W^* \) and \( V(A) \subseteq W^* \). Let \( T^* = T \cap W \). Then by the same argument as above for calculating \( |W_Q| \) and \( c(Q - W_Q) \), and noting that \( S \subseteq W \), we have

\[
|W^*| = 2(n(G_0) - 1) + |V(A)| + f(G_0) + 2f_s(G_0) + |T^*| \\
= 240 + 39 + 49 + 42 + |T^*|
\]

and

\[
c(G - W^*) = (c(G_0) - |T^*|) + 3f_s(G_0) + 2|T^*| + 1 = 168 + 63 + |T^*| + 1.
\]

As \( |T^*| \leq 43 \) by Fact 1, we get

\[
h(W^*) = 1.5c(G - W^*) - |W^*| = 1.5(232 + |T^*|) - (370 + |T^*|) \\
= -22 + 0.5|T^*| \leq -22 + 21.5 \leq -0.5,
\]

giving a contradiction. The proof of Theorem 1 is now completed. \( \square \)

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