Dynamical Evolution of a Cylindrical Shell with Rotational Pressure

Masafumi Seriu*
Department of Physics,
Faculty of Engineering, University of Fukui
Fukui 910-8507, Japan

We prepare a general framework for analyzing the dynamics of a cylindrical shell in the spacetime with cylindrical symmetry. Based on the framework, we investigate a particular model of a cylindrical shell-collapse with rotational pressure, accompanying the radiation of gravitational waves and massless particles. The model has been introduced previously but has been awaiting for proper analysis. Here the analysis is put forward: It is proved that, as far as the weak energy condition is satisfied outside the shell, the collapsing shell bounces back at some point irrespective of the initial conditions, and escapes from the singularity formation.

The behavior after the bounce depends on the sign of \( p_z \), the shell pressure in the z-direction. When \( p_z \geq 0 \), the shell continues to expand without re-contraction. On the other hand, when \( p_z < 0 \) (i.e. it has a tension), the behavior after the bounce can be more complicated depending on the details of the model. However, even in this case, the shell never reaches the zero-radius configuration.

PACS numbers: 04.20.Dw; 04.20.-q; 04.70.Bw

I. INTRODUCTION

Investigations on how the spacetime structures would be during and after gravitational collapses are important topics in gravitational physics and spacetime physics. Nonlinear nature of gravity is making our understanding of the phenomena very difficult, both conceptually and technically.

Steady development of numerical methods is contributing to our deeper understanding on the phenomena. However, it does not mean the analytical investigations are becoming less meaningful. On the contrary, the more the numerical results we get, the more variety of analytical investigations is needed to interpret them and to reach the whole understanding of the phenomena.

In this respect, we have not yet obtained enough variety of mathematical models for gravitational collapses which can be studied, mainly by analytical methods. Indeed, the cases of just spherical collapses with various matter content is already complicated enough, and investigations are still going on [1]. It can be inferred from this situation that constructing and analyzing other types of models may not be an easy task.

As the next step in this direction, cylindrical collapses have been investigated considerably. This class of collapsing models are important because elongated matter distribution can drastically alter the destination of the gravitational collapse. Already in the Newtonian gravity, the Jeans scale plays a vital role and the elongated matter longer than its Jeans scale tends to split into smaller scale objects. In the case of general relativity, there is the so-called “hoop conjecture” [2]; black holes are formed when and only when the matter of mass \( M \) is contained in a compact region whose circumference \( C \) in any direction always satisfies \( C \leq 2\pi \cdot \frac{GM}{c^2} \). Therefore it is of great importance to study the fate of the collapsing elongated matter, such as cylindrical collapses.

The numerical investigation of the collapse of the non-rotating dust spheroid by Shapiro and Teukolsky [3] suggested that the naked singularities could emerge when sufficiently elongated object collapses. This numerical result motivated further analytical investigations on the cylindrical collapses. Here, in view of the cosmic censorship hypothesis [4], investigations on more “natural” conditions, rather than non-rotating cases, are of significance. In particular, the effect of rotation and the effect of matter pressure should be studied. Then, an analytical model of collapsing cylindrical shell made of counter-rotating dust particles was investigated by Apostolatos and Thorne [5]. (Here “counter-rotating particles” indicates the situation in which the half of the particles are rotating in the opposite direction to the other half of the particles.) It takes into account the rotation of particles through the rotational pressure, while the net angular-momentum of the shell is kept zero. By avoiding the frame-dragging effect in this manner, the analysis can be relatively simple. It turned out that even a small amount of rotation prevents the singularity formation as far as this particular model concerned.

Another analytical model of cylindrical gravitational collapses was presented and briefly studied by Pereira and Wang [6, 7]. This model has interesting features in that not only has it taken into account both the rotation and the pressure effect, but it also tries to take into account the radiation of gravitational waves and massless particles from the collapsing region. Indeed, the numerical work suggests that the gravitational radiation
becomes important after the bounce of the cylindrical shell [8]. In spite of several interesting features, however, this model has not yet been properly investigated so far.

Since this model is what is investigated in this paper, let us briefly recall their work [6, 7]. Firstly they prepare general formulas in the coordinate system \(\{t, r, z, \phi\}\) for matching arbitrary two cylindrical spacetimes. Here junction conditions [9] play the central role. Then they apply the formulas to look at a particular model. In this model, an interior flat spacetime and an exterior cylindrical spacetime are matched together at \(r = R_0(t)\). The geometry of the exterior spacetime is so chosen that there is out-going flux which can be interpreted as gravitational waves and massless particles (see Sec.III for the details of this model). Then a dynamical equation for the shell is explicitly derived (see Eq.(39) in Sec.IV). Since this non-linear equation is complicated enough, a very special class of solutions is searched for, and three types of special solutions are claimed to be found. They are the solutions of (i) a collapsing shell which bounces back at some finite radius, followed by an eternal expansion afterwards, (ii) an eternally expanding shell, and (iii) a collapsing shell whose radius reaches zero, resulting in a line-like singularity formation [6, 7].

The solution (iii) looks in particular interesting, since it reminds us of the result of numerical calculations by Shapiro and Teukolsky [3], though the spacetime structures in these two cases are not the same. Considering several interesting features of the model, it is quite regrettable that their investigation is a preliminary one in nature with the limitation to the very special class of solutions, and moreover it turns out that the derivation of the special solution faces with crucial problems, which could totally alter the main results (see Sec.IV for the detailed account on this point). Thus it is meaningful to analyze this model in detail.

With these situations in mind, the present paper aims mainly two goals.

Firstly, we set up a general framework for analyzing cylindrical shell collapses in the same spirit as Pereira and Wang [6], but paying more attention to the mutual relations of three kinds of natural frames of reference. They are (1) the ortho-normal frame \(\{\hat{e}, A\}_{A=0,1,2,3}\) associated with the cylindrical spacetime geometry, (2) the ortho-normal frame \(\{\hat{e}_n, \hat{e}_r, \hat{e}_z, \hat{e}_\phi\}\) associated with a shell \(\Sigma\) in question (here \(\hat{e}_n\) is the normal unit 4-vector to the shell \(\Sigma\) while \(\hat{e}_r\) is the 4-vector tangent to the world-line of an observer located on the shell), and (3) the coordinate frame \(\{\partial_t, \partial_r, \partial_z, \partial_\phi\}\). Computations based on the frame (1) can be carried out systematically, while the frame (2) is more suitable for interpreting the results. Therefore the bridge between them would be helpful. The frame (3) has been also considered since it is often convenient for actual computations.

The second purpose of this paper is to analyze the above-mentioned model, based on the framework we prepare, and investigate the dynamical behavior of the collapsing shell. We assume that the weak energy condition [10] is satisfied outside the shell. It means, roughly speaking, that the collapsing matter behaves as normal source of gravity. It shall be proved that, the collapsing shell bounces back at some point irrespective of the initial conditions, and escapes from the singularity formation. The behavior after the bounce depends on the sign of the shell pressure in the z-direction, \(p_z\). When \(p_z \geq 0\), the shell continues accelerated expansion, and its velocity asymptotically tends to the light velocity. On the other hand, when \(p_z < 0\) (i.e. tension), the behavior after the bounce could be more complicated depending on the details of the model. In either case, the shell never reaches the zero-radius configuration, contrary to the result (the solution (iii) mentioned above) of the preliminary analysis in Refs. [6] and [7].

In Sec.II, the fundamental formulas on the cylindrical spacetime and the cylindrical shell in it are derived. We in particular pay attention to the extrinsic curvature of the shell, which plays the central role for studying the shell dynamics. The junction conditions, which is another essential part for the shell dynamics, are also investigated here. Based on the framework so prepared, then, we investigate in detail a particular model of a collapsing cylinder in Sec.III. In particular, we prove a theorem that the shell never reaches the singularity in this model as far as the weak energy condition is satisfied outside the shell. Section IV is devoted to the detailed comparison between the present analysis and the one reported in Refs. [6] and [7], pinpointing why the latter had been led to the virtual singular solution. Section V is devoted for summary.

## II. GEOMETRY FOR A MOVING CYLINDRICAL SHELL IN A CYLINDRICAL SPACETIME

### A. The metric and frame-vectors

#### 1. The metric

We consider a spacetime with cylindrical symmetry defined by a line-element,

\[
 ds^2 = - T(t, r)^2 dt^2 + R(t, r)^2 dr^2 + Z(t, r)^2 dz^2 + \Phi(t, r)^2 d\phi^2 . \tag{1}
\]

(By a suitable coordinate transformation \((t, r) \mapsto (\hat{t}, \hat{r})\), one can actually assume \(T = R\), but we here keep the form Eq.(1).)

Looking at Eq.(1), the theory of differential forms is efficiently applied [12], by introducing natural 1-forms, \(\hat{\theta}^0 := T(t, r) \, dt, \quad \hat{\theta}^1 := R(t, r) \, dr, \quad \hat{\theta}^2 := Z(t, r) \, dz, \quad \hat{\theta}^3 := \Phi(t, r) \, d\phi \). \tag{2}

The local ortho-normal frame \(\{\hat{e}, A\}_{A=0,1,2,3}\) is then defined as the dual to the 1-forms \(\{\hat{\theta}^A\}_{A=0,1,2,3}\).
Fundamental geometrical quantities, such as curvatures, w.r.t. (with respect to) the frame \(\{\mathbf{e}_A\}_{A=0,1,2,3}\) are derived and collected in Appendix A for the convenience of later use and future applications.

Now let \(\Sigma\) be a timelike 3-surface with cylindrical spatial symmetry embedded in a cylindrical spacetime described by the metric Eq.(1). Due to its symmetry, the surface \(\Sigma\) is characterized by a single equation \(r = \rho(t)\). Let \(\tau\) be a proper time of an observer located on the surface with \(dz = d\theta = 0\). Then, the metric Eq.(1) yields a relation between \(d\tau\) and \(dt\),

\[
d\tau^2 = T^2 (1 - \frac{R^2}{T^2} \rho^2) dt^2 .\tag{3}
\]

We can use \((\tau, z, \phi)\) as a coordinate system on \(\Sigma\), which defines ortho-normal vectors tangent to \(\Sigma\),

\[
\hat{e}_\tau := \partial_\tau = \frac{dt}{d\tau} \partial_t + \frac{d\rho}{dt} \partial_{\rho} = (X, \hat{\rho}, 0, 0)_{(\tau z \phi)} , \tag{4}
\]

and

\[
\hat{e}_z := \frac{1}{Z} \partial_z , \hat{e}_\phi := \frac{1}{\Phi} \partial_\phi . \tag{5}
\]

Here “\(\circ\)” indicates the derivative w.r.t. \(\tau\) and \(X := dt/d\tau\).

A normal unit vector \(\hat{n}\) of the surface \(\Sigma\), which may also be written as \(\hat{e}_n\) or \(\partial_n\), and its dual one-form are given by, respectively,

\[
\hat{e}_n = \hat{n} = \partial_n = \left(\frac{R}{T} \hat{\rho}, \frac{T}{R} \rho, X, 0, 0\right)_{(\tau z \phi)} ,
\]

\[
\hat{n}_{\mu} = XTR(-\hat{\rho}/X, 1, 0, 0)_{(\tau z \phi)} . \tag{6}
\]

(The suffix \((\tau z \phi)\) indicates the frame employed.) We note that the set of vectors \(\{\hat{e}_n, \hat{e}_\tau, \hat{e}_z, \hat{e}_\phi\}\) forms the local ortho-normal frame located on \(\Sigma\).

2. Summary of notations

At this stage, let us summarize notations employed throughout this paper.

1. Capital Latin indices \(A, B, C, \ldots\) are reserved for the indices w.r.t. the local ortho-normal frame \(\{\mathbf{e}_A\}_{A=0,1,2,3}\), and runs from 0 to 3. Einstein’s summation convention is adopted throughout the paper.

2. The first few Greek letters \(\alpha, \beta, \gamma, \ldots\) are reserved for the indices w.r.t. the ortho-normal frame \(\{\hat{e}_n, \hat{e}_\tau, \hat{e}_z, \hat{e}_\phi\}\), and take the values \(n, \tau, z\) or \(\phi\); The later Greek letters \(\mu, \nu, \ldots\) are used for the indices w.r.t. the coordinate bases \(\{\partial_\mu, \partial_\nu, \partial_z, \partial_\phi\}\).

3. Since several frames appear in the paper, we indicate the frame explicitly when necessary. For instance, \((p, q, r, s)_{(\tau z \phi)} := p\partial_\tau + q\partial_r + r\partial_z + s\partial_\phi\) for a vector and \((p, q, r, s)_{(\tau z \phi)} := pdt + qdr + rdz + sdd\phi\) for a 1-form.

4. The symbols \(\tilde{F}, F^\circ\) and \(F^{\circ}\) denote the partial derivatives of a function \(F\) w.r.t. \(t, r\) and \(\tau\), respectively. In the same manner, \(\partial_\tau F\) indicate the directional derivative of \(F\) w.r.t. the vector \(\hat{n}\).

3. Frequently used formulas

We here summarize essential formulas frequently used below. We note the relations,

\[
X := \frac{dt}{d\tau} = \frac{1}{T} \sqrt{1 - \frac{R^2}{T^2} \rho^2} , \quad \circ := \frac{d\rho}{dt} = \rho \frac{dt}{d\tau} = X\rho . \tag{7}
\]

Therefore it follows

\[
X^2T^2 = 1 + R^2 \circ^2 . \tag{8}
\]

For any function \(F(t, r)\) defined in the neighborhood of \(\Sigma\), it follows

\[
\tilde{F} = XT^2 \circ - TR \circ \partial_r F ,
\]

\[
F' = - R^2 \circ \circ \partial_r F + XTR \circ \partial_r F . \tag{10}
\]

Each derivative in Eqs.(9) and (10) is understood to be estimated on \(\Sigma\).

B. Extrinsic curvature of the surface \(\Sigma\)

1. In the ortho-normal frame \(\{\hat{e}_n\}_{n=\tau, z, \phi}\)

We now compute the extrinsic curvature of \(\Sigma\), which later plays the essential role in analyzing the shell dynamics. The extrinsic curvature is handled mainly in the ortho-normal frame \(\{\hat{e}_\tau, \hat{e}_z, \hat{e}_\phi\}\) in this paper. To remind us of it, we write \(K_{\tau\tau\tau}\) etc., rather than \(K_{\tau\tau}\) etc.

There is a standard formula which relates \(K_{\tau\alpha\beta}\) with the Christoffel symbol \(\Gamma\) and the normal vector \(\hat{n}\),

\[
K_{\tau\alpha\beta} = -\hat{n}_\gamma \Gamma^\gamma_{\alpha\beta} = -\Gamma_{\alpha\beta}^\gamma = -E^A_{\alpha\beta}E^B_{\gamma}E^C_{\mu}\Gamma^\gamma_{AB} - E^A_{\mu\beta}E^B_{\nu}\partial_\mu E^B_{\nu} , \tag{11}
\]

where in the last line, Eq.(B11) in Appendix B has been used to express the Christoffel symbol \(\Gamma^\gamma_{\alpha\beta}\) in terms of \(\Gamma^\gamma_{AB}\) since the latter, given in Eq.(A3), is much easier to compute.

Let us first compute \(K_{\tau\tau\tau}\),

\[
K_{\tau\tau\tau} = -E^A_{\tau\tau}E^B_{\tau}E^C_{\rho}\Gamma^\rho_{AB} - E^A_{\rho\rho}E^B_{\tau}\partial_\rho E^B_{\tau} \tag{12}
\]
The final result is

\[ K_{e,e} = -\frac{R}{2XT} \rho - \frac{R}{2XT} \dot{\rho} (\ln R^2)^o + \frac{1}{2} (T^2 X^2 - 1) \partial_n (\ln R^2) + \frac{X^2 T^2}{2} (X^2 T^2 - 1) \partial_n \ln (T^2/R^2) - \frac{X T R}{2} \rho (X^2 T^2 - 1) (\ln (T^2/R^2))^o . \]  

\[ (13) \]

We note that \( K_{e,e} \) contains the second time-derivative of \( \rho \) as is seen in Eq.(13), so that \( K_{e,e} \) plays the most important role in the shell dynamics. In view of its essential significance, the detailed derivation of Eq.(13) is shown in Appendix C. It is by far easier to compute \( K_{e,e} \) and \( K_{e,e} \). Combining Eq.(11) with Eqs.(B6) and (B10), we get

\[ K_{e,e} = \partial_n \ln Z . \]  

\[ (14) \]

\[ K_{e,e} = \partial_n \ln \Phi . \]  

\[ (15) \]

After some computations (see Appendix C for more details), we finally reach the result,

\[ K_{e,e} = -X^2 T R \dot{\rho} + (X^2 T^2 - 1) \frac{XT}{2R} \left( (\ln T^2)'^o + \frac{R^2}{T^2} \dot{\rho} (\ln R^2) \right) + \frac{1}{2} X^2 T R \dot{\rho} (\ln (T^2/R^2))^o . \]  

\[ (16) \]

Here the \( \tau \)-derivative in the last term on the R.H.S. (right-hand side) has been purposely left untouched to make the formula look simpler; the term will vanish whenever \( T = R \), as is the case for the model we analyze later.

C. Junction Conditions

Discontinuity in the extrinsic curvature across a 3-surface implies that the energy-momentum is accumulated on the shell: this is analogous to the relation of the discontinuity in the electric field across a shell with the electric charge there.

Geometrical discontinuity across a 3-surface \( \Sigma \) is described by a discontinuity-tensor induced on \( \Sigma \) defined as

\[ \kappa S_{\alpha\beta} = [K_{\alpha\beta}] - h_{\alpha\beta} [K] , \]  

\[ (17) \]

where \( [Q] := Q_+ - Q_- \), the difference of \( Q \) across the surface; \( \kappa := 8\pi G \) is the Einstein’s gravitational constant.

Then the discontinuity is governed by the junction conditions [9],

I) \( ds^2_{\Sigma} = ds^2_{\Sigma} \)  

II) \( D_{\beta} S_{\alpha}^{\beta} = [T_{\mu\nu} E_{\mu}^{\alpha} \dot{n}^\nu] \) .

\[ \]  

\[ (18) \]

Let us now prepare more explicit expression for Eq.(18). The metric induced on the surface \( \Sigma \) is

\[ ds^2_{\Sigma} = -dt^2 + 2dz^2 + \Phi^2 d\phi^2 = -\Theta^\tau \otimes \Theta^\tau + \Theta^z \otimes \Theta^z + \Theta^0 \otimes \Theta^\phi , \]

where \( \Theta^\tau = dt, \Theta^z = Zdz, \) and \( \Theta^\phi = \Phi d\phi \) form a set of orthonormal 1-forms on \( \Sigma \) dual to the bases \( \{ \hat{e}_\alpha \}_\alpha = r,z,\phi \). Thus the discontinuity-tensor in this frame becomes

\[ S_{e,e} = -\frac{1}{\kappa} ([K_{e,e}] + \rho [K_{e,e}]) =: \epsilon , \]

\[ S_{e,e} = -\frac{1}{\kappa} ([K_{e,e}] - \rho [K_{e,e}]) =: \rho_z , \]

\[ S_{e,e} = -\frac{1}{\kappa} ([K_{e,e}] - \rho [K_{e,e}]) =: \rho_\phi . \]  

\[ (19) \]

where \( \epsilon, \rho_z \) and \( \rho_\phi \) are interpreted as the energy density of the shell, the pressure in the \( z \)-direction and pressure in the \( \phi \)-direction, respectively. It is easy to get

\[ \Gamma_{e,e}^{e,e} = (\ln Z)_{\sigma}^{\nu}, \Gamma_{e,e}^{e,e} = (\ln \Phi)^{\nu}, \Gamma_{e,e}^{e,e} = (\ln Z)_{\nu}^{\sigma}, \]

\[ \Gamma_{e,e}^{e,e} = (\ln \Phi)^{\nu}, \text{ others} = 0 . \]

(We note \( \{ \hat{e}_e \} \) forms the non-coordinate bases and, in particular, \( \Gamma_{e,e}^{e,e} = 1 \).

Thus

\[ D_{\beta} S_{\alpha}^{\beta} = (-\epsilon - (\epsilon + \rho_z)(\ln Z)^o - (\epsilon + \rho_\phi)(\ln \Phi)^o, 0, 0) \]  

\[ (20) \]
in the \((\theta_\tau, \theta_\theta, \theta_\phi)\)-coordinate. Then (II) in Eq.(18) reduces to
\[
\dot{\xi} + (\epsilon + p_z)(\ln Z)^\circ + (\epsilon + p_\phi)(\ln \Phi)^\circ = -[T_{\mu \nu}\ E^\mu_\tau\ \dot{n}^\nu] ,
\]
with \(\dot{n}^\nu\) and \(E^\mu_\alpha\) being given in Eq.(6) and Eq.(B6), respectively.

Noting that
\[
S_\alpha = \dot{\theta}_\tau \otimes \dot{\theta}_\tau + p_z \dot{\theta}_\theta \otimes \dot{\theta}_\tau + p_\phi \dot{\theta}_\phi \otimes \dot{\theta}_\phi = \epsilon d\tau \otimes d\tau + p_z Z(\tau)^2 dz \otimes dz + p_\phi \Phi(\tau)^2 d\phi \otimes d\phi ,
\]
We also get
\[
S_{ab} = \text{diag}(\epsilon, Z(\tau)^2 p_z, \Phi(\tau)^2 p_\phi)_{(\tau, z, \phi)} .
\]
It is straightforward to get
\[
\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}(Z^2)^\circ , \quad \Gamma^\alpha_{\phi\beta} = \frac{1}{2}(\Phi^2)^\circ , \quad \Gamma^\alpha_{\tau\tau} = \Gamma^\tau_{\tau\tau} = (\ln Z)^\circ ,
\]
\[
\Gamma^\alpha_{\theta\phi} = \Gamma^\phi_{\theta\alpha} = (\ln \Phi)^\circ , \text{ others } = 0 ,
\]
and
\[
D_b S^b_{\alpha} = (- \dot{\xi} - (\epsilon + p_z)(\ln Z)^\circ - (\epsilon + p_\phi)(\ln \Phi)^\circ , 0, 0)
\]
in the \((\tau, z, \phi)\)-coordinate, which is the same expression as in Eq.(20). Therefore, (II) in Eq.(18) reduces to Eq.(21) in the \((\tau, z, \phi)\)-frame also.

### III. ANALYSIS OF A COLLAPSING SHELL MODEL WITH ROTATIONAL PRESSURE AND GRAVITATIONAL RADIATION

Having prepared all the necessary formulas in the preceding sections, let us now reanalyze the cylindrical collapsing-shell model considered by Pereira and Wang [6]. The spacetime to be analyzed is constructed by matching two spacetimes \(M_\pm\) described by the metrics \(ds_\pm\) at the timelike surface \(\Sigma\) (the suffix \(-\) is for the interior geometry and \(+\) is for the exterior geometry): The metrics \(ds_\pm\) are given by
\[
ds_+^2 = -dt_+^2 + dr_+^2 + dz_+^2 + r_+^2 d\phi_+^2 , \quad \Gamma^\alpha_{\beta\gamma} = \frac{1}{2}(\Phi^2)^\circ , \quad \Gamma^\alpha_{\tau\tau} = \Gamma^\tau_{\tau\tau} = (\ln Z)^\circ ,
\]
\[
ds_-^2 = e^{2\gamma}(\xi)(-dt_-^2 + dr_-^2 + dz_-^2 + r_-^2 d\phi_-^2)
\]
where \(\gamma = \gamma(\xi)\), a function of \(\xi := t_+ - r_+\) only: The surface \(\Sigma\) is assumed to be described by \(r_\pm = \rho_\pm(t)\). The metric \(ds_\pm\) has the form of \(ds\) in Eq.(1) with
\[
T(t_+, r_+) = R(t_+, r_+) = e^{\gamma(\xi)} , \quad Z(t_+, r_+) = 1 , \quad \Phi(t_+, r_+) = r_+ .
\]
The C-energy (cylindrical energy) [5, 11] of this spacetime is given by
\[
C = \frac{1}{8\kappa}(1 - e^{-2\gamma}) ,
\]
so that we assume \(\gamma \geq 0\) from now on.

The interior spacetime is obviously flat, while the exterior geometry is given by, with the help of Eq.(A8) and noting that \(\dot{\gamma} = -\gamma'\), \(\ddot{\gamma} = \gamma''\),
\[
R_{00} = \frac{\gamma'}{r_+} e^{-2\gamma} ,
\]
\[
R_{01} = \frac{\dot{\gamma}}{r_+} e^{-2\gamma} = -\frac{\gamma'}{r_+} e^{-2\gamma} ,
\]
others = 0 .

Therefore \(R = 0\) and \(G_{00} = G_{11} = -G_{01} = -G_{10} = \frac{\gamma'}{r_+} e^{-2\gamma}\), or
\[
G_{\alpha} = \frac{\gamma'}{r_+} e^{-2\gamma}(\dot{\phi}_0 \otimes \dot{\phi}_0 - \dot{\phi}_1 \otimes \dot{\phi}_1 - \dot{\phi}_1 \otimes \dot{\phi}_0 + \dot{\phi}_0 \otimes \dot{\phi}_1) + \frac{\gamma'}{r_+}(dt_+ \otimes dt_+ + dr_+ \otimes dr_+ + dr_+ \otimes dt_+)
\]

Thus [6]
\[
G_{ab} = \frac{\gamma'}{\alpha^2 r_+} k_\alpha k_\beta ,
\]
where \(k_\mu = (\alpha, -\alpha, 0, 0)_{(\tau z \phi)}\) is a null-vector.

Combined with the Einstein equation, Eq.(25) implies \(\gamma' \geq 0\) when the weak energy condition for matter is imposed. Since we are interested in the behavior of the collapsing shell in more or less “natural” conditions in view of the cosmic censorship conjecture [4], it is reasonable to assume \(\gamma \geq 0\) and \(\gamma' \geq 0\) from now on.

It may be appropriate to make some remarks on the energy condition. Without any condition imposed on matter, any strange spacetime would become possible. Indeed, one can always claim that arbitrary spacetime is the solution of the Einstein equation if the matter with energy-momentum tensor \(T_{ab} := \frac{1}{2}G_{ab}\) is prepared. Therefore it is essential to assume some kind of energy condition to make any meaningful arguments [10]. The weak energy condition is one of such appropriate conditions widely accepted, so we assume the condition here to make the definite arguments below. Of course, there is still a room of considering the violation of the energy condition at the shorter scale, due to the Casimir effect for instance. Though such a possibility is certainly interesting to study, we do not consider it in this paper for definiteness and only investigate the purely classical situations.

We now impose the junction conditions Eq.(18) for two geometries \(M_\pm\):

**Junction Condition (I)**

We impose
\[
ds_-^2|_{r_- = \rho_-(t_-)} = ds_+^2|_{r_+ = \rho_+(t_+)} .
\]
When \( dz_\pm = d\phi_\pm = 0 \), it implies

\[
\frac{dt^2_+}{dt^2_-} = \frac{dt^2_+ - dt^2_-}{e^{2\gamma_+}(dt^2_+ - dt^2_-)} = (1 - \rho_+^2)dt^2_+ - \rho_+^2 dt^2_-
\]

where \( \gamma_+ := \gamma(t_+, \rho_+(t_+)) \). By plugging the relation \( \rho^2 = (\frac{dt_+}{dt_-})^2 \rho_+^2 \) into the above relation and by solving the latter w.r.t. \( \frac{dt_+}{dt_-} \), we get an important formula

\[
\frac{dt_+}{dt_-} = \{(1 - e^{2\gamma_+})\rho_+^2 + e^{2\gamma_+}\}^{-1/2} =: \Delta^{-1} . \tag{27}
\]

Taking the \( t \)-derivative of the above relation, one finds

\[
\frac{d^2 t_+}{dt^2_-} = -\frac{1}{\Delta^4} \left\{ \rho_+(1 - e^{2\gamma_+})\rho_+ - e^{2\gamma_+}\rho_+ (1 - \rho_+^2) \right\} . \tag{28}
\]

Noting the relation,

\[
\frac{d}{dt_+}\Delta = \frac{1}{\Delta} \rho_+ - \frac{e^{2\gamma_+}}{\Delta} (\rho_+ \rho_+ + \gamma_+(1 - \rho_+^2)) , \tag{29}
\]

one also finds

\[
\rho_-(t_-) = \Delta^{-1} \frac{dt_+}{dt_-} (\Delta^{-1} \rho_+) = \frac{e^{2\gamma_+}}{\Delta^2} \left\{ \rho_+ + \gamma'_+(1 - \rho_+^2) \right\} . \tag{30}
\]

### Junction Condition (II)

one finds

\[
[K_{e,e}] = -\frac{e^{-\gamma_+}}{\Delta(1 - \rho_+^2)^{3/2}} \left\{ (\Delta - 1)\rho_+ + \gamma'_+(1 - \rho_+^2)\right\} , \\
[K_{e,e}] = 0 , \\
[K_{e,e}] = -e^{-\gamma_+} \frac{\Delta - 1}{\rho_+\sqrt{1 - \rho_+^2}} .
\]

Equation (19) then yields

\[
\kappa \epsilon = e^{-\gamma_+} \frac{\Delta - 1}{\rho_+\sqrt{1 - \rho_+^2}} , \tag{31}
\]

\[
\kappa p_z = \frac{e^{-\gamma_+}}{\Delta(1 - \rho_+^2)^{3/2}} \left\{ (\Delta - 1)\rho_+ + \gamma'_+(1 - \rho_+^2)\right\} , \tag{32}
\]

\[
\kappa p_\phi = \frac{e^{-\gamma_+}}{\Delta(1 - \rho_+^2)^{3/2}} \left\{ (\Delta - 1)\rho_+ + \gamma'_+(1 - \rho_+^2)\right\} . \tag{33}
\]

with \( \Delta = [\rho_+^2 + e^{2\gamma_+}(1 - \rho_+^2)]^{1/2} \).

Let us find the expressions for \( K_{e,e} \) (Eqs.(14), (15) and (16)) in this model, and then derive the formulas for \( \epsilon, p_z \) and \( p_\phi \) (Eq.(19)).

First of all, the factor \( X \) becomes

\[
X = \frac{e^{-\gamma_+}}{\sqrt{1 - \rho_+^2}} ,
\]

implying that

\[
\rho_+^2 < 1 .
\]

Then, noting the relation \( X_- = \Delta X_+ , \) Eq.(16) with \( T_- = R^2 = e^{2\gamma_+} \) yields

\[
K_{e,e} |_+ = -X_+^3 e^{2\gamma_+}\rho_+ + \gamma'_+ X_+^3 e^{2\gamma_+} \rho_+^2 (1 - \rho_+) , \\
K_{e,e} |_- = -X_-^3 \frac{\rho_-}{\Delta} e^{2\gamma_+} \rho_+ - \gamma'_+ X_-^3 e^{2\gamma_+} (1 - \rho_+^2) \rho_+ ,
\]

where Eq.(30) has been used in the last line. Noting also

\[
K_{e,e} = 0 , \\
K_{e,e} = \partial_n \ln r = \frac{X}{r} ,
\]

Noting the relation \( \Delta^2 - 1 = (1 - \rho_+^2)(e^{2\gamma_+} - 1) \), we see

\[
\gamma_+ = 0 \iff \Delta = 1 \iff \epsilon = 0 . \tag{34}
\]

When \( \gamma \equiv 0, \Delta = 1 \) and \( \epsilon = p_z = p_\phi = 0 \), as it should

\[
> \iff \Delta = 1 \iff \epsilon = 0 .
\]
be.
Setting
\[ P_0 := \kappa p_z \]

\[
\dot{\rho}_+ = \frac{\Delta}{\rho_+} - \frac{\rho_+^2}{\rho_+} + \gamma_+ \frac{\dot{V}}{\Delta - 1} (1 - \dot{\rho}_+) [(\Delta + 1) \dot{\rho}_+ + 1] + \frac{\Delta}{\Delta - 1} (1 - \rho_+^2)^{3/2} e^{\gamma_+} P_0 .
\] (35)

To make the statements concise, we indicate the first, the second and the third terms on the R.H.S. of Eq.(35) by the symbols, \([1]\), \([2]\) and \([3]\), respectively.

The term \([3]\) indicates that when the shell has a positive pressure in the z-direction \((P_0 > 0)\), it causes more acceleration in the expanding direction, while the tension of the shell \((P_0 < 0)\) works to reduce the acceleration. However the term \([3]\) does not change the whole dynamics so much: The term could become important only when \(\Delta \sim 1\), i.e. when \(\dot{\rho}_+ \sim \pm 1\). Setting \(\dot{\rho}_+ = \pm (1 - \delta)\) \((\delta > 0)\), the term behaves as \(-e^{\gamma_+}/\rho_+\delta^{1/2}\), so that the contribution of the term is not significant. Indeed, we shall show below a theorem that under the dynamical equation Eq.(35), the shell never reaches the singular configuration \(\rho_+ = 0\) irrespective of the initial conditions nor the value of \(P_0\).

First of all, we note the following obvious fact:

**Lemma 1**

\[
\dot{\rho}_+ = \frac{\Delta}{\rho_+} - \frac{\rho_+^2}{\rho_+} + \gamma_+ \frac{V}{\Delta - 1} (1 + V) [(\Delta + 1) V - 1] + \frac{\Delta}{\Delta - 1} (1 - V^2)^{2/3} e^{\gamma_+} P_0 .
\] (36)

with \(\Delta = [V^2 + e^{2\gamma_+} (1 - V^2)]^{1/2}\).

The present model has been constructed by giving metrics Eqs. (22) and (23), and there is no particular physical image regarding the matter content forming the shell. However, one can infer that the term \([1]\) comes from the angular momentum effect since it is a decreasing function of \(\rho_+\). On the other hand, the term \([2]\) may be identified with what corresponds to Newtonian gravity under suitable conditions, since it is proportional to \(\gamma_+\).

Let us pay attention to the last factor in the term \([2]\), \(f(V) := (\Delta + 1) V - 1\). The function \(f(V)\) is obviously monotonic, increasing, continuous function of \(V\), with \(f(0) = -1\) and \(f(1) = 1\), so that \(f(V)\) has only one zero in \((0, 1)\).

When the weak energy condition is satisfied, and the shell is not moving so fast \((V \ll 1)\), then \(f(V) < 0\), so that the term \([2]\) yields an attractive force, which is consistent with the attractive nature of Newtonian gravity. A curious feature arises, however, when \(V \sim 1\). Then \(f(V) > 0\), and the same term now yields a repulsive force. Due to this characteristic behavior of \(f(V)\), the shell bounces back at some point, and never collapses to singularities as we shall see soon.

Before proving this statement, let us first get rough understanding on what happens when \(\rho_+\) becomes small. The term \([1]\) is the repulsive force, produced by an infinite effective potential barrier near \(\rho_+ = 0\) (angular momentum effect). In case the shell would have ever reach \(\rho_+ = 0\), it should have been this repulsive effect becomes negligible compared to other terms, i.e. when \(\rho_+\) approaches \(-1\) \((V \rightarrow 1)\) more rapidly than \(\rho_+\) approaches to zero. However, in this case, the term \([2]\) causes a very strong repulsive effect since \(f(V) > 0\) and \(\Delta \sim 1\), while

**Proof:**

Noting that \(\Delta > 1\) and that the terms \([2]\) and \([3]\) are non-negative when \(\dot{\rho}_+ \geq 0\), it follows that \(\dot{\rho}_+ > \frac{1 - e^{\gamma_+}}{\rho_+} > 0\), so that the claim follows. \(\square\)

In particular, when \(|\dot{\rho}_+| \ll 1\), Eq.(35) becomes \(\dot{\rho}_+ \sim e^{\gamma_+}/\rho_+\), so that the effective potential behaves like \(C_{eff} \sim -e^{\gamma_+} \log \rho_+ + \text{const.}\), in the region where \(\gamma_+\) does not change so drastically. Compared with the dust case \([5]\), where \(C_{eff} \sim a/\rho_+^2 + b \ln \rho_+ + \text{const.}\) \((a, b\) are constants), the tendency of expansion of the present model is obvious.

Next, let us investigate the contracting shell. To avoid any confusion, let us set \(V := -\dot{\rho}_+\), so that \(0 < V < 1\). Then Eq.(35) reads

Eq.(32) yields the dynamical equation of the shell,

\[
\text{Consider the case } P_0 \geq 0. \text{ Once the shell satisfies the condition } \dot{\rho}_+ \geq 0, \text{ it expands forever afterwards.}
\]

**Proof:**

Note that \(\Delta > 1\) and that the terms \([2]\) and \([3]\) are non-negative when \(\dot{\rho}_+ \geq 0\), it follows that \(\dot{\rho}_+ > \frac{1 - e^{\gamma_+}}{\rho_+} > 0\), so that the claim follows. \(\square\)
the term \([3]\) becomes negligible. This behavior can be analyzed in more detail as follows. Let us set \(V = 1 - \delta\) \((\delta > 0)\). Then \(\Delta = 1 + (e^{\gamma_+} - 1)\delta + O(\delta^2)\), and one finds \([1]\) \(\sim \frac{\delta}{\rho_+}\), \([2]\) \(\sim \delta^{-1}\), and \([3]\) \(\sim \delta^{1/2}\), so that

\[
\frac{\rho_+\dot{\rho}_+}{\rho_+^2} \sim \frac{2\gamma_+\rho_+}{e^n - 1} \delta.
\]

Therefore, when \(\delta/\rho_+ \to 0\), then the term \([2]\) causes a very strong repulsive effect.

In summary, the shell never reaches \(\rho_+ = 0\) due to the strong repulsive force caused by the term \([1]\) (when \(V\) is not so large), or by the term \([2]\) (when \(V \sim 1\)).

To make the statements below as concise as possible, let us first define the “core-region”. Consider the phase space \(\Gamma := \{(\rho_+, \rho_+) | \rho_+ \geq 0, -1 \leq \dot{\rho}_+ \leq 1\}\). It is understood that suitable topology is endowed on \(\Gamma\).[13] At each time \(t\), the region \(C\) is defined such as (i) \([2] + [3] > 0\) and \(\dot{\rho}_+ < 0\) on \(C\) and (ii) \(C\) is connected, and contains a neighborhood of \((\rho_+ = 0, \dot{\rho}_+ = -1)\). In other words, \(C\) consists of phase points that are “close enough” to the singularity-emerging point \((\rho_+ = 0, \dot{\rho}_+ = -1)\).

**Definition**

We call the above-mentioned region \(C\) in \(\Gamma\) the core-region (at \(t\)), for brevity.

We also say that the shell “enters (or, leaves) the core-region at \(t = t_+\)” if at \(t = t_+ - 0\) the shell’s phase point is outside (or, inside) the core-region, and at \(t = t_+\), the phase point of the shell is on the edge of the core-region; \(([2] + [3] = 0\) and \(\dot{\rho}_+ < 0\)).

It is clear (from the consideration of the order of magnitude of the terms \([1]\), \([2]\) and \([3]\) when \(V \sim 1\)) that, if the shell would have ever reached the singularity, the phase point of the shell should have reached the point \((\rho_+ = 0, \dot{\rho}_+ = -1)\) through the core-region.

Now we can show

**Lemma 2**

*Under the dynamical equation Eq.(35), once the shell enters the core-region at \(t = t_+\), it bounces back without reaching zero-radius. Indeed, \(\rho_+(t)\) is bounded from below as*

\[
\rho_+(t) > \rho_+(t_+)\sqrt{1 - \dot{\rho}_+(t_+)^2}.
\]

**Proof:**

Noting that \(\dot{\rho}_+ > 1 - \rho_+^2\) when the shell is in the core-region, it suffices to show the claim holds even for

\[
\dot{\rho} = 1 - \rho_+^2 / \rho,
\]

with arbitrary initial conditions \(\rho(0) = a(>0)\) and \(\dot{\rho}(0) = -b\) (where \(a > 0\) and \(1 > b > 0\)). However, its solution is given by \(\rho(t) = \left((t - ab)^2 + a^2(1 - b^2)^2\right)^{1/2}\), so that \(\rho_+(t) > \rho(t) \geq a\sqrt{1 - b^2} > 0\). Thus the claim follows. \(\square\)

**Theorem 1**

*Under the dynamical equation (with or without \(P_0\) Eq.(35), the shell never reaches \(\rho_+ = 0\) irrespective of its initial conditions.*

**Proof:**

It suffices only to consider the contracting phase. Then if the shell would have ever reached \(\rho_+ = 0\), the phase point of the shell should have reached the point \((\rho_+ = 0, \dot{\rho}_+ = -1)\) through the core-region. However, Lemma 2 indicates that this never happens. \(\square\)

Thus the line-like singularity never forms in this model irrespective of the value \(P_0\), as far as the weak energy condition is satisfied outside the shell.

The behavior in the expanding phase (and/or after the bounce) varies depending on the value \(P_0\). When \(P_0 \geq 0\), \(\dot{\rho}_+ > 0\) in the expanding phase, so that the shell continues to be accelerated and expand, and its velocity asymptotically approaches to the light velocity. On the other hand, the case \(P_0 < 0\) is negative, thus it might be possible that \(\dot{\rho}\) turns to negative when the shell velocity is not very large. Therefore, the oscillating behavior can arise, which might be interesting to investigate further. (However, even in this case, the shell never collapses to zero-radius due to Theorem 1.) Once the shell velocity becomes sufficiently large in the expanding phase, however, the shell continues to be accelerated and expands forever just as the case of \(P_0 \geq 0\). This is because, when the shell velocity becomes close to 1, by setting \(\dot{\rho}_+ = 1 - \delta (\delta > 0, \delta \ll 1)\), the term \([2]\) behaves as \(\sim \delta^{-1}\) and become dominant, while the terms \([1]\) and \([2]\) behave as \(\sim \delta\).

Now let us illustrate what has been proved rigorously above by some numerical investigations. We set \(P_0 = 0\) for simplicity.

Figure 1 shows a typical behavior of the contracting shell (thinner curve) along with \(p_\phi\) (thicker curve). As an example, we have set \(\gamma_+ (r_+ - t_+) = \frac{1}{100} (r_+ - t_+ + 100)^3\). Initial conditions have been set \(\rho_+(0) = 0.1\) and \(\dot{\rho}_+(0) = -0.999\) (i.e. very close to the light velocity). The bouncing behavior discussed above is obviously seen in the figure. It is also seen that the turning point of the shell coincides with the peak of the rotational pressure.

Figure 2 shows the behavior of the same model with the initial conditions \(\rho_+(0) = 0.1\) and \(\dot{\rho}_+(0) = -0.1\) (i.e. relatively mild contraction). Again, evolutions of \(\rho_+(t)\) (thinner curve) and \(p_\phi\) (thicker curve) are indicated. Due to the mild initial conditions, the shell soon bounces back. Just as the previous case, the turning point of the shell coincides with the peak of the rotational pressure.
FIG. 1: Typical evolutions of the shell-radius (thinner curve) and the rotational pressure $p_\phi$ (thicker curve). We have set $\gamma_+(r_+ - t_+) = 100(r_+ - t_+ + 100)^3$. Initial conditions are $\rho_+(0) = 0.1$ and $\dot{\rho}_+(0) = -0.999$. The vertical line indicates $100\rho_+$ and $p_\phi$, while the horizontal line indicates $t_+$.

![Graph](image1)

FIG. 2: The same model as FIG. 1 with initial conditions, $\rho_+(0) = 0.1$ and $\dot{\rho}_+(0) = -0.1$. The vertical line indicates $10\rho_+$ (for the thinner curve) and $p_\phi$ (for the thicker curve), while the horizontal line indicates $t_+$.

![Graph](image2)

Noting that $[K_{\varepsilon, \varepsilon_\phi}] = 0$, Eq.(19) with $p_z = 0$ implies $\varepsilon = p_\phi$, so that

$$\kappa \varepsilon = \kappa p_\phi - \kappa p_z = \frac{e^{-\gamma_+(\Delta - 1)}}{\rho_+(1 - \dot{\rho}_+^2)^{1/2}} - P_0$$

$$\kappa p_\phi = \frac{e^{-\gamma_+(\Delta - 1)}}{\rho_+(1 - \dot{\rho}_+^2)^{1/2}}. \quad (37)$$

This relation provides another confirmation of the validity of the condition $\gamma > 0$ (see Eq.(34)).

Finally let us note that the relation Eq.(37) reveals a curious character of the model. We first consider the case $P_0 = 0$. In the case of the counter-rotating dust cylinder, there is a relation $\frac{\dot{\rho}_+}{\dot{\rho}_-} = v^2$, where $v$ is the velocity of the dust particle relative to an observer “standing still” on the shell [5]. Comparing this with Eq.(37) ($P_0 = 0$), the present model corresponds to the case of $v = 1$, so that the present model could be interpreted as a cylinder shell consisting of the counter-rotating massless particles, though the two models are not exactly the same. The situation in which massless particles are confined to a compact region arises naturally in superconducting objects, so that the present model should have some relevance. On the other hand, when $p_z < 0$, it behaves more or less like a normal matter and it is consistent with the attractive nature of the term [3]. On the other hand, when $p_z > 0$, Eq.(37) rather looks like a cosmic string. However, the cylinder tends to expand in the present case, contrary to the case of cosmic strings. It may not appropriate at present to pursue such comparisons too further.
IV. COMPARISON WITH THE RESULTS OF THE PRECEDING PAPER

In view of our results above, we need to investigate whether the singularity-forming solution claimed in Ref.[6] and its correction Ref.[7] is a valid one. It turns out that the formal solution in question is not a relevant one and should be discarded.

By comparing one by one our formulas with the corresponding ones shown in Refs.[6] and [7], it turns out that some formulas in Ref.[6] along with its correction Ref.[7] still need slight modifications (see below and Appendix D for more details). However, we first derive the above conclusion faithfully based on the formulas in Refs.[7] and [6]. Then we also show that the same conclusion follows based on our formulas, too, even with the $P_0$-term.

Firstly their functions $f$, $g$, $h$, $l$, $R_0(T)$, $R_0'(T)$, $R_0''(T)$ and $b(T-R)$ correspond to ours as

$$f := T^2, \quad g := R^2, \quad h := Z^2, \quad l := \Phi^2,$$

$$R_0(T) := \rho(t), \quad R_0'(T) := \dot{\rho}(t),$$

$$R_0''(T) := \ddot{\rho}(t), \quad b(T-R) = -2\gamma(t-r). \quad (38)$$

In particular, let us note that the sign of the function $b$ is opposite to our function $\gamma$. Thus we should understand that $b \leq 0$ and $b' \leq 0$. They correspond to the physical conditions $\gamma \geq 0$ and $\gamma' \geq 0$ (see arguments after Eq.(24) and Eq.(25)).

The detailed comparison is shown in Appendix D. Here it suffices to pay attention only to their dynamical equation shown in Ref.[7] (which is the correction of the original Eq.(30) of Ref.[6]):

$$R_0'' = (1 - R_0'^2) \left\{ \frac{\Delta}{R_0} + \frac{b'(\xi_0)(1 - R_0')}{2(\Delta - 1)} (R_0' - \Delta) \right\}, \quad (39)$$

where $R_0$ and its derivatives correspond to our $\rho_+(t)$ and its $t$-derivatives (see Eq.(38)), and $b(\xi_0)$ corresponds to our $-2\gamma + \Delta := \left[ R_0'^2 + e^{-\rho}(1 - R_0'^2) \right]^{1/2}$.

If the terms are rearranged to look as similar to Eq.(35) (with $P_0 = 0$) as possible, their formula Eq.(39) reads

$$R_0'' = \Delta \frac{1 - R_0'^2}{R_0} + \frac{(-b'(\xi_0))}{2(\Delta - 1)} (1 - R_0'^2) (1 - R_0')(\Delta - R_0'). \quad (40)$$

(Compared with Eq.(35) ($P_0 = 0$), the factors $(1 - R_0'^2)$ and $(\Delta - R_0')$ of the second term should be replaced by $R_0'$ and $((\Delta + 1)R_0 + 1)$, respectively.)

Based on Eq.(39), Ref. [7] (just as Ref. [6]) tries to search for special solutions for $R_0$ satisfying

$$R_0'' = \beta \frac{1 - R_0'^2}{R_0}, \quad (41)$$

$$\beta := R_0' \left[ \frac{\Delta}{R_0} + \frac{(-b')(1 - R_0')}{2(\Delta - 1)} (\Delta - R_0') \right], \quad (42)$$

where $\beta$ is assumed to be an arbitrary constant.

As is claimed in Ref.[6], Eq.(41) itself is formally satisfied by

$$R_0'(T) = -(1 - e^{-2\beta(T-T_0)})^{1/2}, \quad (43)$$

or integrating once more,

$$R_0(T) = \left\{ R_0 - (T - T_0) \right\} + \frac{1}{\beta} \left\{ (1 - e^{-2\beta(T-T_0)})^{1/2} - \ln[1 + (1 - e^{-2\beta(T-T_0)})^{1/2}] \right\}. \quad (44)$$

Here two integral constants $T_0$ and $R_0$ have appeared corresponding to the fact that the differential equation Eq.(41) is of the 2nd order. The constant $T_0$ indicates the time when $R_0'$ vanishes, while $R_*$ (along with $T_0$) determines the time $T^*$ when $R_0$ vanishes ($T^* \leq T_0$). These constants $T_0$ and $R_0$ are (indirectly) determined by the initial conditions for solving Eq.(41).

Equation (44) would imply a line-like singularity formation, reaching $R_0 = 0$ at some finite time $T^*$, so that its implication would be significant. We show below, however, that Eq.(44) is not the solution of the original differential equation Eq.(40), irrespective of the parameters $T_0$ and $R_*$, i.e. independent of the initial conditions.

To see this conclusion, let us first define $B(T)$ to be $-1$ times the R.H.S. of Eq.(42),

$$B(T) := (-R_0') \left[ \frac{\Delta}{R_0} + \frac{(-b')(1 - R_0')}{2(\Delta - 1)} (\Delta - R_0') \right]. \quad (45)$$

What we show below is that $B(T)$ varies (actually diverges) along the collapsing solution Eq.(44) so that the starting assumption $\beta = \text{constant}$ (Eq.(42)) is incompatible with the dynamical equation Eq.(41). Causing such self-inconsistency, then, Eq.(44) is not the solution of the original dynamical equation, Eq.(40).

Now we prove

**Theorem 2**

The quantity $B(T)$ diverges, and does not remain constant along the collapsing solution Eq.(44).

**Proof:**

Let us look at the definition of $B(T)$ (Eq.(45)). Both of the terms in the brackets [ ] are positive, since $-1 < R_0' < 0$, $b' < 0$ and $\Delta > 1$. Thus $B(T) > 0$ and accordingly, Eq.(42) implies that $\beta < 0$. We then get the estimation

$$B(T) > (-R_0') \frac{\Delta}{R_0} > \frac{(-R_0')}{R_0},$$

so that

$$\lim_{T \to T^*} B(T) \geq \lim_{T \to T^*} \frac{(-R_0')}{R_0}.$$
where \( T^* \) is the shell-collapsing time, \( \lim_{T \to T^*} R_0(T) = 0 \).

In the case \( \lim_{T \to T^*} R_0'(T) \neq 0 \), the above estimation implies that \( B(T) \to \infty \) as \( T \to T^* \).

In the case \( \lim_{T \to T^*} R_0'(T) = 0 \), the estimation further goes as

\[
\lim_{T \to T^*} B(T) \geq \lim_{T \to T^*} \frac{(-R_0')}{R_0} = \frac{(-R_0'')}{R_0}
\]

where de L'Hôpital's theorem has been applied in the last step. With the help of Eq.(41), then, we get

\[
\lim_{T \to T^*} B(T) \geq (-\beta) \lim_{T \to T^*} \frac{1 - R_0'(T)^2}{R_0'(T)}
\]

Thus we get \( \lim_{T \to T^*} B(T) = \infty \). \( \square \)

The situation is the same for our correct equation, Eq.(35) (with \( P_0 > 0 \)), too. Just as Eqs.(41) and (42), we can ask whether the collapsing solution is allowed for the equation

\[
\dot{\lambda}_+ = \beta \frac{1 - \rho_+^2}{\rho_+^3}, \tag{46}
\]

\[
\beta : = \dot{\lambda}_+ \frac{\Delta}{\rho_+^3} + \frac{\gamma' \rho_+}{(\Delta - 1)(1 + \rho_+)} \{(\Delta + 1) \dot{\rho}_+ + 1\}
\]

\[
+ \frac{\Delta}{\Delta - 1} (1 - \rho_+^2)^{1/2} e^{\gamma_+} P_0 \tag{47}
\]

with \( \beta \) being a constant; \( \Delta :=\) \( [(-\dot{\rho}_+)^2 + e^{\gamma_+} \{1 - (-\dot{\rho}_+)^2\}]^{1/2} \). Equation (46) itself has a formal solution, which is just the same as Eqs.(43) and (44). Though more care is required, essentially the same argument as above follows and it turns out that the formal solution just like Eq.(44) is not the solution of the original dynamical equation Eq.(35). To show this fact, we first introduce some symbols.

**Definition**

(a) Let \( \tilde{B} \) be \(-1\) times the R.H.S. of Eq.(47):

\[
\tilde{B}(t) := V \times \left[ \frac{\Delta}{\rho_+^3} + \frac{\gamma' V}{(\Delta - 1)(1 - V)} \{(\Delta + 1)V - 1\} \right.
\]

\[
\left. + \frac{\Delta}{\Delta - 1}(1 - V^2)^{1/2} e^{\gamma_+} P_0 \right] , \tag{48}
\]

where \( V := -\dot{\rho}_+ (0 < V < 1) \).

(b) Let \( \tilde{D} \) and \( \tilde{E} \) be the second and the third terms in the brackets \([\ ]\) in Eq.(48), respectively:

\[
\tilde{D}(t) : = \frac{\gamma' V}{(\Delta - 1)(1 - V)} \{(\Delta + 1)V - 1\} .
\]

\[
\tilde{E}(t) : = \frac{\Delta}{\Delta - 1}(1 - V^2)^{1/2} e^{\gamma_+} P_0 .
\]

(c) Let \( f(V) := (\Delta + 1)V - 1 \). The sign of \( f(V) \) determines the sign of \( \tilde{D}(t) \). As is already discussed in the previous section (after Lemma 1), the function \( f(V) \) is monotonic, increasing, continuous on \([0, 1]\), with \( f(0) = -1 \) and \( f(1) = 1 \), so that \( f(V) \) has only one zero, say \( V^* \), in \((0, 1)\). When \( \tilde{D} \) is looked as a function of \( V \), then, \( \tilde{D} \leq 0 \) for \( V \in (0, V^*) \) and \( \tilde{D} > 0 \) for \( V \in (V^*, 1) \).

(d) Let \( t^* \) be the shell-collapsing time (which is assumed to exist in the present context), i.e. \( t^* \) is the time such that \( \lim_{t \to t^*} \rho_+(t) = 0 \). (It does not matter whether \( t^* \) is finite or infinite in the argument below.)

We now prove a theorem (Theorem 3 below) corresponding to Theorem 2 step by step. We first note

**Lemma 3**

\(|\tilde{E}|/\tilde{D} \to 0 \) as \( V \to 1 \).

**Proof:**

When \( V \) is close to 1, setting \( V = 1 - \delta \) (\( \delta > 0 \)), then \( \Delta = 1 + (e^{\gamma_+} - 1)\delta + O(\delta^2) \), and one finds \( 1/\tilde{D} = O(\delta^2) \) and \( 1/\tilde{E} = O(\delta^{3/2}) \), so that \(|\tilde{E}|/\tilde{D} = O(\delta^{1/2}) \). Then the claim follows. \( \square \)

Now let us confirm

**Lemma 4**

It follows that \( \lim_{t \to t^*} V(t) < 1 \).

**Proof:**

Studying the solution of Eq.(46) (cf. Eq.(43)), we see \( V < 1 \) (i.e. \( \dot{\rho}_+(t) > -1 \)). Suppose \( \lim_{t \to t^*} \rho(t) = 1 \). Since \( f(V) \) is continuous on \([0, 1]\), then, it means \( \lim_{t \to t^*} f(V(t)) = 1 \), so that \( \tilde{D}(t) \to \infty \) as \( t \to t^* \) (note \( \Delta > 1 \)). Considering the limiting behavior of \( \tilde{B}(t) = V(t)[\dot{\rho}_+ + \tilde{D}(t)(1 + \tilde{E}(t)/\tilde{D}(t))] \) as \( t \to t^* \), Lemma 3 implies \( \tilde{B}(t) \to \infty \) as \( t \to t^* \). Since \( \tilde{\beta} (= -\tilde{B}(t)) \) is a constant by assumption, this should not happen. Thus \( \lim_{t \to t^*} V(t) \neq 1 \). \( \square \)

**Corollary of Lemma 4**

The quantities \( \tilde{D}(t) \) and \( \tilde{E}(t) \) are bounded, \(|\tilde{D}(t)| < \infty \) and \(|\tilde{E}(t)| < \infty \). Furthermore, \( \lim_{t \to t^*} |\tilde{D}(t)| < \infty \) and \( \lim_{t \to t^*} |\tilde{E}(t)| < \infty \).
Proof:
We note that $\dot{D}$ and $\dot{E}$ become unbounded only when $V \to 1$ and that $\dot{D}$ and $\dot{E}$ are continuous as functions of $V$. Now, the solution of Eq.(46) (cf. Eq.(43)) implies $V < 1$. Furthermore Lemma 4 says lim$_{t \to t^*} V(t) < 1$. Then the claim follows. \[]

Then we can show

Lemma 5
The constant parameter $\tilde{\beta}$ should be set negative, $\tilde{\beta} < 0$.

Proof:
We note that lim$_{t \to t^*}$ $|\dot{D}(t)| < \infty$ and lim$_{t \to t^*}$ $|\dot{E}(t)| < \infty$ due to Corollary of Lemma 4. On the other hand, lim$_{t \to t^*}$ $\frac{\Delta}{\rho_+} = \infty$. Then one can find appropriate $t_1$ and $t_2$ ($t_1 < t_2 < t^*$) such that $\frac{\Delta}{\rho_+} > |\dot{D}(t)|+|\dot{E}(t)|$ on $I_1 := (t_1, t_2)$.

Thus $\tilde{B}(t) > 0$ on $I_1$. Since $\tilde{\beta} (= -\tilde{B}(t))$ is a constant by assumption, this means that $\tilde{\beta}$ should be set negative from the outset. \[]

Finally we prove

Theorem 3
The quantity $\tilde{B}(T)$ diverges, and does not remain constant along the collapsing solution (corresponding to Eq.(44)).

Proof:
Let us set $\tilde{C}(t) := \tilde{B}(t) - (\rho_+) [\dot{D}(t) + \dot{E}(t)]$. We note $\tilde{C}(t) = (\rho_+) \frac{\Delta}{\rho_+} > 0$ due to Eq.(48). Then it follows $\tilde{C}(t) \geq \frac{(|\dot{D}(t)| + |\dot{E}(t)|)}{\rho_+}$. Noting $\tilde{\beta} < 0$ (Lemma 5), then, exactly the same argument as the proof of Theorem 2 can be applied, and one concludes lim$_{t \to t^*} \tilde{C}(t) = \infty$. On the other hand, it follows

$$\lim_{t \to t^*} \tilde{C}(t) \leq \lim_{t \to t^*} |\tilde{B}(t)|$$

$$+ \lim_{t \to t^*} |(\rho_+) \dot{D}(t)| + \lim_{t \to t^*} |(\rho_+) \dot{E}(t)| .$$

Due to Corollary of Lemma 4, then, it follows lim$_{t \to t^*}$ $|\tilde{B}(t)| = \infty$. \[]

We conclude that there is no relevant solution indicating the singularity formation in the present model.

V. SUMMARY

In this paper, we have studied the dynamics of a cylindrical shell in the spacetime cylindrical symmetry.

We have started with constructing a general framework for analyzing a cylindrical spacetime and a shell in it, which might be useful for future investigations. Based on the framework, we have investigated a cylindrical shell-collapse model which accompanies the out-going radiation of gravitational waves and massless particles. This model had been introduced by Pereira and Wang [6, 7], but its proper analysis had been awaited. This model could be interpreted as a thin shell filled with radiation. In connection with the cosmic-censorship hypothesis, we are mostly interested in the collapse of “normal” matter, so that the weak energy condition has been assumed outside the shell.

It has been proved that, as far as the weak energy condition is satisfied outside the shell, the shell bounces due to the rotational-pressure effect. After the bounce, it continues to expand without re-contraction when the pressure of the shell in the $z$-direction $p_z$ satisfies $p_z \geq 0$, while in the case $p_z < 0$, the behavior after the bounce can be more complicated. However in either case, the shell never reaches the zero-radius configuration and it escapes from the line-like singularity formation.

Just as the case of a shell filled with counter-rotating dust particles considered by Apocatopolas and Thorne [5], the present case also shows a bouncing due to the effect of rotational pressure, while the eternal expansion after the bounce (when $p_z \geq 0$) is a unique feature of the present model. We have also performed numerical investigations which reveal explicit behaviors of the shell.

Acknowledgments

The author would like to thank S. Jhingan for valuable discussions at the beginning stage of this work. He also thanks H. Sakuragi for valuable discussions on numerical investigations. Important part of work has been completed during the author’s stay at Institute of Cosmology, Tufts University. He is grateful for its nice hospitality. In particular he would like to thank A. Vilenkin for helpful suggestions on the interpretation of the results. This work has been supported by the Japan Ministry of Education, Culture, Sports, Science and Technology with the grant #14740162.

APPENDIX A: SUMMARY OF FUNDAMENTAL GEOMETRICAL QUANTITIES FOR CYLINDRICAL SPACETIMES

Based on the metric Eq.(1), it is convenient to introduce the 1-forms as Eq.(2). Taking the exterior derivative
of $\hat{\theta}^A$’s, we get

\[
\begin{align*}
    d\hat{\theta}^0 &= -\frac{T'}{TR} \hat{\theta}^0 \wedge \hat{\theta}^1, \\
    d\hat{\theta}^2 &= \frac{Z}{TZ} \hat{\theta}^0 \wedge \hat{\theta}^2 + \frac{Z'}{RZ} \hat{\theta}^1 \wedge \hat{\theta}^2, \\
    d\hat{\theta}^3 &= \frac{\Phi}{T \Phi} \hat{\theta}^0 \wedge \hat{\theta}^3 + \frac{\Phi'}{R \Phi} \hat{\theta}^1 \wedge \hat{\theta}^3.
\end{align*}
\]

Here “.$” indicates the partial derivative w.r.t. $t$, while “$'$” means the same w.r.t. $r$. With the help of the first Cartan structure-equation $d\hat{\theta}^A + \omega^A_B \wedge \hat{\theta}^B = 0$, we thus get

\[
\begin{align*}
    \omega^0_1 &= \frac{T'}{TR} \hat{\theta}^0 + \frac{\hat{R}}{TR} \hat{\theta}^1, \quad \omega^0_2 = \frac{\hat{Z}}{TZ} \hat{\theta}^2, \\
    \omega^0_3 &= \frac{\hat{\Phi}}{T \Phi} \hat{\theta}^3, \quad \omega^1_2 = -\frac{Z'}{RZ} \hat{\theta}^2, \\
    \omega^1_3 &= -\frac{\Phi'}{R \Phi} \hat{\theta}^3, \quad \omega^2_3 = 0.
\end{align*}
\]

On account of the relation, $\omega^A_B = \Gamma^A_{BC} \hat{\theta}^C$, we also get [14]

\[
\begin{align*}
    \Gamma^0_{10} &= \frac{T'}{TR}, \quad \Gamma^0_{11} = \frac{\hat{R}}{TR} = \frac{1}{XTR}(\hat{R} - \hat{\rho} R'), \\
    \Gamma^0_{22} &= \frac{\hat{Z}}{TZ} = \frac{1}{XTZ}(\hat{\Phi} - \hat{\rho} \Phi'), \\
    \Gamma^0_{33} &= \frac{\Phi'}{T \Phi} = \frac{1}{X T \Phi}(\hat{\Phi} - \hat{\rho} \Phi'), \\
    \Gamma^1_{22} &= \frac{Z'}{RZ}, \quad \Gamma^1_{33} = \frac{\Phi'}{R \Phi}, \\
    \text{others} &= 0.
\end{align*}
\]

Here “.$” indicates the derivative w.r.t. the proper-time $\tau$ of an observer on the shell, and $X := dt/dr$; they are evaluated on the shell in consideration.

Taking the exterior-derivatives of $\omega^A_B$’s, we get

\[
\begin{align*}
    d\omega^0_1 &= \frac{1}{TR} \left\{ \left( \frac{\hat{R}}{T} \right)' - \left( \frac{T'}{R} \right) \right\} \hat{\theta}^0 \wedge \hat{\theta}^1, \\
    d\omega^0_2 &= \frac{1}{TZ} \left( \frac{\hat{Z}}{T} \right)' \hat{\theta}^0 \wedge \hat{\theta}^2 + \frac{1}{RZ} \left( \frac{\hat{Z}}{T} \right)' \hat{\theta}^1 \wedge \hat{\theta}^2, \\
    d\omega^0_3 &= \frac{1}{T \Phi} \left( \frac{\hat{\Phi}}{T} \right)' \hat{\theta}^0 \wedge \hat{\theta}^3 + \frac{1}{R \Phi} \left( \frac{\hat{\Phi}}{T} \right)' \hat{\theta}^1 \wedge \hat{\theta}^3, \\
    d\omega^1_2 &= -\frac{1}{TZ} \left( \frac{Z'}{R} \right)' \hat{\theta}^0 \wedge \hat{\theta}^2 - \frac{1}{RZ} \left( \frac{Z'}{R} \right)' \hat{\theta}^1 \wedge \hat{\theta}^2, \\
    d\omega^1_3 &= -\frac{1}{T \Phi} \left( \frac{\Phi'}{T} \right)' \hat{\theta}^0 \wedge \hat{\theta}^3 - \frac{1}{R \Phi} \left( \frac{\Phi'}{T} \right)' \hat{\theta}^1 \wedge \hat{\theta}^3, \\
    d\omega^2_3 &= 0.
\end{align*}
\]

Thus, by the second Cartan structure-equation $\Omega^A_B = d\omega^A_B + \omega^A_C \wedge \omega^C_B$, we find

\[
\begin{align*}
    \Omega^0_1 &= \frac{1}{TR} \left\{ \left( \frac{\hat{R}}{T} \right)' - \left( \frac{T'}{R} \right) \right\} \hat{\theta}^0 \wedge \hat{\theta}^1, \\
    \Omega^0_2 &= \left\{ \frac{1}{TZ} \left( \frac{\hat{Z}}{T} \right)' - \frac{1}{R^2 TZ} \right\} \hat{\theta}^0 \wedge \hat{\theta}^2 + \left\{ \frac{1}{RZ} \left( \frac{\hat{Z}}{T} \right)' - \frac{\hat{R} Z'}{TR RZ} \right\} \hat{\theta}^1 \wedge \hat{\theta}^2, \\
    \Omega^0_3 &= \left\{ \frac{1}{T \Phi} \left( \frac{\hat{\Phi}}{T} \right)' - \frac{1}{R^2 T \Phi} \right\} \hat{\theta}^0 \wedge \hat{\theta}^3 + \left\{ \frac{1}{R \Phi} \left( \frac{\hat{\Phi}}{T} \right)' - \frac{1}{T R \Phi} \right\} \hat{\theta}^1 \wedge \hat{\theta}^3, \\
    \Omega^1_2 &= -\left\{ \frac{1}{TZ} \left( \frac{Z'}{R} \right)' - \frac{1}{R^2 T Z} \right\} \hat{\theta}^0 \wedge \hat{\theta}^2 - \left\{ \frac{1}{RZ} \left( \frac{Z'}{R} \right)' - \frac{\hat{R} Z'}{T R^2 RZ} \right\} \hat{\theta}^1 \wedge \hat{\theta}^2, \\
    \Omega^1_3 &= -\left\{ \frac{1}{T \Phi} \left( \frac{\Phi'}{R} \right)' - \frac{1}{R T \Phi} \right\} \hat{\theta}^0 \wedge \hat{\theta}^3 - \left\{ \frac{1}{R \Phi} \left( \frac{\Phi'}{R} \right)' - \frac{1}{T R^2 \Phi} \right\} \hat{\theta}^1 \wedge \hat{\theta}^3, \\
    \Omega^2_3 &= \left( \frac{1}{T^2 Z \Phi} - \frac{1}{R^2 Z' \Phi} \right) \hat{\theta}^2 \wedge \hat{\theta}^3.
\end{align*}
\]
Finally, the scalar curvature becomes, independently of the frame $\{\hat{e}_A\}$,

$$
R^0_{101} = \frac{1}{TR} \left\{ \left( \frac{\dot{R}}{T} \right) - \left( \frac{T'}{R} \right) \right\}, \quad R^0_{202} = \frac{1}{TZ} \left( \frac{\dot{Z}}{T} \right) - \frac{TZ'}{R^2TZ'}, \quad R^0_{212} = \frac{1}{RZ} \left( \frac{\dot{Z}}{T} \right) - \frac{\dot{R}Z'}{TRZR'},
$$

$$
R^0_{303} = \frac{1}{T\Phi} \left( \frac{\dot{\Phi}}{T} \right) - \frac{1}{R^2} T'\Phi', \quad R^0_{313} = \frac{1}{R\Phi} \left( \frac{\dot{\Phi}}{T} \right) - \frac{1}{TR\Phi} \frac{\dot{R}\Phi'}{R}, \quad R^1_{202} = -\frac{1}{TZ} \left( \frac{Z'}{R} \right) + \frac{1}{T^2ZR'}, \quad R^1_{303} = -\frac{1}{R\Phi} \left( \frac{\dot{\Phi}}{T} \right) + \frac{1}{TR\Phi} \frac{T\Phi'}{T}, \quad R^1_{313} = -\frac{1}{R\Phi} \left( \frac{\dot{\Phi}}{T} \right) + \frac{1}{T^2R^2\Phi},
$$

Thus, the Ricci curvature in the frame $\{\hat{e}_A\}$ becomes

$$
R_{00} = -\frac{1}{T} \left\{ \left( \frac{\dot{R}}{T} \right) + \frac{1}{Z} \left( \frac{\dot{Z}}{T} \right) + \frac{1}{\Phi} \left( \frac{\dot{\Phi}}{T} \right) \right\} + \frac{1}{TR} \left( \frac{T'}{R} + \frac{T'}{T^2R} \left( \frac{Z'}{Z} + \frac{\Phi'}{\Phi} \right) \right),
$$

$$
R_{01} = R_{10} = -\frac{1}{T} \left\{ \left( \frac{\dot{Z}}{T} \right) + \frac{1}{\Phi} \left( \frac{\dot{\Phi}}{T} \right) \right\} + \frac{1}{TR} \left( \frac{T'}{R} + \frac{Z'}{Z} + \frac{\Phi'}{\Phi} \right),
$$

$$
R_{11} = \frac{1}{TR} \left( \frac{\dot{R}}{T} \right) + \frac{1}{T^2R} \left( \frac{\dot{Z}}{R} + \frac{\dot{\Phi}}{\Phi} \right) - \frac{1}{T} \left\{ \left( \frac{T'}{R} \right) + \frac{1}{Z} \left( \frac{Z'}{R} \right) + \frac{1}{\Phi} \left( \frac{\Phi'}{R} \right) \right\},
$$

$$
R_{22} = \frac{1}{T} \left( \frac{\dot{Z}}{T} \right) + \frac{1}{T^2Z} \left( \frac{\dot{R}}{R} + \frac{\dot{\Phi}}{\Phi} \right) - \frac{1}{T} \left\{ \left( \frac{T'}{R} \right) + \frac{1}{Z} \left( \frac{Z'}{R} \right) + \frac{1}{\Phi} \left( \frac{\Phi'}{R} \right) \right\},
$$

$$
R_{33} = \frac{1}{T\Phi} \left( \frac{\dot{\Phi}}{T} \right) + \frac{1}{T^2\Phi} \left( \frac{\dot{R}}{R} + \frac{\dot{Z}}{Z} \right) - \frac{1}{\Phi} \left\{ \left( \frac{\Phi'}{R} \right) + \frac{1}{R^2\Phi} \left( \frac{T'}{T} + \frac{Z'}{Z} \right) \right\},
$$

others $= 0$.

Finally, the scalar curvature becomes, independently of the frame,

$$
R = \frac{2}{T} \left\{ \left( \frac{\dot{R}}{T} \right) + \frac{1}{Z} \left( \frac{\dot{Z}}{T} \right) + \frac{1}{\Phi} \left( \frac{\dot{\Phi}}{T} \right) \right\} + \frac{2}{T^2} \left( \frac{\dot{R}\dot{Z}}{RZ} + \frac{\dot{R}\dot{\Phi}}{R\Phi} + \frac{\dot{Z}\dot{\Phi}}{Z\Phi} \right) - \frac{2}{R} \left\{ \left( \frac{T'}{R} \right) + \frac{1}{Z} \left( \frac{T'}{Z} \right) + \frac{1}{\Phi} \left( \frac{\Phi'}{R} \right) \right\},
$$

$$
R^{00} = \frac{1}{T} \left\{ \left( \frac{\dot{R}}{T} \right) + \frac{1}{Z} \left( \frac{\dot{Z}}{T} \right) + \frac{1}{\Phi} \left( \frac{\dot{\Phi}}{T} \right) \right\} + \frac{1}{TR} \left( \frac{T'}{R} \right) + \frac{1}{T^2ZR} \left( \frac{Z'}{Z} + \frac{\Phi'}{\Phi} \right),
$$

$$
R^{01} = R^{10} = \frac{1}{T} \left\{ \left( \frac{\dot{Z}}{T} \right) + \frac{1}{\Phi} \left( \frac{\dot{\Phi}}{T} \right) \right\} + \frac{1}{TR} \left( \frac{T'}{R} \right) + \frac{Z}{Z} + \frac{\Phi}{\Phi},
$$

$$
R^{11} = \frac{1}{T} \left( \frac{\dot{R}}{T} \right) + \frac{1}{T^2R} \left( \frac{\dot{Z}}{R} + \frac{\dot{\Phi}}{\Phi} \right) - \frac{1}{T} \left\{ \left( \frac{T'}{R} \right) + \frac{1}{Z} \left( \frac{Z'}{R} \right) + \frac{1}{\Phi} \left( \frac{\Phi'}{R} \right) \right\},
$$

$$
R^{22} = \frac{1}{T} \left( \frac{\dot{Z}}{T} \right) + \frac{1}{T^2Z} \left( \frac{\dot{R}}{R} + \frac{\dot{\Phi}}{\Phi} \right) - \frac{1}{T} \left\{ \left( \frac{T'}{R} \right) + \frac{1}{Z} \left( \frac{Z'}{R} \right) + \frac{1}{\Phi} \left( \frac{\Phi'}{R} \right) \right\},
$$

$$
R^{33} = \frac{1}{T\Phi} \left( \frac{\dot{\Phi}}{T} \right) + \frac{1}{T^2\Phi} \left( \frac{\dot{R}}{R} + \frac{\dot{Z}}{Z} \right) - \frac{1}{\Phi} \left\{ \left( \frac{\Phi'}{R} \right) + \frac{1}{R^2\Phi} \left( \frac{T'}{T} + \frac{Z'}{Z} \right) \right\},
$$

APPENDIX B: NON-COORDINATE BASES

Throughout this paper, we always consider a cylindrical spacetime defined by the metric Eq.(1). The coordinates $\{x^\mu\} := \{t, r, z, \phi\}$ define the coordinate bases and their dual 1-forms,

$$
e_\mu := \partial_\mu, \quad \theta^\mu := dx^\mu . \quad (B1)
$$

(Here the Greek letters $\mu, \nu, \cdots$ indicate the indices w.r.t. the coordinates $\{t, r, z, \phi\}$.)

On the other hand, a cylindrical shell embedded in the spacetime naturally defines a local ortho-normal frame $\{\hat{e}_A\}$ (see Sec.II A),

$$
\hat{e}_n = \hat{n}_n = \left( \frac{R}{T}, \frac{T}{R}, X, 0, 0 \right)_{(trz\phi)},
$$

$$
\hat{e}_r = \partial_r = (X, \rho, 0, 0)_{(trz\phi)}, \quad (B2)
$$

$$
\hat{e}_z = (0, 0, 1/Z, 0)_{(trz\phi)}, \quad \hat{e}_\phi = (0, 0, 0, 1/\Phi)_{(trz\phi)}, \quad (B3)
$$

where the symbol $\hat{e}$ implies the normalized frame-vector. (The first few Greek letters $\alpha, \beta, \gamma, \cdots$ are used for the indices w.r.t. the above ortho-normal frame, and they take the values $n, \tau, z \text{ or } \phi$. A set of 1-forms $\{\hat{\theta}^\alpha\}$ dual
to \{\hat{e}_\alpha\} is also introduced.

As the third set of frames, the metric given in Eq.(1) defines a natural set of 1-forms \{\hat{\theta}^A\},

\[
\begin{align*}
\hat{\theta}^0 &= T(t,r)dt , & \hat{\theta}^1 &= R(t,r)dr , \\
\hat{\theta}^2 &= Z(t,r)dz , & \hat{\theta}^3 &= \Phi(t,r)d\phi , 
\end{align*}
\]

(B4)

along with their dual ortho-normal bases \{\hat{\epsilon}_A\}. (The capital Latin letters \(A, B, C, \cdots\) indicate the indices w.r.t. the frame determined by Eq.(B4), and they take the values \(0 - 3\).)

Now we summarize the relations among \{\epsilon_\mu\}, \{\hat{\epsilon}_A\}, and \{\theta^\alpha\}, and those among \{\theta^\mu\}, \{\hat{\theta}^A\} and \{\hat{\epsilon}_A\}.

Firstly, Eq.(B3) gives the relation between \{\epsilon_\mu\} and \{\hat{\epsilon}_A\}, and those between \{\theta^\mu\} and \{\hat{\theta}^A\}

\[
\hat{\epsilon}_A = e_\mu \mu \theta^\mu , \quad \hat{\theta}^A = E^{\alpha}_\mu \theta^\mu . \tag{B5}
\]

Here \(E^{\alpha}_\mu\) and \(E^\beta_\nu\) defined below are mutually inverse as matrices:

\[
\begin{align*}
E^\alpha_\mu &= \begin{pmatrix}
-T^{-1}R^\rho_\sigma X & 0 & 0 \\
X T R^{-1} & 0 & 0 \\
0 & 0 & Z^{-1} \\
0 & 0 & 0 & \Phi^{-1}
\end{pmatrix} , \\
E^\beta_\mu &= \begin{pmatrix}
-T R^\rho_\sigma X T R & 0 & 0 \\
X T^2 & -R^\rho_\sigma & 0 \\
0 & 0 & Z \\
0 & 0 & 0 & \Phi
\end{pmatrix} . \tag{B6}
\end{align*}
\]

Secondly, Eq.(B4) gives the relation between \{\epsilon_\mu\} and \{\hat{\epsilon}_A\}, and those between \{\theta^\mu\} and \{\hat{\theta}^A\}

\[
\hat{\theta}^A = E^\alpha_\mu \theta^\mu , \quad \hat{\epsilon}_A = e_\mu \epsilon^\alpha , \quad e_\mu = \epsilon_A E^{\alpha}_\mu . \tag{B7}
\]

Here \(E^A_\mu\) and \(E^\alpha_\mu\) defined below are mutually inverse as matrices:

\[
\begin{align*}
E^A_\mu &= \text{diag}(T, R, Z, \Phi) , \\
E^\alpha_\mu &= \text{diag}(1/T, 1/R, 1/Z, 1/\Phi) . \tag{B8}
\end{align*}
\]

Finally, Eqs.(B5) and (B7) give the relation between \{\hat{\epsilon}_A\} and \{\hat{\epsilon}_A\}, and those between \{\hat{\theta}^A\} and \{\hat{\theta}^A\}

\[
\hat{\epsilon}_A = \epsilon_A E^A_\epsilon , \quad \hat{\theta}^A = E^{\alpha}_\mu \hat{\theta}^\mu . \tag{B9}
\]

Here \(E^A_\epsilon, E^\alpha_\mu\) and \(E^{\alpha}_\mu\) given below are mutually inverse as matrices:

\[
\begin{align*}
E^A_\epsilon &= \begin{pmatrix}
R^\rho_\sigma X T & 0 & 0 \\
X T R^{-1} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} , \\
E^{\alpha}_\mu &= \begin{pmatrix}
-R^\rho_\sigma X T & 0 & 0 & 0 \\
X T & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} . \tag{B10}
\end{align*}
\]

Now noting the relation \[12,\]

\[
\nabla \epsilon_a \hat{\epsilon}_\beta = \Gamma^\gamma_{\alpha\beta} \epsilon^\gamma ,
\]

along with Eq.(B9), we get

\[
\Gamma^\gamma_{\alpha\beta} = E^A_\epsilon E^\beta_\gamma E^C_\alpha E^B_\gamma + E^A_\epsilon E^\gamma_\beta \nabla \epsilon_A E^B_\gamma
\]

\[
= E^A_\epsilon E^\beta_\gamma E^C_\alpha E^B_\gamma + E^\mu_\alpha E^\gamma_\beta \partial_\mu E^B_\gamma . \tag{B11}
\]

Equation (B11) is the basis for calculating the extrinsic curvature.

**APPENDIX C: DERIVATION OF THE FORMULA FOR \(K_{\epsilon\epsilon}\)**

Here derivations of the formula for the extrinsic curvatures (Eqs.(13) and (16)) are purposefully shown in detail. This is to ensure anyone interested in this topic reproducing all the results presented in this paper without any difficulty and to motivate him to go ahead the point where this paper ends.

In order to derive Eq.(13), we note that the first term on the R.H.S. in Eq.(12) becomes, with the help of Eqs.(A3) and (B10),

\[
\begin{align*}
-E^A_\tau E^B_\tau E^\gamma_\alpha E^C_\beta G_{AB} &= -E^1_\tau E^0_\tau E^\alpha_\gamma E^0_\alpha G_{11} \\
&= (R^\rho_\sigma) \left\{ \frac{XT}{2RT} (\ln T^2)^{\rho} + \frac{R}{2XT} \hat{\rho} \left[ (\ln R^2)^\sigma - \rho (\ln R^2)^\rho \right] \right\} . \tag{C1}
\end{align*}
\]

Similarly, with the help of Eqs.(B6) and (B10), the second term on the R.H.S. in Eq.(12) is calculated as

\[
\begin{align*}
-E^\mu_\alpha E^\nu_\beta \partial_\mu E^B_\tau &= -E^\mu_{\alpha_0} (E^\tau_\epsilon \partial_\epsilon E^0_\tau + E^\gamma_\epsilon \partial_\epsilon E^\gamma_\tau) - E^\mu_1 (E^\tau_\epsilon \partial_\epsilon E^1_\tau + E^\gamma_\epsilon \partial_\epsilon E^\gamma_\tau) \\
&= R \hat{\rho} (XT)^\rho - XT (R \hat{\rho})^\rho .
\end{align*}
\]

The above expression can be further modified with the help of the following formulas that can be derived straightforwardly:

\[
\begin{align*}
X' &= \frac{R^2}{2XT^2} \hat{\rho} \hat{\rho} - \frac{X}{2} (\ln T^2)^{\rho} + \frac{R^2}{2XT^2} \hat{\rho}^\rho (\ln R^2)^\rho , \\
X' &= -\frac{T^2 X^3}{2} (\ln T^2)' + \frac{X R^2}{2} \hat{\rho}^\rho (\ln R^2)' . \tag{C2}
\end{align*}
\]
After some calculations we then get

\[ K_{\varepsilon,\varepsilon_\gamma} = -\frac{R}{XT} \rho + \frac{(XT - 1)}{2} R \rho (\ln R^2) + \frac{XTR}{2} \rho^2 (\ln T^2)' - \frac{R^3}{2XT} \rho^4 (\ln R^2)' \]

\[ =: (I) + (II) + (III) + (IV) . \]

The last two terms together yield

\[ (III) + (IV) = \frac{XTR}{2} \rho^2 \left \{ XTR \partial_n(\ln T^2/R^2) - R^2 \rho (\ln T^2/R^2) \right \} + \frac{R}{2XT} \rho^2 \left \{ XTR \partial_n(\ln R^2) - R^2 \rho (\ln R^2) \right \} \]

\[ =: (V_1) + (V_2) + (V_3) + (V_4) , \]

where the formulas in Eq.(10) have been used. Noting that \((II) + (V_4) = -\frac{R}{2XT} \rho (\ln R^2)\), thus, the calculations go as

\[ K_{\varepsilon,\varepsilon_\gamma} = (I) + (II) + (III) + (IV) \]

\[ = (I) + \{(II) + (V_4)\} + (V_3) + \{(V_1) + (V_2)\} , \]

yielding Eq.(13).

Now we derive Eq.(16). To start with, let us note the following formula derived straightforwardly from Eq.(7):

\[ \dot{X} = -\frac{X^3T^2}{2} (\ln T^2)' + \frac{X^3R^2}{2} \rho^2 (\ln R^2) + X^3R^2 \dot{\rho} . \]

Taking the \(t\)-derivative of the relation \(\rho = X \dot{\rho}\) (Eq.(7)), then, we obtain,

\[ \ddot{\rho} = \frac{1}{X^4T^2} \rho^\circ + \frac{1}{2X^2} \rho (\ln T^2) + \frac{R^2}{2X^4T^2} \rho^3 (\ln R^2) . \]

We should return to the basic equation Eq.(12) along with the calculations Eqs.(C1) and (C2). Noting the relations

\[ (XT) = X((TX) + \dot{\rho}(TX)) = \frac{XT}{2} (X^2T^2 - 1)(\ln(R^2/T^2)) + X^4T^2 \dot{\rho} , \]

\[ (\rho) = \rho + \rho^\circ \rho = \rho + \frac{R}{2} \rho^\circ , \]

\[ = X^4T^2 \dot{\rho} + \frac{1}{2} X^3T^2 R \dot{\rho} (\ln(R^2/T^2)) , \]

we finally reach Eq.(16).

APPENDIX D: DETAILED COMPARISON WITH THE FORMULAS IN THE PRECEDING PAPER

1. Extrinsic curvature

Here we point out the discrepancy of the formulas reported in Ref. [6] and its correction Ref. [7] with our formulas.

Let us note once again that their functions \(f, g, h, l, R_0(T), R_0'(T), R_0''(T)\) and \(b(T - R)\) correspond to ours as

\[ f := T^2, g := R^2, h := Z^2, l := \Phi^2, \]

\[ R_0(T) := \rho(t), R_0'(T) := \dot{\rho}(T), \]

\[ R_0''(T) := \ddot{\rho}(T), b(T - R) = -2\gamma(t - r) . \]

They show the expression for the \((\tau \tau)\)-component of the extrinsic curvature (see the first formula in Eq.(10) of Ref.[6]), which we now denote \(K_{\tau \tau}^{(PW)}\), as

\[ K_{\tau \tau}^{(PW)} = -\frac{1}{2} \left \{ \frac{f g}{g} (f g)^{1/2} + \frac{f R}{g} - 2g \left \{ \frac{f R}{g} - 2g \right \} R_0'(T) + \left \{ \frac{d f R}{g} - \frac{g R}{g} \right \} (R_0'(T))^2 + \frac{g R}{g} R_0''(T) - 2R_0''(T) \right \} . \]

It is easily checked that their result is translated into our notation as

\[ K_{\tau \tau}^{(PW)} = \frac{X^3TR}{2T} \dot{\rho} + \frac{XTR}{2T} (\ln R^2) + \frac{XT}{2R} (\ln T^2)' - \frac{1}{2} X^2TR \dot{\rho} (\ln(T^2/R^2)) . \]

We realize, comparing Eq.(D3) with Eq.(16), we realize,

\[ K_{\tau \tau}^{(PW)} = -K_{\varepsilon,\varepsilon_\gamma} + \frac{X^3R^3}{2R} \left \{ (\ln T^2)' + \frac{R^2}{T^2} \rho (\ln R^2) \right \} \]

\[ = -K_{\varepsilon,\varepsilon_\gamma} + \frac{X^2T^2}{2} \partial_n \ln R^2 + \frac{X^4T^4}{2} \partial_n \ln(T^2/R^2) - \frac{X^3R^3}{2R} \rho (\ln(T^2/R^2)) . \]
Here we have made use of Eq.(10) to reach the last line in Eq.(D4):

\[
\frac{(\ln T')'}{T'} + \frac{R^2}{T^2} \rho (\ln R')' = -R^2 \hat{\rho} (\ln(T'/R'))' + XTR \partial_n \ln(T'/R') + \frac{R}{XT} \partial_n \ln R'.
\]

On the other hand, their expressions for the \(zz\)-component and the \((\phi\phi)\)-component of the extrinsic curvature, \(K_{zz}^{(PW)}\) and \(K_{\phi\phi}^{(PW)}\), (see the second and the third formulas in Eq.(10) of Ref.[6]) are related to our \(K_{e_z,e_z}\) and \(K_{e_\phi,e_\phi}\) as

\[
K_{zz}^{(PW)} = -\frac{1}{2} \left( \frac{fg}{f - g R_0'(T)^2} \right)^{1/2} \left( \frac{h_n R}{g} - \frac{h_T R_0'(T)}{f} \right)
= -Z^2 \partial_n \ln Z
= -(E^{e_z,e_z})^{2} K_{e_z,e_z}.
\]

\[
K_{\phi\phi}^{(PW)} = -\frac{1}{2} \left( \frac{fg}{f - g R_0'(T)^2} \right)^{1/2} \left( \frac{l_R}{g} - \frac{l_T R_0'(T)}{f} \right)
= -\Phi^2 \partial_n \ln \Phi
= -(E^{e_\phi,e_\phi})^{2} K_{e_\phi,e_\phi}.
\]

\(\text{(D5)}\)

\(\text{(D6)}\)

It turns out that \(K_{zz}^{(PW)}\) and \(K_{\phi\phi}^{(PW)}\) are totally consistent with our \(K_{e_z,e_z}\) and \(K_{e_\phi,e_\phi}\), taking into account that the former is based on the coordinate frame \(\{\partial_a\}_{a=n,v,z,\phi,\theta}\), while the latter is based on the ortho-normal frame \(\{\hat{e}_a\}_{a=n,v,z,\phi}.\) [On the other hand, \(K_{\tau\tau}^{(PW)}\) is the \((\tau\tau)\)-component (and not \((tt)\)-component), and \(\partial_r = \hat{e}_r\) is a member of the ortho-normal frame \(\{\hat{e}_a\}_{a=n,v,z,\phi}.\) Therefore \(K_{\tau\tau}^{(PW)}\) is what should correspond to our \(K_{e_r,e_r}\) (up to sign) without any further conversion factor.]

Looking at Eqs.(D4)-(D6), we realize

(0) Their definition for the extrinsic curvature and ours differ by the overall sign, which is just the matter of definition and is not essential.

(1) The last three terms in the last line on the R.H.S. in Eq.(D4) are the essential discrepancies in the \((\tau\tau)\)-component of the extrinsic curvature.

(2) There is no essential discrepancy regarding the \((zz)\)-component and the \((\phi\phi)\)-component of the extrinsic curvature.

Their expression Eq.(D2) can be modified so as to coincide with our result \(K_{e_r,e_r}\):

\[
K_{\tau\tau}^{(PW,new)} = -K_{e_r,e_r}
= -\frac{1}{2} \left( \frac{fg}{f - g R_0'(T)^2} \right)^{3/2} \times \left\{ \left( \frac{f_T - g_T R_0'(T)}{g} \right) R_0'(T) + \left( \frac{2f^2 R - g^2 R}{g} \right) (R_0'(T))^2 + \frac{2g_T}{f} R_0'(T)^3 - 2R_0''(T) \right\}.
\]

\(\text{(D7)}\)

Comparing Eq.(D7) with Eq.(D2), we realize that the term \(-\frac{f_T}{g}\) in Eq.(D2) should be removed and that the numerical factor “2” in front of the factor \(\frac{g_T}{f}\) should be corrected to “1”.

2. Formulas for the pressures \(p_z\) and \(p_\phi\)

The expression for \(\frac{dt}{dx}\) (corresponding to our \(\frac{dt}{dx}\)) in Ref.[6] has been corrected in Ref.[7], and the new expression coincides with our Eq.(27). However, the expressions for \(\frac{dx}{dt}\) and \(v_0'(t)\) (corresponding to \(\frac{dx}{dt}\) and \(\hat{\rho}_-(t_-)\), respectively) in Eq.(28) of Ref.[6] are not mentioned in Ref.[7]; they actually should be corrected as our Eqs.(28) and (30), respectively. The expression for \(p_z\) has been corrected in Ref.[7] as

\[
\kappa p_z = \frac{\frac{\epsilon t(x_0/\xi_0)^2}{\Delta(1 - R_0'^2)^{3/2}}}{\Delta(1 - R_0'^2)^{3/2}} \left\{ (1 - \Delta) \left[ \frac{\Delta}{R_0}(1 - R_0'^2) - R_0'' \right] - \frac{\epsilon t'(x_0)}{2} (1 - R_0')(1 - R_0'^2)(R_0' - \Delta) \right\}.
\]

\(\text{(D8)}\)

However, it still needs modification (see our Eq.(32)), which is equivalent to replace the last two factors \((1 - R_0'^2)(R_0' - \Delta)\) in Eq.(D8) with \(-R_0'[(\Delta + 1)R_0' + 1].\) The new expression of \(p_z\) influences the dynamical equation for the shell, since the latter is obtained by setting \(p_z = 0\) (They only consider the case \(p_z = 0\).) The rest is discussed in Sec.IV.

The expressions \(p_\phi\) in Eq.(29) of Ref.[6] should also be modified as our Eq. (33).
[1] For the basic accounts on the spherical collapses, see, e.g. P.S. Joshi, Global Aspects in Gravitation and Cosmology (Clarendon, Oxford, 1993).

[2] K. S. Thorne, in Magic Without Magic: John Archibald Wheeler (J. Klauder (ed.), Freeman, San Francisco, 1972), p.1.

[3] S. L. Shapiro and S. A. Teukolsky, Phys. Rev. Lett. 66, 994 (1991).

[4] R. Penrose, Riv. Nuovo Cimento 1 (Numero Special), 252 (1969).

[5] T. A. Apostolatos and K. S. Thorne, Physical Review D46, 2435 (1992).

[6] P. R. C. T. Pereira and A. Wang, Physical Review D62, #124001 (2000).

[7] P. R. C. T. Pereira and A. Wang, Physical Review D67, #129902(E) (2003).

[8] T. Piran, Phys. Rev. Lett., 1085 (1978).

[9] W. Israel, Il Nuovo Cimento B44, 1 (1966); B48, 463(E) (1967).

[10] See, for instance, R. M. Wald, General Relativity, § 9-2.

[11] K. S. Thorne, Phys. Rev. 138, B251 (1965).

[12] See, for instance, M. Nakahara, Geometry, Topology and Physics (Institute of Physics Publishing, Bristol, 1990), Chapter 7.

[13] The most natural topology is the one induced from the canonical structure read by the Einstein-Hilbert action constructed from the metric Eq.(23). The detailed topological property is not very important in the argument below, so that it suffices to imagine $\Gamma$ as $(\text{radial} - \text{coordinate}) \times \mathbb{R}$.

[14] We recall that $\{\hat{e}_A\}_{A=0,1,2,3}$ forms the non-coordinate bases and $\Gamma^A_{BC}$'s are not tensors, so that there is no surprise in that $\Gamma^0_{01} = 0$ ($\neq \Gamma^0_{10}$).