THE DIFFERENTIAL GEOMETRY OF COMPOSITION SEQUENCES OF DIFFERENTIABLE MANIFOLDS.

Abstract.
Let $F_0 = B, F_1, \ldots, F_n$ be a sequence of differentiable manifolds and $G_l, l \geq 1$ a Lie subgroup of the group of diffeomorphisms of $F_1$, $H_l$ a central subgroup of $G_1$. We denote by $K_l$ the quotient of $G_l$ by $H_l$. We suppose also given a locally trivial principal bundle $p_{K_l}$ over $F_{l-1}$ which typical fiber is $K_l$. In this paper we study the differential geometry of the problem of the existence of a composition sequence of manifolds $M_n \to M_{n-1} \to \cdots \to M_0 = B$, where each map $f_l : M_l \to M_{l-1}$ is a locally trivial differentiable bundle over $M_{l-1}$ which typical fiber is $F_l$, and such that the image of the transitions functions is contained in $G_l$. We associated to this problem a tower of gerbes, and we study its differential geometry: that is we define the notion of connective structure and curvature.

Introduction.

Let $F_0 = B, F_1, \ldots, F_n$ be a sequence of differentiable manifolds and $G_l, l \geq 1$ a Lie subgroup of the group of diffeomorphisms of $F_1$, $H_l$ a central subgroup of $G_1$. We denote by $K_l$ the quotient of $G_l$ by $H_l$. We suppose also given a locally trivial principal bundle $p_{K_l}$ over $F_{l-1}$ which typical fiber is $K_l$. In this paper we study the differential geometry of the problem of the existence of a composition sequence of manifolds $M_n \to M_{n-1} \to \cdots \to M_0 = B$, where each map $f_l : M_l \to M_{l-1}$ is a locally trivial differentiable bundle over $M_{l-1}$ which typical fiber is $F_l$, and such that the image of the transitions functions is contained in $G_l$. We can associate to $f_l$ a bundle over $M_{l-1}$ which typical fiber is $G_l$, we suppose also that the quotient of the restriction of this bundle to a fiber of $f_{l-1}$ by $H_l$ is $p_{K_l}$. We associated to this problem a tower of gerbes, and we study its differential geometry.

The construction of those sequences of fibrations fits in the general problem of the construction of structures in mathematics. There are essentially two ways to obtain new structures:

One is the completion procedure, which is used generally to create space in which unsolved problems have a solution. It also shows the link between algebra and analysis.

The other essential manner to construct structures in mathematics is the gluing procedure, which is the bridge from analysis to geometry. It is the one
involved in our problem. To solve the gluing problem, that is to find obstructions to define global objects which are defined locally, one needs cohomology theories interpreted geometrically.

More practically, consider a topological manifold $M$, it is built by gluing open subsets of a vector space say $E$, to study $M$, we can for example generalize tools used to study $E$. For instance, we can define the notion of continuous functions. Can functions defined locally be defined globally? This problem is solved using a $1-\text{cohomology theory}$.

The notion of sheaf of functions leads to the notion of locally trivial bundles. which are particular case of sheaves of functions. One can also try to solve the following problem. Can sheaves of bundles defined locally be defined globally? To solve this problem, one needs the notion of stacks defined by Giraud [4]. The obstructions to solve this problem is given by a $2-\text{cohomology theory}$.

More generally, one can try to define the notion of sheaf of stacks ($3-\text{sheaf}$) which would lead to the notion of a $3-\text{cohomology theories}$, the notion of $n-\text{sheaf}$, which would lead to the notion of a $n-\text{dimension cohomology theory}$.

To define a notion of $n-\text{sheaf}$, one needs to define first, the notion of $n-\text{category}$, which is not well understood nowadays.

The gluing procedure in the $1-\text{dimensional case}$ is the theory of sheaves. It is used to construct Hilbert schemes (see Gottsche [5]).

In the two dimensional case, it has been established by Giraud, it is the theory of stacks.

\textbf{I. The first lifting problem.}

Let $B$ and $F$ be two differentiable manifolds, $G$ a Lie subgroup of diffeomorphisms of $F$, $H$ a subgroup of $G$ central in $G$, we denote by $K$ the quotient $G/H$. We suppose defined on $B$ a locally trivial $K-\text{principal bundle}$. We want to study the differential geometry of the following problem: Classify the locally trivial bundles over $B$, which typical fiber is $F$ and such that the family of elements $g_{ij}$ which define each trivialization $(U_i, g_{ij}, i, j \in I)$ is included in $G$. This means that the $Diff(F)$ bundle associated to the bundle has a $G$-reduction.

We can associate also to a solution of our problem a principal bundle $p_G$ over $B$ which typical fiber is $G$. We suppose moreover that the quotient of $p_G$ by $H$ is $p_K$. This problem will be called the first lifting problem.

We have the exact sequence $1 \to H \to G \to K \to 1$. We will assume that the map $G \to K$ has local sections.

Our problem is equivalent to the following: study the existence of a bundle over $B$ which typical fiber is $G$, and which quotient by $H$ is $p_K$. To solve it, we will define a gerbe over $B$ which classifying cocycle is the obstruction of the existence of such a bundle. This problem will be called the principal problem associated to the first lifting problem.

Let’s recall now some facts from gerbe theory.

\textbf{Definition 1.}
Let $B$ be a manifold, a sheaf $S$ of categories on $B$, is a map $U \to S(U)$, where $U$ is an open set of $B$, and $S(U)$ category which satisfies the following properties:
- To each inclusion $U \to V$, there exists a map $r_{U,V} : S(V) \to S(U)$ such that $r_{U,V} \circ r_{V,W} = r_{U,W}$.
- Gluing conditions for objects,
  Consider a covering family $(U_i)_{i \in I}$ of an open set $U$ of $B$, and for each $i$, an object $x_i$ of $S(U_i)$, suppose that there exists a map $g_{ij} : r_{U_i \cap U_j, U_j}(x_i) \to r_{U_i \cap U_j, U_i}(x_j)$ such that $g_{ij}g_{jk} = g_{ik}$, then there exists an object $x$ of $C(U)$ such that $r_{U_i,U}(x) = x_i$
  Gluing conditions for arrows,
  Consider two objects $P$ and $Q$ of $S(B)$, then the map $U \to Hom(r_{U,B}(P),r_{U,B}(Q))$ is a sheaf.
Moreover, if the following conditions are satisfied the sheaf of categories $S$ is called a gerbe

$G1$ There exists a covering family $(U_i)_{i \in I}$ of $B$ such that for each $i$ the category $S(U_i)$ is not empty

$G2$ Let $U$ be an open set of $B$, for each objects $x$ and $y$ of $U$, there exists a covering $(U_i)_{i \in I}$ of $U$ such that $r_{U_i,U}(x)$ and $r_{U_i,U}(y)$ are isomorphic

$G3$ Every arrow of $S(U)$ is invertible, and there exists a sheaf $A$ in groups on $B$, such that for each object $x$ of $S(U)$, $Hom(x,x) = A(U)$, an this family of isomorphisms commute with the restriction maps.

The sheaf $A$ is called the band of the gerbe $S$, in the sequel, we will consider only gerbes with commutative band.

**Notation.**

for a covering family $(U_i)_{i \in I}$ of $B$, and an object $x_i$ of $S(U_i)$, we denote by $x^i_{i_1...i_n}$ the element $r_{U_{i_1...i_n} \cap U_{i_1}}(x_i)$, and by $U_{i_1...i_n}$ the intersection $U_{i_1} \cap ... \cap U_{i_n}$.

Let endows $B$ with a gerbe $S$ with band $A$, one can associates the following two Cech cocycle to $B$. Consider a covering family $(U_i)_{i \in I}$ such that for each $i$, $S(U_i)$ is not empty, let $x_i$ and $x_j$ be respectively objects of $S(U_i)$ and $S(U_j)$, we consider a map $g_{ij} : x^j_{ij} \to x^i_{ij}$, the map $t_{ijk} = g_{ij}g_{jk}g_{ki}$ of $Aut(x^i_{ijk})$ is the classifying 2 Cech cocycle.

**Theorem 1.** see [4]

Two gerbes defined over $B$ with band $A$ are isomorphic if and only if their associated 2–cocycles define the same cohomological class. Conversely for each 2–Cech $A$ cocycle $c$, one can define a gerbe which associated cocycle is $c$.

We define now on $B$ the sheaf of categories used to solve our problem. For each open set $U$ of $B$, we denote by $C(U)$ the category which objects are local trivial bundles over $U$ which typical fiber is $F$, such that the $Diff(F)$ bundle associated has a $G$–reduction. This means that the image of the transitions
functions are elements of $G$. Moreover we suppose that the quotient of this reduction by $H$ is the restriction $p_{KU}$, of $p_K$ to $U$. A map between two objects of $C(U)$, is an isomorphism of bundle which induces a map between their associated principal bundles which pushes forward to the identity of $p_{KU}$. This means that it is an isomorphism which preserves and acts on each fiber by an element of $H$.

The map 
\[ U \rightarrow C(U) \]
defines on $B$ a gerbe which band is $H'$, the sheaf of differentiable $H$ functions defined on $B$.

Let 
\[ k_{ij} : U_i \cap U_j \rightarrow K \]
be the the transitions functions of the bundle $p_K$ associated to the covering family $(U_i)_{i \in I}$.

Consider a map 
\[ g_{ij} : U_i \cap U_j \rightarrow G \]
which pushes forward to $k_{ij}$. The map $g_{ij}g_{jk}g_{ki}$ defines the classifying cocycle $c_H$ of the gerbe $C$.

We can also defined the principal gerbe $C_P$ associated to our problem, for each open set $U$ of $B$, we will denote by $C_P(U)$, the set of locally trivial bundles defined on $U$ which typical fiber is $G$, and which pushes forward to the restriction of $p_K$ to $U$.

**Connections on principal bundles.**

Let $\mathcal{K}$, $\mathcal{G}$ and $\mathcal{H}$ be respectively the Lie algebra of $K$, $G$ and $H$. A connection on $p_K$ is a 1–form $\omega : TB \rightarrow \mathcal{K}$ such that:

1. For each fundamental vector field $A_B$ defined by the element $A$ of $\mathcal{K}$, we have: $\omega(A_B(x)) = A$,

2. For each element $g$ of $K$, we have $\omega(g^*(X)) = \text{Ad}(g^{-1})(X)$.

The curvature of $\omega$ is the 2–form $d\omega + \frac{1}{2}[\omega, \omega]$.

**The notion connective structure of the principal gerbe.**

Let $\omega$ be a connection on $p_K$, and $e_U$ an element of $C_P(U)$, recall that is a principal bundle over $U$ which typical fiber is $G$. We denote by $Co(e_U, \omega)$ the family of connections defined on $e_U$ which push forward to the restriction of $\omega$ to $U$. Consider the map $p_1 : \mathcal{G} \rightarrow \mathcal{K}$, and the map $p_2 : e_U \rightarrow p_{K,U}$ where $p_{K,U}$ is the restriction of $p_K$ to $U$. the fact that the connection $a$ defined on $e_U$ pushes forward to the restriction $\omega_U$ of $\omega$ to $U$ means that $p_2^*(\omega_U) = p_1(a)$.

The map $U \rightarrow Co(e_U, \omega)$ is called the connective structure associated to $\omega$. 

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Supposed that \(((U_i)_{i \in I}, g_{ij})\) is a trivialization of the bundle \(p_K\). Let \(e_i\) be an element of \(C_P(U_i)\) endowed with a connection \(w_i\) over the restriction of \(\omega\) to \((p_K)_{U_i}\), the restriction of \(p_K\) to \(U_i\).

There exists a map \(g_{ij}\) between \(e_i^j\) and \(e_i^j\) the respective restrictions of \(e_j\) and \(e_i\) to \(U_{ij}\).

Consider the form \(w_j - g_{ij}^* w_i\), (it pushes forward to a form \(a_{ij}\) defined on \(T U_{ij}\) since \(H\) is central in \(G\)), and \(e_{ij}^k\) the restriction of \(e_i\) to \(U_{ijk}\). On \(U_{ijk}\), we have the restriction of the forms \(w_j - g_{ij}^* w_i\), \(w_k - g_{jk}^* w_j\) and \(w_k - g_{ik}^* w_i\), define respectively on \(e_{ij}^k\), \(e_k^i\), we can define the form

\[
g_{jk}^*(w_j - g_{ij}^* w_i) + (w_k - g_{jk}^* w_j) - (w_k - g_{ik}^* w_i) =
\]

\[-(g_{ij}g_{jk})^* w_i + g_{ik}^* w_i = g_{ik}^*(w_i - (g_{ij}g_{jk}g_{ki})^* w_i) = g_{ik}^*(w_i - c_{ijk}^* w_i) = e_{ij}^k dc_{ijk}\]

**Proposition 1.** The family \(d(a_{ij})\) is a 1-cocycle.

**Proof.**

On \(U_{ijk}\), we have \(a_{jk} - a_{ik} + a_{ij} = e_{ij}^{-1} dc_{ijk}\), since \(d(e_{ijk}^{-1} dc_{ijk}) = 0\), we obtain the result.

Consider now the curvature \(K(w_i)\) of the connection \(w_i\), it is the form \(w_i + \frac{1}{2}[w_i, w_i]\). For any other form \(w_i'\) over \(\omega\), let \(a_i = w_i - w_i'\), we have \(K(w_i) = K(w_i') + da_i\), this implies that \(dK(w_i)\) is not depend of the form over \(\omega_i\).

To set in a conceptual theory, one as to define the cotangent and tangent spaces of the principal gerbe.

**Definition 2.**

Let \(U\) be an open set of \(B\), an element of the tangent space of \(C(U)\) will be a family of vectors fields \((X_e)\) where \(e\) is an object of \(C(U)\), such that if \(e\) and \(f\) are objects of \(C(U)\), then there exists a map \(g : e \to f\) such that \(Tg(X_e) = X_f\).

An element of the cotangent of will be a family of forms \((\alpha_e)\) such that for each vector field \((X_e)\), we have \(\alpha_f(X_f) = \alpha_e(X_e)\), one can define in an obvious way \(\Lambda^n C\).

A \(n\)-tangent field of \(C(U)\) \(nTC(U)\) will be a family of of \(n\)-vector fields \(((X^1_e, \ldots, X^n_e)\) \(\in Obj(C(U))\) where \(X^i_e\) is a vector field tangent to the element \(e\) of \(C(U)\) such for each object \(e\) and \(f\) of \(C(U)\), there is a map \(g : e \to f\) such that \(Tg(X^i_e) = X^i_f\).

one deduce that the family of \(3\)-form \(dK(w_i)\) define a form \(\Omega\) on \(3TC\) which takes value in \(G\) this form is called the curvature of the associated gerbe.

**II. The generalization.**

Let \(B = F_0, F_1, \ldots, F_n\) be a sequence of manifolds. For each \(F_i\), we consider a Lie subgroup \(G_i\) of diffeomorphisms of \(F_i\), and a central subgroup \(H_i\) of \(G_i\). Let denote by \(K_i\) the quotient of \(G_i\) by \(H_i\), we also suppose given locally trivial
principal bundles $p_{Ki}$ over $F_{i-1}$ which typical fiber is $K_i$. We will denote by $\mathcal{H}_i, \mathcal{G}_i$ and $K_i$ the respective Lie algebras of $H_i, G_i$ and $K_i$.

We want to study the differentiable geometry of the following problem:

classify sequences $M_n \rightarrow M_{n-1} \rightarrow M_1 \rightarrow M_0 = B$, where each map $f_i : M_{i+1} \rightarrow M_i$ is locally trivial differentiable bundle which typical fiber is $F_{i+1}$, such that the transitions functions are elements of $G_{i+1}$, so we can define a locally trivial principal bundle $p_{MG_{i+1}}$ over $M_i$ which typical fiber is $G_i$, and consider is quotient $p_{M^{Ki}{i+1}}$ by $H_{i+1}$. We suppose that the restriction $p_{FK_{i+1}}$ of $p_{MG_{i+1}}$ to a fiber of $f_{i-1}$ (naturally diffeomorphic to $F_i$) is $p_{Ki+1}$. We will call this problem the $n$–lifting problem.

We will associate to our problem, a tower of gerbes, and cocycles which will be the obstructions to solve it.

First we define on $B$ the gerbe $C_1$ associated to the first lifting problem defined by the data $F_1, H_1, G_1$ and $K_1$.

Let $U$ be an open set of $B$ and $e_1(U)$ an element of $C_1(U)$, we can associate to $e_1(U)$, the gerbe $C_2(e_1)$ which solve the first lifting problem which data are $F_2, G_2$ and $K_2$, over the basis $e_1(U)$. Remark that $p_{K_2}$ induces naturally a $K_2$–principal bundle $p_{e_1}$ over $e_1(U)$. The quotient of solutions the principal $G_2$–bundles associated to the solutions of this first lifting problem by $H_2$ must be restriction of $p_{e_1}$.

Suppose that we have defined the gerbes $C_1, C_2(e_1),.., C_{i-1}(e_{i-1})$, let $e_i$ be an object of $C_i(e_{i-1})$, we define the gerbe $C_{i+1}(e_i)$ which solve the first lifting problem associated to $F_{i+1}, G_{i+1}, K_{i+1}$ with basis $e_i$. Remark that $p_{K_{i+1}}$ induces naturally a $K_{i+1}$–principal bundle over $e_i$, we denote it by $p_{e_i}$. We suppose that quotient of the principal $G_{i+1}$ bundles associated to the solutions of this first lifting problem by $H_{i+1}$ are restriction of $p_{e_i}$.

We have assumed that the group $H_i$ is central in $G_i$, let $(U^i_{jk}, K^i_{jk})$ be the trivialization of the bundle $p_{Ki}$, one defines over $F_{i-1}$ a locally trivial bundle $Lie_{i-1}$ which typical fiber is $H_i$ and which trivialization maps are:

$$U^i_{jk} \times G_i \rightarrow U^i_{jk} \times \mathcal{G}_i$$

where $g^i_{jk}$ is an element of $G_i$ over $k^i_{jk}$.

this sheaf $Lie_{i-1}$ is defined for the first lifting problem associated to $F_i, H_i, G_i$ and $K_i$ which basis space is $F_{i-1}$.

Suppose that we have defined a sheaf over $B$, $Lie_{i-1..1}(F_i, H_i, G_i)$ associated to the $i$–lifting problem defined by the family of groups $F_1,..., F_{i-1}, H_1,..., H_{i-1}, G_1,..., G_i$. We can define the $i$–lifting problem associated to the family $F'_1 = F_1,..., F'_{i-1} = F_{i-1}, F'_i = Lie_{i-1} H'_{i} = H_1,..., H'_{i-1} = H_{i-1}, H'_i, G_1,..., G_{i-1}, G'_i$, where $G'_i$ and $H'_i$ are the groups of automorphisms of $Lie_{i-1}$ which elements project respectively onto elements of $G_i$ and $H_i$. We set $Lie_{i+1..1}(F_{i+1}, G_{i+1}, H_{i+1}) = Lie_{i+1}(F'_i, G'_i, H'_i)$.
The classifying cocycle.

We will now define a $n + 1$–cocycle associated to the tower of gerbes.

For each manifold $F_i$, we will consider a trivialization $((U^i_j, k^i_{j_1j_2}, j \in J)$ of the bundle $p_K$. Recall that $B = F_0$.

For each open set $U^0_{j_1}$, consider an object $e^1_{j_1}$ of $C_1(U^0_{j_1})$. There exists a map $g^1_{j_1j_2}$ over $k^1_{j_1j_2}$ between the respective restrictions of $e^1_{j_2}$ and $e^1_{j_1}$ to $U^0_{j_1j_2}$.

The map $c^1_{j_1j_2j_3} = g^1_{j_1j_2}g^1_{j_2j_3}g^1_{j_1j_3}$ is an automorphism of $e^1_{j_1j_3}$ the restriction of $e^1_{j_3}$ to $U^0_{j_1j_2j_3}$ of identified to $U^0_{j_1j_2j_3} \times F_1$. The chain $c^1_{j_1j_2j_3}$ can be viewed as a map $U^0_{j_1j_2j_3} \rightarrow H_1$. Since $C_1$ is a gerbe, the family $c^1_{j_1j_2j_3}$ is a $2$–$H_1$ cocycle.

The map $c^1_{j_1j_2j_3}$ acts on the category of open sets of $U^0_{j_1j_2j_3} \times F_1$.

The covering which defines the trivialization of $e^1_{j_1j_2j_3}$ can be considered as $U^0_{j_1j_2j_3} \times U^1$. The map $c^1_{j_1j_2j_3}$ induces a functor $(c^1_{j_1j_2j_3})^*$ between the categories $C_2(U^0_{j_1j_2j_3} \times U^1)$ and $C_2(U^0_{j_1j_2j_3} \times c^1_{j_1j_2j_3}(U^1))$. Since we have assumed $H_1$ central in $G_1$, we can consider $c^1_{j_1j_2j_3a}$, $c^1_{j_1j_3a}$, $c^1_{j_1j_3a}$ as morphisms of $U^0_{j_1j_2j_3} \times F_1$, this enable to compose the functor $(c^1_{j_1j_2j_3})^*$, $(c^1_{j_1j_3a})^*$, $(c^1_{j_1j_3a})^*$, and to define the Cech boundary $d((c^1_{j_1j_2j_3})^*)$.

Since $c^1_{j_1j_2j_3}$ is a $2$–cocycle, the map $d((c^1_{j_1j_2j_3})^*)$ coincide with the restriction of an element $c^1_{j_1j_2j_3a}$ of the set of differentiable functions $U^0_{j_1j_2j_3a} \times U^1 \rightarrow H_2$ which acts by automorphism on the elements of the category $C_2(U^0_{j_1j_2j_3a} \times U^1)$.

**Proposition 1.**

The family $c^1_{j_1j_2j_3a}$ defines a $3$–Cech cocycle.

**Proof.**

We have $d((c^1_{j_1j_2j_3})^*) = \sum_{i=1}^{5} (-1)^{i}c^1_{j_1\ldots j_i\ldots j_5} = \sum_{i=1}^{5} (-1)^{i} \sum_{k=1}^{3} (-1)^{k}(c^1_{j_1\ldots j_k\ldots j_i})^* = 0$.

Supposed now define the $i + 1$ cocycle associated to the $i$–lifting problem. It is a family of maps $c^1_{j_1\ldots j_i+2} : U^0_{j_1\ldots j_i+2} \times \ldots \times U^0_{i-1} \times F_1$. The map $c^1_{j_1\ldots j_i+2}$ defines a functor $(c^1_{j_1\ldots j_i+2})^*$ between the categories $C_{i+1}(U^0_{j_1\ldots j_i+2} \times U^1 \times \ldots \times U^1)$, and the category $C_{i+1}(U^0_{j_1\ldots j_i+2} \times U^1 \times \ldots \times U^1)$, where $U^1$ is an open set of $F_1$. Since we have supposed that $H_i$ is central in $G_i$, we can consider the Cech boundary $d((c^1_{j_1\ldots j_i+2})^*)$. Since the map $c^1_{j_1\ldots j_i+2}$ is an $i$–cocycle, we can consider $d((c^1_{j_1\ldots j_i+2})^*)$ as a differentiable map $c^1_{j_1\ldots j_i+2} : U^1_{j_1\ldots j_i+2} \times U^1 \times \ldots \times U^1 \rightarrow H_{i+1}$.

**Proposition 2.** The family of maps $c^1_{j_1\ldots j_i+3}$ define a $j + 2$ cocycle.

**Proof.**

We have

\[ d(c^1_{j_1\ldots j_i+3}) = \sum_{k=1}^{k=i+4} (-1)^{k}c^1_{j_1\ldots j_kj_i+4} = \sum_{k=1}^{k=i+4} (-1)^{k} \sum_{l=1}^{l=i+3} (-1)^{l}(c^1_{j_1\ldots j_kj_i+4})^* = 0 \]
We have seen how to each \( n \)-lifting problem one can associate a \( n+1 \)-cocycle \( H_n \), is it true that fact conversely to each \( n+1 - H_n \), one can define a \( n+1 \)-lifting problem which \( n+1 \)-classifying cocycle is the given cocycle? If not which conditions must satisfy and \( n+1 \)-cocycle to be the classifying cocycle of a \( n+1 \)-lifting problem? We will first study the first lifting problem.

Recall that if we have the following exact sequence \( 1 \rightarrow S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow 1 \) of sheaves defined on \( B \) such that \( S_1 \) is a commutative sheaf, there exists the following exact sequence in cohomology.

\[
0 \rightarrow H^0(B, S_1) \rightarrow H^0(B, S_2) \rightarrow H^0(B, S_3) \\
\rightarrow H^1(B, S_1) \rightarrow H^1(B, S_2) \rightarrow H^1(B, S_3) \rightarrow H^2(B, S_3)
\]

denote by \( \delta_1 \) the map \( H^1(B, S_3) \rightarrow H^2(B, S_3) \), we have

Proposition 3.

A \( 2-H_1 \) cocycle \( c_{H_1} \) defined on \( B \) is the classifying cocycle of a first lifting problem, if and only if there exists an element \( c_{K_1} \) of \( H^1(B, K_1) \) such that \( c_{H_1} = \delta_1(c_{K_1}) \).

Proof.

It’s enough to remark that each \( c_{K_1} \) cocycle \( c \) define a principal bundle over \( B \), such that \( \delta_1(c) \) is the classifying cocycle of the first lifting problem associated to this bundle.

Remark.

Saying that \( \delta_1(c_{K_1}) = c_{H_1} \) means that \( c_{H_1} \) is deduced from \( c_{K_1} \) in the same way that \( \delta_1 \) is defined.

Generally the map \( \delta_1 \) is not injective. To see this consider a \( G \)-principal bundle \( p_G \) over \( B \), \( H \) a central subgroup of \( G \). Denote by \( K \) the quotient of \( G \) by \( H \) and by \( c \) the cocycle which defines the quotient \( p_K \) of \( p_G \) by \( H \), then \( \delta_1(c) \) is trivial.

Now suppose that the sheaves \( S_1 \), \( S_2 \) and \( S_3 \) are commutative, we can write the following long exact sequence

\[
0 \rightarrow H^0(B, S_1) \rightarrow H^0(B, S_2) \rightarrow H^0(B, S_3) \\
\rightarrow H^1(B, S_1) \rightarrow H^1(B, S_2) \rightarrow H^1(B, S_3) \rightarrow H^2(B, S_3) \\
.. \rightarrow H^n(B, S_1) \rightarrow H^n(B, S_2) \rightarrow H^n(B, S_3) \rightarrow H^n(B, S_3)
\]

denote by \( \delta_n \) the map \( H^n(B, S_3) \rightarrow H^{n+1}(B, S_3) \), we have

Proposition 4.

The \( n+1 \)-cocycle \( c_{H_n} \) of the \( n \)-lifting problem is the image of \( c_{H_{n-1}} \) by \( \delta_n \).

Proof.

Straightforward computations.

Definition 1.
We will say that the $n + 1$–tower is trivial, if and only if its associated cocycle $c_{H_n}$ vanishes.

**The principal tower of gerbes associated.**

We will define the notion of connective structure and curvature related to the tower of gerbes associated to our problem. First we define the principal tower associated to our problem.

Recall that we can associate to $C_1$ a principal gerbe $C_1 P$ defined as follows:

Let $U$ be an open set of $B$, we consider the category $C_1 P(U)$ of $G_1$ principal bundles over $U$ such that the quotient of each of its elements by $H_1$ is the restriction of $p_{K_1}$ to $U$.

Let $e_1(U)$ be an element of $C_1(U)$, we consider an open set $U^1$ of $e_1(U)$, and an element $e_2$ of $C_2(U^1)$, it is an $F_2$ bundle over $U^1$, we denote by $p_{U^1} : e_2 \to U^1$, the canonical projection, we denote by $p_{U^1} : U^1 \to U$ the restriction of the canonical projection of $e_1(U)$ to $U^1$.

We can also defined over $e_2$ the principal bundle $e_{2\{P}$ associated to the first lifting problem defined by $F_2$, $H_2$ and $G_2$ which typical fiber is $G_3$, the elements of the category $C_2 P(U^1)$, are fiber product over $p_{U^1}(U^1)$ of $e_{2\{P}$ and $e_{1\{P}$, where $e_{1\{P}$ is a principal bundle over $p_{U^1}(U^1)$ associated to the first lifting problem defined by the data $F_1$, $H_1$ and $G_1$. Suppose that $U^1 = V \times V^1$ where $V$ and $V^1$ are open sets of respectively $B$ and $F_1$ such that the respective restriction of $p_{K_1}$ and $p_{K_2}$ to $V$ and $V^1$ are trivial. Then the elements of $C_2 P(U^1)$ are isomorphic to $(V \times G_1) \times (V_1 \times G_2)$. The set of morphisms of elements of $C_2 P(U^1)$ is given by the natural action of $H_2$.

Suppose defined the category $C_{i\{P}(U^{i-1})$, where $U^{i-1}$ is an open set of an element $C_i(U^{i-2})$, let $U^i$ be an open set of an object of $C_i(U^{i-1})$, we can define on $U^i$ the principal bundle $e_{i+1\{P}$ which solve the first lifting problem associated to $F_{i+1}$, $G_{i+1}$ and $H_{i+1}$. It is a principal bundle $e_{i\{P}$ over $U^i$ which typical fiber is $G_{i+1}$, denote by $p_{U^i}$ the canonical map $p_{U^i} : U^i \to U^{i-1}$, the elements of $C_{i+1\{P}(U^i)$, are fiber product of $e_{i\{P}$ and $e_{i-1\{P}$ over $p_{U^i}(U^{j-1})$ where $e_{i-1\{P}$ is an element of $C_{i\{P}(p_{U^i}(U^j))$. The automorphisms group of elements of $C_{i+1\{P}(U^i)$ are induced by the action of $H_{i+1}$.

Locally, an object of $C_{i\{P}(U^{i-1})$ is of the form $U \times G_1 \times (U^1 \times G_2) \times \ldots \times (U^{i-1} \times G_i)$.

**Definition 1.**

The $n$–tangent space of the tower of gerbes $C_P$, will be the disjoint unions of the tangent space of the gerbes $C_{i\{P}$.

**The differential geometry of the principal tower of gerbes.**

Suppose defined on each bundle $p_{K_i}$, a connection $w_i$. We have already define the notion of connective structure for the first lifting problem. Recall that for each element $e_1$ of $C_1 P(U)$, it is the family of connections defined on $e_1$ which project to the restriction of $w_1$ to the restriction of $p_{K_1}$ to $U$.  

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Suppose defined the notion of connective structure for the $i$–lifting problem, for each object $e_{iP}$ of $C_{tP}(U^{i-1})$, (we can suppose $e_{iP}$ to be isomorphic to $U \times G_1 \times (U^1 \times G_2) \times \ldots \times (U^{i-1} \times G_i)$) it is a family of connection defined on $e_{iP}$, for an element $e_{i+1P}$ of $C_{i+1P}(U^i)$, we will defined the the set $Co(e_{i+1P})$, as the set of $G_1 \times \ldots \times G_{i+1}$ connections which project to elements of $Co(e_i(p_{1U}(U^i)))$, where $e_i(p_{1U}(U^i))$ is the canonical projection of $e_{i+1P}$ over an element of $C_{iP}(U^i)(U_i)$, consider the projection of $U^i$ to $F_i$, and the restriction $p_{jK_i}$ of $p_{jK}$, to the image of the last projection, we suppose also that elements of $Co(e_{i+1P})$ project to the restriction of $w_{i+1}$.

If this element is isomorphic to $U \times G_1 \times \ldots \times U^{i-1} \times G_i$, then an element of $Co(e_{iP})$ is a connection which project to a connection of $Co(e_{i-1P})$, where $e_{i-1P} = U \times G_1 \times \ldots \times U^{i-2} \times G_{i-2}$ and which project to a connection of $U^{i-1} \times G_i$, which project to the restriction of $w_i$ to $U^{i-1} \times K_i$.

**The sequence of curvatures associated.**

For the gerbe associated to the first lifting problem, we have already defined the curvature, it is a Lie($1$) $3$–form. Let us recall it’s definition.

Consider a trivialization $(U_{i1}, k_{i1} - k)$ of the bundle $p_{K1}$, over $U_{i1} \times G_1$ we choose a connection $w_{i1}$ which project to the restriction of $w_i$ to $U_{i1} \times G_1$, we consider the curvature $K(w_{i1})$ of this connection, then $dK(w_{i1})$ defines the requested $3$–form.

Consider now an element $e_i$ of $C_{iP}(U^{i-1})$, isomorphic to $U_{j1} \times \ldots \times U_{j3} \times G_1 \times (U^1 \times G_2) \times \ldots \times (U^{i-1} \times G_i)$, and $\hat{w}_i$ an element of $Co(e_i)$, if $\hat{w}'_i$ is another element of $Co(e_i)$, then $\hat{w}_i - \hat{w}'_i$ is an $H_1 \oplus \ldots \oplus H_n$ form $a$, we have $K(w_i) = K(w'_i) + da$, this implies that $d(K(w_i))$ is independent of the chosen element of $Co(e_i)$ over $w_i$.

**Definition 2.**

The form curvature $dK(\hat{w}_i)$, defines a $3$–form $\Omega_i$ on $3TC_{iP}(U^{i-1})$.

**III. Holonomy and parallel displacement.**

We will precise the differential geometry of the $n$–lifting problem by defining notions of holonomy and covariant derivative.

Suppose given a connection $\omega$ on $p_{K1}$, for each open set $U$ of $B$ and each couple of objects $(e_1, e_2)$ of objects of $C_1(U)$, one can consider connections $w_1$ and $w_2$ defined respectively on $e_1$ and $e_2$ which push forward to the restriction of $\omega$ to the restriction of $p_{K1}$ to $U$, unfortunately each isomorphism between $e_1$ and $e_2$ does not transforms $\omega_1$ to $\omega_2$. To avoid this problem, we are going to enlarge the distribution which defines the connection $\omega_1$ and $\omega_2$, to a transitive uniform distribution studied by P. Molino in his thesis, this distribution will behave naturally in respect to the morphisms between objects, generalizing the work of the previous author, will define first, the notion of holonomy and covariant derivative for the principal gerbe $C_1$ and generalizing those notions for the principal tower associated to the last $n$–lifting problem. First let recall some results from the work of P. Molino.
Let \( p : E \to B \), be a \( K \)-principal locally trivial bundle over \( B \) which typical fiber is \( K \), a transitive distribution \( \mathcal{D} \) (TD) on \( E \) will be a right invariant distribution defined on \( E \) such that \( d_p(p(D_x)) = T_{p(x)}B \).

We will denote by \( u \) be the map defined by \( u(X) = A \), where \( X \) a vertical vector tangent to \( x \) and \( A \) the element of \( K \) such that \( A^*(x) = X \), where \( A^* \) is the fundamental vector field defined by \( A \). We will also denote \( V_x \) the subset of vertical vector fields at \( x \). The transitive distribution is said to be freely uniform, if \( u(\mathcal{D} \cap V_x) \) does not depend of \( x \), this image is thus a normal subgroup \( L \) of \( K \).

In the sequel, we will only consider free transitive uniform distributions.

**Definition 1.**

An horizontal curve of \( \mathcal{D} \) is a curve \( c : I \to E \) such that, \( c'(t) = \frac{d}{dt} c(t) \) is an element of \( \mathcal{D}_{c(t)} \).

Let, \( x \) be an element of \( E \), the holonomy group of \( \mathcal{D} \) at \( x \), is the set of elements \( k \) of \( K \) such that \( x \) and \( xk^{-1} \) can be joined by an horizontal curve.

Let denote by \( M \) the quotient of \( K \) by \( L, U : K \to M \) the projection map, and \( d : K \to M \) the induced Lie algebra morphism, one can define on \( E \), a \( 1-\)\( M \) form as follows, each element \( v \) of \( T_xE \) can be written \( v = v_1 + v_2 \), where \( v_1 \) is vertical, and \( v_2 \) is an element of \( \mathcal{D}_x \), this decomposition is not unique unless that \( \mathcal{D} \) is a connection, but the projection \( \omega(v) \) of \( u(v_1) \) to \( M \) defined on \( E \) a \( 1- \)\( form \) which kernel is \( \mathcal{D} \).

Let \( V \) be a vector space endowed with a \( K \)-action defined by the representation \( S : K \to GL(V) \), we will denote by \( s : K \to gl(V) \) the induced Lie algebras representation, by \( V' \) the quotient of \( V \) by \( s(L) \), and by \( s' : V \to V' \) the projection map where \( L \) is the Lie algebra of \( L \). We consider a \( p-\)tensorial form \( \alpha : TE \to V \). We denote \( \alpha \), the \( V' \) \( p-\)tensorial form defined by \( \check{\alpha}(v_1, ..., v_p) = v'(\alpha(v_1, ..., v_p)) \) where \( v_1, ..., v_p \) are elements of \( T_xE \), write \( v_i = w_i^1 + w_i^2 \), where \( w_i^1 \) is a vertical vector and \( w_i^2 \) is an element of \( \mathcal{D}_x \) we will define the covariant derivative, \( \nabla_{\mathcal{D}} \) of \( \check{\alpha} \) by \( \nabla_{\mathcal{D}}(v_1, ..., v_p) = s'(d\check{\alpha}(w_1^2, ..., w_p^2)) \).

We have the formula \( \nabla_{\mathcal{D}}(\check{\alpha}) = d\check{\alpha} + \omega \check{\alpha} \). We will denote by \( \nabla_{\mathcal{D}} \), the curvature of \( \mathcal{D} \).

We will now apply this notion of TD to the tower of gerbe. First we consider the case of the gerbe \( C_1 \). Let endow \( p_{K_1} \) with a connection \( w_1, e_1 \) and consider an object \( e_1 \) of \( C_1(U) \), where \( U \) is an open set of \( B \). We can define on \( e_1 \) the transitive distribution \( \mathcal{D}_{e_1} \), the pull-back of the distribution which define the connection \( w_1 \) by the canonical map \( e_1 \to p_{K_1}\{U \} \), if \( e_1' \) is an other object of \( C(U) \), then each map \( f : e_1 \to e_1' \) transforms \( \mathcal{D}_{e_1} \) to \( \mathcal{D}_{e_1'} \). The TD \( \mathcal{D}_{e_1} \) is uniform, its uniform group is \( H_1 \).

**Definition 2.**

For each element \( z \) of \( e_1 \), we will denote by \( Hol(z, e_1, w_1) \), the holonomy group of the transitive distribution \( \mathcal{D}_{e_1} \) in \( z \). Since we have supposed \( H_1 \) to be central in \( G_1 \), this group depends only of the projection of \( z \) to \( p_{K_1} \).
The holonomy of the tower.

We will define recursively the notion of holonomy of the tower of gerbes, by defining on each element of $C_{i+1}p(U^i)$ a free transitive distribution. We suppose defined on each bundle $p_K$, a connection $w_i$.

Let $U^1$ be an open set of $e_1$, recall and object $e_2^2$, of $C_{2p}(U^1)$, are fibers product over $p_{1U}(U^1)$, of $e_{2p}$ and $e_{1p}$, where $p_{2p}$ is a principal bundle associated to an object of $C_2(U^1)$ associated to the first lifting problem defined by $H_3, F_2$ and $G_2$, and $e_{1p}$ is the principal bundle over $p_{1U}(U^1)$ of the first lifting problem associated to the data $H_1, F_1$ and $G_1$.

We have two canonical projections $f_1 : e_2^2 \rightarrow e_{1p}$, and $f_2 : e_2^2 \rightarrow e_{2p}$.

The connection $w_2$ on $p_K$ can be pull back to a connection $w'_2$ on the quotient of $e_{2p}$ by $H_2$. Denote by $X_{e_{2p}}$ the pull back to $e_{2p}$ of the distribution which defines the connection $w'_2$ to $e_{2p}$, we will define

$$D_{2, e_2^2} = f_1^*D_{1, e_{1p}} + f_2^*X_{e_{2p}}.$$ 

Definition 3.

Let $z$ be an element of $e_2^2$. We denote by $Hol(e_2^2, D_2, z, w_1, w_2)$ the holonomy of the transitive distribution $D_{2, e_2^2}$ in $z$.

Suppose defined the distribution $D_{1, e_{1p}}$ and the holonomy $Hol(e_{1p}, D_1, z, w_1, \ldots, w_i)$ where is an element of the object $e_{1p}$, of $C_{i+1p}(U^{i-1})$.

Recall that an object $e^{i+1}$ of $C_{i+1p}(U^i)$ is a fiber product over $p_{iU}(U^i)$ of $e_{ip}$ and $e_{i+1p}$, where $e_{ip}$ is an element of $C_{i+1p}(U^{i-1})$, and $e_{i+1p}$ is a bundle over $U^i$ which solve the first lifting problem associated to $F_{i+1}, H_{i+1}$ and $G_{i+1}$, we have the canonical projection $f_1 : e^{i+1} \rightarrow e_{ip}$, and $f_2 : e^{i+1} \rightarrow e_{i+1p}$. The connection $w_{i+1}$ on $p_{K_{i+1}}$ induces on the quotient of $e_{i+1p}$ by $H_{i+1}$ a connection $\omega'_{i+1}$. Denote by $X_{e_{i+1}}$ the pull back of the distribution which defines the connection $\omega'_{i+1}$ to $e_{i+1p}$, we can define

$$D_{i+1, e^{i+1}} = f_1^*(D_{i, e_{1p}}) + f_2^*(X_{i+1}).$$

Definition 4.

For each element $z$ of $e^{i+1}$, we denote by $Hol(e^{i+1}, D_{i+1}, z, w_1, \ldots, w_{i+1})$, the holonomy of the transitive distribution $D_{i+1, e^{i+1}}$ in $z$.

Parallel Displacement.

We will define the notion of parallel displacement for the tower of gerbes.

Definition 5.

Given a connection $w_1$ on the principal bundle $p_K$, and $z c : I = [0, 1] \rightarrow p_K$, a differentiable curve, A field $X$ is parallel along $c$, if and only if $\nabla_{w_1} \frac{d}{dt}X(c(t)) = 0$, where $\nabla_{w_1}$ is the covariant derivative associated to the connection $w_1$. 

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The parallel displacement along $c$, is the map $s_{c(t)}: T_{c(0)pK_1} \rightarrow T_{c(1)pK_1}$, such that $s_{c(t)}(v_0)$ is the vector $v_1$ of $T_{c(1)pK_1}$ such that $v_0 = X(c(0)), v_1 = X(c(1))$ where $X$ is a parallel vector field defined on $c$, (we suppose $c$ endowed the pull-back of the connection $w_1$).

**Definition 6.**
Consider an object $e_k$ of the category $C_k P(U^k)$ of the principal tower, and a connection $\omega$ of $Co(e_k)$. For each element $z$ of $e_k$, for each curve $c: I = [0,1] \rightarrow e_k$, one can define the parallel displacement $s_{c,\omega}$ along $c$, the parallel displacement along $c$ is the set of maps $s_{c,\omega}, \omega \in Co(e_k)$.

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