50 Years of the Golomb–Welch Conjecture

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Abstract—Since 1968, when the Golomb–Welch conjecture was raised, it has become the main motive power behind the progress in the area of the perfect Lee codes. Although there is a vast literature on the topic and it is widely believed to be true, this conjecture is far from being solved. In this paper, we provide a survey of papers on the Golomb–Welch conjecture. Further, new results on Golomb–Welch conjecture dealing with perfect Lee codes of large radii are presented. Algebraic ways of tackling the conjecture in the future are discussed as well. Finally, a brief survey of research inspired by the conjecture is given.

Index Terms—error correction codes, perfect Lee codes, Golomb–Welch conjecture, tilings.

I. INTRODUCTION

In this paper we deal with codes in the Lee metric. This metric was introduced in [1] and [2] for transmission of signals taken from $\mathbb{GF}(p)$ over noisy channels. It was generalized for $\mathbb{Z}_m$ in [3]. The interest in Lee codes is due to many applications of them. For example, constrained and partial-response channels [4], flash memory [5], interleaving schemes [6], placement of resources in the computer architecture that minimizes access time by processing elements [7], multidimensional burst-error-correction [8], and error-correction in the rank modulation scheme for flash memories [9].

50 years ago, Golomb and Welch [1] raised a conjecture on the existence of perfect $e$-error-correcting codes in the Lee metric. This conjecture lies at the very center of interests in the area of perfect codes in the Lee metric. This conjecture was introduced in [1] and [2] for transmission of signals taken from $\mathbb{GF}(p)$ over noisy channels. It was generalized for $\mathbb{Z}_m$ in [3]. The interest in Lee codes is due to many applications of them. For example, constrained and partial-response channels [4], flash memory [5], interleaving schemes [6], placement of resources in the computer architecture that minimizes access time by processing elements [7], multidimensional burst-error-correction [8], and error-correction in the rank modulation scheme for flash memories [9].

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In [10], Golomb and Welch proved that for each fixed $n$ there exists an $e_n$, $e_n$ unspecified, such that for all $e > e_n$ there is no perfect $e$-error correcting code in $\mathbb{Z}_n$ with the Lee metric. In Section 3 we present the first explicit upper bound on $e_n$. More precisely, we show that the condition periodic can be dropped in the Post [11] and Lepistö [12] bounds (see Theorem[13] and Theorem[14]). Finally, we exhibit how a linear programming technique can be used to obtain another bound on $e_n$, cf. Corollary[15]. Combining these three statements we obtained Theorem[16] that summarizes our new results on the Golomb–Welch conjecture.

Although the conjecture has been tackled in various ways, using different techniques, it seems to us that none of them is powerful enough to entirely solve the conjecture. We believe that a new approach has to be developed. Therefore, possible avenues how to attack the conjecture are discussed in Section 4. Using the so-called polynomial method, a necessary condition for the existence of a tiling of $\mathbb{Z}^n$ by translates of a tile $V$ is proved (see Theorem[17]). We guess that this is a first necessary condition for a generic (arbitrary) tile. Further, we exhibit usage of Fourier analysis in this area; we provide a sufficient condition for a tile $V$ such that each translational tiling of $\mathbb{Z}^n$ by $V$ is periodic (see Theorem[18]). In our quest to prove the Golomb–Welch conjecture we have dealt with tiles of prime size. Later we started to be interested in these tiles on its own right. Now it seems that a part of our research on prime tiles might contribute back to the Golomb–Welch conjecture. In this regard, first we reprove a statement that each tiling of $\mathbb{Z}^n$ by translates of a tile of prime size has to be periodic. In fact, we conjecture that each such tiling has to be even a lattice one (see Conjecture[19]). We prove our conjecture for tiles of size at most 7.

In Section[20] we cover results inspired by the Golomb–Welch conjecture. First we describe several generalizations and modification of the conjecture, and then a brief survey of the results on quasi-perfect Lee codes will be given.

In the last section we summarize our discussion on the methods used and the methods proposed in this paper to solve the Golomb–Welch conjecture.

A. Terminology and Basic Concepts

As usual, let $\mathbb{Z}$ be the set of all integers, $\mathbb{Z}_q$ denote the ring of integers modulo $q$, and let $T^n$ stand for the $n$-fold Cartesian product of a set $T$. A Lee code is a subset of the metric space $(\mathbb{Z}^n, \delta_L)$, where $C = \mathbb{Z}_q^n$, or $C = \mathbb{Z}^n$, and $\delta_L$ is the Lee metric (= the Manhattan metric, the zig-zag metric, the $l^1$-norm). That is, for any two words $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$, $\delta_L(u, v) = \sum_{i=1}^{n} |u_i - v_i|$ for $u, v \in \mathbb{Z}^n_q$, $\delta_L(u, v) = \sum_{i=1}^{n} |u_i - v_i|$ for $u, v \in \mathbb{Z}^n$.

A Lee code $C$ is an $e$-error-correcting code if any two distinct elements of $C$ have distance at least $2e + 1$. An $e$-error-correcting Lee code is further called perfect if for each $x \in \mathbb{Z}^n_q$ (or $x \in \mathbb{Z}^n$), there exists a unique element $c \in C$ such that $\delta_L(x, c) \leq e$. A perfect $e$-error-correcting Lee code in $\mathbb{Z}^n_q$ and in $\mathbb{Z}^n$ will be called PL$(n, e, q)$ and PL$(n, e)$, respectively. These codes are also termed perfect $e$-error-correcting code of block size $n$ over $\mathbb{Z}_q$ (over $\mathbb{Z}$). If $q \geq 2e + 1$, a PL$(n, e, q)$-code is said to be over a large alphabet, otherwise it is said
to be over a small alphabet. A set $S \subseteq \mathbb{Z}^n$ is $q$-periodic if it is periodic with the period $q$ along all coordinate axes. A $PL(n, e, q)$-code $C$ is $(q, e)$-periodic (resp. lattice, linear) if $C$ is a $(q, e)$-periodic set in $\mathbb{Z}^n$ (resp. a subgroup of the additive group $\mathbb{Z}^n$ of full rank).

It is very common to define error-correcting Lee codes using the language of tilings. In this setting it is not difficult to see that to know all about $PL(n, e, q)$-codes with large alphabets, it suffices to study $PL(n, e)$-codes. Indeed, consider the Lee spheres
\[
S(n, e, q) = \{ x \in \mathbb{Z}^n : \delta_L(x, 0) \leq e \}
\]
and
\[
S(n, e) = \{ x \in \mathbb{Z}^n : \delta_L(x, 0) = |x_1| + \cdots + |x_n| \leq e \}
\]
of radius $e$. Then $PL(n, e, q)$-codes and periodic $PL(n, e)$-codes can be naturally identified with tilings of $\mathbb{Z}^n$ and of $\mathbb{Z}^n$ by translates of $S(n, e, q)$ and $S(n, e)$, respectively.

If $q \geq 2e + 1$, then the natural projection map $\mathbb{Z}^n \rightarrow \mathbb{Z}_q^n$ restricts to a bijection from $S(n, e)$ to $S(n, e, q)$. Any tiling of $\mathbb{Z}_q^n$ by $S(n, e, q)$ will then pull back via the projection to a periodic tiling of $\mathbb{Z}^n$ by $S(n, e)$. Then a $PL(n, e, q)$-code induces a periodic $PL(n, e)$-code that is a disjoint union of cosets of $q\mathbb{Z}^n \subset \mathbb{Z}^n$. Conversely, any such periodic $PL(n, e)$-code clearly comes from a $PL(n, e, q)$-code. The following proposition states in a formal way that $PL(n, e)$-codes provide full information about $PL(n, e, q)$-codes.

**Proposition 1.** For $q \geq 2e + 1$, there exists a natural bijection between $PL(n, e, q)$-codes and $q$-periodic $PL(n, e)$-codes that is a union of cosets of $q\mathbb{Z}^n \subset \mathbb{Z}^n$, given by taking the image or the inverse image with respect to the projection map $\mathbb{Z}^n \rightarrow \mathbb{Z}_q^n$.

Since a $PL(n, e)$-code can be seen as a partition of $\mathbb{Z}^n$, only a small step is needed to get a geometrical interpretation of $PL(n, e)$-codes. Let $\mathbb{R}$ be the set of real numbers. Consider the $n$-dimensional space $\mathbb{R}^n$ endowed with the Lee metric $\delta_L$.

The $n$-cube centered at $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is the set $C(x) = \{ y = (y_1, \ldots, y_n) : |y_i - x_i| \leq \frac{1}{2} \}$. By a Lee sphere of radius $e$ in $\mathbb{R}^n$ centered at $0$, $L(n, e)$, we understand the union of $n$-cubes centered at $y$, where $\delta_L(y, 0) \leq e$, and $y$ has integer coordinates. Finally, a Lee sphere of radius $e$ in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ is the set $x + L(n, e) = \{ x + l : l \in L(n, e) \}$. Clearly, a $PL(n, e)$-code exists if and only if there is a tiling of $\mathbb{R}^n$ by Lee spheres $L(n, e)$. The Lee spheres $L(2, 1), L(2, 2), L(3, 1),$ and $L(3, 2)$ are depicted in Figure 1.

The advantage of understanding a $PL(n, e)$-code as a tiling of $\mathbb{R}^n$ of $L(n, e)$ is in the possibility of applying deep results in geometry to Lee codes.

We note that it is common in Coding theory to call the sets $S(n, e) = \{ x \in \mathbb{Z}^n : d(x, 0) \leq e \}$ and $B(n, e) = \{ x \in \mathbb{Z}^n : d(x, 0) = e \}$ the sphere and the boundary of this sphere although in other parts of mathematics they are termed the ball and the sphere. In order not to go against the long time tradition we have decided to stick with this imprecise terminology.

**II. THE GOLUMB–WELCH CONJECTURE**

In this section we state the Golomb–Welch conjecture and survey related results. We do not cover here $PL(n, e, q)$-codes over small alphabets.

In their seminal paper Golomb and Welch [3] discuss at great length the existence of $PL(n, e, q)$-codes. They constructed $PL(n, e, q)$-codes for parameters $(n, e, q) = (1, e, 2e + 1), (2, e, e^2 + (e + 1)^2)$, and $(n, 1, 2n + 1)$. In the last paragraph of Section 3 in [3] it is conjectured that there are no tilings of $\mathbb{Z}_q^n$ by Lee spheres over large alphabet for other values of $(n, e)$. We note that in [3] $\mathbb{Z}_q^n$ is called $n$-dimensional space while $\mathbb{Z}^n$ is termed $n$-dimensional Euclidean space.

**Conjecture 2** (Golomb–Welch, weak version, Section 3 [3]). There is no $PL(n, e, q)$-code over large alphabets for $n \geq 3$ and $e \geq 2$.

In Section 7, Golomb and Welch formulate their conjecture in terms of tiling $n$-dimensional Euclidean space. Thus, with respect to Proposition 1 the following conjecture is a natural strengthening:

**Conjecture 3** (Golomb–Welch, strong version, Section 7 [3]). There is no $PL(n, e)$-code for $n \geq 3$ and $e \geq 2$.

A set $S \subset \mathbb{Z}^n$ is fully periodic if the set of those elements which shift $S$ into itself is a subgroup of $\mathbb{Z}^n$ of finite index. We point out that if the following Lagarias–Wang conjecture is true then Conjectures 2 and 3 are equivalent.

**Conjecture 4** (Lagarias–Wang [10]). If $V$ tiles $\mathbb{Z}^n$ by translations, then $V$ admits a fully periodic tiling, i.e., a $q$-periodic tiling for sufficiently large $q$.

So far Conjecture 4 has been proved for tiles $V$ of prime size [11], for any $V \subset \mathbb{Z}^2$ [12], and for some other special types of tiles.

**A. Survey of Results on the Golomb–Welch Conjecture**

To provide a support for their conjecture, Golomb and Welch [3] show that there is no $PL(n, e)$-code for $(n, e) = (3, 2)$ and also for large $e$. Their basic idea for proving nonexistence of $PL(n, e)$-codes for sufficiently large $e$ is that such a code will induce a dense packing of $\mathbb{R}^n$ by cross-polytopes. The following theorem then follows from the known fact that there is no tiling of $\mathbb{R}^n$ by regular cross-polytopes.

Fig. 1. Figure of $L(2, 1), L(2, 2), L(3, 1),$ and $L(3, 2)$.
Theorem 5 ([13]). For \( n \geq 3 \) there exists \( e_n, e_n \), not specified, such that for any \( e > e_n \) there is no \( PL(n, e) \)-code.

For a more detailed explanation of their idea, see the beginning of Section III. Theorem 5 is not explicit, and not even effective in the sense that it only shows that such a constant \( e_n \) exists. A first explicit bound on \( e_n \), in the case of periodic codes, has been given by Post [13]. He showed, by counting low-dimensional cross-sections, that \( PL(n, e) \)-codes do not exist for \( 3 \leq n \leq 5 \) and \( e \geq n - 1, q \geq 2e + 1 \) and \( n \geq 6, e \geq \frac{\sqrt{5} - 1}{2} \sqrt{n} - \frac{3}{4} \sqrt{2} - \frac{1}{2} \), \( q \geq 2e + 1 \). The result of Post was asymptotically improved by Astola [14], and later by Lepistö [15] who obtained:

Theorem 6 ([15]). For any \( n, e, q \) satisfying \( n < (e + 2)^2/2 \), \( e \geq 285 \), and \( q \geq 2e + 1 \), there is no \( PL(n, e, q) \)-code.

An outline of Post’s and Lepisto’s proofs will be provided in the next section. Developing and refining their ideas we will show that the condition periodic can be dropped from both their results, cf. Theorem 13 and Theorem 15. Also, by using a linear programming method, we obtained a further slight improvement on the bound of \( e_n \), see Corollary 23. The next theorem is a direct combination of these three bounds on \( e_n \).

Theorem 7. There is no \( PL(n, e) \)-code for

\[
3 \leq n \leq 74 \quad \text{and} \quad \max \left\{ \frac{\sqrt{2}}{2} n - \frac{3}{4} \sqrt{2} - \frac{1}{2}, 2 \right\} \leq e,
\]

\[
75 \leq n \leq 405 \quad \text{and} \quad \max \left\{ 18, \sqrt{2n + 20} \right\} \leq e \leq \frac{n - 21}{3},
\]

\[
or \quad \frac{\sqrt{2}}{2} n - \frac{3}{4} \sqrt{2} - \frac{1}{2} \leq e,
\]

\[
406 \leq n \leq 876 \quad \text{and} \quad \sqrt{2n + 40} \leq e \leq \frac{n - 21}{3} \quad \text{or} \quad 285 \leq e,
\]

\[
876 \leq n \quad \text{and} \quad \sqrt{2n + 40} \leq e.
\]

It seems that the most difficult case of the Golomb–Welch conjecture is that of \( e = 2 \). The nonexistence of \( PL(6, 2) \)-codes has been shown in [16]. A step forward in this direction has been made by the second author of this paper (see [17]). He proved that if the volume of the sphere \( |S(n, 2)| = 2n^2 + 2n + 1 \) is prime and a certain number-theoretic condition is satisfied, then \( PL(n, 2) \)-codes do not exist. It turns out that this condition is not restrictive as, e.g., out of 12706 numbers \( n \leq 10^5 \) with \( p = 2n^2 + 2n + 1 \) prime, only 4 numbers \( n \) do not satisfy the condition. However, it is not known if there are infinity many \( n \) with \( p = 2n^2 + 2n + 1 \) prime.

A special case, the nonexistence of linear \( PL(n, 2) \)-codes is proved in [18] for \( n \leq 12 \). The proof is based on the nonexistence of a homomorphism \( \phi : \mathbb{Z}^n \rightarrow G \), an abelian group of order \( |S(n, 2)| \) such that a restriction of \( \phi \) to \( S(n, 2) \) would be a bijection to \( G \). A similar approach has been used in [19] to show the nonexistence of linear \( PL(n, 3) \)-codes for some values of \( n = 12, 21 \mod 27 \), and the nonexistence of linear \( PL(n, 4) \)-codes for some values of \( n = 3, 5, 21, 23 \mod 27 \).

As to the weak version of the Golomb–Welch conjecture, Conjecture 2 the nonexistence of \( PL(n, e, q) \)-codes has been proved for several special cases of \( q \). A list of such cases is given in [20]. To illustrate this type of conditions, here we mention two of them (see [20]): There is no \( PL(n, e) \)-code for \( e = 2, q = p^k \), \( p \) is a prime, \( p \neq 13 \), \( p < \sqrt{|S(n, 2)|} \); and \( e = 3, q \geq 7 \) is not divisible by a prime \( p \equiv 1, 3, 5, 7, 9 \mod 20 \).

Now we turn our attention to the case of small dimension \( n \). For \( 3 \leq n \leq 5 \), the Golomb–Welch conjecture has been proved for all \( e \geq 2 \). In [21], by an elegant “picture says it all” approach it is shown that there is no tiling of \( \mathbb{R}^3 \) by Lee spheres. A further extension of the result has been provided in [30] (see Section V-B). The same result, using an exhaustive computer search, was proved in [22] for \( \mathbb{R}^4 \). It seems that the used algorithm is not computationally feasible for \( n \geq 5 \). Finally, by an algebraic approach based on the nonexistence of \( PL(n, 2) \)-codes, it was proved analytically that, for \( 3 \leq n \leq 5 \), there is no tiling of \( \mathbb{R}^n \) by Lee spheres [23].

III. THE GOLOMB–WELCH CONJECTURE FOR LARGE RADIUS

In this section we study ways of proving the nonexistence of \( PL(n, e) \)-codes, in the case when \( e \) is sufficiently large. Why would \( e \) be large prevent the Lee sphere \( S(n, e) \) from tiling \( \mathbb{Z}^n \)? The intuition is that as \( e \) grows for fixed \( n \), the sphere \( S(n, e) \) becomes more and more similar to the convex hull of \( \{(0, …, 0, ±1, 0, …, 0)\} \). This polytope is the dual of the \( n \)-cube, and is called a cross-polytope.

![Fig. 2. Figure of a cross-polytope in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \)](image)

For \( n \geq 3 \), it is well-known that the cross-polytope does not tile \( \mathbb{R}^n \) by translations. If \( n \neq 4 \), then this fact can be immediately obtained by computing the angle of two adjacent faces, and for \( n = 4 \), taking care of the orientation gives this fact. Using a compactness argument on the space of local configurations, it can be shown that the packing density of a bounded set that does not tile \( \mathbb{R}^n \) is bounded away from 1. On the other hand, a \( PL(n, e) \)-code (even a \( QPL(n, e) \)-code, see Section V-B for the definition of \( QPL(n, e) \)-code) for large \( e \) induces a translational packing of a \( n \)-dimensional cross-polytope with high density. Thus we obtain Theorem 5.

Theorem 5 ([24]). For \( n \geq 3 \) there exists \( e_n, e_n \), not specified, such that for any \( e > e_n \) there is no \( PL(n, e) \)-code.

Remark 8. We note that in [3], the theorem was actually proved only for \( n = 3 \) and \( n \geq 5 \). However, it is not difficult to see that the same argument holds also for \( n = 4 \). Indeed, although there is a tiling of \( \mathbb{R}^4 \) by the \( 4 \)-dimensional cross-polytope, known as the 16-cell honeycomb, this tiling is not by translations.
The idea of the proof of Theorem\[ has been used by several authors (see e.g. \[18\]), where, applying the idea, it is proved that for each \( n \) there are only finitely many values of \( e \) for which quasi-perfect Lee code might exist.

### A. Post’s Bound

The first ever effective result on the nonexistence of \( PL(n,e,q) \)-codes for large \( e \) was obtained by Post.

**Theorem 9** ([13]). For any \( n, e, q \) satisfying \( n \geq 6, e \geq \frac{\sqrt{2}}{4} n - \frac{3}{4} \sqrt{2} - \frac{1}{2} \) and \( q \geq 2e + 1 \), there is no \( PL(n,e,q) \)-code.

Post obtained this theorem by focusing on local configurations of the tiling at the boundary of the Lee spheres. Let us be more specific.

**Definition 10.** A \( k \)-dimensional sector is a subset of \( Z^n \) or \( Z_q^n \) that is a translate of

\[
\{ \mathbf{x} : x_i \in \{0,1\} \text{ if } i \in \{i_1, \ldots, i_k\}, \text{ } x_i = 0 \text{ otherwise}\},
\]

where \( \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\} \).

Clearly, a \( k \)-dimensional sector has cardinality \( 2^k \), and a non-empty intersection of a \( k \)-dimensional sector and a Lee sphere always has cardinality \( \sum_{i=0}^{t} \binom{k}{i} \) for some \( 0 \leq t \leq k \). In [13], Post focused on the 6-dimensional sectors in \( Z^n_q \). A 6-dimensional sector consists of 64 unit cubes (or points if we agree to work in \( Z^n \)), and a Lee sphere can cover either 0, 1, 7, 22, 42, 57, 63, 64 of them. Ignore the case when the 6-dimensional sector is disjoint from or covered entirely by the Lee sphere. Let us say that a pair \((S,T)\) of a Lee sphere \( S \) and a 6-dimensional sector \( T \) is of type \( i \) if \( |S \cap T| = i \). For a fixed sector \( T \), there are only a handful of ways it can be covered completely: one type 63 and one type 1, one type 42 and one type 7 and fifteen type 1, etc. Given a tiling of \( Z^n \) by Lee spheres \( S(n,e) \). We denote by \( t_{T,i} \) the number of type \( i \) pairs \((S,T)\) with given sector \( T \). Listing all possible combinations, Post proves the following.

**Lemma 11** ([13] p. 377]). Given a tiling of \( Z^n \) by \( S(n,e) \), \( n \geq 6 \), for any 6-dimensional sector \( T \),

\[
t_{T,1} - t_{T,7} - 10t_{T,22} + 10t_{T,42} + t_{T,57} - t_{T,63} \geq 0.
\]

On the other hand, let us count the number of type \( i \) pairs for a fixed Lee sphere \( S \). We know quite well the shape of \( S(n,e) \), and it is a matter of computation to count 6-dimensional sectors \( T \) for which \( |S(n,e) \cap T| = i \). Denote by \( g_i \) the number of type \( i \) pairs \((S(n,e),T)\) \[1\]

**Lemma 12** ([13] p. 378–379]). If \( n \geq 6 \) and \( e \geq \frac{\sqrt{2}}{4} n - \frac{3}{4} \sqrt{2} - \frac{1}{2} \), then

\[
g_1 - g_7 - 10g_{22} + 10g_{42} + g_{57} - g_{63} < 0.
\]

Suppose now that \( C \) is a \( PL(n,e,q) \)-code, for \( q \geq 2e + 1 \) and \( n \geq 6, e \geq \frac{\sqrt{2}}{4} n - \frac{3}{4} \sqrt{2} - \frac{1}{2} \). Let \( t_i \) be the total number of type \( i \) pairs \((S,T)\) in \( Z^n_q \), which is clearly finite. Then we count \( t_i \) in two ways as

\[
t_i = \sum_{6\text{-dim. sect. } T} t_{T,i} = |C| \cdot g_i.
\]

Thus from Lemmas \[11\] and \[12\] it follows that

\[
t_1 - t_7 - 10t_{22} + 10t_{42} + t_{57} - t_{63} = \sum_{T} (t_{T,1} - t_{T,7} - 10t_{T,22} + 10t_{T,42} + t_{T,57} - t_{T,63}) \geq 0,
\]

\[
t_1 - t_7 - 10t_{22} + 10t_{42} + t_{57} - t_{63} = |C|(g_1 - g_7 - 10g_{22} + 10g_{42} + g_{57} - g_{63}) < 0,
\]

This is clearly a contradiction, and hence Theorem\[9\] is proved.

It is worth noting that if we can somehow replace the number \( t_i \) of type \( i \) pairs by a notion of density, we would be able to obtain the same theorem even if we are in \( Z^n \) instead of \( Z^n_q \), i.e., if we drop the periodicity condition.

**Theorem 13.** For any \( n, e \) satisfying \( n \geq 6 \) and \( e \geq \frac{\sqrt{2}}{4} n - \frac{3}{4} \sqrt{2} - \frac{1}{2} \), there is no \( PL(n,e) \)-code.

**Proof.** Let \( C \) be a \( PL(n,e) \)-code in \( Z^n \). Denote by \( B_n(N) \) the \( n \)-dimensional box \([-N,N]^n\). Fix \( n \) and \( e \), and let \( N \) be an integer variable that is sufficiently large. Let \( t_i \) be the number of type \( i \) pairs \((S,T)\) where \( S \) is a Lee sphere centered at a codeword in \( B_n(N) \). Counting with respect to \( S \), we immediately have

\[
t_i = |C \cap B_n(N)| \cdot g_i.
\]

On the other hand, if we let \( t'_i \) be the number of type \( i \) pairs \((S,T)\) where \( T \) is a sector contained in \( B_n(N-e) \), then

\[
t'_i = \sum_{T \subseteq B_n(N-e)} t_{T,i}.
\]

If \((S,T)\) is a pair of type \( i \geq 1 \) and \( T \subseteq B_n(N-e) \), then the center of \( S \) is in \( B_n(N) \). Thus \( t_i \geq t'_i \) and their difference is at most the number of pairs \((S,T)\) with \( T \cap (B_n(N+e) \setminus B_n(N-e)) \neq \emptyset \). Hence

\[
0 \leq t_i - t'_i \leq \frac{n}{6} \cdot |B_n(N+e) \setminus B_n(N-e)|
\]

\[
= O(N^{n-1}).
\]

Here, \( \binom{n}{6} \) represents the different possible orientations of the sectors, and we include \( 2^n \) because each 6-dimensional sector has non-empty intersection with at most \( 2^6 \) spheres.

From Lemma\[11\] we have

\[
t'_1 - t'_7 - 10t'_{22} + 10t'_{42} + t'_{57} - t'_{63} \geq 0.
\]

From Lemma\[12\] and the fact that a negative integer is at most \(-1\), we have

\[
t_1 - t_7 - 10t_{22} + 10t_{42} + t_{57} - t_{63} \leq -|C \cap B_n(N)|
\]

\[
= -\frac{|B_n(N)|}{|S(n,e)|} + O(N^{n-1}) = -\frac{2^n}{|S(n,e)|} N^n + O(N^{n-1}).
\]
by taking the sum over all codewords in $B_n(N)$, because

$$B_n(N - e) \subset \bigcup_{a \in C \cap B_n(N)} (a + S(n, e)) \subset B_n(N + e).$$

Equations (1), (2), (3) contradict each other as $N$ tends to infinity.

**Remark 14.** It might be possible to improve the constant $\sqrt{2}/2$ by counting higher-dimensional sectors, or shapes other than $2 \times \cdots \times 2 \times 1 \times \cdots \times 1$ boxes. However, it would require much more computing, and it seems unlikely that it gives a bound that is better than linear.

**B. Lepistõ’s Bound**

Lepistõ [15] proved a bound much stronger than Post’s (see Theorem 6) by modifying an argument of Astola [14].

**Theorem 6 ([15]).** For any $n, e, q$ satisfying $n < (e + 2)^2/2.1$, $e \geq 285$, and $q \geq 2e + 1$, there is no $PL(n, e, q)$-code.

The proof is much more complicated, and thus we only outline the main idea of the proof. Lepistõ considers the set

$$\Lambda(e, s) = \{ x \in \mathbb{Z}_0^n (\text{or } \mathbb{Z}^n) : \delta_L(x, 0) = e + 2, -s < x_i \leq s \text{ for all } i \}.$$  

**Lemma 15 ([15] Lemmas 2b, 10).** If $C \subseteq \Lambda(e, s)$ is an $e$-error-correcting Lee code with at least two elements, where $q \geq 2s$ and $s \geq 2$, then there exist two codewords with Lee distance at most

$$\frac{e + 2}{|C| - 1} \left( |C| \left( 2 - \frac{e + 2}{n} \right) + 4s - 6 \right).$$

In particular, this quantity is at least $2e + 2$, since any two elements in $\Lambda(e, s)$ have even distance.

On the other hand, a standard averaging argument shows the existence of a translate of $\Lambda(e, s)$ with many codewords.

**Lemma 16 ([15] Lemma 1b).** Let $C$ be a $PL(n, e, q)$-code, where $q \geq 2s$ and $e \geq 2$. Then there exists an $a \in \mathbb{Z}_0^n$ such that

$$|(a + \Lambda(e, s)) \cap C| \geq \frac{|\Lambda(e, s)|}{|S(n, e, q)| - |S(n, e - 2, q)|}.$$  

If $C$ is an $e$-error-correcting Lee code, then $(x + \Lambda(e, s)) \cap C$ is an $e$-error-correcting Lee code contained in $x + \Lambda(e, s)$. Lepistõ then uses Lemma 15 to arrive at a contradiction in the case $n < (e + 2)^2/2.1$ and $e \geq 285$. After Lemmas 15 and 16, the proof is purely about estimating $|\Lambda(e, s)|$ and $|S(n, e, q)|$.

As in the case of Post’s result, we can easily extend this to the case in $\mathbb{Z}^n$ by replacing counting the number by computing a density.

**Lemma 17.** Let $C$ be a $PL(n, e)$-code, where $e \geq 2$. Then there exists an $a \in \mathbb{Z}_0^n$ such that

$$|(a + \Lambda(e, s)) \cap C| \geq \frac{|\Lambda(e, s)|}{|S(n, e)| - |S(n, e - 2)|}.$$  

**Proof.** We make a similar argument as in Theorem 13. We count the cardinality $t$ of the set

$$\{(a, x) : x - a \in \Lambda(e, s), x \in C \cap B_n(N)\}.$$  

With respect to $x$, we count

$$t = |\Lambda(e, s)| |C \cap B_n(N)| = \frac{2^n|\Lambda(e, s)|}{|S(n, e)|} N^n + O(N^{n-1}),$$

as in Theorem 13 because the size of $B_n(N)$ is about $2^n N^n$ and $C$ has density $1/|S(n, e)|$.

On the other hand, we can count $t$ with respect to $a$. Here, note that if $a \in C + S(n, e - 2)$ or $a \notin B_n(N + e + 3)$ then $a + \Lambda(e, s)$ and $C \cap B_n(N)$ are always disjoint. Thus

$$t = \sum_{a \in B_n(N + e + 3) \setminus (C + S(n, e - 2))} \left( \frac{|\Lambda(e, s)|}{|S(n, e)|} - \frac{|S(n, e - 2)|}{|S(n, e)|} \right) 2^n N^n + O(N^{n-1}).$$

From Equations (4) and (5) it immediately follows that

$$\max_a |(a + \Lambda(e, s)) \cap C| \geq \frac{|\Lambda(e, s)|}{|S(n, e)| - |S(n, e - 2)|}$$

after taking the limit $N \to \infty$.

We may use Lemma 17 instead of Lemma 16. The proof of Theorem 6 uses more that just Lemmas 15 and 16. However, other lemmas use essentially the same ideas, and using the density trick, they can all be modified to take $PL(n, e, q)$-codes into account. Because the complicated structure of Lepistõ’s proof, it is nearly impossible to give a more detailed account of the modification without unraveling technical details.

**Theorem 18.** For any $n, e$ satisfying $n < (e + 2)^2/2.1$ and $e \geq 285$, there is no $PL(n, e)$-code.

**Proof.** Let $C$ be a $PL(n, e)$-code. Lemma 17 implies that there exists an $a \in \mathbb{Z}^n$ such that

$$\alpha = |(a + \Lambda(e, s)) \cap C| \geq \frac{|\Lambda(e, s)|}{|S(n, e)| - |S(n, e - 2)|}.$$  

Then $(a + \Lambda(e, s)) \cap C$ is an $e$-error correcting Lee code, and hence Lemma 15 shows that

$$2e + 2 \geq \frac{e + 2}{\alpha - 1} \left( 2 - \frac{e + 2}{n} \right) + 4s - 6.$$  

The two inequalities (6) and (7), with Lepistõ’s estimates give a contradiction.

**C. Linear Programming**

We would like to sketch one more method that can be used to prove nonexistence of $PL(n, e)$-codes for large $e$. The idea originates from Golomb and Welch’s observation that $PL(n, e)$-codes induce translational packings of $\mathbb{R}^n$ by cross-polytopes.

It is extremely difficult to obtain an effective upper bound for the packing density of an arbitrary tile. In the case of $n$-dimensional spheres, Cohn and Elkies [24] developed a tool for proving upper bounds for packing densities, which eventually determined the densest sphere packing in dimensions 8 [25] and 24 [26].
Theorem 19 (24 Theorem B.1). Let $V \subseteq \mathbb{R}^n$ be a convex body, symmetric with respect to the origin, $f : \mathbb{R}^n \to \mathbb{R}$ be a nonzero function, and $\hat{f}(t)$ denote its Fourier transform. Assume that:

1. $|f|$ and $|\hat{f}|$ decay faster than $|x|^{-n-\varepsilon}$ for some $\varepsilon > 0$.
2. $f(x) \leq 0$ for $x \notin V + V$.
3. $\hat{f}(t) \geq 0$ for all $t$.

Then packings of $\mathbb{R}^n$ by translates of $V$ have density at most

$$\frac{\text{vol}(V)f(0)}{2^n \hat{f}(0)},$$

where $\text{vol}(V)$ stands for the volume of $V$.

Instead of applying this theorem directly to the cross-polytope, we use a discrete analogue of the theorem and apply it to the discrete Lee sphere. The proof is essentially the same, but using functions $\mathbb{Z}^n \to \mathbb{R}$ and not scaling the tile by 2, because there is no good notion of convexity.

Theorem 20. Let $V \subseteq \mathbb{Z}^n$ be a finite subset, symmetric with respect to the origin. Suppose $f : \mathbb{Z}^n \to \mathbb{R}$ is a nonzero function such that:

1. $|f|$ decay faster than $|x|^{-n-\varepsilon}$ for some $\varepsilon > 0$.
2. $f(x) \leq 0$ for $x \notin V + V$.
3. $\hat{f}(t) = \sum_{x \in \mathbb{Z}^n} f(x)e^{-2\pi i n \cdot t} \geq 0$ for all $t$.

Then packings of $\mathbb{Z}^n$ (not of $\mathbb{R}^n$) by translates of $V$ have density at most

$$\frac{|V||f(0)|}{\hat{f}(0)}.$$

In particular, if $|V||f(0) - \hat{f}(0)|$ then $V$ does not tile $\mathbb{Z}^n$ by translations. It is noteworthy that if $f = \chi_V * \chi_V$, where $\chi_V$ is the characteristic function of $V$ and $*$ denotes the convolution, then all conditions are satisfied and $|V||f(0)|/\hat{f}(0) = 1$.

Checking whether there exists such a function $f$ with $|V||f(0)|/\hat{f}(0)$ is now a linear programming problem. But in its current form, the problem is not easily computable since the restrictions are complicated. Thus we consider one special situation in which the conditions become much simpler.

Denote $\tilde{g}(x) = g(-x)$. For a function $g$ with fast decay, we use a linear perturbation $f = (\chi_V * \epsilon g) * (\chi_V * \epsilon g)$ for $0 < \epsilon \ll 1$. Then conditions (0) and (2) are automatically satisfied, and (1) also is satisfied up to first order of $\epsilon$ if and only if $|\chi_V * \tilde{g}(x)| \geq 0$ for all $x \notin V + V$. We then obtain the following corollary.

Corollary 21. If there exists a function $g : \mathbb{Z}^n \to \mathbb{R}$ satisfying the following conditions, then there is no $PL(n, e)$-code:

1. $|g|$ decays faster than $|x|^{-n-\varepsilon}$ for some $\varepsilon > 0$.
2. $g(x) = 0$ for $x \in S(n, e)$.
3. $g(x) = \chi_{S(n, e)}(x) \geq 0$ for $x \notin S(n, 2e)$.
4. $\sum_{x \in \mathbb{Z}^n} g(x) < 0$. (The sum converges absolutely by (0).)

We note that this corollary has a separate elementary proof that does not appeal to Theorem 20. But it is clear that, while difficult to use, Theorem 20 is much stronger than Corollary 21 by itself.

Nevertheless, Corollary 21 immediately yields a significant bound. Denote by $G$ the isometry group of $\mathbb{Z}^n$, generated by permutations of axes and reflections and consisting of $2^n \cdot n!$ elements. We also introduce the notation

$$(m_1^{\alpha_1}, m_2^{\alpha_2}, \ldots, m_k^{\alpha_k}) = (m_1, m_2, \ldots, m_k, 0, \ldots, 0) \in \mathbb{Z}^n,$$

which makes sense for $\sum \alpha_i \leq n$. The following proposition can be proven by explicit calculation.

Proposition 22. Assume that $n \geq e$ and $\alpha \leq 2e + 2$. Define the function $h : \mathbb{Z}^n \to \mathbb{R}$ as

$$h(x) = \begin{cases} -1 & \text{if } x = (e + 1), \\ \frac{4(n-e-2)(n-e-1)}{e(e+3)(2n-3e-3)} & \text{if } x = (e+1, 0^e), \\ \frac{4(n-e-1)}{2(e+1)(e+1)} & \text{if } x = (e, 0^e), \\ 0 & \text{otherwise}. \end{cases}$$

Then the function $g : \mathbb{Z}^n \to \mathbb{R}$ defined by

$$g(x) = \frac{1}{|G|} \sum_{\gamma \in G} h(\gamma \cdot x)$$

satisfies conditions (0), (1), and (2) of Corollary 21.

Note that $\sum_{x \in \mathbb{Z}^n} g(x) = \sum_{x \in \mathbb{Z}^n} h(x)$. Thus if $e \geq 1$ and $n \geq 2e + 2$ and

$$\frac{4(n-e-1)}{2n-3e-3} \left(\frac{n-e-2}{e(e+3)} + \frac{1}{e+1} \right) + \frac{(2e+1)(e+1)}{2e(n-2e-1)} < 1,$$

then there is no $PL(n, e)$-code. Computing the interval of $n$ for which the inequality is satisfied gives the following corollary.

Corollary 23. If $e \geq 18$ and $3e + 21 \leq n \leq \frac{1}{2}e^2 - 20$, then there is no $PL(n, e)$-code.

For $n \leq 3e + 21$, it is likely that another choice of $g$, which takes more care of the case when $n$ is small compared to $e$, would prove the nonexistence of $PL(n, e)$-codes.

It is curious that this method gives almost the same bound as Lepistö’s. We do not have a good explanation for this, but we also do not think the method itself is equivalent to Lepistö’s. On the other hand, numerical experiments suggest that it is unlikely that Corollary 21 by itself, with a clever choice of function, is powerful enough to resolve the Golomb–Welch conjecture.

IV. ALGEBRAIC APPROACHES TO TRANSLATIONAL TILING PROBLEMS

As indicated in Section II discussed in detail in Section III and in following Section IV-A there is a wide variation of methods and techniques how the Golomb–Welch conjecture has been attacked. Unfortunately, it seems to us that none of these approaches is powerful enough to solve the conjecture.

We guess that to solve the Golomb–Welch conjecture new methods and techniques have to be introduced, new conditions under which there exists a (periodic/lattice) tiling of $\mathbb{Z}^n$ by translates of a finite set $V$ will have to be found. Along this line we provide in this section a necessary condition for the
existence of a tiling of \( \mathbb{Z}^n \) by a generic (arbitrary) tile \( V \). This condition is proved by the so called polynomial method. In the second part of this section we briefly mention applications of Fourier analysis in tilings by translates. We state there (without providing a proof) a condition for a tile such that all tilings by this tile are periodic. Most likely this condition cannot be applied to the Golomb–Welch conjecture. However we believe that further development of Fourier analysis methods might contribute to the solution of the Golomb–Welch conjecture in an essential way. Finally, in our quest to solve the Golomb–Welch conjecture we have focused also on tilings by translates of a tile of prime size. Later we looked at this types of tiling in its own right. Hence, we have (re)proved the statement that each tiling by translates of a tile of prime size is periodic, and also that if there is a tiling by a tile of prime size then there is also a lattice tilings by this tile. We believe that this statement can be further strengthen to: All tilings by a tile of prime size are lattice one. A brief outline of the proof of this conjecture for tiles of small size is given. We guess that it might be possible to prove additional properties of these lattice tilings that will show that the Golomb–Welch conjecture is true in the case when the corresponding Lee sphere \( S(n, e) \) is of prime size.

### A. Polynomial Method

We describe the Polynomial method that has been originally introduced by Barnes [27] who applied this method to tilings of a box with bricks [28]. Later, the same method has been rediscovered independently in [29] and [30], where the authors focus on Nivat’s conjecture. Therefore results in [29] and [30] overlap only in Theorem 33 (see the subsection on tiles of prime size).

Let \( \mathcal{T} = \{ V + 1 : 1 \in \mathcal{L} \} \) be a tiling of \( \mathbb{Z}^n \) by translates of \( V \). We define a linear map \( T_\mathcal{T} : \mathbb{Z}[x_1^\pm, \ldots, x_n^\pm] \to \mathbb{Z} \), where \( \mathbb{Z}[x_1^\pm, \ldots, x_n^\pm] \) is the commutative ring of Laurent polynomials generated by \( x_1^\pm, \ldots, x_n^\pm \) such that, for every \((a_1, \ldots, a_n) \in \mathbb{Z}^n\),

\[
T_\mathcal{T}(x_1^{a_1} \cdots x_n^{a_n}) = \begin{cases} 1 & \text{if } (a_1, \ldots, a_n) \in \mathcal{L} \\ 0 & \text{otherwise} \end{cases}
\]

If the tiling \( \mathcal{T} \) is clear from the context we will drop the subscript and write simply \( T \). We note that \( T \) is uniquely determined as the monomials \( x_1^{a_1} \cdots x_n^{a_n} \) form a basis of the ring as a \( \mathbb{Z} \)-module. Let \( Q_V \in \mathbb{Z}[x_1^\pm, \ldots, x_n^\pm] \) be a polynomial associated with \( V \),

\[
Q_V(x_1, \ldots, x_n) = \sum_{(a_1, \ldots, a_n) \in (-V)} x_1^{a_1} \cdots x_n^{a_n}.
\]

Then for any monomial \( x_1^{m_1} \cdots x_n^{m_n} \),

\[
T(x_1^{m_1} \cdots x_n^{m_n})Q_V = \sum_{(a_1, \ldots, a_n) \in (-V)} |\{(a_1 + m_1, \ldots, a_n + m_n)\} \cap \mathcal{L}| \\
= \left|\left(-V + (m_1, \ldots, m_n)\right) \cap \mathcal{L}\right| = 1.
\]

Since the map \( T \) is linear and any polynomial is a linear combination of monomials, we can immediately extend this equality to

\[
T(PQ_V) = P(1, \ldots, 1)
\]

for any polynomial \( P \in \mathbb{Z}[x_1^\pm, \ldots, x_n^\pm] \).

In what follows we will present results proved by utilizing properties of the linear map \( T \) and the polynomial \( Q_V \), i.e., by using the polynomial method.

We start with a technical statement that will be used in the proof of Conjecture 36 for tiles of small size:

**Theorem 24.** Let \( \mathcal{T} \) be a tiling of \( \mathbb{Z}^n \) by translates of \( V \), and let \( a \) be an integer relatively prime to \( |V| \). Then, for any polynomial \( P \in \mathbb{Z}[x_1^\pm, \ldots, x_n^\pm] \), we have

\[
T(PQ_V(x_1^a, \ldots, x_n^a)) = P(1, \ldots, 1).
\]

**Proof.** We start with the case \( a > 0 \). Since the map \( T \) is linear, we only need to prove \( T(MQ(x_1^a, \ldots, x_n^a)) = 1 \) for any monomial \( M \). To see this it suffices to show

\[
T(MQ(x_1^a, \ldots, x_n^a)) \equiv 1(\text{mod } a).
\]

Indeed, we have

\[
T(MQ(x_1^a, \ldots, x_n^a)Q_V) = \sum_{v \in (-V)} T(M \cdot x_1^{a_1} \cdots x_n^{a_n} \cdot Q_V(x_1^a, \ldots, x_n^a)) \\
\geq \sum_{v \in (-V)} 1 = |V|,
\]

because the map \( T \) takes polynomials with nonnegative coefficients to nonnegative values, \( T(MQ_V(x_1^a, \ldots, x_n^a)) \geq 1 \) for all monomials \( M \). On the other hand,

\[
T(MQ(x_1^a, \ldots, x_n^a)Q_V) = Q_V(1^a, \ldots, 1^a) = |V|.
\]

It follows that the equality holds for every term in (8).

For some fixed \( v \in (-V) \), we have \( T(M \cdot x_1^{a_1} \cdots x_n^{a_n} \cdot Q(x_1^a, \ldots, x_n^a)) = 1 \) for every monomial \( M \). Therefore \( T(MQ(x_1^a, \ldots, x_n^a)) = 1 \) for every monomial \( M \).

The congruence \( T(MQ(x_1^a, \ldots, x_n^a)) \equiv 1(\text{mod } a) \) will be proved by induction on the total number \( k \) of prime factors of \( a \). As noted above, \( T(PQ_V) = P(1, \ldots, 1) \) for any polynomial \( P \in \mathbb{Z}[x_1^\pm, \ldots, x_n^\pm] \). Let \( a = p, \) where \( p \) is a prime. Then

\[
T(MQ_V(x_1^p, \ldots, x_n^p)) = T(MQ_V^p) = T(MQ_V^{p-1}Q_V) \\
= (Q_V(1, \ldots, 1))^{p-1} = |V|^{p-1} \equiv 1(\text{mod } p).
\]

since \( T(RQ_V) = R(1, \ldots, 1) \) for any polynomial \( R \). For \( k > 1 \), let \( q \) is a prime factor of \( a \). By induction hypothesis we have

\[
T(M(Q(x_1^q, \ldots, x_n^q)) \equiv 1(\text{mod } q) \]

which in turn implies

\[
T(PQ_V(x_1^q, \ldots, x_n^q)) = P(1, \ldots, 1) \]

for any polynomial \( P \in \mathbb{Z}[x_1^\pm, \ldots, x_n^\pm] \). Hence,

\[
T(MQ_V(x_1^q, \ldots, x_n^q)) = T(MQ_V((x_1^q, \ldots, x_n^q))) \\
= T(MQ_V^{-1}(x_1^q, \ldots, x_n^q)Q_V(x_1^q, \ldots, x_n^q)) \\
= (Q_V(1, \ldots, 1))^{-1} = |V|^{-1} \equiv 1(\text{mod } q)
\]

for any prime factor \( q \) of \( a \). Now \( T(MQ_V(x_1^a, \ldots, x_n^a)) \equiv 1(\text{mod } a) \) follows from the fact that if \( F \equiv 1(\text{mod } q) \) then
$F \equiv 1 \pmod{q^t}$ for any $t \in \mathbb{N}$, and from the Chinese Reminder Theorem.

To finish the proof we need to show that, for any $a > 0$, it is

$$T(PQV(x_1^a, \ldots, x_n^a)) = P(1, \ldots, 1)$$

for any polynomial $P \in \mathbb{Z}[x_1^\pm 1, \ldots, x_n^\pm 1]$. Again, it is sufficient to prove it for monomials. We first show

$$T(MQV(x_1^{-a}, \ldots, x_n^{-a})) \leq 1$$

for any monomial $M$. Suppose that

$$T(Mx_1^{-a_1} \cdots x_n^{-a_n}) = T(Mx_1^{-a_1} \cdots x_n^{-a_n}) = 1$$

for some distinct $v, u \in (-V)$. Then letting $M' = Mx_1^{(v_1-u_1)} \cdots x_n^{(v_n-u_n)}$, we get

$$T(M'QV) \geq T(M'x_1^{-a_1} \cdots x_n^{-a_n}) + T(M'x_1^{-a_1} \cdots x_n^{-a_n}) = 2$$

which contradicts the original property of $QV$. Thus

$$T(MQV(x_1^{-a}, \ldots, x_n^{-a})) \leq 1$$

for all monomials $M$.

Consider the polynomial $MQV(x_1^{-a}, \ldots, x_n^{-a})$. Because $T(MQV(x_1^{-a}, \ldots, x_n^{-a})QV) = QV(1, \ldots, 1) = |V|$ and

$$T(MQV(x_1^{-a}, \ldots, x_n^{-a})) \leq \sum_{v \in V} 1 = |V|,$$

all terms must attain equality. It follows that

$$T(MQV(x_1^{-a}, \ldots, x_n^{-a})) = 1$$

for all monomials $M$.

As an immediate consequence we get:

**Corollary 25.** Let $T = \{V + 1 : 1 \in L\}$ be a tiling of $\mathbb{Z}^n$ by translates of $V$, and let $a$ be an integer relatively prime to $|V|$. Then $T_a = \{aV + 1 : 1 \in L\}$ is a tiling of $\mathbb{Z}^n$ by translates of a “blowout” tile $aV = \{av : v \in V\}$.

**Proof.** Set $S = aV$. Then

$$QS(x_1, \ldots, x_n) = \sum_{(v_1, \ldots, v_n) \in (-V)} x_1^{v_1} \cdots x_n^{v_n} = QV(x_1^a, \ldots, x_n^a).$$

By Theorem 23

$$T(MQS) = T(MQV(x_1^a, \ldots, x_n^a)) = M(1, \ldots, 1) = 1$$

for any monomial $M$. Thus, for any $x \in \mathbb{Z}^n$,

$$|(-S + x) \cap L| = 1,$$

that is, $T_a = \{aV + 1 : 1 \in L\}$ is a tiling of $\mathbb{Z}^n$ by translates of $aV$.

The following corollary can be found in [11]. We provide here a short proof of this result.

**Corollary 26.** (11) Let $T = \{V + 1 : 1 \in L\}$ be a tiling of $\mathbb{Z}^n$ by translates of $V$, and let $a$ be an integer relatively prime to $|V|$. Then $1 + a(v - w) \notin L$ for each $1 \in L$ and $v \neq w \in V$.

**Proof.** By Corollary 25 $T_a = \{aV + 1 : 1 \in L\}$ is a tiling of $\mathbb{Z}^n$ by translates of $aV$, hence $\mathbb{Z}^n = aV + L$. Assume that $1 + a(v - w) \in L$. Then

$$1 + aV = aV + [1 + a(v - w)]$$

but also

$$1 + aV = aV + 1.$$

That is, $1 + aV \in \mathbb{Z}^n$ would be covered by two distinct tiles of $T_a$.

To start building a theory of tilings of $\mathbb{Z}^n$ by translates of a finite tile, and to further exhibit the strength of this method, at the end of this section we provide a necessary condition for the existence of a tiling of $\mathbb{Z}^n$ by translates of a generic (arbitrary) tile $V$.

We start by recalling a famous theorem of Hilbert that will be applied in the proof of this condition.

**Theorem 27** (Nullstellensatz). Let $J$ be an ideal in $\mathbb{C}[x_1, \ldots, x_n]$ and $S \subseteq \mathbb{C}^n$. Denote by $\mathcal{V}(J)$ the set of all common zeros of polynomials in $J$, and by $\mathcal{I}(S)$ the set of all polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ that vanish at all elements of $S$. Then

$$\mathcal{I}(\mathcal{V}(J)) = \sqrt{J} = \{f \in \mathbb{C}[x_1, \ldots, x_n] : f^n \in J \text{ for some } n \geq 1\}.$$  

We can directly apply Hilbert’s Nullstellensatz to prove a Laurent polynomial version of Nullstellensatz.

**Lemma 28.** Let $\{f_i\}_{i \in I} \subseteq \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a set of Laurent polynomials such that there exists no $(x_1, \ldots, x_n) \in (\mathbb{C} \setminus \{0\})^n$ with $f_i(x_1, \ldots, x_n) = 0$ simultaneously for all $i \in I$. Then there exist Laurent polynomials $p_1, \ldots, p_k$ and indices $i_1, \ldots, i_k \in I$ such that

$$f_{i_1}p_1 + \cdots + f_{i_k}p_k = 1.$$

**Proof.** For each $i \in I$, consider a sufficiently large positive integer $n_i$ which makes $(x_1 \cdots x_n)^{n_i-1}f_i \in \mathbb{C}[x_1, \ldots, x_n]$. Then $g_i = (x_1 \cdots x_n)^{n_i-1}f_i$ is not only a polynomial, but also a multiple of $x_1 \cdots x_n$. Consider the ideal $J \subseteq \mathbb{C}[x_1, \ldots, x_n]$ generated by the polynomials $g_i$. By the condition, there is no $x \in (\mathbb{C} \setminus \{0\})^n$ that makes $g_i(x) = 0$ for all $i \in I$. On the other hand, $g_i(x) = 0$ if any one of $x_1, \ldots, x_n$ is zero since the polynomial is a multiple of $x_1 \cdots x_n$. Thus it follows that

$$\mathcal{V}(J) = \{(x_1, \ldots, x_n) \in \mathbb{C}^n : x_1x_2 \cdots x_n = 0\}.$$  

By Hilbert’s Nullstellensatz, $x_1 \cdots x_n \in \mathcal{I}(\mathcal{V}(J)) = \sqrt{J}$, i.e., there exists a positive integer $m$ for which $(x_1 \cdots x_n)^m \in J$.

Let $q_1, \ldots, q_k$ and $i_1, \ldots, i_k$ be the polynomials and indices which make

$$(x_1 \cdots x_n)^m = g_{i_1}q_1 + \cdots + g_{i_k}q_k$$

$$= (x_1 \cdots x_n)^{n_1i_1}f_{i_1}q_1 + \cdots + (x_1 \cdots x_n)^{n_ki_k}f_{i_k}q_k.$$  

Then dividing both sides by $(x_1 \cdots x_n)^m$, we get

$$1 = \frac{q_1}{(x_1 \cdots x_n)^{m-n_{i_1}}} + \cdots + \frac{q_k}{(x_1 \cdots x_n)^{m-n_{i_k}}}$$

The following statement is the main theorem of this subsection.
Theorem 29. Let $V \subseteq \mathbb{Z}^n$ be a tile with at least 2 elements. Then there is a tiling of $\mathbb{Z}^n$ by translates of $V$ only if there exists $(x_1, \ldots, x_n) \in (\mathbb{C} \setminus \{0\})^n$ such that $Q_V(x_1^a, \ldots, x_n^a) = 0$ simultaneously for all $a$ relatively prime to $|V|$.

Proof. Assume that there is no $(x_1, \ldots, x_n) \in (\mathbb{C} \setminus \{0\})^n$ such that $Q_V(x_1^a, \ldots, x_n^a) = 0$ simultaneously for all $a$ relatively prime to $|V|$. By Lemma 28 we obtain Laurent polynomials $P_1, \ldots, P_l$ and integers $a_1, \ldots, a_l$ relatively prime with $|V|$ for which

\[ P_1Q(x_1^{a_1}, \ldots, x_n^{a_1}) + \cdots + P_lQ(x_1^{a_l}, \ldots, x_n^{a_l}) = 1. \tag{9} \]

Replacing all $x_1, \ldots, x_n$ with 1, we get

\[ P_1(1, \ldots, 1) + \cdots + P_l(1, \ldots, 1) = 1/|V|. \tag{10} \]

Suppose that there exists a tiling of $\mathbb{Z}^n$ by translates of $V$. By (9) and (10), for any monomial $M$,

\[ T(M) = T(MP_1Q(x_1^{a_1}, \ldots, x_n^{a_1}) + \cdots + MP_lQ(x_1^{a_l}, \ldots, x_n^{a_l})) = T(MP_1Q(x_1^{a_1}, \ldots, x_n^{a_1}) + \cdots + T(MP_lQ(x_1^{a_1}, \ldots, x_n^{a_1})) \]

\[ T(P_1(1, \ldots, 1) + \cdots + P_l(1, \ldots, 1)) = 1/|V| + \cdots + 1/|V| \]

Because this is not an integer, as $|V| \geq 2$, we arrive at a contradiction. \hfill \Box

Finally we note that it can be proved that there exists a common zero $(x_1, \ldots, x_n) \in (\mathbb{C} \setminus \{0\})^n$ to $Q_V(x_1^a, \ldots, x_n^a) = 0$ for all $a$ relatively prime to $|V|$ if and only if there is a common zero $(x_1, \ldots, x_n)$ with $|x_i| = 1$ for all $i$. Therefore we have a slightly stronger statement.

Theorem 30. Let $V \subseteq \mathbb{Z}^n$ be a tile with at least 2 elements. There exists a tiling of $\mathbb{Z}^n$ by translates of $V$ only if there exists a $(x_1, \ldots, x_n) \in \mathbb{C}^n$ with $|x_i| = 1$ for all $i$, such that $Q_V(x_1^a, \ldots, x_n^a) = 0$ for all $a$ relatively prime to $|V|$.

Proof. As the proof is tedious, we only provide a sketch. We need only to prove that if there is a common solution $x \in (\mathbb{C} \setminus \{0\})^n$ to $Q_V(x_1^a, \ldots, x_n^a) = 0$, then there is also a common solution with $|x_i| = 1$ for all $i$. If we write

\[ Q_V(x_1, \ldots, x_n) = m_1 + m_2 + \cdots + m_{|V|}, \]

where $m_i$ are monomials in $x_1, \ldots, x_n$, then we can also write

\[ Q_V(x_1^a, \ldots, x_n^a) = m_1 + m_2 + \cdots + m_{|V|}. \]

Because this is zero for all $a = k|V| + 1$,

\[ m_1|V|^k + \cdots + m_{|V|}(m_{|V|})^k = 0 \]

for all $k \in \mathbb{Z}$. Thus if we group $m_i$ by its $|V|$-th powers, the powers of $m_i$ contained in a single group shall add up to 0. This means that if we replace $m_i$ with $m_i/|m_i|$, their powers still add up to zero. Therefore if $(x_1, \ldots, x_n)$ is a common solution, then

\[ \left( \frac{x_1}{|x_1|}, \ldots, \frac{x_n}{|x_n|} \right) \]

is also a common solution. \hfill \Box

Example 31. To demonstrate that the above condition is only a necessary one, consider the Lee sphere $V = S(3, 2)$. We know that there is no $PL(3, 2)$-code, i.e., no tiling of $\mathbb{Z}^3$ by $S(3, 2)$. However there is a common root of $Q_V(x_1^a, y^a, z^a) = 0$ for all $a | 3$. In particular take $a = 3$, $y = e^{2\pi i/3}$, and $z = e^{2\pi i/3}$ for example.

One of the main strength of the above theorem is that it is not limited by a special size or by a special shape of the tile. On the other hand, it is difficult to see whether the system has a common root except in special cases. We will see, Theorem 32 that the conditions simplifies if the size of the tile is prime. If it is composite, the condition is hard to interpret. Therefore, it will require additional research to enable one to apply this theorem toward the Golomb–Welch conjecture.

B. Fourier Analysis in Tilings

To our best knowledge Fourier analysis has been used first time in the area of tilings by translates by Lagarias and Wang [10], and then by Kolountzakis and Lagarias [31]. In both these papers a tiling of the line by a function is studied. We note for the interested reader that an introduction to the application of Fourier analysis in tilings has been given in [32]. Using methods described in [32] we have found a sufficient condition for a generic (arbitrary) tile $V$ such that each tiling of $\mathbb{Z}^n$ by $V$ is periodic.

Theorem 32. Let $V$ be a tile. Suppose there exist only finitely many $(z_1, \ldots, z_n)$ with $|z| = 1$ that satisfy

\[ Q_V(z_1^k, \ldots, z_n^k) = 0 \]

simultaneously for all $k$ with $\gcd(k, |V|) = 1$. Then every tiling by $V$ is periodic.

A proof of this result is rather involved. It is a part of a manuscript where we describe our results on translational tilings obtained by Fourier analysis [33]. The above theorem illustrates possibilities how Fourier analysis can contribute to a solution of the Golomb–Welch conjecture.

C. Tiles of Prime Size

As we have seen in Section [14], algebraic properties of the tile $V$ place interesting restrictions on the translational tiling. In this subsection, we focus on a particularly restrictive case, when the size of the tile is prime. As we have already seen, the conclusion of Corollary 25 holds for all integers $a$ relatively prime to $|V|$. This suggests that there are more restrictions on the tiling when $|V|$ has fewer prime divisors. But in the extreme case, when $|V|$ is a prime number, the situation becomes more interesting. For more details of the results presented in the current section, we refer the reader to [29].

The following theorem has been first proved by Szegedy [11]. However, using the language of polynomials makes the proof more natural. We note that an identical proof can be found in [30].

Theorem 33. ([11], [30], [29]) Let $V \subseteq \mathbb{Z}^n$ be a finite set, and $\mathcal{T} = \{ V + l : l \in \mathbb{L} \}$ be a tiling of $\mathbb{Z}^n$ by translates of
If \( |V| = p \) is prime, then \( p(v - w) \) is a period of \( T \) for any \( v, w \in V \).

**Proof.** For any monomial \( M \),
\[
T(MQ_V(x_1^n, \ldots, x_n^n)) = T(MQ_V'x_1) = T(MQ_V'Q_V) = (Q_V(1, \ldots, 1))^{p-1} = p^{p-1} \equiv 0 \pmod{p}
\]

since \( T(RQ_V) = R(1, \ldots, 1) \) for any polynomial \( R \). On the other hand, by definition
\[
T(MQ_V(x_1^n, \ldots, x_n^n)) = \sum_{x \in V} T(Mx_1^{p-1} \cdots x_n^{p-1}) = \sum_{x \in V} T(Mx_1^{p-1} \cdots x_n^{p-1}) = 0.
\]

Since the sum of \( |V| = p \) numbers, each of which is either 0 or 1, is a multiple of \( p \), we conclude that the numbers are either all 0 or all 1. Hence for any \( v = (v_1, \ldots, v_n) \) and \( w = (w_1, \ldots, w_n) \) in \( V \)
\[
T(Mx_1^{p-1} \cdots x_n^{p-1}) = T(Mx_1^{p-1} \cdots x_n^{p-1}) = 0.
\]

It follows that, for any \( x \in \mathbb{Z}^n \), the point \( x \) is in \( \mathcal{L} \) if and only if \( x + p(v - w) \) is in \( \mathcal{L} \). Therefore, \( p(v - w) \) is a period of \( T \).

This theorem already turns the problem of finding all tilings into a finite computation problem. But, if one is interested only in checking existence of tilings, as in the case of the Golomb–Welch conjecture, the problem becomes much simpler. The following theorem is stated in [11].

**Theorem 34.** ([17]) Let \( V = \{0, v_1, \ldots, v_{p-1}\} \subset \mathbb{Z}^n \) be a prime size tile, and suppose that \( \{v_1, \ldots, v_{p-1}\} \) generate \( \mathbb{Z}^n \) as an abelian group. Then there is a tiling of \( \mathbb{Z}^n \) by translates of \( V \) if and only if there is a lattice tiling of \( \mathbb{Z}^n \) by translates of \( V \), i.e., there is group homomorphism \( \phi : \mathbb{Z}^n \to \mathbb{Z}_p \) that restricts to a bijection \( V \to \mathbb{Z}_p \).

**Proof.** Suppose there is any tiling of \( \mathbb{Z}^n \) by translates of \( V \). From Theorem 29 we get a common nonzero solution to \( Q_V(x_1^n, \ldots, x_n^n) = 0 \) for \( 1 \leq a \leq p - 1 \). Letting \( m_k = x_1^{v_1} \cdots x_n^{v_n} \), we can write \( \sum_{k=0}^{p-1} m_k = 0 \) for all \( 1 \leq a \leq p - 1 \). It follows inductively that the elementary symmetric polynomials are \( \sum_{i_1 < i_2 < \cdots < i_p} m_{i_1} \cdots m_{i_p} = 0 \) for \( 1 \leq a \leq p - 1 \). That is, \( (X - m_0) \cdots (X - m_{p-1}) = X^P - P \) for some \( P \in \mathbb{C} \), and because \( m_0 = 1 \) since \( v_0 = 0 \), we further obtain \( P = 1 \). Hence
\[
\{x_1^{v_1} \cdots x_n^{v_n}\}_{0 \leq k \leq p} = \{1, e^{2\pi i/p}, \ldots, e^{2\pi (p-1)/p}\}.
\]

Because \( \{v_1, \ldots, v_{p-1}\} \) generate \( \mathbb{Z}^n \), all \( x_1, \ldots, x_n \) have to be powers of \( e^{2\pi i/p} \). If we write \( x_k = e^{2\pi i a_k/p} \), then the homomorphism
\[
\mathbb{Z}^n \to \mathbb{Z}_p; \quad (y_1, \ldots, y_n) \mapsto \sum_{i=1}^n a_i y_i
\]
restricts to a bijection \( \mathbb{Z}^n \to \mathbb{Z}_p \).

Because the Lee sphere \( S(n, e) \) always contains 0 and generates \( \mathbb{Z}^n \), we have:

**Corollary 35.** Suppose \( n, e \geq 1 \) and \( |S(n, e)| = p \) is prime. Then every \( PL(n, e) \)-code is periodic with period \( p \) in every direction. Moreover, there is a \( PL(n, e) \)-code if and only if there is a linear \( PL(n, e) \)-code.

Thus, in this case, the task of proving the Golomb–Welch conjecture reduces to verifying that there is no homomorphism \( \mathbb{Z}^n \to \mathbb{Z}_p \) that restricts to a bijection \( S(n, e) \to \mathbb{Z}_p \). This was used in [17] to prove nonexistence of \( PL(n, 2) \)-codes for special \( n \). The primality of \( |S(n, e)| \) heavily depends on \( n \) and \( e \). It is very likely that \( |S(n, 2)| = 2n^2 + 2n + 1 \) is prime for infinitely many \( n \), while \( |S(n, 3)| = (2n+1)(2n^2+2n+3)/3 \) is never prime for \( n \geq 2 \).

The restrictiveness of tilings by prime size tiles raises the natural question: Can we classify all such tilings in some sense? This is not a question directly related to the Golomb–Welch conjecture. However we think this illustrates well the strength of using polynomials in tiling problems.

**Conjecture 36.** Let \( V = \{0, v_1, \ldots, v_{p-1}\} \) be a prime size tile, and suppose that \( v_1, \ldots, v_{p-1} \) generate \( \mathbb{Z}^n \) as an abelian group. Then all tilings of \( \mathbb{Z}^n \) by translates \( V \) are lattice.

It turns out there is “universal” tile for this conjecture. Consider the semi-cross \( V_{p-1} = \{0, e_1, \ldots, e_{p-1}\} \subset \mathbb{Z}^{p-1} \). Given any tile \( V \) satisfying the assumptions of Conjecture 36, define a homomorphism \( \phi : \mathbb{Z}^n \to \mathbb{Z}^p \) by \( \phi(e_i) = v_i \). Observe that if \( T = \{V + 1: 1 \in \mathcal{L}\} \) is a tiling of \( \mathbb{Z}^n \) by \( V \), then \( T_0 = \{V_{p-1} + 1: 1 \in \phi^{-1}(\mathcal{L})\} \) is a tiling of \( \mathbb{Z}^{p-1} \) by \( V_{p-1} \). It is clear that if \( \phi^{-1}(\mathcal{L}) \) is a lattice, then \( \mathcal{L} \) is also a lattice. Therefore the following conjecture, which is a special case, is actually equivalent to Conjecture 36.

**Conjecture 37.** For \( p \) a prime, any tiling of \( \mathbb{Z}^{p-1} \) by the semi-cross \( V_{p-1} = \{0, e_1, \ldots, e_{p-1}\} \) is lattice.

We note that an equivalent conjecture, called Corrádi’s conjecture, is stated in [34] in the context of factorization of abelian groups. In the same paper, the conjecture is also verified up to \( p \leq 7 \). However, the proof given by Szőcs relies on a computer search for large cliques in a certain graph. Here we outline a readable proof that makes use of polynomials. Again, more details are presented in [29].

To facilitate our discussion, we introduce new notions and notations, and state several auxiliary results. Let \( T = \{V_{p-1} + 1: 1 \in \mathcal{L}\} \) be a tiling of \( \mathbb{Z}^{p-1} \) by semi-crosses. We use the terminology of coding theory: The elements of \( \mathbb{Z}^{p-1} \) will be called *words* and the elements of \( \mathcal{L} \), the centers of semi-crosses in \( T \), will be called *codewords*.

By a word of type \( [m_1^{a_1}, \ldots, m_s^{a_s}] \) we mean a word having \( a_1 \) coordinates equal to \( m_1, \ldots, a_s \) coordinates equal to \( m_s \), the other coordinates equal to zero. Let \( W, Z \) be words, and the word \( Z - W \) is of type \( [m_1^{a_1}, \ldots, m_s^{a_s}] \). Then \( Z \) will be called a word of type \( [m_1^{a_1}, \ldots, m_s^{a_s}] \) with respect to \( W \). Further, we will say that a word \( V \) is covered by a codeword \( W \) if \( V \) belongs to the semi-cross centered at \( W \). Finally, two words \( A \) and \( B \) coincide in \( t \) coordinates, if they have the same value in \( t \) nonzero coordinates.

The following lemma facilitates analyzing possible configurations of codewords. The proof is essentially computing the elementary symmetric polynomials \( e_k = \sum x_i \cdots x_k \in \mathbb{Z}_p \) in terms of the polynomials \( s_k = \sum x_i \), and using Theorem 24.

**Lemma 38.** Let \( T \) be a tiling of \( \mathbb{Z}^{p-1} \) by semi-crosses, where
\( p \) is a prime. Then for each \( k < p \),
\[
T\left( \sum_{i_1 < \cdots < i_k} x_{i_1}x_{i_2}\cdots x_{i_k} \right) = \frac{(p-1)^k - (-1)^k}{p} + (-1)^k T(1).
\]

In other words, if \( O \) is a codeword then there are \( \frac{1}{p}\binom{p-1}{k} \) codewords of type \([k]\), otherwise there are \( \frac{1}{p}\binom{p-1}{k} - (-1)^k \) codewords of type \([k]\).

In fact, one can calculate the number of codewords of type \([m_1^n, \ldots, m_s^n]\) depending on whether \( 0 \in \mathbb{Z}^{p-1} \) is codeword. However we do not need the lemma in this generality.

Let us write \( i = (1, \ldots, 1) \). For \( k = p - 1 \) we see that a word \( w \) is a codeword if and only if \( w + (1, \ldots, 1) \) is a codeword. For \( k = 2 \), we see that if \( w \) is a codeword then there are \( t = \frac{p-1}{2} \) codewords \( u_1, \ldots, u_t \) of type \([2]\) with respect to \( w \), and \( t \) codewords \( u'_1, \ldots, u'_t \) of type \([-2]\) with respect to \( w \). Since they cannot share 1 or \(-1\) at the same place,
\[
\sum_{i=1}^{t} (u_i - w) = i, \quad \sum_{i=1}^{t} (u'_i - w) = -i.
\]
Let us denote \( U^+_2(w) = \{u_1, \ldots, u_t\} \) and \( U^-_2(w) = \{u'_1, \ldots, u'_t\} \).

It turns out that it is useful to denote codewords in terms of cyclic shifts. Define \( \pi(a_1, \ldots, a_{p-1}) = (a_2, \ldots, a_{p-1}, a_1) \) and write \( \langle w \rangle = \{w, \pi(w), \pi^2(w), \ldots\} \), the set of all cyclic shifts of \( w \).

**Theorem 39.** Conjecture \( \mathbb{N} \) is true for \( p = 2, 3 \).

**Proof.** For \( p = 2 \), it is obvious. For \( p = 3 \), simply note that all words of the lattice generated by \((3, 0)\) and \((1, 1)\) are codewords. As the determinant of the matrix consisting of the two vectors equals to 3, the proof follows. \( \square \)

**Theorem 40.** Conjecture \( \mathbb{N} \) is true for \( p = 5 \).

**Proof.** Suppose that \( w \) is a codeword. There are 6 codewords of type \([2]\), each of them covered by a codeword either of type \([2^2]\) or of type \([2^2, -1]\). Hence, as there are 2 codewords of type \([2]\), there has to be 4 codewords of type \([2, -1]\). We denote this set by \( U^+_2(w) \), and likewise, the set of 4 codewords of type \([-2, 1]\) by \( U^-_2(w) \).

Because \( U^+_2(w) \) has two words, we may assume without loss of generality that \( U^+_2(w) = \{(1, 0, 1, 0)\} \). Also \( U^-_2(w) \) has 4 elements. Thus we can again assume without loss of generality that \( U^-_2(w) = \{(1, 1, -1, 0)\} \).

By casework near \( w \), it can be proved that if \( a \in U^+_2(w) \) or \( a \in U^-_2(w) \) then \( U^+_2(w + a) = U^+_2(w) \) and \( U^-_2(w + a) = U^-_2(w) \). It follows that all words of the lattice generated by \((1, 0, 1, 0), (1, 1, -1, 0)\) and \((5, 0, 0, 0)\) are codewords. This lattice has determinant 5. \( \square \)

Using similar ideas and methods, but considering many more cases, the following theorem can be proved.

**Theorem 41.** Conjecture \( \mathbb{N} \) is true for \( p = 7 \).

We note that in the proof of the above theorems we have not used explicitly the fact that \( p \) is a prime, other than in Lemma \( \mathbb{N} \). We believe that the property distinguishing tilings by semi-crosses of prime size from others is that of being cyclic. A tiling \( T = \{V + 1 : \mathbf{1} \in \mathcal{L}\} \) is called cyclic if there is reordering of coordinates such that, for each codeword \( \mathbf{1} \),
\[
\mathbf{1} \in \mathcal{L} \Rightarrow \langle \mathbf{1} \rangle \subset \mathcal{L};
\]
that is, for any codeword, also all its cyclic shifts are codewords. Indeed, for \( p > 2 \) a prime, the only lattice tiling (up to permutation of coordinates) of \( \mathbb{Z}^{p-1} \) by semi-crosses is
\[
\mathcal{L} = \{ \mathbf{1} \in \mathbb{Z}^{p-1} : p | 1 + 2t_2 + \cdots + (p-1)t_{p-1} \},
\]
which is cyclic. On the other hand, if \( n \) is not a prime, it can be proven that no lattice tiling of \( \mathbb{Z}^{n-1} \) by semi-crosses is cyclic.

For the sake of completeness we note that for any \( n \) there is a unique, up to a congruency, lattice tiling \( \mathbb{Z}^{n-1} \) by semi-crosses. This follows from a statement in \( \mathbb{N} \), that there is a lattice tiling of \( \mathbb{Z}^n \) by a tile \( V \) if and only if there is a homomorphism \( \phi : \mathbb{Z}^n \rightarrow G \), an additive group of order \( |V| \), such that a restriction of \( \phi \) to \( V \) is a bijection, and from the symmetry of the semi-cross.

We guess that finding additional properties about the lattice tilings (assuming Conjecture \( \mathbb{N} \) is true) will enable one to prove the Golomb–Welch conjecture in the case when \(|S(n, e)| \) is a prime.

**V. Research Inspired by the Golomb–Welch Conjecture**

By Google Scholar there are 191 papers citing \( \mathbb{N} \) (1970) and 83 papers citing \( \mathbb{N} \) (1968); this includes papers that cites both. In this section we describe only a few of these papers.

It is very common in mathematics to generalize a problem in order to be able to solve it. Also in the case of the Golomb–Welch conjecture there are several modifications and generalizations. However, to the best of our knowledge, so far none of these generalizations has contributed to the solution of the Golomb–Welch conjecture itself. In the first part of this section we describe some of these generalizations, in the second we will look at generalizations of perfect Lee codes.

A. Generalizations of Perfect Lee Codes and of the Golomb–Welch Conjecture

We start this subsection with a strengthening of the Golomb–Welch conjecture. As mentioned above, the Golomb–Welch conjecture has been proved for all pairs \((n, e)\) where \( 3 \leq n \leq 5 \). In fact all pertinent results proved a stronger statement: There is no tiling of \( \mathbb{R}^n \) with Lee spheres of radii at least two, even with different radii. Still a stronger conjecture has been raised in \( \mathbb{N} \). For obvious reasons it has been formulated in terms of tilings rather than Lee codes.

**Conjecture 42 (\( \mathbb{N} \)).** For \( n \geq 3 \), there does not exist a tiling of \( \mathbb{R}^n \) with Lee spheres of radius at least 1 such that the radius of at least one of them is at least 2.

In the same paper the conjecture is proved for \( n = 3 \).

Now we focus on diameter-d perfect Lee codes, which constitute a generalization of perfect \( e \)-error-correcting Lee codes. Ahlswede et al. (see \( \mathbb{N} \)) introduced diameter perfect codes.
codes for distance regular graphs. Later the notion has been extended to metric spaces. Let $(M, \delta)$ be a metric space. Then a set $C \subseteq M$ is a diameter-$d$ code if $\delta(u, v) \geq d$ for any $u, v \in C$, and a set $A \subseteq M$ is an anticode of diameter $d$ if $\delta(u, v) \leq d$ for all $u, v \in A$. Further, let $S = \{S_i : i \in I\}$ be a family of subsets of an underlying set $M$. Then a set $T \subseteq M$ is called a transversal of $S$ if there is a bijection $f : I \to T$ so that $f(i) \in S_i$. In what follows we restrict ourselves to $M = \mathbb{Z}^n$.

**Definition 43.** Let $C \subseteq \mathbb{Z}^n$. Then $C$ is a diameter-$d$ perfect Lee code in $\mathbb{Z}^n$ if $C$ is a diameter-$d$ code, and there is a tiling $T$ of $\mathbb{Z}^n$ by translates of the anticode of diameter $d - 1$ of maximum size such that $C$ is a transversal of $T$. The diameter-$d$ perfect Lee code in $\mathbb{Z}^n$ will be denoted by $DPL(n, d)$.

Any error-correcting perfect Lee code is also a diameter perfect Lee code. Indeed, it is easy to see that, for $d$ even, the anticode of diameter $d$ of the Lee sphere $S(n, r)$ with $r = \frac{d}{2}$. Thus, for $d$ odd, $PL(n, d)$-codes are $DPL(n, d)$-codes where $e = \frac{d - 1}{2}$. It was proved in [37] that, for $d$ odd, the anticode of diameter $d$ of maximum size is the double-sphere $DS(n, e) = S(n, e) \cup (S(n, e) + e_1)$ with $e = \frac{d - 1}{2}$.

Etzion [38] asks whether the Golomb–Welch conjecture can be generalized to: Other than Minkowski’s lattice [39] $DS(n, 6)$, are there $DPL(n, d)$-codes with $n \geq 3$, and $d > 4$? Buzaglo and Etzion [40] partially proved the conjecture by showing that there is no $DPL(n, 2r + 2)$-code for $r > 2n - 4$ where $n > 2$. Further generalization of Etzion’s conjecture is given in [41], where the notion of a perfect distance-dominating set in a graph is introduced. This notion generalizes notions of perfect error-correcting codes, perfect diameter codes, perfect codes in graphs [42], and perfect dominating sets [43].

The Lee (Manhattan) metric is a special case of $l_p$ metric for $p = 1$. We note that the nonexistence of some perfect codes in $l_p$ metric, $1 \leq p < \infty$ was shown in [19].

**B. Quasi-Perfect Lee codes and $PL(n, 1, q)$-codes**

As mentioned in Introduction an interest in perfect codes in the Lee metric is due to their various applications. As it is widely believed that the Golomb–Welch conjecture is true, i.e., that there are no $PL(n, e)$-codes for $n \geq 3$ and $e > 1$, codes “close” to perfect codes have been introduced and studied. To the best of our knowledge, quasi-perfect Lee codes have been looked at first time in [44]. A code $C \subseteq \mathbb{Z}^n$ ($C \subseteq \mathbb{Z}_q^n$) is called quasi-perfect if the minimum distance of $C$ is $2e + 1$ or $2e + 2$ and each $x \in \mathbb{Z}^n$ ($\mathbb{Z}_q^n$) is at distance at most $e + 1$ from at least one codeword $y \in C$. Quasi-perfect Lee codes in $\mathbb{Z}^n$ are denoted $QPL(n, e)$ and $QPL(n, e, q)$. In [44] $QPL(2, e, q)$-codes have been constructed for all $e > 1$ and all $2e^2 + 2e + 2 \leq q < 2(e + 1)^2 + 2(e + 1) + 1$. A fast algorithm for decoding these codes was presented in [45]. The first $QPL(n, e)$-code with $n > 2$ has been constructed in [18], namely it is shown there that there is a $QPL(3, e)$-code for all $1 \leq e \leq 6$. Unfortunately, it is also proved there that for each $n$ there are only finitely many values of $e$ such that there is a linear $QPL(n, e)$-code. Thus, the property for a code to be a quasi-perfect code in the Lee metric is still too restrictive. The first construction of $QPL(n, e, q)$-codes for infinitely many $n$, based on Cayley graph, has been recently presented in [46] and [47]. In [48] it was shown that these Cayley graphs are in fact Ramanujan graphs. Another construction of $QPL(n, e)$-codes for (possibly infinitely many) $n \equiv 1 (\mod 6)$ has been provided in [49], where also a construction of quasi-perfect codes under $l_p$ metric is given.

$PL(n, 1, q)$-codes constitute the only known class of perfect $e$-error-correcting codes for $n \geq 3$. Therefore, with respect to possible applications, these codes have been looked at more closely. It is stated in [3] that $PL(n, 1, q)$-codes might exist for $q > 2n + 1$ if $2n + 1$ is a perfect square; to support this claim a $PL(4, 1, 3)$-code is constructed there. A complete answer to the question in the case of linear (lattice) codes is given in [49]. It is proved there that: Let $2n + 1 = p_1^{a_1} \cdots p_k^{a_k}$ be the prime factorization of $2n + 1$ and let $p = \prod_{i=1}^k p_i$. Then a linear $PL(n, 1, q)$-code exists if and only if $p | q$. In particular, the smallest $q$, for which there exists a linear $PL(n, 1, q)$-code, equals $p$.

Szabo [50] showed that if $2n + 1$ is not a prime then there exists a non-linear but periodic $PL(n, 1)$-code. Therefore, in this case, there exists a non-linear $PL(n, 1, q)$-code for suitable values of $q$; a characterization of such $q$’s has not been given yet. If $2n + 1$ is a prime then there is the following conjecture.

**Conjecture 44 ([49]).** If $2n + 1$ is a prime then each $PL(n, 1, q)$-code is linear, and it is a periodic extension of the unique, up to a congruence, $PL(n, 1, 2n + 1)$-code.

This conjecture has been proved in [51] for $n = 2, 3$ and in [52] for $n = 5$. Finally, we note that non-periodic $PL(n, 1)$-codes have been constructed in [53].

$PL(n, 1)$-code can be obviously seen as a tiling of the Euclidian space by crosses with arms of length one. In [54], crosses with arms of length half are considered. These crosses might be scaled by two to form a discrete shape. A tiling with this shape is also known as a perfect dominating set. Buzaglo and Etzion prove that a tiling for such a shape exists if and only if $n = 2^m - 1$ or $n = 3^t - 1$, where $t > 0$. The authors also show a strong connection of these tilings to binary and ternary perfect codes in the Hamming scheme.

**VI. CONCLUSIONS**

50 years ago, Golomb and Welch [2] raised a conjecture whose strong version claims that there is no $PL(n, e)$-code for $n \geq 3$ and $e > 1$. In spite of great effort and plenty of papers on the topic, this conjecture is still far from being solved.

To provide a support for their conjecture, Golomb and Welch [3] show that for $n \geq 3$ there exists $e_n$, $e_n$ not specified, such that for any $e > e_n$ there is no $PL(n, e)$-code. For $3 \leq n \leq 5$, the Golomb–Welch conjecture has been proved for all $e \geq 2$ (see [21], [26], and [23]).

It seems that the most difficult case of the Golomb–Welch conjecture is that of $e = 2$. First, the case $e = 2$ is a threshold
case as there is a $PL(n, 1)$-code for all $n$. Second, in [23], the proof of nonexistence of $PL(n, e)$-codes for $3 \leq n \leq 5$ and all $e \geq 2$ has been based on the nonexistence of $PL(n, 2)$-codes for the given $n$. So far the strongest result in this direction is due to Kim [17], where non-existence of $PL(n, 2)$-code is proved for (likely infinitely) many values of $n$. In addition, the nonexistence of linear $PL(n, 2)$ codes for $n \leq 12$ has been proved in [18].

As to the weak version of the Golomb–Welch conjecture, the nonexistence of periodic $PL(n, e)$-codes has been proved by Post in [13] for $n \geq 6$, $e \geq \sqrt{2n} - \frac{3}{2}\sqrt{2} - \frac{3}{4}$. This result of Post was asymptotically improved by Lepistö [15] who showed that there is no periodic $PL(n, e)$-code for $n < (e + 2)^2/2.1$, $e \geq 285$. Further, the proof of nonexistence of $PL(n, e)q$-codes for specific values of $q$ (i.e. for specific periods for $PL(n, e)$-codes) can be found in [20].

As a main part of this paper we provided new results on the Golomb–Welch conjecture. It is proved here that the condition $P_{L}$ can be dropped from both, the result of Post and the result of Lepistö. In addition, we showed (see Corollary [23]) that $PL(n, e)$-codes do not exist for $e \geq 18$ and $3e + 21 \leq n \leq \frac{1}{2}e^2 - 20$.

The above given results have been proved by a variety of clever methods. Anyway, we feel that none of them is strong enough to prove the Golomb–Welch conjecture in its entirety.

In greater detail, Golomb and Welch prove Theorem 5 by using the fact that a tiling of $\mathbb{R}^n$ by the sphere $S(n, e)$, $e$ large enough, induces a packing of $\mathbb{R}^n$ by translates of the cross-polytope with an arbitrarily high density smaller than 1. On the other hand, it is well-known that for $n \geq 3$, the cross-polytope does not tile $\mathbb{R}^n$ by translations, and it can be shown that the packing density of a bounded set that does not tile $\mathbb{R}^n$ is bounded away from 1. To get an explicit bound on $e$ one would need to have an upper bound on the packing density of the cross-polytope. Unfortunately, this is a very difficult question, and such density is known only for $n = 3$ due to Minkowski [39]. We note that the idea of the proof of Theorem 5 has been used by several authors (see e.g. [18]) for generalizations of Lee codes.

In [13], to obtain an upper bound on $e_n$, Post shows the nonexistence of periodic $PL(n, e)$ codes for $3 \leq n \leq 5$ by proving an inequality for the number of intersections of 3-dimensional sectors with Lee spheres. To get the nonexistence of $PL(n, e)$ codes for $n \geq 6$, $e \geq \sqrt{2n} - \frac{3}{2}\sqrt{2} - \frac{3}{4}$ Post considers 6-dimensional sectors. It is likely, that dealing with sectors of dimension $> 6$, would provide a better bound on $e$. However, the number of types how a $k$-dimensional sector can be covered by Lee spheres grows very fast with $k$; thus to get a needed inequality for the number of intersections of $k$-dimensional sectors with Lee spheres would be extremely difficult.

The method used by Lepistö in [15] to prove the nonexistence of $PL(n, e, q)$-codes for any $n, e, q$ satisfying $n < (e + 2)^2/2.1$, $e \geq 285$, and $q \geq 2e + 1$, is technically very involved. At the moment we do not see a way how this method could be used to prove the nonexistence of $PL(n, e)$ codes for additional values of $(n, e)$.

As for the presented linear programming approach, numerical experiments suggest that it is unlikely that Corollary [21] by itself, with a clever choice of function $g$, is powerful enough to resolve the Golomb–Welch conjecture.

We guess that the methods used in [21], [36], and [23] to prove the Golomb–Welch conjecture for small values of $n$, cannot be applied for $n \geq 6$. The method of [21] seems to be applicable only for $n = 3$, the method of [36] is likely computationally infeasible for for $n \geq 4$, and the method applied in [23] leads for slightly bigger $n$ to a system of too many equations.

It is very common in mathematics to generalize a problem in order to be able to solve it. Also in the case of the Golomb–Welch conjecture there are several modifications and generalizations. However, to the best of our knowledge, so far none of these generalization has contributed to the solution of the Golomb–Welch conjecture itself. Some of these generalization have been described in Section [V].

With respect to above stated, we guess that essentially new methods are needed to prove the Golomb–Welch conjecture. Therefore, in Section [V] we have described two new avenues how to attack this conjecture. Using a polynomial method introduced originally by Barnes [27], a necessary condition (see Theorem [29]) for the existence of a tiling of $\mathbb{Z}^n$ by translates of a tile $V$ is proved in Section [IV-A]. We believe this is the first necessary condition for a generic (arbitrary) tile. However, it is difficult to see whether the system has a common root except in special cases. Therefore, it will require additional research to enable one to apply this theorem toward the Golomb–Welch conjecture.

In Section [IV-B] using a Fourier analysis method introduced by Lagarias and Wang [10], we have found a sufficient condition for a generic (arbitrary) tile $V$ such that each tiling of $\mathbb{Z}^n$ by $V$ is periodic (see Theorem [32]). This theorem illustrates possibilities how Fourier analysis can contribute to a solution of the Golomb–Welch conjecture. Therefore we plan to work on further development of this method.

We guess that it will require a great effort to completely solve the Golomb–Welch conjecture. Hence it would be also nice to solve the conjecture at least for a common special case. In Section [IV-C] we focus on tiles of prime size. The reason is that these tiles have several specific properties. Therefore it looks to us promising to try to prove the Golomb–Welch conjecture for these tiles. We note that the interested reader can find more details on the results presented in this section in [29].

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