Low Space External Memory Construction of the Succinct Permuted Longest Common Prefix Array

German Tischler
Max Planck Institute of Molecular Cell Biology and Genetics, Pfotenhauerstraße 108, 01037 Dresden, Germany
tischler@mpi-cbg.de

Abstract. The longest common prefix (LCP) array is a versatile auxiliary data structure in indexed string matching. It can be used to speed up searching using the suffix array (SA) and provides an implicit representation of the topology of an underlying suffix tree. The LCP array of a string of length $n$ can be represented as an array of length $n$ words, or, in the presence of the SA, as a bit vector of $2^n$ bits plus asymptotically negligible support data structures. External memory construction algorithms for the LCP array have been proposed, but those proposed so far have a space requirement of $O(n)$ words (i.e. $O(n \log n)$ bits) in external memory. This space requirement is in some practical cases prohibitively expensive. We present an external memory algorithm for constructing the $2^n$ bit version of the LCP array which uses $O(n \log \sigma)$ bits of additional space in external memory when given a (compressed) BWT with alphabet size $\sigma$ and a sampled inverse suffix array at sampling rate $O(\log n)$. This is often a significant space gain in practice where $\sigma$ is usually much smaller than $n$ or even constant. We also consider the case of computing succinct LCP arrays for circular strings.

1 Introduction

The suffix array (SA) and longest common prefix array (LCP) were introduced as a lower memory variant of the suffix tree (cf. [27]) for exact string matching using a precomputed index (cf. [19]). For a text of length $n$ both can be computed in linear time in internal memory (IM) (cf. [17,18,5]) and require $n$ words of memory each. For large texts the space requirements of SA and LCP in IM can be prohibitive. Compressed and succinct variants including compressed suffix arrays (see e.g. [13,22,12]), the FM index and variants (see [9,10,11]) and succinct LCP arrays (see [23]) use less space, but for practicality it is also crucial to be able to construct these data structures using affordable space requirements. Construction algorithms for compressed suffix arrays and the Burrows Wheeler transform (BWT, see [4]) using $o(n \log n)$ bits of space in IM (assuming $\sigma \in o(n)$) were introduced (see e.g. [14,21]). It is still unclear whether these algorithms scale well in practice. At the very least they require an amount of IM which
is several times larger than what is needed for the input text. External memory solutions for constructing the suffix array and LCP array have also been presented (see e.g. [10,15]). These algorithms require \(O(n)\) words \(O(n \log n)\) bits of external memory (EM). However, as for their IM pendants, this space requirement is large if the algorithms are used as a vehicle to obtain a compressed representation. Recently algorithms for constructing the BWT in EM without explicitly constructing a full suffix array were designed and implemented (see [8,25]). In this paper we present an algorithm for constructing a succinct LCP array in EM based on a BWT and sampled inverse suffix array while using \(O(n \log \sigma)\) instead of \(O(n \log n)\) bits of space in EM. Both, BWT and sampled inverse suffix array can be produced in space \(O(n \log \sigma)\) in external memory by the algorithm presented in [25,26]. In the final section of this paper we consider the extension of our algorithm to circular strings.

2 Definitions

Let \(\Sigma\) denote a totally ordered and ranked alphabet. w.l.o.g. we assume \(\Sigma = \{0, 1, \ldots, \sigma - 1\}\) for some \(\sigma > 0\). Further let \(s = s_0s_1 \ldots s_{n-1}\) denote a string of length \(|s| = n > 0\) over \(\Sigma\) s.t. the last symbol of \(s\) is the minimal symbol in \(s\) and does not appear elsewhere in \(s\). We use \(s[i]\) to denote \(s_i\) and \(s[i \ldots j]\) for \(s_is_{i+1} \ldots s_j\) for \(0 \leq i \leq j < n\). \(s[i \ldots j]\) denotes the empty string for \(i > j\). The \(i\)’th suffix of \(s\) denoted by \(\tilde{s}_i\) is the string \(s[i \ldots n - 1]\). Suffix \(\tilde{s}_i\) is smaller than \(\tilde{s}_j\) (denoted by \(\tilde{s}_i < \tilde{s}_j\)) if for the smallest \(k\) s.t. \(s[i+k] \neq s[j+k]\) we have \(s[i+k] < s[j+k]\). The suffix array \(SA\) of \(s\) is the permutation of \(0, 1, \ldots, n - 1\) s.t. \(\tilde{s}_{SA[i-1]} < \tilde{s}_{SA[i]}\) for \(i = 1, 2, \ldots, n - 1\). For two suffixes \(\tilde{s}_i\) and \(\tilde{s}_j\) with \(i \neq j\) the longest common prefix \(1cp(i, j)\) of the two is \(s[i + i + \ell - 1]\) for the smallest \(\ell\) s.t. \(s[i + \ell] \neq s[j + \ell]\). The array \(LCP\) of \(s\) is defined by \(LCP[i] = |1cp(SA[i-1], SA[i])|\) for \(i > 0\) and \(LCP[0] = 0\). The inverse suffix array \(ISA\) of \(s\) is defined by \(ISA[SA[i]] = i\) for \(0 \leq i < n\). The permuted LCP array \(PLCP\) of \(s\) is given by \(PLCP[i] = LCP[ISA[i]]\) for \(0 \leq i < n\) and \(PLCP[i] = 0\) otherwise. The Burrows Wheeler transform \(BWT\) of \(s\) is defined by \(BWT[i] = s((SA[i] + n - 1) \mod n)\) for \(0 \leq i < n\). Let \(C\) be the array of length \(\sigma\) s.t. \(C[a] = |\{i \mid s[i] = a\}|\) for \(a \in \Sigma\) and let \(D\) be an array of length \(\sigma + 1\) s.t. \(D[a] = \sum_{i < a} C[i]\) for \(0 \leq a \leq \sigma\). For a sequence \(t = t_0, t_1, \ldots, t_{k-1}\) for some \(k \geq 0\) let \(RANK_t(a, j) = |\{i | 0 \leq i < \min(j, k), t_i = a\}|\), i.e. the number of \(a\) elements in \(t\) up to but excluding index \(j\) and let \(SELECT_t(a, j) = \min\{i | RANK_t(a, i+1) = j + 1\}\) if \(0 \leq j < RANK_t(a, k)\) and undefined otherwise. \(LF\) is defined by \(LF(r) = ISA[(SA[r] + n - 1) \mod n]\). \(B\) is defined by \(B(a, i) = D[a] + RANK_{BWT}(a, i)\) for \(a \in \Sigma, 0 \leq i \leq n\) and
The first linear time algorithm for computing the LCP array from the suffix array and text appeared in [18]. One of the main combinatorial properties used by this algorithm is the fact that \( \text{PLCP}[i] \geq \text{PLCP}[i - 1] - 1 \) for \( 0 < i < n \). This property is also used in [23] to obtain a representation of the PLCP array using \( 2n + o(n) \) bits while allowing constant time access.

Let \( \zeta(0) = 1 \) and \( \zeta(i) = 0 \zeta(i - 1) \) for \( i > 0 \). The \( 2n \) bits in the data structure are the bit sequence \( K = \eta(n - 1) \) given by \( \eta(0) = \zeta(\text{PLCP}[0] + 1) \) and \( \eta(i) = \eta(i - 1)\zeta(\text{PLCP}[i] - \text{PLCP}[i - 1] + 1) \) for \( 0 < i < n \). The \( o(n) \) additional bits are used for a select index (cf. [20]) on \( K \). \( K \) stores the sequence of pairwise differences of adjacent PLCP values shifted by 1 in unary representation (the number \( i \) is represented as \( i \) zero bits followed by a 1 bit). The value \( \text{PLCP}[i] \) can be retrieved as \( \text{SELECT}_{K}(1, i) - 2(i + 1) - 1 \).

In [2] Beller et al present an algorithm for computing the LCP array in IM using a wavelet tree (see [12]). This algorithm runs for \( \ell_m + 1 \) rounds where \( \ell_m \) is the maximum LCP value produced. Round \( i \) for \( 0 \leq i \leq \ell_m \) sets \( \text{LCP}[r] \) for exactly those ranks \( r \) s.t. \( \text{LCP}[r] = i \), i.e. the values are produced in increasing order.

4 Computing the succinct PLCP array

In this section we modify the algorithm by Beller et al (cf. [2]) to produce the succinct \( 2n \) bit PLCP bit vector in EM. The main idea is to use the fact that the algorithm produces the LCP values in increasing order. It starts with a tuple \((\epsilon, (0, n))\) which denotes the empty word and the corresponding rank interval on the suffix array (the lower end 0 is included, the upper \( n \) is excluded). Round \( i \) takes the tuples from the previous round (or the start tuple for round 0) and considers all possible extensions by one symbol via backward search (cf. [9]), i.e. it produces \((aw, (l', r'))\) from \((w, (l, r))\) for each \( aw \) appearing in \( s \). All suffixes considered in round \( i \) starting by \( aw \) in the rank interval \((l', r')\) have a common prefix of length \( i + 1 \), while the suffixes at ranks \( l' - 1 \) and \( l' \) (for \( l \neq 0 \)) as well as at ranks \( r' - 1 \) and \( r' \) (for \( r' < n \)) have a common prefix of at most length \( i \). Based on this insight we can set \( \text{LCP}[l'] \) and \( \text{LCP}[r'] \) to \( i \), if they have not already been set in a previous round. In the tuples the first (string) component is only provided for the sake of exposition, the algorithm does not require or use it. In addition the algorithm prunes away intervals when a respective LCP
value (Beller et al use the upper bound $r'$ for setting new values in [2], we in this paper use the lower bound $l'$ as it simplifies the transition to EM) is already set.

The succinct PLCP array $K$ contains $n$ zero and $n$ one bits. The one bits mark positions in the text (remember PLCP is in text order). The zero bits encode the differences between adjacent PLCP values shifted by 1. For computing this bit vector assume that we start off with a vector of $n$ one bits. The information we need in addition is in front of which 1 bit we have to insert how many 0 bits. If $PLCP[i]$ is not smaller than $PLCP[i - 1]$, then we have to add $PLCP[i] - PLCP[i - 1] + 1$ zero bits just in front of the $i + 1$st 1 bit. In the algorithm we can achieve this by starting to add 0 bits for ranks which did not have their value set in a previous round but which do have the value for the rank of the previous position set in the current round. We call this adding a rank to the active set. We stop adding 0 bits for a rank in the round in which the value for the rank itself gets set, which we call removing a rank from the active set. Figure 1 shows an algorithm implementing this approach in IM. A wavelet tree (cf. [12]) for $BWT$ can be used to compute the backstep, rank and select functions in time $O(\log \sigma)$ and to determine the set of symbols occurring in any index interval on $BWT$ in time $O(\log \sigma + o)$ where $o$ is the number of distinct symbols in the interval.

In the following we show how to adapt this algorithm so it becomes usable in EM and requires no more than $O(n \log \sigma)$ space in EM while using $O(\sigma \log n)$ bits of IM. This means we need to make sure that all data structures used in EM are accessed in a purely sequential way and none use $\omega(n \log \sigma)$ space. In particular we need to consider the representation and access patterns of the queues $Q$ and $NQ$, the Burrows Wheeler transform $BWT$ of $s$, the sets $S$, $T$ and $activeSet$ and the counter array for zero bits $PD$.

For some of the representations we will use Elias $\gamma$ code (cf. [7]) and the following result proven in [26].

**Lemma 1 ([26]).** Let $G$ denote an array of length $\ell$ such that $G[i] \in \mathbb{N}$ for $0 \leq i < \ell$ and $\sum_{i=0}^{\ell-1} G[i] = s$ for some $s \in \mathbb{N}$. Then the $\gamma$ code for $G$ takes $O(\ell + s)$ bits.

This means we can represent any strictly increasing sequence $x_0, x_1, \ldots, x_{k-1}$ of numbers from 0, 1, \ldots, $N$ for $N \in O(n)$ and $k > 0, k \in O(n)$ in $O(n)$ bits by storing the differences $x_i - x_{i-1}$ for $i = 0, 1, \ldots, k - 1$ in $\gamma$ code were we assume $x_{-1} = -1$.

- The queue $NQ$ is not produced in increasing order in the algorithm as stated in Figure 1 (meaning if $(l_1, r_1)$ is enqueued right after $(l_0, r_0)$ then we cannot assume $l_1 \geq r_0$). If however the queue $Q$ is in increasing order
PLCP\text{INTERNAL}(\text{BWT}, n, \text{ISA})

1. \((Q, \text{activeSet}, S, PD) \leftarrow (\emptyset, \emptyset, \emptyset, \emptyset)\)
2. \text{Q.\text{ENQUE}}((0, n))
3. \textbf{while} \text{Q.\text{EMPTY}}() = \text{false} \textbf{do}
   \hspace{1em} > \text{Queue for next round and ranks set in this round}
   \hspace{1em} (NQ, T) \leftarrow (\emptyset, \emptyset)
4. \textbf{while} \text{Q.\text{HASNEXT}}() \textbf{do}
   \hspace{1em} (1, r) \leftarrow \text{Q.\text{NEXT}}()
5. \hspace{1em} \textbf{foreach} \text{sym} \in \{\text{BWT}[i] \mid 1 \leq i < r\} \textbf{do}
6. \hspace{1em} (1', r') \leftarrow \text{BACKSTEP}(\text{sym}, (1, r))
7. \hspace{1em} \textbf{if} \text{S.\text{CONTAINS}}(1') = \text{false} \textbf{then}
8. \hspace{1em} \textbf{get} 1'\text{src} \text{ s.t. } \text{LF}(1'\text{src})=1'
9. \hspace{1em} \textbf{this is the smallest } i \text{ s.t. } l \leq i < r \text{ and } \text{BWT}[i]=\text{sym}
10. \hspace{1em} 1'\text{src} \leftarrow \text{SELECT}_{\text{BWT}}(\text{sym}, \text{RANK}_{\text{BWT}}(\text{sym}, 1))
11. \hspace{1em} \textbf{mark} 1' \text{ as to be set in this round}
12. \hspace{1em} T.\text{INSERT}(1')
13. \hspace{1em} \textbf{if} \text{S.\text{CONTAINS}}(1'\text{src}) = \text{false} \textbf{then}
14. \hspace{1em} \text{activeSet.\text{INSERT}}(1'\text{src})
15. \hspace{1em} NQ.\text{ENQUE}((1', r'))
16. \textbf{Increment number of 0 bits for ranks in active set}
17. \textbf{foreach} \text{r} \in \text{activeSet} \textbf{do}
18. \hspace{1em} \textbf{if} \text{PD.\text{CONTAINS}}(\text{r}) \textbf{then}
19. \hspace{1em} PD[\text{r}] \leftarrow \text{PD}[\text{r}] + 1
20. \hspace{1em} \textbf{else} PD[\text{r}] \leftarrow 1
21. \hspace{1em} \textbf{Remove ranks set in this round from active list}
22. \hspace{1em} \textbf{and update set of ranks finished}
23. \hspace{1em} \textbf{foreach} \text{r} \in T \textbf{do}
24. \hspace{1em} \textbf{if} \text{activeSet.\text{CONTAINS}}(\text{r}) \textbf{then}
25. \hspace{1em} \text{activeSet.\text{REMOVE}}(\text{r})
26. \hspace{1em} S.\text{INSERT}(\text{r})
27. \hspace{1em} Q \leftarrow NQ
28. \textbf{Produce succinct bit vector in text order}
29. \hspace{1em} (i, K) \leftarrow (0, \emptyset)
30. \hspace{1em} \textbf{for} \text{p} \leftarrow 0 \textbf{to} n - 1 \textbf{do}
31. \hspace{1em} r \leftarrow \text{ISA}[\text{p}]
32. \hspace{1em} \textbf{if} \text{PD.\text{CONTAINS}}(\text{r}) \textbf{then}
33. \hspace{1em} \textbf{for} \text{j} \leftarrow 1 \textbf{to} \text{PD}[\text{r}] \textbf{do}
34. \hspace{1em} K[\text{i++}] \leftarrow \text{false}
35. \hspace{1em} K[\text{i++}] \leftarrow \text{true}
36. \hspace{1em} \textbf{return } B

\textbf{Fig. 1.} Internal memory version of PLCP computation algorithm
and we consider only the intervals produced by extensions with a fixed symbol $\text{sym}$, then those extension intervals are in increasing order. The $B$ function is for a fixed first argument $\text{sym}$ monotonously increasing in it’s second argument and has a maximum value of $D[\text{sym}+1]$ which is only reached as an (excluded) right end of any BACKSTEP call and at the same time the (included) minimum left end of calls for BACKSTEP with first parameter $\text{sym}+1$ (if any such exist in $s$). This means if we replace $\text{NQ.enqueue}((l',r'))$ by $\text{NQ.enqueue}((\text{sym},(l',r')))$, sort $\text{NQ}$ stably by the first ($\text{sym}$) component and subsequently drop the first component then the resulting list of intervals will be in sorted order. The sorting can be performed using $O(\log \sigma)$ rounds of bucket sorting along the bit representation of the first component, each of which takes $O(n)$ time as we can never have more than $n$ elements in the queue. During the whole sorting procedure the elements for each single first component will stay in ascending order concerning their second component, which allows us to store the second component using differential $\gamma$ code. The sequence of lower interval bounds and the one of upper interval bounds both form strictly increasing sequences. Starting the difference coding for the sequences for $\text{sym}$ at $D[\text{sym}]-1$ ensures that for both sequences the sum of the stored numbers does not exceed $n$, so we can store them using $O(n)$ bits according to Lemma 1.

- The $T$ set stores a subset of the lower interval bounds produced for $\text{NQ}$. We can thus use similar steps to produce it in sorted order while requiring $O(n \log \sigma)$ bits of space in EM and $O(\sigma \log n)$ bit of IM.

- The values added to $\text{activeSet}$ in line 18 can easily be added in increasing order by first storing them in a heap data structure for each source interval $(l,r)$ and writing the values out in order at the end of the handling of $(l,r)$. This takes space $O(\sigma \log n)$ in IM while the runtime for this is bounded by $O(n \log \sigma)$ for each round (the heap depth is bounded by $\log \sigma$ as we never insert more than $\sigma$ elements into any heap and the total number of elements added is bounded by $n$). The values in increasing order can again be stored using differential $\gamma$ code in $O(n)$ bits. As soon as we have the set of newly added values for a round we can merge it into the set of previously added values, which can be stored in the same way. Storing $\text{activeSet}$ in this way requires $O(n)$ bits of space in EM.

- For each source interval $(l,r)$ the set of symbols in $\{\text{BWT}[i] \mid 1 \leq i < r\}$, the target intervals $(l',r')$ and the respective $1'src$ values can be computed during a linear scan of the BWT sequence streamed from EM while keeping track of the values of the $\text{rank}$ function for each symbol. This requires $O(\sigma \log n)$ bits of space in IM. We keep tu-
binUnBucketSort(\(K, A, m\))

1. \((cnt[0], cnt[1]) \leftarrow (0, 0)\)
2. for \(i \leftarrow 0\) to \(m - 1\) do
3. \(cnt[K[i]] \leftarrow cnt[K[i]] + 1\)
4. \((cnt[0], cnt[1]) \leftarrow (0, cnt[1])\)
5. for \(i \leftarrow 0\) to \(m - 1\) do
6. \(B[i] \leftarrow A[cnt[K[i]]]\)
7. \(cnt[K[i]] \leftarrow cnt[K[i]] + 1\)
8. return \(B\)

Fig. 2. Inverse binary bucket sorting for key vector \(K\) and data vector \(A\), both of length \(m\)

Figures (sym, \(l', 1'src\)) in an AVL tree (cf. [1]) where only the first (sym) component is used as the key. While scanning BWT we insert (sym, \(B(sym, l'src), B(sym, l'src) + 1, 1'src\)) upon first encountering sym at index \(l'src\) in \((l, r)\) and update the third component accordingly whenever we find another instance of sym in the source interval. With the same reasoning as above for the heap used while handling activeSet this takes time \(O(n \log \sigma)\) for one round.

– The accesses to S in line 17 are in ascending index order and updating S in line 30 while scanning S and T can read both sequences in linear ascending order, which is suitable for EM. Accessing S at \(l'\) in line 10 is somewhat more challenging. As shown above the \(l'\) values in each round are only increasing when we look at a single symbol sym. We can obtain the bits we need to see in the required order using the following steps. First compute the sequence of \(l'\) values we need to access in ascending order. This can be done as described above for producing \(NQ\), i.e. produce a set of pairs (sym, \(l'\)), sort it by the first component while using differential \(\gamma\) code for representing the second components and then drop the first component. This takes time \(O(n \log \sigma)\) and space \(O(n \log \sigma)\) bits in EM. It gives us the set of required \(l'\) values in increasing order and thus makes it easy to determine whether S does or does not contain the respective values, which we store as a bit vector in EM. This bit vector has as many bits as \(l'\) values relevant in the current round, which is \(O(n)\). Now we have the relevant bits, but they are in the wrong order, as we sorted the \(l'\) values by the respective sym values. We can reorder the bits by inverse sorting them using the original order of the sym values. Figure 2 shows an algorithm which performs inverse sorting of a sequence A given a binary key vector \(K\). It does this by first determining how many 0 and 1 bits there are in the key vector (lines 1-3) and then rebuilding the original sequence by scanning \(K\) and taking elements from the 0 and 1 regions of the sorted
sequence in accordance with the key bits encountered (lines 5-7). This inverse binary bucket sorting can be extended to inverse radix sorting for non-binary keys. It requires time $O(n \log \sigma)$ (we need $\log \sigma$ rounds of inverse bucket sorting) and space $O(n \log \sigma)$ bits in EM.

- The $\mathbf{PD}$ array can be represented as a bit vector in EM. We initialise it as a vector of $n$ one bits. Adding one to index $r$ is done by inserting a zero bit just ahead of the $k + 1$’st one bit. We scan $\mathit{activeSet}$ and $\mathbf{PD}$ linearly for updating $\mathbf{PD}$ where $\mathbf{PD}$ has at most $2n$ bits at any time. So updating $\mathbf{PD}$ in each round takes $O(n)$ time and storing $\mathbf{PD}$ takes $O(n)$ bits in EM.

Overall each round of the algorithm up to line 32 takes time $O(n \log \sigma)$ and we need $O(n \log \sigma)$ bits of space in EM. In the worst case the maximum LCP value is $n - 2$ (which is e.g. reached $s[i] = 1$ for $0 \leq i < n - 1$ and $s[n - 1] = 0$), so the worst case run time of the algorithm is $O(n^2 \log \sigma)$. In the average case (cf. [24]) the maximum value is in $O(\log \sigma n)$, which gives this part of the algorithm a run time of $O(n \log n \log \sigma)$ on average.

This leaves us with the issue that the procedure above so far produces the difference between PLCP values in rank instead of position order. This is set right by lines 33-39 in Figure 1, however it uses a complete inverse suffix array and requires random access to the $\mathbf{PD}$ array. Given a sampled inverse suffix array at sampling rate $\bar{f} \in O(\log n)$ taking $O(n)$ bits, the $\mathbf{BWT}$ and the $\mathbf{PD}$ bit vector we can produce the final PLCP bit vector using the following steps:

1. Create pairs $(\mathit{ISA}[i], i)$ for $i = 0, 1, \ldots, \lceil \frac{n}{\bar{f}} \rceil - 1$ in EM from the sampled inverse suffix array (both components are stored as $O(\log n)$ bit block code) and sort these pairs by their first (rank) component using radix sort. This takes space $O(n)$ bits in EM and time $O(\frac{n}{\bar{f}} \log n) = O(n)$. After sorting annotate each tuple with one bit set to true as third component (marks the tuple as active), the number 0 stored in $\gamma$ code as fourth component (stores the number of PLCP values added to the tuple so far) and an empty vector of $\gamma$ coded numbers as the fifth component.

2. For $\bar{f}$ rounds do the following: perform an LF operation on the tuples (map $(r, p, a, b, c)$ to $(\mathbf{BWT}[r], \mathbf{LF}(r), p', a, b, c)$ where $p' = (p + n - 1) \mod n$ if $a$ is true and $p$ otherwise) by scanning the $\mathbf{BWT}$ and computing LF as described above while tracking the B function using $O(\sigma \log n)$ bits of IM. Sort the resulting tuples by the first component and drop the first component. This restores the sorted order according to the rank of the tuples and takes time $O(n \log \sigma)$ and space $O(n \log \sigma)$ bits in EM.

Note that for each active (third component is true) tuple in the list we
retain the invariant that for a first component $r$ we have $\text{SA}[r]$ as the second component. Scan the tuples and the $\PD$ bit vector and copy the respective (matching rank) values into tuples marked as active by inserting the value $\PD(r)$ at the front of the vector of $\gamma$ coded values in component five and incrementing the counter for appended values (fourth component) by one. This takes time $O(n)$ and again space $O(n \log \sigma)$ bits in EM. In another scan mark tuples s.t. their second component $p$ is divided by $\int$ as inactive. Note that at the end of each round we have the following property: Let $(r, p, a, c, (v_0, v_1, \ldots, v_{c-1}))$ be a tuple in our list. Then for $i = 0, 1, \ldots, c-1$ we have $v_i = \text{PLCP}[p+i] - \text{PLCP}[p+i-1] + 1$.

3. Sort the tuples by the second component (position) using a $\log n$ round radix sort taking $O(n)$ time and $O(n \log \sigma)$ bits of space. Let $t_0, t_1, \ldots, t_{\lceil \frac{n}{\int} \rceil - 1}$ be the sequence of tuples we have obtained. Then for each $t_i = (r, p, a, b, c)$ with $0 \leq i \leq \lceil \frac{n}{\int} \rceil - 1$ we now have $p = i \int$, $a = \text{false}$, $b$ represents $\min(n - p, \int)$ and $c$ is the sequence $v_0, v_1, \ldots, v_{b-1}$ s.t. $v_i = \text{PLCP}[p+i] - \text{PLCP}[p+i-1] + 1$.

4. Initialise an empty bit vector $K$. Scan the tuples and for each tuple do the following: let $c$ denote the number stored in the fourth component and let $v_0, v_1, \ldots, v_{c-1}$ be the (decoded) numbers stored in the fifth component. For $i$ in $0, 1, \ldots, c-1$ append $v_i$ zero bits to $K$ and then 1 one bit.

The bit vector $K$ is by construction the succinct $2n$ bit representation of the PLCP array. The whole reordering takes time $O(n \log n \log \sigma)$, $O(n \log \sigma)$ bits of space in EM and $O(\sigma \log n)$ bits of space in IM. Each tuple at maximum uses $\log \sigma$ bits for the symbol intermediately introduced in step 2, $O(\log n)$ bits for rank and position and $O(\log n)$ bits for storing the number of $\PD$ values copied into the tuple so far. The sum over all stored $\gamma$ values in the last component of the tuples is bounded by $n$ and reaches $n$ at the end of the procedure.

We summarise the run time and space requirements of the EM algorithm in the following Theorem.

**Theorem 1.** The succinct $2n$ bit PLCP representation for a string $s$ of length $n$ can, given its BWT and sampled suffix array of sampling rate $\int \in O(\log n)$, be constructed in worst case time $O(n^2 \log \sigma)$ and average time $O(n \log n \log \sigma)$ using $O(n \log \sigma)$ bits of space in EM and $O(\sigma \log n)$ bits of space in IM.

5 Reducing Internal Memory Usage

While the algorithm of the previous section has space requirements in $O(n \log \sigma)$ bits in external memory, the need for $O(\sigma \log n)$ bits in IM may
be considered as too large in some situations, even though it is not an
obstacle in practice. We can modify the algorithm to use less space in in-
ternal memory, as we show in the following. A suitable reformulation of
the algorithm is given in Figure 3. The algorithm as shown only reformu-
lates the computation of the bit vector up to the point were it is translated
from rank to position order. The crucial point about the reformulation is
to compute the LF and backstep functions without keeping track of the
value of the rank function in IM for each single symbol in Σ. Observe
that given a set of ranks \( R \) we can compute the set of ranks \( R_{LF} \) defined
by
\[
R_{LF} = \{ r' \mid r' = LF(r), \ r \in R \}
\]
using the following steps: create a bit vector \( R_B \) of length \( n \) s.t. \( R_{B,r} = 1 \) iff \( r \in R \) and then construct the
sequence of pairs \( P_R = (BWT_0, R_{B,0}) (BWT_1, R_{B,1}) \ldots (BWT_{n-1}, R_{B,n-1}) \). Sort
\( P_R \) by the first (symbol) component in a stable way using radix sort in
time \( O(n \log \sigma) \). It is easy to see that the second (bit) component of the
sorted vector represents \( R_{LF} \) by virtue of marking the respective ranks by
1 bits. This method can be extended to computing the backstep function
for a given set of intervals and all possible extensions of the respective
intervals on the left. To this end observe that for a given interval \( [l, r] \) an extension is possible by exactly those symbols contained in the
set given by \( \{ a \mid a = BWT_i \ for \ some \ l \leq i < r \} \), the lower bound \( l' \) of
\( (l', r') = \text{backstep}(a, (l, r)) \) for any such symbol is given by \( l' = LF(l_{src}) \)
where \( l' \) is the smallest number s.t. \( l \leq l_{src} < r \) and \( BWT_{l_{src}} = a \) and \( r' - l' \)
equals the number of \( a \) symbols in the sequence \( BWT_i, BWT_{i+1}, \ldots, BWT_{r-1} \).
The depicted algorithm computes all extensions of a given set of intervals
by the backstep function using the following steps. Assume a list of inter-
vals \( L = (l_0, r_0), (l_1, r_1), \ldots, (l_{m-1}, r_{m-1}) \) is given s.t. \( l_0 = 0, r_{i-1} = l_i \)
for \( i = 1, 2, \ldots, m-1 \) and \( r_{m-1} = n \). In particular the intervals partition
the index space \( 0, 1, \ldots, n-1 \). \( L \) can be stored using \( O(n) \) bits in external
memory using either \( \gamma \) code for storing the increasing sequences of lower
and upper bounds using differential encoding or by storing two bit vectors
of length \( n \) marking the start and end of the intervals. For each interval
\( (l_i, r_i) \) in ascending order do the following to produce a sequence \( Z \):

1. extract the sequence \( B = BWT_{l_1}, BWT_{l_1+1}, \ldots, BWT_{r-1} \) to \( B_S \) and sort it in
time \( O((r - l) \log \sigma) \) using radix sort
2. in a single linear scan of \( B_S \) mark the first occurrence of symbol \( a \) in
\( B_S \) with the number of times it occurs in \( B_S \), i.e. \( |\{ i \mid 0 \leq i < r - l \) and \( B_{Si} = a \}| = |\{ i \mid 0 \leq i < r - l \) and \( B_i = a \}| = |\{ i \mid l \leq i < r \) and \( BWT_i = a \}|. \) The rest of the character instances are marked
with zero. The attached numbers are stored using \( \gamma \) code. The numbers
stored obviously sum up to \( r - l \). Let the obtained sequence be called
\( B_M \).
3. append $B_M$ to $Z$.

Then sort $Z$ stably by the first (symbol) component using radix sort in time $O(n \log \sigma)$. Let $Z_S = (a_0, v_0), (a_1, v_1), \ldots, (a_n, v_n)$ denote the resulting sorted sequence. Further let $J = \{ j \mid v_j \neq 0 \} = j_0, j_1, \ldots, j_{k-1}$ and $I = (j_0, v_{j_0}), (j_1, v_{j_1}), \ldots, (j_{k-1}, v_{j_{k-1}})$. Let

$$\text{backstep}^*(a, L) = \text{backstep}(a, (l_0, r_0)), \ldots, \text{backstep}(a, (l_{m-1}, r_{m-1}))$$

for $a \in \Sigma$ and

$$\text{backstep}^*(L) = \text{backstep}^*(0, L), \ldots, \text{backstep}^*(\sigma - 1, L).$$

Let the filter function $\text{flt}$ be defined by

$$\text{flt}((\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_z, \beta_z)) = \begin{cases} (\alpha_1, \beta_1), \text{flt}((\alpha_2, \beta_2), \ldots, (\alpha_z, \beta_z)) & \text{if } \alpha_1 \neq \beta_1 \\ \text{flt}((\alpha_2, \beta_2), \ldots, (\alpha_z, \beta_z)) & \text{otherwise} \end{cases}$$

Following the same pattern as computing the LF function by attaching the BWT symbols to a bit vector it is straightforward to see that $I$ is exactly the sequence of intervals $\text{flt}(\text{backstep}^*(L))$, i.e. all non empty extensions of intervals in $L$ in ascending order. In consequence we obtain the following result.

**Lemma 2.** Given $BWT$ and a sorted, non overlapping list of intervals $L$ drawn from $[0, n)$ s.t. both $BWT$ and $L$ can be decoded in constant time per element the sorted sequence of intervals $\text{flt}(\text{backstep}^*(L))$ can be computed in time $O(n \log \sigma)$ and space $O(n \log \sigma)$ bits in EM and $O(\log n + \log \sigma)$ in IM.

In each round we activate ranks $r$ s.t. $\text{LF}(r)$ gets set in this round while $r$ itself has not already been set in a previous round. We keep a bit vector $S$ in external memory marking the indices of ranks for which we already observed the corresponding LCP value in a previous round. Remember that a rank $l'$ gets set on $S$ in the first round in which $l'$ appears as a result interval lower bound of a call to $\text{backstep}(a, (l, r))$ for any arguments $a, l$ and $r$. The result intervals for the $\text{backstep}$ operation are encoded in the sequence $Z$ in the algorithm in Figure 3 after it has been sorted in line 21. Interval start points are marked by such tuples which have a non zero count (second component) attached. The information whether or not a rank will be newly set in $S$ in the current round is encoded in the sequence $Z'$ in lines 22–25 of the algorithm. We perform an inverse LF mapping on $Z'$ by performing an inverse sorting of $Z'$ using BWT as key sequence. This allows us to determine which ranks need to be activated by combining information.
from the sequence $S$ and $Z'$ during a linear scan of the two sequences (lines 28 – 32). The active set can be stored as a bit vector marking active ranks. The algorithm produces the indices of newly activated ranks in increasing order, so merging them into the already existing set is trivially performed in linear time $O(n)$. We keep the encoding of the $PD$ vector from the previous section. Updating it by incrementing the counts for active ranks is straightforward and takes time $O(n)$. Finally the algorithm cleans the active set, sets the new ranks in $S$ and computes the input intervals for the next round in lines 34 – 41. Again all of this is easily performed in time $O(n)$. The space usage in internal memory is reduced to $O(\log n + \log \sigma)$ (plus what is necessary to allow buffering for external memory).

Observe that in the reordering of values from rank to position order in the previous section the part taking the most IM is step 2. This is $O(\sigma \log n)$ bits. This is again caused by keeping track of the $B$ function for each symbol of the alphabet while scanning the $BWT$ to compute an $LF$ mapping. As described above we can perform this $LF$ mapping in EM while using $O(\log n + \log \sigma)$ in IM without asymptotically using more space in EM or time. This leads us to the following result.

**Theorem 2.** The succinct $2n$ bit PLCP representation for a string $s$ of length $n$ can, given it’s $BWT$ and sampled suffix array of sampling rate $f \in O(\log n)$, be constructed in worst case time $O(n^2 \log \sigma)$ and average time $O(n \log n \log \sigma)$ using $O(n \log \sigma)$ bits of space in EM and $O(\log \sigma + \log n)$ bits of space in IM.

6 Improvement of Worst Case

While on average our algorithm has a run time of $O(n \log n \log \sigma)$ as the LCP values are $O(\log n)$ on average, we often see cases in practice where, while most of the LCP values are small (in the order of $\log n$), there are some significantly larger values as well. In this case an easy adaption of our algorithm is to stop the computation of the $PD$ vector after a certain number of rounds (say $3 \log n$) and compute the missing values using the algorithm presented in [15]. This adaption can be performed using the following steps before reordering the $PD$ bit vector.

1. Erase all zero bits from the $PD$ bit vector corresponding to ranks which are still in the active set. This removes incomplete values from $PD$ for such ranks $r$ where $LCP[r]$ was not yet reached but $LCP[LF(r)]$ was. This filtering takes time $O(n)$.
2. Compute a list $S_{im}$ (irreducible missing) of ranks $r$ in $S$ s.t. $r = 0$ or $r > 0$ and $BWT[r - 1] \neq BWT[r]$ in time $O(n)$ and space $O(n)$ bits of EM. In the following let $n_{im} = |S_{im}|$. 
PLCP external (BWT, n, ISA)

1. (Q, S) ← (∅, bit vector of n false bits)
2. Q.enqueue((0, n))
3. while |{i | S_i = true}| < n do
   4. Z ← empty sequence
   5. while Q.hasNext() do
      6. (r_l, r_h) ← Q.next()
      7. ⊲ extract sub sequence of BWT for interval [r_l, r_h]
      8. A ← BWT_r_l, BWT_r_l+1, ..., BWT_r_h−1
      9. sort A in time O(|A| log σ)
     10. ⊲ attach count to first occurence of each symbol and append to Z
    11. ℓ ← 0
    12. while ℓ < r_h − r_l do
       13. (h, a) ← (ℓ + 1, A[ℓ])
       14. ⊲ find end of range for same symbol
       15. while h < r_h − r_l and A_h = a do
          16. h ← h + 1
       17. Z.append((a, h − ℓ))
       18. for i ← 1 to (h − ℓ) − 1 do
          19. Z.append((a, 0))
       20. ℓ ← h
    21. sort Z by symbol component in time O(n log σ)
    22. ⊲ construct bit vector Z’ marking ranks which will get set in this round
    23. for r ← 0 to n − 1 do
       24. (a, c) ← Z_r
       25. Z’_r ← (c ≠ 0 and S_r = false)
    26. ⊲ perform LF^(-1) mapping on Z’
    27. inverse sort Z’ using BWT
    28. ⊲ activate ranks for this round
    29. for r ← 0 to n − 1 do
       30. ⊲ if rank r not yet set but LF(r) will be set in this round
       31. if Z’_r = true and S_r = false then
          32. activate r
       33. increment count for active ranks
    34. ⊲ update S and active set, construct intervals for next round
    35. NQ ← ∅
    36. for r’ ← 0 to n − 1 do
       37. (a, c) ← Z_{r’}
       38. if c ≠ 0 then
          39. deactivate r’ and set S_r’
       40. NQ.enqueue((r’, r’ + c))
    41. Q ← NQ

Fig. 3. Low internal memory variant PLCP external
3. Compute the list $S_{imlf}$ containing the ranks in $S_{im}$ and in addition for each rank $r \in S_{im}$ also LF$(r)$. This takes time $O(n \log \sigma)$ and space $O(n \log \sigma)$ in EM where we use a scan over BWT and a subsequent sorting by a symbol component as described above for computing the LF function for a set of ranks. This steps adds all ranks for the previous position of a rank in $S_{im}$, which we need for computing differences between PLCP values for positions $p$ in $S_{im}$ and the respective previous positions $p-1$.

4. For each rank $r > 0$ in $S_{imlf}$ add $r-1$ to $S_{imlf}$ in time $O(n)$. We need these ranks for computing LCP values because LCP$[r]$ is defined by comparing the suffixes at the ranks $r$ and $r-1$.

5. Convert $S_{imlf}$ to block code using $O(\log n)$ bits per rank in time $O(n)$ and space $O(n_{im} \log n)$ bits in EM.

6. Given a sampled inverse suffix array of sampling rate $f \in O(\log n)$ use a method similar to reordering the PLCP difference values above to annotate each rank in $S_{imlf}$ with the corresponding position in time $O(n \log n \log \sigma)$ and space $O(n \log \sigma + n_{im} \log n)$ bits in EM.

7. Sort the resulting tuples by rank in time $O(n_{im} \log n)$ and space $O(n_{im} \log n)$ bits in EM.

8. For each $(r, p)$ in the tuples s.t. there is some tuple $(r-1, p')$ construct $(r, p = SA[r], r-1, p' = SA[r-1])$ in time $O(n_{im})$ and space $O(n_{im} \log n)$ EM bits.

9. Annotate the tuples with the respective LCP value between rank $r$ and $r-1$ stored in block code using a sparse version the algorithm presented in [15]. This requires the text $s$, which, if necessary, can be reconstructed from the BWT and an inverse sampled suffix array at sampling rate $f \in O(\log n)$ in time $O(n \log n \log \sigma)$ and space $O(n \log \sigma)$ in EM. Given $M \in O(n)$ words of IM (i.e. O($M \log n$) bits) of IM this requires time $O(\frac{n^2}{M \log n} + n \log \frac{n}{M \log n})$ using a disk block size of $B$ words (see [15]).

Drop the $r-1$ and $p' = SA[r-1]$ components from the tuples.

10. Sort the tuples by position. Drop all tuples for positions $p$ s.t. $p > 0$ and there is no tuple for $p-1$. For the rest replace the LCP component by the difference of the values for $p$ and $p-1$ if $p > 0$.

11. Sort the tuples by rank (time $O(n_{im} \log n)$) and insert the computed values into the PD bit vector (time $O(n)$).

Using this hybrid algorithm we can obtain a trade off between the faster worst case run time of the algorithm presented in [15] given sufficient IM and the reduced EM space usage of our algorithm presented above. In this second stage of the hybrid algorithm we are generally only interested in computing values for so called irreducible LCP values (cf. [16]) as only such values produce 0 bits in the succinct PLCP vector. The sum over all
irreducible LCP values for any string of length \( n \) is bounded by \( 2n \log n \) (see [16]). This bound is reached for de Bruijn strings (cf. [16]), however in this setting each irreducible LCP value is \( \Theta(\log n) \). If we run the algorithm from the previous Section 4 for \( O(\log^2 n) \) rounds, then all LCP values which remain unset must have a value of \( \Omega(\log^2 n) \), which means there are \( O\left(\frac{n}{\log n}\right) \) such values and consequently the hybrid algorithm runs in worst case time \( O(n \log^2 n \log \sigma) \) while using \( O(n \log \sigma) \) space in EM and \( O\left(\frac{n}{\log n}\right) \) bits in IM.

**Theorem 3.** Given the BWT and sampled inverse suffix array of sampling rate \( f \in O(\log n) \) for a string \( s \) of length \( n \) over an alphabet of size \( \sigma \) the succinct permuted LCP array for \( s \) can be computed in time \( O(n \log^2 n \log \sigma) \) while using \( O(n \log \sigma) \) bits of space in EM and \( O\left(\frac{n}{\log n}\right) \) bits of space in IM.

As the bound of \( 2n \log n \) for the sum over the irreducible LCP values of a string is obtained for LCP values which are all of length \( \sigma \), the interesting question remains whether there is a smaller upper bound for the sum of the irreducible LCP values when only LCP values in \( \omega(\log n) \) are considered in the sum.

7 Circular strings

In this section we relax the original requirement of a unique terminator symbol in \( s \), i.e. we no longer require that \( s_{n-1} < s_i \) for all \( i < n-1 \). Let \( \hat{s} = \hat{s}_0\hat{s}_1\ldots \) be the infinite string defined by \( \hat{s}_i = s_{i \mod n} \). Further let \( \hat{s}[i\ldots] = \hat{s}_i\hat{s}_{i+1}\ldots \) for \( i \geq 0 \), i.e. the suffix of \( \hat{s} \) starting from index \( i \). We define that for two indices \( i, j \) the relation \( \hat{s}[i\ldots] < \hat{s}[j\ldots] \) holds if either there is some \( l \) s.t. \( \hat{s}_{i+l} < \hat{s}_{j+l} \) or \( \hat{s}[i\ldots] = \hat{s}[j\ldots] \) and \( i < j \). According to this definition we either have \( \hat{s}[i\ldots] < \hat{s}[j\ldots] \) or \( \hat{s}[j\ldots] < \hat{s}[i\ldots] \) for \( i \neq j \) and in consequence there is a unique permutation \( \text{SA} = \text{SA}_0, \text{SA}_1, \ldots, \text{SA}_{n-1} \) of \( 0, 1, \ldots, n-1 \) s.t. \( \hat{s}[\text{SA}_i\ldots] < \hat{s}[\text{SA}_j\ldots] \) for \( 0 < i < n \) and we can define \( \text{BWT}[i] = \hat{s}_{\text{SA}_i+n-1} \). When defining a longest common prefix array for circular strings we face the issue of identical suffixes even when they start at different indices and thus infinite values in the array. These (infinite values) obviously occur in exactly such cases when \( s \) is an integer power of a string shorter than \( s \) (i.e. there is some string \( w \) s.t. \( s = ww\ldots w \) which we write as \( w^k \) if \( s \) consists of \( k \) copies of \( w \) juxtaposed). This case is easily detectable by scanning the BWT and determining whether there is some \( k \) dividing \( n \) s.t. for each \( i \) in \( 0, 1, \ldots, \frac{n}{k} - 1 \) we have \( \text{BWT}[ik + 0] = \text{BWT}[ik + 1] = \ldots = \text{BWT}[ik + k - 1] \). Figure 4 shows a linear time algorithm for detecting the maximum period \( p \) of \( s \) s.t. \( s = s[0 \ldots p - 1]^\frac{k}{p} \). For obtaining a meaningful LCP array for a string \( s = w^e \) for \( e > 1 \) we may choose to shrink it’s BWT.
detectPeriod(BWT, n)
1 (e, i) ← (∞, 0)
2 while e > 1 and i < n do
3 (j, c) ← (i + 1, BWT[i])
4 while j < n and c = BWT[j] do
5 j ← j + 1
6 if e = ∞ then
7 (e, i) ← (j − i, j)
8 else (e, i) ← (gcd(j − i, e), j)
9 return n

Fig. 4. Linear time algorithm for detecting maximum period p s.t. the string of length n underlying BWT equals w p for some word w

array to that of a single base factor w by keeping every e’th symbol and discarding the symbols at the other indices.

In the following we assume that s is not an integer power of a word shorter than s and has length n > 1, i.e. s contains at least two different distinct symbols. As shown above this implies that for 0 ≤ i < j < n there is always some 0 ≤ l < n s.t. s i+l ≠ s j+l. In consequence there is a well defined array LCP = LCP0, LCP1, . . . , LCP n−1 given by LCP0 = 0 and LCP1 = l for i = 1, 2, . . . , n − 1 where l is the smallest number s.t. s i+1+1 = s j+1+l. Note that setting LCP0 = 0 is consistent with the scheme for the other ranks as the suffixes at ranks 0 and n − 1 start with different symbols, i.e. the length of their longest common prefix is 0. This also guarantees that the LCP array contains the value 0 at least once. Based on the arrays SA and LCP we can define the array ISA of length n by ISA SA i = i for i = 0, 1, . . . , n − 1 and PLCP = PLCP0, PLCP1, . . . , PLCP n−1 by PLCPi = LCP ISA i. The property of PLCPi − PLCPi−1 ≥ −1 still holds with the same arguments as in the non circular case, in fact this can even be extended to PLCP0 − PLCP n−1 ≥ −1 as the position 0 has no special meaning in the circular case. Note however that we loose one feature crucial for the 2n bit succinct PLCP representation in the transition to circular strings and this is the guarantee of PLCP n−1 = 0 which stems from the unique terminator symbol ensuring that no other suffix relevant for the computation of LCP starts with the same symbol as the one at position n − 1. As an example consider the string abbab with the PLCP array 2, 1, 0, 0, 3 which would translate to the bit vector 0001110100001 of length 13 > 10 = 2n. Note that given SA and a select dictionary on the bit vector we can correctly decode the respective LCP values, however the vector is too long for the 2n bit bound. The reason for the excessive length is precisely the fact that the PLCP array does not end with a 0 value. If we start off with the word babba which is a rotation of
and consequently has the same BWT then the PLCP array is rotated to 3, 2, 1, 0, 0 with the bit vector 0000111101 of length 10 = 2n. We chose babba because it shifts the positions by 1 from abbab and thus moves the last 0 at position n - 2 in the PLCP array of abbab to position n - 1 in the array for babba. We can obtain PLCP_i for abbab by decoding PLCP_{(i+1) mod n} for babba from the succinct PLCP bit vector for babba. Suitable ranks \hat{r} s.t. LCP_{\hat{r}} = 0 can be found by checking the D array. Having chosen one such rank \hat{r} we can deduce the respective position \hat{p} by using a sampled inverse suffix array and the BWT in time \(O(n \log \sigma \log n)\) if the sampling rate is \(O(\log n)\) while using \(O(n \log \sigma)\) bits in EM and \(O(\log \sigma + \log n)\) bits in IM.

For computing the succinct PLCP bit vector of a string using BWT and a sampled inverse suffix array observe that the algorithm we presented in Section 4 and 5 has no knowledge about positions until it reaches the stage of reordering the values from rank to position order. All the generated values are purely differential (i.e. \(\text{PLCP}_i - \text{PLCP}_{(i+n-1) \mod n}\) for \(0 \leq i < n\)), in particular there is no special handling of position 0. The algorithm produces the bit vector 1110100001 for the input abbab which we need to rotate to 0000111101 as described above to obtain correct PLCP values while taking the employed position shift into account during decoding. The hybrid algorithm can also be adapted for circular strings without asymptotically modifying it’s runtime or space usage. In step 9 we need to take care of the fact that the comparison of two suffixes may extend beyond the end of s. Due to our pre conditions however we can guarantee that the longest common prefix of two different suffixes is always shorter than n symbols. This means that two runs over the set of blocks the text is decomposed into in the original algorithm are sufficient, where in the second run no more tuples are added but we only handle such tuples where the comparison extends across block boundaries. When accessing the text we need to use it’s circular extension for comparisons. In step 10 we need to handle the pair of positions \((n - 1, 0)\) if both positions are present. Asymptotically we keep the same time bound for the hybrid algorithm as we extend the amount of work done in step 9, by a constant factor 2 and in step 10 by a finite amount. This gives us the following result.

**Theorem 4.** Given the circular BWT and sampled inverse suffix array of sampling rate \(f \in O(\log n)\) for a circular string \(\hat{s}\) deduced from a string s of length n over an alphabet of size \(\sigma\) the succinct permuted LCP array for \(\hat{s}\) can be computed in time \(O(n \log^2 n \log \sigma)\) while using \(O(n \log \sigma)\) bits of space in EM and \(O(\frac{n}{\log n})\) bits of space in IM.

For the sake of this theorem the succinct permuted LCP array denotes the shifted version plus respective position shift described above. If the input
string $s$ is an integer power of a shorter string $s'$ s.t. $s'$ is not itself an integer power of a shorter string, then the succinct permuted LCP array is constructed using $s'$.

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