HYBRID SUBCONVEXITY BOUNDS FOR TWISTED L-FUNCTIONS ON GL(3)

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Abstract. Let $q$ be a large prime, and $\chi$ the quadratic character modulo $q$. Let $\phi$ be a self-dual Hecke–Maass cusp form for $SL(3, \mathbb{Z})$, and $u_j$ a Hecke–Maass cusp form for $\Gamma_0(q) \subseteq SL(2, \mathbb{Z})$ with spectral parameter $t_j$. We prove the hybrid subconvexity bounds for the twisted $L$-functions

$$L(1/2, \phi \times u_j \times \chi) \ll_{\phi, \varepsilon} (qt_j)^{3/2 - \theta + \varepsilon}, \quad L(1/2 + it, \phi \times \chi) \ll_{\phi, \varepsilon} (qt)^{3/4 - \theta/2 + \varepsilon},$$

for any $\varepsilon > 0$, where $\theta = 1/23$ is admissible.

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1. Introduction

Bounding $L$-functions on their critical lines is one of the central problems in analytic number theory. For $GL(1)$ $L$-functions, subconvexity bounds are due to Weyl [29] in the $t$-aspect, and Burgess [3] in the $q$-aspect. Hybrid bounds for Dirichlet $L$-functions are given by Heath-Brown [9, 10]. For $GL(2)$ $L$-functions, in the weight aspect, this was achieved in Peng [27]. In the conductor aspect, Conrey–Iwaniec [4] used the cubic moment to give a strong subconvexity bound. And recently, Young [30] generalized their method to obtain a Weyl-type hybrid subconvexity bounds for twisted $L$-functions. In the level aspect, this was first given by Duke–Friedlander–Iwaniec [5]. Subconvexity bounds for Rankin–Selberg $L$-functions on $GL(2) \times GL(2)$ were known due to Sarnak [28], Kowalski–Michel–Vanderkam [14], and Lau–Liu–Ye [16], etc. Now for $L$-functions on $GL(1)$ and $GL(2)$, this was solved completely, due to the work of Michel–Venkatach [20] and many other important contributions on the way. For $GL(3)$ $L$-functions, Li [18] gave the first subconvexity bound in the $t$-aspect for self-dual forms. Recently, Mckee–Sun–Ye [19] improved Li’s results. Blomer [1] considered the conductor aspect for twisted $L$-functions on $GL(3)$. On the other hand, in a series of papers [24, 25, 26], Munshi used the circle method and $GL(3)$ Voronoi formula to give the subconvexity bounds. So far, there are mainly two methods to solve the subconvexity problem for $GL(3)$ $L$-functions: the moment method and the circle method. They work in different situations.

In this paper, we consider certain types of twisted $L$-functions of degree 3 and 6 in both $q$ and $t$ aspects. More precisely, let $q$ be a large prime, and $\chi$ the primitive quadratic character modulo $q$. Let $u_j$ be an even Hecke–Maass cusp newform with spectral parameter $t_j$ of level $q' | q$. We denote the Hecke eigenvalues by $\lambda_j(n)$. Let $\phi$ be a self-dual Hecke–Maass form of type $(\nu, \nu)$ for $SL(3, \mathbb{Z})$, with
Fourier coefficients $A(m,n) = A(n,m)$, normalized so that the first Fourier coefficient $A(1,1) = 1$. We define the $L$-function

$$L(s, \phi) = \sum_{n=1}^{\infty} \frac{A(1,n)}{n^s},$$

for $\text{Re}(s) > 1$. The twisted $L$-functions

$$L(s, \phi \times \chi) = \sum_{n=1}^{\infty} \frac{A(m,n)\lambda_j(n)\chi(n)}{(m^2n)^s},$$

for $\text{Re}(s) > 1$, and can be continued to an entire function with a functional equation of conductor $q^3$. Similarly, we define the Rankin–Selberg $L$-function

$$L(s, \phi \times u_j \times \chi) = \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(m,n)\lambda_j(n)\chi(n)}{(m^2n)^s},$$

for $\text{Re}(s) > 1$, and can be continued to an entire function with conductor $q^6$.

Our main result is

**Theorem 1.1.** With notation as above, we have

$$L(1/2, \phi \times u_j \times \chi) \ll_{\phi, \varepsilon} (qt_j)^{3/2-\theta+\varepsilon},$$

and

$$L(1/2 + it, \phi \times \chi) \ll_{\phi, \varepsilon} (qt)^{3/4-\theta/2+\varepsilon},$$

for any $\varepsilon > 0$, where $\theta = (35 - \sqrt{1057})/56$.

In order to prove Theorem 1.1, we will use two different methods to show the following two theorems. And with some modifications, we will give the proof of Theorem 1.1 in the end of §10.

**Theorem 1.2.** With notation as above, we have

$$L(1/2, \phi \times u_j \times \chi) \ll_{\phi, \varepsilon} q^{5/4+\varepsilon}t_j^{3/2+\varepsilon},$$

and

$$L(1/2 + it, \phi \times \chi) \ll_{\phi, \varepsilon} q^{5/8+\varepsilon}t^{3/4+\varepsilon},$$

for any $\varepsilon > 0$.

and

**Theorem 1.3.** With notation as above, we have

$$L(1/2, \phi \times u_j \times \chi) \ll_{\phi, \varepsilon} q^{4+\varepsilon}t_j^{4/3+\varepsilon},$$

and

$$L(1/2 + it, \phi \times \chi) \ll_{\phi, \varepsilon} q^{2+\varepsilon}t^{2/3+\varepsilon},$$

for any $\varepsilon > 0$.

**Remark 1.** Theorem 1.1 gives the first hybrid subconvexity bound for $GL(3)$ $L$-functions. Note that the convexity bound for $L(1/2, \phi \times u_j \times \chi)$ is $(qt_j)^{3/2+\varepsilon}$, and for $L(1/2 + it, \phi \times \chi)$ is $(qt)^{3/4+\varepsilon}$. Theorem 1.2 is crucial, which is a generalization of Blomer’s results in [1], since the bounds there are subconvexity in the $q$-aspect and convexity in the $t$-aspect. So any bound which is subconvexity in term of $t$ and of polynomial growth in term of $q$ is sufficient to get a hybrid subconvexity bound by combining with Theorem 1.2. Theorem 1.3 is a generalization of Li’s results in [18] and Mckee–Sun–Ye’s improvements in [19].

**Remark 2.** Let $f$ be a weight $2k$ holomorphic modular form for $\Gamma_0(q)$. One may prove

$$L(1/2, \phi \times f \times \chi) \ll_{\phi, \varepsilon} (qk)^{3/2-\theta+\varepsilon}.$$  

The proof of the above result is similar to Theorem 1.1 see Li [18] Appendix] for example. One can also think about the hybrid subconvexity bounds for $GL(3)$ $L$-functions in other cases, such as Munshi [22, 23, 26]. We will get into these topics elsewhere.
We end the introduction with a brief outline of the proof of our theorems. In our work, we will assume \( q \ll T^B \), for some fixed \( B > 0 \). Note that Blomer’s method showed an upper bound of the form \( T^A q^{5/4+\varepsilon} \). To prove our theorems, the basic idea is similar to Li [15] and Blomer [1]. We consider the average of \( L(1/2, \phi \times u_j \times \chi) \) over the spectrum of the Laplacian on \( \Gamma_0(q)\backslash \mathbb{H} \), see Proposition 3.1 below. Then and our results follow from a theorem of Lapid [15], which shows that \( L(1/2, \phi \times u_j \times \chi) \) is always a non-negative real number. (We can drop all but one term to obtain an individual bound; similarly for \( L(1/2 + it, \phi \times \chi) \).) To prove Proposition 7.1 which is strong in \( q \)-aspect, after applying the approximate functional equations for the Rankin–Selberg \( L \)-functions, the \( GL(2) \) Kuznetsov formula, and the \( GL(3) \) Voronoi formula, we are led to bound \( S_\sigma(q, N; \delta) \), see (3.11). To estimate \( S_\sigma(q, N; \delta) \), we will use the hybrid large sieve inequality and many results in Conrey–Iwaniec [4], Blomer [1], and Young [30]. This is inspired by Young [30]. However, this will not give us subconvexity bounds in both \( q \) and \( t \) aspects. In order to prove results as in Theorem 1.1 we still need to handle the case \( q \) is much smaller than \( t \). That is, we will need a result as in Theorem 1.3 which is strong in the \( t \)-aspect and will follow from Proposition 7.1. Now, to prove Proposition 3.1 it turns out that Li’s method is still working. The key point here is that we can have a second application of the Voronoi formula. To get a better bound, we will also use an \( n \)-th-order asymptotic expansion of a weighted stationary phase integral as Mckee–Sun–Ye [19] did. Throughout the paper, \( e(x) \) means \( e^{2\pi ix} \), negligible means \( O(T^{-A}) \) for any \( A > 0 \), and \( \varepsilon \) is an arbitrarily small positive number which may not be the same in each occurrence.

2. Preliminaries

In this section, we introduce notation and recall some standard facts of automorphic forms on \( GL(2) \) and \( GL(3) \).

2.1. Automorphic forms. We start by reviewing automorphic forms for \( \Gamma_0(q) \). Let \( \mathbb{H} \) be the upper half-plane. Let \( \mathcal{A}(\Gamma_0(q)\backslash \mathbb{H}) \) denote the space of automorphic functions of weight zero, i.e., the functions \( f: \mathbb{H} \to \mathbb{C} \) which are \( \Gamma_0(q) \)-periodic. Let \( \mathcal{L}(\Gamma_0(q)\backslash \mathbb{H}) \) denote the subspace of square-integrable functions with respect to the inner product

\[
\langle f, g \rangle = \int_{\Gamma_0(q)\backslash \mathbb{H}} f(z)\overline{g(z)}d\mu z, \tag{2.1}
\]

where \( d\mu z = y^{-2}dx\,dy \) is the invariant measure on \( \mathbb{H} \). The Laplace operator

\[
\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \tag{2.2}
\]

acts in the dense subspace of smooth functions in \( \mathcal{L}(\Gamma_0(q)\backslash \mathbb{H}) \) such that \( f \) and \( \Delta f \) are both bounded; it has a self-adjoint extension which yields the spectral decomposition

\[
\mathcal{L}(\Gamma_0(q)\backslash \mathbb{H}) = \mathbb{C} \oplus \mathcal{C}(\Gamma_0(q)\backslash \mathbb{H}) \oplus \mathcal{E}(\Gamma_0(q)\backslash \mathbb{H}).
\]

Here \( \mathbb{C} \) is the space of constant functions, \( \mathcal{C}(\Gamma_0(q)\backslash \mathbb{H}) \) is the space of cusp forms and \( \mathcal{E}(\Gamma_0(q)\backslash \mathbb{H}) \) is the space of Eisenstein series.

We choose an orthonormal basis \( \mathcal{B}(q) \) of even Hecke–Maass forms of level \( q \) as follows: for each even newform \( u_j \) of level \( q \) we choose an orthonormal basis \( \mathcal{V}(u_j) \) of the space generated by \( \{ u_j(dz) : d|(q/q') \} \) containing \( u_j/\|u_j\| \), and let \( \mathcal{B}(q) \) be the union of all \( \mathcal{V}(u_j) \) for \( u_j \) ranging over the newforms of level dividing \( q \). Let \( \mathcal{B}^\ast(q) \) be the subset of all newforms in \( \mathcal{B}(q) \). Each \( u_j \in \mathcal{B}(q) \) with spectral parameter \( \lambda_j \) has a Fourier expansion

\[
u_j(z) = \sum_{n \neq 0} \rho_j(n)W_{s_j}(nz),
\]

where \( W_{s}(z) \) is the \( GL(2) \) Whittaker function given by

\[
W_s(z) := 2|y|^{1/2}K_{s-1/2}(2\pi|y|)e(x),
\]
and $K_s(y)$ is the $K$-Bessel function with $s = 1/2 + it$. We have the Hecke operators acting on $u_j$ with
\[
(T_n u_j)(z) := \frac{1}{\sqrt{n}} \sum_{ad = n, b \equiv \ell \mod d} u_j \left( \frac{a z + b}{d} \right) = \lambda_j(n) u_j(z),
\] for all $n$ with $(n, q) = 1$. We have
\[
\rho_j(\pm n) = \rho_j(\pm 1) \lambda_j(n) n^{-1/2},
\] if $n > 0$. Moreover, the reflection operator $R$ defined by $(Ru_j)(z) = u_j(\bar{z})$ commutes with $\Delta$ and all $T_n$, so that we can also require
\[
Ru_j = \epsilon_j u_j.
\] Since $R$ is an involution, the space $C(\Gamma_0(q)\backslash \mathbb{H})$ is split into even and odd cusp forms according to $\epsilon_j = 1$ and $\epsilon_j = -1$. We define
\[
\omega_j := \frac{4\pi}{\cosh(\pi t_j)} |\rho_j(1)|^2.
\] By [12] Theorem 2], we have
\[
\omega_j^* := \frac{4\pi}{\cosh(\pi t_j)} \sum_{f \in V(u_j)} |\rho_f(1)|^2 \gg q^{-1}(qt_j)^{-\varepsilon}.
\] The Eisenstein series $E_a(z, s)$ is defined by
\[
E_a(z, s) := \sum_{\gamma \in \Gamma_a \backslash \Gamma_0(q)} \text{Im}(\sigma_a^{-1} \gamma z)^s.
\] It has the following Fourier expansion
\[
E_a(z, s) = \delta_a y^s + \varphi_a(s)y^{1-s} + \sum_{n \neq 0} \varphi_a(n, s) W_s(nz),
\] where $\delta_a = 1$ if $a \sim \infty$, or $\delta_a = 0$ otherwise. Let
\[
\eta(n, s) = \sum_{ad = n, \ell} \left( \frac{d}{n} \right)^{s-1/2}.
\] The Eisenstein series $E_a(z, s)$ is even, and we have
\[
T_n E_a(z, s) = \eta(n, s) E_a(z, s),
\] if $(n, q) = 1$. Write $\eta(n) = \eta(n, 1/2 + it)$ and $E_a,\gamma(z) = E_a(z, 1/2 + it)$. We define
\[
\omega_a(t) := \frac{4\pi}{\cosh(\pi t)} |\varphi_a(1, 1/2 + it)|^2.
\] And, by [4] p. 1188], we have
\[
\omega^*(t) := \sum_a \omega_a(t) \gg q^{-1-\varepsilon} \min(|t|^{-\varepsilon}, |t|^2).
\]

Now we recall some background on Maass forms for $SL(3, \mathbb{Z})$. We will follow the notation in Goldfeld’s book [7]. Let $\phi$ be a Maass form of type $(\nu_1, \nu_2)$. We have the following Fourier–Whittaker expansion
\[
\phi(z) = \sum_{\gamma \in U_2(\mathbb{Z}) \backslash \Gamma_0(q)} \sum_{m_1, m_2} \frac{A(m_1, m_2)}{m_1 |m_2|} \sum_{m_1, m_2 \neq 0} W_j \left( M \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} z, \nu_1, \nu_2, \psi_{1,1} \right),
\] (2.9)
where $U_2(\mathbb{Z})$ is the group of $2 \times 2$ upper triangular matrices with integer entries and one on the diagonal, $W_j(z, \nu_1, \nu_2, \psi_{1,1})$ is the Jacquet–Whittaker function, and $M = \text{diag}(m_1 |m_2|, m_1, 1)$. From now on, let $\phi$ be a self-dual Hecke–Maass form of type $(\nu, \nu)$ for $SL(3, \mathbb{Z})$, normalized to have the first Fourier coefficient $A(1, 1) = 1$. For later purposes, we record the Hecke relation
\[
A(m, n) = \sum_{d |(m,n)} \mu(d) A \left( \frac{m}{d}, 1 \right) A \left( 1, \frac{n}{d} \right).
\] (2.10)
Moreover, the Rankin–Selberg theory implies the bound
\[ \sum_{n \leq x} |A(1, n)|^2 \ll x, \] (2.11)
for all \( x \geq 1 \). We will also need the following estimate (see Blomer [1, Eq. (10) and (11)])
\[ \sum_{n \leq x} |A(na, b)|^2 \ll x(ab)^{7/16+\varepsilon}, \text{ and } \sum_{n \leq x} |A(na, b)| \ll x(ab)^{7/32+\varepsilon}. \] (2.12)

2.2. \( L \)-functions and the approximate functional equations. The \( L \)-function attached to \( \phi \) is \( L(s, \phi) = \sum_{n=1}^{\infty} A(1, n)n^{-s} \), and the completed \( L \)-function is given by
\[ \Lambda(s, \phi) = \pi^{-3s/2} \prod_{j=1}^{3} \Gamma \left( \frac{s - \alpha_j}{2} \right) L(s, \phi). \]
where \( \alpha_1 = 3\nu - 1, \alpha_2 = 0, \) and \( \alpha_3 = 1 - 3\nu \). The \( L \)-function attached to the twist \( \phi \times \chi \) is
\[ L(s, \phi \times \chi) = \sum_{n=1}^{\infty} \frac{A(1, n)\chi(n)}{n^s}, \]
whose completed version is
\[ \Lambda(s, \phi \times \chi) = q^{3s/2}L_{\infty}(s, \phi \times \chi)L(s, \phi \times \chi), \]
where
\[ L_{\infty}(s, \phi \times \chi) = \pi^{-3s/2} \prod_{j=1}^{3} \Gamma \left( \frac{s + \delta - \alpha_j}{2} \right), \]
with \( \delta = 0 \) or \( 1 \) according to whether \( \chi(-1) = 1 \) or \(-1 \). Then \( \Lambda(s, \phi \times \chi) \) is entire, and its functional equation is
\[ \Lambda(s, \phi \times \chi) = \Lambda(1 - s, \phi \times \chi). \]
Note that the root number of \( \Lambda(s, \phi \times \chi) \) is \( 1 \).

Next we consider the Rankin–Selberg convolution of \( \phi \) with \( u_j \times \chi \) given by
\[ L(s, \phi \times u_j \times \chi) = \sum_{m,n=1}^{\infty} \frac{A(m, n)\lambda_j(m)\chi(n)}{(m^2n)^s}. \]
The completed version of \( L(s, \phi \times u_j \times \chi) \) is
\[ \Lambda(s, \phi \times u_j \times \chi) = q^{3s}L_{\infty}(s, \phi \times u_j \times \chi)L(s, \phi \times u_j \times \chi), \]
where
\[ L_{\infty}(s, \phi \times u_j \times \chi) = \pi^{-3s} \prod_{j=1}^{3} \prod_{\pm} \Gamma \left( s + \delta \pm \frac{it_j - \alpha_j}{2} \right). \]
The function \( \Lambda(s, \phi \times u_j \times \chi) \) is entire, and its functional equation is
\[ \Lambda(s, \phi \times u_j \times \chi) = \Lambda(1 - s, \phi \times u_j \times \chi). \]
Note that again the root number is \( 1 \). Finally, we consider the convolution with the Eisenstein series
which is defined as
\[ L(s, \phi \times E_{\alpha, t} \times \chi) = \sum_{m,n=1}^{\infty} \frac{A(m, n)\eta_t(n)\chi(n)}{(m^2n)^s}. \]
By the definition of \( \eta_t(n) \) (2.4), by comparing the Euler products, we know that
\[ L(s, \phi \times E_{\alpha, t} \times \chi) = L(s+it, \phi \times \chi)L(s-it, \phi \times \chi). \]

Now we consider the approximate functional equations for \( L(s, \phi \times u_j \times \chi) \) and \( L(s, \phi \times E_{\alpha, t} \times \chi) \).
We use the results from Blomer [1, §2]. Let
\[ G(u) = e^{au^2}. \] (2.13)
We have the following approximate functional equations, (see [13, Theorem 5.3]).
Lemma 2.1. We have

\[
L(1/2, \phi \times u_j \times \chi) = 2 \sum_{m,n=1}^{\infty} A(m,n) \lambda_j(n) \chi(n) \left( \frac{m^2 n}{q^3} \right),
\]

where

\[
V_i(y) = \frac{1}{2\pi i} \int_{(3)} (\pi^3 y)^{-u} \prod_{j=1}^{3} \frac{\Gamma \left( \frac{1/2 + u + \delta \pm it - \alpha_j}{2} \right)}{\Gamma \left( \frac{1/2 + \delta + it - \alpha_j}{2} \right)} G(u) \frac{du}{u}.
\]

And similarly, we have

\[
L(1/2, \phi \times E_n, t \times \chi) = 2 \sum_{m,n=1}^{\infty} A(m,n) \eta_j(n) \chi(n) \left( \frac{m^2 n}{q^3} \right).
\]

We see that \( V_i(y) \) has the following properties which effectively limit the terms in \( \text{(2.14)} \) and \( \text{(2.15)} \) with \( m^2 n \ll (q(1 + |t_j|))^{3+\varepsilon} \) and \( (q(1 + |t_j|))^{3+\varepsilon} \) respectively. Note that we can separate the variables \( t \) and \( y \) in \( V_i(y) \) by the second part of the following lemma. Moreover, we see the \( u \)-integral can be easily handled now. In our later application, we will take \( U = \log^2 (qT) \).

Lemma 2.2. (i) We have

\[
y^k V_i^{(k)}(y) \ll \left( 1 + \frac{y}{(1 + |t|)^s} \right)^{-A},
\]

and

\[
y^k V_i^{(k)}(y) = \delta_k + O \left( \frac{y}{(1 + |t|)^s} \right)^{\alpha},
\]

where \( \delta_0 = 1, \delta_k = 0 \) if \( k > 0 \), and \( 0 < \alpha \leq (1/2 - |\text{Re}(3\nu - 1)|)/3 \) (for example, we can take \( \alpha = 3/32 \)).

(ii) For any \( 1 < U \ll T^\varepsilon, \varepsilon > 0, \) and \( |t - T| \ll T^{1-2\varepsilon} \), we have the following approximation

\[
V_i(y) = \sum_{k=0}^{K/2} \sum_{l=0}^{K} \frac{1}{2\pi i} \int_{\varepsilon - iU}^{\varepsilon + iU} \frac{(t^2 - T^2)^{l}}{T^{3}} V_k, l \left( \frac{y}{T^3} \right) + O \left( y^{-\varepsilon} (1 + |T|)^\varepsilon e^{-U} \right)
\]

\[
+ O \left( \left( \frac{1 + |t - T|}{T} \right)^{K+1} \left( \frac{1 + |U|}{T^3} \right)^{-A}, \right),
\]

where

\[
V_k, l(t) = \frac{1}{2\pi i} \int_{-iU}^{iU} \frac{P_k(u)(2\pi)^{-3u} y^{-u} G(u)}{u} \frac{du}{u},
\]

for some polynomial \( P_k, l \).

Proof. (i) See Iwaniec–Kowalski [13 Proposition 5.4].

(ii) By Stirling’s formula and contour shifts as in Iwaniec–Kowalski [13 p. 100], we have (see Blomer [11 Lemma 1])

\[
V_i(y) = \frac{1}{2\pi i} \int_{\varepsilon - iU}^{\varepsilon + iU} \frac{\pi^{-3u}}{\prod_{j=1}^{3} \Gamma \left( \frac{1/2 + u + \delta \pm it - \alpha_j}{2} \right)} y^{-u} G(u) \frac{du}{u} + O \left( y^{-\varepsilon} (1 + |T|)^\varepsilon \right).
\]

The rest of the proof is following very closely to Young [30] §5. At first, by Stirling’s formula, if \( |\text{Im}(z)| \to \infty \) (with fixed real part), but \( |u| \ll |z|^{1/2} \), then

\[
\frac{\Gamma(z + u)}{\Gamma(z)} = z^u \left( 1 + \sum_{k=1}^{K} \frac{P_k(u)}{z^k} + O \left( \frac{(1 + |u|)^{2K+2}}{|z|^{K+1}} \right) \right),
\]

for certain polynomials \( P_k(u) \) of degree \( 2k \). So for \( |\text{Im}(u)| \ll U, \) and \( t > T \), we have that

\[
\prod_{j=1}^{3} \frac{\prod_{j=1}^{K} \Gamma \left( \frac{1/2 + u + \delta \pm it - \alpha_j}{2} \right)}{\Gamma \left( \frac{1/2 + \delta + it - \alpha_j}{2} \right)} = \left( \frac{t}{2} \right)^{3u} \left( 1 + \sum_{k=1}^{K/2} P_{2k}(u) \frac{t^{2k}}{2^{K+1}} + O \left( \frac{(1 + |u|)^{2K+2}}{t^{K+1}} \right) \right),
\]

\[
\prod_{j=1}^{3} \frac{\prod_{j=1}^{K} \Gamma \left( \frac{1/2 + u + \delta \pm it - \alpha_j}{2} \right)}{\Gamma \left( \frac{1/2 + \delta + it - \alpha_j}{2} \right)} = \left( \frac{t}{2} \right)^{3u} \left( 1 + \sum_{k=1}^{K/2} P_{2k}(u) \frac{t^{2k}}{2^{K+1}} + O \left( \frac{(1 + |u|)^{2K+2}}{t^{K+1}} \right) \right),
\]
for a different collection of $P_k(u)$. Note that, in fact, the factor $(t/2)^{3u}$ is $((t/2)^2)^{3u/2}$, which is even as a function of $t$. For convenience, set $P_0(u) = 1$. Hence
\begin{equation}
V_t(y) = \sum_{k=0}^{K/2} t^{-2k} \frac{1}{2\pi i} \int_{-iU}^{+iU} P_{2k}(u) \left( \frac{t}{2\pi} \right)^{3u} y^{-u} G(u) \frac{du}{u}
\end{equation}
(2.17)
where the extra factor $(1 + \frac{y}{t})^{-A}$ arises from moving the contour to Re$(u) = A$ if $y \geq t^3$, and to Re$(u) = -1/4$ if $y \leq t^3$ (here we use the fact $|\text{Re}(\alpha_j)| \leq 7/32$). We further refine (2.17) by approximating $t$ by $T$. Since in our application $h(t)$ is very small unless $|t - T| \ll M \log^2 T$, where $M \ll T^{1/2}$ and $T$ large, our assumption $|t - T| \ll T^{1-2\varepsilon}$ is flexible enough. Note that $\left| \frac{u^2 - T^2}{T^2} \right| \ll \left| \frac{u - T}{T} \right| \ll T^{-\varepsilon}$. By Taylor expansion, we have
\begin{equation}
t^{3u} = T^{3u} e^{3u \log(1 + \frac{t^2}{T^2})} = T^{3u} \sum_{l=0}^{K} \frac{Q_l(u)}{T^l} \left( \frac{t^2 - T^2}{T^2} \right)^l
\end{equation}
(2.18)
for certain polynomial $Q_l(u)$ of degree $\leq l$. So, by (2.17) and (2.18), we prove this lemma. \hfill \Box

2.3. The Kuznetsov formula for $\Gamma_0(q)$. The two central tools we need in this paper are the Kuznetsov formula for $\Gamma_0(q)$ and the Voronoi formula for $SL(3, \mathbb{Z})$. In this subsection, we recall the Kuznetsov formula, and then in the next subsection, we will review the Voronoi formula. As usual, let
\[ S(a, b; c) = \sum_{d|c}^* e \left( \frac{ad + bd}{c} \right) \]
be the classical Kloosterman sum. For any $m, n \geq 1$, and any test function $h(t)$ which is even and satisfies the following conditions:
(i) $h(t)$ is holomorphic in $|\text{Im}(t)| \leq 1/2 + \varepsilon$,
(ii) $h(t) \ll (1 + |t|)^{-2-\varepsilon}$ in the above strip,
we have the following Kuznetsov formula (see [4] Eq. (3.17)) for example).

**Lemma 2.3.** We have
\begin{equation}
\sum' h(t_j) \omega^*_j \lambda_j(m) \lambda_j(n) + \frac{1}{4\pi} \int_{-\infty}^{\infty} h(t) \omega^*(t) \eta_t(m) \eta_t(n) dt
\end{equation}
\[ = \frac{1}{2} \delta_{m,n} H + \frac{1}{2} \sum_{q|c} \sum_{\pm} S(n, \pm m; c) H^\pm \left( \frac{4\pi \sqrt{mn}}{c} \right), \]
where $\sum'$ restricts to the even Hecke–Maass cusp forms, $\delta_{m,n}$ is the Kronecker symbol,
\begin{align*}
H &= \frac{2}{\pi} \int_0^{\infty} h(t) \tanh(\pi t) dt,
H^+(x) &= 2i \int_{-\infty}^{\infty} J_{2i\varepsilon}(x) \frac{h(t)x}{\cosh(\pi t)} dt, \\
H^-(x) &= \frac{4}{\pi} \int_{-\infty}^{\infty} K_{2i\varepsilon}(x) \sinh(\pi t) h(t) dt,
\end{align*}
(2.19)
and $J_{\nu}(x)$ and $K_{\nu}(x)$ are the standard $J$-Bessel function and $K$-Bessel function respectively.
2.4. The Voronoi formula for \( SL(3, \mathbb{Z}) \). Let \( \psi \) be a smooth compactly supported function on \((0, \infty)\), and let \( \hat{\psi}(s) := \int_{0}^{\infty} \psi(x) x^{s} \frac{dx}{x} \) be its Mellin transform. For \( \sigma > 7/32 \), we define

\[
\Psi^{\pm}(x) := x \frac{1}{2\pi i} \int_{(\sigma)} (\pi^{3} x)^{-s} \prod_{j=1}^{3} \Gamma \left( \frac{s+\alpha_{j}}{2} \right) \hat{\psi}(1-s) ds
\]

\[
\pm x \frac{1}{2\pi i} \int_{(\sigma)} (\pi^{3} x)^{-s} \prod_{j=1}^{3} \Gamma \left( \frac{1+s+\alpha_{j}}{2} \right) \hat{\psi}(1-s) ds.
\]

(2.20)

Here \( \alpha_{j} \) has the same meaning as above, that is, \( \alpha_{1} = 3\nu - 1, \alpha_{2} = 0, \) and \( \alpha_{3} = 1 - 3\nu \). Note that changing \( \psi(y) \) to \( \psi(y/N) \) for a positive real number \( N \) has the effect of changing \( \Psi^{\pm}(x) \) to \( \Psi^{\pm}(xN) \). The Voronoi formula on \( GL(3) \) was first proved by Miller and Schmid \[21\]. The present version is due to Goldfeld and Li \[8\] with slightly renormalized variables (see Blomer \[1, \text{Lemma 3}\]).

**Lemma 2.4.** Let \( c, d, \bar{d} \in \mathbb{Z} \) with \( c \neq 0 \), \( (c, d) = 1 \), and \( d\bar{d} \equiv 1 \pmod{c} \). Then we have

\[
\sum_{n=1}^{\infty} A(m,n)e \left( \frac{nd\bar{d}}{c} \right) \psi(n) = \frac{c^{3/2}}{2} \sum_{\pm} \sum_{n_{1}|m} \sum_{n_{2}=1}^{\infty} A(n_{2},n_{1}) S \left( \frac{mc_{\pm n_{2}}}{n_{1}} \right) \Psi^{\pm} \left( \frac{n_{1}^{2}n_{2}}{c^{3}m} \right).
\]

To prove Theorem 1.3 by applying Lemma 2.4, we need to know the asymptotic behaviour of \( \Psi^{\pm} \). This will be done in \[11\] Our work differs from Blomer \[1\] in the nature of the weight function \( \Psi^{\pm} \). We will use the method of Young \[30\]. See Young \[30, \text{§8}\] for a more detailed discussion of this method. However, to prove Theorem 1.3 we will need the following asymptotic formula for \( \Psi^{\pm} \).

**Lemma 2.5.** Suppose \( \psi(y) \) is a smooth function, compactly supported on \([N, 2N]\). Let \( \Psi^{\pm}(x) \) be defined as in (2.20). Then for any fixed integer \( K \geq 1 \), and \( xN \gg 1 \), we have

\[
\Psi^{\pm}(x) = x \int_{0}^{\infty} \psi(y) \sum_{\ell=1}^{K} \gamma_{\ell} \left( 3(xy)^{1/3} \right) dy + O \left( (xN)^{1-K/3} \right),
\]

where \( \gamma_{\ell} \) are constants depending only on \( \alpha_{1}, \alpha_{2}, \alpha_{3}, \) and \( K \).

**Proof.** See Li \[17, \text{Lemma 6.1}\] and Blomer \[1, \text{Lemma 6}\]. \(\square\)

2.5. The stationary phase lemma. In this subsection, we will recall a result in Mckee–Sun–Ye \[19\], which will be used to prove Theorem 1.3. Let \( f(x) \) be a real function, \( 2n + 3 \) times continuously differentiable for \( \alpha \leq x \leq \beta \). Suppose that \( f'(x) \) changes signs only at \( x = \gamma \), from negative to positive, with \( \alpha < \gamma < \beta \). Let \( g(x) \) be a real function, \( 2n + 1 \) times continuously differentiable for \( \alpha \leq x \leq \beta \). Denote

\[
H_{1}(x) := \frac{g(x)}{2\pi if'(x)}, \quad \text{and} \quad H_{1}(x) := -\frac{H'_{1-1}(x)}{2\pi if'(x)}.
\]

(2.21)

Let

\[
\lambda_{k} := \frac{g(k)(\gamma)}{k!}, \quad \text{for} \quad k = 2, \ldots, 2n + 2,
\]

(2.22)

\[
\eta_{k} := \frac{g(k)(\gamma)}{k!}, \quad \text{for} \quad k = 0, \ldots, 2n,
\]

(2.23)

and \( \varpi_{k} \) be defined by the Taylor expansion of \( g(x) \frac{df}{dx} \), where \( y = h(x - \gamma) \) with \( f(x) - f(\gamma) = \lambda_{2} h^{2}(x - \gamma) \) such that \( y = h(x - \gamma) \) has the same sign as that of \( x - \gamma \). By \[19, \text{Lemma 3.4}\] we have

\[
\varpi_{k} = \eta_{k} + \sum_{\ell=0}^{k-1} \sum_{j=1}^{k-\ell} C_{k\ell j} \lambda_{3} \cdots \lambda_{n_{j}},
\]

(2.24)

for some constant coefficients \( C_{k\ell j} \). See \[19, \text{§2 and §3}\] for more details.
Lemma 2.6. Let $f(x)$, $g(x)$, and $H_k(x)$ be defined as above. Suppose that there are positive parameters $M_0$, $N_0$, $T_0$, $U_0$, with

$$M_0 \geq \beta - \alpha,$$

(2.25)

and positive constants $C_r$ such that for $\alpha \leq x \leq \beta$, $|f^{(r)}(x)| \leq C_r \frac{T_0}{M_0}$, for $r = 2, 3, ..., 2n + 3$, $f''(x) \geq C_2 M_0^2$, and

$$|g^{(s)}(x)| \leq C_s \frac{U_0}{N_0^s}$$

for $s = 0, 1, 2, ..., 2n + 1$. If $T_0$ is sufficiently large comparing to the constants $C_r$, we have for $n \geq 2$ that

$$\int_\alpha^\beta g(x)e(f(x))dx = \frac{e(f(\gamma) + \frac{1}{8})}{\sqrt{f''(\gamma)}} \left( g(\gamma) + \sum_{j=1}^n \frac{(-1)^j(2j-1)!!}{(4\pi i \lambda_2)^j} \right) + \left[ e(f(x)) \cdot \sum_{i=1}^{n+1} H_i(x) \right]_0^\beta + O\left( \frac{U_0 M_0^{2n+4}}{T_0^{n+2} N_0^{n+1}} \left( \frac{M_0}{N_0} + 1 \right) \left( \frac{1}{(\gamma - \alpha)^n + 2} + \frac{1}{(\beta - \gamma)^n + 2} \right) \right)

+ O\left( \frac{U_0 M_0^{2n+4}}{T_0^{n+2} N_0^{n+1}} \left( \frac{M_0}{N_0} + 1 \right) \left( \frac{1}{(\gamma - \alpha)^n + 2} + \frac{1}{(\beta - \gamma)^n + 2} \right) \right)

+ O\left( \frac{U_0 M_0^{2n+4}}{T_0^{n+2} N_0^{n+1}} \left( \frac{M_0}{N_0} + 1 \right) \left( \frac{1}{(\gamma - \alpha)^n + 2} + \frac{1}{(\beta - \gamma)^n + 2} \right) \right)

+ O\left( \frac{U_0 M_0^{2n+4}}{T_0^{n+2} N_0^{n+1}} \left( \frac{M_0}{N_0} + 1 \right) \left( \frac{1}{(\gamma - \alpha)^n + 2} + \frac{1}{(\beta - \gamma)^n + 2} \right) \right).

(2.29)

Proof. See McKee–Sun–Ye [19, Theorem 3.6].

3. Initial setup of Theorem 1.2

We are now ready to start with the proof of Theorem 1.2. As indicated in the introduction, both results follow rather easily from the following bound.

Proposition 3.1. With notation as above, for any $\varepsilon > 0$, $T$ large, and $M \asymp T^{1/2}$, we have

$$\sum_{\substack{u_j \in B^*(q)}} L(1/2, \phi \times u_j \times \chi) + \frac{1}{4\pi} \int_{T-M}^{T+M} |L(1/2 + it, \phi \times \chi)|^2 dt \ll q^{5/4} TM(qT)^\varepsilon.$$

Theorem 1.2 is followed from the above proposition. The key ingredient is Lapid’s theorem [15] about the nonnegativity of $L(1/2, \phi \times u_j \times \chi)$. See Blomer [11, §4] for more details.

To prove Proposition 3.1, we introduce the spectrally normalized first moment of the central values of $L$-functions

$$\mathcal{M} := \sum_{u_j \in B^*(q)} h(t_j)\omega_j^* L(1/2, \phi \times u_j \times \chi) + \frac{1}{4\pi} \int_{-\infty}^\infty h(t)\omega^*(t)|L(1/2 + it, \phi \times \chi)|^2 dt,$$

(3.1)

where

$$h(t) := \frac{1}{\cosh \left( \frac{t-T}{M} \right)} + \frac{1}{\cosh \left( \frac{t+T}{M} \right)}.$$

(3.2)
Here we choose the above weight because we can use Young’s results \[30\] directly. However, maybe the following Li’s weight function
\[ h(t) = e^{-\frac{(\log T)^2}{M^2}} + e^{-\frac{(\log T)^2}{M^2}} \]
will work too. By (2.5) and (2.8), we have
\[ \sum_{u_j \in B^*(q)} L(1/2, \phi \times u_j \times \chi) + \frac{1}{4\pi} \int_{T-M}^{T+M} |L(1/2 + it, \phi \times \chi)|^2 dt \ll M q^{1+\varepsilon}, \]
for any \( \varepsilon > 0 \). Therefore, to prove Proposition \[3.1\] we just need to prove
\[ M \ll_{\phi, \varepsilon} q^{1/4} T M(qT) \varepsilon. \] (3.3)
Applying Lemma \[2.2\] to \( M \), we have
\[ M = 2 \sum_{u_j \in B^*(q)} h(t_j) \omega_j \sum_{m,n=1} A(m,n) \lambda_j(n) \chi(n) V_j \left( \frac{m^2 n}{q^3} \right) + \frac{1}{4\pi} \int_{-\infty}^{\infty} h(t) \omega(t) \sum_{m,n=1} A(m,n) \eta(n) \chi(n) V_t \left( \frac{m^2 n}{q^3} \right) dt. \]
By Lemma \[2.2\] we can truncate the \( m,n \)-sums at
\[ m^2 n \leq (qT)^{3+\varepsilon} \]
at the cost of a negligible error. Now we handle the weight \( V_t(y) \). By Lemma \[2.2\] (we choose \( U = \log^2 (qT) \)), to prove (3.3), we need to prove
\[ 2 \sum_{m^2 n \leq (qT)^{3+\varepsilon}} \frac{A(m,n) \chi(n)}{(m^2 n)^{1/2+2u}} \left( \sum_{u_j \in B^*(q)} h_{k,l}(t_j) \omega_j^* \lambda_j(n) \right) + \frac{1}{4\pi} \int_{-\infty}^{\infty} h_{k,l}(t) \omega^*(t) \eta_l(n) dt \ll_{\phi, \varepsilon} q^{1/4} T M(qT) \varepsilon, \]
uniformly in \( u \in [\varepsilon - i \log^2 (qT), \varepsilon + i \log^2 (qT)] \), where
\[ h_{k,l}(t) = t^{-2kT-2l} (t^2 - 2) t^l h(t). \] (3.4)
Now we apply the Kuznetsov formula with \( m = 1 \) (note that \( m \) has different meaning here), we arrive at bounding
\[ \sum_{m^2 n \leq (qT)^{3+\varepsilon}} \frac{A(m,n) \chi(n)}{(m^2 n)^{1/2+u}} \left( \delta_{n,1} H + \sum_{q | c} \frac{1}{c} \sum_{\pm} S(n, \pm 1; c) H^\pm \left( \frac{4\pi \sqrt{n}}{c} \right) \right), \]
where \( H, H^\pm \) are defined as in (2.19) with \( h(t) = h_{k,l}(t) \). We will only deal with the case \( k = l = 0 \), since the others can be handled similarly. By (2.11), and the fact \( H \ll T M T^\varepsilon \), we know the diagonal term is bounded by
\[ D := \sum_{m^2 n \leq (qT)^{3+\varepsilon}} \frac{A(m,n) \chi(n)}{(m^2 n)^{1/2+u}} \delta_{n,1} H \ll \sum_{m^2 \leq (qT)^{3+\varepsilon}} \frac{|A(m,1)|}{m} |H| \ll T M(qT)^\varepsilon. \] (3.5)
Now we need to bound the off-diagonal terms
\[ R^\pm := \sum_{m^2 n \leq (qT)^{3+\varepsilon}} \frac{A(m,n) \chi(n)}{(m^2 n)^{1/2+u}} \sum_{q | c} \frac{1}{c} S(n, \pm 1; c) H^\pm \left( \frac{4\pi \sqrt{n}}{c} \right). \] (3.6)
By the argument in Blomer \[1, \S 5\], it is then enough to show that
\[ \frac{1}{N^{1/2}} \sum_{m^2 \leq (qT)^{3+\varepsilon}} \frac{|A(1,m)|}{m^{3/2}} |S_\sigma(q, N; \delta)| \ll q^{1/4} T M(qT)^\varepsilon, \] (3.7)
for $\sigma \in \{ \pm \}$, where

$$S_{\sigma}(q, N; \delta) := \sum_{q|c} \frac{1}{c} \sum_{n} A(n, 1) \chi(n) \mathcal{S}(\delta n, \sigma; c) \psi_{\sigma}\left(\frac{n}{N}, \frac{\sqrt{\delta N}}{c}\right),$$

(3.8)

with

$$\psi_{\sigma}(y; D) := \begin{cases} w(y)y^{-v}H^{+}(4\pi \sqrt{yD}), & \text{if } \sigma = 1, \\ w(y)y^{-v}H^{-}(4\pi \sqrt{yD}), & \text{if } \sigma = -1, \end{cases}$$

(3.9)

$w$ a suitable fixed smooth function with support in $[1, 2]$, and

$$N \leq (qT)^{3+\varepsilon}. \quad (3.10)$$

Here we suppress the dependence on $u$ in $\psi_{\sigma}(y; D)$. As Blomer [11, §5] did, by the Voronoi formula, we have

$$S_{\sigma}(q, N; \delta) = \frac{\pi^{3/2}}{2} \sum_{q|c} \frac{1}{c^2} \sum_{c_1|c} c_1 \sum_{n_1|n_1} \sum_{n_2|n_2} \infty A(n_2, n_1) \frac{\sigma(n_2, n_1)}{n_1 n_2} \times \Psi_{\sigma}^{\pm}(\frac{n_2^2 n_2 N}{c_1}; \sqrt{\delta N}) \mathcal{T}_{\delta, c_1, n_1, n_2}^{\pm, \sigma}(c, q), \quad (3.11)$$

where

$$\mathcal{T}_{\delta, c_1, n_1, n_2}^{\pm, \sigma}(c, q) = \sum_{d(c_1)} e\left(\frac{\sigma d}{c}\right) \sum_{a(c)} \chi(a) e\left(\frac{d}{c_1}\right) \times e\left(\frac{d}{c_1} \frac{\delta_1}{n_1 n_1}\right) \sum_{b(\delta_1)} e\left(\frac{bf + n_2^2 \delta_1}{c_1 n_1}\right), \quad (3.12)$$

and $\Psi_{\sigma}^{\pm}(x; D)$ is defined as in (2.20) with $\psi(x) = \psi_{\sigma}(x; D)$.

We end this section by truncating the $c$-sum. Define

$$S_{\sigma}(q, N; C; \delta) = \sum_{c \in C} \frac{1}{c} \sum_{n} A(n, 1) \chi(n) \mathcal{S}(\delta n, \sigma; c) \psi_{\sigma}\left(\frac{n}{N}, \frac{\sqrt{\delta N}}{c}\right).$$

(3.13)

Using the weak bound $H^{\pm}(y) \ll T^{3/4}$, and the Weil bound for Kloosterman sums, we have

$$S_{\sigma}(q, N; C; \delta) \ll T(qT)^{33/8+\varepsilon} C^{-1/4+\varepsilon},$$

which is good enough if $C$ is a large power of $qT$. Therefore, it suffices to bound $S$ with $C \ll (qT)^{B}$ for some large but fixed $B$.

4. Analytic separation of variables

Our goal in the section is to handle $\Psi_{\sigma}^{\pm}(x; D)$. We follow the approach in Young [30]. The following argument will need the stationary phase method. We’ll use the following lemma (see [2, Lemma 8.1 and Proposition 8.2]).

**Lemma 4.1.** Suppose that $w$ is a smooth weight function with compact support on $[X, 2X]$, satisfying $w^{j}(t) \ll X^{-j}$, for $X \gg 1$ (in particular, $w$ is inert with uniformity in $X$). Also suppose that $\phi$ is smooth and satisfies $\phi^{(j)}(t) \ll \frac{1}{X}$ for some $Y \gg X^{\varepsilon}$. Let

$$I = \int_{-\infty}^{\infty} w(t) e^{i \phi(t)} dt. \quad (4.1)$$

(1) If $\phi^{(j)}(t) \gg \frac{1}{X}$ for all $t$ in the support of $w$, then $I \ll A \frac{1}{X}$ for $A$ arbitrarily large.

(2) If $\phi^{(j)}(t) \gg \frac{1}{X}$ for all $t$ in the support of $w$, and there exists $t_0 \in \mathbb{R}$ such that $\phi'(t_0) = 0$ (note $t_0$ is necessarily unique), then

$$I = \frac{e^{i \phi(t_0)}}{\sqrt{\phi''(t_0)}} F(t_0) + O(Y^{-A}), \quad (4.2)$$

(4.2)
where $F$ is an inert function (depending on $A$, but uniformly in $X$ and $Y$) supported on $t_0 \asymp X$.

Here, following Young [30], we say a smooth function $f(x_1, \ldots, x_n)$ on $\mathbb{R}^n$ inert if
\[ x_1^{k_1} \cdots x_n^{k_n} f(x_1, \ldots, x_n) \ll 1, \quad \text{(4.3)} \]
with an implied constant depending on $k_1, \ldots, k_n$ and with the superscript denoting partial differentiation. Now, we recall some results about $H^\pm$ from Young [30] §7.

**Lemma 4.2.** Let $H^+$ be given by (2.19). There exists a function $g$ depending on $T$ and $M$ satisfying $g^{(j)}(y) \ll_{j,A} (1 + |y|)^{-A}$, so that
\[ H^+(x) = MT \int_{|v| \leq \frac{x}{2M}} \cos(x \cosh(v)) e \left( \frac{vT}{\pi} \right) g(Mv) dv + O(T^{-A}). \quad \text{(4.4)} \]
Furthermore, $H^+(x) \ll T^{-A}$ unless $x \gg MT^{1-\varepsilon}$. And if $x \gg MT^{1-\varepsilon}$, then $H^+(x) \ll TMx^{-1/2}$.

**Proof.** See Young [30] Lemma 7.1. And the upper bound for $H^+$ when $x \gg MT^{1-\varepsilon}$ comes from (4.3) and Lemma 4.1. 

**Lemma 4.3.** Let $H^-$ be given by (2.19). There exists a function $g$ depending on $T$ and $M$ satisfying $g^{(j)}(y) \ll_{j,A} (1 + |y|)^{-A}$, so that
\[ H^-(x) = MT \int_{|v| \leq \frac{x}{2M}} \cos(x \sinh(v)) e \left( \frac{vT}{\pi} \right) g(Mv) dv + O(T^{-A}). \quad \text{(4.5)} \]
Furthermore, $H^-(x) \ll (x + T)^{-A}$ unless $x \asymp T$. And if $x \asymp T$, then we have $H^-(x) \ll T^{1+\varepsilon}$.

**Proof.** See Young [30] Lemma 7.2. And the upper bound for $H^-$ when $x \asymp T$ is an easy consequence of (4.5).

Define
\[ \tilde{\Psi}^\pm_{\sigma}(x; D) := e \left( \mp \sigma \frac{x}{D^2} \right) \Psi^\pm_{\sigma}(x; D), \quad \text{(4.6)} \]
and
\[ \Upsilon(t) = \Upsilon_{X,D}(t) := \int_0^\infty w \left( \frac{x}{X} \right) \tilde{\Psi}^\pm_{\sigma}(x; D)x^{it} \frac{dx}{x}, \quad \text{(4.7)} \]
where $w$ is a fixed smooth function supported on $[1/2, 3]$, and with value 1 on $[1, 2]$. Now together with the Mellin technique, we can prove the following lemma, which will help us to separate the variables.

**Lemma 4.4.** Let $x \asymp X$ with $X \gg (qT)^{-B}$ for some large but fixed $B$.

(i) We have $\tilde{\Psi}^\pm_{\sigma}(x; D) \ll T^{-A}$ unless
\[ \begin{cases} D \gg TM^{1-\varepsilon}, & \text{if } \sigma = 1, \\ D \asymp T, & \text{if } \sigma = -1. \end{cases} \quad \text{(4.8)} \]

(ii) When $X \ll T^\varepsilon$, if $D$ satisfies (4.8), then we have
\[ x^k \frac{d^k}{dx^k} \tilde{\Psi}^\pm_{\sigma}(x; D) \ll_{k,\varepsilon} \begin{cases} X^{2/3}TM^{-1/2}T^\varepsilon, & \text{if } \sigma = 1, \\ X^{2/3}T^{1+\varepsilon}, & \text{if } \sigma = -1. \end{cases} \quad \text{(4.9)} \]

Note that here the $\varepsilon$ on the right hand side may depend on $k$. Furthermore, if $x \in [X, 2X]$, we have
\[ \widetilde{\Psi}^\pm_{\sigma}(x; D) = \frac{1}{2\pi i} \int_{-T}^{T} \Upsilon(t)x^{-it} dt + O(T^{-A}). \quad \text{(4.10)} \]

And for $|t| \ll T^\varepsilon$, we have
\[ \Upsilon(t) \ll \begin{cases} X^{2/3}TM^{-1/2}T^\varepsilon, & \text{if } \sigma = 1, \\ X^{2/3}T^{1+\varepsilon}, & \text{if } \sigma = -1. \end{cases} \quad \text{(4.11)} \]
(iii) When $X \gg T^\epsilon$, we have

$$
\Psi^\pm_{\sigma}(x; D) = \sum_{\ell=1}^{K} \gamma_{\ell} \frac{x^{5/6}}{\ell^{1/3}} L(x; D) + O(T^{-A}),
$$

(4.12)

where $L$ is a function that takes the form

$$
L(x; D) = \int_{|t| \ll U} \lambda_{X,T}(t) \left( \frac{x}{D^2} \right)^{it} dt
$$

(4.13)

with the following parameters. Here $\lambda_{X,T}(t) \ll 1$ does not depend on $x$ and $D$. If $\sigma = 1$, then $U = T^2/D$; furthermore, $L$ vanishes unless

$$
X \simeq D^3, \quad \text{and} \quad D \gg MT^{1-\epsilon}.
$$

(4.14)

If $\sigma = -1$, then $U = T^{2/3}X^{1/3}D^{-2/3}$; in addition, $L$ vanishes unless

$$
X \ll D^3M^{-\epsilon}, \quad \text{and} \quad D \asymp T.
$$

(4.15)

Proof. We first handle the case $X \gg T^\epsilon$. By Blomer [11, Lemma 6], we have

$$
\Psi^\pm_{\sigma}(x; D) = x \int_{0}^{\infty} \psi_{\sigma}(y; D) \sum_{\ell=1}^{K} \frac{\gamma_{\ell}}{(xy)^{\ell/3}} e \left( \pm (3xy)^{1/3} \right) dy + O(T^{-A}),
$$

(4.16)

for some constants $\gamma_{\ell}$ depending only on $\alpha_1, \alpha_2, \alpha_3$. Recall the definition of $\psi_{\sigma}(y; D)$ (4.9). By Lemmas 4.2 and 4.3, we arrive at

$$
\sum_{\ell=1}^{K} \frac{xMT \gamma_{\ell}}{\ell^{1/3}} \int_{|v| \ll M} g(Mv) e \left( \frac{vT}{\pi} \right) \left( \int_{0}^{\infty} w(y)y^{-u} e \left( 2\sqrt{y}D\phi_{\sigma}(v) \pm 3(xy)^{1/3} \right) y^{-\ell/3} dy \right) dv,
$$

where $\phi_{\sigma}(v) = \pm \cosh(v)$ for $\sigma = 1$, and $\phi_{\sigma}(v) = \pm \sinh(v)$ for $\sigma = -1$.

The $y$-integral can be analyzed by stationary phase. By Lemma 4.1, we know the above integral is small unless a stationary point exists, which implies

$$
|\phi_{\sigma}(v)| = \pm \phi_{\sigma}(v) \asymp X^{1/3}/D.
$$

(4.17)

In addition, since $|v| \ll M^2/M$, we have $X \asymp D^3$ if $\sigma = 1$, and $X \ll D^3M^{-\epsilon}$ if $\sigma = -1$.

At this point, we can restrict the size of $X$. Recall that in our application, $D = \sqrt{3}N/c$ and $N$ satisfying (2.10), we have $D \ll (qT)^2$. Hence we can assume that $X \ll (qT)^6$. Otherwise, we get $\Psi^\pm_{\sigma}(x; D) \ll T^{-A}$.

Now we consider the range of $D$ to make $\Psi^\pm_{\sigma}(x; D)$ be not negligible. At first, by the above argument, we can restrict ourself to the case $(qT)^{-2} \ll X \ll (qT)^6$. By (2.20) and Parseval’s formula, we have

$$
\Psi^\pm_{\sigma}(x; D) = x \int_{0}^{\infty} \psi_{\sigma}(y; D) g^\pm(\pi^3xy)dy,
$$

(4.18)

where

$$
g^\pm(y) = \frac{1}{2\pi i} \int_{(c)} G^\pm(s)y^{-s} ds
$$

is the inverse Mellin transform of

$$
G^\pm(s) = \prod_{j=1}^{3} \Gamma \left( \frac{s+\alpha_{j}}{2} \right) \pm \frac{1}{\sqrt{3}} \prod_{j=1}^{3} \Gamma \left( \frac{1+s+\alpha_{j}}{2} \right).
$$

Now, by Lemmas 4.2 and 4.3, we can assume that $D \gg MT^{1-\epsilon}$ if $\sigma = 1$, and $D \asymp T$ if $\sigma = -1$. Otherwise, we have $\Psi^\pm_{\sigma}(x; D) \ll T^{-A}$. Thus we give the proof of part (i).

Assuming (4.17), the stationary point at $y_0 = x^2(\phi_{\sigma}(v)D)^{-6} \asymp 1$, so, by Lemma 4.1, we have

$$
\int_{0}^{\infty} w(y)y^{-u} e \left( 2\sqrt{y}D\phi_{\sigma}(v) \pm 3(xy)^{1/3} \right) y^{-\ell/3} dy
$$

$$
= x^{-1/6} e \left( \frac{\pm x}{\phi_{\sigma}^2(v)D^2} \right) w_1(v) + O(T^{-A}),
$$

where $w_1(v)$ is a function.
where $w_1$ is inert in terms of $v$, and $w_1$ has support on \((4.17)\). The fact that $w_1$ is inert in terms of $v$ needs some discussion. We naturally obtain an inert function in terms of $\phi_\sigma(v)$, but since $\phi_\sigma(v)$ has bounded derivatives for $|v| \leq 1$, we do get an inert function of $v$. Hence, to bound $\Psi_\sigma^\pm(x; D)$, we only need to estimate
\[
\sum_{\ell=1}^{K} \frac{x^{5/6} M T^{\gamma_\ell}}{x^{\ell/3}} e \left( \mp \sigma \frac{x}{D^2} \right) \Phi_\sigma \left( \frac{x}{D^2} \right),
\]
where
\[
\Phi_\sigma(y) = \int_{|v| < \frac{M}{x}} g(Mv) e \left( \frac{vT}{\pi} \right) e \left( \frac{\pm y (\sigma - \phi_\sigma^{-2}(v))}{2} \right) w_1(v) dv.
\]
Finally, we shall use the Mellin technique to analyze $\Phi_\sigma(y)$. By the same proof as Young \[30\], we have
\[
y \asymp Y \asymp X/D^2 \asymp \{ X^{1/3}, \quad \text{if } \sigma = 1,
\]
\[
XT^{-2}, \quad \text{if } \sigma = -1,
\]
we have
\[
\Phi_\sigma(y) = \frac{1}{T} \int_{|t| \ll U} \lambda_{Y,T}(t) y^t dt + O(T^{-A}),
\]
where $\lambda_{Y,T}$ and $U$ depend on $Y, T$. Precisely, we have $\lambda_{Y,T}(t) \ll 1$, and
\[
\left\{ \begin{array}{ll}
U = T^2/Y, & \text{if } \sigma = 1, \\
U = Y^{1/3}T^{2/3}, & \text{if } \sigma = -1.
\end{array} \right.
\]
(Note that the assumption $Y \gg 1$ in Young \[30\] Lemma 8.2 is not used. Here we just need $Y/|v_0|^2 \gg T^\epsilon$, where $|v_0| = x^{1/3}/D$ in the proof. And we can derive this from $Y/|v_0|^2 \asymp X/(D|v_0|)^2 \asymp X^{1/3} \gg T^\epsilon$.) Note that in either case, we have $U \gg T^\epsilon$. Now, by \((4.17)\), we have $Y \asymp D$ if $\sigma = 1$. We prove part (iii).

Finally, we deal with the case $X \ll T^\epsilon$. By Blomer \[1\] Lemma 7, for $D$ satisfying \((4.8)\), we have
\[
x^k \frac{d^k}{dx^k} \hat{\Psi}_\sigma^\pm(x; D) \ll_k (1 + x^{1/3})^k x^{2/3} \| \psi_\sigma \|_\infty.
\]
Now, by \((3.9)\) and Lemmas \(4.2\) and \(4.3\) we prove the upper bound \((4.9)\). Next, we want to use the Mellin technique to separate the variables. Recall that
\[
\Upsilon(t) = \int_0^\infty w \left( \frac{x}{X} \right) \hat{\Psi}_\sigma^\pm(x; D) x^t \frac{dx}{x}.
\]
Note that for $|t| \gg T^\epsilon$, (taking $\epsilon > 2e$), we have $t/x \gg T^\epsilon$. So using integral by parts many times, for $|t| \gg T^\epsilon$, we have $\Upsilon(t) \ll (tT)^{-A}$. By the Mellin inversion, for $x \in [X, 2X]$, we have
\[
\hat{\Psi}_\sigma^\pm(x; D) = \frac{1}{2\pi} \int_{-T^\epsilon}^{T^\epsilon} \Upsilon(t) x^{-t} dt = \frac{1}{2\pi} \int_{-T^\epsilon}^{T^\epsilon} \Upsilon(t) x^{-t} dt + O(T^{-A}).
\]
And for $|t| \ll T^\epsilon$, the upper bound \((4.11)\) of $\Upsilon(t)$ is a consequence of \((4.7)\) and \((4.23)\). Thus we prove part (ii). This finishes the proof of the lemma. \(\square\)

Lemma \(4.4\) is good enough to give a nice bound for the terms related the $K$-Bessel function. However, we don’t know how to apply both the large sieve inequalities and a second use of Voronoi formula when we want to bound the terms related to the $J$-Bessel function. So, on the one hand, in the following sections, we will get a bound without using the Voronoi formula twice. This result is good in $q$-aspect and not too bad in $t$-aspect. And then, on the other hand, in \[8\] we will use another method to deal with the integral transforms that appear on the right hand side of the Voronoi formula. This will be done by following Blomer \[1\] §3, Li \[18\] §4, and Mckee–Sun–Ye \[19\] §6. By doing this, we will obtain a bound which is good in the $t$-aspect, and not too bad in the $q$-aspect. Then combining these two bounds, one can get a hybrid subconvexity bound.
5. Applying the large sieve

Let

$$H(w; q) = \sum_{u, v \pmod{q}} \chi_q(uv(u + 1)(v + 1))e_q((uv - 1)w).$$  \hspace{1cm} (5.1)$$

By Conrey–Iwaniec \[4\] Eq. (11.7), we have

$$H(w; q) = \sum_{q_1 q_2 = q} \mu(q_1)\chi_{q_1}(-1)H^*(\overline{q_1} w; q_2),$$  \hspace{1cm} (5.2)$$

where

$$H^*(w; q) = \sum_{u, v \pmod{q}} \chi_q(uv(u + 1)(v + 1))e_q((uv - 1)w),$$  \hspace{1cm} (5.3)$$

and from \[4\] Eq. (11.9), we have

$$H^*(w; q) = \frac{1}{\varphi(q)} \sum_{\psi \pmod{q}} \tau(\psi)g(\chi, \psi)\psi(w),$$  \hspace{1cm} (5.4)$$

where \(\tau(\psi)\) is the Gauss sum, and \(g(\chi, \psi) \ll q^{1+\varepsilon}\).

We first recall the following hybrid large sieve.

**Lemma 5.1.** Suppose \(U \geq 1\), and let \(a_n\) be a sequence of complex numbers. Then

$$\left| \sum_{n \leq N} a_n \psi(n) n^it \right|^2 dt \ll (qU + N) \sum_{n \leq N} |a_n|^2.$$  \hspace{1cm} (5.5)$$

**Proof.** See Gallagher \[6\] Theorem 2]. \(\square\)

**Lemma 5.2.** Suppose that \(q\) is squarefree. Let \(a_1, a_2, b_1, b_2 \in \mathbb{Z}, c \in \mathbb{Z}\), such that \((b_1 b_2 c, q) = (a_1 a_2, c) = 1\). Let \(N_2, D_2 \geq 1\). Suppose \(\alpha_d, \beta_n \in \mathbb{C}\) with \(|\alpha_d| \leq 1\) for \(1 \leq d \leq D_2\), \(1 \leq n \leq N_2\), and \(U \geq 1\). Then for any \(\varepsilon > 0\), we have

$$\left| \int_{|t| \leq U} \left| \sum_{n \leq N_2, d \leq D_2} \alpha_d \beta_n c(a_1 a_2 d n / c) H(b_1 b_2 d n; q) \left( \frac{n}{d} \right)^it \right| dt \right| \ll \frac{q^{1/2+\varepsilon}}{c^{1/2-\varepsilon}} (qcU + D_2)^{1/2} D_2^{1/2} (qcU + N_2)^{1/2} \|\beta\|,$$

where as usual \(\|\beta\| = (\sum |\beta_n|^2)^{1/2}\).

This is a variation of \[4\] Lemma 11.1]. We combine the ingredients in both \[4\] Lemma 13] and \[30\] Lemma 9.2].

**Proof.** By (5.2), we have

$$\left| \int_{|t| \leq U} \left| \sum_{n \leq N_2, d \leq D_2} \alpha_d \beta_n c(a_1 a_2 d n / c) H(b_1 b_2 d n; q) \left( \frac{n}{d} \right)^it \right| dt \right| \leq \sum_{q_1 q_2 = q} \left| \int_{|t| \leq U} \left| \sum_{n \leq N_2, d \leq D_2} \alpha_d \beta_n c(a_1 a_2 d n / c) H^*(b_1 b_2 d n; q_1) \left( \frac{n}{d} \right)^it \right| dt \right|.$$
We just handle the case \( q_2 = q \), since the other cases turn out to have a smaller upper bound. By (5.4), we have

\[
\int_{|t| < U} \left| \sum_{n \leq N_2, d \leq D_2 \atop \gcd(n, qc) = 1} \alpha_d \beta_n \epsilon(a_1 \overline{\nu_2 d n / c}) H^* (b_1 \overline{\nu_2 d n / q} \left( \frac{n}{d} \right)^i) \right| dt
\]

\[
\ll \int_{|t| < U} \frac{1}{\varphi(qc)} \sum_{\psi(q) \psi(c)} \tau(\psi) \tau(\overline{\psi}) g(\chi, \psi) \sum_{n \leq N_2, d \leq D_2 \atop \gcd(n, qc) = 1} \alpha_d \beta_n \psi_d (d n) \left( \frac{n}{d} \right)^i dt
\]

\[
\ll q^{1/2+\varepsilon} \left( \int_{|t| < U} \sum_{\psi(qc)} \left| \sum_d \alpha_d \psi_d (d n) d^{-i} \right|^2 dt \right)^{1/2} \left( \int_{|t| < U} \sum_n \beta_n \psi_d (n n^i) \left| \sum_{d} dt \right|^2 dt \right)^{1/2},
\]

where \(|\alpha_d| \leq 1\). Now after applying Lemma 5.1 we prove the lemma. \(\square\)

6. PROOF OF THEOREM 1.2

Denote \( R_q(n) = S(n, 0; q) \) the Ramanujan sum. By a long and complicated computation, Blomer \([\text{I}], \text{Eq. (51)}\) gave

\[
S_\sigma(q, N; \delta) = \sum_{\pm \delta_0, c_1' c_2'} A(n_2, n_1' f_2 c_1') \varphi(f_1 f_2 d_2 c_1'^i d_1' c_2'^j) \left( \delta_0, c_1' c_2' \right) \psi_\sigma \left( \frac{n_2 (n_1')^2 N}{(f_1 f_2 d_2 c_1')^2 f_2} q \right),
\]

where \(\sigma\) is the Euler function, \(\gamma = \pi^{3/2} \chi(-1)/2\), and

\[
h = (f_1 f_2 d_2', q) = (d_2', q), \quad k = (n_2 (n_1' f_2 c_2')^2 c_2, q/h), \quad \ell = q/(hk).
\]

We summary the relations of these variables and previous variables here, although we don’t need them in this section (see Blomer \([\text{I}], \text{§6}\])

\[
c = qr, \quad c_2 = c/c_1, \quad \delta_0 = (\delta, r), \quad \delta' = \delta_0/c_1,
\]

\[
c' = c/\delta_0, \quad c_2' = c_2/\delta_0, \quad r' = r/\delta_0,
\]

\[
n_1' = n_1/f_2, \quad c_1' = c_1/r', \quad f_1 f_2 d_2' = r'.
\]

As Blomer \([\text{I}], \text{§7}\) did, in the \(q\)-aspect, one can use the decay conditions of \(\Psi_\sigma\) to show that several variables can be dropped. But in our case, things become much more complicated. However, the argument is similar to Blomer \([\text{II}], \text{§7}\), and we need to track the dependence on \(T\) and \(M\). One can see that the argument in (6.2) \([\text{6.3}\]) is similar to (6.1) \([\text{6.1}\]) and even easier. In the \(q\)-aspect, results in (6.2) \([\text{6.3}\]) are better. However, it seems that to get a good bound in the \(t\)-aspect, we have to use the large sieve in all cases.

6.1. The main case. We first deal with the main case, that is, \(c_1' = q, \quad c_2' = h = k = 1\). This is the most important case (at least in the \(q\)-aspect), so we give the details of the treatment of this case. Denote these terms in (6.1) as \(S_\sigma^1(q, N; \delta)\). Note that we have \((d_2' n_1' n_2, q) = 1\). Write \(f_1 = n_1' g\).
Then we have

\[
S_\delta^t(q, N; \delta) = \sum_{\pm} \sum_{\delta_0 \delta = \delta} \frac{\mu(\delta_0)\chi(\delta)}{\delta_0} \sum_{\substack{g,n_1',f_2,d_2' \\
\varphi(n_1'f_2)n_1}} d_2' \mu(f_2) \frac{A(n_2,n_1'f_2)}{n_2} e \left( \pm \sigma n_1'f_2n_2\delta_0\delta'q \right) \frac{g\delta'}{\delta} \times H(\mp \sigma gd_2n_2n_1'f_2\delta_0\delta', q) \tilde{\Psi}^\pm_{\sigma} \left( \frac{n_2sN}{(gd_2'q)^3n_1'f_2}, \frac{\sqrt{\delta}N}{qgn_1'f_2d_2'\delta_0} \right). 
\]

(6.4)

Since we have \((n_1'f_2\delta_0\overline{d_2q}, g\delta') = 1\), let

\[
s = (n_2, g\delta'), \quad n_2 = n_2's, \quad (n_2', g\delta'/s) = 1.
\]

We cancel the factor \(s\) from the numerator and denominator of the exponential getting

\[
S_\delta^t(q, N; \delta) = \sum_{\pm} \sum_{\delta_0 \delta = \delta} \frac{\mu(\delta_0)\chi(\delta)}{\delta_0} \sum_{\substack{g,n_1',f_2 \\
\varphi(n_1'f_2)n_1}} \frac{\mu(f_2)}{n_1'} \sum_{s_1|s\delta} \frac{1}{s} \times \sum_{d_2'} \sum_{(d_2', \phi g n_1'f_2) = 1} \frac{d_2' A(n_2's, n_1'f_2)}{n_2'} e \left( \pm \sigma n_1'f_2n_2\delta_0\delta'q \right) \frac{g\delta'/s}{\delta} \times H(\mp \sigma gd_2n_2n_1'f_2\delta_0\delta', q) \tilde{\Psi}^\pm_{\sigma} \left( \frac{n_2sN}{(gd_2'q)^3n_1'f_2}, \frac{\sqrt{\delta}N}{qgn_1'f_2d_2'\delta_0} \right).
\]

(6.5)

The main actors in (6.5) are the variables \(d_2'\) and \(n_2'\). We open the coprimality condition \((d_2', n_2') = 1\) by Möbius inversion. We introduce a new variable \(r\) and get

\[
S_\delta^t(q, N; \delta) \ll (qT)^\varepsilon \sup_{N_2,D_2} \sum_{\pm} \sum_{\delta_0 \delta = \delta} \frac{1}{\delta_0} \sum_{\substack{g,n_1',f_2,r \\
\varphi(n_1'f_2)\varphi(n_1') = 1}} \frac{1}{\varphi(n_1'f_2)} \times \sum_{s_1|s\delta} \frac{1}{s} \times \sum_{(d_2', \phi g n_1'f_2) = 1} \frac{d_2' A(n_2'rs, n_1'f_2)}{n_2'} e \left( \pm \sigma n_1'f_2n_2\delta_0\delta'q \right) \frac{g\delta'/s}{\delta} \times H(\mp \sigma gd_2n_2n_1'f_2\delta_0\delta', q) \tilde{\Psi}^\pm_{\sigma} \left( \frac{n_2sN}{(gd_2'q)^3n_1'f_2}, \frac{\sqrt{\delta}N}{qgn_1'f_2d_2'\delta_0} \right). \]

By Lemma 3.3 we can assume that

\[
\begin{align*}
1 & \leq D_2 \leq \frac{\sqrt{\delta}N}{qgn_1'f_2d_2'\delta_0}, & \text{if } \sigma = 1, \\
1 & \leq D_2 \leq \frac{\sqrt{\delta}N}{qgn_1'f_2d_2'\delta_0}, & \text{if } \sigma = -1,
\end{align*}
\]

(6.6)

and

\[
\begin{align*}
1 & \leq N_2 \leq \frac{(\delta^3N)^{1/2}}{(n_1'f_2)^2r_0\delta_0^3}, & \text{if } \sigma = 1, \\
1 & \leq N_2 \leq \frac{(\delta^3N)^{1/2}}{(n_1'f_2)^2r_0\delta_0^3}, & \text{if } \sigma = -1.
\end{align*}
\]

(6.7)
Now we consider the case $\sigma = -1$, and $x = \frac{n'N}{(g_2f_2)\nu'f_1f_2} \gg T^\varepsilon$. By Lemma 4.4, we infer

$$
S_\sigma(q, N; \delta) \ll (qT)^\varepsilon M \sum_{\pm} \sum_{d_0 \delta = \delta} \frac{1}{\delta^0} \sum_{g_1, \delta' = \delta} \sum_{s \mid (g_2s)^{3/2}(n_1)^{7/2}r} \frac{1}{(qf_2s)^{3/2}(n_1)^{7/2}r} \times \sup_{D_2, N_2} \left( \frac{N_1^1}{q}\sum_{q^{1/2}N_1^1/2} \int \left| \sum_{s_2 \leq N_2} \sum_{d_2 \leq 2} \alpha(d_2')\beta(n_2') \times A(n_2', s_2, s_1')e \left( \pm \sigma \frac{n_1'f_2s_2n_2'D_2}{g_1'D_2} q \right) \frac{(n_1')^i}{d_2} \right| dt \right).
$$

Note that $U \asymp x^{1/3} \ll T^{1+\varepsilon}/M$, and from (2.12) we have

$$
\left( \sum_{n_2' \geq N_2} |A(n_2', s_1')|^2 \right)^{1/2} \ll (qT)^{\varepsilon} N_2'^{1/2}(rsn_1'f_2)^{7/32}.
$$

Applying Lemma 5.2, we obtain

$$
\frac{1}{N_1^1/2} S_\sigma(q, N; \delta) \ll (qT)^\varepsilon M q^{-1r^{7/32}} \sup_{D_2, N_2} (qU + D_2)^{1/2}(qU\delta + N_2)^{1/2}
$$

$$
\ll (qT)^\varepsilon M q^{-1r^{7/32}} \left( \frac{qT}{M} + \frac{(qT)^{1/2}}{r} \right)^{1/2} \left( \frac{qT}{M} + \frac{(qT)^{3/2}}{rM^{3/2}} \right)^{1/2}
$$

$$
\ll (qT)^\varepsilon \left( T\delta^{1/2}\gamma^{7/32} + q^{1/4}T^{5/4}M^{-1} \right)
$$

$$
\ll (qT)^\varepsilon \left( T\delta^{1/2}(qT)^{7/64} + q^{1/4}TM \right),
$$

provided $T^{1/8+\varepsilon} \ll M \ll T^{1/2}$. Here we use the fact $r \ll (qT)^{1/2+\varepsilon}$, which is a consequence of (6.6). Note that if $\sigma = 1$, and $x \gg T^\varepsilon$, then the same argument will give

$$
\frac{1}{N_1^1/2} S_\sigma(q, N; \delta) \ll (qT)^\varepsilon M q^{-1r^{7/32}} \sup_{D_2, N_2} (qU + D_2)^{1/2}(qU\delta + N_2)^{1/2}
$$

$$
\ll (qT)^\varepsilon M q^{-1r^{7/32}} \left( \frac{qT}{M} + \frac{(qT)^{1/2}}{r} \right)^{1/2} \left( \frac{qT}{M} + \frac{(qT)^{3/2}}{rM^{3/2}} \right)^{1/2}
$$

$$
\ll (qT)^\varepsilon \left( T\delta^{1/2}\gamma^{7/32} + q^{1/4}TM \right) + q^{1/4}TM
$$

$$
\ll (qT)^\varepsilon \left( T\delta^{1/2}(qT)^{7/64} + q^{1/4}TM \right),
$$

provided $M \asymp T^{1/2}$. This will not give us a subconvexity bound in the $t$-aspect, so we have to sum over $n_2$ non-trivially.

Now we consider the case $x \ll T^\varepsilon$, i.e., $N_2 \ll \frac{(\delta N)^{1/2}r}{(n_1'f_2)\nu'f_1f_2}$. Note that the upper bound for $N$ implies that this will happen only if $q \gg T^{1-\varepsilon}$. By Lemma 4.4, for both $\sigma = \pm 1$, we have

$$
S_\sigma(q, N; \delta) \ll (qT)^\varepsilon T \sum_{\pm} \sum_{d_0 \delta = \delta} \frac{1}{\delta^0} \sum_{g_1, \delta' = \delta} \sum_{s \mid (g_2s)^{3/2}(n_1)^{7/2}r} \frac{1}{(qf_2s)^{3/2}(n_1)^{7/2}r} \times \sup_{D_2, N_2} \left( \frac{N_1^1}{q}\sum_{q^{1/2}N_1^1/2} \sum_{D_2, N_2} \sum_{d_2, (n_2', g_1') = 1} \sum_{s \mid (g_2s)^{3/2}(n_1)^{7/2}r} \frac{1}{(qf_2s)^{3/2}(n_1)^{7/2}r} \right)
$$

$$
\times A(n_2'rs, s_1'f_2)e \left( \pm \sigma \frac{n_1'f_2s_2n_2'D_2}{g_1'D_2} q \right) \frac{(n_1')^i}{d_2} \left| \frac{(n_1')^i}{d_2} \right| H(\pm \sigma g_1D_2^2n_2sn_1f_2\delta q, q).
$$
Now by Blomer [11, Lemma 13], we have
\[
\frac{1}{N^{1/2}}S^q_\sigma(q, N; \delta) \ll (qT)^\epsilon q^{-1/32} \sup_{D_2, N_2} (q + D_2)^{1/2}(q\delta + N_2)^{1/2}
\ll (qT)^\epsilon q^{-1/32} \left( q + \frac{(qT)^{1/2}}{r} \right)^{1/2} \left( q\delta + \frac{q^{3/2}}{rT^{3/2}} \right)^{1/2}
\ll (qT)^\epsilon \left( T\delta^{1/2}(qT)^{7/64} + (qT)^{1/4} \right).
\]

Hence in both cases, we prove (3.7). This finishes the estimation of $S_\sigma(x; D)$ when $\sigma = -1$, and also when $\sigma = 1$ and $x \ll T^\epsilon$. In next section, we will focus on the case $\sigma = 1$ and $x \gg T^\epsilon$.

6.2. The case $c_2^\prime = q$. In this subsection, we prove the terms attached with $c_2^\prime = q$, $c_1^\prime = 1$ has a good bound, that is, we show that we can assume $c_2^\prime = 1$. Denote these terms in (6.1) as $S^q_\sigma(q, N; \delta)$.

Let $D = \frac{\sqrt{T}}{q f_1 f_2 d_2^\delta s_0}$. By Lemma 4.1, we can assume
\[
\begin{cases}
D \gg TM^{1-\epsilon}, & \text{if } \sigma = 1, \\
D \asymp T, & \text{if } \sigma = -1.
\end{cases}
\]

Hence, by (6.10), we have
\[
x = \frac{n_2(n_1)^2 N}{(f_1 f_2^2 c_1^2)^3 f_2} = \frac{n_2(n_1)^2 f_2^2 D_0^3 (qD)^3}{(D^3 N)^{1/2}} \gg T^\epsilon
\]

By the condition $(f_1 f_2 d_2^2, q\delta) = 1$, we have $h = (d_2^2, q) = 1$, and then $k = q$ and $\ell = 1$. Hence, after writing $n_1^2 g = f_1$, we have
\[
S_\sigma^q(q, N; \delta) = \gamma \sum_{\pm} \sum_{\delta_0, \delta_0^\prime = \delta} \frac{\mu(\delta_0) \chi(\delta)}{\delta_0} \sum_{g, n_1^2, f_2, d_2^2} \sum_{(n_2, d_2^2) = 1} \frac{1}{\varphi(n_1^2 f_2)} \sum_{g \neq (n_2, d_2^2) = 1} \frac{d_2^2 \mu(f_2)}{q}
\]
\[
\times A(n_2, n_1 f_2) e^{\left( \pm \sigma n_1^2 q^2 f_2 n_2 d_0 d_2^2 \psi \right) \frac{n_2 N}{(gd_2^2)^3 n_1^2 f_2 \sqrt{n_2 s}}},
\]

Since we have $(n_1^2 q^2 f_2 d_0 d_2^2, g\delta^\prime) = 1$, let
\[
s = (n_2, g\delta^\prime), \quad n_2 = n_2^\prime s, \quad (n_2^\prime, g\delta^\prime / s) = 1.
\]

We cancel the factor $s$ from the numerator and denominator of the exponential, and open the coprimality condition $(n_2^2, d_2^2) = 1$ by Möbius inversion (introduce the new variable $r$), we get
\[
S_\sigma^q(q, N; \delta) = \gamma \sum_{\pm} \sum_{\delta_0, \delta_0^\prime = \delta} \frac{\mu(\delta_0) \chi(\delta)}{\delta_0} \sum_{g, n_1^2, f_2, r} \sum_{(g n_1^2 f_2, q\delta^\prime) = 1} \frac{\mu(f_2) \mu(r)}{n_1^2 \varphi(n_1 f_2)} \sum_{s | g\delta^\prime} \frac{1}{s}
\]
\[
\times \sum_{d_2^2} \sum_{(d_2^2, qgn_1^2 f_2) = 1} d_2 A(n_2^2 rs, n_1^2 f_2) \frac{n_2^2 N}{q n_2^2},
\]
\[
\times e^{\left( \pm \sigma n_1^2 q^2 f_2 n_2^2 d_0 d_2^2 \psi \right) \frac{n_2 N}{(gd_2^2)^3 n_1^2 f_2 \sqrt{n_2 s}}},
\]

By the coprimality condition $(n_2^2, d_2^2) = 1$. Denote these terms in (6.1) as $\tilde{\Psi}$.
Then by Lemma 4.4, we have
\[
\frac{1}{N^{1/2}} S^\sigma_H(q, N; \delta) \ll (qT)^\gamma \sum_{\pm} \sum_{\delta_0 = \delta} \frac{1}{\delta_0} \sum_{g, n_1, f_2, r} \frac{1}{(g f_2)^{3/2} (n_1')^{3/2}} \sum_{s | gk} \frac{1}{s^{1/2}} \\
\times \sup_{D_2, N_2} \frac{M}{q D_2^{1/2} N_2^{1/2}} \int_{|t| < U} \left| \sum_{d_2' \leq D_2} \sum_{n_1' \leq N_2} \alpha(d_2') \beta(n_1') \sum_{s | gk} \right| dt.
\]

Here we can assume that
\[
\begin{cases}
1 \leq D_2 \leq \frac{\sqrt{3N}}{q g n_1 f_2 r \delta_0 T^{1/2}}, & \text{if } \sigma = 1, \\
1 \leq D_2 = \frac{\sqrt{3N}}{q g n_1 f_2 r \delta_0 T^{1/2}}, & \text{if } \sigma = -1,
\end{cases}
\]
and
\[
\begin{cases}
1 \leq N_2 \leq \frac{\delta_0 (N_1)^{1/2}}{q (n_1 f_2)^2 \delta_0 T^{1/2}}, & \text{if } \sigma = 1, \\
1 \leq N_2 \leq \frac{\delta_0 (N_1)^{1/2}}{q (n_1 f_2)^2 \delta_0 T^{1/2}}, & \text{if } \sigma = -1.
\end{cases}
\]

Note that in both cases, we have \( U \ll T^{1+\varepsilon}/M \). We can use the multiplicative characters to separate the variables in the exponential function, together with Cauchy–Schwarz inequality and Lemma 5.1, we obtain
\[
\frac{1}{N^{1/2}} S^\sigma_H(q, N; \delta) \ll (qT)^\gamma \sum_{\pm} \sum_{\delta_0 = \delta} \frac{1}{\delta_0} \sum_{g, n_1, f_2, r} \frac{1}{(g f_2)^{3/2} (n_1')^{3/2}} \sum_{s | gk} \frac{1}{s^{1/2}} \\
\times \sup_{D_2, N_2} \frac{M}{q} \frac{(g \delta'/s)^{1/2} (U + D_2)^{1/2} (U + N_2)^{1/2} (r s n_1 f_2)^{7/32}}{}
\]

In the case \( \sigma = -1 \), we have
\[
\frac{1}{N^{1/2}} S^\sigma_H(q, N; \delta) \ll (qT)^\gamma \sum_{\pm} \sum_{\delta_0 = \delta} \frac{1}{\delta_0} \sum_{g, n_1, f_2, r} \frac{1}{(g f_2)^{3/2} (n_1')^{3/2}} \sum_{s | gk} \frac{1}{s^{1/2}} \\
\times \sup_{D_2, N_2} \frac{M}{q} \frac{(g \delta'/s)^{1/2} (T + T^{5/4}) (r s n_1 f_2)^{7/32}}{}
\]

provided \( T^{1+\varepsilon}/M \ll T^{1/2} \), which is good enough for our purpose by (3.7). From now on we assume \( c_2' = 1, c_1' = q \). In the case \( \sigma = 1 \), the same argument will give
\[
\frac{1}{N^{1/2}} S^\sigma_H(q, N; \delta) \ll (qT)^\gamma \delta^{1/2} T M,
\]
provided \( M \asymp T^{1/2} \).

6.3. **The case** \( h = q \). Next we show that \( h = q \) is negligible. Denote these terms in (6.1) as \( S^\sigma_H(\gamma, N; \delta) \). In this case, we have \( k = \ell = 1 \) and \( q | d_2' \). Write \( f_1 = gn_1', f_2 = gd_2'' \). We get
\[
S^\sigma_H(q, N; \delta) = \gamma \sum_{\pm} \sum_{\delta_0 = \delta} \frac{\mu(\delta_0)}{\delta_0} \sum_{g, n_1, f_2, r} \frac{\mu(f_2) \mu(r)}{n_1' \varphi(n_1 f_2)} \sum_{s | gk} \frac{1}{s}
\times q^2 \chi_q(-1) \sum_{d_2''} \frac{d_2'' A(n_2' r s, n_1' f_2)}{n_2'}
\]

\[
\times \left( \pm \frac{n_1' f_2 d_2'' \delta_0 q^2 g_\delta'/s}{g \delta'/s} \right)^{\phi_\delta'/s} \left( \frac{n_2' s N}{(g d_2'' q^2)^{3/2} n_1' f_2} \right)^{\phi_\delta'/s} \sqrt{s N}.
\]

(6.9)
Let \( s = (n_2, g_δ') \) and \( n_2 = n'_2s \) as before. After opening the condition \((n'_2, d''_2) = 1\) by Möbius inversion, we obtain

\[
S^k_\sigma(q, N; δ) = \gamma \sum_{\delta_0 \delta' = δ} \mu(δ_0) \chi(δ) \mu(δ_2) \sum_{g, n'_1, f_2, r} \frac{1}{n'_1 \varphi(n'_1 f_2)} \sum_{s | g δ'} \frac{1}{s} \times q^2 \chi_q(-1) \sum_{d''_2} \frac{d''_2 A(n'_2 r s, n'_1 f_2)}{n'_2} \times e\left( \pm \frac{n'_1 f_2 n'_2 δ_0 δ''_2 q^2}{g_δ'/s} \right) \frac{n'_2 s N}{(g δ''_2 q^3)^3 r^2 n'_1 f_2} \frac{\sqrt{δ_0 N}}{q^2 g_δ n'_1 f_2 d''_2 r δ_0}.
\]

Now we consider the sum over \( d''_2 \leq D_2, n'_2 \leq N_2, \) and we can assume

\[
\begin{cases}
1 \leq D_2 \leq \sqrt[3]{N}, & \text{if } \sigma = 1, \\
1 \leq D_2 \leq \sqrt[3]{q}, & \text{if } \sigma = -1,
\end{cases}
\]

and

\[
\begin{cases}
1 \leq N_2 \leq \frac{1}{(n'_1 f_2)^3 r δ''_2 q^3}, & \text{if } \sigma = 1, \\
1 \leq N_2 \leq \frac{1}{(n'_1 f_2)^3 r δ''_2 q^3}, & \text{if } \sigma = -1.
\end{cases}
\]

The following argument will depend on the size of \( x = \frac{n'_2 s N}{(g δ''_2 q^3)^3 r^2 n'_1 f_2}. \) If \( x \gg T^ε \) and \( \sigma = -1, \) then by Lemma 4.4 we have

\[
\frac{1}{N^{1/2}} S^k_\sigma(q, N; δ) \ll \sum_{s} \sum_{δ_0 δ' = δ} \sum_{g, n'_1, f_2, r} \frac{1}{δ_0} \sum_{g δ'} \frac{1}{s} \sup_{D_2, N_2} \int_{|t| \leq U} \frac{M}{q(D_2 N_2)^{1/2}} \sum_{d''_2 \leq D_2} \sum_{n_2 \leq N_2} \alpha(d''_2) \beta(n'_2) \times A(n'_2 r s, n'_1 f_2) e\left( \pm \frac{n'_1 f_2 n'_2 δ_0 δ''_2 q^2}{g_δ'/s} \right) \left( \frac{n'_2}{d''_2} \right)^{1/2} dt.
\]

Hence by Lemma 5.1 again, we have

\[
\frac{1}{N^{1/2}} S^k_\sigma(q, N; δ) \ll \sup_{D_2, N_2} (q T)^{ε/32} \left( U + D_2 \right)^{1/2} (U N + N_2)^{1/2} \ll (q T)^{ε/32} \left( T^{7/32} + T^{5/4} M^{1/4} \right) \ll (q T)^{ε/2} T M^{1/2},
\]

provided \( T^{1/8+ε} \ll M \ll T^{1/2}. \) And again, if \( x \gg T^ε \) and \( \sigma = 1, \) the same argument shows that

\[
\frac{1}{N^{1/2}} S^k_\sigma(q, N; δ) \ll (q T)^{ε/32} \left( T^{7/32} + T^{5/4} M^{1/2} \right) \ll (q T)^{ε/2} T M^{1/2},
\]

provided \( M \sim T^{1/2}. \)
Now if $x \ll T^\varepsilon$, then by Lemma 4 for both $\sigma = \pm 1$, we have

$$
\frac{1}{N^{1/2}} S^2_\sigma(q, N; \delta) \ll \sum_{\pm} \sum_{\delta_0} 1 \frac{1}{\delta_0} \sum_{g,n_1',f_2,r} 1 \frac{1}{(gf_2)^{3/2}(n_1')^{5/2}T} \sum_{s|\delta_0} 1 \frac{1}{s^{1/2}}
\times \sup_{D_2, N_2} \frac{1}{q(D_2 N_2)^{1/2}} \sum_{n_2' \geq N_2} \sum_{\delta_2' \leq D_2} \alpha(d_2') \beta(n_2')
\times A(n_2' rs, n_1' f_2) e \left( \pm \sigma \frac{n_1' f_2 n_1' d_0 d_2 q}{q\delta/s} \right).
$$

Hence by Blomer $[1]$, Lemma 13, we have

$$
\frac{1}{N^{1/2}} S^2_\sigma(q, N; \delta) \ll \sup_{D_2, N_2} (qT)^\varepsilon \frac{T}{q} t^{7/32} (1 + D_2)^{1/2} (\delta + N_2)^{1/2}
\ll (qT)^\varepsilon t^{1/2} \left( \frac{T}{q} t^{7/32} + \frac{T^{1/4}}{q^{1/4}} \right) \ll (qT)^\varepsilon T M \delta^{1/2},
$$

provided $T^{1/8+\varepsilon} \ll M \ll T^{1/2}$. This proves (3.7) in this case. So from now on, we can assume $c_2' = h = 1$.

6.4. **The case $k = q$.** Now, we show that we can also exclude the case $k = q$. First note that we can simplify $k = (n_2 n_1', q)$. Hence we distinguish two cases and show that the contribution of $q|n_1'$ and $q|n_2$ is negligible. We first deal with the case $q|n_1'$. Denote these terms in (6.1) as $S^2_\sigma(q, N; \delta')$. As before, we have

$$
\delta_0 \quad \sum_{\pm} \sum_{\delta_0} \frac{1}{\delta_0} \sum_{g,n_1',f_2,r} \frac{1}{\delta_0} \sum_{s|\delta_0} \frac{1}{s} \quad \sum_{\pm} \sum_{\delta_0} \frac{1}{\delta_0} \sum_{g,n_1',f_2,r} \frac{1}{\delta_0} \sum_{s|\delta_0} \frac{1}{s}
$$

and

$$
\sum_{d_2|\delta_0} \sum_{d_2|\delta_0} \frac{d_2}{q n_1'} A(n_2' rs, n_1' q f_2)
$$

$$
e \left( \pm \sigma \frac{q^2 n_1' f_2 n_1' d_0 d_2 q}{q\delta/s} \right) \Psi_\sigma \left( \frac{n_1' s N}{(q d_2 q)^{1/2} r q n_1' f_2}, \frac{\sqrt{\delta N}}{q^2 q n_1' f_2 d_2 d_0} \right).
$$

Now we consider the sum over $d_2' \leq D_2, n_2' \leq N_2$, and by Lemma 13, we can assume

$$
\left\{ \begin{array}{l}
1 \leq D_2 \leq \frac{q^3}{\sqrt{N}} T, \\
1 \leq D_2 \leq \frac{q^3}{\sqrt{N}} T M^3, \\
1 \leq N_2 \leq \frac{q^3}{\sqrt{N}} T, \\
1 \leq N_2 \leq \frac{q^3}{\sqrt{N}} T M^3, \\
1 \leq N_2 \leq \frac{q^3}{\sqrt{N}} T M^3, \\
1 \leq N_2 \leq \frac{q^3}{\sqrt{N}} T M^3,
\end{array} \right.
$$

and

$$
\left\{ \begin{array}{l}
1 \leq D_2 \leq \frac{q^3}{\sqrt{N}} T, \\
1 \leq D_2 \leq \frac{q^3}{\sqrt{N}} T M^3, \\
1 \leq N_2 \leq \frac{q^3}{\sqrt{N}} T, \\
1 \leq N_2 \leq \frac{q^3}{\sqrt{N}} T M^3, \\
1 \leq N_2 \leq \frac{q^3}{\sqrt{N}} T M^3, \\
1 \leq N_2 \leq \frac{q^3}{\sqrt{N}} T M^3,
\end{array} \right.
$$

and
If \( \sigma = -1 \), then by Lemma 4.3 we have

\[
\frac{1}{N^{1/2}} S_{\sigma}^2(q, N; \delta) \ll \sum_{\pm \delta_0} \sum_{\delta_0' = \delta} \frac{1}{\delta_0} \sum_{g, n_1', f_2, r} \frac{1}{(g n_1 q_2 f_2)^{3/2r}} \sum_{s, g_0} \frac{1}{s^{1/2}} \times \sup_{D_2, N_2} \frac{M}{q^3(D_2 N_2)^{1/2}} \int_{|t| \ll U} \sum_{d_2 \leq D_2} \sum_{n_2 \leq N_2} \alpha(d_2) \beta(n_2') \times A(n_2' r s, n_1' q f_2) e \left( \pm \sigma \frac{q_2 f_2 n_1' n_2' d_2 d_2' q}{g_0' / s} \right) \left( \frac{n_1' g}{d_2} \right) \left( \frac{\delta_2}{\delta_2' / s} \right) dt.
\]

Hence by Lemma 5.4 again, we have

\[
\frac{1}{N^{1/2}} S_{\sigma}^2(q, N; \delta) \ll \sup_{D_2, N_2} (qT)^{\varepsilon} M \frac{q^3}{q^3} r^{7/32} (U + D_2)^{1/2} (\delta U + N_2)^{1/2} \ll (qT)^{\varepsilon} TM \delta^{1/2},
\]

provided \( T^{1/8+\varepsilon} \ll M \ll T^{1/2} \). If \( \sigma = 1 \), we have

\[
\frac{1}{N^{1/2}} S_{\sigma}^2(q, N; \delta) \ll (qT)^{\varepsilon} \delta^{1/2} \left( \frac{T}{q^3} r^{7/32} + \frac{T^{5/4} M^{1/2}}{q^3} \right) \ll (qT)^{\varepsilon} TM \delta^{1/2},
\]

provided \( M \approx T^{1/2} \).

From now on we assume \( (q, n_1') = 1 \). Since \( c_1' = q \) and \( n_1' | f_1 c_1' \), we have \( n_1' | f_1 \). We write

\( n_1' g = f_1 \).

Now we treat the case \( q | n_2 \). Denote these terms in (6.1) as \( S_{\sigma}^{22}(q, N; \delta) \). Write \( n_2 = q n_2' \). By a similar argument, we get

\[
S_{\sigma}^{22}(q, N; \delta) = \gamma \sum_{\pm \delta_0} \sum_{\delta_0' = \delta} \frac{\mu(\delta_0)}{\delta_0} \chi(\delta) \sum_{g, n_1', f_2, r} \frac{\mu(f_2)}{n_1' \varphi(n_1' f_2)} \sum_{s, g_0} \frac{1}{s} \times \frac{1}{q} \sum_{d_2} \sum_{d_2'} \frac{d_2 A(n_2' r q s, n_1' q f_2)}{n_2'} \times e \left( \pm \sigma \frac{n_1' f_2 n_2' q d_2 d_2' q}{g_0' / s} \right) \Phi \left( \frac{n_1' q g n_2' f_2}{(q g^2 q_2')^{3/2}} \sqrt{d_2 q} \right).
\]

We consider the sum over \( d_2' \approx D_2, n_2' \approx N_2 \), and by Lemma 4.4 we can assume

\[
\begin{aligned}
1 & \leq D_2 \leq \frac{\sqrt{N}}{q n_1' f_2 r_0 q_2 T}, & \text{if } \sigma = 1, \\
1 & \leq D_2 \leq \frac{\sqrt{N}}{q n_1' f_2 r_0 q_2 T}, & \text{if } \sigma = -1,
\end{aligned}
\]

and

\[
\begin{aligned}
1 & \leq N_2 \leq \frac{(\delta^3 N)^{1/2}}{q^2 n_1' f_2 r_0 q_2 T}, & \text{if } \sigma = 1, \\
1 & \leq N_2 \leq \frac{(\delta^3 N)^{1/2}}{q^2 n_1' f_2 r_0 q_2 T}, & \text{if } \sigma = -1.
\end{aligned}
\]
If $x \gg T^\varepsilon$ and $\sigma = -1$, then by Lemma 4.4 we have
\[
\frac{1}{N^{1/2}}S^{\sharp\sharp}_\sigma(q, N; \delta) \ll \sum_{\delta_0 \delta' = \delta} \frac{1}{\delta_0} \sum_{g, n_1, f_2, r} \frac{1}{(gf_2)^{3/2}(n_1')^{3/2}} \sum_{s | g'} \frac{1}{s^{1/2}} \times \sup_{D_2, N_2} \frac{M}{q^2(D_2N_2)^{1/2}} \int_{|t| \in U} \left| \sum_{d_2' \in D_2} \sum_{n_2' \in N_2} \alpha(d_2') \beta(n_2') \times A(n_2'rq, n_1'f_2) e \left( \pm \sigma n_1'f_2n_2'q_0d_2'q / g^\delta / s \right) \right| dt.
\]

The same argument shows that
\[
\frac{1}{N^{1/2}}S^{\sharp\sharp}_\sigma(q, N; \delta) \ll (qT)^\varepsilon q^{-1} T M,
\]
provided $T^{1/8+\varepsilon} \ll M \ll T^{1/2}$. And if $x \gg T^\varepsilon$ and $\sigma = 1$, we get
\[
\frac{1}{N^{1/2}}S^{\sharp\sharp}_\sigma(q, N; \delta) \ll (qT)^\varepsilon q^{-1} T M,
\]
provided $M \asymp T^{1/2}$ again. If $x \ll T^\varepsilon$, then by Lemma 4.4 again, for both $\sigma = \pm 1$, we have
\[
\frac{1}{N^{1/2}}S^{\sharp\sharp}_\sigma(q, N; \delta) \ll \sum_{\delta_0 \delta' = \delta} \frac{1}{\delta_0} \sum_{g, n_1, f_2, r} \frac{1}{(gf_2)^{3/2}(n_1')^{3/2}} \sum_{s | g'} \frac{1}{s^{1/2}} \times \sup_{D_2, N_2} \frac{T}{q^2(D_2N_2)^{1/2}} \int_{|t| \in U} \left| \sum_{d_2' \in D_2} \sum_{n_2' \in N_2} \alpha(d_2') \beta(n_2') \times A(n_2'rq, n_1'f_2) e \left( \pm \sigma n_1'f_2n_2'q_0d_2'q / g^\delta / s \right) \right| dt.
\]

A better bound will show up under the assumption $T^{1/8+\varepsilon} \ll M \ll T^{1/2}$.

6.5. Conclusion. From the above discussion, we can take $M \asymp T^{1/2}$. This proves Proposition 3.1 and hence Theorem 1.2. On the other hand, recalling $R^\pm$ in (3.9), we have
\[
R^- \ll q^{1/4} T M (qT)^\varepsilon,
\]
provided $T^{1/8+\varepsilon} \ll M \ll T^{1/2}$. In the rest of this paper, we will use another method to bound $R^+$, and then prove Theorem 1.3 and Theorem 1.1.

7. Initial setup of Theorem 1.3

As in Section 3, we will use the moment method to prove Theorem 1.3. Since $q^{5/4} T^{3/2} \leq q^4 T^{3/4}$ if $q \geq T^{2/33}$, we know that Theorem 1.3 follows from Theorem 1.2 if $q \geq T^{2/33}$. To prove Theorem 1.3, we only need to consider the case $q \leq T^{2/33}$. However, in the most part of our following arguments, we just require $q \leq T^{1/4}$, since we will need this in the proof of Theorem 1.1 see the end of 4.10. Similarly, at first, we will prove the following proposition.

Proposition 7.1. With notation as before, for any $\varepsilon > 0$, and $T$ large, assuming
\[
q \ll T^{1/6}, \quad \text{and} \quad T^{1/3+\varepsilon} \ll M \ll T^{1/2},
\]
we have
\[
\sum_{n_1 \in B'_{\ast}(q)} L(1/2, \phi \times u_j \times \chi) + \frac{1}{4\pi} \int_{T-M}^{T+M} \left| L(1/2 + it, \phi \times \chi) \right|^2 dt \ll_{\phi, \varepsilon} q^4 T M (qT)^\varepsilon.
\]
It's easy to see that Theorem 1.3 will follow from Proposition 7.1. And as in [3] it suffices to prove
\[ \mathcal{R}^\pm \ll q^3 TM(qT)^\varepsilon, \]  
(7.2)
provided \( T^{1/3+\varepsilon} \ll M \ll T^{1/2} \). Recall that \( \mathcal{R}^\pm \) is defined as in (3.6). Note that we have (6.10), which gives a better bound for \( \mathcal{R}^- \). So we only need to prove (7.2) for \( \mathcal{R}^+ \). As Blomer [1, §5] did, opening the Kloosterman sum, splitting the \( n \)-sum in to residue classes mod \( c \), and detecting the summation congruence condition by primitive additive characters, it suffices to prove, for \( T^{1/3+\varepsilon} \ll M \ll T^{1/2} \), (see Blomer [1, p. 1406–1407])
\[ \sum_{m^2 \delta^3 \leq (qT)^{3+\varepsilon} \atop (\delta, q) = 1; |\mu(t)| = 1} |A(1, m)| |S(q, N; \delta)| \ll q^3 TM(qT)^\varepsilon, \]  
(7.3)
where
\[ S(q, N; \delta) := \sum_{q \mid c} \frac{1}{c} \sum_{a \mid c} \sum_{b(\delta)} e\left(\frac{\delta}{c}\right) \sum_{a(c)} \chi(a) e\left(-\frac{\delta a}{c_1}\right) e\left(\frac{\delta da}{c}\right) \]  
(7.4)
\[ \times \sum_n A(n, 1) e\left(\frac{bn}{c_1}\right) v\left(\frac{n}{N}\right) n^{-1/2} u H^+ \left(\frac{4\pi T \delta n}{c}\right), \]
where \( v \) a suitable fixed smooth function with support in [1, 2], and \( N \) satisfying (8.0). Note that as pointing out in the end of [3] we can restrict the \( c \)-sum to \( c \leq (qT)^B \), for some fixed \( B > 0 \). Now we want to use the Voronoi formula to deal with the \( n \)-sum in (7.2). Before we do this, we need to give an asymptotic formula for \( H^+ \). These will be done in the next section.

8. Integral transforms and special functions

In this section, we follow Blomer [1, §3], Li [18, §4], and McKee–Sun–Ye [19, §6] to give an estimate for the \( n \)-sum in (7.4). As in Li [18, Proposition 4.1], we will give an asymptotic formula for \( H^+ \) at first. We shall follow Li [18, §4] and McKee–Sun–Ye [19, §6] more closely, so readers who are familiar with their works can safely skip this section at a first reading. As Li [18, §4] did, we have
\[ H^+(x) = H^+_1(x) + H^+_2(x) + O(T^{-A}), \]  
(8.1)
where
\[ H^+_1(x) = \frac{4TM}{\pi} \int_{T=-\infty}^{T} \int_{-\infty}^{\infty} \frac{1}{\cosh t} \cos(x \cosh \zeta) e\left(\frac{tM\zeta}{\pi}\right) e\left(\frac{T\zeta}{\pi}\right) dt d\zeta, \]  
(8.2)
and
\[ H^+_2(x) = \frac{4M^2}{\pi} \int_{T=-\infty}^{T} \int_{-\infty}^{\infty} \frac{t}{\cosh t} \cos(x \cosh \zeta) e\left(\frac{tM\zeta}{\pi}\right) e\left(\frac{T\zeta}{\pi}\right) dt d\zeta. \]  
(8.3)
In the following we only treat \( H^+_1(x) \), since \( H^+_2(x) \) is a lower order term which can be handled in a similar way. It is clear that
\[ H^+_1(x) = \frac{4MT}{\pi} \int_{-\infty}^{T} \hat{k} \left(\frac{-M\zeta}{\pi}\right) \cos(x \cosh \zeta) e\left(\frac{T\zeta}{\pi}\right) d\zeta \]
\[ = 4T \int_{-\infty}^{MT\pi} \hat{k}(\zeta) \cos \left(x \cosh \frac{\zeta \pi}{M}\right) e\left(-\frac{T\zeta}{M}\right) d\zeta, \]  
(8.4)
by making a change of variable \( -\frac{M\zeta}{\pi} \rightarrow \zeta \), here
\[ k(t) = \frac{1}{\cosh t}, \]  
(8.5)
and
\[ \hat{k}(\zeta) = \int_{-\infty}^{\infty} k(t) e(-t\zeta) dt, \]  
(8.6)
is its Fourier transform. Since \( \hat{k}(\zeta) \) is a Schwartz class function, one can extend the integral in \( [8, 4] \) to \((-\infty, \infty)\) with a negligible error term. Now let

\[
W(x) := T \int_{\mathbb{R}} \hat{k}(\zeta) \cos \left( x \cosh \frac{\zeta \pi}{M} \right) e \left( - \frac{T \zeta}{M} \right) d\zeta,
\]

then we have

\[
H_{\pm}^+(x) = 4W(x) + O(T^{-A}).
\]

**Lemma 8.1.**

(i) For \(|x| \leq T^{1-\varepsilon} M\), we have

\[
W(x) \ll_{\varepsilon, A} T^{-A}.
\]

(ii) Assume \( MT^{1-\varepsilon} \leq x \leq T^2 \), and \( T^{1/3+2\varepsilon} \leq M \leq T^{1/2} \). Let \( L_1, L_2 \in \mathbb{Z}_+ \). We have

\[
W(x) = \frac{MT}{\sqrt{x}} \sum_{\pm} e \left( \mp \frac{x}{2\pi} \pm \frac{T^2}{\pi x} \right) \sum_{l=0}^{L_1} \sum_{0 \leq l_1 \leq 2l} \sum_{0 \leq l_2 \leq L_2} c_{l_1, l_2} M^{2l-1} T^{4l_2-l_1} \quad \text{or}
\]

\[
\times \left[ \hat{k}(2^{l_1-l_2}) \left( \mp \frac{2MT}{\pi x} \right) - \frac{\pi^6}{6!} \pi^6 (\hat{k}(y))(2^{l_1-l_2}) \left( \mp \frac{2MT}{\pi x} \right) \right] + O \left( \frac{TM}{\sqrt{x}} \left( \frac{T^4}{x^3} \right)^{L_2+1} + T \left( \frac{M}{\sqrt{x}} \right)^{2L_2+3} + \frac{T^3 x^3}{M^{18}} \right)
\]

where \( c_{l_1, l_2} \) are constants depending only on the indices.

**Proof.** See Mckee–Sun–Ye [19, Proposition 6.1].

Now we estimate \( S(q, N; \delta) \). Let \( x = \frac{4\sqrt{MN}}{3} \) in the above lemma. Assume \( MT^{1-\varepsilon} \leq x \leq T^2 \). By choosing \( L_1, L_2 \) large enough (depending on \( \varepsilon \)) in \( [8, 9] \), the contribution to \( S \) from the first two error terms can be made as small as desired. We need to estimate the contribution from the last error term. By the support of \( v \), we may assume that \( x \ll \frac{(qT)^{3/2+\varepsilon}}{e} \). Note that for \( q \ll T^{1/4} \), we always have \( x \leq T^2 \). So the contribution from the last error term is bounded by

\[
\sum_{q \leq c} \frac{1}{q^2} \sum_{c} \sum_{t} |S(\delta a, 1; c)| \sum_{n \leq N} |A(n, 1)| N^{3/2} M^{18} c^3
\]

\[
\ll (qT)^{\varepsilon} \frac{T(qT)^6}{M^{18}} \sum_{q \leq c} \frac{1}{c^{1/2}} \ll (qT)^{\varepsilon} q^{7/2} TM \frac{T^6}{M^{19}} \ll (qT)^{\varepsilon} q^{3/2} TM,
\]

provided \( T^{1/3+2\varepsilon} \leq M \leq T^{1/2} \) and \( q \leq T^{1/6} \). In the finite series \( [8, 8] \), with our assumptions, we always have

\[
M^{2l-1} T^{4l_2-l_1} \ll 1.
\]

All the terms in \( [8, 8] \) are similar, and can be estimated in a similar way, so we will only work with the first term, that is, the term with \( l = l_1 = l_2 = 0 \). We are led to estimate

\[
\hat{S}(q, N; \delta) := \frac{TM}{(\delta T)^{1/4}} \sum_{q \leq c} \sum_{c} \sum_{t} e \left( \frac{\delta}{c} \right) \sum_{n} A(n, 1) \psi(n),
\]

\[
\times \sum_{n} \frac{\hat{b}_n}{c_1} \psi(n),
\]

where

\[
C = \frac{\sqrt{\delta N}}{T^{1-\varepsilon} M}.
\]
and
\[ \psi(y) = y^{-3/4-u}v \left( \frac{y}{N} \right) \sum_{\pm} e \left( \mp \frac{2\sqrt{\delta y}}{c} \pm \frac{T^2 c}{4\pi^2 \sqrt{\delta y}} \right) \hat{k} \left( \mp \frac{MTc}{2\pi^2 \sqrt{\delta y}} \right). \]  
(8.12)

Now we apply the Voronoi formula for the \( n \)-sum in (8.10), getting
\[ S(q, N; \delta) = \frac{T M}{\delta^{1/2}} \sum_{\frac{q}{c} \in C} \sum_{c \leq C} \frac{1}{c^{3/2}} \sum_{c_1} \sum_{c_2 \geq 1} \sum_{d(c)} \chi(a) e \left( \frac{\bar{b}q}{c} \right) e \left( \frac{\delta d a}{c} \right) \]  
\times \frac{c_1 \pi^{3/2}}{2} \sum_{n_1 | c_1, n_2 \geq 1} \frac{A(n_2, n_1)}{n_1 n_2} S(b, \pm n_2; c_1/n_1) \Psi^\pm \left( \frac{n_2^2 n_2}{c_1^3} \right), \]  
(8.13)

where \( \Psi^\pm(x) \) is defined as in (2.20) with \( \psi \) in (8.12), and \( \cal{T}_{c_1, n_1, n_2}^{\pm, \delta}(c, q) = \cal{T}_{c_1, n_1, n_2}^{\pm, \sigma}(c, q) \) with \( \sigma = 1 \).

Now, we will deal with \( \Psi^\pm(x) \), where \( x = \frac{n_2^2 n_2}{c_1^3} \). Since for \( q \ll T^{1/4}, \) we have
\[ xN = \frac{n_2^2 n_2}{c_1^3} N \geq NC^{-3} \geq M^3 T^{1-\varepsilon}. \]

By Lemma 2.5, we have
\[ \Psi^\pm(x) = \gamma_1 x^{2/3} \sum_{\sigma \in \{\pm\}} \int_0^\infty a_\sigma(y) e \left( \frac{\sigma 2\sqrt{\delta y}}{c} \pm 3(xy)^{1/3} \right) dy + \text{lower order terms}, \]
where
\[ a_\sigma(y) = y^{-13/12-u} v \left( \frac{y}{N} \right) e \left( -\sigma \frac{T^2 c}{4\pi^2 \sqrt{\delta y}} \right) \hat{k} \left( \sigma \frac{MTc}{2\pi^2 \sqrt{\delta y}} \right). \]

Note that for \( \Psi^+, \) when \( \sigma = 1 \) has no stationary points, so the contribution to \( \hat{S} \) is negligible; so does \( \Psi^- \) with \( \sigma = -1. \) Hence, we have
\[ \Psi^\pm(x) = \gamma_1 x^{2/3} \int_0^\infty a^\pm(y) e \left( \mp \frac{2\sqrt{\delta y}}{c} \pm 3(xy)^{1/3} \right) dy + \text{lower order terms}, \]  
(8.14)

where
\[ a^\pm(y) = v \left( \frac{y}{N} \right) e \left( \pm \frac{T^2 c}{4\pi^2 \sqrt{\delta y}} \right) \hat{k} \left( \mp \frac{MTc}{2\pi^2 \sqrt{\delta y}} \right) y^{-13/12-u}. \]  
(8.15)

By (8.10), to prove Proposition 7.1 we only need to show
\[ \hat{R} = \hat{R}(q, N; \delta) := \sum_{\frac{q}{c} \in C} \sum_{c \leq C} \frac{1}{c^{3/2}} \sum_{c_1} \sum_{n_1 | c_1, n_2 \geq 1} \frac{A(n_2, n_1)}{n_1 n_2} \]  
\times \mathcal{T}_{c_1, n_1, n_2}^{\pm, \delta}(c, q) \Psi^\pm_0 \left( \frac{n_2^2 n_2}{c_1^3} \right) \ll (qT)^{\varepsilon} q^3, \]  
(8.16)

where
\[ \Psi^\pm_0(x) := x^{2/3} \int_0^\infty a^\pm(y) e \left( \mp \frac{2\sqrt{\delta y}}{c} \pm 3(xy)^{1/3} \right) dy. \]  
(8.17)

Now we will use the stationary phase method to deal with (8.17). Denote
\[ \phi(y) = \mp \frac{2\sqrt{\delta y}}{c} \pm 3(xy)^{1/3}. \]  
(8.18)

By the first derivative of \( \phi \) and the support of \( y, \) we know \( \Psi^\pm_0(x) \) is negligible unless
\[ \frac{2}{3} \frac{(\delta^3 N)^{1/2}}{c^3} \leq x \leq 2 \frac{(\delta^3 N)^{1/2}}{c^3}, \]  
that is,
\[ \frac{2}{3} \frac{(\delta^3 N)^{1/2}}{n_2^2 c^3} \leq n_2 \leq 2 \frac{(\delta^3 N)^{1/2}}{n_2^2 c^3}. \]  
(8.19)
Since the support of $a^\pm$ is in $[N, 2N]$, we have
\[
\int_0^\infty a^\pm(y)c \left( \mp \frac{2\sqrt{cy}}{c} \pm (xy)^{1/3} \right) dy = \int_{\frac{x_0}{2}}^{\frac{x_0}{2}+}\left( \mp \frac{2\sqrt{cy}}{c} \pm (xy)^{1/3} \right) dy.
\] (8.20)

There is a stationary phase point $y_0 = x^2\varepsilon^6/\delta^3$ such that $\phi'(y_0) = 0$. Note that we have
\[
N \leq y \leq 2N, \quad \text{and} \quad c \ll C = \frac{\sqrt{3N}}{T^{1-\varepsilon}M}.
\]

Write $a(y) = a^\pm(y)$. Let $n_0 \in \mathbb{N}$ which will be chosen later. Simple calculus estimates give us
\[
\phi^{(r)}(y) \ll \frac{\sqrt{3N}}{c}N^{-r}, \quad \text{for} \quad r = 2, \ldots, 2n_0 + 3,
\]
and
\[
a^{(r)}(y) \ll N^{-13/12} \left( \frac{\delta^{1/2}N^{3/2}}{T^2c} \right)^{-r}, \quad \text{for} \quad r = 0, 1, \ldots, 2n_0 + 1,
\]
for $y \asymp N$. To apply Lemma 2.6 set
\[
M_0 = 10N, \quad T_0 = \frac{\sqrt{\delta N}}{c}, \quad N_0 = \frac{\delta^{1/2}N^{3/2}}{T^2c}, \quad U_0 = N^{-13/12}.
\] (8.21)

Note that $\phi''(y) \gg T_0M_0^{-2}$ for $y \in \left[ \frac{1}{4}x^2\varepsilon^6/\delta^3, \frac{9}{4}x^2\varepsilon^6/\delta^3 \right]$, and the condition $N_0 \geq M_0^{1+\varepsilon}/\sqrt{T_0}$ is consistent with our assumption $c \leq C$ when $M \geq T^{1/3+2\varepsilon}$.

We are ready to apply Lemma 2.6 (where we take $n = n_0$). The main term of the integral in (8.20) is
\[
e\frac{e^{(\phi(y_0) + 1/8)}}{\sqrt{\phi''(y_0)}} \left( a(y_0) + \sum_{j=1}^{n_0} \varpi_{2j} \frac{(-1)^j(2j-1)!!}{(4\pi i\lambda_2)^j} \right),
\] (8.22)
where $\lambda_2 = |\phi''(y_0)|/2$. Notice we have used
\[
\gamma - \alpha \asymp \beta - \gamma \asymp M_0,
\]
with $\alpha = \frac{1}{4}x^2\varepsilon^6/\delta^3$, $\beta = \frac{9}{4}x^2\varepsilon^6/\delta^3$, and $\gamma = y_0 = x^2\varepsilon^6/\delta^3$. To save time in estimates, notice that there are no boundary terms here. That is, the terms related to $H_i$ in Lemma 2.6 will vanish. This is due to the compact support of $a$, with itself and all of its derivatives zero at $\frac{1}{4}x^2\varepsilon^6/\delta^3$ and $\frac{9}{4}x^2\varepsilon^6/\delta^3$. The sum of the four error terms in Lemma 2.6 can be simplified to
\[
O \left( \frac{U_0M_0^{2n_0+2}}{T_0^{n_0+1}N_0^{2n_0+1}} \right) = O \left( e^{4n_0+2}2^{4n_0+2}N^{-\frac{n_0}{2}}N^{-\frac{13}{2}}\delta^{-\frac{n_0}{2}}N^{-\frac{13}{2}}\delta^{-\frac{n_0}{2}-1} \right).
\] (8.23)

This estimate uses the current assumptions on $c$, and the size of $N$ compared to $q$ and $T$. Note that
\[
M_0 \gg N_0, \quad \text{if} \quad q \ll T^{1/4}.
\]

We now need to deal with the $\varpi_{2j}$ terms in (8.22). Recall the expression for $\varpi_{2j}$ in equation (2.24). Here we take $2 \leq 2j \leq 2n_0$. One can see from (2.24) that the main term from $\varpi_{2j}$ is $a^{(2j)}(y_0)$. (a given in equation (8.15), and $\phi$, above (8.18), take the place of $g$ and $f$ in Lemma 2.6. Further $y_0$ takes the place of $g$.) Using the above estimates, we have
\[
\varpi_{2j} - \frac{a^{(2j)}(y_0)}{(2j)!} = O \left( \frac{U_0}{M_0N_0^{2j-1}} \right).
\] (8.24)

The constant ultimately depends on $n_0$ and we have used $M_0 \gg N_0$. To estimate this error term contribution to $\hat{R}$, we must divide by $\lambda_2^{1+1/2}$ and sum over $j$. (See (8.22).) Since $y_0 \asymp N$, we have $\lambda_2 \asymp \frac{\delta^{1/2}}{cN^{1/2}}$. We have then that this contribution is
\[
O \left( N^{-\frac{3j}{4}} \left( \frac{T^2c}{\delta^{1/2}N^2} \right)^{2j-1} \left( \frac{cN^2}{\delta^{1/2}} \right)^{j+1} \right) = O \left( 3^{j-\frac{1}{2}}T^{4j-2}N^{-\frac{3j}{4}-\frac{1}{2}}\delta^{-\frac{3j}{4}+\frac{1}{2}} \right).
\] (8.25)

We must now estimate the $a^{(2j)}(y_0)$ term in $\varpi_{2j}$ in (8.22). Let $i_j$ be the number of times $v \left( \frac{y^j}{N} \right)$ is differentiated plus the number of times $y^{-13/12-u}$ is differentiated. So at every differentiation either
the factor $\frac{1}{y}$ comes out, or up to a constant, the factor $\frac{1}{y}$ comes out. Notice that $\frac{1}{y} \approx \frac{1}{2}$. Let $i_2$ be the number of times $\hat{k\left(\frac{MTc}{2\pi y^{2/3}}\right)}$ is differentiated, and denote $i_3$ to be the number of times $e\left(\frac{-T^2c}{4\pi^2y^{3/2}}\right)$ is differentiated. Then $i_1 + i_2 + i_3 = 2j$, and neglecting coefficients (which ultimately depend on $n_0$), $a^{(2j)}(y_0)$ is the sum over all combinatorial possibilities of

$$N^{-3/2-i_1}\left(\frac{MTc}{\delta^{1/2}N^{3/2}}\right)^{i_2}\left(\frac{T^2c}{\delta^{3/2}N^{3/2}}\right)^{i_3}.$$  

The main term is when $i_3 = 2j$ and we will estimate this separately, below. So we can assume in all terms, now, that $i_1 + i_2 \geq 1$. To estimate this error term, which is all but one term in $a^{(2j)}(y_0)$, as before, in [8.22], we must divide by $\lambda_2^{\frac{1}{2}+\frac{4}{N^{3/2}}}$ where $\lambda_2 \approx \frac{1}{cN^{3/2}}$ with our assumption on $y_0$. We have then a sum of error terms which are all

$$O\left(M^{i_2} e^{j+i_2+\frac{j}{2}T^{i_2}+2i_3} N^{2j-i_1-\frac{j}{2}i_2-\frac{i}{2}i_3-\frac{i}{2}\delta-\frac{i}{2}(j+i_2+i_3+\frac{j}{2})}\right).$$

Using $i_3 = 2j - i_1 - i_2$, the above bound will be

$$O\left(M^{i_2} e^{j-i_1+\frac{j}{2}T^{j-2i_1-2i_2} N^{-\frac{2}{3}j-i_1-\frac{2}{3}i_2-\frac{2}{3}i_3-\frac{2}{3}\delta-\frac{2}{3}(j+i_2+i_3+\frac{j}{2})}\right).$$

(8.26)

We will bound the contribution of all these error terms to $\tilde{R}$ in the next section.

This leaves the main term of $a^{(2j)}(y_0)$ (where $i_3 = 2j$ and $i_1 = i_2 = 0$) which is

$$a_{2j}(y_0) := \alpha_j \left(\frac{T^2c}{\delta^{1/2}y_0^{3/2}}\right)^{2j} v\left(\frac{y_0}{N}\right) \frac{\hat{k\left(\frac{MTc}{2\pi^2\sqrt{\delta}y_0^{3/2}}\right)}}{e\left(\frac{-T^2c}{4\pi^2\sqrt{\delta}y_0^{3/2}}\right)^{y_0^{13/12-u}}}$$

(8.27)

where the constant $\alpha_j$ depends on $j$ which ultimately can be bounded in terms of $n_0$. As Li [18 §4] did, we can not bound these terms trivially. Instead, we will apply the Voronoi formula a second time. This will be done in section 10.

9. Contribution from the error terms

In this section, we will bound the contribution of these error terms [8.23, 8.25], and [8.26] to $\tilde{R}$, see [8.16]. To do this, we need to recall the result of Blomer [1] for $T_{c_1,n_1,n_2}(c,q)$.

Lemma 9.1. Assume $(q, \delta) = 1$, $\delta_0|c_2$, $(r', c_2) = 1$. Then

$$T_{c_1,n_1,n_2}(c,q) = e\left(\pm \frac{n_2^2c_2(k)^2c_2}{c'd'}\right) \frac{\varphi(c_1)\varphi(c_1/n_1)}{\varphi(c')} \frac{\mu(\delta)}{\delta} r' q \chi(-1)$$

$$\times \sum_{\substack{f_1, f_2, f_2' = r' \\ (f_1, f_2, f_2') = 1 \\ (f_1, f_2, f_2') = 1, f_2 = 1 \delta_0 = 1, f_2 = 1 \delta_0 = 1}} \mu(f_2) f_1 e\left(\pm \frac{(n_1c_2)^2f_2n_2\delta_0^{c_2}}{f_1^{c'}}\right) \chi_{c_1,n_1,n_2}(c,q),$$

(9.1)

where

$$\chi_{c_1,n_1,n_2}(c,q) := \sum_{g_5 \equiv q_0(q)} \chi(g_5g_6) \chi(g_5r' + c_2g_6r' + c_2n_2c_2')$$

$$\times \chi(r'g_6 \equiv n_2c_2') e\left(\delta n_1c_2'g_5 + c_2g_6q\right).$$

(9.2)

Furthermore, we have

$$\chi_{c_1,n_1,n_2}(c,q) = \chi h\left(\frac{h}{\varphi(k)}\right) R_k(n_2c_2') R_k(c_2) R_k(n_1c_2') H(e\left(\frac{r'k}{\varphi(k)}\delta n_2c_2') c_2g_6, k, \ell).$$

(9.3)

Recall that $h, k, \ell$ are defined in [6.22], and the relations of the variables are summarized in [8.3]. If one of the conditions $\delta_0|c_2$ and $(r', c_2) = 1$ is not satisfied, then $T_{c_1,n_1,n_2}(c,q) = 0$.

Proof. See Blomer [1] Lemma 12, eq. (50), and §7. }
By the definition of $\mathcal{V}_{c_1,n_1,n_2}^{\pm,\delta}(c,q)$, we have the following trivial bound

$$|\mathcal{V}_{c_1,n_1,n_2}^{\pm,\delta}(c,q)| \leq q^2.$$ 

A trivial estimate shows

$$\mathcal{T}_{c_1,n_1,n_2}^{\pm,\delta}(c,q) \ll \frac{\delta_0 r^2 q^3}{c_2 n_1^2} \tau_3(c).$$

The contribution to (8.16) from the error term in (8.25) is bounded by

$$E_1 = \sum_{q \mid c \mid c \in \mathcal{C}} \frac{1}{c^2} \sum_{c_1 \mid c} c_1 \sum_{n_1 \mid c_1} \sum_{n_2} \frac{|A(n_2, n_1)|}{n_1 n_2} \frac{\delta_0 r^2 q^3}{c_2 n_1^2} \tau_3(c) \left( \frac{n_1^2 n_2}{c_1^2} \right)^{\frac{3}{5}} c^{3n_0 + 2} T^{4n_0 + 2} N^{-\frac{3}{2} n_0 - \frac{5}{2} \delta - \frac{3}{2} n_0 - 1}$$

$$\leq (qT)^{\varepsilon} T^{4n_0 + 2} N^{-\frac{3}{2} n_0 - \frac{13}{2} \delta - \frac{3}{2} n_0 - 1} q \sum_{q \mid c \mid c \in \mathcal{C}} c^{3n_0 + \frac{1}{2}} \sum_{c_1 \mid c} c_1 \sum_{n_1 \mid c_1} \sum_{n_2} \frac{|A(n_2, n_1)|}{n_1^{2/3}} \frac{1}{n_2^{1/3}} \frac{c^{3n_0 + \frac{1}{2}}}{n_1^{2/3}} c_1^2 c_2,$$

where $n_2$ satisfies (8.19). We have

$$E_1 \ll (qT)^{\varepsilon} T^{4n_0 + 2} N^{-\frac{3}{2} n_0 - \frac{13}{2} \delta - \frac{3}{2} n_0 - 1} q \sum_{q \mid c \mid c \in \mathcal{C}} c^{3n_0 + \frac{1}{2}} \sum_{c_1 \mid c} c_1 \sum_{n_1 \mid c_1} \frac{1}{n_1^{2/3}} \frac{c^{3n_0 + \frac{1}{2}}}{n_1^{2/3}} c_1^2 c_2,$$

$$\leq (qT)^{\varepsilon} T^{4n_0 + 2} N^{-\frac{3}{2} n_0 - \frac{13}{2} \delta - \frac{3}{2} n_0} c^{3n_0 + \frac{1}{2}}.$$

By (8.11) and (3.10), (noting that we can let the $\varepsilon$ in the upper bound of $C$ be much smaller than the $\varepsilon$ in the lower bound of $M$), we have

$$E_1 \ll (qT)^{\varepsilon} q^{3/2} T^{n_0 + 1} M^{-3n_0 - 5/2} \ll (qT)^{\varepsilon} q^{3/2},$$

provided $T^{1/3 + \varepsilon} \ll M \ll T^{1/2}$ and $n_0 \geq 1/\varepsilon$. Similarly, the contribution to (8.16) from the error term in (8.25) is bounded by

$$E_2 = \sum_{q \mid c \mid c \in \mathcal{C}} \frac{1}{c^2} \sum_{c_1 \mid c} c_1 \sum_{n_1 \mid c_1} \sum_{n_2} \frac{|A(n_2, n_1)|}{n_1 n_2} \frac{\delta_0 r^2 q^3}{c_2 n_1^2} \tau_3(c)$$

$$\times \left( \frac{n_1^2 n_2}{c_1^2} \right)^{\frac{3}{5}} c^{3j - \frac{3}{2} j - 2 T^{4j - 2} N^{-\frac{3}{2} j - \frac{1}{2} j - \frac{3}{2} j + \frac{1}{4} q} \sum_{q \mid c \mid c \in \mathcal{C}} c^{3j - 1}$$

$$\ll (qT)^{\varepsilon} T^{4j - 2} N^{-\frac{3}{2} j + \frac{1}{2} j - \frac{3}{2} j + \frac{1}{4} q} \sum_{q \mid c \mid c \in \mathcal{C}} c^{3j} \ll (qT)^{\varepsilon} T^{4j - 2} M^{-3j} \ll (qT)^{\varepsilon} q^{3/2} T^{j - 2} M^{-3j} \ll (qT)^{\varepsilon} q^{3/2}.$$
for \(0 \leq j \leq n_0\), provided \(T^{1/3} \ll M \ll T^{1/2}\). Finally, the contribution to (8.10) from the error term in (8.26) is bounded by

\[
E_3 = \sum_{q \mid c, c \in C} \frac{1}{c^3} \sum_{c_1 | c} c_1 \sum_{n_1 | c_1, n_2} \left| A(n_2, n_1) \right| \frac{2^r q}{c_1^2 n_1} \frac{\tau_3(c)}{n_1 n_2} \frac{n_1^2 n_2}{c_1^2} \frac{1}{T^{3/2}} \\
\times M^{1/2} c^{3j-i+1/2} T^{4j-2i_1-i_2} N^{-\frac{1}{2}} j^{3j-2i_1-i_2} \delta^{-\frac{1}{2} j+\frac{1}{2} i_1} \leq (qT)^{\varepsilon} q^{\frac{1}{2}} M^{-3j+i_1+i_2-1} T^{j-i-1} N^{-2i_1+i_2} \delta^{\frac{1}{2}} \delta^{\frac{1}{2}},
\]

for \(1 \leq i_1 + i_2 \leq 2j\), provided \(T^{1/3} \ll M \ll T^{1/2}\). We finish the estimate of these error terms.

### 10. Completion of the proof of Theorem 1.3 and Theorem 1.1

In this section, we will give the proof of Theorem 1.3 and Theorem 1.1. At first, we will estimate the contribution to \(\mathcal{R}\) of \(a_{2j}(y_0)\) in (8.10). To bound this, we only need to estimate

\[
\mathcal{R}_j = \mathcal{R}_j(q, N; \delta) := \sum_{c \in C} \sum_{c_1 | c} c_1 \sum_{n_1 | c_1, n_2 \geq 1} A(n_2, n_1) \frac{1}{n_1 n_2} \times x^{2/3} a_{2j}(y_0) \frac{\lambda_{j, N}^{\pm, \delta}}{\lambda_{j, N}^{1/2}} \frac{\kappa T_j^{\pm, \delta}}{c_1, n_1, n_2}(c, q),
\]

for all \(0 \leq j \leq n_0\), where \(x = \frac{n_1^2 n_2}{c_1}\), \(y_0 = \frac{x^2}{\delta}\), and \(\lambda_2 = \frac{x^{j/2}}{12c_0^{3/2}}\). Inserting these values, together with (8.27), we have

\[
\mathcal{R}_j = 12jT^{1/2} \delta^{2j + 3/4 + 3u} \sum_{c \in C} \sum_{c_1 | c} c_1 \sum_{n_1 | c_1, n_2 \geq 1} A(n_2, n_1) \frac{1}{n_1 n_2} \times x^{2/3} a_{2j}(y_0) \frac{\lambda_{j, N}^{\pm, \delta}}{\lambda_{j, N}^{1/2}} \frac{\kappa T_j^{\pm, \delta}}{c_1, n_1, n_2}(c, q),
\]

where

\[
\mathcal{R}_j = \frac{1}{q} \sum_{f_1, f_2 \in C/q \delta_0} \frac{\mu(\delta_0) \chi(\delta)}{\delta_0 \delta_0 + 2 + 6u} \sum_{c_1 \mid c_1} \left( \frac{c_1^{3j}}{c_1^{2j} \delta_1} \right)^3 \frac{\varphi(f_1 f_2 c_1 c_2')(f_1 c_1^3)}{\varphi(f_1 f_2 c_1 c_2')},
\]

and

\[
\mathcal{R}_j = \frac{1}{T^{1/2}} \delta^{2j + 3/4 + 3u} \sum_{f_1, f_2 \in C/q \delta_0} \frac{\mu(\delta_0) \chi(\delta)}{\delta_0 \delta_0 + 2 + 6u} \sum_{c_1 \mid c_1} \left( \frac{c_1^{3j}}{c_1^{2j} \delta_1} \right)^3 \frac{\varphi(f_1 f_2 c_1 c_2')(f_1 c_1^3)}{\varphi(f_1 f_2 c_1 c_2')},
\]

By Lemma 9.1 after some simplification, we have

\[
\mathcal{R}_j = \frac{1}{q} \sum_{f_1, f_2 \in C/q \delta_0} \frac{\mu(\delta_0) \chi(\delta)}{\delta_0 \delta_0 + 2 + 6u} \sum_{c_1 \mid c_1} \left( \frac{c_1^{3j}}{c_1^{2j} \delta_1} \right)^3 \frac{\varphi(f_1 f_2 c_1 c_2')(f_1 c_1^3)}{\varphi(f_1 f_2 c_1 c_2')},
\]

and

\[
\mathcal{R}_j = \frac{1}{T^{1/2}} \delta^{2j + 3/4 + 3u} \sum_{f_1, f_2 \in C/q \delta_0} \frac{\mu(\delta_0) \chi(\delta)}{\delta_0 \delta_0 + 2 + 6u} \sum_{c_1 \mid c_1} \left( \frac{c_1^{3j}}{c_1^{2j} \delta_1} \right)^3 \frac{\varphi(f_1 f_2 c_1 c_2')(f_1 c_1^3)}{\varphi(f_1 f_2 c_1 c_2')},
\]

for \(1 \leq i_1 + i_2 \leq 2j\), provided \(T^{1/3} \ll M \ll T^{1/2}\). We finish the estimate of these error terms.
where \( \gamma_j = 12^{j+1} \chi(-1) \alpha_j \). Recall that the relations of the new variables and the old variables are

\[
c = c'_1 f_1 f_2 d'_2 \delta_0, \quad c_1 = c'_1 f_1 f_2 d'_2, \quad c_2 = c'_2 \delta_0, \quad r = f_1 f_2 d'_2 \delta_0, \quad n_1 = n'_1 f_2.
\]

Note that \( u \in [\varepsilon - i \log^2(qT), \varepsilon + i \log^2(qT)] \), so the appearance of \( u \) in the exponents is harmless. As in \( \text{[103]} \) we have the following four cases to handle. Since all these cases are similar, we will only deal with the main case, that is the case \( c'_1 = q, \ c'_2 = h = k = 1 \). Denote these terms in \( \text{[104]} \) as \( \tilde{R}_j \). Note that we have \( (d'_2 n'_1 n_2, q) = 1 \). Write \( f_1 = n'_1 g \). Then we have

\[
\tilde{R}_j = \gamma_j q^{3j-1} T^{4j} \delta^{3j+3/4+3u} \sum_{\delta_0' = \delta} \frac{\mu(\delta_0) \chi(\delta)}{\delta_0^{3j+2+6u}} \sum_{g n'_1 f_2 \in \mathbb{C}/(q \delta_0)} \left( \sum_{g n'_1 f_2 \equiv 1, \mu^2(g n'_1) = 1} (d'_2)^3 \right) \nonumber
\]

\[
\times \sum_{n'_2 = 1} A(n_2, n'_1 f_2) \frac{n'_1 f_2 d_0 d'_2 g n_2}{g \delta'} \left( \frac{MTqgd'_2}{2\pi^2 \delta_0 f_2 n'_1 n_2} \right) e\left( -\frac{T^2 q g d'_2}{4\pi^2 \delta_0 f_2 n'_1 n_2} \right). \tag{10.6}
\]

To remove the coprime condition \( (n_2, d'_2) = 1 \), we split the \( n_2 \)-sum into residue classes mod \( d'_2 \), and then detect the summation congruence condition by additive characters mod \( d'_2 \), getting

\[
\tilde{R}_j \ll (qT)^\varepsilon q^{3j-1} T^{4j} \delta^{3j+3/4} \sum_{\delta_0' = \delta} \frac{1}{\delta_0^{3j+2}} \sum_{g n'_1 f_2 \in \mathbb{C}/(q \delta_0)} \left( \sum_{g n'_1 f_2 \equiv 1, \mu^2(g n'_1) = 1} (d'_2)^3 \right) \sum_{a_1(d'_2)} \sum_{b_1(d'_2)} \left( -\frac{b_1 a_1}{d'_2} \right) \nonumber
\]

\[
\times \sum_{n'_2} A(n_2, n'_1 f_2) \frac{n_2 n_1}{d'_2} \left( \frac{b_1 n_2}{d'_2} \right) e\left( -\frac{n'_1 f_2 d_0 d'_2 g n_2}{g \delta'} \right) \left( \frac{MTqgd'_2}{2\pi^2 \delta_0 f_2 n'_1 n_2} \right) e\left( -\frac{T^2 q g d'_2}{4\pi^2 \delta_0 f_2 n'_1 n_2} \right). \nonumber
\]

Now by \( \text{[5.1]}, \text{[5.2]}, \text{[5.4]} \), we have

\[
\tilde{R}_j \ll \tilde{R}_j^{1,1} + \tilde{R}_j^{1,2}, \quad \tag{10.7}
\]
where

\[ \tilde{R}_j^{1.1} := (qT)^{3j} q^{3j-1} T^{3j} \delta^{3j+3/4} \sum_{\pm} \sum_{\delta_0, \delta' = 0} \frac{1}{\delta_0^{3j+2}} \sum_{g n'_1 f_2 \equiv C/(\varphi q_0)} \sum_{(g n'_1 f_2, g \delta') = 1} (d'_2)^{3j} \frac{1}{d'_2} \sum_{b_1(d'_2)} |S(0, -b_1; d'_2)| \]

\[ \times \frac{g^{3j-1}}{f_2^{3j+2}} \sum_{(d'_2)^{3j} \frac{1}{d'_2} \sum_{b_1(d'_2)} |S(0, -b_1; d'_2)|} (\delta')^3 N \]

\[ \times v \left( \frac{(n'_1 f_2)^4 \delta_0^2 n_2^2}{(\delta')^3 N} \right) \tilde{k} \left( \frac{MT q g d' g'}{2\pi^2 \delta_0 f_2 n_1^2 n_2} \right) e \left( \frac{-T^2 q g d' g'}{4\pi^2 \delta_0 f_2 n_1^2 n_2} \right), \] (10.8)

and

\[ \tilde{R}_j^{1.2} := (qT)^{3j} q^{3j+4} T^{3j} \delta^{3j+3/4} \sum_{\pm} \sum_{\delta_0, \delta' = 0} \frac{1}{\delta_0^{3j+2}} \sum_{g n'_1 f_2 \equiv C/(\varphi q_0)} \sum_{(g n'_1 f_2, g \delta') = 1} (d'_2)^{3j} \frac{1}{d'_2} \sum_{b_1(d'_2)} |S(0, -b_1; d'_2)| \]

\[ \times \frac{1}{\varphi(q)} \sum_{\psi(q)} \sum_{n_2} \frac{A(n_2, n'_1 f_2)}{n_2^{3j+1+2u}} e \left( \frac{b_1 n_2}{d'_2} \right) e \left( \frac{\pm n'_1 f_2 d_0 \delta_0 q n_2}{g \delta'} \right) \psi(n_2) \]

\[ \times v \left( \frac{(n'_1 f_2)^4 \delta_0^2 n_2^2}{(\delta')^3 N} \right) k \left( \frac{MT q g d' g'}{2\pi^2 \delta_0 f_2 n_1^2 n_2} \right) e \left( \frac{-T^2 q g d' g'}{4\pi^2 \delta_0 f_2 n_1^2 n_2} \right), \] (10.9)

We will focus on \( \tilde{R}_j^{1.2} \), since it turns out that \( \tilde{R}_j^{1.1} \) is easier and has a better upper bound. At first we need to remove the factor \( \psi(n_2) \) in the innermost sum of (10.10). Again, we split the \( n_2 \)-sum into residue classes mod \( q \), and then detect the summation congruence condition by additive characters mod \( q \), getting

\[ \tilde{R}_j^{1.2} = (qT)^{3j} q^{3j+4} T^{3j} \delta^{3j+3/4} \sum_{\pm} \sum_{\delta_0, \delta' = 0} \frac{1}{\delta_0^{3j+2}} \sum_{g n'_1 f_2 \equiv C/(\varphi q_0)} \sum_{(g n'_1 f_2, g \delta') = 1} (d'_2)^{3j} \frac{1}{d'_2} \sum_{b_1(d'_2)} |S(0, -b_1; d'_2)| \]

\[ \times \frac{g^{3j-1}}{f_2^{3j+2}} \sum_{(d'_2)^{3j} \frac{1}{d'_2} \sum_{b_1(d'_2)} |S(0, -b_1; d'_2)|} (\delta')^3 N \]

\[ \times v \left( \frac{(n'_1 f_2)^4 \delta_0^2 n_2^2}{(\delta')^3 N} \right) \tilde{k} \left( \frac{MT q g d' g'}{2\pi^2 \delta_0 f_2 n_1^2 n_2} \right) e \left( \frac{-T^2 q g d' g'}{4\pi^2 \delta_0 f_2 n_1^2 n_2} \right), \]
is the Kloosterman sum with character \( \psi \). Note that \( S(0, -b_1; d_2') \) is related to the Ramanujan sum, and \( S_\psi (0, -b_2; q) \) is related to the Gauss sum, inserting the upper bound for these sums, we have

\[
R_j^{1,2} \ll (qT)^{3/4}q^{3j+1}T^{4j}d_j^{3j+3/4} \sum_{\delta_0, \ell = \delta_0} \frac{1}{\delta_0^{j+2}} \sum_{n \equiv \delta_0 \pmod{\delta}} \sum_{\psi_0, \gamma \psi_0 / (\psi_0, \gamma)} \frac{(d')^j}{d'^2} \sum_{b_1, b_2} (b_1, d') \frac{1}{q} \sum_{b_2 (q)}
\]

\[
(10.10)
\]

Now we will handle the inner \( n_2 \)-sum, that is,

\[
\sum_{n_2} A(n_2, n'_2) e \left( \frac{b' n_2}{c'} \right) w_j(n_2),
\]

where

\[
w_j(y) := \frac{1}{y^{3j+1+2u}} v \left( \frac{n_1 f_2}{(\delta')^3} \right) \tilde{k} \left( \frac{M T q_{d_2} d'_2 \delta'}{2\pi^2 \delta_0 n_1 n_2} \right) e \left( -\frac{T^2 q_{d_2} d'_2 \delta'}{4\pi^2 \delta_0 n_1 n_2} \right),
\]

and

\[
\frac{b'}{c'} := \frac{b_1 q_{d_2} d'_2 + b_2 d'_2 q_{d_2} \pm n_1 f_2 \delta_0}{d'_2 q_{d_2}}, \quad \text{with} \quad (b', c') = 1, \quad \text{and} \quad c' |d'_2 q_{d_2} \delta'.
\]

We apply the Voronoi formula on \( GL(3) \) a second time, getting

\[
\sum_{n_2} A(n_2, n'_2) e \left( \frac{b' n_2}{c'} \right) w_j(n_2)
\]

\[
= \frac{c' \pi^{3/2}}{2} \sum_{l_1} \sum_{n'_1 f_2 l_1 = 1} A(l_2, l_1) l_1 l_2 S \left( \frac{1}{n'_1 f_2 l_1} \right) \mathcal{W}^\pm_j \left( \frac{l_1^2 l_2}{(c')^3 n'_1 f_2} \right),
\]

where \( \mathcal{W}^\pm_j \) is defined by (2.20) with \( \psi = w_j \). By the support of \( v \), we know \( w_j \) is supported in \([Y, \sqrt{2}Y]\), with

\[
Y := \frac{\sqrt{\delta_0 \delta_2}}{(n_1 f_2)^2 \delta_0} \geq 1.
\]

By the facts \( c' \leq d'_2 q_{d_2} \delta' \) and \( \delta_0 n'_1 f_2 d'_2 \ll C \), the bounds for \( N \) and \( C \), i.e., (3.10) and (8.11), and the bounds for \( q \) and \( M \), i.e., (7.1), writing \( x = \frac{l_1^2 l_2}{(c')^3 n'_1 f_2} \), we have

\[
x Y = \frac{l_1^2 l_2}{(c')^3 n'_1 f_2} \frac{\sqrt{\delta_0 \delta_2}}{\sqrt{C^3 \delta'}} \gg \frac{\sqrt{\delta_0 \delta_2}}{C^3 \delta'} \gg \frac{M^3 T^{-\varepsilon}}{q^3} \gg T^c.
\]

Now by Lemma 2.5, we have

\[
\mathcal{W}^\pm_j(x) = x \int_0^\infty w_j(y) \sum_{\ell = 1}^K \frac{\gamma \ell}{(xy)^{1/3}} e \left( \pm 3(xy)^{1/3} \right) dy + O \left( T^{-A} \right),
\]

for some large \( K \) and \( A \). We will only deal with the term with \( \ell = 1 \), since the others can be handled similarly. By (10.12), we are led to estimate

\[
\mathcal{W}^+_{j,0}(x) := x^{2/3} \int_0^\infty b(y) e (\phi_1(y)) dy, \quad \text{and} \quad \mathcal{W}^-_{j,0}(x) := x^{2/3} \int_0^\infty b(y) e (\phi_2(y)) dy,
\]
where
\[ b(y) := \frac{1}{y^{3j+4+2u}} e \left( \frac{(n_1' f_2)^3 \delta_0^3 \delta_0}{2\pi^2 \delta_0 f_2 n_1' y} \right) \]
and
\[ \phi_1(y) := -\frac{T^2 qgd_2^2 \delta'}{4\pi^2 \delta_0 f_2 n_1' y} + 3(xy)^{1/3}, \quad \text{and} \quad \phi_2(y) := -\frac{T^2 qgd_2^2 \delta'}{4\pi^2 \delta_0 f_2 n_1' y} - 3(xy)^{1/3}. \]
By the support of \( v \), we have
\[ \int_0^\infty b(y)e(\phi_i(y)) \, dy = \int_{\sqrt{2} Y}^\infty b(y)e(\phi_i(y)) \, dy, \quad \text{for} \ i = 1, 2. \]
For \( y \in [Y, \sqrt{2}Y] \), we have
\[ |\phi_i^{(r)}(y)| \leq C_r T_i/M_1^r, \quad |\phi^{(s)}(y)| \leq C_s U_1/N_1^s, \]
where
\[ T_1 = \max \left( \frac{T^2 qgd_2^2 \delta'}{\delta_0 f_2 n_1' Y}, (xy)^{1/3} \right), \quad M_1 = Y, \quad U_1 = \frac{1}{Y^{3j+4/3}}, \quad N_1 = Y. \]
Since we have
\[ \phi_i'(y) = \frac{T^2 qgd_2^2 \delta'}{4\pi^2 \delta_0 f_2 n_1' y^2} + x^{1/3} y^{-2/3} \ll T_1/M_1, \]
by partial integration \( r \) times, we have
\[ W_{j,0}^+(x) \ll \frac{x^{2/3} U_1}{T_1^r} \ll \frac{x^{2/3} U_1}{T_1^{r-2}} \ll T^{-A}, \]
for sufficient large \( r \). So \( W_{j,0}^+(x) \) is negligible.

Now we turn to \( W_{j,0}^-(x) \). Since
\[ \phi_2''(y) = \frac{T^2 qgd_2^2 \delta'}{4\pi^2 \delta_0 f_2 n_1' y^2} - x^{1/3} y^{-2/3}, \]
if
\[ x \geq \frac{T^6(qgd_2 n_1' f_2 \delta_0)^3(n_1' f_2)^2}{10\pi^6(\delta')^3 N^2}, \quad \text{or} \quad x \leq \frac{T^6(qgd_2 n_1' f_2 \delta_0)^3(n_1' f_2)^2}{1100\pi^6(\delta')^3 N^2}, \]
one has
\[ |\phi_2''(y)| \gg T_1/M_1. \]
By the same argument, we show \( W_{j,0}^-(x) \) is negligible. For the remaining case
\[ \frac{T^6(qgd_2 n_1' f_2 \delta_0)^3(n_1' f_2)^2}{1100\pi^6(\delta')^3 N^2} \leq x \leq \frac{T^6(qgd_2 n_1' f_2 \delta_0)^3(n_1' f_2)^2}{10\pi^6(\delta')^3 N^2}, \quad \text{i.e.} \quad \frac{L_2}{1100} \leq l_2 \leq \frac{L_2}{10}, \]
with
\[ L_2 = \frac{T^6(qgd_2 n_1' f_2 \delta_0)^3(n_1' f_2)^3(\delta')^3}{\pi^6(\delta')^3 N^2 l_1^2}, \]
we have
\[ |\phi_2''(y)| \gg T_1/M_1^2, \]
for any \( y \in [Y, \sqrt{2}Y] \). Note that in this case we have
\[ T_1 \approx \frac{T^2 qgd_2^2 \delta'}{\delta_0 f_2 n_1' Y}, \quad \text{and} \quad x \approx T_3/Y. \]
Therefore, by the second derivative test [11, Lemma 5.1.3 or Lemma 5.5.6], we have
\[ W_{j,0}^-(x) \ll \frac{x^{2/3} U_1}{(T_1/M_1^2)^{1/2}} \ll \frac{T^{3/2}}{Y^{3j+1}} \ll T^3(qgd_2)^{3/2}(n_1' f_2)^{6j+2} \delta_0^{6j+2} \delta_0^2 (\delta^3 N)^{-\frac{2j}{3} - \frac{2}{4}}. \]
Combining (10.10), (10.14), and (10.17), and invoking the trivial bound for the Kloosterman sum in (10.14), one concludes that

\[
\hat{R}^{1,2}_j \ll (qT)^{\epsilon_2} q^{3j+1/T} T^{4j} \delta^{3j+3/4} \sum_{d' \leq \delta} \sum_{d'_2 \leq C/q} \frac{1}{d'_2} \sum_{g n_1 f_2 \in \mathcal{C}/(q\delta_0)} \sum_{(g n_1 f_2, q\delta_0) = 1} g^{3j-1} f_2^{3j+2} (n_1')^{3j+3} \\
\times \sum_{d'_2 \leq C/(q g n_1' f_2)} \frac{(d'_2)^{\delta_0}}{d'_2} \sum_{b_1(d'_2)} \sum_{b_2(q)} \frac{1}{q} \sum_{c'} \sum_{l_1 c' n_1' l_2 = 1} |A(l_2, l_1)| l_1 l_2 \\
\times \frac{n_1' f_2 c'}{l_1} \left( q g d'_2 \right)^{3/2} (n_1 f_2)^{\delta_0} \delta_0 \delta_0 \delta_0 \left( \delta_0 N \right)^{-\frac{3j}{2}} \frac{g}{\delta_0} \frac{q}{2}.
\]

This proves Proposition 7.1, and hence, Theorem 1.3.

By (2.12), we have

\[
\hat{R}^{1,2}_j \ll (qT)^{\epsilon_2} q^{3j+9/2} T^{4j+3} N^{-3j/2-5/4} \delta^{-3j/2+1/2}.
\]

And by (8.11) and (8.10), we have

\[
\hat{R}^{1,2}_j \ll (qT)^{\epsilon_2} \delta^{11/4} NT^{-3j-2} M^{-3j-9/2} \ll (qT)^{\epsilon} q^{3j+3/2} M^{-3j-9/2} \ll (qT)^{\epsilon_3} T^{1/3+\epsilon} \leq M \leq T^{1/2}. \tag{10.18}
\]

This proves Theorem 1.1 and hence, Theorem 1.3.

Proof of Theorem 1.1 One can use Theorem 1.2 and Theorem 1.3 directly to give a hybrid subconvexity bound with \( \theta = 1/70 \). To get a better bound, that is, to prove Theorem 1.1 with \( \theta = (35 - \sqrt{1057})/56 \), which we fix from now on, we will modify the proof of Theorem 1.3. At first, note that if \( q \geq T^{\theta/(1/4 - \theta)} = T^{(\sqrt{1057} - 23)/44} \), then

\[
q^{5/4} T^{3/2} \leq (qT)^{3/2 - \theta}.
\]

Hence in this case, Theorem 1.1 follows from Theorem 1.2. Now we assume

\[
q \leq T^{\theta/(1/4 - \theta)} = T^{(\sqrt{1057} - 23)/44} < T^{1/4}. \tag{10.19}
\]

As in [7], we only need to prove

\[
\sum_{u_j \in \mathbb{N}^n(q)} L(1/2, \phi \times u_j \times \chi) + \frac{1}{4\pi} \int_{T - M}^{T + M} |L(1/2 + it, \phi \times \chi)|^2 dt \ll_{\phi, \epsilon} q^{3/2-\theta} T M (qT)^{\epsilon},
\]

provided

\[
T^{1/2-\theta} \ll M \ll T^{1/2}. \tag{10.20}
\]

As in the proof of Proposition 7.4, we only need to bound (8.9), (9.4), (9.5), and (9.6), and under our new assumptions (10.19) and (10.20). It’s easy to see that the bound in (8.9) will be \( q^{1/2-\theta} T M (qT)^{\epsilon} \) now. Moreover, (9.4), (9.5), and (9.6) are easy to handle too. Now we consider (10.18). By (10.20), we have

\[
\hat{R}^{1,2}_j \ll (qT)^{\epsilon_2} q^{3j+1/T} M^{-3j-2} \ll (qT)^{\epsilon} q^{3j+1/T} M^{-3j-2} \ll (qT)^{\epsilon} q^{2+\theta} T^{2\theta - \frac{5}{2}}.
\]

So we want \( q \leq T^{(\frac{4-2\theta}{4} + \theta)} \), which coincides with (10.19) with our choice of \( \theta \). Now we complete the proof of Theorem 1.1.

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