Bakry–Émery black holes

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Abstract
Scalar–tensor gravitation theories, such as the Brans–Dicke family of theories, are commonly partly described by a modified Einstein equation in which the Ricci tensor is replaced by the Bakry–Émery–Ricci tensor of a Lorentzian metric and scalar field. In physics this formulation is sometimes referred to as the ‘Jordan frame’. Just as, in General Relativity, natural energy conditions on the stress–energy tensor become conditions on the Ricci tensor, in scalar–tensor theories expressed in the Jordan frame natural energy conditions become conditions on the Bakry–Émery–Ricci tensor. We show that, if the Bakry–Émery tensor obeys the null energy condition with an upper bound on the Bakry–Émery scalar function, there is a modified notion of apparent horizon which obeys analogues of familiar theorems from General Relativity. The Bakry–Émery modified apparent horizon always lies behind an event horizon and the event horizon obeys a modified area theorem. Under more restrictive conditions, the modified apparent horizon obeys an analogue of the Hawking topology theorem in four spacetime dimensions. Since topological censorship is known to yield a horizon topology theorem independent of the Hawking theorem, in an appendix we obtain a Bakry–Émery version of the topological censorship theorem. We apply our results to the Brans–Dicke theory, and obtain an area theorem for horizons in that theory. Our theorems can be used to understand behaviour observed in numerical simulations by Scheel et al (1995 Phys. Rev. D 51 4236–49) of dust collapse in Brans–Dicke theory.

Keywords: black holes, scalar–tensor, Brans–Dicke, Bakry–Émery, horizon

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1. Introduction

It is a fortunate circumstance that many of the tools of Riemannian comparison geometry carry over to the Lorentzian case. In particular, the basic scalar Riccati equation estimate for the mean curvature of hypersurfaces becomes an estimate for the Raychaudhuri equation, which
leads to several important theorems in general relativity, including singularity theorems and much of the general theory of black holes and of cosmology as well.

In the past decade, a comparison theory for the Bakry–Émery–Ricci tensor has emerged ([17], see also [14]). The Bakry–Émery–Ricci tensor (or simply Bakry–Émery tensor) arises in Riemannian manifolds that have a preferred scalar function \( f \) or, equivalently, a distinguished measure, and is defined by

\[
R_{ij} := R_{ij}[g] + \nabla_i \nabla_j f,
\]

where \( R_{ij}[g] = \text{Ric}[g] \) is the Ricci tensor of the Riemannian metric \( g \) and \( \nabla_i \nabla_j f = \text{Hess} f \) is the Hessian defined by the Levi-Civita connection \( \nabla \) of \( g \).

Case [3] has shown that certain aspects of the Bakry–Émery comparison theory carry over to the case of a Lorentzian metric, allowing him to prove a Bakry–Émery version of the Hawking–Penrose singularity theorem for general relativity. It was observed in [18] that this leads to a Hawking–Penrose theorem for scalar–tensor gravitation theories in the so-called Jordan frame formulation. It then seems reasonable to ask whether more of the Bakry–Émery theory can be applied to the Lorentzian case, and whether this can be used to bring the theory of black holes in scalar–tensor theories to a state of development similar to that which has been achieved in the case of pure general relativity.

Here we show that significant portions of the theory of black holes can be adapted to a (Lorentzian) Bakry–Émery formulation. In general relativity, energy conditions (positivity conditions on components of the Ricci or Einstein tensor) lead to theorems that govern apparent and event horizons. We show that when energy conditions are applied instead to Bakry–Émery–Ricci tensor (or, as the case may be, the Bakry–Émery version of the Einstein tensor), and when \( f \) obeys appropriate conditions as well, one can prove analogous theorems governing the so-called \( f \)-modified apparent horizon.

We refer the reader to [13] and [16] for the standard approach to the theory of black holes, including definitions and appropriate background needed below. One first needs the notion of an exterior region that is asymptotically flat [16, chapter 11] and strongly asymptotically predictable [16, p 299]. These exterior regions are called domains of outer communications and are defined by

\[
D := I^+(\mathcal{I}^-) \cap I^-(\mathcal{I}^+).
\]

Here \( \mathcal{I} := \mathcal{I}^- \cup \mathcal{I}^+ \) denotes a connected component of conformal infinity (see [16, chapter 11]), where \( \mathcal{I}^+ \) is future-null infinity, \( \mathcal{I}^- \) is past-null infinity, and \( \mathcal{I}^0 \) is spatial infinity. Also, \( I^+ (X) \) denotes the chronological future (past) of the set \( X \); dually, \( I^- (X) \) denotes the chronological past of \( X \). Finally, we will augment asymptotic flatness by also requiring that the directional derivative \( \nabla_l f \) of \( f \) in any future-directed null direction \( l \) must vanish on approach to \( \mathcal{I}^- \).

Now consider a closed, spacelike, co-dimension 2 surface embedded in \((M, g, f)\). There are two linearly independent future-null vector fields orthogonal to this surface, denoted \( \ell^{(i)} \) for \( i \in \{1, 2\} \) (we sometimes write \( l := I^{(1)}, k := I^{(2)} \) instead). The surface is called \( f \)-trapped if

\[
\theta_f^{(i)} := \theta^{(i)} - \nabla_{\ell^{(i)}} f \leq 0
\]

everywhere on the surface for each \( i \in \{1, 2\} \), where \( \theta^{(i)} \) is the expansion scalar associated to \( \ell^{(i)} \). We will not always repeat the definitions of the terms trapped surface, marginally trapped surface, outer trapped surface, apparent horizon, etc, as these are well-explained in standard texts, e.g., [16, chapter 12] and [13, chapter 9], but we will modify these terms with an \( f \) (e.g., \( f \)-trapped surface) to mean that the condition \( \theta_f^{(i)} = 0 \) or \( \theta_f^{(i)} \leq 0 \), as appropriate, in the conventional definition is replaced by \( \theta_f^{(i)} = 0 \) or \( \theta_f^{(i)} \leq 0 \), respectively.
We obtain three general theorems related to well-known theorems in general relativity. The spirit of these theorems is that what is true for apparent horizons under standard energy conditions is true for \( f \)-apparent horizons under Bakry–Émery energy conditions together with additional conditions on \( f \). The reasonableness of these additional conditions is then the interesting question and one for which considerations of physical applicability are important. With this in mind, a fourth theorem casts our results in the realm of scalar–tensor gravitation theory.

**Theorem 1.1.** Let \( D \) be an asymptotically flat and strongly asymptotically predictable connected component of a spacetime \((M, g, f)\). Assume that the \( f \)-modified null energy condition

\[
Rf_{ij} l^i l^j \geq 0 \tag{1.4}
\]

holds for all future-null vectors \( l \) in \( TM \) and that \( \nabla f \to 0 \) on approach to \( \mathcal{I}^+ \). Assume also that there is a \( k \in \mathbb{R} \) such that \( f \leq k \). Then no closed \( f \)-trapped surface and no \( f \)-apparent horizon intersects \( D \).

That is, \( f \)-apparent horizons must lie behind event horizons when the conditions of the theorem hold.

In the standard theory, the Hawking area theorem arises from the same basic analysis, essentially an estimate for the Raychaudhuri equation, that underlies the proof that apparent horizons lie behind event horizons. In the present case, we define the \( f \)-surface area or \( f \)-volume of a spacelike submanifold \( S \) as

\[
A_f[S] := \int_S e^{-f} \, dS, \tag{1.5}
\]

whenever the integral converges, where \( dS \) is the volume element induced on \( S \) by the spacetime metric \( g \). Then we have the following result:

**Theorem 1.2.** In a strongly asymptotically predictable spacetime, assume that \( f \leq k \) and that \( Rf_{ij} l^i l^j \geq 0 \) for every null \( l \). Let \( \mathcal{H} \) be the black hole event horizon and let \( \Sigma_1 \) and \( \Sigma_2 \) be Cauchy surfaces intersecting \( \mathcal{H} \) in closed surfaces \( S_1 := \Sigma_1 \cap \mathcal{H} \) and \( S_2 := \Sigma_2 \cap \mathcal{H} \), such that \( S_2 \) lies everywhere to the future of \( S_1 \). Then \( A_f[S_2] \geq A_f[S_1] \).

Apparent horizons in general relativity have a stability property analogous to the stability of those minimal surfaces in Riemannian geometry which are genuine local minimizers of the area functional. In particular, there is a stability operator for apparent horizons, whose spectrum must be nonnegative. In general relativity, this leads to the Hawking horizon topology theorem [13, theorem 9.3.2] (see [11] for a theorem in general dimension). Likewise in the present case there is a stability operator for \( f \)-apparent horizons and, in four spacetime dimensions at least, an associated Hawking-type topology theorem, when certain conditions are imposed on \( f \). For example, we have

**Theorem 1.3.** Consider an \( n = 4 \) dimensional spacetime. Assume that, for every pair of future-timelike vectors \( v, w \), the \( f \)-modified Einstein tensor

\[
G^f_{\mu \nu} := R^f_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R^f, \tag{1.6}
\]

(where \( R^f := g^{\mu \nu} R^f_{\mu \nu} \) obeys

\[
G^f_{\mu \nu} | v^\mu w^\nu \geq 0, \tag{1.7}
\]

on an outer \( f \)-apparent horizon \( S \), so

\[
\theta_f|_S := \theta - \nabla_{\lambda} f|_S = 0, \tag{1.8}
\]
where \( l \) is the outbound null direction orthogonal to \( S \). Assume further that \( f \) obeys
\[
\Box f|_S \geq 0. \quad (1.9)
\]
and
\[
\nabla_l f|_S = 0. \quad (1.10)
\]
Then every such outer \( f \)-apparent horizon \( S \) is either a 2-sphere or a torus with induced metric \( e^{2f} \delta \), where \( \delta \) is a flat metric.

Of course, although by (1.8), any outer \( f \)-apparent horizon \( S \) with \( \nabla_l f|_S = 0 \) has \( \theta = 0 \), the original Hawking topology theorem cannot be directly applied. First, an outer \( f \)-apparent horizon need not be an outer apparent horizon but, moreover, the energy condition (1.7) does not imply that \( G_{\mu\nu}|_S v^\mu w^\nu \geq 0 \). As well, we draw attention to the borderline case of the toroidal horizon. In General Relativity, it was known that toroidal topology was only possible in the presence of the dominant energy condition if the induced metric on the horizon were flat. (Even this possibility was later ruled out entirely [8] for outermost apparent horizons.) We see from theorem 1.3 that in the present case the horizon metric is once again completely determined up to choice of flat torus metric \( \delta \), but is now \( e^{2f} \delta \) so in general it is not flat.

As the conditions on \( f \) in theorem 1.3 may appear not to be optimal, we should ask whether they are at least physical. To examine this question, we consider scalar–tensor gravitation theory. Scalar–tensor theories are in a sense almost ubiquitous in modern physics. They arise, for example, whenever a Kaluza–Klein type reduction from higher dimensions is employed, including in string theory. The prototypical scalar–tensor theory is Brans–Dicke gravitation theory formulated in four spacetime dimensions, for which the above theorems lead to the following result:

**Theorem 1.4.** For some \( \omega \in (-\frac{3}{2}, \infty) \), assume that \((M^4, g, \varphi)\) is a solution of the Brans–Dicke equations of gravitation in the Jordan frame formulation\(^1\) such that the matter stress–energy tensor \( T_{\mu\nu} \) obeys \( T_{\mu\nu}|_l v^\mu v^\nu \geq 0 \) for all future-null vectors \( l \). Define \( f = -\log \varphi \) and assume that \( \varphi \geq C > 0 \). Then

(i) any Jordan frame \( f \)-apparent horizon \( S \) lies behind an event horizon,
(ii) any Einstein frame apparent horizon must lie behind an event horizon, and
(iii) for \( S_1, S_2 \) as in theorem 1.2, we have
\[
\int_{S_1} \varphi \, dS \leq \int_{S_2} \varphi \, dS, \quad (1.11)
\]
where \( dS \) is the area element induced by \( g \).\(^2\)
Moreover, assume that \( S \) is an outer \( f \)-apparent horizon such that
\[
0 \geq T|_S := g^{\mu\nu} T_{\mu\nu}|_S, \quad (1.12)
\]
and
\[
0 \leq \left[ T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right] \delta^\mu^\nu \quad \left( \frac{3}{3 + 2\omega} \right)_{S} \quad (1.13)
\]
for every choice of future-timelike vectors \( u, v \) in \( T_p M, p \in S \), and assume that \( \nabla_l f|_S \equiv -\frac{1}{\varphi} \nabla_l \varphi|_S = 0 \), where \( l \) is the null vector in (1.8). Then

\(^1\) The Jordan frame formulation is that in which the field equations are as presented in section 5. There is also the Einstein frame formulation, in which the field equations resemble those of general relativity. These formulations are related by a conformal transformation. See [6] for details.

\(^2\) That is, \( dS \) is the Jordan frame area element, so \( \varphi dS \) is the Einstein frame area element. Therefore, statement (iii) says that the Einstein frame horizon cross-section area increases.
(iv) S is either a 2-sphere or a 2-torus, and if it is a 2-torus then it has induced metric $\delta/\phi^2$, where $\delta$ is a flat metric on the torus.

Condition (1.12) holds for perfect fluids of mass-energy density $\rho$ and pressure $p$ if $\rho \geq 3p$, for free scalar fields $\psi$ if $(\partial \psi/\partial t)^2 \leq |\nabla \psi|^2$, and for massless Maxwell fields always. The condition that $\nabla l \phi|_S = 0$ is reasonable at least in the static case.

Parts (i)–(iii) of this theorem are in fact unsurprising. The conformal transformation from the Jordan to the Einstein frame formulation (see (2.10), (2.12)) preserves the assumptions, maps the Jordan frame horizon area element $d\tilde{S}$ to $\phi dS$, and maps the Jordan frame $f$-modified horizon expansion scalar to the Einstein frame unmodified expansion scalar so as to preserve the sign (see remark 2.6). In this light, parts (i)–(iii) are easily understood. They lend confidence to the underlying Bakry–Émery theorems on which they are based and which do not necessarily have a physical context. Furthermore, the phenomenon in statement (ii) of theorem 1.4 has been observed in an interesting numerical study of the collapse of collisionless dust in Brans–Dicke theory [15]. The stress–energy tensor of such a model obeys the null energy condition. However, the numerical evolution reveals that the null components of the Jordan frame Ricci tensor are sometimes negative [15, figure 10], and the apparent horizon with respect to the Jordan frame metric sometimes lies outside the event horizon [15, figure 9]. Nonetheless, the Einstein frame apparent horizon always lies behind the event horizon [15, figures 11 and 12].

This paper is organized as follows. Section 2 contains the proof of theorem 1.1, while theorem 1.3 is proved in section 3. Section 4 is devoted to Brans–Dicke theory and the proof of theorem 1.4. An appendix is devoted to a simple Bakry–Émery version of the topological censorship theorem [7], based on an easy modification of an extant proof [12]. This produces independent constraints on horizon topology as an immediate corollary (cf [5, 10]).

2. $f$-trapped surfaces and $f$-apparent horizons

We begin with a closed spacelike surface $S$ of co-dimension 2 in a spacetime of dimension $n$, thus $\dim S = n - 2$. There are two congruences of null geodesics issuing orthogonally from $S$, called the outgoing and ingoing null congruences. As in the introductory section, we denote the tangent fields to these congruences by $l^{(i)}$ and their respective expansion scalars by $\theta^{(i)}$ with $i = 1$ for the outgoing congruence (we also use the notation $\theta^{(1)} = \theta$) and $i = 2$ for the ingoing congruence (we also write $\theta^{(2)} = \kappa$). Then $S$ is outer trapped if $\theta^{(i)} < 0$ for both $i = 1$ and $i = 2$. If

$$\theta_f^{(i)} := \theta^{(i)} - \nabla_{l^{(i)}} f < 0$$

(2.1)

then $S$ is outer $f$-trapped if $i = 1$ and $f$-trapped if $i = 1, 2$. We say $S$ is marginally $f$-trapped if the strict inequality in (2.1) is replaced by the closed inequality $\leq 0$.

The key to the issue is the following lemma (see also [3]):

**Lemma 2.1.** Assume that $f$ is a smooth function obeying $f \leq k$ for some constant $k$ and say that

$$\text{Ric}^f (l, l) \geq 0$$

(2.2)

along every null geodesic $\gamma$ with tangent field $l = \frac{d\gamma}{ds}$ and belonging to a null geodesic congruence issuing from $S$. If $\theta_f < 0$ on $S$, then there is a focal point to $S$ at some $s > 0$ along $\gamma$.

**Remark 2.2.** The result holds for timelike $\gamma$ too, provided (2.2) holds for every timelike geodesic congruence issuing from $S$, with tangent field $l$.

**Proof.** The Riccati equation governing $\theta$ is known as the Raychaudhuri equation. Dropping a term of definite sign, it yields the inequality

$$\text{Ric}^f (l, l) \geq 0$$

(2.2)
where \( m = n - 2 \) for a null congruence \( (m = n - 1 \) for a timelike congruence). Adding \( -\nabla_i \nabla_i f \) to each side and writing \( \text{Ric}^f := (\text{Ric} + \nabla \nabla f)(l, l) = \text{Ric}(l, l) + \nabla_i \nabla_i f \) (using that \( \nabla_l l = 0 \)), then

\[
\frac{d\theta}{ds} \leq -\frac{\text{Ric}(l, l) - \theta^2}{m},
\]

where on the left-hand side we used the definition of \( \theta \) as in (2.1) and in the last line we used (2.2). For economy of notation, we suppress the geodesic \( \gamma \) and simply write \( f \). The inequality in passing from the middle line on the right-hand side to the last line is an equality iff, along the null geodesic \( \gamma \) under consideration, \( \text{Ric}^f(l, l) = 0 \) and \( \nabla_l f = 0 \) (the first line is an equality iff the shear of the congruence containing \( \gamma \) vanishes).

Then we can rewrite (2.4) as

\[
\frac{d}{ds} \left( \frac{m}{e^{\frac{m}{\theta} \theta_f}} \right) \geq e^{-\frac{m}{\theta}}.
\]

Now assume that \( f \) is bounded above by \( k \). Then, integrating (2.5) over \( s \in [0, t] \), for \( t \) small enough so that the left-hand side of (2.5) remains well-defined, we obtain

\[
-\frac{m}{e^{\frac{m}{\theta} \theta_f(t)}} \geq \int_0^t e^{-\frac{m}{\theta} \theta_f(s)} ds + \frac{m}{e^{\frac{m}{\theta} \theta_f(0)}},
\]

This is an equality at \( t = 0 \), and then there is an interval \( t \in [0, T) \) such that each side is negative since \( \theta_f(0) < 0 \) for an \( f \)-trapped surface. There will then be some \( t = T_1 > 0 \) such that the right-hand side of (2.6) approaches 0 from below as \( t \nearrow T_1 \). Thus the left-hand side must tend to 0 from below as \( t \nearrow T_2 \) for some \( 0 < T_2 < T_1 \). Since \( f \leq k \), then necessarily \( \theta_f(t) \rightarrow -\infty \) as \( t \nearrow T_2 \). But because \( \theta_f(t) = \theta - \nabla_l f \) and \( f \) is smooth, then \( \theta \rightarrow -\infty \) as \( t \nearrow T_2 \).

**Corollary 2.3.** In a strongly asymptotically predictable and asymptotically flat spacetime, no \( f \)-trapped surface meets \( D \).

The proof is standard:

**Proof.** Say \( S \) is \( f \)-trapped. By way of contradiction, if \( S \cap D \) were nonempty, there would exist causal curves from \( S \) to \( I \). Since these causal curves cannot approach spatial infinity \( \partial I \), the boundary \( \partial I^+ (S) \) therefore would meet \( I \), say at a point \( q \in I \). Since spacetime is strongly asymptotically predictable, then \( \partial I^+ (S) \) would contain a past-null geodesic generator beginning at \( q \in I \) and which does not end until it reaches \( S \). Reversing direction, this gives a future-inextendible null geodesic \( \gamma : [0, \infty) \rightarrow M \) lying always on \( \partial I^+(S) \). It therefore cannot contain a focal point to \( S \), contradicting lemma 2.1.

The case of a marginally trapped surface is somewhat more subtle, and it is here that the asymptotic flatness of \( f \)-strongly asymptotically predictable domains is needed.

**Proposition 2.4.** Let \( D \) be an asymptotically flat, strongly asymptotically predictable domain of outer communications and assume the conditions of lemma 2.1 hold on \( D \). Assume further that \( \nabla_l f \rightarrow 0 \) on approach to \( I^+ \) for any null \( l \). Then no marginally \( f \)-trapped closed spacelike surface \( S \) intersects \( D \).
Sketch of proof. The standard proof is given, e.g., in [16, proposition 12.2.3, p 310] (see also [13, section 9.2, p 320]), but has a small technical gap, so we follow instead the proof given in [4, theorem 6.1]. We restrict ourselves to outlining the logic of that proof and discussing the modification required to accommodate non-zero \( f \).

All versions of the proof proceed by way of contradiction by assuming that the future of \( S \) meets \( \mathcal{I} \). As this future does not meet a neighbourhood of spatial infinity \( \mathcal{I}^0 \), there is then a point \( q \in \mathcal{I}^+ \cap \partial \mathcal{I}^+ (S) \). Using the asymptotic predictability, \( q \) can be reached from \( S \) by a null geodesic \( \gamma \), which has no focal point to \( S \) except possibly at \( q \) (this latter possibility is not accounted for in older versions of the proof).

In [4], a smooth spacelike surface \( S^+ \subset \mathcal{I} \) through \( q \) is constructed. No point of this surface lies in the chronological future of \( S \) (at least, for suitably chosen \( q \) and \( S^+ \): in the presence of focal points on \( \mathcal{I}^+ \), the choice of \( q \) and \( S^+ \) is explained in [4]). The boundary of the past \( \partial \mathcal{I}^- (S^+ \) of this surface will contain \( \gamma \). The expansion scalar, call it \( \theta^{(1)} \), of \( \partial \mathcal{I}^- (S^+) \) along \( \gamma \) is computed using the unnumbered equation in the proof of theorem 6.1 in [4], where it is shown to be positive and bounded away from 0. That is, there is an \( s_0 \in \mathbb{R} \) such that we can write \( \theta^{(1)} |_{\gamma(t)} \geq C > 0 \) for any \( s \geq s_0 \).

On the other hand, by equation (2.6), the \( f \)-modified expansion, say \( \theta^{(2)}_f (t) \) of the outward null congruence from \( S \) obeys

\[
\theta^{(2)}_f (t) \leq \frac{m e^{-\frac{2 \Phi(t)}{m}} - \epsilon}{t e^{-\frac{\Phi(t)}{m}} + t e^{-\frac{\Phi(t)}{m}}} \leq 0,
\]

and so since \( \nabla_{\gamma} f \to 0 \) near \( \mathcal{I} \) (i.e., as \( t \to \infty \) along the congruence), then the ordinary expansion scalar obeys \( \theta^{(2)}_f (t) \leq \epsilon \), where we can make \( \epsilon > 0 \) be arbitrarily small by choosing \( t \) sufficiently large. In particular, eventually

\[
\theta^{(2)} \leq \epsilon < C \leq \theta^{(1)}.
\]

The problem is that \( \mathcal{I}^+ (S) \) cannot intersect \( \mathcal{I}^- (S^+) \) though the boundaries of these sets share a common generating curve \( \gamma \), so the boundaries of \( \mathcal{I}^+ (S) \) and \( \mathcal{I}^- (S^+) \) must ‘bend away from each other’ along \( \gamma \). For this to happen, we must have \( \theta^{(2)} \geq \theta^{(1)} \), contradicting (2.8). The resolution of this contradiction is that \( \partial \mathcal{I}^+ (S) \) cannot meet \( \mathcal{I}^+ \), establishing the proposition.

In [4], the argument in the last paragraph above instead proceeds by appeal to the maximum principle. The same could have been done here. In the [4] (i.e., no \( f \) case), one obtains 2.8 with \( \epsilon = 0 \), implying both that \( \theta^{(2)} \leq 0 < \theta^{(1)} \). The geometric maximum principle for smooth hypersurfaces [9, theorem 2.1], then implies that \( \partial \mathcal{I}^+ (S) \) and \( \partial \mathcal{I}^- (S^+) \) would necessarily coincide near \( \gamma \) and would have expansion scalar \( \theta = \theta^{(1)} = \theta^{(2)} = 0 \), which contradicts another implication of (2.8) with \( \epsilon = 0 \), which is that \( \theta^{(1)} \geq C > 0 \). In the present case (i.e., with \( f \)), because \( \epsilon > 0 \), we cannot appeal directly to [9, theorem 2.1]. Nonetheless, from (2.8) we have \( \theta^{(2)} \leq \epsilon < \theta^{(1)} \) and so, by appealing to [9, theorem 2.2], we have \( \theta^{(1)} = \theta^{(2)} = \epsilon \) locally near \( \gamma \). But (2.8) also implies that \( \theta^{(1)} \geq C > \epsilon \) at points along \( \gamma \), so again we have a contradiction which establishes the proposition.

Mimicking the standard analysis, we define a total \( f \)-trapped region to be the union of all \( f \)-trapped surfaces and define the \( f \)-apparent horizon to be its boundary. When this boundary is smooth, it has outbound expansion \( \theta^{(1)}_f = 0 \); the proof is standard (e.g., [16, theorem 12.2.5, mutatis mutandis]). Therefore the proof of proposition 2.4 applies in this case, so an \( f \)-apparent horizon must lie entirely outside \( \mathcal{D} \) (i.e., it coincides with, or lies behind, a black hole event horizon). This completes the proof of theorem 1.1.

We remark that, if the condition (2.2) holds for all timelike \( l \) as well, if \( f \) is bounded above, if an \( f \)-modified version of the null generic condition holds, and if there are no closed
timelike curves, then Case’s singularity theorem [3] guarantees that the future of the trapped region is nonspacelike geodesically incomplete.

We now briefly turn attention back to event horizons and the proof of theorem 1.2. A well-known result in the standard theory is that, when $R_{ij}l^il^j \geq 0$ for all null $l$, then the null geodesic generators of an event horizon have $\theta \geq 0$ [13, lemma 9.2.2]. Here we have an analogous result:

**Lemma 2.5.** Assume that $f \leq k$ and the $f$-null energy condition (2.2) holds as in lemma 2.1. Then the $f$-modified expansion of the null geodesic generators of the event horizon $H$ of a future asymptotically predictable spacetime obey $\theta_f \geq 0$.

**Sketch of proof.** The proof is the same as that of [13, lemma 9.2.2] or [16, proof of theorem 12.2.6, first paragraph], but since we have condition (2.2), we use $\theta_f$. If $\theta_f(p) < 0$ at some $p \in H$, then one can deform $H$ slightly outward so as to intersect $D$, keeping $\theta_f < 0$ somewhere, say at $p'$. But then $\partial I^+(p')$ will intersect $I^+$, say at $q$, and, using asymptotic predictability, one can trace back from $q$ to construct a null geodesic generator of $\partial I^+(p')$ which extends from $p'$ to $q$. This generator cannot have a focal point, contradicting lemma 2.1. □

Then we are in a position to prove theorem 1.2.

**Proof of theorem 1.2.** Again, following the standard proof ([13, proposition 9.2.7], [16, proof of theorem 12.2.6, second paragraph]), we observe that there is an injective (but possibly not surjective) map $\psi : S_1 \to S_2$ by transport along the null geodesic generators of $H$. By lemma 2.5 we have that $\theta_f \geq 0$ along these generators. But, differentiating equation (1.5) along any of these null geodesic generators, say with tangent vector $l = \frac{d}{ds}$, we have

$$\frac{dA_l}{ds} = \int_S (\theta - \nabla_l f) e^{-f} \, dS \equiv \int_S \theta_l e^{-f} \, dS.$$  

(2.9)

This shows that $A_f[S_1] \leq A_f[\psi(S_1)] \leq A_f[S_2]$. □

Proposition 2.4 can also be obtained from standard results in [4, 13], and [16] which rely on the usual Raychaudhuri equation (2.3), without invoking the $f$-modified equation (2.4) and the associated estimate that appears in the proof of lemma 2.1. We now outline that argument. Let

$$\tilde{g}_{ij} := e^{-\frac{2f}{\epsilon}} g_{ij}.$$  

(2.10)

Then the expansion scalar of the $l$-congruence transforms as

$$\tilde{\theta} = e^{\frac{2f}{\epsilon}} (\theta - \nabla_l f) = e^{\frac{2f}{\epsilon}} \theta_f.$$  

(2.11)

**Remark 2.6.** $\theta_f$ and $\tilde{\theta}$ are either both positive, both negative, or both zero.

The prefactor $e^{\frac{2f}{\epsilon}}$ ensures that $\tilde{\theta}$ is the expansion scalar of a congruence of null geodesics, not just pregeodesics, with respect to $\tilde{g}$. The conditions $f \leq k$ and $\nabla_l f \to 0$ in the above theorems now ensure that the conformal rescaling (2.10) preserves asymptotic flatness and future asymptotic predictability (e.g., unless $f \leq k$, $g$-complete geodesics might not be $\tilde{g}$-complete).

Now $\tilde{\theta}$ is governed by the ordinary Raychaudhuri equation (without $f$-terms) for the rescaled metric $\tilde{g}$. A standard conformal transformation formula shows that the Ricci curvature $\tilde{R}_{ij}$ of $\tilde{g}$ is given in terms of the Ricci curvature $R_{ij}$ of $g$ by
\[ \hat{R}_{ij} = R_{ij} + \nabla_i \nabla_j f + \frac{1}{(n-2)} [\nabla_i \varphi \nabla_j \varphi + g_{ij}(\Delta f - |\nabla f|^2)] \]
\[ = \hat{R}_{ij}^0 + \frac{1}{(n-2)} [\nabla_i \varphi \nabla_j \varphi + g_{ij}(\Delta f - |\nabla f|^2)]. \quad (2.12) \]

From this it is easy to see that, for \( l \) a null vector, then \( \hat{R}_{ij}^0 \parallel l \parallel \geq 0 \) implies \( \hat{R}_{ij} \parallel l \parallel \geq 0 \). Thus, our assumptions on \( g \) imply that the null energy condition holds for \( \hat{R}_{ij} \) at every point, and then the usual analysis shows that apparent horizons in \( \hat{g} \) lie behind event horizons. But by (2.10) \( f \)-trapped or marginally \( f \)-trapped regions with respect to \( g \) are trapped or, respectively, marginally trapped with respect to \( \hat{g} \), and since event horizons are conformally invariant, it follows that \( f \)-apparent horizons with respect to \( g \) lie behind event horizons.

### 3. Stability and the Hawking topology theorem

Consider now an arbitrary variation \( \Phi : S \times I \rightarrow M \) of a closed, spacelike 2-surface \( S \) embedded in spacetime \( M \), where \( I \subseteq \mathbb{R} \) is an open interval containing 0. We let \( x_0(\sigma) := \Phi(p, \sigma) \) be the image of \((p, \sigma), p \in S \). Varying \( \sigma \), this yields a curve such that \( x(0) = p \in S \), and we define \( q := \frac{\partial x}{\partial \sigma} \) to be the tangent field to this curve. We can specify the variation by specifying \( q \). We write
\[
q := q^\parallel + q^\perp,
q^\perp := bl - 2uk, \quad (3.1)
\]
where \( q^\parallel \) is tangent to the leaves \( S_\sigma \) of the variation, \( l \) and \( k \) are linearly independent future-null vectors in the normal bundle \( NS_\sigma \) to the leaves (defined first on \( S \), then parallel transported to a neighbourhood of \( S \)—see [1] for further details) and are normalized so that \( k \cdot l = -2 \), and \( b, u : S \rightarrow \mathbb{R} \) are arbitrary functions on \( S \). We define the vector second fundamental form \( K : TS_\sigma \times TS_\sigma \rightarrow NS_\sigma \) by
\[
K(X, Y) = -(\nabla_X Y) \perp, X, Y \in TS_\sigma, \quad (3.2)
\]
where the superscript \( \perp \) denotes projection into \( NS_\sigma \). The mean curvature vector of \( S_\sigma \) is then \( H = \text{tr}_g K \equiv h^{AB} K_{AB} \), where \( h_{\mu\nu} := g_{\mu\nu} + \frac{1}{2}(k_\mu l_\nu + l_\mu k_\nu) \) is the first fundamental form or induced metric on \( S \). The null expansion scalars are \( \theta^{(1)} := \theta := H \cdot l \) and \( \theta^{(2)} := \kappa := H \cdot k \), and we have \( H = -\frac{1}{2}(\theta k + \kappa l) \).

Then the variation of the expansion scalar \( \theta \) along integral curves of \( q \) is given (in the notation of (1)) by
\[
\delta_q \theta = a \theta + q^\perp \theta \left[ -2b(k_{AB}l^A l^B + G_{\mu
u}l^\mu l^\nu) - 2k^A s_{AB} - 2s^A D_A \right] - \left[ \frac{1}{2} R_{\mu\nu} - \frac{1}{2} H^2 - G_{\mu\nu} l^\mu l^\nu + VG_{\mu\nu} l^\mu l^\nu - l^\mu \sigma_A - D_A s^\mu \right] u. \quad (3.3)
\]
Here \( G_{\mu\nu} \) is the spacetime Einstein tensor, \( R_{\mu\nu} \) is the scalar curvature of \( S_\sigma \), \( \nabla_{S_\sigma} \) := \( D_A D_A \) is the Laplacian on \( S_\sigma \), \( D_A \) is the connection on \( S_\sigma \) capital Latin indices run over a basis for \( TS_\sigma \), \( \{e_A\} \) is such a basis (say orthonormal), \( s_A := -\frac{1}{2} k_\mu \nabla_{e_A} l^\mu \), and \( a := -\frac{1}{2} k_\mu \frac{a}{a^\mu} |a = 0|^\mu \). As well, we have changed basis for \( NS_\sigma \) from \( \{l, k\} \) to \( \{l, w\} \), where
\[
w := VL + \frac{1}{2} k, \quad (3.4)
\]
with \( V \geq 0 \) an arbitrary function on \( S \), so that \( w \) is future-causal. We will also make use of the combination
\[
v := VL - \frac{1}{2} k. \quad (3.5)
\]
If \( V \neq 0 \) then \( v \) is spacelike. Finally, since \( H = -\frac{1}{2}(\theta k + \kappa l) \), then \( H^2 = -\theta \kappa \).
We can choose the variation such that $a = 0$, $u = 0$, $q^\parallel = 0$, and $b = 1$. This yields the Raychaudhuri equation
\begin{align*}
\nabla_i \theta &= - (G_{\mu \nu} + K_{\mu \alpha \beta} K^{\alpha \beta}) l^\mu l^\nu \\
&= - (R_{\mu \nu} + K_{\mu \alpha \beta} K^{\alpha \beta}) l^\mu l^\nu \\
&\geq - R_{\mu \nu} l^\mu l^\nu - \frac{1}{m} \theta^2,
\end{align*}
upon dropping a term of definite sign (cf equation (2.3)). A different choice is $a = 0$, $q^\parallel = 0$, $b = uV$, with $u : S \to \mathbb{R}$ to be chosen later, so that now $q = uv$. This yields
\begin{equation}
\delta_w \theta = - \Delta_S u + 2 \Delta D_A u + \left( \frac{1}{2} R_S + \frac{1}{2} \theta \kappa - G_{\mu \nu} l^\mu w^\nu - V K_{\mu \alpha \beta} K^{\alpha \beta} l^\mu l^\nu - s^A s_A + D_A s^A \right) u.
\end{equation}

This should be compared to equation (5) in [1], but note we have been using $w$ in place of the vector $u$ in [1] to avoid confusion with the function $u$. Also, here we keep the $- \frac{1}{2} H^2 u \equiv \frac{1}{2} \theta \kappa u$ term which is obviously zero at a marginally trapped surface (so this term is dropped in [1]). We do this because, in the present case, we want instead to consider marginally outer $f$-trapped surfaces, where $\theta_l = 0$. As with equation (5) in [1], from here onward we will evaluate (3.7) only on the surface $S$ (so $\sigma = 0$), which is an outer $f$-apparent horizon, meaning that it is the smooth boundary of a region whose points lie on outer $f$-trapped surfaces, and thus $S$ is outer marginally $f$-trapped. Then we have
\begin{equation}
\delta_{\alpha \beta} \theta = - \Delta_S u + 2 \Delta D_A u + \left( \frac{1}{2} R_S + \frac{1}{2} \theta \kappa - G_{\mu \nu} l^\mu w^\nu - V K_{\mu \alpha \beta} K^{\alpha \beta} l^\mu l^\nu - s^A s_A + D_A s^A \right) u.
\end{equation}

Now the idea is to replace various terms with $f$-modified terms. Here and in what follows, we remind the reader that subscripts $l$ and $k$ refer to the null basis vectors and are not indices ranging over a set of values. We begin with
\begin{equation}
\delta_{\alpha \beta} \theta \equiv u \nabla_i \theta = u \nabla_i \theta_f + u \nabla_i \nabla_f.
\end{equation}

Moving to the right-hand side of (3.8), we define
\begin{align*}
G_{\mu \nu}^l :&= R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R^l =: R_{\mu \nu} + \nabla_{\mu} \nabla_{\nu} f - \frac{1}{2} g_{\mu \nu} (R + \Box f) \\
&= G_{\mu \nu} + \nabla_{\mu} \nabla_{\nu} f - \frac{1}{2} g_{\mu \nu} \Box f,
\end{align*}
where we write $\Box f$ for the d’Alembertian of $f$ (i.e., for the scalar Laplacian $\Box f = \nabla^\nu \nabla_{\nu} f$ in Lorentzian signature).

Then
\begin{align*}
G_{\mu \nu} l^\mu w^\nu &= G_{\mu \nu}^l l^\mu w^\nu - (\nabla_{\mu} \nabla_{\nu} f) l^\mu w^\nu - \frac{1}{2} \Box f \\
&= G_{\mu \nu}^l l^\mu w^\nu - \nabla_{\mu} \nabla_{\nu} f + \nabla_{\mu} l \cdot \nabla_{\nu} f - \frac{1}{2} \Box f \\
&= G_{\mu \nu}^l l^\mu w^\nu - \nabla_{\mu} \nabla_{\nu} f + \frac{1}{2} \nabla_{\mu} f \cdot \nabla_{\nu} f - \frac{1}{2} \Box f.
\end{align*}

Here we used that $\nabla_l = 0$.

If we insert (3.11) and (3.9) into (3.8), using (3.5) and (3.4) to combine the $\nabla_{\mu} \nabla_{\nu} f$ term from (3.9) with the $\nabla_{\nu} \nabla_l f$ term from (3.11), we obtain
\begin{align*}
\nabla_{\nu} \left[ \left( \frac{1}{2} R_S + \frac{1}{2} \theta \kappa - G_{\mu \nu} l^\mu w^\nu + \frac{1}{2} (k^\mu l^\nu + l^\mu k^\nu) \nabla_{\nu} \nabla_{\nu} f \\
+ \frac{1}{2} \nabla_{\mu} f \cdot \nabla_{\nu} f + \frac{1}{2} \Box f - V K_{\mu \alpha \beta} K^{\alpha \beta} l^\mu l^\nu - s^A s_A + D_A s^A \right) u \right]
= - \Delta_S h + 2 \Delta D_A u + \left( \frac{1}{2} R_S + \frac{1}{2} \theta \kappa - G_{\mu \nu} l^\mu w^\nu + h^\mu \nabla_{\nu} \nabla_{\nu} f \\
+ \frac{1}{2} \nabla_{\mu} f \cdot \nabla_{\nu} f - \frac{1}{2} \Box f - V K_{\mu \alpha \beta} K^{\alpha \beta} l^\mu l^\nu - s^A s_A + D_A s^A \right) u,
\end{align*}
using that $\frac{1}{2} (k^\mu l^\nu + l^\mu k^\nu) \nabla_{\mu} \nabla_{\nu} f = h^\mu \nabla_{\nu} \nabla_{\nu} f - g^{\mu \nu} \nabla_{\mu} \nabla_{\nu} f = h^\mu \nabla_{\mu} \nabla_{\nu} f - \Box f$. 

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The \( h^{\mu \nu} \nabla_\mu \nabla_\nu f \) term can also be expanded. It yields
\[
\begin{align*}
  h^{\mu \nu} \nabla_\mu \nabla_\nu f &= h^{\mu \nu} \left[ h_{\mu \nu} + \frac{1}{2} \left( \sum_{i=1}^{n} \epsilon_{\mu \nu i} \epsilon_{\lambda \sigma i} \right) \right] \nabla^\alpha f \\
  &= \Delta_S f - \frac{1}{2} \left( h^{\mu \nu} \nabla_\mu l_\nu + h^{\mu \nu} \nabla_\nu l_\mu f \right) \\
  &= \Delta_S f + \frac{1}{2} \theta \nabla_l f + \frac{1}{2} \kappa \nabla_l f \\
  &= \Delta_S f + \nabla_l f \nabla_v f + \frac{1}{2} \theta \nabla_l f + \frac{1}{2} \kappa \nabla_l f.
\end{align*}
\]

To make further progress, we define
\[
K^f_{\mu \nu} := K_{\mu \nu} - \frac{1}{m} h_{\mu \nu} \nabla_\mu f,
\]
where \( m = n - 2 = \text{rank}(h_{\mu \nu}) \). This will not be used to replace the \( K_{\mu \nu} K^{AB} l^{\mu \nu} \) term, which is already a square. Rather, we take the trace and define
\[
H^f_\mu := H_\mu - \nabla_\mu f, \quad \theta^f := H^f \cdot l, \quad \kappa^f := H^f \cdot k,
\]
so \( \theta^f \) and \( \kappa^f \) are the \( f \)-modified expansions of the \( l \) and \( k \) null congruences, respectively. Then observe that the \( \frac{1}{2} \theta \kappa \) term in (12.6) becomes
\[
\frac{1}{2} \theta \kappa = \frac{1}{2} (\theta^f + \nabla_l f)(\kappa^f + \nabla_k f).
\]

Inserting (13.13) and (13.18) into (12.6), we obtain
\[
u \nabla \big|_S \theta f = - \Delta_S u + 2 \delta^A D_A u + \left[ \frac{1}{2} R_S + \frac{1}{2} \nabla_l f \nabla_k f - G^f_{\mu \nu} l^\mu w^\nu - \frac{1}{4} \Delta f - V K_{\mu \nu} K_e^{AB} l^{\mu \nu}
- s_A s^A + D_A s^A + \Delta_S f \right] u
\]
\[
= - \Delta_S u + 2 \delta^A D_A u + \left[ \frac{1}{2} R_S + \left( \frac{1}{2} \nabla_k f + \kappa^f \right) \nabla_l f - G^f_{\mu \nu} l^\mu w^\nu - \frac{1}{4} \Delta f
- V K_{\mu \nu} K_e^{AB} l^{\mu \nu} - s_A s^A + D_A s^A + \Delta_S f \right] u
\]
where in the last equality we used that \( \nabla_k l \big|_S = 0 \) by construction (see [1]) and \( \theta^f \big|_S = 0 \). Thus we obtain the \( f \)-stability operator
\[
\tilde{L}^f_\psi := \left[ L^f - \frac{1}{4} \Delta f + \left( \frac{1}{2} \nabla_k f + \kappa^f \right) \nabla_l f \right] \psi
\]
\[
L^f_\psi := - \Delta_S \psi + 2 \delta^A D_A \psi + \left[ \frac{1}{2} R_S - G_{\mu \nu} l^\mu w^\nu - V K_{\mu \nu} K_e^{AB} l^{\mu \nu} - s_A s^A + D_A s^A + \Delta_S f \right] \psi.
\]

Here \( \tilde{L}^f_\psi \) is the stability operator \( L_\psi \) of [1, equation (5)] with \( G_{\mu \nu} \) replaced by \( G^f_{\mu \nu} \), and with the divergence term \( D_A s^A \) modified to become \( D_A s^A + \Delta_S f = D_A (s^A + D^A) \) (as well, [1] writes \( L_\psi \), not \( L_\psi \)). It is noted in [1] that operators of this form, though not self-adjoint, have a real principal eigenvalue \( \lambda_1 \leq \text{Re}(\lambda) \) where \( \lambda \) is any other eigenvalue, and the eigenfunction corresponding to \( \lambda_1 \) is positive. The operator \( \tilde{L}^f_\psi \) is further modified as in (20) to produce the \( f \)-stability operator.

**Proof of theorem 1.3.** Under assumptions (1.9), (1.10), we have
\[
\tilde{L}^f_\psi \psi \leq L^f_\psi \psi \leq - \Delta_S \psi + 2 \delta^A D_A \psi + \left[ Q^f - s_A s^A + D_A s^A \right] \psi
\]
\[
Q^f := Q + \Delta_S f := \frac{1}{2} R_S - G_{\mu \nu} l^\mu w^\nu - V K_{\mu \nu} K_e^{AB} l^{\mu \nu} + \Delta_S f,
\]
when \( \psi \) is a positive function. Following the reasoning in [11], we note that the surface \( \Sigma \) cannot be \( f \)-trapped unless the principal eigenvalue, say \( \lambda_1 \), of the operator on the right-hand
side of (3.22) is positive. If \( \phi_1 > 0 \) belongs to the eigenspace (we can arrange that \( \phi_1 \) is positive), then using \( \psi = \phi_1 \) in (3.22) and writing \( u = \log \phi_1 \), we get

\[
- \Delta_s u + Q + D_\mu s^\mu - |s - Du|^2 \geq 0,
\]

where we have completed the square on the terms \( 2s^\mu D_\mu \psi - s^4 s_\lambda \) in (3.22).

If we fix the spacetime dimension be \( n = 4 \) and integrate (3.24) over \( S \), which is now a closed 2-surface \( S \), and use of the divergence theorem, then we obtain \( \int_S R_s dS \geq 0 \). Thus the Euler characteristic of \( S \) is nonnegative and is positive unless the conditions \( G^f_{\mu \nu} w^\nu, s = Du, \) and \( K_{\mu AB} b^\mu = 0 \) all hold pointwise on \( S \) and for all null \( l \) and timelike \( w \). Hence, \( S \) is either a 2-sphere or a 2-torus.

It remains to prove that in the latter case, the induced metric is \( e^{2/\delta} \). To do so, we follow the argument [11], which is given in arbitrary dimension. We maintain arbitrary dimension for as long as possible, to gain insight into that case.

Multiplying (3.24) by \( \psi^2 \) for \( \psi \in C^\infty (S) \) and using some simple identities, we have

\[
Q^f \psi^2 - |s - Du|^2 \geq - \psi^2 D \cdot (s - Du)
= - D \cdot (\psi^2 (s - Du)) + 2\psi (D\psi) \cdot (s - Du)
\geq - D \cdot (\psi^2 (s - Du)) - 2|\psi||D\psi||s - Du|
\geq - D \cdot (\psi^2 (s - Du)) - |D\psi|^2 - |s - Du|^2 \psi^2.
\]

Re-arranging terms and integrating, we obtain

\[
\int_S (Q^f \psi^2 + |D\psi|^2) \geq 0
\]

for any \( \psi \in C^\infty (S) \). Now by the Rayleigh formula, the lowest eigenvalue \( \hat{\lambda}_1 \) of the self-adjoint operator \( \hat{L} = -\Delta_s + Q^f \) is given by

\[
\hat{\lambda}_1 = \inf_{\psi} \frac{\int_S (Q^f \psi^2 + |D\psi|^2)}{\int_S \psi^2}.
\]

where the infimum is over all \( \psi \in C^\infty (S) \setminus \{0\} \) (i.e., excluding the zero function). Thus, by (3.26), we see that \( \hat{\lambda}_1 \geq 0 \). Let \( \hat{\phi}_1 \) denote a corresponding eigenfunction, chosen so that \( \hat{\phi}_1 > 0 \).

As above, let \( h_{\mu \nu} \) be the metric induced by \( g_{\mu \nu} \) on \( S \).

\[
\hat{h}_{\mu \nu} := \psi^{2/(n-3)} h_{\mu \nu} := (e^{-\hat{\phi}_1})^{2/(n-3)} h_{\mu \nu}.
\]

The scalar curvature \( \hat{R}_S \) of \( \hat{h} \) is given in terms of the scalar curvature \( R_S \) of \( \hat{h} \) by a standard formula:

\[
\hat{R}_S = \varphi - \frac{n}{n-1} \left( R_S - \frac{2}{\varphi} \Delta_\varphi \varphi + \frac{n - 2}{n - 3} \frac{|D\varphi|^2}{\varphi^2} \right)
= (e^{-\hat{\phi}_1})^{-\frac{n}{n-1}} \left[ R_S + 2\Delta \hat{\varphi}_1 - 2 \frac{\Delta \hat{\varphi}_1}{\hat{\phi}_1} + 2 \frac{|D\hat{\varphi}_1|^2}{\hat{\phi}_1^2} - \frac{n - 4}{n - 3} \frac{|D\hat{\varphi}_1|^2}{\hat{\phi}_1^2} - \frac{n - 4}{n - 3} \frac{|D\hat{\varphi}_1|^2}{\hat{\phi}_1^2} \right],
\]

where in the middle equality we used that \( \varphi = e^{-\hat{\phi}_1} \) and in the final equality we used that \( -\Delta \hat{\varphi}_1 + Q^f \hat{\phi}_1 = \hat{\lambda}_1 \hat{\phi}_1 \). As well, we used (3.23) and defined \( |K(l)|^2 := K_{\mu AB} K^{\mu AB} l^P \).

Now \( \hat{\lambda}_1 \geq 0 \). Thus, when \( n = 4 \) and when the energy condition (1.7) holds, we see that \( \hat{R}_S \geq 0 \) pointwise. Then \( \chi (S) = 0 \implies \hat{R}_S \geq 0 \), which in turn implies that \( \hat{\lambda}_1 = 0 \).

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every future-null $l$ and future-timelike $w$, $K_{\mu\nu}l^\mu l^\nu = 0$ for every future-null $l$ outbound from $S$, and $Dp = 0$. But then the middle line in (3.29) collapses to
\begin{equation}
0 = R_S + 2\Delta_S f. \tag{3.30}
\end{equation}
But in two dimensions, $e^{2f}(R_S + \Delta_S f)$ is the scalar curvature of the metric $e^{-2f}g$, and so $e^{-2f}g$ is a flat metric. \hfill \Box

4. Scalar–tensor theory

The Brans–Dicke theory [2, 6] is actually a family of theories in four spacetime dimensions parametrized by a real number $\omega \in (-\frac{1}{2}, \infty)$, and containing a metric $g_{ij}$ and scalar field $\varphi > 0$ as ‘gravitational variables’. For a given non-gravitational stress–energy tensor $T_{ij}$ whose trace is $T := g^{ij}T_{ij}$ and for a fixed value of $\omega$ these variables are solutions of the system
\begin{equation}
G_{ij} = \frac{1}{\varphi}(\nabla_i \nabla_j \varphi - g_{ij} \Box \varphi) + \frac{8\pi}{\varphi} T_{ij} + \frac{\omega}{\varphi^2} \left( \nabla_i \varphi \nabla_j \varphi - \frac{1}{2} g_{ij} |\nabla \varphi|^2 \right), \tag{4.1}
\end{equation}
\begin{equation}
\Box \varphi = \frac{8\pi T}{3 + 2\omega}. \tag{4.2}
\end{equation}
The form of these equations is of course not invariant under conformal transformations. This particular form is called the Jordan frame form of the theory in the physics literature. Using $f := -\log \varphi$ (4.3) and carrying out some straightforward manipulations including a substitution using (4.2), equation (4.1) can be brought to the form
\begin{equation}
G'_{ij} = 8\pi e^f \left[ T_{ij} - \frac{1}{2} g_{ij} \frac{T}{(3 + 2\omega)} \right] + 8\pi (1 + \omega) S_{ij}, \tag{4.4}
\end{equation}
\begin{equation}
S_{ij} = \frac{1}{8\pi} \left( \nabla_i f \nabla_j f - \frac{1}{2} g_{ij} |\nabla f|^2 \right), \tag{4.5}
\end{equation}
where $G'_{ij}$ is the Bakry–Émery Einstein tensor defined in (3.10) and $S_{ij}$ is the stress–energy tensor of a free scalar field. We caution that $|\nabla f|^2 := g^{-1}(df, df)$ can be negative since $g$ is Lorentzian.

**Proof of theorem 1.4.** Let $v$ be a future-timelike vector and define $\tau_l := S_{ij} u^i v^j$. Then it is easy to verify that $\tau_l \tau^l \leq 0$ and $\tau_l u^l \geq 0$, and as this is the case for any future-timelike $v$, then $\tau$ is past-causal. This in turn implies that $S_{ij} u^i v^j \geq 0$ for any future-timelike (or, by continuity, future-null) $u$ and $v$. Thus $S_{ij}$ obeys the dominant energy condition. If as well $T_{\mu\nu}$ obeys condition (1.13) for all future-timelike vectors $u$ and $v$, we then have $G'_{\mu\nu} u^\mu v^\nu \geq 0$.

Proof of (i). From the last paragraph, it follows that $S_{ij}$ obeys the null energy condition $S_{ij} l^i l^j \geq 0$ for all null vectors $l$. By assumption, $T_{ij}$ also obeys the null energy condition. Then from (4.4) and (1.6), we see that $R'_{l^i l^j} \geq 0$ for all null vectors $l$. Furthermore, $\varphi \geq C > 0 \Rightarrow f \leq \log(1/C)$ so $f$ is bounded above. Then, invoking theorem 1.1, we see that any outer $f$-apparent horizon must lie behind an event horizon, proving part (i) of theorem 1.4.

Proof of (ii). We see from (2.12) that the conformal transformation (2.20) eliminates the Hessian term from the Ricci tensor and thus from the field equation (4.1). Thus, the Einstein frame metric is the metric $\tilde{g}$ obtained from $g$ using the $n = 4$ case of (2.10). By remark 2.6, the
Jordan frame $f$-trapped and Einstein frame trapped regions correspond. Thus, the boundaries of these regions correspond. Then the result follows from part (i).

Proof of (iii). Invoke theorem 1.2 and observe that (4.3) implies that $e^{-f} = \varphi$ so $A_f[S] = \int_S \varphi \, dS$.

Proof of (iv). Now note that

$$
\Box f = -\Box \log \varphi = -\frac{1}{\varphi} \Box \varphi + \frac{|d\varphi|^2}{\varphi^2} = -\frac{8\pi e^{-f} T}{3 + 2\omega} + |df|^2 \geq |df|^2
$$

when $T|_S \leq 0$ and $\omega \geq -3/2$. Because $|\cdot|$ is a Lorentzian norm, $|df|^2$ could be negative. However, it is not, since $\nabla_i \varphi|_S = 0 \iff \nabla_i f|_S = 0 \Rightarrow |df|^2 \equiv h^{AB} D_A f D_B f|_S - \nabla_i f \nabla_i f|_S = h^{AB} D_A f D_B f|_S \geq 0$. Thus $\Box f|_S \geq 0$, and we can invoke theorem 1.3.

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Appendix. Bakry–Émery topological censorship

We recall that the principle of topological censorship governs the topology of black hole horizons:

**Proposition A.1.** Consider a four-dimensional, asymptotically flat spacetime with a globally hyperbolic connected component $\mathcal{D}$ of the domain of outer communications and non-empty event horizon. If the principle of topological censorship holds, each connected component of the event horizon is a 2-sphere.

The principle of topological censorship, also called active topological censorship, states that any causal curve in an asymptotically flat domain of outer communications $\mathcal{D}$ and beginning and ending on $\mathcal{I}$ is fixed-endpoint homotopic to a causal curve on $\mathcal{I}$. A theorem [7] states that the principle holds whenever a globally hyperbolic $\mathcal{D}$ obeys the null energy condition or the weaker averaged null energy condition (ANEC).\(^3\)

**Proof of proposition A.1.** See [5, 10].

The ANEC may be stated as the requirement that

$$
\int_{-\infty}^{\infty} \text{Ric}(\eta', \eta') \, d\lambda \geq 0
$$

along every affinely parametrized complete null geodesic $\eta : (-\infty, \infty) \to M$, where the integral in (A.1) is taken over an affine parameter. In (A.1), by $\gg 0'$, we include the case that the integral diverges, provided in the latter case the improper integral in (A.1), when replaced by an integral over $[a, b]$, is nonnegative for all choices of $a$ and $b$ with $a \leq -c$, $b \geq c$, for some $c$.

\(^3\) For other formulations, including modified formulations for asymptotically anti-de Sitter spacetimes, see [10].
If we replace the Ricci tensor in (A.1) by the Bakry–Émery tensor and use that \( \eta \) is geodesic so that \( \nabla_{\eta'} \eta' = 0 \), then the left-hand side is replaced by

\[
\int_{-\infty}^{\infty} \text{Ric}^f(\eta', \eta') \, d\lambda = \int_{-\infty}^{\infty} \text{Ric}(\eta', \eta') \, d\lambda + \nabla_{\eta} f \big|_{-\infty}^{\infty}.
\] (A.2)

Indeed, more generally, given a vector field \( w \) we can use instead the harmonic Ricci tensor

\[
\text{Ric}_w[g] := \text{Ric}[g] + \frac{1}{2} \xi_w g,
\] (A.3)

which reduces to the Bakry–Émery tensor when \( w = \nabla f \), and then

\[
\int_{-\infty}^{\infty} \text{Ric}_w(\eta', \eta') \, d\lambda = \int_{-\infty}^{\infty} \text{Ric}(\eta', \eta') \, d\lambda + \eta' \cdot w \big|_{-\infty}^{\infty}.
\] (A.4)

This suggests the following definition, which we formulate so as to include both infinite and semi-infinite geodesics:

**Definition A.2.** Given a future-directed causal geodesic \( \eta : I \to M \) with \( I = (-\infty, \infty) \), we say a vector field \( w \) is net decreasing with respect to \( \eta \) if \( \lim_{\lambda \to -\infty} \eta'(\lambda) \cdot (w \circ \eta)(\lambda) \leq \lim_{\lambda \to +\infty} \eta'(\lambda) \cdot (w \circ \eta)(\lambda) \). If instead \( I = [0, \infty) \), we say a vector field \( w \) is net decreasing with respect to \( \eta \) if \( \lim_{\lambda \to -\infty} \eta'(\lambda) \cdot (w \circ \eta)(\lambda) \leq \lim_{\lambda \to \infty} \eta'(\lambda) \cdot (w \circ \eta)(\lambda) \).

Net decreasing vector fields include various special cases of interest. For example, if \( w \) vanishes on a connected component \( \mathcal{I} \) of conformal infinity then it is net decreasing along any causal geodesic \( \eta : (-\infty, \infty) \to M \) beginning and ending on \( \mathcal{I} \). In an asymptotically flat spacetime, \( w \) is also net decreasing along any \( \eta : (-\infty, \infty) \to M \) if it is future-causal at \( \mathcal{I}^+ \) and past-causal at \( \mathcal{I}^- \). In cases such as those just discussed, we say that \( w \) is net decreasing on the domain of outer communications \( \mathcal{D} \). If \( \mathcal{D} \) has a Cauchy surface \( \Sigma \) such that \( w \) is past-causal (including possibly vanishing) along it and future-causal (again, possibly vanishing) at \( \mathcal{I}^- \), then \( w \) is net decreasing along any null geodesic \( \eta : [0, \infty) \to M \) from \( \eta(0) \in \Sigma \) to \( \mathcal{I}^- \), and then we say that \( w \) is net decreasing in the future development of \( \Sigma \).

Therefore, along any geodesic \( \eta : I \to M \) along which \( w \) is net decreasing, we have

\[
\int_I \text{Ric}(\eta', \eta') \, d\lambda \geq \int_I \text{Ric}_w(\eta', \eta') \, d\lambda.
\] (A.5)

We are thus led to a second definition:

**Definition A.3.** We say that the \( w \)-averaged null energy condition (or \( w \)-ANEC) is obeyed along an infinite or semi-infinite null geodesic \( \eta : I \to M, I = [0, \infty) \) or \( I = (-\infty, \infty) \), if, for a vector field \( w \) defined along \( \eta \), we have

\[
\int_I \text{Ric}_w(\eta', \eta') \, d\lambda \geq 0.
\] (A.6)

We say that a spacetime \((M, g)\) obeys the \( w \)-ANEC if the \( w \)-ANEC is obeyed along every null geodesic \( \eta : (-\infty, \infty) \to M \). We say that a spacetime \((M, g)\) obeys the \( w \)-ANEC to the future of a Cauchy surface \( \Sigma \) if

\[
\int_0^{\infty} \text{Ric}_w(\eta', \eta') \, d\lambda \geq 0
\] (A.7)

along all complete future-null geodesics \( \eta : [0, \infty) \to M \) such that \( \eta(0) \in \Sigma \). In the special case that \( w \) is a gradient vector field \( w = \nabla f \), we also say that \((M, g)\) obeys the \( f \)-averaged null energy condition (or \( f \)-ANEC).

Note that these conditions reduce to the familiar ANEC or averaged null energy conditions when \( w \equiv 0 \). Also note that (2.2) implies (A.6) or (A.7) (keeping in mind the remarks after (A.1)). Combining these definitions, we immediately have the following lemma:
Proposition A.4. Let \( w \) be a net decreasing vector field on a domain of outer communications \( D \) of an asymptotically flat spacetime \((M, g)\) which is causally continuous at spatial infinity \( \partial I^0 \) and such that every null geodesic visible from \( I^+ \) is future-complete in \((M, g)\). If the generic curvature condition (as defined in [12] or [13]) and \( w \)-weighted ANEC hold along every null geodesic \( \eta : (-\infty, \infty) \rightarrow D \), then the topological censorship theorem, in the form of [12, theorem 2], holds on \( D \).

Proof. By the assumptions and by (A.5), we have that ANEC holds in \( D \). By [12, remark 2.2], then every complete null geodesic in \( D \) will have a pair of conjugate points. Then the conditions of [12, theorem 2] hold. □

This version of topological censorship is sometimes viewed as being less than fully satisfactory because of its reliance on the generic curvature condition. We can remove this assumption, but then we must instead assume that \( w \) is net decreasing from a Cauchy surface and that the \( w \)-ANEC condition holds to the future of that Cauchy surface. This then yields the version of topological censorship found in [7], see also [10], with the ANEC assumption replaced by the \( w \)-ANEC condition and the net decreasing condition for \( w \).

References

[1] Andersson L, Mars M and Simon W 2008 Stability of marginally outer trapped surfaces and existence of marginally outer trapped tubes Adv. Theor. Math. Phys. 12 853–88
[2] Brans C and Dicke R H 1961 Mach’s principle and a relativistic theory of gravitation Phys. Rev. 124 925–35
[3] Case J S 2010 Singularity theorems and the Lorentzian splitting theorem for the Bakry–Émery–Ricci tensor J. Geom. Phys. 60 477–90
[4] Chrusciel P T, Galloway G J and Solis D 2009 Topological censorship for Kaluza–Klein spacetimes Ann. Henri Poincaré 10 893–912
[5] Chrusciel P T and Wald R M 1994 On the topology of stationary black holes Class. Quantum Grav. 11 L147–52
[6] Fararo V 2004 Cosmology in Scalar–Tensor Gravity (Dordrecht: Kluwer)
[7] Friedman J L, Schleich K and Witt D M 1993 Topological censorship Phys. Rev. Lett. 71 1486–9
Friedman J L, Schleich K and Witt D M 1995 Phys. Rev. Lett. 75 1872 (erratum)
[8] Galloway G J 2008 Rigidity of marginally trapped surfaces and the topology of black holes Commun. Anal. Geom. 16 217–29
[9] Galloway G J 2000 Maximum principles for null hypersurfaces and null splitting theorems Ann. Henri Poincaré 1 543–67
[10] Galloway G J, Schleich K, Witt D M and Woolgar E 1999 Topological censorship and higher genus black holes Phys. Rev. D 60 104039
[11] Galloway G J and Schoen R 2006 A generalization of Hawking’s topology theorem to higher dimensions Commun. Math. Phys. 266 571–6
[12] Galloway G J and Woolgar E 1997 The cosmic censor forbids naked topology Class. Quantum Grav. 14 L1–L7
[13] Hawking S W and Ellis G F R 1973 The Large Scale Structure of Space-Time (Cambridge: Cambridge University Press)
[14] Lott J 2003 Some geometric properties of the Bakry–Émery–Ricci tensor Comment. Math. Helv. 78 865–83
[15] Scheel M A, Shapiro S L and Teukolsky S A 1995 Collapse to black holes in Brans–Dicke theory. II. Comparison with general relativity Phys. Rev. D 51 4236–49
[16] Wald R M 1984 General Relativity (Chicago, IL: University of Chicago Press)
[17] Wei G and Wylie W 2009 Comparison theory for the Bakry–Émery Ricci tensor J. Differ. Geom. 83 337–405
[18] Woolgar E 2013 Scalar–tensor gravitation and the Bakry–Émery–Ricci tensor Class. Quantum Grav. 30 085007