Abstract. The purpose of this article is to approximately compute the eigenvalues of the symmetric Dirichlet Laplacian within an interval $(0, \Lambda)$. A domain decomposition Ritz method, PU-CPI, is proposed. This method can be used in distributed computing environments where communication is expensive, e.g., in clusters running on cloud computing services or networked workstations. The Ritz space of PU-CPI is obtained from local subspaces consistent with a decomposition of the domain into subdomains. These local subspaces are constructed independently of each other using only data related to the corresponding subdomain. Relative eigenvalue error is analysed. Numerical examples on a cluster of workstations validate the error analysis and the performance of the method.

Key words. eigenvalue problem, subspace method, dimension reduction, domain decomposition.

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1. Introduction. Assume that $\Omega \subset \mathbb{R}^d, d = 2, 3$, is a bounded domain with Lipschitz boundary, and that $V \subset H^1_0(\Omega)$ is a closed subspace. Consider the following eigenproblem: Find $(\lambda_j, u_j) \in \mathbb{R}^+ \times V \setminus \{0\}$ such that

$$\int_{\Omega} \nabla u_j \cdot \nabla w \, dx = \lambda_j \int_{\Omega} u_j w \, dx \quad \text{and} \quad \|u_j\|_{L^2(\Omega)} = 1$$

for each $w \in V$. Here $\mathbb{R}^+ := (0, \infty)$. The eigenvalues $\lambda_j$ are numbered in non-decreasing order and repeated by their multiplicities. The purpose of the paper is to compute all eigenvalues within the spectral interval of interest $(0, \Lambda)$ for $\Lambda \in \mathbb{R}^+$ to a given accuracy.

If $V$ is a finite element space, it may happen that (1.1) cannot be solved using a single workstation. Then a distributed solution method has to be used, and there are two options: firstly, a parallel eigenvalue iteration (e.g. shift-and-invert Lanczos) together with a parallel solver for the shifted linear system, see e.g. [2]. Secondly, one can use a Domain Decomposition (DD) method such as AMLS [7], RS-DDS [19], or a CMS variant proposed in [15], see also [5–7,18].

We propose a novel DD eigensolver, Partition of Unity Condensed Pole Interpolation (PU-CPI) method that does not require any communication between its distributed tasks. Communication only takes place at the beginning and the end between master and workers. This is not the case with parallel iterative eigenvalue solvers and existing DD methods. The PU-CPI method can be used, e.g., on a cluster running in a cloud computing service or on networked workstations. As a test problem, we compute the lowest 200 eigenvalues of (1.1) when $V$ is a tetrahedral first-order finite element space. The domain is three-dimensional and the resulting algebraic eigenvalue problem has approximately $10^7$ unknowns. The PU-CPI computation took less than two hours on a cluster of 26 networked workstations; further details are given in Section 6.
As most modern eigensolvers, the PU-CPI is a Ritz method. Instead of (1.1), we solve the problem: Find \((\tilde{\lambda}_j, \tilde{u}_j) \in \mathbb{R}^+ \times \tilde{V} \setminus \{0\}\) such that for each \(w \in \tilde{V}\)

\[
\int_{\Omega} \nabla \tilde{u}_j \cdot \nabla w \, dx = \tilde{\lambda}_j \int_{\Omega} \tilde{u}_j w \, dx \quad \text{and} \quad \|\tilde{u}_j\|_{L^2(\Omega)} = 1,
\]

where the method subspace \(\tilde{V} \subset \mathcal{V}\) is finite-dimensional. The eigenvalues \(\tilde{\lambda}_j\) are in non-decreasing order and repeated according to their multiplicities. We assume that (1.2) can be solved exactly, and focus on the construction of \(\tilde{V}\) and estimating the relative error between the corresponding eigenvalues of (1.1) and (1.2). We refer the readers interested in the finite element discretisation error in eigenvalue problems to [8].

Let \(\{U^{(p)}\}_{p=1}^M, U^{(p)} \subset \Omega\), be a open cover of \(\Omega\). The PU-CPI method subspace \(\tilde{V}\) is obtained by combining local method subspaces associated to \(U^{(p)}\) similarly to the partition of unity method [22]. We proceed to outline the construction of the local subspace on \(U = U^{(p)}\) based on the representation formula

\[
u_j|_U = Z_U(\lambda_j)u_{B,j} + g_j \quad \text{for} \quad g_j \in E_\Lambda(U) \quad \text{and} \quad \lambda_j \in (0, \Lambda).
\]

The main ingredients of the above formula are the finite-dimensional subspace \(E_\Lambda(U)\) and the operator-valued function \(Z_U\), defined using an extended domain \(\tilde{U}, U \subset \tilde{U} \subset \Omega\). Operator \(Z_U(t)\) is a boundary-interior mapping of functions on \(\partial \tilde{U}\) to functions on \(\tilde{U}\). Detailed definitions are given in Section 3. The local method subspace is constructed from \(E_\Lambda\) and a low-rank approximation to \(Z_U(t)\) for \(t \in (0, \Lambda)\) obtained by interpolation, linearisation, and Singular Value Decomposition (SVD). The extended domain \(\tilde{U}\) is required to make \(Z_U(t)\) a compact operator. An analogous approach is proposed in [3] for multiscale and in [24] for parameter dependent source problems. Moreover, our techniques are similar to [17], where the solution of a family of algebraic eigenvalue problems with \(2 \times 2\)-block structure is considered.

In our implementation of PU-CPI, the computation proceeds in three steps:

1. Preparation of the computational grid on the master.
2. Distributed computation of the local subspaces by workers.
3. Assembly and solution of the reduced eigenvalue problem on the master.

The article is organised as follows. We begin by reviewing the preliminaries and error analysis of Ritz methods. In Section 3, we construct the local subspace for a single subdomain and derive the local error estimate for it. In Section 4, we combine the local subspaces to the method subspace \(\tilde{V}\) and introduce the global error estimates. Section 5 is devoted to standard first order finite element space. We conclude the article with numerical examples in Section 6, followed by a discussion.

2. Background. Let \(\Omega', \Omega \subset \mathbb{R}^d\) for \(d = 2, 3\) be open bounded set with Lipschitz boundary such that \(\Omega' \subset \Omega\). The inner products for \(H^1(\Omega')\) and \(H^1_0(\Omega')\) are

\[
(f, g)_{H^1(\Omega')} := (\nabla f, \nabla g)_{L^2(\Omega'; \mathbb{R}^d)} + (f, g)_{L^2(\Omega')},
\]

\[
(f, g)_{H^1_0(\Omega')} := (\nabla f, \nabla g)_{L^2(\Omega'; \mathbb{R}^d)}.
\]

The corresponding norms are denoted by \(\| \cdot \|_{H^1(\Omega')}\) and \(\| \cdot \|_{H^1_0(\Omega')}\), respectively.

In the following, we discuss a subspace method for the eigenproblem related to the Laplace operator and its finite element discretisation, treated using the formulation in (1.1) with different choices of the space \(\mathcal{V}\). The Laplace operator is treated by setting \(\mathcal{V} = H^1_0(\Omega)\), and the solution \((\lambda, u) \in \mathbb{R}^+ \times H^1_0(\Omega) \setminus \{0\}\) (1.1) is required to
satisfy
\begin{equation}
- \Delta u = \lambda u \quad \text{in } L^2(\Omega) \quad \text{and} \quad \|u\|_{L^2(\Omega)} = 1.
\end{equation}

The finite element discretisation of (2.1) is obtained when \( V = V_h \), where
\begin{equation}
V_h := \{ w \in H^1_0(\Omega) \mid w|_K \in P^1(K) \quad \text{for all } K \in T_h \}
\end{equation}
is the finite element space related to a conforming partition of \( \Omega \) into simplices \( T_h \). Here \( P^1(K) \) denotes the space of first-order polynomials on \( K \subset \mathbb{R}^d \).

We work with restrictions of functions from the space \( V \subset H^1_0(\Omega) \) to a subdomain \( \Omega' \subset \Omega \). Denote
\[ V(\Omega') := \{ w|_{\Omega'} \mid w \in V \} \quad \text{and} \quad \text{tr} V(\Omega') := \{ \gamma_{\partial \Omega'} w \mid w \in V(\Omega') \}, \]
where \( \gamma_{\partial \Omega'} \in \mathcal{B}(H^1(\Omega'), H^{1/2}(\Omega')) \) is the trace operator on \( H^1(\Omega') \). The space of functions with homogeneous boundary values is denoted by
\[ V_0(\Omega') := \{ w \in V(\Omega') \mid \gamma_{\partial \Omega'} w = 0 \}. \]
The spaces \( V(\Omega') \), \( V_0(\Omega') \) inherit their inner products and norms from spaces \( H^1(\Omega') \), \( H^1_0(\Omega') \), respectively.

For \( \text{tr} V(\Omega') \), we use the norm
\begin{equation}
\|f\|_{\text{tr} V(\Omega')} := \frac{1}{\sqrt{2}} \min_{w \in V(\Omega')} \|w\|_{H^1(\Omega')}.
\end{equation}

We make a standing assumption that all these spaces are complete. This holds, e.g., if \( V = H^1_0(\Omega) \) or it is finite dimensional.

### 2.1. Subspace methods.
Convergence results for Ritz methods that relate the eigenvalues of (1.1) and (1.2) have been studied by several authors, see e.g. [4, 8, 12].

We apply an estimate adapted from [21, Theorem 3.2], where the relative eigenvalue error is bounded by the approximability of the corresponding eigenfunction in the method subspace \( \tilde{V} \).

When (1.1) is posed in a closed method subspace \( W \subset H^1_0(\Omega') \) instead of \( V \), we denote the set of eigenvalues as \( \sigma(W) \). Further, the spectral gap of \( V \) on \((0, \Lambda)\) is defined as
\begin{equation}
\rho_{\Lambda} := \min_{\lambda, \mu \in \sigma(V) \cap (0, \Lambda)} |\mu - \lambda|.
\end{equation}

**Proposition 2.1.** Let \( \rho_{\Lambda} \) be as defined in (2.4), \( \tilde{V} \subset V \) a finite-dimensional subspace, \( 1 \leq j \leq \#(\sigma(V) \cap (0, \Lambda)) \), and \((\lambda_j, u_j) \in \sigma(V) \times V \setminus \{0\} \) eigenpair of (1.1) corresponding to a simple eigenvalue \( \lambda_j \). Assume that the Hausdorff distance
\begin{equation}
\text{dist} \left( \sigma(\tilde{V}) \cap (0, \Lambda), \sigma(V) \cap (0, \Lambda) \right) \leq \frac{1}{2} \rho_{\Lambda}.
\end{equation}
Then there exists \( \tilde{\lambda} \in \sigma(\tilde{V}) \) and \( C(\lambda_j) \equiv C(\lambda_j; V) \) such that
\begin{equation}
\frac{|\lambda_j - \tilde{\lambda}|}{\lambda_j} \leq C(\lambda_j) \min_{v \in \tilde{V}} \|u_j - v\|_{H^1_0(\Omega')}^2.
\end{equation}
The statement in [21] does not give a condition for the subspace \( \widetilde{V} \), but features a multiplicative term dependent on \( \widetilde{V} \) and \( V \) that has to be somehow controlled. Here we introduce assumption (2.5) which holds when the method related to \( \widetilde{V} \) has entered the asymptotic spectral convergence region. Observe that [21] uses different normalisation of eigenfunctions which affects \( C(\lambda_j) \), but is cancelled later in Theorem 4.2.

### 2.2. The PU-CPI method subspace.
Let \( \{U^{(p)}\}_{p=1}^M \), \( U^{(p)} \subset \Omega \), be an open cover of the domain \( \Omega \subset \mathbb{R}^d \) such that each \( U^{(p)} \) has Lipschitz boundary. Next, we describe how the PU-CPI method subspace \( \widetilde{V} \) is constructed from the local method subspaces \( V(U^{(p)}) \subset \mathcal{V}(U^{(p)}) \).

For \( p = 1, \ldots, M \), let the stitching operators \( R^{(p)} \in \mathcal{B}(\mathcal{V}(U^{(p)}), \mathcal{V}) \) satisfy

\[
(R^{(p)}w^{(p)})|_{\Omega \setminus U^{(p)}} = 0 \quad \text{and} \quad \sum_{p=1}^M R^{(p)}(w|_{U^{(p)}}) = w
\]

for each \( w^{(p)} \in \mathcal{V}(U^{(p)}) \) and \( w \in \mathcal{V} \). Suitable operators \( \{R^{(p)}\}_{p=1}^M \) for \( \mathcal{V} = H^1_0(\Omega) \) can be obtained by multiplication with a partition of unity as in [22]. The PU-CPI method subspace \( \widetilde{V} \equiv \widetilde{V}(\{V(U^{(p)})\}_{p=1}^M) \) is defined as

\[
\widetilde{V} := \left\{ w \in \mathcal{V} \mid w = \sum_{p=1}^M R^{(p)}w^{(p)} \quad \text{for} \quad w^{(p)} \in V(U^{(p)}) \right\}
\]

where \( \widetilde{V}(U^{(p)}) \subset \mathcal{V}(U^{(p)}) \) has a low dimension. If \( \widetilde{V} \) satisfies (2.7) and the assumptions of Proposition 2.1, the eigenvalue error depends on approximation properties of the local subspaces. Using a similar technique as in [22] gives

\[
\min_{v \in \mathcal{V}} \|u_j - v\|_{H^1_0(\Omega)} \leq \|G\|_{L^\infty(\Omega)} \left( \sum_{p=1}^M \mathcal{E}(u_j, U^{(p)}) \right)^{1/2},
\]

where function \( \mathcal{E} \) is the local approximation error,

\[
\mathcal{E}(u, U^{(p)}) := \min_{w \in \mathcal{V}(U^{(p)})} \int_{U^{(p)}} \left| \nabla R^{(p)}(u|_{U^{(p)}} - w) \right|^2 \, dx
\]

and \( G : \Omega \rightarrow \{1, \ldots, M\} \) is defined as

\[
G(x) = \#\{p \mid x \in U^{(p)}\}.
\]

The aim is to design the local method subspaces \( \widetilde{V}(U^{(p)}) \) so that both \( \text{dim}(\widetilde{V}(U^{(p)})) \) and \( \mathcal{E}_{j,p} \equiv \mathcal{E}(u_j, U^{(p)}) \) are small.

### 3. Local method subspace.
A local method subspace \( \widetilde{V}(U^{(p)}) \subset \mathcal{V}(U^{(p)}) \) for a single subdomain \( U^{(p)} \) is designed next. For notational convenience, denote \( U = U^{(p)} \), \( R = R^{(p)} \), and let \( (\lambda, u) \) be some solution to (1.1) satisfying \( \lambda < \Lambda \).

#### 3.1. Extended subdomain.
Given \( r > 0 \) and \( U \subset \Omega \), let \( \hat{U} \subset \Omega \) be a domain satisfying

\[
\{ x \in \Omega \mid \text{dist}(x, U) < r \} \subset \hat{U}.
\]
Any such $\hat{U}$ is called an $r$-extension of $U$ and we make it a standing assumption that both $U$ and $\hat{U}$ have Lipschitz boundaries. In the following, $\hat{U}$ is fixed unless otherwise stated. The effect of the parameter $r$ is numerically studied in Section 6. As shown in the next section, an essential component of the PU-CPI method is an operator-valued function $Z_\hat{U} : (0, \Lambda) \to B(\text{tr}V(\hat{U}), V(\hat{U}))$. It will be shown that $Z_\hat{U}$ is analytic, and due to the use of the $r$-extension and elliptic regularity also compact operator-valued.

3.2. Eigenfunction representation formula. We represent $u|_{\hat{U}}$ as a function of the boundary trace $\gamma_{\partial \hat{U}}(u|_{\hat{U}})$. By (1.1), $u|_{\hat{U}}$ satisfies

$$\int_{\hat{U}} (\nabla u|_{\hat{U}} \cdot \nabla w - \lambda u|_{\hat{U}} w) \, dx = 0 \tag{3.2}$$

for each $w \in \mathcal{V}_0(\hat{U})$. We assume that there exists a right inverse $E \in B(\text{tr}V(\hat{U}), V(\hat{U}))$ of $\gamma_{\partial \hat{U}}$ satisfying

$$E : \text{tr}V(\hat{U}) \to \{ v \in V(\hat{U}) \mid v|_{\hat{U}} = 0 \}. \tag{3.3}$$

Such $E$ exists if $U \neq \hat{U}$ and either $\mathcal{V} = H^1_0(\Omega)$ or a sufficiently fine finite element mesh for $\Omega$ is used for the construction of $\mathcal{V}$.

Equation (3.2) is solved by decomposing

$$u|_{\hat{U}} = u_0 + Eu_B \quad \text{where} \quad u_0 \in \mathcal{V}_0(\hat{U}) \quad \text{and} \quad u_B := \gamma_{\partial \hat{U}}(u|_{\hat{U}}). \tag{3.4}$$

It follows from (3.3) that $u|_{\hat{U}} = u_0|_{\hat{U}}$. Using the decomposition in (3.4), (3.2) gives

$$\int_{\hat{U}} (\nabla u_0 \cdot \nabla w - \lambda u_0 w) \, dx = -\int_{\hat{U}} (\nabla w \cdot \nabla Eu_B - \lambda w Eu_B) \, dx \tag{3.5}$$

for each $w \in \mathcal{V}_0(\hat{U})$, which defines $u_0$ as a function of $\lambda$ and $u_B$. We proceed as in [17] and use an $L^2(\hat{U})$-orthonormal eigenbasis expansion to solve (3.5). Let $(\mu_k, v_k) \in \mathbb{R}^+ \times \mathcal{V}_0(\hat{U}) \setminus \{0\}$ be such that

$$\int_{\hat{U}} \nabla v_k \cdot \nabla w \, dx = \mu_k \int_{\hat{U}} v_k w \, dx \quad \text{and} \quad ||v_k||_{L^2(\hat{U})} = 1 \tag{3.6}$$

for each $w \in \mathcal{V}_0(\hat{U})$. Assume that $\{\mu_k\}_k \subset \mathbb{R}^+$ are indexed in non-decreasing order and repeated according to their multiplicities. The set $\{v_k\}_k$ is chosen as an $L^2(\hat{U})$-orthonormal basis of $\mathcal{V}_0(\hat{U})$, hence $\{v_k/\sqrt{\mu_k}\}_k$ is an $H^1_0(\hat{U})$-orthonormal basis of $\mathcal{V}_0(\hat{U})$. To solve $u_0$ from (3.5), expand in $H^1_0(\hat{U})$

$$u_0 = \sum_{j=1}^{\dim(\mathcal{V}_0(\hat{U}))} \alpha_j v_j \quad \text{where each} \quad \alpha_j \in \mathbb{R}. \tag{3.7}$$

Using this expansion with (3.5) and setting $w = v_k$ in (3.6), the orthogonality of the eigenfunctions gives

$$\alpha_k(\mu_k - \lambda) = -\int_{\hat{U}} (\nabla v_k \cdot \nabla Eu_B - \lambda v_k Eu_B) \, dx. \tag{3.8}$$

If $\lambda \notin \sigma(\mathcal{V}_0(\hat{U}))$, $u_0$ is determined by solving $\alpha_k$ for $k = 1, \ldots, \dim \mathcal{V}_0(\hat{U})$. To treat any $\lambda \in (0, \Lambda)$, we split the coefficients $\alpha_k$ into two groups using the parameter $\Lambda > \Lambda$ and $K : \mathbb{R}^+ \to \mathbb{N}$, given by

$$K(t) := \# \{ \mu_k \in \sigma(\mathcal{V}(\hat{U})) \mid \mu_k \leq t \}.$$
Since $\lambda \in (0, \Lambda)$ the coefficients $\alpha_k$ in (3.7) for $k > K(\hat{\Lambda})$ are obtained from (3.8). We have now proved the following lemma:

**Lemma 3.1.** Let $\hat{\Lambda} > \Lambda > 0$. Assume that $(\lambda, u) \in (0, \Lambda) \times \mathcal{V}$ and $U \subset \tilde{U} \subset \Omega$ satisfy (1.1) and (3.1), respectively. Then we have the following orthogonal splitting in $L^2(\tilde{U})$ and in $H^1_0(\tilde{U})$:

$$
(3.9) \quad u|_U = \sum_{k=1}^{K(\hat{\Lambda})} \alpha_k v_k|_U + (Z(\lambda)u_B)|_U,
$$

where $u_B = \gamma_{\partial\tilde{U}} u$, $\{\alpha_k\}_{k=1}^{K(\hat{\Lambda})} \subset \mathbb{R}$, and $Z : (0, \Lambda) \to \mathcal{B}(tr\mathcal{V}(\tilde{U}), \mathcal{V}_0(\tilde{U}))$ is defined as

$$
(3.10) \quad Z(t)w_B := \sum_{k=K(\hat{\Lambda})+1}^{\dim(\mathcal{V}_0(\tilde{U}))} \frac{v_k}{\mu_k - t} \int_{\tilde{U}} (\nabla v_k \cdot \nabla Ew_B + tv_k Ew_B) \, dx.
$$

The sum converges in $L^2(\tilde{U})$ and $H^1_0(\tilde{U})$.

There are many ways of showing that $Z(t) \in \mathcal{B}(tr\mathcal{V}(\tilde{U}), \mathcal{V}_0(\tilde{U}))$ for $t \in (0, \Lambda)$; e.g., by using Lemma 3.6. The function $Z$ depends implicitly on $\hat{\Lambda}, \tilde{U}$ and $U$ in addition to $t$.

**3.3. Evaluation of $Z$.** Following the approach used in [17], we discuss how $Z$ can be evaluated given $K(\hat{\Lambda})$ lowest eigenmodes of (3.6). Denote

$$
E_{\hat{\Lambda}} := \text{span}\{v_1, \ldots, v_{K(\hat{\Lambda})}\} \text{ for } (v_k, \mu_k) \text{ satisfying (3.6)}.
$$

Fix $t \in (0, \Lambda)$, $w_B \in tr\mathcal{V}(\tilde{U})$, and solve the auxiliary problem: Find $\hat{z}_0(t) \in \mathcal{V}_0(\tilde{U})$ such that

$$
(3.11) \quad \int_{\tilde{U}} (\nabla \hat{z}_0(t) \cdot \nabla w - t\hat{z}_0(t)w) \, dx = -\int_{\tilde{U}} (\nabla w \cdot \nabla Ew_B - tw Ew_B) \, dx
$$

for each $w \in \mathcal{V}_0(\tilde{U})$. As in Section 3.2, each solution admits the orthogonal splitting

$$
\hat{z}_0(t) = \sum_{k=1}^{K(\hat{\Lambda})} \alpha_k v_k + Z(t)w_B.
$$

For the exceptional $t \in \sigma(\mathcal{V}_0(\tilde{U}))$ some $\alpha_k$’s cannot be uniquely solved from (3.11). After $\hat{z}_0(t)$ has been solved from (3.11), $Z(t)w_B$ can be evaluated as $Z(t)w_B = P\hat{z}_0(t)$, where $P \in \mathcal{B}(\mathcal{V}_0(\tilde{U}))$ is the $L^2(\tilde{U})$ orthogonal projection onto the orthocomplement of $E_{\hat{\Lambda}}$.

**3.4. The complementing subspace.** Our aim is to design the finite-dimensional subspace $\tilde{\mathcal{V}}(U)$ such that the local approximation error in (2.9), namely

$$
\min_{v \in \tilde{\mathcal{V}}(U)} \int_{\tilde{U}} |\nabla [R(u|_U - v)]|^2 \, dx,
$$

can be made arbitrarily small for any $(\lambda, u) \in (0, \Lambda) \times \mathcal{V}$ satisfying (1.1). For $w_B \in tr\mathcal{V}(\tilde{U})$ and $t \in (0, \Lambda)$, denote

$$
(3.12) \quad Z_U(t)w_B = (Z(t)w_B)|_U.
$$
By Lemma 3.1,
\[
(3.13) \quad u|_U = \sum_{k=1}^{K(\lambda)} \alpha_k v_k|_U + Z_U(\lambda)u_B,
\]
for some real-valued \(\alpha_k\)'s. We construct \(\tilde{V}(U)\) according to the splitting in (3.13) as
\[
(3.14) \quad \tilde{V}(U) = E_{\tilde{A}}(U) \oplus W(U) \quad \text{where} \quad E_{\tilde{A}}(U) := \text{span}\{v_1|_U, \ldots, v_{K(\lambda)}|_U\},
\]
and \(\oplus\) denotes the orthogonal direct sum in \(\mathcal{V}\). The space \(W(U)\) is called the local complementing subspace. Let
\[
(3.15) \quad e_U(W(U)) := \sup_{\ell \in (0,\Lambda)} \inf_{v \in W(U)} \int_U |\nabla [R(Z_U(t)w_B - v)]|^2 \, dx \left/ ||u||_{H^1(U)}^2 \right.,
\]
where \(w_B = \gamma_0w\). As the first term on the right hand side of (3.13) is included in \(\tilde{V}(U)\), the local approximation error of \(u\) on \(U\) has the estimate
\[
(3.16) \quad \mathcal{E}(u, U) \equiv \min_{v \in W(U)} \int_U |\nabla [R(u|_U - v)]|^2 \, dx \leq e_U(W(U)) ||u||_{H^1(U)}^2,
\]
for each \((\Lambda, u) \in (0, \Lambda) \times \mathcal{V}\) satisfying (1.1).

Next, we design the local complementing subspace \(W(U)\) so that the local approximation error in (2.9) can be made arbitrarily small. We begin with the interpolation step. Denote the a set of \(N \geq 1\) Chebyshev nodes on the interval \((0, \Lambda)\) as \(\{\xi_i\}_{i=1}^N \subset (0, \Lambda)\). Define \(\tilde{Z} : (0, \Lambda) \to B(tr\mathcal{V}(\tilde{U}), \mathcal{V}_0(\tilde{U}))\) as
\[
(3.17) \quad \tilde{Z}(t) = \sum_{i=1}^N \ell_i(t)Z(\xi_i) \quad \text{where} \quad \ell_i(t) = \prod_{\substack{1 \leq j \leq N \\xi_i \neq \xi_j}} \frac{t - \xi_j}{\xi_i - \xi_j} \quad \text{for} \quad i = 1, \ldots, N
\]
are the Lagrange interpolation polynomials. Due to analyticity of the operator-valued function \(Z\), this approximation will convergence for sufficiently large \(\Lambda\). We proceed with a linearisation step. Define a linear operator\(^1\)
\[
(3.18) \quad B \in B(tr\mathcal{V}(\tilde{U}; \mathbb{R}^N), \mathcal{V}(U)) \quad \text{as} \quad Bv_B := [Z_U(\xi_1) \ldots Z_U(\xi_N)]v_B.
\]
Here \(\tilde{Z}_U(t)w_B = (\tilde{Z}(t)w_B)|_U\) for all \(w_B \in tr\mathcal{V}(\tilde{U})\) and \(t \in (0, \Lambda)\). Furthermore,
\[
B\ell(t)w_B = B \begin{bmatrix} \ell_1(t)w_B & \ldots & \ell_N(t)w_B \end{bmatrix}^T = \tilde{Z}_U(t)w_B,
\]
and hence, \(\text{range}(\tilde{Z}_U(t)) \subseteq \text{range}(B)\) for any \(t \in (0, \Lambda)\).

We continue with the finite-rank approximation step. Given a finite-rank operator \(\tilde{B} \in B(tr\mathcal{V}(\tilde{U}; \mathbb{R}^N), \mathcal{V}(U))\), the complementing subspace is fixed as \(W(U) := \text{range}(\tilde{B})\). We show that \(e_U(W(U))\) in (3.15) is bounded from above by
\[
\|R(Z_U(t) - \tilde{Z}_U(t))\|_{B(tr\mathcal{V}(\tilde{U}), \mathcal{V})} \quad \text{and} \quad \|B - \tilde{B}\|_*,
\]
resulting in Theorem 4.2.

If \(\mathcal{V} = H_0^1(\Omega)\) each of the operators \(Z_U(\xi_i), i = 1, \ldots, N\) is compact, which makes finding \(\tilde{B}\) feasible:

\(^1\)Here \(tr\mathcal{V}(\tilde{U}; \mathbb{R}^N)\) is defined as \([tr\mathcal{V}(\tilde{U})]^N\) and equipped with the natural Hilbert space norm.
LEMMA 3.2. Let $U \subset \hat{U} \subset \Omega \subset \mathbb{R}^d$ be as in (3.1), $\mathcal{V} = H^1_0(\Omega)$, and $Z_U : (0, \Lambda) \rightarrow \mathcal{B}(tr\mathcal{V}(\hat{U}; \mathbb{R}^N), \mathcal{V}(U))$ be as defined in (3.12). In addition, assume that $\hat{U}$ is a convex polygonal ($d = 2$) or convex polyhedral domain ($d = 3$). Then $Z_U(t)$ is a compact operator from $tr\mathcal{V}(\hat{U})$ to $\mathcal{V}(U)$ for all $t \in (0, \Lambda) \setminus \sigma(V_0(\hat{U}))$.

This lemma is proved below.

Representing $u|_U$ in terms of $\gamma_{B_U}u$ is motivated by Lemma 3.2, keeping in mind that compact operators can be uniformly approximated by finite-rank operators. Further, the same holds for $B$ in (3.18) since the number of Chebyshev nodes $N$ is finite. We need the following proposition:

PROPOSITION 3.3. Let $U, \mathcal{X}, \mathcal{Y}$ be Banach spaces, $T \in \mathcal{B}(U, \mathcal{Y})$, range($T$) $\subset \mathcal{X}$, and $\mathcal{X}$ continuously embedded in $\mathcal{Y}$. Then $T \in \mathcal{B}(U, \mathcal{X})$. In addition, if the embedding $\mathcal{X} \subset \mathcal{Y}$ is compact, then $T$ is a compact operator from $U$ to $\mathcal{Y}$.

Proof. Let $u_j \rightarrow u$ in $U$ and $Tu_j \rightarrow x$ in $\mathcal{X}$. Since $T : U \rightarrow \mathcal{Y}$ is bounded, $Tu_j \rightarrow Tu$ in $\mathcal{Y}$. As $\mathcal{X}$ is continuously embedded in $\mathcal{Y}$, $x = Tu$ as equality in $\mathcal{X}$. We have now shown that $T : U \rightarrow \mathcal{X}$ is a closed linear operator. The first claim follows from the closed graph theorem. The second claim follows since the composition of a compact operator and a bounded operator is compact.

Hence, if $\mathcal{V} = H^1_0(\Omega)$ and $t \in (0, \Lambda)$, the compactness of $Z_U(t) \in \mathcal{B}(tr\mathcal{V}(\hat{U}; \mathbb{R}^N), \mathcal{V}(U))$ follows by showing that $Z_U(t)w_B \in H^2(U)$ for all $t \in (0, \Lambda)$ and $w_B \in tr\mathcal{V}(\hat{U})$. Due to standing assumptions made on $U$, $H^2(U)$ is compactly embedded in $H^1(U)$; see, e.g., [1, Th. 6.3].

PROPOSITION 3.4. Let the domains $U \subset \hat{U} \subset \Omega \subset \mathbb{R}^d$ be as in (3.1). Let $\mathcal{V} = H^1_0(\Omega)$, and $u \in \mathcal{V}(\hat{U})$ such that $\Delta u \in L^2(\hat{U})$. Assume that one of the following holds:

(i) $\partial \Omega \cap \partial \hat{U} = \emptyset$;
(ii) $d = 2$, $\hat{U}$ is a convex polygonal domain, and $\partial \Omega \cap \partial \hat{U} \neq \emptyset$; or
(iii) $d = 3$, $\hat{U}$ is a convex polyhedral domain, and $\partial \Omega \cap \partial \hat{U} \neq \emptyset$.

Then $u|_U \in H^2(U)$.

Cases (i) and (ii) are illustrated in Figure 1.

Proof. If (i) holds, the claim follows from the interior regularity estimate; see, e.g., [14, Ch. 6.3]. Assume that (ii) or (iii) holds. Let $\varphi \in C^\infty(\hat{U})$ be a cut-off function satisfying $\varphi = 1$ in $U$ and $\varphi = 0$ on $\partial \hat{U} \setminus \partial \Omega$. The function $\varphi u \in H^1_0(\hat{U})$ satisfies $\Delta(\varphi u) \in L^2(\hat{U})$ by a straightforward computation. By [16, Ch. 2.4 & 2.6] and assumptions (ii), (iii), we have $\varphi u \in H^2(\hat{U})$. The claim follows from $\varphi u|_U = u|_U$.

We complete this section by giving a proof of Lemma 3.2.

Proof of Lemma 3.2. Fix $t \in (0, \Lambda) \setminus \sigma(V_0(\hat{U}))$ and $w_B \in tr\mathcal{V}(\hat{U})$. Let $\tilde{z} \in \mathcal{V}(\hat{U})$ be the variational solution of

\[
\begin{aligned}
(\Delta + t)\tilde{z} &= 0 \quad \text{in } \hat{U}, \\
\tilde{z} &= w_B \quad \text{on } \partial \hat{U},
\end{aligned}
\]

obviously satisfying $\tilde{z} \in H^1(\hat{U})$ and $\Delta \tilde{z} \in L^2(\hat{U})$. Similar to Section 3.2, decompose $\tilde{z} = \tilde{z}_0 + Ew_B$, where $E$ satisfies (3.3). As $t \not\in \sigma(V_0(\hat{U}))$,

\[
\tilde{z}_0 = Z(t)w_B + \sum_{k=1}^{K(\Lambda)} \frac{v_k}{\mu_k - t} \int_{\hat{U}} (-\nabla v_k \cdot \nabla Ew_B + tv_k Ew_B) \, dx.
\]
Further, using $\hat{z} = \hat{z}_0 + Ew_B$ gives

$$Z(t)w_B + Ew_B = \hat{z} - \sum_{k=1}^{K(\Lambda)} \frac{v_k}{\mu_k - t} \int_{\hat{U}} (-\nabla v_k \cdot \nabla Ew_B + tv_k Ew_B) \, dx.$$ 

Since the sum on the right hand side has a finite number of terms where $\Delta v_k \in L^2(\hat{U})$, it follows that $\Delta (Z(t)w_B + Ew_B) \in L^2(\hat{U})$. Using Proposition 3.4 and the property $(Ew_B)|_{\hat{U}} = 0$ gives $(Z(t)w_B + Ew_B)|_{\hat{U}} = Z_U(t)w_B \in H^2(\hat{U})$. Proposition 3.3 completes the proof.

Remark 3.5. The assumption of convexity for the compactness of $Z_U(t)$ can be relaxed using a more technical variant of Proposition 3.4 stated in weighted Sobolev spaces, see e.g., [23]. In addition, Lemma 3.2 can be extended to cover all values $t \in (0, \Lambda)$.

3.5. Interpolation error. Next, we study how the error terms

$$(3.19) \quad e_0 := \left\Vert \left( \hat{Z}(t) - Z(t) \right) w_B \right\Vert_{L^2(\hat{U})} \quad \text{and} \quad e_1 := \left\Vert \left( \hat{Z}(t) - Z(t) \right) w_B \right\Vert_{H^1_0(\hat{U})},$$

depend on $N$ and $\Lambda$. By Lemma 3.1, the function $Z(t)w_B \in V_0(\hat{U})$ admits the expansion

$$Z(t)w_B = \sum_{k=K(\Lambda)+1}^{\dim(V_0(\hat{U}))} c_{1,k}(w_B) \frac{\mu_k - t}{v_k},$$

where the coefficients $c_{0,k} : trV(\hat{U}) \to \mathbb{R}$ and $c_{1,k} : trV(\hat{U}) \to \mathbb{R}$ are defined as

$$(3.20) \quad c_{0,k}(w_B) := (v_k, Ew_B)_{L^2(\hat{U})} \quad \text{and} \quad c_{1,k}(w_B) := -\langle \nabla v_k, \nabla Ew_B \rangle_{L^2(\hat{U}; \mathbb{R}^d)}$$

for $k = 1, \ldots, \dim(V_0(\hat{U}))$. A technical estimate related to these coefficients is given in the following lemma.

**Lemma 3.6.** Let $c_{0,k}(w_B)$ and $c_{1,k}(w_B)$ be as in (3.20). Then

$$\sum_{k=K(\Lambda)+1}^{\dim(V_0(\hat{U}))} c_{0,k}^2(w_B) \leq \left\Vert Ew_B \right\Vert_{L^2(\hat{U})}^2 \quad \text{and} \quad \sum_{k=K(\Lambda)+1}^{\dim(V_0(\hat{U}))} \frac{c_{1,k}^2(w_B)}{\mu_k} \leq \left\Vert \nabla (Ew_B) \right\Vert_{L^2(\hat{U}; \mathbb{R}^d)}^2.$$
Proof. We only prove the latter inequality. For any $u \in \mathcal{V}(\hat{U})$ define $P_1 u \in \mathcal{V}_0(\hat{U})$ uniquely by the Riesz representation theorem on the Hilbert space $\mathcal{V}_0(\hat{U})$, requiring

$$(\nabla P_1 u, \nabla v)_{L^2(\hat{U}; \mathbb{R}^d)} = (\nabla u, \nabla v)_{L^2(\hat{U}; \mathbb{R}^d)}$$

for each $v \in \mathcal{V}_0(\hat{U})$.

Then the mapping $u \mapsto P_1 u$ is linear, it satisfies $P_1^2 = P_1$, and $\|\nabla (P_1 E w_B)\|_{L^2(\hat{U}; \mathbb{R}^d)} \leq \|\nabla E w_B\|_{L^2(\hat{U}; \mathbb{R}^d)}$. Since $P_1 u$ is uniquely defined, it also follows that $P_1 u = u$ for all $u \in \mathcal{V}_0(\hat{U})$. Hence, $P_1$ is a projection on $\mathcal{V}(\hat{U})$ with range$(P_1) = \mathcal{V}_0(\hat{U})$. Since $\{v_k/\sqrt{\mu_k}\}_k$ is orthonormal basis of $\mathcal{V}_0(\hat{U})$, we have

$$P_1 E w_B = - \sum_{k=1}^{\dim(\mathcal{V}_0(\hat{U}))} \frac{v_k}{\sqrt{\mu_k}} c_{1,k}(w_B) \sqrt{\mu_k}.$$ 

The claim follows using Parseval’s identity.

Denote

$$f_{m,k}(t) = t^{1-m}(\mu_k - t)^{-1}$$

for $m = 0, 1$ and $k = K(\hat{\Lambda}) + 1, \ldots, \dim(\mathcal{V}_0(\hat{U}))$. Recalling (3.17), we have

$$\begin{align*}
(Z(t) - \hat{Z}(t)) w_B &= \sum_{k=K(\hat{\Lambda})+1}^{\dim(\mathcal{V}_0(\hat{U}))} \left( f_{1,k}(t) - \sum_{i=1}^{N} \ell_i(t) f_{1,k}(\xi_i) \right) c_{1,k}(w_B) v_k \\
&+ \sum_{k=K(\hat{\Lambda})+1}^{\dim(\mathcal{V}_0(\hat{U}))} \left( f_{0,k}(t) - \sum_{i=1}^{N} \ell_i(t) f_{0,k}(\xi_i) \right) c_{0,k}(w_B) v_k.
\end{align*}
$$

(3.21)

Observe that the expressions in parentheses in (3.21) are Lagrange interpolation errors with Chebyshev nodes $\{\xi_i\}_{i=1}^{N} \subset (0, \hat{\Lambda})$. The derivatives of $f_{m,k}$ satisfy

$$1 \frac{d^N f_{m,k}(t)}{dt^N}(t) = \frac{\mu_k^{1-m}}{(\mu_k - t)^{(N+1)}}$$

(3.22)

Hence, we have the estimate for $k = K(\hat{\Lambda}) + 1, \ldots, \dim(\mathcal{V}_0(\hat{U}))$

$$\mu_k^{1+m} \| f_{m,k}(\cdot) - \sum_{i=1}^{N} \ell_i(\cdot) f_{m,k}(\xi_i) \|_{L^\infty(0, \hat{\Lambda})}^2 \leq \frac{\mu_k^{3-m} \Lambda^{2N}}{4^{2N-1} (\mu_k - \hat{\Lambda})^{2(N+1)}}$$

(3.23)

for $m = 0, 1$; see, e.g., [13, Ch. 3.3]. We are now in the position to give an estimate for the error terms $e_0$ and $e_1$:

**Lemma 3.7.** Let $t \in (0, \hat{\Lambda})$, $w_B \in tr\mathcal{V}(\hat{U})$, and $\hat{Z}(t)$ be as in (3.17). Then the error terms in (3.19) satisfy

$$e_1 \leq 12 \left[ 4(\eta - 1) \right]^{-N-1} (\eta^{l+1} \Lambda^{l-1} + \eta^{l+2} \Lambda^l)^{1/2} \| E w_B \|_{\mathcal{V}(\hat{U})}$$

for $l = 0, 1$ and $\eta := \hat{\Lambda}/\Lambda$.

In [17], the parameter $\eta$ is called the **oversampling parameter**. Observe that for $\eta > 5/4$, $e_1$ and $e_0$ converge to zero as $N \to \infty$. 


Proof. As the estimates for \( l = 0, 1 \) follow from similar arguments, we only consider \( l = 1 \). By triangle inequality, Parseval’s identity, and (3.21), we have

\[
\frac{1}{2} e_l^2 \leq \sum_{k=K(\bar{\Lambda})+1}^{\dim(V_0(\bar{U}))} \mu_k^2 \left( f_{1,k}(t) - \sum_{i=1}^{N} \ell_i(t) f_{1,k}(\xi_i) \right)^2 \frac{c_{1,k}(w_B)}{\mu_k^2},
\]

(3.24)

\[
+ \sum_{k=K(\bar{\Lambda})+1}^{\dim(V_0(\bar{U}))} \mu_k \left( f_{0,k}(t) - \sum_{i=1}^{N} \ell_i(t) f_{0,k}(\xi_i) \right)^2 \frac{c_{0,k}(w_B)}{\mu_k^2}.
\]

We proceed to estimate the right hand side of (3.23). Since \( \mu_k \geq \eta \Lambda = \tilde{\Lambda} > \Lambda \), we have

\[
\frac{\mu_k^{2-m} \Lambda^{2N}}{4^{2N-1}(\mu_k - \Lambda)^{2(N+1)}} \leq \frac{\Lambda^{1-m}}{4^{2N-1}\eta^{2N+m-1}} \left( \frac{\mu_k}{\mu_k - \Lambda} \right)^{2(N+1)}
\]

and \( \mu_k(\mu_k - \Lambda)^{-1} = (1 - \Lambda/\mu_k)^{-1} \leq \eta(\eta - 1)^{-1} \), recalling \( \eta > 1 \). Hence,

\[
\mu_k^{1+m}\|f_{m,k}(\cdot) - \sum_{i=1}^{N} \ell_i(\cdot) f_{m,k}(\xi_i)\|_{L^2(0,\Lambda)}^2 \leq \frac{\Lambda^{1-m}\mu_k^{3-m}}{4^{2N-1} \eta^{2N+m-1}} \left( \frac{1}{\eta - 1} \right)^{2(N+1)}.
\]

(3.25)

Using Lemma 3.6 and (3.25) together with (3.24) gives

\[
e_l^2 \leq \frac{2\eta^{l+1}}{4^{2N-1}(\eta - 1)^{2N+2}} \left( \Lambda^{l-1}\|\nabla(E w_B)\|_{L^2(\bar{U})}^2 + \eta \Lambda \|E w_B\|_{L^2(\bar{U})}^2 \right)
\]

for \( l = 1 \).

Carrying out similar argumentation leads to the same formula for \( l = 0 \). Estimating the coefficient completes the proof.

We conclude this subsection by using Lemma 3.7 to obtain an upper bound for the local interpolation error:

Theorem 3.8. Let \( Z, \tilde{Z} : (0, \Lambda) \to \mathcal{B}(tr\mathcal{V}(\bar{U}; \mathbb{R}^N), \mathcal{V}(\bar{U})) \) be as defined in (3.10) and (3.17), respectively. In addition, define \( Z_U, \tilde{Z}_U : (0, \Lambda) \to \mathcal{B}(tr\mathcal{V}(\bar{U}; \mathbb{R}^N), \mathcal{V}(\bar{U})) \) as \( Z_U(t)w = (Z(t)w)|_U \) and \( \tilde{Z}_U(t)w = (\tilde{Z}(t)w)|_U \), respectively. Then for \( t \in (0, \Lambda) \)

\[
\|\tilde{Z}_U(t) - Z_U(t)\|_{\mathcal{B}(tr\mathcal{V}(\bar{U}), \mathcal{V})} \leq C_E e(\eta, N),
\]

where \( C_E = C_E(\mathcal{V}, U, \bar{U}) := \|E\|_{\mathcal{B}(tr\mathcal{V}(\bar{U}), \mathcal{V}(\bar{U}))} \) and

\[
e(\eta, N) = 12\eta [4(\eta - 1)]^{-1/2} \left( 2 + \eta \Lambda + \frac{1}{\eta \Lambda} \right)^{1/2}.
\]

Recall that \( Z_U, \tilde{Z}_U \) depend implicitly on \( \Lambda, N \). We expect the constant \( C_E \) to be inversely proportional to the extension radius \( r \). Note that for \( \eta > 5/4 \), increasing the number of interpolation points \( N \) decreases the error exponentially.

3.6. Low-rank approximation error. Recall the definitions of the operator \( B \in \mathcal{B}(tr\mathcal{V}(\bar{U}; \mathbb{R}^N), \mathcal{V}(\bar{U})) \) in (3.18) and \( \mathcal{W}(U) \),

\[
Bu_B := [Z_U(\xi_1) \ldots Z_U(\xi_N)] v_B \text{ and } \mathcal{W}(U) := \text{range}(\tilde{B}),
\]
where \( \hat{B} \in \mathcal{B}(\text{tr}\mathcal{V}(\hat{U};\mathbb{R}^N), \mathcal{V}(U)) \) is a finite-rank operator. Next, we relate the error term in (3.15) to the operator norm of \( B - \hat{B} \). We define

\[
\|w_B\|_{\text{tr}\mathcal{V}(\hat{U};\mathbb{R}^N)} := \left( \sum_{i=1}^{N} \|w_{B,i}\|_{\text{tr}\mathcal{V}(\hat{U})}^2 \right)^{1/2}
\]

and

\[
|w|_{\mathcal{V}(U)} := \left( \int_U |\nabla (Rw)|^2 \, dx + \int_U w^2 \, dx \right)^{1/2}.
\]

We proceed with a technical lemma:

**Lemma 3.9.** For any \( t \in (0, \Lambda) \) and \( w \in \mathcal{V}(\hat{U}) \)

\[
\|\ell(t)w_B\|_{\text{tr}\mathcal{V}(\hat{U};\mathbb{R}^N)} \leq \frac{\Lambda_N}{\sqrt{2}} \|w\|_{H^1(\hat{U})},
\]

where \( w_B = \gamma_{\hat{U}}w \) and \( \Lambda_N := \max_{t \in [0, \Lambda]} \sum_{i=1}^{N} |\ell_i(t)| \) is the Lebesgue constant related to the Chebyshev nodes \( \{\xi_i\}_{i=1}^{N} \subset (0, \Lambda) \).

For the estimate of the Lebesgue constant see, e.g., [11].

**Proof.** Using definitions (3.26) and (2.3),

\[
\|\ell(t)w_B\|_{\text{tr}\mathcal{V}(\hat{U};\mathbb{R}^N)}^2 = \|w_B\|_{\text{tr}\mathcal{V}(\hat{U})}^2 \sum_{i=1}^{N} |\ell_i(\lambda)|^2 \leq \frac{1}{2} \|w\|_{H^1(\hat{U})}^2 \sum_{i=1}^{N} |\ell_i(\lambda)|^2.
\]

The proof is completed by observing that \( \sum_{i=1}^{N} |\ell_i(t)|^2 \leq \left( \sum_{i=1}^{N} |\ell_i(t)| \right)^2 \leq \Lambda_N^2. \)

We are now in the position to give an upper bound for the error term \( e_U(W(U)) \) in (3.15).

**Theorem 3.10.** Let \( \{\xi_i\}_{i=1}^{N} \) be the Chebyshev nodes on \((0, \Lambda)\), \( Z_U(t) \) be as in (3.12), and \( e_U \) as defined in (3.15). Further, let \( R \in \mathcal{B}(\mathcal{V}(U), \mathcal{V}) \) be a stitching operator as defined in Section 2.2, and \( B \in \mathcal{B}(\text{tr}\mathcal{V}(\hat{U};\mathbb{R}^N), \mathcal{V}(U)) \) be as defined in (3.18). For any \( \hat{B} \in \mathcal{B}(\text{tr}\mathcal{V}(U;\mathbb{R}^N), \mathcal{V}(U)) \),

\[
e_U(W(U))^{1/2} \leq \frac{1}{\sqrt{2}} \left[ C_E e(\eta, N) \|R\|_{\mathcal{B}(\mathcal{V}(U), \mathcal{V})} + \Lambda_N \|B - \hat{B}\|_* \right],
\]

where \( W(U) = \text{range}(\mathcal{B}(\mathcal{V}(U); \mathcal{V})) \), and \( C_E, e(\eta, N) \) are as defined in Theorem 3.8. Here we denote \( \| \cdot \|_* := \| \cdot \|_{\mathcal{B}(\text{tr}\mathcal{V}(\hat{U};\mathbb{R}^N), \mathcal{V}(U))}. \)

**Proof.** Let \( w \in H^1(\hat{U}) \) and \( w_B = \gamma_{\hat{U}}w \). Observe that

\[
\inf_{v \in W(U)} \left( \int_U |\nabla (Z_U(t)w_B - v)|^2 \, dx \right)^{1/2} \leq \frac{1}{\sqrt{2}} \|R\|_{\mathcal{B}(\mathcal{V}(U), \mathcal{V})} \|(Z_U(t) - \hat{Z}_U(t))\|_{\mathcal{B}(\text{tr}\mathcal{V}(\hat{U}), \mathcal{V})} \|w\|_{H^1(\hat{U})} + \inf_{v \in W(U)} \left( \int_U |\nabla (\hat{Z}_U(t)w_B - v)|^2 \, dx \right)^{1/2}
\]
Lemma 3.9 completes the proof.

The first term on the right hand side is bounded by using Theorem 3.8. Choosing \( v = \hat{B}t(t)w_B \in \mathcal{W}(U) \) and recalling \( \hat{Z}_U(t)w_B = Bt(t)w_B \) yields

\[
\inf_{v \in \mathcal{W}(U)} \left( \int_U |\nabla[R(\hat{Z}_U(t)w_B - v)]|^2 \, dx \right)^{1/2} \leq \|B - \hat{B}\| \|\ell(t)w_B\|_{\mathcal{W}(U;\mathcal{R}^N)}.
\]

Lemma 3.9 completes the proof.

We have now constructed the local subspace \( \tilde{\mathcal{V}}(U) \) and estimated the local approximation error \( \mathcal{E}(u, U) \) for a subdomain \( U = U^{(p)} \subset \Omega \) via (3.16). The error estimate for the global reduced problem follows by using the stitching operators.

4. Partition of Unity CPI. We proceed to define the local subspaces \( \tilde{\mathcal{V}}(U^{(p)}) \) used in the PU-CPI method and to derive a relative eigenvalue error estimate.

We extend the notation of Section 3 to the case of several subdomains \( \{U^{(p)}\}_{p=1}^M \), and we set \( U = U^{(p)} \) for \( p \in \{1, \ldots, M\} \). Denote the \( r \)-extension of \( U^{(p)} \) by \( \hat{U}^{(p)} \) as in (3.1). Let \( (\mu_k^{(p)}, v_k^{(p)}) \in \mathbb{R}^+ \times \mathcal{V}_0(\hat{U}^{(p)}) \setminus \{0\} \) satisfy

\[
\int_{\hat{U}^{(p)}} \nabla v_k^{(p)} \cdot \nabla w \, dx = \mu_k^{(p)} \int_{\hat{U}^{(p)}} v_k^{(p)} w \, dx,
\]

for each \( w \in \mathcal{V}_0(\hat{U}^{(p)}) \) as in (3.6). The functions \( v_k^{(p)} \) are chosen orthonormal in \( L^2(\hat{U}^{(p)}) \), \( \mu_k^{(p)} \) are in non-decreasing order, and \( K^{(p)}(\hat{\lambda}) := \# \{ \, k \in \mathbb{N} \mid \mu_k^{(p)} \leq \hat{\lambda} \, \} \). Similarly to (3.14), the local subspaces are \( \tilde{\mathcal{V}}(U^{(p)}) = \hat{E}_\Lambda(U^{(p)}) \oplus \mathcal{W}(U^{(p)}) \), where

\[
(4.1) \quad E_\Lambda(U^{(p)}) = \text{span}\{v_1^{(p)}|_{U^{(p)}}, \ldots, v_{K^{(p)}}^{(p)}|_{U^{(p)}}\}.
\]

Define \( Z^{(p)} : (0, \Lambda) \to \mathcal{B}(\mathcal{V}(\hat{U}^{(p)}), \mathcal{V}_0(\hat{U}^{(p)}) \} \) by replacing \( \mu_k, v_k, \) and \( E \) in (3.12) by \( \mu_k^{(p)}, v_k^{(p)}, \) and the right inverse of the trace operator \( E^{(p)} : \text{tr}\mathcal{V}(\hat{U}^{(p)}) \to \{ \, v \in \mathcal{V}(\hat{U}^{(p)}) \mid v_{U^{(p)}} = 0 \} \). Recall that the existence of \( E^{(p)} \) is a structural assumption made on \( \mathcal{V}, U^{(p)}, \) and \( \hat{U}^{(p)} \).

Let \( Z_{U^{(p)}} : (0, \Lambda) \to \mathcal{B}(\mathcal{V}(\hat{U}^{(p)}), \mathcal{V}(U^{(p)}) \} \) be defined as in (3.12) and

\[
(4.2) \quad B^{(p)} = \mathcal{B}(\text{tr}\mathcal{V}(\hat{U}^{(p)}), \mathcal{V}(U^{(p)})) \quad \text{such that}
\]

\[
B^{(p)} = [Z_{U^{(p)}}(\xi_1) \ldots Z_{U^{(p)}}(\xi_N)].
\]

We choose the complementing subspace as \( \mathcal{W}(U^{(p)}) = \text{range}(\hat{B}^{(p)}) \), where \( \hat{B}^{(p)} \in \mathcal{B}(\text{tr}\mathcal{V}(\hat{U}^{(p)}); \mathcal{R}^N), \mathcal{V}(U^{(p)}) \} \) will later be a low-rank approximation of \( B^{(p)} \).

**Assumptions 4.1.** Let \( \Lambda > 0 \) and \( (\lambda_j, u_j) \in (0, \Lambda) \times \mathcal{V} \) satisfy (1.1). Make the same assumptions as in Proposition 2.1. Let \( \hat{\Lambda} = \eta \Lambda \) for \( \eta > 1 \), and let \( (\xi_i)_{i=1}^N \) be the Chebyshev interpolation points of \( (0, \Lambda) \).

**Theorem 4.2.** Make Assumptions 4.1. For \( p = 1, \ldots, M \), let \( E_\Lambda(U^{(p)}) \) be as defined in (4.1), \( R^{(p)} \in \mathcal{B}(\mathcal{V}(U^{(p)}), \mathcal{V}) \) satisfy the assumptions of Section 2.2, \( B^{(p)} \) be as defined in (4.2), and \( \hat{B}^{(p)} \in \mathcal{B}(\text{tr}\mathcal{V}(\hat{U}^{(p)}); \mathcal{R}^N), \mathcal{V}(U^{(p)}) \} \). Define the PU-CPI method subspace \( \tilde{\mathcal{V}} \) as in (2.7) using the local subspaces \( \tilde{\mathcal{V}}(U^{(p)}) = E_\Lambda(U^{(p)}) \oplus \mathcal{W}(U^{(p)}) \) and the local complementing subspaces \( \mathcal{W}(U^{(p)}) = \text{range}(\hat{B}^{(p)}) \).

Then there exists \( \hat{\lambda} \in \sigma(\hat{\mathcal{V}}) \) such that

\[
\frac{|\lambda_j - \hat{\lambda}|}{\lambda_j} \leq C_M(\lambda_j) \max_{p=1,\ldots,M} \left[ A_N^2 \|B^{(p)} - \hat{B}^{(p)}\|_*^2 + C_{\xi_N}(\eta, N)^2 \|R^{(p)}\|_B^2 \|B(\mathcal{W}(U^{(p)}), \mathcal{V})\|^2 \right],
\]
where \( e(\eta,N) \) and \( \| \cdot \|_a \) are as defined in Theorems 3.8 and 3.10, respectively. The constants \( C_M(\lambda_j) \) and \( C_{E(p)} \) are defined as

\[
C_M(\lambda_j) := C(\lambda_j)(\lambda_j + 1)\|\hat{G}\|_\infty^4 \quad \text{and} \quad C_{E(p)} := \|E(p)\|_{\mathcal{B}(\hat{V}(U(p)),\hat{V}(U(p)))},
\]

where \( C(\lambda_j) \) is as defined Proposition 2.1. The counting function \( \hat{G} : \Omega \to \{1, \ldots, M\} \) is defined as \( \hat{G}(x) := \#\{p \mid x \in \bar{U}(p)\} \).

Proof. Proposition 2.1 together with (2.8) and (3.16) gives

\[
\frac{|\lambda_j - \hat{\lambda}|}{\lambda_j} \leq C(\lambda_j)\|G\|_\infty^2 \sum_{p=1}^M \mathcal{E}(u_j, U(p)) \leq C(\lambda_j)\|G\|_\infty^2 \sum_{p=1}^M e_U \left( \mathcal{W}(U(p)) \right) \|u_j|_{\hat{G}(p)}\|_{H^1(\hat{G}(p))}^2
\]

for the local complementing subspaces \( \mathcal{W}(U(p)) = \text{range}(\hat{B}(p)) \) constructed in Section 3 for \( U = U(p) \). Estimating the sum similarly with (2.8) and observing that \( \|u_j\|_{H^1(\Omega)} = (\lambda_j + 1)\|u_j\|_{L^2(\Omega)} = (\lambda_j + 1) \), gives

\[
\frac{|\lambda_j - \hat{\lambda}|}{\lambda_j} \leq C(\lambda_j)(\lambda_j + 1)\|G\|_\infty^2 \|\hat{G}\|_\infty^2 \max_{p=1,\ldots,M} e_U \left( \mathcal{W}(U(p)) \right).
\]

Since \( U(p) \subset \bar{U}(p) \) we have \( \|G\|_\infty \leq \|\hat{G}\|_\infty \). Theorem 3.10 completes the proof. \( \Box \)

In the practical application of the PU-CPI method, the foremost challenge is to define the low-rank approximating operators \( \hat{B}(p) \) and to efficiently construct a basis for the local complementing subspaces \( \mathcal{W}(U(p)) = \text{range}(\hat{B}(p)) \). In Section 5, we use the finite element method, i.e., \( \mathcal{V} = \mathcal{V}_h \), and use singular value decomposition for this purpose.

5. Finite element realisation of PU-CPI. Define the set function (i.e., open interior of closure) \( \text{intc} : A \mapsto B \) as \( A = \text{int}(\overline{B}) \) for \( B \subset \mathbb{R}^d \). A finite family of sets \( \{K_i\}_i \subset \Omega \) is called a triangular or a tetrahedral partition of \( \Omega \), if \( K_i \subset \Omega \) are open simplicial sets satisfying \( \Omega = \text{intc}(\cup_i K_i) \) and \( K_i \cap K_j = \emptyset \) for \( i \neq j \). We make a standing assumption that partitions do not contain hanging nodes.

We consider the FE discretisation of (2.1) under the following assumptions.

Assumptions 5.1.

(i) Let \( \{\mathcal{T}_h\}_h \) be a family of shape regular triangular or a tetrahedral partitions of \( \Omega \) with mesh size \( h = \max_{K \in \mathcal{T}_h} \text{diam}(K) \) in the sense of [9].

(ii) Let

\[
(\mathcal{V}_h = \mathcal{V} = \{ \mathcal{W}_h = \{ w \in H^1_0(\Omega) \mid w|_{K} \in P^1(K) \quad \text{for all } K \in \mathcal{T}_h \}, \text{ and } \{ \psi_i \}_i \text{ the nodal basis functions of } \mathcal{V}_h. \}
\]

We call \( x \) the coordinate vector of \( w \in \mathcal{V}_h \) and define the one-to-one correspondence \( x \sim w \) when \( w = \sum x_i \psi_i \). The same convention is used in all subspaces of \( \mathcal{V}_h \).

An open cover \( \{U(p)\}_{p=1}^M \) is constructed by dividing the vertices of the partition \( \mathcal{T}_h \) into nonempty disjoint sets \( \{N_p\}_{p=1}^M \) using, e.g., METIS [20]. The set \( U(p) \) is obtained as

\[
U(p) = \text{intc} \{ K \in \mathcal{T}_h \mid K \text{ has at least one vertex index in } N_p \}
\]

\[\text{Observe that sets } \{U(p)\}_p \text{ consist of simplices in partition of } \mathcal{T}_h. \text{ Thus the diameter of each } U(p) \text{ is always larger than } h, \text{ linking the scale } h \text{ and scales of } U(p) \text{'s.} \]
The $r$-extension of a subdomain $U^{(p)}$ is chosen as

\[(5.3) \quad \hat{U}^{(p)} = \text{intc} \left\{ K \in \mathcal{T}_h \mid \text{dist}(K, U^{(p)}) \leq r \right\}.\]

An example of an open cover and the related $r$-extensions is given in Figure 2. Note that our definition allows very exotic open covers, not all of which are computationally meaningful.

We proceed to define bases for the subspaces defined on $U \equiv U^{(p)}$ and $\hat{U} \equiv \hat{U}^{(p)}$:

\[(5.4) \quad \mathcal{V}_h(\hat{U}) = \text{span} \{ \psi_{\hat{U}}^1, \ldots, \psi_{\hat{U}}^n \}, \quad \mathcal{V}_h(U) = \text{span} \{ \psi_U^1, \ldots, \psi_U^n \}.\]

We further assume that the basis functions are ordered so that

\[(5.5) \quad \text{tr} \mathcal{V}_h(\hat{U}) = \text{span} \{ \psi_{\hat{U}}^1|_{\partial \hat{U}}, \ldots, \psi_{\hat{U}}^n|_{\partial \hat{U}} \}, \quad \text{tr} \mathcal{V}_h(U) = \text{span} \{ \psi_U^1|_{\partial U}, \ldots, \psi_U^n|_{\partial U} \}, \quad \mathcal{V}_{h0}(\hat{U}) = \text{span} \{ \psi_{\hat{U}}^1_{n_B+1}, \ldots, \psi_{\hat{U}}^n \}, \quad \mathcal{V}_{h0}(U) = \text{span} \{ \psi_U^1_{n_B+1}, \ldots, \psi_U^n \}.\]

Denote $n_I = n - n_B$ and $\hat{n}_I = \hat{n} - \hat{n}_B$ and assume that $n_I, \hat{n}_B, \hat{n}_I, \text{ and } n_B$ all are non-zero. Because of the ordering in (5.5), it is natural to split the coordinate vectors $x \in \mathbb{R}^{\hat{n}}$ to the boundary and interior coordinates as

\[(5.6) \quad x = \begin{bmatrix} x_B \\ x_I \end{bmatrix} \quad \text{where} \quad x_B \in \mathbb{R}^{\hat{n}_B} \quad \text{and} \quad x_I \in \mathbb{R}^{\hat{n}_I}.\]

This splitting is applied to $\hat{n} \times \hat{n}$-matrices as follows

\[(5.7) \quad A = \begin{bmatrix} A_{BB} & A_{BI} \\ A_{IB} & A_{II} \end{bmatrix},\]

where $A_{BB} \in \mathbb{R}^{\hat{n}_B \times \hat{n}_B}, A_{BI} \in \mathbb{R}^{\hat{n}_B \times \hat{n}_I}, A_{IB} \in \mathbb{R}^{\hat{n}_I \times \hat{n}_B}$ and $A_{II} \in \mathbb{R}^{\hat{n}_I \times \hat{n}_I}$. Let $E_h: \text{tr} \mathcal{V}_h(\hat{U}) \to \mathcal{V}_h(\hat{U})$ be defined as

\[E_h w_B = \sum_{l=1}^{\hat{n}_B} x_{B l}^I \psi_{\hat{U}}^l \quad \text{where} \quad x_B = \begin{bmatrix} x_{B1} \\ \vdots \\ x_{B \hat{n}_B} \end{bmatrix} \sim w_B.\]

That is, $E_h$ is a right inverse of the trace operator that satisfies $(E_h w_B)|_U = 0$. 

---

**Fig. 2.** A partitioning of a cuboid with four subdomains and a visualisation of an extended subdomain on one part. Surface triangles belonging to several $U^{(p)}$ and to set $\Gamma$ defined in (5.4) are visualised in white.
5.1. Evaluation of the trace norm. We discuss evaluation of the norm of $tr\mathcal{V}_h(\hat{U})$ required to construct $\hat{B}$ in practice.

Lemma 5.2. Let $\mathcal{V}_h(\hat{U})$, $tr\mathcal{V}_h(\hat{U})$ be as defined in (5.4) and assume that (5.5) holds. Define $K \in \mathbb{R}^{n \times n}$ as

$$K_{ij} = \int_{\hat{U}} \left( \nabla \psi_i^0 \cdot \nabla \psi_j^0 + \psi_i^0 \psi_j^0 \right) \, dx \quad \text{for } i, j = 1, \ldots, n$$

and then split $K$ into $K_{BB}, K_BI$, and $K_{II}$ according to (5.7). Then for any $f \in tr\mathcal{V}_h(\hat{U})$,

$$(5.8) \quad \|f\|_{tr\mathcal{V}_h(\hat{U})} = (x_B^T S x_B)^{1/2} \quad \text{where } x_B \sim f \quad \text{and } S = K_{BB} - K_{BI} K_{II}^{-1} K_{BI}^T.$$

Proof. Observe that for $v_1, v_2 \in \mathcal{V}_h(\hat{U})$ it holds

$$(v_1, v_2)_{H^1(\hat{U})} = v_2^T K v_1 \quad \text{where } v_1 \sim v_1, v_2 \sim v_2.$$

Using the splitting (5.6) and unitary equivalence (5.9) gives

$$\|f\|_{tr\mathcal{V}_h(\hat{U})} = \frac{1}{2} \min_{y_I \in \mathbb{R}^n} \left[ x_B^T \begin{bmatrix} K_{BB} & K_{BI} \\ K_{BI}^T & K_{II} \end{bmatrix} K_{II}^{-1} \begin{bmatrix} K_{BB} & K_{BI} \\ K_{BI}^T & K_{II} \end{bmatrix} y_I \right] \quad \text{where } x_B \sim f.$$

Direct calculation gives $y_I = -K_{II}^{-1} K_{BI}^T x_B$. Hence,

$$\|f\|_{tr\mathcal{V}_h(\hat{U})}^2 = x_B^T \begin{bmatrix} 1 & -K_{II}^{-1} K_{BI}^T \\ -K_{BI} K_{II}^{-1} K_{BI}^T & K_{II} \end{bmatrix} \begin{bmatrix} K_{BB} & K_{BI} \\ K_{BI}^T & K_{II} \end{bmatrix} \begin{bmatrix} x_B \\ y_I \end{bmatrix} = x_B^T S x_B,$$

which completes the proof. \hfill \square

Remark 5.3. The matrix $S$ defined in (5.8) is dense and expensive to construct. To circumvent this, consider the linear system

$$\begin{bmatrix} K_{BB} & K_{BI} \\ K_{BI}^T & K_{II} \end{bmatrix} \begin{bmatrix} y_B \\ y_I \end{bmatrix} = \begin{bmatrix} x_B \\ 0 \end{bmatrix}.$$}

By direct calculation $S y_B = x_B$. Since $K$ is invertible, so is $S$. Hence,

$$(5.10) \quad S^{-1} x_B = F_B K^{-1} F_B x_B \quad \text{where } F_B \in \mathbb{R}^{n \times n}, (F_B)_{ij} = \delta_{ij}.$$

Using the equation above, the action of $S^{-1}$ can be efficiently computed by storing the Cholesky factorisation of $K$. Due to this, our implementation of PU-CPI method subspace uses $S^{-1}$ instead of $S$.

5.2. Stitching operators. In Section 2.2, the open cover $\{U^{(p)}\}_p$ is related to a family of stitching operators $\{R^{(p)}\}_p$, $R^{(p)} : \mathcal{V}(U^{(p)}) \to \mathcal{V}$. For $V = \mathcal{V}_h$ we define the stitching operator $R_h : \mathcal{V}_h(U) \to \mathcal{V}_h$ corresponding the subdomain $U = U^{(p)}$ by

$$(5.11) \quad (R_{h})_{|_{\Omega \setminus U}} = 0 \quad \text{and } (R_h w)|_U = \sum_{l=n_B+1}^{n} \psi_l^U x_l.$$

Even though $\{\psi_l^U\}_{l=n_B+1}^n$ is a basis of $\mathcal{V}_h_0(U)$, the embedding $\mathcal{V}_h_0(U)$ into $\mathcal{V}_h$ by zero extension makes it possible to regard $R_h w$ as element of $\mathcal{V}_h$. The PU-CPI error estimate in Theorem 4.2 depends on $\|R_h\|_{\mathcal{V}(\mathcal{V}_h(U), \mathcal{V}_h)}$, which we estimate next.

---

3In our implementation of the stitching operator, we select the basis functions $\{\psi_l^U\}$ from the set $\{\psi_l\}_l$ to avoid changing bases. Keeping track of the related indexing is challenging and not discussed here nor in the following.
Lemma 5.4. Let $U \subset \Omega$ be defined similarly to (5.2) and $R_h \in \mathcal{B}(\mathcal{V}_h(U), \mathcal{V}_h)$ as in (5.11). Under Assumptions 5.1 there exists constant $C_R = C_R(\{T_h\}_h)$ such that

$$\|R_h\|_{\mathcal{B}(\mathcal{V}_h(U), \mathcal{V}_h)} \leq C_R h^{-1}.$$

Proof. Recall that $\mathcal{V}_h$ and $\mathcal{V}_h(U)$ inherit their norms from $H^1_0(\Omega)$ and $H^1(U)$, respectively. Let $w \in \mathcal{V}_h(U)$ and $x \sim w$. By the inverse inequality in, e.g., [10, Section 4.5] there exists constant $C_{inv} := C_{inv}(\{T_h\}_h)$, independent of $h$, such that

$$\|R_h w\|_{H^1_0(\Omega)} = \|\nabla R_h w\|_{L^2(\Omega; \mathbb{R}^d)} \leq C_{inv} h^{-1} \|R_h w\|_{L^2(\Omega)}.$$

Observe that $\text{supp}(R_h w) \subset U$ for each $w \in \mathcal{V}_h(U)$. The following norm equivalence is given, e.g., in [10, Lemma 6.2.7]:

$$c_1 h^{d/2} |x| \leq \|w\|_{L^2(U)} \leq C_1 h^{d/2} |x| \quad \text{where} \quad |x| = (x^T x)^{1/2}$$

for any $w \in \mathcal{V}_h(U)$, $x \sim w$, and constants $c_1 = c_1(\{T_h\}_h), C_1 = C_1(\{T_h\}_h)$. Using (5.12) and the definition (5.11) gives

$$\|R_h w\|_{L^2(U)} \leq C_1 h^{d/2} \left( \sum_{d=n_B+1}^{n} x_d^2 \right)^{1/2} \leq C_1 h^{d/2} |x| \leq C_1 c_1^{-1} \|w\|_{L^2(U)}.$$

5.3. The local complementing subspace. We proceed to construct a basis for the local complementing subspace $\mathcal{V}_h(U)$. To this end, we represent the linear operators $Z_h(t)$ and $B_h$ as matrices using the bases of $tr\mathcal{V}_h(\hat{U}), \mathcal{V}_{h0}(\hat{U})$ and $\mathcal{V}_h(U)$ defined in (5.4)–(5.5). Denote by $A, M \in \mathbb{R}^{N \times N}$ the stiffness and mass matrices of the FE-discretised version of (3.2), respectively. Both of these matrices are splitted as in (5.7). Following Section 3.3, the matrix representation of $Z_h(t)$ is $Z_h : (0, \Lambda) \to \mathbb{R}^{\tilde{n} \times \tilde{n}}$ given by

$$Z_h(t) := P_h (A_{II} - t M_{II})^{-1} (-A_{BI}^T + t M_{BI}^T),$$

where $Z_h(t)$ is real analytic for all $t \in (0, \Lambda)$. Here $\dagger$ is the Moore-Penrose pseudo-inverse and $P_h := I - \sum_{k=1}^{K} v_k v_k^T M_{II}$, where $v_k \sim v_k$ for eigenfunctions $v_k \in \mathcal{V}_h(\hat{U})$ of (3.6) \footnote{This is another way to define $Z_h$ for all $t \in (0, \Lambda)$ compared to Section 3.2, also used in [17].}. The matrix representation of the operator $B_h$, defined in (3.18), in the natural basis of the cartesian product space $tr\mathcal{V}_h(\hat{U}; \mathbb{R}^N)$ is

$$B_h = F_U \begin{bmatrix} Z_h(\xi_1) & \cdots & Z_h(\xi_N) \end{bmatrix} \in \mathbb{R}^{n \times \tilde{n} \times \tilde{n}} \times \tilde{n},$$

where $n = \dim(\mathcal{V}_h(U))$ and $F_U \in \mathbb{R}^{n \times \tilde{n} \times \tilde{n}}$ is the matrix representation of the restriction operator $F_U : \mathcal{V}_h(\hat{U}) \to \mathcal{V}_h(U)$ given by $F_U v = v|_U$ in bases (5.4)–(5.5). The norm of the cartesian product space $tr\mathcal{V}_h(\hat{U}; \mathbb{R}^N)$ in terms of coordinate vectors is given by

$$\|v_B\|_{tr\mathcal{V}_h(\hat{U}; \mathbb{R}^N)} = \|(1_N \otimes S^{1/2}) x_B\|_2 \quad \text{for} \quad x_B \sim v_B$$

by Lemma 5.2. Here $1_N \in \mathbb{R}^{N \times N}$ is the identity matrix and $\otimes$ denotes the Kronecker product. Finally, observe that $\|w\|_{\mathcal{V}_h(U)} = \|K_R^{1/2} x\|_2$ with $x \sim w$ and the symmetric,
operators are obtained by computing the SVD of the
\[ R(5.17) \]
respectively. Then
\[ \| \cdot \| \]
has the singular values
\[ \sigma \]
min
\[ (5.16) \]
positive definite matrix
\[ K_R \in \mathbb{R}^{n \times n} \]
defined as
\[ (K_R)_{lm} = \int_U (\nabla (R_h \psi_l^U) \cdot \nabla (R_h \psi_m^U) + \psi_l^U \psi_m^U) \, dx. \]

It is well–known that the finite–dimensional operator \( B_h \in B(\text{tr} \mathcal{V}_h(\overline{U}; \mathbb{R}^N), \mathcal{V}_h R(U)) \) has the singular values \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0 \) and for \( k < n \) there exists rank \( k \) operators \( B_{hk} \) satisfying
\[ \min_{\text{rank}(T) \leq k} \| B_h - T \|_* = \| B_h - B_{hk} \|_* = \sigma_{k+1}, \]
where \( \| \cdot \|_* = \| \cdot \|_{B(\text{tr} \mathcal{V}_h(\overline{U}; \mathbb{R}^N), \mathcal{V}_h R(U))} \). Here, we have used the fact that \( n < \hat{n} \). These operators are obtained by computing the SVD of the \( \mathbb{R}^{n \times \hat{n}} \)–matrix
\[ C := K_R^{1/2} B_h (I_N \otimes S^{-1/2}) = \sum_{i=1}^n \sigma_i u_i v_i^T, \]
where \( \{ u_i \}_{i=1}^n \subset \mathbb{R}^n \) and \( \{ v_i \}_{i=1}^\hat{n} \subset \mathbb{R}^\hat{n} \) are left– and right–singular vectors of \( C \), respectively. Then
\[ (5.17) \]
as can be seen from the definition of the operator norm \( \| \cdot \|_* \) by a change of variables.

Let the local complementing subspace be \( \mathcal{W}_h(U) = \text{range}(B_{hk}) \) for \( B_{hk} \) given in
\[ (5.17) \]
The basis for \( \mathcal{W}_h(U) \) is obtained from the first \( k \) left–singular vectors \( \{ u_i \}_i \) of the matrix \( C \) as
\[ (5.18) \]
In practice, the vectors \( \{ u_i \}_i \) are computed by solving the largest \( k \) eigenpairs of the \( \mathbb{R}^{n \times n} \)–matrix
\[ (5.19) \]
using the Lanczos iteration with the mapping \( \mathbf{x} \mapsto CC^T \mathbf{x} \). There are two reasons for using the dual approach. First, the dimension of \( CC^T \) is independent of \( N \). Second, an explicit construction of \( S \) is avoided by utilising Remark 5.3.

Combining the above discussion with Theorem 4.2 yields an estimate for the relative eigenvalue error.

**Theorem 5.5.** Make Assumptions 4.1 and let \( \overline{\mathcal{V}}_h \) satisfy Assumptions 5.1. Let the stitching operators \( B_{h(p)} \), \( p \in \{ 1, \ldots, M \} \), be defined as in (5.11) for \( U = U^{(p)} \). Let the singular values \( \sigma_1^{(p)} \geq \sigma_2^{(p)} \geq \ldots \geq \sigma_{n(p)}^{(p)} \) and left–singular vectors \( \{ u_i^{(p)} \}_{i=1}^{n(p)} \) be defined as above for \( U = U^{(p)} \). The local complementing subspaces are defined as
\[ \mathcal{W}_h(U^{(p)}) := \left\{ \sum_i y_i \psi_i^{U(p)} \mid y \in K_R^{(p)-1/2} \text{span}\{ u_1^{(p)}, \ldots, u_{k(p)}^{(p)} \} \right\}, \]

\(^5\)In practice, the square roots \( K_R^{1/2} \) are replaced by the Cholesky factors of \( K_R \).
where \( \{ k^{(p)} \}_{p=1}^{M} \) are local cut-off indices, \( \{ \psi_{l}^{(p)} \}_{l} \) is a basis of \( \mathcal{V}_{h}(U^{(p)}) \), and \( R^{(p)}_{R} \) defined as in (5.15) for \( U = U^{(p)} \) and \( R = R^{(p)} \). Define the local subspaces as \( \tilde{\mathcal{V}}_{h}(U^{(p)}) := E_{\lambda}^{(p)}(U^{(p)}) \oplus W_{h}(U^{(p)}) \), where \( E_{\lambda}^{(p)}(U^{(p)}) \) is as in (4.1), and the associated PU-CPI method subspace \( \tilde{\mathcal{V}}_{h} \) as in (2.7).

Then there exists \( \tilde{\lambda} \in \sigma(\tilde{\mathcal{V}}_{h}) \) such that

\[
\frac{|\lambda_{j} - \tilde{\lambda}|}{\lambda_{j}} \leq C_{M}(\lambda_{j}) \max_{p=1, \ldots, M} \left[ \Lambda_{N}(\sigma_{k_{h}^{(p)}+1})^{2} + C_{E}^{2}C_{R}^{2}h^{-2}e^{2}(\eta, N) \right],
\]

where \( \Lambda_{N}, C_{M}(\lambda_{j}) \) and \( C_{E}^{2}(p) \) are as defined in Theorem 4.2, \( C_{R} \) as in Lemma 5.4, and \( e(\eta, N) \) as in Theorem 3.8.

### 5.4. Assembly of the PU-CPI Ritz eigenproblem.

The remaining task is to solve the global Ritz eigenvalue problem (1.2) posed in the PU-PCI method subspace \( \tilde{\mathcal{V}}_{h} \). Let \( \{ \varphi_{l}^{(p)} \}_{l} \) be a basis of the space \( R^{(p)}_{h} \tilde{\mathcal{V}}_{h}(U^{(p)}) \subset \mathcal{V}_{h} \) and denote \( n^{(p)} := \dim(R^{(p)}_{h} \tilde{\mathcal{V}}_{h}(U^{(p)})) \). Then the ordered set

\[
\{ \varphi_{l}^{(p)} \mid l = 1, \ldots, n^{(p)}, \ p = 1, \ldots, M \} = \{ \phi_{k} \mid k = 1, \ldots, \sum_{p=1}^{M} n^{(p)} \}
\]

is a basis for the PU-CPI method subspace \( \tilde{\mathcal{V}}_{h} \) defined in (2.7) with dimension \( \tilde{n} := \sum_{p=1}^{M} n^{(p)} \). The ordering in (5.20) defines an integer–valued function \( \sigma(p, l) \) satisfying

\[
\varphi_{l}^{(p)} = \phi_{\sigma(p,l)} \quad \text{for} \ l = 1, \ldots, n^{(p)}, \ p = 1, \ldots, M.
\]

Next, we assemble the matrices \( A, M \) in the global eigenproblem: find \( (\tilde{\lambda}_{k}, \tilde{\mathbf{v}}_{k}) \in \mathbb{R}^{+} \times \mathbb{R}^{n} \) such that

\[
A_{\tilde{\mathbf{v}}_{k}} = \tilde{\lambda}_{k}M_{\tilde{\mathbf{v}}_{k}},
\]

where \( A_{lm} = (\nabla \phi_{l}, \nabla \phi_{m})_{L^{2}(\Omega; \mathbb{R}^{i})} \) and \( M_{lm} = (\phi_{l}, \phi_{m})_{L^{2}(\Omega)} \). In our early numerical experiments, a straightforward assembly of \( A \) and \( M \) proved to be time consuming. Next, we outline a more efficient and numerically more stable strategy.

We only study the entries of \( A \) since the entries of \( M \) are computed similarly. The entries of \( A \) are obtained by computing

\[
A_{\sigma(p,l),\sigma(q,m)} = \int_{\Omega} \nabla \varphi_{l}^{(p)} \cdot \nabla \varphi_{m}^{(q)} \, dx
\]

for each \( l = 1, \ldots, n^{(p)}, m = 1, \ldots, n^{(q)} \) and \( p, q \in \{ 1, \ldots, M \} \). If \( p \neq q \) in (5.21),

\[
A_{\sigma(p,l),\sigma(q,m)} = \int_{\Gamma} \nabla \varphi_{l}^{(p)} \cdot \nabla \varphi_{m}^{(q)} \, dx,
\]

where the overlap set \( \Gamma \subset \Omega \) is defined as

\[
\Gamma = \text{intc}\{ K \in T_{h} \mid K \text{ has vertex indices in at least two sets } N_{p} \},
\]

see Figure 2. The off-diagonal entries in (5.22) can be computed if the functions \( \{ \varphi_{l} \}_{l=1}^{\tilde{n}} \) are known.
If $p = q$ in (5.21),

\begin{equation}
A_{\sigma(p,l),\sigma(p,m)} = \int_{U(p)} \nabla \varphi_l^{(p)} \cdot \nabla \varphi_m^{(p)} \, dx.
\end{equation}

To store the minimal amount of data, the basis functions $\{\varphi_l^{(p)}\}_{l=1}^{n(p)}$ are solutions of the symmetric eigenvalue problem

\begin{equation}
\int_{U(p)} \nabla \varphi_l^{(p)} \cdot \nabla \varphi_m^{(p)} \, dx = d_l^{(p)} \int_{U(p)} \varphi_l^{(p)} \varphi_m^{(p)} \, dx \quad \text{and} \quad \|\varphi_l^{(p)}\|_{L^2(U(p))} = 1
\end{equation}

for eigenvalues $d_l^{(p)} \in \mathbb{R}^+$ and for each $l, m = 1, \ldots, n^{(p)}$. Thus, for each $p$,

\begin{equation*}
A_{\sigma(p,l),\sigma(p,m)} = d_l^{(p)} \delta_{lm} \quad \text{and} \quad M_{\sigma(p,l),\sigma(p,m)} = \delta_{lm}.
\end{equation*}

To summarise, the matrices $A$ and $M$ can be fully characterised based on the data

\begin{equation*}
\{\varphi_l|_{\Gamma}\}_{l=1}^{n(p)}, \quad \{\nabla \varphi_l|_{\Gamma}\}_{l=1}^{n(p)}, \quad \text{and} \quad \{d_l^{(p)}\}_{l=1}^{n(p)} \quad \text{for} \quad p = 1, \ldots, M.
\end{equation*}

If needed, restrictions of the basis functions are can be stored, e.g., on some inner surface to visualise the eigenfunctions.

### 5.5. Overview of the PU-CPI algorithm

The PU-CPI is intended for distributed computing environment with a single master and multiple workers. The input data for the algorithm is specified in Table 1.

| Parameter | Description |
|-----------|-------------|
| $\Lambda$ | Spectral interval of interest $(0, \Lambda)$ |
| $N$       | Number of interpolation points |
| $\eta$    | Oversampling parameter |
| $T_h$     | Triangular $(d=2)$ or tetrahedral $(d=3)$ partition of $\Omega$ |
| $M$       | Number of subdomains |
| $r$       | Extension radius |
| $tol$     | Cut-off tolerance for singular values |

The cut-off tolerance is used to determine the parameters $k^{(p)}$ in Theorem 5.5 so that $\sigma_{k^{(p)}+1} \leq tol$. Theorem 5.5 gives the error estimate: for any $\lambda_j \in \sigma(V_h) \cap (0, \Lambda)$ there exists $\hat{\lambda} \in \sigma(\hat{V}_h)$ such that

\begin{equation*}
\frac{|\lambda_j - \hat{\lambda}|}{\lambda_j} \leq C \left[ tol^2 + e^2(\eta, N) \right] \quad \text{for some constant} \quad C.
\end{equation*}

The PU-CPI proceeds in three steps:

1. **(work division)** METIS is used to partition the vertices of $T_h$ into $M$ subsets by the master. The submeshes defining $U^{(p)}$ and $\hat{U}^{(p)}$ are created from these vertex sets as explained in Section 5. The submeshes defining $U^{(p)}$ and $\hat{U}^{(p)}$ for $p = 1, \ldots, M$ are submitted to workers.

2. **(distributed computation)** Each worker receives a submesh and computes a basis for $R_h^{(p)}V_h(U^{(p)})$ in the following steps (i)–(v), where all matrices refer to the subdomain $U^{(p)}$. 


(i) Assemble the stiffness and mass matrices $A, M$ related to $V_h(\tilde{U}(p))$. Split $A, M$ to interior and boundary parts according to (5.7). Compute the $K(\tilde{\Lambda})$ lowest eigenpairs $(\mu_k, v_k)$ of the pencil $(A_{II}, M_{II})$, and form the projection $P_h = I - \sum_{k=1}^{K(\tilde{\Lambda})} v_k v_k^T M_{II}$.

(ii) Construct the matrices $K_R$ as in (5.15), $F_U$ as in (5.14), $K$ as in Lemma 5.2, and $F_B$ as in (5.10).

(iii) Compute the largest eigenpairs $(\sigma_j^2, c_k)$ of

$$CC^T = L^T \left( \sum_{i=1}^{N} F_U Z_h(\xi_i) S^{-1} Z_h(\xi_i)^T F_U^T \right) L$$

using Lanczos iteration. The action $x \mapsto S^{-1}x$ is evaluated as explained in Remark 5.3.

(iv) An auxiliary basis for $R_h(p) V_h(\tilde{U}(p))$ is obtained from column vectors of $Q$,

$$Q := R \begin{bmatrix} F_U v_1, \ldots, F_U v_{K(\tilde{\Lambda})}, L^{-T} c_1, \ldots, L^{-T} c_K \end{bmatrix},$$

where $R$ is the matrix representation of $R_h(p)$ restricted to $V_{h0}(U(p))$. To satisfy (5.24), we solve the diagonal matrix $D$ and the invertible matrix $V$ from the eigenvalue problem

$$Q^T A_0 Q V = Q^T M_0 Q V D \text{ and } V^T Q^T M_0 Q V = I,$$

where $A_0$ and $M_0$ are stiffness and mass matrix in $V_{h0}(U(p))$. The final subspace is obtained from the columns of $Q = QV$.

(v) Submit $\text{diag}(D)$ and $Q(:, n_T)$ to the master. Here $n_T$ is set of those vertex indices that lie on $\Gamma$.

**Step 3. (Solution of the global PU-CPI eigenproblem)** The master solves (1.2) posed in the method subspace $\tilde{V}_h$. The required matrices are constructed as outlined in Section 5.4 and the resulting problem solved using the Lanczos iteration.

6. **Numerical examples.** We give numerical examples validating the theoretical results and demonstrating the potential of PU-CPI variant of Section 5. For this purpose, we use a cluster of 26 desktop computers of which 24 had a Xeon E3-1230 CPU, and two were equipped with Xeon W-2133. There was 32 GB of RAM in all but one workstation which had 64 GB. Because solving the smallest eigenvalues of the global Ritz eigenvalue problem (1.2) posed in the PU-PCI method subspace $\tilde{V}_h$ using shift-and-invert Lanczos iteration requires lots of memory, the workstation with 64 GB of RAM acted as the master. All data were transferred over NFS, and distributed tasks were launched using GNU parallel [25]. All computations were done using MATLAB R2019a mostly during office hours, when the computers were also in use. This results to conservative run-time estimates for PU-CPI.

We study the behaviour and convergence of the method using the domain

$$\Omega = F((0,1)^3) \text{ where } F : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 + 0.4x_3(2x_1 - 1) \\ x_2 + 0.4x_3(2x_2 - 1) \\ x_3 \end{bmatrix},$$

The homogenous Dirichlet boundary condition is imposed on $\partial U(p) \cap \partial \Omega$ and this has been communicated to the worker.
see Figure 2. As in Section 5, problem (1.1) is posed in the space \( V \equiv V_h \), where \( V_h \) is the finite element space of piecewise linear function over tetrahedral partition \( T_h \) of domain \( \Omega \). The mesh parameter values \( h \) are varied by mapping different uniform tetrahedral meshes of \((0,1)^3\) with \( F \). The open cover \( \{ U^{(p)} \}_p \) of \( \Omega \) is constructed by splitting the vertex indices of the \( T_h \) into disjoint sets \( \{ N_p \}_p \) using METIS as explained in Section 5. The subdomains produced in this manner can have significantly different shapes and sizes in a way that cannot be controlled. We observed that choosing the extension radius \( r^{(p)} \) proportional to the diameter of the corresponding subdomain \( U^{(p)} \) is beneficial for keeping the dimension of the sub–problems reasonable. This is done heuristically: define the empirical radius of \( U^{(p)} \) by

\[
r^{(p)}_c = \frac{1}{2} \left( \max_{m \in N_p} u^T x_m - \min_{m \in N_p} u^T x_m \right),
\]

where \( u \) is the first principal component of the coordinate vector set \( \{ x_m \}_{m \in N_p} \). Unless otherwise stated, we choose the extension radius for subdomain \( U^{(p)} \) as \( r^{(p)} = 0.2 r^{(p)}_c \).

Intuitively speaking, we have observed that PU-CPI works best when the subdomains \( U^{(p)} \) touch each other as little as possible. So as to domain \( \Omega \) in (6.1), we observed that METIS produces subdomains that have significant intersections compared to their diameters. This represents the worst–case behaviour of PU-CPI.

Throughout this section, we approximate 200 lowest eigenvalues of problem (1.1), and the parameter \( \Lambda \) is chosen accordingly. While experimenting with PU-CPI, it appears that choosing \( N = 5 \) and \( \eta = 2.5 \) makes the interpolation error smaller than \( 10^{-10} \) for all mesh sizes \( h \) used. Hence, these values were kept fixed, and the dependency of the relative eigenvalue error on \( N \) and \( \eta \) was not investigated. We focus on the effect of cut-off tolerance of singular values, number of subdomains, problem size, and the extension radius on computational load and accuracy.

### 6.1. Varying mesh density.

The eigenvalue problem (1.1) was solved with different mesh parameters \( h \). Subdomains with about 5000 vertices were used except for the three densest meshes. For these meshes, a smaller number of larger subdomains was required to decrease \( \text{dim}(\tilde{V}_h) \), so that the eigenvalue problem (1.2) posed in space \( \tilde{V}_h \) could be solved by the master workstation. Since METIS failed to partition the densest mesh, it was manually divided into cube-shaped subdomains.

The results are shown in Table 2. The maximum relative eigenvalue error was estimated by comparing PU-CPI against shift-and-invert Lanczos solution of (1.1) using MATLAB’s \texttt{eigs} function with a tolerance of \( 10^{-10} \). The sparsity of the matrices produced by PU-CPI is shown in Figure 4. A breakdown of time required by each step of PU-CPI is shown in Table 3. The comparable values \( t_{\text{CPI}} \) and \( t_{\text{FEM}} \) are the wall clock times (in seconds) spent after the mesh structure was constructed. For fair comparison, standard FE solution uses MATLAB’s \texttt{eigs} function with a tolerance of \( 10^{-4} \). In addition, \( t_{\text{CPI}} \) includes file I/O times and network delays, where as \( t_{\text{FEM}} \) includes the time required to assemble the full stiffness and mass matrices.

### 6.2. Effect of subdomain extension.

When using a larger extension radius \( r^{(p)} \), the singular values \( \sigma_k^{(p)} \) of \( \mathbf{C} \) in (5.19) are expected to decay faster. This effect is studied using meshes with 54,872 and 195,112 of Degrees–Of–Freedom (DOF). In both cases, the singular values \( \sigma_k^{(p)} \) were computed for a single subdomain with extension radius \( r^{(p)} = 0.2 r^{(p)}_c, 0.6 r^{(p)}_c \) and \( r^{(p)}_c \). The results are shown in Figure 6, and the extended subdomains with different radii are visualised in Figure 3. As expected, the
Fig. 3. Examples of extended subdomains $\tilde{U}^{(p)}$ for extension radii $0.2r_c^{(p)}, 0.6r_c^{(p)}$, and $r_c^{(p)}$. The figure depicts a cross-section where $U^{(p)}$ is colored in dark blue.

Fig. 4. Sparsity pattern of the PU-CPI stiffness and mass matrices corresponding to (1.2) posed on $\tilde{V}_h$ with $\dim(V_h) = 195112$, $\dim(\tilde{V}_h) = 492$ and $M = 44$.

Table 2

| dim($V_h$) | dim(\tilde{V}_h) | $M$ | rel. fill-in | max rel error | $\text{avg}_p \{\dim(\tilde{V}(U^{(p)}))\}$ |
|-----------|-----------------|----|-------------|---------------|---------------------------------|
| 54 872    | 2713            | 13 | 436.0       | $4.28 \times 10^{-3}$ | 209               |
| 110 592   | 3777            | 25 | 291.9       | $1.90 \times 10^{-4}$ | 151               |
| 195 112   | 4920            | 44 | 171.9       | $2.99 \times 10^{-4}$ | 112               |
| 314 432   | 6048            | 69 | 121.1       | $3.73 \times 10^{-4}$ | 88                |
| 474 552   | 7686            | 103| 85.6        | $3.47 \times 10^{-4}$ | 75                |
| 681 472   | 9791            | 146| 76.9        | $3.81 \times 10^{-4}$ | 67                |
| 941 192   | 12 398          | 200| 68.2        | -              | 62                 |
| 1 259 712 | 15 587          | 267| 60.8        | -              | 58                 |
| 1 643 032 | 19 735          | 346| 59.9        | -              | 57                 |
| 2 097 152 | 24 276          | 440| 56.3        | -              | 55                 |
| 2 628 072 | 30 124          | 549| 56.4        | -              | 55                 |
| 3 241 792 | 12 820          | 150| 26.7        | -              | 85                 |
| 5 000 211 | 19 261          | 250| 24.7        | -              | 77                 |
| 10 360 232| 29 124          | 308| 22.8        | -              | 95                 |

Relative fill-in is the ratio of the number of non-zeros in the stiffness matrices from spaces $\tilde{V}_h$ and $V_h$. The last column is the average size of the dimension of the local method subspaces. Relative error is not given when the problem could not be solved using MATLAB eigs on a single workstation.

Singular values decay much faster for larger $r_c^{(p)}$. This comes at higher computational cost due to increase in the extended subdomain DOFs. At the same time, the faster decay of singular values leads to smaller $\dim(V_h)$.

6.3. The effect of the cut-off tolerance of singular values. The computations were performed using three different mesh densities and several values of $tol$. The maximum relative eigenvalue error and $\dim(V_h)$ are shown in Figure 5. Additionally, relative error for each of the 200 lowest eigenvalues are detailed in Figure 7. These results verify the linear relationship between $tol^2$ and the relative eigenvalue error predicted in Section 5.5. In this examples, choosing $tol = 1$ already produces relative eigenvalue error smaller than 1%.
Table 3

| dim($\tilde{V}_h$) | avg. $t_{sub}$ | METIS | $r$-ext. | $t_{red}$ | $t_{CPI}$ | $t_{FEM}$ |
|-------------------|----------------|--------|----------|----------|----------|----------|
| 54 872            | 32.3           | 1.6    | 6.5      | 13.3     | 105.1    | 53.9     |
| 110 592           | 35.5           | 3.4    | 13.6     | 19.2     | 130.0    | 142.2    |
| 195 112           | 48.8           | 6.2    | 31.7     | 25.2     | 177.2    | 307.5    |
| 314 432           | 59.1           | 10.3   | 48.6     | 33.8     | 227.3    | 661.7    |
| 474 552           | 63.8           | 16.7   | 78.3     | 43.5     | 303.7    | 1264.7   |
| 681 472           | 68.0           | 24.1   | 123.8    | 56.1     | 429.7    | 1859.7   |
| 941 192           | 71.2           | 34.7   | 189.7    | 79.4     | 610.8    |          |
| 12 597 12         | 75.5           | 47.2   | 263.8    | 107.0    | 772.7    |          |
| 1 643 032         | 75.8           | 65.0   | 357.6    | 151.6    | 1030.5   |          |
| 2 097 152         | 79.9           | 85.7   | 511.0    | 212.5    | 1397.2   |          |
| 2 628 072         | 79.6           | 107.3  | 671.3    | 302.4    | 1781.5   |          |
| 3 241 792         | 378.9          | 114.8  | 1562.0   | 165.5    | 4509.7   |          |
| 5 000 211         | 385.3          | 327.4  | 1680.4   | 306.1    | 6312.9   |          |
| 10 360 232        | 344.4          | -      | 1365.4   | 690.3    | 6525.7   |          |

Fig. 5. Maximum relative eigenvalue error using three mesh densities. Left panel: Given as a function of the cut-off tolerance for singular values $tol^2$.
Right panel: Given as a function of $\dim(\tilde{V}_h)$ in the same sample points.

7. Conclusions. PU-CPI method for the approximate solution of eigenvalues in $(0, \Lambda)$ of the Laplace operator on domain $\Omega$ is proposed. PU-CPI is a Ritz method where the method subspace $\tilde{V}$ is constructed from the local method subspaces $\{\tilde{V}(U^{(p)})\}_p$ of $U^{(p)} \subset \Omega$ as stated in (2.7). Since the local subspaces are independent of each other, PU-CPI can be used in distributed computing environments where communication is costly. Failed distributed tasks can be restarted, making the implementation of PU-CPI very robust.

Let $(u, \lambda)$ be solution of (3.20) for $\lambda \in (0, \Lambda)$. According to Proposition 2.1 and (2.8), the local method subspaces should be designed to approximate $u|_{U^{(p)}}$. Local information on $u|_{U^{(p)}}$ is obtained in terms of the operator-valued function $Z_U$ in Lemma 3.1. Since $Z_U$ is compact operator-valued by Lemma 3.2, its values can be
Fig. 6. Effect of the extension radius on the decay of singular values \(\{\sigma_j\}_{j=1}^{200}\) for the cube with 54,872 (left panel) and 195,112 (right panel) DOFs. In these experiments, extended subdomain DOFs range between 10,610 – 32,423 and 12,644 – 49,659, respectively.

Fig. 7. Relative eigenvalue errors for the 200 lowest modes for three mesh densities. The cut-off tolerance for singular values \(\text{tol}^2 = 0.1\) (left panel) and \(\text{tol}^2 = 0.001\) (right panel).

efficiently low-rank approximated.

The local method subspace for the single subdomain \(U \equiv U^{(p)}\) is designed to approximate range of \(Z_U\) in the sense of (3.15). This approximation makes use of interpolation, linearisation, and low-rank approximation as explained in Section 3.4. Local approximation error is estimated in Theorems 3.8 and 3.10. Theorem 4.2 combines these estimates to bound the global relative eigenvalue error.

An example of low-rank approximation is given for first-order FEM in Theorem 5.5. The key ingredient is Lemma 5.2 and Remark 5.3 that allow numerical treatment of a required boundary trace norm. A basis for the local method subspace is obtained from eigenvectors of \(CC^T\) in (5.19). The dimension of \(CC^T\) is independent of the number of interpolation points \(N\) and the oversampling parameter \(\eta\).

Finally, numerical examples validating the theoretical results and demonstrating the potential of PU-CPI are given in Section 6. The authors could use inexpensive networked workstations to solve an eigenvalue problem ten times as large as straightforwardly solvable on a single workstation. In contrary to using a supercomputer, such networked workstations are widely available.

Acoustic eigenvalues problem, for example, benefit from treatment of more general boundary conditions. The authors have implemented PU-CPI for mixed homogeneous Dirichlet and Neumann boundary conditions, and the error analysis extends to this case.
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