EQUIVARIANT FUNCTIONS AND VECTOR-VALUED MODULAR FORMS

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Abstract. For any discrete group $\Gamma$ and any 2-dimensional complex representation $\rho$ of $\Gamma$, we introduce the notion of $\rho$-equivariant functions, and we show that they are parameterized by vector-valued modular forms. We also provide examples arising from the monodromy of differential equations.

1. Introduction

Throughout this paper, by a discrete group, we mean a finitely generated Fuchsian group of the first kind, acting on the upper half-plane $\mathbb{H} = \{ z \in \mathbb{C} : \text{Im} z > 0 \}$. Let $\Gamma$ be such a group. Let $\rho : \Gamma \rightarrow \text{GL}_2(\mathbb{C})$ be a 2-dimensional complex representation of $\Gamma$. A meromorphic function $h$ on $\mathbb{H}$ is called a $\rho$-equivariant function with respect to $\Gamma$ if

$$h(\gamma \cdot z) = \rho(\gamma) \cdot h(z) \quad \text{for all } z \in \mathbb{H}, \gamma \in \Gamma,$$

where the action on both sides is by linear fractional transformations. The set of $\rho$-equivariant functions for $\Gamma$ will be denoted by $E_\rho(\Gamma)$.

In the case $\rho$ is the defining representation of $\Gamma$, that is $\rho(\gamma) = \gamma$ for all $\gamma \in \Gamma$, then elements of $E_\rho(\Gamma)$ are simply called equivariant functions. These were studied extensively in [1, 2, 3] and have various connections to modular forms, quasi-modular forms, elliptic functions and to sections of the canonical line bundle of $X(\Gamma) = \Gamma \backslash \mathbb{H}$. In particular, one shows that the set of equivariant functions for a discrete group $\Gamma$ without the trivial one $h_0(z) = z$ has a vector space structure isomorphic to the space of weight 2 automorphic forms for $\Gamma$.

In this paper, we will treat the general case where $\rho$ is an arbitrary 2-dimensional complex representation of $\Gamma$. The main result of this paper states that every $\rho$-equivariant function is parameterized by a 2-dimensional vector-valued modular form for $\rho$. More precisely, if
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$F = (f_1, f_2)^t$ is a vector-valued modular form for $\Gamma$ and $\rho$ (see Section 2), then $h_F = f_1/f_2$ is a $\rho$-equivariant function for $\Gamma$. We will show that, in fact, every $\rho$-equivariant function arises in this way. To achieve this parametrization, we use the fact that the Schwarz derivative of a $\rho$-equivariant function is a weight 4 automorphic form for $\Gamma$, in addition to the knowledge of the existence of global solutions to a certain second degree differential equation.

Finally, in the last section, we construct examples of $\rho$-equivariant functions when $\rho$ is the monodromy representation of second degree ordinary differential equations.

2. Vector-valued modular forms

The theory of vector-valued modular forms was introduced long ago as a higher dimensional generalization of the classical (scalar) modular forms. Let $\Gamma$ be a discrete group and let $\rho : \Gamma \rightarrow \text{GL}_n(\mathbb{C})$ be an $n$-dimensional complex representation of $\Gamma$. A vector-valued modular form of integral weight $k$ for $\Gamma$ and representation $\rho$ is an $n$-tuple $F(z) = (f_1(z), \ldots, f_n(z))^t$ of meromorphic functions on the complex upper half-plane $\mathbb{H}$ satisfying

\begin{equation}
F(z)|_{k\gamma} = \rho(\gamma) F(z) \quad \text{for all } \gamma \in \Gamma, \ z \in \mathbb{H},
\end{equation}

and some growth conditions at the cusps that are similar to those for classical automorphic forms. The slash operator in (2.1) is defined as usual by $F|_{k\gamma}(z) = (cz + d)^{-k}F(\gamma \cdot z)$ where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The set of these vector-valued modular forms will be denoted by $V_\rho(\Gamma)_k$, and we will refer to them as $\rho$-VMF of weight $k$.

We omit the multiplier system since it will not have any effect in this paper. If $F(z)$ is holomorphic, then it will be called a holomorphic vector-valued modular form.

The theory of vector-valued modular forms is fairly well understood when $\Gamma$ is the modular group. See for instance [4] and the references therein. However, except for the genus 0 subgroups of the modular group, it is not even clear that nonzero vector-valued modular forms exist. In connection with equivariant functions, we have the following straightforward proposition.

**Proposition 2.1.** Let $\Gamma$ be a discrete group, and $\rho$ an arbitrary 2-dimensional complex representation of $\Gamma$. If $F(z) = (f_1(z), f_2(z))^t$ is a vector-valued modular form for $\rho$ of a certain weight, then $f_1(z)/f_2(z)$ is a $\rho$-equivariant function for $\Gamma$. 
We will prove below that every $\rho$-equivariant function arises in this way.

3. Differential equations

Let $D$ be a domain in $\mathbb{C}$ and let $f$ be a meromorphic function on $D$. Its Schwarz derivative, $S(f)$, is defined by

$$S(f) = \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2.$$

This is an important tool in projective geometry and differential equations. The main properties that will be useful to us are summarized as follows (see [5] for more details):

**Proposition 3.1.** We have

1. If $y_1$ and $y_2$ are two linearly independent solutions to a differential equation $y'' + Qy = 0$ where $Q$ is a meromorphic function on $D$, then $S(y_1/y_2) = 2Q$.
2. If $f$ and $g$ are two meromorphic functions on $D$, then $S(f) = S(g)$ if and only if $f = ag + b$ for some $(a b c d) \in GL_2(\mathbb{C})$.
3. $S(f \circ \gamma)(z) = (cz+d)^4 S(f)$ provided $\gamma \cdot z \in D$, where $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$.

In particular, we have

**Proposition 3.2.** If $f$ is a $\rho$-equivariant for a discrete group $\Gamma$, then $S(f)$ is an automorphic form of weight 4 for $\Gamma$.

Now, consider the second order ordinary differential equation (ODE)

$$x'' + P x' + Q x = 0,$$

where $P$ and $Q$ are holomorphic functions in $D$. This ODE has two linearly independent holomorphic solutions in $D$ if $D$ is simply connected. For a fixed $z_0 \in D$, set

$$y(z) = x(z) \exp \left( \frac{1}{2} \int_{z_0}^{z} P(w) dw \right).$$

The above ODE reduces to an ODE in normal form

$$y'' + gy = 0,$$

with

$$g = Q - \frac{1}{2} P' - \frac{1}{4} P^2.$$
When the domain $D$ is not simply connected, we may not expect to find global solutions to (3.1) on $D$. However, under some conditions on $g$, global solutions do exist as it is illustrated in the following theorem which will be crucial for the rest of this paper.

**Theorem 3.3.** Let $D$ be a domain in $\mathbb{C}$. Suppose $h$ is a nonconstant meromorphic function on $D$ such that $S(h)$ is holomorphic in $D$, and let $g = \frac{1}{2}S(h)$. Then the differential equation

$$y'' + gy = 0$$

has two linearly independent holomorphic solutions in $D$.

**Proof.** Let $\{U_i, i \in I\}$ be a covering of $D$ by open discs with $\dim V(U_i) = 2$ for all $i \in I$ where $V(U_i)$ denotes the space of holomorphic solutions to $y'' + gy = 0$ on $U_i$. Choose $L_i$ and $K_i$ to form a basis for $V(U_i)$. Using property (1) of Proposition 3.1 we have $S(K_i/L_i) = 2g = S(h)$ on $U_i$. Now, using property (2) of Proposition 3.1 we have $K_i/L_i = \alpha_i \cdot h$ for $\alpha_i \in \text{GL}_2(\mathbb{C})$. In the meantime, on each connected component $W$ of $U_i \cap U_j$, we have

$$(K_i, L_i)^t = \alpha_W(K_j, L_j)^t, \quad \alpha_W \in \text{GL}_2(\mathbb{C}),$$

since each of $(K_i, L_i)$ and $(K_j, L_j)$ is a basis of $V(W)$. Hence, on $W$ we have

$$\frac{K_i}{L_i} = \alpha_W \cdot \frac{K_j}{L_j},$$

and therefore

$$\alpha_i h = \alpha_W \alpha_j h.$$  

It follows that

$$\alpha_i \alpha_j^{-1} = \alpha_W$$

as $h$ is meromorphic and nonconstant and thus it takes more than three distinct values on the domain $D$. Therefore, $\alpha_W$ does not depend on $W$. Moreover, on $U_i \cap U_j$ we have

$$\alpha_i^{-1}(K_i, L_i)^t = \alpha_j^{-1}(K_j, L_j)^t.  \quad (3.2)$$

If we define $f_1$ and $f_2$ on $U_i$ by

$$(f_1, f_2)^t = \alpha_i^{-1}(K_i, L_i)^t,$$

then using (3.2), we see that $f_1$ and $f_2$ are well defined all over $D$ and they are two linearly independent solutions to $y'' + gy = 0$ on all of $D$ as they are linearly independent over $U_i$. \qed
4. The Correspondence

In this section, we will prove that every $\rho-$equivariant function arises from a vector-valued modular form as in Proposition 2.1. We start with the following property of the slash operator known as Bol’s identity.

**Proposition 4.1.** Let $r$ be a nonnegative integer, $F(z)$ a complex function and $\gamma \in \text{SL}_2(\mathbb{C})$, then

$$(F|_{-r}\gamma)^{(r+1)}(z) = F^{(r+1)}|_{r+2}\gamma(z).$$

As a consequence, we have

**Corollary 4.2.** Let $r$ be a nonnegative integer, $g$ a $\Gamma-$automorphic form of weight $2(r + 1)$ and $D$ a domain in $\mathbb{H}$ that is stable under the action of $\Gamma$. Denote by $V_r(D)$ the solution space on $D$ to the differential equation

$$f^{(r+1)} + gf = 0.$$

Then for all $\gamma \in \Gamma$,

$$f \in V_r(D) \text{ if and only if } f|_{-r}\gamma \in V_r(D).$$

**Corollary 4.3.** The operator $|_{-r}$ provides a representation $\rho_r$ of $\Gamma$ in $\text{GL}(V_r)$. Moreover, if $f_1, f_2, \ldots, f_{r+1}$ form a basis of $V_r$ (if the basis exists), then

$$F = (f_1, f_2, \ldots, f_{r+1})^t$$

behaves as a $\rho_r-$VMF of weight $-r$ for $\Gamma$.

We state the main result of this paper. Recall from Proposition 2.1 that if $F = (f_1, f_2)^t$ is a $\rho-$VMF, then $h_F = f_1/f_2$ is a $\rho-$equivariant function.

**Theorem 4.4.** The map

$$V_\rho(\Gamma)|_{-1} \rightarrow E_\rho(\Gamma)$$

$$F \mapsto h_F$$

is surjective.

**Proof.** Suppose that $h$ is a $\rho-$equivariant function for $\Gamma$. According Proposition 3.2 its Schwarz derivative $S(h)$ is an automorphic form of weight 4 for $\Gamma$. Let $g = \frac{1}{2} S(h)$ and $D$ the complement in $\mathbb{H}$ of the set of poles of $g$. Then $D$ is a domain that is stable under $\Gamma$ since $g$ is an automorphic form for $\Gamma$. 
Using the same notation as in the previous section, we have, for \( r = 1 \), \( S(f_1/f_2) = S(h) \) where \( \{f_1, f_2\} \) are two linearly independent solutions in \( V(D) \) provided by Theorem 3.3. Hence, by Proposition 3.1

\[
\frac{f_1}{f_2} = \alpha \cdot h, \quad \alpha \in \text{GL}_2(\mathbb{C}).
\]

Also, using Corollary 4.2 with \( r = 1 \), we deduce that \( F_1 = (f_1, f_2)^t \) is a \( \rho_1 \)-VMF of weight \(-1\) for \( \Gamma \). Therefore,

\[
\alpha^{-1}\rho_1\alpha = \rho.
\]

Hence \( F = \alpha^{-1}F_1 \) is a \( \rho \)-VMF of weight \(-1\) for \( \Gamma \) with \( h_F = h \) on \( D \). Since \( g \) has only double poles, then by looking at the form of the solutions near a singular point, and using the fact that \( f_1 \) and \( f_2 \) are holomorphic and thus single-valued, we see that \( f_1 \) and \( f_2 \) can be extended to meromorphic functions on \( \mathbb{H} \). \( \square \)

**Remark 4.5.** If \( f \) is an automorphic form of weight \( k + 1 \) for \( \Gamma \), then \( (f_1, f_2) \rightarrow (ff_1, ff_2) \) yields an isomorphism between \( V_{\rho}(\Gamma)_{-1} \) and \( V_{\rho}(\Gamma)_k \), and therefore, the above surjection in the theorem extends to \( V_{\rho}(\Gamma)_k \) whenever it is nontrivial.

### 5. Examples

In this section, we shall construct examples of \( \rho \)-VMF’s and of \( \rho \)-equivariant functions when \( \rho \) is the monodromy representation of a differential equation.

Let \( U \) be a domain in \( \mathbb{C} \) such that \( \mathbb{C} \setminus U \) contains at least two points. The universal covering of \( U \) is then \( \mathbb{H} \) as it cannot be \( \mathbb{P}_1(\mathbb{C}) \) because \( U \) is noncompact and it cannot be \( \mathbb{C} \) because of Picard’s theorem.

Let \( \pi : \mathbb{H} \rightarrow U \) be the covering map. We consider the differential equation on \( U \)

\[
y'' + Py' + Qy = 0
\]

where \( P \) and \( Q \) are two holomorphic functions on \( U \). This differential equation has a lift to \( \mathbb{H} \)

\[
y'' + \pi^*Py' + \pi^*Qy = 0.
\]

Let \( V \) be the solution space to (5.2) which is a 2-dimensional vector space since \( \mathbb{H} \) is simply connected. Let \( \gamma \) be a covering transformation in \( \text{Deck}(\mathbb{H}/U) \) which is isomorphic to the fundamental group \( \pi_1(U) \) and let \( f \in V \). Then \( \gamma^*f = f \circ \gamma^{-1} \) is also a solution in \( V \). This defines the monodromy representation of \( \pi_1(U) \):

\[
\rho : \pi_1(U) \rightarrow \text{GL}(V).
\]
If $f_1$ and $f_2$ are two linearly independent solutions in $V$, we set $F = (f_1, f_2)$. Then we have

$$F \circ \gamma = \rho(\gamma)F.$$ 

Therefore, the quotient $f_1/f_2$ is a $\rho$–equivariant function on $\mathbb{H}$ for the group $\pi_1(U)$ which is a torsion-free discrete group.

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