Axion-fermion coupling and dyon charge as physical signatures of a space-time inner symmetry

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Abstract In this paper we intend to complement the identification given in Kuerten and Fernandes-Silva (Mod Phys Lett A 33:1850092, 2018) which relates the axion to a metric spinor phase by means of Maxwell’s theory in the Infeld-van der Waerden’s γ-formalism. Thus, we obtain two alternative identifications: The first focuses on Dirac’s theory so that when obtaining an axion-like phase-fermion coupling, we achieve the first identification, and the last one investigates the phase behavior under Peccei–Quinn rotations in order to show that the phase changes as an axion pseudoparticle. With the formal aspects established, we also study the semiclassical fermion-photon system to demonstrate that the magnetic monopole current defined in Kuerten and Fernandes-Silva (Mod. Phys. Lett. A. 33:1850092) has dyon charge in flat universe and acquires a Witten effect form when there is a demand for chiral symmetry.

1 Introduction

Axion theories have provided several advances in theoretical and experimental physics, such as quantum chromodynamics, condensed matter, string theory and dark matter studies [2–7]. Peccei and Quinn [8, 9] introduced the axion field to solve the strong problem in CP symmetry. In Ref. [1], the so-called axion electrodynamics has played an important role to identify the axion with a metric spinor phase. Introduced by Wilczek [10], this new electrodynamics finds basis on the Lagrangian term

$$α F^{μν} F_{μν}^*, \quad (1)$$

where $F_{μν}$ is the Maxwell tensor, $F_{μν}^*$ is its Hodge dual and $α$ the axion field. As usual, in trivial topology cases, we have $α F^{μν} F_{μν}^* = 0$. Term (1) modifies Maxwell inhomogeneous equations. Since then, some generalizations have been elaborated to extend the Wilczek’s theory, and Tiwari establishes a local dual invariant electrodynamics (LDIE) which naturally includes an axion electrodynamics [11–14]. Instead of the original axion electrodynamics formulation, the LDIE also modifies homogeneous equations. The non-observation of magnetic monopoles is justified by demanding that the axionic correction cancels the magnetic monopole term.

In [1] we see the LDIE that satisfies the Maxwell equations is identical to that found in the γ-formalism, when magnetic monopole currents are properly defined. Those magnetic currents got inspiration from the definition of geometrical sources for Infeld-van der Waerden electromagnetic fields given in [15]. In Infeld-van der Waerden’s formalisms [16–23], the usual metric spinor $ε_{AB}$, used in the ε-formalism admits phase and scale transformations that do not affect the fundamental metric structure $g_{μν}$. In fact, taking into account $ε_{AB} \mapsto |γ| e^{θi} ε_{AB}$ and $Σ^{AA'} \mapsto |γ|^{-1} Σ^{AA'}$, the metric tensor transforms as $g_{μν} \mapsto g_{μν}$, where $Σ^{AA'}$ is a connecting object component (also known as Infeld-van der Waerden symbol). Therefore, the γ-formalism defines the metric spinor and the connecting object in the following manner

$$γ_{AB} \equiv |γ| e^{θi} ε_{AB} \text{ and } Σ^{AA'} \equiv |γ|^{-1} Σ^{AA'}.$$  \quad (2)

Thus, each γε-formalism does have base on its metric spinor: constant in the ε-formalism and depending locally on the space-time coordinates in the γ-formalism.

Formally, the Infeld-van der Waerden’s formalisms are based on the homomorphism $h : \mathcal{W}(2, \mathbb{C}) \to SO^*_2(1, 3)$ between the Weyl $\mathcal{W}(2, \mathbb{C})$ and the orthochronous proper Lorentz group $SO^*_2(1, 3)$. Sometimes classical world theories can be rewritten in a 2-spinor version, since there is a homomorphism two to one which relates the linear group of unimodular complex $2 \times 2$ matrices $SL(2, \mathbb{C})$ to the group $SO^*_2(1, 3)$. The group $\mathcal{W}(2, \mathbb{C})$ is longer than $SL(2, \mathbb{C})$ and contains a $U(1)$ freedom, which is symbolically stated by $h(e^{iλ} g) = h(g) \in SO^*_2(1, 3)$ where $g \in \mathcal{W}(2, \mathbb{C})$ [24].

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Historically, Infeld and van der Waerden considered the Weyl’s work [25] to implement Dirac’s theory in general relativity. Weyl studied the relationship between the tetrad formalism for curved space-time and the parameter of the Dirac 4-spinor phase transformation to conclude that if the tetrad varies, the parameter varies as well [24, 25]. Originally, the theory provided a geometrical origin of the electromagnetic potential, since it would lead to an imaginary part of the spinor connection trace that would satisfy the Weyl’s principle of gauge invariance. Unfortunately, there was no consolidation of this idea and it should not be understood on its original form [26]. In this interpretation, as the scale/phase couples with each type of fermion, the formalism would imply a relation between electric charge and spin. However, the neutron disabled this idea, because it has spin but no electric charge. Furthermore, the interpretation of the imaginary part of the spinor connection trace impaired some investigations over Maxwell’s theory in the \( \gamma \)-formalism. Nowadays, the physical significance of the phase is free to be reinterpreted on a modern perspective.

In [1], it considered the potential as an external physical entity such that it established the identification \( \Theta \sim \alpha \). However, the identification solely was based on the Maxwell’s theory. Therefore, in the present study, we will take other ways that lead to such identification. Wanting to repeat a similar result with the derived in [1], in here, we will investigate Dirac’s theory in the \( \gamma \)-formalism and so to show that the space-time phase also transforms as an axion pseudoparticle. Besides these identifications, we also consider the fermion-photon system and demonstrate that the magnetic monopole source of [1] carries a dyon charge in Minkowski space-time as also, on a specific case, it has a Witten effect form where the space-time phase assumes the axion role.

We will also use \( h = c = 1 \) as the metric signature (+−−−). Round/square brackets will indicate the index symmetry/antisymmetry. The paper’s organization will be as follows. In section 2, we will review the axion-electrodynamics provided by the \( \gamma \)-formalism. In section 3, we must obtain the axion-fermion couplings from \( \gamma \)-formalism and establish the Weyl-Peccei Quinn transformations to identify the phase with the axion again. In section 4, we will use the Maxwell-Dirac system to explicitly obtain a magnetic monopole 2-spinor form and derive its effective magnetic charge. In addition, we will show that such charge has a dyon structure and it assumes a quantized Witten form when requiring a chiral invariance.

### 2 Identifying axion with metric spinor phase: the photon case

In the present section, we will review the results obtained in [1]. The Lagrangian that provides the axion electrodynamics is given by

\[
2F^{\mu
\nu}F_{\mu\nu} + 4\alpha F^{\mu
\nu}F^*_{\mu\nu} + A^\mu j_\mu, \tag{4}
\]

where \( \mathcal{L}_M = 2F^{\mu
\nu}F_{\mu\nu} \) is the usual Maxwell Lagrangian, \( \mathcal{L}_{CS} = 4\alpha F^{\mu
\nu}F^*_{\mu\nu} \) is the Chern-Simons term and \( \mathcal{L}_1 = A^\mu j_\mu \) is an interaction term between the gauge potential \( A^\mu \) and the electric current density \( j_\mu \). In the 3-vector notation, we have \( F^{\mu\nu}F_{\mu\nu} = 2(B^2 - E^2) \) and \( F^{\mu\nu}F^*_{\mu\nu} = 4E \cdot B \), where \( E \) and \( B \) are the electric and magnetic fields, respectively. The field equations derived from (4) are

\[
\partial^\mu F_{\mu\nu} = 4\pi j_\nu + (\partial^\mu \alpha) F^*_{\mu\nu} \quad \text{and} \quad \partial^\mu F^*_{\mu\nu} = 0. \tag{5}
\]

The first expression ensures the effect caused due to an axion field. Alternatively, in [11], it occurs a generalization of the axion electrodynamics to contain a local duality symmetry. In this case, it is

\[
\partial^\mu F_{\mu\nu} = 4\pi j_\nu + (\partial^\mu \alpha) F^*_{\mu\nu} \quad \text{and} \quad \partial^\mu F^*_{\mu\nu} = 4\pi m_\nu - (\partial^\mu \alpha) F_{\mu\nu}, \tag{6}
\]

where \( m_\nu \) is a magnetic monopole current density.

In general, system (6) is invariant under the dual transformations \( \mathcal{\hat{\Theta}} \mapsto \mathcal{\hat{\Theta}}, \mathcal{C} \mapsto \mathcal{\hat{C}} \) and \( \partial_\nu \alpha \mapsto \partial_\nu \alpha + \partial_\mu \phi \), where \( \mathcal{\hat{\Theta}}, \mathcal{\hat{C}} \) and \( \mathcal{\hat{\Sigma}} \), respectively, are given by

\[
\mathcal{\hat{\Theta}} = \begin{pmatrix} F_{\mu\nu} \\ F^*_{\mu\nu} \end{pmatrix}, \quad \mathcal{\hat{C}} = \begin{pmatrix} j_\nu \\ m_\nu \end{pmatrix} \quad \text{and} \quad \mathcal{\hat{\Sigma}} = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}, \tag{7}
\]

with \( \phi = \phi(x^\mu) \). It is worth reporting that (6) assumes pattern (5) by putting out the right side of the second expression, so that \( (\partial^\mu \alpha) F^*_{\mu\nu} = 4\pi m_\nu \). We can understand this fact as a solution for the non-observation of the magnetic monopole in nature. Still by (6), it is clear that the dual axion electrodynamics that satisfies the Maxwell electromagnetism provides

\[
(\partial^\mu \alpha) F^*_{\mu\nu} = 0 \quad \text{and} \quad (\partial^\mu \alpha) F_{\mu\nu} = 4\pi m_\nu. \tag{8}
\]
On the other hand, the Clifford algebra $Cl(3,1)$ in the $\gamma$-formalism is represented by

$$g_{\mu\nu} = \Upsilon^{AA'} \gamma^B B' \gamma_{AY} A'B'. \quad \text{(9)}$$

There is an adoption of Einstein’s convention, and each spinor index runs from 0 to 1 ($0'$ to 1'). The metric spinor $\gamma_{AB}$ and the Infeld-van der Waerden symbols $\Upsilon^{AA'}$ are defined by (2) (for a review about the objects, see [18, 21]). We use the metric spinor to lower (or raise) the spinor indexes: $\xi_A = \gamma_{BA} \xi^B, \xi^A = \gamma^{AB} \xi_B$. The object $\gamma_{AB}$ is a skew-symmetric spinor component. In the matrix form ($\gamma_{AB}$), it is

$$(\gamma_{AB}) = \begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix}, \quad \text{with} \quad \gamma = |\gamma|e^{\Theta i}, \quad \text{(10)}$$

where $|\gamma|$ and $\Theta$ are real-valued functions of $x^\mu$. Spinors and tensors are related by using a Hermitian matrix set $\Upsilon$, such as $v_\mu = \Upsilon^{AA'} v_{AA'}$ and $v_{AA'} = \Upsilon^{AA'} v_\mu$. In general, we will denote a complex conjugation by $(S^\mu_{AB})^* = S^{\mu*}_{AB}$, for any $S^\mu_{AB}$.

The covariant derivative of some generic spinors $\xi^A$ and $\zeta_A$ is given as it follows

$$\nabla_\mu \xi^A = \partial_\mu \xi^A + \Xi_{\mu B} ^{A} \xi^B \quad \text{and} \quad \nabla_\mu \zeta_A = \partial_\mu \zeta_A - \Xi_{\mu A} ^{B} \zeta_B, \quad \text{(11)}$$

with $\Xi_{\mu A} ^{B}$ being a spinor connection. In the $\gamma$-formalism, the covariant derivative $\nabla_{AA'}$ is taken by using the Infeld-van der Waerden symbols: $\nabla_{AA'} = \Upsilon^{AA'} \nabla_\mu$ and $\nabla^AA' = \gamma^{AB} \gamma^{\mu} B' \gamma^{*}_{BB'} \nabla_\mu$. In Minkowski universe, $\nabla_\mu = \partial_\mu$ and $\Upsilon^{AA'} = \sigma^{\mu}_{AA'}/\sqrt{2}$ since $|\gamma| = 1$. Here $\sigma^{\mu}_{AA'} = (\parallel, \sigma^{i}_{AA'})$ where $\sigma^{i}_{AA'}$ are the Pauli matrices and $\parallel$ the $2 \times 2$ unity matrix. The complex component $\Xi_{\mu A} ^{B}$ can be written as [21]

$$\Xi_{\mu A} ^{B} = \partial_\mu \ln(|\gamma|) - 2i \Xi_{\mu} , \quad \text{with} \quad \Xi_{\mu} = -(1/2)\text{Im}\Xi_{\mu A} ^{A}. \quad \text{(12)}$$

In the $\gamma$-formalism, the metric compatibility $\nabla_\mu g_{\nu\mu} = 0$ leads to the eigenvalue equations [21]

$$\nabla_\mu \gamma_{AB} = i\beta_{\mu} \gamma_{AB} \quad \text{and} \quad \nabla_\mu \gamma^{AB} = -i\beta_{\mu} \gamma^{AB}, \quad \text{(13)}$$

where $\beta_{\mu}$ is defined by

$$\beta_{\mu} = \partial_\mu \Theta + 2 \Xi_{\mu}. \quad \text{(14)}$$

In flat space-time, we can consider the choice $\Xi_{\mu A} ^{B} = 0$. Thus, $\Xi_{\mu} = 0$ and $|\gamma| = \text{const} > 0$, so that $\beta_{\mu} = \partial_\mu \Theta$. Here, we will assume $|\gamma| = 1$. It is usual the spinors $\xi^A$ and $\zeta_A$ transform under the action of the generalized Weyl group, which has the following component form:

$$\Delta_{A} ^{B} = \sqrt{\rho} e^{i\lambda} \xi_{A} ^{B}, \quad \text{(15)}$$

where $\rho > 0$ is a real function and $\lambda$ the gauge parameter of the group.

The 2-spinor version of the Maxwell tensor and its Hodge dual in the $\gamma$-formalism are [21, 28, 29]

$$2F_{AA'B'B'} = \gamma_{AB} f_{A'B'} + \gamma_{A'B'} f_{AB} \quad \text{and} \quad 2F'^{*}_{AA'B'B'} = i(\gamma_{AB} f_{A'B'} - \gamma_{A'B'} f_{AB}), \quad \text{(16)}$$

in which $f_{AB} = f_{(AB)}$ is named Maxwell spinor. It is convenient to define the following complex linear combination: $F^{(\pm)}_{\mu\nu} = F_{\mu\nu} \pm iF'_{\mu\nu}$ so that by taking expression (16), we find

$$F^{(+)}_{AA'B'B'} = \gamma_{AB} f_{A'B'} \quad \text{and} \quad F^{(-)}_{AA'B'B'} = \gamma_{AB} f_{A'B'}. \quad \text{(17)}$$

Let us then take the complex version of Maxwell’s equations, i.e.,

$$\nabla^{\mu} F^{(\pm)}_{\mu\nu} = 4\pi j_{\nu}. \quad \text{(18)}$$

If we consider (17) as well as eigenvalue equation (13), complex form (18) yields

$$\nabla_{\mu} f_{AB} = 2\pi j_{AA'} + i\beta_{A} ^{B} f_{AB} \quad \text{(19)}$$

In the $\epsilon$-formalism, as equation (18) with magnetic source term $\nabla^{\mu} F^{(\pm)}_{\mu\nu} = 4\pi (j_{\nu} \pm im_{\nu})$ provides $\nabla_{\mu} f_{AB} = 2\pi (j_{AA'} + im_{AA'})$, it follows the definition elaborated in [1], namely

$$\beta_{A} ^{B} f_{AB} = 2\pi m_{AA'}. \quad \text{(20)}$$

In world representation, definition (20) implies

$$\beta_{\mu} ^{\nu} F^{(\pm)}_{\mu\nu} = 4\pi m_{\nu}, \quad \text{(21)}$$

Expression (21) is the master equation to identify the space-time phase with the axion field.
In Minkowski space-time, the $\beta$-vector is $\beta_\mu = \partial_\mu \Theta$ so that Eq. (21) becomes $(\partial^\mu \Theta) F^{(\pm)}_{\mu \nu} = 4\pi m_e$. Now, if we specify $F^{(\pm)}_{\mu \nu}$ at terms of the Maxwell tensor and its Hodge dual, we obtain the following pair of equations
\[(\partial^\mu \Theta) F^{(\pm)}_{\mu \nu} = 0 \text{ and } (\partial^\mu \Theta) F^{(\pm)}_{\mu \nu} = 4\pi m_e. \tag{22}\]
It is clear that expressions (22) and (8) are formally identical so that $\Theta \sim \alpha$ establishes in [1].

On the Lagrangian viewpoint in flat universe, the Maxwell Lagrangian in the $\gamma$-formalism is given by
\[\mathcal{L}_M = 2 F^{\mu \nu} F_{\mu \nu} = \text{Re} \left[ \gamma^{AC} \gamma^{BD} f_{AB} f_{CD} \right]. \tag{23}\]
If we explicit the $\gamma$-terms, the right side of (23) leads to
\[\text{Re} \left[ \epsilon^{AC} \epsilon^{BD} f_{AB} f_{CD} \right] \cos(2\Theta) + \text{Im} \left[ \epsilon^{AC} \epsilon^{BD} f_{AB} f_{CD} \right] \sin(2\Theta). \tag{24}\]
By taking the approximation $\Theta \simeq 0 (\gamma^{AB} \simeq \epsilon^{AB})$, we have
\[\text{Re} \left[ \epsilon^{AC} \epsilon^{BD} f_{AB} f_{CD} \right] \simeq 2 F^{\mu \nu} F_{\mu \nu} \text{ and } \text{Im} \left[ \epsilon^{AC} \epsilon^{BD} f_{AB} f_{CD} \right] \simeq 2 F^{\mu \nu} F^{\ast}_{\mu \nu}, \tag{25}\]
whence, due to (24), we find of (23) an effective Chern-Simons theory:
\[\mathcal{L}_M \simeq 2 F^{\mu \nu} F_{\mu \nu} + 4\Theta F^{\mu \nu} F^{\ast}_{\mu \nu}. \tag{26}\]
Hence, when $\Theta \simeq 0$ the invariant form $\text{Re} \left[ f^{AB} f_{AB} \right]$ in the $\gamma$-formalism generates an axion-like space-time phase-photon coupling.

### 3 Axion-like phase-fermion coupling and Peccei–Quinn pseudoparticle behavior

Similarly, as we have seen for Maxwell’s theory, in this section we will study the Dirac’s theory to show that the interactions between the phase and some fermions are axion-like. Afterward, we will generalize the Weyl transformation to a general spin transformation that carries Peccei–Quinn and Weyl spin rotations simultaneously. With the Weyl-PQ transformation established, we will demonstrate that the space-time phase behaves as a pseudoscalar (such as an axion-like pseudoparticle) under the action of PQ group.

#### 3.1 Axion-like phase-fermion coupling

For a comprehensive review about Dirac’s theory in the Infeld-van der Waerden’s formalism, see [18, 20, 30]. Let us follow Ref. [28] to present the first pair of Dirac equations. In the 2- spinor formalism, the fermion dynamics, in a generic space-time, is described by the pair
\[\nabla^{AA'} \psi_A = \mu \chi_A' \text{ and } \nabla_{AA'} \chi^{A'} = -\mu \psi_A, \tag{27}\]
where $\psi_A$ and $\chi^{A'}$ are Weyl 2-spinors with opposite chiralities, and $\mu = m/\sqrt{2}$ the normalized rest mass.

In the $\gamma$-formalism, because we have eigenvalue equation (13), duo (27) is equivalent to
\[\nabla_{AA'} \psi^{A'} + i \beta_{AA'} \psi^{A'} = -\mu \chi^{A'} \text{ and } \nabla^{AA'} \chi^{A'} + i \beta^{AA'} \chi^{A'} = \mu \psi^{A'}. \tag{28}\]
The Dirac fields which satisfy (27) and (28), respectively (as well as their complex conjugates), are $\mathcal{D} = \left\{ (\psi_A, \chi^{A'}), (\chi_A, \psi^{A'}) \right\}$ and $\mathcal{D}^\Phi = \left\{ (\psi^{A'}, \chi_A), (\chi^{A'}, \psi^{A'}) \right\}$. By using the metric spinors, depending on the formalism considered, $\mathcal{D}^\Phi$ is obtained from $\mathcal{D}$. In the $\epsilon$-formalism, when putting out the $\beta$-terms, we obtain the equivalent equations of (28).

If we also consider (14) and (28) as reference [31], where the axion-fermion coupling has been explicitly written in the index 2-spinor formalism, we clearly see an axion-like coupling for the system $\mathcal{D}^\Phi$. In fact, the $\beta$-terms of (28) are decomposed as
\[\beta_{AA'} \psi^{A'} = \psi^A \delta_{AA'} \Theta + 2 \Xi_{AA'} \psi^{A'} \text{ and } \beta^{AA'} \chi^{A'} = \chi^A \delta_{AA'} \Theta + 2 \Xi^{AA'} \chi^{A'}. \tag{29}\]
where we used (14) to decompose the $\beta$-terms and [31] to identify the axion-like couplings.

It is important mentioning that solely $\mathcal{D}^\Phi$ couples in an explicit way with $\Theta$, while $\mathcal{D}$ does not. Usually, the 4-component Dirac spinor is defined by taking $\mathcal{D}_1 = (\psi_A, \chi^{A'})$, since $\mathcal{D}_1 \rightarrow e^{i\beta} \mathcal{D}_1$ under the action of the Weyl group $\Delta^{AB}_A$ (where $\beta = 1$ in (15)). On the other hand, $\mathcal{D}_1^\Phi = (\psi^{A'}, \chi_A)$ transforms as $\mathcal{D}_1^\Phi \rightarrow e^{-i\lambda} \mathcal{D}_1^\Phi$. We can invert this election for a sign redefinition $\lambda \rightarrow -\lambda$ in (15).

On the Lagrangian viewpoint, the axion-like coupling is seen through the manipulation
\[\psi_A \nabla^{AA'} \psi_A + \chi^A \nabla_{AA'} \chi^{A'} = \psi_A \nabla_{AA'} \psi_A + \chi^A \nabla^{AA'} \chi^{A'} + i \psi^A \delta_{AA'}^A \Theta + i \chi^A \delta_{AA'} \Theta \subset \mathcal{L}_D. \tag{30}\]
where \( \mathcal{L}_D \) is the Dirac Lagrangian. It follows then by (14)

\[
\psi^A \bar{\psi}^A \beta_{\alpha \alpha'} + \chi_{\alpha} \chi_{\alpha'} \gamma^{\alpha \alpha'} = \psi^A \bar{\psi}^A \Theta + \chi_{\alpha} \chi_{\alpha} \tilde{\Theta} + 2 \left( \psi^A \bar{\psi}^A \Sigma_{\alpha \alpha'} + \chi_{\alpha} \chi_{\alpha} \Sigma^{\alpha \alpha'} \right)
\]

Decomposition (31) naturally provides an axion-like interaction term in Dirac’s theory.

3.2 Weyl-PQ spin transformations: space-time phase as a Peccei–Quinn pseudoparticle

The space-time phase has been identified here with the axion field based on its coupling with Dirac field, as with the Maxwell field in [1]. Individually, those identifications are not enough, since the behavior of \( \Theta \) under chiral rotation has not been established. As we have pointed out in the previous subsection, the Weyl 2-spinors, according to the action of the Weyl group, defines the Dirac spinor. However, we have the chiral rotations too, which rotate the Dirac spinors in the following manner:

\[
\psi \rightarrow \tilde{\psi} = e^{i\chi} \psi.
\]

In here, we will denote, respectively, by \( \chi = \tilde{\eta} \) and \( \eta = \tilde{\eta} \), the Weyl and PQ transformations. The original Peccei–Quinn procedure [8] states that, under a chiral rotation, an axion-like pseudoscalar \( \theta \) transforms as

\[
\theta \rightarrow \tilde{\theta} = \theta - 2\chi.
\]

Transformation (33) solves the CP-problem. Our strategy will be to identify \( \Theta \) with \( \alpha \), using a similar idea to the one developed by Infeld and van der Waerden.

Long ago, Infeld and van der Waerden considered the gauge behavior of the electromagnetic potential \( A_\mu \), i.e.,

\[
A_\mu \rightarrow A_\mu - \partial_\mu \lambda.
\]

Once they knew that the object \( \Xi_\mu \) transforms as \( \Xi_\mu \rightarrow \Xi_\mu - (1/2) \text{Im} \left[ \partial_\mu \ln(\det \Delta_\alpha^B) \right] \) (see for example Ref. [21]), they would put the determinant \( \det \Delta_\alpha^B = e^{2i\lambda} \) of the Weyl group and so to find

\[
\Xi_\mu \rightarrow \Xi_\mu - \partial_\mu \lambda.
\]

Because (34) look likes (35), Infeld and van der Waerden suggested the identification \( \Xi_\mu \sim A_\mu \).

Our strategy is to verify if \( \Theta \) behaves as (33) when considering a PQ transformation. A similar problem was solved in [32], where the author worked \( \Xi_\mu \) as being a mixture of polar and axial vectors. Our first step is to find spin transformations that represent simultaneously Weyl and PQ rotations. In the 2-spinor language, the translation for chiral rotation is

\[
\tilde{\psi}_A = e^{i\chi} \psi_A \quad \text{and} \quad \tilde{\chi}_A = e^{-i\chi} \chi_A.
\]

Then we need to obtain the spin transformations where, in general, \( \psi_A = e^{i(\lambda+\xi)} \) and \( \chi_A = e^{i(\lambda-\zeta)} \chi_A \). The notation \( (\tilde{\eta}) \) denotes the composition of the Weyl \( (\tilde{\eta}) \)-PQ \( (\tilde{\eta}) \) transformations.

Let us consider the general spin transformation of the metric spinor:

\[
\gamma_{AB} = A^C \Delta_B^D \gamma_{CD}.
\]

Now, we decompose (36) as it follows

\[
\Delta_\alpha^B = \Delta_\alpha^C \Delta_*^B = \sqrt{\Delta_\alpha^C \Delta_*^B},
\]

where \( \Delta_\alpha^C \) and \( \Delta_*^B \) are the determinants of the Weyl \( (\Delta_\alpha^B) \) and PQ \( (\Delta_*^B) \) rotations. Under a Weyl transformation, we have \( \gamma_{AB} = \Delta_\alpha^B \gamma_{AB} \) while under a PQ transformation, we suppose

\[
\gamma_{AB} = \Delta_*^B (\gamma_{AB}).
\]

Through the previous considerations, metric spinor transformation (36) becomes

\[
\gamma_{AB} = \Delta_\alpha^\gamma_{AB} \Delta_*^\gamma_{AB}.
\]

As we will verify later, a PQ transformation can be implemented by demanding

\[
\gamma_{AB} = \Delta_\alpha^\gamma_{AB} \Leftrightarrow \tilde{\gamma} = (\Delta^*)^{-1} \gamma.
\]

From (40), it is clear \( \gamma_{AB} = \tilde{\gamma}_{AB} \). By implementing \( \gamma_{AB} \gamma_{AB} = \gamma_{AB} \gamma_{AB} \), we deduce the expressions

\[
\gamma_{AB} = (\Delta^*)^{-1} \gamma_{AB} \quad \text{and} \quad \tilde{\gamma}_{AB} = \Delta_*^\gamma_{AB}.
\]

The transformation of generic spinors \( \psi_A \) and \( \mu^A \) is then given by

\[
\psi_A = \sqrt{\Delta^\gamma \Delta_*^\gamma} \psi_A \quad \text{and} \quad \mu^A = \sqrt{\Delta^\gamma \Delta_*^\gamma} \mu^A.
\]
Since $\Delta^*=e^{2i\xi}$ and $\Delta^*=e^{2i\xi}$, the 2-component fermions transform as
\[ \psi_A = e^{i(x+\xi)}\psi_A \text{ and } \bar{\psi}^{\prime} = e^{i(x-\xi)}\chi^{A'}, \] (43)
or particularly $\tilde{\psi}_A = e^{i\xi}\psi_A (\tilde{\chi}^{A'} = e^{i\xi}\chi^{A'})$ (Weyl) and $\bar{\psi}_A = e^{i\epsilon}\bar{\psi}_A (\tilde{\chi}^{A'} = e^{-i\xi}\chi^{A'})$ (PQ).

As we have observed, the Weyl-PQ transformation is possible if it satisfies $\tilde{\chi} = (1/\Delta^*)\gamma$. In Minkowski space-time, we have $\gamma = e^{i\epsilon}$, such that under a PQ rotation, we obtain $e^{i\epsilon} = (e^{i\epsilon}/\Delta^*)$. Therefore $\tilde{\Theta} = \Theta + i \ln \Delta^* + 2n\pi$, so that
\[ \Theta \Leftrightarrow \Theta - 2\xi + 2n\pi, \quad n \in \mathbb{Z}, \] (44)
PQ-behavior

since $\Delta^* = e^{2i\xi}$, with $\mathbb{Z}$ being the set of integers. We have demonstrated that $\Theta$ satisfies PQ requirement (33). We must note that the invariant form $\pi^A\pi_A$ is broken in our formulation: $\pi^A\pi_A = e^{2i\xi}\mu^A\nu_A$. This property is a formal declaration of no chiral symmetry for massive fermion terms.

4 Maxwell-Dirac system: magnetic monopole current and its dyonic charge

When coupled with Maxwell fields, the Dirac equations in curved space-times are taken when substituting $\nabla_{AA'}$ by $\nabla_{AA'} - ieA_{AA'}$ in (28), with $A_\mu$ being an electromagnetic potential component. It follows expression (19) rewritten as
\[ \left[ \nabla_{AA'} - i \left( eA_{AA'} + \partial_{AA'} + 2\pi A_{AA'} \right) \right] f_{AB} = e(\psi_A\psi_A' + \chi_A\chi_A'), \] (45)
where we used (14) and the electric current density [28]
\[ j_{AA'} = q(\psi_A\psi_A' + \chi_A\chi_A'), \quad \text{with } q \doteq \frac{e}{2\pi}, \] (46)
being $e$ the electric charge.

Let us consider equations (22) in curved space-time:
\[ \nabla^\mu F^{(\pm)}_{\mu\nu} = 4\pi j_\nu \quad \text{and} \quad \beta^\mu F^{(\pm)}_{\mu\nu} = 4\pi m_\nu. \] (47)

If we apply the covariant derivative on the last one of (47), we will obtain
\[ W^{\mu\nu} F_{\mu\nu} + 8\pi(\beta^\mu j_\mu + \nabla^\mu m_\mu) = 0 \quad \text{and} \quad W^{\mu\nu} F_{\mu\nu}^* = 0, \] (48)
where $W_{\mu\nu} \doteq 2\partial_{\mu} \pi_{\nu1}$ is an Infeld-van der Waerden curvature bivector. Still, $\beta^{[\mu}F^{\nu]} = 0$ implies $\beta^\mu m_\mu = 0$. In flat space-time, when remembering the definition $\pi_{\mu1} = -(1/2)\text{Im} \pi_{\mu1}A^A$, the first equation of (48) leads to $\partial^\mu m_\mu = -j_\mu \partial^\mu \Theta$.

If we require the null divergence for electric sources, namely $\partial^\mu j_\mu = 0$, it follows
\[ m_\mu = -(\Theta - C) j_\mu, \] (49)
where $C$ is a constant. Once $\beta^\mu m_\mu = 0$ and, in general, $\Theta (x^\mu) \neq C$, in Minkowski universe $-(\Theta - C) j_\mu \partial^\mu \Theta = 0$, which implies $j_\mu \partial^\mu \Theta = 0$ so that $\partial^\mu m_\mu = 0$. Therefore, in flat background, we have the divergence equations
\[ \partial^\mu j_\mu = 0 = \partial^\mu m_\mu, \] (50)
as well as the orthogonality relationships
\[ j_\mu \partial^\mu \Theta = 0 = m_\mu \partial^\mu \Theta. \] (51)

If we take (46), the 2-spinor form of (49) yields
\[ m_{AA'} = g(\chi_A\chi_A'), \quad \text{with } g \doteq -\frac{e}{2\pi}(\Theta - C). \] (52)
The term $g$ in (52) can be understood as an effective charge and interpreted as a dyonic charge [33, 34]. Under a PQ transformation, our dyon charge behaves as
\[ g \mapsto -\frac{e}{2\pi}(\Theta - 2\xi + 2n\pi - C). \] (53)
When $\xi = n\pi$ with $n \in \mathbb{Z}$, we have $\tilde{g} = g$. Another case, where $\tilde{g} = g$, is when $2\xi = C = 2n\pi$. In the last one, we derive a quantized dyon charge given by
\[ g = -e\left(\frac{\Theta}{2\pi} - n\right). \] (54)
Expression (54) is identical to the found in the named Witten effect [35].
From usual axion electrodynamics, the Witten effect is obtained when demanding \( \nabla \cdot \mathbf{B} \neq 0 \). However, we must note that our charge \( (54) \) is not associated with the monopole equation. To be specific, since \((\partial^\mu \Theta) F_{\mu
u} = 4\pi m_\psi \) and \( \partial^\mu F^{\mu\nu}_* = 0 \), the monopole density is proportional to \((\nabla \Theta) \cdot \mathbf{E}\) while \( \nabla \cdot \mathbf{B} = 0 \). Hence, the global definition \( \mathbf{B} = \nabla \times \mathbf{A} \) for magnetic field remains valid.

5 Concluding remarks and outlook

By means of [1], we recapitulated that Maxwell’s theory generates an axion electrodynamics when defining magnetic monopole currents in a suitable way. Especially in this article, we propose to promote the idea which conjectures the axion as a physical manifestation of a space-time internal freedom, namely a phase in the metric spinor. In view of others interactions of the axion field, we obtained axion-like space-time phase-fermion couplings in the usual Dirac’s theory. Thus, we have shown through Maxwell (Ref. [1]) and Dirac theories that Infeld-van der Waerden’s \( \gamma \)-formalism provides an axionic classical sector, where a metric spinor phase played the axion role.

However, under PQ rotations, the behavior of this phase was not established. So, we elaborated a Weyl-PQ approach in which the PQ rotations are implemented in the usual spin transformations, concluding that the phase behaves geometrically as an axion pseudoparticle. On the final part of this investigation, we studied the Dirac-Maxwell system to obtain a 2-spinor form for the magnetic monopole current. The proposition showed that the monopole has a dyon charge and acquires a Witten effect pattern in pseudoparticle. On the final part of this investigation, we studied the Dirac-Maxwell system to obtain a 2-spinor form for the magnetic monopole current. The proposition showed that the monopole has a dyon charge and acquires a Witten effect pattern in pseudoparticle. On the final part of this investigation, we studied the Dirac-Maxwell system to obtain a 2-spinor form for the magnetic monopole current. The proposition showed that the monopole has a dyon charge and acquires a Witten effect pattern in pseudoparticle.

In accordance with theoretical scope, it is indispensable to research some experimental and phenomenological implications as, for example, we suggest in what follows:

**Magnetic (dyon) charges bounds in local dual axion electrodynamics.** Recently, stronger bounds on magnetic and dyon charges have been analyzed in Refs. [36, 37] using non-associative quantum mechanics. As can be seen in [37], when considering dyonic charge for protons or proton–electrons, the authors found the bounds \( g \leq 7 \times 10^{-36} g_{Dirac} \) (protons) and \( g \leq 1.5 \times 10^{-8} g_{Dirac} \) (proton–electrons) by means of the average magnetic field of the Moon, being \( g_{Dirac} \), the Dirac magnetic charge. The results obtained in [37] (as also in [36]) are derived using the Gauss law for \( \nabla \cdot \mathbf{B} \neq 0 \), namely \( \mathcal{G} \sim \int \nabla \cdot (\mathbf{B}(r)) r^2 dr \), when assuming a static magnetic field and \( \mathcal{G} \) spherically symmetric. However, the local dual axion electrodynamics claims that the magnetic monopole density is absent, in the sense of \( \nabla \cdot \mathbf{B} = 0 \), because \((\nabla \Theta) \cdot \mathbf{E} \) eliminates the monopole term. Therefore the correct formula to use would be

\[
\mathcal{G} \sim \int \int [((\nabla \Theta) \cdot \mathbf{E})] dV. \tag{55}
\]

Expression (55) may be used posteriorly to find the bounds of magnetic (dyon) charge in local dual axion electrodynamics similarly as done in Refs. [36, 37].

**Axion-photon conversion.** The conversion of the photons into axions is obtained when considering the Euler–Heisenberg’s theory of quantum electrodynamics [38]. The theory which describes this process takes into account the Lagrangian \( \mathcal{L}_M + \mathcal{L}_{CS} + \mathcal{L}_{EH} \), where the term \( \mathcal{L}_{EH} \sim \left( F_{\mu
u} F_{\mu
u} \right)^2 + (7/4) \left( F_{\mu
u} F^{\mu\nu}_* \right)^2 \) represents the Euler–Heisenberg’s theory in the weak field regime. In Ref. [1], we considered the Maxwell’s theory in the \( \gamma \)-formalism to show that, under the approximation of \( \Theta \) very small, it effectively behaves as a Maxwell-Chern-Simons theory. It is necessary note that our results only provided formal correspondences between the axion and the phase without presenting any substantial modification on the phenomenological viewpoint.

We believe that to verify some modification in current or future physics, we could consider the effective Euler–Heisenberg’s theory in the \( \gamma \)-formalism to observe a possible correction in quantum sector since the term \( \mathcal{L}_M + \mathcal{L}_{EH} \) would approximate to \( \mathcal{L}_M + \mathcal{L}_{CS} + \mathcal{L}_{EH} + \Delta \mathcal{L}_{EH} \), with \( \Delta \mathcal{L}_{EH} \) the correction in quantum level. Private calculations indicate

\[
\mathcal{L}_{EH} \sim (\text{Re} \Sigma)^2 \left[ \cos^2(2\Theta) + \frac{7}{4} \sin^2(2\Theta) \right] + (\text{Im} \Sigma)^2 \left[ \sin^2(2\Theta) + \frac{7}{4} \cos^2(2\Theta) \right] - \frac{3}{2} \text{Re} \Sigma \text{Im} \Sigma \cos(2\Theta) \sin(2\Theta), \tag{56}
\]

where \( \Sigma = \varepsilon^{AC} \varepsilon^{BD} f_{AB} f_{CD} \). Then, similarly as we proceed to derive (26), if we consider the approximation \( \Theta \approx 0 \), we obtain \( \Delta \mathcal{L}_{EH} \propto \Theta F_{\mu\nu} F_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \) so that, perhaps, we can confront the implications of this term with, for example, astrophysical phenomena (for instance, see [39] for a recent work).

Data availability statement Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.
References

1. A.M. Kuerten, A. Fernandes-Silva, Axion electrodynamics from Infeld-van der Waerden formalisms. Mod. Phys. Lett. A 33(16), 1850092 (2018). [arXiv:1711.09496 [gr-qc]]
2. X.-L. Qi, S.-C. Zhang, The quantum spin Hall effect and topological insulators. Phys. Today 63(1), 33–38 (2010). [arXiv:1001.1602 [cond-mat.mtrl-sci]]
3. I. Bakas, Solitons of axion-dilaton gravity. Phys. Rev. D 54, 6424 (1996). [hep-th/9605043]
4. J. Preskill, M. Wise, F. Wilczek, Cosmology of the invisible axion. Phys. Lett. B 120, 127 (1983)
5. L.F. Abbott, P. Sikivie, A cosmological bound on the invisible axion. Phys. Lett. B 120, 133 (1983)
6. U. Buyukcam, J.G. Cardoso, Null Infeld-van der Waerden electromagnetic fields from geometric sources. Int. J. Theor. Phys. 50, 699–705 (2011)
7. M. Carmeli, S. Malin, Theory of Spinors, An Introduction (Word Scientific, Singapore, 2000)
8. J.G. Cardoso, Wave equations for classical two-component Dirac fields in curved spacetimes without torsion. Class. Quant. Grav. 23, 283 (1979)
9. T. Toyoda, A unified theory of the fermi interaction. Nuc. Phys. B 86, 661 (1958)
10. D. Zwanziger, Quantum field theory of particles with both electric and magnetic charges. Phys. Rev. 176, 1489 (1968)
11. J. Schwinger, Magnetic charge and quantum field theory. Phys. Rev. 144, 1087 (1966)
12. E. Witten, Dyons of charge $e^\theta/2\pi$. Phys. Lett. B 86, 283 (1979)
13. M. Bojowald, S. Brahma, U. Buyukcam, J. Guglielmon, M. van Kuppeveld, Small magnetic charges and monopoles in non-associative quantum mechanics. Phys. Rev. Lett. 121(20), 201602 (2018). [arXiv:1810.06540 [hep-th]]
14. A. Addazi, A. Marcianò, Solar system and atomic stronger bounds on exotic dyonic matter and non-associative quantum mechanics. EPL 133, 30004 (2021)
15. G. Raffelt, L. Stodolsky, Mixing of the photon with low mass particles. Phys. Rev. D 37, 1237 (1988)
16. C. Desert, D. Dunsky, B.R. Safdi, Upper limit on the axion-photon coupling from magnetic white dwarf polarization. Phys. Rev. D 105, 103034 (2022). [arXiv:2203.04319 [hep-ph]]