Non-Archimedean Geometry and Physics on Adelic Spaces *

Branko DRAGOVICH†
Institute of Physics
P.O.Box 57, 11001 Belgrade, Serbia and Montenegro

Abstract

This is a brief review article of various applications of non-Archimedean geometry, \( p \)-adic numbers and adeles in modern mathematical physics.

1 Introduction

It is well known that theoretical physics is strongly related to mathematics. Space, time and matter are basic concepts in all physical theories. They have become usually profound and gradually unified in new theories using more general mathematical tools. For example, transition from nonrelativistic to relativistic kinematics required to pass from Euclidean space and time to the Minkowski space. To describe phenomena in strong gravitational fields and accelerated frames, general theory of relativity was discovered, where space-time is described by pseudo-Riemannian geometry which is related to the distribution of matter. Dynamics in quantum mechanics can be regarded as motion of a particle in a phase space \((x, k)\) with symplectic geometry and the Heisenberg uncertainty \(\Delta x^i \Delta k^j \geq \frac{\hbar}{2} \delta^{ij}\), where \(\hbar = \frac{h}{2\pi}\) is the reduced Planck constant. In recent years, noncommutative geometry based on relation \([x^i, x^j] \neq 0\) has attracted a significant interest in quantum theory.

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†e-mail address: dragovich@phy.bg.ac.yu
According to various considerations, which take together quantum and gravitational principles, there is a restriction on empirical accuracy of physical variables due to the relation

$$\Delta x \geq \ell_0 = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{\frac{33}{2}} \text{cm}, \tag{1.1}$$

where $\Delta x$ is an uncertainty measuring a distance, $\ell_0$ is the Planck length, $G$ is Newton’s gravitational constant and $c$ is the speed of light in vacuum. The uncertainty (1.1) means that one cannot measure distances smaller than $\ell_0$. Since this result is derived assuming that space-time consists of real points and has an Archimedean geometry, it becomes desirable to employ also non-Archimedean geometry based on $p$-adic numbers. Quite natural framework to consider simultaneously real (Archimedean) and $p$-adic (non-Archimedean) spaces is by means of an adelic space.

In this paper, at an introductory level I briefly review some basic characteristics of non-Archimedean geometry, $p$-adic numbers and adeles, as well as their use in some parts of modern mathematical physics.

## 2 Non-Archimedean Geometry and $p$-Adic Numbers

Recall that having two segments of straight line of different lengths $a$ and $b$, where $a < b$, one can overpass the longer $b$ by applying the smaller $a$ some $n$-times along $b$. In other words, if $a$ and $b$ are two positive real numbers and $a < b$ then there exists an enough large natural number $n$ such that $na > b$. This is an evident property of the Euclidean spaces (and the field of real numbers), which is known as Archimedean postulate, and can be extended to the standard Riemannian spaces. One of the axioms of the metric spaces is the triangle inequality which reads:

$$d(x, y) \leq d(x, z) + d(z, y), \tag{2.1}$$

where $d(x, y)$ is a distance between points $x$ and $y$. However, there is a subclass of metric spaces for which triangle inequality is stronger in such way that:

$$d(x, y) \leq \max\{d(x, z), d(z, y)\} \leq d(x, z) + d(z, y). \tag{2.2}$$

Metric spaces with strong triangle inequality (2.2) are called non-Archimedean or ultrametric spaces.
Since a measurement means quantitative comparison of a given observable with respect to a fixed value taken as its unit, it follows that a realization of the Archimedean postulate is practically equivalent to the measurements of distances. According to the uncertainty (1.1), it is not possible to measure distances shorter than $10^{-33} \text{cm}$ and consequently there is no place for an Archimedean geometry beyond the Planck length. By this way, standard approach to quantum gravity, which is based only on Archimedean geometry and real numbers, predicts its own breakdown at the Planck scale. Hence, a new approach, which takes into account not only Archimedean but also a non-Archimedean geometry, seems to be quite necessary. The most natural ambient to realize both of these geometries is an adelic space, which is a whole of real and all $p$-adic spaces.

Any set with trivial metric ($d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$) is a simple example of the non-Archimedean space. Set of all real polynomials $\mathbb{R}[X]$, for which metric is defined by a suitable valuation, is another example of the non-Archimedean space. Namely, for a nonzero polynomial $f \in \mathbb{R}[X]$ given by $f = a_nX^n + \cdots + a_1X + a_0, \ a_n \neq 0$, one can define the degree of $f$ as $d(f) = n$, and $d(f) = -\infty$ if $f$ is the zero polynomial. Then a norm can be introduced as $|f| = \rho^{d(f)}$ if $f \neq 0$, and $|f| = 0$ if $f = 0$, where $\rho$ is a real number greater than 1. Since $|f + g| \leq \max\{|f|, |g|\}$ and $|fg| = |f||g|$, where $f, g \in \mathbb{R}[X]$, this is a non-Archimedean valuation which gives a non-Archimedean geometry by $d(f, g) = |f - g|$. This can be extended to the field $\mathbb{R}(X)$ of rational functions. On the field of complex numbers $\mathbb{C}$ there exist infinitely many inequivalent non-Archimedean valuations which make $\mathbb{C}$ into complete valued fields (see [1] p. 46) with non-Archimedean geometries. Hyperreal numbers in nonstandard analysis also have non-Archimedean properties. In the sequel we will restrict to the spaces of $p$-adic numbers as presently the most important class of non-Archimedean geometries. In the rest of this section, a brief review of some basic properties of $p$-adic numbers and their functions will be presented.

There are physical and mathematical reasons to introduce $p$-adic numbers starting with the field of rational numbers $\mathbb{Q}$ and performing completions with respect to its non-Archimedean valuations. From physical point of view, numerical results of all experiments and observations are some rational numbers. From algebraic point of view, $\mathbb{Q}$ is the simplest number field of characteristic 0. Recall that any $0 \neq x \in \mathbb{Q}$ can be presented as infinite expansions into the two essentially different forms [1]:

$$x = \sum_{k=n}^{-\infty} a_k10^k, \ a_k = 0, 1, \cdots, 9, \ a_n \neq 0,$$ \quad (2.3)
\[ x = \sum_{k=m}^{\infty} b_k p^k, \quad b_k = 0, 1, \ldots, p - 1, \quad b_m \neq 0, \quad (2.4) \]

where (2.3) is the ordinary one to the base 10, (2.4) is another to the base \( p \) (\( p \) is any prime number), and \( n, m \) are some integers which depend on \( x \). The above representations (2.3) and (2.4) exhibit the usual repetition of digits, however the expansions are in the mutually opposite directions. The series (2.3) and (2.4) are convergent with respect to the metrics induced by the usual absolute value \(| \cdot |_\infty\) and \( p \)-adic absolute value (\( p \)-adic norm, \( p \)-adic valuation) \(| \cdot |_p\), respectively. Due to the Ostrowski theorem these valuations exhaust all possible inequivalent non-trivial norms on \( \mathbb{Q} \). Performing completions to all non-trivial norms, \( i.e. \) allowing all possible combinations for digits, one obtains standard representation of real and \( p \)-adic numbers in the form (2.3) and (2.4), respectively. Thus, the field of real numbers \( \mathbb{R} \) and the fields of \( p \)-adic numbers \( \mathbb{Q}_p \) exhaust all number fields which may be obtained by completion of \( \mathbb{Q} \) and consequently contain \( \mathbb{Q} \) as a dense subfield. Since \( p \)-adic norm of any term in (2.4) is \(|b_k p^k|_p = p^{-k}\) if \( b_k \neq 0\), geometry of \( p \)-adic numbers is the non-Archimedean one, \( i.e. \) strong triangle inequality \(|x + y|_p \leq \max(|x|_p, |y|_p)\) holds and \(|x|_p = p^{-m}\). \( \mathbb{R} \) and \( \mathbb{Q}_p \) have many distinct algebraic and geometric properties.

Unlike the real case, there are different algebraic extensions for all orders of \( p \)-adic algebraic equations. Algebraic closure of \( \mathbb{Q}_p \) is an infinite \( \mathbb{Q}_p \)-vector space and it is not metrically complete. After completion it becomes the field of \( p \)-adic complex numbers \( \mathbb{C}_p \), which is also algebraically closed, and is a \( p \)-adic analogue of the ordinary \( \mathbb{C} \). This \( \mathbb{C}_p \) has much richer structure than \( \mathbb{C} \) and offers many new possibilities in related analysis and possible applications.

It is often of interest for applications the ring of \( p \)-adic integers \( \mathbb{Z}_p = \{ x \in \mathbb{Q}_p : |x|_p \leq 1 \} \), \( i.e. \) \( p \)-adic integers have the representation \( x = x_0 + x_1 p + x_2 p^2 + \cdots \).

\( p \)-Adic numbers can be suitably visualized by means of trees \([2]\) and as fractals in the Euclidean spaces \([3]\). \( \mathbb{Z}_p \) has property \( \mathbb{Z}_p \supset p \mathbb{Z}_p \supset p^2 \mathbb{Z}_p \supset p^3 \mathbb{Z}_p \cdots \) being a natural mathematical tool to investigate physical structures with a hierarchy.

\( \mathbb{Z}_p \) is topologically compact and complete, and \( \mathbb{Z} \) is dense in \( \mathbb{Z}_p \). \( \mathbb{Q}_p \) is locally compact, separable and totally disconnected complete topological space. Some \( p \)-adic spaces have rather exotic properties: (i) all triangles are isosceles and unequal side is the shortest one, (ii) two discs cannot partially intersect, and (iii) every point of a disc may be regarded as its center.

There is no natural ordering on \( \mathbb{Q}_p \). However one can introduce a linear order on \( \mathbb{Q}_p \) in the following way: \( x < y \) if \(|x|_p < |y|_p\), or if \(|x|_p = |y|_p\) then there exists...
such index \( r \geq 0 \) that digits satisfy \( x_0 = y_0, x_1 = y_1, \ldots, x_{r-1} = y_{r-1}, x_r < y_r \). Here, \( x_k \) and \( y_k \) are digits related to \( x \) and \( y \), respectively, in their expansions of the form \( x = p^m(x_0 + x_1 p + x_2 p^2 \cdots) \). This ordering is very useful in calculation of \( p \)-adic path integrals by the time discretization method.

There are primary two kinds of analyses on \( \mathbb{Q}_p \) which are of interest for physics, and they are based on two different mappings: \( \mathbb{Q}_p \rightarrow \mathbb{Q}_p \) and \( \mathbb{Q}_p \rightarrow \mathbb{C} \). We use both, in classical and quantum \( p \)-adic models, respectively.

Elementary \( p \)-adic valued functions and their derivatives are defined by the same power series (i.e with the same rational coefficients but \( p \)-adic arguments) as in the real case. Their regions of convergence are determined by means of \( p \)-adic norm and they are usually restricted to \(|x|_p < 1\). It is worth noting that \( \sum_{n=0}^{\infty} P_k(n)n!x^n \), where \( P_k(n) \) is a polynomial in \( n \) of degree \( k \) with integer coefficients, converges on \(|x|_p \leq 1\) for all \( p \) and their rational summation is investigated in Ref. [4]. As a definite \( p \)-adic valued integral of an analytic function \( f(x) = f_0 + f_1 x + f_2 x^2 + \cdots \) one takes difference of the corresponding antiderivative in end points, \( i.e. \)

\[
\int_a^b f(x) = \sum_{n=0}^{\infty} \frac{f_n}{n+1} (b^{n+1} - a^{n+1}).
\] (2.5)

Usual complex-valued functions of \( p \)-adic variable, which are employed in mathematical physics, are: \( i \) an additive character \( \chi_p(x) = \exp 2\pi i \{x\}_p \), where \( \{x\}_p \) is the fractional part of \( x \in \mathbb{Q}_p \), \( ii \) a multiplicative character \( \pi_s(x) = |x|^s_p \), where \( s \in \mathbb{C} \), and \( iii \) locally constant functions with compact support, like \( \Omega(|x|_p) \), where

\[
\Omega(|x|_p) = \begin{cases} 
1, & |x|_p \leq 1, \\
0, & |x|_p > 1.
\end{cases}
\] (2.6)

There is well defined Haar measure and integration [5]. So, we have

\[
\int_{\mathbb{Q}_p} \chi_p(ayx) \, dx = \delta_p(ay) = \left| a \right|_p^{-1} \delta_p(y), \quad a \neq 0,
\] (2.7)

\[
\int_{\mathbb{Q}_p} \chi_p(\alpha x^2 + \beta x) \, dx = \lambda_p(\alpha) \left| 2\alpha \right|_p^{-\frac{1}{2}} \chi_p \left( -\frac{\beta^2}{4\alpha} \right), \quad \alpha \neq 0,
\] (2.8)

where \( \delta_p(u) \) is the \( p \)-adic Dirac \( \delta \) function. The number-theoretic function \( \lambda_p(x) \) in (2.8) is a map \( \lambda_p : \mathbb{Q}_p^* \rightarrow \mathbb{C} \) defined as follows:

\[
\lambda_p(x) = \begin{cases} 
1, & m = 2j, \quad p \neq 2, \\
\left( \frac{m}{p} \right), & m = 2j + 1, \quad p \equiv 1 (\text{mod } 4), \\
i \left( \frac{m}{p} \right), & m = 2j + 1, \quad p \equiv 3 (\text{mod } 4),
\end{cases}
\] (2.9)
\[
\lambda_2(x) = \begin{cases} 
\frac{1}{\sqrt{2}}[1 + (-1)^{x_1}i], & m = 2j, \\
\frac{1}{\sqrt{2}}(-1)^{x_1+x_2}[1 + (-1)^{x_2}i], & m = 2j + 1,
\end{cases} \tag{2.10}
\]

where \(x\) is presented as \(x = p^m (x_0 + x_1 p + x_2 p^2 + \cdots)\), and \(m, j \in \mathbb{Z}\). \(\left(\frac{a}{p}\right)\) is the Legendre symbol defined as

\[
\left(\frac{a}{p}\right) = \begin{cases} 
1, & \text{if } a \equiv y^2 \pmod{p}, \\
-1, & \text{if } a \not\equiv y^2 \pmod{p}, \\
0, & \text{if } a \equiv 0 \pmod{p},
\end{cases} \tag{2.11}
\]

and \(\mathbb{Q}_p^* = \mathbb{Q}_p \setminus \{0\}\). It is often sufficient to use their standard properties:

\[
\lambda_p(a^2 x) = \lambda_p(x), \quad \lambda_p(x) \lambda_p(-x) = 1, \quad \lambda_p(x) \lambda_p(y) = \lambda_p(x + y) \lambda_p(x^{-1} + y^{-1}),
\]

\[|\lambda_p(x)|_\infty = 1, \quad a \neq 0. \tag{2.12}\]

Recall that the real analogues of (2.7) and (2.8) have the same form, \(i.e.\)

\[
\int_{\mathbb{Q}_\infty} \chi_\infty(axy) \, dx = \delta_\infty(ay) = |a|_\infty^{-1} \delta_\infty(y), \quad a \neq 0, \tag{2.13}\]

\[
\int_{\mathbb{Q}_\infty} \chi_\infty(\alpha x^2 + \beta x) \, dx = \lambda_\infty(\alpha) |2\alpha|_\infty^{\frac{1}{2}} \chi_\infty \left( -\frac{\beta^2}{4\alpha} \right), \quad \alpha \neq 0, \tag{2.14}\]

where \(\mathbb{Q}_\infty \equiv \mathbb{R}\), \(\chi_\infty(x) = \exp(-2\pi i x)\) is an additive character in the real case and \(\delta_\infty\) is the ordinary Dirac \(\delta\) function. Function \(\lambda_\infty(x)\) is defined as

\[
\lambda_\infty(x) = \sqrt{\frac{\text{sign } x}{i}}, \quad x \in \mathbb{R}^* = \mathbb{R} \setminus \{0\} \tag{2.15}\]

and exhibits the same properties (2.12).

For a more information on usual properties of \(p\)-adic numbers and related analysis one can see [6, 1, 5, 7, 8, 9].

3 Adeles and Their Functions

Real and \(p\)-adic numbers are unified in the form of adeles. An adele \(x [9, 10, 11]\) is an infinite sequence

\[
x = (x_\infty, x_2, \cdots, x_p, \cdots), \tag{3.1}\]

where \(x_\infty \in \mathbb{R}\) and \(x_p \in \mathbb{Q}_p\) with the restriction that for all but a finite set \(S\) of primes \(p\) one has \(x_p \in \mathbb{Z}_p\). Rational numbers are naturally embedded in the space
of adeles. Componentwise addition and multiplication are ordinary arithmetical operations on the ring of adeles $\mathcal{A}$, which can be regarded as

$$\mathcal{A} = \bigcup_S \mathcal{A}(S), \quad \mathcal{A}(S) = \mathbb{R} \times \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p.$$  \hfill (3.2)

$\mathcal{A}$ is a locally compact topological space.

There are also two kinds of analyses over topological ring of adeles $\mathcal{A}$, which are generalizations of the corresponding analyses over $\mathbb{R}$ and $\mathbb{Q}_p$. The first one is related to the mapping $\mathcal{A} \to \mathcal{A}$ and the other one to $\mathcal{A} \to \mathbb{C}$. In complex-valued adelic analysis it is worth mentioning an additive character

$$\chi(x) = \chi_\infty(x_\infty) \prod_p \chi_p(x_p),$$  \hfill (3.3)

a multiplicative character

$$|x|^s = |x_\infty|^s \prod_p |x_p|^s, \quad s \in \mathbb{C},$$  \hfill (3.4)

and elementary functions of the form

$$\phi(x) = \phi_\infty(x_\infty) \prod_{p \in S} \phi_p(x_p) \prod_{p \notin S} \Omega(|x_p|),$$  \hfill (3.5)

where $\phi_\infty(x_\infty)$ is an infinitely differentiable function on $\mathbb{R}$ such that $|x_\infty|^n \phi_\infty(x_\infty) \to 0$ as $|x_\infty| \to \infty$ for any $n \in \{0, 1, 2, \cdots\}$, and $\phi_p(x_p)$ are some locally constant functions with compact support. All finite linear combinations of elementary functions (3.5) make the set $S(\mathcal{A})$ of the Schwartz-Bruhat adelic functions. The Fourier transform of $\phi(x) \in S(\mathcal{A})$, which maps $S(\mathcal{A})$ onto $\mathcal{A}$, is

$$\tilde{\phi}(y) = \int_\mathcal{A} \phi(x) \chi(xy) dx,$$  \hfill (3.6)

where $\chi(xy)$ is defined by (3.3) and $dx = dx_\infty dx_2 dx_3 \cdots$ is the Haar measure on $\mathcal{A}$.

One can define the Hilbert space on $\mathcal{A}$, which we will denote by $L_2(\mathcal{A})$. It contains infinitely many complex-valued functions of adelic argument (for example, $\Psi_1(x), \Psi_2(x), \cdots$) with scalar product

$$(\Psi_1, \Psi_2) = \int_\mathcal{A} \bar{\Psi}_1(x) \Psi_2(x) dx$$

and norm

$$||\Psi|| = (\Psi, \Psi)^{\frac{1}{2}} < \infty.$$
A basis of $L_2(\mathcal{A})$ may be given by the set of orthonormal eigenfunctions in a spectral problem of the evolution operator $U(t)$, where $t \in \mathcal{A}$. Such eigenfunctions have the form

$$\psi_{S,\alpha}(x, t) = \psi_\infty(x, t) \prod_{p \in S} \psi_{\alpha_p}(x_p, t_p) \prod_{p \not\in S} \Omega(|x_p|), \quad (3.7)$$

where $\psi_\infty \in L_2(\mathbb{R})$ and $\psi_{\alpha_p} \in L_2(\mathbb{Q}_p)$ are eigenfunctions in ordinary and $p$-adic cases, respectively. Indices $n, \alpha_2, \cdots, \alpha_p, \cdots$ are related to the corresponding real and $p$-adic eigenvalues of the same observable in a physical system. $\Omega(|x_p|)$ is an element of $L_2(\mathbb{Q}_p)$, defined by (2.6), which is invariant under transformation of an evolution operator $U_p(t)$ and provides convergence of the infinite product (3.7). For a fixed $S$, states $\psi_{S,\alpha}(x, t)$ in (3.7) are eigenfunctions of $L_2(\mathcal{A}(S))$, where $\mathcal{A}(S)$ is a subset of adeles $\mathcal{A}$ defined by (3.2). Elements of $L_2(\mathcal{A})$ may be regarded as superpositions $\Psi(x) = \sum_S C(S)\Psi_S(x)$, where $\Psi_S(x) \in L_2(\mathcal{A}(S))$, i.e.

$$\Psi_S(x) = \Psi_\infty(x) \prod_{p \in S} \Psi_p(x_p) \prod_{p \not\in S} \Omega(|x_p|), \quad x \in \mathcal{A}, \quad (3.8)$$

and $\sum_S |C(S)|^2 = 1$.

Theory of $p$-adic generalized functions is presented in Ref. [5]. Construction of generalized functions on adelic spaces is a hard task, but there is some progress within adelic quantum mechanics [12].

4 Quantum Mechanics on Adelic Spaces

There are a number of reasons to use $p$-adic numbers and adeles in investigation of mathematical and theoretical aspects of modern quantum physics. Some primary of them are: (i) the field of rational numbers $\mathbb{Q}$, which contains all observational and experimental numerical data, is a dense subfield not only in $\mathbb{R}$ but also in the fields of $p$-adic numbers $\mathbb{Q}_p$, (ii) there is an analysis [5] within and over $\mathbb{Q}_p$ like that one related to $\mathbb{R}$, (iii) general mathematical methods and fundamental physical laws should be invariant [13] under an interchange of the number fields $\mathbb{R}$ and $\mathbb{Q}_p$, (iv) there is a quantum gravity uncertainty $\Delta x$ (see (1.1)), when measures distances around the Planck length $\ell_0$, which restricts priority of Archimedean geometry based on real numbers and gives rise to employment of non-Archimedean geometry related to $p$-adic numbers, (v) it seems to be quite reasonable to extend standard Feynman’s path integral method to non-Archimedean spaces, and (vi) adelic quantum mechanics [14] is consistent with all the above assertions.
In order to investigate adelic quantum theory in a systematic way it is natural to start by formulation of adelic quantum mechanics. According to [14], adelic quantum mechanics contains three main ingredients \((L_2(A), W(z), U(t))\) where: (i) \(L_2(A)\) is an adelic Hilbert space, (ii) \(W(z)\) denotes Weyl quantization of complex-valued functions on adelic classical phase space, and (iii) \(U(t)\) is the unitary representation of an evolution operator on \(L_2(A)\).

Canonical noncommutativity in \(p\)-adic case can be represented by the Weyl operators \((\hbar = 1)\)

\[
\hat{Q}_p(\alpha)\psi_p(x) = \chi_p(\alpha x)\psi_p(x) \quad (4.1)
\]

\[
\hat{K}_p(\beta)\psi_p(x) = \psi_p(x + \beta) \quad (4.2)
\]

in the form

\[
\hat{Q}_p(\alpha)\hat{K}_p(\beta) = \chi_p(\alpha\beta)\hat{K}_p(\beta)\hat{Q}_p(\alpha). \quad (4.3)
\]

It is possible to introduce the product of unitary operators

\[
\hat{W}_p(z) = \chi_p(-\frac{1}{2}qk)\hat{K}_p(\beta)\hat{Q}_p(\alpha), \quad z \in Q_p \times Q_p, \quad (4.4)
\]

which is a unitary representation of the Heisenberg-Weyl group. Recall that this group consists of the elements \((z, \alpha)\) with the group product

\[
(z, \alpha) \cdot (z', \alpha') = (z + z', \alpha + \alpha' + \frac{1}{2}B(z, z')), \quad (4.5)
\]

where \(B(z, z') = -kq' + qk'\) is a skew-symmetric bilinear form on the phase space.

Dynamics of a \(p\)-adic quantum model is described by a unitary evolution operator \(U_p(t)\) in terms of its kernel \(K^{(p)}_t(x, y)\)

\[
U_p(t)\psi^{(p)}(x) = \int_{Q_p} K^{(p)}_t(x, y)\psi^{(p)}(y)dy. \quad (4.6)
\]

In this way, \(p\)-adic quantum mechanics [15] is given by a triple

\[
(L_2(Q_p), W_p(z_p), U_p(t_p)). \quad (4.7)
\]

Keeping in mind that ordinary quantum mechanics can be also given as the analogue of (4.7), ordinary and \(p\)-adic quantum mechanics can be unified in the form of the above-mentioned adelic quantum mechanics [14].

Adelic evolution operator \(U(t)\) is defined by

\[
U(t)\psi(x) = \int_{A} K_t(x, y)\psi(y)dy = \prod_v \int_{Q_v} K^{(v)}_t(x_v, y_v)\psi^{(v)}(y_v)dy_v. \quad (4.8)
\]
where \( v = \infty, 2, 3, \ldots, p, \ldots \). The eigenvalue problem for \( U(t) \) reads

\[
U(t)\psi_\alpha(x) = \chi(E_\alpha t)\psi_\alpha(x),
\]

(4.9)

where \( \psi_\alpha \) are adelic eigenfunctions, \( E_\alpha = (E_\infty, E_2, \ldots, E_p, \ldots) \) is the corresponding adelic energy. Note that any adelic eigenfunction has the form (3.7).

A suitable way to compute \( p \)-adic propagator \( K_p(x'', t''; x', t') \) is to use Feynman’s path integral method, i.e.

\[
K_p(x'', t''; x', t') = \int_{x', t'}^{x'', t''} \chi_p \left( \frac{-1}{\hbar} \int_{t'}^{t''} L(\dot{q}, q, t) dt \right) Dq.
\]

(4.10)

It has been evaluated [16, 17] for quadratic Lagrangians in the same way for real and \( p \)-adic cases, and the following exact general expression for propagator is obtained:

\[
K_v(x'', t''; x', t') = \lambda_v \left( -\frac{1}{2\hbar} \frac{\partial^2 \bar{S}}{\partial x'' \partial x'} \right) \left| \frac{1}{\hbar} \frac{\partial^2 \bar{S}}{\partial x'' \partial x'} \right|^{\frac{1}{2}} \chi_v(\frac{-1}{\hbar} \bar{S}(x'', t''; x', t')),
\]

(4.11)

where \( \lambda_v \) functions satisfy relations (2.12) and \( \bar{S}(x'', t''; x', t') \) is the action for classical trajectory. When one has a system with more than one dimension with uncoupled spatial coordinates, then the total \( v \)-adic propagator is the product of the corresponding one-dimensional propagators.

As an illustration of \( p \)-adic and adelic quantum-mechanical models, the following one-dimensional systems with the quadratic Lagrangians were considered: a free particle and a harmonic oscillator [5, 14], a particle in a constant field [18], a free relativistic particle [19] and a harmonic oscillator with time-dependent frequency [20].

Adelic quantum mechanics takes into account ordinary as well as \( p \)-adic quantum effects and may be regarded as a starting point for construction of a more complete quantum cosmology and string/M-theory. In the low-energy limit adelic quantum mechanics effectively becomes the ordinary one [19].

5 Adelic Quantum Cosmology

The main task of quantum cosmology is to describe the very early stage in the evolution of the Universe. At this stage, the Universe was in a quantum state, which should be described by a wave function. Usually one takes it that this wave function is complex-valued and depends on some real parameters. Since quantum
cosmology is related to the Planck scale phenomena it is natural to reconsider its foundations. We maintain here the standard point of view that the wave function takes complex values, but we treat its arguments in a more complete way. Namely, we regard space-time coordinates, gravitational and matter fields to be adelic, i.e. they have real as well as $p$-adic properties simultaneously.

As there is no appropriate complex-valued $p$-adic Schrödinger equation, so there is not also $p$-adic generalization of the Wheeler - De Witt equation for cosmological models. Instead of differential approach, Feynman’s path integral method was exploited [21] and minisuperspace cosmological models are investigated as models of adelic quantum mechanics [22, 23].

Adelic minisuperspace quantum cosmology is an application of adelic quantum mechanics to the cosmological models. In the path integral approach to standard quantum cosmology, the starting point is Feynman’s path integral method, i.e. the amplitude to go from one state with intrinsic metric $h'_{ij}$ and matter configuration $\phi'$ on an initial hypersurface $\Sigma'$ to another state with metric $h''_{ij}$ and matter configuration $\phi''$ on a final hypersurface $\Sigma''$ is given by the path integral

$$K_\infty(h''_{ij}, \phi'', \Sigma''; h'_{ij}, \phi', \Sigma') = \int \mathcal{D}(g_{\mu\nu})_{\infty} \mathcal{D}(\Phi)_{\infty} \chi_{\infty}(-S_{\infty}[g_{\mu\nu}, \Phi]),$$  \hspace{1cm} (5.1)

over all four-geometries $g_{\mu\nu}$ and matter configurations $\Phi$, which interpolate between the initial and final configurations. In (5.1) $S_{\infty}[g_{\mu\nu}, \Phi]$ is an Einstein-Hilbert action for the gravitational and matter fields. This action can be calculated using metric in the standard 3+1 decomposition

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(N^2 - N_i N^i)dt^2 + 2N_i dx^i dt + h_{ij}dx^i dx^j,$$  \hspace{1cm} (5.2)

where $N$ and $N_i$ are the lapse and shift functions, respectively. To perform $p$-adic and adelic generalization we make first $p$-adic counterpart of the action using form-invariance under change of real to the $p$-adic number fields. Then we generalize (5.1) and introduce $p$-adic complex-valued cosmological amplitude

$$K_p(h''_{ij}, \phi'', \Sigma''; h'_{ij}, \phi', \Sigma') = \int \mathcal{D}(g_{\mu\nu})_p \mathcal{D}(\Phi)_p \chi_p(-S_p[g_{\mu\nu}, \Phi]).$$  \hspace{1cm} (5.3)

Since the space of all three-metrics and matter field configurations on a three-surface, called superspace, has infinitely many dimensions, in computation one takes an approximation. A useful approximation is to truncate the infinite degrees of freedom to a finite number $q_\alpha(t)$, ($\alpha = 1, 2, ..., n$). In this way, one obtains a minisuperspace model. Usually, one restricts the four-metric to be of the form (5.2),
with $N^i = 0$ and $h_{ij}$ approximated by $q_\alpha(t)$. For the homogeneous and isotropic cosmologies the metric is a Robertson-Walker one, which spatial sector reads

$$h_{ij}dx^i dx^j = a^2(t)d\Omega^2_3 = a^2(t)\left[d\chi^2 + \sin^2(\chi)(d\theta^2 + \sin^2\theta d\varphi^2)\right], \quad (5.4)$$

where $a(t)$ is a scale factor. If we use also a single scalar field $\phi$, as a matter content of the model, minisuperspace coordinates are $a$ and $\phi$. More generally, models can be homogeneous but also anisotropic ones.

For the boundary condition $q_\alpha(t'') = q''_\alpha$, $q_\alpha(t') = q'_\alpha$ in the gauge $N = 1$, we have $v$-adic minisuperspace propagator

$$K_v(q''_\alpha, t''; q'_\alpha, t') = \int \mathcal{D}q_\alpha \chi_v(-S_v[q_\alpha]), \quad (5.6)$$

where

$$S_v[q_\alpha] = \int_{t'}^{t''} dt \left[\frac{1}{2} f_{\alpha\beta}(q) \dot{q}_\alpha \dot{q}_\beta - U(q)\right], \quad (5.7)$$

is an ordinary quantum-mechanical propagator between fixed minisuperspace coordinates $(q'_\alpha, q''_\alpha)$ in fixed times. $S_v$ is the $v$-adic action of the minisuperspace model, i.e.

$$K_v(q''_\alpha, t''; q'_\alpha, t') = \int \mathcal{D}q_\alpha \chi_v(-S_v[q_\alpha]), \quad (5.6)$$

where $f_{\alpha\beta}$ is a minisuperspace metric $(ds^2_m = f_{\alpha\beta}dq^\alpha dq^\beta)$ with an indefinite signature $(-, +, +, \ldots)$. This metric includes spatial (gravitational) components and also matter variables for the given model.

The standard minisuperspace ground-state wave function in the Hartle-Hawking (no-boundary) proposal [24] is defined by functional integration in the Euclidean version of

$$\Psi_\infty[h_{ij}] = \int \mathcal{D}(g_{\mu\nu})_{\infty} \mathcal{D}(\Phi)_{\infty} \chi_{\infty}(-S_{\infty}[g_{\mu\nu}, \Phi]), \quad (5.8)$$

over all compact four-geometries $g_{\mu\nu}$ which induce $h_{ij}$ at the compact three-manifold. This three-manifold is the only boundary of the all four-manifolds. Extending Hartle-Hawking proposal to the $p$-adic minisuperspace, an adelic Hartle-Hawking wave function is the infinite product

$$\Psi[h_{ij}] = \prod_v \int \mathcal{D}(g_{\mu\nu})_v \mathcal{D}(\Phi)_v \chi_v(-S_v[g_{\mu\nu}, \Phi]), \quad (5.9)$$

where path integration must be performed over both, Archimedean and non-Archimedean geometries. If evaluation of the corresponding functional integrals for a minisuperspace model yields $\Psi(q_\alpha)$ in the form (3.7), then we say that such cosmological model
is a Hartle-Hawking adelic one. It is shown [21] that the de Sitter minisuperspace model in $D = 4$ space-time dimensions is the Hartle-Hawking adelic one.

It is shown in [22, 23] that $p$-adic and adelic generalization of the minisuperspace cosmological models can be successfully performed in the framework of $p$-adic and adelic quantum mechanics [14] without use of the Hartle-Hawking approach. The following cosmological models are investigated: the de Sitter model, model with a homogeneous scalar field, anisotropic Bianchi model with three scale factors and some two-dimensional minisuperspace models. As a result of $p$-adic effects and adelic approach, in these models there is some discreteness of minisuperspace and cosmological constant. This kind of discreteness was obtained for the first time in the context of the Hartle-Hawking adelic de Sitter quantum model [21].

6 Adelic String/M-theory

A notion of $p$-adic string was introduced in [25], where the hypothesis on the existence of non-Archimedean geometry at the Planck scale was made, and string theory with $p$-adic numbers was initiated. In particular, generalization of the usual Veneziano and Virasoro-Shapiro amplitudes with complex-valued multiplicative characters over various number fields was proposed and $p$-adic valued Veneziano amplitude was constructed by means of $p$-adic interpolation. Very successful $p$-adic analogues of the Veneziano and Virasoro-Shapiro amplitudes were proposed in [26] as the corresponding Gel’fand-Graev [9] beta functions. Using this approach, Freund and Witten obtained [27] an attractive adelic formula, which states that the product of the crossing symmetric Veneziano (or Virasoro-Shapiro) amplitude and its all $p$-adic counterparts equals unit (or a definite constant). This gives possibility to consider an ordinary four-point function, which is rather complicate, as an infinite product of its inverse $p$-adic analogues, which have simple forms. These first papers induced an interest in various aspects of $p$-adic string theory (for a review, see [28, 5]). A recent interest in $p$-adic string theory has been mainly related to the generalized adelic formulas for four-point string amplitudes [29], the tachyon condensation [30], nonlinear dynamics [31] and the new attractive adelic approach [32].

Like in the ordinary string theory, the starting point in $p$-adic string theory is a construction of the corresponding scattering amplitudes. Recall that the ordinary crossing symmetric Veneziano amplitude can be presented in the following forms:
\[ A_\infty(k_1, \cdots, k_4) \equiv A_\infty(a, b) \]

\[ = g^2 \int_\mathbb{R} |x|_\infty^{a-1} |1 - x|_\infty^{b-1} dx \quad (6.1) \]

\[ = g^2 \left[ \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} + \frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)} + \frac{\Gamma(c)\Gamma(a)}{\Gamma(c+a)} \right] \quad (6.2) \]

\[ = g^2 \frac{\zeta(1-a)}{\zeta(a)} \frac{\zeta(1-b)}{\zeta(b)} \frac{\zeta(1-c)}{\zeta(c)} \quad (6.3) \]

\[ = g^2 \int D\mathbf{x} \exp \left( -\frac{i}{2\pi} \int d^2\sigma \partial^\alpha X_\mu \partial_\alpha X^\mu \right) \prod_{j=1}^4 \int d^2\sigma_j \exp \left( ik^{(j)} X^\mu \right), \quad (6.4) \]

where \( \hbar = 1 \), \( T = 1/\pi \), and \( a = -\alpha(s) = -1 - \frac{s}{2}, \ b = -\alpha(t), \ c = -\alpha(u) \) with the condition \( s + t + u = -8 \), i.e. \( a + b + c = 1 \).

To introduce a \( p \)-adic Veneziano amplitude one can consider \( p \)-adic analogues of all the above four expressions. \( p \)-Adic generalization of the first expression was proposed in [26] and it reads

\[ A_p(a, b) = g_p^2 \int_{Q_p} |x|_p^{a-1} |1 - x|_p^{b-1} dx, \quad (6.5) \]

where \( | \cdot |_p \) denotes \( p \)-adic absolute value. In this case only string world-sheet parameter \( x \) is treated as \( p \)-adic variable, and all other quantities maintain their usual (real) valuation. An attractive adelic formula of the form

\[ A_\infty(a, b) \prod_p A_p(a, b) = 1 \quad (6.6) \]

was found [27], where \( A_\infty(a, b) \) denotes the usual Veneziano amplitude (6.1). A similar product formula holds also for the Virasoro-Shapiro amplitude. These infinite products are divergent, but they can be successfully regularized. Unfortunately, there is a problem to extend this formula to the higher-point functions.

\( p \)-Adic analogues of (6.2) and (6.3) were also proposed in [25] and [33], respectively. In these cases, world-sheet, string momenta and amplitudes are manifestly \( p \)-adic. Since string amplitudes are \( p \)-adic valued functions, it is not so far enough clear their physical interpretation.

Expression (6.4) is based on Feynman’s path integral method, which is generic for all quantum systems and has successful \( p \)-adic generalization. \( p \)-Adic counterpart of (6.4) is proposed in [32] and has been partially elaborated in [34] and [35]. Note that in this approach, \( p \)-adic string amplitude is complex-valued, while not only the
world-sheet parameters but also target space coordinates and string momenta are \( p \)-adic variables. Such \( p \)-adic generalization is a natural extension of the formalism of \( p \)-adic [15] and adelic [14] quantum mechanics to string theory. This is a promising subject and should be investigated in detail, and applied to the branes and M-theory, which is presently the best candidate for the fundamental physical theory.

7 Concluding Remarks

One of the very interesting and fruitful recent developments in string theory has been noncommutative geometry and the corresponding noncommutative field theory, which may be regarded as a deformation of the ordinary one in which field multiplication is replaced by the Moyal (star) product

\[
(f \star g)(x) = \exp \left[ \frac{i\hbar}{2} \theta^{ij} \frac{\partial}{\partial y^i} \frac{\partial}{\partial z^j} \right] f(y)g(z)\big|_{y=z=x},
\]

(7.1)

where \( x = (x^1, x^2, \cdots, x^d) \) is a spatial point, and \( \theta^{ij} = -\theta^{ji} \) are noncommutativity parameters. Replacing the ordinary product between noncommutative coordinates by the Moyal product (7.1) we have

\[
x^i \star x^j - x^j \star x^i = i\hbar \theta^{ij},
\]

(7.2)

which resembles the usual Heisenberg algebra. It is worth noting that one can introduce [35] the Moyal product in \( p \)-adic quantum mechanics and it reads

\[
(f \star g)(x) = \int_{\mathcal{Q}_p^d} \int_{\mathcal{Q}_p^d} dk dk' \chi_p(-x^i k_i + x^j k'_j) + \frac{1}{2} k_i k'_j \theta^{ij}) \hat{f}(k)\hat{g}(k'),
\]

(7.3)

where \( d \) denotes spatial dimensionality, and \( \hat{f}(k), \hat{g}(k') \) denote the Fourier transforms of \( f(x) \) and \( g(x) \). Some real, \( p \)-adic and adelic aspects of the noncommutative scalar solitons [36] are investigated in Ref. [37].

A natural extension of adelic quantum mechanics to adelic field theory is considered [38], as well. \( p \)-Adic quantum mechanics with \( p \)-adic valued wave functions has been investigated and presented in [5, 39, 40]. There have been also applications of \( p \)-adic numbers and adèles relevant for some other directions of mathematical research in physics, but I restricted this review to some aspects of quantum theory.

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