Stabilization for a Flexible Beam With Tip Mass and Control Matched Disturbance

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Abstract—This article is concerned with the output feedback exponential stabilization for a flexible beam with tip mass. When there is no disturbance, it is shown that only one noncollocated measurement is enough to exponentially stabilize the original system by constructing an infinite-dimensional Luenberger state observer to track the state and designing an estimated state-based output feedback control law. This essentially improves the existing result in (Conrad et al. 1998) where two collocated measurements, including high-order feedback were adopted. In the case that boundary internal uncertainty and external disturbance are considered, an infinite-dimensional disturbance estimator is constructed to estimate the state and total disturbance in real time. By virtue of the estimated state and estimated total disturbance, an output feedback control law is designed to exponentially stabilize the original system while guaranteeing the boundedness of the closed-loop system. Some illustration simulations are presented.

Index Terms—Beam equation, disturbance estimator, exponential stability, Riesz basis.

I. INTRODUCTION

This article is concerned with the dynamic stabilization for a flexible beam with tip mass, which describes the spacecraft control laboratory experiment (SCOLE) model in the sense that the moment of inertia at \( x = 1 \) is neglected. Such system with boundary control matched internal uncertainty and external disturbance can be written mathematically as follows:

\[
\begin{align*}
\dot{w}_t(x,t) + w_{xx}(x,t) = 0, & \quad x \in (0, 1), \quad t > 0, \\
w(0,t) = w_x(0, t) = w_x(1, t) = 0, & \quad t \geq 0, \\
-w_{xx}(1, t) + mw_{tt}(1, t) = u(t) + f(t), & \quad t \geq 0, \\
y(t) = (w_x(0, t), w(1, t))^T, & \quad t \geq 0
\end{align*}
\]

where \( w(x, t) \) is the displacement of the beam at position \( x \) and time \( t \), \( m > 0 \) is the tip mass, \( u(t) \) is the boundary shear control at the free end of the beam, \( F(t) = w(t), w_x(t) + d(t), v : H^2(0, 1) \times L^2(0, 1) \rightarrow \mathbb{R} \) is the internal uncertainty, \( d(t) \) is the external disturbance, and \( y(t) \) is the observation with \( w_{xx}(0, t) \) being bending strain at \( x = 0 \). System (1) can be derived by Hamilton principle [10], [11].

We consider system (1) in the state Hilbert space \( \mathcal{H}_1 = H_2^2(0, 1) \times L^2(0, 1) \times \mathbb{R} \) with inner product induced norm given by

\[
\| (f, g, \eta) \|_{\mathcal{H}_1} = \int_0^1 [|f''(x)|^2 + |g(x)|^2]dx + \frac{1}{2} |\eta|^2, \quad (f, g, \eta) \in \mathcal{H}_1,
\]

where \( H_2^2(0, 1) = \{ f \in H^2(0, 1) : f(0) = f'(0) = 0 \} \). Define the operator \( A_1 : \mathcal{D}(A_1) \subset \mathcal{H}_1 \rightarrow \mathcal{H}_1 \) by \( A_1(f, g, \eta) = (g, -f') \), \( f'''(1) \) \( \forall (f, g, \eta) \in \mathcal{D}(A_1), \mathcal{D}(A_1) = \{ (f, g, \eta) \in (H^4(0, 1) \cap H_2^2(0, 1)) \times H_2^2(0, 1) \times \mathbb{R} \}, \|

It is routine to check that \( A_1 \) is a skew-adjoint operator and thereby it generates a unity group. System (1) is then abstractly written as follows:

\[
\frac{d}{dt}(w(t), w_x(t), mw_t(t)) = A_1(w(t), w_x(t), mw_t(t)) + B_1[u(t) + F(t)]
\]

where \( B_1 = (0, 0, 1)^T \) is a bounded linear operator.

Proposition 1.1: Suppose that \( v : H^2(0, 1) \times L^2(0, 1) \rightarrow \mathbb{R} \) is continuous and satisfies global Lipschitz condition in \( H^2(0, 1) \times L^2(0, 1) \). Then, for any \((w_0(t), w_x(0, t), mw_t(1, t)) \in \mathcal{H}_1, u, d \in E_{loc}^2(0, \infty)\), there exists a unique global solution (mild solution) to (1) such that \((w(t), w_x(t), mw_t(t)) \in C(0, \infty) \mathcal{H}_1\).

Proof: The proof can be obtained directly from the procedure of [35], Proposition 1.1.

In the case that the disturbance is not considered \((F(t) \equiv 0)\), system (1) has been extensively discussed [1], [6], [15], [28], [30], [32]. It is well known that if \( m = 0 \) (no tip mass is considered), then (1) devolves into Euler–Bernoulli beam equation with shear boundary control, and the boundary velocity feedback \( u(t) = -\alpha w_t(1, t) \) has been proved to exponentially stabilize the original system by virtue of multiplier method or Riesz basis approach [3], [14]. However, in presence of tip mass, boundary velocity feedback \( u(t) = -\alpha w_t(1, t) \) can be regarded as a compact perturbation of the free system \((u(t) = 0)\) [30]; such a compact perturbation makes the closed-loop system strongly stable [28] but cannot guarantee the exponential stability [32]. In order to exponentially stabilize the system, apart from the velocity, high-order feedback, such as \( w_{xx}(1, t) \) should also be considered [30]. In [6], it was proved by virtue of energy multiplier approach that for any \( \beta > 0 \) the closed-loop system under the feedback control law \( y(t) = -\alpha w_t(1, t) + \beta w_{xx}(1, t) \) is exponentially stable. Furthermore, the authors in [6] also proved that there exist a set of generalized eigenfunctions of the closed-loop system that forms a Riesz basis provided \( m = \alpha \beta \). By virtue of their Riesz basis generation theory in [14], Guo [15] proved that the Riesz basis property indeed holds for all the cases \( m, \alpha, \beta > 0 \).

The aforementioned literatures considered colocated control and observation, where the actuators and sensors lie in the same boundary place \( x = 1 \). The colocated design approach was first introduced in circuit theory in 1950s [13]. However, in practical control, the performance of closed-loop system under colocated output feedback may be not so good [5]; noncolocated control design has been widely used [5], [33]. Since the closed-loop system under noncolocated design is usually nondissipative, it is hard to apply the traditional Lyapunov method or multiplier approach to discuss the stability. To overcome this difficulty, instead of direct output feedback control design, compensator-based controller design method is a possible choice. In 1975, Gressang and Lamont [12] first presented in semigroup framework a generalized Luenberger stabilizing compensator for infinite-dimensional systems with bounded input and output operators. The authors in [7] and [27] discussed finite-dimensional compensators for infinite-dimensional linear system with unbounded input and bounded/unbounded output operators. By

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The internal uncertainty is taken into consideration.
2) The high-order collocated feedbacks \( w_{xx} \) and \( w_{xxx} \) are not used.
3) We consider arbitrary \( m, \alpha, \beta > 0 \) while Li et al. [29] just solved the special case \( \alpha, \beta > 0 \), and \( m = \alpha \beta \).
4) Since no additional state observer is used, our control strategy is concise and energy saving, and our method provides an idea to improve the results of [35] and [36], see Remark 2.1.

The rest of this article is organized as follows. In Section II, we present an infinite-dimensional disturbance estimator for system (1) to estimated total disturbance and state in real time. An estimated disturbance and estimated state-based control law is then designed. In Section III, the exponential stability of a couple subsystem including the original equation of the closed-loop is concluded. The other state of the closed-loop system is proved to be bounded. In Section IV, we present some simulations. Finally, Section V concludes this article.

In the appendix, we give the observer and controller design for system (1) without disturbance.

II. DISTURBANCE ESTIMATOR AND CONTROLLER DESIGN

In this section, we shall design an infinite-dimensional disturbance estimator to estimate the total disturbance \( F(t) \) of system (1). We first introduce an auxiliary system to transfer the total disturbance \( F(t) \) into an exponentially stable system

\[
\begin{align*}
& l_t(x,t) + l_{xxxx}(x,t) = 0 \\
& l(0,t) = l_x(1,t) = 0 \\
& l_x(0,t) - c_1 l_x(0,t) + \gamma_l(t) + w_{xx}(0,t) = 0 \\
& -l_{xx}(1,t) + m_l(1,t) = u(t)
\end{align*}
\]

where \( c, \gamma > 0 \) are tuning parameters. Obviously, the auxiliary system (2) is the same as the observer (24) in the Appendix designed for the system without disturbance.

Set \( \tilde{x}(t) = l(0,t) - w(t) \) to get

\[
\begin{align*}
\tilde{l}_{tt}(x,t) + \tilde{l}_{xxxx}(x,t) &= 0 \\
\tilde{l}(0,t) &= \tilde{l}_x(1,t) = 0 \\
\tilde{l}_x(0,t) &= c_1 \tilde{l}_x(0,t) + \gamma_l(t) \tilde{l}(0,t) \\
-\tilde{l}_{xx}(1,t) + m_l(1,t) &= -F(t)
\end{align*}
\]

We consider system (3) in the state Hilbert space \( H_2 = H^2_2(0,1) \times L^2(0,1) \times R \). \( H^2_2(0,1) = \{ f \in H^2(0,1), f(0) = 0 \} \), with inner product induced norm given by \( \| (f, g, \eta) \|^2_{H_2} = \int_0^1 |f'(x)|^2 + (g(x)^2)|dx| + |f(0)|^2 + \|\eta\|^2_2, (f, g, \eta) \in H_2 \).

Then, (3) is written abstractly by

\[
\frac{d}{dt} (\tilde{l}(t), \tilde{l}_x(t), m_l(t)) = \mathbf{A}_2 F(t)
\]

where \( \mathbf{A}_2 : D(\mathbf{A}_2)(\subset H_2) \to H_2 \) is defined by \( A_2(f, g, \eta) = (g, -f''(0), f''(0)) \) for \( (f, g, \eta) \in D(\mathbf{A}_2) \). \( D(\mathbf{A}_2) = \{ (f, g, \eta) \in (H^4(0,1) \times H^2_2(0,1)) \times H^2_2(0,1) \times R | f''(0) = cg'(0) + g'f(0), f''(1) = 0, \eta = mg(1) \} \), and \( B_0 = (0, 0, 1)^2 \) is a bounded linear operator. Use [35, Lemmas A.1 and A.2], we derive the following lemma.

Lemma 2.1: Assume that \( c \in L^\infty(0, \infty) \) (or \( c \in L^\infty(\infty, \infty) \)), \( v : H^2(0,1) \times L^2(0,1) \to R \) is continuous and system (1) admits a unique bounded solution \( (w(t), w_x(t), w_{xx}(t)) \in C(0, \infty; H^2(0,1) \times L^2(0,1)) \). Then, for any initial value \( (\tilde{l}(0), \tilde{l}_x(0), m_l(1)) \in H_2 \), there exists a unique solution to system (3) such that \( (\tilde{l}(t), \tilde{l}_x(t), m_l(1, t)) \in C(0, \infty; H_2) \) and \( \| (\tilde{l}(t), \tilde{l}_x(t), m_l(1,t)) \|_{H_2} < \infty \). Moreover, if
limit as $t \to \infty$, $v(w(t), t) = 0$ and $\tilde{d} \in L^2(0, \infty)$, then
\[ \lim_{t \to \infty} \| \ddot{\tilde{u}}(t), \dot{\tilde{y}}(t), m_t(t), \dot{m}_t(t) \| = 0. \]

For error system (3), we design the following observer:
\[
\begin{align*}
\dot{z}_{se}(t) &= z_{se}(t), \\
\dot{z}(t) - \tilde{z}(t) &= 0, \\
\dot{z}(0) &= 0, \\
\dot{z}_{se}(0) &= c \hat{z}_{se}(0) + \gamma \hat{z}_w(0).
\end{align*}
\]

Set $\dot{z}(t) = z(t) - \tilde{z}(t)$ to derive
\[
\begin{align*}
\dot{z}_{se}(t) + \dot{z}_{se}(t) &= 0, \\
\dot{z}(0) &= \hat{z}(0) + \hat{z}(0) = 0, \\
\dot{z}_{se}(0) &= c \hat{z}_{se}(0) + \gamma \hat{z}_w(0).
\end{align*}
\]

Consider system (6) in the space $H = H_0^2(0, 1) \times L^2(0, 1)$, $H_0^2(0, 1) = \{ f \in H^2(0, 1) | f(0) = f(1) = 0 \}$ with inner product induced norm $\| f \|_2 = \int_0^1 |f(x)|^2 dx + |f(0)|^2$. Define $A = \{ f, g \} = \{ (f, g) \in D(\alpha), D(\alpha) = \{ f, g \} \in (H_0^2(0, 1) \times H_0^2(0, 1)) | f(0) = f(1) = 0, f''(0) = \gamma f(0) + \gamma f''(0) \}$. Then, system (6) can be written abstractly as $\dot{z}(t), \dot{z}_w(t) = 0$.

**Lemma 2.2:** There is a sequence of generalized eigenfunctions of $A$, which forms Riesz basis for $H$; $A$ is a generator of exponentially stable $C_0$-semigroup.

Because of the page limitation, the proof Lemma 2.2 is given in a full version.\(^1\) The main idea of the proof is based on the comparison method developed by Guo and Yu [14] and Guo and Wang [20]. Similar to [35, Lemma 2.3], we obtain the following lemma.

**Lemma 2.3:** Assume that $\langle \hat{z}(t), 0 \rangle, \langle \hat{z}_w(t), 0 \rangle \in D(\alpha)$. The solution of (6) satisfies $\| \hat{z}_{se}(t) \| \leq M e^{-\mu t}$ for some constant $M, \mu > 0$.

We put systems (2) and (5) together to derive the following infinite-dimensional error estimator:
\[
\begin{align*}
\dot{l}_t(x, t) + l_{se}(x, t) &= 0, x \in (0, 1), \\
\dot{l}(0, t) &= l_{se}(1, 0), \\
\dot{l}(0) &= c \hat{z}_w(0, t) + w_x(0, t), \\
-\dot{l}_{se}(t) + \dot{l}_{se}(t) &= u(t), \\
\dot{z}(0, t) &= z_{se}(0, t) + \gamma \hat{z}_w(0, t).
\end{align*}
\]

By Lemma 2.3, for $(\hat{z}(t), 0, \hat{z}_w(t), 0) \in D(\alpha)$, $F(t) = \hat{l}_t(x, t, t) - m \hat{l}_t(t, t) = z_{se}(t) - m \hat{z}_t(t, t) = \hat{z}_w(t, t)$, which means that the observer system (5) is a total disturbance estimator. Moreover, (6) decays exponentially, we obtain $\hat{w}(t) = l(t, t) - \hat{l}(t, t) + \hat{z}(t, t) \approx l(t, t) - \hat{z}(t, t)$, this implies that $l(t, t) - \hat{z}(t, t)$ is the estimate of $\hat{w}(t, t)$.

Since the state feedback $u(t) = -w_x(1, t) + \beta w_{se}(1, t)$ exponentially stabilizes system (1) without disturbance, $z_{se}(1, t) - m \hat{z}_t(t, 1) - \hat{z}_w(t, 1)$ is the estimate of total disturbance $F(t)$, and $w(t)$ is estimated by $l(t, t) - \hat{z}(t, t)$, it is natural to design the following controller:
\[
\begin{align*}
u(t) &= -z_{se}(1, t) + m \hat{z}_t(t, 1) - \alpha [l(t, t) - \hat{z}(t, t)] \\
&+ \beta [l_{se}(1, t) - z_{se}(1, t)].
\end{align*}
\]

With the controller (8), we derive the closed-loop system
\[
\begin{align*}
u(w(t), t) + w_{se}(x, t) &= 0, \\
\dot{w}(0, t) &= w_x(0, t) = w_x(1, t) = 0, \\
-w_x(1, t) + m \hat{w}(1, t) &= -z_{se}(1, t) + m \hat{z}_t(1, t) \\
-\alpha [l(t, t) - z(1, t)] + \beta (l_{se}(1, t) - z_{se}(1, t)].
\end{align*}
\]

**Remark 2.1:** In [35] and [36], the disturbance estimators [35, Eq. (6)] and [36, Eq. (2.4)] were only used to estimate the total disturbances; in order to derive estimated states, the authors designed additional Luenberger state observers [35, Eq. (24)] and [36, Eq. (3.1)] of the original systems by compensating the total disturbance. However, we observe that the state of the original system can also be estimated by the disturbance estimator. Therefore, no additional state observer similar to [35, Eq. (24)] and [36, Eq. (3.1)] is designed. Based on this, in control law (8), we directly used the estimated state stemmed from the disturbance estimator. This makes our control strategy more concise and energy saving, and our method provides an idea to simplify [35] and [36].

### III. STABILITY OF THE CLOSE-LOOP SYSTEM

In this section, our objective is to verify that the state $(w(t), w_x(t), w_{se}(t))$ of the closed-loop system (9) is exponentially stable while guaranteeing the boundedness of the other variables. To this end, we first write $w$-part of (9) and (6) together to get
\[
\begin{align*}
u(t, x) + w_{se}(x, t) &= 0, \\
\dot{w}(0, t) &= w_x(0, t) = w_x(1, t) = 0, \\
-w_x(1, t) + m \hat{w}(1, t) &= -z_{se}(1, t) + m \hat{z}_t(1, t) \\
+ \beta w_{se}(1, t) - \hat{z}_{se}(1, t) &= \hat{z}(t, t) \\
\hat{z}(0, t) &= \hat{z}_w(t, t), \\
\hat{z}_w(t) &= \hat{z}_w(0, t) + \gamma \hat{z}_w(0, t).
\end{align*}
\]

It is natural to choose the state Hilbert space of system (10) as $\mathcal{H} = \mathbb{H} \times \mathbb{R}^2$. Define the operator $A_1 : D(A_1) \subset \mathcal{H} \to \mathcal{H}$ by $A_1 = \{ (f, g, \eta, p, q) = (g, -f', -\eta \beta^{-1} + \beta^{-1}(\alpha - m \beta^{-1} q)g) | f, g, p, q \in (H_0^2(0, 1), \mathbb{R}^2) \}$. System (10) is abstractly described by $\dot{X}(t) = A_1 X(t)$, where $X(t) = (w(t), w_x(t), w_{se}(t), \hat{z}_w(t), \hat{z}_w(0, t), \hat{z}_w(0, t))$. It is easily seen that the operator $A_1$ is not dissipative in the current inner product; it seems difficult to find an equivalent inner product in $X_1$ to make $A_1$ dissipative; the multiplier method may not be effective to verify the stability of the semigroup $e^{A_1 t}$. Instead, we shall use Riesz basis approach to prove the stability, the key step is to find out the complicated but important relations (15) and (16) between sequences of generalized eigenfunctions.

**Theorem 3.1:** System (10) is governed by an exponentially stable $C_0$-semigroup.

**Proof:** It is routine to show that $A_1^{-1}$ exists and is compact on $X_1$, that is, $A_1$ is a discrete operator and the spectrum consists of eigenvalues.

\[\text{[Online]. Available: https://arxiv.org/abs/2011.07848}\]
By the same procedure as the proof of Theorem 6.3, we can obtain
\[ \sigma(A_1) = \sigma(A) \cup \sigma(A) \]
Next, we shall show that the generalized eigenfunction of \( A_1 \) forms a Riesz basis for \( X_1 \). Let \( \{ \lambda_n, \phi_n \}_n \) and \( \{ \lambda_{1n}, \phi_{1n} \}_n \) be, respectively, the eigenvalues of \( A \) and \( A_1 \), \( \lambda_n = \tau_n^2 \), \( \lambda_{1n} = \tau_n^2 \).
Let \((2\tau_n^2 e^{-\tau_n} \phi_n, 2\tau_n e^{-\tau_n} \phi_n)_{n=1}^\infty \) and \((2\tau_n^2 e^{-\tau_n} f_n, 2\tau_n e^{-\tau_n} f_n)_{n=1}^\infty \) be the generalized eigenfunctions corresponding to \( \{ \lambda_n \}_n \) and \( \{ \lambda_{1n} \}_n \), respectively. As a result, \((0, 0, 2\tau_n^2 e^{-\tau_n} \phi_n, 2\tau_n e^{-\tau_n} \phi_n)_{n=1}^\infty \) forms a Riesz basis for \( H \) and \( H_1 \), respectively. Therefore, \((0, 0, 2\tau_n^2 e^{-\tau_n} \phi_n, 2\tau_n e^{-\tau_n} \phi_n)_{n=1}^\infty \) forms a Riesz basis for \( H \times H \), equivalent to that of \((0, 0, 2\tau_n^2 e^{-\tau_n} \phi_n, 2\tau_n e^{-\tau_n} \phi_n)_{n=1}^\infty \), \((0, 0, 2\tau_n^2 e^{-\tau_n} \phi_n, 2\tau_n e^{-\tau_n} \phi_n)_{n=1}^\infty \) forms a Riesz basis for \((L^2(0,1)^2 \times C)^2\).
Let \( \lambda = \tau_2^2 \in \sigma(A) \) and \((2\tau_2^2 - \tau_2^2 f, 2\tau_2 e^{-\tau_2} f, -2\tau_2^2 e^{-\tau_2} f_1, 2\tau_2 e^{-\tau_2} f_1) \) be the corresponding eigenfunction.
If \( \phi = 0 \), then \( \lambda \in \sigma(A) \). Hence, in this case the eigenvalues \( \{ \lambda_{1n} \}_n \) corresponds to the eigenfunction \((2\tau_n^2 e^{-\tau_n} f_n, 2\tau_n e^{-\tau_n} f_n)_{n=1}^\infty \) of \( H_1 \), where
\[ \phi_n = \sinh \tau_n x - 1 + \sin \tau_n x \cos \tau_n x - \cos \tau_n x \sin \tau_n x + \frac{2\tau_n}{\alpha \tau_n + \gamma} \sinh \tau_n x + \sin \tau_n x(1 - \cos \tau_n x \sin \tau_n x) - \sin \tau_n x \cos \tau_n x = \frac{2\tau_n}{\alpha \tau_n + \gamma} \sinh \tau_n x + \sin \tau_n x \sin \tau_n x. \]
Then, there holds \((1 + i\beta z^2) P_n(1) = 2\tau_n^2 P_n \), where
\[ P_n = (\beta z_2^2 - \tau_2^2) \left[ \sinh \tau_n x + \sin \tau_n x(1 + \cos \tau_n x \sin \tau_n x) + \frac{\tau(\sin \tau_n x \cos \tau_n x + \cosh \tau_n x \sin \tau_n x)}{\alpha \tau_n^2 + \beta} \right]. \]
Denote by \((2\tau_1^2 e^{-\tau_1^2} f_{1n}, 2\tau_1 e^{-\tau_1^2} f_{1n})_{n=1}^\infty \) the eigenfunction of \( A_1 \) corresponding to \( \lambda_{1n} \).
Then, we have
\[ \begin{cases} f_1(0) = 0, \quad f_1(1) = 0, \\ f_{1n}(0) = 0, \quad f_{1n}(1) = 0, \quad (1 + i\beta z^2) f_{1n}(1) = (\alpha \tau_n^2 - \tau_2^2) f_{1n}(1) + 2\tau_n P_n. \end{cases} \]
The solution of (13) is as follows:
\[ f_n(x) = b_{11} \cos \tau_n x - \sin \tau_n x + b_{12} \sin \tau_n x - \cos \tau_n x \]
where \( b_{11} = \frac{\tau \sin \tau_n x + \cos \tau_n x}{U_n}, \quad b_{12} = \frac{\tau \cos \tau_n x + \cos \tau_n x}{U_n}, \quad U_n = \tau_n \cdot (1 + \cos \tau_n x \cos \tau_n x) \]
and \( \sin \tau_n x - \sin \tau_n x \sin \tau_n x \).
Hence, we have
\[ \begin{align*} 2\tau_1^2 e^{-\tau_1^2} f_{1n}(x) &= -e^{-\tau_1^2} + \cos \tau_1^2 x \\
2\tau_2 e^{-\tau_2^2} f_{1n}(x) &= i(e^{-\tau_2^2} + \cos \tau_2^2 x) \\
-\sin \tau_2^2 x + O(1)^{-1} \end{align*} \]
where \( p = n+1/4 \). Denote \( Q = \begin{pmatrix} I_1 & J \end{pmatrix} \) with \( J = \begin{pmatrix} 1 & 0 \end{pmatrix} \).
Then, \( Q \) is a bounded linear operator and it has bounded inverse. Moreover, we obtain the following relations:
\[ \begin{align*} (2\tau_1^2 e^{-\tau_1^2} f_{1n}, 2\tau_1 e^{-\tau_1^2} f_{1n}) &= \frac{2\beta_1}{\lambda_1^2 + \beta_2} e^{-\tau_1^2} f_{1n}(1, 0, 0) \\
\tau_1 e^{-\tau_1^2} f_{1n} &= \frac{2\beta_1}{\lambda_1^2 + \beta_2} e^{-\tau_1^2} f_{1n}(1, 0, 0) \\
\tau_2 e^{-\tau_2^2} f_{1n}(1) &= \frac{2\beta_1}{\lambda_1^2 + \beta_2} e^{-\tau_1^2} f_{1n}(1, 0, 0) \end{align*} \]
where \( \tau_1 e^{-\tau_1^2} f_{1n} \) is bounded. Hence, \( \lambda_1 \) and \( \lambda_2 \) are the eigenvalues of \( A_1 \).
eigenfunctions of $A$. To overcome this difficulty, the key step is to find out the relations (15) and (16).

By Theorem 3.1, it follows that, there exist two positive constants $M_{A_1}$ and $\omega_{A_1}$ such that:

$$\|e^{A_1t}\|_{H_1} \leq \frac{M_{A_1}}{M_1} e^{\omega_{A_1} t}, \quad t \geq 0. \quad (17)$$

**Theorem 3.2**: Assume that $d \in L^\infty(0, \infty)$ (or $d \in L^2(0, \infty)$) and $v : H^2(0, 1) \times L^2(0, 1) \rightarrow \mathbb{R}$ is continuous. For any initial value $(v(0), w_0(\cdot), l_0(\cdot), \tilde{l}_0(\cdot), \tilde{z}_0(\cdot), \tilde{z}_1(\cdot), z_0(\cdot)) \in \mathcal{H}$, the function $x(t) = (l(t), w(t), \tilde{l}(t), \tilde{z}(t), z(t)) \in C(0, \infty; H^2(0, 1) \times L^2(0, 1))$, satisfying properties $z(1) = l(1) - w(1)$, $d(t)$ with

$$\begin{align*}
& \int_0^1 (|w_t(t)|^2 + |w_{xx}(t)|^2)dt + |l(t)|^2 + |w(l(t), t)|^2 + |z(t)|^2 + |\tilde{z}(t)|^2 \\
& + |l_1(t) - w_1(t)|^2 < \infty
\end{align*}
$$

where $M_3$ and $\gamma_3$ are two positive constants. If in addition $v(0, 0) = 0$ and $d \in L^2(0, \infty)$, then

$$\lim_{t \to \infty} \int_0^1 (|l(t)|^2 + |w(t)|^2 + |z(t)|^2 + |\tilde{z}(t)|^2)dt = 0.$$  

**Proof**: Fix the initial value $(l(0), w(0), l_0(\cdot), w_0(\cdot), \tilde{l}(\cdot), \tilde{l}_0(\cdot), z(\cdot), z_0(\cdot), \tilde{z}(\cdot), \tilde{z}_0(\cdot)) \in \mathcal{H} \times \mathcal{H}$, satisfying properties $z(1) = l(1) - w(1)$. By Theorem 3.1, it follows that:

$$\begin{align*}
& \int_0^1 (|w_t(t)|^2 + |w_{xx}(t)|^2)dt + |l(t)|^2 + |w(l(t), t)|^2 + |z(t)|^2 + |\tilde{z}(t)|^2 \\
& + |l_1(t) - w_1(t)|^2 \leq 0.
\end{align*}
$$

The continuity and exponential decay of $(w(\cdot, t), w^*_1(\cdot, t))$ on $H^2(0, 1) \times L^2(0, 1)$ is, therefore, obtained. Since $v$ is continuity, we derive $v(w(\cdot, t), w^*_1(\cdot, t)) \in L^\infty(0, \infty)$. By Lemma 2.1, we obtain $(\tilde{l}(t), \tilde{l}_1(t), \tilde{m}_t(1), \tilde{m}_{t1}(1)) \in C(0, \infty; H^2(0, 1) \times L^2(0, 1) \times (H^2(0, 1) \times L^2(0, 1))^2)$ and

$$\begin{align*}
& \int_0^1 (|\tilde{l}(t)|^2 + |\tilde{l}_1(t)|^2 + |\tilde{m}_t(1)|^2 + |\tilde{m}_{t1}(1)|^2)dt < \infty.
\end{align*}
$$

This indicates that (22) has a unique solution and $(w(\cdot, t), w^*_1(\cdot, t))$ is continuous and exponential stability of $(l(\cdot, t), w(\cdot, t), \tilde{l}(\cdot, t), \tilde{z}(\cdot, t), \tilde{z}_1(\cdot, t), \tilde{z}_0(\cdot, t)) \in C(0, \infty; H^2(0, 1) \times L^2(0, 1) \times (H^2(0, 1) \times L^2(0, 1))^2)$ and (19) is derived.

**Remark 3.3**: Theorem 3.2 not only gives the exponential stability of $(w(\cdot, t), w^*_1(\cdot, t))$ and boundedness of $(l(\cdot, t), \tilde{l}(\cdot, t), z(\cdot, t), \tilde{z}(\cdot, t), \tilde{z}_1(\cdot, t), \tilde{z}_0(\cdot, t))$, but also the exponential stability of $-l_{xxx}(1, t) + m\beta^{-1}w_1(1, t) + z_{xx}(1, t)$ and the boundedness of $l_1(1, t) - w_1(1, t)$. The reason is that

$$\begin{align*}
& \int_0^1 (|w_t(t)|^2 + |w_{xx}(t)|^2)dt + |l(t)|^2 + |w(l(t), t)|^2 + |z(t)|^2 + |\tilde{z}(t)|^2 \\
& + |l_1(t) - w_1(t)|^2 \leq M_{A_1} e^{\omega_{A_1} t}
\end{align*}
$$

where $M_{A_1}$ and $\omega_{A_1}$ are two positive constants, $\gamma_2 = 2\omega_{A_1}$ and $M_2 = M_{A_1} ||(w(0), w_0(\cdot), l_0(\cdot), \tilde{l}_0(\cdot), z_0(\cdot), \tilde{z}_0(\cdot), \tilde{z}_1(\cdot), \tilde{z}_1(\cdot), z(\cdot), \tilde{z}(\cdot), z_0(\cdot), \tilde{z}_0(\cdot))||_{H_1}^2$ is (18).
existence and boundedness of \((\hat{p}(\cdot, t), \hat{p}_t(\cdot, t))\) are thereby derived by viewing \(v(w(\cdot, t), w_t(\cdot, t)) + d(t)\) as the boundary input.

**Remark 3.5:** In Theorem 3.2, the differences and improvements with respect to the results in [29] mainly lie in the following.

1) The internal uncertainty is taken into consideration, while the work in [29] just studies the case of \(v(w, w_t) = 0\).
2) Only “low order” measurements \(w(0, t)\) and \(w_{xx}(0, t)\) are adopted, while the work in [29] the velocity \(w_t(1, t)\) as well as the high-order collocated feedback \(w_{xxxt}(1, t)\) were used.
3) We consider arbitrary \(m, \alpha, \beta > 0\), while the work in [29] just solved the special case \(\alpha, \beta > 0, \text{ and } m = \alpha\beta\).

**IV. NUMERICAL SIMULATION**

In order to illustrate the effectiveness of the proposed feedback control, we present in this section some numerical simulations for the closed-loop system (9). Finite difference scheme is adopted and the numerical results are programmed in MATLAB. The steps of time and the space step are chosen as 1/2000 and 1/10, respectively. We take the nonlinear internal uncertainty \(v(w(\cdot, t), w_t(\cdot, t)) = e^{\cos(w(1, t))} + 2\sin(w(1, t))\), and the external disturbance \(d(t) = \sin(3t)\). We chose the tip mass \(m = 5\). The turning parameters are taken as \(c = \alpha = 1, \beta = \gamma = 2\), the initial values are chosen by

\[
\begin{align*}
w(x, 0) &= x^2, w_t(x, 0) = 0 \\
l(x, 0) &= 3x^3, l_t(x, 0) = 0 \\
z(x, 0) &= 2x, z_t(x, 0) = 0.
\end{align*}
\]

Figs. 1, 3, and 5 show the displacements \(w(x, t), l(x, t),\) and \(z(x, t),\) respectively. Fig. 2, 4, and 6 show the velocities \(w_t(x, t), l_t(x, t),\) and \(z_t(x, t),\) respectively. Fig. 7 is \(\eta(t) = m\beta^{-1}w_t(1, t) - w_{xxx}(1, t)\), and Fig. 8 is \(\phi(t) = m[l_t(1, t) - w_t(1, t)]\). Fig. 9 shows the time response of the input, and Fig. 10 shows the time response of
infinite-dimensional Luensberger state observer described by
\[
\begin{align*}
\hat{w}(x,t) + \tilde{w}_{xx}(x,t) &= 0 \\
\hat{w}(0,t) &= \tilde{w}_{xx}(1,t) = 0 \\
\tilde{w}_{xx}(0,t) &= c\tilde{w}_{xx}(0,t) + \gamma \tilde{w}(0,t) + w_{xx}(0,t) \\
-w_{xx}(1,t) + m\tilde{w}_{tt}(1,t) &= u(t)
\end{align*}
\] (24)
where \(c, \gamma > 0\) are tuning parameters.

Set \(\tilde{w}(x,t) = w(x,t) - w(t,x)\) to derive
\[
\begin{align*}
\hat{w}(x,t) + \tilde{w}_{xx}(x,t) &= 0 \\
\hat{w}(0,t) &= \tilde{w}_{xx}(1,t) = 0 \\
\tilde{w}_{xx}(0,t) &= c\tilde{w}_{xx}(0,t) + \gamma \tilde{w}(0,t) \\
-w_{xx}(1,t) + m\tilde{w}_{tt}(1,t) &= 0
\end{align*}
\] (25)
which is written abstractly by
\[
\frac{d}{dt}(\tilde{w}(\cdot,t), \tilde{w}(\cdot,t), m\tilde{w}_{tt}(1,t)) = A_2(\tilde{w}(\cdot,t), \tilde{w}(\cdot,t), m\tilde{w}_{tt}(1,t)).
\] (26)

**Lemma 6.4:** There exist a sequence of generalized eigenfunctions of operator \(A_2\) that forms Riesz basis for \(H_2\). Moreover, \(A_2\) generates an exponentially stable \(C_0\)-semigroup on \(H_2\).

We design an estimated state-based output feedback controller
\[
u(t) = -\alpha \tilde{w}(1,t) + \beta \tilde{w}_{xx}(1,t)
\] (27)
to derive the closed-loop system
\[
\begin{align*}
w_{tt}(x,t) + w_{xx}(x,t) &= 0 \\
\hat{w}(0,t) &= w_{xx}(0,t) = w_{xx}(1,t) = 0 \\
-w_{xx}(1,t) + m\tilde{w}_{tt}(1,t) &= -\alpha \tilde{w}(1,t) + \beta \tilde{w}_{xx}(1,t) \\
\tilde{w}(x,t) + \tilde{w}_{xx}(x,t) &= 0 \\
\hat{w}(0,t) &= \tilde{w}_{xx}(1,t) = 0 \\
\tilde{w}_{xx}(0,t) &= c\tilde{w}_{xx}(0,t) + \gamma \tilde{w}(0,t) + w_{xx}(0,t) \\
-w_{xx}(1,t) + m\tilde{w}_{tt}(1,t) &= -\alpha \tilde{w}(1,t) + \beta \tilde{w}_{xx}(1,t)
\end{align*}
\] (28)
which is equivalent to
\[
\begin{align*}
w_{tt}(x,t) + w_{xx}(x,t) &= 0 \\
\hat{w}(0,t) &= w_{xx}(0,t) = w_{xx}(1,t) = 0 \\
-w_{xx}(1,t) + m\tilde{w}_{tt}(1,t) &= -\alpha \tilde{w}(1,t) + \beta \tilde{w}_{xx}(1,t) \\
\tilde{w}(x,t) + \tilde{w}_{xx}(x,t) &= 0 \\
\hat{w}(0,t) &= \tilde{w}_{xx}(1,t) = 0 \\
\tilde{w}_{xx}(0,t) &= c\tilde{w}_{xx}(0,t) + \gamma \tilde{w}(0,t) + w_{xx}(0,t) \\
-w_{xx}(1,t) + m\tilde{w}_{tt}(1,t) &= -\alpha \tilde{w}(1,t) + \beta \tilde{w}_{xx}(1,t)
\end{align*}
\] (29)
Consider system (29) in Hilbert state space \(\mathcal{H} = H_1 \times H_2\), where \(H_1 = H^1_0(0,1) \times L^2(0,1) \times C\). The norm is given by
\[
\|\cdot\|_{H^1_0(0,1) \times L^2(0,1) \times C} = \int_0^1 \|\cdot\|^2 + \|\cdot\|^2 dx + \frac{\|\cdot\|^2}{m \beta^2}
\]
Define the operator \(A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}\) by \(A(f,g,\eta) = \{g, -f, -\eta \beta^{-1} - \beta^{-1}(\alpha - \beta^{-1})\gamma(1)\} \forall (f,g,\eta) \in D(A)\),
\[
D(A) = \{(f,g) \in H^1_0(0,1) \cap H^2(0,1) \times H^2_0(0,1) | f'(1) = 0, \eta = -f''(1) + m \beta^{-1} g(1)\}
\]
It follows that \(A\) generates an exponentially stable \(C_0\)-semigroup, and there exist a sequence of generalized eigenfunctions of \(A\) that forms Riesz basis for \(H_1\). Furthermore, we define the operator \(A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}\) by \(A(f,g,\eta,\phi,\psi,\nu,\nu) = \{g, -f, -\eta \beta^{-1} - \beta^{-1}(\alpha - \beta^{-1})\gamma(1) - \beta^{-1} \phi''(1) - \alpha \beta^{-1} \psi(1), \psi, -f, \phi''(1)\} \in D(A)\),
\[
D(A) = \{(f,g,\eta,\phi,\psi,\nu,\nu) \in H^2_0(0,1) \times H^2_0(0,1) \times H^2(0,1) \cap \mathcal{H} | f'(1) = 0, \eta = -f''(1) + m \beta^{-1} g(1) - \phi''(1)\}
\]
where \(\mathcal{H} = H_1 \times H_2\).
**System (29)** is abstractly described by \( \frac{d}{dt} Z(t) = A Z(t) \), where \( Z(t) = (w(t), \dot{w}(t), -w_{xx}(t), -\dot{w}_{xx}(t)) = m \beta + w_{x}(t), \dot{w}_{x}(t), m \dot{w}(t), w(t) \).

**Theorem 6.3:** The operator \( A \) generates an exponentially stable \( C_0 \)-semigroup on \( H \); there exist two positive constants \( M_A \) and \( \omega \) such that \( \| e^{tA} \|_H \leq M_A e^{-\omega t}, t \geq 0 \). Moreover, for any initial condition \( (w_0(t), \dot{w}_0(t), -w_{x0}(t), -\dot{w}_{x0}(t)) + m \beta + w_{x0}(t), \dot{w}_{x0}(t), m \dot{w}_{x0}(t), w_{x0}(t)) \in \mathcal{H} \), the state \( (w(t), \dot{w}(t), -w_{xx}(t), -\dot{w}_{xx}(t)) \) satisfies \( \| (w(t), \dot{w}(t), -w_{xx}(t), -\dot{w}_{xx}(t)) \|_H \leq M_A e^{-\omega t} \), where \( M_A \) is a positive constant.

In presence of disturbance, we consider the special case \( F(t) \equiv F \). With the same design, the closed-loop system has the boundary condition \( -\dot{w}_{xx}(t, 1) + m \beta \dot{w}(t, 1) = -F \) and then \( w(x, t) = (\frac{\partial}{\partial x} \frac{x^2}{2} + \frac{x^2}{2}) F, \dot{w}(x, t) = (\frac{x^2}{2} - \frac{x^2}{2} - x) \gamma F \) is an unstable solution. This means that when the disturbance is considered, the controller should be redesigned.

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