Hamiltonian $U(2)$-actions and Szegö kernel asymptotics

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Abstract. In this paper we shall review some recent results on the asymptotic expansion of the equivariant components of an algebro geometric Szegö kernel determined by the linearization of a Hamiltonian action of $U(2)$ (with certain assumptions). We shall build on the techniques developed in [13], [1], and [11], and therefore ultimately on the microlocal description of the Szegö kernel as a Fourier integral operator in [3].

1. Introduction
Suppose that $M$ is a connected projective manifold of complex dimension $d$, and let $A$ be an ample line bundle on it. Let $h$ be an Hermitian metric on $A$, such that the unique connection on $A$ compatible with both the complex structure and the Hermitian metric has curvature $\Theta = -2i\omega$, where $\omega$ a Hodge form. In particular, $\omega^d$ is a volume form on $M$; we shall denote by $dV_M$ the associated density.

Let $A^\vee$ be the dual line bundle to $A$, endowed with the induced metric, and consider the unit circle bundle $X \subseteq A^\vee$; thus $X$ is a principal $S^1$-bundle on $M$, with projection $\pi : X \rightarrow M$. The connection on $A$ corresponds to a (normalized) connection 1-form $\alpha$ on $X$, such that $d\alpha = 2\pi^*(\omega)$. Then $(\alpha/2\pi) \wedge \pi^*(\omega)^d$ is a volume form on $X$; $dV_X$ will denote the associated density.

Since $\omega$ is Kähler, $(X, \alpha)$ is a contact CR manifold. Let us denote by $H(X) \subseteq L^2(X)$ its Hardy space, by $\Pi : L^2(X) \rightarrow H(X)$ the orthogonal projector (the Szegö projector), and by $\Pi \in D'(X \times X)$ its distributional kernel (the Szegö kernel). It is a well-known foundational result, due to Boutet de Monvel and Sjöstrand, that $\Pi$ is a Fourier integral operator with complex phase ([3], [2], [11]).

Since the CR structure of $X$ is $S^1$-invariant, there is a naturally induced unitary representation of $S^1$ on $H(X)$; therefore, $H(X)$ splits unitarily and equivariantly as a direct sum of isotypical components:

$$H(X) = \bigoplus_{k \in \mathbb{Z}} H(X)_k,$$

where $H_k(X) \subseteq H(X)$ is the subspace of CR functions that transform like the character $e^{ik\theta}$. We have $H(X)_k = 0$ if $k < 0$, and for $k \geq 0$ there is a natural unitary isomorphism between $H(X)_k$ and $H^0(M, A^{\otimes k})$. In particular, every $H(X)_k$ is finite-dimensional.
Thus we have
\[ \Pi = \sum_{k \geq 0} \Pi_k, \]
where \( \Pi_k : L^2(X) \rightarrow H(X)_k \) is the orthogonal projector. The distributional kernel of \( \Pi_k \) (or reproducing kernel of \( H(X)_k \)) is a function \( \Pi_k \in C^\infty(X \times X) \), and its asymptotic behavior has been extensively investigated, starting with the on-diagonal asymptotics of \([12],[4],[13]\). The most relevant approach for our discussion is the one developed in \([13],[1],[11]\) (we refer to the introductions of \([6]\) and \([7]\) for a somewhat wider discussion and reference list).

Specifically, as \( k \rightarrow +\infty \) we have
(i) if \( x, y \in X \), and \( x \notin S^1 \cdot y \), then
\[ \Pi_k(x,y) = O(k^{-\infty}); \]
(ii) if \( x, y \in X \) and \( g, h \in S^1 \), then (denoting \( r_g : X \rightarrow X \) the action of \( g \) on \( X \))
\[ \Pi_k(r_g(x),r_h(y)) = g^k h^{-k} \Pi_k(x,y); \]
(iii) uniformly on \( x \in X \), there is an asymptotic expansion of the form
\[ \Pi_k(x,x) \sim \left( \frac{k}{\pi} \right)^d \cdot \left[ 1 + \sum_{j \geq 1} k^{-j} a_j(m_x) \right], \]
where \( a_j : M \rightarrow \mathbb{R} \) is \( C^\infty \), and we have set \( m_x := \pi(x) \). Each \( a_j \) is describable in terms of the curvature tensor of \( M \) (for instance, as first remarked by \([8]\), \( a_1 \) is the scalar curvature).

On the other hand, when \( x \rightarrow S^1 \cdot y \) at a suitable pace, there is an asymptotic expansion which captures an exponential decrease of \( \Pi_k(x,y) \) away from \( (\pi \times \pi)^{-1}(\Delta_M) \), in a family of shrinking neighborhoods of the latter (\( \Delta_M \) is the diagonal in \( M \times M \)). To express this, following \([1]\) and \([11]\) let us define \( \psi_2 : \mathbb{C}^d \times \mathbb{C}^d \rightarrow \mathbb{C} \) by
\[ \psi_2(v_1,v_2) := -i \omega_0(v_1,v_2) - \frac{1}{2} \|v_1 - v_2\|^2. \]

In so-called Heisenberg local coordinates (henceforth, HLC’s) centered at \( x \in X \), we have
\[ \Pi_k \left( x + \left( \theta_1, \frac{v_1}{\sqrt{k}} \right), x + \left( \theta_2, \frac{v_2}{\sqrt{k}} \right) \right) \sim \left( \frac{k}{\pi} \right)^d \cdot e^{k(\theta_1 - \theta_2) + \psi_2(v_1,v_2)} \cdot \left[ 1 + \sum_{j \geq 1} k^{-j/2} \cdot R_j(m_x;v_1,v_2) \right]. \tag{2} \]

Here \( R_j(m_x;\cdot,\cdot) \) is a polynomial of degree \( \leq 3j \). To appreciate the significance of (2), recall that in the expression \( x + (\theta,v) \) the coordinate \( \theta \) expresses the action of \( e^{i\theta} \in S^1 \), i.e. \( x + (\theta,v) = r_{e^{i\theta}}(x + (0,v)) \), while \( v \) represents an ‘horizontal’ displacement from \( x \) (see \([11]\) for a detailed description of HLC’s). Hence, \( v_1 - v_2 \) represents a displacement from \( (\pi \times \pi)^{-1}(\Delta_M) \). For fixed \( C > 0 \) and \( \epsilon \in (0,1/6) \), the asymptotic expansion (2) holds uniformly for \( \|v_j\| \leq C k^\epsilon \).

The general issue we are involved with is the following: how does the previous analysis carry over when the standard \( S^1 \)-action is replaced by a more general action of a connected compact Lie group \( G \) on \( X \), yielding a unitary representation on \( H(X) \)?

As a motivation for the following arguments, let us remark that the standard circle action \( r : S^1 \times X \rightarrow X \) descends to the trivial action on \( M \), and the latter may be viewed as a
Hamiltonian action with constant moment map equal to 1. Thus we may interpret $r$ as the contact lift of the trivial action on $M$ associated to the moment map $\Phi_{S^1} \equiv 1$.

The latter interpretation yields a much more general setting for the previous result. Namely, given any Hamiltonian action of a connected compact Lie group $G$, with moment map $\Phi_G : M \to g^\vee$, we have a built-in infinitesimal contact action of the Lie algebra $g$ on $X$. Explicitly, for any $\xi \in g$ let $\xi_M \in X(M)$ be the associated Hamiltonian vector field on $M$, and let $\xi_M \in X(X)$ be its horizontal lift to $X$. Then

$$\xi_X := \xi_M^g - (\Phi_G, \xi) \partial_\theta$$

is a contact vector field on $X$; here $\partial_\theta$ is the generator of the $S^1$-action.

Let us assume, as is the case in many natural and interesting situations, that this infinitesimal action is the differential of a genuine contact action $\tilde{\mu}$ of $G$ on $X$. For instance, when $G = S^1$, $\mu$ is trivial and $\Phi_G = 1$, as we have remarked we recover the (reverse) standard action on $X$. Let us also assume that $\mu$ is holomorphic. Then $\tilde{\mu}$ preserves the contact and CR structures of $X$, and induces a unitary representation on $H(X)$.

Hence, by the Theorem of Peter and Weyl, $H(X)$ splits equivariantly and unitarily over the irreducible representations of $G$, which are all finite dimensional:

$$H(X) = \bigoplus_{\mathcal{P} \in G} H(X, \mathcal{P}).$$

(3)

If $0 \notin \Phi_G(M)$, then every isotypical component $H(X, \mathcal{P})$ is finite-dimensional. Therefore, the corresponding projection operator $\Pi_{\mathcal{P}} : L^2(X) \to H(X, \mathcal{P})$ is smoothing, i.e., its distributional integral kernel is in fact a $C^\infty$ function $\Pi_{\mathcal{P}} \in C^\infty(X \times X)$. We are interested in the pointwise asymptotics of $\Pi_{\mathcal{P}}$ as $\mathcal{P} \to \infty$ along a ‘ray’ in weight space, in a sense to be specified.

When $G$ is a torus, this problem has been investigated in [9], [10], [5]. The cases where $G = U(2)$ and $G = SU(2)$ have been studied in [6] and [7], respectively. We shall give an overview of some of the results in [6].

As is well-known, the irreducible representations of $U(2)$ are indexed by the pairs $\mathcal{P} = (\nu_1, \nu_2) \in \mathbb{Z}^2$ with $\nu_1 > \nu_2$, the irreducible representation associated to $\mathcal{P}$ being $V_{\mathcal{P}} := \text{det}^{\nu_2} \otimes \text{Sym}^{\nu_1-\nu_2+1}(\mathbb{C}^2)$. We shall fix a weight $\mathcal{P}$, and consider the asymptotics of $\Pi_k V_{\mathcal{P}}(x, y)$ when $k \to +\infty$.

Let us set from now on $G = U(2)$ and $g = \mathfrak{su}(2)$, and let $O \subset g$ be the coadjoint orbit through $\mathcal{P}$ (with some abuse of language, we view $\mathcal{P}$ as an element of $g$, and identify $g \cong g^\vee$ equivariantly). Furthermore, let $T \leq G$ be the standard maximal torus, with Lie algebra $\mathfrak{t} \subset g$. The restricted action of $T$ is also Hamiltonian, and its moment map $\Phi_T : M \to \mathfrak{t}$ is the composition of $\Phi_G$ with the orthogonal projection $g \to \mathfrak{t}$.

We shall make the following assumptions:

(i) $0 \notin \Phi_T(M)$, and therefore also $0 \notin \Phi_G(M)$;

(ii) $\Phi_G$ is transverse to the ray $\mathbb{R}_{+} \cdot \mathcal{P}$, or equivalently to the cone over the coadjoint orbit, $\mathbb{R}_{+}O$;

(iii) $\Phi_T$ is transverse to $\mathbb{R}_{+} \mathcal{P}$.

For example, consider the unitary representation of $G$ on $\mathbf{C}^4 \cong \mathbf{C}^2 \times \mathbf{C}^2$ given by $A \cdot (Z, W) := (AZ, AW)$. By restriction we obtain a contact action $\tilde{\mu}$ on $S^7$, and by passing to projective space an Hamiltonian action $\mu$ on $\mathbf{P}^3$. The positive line bundle $A$ is of course the hyperplane line bundle, $X = S^7$ and $\pi : S^7 \to \mathbf{P}^3$ is the Hopf map. All the previous assumptions are then satisfied for any pair $\mathcal{P}$ with $\nu_1 > \nu_2$. An explicit plethysm computation yields that $\dim H_{\mathcal{P}}(X) = O(k^2)$ (see [6] for a detailed discussion of explicit examples).

To begin with, $\Pi_k \mathcal{P}(x, x)$ does not have a uniform asymptotic expansion in this case, but rather it concentrates over a certain hypersurface in $M$. More precisely, let us set
under the previous assumptions, \( M_G^C := \pi^{-1}(\mathbb{R}_+ \mathcal{O}) \). Under the previous assumptions, \( M_G^C \subseteq M \) is a connected real hypersurface in \( M \), and \( M \setminus M_G^C \) has two connected components, that we shall euphemistically call the ‘inside’, \( A \), and the ‘outside’, \( B \). Explicitly, if \( m \in M \) the image in \( t \cong \mathbb{R}^2 \) of the orbit \( G \cdot m \) is a closed segment \( J_m \) on the line \( x + y = \text{trace}(\Phi_G(m)) \). Then \( m \in B \) if and only if \( \mathbb{R}_+ \overrightarrow{v} \) does not intersect \( J_m \), \( m \in M_G^C \) if and only if \( \mathbb{R}_+ \overrightarrow{v} \) intersects \( J_m \) in an endpoint, and \( m \in A \) if and only if \( \mathbb{R}_+ \overrightarrow{v} \) intersects \( J_m \) in an interior point.

Then \( \Pi_k \overrightarrow{v}(x, y) = O(k^{-\infty}) \), unless \((m_x, m_y) \in M_G^C \times M_G^C \), and \( x \in G \cdot y \). Rapid decrease holds uniformly for pairs \((x, y)\) converging from the ‘outside’ \( B \) at a sufficiently slow pace to the locus where the previous conditions are satisfied.

We are thus led to consider the asymptotics of \( \Pi_k \overrightarrow{v}(x, x) \) when \( m_x \in M_G^C \). Under the previous assumptions, the action of \( G \) on the inverse image \( X_G^C := \pi^{-1}(M_G^C) \subset X \) is locally free. We shall make the stronger simplifying assumption that \( G \) acts freely on \( X_G^C \) (this is the case in the examples discussed in [6]). Then \( \Pi_k \overrightarrow{v}(x, x) \) admits an asymptotic expansion for \( k \to +\infty \) of the following form:

\[
\Pi_k \overrightarrow{v}(x, x) \sim \frac{C}{\|\Phi_G(m)\|^{d+1/2}} \cdot D_{\overrightarrow{v}}(m) \cdot \left( \frac{\|\overrightarrow{v}\| \cdot k}{\pi} \right)^{d-1/2} \\
\cdot \left[ 1 + \sum_{j \geq 1} k^{-j/4} a_j(m_x) \right],
\]

for appropriate \( C^\infty \) functions \( a_j \). Here \( C \) is a universal constant, and \( D_{\overrightarrow{v}}(m) \) is a distortion function in the normal direction to \( M_G^C \) (we refer to [6] for precise definitions).

We can give a rescaled version of the previous result, at least for ‘horizontal displacements’ along direction perpendicular to the orbits of the complexified action. Namely, if \( x \in X_G^C \) and \( \mathbf{v}_j \in T_{m_x}M, j = 1, 2 \), satisfy \( h_{m_x}(\mathbf{v}_j, \xi_M(m)) = 0 \) for every \( \xi \in \mathfrak{g} \), then the previous pointwise asymptotics may be refined as follows:

\[
\Pi_k \overrightarrow{v} \left( x + \frac{\mathbf{v}_1}{\sqrt{k}} x + \frac{\mathbf{v}_2}{\sqrt{k}} \right) \sim \frac{C \cdot e^{\psi_2(\mathbf{v}_1, \mathbf{v}_2)/\lambda} \cdot D_{\overrightarrow{v}}(m_x)}{\|\Phi_G(m)\|^{d+1/2}} \cdot D_{\overrightarrow{v}}(m) \cdot \left( \frac{\|\overrightarrow{v}\| \cdot k}{\pi} \right)^{d-1/2} \\
\cdot \left[ 1 + \sum_{j \geq 1} k^{-j/4} a_j(\overrightarrow{v}, m_x; \mathbf{v}_1, \mathbf{v}_2) \right],
\]

where \( a_j(\overrightarrow{v}, m_x; \cdot, \cdot) \) is a polynomial of degree \( \leq \lceil 3j/2 \rceil \); the asymptotic expansion holds uniformly for \( \|\mathbf{v}_j\| \leq C^\epsilon k^\epsilon \), for any \( C > 0 \) and \( \epsilon \in (0, 1/6) \).

For normal displacements, there is an asymptotic expansion whose terms are expressed by a less terse integral formula; however, it can be used to obtain an estimate on the dimension of the isotypical component:

\[
\dim H_k \overrightarrow{v}(X) \geq C \cdot \left( \frac{k \|\overrightarrow{v}\|}{\pi} \right)^{d-1} \cdot \int_{M_G^C} \frac{D_{\overrightarrow{v}}(m)}{\|\Phi_G(m)\|^d} dV_{M_G^C}(m) + O\left(k^{d-3/4}\right).
\]

Although our presentation is limited to the Kähler setting for the sake of simplicity, the previous results may be naturally extended to the more general almost complex framework of [2] and [11].
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