Note on Partitions with Even Parts below Odd Parts*

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1. INTRODUCTION

The study of integer partitions in which all even parts are smaller than odd parts with some additional restrictions was first considered by Andrews [1]. These partitions later attracted extensive research interests, with follow-ups by Andrews [2], Bringmann and Jennings-Shaffer [3] and the author [4]. In particular, Andrews defined the following partition set in [1].

Definition 1. We denote by $EO^*$ the set of partitions with no even parts such that each different part appears an even number of times (in which we tacitly assume that $0$ is the largest even part) or partitions with all even parts smaller than odd parts such that only the largest even part appears an odd number of times.

For example, 6 has four partitions in $EO^*$, namely, $1 + 1 + 1 + 1 + 1$, $2 + 2 + 2$, $3 + 3$ and 6. Andrews showed that $EO^*$ satisfies the generating function identity

$$
\sum_{\pi \in EO^*} q^{||\pi||} = \frac{(q^4; q^4)_{\infty}}{(q^2; q^4)^2_{\infty}}
$$

where $||\pi||$ denotes the sum of all parts in $\pi$. Throughout, we adopt the standard $q$-Pochhammer symbol for $n \in \mathbb{N} \cup \{\infty\}$,

$$(A; q)_n := \prod_{k=0}^{n-1} (1 - Aq^k).$$

One suggestion in [1, Problem 4] is to “undertake a more extensive investigation of the properties of $EO^*$.” Our first objective in this note is about two disjoint subsets of $EO^*$ separated by the residue classes of the largest even part modulo 4.

Theorem 1. Let $eo_0^*(n)$ and $eo_2^*(n)$ denote the number of partitions of $n$ in $EO^*$ with largest even part congruent to 0 and 2 modulo 4, respectively. Then

$$
eo_0^*(n) \begin{cases} 
eq eo_2^*(n) & \text{if } n \text{ is not divisible by 4;} \\
> eo_2^*(n) & \text{if } n \text{ is divisible by 4.}
\end{cases}
$$

There are many natural extensions of integer partitions, among which the most important one is the overpartition [5]. An overpartition is an integer partition in which the first occurrence of each distinct part may be overlined. We have the following overpartition analog of $EO^*$.

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**Definition 2.** We denote by $\mathcal{EO}^*$ the set of overpartitions with no even parts such that each different part appears an even number of times (in which we tacitly assume that 0 is the largest even part) or overpartitions with all even parts smaller than odd parts such that only the largest even part appears an odd number of times.

For example, 6 has eight partitions in $\mathcal{EO}^*$, namely, $1 + 1 + 1 + 1 + 1 + 1$, $\overline{3} + 1 + 1 + 1 + 1$, $2 + 2 + 2$, $\overline{2} + 2 + 2$, $3 + 3$, $\overline{3} + 3$, 6 and $\overline{6}$. Our next objective is to give an overpartition analog of Theorem 1.

**Theorem 2.** Let $\text{eo}^*_0(n)$ and $\text{eo}^*_2(n)$ denote the number of overpartitions of $n$ in $\mathcal{EO}^*$ with largest even part congruent to 0 and 2 modulo 4, respectively. Then

$$
\text{eo}^*_0(n) = \begin{cases} 
\text{eo}^*_2(n) & \text{if } n \text{ is not divisible by } 4; \\
> \text{eo}^*_2(n) & \text{if } n \text{ is divisible by } 4.
\end{cases}
$$

2. **PROOF OF THEOREM 1**

We first establish a generating function identity for $\text{eo}^*_0(n) - \text{eo}^*_2(n)$.

**Theorem 3.** The following equality holds:

$$
\sum_{n \geq 0} (\text{eo}^*_0(n) - \text{eo}^*_2(n)) q^n = \frac{(-q^4; q^4)^\infty}{(q^4; q^8)^\infty}.
$$

Our proof of Theorem 3 relies on the $q$-binomial theorem [6, Equation (II.3)].

**Lemma 1 (q-Binomial theorem).** The following equality holds:

$$
\sum_{n \geq 0} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_\infty}{(z; q)_\infty}.
$$

**Proof of Theorem 3.** Let $k$ be a nonnegative integer. We first notice that the generating function for partitions in $\mathcal{EO}^*$ with largest even part equal to $2k$ is given by

$$
\frac{q^{2k}}{(q^4; q^4)_k} \cdot \frac{1}{(q^{4k+2}; q^4)^\infty},
$$

where the first multiplicand comes from all even parts and the second multiplicand comes from all odd parts. It follows that

$$
\sum_{n \geq 0} (\text{eo}^*_0(n) - \text{eo}^*_2(n)) q^n = \sum_{k \geq 0} (-1)^k \frac{q^{2k}}{(q^4; q^4)_k (q^{4k+2}; q^4)^\infty} = \frac{1}{(q^2; q^4)^\infty} \sum_{k \geq 0} \frac{(q^2; q^4)_k (-q^2)^k}{(q^4; q^4)_k}.
$$

Finally, applying the $q$-binomial theorem (2) with $a \to q^2$, $z \to -q^2$ and $q \to q^4$ yields

$$
\sum_{n \geq 0} (\text{eo}^*_0(n) - \text{eo}^*_2(n)) q^n = \frac{1}{(q^2; q^4)^\infty} \frac{(-q^4; q^4)_\infty}{(-q^2; q^4)^\infty} = \frac{(-q^4; q^4)_\infty}{(q^4; q^8)^\infty}.
$$

We therefore arrive at Theorem 3.

**Proof of Theorem 1.** We simply notice that the right-hand side of (1) is a series of $q^4$ with positive coefficients. Therefore, $\text{eo}^*_0(n) - \text{eo}^*_2(n)$ is positive when $n$ is a multiple of 4 and zero otherwise. This proves Theorem 1.
3. PROOF OF THEOREM 2

For $EO$, we will establish a bivariate generating function identity. Let $o(\pi)$ count the number of overlined parts in $\pi$ for any $\pi \in EO$. We also assign a weight $w(\pi)$ to each $\pi$ by

$$w(\pi) = \begin{cases} 1 & \text{if the largest even part of } \pi \text{ is divisible by } 4; \\ -1 & \text{if the largest even part of } \pi \text{ is not divisible by } 4. \end{cases}$$

**Theorem 4.** The following equality holds:

$$\sum_{\pi \in EO} w(\pi) z^{o(\pi)} q^{|\pi|} = \frac{(-q^4; q^4)_\infty (-zq^4; q^8)^2}{(q^4; q^8)_\infty^2}.$$  \hfill (3)

For its proof, we require the Bailey–Daum sum, also known as the $q$-Kummer sum [6, Equation (II.9)].

**Lemma 2** (Bailey–Daum sum). We have

$$\sum_{n \geq 0} \frac{(a; q)_n (b; q)_n}{(q; q)_n (aq/b; q)_n} (-q/b)^n = \frac{(-q; q)_\infty (aq; q^2)_\infty (aq^2/b^2; q^2)_\infty}{(-q/b; q)_\infty (aq/b; q)_\infty}.$$  \hfill (4)

**Proof of Theorem 4.** First, it is a simple observation that the generating function for overpartitions in $EO$ with no even parts is

$$\frac{(-zq^2; q^4)_\infty}{(q^2; q^4)_\infty}.$$

Let $k$ be a positive integer. We then notice that the generating function for overpartitions in $EO$ with largest even part equal to $2k$ is given by

$$\frac{(1+z)q^{2k}(-zq^4; q^4)_{k-1}}{(q^4; q^4)_k} \frac{(-zq^{4k+2}; q^4)_\infty}{(q^{4k+2}; q^4)_\infty},$$

where, again, the first multiplicand comes from all even parts and the second multiplicand comes from all odd parts. Hence

$$\sum_{\pi \in EO} w(\pi) z^{o(\pi)} q^{|\pi|} = \frac{(-zq^2; q^4)_\infty}{(q^2; q^4)_\infty} + \sum_{k \geq 1} (-1)^k \frac{(1+z)q^{2k}(-zq^4; q^4)_{k-1}}{(q^4; q^4)_k (q^{4k+2}; q^4)_\infty} \frac{(-zq^{4k+2}; q^4)_\infty}{(q^{4k+2}; q^4)_\infty}.$$

It follows by the Bailey–Daum sum (4) with $a \rightarrow -z$, $b \rightarrow q^2$ and $q \rightarrow q^4$ that

$$\sum_{\pi \in EO} w(\pi) z^{o(\pi)} q^{|\pi|} = \frac{(-zq^2; q^4)_\infty}{(q^2; q^4)_\infty} \frac{(-zq^4; q^4)_\infty (-zq^8)^2}{(q^4; q^8)_\infty^2} = \frac{(-q^4; q^4)_\infty (-zq^4; q^8)^2}{(q^4; q^8)_\infty^2}. $$

Thus, Theorem 4 holds.

**Proof of Theorem 2.** Taking $z = 1$ in (3), we have

$$\sum_{n \geq 0} (\overline{e}_0(n) - \overline{e}_2(n)) q^n = \sum_{\pi \in EO} w(\pi) q^{\pi} = \frac{(-q^4; q^4)_\infty (-q^4; q^8)^2}{(q^4; q^8)_\infty}.$$  \hfill (5)

This is an overpartition analog of (1). We notice as well that the right-hand side of the above identity is a series of $q^4$ with positive coefficients. Theorem 2 therefore follows. \hfill □
Remark. We further take $z = 0$ in (3). Notice that the left-hand side becomes
\[ \sum_{\pi \in \mathcal{O}} w(\pi) q^{\ell(\pi)}, \]
which is simply the generating function of $\text{eo}_0^*(n) - \text{eo}_2^*(n)$. On the other hand, the term $(-zq^4; q^8)_{\infty}^2$ in the numerator of the right-hand side vanishes. Therefore, Theorem 4 reduces to Theorem 3 when $z = 0$.

4. FINAL REMARKS

It would be interesting to see combinatorial proofs of Theorems 1 and 2. Especially, for any nonegative integer $n$, we have $\text{eo}_0^*(4n + 2) = \text{eo}_2^*(4n + 2)$ and $\overline{\text{eo}}_0^*(4n + 2) = \overline{\text{eo}}_2^*(4n + 2)$. This indicates the existence of bijections between partitions enumerated by $\text{eo}_0^*(4n + 2)$ and $\text{eo}_2^*(4n + 2)$, as well as overpartitions enumerated by $\overline{\text{eo}}_0^*(4n + 2)$ and $\overline{\text{eo}}_2^*(4n + 2)$. It would be appealing to find explicit constructions of such bijections.

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