BRIDGE SPHERES FOR THE UNKNOT ARE TOPOLOGICALLY MINIMAL

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Topologically minimal surfaces were defined by Bachman as topological analogues of geometrically minimal surfaces, and one can associate a topological index to each topologically minimal surface. We show that an \((n + 1)\)-bridge sphere for the unknot is a topologically minimal surface of index at most \(n\).

1. Introduction

Let \(S\) be a closed orientable separating surface embedded in a 3-manifold \(M\). The structure of the set of compressing disks for \(S\), such as how a pair of compressing disks on opposite sides of \(S\) intersects, reveals some topological properties of \(M\). For example, if \(S\) is a minimal genus Heegaard surface of an irreducible manifold \(M\) and \(S\) has a pair of disjoint compressing disks on opposite sides, then \(M\) contains an incompressible surface [Casson and Gordon 1987].

The disk complex \(\mathcal{D}(S)\) of \(S\) is a simplicial complex defined as follows.

- Vertices of \(\mathcal{D}(S)\) are isotopy classes of compressing disks for \(S\).
- A collection of \(k + 1\) vertices forms a \(k\)-simplex if there are representatives for each that are pairwise disjoint.

The disk complex of an incompressible surface is empty. A surface \(S\) is strongly irreducible if \(S\) compresses to both sides and every compressing disk for \(S\) on one side intersects every compressing disk on the opposite side. So the disk complex of a strongly irreducible surface is disconnected. Extending these notions, Bachman [2010] defined topologically minimal surfaces, which can be regarded as topological analogues of (geometrically) minimal surfaces.

A surface \(S\) is topologically minimal if \(\mathcal{D}(S)\) is empty or \(\pi_i(\mathcal{D}(S))\) is nontrivial for some \(i\). The topological index of \(S\) is 0 if \(\mathcal{D}(S)\) is empty, and the smallest \(n\) such that \(\pi_{n-1}(\mathcal{D}(S))\) is nontrivial, otherwise.

Topologically minimal surfaces share some useful properties. For example, if an irreducible manifold contains a topologically minimal surface and an incompressible surface, the manifold is virtually fibered.

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surface, then the two surfaces can be isotoped so that any intersection loop is essential in both surfaces. There exist topologically minimal surfaces of arbitrarily high index [Bachman and Johnson 2010], and see also [Lee 2015] for possibly high index surfaces in \((\text{closed orientable surface}) \times I\). In this paper we consider bridge splittings of 3-manifolds, and show that the simplest bridge surfaces, bridge spheres for the unknot in \(S^3\), are topologically minimal. The main idea is to construct a retraction from the disk complex of a bridge sphere to \(S^{n-1}\) as in [Bachman and Johnson 2010] and [Lee 2015].

**Theorem 1.1.** An \((n + 1)\)-bridge sphere for the unknot is a topologically minimal surface of index at most \(n\).

In particular, the topological index of a 3-bridge sphere for the unknot is two. We conjecture that the topological index of an \((n + 1)\)-bridge sphere for the unknot is \(n\). There is another conjecture that the topological index of a genus \(n\) Heegaard surface of \(S^3\) is \(2n - 1\). This correspondence may be due to the fact that a genus \(n\) Heegaard splitting of \(S^3\) can be obtained as a 2-fold covering of \(S^3\) branched along an unknot in \((n + 1)\)-bridge position.

## 2. Bridge splitting

For a closed 3-manifold \(M\), a **Heegaard splitting** \(M = V^+ \cup_S V^-\) is a decomposition of \(M\) into two handlebodies \(V^+\) and \(V^-\) with \(\partial V^+ = \partial V^- = S\). The surface \(S\) is called a **Heegaard surface** of the Heegaard splitting.

Let \(K\) be a knot in \(M\) such that \(V^\pm \cap K\) is a collection of \(n\) boundary-parallel arcs \(\{a^+_1, \ldots, a^+_n\}\) in \(V^+\). Each \(a^+_i\) is called a **bridge**. The decomposition

\[
(M, K) = (V^+, V^+ \cap K) \cup_S (V^-, V^- \cap K)
\]

is called a **bridge splitting** of \((M, K)\), and we say that \(K\) is in \(n\)-**bridge position** with respect to \(S\). A bridge \(a^{\pm}_i\) cobounds a **bridge disk** \(\Delta^{\pm}_i\) with an arc in \(S\). We can take the bridge disks \(\Delta^{+}_i\) \((i = 1, \ldots, n)\) to be mutually disjoint, and similarly for \(\Delta^{-}_i\) \((i = 1, \ldots, n)\). By a **bridge surface**, we mean \(S - K\). The set of vertices of \(\mathcal{D}(S - K)\) consists of compressing disks for \(S - K\) in \(V^+ - K\) and \(V^- - K\).

Two bridge surfaces \(S - K\) and \(S' - K\) are equivalent if they are isotopic in \(M - K\). An \(n\)-bridge position of the unknot in \(S^3\) is unique for every \(n\) [Otal 1982], so for \(n \geq 2\) it is **perturbed**, i.e., there exists a pair of bridge disks \(\Delta_i^+\) and \(\Delta_j^-\) such that the arcs \(\Delta_i^+ \cap S\) and \(\Delta_j^- \cap S\) intersect at one endpoint. The uniqueness also holds for 2-bridge knots [Scharlemann and Tomova 2008] and torus knots [Ozawa 2011]. However, there are 3-bridge knots that admit multiple 3-bridge spheres [Birman 1976; Montesinos 1976].
3. Proof of Theorem 1.1

Let $S^3$ be decomposed into two 3-balls $B^+$ and $B^-$ with common boundary $S$. Let $K$ be an unknot in $S^3$ which is in $(n+1)$-bridge position with respect to $S$. Then $K \cap B^\pm$ is a collection of $n+1$ bridges $a_i^\pm$ ($i = 1, \ldots, n+1$) in $B^\pm$. We assume that the bridges are arranged with $a_i^+$ adjacent to $a_{i-1}^+$ and $a_{i+1}^-$, $a_i^-\pm$ adjacent to $a_{i-1}^-$ and $a_{i+1}^+$, with $a_i^\pm$ adjacent to $a_{i-1}^\mp$ and $a_{i+1}^\mp$ for $2 \leq i \leq n$, and with $a_n^\pm$ and $a_{n+1}^\mp$. Let $\{\Delta_i^\pm\}$ be a collection of disjoint bridge disks $\Delta_i^\pm$ for $a_i^\pm$ with $\Delta_i^\mp \cap S = b_i^\mp$. We assume that $\text{int } b_i^+ \cap \text{int } b_j^- = \emptyset$ for any $i$ and $j$. See Figure 1 for an example.

Let $P$ be the $(2n+2)$-punctured sphere $S - K$. We define compressing disks $D_i^\pm$ ($i = 1, \ldots, n$) for $P$ in $B^+ - K$ as follows. Let $D_1^+$ be a disk in $B^+ - K$ such that $\partial D_1^- = \partial N(b_1^+)$, where $N(b_1^+)$ is a neighborhood of $b_1^+$ taken in $S$. Similarly, other disks are defined so as to satisfy the following.

$$
\partial D_i^- = \partial N(b_i^-), \\
\partial D_2^+ = \partial N(b_1^+ \cup b_1^- \cup b_2^+), \\
\partial D_2^- = \partial N(b_1^- \cup b_1^+ \cup b_2^-), \\
\vdots \\
\partial D_i^+ = \partial N(b_1^+ \cup b_1^- \cup \cdots \cup b_{i-1}^+ \cup b_{i-1}^- \cup b_i^+), \\
\partial D_i^- = \partial N(b_1^- \cup b_1^+ \cup \cdots \cup b_{i-1}^- \cup b_{i-1}^+ \cup b_i^-), \\
\vdots \\
\partial D_n^+ = \partial N(b_1^+ \cup b_1^- \cup \cdots \cup b_{n-1}^+ \cup b_{n-1}^- \cup b_n^+), \\
\partial D_n^- = \partial N(b_1^- \cup b_1^+ \cup \cdots \cup b_{n-1}^- \cup b_{n-1}^+ \cup b_n^-).
$$

The $\partial D_i^\pm$'s in $P$ are depicted in Figure 2.

Now we define subsets $C_i^\pm$ ($i = 1, \ldots, n$) of the set of vertices of $\mathcal{D}(P)$ as
follows. For odd \(i\), let
\[
\begin{align*}
C_i^+ &= \{D_i^+\}, \\
C_i^- &= \text{essential disks in } B^- - K \text{ that intersect } D_i^+ \\
&\quad \text{and are disjoint from } D_1^+, D_3^+, \ldots, D_{i-2}^+.
\end{align*}
\]
For even \(i\), let
\[
\begin{align*}
C_i^+ &= \text{essential disks in } B^+ - K \text{ that intersect } D_i^- \\
&\quad \text{and are disjoint from } D_2^-, D_4^-, \ldots, D_{i-2}^-,
\end{align*}
\]
\[
C_i^- = \{D_i^-\}.
\]
Note that for all \(i\), \(D_i^\pm\) belongs to \(C_i^\pm\).

**Lemma 3.1.** The collection \(\{C_i^\pm\} (i = 1, \ldots, n)\) is a partition of the set of essential disks in \(B^\pm - K\).

**Proof.** First we show that \(\{C_i^+\} (i = 1, \ldots, n)\) is a partition of the set of essential disks in \(B^+ - K\). We show that any essential disk in \(B^+ - K\) belongs to one and only one \(C_i^+\).

An essential disk in \(B^+ - K\) that intersects \(D_2^-\) belongs to \(C_2^+\) by definition. Let \(E_2 = N(b_1^- \cup b_1^+ \cup b_2^-)\) be the disk in \(S\) such that \(\partial E_2 = \partial D_2^-\).

**Claim 1.** If an essential disk \(D\) in \(B^+ - K\) is disjoint from \(D_2^-\) and \(\partial D\) is in \(E_2\), then \(D\) is isotopic to \(D_1^+ \in C_1^+\).

**Proof of Claim 1.** We assume that \(D\) intersects \(D_1^+\) transversely and minimally, so \(D \cap D_1^+\) consists of arc components. Let \(E_1 = N(b_1^+)\) be the disk in \(S\) such that \(\partial E_1 = \partial D_1^+\). See Figure 3. Suppose that \(D \cap D_1^+ \neq \emptyset\). Consider an outermost
Figure 3. $D_1^+$ in $C_1^+$.

Disk $\Delta$ of $D$ cut off by an outermost arc of $D \cap D_1^+$. By the minimality of $|D \cap D_1^+|$, $\Delta$ cannot lie in the 3-ball $B$ bounded by $D_1^+ \cup E_1$ containing $a_1^+$. So $\Delta$ lies outside of $B$. Let $\overline{D}$ be one of the disks obtained from $D_1^+$ by surgery along $\Delta$ such that $\partial \overline{D}$ bounds a disk $\overline{E}$ in $E_2 - E_1$. Let $p$ be the point $a_2^+ \cap (E_2 - E_1)$ and $q$ be the point $a_3^+ \cap (E_2 - E_1)$.

Suppose $E$ contains $p$. Then the sphere $\overline{D} \cup E$ intersects $a_2^+ \cup b_2^+$ in a single point after a slight isotopy of $\text{int} \ b_2^+$ into $B^-$, a contradiction. So $E$ does not contain $p$, and by similar reasoning $E$ does not contain $q$. Then $E$ is an inessential disk in $E_2 - E_1 - K$, so we can reduce $|D \cap D_1^+|$, a contradiction. Hence $D \cap D_1^+ = \emptyset$. Let $E$ be the disk in $E_2$ such that $\partial E = \partial D$. If $\partial E$ is in $E_1$, then $D$ is isotopic to $D_1^+$. Suppose $\partial E$ is in $E_2 - E_1$. Then $E$ contains neither $p$ nor $q$, since otherwise $D \cup E$ intersects $a_2^+ \cup b_2^+$ or $a_3^+ \cup b_3^+$ in a single point as above. So we get the conclusion that $D$ is isotopic to $D_1^+$.

Therefore if an essential disk in $B^+ - K$ is disjoint from $D_2^-$ and its boundary is in $S - E_2$, then it belongs to one of $C_3^+, \ldots, C_n^+$.

An essential disk in $B^+ - K$ that is disjoint from $D_2^-$ and intersects $D_4^-$ belongs to $C_4^+$ by definition. Let $E_4 = N(b_1^+ \cup b_1^- \cup \cdots \cup b_3^- \cup b_3^+ \cup b_4^+)$ be the disk in $S$ such that $\partial E_4 = \partial D_4^-$. Let $D$ be an essential disk in $B^+ - K$ that is disjoint from $D_2^-$ and $D_4^-$ and such that $\partial D \subset S - E_2$.

Claim 2. If $\partial D$ is in $E_4$ (hence in $E_4 - E_2$), then $D$ is isotopic to $D_3^+ \in C_3^+$.

Proof of Claim 2. We assume that $|D \cap D_3^+|$ is minimal up to isotopy, so $D \cap D_3^+$ consists of arc components. Let $E_3 = N(b_1^+ \cup b_1^- \cup b_2^+ \cup b_2^- \cup b_3^+)$ be the disk in $S$ such that $\partial E_3 = \partial D_3^+$. See Figure 4.

Suppose that $D \cap D_3^+ \neq \emptyset$. Consider an outermost disk $\Delta$ of $D$ cut off by an outermost arc of $D \cap D_3^+$. Without loss of generality, we assume that $\partial \Delta \cap S$ lies in $E_3 - E_2$. Let $\overline{D}$ be one of the disks obtained from $D_3^+$ by surgery along $\Delta$ such
that \( \partial D \) bounds a disk \( \overline{E} \) in \( E_3 - E_2 \). Let \( p \) be the point \( a_2^+ \cap (E_3 - E_2) \) and \( q \) be the point \( a_3^+ \cap (E_3 - E_2) \).

Suppose \( \overline{E} \) contains \( p \). Then the sphere \( \overline{D} \cup \overline{E} \) intersects \( a_2^+ \cup b_2^+ \) in a single point after a slight isotopy, a contradiction. So \( \overline{E} \) does not contain \( p \), and similarly \( \overline{E} \) does not contain \( q \). Then \( \overline{E} \) is an inessential disk in \( E_3 - E_2 - K \), so we can reduce \( |D \cap D_3^+| \) , a contradiction. Hence \( D \cap D_3^+ = \emptyset \). Then, reasoning as we did for Claim 1, we see that \( D \) is isotopic to \( D_3^+ \).

\[ \square \]

Therefore if an essential disk in \( B^+ - K \) is disjoint from \( D_2^- \) and \( D_4^- \) and its boundary is in \( S - E_4 \), then it belongs to one of \( C_5^+, \ldots, C_n^+ \).

In general, let \( E_{2i} = N(b_1^+ \cup \cdots \cup b_{2i-1}^+ \cup b_{2i-1}^- \cup b_{2i}^-) \) be the disk in \( S \) such that \( \partial E_{2i} = \partial D_{2i}^- \). Let \( D \) be an essential disk in \( B^+ - K \) that is disjoint from \( D_2^-, D_4^-, \ldots, D_{2i-2}^+ \) and such that \( \partial D \subset S - E_{2i-2} \).

- If \( \partial D \subset E_{2i} - E_{2i-2} \), then \( D \) is isotopic to \( D_{2i-1}^+ \in C_{2i-1}^+ \).
- If \( D \) intersects \( D_{2i}^- \), then \( D \) belongs to \( C_{2i}^+ \) by definition.
- If \( \partial D \subset S - E_{2i} \), then \( D \) belongs to one of \( C_{2i+1}^+, \ldots, C_n^+ \).

An inductive argument in this way leads to the conclusion that any essential disk in \( B^+ - K \) belongs to one and only one \( C_i^+ \). A similar argument shows that \( \{C_i^-\} \) \((i = 1, \ldots, n)\) is a partition of the set of essential disks in \( B^- - K \). \[ \square \]

The collection of disks \( \{D_1^+, D_1^-, \ldots, D_n^+, D_n^-\} \) spans an \((n - 1)\)-sphere \( S^{n-1} \) in \( \mathcal{D}(P) \). There is no edge in \( \mathcal{D}(P) \) connecting \( C_i^+ \) and \( C_i^- \) by definition. There exists an edge in \( \mathcal{D}(P) \) connecting \( C_i^+ \) and \( C_j^+ \) for \( i \neq j \), e.g., an edge between \( D_i^+ \) and \( D_j^+ \), and there exists an edge in \( \mathcal{D}(P) \) connecting \( C_i^+ \) and \( C_j^- \) for \( i \neq j \), e.g., an edge between \( D_i^+ \) and \( D_j^- \). Hence if we define a map \( \bar{r} \) from the set of vertices of \( \mathcal{D}(P) \) to the set of vertices of \( S^{n-1} \) by

\[
\bar{r}(v) = D_i^+ \quad \text{if} \quad v \in C_i^+,
\]
then $\tilde{r}$ extends to a continuous map from the 1-skeleton of $\mathcal{D}(P)$ to the 1-skeleton of $S^{n-1}$. Since higher-dimensional simplices of $\mathcal{D}(P)$ are determined by 1-simplices, the map $\tilde{r}$ can be extended to a retraction $r : \mathcal{D}(P) \to S^{n-1}$. Hence $\pi_{n-1}(\mathcal{D}(P)) \neq 1$, and the topological index of $P$ is at most $n$.

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