ON CONVERGENCE OF STRINGY MOTIVES OF WILD $p^n$-CYCLIC QUOTIENT SINGULARITIES

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Abstract. The wild McKay correspondence, a variant of the McKay correspondence in positive characteristics, shows that stringy motives of quotient varieties equal some motivic integrals on the moduli space of of the Galois covers of a formal disk. In this paper, we determine when the integrals converge for the the case of cyclic groups of prime power order. As an application, we give a criterion for the quotient variety being canonical or log canonical.

1. Introduction

Let $k$ be an algebraically closed field of characteristic $p > 0$, $G = \mathbb{Z}/p^n\mathbb{Z}$ the cyclic group of order $p^n$, and $V$ a $G$-representation of dimension $d$. We identify $V$ with the $d$-dimensional affine space $\mathbb{A}^d$. Yasuda proved the wild McKay correspondence theorem [9, Corollary 16.3], which shows that the stringy motive $M_{st}(V/G)$ of the quotient variety $V/G$ equals the integral of the form $\int_{G-Cov(D)} L^{d-\sigma}$. Here $G$-Cov(D) denotes the moduli space of $G$-covers of $D = \text{Spec } k[[t]]$ and $\sigma$ the $\sigma$-function associated to given $G$-representation $V$. Yasuda and the author [4] give an explicit formula for the $\sigma$-function, but it is not easy to compute the integrals in general.

The subject of this paper is the convergence of the integrals $\int_{G-Cov(D)} L^{d-\sigma}$. When $n \leq 2$, we already have criteria [8, 4]. We generalize them to general $n$ and then apply it to study singularities of the quotient varieties $V/G$.

The moduli space $G$-Cov(D) can be described by the Artin–Schreier–Witt theory and hence the integral can be written as an infinite series of the form

$$\int_{G-Cov(D)} L^{d-\sigma} = \sum_{f} [G$-$Cov(D; f)]L^{d-\sigma}|_{G-Cov(D; f)},$$

where $G$-Cov(D) $= \bigsqcup G$-Cov(D; f) is a stratification. Moreover, the value $\sigma(E)$ of the $\sigma$-function at $E \in G$-Cov(D; f) can be written in terms of (upper) ramification jumps of the $G$-extension $L/k((t))$ corresponding to $E$. By considering the infinite series above as one of functions with upper ramification jumps as variables, we see that the integral $\int_{G-Cov(D)} L^{d-\sigma}$ converges if and only if some linear function tends to $-\infty$. For an indecomposable $G$-representation $V$ of dimension $d$, we define the following invariants:

$$s_d^{(m)} := \sum_{0 \leq i_0 + i_1 \cdots + i_{n-1} < d, 0 \leq i_0, i_1, \ldots, i_{n-1} < p} \sum_{0 \leq m \leq n - 1} i_m \quad (0 \leq m \leq n - 1),$$

$$D_V^{(m)} := p^{n-1} \left( s_d^{(m)} - (p-1) \sum_{l=0}^{m-1} p^{m-l-1}s_d^{(l)} \right) \quad (0 \leq m \leq n - 1).$$

We generalize them to decomposable ones in the way that they become additive for direct sums.

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Theorem 1.1 (Theorem 4.1). Let $V$ be a $G$-representation of dimension $d$. The integral
\[ \int_{G-Cov(D)} L^{d-\nu} \] converges if and only if the inequalities
\[ 1 - p^{n-m} - \sum_{i=1}^{n-1} \frac{D^{(i)}_V}{p^{2n-1-i}} < 0 \quad (m = 0, 1, \ldots, n-1) \]
hold.

Using the wild McKay correspondence, we can study singularities of the quotient $V/G$; for instance, we have the following simple criterion.

Theorem 1.2 (Theorem 5.1). Assume that $V$ is an effective indecomposable $G$-representation of dimension $d$ which has no pseudo-reflection. Let $X = V/G$ be the quotient variety. The following holds:

1. $X$ is canonical if $d \geq p - 1 + p^n$. Furthermore, if there is a log resolution of $X$, then the converse is also true.

2. $X$ is log canonical if and only if $d \geq p - 1 + p^n$.

We know when given $G$-representation has pseudo-reflections [4] Lemma 4.6. Note that an effective indecomposable $G$-representation of dimension $d$ has no pseudo-reflection if and only if $d > 1 + p^{n-1}$.

The outline of this paper is as follows. In Section 2, we review basic facts for motivic integrals, the moduli space $G$-Cov($D$), the $\nu$-functions, and singularities. We discuss convergence of the integral over connected $G$-covers in Section 3 and then apply it to the integral $\int_{G-Cov(D)} L^{d-\sigma}$. As application of the main theorem, we give some criteria for whether the quotient variety $V/G$ is canonical (resp. log canonical) or not in Sections 5 and 6.

Notation and convention. Unless otherwise noted, we follow the notation below. We denote by $k$ an algebraically closed field of characteristic $p > 0$, and by $K = k((t))$ the field of formal Laurent power series over $k$. We set $G = \langle \sigma \rangle$ a cyclic group of order $p^n$.

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2. Preliminaries

2.1. Motivic integration and stringy motives. To state the wild McKay correspondence theorem, we briefly review motivic integration and define stringy motives.

Let $X$ be a $k$-variety $X$ of dimension $d$. We denote by $J_n X$ the space of $n$-jets and by $J_\infty X$ the space of arcs. The motivic measure $\mu_X$ on $J_\infty X$ takes values in the ring $\tilde{\mathcal{M}}$, which is a version of the completed Grothendieck ring of varieties (see [6] for details). The element of $\tilde{\mathcal{M}}$ defined by $Y$ is denoted by $[Y]$. We write $\mathcal{L} = \{ A^d_k \}$. Let $\pi_n : J_n X \to J_n X$ be the truncation map. We call a subset $C \subset J_n X$ stable if there exists $n \in \mathbb{N}$ such that $\pi_n(C) \subset J_n X$ is constructible, $C = \pi_n^{-1}(\pi_n C)$, and the map $\pi_{m+1}(C) \to \pi_m(C)$ is a piecewise trivial $A^d_k$-bundle for every $m \geq n$. We define the measure $\mu_X(C)$ of a stable subset $C \subset J_n X$ by $\mu_X(C) := [\pi_n(C)]\mathcal{L}^{-nd}$ for $n \gg 0$. For a more general measurable subset, we define its measure as the limit of ones of stable subsets. For a measurable subset $C \subset J_n X$ and a function $F : C \to \mathbb{Z} \cup \{ \infty \}$ such that every fiber is constructible, we define
\[ \int_C L^F := \sum_{m \in \mathbb{Z}} \mu_X(F^{-1}(m))\mathcal{L}^m. \]

We assume that $X$ is normal and its canonical sheaf $\omega_X$ is invertible. We then define the $\omega$-Jacobian ideal $J_X \subset \mathcal{O}_X$ by $J_X \omega_X = \text{Im} \left( \bigwedge^d \Omega_X/k \to \omega_X \right)$ and the stringy motive $M_{st}(X)$ of $X$ by
\[ M_{st}(X) := \int_{J_n X} \mathcal{L}^{-nd} J_X. \]
Here ord denotes the order function associated to an ideal sheaf.

**Remark 2.1.** In our situation where $G = \mathbb{Z}/p^n\mathbb{Z}$ acts on $A^d_k$ linearly, the quotient variety $X = A^d_k/G$ is 1-Gorenstein, that is, the canonical sheaf $\omega_X$ is invertible (see \[1\] Theorem 3.1.8).

### 2.2. The wild McKay correspondence

Yasuda proved the following.

**Theorem 2.2** (the wild McKay correspondence \[9\] Corollary 16.3). Let $G$ be an arbitrary finite group. Assume that $G$ acts on $A^d_k$ linearly and effectively and that $G$ has no pseudo-reflection. Then we have

$$M_{\text{st}}(A^d_k/G) = \int_{G-\text{Cov}(D)} \mathbb{I}^d_{-\nu}.$$

Here $G-\text{Cov}(D)$ denotes the moduli space of $G$-covers of $D = \text{Spec } k[[t]]$, and $\nu$ is the $\nu$-function $\nu: G-\text{Cov}(D) \to \mathbb{Q}$ associated to the $G$-action on $A^d_k$.

By a $G$-cover $E$ of $D$, we mean the normalization of $E$ in an étale $G$-cover $E'$ of $D' = \text{Spec } K$. Note that the $\nu$-function depends on the given $G$-representation. We sometimes write the $\nu$-function as $\nu_V$, referring to the $G$-representation $V$ in question.

Let us consider the case $G = \mathbb{Z}/p^n\mathbb{Z}$, which is of our principal interest.

We can describe the moduli space $G-\text{Cov}(D)$ by using the Artin–Schreier–Witt theory; there is a one-to-one correspondence between $G$-covers $E$ and reduced Witt vectors $f = (f_0, f_1, \ldots, f_{n-1}) (f_m \in k[[t]])$. Here reduced means each $f_m$ is of the form $f_m = \sum_{j=0}^m a_{mj} t^{-j}$ ($a_{mj} \in k$). Moreover, we stratify $G-\text{Cov}(D)$ as follows:

$$G-\text{Cov}(D) = \bigsqcup_j G-\text{Cov}(D; j),$$

$$G-\text{Cov}(D; j) \leftrightarrow \{ f = (f_0, f_1, \ldots, f_{n-1}) \mid \text{ord } f_m = -m (m = 0, 1, \ldots, n-1) \},$$

where $j = (j_0, j_1, \ldots, j_{n-1})$ is an $n$-tuple of positive integers with $p \nmid j_m$ or $-\infty$. Note that we can identify $G-\text{Cov}(D; j)$ with $\prod_{m \neq -\infty} \mathbb{G}_m \times_k A^{m-1-n}/p]$. See \[1\] Section 2 for details.

For the definition of $\nu$-function, see \[7\] Definition 5.4]. In our case $G = \mathbb{Z}/p^n\mathbb{Z}$, we can compute the value $\nu_V(E)$ as follows:

**Theorem 2.3** (\[4\] Theorem 3.3]). Let $E$ be a connected $G$-cover and $L/K$ the corresponding $G$-extension. Assume that the $G$-extension $L/K$ is defined by an equation $\varphi(g) = f$, where $f$ is a reduced Witt vector with $-j_m = \text{ord } f_m$ ($j_m \in \mathbb{N} \cup \{-\infty\}$). Note that $j_0 \neq -\infty$ since $E$ is connected. Put

$$u_i = \max\{ p^{l-1-m} j_m \mid m = 0, 1, \ldots, l-1 \},$$

$$l_i = u_0 + (u_1 - u_0) + \cdots + (u_l - u_{l-1}) p.$$  

Then

$$\nu_V(E) = \sum_{0 \leq b_i + b_l + \cdots + b_{n-1} < d, \ 0 \leq b_0, b_1, \ldots, b_{n-1} < p} \frac{[i_0 p^{n-1} l_0 + i_1 p^{n-2} l_1 + \cdots + i_{n-1} l_{n-1}]}{p^n}.$$  

When $E$ is not connected with a connected component $E'$ and the stabilizer subgroup $G' \subset G$, then we have

$$\nu_V(E) = \nu_V(E'),$$

where $V'$ is the restriction of $V$ to $G'$.

**Remark 2.4.** In the situation of Theorem 2.3, we remark that

1. $u_i$ (resp. $l_i$) are the upper (resp. lower) ramification jumps of the extension $L/K$,
2. the function $\nu_V$ is constant on each stratum $G-\text{Cov}(D; j)$.
2.3. **Singularities.** We can study singularities of quotient varieties via the wild McKay correspondence.

**Proposition 2.5** ([6] Proposition 6.6, [9] Corollary 16.4). Let $X = \mathbb{A}^d_k / G$ be the quotient by a finite group $G$. If the integral $\int_{G\text{-Cov}(D)} \prod d^{-\nu_V}$ converges, then $X$ is canonical. Furthermore, if there is a log resolution of $X$, then the converse is also true.

Furthermore, we can also estimate discrepancies of quotient varieties $X = \mathbb{A}^d_k / G$ by computing the integral $\int_{G\text{-Cov}(D)} \prod d^{-\nu_V}$. See [5] for details (see also [4] Section 4.4) for the case $G = \mathbb{Z}/p^n\mathbb{Z}$.

### 3. Integrals over connected $G$-covers

In this section, we consider integrals over connected $G$-covers, $\int_{G\text{-Cov}^0(D)} \prod d^{-\nu_V}$, where $G\text{-Cov}^0(D) = \bigsqcup_{j \in \mathbb{Z}} G\text{-Cov}(D; j)$ denotes the set of connected $G$-covers of the formal disk $D = \text{Spec} \mathbb{k}[[t]]$. As we see in Theorem 2.3 for a connected $G$-cover $E$, the value $\nu_V(E)$ is determined by the upper ramification jumps of the corresponding $G$-extension $L/K$. By abuse of notation, let us consider $\nu_V$ as a function in variables $u = (u_0, u_1, \ldots, u_n)$. The following is well-known (see, for example, [3] Lemma 3.5).

**Lemma 3.1.** Let $u = (u_0, u_1, \ldots, u_{n-1})$ be an increasing sequence of positive integers. Then $u$ occurs as the set of upper ramification jumps of a $G$-extension of $K$ if and only if the following conditions hold:

1. $p \nmid u_0$, and
2. for $1 \leq i \leq n - 1$, either
   2.a. $u_i = pu_{i-1}$ or
   2.b. both $u_i > pu_{i-1}$ and $p \nmid u_i$.

We denote by $\mathcal{U}$ the set of increasing sequences of positive integers satisfying the conditions of Lemma 3.1. For $u = (u_0, u_1, \ldots, u_{n-1})$, set

$$\mathcal{J}(u) := \{ j = (j_0, j_1, \ldots, j_{n-1}) \mid u_m = \max\{p^{m-1-i}j_i \mid i = 0, 1, \ldots, m - 1\} \}. $$

Then we obtain

$$\int_{G\text{-Cov}^0(D)} \prod d^{-\nu_V} = \sum_{u \in \mathcal{U}} \left( \sum_{j \in \mathcal{J}(u)} [G\text{-Cov}(D; j)] \right) \prod d^{-\nu_V(u)}.$$

In addition, by definition, we have

$$\dim \sum_{j \in \mathcal{J}(j)} [G\text{-Cov}(D; j)] = d + u_0 - \lfloor u_0/p \rfloor + u_1 - \lfloor u_1/p \rfloor + \cdots + u_{n-1} - \lfloor u_{n-1}/p \rfloor.$$

Therefore, it is enough to study the asymptotic behavior of the function

$$\nu_V(u_0, u_1, \ldots, u_{n-1})$$

in variables $u_0, u_1, \ldots, u_{n-1}$.

Let us define some invariants to study the function $\nu_V$.

**Definition 3.2.** For a positive integer $d$ ($d \leq p^n$) and a non-negative integer $m$ ($m \leq n-1$), we define

$$S_d^{(m)} := \sum_{0 \leq l_0 + l_1 + \cdots + l_{m-1} < d} l_m.$$

Namely, $S_d^{(m)}$ is the sum of the $(m + 1)$-th digits of the integers $0, 1, \ldots, d - 1$ in base-$p$ notation. We can write them explicitly.
Lemma 3.3. Let $d = d_0 + d_1 p + \cdots + d_{n-1} p^{n-1}$ (0 \leq d_m < p$ for $m = 0, 1, \ldots, n-2; 0 \leq d_{n-1}$). Then the equality

$$S_d^{(m)} = p^m S_d^{(0)} + \sum_{i=0}^{m-1} p^i d_i d_m \quad (m > 0)$$

holds. In addition, we have

$$S_d^{(0)} = (d_1 + d_2 p + \cdots + d_{n-1} p^{n-2}) \left( \frac{p(p-1)}{2} + \frac{d_0(d_0-1)}{2} \right).$$

Proof. The second equality is obvious. Let $q = d_1 + d_2 p + \cdots + d_{n-1} p^{n-2}$. By definition, we have

$$S_d^{(m)} = \sum_{0 \leq k_0+k_1+\cdots+k_{n-1} < d} \sum_{0 \leq k_0+k_1+\cdots+k_{n-1} < q} \sum_{0 \leq l_0+l_1+\cdots+l_{n-1} < p} p^{i_0} d_{l_0} d_{l_1} \cdots d_{l_{n-1}} = p q^{(m-1)} + d_0 d_m.$$

Hence we obtain the first equality by induction.

Definition 3.4. Let $V$ be an indecomposable $G$-representation of dimension $d$. We define

$$D_V^{(m)} := p^{m-1} \left( S_d^{(m)} - (p-1) \sum_{i=0}^{m-1} p^{m-1-i} S_d^{(1)} \right).$$

For decomposable $G$-representations, we define the invariants $D_V^{(m)}$ in the way that they become additive for direct sums.

Lemma 3.5. For integers $q_m$ and $r_m$ ($m = 0, 1, \ldots, n-1$), we have

$$\nu_V(q_0 p^n + r_0, q_1 p^n + r_1, \ldots, q_{n-1} p^n + r_{n-1}) = \sum_{m=0}^{n-1} D_V^{(m)} q_m + \nu_V(r_0, r_1, \ldots, r_{n-1}).$$

Proof. Since the function $\nu_V$ and the invariants $D_V^{(m)}$ are additive with respect to direct sums of $G$-representations, we may assume that $V$ is indecomposable of dimension $d$.

By direct computing with Theorem 2.3, we obtain

$$\nu_V(q_0 p^n + r_0, q_1 p^n + r_1, \ldots, q_{n-1} p^n + r_{n-1}) = \sum_{0 \leq k_0+k_1+\cdots+k_{n-1} < d} \sum_{0 \leq l_0+l_1+\cdots+l_{n-1} < p} \nu_V(r_0, r_1, \ldots, r_{n-1}).$$

Moreover, we have

$$\sum_{m=0}^{n-1} p^{m-1} q_m (0 + (q_1 - q_0) p + \cdots + (q_m - q_{m-1}) p^m)$$

$$= p^{n-1} q_0 + p^{n-2} q_1 \pm (p-1) q_0 + \pm \cdots + q_{n-1} (-1)^{n-1} (p-1) q_{n-1} + \pm p^{n-2} q_{n-2} + p^{n-1} q_{n-1}$$

$$= p^{n-1} q_0 + \pm p^{n-2} q_1 \pm (p-1) q_0 + \pm \cdots + (p-1) q_{n-1} + \pm p^{n-2} q_{n-2} + p^{n-1} q_{n-1}$$

$$= p^{n-1} q_0 + \pm p^{n-2} q_1 \pm (p-1) q_0 + \pm \cdots + (p-1) q_{n-1} + \pm p^{n-2} q_{n-2} + p^{n-1} q_{n-1}$$

$$= \nu_V(r_0, r_1, \ldots, r_{n-1}).$$
The above equalities together with the definition of $D_V^{(m)}$ show the lemma.

We state the following as a conclusion of this section.

**Theorem 3.6.** Let $V$ be a $G$-representation of dimension $d$ (not necessarily indecomposable). The integral $\int_{G-\text{Cov}^0(D)} \mathbb{L}^{d-\text{poly}}$ on the space $G-\text{Cov}^0(D)$ of the connected $G$-covers converges if and only if the strict inequalities

$$1 - \frac{1}{p^{n-m}} - \sum_{l=m}^{n-1} \frac{D_V^{(l)}}{p^{2n-1-l}} < 0 \quad (m = 0, 1, \ldots, n-1)$$



hold. If the inequalities $\leq 0$ hold, then the integral $\int_{G-\text{Cov}^0(D)} \mathbb{L}^{d-\text{poly}}$ has terms of dimension bounded above.

**Proof.** It is obvious that the integral $\int_{G-\text{Cov}^0(D)} \mathbb{L}^{d-\text{poly}}$ converges if and only if $[1]$ tends to $-\infty$ as the all variables $u_m$ increase.

From Lemma 3.3 we have

$$u_0 - [u_0/p] + u_1 - [u_1/p] + \cdots + u_{n-1} - [u_{n-1}/p] - u_V(u_0, u_1, \ldots, u_{n-1})$$

$$\equiv_{\text{bdd}} u_0 - u_0/p + u_1 - u_1/p + \cdots + u_{n-1} - u_{n-1}/p - \sum_{m=0}^{n-1} D_V^{(m)} u_m/p^n$$

$$= \sum_{m=0}^{n-1} \left( 1 - \frac{1}{p} - \frac{D_V^{(m)}}{p^n} \right) u_m =: f(u),$$

where $\equiv_{\text{bdd}}$ means equivalence modulo bounded functions. What we want to study is the limit of the function $f(u)$. Thus we consider $f(u)$ as a function defined on

$$\mathcal{U} := \{ u = (u_0, u_1, \ldots, u_{n-1}) \in \mathbb{R}^n \mid u_0 \geq 1, u_i \geq pu_{i-1} (i = 1, 2, \ldots, n-1) \}$$

instead of $\mathcal{U}$. For $t \in \mathbb{R}_{\geq 1}$, let $\mathcal{U}_t := \mathcal{U} \cap \{ u_{n-1} = t \}$ be the intersection of the polyhedron $\mathcal{U}$ and the hyperplane $u_{n-1} = t$. Assume $t \geq p^{n-1}$ so that $\mathcal{U}_t$ becomes non-empty. Obviously, $\mathcal{U}_t$ is bounded, that is, it is a polytope. Since $f$ is a linear function, thus the maximum value of $f|_{\mathcal{U}_t}$ is attained at the one of its vertices $(1, p \cdot \ldots \cdot p^{n-2}, t), (1, p \cdot \ldots \cdot p^{n-3}, t/p, t), \ldots, (t/p^{n-1}, \ldots, t/p, t) \in \mathcal{U}_t$.

By substituting, we have

$$f(1, p \cdot \ldots \cdot p^{m-1}, t/p^{n-1-m}, \ldots, t/p, t) \equiv_{\text{bdd}} \sum_{l=m}^{n-1} \left( 1 - \frac{1}{p} - \frac{D_V^{(l)}}{p^n} \right) t$$

$$= \left( 1 - \frac{1}{p^{n-m}} - \sum_{l=m}^{n-1} \frac{D_V^{(l)}}{p^{2n-1-l}} \right) t.$$
hold. If the inequalities \(0 \leq \alpha \leq 1\) hold, then the integral \(\int_{G_{\text{Cov}}(D)} \mathbb{L}^{-d_{\alpha}}\) has terms of dimension bounded above.

**Proof.** We prove by induction on \(n\). The case \(n = 1\) is just [6 Proposition 6.9]. Let \(H = \mathbb{Z}/p^{n-1}\mathbb{Z}\) be the subgroup of \(G\) of index \(p\) and \(W\) the restriction of \(V\) to \(H\). Let us divide the integral as follows:

\[
\int_{G_{\text{Cov}}(D)} \mathbb{L}^{-d_{\alpha}} = \int_{H_{\text{Cov}}(D)} \mathbb{L}^{-d_{\alpha}} + \int_{G_{\text{Cov}}^2(D)} \mathbb{L}^{-d_{\alpha}}.
\]

Note that the necessary and sufficient condition on convergence of \(\int_{H_{\text{Cov}}(D)} \mathbb{L}^{-d_{\alpha}}\) is given by the induction hypothesis, and one of \(\int_{G_{\text{Cov}}^2(D)} \mathbb{L}^{-d_{\alpha}}\) is by Theorem 3.6. From the lemma below, we have

\[
1 - \frac{1}{p^{n-m}} \sum_{l=m}^{n-1} \frac{D_V^{(l)}}{p^{n-1-l}} = 1 - \frac{1}{p^{n-m}} \sum_{l=m}^{n-1} \frac{pD_W^{(l-1)}}{p^{n-1-l}} = 1 - \frac{1}{p^{n-1-(m-1)}} \sum_{l=m-1}^{n-2} \frac{D_W^{(l)}}{p^{2(n-1)-l}}
\]

for \(m = 1, 2, \ldots, n-1\). Therefore, the convergence of the integral \(\int_{H_{\text{Cov}}(D)} \mathbb{L}^{-d_{\alpha}}\) implies that of the integral \(\int_{G_{\text{Cov}}^2(D)} \mathbb{L}^{-d_{\alpha}}\), and hence the proof is completed. \(\square\)

**Lemma 4.2.** In the situation of Theorem 4.1 for \(m = 1, 2, \ldots, n-1\), we have \(D_V^{(m)} = pD_W^{(m-1)}\).

**Proof.** Since the invariants \(D_V^{(m)}\) are additive with respect to direct sum, thus we may assume that \(V\) is indecomposable. Moreover, by [4 Lemma 4.5], we obtain

\[
W \simeq W_{q+1} \oplus W_q^p - r,
\]

where \(d = r + gp\) (\(0 \leq r < p\)) and \(W_q\) denotes the indecomposable \(G\)-representation of dimension \(e\). Put \(d = d_0 + d_1p + \cdots + d_{n-1}p^{n-1}\) (\(0 \leq d_m < p\) for \(m = 0, 1, \ldots, n-2\)). Note that \(r = d_0\) and \(q = d_1 + d_2p + \cdots + d_{n-1}p^{n-2}\). By definition, we have

\[
S_{q+1}^{(m-1)} = \sum_{0 \leq i_0 + \cdots + i_{n-1}p^{n-2}}^{i_m-1} \sum_{0 \leq k_0, k_1, \ldots, k_{n-2}} \frac{1}{p^{n-1}} < p, < q + 1,
\]

and hence

\[
rS_{q+1}^{(m-1)} + (p-r)S_q^{(m-1)} = pS_q^{(m-1)} + rd_m.
\]

Therefore, combining [4], we obtain

\[
S_d^{(m)} = rS_{q+1}^{(m-1)} + (p-r)S_q^{(m-1)}.
\]

Now the lemma follows from the definition of \(D_V^{(m)}\). \(\square\)

**Corollary 4.3.** Let \(X := V/G\) be the quotient variety.
(1) $X$ is canonical if the strict inequalities $1 - 1/p^{n-m} - \sum_{l=m}^{n-1} D^{(l)}_V/p^l < 0$ hold. Furthermore, if there is a log resolution of $X$, then the converse is also true.

(2) $X$ is log canonical if and only if the inequalities $1 - 1/p^{n-m} - \sum_{l=m}^{n-1} D^{(l)}_V/p^l \leq 0$ hold.

Proof. (1). If the strict inequalities hold, then the integral $\int_{G_{\text{Conv}(D)}} L^{d-\psi}$ and hence the stringy motive $M_d(X)$ converges. From [8] Proposition 6.6], we obtain the claim.

(2). Holding the inequalities is equivalent to the integral $\int_{G_{\text{Conv}(D)}} L^{d-\psi}$ has terms of dimensions bounded above. Hence, from [9] Corollary 16.4 (1), we obtain desired conclusion.

5. INDECOMPOSABLE CASES

We give more precise estimation for the indecomposable cases.

**Theorem 5.1.** Assume that $V$ is an indecomposable $G$-representation of dimension $d$ which has no pseudo-reflection. Let $X := V/G$ be the quotient variety.

(1) $X$ is canonical if $d \geq p + p^{n-1}$. Furthermore, if there is a log resolution of $X$, then the converse is also true.

(2) $X$ is log canonical if and only if $d \geq p - 1 + p^{n-1}$.

**Lemma 5.2.** We consider the invariants $D^{(l)}_V$ as functions in variable $d$. Then the sum $\sum_{l=m}^{n-1} D^{(l)}_V/p^l$ is strictly increasing.

Proof. By definition, we have

$$
\sum_{l=m}^{n-1} p^l D^{(l)}_V = \sum_{l=m}^{n-1} p^l \left( - (p-1) \sum_{j=m}^{l-1} p^{l-j} S^{(j)}_d + S^{(l)}_d \right)
$$

$$
= p^{n-1} (p^m S^{(m)}_d + (- (p-1) p^{m-(m+1)} + p^{m+1}) S^{(m+1)}_d + \cdots + (- (p-1) p^{m-(n-1)} + p^{(m+1)-(n-1)} + \cdots + p^{(n-2)-(n-1)} + p^{n-1}) S^{(n-1)}_d )
$$

$$
= p^{n-1} \sum_{l=m}^{n-1} \left( - (p-1) \sum_{j=m}^{l-1} p^{l-j} + p^l \right) S^{(l)}_d
$$

$$
= p^{n-1} \sum_{l=m}^{n-1} (p^{m-l} - 1 + p^l) S^{(l)}_d
$$

$$
\geq p^{n-1} \left( p^{n-1} S^{(n-1)}_d \right) = p^{n-1} D^{(n-1)}_V.
$$

Since $D^{(n-1)}_V = p^{n-1} S^{(n-1)}_d$ is strictly increasing with respect to $d$, thus we get the desired conclusion.

**Remark 5.3.** From the proof above, $\sum_{l=m}^{n-1} p^l D^{(l)}_V$ is monotone decreasing with respect to $m$. Since $D^{(l)}_V$ is non-negative, thus $D^{(l)}_V$ are all non-negative.

Proof of Theorem 5.1. The cases $n = 1$ and $n = 2$ are proved in [8] and [4] respectively. We assume that $n \geq 3$. It is enough to show that the inequalities in Corollary 4.3 hold.

(2). We consider the case $d = p - 1 + p^{n-1}$. By direct computation, we obtain

$$
S^{(0)}_d = \frac{p^{n-1}(p-1)}{2} + \frac{(p-1)(p-2)}{2},
$$

$$
S^{(m)}_d = \begin{cases} p^{n-1}(p-1)/2 & \text{if } 0 < m < n-1, \\ p-1 & \text{if } m = n-1, \end{cases}
$$
and hence
\[ D_V^{(n-1)} = p^{n-1}(p-1), \]
\[ D_V^{(n-2)} = p^{n-1}(p-1) \left( \frac{p^{n-1}}{2} - \frac{p-1}{p} \right). \]

For \( m = n-1 \), we have
\[ 1 - \frac{1}{p^{m-1}} = \frac{D_V^{(n-1)}}{p^{n-1}} = 1 - \frac{p-1}{p} = 0. \]

Since \( D_V^{(m)} \) are strictly increasing with respect to \( d \), from Corollary 4.3 (2), thus \( X \) is not log canonical when \( d < p-1 + p^{n-1} \).

On the other hand, for \( m = 0, 1, \ldots, n-2 \), we have
\[
1 - \frac{1}{p^{m-1}} = \sum_{l=m}^{n-1} \frac{D_V^{(l)}}{p^{2m-1-l}} < 1 - \sum_{l=m}^{n-1} \frac{D_V^{(l)}}{p^{2m-1-l}} \leq 1 - \frac{D_V^{(n-2)}}{p^{m+1}} \frac{D_V^{(n)}}{p^n}.
\]

Thus we obtain
\[
1 - \left( \frac{D_V^{(n-2)}}{p^{m+1}} + \frac{D_V^{(n-2)}}{p^n} \right) = 1 - \frac{p-1}{p} \left( \frac{p^{n-1}}{2} - \frac{p-1}{p} \right) = \frac{2 - p(p-1)(p^{n-1} - 4)}{2p^3}.
\]

Therefore, we see that the inequalities in Corollary 4.3 (2) hold when \( d \geq p-1 + p^{n-1} \). Hence the quotient \( X = V/G \) is log canonical except when \( (p, n) = (2, 3) \). We remark that if \( (p, n) = (2, 3) \), the \( G \)-representation \( V \) has pseudo-reflections.

(1) We consider the case \( d = p + p^{n-1} \). Similarly we have
\[
1 - \frac{1}{p^{m-1}} = \sum_{l=m}^{n-1} \frac{D_V^{(l)}}{p^{2m-1-l}} < 1 - \sum_{l=m}^{n-1} \frac{D_V^{(l)}}{p^{2m-1-l}} \leq 1 - \frac{D_V^{(n-1)}}{p^{m}}.
\]

By direct computing, we obtain
\[ S_d^{(n-1)} = p, D_V^{(n-1)} = p^n, \]

and hence
\[
1 - \frac{1}{p^{m-1}} = \sum_{l=m}^{n-1} \frac{D_V^{(l)}}{p^{2m-1-l}} < 1 - \frac{p^n}{p^n} = 0.
\]

Therefore, then the quotient \( X = V/G \) is canonical if \( d \geq p + p^{n-1} \).

6. Finite groups whose \( p \)-Sylow subgroup is cyclic

We can slightly generalize our main result as follows.

**Theorem 6.1.** Let \( \tilde{G} \) be a finite group whose \( p \)-Sylow subgroup is \( G \cong \mathbb{Z}/p^n \mathbb{Z} \), and \( \tilde{V} \) a \( \tilde{G} \)-representation. Assume that the restriction \( V \) of \( \tilde{V} \) to \( G \) has no pseudo-reflection. Let \( \tilde{X} := \tilde{V}/\tilde{G} \) and \( X := V/G \) be the quotient varieties.

(1) \( \tilde{X} \) is log terminal if the inequalities \( 1 - 1/p^{m-1} - \sum_{l=m}^{n-1} D_V^{(l)}/p^{2m-1-l} < 0 \) \( (m = 0, 1, \ldots, n-1) \) hold.

(2) \( \tilde{X} \) is log canonical if and only if the inequalities \( 1 - 1/p^{m-1} - \sum_{l=m}^{n-1} D_V^{(l)}/p^{2m-1-l} \leq 0 \) \( (m = 0, 1, \ldots, n-1) \) hold.
Proof. Let $\pi : X \to \tilde{X}$ be the canonical projection.

(1). If the inequalities hold, then $X$ is canonical. From [5] Theorem 6.5, we see that $\tilde{X}$ is log terminal.

(2). $X$ is log canonical if and only if the inequalities hold. If $\tilde{X}$ is log canonical, from the contraposition of [5] Theorem 6.4], $X$ is log canonical. Conversely, with similar proof as in [5] Theorem 6.5], we see that if $X$ is log canonical, then $\tilde{X}$ is log canonical. □

Remark 6.2. There are only finitely many indecomposable $G$-representations up to isomorphism. Moreover, an explicit formula for the number of non-isomorphic ones is given in [2].

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