Jordan Matsuo algebras over fields of characteristic 3

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Abstract

The Matsuo algebra associated with a connected Fischer space is shown to be a Jordan algebra over a field of characteristic 3 if and only if the Fischer space is isomorphic to either the affine space of order 3 or the Fischer space associated with the symmetric group. The proof uses a characterization of the affine spaces of order 3 and equivalence of Jordan and linearized Jordan identities over a field of characteristic 3 in case the algebra is spanned by idempotents.

1 Introduction

From any partial triple system \((\mathcal{P}, \mathcal{L})\), a family of commutative nonassociative algebras parametrized by one parameter \(\delta\) was constructed in \[\text{Mat03}\] (cf. \[\text{Mat05}\]). Such algebras over a field \(\mathbb{F}\) are called the Matsuo algebras associated with \((\mathcal{P}, \mathcal{L})\) and denoted by \(M((\mathcal{P}, \mathcal{L}), \delta, \mathbb{F})\), where \(\delta \in \mathbb{F}\). If \((\mathcal{P}, \mathcal{L})\) belongs to Fischer spaces, the geometric counterparts to 3-transposition groups, and if \(2\delta \neq 0, 1\), then the associated Matsuo algebra becomes an axial algebra of Jordan type \(2\delta\) \([\text{HRS15}]\).

Recently, T. De Medts and F. Rehren considered the problem of classifying Jordan Matsuo algebras associated with Fischer spaces in \[\text{DR17}\]. Note that the parameter \(\delta\) must be \(\frac{1}{4}\) if the associated Matsuo algebra is Jordan.

The purpose of this paper is to investigate this classification over fields of characteristic 3. Our main result is the following theorem.

**Theorem.** Let \((\mathcal{P}, \mathcal{L})\) be a connected Fischer space of rank \(n\) and \(\mathbb{F}\) a field of characteristic 3. Then the associated Matsuo algebra \(M((\mathcal{P}, \mathcal{L}), \frac{1}{4}, \mathbb{F})\) is Jordan if and only if \((\mathcal{P}, \mathcal{L})\) is isomorphic to either the Fischer space associated with the symmetric group \((\text{Sym}(n+1), (1,2)^{\text{Sym}(n+1)})\) or the \((n-1)\)-dimensional affine space \(\text{AG}(n-1, 3)\) of order three.

By combining this result and the argument of \[\text{DR17}\], the classification of finite-dimensional Jordan Matsuo algebras associated with Fischer spaces over all fields of characteristic not 2 is completed. (The second case in Theorem had been erroneously omitted in \[\text{DR17}\].)

\[^{1}\]The factor \(\frac{1}{16}\) mentioned in page 339 of \[\text{DR17}\] is in fact \(\frac{1}{4}\). As it is 0 in a field of characteristic 3, one cannot conclude that the algebra considered there is not Jordan over a field of characteristic 3, and it is indeed Jordan as we will see in the present paper.
Section 2 recalls the definitions of 3-transposition groups, Fischer spaces and Matsuo algebras. Section 3 gives a characterization of affine spaces of order three by means of their automorphisms. This characterization is used in section 5. Section 4 shows that the Matsuo algebras associated with the affine spaces of order 3 is Jordan over a field of characteristic 3. In Section 5, the proof of Theorem is completed.

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2 3-transposition groups, Fischer spaces and Matsuo algebras

A 3-transposition group is a pair \((G, D)\) where \(G\) is a group and \(D\) is a normal subset of \(G\) satisfying \(|g| = 2\) and \(|gh| \leq 3\) for all \(g, h \in D\). The rank of \((G, D)\) is the minimum size of subsets of \(D\) generating \((G, D)\) and denoted by \(r(G, D)\). If \(r(G, D) < \infty\), then \(G\) is finite.

Let \(S\) be a subset of a group \(G\). The noncommuting graph of \(S\) is a graph \(S\) with vertices \(S\) and edges \(\{g, h\}\) for \(g, h \in S\) with \(gh \neq hg\). A 3-transposition group \((G, D)\) is said to be connected if the noncommuting graph of \(D\) is connected. In this paper, for a group \(G\), \(g \in G\) and \(S \subset G\), \(g^G\) and \(S^G\) denote \(\{g^h : h \in G\}\) and \(\bigcup_{g \in S} g^G\) respectively.

For the symmetric group \(\text{Sym}(n)\), the pair \((\text{Sym}(n), (1,2)^{\text{Sym}(n)})\) is a 3-transposition group of rank \(n-1\). Set \(\tilde{G} = \langle \tilde{g}_1, \tilde{g}_2, \tilde{g}_3, \tilde{g}_4 \mid \tilde{g}_i^2 = 1, 1 \leq i \leq 4 \rangle\) and \(\tilde{D} = \{\tilde{g}_1, \tilde{g}_2, \tilde{g}_3, \tilde{g}_4\}^G\). Then the pair \((G, D)\) with \(G = \tilde{G}/\langle g^h(h^g)^{-1} \mid g, h \in \tilde{D} \rangle\) and \(D = \{g_1, g_2, g_3, g_4\}^G\) is a 3-transposition group of rank 4, where \(g_i\) is the image of \(\tilde{g}_i\). This group \(G\) is called \(M.\ Hall’s group\ 3^{10} : 2\) (this group was defined in [Hal62]).

A partial triple system is a pair \((\mathcal{P}, \mathcal{L})\) where \(\mathcal{P}\) is a set and \(\mathcal{L}\) is a set of 3-subsets of \(\mathcal{P}\) such that \(|l_1 \cap l_2| \leq 1\) for all distinct \(l_1, l_2 \in \mathcal{L}\). We call an element of \(\mathcal{P}\) a point and an element of \(\mathcal{L}\) a line. A subsystem of \((\mathcal{P}, \mathcal{L})\) is a partial triple system \((\mathcal{P}', \mathcal{L}')\) with \(\mathcal{P}' \subset \mathcal{P}\) and \(\mathcal{L} \supset \mathcal{L}' \supset \{l \in \mathcal{L} \mid |l \cap \mathcal{P}'| \geq 2\}\). A subsystem of \((\mathcal{P}, \mathcal{L})\) is said to be generated by a subset \(S\) of \(\mathcal{P}\) if it is the minimum subsystem whose point set contains \(S\). A plane of a partial triple system \((\mathcal{P}, \mathcal{L})\) is a subsystem of \((\mathcal{P}, \mathcal{L})\) generated by \(l_1 \cup l_2\) for a pair \((l_1, l_2)\) of distinct and intersecting lines. The rank of \((\mathcal{P}, \mathcal{L})\) is the minimum size of subsets of \(\mathcal{P}\) generating \((\mathcal{P}, \mathcal{L})\) itself. The rank of \((\mathcal{P}, \mathcal{L})\) is denoted by \(r(\mathcal{P}, \mathcal{L})\). Two partial triple systems are said to be isomorphic if a bijection between the point sets induces a bijection between the line sets.
Two distinct points \(p\) and \(q\) are said to be \textit{collinear} if both \(p\) and \(q\) are contained in a common line. Let \(\sim\) be the equivalence relation on \(P\) generated by \(p \sim q\) for two collinear points \(p\) and \(q\). For a \(\sim\)-equivalence class \(P'\) of \(P\), we call the subsystem generated by \(P'\) a \textit{connected component} of \((P, L)\). The point set of a connected component of \((P, L)\) is a \(\sim\)-equivalence class of \(P\). \((P, L)\) is said to be \textit{connected} if it has a unique connected component.

We recall the two important examples of partial triple systems. The \textit{dual affine plane} \(DA(2, 2)\) of order two is the partial triple system obtained by discarding a point and all lines including the point from the projective plane of order two. Thus for the point set \(\{p, q, r, s, t, u\}\), we may take the line set to be \(\{\{p, q, r\}, \{p, t, u\}, \{q, r, t\}, \{q, s, u\}\}\). The second example is the \(n\)-dimensional \textit{affine space} \(AG(n, 3)\) of order three. The points are the vector of \(\mathbb{F}_3^n\) and the lines are the 1-flats of \(\mathbb{F}_3^n\). Thus for the point set \(\mathbb{F}_3^n\), we may take the line set to be \(\{\{x, y, -x - y\} \mid x, y \in \mathbb{F}_3^n, x \neq y\}\).

A \textit{Fischer space} is a partial triple system whose planes are isomorphic to either \(DA(2, 2)\) or \(AG(2, 3)\). A Fischer space is said to be \textit{of affine type} if its planes are isomorphic to \(AG(2, 3)\).

Let \((G, D)\) be a 3-transposition group. The pair \((P, L)\) with \(P = D\) and \(L = \{\{c, d, c'^d\} \mid c, d \in D, |cd| = 3\}\) is a Fischer space. We call this Fischer space the Fischer space associated with \((G, D)\) and denote it by \(FS(G, D)\). Let \(FSS_n\) denote \(FS(Sym(n), (1, 2)^{Sym(n)})\). Note that \(r(\text{FS}(G, D)) = r(G, D)\) and \(FS(G, D)\) is connected if and only if \((G, D)\) is connected.

Let \((P, L)\) be a Fischer space. For each \(p \in P\), the \textit{associated transposition} \(\tau(p)\) is the automorphism of \((P, L)\) which fixes the points not collinear with \(p\) and switches \(q\) and \(r\) if \(\{p, q, r\} \in L\). Note that the mapping \(\tau\) from \(P\) to the automorphism group of \((P, L)\) satisfies the following properties:

\begin{itemize}
  \item \(\tau(p)^2 = \text{id}\) for all \(p \in P\).
  \item \(\tau(p)\tau(q) = \tau(q)\tau(p)\) if \(p\) and \(q\) are not collinear.
  \item \(\tau(p)\tau(q) = \tau(q)\tau(p)\) if \(p\) and \(q\) are collinear.
  \item \(\tau(p^{\tau(q)}) = \tau(p)\tau(q)\) for all \(p, q \in P\), where \(p^f = f(p)\) for \(p \in P\) and an automorphism \(f\) of \((P, L)\).
\end{itemize}

The pair \((G, D)\) with \(G = \langle \tau(p) \mid p \in P \rangle\) and \(D = \{\tau(p) \mid p \in P\}\) is a 3-transposition group. If \((P, L)\) is connected, then the associated Fischer space \(FS(G, D)\) is isomorphic to \((P, L)\). So, for a Fischer space \((P, L)\), there exists a 3-transposition group \((G, D)\) such that \((P, L)\) is isomorphic to \(FS(G, D)\). For example, \(AG(n, 3)\) is isomorphic to \(FS(\mathbb{F}_3^n \times \mathbb{F}_3^n, \{(v, -1) \mid v \in \mathbb{F}_3^n\})\), where the action of \(\mathbb{F}_3^n\) is the scalar multiplication of \(\mathbb{F}_3^n\).
Let \((P, L)\) be a partial triple system and \(F\) a field with \(\text{ch}(F) \neq 2\). Choose a constant \(\delta \in F\). Then, the Matsuo algebra \(M((P, L), \delta, F)\) associated with \((P, L)\) is the \(F\)-space \(\bigoplus_{p \in P} Fa(p)\) with the multiplication given as follows:

(i) \(a(p)a(p) = a(p)\),

(ii) \(a(p)a(q) = 0\) if \(p\) and \(q\) are not collinear,

(iii) \(a(p)a(q) = \delta(a(p) + a(q) - a(r))\) if \(p\) and \(q\) are collinear and \(\{p, q, r\}\) is the common line.

In [HRS15], it is proved that \(M((P, L), \delta, F)\) is a primitive axial algebra of Jordan type \(2\delta\) if \((P, L)\) is a Fischer space and \(\delta \neq 0, \frac{1}{2}\).

Let \(\{(P_i, L_i)\}_{i \in I}\) be the set of connected components of a Fischer space \((P, L)\). Then

\[
M((P, L), \frac{1}{4}, F) \cong \bigoplus_{i \in I} M((P_i, L_i), \frac{1}{4}, F).
\]

From now on, we assume that Fischer spaces are connected.

3 The structure of affine spaces as Fischer spaces

In this section, we give a characterization of affine spaces using the mapping \(\tau\) defined for each Fischer space in the preceding section.

First, let us give an alternative construction of \(AG(n - 1, 3)\). Let \(I_n = \{1, 2, \ldots, n\}\), \(R_n^{(k)}\) be the set of the sequences \((i_{-1}, i_1, \ldots, i_k)\) of the elements of \(I_n\) and \(R_n = \bigcup_{k=0}^{n} R_n^{(k)}\). If a sequence \(x \in R_n\) is in \(R_n^{(k)}\), we say that the length of \(x\) is \(k\). Let \(\rho : R_n \to \text{Map}(R_n, R_n)\) be the mapping defined by setting

\[
\rho(j_{-1}, j_1, \ldots, j_k)(i_{-1}, i_1, \ldots, i_m) = (i_{-1}, i_1, \ldots, i_m, j_k, \ldots, j_1, j_{-1}, j_1, \ldots, j_k)
\]

for \((j_{-1}, j_1, \ldots, j_k), (i_{-1}, i_1, \ldots, i_m) \in R_n\).

Let \(\approx\) be the equivalence relation on \(R_n\) generated by the following elementary equivalences:

- \(\rho(x_k) \cdots \rho(x_1)(x_1) \approx \rho(x_k) \cdots \rho(x_2)(x_1)\) for \(k \geq 1\) and \(x_1, \ldots, x_k \in R_n\).
- \(\rho(x_k) \cdots \rho(x_2)\rho(x_2)(x_1) \approx \rho(x_k) \cdots \rho(x_3)(x_1)\) for \(k \geq 2\) and \(x_1, \ldots, x_k \in R_n\).
- \(\rho(x_k) \cdots \rho(x_2)(x_1) \approx \rho(x_k) \cdots \rho(x_3)\rho(x_1)(x_2)\) for \(k \geq 2\) and \(x_1, \ldots, x_k \in R_n\).
- \(\rho(x_k) \cdots \rho(x_2)(x_1) \approx \rho(x_k) \cdots \rho(x_3)\rho(x_2)\rho(x_3)\rho(x_4)(x_1)\) for \(k \geq 4\) and \(x_1, \ldots, x_k \in R_n\).
Let $Q_n = R_n/\approx$, $f$ the natural surjection from $R_n$ to $Q_n$ and $q_i = f(i)$ for $i \in I_n$. Then the mapping $\rho$ induces a mapping $\sigma: Q_n \to \text{Map}(Q_n, Q_n)$ because $\rho(y)(x) \approx \rho(x)(y) \approx \rho(x)(z) \approx \rho(z)(x)$ if $y \approx z$. Set $M_n = \{(p, q, p^{\sigma(q)}) \mid p, q \in Q_n, p \neq q\}$, where $p^{\sigma(q)} = \sigma(q)(p)$. Then, the pair $(Q_n, M_n)$ is a partial triple system.

**Proposition 3.1.** $(Q_n, M_n)$ is isomorphic to $AG(n-1, 3)$.

In order to show this proposition, let us consider the set $S_n$ of sequences $(i_1, i_2, \ldots, i_m) \in R_n$ satisfying the following conditions:

- $|\{j \in \{-1, 1, \ldots, m\} \mid i_j = i\}| \leq 2$ for all $i \in I_n$,
- $\forall j \in \{-1, 1, \ldots, m - 2\}, i_j \leq i_{j+2}$,
- $\forall k \in I_n, |\{j \in \{-1, 1, \ldots, m\} \mid i_j = k\}| \leq 2$,
- $i_j = i_k \Rightarrow j - k = \pm 2, j \in 2\mathbb{Z}$ and $i_j > i_l$ for all $l \in 2\mathbb{Z} + 1$.

**Lemma 3.2.** The restriction $f|_{S_n}: S_n \to Q_n$ is surjective.

**Proof.** First note the following invariance of the surjection $f: R_n \to Q_n$:

1. $f(i_{-1}, \ldots, i_j, i_{j+1}, i_{j+2}, \ldots, i_m) = f(i_{-1}, \ldots, i_{j+2}, i_{j+1}, i_j, \ldots, i_m)$ for all $j \in \{-1, 1, \ldots, m - 2\}$.
2. $f(i_{-1}, i_1 = i_{-1}, i_2, \ldots, i_m) = f(i_{-1}, i_2, \ldots, i_m)$.
3. $f(i_{-1}, \ldots, i_j, i_{j+1} = i_j, \ldots, i_m) = f(i_{-1}, \ldots, i_{j-1}, i_{j+2}, \ldots, i_m)$, for all $j \in \{1, 2, \ldots, m - 1\}$.
4. $f(i_{-1}, \ldots, i_j, i_{j+1}, i_{j+2} = i_j, \ldots, i_m) = f(i_{-1}, \ldots, i_{j+1}, i_{j}, i_{j+1}, \ldots, i_m)$ for all $j \in \{1, 2, \ldots, m - 2\}$.

Let $x \in Q_n$ and $(i_{-1}, i_1, \ldots, i_m) \in R_n$ be a sequence satisfying $f(i_{-1}, i_1, \ldots, i_m) = x$. Firstly, if $i_j = i_k = i = s$ and $j, k$ and $l$ are distinct, then by the invariance above, there exists a shorter sequence $y$ such that $f(y) = x$. Hence we may assume $|\{j \mid i_j = s\}| \leq 2$ for all $s \in I_n$. Secondly, suppose $i_j = i_k = s$ and $j$ or $k$ is odd. Then, we can reduce $|\{r \mid i_r = s\}|$ by (1), (2), and (3). So, we may assume $i_j = i_k, j \neq k \Rightarrow j, k \in 2\mathbb{Z}$. Thirdly, by (1) and (4), if $i_j = i_k = s, i_l = t$ and $l$ is odd, then the value of $f$ does not change if we make $i_j = i_k = t$ and $i_l = s$. So we may assume $i_j > i_l$ for all $l \in 2\mathbb{Z} + 1$ if there exists $k$ such that $i_j = i_k$ and $j \neq k$. Finally, by (1), we may assume $\forall j \in \{-1, 1, \ldots, m - 2\}, i_j \leq i_{j+2}$. Therefore, we may assume the sequence $(i_{-1}, i_1, \ldots, i_m)$ is an element of $S_n$. So, for all $x \in Q_n$, there exist an element $y$ of $S_n$ such that $x = f(y)$. Therefore $f|_{S_n}$ is surjective.

**Lemma 3.3.** $|S_n| = 3^{n-1}$. 

Proof. Set $T_k = \{(i_1, i_1, \ldots, i_m) \in S_n \mid \{i_1, i_1, \ldots, i_m\} = \{1, \ldots, k\}\}$ for $1 \leq k \leq n$. For $x = (i_1, i_1, \ldots, i_m) \in S_n$, set $s(x) = \{|j \mid i_j = i_{j+2}\}$ and $t(x) = \{|j \in 2Z \mid i_j \neq i_{j+2}\}$. Then, for $x \in T_k$, $2s(x) + t(x) = m' = k - s(x) - t(x) - 1$ if $m = 2m'$ and $2s(x) + t(x) = m' = k - s(x) - t(x) - 2$ if $m = 2m' + 1$. Since $x \in T_k$ is determined by $\{i_{2j} \mid i_{2j} \neq i_{2j+2}\}$,

$$|T_k| = \sum_{0 \leq t \leq k \atop 0 \leq k \neq 3Z} \binom{k}{t}$$

$$= \frac{1}{2} \sum_{0 \leq t \leq k, k+t \in 3Z} \binom{k}{t}$$

$$= \frac{1}{2} \left( \sum_{t=0}^{k} \binom{k}{t} - \sum_{0 \leq t \leq k, t \in 3Z} \binom{k}{t} \right)$$

$$= \frac{1}{2} (2^k - \frac{1}{3}(2^k + \omega^k(1 + \omega)^k + \omega^{2k}(1 + \omega^2)^k))$$

$$= \frac{1}{3} (2^k - (-1)^k),$$

where $\omega = \frac{-1 - \sqrt{3}}{2}$. Therefore,

$$|S_n| = \sum_{k=1}^{n} \binom{n}{k} |T_k| = 3^{n-1}.$$

\[\square\]

Proof of Proposition 3.1. Let $\{e_1, \ldots, e_{n-1}\}$ be a basis of $\mathbb{F}_3^{n-1}$ and $e_n = 0 \in \mathbb{F}_3^{n-1}$. By the structure of $AG(n, 3)$, the mapping from $Q_n$ to $\mathbb{F}_3^{n-1}$ such that $q_{\sigma(n_1) \cdots \sigma(n_k)} \mapsto e_{e_{i_1}^{\sigma(n_1)} \cdots e_{i_k}^{\sigma(n_k)}}$ is well-defined and surjective. By the lemmas above, $|Q_n| \leq |S_n| = 3^{n-1}$ and hence this map is bijective. It is easy to see that this mapping induces a surjection from $M_n$ to the set of lines of $AG(n-1, 3)$. So, this map induces an isomorphism of Fischer spaces. \[\square\]

The following corollary gives the characterization of affine spaces mentioned at the beginning of this section.

**Corollary 3.4.** Let $(P, L)$ be a connected Fischer space of affine type. If $(P, L)$ is of rank $n$ and $\tau(x)\tau(y)\tau(z) = \tau(z)\tau(y)\tau(x)$ for all $x, y, z \in P$, then $(P, L)$ is isomorphic to $AG(n-1, 3)$.

**Proof.** Let $\{p_1, \ldots, p_n\}$ be a subset of $P$ generating $(P, L)$. Then, by the structure of $(P, L)$, there exists a surjection $\pi : \mathbb{F}_3^{n-1} \to P$ such that $e_i \mapsto p_i$ and it induces the quotient map from $AG(n-1, 3) \cong (Q_n, M_n)$ to $(P, L)$. It suffices to show that $\pi$ is injective. Let $v, w \in \mathbb{F}_3^{n-1}$ and $\pi(v) = \pi(w)$. Then $\pi(u) = \pi(u + v - w)$ for all $u \in \mathbb{F}_3^{n-1}$. Let $S$ be a subset of $\mathbb{F}_3^{n-1}$ such that the image of $S$ is a basis of $\mathbb{F}_3^{n-1}/\mathbb{F}_3(v - w)$. Then, $(P, L)$ is generated by
{π(u) | u ∈ S} ∪ {π(0)}. Since (P, L) is of rank n, v = w. Therefore π is injective. □

4 Matsuo algebras associated with affine spaces

Recall that a Jordan algebra is a commutative nonassociative algebra A satisfying \((a^2b)a = a^2(ba)\) for all \(a, b \in A\). It is known that the linearized Jordan identity

\[((xz)y)w + ((zw)y)x + ((wx)y)z - (xz)(yw) - (zw)(yx) - (wx)(yz) = 0\]

holds for all \(x, y, z, w \in A\) if A is a Jordan algebra over a field of characteristic not 2. (For example, see [McC04], Proposition 1.8.5 (1).) Conversely, a commutative nonassociative algebra A over a field of characteristic not 3 is a Jordan algebra if the linearized Jordan identity holds for all \(x, y, z, w \in A\). From now on, let \(J(x, y, z, w)\) denote the left-hand side of the linearized Jordan identity.

**Lemma 4.1.** Let \(F\) be a field of characteristic 3 and A a commutative nonassociative algebra over \(F\) spanned by idempotents. Then A is Jordan if the linearized Jordan identity holds for all \(x, y, z, w \in A\).

**Proof.** Suppose \(a = \sum_{i=1}^{n} x_i a_i \in A\), where \(x_1, \ldots, x_n \in F\) and \(a_1, \ldots, a_n \in A\) are idempotents. Then, for all \(b \in A\),

\[(a^2b)a - a^2(ba) = \sum_{1 \leq i \leq n} x_i^3((a_i^2)b)a_i - a_i^2(ba_i))
+ \sum_{1 \leq i, j \leq n, i \neq j} x_i^2 x_j^2 ((a_i^2 a_j) a_j) + 2((a_i a_j) b a_i) - a_i^2 (ba_j) - 2(a_i a_j) (ba_i)
+ 2 \sum_{1 \leq i < j < k \leq n} x_i x_j x_k J(a_i, a_j, b, a_k)
= \sum_{1 \leq i} x_i^3 ((a_i b)a_i - a_i (ba_i)) + \sum x_i^2 x_j J(a_i, a_j, b, a_k)
= 0.

Since A is spanned by idempotents, the Jordan identity holds for all \(a, b \in A\). □

Let \(F\) be a field of characteristic 3. Let us show that \(M(AG(n, 3), \frac{1}{7}, F)\) is Jordan. To this end, it suffices to verify the linearized Jordan identity because Matsuo algebras are spanned by idempotents. Since this identity is linear, it suffices to verify \(J(a(x), a(y), a(z), a(w)) = 0\) for \(x, y, z, w \in F_3^n\). Since \(a(x)a(y) = a(x) + a(y) - a(-x - y)\) for all \(x, y \in F_3^n\),

\[J(a(x), a(y), a(z), a(w)) = 3a(x + y + z + w) - 3a(y - x - z - w) = 0.\]
such an element of $D \in \text{Sym}_G$ of subsets of $T$ that the noncommuting graph of it is a line. Let $T_\langle D \rangle$ be a 3-transposition group such that $(P, L)$ is isomorphic to $FSS_5$ and $AG(3, 3)$ cannot give rise to Jordan algebras.

Since these six 3-transpositions of $3^{10} : 2$ are distinct, the linearized Jordan identity does not hold. So, the Matsuo algebra associated with $(P, L)$ is not Jordan. By similar calculation for the other cases, the four Fischer spaces of rank 4 except $FSS_5$ and $AG(3, 3)$ cannot give rise to Jordan algebras.

Suppose $n \geq 5$. Let $(P, L)$ be a Fischer space of rank $n$ with the associated Matsuo algebra Jordan.

Remark 4.2. If $\text{ch}(\mathbb{F}) \neq 3$ and $\{x, y, z, w\}$ generates $AG(3, 3)$, then

$$J(a(x), a(y), a(z), a(w)) = \frac{3}{64}(a(x + y + z + w) - a(y - x - z - w)) \neq 0.$$ 

So, for $n \geq 3$, $M(AG(n, 3), \frac{1}{4}, \mathbb{F})$ is Jordan only if $\text{ch}(\mathbb{F}) = 3$.

5 Proof of Theorem

It is proved that the Matsuo algebras associated with $FS(\text{Sym}(n), (1, 2)^{\text{Sym}(n)})$ and $AG(2, 3)$ are Jordan algebras if $\delta = \frac{1}{4}$ in [DR17]. By this result and the result in the preceding section, it is already proved that the Matsuo algebras associated with $FSS_n$ or $AG(n, 3)$ are Jordan. So it suffices to show that a Fischer space of rank $n$ is isomorphic to $FSS_{n+1}$ or $AG(n - 1, 3)$ if the associated Matsuo algebra is Jordan.

We proceed case-by-case for Fischer spaces of rank at most 4. Fischer spaces of rank at most 3 are isomorphic to $FSS_2$, $FSS_3$, $FSS_4$ or $AG(2, 3)$.

The Fischer spaces of rank 4 are classified in [CH95], Proposition 3.3. They are associated with the six 3-transposition groups, including $\text{Sym}(5)$, M. Hall’s $3^{10} : 2$, and $\mathbb{F}_3^4 \rtimes \mathbb{F}_3^\times$. Let $(P, L)$ be a Fischer space associated with M. Hall’s $3^{10} : 2$ and $x, y, z, w$ be 3-transpositions generating $3^{10} : 2$. Then,

$$J(a(x), a(y), a(z), a(w)) = \frac{1}{64}(a(x^{zyw}) + a(x^{wyz}) + a(x^{yzw}) - a(x^{zw}) - a(x^{wy}) - a(z^{wy}))$$

Since these six 3-transpositions of $3^{10} : 2$ are distinct, the linearized Jordan identity does not hold. So, the Matsuo algebra associated with $(P, L)$ is not Jordan. By similar calculation for the other cases, the four Fischer spaces of rank 4 except $FSS_5$ and $AG(3, 3)$ cannot give rise to Jordan algebras.

Suppose $n \geq 5$. Let $(P, L)$ be a Fischer space of rank $n$ with the associated Matsuo algebra Jordan.

Suppose $(P, L)$ has a subspace of $(P, L)$ associated with $\text{Sym}(5)$. Let $(G, D)$ be a 3-transposition group such that $(P, L)$ is isomorphic to $FS(G, D)$. To show that $G \cong \text{Sym}(n + 1)$, let us construct a sequence $T_4 \subseteq T_5 \subseteq \cdots$ of subsets of $D$ such that the noncommuting graph $T_r$ of $T_r$ is a line and $\langle T_r \rangle \cong \text{Sym}(|T_r|)$. By the assumption, $G$ has a subgroup generated by 4-subset of $D$ and isomorphic to $\text{Sym}(5)$. So there exists a 4-subset of $D$ such that the noncommuting graph of it is a line. Let $T_4$ be such a 4-subset. Assume that the set $T_r$ is a subset of $D$ satisfying the conditions above. If $T_r$ generates $G$, then set $T_{r+1} = T_r$. Suppose that $T_r$ does not generate $G$. Then, for some $h \in D \setminus \langle T_r \rangle$, the noncommuting graph $T_r(h)$ of $T_r \cup \{h\}$ is connected. Let $h$ be such an element of $D$. Since a rank 3 quotient of $(H_3, E_3)$ cannot be a subgroup of $\text{Sym}(n)$, all subgroups generated by 4 elements of $T_r \cup \{h\}$ are isomorphic
to $\text{Sym}(5)$. In [DR17], it is proved that the noncommuting graph $\mathcal{T}$ of $T$ is not $\tilde{A}_{r-1}$ for each $T \subset D$ with $r(\langle T \rangle) \geq 4$. Hence the number of elements of $T_r$ which do not commute with $h$ is at most two and these two elements do not commute. So there exists an element $h' \in D$ such that $\langle T_r \cup \{h'\} \rangle = \langle T_r \cup \{h\} \rangle$ and $T_r(h')$ is a line. Therefore $\langle T_r \cup \{h'\} \rangle$ is isomorphic to $\text{Sym}(r+2)$. Let $T_{r+1} = T_r \cup \{h'\}$. Then, since $|G| < \infty$, $G$ is generated by $T_r$ for some $r$ and hence $(G,D)$ is isomorphic to $(\text{Sym}(m), (1,2)^{\text{Sym}(m)})$ for some $m$. Since $(\mathcal{P}, \mathcal{L})$ is of rank $n$, $m = n + 1$.

Suppose that all subspaces of $(\mathcal{P}, \mathcal{L})$ of rank 4 are not isomorphic to $FSS_5$. Then, all subspaces of rank 4 must be isomorphic to $AG(3,3)$. So $(\mathcal{P}, \mathcal{L})$ is of affine type and $\tau(x) \tau(y) \tau(z) = \tau(z) \tau(y) \tau(x)$ for all $x, y, z \in \mathcal{P}$. Then, by Corollary 3.4, $(\mathcal{P}, \mathcal{L})$ must be isomorphic to $AG(n-1,3)$. The proof of Theorem is completed.

**Remark 5.1.** When $\text{ch}(\mathbb{F}) \neq 3$, the Matsuo algebra associated with a Fischer space $(\mathcal{P}, \mathcal{L})$ becomes Jordan if and only if $(\mathcal{P}, \mathcal{L})$ satisfies the condition (i) or it is $AG(2,3)$. See [DR17].

**Remark 5.2.** Let $(G,D)$ be a connected 3-transposition group of rank $n$ such that the Matsuo algebra associated with $FS(G,D)$ is Jordan. Then, by the results above, the central quotient of $(G,D)$ is isomorphic to $(\text{Sym}(n+1), (1,2)^{\text{Sym}(n+1)})$ or $(\mathbb{F}_3^{n-1} \rtimes \mathbb{F}_3^n, \{(v,-1) \mid v \in \mathbb{F}_3^{n-1}\})$.

**Remark 5.3.** This theorem can be generalized to the infinite-dimensional case. In this generalization, a Fischer space with the associated Matsuo algebras being Jordan is isomorphic to the Fischer space associated with the inductive limit of symmetric groups of finite degrees or that of finite-dimensional affine spaces.

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