Abstract

In federated learning, participating clients typically possess non-i.i.d. data, posing a significant challenge to generalization to unseen distributions. To address this, we propose a Wasserstein distributionally robust optimization scheme called WAFL. Leveraging its duality, we frame WAFL as an empirical surrogate risk minimization problem, and solve it using a local SGD-based algorithm with convergence guarantees. We show that the robustness of WAFL is more general than related approaches, and the generalization bound is robust to all adversarial distributions inside the Wasserstein ball (ambiguity set). Since the center location and radius of the Wasserstein ball can be suitably modified, WAFL shows its applicability not only in robustness but also in domain adaptation. Through empirical evaluation, we demonstrate that WAFL generalizes better than the vanilla FedAvg in non-i.i.d. settings, and is more robust than other related methods in distribution shift settings. Further, using benchmark datasets we show that WAFL is capable of generalizing to unseen target domains.

1 Introduction

Federated learning (FL) \cite{maciel2018federated,carey2020federated} has emerged as a cutting-edge technique in distributed and privacy-preserving machine learning. The nature of non-i.i.d. data in clients’ devices poses an important challenge to FL commonly called statistical heterogeneity. The global model trained on this data using the de facto FedAvg algorithm \cite{maciel2018federated} has been shown to generalize poorly to individual clients’ data, and further to unseen distributions on new clients as they enter the network.

Several solutions to data heterogeneity have been proposed. Personalized FL \cite{chefer2017personalized,nguyen2018personalized} and multi-task FL \cite{yang2019federated,chen2019federated} are client-adaptive approaches, where a personalized model is adapted to each client from the
global model. From another perspective, distributionally robust FL trains a model using a worst-case objective over an ambiguity set. This approach is client-uniform because a single global model is judiciously learned to deliver uniformly good performance not only for all training clients but also for new/unseen clients with unknown data distributions. It is specifically useful when test distributions drift away from the training distributions.

A natural question when designing distributionally robust FL frameworks is generalization: How can minimizing the training error also bound the test error? In FL, Mohri et al. proposed agnostic FL where a model is designed to be robust against any distribution that lies inside the convex hull of the clients’ distributions. Reisizadeh et al. applied the general affine covariate shift – used in the standard adversarial robust training – into FL training. In characterizing the generalization bounds, while Mohri et al. relied on the standard Rademacher complexity, Reisizadeh et al. use the margin-based technique developed by Bartlett et al.

In this work, we take a different approach called WAsserstein distributionally robust FL (WAFL for short). The ambiguity set in WAFL is a Wasserstein ball of all adversarial distributions in close proximity to the nominal data distribution at the center. Our main contributions are:

- We propose WAFL, a Wasserstein distributionally robust optimization problem for FL. To make WAFL amenable to distributed optimization, we transform the original problem into a minimization of the empirical surrogate risk and solve it using a local SGD-based algorithm with convergence guarantees.
- We demonstrate WAFL’s flexibility in robustness and domain adaptation by adjusting its hyper-parameters related to the Wasserstein ball’s center and radius. We show how WAFL’s output can reduce the test error by bounding its excess risk, and call this the robust generalization bound as it is applicable to all adversarial distributions inside the Wasserstein ball.
- Experimentally, we show WAFL’s significant improvement over the de facto FedAvg and other robust FL methods both in scenarios with adversarial attacks and in applications of multi-source domain adaptation.

2 Related Work

Federated learning was introduced in response to three challenges of machine learning at scale: massive data quantities at the edge, communication-critical networks of participating devices, and privacy-preserving learning without central data storage. The de facto FedAvg algorithm based on local stochastic gradient descent (SGD) and averaging is often considered a baseline in FL.

Most challenges of FL are categorized into systems heterogeneity and statistical heterogeneity. The former focuses on communication problems such as connection loss and bandwidth minimization. This motivated some prior works to design more communication-efficient methods. On the other hand, statistical heterogeneity is concerned with clients’ non-i.i.d. data, which is the main cause behind aggregating very different models leading to one that does not perform well on any data distribution. To address this, many ideas have been introduced. Li et al. provided much theoretical analysis of FL non-i.i.d. settings. Zhao et al. proposed an FL framework which globally shares a small subset of data among clients to train the model with non-i.i.d. data. Furthermore, some studied on multi-task FL frameworks in which each client individually learns its own data pattern while borrowing information from other clients, while several personalized FL models have also been developed in response to distribution shifts.

Wasserstein distributionally robust optimization (WDRO) aims to learn a robust model against adversarially manipulated data. An unknown data distribution is assumed to lie within a Wasserstein ball centered around the empirical distribution. WDRO has received attention as a promising tool for training parametric models, both in centralized and federated learning settings.

In centralized learning, many studies have proposed solutions based on WDRO problems for certain machine learning tasks. For instance, Shafieezadeh Abadeh et al. considered a robust logistic regression model under the assumption that the probability distributions lie in a Wasserstein ball. Chen and Paschalidis, Blanchet et al., Gao et al. leveraged WDRO to recover regularization formulations in classification and regression. Gao and Kleywegt proposed a minimizer based on a tractable approximation of the local worst-case risk. Esfahani and Kuhn used WDRO to formulate the search for the largest perturbation range as an optimization problem and solve its dual.
problem. Sinha et al. [28] introduced a robustness certificate based on a Lagrangian relaxation of the loss function which is provably robust against adversarial input distributions within a Wasserstein ball centered around the original input distribution. Lau and Liu [29] suggested using the notion of Wasserstein barycenter to construct the nominal distribution in WDRO problems.

In the context of FL, several works have studied robustness from different perspectives. For example, Reisizadeh et al. [13] proposed an adversarial robust training method called FedRobust based on a minimax formulation involving the Wasserstein distance. Deng et al. [14] proposed DRFA, a communication-efficient distributionally robust algorithm based on periodic averaging techniques. Mohri et al. [11] and Du et al. [12] introduced agnostic FL frameworks using two-player adversarial minimax games between the learner and the adversary to achieve fairness.

3 Wasserstein Robust Federated Learning

3.1 Expected Risk and Empirical Risk Minimization in Federated Learning

Consider $m$ clients where each client $i \in [m] := \{1, \ldots, m\}$ has its data generating distribution $P_i$ supported on domain $Z_i := (X_i, Y_i)$. Consider the parametrized hypothesis class $\mathcal{H} = \{h_\theta \mid \theta \in \mathbb{R}^d\}$, where each member $h_\theta$ is a mapping from $X_i$ to $Y_i$ parametrized by $\theta$. With $z_i := (x_i, y_i) \in Z_i$, we use $\ell(z_i, h_\theta)$, shorthand for $\ell(y_i, h_\theta(x_i))$, to represent the cost of predicting $h_\theta(x_i)$ when the ground-truth label is $y_i$. For example, if $h_\theta(x_i) = \theta^T x_i$ and $y_i \in \mathbb{R}$, a square loss $\ell(z_i, h_\theta) = \ell(y_i, h_\theta(x_i)) = (\theta^T x_i - y_i)^2$ can be considered. In FL, all clients collaborate with a server to find a global model $\theta$ such that the weighted sum of risks is minimized:

$$\min_{\theta \in \mathbb{R}^d} \sum_{i=1}^m \lambda_i E_{Z_i \sim P_i} [\ell(Z_i, h_\theta)],$$

where $E_{Z_i \sim P_i} [\ell(Z_i, h_\theta)]$ is client $i$’s expected risk and $\lambda_i \geq 0$ represents the relative “weight” of client $i$ satisfying $\sum_{i=1}^m \lambda_i = 1$. Therefore, $\lambda = [\lambda_1, \ldots, \lambda_m]^T$ belongs to the simplex $\Delta := \{\lambda \in \mathbb{R}^m : \lambda \succeq 0 \text{ and } \lambda^T 1_m = 1\}$. Define by $P_\lambda := \sum_{i=1}^m \lambda_i P_i$ the mixed clients’ distribution over $m$ domains $Z := \{Z_1, \ldots, Z_m\}$. We denote by $Z \sim P_\lambda$ a random data point $Z$ generated by $P_\lambda$, which means that the domain of client $i$ is chosen with probability $P(Z = Z_i) = \lambda_i$ first, then a data point $z_i \in Z_i$ is selected with probability $P(Z_i = z_i), Z_i \sim P_i$.

While the underlying distributions $P_i$ are unknown, clients have access to finite observations $z_i \in [n_i]$. We abuse the notation $[n_i]$ to denote the set of client $i$’s both observable data points and their indices. Let $\hat{P}_{n_i} := \frac{1}{n_i} \sum_{z_i \in [n_i]} \delta_{z_i}$ be the empirical distribution of $P_i$, where $\delta_{z_i}$ is the Dirac point mass at $z_i$. In general, we use the notation $\hat{\cdot}$ for quantities that are dependent on training data. Define by $\hat{P}_\lambda := \sum_{i=1}^m \lambda_i \hat{P}_{n_i}$, the mixed empirical distribution of $n := \sum_{i=1}^m n_i$ training data from $m$ clients. The empirical risk minimization (ERM) problem of (1) is:

$$\min_{\theta \in \mathbb{R}^d} \left\{ E_{Z \sim \hat{P}_\lambda} [\ell(Z, h_\theta)] = \sum_{i=1}^m \lambda_i E_{Z_i \sim \hat{P}_{n_i}} [\ell(Z_i, h_\theta)] = \sum_{i=1}^m \frac{n_i}{n} \left( \frac{1}{n_i} \sum_{z_i \in [n_i]} \ell(z_i, h_\theta) \right) \right\},$$

where $\lambda_i = n_i/n$ is typically chosen in the ERM of the standard FL. A detailed discussion on how to choose more appropriate values for $\lambda_i$ is found later in Secs. 5 and 6.

3.2 Wasserstein Robust Risk in Federated Learning

Models resulting from (2) have been shown to be vulnerable to adversarial attacks and to lack of robustness to distribution shifts. We consider a robust variant of the ERM framework involving the worst-case risk with respect to the $p$-Wasserstein distance between two probability measures. Given a set $Z$, define $d : Z \times Z \to [0, \infty)$ to be the cost of “transportation” between its two points. Suppose $P$ and $Q$ are two distributions on $Z$. Let $\Pi(P, Q)$, called their couplings, be the set of joint probability measures $\pi$ on $Z \times Z$ whose marginals are $P$ and $Q$. In other words, $\pi(A, Z) = P(A)$

\footnote{The function $d$ must satisfy non-negativity, lower semi-continuity and $d(z, z) = 0, \forall z \in Z$.}
and \( \pi(Z, A) = Q(A), \forall A \subset Z \). The \( p \)-Wasserstein distance between \( P \) and \( Q \) is defined as

\[
W_p(P, Q) = \inf_{\pi \in \Pi(P, Q)} \left( E(Z, Z') \sim \pi \left[ d^p(Z, Z') \right] \right)^{1/p}.
\]

This distance represents the minimum cost of transporting one distribution to another, where the cost of moving a unit point mass is determined by the ground metric on the space of uncertainty realizations. In this work, we mainly work with \( p = 2 \).

Let \( B_\rho(P, \rho) := \{ Q : W_p(P, Q) \leq \rho \} \) denote the Wasserstein ball centered at \( P \) (i.e., nominal distribution) and having radius \( \rho \geq 0 \). We modify (2) into the following Wasserstein robust risk minimization in FL:

\[
WAFL: \min_{\theta \in \mathbb{R}^d} \left\{ \sup_{Q \in B_\rho(P, \rho)} E_{Z' \sim Q} [\ell(Z', h_\theta)] \right\}.
\]

There are several merits to this framework. First, the ambiguity set \( B_\rho(P, \rho) \) contains all (continuous or discrete) distributions \( Q \) that can be converted from the (discrete) nominal distribution \( \hat{P}_\lambda \) at a bounded transportation cost \( \rho \). Second, Wasserstein distances can be approximated from the samples. Based on the non-asymptotic convergence results of Fournier and Guillin [30], we can specify a suitable value for \( \rho \) to probabilistically bound \( W_p(P, Q) \) by the distance between their empirical distributions \( W_p(\hat{P}, \hat{Q}) \) (e.g., for multi-source domain adaptation).

In any robust optimization problem, the ambiguity set is a key ingredient to defining the level of robustness. We will compare WAFL in (4) with other approaches in terms of their ambiguity set, showing that the Wasserstein ambiguity set can easily be adjusted to cover other ambiguity sets, making WAFL more general and flexible than existing methods.

**Agnostic FL.** Using this approach, existing techniques [11, 14] minimize the worst-case loss

\[
\max_{\lambda \in \Delta} E_{Z \sim \hat{P}_\lambda} [\ell(Z, h_\theta)],
\]

hence its distributional ambiguity set is \( Q_\Delta := \{ \hat{P}_\lambda : \lambda \in \Delta \} \). While Agnostic FL’s ambiguity set is the static convex hull of \( \{ \hat{P}_\lambda \}_{i \in [m]} \), WAFL’s ambiguity set \( B(\hat{P}_\lambda, \rho) \) can be adjusted by controlling the robustness level \( \rho \) and by positioning the ball center using \( \lambda \), which is useful for domain adaptation. Furthermore, by controlling \( \rho \) and \( \lambda \), we can flexibly enlarge \( B(\hat{P}_\lambda, \rho) \) to cover \( Q_\Delta \), or shrink it down to sufficiently include an arbitrary distribution \( Q \) that is outside of the convex hull for domain adaptation (see Fig. 1).

**Adversarial robust FL.** Reisizadeh et al. [13] combined a general affine covariate shift in standard adversarial robust training with FL. Most existing techniques under this approach [31, 33] define an adversarial perturbation \( u \) at a data point \( Z \) and minimize the worst-case loss over all perturbations:

\[
\max_{u \in \mathcal{U}} E_{Z \sim \hat{P}_\lambda} [\ell(Z + u, h_\theta)],
\]

where the ambiguity set is \( \mathcal{U} := \{ u \in \mathbb{R}^{d+1} : \|u\| \leq \epsilon \} \). In Appendix A, we show that the Wasserstein ambiguity set can also contain the perturbation points induced by the solution to this adversarial robust training problem.

### 3.3 WAFL: Algorithm Design and Convergence Analysis

The original form of WAFL in (4) is not friendly for distributed algorithm design. Fortunately, the Wasserstein robust risk (or \( B(\hat{P}_\lambda, \rho) \)-worst-case risk) has its dual formulation as follows [26, 28]

\[
\sup_{Q \in B(\hat{P}_\lambda, \rho)} E_{Z' \sim Q} [\ell(Z', h_\theta)] = \inf_{\gamma \geq 0} \left\{ \gamma \rho^2 + E_{Z \sim \hat{P}_\lambda} [\phi_\gamma(Z, \theta)] \right\},
\]

where \( \phi_\gamma(z, \theta) := \sup_{\zeta \in Z} [\ell(\zeta, h_\theta) - \gamma d^2(\zeta, z)] \), and \( d^2(z, z') = \|x - x'\|^2 + \kappa|y - y'|^2, \kappa > 0 \).

The crux of using the dual is that the inner supremum problem (finding \( \phi_\gamma \)) is easily solvable when...
Algorithm 1 Local SGD for WAFL

\begin{algorithm}
\begin{algorithmic}[1]
\State \textbf{for} $t = 0, 1, ..., T - 1$ \textbf{do} \hspace{1cm} // Global rounds
\State Sample a subset of clients $S_t \subseteq [m]$ \hfill
\State \textbf{for} client $i \in S_t$ in parallel \textbf{do} \hfill
\State Set local parameters: $\theta_i^{(t,0)} = \theta^t$ \hfill
\State \textbf{for} $k = 0, 1, ..., K - 1$ \textbf{do} \hfill // Local rounds
\State Sample a mini-batch $D_i$ from client $i$’s dataset \hfill
\State $\theta_i^{(t,k+1)} = \theta_i^{(t,k)} - \frac{\eta}{|D_i|} \sum_{z_i \in D_i} \nabla_{\theta} \phi(z_i, \theta_i^{(t,k)})$ \hfill
\State Send $\theta_i^{(t,K)}$ to server \hfill
\State \textbf{Server:} update $\theta^{(t+1)} = \sum_{i \in S_t} \lambda_i \theta_i^{(t,K)} / \sum_{i \in S_t} \lambda_i$ \hfill
\end{algorithmic}
\end{algorithm}

its objective is well-conditioned: if $z \mapsto \ell(z, \cdot)$ is $L_{zz}$-smooth and $z \mapsto d(z, \cdot)$ is 1-strongly convex, setting $\gamma > L_{zz}$ ensures that $\zeta \mapsto \ell(\zeta, h_\theta) - \gamma d^2(\zeta, z)$ is strongly concave, and using gradient ascent for the inner supremum problem (for finding $\phi_\gamma$) enjoys linear convergence. Therefore, instead of finding the optimal $\gamma^*$ to $[5]$ that may not satisfy $\gamma^* > L$, we set $\gamma > L$ as a control hyperparameter to ensure there exists a unique solution $z_\gamma^*$ to $\sup_{\zeta \in \mathcal{Z}} [\ell(\zeta, h_\theta) - \gamma d^2(\zeta, z)]$ for each $z_i$, and thus $\nabla_{\theta} \phi(\cdot, z_i, \cdot) = \nabla_{\theta} \phi(z_i^*, h_\theta)$ \cite[Lemma 1]{WAFL}. We will characterize the effect of sub-optimality of $\gamma$ to the excess risk in Lemma 4.3. Then, we obtain the following client-decomposable problem, which is amenable to distributed algorithm design:

$$
\min_{\theta \in \mathbb{R}^d} \left\{ \mathbb{E}_{Z \sim \pi_Z} \left[ \phi_{\gamma}(Z, \theta) \right] = \sum_{i=1}^m \lambda_i \mathbb{E}_{Z \sim \pi_Z} \left[ \phi_{\gamma}(Z_i, \theta) \right] \right\}.
$$

This motivates the development of Algorithm 1 for solving (6). The structure of WAFL is similar to FedAvg with $T$ communication rounds and three additional key components. First, \textit{client sampling} (line 1) refers to the partial participation of clients in each global round. Second, each client performs $K$ \textit{local steps} (line 6) before sending its local model to the server. Finally, \textit{stochastic approximation} of a client’s gradient using a mini-batch (lines 6 and 7) is necessary when the data size is large. The other hand, in WAFL and FedAvg is that WAFL aims to minimize the risk with respect to the surrogate loss $\phi_\gamma$, rather than $\ell$. We show that the convergence of WAFL can be similarly characterized as that of FedAvg, the de facto FL algorithm based on local SGD updates \cite{FedAvg}. In FedAvg optimization, we seek to establish the convergence when using the original loss function $\ell$. On the other hand, in WAFL the convergence is with respect to the surrogate loss $\phi_\gamma$, through which the local and global risks are defined by $F_\ell(\theta) := \mathbb{E}_{Z \sim \pi_Z} \left[ \phi_{\gamma}(Z_i, \theta) \right]$ and $F(\theta) := \mathbb{E}_{Z \sim \pi_Z} \left[ \phi_\gamma(Z_i, \theta) \right]$, respectively.

We first make the following assumptions, common to analyses of Wasserstein-robust optimization \cite{WAFL}. Unless stated otherwise, all distances are the Euclidean norm.

\textbf{Assumption 3.1.} The function $\ell : \mathcal{Z} \times \mathcal{H} \to \mathbb{R}$ is continuous, and $d(\cdot, z_0)$ is 1-strongly convex, for all $z_0 \in \mathcal{Z}$.

\textbf{Assumption 3.2.} The loss function $\ell : \mathcal{Z} \times \mathcal{H} \to \mathbb{R}$ is Lipschitz continuous as follows

\begin{itemize}
  \item[(a)] $\|\ell(z, h_\theta) - \ell(z', h_\theta)\| \leq L_z \|z - z'\|$;
  \item[(b)] $\|\ell(z, h_\theta) - \ell(z, h_\theta')\| \leq L_\theta \|\theta - \theta'\|$.
\end{itemize}

\textbf{Assumption 3.3.} The loss function $\ell : \mathcal{Z} \times \mathcal{H} \to \mathbb{R}$ is Lipschitz smooth as follows

\begin{itemize}
  \item[(a)] $\|\nabla_{\theta} \ell(z, h_\theta) - \nabla_{\theta} \ell(z', h_\theta)\| \leq L_{\theta \theta} \|\theta - \theta'\|$;
  \item[(b)] $\|\nabla_z \ell(z, h_\theta) - \nabla_z \ell(z', h_\theta)\| \leq L_{zz} \|z - z'\|$;
  \item[(c)] $\|\nabla_{\theta} \ell(z, h_\theta) - \nabla_{\theta} \ell(z', h_\theta)\| \leq L_{\theta \theta} \|z - z'\|$;
  \item[(d)] $\|\nabla_z \ell(z, h_\theta) - \nabla_z \ell(z', h_\theta)\| \leq L_{zz} \|\theta - \theta'\|$.
\end{itemize}

Given Assumption 3.3, it has been shown that the mapping $\theta \mapsto \phi_\gamma(\cdot, \cdot)$ is $L$-smooth with $L = L_{\theta \theta} + \frac{L_{\theta \theta} L_{zz}}{L_{zz}}$, $\gamma > L_{zz}$ \cite{WAFL}, more detail in Lemma C.1 in Appendix C. In addition, we make the following assumptions common to FL analysis \cite{FL}.

\textbf{Assumption 3.4.} The unbiased stochastic approximation of $\nabla F_\ell(\theta)$, denoted by $g_{\phi_\gamma}(\theta) := \nabla_{\phi} \phi_\gamma(z_i, \cdot), z_i \sim P_i$, has $\sigma^2$-uniformly bounded variance, i.e., $\mathbb{E} \left[ \|g_{\phi_\gamma}(\theta) - \nabla F_\ell(\theta)\|^2 \right] \leq \sigma^2$.

\textbf{Assumption 3.5.} The difference between the local gradient $\nabla F_\ell(\theta)$ and the global gradient $\nabla F(\theta)$ is $\Omega$-uniformly bounded, i.e., $\max_i \sup_{\theta} \|\nabla F_\ell(\theta) - \nabla F(\theta)\| \leq \Omega$. 


Assuming complete participation of clients in every round ($S_t = [m], \forall t$), using standard techniques in [6], we have:

**Theorem 3.6 (WAFL’s convergence for convex loss function).** Let Assumptions [3.1, 3.5] hold and the mapping $\theta \mapsto \ell(z, h_\theta)$ be convex. Denote by $\hat{\theta}^{(t,k)}$ the “shadow” sequence, defined as $\hat{\theta}^{(t,k)} = \sum_{i=1}^{m} \lambda_i \theta_i^{(k)}$ and by $\theta^*$ the optimal solution to $\min_{\theta \in \mathbb{R}^d} F(\theta)$. If the learning rate $\eta$ is at most

$$\min \left\{ \frac{1}{3L} : \frac{D}{2\sqrt{KT} \lambda} \leq \frac{48^{2/3} K 2^{T/3} L^{2/3} \bar{\sigma}^2}{40^{2/3} K T^{2/3} L^{2/3} \bar{\sigma}^2} \right\},$$

then we have

$$\mathbb{E} \left[ \frac{1}{KT} \sum_{t=0}^{T-1} \sum_{k=0}^{K-1} F(\hat{\theta}^{(t,k)}) - F(\theta^*) \right] \leq \mathcal{O} \left( \frac{LD^2}{KT} + \frac{\hat{\sigma} DA^{\frac{2}{3}}}{\sqrt{KT}} + \frac{L^{\frac{2}{3}} \bar{\sigma}^2 D^{\frac{2}{3}}}{K^{\frac{2}{3}} T^{\frac{2}{3}}} + \frac{L^{\frac{2}{3}} \bar{\sigma}^2 D^{\frac{2}{3}}}{T^{\frac{2}{3}}} \right),$$

where $\hat{\sigma}^2 := \frac{\sigma^2 |D|}{|F|}$, $D := \|\theta(0) - \theta^*\|$ and $\Lambda := \sum_{i=1}^{m} \lambda_i^2$.

### 4 Robust Generalization Bounds

We show the generalization and robustness properties of WAFL’s output by bounding its excess risk. Denote the loss class by $\mathcal{F} := \ell \circ \mathcal{H} := \{z \mapsto \ell(z, h), h \in \mathcal{H}\}$, where we use $f$ (resp. $f_0$) $\in \mathcal{F}$ to represent a generic loss (resp. a loss function parametrized by $\theta$).

**Definition 4.1.** Denote the expected risk and surrogate of Wasserstein robust risk of an arbitrary $f$, respectively, as

$$\mathcal{L}(P_\lambda, f) := \mathbb{E}_{Z \sim P_\lambda} \left[ \ell(Z, h) \right] \quad \text{and} \quad \mathcal{L}^\gamma_\rho(P_\lambda, f) := \mathbb{E}_{Z \sim P_\lambda} \left[ \phi_{\gamma}(Z, f) \right] + \gamma \rho^2.$$

Then their excess risks are defined respectively as follows

$$\mathcal{E}(P_\lambda, f) := \mathcal{L}(P_\lambda, f) - \inf_{f' \in \mathcal{F}} \mathcal{L}(P_\lambda, f') \quad \text{and} \quad \mathcal{E}^\gamma_\rho(P_\lambda, f) := \mathcal{L}^\gamma_\rho(P_\lambda, f) - \inf_{f' \in \mathcal{F}} \mathcal{L}^\gamma_\rho(P_\lambda, f').$$

If a distribution $Q$ is in the ambiguity set $\mathcal{B}(P_\lambda, \rho)$, we can bound its excess risk $\mathcal{E}(Q,f)$ as follows.

**Lemma 4.2.** Let Assumption 3.2 hold and $\gamma \geq L_z/\rho$. For all $f \in \mathcal{F}$ and all $Q \in \mathcal{B}(P_\lambda, \rho)$,

$$\mathcal{E}^\gamma_\rho(P_\lambda, f) - g(\rho, \gamma) \leq \mathcal{E}(Q,f) \leq \mathcal{E}^\gamma_\rho(P_\lambda, f) + g(\rho, \gamma),$$

where $g(\rho, \gamma) := 2L_z^2 \rho + \gamma^2$ and $\gamma^* := \arg\min_{\gamma \geq 0} \mathcal{L}^\gamma_\rho(P_\lambda, f)$.

**Remark 4.3.** Lemma 4.2 shows that the lower and upper bounds for $\mathcal{E}(Q,f)$ can be analyzed using $\mathcal{E}^\gamma_\rho(P_\lambda, f)$ and a two-component error term $g(\rho, \gamma)$ capturing the impact of the control parameters $\rho$ and $\gamma$. Particularly, the first component, $2L_z^2 \rho$, says that when $\rho$ is increased – to allow for a larger Wasserstein distance between the nominal $P_\lambda$ and any worst-case distribution $Q$ – the difference between $\mathcal{E}(Q,f)$ and $\mathcal{E}^\gamma_\rho(P_\lambda, f)$ increases, and this error is amplified at most by the Lipschitz constant $L_z$ of the mapping $z \mapsto \ell(z, \cdot)$. The second component, $|\gamma - \gamma^*|\rho^2$, addresses the sub-optimality error of a chosen value of $\gamma$, which is amplified when $\gamma$ is drifted away from the optimal $\gamma^*$. Note that $\mathcal{L}^\gamma_\rho(P_\lambda, f)$ is the same as $\mathcal{B}(P_\lambda, \rho)$-worst-case risk thanks to the strong duality in [5], obtained with $\rho > 0$.

Denote by $\hat{\theta}^* \in \Theta$ an $\epsilon$-minimizer to the surrogate ERM, i.e., $\mathbb{E}_{Z \sim \hat{P}_\lambda} \left[ \phi_{\gamma}(Z, f_{\hat{\theta}^*}) \right] \leq \inf_{\theta \in \Theta} \mathbb{E}_{Z \sim \hat{P}_\lambda} \left[ \phi_{\gamma}(Z, f_\theta) \right] + \epsilon$, where $\Theta \subset \mathbb{R}^d$ is a parameter class, we obtain the following.

**Theorem 4.4 (Robust generalization bounds).** Let Assumptions 3.2 and 3.3 hold, $\gamma \geq \max\{L_z, L_z/\rho\}$, and $|\ell(z, h)| \leq M_t$. We have the following result for all $Q \in \mathcal{B}(P_\lambda, \rho)$

$$\mathcal{E}(Q,f_{\hat{\theta}^*}) \leq \sum_{i=1}^{m} \frac{48\mathcal{E}(\Theta)}{\sum n_i} + 2M_t \sqrt{\frac{\log^2(2m/\delta)}{n_i}} + \epsilon + g(\rho, \gamma)$$

with probability at least $1 - \delta$, where $\mathcal{E}(\Theta) := L_\theta \int_0^\infty \sqrt{\log \mathcal{N}(\Theta, \|\cdot\|_\Theta, \epsilon)} d\epsilon$ and $\mathcal{N}(\Theta, \|\cdot\|_\Theta, \epsilon)$ denotes the $\epsilon$-covering number of $\Theta$ w.r.t a metric $\|\cdot\|_\Theta$ as the norm on $\Theta$.
The proof of Theorem 4.4 leverages Lemma 4.2 to bound $\mathbb{E}(Q, f)$, $\forall Q \in \mathcal{B}(P_\lambda, \rho)$, based on the bound of $\mathcal{E}_\rho(P_\lambda, f_{\hat{\rho}})$. The result shows using WAFL to minimize the surrogate of Wasserstein robust empirical risk also controls robustness and generalization. For example, $\mathcal{H} = \{ (\theta, \cdot) : \theta \in \Theta \}$, $\Theta = \{ \theta \in \mathbb{R}^d : \|\theta\|_2 \leq C \}$. The diameter of $\Theta$ is $\sup_{\theta, \theta' \in \Theta} \|\theta - \theta'\| = 2C$, thus $\mathcal{N}(\Theta, \| \cdot \|_2, \epsilon) = (1 + 2C/\epsilon)^d$, and $\mathcal{C}(\Theta) \leq 3CL_\theta \epsilon d/2$ [37].

Generally, the radius of Wasserstein ball $\rho$ can be considered a hyperparameter that needs fine-tuning (e.g., through cross-validation). In principle, $\rho$ should not be too large to become over-conservative, which can hurt the empirical average performance, but also not too small to become similar to the ERM, and thus can lack robustness. From a statistical standpoint, we are interested in learning how to scale $\rho$ w.r.t. the sample size $n_i, i \in [m]$, such that the generalization of the WAFL solution $\hat{\theta}^\rho$ w.r.t. the true distribution $P_\lambda$ is guaranteed, while still ensuring robustness w.r.t. all distributions inside the Wasserstein ball. Using the result from Fournier and Guillin [30] showing that $\rho^\delta_i$ can be probabilistically computed using linear or convex programming especially with entropic regularization and the Sinkhorn algorithm [40].

5 Choosing $\lambda$: Applications

We focus on two applications: multi-source domain adaptation and generalization to all client distributions. We provide insights on choosing the weights $\lambda$ for these applications.

Multi-source domain adaptation: Consider the multi-source domain distribution $P_\lambda$ [38]. Lee and Raginsky [37] show that solving the minimax risk with the Wasserstein ambiguity set can help transfer data/knowledge from the source domain $P_\lambda$ to a different but related target domain $Q$. They bound the distance $W_p(P_\lambda, Q)$ using the triangle inequality

$$W_p(P_\lambda, Q) \leq W_p(P_\lambda, \hat{P}_\lambda) + W_p(\hat{P}_\lambda, \hat{Q}) + W_p(\hat{Q}, Q),$$

where $\hat{P}_\lambda$ and $\hat{Q}$ are the empirical versions of $P_\lambda$ and $Q$, respectively. While $W_p(P_\lambda, \hat{P}_\lambda)$ and $W_p(\hat{P}_\lambda, \hat{Q})$ can be probabilistically bounded with a confidence parameter $\delta \in (0, 1)$ according to Fournier and Guillin [30], $W_p(\hat{P}_\lambda, \hat{Q})$ can be deterministically computed using linear or convex programming [39]. In the FL context, in order to have a better bound for $W_2(P_\lambda, Q)$ similar to (7), it is straightforward to choose $\lambda = \arg \min_{\lambda \in \Delta} W_2(\hat{P}_\lambda, \hat{Q})$. To relax this problem into a form solvable using existing approaches, observe that $W_2(\hat{P}_\lambda, Q) \leq \sum_{i=1}^m \lambda_i W_2(\hat{P}_{n_i}, Q)$ due to the convexity of the Wasserstein distance. We then consider the following upper bound to $\min_{\lambda \in \Delta} W_2(\hat{P}_\lambda, \hat{Q})$:

$$\min_{\lambda \in \Delta} \sum_{i=1}^m \lambda_i W_2^2(\hat{P}_{n_i}, \hat{Q}) =: \rho^*^2,$$

which is a linear program, considering each $W_2^2(\hat{P}_{n_i}, \hat{Q})$ can be found by efficiently solving convex programs especially with entropic regularization and the Sinkhorn algorithm [40].

Corollary 5.1. Denote the solution to (8) by $\lambda^*$, and assume that domain $Q$ generates $n_Q$ i.i.d. data points. With probability at least $1 - \delta$, we have

$$W_2(P_\lambda, Q) \leq W_2(P_\lambda, \hat{P}_\lambda) + W_2(\hat{P}_\lambda, \hat{Q}) + W_2(Q, \hat{Q}) \leq \sqrt{\sum_{i=1}^m \lambda_i^2 \hat{\rho}_{n_i}^2} + \rho^* + \hat{\rho}_{n_Q}^d/2.$$

The proof of this corollary is similar to that of Corollary 5.1 in Appendix B.

Covering all client distributions in the Wasserstein ball: Suppose we want to cover all client distributions inside a Wasserstein ball so that the generalization and robustness result by WAFL in Theorem 4.4 is applicable to all clients’ distributions. We show in Appendix B that this is a problem of finding $\lambda$ such that the Wasserstein distance between $P_\lambda$ and $P_j, \forall j$, is as small as possible.
6 Experiments

We aim to show four key results through numerical experiments. First, we show the relationship between the hyperparameter \(\gamma\) and the traditional worst-case perturbation \(\rho\) used in distributionally robust learning. Second, we investigate the effect of \(\gamma\) on WAFL’s performance in two data settings. Third, we provide an extensive comparison of WAFL with other robust baselines and with FedAvg in scenarios with varying degrees of attack in an FL network. Finally, we perform several experiments in multi-source domain adaptation to illustrate the findings in Sec. 5.

Experimental settings. We design two non-i.i.d. FL settings. First, we use the MNIST dataset to distribute to 100 clients and employ a multinomial logistic regression model in a convex setting. We then use CIFAR-10 to distribute to 20 clients and employ a CNN model in McMahan et al. in a non-convex setting. In the following experiments, we randomly sample \(|S_t| = 10\) clients to participate in training at each communication round. When the stochastic gradient is calculated, we use a batch size of \(|D_t| = 64\). For a fair comparison, we use the same number of global and local optimization rounds for each algorithm (\(T = 200, K = 2\)). More detail can be found in Appendix G.1.

Effect of \(\gamma\) on the worst-case risk perturbations. Define the (squared) average worst-case perturbation as \(\hat{\rho}^2 = \mathbb{E}_{Z \sim \hat{P}_0}|d^2(Z, Z)|\), where \(Z\) is the adversarial example of \(Z\) as a solution to \(\phi_\gamma((Z, \cdot))\). Fig. 2 depicts the relationship between \(\hat{\rho}\) and the predetermined \(\gamma\) in the two data settings, and shows that smaller \(\gamma\) corresponds to larger \(\hat{\rho}\). This allows us to indirectly control the amount of worst-case perturbation through the change of the hyperparameter \(\gamma\) in the opposite direction. In other words, \(\gamma\) is a hyperparameter that needs fine-tuning in order to obtain the best performance, and setting a sufficiently large \(\gamma\) provides a moderate level of robustness (smaller \(\rho\) by duality) while ensuring \(\phi_\gamma\) can be solved fast using gradient methods (Sec. 3.3).

Effect of \(\gamma\) on the generalizability and robustness of WAFL. Consider \(\hat{P}_\lambda\) and \(\hat{Q}\) as the empirical distributions of training and test samples, respectively. By controlling the hyperparameter \(\gamma\), we aim to train a global model robust to any test distribution \(\hat{Q}\). To do so, we design two scenarios. In the clean data scenario, the global model is trained with different values of \(\gamma\) and evaluated on clients’ hold-out test data. In the distribution shift scenario, the training process is the same, but the hold-out test data go through distribution shifts. To obtain these shifts, we employ the common PGD attack under the \(l_\infty\)-norm to generate an \(\epsilon\)-level perturbation of clients’ test data. We fix the number of gradient steps to generate adversarial examples, and use \(t_{\text{adv}} = 40, \epsilon = 0.3, \alpha = 0.01\) for MNIST, and \(t_{\text{adv}} = 10, \epsilon = 8/255, \alpha = 2/255\) for CIFAR-10. We note that this setting is similar to that involving adversarial poisoning attacks, whose main purpose is to increase the Wasserstein distance between \(\hat{P}_\lambda\) and \(\hat{Q}\), thereby helping to verify the robustness of WAFL.

Fig. 3 shows the performance of WAFL and FedAvg in the two scenarios. Under clean data, the distance between \(\hat{P}_\gamma\) and \(\hat{Q}\) is relatively small, therefore requiring a lower amount of robustness (large \(\gamma\)). By carefully fine-tuning \(\gamma\) in the ranges [0.5, 1] for MNIST and [10, 20] for CIFAR-10, WAFL enjoys the same or even better performance as FedAvg. The benefit of \(\gamma\) emerges most clearly under distribution shift. In this scenario, \(\hat{P}_\gamma\) and \(\hat{Q}\) grow further apart, requiring a larger ambiguity set \(B(\hat{P}_\lambda, \hat{\rho})\) (or, equivalently, a smaller \(\gamma\)) to ensure robustness. Meanwhile, too small \(\gamma\) may violate the assumption that \(\gamma > L_{zz}\) and can hurt WAFL’s performance as \(\hat{\rho}\) becomes too large, as demonstrated in Sec. 4. In later experiments, we choose \(\gamma = 0.5\) for MNIST and \(\gamma = 10\) for CIFAR-10.

Comparison with other robust methods. We compare WAFL with FedAvg and four robust baselines in FL: FedPGM, FedFGSM, distributionally robust FedAvg [14] DRFA and agnostic FL [11] AFL, FedPGM and FedFGSM are FedAvg with adversarial training using the PGD method and the FGSM method on local clients, respectively. In each local update of FedPGD and FedFGSM, all clients solve \(\delta^{\ast} = \arg\max_{\|\delta\|_{l_\infty} \leq \epsilon} \{\ell(h_\gamma(z + \delta), y)\}\) using projection onto an \(l_\infty\)-norm to find the worst-case perturbation \(\delta\). While FedPGD uses \(t_{\text{adv}}\) gradient steps to find \(\delta^{\ast}\), FedFGSM uses only one gradient step. We use the same value of \(t_{\text{adv}}\) when training using WAFL and FedPGM. On the other hand, DRFA and AFL both aim to achieve robustness by changing the clients’ weights \(\lambda_i\).
We note that the global accuracy decreases monotonically with the percentage of attacked clients. All algorithms are fine-tuned to obtain their best performance on the target datasets. While FedAvg, by definition a non-robust method, unsurprisingly suffers the largest performance drop, WAFL outperforms all baselines, retaining over 50% accuracy on MNIST and nearly 45% on CIFAR-10 even when 80% of clients experience distribution shifts. We observe that the performance of FedPGD and FedFGSM is much better than DRFA and AFL, and is the closest to WAFL. This suggests that adjusting the clients’ weights may not notably help with achieving robustness.

Furthermore, we provide a comparison between the performance of WAFL with different p values and other baselines in Appendix G.3 to show that the duality result in [5] suffices with any ℓp norm.

**Domain adaptation.** Sec. 5 describes WAFL’s capability in multi-source domain adaptation by solving a linear program in λ. We empirically demonstrate that capability using three digit recognition datasets including MNIST (mt) [41], USPS (up) [43] and SVHN (sv) [44]. We convert all images to have the size of 28 × 28 × 3. More information about these datasets can be found in Appendix G.1. We then train a global multinomial logistic regression model on two source domain datasets, and evaluate it using the remaining dataset as the target domain. To solve the linear program in (6), we estimate the Wasserstein distance by leveraging the computational methods introduced in [40, 45], and solve the linear program using SciPy. For comparison, we use FedAvg in two scenarios: λi = ni/n and λi = 1/m. We also employ AFL and DRFA, both of which can vary the λi to achieve robustness. All algorithms are fine-tuned to obtain their best performance on the target datasets.

The accuracies on the target domains are presented in Table 1. In all three scenarios, WAFL outperforms all other methods, especially in the settings mt, sv → up and sv, up → mt, where WAFL’s accuracy exceeds the second-best accuracy (achieved by DRFA) by five percentage points. We note that the sv dataset is the most different from the other two, measured by the Wasserstein distance, which is why generalization to sv’s domain is the most difficult.

https://docs.scipy.org/doc/scipy/reference/optimize.html
Table 1: Accuracies on target domains.

| $\lambda$ | $\text{mt, sv} \rightarrow \text{up}$ | $\text{mt, up} \rightarrow \text{sv}$ | $\text{up, sv} \rightarrow \text{mt}$ | Avg |
|-----------|----------------|----------------|----------------|-----|
| $n_i/n$   | 59.0           | 14.1           | 16.1           | 29.7|
| $1/m$     | 58.7           | 14.9           | 52.1           | 41.6|
| AFL       | 60.1           | 15.0           | 52.4           | 42.5|
| DRFA      | 61.6           | 15.1           | 53.0           | 43.2|
| WAFL      | 65.6           | 16.6           | 58.1           | 46.7|

7 Conclusion

In this paper, we apply the Wasserstein distributionally robust training method to federated learning to handle statistical heterogeneity. We first remodel the duality of the worst-case risk to an empirical surrogate risk minimization problem, and then solve it using a local SGD-based algorithm with convergence guarantees. We show that WAFL is more general in terms of robustness compared to related approaches, and obtains an explicit robust generalization bound with respect to all unknown distributions in the Wasserstein ambiguity set. Through numerical experiments, we demonstrate that WAFL generalizes better than the standard FedAvg baseline in non-i.i.d. settings, and outperforms other robust FL methods in scenarios with distribution shifts and in applications of multi-source domain adaptation.
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Appendix

A  Adversarial Robust FL’s Ambiguity Set vs Wasserstein Ball

We show that using the Wasserstein ambiguity set contains the perturbation points induced by the solution to the Adversarial Robust FL approach. As we present in Sec. 3, existing techniques for adversarial training robust models [31,35,46] define an adversarial perturbation $u$ at a data point $Z$, and minimize the following worst-case loss over all possible perturbations

$$\max_{u \in \mathcal{U}} \mathbb{E}_{Z \sim \hat{P}_\lambda} [\ell(Z + u, h_\theta)],$$

(9)

where the ambiguity set $\mathcal{U} := \{u \in \mathbb{R}^{d+1} : \|u\| \leq \epsilon\}$. To compare this approach with Wasserstein-robust FL, we relate the above problem to its counterpart defined in the probability space of input as follows

$$\max_{Q \in \mathcal{Q}(\epsilon)} \mathbb{E}_{\tilde{Z} \sim \hat{P}_\lambda} [\ell(\tilde{Z}, h_\theta)],$$

(10)

where $\mathcal{Q}(\epsilon) := \{\tilde{Q} : \mathbb{P}[\|\tilde{Z} - Z\| \leq \epsilon] = 1, Z \sim \tilde{P}_\lambda, \tilde{Z} \sim \tilde{Q}\}$. Considering $u^*$ as a solution to problem (9), we see that the distribution $Q'$ of perturbation points (i.e., $\tilde{Z} := (Z + u^*) \sim Q'$) in problem (9) belongs to the feasible set $\mathcal{Q}(\epsilon)$ in problem (10). Next, consider an arbitrary distribution $\tilde{Q} \in \mathcal{Q}(\epsilon)$ in problem (10), with any $\tilde{Z} \sim \tilde{Q}$ and $Z \sim \tilde{P}_\lambda$, we have

$$\|\tilde{Z} - Z\| \leq \epsilon \implies \mathbb{E}_{Z \sim \hat{P}_\lambda, \tilde{Z} \sim \tilde{Q}} [\|\tilde{Z} - Z\|] \leq \epsilon \implies \inf_{\pi \in \Pi(\tilde{P}_\lambda, \tilde{Q})} \mathbb{E}_{(Z, Z') \sim \pi} [\|\tilde{Z} - Z\|] \leq \epsilon,$$

which implies that $W_1(\tilde{P}_\lambda, \tilde{Q}) \leq \epsilon, \forall \tilde{Q} \in \mathcal{Q}(\epsilon)$, and thus $\mathcal{Q}(\epsilon) \subset B_1(\tilde{P}_\lambda, \epsilon)$. We have shown that the Wasserstein ambiguity set contains the perturbation points induced by the solution to the adversarial robust training problem (9).

B  Choosing $\lambda$: generalizing to all client distributions

We show that by calibrating appropriate $\lambda$ value, our proposed algorithm will be capable of generalizing to all client distributions. Suppose we want to cover all client distributions inside a Wasserstein ball so that the generalization and robustness result by WAFL in Theorem 4.4 is applicable to all clients’ distributions. This is the problem of finding $\lambda$ such that the Wasserstein distance between $P_\lambda$ and $P_j, \forall j$, is as small as possible. Instead of directly finding the minimum Wasserstein radius that cover all client distributions, we will leverage the popular Wasserstein barycenter problem [39]. Specifically, consider the problem

$$\min_{\lambda \in \Delta} W_2^2(\tilde{P}_\lambda, P^\bullet) \quad \text{s.t.} \quad P^\bullet = \arg \min_{Q \in \mathcal{P}} \sum_{i=1}^m \lambda_i W_2^2(\tilde{P}_{n_i}, Q),$$

(11)

where $P^\bullet$ is the Wasserstein bary center w.r.t the solution $\lambda$ to this problem. Even though the solution is not straightforward, we propose to solve its tractable upper-bound:

$$\min_{\lambda \in \Delta, Q \in \mathcal{P}} \sum_{i=1}^m \lambda_i W_2^2(\tilde{P}_{n_i}, Q).$$

(12)

This is a bi-convex problem, which is convex w.r.t to $\lambda$ (resp. $Q$) when fixing $Q$ (resp. $\lambda$). Thus, we can use alternative minimization [47] to find a local solution to this problem. Denoting $\hat{\lambda}$ as the solution to (11) and $(\lambda^*, P^*)$ as a local solution to (12), we obtain

$$W_2^2(\tilde{P}_\lambda, P^\bullet) \leq W_2^2(\tilde{P}_{\lambda^*}, P^*) \leq \sum_{i=1}^m \lambda_i^* W_2^2(\tilde{P}_{n_i}, P^*) =: \rho^2.$$  

(13)

Corollary B.1. For all client $j \in [m]$, with probability at least $1 - \delta$, we have

$$W_2(P_{\lambda^*}, P_j) \leq W_2(P_{\lambda^*}, \tilde{P}_{\lambda^*}) + W_2(\tilde{P}_{\lambda^*}, P^*) + W_2(P^*, \tilde{P}_{n_j}) + W_2(\tilde{P}_{n_j}, P_j) \leq \sqrt{\sum_{i=1}^m \frac{\lambda_i^{\beta/3m} + \rho^* + \rho^* \lambda_j + \rho_j^{\beta/2}}{\lambda_j}}.$$
Proof. The first line is by triangle inequality. The second line is by following facts: (i) $\sum_{i=1}^{m} \frac{\lambda_i^2 \rho_{i,m}}{\lambda_i} \leq \frac{\delta}{2}$ according to (36), (ii) $W_2(P^*, \widetilde{P}_{ij}) = \frac{\lambda_i W_2(P^*, \widetilde{P}_{ij})}{\lambda_i} \leq \frac{\rho_{i,m}^2}{\lambda_i}$, and (iii) $\sum_{i=1}^{m} \lambda_i W_2(P_{ij}, \widetilde{P}_{ij}) \leq \frac{\rho_{i,m}^2}{\lambda_i}$, and (iv) using union bound. \qed

C Proof of Theorem 3.6

Our proof is based on the analysis of local SGD for FL presented in [36].

Fix some $z_i$. Define $\varphi(\zeta; \theta; z_i) := \ell(\zeta, h_\theta) - \gamma d(\zeta, z_i)$. Since $\ell$ is $L_{zz}$-smooth and $d$ is 1-strongly convex, $\varphi(\zeta; \theta; z_i)$ is $(\gamma - L_{zz})$-strongly concave with respect to $\zeta$, given that $\gamma > L_{zz}$.

Lemma C.1. Let $z_i^* = \arg \max_{\zeta \in \mathcal{Z}} \varphi(\zeta; \theta; z_i)$. Therefore, $\varphi_\gamma(\theta; z_i) = \varphi(z_i^*, \theta; z_i)$. Let $\ell$ satisfy Assumption 3.3. Then $\varphi_\gamma$ is differentiable, and$$\|\nabla \varphi_\gamma(z_i, \theta) - \nabla \varphi_\gamma(z_i, \theta')\| \leq L\|\theta - \theta'\|,$$with $L = L_{\theta\theta} + \frac{L_{\theta\theta} L_{zz}}{\gamma - L_{zz}}$ when $\gamma > L_{zz}$.

The proof can be found in [28]. Lemma C.1 implies that $\varphi_\gamma$ is $L$-smooth.

Define $g_i(\theta) := \frac{1}{|D_i|} \sum_{z_i \in D_i} \nabla \theta \varphi_\gamma(z_i, \theta)$, then we have

$$E\left[\|g_i(\theta) - \nabla F_i(\theta)\|^2\right] = E\left[\frac{1}{|D_i|} \sum_{z_i \in D_i} \nabla \theta \varphi_\gamma(z_i, \theta) - \nabla F_i(\theta)\|^2\right] \leq \frac{1}{|D_i|} E\left[\|g_i(\theta) - \nabla F_i(\theta)\|^2\right] \leq \frac{\sigma^2}{|D_i|} := \tilde{\sigma}^2 \quad \text{(by Assumption 3.4)} \quad (14)$$

With the shadow sequence $\tilde{\theta}^{(t+k)} = \sum_{i=1}^{m} \lambda_i \theta^{(t+k)}$, we have

$$\tilde{\theta}^{(t,k+1)} = \sum_{i=1}^{m} \lambda_i \theta^{(t,k+1)} = \sum_{i=1}^{m} \lambda_i \left(\theta^{(t,k)} - \eta g_i(\theta^{(t,k)})\right) = \tilde{\theta}^{(t,k)} - \eta \sum_{i=1}^{m} \lambda_i g_i(\theta^{(t,k)}).$$

Lemma C.2. If the client learning rate satisfies $\eta \leq \frac{1}{3L}$, then

$$\frac{1}{K} \sum_{k=0}^{K-1} E\left[F(\tilde{\theta}^{(t,k)}) - F(\theta^*)\right] \leq 2\eta \tilde{\sigma}^2 \left(\sum_{i=1}^{m} \lambda_i^2\right) + L \sum_{i=1}^{m} \lambda_i \sum_{k=0}^{K-1} \frac{1}{K} E\left[\|\theta^{(t,k)} - \tilde{\theta}^{(t,k)}\|^2\right]$$

\[ + \frac{1}{2\eta K} \left(\|\theta(t) - \theta^*\|^2 - E\left[\|\theta(t+1) - \theta^*\|^2\right]\right). \quad (15)\]

Proof. Since $\tilde{\theta}^{(t,k+1)} = \tilde{\theta}^{(t,k)} - \eta \sum_{i=1}^{m} \lambda_i g_i(\theta^{(t,k)})$, by parallelogram law

$$\sum_{i=1}^{m} \lambda_i \left(g_i(\theta^{(t,k)}), \tilde{\theta}^{(t,k+1)} - \theta^*\right) = \frac{1}{2\eta} \left(\|\tilde{\theta}^{(t,k)} - \theta^*\|^2 - \|\tilde{\theta}^{(t,k+1)} - \tilde{\theta}^{(t,k)}\|^2 - \|\tilde{\theta}^{(t,k+1)} - \theta^*\|^2\right).$$

Fact: $F_i(\theta) = E_{Z_i \sim P_i} \left[\varphi_i(Z_i, \theta)\right]$ is Lipschitz smooth with $L = L_{\theta\theta} + \frac{L_{\theta\theta} L_{zz}}{\gamma - L_{zz}}$ when $\gamma > L_{zz}$. With the assumption that $\theta \mapsto \ell(z, h_\theta)$ is convex, we have $\theta \mapsto F_i(\theta)$ is convex.

Since $F_i$ is convex and $L$-smooth,

$$F_i(\tilde{\theta}^{(t,k+1)}) \leq F_i(\theta^{(t,k)}) + \left(\nabla F_i(\theta^{(t,k)}), \tilde{\theta}^{(t,k+1)} - \theta^{(t,k)}\right) + \frac{L}{2} \|\tilde{\theta}^{(t,k+1)} - \theta^{(t,k)}\|^2 \leq F_i(\theta^*) + \left(\nabla F_i(\theta^{(t,k)}), \tilde{\theta}^{(t,k+1)} - \theta^*\right) + \frac{L}{2} \|\tilde{\theta}^{(t,k+1)} - \theta^{(t,k)}\|^2 \leq F_i(\theta^*) + \left(\nabla F_i(\theta^{(t,k)}), \tilde{\theta}^{(t,k+1)} - \theta^*\right) + L \|\tilde{\theta}^{(t,k+1)} - \tilde{\theta}^{(t,k)}\|^2 + L \|\theta^{(t,k)} - \tilde{\theta}^{(t,k)}\|^2. \quad (16)$$
From (15) and (16), we have
\[
F(\tilde{\theta}^{(t,k+1)}) - F(\theta^*) = \sum_{i=1}^{m} \lambda_i \left( F_i(\tilde{\theta}^{(t,k+1)}) - F(\theta^*) \right)
\leq \sum_{i=1}^{m} \lambda_i \left( \nabla F_i(\theta_i^{(t,k)}) - g_i(\theta_i^{(t,k)}, \tilde{\theta}^{(t,k+1)} - \theta^*) \right) + L \| \tilde{\theta}^{(t,k+1)} - \tilde{\theta}^{(t,k)} \|^2 + L \sum_{i=1}^{m} \lambda_i \| \theta_i^{(t,k)} - \tilde{\theta}^{(t,k)} \|^2
+ \frac{1}{2\eta} \left( \| \tilde{\theta}^{(t,k)} - \theta^* \|^2 - \| \tilde{\theta}^{(t,k+1)} - \tilde{\theta}^{(t,k)} \|^2 - \| \tilde{\theta}^{(t,k+1)} - \theta^* \|^2 \right) .
\]
(17)
We have
\[
E \left[ \sum_{i=1}^{m} \lambda_i \left( \nabla F_i(\theta_i^{(t,k)}) - g_i(\theta_i^{(t,k)}, \tilde{\theta}^{(t,k+1)} - \theta^*) \right) \right]
= E \left[ \sum_{i=1}^{m} \lambda_i \left( \nabla F_i(\theta_i^{(t,k)}) - g_i(\theta_i^{(t,k)}, \tilde{\theta}^{(t,k+1)} - \tilde{\theta}(t,k)) \right) \right]
\text{ (since } E[g_i(\theta_i^{(t,k)})] = \nabla F_i(\theta_i^{(t,k)}) \text{ given } \tilde{\theta}(t,k), \theta^*)
\leq 3 \eta \cdot E \left[ \sum_{i=1}^{m} \lambda_i \| \nabla F_i(\theta_i^{(t,k)}) - g_i(\theta_i^{(t,k)}) \|^2 \right] + \frac{1}{6\eta} E \left[ \| \tilde{\theta}^{(t,k+1)} - \tilde{\theta}^{(t,k)} \|^2 \right]
\text{ (by Peter Paul inequality)}
\leq 2\eta \bar{\sigma}^2 \sum_{i=1}^{m} \lambda_i^2 + \frac{1}{6\eta} E \left[ \| \tilde{\theta}^{(t,k+1)} - \tilde{\theta}^{(t,k)} \|^2 \right].
\]
(18)
Plugging (18) back to the conditional expectation of (17), and noting that \( \eta \leq \frac{1}{3\bar{\sigma}} \), we have
\[
E \left[ F(\tilde{\theta}^{(t,k+1)}) - F(\theta^*) \right] + \frac{1}{2\eta} \left( E \left[ \| \tilde{\theta}^{(t,k+1)} - \theta^* \|^2 \right] - E \| \tilde{\theta}^{(t,k)} - \theta^* \|^2 \right)
\leq 2\eta \bar{\sigma}^2 \sum_{i=1}^{m} \lambda_i^2 - \left( \frac{1}{3\eta} - L \right) E \left[ \| \tilde{\theta}^{(t,k+1)} - \tilde{\theta}^{(t,k)} \|^2 \right] + L \sum_{i=1}^{m} \lambda_i \| \theta_i^{(t,k)} - \tilde{\theta}^{(t,k)} \|^2
\leq 2\eta \bar{\sigma}^2 \sum_{i=1}^{m} \lambda_i^2 + L \sum_{i=1}^{m} \lambda_i \| \theta_i^{(t,k)} - \tilde{\theta}^{(t,k)} \|^2
\]
By convexity of \( F \) and telescoping \( k \) from 0 to \( K - 1 \), we have
\[
\frac{1}{K} \sum_{k=0}^{K-1} E \left[ F(\tilde{\theta}^{(t,k)}) - F(\theta^*) \right] \leq 2\eta \bar{\sigma}^2 \sum_{i=1}^{m} \lambda_i^2 + L \sum_{i=1}^{m} \lambda_i \sum_{k=0}^{K-1} \frac{1}{K} E \left[ \| \theta_i^{(t,k)} - \tilde{\theta}^{(t,k)} \|^2 \right]
+ \frac{1}{2\eta K} \left( \| \tilde{\theta}^{(t,0)} - \theta^* \|^2 - E \left[ \| \tilde{\theta}^{(t,K)} - \theta^* \|^2 \right] \right).
\]
Since \( \tilde{\theta}^{(t,0)} = \theta^{(t)} \) and \( \tilde{\theta}^{(t,K)} = \theta^{(t+1)} \), we complete the proof. \( \square \)

**Lemma C.3 (Bounded client drift).** Assuming the client learning rate satisfies \( \eta \leq \frac{1}{3\bar{\sigma}} \), we have
\[
E \left[ \| \theta_i^{(t,k)} - \tilde{\theta}^{(t,k)} \|^2 \right] \leq \bar{\eta}^2 (24K^2\bar{\Omega}^2 + 20K\bar{\Omega}^2).
\]
where \( \bar{\Omega}^2 := \max \{ \bar{\sigma}^2, \Omega^2 \} \).

**Proof.**
\[
E \left[ \| \theta_i^{(t,k+1)} - \theta_2^{(t,k+1)} \|^2 \right] = E \left[ \| \theta_1^{(t,k)} - \theta_2^{(t,k)} - \eta \left( g_1(\theta_1^{(t,k)}) - g_2(\theta_1^{(t,k)}) \right) \|^2 \right]
= \| \theta_1^{(t,k)} - \theta_2^{(t,k)} \|^2 - 2\eta \left( \nabla F_2(\theta_1^{(t,k)}) - g_2(\theta_1^{(t,k)}) \right) \cdot \left( \nabla F_1(\theta_1^{(t,k)}) - \nabla F_2(\theta_1^{(t,k)}) \right)
- 2\eta \left( \nabla F_1(\theta_1^{(t,k)}) - \nabla F_2(\theta_2^{(t,k)}) \right) \cdot \left( \theta_1^{(t,k)} - \theta_2^{(t,k)} \right)
+ \eta^2 \| g_1(\theta_1^{(t,k)}) - g_2(\theta_2^{(t,k)}) \|^2.
\]
(19)
The second term (and similarly for the third term) is bounded as follows

\[- \left\langle g_1(\theta_1^{(t,k)}), \nabla F_1(\theta_1^{(t,k)}), \theta_1^{(t,k)} - \theta_2^{(t,k)} \right\rangle \]

\[\leq \frac{1}{6\eta K} \left\| \theta_1^{(t,k)} - \theta_2^{(t,k)} \right\|^2 + \frac{3\eta K}{2} \left\| g_1(\theta_1^{(t,k)}) - \nabla F_1(\theta_1^{(t,k)}) \right\|^2 \text{ (by Peter Paul inequality)}\]

\[= \frac{1}{6\eta K} \left\| \theta_1^{(t,k)} - \theta_2^{(t,k)} \right\|^2 + \frac{3\eta K}{2} \sigma^2 \text{ (by (14))}\]

Since \( \max, \sup_{\theta} \|\nabla F_1(\theta) - \nabla F(\theta)\| \leq \Omega \text{ (Assumption 3.5)} \), the 4th-term is bounded as

\[- \left\langle \nabla F_1(\theta_1^{(t,k)}), \nabla F_2(\theta_2^{(t,k)}), \theta_1^{(t,k)} - \theta_2^{(t,k)} \right\rangle \]

\[\leq - \left\langle \nabla F(\theta_1^{(t,k)}), \nabla F(\theta_2^{(t,k)}), \theta_1^{(t,k)} - \theta_2^{(t,k)} \right\rangle + 2\Omega \left\| \theta_1^{(t,k)} - \theta_2^{(t,k)} \right\| \]

\[\leq - \frac{1}{L} \left\| \nabla F(\theta_1^{(t,k)}) - \nabla F(\theta_2^{(t,k)}) \right\|^2 + 2\Omega \left\| \theta_1^{(t,k)} - \theta_2^{(t,k)} \right\|^2 \text{ (by smoothness and convexity)}\]

\[\leq - \frac{1}{L} \left\| \nabla F(\theta_1^{(t,k)}) - \nabla F(\theta_2^{(t,k)}) \right\|^2 + \frac{1}{6\eta K} \left\| \theta_1^{(t,k)} - \theta_2^{(t,k)} \right\|^2 + 6\eta K\Omega^2 \text{ (by Young’s inequality)}\]

The last term is bounded as follows

\[\left\| g_1(\theta_1^{(t,k)}) - g_2(\theta_2^{(t,k)}) \right\|^2 \]

\[\leq 5 \left( \left\| g_1(\theta_1^{(t,k)}) \right\|^2 + \left\| \nabla F_1(\theta_1^{(t,k)}) - \nabla F(\theta_1^{(t,k)}) \right\|^2 + \left\| \nabla F(\theta_1^{(t,k)}) - \nabla F(\theta_2^{(t,k)}) \right\|^2 \right) \]

\[+ \left\| \nabla F(\theta_2^{(t,k)}) \right\|^2 + \left\| g_2(\theta_2^{(t,k)}) - \nabla F_2(\theta_2^{(t,k)}) \right\|^2 \]

\[\leq 5 \left\| \nabla F(\theta_1^{(t,k)}) - \nabla F(\theta_2^{(t,k)}) \right\|^2 + 10(\tilde{\sigma}^2 + \Omega^2) \text{ (by (14) and Assumption 3.5)}\]

Substituting the above four bounds back to (19) gives (note that \( \eta \leq \frac{1}{4L} \))

\[E \left[ \left\| \theta_1^{(t+1)} - \theta_2^{(t+1)} \right\|^2 \right] \leq \left( 1 + \frac{1}{K} \right) \left\| \theta_1^{(t,k)} - \theta_2^{(t,k)} \right\|^2 - \frac{2}{L} \left\| \nabla F(\theta_1^{(t,k)}) - \nabla F(\theta_2^{(t,k)}) \right\|^2 \]

\[+ 6\eta^2 K\tilde{\sigma}^2 + 12\eta^2 K\Omega^2 + 10\eta^2 (\tilde{\sigma}^2 + \Omega^2) \]

\[\leq \left( 1 + \frac{1}{K} \right) \left\| \theta_1^{(t,k)} - \theta_2^{(t,k)} \right\|^2 + 6\eta^2 K\tilde{\sigma}^2 + 12\eta^2 K\Omega^2 + 10\eta^2 (\tilde{\sigma}^2 + \Omega^2). \]

Unrolling recursively, we obtain

\[E \left[ \left\| \theta_1^{(t+1)} - \theta_2^{(t+1)} \right\|^2 \right] \leq \left( 1 + \frac{1}{K} \right)^{K-1} \left[ \left\| \theta_1^{(t,k)} - \theta_2^{(t,k)} \right\|^2 + 6\eta^2 K\tilde{\sigma}^2 + 12\eta^2 K\Omega^2 + 10\eta^2 (\tilde{\sigma}^2 + \Omega^2) \right] \]

\[\leq 12\eta^2 K^2\tilde{\sigma}^2 + 24\eta^2 K^2\Omega^2 + 20\eta^2 K(\tilde{\sigma}^2 + \Omega^2) \]

\[\leq \eta^2 (24K^2\tilde{\sigma}^2 + 20K\Omega^2). \]

where we use the fact that \( \left( 1 + \frac{1}{K} \right)^{K-1} \leq K(e - 1) \leq 2K \), and \( \tilde{\Omega}^2 := \max \{ \tilde{\sigma}^2, \Omega^2 \} \).

By convexity, for any \( i \),

\[E \left[ \left\| \theta_1^{(t,k+1)} - \bar{\theta}^{(t,k+1)} \right\|^2 \right] \leq \eta^2 (24K^2\tilde{\Omega}^2 + 20K\bar{\Omega}^2). \]

Substituting the result of Lemma C.3 into Lemma C.2 and telescoping over \( t \), we obtain

\[E \left[ \frac{1}{T} \sum_{t=0}^{T-1} \frac{1}{K} \sum_{k=0}^{K-1} F(\bar{\theta}^{(t,k)}) - F(\theta^*) \right] \leq \frac{D^2}{2\eta KT} + 2\eta^2 \Lambda + \eta^2 L(24K^2\tilde{\Omega}^2 + 20K\bar{\Omega}^2), \]
where \( D := \| \theta^{(0)} - \theta^* \| \), \( \Lambda := \sum_{i=1}^{m} \lambda_i^2 \). By optimizing \( \eta \) on the R.H.S, we obtain
\[
\mathbb{E} \left[ \frac{1}{KT} \sum_{t=0}^{T-1} \sum_{k=0}^{K-1} F(\tilde{g}^{(t,k)}) - F(\theta^*) \right] \leq O \left( \frac{L D^2}{KT} + \frac{\bar{D} \Lambda^2}{\sqrt{KT}} + \frac{L^4 \bar{\Omega}^2 D^4}{K^2 T^4} + \frac{L^4 \bar{\Omega}^2 D^4}{T^4} \right),
\]
when
\[
\eta = \min \left\{ \frac{1}{3L}, \frac{D}{2\sqrt{KT} \Lambda^2}, \frac{D^3}{48 \bar{\Omega}^2 T^4 L^2 \bar{\Omega}^2}, \frac{D^3}{40 \bar{\Omega}^2 K T^2 L \bar{\Omega}^2} \right\}.
\]

### D Proof of Lemma 4.2

We first prove the following fact:

**Fact 1:**

(a) \( \mathcal{L}(Q, f) \leq \mathcal{L}_\rho^*(P_\lambda, f), \quad \forall f \in \mathcal{F}, Q \in \mathcal{B}(P_\lambda, \rho) \).

(b) \( \inf_{f' \in \mathcal{F}} \mathcal{L}(Q, f') \leq \inf_{f' \in \mathcal{F}} \mathcal{L}_\rho^*(P_\lambda, f'), \quad \forall Q \in \mathcal{B}(P_\lambda, \rho) \).

For (a), we have
\[
\mathcal{L}(Q, f) \leq \sup_{P' \in \mathcal{B}(P_\lambda, \rho)} \mathcal{L}(P', f) = \inf_{\gamma \geq 0} \left\{ \gamma \rho^2 + \mathbb{E}_{Z \sim P_\lambda} \left[ \phi_\gamma(Z, f) \right] \right\} 
\]
\[
\leq \gamma \rho^2 + \mathbb{E}_{Z \sim P_\lambda} \left[ \phi_\gamma(Z, f) \right] =: \mathcal{L}_\rho^*(P_\lambda, f),
\]
where the equality is due to strong duality result by Gao and Kleywegt [26].

For (b), defining \( f_{P_\lambda} := \arg \min_{f' \in \mathcal{F}} \mathcal{L}_\rho^*(P_\lambda, f') \), we have
\[
\inf_{f' \in \mathcal{F}} \mathcal{L}(Q, f') \leq \mathcal{L}(Q, f_{P_\lambda}) \leq \sup_{P' \in \mathcal{B}(P_\lambda, \rho)} \mathcal{L}(P', f_{P_\lambda}) \quad (20)
\]
\[
= \inf_{\gamma \geq 0} \left\{ \gamma \rho^2 + \mathbb{E}_{Z \sim P_\lambda} \left[ \phi_\gamma(Z, f_{P_\lambda}) \right] \right\} \quad (21)
\]
\[
\leq \gamma \rho^2 + \mathbb{E}_{Z \sim P_\lambda} \left[ \phi_\gamma(Z, f_{P_\lambda}) \right] \quad (22)
\]
\[
= \inf_{f' \in \mathcal{F}} \mathcal{L}_\rho^*(P_\lambda, f'). \quad (23)
\]

We next prove the second fact:

**Fact 2:**

(a) \( \mathcal{L}_\rho^*(P_\lambda, f) \leq \mathcal{L}(Q, f) + 2L_\rho^2 + |\gamma - \gamma^*| \rho^2, \quad \forall f \in \mathcal{F}, Q \in \mathcal{B}(P_\lambda, \rho) \).

(b) \( \inf_{f' \in \mathcal{F}} \mathcal{L}_\rho^*(P_\lambda, f') \leq \inf_{f' \in \mathcal{F}} \mathcal{L}(Q, f') + 2L_\rho^2 + |\gamma - \gamma^*| \rho^2. \)

For (a), we have:
\[
\mathcal{L}_\rho^*(P_\lambda, f) = \left\{ \sup_{P' \in \mathcal{B}(P_\lambda, \rho)} \mathcal{L}(P', f) \right\} + \left\{ \mathcal{L}_\rho^*(P_\lambda, f) - \sup_{P' \in \mathcal{B}(P_\lambda, \rho)} \mathcal{L}(P', f) \right\}
\]
\[
\leq \left\{ \mathcal{L}(Q, f) + 2L_\rho^2 \right\} + \left\{ \mathbb{E}_{Z \sim P_\lambda} \left[ \phi_\gamma(Z, f) \right] + \rho^2 \gamma - \min_{\gamma \geq 0} \left\{ \rho^2 \gamma + \mathbb{E}_{Z \sim P_\lambda} \left[ \phi_\gamma(Z, f) \right] \right\} \right\}
\]
\[
\leq \mathcal{L}(Q, f) + 2L_\rho^2 + \rho^2 (\gamma - \gamma^*) + \mathbb{E}_{Z \sim P_\lambda} \left[ \phi_\gamma(Z, f) - \phi_\gamma^*(Z, f) \right]
\]
\[
= \mathcal{L}(Q, f) + 2L_\rho^2 + \rho^2 (\gamma - \gamma^*) + \mathbb{E}_{Z \sim P_\lambda} \left[ \sup_{\zeta \in Z} \left\{ \ell(\zeta, h) - \gamma d(\zeta, Z) \right\} - \sup_{\zeta \in Z} \left\{ \ell(\zeta, h) - \gamma^* d^2(\zeta, Z) \right\} \right]
\]
\[
= \mathcal{L}(Q, f) + 2L_\rho^2 + (\gamma - \gamma^*) \left( \rho^2 - \mathbb{E}_{Z \sim P_\lambda} \left[ \sup_{\zeta \in Z} d^2(\zeta, Z) \right] \right)
\]
\[
\leq \mathcal{L}(Q, f) + 2L_\rho^2 + |\gamma - \gamma^*| \rho^2,
\]
where the first inequality is due to Proposition D.1 and the last inequality is because we choose \( \gamma \geq L_z/\rho \) and that fact that \( \gamma^* \leq L_z/\rho \) by Lemma 1 of Lee and Raginsky [37].

For (b), defining \( f_Q := \arg\min_{f \in \mathcal{F}} \mathcal{L}(Q, f) \), we have

\[
\inf_{f' \in \mathcal{F}} \mathcal{L}_\rho^\gamma(P_\lambda, f') \leq \mathcal{L}_\rho^\gamma(P_\lambda, f_Q) \\
\leq \mathcal{L}(Q, f_Q) + 2L_z\rho + |\gamma - \gamma^*|\rho^2 \tag{25} \\
= \inf_{f' \in \mathcal{F}} \mathcal{L}(Q, f') + 2L_z\rho + |\gamma - \gamma^*|\rho^2, \tag{26}
\]

where the second line is due to Fact 2(a).

Combining all facts, we complete the proof. Specifically, by adding two inequalities in Fact 1(a) and Fact 2(b), we obtain the upperbound of Lemma 4.2. Similarly, adding two inequalities in Fact 1(b) and Fact 2(a), we obtain the lowerbound of this lemma.

Finally, we provide the proof of the following proposition that was used in proving Fact 2(a).

**Proposition D.1.** Let Assumption 3.2 (a) holds. For any \( f \in \mathcal{F} \) and for all \( Q \in \mathcal{B}(P_\lambda, \rho) \), we have

\[
\sup_{P' \in \mathcal{B}(P_\lambda, \rho)} \mathcal{L}(P', f) \leq \mathcal{L}(Q, f) + 2L_z\rho.
\]

**Proof.** Denote \( P^* := \arg\max_{P' \in \mathcal{B}(P_\lambda, \rho)} \mathcal{L}(P', f) \). We have

\[
\sup_{P' \in \mathcal{B}(P_\lambda, \rho)} \mathcal{L}(P', f) = \mathcal{L}(Q, f) + \sup_{P' \in \mathcal{B}(P_\lambda, \rho)} \mathcal{L}(P', f) - \mathcal{L}(Q, f) \\
\leq \mathcal{L}(Q, f) + |\mathcal{L}(P^*, f) - \mathcal{L}(Q, f)|, \\
\leq \mathcal{L}(Q, f) + L_z [\mathbb{E}_{Z \sim P^*} \ell(Z, h)/L_z] - \mathbb{E}_{Z \sim Q} \ell(Z, h)/L_z] \\
\leq \mathcal{L}(Q, f) + L_z W_1(P^*, Q) \\
\leq \mathcal{L}(Q, f) + L_z [W_2(P^*, P_\lambda) + W_2(P_\lambda, Q)] \\
\leq \mathcal{L}(Q, f) + L_z 2\rho, \tag{27}
\]

where the fourth line is due to the Kantorovich-Rubinstein dual representation theorem, i.e.,

\[
W_1(P, Q) = \sup_h \{ \mathbb{E}_{Z \sim P} [h(Z)] - \mathbb{E}_{Z \sim Q} [h(Z)] : h(\cdot) \text{ is 1-Lipschitz} \}
\]

and the fifth line is due to \( W_1(P^*, Q) \leq W_2(P^*, Q) \) and triangle inequality. \hfill \qed

**E Proof of Theorem 4.4**

**Proof.** To simplify notation, we denote \( \Phi := \phi_\gamma \circ \mathcal{F} = \{ z \mapsto \phi_\gamma(z, f) : f \in \mathcal{F} \} \) where \( \mathcal{F} = \{ f_\theta, \theta \in \Theta \subset \mathbb{R}^d \} \), which represents the composition of \( \phi_\gamma \) with each of the loss function \( f_\theta \) parametrized by \( \theta \) belonging to the parameter class \( \Theta \).

Defining \( f_{P_\lambda} \in \arg\min_{f \in \mathcal{F}} \mathcal{L}_\rho^\gamma(P_\lambda, f) \) and \( \hat{\theta}^* \in \arg\min_{\theta \in \Theta} \mathbb{E}_{Z \sim \hat{P}_\lambda} [\phi_\gamma(Z, f_\theta)] \) such that \( \mathcal{L}_\rho^\gamma(\hat{P}_\lambda, f_{\hat{\theta}^*}) = \inf_{\theta \in \Theta} \left[ \mathbb{E}_{Z \sim \hat{P}_\lambda} [\phi_\gamma(Z, f_\theta)] + \gamma \rho^2 \right] \), we decompose the excess risk as follows:
where the first inequality is due to optimization error and definition of $\hat{\theta}^*$. The second inequality is due to the fact that $|\sum_{i=1}^m 1_i a_i| \leq \sum_{i=1}^m 1_i |a_i|$, $\forall a_i \in \mathbb{R}$ and $\lambda_i \geq 0$. The third inequality is because pushing the sup inside increases the value. For the last inequality, using the facts that (i) $|\sigma_i(z,f)| \leq M_k$ due to $-M_k \leq \ell(z,h) \leq \phi_i(z,f) \leq \sup_{z,h \in Z} \ell(z,h) \leq M_k$ and (ii) the Rademacher complexity of the function class $\Phi$ defined by $\mathcal{R}_i(\Phi) = \mathbb{E}[\sup_{\phi_i \in \Phi} \frac{1}{n_i} \sum_{k=1}^{n_i} \sigma_k \phi_i(Z_k, f_0)]$ where the expectation is w.r.t both $Z_k \overset{i.i.d.}{\sim} P_i$ and i.i.d. Rademacher random variable $\sigma_k$ independent of $Z_k$, $\forall k \in [n]$, we have

$$\sup_{\phi_i \in \Phi} |\mathbb{E}_{Z_i \sim P_i} [\phi_i(Z_i, f_0)] - \mathbb{E}_{Z_i \sim \hat{P}_i} [\phi_i(Z_i, f_0)]| \geq 2\mathcal{R}_i(\Phi) + M_k \sqrt{\frac{2 \log(2m/\delta)}{n_i}}$$

(29)

with probability $\leq \delta / m$ due to the standard symmetrization argument and McDiarmid’s inequality [48, Theorem 26.5].

Multiplying $\lambda_i$ to both sides of (29), summing up the inequalities over all $i \in [n]$, and using union bound, we obtain (28).

Define a stochastic process $(X_{\phi_i})_{\phi_i \in \Phi}$

$$X_{\phi_i} := \frac{1}{\sqrt{n_i}} \sum_{k=1}^{n_i} \sigma_k \phi_i(Z_k, f_0)$$

which is zero-mean because $\mathbb{E} [X_{\phi_i}] = 0$ for all $\phi_i \in \Phi$. To upper-bound $\mathcal{R}_n(\Phi)$, we first show that $(X_{\phi_i})_{\phi_i \in \Phi}$ is a sub-Gaussian process with respect to the following pseudometric

$$\|\phi_i - \phi'_i\|_\infty := \sup_{z \in Z} |\phi_i(z, f_0) - \phi'_i(z, f_0)|.$$  

(30)

For any $t \in \mathbb{R}$, using Höfdding inequality with the fact that $\sigma_k, k \in [n]$, are i.i.d. bounded random variable with sub-Gaussian parameter 1, we have

$$\mathbb{E} \left[ \exp \left( t \left( X_{\phi_i} - X_{\phi'_i} \right) \right) \right] = \mathbb{E} \left[ \exp \left( \frac{t}{\sqrt{n_i}} \sum_{k=1}^{n_i} \sigma_k (\phi_i(Z_k, f_0) - \phi(Z_k, f_0)) \right) \right]$$

$$= \left( \mathbb{E} \left[ \exp \left( \frac{t}{\sqrt{n_i}} \sigma_1 (\phi_i(Z_1, f_0) - \phi(Z_1, f_0)) \right) \right] \right)^{n_i} \leq \exp \left( \frac{t^2 \|\phi_i - \phi'_i\|_\infty^2}{2} \right).$$

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Then, invoking Dudley entropy integral, we have
\[ \sqrt{m_n} \mathcal{R}_l(\Phi) = \mathbb{E} \sup_{\phi_\gamma \in \Phi} X_{\phi_\gamma} \leq 12 \int_0^\infty \sqrt{\log \mathcal{N}(\Theta, \| \cdot \|_\infty, \epsilon)} d\epsilon \] (31)

We will show that when \( \theta \mapsto \ell(z, h_\theta) \) is \( L_\theta \)-Lipschitz by Assumption 3.2, then \( \theta \mapsto \phi_\gamma(z, f_\theta) \) is also \( L_\theta \)-Lipschitz as follows.
\[
\left| \phi_\gamma(z, f_\theta) - \phi_\gamma(z, f_{\theta'}) \right| = \left| \sup_{\zeta \in Z} \inf_{\zeta' \in Z} \left\{ \ell(\zeta, h_\theta) - \gamma d(\zeta, z) - \ell(\zeta', h_{\theta'}) + \gamma d(\zeta', z) \right\} \right|
\leq \left| \sup_{\zeta \in Z} \left\{ \ell(\zeta, h_{\theta'}) - \ell(\zeta, h_{\theta}) \right\} \right|
\leq \sup_{\zeta \in Z} \left| \ell(\zeta, h_{\theta}) - \ell(\zeta, h_{\theta'}) \right|
\leq L_\theta \| \theta - \theta' \|
\]
which implies
\[ \| \phi_{\gamma} - \phi'_{\gamma} \|_{\infty} \leq L_\theta \| \theta - \theta' \|. \]

Therefore, by contraction principle [48], we have
\[ \mathcal{N}(\Phi, \| \cdot \|_\infty, \epsilon) \leq \mathcal{N}(\Theta, \| \cdot \|, \epsilon/L_\theta). \] (32)

Substituting (32) and (31) into (28), we obtain
\[ \mathcal{E}_\rho^\gamma(P_\lambda, f_{\bar{\theta}_n}) \leq \sum_{i=1}^m \lambda_i \left[ \frac{48\mathbb{E}(\Theta)/\sqrt{n_i}}{n_i} + 2Ml \frac{2\log(2m/\delta)}{n_i} \right] + \epsilon, \] (33)
which will be substituted into the upper-bound in Lemma 4.2 to complete the proof.

\[ \square \]

F Proof of Corollary 4.5

We now present how we adapt the result from Fournier and Guillin [30] to prove Corollary 4.5.

Proposition F.1 (Measure concentration [30, Theorem 2]). Let \( P \) be a probability distribution on a bounded set \( Z \). Let \( \hat{P}_n \) denote the empirical distribution of \( Z_1, \ldots, Z_n \overset{i.i.d}{\sim} P \). Assuming that there exist constants \( a > 1 \) such that \( A := \mathbb{E}_{Z \sim P} \exp(\|Z\|^a) \) is finite (i.e., \( P \) is a light-tail distribution). Then, for any \( \rho > 0 \),
\[ P \left[ W_p(\hat{P}_n, P) \geq \rho \right] \leq \begin{cases} c_1 \exp \left( -c_2 n \rho^{\max\{d/p, 2\}} \right) & \text{if } \rho \leq 1 \\ c_1 \exp \left( -c_2 n \rho^a \right) & \text{if } \rho > 1 \end{cases} \]
where \( c_1, c_2 \) are constants depending on \( a, A \) and \( d \).

As a consequence of this proposition, for any \( \delta > 0 \), we have
\[ P \left[ W_2(\hat{P}_n, P) \leq \rho_n^\delta \right] \geq 1 - \delta \] where \( \rho_n^\delta := \begin{cases} \left( \frac{\log(c_1/\delta)}{c_2 n} \right)^{\min\{2/d, 1/2\}} & \text{if } n \geq \frac{\log(c_1/\delta)}{c_2} \\ \left( \frac{\log(c_1/\delta)}{c_2 n} \right)^{1/\alpha} & \text{if } n < \frac{\log(c_1/\delta)}{c_2} \end{cases} \] (34)

In Proposition F.1, Fournier and Guillin [30] show that the empirical distribution \( \hat{P}_n \) converges in Wasserstein distance to the true \( P \) at a specific rate. This implies that judiciously scaling the radius of Wasserstein balls according to (34) provides natural confidence regions for the data-generating distribution \( P \).

By the duality of transport cost [49, p.261], we have
\[ W_p(\mu, \nu) = \sup_{\varphi(x)+\psi(y) \leq d_\nu(x, y)} \int \varphi d\mu + \psi d\nu = \sup_{\varphi(x)+\psi(y) \leq d_\nu(x, y)} T_f(\mu, \nu), \forall p \geq 1, \]
which is the supremum of linear functionals \( T_f : \mathcal{P} \times \mathcal{P} \to \mathbb{R} \) defined by \( T_f(\mu, \nu) = \langle (\mu, \nu), (\varphi, \psi) \rangle \); therefore, \( (\mu, \nu) \mapsto W_p^p(\mu, \nu) \) is convex, \( \forall p \geq 1 \). Thus we have

\[
W_2^2(P, \hat{P}_\lambda) = W_2^2\left(\sum_{i=1}^{m} \lambda_i (\hat{P}_{n_i}, P_i)\right) \leq \sum_{i=1}^{m} \lambda_i W_2^2(\hat{P}_{n_i}, P_i).
\]  

(35)

Then, we have

\[
P\left[W_2(P, \hat{P}_\lambda) \geq \sqrt{\sum_{i=1}^{m} \lambda_i \hat{\rho}_{n_i}^{\delta/m}} \right] = P\left[W_2^2(P, \hat{P}_\lambda) \geq \sum_{i=1}^{m} \lambda_i \hat{\rho}_{n_i}^{\delta/m} \right] \\
\leq P\left[\sum_{i=1}^{m} \lambda_i W_2^2(\hat{P}_{n_i}, P_i) \geq \sum_{i=1}^{m} \lambda_i \hat{\rho}_{n_i}^{\delta/m} \right] \\
\leq \sum_{i=1}^{m} P\left[W_2^2(\hat{P}_{n_i}, P_i) \geq \hat{\rho}_{n_i}^{\delta/m} \right] \\
= \sum_{i=1}^{m} P\left[W_2(\hat{P}_{n_i}, P_i) \geq \hat{\rho}_{n_i}^{\delta/2m} \right] \\
\leq \sum_{i=1}^{m} \frac{\delta}{2m} = \frac{\delta}{2},
\]

(36)

where the first inequality is due to [35], the second inequality is due to the union bound, and the last inequality is due to Proposition F.1 and (34).

According to (27), by setting \( \rho = \left(\sum_{i=1}^{m} \lambda_i \hat{\rho}_{n_i}^{\delta/m}\right)^{1/2} \) in Theorem 4.4 and using union bound, we complete the proof.

G Additional Experimental Settings And Results

G.1 Datasets

| Dataset | \( m \) | Total samples | Num labels / client | Samples / client | \( \text{Mean} \) | Std |
|---------|------|---------------|---------------------|-----------------|----------------|-----|
| CIFAR-10 | 20   | 43,098        | 3                   | 2154            | 593.8          |     |
| MNIST   | 100  | 70,000        | 2                   | 700             | 313.4          |     |

For robustness-related experiments, we distribute all datasets to clients as follows:

- **MNIST**: A handwritten digit dataset [41] including 70,000 instances belonged to 10 classes. We distribute dataset to \( m = 100 \) clients and each client has a different local data size with only 2 of the 10 classes.

- **CIFAR-10**: An object recognition dataset [42] including 60,000 colored images belonged to 10 classes. We partition the dataset to \( m = 20 \) clients and there are 3 labels per client. Each client has a different local data size.

We standardize and randomly split all datasets with 75% and 25% for training and testing, respectively. The statistics of all datasets are summarized in Table 2.

For domain adaptation experiments, we use a set of three digit recognition datasets including MNIST [41], USPS [43], and SVHN [44]. In particular, we choose two datasets for training and one for testing. The statistics of all datasets are summarized in Table 3.
Table 3: Statistics of all datasets using in the domain adaptation experiments.

| Dataset   | Original Size | Total samples | Num labels / client | Samples / client Training | Testing |
|-----------|---------------|---------------|---------------------|---------------------------|---------|
| MNIST     | 28x28         | 70,000        | 10                  | 60,000                    | 10,000  |
| USPS      | 16x16         | 9,298         | 10                  | 7,291                     | 2,007   |
| SVHN      | 32x32         | 89,289        | 10                  | 63,257                    | 26,032  |

G.2 Models

The details of models for each dataset is provided as follows:

- **MNIST**: We use a multinomial logistic regression model (MLR) with a cross-entropy loss function and an $L_2$-regularization term.
- **CIFAR-10**: We use a CNN model employed in McMahan et al. [2].
- **Three digit recognition datasets (MNIST, USPS, SVHN)**: We use a multinomial logistic regression model (MLR) with a cross-entropy loss function and an $L_2$-regularization term.

In all settings, we set the number of local epochs to $K = 2$ and the number of communication rounds to $T = 200$. For domain adaptation applications, we assign one source domain to one client. For other experiments, we randomly sample 10 clients to participate in training the global robust model in each communication round. All experiments were conducted using PyTorch [50].

G.3 Comparison between WAFL (with $p = 1$ and $p = 2$) and other methods on MNIST

![Figure 5: Comparison between WAFL (with $p = 1$ and $p = 2$) and other methods on MNIST.](image)

In an additional experiment, we train WAFL using $p = 1$. The duality result in (5) requires only that the distance metric $d$ continuous and convex in its first argument [28]. Therefore, any $\ell_p$ norm would suffice. The use of the $\ell_2$ norm ensures that $d$ is 1-strongly convex, implying that solving for $\phi_\gamma$ enjoys linear convergence. As depicted in Fig. 5, WAFL’s performance when $p = 1$ is close but not as good as when $p = 2$.

G.4 Convergence of WAFL

We verify the convergence of WAFL under two cases: clean data (no attacked clients) and distribution shifts (where 40% of clients are attacked). In each case, we use two datasets: MNIST and CIFAR-10 and employ the same setup as in Sec. 6. Specifically, for MNIST, we distribute the dataset to 100 clients and set $\gamma = 0.05$. For CIFAR-10, we use 20 clients and set $\gamma = 0.5$. We use $T = 200$ communication iterations.

To show WAFL’s convergence, we plot both the original loss (using the function $\ell$) and global accuracy in Fig. 6.
Figure 6: Convergence of WAFL.