Abstract

In this paper we continue the study of triangular matrix categories $A = \begin{bmatrix} T & 0 \\ M & U \end{bmatrix}$ initiated in León-Galeana et al. (2022). First, given an additive category $\mathcal{C}$ and an ideal $\mathcal{I}_B$ in $\mathcal{C}$, we prove a well known result that there is a canonical recollement

$$\text{Mod}(\mathcal{C}/\mathcal{I}_B) \preccurlyeq \text{Mod}(\mathcal{C}) \preccurlyeq \text{Mod}(\mathcal{B})$$

We show that given a recollement between functor categories we can induce a new recollement between triangular matrix categories, this is a generalization of a result given by Chen and Zheng in (J. Algebra, 321 (9), 2474–2485 2009, [Theorem 4.4]). In the case of dualizing $K$-varieties we can restrict the recollement we obtained to the categories of finitely presented functors. Given a dualizing variety $\mathcal{C}$, we describe the maps category of $\text{mod}(\mathcal{C})$ as modules over a triangular matrix category and we study its Auslander-Reiten sequences and contravariantly finite subcategories, in particular we generalize several results from Martínez-Villa and Ortíz-Morales (Inter. J Algebra, 5 (11), 529–561 2011). Finally, we prove a generalization of a result due to Smalø (2011, [Theorem 2.1]), which give us a way of construct functorially finite subcategories in the category $\text{Mod}\left(\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}\right)$ from those of $\text{Mod}(\mathcal{T})$ and $\text{Mod}(\mathcal{U})$.

Keywords Auslander-Reiten theory · Dualizing varieties · Functor categories · Recollements · Triangular matrix rings.
1 Introduction

A recollement of abelian categories is an exact sequence of abelian categories where both the inclusion and the quotient functors admit left and right adjoints. Recollements first appeared in the context of triangulated categories in the construction of the category of perverse sheaves on a singular space by Beilinson, Bernstein and Deligne (see [10]); they axiomatized Grothendieck’s six functors for derived categories of sheaves. In representation theory, recollements were used by Cline, Parshall and Scott to study module categories of finite dimensional algebras over a field (see [28]). They appear in connection with quasi-hereditary algebras and highest weight categories. Recollements of triangulated categories also have appeared in the work of Angeleri-Hügel, Koenig and Liu in connection with tilting theory and homological conjectures of derived categories of rings (see [1, 2] and [3]).

In the context of abelian categories, recollements were studied by Franjou and Pirashvili in [14], motivated by the work of MacPherson-Vilonen in derived category of perverse sheaves (see [24]). Several homological properties of recollements of abelian categories have been amply studied (see [30–32, 35]).

It should be noted that recollements of abelian categories appear naturally in various settings in representation theory. For example any idempotent element $e$ in a ring $R$ with units induces a canonical recollement between the module categories over the rings $R$, $R/ReR$ and $eRe$. In fact, in [31], Psaroudakis and Vitoria, studied recollements of module categories and they showed that a recollement whose terms are module categories is equivalent to one induced by an idempotent element.

In the context of comma categories we point out that Chen and Zhen (see [11]), studied conditions under which a recollement relative to abelian categories induces a new recollement relative to abelian categories and comma categories and they applied their results to deduce results about recollements in categories of modules over triangular matrices rings. On the other hand, rings of the form $T = [\begin{array}{cc} T & 0 \\ M & U \end{array}]$ where $T$ and $U$ are rings and $M$ is a $T$-$U$-bimodule have appeared often in the study of the representation theory of artin rings and algebras (see for example [7, 12, 16–18]). Such a rings appear naturally in the study of homomorphic images of hereditary artin algebras. These types of algebras have been amply studied. For example, Zhu considered the triangular matrix algebra $\Lambda := [\begin{array}{cc} T & 0 \\ M & U \end{array}]$ where $T$ and $U$ are quasi-hereditary algebras and he proved that under suitable conditions on $M$, $\Lambda$ is quasi-hereditary algebra (see [38]). In the paper [37], for an artin algebra $A$ the triangular matrix algebra of rank two $T_2(A) = [\begin{array}{ccc} A & A \\ 0 & A \end{array}]$ was extended to the one of rank $n$, $T_n(A) = [\begin{array}{cccc} A & A & \cdots & A \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & A \end{array}]$ and obtained that there is a relation between the morphism category and the module category of the corresponding matrix algebra.

Also, in this direction let us recall the following result due to Smalø: Let $T$ and $U$ be artin algebras, consider the matrix algebra $\Lambda := [\begin{array}{cc} T & 0 \\ M & U \end{array}]$ and denote by $\text{mod}(\Lambda)^{\mathcal{X}}_{\mathcal{Y}}$ the full subcategory of $\text{mod}(\Lambda)$ whose objects are $U$-morphisms $f : M \otimes_T A \rightarrow B$ with $A \in \mathcal{X}$.
and \( \mathcal{B} \in \mathcal{Y} \) where \( \mathcal{X} \subseteq \text{mod} (T) \) and \( \mathcal{Y} \subseteq \text{mod} (U) \) are subcategories. Then, \( \text{mod} (\mathcal{A})^\mathcal{X} \) is functorially finite in \( \text{mod} (\mathcal{A}) \) if and only if \( \mathcal{X} \) and \( \mathcal{Y} \) are functorially finite in \( \text{mod} (T) \) and \( \text{mod} (U) \) respectively (see [36, Theorem 2.1]).

In the paper [22], given two preadditive categories \( \mathcal{U} \) and \( \mathcal{T} \) and \( M \in \text{Mod} (\mathcal{U} \otimes \mathcal{T}^{\text{op}}) \) we constructed the matrix category \( \mathcal{A} := \begin{bmatrix} \mathcal{T} & 0 \\ M & \mathcal{U} \end{bmatrix} \) and we studied several of its properties. In particular we proved that there exists and equivalence of categories \( \text{Mod}(\mathcal{T}), \text{GMod}(\mathcal{U}) \cong \text{Mod}(\mathcal{A}) \). In this paper, one of the main results is a generalization of the result in [11, theorem 4.4], that given a recollement between functor categories we can induce a recollement between modules over certain triangular matrix categories \( \text{Mod} \left( \begin{bmatrix} \mathcal{T} & 0 \\ M & \mathcal{U} \end{bmatrix} \right) \). We also show that in the case of dualizing \( K \)-varieties we can restrict that recollement to the category of finitely presented modules \( \text{mod} (\mathcal{A}) \) (see Theorems 4.1 and 4.12). Finally, we prove an analogous of the Smalø’s result mentioned above, but for the context of dualizing varieties (see Theorem 8.3). We now give a brief description of the contents on this paper.

In Section 2, we recall basic concepts and properties of the category \( \text{Mod}(\mathcal{C}) \), and some properties of dualizing varieties and comma categories that will be use throughout the paper. In Section 3, we recall the notion of recollement and we show that there is a recollement coming from a triple adjoint defined by M. Auslander in [5]. This result is well known (see for example [32, Example 3.12]), but we give a proof by the convenience of the reader.

In Section 4, we show how construct recollements in the category of modules over triangular matrix categories \( \text{Mod} \left( \begin{bmatrix} \mathcal{T} & 0 \\ M & \mathcal{U} \end{bmatrix} \right) \). In this section we prove that given a recollement in functor categories we can induce a recollement between modules over triangular matrix categories (see ([11, Theorem 4.4])).

In Section 5, we study the category maps(\( \text{Mod}(\mathcal{C}) \)) := \( \left( \text{Mod}(\mathcal{C}), \text{Mod}(\mathcal{C}) \right) \) of maps of the category \( \text{Mod}(\mathcal{C}) \) and we give in this setting a description of the functor \( \widehat{\Theta} : \left( \text{Mod}(\mathcal{T}), \text{GMod}(\mathcal{U}) \right) \rightarrow \left( \text{Mod}(\mathcal{U}^{\text{op}}), \text{GMod}(\mathcal{T}^{\text{op}}) \right) \) constructed in [22, Proposition 4.9] (see Proposition 5.2). We also give a description of the projective and injective objects of the category maps(\( \text{mod}(\mathcal{C}) \)) when \( \mathcal{C} \) is a dualizing variety and we also describe its radical (see Proposition 5.5).

In Section 6, we study the Auslander-Reiten translate in comma categories. So, we construct \( (-)^* : \left( \text{Mod}(\mathcal{T}), \text{GMod}(\mathcal{U}) \right) \rightarrow \left( \text{Mod}(\mathcal{U}^{\text{op}}), \text{GMod}(\mathcal{T}^{\text{op}}) \right) \) and we describe how it acts on the category maps(\( \text{mod}(\mathcal{T}), \text{GMod}(\mathcal{U}) \)), of finitely generated projective objects (see Propositions 6.4, 6.5 and 6.6). We also describe the action of the Auslander-Reiten translate in the category of maps maps(\( \text{mod}(\mathcal{C}) \)) := \( \left( \text{mod}(\mathcal{C}), \text{mod}(\mathcal{C}) \right) \) when \( \mathcal{C} \) is a dualizing variety (see Proposition 6.12).

In Section 7, we generalize some results from [25]. We consider a dualizing \( K \)-variety \( \mathcal{C} \) and we study \( \mathcal{A} = \begin{bmatrix} \mathcal{C} & 0 \\ \text{Hom} \mathcal{C} & \mathcal{C} \end{bmatrix} \). We construct almost split sequences in \( \text{mod}(\mathcal{A}) \) that arise from almost split sequences in \( \text{mod}(\mathcal{C}) \) (see Propositions 7.1 and 7.5). We also study almost split sequences in \( \text{mod}(\text{mod}(\mathcal{C})^{\text{op}}) \) coming from almost split sequences with certain conditions in maps(\( \text{mod}(\mathcal{C}) \)) (see Proposition 7.7).

In Section 8, we study functorially finite subcategories in \( \text{Mod} \left( \begin{bmatrix} \mathcal{T} & 0 \\ M & \mathcal{U} \end{bmatrix} \right) \). In particular, we prove a generalization of the result of Smalø in [36, Theorem 2.1], which give us a way of construct functorially finite subcategories in the category \( \text{Mod} \left( \begin{bmatrix} \mathcal{T} & 0 \\ M & \mathcal{U} \end{bmatrix} \right) \) of modules over a triangular matrix category (see Corollary 8.3). Finally, we see that the categories of
monomorphisms and epimorphisms in maps(mod(C)) = (mod(C), mod(C)) are funtorially finite (see Proposition 8.9).

2 Preliminaries

2.1 Categorical Foundations and Notations

An arbitrary category $\mathcal{C}$ is **skeletally small** if there is a full subcategory $\mathcal{C}'$ of $\mathcal{C}$ such that the class of objects of $\mathcal{C}'$ is a set and every object of $\mathcal{C}$ is isomorphic to an object in $\mathcal{C}'$. We recall that a category $\mathcal{C}$ together with an abeliangroup structure on each of the sets of morphisms $\mathcal{C}(C_1, C_2)$ is called **preadditive category** provided all the composition maps $\mathcal{C}(C, C') \times \mathcal{C}(C', C'') \to \mathcal{C}(C, C'')$ in $\mathcal{C}$ are bilinear maps of abelian groups. A covariant functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ between preadditive categories $\mathcal{C}_1$ and $\mathcal{C}_2$ is said to be **additive** if for each pair of objects $C$ and $C'$ in $\mathcal{C}_1$, the map $F : \mathcal{C}_1(C, C') \to \mathcal{C}_2(F(C), F(C'))$ is a morphism of abelian groups. Let $\mathcal{C}$ and $\mathcal{D}$ be preadditive categories and $\text{Ab}$ the category of abelian groups. A functor $F : \mathcal{C} \times \mathcal{D} \to \text{Ab}$ is called **biadditive** if $F(f + f', g) = F(f, g) + F(f', g)$ $\forall f, f' \in \mathcal{C}(C, C'), \forall g \in \mathcal{D}(D, D')$ and $F(f, g + g') = F(f, g) + F(f, g')$ $\forall f \in \mathcal{C}_1(C, C'), \forall g, g' \in \mathcal{D}(D, D')$.

If $\mathcal{C}$ is a preadditive category we always considerer its opposite category $\mathcal{C}^{\text{op}}$ as a preadditive category by letting $\mathcal{C}_1^{\text{op}}(C', C) = \mathcal{C}(C, C')$. We follow the usual convention of identifying each contravariant functor $F$ from a category $\mathcal{C}$ to $\mathcal{D}$ with the covariant functor $F$ from $\mathcal{C}^{\text{op}}$ to $\mathcal{D}$. An **additive category** is a preadditive category $\mathcal{C}$ such that every finite family of objects in $\mathcal{C}$ has a coproduct.

2.2 The Category $\text{Mod}(\mathcal{C})$

Throughout this section $\mathcal{C}$ will be an arbitrary skeletally small preadditive category, and $\text{Mod}(\mathcal{C})$ will denote the category of covariant functors from $\mathcal{C}$ to the category of abelian groups $\text{Ab}$, called the category of $\mathcal{C}$-modules. This category has as objects the functors from $\mathcal{C}$ to $\text{Ab}$, and and a morphism $f : M_1 \to M_2$ of $\mathcal{C}$-modules is a natural transformation. We sometimes we will write for short, $\mathcal{C}(\mathcal{C}, -)$ instead of $\text{Hom}_\mathcal{C}(\mathcal{C}, -)$ and when it is clear from the context we will use just $\mathcal{C}(\mathcal{C}, -)$.

We now recall some of properties of the category $\text{Mod}(\mathcal{C})$, for more details consult [5]. The category $\text{Mod}(\mathcal{C})$ is an abelian with the following properties: For each $C$ in $\mathcal{C}$, the $C$-module $(C, -)$ given by $(C, -)(X) = \mathcal{C}(C, X)$ for each $X$ in $\mathcal{C}$, has the property that for each $\mathcal{C}$-module $M$, the map $((C, -), M) \to M(C)$ given by $f \mapsto f_C(1_C)$ for each $\mathcal{C}$-morphism $f : (C, -) \to M$ is an isomorphism of abelian groups. We will often consider this isomorphism an identification. Hence

1. The functor $P : \mathcal{C} \to \text{Mod}(\mathcal{C})$ given by $P(C) = (C, -)$ is fully faithful.
2. For each family $\{C_i\}_{i \in I}$ of objects in $\mathcal{C}$, the $\mathcal{C}$-module $\coprod_{i \in I} P(C_i)$ is a projective $\mathcal{C}$-module.
3. Given a $\mathcal{C}$-module $M$, there is a family $\{C_i\}_{i \in I}$ of objects in $\mathcal{C}$ such that there is an epimorphism $\coprod_{i \in I} P(C_i) \to M \to 0$. We say that $M$ is finitely generated if such family is finite.
4. A **finitely generated projective** $\mathcal{C}$-module is a direct summand of $\coprod_{i \in I} P(C_i)$ for some finite family of objects $\{C_i\}_{i \in I}$ in $\mathcal{C}$. 
2.3 Change of Categories

The results that appear in this subsection are directly taken from [5]. Let \( \mathcal{C} \) be a skeletally small preadditive category. There is a unique (up to isomorphism) functor \( \otimes_{\mathcal{C}} : \text{Mod}(\mathcal{C}^{\text{op}}) \times \text{Mod}(\mathcal{C}) \rightarrow \text{Ab} \) called the tensor product. The abelian group \( \otimes_{\mathcal{C}}(A, B) \) is denoted by \( A \otimes_{\mathcal{C}} B \) for all \( \mathcal{C}^{\text{op}} \)-modules \( A \) and all \( \mathcal{C} \)-modules \( B \).

**Proposition 2.1.** The tensor product has the following properties:

1. (a) For each \( \mathcal{C} \)-module \( B \), the functor \( \otimes_{\mathcal{C}} B : \text{Mod}(\mathcal{C}^{\text{op}}) \rightarrow \text{Ab} \) given by \( (\otimes_{\mathcal{C}} B)(A) = A \otimes_{\mathcal{C}} B \) for all \( \mathcal{C}^{\text{op}} \)-modules \( A \) is right exact.

2. (b) For each \( \mathcal{C}^{\text{op}} \)-module \( A \), the functor \( A \otimes_{\mathcal{C}} : \text{Mod}(\mathcal{C}) \rightarrow \text{Ab} \) given by \( (A \otimes_{\mathcal{C}})(B) = A \otimes_{\mathcal{C}} B \) for all \( \mathcal{C} \)-modules \( B \) is right exact.

3. For each \( \mathcal{C}^{\text{op}} \)-module \( A \) and each \( \mathcal{C} \)-module \( B \), the functors \( A \otimes_{\mathcal{C}} \) and \( \otimes_{\mathcal{C}} B \) preserve arbitrary sums.

4. For each object \( C \) in \( \mathcal{C} \) we have \( A \otimes_{\mathcal{C}} (C, -) = A(C) \) and \( (C, -) \otimes_{\mathcal{C}} B = B(C) \) for all \( \mathcal{C}^{\text{op}} \)-modules \( A \) and all \( \mathcal{C} \)-modules \( B \).

Suppose now that \( \mathcal{C}' \) is a subcategory of the skeletally small preadditive category \( \mathcal{C} \). We use the tensor product of \( \mathcal{C}' \)-modules, to describe the left adjoint \( \mathcal{C} \otimes_{\mathcal{C}'} \) of the restriction functor \( \text{res}_{\mathcal{C}'} : \text{Mod}(\mathcal{C}) \rightarrow \text{Mod}(\mathcal{C}') \).

Define the functor \( \mathcal{C} \otimes_{\mathcal{C}'} : \text{Mod}(\mathcal{C}') \rightarrow \text{Mod}(\mathcal{C}) \) by \( (\mathcal{C} \otimes_{\mathcal{C}'} M)(C) = (C, -) \otimes_{\mathcal{C}'} M \) for all \( M \in \text{Mod}(\mathcal{C}') \) and \( C \in \mathcal{C} \). Using the properties of the tensor product it is not difficult to establish the following proposition.

**Proposition 2.2.** [5, Proposition 3.1] Let \( \mathcal{C}' \) a subcategory of the skeletally small preadditive category \( \mathcal{C} \). Then the functor \( \mathcal{C} \otimes_{\mathcal{C}'} : \text{Mod}(\mathcal{C}') \rightarrow \text{Mod}(\mathcal{C}) \) satisfies:

1. \( \mathcal{C} \otimes_{\mathcal{C}'} \) is right exact and preserves sums;
2. The composition \( \text{Mod}(\mathcal{C}') \xrightarrow{\mathcal{C} \otimes_{\mathcal{C}'}} \text{Mod}(\mathcal{C}) \xrightarrow{\text{res}_{\mathcal{C}'}} \text{Mod}(\mathcal{C}') \) is the identity on \( \text{Mod}(\mathcal{C}') \);
3. For each object \( C' \in \mathcal{C}' \), we have \( \mathcal{C} \otimes_{\mathcal{C}'} C' (C', -) = \mathcal{C} (C', -) \);
4. For each \( \mathcal{C}' \)-module \( M \) and each \( \mathcal{C} \)-module \( N \), the restriction map

\[
\mathcal{C} (C \otimes_{\mathcal{C}'} M, N) \rightarrow C' (M, N \mid_{C'})
\]

is an isomorphism;
5. \( \mathcal{C} \otimes_{\mathcal{C}'} \) is a fully faithful functor.

Having described the left adjoint \( \mathcal{C} \otimes_{\mathcal{C}'} \) of the restriction functor \( \text{res}_{\mathcal{C}'} : \text{Mod}(\mathcal{C}) \rightarrow \text{Mod}(\mathcal{C}') \), we now describe its right adjoint.

Let \( \mathcal{C}' \) be a full subcategory of the category \( \mathcal{C} \). Define the functor \( \mathcal{C}' (C, -) : \text{Mod}(\mathcal{C}') \rightarrow \text{Mod}(\mathcal{C}) \) by \( \mathcal{C}' (C, M)(X) = \mathcal{C}' (X, -) \mid_{\mathcal{C}'} M \) for all \( \mathcal{C}' \)-modules \( M \) and all objects \( X \) in \( \mathcal{C} \). We have the following proposition.

**Proposition 2.3.** [5, Proposition 3.4] Let \( \mathcal{C}' \) a subcategory of the skeletally small preadditive category \( \mathcal{C} \). Then the functor \( \mathcal{C}' (C, -) : \text{Mod}(\mathcal{C}') \rightarrow \text{Mod}(\mathcal{C}) \) has the following properties:

1. \( \mathcal{C}' (C, -) \) is left exact and preserves inverse limits;
2. The composition \( \text{Mod}(\mathcal{C}') \xrightarrow{\mathcal{C}' (C, -)} \text{Mod}(\mathcal{C}) \xrightarrow{\text{res}_{\mathcal{C}'}} \text{Mod}(\mathcal{C}') \) is the identity on \( \text{Mod}(\mathcal{C}') \);
3. For each $\mathcal{C'}$-module $M$ and $\mathcal{C}$-module $N$, the restriction map
\[
\mathcal{C} \left( N, \mathcal{C'}(C, M) \right) \longrightarrow \mathcal{C'}(N |_{\mathcal{C'}}, M)
\]
is an isomorphism;

4. $\mathcal{C'}(\mathcal{C}, -)$ is a fully faithful functor.

### 2.4 Dualizing Varieties and Krull-Schmidt Categories

Let $\mathcal{C}$ be a preadditive category, we denote by $\text{proj}(\text{Mod}(\mathcal{C}))$ the full subcategory of $\text{Mod}(\mathcal{C})$ consisting of all finitely generated projective $\mathcal{C}$-modules. Let $\mathcal{C}$ be an additive category, it is said that $\mathcal{C}$ is a category in which idempotents split if given $e : C \rightarrow C$ an idempotent endomorphism of an object $C \in \mathcal{C}$, then $e$ has a kernel in $\mathcal{C}$. It is well known that for a preadditive category $\mathcal{C}$ the category $\text{proj}(\text{Mod}(\mathcal{C}))$ is a skeletally small additive category in which idempotents split, the functor $P : \mathcal{C} \rightarrow \text{proj}(\text{Mod}(\mathcal{C}))$ given by $P(C) = \mathcal{C}(C, -)$, is fully faithful and induces by restriction an equivalence $\text{Mod}(\text{proj}(\text{Mod}(\mathcal{C}))^{op}) \simeq \text{Mod}(\mathcal{C})$. We recall the following notion given by Auslander in [5]. A variety is a skeletally small, additive category in which idempotents split.

Given a ring $R$, we denote by $\text{Mod}(R)$ the category of left $R$-modules and by $\text{mod}(R)$ the full subcategory of $\text{Mod}(R)$ consisting of the finitely generated $R$-modules.

To fix the notation, we recall known results on functors and categories that we use through the paper, referring for the proofs to the papers by Auslander and Reiten [4–6].

**Definition 2.4.** Let $R$ be a commutative artin ring. An $R$-variety $\mathcal{C}$, is a variety such that $\mathcal{C}(C_1, C_2)$ is an $R$-module, and the composition is $R$-bilinear. An $R$-variety $\mathcal{C}$ is **Hom-finite**, if for each pair of objects $C_1, C_2$ in $\mathcal{C}$, the $R$-module $\mathcal{C}(C_1, C_2)$ is finitely generated. We denote by $(\mathcal{C}, \text{mod}(R))$, the full subcategory of $(\mathcal{C}, \text{Mod}(R))$ consisting of the $\mathcal{C}$-modules such that; for every $C$ in $\mathcal{C}$ the $R$-module $M(C)$ is finitely generated.

Suppose $\mathcal{C}$ is a Hom-finite $R$-variety. If $M : C \longrightarrow \text{Ab}$ is a $\mathcal{C}$-module, then for each $C \in \mathcal{C}$ the abelian group $M(C)$ has a structure of $\text{End}_\mathcal{C}(C)^{op}$-module and hence as an $R$-module since $\text{End}_\mathcal{C}(C)$ is an $R$-algebra. Further if $f : M \longrightarrow M'$ is a morphism of $\mathcal{C}$-modules it is easy to show that $f_C : M(C) \longrightarrow M'(C)$ is a morphism of $R$-modules for each $C \in \mathcal{C}$. Then, $\mathcal{C}(R)$ is an $R$-variety, which we identify with the category of covariant functors $(\mathcal{C}, \text{Mod}(R))$. Moreover, the category $(\mathcal{C}, \text{mod}(R))$ is abelian and the inclusion $(\mathcal{C}, \text{mod}(R)) \rightarrow (\mathcal{C}, \text{Mod}(R))$ is exact.

**Definition 2.5.** Let $\mathcal{C}$ be a Hom-finite $R$-variety. We denote by $\text{mod}(\mathcal{C})$ the full subcategory of $\text{Mod}(\mathcal{C})$ whose objects are the **finitely presented functors**. That is, $M \in \text{mod}(\mathcal{C})$ if and only if, there exists an exact sequence in $\text{Mod}(\mathcal{C})$
\[
P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,
\]
where $P_1$ and $P_0$ are finitely generated projective $\mathcal{C}$-modules.

It is easy to see that $P_0$ and $P_1$ can be chosen as the representables. Then a functor $M$ is finitely presented if there exists an exact sequence
\[
\mathcal{C}(-, C_1) \longrightarrow \mathcal{C}(-, C_0) \longrightarrow M \longrightarrow 0,
\]
with $C_0, C_1 \in \mathcal{C}$. It was proved in [15] that $\text{mod}(\mathcal{C})$ is abelian if and only if $\mathcal{C}$ has pseudokernels, this fact was later rediscovered by Auslander in [6].
Consider the functors \( \mathbb{D}_{C^{op}} : (C^{op}, \text{mod}(R)) \to (C, \text{mod}(R)) \), and \( \mathbb{D}_C : (C, \text{mod}(R)) \to (C^{op}, \text{mod}(R)) \), which are defined as follows: for any object \( C \) in \( C \), \( \mathbb{D}(M)(C) = \text{Hom}_R(M(C), I(R/r)) \), with \( r \) the Jacobson radical of \( R \), and \( I(R/r) \) is the injective envelope of \( R/r \). The functor \( \mathbb{D}_C \) defines a duality between \( (C, \text{mod}(R)) \) and \( (C^{op}, \text{mod}(R)) \).

We know that since \( C \) is Hom-finite, \( \text{mod}(C) \) is a subcategory of \( (C, \text{mod}(R)) \). Then we have the following definition due to Auslander and Reiten (see [6]).

**Definition 2.6.** An Hom-finite \( R \)-variety \( C \) is **dualizing**, if the functor

\[
\mathbb{D}_C : (C, \text{mod}(R)) \to (C^{op}, \text{mod}(R))
\]

induces a duality between the categories \( \text{mod}(C) \) and \( \text{mod}(C^{op}) \).

It is clear from the definition that for dualizing categories \( C \) the category \( \text{mod}(C) \) has enough injectives. To finish, we recall the following definition:

**Definition 2.7.** An additive category \( C \) is **Krull-Schmidt**, if every object in \( C \) decomposes in a finite sum of objects whose endomorphism ring is local.

Assume that \( R \) is a commutative ring and \( R \) is a dualizing \( R \)-variety. Since the endomorphism ring of each object in \( C \) is an artin algebra, it follows that \( C \) is a Krull-Schmidt category [6, p.337], moreover, we have that for a dualizing variety the finitely presented functors have projective covers [5, Cor. 4.13], [21, Cor. 4.4].

### 2.5 Tensor Product of Categories

If \( C \) and \( D \) are preadditive categories, B. Mitchell defined in [27] the **tensor product** \( C \otimes D \) of two preadditive categories, whose objects are those of \( C \times D \) and the abelian group of morphism from \( (C, D) \) to \( (C', D') \) is the ordinary tensor product of abelian groups \( C(C, C') \otimes_Z D(D, D') \). Since that the tensor product of abelian groups is associative and commutative and the composition in \( C \) and \( D \) is \( Z \)-bilinear then the bilinear composition in \( C \otimes D \) is given as follows:

\[
(f_2 \otimes g_2) \circ (f_1 \otimes g_1) := (f_2 \circ f_1) \otimes (g_2 \circ g_1)
\]

for all \( f_1 \otimes g_1 \in C(C, C') \otimes D(D, D') \) and \( f_2 \otimes g_2 \in C(C', C'') \otimes D(D', D'') \).

### 2.6 The Path Category of a Quiver

A quiver \( \Delta \) consists of a set of vertices \( \Delta_0 \) and a set of arrows \( \Delta_1 \) which is the disjoint union of sets \( \Delta(x, y) \) where the elements of \( \Delta(x, y) \) are the arrows \( \alpha : x \to y \) from the vertex \( x \) to the vertex \( y \). Given a quiver \( \Delta \) its path category \( \text{Pth}\Delta \) has as objects the vertices of \( \Delta \) and the morphisms \( x \to y \) are paths from \( x \) to \( y \) which are by definition the formal compositions \( \alpha_n \cdots \alpha_1 \) where \( \alpha_1 \) stars in \( x, \alpha_n \) ends in \( y \) and the end point of \( \alpha_i \) coincides with the start point of \( \alpha_{i+1} \) for all \( i \in \{1, \ldots, n-1\} \). The positive integer \( n \) is called the length of the path. There is a path \( \xi_x \) of length 0 for each vertex to itself. The composition in \( \text{Pth}\Delta \) of paths of positive length is just concatenations whereas the \( \xi_x \) act as identities.

Given a quiver \( \Delta \) and a field \( K \), an additive \( K \)-category \( K\Delta \) is associated to \( \Delta \) by taking as the indecomposable objects in \( K\Delta \) the vertices of \( \Delta \) and hence an arbitrary object of \( K\Delta \) is a direct sum of indecomposable objects. Given \( x, y \in \Delta_0 \) the set of maps from \( x \) to \( y \) is given by the \( K \)-vector space with basis the set of all paths from \( x \) to \( y \). The composition
in $K \Delta$ is of course obtained by $K$-linear extension of the composition in $P^\Delta$, that is, the product of two composable paths is defined to be the corresponding composition, the product of two non-composable paths is, by definition, zero. In this way we obtain an associative $K$-algebra which has unit element if and only if $\Delta_0$ is finite (the unit element is given by $\sum_{x \in \Delta_0} \xi_x$).

In $K \Delta$, we denote by $K \Delta^+$ the ideal generated by all arrows and by $(K \Delta^+)^n$ the ideal generated by all paths of length $\geq n$.

Given vertices $x, y \in \Delta_0$, a finite linear combination $\sum_w \lambda_w w, \lambda_w \in K$ where $w$ are paths of length $\geq 2$ from $x$ to $y$, is called a relation on $\Delta$. It can be seen that any ideal $I \subset (K \Delta^+)^2$ can be generated, as an ideal, by relations. If $I$ is generated as an ideal by the set $\{\rho_i \mid i\}$ of relations, we write $I = \langle \rho_i \mid i \rangle$.

Given a quiver $\Delta = (\Delta_0, \Delta_1)$ a representation $V = (V_x, f_\alpha)$ of $\Delta$ over $K$ is given by vector spaces $V_x$ for all $x \in \Delta_0$, and linear maps $f_\alpha : V_x \to V_y$, for any arrow $\alpha : x \to y$. The category of representations of $\Delta$ is the category with objects the representations, and a morphism of representations $h = (h_x) : V \to V'$ is given by maps $h_x : V_x \to V'_x$ such that $h_x f_\alpha = f_\alpha' h_x$ for any $\alpha : x \to y$. The category of representations of $\Delta$ is denoted by $\text{Rep}(\Delta)$.

Given a set of relations $\langle \rho_i \mid i \rangle$ of $\Delta$, we denote by $K \Delta / \langle \rho_i \mid i \rangle$ the path category given by the quiver $\Delta$ and relations $\rho_i$. The category of functors $\text{Mod}(K \Delta / \langle \rho_i \mid i \rangle) = (K \Delta / \langle \rho_i \mid i \rangle, \text{Mod}(K))$ can be identified with the representations of $\Delta$ satisfying the relations $\rho_i$ which is denoted by $\text{Rep}(\Delta, \{\rho_i \mid i\})$, (see [34, p. 42]).

### 2.7 Quotient Category and the Radical of a Category

A two sided ideal $I(-, ?)$ is an additive subfunctor of the two variable functor $C(-, ?) : \mathcal{C}^{op} \otimes \mathcal{C} \to \text{Ab}$ such that: (a) if $f \in I(X, Y)$ and $g \in C(Y, Z)$, then $gf \in I(X, Z)$; and (b) if $f \in I(X, Y)$ and $h \in C(U, X)$, then $fh \in I(U, Y)$. If $I$ is a two-sided ideal, then we can form the quotient category $\mathcal{C}/I$ whose objects are those of $\mathcal{C}$, and where $(C/I)(X, Y) := C(X, Y)/I(X, Y)$. Finally the composition is induced by that of $\mathcal{C}$ (see [27]). There is a canonical projection functor $\pi : \mathcal{C} \to \mathcal{C}/I$ such that:

1. $\pi(X) = X$, for all $X \in \mathcal{C}$.
2. For all $f \in C(X, Y)$, $\pi(f) = f + I(X, Y) := \bar{f}$.

Based on the Jacobson radical of a ring, we introduce the radical of a preadditive category. This concept goes back to work of Kelly (see [20]).

**Definition 2.8.** The (Jacobson) radical of an preadditive category $\mathcal{C}$ is the two-sided ideal $\text{rad}_C$ in $\mathcal{C}$ defined by the formula

$$\text{rad}_C(X, Y) = \{h \in C(X, Y) \mid 1_X - gh \text{ is invertible for any } g \in C(Y, X)\}$$

for all objects $X$ and $Y$ of $\mathcal{C}$.

### 2.8 Comma Categories

If $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $F : \mathcal{A} \to \mathcal{B}$ is an additive functor. The comma category $(\mathcal{B}, F \mathcal{A})$ is the category whose objects are triples $(\mathcal{B}, f, A)$ where $f : B \to FA$;
and whose morphisms between the objects \((B, f, A)\) and \((B', f', A')\) are pair \((\beta, \alpha)\) of morphisms in \(B \times A\) such that the diagram

\[
\begin{array}{ccc}
B & \xrightarrow{\beta} & B' \\
\downarrow{f} & & \downarrow{f'} \\
FA & \xrightarrow{F \alpha} & FA'
\end{array}
\]

is commutative in \(B\) (see [13]).

### 3 Recollements in Functor Categories Induced by an Auslander’s Triple Adjoint

We recall some basic definitions. Consider functors \(F : C \to D\) and \(G : D \to C\). We say that \(F\) is **left adjoint** to \(G\) or that \(G\) is **right adjoint** to \(F\), and that \((F, G)\) is an adjoint pair if there is a natural equivalence

\[
\eta = \left\{ \eta_{X,Y} : \eta_{X,Y} : \text{Hom}_D(FX, Y) \to \text{Hom}_C(X, GY) \right\}_{X \in C, Y \in D}
\]

between the functors \(\text{Hom}_D(F(\_), \_\_)\) and \(\text{Hom}_C(\_, G(\_))\).

**Definition 3.1.** Let \(A, B\) and \(C\) be abelian categories

(a) The diagram

\[
\begin{array}{ccc}
C & \xrightarrow{i^*} & A & \xleftarrow{j_*} & B \\
\downarrow{i_*} & & \downarrow{j^!} & & \downarrow{j_*}
\end{array}
\]

is a called a **left recollement** if the additive functors \(i^*, i_*, j_!\) and \(j^!\) satisfy the following conditions:

- **(LR1)** \((i^*, i_*)\) and \((j_!, j^!))\) are adjoint pairs;
- **(LR2)** \(j^!i_* = 0\);
- **(LR3)** \(i_!, j_!\) are full embedding functors.

(b) The diagram

\[
\begin{array}{ccc}
C & \xrightarrow{i_*} & A & \xleftarrow{j^*} & B \\
\downarrow{i^!} & & \downarrow{j_*} & & \downarrow{j^*}
\end{array}
\]

is called a **right recollement** if the additive functors \(i_!, i^!, j^*\) and \(j_*\) satisfy the following conditions:

- **(RR1)** \((i_!, i^!)\) and \((j^*, j_*)\) are adjoint pairs;
- **(RR2)** \(j^*i_! = 0\);
- **(RR3)** \(i_!, j_*\) are full embedding functors.
**Definition 3.2.** Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be abelian categories. Then the diagram

![Diagram](image)

is called a **recollement**, if the additive functors $i^*, i_*=i_1, i_1$, $j_1, j^1 = j^*$ and $j_*$ satisfy the following conditions:

1. **(R1)** $(i^*, i_*) = (i_1, i_1^1)$ and $(j_1, j^1 = j^*, j_*)$ are adjoint triples, i.e. $(i^*, i_*)$, $(i_1, i_1^1)$ $(j_1, j^1)$ and $(j^*, j_*)$ are adjoint pairs;
2. **(R2)** $j^*i_* = 0$;
3. **(R3)** $i_*, j_1, j_*$ are full embedding functors.

Let $\mathcal{C}$ be an additive category and $\mathcal{B}$ be a full additive subcategory of $\mathcal{C}$. Maurice Auslander introduced in [5] three functors such that, according with Propositions 2.2 and 2.3, together form an adjoint triple $(\mathcal{C} \otimes \mathcal{B}, \text{res}_\mathcal{B}, \mathcal{B}(\mathcal{C}, -))$

![Diagram](image)

In this subsection we show how to extend the adjoint triple (2) to a recollement of functor categories. Similar results are given in [35], but we present them in a slightly different way.

Before defining a new triple adjoint, we need some preparatory results. Let $\mathcal{B}$ a full additive subcategory of $\mathcal{C}$. For all pair of objects $C, C' \in \mathcal{C}$ we denote by $I_B(C, C')$ the abelian subgroup of $\mathcal{C}(C, C')$ whose elements are morphism $f : C \to C'$ which factor through $\mathcal{B}$. It is not hard to see that under these conditions $I_B(-, ?)$ is a two sided ideal of $\mathcal{C}(-, ?)$ (here we use that $\mathcal{B}$ have finite coproducts). Thus we can considerer the quotient category $\mathcal{C}/I_B$.

The canonical functor $\pi : \mathcal{C} \to \mathcal{C}/I_B$ induces an exact functor by restriction $\text{res}_\mathcal{C} : \text{Mod}(\mathcal{C}/I_B) \to \text{Mod}(\mathcal{C})$ defined by $\text{res}_\mathcal{C}(F) := F \circ \pi$ for all $F \in \text{Mod}(\mathcal{C}/I_B)$. Thus,

1. **(i)** $\text{res}_\mathcal{C}(F)(C) = F(C)$ for all $C \in \mathcal{C}$.
2. **(ii)** $\text{res}_\mathcal{C}(F)(f) = F(\overline{f})$ for all $f \in \mathcal{C}(C, C')$.

We denote by $\mathcal{K}$ the full subcategory of $\text{Mod}(\mathcal{C})$ whose objects are the functors in $\text{Mod}(\mathcal{C})$ that vanish in $\mathcal{B}$. That is, $\mathcal{K} = \{F \in \text{Mod}(\mathcal{C}) \mid \text{res}_\mathcal{B}(F) = 0\}$. We prove that $\mathcal{K}$ is isomorphic to the category $\text{Mod}(\mathcal{C}/I_B)$.

Now, we define a functor $\alpha : \mathcal{K} \to \text{Mod}(\mathcal{C}/I_B)$ given by $\alpha(G)(C) = G(C)$ for all $G \in \mathcal{K}$ and for all $C \in \mathcal{C}$ and also $\alpha(G)(\overline{f}) = G(f)$ for all $\overline{f} = f + I_B(C, C') \in \mathcal{C}/I_B(C, C')$.

**Lemma 3.3.** The functors $\text{res}_\mathcal{C} : \text{Mod}(\mathcal{C}/I_B) \to \text{Mod}(\mathcal{C})$ and $\alpha : \mathcal{K} \to \text{Mod}(\mathcal{C}/I_B)$ induce an isomorphism of categories between the categories $\text{Mod}(\mathcal{C}/I_B)$ and $\mathcal{K}$. In this way $\text{res}_\mathcal{C}$ is a full embedding functor which essential image is $\mathcal{K}$, and $\mathcal{K} = \text{Ker}(\text{res}_\mathcal{B})$. 

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Proof First we note that \( \text{res}_C(F) \in \mathcal{C} \). Indeed, consider \( B \in \mathcal{B} \). Then \( 1_{\text{res}_C(F)(B)} = \text{res}_C(F)(1_B) = F(1_B) = F(1_B + I_B(B, B)) = F(0) = 0 \). Thus \( \text{res}_C(F)(B) = 0 \).

Now, let \( F \in \text{Mod}(\mathcal{C}/I_B) \). Then for all \( C \in \mathcal{C} \) we get that \( (\alpha \circ \text{res}_C)(F)(C) = F(C) \); and for all \( f \in (\mathcal{C}/I_B)(C, C') \) we obtain \( (\alpha \circ \text{res}_C)(F)(f) = (\text{res}_C(F))(f) = \text{res}_C(F)(f) = F(f) \). It follows that \( \alpha \circ \text{res}_C = 1_{\text{Mod}(\mathcal{C}/I_B)} \), and similarly we get that \( \text{res}_C \circ \alpha = 1_{\mathcal{C}} \). The rest of the proof is clear.

Now we will construct a triple adjoint \( (\mathcal{C}/I_B \otimes \mathcal{C}, \text{res}_C, \mathcal{C}(\mathcal{C}/I_B, -)) \): \[
\begin{array}{ccc}
\mathcal{C}(\mathcal{C}/I_B, -) & \xrightarrow{\text{res}_C} & \text{Mod}(\mathcal{C}) \\
\text{Mod}(\mathcal{C}/I_B) & \xrightarrow{\mathcal{C}/I_B \otimes \mathcal{C}} & \mathcal{C}/I_B \otimes \mathcal{C}
\end{array}
\] (3)

In order to construct this, we will need some preparatory results.

**Lemma 3.4.** Let \( C, C' \in \mathcal{C} \).

(i) Assume \( f \in \text{Hom}_C(C, C') \). Then we have a morphism of \( \mathcal{C}^{op}\text{-modules} \)
\[
\frac{C}{I_B}(-, f) : \frac{C}{I_B}(-, C) \rightarrow \frac{C}{I_B}(-, C')
\]
such that for all \( Z \in \mathcal{C} \)
\[
\frac{C}{I_B}(Z, f) : \frac{C}{I_B}(Z, C) \rightarrow \frac{C}{I_B}(Z, C')
\]

(ii) \( \overline{f} = f + I_B(C, C') = f' + I_B(C, C') \in \mathcal{C}(C, C')/I_B(C, C') \) implies that \( \frac{C}{I_B}(-, f) = \frac{C}{I_B}(-, f') \).

The same holds in the category of \( \mathcal{C}\text{-modules} \).

Proof Straightforward. \( \square \)

Now, we define two functors. The functor \( \frac{C}{I_B} \otimes \mathcal{C} : \text{Mod}(\mathcal{C}) \rightarrow \text{Mod}(\mathcal{C}/I_B) \) as follows:
\[
\left( \frac{C}{I_B} \otimes \mathcal{C} M \right)(C) := \frac{C}{I_B}(-, C) \otimes_C M \text{ for all } M \in \text{Mod}(\mathcal{C}) \text{ and } \left( \frac{C}{I_B} \otimes \mathcal{C} \right)(f) = \frac{C}{I_B}(-, f) \otimes_C M \text{ for all } f \in \text{Mod}(\mathcal{C}/I_B) \text{ and } \frac{C}{I_B}(C, C') \in \mathcal{C}(C, C')/I_B(C, C').
\]

And we define the functor \( \mathcal{C}(\frac{C}{I_B}, -) : \text{Mod}(\mathcal{C}) \rightarrow \text{Mod}(\mathcal{C}/I_B) \) by: \( \mathcal{C}(\frac{C}{I_B}, M)(C) = \mathcal{C} \left( \frac{C}{I_B}(-, C), M \right) \text{ for all } M \in \text{Mod}(\mathcal{C}) \text{ and } \frac{C}{I_B}(C, C') \in \mathcal{C}(C, C')/I_B(C, C') \) for all \( f \in \text{Mod}(\mathcal{C}/I_B) \) and \( \frac{C}{I_B}(f, -) \).

We establish the following proposition.

**Proposition 3.5.** Let \( \mathcal{C} \) be an additive category and \( \mathcal{B} \) be a full additive subcategory of \( \mathcal{C} \). Then the functors \( \mathcal{C}/I_B \otimes \mathcal{C} : \text{Mod}(\mathcal{C}) \rightarrow \text{Mod}(\mathcal{C}/I_B) \) and \( \mathcal{C}(\frac{C}{I_B}, -) : \text{Mod}(\mathcal{C}) \rightarrow \text{Mod}(\mathcal{C}/I_B) \) satisfy:

(i) \( \frac{C}{I_B} \otimes \mathcal{C} \) is right exact and preserves sums.

(ii) \( \mathcal{C}(\frac{C}{I_B}, C, -) = \frac{\mathcal{C}(C, -)}{I_B(C, -)}. \)
(iii) For all $M \in \text{Mod}(C)$ and $N \in \text{Mod}(C/I_B)$ there exists a natural isomorphism
\[
\left( \frac{C}{I_B} \otimes_C M, N \right) \cong (M, \text{res}_C(N)).
\]
(iv) $C(\frac{C}{I_B}, -)$ is left exact.
(v) For all $M \in \text{Mod}(C)$ and $N \in \text{Mod}(C/I_B)$ there exists a natural isomorphism
\[
\frac{C}{I_B} \left( N, C(\frac{C}{I_B}, M) \right) \rightarrow C \left( \text{res}_C(N), M \right).
\]

Proof Similar to the proof of [5, Proposition 3.1] and [5, Proposition 3.4].

We note that in Proposition 3.5, we have computed the functors on the left side of the recollement and those functors are induced by the diagram (2). This is a particular case of a more general result (see Remark 2.5 of [32]).

The following proposition is well known, see example 3.12 in [32]. However, for completeness we include a proof.

**Proposition 3.6.** Let $C$ be an additive category and $B$ be a full additive subcategory of $C$. Then there is a recollement:

\[
\begin{array}{ccc}
\text{Mod}(C/I_B) & \xrightarrow{\text{res}_C} & \text{Mod}(C) \\
\downarrow & & \downarrow \\
C(\frac{C}{I_B}, -) & & B(C, -)
\end{array}
\]

Proof We use the notation given in diagram (2).

(R1) By Propositions 2.2, 2.3 and 3.5, the triples $\left( C \otimes_B, \text{res}_B, B(C, -) \right)$ and \( \left( \frac{C}{I_B} \otimes_C, \text{res}_C, C(\frac{C}{I_B}, -) \right) \) are adjoint triples.

(R2) By Lemma 3.3, we have $\text{res}_B \text{res}_C = 0$.

(R3) By Lemma 3.3 and Propositions 2.2 and 2.3, $\text{res}_C, C \otimes_B$ and $B(C, -)$ are full embedding functors.

It is worth to mention that in the context of dualizing $K$-varieties, Y, Ogawa in [35, Theorem 2.5] have proved the following result.

**Theorem 3.7.** Let $C$ be a dualizing $K$-variety and $B$ a functorially finite subcategory of $C$. Then the recollement in Proposition 3.6, restricts to a recollement:

\[
\begin{array}{ccc}
\text{mod}(C/I_B) & \xrightarrow{\text{res}_C} & \text{mod}(C) \\
\downarrow & & \downarrow \\
C(\frac{C}{I_B}, -) & & B(C, -)
\end{array}
\]

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4 Another Recollement

Our purpose in this section is to prove the following Theorem which generalizes the results given by Q. Chen and M. Zheng in [11, Theorem 4.4]. In order to state the result we need the following construction which will be given in detail in Definition 4.4. Given an additive functor $F : \text{Mod}(\mathcal{R}) \to \text{Mod}(\mathcal{S})$ and $M \in \text{Mod}(\mathcal{T} \otimes T^{op})$, there exists another bimodule in $\text{Mod}(S \otimes T^{op})$ denoted by $F(M)$.

**Theorem 4.1.** Let $\mathcal{R}$, $\mathcal{S}$, $\mathcal{C}$ and $\mathcal{T}$ be preadditive categories and $M \in \text{Mod}(\mathcal{R} \otimes T^{op})$.

(a) If the diagram

$$
\begin{array}{ccc}
\text{Mod}(\mathcal{C}) & \xrightarrow{i^*} & \text{Mod}(\mathcal{S}) \xrightarrow{i_*} \text{Mod}(\mathcal{R}) \\
\text{Mod}(\mathcal{A}^1) & \xleftarrow{j_1} & \text{Mod}(\mathcal{T}) \xleftarrow{j_!} \\
\end{array}
$$

is a left recollement, then there is a left recollement

$$
\begin{array}{ccc}
\text{Mod}(\mathcal{C}) & \xrightarrow{i^*} & \text{Mod}(\mathcal{A}^1) \xrightarrow{i_*} \text{Mod}(\mathcal{A}) \\
\text{Mod}(\mathcal{C}) & \xrightarrow{j^*} & \text{Mod}(\mathcal{R}) \xleftarrow{j_*} \\
\end{array}
$$

where $\mathcal{A} := \left( \begin{array}{cc} \mathcal{T} & 0 \\ M & \mathcal{R} \end{array} \right)$ and $\mathcal{A}^1 := \left( \begin{array}{cc} \mathcal{T} & 0 \\ \mathcal{C} & \mathcal{T} \end{array} \right)$ are matrix categories and $j_!(M)$ is a bimodule constructed as in Definition 4.4.

(b) If the diagram

$$
\begin{array}{ccc}
\text{Mod}(\mathcal{C}) & \xrightarrow{i^*} & \text{Mod}(\mathcal{S}) \xrightarrow{i_*} \text{Mod}(\mathcal{R}) \\
\text{Mod}(\mathcal{A}^*) & \xleftarrow{j_!} & \text{Mod}(\mathcal{A}) \xleftarrow{j_*} \\
\end{array}
$$

is a right recollement, then there is a right recollement

$$
\begin{array}{ccc}
\text{Mod}(\mathcal{C}) & \xrightarrow{i^*} & \text{Mod}(\mathcal{A}^*) \xrightarrow{i_*} \text{Mod}(\mathcal{A}) \\
\text{Mod}(\mathcal{C}) & \xrightarrow{j^*} & \text{Mod}(\mathcal{R}) \xleftarrow{j_*} \\
\end{array}
$$

where $\mathcal{A} := \left( \begin{array}{cc} \mathcal{T} & 0 \\ M & \mathcal{R} \end{array} \right)$ and $\mathcal{A}^* := \left( \begin{array}{cc} \mathcal{T} & 0 \\ \mathcal{C} & \mathcal{T} \end{array} \right)$ are matrix categories and $j_!(M)$ is a bimodule constructed as in Definition 4.4.

We adapt the arguments given in [11] to prove Theorem 4.1. Thus, we first recall some notation and results of [22].

In [22] the notion of triangular matrix category was introduced. For convenience of the reader, we recall briefly these concepts. Let $\mathcal{R}$, $\mathcal{T}$ preadditive categories and $M \in \text{Mod}(\mathcal{R} \otimes T^{op})$, the **triangular matrix category** $\mathcal{A} = \left[ \begin{array}{cc} \mathcal{T} & 0 \\ M & \mathcal{R} \end{array} \right]$ is defined as follows.

(a) The class of objects of this category are matrices $\left[ \begin{array}{cc} T & 0 \\ M & R \end{array} \right]$ where the objects $T$ and $R$ are in $\mathcal{T}$ and $\mathcal{R}$ respectively.
(b) Given a pair of objects \([\begin{bmatrix} T & 0 \\ M & R \end{bmatrix}, \begin{bmatrix} T' & 0 \\ M & R' \end{bmatrix}]\) in \(\Lambda\),
\[
\text{Hom}_\Lambda \left( \begin{bmatrix} T & 0 \\ M & R \end{bmatrix}, \begin{bmatrix} T' & 0 \\ M & R' \end{bmatrix} \right) := \begin{bmatrix} \text{Hom}_T(T, T') & 0 \\ \text{Hom}_R(R, R') \end{bmatrix}.
\]

The composition is given by
\[
\circ : \begin{bmatrix} T(T', T'') & 0 \\ M(R', T'') & \text{Hom}_R(R', R'') \end{bmatrix} \times \begin{bmatrix} T(T', T') & 0 \\ M(R', R') & \text{Hom}_R(R, R') \end{bmatrix} \rightarrow \begin{bmatrix} T(T, T'') & 0 \\ M(R', T) & \text{Hom}_R(R, R'') \end{bmatrix}
\]
\[
\left( \begin{bmatrix} f_{12} \\ m_{21} \\ 0 \\ r_2 \\ 0 \\ f_{11} \\ m_{11} \\ 0 \\ r_1 \end{bmatrix}, \begin{bmatrix} f_{11} \\ m_{11} \\ 0 \\ r_1 \end{bmatrix} \right) \mapsto \begin{bmatrix} f_{20f_{11}} \\ m_{20m_{11}} + r_2m_1 + r_2f_{11} \\ 0 \\ r_2 \end{bmatrix}.
\]

We recall that \(m_2 \bullet t_1 := M(1_{R^0} \otimes t_1^{op})(m_2)\) and \(r_2 \bullet m_1 = M(r_2 \otimes 1_T)(m_1)\), and given an object \(\begin{bmatrix} T & 0 \\ M & R \end{bmatrix} \in \Lambda\), the identity morphism is given by \(1_{\begin{bmatrix} T & 0 \\ M & R \end{bmatrix}} := \begin{bmatrix} 1_T & 0 \\ 0 & 1_R \end{bmatrix}\).

In [22, Theorem 3.17] it is proved the following result.

**Theorem 4.2.** Let \(\mathcal{R}\) and \(\mathcal{T}\) be preadditive categories and \(M \in \text{Mod}(\mathcal{R} \otimes \mathcal{T}^{op})\). Then, there exists a functor \(G_1 : \text{Mod}(\mathcal{R}) \rightarrow \text{Mod}(\mathcal{T})\) for which there is an equivalence of categories
\[
\left(\text{Mod}(\mathcal{T}), \text{Mod}(\mathcal{R})\right) \cong \text{Mod}\left(\begin{bmatrix} T & 0 \\ M & R \end{bmatrix}\right).
\]

In the remark below we briefly recall the definition of the functor \(G_1\) mentioned above.

**Remark 4.3.** Let \(\mathcal{R}, \mathcal{S}, \mathcal{T}\) preadditive categories and consider an additive functor \(M \in \text{Mod}(\mathcal{R} \otimes \mathcal{T}^{op})\). For all \(T \in \mathcal{T}\) we have the functor \(M_T : \mathcal{R} \rightarrow \text{Ab}\) defined as follows:
1. \(M_T(R) := M(R, T), \text{for all } R \in \mathcal{R}\).
2. \(M_T(r) := M(r \otimes 1_T) : M_T(R) \rightarrow M_T(R'), \text{for all } r \in \text{Hom}_\mathcal{R}(R, R')\).

Also for all \(t \in \text{Hom}_\mathcal{T}(T, T')\) we have a morphism of \(\mathcal{R}\)-modules \(\tilde{t} : M_T \rightarrow M_T\) such that \(\tilde{r} = \{[\tilde{r}]_R : M_T(R) \rightarrow M_T(R)\}_{R \in \mathcal{R}}\), where \([\tilde{r}]_R = M(1_R \otimes t^{op}) : M(T, R') \rightarrow M(T, R)\).

So we have the functor \(G_1 : \text{Mod}(\mathcal{R}) \rightarrow \text{Mod}(\mathcal{T})\) as follows:
1. \(G_1(B)(T) := \text{Hom}_{\text{Mod}(\mathcal{R})}(M_T, B), \text{for all } B \in \text{Mod}(\mathcal{R})\) and for all \(T \in \mathcal{T}\). Moreover \(G_1(B)(r) := \text{Hom}_{\text{Mod}(\mathcal{R})}(\tilde{r}, B)\) for all \(B \in \text{Mod}(\mathcal{R})\) and for all \(r \in \text{Hom}_\mathcal{T}(T, T')\).
2. If \(\eta : B \rightarrow B'\) is a morphism of \(\mathcal{R}\)-modules, \(G_1(\eta) : G_1(B) \rightarrow G_1(B')\) is defined by \(G_1(\eta) = \{[G_1(\eta)]_T : G_1(B)(T) \rightarrow G_1(B')(T)\}_{T \in \mathcal{T}}\) with \([G_1(\eta)]_T := \text{Hom}_{\text{Mod}(\mathcal{R})}(M_T, \eta).\)

Similarly, given \(M \in \text{Mod}(\mathcal{R} \otimes \mathcal{T}^{op})\) we have \(\overline{M} \in \text{Mod}(\mathcal{T}^{op} \otimes \mathcal{R})\) and a functor \(\overline{G_1} : \text{Mod}(\mathcal{T}^{op}) \rightarrow \text{Mod}(\mathcal{R})\) (see Section 4 in [22]).

**Definition 4.4.** Let \(F : \text{Mod}(\mathcal{R}) \rightarrow \text{Mod}(\mathcal{S})\) an additive functor and \(M \in \text{Mod}(\mathcal{R} \otimes \mathcal{T}^{op})\). We define a bimodule in \(\text{Mod}(\mathcal{S} \otimes \mathcal{T}^{op})\) denoted by \(N := F(M)\) as follows, the functor \(N : F(M) : \mathcal{S} \otimes \mathcal{T}^{op} \rightarrow \text{Ab}\) is given by:
1. \(N(S, T) := F(M_T)(S)\) for all \((S, T) \in \mathcal{S} \otimes \mathcal{T}^{op}\).
(ii) Let $g \otimes t^o : (S, T) \rightarrow (S', T')$ where $g : S \rightarrow S'$ in $S$ and $t : T \rightarrow T$ in $T$. Since $\bar{i} : M_T \rightarrow M_{T'}$ is a morphism of $R$-modules, then $F(\bar{i}) : F(M_T) \rightarrow F(M_{T'})$ is a morphism of $S$-modules. Thus we have the following commutative diagram.

\[
\begin{array}{ccc}
F(M_T)(S) & \xrightarrow{[F(\bar{i})]_S} & F(M_{T'})(S) \\
\xdownarrow{F(M_T)(g)} & & \xdownarrow{F(M_{T'})(g)} \\
F(M_T)(S') & \xrightarrow{[F(\bar{i})]_{S'}} & F(M_{T'})(S')
\end{array}
\]

Hence we define $N(g \otimes t^o) := F(M_{T'})(g) \circ [F(\bar{i})]_S = [F(\bar{i})]_{S'} \circ F(M_T)(g)$.

Now, that we have a bimodule $N \in \text{Mod}(S \otimes T^o)$ and with this bimodule we define a functor $G_2$ similar to $G_1$. For convenience of the reader we repeat its construction.

**Remark 4.5.** We define a functor $G_2 : \text{Mod}(S) \rightarrow \text{Mod}(T)$ as follows:

1. $G_2(L)(T) := \text{Hom}_{\text{Mod}(S)}(N_T, L)$, for all $L \in \text{Mod}(S)$ and for all $T \in T$, where $N_T := F(M_T) \in \text{Mod}(S)$. Moreover $G_2(L)(t) := \text{Hom}_{\text{Mod}(S)}(F(\bar{i}), L)$ for all $L \in \text{Mod}(S)$ and for all $t \in \text{Hom}_T(T', T)$.

2. If $\gamma : L \rightarrow L'$ is a morphism of $S$-modules we define $G_2(\gamma) : G_2(L) \rightarrow G_2(L')$ as:
   
   $G_2(\gamma) = \{[G_2(\gamma)]_T := \text{Hom}_{\text{Mod}(S)}(N_T, \gamma) : G_2(L)(T) \rightarrow G_2(L')(T)\}_{T \in T}$

Since $N = F(M) \in \text{Mod}(S \otimes T^o)$, we get the category $\left(\text{Mod}(T), G_2 \text{Mod}(S)\right)$, and we have an equivalence of categories

\[\left(\text{Mod}(T), G_2 \text{Mod}(S)\right) \cong \text{Mod}(\frac{T}{\text{Hom}_T(T', T)}).
\]

Suppose that we have two additive functors $F : \text{Mod}(R) \rightarrow \text{Mod}(S)$ and $G : \text{Mod}(S) \rightarrow \text{Mod}(R)$ and a bimodule $M \in \text{Mod}(R \otimes T^o)$.

Then, for all $B \in \text{Mod}(R)$ and $T \in T$, $F$ defines a mapping

\[F_{M, B} : \text{Hom}_{\text{Mod}(R)}(M_T, B) \rightarrow \text{Hom}_{\text{Mod}(S)}(FM_T, FB), \ f \mapsto F(f).
\]

Similarly, for all $L \in \text{Mod}(S)$ and $T \in T$, $G$ defines a mapping

\[G_{N, L} : \text{Hom}_{\text{Mod}(S)}(N_T, L) \rightarrow \text{Hom}_{\text{Mod}(R)}(GN_T, GL), \ g \mapsto G(g)
\]

where $N = F(M) \in \text{Mod}(S \otimes T^o)$ is as given in Definition 4.4.

In this way we have the following Lemma.

**Lemma 4.6.** Let $M \in \text{Mod}(R \otimes T^o)$ be. Consider two additive functors $F : \text{Mod}(R) \rightarrow \text{Mod}(S)$ and $G : \text{Mod}(S) \rightarrow \text{Mod}(R)$. Then, the following conditions hold.

(a) For all $B \in \text{Mod}(R)$, the family $F_{M, B} := \{F_{M, T, B} : G_1(B)(T) \rightarrow (G_2 \circ F)(B)(T)\}_{T \in T}$ is natural transformation, that is, $F_{M, B} : G_1(B) \rightarrow (G_2 \circ F)(B)$ is a morphism of $T$-modules.

(b) $\xi := F_{M, -} : G_1 \rightarrow G_2 \circ F$ is a natural transformation.

(c) Suppose that $M_T = G(N_T)$ for all $T \in T$. Then for all $L \in \text{Mod}(S)$, the family $G_{N, L} := \{G_{N, T, L} : G_2(L)(T) \rightarrow (G_1 \circ G)(L)(T)\}_{T \in T}$ is a natural transformation, that is, $G_{N, L} : G_2(L) \rightarrow (G_1 \circ G)(L)$ is a morphism of $T$-modules.

(d) $\rho := G_{N, -} = G_2 \rightarrow G_1 \circ G$ is a natural transformation.
Proof Since $F(M_T) = N_T$, for all $B \in \text{Mod}(\mathcal{R})$ and $T \in \mathcal{T}$ we have $G_1(B)(T) = \text{Hom}_{\text{Mod}(\mathcal{R})}(M_T, B)$ and $(G_2 \circ F)(B)(T) = \text{Hom}_{\text{Mod}(S)}(F(M_T), F(B))$.

(a) Let $t \in \text{Hom}_{\mathcal{T}}(T, T')$ and $B \in \text{Mod}(S)$. We have to show that the following diagram commutes

\[
\begin{array}{ccc}
G_1(B)(T) & \xrightarrow{F_{M_T, B}} & G_2(B)(T) \\
\downarrow G_1(B)(t) & & \downarrow G_2(B)(t) \\
G_1(B)(T') & \xrightarrow{F_{M_{T'}, B}} & G_2(B)(T').
\end{array}
\]

Note that if $\phi \in G_1(B)(T)$ then

\[(G_2(B)(t)(\phi) \circ F_{M_{T'}, B})(\phi) = \text{Hom}_{\text{Mod}(S)}(F(t), FB)(\phi) = F(\phi)F(t)\]

and

\[(F_{M_{T'}, B} \circ G_1(B)(t))(\phi) = F_{M_{T'}, B} \left( \text{Hom}_{\text{Mod}(\mathcal{R})}(t, B)(\phi) \right) = F_{M_{T'}, B}(\phi t) = F(\phi t).
\]

Proving that the diagram commutes.

(b) Let $f \in \text{Hom}_{\text{Mod}(\mathcal{R})}(B, B')$ then we have to show the following diagram commutes

\[
\begin{array}{ccc}
G_1(B) & \xrightarrow{F_{M, B}} & G_2 F(B) \\
\downarrow G_1(f) & & \downarrow G_2 F(f) \\
G_1(B') & \xrightarrow{F_{M, B'}} & G_2 F(B').
\end{array}
\]

For all $T \in \mathcal{T}$ and for all $\phi \in G_1(B)(T)$ we obtain the equalities.

\[((G_2 F(f))_T \circ F_{M_{T'}, B})(\phi) = [G_2 F(f)]_T (F_{M_{T'}, B}(\phi)) = \text{Hom}_{\text{Mod}(S)}(F(M_T), F(f))(\phi) = F(f)F(\phi)\]

and

\[((F_{M_{T'}, B'} \circ [G_1(f)]_T)(\phi) = F_{M_{T'}, B'} \left( \text{Hom}_{\text{Mod}(\mathcal{R})}(M_{T'}, f)(\phi) \right) = F(f)\]

Proving that the previous diagram is commutative.

(c) and (d). Suppose that $M_T = G(N_T)$ for all $T \in \mathcal{T}$. Then, for all $L \in \text{Mod}(S)$ and $T \in \mathcal{T}$ we have $G_2(L)(T) = \text{Hom}_{\text{Mod}(S)}(N_T, L)$ and $(G_1 \circ G)(L)(T) = \text{Hom}_{\text{Mod}(\mathcal{R})}(G(N_T), G(L))$. Therefore the proof of (c) and (d) is similar to (a) and (b). □

Definition 4.7. [11, Definition 3.2] Let $G_1 : \mathcal{A} \to \mathcal{B}$, $G_2 : \mathcal{B} \to \mathcal{B}$, $F : \mathcal{A} \to \mathcal{B}$ and $H : \mathcal{B} \to \mathcal{A}$ be additive functors. Assume that $(F, H)$ is an adjoint pair, with $\eta$ being the adjugant equivalence. We say that the pair $(G_1, G_2)$ is compatible with the adjoint pair $(F, H)$ if there exist two natural transformations

\[\xi : G_1 \to G_2 F\]

and

\[\rho : G_2 \to G_1 H\]

such that $\rho Y$ is a monomorphism and $G_1(\eta_X, Y(f)) = \rho YG_2(f)\xi_X$ for every $X \in \mathcal{A}$, $Y \in \mathcal{B}$ and $f \in \text{Hom}_{\mathcal{B}}(FX, Y)$.

The proof of Theorem 4.1 is based in the following result.
**Theorem 4.8.** [11, Theorem 3.6] Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ be abelian categories, and let $G_1 : \mathcal{A} \to \mathcal{D}$ and $G_2 : \mathcal{B} \to \mathcal{D}$ be left exact additive functors.

(a) If the diagram

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{i^*} & \mathcal{A} \\
i_* & \xrightarrow{\downarrow} & \downarrow j \\
& \xrightarrow{j^*} & \mathcal{B}
\end{array}
$$

is a left recollement where, $(G_2, G_1)$ is compatible with the adjoint $(j_!, j^!)$, then there is a left recollement

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{i^*} & (\mathcal{D}, G_1 \mathcal{A}) \\
i_* & \xrightarrow{\downarrow} & \downarrow j \\
& \xrightarrow{j^*} & (\mathcal{D}, G_2 \mathcal{B})
\end{array}
$$

(b) If the diagram

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{i} & \mathcal{A} \\
i^* & \xrightarrow{\downarrow} & \downarrow j \\
& \xrightarrow{j^!} & \mathcal{B}
\end{array}
$$

is a right recollement, where $(G_1, G_2)$ is compatible with the adjoint $(j^!, j_*)$, then there is a right recollement

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{i} & (\mathcal{D}, G_1 \mathcal{A}) \\
i^* & \xrightarrow{\downarrow} & \downarrow j \\
& \xrightarrow{j^!} & (\mathcal{D}, G_2 \mathcal{B})
\end{array}
$$

In order to prove Theorem 4.1 we need the following result which generalizes [11, Lemma 4.2].

**Lemma 4.9.** Let $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$ be preadditive categories and $F : \text{Mod}(\mathcal{R}) \to \text{Mod}(\mathcal{S})$ and $G : \text{Mod}(\mathcal{S}) \to \text{Mod}(\mathcal{R})$ be additive functors. For $M \in \text{Mod}(\mathcal{R} \otimes \mathcal{T}^{\text{op}})$ consider the additive functors $\mathcal{G}_1 : \text{Mod}(\mathcal{R}) \to \text{Mod}(\mathcal{T})$ and $\mathcal{G}_2 : \text{Mod}(\mathcal{S}) \to \text{Mod}(\mathcal{T})$ as we have defined in Remarks 4.3 and 4.5 where $N_T = F(M_T)$. If $(F, G)$ is an adjoint pair and its unit $\varepsilon : 1_{\text{Mod}(\mathcal{R})} \to GF$ satisfies $\varepsilon_{M_T} = 1_{M_T}$ for all $T \in \mathcal{T}$, then the pair $(\mathcal{G}_1, \mathcal{G}_2)$ is compatible with $(F, G)$.

**Proof** Since $(F, G)$ is an adjoint pair there exist a natural equivalence

$$
\eta := \{\eta_{B, L} : \text{Hom}_{\text{Mod}(\mathcal{S})}(FB, L) \to \text{Hom}_{\text{Mod}(\mathcal{R})}(B, GL)\}_{B \in \text{Mod}(\mathcal{R}), L \in \text{Mod}(\mathcal{S})}
$$

By Lemma 4.6 (b), we have natural transformations $\xi := F_{M, -} : \mathcal{G}_1 \to \mathcal{G}_2 F$. Since $\varepsilon_{M_T} = 1_{M_T}$, we have that $G(N_T) = M_T$ for all $T \in \mathcal{T}$ and by Lemma 4.6 (d), we have a natural transformation $\rho := G_{N_T, -} = \mathcal{G}_2 \to \mathcal{G}_1 G$. First we will see that for all $L \in \text{Mod}(\mathcal{S})$ the morphism

$$
\rho_L : \mathcal{G}_2(L) \to \mathcal{G}_1 G(L)
$$

is a monomorphism in $\text{Mod}(\mathcal{T})$. Indeed, for $T \in \mathcal{T}$ we have to show that

$$
[\rho_L]_T := G_{N_T, L} : \text{Hom}_{\text{Mod}(\mathcal{S})}(N_T, L) \to \text{Hom}_{\text{Mod}(\mathcal{R})}(M_T, G(L))
$$
is a monomorphism. Consider the morphism
\[ \eta_{M_T, L} : \text{Hom}_{\text{Mod}}(S)(F(M_T), L) \to \text{Hom}_{\text{Mod}}(\mathcal{R})(M_T, G(L)). \]
We assert that \( \eta_{M_T, L} = G_{N_T, L}. \) Indeed, let \( \beta \in \text{Hom}_{\text{Mod}}(S)(F(M_T), L). \) Then we have the following commutative diagram
\[
\begin{array}{ccc}
\text{Hom}_{\text{Mod}}(S)(F(M_T), F(M_T)) & \xrightarrow{\eta_{M_T, F(M_T)}} & \text{Hom}_{\text{Mod}}(\mathcal{R})(M_T, GF(M_T)) \\
\downarrow & & \downarrow \\
\text{Hom}_{\text{Mod}}(S)(F(M_T), L) & \xrightarrow{\eta_{M_T, L}} & \text{Hom}_{\text{Mod}}(\mathcal{R})(M_T, G(L)).
\end{array}
\]
Then
\[
G(\beta) \circ \left( \eta_{M_T, F(M_T)}(1_{F(M_T)}) \right) = \left( \text{Hom}_{\text{Mod}}(\mathcal{R})(M_T, G(\beta)) \circ \eta_{M_T, F(M_T)} \right)(1_{F(M_T)})
\]
\[
= \left( \eta_{M_T, L} \circ \text{Hom}_{\text{Mod}}(S)(N_T, \beta) \right)(1_{F(M_T)})
\]
\[
= \eta_{M_T, L} \left( \text{Hom}_{\text{Mod}}(S)(N_T, \beta) \right)(1_{F(M_T)})
\]
\[
= \eta_{M_T, L}(\beta \circ 1_{F(M_T)})
\]
\[
= \eta_{M_T, L}(\beta).
\]
Since \( 1_{M_T} = \varepsilon_{M_T} = \eta_{M_T, F(M_T)}(1_{F(M_T)}) \) then \( G(\beta) = \eta_{M_T, L}(\beta). \) Since \( G_{N_T, L}(\beta) = G(\beta) \) and \( \eta_{M_T, L} \) is injective, it follows that \( G_{N_T, L} \) is injective, for all \( T \in \mathcal{T}. \) Proving that \( \rho_L \) is a monomorphism for each \( L \in \text{Mod}(S). \)

Now we have to show that \( \mathcal{G}_1(\eta_{B, L}(f)) = \rho_L \mathcal{G}_2(f) \xi_B \) for all \( f \in \text{Hom}_{\text{Mod}}(S)(F(B), L). \) That is, we have to show that
\[
\left[ \mathcal{G}_1(\eta_{B, L}(f)) \right]_T = \rho_{N_T, L}[\mathcal{G}_2(f)]_T \xi_{M_T, B} \quad \forall T \in \mathcal{T}.
\]
Let \( \alpha \in \text{Hom}_{\text{Mod}}(\mathcal{R})(M_T, B). \) It follows from the following commutative diagram
\[
\begin{array}{ccc}
\text{Hom}_{\text{Mod}}(S)(F(B), L) & \xrightarrow{\eta_{B, L}} & \text{Hom}_{\text{Mod}}(\mathcal{R})(B, G(L)) \\
\downarrow & & \downarrow \\
\text{Hom}_{\text{Mod}}(S)(F(M_T), L) & \xrightarrow{\eta_{M_T, L}} & \text{Hom}_{\text{Mod}}(\mathcal{R})(M_T, G(L)).
\end{array}
\]
that \( G(f \circ F(\alpha)) = \eta_{M_T, L}(f \circ F(\alpha)) = \eta_{B, L}(f) \circ \alpha. \) We note that
\[
\left[ \mathcal{G}_1(\eta_{B, L}(f)) \right]_T(\alpha) = \text{Hom}_{\text{Mod}}(\mathcal{R})(M_T, \eta_{B, L}(f))(\alpha) = \eta_{B, L}(f) \circ \alpha.
\]
On the other hand,
\[
\left( \rho_{N_T, L} \circ [\mathcal{G}_2(f)]_T \circ \xi_{M_T, B} \right)(\alpha) = (G_{N_T, L} \circ \text{Hom}_{\text{Mod}}(S)(F(M_T), f))(F_{M_T, B}(\alpha))
\]
\[
= (G_{N_T, L} \circ \text{Hom}_{\text{Mod}}(S)(F(M_T), f))(F(\alpha))
\]
\[
= G_{N_T, L}(f \circ F(\alpha))
\]
\[
= G(f \circ F(\alpha)).
\]
This proves that \( (\mathcal{G}_1, \mathcal{G}_2) \) is compatible with \( (F, G). \)

We recall the following general fact about adjoint functors, for a proof see [19, Proposition 7.6] in p. 67.
Lemma 4.10. Let $F$ be a full embedding functor. If $(F, G)$ is an adjoint pair, then there exists a functor $G'$ such that $(F, G')$ is an adjoint pair, where $G' \cong G$ and the unit $\epsilon' : 1 \to G'F$ is the identity.

Now, we are ready to proof Theorem 4.1.

Proof of Theorem 4.1. We only prove (a), since (b) is similar. Consider

\[
\begin{array}{c}
\text{Mod}(C) \xleftarrow{i^*} \text{Mod}(S) \xrightarrow{j_1} \text{Mod}(R) \\
\downarrow{i_*} \quad \quad \quad \quad \downarrow{j^*}
\end{array}
\]

a left recollement and $M \in \text{Mod}(R \otimes T^{op})$. Set $N = j_1(M)$ as in Definition 4.4, and consider the additive functors $G_1 : \text{Mod}(R) \to \text{Mod}(T)$ and $G_2 : \text{Mod}(S) \to \text{Mod}(T)$ as defined in Remarks 4.3 and 4.5. Since $j_1$ is a full embedding, by Lemma 4.10, we may assume that the unit $\epsilon : 1 \to j_1^*j_1$, of the adjoint pair $(j_1, j_1^*)$, is the identity. In particular, we have that $\epsilon_{M_T} = 1_{M_T}$. Thus, from Lemma 4.9, the pair $(G_1, G_2)$ is compatible with $(j_1, j_1^*)$ and the rest follows from Theorems 4.8 and 4.2.

We note that recollement can been seen as the gluing of a left recollement and a right recollement. Since $i_* = i^*$ and $j^* = j^*$, it follows that $i_* = i^*$ and $j_1 = j_1^*$. For any $M \in \text{Mod}(R \otimes T^{op})$, consider the matrix categories $\Lambda := (\begin{array}{c} T \\ 0 \\ R \end{array})$ and $\Lambda^1 := (\begin{array}{c} T \\ j_1(M) \\ 0 \\ S \end{array})$.

It follows, that if the diagram

\[
\begin{array}{c}
\text{Mod}(C) \xleftarrow{i^*} \text{Mod}(S) \xrightarrow{j_1} \text{Mod}(R) \\
\downarrow{i_*} \quad \quad \quad \quad \downarrow{j^*}
\end{array}
\]

is a recollement, then there is a recollement

\[
\begin{array}{c}
\text{Mod}(C) \xleftarrow{i^*} \text{Mod}(\Lambda^1) \xrightarrow{j_1} \text{Mod}(\Lambda). \\
\downarrow{i_*} \quad \quad \quad \quad \downarrow{j^*}
\end{array}
\]

Recall that two preadditive categories $C$ and $D$ are called Morita equivalent if the functor categories $\text{Mod}(C)$ and $\text{Mod}(D)$ are equivalent.

Corollary 4.11. Let $R$, $S$, and $T$ preadditive categories, and $M \in \text{Mod}(R \otimes T^{op})$. If $R$ is Morita equivalent to $S$, then there exists a functor $N \in \text{Mod}(S \otimes T^{op})$ making the triangular matrix category $(\begin{array}{c} T \\ 0 \\ R \end{array})$ Morita equivalent to $(\begin{array}{c} T \\ 0 \\ S \end{array})$.

Proof Suppose that we have mutually inverse functors $F : \text{Mod}(R) \to \text{Mod}(S)$ and $G : \text{Mod}(S) \to \text{Mod}(R)$. Since Ker$(G) = 0$ we have a recollement

\[
\begin{array}{c}
0 \xrightarrow{} \text{Mod}(S) \xrightarrow{G} \text{Mod}(R) \\
\downarrow{F} \quad \quad \quad \quad \downarrow{F}
\end{array}
\]

Let $M \in \text{Mod}(R \otimes T^{op})$. By Theorem 4.1 we have a recollement

\[
\begin{array}{c}
0 \xrightarrow{} \text{Mod}(\Lambda^1) \xrightarrow{j_1} \text{Mod}(\Lambda). \\
\downarrow{i_*} \quad \quad \quad \quad \downarrow{j^*}
\end{array}
\]

where $\Lambda := (\begin{array}{c} T \\ 0 \\ R \end{array})$ and $\Lambda^1 := (\begin{array}{c} T \\ F(M) \\ 0 \\ S \end{array})$. Then we conclude that $\Lambda := (\begin{array}{c} T \\ 0 \\ R \end{array})$ and $\Lambda^1 := (\begin{array}{c} T \\ F(M) \\ 0 \\ S \end{array})$ are Morita equivalent. \qed
The following result tell us that under certain conditions we can restrict the recollements in Theorem 4.1 to the category of finitely presented modules.

**Theorem 4.12.** Let $\mathcal{R}, \mathcal{S}, \mathcal{C}$ and $\mathcal{T}$ be dualizing $K$-varieties and $M \in \text{Mod}(\mathcal{R} \otimes_K \mathcal{T}^{op})$ such that $M_T \in \text{mod}(\mathcal{R})$ and $M_R \in \text{mod}(\mathcal{T}^{op})$ for all $T \in \mathcal{T}$ and $R \in \mathcal{U}$. Consider the matrix categories $\Lambda := \left( \begin{array}{cc} T & 0 \\ M_R & \mathcal{R} \end{array} \right)$, $\Lambda^! := \left( \begin{array}{cc} T & 0 \\ j_!(M) & \mathcal{S} \end{array} \right)$, where the bimodules $j_!(M)$ and $j_*(M)$ are constructed as in Definition 4.4. Moreover suppose that $j_!(M)_{S}, \ j_* (M)_S \in \text{mod}(T^{op})$ for all $S \in \mathcal{S}$.

(a) **If the diagram**

\[
\begin{array}{ccc}
\text{mod}(\mathcal{C}) & \xrightarrow{i^*} & \text{mod}(\mathcal{S}) \\
\downarrow{i_*} & & \downarrow{j_!} \\
\text{mod}(\mathcal{R}) & \xleftarrow{j_!} & \text{mod}(\mathcal{R})
\end{array}
\]

is a left recollement, then there is a left recollement

\[
\begin{array}{ccc}
\text{mod}(\mathcal{C}) & \xrightarrow{i^*} & \text{mod}(\Lambda^!) \\
\downarrow{i_*} & & \downarrow{j_!} \\
\text{mod}(\Lambda) & \xleftarrow{j_!} & \text{mod}(\Lambda)
\end{array}
\]

(b) **If the diagram**

\[
\begin{array}{ccc}
\text{mod}(\mathcal{C}) & \xrightarrow{i} & \text{mod}(\mathcal{S}) \\
\downarrow{i^!} & & \downarrow{j_*} \\
\text{mod}(\mathcal{R}) & \xleftarrow{j_*} & \text{mod}(\mathcal{R})
\end{array}
\]

is a right recollement, then there is a right recollement

\[
\begin{array}{ccc}
\text{mod}(\mathcal{C}) & \xrightarrow{i^!} & \text{mod}(\Lambda^*) \\
\downarrow{i_*} & & \downarrow{j_*} \\
\text{mod}(\Lambda) & \xleftarrow{j_*} & \text{mod}(\Lambda)
\end{array}
\]

**Proof** First, we note that by the Definition 4.4, we have that $j_!(M)_T = j_!(M_T)$; and since $M_T \in \text{mod}(\mathcal{R})$ and $j_! : \text{mod}(\mathcal{R}) \rightarrow \text{mod}(\mathcal{S})$ we have that $j_!(M)_T \in \text{mod}(\mathcal{S})$. Similarly $j_*(M)_T \in \text{mod}(\mathcal{S})$. Then by [22, Proposition 6.3], we have equivalences

\[
\left( \text{mod}(\mathcal{T}), \mathcal{G}_1 \text{mod}(\mathcal{R}) \right) \overset{\sim}{\rightarrow} \text{mod}\left( \left[ \begin{array}{cc} T & 0 \\ M_R & \mathcal{R} \end{array} \right] \right),
\]

and

\[
\left( \text{mod}(\mathcal{T}), \mathcal{G}_2 \text{mod}(\mathcal{S}) \right) \overset{\sim}{\rightarrow} \text{mod}\left( \left[ \begin{array}{cc} T & 0 \\ j_!(M) & \mathcal{S} \end{array} \right] \right),
\]

where $\mathcal{G}_1 : \text{Mod}(\mathcal{R}) \rightarrow \text{Mod}(\mathcal{T})$ and $\mathcal{G}_2 : \text{Mod}(\mathcal{S}) \rightarrow \text{Mod}(\mathcal{T})$ are defined in Remarks 4.3 and 4.5. Under our conditions we have that they restrict well to $\mathcal{G}_1 |_{\text{mod}(\mathcal{R})} : \text{mod}(\mathcal{R}) \rightarrow \text{mod}(\mathcal{T})$ and $\mathcal{G}_2 |_{\text{mod}(\mathcal{S})} : \text{mod}(\mathcal{S}) \rightarrow \text{mod}(\mathcal{T})$. It can be seen as in the proof of Theorem 4.1, that they are compatible with $(j_!, j^!)$ and the rest follows from Theorem 4.8. \qed

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5 The Maps Category

Assume that \( C \) is an \( R \)-variety. The maps category, \( \text{maps}(C) \) is defined as follows. The objects in \( \text{maps}(C) \) are triples \((f_1, A_1, A_0)\) where \( f_1 : A_1 \to A_0 \) is a morphism in \( C \), and the maps are pairs \((h_1, h_0) : (f_1, A_1, A_0) \to (g_1, B_1, B_0)\), such that the following square commutes

\[
\begin{array}{ccc}
A_1 & \xrightarrow{f_1} & A_0 \\
\downarrow{h_1} & & \downarrow{h_0} \\
B_1 & \xrightarrow{g_1} & B_0.
\end{array}
\]

In this section, we study the category \( \text{maps}(\text{Mod}(C)) := \left( \text{Mod}(C), \text{Mod}(C) \right) \) of maps of the category \( \text{Mod}(C) \) and we give in this setting a description of the functor \( \hat{\Theta} : \left( \text{Mod}(T), \text{GMod}(\mathcal{U}) \right) \to \left( \text{Mod}(\mathcal{U}^{op}), \text{GMod}(\mathcal{T}^{op}) \right) \) constructed in [22, Proposition 4.9] (see Proposition 5.2), this functor will be useful when we compute the injective objects in \( \text{maps}(\text{mod}(C)) \) (see Proposition 5.5), since it is a duality. We also give a description of the projective and injective objects of the category \( \text{maps}(\text{mod}(C)) \) when \( C \) is a dualizing variety and we also describe its radical (see Proposition 5.5). Let \( B \) be a dualizing \( K \)-variety and consider the category \( \left[ \begin{smallmatrix} 0 \\ \text{Hom} \end{smallmatrix} \right] \) of triangular matrices with \( M := \text{Hom} : C \otimes \mathcal{C}^{op} \to \text{Ab} \) defined as \( M_T = \text{Hom}_C(T, -) \in \text{Mod}(C) \) for \( T \in \mathcal{C}^{op} \) and \( M_U = \text{Hom}_C(\mathcal{C}, U) \in \text{Mod}(\mathcal{C}^{op}) \) for \( U \in \mathcal{C} \) (see [22, Proposition 2.8] for the notation). In [22, Corollary 6.11(a)], we proved that \( \left[ \begin{smallmatrix} 0 \\ \text{Hom} \end{smallmatrix} \right] \) is a dualizing category. We will show in this section, that the category \( \text{mod}\left( \left[ \begin{smallmatrix} 0 \\ \text{Hom} \end{smallmatrix} \right] \right) \) is equivalent to the category \( \text{maps}(\text{mod}(C)) \). Some results in this section are generalizations of results given in the chapter 3 of [8]. Throughout this section \( K \) will denote an arbitrary field.

**Definition 5.1.** Define a functor

\[
\begin{array}{ccc}
\left( \text{Mod}(C), \text{Mod}(C) \right) & \xrightarrow{\hat{\Theta}_C} & \left( \text{Mod}(\mathcal{C}^{op}), \text{Mod}(\mathcal{C}^{op}) \right)
\end{array}
\]

in objects as \( \hat{\Theta}_C(A \xrightarrow{f} B) = \text{D}_{\mathcal{C}}(B) \xrightarrow{\text{D}_{\mathcal{C}}(f)} \text{D}_{\mathcal{C}}(A) \) and if \((\alpha, \beta) : (A, f, B) \to (A', f', B')\) is a morphism in \( \text{maps}(\text{Mod}(C)) \) then \( \hat{\Theta}_C(\alpha, \beta) = (\text{D}_{\mathcal{C}}(\beta), \text{D}_{\mathcal{C}}(\alpha)) \).

First we have the following result, which tell us that we can identify the functor \( \hat{\Theta} : \left( \text{Mod}(T), \text{GMod}(\mathcal{U}) \right) \to \left( \text{Mod}(\mathcal{U}^{op}), \text{GMod}(\mathcal{T}^{op}) \right) \) constructed in [22, Proposition 4.9] with \( \hat{\Theta} \).

**Proposition 5.2.** Let \( C \) be a \( K \)-variety and \( \text{maps}(\text{Mod}(C)) := \left( \text{Mod}(C), \text{Mod}(C) \right) \) the maps category. Let \( M := \text{Hom} \in \text{Mod}(C \otimes_K \mathcal{C}^{op}) \) be where \( M_T = \text{Hom}_C(T, -) \in \text{Mod}(C) \) for \( T \in \mathcal{C}^{op} \) and \( M_U = \text{Hom}_C(\mathcal{C}, U) \in \text{Mod}(\mathcal{C}^{op}) \) for \( U \in \mathcal{C} \) and consider the induced
functors $\mathcal{G} : \text{Mod}(\mathcal{C}) \to \text{Mod}(\mathcal{C}), \mathcal{G}^\circ : \text{Mod}(\mathcal{C}^{\text{op}}) \to \text{Mod}(\mathcal{C}^{\text{op}})$ (see Remark 4.3). Then, there exists isomorphisms $J_1$ and $J_2$ such that the following diagram commutes

$$
\begin{array}{ccc}
\text{Mod}(\mathcal{C}), \text{GMod}(\mathcal{C}) & \xrightarrow{\widehat{\Theta}} & \text{Mod}(\mathcal{C}^{\text{op}}), \text{GMod}(\mathcal{C}^{\text{op}}) \\
J_1 & & J_2 \\
(\text{Mod}(\mathcal{C}), \text{Mod}(\mathcal{C})) & \xrightarrow{\overline{\Theta}_C} & (\text{Mod}(\mathcal{C}^{\text{op}}), \text{Mod}(\mathcal{C}^{\text{op}}))
\end{array}
$$

where $\widehat{\Theta}$ is the functor defined in [22, Proposition 4.9].

**Proof** Let us define $J_1 : \big(\text{Mod}(\mathcal{C}), \text{Mod}(\mathcal{C})\big) \to \big(\text{Mod}(\mathcal{C}), \text{GMod}(\mathcal{C})\big)$. For this, consider the Yoneda isomorphism $\theta^{-1}_{B, T} : B(T) \to \text{Hom}_{\text{Mod}(\mathcal{C})}(\text{Hom}_C(T, -), B)$. Let $f : A \to B \in \text{maps}(\text{Mod}(\mathcal{C}))$ we set $J_1(f) := \widehat{f} : A \to \mathcal{G}(B)$ where for $T \in \mathcal{C}$ we have that $\widehat{f}_T := \theta^{-1}_{B, T} \circ f_T : A(T) \to \text{Hom}_{\text{Mod}(\mathcal{C})}(\text{Hom}_C(T, -), B) = \mathcal{G}(B)(T)$.

It is easy to see that $J_1^{-1} : \big(\text{Mod}(\mathcal{C}), \text{GMod}(\mathcal{C})\big) \to \big(\text{Mod}(\mathcal{C}), \text{Mod}(\mathcal{C})\big)$ is defined as follows: for $f : A \to \mathcal{G}(B)$ we set $J_1^{-1}(f) = f' : A \to B$ where for $T \in \mathcal{T}$ we have that $f'_T := \theta_{B, T} \circ f_T : A(T) \to B(T)$.

In a similar way is defined $J_2$ and direct computations show that $\widehat{\Theta} \circ J_1 = J_2 \circ \overline{\Theta}_C$. \hfill $\Box$

**Corollary 5.3.** Let $\mathcal{C}$ be a dualizing variety. Then with the conditions as in Proposition 5.2, we have a commutative diagram with $J_1$ and $J_2$ isomorphisms

$$
\begin{array}{ccc}
\text{mod}(\mathcal{C}), \text{Gmod}(\mathcal{C}) & \xrightarrow{\widehat{\Theta}} & \text{mod}(\mathcal{C}^{\text{op}}), \text{Gmod}(\mathcal{C}^{\text{op}}) \\
J_1 & & J_2 \\
\text{mod}(\mathcal{C}), \text{mod}(\mathcal{C}) & \xrightarrow{\overline{\Theta}_C} & \text{mod}(\mathcal{C}^{\text{op}}), \text{mod}(\mathcal{C}^{\text{op}})
\end{array}
$$

**Proof** It follows from Proposition 5.2 and [22, Proposition 6.4]. \hfill $\Box$

**Proposition 5.4.** Let $\mathcal{C}$ be a $K$-variety and consider the category $\Lambda = \left[ \begin{array}{cc} \mathcal{C} & 0 \\ \text{Hom}_C & \end{array} \right]$.

(i) There is an equivalence of categories

$$
\text{Mod}(\Lambda) \tilde{\to} \text{maps}(\text{Mod}(\mathcal{C})).
$$

(ii) If $\mathcal{C}$ is dualizing, there is an equivalence of categories

$$
\text{mod}(\Lambda) \tilde{\to} \text{maps}(\text{mod}(\mathcal{C})).
$$

**Proof** (i) Is proved in [22, Theorem 3.17] that $\text{Mod}(\Lambda)$ is equivalent to the comma category $\left(\text{Mod}(\mathcal{C}), \text{GMod}(\mathcal{C})\right)$. By Proposition 5.2, the category $\text{Mod}(\left[ \begin{array}{cc} \mathcal{C} & 0 \\ \text{Hom}_C & \end{array} \right])$ is equivalent to the category which objects are morphisms of $\mathcal{C}$-modules $A \xrightarrow{f} B$, with $A, B \in \text{Mod}(\mathcal{C})$. In this way we have the equivalence of categories

$$
\mathcal{S} \circ J_1 : \text{maps}(\text{Mod}(\mathcal{C})) \to \text{Mod}(\left[ \begin{array}{cc} \mathcal{C} & 0 \\ \text{Hom}_C & \end{array} \right]).
$$
(ii) Note that $\mathbf{Hom}_C = \text{Hom}(C, -) \in \text{mod}(\mathcal{C})$ and $\mathbf{Hom}_{C'} = \text{Hom}(-, C') \in \text{mod}(\mathcal{C}^{op})$, for all $C, C' \in \mathcal{C}$. Therefore, the equivalence follows from the fact that $\mathcal{C}$ is dualizing, by Corollary 5.3 and [22, Proposition 6.3].

In the following we will write $(C, -)$ and $(-, C)$ instead of $\text{Hom}_C(C, -)$ and $\text{Hom}_C(-, C)$.

**Proposition 5.5.** Let $\mathcal{C}$ be a $K$-variety. Then, the following conditions hold.

(i) \begin{align*}
\text{rad} & \left( \begin{bmatrix}
\text{Hom}_C c_0 & 0 \\
0 & \text{Hom}_C c_1
\end{bmatrix}, \begin{bmatrix}
\text{Hom}_C c_0' & 0 \\
0 & \text{Hom}_C c_1'
\end{bmatrix} \right) = \begin{bmatrix}
\text{rad}_{\mathcal{C}}(c_0, c_0') & 0 \\
\text{Hom}_{\mathcal{C}}(c_0, c_0') & \text{rad}_{\mathcal{C}}(c_1, c_1')
\end{bmatrix}
\end{align*}

(ii) Suppose that $\mathcal{C}$ is a dualizing variety.

(iia) The indecomposable projective objects in $\text{maps}(\text{mod}(\mathcal{C}))$ are objects which isomorphic to: objects of the form $((C, -), (1_C, -), (C, -))$ where $C$ is an indecomposable object in $\mathcal{C}$; and to $\left(0, 0, (C, -)\right)$ where $C$ is an indecomposable object in $\mathcal{C}$.

(iib) The indecomposable injective objects in $\text{maps}(\text{mod}(\mathcal{C}))$ are objects which are isomorphic to: objects of the form $\left(D_{\mathcal{C}^{op}}(-, C), D_{\mathcal{C}^{op}}(-, 1_C), D_{\mathcal{C}^{op}}(-, C)\right)$ where $C$ is an indecomposable object in $\mathcal{C}$; and to $\left(D_{\mathcal{C}^{op}}(-, C), 0, 0\right)$ where $C$ is an indecomposable object in $\mathcal{C}$.

**Proof** (i) It follows from [22, Proposition 3.8].

(iiia) Let $P = \left(\begin{bmatrix}
\text{Hom}_C c_0 & 0 \\
0 & \text{Hom}_C c_1
\end{bmatrix}, -\right)$ a projective object in $\text{Mod}\left(\begin{bmatrix}
\text{Hom}_C c_0 & 0 \\
0 & \text{Hom}_C c_1
\end{bmatrix}\right)$. Consider the object $g : (C, -) \to G(\mathbf{Hom}_C \bigsqcup (C', -))$. Then, by the equivalence $S : \text{maps}(\text{mod}(\mathcal{C})) \to \text{Mod}\left(\begin{bmatrix}
\text{Hom}_C c_0 & 0 \\
0 & \text{Hom}_C c_1
\end{bmatrix}\right)$ we get by [22, Proposition 5.5],

$S\left((C, -), g, \mathbf{Hom}_C \bigsqcup (C', -)\right) \cong P$.

Moreover, we have the following commutative diagram, where the vertical maps are isomorphisms in $\text{maps}(\text{mod}(\mathcal{C}))$.

Since $\mathcal{C}$ is a Krull-Schmidt category, $C$ and $C'$ decomposes as $C = \bigoplus_{i=1}^n C_i$ and $C' = \bigoplus_{i=1}^n C_i'$ such that $\text{End}_C(C_i)$ and $\text{End}_C(C_i')$ are local rings. Thus, we have decompositions

$\left((C, -), 1_{(C,-)}, (C, -)\right) = \bigoplus_{i=1}^n (C_i, -)$. 

\begin{align*}
\text{Hom}_C c_0 & \text{Hom}_C c_1 \\
0 & \text{Hom}_C c_1'
\end{bmatrix}
\begin{bmatrix}
\text{Hom}_C c_0' & 0 \\
0 & \text{Hom}_C c_1'
\end{bmatrix} = \begin{bmatrix}
\text{rad}_{\mathcal{C}}(c_0, c_0') & 0 \\
\text{Hom}_{\mathcal{C}}(c_0, c_0') & \text{rad}_{\mathcal{C}}(c_1, c_1')
\end{bmatrix}
\end{align*}
and
\[(0, 0, (C',-)) = \prod_{j=1}^{m} (0, 0, (C'_j, -))\]

for which \(\text{End}_{\text{maps}(\mathcal{C})}(0, 0, (C'_j, -)) \cong \text{End}_{\mathcal{C}}(C'_j)\) and also we have the isomorphism \(\text{End}_{\text{maps}(\mathcal{C})}(((C_i, -), 1_{(C_i, -)}, (C_i, -))) \cong \text{End}_{\mathcal{C}}(C_i)\).

(iib) By Proposition 5.4(ii) we have that \(\text{mod}(\Lambda) \xrightarrow{\sim} \text{maps}(\text{mod}(\mathcal{C}))\). Hence, by Proposition 5.3, and [22, Theorem 6.10], we have that \(\text{mod}(\mathcal{C})\) is a duality. By (iia), the indecomposable projective objects in \(\text{maps}(\text{mod}(\mathcal{C}))\) are isomorphic to objects of the form \((0, 0, (-, C))\) where \(C\) is an indecomposable object in \(\mathcal{C}\); and to \((0, 0, (-, C))\) where \(C\) is a indecomposable object in \(\mathcal{C}\). Then applying \(\overline{\Theta}_{\mathcal{C}}\) to this objects we obtain the indecomposable injective objects in \(\text{maps}(\text{mod}(\mathcal{C}))\).

5.1 Example

Let \(K\) be a field. We will describe a triangular matrix category \(\Lambda\) such that the category \(\text{Mod}(\Lambda)\) is equivalent to the category \(\text{maps}(\text{Ch}(\text{Mod}(K)))\), where \(\text{Ch}(\text{Mod}(K))\) is the category of chain complexes in \(\text{Mod}(K)\). In this example we will use the notation given in Section 2.6.

Let \(\Lambda = (\Lambda_0, \Lambda_1)\) be the quiver with \(\Lambda_0 = \mathbb{Z}\) and \(\Lambda_1 = \{\alpha_i : i \rightarrow i + 1 \mid i \in \mathbb{Z}\}\), with the set of relations \(\rho = \{\alpha_i \alpha_{i-1} \mid i \in \mathbb{Z}\}\)

\[\cdots \xrightarrow{i-1} i \xrightarrow{\alpha_{i-1}} i \xrightarrow{\alpha_i} i + 1 \xrightarrow{} \cdots\]

and consider the path category \(\mathcal{C} = K\Delta/(\rho)\). Then the category \(\text{Mod}(\mathcal{C})\) is equivalent to \(\text{Ch}(\text{Mod}(K))\).

On the other hand, let \(\Lambda = (\Lambda_0, \Lambda_1)\) be the quiver with \(\Lambda_0 = (\Lambda_0 \times \{1\}) \cup (\Lambda_0 \times \{2\})\) and \(\Lambda_1 = (\Lambda_1 \times \{1\}) \cup (\Lambda_1 \times \{2\})\) with relations given by the set

\[\tilde{\rho} = \{(\alpha_{i+1}, 1)(\alpha_i, 1), (\alpha_{i+1}, 2)(\alpha_i, 2), (\alpha_i, 2)\beta_i - \gamma_i, \beta_{i+1}(\alpha_i, 1) - \gamma\}_{i \in \mathbb{Z}}\]

\[\cdots \xrightarrow{} (i-1, 1) \xrightarrow{(\alpha_{i-1}, 1)} (i, 1) \xrightarrow{(\alpha_i, 1)} (i + 1, 1) \xrightarrow{} \cdots\]

\[\beta_i \downarrow \gamma_i \downarrow \beta_i \downarrow \gamma_i \downarrow \beta_i \downarrow \gamma_i \downarrow \beta_i \downarrow \gamma_i \]

\[\cdots \xrightarrow{} (i-1, 2) \xrightarrow{(\alpha_{i-1}, 2)} (i, 2) \xrightarrow{(\alpha_i, 2)} (i + 1, 2) \xrightarrow{} \cdots\]

Then we can construct the path category \(K\tilde{\Delta}/(\tilde{\rho})\). Consider \(\text{Hom} : K\Delta/(\rho) \otimes (K\Delta/(\rho))^{op} \rightarrow \text{Mod}(K)\), then the category \(\Lambda = \left[\begin{array}{c} K\Delta/(\rho) \\ \text{Hom} \end{array} \right]_{K\Delta/(\rho)}\) is equivalent to the path category \(K\tilde{\Delta}/(\tilde{\rho})\).
Indeed, we have two inclusion functors

\[ \Phi_1, \Phi_2 : K \Delta/\langle \rho \rangle \rightarrow K \tilde{\Delta}/\langle \tilde{\rho} \rangle \]

defined as follows: for \( i \in \Delta \) and \( \alpha_i : i \rightarrow i + 1 \) we set \( \Phi_1(i) = (i, 1) \) and \( \Phi_1(\alpha_i) = (\alpha_i, 1) \); and \( \Phi_2(i) = (i, 2) \) and \( \Phi_2(\alpha_i) = (\alpha_i, 2) \).

We establish a functor

\[ \Phi : \Lambda \rightarrow K \tilde{\Delta}/\langle \tilde{\rho} \rangle \]

defined on objects by \( \Phi \left( \begin{bmatrix} i & 0 \\ \text{Hom} & j \end{bmatrix} \right) = (i, 1) \oplus (j, 2) \), for all \( i, j \in \mathbb{Z} \). Similarly, \( \Phi \) can be defined in morphisms. Now, it is easy to see that \( \Phi \) is an equivalence of categories (for more details see the last example in Section 7 in [22]). Then \( \text{Mod}(\Lambda) \) is equivalent to \( \text{Mod}(K \tilde{\Delta}/\langle \tilde{\rho} \rangle) \). But the modules in the category \( \text{Mod}(K \tilde{\Delta}/\langle \tilde{\rho} \rangle) \) are given by \( K \)-representations of the form

\[
\begin{array}{c}
\cdots \rightarrow V_{-1} \xrightarrow{f_{-1}} V_0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} \cdots \\
\downarrow h_{-1} \quad \downarrow h_0 \quad \downarrow h_1 \\
\cdots \rightarrow W_{-1} \xrightarrow{g_{-1}} W_0 \xrightarrow{g_0} W_1 \xrightarrow{g_1} \cdots
\end{array}
\]

where \( f_{i+1}f_i = 0 \) for all \( i \in \mathbb{Z} \) and such that all the squares are commutative. Hence, \( \text{Mod}(K \tilde{\Delta}/\langle \tilde{\rho} \rangle) \) is the category of morphisms of chain complexes in \( \text{Mod}(K) \). That is, \( \text{Mod}(K \tilde{\Delta}/\langle \tilde{\rho} \rangle) \) coincides with maps \( \text{Ch}(\text{Mod}(K)) \).

### 6 Auslander-Reiten Translate in the Category of Maps

Let \( R \) be a commutative ring. Almost split sequences for dualizing varieties were studied by M. Auslander and Idun Reiten in \( \text{mod}(C) \) for a \( R \)-dualizing variety \( C \) as a generalization of the concept of almost split sequences for \( \text{mod}(\Lambda) \) for an artin \( R \)-algebra (see [6]). The crucial ingredient is the explicit construction of the Auslander-Reiten translate \( \tau \) by taking the dual of the transpose \( D \text{Tr} M \) of a finitely presented \( C \)-module \( M \). The duality \( D_C : \text{mod}(C) \rightarrow \text{mod}(C^{opp}) \) for dualizing varieties \( C \) is given in Definition 2.6 and the transpose \( \text{Tr} : \text{mod}(C) \rightarrow \text{mod}(C^{opp}) \) is defined as follows: consider the functor \( (\cdot)^* : \text{mod}(C) \rightarrow \text{mod}(C^{opp}) \) defined by \( (M)^*(C) = (M, (C, -)) \), for all \( M \in \text{mod}(C) \), and \( C \in C \), then take a minimal projective resolution for \( M : (X, -) \xrightarrow{(f, -)} (Y, -) \rightarrow M \rightarrow 0 \), thus \( \text{Tr}M := \text{Coker}((f, -)^*) \).

For that reason in this section, we study the transpose and the dual in the category of maps. In the same way as in the classic case, we have a duality \( (-)^* : \text{proj} \left( \text{mod}(C), \text{mod}(C) \right) \rightarrow \text{proj} \left( \text{mod}(C^{opp}), \text{mod}(C^{opp}) \right) \) between the projectives in the category of maps (see Proposition 6.7).

One of the main results in this section is to describe the Auslander-Reiten translate in the category of maps which will be denoted by \( \text{Tau} \). In particular, we show that if \( f : C_1 \rightarrow C_2 \) is a morphism in \( \text{mod}(C) \) such that there exists exact sequence

\[ C_1 \xrightarrow{f} C_2 \rightarrow C_3 \rightarrow 0 \]  

with \( C_3 \neq 0 \) and \( C_3 \) not projective. Then

\[ \text{Tau}(C_1, f, C_2) = (D_{C^{opp}}(Y), D_{C^{opp}}(g), D_{C^{opp}} \text{Tr}(C_3)) \]
for some morphism $g : \text{Tr}(C_3) \longrightarrow Y$ such that there exists an exact sequence
\[
0 \longrightarrow \mathcal{D}_{\text{C}} \text{Tr}(C_1) \longrightarrow \mathcal{D}_{\text{C}}(Y) \longrightarrow \mathcal{D}_{\text{C}}(\text{g}) \longrightarrow \mathcal{D}_{\text{C}} \text{Tr}(C_3)
\]
where Tr denotes the Auslander-Reiten translate in mod($\mathcal{C}$) (see Theorem 6.12).

In order to have all the ingredients to prove the above result, we consider $\Lambda = \left[ \begin{array}{cc} T & 0 \\ M & U \end{array} \right]$ the matrix triangular category. Now, we recall the construction of a functor $(-)^* : \text{Mod}(\Lambda) \longrightarrow \text{Mod}(\Lambda^{op})$ which is a generalization of the functor $\text{Mod}(\Lambda) \longrightarrow \text{Mod}(\Lambda^{op})$ given by $M \mapsto \text{Hom}_\Lambda(M, \Lambda)$ for all $\Lambda$-modules $M$, where $\Lambda$ is an artin algebra.

For each $\Lambda$-module $M$ define $M^* : \Lambda^{op} \longrightarrow \text{Ab}$ by
\[
M^* \left( \left[ \begin{array}{c} T \\ M & U \end{array} \right] \right) := \text{Hom}_{\text{Mod}(\Lambda)}(M, \text{Hom}_\Lambda \left( \left[ \begin{array}{c} T \\ M & U \end{array} \right], - \right))
\]
and $M^*$ is defined in morphisms in the obvious way. Then we have a contravariant functor $(-)^* : \text{Mod}(\Lambda) \longrightarrow \text{Mod}(\Lambda^{op})$. Now, taking into account that we have equivalences given in [22, Theorem 3.17], we define a contravariant functor $\Psi := (T^* \circ \bar{S})^{-1} \circ (-)^* \circ S : \left( \text{Mod}(T), \mathcal{G} \text{Mod}(\mathcal{U}) \right) \longrightarrow \left( \text{Mod}(U^{op}), \mathcal{G} \text{Mod}(T^{op}) \right)$ (see discussion before [22, Theorem 4.11] for the notation) which we will denote also by $(-)^*$, such that the following diagram commutes up to a natural equivalence

\[
\begin{array}{ccc}
\left( \text{Mod}(T), \mathcal{G} \text{Mod}(\mathcal{U}) \right) & \xrightarrow{S} & \text{Mod}(\Lambda) \\
\Psi = (-)^* & & \downarrow (-)^* \\
\left( \text{Mod}(U^{op}), \mathcal{G} \text{Mod}(T^{op}) \right) & \xrightarrow{T^* \circ \bar{S}} & \text{Mod}(\Lambda^{op}).
\end{array}
\]

**Remark 6.1.** It is easy to show that if $P := \text{Hom}_\Lambda \left( \left[ \begin{array}{c} T \\ M & U \end{array} \right], - \right) : \Lambda \rightarrow \text{Ab}$, then $P^* := \text{Hom}_\Lambda \left(-, \left[ \begin{array}{c} T \\ M & U \end{array} \right] \right)$.

**Remark 6.2.** We recall the following result given in [22, Proposition 5.5]:

(a) Consider the projective $\Lambda$-module, $P := \text{Hom}_\Lambda \left( \left[ \begin{array}{c} T \\ M & U \end{array} \right], - \right) : \Lambda \rightarrow \text{Ab}$ and the morphism of $T$-modules $g : \text{Hom}_T(T, -) \longrightarrow \mathcal{G} \left( M_T \cup \text{Hom}_T(U, -) \right)$ given by $g := \left\{ [g]_T' : \text{Hom}_T(T, T') \longrightarrow \text{Hom}_T(M_{T'}, M_T \cup \text{Hom}_T(U, -)) \right\}_{T' \in T}$, with $[g]_T(T) := \left[ \begin{array}{c} 0 \\ g \end{array} \right] : M_{T'} \rightarrow M_T \cup \text{Hom}_T(U, -)$ for all $t \in \text{Hom}_T(T, T')$. Then
\[
P \cong \left( \text{Hom}_T(T, -) \right) \cup \left[ \begin{array}{c} 0 \\ g \end{array} \right] (M_T \cup \text{Hom}_T(U, -)).
\]

The reader should observe that there is a $g$ under the coproduct symbol. This is because the action of the functor $\left( \text{Hom}_T(T, -) \right) \cup \left[ \begin{array}{c} 0 \\ g \end{array} \right] (M_T \cup \text{Hom}_T(U, -)) : \Lambda \rightarrow \text{Ab}$ on morphisms is twisted by $g$ (for details, see discussion before [22, Remark 3.9]).
(b) Consider the projective $\overline{A}$-module, $P := \text{Hom}_\mathbb{P}\left(\begin{bmatrix} U & 0 \\ M & T \end{bmatrix}, \_\right) : \overline{A} \rightarrow \text{Ab}$. Then

$$P \cong \left(\text{Hom}_{\mathcal{U}}(-, U)\right) \amalg \left(\text{Hom}_{\mathcal{T}}(-, T)\right),$$

where $\overline{g} := \left\{ [\overline{f}]_{U'} : \text{Hom}_{\mathcal{U}}(U', U) \rightarrow \text{Hom}_{\mathcal{T}^{op}}(M_{U'}, M_U \amalg \text{Hom}_{\mathcal{T}}(-, T)) \right\}_{U' \in \mathcal{U}}$.

with $[\overline{f}]_{U'}(u) := \left[ \begin{smallmatrix} \overline{u} \\ 0 \end{smallmatrix} \right] : M_{U'} \rightarrow M_U \amalg \text{Hom}_{\mathcal{T}}(-, T)$ for all $u \in \text{Hom}_{\mathcal{U}}(U', U)$.

By Section 5 in [22], we know that there exists a functor $\mathcal{F} : \text{Mod}(\mathcal{T}) \rightarrow \text{Mod}(\mathcal{U})$ such that $\mathcal{F}$ is left adjoint to $\mathcal{G}$. That is, there exist a natural bijection

$$\phi_{A,B} : \text{Hom}_{\text{Mod}(\mathcal{U})}(\mathcal{F}(A), B) \rightarrow \text{Hom}_{\text{Mod}(\mathcal{T})}(A, \mathcal{G}(B)).$$

**Proposition 6.3.** Consider the isomorphism of categories given in [22, Proposition 5.4]

$$H : \left(\mathcal{F}(\text{Mod}(\mathcal{T})), \text{Mod}(\mathcal{U})\right) \rightarrow \left(\text{Mod}(\mathcal{T}), \mathcal{G}(\text{Mod}(\mathcal{U}))\right).$$

and the object $\left[ \begin{smallmatrix} 1_{\mathcal{T}} \\ 0 \end{smallmatrix} \right] : \mathcal{F}(\text{Hom}_{\mathcal{T}}(T, -)) = \mathcal{M}_T \rightarrow \mathcal{M}_T \amalg \text{Hom}_{\mathcal{U}}(U, -)$ (see [22, Lemma 5.8]), in the category $\left(\mathcal{F}(\text{Mod}(\mathcal{T})), \text{Mod}(\mathcal{U})\right)$. Then $H\left(\left[ \begin{smallmatrix} 1_{\mathcal{T}} \\ 0 \end{smallmatrix} \right] \right)$ corresponds to the object $g : \text{Hom}_{\mathcal{T}}(T, -) \rightarrow \mathcal{G}(\mathcal{M}_T \amalg \text{Hom}_{\mathcal{U}}(U, -))$ in the category $\left(\text{Mod}(\mathcal{T}), \mathcal{G}(\text{Mod}(\mathcal{U}))\right)$.

**Proof** Let $h : \mathcal{F}(A) \rightarrow B$ be an object in $\left(\mathcal{F}(\text{Mod}(\mathcal{T})), \text{Mod}(\mathcal{U})\right)$ and consider the bijection $\phi_{A,B} : \text{Hom}_{\text{Mod}(\mathcal{U})}(\mathcal{F}(A), B) \rightarrow \text{Hom}_{\text{Mod}(\mathcal{T})}(A, \mathcal{G}(B))$.

By definition, we have that $H(A, h, B) := (A, \phi_{A,B}(h), B)$. Then we have

$$\phi := \phi_{\text{Hom}_{\mathcal{T}}(T, -), \mathcal{M}_T} : \text{Hom}_{\text{Mod}(\mathcal{U})}(\mathcal{M}_T, \mathcal{M}_T) \rightarrow \text{Hom}_{\text{Mod}(\mathcal{T})}(\text{Hom}_{\mathcal{T}}(T, -), \mathcal{G}(\mathcal{M}_T))$$

because $\mathcal{F}(\text{Hom}_{\mathcal{T}}(T, -)) = \mathcal{M}_T$ (see [22, Lemma 5.8(i)]). By [Theorem 6.3] [29], we have that in this case the isomorphism $\phi$ coincides with the Yoneda isomorphism. Then for $1_{\mathcal{M}_T} : \mathcal{M}_T \rightarrow \mathcal{M}_T$ we have that $\phi(1_{\mathcal{M}_T}) : \text{Hom}_{\mathcal{T}}(T, -) \rightarrow \mathcal{G}(\mathcal{M}_T)$ is such that for $T' \in \mathcal{T}$

$$\phi(1_{\mathcal{M}_T})_{T'} : \text{Hom}_{\mathcal{T}}(T, T') \rightarrow \mathcal{G}(\mathcal{M}_T)(T') = \text{Hom}(\mathcal{M}_{T'}, \mathcal{M}_T)$$

is defined as $\phi(1_{\mathcal{M}_T})_{T'}(t) = \left(\mathcal{G}(\mathcal{M}_T)(t)\right)(1_{\mathcal{M}_T}) = \text{Hom}_{\text{Mod}(\mathcal{U})}(\overline{t}, \mathcal{M}_T)(1_{\mathcal{M}_T}) = \overline{t}$.

Since $\mathcal{G}(\mathcal{M}_T \amalg \text{Hom}_{\mathcal{U}}(U, -)) = \mathcal{G}(\mathcal{M}_T) \amalg \mathcal{G}(\text{Hom}_{\mathcal{U}}(U, -))$, we can see $g$ as follows

$$g = \left[ \begin{smallmatrix} g_1 \\ g_2 \end{smallmatrix} \right] : \text{Hom}_{\mathcal{T}}(T, -) \rightarrow \mathcal{G}(\mathcal{M}_T) \amalg \mathcal{G}(\text{Hom}_{\mathcal{U}}(U, -)).$$

It is straightforward to show that $g_1 = \phi(1_{\mathcal{M}_T})$ and $g_2 = 0$. Then

$$g = \left[ \begin{smallmatrix} \phi(1_{\mathcal{M}_T}) \\ 0 \end{smallmatrix} \right] = \left[ \begin{smallmatrix} \phi(1_{\mathcal{M}_T}) \\ \phi(0) \end{smallmatrix} \right] = \phi\left(\left[ \begin{smallmatrix} 1_{\mathcal{T}} \\ 0 \end{smallmatrix} \right] \right). \quad \square$$

In the next proposition we get a description of how the functor $(-)^*$ act on projective objects where $(-)^* : \left(\text{Mod}(\mathcal{T}), \mathcal{G}\text{Mod}(\mathcal{U})\right) \rightarrow \left(\text{Mod}(\mathcal{U}^{op}), \mathcal{G}\text{Mod}(\mathcal{T}^{op})\right)$.

**Proposition 6.4.** Consider the projective objects

$$\text{Hom}_{\mathcal{T}}(T, -) \rightarrow \mathcal{G}(\mathcal{M}_T \amalg \text{Hom}_{\mathcal{U}}(U, -)) \in \left(\text{Mod}(\mathcal{T}), \mathcal{G}\text{Mod}(\mathcal{U})\right)$$

\[
\begin{align*}
\text{Hom}_{\mathcal{T}}(T, -) & \rightarrow \mathcal{G}(\mathcal{M}_T \amalg \text{Hom}_{\mathcal{U}}(U, -)) \\
& \in \left(\text{Mod}(\mathcal{T}), \mathcal{G}\text{Mod}(\mathcal{U})\right).
\end{align*}
\]
\[
\text{Hom}_\mathcal{U}(-, U) \xrightarrow{\overline{g}} \overline{\mathcal{G}}(M_U \sqcup \text{Hom}_\mathcal{T}(-, T)) \in \left( \text{Mod}(\mathcal{O}^{op}), \overline{\mathcal{G}}\text{Mod}(T^{op}) \right)
\]
as in Remark 6.2. Then \( g^* = \overline{g} \), that is:
\[
\left( \text{Hom}_\mathcal{T}(T, -) \xrightarrow{\overline{g}} \mathcal{G}(M_T \sqcup \text{Hom}_\mathcal{U}(U, -)) \right)^* = \text{Hom}_\mathcal{U}(-, U) \xrightarrow{\overline{g}} \overline{\mathcal{G}}(M_U \sqcup \text{Hom}_\mathcal{T}(-, T)).
\]

**Proof** Consider the equivalences given in [22, Theorem 3.17], and the induced by the functor \( \mathcal{T} : \Lambda^{op} \rightarrow \overline{\Lambda} \) given in [22, Proposition 4.4],
\[
\mathcal{F} : \left( \text{Mod}(\mathcal{T}), \mathcal{G}\text{Mod}(\mathcal{U}) \right) \rightarrow \text{Mod}(\Lambda), \quad \mathcal{F} : \left( \text{Mod}(\mathcal{U}^{op}), \overline{\mathcal{G}}\text{Mod}(T^{op}) \right) \rightarrow \text{Mod}(\overline{\Lambda})
\]
and \( \mathcal{T}^* : \text{Mod}(\overline{\Lambda}) \rightarrow \text{Mod}(\Lambda^{op}) \).

Since \( P = \mathcal{F}\left( \text{Hom}_\mathcal{T}(T, -) \xrightarrow{\overline{g}} \mathcal{G}(M_T \sqcup \text{Hom}_\mathcal{U}(U, -)) \right) = \text{Hom}_\Lambda\left( \begin{bmatrix} T & 0 \\ M & U \end{bmatrix}, - \right) : \Lambda \rightarrow \text{Ab} \), then \( P^* := \text{Hom}_\Lambda\left( -, \begin{bmatrix} T & 0 \\ M & U \end{bmatrix} \right) \in \text{Mod}(\Lambda^{op}) \) (see Remark 6.1). We also have that
\[
\mathcal{F}\left( \text{Hom}_\mathcal{U}(-, U) \rightarrow \overline{\mathcal{G}}(M_U \sqcup \text{Hom}_\mathcal{T}(-, T)) \right) := \text{Hom}_{\overline{\Lambda}}\left( \begin{bmatrix} U & 0 \\ M & T \end{bmatrix}, - \right).
\]

It is straightforward to see that \( \mathcal{T}^*\left( \mathcal{F}\left( \text{Hom}_\mathcal{U}(-, U) \rightarrow \overline{\mathcal{G}}(M_U \sqcup \text{Hom}_\mathcal{T}(-, T)) \right) \right) \cong \text{Hom}_\Lambda\left( -, \begin{bmatrix} T & 0 \\ M & U \end{bmatrix} \right) \). We conclude that \( \text{Hom}_{\overline{\Lambda}}\left( \begin{bmatrix} U & 0 \\ M & T \end{bmatrix}, - \right) \circ \mathcal{T} \cong \text{Hom}_\Lambda \left( -, \begin{bmatrix} T & 0 \\ M & U \end{bmatrix} \right) \).

Now that we know how \((-)^*\) act on projective objects, the following proposition tell us how it act on morphism between projective objects in the category \( \left( \text{Mod}(\mathcal{T}), \mathcal{G}\text{Mod}(\mathcal{U}) \right) \).

**Proposition 6.5.** Consider the morphism between projectives in the comma category \( \left( \text{Mod}(\mathcal{T}), \mathcal{G}\text{Mod}(\mathcal{U}) \right) \) given by the diagram
\[
\begin{array}{ccc}
\text{Hom}_\mathcal{T}(T, -) & \xrightarrow{\alpha} & \text{Hom}_\mathcal{T}(T', -) \\
\downarrow f & & \downarrow f' \\
\mathcal{G}(M_T \sqcup \text{Hom}_\mathcal{U}(U, -)) & \xrightarrow{\mathcal{G}(\beta)} & \mathcal{G}(M_{T'} \sqcup \text{Hom}_\mathcal{U}(U', -)).
\end{array}
\]

Then \( \beta = \begin{bmatrix} \rho(a) \\ 0 \end{bmatrix}_{\text{Hom}_\mathcal{U}(U, -)} a_{T2} \) and via the functor \((-)^* : \left( \text{Mod}(\mathcal{T}), \mathcal{G}\text{Mod}(\mathcal{U}) \right) \rightarrow \left( \text{Mod}(\mathcal{O}^{op}), \overline{\mathcal{G}}\text{Mod}(T^{op}) \right) \) the previous morphism corresponds to
\[
\begin{array}{ccc}
\text{Hom}_\mathcal{U}(-, U') & \xrightarrow{\overline{\alpha} = \text{Hom}_\mathcal{U}(-, u)} & \text{Hom}_\mathcal{U}(-, U) \\
\downarrow \overline{\mathcal{T}} & & \downarrow \mathcal{T} \\
\overline{\mathcal{G}}(M_{U'} \sqcup \text{Hom}_\mathcal{T}(-, T')) & \xrightarrow{\overline{\mathcal{G}}\left( \begin{bmatrix} b_{12} \\ 0 \end{bmatrix}_{\text{Hom}_\mathcal{T}(-, T')} \right)} & \overline{\mathcal{G}}(M_U \sqcup \text{Hom}_\mathcal{T}(-, T))
\end{array}
\]
in the category \( \left( \text{Mod}(\mathcal{O}^{op}), \overline{\mathcal{G}}\text{Mod}(T^{op}) \right) \) with \( b_{12} := \Psi^{-1}(\Theta(a_{12})) \) where the morphisms \( \Theta : \text{Hom}_{\text{Mod}(\mathcal{U})}(\text{Hom}_\mathcal{U}(U, -), M_{T'}) \rightarrow M(U, T') \) and \( \Psi : \text{Hom}_{\text{Mod}(T^{op})}(\text{Hom}_\mathcal{T}(-, T'), M_U) \rightarrow M(U, T') \) are the Yoneda isomorphisms.
Proof Consider $\beta : M_T \amalg \text{Hom}_\mathcal{U}(U, -) \longrightarrow M_{T'} \amalg \text{Hom}_\mathcal{U}(U', -)$ and the following commutative diagram

$$
\begin{array}{ccc}
\text{Hom}_\mathcal{T}(T, -) & \xrightarrow{\alpha} & \text{Hom}_\mathcal{T}(T', -) \\
\downarrow f & & \downarrow f' \\
\mathbb{G}(M_T \amalg \text{Hom}_\mathcal{U}(U, -)) & \xrightarrow{\mathbb{G}(\beta)} & \mathbb{G}(M_{T'} \amalg \text{Hom}_\mathcal{U}(U', -))
\end{array}
$$

By adjunction and Proposition 6.3, we have the following commutative diagram

$$
\begin{array}{ccc}
M_T = \mathcal{F}(\text{Hom}_\mathcal{T}(T, -)) & \xrightarrow{\mathcal{F}(\alpha)} & M_{T'} = \mathcal{F}(\text{Hom}_\mathcal{T}(T', -)) \\
\downarrow [1] & & \downarrow [1] \\
M_T \amalg \text{Hom}_\mathcal{U}(U, -) & \xrightarrow{\beta} & M_{T'} \amalg \text{Hom}_\mathcal{U}(U', -).
\end{array}
$$

Since $\beta = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ where $a_{11} : M_T \longrightarrow M_{T'}, a_{12} : \text{Hom}_\mathcal{U}(U, -) \longrightarrow M_{T'}, a_{21} : M_T \longrightarrow \text{Hom}_\mathcal{U}(U', -)$ and $a_{22} : \text{Hom}_\mathcal{U}(U, -) \longrightarrow \text{Hom}_\mathcal{U}(U', -)$ we have that

$$
\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathcal{F}(\alpha) \\ 0 \end{bmatrix}.
$$

Therefore, $a_{11} = \mathcal{F}(\alpha)$ and $a_{21} = 0$. By Yoneda, $a_{12}$ is determined by an element $m \in \text{M}(U, T')$ and $a_{22} := \text{Hom}_\mathcal{U}(u, -) : \text{Hom}_\mathcal{U}(U, -) \longrightarrow \text{Hom}_\mathcal{U}(U', -)$ and $\alpha = \text{Hom}_\mathcal{T}(t, -) : \text{Hom}_\mathcal{T}(T, -) \longrightarrow \text{Hom}_\mathcal{T}(T', -)$ for some $u : U' \longrightarrow U$ and $t : T' \longrightarrow T$.

We define $\bar{\beta} = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{bmatrix} : M_{U'} \amalg \text{Hom}_\mathcal{T}(-, T') \longrightarrow M_U \amalg \text{Hom}_\mathcal{T}(-, T)$, where $\bar{u} : M_{U'} \longrightarrow M_U$, $\text{Hom}_\mathcal{T}(-, t) : \text{Hom}_\mathcal{T}(-, T') \longrightarrow \text{Hom}_\mathcal{T}(-, T)$ and $b_{12} : \text{Hom}_\mathcal{T}(-, T') \longrightarrow M_U$ is defined as $b_{12} := \Psi^{-1}(\Theta(a_{12}))$ where the morphisms $\Theta : \text{Hom}_\text{Mod}(\mathcal{U}) \big(\text{Hom}_\mathcal{U}(U, -), M_{T'}\big) \longrightarrow M(U, T')$ and $\Psi : \text{Hom}_\text{Mod}(\mathcal{T}^{op}) \big(\text{Hom}_\mathcal{T}(-, T'), M_U\big) \longrightarrow M(U, T')$ are the Yoneda isomorphisms.

Since the following diagram commutes

$$
\begin{array}{ccc}
M_{U'} = \mathcal{F}(\text{Hom}_\mathcal{U}(-, U')) & \xrightarrow{\mathcal{F}(\text{Hom}_\mathcal{U}(-, u)) = \bar{u}} & M_U = \mathcal{F}(\text{Hom}_\mathcal{U}(-, U)) \\
\downarrow [1] & & \downarrow [1] \\
M_{U'} \amalg \text{Hom}_\mathcal{T}(-, T') & \xrightarrow{\bar{\beta}} & M_U \amalg \text{Hom}_\mathcal{T}(-, T),
\end{array}
$$

by adjunction we have the following commutative diagram

$$
\begin{array}{ccc}
\text{Hom}_\mathcal{U}(-, U') & \xrightarrow{\bar{\sigma} = \text{Hom}_\mathcal{U}(-, u)} & \text{Hom}_\mathcal{U}(-, U) \\
\downarrow \bar{f} & & \downarrow \bar{f} \\
\mathcal{G}(M_{U'} \amalg \text{Hom}_\mathcal{T}(-, T')) & \xrightarrow{\mathcal{G}(\bar{u}) \begin{bmatrix} b_{12} \\ \text{Hom}_\mathcal{T}(-, t) \end{bmatrix}} & \mathcal{G}(M_U \amalg \text{Hom}_\mathcal{T}(-, T))
\end{array}
$$

Via the functor $\mathcal{G} : \big(\text{Mod}(\mathcal{U}^{op}), \mathcal{G}\text{Mod}(\mathcal{T}^{op})\big) \longrightarrow \text{Mod}(\overline{\mathcal{A}})$ we have the morphism in $\text{Mod}(\overline{\mathcal{A}})$:

$$
\overline{\alpha} \overline{\beta} : \text{Hom}_\mathcal{U}(-, U') \amalg \overline{\mathcal{T}}(M_{U'} \amalg \text{Hom}_\mathcal{T}(-, T')) \longrightarrow \text{Hom}_\mathcal{U}(-, U) \amalg \overline{\mathcal{T}}(M_U \amalg \text{Hom}_\mathcal{T}(-, T)).
$$
where for $\begin{bmatrix} U_1 & 0 \\ M & T_1 \end{bmatrix} \in \Lambda$, we have $[\begin{array}{c} \alpha \\ \beta \end{array}] = \begin{bmatrix} U_1 & 0 \\ M & T_1 \end{bmatrix}$, we have $\alpha U_1 \leq \beta T_1$ with $\beta T_1 = [\begin{array}{c} [\alpha] \\ \alpha \leq [\beta] \\ [\beta] T_1 \end{array}].$ Now, it is straightforward to show that there is an isomorphism

$$\overline{\alpha} \leq \overline{\beta} \simeq \text{Hom}_A \left( \left[ \begin{array}{c} u^{op} \\ 0 \\ v^{op} \end{array} \right], - \right), \text{Hom}_A \left( \left[ \begin{array}{c} U' \\ 0 \\ M' \end{array} \right], - \right) \rightarrow \text{Hom}_A \left( \left[ \begin{array}{c} U \\ 0 \\ M \end{array} \right], - \right),$$

where $\left[ \begin{array}{c} u^{op} \\ 0 \\ v^{op} \end{array} \right] : \left[ \begin{array}{c} U' \\ 0 \\ M' \end{array} \right] \rightarrow \left[ \begin{array}{c} U \\ 0 \\ M \end{array} \right]$ is a morphism in $\Lambda$.

It is easy to show that $(\alpha \leq \beta)^* \simeq (\overline{\alpha} \leq \overline{\beta}) \circ T$, and this proves the proposition.

Given an abelian category $\mathcal{A}$ let us denote by $\text{proj}(\mathcal{A})$ the full subcategory of finitely generated projective objects.

**Proposition 6.6.** Let us denote by $\text{proj}\left( \left( \text{Mod}(\mathcal{T}), \mathbb{G}\text{Mod}(\mathcal{U}) \right) \right)$ the category of finitely generated projective objects in $\left( \text{Mod}(\mathcal{T}), \mathbb{G}\text{Mod}(\mathcal{U}) \right)$.

(a) Then we have a duality

$$(\cdot)^* : \text{proj}\left( \left( \text{Mod}(\mathcal{T}), \mathbb{G}\text{Mod}(\mathcal{U}) \right) \right) \rightarrow \text{proj}\left( \left( \text{Mod}(\mathcal{U}^{op}), \mathbb{G}\text{Mod}(\mathcal{T}^{op}) \right) \right)$$

(b) Suppose that $\mathcal{U}$ and $\mathcal{T}$ are dualizing varieties and $M_T \in \text{mod}(\mathcal{U})$ and $M_U \in \text{mod}(\mathcal{T}^{op})$ for all $U \in \mathcal{U}$ and $T \in \mathcal{T}^{op}$. Then we have a duality

$$(\cdot)^* : \text{proj}\left( \left( \text{mod}(\mathcal{T}), \mathbb{G}\text{mod}(\mathcal{U}) \right) \right) \rightarrow \text{proj}\left( \left( \text{mod}(\mathcal{U}^{op}), \mathbb{G}\text{mod}(\mathcal{T}^{op}) \right) \right)$$

**Proof.**

(a) It is known that the functor $(\cdot)^*$ restricts to a duality $(\cdot)^* : \text{proj}(\text{Mod}(\mathcal{A}))$ (see [6] in page 337). Since $\left( \text{Mod}(\mathcal{T}), \mathbb{G}\text{Mod}(\mathcal{U}) \right) \simeq \text{Mod}(\mathcal{A})$ and $\left( \text{Mod}(\mathcal{U}^{op}), \mathbb{G}\text{Mod}(\mathcal{T}^{op}) \right) \simeq \text{Mod}(\mathcal{A}^{op})$ we have the result.

(b) Since $\text{proj}(\text{Mod}(\mathcal{A})) \subset \text{proj}(\text{Mod}(\mathcal{A}))$, we have that $\text{proj}(\text{mod}(\mathcal{A})) = \text{proj}(\text{Mod}(\mathcal{A}))$. Since $\left( \text{mod}(\mathcal{T}), \mathbb{G}\text{mod}(\mathcal{U}) \right) \simeq \text{mod}(\mathcal{A})$ and $\left( \text{mod}(\mathcal{U}^{op}), \mathbb{G}\text{mod}(\mathcal{T}^{op}) \right) \simeq \text{mod}(\mathcal{A}^{op})$. The result follows from (a).

With help of the previous propositions, the next proposition tell us how $(\cdot)^*$ act on the projective objects of the maps category $\left( \text{Mod}(\mathcal{C}), \text{Mod}(\mathcal{C}) \right)$. In the following we will write $\mathbb{C}(C, -)$ and $\mathbb{C}(-, C)$ instead of $\text{Hom}_\mathbb{C}(C, -)$ and $\text{Hom}_\mathbb{C}(-, C)$.

**Proposition 6.7.** Let $M := \text{Hom} \in \text{Mod}(\mathcal{C} \otimes \mathcal{C}^{op})$ be where $M_T = \text{Hom}_\mathbb{C}(\mathcal{T}, -) \in \text{Mod}(\mathcal{C})$ for $T \in \mathcal{C}^{op}$ and $M_U = \text{Hom}_\mathbb{C}(\mathcal{C}, -) \in \text{Mod}(\mathcal{C}^{op})$ for $U \in \mathcal{C}$. Consider the induced functor $\mathbb{G} : \text{Mod}(\mathcal{C}) \rightarrow \text{Mod}(\mathcal{C})$ and the duality

$$(\cdot)^* : \text{proj}\left( \left( \text{Mod}(\mathcal{C}), \mathbb{G}\text{Mod}(\mathcal{C}) \right) \right) \rightarrow \text{proj}\left( \left( \text{Mod}(\mathcal{C}^{op}), \mathbb{G}\text{Mod}(\mathcal{C}^{op}) \right) \right)$$

given in Proposition 6.6 and the isomorphisms $J_1$ and $J_2$ given in Proposition 5.2. Then we get a duality, $(\cdot)^* : \text{proj}(\text{mod}(\mathcal{C}), \text{mod}(\mathcal{C})) \rightarrow \text{proj}(\text{mod}(\mathcal{C}^{op}), \text{mod}(\mathcal{C}^{op}))$
defined as $J_2^{-1} \circ (-)^* \circ J_1$, which will be denoted as $(-)^*$. Moreover, maps of the form $C(C_1, -) \to \bigoplus_{C(C_2, -)}$ are projective in $\left(\text{mod}(C), \text{mod}(C)\right)$ and

\[
\left( C(C_1, -) \to \bigoplus_{C(C_2, -)} \right)^* = C(-, C_2) \to \bigoplus_{C(-, C_1)}.
\]

**Proof** Let us consider $C(C_1, -) \to \bigoplus_{C(C_2, -)}$ and the Yoneda isomorphism

\[ Y_{C'} : C(C_1, C') \to \bigoplus_{C(C_2, C')} \to \mathcal{G}\left( C(C_1, -) \bigoplus_{C(C_2, -)} \right)(C'). \]

We assert that the morphism

\[ J_1(f) := \hat{f} : C(C_1, -) \to \mathcal{G}\left( C(C_1, -) \bigoplus_{C(C_2, -)} \right) \]

is a projective in the category $\left(\text{Mod}(C), \text{G}(\text{Mod}(C))\right)$. Indeed, using that $M := \text{Hom} \in \text{Mod}(C \otimes C^{op})$ where $M_T = \text{Hom}_C(T, -) \in \text{Mod}(C)$ for $T \in C^{op}$ and $M_U = \text{Hom}_C(-, U) \in \text{Mod}(C^{op})$ for $U \in C$ and the description given in Remark 6.2, we have that the projective $g$ given in Remark 6.2, coincides with $\hat{f}$. Then, we have the commutative diagram

\[
\begin{array}{ccc}
C(C_1, -) & \xrightarrow{\text{[1 \ 0]}} & \bigoplus_{C(C_2, -)} \\
\| & & \| \\
C(C_1, -) & \xrightarrow{\mathcal{G}\left( C(C_1, -) \bigoplus_{C(C_2, -)} \right)} & \mathcal{G}\left( C(C_1, -) \bigoplus_{C(C_2, -)} \right).
\end{array}
\]

Therefore we have that $J_3^{-1}(\mathcal{G}) = C(-, C_2) \to \mathcal{G}(M_C \bigoplus_{C(C_2, -)}) \to \mathcal{G}(M_C \bigoplus_{C(C_2, -)}(C_1))$. Therefore, by Proposition 6.4 we have the following equalities

\[
\left( C(C_1, -) \xrightarrow{\text{[0 \ 1]}} \bigoplus_{C(C_2, -)} \right)^* = C(-, C_2) \to \mathcal{G}(M_C \bigoplus_{C(C_2, -)}) \to \mathcal{G}(M_C \bigoplus_{C(C_2, -)}(C_1))
\]

\[
= C(-, C_2) \xrightarrow{\text{[0 \ 1]}} \bigoplus_{C(-, C_1)}.
\]

Now that we know how $(-)^*$ acts on projective objects in the maps category, the next proposition tells us how it act on morphism between projectives in the maps category $\left(\text{Mod}(C), \text{Mod}(C)\right)$. This is important because it is an ingredient that we need in order to define the Auslander-Reiten translate in the category $\left(\text{mod}(C), \text{mod}(C)\right)$. 

\[ \square \]

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Proposition 6.8. Let $M := \overline{\text{Hom}} \in \text{Mod}(\mathcal{C} \otimes \mathcal{C}^{op})$ be where $M_T = \text{Hom}_{\mathcal{C}}(T, -) \in \text{Mod}(\mathcal{C})$ for $T \in \mathcal{C}^{op}$ and $M_U = \text{Hom}_{\mathcal{C}}(-, U) \in \text{Mod}(\mathcal{C}^{op})$ for $U \in \mathcal{C}$. Consider the induced functor $\mathcal{G} : \text{Mod}(\mathcal{C}) \to \text{Mod}(\mathcal{C})$ and a map between projectives in the category $\left(\text{Mod}(\mathcal{C}), \text{Mod}(\mathcal{C})\right)$

\[
\begin{array}{ccc}
C(C_1, -) & \xrightarrow{a_{11}} & C(C'_1, -) \\
\downarrow[\delta] & & \downarrow[\delta'] \\
C(C_1, -) \amalg C(C_2, -) & \xrightarrow{\beta} & C(C'_1, -) \amalg C(C'_2, -)
\end{array}
\]

where $\beta = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$. Then, applying $(-)^*$ we get the following map in the category $\left(\text{Mod}(\mathcal{C}^{op}), \text{Mod}(\mathcal{C}^{op})\right)$

\[
\begin{array}{ccc}
C(-, C'_2) & \xrightarrow{a_{22}^*} & C(-, C_2) \\
\downarrow[\delta] & & \downarrow[\delta'] \\
C(-, C'_2) \amalg C(-, C'_1) & \xrightarrow{\bar{\beta}} & C(-, C_2) \amalg C(-, C_1)
\end{array}
\]

where $\bar{\beta} = \begin{bmatrix} a_{22} & a_{22}^* \\ 0 & a_{11}^* \end{bmatrix}$.

Proof By Propositions 5.2 and 6.7, we have that the given diagram corresponds in the category $\left(\text{Mod}(\mathcal{C}), \mathcal{G}\text{Mod}(\mathcal{C})\right)$ to

\[
\begin{array}{ccc}
\text{Hom}_{\mathcal{C}}(C_1, -) & \xrightarrow{a_{11}=\text{Hom}_{\mathcal{C}}(t, -)} & \text{Hom}_{\mathcal{C}}(C'_1, -) \\
\downarrow[f] & & \downarrow[f'] \\
\mathcal{G}(M_{C_1} \amalg \text{Hom}_{\mathcal{C}}(C_2, -)) & \xrightarrow{\mathcal{G}(\beta)} & \mathcal{G}(M_{C'_1} \amalg \text{Hom}_{\mathcal{C}}(C'_2, -)).
\end{array}
\]

Then $\bar{\beta} = \begin{bmatrix} \Psi(a_{11}) & a_{12} \\ 0 & \text{Hom}_{\mathcal{C}}(a, -) \end{bmatrix}$ and it can be seen that $F(a_{11}) = a_{11}$. By Proposition 6.5, via the functor $(-)^* : \left(\text{Mod}(\mathcal{C}), \mathcal{G}\text{Mod}(\mathcal{C})\right) \to \left(\text{Mod}(\mathcal{C}^{op}), \mathcal{G}\text{Mod}(\mathcal{C}^{op})\right)$ the previous diagram corresponds to

\[
\begin{array}{ccc}
\text{Hom}_{\mathcal{C}}(-, C'_2) & \xrightarrow{\text{Hom}_{\mathcal{C}}(-, u)} & \text{Hom}_{\mathcal{C}}(-, C_2) \\
\downarrow[\bar{f}] & & \downarrow[\bar{f}'] \\
\mathcal{G}(M_{C'_2} \amalg \text{Hom}_{\mathcal{C}}(-, C'_1)) & \xrightarrow{\mathcal{G}(\bar{\beta} \begin{bmatrix} b_{12} \\ 0 \end{bmatrix} \text{Hom}_{\mathcal{C}}(-, t)))} & \mathcal{G}(M_{C_2} \amalg \text{Hom}_{\mathcal{C}}(-, C_1)).
\end{array}
\]

But in this case, as above, we get that $\bar{u} : M_{C'_2} \to M_{C_2}$ coincides with $\text{Hom}_{\mathcal{C}}(-, u)$. Now, by Proposition 6.5, we obtain the equality $b_{12} := \Psi^{-1}(\Theta(a_{12}))$ where the morphisms $\Theta : \text{Hom}_{\text{Mod}(\mathcal{C})}(\text{Hom}_{\mathcal{C}}(C_2, -), M_{C'_1}) \to M_{C'_1}(C_2) = \mathcal{C}(C'_1, C_2)$ and $\Psi : \text{Hom}_{\text{Mod}(\mathcal{C}^{op})}(\text{Hom}_{\mathcal{C}}(-, C'_1), M_{C_2}) = \text{Hom}_{\text{Mod}(\mathcal{C}^{op})}(\text{Hom}_{\mathcal{C}}(-, C'_1), \mathcal{C}(-, C_2)) \to \mathcal{C}(C'_1, C_2)$ are the Yoneda isomorphisms. Then we conclude that $b_{12} = a_{12}^*$. 

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Then, composing the last diagram with the inverse of the Yoneda isomorphisms $Y_1 : M_{C_1} \cong \text{Hom}_C(\mathfrak{C}, C_1') \rightarrow (\mathcal{M}_{C_1} \cong \text{Hom}_C(\mathfrak{C}, C_1))$, and $Y_2 : M_{C_2} \cong \text{Hom}_C(\mathfrak{C}, C_1) \rightarrow (\mathcal{M}_{C_2} \cong \text{Hom}_C(\mathfrak{C}, C_1))$ we get the required diagram.

The following proposition tell us how to compute projective resolutions in the category of maps of an abelian category with projective covers $C$.

**Proposition 6.9.** Let $C$ be an abelian category with projective covers and let $f : A \rightarrow B$ be a morphism in $C$.

(i) Assume that $\text{Coker}(f) \neq 0$. Construct the following diagram

\[
\begin{array}{c}
P_0 \xrightarrow{[1]} P_0 \oplus Q_0 \xrightarrow{[0,1]} Q_0 \rightarrow 0 \\
\downarrow \alpha \downarrow \gamma \downarrow \beta' \downarrow \beta \\
A \xrightarrow{f} B \rightarrow C \rightarrow 0
\end{array}
\]

where $\alpha : P_0 \rightarrow A$ and $\beta : Q_0 \rightarrow C$ are projective covers $\beta' : Q_0 \rightarrow B$ is the induced morphism by the projectivity of $Q_0$ and $\gamma = (f, \beta')$. Then the morphism $(\alpha, \gamma) : (P_0, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, P_0 \oplus Q_0) \rightarrow (A, f, B)$ given by the following diagram

\[
\begin{array}{c}
P_0 \\
\downarrow (\alpha, \gamma)
\end{array} \xrightarrow{P_0} \begin{array}{c}
A \\
\downarrow B
\end{array} \rightarrow 0
\]

is a projective cover of the object $f : A \rightarrow B$ in the category $\text{maps}(C)$.

(ii) Assume that $\text{Coker}(f) = 0$. Consider $\alpha : P_0 \rightarrow A$ the projective cover of $A$. Then, the morphism $(\alpha, f) : (P_0, 1, P_0) \rightarrow (A, f, B)$ is a projective cover of $(A, f, B)$.

**Proof** Let us see that $(\alpha, \gamma)$ is minimal. Indeed, let

\[
(\ast) : (\theta_1, \theta_2) : \left( P_0, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \rightarrow \left( P_0, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)
\]

such that $(\alpha, \gamma)(\theta_1, \theta_2) = (\alpha, \gamma)$. Then $\theta_1 = \alpha$ and $\gamma\theta_2 = \gamma$. Since $\alpha$ is projective cover, we have that it is minimal and hence $\theta_1$ is an isomorphism. Now, since $(\ast)$ is a morphism in the category of maps, we have the following commutative diagram

\[
\begin{array}{c}
P_0 \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} P_0 \oplus Q_0 \\
\downarrow \theta_1 \downarrow \theta_2 \\
P_0 \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} P_0 \oplus Q_0
\end{array}
\]

If $\theta_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ we have that $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \theta_1 \\ 0 \end{bmatrix}$. Therefore, we conclude that $a_{11} = \theta_1$ and $a_{21} = 0$. Now, $(0, \beta)\theta_2 = \pi \gamma\theta_2 = \pi \gamma = (0, B)$ and then $(0, \beta) \begin{bmatrix} \theta_1 \\ 0 \end{bmatrix} = (0, \beta a_{22}) = (0, \beta)$. Hence, $\beta = \beta a_{22}$ and since $\beta$ is minimal, we have that $a_{22}$ is an isomorphism. Since $a_{12} : Q_0 \rightarrow P_0$, we have the morphism $a_{12}^{-1} a_{12}a_{22}^{-1} : Q_0 \rightarrow P_0$. Now it is easy to show that $\theta_2^{-1} = \begin{bmatrix} a_{11}^{-1} & a_{11}^{-1}a_{12}a_{22}^{-1} \\ 0 & a_{22}^{-1} \end{bmatrix}$. Hence $(\theta_1, \theta_2)$ is an isomorphism, proving that $(\alpha, \gamma)$ is
minimal. Now, it is easy to show that \( \left( P_0, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, P_0 \oplus Q_0 \right) \) is a projective object in maps(\( C \)), then we conclude that \((\alpha, \gamma)\) is a projective cover.

(ii) Similar to (i).

We define the transpose which is needed to get the Auslander-Reiten translate.

**Definition 6.10.** (Transpose) Let \( f : A \longrightarrow G(B) \) an object in \( \left( \text{mod}(T), \mathbb{C}\text{mod}(U) \right) \). Consider a minimal projective presentation

\[
\begin{array}{ccc}
\text{Hom}_T(T, -) & \xrightarrow{(\alpha, \beta)} & \text{Hom}_T(T', -) \\
G(M_T \oplus \text{Hom}_U(U, -)) & \downarrow & G(M_{T'} \oplus \text{Hom}_U(U', -)) \\
A & \downarrow & 0 \\
G(B) & \downarrow & \\
\end{array}
\]

By applying the functor \((-)^*\) given in Proposition 6.6, we define \( \overline{\text{TR}}(A, f, B) := \text{Coker}(\alpha, \beta)^* \).

Almost split sequences for dualizing varieties were studied by M. Auslander and Idun Reiten in \( \text{mod}(C) \) for a \( R \)-dualizing variety. The crucial ingredient is the explicit construction of the Auslander-Reiten translate \( \tau \) by taking the dual of the transpose \( D\text{Tr}M \) of a finitely presented \( C \)-module \( M \).

Let \( K \) be a field. In the case \( U \) and \( T \) are dualizing \( K \)-varieties and \( M \in \text{Mod}(U \otimes T^{op}) \) satisfies that \( M_T \in \text{mod}(U) \) and \( M_U \in \text{mod}(T^{op}) \) for all \( T \in T, U \in U \), we have that \( \left( \text{mod}(T), \mathbb{C}\text{mod}(U) \right) \simeq \text{mod}(\Lambda) \) where \( \Lambda \) is a dualizing \( K \)-variety (see [22, Theorem 6.10]). So, in order to define the Auslander-Reiten translate in the comma category \( \left( \text{mod}(T), \mathbb{C}\text{mod}(U) \right) \) we use the description of the transpose and the dual in the category \( \left( \text{mod}(T), \mathbb{C}\text{mod}(U) \right) \). Now, we define the Auslander-Reiten translate.

**Definition 6.11.** (Auslander-Reiten translate) Suppose that \( U \) and \( T \) are dualizing \( K \)-varieties, \( M \in \text{Mod}(U \otimes T^{op}) \) satisfies that \( M_T \in \text{mod}(U) \) and \( M_U \in \text{mod}(T^{op}) \) for all \( T \in T, U \in U^{op} \). For an object \( f : A \longrightarrow G(B) \) in \( \left( \text{mod}(T), \mathbb{C}\text{mod}(U) \right) \) we construct the exact sequence as in Definition 6.10

\[
\begin{array}{ccc}
Q^* & \xrightarrow{(\alpha, \beta)^*} & P^* \\
& \longrightarrow & \text{Coker}(\alpha, \beta)^* \\
& \longrightarrow & 0
\end{array}
\]

Considering the duality \( \hat{\Theta} : \left( \text{mod}(U^{op}), \mathbb{C}\text{mod}(T^{op}) \right) \longrightarrow \left( \text{mod}(T), \mathbb{C}\text{mod}(U) \right) \) given in [22, Theorem 6.10], we define the Auslander-Reiten translate \( \text{Tau} : \left( \text{mod}(T), \mathbb{C}\text{mod}(U) \right) \longrightarrow \left( \text{mod}(T), \mathbb{C}\text{mod}(U) \right) \) as

\[
\text{Tau}(A, f, B) := \hat{\Theta} \left( \text{Coker}(\alpha, \beta)^* \right).
\]

By Proposition 5.4(ii), we get that maps(\( \text{mod}(C) \)) \( \simeq \text{mod}(\Lambda) \) where \( \Lambda = \begin{bmatrix} C & 0 \\
\text{Hom}_C & C \end{bmatrix} \) is a dualizing \( K \)-variety. Then, we are able to describe that Auslander-Reiten translate in the category maps(\( \text{mod}(C) \)).

Let \( C \) be a dualizing \( K \)-variety. It is well known that under this conditions \( C \) is Krull-Schmidt (see page 318 in [5]) and therefore we have the every \( C \)-module in \( \text{mod}(C) \) has a
minimal projective presentation (see for example [26, Lemma 2.1]). Then we can define the transpose in \( \text{mod}(C) \) which we will denote by \( \text{Tr} \). That is, \( \text{Tr} : \text{mod}(C) \to \text{mod}(C^{op}) \).

**Theorem 6.12.** Let \( C \) be a dualizing \( K \)-variety and consider the equivalence \( \text{mod}(\Lambda) \sim \text{maps}(\text{mod}(C)) \) given in Proposition 5.4(ii) and the duality \( \mathbb{D}_{C^{op}} : \text{mod}(C^{op}) \to \text{mod}(C) \). Let \( f : C_1 \to C_2 \) be a morphism in \( \text{mod}(C) \) such that there exists exact sequence

\[
C_1 \to C_2 \to C_3 \to 0 \text{ with } C_3 \neq 0 \text{ and } C_3 \text{ not projective. Then}
\]

\[
\text{Tau}(C_1, f, C_2) = (\mathbb{D}_{C^{op}}(Y), \mathbb{D}_{C^{op}}(g), \mathbb{D}_{C^{op}}\text{Tr}(C_3))
\]

for some morphism \( g : \text{Tr}(C_3) \to Y \) such that there exists an exact sequence

\[
0 \to \mathbb{D}_{C^{op}}\text{Tr}(C_1) \to \mathbb{D}_{C^{op}}(Y) \xrightarrow{\mathbb{D}_{C^{op}}(g)} \mathbb{D}_{C^{op}}\text{Tr}(C_3)
\]

**Proof** Since \( C \) is an additive category and with splitting idempotent, we have that every finitely presented projective \( C \)-module \( P \) is of the form \( \text{Hom}_C(C, -) \) for some object \( C \in C \) (see [5]). Then in all what follows whenever we write a projective \( C \)-module \( P \), we mean a projective module of the form \( \text{Hom}_C(C, -) \) for some object \( C \in C \).

Let \( f : C_1 \to C_2 \) a morphism in \( \text{mod}(C) \) and \( C_3 = \text{Coker}(f) \), following Proposition 6.9, we construct a minimal projective presentation of \( (C_1, f, C_2) \)

\[
\begin{array}{ccccccccc}
P_1 & \xrightarrow{\lambda_1} & P_0 & \xrightarrow{\lambda_0} & C_1 & \to & 0 \\
& & \downarrow{\begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}} & & \downarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} & & \downarrow{f} \\
P_1 \oplus Q_1 & \to & P_0 \oplus Q_0 \xrightarrow{\gamma_0} & C_2 & \to & 0,
\end{array}
\]

where

\[
P_1 \xrightarrow{\lambda_1} P_0 \xrightarrow{\lambda_0} C_1 \to 0, \quad Q_1 \xrightarrow{b} Q_0 \xrightarrow{c} C_3 \to 0
\]

are minimal projective presentation of \( C_1 \) and \( C_3 \) respectively. Applying \((-)^*\) in the category maps(\( \text{mod}(C) \)) we get

\[
(P_0 \to P_0 \oplus Q_0)^* \to (P_1 \to P_1 \oplus Q_1)^* \to \text{TR}(C_1, f, C_2) \to 0
\]

By Proposition 6.8, the last exact sequence is represented by the following diagram

\[
\begin{array}{ccccccccc}
Q_0^* & \xrightarrow{b^*} & Q_1^* & \to & \text{Tr}(C_3) & \to & 0 \\
& & \downarrow{\begin{bmatrix} 1 & a^* \\ 0 & b^* \end{bmatrix}} & & \downarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} & & \downarrow{g} \\
Q_0^* \oplus P_0^* & \to & Q_1^* \oplus P_1^* \to & Y & \to & 0.
\end{array}
\]
where by definition $\overline{TR}(C_1, f, C_2) = (\text{Tr}(C_3), g, Y)$ with $\text{Tr}$ the transpose in mod($C$). Then we can complete to the diagram

$$
\begin{array}{cccc}
Q_0^* & \xrightarrow{b^*} & Q_1^* & \xrightarrow{\text{Tr}} \text{Tr}(C_3) \\
\begin{bmatrix} 1 \\ 0 \end{bmatrix} & & \begin{bmatrix} b^* \\ a^* \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
Q_0^* \oplus P_0^* & \xrightarrow{\lambda_i^*} & Q_1^* \oplus P_1^* & \xrightarrow{} Y \\
\begin{bmatrix} 0 & 1 \end{bmatrix} & & \begin{bmatrix} 0 & 1 \end{bmatrix} & \begin{bmatrix} \delta \end{bmatrix} \\
P_0^* & \xrightarrow{\lambda_i^*} & P_1^* & \xrightarrow{\text{Tr}} \text{Tr}(C_1) \\
0 & & 0 & 0
\end{array}
$$

Applying the duality $\hat{\Theta}' : \left( \text{mod}(C^{op}), \overline{\text{C}}(\text{mod}(C^{op})) \right) \rightarrow \left( \text{mod}(C), \overline{\text{C}}(\text{mod}(C)) \right)$ (see [22, Theorem 6.10] and Proposition 5.3) to the previous diagram we get that the Auslander-Reiten translation of $(C_1, f, C_2)$ is the map $(\overline{\text{D}}_{C^{op}}(Y), \overline{\text{D}}_{C^{op}}(g), \overline{\text{D}}_{C^{op}} \text{Tr}(C_3))$.

7 Almost Split Sequences in the Maps Category

Dualizing $K$-varieties were introduced by Auslander and Reiten as a generalization of artin $K$-algebras (see [6]). It is well-known that the existence of almost split sequences is quite useful in the representation theory of artin algebras. A $K$-variety $A$ being dualizing ensures that the category mod($A$) of finitely presented functors in Mod($A$) has almost split sequences (see Theorem 7.1.3 in [33]). From a given dualizing $K$-variety $A$ there are some known constructions of dualizing $K$-varieties such as mod($A$), the functorially finite Krull-Schmidt categories of $A$, residue categories $A/1_A$ of $A$ module the ideal $(1_A)$ of $A$ generated by the identity morphism $1_A$ of an object $A \in A$ and the category $C^b(\text{mod}(A))$ of bounded complexes over mod($A$) (see [9]).

In all this section $K$ will be an arbitrary field. Let $C$ be a dualizing $K$-variety, and $A = \left[ \begin{array}{c} C \\ \text{Hom} \end{array} \right]$. Now, we consider almost split sequences in mod($A$) that arise from almost split sequences in mod($C$). That is, we consider almost split sequences in mod($A$) $\simeq$ maps(mod($C$)) of the form

$$
0 \rightarrow (N_1, g, N_2) \overset{(j_2, j_1)}{\rightarrow} (E_1, h, E_2) \overset{(\pi_1, \pi_2)}{\rightarrow} (M_1, f, M_2) \rightarrow 0,
$$

such that $(M_1, f, M_2)$ is one of the following cases $(M, 1_M, M), (M, 0, 0), (0, 0, M)$, with $M$ a non projective indecomposable $C$-module, and $(N_1, g, N_2)$ is one of the following cases $(N, 1_N, N), (N, 0, 0), (0, 0, N)$, with $N$ a non injective indecomposable $C$-module. The following which is a generalization of [25, Theorem 3.1(a), Theorem 3.2 (a)].

**Proposition 7.1.** Let $C$ be a dualizing $K$-variety.

1. Let $0 \rightarrow \tau M \overset{\iota}{\rightarrow} E \overset{\pi}{\rightarrow} M \rightarrow 0$ be an almost split sequence of $C$-modules. Then the exact sequences in maps(mod($C$)):
(i) \[ 0 \to (\tau M, 0, 0) \xrightarrow{(j, 0)} (E, \pi, M) \xrightarrow{(\pi, 1_M)} (M, 1, M, M) \to 0, \]

(ii) \[ 0 \to (\tau M, 1, \tau M, \tau M) \xrightarrow{\left(1, \tau, \tau, j\right)} (\tau M, j, E) \xrightarrow{(0, \pi)} (0, 0, M) \to 0, \]

are almost split.

(2) Let \( 0 \to N \xrightarrow{j} E \xrightarrow{\pi} \tau^{-1} N \to 0 \) an almost split sequence of \( C \)-modules. Then the exact sequences in maps\((\text{mod}(C)):\)

\[
\begin{align*}
(i) & \quad 0 \to (N, 1, N) \xrightarrow{(1, j)} (N, j, E) \xrightarrow{(0, \pi)} (0, 0, \tau^{-1} N) \to 0 \\
(ii) & \quad 0 \to (N, 0, 0) \xrightarrow{(j, 0)} (E, \pi, \tau^{-1} N) \xrightarrow{(\pi, 1, \tau^{-1} N)} (\tau^{-1} N, 1, \tau^{-1} N, \tau^{-1} N) \to 0
\end{align*}
\]

are almost split.

**Proof** (1) (i) Since \( \pi : E \to M \) does not splits, the map \( (\pi, 1_M) : (E, \pi, M) \to (M, 1_M, M) \) does not split. Let \((q_1, q_2) : (X_1, f, X_2) \to (M, 1_M, M)\) be a map that is not a splittable epimorphism. Then \( q_2 f = 1_M q_1 = q_1 \).

We claim that \( q_1 \) is not a splittable epimorphism. Indeed, if \( q_1 \) is a splittable epimorphism, then there exists a morphism \( s : M \to X_1 \), such that \( q_1 s = 1_M \). Thus, we have a morphism \((s, f s) : (M, 1_M, M) \to (X_1, f, X_2)\) and we get that \((q_1, q_2) \circ (s, f s) = (1_M, 1_M) = (M, 1_M, M) : (M, 1_M, M) \to (M, 1_M, M)\) and hence \((q_1, q_2) : (X_1, X_2, f) \to (M, 1_M, M)\) is a splittable epimorphism which is a contradiction.

Since \( \pi : E \to M \) is a right almost split morphism, there exists a map \( h : X_1 \to E \) such that \( \pi h = q_1 \), and \( q_2 f = q_1 = \pi h \). Thus, we have a morphism \((h, q_2) : (X_1, f, X_2) \to (E, \pi, M)\), and the following commutative diagram

\[
\begin{tikzcd}
(X_1, f, X_2) \arrow{r}{(h, q_2)} \arrow{d}{(q_1, q_2)} & (E, \pi, M) \arrow{r}{(\pi, 1_M)} & (M, 1_M, M)
\end{tikzcd}
\]

That is, we get a lifting \((h, q_2) : (X_1, f, X_2) \to (E, \pi, M)\) of \((q_1, q_2)\) and we have proved that \((\pi, 1_M)\) is right almost split and thus \( \tau(M, 1_M, M) = (\tau M, 0, 0) \).

(ii). Similar to (i).

**Proposition 7.2.** Given a minimal projective presentation \( P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0 \) of \( M \) in \( \text{mod}(C) \), we have that the following diagram

\[
\begin{tikzcd}
P_1 \arrow{r}{d_1} \arrow{d}{[1, 0]} & P_0 \arrow{r}{d_0} \arrow{d}{1} & M \arrow{r}{0} & 0 \\
P_1 \amalg P_0 \arrow{r}{[d_1, 1]} & P_0 \arrow{r}{0} & 0 \arrow{r}{0}
\end{tikzcd}
\]

is a minimal projective presentation of \( M \to 0 \) in the category \( \text{maps}(\text{mod}(C)) \).

**Proof** Is easy to see that \( 1 : P_0 \to P_0 \) is a minimal projective cover of \( M \to 0 \). Making the construction of Proposition 6.9, we get the required diagram.
Proposition 7.3. Let $C$ be a dualizing $R$-variety for some commutative artin ring $R$. Let $C$ be an indecomposable non projective object and Let

\[ \eta : 0 \rightarrow DTrC \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0 \]

a non split exact sequence such that every non isomorphism $C \rightarrow C$ factors through $g$. Then $\eta$ is an almost split sequence.

Proof The proof of [8, 2.1] in page 147, can be adapted for this setting. \qed

Proposition 7.4. Let $C$ be a dualizing $K$-variety and $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ an almost split sequence in $\text{mod}(C)$. Then $A \cong DTr(C)$.

Proof See [33, Proposition 7.1.4] in page 90. \qed

We know that $\Lambda = \begin{bmatrix} C & 0 \\ \text{Hom}_C & C \end{bmatrix}$ is a dualizing $K$-variety (see [22, Theorem 6.10]) if $C$ is dualizing; and therefore by Proposition 7.4, we have that the first term of an almost split sequence in $\text{maps}(\text{mod}(C)) \simeq \text{mod}(\Lambda)$ (see Proposition 5.4) is determined by the ending term.

The following which is a generalization of [25, Theorem 3.1(b), Theorem 3.2 (b)].

Proposition 7.5. (i) Let $0 \rightarrow \tau M \xrightarrow{j} E \xrightarrow{\pi} M \rightarrow 0$ be an almost split sequence in $\text{mod}(C)$. Given a minimal projective resolution $P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \rightarrow 0$, we obtain a commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \tau (M) & \xrightarrow{j} & E & \xrightarrow{\pi} & M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \tau (M) & \xrightarrow{\mu \in \text{D}_{\text{C}}(q)} & \text{D}_{\text{C}}(P_1^*) & \xrightarrow{\text{D}_{\text{C}}(d_1^*)} & \text{D}_{\text{C}}(P_0^*) & \rightarrow & 0
\end{array}
\]

Then the exact sequence

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \text{D}_{\text{C}}(P_1^*) & \rightarrow & \text{D}_{\text{C}}(P_1^*) \sqcup M & \rightarrow & M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \text{D}_{\text{C}}(P_0^*) & \rightarrow & \text{D}_{\text{C}}(P_0^*) & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]

is an almost split sequence in $\text{maps}(\text{mod}(C))$.

(ii) Let $0 \rightarrow N \xrightarrow{j} E \xrightarrow{\pi} \tau^{-1}N \rightarrow 0$ an almost split sequence in $\text{mod}(C)$. Given a minimal injective resolution $0 \rightarrow N \xrightarrow{q_0} I_0 \xrightarrow{q_1} I_1$, we obtain a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & N & \xrightarrow{j} & E & \xrightarrow{\pi} & \tau^{-1}(N) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
(\text{D}_{\text{C}}(I_0))^* & \rightarrow & (\text{D}_{\text{C}}(q_1))^* & \rightarrow & (\text{D}_{\text{C}}(I_0))^* & \rightarrow & \tau^{-1}(N) & \rightarrow & 0
\end{array}
\]
Then the exact sequence

\[
\begin{array}{ccccccccc}
0 & \rightarrow & 0 & \rightarrow & (\mathbb{D}_C(I_0))^* & \rightarrow & (\mathbb{D}_C(I_0))^* & \rightarrow & 0 \\
& \downarrow & & & \uparrow 1 & & \downarrow & & \\
0 & \rightarrow & N & \rightarrow & (\mathbb{D}_C(I_1))^* \oplus N & \rightarrow & (\mathbb{D}_C(I_1))^* & \rightarrow & 0 \\
& \downarrow & \begin{bmatrix} 0 \\ 1 \end{bmatrix} & & & \begin{bmatrix} 0 \\ 1 \end{bmatrix} & & & \\
& & (\mathbb{D}_C(q_1))^* & & (\mathbb{D}_C(q_1))^* & & & \\
\end{array}
\]

is an almost split sequence in maps(mod(C)).

**Proof** (i). Let \( 0 \rightarrow \tau M \xrightarrow{j} E \xrightarrow{\pi} M \rightarrow 0 \) be an almost split sequence of \( C \)-modules in mod(\( C \)) and consider a minimal projective presentation \( P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \rightarrow 0 \) for \( M \). Then we get \( P_0^* \xrightarrow{d_1^*} P_1^* \xrightarrow{q} \text{Tr}(M) \rightarrow 0 \) and applying \( \mathbb{D}_{C^{\text{op}}} \) we get

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathbb{D}_{C^{\text{op}}}(M) & \xrightarrow{u=\mathbb{D}_{C^{\text{op}}}(q)} & \mathbb{D}_{C^{\text{op}}}(P_1^*) & \xrightarrow{\mathbb{D}_{C^{\text{op}}}(d_1^*)} & \mathbb{D}_{C^{\text{op}}}(P_0^*) \\
& & \mathbb{D}_{C^{\text{op}}}(P_1^*) & & \mathbb{D}_{C^{\text{op}}}(P_0^*) & & \\
\end{array}
\]

Since \( \mathbb{D}_{C^{\text{op}}}(P_1^*) \) is injective there exists a map \( f : E \rightarrow \mathbb{D}_{C^{\text{op}}}(P_1^*) \) such that \( fj = u \). Then we have the following commutative diagram

\[
(\ast) : \quad \begin{array}{cccccc}
0 & \rightarrow & \tau(M) & \xrightarrow{j} & E & \xrightarrow{\pi} & M & \rightarrow & 0 \\
& \downarrow & & \downarrow f & & \downarrow h & & \\
0 & \rightarrow & \tau(M) & \xrightarrow{u=\mathbb{D}_{C^{\text{op}}}(q)} & \mathbb{D}_{C^{\text{op}}}(P_1^*) & \xrightarrow{\mathbb{D}_{C^{\text{op}}}(d_1^*)} & \mathbb{D}_{C^{\text{op}}}(P_0^*) \\
& & \mathbb{D}_{C^{\text{op}}}(P_1^*) & & \mathbb{D}_{C^{\text{op}}}(P_0^*) & & \\
\end{array}
\]

We note that \( h \neq 0 \) since the upper exact sequence does not split. Next, we will show that the following diagram defines an almost split sequence

\[
(\ast\ast) : \quad \begin{array}{cccccc}
0 & \rightarrow & \mathbb{D}_{C^{\text{op}}}(P_1^*) & \rightarrow & \mathbb{D}_{C^{\text{op}}}(P_1^*) \oplus M & \rightarrow & M & \rightarrow & 0 \\
& \downarrow & \mathbb{D}_{C^{\text{op}}}(d_1^*) & & \mathbb{D}_{C^{\text{op}}}(d_1^*),h & & & & \\
0 & \rightarrow & \mathbb{D}_{C^{\text{op}}}(P_0^*) & \rightarrow & \mathbb{D}_{C^{\text{op}}}(P_0^*) \oplus M & \rightarrow & M & \rightarrow & 0 \\
& & \mathbb{D}_{C^{\text{op}}}(P_0^*) & & \mathbb{D}_{C^{\text{op}}}(P_0^*) & & & & \\
\end{array}
\]

Indeed, since \( h \neq 0 \) we have that exact sequence in the category maps(mod(C)) does not split. By Proposition 7.2, we have that the following diagram

\[
\begin{array}{cccccc}
P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & M & \rightarrow & 0 \\
& \downarrow & \begin{bmatrix} 0 \\ 1 \end{bmatrix} & & \downarrow 1 & & \\
P_1 \oplus P_0 & \xrightarrow{[d_1,1]} & P_0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

is a minimal projective presentation of \( M \rightarrow 0 \).

By Proposition 6.8, applying \((-)^*\) we have the diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & P_0^* & \rightarrow & P_0^* \oplus P_1^* & & \\
& \downarrow & \begin{bmatrix} 1 \\ 0 \end{bmatrix} & & \begin{bmatrix} 1 \\ 0 \end{bmatrix} & & \\
P_0^* & \rightarrow & P_0^* \oplus P_1^* & & & & \\
\end{array}
\]
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We have the following exact sequence \( P_0^* \xrightarrow{d^*_1} P_0^* \sqcup P_1^* \xrightarrow{[d^*_1, -1]} P_1^* \rightarrow 0 \). Then we have the exact sequence in maps(mod(\(C\))

\[
\begin{array}{cccccc}
0 & \rightarrow & P_0^* & \rightarrow & P_0^* & \rightarrow & 0 \\
& & \downarrow{d^*_1} & & \downarrow{[d^*_1, -1]} & & \downarrow{d^*_1} \\
P_0^* & \rightarrow & P_0^* \sqcup P_1^* & \rightarrow & P_1^* & \rightarrow & 0
\end{array}
\]

Therefore the \( \overline{TR}(M, 0, 0) = (P_0^*, d^*_1, P_1^*) \). Now by Proposition 5.2, applying duality \( D = \overline{\Theta} \) in the maps category, we have that \( D(P_0^* \rightarrow P_1^*) \) is given by the map \( D_{\text{Cop}}(d^*_1) : D_{\text{Cop}}(P_0^*) \rightarrow D_{\text{Cop}}(P_1^*) \).

In this way, we conclude that almost split sequences in the category maps(mod(\(C\))) that have ending term \( M \rightarrow 0 \) must have first term \( \text{TR} \geq 0 \). Now, by Proposition 7.3 in order to show that the diagram (**) define an almost exact sequence in maps(mod(\(C\))) is enough to see that every not isomorphism \( (\alpha, 0) : (M, 0, 0) \rightarrow (M, 0, 0) \) factors through \( ([0, 1], 0) : \left( D_{\text{Cop}}(P_0^*) \sqcup M, [D_{\text{Cop}}(P_0^*), h], D_{\text{Cop}}(P_0^*) \right) \rightarrow (M, 0, 0) \).

Indeed, we have that \( \alpha : M \rightarrow M \) is not an isomorphism. Since \( 0 \rightarrow \tau M \rightarrow E \xrightarrow{\pi} M \rightarrow 0 \) is an almost split sequence of \( C \)-modules, we have that there exists \( \alpha' : M \rightarrow E \) such that \( \alpha = \pi \alpha' \). Considering the exact diagram (\(*\)), we have that

\[
h \circ \alpha = D_{\text{Cop}}(d^*_1) \circ f \circ \alpha'.
\]

Since \( D_{\text{Cop}}(d^*_1, h)[\begin{bmatrix} -f_{\alpha'} \\
\alpha \end{bmatrix} = h \circ \alpha - D_{\text{Cop}}(d^*_1) \circ f \circ \alpha = 0 \), we have the following diagram is commutative

\[
\begin{array}{cccccc}
M & \xrightarrow{[\begin{bmatrix} -f_{\alpha'} \\
\alpha \end{bmatrix}} & D_{\text{Cop}}(P_1^*) \sqcup M & \xrightarrow{[0, 1]} & M \\
& & \downarrow{[D_{\text{Cop}}(d^*_1), h]} & & \downarrow{0} \\
0 & \rightarrow & D_{\text{Cop}}(P_0^*) & \rightarrow & 0.
\end{array}
\]

Now, since \( [0, 1][\begin{bmatrix} -f_{\alpha'} \\
\alpha \end{bmatrix} = \alpha \), we conclude that the last diagram is the same as the morphism \( (\alpha, 0) : (M, 0, 0) \rightarrow (M, 0, 0) \), proving the required condition. Therefore, by Proposition 7.3 we conclude that the diagram (**) defines an almost split sequence in the category maps(mod(\(C\)).

(\(ii\)) Similar to (\(i\)).

Now, we define a functor which will give us a relation between almost split sequences in mod(mod(\(C\))) and almost split sequences in mod(\(A\)).

**Definition 7.6.** Let \( \Phi : \text{maps(mod}(C)) \rightarrow \text{mod(mod}(C)^{\text{op}}) \) given by

\[
\Phi(A_1 \xrightarrow{f} A_0) = \text{Coker}\left( (\cdot, A_1) \xrightarrow{(-, f)} (-, A_0) \right),
\]

where \( A_1 \) and \( A_0 \) are objects in mod(\(C\)).

We can see now that the functor \( \Phi \) preserves almost split sequences. The following is a generalization of [25, Theorem 3.4]
Theorem 7.7. Let

\[ 0 \rightarrow (N_1, g, N_2) \xrightarrow{(j_1, j_2)} (E_1, h, E_2) \xrightarrow{(p_1, p_2)} (M_1, f, M_2) \rightarrow 0 \]

be an almost split sequence in maps(mod(\mathcal{C})), such that g, f, are neither splittable epimorphisms nor splittable monomorphisms. Then the exact sequence

\[ 0 \rightarrow G \overset{\rho}{\rightarrow} H \overset{\sigma}{\rightarrow} F \rightarrow 0 \]

obtained from the commutative diagram:

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
(-, N_1) & (-, N_2) & G \\
\rho \downarrow & \downarrow & \downarrow \\
(-, E_1) & (-, E_2) & H \\
\sigma \downarrow & \downarrow & \downarrow \\
(-, M_1) & (-, M_2) & F \\
0 & 0 & 0
\end{array}
\]

is an almost split sequence in mod(mod(\mathcal{C}^{op})).

Proof Same proof given in [25, Theorem 3.4] works for this setting.

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8 Functorially Finite Subcategories

Let \( \mathcal{C} \) be an arbitrary category. Let \( \mathcal{X} \) be a subcategory of \( \mathcal{C} \). A morphism \( f : X \rightarrow M \) in \( \mathcal{C} \) with \( X \in \mathcal{X} \) is a right \( \mathcal{X} \)-approximation of \( M \) if \( \text{Hom}_\mathcal{C}(Z, X) \rightarrow \text{Hom}_\mathcal{C}(Z, M) \) is surjective for every \( Z \in \mathcal{X} \). Dually, a morphism \( g : M \rightarrow X \) with \( X \in \mathcal{X} \) is a left \( \mathcal{X} \)-approximation if \( \text{Hom}_\mathcal{C}(X, Z) \rightarrow \text{Hom}_\mathcal{C}(M, Z) \) is surjective for every \( Z \in \mathcal{X} \).

A subcategory \( \mathcal{X} \) of \( \mathcal{C} \) is contravariantly (covariantly) finite in \( \mathcal{C} \) if every object \( M \in \mathcal{C} \) has a right (left) \( \mathcal{X} \)-approximation; and \( \mathcal{C} \) is functorially finite if it is both contravariantly and covariantly finite.

8.1 Functorially Finite Subcategories in mod(\( \mathcal{A} \)) and a Result of Smalø

In this subsection we prove a result that generalizes the given by S. O. Smalø in [36, Theorem 2.1] and we will see some implications that it give us respect the category mod(\( \mathcal{C} \)) for a dualizing variety \( \mathcal{C} \). The following result is essentially given by Smalø in [36, Theorem 2.1]. We give a proof for the convenience of the reader.

Theorem 8.1. Let \( \mathcal{A} \) and \( \mathcal{B} \) be abelian categories and \( G : \mathcal{B} \rightarrow \mathcal{A} \) a covariant functor. Consider the comma category \( (G(\mathcal{B}), \mathcal{A}) \) and \( \mathcal{X} \subseteq \mathcal{A} \) and \( \mathcal{Y} \subseteq \mathcal{B} \) subcategories containing the zero object. We denote by \( \mathcal{D}^\mathcal{X}_\mathcal{Y} \) the full subcategory of \( (G(\mathcal{B}), \mathcal{A}) \) whose objects are: the morphisms \( g : G(\mathcal{B}) \rightarrow \mathcal{A} \) with \( B \in \mathcal{Y} \) and \( A \in \mathcal{X} \). Then \( \mathcal{D}^\mathcal{X}_\mathcal{Y} \) is covariant finite in \( (G(\mathcal{B}), \mathcal{A}) \) if and only if \( \mathcal{X} \subseteq \mathcal{A} \) and \( \mathcal{Y} \subseteq \mathcal{B} \) are covariant finite subcategories.
Proof ($\iff$). Let $g : G(B) \to A$ an object in $(G(B), A)$. Since $\mathcal{Y}$ is covariant finite, there exists an $\mathcal{Y}$-left approximation $\alpha_B : B \to Y_B$. Then we have the following pushout diagram in $\mathcal{A}$

\[
\begin{array}{ccc}
G(B) & \xrightarrow{G(\alpha_B)} & G(Y_B) \\
\downarrow{g} & & \downarrow{g'} \\
A & \xrightarrow{\delta} & C
\end{array}
\]

Since $\mathcal{X}$ is covariant finite in $\mathcal{A}$ we have a $\mathcal{X}$-left approximation $\beta_C : C \to X_C$. Then we have the following commutative diagram

\[
\begin{array}{ccc}
G(B) & \xrightarrow{G(\alpha_B)} & G(Y_B) \\
\downarrow{g} & & \downarrow{\beta_C g'} \\
A & \xrightarrow{\beta_C \delta} & X_C
\end{array}
\]

Then, we have the object $(Y_B, \beta_C g', X_C) \in \mathcal{D}_X^{\mathcal{Y}}$. We assert that $(\alpha_B, \beta_C \delta) : (B, g, A) \to (Y_B, \beta_C g', X_C)$ is a $\mathcal{D}_X^{\mathcal{Y}}$-left approximation of $(B, g, A)$.

Indeed, let $(\lambda, \phi) : (B, g, A) \to (B', f, A')$ a morphism in $(G(B), A)$ with $B' \in \mathcal{Y}$ and $A' \in \mathcal{X}$.

Since $\lambda : B \to B'$ is a morphism in $B$ with $B' \in B$, there exists a morphism $\lambda' : Y_B \to B'$ such that $\lambda = \lambda' \circ \alpha_B$. Then we get the following commutative diagram

\[
\begin{array}{ccc}
G(B) & \xrightarrow{G(\alpha_B)} & G(Y_B) \\
\downarrow{g} & & \downarrow{fG(\lambda')} \\
A & \xrightarrow{\phi} & A'
\end{array}
\]

Since the diagram $(\ast)$ is pushout, there exists a morphism $\sigma : C \to A'$ such that $\sigma g' = f G(\lambda')$ and $\sigma \delta = \phi$. Since $A' \in \mathcal{X}$ and $\beta_C : C \to X_C$ is a $\mathcal{X}$-left approximation of $C$, there exists a morphism $\sigma' : X_C \to A'$ such that $\sigma = \sigma' \beta_C$. We have the diagram

\[
\begin{array}{ccc}
G(B) & \xrightarrow{G(\alpha_B)} & G(Y_B) \\
\downarrow{g} & & \downarrow{\beta_C g'} & \downarrow{G(\lambda')} \\
A & \xrightarrow{\beta_C \delta} & X_C & \xrightarrow{\sigma'} & A'
\end{array}
\]

which is commutative since $f G(\lambda') = \sigma g' = \sigma' \beta_C g'$. Moreover we have that $\sigma' \beta_C \delta = \sigma \delta = \phi$ and since $\lambda = \lambda' \circ \alpha_B$ we have that $G(\lambda) = G(\lambda') G(\alpha_B)$. Therefore, the morphism $(\lambda, \phi)$ factor through the morphism $(\alpha_B, \beta_C \delta)$. Proving that $\mathcal{D}_X^{\mathcal{Y}}$ is covariant finite in $(G(B), A)$.

$(\implies)$. Let us suppose that $\mathcal{D}_X^{\mathcal{Y}}$ is covariant finite in $(G(B), A)$. Let $B$ be and object in $\mathcal{B}$. Consider the object $g : G(B) \to 0$ in $(G(B), A)$. Since $\mathcal{D}_X^{\mathcal{Y}}$ is covariant finite, there exists and object $\beta : G(Y) \to X$ with $Y \in \mathcal{Y}$ and $X \in \mathcal{X}$ and a morphism $(\lambda, \theta) : (B, 0, 0) \to (Y, \beta, X)$ which is a left $\mathcal{D}_X^{\mathcal{Y}}$-approximation. We assert that $\lambda : B \to Y$ is a
left $\mathcal{Y}$-approximation. Indeed, let $\gamma : B \to Y'$ a morphism in $B$ with $Y' \in \mathcal{Y}$. Then we have the following commutative diagram

$$
\begin{array}{ccc}
G(B) & \xrightarrow{G(\gamma)} & G(Y') \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\end{array}
$$

Since $(\lambda, \theta) : (B, 0, 0) \to (Y, \beta, X)$ is a left $D^X_\lambda$-approximation, there exists $(\psi, \tau) : (Y, \beta, X) \to (Y', 0, 0)$ such that $(\gamma, 0) = (\psi, \tau) \circ (\lambda, \theta)$. Therefore, we get that $\gamma = \psi \lambda$, proving that $\lambda : B \to Y$ is a left $\mathcal{Y}$-approximation. Thus, $\mathcal{Y}$ is covariantly finite in $B$.

Similarly, $\mathcal{X}$ is covariantly finite in $A$.

**Theorem 8.2.** Let $A$ and $B$ be abelian categories and $F : A \to B$, $G : B \to A$ a covariant functors such that $G$ is left adjoint to $F$. Consider the comma category $(B, F(A))$ and $\mathcal{X} \subseteq A$, $\mathcal{Y} \subseteq B$ subcategories containing the zero object. We denote by $C^\mathcal{Y}_\mathcal{X}$ the full subcategory of $(B, F(A))$ whose objects are: the morphisms $f : B \to F(A)$ with $B \in \mathcal{Y}$ and $A \in \mathcal{X}$. Then $C^\mathcal{Y}_\mathcal{X}$ is functorially finite in $(B, F(A))$ if and only if $\mathcal{X} \subseteq A$ and $\mathcal{Y} \subseteq B$ are functorially finite.

**Proof** We have isomorphism $\phi_{B,A} : \text{Hom}_A(G(B), A) \to \text{Hom}_B(B, F(A))$. This defines an isomorphism between the comma categories $\phi : \left( G(B), A \right) \to \left( B, F(A) \right)$. Denote by $D^\mathcal{Y}_\mathcal{X}$ the full subcategory of $(G(B), A)$ whose objects are: the morphisms $g : G(B) \to A$ with $B \in \mathcal{Y}$ and $A \in \mathcal{X}$ and $C^\mathcal{Y}_\mathcal{X}$ the full subcategory of $(B, F(A))$ whose objects are: the morphisms $f : B \to F(A)$ with $B \in \mathcal{Y}$ and $A \in \mathcal{X}$. Then we have an isomorphism $\phi : D^\mathcal{Y}_\mathcal{X} \to C^\mathcal{Y}_\mathcal{X}$. By Theorem 8.1, and its dual we get that $D^\mathcal{Y}_\mathcal{X}$ is covariant finite in $(B, F(A))$ and $C^\mathcal{Y}_\mathcal{X}$ is contravariantly finite in $(B, F(A))$. Then, via the isomorphism $\phi$ we get that $C^\mathcal{Y}_\mathcal{X}$ is functorially finite in $(B, F(A))$.

The following result that generalizes the given by S. O. Smalø in [36, Theorem 2.1].

**Corollary 8.3.** Let $\mathcal{U}$ and $\mathcal{T}$ preadditive categories and $M \in \text{Mod}(\mathcal{U} \otimes \mathcal{T}^{op})$ and consider $\Lambda = \left[ \begin{array}{c} T \\ M \end{array} \right] \in \text{Mod}(\mathcal{U})$. If $\mathcal{X} \subseteq \text{Mod}(\mathcal{U})$ and $\mathcal{Y} \subseteq \text{Mod}(\mathcal{T})$ are subcategories, denote by $(\mathcal{Y}, \mathcal{G}\mathcal{X})$ the full subcategory of $(\text{Mod}(\mathcal{T}), \mathcal{G}\text{Mod}(\mathcal{U}))$ whose objects are morphisms of $\mathcal{T}$-modules $f : B \to \mathcal{G}(A)$ with $B \in \mathcal{Y}$ and $A \in \mathcal{X}$. Then $(\mathcal{Y}, \mathcal{G}\mathcal{X})$ is a covariantly (contravariantly, functorially) finite subcategory of $\text{Mod}(\left[ \begin{array}{c} T \\ M \end{array} \right])$ if and only if $\mathcal{Y} \subseteq \text{Mod}(\mathcal{T})$ and $\mathcal{X} \subseteq \text{Mod}(\mathcal{U})$ are covariantly (contravariantly, functorially) finite.

**Proof** By Section 5 in [22], we have left adjoint $\mathcal{F} : \text{Mod}(\mathcal{T}) \to \text{Mod}(\mathcal{U})$ to $\mathcal{G}$ and by [22, Theorem 3.17], there is equivalence $\left( \text{Mod}(\mathcal{T}), \mathcal{G}\text{Mod}(\mathcal{U}) \right) \simeq \text{Mod}(\left[ \begin{array}{c} T \\ M \end{array} \right])$. The result follows from Theorem 8.1 its dual and Theorem 8.2.

**Corollary 8.4.** Let $\mathcal{U}$ and $\mathcal{T}$ dualizing $K$-varieties and $M \in \text{mod}(\mathcal{U} \otimes \mathcal{T}^{op})$ such that $M_T \in \text{mod}(\mathcal{U})$ and $M_U \in \text{mod}(\mathcal{T})$ for all $U \in \mathcal{U}$ and $T \in \mathcal{T}$ and consider $\Lambda = \left[ \begin{array}{c} T \\ M \end{array} \right]$. If
\( \mathcal{X} \subseteq \text{mod}(\mathcal{U}) \) and \( \mathcal{Y} \subseteq \text{mod}(\mathcal{T}) \) are subcategories, denote by \((\mathcal{Y}, \mathcal{G}\mathcal{X})\) the full subcategory of \( \left( \text{mod}(\mathcal{T}), \mathcal{G}\text{mod}(\mathcal{U}) \right) \) whose objects are morphisms of \( T \)-modules \( f : B \rightarrow \mathcal{G}(A) \) with \( B \in \mathcal{Y} \) and \( A \in \mathcal{X} \). Then \((\mathcal{Y}, \mathcal{G}\mathcal{X})\) is a covariantly (contravariantly, functorially) finite subcategory of \( \text{mod}\left( \begin{bmatrix} T & 0 \\ M & U \end{bmatrix} \right) \) if and only if \( \mathcal{Y} \subseteq \text{mod}(\mathcal{T}) \) and \( \mathcal{X} \subseteq \text{mod}(\mathcal{U}) \) are covariantly (contravariantly, functorially) finite.

**Proof** By [22, Proposition 6.3] we have an equivalence \( \left( \text{mod}(\mathcal{T}), \mathcal{G}\text{mod}(\mathcal{U}) \right) \simeq \text{mod}\left( \begin{bmatrix} T & 0 \\ M & U \end{bmatrix} \right) \). We also have an adjoint pair \( (\mathcal{F}^*, \mathcal{G}^*) \) by discussion after [22, Lemma 5.1]. Therefore the result follows from Theorem 8.1 its dual and Theorem 8.2.

The following is the generalization of a result of Smalø [36, Corollary 2.2].

**Corollary 8.5.** Let \( \mathcal{U} \) and \( \mathcal{T} \) dualizing \( K \)-varieties and \( M \in \text{Mod}(\mathcal{U} \otimes T^{op}) \) such that \( M_T \in \text{mod}(\mathcal{U}) \) and \( M_U \in \text{mod}(\mathcal{T}) \) for all \( U \in \mathcal{U} \) and \( T \in \mathcal{T} \) and consider \( \Lambda = \begin{bmatrix} T & 0 \\ M & U \end{bmatrix} \). If \( \mathcal{X} \subseteq \text{mod}(\mathcal{U}) \) and \( \mathcal{Y} \subseteq \text{mod}(\mathcal{T}) \) are functorially finite which are closed under extensions. Then \((\mathcal{Y}, \mathcal{G}\mathcal{X})\) is functorially finite subcategory of \( \text{mod}\left( \begin{bmatrix} T & 0 \\ M & U \end{bmatrix} \right) \) which is closed under extensions and moreover \((\mathcal{Y}, \mathcal{G}\mathcal{X})\) has almost split sequences.

**Proof** By [22, Theorem 6.10], it follows that \( \Lambda \) is a dualizing \( R \)-variety and therefore \( \text{mod}(\Lambda) \) is Krull-Schmidt. Since \( \mathcal{X} \subseteq \text{mod}(\mathcal{U}) \) and \( \mathcal{Y} \subseteq \text{mod}(\mathcal{T}) \) are closed under extensions, this also holds for \((\mathcal{Y}, \mathcal{G}\mathcal{X})\). Thus, \((\mathcal{Y}, \mathcal{G}\mathcal{X})\) is a Krull-Schmidt subcategory of \( \text{mod}(\mathcal{T}), \mathcal{G}\text{mod}(\mathcal{U}) \simeq \text{mod}(\Lambda) \), and the rest of the proof follows from [23, Corollary 3.5].

In the following results \( K \) will be an arbitrary field.

**Corollary 8.6.** Let \( C \) be a dualizing \( K \)-variety and \( \mathcal{X} \subseteq \text{mod}(C) \) be a subcategory. Denote by \( \text{maps}(\mathcal{X}) \) the full subcategory of \( \text{maps}(\text{mod}(C)) \) whose objects are morphisms of \( C \)-modules \( f : B \rightarrow A \) with \( A, B \in \mathcal{X} \). Then \( \mathcal{X} \) is contravariantly (covariantly, functorially) finite in \( \text{mod}(C) \) if and only if \( \text{maps}(\mathcal{X}) \) is contravariantly (covariantly, functorially) finite in \( \text{maps}(\text{mod}(C)) \).

### 8.2 Functorially Finite Subcategories in \( \text{mod}(\text{mod}(C)) \)

Let \( C \) be a dualizing \( K \)-variety. By Corollary 8.6 there is a close relation between contravariantly, covariantly and functorially subcategories in \( \text{mod}(\text{mod}(C)) \) and contravariantly, covariantly and functorially subcategories in maps(\( \text{mod}(C) \)).

Now, consider the matrix category \( \Lambda := \begin{bmatrix} C & 0 \\ \Hom & C \end{bmatrix} \). There is an equivalence of categories

\[ \Psi : \text{mod}\left( \begin{bmatrix} C & 0 \\ \Hom & C \end{bmatrix} \right) \rightarrow \text{maps}(\text{mod}(C)). \]

Recall, we have the functor (see Definition 7.6)

\[ \Phi : \text{maps}(\text{mod}(C)) \rightarrow \text{mod}(\text{mod}(C)^{op}) \]

given by \( \Phi( A_1 \xrightarrow{f} A_0) = \text{Coker}\left( (\xrightarrow{-, A_1}, A_1) \xrightarrow{(\xrightarrow{-,-}, A_0)} \right) \).

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Now, we list some ways to get functorially finite subcategories in maps$(\text{mod}\mathcal{A})$. Obviously $\Psi(\mathcal{X})$ is functorially finite in maps$(\text{mod}(\mathcal{C}))$ if $\mathcal{X}$ is a functorially finite subcategory in mod$\left(\begin{bmatrix} \mathcal{C} & \mathcal{0} \\ \text{Hom} & \mathcal{C} \end{bmatrix}\right)$.

In this part we will see that some properties like: contravariantly, covariantly, functorially finite subcategories of maps$(\text{mod}(\mathcal{C}))$, are preserved by the functor $\Phi$.

The following result is a generalization of a result in [25].

**Theorem 8.7.** Let $\mathcal{C} \subset \text{maps}(\text{mod}(\mathcal{C}))$ be a subcategory. Then the following statements hold:

(a) If $\mathcal{C}$ is contravariantly finite in maps$(\text{mod}(\mathcal{C}))$, then $\Phi(\mathcal{C})$ is a contravariantly finite subcategory of mod$(\text{mod}(\mathcal{C}))^{\text{op}}$.

(b) If $\mathcal{C}$ is covariantly finite in maps$(\text{mod}(\mathcal{C}))$, then $\Phi(\mathcal{C})$ is a covariantly finite subcategory of mod$(\text{mod}(\mathcal{C}))^{\text{op}}$.

(c) If $\mathcal{C}$ is functorially finite in maps$(\text{mod}(\mathcal{C}))$, then $\Phi(\mathcal{C})$ is a functorially finite subcategory of mod$(\text{mod}(\mathcal{C}))^{\text{op}}$.

**Proof** Same proof as in [25, Theorem 3.8] \(\square\)

**Remark 8.8.** We can define the functor $\Phi'$ : maps$(\text{mod}(\mathcal{C})) \rightarrow \text{mod}(\text{mod}(\mathcal{C}))$ as:

$$\Phi' : (A_1 \xrightarrow{f} A_0) = \text{Coker}((A_0, -) \xrightarrow{(f,-)} (A_1, -)).$$

The same properties: contravariantly, covariantly, functorially finite subcategories of maps$(\text{mod}(\mathcal{C}))$, are preserved by the functor $\Phi'$.

On the other hand, if $\mathcal{X}$ is a functorially finite subcategory of mod$(\mathcal{C})$, we get that maps$(\mathcal{X}')$ is a functorially finite subcategory of maps$(\text{mod}(\mathcal{C}))$, by Corollary 8.6. Thus we have a way to get contravariantly (covariantly, functorially) finite subcategories in maps$(\text{mod}(\mathcal{C}))$ from the ones of mod$(\mathcal{C})$, which of course are in bijective correspondence with ones in mod$\left(\begin{bmatrix} \mathcal{C} & \mathcal{0} \\ \text{Hom} & \mathcal{C} \end{bmatrix}\right)$ by the equivalence $\Psi$. Thus, we have induced maps:

$$\begin{array}{ccc}
\{\text{functorially finite subcategories } \mathcal{C} \subset \text{maps}(\text{mod}(\mathcal{C}))\} & \xrightarrow{\Phi} & \{\text{functorially finite subcategories } \mathcal{E} \subset \text{mod}(\text{mod}(\mathcal{C}))^{\text{op}}\} \\
\downarrow & & \downarrow \\
\{\text{functorially finite subcategories } \mathcal{F} \subset \text{maps}(\text{mod}(\mathcal{C}))^{\text{op}}\}
\end{array}$$

Finally, we have the following examples of functorially finite subcategories of the category maps$(\text{mod}(\mathcal{C}))$.

We denote by $\text{e}\text{maps}(\text{mod}(\mathcal{C}))$ the full subcategory of maps$(\text{mod}(\mathcal{C}))$ consisting of all maps $(M_1, f, M_2)$, such that $f$ is an epimorphism and by $\text{m}\text{maps}(\text{mod}(\mathcal{C}))$ the full subcategory of maps$(\text{mod}(\mathcal{C}))$ consisting of all maps $(M_1, f, M_2)$, such that $f$ is an monomorphism.
Proposition 8.9. The categories emaps(\text{mod}(\mathcal{C})) and mmaps(\text{mod}(\mathcal{C})) are functorially finite in maps(\text{mod}(\mathcal{C})).

Proof The proof given in [25, Theorem 3.12] works for this setting.

Acknowledgments The authors thank project PAPIIT-Universidad Nacional Autónoma de México IA105317. The authors are very grateful for the referee’s valuable comments and suggestions, which have improved the quality and readability of the article.

References

1. Hügel, L.A., Koenig, S., Liu, Q.: Recollements and tilting objects. J. Pure. Appl. Algebra 215(4), 420–438 (2011)
2. Hügel, L.A., Koenig, S., Liu, Q.: On the uniqueness of stratifications of derived modules categories. J. Algebra, 120–137 (2012)
3. Hügel, L.A., Koenig, S., Liu, Q.: Jordan hölder theorems for derived module categories of piecewise hereditary algebras. J. Algebra 352, 361–381 (2012)
4. Auslander, M.: Representation dimension of artin algebras. Queen mary college mathematics notes, 1971 republished in selected works of maurice auslander: part 1. amer. math. soc., Providence (1999)
5. Auslander, M.: Representation theory of artin algebras i. Comm. Algebra 1(3), 177–268 (1974)
6. Auslander, M., Reiten, I.: Stable equivalence of dualizing R-varieties. Adv. in Math. 12(3), 306–366 (1994)
7. Auslander, M., Platzeck, M.I., Reiten, I.: Coxeter functors without diagrams. Trans. Amer. Math. Soc. 250, 1–46 (1979)
8. Auslander, M., Reiten, I., Smalø., S.: Representation theory of artin algebras. Studies in advanced mathematics 36. Cambridge University Press, Cambridge (1995)
9. Bautista, R., Souto Salorio, M.J., Zuazua, R.: Almost split sequences for complexes of fixed size. J. Algebra 287(1), 140–168 (2005)
10. Beilinson, A., Bernstein, J., Deligne, P: Faisceaux Pervers. In: Analysis and topology on singular spaces i. luminy, astérisque 100, soc. math. france, 5-171 (1982)
11. Chen, Q., Zheng, M.: Recollements of abelian categories and special types of comma categories. J. Algebra. 321(9), 2474–2485 (2009)
12. Dlab, V., Ringel, C.M.: Representations f graphs and algebras. Memoirs of the A.M.S. No. 173 (1976)
13. Fossum, R.M., Griffith, P.A., Reiten, I.: Trivial extensions of abelian categories. Lecture notes in mathematics no. 456. Springer, Berlin (1975)
14. Franjou, V., Pirashvili, T.: Comparison of abelian categories recollements. Documenta Math. 9, 41–56 (2004)
15. Frey, P.: Representations in abelian categories. In: Proceedings of the conference on categorical algebra, La Jolla, 95-120 (1966)
16. Gordon, R., Green, E.L.: Modules with cores and amalgamations of indecomposable modules. Memoirs of the A.M.S. No. 187 (1978)
17. Green, E.L.: The representation theory of tensor algebras. J. Algebra 34, 136–171 (1975)
18. Green, E.L.: On the representation theory of rings in matrix form. Pacific J Math. 100(1), 123–138 (1982)
19. Hilton, P.J., Stammbuch, U. A Course in Homological Algebra, Second edition. Graduated Texts in Mathematics. Springer, Heidelberg (1997)
20. Kelly, G.M.: On the radical of a category. J. Aust. Math. Soc. 4, 299–307 (1964)
21. Krause, H.: Krull-Schmidt categories projective covers. Expo. Math. 33, 535–549 (2015)
22. León-Galeana, A., Ortiz-Morales, M., Santiago-Vargas, V.: Triangular Matrix Categories I: Dualizing Varieties and generalized one-point extensions To appear in Algebr. Represent. Theory. (2022)
23. Liu, S., Ng, P., Paquette, C.: Almost Split Sequences and Approximations. Algebr Represent Theor 16(6), 1809–1827 (2013)
24. MacPherson, R., Vilonen, K.: Elementary construction of perversive sheaves. Invent. Math. 84, 403–436 (1986)
25. Martínez-Villa, R., Ortiz-Morales, M.: Tilting theory and functor categories III: the maps category. Inter. J Algebra. 5(11), 529–561 (2011)
26. Mendoza, O., Ortíz, M., Sáenz, E.C., Santiago, V.: A generalization of the theory of standardly stratified algebras i: Standardly stratified ringoids. Glasg. Math. J., 1–36. https://doi.org/10.1017/S0017089520000476 (2020)
27. Mitchell, B.: Rings with several objects. Adv. in Math 8, 1–161 (1972)
28. Parshall, B., Scott, L.L.: Derived categories, quasi-hereditary algebras, and algebraic groups. Proc. of the Ottawa-Moosone Workshop in algebra (1987), Math. Lect. Note Series, Carleton University and Universite d’Ottawa, 1–111 (1988)
29. Popescu, N., Abelian categories with applications to rings and modules. London Mathematical Society Monographs No. 3. Academic Press, London., New York. MR 0340375 (1973)
30. Psaroudakis, C.: Homological theory of recollements of abelian categories. J. Algebra 398, 63–110 (2014)
31. Psaroudakis, C., Vitoria, J.: Recollements of module categories. J. Appl. Categor. Struc. 22, 579–593 (2014)
32. Psaroudakis, C.: A representation-theoretic approach to recollements of abelian categories. Contemp. Math. of Amer. Math. Soc. 716, 67–154 (2018)
33. Reiten, I.: The use of almost split sequences in the representation theory of artin algebras. Lecture Notes in Math, vol. 944, pp. 29–104. Springer, Berling (1982)
34. Ringel, C.M.: Tame Algebras and Integral Quadratic. Forms Lecture Notes in Mathematics. Springer, Berlin (1984)
35. Ogawa, Y.: Recollements for dualizing k-varieties and Auslander’s formulas. Appl. Categor. Struc. https://doi.org/10.1007/s10485-018-9546-y (2018)
36. Smalo, S.O.: Functorial finite subcategories over triangular matrix rings. Proceedings of the American Mathematical Society 3, 111 (1991)
37. Zhang, P.: Monomorphism categories, cotilting theory and Gorenstein–projective modules. J. Algebra 339, 181–202 (2011)
38. Zhu, B.: Triangular matrix algebras over quasi-hereditary algebras. Tsukuba J. Math. 25(1), 1–11 (2001)

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