Investigation of cracked rotor analytically and numerically

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Abstract. The aim of this paper is analyzing the cracked Jeffcott with an offset disk rotor system. Mathematical modeling of a Jeffcott rotor with a crack while the depths of crack changes allows analyzing the dynamic behavior of the rotor. The effect of changing speeds has been investigated on the dynamic parameters of journal bearings (stiffness and damping) and then on Sommerfeld number. The relation of the dynamic response with the static response of rotor with cracks (dynamic load factor DLF) and with the ratio of depth crack to radius rotor has been found. The results have obtained analytically and numerically by using MATLAB and ANSYS, respectively. The results show when crack’s depths increase the DLF increases and reach to maximum when a/R = 0.8.

Keywords: Rotor Dynamics, Cracked Rotor, Dynamic load factor, DLF.

Introduction
In the last years, the dynamics of rotors become very important in the design of rotary machines, because of the high cost of maintenance of these machines based on vibration. The rotor is the most important part in rotary machinery, which gives the main function of the machine, because of its widely used in industry and power plants so this work focuses on the study of an uncracked and cracked rotor.

There are many excitation forces sources may be encountered in the rotary machines. The mass eccentricity, which produces unbalance force, is the most known excitation source in the rotor bearing systems. The unbalance excitation is asynchronous harmonic excitation, where its frequency coincides with the rotation speed of the rotor.

To find the critical speed, the harmonic response due to unbalance mass can be calculated at disk location where the maximum response occurs and therefore the critical speed will be the rotor speed at the maximum response displacement [1].

The critical speed of the rotor supported on the fluid film journal bearings cannot be defined as in the case of a rigid bearing rotor, because the stiffness coefficients values are depending on the rotor speed. The ability to recognize the crack at early stages of growth is imperative for reducing maintenance cost and time. Early crack detection makes the operator think for repair with no need of prematurely take the machine for an extended period of operation [2]. Prevention to occur cracks it is very important in design. The structure of any device is treated either chemically by adding some elements or physical (heat treatment) to make the structure more stiffness and resist cracking, corrosion and other defects [3-7]. Researchers were interested especially, in resistance to cracks in different fields, including surface microcracks in the rolling of thick sheet [8] and studied the possibility of strengthening factors resistant to cracks in immersion lenses [9].
In this paper, the DLF has been found by dividing the dynamic maximum displacement response on the static response for analytical and numerical methods. The aim of a standard rotor dynamics analysis and design checking is to help to characterize the transverse dynamic design characteristics but analysis of some rotary equipment may need analysis specific to the unit [10-14], a general method has used for performing the standard lateral analysis of vibration by using FEA with selected BEAM188 element for shaft and COMBI214 element for bearings in the ANSYS [15].

**Fundamental Equations**

**Fluid Film Bearings.** The rotor supported by a fluid film which doing countless things more than the friction losses. In Fig. 1, the bearing center C and the journal center \( \bar{C} \) will form an attitude of the bearing and makes the angle with the vertical load (W), the clearance \( h \) will be extended between two values. Table 1 shows the dimension of the selected model and properties of the shaft’s material (AISI4140).

![Figure 1. Bearings representation by rotor springs and dampers and sliding bearing: eccentricity between shaft and bearing.](image)

| Description                              | Dimensions of the selected model       |
|------------------------------------------|----------------------------------------|
| Total shaft length (m)                   | 0.654                                  |
| Shaft diameter (m)                       | 0.048                                  |
| Disk diameter (m)                        | 0.34                                   |
| Distances between disk and bearings (m)  | L1 = 0.24, L2 = 0.414                   |
| Crack depths (m)                         | 0.005, 0.01, 0.015, 0.020              |
| Distances between the left side of disk and crack (m) | 0.01                                  |
| Disk thickness (m)                       | 0.02                                   |
| Total rotor mass (Kg)                    | 23.25                                  |
| Young Modulus (E) (N/m2)                 | 2.05×1011                              |
| Poisson's Ratio (\( \bar{\sigma} \))     | 0.29                                   |
| Density (ρ) (Kg/m3)                      | 7850                                   |

From the bearing geometry, speed, eccentricity, pressure and attitude angle, the Sommerfeld parameter derived to give an indication about the bearing eccentricity as [16]:

\[
S = \frac{(\mu DLN)}{W} \left( \frac{r}{C_i} \right)^2
\]

The radial and tangential forces \( F_r, F_t \) are:
The force $F_t$ opposes the sliding motion and the power lost is $F_t \cdot \Omega D/2$. The resultant force on the bearing is opposite to the applied load on the rotor:

$$F = \sqrt{F_r^2 + F_i^2} = \frac{\pi D \mu \Omega \omega^3}{8h^2(1-\varepsilon^2)^{3/2}} \left( \frac{16}{\pi^2} - 1 \right) \varepsilon + 1/2.$$

If the load on the bearing is known, then the modified Sommerfeld number, given by [17]:

$$S_x = \frac{D \mu \Omega \omega^3}{4Fh^2}.$$

The vertical resultant force is known, and then the bearing eccentricity may be obtained by rearranging Eq. 3 to give Eq. 5, where $S_x$ from Eq. 4, called modified Sommerfeld number is known for a speed, load, and oil viscosity, [18]:

$$\varepsilon^8 - 4\varepsilon^6 + (6 - S_x^2(16 - \pi^2))\varepsilon^4 - (4 + \pi^2 S_x^2)\varepsilon^2 + 1 = 0.$$

The values of eccentricity ratio $\varepsilon$ is $\tilde{C} - C/h$ always taken between 0-1 so the value of $\varepsilon$ has been found by Newton-Raphson Method from zero to 6000 RPM using MATLAB. Then find the relation between modified Sommerfeld number and all of the stiffness and damping with rotation speed. The displacement ought to be small because a linear analysis does not include any limitations on the displacement. We considered short journal bearings, so the matrices are $2 \times 2$ for stiffness and damping matrices we found as, [16]:

$$K = \frac{F}{h} \begin{bmatrix} a_{xx} & a_{xy} \\ a_{yx} & a_{yy} \end{bmatrix} = \begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix},$$

$$c = \frac{F}{h\times\Omega} \begin{bmatrix} b_{xx} & b_{xy} \\ b_{yx} & b_{yy} \end{bmatrix} = \begin{bmatrix} c_{xx} & c_{xy} \\ c_{yx} & c_{yy} \end{bmatrix}.$$

All parameters in Eq. 6, Eq. 7, were taken from [1, 16].

Response of Rotor Supported by Two Journal Bearings. There are two known coordinates, fixed coordinates, and rotating coordinates, and there are two types of stiffness depending on these types of coordinates system. For breathing crack case (opening and closing), it assumed coordinates lies on the rotor and rotates with it. After that, the reduction in stiffness could be gating in both $\zeta$ and $\eta$ directions and transfer the stiffness matrix to the fixed coordinates with joining system inertia to stiffness for finding the equation of motion of fixed $(x, y)$ coordinates Fig. 2. The stiffness matrix for rotating coordinates ($\zeta, \eta$) for the uncracked shaft is $[K]$. The reduction of stiffness due to crack is $[K_c(\theta)]$, where $\theta$ is the angle between rotor response and crack axis. The stiffness of cracked rotor is:

$$[K] - [K_c(\theta)] = [K_{cr}].$$

The stiffness matrix of the cracked rotor with fixed coordinates after using the transformation matrix to convert it from rotating coordinates is [19]:

$$[K_{cr}] = [K] - [K_c(\theta, t)].$$
Then, the equation could be written as:

\[
[M]\{\dot{y}\} + [D] + [G]\{\dot{y}\} + [K - \{K_c(\theta)\}]\{y\} = Q_u
\] (10)

where \(Q_u\) is the out of balance force, the gyroscopic and damping have been considered in the skew-symmetric matrix \([G]\) and in symmetric positive semi-definite matrix \([D]\).

When the damping matrix \([D]\) in the rotor is axisymmetric then will produce a skew symmetric contribution to uncracked matrix \([K]\).

**Figure 2.** Cross-section of the cracked area and Breathing crack model.

Equation (10) is nonlinear because \([K_c(t)]\) is nonlinear and changes with the time \(t\). That means, it depends on speed and crack depth, [19].

The flexibilities have calculated for crack depths \(a/R\). In case of small crack \((a/R < 0.5)\), the flexibility \(\bar{C}_{44}\) is much less than \(\bar{C}_{55}\), so for this reason, the cross flexibility neglected. This gives advantages for representation of crack flexibility by one parameter \(\bar{C}_{55}\) only, so it is easy to get the analytical results.

To get the deflection at a location near the disc in the case of open crack one can write [10]:

\[
\begin{bmatrix}
\epsilon \\
\eta
\end{bmatrix} =
\begin{bmatrix}
G_o + \Delta g_\epsilon & 0 \\
0 & G_o + \Delta g_\eta
\end{bmatrix}
\begin{bmatrix}
F_\epsilon \\
F_\eta
\end{bmatrix}
\] (11)

Where \(G_o\) is the flexibility of the uncracked shaft, \(\Delta g_\epsilon, \Delta g_\eta\) are the additional flexibilities due to crack. The rotation of the disc taking the shaft with crack to the compression zone. The crack is going to close because of the weight, therefore \(\Delta g_\epsilon, \Delta g_\eta\) will be zero, and then the shaft is rounding again. This behavior is shown in Fig. 2.

The stiffness matrix \([K(t)]\) dependents on two-fold time. The first refers to transformation from rotor fixed to the inertial reference system, while the second refers to steering function \(f(t)\) which takes two values 1 or 0 depends on the position of a crack in compression or tension zone. But \(f(t)\) also depends on the shaft deflection (x), then the equation of motion for a shaft with breathing crack could be written as [10]:

\[
\begin{bmatrix}
M & 0 \\
0 & M
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} +
\begin{bmatrix}
c & 0 \\
0 & c
\end{bmatrix}
\begin{bmatrix}
\dot{\epsilon} \\
\dot{\eta}
\end{bmatrix} +
\begin{bmatrix}
K_{11} & K_{12} \\
K_{21} & K_{22}
\end{bmatrix}
\begin{bmatrix}
\epsilon \\
\eta
\end{bmatrix} =
\begin{bmatrix}
mg \\
0
\end{bmatrix} + m\Omega^2
\begin{bmatrix}
\cos(\beta + \Omega t) \\
\sin(\beta + \Omega t)
\end{bmatrix}
\] (12)

The out of balance response (unbalance response) can be written as follows:

\[
r^* = \frac{x + iy}{u}
\] (13)
Static Amplitude Response of Rotor. The calculations of static response have been done at the critical speed values for the uncracked model and all cases of cracks like single transverse crack with many depths, for static loading, the system excitation unbalance force vector \( Q \) is constant, the static response could be obtained from the stiffness matrix and force vector as:

\[
\begin{bmatrix}
Q_1 \\
Q_2
\end{bmatrix} =
\begin{bmatrix}
K_1 & K_{12} \\
K_{21} & K_2
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}.
\]

Then

\[
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
K_1 & K_{12} \\
K_{21} & K_2
\end{bmatrix}^{-1}
\begin{bmatrix}
Q_1 \\
Q_2
\end{bmatrix}.
\]

The stiffness of the uncracked rotor shaft is [20]:

\[
K = \frac{3EI(L_1^3 + L_2^3)}{L_1^3 \times L_2^3}.
\]

For harmonic motion, equation becomes. As got the two equations of motion:

\[
m\ddot{x} + K_1 x + K_{12} y = m_n \Omega^2 \cos \Omega t,
\]

\[
m\ddot{y} + K_2 y + K_{21} x = m_n \Omega^2 \sin \Omega t.
\]

where

\[
K_1 = \frac{K[(K_{xx} + i\Omega C_{xx})(K_{yy} + K + i\Omega C_{yy}) - (K_{xy} + i\Omega C_{xy})(K_{yx} + i\Omega C_{yx})]}{(K_{xx} + K + i\Omega C_{xx})(K_{yy} + K + i\Omega C_{yy}) - (K_{xy} + i\Omega C_{xy})(K_{yx} + i\Omega C_{yx})},
\]

\[
K_2 = \frac{K[(K_{yy} + i\Omega C_{yy})(K_{xx} + K + i\Omega C_{xx}) - (K_{xy} + i\Omega C_{xy})(K_{yx} + i\Omega C_{yx})]}{(K_{xx} + K + i\Omega C_{xx})(K_{yy} + K + i\Omega C_{yy}) - (K_{xy} + i\Omega C_{xy})(K_{yx} + i\Omega C_{yx})},
\]

\[
K_{12} = \frac{K^2(K_{xy} + i\Omega C_{xy})}{(K_{xx} + K + i\Omega C_{xx})(K_{yy} + K + i\Omega C_{yy}) - (K_{xy} + i\Omega C_{xy})(K_{yx} + i\Omega C_{yx})},
\]

\[
K_{21} = \frac{K^2(K_{yx} + i\Omega C_{yx})}{(K_{xx} + K + i\Omega C_{xx})(K_{yy} + K + i\Omega C_{yy}) - (K_{xy} + i\Omega C_{xy})(K_{yx} + i\Omega C_{yx})}.
\]

K1, K2, K12 and K21 are depend on the stiffness of rotor and bearings. The equivalent mass is;

\[
m = \frac{17}{35} \times m_s + m_d,
\]

where

\[
Q_1 = m_n(e)\Omega^2, W = m \times g, \Omega = \frac{2\pi N}{60}, Q_2 = m_n(e)\Omega^2 + W
\]

\[
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
K_1 & K_{12} \\
K_{21} & K_2
\end{bmatrix}^{-1}
\begin{bmatrix}
m_n(e)\Omega^2 \\
m_n(e)\Omega^2 + W
\end{bmatrix}.
\]

The net static response as an orbit will be as:
\[ r_s = \sqrt{x^2 + y^2} \],

The Dynamic Load Factor:

\[ DLF = \frac{r^*}{r_s} \].

**Modeling and Design Data Input by ANSYS**

The modeling features for the rotor and bearing support flexibility are described in this paper and show how element BEAM188, COMBI214 are used to model the shaft with bearings and MASS21 for disk to model the masses. The stiffness and damping of both bearings with cross-coupling directions and stiffness with damping of the shaft have been considered in this analysis including their variations with the changing of the rotational speed of the rotor to get accurate results [15].

**Results**

The DLF has been found by dividing the dynamic maximum displacement response on the static response for analytical and numerical methods. The response displacement DLF increases when the cracks depths ratios increase as shown in Fig. 3. The results have been compared and listed in Table 2. The percentage of error has been found for DLF, which has found from analytical and numerical, the results were in good agreement with a maximum percentage of error 2.3 %.

![Figure 3. Dynamic load factor (DLF) versus crack depths ratio of a cracked rotor.](image)

| a/R | Analytical Max.Stat. disp. (mm) | Analytical Max.Dyn. disp. (mm) | Analytical Max.Dyn. response (mm) | Analytical Crit.Speed (RPM) | Numerical Max.Stat. disp. (mm) | Numerical Max.Dyn. disp. (mm) | Numerical Max.Dyn. response (mm) | Numerical Crit.Speed (RPM) | DLF Error % |
|-----|--------------------------------|--------------------------------|----------------------------------|-----------------------------|-------------------------------|-------------------------------|-------------------------------|-----------------------------|--------------|
| 0   | 0.05243                        | 0.05155                        | 0.0409                           | 0.0396                      | 6400                          | 6150                          | 0.780                         | 0.7681                      | 0.780        |
| 0.2 | 0.05474                        | 0.052801                       | 0.0430                           | 0.0406                      | 6200                          | 6050                          | 0.785                         | 0.7689                      | 0.785        |
| 0.4 | 0.05706                        | 0.05352                        | 0.0464                           | 0.0426                      | 6060                          | 5800                          | 0.813                         | 0.7959                      | 0.7959       |
| 0.6 | 0.05806                        | 0.05785                        | 0.0543                           | 0.0529                      | 5800                          | 5600                          | 0.935                         | 0.9144                      | 0.9144       |
| 0.8 | 0.06433                        | 0.06217                        | 0.0620                           | 0.0585                      | 5300                          | 5400                          | 0.963                         | 0.9409                      | 0.9409       |

Table 2 Analytical and Numerical Response DLF of a cracked rotor.
Summary
The Dynamic Load Factor (DLF) is a very important engineering concept, it in a simple manner the ratio of maximum dynamic response to the static displacement that will be produced by steady force with the same value as the peak magnitude of the acting force. The factor could be extended to response displacement ratio, as a principle to engineering judgment in the early stages of a design, is regarded as an important value [21-23]. In this paper, the cracked Jeffcott with an offset disk rotor system has been considered. The dynamic behavior of the rotor with the help of mathematical modeling of a Jeffcott rotor with a crack while the depths of crack changes has been done. The DLF has been found for analytical and numerical methods. The response displacement DLF increases when the cracks depths ratios increase.

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