Strong Solutions for 1D Compressible Navier-Stokes/Allen-Cahn System with Phase Variable Dependent Viscosity

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Abstract

This paper is concerned with a non-isentropic compressible Navier-Stokes/Allen-Cahn system with phase variable dependent viscosity \( \eta(\chi) = \chi^\alpha \) and temperature dependent heat-conductivity \( \kappa(\theta) = \theta^\beta \). We show the global existence of strong solutions under some assumptions on growth exponent \( \alpha \) and initial data. It is worth noting that the initial data could be large if \( \alpha \geq 0 \) is small, and the growth exponent \( \beta > 0 \) can be arbitrary large.

Key Words: Navier-Stokes/Allen-Cahn; phase variable dependent viscosity; existence; global solutions.

1 Introduction

In this paper, we investigate a diffuse interface model for two-phase flows of viscous compressible fluids, which was proposed by Blesgen [3]. This model can be used to describe topological transitions on the interface such as droplet coalescence or droplet break-up. A lot of attentions have been paid to diffuse interface models because of their clear background and their applications in numerical simulations. Great progresses have been achieved in the studies of incompressible case, i.e. \( \rho = \text{const.} \), see [1, 17, 18] for example. The researches on compressible diffuse interface models mainly focus on Navier-Stokes/Allen-Cahn system [3, 19] and Navier-Stokes/Cahn-Hilliard system [2, 28]. The theoretical analysis of compressible Navier-Stokes/Allen-Cahn model began with Feireisl et al. [16] and Kotschote [24]. They

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proved the existence of weak solutions for isentropic system and local strong solutions for non-isentropic system, respectively. Here, we are interested in non-isentropic compressible Navier-Stokes/Allen-Cahn system, which was simplified by Chen et al. \[5\] into the following form

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
\rho u_t + \rho (u \cdot \nabla) u - 2\eta(\chi)\text{div} u - \lambda(\chi)\nabla \text{div} u &= -\text{div}\left(\delta \nabla \chi \otimes \nabla \chi - \frac{\delta}{2} |\nabla \chi|^2 + \theta \frac{\partial p}{\partial \theta}\right), \\
\rho \chi_t + \rho (u \cdot \nabla) \chi &= -\mu, \\
\mu &= \frac{\rho}{\delta}(\chi^3 - \chi) - \delta \Delta \chi, \\
c_v(\rho \theta_t + \rho u \cdot \nabla \theta) + \theta \rho_0 \text{div} u - \text{div}(\kappa(\theta) \nabla \theta) &= 2\eta(\chi)|\nabla u|_2^2 + \lambda(\chi)(\text{div} u)^2 + \mu^2,
\end{align*}
\]

where \(\rho, u, \chi, \theta\) represent the total density, the mean velocity of the fluid mixture, the phase field variable. Moreover, \(\mu\) is the chemical potential, \(\delta\) and \(c_v\) are related to the thickness of the interfacial region and the heat capacity at constant volume. The viscosity coefficients \(\eta(\chi)\) and \(\lambda(\chi)\) satisfy \(\eta > 0, \lambda + \frac{2}{N} \eta > 0\). \(\kappa(\theta)\) is the heat-conductivity, and the pressure \(p = R\rho \theta\) with \(R > 0\).

In this paper, we assume that the viscosity coefficients satisfy \(\lambda(\chi) = -\eta(\chi)\), then the above system in 1D becomes

\[
\begin{align*}
\rho_t + (\rho u)_\bar{x} &= 0, \\
\rho u_t + \rho uu_\bar{x} + (R\rho \theta)_\bar{x} &= (\eta(\chi)u_\bar{x})_\bar{x} - \frac{\delta}{2} \left(\chi^3_\bar{x}\right)_\bar{x}, \\
\rho \chi_t + \rho \chi_\bar{x}_\bar{x} &= -\mu, \\
\mu &= -\delta \chi_\bar{x}_\bar{x} + \frac{\rho}{\delta}(\chi^3 - \chi), \\
c_v(\rho \theta_t + \rho u \theta_\bar{x}) + R\rho \theta u_\bar{x} - (\kappa(\theta)\theta_\bar{x})_\bar{x} &= \eta u^2_\bar{x} + \mu^2
\end{align*}
\]

for \((\bar{x}, t) \in (0, 1) \times (0, +\infty)\). We supplement (1.1) with initial value conditions

\[
(\rho, u, \chi, \theta)|_{t=0} = (\rho_0, u_0, \chi_0, \theta_0), \quad \bar{x} \in (0, 1) \quad (1.2)
\]

and boundary value conditions

\[
(u, \chi_\bar{x}, \theta_\bar{x})|_{\bar{x}=0,1} = (0, 0, 0), \quad t \geq 0. \quad (1.3)
\]

Without loss of generality, we assume \(\int_0^1 \rho_0(\bar{x})d\bar{x} = 1\). Then in Lagrange coordinates \(x = \)
\[
\int_0^1 \rho(\xi, t) d\xi, \text{ the system (1.1)–(1.3) can be rewritten as }
\]

\[
\begin{align*}
\frac{\partial v}{\partial t} &= u_x, \\
\frac{\partial u}{\partial t} + \left(\frac{\theta}{v}\right)_x &= \left(\frac{\eta(\chi) u_x}{v}\right)_x - \frac{1}{2} \left(\frac{\chi_x^2}{v^2}\right)_x, \\
\chi_t &= -\nu \mu, \\
\mu &= -\left(\frac{\chi_x}{v}\right)_x + (\chi^3 - \chi), \\
\theta_t + \frac{\theta}{v} u_x &= \left(\frac{\kappa(\theta) \theta_x}{v}\right)_x + \frac{\eta \mu_x^2}{v} + \nu \mu^2
\end{align*}
\]

with initial and boundary value conditions

\[
\begin{align*}
(v, u, \chi, \theta) \bigg|_{t=0} &= (v_0, u_0, \chi_0, \theta_0), \quad x \in (0, 1), \\
(u, \chi_x, \theta_x) \bigg|_{x=0, 1} &= (0, 0, 0), \quad t \geq 0.
\end{align*}
\]

Here \( v = \frac{1}{\rho} \) represents specific volume. In the following, we choose

\[
\eta(\chi) = \tilde{\eta} \chi^\alpha, \quad \kappa(\theta) = \tilde{\kappa} \theta^\beta,
\]

with \( \alpha \geq 0, \beta > 0 \), and the constants \( R = c_v = \tilde{\eta} = \tilde{\kappa} = \delta = 1 \).

Before introducing our main result, we first give a brief review on some related works. For isentropic compressible NSAC model with constant viscosity, Ding et al. [14] and Chen et al. [4] obtained the 1D global well-posedness without vacuum and with vacuum, respectively. Later, Ding et al. [15] derived the existence and uniqueness of global strong solutions with free boundary condition. Very recently, Chen and Zhu [10] assumed that the viscosity coefficient satisfied

\[
\eta(\rho, \chi) = 1 + \rho^\alpha \chi^\beta.
\]

They proved the existence and uniqueness of global classical solutions when \( 2 \leq \alpha \leq \gamma \) and \( \beta = 0 \). In the case of \( \beta \geq 1 \), they obtained a blow-up criterion for strong solutions. In [4,10], the phase variable \( \chi \) was assumed to satisfy Dirichlet boundary conditions, i.e. \( \chi|_{x=0,1} = 0 \). For non-isentropic compressible NSAC system with constant viscosity, Chen et al. [5,6] studied global strong solutions of initial-boundary value problem and Cauchy problem. Besides, there are also some researches on asymptotic behavior, weak solutions, stationary solutions and wave problem, one can find them in the references [7,9,11,25,29,30,36].

The main purpose of this paper is to deal with phase variable dependent viscosity. Since this model is governed by the full compressible Navier-Stokes equation coupled with Allen-Cahn
equations, let’s do some review on the full compressible Navier-Stokes equations. When both the viscosity $\eta$ and the heat-conductivity $\kappa$ are positive constants, the analysis mainly relies on the upper and lower bounds of specific volume $v$ and temperature $\theta$. The proof was built upon a representation of specific volume $v$, which was obtained by Kazhikhov et al. \[26,27\]. When the viscosity $\eta$ is a constant or depends only on $v$, mass conservation equation and momentum conservation equation imply

$$\left(\frac{\eta(v)v_x}{v}\right)_t = u_t + P_x, \quad (1.7)$$

which was observed by Kanel in \[22\]. By virtue of (1.7), one can get global well-posedness of solutions with large initial data, see \[8,12,13,20,23,31,34\] and references therein. When the viscosity $\eta$ depends on the temperature $\theta$ and the specific volume $v$, the identity (1.7) becomes

$$\left(\frac{\eta(v,\theta)v_x}{v}\right)_t = u_t + P_x + \frac{\eta_\theta(v,\theta)}{v}(\theta v_x - u_x \theta_x). \quad (1.8)$$

It is clear that the temperature dependent viscosity has a strong influence on the solution. And as pointed out in \[21\], such a dependence turns out to be challenging. Later, Wang and Zhao \[35\] considered the following case

$$\eta(v,\theta) = \tilde{\eta} h(v) \theta^\alpha, \quad \kappa(\theta) = \tilde{\kappa} h(v) \theta^\beta$$

under some structure assumptions. But their results excluded the case of $h \equiv const$. Recently, Sun, Zhang and Zhao \[32\] assumed that

$$\eta(\theta) = \theta^\alpha, \quad \kappa(\theta) = \theta^\beta,$$

with $\alpha \geq 0, \beta \geq 0$ and the initial data $v_0 \geq V_0, \theta_0 \geq V_0$ for some constant $V_0 \geq 0$. When $\alpha$ is small, they obtained the existence and uniqueness of strong solutions. Thanks to the ideas in \[32\], we can handle our problem for the case $\eta(\chi) = \chi^\alpha$.

Our main result is the following global-in-time existent theorem.

**Theorem 1.1** For given positive numbers $M_0$ and $V_0$, assume that

$$v_0 \geq V_0, \quad V_0 \leq \chi_0 \leq 1, \quad \theta_0 \geq V_0, \quad \|(v_0,u_0,\theta_0)\|_{H^2} + \|\chi_0\|_{H^1} \leq M_0. \quad (1.9)$$

Then there exists a positive constant $\epsilon_0 > 0$, depending only on $M_0$, $V_0$ and $\beta$, such that the problem (1.4)–(1.6) with $0 \leq \alpha \leq \epsilon_0$ and $\beta > 0$ admits a unique global strong solution $(v,u,\chi,\theta)$ on $(x,t) \in [0,1] \times [0, +\infty)$, satisfying

$$\inf_{(x,t) \in [0,1] \times [0, +\infty)} \{v(x,t), \theta(x,t)\} > 0, \quad \sup_{(x,t) \in [0,1] \times [0, +\infty)} \{v(x,t), \theta(x,t)\} < \infty,$$
$V_0 \leq \chi(x, t) \leq 1, \quad (x, t) \in [0, 1] \times [0, +\infty),$

and

$(v, u, \theta) \in C([0, \infty); H^2), \quad \chi \in C([0, \infty); H^3),$

$v_x \in L^2(0, \infty; H^1), \quad (u_x, \chi_x, \theta_x) \in L^2(0, \infty; H^2).$

**Remark 1.1**

(i) Even though we can just handle the case when $\alpha$ is small and require $\chi$ has positive lower bound, but this is a first result on global strong solutions with phase variable dependent viscosity, which is more in line with physical reality.

(ii) The strong solutions exist on whole time interval $[0, +\infty)$, so all the bounds in a priori estimates are time-independent. This makes it possible to consider long time behavior of the solutions.

(iii) The theorem is also correct if one replaces $\eta = \eta(\chi) = \chi^\alpha$ by $\eta = \eta(\theta) = \theta^\alpha$.

The key step of the proof is to obtain lower and upper bounds of the specific volume $v$. We apply the argument developed by Kazhikhov [26] to derive a presentation of specific volume (Lemma 2.2). In our model, (1.8) becomes

$$\left(\frac{\eta(\chi)v_x}{v}\right)_t = u_t + P_x + \frac{1}{2}\left(\frac{x^2}{v^2}\right)_x + \frac{\alpha x^{\alpha-1}}{v}(\chi v_x - u_x\chi_x).$$

(1.10)

The last term in (1.10) is highly nonlinear. Fortunately, we can use the smallness of $\alpha$ to control it as in [32]. Moreover, comparing with (1.8), we need to deal with the high-order strongly nonlinear term $\left(\frac{x^2}{v^2}\right)_x$ (Lemma 2.4, Lemma 2.7).

The structure of this paper is as follows. In Section 2, we do some a priori estimates which are independent with $T$. Then We finish the proof of Theorem 1.1 by using continuity method in Section 3.

## 2 A priori estimates

For given some positive constants $m_i(i = 1, 2, 3)$ and $N$, we define the set

$$X(0, T; m_1, m_2, m_3, N) := \left\{ (v, u, \chi, \theta) : (v, u, \theta) \in C([0, T]; H^2), \chi \in C([0, T]; H^3),
(v_x, \chi_x) \in L^\infty(0, T; H^1),
(v_t, u_t) \in L^2(0, T; H^1),
(u_x, \theta_x, \chi_x) \in L^2(0, T; H^2),
E(0, T) \leq N^2,
v \geq m_1, \chi \geq m_2, \theta \geq m_3, \forall (x, t) \in [0, 1] \times [0, T]. \right\}$$
Multiplying \((1)\) then the proof of Lemma 2.2 is finished. □

Lemma 2.1 Let \((v, u, \chi, \theta) \in X(0, T; m_1, m_2, m_3, N)\) be a solution to the problem \((1.4)-(1.6)\) on \([0, 1] \times [0, T]\). Then

\[
\sup_{0 \leq t \leq T} \int_0^1 \left( \frac{u^2}{2} + \frac{(\chi^2 - 1)^2}{4} + \frac{x_x^2}{2v} + \Phi(v) + \Phi(\theta) \right) dx + \int_0^T W(\tau) d\tau \leq E_0, \quad (2.2)
\]

where

\[
\Phi(s) = s - \ln s - 1, \quad W(t) = \int_0^1 \left( \frac{\theta \theta_x^2}{v \theta^2} + \frac{\eta(\chi) u_x^2}{v \theta} + \frac{v \mu^2}{\theta} \right) dx,
\]

and

\[
E_0 = \int_0^1 \left( \frac{u_0^2}{2} + \frac{(\chi_0^2 - 1)^2}{4} + \frac{x_{0x}^2}{2v_0} + \Phi(v_0) + \Phi(\theta_0) \right) dx.
\]

Proof. From \((1.6)\), multiplying \((1.4)_3\) by \(\mu\), adding \((1.4)_1\) and \((1.4)_5\) integrating them over \([0, 1]\) by parts, combining with \((2.1)\), we have

\[
\int_0^1 v dx = 1, \quad \int_0^1 \left( \theta + \frac{u^2}{2} + \frac{(\chi^2 - 1)^2}{4} + \frac{x_x^2}{2v} \right) dx = 1. \quad (2.3)
\]

Multiplying \((1.4)_1, (1.4)_2, (1.4)_3\) and \((1.4)_5\) by \(1 - v^{-1}, u, \mu\) and \(1 - \theta^{-1}\), respectively, integrating by parts over \([0, 1]\), adding them together and combining the boundary condition \((1.6)\), we get

\[
\frac{d}{dt} \int_0^1 \left( \frac{u^2}{2} + \frac{(\chi^2 - 1)^2}{4} + \frac{x_x^2}{2v} + (v - \ln v) + (\theta - \ln \theta) \right) dx + W(t) = 0.
\]

then the proof of Lemma 2.1 is finished. □

Lemma 2.2 Let \((v, u, \chi, \theta) \in X(0, T; m_1, m_2, m_3, N)\) be a solution to the problem \((1.4)-(1.6)\) on \([0, 1] \times [0, T]\), we define \(\eta_0 := \eta(\chi_0)\), then for any \(t \geq 0\), there is a \(\alpha_0(t) \in (0, 1)\) such that

\[
v = B(t)D(x, t) + \int_0^t \frac{B(t)D(x, \tau)}{B(\tau)D(x, \tau)} v(x, \tau)J(x, \tau) d\tau, \quad (2.4)
\]
where
\[
  g(x, t) := - \left[ u \left( \frac{1}{\eta} \right) + \frac{\theta}{v} \left( \frac{1}{\eta} \right) + \frac{\chi_x^2}{2\eta v^2} \left( \frac{1}{\eta} \right) + \frac{\eta_x u_x}{\eta v} \right],
\]
(2.5)
\[
  B(t) := \exp \left\{ - \int_0^t \int_0^1 \left( \frac{\theta + u^2}{\eta} + \frac{\chi_x^2}{2\eta v} \right) dxd\tau \right\},
\]
(2.6)
\[
  D(x, t) := v_0(x) \exp \left\{ \int_0^x \frac{u}{\eta} dy - \int_0^x \frac{u_0}{\eta_0} dy + \int_0^1 \int_0^1 v_0 \left( \int_0^x \frac{u_0}{\eta_0} dy \right) dx \right\},
\]
(2.7)
\[
  J(x, t) := \left( \frac{\theta}{\eta v} + \frac{\chi_x^2}{2\eta v^2} + \int_0^x gdy \right)(x, t) - \int_0^t \int_0^1 \left( \frac{\theta}{\eta v} + \frac{\chi_x^2}{2\eta v^2} + \int_0^x gdy \right)(x, t)dx.
\]
(2.8)

**Proof.** For \( g = g(x, t) \) given by (2.5), it follows from (1.3.1) and (1.4.2) that
\[
  \left( \frac{u}{\eta} \right)_t + \left( \frac{\theta}{\eta v} + \frac{\chi_x^2}{2\eta v^2} + \int_0^x gdy \right)_x + g(x, t) = \left( \frac{u_x}{v} \right)_x = (\ln v)_x.
\]
(2.9)

Define
\[
  \varphi(x, t) := \int_0^x \left( \frac{u}{\eta} - \frac{\theta}{\eta v} - \frac{\chi_x^2}{2\eta v^2} - \int_0^x g(y, \tau) d\tau \right) dy + \int_0^x \frac{u_0}{\eta_0} dy,
\]
(2.10)
where \( \eta_0 := \eta(\chi_0) \). Then we have
\[
  \varphi_t = \frac{u_x}{v} - \frac{\theta}{\eta v} - \frac{\chi_x^2}{2\eta v^2} - \int_0^x g(y, t) dy,
\]
\[
  \varphi_x = \frac{u}{\eta}.
\]
Combining with (1.4.1), we arrive at
\[
  (v\varphi)_t - (u\varphi)_x = u_x - \frac{\theta + u^2}{\eta} - \frac{\chi_x^2}{2\eta v} - v \int_0^x g(y, t) dy.
\]
(2.11)

Integrating (2.11) over \([0, 1] \times [0, t]\) and using (1.6), we get
\[
  \int_0^1 (v\varphi)dx - \int_0^1 (v_0\varphi_0)dx = - \int_0^1 \int_0^1 \left( \frac{\theta + u^2}{\eta} + \frac{\chi_x^2}{2\eta v} + v \int_0^x g(y, t) dy \right) dx d\tau.
\]

On one hand, thanks to the mean value theorem and (2.3), there is a \( \alpha_0(t) \) for any \( t \geq 0 \) such that
\[
  \varphi(\alpha_0(t), t) = \int_0^1 (v\varphi)dx
\]
(2.12)
\[
  = \int_0^1 v_0 \left( \int_0^x \frac{u_0}{\eta_0} dy \right) dx - \int_0^1 \int_0^1 \left( \frac{\theta + u^2}{\eta} + \frac{\chi_x^2}{2\eta v} + v \int_0^x g(y, t) dy \right) dx d\tau.
\]
On the other hand, it follows from (2.10) that
\[
  \varphi(\alpha_0(t), t) = \ln v(\alpha_0(t), t) - \ln v_0(\alpha_0(t)) - \int_0^1 \left( \frac{\theta}{\eta v} \right)(\alpha_0(t), \tau) d\tau
\]
\[
  - \int_0^1 \left( \frac{\chi_x^2}{2\eta v^2} \right)(\alpha_0(t), \tau) d\tau - \int_0^x \int_0^{\alpha_0(t)} g(y, \tau) dy d\tau + \int_0^{\alpha_0(t)} \frac{u_0}{\eta_0} (y) dy.
\]
(2.13)
Hence, collecting (2.12) and (2.13), we have
\[
\ln v(\alpha_0(t), t) - \ln v_0(\alpha_0(t)) = \int_0^t \int_0^1 v_0 \left( \int_0^x \frac{u_0}{\eta_0} dy \right) dx - \int_0^t \int_0^1 \left( \theta + \frac{u_2^2}{\eta} + \frac{\chi_2^2}{2\eta v} + v \int_0^x g(y, t) dy \right) dx dt
\]
\[
+ \int_0^t \frac{\theta}{\eta v} (\alpha_0(t), \tau) d\tau + \int_0^t \left( \frac{\chi_2^2}{2\eta v^2} \right) (\alpha_0(t), \tau) d\tau
\]
\[
+ \int_0^t \int_0^{\alpha_0(t)} g(y, \tau) d\tau dy - \int_0^t \int_0^{\alpha_0(t)} \left( \frac{u_0}{\eta_0} \right) (y) dy.
\]
By virtue of (2.14), we integrate (2.9) over \([\alpha_0(t), x] \times [0, t]\) to deduce
\[
\ln \frac{v(x, t)}{v_0(x)} = \ln v(\alpha_0(t), t) - \ln v_0(\alpha_0(t)) + \int_{\alpha_0(t)}^x \left[ \left( \frac{u}{\eta} \right) (y, t) - \left( \frac{u_0}{\eta_0} \right) (y) \right] dy
\]
\[
+ \int_0^t \int_0^x g(y, \tau) d\tau dy + \int_0^t \left[ \left( \frac{\theta}{\eta v} \right) (x, \tau) - \left( \frac{\theta}{\eta v} \right) (\alpha_0(t), \tau) \right] d\tau
\]
\[
+ \int_0^t \left[ \left( \frac{\chi_2^2}{2\eta v^2} \right) (x, \tau) - \left( \frac{\chi_2^2}{2\eta v^2} \right) (\alpha_0(t), \tau) \right] d\tau
\]
\[
= \int_0^t \left( \frac{\theta}{\eta v} \right) (x, \tau) d\tau + \int_0^t \left( \frac{\chi_2^2}{2\eta v^2} \right) (x, \tau) d\tau + \int_0^t \int_0^x g(y, \tau) d\tau dy
\]
\[
- \int_0^t \int_0^1 v \left( \int_0^x g(y, \tau) dy \right) dx dt - \int_0^t \int_0^1 \left( \frac{\theta + u_2^2}{\eta} + \frac{\chi_2^2}{2\eta v} \right) dx dt
\]
\[
+ \int_{\alpha_0(t)}^x \left[ \left( \frac{u}{\eta} \right) (y, t) - \left( \frac{u_0}{\eta_0} \right) (y) \right] dy + \int_0^t v_0 \left( \int_0^x \frac{u_0}{\eta_0} dy \right) dx,
\]
which implies
\[
v(x, t) = A(x, t)B(t)D(x, t). \quad (2.15)
\]
Here \(B(x, t)\) and \(D(x, t)\) are given in (2.6) and (2.7). Besides,
\[
A(x, t) := \exp \left\{ \int_0^t \left( \frac{\theta}{\eta v} \right) (x, \tau) d\tau + \int_0^t \left( \frac{\chi_2^2}{2\eta v^2} \right) (x, \tau) d\tau + \int_0^t \int_0^x g(y, \tau) d\tau dy \right\}.
\]
Noting that
\[
\frac{d}{dt} A(x, t) = A(x, t)J(x, t) = \frac{v(x, t)J(x, t)}{B(t)D(x, t)}, \quad (2.16)
\]
where \(J(x, t)\) is given in (2.8), integrating (2.16) over \((0, t)\) yields
\[
A(x, t) = 1 + \int_0^t \frac{v(x, \tau)J(x, \tau)}{B(\tau)D(x, \tau)} d\tau.
\]
By inserting the above equality into (2.15), we derive (2.4). □
Lemma 2.3 There exist two positive constants $C_0$ and $\varepsilon_1$, depending only on $\beta$, $V_0$ and $M_0$, such that if $(v, u, \chi, \theta) \in X(0, T; m_1, m_2, m_3, N)$ is a solution of the problem (1.4)–(1.6) on $(0, T)$, satisfying

$$m_2^{-\alpha} \leq 2, \quad (2N)^\alpha \leq 1, \quad \alpha H(m_1, m_2, m_3, N) \leq \varepsilon_1,$$

with $H(m_1, m_2, m_3, N) \triangleq (1 + m_1^{-1} + m_2^{-1} + m_3^{-1} + N)^8$, then

$$C_0 \leq v(x, t) \leq C_0^{-1}, \quad \forall (x, t) \in [0, 1] \times [0, T].$$

Proof. From (2.2), we get

$$\int_0^1 \left( \frac{u^2}{2} + \frac{(\chi^2 - 1)^2}{4} + \frac{\chi_x^2}{2v} + \theta - \ln \theta - 1 \right) dx \leq E_0.$$

Thanks to (2.3) and Jessen’s inequality, we have

$$- \ln \bar{\theta} = - \ln \left( \int_0^1 \theta dx \right) \leq - \int_0^1 \ln \theta dx \leq E_0.$$

It implies that

$$\bar{\theta} \geq e^{-E_0} =: \gamma_1 \in (0, 1).$$

Thus, we arrive at

$$\bar{\theta} := \int_0^1 \theta dx \in [\gamma_1, 1].$$

First, we estimate $D(x, t)$. From (2.3) and (2.17), it holds that

$$\left| \int_{\alpha_0(t)}^x \frac{u}{\eta} dy \right| \leq \int_0^1 \frac{|u|}{\chi^\alpha} dy \leq 2 \|u\|_{L^2} \leq C.$$

Thus, we have

$$C^{-1} \leq D(x, t) \leq C, \quad \forall (x, t) \in [0, 1] \times [0, T].$$

Next, we estimate $B(t)$ by using (2.3) and (2.19). It follows from the Sobolev’s inequality that

$$\|\chi - \bar{\chi}(0)\|_{L^\infty} \leq \|\chi_\delta(0)\|_{L^2} \leq N, \quad \forall t \in [0, T].$$

In addition, we have

$$\bar{\chi} \leq \int_0^1 |\chi| dx \leq \int_0^1 \chi^2 dx + 1 = \int_0^1 (\chi^2 - 1) dx + 2 \leq \int_0^1 (\chi^2 - 1)^2 dx + 3 \leq 7, \quad \forall t \in [0, T].$$

Thus, we show that $\|\chi\|_{L^\infty(\Omega_T)} \leq 7 + N \leq 2N$. Noting that

$$\int_0^1 \left( \frac{\theta + u^2}{\eta} + \frac{\chi_x^2}{2\eta^2} \right) dx \leq 2m_2^{-\alpha} \int_0^1 \left( \frac{\theta + u^2}{2} + \frac{\chi_x^2}{2v} \right) dx \leq 4,$$
Furthermore, we get
\[ e^{-4t} \leq B(t) \leq e^{-\gamma_1 t}. \] (2.21)

Furthermore, we get
\[ e^{-4(t-r)} \leq \frac{B(t)}{B(\tau)} \leq e^{-\gamma_1(t-r)}. \]

In terms of the definition of \( g \), by (2.17), we get
\[
\left| \int_0^t g dy \right| \leq ||v||_{L^\infty} \int_0^1 \left| u \left( \frac{1}{\eta} \right)_t + \theta \left( \frac{1}{v} \right)_x + \frac{\chi^2}{2\eta v} \right| \, dx \\
\leq \alpha ||v||_{L^\infty} \int_0^1 \left( ||\chi^{\alpha-1} \chi_x u|| + ||\chi^{\alpha-1} \chi_x \theta|| + ||\chi^{\alpha-1} \chi_x \chi_x|| + \frac{||u_x||}{\chi v} \right) \, dx \\
\leq 2\alpha N m^{-\alpha} \left( \frac{1}{m_2} ||\chi||_{L^2} ||u||_{L^2} + \frac{1}{m_1 m_2} ||\theta||_{L^2} ||\chi||_{L^3} + \frac{1}{m_1 m_2} ||\chi||_{L^\infty}^3 \right) + \frac{2\alpha N}{m_1 m_2} ||u_x||_{L^2} ||\chi||_{L^2}^2 \\
\leq C\alpha H(m_1, m_2, m, N) + \alpha N ||\chi||_{L^2}^2,
\] where we have used Cauchy-Schwartz’s inequality and the following facts
\[
||v||_{L^\infty} \leq ||v||_{L^2} + ||v_x||_{L^2} \leq 1 + N \leq 2N, \quad ||\chi||_{L^\infty} \leq ||\chi||_{L^2} \leq N^3, \\
||\theta||_{L^2} \leq ||\theta - \tilde{\theta}||_{L^2} + ||\tilde{\theta}||_{L^2} \leq ||\chi||_{L^2} + 1 \leq N + 1 \leq 2N, \quad ||u||_{L^2} \leq ||u_x||_{L^2} \leq N.
\]

In a similar manner, we can deduce that
\[
\left| \int_0^t v \left( \int_0^x g \, dy \right) \, dx \right| \leq ||v||_{L^\infty} \int_0^1 |g| \, dx \leq C\alpha H(m_1, m_2, m, N) + \alpha N ||\chi||_{L^2}^2.
\] (2.23)

Let \( f_+ := \max\{f, 0\} \). Thanks to the mean value theorem, (2.3) and (2.19), we have
\[
\left( \frac{\theta^{\beta+1}}{\theta^{\beta+1}} (t) - \frac{\theta^{\beta+1}}{\theta^{\beta+1}} (x, t) \right)_+ \leq C \left( \int_0^1 \frac{\theta^{\beta+1} \theta^{\beta+1} \theta^{\beta+1}}{\sqrt{\theta^{\beta+1}}} \, dx \right)^{\frac{1}{2}} \left( \int_0^1 \sqrt{\theta^{\beta+1}} \, dx \right)^{\frac{1}{2}} \leq CW^{1/2}(t),
\]
where \( \chi(\theta) = 1 \) if \( \theta \leq \tilde{\theta} \), and \( \chi(\theta) = 0 \) if \( \theta > \tilde{\theta} \). Then by Young’s inequality, we get
\[
\min_{x \in [0,1]} \theta(x, t) \geq C_1 - C_2 W(t).
\] (2.24)

Using (2.2), (2.20), (2.21)–(2.22), (2.23) and (2.76), we infer from (2.4) and (2.17) that
\[
v(x, t) \geq C^{-1} \int_0^t e^{-4(t-r)} \min_{x \in [0,1]} \theta \, d\tau - C\alpha N \int_0^t e^{-\gamma_1(t-r)} ||\chi||_{L^2}^2 \, d\tau \\
- \int_0^t Cae^{-\gamma_1(t-r)} H(m_1, m_2, m, N) \, d\tau \\
\geq C^{-1} \int_0^t e^{-4(t-r)} [C_1 - C_2 W(t)] \, d\tau - C\alpha H(m_1, m_2, m, N) \\
\geq \frac{C_1}{4C} (1 - e^{-4t}) - \frac{C_2}{C} \int_0^t e^{-4(t-r)} W(\tau) \, d\tau - C\varepsilon_1.
\]
In view of (2.2), we get
\[
\int_0^t e^{-4(t-\tau)} W(\tau) \, d\tau = \int_0^t e^{-4(t-\tau)} W(\tau) \, d\tau + \int_t^\infty e^{-4(t-\tau)} W(\tau) \, d\tau \\
\leq e^{-2t} \int_0^t W(\tau) \, d\tau + \int_t^\infty W(\tau) \, d\tau \\
\to 0, \quad \text{as} \quad t \to \infty.
\]

Hence, we can choose a $\bar{T}$ sufficiently large such that
\[
\frac{C_1 e^{-4t}}{4C} + \frac{C_2}{C} \int_0^t e^{-4(t-\tau)} W(\tau) \, d\tau \leq \frac{C_1}{16C}, \quad t \geq \bar{T}.
\]

Then we get
\[
v(x, t) \geq \frac{C_1}{8C}, \quad \forall x \in [0, 1], \ t \geq \bar{T}, \quad (2.25)
\]

provided $\varepsilon_1 > 0$ is chosen to be small enough such that $\varepsilon_1 \leq \min \{1, C_1/(16C)\}$.

For $(x, t) \in [0, 1] \times [0, \bar{T}]$, by (2.17), (2.20), (2.21)–(2.22) and (2.23), we derive from (2.4) that
\[
v(x, t) \geq B(t)D(x, t) - C\alpha \int_0^{\bar{T}} e^{-\gamma(t-\tau)} [N||x||_2^2 + H(m_1, m_2, m_3, N)] \, d\tau \\
\geq C_3^{-1} e^{-4\bar{T}} - C_3\alpha H(m_1, m_2, m_3, N) \\
\geq C_3^{-1} e^{-4\bar{T}} - C_3\varepsilon_1 \geq \frac{e^{-4\bar{T}}}{2C_3}.
\]  

provided $\varepsilon_1$ is chosen to be such that $\varepsilon_1 \leq e^{-4\bar{T}}/(2C^2_3)$. Combining (2.25) and (2.26) gives
\[
v(x, t) \geq C_0 := \min \left\{ \frac{C_1}{8C}, \frac{e^{-4\bar{T}}}{2C_3} \right\}, \quad (2.27)
\]

provided $\alpha H(m_1, m_2, m_3, N) \leq \varepsilon_1$ with $\varepsilon_1 \leq \min \{1, C_1/(16C), e^{-4\bar{T}}/(2C^2_3)\}$.

In what follows, we will deduce the upper bounds of $v$. Combining $v$ (2.4) with (2.20), (2.21)–(2.23), we have
\[
v(x, t) \leq C + C \int_0^t e^{-\gamma(t-\tau)} \left( \max_{x \in [0, 1]} \theta(x, \tau) + \max_{x \in [0, 1]} \left( \frac{x}{v} \right)^2 \max_{x \in [0, 1]} v(x, \tau) \right) \, d\tau \\
+ CaH(m_1, m_2, m_3, N).
\]  

Noting that for $\beta > 0$, there holds
\[
\theta^{\frac{\beta_1}{\gamma}(x, t) - \tilde{\theta}^{\frac{\beta_1}{\gamma}}(t)} \leq C \left( \int_0^1 \frac{\theta^{\beta_1}}{v\theta^2} \, dx \right)^\frac{1}{2} \left( \int_0^1 v^{\gamma} \, dx \right)^\frac{1}{2} \leq CW^{\frac{\gamma}{2}}(t) \max_{x \in [0, 1]} v^\frac{\gamma}{2}(x, t).
\]
Thus, by Young’s inequality, we get
\[
\theta(x, t) \leq C + CW(t) \max_{x \in [0, 1]} v(x, t). \tag{2.29}
\]

Moreover, by (2.2), for any \((x, t) \in [0, 1] \times [0, T]\), it holds that
\[
|\chi(x, t)| \leq \left| \int_0^1 (\chi(x, t) - \chi(y, t))dy \right| + \left| \int_0^1 \chi(y, t)dy \right|
\leq \int_0^1 \left( \int_y^\infty \chi_s(\xi) d\xi \right) dy + C
\leq \int_0^1 |\chi_s| dy + C \leq \left( \int_0^1 \frac{\chi_s^2}{v} dy \right)^\frac{1}{2} \left( \int_0^1 v dy \right)^\frac{1}{2} + C
\leq C. \tag{2.30}
\]

In view of (1.24), (2.2), (2.27) and (2.30), we derive that
\[
\max_{x \in [0, 1]} \left( \frac{\chi_s}{v} \right)^2 \leq C \int_0^1 \frac{\chi_s^2}{v} \left( \frac{\chi_{ss}}{v} \right) dx
= C \int_0^1 \frac{\chi_s^2}{v} \left[ \frac{\chi_s}{v} \right] - (\chi^3 - \chi) + (\chi^3 - \chi) dx
\leq C \int_0^1 \max_{x \in [0, 1]} \left( \frac{\chi_s}{v} \right) |\mu| dx + C \int_0^1 \left( \frac{\chi_s}{v} \right) (\chi^3 - \chi) dx
\leq \tilde{\varepsilon} \max_{x \in [0, 1]} \left( \frac{\chi_s}{v} \right)^2 \int_0^1 \frac{\theta}{v} dx + C \int_0^1 \frac{v\mu^2}{\theta} dx + C
\leq \tilde{\varepsilon} C_0^{-1} \max_{x \in [0, 1]} \left( \frac{\chi_s}{v} \right)^2 + CW(t) + C,
\]

where we have used Cauchy-Schwartz’s inequality. We can choose \(\tilde{\varepsilon}\) small enough such that \(\tilde{\varepsilon} C_0^{-1} \leq 1/2\), then
\[
\max_{x \in [0, 1]} \left( \frac{\chi_s}{v} \right)^2 \leq C + CW(t). \tag{2.31}
\]

Combining (2.29) with (2.31), we get
\[
\max_{x \in [0, 1]} \theta(x, \tau) \max_{x \in [0, 1]} \left( \frac{\chi_s}{v} \right)^2 \max_{x \in [0, 1]} v(x, \tau)
\leq C + CW(t) \max_{x \in [0, 1]} v(x, t) + (C + CW(t)) \max_{x \in [0, 1]} v(x, t)
\leq C + C(1 + W(t)) \max_{x \in [0, 1]} v(x, t).
\]

It follows from (2.28) that
\[
v(x, t) \leq C + C \int_0^t e^{-\gamma(t-\tau)} \left[ C + C(1 + W(\tau)) \max_{x \in [0, 1]} v(x, t) \right] d\tau
\leq C + C \int_0^t \left( e^{-\gamma(t-\tau)} + W(\tau) \right) \max_{x \in [0, 1]} v(x, t) d\tau.
\]

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Combining the above inequality with (2.2) and Gronwall’s inequality leads to the desired upper bound of specific volume. This, together with (2.27), finishes the proof of Lemma 2.3 □

Base on the upper and lower bounds of $v$, we can derive the bounds of $\chi$, one can find the proof in Appendix. The following lower order a priori estimates are crucial for the high order estimates which require time-independent bounds.

**Lemma 2.4** Let the conditions of Lemma 2.3 be in force. Then there hold

$$
||u_x, \chi_t, \chi^2 - 1||_{L^1} + \int_0^1 \frac{\chi^2}{v} \, dx + \int_0^1 (\chi^2 - 1)^2 \, dx \leq CW^\frac{3}{2}(t), \quad t \geq 0,
$$

and

$$
\int_0^T ||\phi^2 - 1||_{L^\infty} ^2 \, dt \leq C.
$$

**Proof.** First, in view of (2.2), (2.3) and (1.4)$_4$, we have

$$
||u_x||_{L^1} = \int_0^1 |u_x| \, dx = \int_0^1 \sqrt{\eta(\chi)} |u_x| \cdot \frac{\sqrt{\eta}}{\sqrt{\theta}} \, dx
\leq C \left( \int_0^1 \eta(\chi) u_x^2 \, dx \right)^{\frac{1}{2}} \left( \int_0^1 \theta \, dx \right)^{\frac{1}{2}}
\leq CW^\frac{3}{2}(t),
$$

and

$$
||\chi_t||_{L^1} = \int_0^1 |\chi_t| \, dx = \int_0^1 \frac{|\chi_t|}{\sqrt{\theta}} \cdot \sqrt{\eta} \, dx \leq C \left( \int_0^1 \frac{\chi_t^2}{\sqrt{\theta}} \, dx \right)^{\frac{1}{2}} \left( \int_0^1 \theta \, dx \right)^{\frac{1}{2}} \leq CW^\frac{3}{2}(t).
$$

Next, it follows from (1.4)$_3$ and (1.4)$_4$ that

$$
(1 - \chi^2) + \left( \frac{X_x}{\chi v} \right)_x \frac{1}{\chi v} = \chi_t.
$$

Integrating the above equality over [0, 1], by using (2.34), we obtain

$$
\int_0^1 (1 - \chi^2) \, dx + \int_0^1 \frac{X_x^2}{\chi^2} \, dx = \int_0^1 \frac{\chi_t}{\chi v} \, dx \leq C||\chi_t||_{L^1} \leq CW^\frac{3}{2}(t).
$$

In view of (1.4)$_3$ and (1.4)$_4$, we have

$$
\frac{X_t}{v} = \left( \frac{X_x}{v} \right)_x - (\chi^3 - \chi).
$$
Multiplying (2.54) by \( \chi \), and using (2.34), (2.36), we get
\[
\int_0^1 \frac{X_5^2}{v} \, dx + \int_0^1 (\chi^2 - 1)^2 \, dx = \int_0^1 (1 - \chi^2) \, dx - \int_0^1 \frac{\chi t}{v} \, dx \\
\leq CW^\frac{1}{2}(t) + C\|\chi_t\|_{L^1} \\
\leq CW^\frac{1}{2}(t).
\]

From (2.35), in a similar manner to (2.36), there holds
\[
\int_0^1 (\chi^2 - 1) \, dx = \int_0^1 \frac{X_5^2}{v\chi^2} \, dx - \int_0^1 \frac{X_t}{\chi v} \, dx \leq C \int_0^1 \frac{X_5^2}{v} \, dx + C\|\chi_t\|_{L^1} \leq CW^\frac{1}{2}(t).
\]

Which together with (2.36) implies \( \|\chi^2 - 1\|_{L^1} \leq CW^\frac{1}{2}(t) \).

Finally,
\[
\left\| \theta^{\frac{1}{2}} - 1 \right\|_{L^\infty} \leq \left\| \theta^{\frac{1}{2}} - \bar{\theta}^{\frac{1}{2}} \right\|_{L^\infty} + \left\| \bar{\theta}^{\frac{1}{2}} - 1 \right\|_{L^\infty}.
\]

For the first item on the right side, using (2.3) and (2.18), for any \( \beta > 0 \), there holds
\[
|\theta^{\frac{1}{2}} - \bar{\theta}| \leq \left| \theta^{\frac{1}{2}} - \bar{\theta}^{1/2} \right| + \left( \int_0^1 \frac{\partial \theta^{\frac{1}{2}}}{\partial t} \, dx \right)^{\frac{1}{2}} + \left( \int_0^1 v \theta \, dx \right)^{\frac{1}{2}} \leq CW^\frac{1}{2}(t). \quad (2.38)
\]

For the second item, it follows from (2.3) and (2.32) that
\[
|1 - \bar{\theta}| \leq |1 - \bar{\theta}| = C \left| 1 - \int_0^1 \theta \, dx \right| \\
= C \int_0^1 \frac{u^2}{2} \, dx + C \int_0^1 \left( \frac{\chi^2 - 1}{4} + \frac{X_5^2}{2v} \right) \, dx \\
\leq C\|u\|_{L^\infty}\|u\|_{L^2} + CW^\frac{1}{2}(t) \\
\leq C \int_0^1 |u_s| \, dx + CW^\frac{1}{2}(t) \leq CW^\frac{1}{2}(t).
\] (2.39)

Combining (2.38), (2.39) with (2.2) gives
\[
\int_0^T \left\| \theta^{\frac{1}{2}} - 1 \right\|_{L^\infty}^2 \, dt \leq C \int_0^T W(t) \, dt \leq C.
\]

\( \Box \)

**Lemma 2.5** Let the conditions of Lemma 2.3 be in force. Then for any \( p > 0 \), there exists a positive constant \( C \), which may depend on \( p \), such that
\[
\int_0^T \int_0^1 \frac{\theta^p \theta^2}{\theta^{p+1}} \, dx \, dt \leq C(p). \quad (2.40)
\]
Proof. We only consider the case that \( p \neq 1 \), since (2.40) with \( p = 1 \) is thanks to (2.2) and (2.18). In fact, multiplying (1.4) by \( \theta^{-p} \) with \( p \neq 1 \) and integrating by parts, we derive that

\[
\frac{1}{p-1} \int_0^1 \frac{d}{dx} \theta^{1-p} dx + p \int_0^1 \frac{\theta^p}{v} dx + \int_0^1 \frac{\eta(x) u_x^2}{v \theta^p} dx + \int_0^1 \frac{\mu^2}{\theta^p} dx
\]

\[
= \int_0^1 (\theta^{-p} - 1) u_x dx + \left( \int_0^1 \ln v dx \right)_t
\]

\[
\leq C(p) \left\| \theta^\frac{1}{2} - 1 \right\|_{L^\infty} \left( \int_0^1 \frac{v \theta^{-p}}{\theta} dx \right)^{\frac{1}{2}} \left( \int_0^1 \frac{\eta(x) u_x^2}{v \theta^p} dx \right)^{\frac{1}{2}}
\]

\[
+ C(p) \left\| \theta^\frac{1}{2} - 1 \right\|_{L^\infty} \left( \int_0^1 |u_x| dx + \left( \int_0^1 \ln v dx \right)_t \right)^2 + C(p) \left( \int_0^1 |u_x| dx \right)^2 + \left( \int_0^1 \ln v dx \right)_t.
\]

(2.41)

Case I. \( p > 1 \).

By virtue of Gronwall’s inequality, (2.32), (2.33) and the fact that \( \| \ln v(t) \|_{L^1} \leq C \) due to (2.18), we can infer from (2.41) that (2.40) holds.

Case II. \( 0 < p < 1 \).

Noting that \( \| \theta^{1-p} \|_{L^1} \leq C + C \| \theta \|_{L^1} \), integrating (2.41) over \([0, T]\), by (2.32), (2.33) and \( \| \ln v(t) \|_{L^1} \leq C \), we easily arrive at (2.40) with \( 0 < p < 1 \).

Combining Case I. with Case II. finishes the proof of Lemma 2.5

Lemma 2.6 Let the conditions of Lemma 2.3 be in force. Then there holds

\[
\int_0^T \| u_s \|_{L^2}^2 dt + \int_0^T \| u \|_{L^2}^2 dt \leq C.
\]

(2.42)

Proof. Integrating (1.4) over \([0, 1] \times [0, T]\), using (2.3), (2.18), (2.33) and the boundary condition (1.6), we find that

\[
\int_0^T \int_0^1 v \mu^2 dx dt + \int_0^T \int_0^1 \frac{\eta(x) u_x^2}{v} dx dt
\]

\[
= \int_0^1 \theta(x, T) dx - \int_0^1 \theta_0(x) dx + \int_0^T \int_0^1 \frac{\theta u_x}{v} dx dt
\]

\[
\leq C + \int_0^T \int_0^1 \frac{(\theta - 1) u_x}{v} dx dt + \int_0^T \left( \int_0^1 \ln v dx \right)_t dt
\]

(2.43)
\[ \begin{align*}
&\leq C + \frac{1}{2} \int_0^T \int_0^1 \frac{\eta(x)u_x^2}{v} \, dx \, dt + \int_0^T \int_0^1 \frac{(\theta^2 - 1)^2(\theta^2 + 1)^2}{\eta(x)^2} \, dx \, dt \\
&\leq C + \frac{1}{2} \int_0^T \int_0^1 \frac{\eta(x)u_x^2}{v} \, dx \, dt + C \int_0^T \left( \|\theta^2 - 1\|_{L^\infty}^2 \int_0^1 (\theta + 1) \, dx \right) \, dt \\
&\leq C + \frac{1}{2} \int_0^T \int_0^1 \frac{\eta(x)u_x^2}{v} \, dx \, dt + C \int_0^T \|\theta^2 - 1\|_{L^\infty}^2 \, dt \\
&\leq C + \frac{1}{2} \int_0^T \int_0^1 \frac{\eta(x)u_x^2}{v} \, dx \, dt.
\end{align*} \]

It implies that Lemma 2.6 holds. \(\square\)

**Lemma 2.7** Let the conditions of Lemma 2.3 be in force. Then it holds that
\[ \int_0^T \left\| \left( \frac{X_x}{v} \right)_x \right\|_{L^2}^2 \, dt + \int_0^T \|\chi^2 - 1\|_{L^2}^2 \, dt \leq C. \tag{2.43} \]

**Proof.** From (1.4)\(_4\) and (2.18), we derive that
\[ \int_0^1 \left| \left( \frac{X_x}{v} \right)_x \right|^2 \, dx \leq 2 \int_0^1 \mu^2 \, dx + 2 \int_0^1 \chi^2 (\chi^2 - 1)^2 \, dx \]
\[ \leq C \|\mu\|_{L^2}^2 + C \int_0^1 (\chi^2 - 1)^2 \, dx \]
\[ \leq C \|\mu\|_{L^2}^2 + C \|\chi^2 - 1\|_{L^\infty} \int_0^1 |\chi^2 - 1| \, dx \]
\[ \leq C \|\mu\|_{L^2}^2 + \tilde{\epsilon}_1 \|\chi^2 - 1\|_{L^2}^2 + C \|\chi^2 - 1\|_{L^1}^2 \]
\[ \leq C \|\mu\|_{L^2}^2 + (C + \tilde{\epsilon}_1) \|\chi^2 - 1\|_{L^2}^2 + \tilde{\epsilon}_1 \|2\chi x\|_{L^2}^2 \]
\[ \leq C \|\mu\|_{L^2}^2 + (C + \tilde{\epsilon}_1) \|\chi^2 - 1\|_{L^2}^2 + \tilde{\epsilon}_1 \tilde{C} \left\| \left( \frac{X_x}{v} \right)_x \right\|_{L^2}^2, \]

where we have used Cauchy-Schwartz’s inequality and the fact \( W^{1,1}((0, 1)) \hookrightarrow L^\infty((0, 1)) \). We can choose \( \tilde{\epsilon} \) is small enough with \( \tilde{\epsilon}_1 \tilde{C} \leq 1/2 \), integrating the above inequality over \([0, T]\) gives
\[ \int_0^T \left\| \left( \frac{X_x}{v} \right)_x \right\|_{L^2}^2 \, dt \leq C \int_0^T \|\mu\|_{L^2}^2 \, dt + C \int_0^T \|\chi^2 - 1\|_{L^1}^2 \, dt \leq C, \tag{2.44} \]
where we have used (2.32) and (2.42).

In view of (1.4)\(_4\), we get
\[ \int_0^T \|\chi^2 - 1\|_{L^2}^2 \, dt \leq C \int_0^T \int_0^1 \frac{1}{\chi^2} \mu^2 \, dx \, dt + C \int_0^T \int_0^1 \frac{1}{\chi^2} \left( \left( \frac{X_x}{v} \right)_x \right)^2 \, dx \, dt \]
\[ \leq C \int_0^T \|\mu\|_{L^2}^2 \, dt + \int_0^T \left\| \left( \frac{X_x}{v} \right)_x \right\|_{L^2}^2 \, dt \leq C. \]

From which and (2.44), Lemma 2.7 holds. \(\square\)
Lemma 2.8  Let the conditions of Lemma 2.3 be in force. Then there holds
\[ \sup_{0 \leq t \leq T} \left\| (v_x(t), \chi_x(t)) \right\|_{L^2}^2 + \int_0^T \int_0^1 \left( v_x^2 + \chi_x^2 + \theta v_x^2 \right) \ dx \ dt \leq C_2. \]  

**Proof.** Based on (1.4) and (1.4)\textsubscript{2}, we deduce that
\[ \left( \frac{\eta(\chi)v_x}{v} \right)_t = \eta \frac{v_x}{v} + \eta \left( \frac{u_x}{v} \right)_x = \left( \frac{\eta u_x}{v} \right)_x + \frac{\eta u_x - \eta u_x}{v} \]
\[ = u_t + \frac{\theta_x}{v} - \frac{\theta v_x}{v^2} + \frac{1}{2} \left( \frac{\chi_x^2}{v^2} \right)_x + \frac{a v_x}{v} (\chi v_x - \chi u_x). \]  

Multiplying (2.46) by \( \eta(\chi)v_x/v \) and integrating by parts over \([0, 1]\) yield
\[ \frac{1}{2} \frac{d}{dt} \left\| \frac{\eta(\chi)v_x}{v} \right\|_{L^2}^2 + \int_0^1 \frac{\eta(\chi)\theta_v^2}{v^3} \ dx \]
\[ = \int_0^1 \frac{\eta(\chi)u_v v_x}{v} \ dx + \int_0^1 \frac{\eta(\chi)v_v \theta_x}{v^2} \ dx + \frac{1}{2} \int_0^1 \left( \frac{\chi_x^2}{v^2} \right)_x \ \frac{\eta(\chi)v_x}{v} \ dx \]
\[ + \int_0^1 \frac{\alpha \eta^2(\chi_v v_x - \chi_x u_x) v_x}{v^2} \ dx \]
\[ = \frac{d}{dt} \int_0^1 \frac{\eta(\chi)u v_x}{v} \ dx - \int_0^1 \left( \eta (\ln v)_x (\eta (\ln v))_x + \eta u (\ln v)_x \right) \ dx + \int_0^1 \frac{\eta(\chi)v_x \theta_x}{v^2} \ dx \]
\[ + \frac{1}{2} \int_0^1 \left( \frac{\chi_x^2}{v^2} \right)_x \ \frac{\eta(\chi)v_x}{v} \ dx + \int_0^1 \frac{\alpha \eta^2(\chi_v v_x - \chi_x u_x) v_x}{v^2} \ dx. \]

Due to the non-slip boundary conditions \( u |_{x=1} = 0 \) and the fact
\[ \int_0^1 \frac{\eta(\chi)uv_x}{v} \ dx \leq \frac{1}{2} \int_0^1 \left( \frac{\eta(\chi)v_x}{v} \right)^2 \ dx + C \int_0^1 u^2 \ dx, \]
we derive after integrating (2.47) over \([0, T]\) that
\[ \frac{1}{4} \sup_{0 \leq t \leq T} \left\| \frac{\eta(\chi)v_x}{v} \right\|_{L^2}^2 + \int_0^T \int_0^1 \frac{\eta(\chi)\theta_v^2}{v^3} \ dx \ dt \]
\[ \leq C + \int_0^T \int_0^1 \left[ \left( \frac{\eta u_x}{v} \right)_x - \eta u (\ln v)_x \right] \ dx \ dt + \int_0^T \int_0^1 \frac{\eta(\chi)v_x \theta_x}{v^2} \ dx \ dt \]
\[ + \frac{1}{2} \int_0^T \int_0^1 \left( \frac{\chi_x^2}{v^2} \right)_x \ \frac{\eta(\chi)v_x}{v} \ dx + \int_0^T \int_0^1 \frac{\alpha \eta^2(\chi_v v_x - \chi_x u_x) v_x}{v^2} \ dx \ dt \]
\[ := C + \sum_{i=1}^4 I_i. \]

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It follows from (2.17) and (2.18) that

\[
I_1 = \int_0^T \int_0^1 \left[ (\eta u)_x - \eta u (\ln v)_x \right] dxdt
\]

\[
\leq C \int_0^T \int_0^1 \left[ \eta u_x^2 + \alpha \chi^{\alpha-1} (|\chi_{xu} u_x| + |\chi_{xv} v_x|) \right] dxdt
\]

\[
\leq C + C\alpha \int_0^T \left( \|u_x\|_{L^2}^2 \left( \|u_x\|_{L^2}^2 + \|v_x\|_{L^2}^2 \right) + \|\chi_{xu} u_x\|_{L^2}^2 + \|\chi_{xv} v_x\|_{L^2}^2 \right) dt
\]

\[
\leq C + C\alpha N^2 \int_0^T \left( \|u_x\|_{L^2}^2 + \left\| \left( \frac{\chi_x}{v} \right)_x \right\|_{L^2}^2 + \|\chi_{xu} u_x\|_{L^2}^2 \right) dt
\]

\[
\leq C + C\alpha H(m_1, m_2, m_3, N) \leq C,
\]

where we have used the fact \(\|u\|_{L^\infty} \leq C\|u\|_{L^2} + C\|u_x\|_{L^2} \leq C\|u_x\|_{L^2}\). We choose \(p = \beta > 0\) in (2.40) and obtain

\[
\int_0^T \int_0^1 \frac{\theta^2}{\theta} dxdt \leq C(\beta) \leq C.
\]

Using (2.18) and (2.50) gives

\[
I_2 = \int_0^T \int_0^1 \frac{\eta(\chi) v_x \theta_x}{v^2} dxdt
\]

\[
\leq \frac{3}{8} \int_0^T \int_0^1 \frac{\eta(\chi) v \theta_x^2}{v^3} dxdt + C \int_0^T \int_0^1 \frac{\eta(\chi) \theta_x^2}{v^3} dxdt
\]

\[
\leq \frac{3}{8} \int_0^T \int_0^1 \frac{\eta(\chi) \theta v_x^2}{v^3} dxdt + C \int_0^T \int_0^1 \frac{\theta_x^2}{\theta} dxdt
\]

\[
\leq \frac{3}{8} \int_0^T \int_0^1 \frac{\eta(\chi) \theta v_x^2}{v^3} dxdt + C.
\]

In view of (2.43), we have

\[
I_3 = \frac{1}{2} \int_0^T \int_0^1 \left( \frac{\chi_x}{v^2} \right)_x \frac{\eta(\chi) v_x}{v} dxdt
\]

\[
\leq \int_0^T \int_0^1 \left( \frac{\chi_x}{v} \right)_x^2 \frac{\eta(\chi) v_x}{v} dxdt + \int_0^T \int_0^1 \left( \frac{\chi_x}{v} \right)_x^2 dxdt
\]

\[
\leq \int_0^T \left\| \frac{\chi_x}{v} \right\|_{L^\infty} \int_0^1 \left( \frac{\eta(\chi) v_x}{v} \right)^2 dxdt + C
\]

\[
\leq \int_0^T \left\| \left( \frac{\chi_x}{v} \right)_x \right\|_{L^2} \left\| \frac{\eta(\chi) v_x}{v} \right\|_{L^2} dxdt + C.
\]
In terms of (2.17), (2.18) and (2.43), then we get

\[ I_4 = \int_0^T \int_0^1 \frac{\alpha \eta^2 (\chi x v_x - \chi u_x) v_x}{v^2} \, dx \, dt \]

\[ \leq C \alpha \int_0^T \int_0^1 \sqrt{7} \left( v_x^2 \chi_x + |\chi_x u_x v_x| \right) \, dx \, dt \]

\[ \leq \frac{3}{8} \int_0^T \int_0^1 \frac{\eta (x) \theta v_x^2}{v^3} \, dx \, dt + C \alpha^2 \int_0^T \int_0^1 \frac{v^3}{\theta} \left( v_x^2 \chi_x^2 + \chi_x^2 v_x^2 \right) \, dx \, dt \]

\[ \leq \frac{3}{8} \int_0^T \int_0^1 \frac{\eta (x) \theta v_x^2}{v^3} \, dx \, dt + C \frac{\alpha^2 N^2}{m_3} \int_0^T \int_0^1 \left( \chi_x^2 + u_x^2 \right) \, dx \, dt \]

\[ \leq \frac{3}{8} \int_0^T \int_0^1 \frac{\eta (x) \theta v_x^2}{v^3} \, dx \, dt + C \, \alpha \left( \alpha \left( \alpha \right) \right) \]

where we have used the fact \( \| (v_x, \chi_x) \|_{L^\infty} \leq C \| (v_x, \chi_x) \|_{H^1} \leq C N \) for \( t \in [0, T] \).

Substituting (2.49)–(2.51) into (2.48), by virtue of Gronwall’s inequality, (2.43) and (2.18), we deduce that

\[ \sup_{0 \leq t \leq T} \| v_x(t) \|_{L^2}^2 + \int_0^T \int_0^1 \theta v_x^2 \, dx \, dt \leq C. \]

From which and (2.33), we get

\[ \int_0^T \int_0^1 v_x^2 \, dx \, dt \leq C \int_0^T \left\| \theta^2 - 1 \right\|_{L^\infty}^2 \| v_x \|_{L^2}^2 \, dt + C \int_0^T \int_0^1 \theta v_x^2 \, dx \, dt \leq C. \]

We rewrite (1.4) as follows

\[ \chi_1 = v \left( \frac{\chi_x}{v} \right)_x - v(\chi^3 - \chi) = \chi_{xx} - \frac{\chi_x v_x}{v} - v(\chi^3 - \chi). \]  

(2.52)

Multiplying (2.52) by \( \chi_{xx} \) and integrating the result over [0, 1] yield

\[ \frac{1}{2} \frac{d}{dt} \| \chi_x \|_{L^2}^2 + \| \chi_{xx} \|_{L^2}^2 = \int_0^1 \frac{\chi_x v_x}{v} \chi_{xx} \, dx + \int_0^1 v(\chi^3 - \chi) \chi_{xx} \, dx \]

\[ \leq \frac{1}{2} \| \chi_{xx} \|_{L^2}^2 + C \int_0^1 \chi_x^2 v_x^2 \, dx + C \int_0^1 \chi^2 (\chi^2 - 1)^2 \, dx \]

\[ \leq \frac{1}{2} \| \chi_{xx} \|_{L^2}^2 + C \left\| \frac{\chi_x}{v} \right\|_{L^\infty}^2 \| v_x \|_{L^2}^2 + C \| \chi^2 - 1 \|_{L^2}^2 \]

\[ \leq \frac{1}{2} \| \chi_{xx} \|_{L^2}^2 + C \left\| \left( \frac{\chi_x}{v} \right)_x \right\|_{L^2}^2 + C \| \chi^2 - 1 \|_{L^2}^2. \]

By virtue of Gronwall’s inequality and (2.43), we arrive at

\[ \sup_{0 \leq t \leq T} \| \chi_x \|_{L^2}^2 + \int_0^T \int_0^1 \chi_{xx}^2 \, dx \, dt \leq C. \]

This completes the proof of Lemma 2.8.
Lemma 2.9 Let the conditions of Lemma [2.3] be in force. Then we have

\[
\sup_{0 \leq t \leq T} \left\| \left( X_{xx}, X_t, \frac{X_x}{v} \right) \right\|_{L^2}^2 + \int_0^T \left\| \left( X_{xt}, \frac{X_x}{v} \right) \right\|_{L^2}^2 \, dt \leq C_3. \tag{2.53}
\]

\textbf{Proof.} Rewrite (1.4) as

\[
\frac{X_t}{v} = \left( \frac{X_x}{v} \right)_t - (\chi^3 - \chi). \tag{2.54}
\]

Differentiating (2.54) with respect to \( x \) yields

\[
\left( \frac{X_x}{v} \right)_t - \left( \frac{X_x}{v} \right)_{xx} = \frac{X_x v_x}{v^2} - \frac{X_x u_x}{v^2} - \chi \left( 3 \chi^2 - 1 \right).
\]

Multiplying the above equation by \( \left( \frac{X_x}{v} \right)_t \) and integrating over \([0, 1]\), we get

\[
\frac{1}{2} \frac{d}{dt} \left( \left\| \left( X_x \right)_t \right\|_{L^2}^2 + \left\| \frac{X_x}{v} \right\|_{L^2}^2 \right)
= \int_0^1 \left( \frac{X_x v_x}{v^2} - \frac{X_x u_x}{v^2} - \chi \left( 3 \chi^2 - 1 \right) \right) \left( \frac{X_x}{v} \right)_t \, dx
\leq \frac{1}{4} \left\| \left( \frac{X_x}{v} \right)_t \right\|_{L^2}^2 + C \int_0^1 \left( \chi^2 v_x^2 + \chi^2 u_x^2 + \chi \chi^2 (3 \chi^2 - 1)^2 \right) \, dx
\leq \frac{1}{4} \left\| \left( \frac{X_x}{v} \right)_t \right\|_{L^2}^2 + C \| \chi \|_{L^infty} \| v_x \|_{L^2}^2 + C \| \chi \|_{L^infty} \| u_x \|_{L^2}^2 + C \| \chi \|_{L^2}^2
\leq \frac{1}{4} \left\| \left( \frac{X_x}{v} \right)_t \right\|_{L^2}^2 + C \| \chi \|_{L^infty} \| \chi \|_{L^2}^2 + C \| \chi \|_{L^2}^2
= \frac{1}{2} \left\| \left( \frac{X_x}{v} \right)_t \right\|_{L^2}^2 + C \| \chi \|_{L^2}^2 + C \| \chi \|_{L^2}^2
\leq \frac{1}{2} \left\| \left( \frac{X_x}{v} \right)_t \right\|_{L^2}^2 + C \| \chi \|_{L^2}^2 + C \| \chi \|_{L^2}^2.
\]

Integrating it over \([0, T]\) gives

\[
\sup_{0 \leq t \leq T} \left\| \left( \frac{X_x}{v} \right)_t \right\|_{L^2}^2 + \int_0^T \left\| \left( \frac{X_x}{v} \right)_t \right\|_{L^2}^2 \, dt \leq C.
\]

Thus, it follows from (2.18) and the above inequality that

\[
\| \chi_{xx} \|_{L^2}^2 = \left\| \left( \frac{X_x}{v} \right)_x + \frac{X_{xx} v_x}{v^2} \right\|_{L^2}^2 \leq C \left\| \left( \frac{X_x}{v} \right)_x \right\|_{L^2}^2 + C \left\| \frac{X_x}{v} \right\|_{L^infty}^2 \| v_x \|_{L^2}^2 \leq C \left\| \left( \frac{X_x}{v} \right)_x \right\|_{L^2}^2 \leq C.
\]

Moreover,

\[
\| \chi \|_{L^2}^2 = \left\| v \left( \frac{X_x}{v} \right)_x - v (\chi^3 - \chi) \right\|_{L^2}^2 \leq C \left\| \left( \frac{X_x}{v} \right)_x \right\|_{L^2}^2 + C \leq C.
\]
Similarly, we can deduce that

\[
\int_0^T \|x_{,tt}\|_{L^2}^2 \, dt = \int_0^T \left\| \left( \frac{X_{,t}}{v} \right)_t + \frac{X_{,t} u_{,x}}{v^2} \right\|_{L^2}^2 \, dt \\
\leq C \int_0^T \left\| \left( \frac{X_{,t}}{v} \right)_t \right\|_{L^2}^2 \, dt + C \int_0^T \left\| \frac{X_{,t}}{v} \right\|_{L^\infty} \|u_x\|_{L^2}^2 \, dt \leq C.
\]

Then we see that (2.53) holds. \(\square\)

**Lemma 2.10** Let the conditions of Lemma 2.9 be in force, then

\[
\sup_{0 \leq t \leq T} \|u_x\|_{L^2}^2 + \int_0^T \|(u_t, u_{xx}, \theta_x)\|_{L^2}^2 \, dt \leq C_4.
\]  

**(Proof.)** We rewrite (1.42) as follows

\[
u_t - \frac{\eta(\chi)}{v} u_{xx} = \frac{\eta_x u_x}{v} - \frac{\eta u_{xx}}{v^2} - \frac{\theta_x}{v} + \frac{\theta v_x}{v} - \frac{1}{2} \frac{x^2}{v^2} x.
\]

Multiplying it by \(u_{xx}\), then integrating the result over \([0, 1] \times [0, T]\), by using (2.18) we have

\[
\frac{1}{2} \frac{d}{dt} \|u_x\|_{L^2}^2 + \int_0^1 \frac{\eta(\chi)}{v} u_{xx}^2 \, dx \\
= \int_0^1 \left( \frac{\eta_x u_x}{v} - \frac{\eta u_{xx}}{v^2} - \frac{\theta_x}{v} + \frac{\theta v_x}{v} - \frac{1}{2} \frac{x^2}{v^2} x \right) u_{xx} \, dx \\
\leq \frac{1}{4} \int_0^1 \frac{\eta(\chi)}{v} u_{xx}^2 \, dx + C \int_0^1 \left( \frac{\eta}{v^2} u_{xx}^2 + \frac{\eta^2}{v^2} u_{xx}^2 + \frac{\theta^2}{v^2} \right) \left( \frac{X_{,t}}{v} \right)^2 \, dx \\
\leq \frac{1}{4} \int_0^1 \frac{\eta(\chi)}{v} u_{xx}^2 \, dx + C \int_0^1 \left( \frac{\eta}{v^2} u_{xx}^2 + \frac{\theta^2}{v^2} \right) \left( \frac{X_{,t}}{v} \right)^2 \, dx.
\]

(2.56)

Using (2.18), (2.42) and (2.45) gives

\[
\int_0^T \int_0^1 \left( u_{x}^2 + u_{xx}^2 \right) \, dx \, dt \leq C + \int_0^T \|u_x\|_{L^\infty} \|v_x\|_{L^2}^2 \, dt \\
\leq C + C \int_0^T \|u_{xx}\|_{L^2} \|u_x\|_{L^2} \, dt \\
\leq \frac{1}{4} \int_0^1 \frac{\eta(\chi)}{v} u_{xx} \, dx \, dt + C.
\]  

(2.57)

In view of (2.19) and (2.45), we derive that

\[
\int_0^T \int_0^1 \left( \theta_x^2 + \theta_{xx}^2 \right) \, dx \, dt \leq C \int_0^T \left( \|\theta_x\|_{L^2}^2 + \|\theta - \bar{\theta}\|_{L^\infty} \|v_x\|_{L^2}^2 \right) \, dt
\]
\[ \leq C \int_0^T \left( \| \theta \|_{L^2}^2 + \| \theta - \bar{\theta} \|_{L^\infty}^2 \right) dt \]

\[ \leq C \int_0^T \| \theta \|_{L^2}^2 dt + C. \]

Thanks to (2.43) and (2.53), we get

\[ \int_0^T \int_0^1 \left( \frac{\chi_x}{v} \right)^2 \left( \frac{\chi_x}{v} \right)^2 \, dx \, dt \leq \int_0^T \left\| \frac{\chi_x}{v} \right\|_{L^\infty}^2 \left\| \left( \frac{\chi_x}{v} \right)_x \right\|_{L^2}^2 \, dt \]

\[ \leq C \int_0^T \left\| \left( \frac{\chi_x}{v} \right)_x \right\|_{L^2}^2 \, dt \leq C. \] (2.58)

Hence, integrating (2.56) over \([0, T]\), and inserting (2.57)–(2.58) into it, we arrive at

\[ \sup_{0 \leq t \leq T} \| \theta \|_{L^2}^2 + \int_0^T \| \theta \|_{L^2}^2 dt \leq C + C \int_0^T \| \theta \|_{L^2}^2 dt. \] (2.59)

Next, we deal with the last term in (2.59). Indeed, if \( \beta > 1 \), one can choose \( p = \beta - 1 \) in (2.40) to obtain \( \| \theta \|_{L^2(0,T; L^2)}^2 \leq C \). With which and (2.59), it holds that

\[ \sup_{0 \leq t \leq T} \| u_x \|_{L^2}^2 + \int_0^T \left( \| u_x \|_{L^2}^2 + \| \theta \|_{L^2}^2 \right) dt \leq C. \] (2.60)

In the case of \( 0 < \beta \leq 1 \), multiplying (1.4)_3 by \( \theta \), by using (1.4)_1, (1.4)_3, (2.32) and (2.39), we have

\[ \frac{1}{2} \frac{d}{dt} \| \theta \|_{L^2}^2 + \int_0^1 \frac{\theta^2 \theta_x^2}{v} \, dx \]

\[ = \int_0^1 \frac{\eta(\chi) u_x^2 \theta}{v} \, dx - \int_0^1 \frac{\theta^2 u_x}{v} \, dx + \int_0^1 v^2 \theta \, dx \]

\[ \leq \int_0^1 \frac{\eta(\chi) u_x^2 \theta}{v} \, dx - \int_0^1 \frac{(\theta^2 - \bar{\theta}^2) u_x}{v} \, dx + (1 - \bar{\theta}^2) \int_0^1 \frac{u_x}{v} \, dx - \int_0^1 \frac{u_x}{v} \, dx + \int_0^1 \frac{\chi_x^2 \theta}{v} \, dx \]

\[ \leq \int_0^1 \frac{\eta(\chi) u_x^2 \theta}{v} \, dx - \int_0^1 \frac{(\theta^2 - \bar{\theta}^2) u_x}{v} \, dx + C \left( \int_0^1 \ln v \, dx \right) + \int_0^1 \frac{\chi_x^2 \theta}{v} \, dx \]

\[ \leq \int_0^1 \eta(\chi) u_x^2 \theta \, dx - \int_0^1 \frac{(\theta^2 - \bar{\theta}^2) u_x}{v} \, dx + \int_0^1 \chi_x^2 \theta \, dx - \left( \int_0^1 \ln v \, dx \right) + CW(t) \]

\[ := \sum_{i=1}^3 J_i - \left( \int_0^1 \ln v \, dx \right) + CW(t). \] (2.61)

It follows from (2.3) and (2.18) that

\[ J_1 = \int_0^1 \frac{\eta(\chi) u_x^2 \theta}{v} \, dx \leq C \| u_x \|_{L^\infty}^2 \| \theta \|_{L^2} \leq C \| u_x \|_{L^2} \| u_x \|_{L^2} \]

\[ \leq \bar{\varepsilon} \| u_x \|_{L^2}^2 + C \bar{\varepsilon}^{-1} \| u_x \|_{L^2}^2. \] (2.62)
For $0 < \beta \leq 1$, it is easy to check that $\theta^{\beta-2} + \theta^\beta \geq 1$. Thus

$$
\left\| \theta^2 - \tilde{\theta}^2 \right\|_{L^\infty} \leq C \int_0^1 |\theta \theta_x| \, dx \leq C \|\theta\|_{L^2} \|\theta \theta_x\|_{L^2}
$$

$$
\leq C \|\theta\|_{L^2} \left( \left\| \left( \theta^{\beta-1} + \theta^\beta \right) \theta_x \right\|_{L^2} + \left\| \theta^\beta \theta_x \right\|_{L^2} \right).
$$

Together with (2.18), (2.32) and Cauchy-Schwartz’s inequality, we get

$$
J_2 = - \int_0^1 \frac{(\theta^2 - \tilde{\theta}^2)u_x}{\nu} \, dx \leq C \left\| \theta^2 - \tilde{\theta}^2 \right\|_{L^\infty} \|u_x\|_{L^1}
$$

$$
\leq C \|\theta\|_{L^2} \left( W^2(t) + \left\| \theta^\beta \theta_x \right\|_{L^2} \right) W^2(t)
$$

$$
\leq \frac{1}{2} \left\| \theta^\beta \theta_x \right\|_{L^2}^2 + CW(t) \|\theta\|_{L^2} + CW(t) \|\theta\|_{L^2}^2
$$

$$
\leq \frac{1}{2} \left\| \theta^\beta \theta_x \right\|_{L^2}^2 + CW(t) + CW(t) \|\theta\|_{L^2}^2.
$$

(2.63)

In view of (2.3) and (2.18), we derive

$$
J_3 = \int_0^1 \frac{\chi^2 \theta}{\nu} \, dx \leq C \|\chi\|_{L^\infty}^2 \int_0^1 \theta \, dx \leq C \|\chi\|_{L^2}^2 + C \|\chi x\|_{L^2}^2.
$$

(2.64)

Hence, putting (2.62), (2.63) and (2.64) into (2.61), integrating over $[0, T]$, and using Gronwall’s inequality, we have

$$
\sup_{0 \leq t \leq T} \|\theta\|_{L^2}^2 + \int_0^T \int_0^1 \theta^\beta \theta_x^2 \, dxdt \leq C(\tilde{\varepsilon}_2) + \tilde{\varepsilon}_2 \int_0^T \|u_{xx}\|_{L^2}^2 \, dt.
$$

Thus

$$
\sup_{0 \leq t \leq T} \|\theta\|_{L^2}^2 + \int_0^T \int_0^1 \theta^\beta \theta_x^2 \, dxdt \leq \sup_{0 \leq t \leq T} \|\theta\|_{L^2}^2 + \int_0^T \int_0^1 (\theta^{\beta-2} + \theta^\beta) \theta_x^2 \, dxdt
$$

$$
\leq C(\tilde{\varepsilon}_2) + \tilde{\varepsilon}_2 \int_0^T \|u_{xx}\|_{L^2}^2 \, dt.
$$

(2.65)

Plugging (2.65) into (2.59) and choosing $\tilde{\varepsilon}_2$ small enough, we conclude that (2.60) also holds for $0 < \beta \leq 1$. Hence, we have shown that (2.60) is valid for any $\beta > 0$.

Finally, it follows from (2.19), (2.45) and (2.60) that

$$
\int_0^T \|\theta \theta_x\|_{L^2}^2 \, dt \leq C \int_0^T \left( \|\theta - \tilde{\theta}\|_{L^\infty} \|\theta\|_{L^2}^2 + \|\theta\|_{L^\infty} \|\theta_x\|_{L^2}^2 \right) \, dt
$$

$$
\leq C \int_0^T \left( \|\theta\|_{L^2}^2 \|\theta_x\|_{L^2}^2 + \|\theta_x\|_{L^2}^2 \right) \, dt
$$

$$
\leq C + C \int_0^T \|\theta_x\|_{L^2}^2 \, dt \leq C.
$$
From which and (2.18), (2.42), (2.43), (2.45), (2.53), (2.60), we get
\[
\int_0^T \|u_t\|_{L^2}^2 \, dt = \int_0^T \left\| \eta \chi u_{xx} + \eta u_x v_x - \theta x + \theta v_x - \frac{1}{2} \left( \frac{\chi^2}{v^2} \right)_x \right\|_{L^2}^2 \, dt
\]
\[
\leq C \int_0^T \left( |u_{xx}|_{L^2}^2 + \alpha^2 |\chi x|_{L^\infty} |u_x|_{L^2}^2 + |u_x|_{L^2}^2 |v_x|_{L^2}^2 + |\theta x|_{L^2}^2 \right) \, dt
\]
\[
+ C \int_0^T \left( |\theta v_x|_{L^2}^2 + \left| \frac{\chi_x}{v} \right|_{L^\infty}^2 \left| \frac{\chi}{v} \right|_{L^2}^2 \right) \, dt
\]
\[
\leq C + C \int_0^T \|\theta v_x\|_{L^2}^2 \, dt \leq C.
\]
This, together with (2.60), finishes the proof of (2.55). □

**Lemma 2.11** Let the conditions of Lemma 2.3 be in force. Then there hold
\[
C_1 \leq \theta(x, t) \leq C_1^{-1}, \quad \forall (x, t) \in [0, 1] \times [0, T],
\]
(2.66)
and
\[
\sup_{0 \leq t \leq T} \|\theta_x(t)\|_{L^2}^2 + \int_0^T \|(\theta_{xx}, \theta_t)\|_{L^2}^2 \, dt \leq C_5.
\]
(2.67)

**Proof.** First, we make use of (2.18) and (2.64) to deduce from (2.61) that
\[
\frac{1}{2} \frac{d}{dt} \|\theta\|_{L^2}^2 + \int_0^1 \frac{\theta \theta_x^2}{v} \, dx
\]
\[
\leq \int_0^1 \frac{\eta \chi u_{xx} \theta}{v} \, dx - \int_0^1 \frac{\theta^2 - \theta_x^2}{v} u_x \, dx + \int_0^1 \frac{\chi_x \theta}{v} \, dx - \left( \int_0^1 \ln v \, dx \right)_t + CW(t)
\]
\[
\leq C |u_x|_{L^\infty}^2 + C \|\theta^2 - \theta_x^2\|_{L^1} |u_x|_{L^1} + C |\chi_x|_{H^1}^2 - \left( \int_0^1 \ln v \, dx \right)_t + CW(t)
\]
\[
\leq C |u_x|_{L^\infty}^2 + C \|\theta^2 - \theta_x^2\|_{L^1} |u_x|_{L^1} W^2(t) + C |\chi_x|_{H^1}^2 - \left( \int_0^1 \ln v \, dx \right)_t + CW(t),
\]
\[
\leq C |u_x|_{H^1}^2 + C \|\theta_{L^2} |\theta|_{L^2} W(t) + C |\chi|_{H^1}^2 - \left( \int_0^1 \ln v \, dx \right)_t + CW(t),
\]
\[
\leq C |u_x|_{H^1}^2 + CW(t) \|\theta\|_{L^2}^2 + C |\theta_x|_{L^2}^2 + C |\chi|_{H^1}^2 - \left( \int_0^1 \ln v \, dx \right)_t + CW(t).
\]
Integrating it over [0, T], using (2.2), (2.42), (2.53), (2.55) and Gronwall’s inequality, we get
\[
\sup_{0 \leq t \leq T} \|\theta\|_{L^2}^2 + \int_0^T \int_0^1 \theta \theta_x^2 \, dx \, dt \leq C.
\]
(2.68)
Next, multiplying (1.4) by \(\theta \theta_t\) and integrating the result over [0, 1] yield
\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \left( \theta \theta_t \right)^2 \frac{1}{v} \, dx + \int_0^1 \theta \theta_t \theta_x^2 \, dx.
\]
and Gronwall’s inequality, we obtain

\[
\int_0^1 \frac{(\theta^\beta \theta_x)^2} {v^2} \, dx + \int_0^1 \left( \frac{\eta(\chi)u_x^2 - \theta u_x} {v} + v \mu^2 \right) \theta^\beta \theta_x \, dx \\
\leq \frac{1} {2} \int_0^1 \theta^\beta \theta_t^2 \, dx + C \int_0^1 (\theta^{\beta+2} u_x^2 + \theta^\beta u_x^4 + \theta^\beta \mu^4) \, dx + C \|u_x\|_{L^\infty} \|\theta\|_{L^\infty}^\beta \int_0^1 \theta^\beta \theta_x^2 \, dx
\]

:= \frac{1} {2} \int_0^1 \theta^\beta \theta_t^2 \, dx + K_1 + K_2. \tag{2.69}

The second term on the right-hand side can be estimated as follows

\[
K_1 = C \int_0^1 (\theta^{\beta+2} u_x^2 + \theta^\beta u_x^4 + \theta^\beta \mu^4) \, dx \\
\leq C \left( (\|\theta\|_{L^\infty}^{\beta+1} \|u_x\|_{L^\infty} \|\theta\|_{L^2} \|u_x\|_{L^2} + \|\theta\|_{L^\infty} \|u_x\|_{L^2}^2 \|u_x\|_{L^2}^2 + \|\theta\|_{L^\infty} \|\mu\|_{L^2} \|u_x\|_{L^2}^2) \right) \\
\leq C \left( 1 + \|\theta\|_{L^\infty}^{\beta+1} \right) \|u_x\|_{L^2} \|u_x\|_{L^2} + C \left( 1 + \|\theta\|_{L^\infty}^{\beta+1} \right) \|\chi\|_{L^\infty}^2 \tag{2.70}
\]

\[
\leq C \left( \|u_x\|_{L^2}^2 + \|u_x\|_{L^2}^2 + \|\chi\|_{L^2} \right) + C \left( \|u_x\|_{L^2}^2 + \|\chi\|_{L^2}^2 \right) \|\theta^\beta \theta_x\|_{L^2}^2,
\]

where we have used (1.4)\(_3\), (2.53), (2.55), (2.68), Young’s inequality and the following Sobolev inequalities

\[
\|u_x\|_{L^\infty} \leq C\|u_x\|_{L^2}, \quad \|u_x\|_{L^\infty}^2 \leq C\|u_x\|_{L^2} \|u_x\|_{L^2},
\]

and

\[
\|\chi\|_{L^\infty}^2 \leq C\|\chi\|_{L^2} \|\chi\|_{L^2}.
\]

Similarly, we have

\[
K_2 = C \|u_x\|_{L^\infty} \|\theta\|_{L^\infty}^\beta \int_0^1 \theta^\beta \theta_x^2 \, dx \\
\leq C \|u_x\|_{L^\infty} \left( 1 + \|\theta\|_{L^\infty}^{\beta+1} \right) \|\theta^\beta \theta_x\|_{L^2} \|\theta^\beta \theta_x\|_{L^2} \tag{2.71}
\]

\[
\leq C \left( \|u_x\|_{L^2}^2 + \|\theta^\beta \theta_x\|_{L^2}^2 \right) \|\theta^\beta \theta_x\|_{L^2} \leq C \left( \|u_x\|_{L^2}^2 + \|\theta^\beta \theta_x\|_{L^2}^2 \right) \|\theta^\beta \theta_x\|_{L^2}.
\]

Thus, substituting (2.70) and (2.71) into (2.69), using (1.4)\(_3\), (2.18), (2.6), (2.53), (2.55), (2.68) and Gronwall’s inequality, we obtain

\[
\sup_{0 \leq t \leq T} \|\theta^\beta \theta_x\|_{L^2}^2 + \int_0^T \int_0^1 \theta^\beta \theta_t^2 \, dx \, dt \leq C. \tag{2.72}
\]

in view of (2.19), noting that

\[
\|\theta\|_{L^\infty}^{\beta+1} = \|\theta^{\beta+1}\|_{L^\infty} \leq \|\theta^{\beta+1}\|_{L^\infty} + C \leq C\|\theta^\beta \theta_x\|_{L^2} + C \leq C,
\]
which implies

$$\theta(x, t) \leq C, \quad \forall (x, t) \in [0, 1] \times [0, T].$$

(2.73)

Moreover, it follows from (2.68) and (2.73) that

$$\int_0^T \int_0^1 (\theta^{\beta+1} - \bar{\theta}^{\beta+1})^2 \, dx \, dt \leq C \int_0^T \int_0^1 \theta^{2\beta} \theta_x^2 \, dx \, dt \leq C \sup_{0 \leq s \leq T} \|\theta\|_{L^\infty}^\beta \int_0^T \int_0^1 \theta^2 \theta_x^2 \, dx \, dt \leq C.$$

(2.74)

From (2.2), (2.3), (2.53) and (2.55), there holds

$$\int_0^T \bar{\theta}_t^2 \, dt = \int_0^T \left\{ \frac{d}{dt} \left[ 1 - \int_0^1 \left( \frac{u^2}{2} + \frac{(\chi^2 - 1)^2}{4} + \frac{\chi_x^2}{2v} \right) \, dx \right] \right\}^2 \, dt$$

$$= \int_0^T \left\{ \int_0^1 \left( -uu_t - (\chi^2 - 1)\chi\chi_t - \frac{\chi\chi_x\chi_t}{v} + \frac{\chi_x^2\chi_t}{2v^2} \right) \, dx \right\}^2 \, dt$$

$$\leq \int_0^T \left( \|u\|_{L^2}^2 \|u_t\|_{L^2}^2 + \|\chi\|_{L^2}^2 \|\chi_t\|_{L^2}^2 + \|\chi_x\|_{L^2}^2 \|\chi_x\chi_t\|_{L^2}^2 + \|\chi_x\|_{L^\infty} \|u_x\|_{L^2}^2 \right) \, dt$$

$$\leq C \int_0^T \left( \|u_t\|_{L^2}^2 + \|\chi\|_{L^2}^2 + \|\chi_t\|_{L^2}^2 + \|u_x\|_{L^2}^2 \right) \, dt$$

$$\leq C.$$

Hence, we have

$$\int_0^T \left| \frac{d}{dt} \int_0^1 (\theta^{\beta+1} - \bar{\theta}^{\beta+1})^2 \, dx \right| \, dt$$

$$\leq C \int_0^T \int_0^1 (\theta^{\beta+1} - \bar{\theta}^{\beta+1})^2 \, dx \, dt + C \int_0^T \left( \|\theta^2 \theta_x^2\|_{L^2}^2 + \|\theta^2 \theta_x^2\|_{H^1}^2 \right) \, dt$$

(2.75)

$$\leq C + C \int_0^T \bar{\theta}_t^2 \, dt \leq C.$$

Combining (2.74) with (2.75), one arrive at

$$\lim_{t \to +\infty} \int_0^1 (\theta^{\beta+1} - \bar{\theta}^{\beta+1})^2 \, dx \, dt = 0.$$

Together with (2.72), we see that, as $t \to +\infty$,

$$\left\| \left( \theta^{\beta+1} - \bar{\theta}^{\beta+1} \right) (t) \right\|_{H^1}^2 \leq C \left\| \left( \theta^{\beta+1} - \bar{\theta}^{\beta+1} \right) (t) \right\|_{L^2}^2 \|\theta^2 \theta_x^2\|_{L^2} \to 0.$$

Then, by (2.19), we conclude that there exists a time $T_0 \gg 1$ such that

$$\theta(x, t) \geq \frac{\gamma_1}{2}, \quad \forall (x, t) \in [0, 1] \times [T_0, +\infty).$$

(2.76)
Let $T_0$ be fixed as in (2.76). Multiplying (1.25) by $\theta^{-p}$ with $p > 2$, and integrating by parts over $[0, 1]$, by (2.18), we have

\[
\frac{1}{p - 1} \frac{d}{dt} \|\theta^{-1}\|_{L^{p-1}}^2 + p \int_0^1 \theta^p \theta_x^2 \, dx + \int_0^1 \frac{\eta(x)u_x^2}{v\theta^p} \, dx + \int_0^1 \frac{v\mu^2}{\theta^p} \, dx
\]

which implies

\[
\frac{d}{dt} \|\theta^{-1}\|_{L^{p-1}} \leq C,
\]

where $C$ is a generic positive constant independent of $p$. Hence, integrating the above inequality over $[0, t]$ and letting $p \to \infty$, we obtain

\[
\theta^{-1}(x, t) \leq C(T_0 + 1) \iff \theta(x, t) \geq [C(T_0 + 1)]^{-1}, \quad \forall (x, t) \in [0, 1] \times [0, T_0].
\]

This, together with (2.73) and (2.76), proves (2.66).

Finally, using (2.66), we get from (2.72) that

\[
\|\theta_x\|_{L^2}^2 = \|\theta^p \theta_x \cdot \theta^{-p}\|_{L^2}^2 \leq C_1^p \|\theta^p \theta_x\|_{L^2}^2 \leq C,
\]

and

\[
\int_0^T \|\theta_x\|_{L^2}^2 \, dt \leq C_1^\frac{p}{2} \int_0^T \|\theta^p \theta_x\|_{L^2} \, dt \leq C.
\]

In view of (2.42), (2.45), (2.53), (2.55), (2.77) and (2.78), we deduce from (1.25) that

\[
\int_0^T \|\theta_{xx}\|_{L^2}^2 \, dt \leq C \int_0^T \left( \theta_t^2 + u_x^2 + \theta_x^4 + \theta^2 v^2 + u_x^2 + \mu^4 \right) \, dx \, dt
\]

\[
\leq C + C \int_0^T \|\theta_x\|_{L^2}^2 \, dt \leq C + C \int_0^T \|\theta_x\|_{L^2} \|\theta_{xx}\|_{L^2} \, dt
\]

\[
\leq \frac{1}{2} \int_0^T \|\theta_{xx}\|_{L^2}^2 \, dt + C.
\]

Together with (2.77) and (2.78), it leads to (2.72).

\[\square\]

**Lemma 2.12** Let the conditions of Lemma 2.3 be in force. Then it holds that

\[
\sup_{0 \leq t \leq T} \|(u_t, u_{xx})\|_{L^2}^2 + \int_0^T \|u_{xx}\|_{L^2}^2 \, dt \leq C_6.
\]
**Proof.** Differentiating (1.4)₂ with respect to \( t \), by (1.4)₁ we find

\[
u_{tt} + \left( \frac{v_{tt} - \theta u_x}{v^2} \right)_x = \left( \left( \frac{\eta}{v} \right)_x \right) u_x + \left( \frac{\eta}{v} u_{tt} \right)_x - \frac{1}{2} \left( \frac{x_t^2}{v^2} \right)_{xx} .
\]

We multiply it by \( u_t \) and integrate the result \([0, 1]\), then

\[
\frac{1}{2} \frac{d}{dt} \|u_t\|_{L^2}^2 + \int_0^1 \frac{\eta(x) u_{tt}^2}{v} \, dx = - \int_0^1 \left( \frac{\eta}{v} \right) u_x u_{xt} \, dx + \int_0^1 \frac{v_{tt} - \theta u_x}{v^2} u_{xt} \, dx + \int_0^1 \left( \frac{\chi}{v} \right) \left( 
\frac{x_t^2}{v^2} \right) \, u_{tt} \, dx
\]

\[
\leq \frac{1}{2} \int_0^1 \frac{\eta(x) u_{tt}^2}{v} \, dx + C \left( \|u_t\|_{L^\infty} \left( \|u_t\|_{L^2}^2 + \|u_{xt}\|_{L^2}^2 + \|\chi\|_{L^2}^2 \right) \right)
\]

\[
\leq \frac{1}{2} \int_0^1 \frac{\eta(x) u_{tt}^2}{v} \, dx + C \left( \|u_t\|_{L^\infty}^2 + \|\chi\|_{L^2}^2 \right)
\]

where we have used (2.53), (2.55) and Cauchy-Schwartz’s inequality. In view of (2.53), (2.55) and (2.72), one has

\[
\sup_{0 \leq t \leq T} \|u_t\|_{L^2}^2 + \int_0^T \|u_{xt}\|_{L^2}^2 \, dt \leq C. \quad (2.80)
\]

As a result, it follows from (1.4)₂ that

\[
\|u_{xx}\|_{L^2}^2 \leq C \left( \|u_t\|_{L^2}^2 + \|\theta x\|_{L^2}^2 + \|\chi_x\|_{L^2}^2 \left( \|\chi_x\|_{L^2}^2 + \|\chi_x\|_{L^2}^2 \right) \right)
\]

\[
+ C \|\chi_{xx}\|_{L^2}^2 \left( \|\chi_{xx}\|_{L^2}^2 + \|\chi_{xx}\|_{L^2}^2 \right) \|\chi_{xx}\|_{L^2}^2 \right) \leq C \|u_{xx}\|_{L^2}^2 + C \leq C \|u_{xx}\|_{L^2} \|u_{xx}\|_{L^2} + C
\]

\[
\leq \frac{1}{2} \|u_{xx}\|_{L^2}^2 + C.
\]

This, together with (2.80), finishes the proof of (2.79). \(\square\)

**Lemma 2.13** Let the conditions of Lemma 2.3 be in force. Then we have

\[
\sup_{0 \leq t \leq T} \|v_{xx}\|_{L^2}^2 + \int_0^T \|\left( v_{xx}, u_{xx} \right) \|_{L^2}^2 \, dt \leq C_7. \quad (2.81)
\]

**Proof.** Differentiating (1.4)₂ with respect to \( x \), we get

\[
u_{xt} - \eta(x) \left( \frac{v_x}{v} \right)_{xt} = - \left( \frac{v_{tx} - \theta v_x}{v^2} \right)_x + \eta_x \left( \frac{u_x}{v} \right) + \left( \frac{\eta u_x}{v} \right)_x - \frac{1}{2} \left( \frac{x_t^2}{v^2} \right)_{xx} . \quad (2.82)
\]

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Multiplying it by $\left(\frac{v_t}{v}\right)_x$ and integrating the result over $[0, 1]$ yield

$$
\frac{1}{2} \frac{d}{dt} \int_0^1 \eta(x) \left| \left(\frac{v_t}{v}\right)_x \right|^2 dx + \int_0^1 \frac{\theta}{v} \left| \left(\frac{v_t}{v}\right)_x \right|^2 dx \\
\leq C \|x\|_{L^\infty} \int_0^1 \left| \left(\frac{v_t}{v}\right)_x \right|^2 dx + \frac{1}{2} \int_0^1 \frac{\theta}{v} \left| \left(\frac{v_t}{v}\right)_x \right|^2 dx + C \|u_{x,x}\|^2_{L^2} + C \int_0^1 v_t^4 dx \\
+ C \int_0^1 \left( \theta_{x}^2 + \theta_{x}^2 v_{x}^2 + \chi_{x}^2 u_{x,x}^2 + \chi_{x}^2 v_{x}^2 + \chi_{x}^2 u_{x}^2 + \left( \frac{\chi_{x}}{v} \right) \left( \frac{\chi_{x}}{v} \right)_x^2 \right) dx \\
\leq \frac{1}{2} \int_0^1 \frac{\theta}{v} \left| \left(\frac{v_t}{v}\right)_x \right|^2 dx + C(\|x\|_{L^2} + \|v_{x,x}\|_{L^2} + \|v_{x,x}\|_{L^2}) \int_0^1 \eta(x) \left| \left(\frac{v_t}{v}\right)_x \right|^2 dx \\
+ C \left( \|u_{x,x}\|^2_{L^2} + \|v_{x,x}\|^2_{L^2} + \|\theta_{x}\|_{H^1}^2 + \|u_{x,x}\|_{H^1}^2 + \left( \frac{\chi_{x}}{v} \right)_x^2 + \left( \frac{\chi_{x}}{v} \right)_{xx}^2 \right) \\
\leq \frac{1}{2} \int_0^1 \frac{\theta}{v} \left| \left(\frac{v_t}{v}\right)_x \right|^2 dx + C(\|x\|_{H^1} + \|v_{x,x}\|_{L^2}) \int_0^1 \eta(x) \left| \left(\frac{v_t}{v}\right)_x \right|^2 dx \\
+ C \left( \|u_{x,x}\|^2_{L^2} + \|v_{x,x}\|^2_{L^2} + \|\theta_{x}\|_{H^1}^2 + \|u_{x,x}\|_{H^1}^2 + \|\chi_{x}\|_{H^1}^2 + \|\chi_{x}\|_{H^1}^2 \right),
$$

where we have used (2.45), (2.53) and the facts

$$
\int_0^1 v_t^4 dx \leq \|v_{x,x}\|^2_{L^\infty} \|v_{x,x}\|^2_{L^2} \leq C \left( \frac{v_t}{v} \right)_{L^\infty} \|v_{x,x}\|^2_{L^2} \\
\leq C \left( \left( \frac{v_t}{v} \right)^2_{L^2} + \left( \frac{v_t}{v} \right)^2_{H^1} \right) \|v_{x,x}\|^2_{L^2} \\
\leq C \left( 1 + \int_0^1 \eta(x) \left| \left(\frac{v_t}{v}\right)_x \right|^2 dx \right) \|v_{x,x}\|^2_{L^2},
$$

and

$$
\left( \frac{\chi_{x}}{v} \right)_{xx}^2 = \left( \frac{\chi_t}{v} + (\chi^3 - \chi) \right)_{xx}^2 \leq C \int_0^1 \left( \chi_{xx}^2 + \chi^2 v_{x}^2 + \chi_{x}^2 \right) dx \\
\leq C \left( \|\chi_{x}\|^2_{L^2} + \|\chi\|^2_{L^\infty} \|v_{x,x}\|^2_{L^2} + \|\chi_{x}\|^2_{L^2} \right) \\
\leq C \left( \|\chi_{x}\|^2_{H^1} + \|\chi_{x}\|^2_{L^2} \right).
$$

Using (2.42), (2.45), (2.53), (2.55), (2.72), (2.79) and Gronwall’s inequality, we get

$$
\sup_{0 \leq t \leq T} \left\| \left(\frac{v_t}{v}\right)_x \right\|^2_{L^2} + \int_0^T \left\| \left(\frac{v_t}{v}\right)_x \right\|^2_{L^2} dt \leq C.
$$
Noting that
\[
\|v_{xx}\|_{L^2}^2 \leq \left\| \left( \frac{v_x}{v} \right)_x \right\|_{L^2}^2 + \left\| \frac{v_x}{v} \right\|_{L^2}^2 \leq C \left\| \left( \frac{v_x}{v} \right)_x \right\|_{L^2}^2 + C\|v_x\|_{L^2}^2 \|v_{xx}\|_{L^2}^2
\]
we have
\[
\sup_{0 \leq t \leq T} \|v_{xx}\|_{L^2}^2 + \int_0^T \|v_{xx}\|_{L^2}^2 \, dt \leq C. \tag{2.83}
\]
In view of (2.45), (2.53), (2.55), (2.72) (2.79), we deduce from (2.82) that
\[
\int_0^T \|u_{xxx}\|_{L^2}^2 \, dt \leq C \int_0^T \left( \|u_x\|^2_{L^2} + \|\theta_{xx}\|^2_{L^2} + \|\theta_{v}\|_{L^2}^2 + \|v_{xx}\|^2_{L^2} + \|v_{xx}\|^2_{L^2} \right) \, dt
\]
\[
+ C \int_0^T \left( \|\chi_{x} u_{xx}\|_{L^2}^2 + \|v_x u_{xx}\|_{L^2}^2 + \|v_x u_{x}\|_{L^2}^2 + \|v_x u_{x}\|_{L^2}^2 \right) \, dt
\]
\[
+ C \int_0^T \left( \|u_{xx}\|^2_{L^2} + \left\| \left( \frac{X_s}{v} \right)_x \right\|_{L^2}^2 + \left\| \left( \frac{X_t}{v} \right)_x \right\|_{L^2}^2 \right) \, dt \leq C.
\]
It combined with (2.83) lead to (2.81). \qed

**Lemma 2.14** Let the conditions of Lemma 2.3 be in force. Then we have
\[
\sup_{0 \leq t \leq T} \left\| \left( X_{xxx}, X_{x}, \left( \frac{X_t}{v} \right)_t \right) \right\|_{L^2}^2 + \int_0^T \left\| \left( X_{xxx}, X_{x}, \left( \frac{X_t}{v} \right)_t \right) \right\|_{L^2}^2 \, dt \leq C_8. \tag{2.84}
\]

**Proof.** First, differentiating (2.54) with respect to \( t \), one has
\[
\left( \frac{X_t}{v} \right)_t = \left( \frac{X_t}{v} \right)_{tx} - (3\chi^2 - 1)X_t. \tag{2.85}
\]
Multiplying it by \( \left( \frac{X_t}{v} \right)_x \) and integrating the result over \([0, 1]\) yield
\[
\left\| \left( \frac{X_t}{v} \right)_t \right\|_{L^2}^2 = - \int_0^1 \left( \frac{X_t}{v} \right)_t \left( \frac{X_t}{v} \right)_{xx} \, dx - \int_0^1 (3\chi^2 - 1)X_t \left( \frac{X_t}{v} \right)_t \, dx
\]
\[
= - \int_0^1 \left( \frac{X_t}{v} \right)_t \left( \frac{X_t}{v} \right)_{xx} \, dx - \int_0^1 \left( \frac{X_t}{v} \right)_t \left( \frac{X_t v_x}{v^2} - \frac{X_x v_t}{v^2} \right)_t \, dx
\]
\[
- \int_0^1 (3\chi^2 - 1)X_t \left( \frac{X_t}{v} \right)_t \, dx, \tag{2.86}
\]
where we have used the facts
\[
\left( \frac{X_t}{v} \right)_x = \left( \frac{X_t}{v} \right)_t + \frac{X_t v_x}{v^2} - \frac{X_x v_t}{v^2}
\]
and
\[
\left( \frac{X_t}{v} \right)_t = \left( \frac{X_t}{v} \right)_x + \frac{X_t v_x}{v^2} - \frac{X_x v_t}{v^2}.
\]
Thus, it follows from (2.86) that

\[
\frac{1}{2} \frac{d}{dt} \left( \left\| \left( \frac{X_s}{v} \right) \right\|_{L^2}^2 + \left\| \left( \frac{X_t}{v} \right) \right\|_{L^2}^2 \right) \\
= - \int_0^t \left( \frac{X_s}{v} \right)_t \left( \frac{X_s v_t}{v^2} - \frac{X_t v_x}{v^2} \right) dx - \int_0^t (3\chi^2 - 1)\chi_t \left( \frac{X_t}{v} \right)_t dx \\
= - \int_0^t \left( \frac{X_s}{v} \right)_t \left( \frac{X_s v_t + X_t v_x}{v^2} \right) dx + 2\frac{X_s v_t}{v^3} - \left( \frac{X_t}{v} \right)_t v_x - \left( \frac{X_t}{v} \right) v_x + \left( \frac{X_t}{v} \right)_t v_x v_t \right) dx \\
- \int_0^t (3\chi^2 - 1)\chi_t \left( \frac{X_t}{v} \right)_t dx \\
\leq \frac{1}{4} \left\| \left( \frac{X_s}{v} \right) \right\|_{L^2}^2 + C \left\| v_x \left( \frac{X_s}{v} \right) \right\|_{L^2}^2 + C \left( \left\| u_x \right\|_{L^\infty}^2 + \left\| \chi_x \right\|_{L^\infty}^2 + \left\| \chi_t \right\|_{L^2}^2 \right) \left\| \left( \frac{X_s}{v} \right)_t \right\|_{L^2}^2 \\
+ C \left( \left\| \chi \right\|_{L^2}^2 + \left\| \chi_t \right\|_{L^2}^2 + \left\| u_x \right\|_{L^2}^2 + \left\| u_x v_x \right\|_{L^2}^2 + \left\| u_{xx} \right\|_{L^2}^2 \right) \\
\leq \frac{1}{4} \left\| \left( \frac{X_s}{v} \right) \right\|_{L^2}^2 + C \left\| \left( \frac{X_s}{v} \right)_t \right\|_{L^2}^2 + C \left( \left\| u_x \right\|_{H^1}^2 + \left\| \chi_{xx} \right\|_{L^2}^2 + \left\| \chi_t \right\|_{H^1}^2 \right) \left\| \left( \frac{X_s}{v} \right)_t \right\|_{L^2}^2 \\
+ C \left( \left\| \chi_{xx} \right\|_{L^2}^2 + \left\| \chi_t \right\|_{L^2}^2 + \left\| u_x \right\|_{L^2}^2 + \left\| u_x v_x \right\|_{L^2}^2 \right), \tag{2.87}
\]

where we have used (2.45) and Cauchy-Schwartz’s inequality. The second term on the right-hand side can be estimated as follows

\[
\left\| \left( \frac{X_s}{v} \right)_t \right\|_{L^2}^2 \leq C \left\| \left( \frac{X_s}{v} \right) \right\|_{L^2} \left\| \left( \frac{X_s}{v} \right)_t \right\|_{L^2} \\
= C \left\| \left( \frac{X_s}{v} \right) \right\|_{L^2} \left\| \left( \frac{X_s}{v} \right) + (\chi^3 - \chi) \right\|_{L^2} \\
\leq \frac{1}{4} \left\| \left( \frac{X_s}{v} \right) \right\|_{L^2}^2 + C \left\| \left( \frac{X_s}{v} \right) \right\|_{L^2}^2 + C \left\| \chi_t \right\|_{L^2}^2.
\]

Putting it into (2.87), using (2.42), (2.45), (2.53), (2.55), (2.79) and Gronwall’s inequality yield

\[
\sup_{0 \leq t \leq T} \left\| \left( \frac{X_s}{v} \right)_t \right\|_{L^2}^2 + \int_0^T \left\| \left( \frac{X_s}{v} \right)_t \right\|_{L^2}^2 dt \leq C, \tag{2.88}
\]

which implies

\[
\left\| \chi_{xx} \right\|_{L^2}^2 \leq \left\| \left( \frac{X_s}{v} \right)_t \right\|_{L^2}^2 \leq C \left\| \left( \frac{X_s}{v} \right) \right\|_{L^2}^2 + C \left\| \chi_{xx} \right\|_{L^\infty}^2 \left\| u_x \right\|_{L^2}^2 \leq C.
\]

Moreover, in view of (2.81), we obtain

\[
\left\| \chi_{xxx} \right\|_{L^2}^2 = \left\| \chi_{xx} + \frac{X_{xx} v_x}{v} + \frac{X_{xx} v_x}{v^2} - \frac{X_{xx} v_x}{v^2} + v_x (\chi^3 - \chi) + v (3\chi^2 - 1) \chi_x \right\|_{L^2}^2 \\
\leq C \left( \left\| \chi_{xx} \right\|_{L^2}^2 + \left\| v_x \right\|_{L^\infty}^2 + \left\| v_x \right\|_{L^2}^2 + \left\| v_x \right\|_{L^2}^2 + \left\| \chi_x \right\|_{L^2}^2 \right) \\
\leq C \left( \left\| \chi_{xx} \right\|_{L^2}^2 + \left\| v_x \right\|_{H^1}^2 + \left\| \chi_x \right\|_{L^2}^2 \right) \leq C.
\]

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From which we have
\[ \int_0^T \| \chi_{xx} \|^2_{L^2} \, dt \leq C \int_0^T \left( \| \chi_{x} \|^2_{L^2} + \| \nu_x \|^2_{H^1} + \| \chi_{x} \|^2_{L^2} \right) \, dt \leq C. \]

Finally, it follows from (2.85) that
\[ \int_0^T \left\| \left( \frac{X}{v} \right)_{xx} \right\|^2_{L^2} \, dt \leq C \int_0^T \left( \left\| \left( \frac{X}{v} \right)_{x} \right\|^2_{L^2} + \| \chi \|^2_{L^2} \right) \, dt \leq C, \tag{2.89} \]

which gives
\[ \int_0^T \| \chi_{xx} \|^2_{L^2} \, dt = \int_0^T \left\| \left( \frac{X}{v} \right)_{x} \right\|^2_{L^2} \, dt \leq C \int_0^T \left( \left\| \left( \frac{X}{v} \right)_{xx} \right\|^2_{L^2} + \| \chi_{x} \|^2_{L^2} + \| \chi \|^2_{L^2} \right) \, dt \leq C + C \int_0^T \| \chi_{xx} \|_{L^2} \| \chi_{x} \|_{L^2} \, dt \leq C + \frac{1}{2} \int_0^T \| \chi_{xx} \|^2_{L^2} \, dt. \]

This, together with (2.88)–(2.89), leads to (2.84). \qed

**Lemma 2.15** Let the conditions of Lemma 2.3 be in force. Then it holds that
\[ \sup_{0 \leq t \leq T} \| (\theta, \theta_t) \|^2_{L^2} + \int_0^T \| (\theta_{xxx}, \theta_{xt}) \|^2_{L^2} \, dt \leq C_9. \tag{2.90} \]

**Proof.** First, differentiating (1.4) with respect to \( t \), by (1.4), we get
\[ \theta_t - \left( \frac{\theta^2 \theta_{xt}}{v} + \frac{\beta \theta^2 - \theta t \theta x}{v^2} - \frac{\theta^2 \theta x u x}{v^2} \right) = -\left( \frac{\theta u x}{v} \right)_t + \left( \frac{v \theta}{v} \right)_x + \left( \frac{v^2 \theta}{v} \right)_x. \]

Multiplying it by \( \theta_t \) and integrating the result over \([0, 1]\) yield
\[
\frac{1}{2} \int_0^T \| \theta_t \|^2_{L^2} \, dt + \int_0^1 \frac{\theta^2 \theta_{xt}^2}{v} \, dx \\
\leq \frac{1}{2} \int_0^1 \theta^2 \theta_{xt}^2 \, dx + C \int_0^1 \left( \theta_t^2 t^2_{x} + \theta_{xx}^2 u_{xx}^2 \right) \, dx + C \int_0^1 \left( \theta^2 |u_x| + |u_x t| + |u_{xx} t| \right) \, dx \\
+ C \int_0^1 \left( |u_x| u_{xx}^2 + |u_x u_{xx}| + |u_{xx}| t^2 + |u_{xx} t| \right) \theta_t \, dx \\
\leq \frac{1}{2} \int_0^1 \theta^2 \theta_{xt}^2 \, dx + C \left( \| \theta_x \|^2_{L^2} \| \theta_t \|^2_{L^2} + \| u_x \|^2_{L^2} \| \theta_t \|^2_{L^2} + \| u_x \|^2_{L^2} \| \theta_t \|^2_{L^2} \right) \\
+ C \left( \| u_x \|^2_{L^2} + \| u_{xx} \|^2_{L^2} + \| u_{xx} u_x \|^2_{L^2} + \| u_{xx} \|^2_{L^2} + \| u_{xx} \|^2_{L^2} + \| u_{xx} \|^2_{L^2} \right) \\
+ C \left( 1 + \| u_x \|^2_{L^2} + \| u_{xx} \|^2_{L^2} \right) \| \theta_t \|^2_{L^2} \\
\leq \frac{1}{2} \int_0^1 \theta^2 \theta_{xt}^2 \, dx + C \left( 1 + \| \theta_x \|^2_{L^2} + \| u_x \|^2_{L^2} + \| u_{xx} \|^2_{L^2} \right) \| \theta_t \|^2_{L^2}. \]

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Thus, there holds
\[
+ C \left( \|u_x\|_{H^1}^2 + \|u_{xx}\|_{L^2}^2 + \|\chi\|_{L^2}^2 + \|\mu\|_{L^2}^2 + \|\mu\|_{L^2}^2 \right)
\]
\[
\leq \frac{1}{2} \int_0^1 \frac{\theta^2 \theta_x^2}{v} dx + C \left( \|\theta_x\|_{H^1}^2 + \|u_x\|_{H^1}^2 + \|\chi\|_{H^1}^2 \right) \|\theta\|_{L^2}^2
\]
\[
+ C \left( \|\theta\|_{L^2}^2 + \|u_x\|_{H^1}^2 + \|u_{xx}\|_{L^2}^2 + \|\chi\|_{H^1}^2 + \|\left( \frac{\chi}{v} \right)\|_{L^2}^2 \right),
\]
where we have used (2.53), (2.55) and (2.72). Combining the above inequality with (2.42), (2.53), (2.55), (2.72), (2.79), (2.84) and Gronwall’s inequality, we have
\[
\sup_{0 \leq t \leq T} \|\theta\|_{L^2}^2 + \int_0^T \|\theta_x\|_{L^2}^2 dt \leq C. \tag{2.91}
\]
Next, it follows from (1.4) that
\[
\|\theta_x\|_{L^2}^2 \leq C \int_0^1 \left( \theta_x^4 + \theta_x^2 v^2 + \theta_x^2 + u_x^2 + u_x^4 + \mu^4 \right) dx
\]
\[
\leq C \left( \|\theta_x\|_{L^2}^2 + \|u_x\|_{L^2}^2 + \|\chi\|_{L^2}^2 \right) + C \tag{2.92}
\]
\[
\leq C \|\theta_x\|_{L^2}^2 + \|\theta_x\|_{L^2}^2 + C \leq \frac{1}{2} \|\theta_x\|_{L^2}^2 + C.
\]
Finally, differentiating (1.4) with respect to \(x\), by (1.4), we obtain
\[
\frac{\theta^2 \theta_{xxx}}{v} = -\theta_{xx} \left( \frac{\theta^2}{v} \right) + \left( -\beta \theta^{-1} \frac{\theta_x^2}{v} + \theta \frac{\theta_x v_x}{v^2} + \theta_x + \frac{\theta}{v} u_x - \frac{\eta(\chi) u_x^2}{v} - v \mu^2 \right). \]
Thus, there holds
\[
\int_0^T \|\theta_{xxx}\|_{L^2}^2 dt \leq C \int_0^T \int_0^1 \left( \theta_x^2 \theta_{xx}^2 + v^2 \theta_x^2 + \theta_x^4 + \theta_v^2 v_x^2 + \theta_x^2 v_{xx}^2 + \theta_x^4 v_x^2 \right) dx dt
\]
\[
+ C \int_0^T \int_0^1 \left( \theta_x^2 + \theta_x^2 u_x^2 + u_x^2 + u_x^2 v_x^2 + \chi x^4 u_x^2 + u_x^2 u_{xx}^2 \right) dx dt
\]
\[
+ C \int_0^T \int_0^1 \left( u_x^4 v_x^2 + v_x^2 \mu^4 + \mu^4 \mu_x^2 \right) dx dt
\]
\[
\leq C.
\]
From which and (2.91), (2.92), we obtain (2.90). \qed

3 Proof of Theorem 1.1

First, the local existence of unique strong solutions can be proved by Banach Fixed Point theorem on (see [33]).
Lemma 3.1 Suppose that (3.9) holds. Then there exists \( T_0 = T_0(\nu_0, \nu_0, \nu_0, M_0) \), depending only on \( \beta, \nu_0 \) and \( M_0 \), such that the initial boundary value problem (3.4)–(3.6) has a unique solution

\[
(v, u, \chi, \theta) \in X(0, T_0; \frac{1}{2}V_0, \nu_0, \frac{1}{2}V_0, KM_0),
\]

where \( K \geq 2 \) is a positive constant.

In what follows, by the a priori estimates established in Section 2 and local well-posedness result, we derive the existence and uniqueness of global strong solutions.

**Proof of Theorem 1.1** It follows from Lemma 3.1 that the initial boundary value problem (3.4)–(3.6) has a unique solution

\[
(v, u, \chi, \theta) \in X(0, t; \frac{1}{2}V_0, \nu_0, \frac{1}{2}V_0, KM_0),
\]

where \( t_1 = T_0(\nu_0, \nu_0, \nu_0, M_0) > 0 \).

Moreover, if we take \( \alpha \leq \alpha_1 \) with \( \alpha_1 \) being small enough such that

\[
(\nu_0)^{-\alpha_1} \leq 2, \quad (2KM_0)^{\alpha_1} \leq 1, \quad \alpha_1 H(\frac{1}{2}V_0, \nu_0, \frac{1}{2}V_0, KM_0) \leq \varepsilon_1,
\]

where \( \varepsilon_1 > 0 \) is chosen in Lemma 2.3 Then we deduce from Lemma 2.1,2.14 that the solution \((v, u, \chi, \theta)\) satisfies

\[
C_0 \leq v \leq C_0^{-1}, \quad \nu_0 \leq \chi \leq 1, \quad C_1 \leq \theta \leq C_1^{-1}, \quad (x, t) \in [0, 1] \times [0, t_1],
\]

and

\[
\sup_{0 \leq t \leq t_1} \|(v, u, \chi, \theta)(t)\|_{L^2}^2 + \int_0^{t_1} \|\chi(t)\|_{L^2}^2 dt \leq C_{10}^2 := \sum_{i=2}^{9} C_i,
\]

where \( C_2, \ldots, C_9 \) are the same ones as in Section 2

Next, taking \((v, u, \chi, \theta)(\cdot, t_1)\) as the initial data and applying Lemma 3.1 again, we can extend the local solution \((v, u, \chi, \theta)\) to the time interval \([t_1, t_1 + t_2]\) with \( t_2 = T_0(C_0, \nu_0, C_1, C_{10}) \). Moreover, we have

\[
v \geq \frac{1}{2}C_0, \quad \nu \geq \nu_0, \quad \theta \geq \frac{1}{2}C_0, \quad (x, t) \in [0, 1] \times [t_1, t_1 + t_2],
\]

and

\[
\sup_{t_1 \leq t \leq t_1 + t_2} \|(v, u, \chi, \theta)(t)\|_{L^2}^2 + \int_{t_1}^{t_1 + t_2} \|\chi(t)\|_{L^2}^2 dt \leq (KC_{10})^2.
\]

Hence, collecting (3.2)–(3.5), we get

\[
v \geq \frac{1}{2}C_0, \quad \nu \geq \nu_0, \quad \theta \geq \frac{1}{2}C_0, \quad (x, t) \in [0, 1] \times [0, t_1 + t_2].
\]
and
\[
\sup_{0 \leq t \leq t_1 + t_2} \| (v, u, \chi, \theta)(t) \|_{H^2}^2 + \int_0^{N_1 + t_2} \| \chi_t \|_{L^2}^2 dt \leq (K^2 + 1) C_{10}^2.
\]

Take \( \alpha \leq \min\{\alpha_1, \alpha_2\} \), where \( \alpha_1 \) is the same as one as in (3.1) and \( \alpha_2 \) is chosen to be such that
\[
(V_0)^{\alpha_2} \leq 2, \quad (2 \sqrt{1 + K^2 C_{10}})^{\alpha_2} \leq 1, \quad \alpha_2 H \left( \frac{1}{2} C_0, V_0, \frac{1}{2} C_1, \sqrt{1 + K^2 C_{10}} \right) \leq \varepsilon_1,
\]
where \( \varepsilon_1 > 0 \) is chosen in Lemma 2.3. Then we infer from Lemma 2.1–2.14 again that the solution \((v, u, \chi, \theta)\) satisfies (3.2) and (3.3) on \([0, t_1 + t_2] \).

Hence, choosing \( \varepsilon_0 = \min\{\alpha_1, \alpha_2\} \) and repeating the above procedure, we see that the initial boundary value problem (1.4)–(1.6) has a unique solution \((v, u, \chi, \theta) \in X(0, +\infty; C_0, V_0, C_1, C_{10})\).

This completes the proof of the global existence of strong solutions. The uniqueness of the solutions can be easily obtained by the standard energy method. \( \square \)

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**A Appendix**

In the proof of local existence, it is easy to show that \( v \in L^\infty(Q_T), A^{-1} \leq v \leq A \) (for some constant \( A > 1 \) and \( Q_T = (0, t) \times (0, T) \)), which leads to the following lemma.

**Lemma A.1** Assume \( v \in L^\infty(Q_T) \), \( A^{-1} \leq v \leq A \), \( \chi \) is a smooth solution to (1.4) with \( V_0 \leq \chi_0 \leq 1 \) and \( \chi_t \big|_{t=0} = 0 \). Then
\[
V_0 \leq \chi(x, t) \leq 1, \quad (x, t) \in [0, 1] \times [0, T_0].
\]

**Proof.** We rewrite the equations (1.4) as
\[
\chi_t = \chi_{xx} - \frac{v_x}{v} \chi_x - v \chi^2 - 1.
\]
Multiplying the above equation by $2\chi$ gives
\[(\chi^2 - 1)_t - (\chi^2 - 1)_{xx} + \frac{\nu}{\chi^2 - 1}_x + 2v(\chi^2 - 1) = -2v(\chi^2 - 1)^2 - 2\chi^2 \leq 0.\]

Then the maximum principle implies $\chi^2 - 1 \leq 0$. Thus, we have
\[\chi(x, t) \leq 1.\]

Define $Y(x, t) = e^{At}\chi(x, t)$. Then $Y$ satisfies
\[Y_t - Y_{xx} + \frac{\nu}{\chi}Y_x + [A + v(\chi^2 - 1)]Y = 0.\]

By the maximum principle and the definition of $Y$, we arrive at
\[\chi(x, t) \geq 0.\]

Then consider the equation
\[(\chi - V_0)_t - (\chi - V_0)_{xx} + \frac{\nu}{\chi}(\chi - V_0)_x = -v(\chi^2 - 1) \geq 0.\]

The maximum principle implies that
\[\chi(x, t) \geq V_0.\]

This completes the proof of Lemma A.1.

\[\square\]

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