The growth and approximation for an analytic function represented by Laplace–Stieltjes transforms with generalized order converging in the half plane

Hong Yan Xu1* and Hua Wang1

Abstract
By utilizing the concept of generalized order, we investigate the growth of Laplace–Stieltjes transform converging in the half plane and obtain one equivalence theorem concerning the generalized order of Laplace–Stieltjes transforms. Besides, we also study the problem on the approximation of this Laplace–Stieltjes transform and give some results about the generalized order, the error, and the coefficients of Laplace–Stieltjes transforms. Our results are extension and improvement of the previous theorems given by Luo and Kong, Singhall, and Srivastava.

MSC: 44A10; 30D15

Keywords: Approximation; Laplace–Stieltjes transform; Half plane; Error

1 Introduction
Laplace–Stieltjes transforms

\[ F(s) = \int_{0}^{+\infty} e^{sx} \, d\alpha(x), \quad s = \sigma + it, \quad (1) \]

where \( \alpha(x) \) is a bounded variation on any finite interval \([0, Y]\) \((0 < Y < +\infty)\), and \( \sigma \) and \( t \) are real variables, as we know, if \( \alpha(t) \) is absolutely continuous, then \( F(s) \) becomes the classical Laplace integral of the form

\[ F(s) = \int_{0}^{\infty} e^{st} g(t) \, dt. \quad (2) \]

If \( \alpha(t) \) is a step-function and satisfies

\[ \alpha(x) = \begin{cases} a_1 + a_2 + \cdots + a_n, & \lambda_n \leq x < \lambda_{n+1}; \\ 0, & 0 \leq x < \lambda_1; \\ \frac{\alpha(x)+\alpha(x-)}{2}, & x > 0, \end{cases} \]

© The Author(s) 2018. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.
where the sequence $\{\lambda_n\}_{0}^{\infty}$ satisfies

$$0 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n \uparrow +\infty,$$

thus $F(s)$ becomes a Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \quad s = \sigma + it.$$  \tag{4}

($\sigma$, $t$ are real variables), $a_n$ are nonzero complex numbers. Obviously, if $\alpha(t)$ is an increasing continuous function which is not absolutely continuous, then the integral (1) defines a class of functions $F(s)$ which cannot be expressed either in the form (2) or (4).

Let a sequence $\{\lambda_n\}_{n=1}^{\infty}$ satisfy (3), and

$$\limsup_{n \to +\infty} (\lambda_{n+1} - \lambda_n) = h < +\infty, \quad \limsup_{n \to +\infty} \frac{n}{\lambda_n} = D < \infty.$$  \tag{5}

Set

$$A_n^* = \sup_{\lambda_n < x \leq \lambda_{n+1}, -\infty < \sigma < +\infty} \left| \int_{\lambda_n}^{x} e^{\sigma y} \, d\alpha(y) \right|,$$

if

$$\limsup_{n \to +\infty} \frac{\log A_n^*}{\lambda_n} = 0,$$  \tag{6}

it is easy to get $\sigma_F^* = 0$, that is, $F(s)$ is analytic in the left half plane; if

$$\limsup_{n \to +\infty} \frac{\log A_n^*}{\lambda_n} = -\infty,$$  \tag{7}

it follows that $\sigma_F^* = +\infty$, that is, $F(s)$ is analytic in the whole plane. For convenience, we use $L_\beta$ to be a class of all the functions $F(s)$ of the form (1) which are analytic in the half plane $\Re s < \beta$ ($-\infty < \beta < +\infty$) and the sequence $\{\lambda_n\}$ satisfies (3) and (5); $L_0$ to be the class of all the functions $F(s)$ of the form (1) which are analytic in the half plane $\Re s < 0$ and the sequence $\{\lambda_n\}$ satisfies (3), (5), and (6); and $L_\infty$ to be the class of all the functions $F(s)$ of the form (1) which are analytic in the whole plane $\Re s < +\infty$ and the sequence $\{\lambda_n\}$ satisfies (3), (5), and (7). Thus, if $-\infty < \beta < 0$ and $F(s) \in L_\beta$, then $F(s) \in L_0$.

In 1963, Yu [26] first proved the Valiron–Knopp–Bohr formula of the associated abscissas of bounded convergence, absolute convergence, and uniform convergence of Laplace–Stieltjes transform. Moreover, Yu [26] also estimated the growth of the maximal molecule $M_{\sigma}(F)$, the maximal term $\mu(\sigma, F)$, by introducing the concepts of the order of $F(s)$, and investigated the singular direction–Borel line of entire functions represented by Laplace–Stieltjes transforms converging in the whole complex plane. After his wonderful works, considerable attention has been paid to the value distribution and the growth of analytic functions represented by Laplace–Stieltjes transforms converging in the whole plane or the half plane (see [1, 3, 4, 6–8, 11–15, 18–25, 27]).
Set
\[
\mu(\sigma, F) = \max_{n \in \mathbb{N}} \left\{ A_n^* e^{\lambda_n \sigma} \right\} \quad (\sigma < 0),
\]
\[
M_n(\sigma, F) = \sup_{0 < x < +\infty, -\infty < a < +\infty} \left| \int_0^x e^{(\sigma + iy) a(y)} \, dy \right| \quad (\sigma < 0).
\]

For \( F(s) \in L_0 \), in view of \( M_n(\sigma, F) \to +\infty \) as \( \sigma \to 0^- \), the concepts of order and type can be usually used in estimating the growth of \( F(s) \) precisely.

**Definition 1.1** If Laplace–Stieltjes transform (1) satisfies \( \sigma F' = 0 \) and
\[
\limsup_{\sigma \to 0^-} \frac{\log^* \log^* M_n(\sigma, F)}{-\log(-\sigma)} = \rho, \quad 0 \leq \rho \leq +\infty,
\]
we call \( F(s) \) of order \( \rho \) in the left half plane, where \( \log^* x = \max(\log x, 0) \). Furthermore, if \( \rho \in (0, +\infty) \), the type of \( F(s) \) is defined by
\[
\limsup_{\sigma \to 0^-} \frac{\log^* M_n(\sigma, F)}{(-\frac{1}{\pi})^\rho} = T, \quad 0 \leq T \leq +\infty.
\]

**Remark 1.1** However, if \( \rho = 0 \) and \( \rho = +\infty \), we cannot estimate the growth of such functions precisely by using the concept of type.

In 2012 and 2014, Luo and Kong [9, 10] investigated the growth of Laplace–Stieltjes transform converging on the whole plane and obtained the following.

**Theorem 1.1** (see [10]) If the L-S transform \( F(s) \in L_\infty \) and is of order \( \rho \) \((0 < \rho < \infty)\), then
\[
\rho = \limsup_{n \to +\infty} \frac{\lambda_n \log \lambda_n}{-\log A_n^*}.
\]

**Theorem 1.2** (see [9]) If the L-S transform \( F(s) \in L_\infty \), then for \( p = 1 \), we have
\[
\limsup_{\sigma \to +\infty} \frac{h(\log M_n(\sigma, F))}{h(\sigma)} - 1 = \limsup_{n \to +\infty} \frac{h(\lambda_n)}{h(-\frac{1}{\lambda_n} \log A_n^*)},
\]
and for \( p = 2, 3, \ldots \), we have
\[
\limsup_{n \to +\infty} \frac{h(\lambda_n)}{h(-\frac{1}{\lambda_n} \log A_n^*)} \leq \limsup_{\sigma \to +\infty} \frac{h(\log M_n(\sigma, F))}{h(\sigma)} \leq \limsup_{n \to +\infty} \frac{h(\lambda_n)}{h(-\frac{1}{\lambda_n} \log A_n^*)} + 1,
\]
where \( h(x) \) satisfies the following conditions:
(i) \( h(x) \) is defined on \([a, +\infty)\) and is positive, strictly increasing, differentiable and tends to \( +\infty \) as \( x \to +\infty \);
(ii) \( \lim_{x \to +\infty} \frac{d h(x)}{d (\log^p x)} = k \in (0, +\infty), p \geq 1, p \in \mathbb{N}^+ \), where \( \log^0 x = x, \log^1 x = \log x, \) and \( \log^{[p]} x = \log(\log^{[p-1]} x) \).
In this paper, the first aim is to investigate the growth of analytic functions represented by Laplace–Stieltjes transforms with generalized order converging in the half plane, and we obtain some theorems about the generalized order $A_0^*$ and $\lambda_0$, which are improvements of the previous results given by Luo and Kong [9, 10]. To state our results, we first introduce the following notations and definitions.

Let $\Gamma_1$ be a class of continuous increasing functions $A$ such that $A(x) \geq 0$ for $x \leq x_0$, $A(x) = A(x_0)$ for $x \leq x_0$ and on $[x_0, +\infty)$ the function $A$ increases to $+\infty$; and $\Gamma_0^0$ be a class such that $\Gamma_0^0 \subset \Gamma_1$ and $A(x(1 + o(1))) = (1 + o(1))A(x)$ as $x \to +\infty$, for $A \in \Gamma_0^0$; further, $A \in \Gamma_0^0$ if $A \in \Gamma_1$ and for any $\eta > 0$, $A(\eta x) = (1 + o(1))A(x)$ as $x \to +\infty$. Obviously, it follows $\Gamma_0^0 \subset \Gamma_0$ and $h(x) \in \Gamma_1$.

**Definition 1.2** Let $F(s) \in L_0$ and $A \in \Gamma_1$, $B \in \Gamma_1$. If

$$\rho_{AB}(F) = \limsup_{\sigma \to 0^-} \frac{A(\log M_{\mu}(\sigma, F))}{B(-\frac{1}{\sigma})},$$

then $\rho_{AB}$ is called generalized order of $F(s)$.

**Remark 1.2** Let $A(x) = \log x$ and $B = \log x$, then $\rho_{AB}(F) = \rho$.

**Remark 1.3** Let $A(x) = \log_p x$ and $B = \log_q x$, then $\rho_{AB}(F) = \rho(p, q)(F)$, where $\rho(p, q)(F)$ is the $(p, q)$-order of $F(s)$ (see [2]).

**Remark 1.4** Let $A(x) = \log x$ and $B = \log \log x$, then $\rho_{AB}(F) = \rho_l(F)$, where $\rho_l(F)$ is the logarithmic order of $F(s)$.

### 2 Results and discussion

For generalized order of Laplace–Stieltjes transform (1), we obtain the following.

**Theorem 2.1** Let $F(s) \in L_0$, $A \in \Gamma_0^0$ and $B \in \Gamma_0^0$ be continuously differentiable, and the function $B$ increase more rapidly than $A$ such that, for any constant $\eta \in (0, +\infty)$,

$$\frac{x}{B^{-1}(\eta A(x))} \to +\infty \quad (x_0 \leq x \to +\infty),$$  \hfill (8)

and

$$A\left(\frac{x}{B^{-1}(\eta A(x))}\right) = (1 + o(1))A(x) \quad (x \to +\infty).$$  \hfill (9)

If

$$\lim_{\sigma \to 0^-} \frac{\log M_{\mu}(\sigma, F)}{-\log(-\sigma)} = +\infty$$  \hfill (10)

and

$$\rho_{AB}(F) = \limsup_{\sigma \to 0^-} \frac{A(\log M_{\mu}(\sigma, F))}{B(-\frac{1}{\sigma})}, \quad 0 \leq \rho_{AB}(F) \leq +\infty,$$
then

\[ \rho_{AB}(F) = \limsup_{n \to +\infty} \frac{A(\lambda_n)}{B\left(\frac{\lambda_n}{\log \lambda_n}\right)}. \]

**Theorem 2.2** Let \( F(s) \in L_0 \), \( A \in \Gamma_{\omega}^0 \), and \( B \in \Gamma_{\omega}^0 \) be continuously differentiable, and the function \( A \) increase more rapidly than \( B \) such that, for any constant \( \eta \in (0, +\infty) \),

\[ \frac{x}{A^{-1}(\eta B(x))} \uparrow +\infty \quad (x_0 \leq x \to +\infty), \tag{11} \]

and

\[ B\left(\frac{x}{A^{-1}(\eta B(x))}\right) = \left(1 + o(1)\right) B(x) \quad (x \to +\infty). \tag{12} \]

If \( F(s) \) satisfies (10) and

\[ \rho_{AB}(F) = \limsup_{\sigma \to 0^-} \frac{A(\log M_n(\sigma, F))}{B(-\frac{1}{\sigma})}, \quad 0 \leq \rho_{AB}(F) \leq +\infty, \]

then

\[ \rho_{AB}(F) = \limsup_{n \to +\infty} \frac{A(\log A^*_n)}{B(\lambda_n)}. \]

If Laplace–Stieltjes transform (1) satisfies \( A^*_n = 0 \) for \( n \geq k + 1 \) and \( A^*_n \neq 0 \), then \( F(s) \) will be said to be an exponential polynomial of degree \( k \) usually denoted by \( p_k \), i.e., \( p_k(s) = \int_0^{\lambda^*_k} \exp(\lambda y) \, d\alpha(y) \). If we choose a suitable function \( \alpha(y) \), the function \( p_k(s) \) may be reduced to a polynomial in terms of \( \exp(s\lambda_i) \), that is, \( \sum_{i=1}^k b_i \exp(s\lambda_i) \). We denote \( \Pi_k \) to be the class of all exponential polynomials of degree almost \( k \), that is,

\[ \Pi_k = \left\{ \sum_{i=1}^k b_i \exp(s\lambda_i) : (b_1, b_2, \ldots, b_k) \in \mathbb{C}^k \right\}. \]

For \( F(s) \in \overline{L_\beta}, -\infty < \beta < 0 \), we denote by \( E_n(F, \beta) \) the error in approximating the function \( F(s) \) by exponential polynomials of degree \( n \) in uniform norm as

\[ E_n(F, \beta) = \inf_{p \in \Pi_n} \|F - p\|_{\beta}, \quad n = 1, 2, \ldots, \]

where

\[ \|F - p\|_{\beta} = \max_{-\infty < \beta < +\infty} \left| F(\beta + it) - p(\beta + it) \right|. \]

In 2017, Singhal and Srivastava [17] studied the approximation of Laplace–Stieltjes transforms of finite order converging on the whole plane and obtained the following theorem.
Theorem 2.3 (see [17]) If Laplace–Stieltjes transform $F(s) \in L_\infty$ and is of order $\rho$ ($0 < \rho < \infty$) and of type $T$, then for any real number $-\infty < \beta < +\infty$, we have
\[
\rho = \limsup_{n \to +\infty} \frac{\lambda_n \log \lambda_n}{-\log E_{n-1}(F, \beta) \exp(-\beta \lambda_n)} = \limsup_{n \to +\infty} \frac{\lambda_n \log \lambda_n}{-\log E_{n-1}(F, \beta)}
\]
and
\[
T = \limsup_{n \to +\infty} \frac{\lambda_n}{\rho} \left( E_{n-1}(F, \beta) \exp(-\beta \lambda_n) \right)^{\frac{\rho}{\lambda_n}} = \limsup_{n \to +\infty} \frac{\lambda_n}{\rho \exp(\rho \beta + 1) \left(E_{n-1}(F, \beta)\right)^{\frac{\rho}{\lambda_n}}}.
\]

In the same year, the author and Kong [20] investigated the approximation of Laplace–Stieltjes transform $F(s) \in L_0$ with infinite order and obtained the following.

Theorem 2.4 (see [20, Theorem 2.5]) If the Laplace–Stieltjes transform $F(s) \in L_0$ and is of infinite order, if $\lambda_n \sim \lambda_{n+1}$, then for any real number $-\infty < \beta < +\infty$, then for any fixed real number $-\infty < \sigma < 0$, we have
\[
\limsup_{\sigma \to -0} \frac{X(\log^+ M_n(\sigma, F))}{\log(-\frac{\beta}{\sigma})} = \rho^* \quad \iff \quad \limsup_{n \to +\infty} \frac{X(\lambda_n)}{\log^+ \left( E_{n-1}(F, \beta) \exp(-\beta \lambda_n) \right)} = \rho^*,
\]
where $0 < \rho^* < \infty$, $X(\cdot)$-order can be seen in [20].

The second purpose of this paper is to study the approximation of Laplace–Stieltjes transform $F(s) \in L_0$ with generalized order, and our results are listed as follows.

Theorem 2.5 Let $F(s) \in L_0$, $A \in \Gamma^0$, and $B \in \Gamma^{0i}$ be continuously differentiable satisfying (8) and (9), and let the function $B$ increase more rapidly than $A$. If $F(s)$ satisfies (10) and
\[
\rho_{AB}(F) = \limsup_{\sigma \to -0} \frac{A(\log M_n(\sigma, F))}{B(-\frac{1}{\sigma})}, \quad 0 \leq \rho_{AB}(F) \leq +\infty,
\]
then for any real number $-\infty < \beta < 0$, we have
\[
\rho_{AB}(F) = \limsup_{n \to +\infty} \frac{A(\lambda_n)}{B\left(\frac{\lambda_n}{\log E_{n-1}(F, \beta) \exp(-\beta \lambda_n)}\right)}.
\]

Theorem 2.6 Under the assumptions of Theorem 2.2, then for any real number $-\infty < \beta < 0$, we have
\[
\rho_{AB}(F) = \limsup_{n \to +\infty} \frac{A(\log E_{n-1}(F, \beta))}{B(\lambda_n)}.
\]

3 Conclusions

Regarding Theorems 2.1 and 2.2, the generalized order of Laplace–Stieltjes transforms are discussed by using the more abstract functions, and some related theorems among $\lambda_n$, $A^*_n$, and the generalized order are obtained. Moreover, we also investigate some properties of approximation on analytic functions defined by Laplace–Stieltjes transforms of generalized order. For the topic of the growth and approximation of Laplace–Stieltjes transforms of generalized order, it seems that this topic has never been treated before. Our theorems are generalization and improvement of the previous results given by Luo and Kong [9, 10], Singhal and Srivastava [17].
4 Methods

To prove our results, we also need to give the following lemmas (see [16]).

Let \( \mathbb{Z}_0 \) denote the set of positive unbounded functions \( \phi \) on \( (-\infty, 0) \) such that the derivative \( \phi' \) is positive, continuous, and increasing to \(+\infty\) on \( (-\infty, 0) \). Thus, if \( \phi \in \mathbb{Z}_0 \), then \( \phi(x) \to \zeta \geq 0 \) and \( \phi'(x) \to 0 \) as \( x \to -\infty \). Let \( \psi \) be the inverse function of \( \phi' \), then \( \psi \) is continuous on \((0, +\infty)\) and increases to 0. Set \( \phi \in \mathbb{Z}_0 \) and \( \psi(x) = x - \frac{x \psi(x)}{\phi'(x)} \). For \(-\infty < x \to 0 < x \), since \( \phi' \) is increasing on \( (-\infty, 0) \), we have

\[
\phi'(x) \phi(x + \iota) - \phi'(x + \iota) \phi(x) < \phi'(x) \left[ \phi(x + \iota) - \phi(x) \right] = \phi'(x) \int_0^{x + \iota} \phi'(t) \, dt
\]

\[
< (x + \iota - x) \phi'(x) \phi'(x + \iota),
\]

that is,

\[
\psi(x) = x - \frac{\phi(x)}{\phi'(x)} < x + \iota - \frac{\phi(x + \iota)}{\phi'(x + \iota)} = \psi(x + \iota).
\]

Thus, it means that \( \psi \) is an increasing function on \( (-\infty, 0) \).

Next, we will prove that \( \psi(x) \to 0 \) as \( x \to 0 \), that is, there is no constant \( \eta < 0 \) such that \( \psi(x) \leq \eta \) for all \( x \in (-\infty, 0) \). Assume that there exist two constants \( \eta, K_1 \) such that \( \psi(x) \leq \eta \) for all \( x \in (-\infty, 0) \) and \( \eta < K_1 < 0 \). Since \( \psi \) is an increasing function and \( \psi(x) \to 0 < x \), then it follows \( \frac{\psi'(x)}{\psi(x)} \leq \frac{1}{x - \eta} \) for \( K_1 \leq x < 0 \). Thus, it follows

\[
\log \phi(x) = \log \phi(K_1) + \int_K^x \frac{\phi'(t)}{\phi(t)} \, dt \leq \log \phi(K_1) + \int_K^x \frac{1}{t - \eta} \, dt
\]

\[
= \log \phi(K_1) + \log \frac{x - \eta}{K_1 - \eta}.
\]

Hence \( \phi(x) \leq \phi(K_1) \frac{x - \eta}{K_1 - \eta} \). In view of \( \phi'(x) \to +\infty \) \( x \to 0 \), we get a contradiction. Thus, it follows \( \psi(x) \to 0 \) as \( x \to 0 \).

Besides, let \( \psi^{-1} \) be the inverse function of \( \psi \). Then \( \psi^{-1} \) is an increasing function on \((-\infty, 0)\) and \( \phi'(\psi^{-1}(\sigma)) \) increases to \(+\infty\) on \((-\infty, 0)\).

**Lemma 4.1** Let \( \phi \in \mathbb{Z}_0 \), then the conclusion that \( \log \mu(\sigma, F) \leq \phi(\sigma) \) for any \( \sigma \in (-\infty, 0) \) holds if and only if \( \log A_n^* \leq -\lambda_n \psi(\phi(x)) \) for all \( n \geq 0 \).

**Proof** Suppose that \( \log \mu(\sigma, F) \leq \phi(\sigma) \) for any \( \sigma \in (-\infty, 0) \), then \( \log A_n^* \leq \phi(\sigma) - \sigma \lambda_n \) for all \( n > 0 \) and \( \sigma \in (-\infty, 0) \). Thus, take \( \sigma = \psi(\lambda_n) \), it follows for all \( n \geq 0 \) that

\[
\log A_n^* \leq \phi(\psi(\lambda_n)) - \lambda_n \psi(\phi(\lambda_n)) = -\lambda_n \psi\left( \psi(\phi(\lambda_n)) \right) = -\lambda_n \psi\left( \psi(\lambda_n) \right).
\]

On the contrary, assume that \( \log A_n^* \leq -\lambda_n \psi(\phi(\lambda_n)) \) for all \( n \geq 0 \). Since, for any \( \sigma < 0 \) and \( x < 0 \),

\[
(\sigma - x) \phi'(x) \leq \int_x^\sigma \phi'(t) \, dt = \phi(x) - \phi(\sigma),
\]
then it follows

\[
\log \mu(\sigma, F) \leq \max \{ -\lambda_n \psi(\lambda_n) + \lambda_n \sigma : n \geq 0 \} \leq \max \{ -t \psi(\phi(t)) + t \sigma : t \geq 0 \}
\]

\[
= \max \{ -\phi'(x) \psi(x) + \sigma + \phi'(x) : x > -\infty \}
\]

\[
= \max \{ (\sigma - x)\phi'(x) + \phi(x) : x > -\infty \} = \phi(\sigma).
\]

Therefore, this completes the proof of Lemma 4.1. \(\square\)

Lemma 4.2 If the L-S transform \(F(s) \in L_\infty\), then for any \(\sigma \ (-\infty < \sigma < 0)\) and \(\varepsilon > 0\), we have

\[
\frac{1}{p} \mu(\sigma, F) \leq M_u(\sigma, F) \leq C \mu((1-\varepsilon)\sigma, F) \frac{1}{-\sigma},
\]

where \(p > 2\) and \(C \neq 0\) are constants.

Proof We will adapt the method as in Yu [26] and Kong and Hong [5]. Set

\[
I(x; \sigma + it) = \int_0^x \exp\left\{ (\sigma + it)y \right\} \, d\alpha(y).
\]

In view of (5), there exists a positive number \(\xi\) satisfying \(0 < \lambda_{n+1} - \lambda_n \leq \xi\) \((n = 1, 2, 3, \ldots)\). Thus, it yields \(e^{-\xi \sigma} < \frac{1}{2}\) for \(\sigma\) sufficiently close to 0\(^{-}\), where \(p > 2\) is a constant. For \(x > \lambda_n\), it follows

\[
\int_{\lambda_n}^x \exp(it) \, d\alpha(y) = \int_{\lambda_n}^x \exp(-\sigma y) \, d\alpha(y; \sigma + it)
\]

\[
= \int_{\lambda_n}^x \exp(-\sigma y) \, d\alpha(y; \sigma + it) dy.
\]

Then, for any \(\sigma < 0\) and any \(x \in (\lambda_n, \lambda_{n+1})\), it yields

\[
\left| \int_{\lambda_n}^x \exp(it) \, d\alpha(y) \right| \leq M_u(\sigma, F) \left[ \exp(-x\sigma) + \exp(-\sigma \lambda_n) + \exp(-x\sigma) - \exp(-\sigma \lambda_n) \right]
\]

\[
\leq 2M_u(\sigma, F) \exp\left\{ -(\lambda_n +\varepsilon) \sigma \right\} \leq pM_u(\sigma, F) \exp\left\{ -\lambda_n \sigma \right\},
\]

which implies

\[
\frac{1}{p} \mu(\sigma, F) \leq M_u(\sigma, F). \tag{14}
\]

On the other hand, for any \(x > 0\), it follows that there exists a positive integer \(n \in \mathbb{N}\), such that \(\lambda_n < x \leq \lambda_{n+1}\). Thus, it follows

\[
\int_0^x \exp\{ (\sigma + it)y \} \, d\alpha(y) = \sum_{k=1}^{n-1} \int_{\lambda_k}^{\lambda_{k+1}} \exp\{ (\sigma + it)y \} \, d\alpha(y) + \int_{\lambda_n}^x \exp\{ (\sigma + it)y \} \, d\alpha(y).
\]

Set \(I_k(x; it) = \int_{\lambda_n}^{\lambda_{k+1}} \exp(it) \, d\alpha(y) \quad (\lambda_k < x \leq \lambda_{k+1})\), then for any real number \(t\) and \(\sigma < 0\), it follows \(|I_k(x; it)| \leq A_k^* \leq \mu(\sigma, F) e^{-\lambda_k \sigma}\). Thus, for any \(\varepsilon \in (0, 1)\) and \(\sigma < 0\), it yields \(|I_k(x; it)| \leq \mu(\sigma, F) e^{-\lambda_k \sigma}\) for any \(t\) and \(\sigma < 0\).
\( \mu((1 - \varepsilon)\sigma, F)e^{-\lambda_k(1 - e)\sigma} \) and

\[
\int_0^\infty \exp\{(\sigma + it)y\} \, d\alpha(y) = \sum_{k=1}^{n-1} \left[ \exp(\lambda_{k+1} \sigma)I_k(\lambda_{k+1}; it) - \sigma \int_{\lambda_k}^{\lambda_{k+1}} \exp(\sigma y)I_k(y; it) \, dy \right]
+ \exp(x\sigma)I_n(x; it) - \sigma \int_{\lambda_{N_1}}^{\infty} \exp(\sigma y)I_n(y; it) \, dy.
\]

Hence, we can deduce

\[
\left| \int_0^\infty \exp\{(\sigma + it)y\} \, d\alpha(y) \right| \leq \sum_{k=1}^{n-1} \mu((1 - \varepsilon)\sigma, F) \exp\{-\lambda_k(1 - \varepsilon)\sigma\} \left( \exp(\lambda_{k+1} \sigma) + \exp(\lambda_{k+1} \sigma) - \exp(\lambda_k \sigma)\right)
+ \mu((1 - \varepsilon)\sigma, F) \sum_{k=1}^{\infty} \exp(\lambda_k \varepsilon \sigma).
\]

In view of (5), for the above \( \varepsilon \), there exists \( N_1 \in \mathbb{N} \), such that, for any \( n > N_1 \), we have \( \lambda_n > \frac{n}{D+\varepsilon} \). Hence it follows for \( \sigma \to 0^- \) that

\[
\sum_{k=1}^{\infty} \exp(\lambda_k \varepsilon \sigma) \leq \sum_{k=1}^{N_1} \exp(\lambda_k \varepsilon \sigma) + \sum_{k=N_1+1}^{\infty} \exp\left\{ \frac{k \varepsilon \sigma}{D+\varepsilon} \right\} < C \frac{1}{-\sigma}, \tag{15}
\]

where \( C \) is a constant on \( \varepsilon \) and (5). Therefore, this lemma is proved from (14) and (15). \( \square \)

4.1 Proofs of Theorems 2.1 and 2.2

4.1.1 The proof of Theorem 2.1

Suppose that \( \varrho := \rho_{A,B}(F) < +\infty \) and

\[
\vartheta = \limsup_{n \to +\infty} \frac{A(\lambda_n)}{B\left(\frac{\lambda_n}{\log A_n}\right)}.
\]

In view of the definition of generalized order and Lemma 4.2, for any \( \varepsilon > 0 \), there exists a constant \( \sigma_0 < 0 \) such that, for all \( 0 > \sigma > \sigma_0 \),

\[
\log \mu(\sigma, F) \leq A^{-1}\left( (\varrho + \varepsilon)B\left(\frac{-1}{\sigma}\right) \right) + \log p,
\]

that is,

\[
\log A_n^* \leq A^{-1}\left( (\varrho + \varepsilon)B\left(\frac{-1}{\sigma}\right) \right) - \lambda_n \sigma + \log p, \quad n \geq 0. \tag{16}
\]

Choosing

\[
-\frac{1}{\sigma} = B^{-1}\left( \frac{1}{\varrho + \varepsilon}A\left( \frac{\lambda_n}{B^{-1}(A(\lambda_n))/\varrho\varepsilon} \right) \right),
\]
we conclude from (9) and (16) that
\[
\log A_n^* \leq \frac{\lambda_n}{B^{-1}(\frac{A(\lambda_n)}{\rho + \varepsilon})} + \frac{\lambda_n}{B^{-1}(\frac{1}{\rho + \varepsilon} A(\frac{\lambda_n}{B^{-1}(\frac{A(\lambda_n)}{\rho + \varepsilon})}))} + \log p
\]
\[
= \frac{(1 + o(1))\lambda_n}{B^{-1}((1 + o(1))\frac{A(\lambda_n)}{\rho + \varepsilon})}, \quad \text{as } n \to +\infty,
\]
which implies
\[
A(\lambda_n) \leq (\rho + \varepsilon) B\left(\frac{(1 + o(1))\lambda_n}{\log A_n^*}\right), \quad \text{as } n \to +\infty.
\]

(17)

Since \( A \in \Gamma^{\Omega_1}_0\), \( B \in \Gamma^{\Omega_1}_0 \) and let \( \varepsilon \to 0^+ \), we can conclude from (17) that \( \vartheta \leq \rho \).
Assume \( \vartheta < \rho \), then we can choose a constant \( \rho_1 \) such that \( \vartheta < \rho_1 < \rho \). Since \( B^{-1}(\frac{A(\lambda_n)}{\rho_1}) \) is an increasing function, then there exists a positive integer \( n_0 \) such that, for \( n \geq n_0 \),
\[
\log A_n^* \leq \frac{\lambda_n}{B^{-1}(\frac{A(\lambda_n)}{\rho_1})} \leq \int_{\lambda_n}^{\lambda_n} \frac{1}{B^{-1}(\frac{A(\lambda_n)}{\rho_1})} dt + K_1,
\]
where here and further \( K_j \) is a constant.

Since \( \phi \in \Xi_0 \), and let
\[
\phi(\sigma) = \int_{-\frac{1}{\rho_1}}^{\frac{1}{\rho_1}} \frac{A^{-1}(\rho_1 B(\tau))}{\tau^2} d\tau + K_2 \quad \text{for } 0 > \sigma \geq \sigma_0.
\]

Then it follows
\[
\phi'(\sigma) = A^{-1}\left(\rho_1 B\left(-\frac{1}{\sigma}\right)\right), \quad \psi(\lambda_n) = \frac{1}{\lambda_n B^{-1}(\frac{A(\lambda_n)}{\rho_1})},
\]
\[
\left[\lambda_n \psi'\left(\psi(\lambda_n)\right)\right]' = \left[\lambda_n \psi(\lambda_n) - \phi\left(\psi(\lambda_n)\right)\right]' = \psi(\lambda_n),
\]
and
\[
-\lambda_n \psi'(\psi(\lambda_n)) = -\int_{\lambda_n}^{\lambda_n} \psi(t) dt + K_2 = \int_{\lambda_n}^{\lambda_n} \frac{1}{B^{-1}(\frac{A(\lambda_n)}{\rho_1})} dt + K_2.
\]

(19)

Thus, in view of (18) and (19), it follows
\[
\log \mu(\sigma, F) \leq \phi(\sigma) = \int_{-\frac{1}{\rho_1}}^{\frac{1}{\rho_1}} \frac{A^{-1}(\rho_1 B(\tau))}{\tau^2} d\tau + K_2
\]
\[
\leq A^{-1}\left(\rho_1 B\left(-\frac{1}{\sigma}\right)\right) \int_{-\frac{1}{\rho_1}}^{\frac{1}{\rho_1}} \frac{dt}{\tau^2} + K_3
\]
\[
\leq -\sigma_1 A^{-1}\left(\rho_1 B\left(-\frac{1}{\sigma}\right)\right) + K_3.
\]

(20)

Since \( A \in \Gamma^{\Omega_1}_0 \), in view of (10), (20) and by applying Lemma 4.2, we can deduce \( \rho_{A,B}(F) \leq \rho_1 \), which implies a contradiction with \( \rho_{A,B}(F) > \rho_1 \). Hence \( \vartheta = \rho_{A,B}(F) \).
If $\rho_{AB}(F) = +\infty$, by using the same argument as above, it is easy to prove that the conclusion is true. Therefore, this completes the proof of Theorem 2.1.

### 4.1.2 The proof of Theorem 2.2

Suppose that $\rho := \rho_{AB}(F) < +\infty$ and

$$
\vartheta_1 = \limsup_{n \to +\infty} \frac{A(\log A_n^{\ast})}{B(\lambda_n)}.
$$

In view of the definition of generalized order and Lemma 4.2, for any $\varepsilon > 0$, there exists a constant $\sigma_0 < 0$ such that, for all $0 > \sigma > \sigma_0$,

$$
\log \mu(\sigma, F) \leq A^{-1}\left(\left(\rho + \varepsilon\right)B\left(\frac{1}{\sigma}\right)\right) + \log p,
$$

that is,

$$
\log A_n^{\ast} \leq A^{-1}\left(\left(\rho + \varepsilon\right)B\left(\frac{1}{\sigma}\right)\right) - \lambda_n \sigma + \log p, \quad n \geq 0. \quad (21)
$$

Choosing $-\frac{1}{\sigma} = \lambda_n$, we conclude from (21) that

$$
\log A_n^{\ast} \leq A^{-1}\left(\left(\rho + \varepsilon\right)B(\lambda_n)\right) + 1 + \log p
\leq (1 + o(1))A^{-1}\left(\left(\rho + \varepsilon\right)B(\lambda_n)\right), \quad \text{as } n \to +\infty. \quad (22)
$$

Since $A \in \Gamma^{0\ast}$, $B \in \Gamma^{0\ast}$ and let $\varepsilon \to 0^+$, we can conclude from (22) that $\vartheta_1 \leq \rho$.

Assume $\vartheta_1 < \rho$, then we can choose a constant $\rho_2$ such that $\vartheta_1 < \rho_2 < \rho$. It means that there exists a positive integer $n_0$ such that, for $n \geq n_0$,

$$
\log A_n^{\ast} \leq A^{-1}(\rho_2 B(\lambda_n)),
$$

that is,

$$
\log \mu(\sigma, F) \leq \max\left\{A^{-1}(\rho_2 B(\lambda_n)) + \lambda_n \sigma : n \geq n_0\right\} + K_5. \quad (23)
$$

In view of (10), the following equation

$$
A^{-1}(\rho_2 B(t)) + t \sigma = 0
$$

has a unique solution $t_1 := t(\sigma)$ such that $t_1 \uparrow +\infty$ as $\sigma \to 0^-$, and for $t \geq t_1$ we can deduce that $A^{-1}(\rho_2 B(t)) + t \sigma \leq 0$. Hence, it follows

$$
\log \mu(\sigma, F) \leq \max\left\{A^{-1}(\rho_2 B(t)) + t \sigma : t_0 \leq t \leq t_1\right\} + K_6
\leq A^{-1}(\rho_2 B(t_1)) + K_6. \quad (24)
$$

In view of

$$
-\frac{1}{\sigma} = \frac{t_1}{A^{-1}(\rho_2 B(t_1))},
$$
it follows from (12) that

$$\mathcal{B}\left(-\frac{1}{\sigma}\right) = \mathcal{B}\left(\frac{t_1}{\mathcal{A}^{-1}(\rho_2 \mathcal{B}(t_1))}\right) = (1 + o(1)) \mathcal{B}(t_1), \quad \sigma \to 0^-. \quad (25)$$

Hence, we can deduce from (24) and (25) that

$$\log \mu(\sigma, F) \leq \mathcal{A}^{-1}\left(\rho_1 (1 + o(1)) \mathcal{B}\left(-\frac{1}{\sigma}\right)\right), \quad \text{as } \sigma \to 0^-,$$

which implies \(\rho_{AB}(F) \leq \rho_2 < \rho_{AB}(F)\) by combining Lemma 4.2 and (10), a contradiction. Therefore, \(\vartheta_1 = \rho_{AB}(F)\).

If \(\rho_{AB}(F) = +\infty\), by using the same argument as above, it is easy to prove that the conclusion is true. Therefore, this completes the proof of Theorem 2.2.

### 4.2 Proofs of Theorems 2.5 and 2.6

#### 4.2.1 The proof of Theorem 2.5

Suppose that \(\rho := \rho_{AB}(F) < +\infty\) and

$$\vartheta_3 = \limsup_{n \to +\infty} \frac{\mathcal{A}(\lambda_n)}{\mathcal{B}(\log \mu_{\infty}(F, \beta) \exp(-\beta \lambda_n))}.$$  

In view of the definition of generalized order and Lemma 4.2, for any \(\varepsilon > 0\), there exists a constant \(\sigma_0 < 0\) such that, for all \(0 > \sigma > \sigma_0\),

$$\log M_a(\sigma, F) \leq \mathcal{A}^{-1}\left((\rho + \varepsilon) \mathcal{B}\left(-\frac{1}{\sigma}\right)\right). \quad (26)$$

Since \(F(s) \in L_0\), and for any constant \(\beta (-\infty < \beta < 0)\), then \(F(s) \in L_\beta\). Hence, for \(\beta < \sigma < 0\) and \(p_k \in \Pi_k\), it follows

$$E_k(F, \beta) \leq \|F - p_k\|_\beta \leq \left|F(\beta + it) - p_k(\beta + it)\right| \leq \left|\int_0^{\lambda_k} \exp\{s\} \, d\alpha(y) - \int_0^{\lambda_k} \exp\{s\} \, d\alpha(y)\right| = \left|\int_{\lambda_k}^{\lambda_{k+1}} \exp\{s\} \, d\alpha(y)\right|. \quad (27)$$

Let

$$I_{j,k}(b;it) = \int_{\lambda_{j,k}}^{b} \exp\{ity\} \, d\alpha(y) \quad (\lambda_{j,k} < b \leq \lambda_{j,k+1}),$$

then \(\left|I_{j,k}(b;it)\right| \leq A^*_j.\) In view of

$$\left|\int_{\lambda_k}^{\lambda_{k+1}} \exp\{(\beta + it)y\} \, d\alpha(y)\right| = \lim_{b \to +\infty} \left|\int_{\lambda_k}^{b} \exp\{(\beta + it)y\} \, d\alpha(y)\right|,$$
where \(-\infty < \beta < 0\), hence

\[
\left| \int_{\lambda_k}^b \exp\{ (\beta + it) y \} \, da(y) \right| = \left| \sum_{j=k}^{n+1-k-1} \int_{\lambda_{j-1}}^{\lambda_j} \exp\{ (\beta + it) y \} \, da(y) + \int_{\lambda_{n+1-k}}^b \exp\{ (\beta + it) y \} \, da(y) \right|
\]

\[
= \left[ \sum_{j=k}^{n+1-k} e^{\beta\lambda_{j+1}} I_j(\lambda_{j+1} \beta) - \beta \int_{\lambda_{j-1}}^{\lambda_j} e^{\beta\lambda_{j+1}} I_j(\lambda_{j+1} \beta) \, dy \right]
\]

\[
+ e^{\beta\lambda_{n+1-k}} I_{n+1-k}(\beta) - \beta \int_{\lambda_{n+1-k}}^b e^{\beta\lambda_{n+1-k}} I_{n+1-k}(\beta) \, dy
\]

\[
\leq \sum_{j=k}^{n+1} \left[ A_n^* e^{\beta\lambda_{j+1}} + A_n^* (e^{\beta\lambda_{j+1}} - e^{\beta\lambda_{j+1}}) \right] + 2e^{\beta\lambda_{n+1-k}} A_n^* - e^{\beta\lambda_{n+1-k}} A_n^*
\]

\[
\leq 2 \sum_{j=k}^{n+1} A_n^* e^{\beta\lambda_{j+1}}.
\]

Therefore, we conclude

\[
\left| \int_{\lambda_k}^\infty \exp\{ (\beta + it) y \} \, da(y) \right| \leq 2 \sum_{n=k}^{\infty} A_n^* \exp\{ \beta\lambda_{n+1} \}, \quad \text{as} \ n \to +\infty. \quad (28)
\]

In view of Lemma 4.2, it follows \( A_n^* \leq pM_\sigma(\sigma,F) e^{-\sigma\lambda_n} \). So, for any \( \sigma \ (\beta < \sigma < 0) \), it yields from (27) and (28) that

\[
E_n(F,\beta) \leq 2 \sum_{k=n+1}^{\infty} \sum_{l=k}^{\infty} \exp\{ (\beta - \sigma)\lambda_k \} \leq 2pM_\sigma(\sigma,F) \sum_{k=n+1}^{\infty} \exp\{ (\beta - \sigma)\lambda_k \}. \quad (29)
\]

In view of (5), we can choose \( h' \ (0 < h' < h) \) such that \( (\lambda_{n+1} - \lambda_n) \geq h' \) for \( n \geq 0 \). Then, for \( \sigma \geq \frac{\beta}{2} \), it follows from (29) that

\[
E_n(F,\beta) \leq M_\sigma(\sigma,F) \sum_{k=n+1}^{\infty} \exp\{ (\beta - \sigma)\lambda_k \}
\]

\[
\leq M_\sigma(\sigma,F) \sum_{k=n+1}^{\infty} \exp\{ (\beta - \sigma)\lambda_k \} \sum_{k=n+1}^{\infty} \exp\{ (\beta - \sigma)\lambda_k \}
\]

\[
= M_\sigma(\sigma,F) \sum_{k=n+1}^{\infty} \exp\{ (\beta - \sigma)\lambda_k \} \left( 1 - \exp\left( \frac{\beta}{2} h' \right) \right)^{-1},
\]

that is,

\[
E_{n-1}(F,\beta) \leq KM_\sigma(\sigma,F) \exp\{ \lambda_n(\beta - \sigma) \}, \quad (30)
\]

where \( K' \) is a constant. Hence, it follows from (26) and (30) that

\[
\log\{E_{n-1}(F,\beta) \exp\{ -\beta\lambda_n \} \} \leq A^{-1} \left( (\rho + \varepsilon) \mathcal{B} \left( -\frac{1}{\sigma} \right) \right) - \lambda_n \sigma + \log K, \quad n \geq 0. \quad (31)
\]
Let
\[-\frac{1}{\sigma} = B^{-1}\left( \frac{1}{\rho + \varepsilon} A \left( \frac{\lambda_n}{B^{-1}(\frac{\lambda_n}{\rho + \varepsilon})} \right) \right),\]
we conclude from (9) and (31) that
\[
\log\left[ E_{n-1}(F, \beta) \exp[-\beta \lambda_n] \right] \leq \frac{\lambda_n}{B^{-1}(\frac{\lambda_n}{\rho + \varepsilon})} + \log K
\]
\[= \frac{(1 + o(1))\lambda_n}{B^{-1}((1 + o(1))\frac{\lambda_n}{\rho + \varepsilon})}, \quad \text{as } n \to +\infty,
\]
which implies
\[
A(\lambda_n) \leq (\rho + \varepsilon)B\left( \frac{(1 + o(1))\lambda_n}{\log[E_{n-1}(F, \beta) \exp[-\beta \lambda_n]]} \right), \quad \text{as } n \to +\infty. \tag{32}
\]
Since \( A \in \Gamma_0^1 \), \( B \in \Gamma_0^1 \) and let \( \varepsilon \to 0^+ \), we can conclude from (32) that \( \vartheta_3 \leq \rho \).

Assume \( \vartheta_3 < \rho \), then we can choose a constant \( \rho_3 \) such that \( \vartheta_3 < \rho_3 < \rho \). Since \( B^{-1}(\frac{A(s)}{\rho_3}) \) is an increasing function, then there exists a positive integer \( n_0 \) such that, for \( n \geq n_0 \),
\[
\log\left[ E_{n-1}(F, \beta) \exp[-\beta \lambda_n] \right] \leq \frac{\lambda_n}{B^{-1}(\frac{\lambda_n}{\rho_3})} \leq \int_{\lambda_n}^{\lambda} \frac{1}{B^{-1}(\frac{A(s)}{\rho_3})} \, dt + K_1. \tag{33}
\]
For any \( \beta < 0 \), then there exists \( p_1 \in \Pi_{n-1} \) such that
\[
\|F - p_1\| \leq 2E_{n-1}(F, \beta). \tag{34}
\]
And since
\[
A_n^* \exp[\beta \lambda_n] = \sup_{\lambda_n < s \leq \lambda_n + 1, \infty < t < \infty} \left| \int_{\lambda_n}^{\infty} \exp\{ity\} \, d\alpha(y) \right| \exp[\beta \lambda_n]
\]
\[\leq \sup_{\lambda_n < s \leq \lambda_n + 1, \infty < t < \infty} \left| \int_{\lambda_n}^{\infty} \exp\{(\beta + it)y\} \, d\alpha(y) \right|
\]
\[\leq \sup_{\infty < t < \infty} \left| \int_{\lambda_n}^{\infty} \exp\{(\beta + it)y\} \, d\alpha(y) \right|,
\]
thus, for any \( p \in \Pi_{n-1} \), it follows
\[
A_n^* \exp[\beta \lambda_n] \leq |F(\beta + it) - p(\beta + it)| \leq \|F - p\| \beta. \tag{35}
\]
Hence, for any \( \beta < 0 \) and \( F(s) \in L_0^1 \), it follows from (34) and (35) that
\[
A_n^* \leq 2E_{n-1}(F, \beta) \exp[-\beta \lambda_n]. \tag{36}
\]
Hence, (18) follows from (33) and (36).
Since $\phi \in \mathbb{Z}_0$, and let
\[
\phi(\sigma) = \int_{-1}^{1} \frac{A^{-1}(\rho_1B(t))}{t^2} \, dt + K_2 \quad \text{for } 0 > \sigma \geq \sigma_0,
\]
and
\[
\phi' = \frac{1}{B^{-1}(\frac{A(\lambda_n)}{\rho_1})}.
\]

By using the same argument as in the proof of Theorem 2.1, we conclude
\[
\log \mu(\sigma, F) \leq \phi(\sigma) = \int_{-1}^{1} \frac{A^{-1}(\rho_1B(t))}{t^2} \, dt + K_2 \\
\leq -\sigma_1A^{-1}\left(\rho_1B\left(-\frac{1}{\sigma}\right)\right) + K_3. \quad (37)
\]
Since $A \in \Gamma^0$, in view of (10), (37) and by applying Lemma 4.2, we can deduce $\rho_{AB}(F) \leq \rho_3$, which implies a contradiction with $\rho_{AB}(F) > \rho_1$. Hence $\vartheta_3 = \rho_{AB}(F)$.

If $\rho_{AB}(F) = +\infty$, by using the same argument as above, it is easy to prove that the conclusion is true. Therefore, this completes the proof of Theorem 2.5.

4.2.2 The proof of Theorem 2.6

By combining the arguments as in the proofs of Theorems 2.2 and 2.5, we can easily prove the conclusion of Theorem 2.6.

Acknowledgements

We thank the referee(s) for reading the manuscript very carefully and making a number of valuable and kind comments which improved the presentation.

Funding

The authors were supported by the National Natural Science Foundation of China (11561033), the Natural Science Foundation of Jiangxi Province in China (20132AB211001, 20151BAB201008), and the Foundation of Education Department of Jiangxi (GJJ160914, GJJ170759, GJJ161603, GJJ170788) of China.

Competing interests

The authors declare that none of the authors have any competing interests in the manuscript.

Authors’ contributions

HYX and HW completed the main part of this article, HYX and HW corrected the main theorems. All authors read and approved the final manuscript.

Publisher’s Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 28 March 2018 Accepted: 18 July 2018 Published online: 24 July 2018

References

1. Baty, C.J.K.: Tauberian theorem for the Laplace–Stieltjes transform. Trans. Am. Math. Soc. 322(2), 783–804 (1990)
2. Juneja, O.P., Kapor, G.P., Bajpai, S.K.: On the $(p, q)$-order and lower $(p, q)$-order of an entire function. J. Reine Angew. Math. 282, 53–67 (1976)
3. Knopp, K.: Über die Konvergenzabszisse des Laplace-Integrals. Math. Z. 54, 291–296 (1951)
4. Kong, Y.Y.: Laplace–Stieltjes transforms of infinite order in the right half-plane. Acta Math. Sin. 55, 141–148 (2012)
5. Kong, Y.Y., Hong, Y.: On the Growth of Laplace–Stieltjes Transforms and the Singular Direction of Complex Analysis. Jinan University Press, Guangzhou (2010)
6. Kong, Y.Y., Huo, Y.Y.: On generalized orders and types of Laplace–Stieltjes transforms analytic in the right half-plane. Acta Math. Sin. 59A, 91–98 (2016)
7. Kong, Y.Y., Sun, D.C.: On the growth of zero order Laplace–Stieltjes transform convergent in the right half-plane. Acta Math. Sci. 28B(2), 431–440 (2008)
8. Kong, Y.Y, Yang, Y.: On the growth properties of the Laplace–Stieltjes transform. Complex Var. Elliptic Equ. 59, 553–563 (2014)
9. Luo, X., Kong, Y.Y.: On the order and type of Laplace–Stieltjes transforms of slow growth. Acta Math. Sci. 32A, 601–607 (2012)
10. Luo, X., Liu, X.Z., Kong, Y.Y.: The regular growth of Laplace–Stieltjes transforms. J. Math. 34, 1181–1186 (2014)
11. Mishkelyavichyus, A.: A Tauberian theorem for the Laplace–Stieltjes integral and the Dirichlet series. Liet. Mat. Rink. 29(4), 745–753 (1989) (Russian)
12. Posiko, O., Sheremeta, M.M.: Logarithms of Laplace–Stieltjes integral and maximum of the integrand. Integral Transforms Spec. Funct. 18(4), 271–283 (2007)
13. Ru, M.: The recent progress in Nevanlinna theory. J. Jiangxi Norm. Univ. Nat. Sci. 42(1), 1–11 (2018)
14. Shang, L.N., Gao, Z.S.: The growth of entire functions of infinite order represented by Laplace–Stieltjes transformation. Acta Math. Sci. 27A(6), 1035–1043 (2007) (in Chinese)
15. Shang, L.N., Gao, Z.S.: The value distribution of analytic functions defined by Laplace–Stieltjes transforms. Acta Math. Sinica (Chin. Ser.) 51(5), 993–1000 (2008)
16. Sheremeta, M.N., Fedynyak, S.I.: On the derivative of a Dirichlet series. Sib. Math. J. 39(1), 181–197 (1998)
17. Singhal, C., Srivastava, G.S.: On the growth and approximation of entire functions represented by Laplace–Stieltjes transformation. Ann. Univ. Ferrara 59(1), 1–12 (2017)
18. Tang, W.J., Cui, Y.Q., Xu, H.Q., Xu, H.Y.: On some q-order and q-type of Taylor–Hadamard product function. J. Jiangxi Norm. Univ. Nat. Sci. 40, 276–279 (2016)
19. Xu, H.Y.: The logarithmic order and logarithmic type of Laplace–Stieltjes transform. J. Jiangxi Norm. Univ. Nat. Sci. 41, 180–183 (2017)
20. Xu, H.Y., Kong, Y.Y.: The approximation of analytic function defined by Laplace–Stieltjes transformations convergent in the left half-plane. Houst. J. Math. 43, 783–806 (2017)
21. Xu, H.Y., Kong, Y.Y.: The approximation of Laplace–Stieltjes transformations with finite order on the left half plane. C. R. Math. 356(4), 63–76 (2018)
22. Xu, H.Y., Liu, S.Y.: The approximation of Laplace–Stieltjes transforms with finite order. J. Inequal. Appl. 2017, 164 (2017)
23. Xu, H.Y., Xuan, Z.X.: The growth and value distribution of Laplace–Stieltjes transformations with infinite order in the right half-plane. J. Inequal. Appl. 2013, 273 (2013)
24. Xu, H.Y., Xuan, Z.X.: The singular points of analytic functions with finite X-order defined by Laplace–Stieltjes transformations. J. Funct. Spaces 2015, Article ID 865096 (2015)
25. Xu, H.Y., Li, C.F., Cao, T.B.: On proximate order and type functions of Laplace–Stieltjes transformations convergent in the right half-plane. Math. Commun. 17, 355–369 (2012)
26. Yu, J.R.: Borel’s line of entire functions represented by Laplace–Stieltjes transformation. Acta Math. Sin. 13, 471–484 (1963) (in Chinese)
27. Zong, C., Wang, L.B., Deng, H.Y., Zhang, X.Y.: The existence of periodic solutions for a class of difference systems with (ϕ1, ϕ2)-Laplacian. J. Jiangxi Norm. Univ. Nat. Sci. 42, 187–193 (2018)