SYMMETRIC INVARIANTS RELATED TO REPRESENTATIONS OF EXCEPTIONAL SIMPLE GROUPS

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To our teacher Ernest B. Vinberg on occasion of his 80th birthday

ABSTRACT. We classify the finite-dimensional rational representations \( V \) of the exceptional algebraic groups \( G \) with \( \mathfrak{g} = \text{Lie} G \) such that the symmetric invariants of the semi-direct product \( \mathfrak{g} \ltimes V \), where \( V \) is an Abelian ideal, form a polynomial ring.

INTRODUCTION

The ground field \( k \) is algebraically closed and of characteristic 0. In 1976, Vinberg et al. classified the irreducible representations of simple algebraic groups with polynomial rings of invariants [KPV76]. Such representations are sometimes called coregular. The most important class of coregular representations of reductive groups is provided by the \( \theta \)-groups introduced and studied in depth by Vinberg; see [Vin76]. Since then the classification of coregular representations of semi-simple groups has attracted much attention. The reducible coregular representations of simple groups have been classified independently by Schwarz [Sch78] and Adamovich–Golovina [AG79], while the irreducible coregular representations of semi-simple groups are classified by Littelmann [Lit89].

For several decades, only rings of invariants of representations of reductive groups were considered. However, invariants of non-reductive groups are also very important in Representation Theory. Let \( S \) be an algebraic group with \( \mathfrak{s} = \text{Lie} S \). The invariants of \( S \) in the symmetric algebra \( S(\mathfrak{s}) \) of \( \mathfrak{s} \) (= symmetric invariants of \( \mathfrak{s} \) or \( S \)) help us to understand the coadjoint action \( (S : \mathfrak{s}^*) \) and in particular, coadjoint orbits, as well as representation theory of \( S \). Several classes of non-reductive groups \( S \) such that \( S(\mathfrak{s})^S \) is a polynomial ring have been found recently; see e.g. [Jos07, PPY07, Pan07b, Yak]. A quest for this type of groups continues. Hopefully, one can find interesting properties of \( S \) and its representations under the assumption that the ring \( S(\mathfrak{s})^S \) is polynomial.

A natural class of non-reductive groups, which is still tractable, is given by a semi-direct product construction; see [2] for details. In [Yak], the following problem has been proposed: to classify all representations \( V \) of simple algebraic groups \( G \) such that the ring of symmetric invariants of the semi-direct product \( \mathfrak{q} = \mathfrak{g} \ltimes V \) is polynomial (in other words, the coadjoint representation of \( \mathfrak{q} \) is coregular). It is easily seen that if \( \mathfrak{q} \) has this property, then \( k[V^*]^G \) is a polynomial ring, too. Therefore, the suitable representations \( (G, V) \) have to be extracted from the lists of [Sch78, AG79]. Some natural representations of \( G = \text{SL}_n \) are studied in [Yak]. Those considerations imply that the \( \text{SL}_n \)-case is very difficult. For this reason, we take here the other end and classify such representations.
(\(G, V\)) for the exceptional algebraic groups \(G\). In a forthcoming article, we provide such a classification for the representations of the orthogonal and symplectic groups. To a great extent, our classification results rely on the theory developed by the second author in [Yak17].

**Notation.** Let \(S\) act on an irreducible affine variety \(X\). Then \(k[X]^S\) is the algebra of \(S\)-invariant regular functions on \(X\) and \(k(X)^S\) is the field of \(S\)-invariant rational functions. If \(k[X]^S\) is finitely generated, then \(X/S := \text{Spec} k[X]^S\). Whenever \(k[X]^S\) is a graded polynomial ring, the elements of any set of algebraically independent homogeneous generators will be referred to as basic invariants. If \(V\) is an \(S\)-module and \(v \in V\), then \(s_v = \{\zeta \in s \mid \zeta \cdot v = 0\}\) is the stabiliser of \(v\) in \(s\) and \(S_v = \{s \in S \mid s \cdot v = v\}\) is the isotropy group of \(v\) in \(S\).

In explicit examples of \([\mathfrak{s}]\) and in Table 1 we identify the representations \(V\) of semisimple groups with their highest weights, using the multiplicative notation and the Vinberg–Onishchik numbering of the fundamental weights [VO88]. For instance, if \(\varpi_1, \ldots, \varpi_n\) are the fundamental weights of a simple algebraic group \(G\), then \(V = \varpi_1^2 + 2\varpi_j\) stands for the direct sum of three simple \(G\)-modules, with highest weights \(2\varpi_1\) (once) and \(\varpi_j\) (twice). If \(H \subset G\) is semi-simple and we are describing the restriction of \(V\) to \(H\) (i.e., \(V|_H\)), then the fundamental weights of \(H\) are denoted by \(\varpi_i\). Write ‘1’ for the trivial one-dimensional representation.

1. Preliminaries on the coadjoint representations

Let \(S\) be an affine algebraic group with Lie algebra \(\mathfrak{s}\). The symmetric algebra \(S(\mathfrak{s})\) over \(k\) is identified with the graded algebra of polynomial functions on \(\mathfrak{s}^*\) and we also write \(k[\mathfrak{s}^*]\) for it. The index of \(\mathfrak{s}\), \(\text{ind}\mathfrak{s}\), is the minimal codimension of \(S\)-orbits in \(\mathfrak{s}^*\). Equivalently, \(\text{ind}\mathfrak{s} = \min_{\xi \in \mathfrak{s}^*} \dim \mathfrak{s}_\xi\). By Rosenlicht’s theorem [VP89, 2.3], one also has \(\text{ind}\mathfrak{s} = \text{tr. deg } k(\mathfrak{s}^*)^S\). The “magic number” associated with \(\mathfrak{s}\) is \(b(\mathfrak{s}) = (\dim \mathfrak{s} + \text{ind}\mathfrak{s})/2\).

Since the coadjoint orbits are even-dimensional, the magic number is an integer. If \(\mathfrak{s}\) is reductive, then \(\text{ind}\mathfrak{s} = \text{rk } \mathfrak{s}\) and \(b(\mathfrak{s})\) equals the dimension of a Borel subalgebra. The Poisson bracket \{ , \} in \(k[\mathfrak{s}^*]\) is defined on the elements of degree 1 (i.e., on \(\mathfrak{s}\)) by \([x, y] := [x, y]\). The centre of the Poisson algebra \(S(\mathfrak{s})\) is \(S(\mathfrak{s})^0 = \{F \in S(\mathfrak{s}) \mid \{F, x\} = 0 \forall x \in \mathfrak{s}\}\. If \(S^0\) is the identity component of \(S\), then \(S(\mathfrak{s})^0 = S(\mathfrak{s})^{S^0}\). The set of \(S\)-regular elements of \(\mathfrak{s}^*\) is \(\mathfrak{s}^*_\text{reg} = \{\eta \in \mathfrak{s}^* \mid \dim S \cdot \eta \geq \dim S \cdot \eta'\text{ for all }\eta' \in \mathfrak{s}^*\}\). We say that \(\mathfrak{s}\) has the codim-2 property if \(\text{codim}(\mathfrak{s}^* \setminus \mathfrak{s}^*_{\text{reg}}) \geq 2\). The following useful result appears in [Pan07b, Theorem 1.2]:

\(\blacklozenge\) Suppose that \(S\) is connected, \(\mathfrak{s}\) has the codim-2 property, and there are homogeneous algebraically independent \(f_1, \ldots, f_l \in k[\mathfrak{s}^*]^S\) such that \(l = \text{ind}\mathfrak{s}\) and \(\sum_{i=1}^l \deg f_i = b(\mathfrak{s})\). Then \(k[\mathfrak{s}^*]^S = k[f_1, \ldots, f_l]\) and \((df_1)_\xi, \ldots, (df_l)_\xi\) are linearly independent if and only if \(\xi \in \mathfrak{s}^*_{\text{reg}}\).

More generally, one can define the set of \(S\)-regular elements for any \(S\)-action on an irreducible variety \(X\); that is, \(X_{\text{reg}} = \{x \in X \mid \dim S \cdot x \geq \dim S \cdot x'\text{ for all }x' \in X\}\).

We say that the action \((S : X)\) has a generic stabiliser, if there exists a dense open subset \(\Omega \subset X\) such that all stabilisers \(s_x, x \in \Omega\), are \(S\)-conjugate. Then any subalgebra \(\mathfrak{s}_x, x \in \Omega\), is called a generic stabiliser (= g.i.g.). The points of \(\Omega\) are said to be \(S\)-generic (or, just group if the context is clear from the context). Likewise, one defines a generic isotropy group (= g.i.g.), which is a subgroup of \(S\). By [Ric72, §4], \((S : X)\) has a generic stabiliser if and only if it has a generic isotropy group. It is also shown therein that g.i.g. always exists if \(S\) is reductive and \(X\) is smooth. If \(H\) is a generic isotropy group for \((S : X)\) and \(\mathfrak{h} = \text{Lie } H\), then we write \(H = \text{g.i.g.}(S : X)\) and \(\mathfrak{h} = \text{g.s.}(S : X)\).
A systematic treatment of generic stabilisers in the context of reductive group actions can be found in [VPS89, §7]. Note that if a generic stabiliser for \( (S : X) \) exists, then any \( S \)-generic point is \( S \)-regular, but not vice versa.

Recall that \( f \in \mathbb{k}[X] \) is called a semi-invariant of \( S \) if \( S \cdot f \subseteq \mathbb{k}f \). A semi-invariant \( f \) is proper if \( f \not\in \mathbb{k}[X]^S \).

2. ON THE COADJOINT REPRESENTATIONS OF SEMI-DIRECT PRODUCTS

In this section, we gather some results on the coadjoint representation that are specific for semi-direct products. In particular, we recall the necessary theory from [Yak17].

Let \( G \subset GL(V) \) be a connected algebraic group with \( g = \text{Lie} G \). The vector space \( g \oplus V \) has a natural structure of Lie algebra, the semi-direct product of \( g \) and \( V \). Explicitly, if \( x, x' \in g \) and \( v, v' \in V \), then
\[
[(x, v), (x', v')] = ([x, x'], x \cdot v' - x' \cdot v).
\]
This Lie algebra is denoted by \( s = g \ltimes V \), and \( V \simeq \{(0, v) \mid v \in V \} \) is an abelian ideal of \( s \). The corresponding connected algebraic group \( S \) is the semi-direct product of \( G \) and the commutative unipotent group \( \exp(V) \simeq V \). The group \( S \) can be identified with \( G \times V \), the product being given by
\[
(s, v)(s', v') = (ss', (s')^{-1} \cdot v + v'), \quad \text{where } s, s' \in G.
\]
In particular, \( (s, v)^{-1} = (s^{-1}, -s \cdot v) \). Then \( \exp(V) \) can be identified with
\[
1 \ltimes V := \{(1, v) \mid v \in V \} \subseteq G \ltimes V.
\]
If \( G \) is reductive, then the subgroup \( 1 \ltimes V \) is the unipotent radical of \( S \), also denoted by \( R_u(S) \).

There is a general formula for the index of \( s \simeq g \ltimes V \), which is due to M. Raïs [Rai78]. Namely, there is a dense open subset \( \Omega \subset V^* \) such that
\[
\text{ind } s = \text{tr. deg } \mathbb{k}(V^*)^G + \text{ind } g_\xi
\]
for any \( \xi \in \Omega \). In particular, if a generic stabiliser for \( (G : V^*) \) exists, then one can take \( g_\xi \) to be a generic stabiliser.

Remark 2.1. There are some useful observations related to the symmetric invariants of the semi-direct product \( s = g \ltimes V \):

(i) The decomposition \( s^* = g^* \oplus V^* \) yields a bi-grading of \( \mathbb{k}[s]^S \) [Pan07b, Theorem 2.3(i)]. If \( H \) is a bi-homogenous \( S \)-invariant, then \( \text{deg}_g H \) and \( \text{deg}_V H \) stand for the corresponding degrees.

(ii) The algebra \( \mathbb{k}[V^*]^G \) is contained in \( \mathbb{k}[s]^S \). Moreover, a minimal generating system for \( \mathbb{k}[V^*]^G \) is a part of a minimal generating system of \( \mathbb{k}[s]^S \) [Pan07b, Sect. 2(A)]. Therefore, if \( \mathbb{k}[s]^S \) is a polynomial ring, then so is \( \mathbb{k}[V^*]^G \).

Proposition 2.2 (Prop. 3.11 in [Yak17]). Let \( G \) be a connected algebraic group acting on a finite-dimensional vector space \( V \). Suppose that \( G \) has no proper semi-invariants in \( \mathbb{k}[s^1]^G \) and \( \mathbb{k}[s^1]^S \) is a polynomial ring in \( \text{ind } s \) variables. For generic \( \xi \in V^* \), we then have

- the restriction map \( \psi : \mathbb{k}[s^1]^S \to \mathbb{k}[g^* \times \{\xi\}]^{G_\xi \ltimes V} \simeq S(g_\xi)^{G_\xi} \) is surjective;
- \( S(g_\xi)^{G_\xi} \) coincides with \( S(g_\xi)^{G_\xi} \);
- \( S(g_\xi)^{G_\xi} \) is a polynomial ring in \( \text{ind } g_\xi \) variables.
Note that $G$ is not assumed to be reductive and $G_ξ$ is not assumed to be connected in the above proposition! We mention also that there are isomorphisms $k[\xi^* \times \{ξ\}]^{G_ξ \times V} \simeq k[\mathfrak{g}_ξ]^{G_ξ}$ for any $ξ \in V^*$; see [Yak17] Lemma 2.5.

From now on, $G$ is supposed to be reductive. The action $(G : V)$ is said to be stable if the union of closed $G$-orbits is dense in $V$. Then a generic stabiliser $\mathfrak{g.s.}(G : V)$ is necessarily reductive.

Consider the following assumptions on $G$ and $V$:

$(\Diamond)$ The action of $(G : V^*)$ is stable, $k[V^*]^G$ is a polynomial ring, $k[\mathfrak{g}_ξ^*]^{G_ξ}$ is a polynomial ring for generic $ξ \in V^*$, and $G$ has no proper semi-invariants in $k[V^*]$.

The following result of the second author was excluded from the final text of [Yak17].

**Theorem 2.3.** Suppose that $G$ and $V$ satisfy condition $(\Diamond)$ and $V^*/G = \mathbb{A}^1$, i.e., $k[V^*]^G = k[F]$ for some homogeneous $F$. Let $H$ be a generic isotropy group for $(G : V^*)$ and $\mathfrak{h} = \text{Lie } H$. Assume further that $D = \{x \in V^* | F(x) = 0\}$ contains an open $G$-orbit, say $G \cdot y$, $\text{ind}_y \mathfrak{g}_y = \text{ind}_y \mathfrak{h}_y =: ℓ$, and $S(\mathfrak{g}_y)^{G_y}$ is a polynomial ring in $ℓ$ variables with the same degrees of generators as $S(\mathfrak{h})^H$. Then $k[\mathfrak{s}^*]^{S}$ is a polynomial ring in $\text{ind } \mathfrak{s} = ℓ + 1$ variables.

**Proof.** If $ℓ = 0$, then $k[\mathfrak{s}^*]^{S} = k[F]$ and we have nothing to do. Assume that $ℓ \geq 1$. Let $\{\mathcal{H}_i | 1 \leq i \leq ℓ\}$ be bi-homogeneous $S$-invariants chosen as in [Yak17] Lemma 3.5(i)]. Assume that $\deg_\mathfrak{g} \mathcal{H}_i \leq \deg_\mathfrak{h} \mathcal{H}_i$ if $i < j$. We will show that these polynomials can be modified in such a way that the new set satisfies the conditions of [Yak17] Lemma 3.5(ii)] and therefore freely generates $k[\mathfrak{s}^*]^{S}$ over $k[F]$.

Notice that $F$ is an irreducible polynomial, because $(\Diamond)$ includes also the absence of proper semi-invariants. Thereby $k[D]^G = k$ and a non-trivial relation over $k[D]^G$ among $\mathcal{H}_i = H_i|_{\mathfrak{g}^* \times D}$ gives also a non-trivial relation among $\mathfrak{h}_i = H_i|_{\mathfrak{g}^* \times \{y\}}$. Recall that $\mathfrak{h}_i \in S(\mathfrak{g}_y)^{G_y}$ by [Yak17] Lemma 2.5]. Assume that $\mathfrak{h}_1, \ldots, \mathfrak{h}_j$ are algebraically independent if $j = d$ and dependent for $j = d + 1$. Then $\mathfrak{H}_d$ is not among the generators of $S(\mathfrak{g}_y)^{G_y}$ and it can be expressed as a polynomial $R(h_1, \ldots, H_d)$. Then also

$$\widetilde{H}_{d+1} - R(\mathfrak{H}_1, \ldots, \mathfrak{H}_d) = 0$$

and we can replace $\mathcal{H}_{d+1}$ by the bi-homogeneous part of $\frac{1}{F}(\mathcal{H}_{d+1} - R(\mathcal{H}_1, \ldots, \mathcal{H}_d))$ of bi-degree $(\deg_\mathfrak{g} \mathcal{H}_{d+1}, \deg_\mathfrak{h} \mathcal{H}_{d+1} - \deg F)$. Clearly, $\sum_i \deg \mathcal{H}_i$ is decreasing and therefore the process will end up at some stage and bring a new set $\{\mathcal{H}_i\}$ satisfying the conditions of [Yak17] Lemma 3.5(ii)]].

**Remark 2.4.** Although Theorem 2.3 asserts that $k[\mathfrak{s}^*]^{S}$ is a polynomial ring under certain conditions, it does not say anything about the (bi-)degrees of the generators $\mathcal{H}_i$. Finding these degrees is not an easy task. Let us say a few words about it.

Suppose that $G$ is semi-simple, $k[\mathfrak{s}^*]^{S}$ is a polynomial ring, and $\mathfrak{s}$ has the codim-2 property. As is mentioned above,

$$\text{ind } \mathfrak{s} = \text{tr. } \deg k(V^*)^G + \text{ind } \mathfrak{g}_ξ$$

with $ξ \in V^*$ generic. Here

$$\text{tr. } \deg k(V^*)^G = \dim V^*/G$$

and

$$k[\mathfrak{s}^*]^{S} = k[\mathcal{H}_1, \ldots, \mathcal{H}_ℓ, F_1, \ldots, F_r],$$

where $ℓ = \text{ind } \mathfrak{g}_ξ$, $r = \dim V^*/G$, and all generators are bi-homogeneous. The generators $F_j$ are elements of $S(V)$. The $\mathfrak{g}$-degrees of the polynomials $\mathcal{H}_i$ are the degrees of basic
invariants in $S(g_{\ell})^{G_{\ell}}$. Furthermore,

$$\sum_{i=1}^{\ell} \deg V H_i + \sum_{j=1}^{r} \deg F_j = \dim V;$$

see [Yak, Section 2] for a detailed explanation. Thus, the only open problem is how to determine the $V$-degrees of the $H_i$. In particular, the problem simplifies considerably, if $\ell$ is small.

3. The classification and table

In this section, $G$ is an exceptional algebraic group, i.e., $G$ is a simple algebraic group of one of the types $E_6, E_7, E_8, F_4, G_2$. We classify the (finite-dimensional rational) representations $(G : V)$ such that the symmetric invariants of $s = g \ltimes V$ form a polynomial ring. This will be referred to as property (FA) for $s$. We also say that $s$ (or just the action $(G : V)$) is good (resp. bad), if (FA) is (resp. is not) satisfied for $s$.

To distinguish exceptional groups and Lie algebras, we write, say, $E_7$ for the group and $E_7$ for the respective algebra; while the corresponding Dynkin type is referred to as $E_7$.

**Example 3.1.** If $G$ is arbitrary semi-simple, then $g \ltimes g$ always has (FA) [Tak71]. Therefore we exclude the adjoint representations from our further consideration.

If $k[g^*]^S$ is a polynomial ring, then so is $k[V^*]^G$ (Remark 2.1(ii)). For this reason, we have to examine all representations of $G$ with polynomial rings of invariants. If $(G : V)$ is a representation of an exceptional algebraic group such that $V \neq g$ and $k[V]^G$ is a polynomial ring, then $V$ or $V^*$ is contained in Table 1. This can be extracted from the classifications in [AG79] or [Sch78]. Furthermore, the algebras $k[V]^G$ and $k[V^*]^G$ (as well as $S(g \ltimes V)^G_{\ltimes V}$ and $S(g \ltimes V^*)^{G_{\ltimes V^*}}$) are isomorphic, so that it is enough to keep track of only $V$ or $V^*$. As in [AG79, Ela72, Yak17], we use the Vinberg–Onishchik numbering of fundamental weights $\varpi_i$. Here $H$ is a generic isotropy group for $(G : V^*)$ and column (FA) refers to the presence of property that $k[g^*]^S$ is a polynomial ring, where $s = g \ltimes V$. We write $q(V/\!/G)$ for the sum of degrees of the basic invariants in $k[V^*]^G$.

| No | $G$ | $V$ | $\dim V$ | $\dim V^*/\!/G$ | $q(V/\!/G)$ | $H$ | ind $s$ (FA) | Ref. |
|----|-----|-----|---------|-----------------|-------------|-----|-------------|------|
| 1a | $G_2$ | $\varpi_1$ | 7 | 1 | 2 | $A_2$ | 3 | + | Eq. 3.1 |
| 1b | $2\varpi_1$ | 14 | 3 | 6 | $A_1$ | 4 | + | [Yak17, Ex. 4.8] |
| 1c | $3\varpi_1$ | 21 | 7 | 15 | $\{1\}$ | 7 | + | Example 3.3 |
| 2a | $F_4$ | $\varpi_1$ | 26 | 2 | 5 | $D_4$ | 6 | - | Example 3.4 |
| 2b | $2\varpi_1$ | 52 | 8 | 22 | $A_2$ | 10 | + | Eq. 3.1 |
| 3a | $E_6$ | $\varpi_1$ | 27 | 1 | 3 | $F_4$ | 5 | + | Example 3.2 |
| 3b | $\varpi_1 + \varpi_5$ | 54 | 4 | 12 | $D_4$ | 8 | - | Example 3.3 |
| 3c | $2\varpi_1$ | 54 | 4 | 12 | $D_4$ | 8 | - | Example 3.3 |
| 3d | $3\varpi_1$ | 81 | 11 | 36 | $A_2$ | 13 | + | Theorem 3.11 |
| 3e | $2\varpi_1 + \varpi_5$ | 81 | 11 | 36 | $A_2$ | 13 | + | Theorem 3.11 |
| 4a | $E_7$ | $\varpi_1$ | 56 | 1 | 4 | $E_6$ | 7 | - | Theorem 3.12 |
| 4b | $2\varpi_1$ | 112 | 7 | 28 | $D_4$ | 11 | - | Example 3.8 |
In Table 4, the group $H$ is always reductive. Since $G$ is semi-simple, this implies that the action $(G : V^*)$ is stable in all cases; see [VP89] Theorem 7.15. The fact that $G$ is semi-simple means also that $G$, as well as $g \ltimes V$, has only trivial characters and therefore has no proper semi-invariants.

We provide below necessary explanations.

**Example 3.2.** Consider item 3a in the table. Here $V^*/G = \mathbb{A}^1$, i.e., $k[V^*]^G = k[F]$ for some homogeneous $F$. The divisor $D = \{ \xi \in V^* \mid F(\xi) = 0 \}$ contains a dense $G$-orbit, say $G \cdot \eta$, whose stabiliser is the semi-direct product $g_\eta = s_0 \ltimes \varpi_4$. A generic isotropy group for $(G : V^*)$ is the exceptional group $\mathbf{F}_4$ and $g_\eta$ is a $\mathbb{Z}_2$-contraction of $\mathcal{F}_4$. By [Pan07b, Theorem 4.7], $S(g_\eta)^{G*}$ is a polynomial ring whose degrees of basic invariants are the same as those for $\mathcal{F}_4$. Therefore, using Theorem 2.3, we obtain that $k[s^*]^S$ is a polynomial ring.

**Example 3.3.** If $0 = h$, then $k[s^*]^S \simeq k[V^*]^G$ [Pan07a, Theorem 6.4]; cf. [Yak17, Example 3.1]. Therefore, item 1c is a good case.

**Example 3.4.** The semi-direct product in 2a is a $\mathbb{Z}_2$-contraction of $\mathcal{E}_6$. This is one of the four bad $\mathbb{Z}_2$-contractions of simple Lie algebras, i.e., $k[s^*]^S$ is not a polynomial ring here; see [Yak17, Section 6.1].

If $G$ is semi-simple, $V$ is a reducible $G$-module, say $V = V_1 \oplus V_2$, then there is a trick that allows us to relate the polynomiality problem for the symmetric invariants of $s = g \ltimes V$ to a smaller semi-direct product. The precise statement is as follows.

**Proposition 3.5.** With $s = g \ltimes (V_1 \oplus V_2)$ as above, let $H$ be a generic isotropy group for $(G : V_1^*)$ and $h = \text{Lie } H$. If $k[s^*]^S$ is a polynomial ring, then so is $k[q^*]^Q$, where $\tilde{q} = h \ltimes (V_2|_H)$.

**Proof.** Consider the (non-reductive) semi-direct product $q = g \ltimes V_2$. Then $s = q \ltimes V_1$. It is assumed that the unipotent radical of $Q$, $1 \ltimes V_2$, acts trivially on $V_1$. If $\xi \in V_1^*$ is generic for $(G : V_1^*)$, then it is also generic for $(Q : V_1^*)$. If $G_\xi = H$, then the corresponding isotropy group in $Q$ is $Q_\xi = H \ltimes V_2$ and $q_\xi = h \ltimes V_2 \simeq \tilde{q}$, where $V_2$ is considered as $H$-module. Since $G$ is semi-simple, all hypotheses of Proposition 2.2 are satisfied for $Q$ in place of $G$ and $V = V_1$. Therefore, $S(q_\xi)^{Q_\xi} \simeq S(q_\xi)^{q_\xi}$ is a polynomial ring. □

**Remark 3.6.** It is easily seen that the passage from $(G : V = V_1 \oplus V_2)$ to $(H : V_2)$, where $H = g.i.g.(G : V_1)$, preserves generic isotropy groups. For generic stabilizers, this appears already in [Ela72, §3].

One can use Proposition 3.5 as a tool for proving that $k[s^*]^S$ is not a polynomial ring.

**Example 3.7.** In case 3b, we take $Q = E_6 \ltimes \varpi_1$ and $V_1 = \varpi_5$. Then $s = (E_6 \ltimes \varpi_1) \ltimes \varpi_5 \simeq E_6 \ltimes (\varpi_1 + \varpi_5)$. If $\xi \in V_1^* = \varpi_1$ is generic, then $q_\xi \simeq \mathcal{F}_4$ and $\varpi_1|_{\mathcal{F}_4} = \varpi_1 + 1$. Therefore, $q_\xi$ is isomorphic to the semi-direct product related to item 2a, modulo a one-dimensional centre. Therefore, $S(q_\xi)^{Q_\xi}$ is not a polynomial ring (see Example 3.4), and we conclude, using Proposition 3.5, that $k[s^*]^S$ is not a polynomial ring, too.

In case 3c, we take the same $Q$ and $V_1 = \varpi_1$. The rest is more or less the same, since $\mathcal{F}_4 \ltimes (\varpi_1|_{\mathcal{F}_4})$ is again a generic isotropy subgroup for $(Q : V_1^*)$.

**Example 3.8.** In case 4b, we take $Q = E_7 \ltimes \varpi_1$ and $V_1 = \varpi_1$. Note that $\varpi_1$ is a symplectic $E_7$-module. If $\xi \in V_1^*$ is generic, then the corresponding stabiliser in $E_7$ is isomorphic to $E_6$ [Har71, Ela72]. Thereby $q_\xi = E_6 \ltimes (\varpi_1|_{E_6}) = E_6 \ltimes (\varpi_1 + \varpi_5 + 21)$. Therefore, $q_\xi$ represents item 3b (modulo a two-dimensional centre) and we have already demonstrated in the previous example that here $k[q_\xi^*]^{Q_\xi}$ is not a polynomial ring!
The output of Examples 3.7 and 3.8 is that there is the tree of reductions to a “root”
bad semi-direct product:
\[(E_6, 2\varpi_1)\]

\[(E_7, 2\varpi_1) \rightarrow (E_6, \varpi_1 + \varpi_5) \rightarrow (F_4, \varpi_1),\]

and therefore all these items represent bad semi-direct products. Using Proposition 3.5
in a similar fashion, one obtains another tree of reductions:

(3.1) \[(E_6, 2\varpi_1 + \varpi_5)\]

\[(E_6, 3\varpi_1) \rightarrow (F_4, 2\varpi_1) \rightarrow (D_4, \varpi_1 + \varpi_3 + \varpi_4) \rightarrow (G_2, \varpi_1).\]

Some details for the passage from \(E_6\) to \(so_8\) and then to \(G_2\) are given below in the proof
of Theorem 3.11. Note that the representations occurring in (3.1) have one and the same
generic isotropy group, namely \(SL_3\). As we will shortly see, tree (3.1) actually consists of
good cases. Here our strategy is to prove that both “crown” \(E_6\)-cases are good. To this
end, we need some properties of the representation \((G_2, \varpi_1)\) related to the “root” case.

**Lemma 3.9.** Let \(v_1\) be a highest weight vector in the \(G_2\)-module \(\varpi_1\) and \(Q := (G_2)_{v_1}\)
the respective isotropy group. Then (i) \(q = \text{Lie } Q\) has the codim-2 property and (ii) the
coadjoint representation of \(Q\) has a polynomial ring of invariants whose degrees of basic
invariants are 2, 3.

**Proof.** A generic isotropy group for \((G_2 : \varpi_1)\) is connected and isomorphic to \(SL_3\) [Har71]
and \(q\) is a contraction of \(sl_2\) (see [VGO90] Ch. 7, §2] for Lie algebra contractions). We
also have \(q = l \times n\), where \(l = sl_2\) and the nilpotent radical \(n\) is a 5-dimensional \(Z\)-graded
non-abelian Lie algebra of the form
\[n(1) \oplus n(2) \oplus n(3) = k^2 \oplus k \oplus k^2.\]

Here \(k^2\) and \(k^2\) are standard \(sl_2\)-modules and \(k\) is the trivial \(sl_2\)-module. Let \(\{a_1, b_1\}\) be
a basis for \(k^2\), \(\{a_2, b_2\}\) a basis for \(k^2\), and \(\{u\}\) a basis for \(k\). Without loss of generality,
we may assume that \([a_1, b_1] = u, [a_1, u] = a_2, \text{ and } [b_1, u] = b_2\).

(i) Since \(q\) is a contraction of \(sl_3\) and \(\text{ind } sl_3 = 2\), we have \(\text{ind } q \geq 2\). On the other hand,
if \(0 \neq \xi \in n(3)^* \subset n^* \subset q^*, \) then \(\dim q^* = 2\). Hence \(\text{ind } q = 2\) and \(n(3)^* \setminus \{0\} \subset q^*_\text{reg}\). Since
\(\dim n(3) = 2\), the last property readily implies that \(q^* \setminus q^*_\text{reg}\) cannot contain divisors.

(ii) It is easily seen that
\[h_1 = 2a_1b_2 - 2b_1a_2 + u^2 \in S(n) \subset S(q)\]
is a \(q\)-invariant. There is also another invariant of degree three. Let \(\{e, h, f\}\) be a standard
basis of \(sl_2\) (i.e., \([e, f] = h, [h, e] = 2e, [h, f] = -2f\) ). We assume that \([e, a_1] = 0\) and
\([f, a_1] = b_1\), which implies \([h, a_1] = a_1\) and \([h, b_1] = -b_1\). Then
\[h_2 = b_2^2e + a_2b_2h - a_2f + u(a_1b_2 - a_2b_1) + \frac{1}{3}u^3\]
is an \(sl_2\)-invariant and in addition, the following Poisson bracket can be computed:
\[\{a_1, h_2\} = a_2b_2(-a_1) - a_2^2(-b_1) + a_2(a_1b_2 - a_2b_1) - ua_2u + u^2a_2 = 0.\]

Since \(l = sl_2\) and \(a_1\) generate \(q\) as Lie algebra, \(h_2\) is also a \(q\)-invariant. The polynomials
\(h_1\) and \(h_2\) are algebraically independent, because \(h_1 \in S(n)\) and \(h_2 \notin S(n)\). By (i), \(q\) has
the \textit{codim}-2 property. Since \( \dim q = 8 \), \( \ind q = 2 \), and \( b(q) = 5 \), we have \( \deg h_1 + \deg h_2 = b(q) \). Therefore, \( h_1 \) and \( h_2 \) freely generate \( S(q)^G \); see (\( \bullet \)) in \S1.

\begin{remark}
In this case, \((G_2)_\nu\) is a so-called \textit{truncated parabolic subgroup}. Symmetric invariants of truncated (bi-)parabolics were intensively studied by Fauquant-Millet and Joseph; see e.g., \cite{PMJ08, Jos07}. Let \( \tau_{tr} \) be a truncated (bi-)parabolic in type A or C. Then \( S(\tau_{tr})^{\nu} \) is a polynomial ring in \( \ind \tau_{tr} \) homogeneous generators and the sum of their degrees is equal to \( b(\tau_{tr}) \). The same properties hold for many truncated (bi-)parabolics in other types \cite{Jos07}. It is very probable that a sufficient condition of \cite{Jos07} is satisfied for \((92)_\nu\). However, we prefer to keep the explicit construction of generators.
\end{remark}

\begin{theorem}
If \( s = E_6 \times (3 \varpi_1) \) or \( s = E_6 \times (2 \varpi_1 + \varpi_5) \), then \( k[s^*]^{S} \) is a polynomial ring.
\end{theorem}

\begin{proof}
Here \( G = E_6 \), \( q = E_6 \), and \( V = 3 \varpi_1 \) or \( 2 \varpi_1 + \varpi_5 \). Accordingly, \( V^* = 3 \varpi_5 \) or \( 2 \varpi_5 + \varpi_1 \). In both cases, \( \ind s = 13 \), \( V^*/G \simeq S_1 \), and a generic isotropy group for \((G : V^*) = SL_3 \). By \cite[Theorem 2.8 and Lemma 3.5(i)]{Yak17}, there are bi-homogeneous irreducible polynomials \( F_1, F_2 \in k[s^*]^S \) such that \( \deg F_1 = 2 \), \( \deg F_2 = 3 \), and their restrictions to \( g^* \times \{ \xi \} \), where \( \xi \in V^* \) is \( G \)-generic, are the basic invariants of \( SL_3 \). Here, for \( g^* \times \{ \xi \} \subset s^* \), we use the isomorphism

\begin{equation}
\tag{3.2}
k[g^* \times \{ \xi \}]^{G \times V} \simeq S(g)^{G \xi};
\end{equation}

see \cite[Lemma 2.5]{Yak17}. By \cite[Lemma 3.5(ii)]{Yak17}, \( F_1 \) and \( F_2 \) generate \( k[s^*]^S \) over \( k[V^*]^G \) if and only if the restrictions \( F_1|_{g^* \times D} \) and \( F_2|_{g^* \times D} \) are algebraically independent for any \( G \)-stable homogeneous divisor \( D \subset V^* \). Let us prove that this is really the case.

- If there is a non-trivial relation for the restrictions of \( F_1 \) and \( F_2 \) to \( g^* \times D \), then this also yields a non-trivial relation for the subsequent restrictions of \( F_1 \) and \( F_2 \) to \( g^* \times \{ \eta \} \), where \( \eta \) is \( G \)-generic in \( D \).

- If \( G \eta \simeq SL_3 \), i.e., \( \eta \) is also \( G \)-generic in \( V^* \), then \( F_1|_{g^* \times \{ \eta \}} \) and \( F_2|_{g^* \times \{ \eta \}} \) are non-zero elements of \( S(sl_3)^{SL_3} \) of degrees 2 and 3, respectively. The invariants of degrees 2 and 3 in \( S(sl_3)^{SL_3} \) are uniquely determined, up to a scalar factor, and they are algebraically independent. Hence \( F_1|_{g^* \times D} \) and \( F_2|_{g^* \times D} \) are algebraic independent for such divisors.

- However, it can happen that a divisor \( D \) contains no “globally” \( G \)-generic points. To circumvent this difficulty, consider three projections from \( V^* \) to its simple constituents, and their (non-trivial!) restrictions to \( D \):

\begin{equation}
\begin{array}{c}
d \\
\vdots \\
p_1 \quad p_2 \quad p_3 \\
\varpi_5 \quad \varpi_5 \quad \varpi_1 \text{ or } \varpi_5.
\end{array}
\end{equation}

For \( \eta = y_1 + y_2 + y_3 \in D \subset V^* \), we have \( p_i(\eta) = y_i \). Since \( D \) is a divisor, at least two of the \( p_i \)'s are dominant. Without loss of generality, we may assume that \( p_1 \) and \( p_2 \) are dominant, whereas \( p_3 \) is not and then \( p_3(D) \) is a divisor in \( p_3(V^*) \). (For, if all \( p_i \)'s are dominant, then \( D \) contains a globally generic point.) Therefore, we can take \( y_1, y_2 \) to be generic in the \( p_i(V^*) \). Then \( y_1 + y_2 \) is \( G \)-generic in \( 2 \varpi_5 \), \( G_{y_1+y_2} = Spin_8 \), and

\begin{equation}
\varpi_1|_{Spin_8} \simeq \varpi_5|_{Spin_8} \simeq \varpi_1 + \varpi_3 + \varpi_4 + 31.
\end{equation}

Here \( G_\eta = (Spin_8)_{y_3} \). Having obtained a \( Spin_8 \)-stable divisor \( p_3(D) \) in the \( Spin_8 \)-module \( \varpi_1 + \varpi_3 + \varpi_4 + 31 \), we consider \( \tilde{V} := \varpi_1 + \varpi_3 + \varpi_4 + 31 \) and \( \tilde{D} := p_3(D) \cap \tilde{V} \). If \( \tilde{D} = \tilde{V} \), then \( (Spin_8)_{y_3} = SL_3 \) for some \( y_3 \), i.e., again \( G_\eta = SL_3 \).

- If \( \tilde{D} \) is a divisor in \( \tilde{V} \), then we can play the same game with \( Spin_8 \) and \( \tilde{D} \). Let \( y_3 = x_1 + x_3 + x_4 \in \tilde{D} \), where \( x_i \in \varpi_i \). Again, at least two of the projections \( \tilde{p}_i : \tilde{V} \to \varpi_i \).
(i = 1, 3, 4) are dominant. Without loss of generality, we may assume that \( \tilde{p}_1 \) and \( \tilde{p}_3 \) are dominant and then \( x_1 \) and \( x_3 \) are generic elements. Then \( (\text{Spin}_8)_{x_1 + x_3} \cong G_2 \), \( (G_2)_{x_4} = (\text{Spin}_8)_{y_1}, \) and \( \tilde{\omega}_1 \mid G_2 \cong \tilde{\omega}_1 + 1 \). The structure of \( G_2 \)-orbits in \( \tilde{\omega}_1 \) shows that either \( x_4 \) is \( G_2 \)-generic and then \( (G_2)_{x_4} \cong \text{SL}_3 \) or \( x_4 \) is a highest weight vector and \( (G_2)_{x_4} \cong (G_2)_{v_1} \); cf. Lemma 3.9.

Thus, for any \( G \)-stable divisor \( D \subset V^* \), there is \( \eta \in D \) such that \( G_\eta = \text{SL}_3 \) or \( (G_2)_{v_1} \), and in both cases \( S(g_\eta)G_n \) is generated by algebraically independent invariants of degrees 2 and 3 (see Lemma 3.9 for the latter). It follows that \( F_1 \mid g^* \times D \) and \( F_2 \mid g^* \times D \) are algebraically independent, and we are done. \( \square \)

Combining Proposition 3.5, Theorem 3.11, and tree (3.1), we conclude that cases 3d, 3e, 2b, and 1a are good. (Note also that we have found one good case related to a representation of the classical Lie algebra \( D_4 \).) Thus, it remains to handle only the semi-direct product \( E_7 \ltimes \varpi_1 \) (case 4a).

**Theorem 3.12.** If \( s = E_7 \ltimes \varpi_1 \), then \( k[s]^S \) is not a polynomial ring.

**Proof.** Here \( G = E_7, g = \varpi_7, \) and \( k[V^*]^G = k[F] \). By [Har71], a generic isotropy group for \( (G : V^*) \) is connected and isomorphic to \( E_6 \), and the null-cone \( D = \{ \xi \in V^* \mid F(\xi) = 0 \} \) contains a dense \( G \)-orbit, say \( G \cdot y \), whose isotropy group \( G_y \) is connected and isomorphic to \( F_4 \ltimes \varpi_1 \).

Assume that \( k[s]^S \) is polynomial. By [Yak17, Theorem 2.8 and Lemma 3.5(i)], since \( \text{ind } s = 7 \) and \( g \) isomorphic to \( E_6 \), there are bi-homogeneous polynomials \( H_1, \ldots, H_6 \) such that \( k[s]^S = k[F, H_1, \ldots, H_6] \) and the \( H_i \mid g^* \times \{ \xi \} 's yield the basic invariants of \( E_6 \) for a generic \( \xi \). Here we again use Proposition 2.2 and isomorphism 3.2. By [Yak17, Lemma 3.5(ii)], the restrictions \( H_i \mid g^* \times D, i = 1, \ldots, 6 \) remain algebraically independent. On the other hand, let us consider further restrictions \( H_i \mid g^* \times \{ y' \}, i = 1, \ldots, 6, \) where \( y' \) belongs to the dense \( G \)-orbit in \( D \). Recall that \( g_{y'} \cong F_4 \ltimes \varpi_1 \) and the latter is a bad \( Z_2 \)-contraction of \( E_6 \); see Example 3.4. Moreover, the algebra of symmetric invariants of \( F_4 \times \varpi_1 \) does not have algebraically independent invariants whose degrees are the same as the degrees of basic invariants of \( E_6 \) [Yak17, Section 6.1]. This implies that \( H_i \mid g^* \times \{ y' \}, i = 1, \ldots, 6 \) must be algebraically dependent for any \( y' \in G \cdot y \).

Let \( \mathcal{L}(H_1 \mid g^* \times \{ y' \}), \ldots, H_6 \mid g^* \times \{ y' \} = 0 \) be a polynomial relation for some \( y' \). Since the \( H_i 's \) are \( G \)-invariant, the relation with the same coefficients holds for all \( y' \in G \cdot y \). Hence, this dependence can be lifted to \( g^* \times G \cdot y \) and then carried over to \( g^* \times D \). This contradiction shows that the ring \( k[s]^S \) cannot be polynomial. \( \square \)

Summarising, we obtain the main result of the article below.

**Theorem 3.13.** Let \( G \) be an exceptional algebraic group, \( V \) a (finite-dimensional rational) \( G \)-module, and \( s = g \ltimes V \). Then \( k[s]^S \) is a polynomial ring if and only if one of the following two conditions is satisfied: (i) \( V = g \); (ii) \( V \) or \( V^* \) represents one of the seven good cases in Table 1.

Moreover, by Raïs’ formula (see §2), one has

\[ \text{tr. deg } k[s]^S = \text{ind } s = \dim V^*/G + \text{rk } H. \]

In the good cases, we neither construct generators nor give their degrees. The reason is that the main results of [Yak17] as well as Theorem 2.3 are purely existence theorems. Yet, as explained in Remark 2.4, a great deal of information on the degrees is available. If \( \ell \) is small, then using ad hoc methods one can determine all the degrees. Let us see how this can be done, for example, with \( \ell = 2 \). Take item 1a. Since \( h = s_{13} \), we have \( k[s]^S = k[V^*]^G[H_1, H_2] \) with \( \deg H_1 = 2, \deg H_2 = 3 \). Set \( a_i = \deg_{V^*} H_i \). It can easily be seen that \( s \) has the codim-2 property. The discussion in Remark 2.4 shows that
\[a_1 + a_2 = 5.\] Following the same strategy as in [Yak17, Prop. 3.10], one can produce an $S$-invariant of bi-degree \((2, 2)\). This implies that \(a_1 = 2\) and \(a_2 = 3\).

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