Universality of the Bottleneck Distance for Extended Persistence Diagrams

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Abstract
The extended persistence diagram is an invariant of piecewise linear functions, introduced by Cohen-Steiner, Edelsbrunner, and Harer. The bottleneck distance has been introduced by the same authors as an extended pseudometric on the set of extended persistence diagrams, which is stable under perturbations of functions. We address the question whether the bottleneck distance is the largest possible stable distance, providing an affirmative answer. Finally, we contrast the bottleneck distance with the interleaving distance of sheaves by showing that the interleaving distance of sheaves is not intrinsic, let alone universal.

1 Introduction
The core idea of topological persistence is to construct invariants of continuous real-valued functions by considering preimages and applying homology with coefficients in a fixed field $K$, or any other functorial invariant from algebraic topology. The most basic incarnation of this idea studies the homology of sublevel sets. Sublevel set persistent homology was introduced in [ELZ02] and is described up to isomorphism by the sublevel set persistence diagram [CSEH07]. Furthermore, the authors of [CSEH07] introduce the bottleneck distance, an extended metric for sublevel set persistence diagrams. An extension of this invariant considers preimages of arbitrary closed intervals [DW07, CdM09]. This is commonly referred to as interlevel set persistent homology. As shown by [CdM09], interlevel set persistent homology of a PL function $f: X \to \mathbb{R}$ on a finite connected simplicial complex, is described up to isomorphism by the extended persistence diagram $\text{Dgm}(f)$ due to [CSEH09]. It is well-known that the operation $f \mapsto \text{Dgm}(f)$ is stable [CSEH09, HKLM19]. That is, for functions $f, g: X \to \mathbb{R}$ as above,
$$d_B(\text{Dgm}(f), \text{Dgm}(g)) \leq ||f - g||_\infty,$$
where $d_B(\text{Dgm}(f), \text{Dgm}(g))$ is the bottleneck distance of $\text{Dgm}(f)$ and $\text{Dgm}(g)$. In this paper we prove that this distance is universal: it is the largest possible stable distance on persistence diagrams realized by functions. As a note of caution, we point out that universality is only achieved by a version of the bottleneck distance in which any vertex contained inside the extended subdiagram has to be matched, as is made precise in Section 2.2.

We obtain this universality result by studying certain functors on a partially ordered set $\mathcal{M}$ with the shape of an unbounded strip depicted in Fig. 1.1. For a PL function $f: X \to \mathbb{R}$,
the associated functor \( h(f) : \mathcal{M} \to \text{Vect}_K \) on the strip \( \mathcal{M} \) can be thought of as gluing together the homology pyramids of \([CdM09, BEMP13]\) associated to \( f \) along neighboring dimensions. We define the associated extended persistence diagram \( \text{Dgm}(f) \) in terms of the functor \( h(f) \) in Section 2.2 as a multiset on the interior \( \text{int} \mathcal{M} \) of \( \mathcal{M} \). Our definition is equivalent to the familiar one from \([CSEH09]\), but clarifies the above mentioned conditions on matchings necessary for a universal bottleneck distance. In Appendix B we show that the ordinary, relative, and extended subdiagrams, as originally defined in \([CSEH09]\), can be obtained by restriction to the corresponding regions depicted in Fig. 1.1.

In practice it may be intractable to compute the entire extended persistence diagram of \( f : X \to \mathbb{R} \). Instead we may prefer to compute the restriction of \( \text{Dgm}(f) \) to some closed upset \( U \subseteq \mathcal{M} \) of \( \mathcal{M} \), where points in \( \mathcal{M} \) are larger than others if they are further up or further left. Now in order for the bottleneck distance of Definition 2.4 below to be universal when applied to such restricted persistence diagrams, we need to impose some restrictions on the upset \( U \subseteq \mathcal{M} \). (This limitation arises from the fact that Lemma 4.6 below is invalid for arbitrary closed upsets \( U \subseteq \mathcal{M} \).) Considering Fig. 1.1, we may slice each of the square regions corresponding to extended subdiagrams in half along their diagonals to obtain a triangulation of the grey shaded region. We say that a closed upset \( U \subseteq \mathcal{M} \) is admissible if it contains the region labeled \( \text{Ext}_0 \) and if it is compatible with this triangulation in the sense that the interior of each triangle is either fully contained in or disjoint from \( U \). Moreover, for an admissible upset \( U \subseteq \mathcal{M} \) we say that a multiset \( \mu : \text{int} U \to \mathbb{N}_0 \) is a realizable persistence diagram if \( \mu = \text{Dgm}(f)|_{\text{int} U} \) for some function \( f \) as above. Universality follows immediately from the following theorem.

**Theorem 1.1.** For an admissible upset \( U \) and any two realizable persistence diagrams \( \mu, \nu : \text{int} U \to \mathbb{N}_0 \) with \( d_B(\mu, \nu) < \infty \), there exists a finite simplicial complex \( X \) and piecewise linear functions \( f, g : X \to \mathbb{R} \) with

\[
\text{Dgm}(f)|_{\text{int} U} = \mu, \quad \text{Dgm}(g)|_{\text{int} U} = \nu, \quad \text{and} \quad \|f - g\|_\infty = d_B(\mu, \nu).
\]
An analogous theorem in the context of sublevel set persistent homology has already been given in [Les15]. As a feature of the bottleneck distance we also highlight that it can be defined generically over an admissible upset $U \subseteq M$ in a straightforward way as in Definition 2.4 while at the same time being universal. In Section 5 we contrast this to interleaving distances of sheaves. While the interleaving distance of derived level set persistence by [Cur14, KS18] is in some sense equivalent to the bottleneck distance by [BG22] and hence universal we show that the interleaving distance of sheaves, which can be seen as a counterpart to derived level set persistence in degree 0, is not intrinsic, let alone universal.

In [BBF21] we provide another construction of the extended persistence diagram, which is closely related to the construction we provide here but requires fewer tameness assumptions. More specifically, in [BBF21] we use (singular) cohomology in place of singular homology and we take preimages of open subsets in place of closed subsets. However, as is implied by Theorem 1.1 it suffices to consider piecewise linear functions on finite simplicial complexes to prove universality for realizable persistence diagrams. For this reason, singular homology and preimages of closed subsets are sufficient in the present context. Moreover, considering preimages of closed subsets, the translation from singular homology to simplicial homology is very straightforward. We use this connection to provide several properties in Section 3 which are essential to the soundness of our computations. We leave it to future work to generalize the results from Section 3 to the weaker tameness assumptions in [BBF21].

2 Preliminaries

In this section, we formalize the requisite notions of relative interlevel set homology and persistence diagrams, building on and extending several ideas that appear in the relevant literature, and aiming for an explicit description of those ideas. In particular, we will assemble all relevant persistent homology in one single functor, which will turn out to be a helpful and elucidating perspective for studying the persistent homology associated to a function.

2.1 Relative Interlevel Set Homology

For a piecewise linear function $f : X \to \mathbb{R}$, the inverse image map $f^{-1}$ provides a monotone map from the poset of compact intervals to $2^X$. The image of this map consists of all interlevel sets of $f$, where an interlevel set is a preimage $f^{-1}(I)$ of a closed interval $I$. Post-composing this map with homology we obtain a functor from the poset of compact intervals to the category of graded vector spaces over $K$:

$$I \mapsto H_*(f^{-1}(I)). \quad (2.1)$$

This invariant is commonly referred to as interlevel set homology. As proposed by [CdM09] for the discretely indexed setting, we consider the following extension of this invariant:

1. For interlevel sets described as a union of two smaller interlevel sets there exist connecting maps from a Mayer–Vietoris sequence, which we would like include in our structure.
2. In addition to absolute homology groups we would like to include relative homology groups as well. More specifically, we aim to include all homology groups of preimages of pairs of closed subspaces of $\mathbb{R} := [-\infty, \infty]$ whose difference is an interval contained in $\mathbb{R}$. By excision, each such relative homology group can be written as $H_n(f^{-1}(I,C))$, where $I$ is a closed interval and $C$ is the complement of an open interval.

Extending the interlevel set homology functor (2.1) in the first direction leads to the notion of a Mayer–Vietoris system as defined in [BGO19, Definition 2.14]. On the other hand, the construction by [BEMP13] gives an extension in the second direction. The invariant we consider combines both extensions to obtain a continuously indexed version of the construction by [CdM09]. Specifically, we encode all information as a functor $h(f)$ on one large poset $M$ (with Mayer–Vietoris systems arising as restrictions to a subposet of $M$).

Any point $u \in M$ should correspond to a pair $(I,C)$ as above and a degree $n$ in such a way that $h(f)(u) = H_n(f^{-1}(I,C))$. The natural symmetry of this parametrization is expressed by an automorphism $T: M \to M$ such that given $h(f)(u) = H_n(f^{-1}(I,C))$ for some $u \in M$ we have $(h(f) \circ T)(u) = H_{n-1}(f^{-1}(I,C))$. In other words, if $u \in M$ corresponds to a pair $(I,C)$ and a degree $n$, then $T(u)$ corresponds to the pair $(I,C)$ and the degree $n - 1$.

Explicitly, $M$ is given as the convex hull of two lines $l_0$ and $l_1$ of slope $-1$ in $\mathbb{R} \times \mathbb{R}$ passing through $-\pi$ respectively $\pi$ on the $x$-axis, as shown in Fig. 2.1. Here $\mathbb{R}$ and $\mathbb{R}^o$ denote the posets given by the orders $\leq$ and $\geq$ on $\mathbb{R}$, respectively. This makes $\mathbb{R}^o \times \mathbb{R}$ the product poset and $M$ a subposet.

The automorphism $T: M \to M$ has the following defining property (also see Fig. 2.2):

Let $u \in M$, $h_0$ be the horizontal line through $u$, let $g_0$ be the vertical line through $u$, let $l_1$ be the horizontal line through $T(u)$, and let $g_1$ be the vertical line through $T(u)$. Then the lines $l_0$, $h_0$, and $g_1$ intersect in a common point, and the same is true for the lines $l_1$, $g_0$, and $h_1$.

We also note that $T$ is a glide reflection along the bisecting line between $l_0$ and $l_1$, and the amount of translation is the distance of $l_0$ and $l_1$. Moreover, as a space, $M/\langle T \rangle$ is a Möbius strip; see also [CdM09].

Now in order to specify a degree for each point in $M$, it suffices to specify all points corresponding to degree 0. More specifically, we will now specify a fundamental domain.
Figure 2.2: Incidences defining $T$.

Figure 2.3: The fundamental domain $D := \uparrow \text{Im } ▲ \setminus T(\uparrow \text{Im } ▲)$.

$D$ with respect to the action of $\langle T \rangle$, which consists of all points corresponding to degree 0. To this end, we embed the extended reals $\mathbb{R}$ into the strip $M$ by precomposing the diagonal map $\Delta : \mathbb{R} \to \mathbb{R}^2, t \mapsto (t, t)$ with the homeomorphism $\arctan : \mathbb{R} \to [-\pi/2, \pi/2]$, yielding a map

$$▲ = \Delta \circ \arctan : \mathbb{R} \to M, \ t \mapsto (\arctan t, \arctan t)$$

such that $\text{Im } ▲$ is a perpendicular line segment through the origin joining $l_0$ and $l_1$, see Fig. 2.1. With this we may define our fundamental domain as

$$D := (\uparrow \text{Im } ▲) \setminus T(\uparrow \text{Im } ▲),$$

see Fig. 2.3. Here $\uparrow \text{Im } ▲$ is the upset of the image of ▲.

Now $D$ provides a tessellation of $M$ as shown in Fig. 2.4, and we assign the degree $n$ to any point in $T^{-n}(D)$. This convention for $T$ is chosen in analogy to the topological suspension, which also decreases the homological degree: a homology class of degree $d$ in the suspension corresponds to a homology class of degree $d - 1$ in the original space.

It remains to assign a pair $(I, C)$ to any point $u \in D$ of the fundamental domain. The following proposition provides such an assignment; a schematic image is shown in Fig. 2.5

**Proposition 2.1.** Let $\mathcal{P}$ denote the set of pairs of closed subspaces of $\mathbb{R}$. Then there is a unique monotone map

$$\rho : D \to \mathcal{P}$$
with the following three properties:

1. For any \( t \in \mathbb{R} \) we have \((\rho \circ \blacktriangle)(t) = (\{t\}, \emptyset)\).

2. For any \( u \in D \cap \partial M \) the two components of \( \rho(u) \) are identical.

3. For any axis-aligned rectangle contained in \( D \) the corresponding joins and meets are preserved by \( \rho \).

For \( u \in D \) we may describe \((I, C) := \rho(u)\) more explicitly as follows. The interval \( I \) is given by taking the downset of \( u \) in the poset \( M \), denoted by \( \downarrow u \), and taking the preimage under the embedding \( \blacktriangle : \mathbb{R} \to M \). Similarly, \( C \) is given by starting with the transformed point \( T^{-1}(u) \), taking its upset, forming the complement of this upset in \( M \), and taking the preimage of the closure under \( \blacktriangle \). Thus, we have the formula

\[
\rho(u) = \blacktriangle^{-1} \left( \downarrow u, \overline{M \setminus \uparrow T^{-1}(u)} \right).
\]
We also note that, if $I$ is a bounded interval, then $C$ must be empty; if $I$ is a proper downset (upset), then so is $C$; and if $\mathbb{R} \setminus C \subseteq \mathbb{R}$, then $I$ must be $\mathbb{R}$. Any other point in the orbit of the point $u$ is assigned the same pair, but a different degree.

Now suppose $f: X \to \mathbb{R}$ is a piecewise linear function with $X$ a finite simplicial complex. For any $n \in \mathbb{Z}$, we obtain a functor

$$D \to \text{vect}_K, \ u \mapsto H_n(f^{-1}(\rho(u))).$$

(2.2)

Here vect$_K$ denotes the category of finite-dimensional vector spaces over $K$. This functor describes the $n$-th layer of the Mayer–Vietoris pyramid of $f$, as introduced in [CdM09] and extensively studied in [BEMP13] and [CdSKM19]. As pointed out in [CdM09], the different layers can be glued together to form one large diagram. More specifically, we may move the functor in (2.2) down by $n$ tiles in the tesselation of $\mathcal{M}$ shown in Fig. 2.4 by precomposition with $T^n$:

$$T^{-n}(D) \to \text{vect}_K, \ u \mapsto H_n(f^{-1}(\rho(T^n(u))).$$

(2.3)

This way we obtain a single functor on each tile $T^{-n}(D)$. We can further extend these functors into one single functor

$$h(f) := h(f; K): \mathcal{M} \to \text{vect}_K.$$  

We will refer to this functor as the relative interlevel set homology of $f$ with coefficients in $K$. For better readability we suppress the field $K$ as an argument of $h$. We define the restriction $h(f)|T_{-n}(D)$ as the functor specified in (2.3).

It remains to specify the linear maps in $h(f)$ between comparable elements from different tiles. In the following we will define these maps as either the zero map or as the boundary operator of a Mayer–Vietoris sequence. To this end, let $u \preceq v \in \mathcal{M}$, with $u$ and $v$ lying in different tiles, and consider the interval $[u, v]$ in the poset $\mathcal{M} \subseteq \mathbb{R}^n \times \mathbb{R}$, which is the intersection of a closed axis-aligned rectangle with the closed strip $\mathcal{M}$. If there is a point $x \in [u, v] \cap \partial \mathcal{M}$, then $h(f)(x) \cong \{0\}$, since $\rho$ assigns to any point in $D \cap \partial \mathcal{M}$ a pair of identical intervals. In this case, we thus have to set $h(f)(u \preceq v) = 0$, as this map factors through $\{0\}$.

Now consider the case $[u, v] \cap \partial \mathcal{M} = \emptyset$, and let $w := T(u)$. As illustrated by Fig. 2.6, we have $v \preceq w$, and thus $u$ and $v$ have to lie in adjacent tiles. For simplicity, we first consider the case $v \in D$, which implies $u \in T^{-1}(D)$ and $w \in D$. In particular, the poset interval $[v, w]$ is an axis aligned rectangle with corners $v = (v_1, v_2)$, $w = (w_1, w_2)$, $(v_1, v_2)$, and $(v_1, w_2)$. We have the meet $v = (w_1, v_2) \land (v_1, w_2)$ and the join $w = (w_1, v_2) \lor (v_1, w_2)$. Since this rectangle is contained in $D$, join and meet are preserved by $\rho$ by Proposition 2.1.3. Moreover, since taking preimages $f^{-1}: \mathbb{R} \to \mathbb{R}$ is a homomorphism of Boolean algebras, $f^{-1}$ also preserves joins and meets, which in this case are the componentwise unions and intersections. Writing $F = f^{-1} \circ \rho$, we get a Mayer–Vietoris sequence

$$\cdots \to H_1(F(w)) \xrightarrow{\partial} H_0(F(v)) \to H_0(F(w_1, v_2)) \oplus H_0(F(v_1, w_2)) \to H_0(F(w)) \to 0$$  

and define $h(f)(u \preceq v) := \partial$.

We note that the existence of the Mayer–Vietoris sequence above is ensured by a very general criterion, given in [tom08 Theorem 10.7.7]. This criterion requires certain triads
to be excisive, which is satisfied here because \( f \) is piecewise linear. In any subsequent applications of the Mayer–Vietoris sequence, we will continue to use this criterion, omitting the straightforward proof that the corresponding triads are excisive.

In the general situation where \( v \in T^{-n}(D) \) for some \( n \in \mathbb{Z} \), the points \( v' := T^n(v) \) and \( w' := T^{n+1}(u) \) lie in \( D \). Using the above arguments with \( v' \) and \( w' \) in place of \( v \) respectively \( w \), we obtain the Mayer–Vietoris sequence

\[
\cdots \to H_{n+1}(F(w')) \xrightarrow{\partial} H_n(F(v')) \to H_n(F(v'_1, v'_2)) \oplus H_n(F(v'_1, w'_2)) \to H_n(F(w')) \to \cdots
\]

and define \( h(f)(u \preceq v) := \partial \).

We refer to \( h(f) \) as the relative interlevel set homology of \( f \) with coefficients in \( K \). We will generalize the construction of \( h(f) \) to pairs \((X, A)\) and show its functoriality in Appendix A.

2.2 The Extended Persistence Diagram

Having defined the relative interlevel set homology as a functor \( h(f) : \mathbb{M} \to \text{vect}_K \), we now formalize the notion of an extended persistence diagram, originally due to [CSEH09], as an invariant of functors \( F : \mathbb{M} \to \text{vect}_K \) vanishing on \( \partial \mathbb{M} \). The persistence diagram of \( F \) is a multiset \( \text{Dgm}(F) = \mu : \text{int} \mathbb{M} \to \mathbb{N}_0 \), which counts, for each point \( v = (v_1, v_2) \in \mathbb{M} \), the maximal number \( \mu(v) \) of linearly independent vectors in \( F(v) \) born at \( v \); for the functor \( h(f) \), these are homology classes. More precisely, we define

\[
\text{Dgm}(F) : \text{int} \mathbb{M} \to \mathbb{N}_0, \ v \mapsto \dim_K F(v) - \dim_K \sum_{u \prec v} \text{Im} F(u) \preceq v).
\]

In the last term, \( u \) ranges over all \( u \in \mathbb{M} \) with \( u \prec v \). Moreover, note that

\[
\sum_{u \prec v} \text{Im} F(u \preceq v) = \left( \bigcup_{x \leq v_1} \text{Im} F((x, v_2) \preceq v) \right) + \left( \bigcup_{y < v_2} \text{Im} F((v_1, y) \preceq v) \right).
\]

Now let \( f : X \to \mathbb{R} \) be a piecewise linear function with \( X \) a finite simplicial complex.
Figure 2.7: A simplicial complex (top) and the extended persistence diagram (bottom) of the associated height function. The numbers next to the dots indicate their multiplicities in the persistence diagram. The complex shown on top is an instance of the construction we use to realize extended persistence diagrams in the proof of Theorem 1.1. Subcomplexes shaded in color correspond to dots in the persistence diagram. The rectangular regions show the support of the corresponding “features” in the relative interlevel set homology.

Definition 2.2 (Extended Persistence Diagram). The extended persistence diagram of \( f \) (over \( K \)) is \( \text{Dgm}(f) := \text{Dgm}(f; K) := \text{Dgm}(h(f)) \).

See Fig. 2.7 for an example. In Appendix B we show that the restriction of \( \text{Dgm}(f) : \text{int} \ M \to \mathbb{N}_0 \) to any of the regions shown in Fig. 1.1 yields the corresponding ordinary, relative, or extended subdiagram as defined in [CSEH09] up to reparametrization. Moreover, we note that \( \text{Dgm}(f) \) is supported in the downset \( \downarrow \text{Im} \Delta \subseteq \text{M} \); in Fig. 1.1 this region is shaded in dark gray. As \( f \) is bounded, \( \text{Dgm}(f) \) is also supported in the union of open squares \( (-\frac{\pi}{2}, \frac{\pi}{2})^2 + \pi \mathbb{Z}^2 \).

We now define the bottleneck distance of extended persistence diagrams. To this end, we first define an extended metric \( d : \text{M} \times \text{M} \to [0, \infty] \) on \( \text{M} \). Following the approach of [MP20], we then define an extended metric on multisets of \( \text{int} \ M \) in terms of \( d \). Let \( d_0 : \mathbb{R} \times \mathbb{R} \to [0, \infty] \) be the unique extended metric on \( \mathbb{R} \) such that for all \( s < t \) we have
the equation
\[ d_0(s, t) = \begin{cases} \tan t - \tan s & [s, t] \cap \left( \frac{\pi}{2} + \pi \mathbb{Z} \right) = \emptyset, \\ \infty & \text{otherwise.} \end{cases} \]

**Definition 2.3.** Writing \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \) for points \( u, v \in M \) we define
\[ d : M \times M \to [0, \infty], \quad (u, v) \mapsto \max\{d_0(u_1, v_1), d_0(u_2, v_2)\} \]
to be the extended metric on \( M \) given by the maximum of \( d_0 \) on each copy of \( \mathbb{R} \).

Using the extended metric \( d : M \times M \to [0, \infty] \) we can now express how a perturbation of a function \( f : X \to \mathbb{R} \) as above may affect its persistence diagram. If we think of the vertices of \( \text{Dgm}(f) \) as “features” of \( f : X \to \mathbb{R} \), then a \( \delta \)-perturbation of \( f : X \to \mathbb{R} \) may cause the corresponding vertices of the persistence diagram to be moved by up to a distance of \( \delta \) away from their original position with respect to \( d : M \times M \to [0, \infty] \). As the persistence diagram \( \text{Dgm}(f) : \text{int} M \to \mathbb{N}_0 \) is undefined at the boundary \( \partial M \), vertices that are \( \delta \)-close to the boundary \( \partial M \) may disappear altogether and moreover, new vertices within a distance of \( \delta \) from \( \partial M \) may appear in the persistence diagram of the perturbation. This intuition is made precise by the [CSEH09, Stability Theorem] for the bottleneck distance of extended persistence diagrams. Completely analogously, when \( U \subseteq M \) is an admissible upset of \( M \) and when we merely compute \( \text{Dgm}(f)|_{\text{int} U} \), then perturbations may cause vertices to disappear in \( \partial U \) and other vertices to appear in close proximity to \( \partial U \).

We follow [MP20] to provide a combinatorial description of the bottleneck distance.

**Definition 2.4 (Matchings and the Bottleneck Distance).** Let \( U \subseteq M \) be an admissible upset and let \( \mu, \nu : \text{int} U \to \mathbb{N}_0 \) be finite multisets. A matching between \( \mu \) and \( \nu \) is a multiset “of edges” \( M : U \times U \to \mathbb{N}_0 \) such that
\[ \text{pr}_1(M)|_{\text{int} U} = \mu, \quad \text{pr}_2(M)|_{\text{int} U} = \nu, \quad \text{and} \quad M|_{\partial U \times \partial U} = 0, \]
where
\[ \text{pr}_1(M) : U \to \mathbb{N}_0, \quad u \mapsto \sum_{v \in U} M(u, v) \]
and
\[ \text{pr}_2(M) : U \to \mathbb{N}_0, \quad v \mapsto \sum_{u \in U} M(u, v) \]
are the projections of \( M : U \times U \to \mathbb{N}_0 \) to the first and to the second component respectively. The norm of a matching \( M \) is defined as
\[ \|M\| := \sup d \left( M^{-1}\left( \mathbb{N} \setminus \{0\} \right) \right) \]
and the bottleneck distance of \( \mu \) and \( \nu \) is
\[ d_B(\mu, \nu) := \inf\{\|M\| \mid M \text{ is a matching between } \mu \text{ and } \nu\}. \]

If we consider the regions in Fig. 1.1 in conjunction with the extended metric \( d : M \times M \to [0, \infty] \) defined in Definition 2.3 above, then we observe the following. While all interior points of regions corresponding ordinary and relative subdiagrams have finite distance to the boundary \( \partial M \), all of the interior points of regions corresponding to extended subdiagrams have distance \( \infty \) to \( \partial M \). As a result, any matching between extended
persistence diagrams of finite norm has to match any vertex contained in the extended subdiagram to a vertex of the other extended persistence diagram. Usually, all three subdiagrams are layered on top of each other in such a way that the anti-diagonal edges of the ordinary and relative subdiagrams bounding $\mathbb{M}$ and the diagonal of the extended subdiagram coincide. For this reason, one may be tempted to match vertices in the extended subdiagram to “phantom vertices” on the diagonal as well. As far as stability is concerned, this is perfectly fine, as the resulting bottleneck distance is only getting smaller, and even necessary, if only the part of the extended subdiagram above the diagonal is known, i.e. the diagonal of the extended subdiagram is contained $\partial U$. However, this distinction is crucial to the universality of the bottleneck distance on (full) extended persistence diagrams and the soundness of Theorem 1.1 in particular.

3 Properties of Relative Interlevel Set Homology

We think of the relative interlevel set homology of a piecewise linear function $f: X \to \mathbb{R}$ (with $X$ a finite complex) as an analogue to the homology of a space. There are various tools to reduce the computation of the homology of a space to smaller subcomputations, and one of them is the relative homology of a pair of spaces. In order to compute the relative interlevel set homology of a function, we develop a counterpart to relative homology in Appendix A. More specifically, given a finite simplicial pair $(X, A)$ and a piecewise linear function $f: X \to \mathbb{R}$ we construct a functor

\[ h(X, A; f) := h(X, A; f; K): \mathbb{M} \to \text{vect}_K, \]

which we refer to as the relative interlevel set homology of the function $f: A \subseteq X \to \mathbb{R}$. Similarly, we write $\text{Dgm}(X, A; f) := \text{Dgm}(h(X, A; f))$ for the corresponding persistence diagram. In the case $A = \emptyset$, we also write $h(X; f) := h(X, \emptyset; f)$ and $\text{Dgm}(X; f) := \text{Dgm}(X, \emptyset; f)$. When the pair $(X, A)$ is clear from the context, we may suppress it as an argument of $h$ or $\text{Dgm}$ and simply write $h(f)$ and $\text{Dgm}(f)$.

One of the most useful properties of relative homology is functoriality. Before we continue, we explain in what sense $h = h(\cdot; K)$ is a functor. To this end, we describe the category of functions $f: A \subseteq X \to \mathbb{R}$ that provides the natural domain for this functor. Denoting this category by $\mathcal{F}_0$, its objects are triples $(X, A; f)$ with $(X, A)$ a finite simplicial pair and $f: X \to \mathbb{R}$ a piecewise linear function. Now let $(X, A; f)$ and $(Y, B; g)$ be objects of $\mathcal{F}_0$. A morphism $\varphi$ from $(X, A; f)$ to $(Y, B; g)$ is a continuous map $\varphi: X \to Y$ with $\varphi(A) \subseteq B$ and the property that the diagram

\[ \begin{array}{ccc}
X & \xrightarrow{\varphi} & Y \\
\downarrow f & & \downarrow g \\
\mathbb{R} & \xrightarrow{\varphi} & \mathbb{R}
\end{array} \]

commutes. We may also write this as $\varphi: (X, A; f) \to (Y, B; g)$. The subscript 0 in the notation $\mathcal{F}_0$ indicates the equivalent condition that $f$ and $g \circ \varphi$ have distance 0 in the supremum norm. The composition and the identities of $\mathcal{F}_0$ are defined in the obvious way. Moreover, as homology is a functor on topological spaces and $h = h(\cdot; K)$ is defined in terms of homology (see Appendix A), this makes $h$ a functor from $\mathcal{F}_0$ to the category.
vect\textsuperscript{M} of functors from \( M \) to \( \text{vect}_K \). As it turns out, this relative interlevel set homology functor \( h : \mathcal{F}_0 \rightarrow \text{vect}_K \) satisfies certain properties analogous to the Eilenberg–Steenrod axioms for homology theories of topological spaces.

For the first property, let \((X, A; f)\) and \((Y, B; g)\) be objects of \( \mathcal{F}_0 \), and let \( \varphi, \psi : (X, A; f) \rightarrow (Y, B; g) \) be morphisms. Then \((X \times [0, 1], A \times [0, 1]; f \circ \text{pr}_1)\) is an object of \( \mathcal{F}_0 \), and we define a fiberwise homotopy \( \eta \) from \( \varphi \) to \( \psi \) to be a morphism

\[
\eta : (X \times [0, 1], A \times [0, 1]; f \circ \text{pr}_1) \rightarrow (Y, B; g)
\]

such that \( \eta(\cdot, 0) = \varphi \) and \( \eta(\cdot, 1) = \psi \). If such a homotopy exists, we say that \( \varphi \) and \( \psi \) are fiberwise homotopic.

**Lemma 3.1** (Homotopy Invariance). If \( \varphi \) and \( \psi \) are fiberwise homotopic, then \( h(\varphi) \sim h(\psi) \).

**Proof.** As \( h = h(\cdot; K) \) is defined in terms of homology, this follows from the homotopy invariance of homology. \( \square \)

**Lemma 3.2** (Mayer–Vietoris Sequence). Let \( X \) be a finite simplicial complex with subcomplexes \( A_0 \subseteq X_0 \subseteq X, A_1 \subseteq X_1 \subseteq X, \) and \( A \subseteq X \), such that

\[
X = X_0 \cup X_1 \quad \text{and} \quad A = A_0 \cup A_1.
\]

Moreover, let \( f : X \rightarrow \mathbb{R} \) be a piecewise linear map. Then there is an exact sequence of functors \( M \rightarrow \text{vect}_K \) given by

\[
\begin{align*}
&h(X_0 \cap X_1, A_0 \cap A_1; f|_{X_0 \cap X_1}) \\
&\downarrow \quad (1) \\
&h(X_0, A_0; f|_{X_0}) \oplus h(X_1, A_1; f|_{X_1}) \\
&\downarrow (1 - 1) \\
&h(X, A; f) \\
&\downarrow \theta \\
&h(X_0 \cap X_1, A_0 \cap A_1; f|_{X_0 \cap X_1}) \circ T \\
&\downarrow (1) \\
&(h(X_0, A_0; f|_{X_0}) \circ T) \oplus (h(X_1, A_1; f|_{X_1}) \circ T),
\end{align*}
\]

where \( 1 \) denotes a map induced by inclusion. We name this the Mayer–Vietoris sequence for \( (A; A_0, A_1) \subseteq (X; X_0, X_1) \) relative to \( f \).

**Proof.** Let \( u \in D \) and \( n \in \mathbb{Z} \). By Lemma A.3, the operation \(- \cap f^{-1}(\rho(u))\) preserves componentwise unions and intersections. Thus, \((X_0 \cap X_1, A_0 \cap A_1) \cap f^{-1}(\rho(u))\) is the componentwise intersection of \((X_0, A_1) \cap f^{-1}(\rho(u))\) and \((X_1, A_1) \cap f^{-1}(\rho(u))\), while \((X, A) \cap f^{-1}(\rho(u))\)
is their union. We obtain the sequence

$$h(X_0 \cap X_1, A_0 \cap A_1; f|_{X_0 \cap X_1}) (T^n(u))$$

as a portion from the corresponding Mayer–Vietoris sequence. Since $D$ is a fundamental domain, we get (3.2) pointwise, at each index of $M$. Moreover, the maps denoted as

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

are natural transformations by the functoriality of homology and the naturality of the boundary operator of the Mayer–Vietoris sequence.

It remains to be shown that $\partial$ is a natural transformation. In part, this follows from the naturality of the above Mayer–Vietoris sequence. However, recall from our construction of $h$ in Appendix A that some of the internal maps of $h(X, A; f)$ and $h(X_0 \cap X_1, A_0 \cap A_1; f|_{X_0 \cap X_1})$ are boundary operators as well. Therefore, we need to check whether certain squares with all maps boundary operators commute. Specifically, given $T^n(v) \preceq T^{n+1}(u)$ for some $u = (u_1, u_2) \preceq v = (v_1, v_2) \in D$ and $n \in \mathbb{Z}$, we have to show that the diagram

$$h(X, A; f) (T^{n+1}(u)) \xrightarrow{\partial_{T^{n+1}(u)}} h(X_0 \cap X_1, A_0 \cap A_1; f|_{X_0 \cap X_1}) (T^{n+2}(u))$$

commutes. By unravelling the definition of the relative interlevel set homology $h$ we may rewrite this square to a more concrete form with all maps boundary operators of some
Mayer–Vietoris sequence:

\[
\begin{align*}
H_{n-1} \left( (X, A) \cap f^{-1}(\rho(u)) \right) & \longrightarrow H_{n-2} \left( (X_0 \cap X_1, A_0 \cap A_1) \cap f^{-1}(\rho(u)) \right) \\
& \quad \uparrow \\
H_n \left( (X, A) \cap f^{-1}(\rho(v)) \right) & \longrightarrow H_{n-1} \left( (X_0 \cap X_1, A_0 \cap A_1) \cap f^{-1}(\rho(v)) \right).
\end{align*}
\]

(3.3)

As \( f \) is piecewise linear, there are subdivisions of \( \mathbb{R} \) and \( X \) such that both components of \( \rho(v_1, u_2) \) and \( \rho(u_1, v_2) \) are subcomplexes of \( \mathbb{R} \) and such that \( f: X \to \mathbb{R} \) is a simplicial map. Thus, we may use the isomorphism from simplicial homology to singular homology to show that the square (3.3) commutes. As it turns out, the boundary operator for the Mayer–Vietoris sequence in simplicial homology, which is defined in terms of the zigzag lemma, and the boundary operator from [tom08, Theorem 10.7.7], which is defined in terms of the long exact sequence of a triple and the suspension isomorphism, commute with the corresponding isomorphisms from simplicial to singular homology of domain and codomain. Thus, we may think of each of the arrows in (3.3) as boundary operators of a Mayer–Vietoris sequence in simplicial homology. Moreover, as the boundary operator of Mayer–Vietoris sequences in simplicial homology is defined in terms of the zigzag lemma, we can prove the commutativity of (3.3) by a diagram chase. To this end, we consider the commutative diagram

\[
\begin{array}{cccc}
\cdots & \longrightarrow & C_n \left( (X, A) \cap f^{-1}(\rho(u)) \right) & \longrightarrow \\
& \downarrow & \downarrow & \downarrow \\
& \cdots & C_n \left( (X_0, A_0) \cap f^{-1}(\rho(v_1, u_2)) \right) & \longrightarrow \\
& \downarrow & \downarrow & \downarrow \\
& \cdots & C_n \left( (X_1, A_1) \cap f^{-1}(\rho(v_1, u_2)) \right) & \longrightarrow \\
& \downarrow & \downarrow & \downarrow \\
& \cdots & C_n \left( (X_1, A_1) \cap f^{-1}(\rho(v_1, u_2)) \right) & \longrightarrow \\
& \downarrow & \downarrow & \downarrow \\
& \cdots & C_n \left( (X, A) \cap f^{-1}(\rho(u)) \right) & \longrightarrow \\
\end{array}
\]

of simplicial chain complexes with coefficients in \( K \). Note that each of the arrows of the outer square of this large diagram (3.4) point in the opposite direction when compared to the corresponding arrows in (3.3). Now each row and each column of (3.4) is a short exact sequence of simplicial chain complexes and moreover, each of the boundary operators from (3.3) arises as a boundary map from the zigzag lemma of the corresponding short exact sequence of simplicial chain complexes in (3.4). With this in mind, the commutativity of (3.3) follows from a diagram chase in (3.4).

**Corollary 3.3** (Excision). Let \( X \) be a finite simplicial complex with subcomplexes \( A \) and \( B \) such that \( X = A \cup B \). Moreover, let \( f: X \to \mathbb{R} \) be a piecewise linear map. Then the inclusion

\[
(A, A \cap B; f|_A) \hookrightarrow (X, B; f)
\]

is a homotopy equivalence.
induces a natural isomorphism \( h(A, A \cap B; f|_A) \xrightarrow{\cong} h(X, B; f) \) in relative interlevel set homology.

**Proof.** We consider the Mayer–Vietoris sequence for \((X; B, A) \subseteq (X; X, A))\) relative to \(f\). Since both \(h(A, A; f|_A)\) and \(h(X, X; f)\) are constantly zero, the isomorphism follows from exactness. \(\square\)

**Corollary 3.4** (Exact Sequence of a Pair). Let \((X, A; f)\) be an object in \(\mathcal{F}_0\). Then we have an exact sequence

\[
h(A; f|_A) \xrightarrow{1} h(X; f) \xrightarrow{1} h(X, A; f) \xrightarrow{\partial} h(A; f|_A) \circ T \xrightarrow{1} h(X; f) \circ T,
\]

where 1 denotes a map induced by inclusion.

**Proof.** This is equivalent to the Mayer–Vietoris sequence for \((A; \emptyset, A) \subseteq (X; X, A))\) relative to \(f\). \(\square\)

The last corollary implies yet another corollary, which will be useful later.

**Corollary 3.5.** Let \((X, A; f)\) be an object in \(\mathcal{F}_0\) such that there is a fiberwise retraction \(r: X \to A\) to the inclusion \(A \subseteq X\): \(r|_A = \text{id}_A\) and \(f \circ r = f\). Then

\[
h(X; f) \cong h(A; f|_A) \oplus h(X, A; f).
\]

In particular, we have \(\text{Dgm}(X; f) = \text{Dgm}(A; f|_A) + \text{Dgm}(X, A; f)\).

**Proof.** We consider the exact sequence from the previous corollary. The map \(r\) yields a left inverse to the map induced by inclusion,

\[
h(A; f|_A) \xrightarrow{1} h(X; f),
\]

so by exactness \(\partial \circ T^{-1} = 0 = \partial\). Thus we obtain the split exact sequence

\[
0 \to h(A; f|_A) \xrightarrow{1} h(X; f) \xrightarrow{1} h(X, A; f) \to 0,
\]

and hence

\[
h(X; f) \cong h(A; f|_A) \oplus h(X, A; f).
\] \(\square\)

**Lemma 3.6** (Additivity). Let \((X, A; f)\) be an object of \(\mathcal{F}_0\), and let \(\{(X_i, A_i) \subseteq (X, A)\mid i = 1, \ldots, n\}\) be a family of pairs of subcomplexes with \(X = \bigsqcup_{i=1}^n X_i\) and \(A = \bigsqcup_{i=1}^n A_i\). Then the inclusions induce a natural isomorphism

\[
\bigoplus_{i=1}^n h(X_i, A_i; f|_{X_i}) \xrightarrow{\cong} h(X, A; f).
\]

**Proof.** This follows from the additivity of homology or by induction from the Mayer–Vietoris sequence Lemma 3.2. \(\square\)

**Lemma 3.7** (Dimension). Let \(f: [0, 1] \to \mathbb{R}\) be a monotone affine map. Then

\[
\text{Dgm}(f) = 1_u,
\]

where \(\rho(u) = ([f(0), f(1)], \emptyset)\).

Here \(1_u\), for \(u \in \mathbb{M}\), is the indicator function \(1_u: v \mapsto \begin{cases} 1 & v = u \\ 0 & v \neq u. \end{cases}\)
4 Universality of the Bottleneck Distance

In this section, we prove our main result, by providing a construction that realizes any given extended persistence diagram as the relative interlevel set homology of a function. We then further extend this construction to realize any \( \delta \)-matching between extended persistence diagrams as a pair of functions on a common domain and with distance \( \delta \) in the supremum norm.

4.1 Lifting Points in \( \mathbb{M} \)

We start by defining the notion of a lift of a point \( u \in \mathbb{M} \).

**Definition 4.1.** A lift of a point \( u \in \mathbb{M} \) is a function \( f : A \subseteq X \to \mathbb{R} \) in the category \( \mathcal{F}_0 \) with

\[
\text{Dgm}(X, A; f) = 1_u|_{\text{int}\mathbb{M}}.
\]

We note that for any point \( u \in \partial \mathbb{M} \) the inclusion of the empty set \( \emptyset \subset \mathbb{R} \) is a lift of \( u \). For technical reasons, we also allow for boundary points in the definition, in order to avoid some case distinctions. As already noted in Section 2.2, the extended persistence diagram \( \text{Dgm}(X, A; f) \) for a function \( f : A \subseteq X \to \mathbb{R} \) in \( \mathcal{F}_0 \) is supported on the intersection \( S \) of the downset \( \downarrow \text{Im} \Delta \subseteq \mathbb{M} \) with the union of open squares \((-\frac{\pi}{2}, \frac{\pi}{2})^2 + \pi \mathbb{Z}^2\). In particular, there are no lifts for any points contained in \( \text{int} \mathbb{M} \setminus S \).

We now construct a lift \( f_u : A_u \subseteq X_u \to \mathbb{R} \) for any point \( u \in S \). To this end, we partition \( S \) into regions shown in Fig. 4.1. We start with the region \( R \subset S \), which is the connected component of \( S \) containing the origin shaded in dark gray in Fig. 4.1. For any point \( u \in R \), a lift \( f_u : [0, 1] \to \mathbb{R} \) is provided by Lemma 3.7. All remaining points of \( S \) are contained in \( I := S \setminus R \). We use a construction that is sketched in Fig. 4.1 and formalized in Appendix C. This figure shows four pairs of geometric simplicial complexes with ambient space \( \mathbb{R}^3 \) and beneath them the strip \( \mathbb{M} \) with some of the regions colored. The relative part of each simplicial pair is given by the solid black line. Four of the regions of \( \mathbb{M} \) are shaded with saturated colors; for a point \( u \) in one of these four regions, the corresponding lift is given by the height function \( r_u : A_u \subset X_u \to [0, 1] \) of the simplicial pair shaded in the same color, post-composed with the appropriate monotone affine map \( b_u : [0, 1] \to \mathbb{R} \); see (C.1) for an explicit formula for \( b_u \). This way we obtain a lift \( f_u := b_u \circ r_u \) for each point \( u \) in one of these four regions. Moreover, for each such \( u \) there is a level-preserving isomorphism \( j_u : [0, 1] \to A_u \), where \( A_u \) is the solid black line. Furthermore, the red simplicial pair and the green simplicial pair admit an isomorphism preserving the levels of the solid black line. Thus, we may think of the domains of our choices of lifts for the saturated red and the saturated green region as being identical. The other regions of \( I \) are shaded in a pale color. For a point in one of these regions, the corresponding lift is given by a higher-or lower-dimensional version of a lift for the region pictured in a more saturated version of the same color. We provide a formal construction of the maps \( r_u : X_u \to [0, 1] \), \( b_u : [0, 1] \to \mathbb{R} \), and \( j_u : [0, 1] \to A_u \subset X_u \) for each \( u \in I \) in Appendix C. Finally, we note that our choices of lifts satisfy the following essential property.

**Lemma 4.2.** The family \( \{(X_u, A_u; f_u) \mid u \in S\} \) of lifts is isometric in the sense that any points \( u, v \in S \) of finite distance \( d(u, v) < \infty \) satisfy

\[
(X_u, A_u) = (X_v, A_v) \quad \text{and} \quad \|f_u - f_v\|_\infty = d(u, v).
\]
4.2 Lifting Multisets

In order to motivate our next definition, we first consider the height function shown in Fig. 2.7 and its persistence diagram. Let \( a = (a_1, a_2) \) be the green vertex of the persistence diagram shown in Fig. 2.7. Letting \( \sigma: \mathbb{R} \to \mathbb{R}, \ t \mapsto \pi - t \) be the reflection at \( \frac{\pi}{2} \), we consider the region \( ([a_2, a_1] \cup \sigma[a_2, a_1])^2 + 2\pi z^2 \). The intersection of \( M \) and this region is the green shaded area in Fig. 4.2. As we can see, both the blue and the red vertex are contained in this green area. As it turns out, this is true for any realizable persistence diagram. More specifically, any realizable persistence diagram \( \mu: \text{int} M \to \mathbb{N}_0 \) satisfies both conditions of the following definition.

**Definition 4.3.** Let \( C \subseteq M \) be a convex subset containing \( R \) and let \( \mu: C \to \mathbb{N}_0 \) be a finite multiset. We say that \( \mu \) is **admissible** if the following two conditions are satisfied:

1. The multiset \( \mu \) contains exactly one point \( a = (a_1, a_2) \in R \).
2. All other points of \( \mu \) are contained in \( I \) as well as \(([a_2, a_1] \cup \sigma[a_2, a_1])^2 + 2\pi \mathbb{Z}^2\). In particular, any realizable persistence diagram is an admissible multiset. As we will see with Lemma 4.5 below, the converse is true as well. Now the notion of a multiset does not distinguish between individual instances of the same element of the underlying set. The following definition enables us to make this distinction.

**Definition 4.4.** A **multiset representation** is a map \( w : S \rightarrow M \) with finite domain \( S \). Its associated multiset is

\[
Dgm(w) : M \rightarrow \mathbb{N}_0, u \mapsto \#w^{-1}(u).
\]

We say \( w : S \rightarrow M \) is **admissible** if \( Dgm(w) \) is. Moreover, if \( U \subseteq M \) is an admissible upset containing the support of \( Dgm(w) \) and if \( \mu : U \rightarrow \mathbb{N}_0 \) is a multiset with \( \mu = Dgm(w)|_U \), then we say that \( w \) is a representation of \( \mu \).

Fig. 4.3 shows a multiset representation \( w : \{0, 1, 2, 3\} \rightarrow M \) of the extended persistence diagram shown in Fig. 2.7. For multiset representations \( w_i : S_i \rightarrow M \) with \( i = 1, 2 \) and a map \( \varphi : S_1 \rightarrow S_2 \), the **norm of \( \varphi \)** is

\[
||\varphi|| := \sup_{s \in S_1} d(w_1(s), (w_2 \circ \varphi)(s)) \in [0, \infty].
\]

Admissible multiset representations and maps of finite norm form a category \( \mathcal{A} \). Such categories are also known as normed or weighted categories, see \[\text{[BdSS17, Section 2.2]}\].

We now extend the isometric family of lifts of points in \( M \) from Lemma 4.2 to admissible multisets, by gluing lifts of the points contained. When gluing these lifts, we have to make certain choices, and we keep track of these choices using representations of multisets and another category \( \mathcal{F} \) of functions on simplicial pairs. The category \( \mathcal{F} \) is defined verbatim the same way as the category \( \mathcal{F}_0 \) from Section 3 except that the commutativity of the diagram in (3.1) is not required. For a morphism \( \varphi : (X, A; f) \rightarrow (Y, B; g) \) in \( \mathcal{F} \) we define the **norm of \( \varphi \)** as

\[
||\varphi|| := ||f - g \circ \varphi||_\infty \in [0, \infty).
\]

\[\text{Figure 4.2: The extended persistence diagram from Fig. 2.7 is admissible.}\]
This makes $\mathcal{F}_0$ the subcategory of $\mathcal{F}$ of all morphisms with vanishing norm. The main goal of this subsection is to specify a norm-preserving functor 

$$F: A \to \mathcal{F}$$

such that for any admissible multiset representation $w: S \to \mathbb{M}$ we have $F(w) = (X, \emptyset; f)$ for some function $f: X \to \mathbb{R}$, and moreover,

$$\beta_0(X) = 1 \quad \text{and} \quad \text{Dgm}(f) = \text{Dgm}(w)|_{\text{int} \mathbb{M}}.$$ 

We now describe the construction of $F(w)$ for any admissible multiset representation $w: S \to \mathbb{M}$. If $w$ is the multiset representation shown in Fig. 4.3 then $F(w)$ will be the height function of the simplicial complex shown in Fig. 2.7.

We start by defining an auxiliary simplicial complex $A$, which we may think of as a \textit{blank booklet} whose pages are indexed or “numbered” by $S \setminus \{s_0\}$, where $s_0$ is the single element of $w^{-1}(R)$. More specifically, we define $A$ as the mapping cylinder $M(\text{pr}_1)$ of the projection $\text{pr}_1: [0,1] \times (S \setminus \{s_0\}) \to [0,1], \ (t,s) \mapsto t$.

In Fig. 2.7 the complex $A$ corresponds to the subcomplex shaded in gray. With some abuse of notation, we view $[0,1] \times (S \setminus \{s_0\})$ as the subspace of $A = M(\text{pr}_1)$ that is the top of the mapping cylinder. Moreover, we refer to $[0,1] \times (S \setminus \{s_0\})$ as the \textit{fore edges} of $A$. Similarly, we view $[0,1]$ as the subspace of $A$ that is the bottom of the mapping cylinder and refer to it as the \textit{spine} of $A$. In Fig. 2.7 the spine $[0,1]$ is shaded in green.

We now extend the booklet $A$ to construct the complex $X$, i.e., the domain of $F(w)$. For each index $s \in S \setminus \{s_0\}$, we glue the complex $X_{w(s)}$, which is associated to the specified lift of $w(s)$, to the fore edge $[0,1] \times \{s\}$ of the booklet $A$ along the map $j_{w(s)}: [0,1] \times \{s\} \cong [0,1] \to X_{w(s)}$. This way we obtain the simplicial pair $(X,A)$. For the example from Fig. 4.3 this amounts to gluing the blue simplicial cylinder as well as the two red horns to the gray shaded booklet in Fig. 2.7.
We now define a simplicial retraction \( r: X \to A \) of \( X \) onto \( A \). Again, we flip through the pages of \( A \) and when we are at the page with index \( s \in S \setminus \{s_0\} \), we retract \( X_w(s) \), which we glued to this page in the construction of \( X \), to the fore edge \([0, 1] \times \{s\}\) via \( r_w(s): X_w(s) \to [0, 1] \cong [0, 1] \times \{s\}\) from Section 4.1 and Appendix C. In other words, the family of retractions \( \{r_w(s): X_w(s) \to [0, 1] | s \in S \setminus \{s_0\}\} \) assembles a retraction \( r: X \to A \). In our example this amounts to orthogonally projecting the blue simplicial cylinder and the two red horns in Fig. 2.7 to the corresponding fore edge of the gray shaded booklet \( A \).

Finally, we construct the function \( f: X \to \mathbb{R} \). We start with defining an affine function \( b: A \to \mathbb{R} \) by specifying its restrictions to the spine \([0, 1]\) and the fore edges \([0, 1] \times (S \setminus \{s_0\})\) of the booklet \( A \). The function \( f: X \to \mathbb{R} \) is then defined as the composition \( f = b \circ r \). The restriction of \( b: A \to \mathbb{R} \) to the spine \([0, 1]\) of the booklet \( A \) is given by \( f_w(s_0): [0, 1] \to \mathbb{R} \) from Section 4.1, which is characterized by \( \text{Dgm}(f_w(s_0)) = 1_{w(s_0)}\) and the assumptions of Lemma 3.7. For each \( s \in S \setminus \{s_0\} \) the restriction of \( b: A \to \mathbb{R} \) to the fore edge \([0, 1] \times \{s\}\) is given by the function \( b_w(s): [0, 1] \times \{s\} \cong [0, 1] \to \mathbb{R} \) from Section 4.1 or rather (C.1). In the example from Fig. 2.7 the function \( b: A \to \mathbb{R} \) is the restriction of the height function to the subcomplex shaded in gray. Finally we set

\[
\begin{align*}
  f := b \circ r \quad \text{and} \quad F(w) := (X, \emptyset; f).
\end{align*}
\]

Since the above constructions are functorial, this defines the desired functor \( F: A \to \mathcal{F} \).

The norms of morphisms are preserved by construction. It remains to show that \( F \) has the desired property:

**Lemma 4.5.** \( \text{Dgm}(f) = \text{Dgm}(w)|_{\text{int} \mathbb{M}} =: \mu. \)

**Proof.** By Corollary 3.5 we have

\[
\text{Dgm}(f) = \text{Dgm}(b) + \text{Dgm}(X, A; f).
\]

By homotopy invariance (Lemma 3.1), we have

\[
h(b) \cong h\left(f_w(s_0)\right) \quad \text{and thus} \quad \text{Dgm}(b) = 1_{w(s_0)}.
\]

Moreover, by excision (Corollary 3.3) and additivity (Lemma 3.6), we have

\[
h(X, A; f) \cong \bigoplus_{s \in S \setminus \{s_0\}} h\left(X_w(s), A_w(s); f_w(s)\right)
\]

and thus

\[
\begin{align*}
\text{Dgm}(X, A; f) &= \sum_{s \in S \setminus \{s_0\}} \text{Dgm}\left(X_w(s), A_w(s); f_w(s)\right) \\
&= \sum_{s \in S \setminus \{s_0\}} 1_{w(s)}|_{\text{int} \mathbb{M}} \\
&= \mu - 1_{w(s_0)}.
\end{align*}
\]

Altogether, we obtain \( \text{Dgm}(f) = \text{Dgm}(b) + \text{Dgm}(X, A; f) = 1_{w(s_0)} + \mu - 1_{w(s_0)} = \mu. \)

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4.3 Universality of the Bottleneck Distance

In order to show Theorem 1.1, we use the following auxiliary result.

**Lemma 4.6 (Matchings with Admissible Projections).** Let $U \subseteq M$ be an admissible upset and let $\mu, \nu: \text{int } U \to N_0$ be two realizable persistence diagrams. Suppose $M: U \times U \to N_0$ is a matching between $\mu$ and $\nu$ of finite norm $\|M\| < \infty$. Then there is a matching $N: U \times U \to N_0$ (between $\mu$ and $\nu$) such that $\|N\| \leq \|M\|$ and both $pr_1(N)$ and $pr_2(N)$ are admissible multisets.

**Proof.** Suppose $pr_1(M): U \to N_0$ is not admissible. By symmetry it suffices to show that there is a matching $N: U \times U \to N_0$ between $\mu$ and $\nu$ with $pr_1(N): U \to N_0$ admissible, $\|N\| \leq \|M\|$, and $pr_2(N) = pr_2(M)$. Now in order to construct $N$ from $M$ we consider the “elements” of $pr_1(M)$ violating the second condition from Definition 4.3 and we replace the corresponding edges in $M$ one by one. Suppose that $u \in (pr_1(M))^{-1}(N \setminus \{0\})$ is a vertex violating the second condition from Definition 4.3 and let $a = (a_1, a_2) \in R$ be the unique vertex with $\mu(a) = 1$. Then we have $u \in \partial U$ necessarily and there is a vertex $v \in \text{int } U$ with $M(u, v) \geq 1$. We assume that $u \in l_0$, the other two cases $u \in l_1$ and $u \in \partial U \setminus \partial M$ are similar.

Now let $\nabla \subset M$ be the triangular region of all points in $([a_2, a_1] \cup \sigma[a_2, a_1])^2 + 2\pi Z^2$ and $M$ that are of finite distance to $u$ (as well as $v$). We have to show there is a point $w \in \nabla \cap \partial U = \nabla \cap l_0$ such that $d(w, v) \leq \|M\|$, since then we may set

$$N := M - 1_{(u,v)} + 1_{(w,v)}$$

to obtain the matching $N: U \times U \to N_0$ with $\|N\| \leq \|M\|$ and $pr_1(N)(u) = pr_1(M)(u) - 1$ and we may continue by induction. Thus, we are done in case we have $d(v, \nabla \cap \partial U) \leq d(v, u) \leq \|M\|$. Now suppose we have $d(v, \nabla \cap \partial U) \geq d(v, u)$. Without loss of generality we assume that $u$ is further up left than any of the points in $\nabla$. In Fig. 4.4 we show the bisector of $u$ and $\nabla \cap \partial U = \nabla \cap l_0$ with respect to $d: M \times M \to [0, \infty]$ in tangential coordinates. Now let $w$ be the upper left vertex of $\nabla$. As $v$ is on or to the upper left of the bisecting line between $u$ and $\nabla \cap \partial U$ it is also to the upper left of the line through $w$ that is parallel to the bisector and hence

$$d(v, w) = d(v, \nabla). \quad (4.1)$$

Now let $a'$ be the unique point in $R$ with $\nu(a') = 1$. Then we have $d(a', a) \geq d(v, \nabla)$ as well as $M(a, a') = 1$. In conjunction with (4.1) we obtain

$$d(v, w) = d(v, \nabla) \leq d(a', a) \leq \|M\|. \quad \Box$$

**Theorem 1.1.** For an admissible upset $U$ and any two realizable persistence diagrams $\mu, \nu: \text{int } U \to N_0$ with $d_B(\mu, \nu) < \infty$, there exists a finite simplicial complex $X$ and piecewise linear functions $f, g: X \to \mathbb{R}$ with

$$Dgm(f)|_{\text{int } U} = \mu, \quad Dgm(g)|_{\text{int } U} = \nu, \quad \|f - g\|_\infty = d_B(\mu, \nu).$$

**Proof.** As $d_B(\mu, \nu) < \infty$, there is a matching between $\mu$ and $\nu$, and as these multisets are finite, the infimum

$$d_B(\mu, \nu) = \inf\{\|M\| \mid M \text{ is a matching between } \mu \text{ and } \nu\}$$

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Figure 4.4: The triangular region $\nabla \subset ([a_2, a_1] \cup \sigma[a_2, a_1])^2 + 2\pi\mathbb{Z}^2$ is shaded in green. This figure is drawn in tangential coordinates, so that the bisector between $u$ and $\nabla \cap \partial U$ with respect to $d: M \times M \to [0, \infty]$ appears as a straight line.

is attained by some matching $M$. Moreover, we may assume that $\text{pr}_i(M)$ and $\text{pr}_2(M)$ are admissible by the previous Lemma 4.6.

We choose representations $w_i: S_i \to M$ of $\text{pr}_i(M)$ for $i = 1, 2$. Moreover, we choose a bijection $\varphi: S_1 \to S_2$ with $||\varphi|| = ||M||$. Now let $(X, \emptyset; f) := F(w_1)$, and let $g'$ be the function of $F(w_2)$. Since $F$ is a functor, $F(\varphi)$ is a homeomorphism, and so the persistence diagrams of $g'$ and $g := g' \circ F(\varphi)$ are identical. In summary, we obtain

$$Dgm(f)|_{\text{int } U} = \mu, \quad Dgm(g)|_{\text{int } U} = \nu, \quad \text{and} \quad \|f - g\|_\infty = ||F(\varphi)|| = ||\varphi|| = ||M|| = d_B(\mu, \nu). \quad \square$$

**Corollary 4.7.** Let $U \subseteq M$ be an admissible upset and consider the set of realizable persistence diagrams $\mu: \text{int } U \to \mathbb{N}_0$ together with the bottleneck distance $d_B$ as an extended metric space. This is then a geodesic extended metric space.

**Proof.** Let $\mu, \nu: \text{int } U \to \mathbb{N}_0$ be two realizable persistence diagrams with $d_B(\mu, \nu) < \infty$. Then there exists a finite simplicial complex $X$ and piecewise linear functions $f, g: X \to \mathbb{R}$ with

$$Dgm(f)|_{\text{int } U} = \mu, \quad Dgm(g)|_{\text{int } U} = \nu, \quad \text{and} \quad \|f - g\|_\infty = d_B(\mu, \nu)$$

by Theorem 1.1. Thus, we may define the geodesic

$$\gamma: [0, 1] \to \mathbb{N}_0^{\text{int } U}, \quad t \mapsto Dgm((1-t)f + tg)|_{\text{int } U}$$

from $\mu$ to $\nu$. \quad \square
5 The Bottleneck Distance in Contrast to Interleaving Distances

For a PL function \( f : X \to \mathbb{R} \) on a finite simplicial complex \( X \) the derived levelset persistence of \( f \) as introduced by [Cur14] is fully determined by the extended persistence diagram \( \text{Dgm}(f) \) (up to isomorphism) and vice versa, see for example [BF22, Theorem 1.5, Proposition 3.34], and [BBF21, Section 3.2.2]. Moreover, the derived interleaving distance by [KS18, Cur14] and the bottleneck distance coincide in this setting by [BG22].

In the following we show that the situation is more subtle, when we restrict to an admissible upset \( U \subseteq M \). Suppose we have \( U := \bigcup_{n=1}^{\infty} T_n(D) = \{(x, y) : y + 2\pi \geq x\} \). Then \( \text{Dgm}(f)|_{\text{int}U} \) contains the same information as the level set barcode of \( f : X \to \mathbb{R} \) in degree 0, see for example [BBF21, Section 3.2.1]. Moreover, the level set barcode fully classifies the pushforward \( f_\ast K_X \) of the sheaf of locally constant \( K \)-valued functions \( K_X \) on \( X \). So one might be tempted to think that the interleaving distance of sheaves on the reals in the sense of [Cur14, Definition 15.2.3] shares similar properties as the bottleneck distance. However, in contrast to Corollary 4.7 the interleaving distance of sheaves is not intrinsic, let alone geodesic. Before we prove this, we fix some notation which we adopt from [BdSS15]. Let \( I \) denote the poset of open intervals of the real numbers \( \mathbb{R} \) and for \( \delta \geq 0 \) we define the monotone map

\[
\Omega_\delta : I \to I, \quad (a, b) \mapsto (a - \delta, a + \delta).
\]

Then we obtain a monotone map

\[
[0, \infty) \to \text{Map}(I, I), \quad \delta \mapsto \Omega_\delta.
\]

Now considering \( I \) as a thin category, we may consider the relation \( \Omega_0 = \text{id} \leq \Omega_\delta \) as a natural transformation, which we denote by \( \Omega_{0 \leq \delta} : \Omega_0 \to \Omega_\delta \).

**Definition 5.1 (\( \delta \)-Interleaving).** Let \( \delta \geq 0 \) and let \( F \) and \( G \) be sheaves on the real numbers. Then a \( \delta \)-interleaving of \( F \) and \( G \) is a pair of natural transformations

\[
\varphi : G \circ \Omega_\delta \to F \quad \text{and} \quad \psi : F \circ \Omega_\delta \to G
\]

such that both triangles in the diagram

\[
\begin{array}{ccc}
F \circ \Omega_{2\delta} & \xrightarrow{\psi \circ \Omega_\delta} & G \circ \Omega_{2\delta} \\
\downarrow \varphi \circ \Omega_\delta & & \downarrow \psi \circ \Omega_\delta \\
F \circ \Omega_\delta & \xrightarrow{\psi} & G \circ \Omega_\delta \\
\downarrow \varphi & & \downarrow \psi \\
F & \xrightarrow{\varphi} & G
\end{array}
\]

commute.

Now let \( F \) be a sheaf on the reals and let \( I = (a, b) \in I \) be an open interval. Then we have the open cover \((\infty, b) \cup (a, \infty) = \mathbb{R}\). Using the same notation as in Appendix D there is a naturally induced map

\[
\nabla_{(a, b), (a, \infty)}(F) : F((\infty, b)) \oplus F((a, \infty)) \to F(I)
\]

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with cokernel \( \kappa_I(F) := \text{coker} \left( \nabla_{(-\infty,b),(a,\infty)}(F) \right) \), which is natural in \( F \). Now suppose we have \( \delta \geq 0 \), then we may apply the functor \( \kappa_I = \text{coker} \left( \nabla_{(-\infty,b),(a,\infty)}(-) \right) \) to the sheaf homomorphism

\[
F \circ \Omega_{0\leq \delta} : F \circ \Omega_{\delta} \to F
\]

to obtain the map

\[
\kappa_I(F \circ \Omega_{0\leq \delta}) : \kappa_I(F \circ \Omega_{\delta}) \to \kappa_I(F).
\]  

(5.2)

**Lemma 5.2.** The map \( \kappa_I(F \circ \Omega_{0\leq \delta}) \) from (5.2) is injective.

**Proof.** We have

\[
\kappa_I(F \circ \Omega_{\delta}) = \text{coker} \left( \nabla_{(-\infty,b),(a,\infty)}(F \circ \Omega_{\delta}) \right) = \text{coker} \left( \nabla_{(-\infty,b+\delta),(a-\delta,\infty)}(F \circ \Omega_{\delta}) \right),
\]

hence the result follows from Lemma D.1. \( \square \)

**Corollary 5.3.** Let \( \delta \geq 0 \), let \( F \) and \( G \) be sheaves on the real numbers, and let

\[
\varphi : G \circ \Omega_{\delta} \to F \quad \text{and} \quad \psi : F \circ \Omega_{\delta} \to G
\]

be a \( \delta \)-interleaving of \( F \) and \( G \). Moreover, let \( I \in \mathcal{I} \) be an open interval. Then the map

\[
\kappa_I(\psi \circ \Omega_{\delta}) : \kappa_I(F \circ \Omega_{2\delta}) \to \kappa_I(G \circ \Omega_{\delta})
\]

is injective.

**Proof.** This follows in conjunction with the commutativity of the left triangle in (5.1). \( \square \)

Now we harness the example by [BLM21, Proposition 3.6] as well as Corollary 5.3 to show that the interleaving distance of sheaves is not intrinsic. (To make computations more convenient we multiply everything with a factor of 2.) To this end, let

\[
C := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + z^2 = 4, |2y - x| \leq 2\}
\]

and let

\[
f : C \to \mathbb{R}, (x, y, z) \mapsto x \quad \text{and} \quad g : C \to \mathbb{R}, (x, y, z) \mapsto y
\]

be the projections to the \( x \)- and the \( y \)-axis respectively, see also Fig. 5.1. We consider the two pushforwards \( f_*K_C \) and \( g_*K_C \) of the sheaf of locally constant \( K \)-valued functions \( K_C \) on \( C \). By [BLM21, Proposition 3.9] the interleaving distance of the Reeb graphs associated
to $f$ and $g$ is 1. Moreover, the interleaving distance of these Reeb graphs is the same as the interleaving distance of the Reeb cosheaves associated to $f$ and $g$ by [dSMIP16]. Now up to isomorphism the pushforward $f_*K_C$ can also be obtained from the Reeb cosheaf associated to $f: C \to \mathbb{R}$ by post-composition with the contravariant functor

$$\Map(-, K): S \mapsto \Map(S, K).$$

Thus, the interleaving distance of $f_*K_C$ and $g_*K_C$ is at most 1 by [BdSS15, Proposition 2.2.11]. Since $|f(p) - g(p)| \leq 2$ for all $p \in C$, the induced intrinsic interleaving distance of $f_*K_C$ and $g_*K_C$ is at most 2. In the following we show that the induced intrinsic interleaving distance is at least 2 and hence equal to 2. Thus, the interleaving distance of sheaves on the reals and the induced intrinsic interleaving distance differ by at least a factor of 2.

**Proposition 5.4.** The induced intrinsic interleaving distance of $f_*K_C$ and $g_*K_C$ is 2.

**Proof.** It remains to show that the induced intrinsic interleaving distance of $f_*K_C$ and $g_*K_C$ is at least 2. To this end, let $\varepsilon := \frac{1}{2^{N+1} + 1}$ for some $N \in \mathbb{N}$ and suppose there is a family of sheaves

$$\{\gamma(t) \mid 0 \leq t \leq 2 - 4\varepsilon\}$$

with $\gamma(t)$ and $\gamma(t')$ being $|t - t'|$-interleaved for all $t, t' \in [0, 2 - 4\varepsilon]$ and $\gamma(0) = f_*K_C$ and $\gamma(2 - 4\varepsilon) = g_*K_C$. We consider the values of $\gamma$ at the finite sequence of consecutive times

$$t_0 := 0,$$
$$t_1 := 1 - \varepsilon,$$
$$t_2 := \frac{3}{2}(1 - \varepsilon),$$
$$t_3 := \frac{7}{4}(1 - \varepsilon),$$
$$\vdots$$
$$t_{n+1} := \frac{2^{n+1} - 1}{2^n}(1 - \varepsilon) \quad \text{for } n = 0, \ldots, N,$$
$$\vdots$$
and
$$t_{N+1} := \frac{2^{N+1} - 1}{2^N}(1 - \varepsilon) = 2 - 4\varepsilon.$$

If we set $\delta_n := t_{n+1} - t_n = \frac{1}{2^n}(1 - \varepsilon)$ for $n = 0, \ldots, N$, then we may also characterize the sequence (5.3) by the equations

$$t_0 = 0, \quad t_{N+1} = 2 - 4\varepsilon, \quad \text{and} \quad \delta_n = 2\delta_{n+1} \quad \text{for } n = 0, \ldots, N - 1.$$ 

For any two consecutive times $t_n$ and $t_{n+1}$ for $n = 0, \ldots, N$ we may choose a $\delta_n$-interleaving of $\gamma(t_n)$ and $\gamma(t_{n+1})$ with interleaving homomorphisms

$$\varphi_n : \gamma(t_{n+1}) \circ \Omega_{\delta_n} \to \gamma(t_n)$$

and

$$\psi_n : \gamma(t_n) \circ \Omega_{\delta_n} \to \gamma(t_{n+1}).$$
By gluing the left triangles of the corresponding interleaving diagrams (5.1) in the special case where $N = 2$ we obtain the commutative diagram

\[ f_* K_C \circ \Omega_{16\varepsilon} \]

Now let $I := (-\varepsilon, \varepsilon)$. By Corollary 5.3 we have the injective map

\[ \kappa_f (\psi_n \circ \Omega_{\delta_n}) : \kappa_f (\gamma (t_n) \circ \Omega_{2\delta_n}) \to \kappa_f (\gamma (t_{n+1}) \circ \Omega_{\delta}) \]

for any $n = 0, \ldots, N$. Moreover, as is illustrated by (5.4), any two consecutive homomorphisms $\psi_n \circ \Omega_{\delta_n} : \gamma (t_n) \circ \Omega_{2\delta_n} \to \gamma (t_{n+1}) \circ \Omega_{\delta}$ are composable. Thus, by applying the
functor $\kappa_I$ to the composition

$$f_* K_C \circ \Omega_{2\delta_0}$$

$$\gamma(t_1) \circ \Omega_{\delta_0}$$

$$\gamma(t_2) \circ \Omega_{\delta_1}$$

$$\gamma(t_N) \circ \Omega_{2\delta_N}$$

$$g_* K_C \circ \Omega_{\delta_N}$$

of sheaf homomorphisms we obtain an injective map

$$\kappa_I(f_* K_C \circ \Omega_{2\delta_0}) \longrightarrow \kappa_I(g_* K_C \circ \Omega_{\delta_N}). \quad (5.5)$$

Now let

$$I' := \Omega_{2\delta_0}(I) = (-2 + \varepsilon, 2 - \varepsilon) \quad \text{and} \quad I'' := \Omega_{\delta_N}(I).$$

Then we have

$$\kappa_I(f_* K_C \circ \Omega_{2\delta_0}) = \kappa_{I'}(f_* K_C) \quad \text{and} \quad \kappa_I(g_* K_C \circ \Omega_{\delta_N}) = \kappa_{I''}(g_* K_C).$$

Moreover, evaluating $f_* K_C$ at the corresponding intervals we obtain the commutative diagram

$$\begin{array}{ccc}
(f_* K_C)(-\infty, 2 - \varepsilon) & \sim & (f_* K_C)(-2 + \varepsilon, \infty) \\
\downarrow & & \downarrow \\
K^2 & \sim & K^2 \\
\downarrow & & \downarrow \\
(f_* K_C)(I') & \sim & (f_* K_C)(I')
\end{array}$$

hence $\kappa_{I'}(f_* K_C) \cong K$. On the other hand, we have $\kappa_{I''}(g_* K_C) \cong \{0\}$ in contradiction to the existence of an injective map as in (5.5). □
**Corollary 5.5.** The interleaving distance of sheaves on the reals taking values in the category of $K$-vector spaces and the induced intrinsic interleaving distance differ by at least a factor of 2.

**Remark 5.6.** In the above argument we didn’t use that $K$ was a field and so the proof applies to any ring, the integers in particular. Thus, Corollary 5.5 also applies to sheaves valued in the category of abelian groups.

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**References**

[BBF21] Ulrich Bauer, Magnus Bakke Botnan, and Benedikt Fuhr. Structure and Interleavings of Relative Interlevel Set Cohomology. arXiv e-prints, 2021.

[BdSS15] Peter Bubenik, Vin de Silva, and Jonathan Scott. Metrics for generalized persistence modules. Found. Comput. Math., 15(6):1501–1531, 2015.

[BdSS17] Peter Bubenik, Vin de Silva, and Jonathan Scott. Interleaving and gromov-hausdorff distance, 2017. Preprint.

[BEMP13] Paul Bendich, Herbert Edelsbrunner, Dmitriy Morozov, and Amit Patel. Homology and robustness of level and interlevel sets. Homology Homotopy Appl., 15(1):51–72, 2013.

[BF22] Ulrich Bauer and Benedikt Fuhr. Categorification of Extended Persistence Diagrams. arXiv e-prints, May 2022.

[BG22] Nicolas Berkouk and Grégory Ginot. A derived isometry theorem for sheaves. Adv. Math., 394:Paper No. 108033, 39, 2022.

[BGO19] Nicolas Berkouk, Grégory Ginot, and Steve Oudot. Level-sets persistence and sheaf theory. arXiv e-prints, Jul 2019.

[BLM21] Ulrich Bauer, Claudia Landi, and Facundo Mémoli. The Reeb graph edit distance is universal. Found. Comput. Math., 21(5):1441–1464, 2021.

[CdM09] Gunnar Carlsson, Vin de Silva, and Dmitriy Morozov. Zigzag persistent homology and real-valued functions. In Proceedings of the Twenty-fifth Annual Symposium on Computational Geometry, SCG ’09, pages 247–256, New York, NY, USA, 2009. ACM.

[CdSKM19] Gunnar Carlsson, Vin de Silva, Sara Kališnik, and Dmitriy Morozov. Parametrized homology via zigzag persistence. Algebr. Geom. Topol., 19(2):657–700, 2019.
[CSEH07] David Cohen-Steiner, Herbert Edelsbrunner, and John Harer. Stability of persistence diagrams. Discrete Comput. Geom., 37(1):103–120, 2007.

[CSEH09] David Cohen-Steiner, Herbert Edelsbrunner, and John Harer. Extending persistence using Poincaré and Lefschetz duality. Found. Comput. Math., 9(1):79–103, 2009.

[Cur14] Justin Michael Curry. Sheaves, cosheaves and applications. ProQuest LLC, Ann Arbor, MI, 2014. Thesis (Ph.D.)–University of Pennsylvania.

[dSMP16] Vin de Silva, Elizabeth Munch, and Amit Patel. Categorified Reeb graphs. Discrete Comput. Geom., 55(4):854–906, 2016.

[DW07] Tamal K. Dey and Rephael Wenger. Stability of critical points with interval persistence. Discrete Comput. Geom., 38(3):479–512, 2007.

[ELZ02] Herbert Edelsbrunner, David Letscher, and Afra Zomorodian. Topological persistence and simplification. volume 28, pages 511–533. 2002. Discrete and computational geometry and graph drawing (Columbia, SC, 2001).

[HKLM19] Shaun Harker, Miroslav Kramár, Rachel Levanger, and Konstantin Mischaikow. A comparison framework for interleaved persistence modules. J. Appl. Comput. Topol., 3(1-2):85–118, 2019.

[KS90] Masaki Kashiwara and Pierre Schapira. Sheaves on manifolds, volume 292 of Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1990. With a chapter in French by Christian Houzel.

[KS18] Masaki Kashiwara and Pierre Schapira. Persistent homology and microlocal sheaf theory. J. Appl. Comput. Topol., 2(1-2):83–113, 2018.

[Les15] Michael Lesnick. The theory of the interleaving distance on multidimensional persistence modules. Found. Comput. Math., 15(3):613–650, 2015.

[MP20] Alex McCleary and Amit Patel. Bottleneck stability for generalized persistence diagrams. Proc. Amer. Math. Soc., 148(7):3149–3161, 2020.

[Spa81] Edwin H. Spanier. Algebraic topology. Springer-Verlag, New York-Berlin, 1981. Corrected reprint.

[tom08] Tammo tom Dieck. Algebraic topology. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008.
A Constructing Relative Interlevel Set Homology

For computations using long exact sequences in homology, a generalization of the relative interlevel set homology for a function \( f : X \to \mathbb{R} \) considered on a pair \( A \subseteq X \) of spaces will be useful. For such considerations we will use an intersection-like operation on pairs \( A \subseteq X \) of spaces. We note that this operation turns out to be different from the componentwise intersection, which is the meet operation in the lattice \((2^X \times 2^X, \subseteq)\).

**Definition A.1.** We say that a pair of sets \((X, A)\) is admissible if \( A \subseteq X \). For two admissible pairs of sets \((X, A)\) and \((Y, B)\), we define \((X, A) \cap (Y, B)\) to be the pair \((X \cap Y, (A \cap Y) \cup (X \cap B))\).

Substituting intersections by products, the above definition becomes very similar to a well known notion of a product of pairs of spaces used in a relative version of the Künneth Theorem, see for example [Spa81, Section 5.3]. Moreover, if we view the partial order on sets given by inclusions as the structure of a category, then the intersection is the categorical product.

**Lemma A.2** (Closedness and Commutativity). For two admissible pairs of sets \((X, A)\) and \((Y, B)\), the pair \((X, A) \cap (Y, B)\) is admissible, and we have \((X, A) \cap (Y, B) = (Y, B) \cap (X, A)\).

Let \((Z, C)\) be another admissible pair.

**Lemma A.3.** Consider an admissible pair of sets \((X, A)\). The operation \((X, A) \cap -\) preserves componentwise unions and intersections of admissible pairs. In other words, for two further admissible pairs of sets \((Y, B)\) and \((Z, C)\), we have the equation

\[
(X, A) \cap (Y \cup Z, B \cup C) = ((X, A) \cap (Y, B)) \cup ((X, A) \cap (Z, C)),
\]

and similarly for intersections. Here, the union on the right hand side of the above equation is to be understood as a componentwise union.

Now let \((X, A)\) be a finite simplicial pair and \( f : X \to \mathbb{R} \) be a piecewise linear map. In the following we construct the relative interlevel set homology \( h(X, A; f) \) of \( f : A \subseteq X \to \mathbb{R} \). As we now have a relative part \( A \) as well, this is an extension of our construction from Section 2. In order to construct the functor \( h(X, A; f) \), we use a similar procedure as in [BBF21, Section 2]. To this end, we set

\[
F' : D \to \text{vect}_K, u \mapsto H_\bullet((X, A) \cap f^{-1}(\rho(u)); K),
\]

where \( \text{vect}_K \) is the category of finite-dimensional graded vector space over \( K \). We construct \( h(X, A; f) : \mathbb{M} \to \text{vect}_K \) in such a way that it carries the same (and more) information as \( F' \) with precomposition by \( T \) taking the place of the degree shift

\[
\Sigma : \text{vect}_K \to \text{vect}_K, M_\bullet \mapsto M_{\bullet-1}.
\]

As an intermediate step, we extend \( F' \) to a functor

\[
F : \mathbb{M} \to \text{vect}_K.
\]
which is $\mathbb{Z}$-equivariant or strictly stable in the sense that

$$F \circ T = \Sigma \circ F.$$  \hspace{1cm} (A.1)

Now as a map into the objects of $\text{vect}_K^\mathbb{Z}$ such a functor $F$ carries no new information in comparison to $F'$. Moreover, by (A.1), most of the information carried by $F$ is redundant and we may discard all redundant information by post-composition with the projection $\text{pr}_0 : \text{vect}_K^\mathbb{Z} \to \text{vect}_K, M \mapsto M_0$.

Now in order to obtain such a strictly stable functor $F$ from $F'$ we need to glue consecutive layers using connecting homomorphisms. To this end, let $R_D := \{(w, \hat{u}) \in D \times T(D) \mid w \preceq \hat{u} \preceq T(w)\}$ as in [BBF21, Definition A.8]. As shown in Fig. A.1, any pair $(w, \hat{u}) \in R_D$ determines an axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in D$ with $T(u) = \hat{u}$. Moreover, as this rectangle is contained in $D$, the corresponding join $w = v_1 \lor v_2$ and meet $u = v_1 \land v_2$ are preserved by $\rho$ by Proposition 2.1.3. Furthermore, since taking preimages is a homomorphism of boolean algebras, $f^{-1}$ also preserves joins and meets, which in this case are the componentwise unions and intersections. Finally, by Lemma A.3, joins and meets are also preserved by $(X, A) \sqcap \cdot$. This means that $(X, A) \sqcap f^{-1}(\rho(v_1))$ is the componentwise intersection of $(X, A) \sqcap f^{-1}(\rho(v_1))$ and $(X, A) \sqcap f^{-1}(\rho(v_2))$, while $(X, A) \sqcap f^{-1}(\rho(w))$ is their union. As $f$ is piecewise linear and $A$ a subcomplex of $X$, the triad $(X, A) \sqcap f^{-1}(\rho(w); \rho(v_1), \rho(v_2))$ of pairs is excisive in each component. Thus, we have the boundary map

$$\partial'_{(w, \hat{u})} : H_\bullet((X, A) \sqcap f^{-1}(\rho(w)); K) \to H_{\bullet-1}((X, A) \sqcap f^{-1}(\rho(u)); K)$$

of the corresponding Mayer–Vietoris sequence as described by [tom08, Theorem 10.7.7].

Now let $\text{pr}_1 : R_D \to D$ and $\text{pr}_2 : R_D \to T(D)$ be the projections to the first and the second component, respectively. Then we have functors $F' \circ \text{pr}_1$ and $\Sigma \circ F' \circ T^{-1} \circ \text{pr}_2$ from $R_D$ to $\text{vect}_K^\mathbb{Z}$ and $\partial'$ is a natural transformation

$$\partial : F' \circ \text{pr}_1 \Rightarrow \Sigma \circ F' \circ T^{-1} \circ \text{pr}_2.$$
Thus, the functor $F' : D \to \text{vect}_K^Z$ and $\partial'$ determine a unique strictly stable functor $F : \mathbb{M} \to \text{vect}_K^Z$ by [BBF21, Proposition A.14]. To obtain a functor of type $\mathbb{M} \to \text{vect}_K$ from $F$ we post-compose $F : \mathbb{M} \to \text{vect}_K^Z$ with the projection $\text{pr}_0 : \text{vect}_K^Z \to \text{vect}_K$ and we define

$$h(X, A; f) := h(X, A; f; K) := \text{pr}_0 \circ F.$$  

### B  Structure of Relative Interlevel Set Homology

In the following we describe the structure of relative interlevel set homology in order to show that our Definition 2.2 of the extended persistence diagram is indeed equivalent to the original definition from [CSEH09]. This relation to extended persistence is also discussed in [CdM09, BEMP13]. Moreover, the authors of [BEMP13] have shown that the restrictions of relative interlevel set homology to each individual tile (called pyramid in that article) can be decomposed into restricted blocks. We show that the relative interlevel set homology decomposes into the following type of indecomposables, see also Fig. B.1.

**Definition B.1 (Block).** For $v \in \text{int} \mathbb{M}$ we define

$$B^v : \mathbb{M} \to \text{Vect}_K, \ w \mapsto \begin{cases} K & w \in (\uparrow v) \cap \text{int}(\downarrow T(v)) \\ \{0\} & \text{otherwise} \end{cases},$$

where $\text{int}(\downarrow T(v))$ is the interior of the downset of $T(v)$ in $\mathbb{M}$. The internal maps are identities whenever both domain and codomain are $K$, otherwise they are zero.

To this end, we make use of a structure theorem from [BBF21, Theorem 3.21], which applies to certain functors from $\mathbb{M}$ to $\text{vect}_K$ and in particular to the relative interlevel set homology $h(f) : \mathbb{M} \to \text{vect}_K$ of a piecewise linear function $f : X \to \mathbb{R}$ on a finite simplicial complex $X$. More specifically, our structure theorem assumes that the following two notions apply to $h(f) : \mathbb{M} \to \text{vect}_K$.

**Definition B.2.** We say that a functor $F : \mathbb{M} \to \text{Vect}_K$ is **sequentially continuous**, if for any decreasing sequence $(u_k)_{k=1}^\infty$ in $\mathbb{M}$ converging to $u$ the natural map

$$F(u) \to \lim_{k} F(u_k)$$

Figure B.1: The indecomposable $B^v : \mathbb{M} \to \text{Vect}_K$. 

is an isomorphism; see also \cite{BBF21} Definition 2.4.

As $f : X \rightarrow \mathbb{R}$ is simplexwise linear on some finite triangulation of $X$ and as singular homology is homotopy invariant, the relative interlevel set homology $h(f) : M \rightarrow \text{vect}_K$ is indeed sequentially continuous.

\textbf{Definition B.3.} We say that a functor $F : M \rightarrow \text{Vect}_K$ vanishing on $\partial M$ is \textit{homological}, if for any axis-aligned rectangle with one corner lying on $l_1$ and the other corners $u \preceq v \preceq w \in M$, the long sequence

\[
\cdots \longrightarrow F(T^{-1}(w)) \longrightarrow F(T(u)) \longrightarrow F(u) \longrightarrow F(v) \longrightarrow F(w) \longrightarrow F(T(v)) \longrightarrow \cdots
\]

is exact; see also Fig. B.2 or \cite{BBF21} Definition C.3.

In \cite{BBF21} Proposition C.4 we also provide the following useful characterization of homological functors.

\textbf{Proposition B.4.} A functor $F : M \rightarrow \text{Vect}_K$ vanishing on $\partial M$ is homological iff for any
axis-aligned rectangle $u \leq v_1, v_2 \leq w \in D$ as shown in Fig. A.1 the long sequence

$$
\cdots \to F(T^{-1}(w)) \to F(u) \to F(v_1) \oplus F(v_2) \overset{(1 \quad -1)}{\to} F(w) \to F(T(u)) \to \cdots
$$

(B.1)

is exact.

Now if we set $F := h(f) : M \to \text{vect}_K$, then the long sequence (B.1) is a Mayer–Vietoris sequence by the construction of $h(f)$ and hence exact. Thus, $h(f) : M \to \text{vect}_K$ is homological. All in all, the relative interlevel set homology $h(f) : M \to \text{vect}_K$ satisfies the assumptions of the following theorem from [BBF21, Theorem 3.21].

**Theorem B.5.** Any sequentially continuous homological functor $F : M \to \text{vect}_K$ decomposes as

$$
F \cong \bigoplus_{v \in \operatorname{int} M} (B^v)^{\oplus \nu(v)},
$$

where $\nu := \operatorname{Dgm}(F)$.

In particular the relative interlevel set homology of $f : X \to \mathbb{R}$ is completely classified by the extended persistence diagram $\operatorname{Dgm}(f)$. We explain how this implies that our definition of the extended persistence diagram is equivalent to the definition from [CSEH09]. This is closely related to [BBF21, Section 3.2], where we also describe a connection to the level set barcode. We consider Fig. B.3 and the restriction of $h(f) : M \to \text{vect}_K$ to the subposet of $M$, which is shaded in blue in this figure. Here each point on the vertical blue line segment to the upper left is assigned the homology space in degree 0 of a sublevel set of $f$, of $X$, or of a pair with $X$ as the first component and a superlevel set as the second component. Up to isomorphism of posets, this is the extended persistent homology of $f : X \to \mathbb{R}$ in degree 0. Similarly, any point on the horizontal blue line segment in the center is assigned the homology space of some pair of preimages in degree 1 and any point on the vertical blue line at the lower right is assigned the homology of some pair in degree 2. By Theorem B.5 the relative interlevel set homology $h(f)$ decomposes into blocks as in Definition B.1. Now the support of each such block intersects exactly one of these blue line segments. We focus on the horizontal blue line segment in the center of Fig. B.3 which carries the extended persistent homology of $f : X \to \mathbb{R}$ in degree 1. Any choice of decomposition of $h(f) : M \to \text{vect}_K$ yields a decomposition of its restriction to this line segment and thus of persistent cohomology in degree 1. Moreover, the support of the contravariant block assigned to any of the black dots in Fig. B.3 intersects this horizontal line segment.

First assume that the black dot on the lower right appears in $\operatorname{Dgm}(f)$. Then the restriction of the associated block to the horizontal blue line segment is a direct summand of the persistent homology of $f : X \to \mathbb{R}$ in degree 1 and the intersection of its support with the blue line segment is the life span of the corresponding feature in the sense that the point of intersection of the right edge marks the birth of a homology class that dies
Figure B.3: The subposet of $\mathcal{M}$ corresponding to extended persistence shaded in blue as well as three vertices contained in the domains corresponding to 1-dimensional relative, extended, and ordinary persistence; see also Fig. 1.1.

as soon as it is mapped into the sublevel set corresponding to the point of intersection of the left edge. Moreover, this life span is encoded by the position of this black dot. Now this particular black dot on the lower right of Fig. B.3 is contained in the triangular region labeled as Ord$_1$ in Fig. 1.1. Furthermore, any vertex of the extended persistence diagram $\text{Dgm}(f)$ contained in the triangular region labeled Ord$_1$ describes a feature of $f: X \to \mathbb{R}$, which is born at some sublevel set and also dies at some sublevel set. Thus, up to reparametrization, the ordinary persistence diagram of $f: X \to \mathbb{R}$ in degree 1 is the restriction of $\text{Dgm}(f): \mathcal{M} \to \mathcal{N}_0$ to the region labeled Ord$_1$ in Fig. 1.1.

Now suppose that the black dot on the upper left in Fig. B.3 appears in $\text{Dgm}(f)$. Then the intersection of the support of the associated contravariant block with the horizontal blue line segment describes the life span of a feature which is born at the homology of $X$ relative to some superlevel set in degree 1 and also dies at some relative homology space. Moreover, this is true for any vertex of $\text{Dgm}(f)$ contained in the triangular region labeled Rel$_1$ in Fig. 1.1. Thus, up to reparametrization, the relative subdiagram of $f: X \to \mathbb{R}$ in degree 1 is the restriction of $\text{Dgm}(f): \mathcal{M} \to \mathcal{N}_0$ to the region labeled Rel$_1$ in Fig. 1.1.

Finally, the black dot to the upper right in Fig. B.3 (if in $\text{Dgm}(f)$), or any other vertex of $\text{Dgm}(f)$ in the square region labeled Ext$_1$ in Fig. 1.1 describes a feature, which is born at the homology of some sublevel set of $f: X \to \mathbb{R}$ and dies at the homology of $X$ relative to some superlevel set. Thus, the extended subdiagram of $f: X \to \mathbb{R}$ in degree 1 is the restriction of $\text{Dgm}(f): \mathcal{M} \to \mathcal{N}_0$ to the square region labeled Ext$_1$ in Fig. 1.1.

As we have analogous correspondences for each line segment of the subposet of $\mathcal{M}$ shaded in blue in Fig. B.3, we obtain a partition of the lower right part of the strip $\mathcal{M}$ into regions, corresponding to ordinary, relative, and extended subdiagrams as labeled in Fig. 1.1, analogous to [CSEH09].
C Constructing Lifts of Points

In this appendix we provide formal definitions for the construction of lifts of points $u \in \mathbb{I}$ from Section 4.1. More specifically, we provide formal definitions of the simplicial complex $X_u$ as well as the maps $r_u : X_u \to [0, 1]$, $b_u : [0, 1] \to \mathbb{R}$, and $j_u : [0, 1] \to A_u \subset X_u$ for each $u \in \mathbb{I}$.

We start with defining the affine function $b_u : [0, 1] \to \mathbb{R}$ for each $u = (u_1, u_2) \in \mathbb{S}$ by the formula

$$
 b_u : [0, 1] \to \mathbb{R}, \begin{cases}
 0 \mapsto \min \text{fauxtang}(\{u_1, u_2\}) \\
 1 \mapsto \max \text{fauxtang}(\{u_1, u_2\}),
\end{cases}
$$

(C.1)

where fauxtan : $\mathbb{R} \to \mathbb{R}$, shown in Fig. C.1, is the unique continuous extension of the restricted ordinary tangent function $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ that is symmetric with respect to translation by $2\pi$ and reflection at $\frac{\pi}{2}$: fauxtan(x) = fauxtan(x + 2\pi) = fauxtan(\pi - x) for all $x \in \mathbb{R}$, and fauxtan($\pm \frac{\pi}{2}$) = $\pm \infty$. As a side note, for each $u \in \mathbb{R}$ we have $\text{Dgm}(b_u) = 1_u$ and hence $b_u = f_u$.

Next we construct the simplicial complex $X_u$ as well as the two simplicial maps $j_u : [0, 1] \to X_u$ and $r_u : X_u \to [0, 1]$ in terms of the corresponding vertex maps for
each \( u \in \mathbb{I} \). To this end, let \( Q := \uparrow (\frac{\pi}{2}, -\frac{\pi}{2}) \) be the principal upset of \((\frac{\pi}{2}, -\frac{\pi}{2})\). Then \( E := Q \setminus T(Q) \) is a fundamental domain for the \( \mathbb{Z} \)-action on \( M \), and the map
\[
\mathbb{N} \times E \to M \setminus Q, (n,v) \mapsto T^{-n}(v)
\]
is a bijection, where 0 is excluded from the natural numbers \( \mathbb{N} \), i.e. \( 0 \notin \mathbb{N} \). Now the intersection
\[
E \cap \left( \left( -\frac{\pi}{2}, \frac{\pi}{2} \right)^2 + \pi \mathbb{Z}^2 \right)
\]
has three connected components; namely
\[
E_1 := E \cap \left( -\frac{3\pi}{2}, -\frac{\pi}{2} \right) \times \left( -\frac{\pi}{2}, \frac{\pi}{2} \right),
\]
\[
E_2 := E \cap \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \times \left( \frac{\pi}{2}, 3\frac{\pi}{2} \right), \quad \text{and}
\]
\[
E_3 := E \cap \left( -\frac{\pi}{2}, \frac{\pi}{2} \right)^2
\]
as shown in Fig. C.2. Altogether we obtain the bijection
\[
\mathbb{N} \times (E_1 \cup E_2 \cup E_3) \xrightarrow{\cong} \mathbb{I}, (n,v) \mapsto T^{-n}(v).
\]

In the following we denote the standard \( n \)-simplex by \( \Delta^n \) for \( n \in \mathbb{N}_0 \). Its vertex set is given by the standard basis \{\( e_0, \ldots, e_n \)\} \( \subset \mathbb{R}^{n+1} \). Now let \( u \in \mathbb{I} \) and let \( n \in \mathbb{N} \) such that \( v := T^n(u) \in E \). We distinguish between three cases depending on which of the three connected components of \( E \) contains \( v \).

For \( v \in E_1 \), we set
\[
X_u := \Lambda_{n+1}^{n+1},
\]
\[
j_u : [0,1] \to X_u, \begin{cases} 0 \mapsto e_0, \\ 1 \mapsto e_{n+1}, \end{cases}
\]
and \( r_u : X_u \to [0,1], e_k \mapsto \begin{cases} 0 & 0 \leq k \leq n \\ 1 & k = n + 1. \end{cases} \)
Here $\Lambda_{n+1}^{n+1}$ is the simplicial complex we obtain from $\partial \Delta^{n+1}$ after removing the facet opposite to the $(n+1)$-st vertex, keeping all other simplices.

For $v \in E_2$, we set

$$X_u := \Lambda_0^{n+1},$$

$$j_u : [0, 1] \to X_u, \quad \begin{cases} 0 \mapsto e_0 \\ 1 \mapsto e_{n+1}, \end{cases}$$

and

$$r_u : X_u \to [0, 1], \quad e_k \mapsto \begin{cases} 0 & k = 0 \\ 1 & 1 \leq k \leq n+1. \end{cases}$$

Here $\Lambda_0^{n+1}$ is the simplicial complex we obtain from $\partial \Delta^{n+1}$ after removing the facet opposite to the 0-th vertex.

For $v \in E_3$ the construction is not as direct as in the previous two cases. We start by setting

$$X_u := (\partial \Delta^{n+1}) \times [0, 1] \subset \mathbb{R}^{n+2} \times \mathbb{R}$$

with the following triangulation (which coincides with the triangulation induced by the corresponding simplicial set). The vertex set of $X_u$ is $\{e_0, \ldots, e_{n+1}\} \times \{0, 1\}$ and a set of pairs in $\{e_0, \ldots, e_{n+1}\} \times \{0, 1\}$ spans a simplex in $X_u$ iff the index of each basis vector that appears in a pair with 0 is at most the index of any basis vector that appears in a pair with 1 and there is one basis vector not appearing in any pair. (The second condition ensures that we are not adding a simplex to $X_u$ that is not even contained in the proclaimed underlying space of $X_u$.) With this triangulation, we may define the simplicial map

$$j_u : [0, 1] \to X_u, \quad \begin{cases} 0 \mapsto (e_0, 0) \\ 1 \mapsto (e_{n+1}, 1). \end{cases}$$

To define the retraction $r_u : X_u \to [0, 1]$ we have to distinguish between two cases depending on which of the two coordinates $v_1$ and $v_2$ of $v = (v_1, v_2)$ is larger than or smaller than the other. For $v_1 \leq v_2$ we set

$$r_u : X_u \to [0, 1], \quad (e_k, l) \mapsto \begin{cases} 0 & 0 \leq k \leq n \\ 1 & k = n+1 \end{cases}$$

and if $v_1 > v_2$, then we define $r_u$ to be the projection

$$r_u := \text{pr}_2 : (\partial \Delta^{n+1}) \times [0, 1] \to [0, 1], \quad (e_k, l) \mapsto l$$

onto the second factor. Here the intuition is the following. When $v_1 = v_2$, then $b_u : [0, 1] \to \mathbb{R}$ is constant, so we can “flip” the retraction $r_u : X_u \to [0, 1]$ without changing the composed map $f_u = b_u \circ r_u$. 

38
D Sections of Sheaves on Intersections

Let $X$ be a topological space, let $U_1 \cup U_2 = X$ be an open cover of $X$, and let $F$ be a sheaf on $X$ with values in the category of abelian groups. Then we may consider the group of sections $F(U_1 \cap U_2)$ of $F$ on the intersection of $U_1$ and $U_2$. Some of these sections can be obtained by restriction from sections of $F$ on $U_1$ or $U_2$. Thus, there is an induced group homomorphism

$$\nabla_{U_1, U_2}(F): F(U_1) \oplus F(U_2) \to F(U_1 \cap U_2).$$

Now suppose we have open subsets $V_1, V_2 \subseteq X$ with $U_i \subseteq V_i$ for $i = 1, 2$. Then we have the commutative diagram

$$
\begin{array}{ccc}
F(V_1) \oplus F(V_2) & \longrightarrow & F(U_1) \oplus F(U_2) \\
\nabla_{V_1, V_2}(F) & & \nabla_{U_1, U_2}(F) \\
F(V_1 \cap V_2) & \longrightarrow & F(U_1 \cap U_2) \\
\downarrow & & \downarrow \\
\text{coker}(\nabla_{V_1, V_2}(F)) & \longrightarrow & \text{coker}(\nabla_{U_1, U_2}(F))
\end{array}
$$

(L.1)

as well as the induced map on cokernels $\alpha: \text{coker}(\nabla_{V_1, V_2}(F)) \to \text{coker}(\nabla_{U_1, U_2}(F))$.

Lemma D.1. The map $\alpha: \text{coker}(\nabla_{V_1, V_2}(F)) \to \text{coker}(\nabla_{U_1, U_2}(F))$ in the above diagram (L.1) is injective.

Proof. Identifying $H^0(U; F)$ with $F(U)$ for any open subset $U \subseteq X$ and using the Mayer-Vietoris sequence for sheaf cohomology we obtain the commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
H^0(X; F) & \longrightarrow & H^0(X; F) \\
\left( \begin{array}{c} 1 \\ -1 \end{array} \right) & \downarrow & \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \\
H^0(V_1; F) \oplus H^0(V_2; F) & \longrightarrow & H^0(U_1; F) \oplus H^0(U_2; F) \\
\nabla_{V_1, V_2}(F) & \downarrow & \nabla_{U_1, U_2}(F) \\
H^0(V_1 \cap V_2; F) & \longrightarrow & H^0(U_1 \cap U_2; F) \\
\downarrow & & \downarrow \\
H^1(X; F) & \longrightarrow & H^1(X; F)
\end{array}
$$

with exact columns. By the exactness of these two columns at $H^0(V_1 \cap V_2; F)$ and

\[\text{see for example [KS90, Remark 2.6.10]}\]
$H^0(U_1 \cap U_2; F)$ respectively we obtain the commutative diagram

\[
\begin{array}{ccc}
\text{coker}(\nabla_{V_1,V_2}(F)) & \xrightarrow{\alpha} & \text{coker}(\nabla_{U_1,U_2}(F)) \\
\downarrow & & \downarrow \\
H^1(X; F) & \longrightarrow & H^1(X; F)
\end{array}
\]

with the two vertical maps monomorphisms of abelian groups as indicated. Thus, the map $\alpha: \text{coker}(\nabla_{V_1,V_2}(F)) \to \text{coker}(\nabla_{U_1,U_2}(F))$ is a monomorphism as well.