Radii and Density Calculations of $^{209}\text{Bi}$ by Using Skyrme-Hartree-Fock Method

Yacobus Yulianto$^{1,*}$, Zaki Su’ud$^{1,b}$

$^1$Nuclear and Biophysics Research Division, Physics Department, Faculty of Mathematics and Natural Science, Institut Teknologi Bandung, Jl. Ganesha 10 Bandung, 40132, Indonesia.

E-mail: $^a$yacyuli@nasi.com, $^b$zakisuud@gmail.com

Abstract. In this study, the densities, the neutron skin thickness, and the rms radii for both proton and neutron of $^{209}\text{Bi}$ are calculated using Hartree-Fock method with Skyrme set parameters, especially SLy4, SkM*, Z$_\sigma$, and SIII set parameters. All obtained results are in good agreement with the related experiment and the other researcher results.

1. Introduction

One of the basic quantities to describe nuclear structure is the nuclear density distribution, which can give detail information on the nuclei internal structure which related to proton and neutron wave function. The other quantities are the nuclear charge radii which represent the most useful observables information in nuclear structure analysis [1]. These observables, which can be determined from the form factors, provide information about the nuclear shape. To study about nuclear structure, especially for nuclei far from stability line, Hartree-Fock method combined with Skyrme interaction provides a very successful microscopic model to describe many nuclear properties, such as nuclear ground states, collective motions, fission barriers, giant resonances, and heavy-ion collisions [2].

In this study, it is investigated the nuclear ground state properties of $^{209}\text{Bi}$ by using Skyrme-Hartree-Fock (SHF) method with Skyrme set parameters, i.e. SLy4, SkM*, Z$_\sigma$, and SIII. It is focused to calculated the charge, mass, proton, and neutron radii, the total energy, and the neutron skin thickness. The nuclear charge density is a most useful observable for analyzing nuclear structure: it provides information about the nuclear shape and can be determined by clear-cut procedures from the cross section for elastic electron scattering [3]. The calculation results have been compared with the experiment and theoretical results of the other researchers.

2. Theory

2.1. Skyrme-Hartree-Fock

In 1950, T.H.R Skyrme [4] first proposed the effective interaction between nucleons by a zero-range potential. In this phenomenological interaction, the nucleons potential can be represented by

\[
\begin{align*}
V_{\text{Skyrme}} &= t_0 (1 + x_0 P_0) \rho \delta (\vec{r}_f) + \frac{1}{2} t_1 (1 + x_1 P_1) \left[ \rho^2 \delta (\vec{r}_f) + \delta (\vec{r}_f) \rho^2 \right] + t_2 (1 + x_2 P_2) \rho \vec{p}_{12} \cdot \delta (\vec{r}_f) \rho_1 \\
&+ \frac{1}{6} t_3 (1 + x_3 P_3) \rho^3 \left( \vec{r}_f \right) \delta (\vec{r}_f) + it_4 \vec{p}_{12} \cdot \delta (\vec{r}_f) \left( \vec{\sigma}_f + \vec{\sigma}_f \right) \times \vec{p}_{12}
\end{align*}
\]  

(1)
where $t_0, t_1, t_2, t_3, x_0, x_1, x_2, x_3, \alpha, \delta W_0$ are the Skyrme set parameters, $\bar{\rho}_{12} = \bar{\rho}_i - \bar{\rho}_j$ is the relative momentum, $P_x$ is the space exchange operator $\vec{r}_i \rightarrow \vec{r}_j$, $\vec{r} = \frac{1}{2} (\vec{r}_i + \vec{r}_j)$, $\rho$ represents the density, $\delta (\vec{r})$ is the delta function, and $\vec{\sigma}$ is the vector of Pauli spin matrices [5].

The Hartree-Fock equations for Skyrme interaction are obtained by writing that the total energy $E$ is stationary with respect to individual variations of the single-particle states $\varphi_i$ where the single-particle wave functions $\varphi_i$ have to satisfy the following set equations

$$\left[ -\vec{\nabla} \cdot \frac{\hbar^2}{2m^*_q} \vec{\nabla} + U_q (\vec{r}) + \vec{W} (\vec{r}) \cdot ( -i ) \left( \vec{\nabla} \times \vec{\sigma} \right) \right] \varphi_i = \epsilon_i \varphi_i$$

(2)

where $q$ stands for the charge of the single-particle state $i$ [1]. The Hartree-Fock method with complete formula can be found in these following references [2, 5, 6].

2.2. The Nuclear Charge Density

To compute the observable charge density from the Hartree-Fock results, one has to take into account that the nucleons themselves have an intrinsic electromagnetic structure [7]. It is needed to fold the proton and neutron densities from the Hartree-Fock method with the intrinsic charge density of the nucleons. Folding becomes a simple product in Fourier space, so that it can be transformed the densities to the so-called form factors

$$F_q (k) = 4\pi \int_0^\infty \! dr \, r^2 j_0 (kr) \rho_q (r)$$

(3)

where $j_0$ is the spherical Bessel function of zeroth order. The charge form factor is then given by

$$F_C (k) = \sum_q \left[ F_q (k) G_{E,q} (k) + F_{\alpha,q} (k) G_{M} (k) \right] \times \exp \left( \frac{1}{8} (kR)^2 \langle \rho^2 \rangle \right)$$

(4)

where $F_{i,q}$ is the form factor of the spin-orbit current $\nabla J$ (accounting for magnetic contributions to the charge density), $G_{E,q}$ are the electric form factors of the nucleons, $G_{M}$ is the magnetic form factor of the nucleons (assumed to be equal for both species), and the exponential factor takes into account an unfolding of the spurious vibrations of the nuclear center of mass in the harmonic approximation (the $\langle \rho^2 \rangle$ therein is the same as in the zero-point energy). The charge density is obtained from the charge form factor by the inverse Fourier-Bessel transform

$$\rho_C (r) = \frac{1}{2\pi^2} \int dk \, k^2 j_0 (kr) F_C (k)$$

(5)

Other information can be drawn directly from the form factor. The form factor is thus given on a grid $\{ k_j, j = 1, \ldots, N_k \}$. All three form parameters defined in the previous subsection require access to the form factor $F_C$ at any value of $k$. These intermediate values are computed by Fourier-Bessel interpolation

$$F (k) = \frac{\sin (kR_{\max})}{kR_{\max}} \sum_{j=1}^{N_k} \frac{1 - kR_{\max}}{j\pi} \left[ 1 - \frac{kR_{\max}}{j\pi} \right]^{-1} F (k_j)$$

(6)

where $R_{\max} = N_k \Delta_r$ is the length of the coordinate grid [4]. The root-mean-square (rms) radii of charge, neutron and proton can be defined as [8]

$$r_q = \langle r_q^2 \rangle^{1/2} = \frac{\left[ \int \vec{r}^2 \rho_q (\vec{r}) d^3\vec{r} \right]^{1/2}}{\int \rho_q (\vec{r}) d^3\vec{r}}$$

(7)

The neutron skin thickness can be defined as the difference between the neutron rms radius and the proton rms radius $t = r_n - r_p$ [8].
3. Research Method
In this research, the calculations of nuclear total energy of $^{209}$Bi have been performed by using Hartree-Fock method with Skyrme forces set parameters, especially SLy4, SkM*, $Z_\sigma$, and SIII. Calculations process were performed using HAFOMN code [5]. Procedure of HAFOMN code can be seen in Figure 1. Calculations results of this research have been compared with experiment [8, 9] and Tel et al calculation results [8]. In this research, the used Skyrme force parameters are SLy4 [11], SkM* [12], $Z_\sigma$ [13], and SIII [14]. The values of each set parameter can be found in Table 1.

Table 1. The SLy4 [11], SkM* [12], $Z_\sigma$ [13], and SIII [14] set parameters for Skyrme interaction.

| Parameter | SLy4 | SkM* | $Z_\sigma$ | SIII |
|-----------|------|------|------------|------|
| $t_0$     | -2488| -2645| -1983.76   | -1128.75 |
| $t_1$     | 486.82| 410  | 362.252    | 395  |
| $t_2$     | -546.3| -135 | -104.27    | -95  |
| $t_3$     | 13777 | 15595| 11861.4    | 14000|
| $x_0$     | 0.834 | 0.09 | 1.1717     | 0.45 |
| $x_1$     | -0.344| 0    | 0          | 0    |
| $x_2$     | -1    | 0    | 0          | 0    |
| $x_3$     | 1.354 | 0    | 1.762      | 1    |
| $W_0$     | 123   | 130  | 123.69     | 120  |
| $\alpha$  | 0.167 | 0.167| 0.25       | 1    |

4. Result and Discussion
4.1. Total energy and Radii Calculation
In this research, it is calculated the total energy for $^{209}$Bi by using Skyrme-Hartree-Fock method and compared with experiment data taken from Atomic Data and Nuclear Data Tables [9]. It can be seen from Table 2 and Table 3 that total energy calculated by $Z_\sigma$ are in very good agreement with experiment results, followed by SkM*, SLy4, and SIII respectively. In this research, after calculating the total energy, it is also investigated the charge, mass, proton, and neutron radii for each nucleus.

Table 2. Total energy calculation results of $^{209}$Bi by using SLy4 and SkM* set parameters.

| Nucleus | Exp [9] (MeV) | This research SLy4 | Discrepancy (MeV) | This research SkM* | Discrepancy (MeV) |
|---------|---------------|-------------------|-------------------|-------------------|-------------------|
| $^{209}$Bi | -1640.229    | -1622.5549        | -17.6741          | -1629.0986        | -11.1304          |

Table 3. Total energy calculation results of $^{209}$Bi by using $Z_\sigma$ and SIII set parameters.

| Nucleus | Exp [9] (MeV) | This research $Z_\sigma$ | Discrepancy (MeV) | This research SIII | Discrepancy (MeV) |
|---------|---------------|-------------------------|-------------------|-------------------|-------------------|
| $^{209}$Bi | -1640.229    | -1641.5865              | 1.3575            | -1620.3451        | -19.8839          |
From Table 4, all parameters are in good agreement with experiment results, however SkM* results are the closest to experiment results, followed by the results of SLy4, Z, and SIII respectively.

**Table 4.** Calculation results of charge radius of $^{209}$Bi (all units are in fm).

| Nucleus | This research | Tel et al [8] | Exp [10] |
|---------|---------------|---------------|----------|
| $^{209}$Bi | 5.539 | 5.531 | 5.462 | 5.597 | 5.5942 | 5.5149 | 5.5211 ± 0.0906 |

**Table 5.** Calculation results of mass and proton radii of $^{209}$Bi (all units are in fm).

| Nucleus | Mass Radius | Proton Radius |
|---------|-------------|---------------|
| $^{209}$Bi | 5.575 | 5.573 | 5.439 | 5.616 | 5.486 | 5.478 | 5.411 | 5.548 |

**Table 6.** Calculation results of neutron radius of $^{209}$Bi (all units are in fm).

| Nucleus | Neutron Radius | Neutron skin thickness |
|---------|----------------|------------------------|
| $^{209}$Bi | 5.634 | 5.635 | 5.457 | 5.661 | 0.148 | 0.157 | 0.046 | 0.113 |

### 4.2. Local Density and Local Potential Distributions

In this research, it is also calculated the distribution of both proton and neutron local densities for each nucleus to understand its ground state properties. The local density distribution for proton and neutron of $^{209}$Bi can be seen in Figure 2, where it can be seen that, in radius 0-1 fm, SLy4 gives the highest results and SIII gives the lowest results. In this area, SLy4 results decline more drastically than the other parameters. In radius 2-5 fm, $Z_\sigma$ gives the highest results and SIII gives the lowest results. The results of SLy4 and SkM* are similar to each other in this area. In radius 5-12 fm, all parameters give similar results. From neutron density profiles, in radius 0-1 fm, $Z_\sigma$ obtains the highest results and SIII obtains the lowest results. In this area, SLy4 slope is more drastically than the other parameters. In radius 2-5 fm, $Z_\sigma$ obtains the highest results and SIII obtains the lowest results. The SLy4 and SkM* results are more similar in this area. In radius 5-12 fm, all parameters obtain similar pattern but $Z_\sigma$ is little bit different from the others.

![Figure 2. Nucleon local density distribution of $^{209}$Bi obtained by each set parameter.](image)

Figure 3 displays the local potential distributions for proton and neutron of $^{209}$Bi. From this figure, it can be seen that SLy4 obtains the deepest valley. The SIII and the SkM* obtain the same depth of valley but the slopes and surface regions for all parameters are good enough similar to each other. It can be summarized that all parameters obtain similar results, especially the slopes and surface regions (in region >6 fm). All parameter used in this research are in good agreement with the related experiment and the other researchers results. Most of SkM* results are closer to the related experiment results than SLy4, $Z_\sigma$, and SIII results. From the nucleon density and potential profiles, it can be seen
that the obtained results of all parameters are similar to each other, especially the slope and the surface regions. From those results above, it can be indicated that Skyrme-Hartree-Fock method can be the useful tool to study the nuclear ground state properties of heavy nuclei.

![Image](image_url)

**Figure 3.** Nucleon local potential distribution of $^{209}$Bi obtained by each set parameter.

5. Conclusion

It has been performed a study to calculate the densities, the neutron skin thickness, and the rms radii for both proton and neutron of $^{209}$Bi by using Hartree-Fock method with Skyrme set parameters, especially SLy4, SkM*, $Z_\sigma$, and SIII set parameters. In this study, all used parameters are in good agreement with the results of the related experiment and the other researchers, where SkM* results are closer collectively to experiment energy than SLy4, $Z_\sigma$ or SIII. The radii calculation of this research with SkM* and SIII set parameters are close enough to Tel et al and the related experiment results. These results indicate that Skyrme-Hartree-Fock method can be the good useful tool to explain the ground state properties of heavy nuclei.

Acknowledgment

The authors are grateful to Prof. P.G. Reinhard (Germany) for discussion and guide in using the HAFOMN code.

References

[1] Reinhard P G and Nazarewicz W arXiv:1601.06324v1
[2] Vautherin D and Brink D M 1972 Phys. Rev. C 5 626
[3] Dreher B, Friedrich J, Merle K, Luhrs G and Rothaas H 1974 Nucl. Phys. A 235 219
[4] Skyrme T H R 1959 Nucl. Phys. 9 615
[5] Reinhard P G 1991 Skyrme-Hartree-Fock Model of the Nuclear Ground State (Computational Nuclear Physics I - Nuclear Structure) ed. Langanke K, Koonin, S E and Maruhn J A (Springer-Verlag Berlin Heidelberg) chapter 2 pp 28-50
[6] Bender M, Heenen P H, and Reinhard P G 2003 Reviews of Modern Physics 75 121
[7] Friar J L and Negele J W 1975 Adv. Nucl. Phys. 8 219
[8] Tel E and Aydin A 2012 J. Fusion Energy 31 73
[9] Wang M et al. 2012 Chinese Physics C 36 1603
[10] Angeli I and Marinova K 2013 Atomic Data and Nuclear Data Tables 99 69
[11] Chabanat E, et al. 1997 Nucl. Phys. A 627 710
[12] Bartel J, Quentin P, and Brack M 1982 Nucl. Phys. A 386 79
[13] Friedrich J and Reinhard P G 1986 Phys. Rev. C 33 335
[14] Beiner M, Flocard H, and Giai N V 1975 Nucl. Phys. A 238 29