VALUATION OF AMERICAN STRANGLE OPTION: VARIATIONAL INEQUALITY APPROACH

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Abstract. In this paper, we investigate a parabolic variational inequality problem associated with the American strangle option pricing. We obtain the existence and uniqueness of $W_2^{2,1}$, $p$, $loc$ solution to the problem. Also, we analyze the smoothness and monotonicity of two free boundaries. Finally, numerical results of the model based on this problem are described and used to show the boundary properties and the price behavior.

1. Introduction. Various strategies have been studied to reduce the risk of options since the financial crisis. According to Chaput and Ederington [1], strangle and straddle account for more than 80% of options strategies. A strangle option is a strategy that holds a position at the same time in both a call and a put with different strike prices but with the same expiry. If we are expecting large movements in underlying assets, but are not sure which direction the movement will be, we can buy or sell them to reduce the risk exposed by a European call or put option. In particular, a straddle option is one of strangle options where the strike price of the call portion is the same as the strike price of the put portion.

Here we focus on American strangle options. The definition of an American type option is an option contract that allows option holders to exercise their rights at any time before expiry. Since option holders in the American option can exercise their rights at any time before expiry, the pricing of such options is often categorized as the optimal stopping problem or the free boundary problem. In this paper, we

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study the parabolic variational inequality associated with the model of American strangle options pricing. In other words, we will investigate $V(t,s)$ satisfying

$$
\begin{align*}
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} s^2 \partial_{ss} V + (r-q)s \partial_s V - rV &= 0, \\
&\text{if } V > (s - K_2)^+ + (K_1 - s)^+, \quad (t,s) \in (0,T] \times (0,\infty), \\
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} s^2 \partial_{ss} V + (r-q)s \partial_s V - rV &\leq 0, \\
&\text{if } V = (s - K_2)^+ + (K_1 - s)^+, \quad (t,s) \in (0,T] \times (0,\infty), \\
V(T,s) &= (s - K_2)^+ + (K_1 - s)^+, \quad s \in [0,\infty),
\end{align*}
$$

(1.1)

where $r, \sigma, K_1, K_2$ are positive constants with $K_1 < K_2$ and $q$ is a constant with $q \geq 0$. In Appendix A, we present the formulation and the financial background of the problem (1.1).

There are various studies on the American strangle option. Chiarella and Ziogas derived the integral equation satisfying the American strangle option in [3] by using the incomplete Fourier transform method. Qiu [10] gave an alternative method to derive the EEP representation of the American strangle option value and analyzed the properties of the option value and the early exercise boundary. Ma et al. [9] construct tight lower and upper bounds for the price of an American strangle. In addition, there are a variety of studies on parabolic variational inequality arising in option pricing. Yang et al. [12], [15] considered parabolic variational inequalities associated with European-style installment call or put option pricing and obtained the existence and uniqueness of $W^{2,1}_{p,\text{loc}}$ solution to the problem and the monotonicity, smoothness and boundedness properties of free boundaries. Also, Chen et al. [2] proved existence and uniqueness of weak solution in variational inequality in the case of American lookback option with fixed strike price. However, the parabolic variational inequalities in the above researches, have only one free boundary.

Of course, Yang and Yi [13] already considered a parabolic variational inequality problem associated with the American-style continuous-installment options with two free boundaries, the lower obstacle of the variational inequality is a monotone function in spatial variables. In the present paper, variational inequality with two free boundaries does not have monotonicity condition on the lower obstacle function in (1.1). The novelty of this paper is that we analyze more general case of this problem.

The contributions of the paper are threefold: (i) we prove the existence and uniqueness of $W^{2,1}_{p,\text{loc}}((0,T] \times (0,\infty)) \cap C([0,T] \times (0,\infty))$ solution to the parabolic variational inequality (1.1). (ii) We prove that two free boundaries (1.1) are monotone and $C^\infty$-regular. (iii) We prove the existence and uniqueness of $W^2_{p,\text{loc}}((0,\infty))$ solution to the stationary problem of (1.1) and utilize it to show that the free boundaries are bounded.

The rest of this paper is organized as follows. In section 2, we prove the existence and uniqueness of $W^{2,1}_{p,\text{loc}}$ solution to problem (1.1). In section 3, we show that the monotonicity and $C^\infty$-regularity of two free boundaries based on the results in section 2. Moreover, we will prove the starting points of the free boundaries. In section 4, we conduct comparative static analysis of variational inequality (1.1). In section 5, we solve the stationary problem arising from American strangle option and use it to show that two free boundaries are bounded. In section 6, we describe the numerical result applying the finite difference scheme. Appendix A is
the formulation of the model. Appendix B shows that the unique solution to the
problem (1.1) coincides with the expected value of the American strangle option.

2. Existence and uniqueness of a solution. We first transform the degenerate
backward parabolic problem (1.1) into a familiar forward non-degenerate parabolic
problem. Setting

$$\tau = T - t, \quad x = \ln \left( \frac{s}{K_2} \right), \quad Y(\tau, x) = \frac{V(t, s)}{K_2},$$

we have

$$\begin{cases}
\partial_\tau Y - \mathcal{LY} = 0, & \text{if } Y > (e^x - 1)^+ + (\kappa - e^x)^+, \quad (\tau, x) \in [0, T) \times \mathbb{R}, \\
\partial_\tau Y - \mathcal{LY} \geq 0, & \text{if } Y = (e^x - 1)^+ + (\kappa - e^x)^+, \quad (\tau, x) \in [0, T) \times \mathbb{R}, \\
Y(0, x) = (e^x - 1)^+ + (\kappa - e^x)^+, & x \in \mathbb{R},
\end{cases}$$

(2.1)

where

$$\kappa := \frac{K_1}{K_2} \in (0, 1)$$

and

$$\mathcal{LY} := \frac{\sigma^2}{2} \partial_{xx} Y + \left( r - q - \frac{\sigma^2}{2} \right) \partial_x Y - rY.$$  

(2.2)

We now consider the problem in the bounded domain $[0, T) \times (-n, n)$:

$$\begin{cases}
\partial_\tau Y_n - \mathcal{LY}_n = 0, & \text{if } Y_n > (e^x - 1)^+ + (\kappa - e^x)^+, \quad (\tau, x) \in [0, T) \times (-n, n), \\
\partial_\tau Y_n - \mathcal{LY}_n \geq 0, & \text{if } Y_n = (e^x - 1)^+ + (\kappa - e^x)^+, \quad (\tau, x) \in [0, T) \times (-n, n), \\
\partial_x Y_n(\tau, -n) = -e^{-n}, \quad \partial_x Y_n(\tau, n) = e^n, & \tau \in [0, T], \\
Y_n(0, x) = (e^x - 1)^+ + (\kappa - e^x)^+, & x \in [-n, n],
\end{cases}$$

(2.3)

where $n \in \mathbb{N}$ with $n > \ln \left( \frac{2}{\kappa} \right)$.

The lower obstacle and terminal condition function of variational inequality (2.1)
are not monotonic for spatial variable $x$. Thus, we appropriately transform the
value function to have monotonicity. The following lemma provides the existence,
uniqueness and properties of a solution to the above problem.

**Lemma 2.1.** For each fixed $n \in \mathbb{N}$ with $n > \ln \left( \frac{2}{\kappa} \right)$, there exists a unique solution

$$Y_n \in C \left( [0, T] \times [-n, n] \right) \cap W^{2,1}_p \left( ((0, T) \times (-n, n)) \setminus \left( B_p(0, 0) \cup B_p(0, \ln \kappa) \right) \right)$$

(2.4)

to the problem (2.3), where $1 < p < \infty$, $\rho > 0$ and $B_p(0, x_0) = \{ (\tau, x) : \tau^2 + (x - x_0)^2 \leq \rho^2 \}$. Furthermore, if $n \in \mathbb{N}$ is large enough, then we have

$$\begin{cases}
(e^x - 1)^+ + (\kappa - e^x)^+ \leq Y_n \leq e^x + \kappa, \\
\partial_\tau Y_n \geq 0, \quad -e^x \leq \partial_x Y_n \leq e^x.
\end{cases}$$

(2.5)

**Proof.** We first define a penalty function $\beta_\varepsilon \in C^\infty(\mathbb{R})$ ($0 < \varepsilon < 1$) satisfying

$$\begin{cases}
\beta_\varepsilon(t) \leq 0, \quad \beta_\varepsilon'(t) \geq 0, \quad \beta_\varepsilon''(t) \leq 0, & \forall t \in \mathbb{R}; \\
\beta_\varepsilon(t) = 0 & \text{if } t \geq \varepsilon; \quad \beta_\varepsilon(0) = -C_0 \quad \text{for } C_0 = 3(r + q)e^n + 3r; \\
\lim_{\varepsilon \to 0} \beta_\varepsilon(t) = 0 & \text{if } t > 0; \quad \lim_{\varepsilon \to 0} \beta_\varepsilon(t) = -\infty & \text{if } t < 0.
\end{cases}$$

(2.6)
Since the functions \((e^x - 1)^+\) and \((\kappa - e^x)^+\) are not smooth enough, we also define a function \(\varphi_\varepsilon \in C^\infty(\mathbb{R})\) satisfying

\[
\begin{cases}
\varphi_\varepsilon(t) \geq 0, & 0 \leq \varphi_\varepsilon'(t) \leq 1, & \varphi_\varepsilon''(t) \geq 0, & \forall t \in \mathbb{R}; \\
\varphi_\varepsilon(t) = t & \text{if } t \geq \varepsilon; \\
\varphi_\varepsilon(t) = 0 & \text{if } t \leq -\varepsilon; \\
\lim_{\varepsilon \to 0} \varphi_\varepsilon(t) = t^+, & \forall t \in \mathbb{R}.
\end{cases}
\]

(2.7)

We then consider the following approximation of the problem (2.3):

\[
\begin{cases}
\partial_x Y_{n,\varepsilon} - \mathcal{L}Y_{n,\varepsilon} + \beta_\varepsilon (Y_{n,\varepsilon} - \varphi_\varepsilon(e^x - 1) - \varphi_\varepsilon(\kappa - e^x)) = 0, \\
(\tau, x) \in [0, T) \times (-n, n), \\
\partial_x Y_{n,\varepsilon}(\tau, -n) = -e^{-n}, & \partial_x Y_{n,\varepsilon}(\tau, n) = e^n, & \tau \in [0, T], \\
Y_{n,\varepsilon}(0, x) = \varphi_\varepsilon(e^x - 1) + \varphi_\varepsilon(\kappa - e^x), & x \in [-n, n],
\end{cases}
\]

(2.8)

By Schauder’s fixed point theorem, the above problem (2.8) has a unique \(W_p^{2,1}\) solution, see [14]. We next claim that

\[
\varphi_\varepsilon(e^x - 1) + \varphi_\varepsilon(\kappa - e^x) \leq Y_{n,\varepsilon} \leq e^x + \kappa
\]

(2.9)

for sufficiently large \(n\). Observe from (2.7) that

\[
t^+ \leq \varphi_\varepsilon(t) \leq (t + \varepsilon)^+, & \forall t \in \mathbb{R}.
\]

(2.10)

Then we deduce from (2.7) and (2.10) that

\[
\partial_x [\varphi_\varepsilon(e^x - 1) + \varphi_\varepsilon(\kappa - e^x)] - \mathcal{L}[\varphi_\varepsilon(e^x - 1) + \varphi_\varepsilon(\kappa - e^x)] \\
+ \beta_\varepsilon (\varphi_\varepsilon(e^x - 1) + \varphi_\varepsilon(\kappa - e^x) - \varphi_\varepsilon(e^x - 1) - \varphi_\varepsilon(\kappa - e^x)) \\
= -\mathcal{L}[\varphi_\varepsilon(e^x - 1) + \varphi_\varepsilon(\kappa - e^x)] + \beta_\varepsilon(0) \\
= -\frac{\sigma^2}{2} [\varphi_\varepsilon''(e^x - 1) + \varphi_\varepsilon''(\kappa - e^x)] e^{2x} + (q - r) [\varphi_\varepsilon'(e^x - 1) - \varphi_\varepsilon'(\kappa - e^x)] e^x \\
+ r [\varphi_\varepsilon(e^x - 1) + \varphi_\varepsilon(\kappa - e^x)] - C_0 \\
\leq 2(q + r)e^x + r [(e^x - 1 + \varepsilon)^+ + (\kappa - e^x + \varepsilon)^+] - C_0 \\
\leq 2(q + r)e^x + r [(e^x + \varepsilon) + (\kappa + \varepsilon)] - C_0 \\
\leq (2q + 3r)e^x + 3r - C_0 \leq 3(q + r)e^n + 3r - C_0 = 0.
\]

Furthermore, we see from the boundary conditions in (2.8) that if \(0 < \varepsilon < \frac{\kappa}{2}\),

\[
\begin{cases}
\varphi_\varepsilon(e^x - 1) + \varphi_\varepsilon(\kappa - e^x) = Y_{n,\varepsilon}(\tau, x), & \tau = 0, \\
\partial_x [\varphi_\varepsilon(e^x - 1) + \varphi_\varepsilon(\kappa - e^x)] = -e^{-n} = \partial_x Y_{n,\varepsilon}(\tau, x), & x = -n, \\
\partial_x [\varphi_\varepsilon(e^x - 1) + \varphi_\varepsilon(\kappa - e^x)] = e^n = \partial_x Y_{n,\varepsilon}(\tau, x), & x = n.
\end{cases}
\]

By the comparison principle, we get

\[
\varphi_\varepsilon(e^x - 1) + \varphi_\varepsilon(\kappa - e^x) \leq Y_{n,\varepsilon}.
\]

(2.11)
On the other hand, it follows from (2.6) and (2.7) that if \(0 < \varepsilon < \frac{\kappa}{2} < \frac{1}{2}\),
\[
\partial_\tau [e^\tau + \kappa] - \mathcal{L} [e^\tau + \kappa] + \beta_\varepsilon (e^\tau + \kappa - \varphi_\varepsilon (e^\tau - 1) - \varphi_\varepsilon (\kappa - e^\tau)) = -\mathcal{L} [e^\tau + \kappa] + \beta_\varepsilon (e^\tau + \kappa - \varphi_\varepsilon (e^\tau - 1) - \varphi_\varepsilon (\kappa - e^\tau))
\]
\[
= q e^\tau + r \kappa + \beta_\varepsilon (e^\tau - \varphi_\varepsilon (e^\tau - 1) + \kappa - \varphi_\varepsilon (\kappa - e^\tau))
\]
\[
\geq q e^\tau + r \kappa + \beta_\varepsilon (1 - \varepsilon) - q e^\tau + r \kappa \geq 0.
\]
Also, from the boundary conditions in (2.8), we get
\[
\frac{\partial z}{\partial n} [e^\tau + \kappa] = e^{-\varepsilon} \geq -e^{-\varepsilon} = \partial_x Y_{n,\varepsilon}(\tau, x), \quad y = n,
\]
\[
\frac{\partial z}{\partial n} [e^\tau + \kappa] = e^{\varepsilon} = \partial_x Y_{n,\varepsilon}(\tau, x), \quad y = n.
\]
By the comparison principle, we obtain
\[
e^{\varepsilon} + \kappa \geq Y_{n,\varepsilon}.
\]
(2.12)

Therefore, the claim (2.9) follows from (2.11) and (2.12).

We see from (2.9) that \(-C_0 \leq \beta_\varepsilon (Y_{n,\varepsilon} - \varphi_\varepsilon (e^\tau - 1) - \varphi_\varepsilon (\kappa - e^\tau)) \leq 0\). Employing a \(C^0\) estimate (see [8, Theorem 6.33]), we get
\[
\|Y_{n,\varepsilon}\|_{C^{0,\alpha/2}([0,T] \times [-n, n])} \leq c
\]
for some constant \(c > 0\) which is independent of \(\varepsilon\). We then apply the method in [5] to deduce that
\[
Y_{n,\varepsilon} \rightarrow Y_n \quad \text{in} \quad W^{2,1}_p ([0,T] \times (-n, n)) \setminus (B_\rho(0,0) \cup B_\rho(0, \ln \kappa))
\]
and
\[
Y_{n,\varepsilon} \rightarrow Y_n \quad \text{in} \quad C ([0,T] \times [-n, n])
\]
as \(\varepsilon \rightarrow 0^+\), where \(Y_n\) is the solution to the problem (2.3). Hence, (2.4) follows immediately from (2.9).

Now let us prove (2.5). For any small \(\delta > 0\), we see from (2.3) and (2.4) that the function \(Y_n(\tau + \delta, x)\) satisfies
\[
\begin{cases}
\partial_\tau Y_n(\tau + \delta, x) - \mathcal{L} Y_n(\tau + \delta, x) = 0, \\
\text{if} \quad Y_n(\tau + \delta, x) > (e^\tau - 1)^+ + (\kappa - e^\tau)^+, \quad (\tau, x) \in [0, T - \delta) \times (-n, n)
\end{cases}
\]
\[
\begin{cases}
\partial_\tau Y_n(\tau + \delta, x) - \mathcal{L} Y_n(\tau + \delta, x) \geq 0, \\
\text{if} \quad Y_n(\tau + \delta, x) = (e^\tau - 1)^+ + (\kappa - e^\tau)^+, \quad (\tau, x) \in [0, T - \delta) \times (-n, n)
\end{cases}
\]
\[
\begin{cases}
\partial_x Y_n(\tau + \delta, -n) = -e^{-\varepsilon}, \quad \partial_x Y_n(\tau + \delta, n) = e^{\varepsilon}, \quad \tau \in [0, T - \delta],
\end{cases}
\]
\[
Y_n(0 + \delta, x) \geq (e^\tau - 1)^+ + (\kappa - e^\tau)^+ = Y_n(0, x), \quad x \in [-n, n].
\]

Applying the monotonicity of solution of variational inequality with respect to initial value (see [5]), we obtain
\[
Y_n(\tau + \delta, x) \geq Y_n(\tau, x), \quad \forall (\tau, x) \in [0, T - \delta) \times (-n, n),
\]
which yields the first inequality in (2.5). For the second inequalities in (2.5), we differentiate (2.8) with respect to \(x\), then we have
Then we conclude from the maximum principle that
\[
(2.3)
\]
and that the set
\[
\partial U \quad (\tau,x) \in [0,T) \times (-n,n),
\]
where \( Z := \partial_x Y_{n,\varepsilon} \) and \( \beta'_\varepsilon(\cdots) = \beta'_\varepsilon(Y_{n,\varepsilon} - \varphi_\varepsilon(e^x - 1) - \varphi_\varepsilon(\kappa - e^x)) \). Letting \( Z_1 := Z + e^x \), we see from (2.13) that \( Z_1 \) satisfies
\[
\begin{align*}
\partial_t Z_1 - \mathcal{L}Z_1 + \beta'_\varepsilon(\cdots)Z_1 \\
= -L[e^x] + \beta'_\varepsilon(\cdots) [\varphi'_\varepsilon(e^x - 1) - \varphi'_\varepsilon(\kappa - e^x) + 1] e^x, \\
\quad (\tau,x) \in [0,T) \times (-n,n), \\
Z_1(\tau,-n) = 0, \\
Z_1(\tau,n) = 2e^n, \\
Z_1(0,x) = [\varphi'_\varepsilon(e^x - 1) - \varphi'_\varepsilon(\kappa - e^x) + 1] e^x, \\
x \in [-n,n],
\end{align*}
\]
Then we conclude from the maximum principle that \( Z_1 \geq 0 \) in \([0,T) \times [-n,n] \), that is, \( \partial_x Y_{n,\varepsilon} \geq -e^x \) in \([0,T) \times [-n,n] \). Similarly, we set \( Z_2 := Z - e^x \) and then deduce that
\[
\begin{align*}
\partial_t Z_2 - \mathcal{L}Z_2 + \beta'_\varepsilon(\cdots)Z_2 \\
= -L[e^x] + \beta'_\varepsilon(\cdots) [\varphi'_\varepsilon(e^x - 1) - \varphi'_\varepsilon(\kappa - e^x) - 1] e^x \leq 0, \\
\quad (\tau,x) \in [0,T) \times (-n,n), \\
Z_2(\tau,-n) = -2e^{-n} \leq 0, \\
Z_2(\tau,n) = 0, \\
Z_2(0,x) = [\varphi'_\varepsilon(e^x - 1) - \varphi'_\varepsilon(\kappa - e^x) - 1] e^x \leq 0, \\
x \in [-n,n].
\end{align*}
\]
It follows from the maximum principle that \( Z_2 \leq 0 \), and hence that \( \partial_x Y_{n,\varepsilon} \leq e^x \). Thus we obtain (2.5).

To prove the uniqueness, suppose that \( Y_n \) and \( \widetilde{Y}_n \) are two solutions to the problem (2.3) and that the set
\[
\mathcal{N} := \left\{ (\tau,x) \in [0,T] \times [-n,n] : Y_n(\tau,x) < \widetilde{Y}_n(\tau,x) \right\}
\]
is non-empty. Observe from (2.4) that if \( (\tau,x) \in \mathcal{N} \),
\[
\widetilde{Y}_n(\tau,x) > (e^x - 1)^+ + (\kappa - e^x)^+,
\]
and hence
\[
\partial_t \widetilde{Y}_n - \mathcal{L}\widetilde{Y}_n = 0.
\]
Therefore, the function \( U := \widetilde{Y}_n - Y_n \) satisfies
\[
\begin{align*}
\partial_t U - \mathcal{L}U &\leq 0, \quad \text{in } \mathcal{N}, \\
\partial_x U &\leq 0, \quad \text{on } \partial_\rho \mathcal{N} \cap ([0,T] \times \{n\} \cup [0,T] \times \{n\}) \\
\partial_x U &\leq 0, \quad \text{on } \partial_\rho \mathcal{N} \setminus ([0,T] \times \{n\} \cup [0,T] \times \{n\}).
\end{align*}
\]
where $\partial_\rho N$ is the parabolic boundary of the domain $N$. Then it follows from the ABP maximum principle (see [11]) that $U \leq 0$ in $N$, which contradicts the definition of the set $N$. Hence $N = 0$ and $Y_n \geq \tilde{Y}_n$. Similarly, we conclude that $Y_n \leq \tilde{Y}_n$, and finally that $Y_n = \tilde{Y}_n$.

**Theorem 2.2.** There exists a unique solution

$$Y \in C([0, T] \times \mathbb{R}) \cap W^{2,1}_p \left(((0, T) \times (-R, R)) \setminus (B_\rho(0, 0) \cup B_\rho(0, \ln \kappa))\right)$$

to the problem (2.1) for all $R > \ln \left(\frac{2}{\kappa}\right), \rho > 0$ and $1 < \rho < \infty$. Moreover, we have

$$(e^x - 1)^+ + (\kappa - e^x)^+ \leq Y \leq e^x + \kappa,$$

(2.14)

$$\partial_x Y \geq 0, \quad -e^x \leq \partial_y Y \leq e^x.$$  

(2.15)

**Proof.** Since the solution $Y_n$ of the problem (2.3) belongs to $W^{2,1}_p_{\text{loc}} \left((0, T) \times (-n, n)\right)$, we can rewrite the problem (2.3) as

$$
\begin{cases}
\partial_t Y_n - \mathcal{L} Y_n = f(\tau, x), & \text{in } [0, T) \times (-n, n), \\
\partial_\xi Y_n(\tau, -n) = e^{-n}, \quad \partial_\xi Y_n(\tau, n) = e^n, & \tau \in [0, T], \\
Y_n(0, x) = (e^x - 1)^+ + (\kappa - e^x)^+, & x \in [-n, n],
\end{cases}
$$

(2.16)

where $f \in L^p_{\text{loc}}((0, T) \times (-n, n))$ and

$$f(\tau, x) = \chi_{\{Y_n = e^x - 1\}}(\tau, x) \cdot (qe^x - r) + \chi_{\{Y_n = \kappa - e^x\}}(\tau, x) \cdot (-qe^x + r\kappa) \quad \text{a.e. in } [0, T) \times (-n, n).$$

(2.17)

Here, $\chi_A$ denotes the characteristic function of the set $A$. Then we see that

$$|f(\tau, x)| \leq c(R) \quad \text{for } (\tau, x) \in [0, T) \times [-R, R],$$

where the constant $c(R)$ depends on $R$, but is independent of $n$. Therefore, it follows from the $W^{2,1}_p$ estimates (see [8]) and (2.4) that for $n > R > \ln \left(\frac{2}{\kappa}\right)$,

$$\|Y_n\|_{W^{2,1}_p((0, T) \times (-R, R))} \leq c \left(\|Y_n\|_{L^\infty([0, T) \times (-R, R))} + \|e^x - 1\|_{C^2([-R, -\rho] \cup [\rho, R])} + \|e^x - 1\|_{C^2([-R, \ln \kappa - \rho] \cup [\ln \kappa + \rho, R])} + \|f\|_{L^\infty((0, T) \times (-R, R))}\right) \leq c(R)$$

for some constant $c(R)$ which is independent of $n$. Letting $n \to \infty$, we deduce that, up to a subsequence,

$$Y_n \to Y^R \quad \text{in } W^{2,1}_p \left(((0, T) \times (-R, R)) \setminus (B_\rho(0, 0) \cup B_\rho(0, \ln \kappa))\right).$$

In addition, we obtain from the Sobolev embedding theorem that

$$Y_n \to Y^R \quad \text{in } C([0, T) \times (-R, R)) \quad \text{and} \quad \partial_\xi Y_n \to \partial_\xi Y^R \quad \text{in } C([0, T) \times (-R, R)).$$

We now define $Y := Y^R$ if $x \in [-R, R]$. Then it is obvious that $Y$ is well-defined and that $Y$ is the solution to the problem (2.1). Since $Y^R, \partial_\xi Y^R \in C([0, T) \times (-R, R))$, we have $Y, \partial_\xi Y \in C([0, T) \times \mathbb{R}$). Furthermore, the $C^2$ estimate yields $Y \in C((0, T] \times \mathbb{R})$ and the inequalities (2.14) and (2.15) follow from (2.4) and (2.5), respectively. The proof of the uniqueness is the same as that of Lemma 2.1.  

$\square$
3. Analysis of the free boundaries. In this section we analyze the free boundaries of variational inequality (2.1). If \( q \neq 0 \), we will show that the variation inequality (2.1) has two free boundaries. Since the two free boundaries interact with each other, it is not easy to prove the monotonicity and smoothness of the free boundaries.

Let \( D = [0, T] \times \mathbb{R} \) be the whole region. We denote
\[
E = \{ (\tau, x) \in D : Y(\tau, x) = (e^x - 1)^+ + (\kappa - e^x)^+ \} \quad \text{(the exercise region)},
\]
\[
C = \{ (\tau, x) \in D : Y(\tau, x) > (e^x - 1)^+ + (\kappa - e^x)^+ \} \quad \text{(the continuation region)}.
\]
The exercise region \( E \) is a disjoint union of three subregions \( E^0, E^A, E^B \) of \( E \), where
\[
E^0 = \{ (\tau, x) \in D : Y(\tau, x) = 0 \}, \quad E^A = \{ (\tau, x) \in D : Y(\tau, x) = (\kappa - e^x)^+ > 0 \}, \quad E^B = \{ (\tau, x) \in D : Y(\tau, x) = (e^x - 1)^+ > 0 \}.
\]

Theorem 2.2 shows that
\[
\partial_\tau Y - (\kappa - e^x) \leq 0 \quad \text{and} \quad \partial_x Y - (e^x - 1) \leq 0.
\]
Hence we can define the free boundaries
\[
A(\tau) = \sup \{ x : x \leq \ln \kappa, Y(\tau, x) = \kappa - e^x \}, \quad \tau > 0 \quad \text{(the continuation region)};
\]
\[
B(\tau) = \inf \{ x : x \geq 0, Y(\tau, x) = e^x - 1 > 0 \}, \quad \tau > 0.
\]

We note that the free boundary \( A(\tau) \) separates \( E^A \) from \( C \) and that the free boundary \( B(\tau) \) separates \( E^B \) from \( C \).

Let us denote
\[
S^A(\tau) = e^{A(\tau)} \cdot K_2 \quad \text{and} \quad S^B(\tau) = e^{B(\tau)} \cdot K_2.
\]

Then, the \( S^A(\tau) \) and \( S^B(\tau) \) are the free boundaries of the variational inequality (1.1). When the underlying asset hits below \( S^A(\tau) \) or above \( S^B(\tau) \), the option holder can benefit by exercising his/her rights. Also, the region surrounded by free boundaries \( S^A(\tau) \) and \( S^B(\tau) \) is the continuation region \( C \) of American strangle options.

**Lemma 3.1.** For any \( \tau \in (0, T) \), \( Y(\tau, \ln \kappa) > 0 \) and \( Y(\tau, 0) > 0 \). Accordingly, we have
\[
\{ 0 < \tau \leq T, x = \ln \kappa \}, \{ 0 < \tau \leq T, x = 0 \} \subset C.
\]

**Proof.** Let us first show that \( Y(\tau, 0) > 0 \) for all \( \tau \in (0, T] \). Suppose the assertion is false. Since \( Y(\tau, 0) \geq (e^x - 1)^+ + (\kappa - e^x)^+|_{x=0} = 0 \), it follows that there exists \( \tau_0 > 0 \) such that \( Y(\tau_0, 0) = 0 \). Then we deduce from \( Y(0, 0) = 0 \) and \( \partial_x Y \geq 0 \) that \( Y(\tau, 0) = 0 \) for all \( 0 \leq \tau \leq \tau_0 \). By Theorem 2.2, we obtain that for \( x \geq 0 \),
\[
Y - (e^x - 1) \geq 0 \quad \text{and} \quad \partial_x (Y - (e^x - 1)) = \partial_x Y - e^x \leq 0.
\]
This forces \( Y(\tau, x) = e^x - 1 \) and, in consequence, \( \partial_x Y(\tau, x) = e^x \) for all \( (\tau, x) \in (0, \tau_0) \times (0, \infty) \). On the other hand, by Theorem 2.2 and the Sobolev embedding theorem, there exists \( \alpha \in (0, 1) \) such that \( Y(\tau, \cdot) \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}) \) for a.e. \( \tau \in (0, T) \). Therefore, there exists \( \tau_1 \in (0, \tau_0) \) such that
\[
\partial_x Y(\tau_1, x) \geq \frac{1}{2}, \quad \forall x \geq -\varepsilon,
\]
for some small \( \varepsilon > 0 \). Then it follows from \( Y(\tau_1, 0) = 0 \) that
\[
Y(\tau_1, -\varepsilon) = -\int_{-\varepsilon}^{0} \partial_x Y(\tau_1, x) \, dx \leq -\frac{\varepsilon}{2} < 0.
\]
This contradicts the fact that \( Y \geq (e^x - 1)^+ + (\kappa - e^x)^+ \geq 0 \).

We next show that \( Y(\tau, \ln \kappa) > 0 \) for all \( \tau \in (0, T] \). If the assertion is not true, then we deduce from \( Y(\tau, \ln \kappa) \geq (e^x - 1)^+ + (\kappa - e^x)^+|_{x=\ln \kappa} = 0 \) that there exists \( \tau_0 > 0 \) such that \( Y(\tau_0, \ln \kappa) = 0 \). Hence we have \( Y(\tau, \ln \kappa) = 0 \) for all \( 0 \leq \tau \leq \tau_0 \).

Observe from Theorem 2.2 that for \( x \leq \ln \kappa \),
\[
Y - (\kappa - e^x) > 0 \quad \text{and} \quad \partial_x (Y - (\kappa - e^x)) = -\partial_x (Y - (\kappa - e^x)) = -\partial_x Y - e^x \leq 0.
\]

We thus get \( Y(\tau, x) = \kappa - e^x \) and \( \partial_x Y(\tau, x) = -e^x \) for all \( (\tau, x) \in (0, \tau_0) \times (-\infty, \ln \kappa) \).

Since \( Y(\tau, \cdot) \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}) \) for a.e. \( \tau \in (0, T) \), there exist \( \tau_2 \in (0, \tau_0) \) and small \( \varepsilon > 0 \) such that
\[
-2\kappa \leq \partial_x Y(\tau_2, x) \leq -\frac{\kappa}{2}, \quad \forall x \in (\ln \kappa - \varepsilon, \ln \kappa + \varepsilon).
\]

Therefore, we obtain
\[
Y(\tau_2, \ln \kappa + \varepsilon) = Y(\tau_2, \ln \kappa + \varepsilon) - Y(\tau_2, \ln \kappa) = \int_{\ln \kappa}^{\ln \kappa + \varepsilon} \partial_x Y(\tau_2, x) \, dx \leq -\frac{\kappa \varepsilon}{2} < 0,
\]
a contradiction.

\[\square\]

**Lemma 3.2.** If \( \tau > 0 \), \( Y \) is always positive, i.e.,
\[
Y(\tau, x) > 0 \quad \text{for all} \quad \tau > 0 \quad \text{and} \quad x \in \mathbb{R}.
\]

**Proof.** We observe from (2.1) and Lemma 3.1 that
\[
\begin{align*}
\partial_x Y - L Y &\geq 0, \quad (\tau, x) \in (0, T] \times (\ln \kappa, 0), \\
Y(\tau, \ln \kappa) &> 0, \quad Y(\tau, 0) > 0, \quad \tau \in (0, T], \\
Y(0, x) &= (e^x - 1)^+ + (\kappa - e^x)^+ > 0, \quad x \in [\ln \kappa, 0].
\end{align*}
\]

Therefore, it follows from the strong maximum principle that
\[
Y(\tau, x) > 0 \quad \text{for} \quad (\tau, x) \in (0, T] \times (\ln \kappa, 0).
\]

Moreover, (2.14) and Lemma 3.1 show that
\[
Y(\tau, x) > 0 \quad \text{for} \quad (\tau, x) \in (0, T] \times ((-\infty, \ln \kappa) \cup [0, \infty)),
\]
which completes the proof. \[\square\]

We now prove the regularity and the monotone property of the free boundaries. In addition, we describe the limiting behavior of the free boundaries as time to maturity goes to zero.

**Theorem 3.3.**

1. If \( q = 0 \), then \( E^B \) is the empty set, and hence \( B(\tau) \) does not exist.
2. If \( q > 0 \), then \( B(\tau) \) is smooth and strictly increasing in \( (0, T] \). Moreover,
\[
B(0) := \lim_{\tau \to 0^+} B(\tau) = \max \left\{ 0, \ln \left( \frac{r}{q} \right) \right\} = \ln \left( \max \left\{ 1, \frac{r}{q} \right\} \right).
\]

**Proof.** (1) Suppose that \( E^B \neq \emptyset \). We observe from the definition of \( E^B \) that
\[
(\partial_x - L) Y = (\partial_x - L)(e^x - 1) = -L[e^x - 1] = q e^x - r \quad \text{in} \quad E^B.
\]

When \( q = 0 \), it follows that \((\partial_x - L) Y = -r < 0 \) in \( E^B \). On the other hand, (2.1) leads to \((\partial_x - L) Y \geq 0 \) in \( E^B \). This is a contradiction. Therefore, we conclude that \( E^B = \emptyset \), and hence that \( B(\tau) \) does not exist.
On the other hand, we discover from (2.1) that
\( B(\tau) \geq B(0) = x_0 > 0, \ \forall \tau \geq 0. \)
It follows from the definition of \( B(\tau) \) that
\[ Y(\tau, x) > e^x - 1, \ \forall (\tau, x) \in (0, T) \times (0, x_0), \]
and hence
\[
(\partial_x - \mathcal{L})[Y - (e^x - 1)] = 0 + \mathcal{L}[e^x - 1] = -q e^x + r \leq -q(e^x - 1) < 0
\]
in \((0, T) \times (0, x_0),\) where we have used the fact that \( r \leq q. \) Then we obtain
\[
\partial_x Y(0, x) < \mathcal{L}[Y(0, x) - (e^x - 1)] = 0, \ \forall x \in (0, x_0),
\]
a contradiction. Therefore, we conclude that \( B(0) = 0 \) in the case \( q \geq r. \)

We now turn to the case \( q < r. \) Suppose that \( B(0) = x_0 > \ln \frac{r}{q}. \) In the same manner we can see that
\[ Y(\tau, x) > e^x - 1, \ \forall (\tau, x) \in (0, T) \times \left( \ln \frac{r}{q}, x_0 \right), \]
and that
\[
(\partial_x - \mathcal{L})[Y - (e^x - 1)] = 0 + \mathcal{L}[e^x - 1] = -q e^x + r < 0
\]
in \((0, T) \times \left( \ln \frac{r}{q}, x_0 \right).\) We thus get
\[
\partial_x Y(0, x) < \mathcal{L}[Y(0, x) - (e^x - 1)] = 0, \ \forall x \in \left( \ln \frac{r}{q}, x_0 \right),
\]
a contradiction. Therefore, \( B(0) \leq \ln \frac{r}{q}. \) To show that \( B(0) = \ln \frac{r}{q}, \) we now suppose that \( B(0) = x_0 < \ln \frac{r}{q}. \) Let \( x_1 := \frac{1}{q} \left( x_0 + \ln \frac{r}{q} \right). \) We deduce that there exists \( \tau_0 > 0 \) such that \( B(\tau_0) \leq x_1, \) and hence
\[ Y = e^x - 1 \ \text{in} \ (0, \tau_0) \times \left( x_1, \ln \frac{r}{q} \right). \]
Then it is clear that
\[
(\partial_x - \mathcal{L})[Y - (e^x - 1)] = 0 \ \text{in} \ (0, \tau_0) \times \left( x_1, \ln \frac{r}{q} \right).
\]
On the other hand, we discover from (2.1) that
\[
(\partial_x - \mathcal{L})[Y - (e^x - 1)] \geq \mathcal{L}[e^x - 1] = -q e^x + r > 0 \ \text{for} \ x < \ln \frac{r}{q}.
\]
We have arrived at a contradiction which proves \( B(0) = \ln \frac{r}{q} \) in the case \( q < r. \) Thus (3.1) is proved.

We now show that \( B(\tau) \) is strictly increasing in \((0, T].\) Conversely, suppose that \( B(\tau) \) is not strictly increasing in \((0, T].\) Then \( B(\tau_1) = B(\tau_2) = x_0 \) for some \( 0 < \tau_1 < \tau_2 \leq T. \) We note from Lemma 3.1 that \( x_0 > 0. \) It follows from the definition of \( B(\tau) \) that
\[ Y(\tau, x_0) = e^{x_0} - 1, \ \forall \tau \in [\tau_1, \tau_2], \]
and hence
\[
\partial_x Y(\tau, x_0) = 0, \ \forall \tau \in (\tau_1, \tau_2).
\]
Moreover, we have
\[(\partial_\tau - \mathcal{L})Y = 0 \quad \text{in} \quad (\tau_1, \tau_2) \times (0, x_0),\]
and, in consequence,
\[(\partial_\tau - \mathcal{L})(\partial_\tau Y) = 0 \quad \text{in} \quad (\tau_1, \tau_2) \times (0, x_0).\]
Since \(\partial_\tau Y \geq 0\), the strong maximum principle implies that either \(\partial_\tau Y \equiv 0\) or \(\partial_{x\tau} Y(\tau, x_0) < 0\) for any \(\tau \in (\tau_1, \tau_2)\). On the other hand, we deduce from the definition of \(B(\tau)\) that \(\partial_\tau Y(\tau, x_0) = e^{x_0\tau}\) for any \(\tau \in [\tau_1, \tau_2]\), and so \(\partial_{x\tau} Y(\tau, x_0) = 0\) for any \(\tau \in (\tau_1, \tau_2)\). This is a contradiction. Hence \(B(\tau)\) is strictly increasing in \((0, T]\).

Let us show that \(B(\tau)\) is continuous in \((0, T]\). Conversely, suppose that \(B(\tau)\) is not continuous in \((0, T]\). Then there exist \(\tau_0 \in (0, T], x_0 \geq 0\) and small \(\varepsilon, \delta_0 > 0\) such that
\[B(\tau_0 - \varepsilon) \leq x_0 \quad \text{and} \quad B(\tau_0 + \varepsilon) \geq x_0 + \delta_0\]
for all \(\varepsilon \in (0, \varepsilon_0)\). It follows from the definition of \(B(\tau)\) that for any \(\varepsilon \in (0, \varepsilon_0)\),
\[Y(\tau_0 - \varepsilon, x) = e^x - 1 \quad \text{for all} \quad x \geq x_0 \quad \text{(3.2)}\]
and
\[Y(\tau_0 + \varepsilon, x) > e^x - 1 \quad \text{for all} \quad 0 \leq x < x_0 + \delta_0. \quad \text{(3.3)}\]
In addition, (3.2) and the continuity of \(Y\) yield
\[Y(\tau_0, x) = e^x - 1 \quad \text{for all} \quad x \geq x_0. \quad \text{(3.4)}\]
Since \(B(\tau)\) is strictly increasing in \((0, T]\), we observe that \(x_0 > B(0) = \max \left\{0, \ln \frac{r}{q}\right\}\).

We deduce from (3.3) that in \((\tau_0, \tau_0 + \varepsilon) \times (x_0, x_0 + \delta_0),\)
\[(\partial_\tau - \mathcal{L})[Y(\tau_0, x) - (e^x - 1)] = 0 + \mathcal{L}[e^x - 1] = -q e^{x_0} + r \leq -q e^{x_0} + r < 0 \quad \text{(3.5)}\]
We thus obtain from (3.4) and (3.5) that
\[\partial_\tau Y(\tau_0, x) < \mathcal{L}[Y(\tau_0, x) - (e^x - 1)] = 0, \quad \forall x \in (x_0, x_0 + \delta_0),\]
which is a contradiction. Therefore, \(B(\tau)\) is continuous in \((0, T]\). Moreover, since \(\partial_\tau Y \geq 0\) and \((e^x - 1)^+ + (\kappa - e^x)^+\) is the lower obstacle, it can be proved that \(B(\tau) \in C^{0,1}(0, T]\) by the method developed by Friedman in [4]. At this point, it follows from the bootstrap argument (see also [6]) that \(B(\tau) \in C^\infty(0, T]\). This completes the proof. \hfill \Box

**Remark 3.1.** If \(q \neq 0\), Theorem 3.3 implies that when the underlying asset increase sufficiently, if it is optimal to exercise the long-term American strangle, it is never optimal to leave un-exercised the short-term American strangle.

**Theorem 3.4.** \(A(\tau)\) is smooth and strictly decreasing in \((0, T]\). Moreover,
\[A(0) := \lim_{\tau \to 0^+} A(\tau) = \begin{cases} \ln \left(\kappa \min \left\{1, \frac{\varepsilon}{q}\right\}\right), & \text{if} \quad q > 0, \\ \ln \kappa, & \text{if} \quad q = 0. \end{cases} \quad \text{(3.6)}\]

**Proof.** We observe from the definition of \(A(\tau)\) that \(A(\tau) \leq \ln \kappa\). Since \(\partial_\tau Y \geq 0\), \(A(\tau)\) is decreasing in \((0, T]\). Hence \(A(0) := \lim_{\tau \to 0^+} A(\tau)\) exists. We first consider the case \(0 \leq q \leq r\). Suppose that \(A(0) = x_0 < \ln \kappa\). Since \(A(\tau)\) is decreasing, we get
\[A(\tau) \leq A(0) = x_0 < \ln \kappa, \quad \forall \tau \geq 0.\]
Then it follows from the definition of $A(\tau)$ that
\[ Y(\tau, x) > \kappa - e^x, \quad \forall (\tau, x) \in (0, T) \times (x_0, \ln \kappa), \]
and hence
\[ (\partial_\tau - \mathcal{L})[Y - (\kappa - e^x)] = 0 + \mathcal{L}[\kappa - e^x] = q e^x - r \kappa. \]
in $(0, T) \times (x_0, \ln \kappa)$. If $q = 0$, then $(\partial_\tau - \mathcal{L})[Y - (\kappa - e^x)] = -r \kappa < 0$. If $0 < q \leq r$, then $(\partial_\tau - \mathcal{L})[Y - (\kappa - e^x)] = q e^x - r \kappa < q \kappa - r \kappa \leq 0$ in $(0, T) \times (x_0, \ln \kappa)$. Thus we deduce that in $(0, T) \times (x_0, \ln \kappa)$,
\[ (\partial_\tau - \mathcal{L})[Y - (\kappa - e^x)] < 0. \]
From this we conclude that
\[ \partial_\tau Y(0, x) < \mathcal{L}[Y(0, x) - (\kappa - e^x)] = 0, \quad \forall x \in (x_0, \ln \kappa), \]
which is a contradiction. Therefore, we obtain $A(0) = \ln \kappa$ in the case $0 \leq q \leq r$.

We now consider the case $q > r$. Suppose that $A(0) = x_0 < \ln \frac{r \kappa}{q}$. We obtain similarly that
\[ Y(\tau, x) > \kappa - e^x, \quad \forall (\tau, x) \in (0, T) \times \left(x_0, \ln \frac{r \kappa}{q}\right), \]
and that
\[ (\partial_\tau - \mathcal{L})[Y - (\kappa - e^x)] = 0 + \mathcal{L}[\kappa - e^x] = q e^x - r \kappa < 0 \]
in $(0, T) \times \left(x_0, \ln \frac{r \kappa}{q}\right)$. Thus we get
\[ \partial_\tau Y(0, x) < \mathcal{L}[Y(0, x) - (\kappa - e^x)] = 0, \quad \forall x \in \left(x_0, \ln \frac{r \kappa}{q}\right), \]
a contradiction. Therefore, $A(0) \geq \ln \frac{r \kappa}{q}$. We now suppose that $A(0) = x_0 > \ln \frac{r \kappa}{q}$.

As in the proof of the previous theorem, there exists $\tau_0 > 0$ such that
\[ (\partial_\tau - \mathcal{L})[Y - (\kappa - e^x)] = 0 \quad \text{in} \quad (0, \tau_0) \times \left(\ln \frac{r \kappa}{q}, \frac{1}{2} \left(\ln \frac{r \kappa}{q} + x_0\right)\right). \quad (3.7) \]
On the other hand, we deduce from (2.1) that
\[ (\partial_\tau - \mathcal{L})[Y - (\kappa - e^x)] \geq \mathcal{L}[\kappa - e^x] = q e^x - r \kappa > 0 \quad \text{for} \quad x > \ln \frac{r \kappa}{q}, \]
which contradicts (3.7). This yields $A(0) \leq \ln \frac{r \kappa}{q}$, and hence $A(0) = \ln \frac{r \kappa}{q}$.

Analysis similar to that in the proof of Theorem 3.3 shows that $A(\tau)$ is strictly decreasing in $(0, T]$. Moreover, the smoothness of $A(\tau)$ follows by the same method as Theorem 3.3.

\begin{flushright} \Box \end{flushright}

**Remark 3.2.** Contrary to Remark 3.1, Theorem 3.4 means that when the underlying asset decrease sufficiently, if it is to exercise the short-term American strangle, the long-term American strangle already has been exercised.

**Remark 3.3.** If $q \neq 0$, Theorem 3.3 and Theorem 3.4 imply that the free boundaries $S^A(\tau)$ and $S^B(\tau)$ are well-defined. If $q = 0$, (1) in Theorem 3.3 implies that the free boundary $B(\tau)$ or $S^B(\tau)$ does not exist. That is, for the American strangle option written on an underlying asset without dividends, although the underlying asset increases enough, the option holder does not exercise his/her right. This phenomena is consistent with American call option with non-dividend. Since the price of the American call option on a non-dividend-paying stock always exceeds its intrinsic value prior to expiration, the early exercise is never optimal.
4. **Comparative static analysis.** In this section we conduct comparative static analysis with respect to important model parameters. First, we analyze the effect of value $\sigma$, the volatility of the American strangle option.

**Lemma 4.1.** Let $Y$ be the solution to (2.1). Then we have
\[ \partial_{xx} Y - \partial_x Y \geq 0. \] (4.1)

**Proof.** We denote
\[ C = \{ (\tau, x) \in D : Y(\tau, x) > (e^x - 1)^+ + (\kappa - e^x)^+ \}, \]
\[ E = \{ (\tau, x) \in D : Y(\tau, x) = (e^x - 1)^+ + (\kappa - e^x)^+ \}, \]
where $D = (0, T] \times \mathbb{R}$. In the continuation region $C$, it follows from (2.1) that
\[ \partial_\tau (\partial_x Y - L Y) = 0, \]
and hence
\[ \partial_\tau (\partial_x Y - Y) - L (\partial_x Y - Y) = 0. \] (4.2)

On the other hand, we observe from Lemma 3.2 that $E \subset ((-\infty, \ln \kappa) \cup (0, \infty)) \times (0, T]$. Hence, we see that in the exercise region $E$,
\[ Y(\tau, x) = \begin{cases} \kappa - e^x & \text{if } x < \ln \kappa, \\ e^x - 1 & \text{if } x > 0, \end{cases} \]
and so
\[ (\partial_x Y - Y)(\tau, x) = \begin{cases} -\kappa & \text{if } x < \ln \kappa, \\ 1 & \text{if } x > 0. \end{cases} \]

We thus get
\[ \partial_\tau (\partial_x Y - Y) - L (\partial_x Y - Y) = \begin{cases} -r \kappa & \text{if } x < \ln \kappa, \\ r & \text{if } x > 0, \end{cases} \] (4.3)
in the exercise region $E$. Combining (4.2) and (4.3) gives
\[ \partial_\tau (\partial_x Y - Y) - L (\partial_x Y - Y) = \begin{cases} 0 & \text{in } C, \\ -r \kappa & \text{in } E \cap \{ x < \ln \kappa \}, \\ r & \text{in } E \cap \{ x > 0 \}. \end{cases} \] (4.4)

For $\delta > 0$, we now set $Y(\tau, x) := Y(\tau, x - \delta)$. We also denote
\[ C_\delta = \{ (\tau, x) \in D : Y_\delta(\tau, x) > (e^{x-\delta} - 1)^+ + (\kappa - e^{x-\delta})^+ \}, \]
\[ E_\delta = \{ (\tau, x) \in D : Y_\delta(\tau, x) = (e^{x-\delta} - 1)^+ + (\kappa - e^{x-\delta})^+ \}. \]
Then we obtain similarly that
\[ \partial_\tau (\partial_x Y_\delta - Y_\delta) - L (\partial_x Y_\delta - Y_\delta) = \begin{cases} 0 & \text{in } C_\delta, \\ -r \kappa & \text{in } E_\delta \cap \{ x < \ln \kappa + \delta \}, \\ r & \text{in } E_\delta \cap \{ x > \delta \}. \end{cases} \] (4.5)
In the region $E \cap \{x < \ln \kappa\}$, we know that $Y(\tau, x) = \kappa - e^x$. Then we deduce from Section 3 that $Y_\delta(\tau, x) = Y(\tau, x - \delta) = \kappa - e^{x-\delta}$, and so

$$\partial_x (\partial_x Y_\delta - Y_\delta) - \mathcal{L}(\partial_x Y_\delta - Y_\delta) = -r\kappa.$$  

Therefore, we have

$$\partial_x [(\partial_x Y - Y) - (\partial_x Y_\delta - Y_\delta)] - \mathcal{L}[(\partial_x Y - Y) - (\partial_x Y_\delta - Y_\delta)] = -r\kappa - (-r\kappa) = 0 \quad \text{in} \quad E \cap \{x < \ln \kappa\}. \tag{4.6}$$

In addition, we notice that

$$\mathcal{C} \cap (\mathcal{E} \cap \{x > \delta\}) = \mathcal{C} \cap ((\mathcal{E} + (0, \delta)) \cap \{x > \delta\}) = \mathcal{C} \cap ((\mathcal{E} \cap \{x > 0\}) + (0, \delta)) \subset \mathcal{C} \cap (\mathcal{E} \cap \{x > 0\}) = \emptyset. \tag{4.7}$$

We now combine (4.4)–(4.7) to discover that

$$\partial_x [(\partial_x Y - Y) - (\partial_x Y_\delta - Y_\delta)] - \mathcal{L}[(\partial_x Y - Y) - (\partial_x Y_\delta - Y_\delta)] \geq 0 \tag{4.8}$$

in the whole domain $D = (0, T] \times \mathbb{R}$. Letting $Y_0(x) := (e^x - 1)^+ + (\kappa - e^x)^+$, we see that $\partial_x Y_0 - Y_0$ is increasing in $\mathbb{R}$, and hence

$$[(\partial_x Y - Y) - (\partial_x Y_\delta - Y_\delta)](0, x) \geq 0, \quad \forall x \in \mathbb{R}. \tag{4.9}$$

Then we conclude from (4.8) and (4.9) that

$$[(\partial_x Y - Y) - (\partial_x Y_\delta - Y_\delta)](\tau, x) \geq 0, \quad \forall (\tau, x) \in D.$$ 

It follows that

$$(\partial_x Y - Y)(\tau, x) \geq (\partial_x Y - Y)(\tau, x - \delta)$$

for all $(\tau, x) \in D$ and $\delta > 0$. This yields the desired inequality (4.1). \qed

Using the above Lemma, we prove the following theorem, the behavior of the free boundaries according to $\sigma$.

**Theorem 4.2.**

1. $A(\tau)$ is decreasing with respect to $\sigma$.
2. $B(\tau)$ is increasing with respect to $\sigma$.

**Proof.** Let $\sigma_1, \sigma_2$ be positive constants such that $\sigma_1 > \sigma_2$. Let $Y_1$ be the solution to (2.1) with $\sigma = \sigma_1$ and let $Y_2$ be the solution to (2.1) with $\sigma = \sigma_2$. Then it follows from (2.1), (2.2) and Lemma 4.1 that $Y_1$ satisfies

$$\begin{cases}
\partial_x Y_1 - \frac{\sigma_1^2}{2} \partial_{xx} Y_1 - (r - q - \frac{\sigma_1^2}{2}) \partial_x Y_1 + r Y_1 = \frac{1}{2}(\sigma_1^2 - \sigma_2^2)(\partial_{xx} Y_1 - \partial_x Y_1) \geq 0, \\
\quad \text{if} \quad Y_1 > (e^x - 1)^+ + (\kappa - e^x)^+, \quad (\tau, x) \in [0, T) \times \mathbb{R}, \\
\partial_x Y_1 - \frac{\sigma_2^2}{2} \partial_{xx} Y_1 - (r - q - \frac{\sigma_2^2}{2}) \partial_x Y_1 + r Y_1 \geq \frac{1}{2}(\sigma_1^2 - \sigma_2^2)(\partial_{xx} Y_1 - \partial_x Y_1) \geq 0, \\
\quad \text{if} \quad Y_1 = (e^x - 1)^+ + (\kappa - e^x)^+, \quad (\tau, x) \in [0, T) \times \mathbb{R}, \\
Y_1(0, x) = (e^x - 1)^+ + (\kappa - e^x)^+ = Y_2(0, x), \quad x \in \mathbb{R}.
\end{cases}$$

By the comparison principle, we have $Y_1 \geq Y_2$. From this we deduce that $\mathcal{E}_1^A \subset \mathcal{E}_2^A$ and $\mathcal{E}_1^B \subset \mathcal{E}_2^B$, where

$$\begin{align*}
\mathcal{E}_j^A := \{(\tau, x) \in D : Y_j(\tau, x) = (\kappa - e^x)^+ > 0\}, \\
\mathcal{E}_j^B := \{(\tau, x) \in D : Y_j(\tau, x) = (e^x - 1)^+ > 0\},
\end{align*}$$

for $j = 1, 2$. \hfill \box
for \( j = 1, 2 \). Therefore, we obtain \( A_1(\tau) \leq A_2(\tau) \) and \( B_1(\tau) \geq B_2(\tau) \), where \( A_j(\tau) \) and \( B_j(\tau) \) are the free boundaries of \( Y_j \) for \( j = 1, 2 \). This completes the proof.

For fixed \( r, q, \sigma \) and maturity \( T \), let us define \( F_p(\tau) \) and \( F_c(\tau) \) as the free boundaries of the degenerate backward parabolic problem arising from the American put option with strike price \( K_1 \) and the American call option with strike price \( K_2 \), respectively. Then, by the following theorem, we can compare \( A(\tau) \) and \( F_p(\tau) \), \( B(\tau) \) and \( F_c(\tau) \), respectively.

**Theorem 4.3.** For any \( \tau \in (0, T] \), we have \( A(\tau) \leq F_p(\tau) \) and \( B(\tau) \geq F_c(\tau) \).

**Proof.** We first prove that \( B(\tau) \geq F_c(\tau) \). Let \( Y_c \) be the solution to the problem

\[
\begin{aligned}
\partial_t Y_c - \mathcal{L} Y_c &= 0, & & \text{if } Y_c > (e^x - 1)^+, \ (\tau, x) \in [0, T) \times \mathbb{R}, \\
\partial_t Y_c - \mathcal{L} Y_c &\geq 0, & & \text{if } Y_c = (e^x - 1)^+, \ (\tau, x) \in [0, T) \times \mathbb{R}, \\
Y_c(0, x) &= (e^x - 1)^+, & & x \in \mathbb{R},
\end{aligned}
\]

which is the forward non-degenerate parabolic problem arising from the model of American call option. In view of

\[ Y(0, x) = (e^x - 1)^+ + (\kappa - e^x)^+ \geq (e^x - 1)^+ = Y_c(0, x), \]

the monotonicity of solution of variational inequality with respect to initial value yields \( Y \geq Y_c \) in \([0, T) \times \mathbb{R}\). This implies that \( B(\tau) \geq F_c(\tau) \) for all \( \tau \in (0, T] \).

To prove \( A(\tau) \leq F_p(\tau) \), setting

\[ \tau = T - t, \quad x = \ln \left( \frac{s}{K_1} \right), \quad \bar{Y}(\tau, x) = \frac{V(t, s)}{K_1}, \]

we have

\[
\begin{aligned}
\partial_t \bar{Y} - \mathcal{L} \bar{Y} &= 0, & & \text{if } \bar{Y} > (e^x - \bar{\kappa})^+ + (1 - e^x)^+, \ (\tau, x) \in [0, T) \times \mathbb{R}, \\
\partial_t \bar{Y} - \mathcal{L} \bar{Y} &\geq 0, & & \text{if } \bar{Y} = (e^x - \bar{\kappa})^+ + (1 - e^x)^+, \ (\tau, x) \in [0, T) \times \mathbb{R}, \\
\bar{Y}(0, x) &= (e^x - \bar{\kappa})^+ + (1 - e^x)^+, & & x \in \mathbb{R},
\end{aligned}
\]

where

\[ \bar{\kappa} := \frac{K_2}{K_1} = \frac{1}{\kappa} \in (1, \infty). \]

Now let \( \bar{Y}_c \) be the solution to the forward non-degenerate parabolic problem arising from the model of American put option:

\[
\begin{aligned}
\partial_t \bar{Y}_c - \mathcal{L} \bar{Y}_c &= 0, & & \text{if } \bar{Y}_c > (1 - e^x)^+, \ (\tau, x) \in [0, T) \times \mathbb{R}, \\
\partial_t \bar{Y}_c - \mathcal{L} \bar{Y}_c &\geq 0, & & \text{if } \bar{Y}_c = (1 - e^x)^+, \ (\tau, x) \in [0, T) \times \mathbb{R}, \\
\bar{Y}_c(0, x) &= (1 - e^x)^+, & & x \in \mathbb{R},
\end{aligned}
\]

Since

\[ \bar{Y}(0, x) = (e^x - \bar{\kappa})^+ + (1 - e^x)^+ \geq (1 - e^x)^+ = \bar{Y}_c(0, x), \]

we have \( \bar{Y} \geq \bar{Y}_c \) in \([0, T) \times \mathbb{R}\), and hence \( A(\tau) \leq F_p(\tau) \) for all \( \tau \in (0, T] \).
5. Stationary problem for American strangle option. In this section, we consider a stationary problem arising in the American strangle option. According to Theorem 3.3, the problem (2.1) is divided into two cases according to the range of $q$:

1. There exist two free boundaries as $q > 0$,
2. There is only one free boundary as $q = 0$.

Therefore, we also need to consider about the stationary problems corresponding to problem (2.1) separately.

(1) For $q > 0$, the stationary problem is

\[
\begin{align*}
-\mathcal{L}V &= 0, \quad \text{if } V > (e^x - 1)^+ + (\kappa - e^x)^+, \quad x \in \mathbb{R}, \\
-\mathcal{L}V &\geq 0, \quad \text{if } V = (e^x - 1)^+ + (\kappa - e^x)^+, \quad x \in \mathbb{R},
\end{align*}
\]

where $\mathcal{L}$ is defined in (2.2).

Theorem 5.1. The variational inequality (5.1) has a unique $W^{2,\text{loc}}$ solution, which is

\[
V(x) = \begin{cases} 
\kappa - e^x, & x < s_p, \\
c_1 e^{n_1 x} + c_2 e^{n_2 x}, & x \in [s_p, s_c], \\
e^x - 1, & x > s_c,
\end{cases}
\]

where

\[
c_1 = \frac{(n_2 - 1)e^{s_p} - n_2}{n_2 - n_1} e^{-n_1 s_c}, \quad c_2 = \frac{(n_1 - 1)e^{s_c} - n_1}{n_1 - n_2} e^{-n_2 s_c},
\]

and $n_1, n_2$ are the positive and negative roots of the algebraic equation

\[
\frac{\sigma^2}{2} n^2 + \left( r - q - \frac{\sigma^2}{2} \right) n - r = 0.
\]

The constants $s_c$ and $s_p$ are defined as

\[
e^{s_p} = \frac{n_2}{n_2 - 1} \cdot \frac{\kappa y^{n_1} + 1}{y^{n_1} + y}, \quad e^{s_c} = e^{s_p} \cdot y.
\]

Here, $y \in (1, +\infty)$ is the unique solution of the algebraic equation $f(y) = 0$ with

\[
f(y) = -(n_1 - 1)n_2(1 + \kappa y^{n_1})(y^{1-n_1} + y^{n_2-n_1}) - n_1(1 - n_2)(1 + \kappa y^{n_2})(y^{1-n_1} + 1).
\]

In fact, $y > \frac{1}{\kappa}$.

Proof. We first consider the following free boundary problem:

\[
\begin{align*}
\mathcal{L}V &= 0, \quad s_p < x < s_c, \\
V(s_p) &= \kappa - e^{s_p}, \quad V'(s_p) = -e^{s_p}, \\
V(s_c) &= e^{s_c} - 1, \quad V'(s_c) = e^{s_c}.
\end{align*}
\]

Then we extend the solution $V$ onto $\mathbb{R}$ by

\[
V(x) = \kappa - e^x \quad \text{if } x \in (-\infty, s_p) \quad \text{and} \quad V(x) = e^x - 1 \quad \text{if } x \in (s_c, +\infty).
\]

Next, we show that $V$ is the unique solution to the variational inequality (5.1). We can let the general solution for (5.5) in the form of

\[
V(x) = c_1 e^{n_1 x} + c_2 e^{n_2 x}, \quad x \in [s_p, s_c].
\]

It is easy to check that $n_1 = 1$ if $q = 0$ and $n_1 > 1$ if $q > 0$. \(\square\)
From (5.5) and (5.7),
\[
V(s_c) = c_1e^{n_1s_c} + c_2e^{n_2s_c} = e^{s_c} - 1,
\]
\[
V'(s_c) = c_1n_1e^{n_1s_c} + c_2n_2e^{n_2s_c} = e^{s_c}.
\] (5.8)

Therefore, \( c_1 \) and \( c_2 \) are given by
\[
c_1 = \frac{(n_2 - 1)e^{s_c} - n_2}{n_2 - n_1}e^{-n_1s_c}, \quad c_2 = \frac{(n_1 - 1)e^{s_c} - n_1}{n_1 - n_2}e^{-n_2s_c}.
\] (5.9)

Similarly,
\[
V(s_p) = c_1e^{n_1s_p} + c_2e^{n_2s_p} = \kappa - e^{s_p},
\]
\[
V'(s_p) = c_1n_1e^{n_1s_p} + c_2n_2e^{n_2s_p} = e^{s_p},
\] (5.10)

and
\[
c_1 = \frac{(1 - n_2)e^{s_p} + \kappa n_2}{n_2 - n_1}e^{-n_1s_p}, \quad c_2 = \frac{(1 - n_1)e^{s_p} + \kappa n_1}{n_1 - n_2}e^{-n_2s_p}.
\] (5.11)

From (5.9) and (5.11), we obtain
\[
\frac{(n_2 - 1)e^{s_c} - n_2}{e^{n_1s_c}} = \frac{(1 - n_2)e^{s_p} + \kappa n_2}{e^{n_1s_p}},
\]
\[
\frac{(n_1 - 1)e^{s_c} - n_1}{e^{n_2s_c}} = \frac{(1 - n_1)e^{s_p} + \kappa n_1}{e^{n_2s_p}}.
\] (5.12)

and
\[
n_2e^{n_1s_p} + \kappa n_2e^{n_1s_c} = (n_2 - 1)e^{s_p+n_1s_p} - (1 - n_2)e^{s_p+n_1s_c},
\]
\[
n_1e^{n_2s_c} + \kappa n_1e^{n_2s_p} = (n_1 - 1)e^{s_c+n_2s_p} - (1 - n_1)e^{s_c+n_2s_c}.
\] (5.13)

Therefore, we have
\[
0 = -(n_1 - 1)n_2 \left(1 + \kappa e^{n_1(s_c-s_p)}\right) \left(e^{(1-n_1)(s_c-s_p)} + e^{(n_2-n_1)(s_c-s_p)}\right)
- n_1(1 - n_2) \left(1 + \kappa e^{n_2(s_c-s_p)}\right) \left(e^{(1-n_2)(s_c-s_p)} + 1\right).
\] (5.14)

Let us define \( y = e^{s_c-s_p} \) and \( f(y) \) as (5.4). Then, we prove that \( f(y) = 0 \) possesses a unique solution \( y \in (1, +\infty) \) and that \( y > \frac{1}{\kappa} \). We observe that
\[
f'(y) = -(n_1 - 1)n_2 \left(1 - n_1\right)y^{n_2-n_1-1} + \kappa + (n_2 - n_1)y^{n_2-n_1-1} + \kappa n_2y^{n_2-1}
- n_1(1 - n_2) \left(1 - n_1\right)y^{n_2-n_2-1} + \kappa(n_2 - n_1 + 1)y^{n_2-n_1-1} + \kappa n_2y^{n_2-1}
- y^{n_2-n_1-1}(n_2 - n_1)y^{n_2-n_1-1} - \kappa n_1(1 - n_2)(n_2 - n_2 + 1)y
+ (1 - n_1)(n_2 - n_1)y^{n_2-n_1-1} + \kappa(1 - n_1)n_2 + \kappa n_2(n_2 - n_1)y^{n_2-1}.
\]

Since \( n_1 > 1 \) and \( n_2 < 0 \),
\[
(1 - n_1)(n_2 - n_1)y^{n_2-n_1-1} > 0, \kappa(1 - n_1)n_2 \geq 0, \kappa n_2(n_2 - n_1)y^{n_2-1} > 0,
\]
and for \( y \in \left(\frac{1}{\kappa}, +\infty\right)\),
\[
y^{n_2-n_1-1}(n_2 - n_1)y^{n_2-n_2-1} - \kappa n_1(1 - n_2)(n_2 - n_2 + 1)y
\geq y^{n_2-n_1-1}(n_2 - n_1)y^{n_2-n_2-1} - n_1(1 - n_2)(n_2 - n_2 + 1))
= y^{n_2-n_1-1}(n_2 - n_1)^2 + n_1(n_2 - 1)) \geq y^{n_2-n_1-1}(n_2 - n_1)^2 + n_1(n_2 - 1)) = y^{n_2-n_1-1}(n_2 - n_1)n_2 > 0.
\]
Therefore, \( f'(y) > 0 \) for \( \frac{1}{\kappa} < y < +\infty \). Moreover, we see that
\[
\begin{align*}
    f \left( \frac{1}{\kappa} \right) &= (n_2 - n_1)(1 + \kappa^{1-n_2})(1 + \kappa^{n_1-1}) < 0, \\
    f(+\infty) &= +\infty.
\end{align*}
\]
Hence, \( f(y) = 0 \) has a unique solution in \( y \in \left( \frac{1}{\kappa}, +\infty \right) \).

Also, for \( 1 < y < \frac{1}{\kappa} \),
\[
\begin{align*}
    f(y) &< -n_1 n_2 (1 + \kappa y^{n_1})(y^{1-n_1} + y^{n_2-n_1}) + n_1 n_2 (1 + \kappa y^{n_2})(y^{1-n_1} + 1) \\
    &= n_1 n_2 (1 - ky)(1 - y^{n_2-n_1}) \leq 0,
\end{align*}
\]
then, in \( y \in \left( 1, \frac{1}{\kappa} \right) \), there is no solution of \( f(y) = 0 \). Hence \( f(y) = 0 \) possesses a unique solution \( y \in (1, +\infty) \) with \( y > \frac{1}{\kappa} \).

Since \( z = e^{s_c - s_p} \) in (5.12), we obtain
\[
e^{s_p} = \frac{n_2}{n_2 - 1} \cdot \kappa y^{n_1} + 1 \cdot y^{n_1} + y, \quad e^{s_c} = e^{s_p} \cdot y.
\]

Next, we show that \( V \) defined as (5.2) is the unique \( W^2_{p, \text{loc}} \) solution to the variational inequality (5.1). By a method similar to Section 2, we can get the uniqueness of the solution of the problem (5.1). Also, it is not hard to check that \( V(x) \in W^2_{p, \text{loc}} \).

Therefore, we only need to show that \( V \) satisfies the variational inequality (5.1).

Firstly, for \( f(y) = 0 \), we obtain
\[
\begin{align*}
    \frac{n_1 - 1}{n_1} \cdot \frac{y + \kappa y^{n_1}}{y + y^{n_1}} &= \frac{n_2 - 1}{n_2} \cdot \frac{y + \kappa y^{n_2}}{y + y^{n_2}}
\end{align*}
\]
and
\[
e^{s_c} = \frac{n_2}{n_2 - 1} \cdot \frac{1 + \kappa y^{n_1}}{1 + y^{n_1-1}} = \frac{n_1}{n_1 - 1} \cdot \frac{1 + \kappa y^{n_2}}{1 + y^{n_2-1}} > \frac{n_1}{n_1 - 1} > 1.
\]

Also, it is not hard to check that \( s_c > 0 \) and \( s_p < \ln \kappa < 0 \). Therefore,
\[
c_1, c_2 > 0
\]
and
\[
V(x) \geq 0 \quad \text{for} \quad x \in [s_p, s_c].
\]

Secondly, by (5.5), we obtain that
\[
\begin{align*}
    \begin{cases}
        -\mathcal{L}(V' - e^x) = \mathcal{L}e^x = -qe^x < 0, & s_p < x < s_c, \\
        V'(x) - e^x = e^x - e^x \leq 0, & x = s_c, \\
        V'(x) - e^x = -e^x - e^x \leq 0, & x = s_p,
    \end{cases}
\end{align*}
\]
and
\[
\begin{align*}
    \begin{cases}
        -\mathcal{L}(-V' - e^x) = \mathcal{L}e^x = -qe^x < 0, & s_p < x < s_c, \\
        -V'(x) - e^x = -e^x + e^x \leq 0, & x = s_c, \\
        -V'(x) - e^x = e^x - e^x \leq 0, & x = s_p.
    \end{cases}
\end{align*}
\]
Applying the maximum principle, we have \( -e^x \leq V'(x) \leq e^x, \quad x \in [s_p, s_c] \). Since
\[
\begin{align*}
    (V(x) - (e^x - 1))' &= V'(x) - e^x \leq 0, \\
    (V(x) - (\kappa - e^x))' &= V'(x) + e^x \geq 0,
\end{align*}
\]
and $V(s_c) = e^{s_c} - 1$, $V(s_p) = \kappa - e^{s_p}$, we get

$$V(x) \geq e^x - 1, \quad V(x) \geq \kappa - e^x.$$  

We see from $V \geq 0$ that

$$V(x) \geq (e^x - 1)^+ + (\kappa - e^x)^+.$$  

Finally, if $x > s_c$,

$$-\mathcal{L}V = -\mathcal{L}(e^x - 1) = qe^x - r.$$  

It is easy to check that

$$\frac{n_1}{n_1 - 1} \cdot \frac{n_2}{n_2 - 1} = \frac{r}{q}.$$  

Therefore, for $x > s_c$,

$$e^{s_c} = \frac{n_2}{n_2 - 1} \cdot \frac{1 + \kappa y^{n_1}}{1 + y^{n_1 - 1}} = \frac{r}{q} \frac{n_1 - 1 + \kappa y^{n_2}}{n_1 - 1 + y^{n_2 - 1}} \geq \frac{r}{q}$$

and

$$-\mathcal{L}V = qe^x - r \geq qe^{s_c} - r \geq 0.$$  

If $x < s_p$,

$$-\mathcal{L}V = -\mathcal{L}(\kappa - e^x) = -qe^x + \kappa r.$$  

For $x < s_p$,

$$e^{s_p} = \frac{n_2}{n_2 - 1} \cdot \frac{1 + \kappa y^{n_1}}{1 + y^{n_1}} = \frac{r}{q} \frac{n_1 - 1 + \kappa y^{n_2}}{n_1 - 1 + y^{n_2}} < \frac{r}{q}$$

and

$$-\mathcal{L}V = -qe^x + \kappa r \geq -qe^{s_p} + \kappa r \geq 0.$$  

Therefore, $V$ is the unique $W^2_{p,\text{loc}}$ solution to problem (5.1).

(2) As $q = 0$, the subregion $\mathcal{E}^B$ of exercise region $\mathcal{E}$ does not exist and for the uniqueness of the solution to the corresponding stationary problem, we need to impose an additional condition, which comes from the properties of strangle options. The problem is

$$(5.15)$$

$$\begin{cases}
-\mathcal{L}V = 0, & \text{if } V > (e^x - 1)^+ + (\kappa - e^x)^+, \ x \in \mathbb{R}, \\
-\mathcal{L}V \geq 0, & \text{if } V = (e^x - 1)^+ + (\kappa - e^x)^+, \ x \in \mathbb{R}, \\
\lim_{x \to \infty} e^{-x}V(x) = 1.
\end{cases}$$

Theorem 5.2. The variational inequality (5.15) has a unique $W^2_{p,\text{loc}}$ solution, which is

$$(5.16)$$

$$V(x) = \begin{cases}
\kappa - e^x, & x < s_p, \\
e^x + \frac{\kappa}{1 - n} e^{n(x-s_p)}, & x \in [s_p, +\infty],
\end{cases}$$

where $n = -2r/\sigma^2$ and $e^{s_p} = \frac{n}{n+1} \frac{\kappa}{2}$. 

Proof. In the first, we solve the free boundary problem (5.17)

$$(5.17)$$

$$\begin{cases}
\mathcal{L}V = 0, & s_p < x, \\
V(s_p) = \kappa - e^{s_p}, \quad V'(s_p) = -e^{s_p}, \\
\lim_{x \to \infty} e^{-x}V(x) = 1.
\end{cases}$$
The general solution form of (5.17) is given by
\[ V(x) = c_1 e^{x} + c_2 e^{nx}, \quad (5.18) \]
where \( n = -2r/\sigma^2 \). Since \( \lim_{x \to \infty} e^{-x} V(x) = 1 \), we obtain \( c_1 = 1 \) and \( V(x) = e^{x} + c_2 e^{nx}, \quad x \in [s_p, +\infty). \)

From \( V(s_p) = \kappa - e^{s_p} \) and \( V'(s_p) = -e^{s_p} \), we have
\[ e^{s_p} + c_2 e^{ns_p} = \kappa - e^{s_p}, \quad e^{s_p} + nc_2 e^{ns_p} = -e^{s_p}. \]

It is easily seen that
\[ e^{s_p} = \frac{n}{n-1} \frac{\kappa}{2}, \quad c_2 = \frac{\kappa}{1-n} e^{-ns_p}, \]
then (5.16) follows if we extend \( V \) by \( V(x) = \kappa - e^{x} \) for any \( x < s_p \). We note that \( e^{s_p} < \kappa \) and \( c_2 > 0 \), and so \( V(x) \geq 0 \).

Next, we prove that \( V \) possessing the form (5.16) is the unique \( W^{2,p}_{\text{loc}} \) solution to the variational inequality (5.15). Similarly in case (1), we only need to prove that \( V \) satisfies (5.15).

First, for \( x < s_p \),
\[ -\mathcal{L}V = -\mathcal{L}(\kappa - e^{x}) = r\kappa \geq 0. \quad (5.19) \]
Secondly, from (5.16), we see that for any \( x > s_p \), there holds
\[ V'(x) = e^{x} + \frac{\kappa}{1-n} e^{n(x-s_p)}. \]

Moreover, for \( x \geq s_p \),
\[ V'(x) = e^{x} + \frac{\kappa}{1-n} e^{n(x-s_p)} - \frac{2}{\kappa} e^{n(x-s_p)} + s_p \geq e^{x}. \]

Since \( V'(s_p) = \kappa - e^{s_p} \), we have \( V(x) \geq \kappa - e^{x}, \quad x \in [s_p, +\infty) \). We also have
\[ V(x) = e^{x} + \frac{\kappa}{1-n} e^{n(x-s_p)} > e^{x} - 1, \quad x \in [s_p, +\infty). \]

Thus, \( V \geq (e^{x} - 1) + (\kappa - e^{x}) \) and \( V \) is the \( W^{2,p}_{\text{loc}} \) solution to variational inequality (5.15).

Applying the comparison principle with respect to initial value of variational inequality, we can obtain the following theorem.

**Theorem 5.3.** There exist constants \( s_p < 0 \) and \( s_c > 0 \) such that
\[ s_p \leq A(\tau) \leq \ln \left( \kappa \min \left\{ 1, \frac{r}{q} \right\} \right) , \quad \ln \left( \max \left\{ 1, \frac{r}{q} \right\} \right) \leq B(\tau) \leq s_c. \]
In fact, the constants \( s_p \) and \( s_c \) are defined in Theorem 5.1 and Theorem 5.2.
6. Numerical methods and results. In this section, we obtain the numerical solution of the value function and free boundaries of the American strangle option by applying the finite difference scheme.

Starting from (2.1), we have
\[
\begin{aligned}
\min \left\{ \frac{\partial_x Y}{\Delta x} - \frac{\sigma^2}{2} \partial_{xx} Y - \left( r - q - \frac{\sigma^2}{2} \right) \partial_t Y + r Y, \right. \\
Y - (e^{\tau} - 1) - (\kappa - e^{\tau})^+ \}
= 0, \quad \text{for } x \in \mathbb{R}, \quad \tau \in (0, T], \quad (6.1) \\
Y(0, x) = (e^{\tau} - 1)^+ + (\kappa - e^{\tau})^+, \quad \text{for } x \in \mathbb{R}.
\end{aligned}
\]

Given mesh size $\Delta \tau$, $\Delta x > 0$, $Y_j^n = Y(n\Delta \tau, j \Delta x)$ represents the value of numerical approximation at $(n\Delta \tau, j \Delta x)$, then the PDE is converted to the following difference equation:
\[
\begin{aligned}
\min \left\{ \frac{Y_j^n - Y_{j+1}^{n-1}}{\Delta \tau} - \frac{\sigma^2}{2} \left[ \frac{Y_{j+1}^{n-1} - 2Y_j^{n-1} + Y_{j-1}^{n-1}}{\Delta x^2} \right] \\
- \left( r - q - \frac{\sigma^2}{2} \right) \frac{Y_{j+1}^{n-1} - Y_{j-1}^{n-1}}{2\Delta x} + rY_j^n, Y_j^n - (e^{\tau} - 1)^+ - (\kappa - e^{\tau})^+ \}
= 0, \quad (6.2) \\
Y_j^0 = (e^{\Delta x} - 1)^+ + (\kappa - e^{\Delta x})^+.
\end{aligned}
\]

It means
\[
\begin{aligned}
Y_j^n = \max \left\{ \frac{1}{1 + r\Delta \tau} \left[ \left(1 - \frac{\sigma^2 \Delta \tau}{2\Delta x^2} \right) Y_j^{n-1} + \frac{\sigma^2 + (r - q - \sigma^2/2) \Delta \tau}{2\Delta x^2} \Delta \tau Y_j^{n-1} \right] \\
+ \frac{\sigma^2 - (r - q - \sigma^2/2) \Delta x}{2\Delta x^2} \Delta \tau Y_j^{n-1}, e^{\Delta x} - 1, \kappa - e^{\Delta x} \right\}, \quad (6.3) \\
Y_j^0 = (e^{\Delta x} - 1)^+ + (\kappa - e^{\Delta x})^+.
\end{aligned}
\]

Choosing $\frac{\sigma \Delta \tau}{\Delta x^2} = 1$, we have
\[
\begin{aligned}
Y_j^n = \max \left\{ \frac{1}{1 + r\Delta \tau} \left[ \left(\frac{1}{2} + \frac{r - q - \sigma^2/2}{2\sigma} \right) Y_j^{n-1} \right] \\
+ \left(\frac{1}{2} - \frac{r - q - \sigma^2/2}{2\sigma} \right) \Delta \tau Y_j^{n-1}, e^{\Delta x} - 1, \kappa - e^{\Delta x} \right\}, \quad (6.4) \\
Y_j^0 = (e^{\Delta x} - 1)^+ + (\kappa - e^{\Delta x})^+.
\end{aligned}
\]

We denote $u = e^{\sqrt{\Delta \tau}}$, $d = u^{-1}$, $\rho = e^{\Delta \tau}$, $p = (pe^{-q\Delta \tau} - d)(u - d)^{-1}$. Applying the Taylor expansion, we see that as $\Delta \tau \to 0^+$, there holds
\[
\frac{1}{2} + \frac{r - q - \sigma^2/2}{2\sigma} \Delta \tau = p + o(\Delta \tau), \quad \frac{1}{1 + r\Delta \tau} = \frac{1}{\rho} + O(\Delta \tau^2) .
\]

Ignoring a higher order of $\sqrt{\Delta \tau}$, we get
\[
\begin{aligned}
Y_j^n = \max \left\{ \frac{1}{\rho} \left[ pV_{j+1}^{n-1} + (1 - p)V_{j-1}^{n-1}, u^j - 1, \kappa - w^j \right] \right\}, \\
Y_j^0 = (w^j - 1)^+ + (\kappa - w^j)^+.
\end{aligned}
\]

\[
Y_j^n = \max \left\{ \frac{1}{\rho} \left[ pV_{j+1}^{n-1} + (1 - p)V_{j-1}^{n-1}, u^j - 1, \kappa - w^j \right] \right\},
\]

\[
Y_j^0 = (w^j - 1)^+ + (\kappa - w^j)^+.
\]
Consider the point \((\tau, x) = (n\Delta \tau, j\Delta x)\), then
\[
Y^n_j = Y(\tau, x),
\]
\[
Y^{n-1}_j = V(\tau - \Delta \tau, x),
\]
\[
Y^{n-1}_{j+1} = V(\tau - \Delta \tau, x + \Delta x),
\]
\[
Y^{n-1}_{j-1} = V(\tau - \Delta \tau, x - \Delta x),
\]
and get Figures 1–5.

**Figure 1.** The change of the option value function \(V(t, s)\) with respect to stock price \(s\) where \(r = 0.05\), \(q = 0.1\), \(\sigma = 0.3\), \(K_1 = 1\) and \(K_2 = 1.5\).

**Figure 2.** The change of the free boundaries \(A(\tau)\) and \(B(\tau)\) with respect to \(\sigma\) where \(r = 0.05\), \(q = 0.05\), \(K_1 = 1\) and \(K_2 = 1.1\).

Figure 1 shows the value function of the American strangle option using the numerical method described above. Since the American strangle option has both the properties of the call option and the put option, the value of the option increases as the underlying asset increases or decreases sufficiently. That is, the value function \(V\) does not hold monotonicity with respect to \(s\). This intuition is consistent with the numerical result in Figure 1.
Figure 3. Compare the free boundary $B(\tau)$ and the free boundary $F_c(\tau)$ with $r = 0.05$, $q = 0.05$, $\sigma = 0.2$, $K_1 = 1$ and $K_2 = 1.1$.

Figure 4. Compare the free boundary $A(\tau)$ and the free boundary $F_p(\tau)$ with $r = 0.05$, $q = 0.05$, $\sigma = 0.2$, $K_1 = 1$ and $K_2 = 1.1$.

Figure 5. Upper and lower bounds of $A(\tau)$ and the free boundary $B(\tau)$, respectively, with $r = 0.05$, $q = 0.05$, $\sigma = 0.2$, $K_1 = 1$ and $K_2 = 1.1$. 
In Figure 2, it can be seen that as the volatility $\sigma$ of the underlying asset increases, $A(\tau)$ decreases and $B(\tau)$ increases. As mentioned in the introduction, the American strangle option can obtain more profits as the underlying asset fluctuates more. That is, as the volatility $\sigma$ of the underlying assets increases, the price of the option increases, so it should be exercised when the underlying asset price is higher or lower than when the volatility is lower. This is consistent with the result of Theorem 4.2.

In addition, the area enclosed by the two free boundaries $A(\tau)$ and $B(\tau)$ is the continuation region $C$ of Problem (2.1) and the complement of this region is the exercise region $E$ of Problem (2.1).

Figure 3 plots the behavior of free boundaries $B(\tau)$ and $F_c(\tau)$ according to $\tau$. Observe that $B(0) = F_c(0), B(\tau) \geq F_c(\tau), B(\tau)$ and $F_c(\tau)$ are increasing. Likewise, Figure 4 shows the behavior of free boundaries $A(\tau)$ and $F_p(\tau)$ according to $\tau$. Observe that $A(0) = F_p(0), A(\tau) \leq F_p(\tau), A(\tau)$ and $F_p(\tau)$ are decreasing. Therefore, Figure 4 and Figure 5 illustrate the result of Theorem 4.3 numerically. The American strangle option is more expensive than the American call option and the American put option because it can obtain benefits both cases in which the underlying asset increases or decreases. Therefore, the American strangle option should be exercised at a higher price than the corresponding American call option and at a lower price than the corresponding American put option. This intuition is also consistent with the numerical results in Figure 3-4.

Figure 5 shows the upper and lower bounds of free boundaries $A(\tau)$ and $B(\tau)$, respectively. In other words, $s_p \leq A(\tau) \leq \ln(\kappa \max(1, \frac{\bar{q}}{\bar{q}}))$ and $\ln(\max(1, \frac{\bar{q}}{\bar{q}})) \leq B(\tau) \leq s_c$. The numerical results coincide with Theorem 5.3.

**Appendix A. Formulation of the model.** An American strangle option whose underlying asset is the stock which has the price $S_t$ given by

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t,$$

(A.1)

where $t$ is calendar time, $r$ represents the risk-free interest rate, $q(\geq 0)$ is the continuous dividend rate, and $\sigma > 0$ is the constant volatility of $S_t$. Also, $(W_t)_{t \geq 0}$ is a standard Brownian motion on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, P)$ with a risk-neutral measure $\mathbb{P}$, where $(\mathcal{F}_t)_{t \geq 0}$ is the natural filtration generated by $(W_t)_{t \geq 0}$.

Let $\mathcal{T}_{t,T}$ be the set of all stopping time in $[t, T]$, and the value of American strangle option price at time $t$ is defined by

$$V(t, S_t) = \max_{\theta \in \mathcal{T}_{t,T}} \mathbb{E} \left[ e^{-r(\theta - t)} \left\{ (S_\theta - K_2)^+ + (K_1 - S_\theta)^+ \right\} \right| \mathcal{F}_t],$$

(A.2)

where $K_1$ and $K_2$ denote the strike price of put and call options, respectively, and we assume $K_1 < K_2$.

It is obvious that

$$V(t, S_t) \geq (S_t - K_2)^+ + (K_1 - S_t)^+.$$  

(A.3)

By the definition of (A.2) and the law of iterated expectations,

$$e^{-rt}V(t, S_t) = \max_{\theta \in \mathcal{T}_{t,T}} \mathbb{E} \left[ e^{-r\theta} \left\{ (S_\theta - K_2)^+ + (K_1 - S_\theta)^+ \right\} \right| \mathcal{F}_t]$$

$$= \max_{\theta \in \mathcal{T}_{t,T}} \mathbb{E} \left[ e^{-r\theta} \left\{ (S_\theta - K_2)^+ + (K_1 - S_\theta)^+ \right\} \right| \mathcal{F}_{t+h}] \mid \mathcal{F}_t]$$

$$\geq \mathbb{E} \left[ \max_{\theta \in \mathcal{T}_{t+h,T}} \mathbb{E} \left[ e^{-r\theta} \left\{ (S_\theta - K_2)^+ + (K_1 - S_\theta)^+ \right\} \right| \mathcal{F}_{t+h}] \mid \mathcal{F}_t]$$

$$= \max_{\theta \in \mathcal{T}_{t+h,T}} \mathbb{E} \left[ e^{-r\theta} \left\{ (S_\theta - K_2)^+ + (K_1 - S_\theta)^+ \right\} \right| \mathcal{F}_{t+h}] \mid \mathcal{F}_t].$$
which yields $e^{-rt}V(t, S_t)$ is a supermartingale under $\mathbb{P}$. By the Itô formula, we can obtain
\[
\partial_t V(t, S_t) + \frac{\sigma^2}{2} S_t^2 \partial_{SS} V(t, S_t) + (r - q) S_t \partial_S V(t, S_t) - r V(t, S_t) \leq 0. \tag{A.4}
\]
When $V(t, S_t) > (S_t - K_2)^+ + (K_1 - S_t)^+$, there exists $h > 0$ small enough such that the optimal stopping time $\theta^* \in T_{t+h, T}$. Thus,
\[
e^{-rt} V(t, S_t) = \max_{\theta \in T_{t, T}} \mathbb{E} \left[ e^{-r\theta} \left\{ (S_\theta - K_2)^+ + (K_1 - S_\theta)^+ \right\} \mid \mathcal{F}_t \right]
= \mathbb{E} \left[ e^{-r\theta^*} \left\{ (S_{\theta^*} - K_2)^+ + (K_1 - S_{\theta^*})^+ \right\} \mid \mathcal{F}_t \right]
= \max_{\theta \in T_{t+h, T}} \mathbb{E} \left[ \{ (S_\theta - K_2)^+ + (K_1 - S_\theta)^+ \} \mid \mathcal{F}_{t+h} \right] \mid \mathcal{F}_t
= \mathbb{E} \left[ e^{-r(t+h)} V(t + h, S_{t+h}) \mid \mathcal{F}_t \right],
\]
which implies that $e^{-r(t+h)} V(t \land \theta^*, S_{t \land \theta^*})$ is a martingale under $\mathbb{P}$. We then apply the Itô formula to get
\[
\partial_t V(t, S_t) + \frac{\sigma^2}{2} S_t^2 \partial_{SS} V(t, S_t) + (r - q) S_t \partial_S V(t, S_t) - r V(t, S_t) = 0. \tag{A.5}
\]
By the definition of $V(t, S_t)$ in (A.2),
\[
V(T, s) = (s - K_2)^+ + (K_1 - s)^+. \tag{A.6}
\]
It follows from (A.3)–(A.5) that $V(t, s)$ satisfies
\[
\begin{align*}
\partial_t V + \frac{\sigma^2}{2} s^2 \partial_{ss} V + (r - q) s \partial_s V - r V &= 0, \\
&\text{if } V > (s - K_2)^+ + (K_1 - s)^+, \ (t, s) \in (0, T] \times (0, +\infty), \\
\partial_t V + \frac{\sigma^2}{2} s^2 \partial_{ss} V + (r - q) s \partial_s V - r V &\leq 0, \\
&\text{if } V = (s - K_2)^+ + (K_1 - s)^+, \ (t, s) \in (0, T] \times (0, +\infty), \\
V(T, s) &= (s - K_2)^+ + (K_1 - s)^+, \ s \in [0, +\infty).
\end{align*} \tag{A.7}
\]

Appendix B. Verification. In this section, we show that the unique solution $V(t, s)$ to the problem (1.1) coincides with the expected value of the American strangle option price, i.e., (A.2) holds. By Theorem 2.2, we know $v(\cdot, \cdot) \in W^{2,1}_{p, \text{loc}}((0, T) \times (0, +\infty)$, so $V(\cdot, \cdot)$ and $\partial_t V(\cdot, \cdot)$ are continuous in $(0, T) \times (0, +\infty)$ by embedding theorem [7]. Since $\partial_t V \leq 0$ and $(S - K_2)^+ + (K_1 - s)^+$ is the lower obstacle, we can obtain from a standard method as explained in Friedman [4] that $\partial_t V(\cdot, \cdot)$ is continuous across $s = S^A(t)$ and $s = S^B(t)$, where $S^A(t)$, $S^B(t)$ are the free boundaries of problem (1.1). Therefore $\partial_t V(\cdot, \cdot)$ is continuous in $(0, T) \times (0, +\infty)$. Moreover, $\partial_t V(\cdot, \cdot)$ and $\partial_{ss} V(\cdot, \cdot)$ are locally bounded in $(0, T) \times (0, +\infty)$. Moreover, $\partial_t V(\cdot, \cdot)$ and $\partial_{ss} V(\cdot, \cdot)$ are locally bounded in $(0, T) \times (0, +\infty)$.

It follows from Itô’s lemma that for any $\theta \in T_{t, T}$,
\[
e^{-r(\theta-t)} V(\theta, S_\theta)
= V(\theta, S_\theta) + \int_t^\theta e^{-r(u-t)} \left( \partial_t V(u, S_u) + \frac{\sigma^2}{2} S_u^2 \partial_{ss} V(u, S_u) \right) du + (r - q) S_u \partial_s V(u, S_u) - r V(u, S_u) \right) du + \int_t^\theta \sigma S_u \partial_s V(u, S_u) dW_u.
\]

(B.1)
If $V(t, S_t) \geq (S_t - K_2)^+ + (K_1 - S_t)^+$, we get

$$V(t, S_t) \geq \mathbb{E} \left[ e^{-r(t-t')} V(\theta, S_\theta) \mid \mathcal{F}_t \right] \geq \mathbb{E} \left[ e^{-r(t-t')} \left( (S_\theta - K_2)^+ + (K_1 - S_\theta)^+ \right) \mid \mathcal{F}_t \right].$$

Therefore,

$$V(t, S_t) \geq \max_{\theta \in T_{t,T}} \mathbb{E} \left[ e^{-r(t-t')} \left( (S_\theta - K_2)^+ + (K_1 - S_\theta)^+ \right) \mid \mathcal{F}_t \right]. \quad (B.2)$$

We define the stopping time $\theta^*$ as follows:

$$\theta^* = \begin{cases} 
\min\{t \in [0, T) \mid V(t, S_t) = (s - K_2)^+ + (K_1 - s)^+\}, & T, \text{ if } V(t, S_t) > (s - K_2)^+ + (K_1 - s)^+, t \in [0, T). 
\end{cases} \quad (B.3)$$

If $S^A(t) \geq S_t$ or $S_t \geq S^B(t)$, then $\theta^* = t$, and

$$V(t, S_t) = (S_t - K_2)^+ + (K_1 - S_t)^+ = \mathbb{E} \left[ e^{-r(t-t')} \left( (S_\theta - K_2)^+ + (K_1 - S_\theta)^+ \right) \mid \mathcal{F}_t \right].$$

If $S^A(t) < S_t < S^B(t)$, choose $\theta^*$ defined in (B.3), then,

$$\begin{cases} 
\partial_t V(u, S_u) + \frac{\sigma^2}{2} S^2_u \partial_{uu} V(u, S_u) + (r - q) S_u \partial_u V(u, S_u) - rV(u, S_u) = 0, & \text{for } u \in (t, \theta^*), \\
V(\theta^*, S_{\theta^*}) = (S_{\theta^*} - K_2)^+ + (K_1 - S_{\theta^*})^+. 
\end{cases} \quad (B.4)$$

By (B.1) and (B.4), we can obtain

$$V(t, S_t) = \mathbb{E} \left[ e^{-r(t-t')} V(\theta^*, S_{\theta^*}) \mid \mathcal{F}_t \right] = \mathbb{E} \left[ e^{-r(t-t')} \left( (S_{\theta^*} - K_2)^+ + (K_1 - S_{\theta^*})^+ \right) \mid \mathcal{F}_t \right].$$

Then we conclude from (B.2) that

$$V(t, S_t) = \max_{\theta \in T_{t,T}} \mathbb{E} \left[ e^{-r(t-t')} \left( (S_\theta - K_2)^+ + (K_1 - S_\theta)^+ \right) \mid \mathcal{F}_t \right].$$

REFERENCES

[1] J. S. Chaput and L. H. Ederington, Volatility trade design, Journal of Futures Markets, 25 (2005), 243–279.
[2] X. Chen, F. Yi and L. Wang, American lookback option with fixed strike price-2-D parabolic variational inequality, J. Differential Equations, 251 (2011), 3063–3089.
[3] C. Chiarella and A. Ziogas, Evaluation of American strangles, Journal of Economic Dynamics and Control, 29 (2005), 31–62.
[4] A. Friedman, Parabolic variational inequalities in one space dimension and smoothness of the free boundary, J. Funct. Anal., 18 (1975), 151–176.
[5] A. Friedman, Variational Principles and Free-Boundary Problems, John Wiley & Sons, Inc., New York, 1982.
[6] L. Jiang, Existence and differentiability of the solution of a two-phase Stefan problem for quasilinear parabolic equations, Acta Math. Sinica, 15 (1965), 749–764.
[7] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural’ceva, Linear and Quasi-Linear Equations of Parabolic Type, American Mathematical Society, Providence, RI, 1968.
[8] G. M. Lieberman, Second Order Parabolic Differential Equations, World Scientific Publishing Co., Inc., River Edge, NJ, 1996.
[9] J. Ma, W. Li and Z. Cui, Valuation of American strangles through an optimized lower-upper bound approach, *Journal of Operations Research Society of China*, 6 (2018), 25–47.
[10] S. Qiu, *American Strangle Options*, Research Report, School of Mathematics, The University of Manchester, 2014.
[11] K. Tso, On an Aleksandrov-Bakel’man type maximum principle for second-order parabolic equations, *Comm. Partial Differential Equations*, 10 (1985), 543–553.
[12] Z. Yang and F. Yi, Valuation of European installment put option: Variational inequality approach, *Communications in Contemporary Mathematics*, 11 (2009), 279–307.
[13] Z. Yang and F. Yi, A variational inequality arising from American installment call options pricing, *J. Math. Anal. Appl.*, 357 (2009), 54–68.
[14] Z. Yang, F. Yi and M. Dai, A parabolic variational inequality arising from the valuation of strike reset options, *J. Differential Equations*, 230 (2006), 481–501.
[15] Z. Yang, F. Yi and X. Wang, A variational inequality arising from European installment call options pricing, *SIAM Journal on Mathematical Analysis*, 40 (2008), 306–326.

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