Interference of the evolutions of Rb and Na Bose-Einstein condensates

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Spin evolution of a mixture of two species of spin-1 Bose-Einstein condensates (Rb and Na) has been investigated beyond the mean field theory. Both analytical expression and numerical results on the populations of spin-components are obtained. The interference is found to depend on the initial states and the inter-species interaction sensitively. A number of phenomena arising from the interference have been predicted. In particular, periodic behavior and the alternate appearances of a zone of oscillation and a long quiet zone are found.

PACS numbers: 03.75. Fi, 03.65. Fd

Optical traps of spin-1 to spin-3 bosonic atoms have been implemented experimentally, which liberate the freedoms of spin of the atoms [1, 2, 3, 4, 5, 6]. At low temperature, the trapped atoms will form a spinor Bose-Einstein condensate. The eigen-states of the system conserve the total spin $S$ and its Z-component $M$. If the system is given at an initial state which is not an eigenstate but a superposition of them, e.g., a Fock-state, the initial state will evolve afterward. The evolution of spins is an exquisite process because it depends not only on the interactions but also very sensitively on the initial states. The study of spin-evolution is not only an attractive topic but also promising in promoting micro-techniques. This topic has been extensively studied experimentally and theoretically. One can measure and calculate the evolution of the average populations of the three spin components $\mu = 1, 0, -1$. This process is called spin mixing or spin dynamics [7, 8, 9, 10, 11, 12, 13, 14]. However, how the spin-evolution of a species would affect the evolution of another species when they are mixed up in a trap is a topic remains untapped.

In a previous work [15], beyond the mean field theory, the spin-evolution of a mixture of two species of spin-1 atoms is in principle possible to be realized. A study of the eigen-states of the mixture has been given in [19]. Where the total spin-states are labeled by good quantum numbers $(S_A, S_B, S, M)$, where $S_A(S_B)$ is the total spin of a species A(B), and $S$ and $M$ are for the whole system. As a generalization of the above two works, in this paper we study the spin-evolution of the mixture. It is expected that the interference due to the inter-species interaction will lead to new physics, which might enrich the charming of this attractive field.

It is assumed that the two species, A and B, contain $N_A$ and $N_B$ spin-1 atoms, respectively. The interspecies interaction is $V_{ij}^{AB} = (c_0^A + c_2^A F_i F_j) \delta (r_i - r_j)$, where $F_i$ is the operator of the spin of the $i-$th atom, $C = A$ or $B$. The inter-species interaction is $V_{ij}^{AB} = (c_0^A + c_2^A F_i F_j) \delta (r_i - r_j)$. Under the single mode approximation (SMA) all the atoms of the same species have the same spatial wave function $\phi_A$ or $\phi_B$. We shall focus at the evolution of the spins, while $\phi_A$ and $\phi_B$ are assumed not to be changed with time. Accordingly, the Hamiltonian responsible for the spin evolution reads

$$H = g_A S_A^2 + g_B S_B^2 + g_{AB} \hat{S}^2$$

where $\hat{S}_A$, $\hat{S}_B$, and $\hat{S}$ are the total spin operators of the species A, B, and the whole of them, respectively.

$$\begin{align*}
g_A &= \frac{1}{2} (c_A^2 \int dr |\phi_A|^4 - c_2^A \int dr |\phi_A|^2 |\phi_B|^2) \\
g_B &= \frac{1}{2} (c_B^2 \int dr |\phi_B|^4 - c_2^B \int dr |\phi_A|^2 |\phi_B|^2) \\
g_{AB} &= \frac{1}{2} c_2^{AB} \int dr |\phi_A|^2 |\phi_B|^2
\end{align*}$$

Accordingly, the eigen-states of the Hamiltonian read

$$|\alpha, M\rangle = |S_A, S_B, S, M\rangle = \sum_{M_A, M_B} C_{SM_A M}^{SM} |\phi_A^{N_A} S_A, M_A\rangle |\phi_B^{N_B} S_B, M_B\rangle$$

where $\alpha$ denotes the set $(S_A, S_B, S)$, $|\phi_{NC}^{S_C M_C}\rangle$ denotes a all-symmetric spin-state of the C-species with spin $S_C$ and its Z-component $M_C$. Then $S_A$ and $S_B$ are coupled to the total spin $S$ via the Clebsch-Gordan coefficients. Evidently, $S_C \leq N_C$, $|S_A - S_B| \leq S \leq S_A + S_B$, and $S \geq M$. Besides, due to the requirement of symmetry, $N_C - S_C$ must be even and $|\phi_{NC}^{S_C M_C}\rangle$ is unique. [17, 18]

When $N_A$, $N_B$, and $M$ are given, the set of eigen-states with $S_A$, $S_B$, and $S$ satisfying the above requirement are complete for the Hamiltonian eq. (1).

Incidentally, we have assumed that the external magnetic field is exactly zero. In experiments, if a weak residual magnetic field exists, in order that the effect of the quadratic Zeeman term can be neglected, from the experience of the single-species evolution, the field should be weaker than $10^{-4}$ Gauss.
A Fock-state of a species is denoted as $|N_{C1}, N_{C0}, N_{C,-1}⟩$, where $N_{Cµ}$ is the number of C-atoms with the spin-component $µ$ (0 or ±1), and $N_{C} = \sum_{µ} N_{Cµ}$. Initially, it is assumed that each species is in a Fock-state. Thus the initial state $|O⟩ = |N_{A1}^{'}, N_{A0}^{'}, N_{A,-1}⟩|N_{B1}^{'}, N_{B0}^{'}, N_{B,-1}⟩$, where $N_{Cµ}'$ are the initial values. The associated formal time-dependent solution is just $|Ψ(t)⟩ = e^{-iHt/\hbar}|O⟩$. When $|O⟩$ is expanded by the eigen-states of $H$ (eq. (3)), the formal solution can be written in a practicable form as

$$|Ψ(t)⟩ = \sum_{α} e^{-iE_{α}t/\hbar}|α, M⟩⟨α, M|O⟩$$  \hspace{1cm} (4)

where $M = N_{A1}^{'1} - N_{A1}^{'1} - N_{B1}^{'1} - N_{B1}^{'1}$, $E_α = g_{A}S_{A}(S_{A} + 1) + g_{B}S_{B}(S_{B} + 1) + g_{A}g_{B}S(S + 1)$. Since the set $|α⟩$ is complete, the above expression is an exact solution of the Hamiltonian if the coefficients $|α, M⟩$ can be exactly calculated. This is given in the appendix. Since $M$ is conserved during the evolution, $|α, M⟩$ is written simply as $|α⟩$ in the follows.

Our aim is to calculate the time-dependence of the populations

$$⟨Ψ(t)|\hat{N}_{α}^{C}|Ψ(t)⟩/N_{C} = P_{α}^{C}(t)$$  \hspace{1cm} (5)

where $\hat{N}_{α}^{C}$ is the operator of number of C-atoms in $µ$. Obviously, $\sum_{µ} P_{α}^{C}(t) = 1$ holds at any time. Inserting $t$ into (5), we have

$$P_{α}^{C}(t) = B_{α}^{C} + O_{α}^{C}(t)$$  \hspace{1cm} (6)

with

$$B_{α}^{C} = \frac{1}{N_{C}} \sum_{α} ⟨α|\hat{N}_{α}^{C}|α⟩⟨O|α⟩^{2}$$  \hspace{1cm} (7)

$$O_{α}^{C}(t) = \frac{2}{N_{C}} \sum_{α < α'} \cos[(E_{α'} - E_{α})t/h]⟨α'|\hat{N}_{α}^{C}|α⟩⟨O|α⟩⟨O|α'⟩$$  \hspace{1cm} (8)

The expression of $⟨α'|\hat{N}_{α}^{C}|α⟩$ is also given in the appendix. The above formulae are a direct generalization of those given in eq. (3) to (7) of [12] for a single species, and they reveal an oscillation described by $O_{α}^{C}(t)$ around a background $B_{α}^{C}$. It is noticeable that $B_{α}^{C}$ depends not at all on the interactions, but only determined by the inherent symmetry and the initial state. In fact, the dynamics affects only the factor $E_{α'} - E_{α}$ in $O_{α}^{C}(t)$ (this would cause a re-scale of time as we shall see). All other factors in (7) and (8) are independent of dynamics. Furthermore, from eq. (8) and the appendix, $⟨α'|\hat{N}_{α}^{A}|α⟩$ is nonzero only if $S_{B} = S_{B}$, it implies that the associated $E_{α'} - E_{α}$ does not depend on $g_{B}$, and therefore $P_{α}^{A}$ does not depend on $g_{B}$. Similarly, $P_{α}^{B}$ does not depend on $g_{A}$.

We choose $^{87}$Rb as A and $^{23}$Na as B. They are confined by a parabolic potential with frequency $ω = 300/\text{sec}$ (unless specified). The parameters for the inter-species interaction are from Table III of ref. [20]. The set of coupled Gross-Pitaevskii equations for the mixture derived in our previous paper [10] has been used to obtain the spatial wave functions. Numerical results on $P_{α}^{C}(t)$ for selected cases are given in the follows.

(1) Interference of two initially fully polarized systems

If both species are polarized and they have the same direction of polarization, no spin-flips would occur due to the conservation of the total magnetization. Therefore only the cases with opposite directions are considered. Let an initial state be hereafter labeled as $(N_{A1}^{'}, N_{A0}^{'}, N_{A,-1}^{'})A(N_{B1}^{'}, N_{B0}^{'}, N_{B,-1}^{'})B$. For the case $(N_{A,0,0}, A(0,0, N_{B})_B$, namely, all A-atoms are up and all B-atoms are down initially, the factors in eq. (3) have simpler forms. They are

$$⟨O|α⟩ = δ_{S_{A},N_{A}}δ_{S_{B},N_{B}}δ_{M,N_{A},N_{B}}C_{N_{A},N_{A},N_{B},N_{B}}^{S,N_{A},N_{B}}$$  \hspace{1cm} (9)

and

$$⟨α'|\hat{N}_{α}^{A}|α⟩$$  \hspace{1cm} (10)

where the quantities inside the brackets are the fractional parentage coefficients given in the appendix [16]. Since both the Clebsch-Gordan and fractional parentage coefficients arise purely from symmetry, it is clear that symmetry is decisive to the evolution.

Due to eq. (3) the factor $⟨O|α⟩⟨O|α'⟩$ in eq. (3) assures $S_{A} = S_{A} = N_{A}$, and $S_{B} = S_{B} = N_{B}$. Accordingly, $E_{α'} - E_{α} = g_{A}g_{B}S'(S' + 1) - S(S + 1)$. It implies three points: (i) The evolutions of $P_{α}^{A}(t)$ and $P_{α}^{B}(t)$ do not depend on $g_{A}$ and $g_{B}$, but on $g_{A}g_{B}$, (ii) The effect of interactions is embodied uniquely via the factor $g_{A}g_{B}$. Therefore the effect is just a re-scale of time, namely, to accelerate or slow down the evolution. The general features of evolutions are determined by inherent symmetry and the initial conditions. (iii) Since $S'(S' + 1) - S(S + 1)$ must be an even integer $I_{e}$, the time-dependent factor in eq. (5) can be rewritten as $\cos(I_{e}g_{A}g_{B}t/h)$. Thus $P_{α}^{A}(t)$ and $P_{α}^{B}(t)$ are strictly periodic with the same period $t_{p} = \pi h/|g_{A}g_{B}|$, and they are symmetric with respect to $t_{p}/2$. So, if the period can be determined, it provides an opportunity to measure the A-B interaction. In what follows, $t_{p}$ is used as the unit of time.

When the initial state is $(40, 0, 0)A(0, 0, 40)_{B}$, $P_{α}^{C}(t)$ are plotted in Fig. 1 against $τ = t/t_{p}$ from 0 to 1.2. Where the strict periodicity together with the symmetry with respect to $τ = 1/2$ are clearly shown. If the inter-species interaction is removed, there would be no spin-flips due to the conservation of the magnetization of each species. Due to the interaction, in the early stage,
there is a very strong process of spin-flips initiated by the collisions between the up A-atoms and down B-atoms. Thereby $P_A^1$ drops rapidly from 1 to a minimum 0.222 at $\tau = \tau_{\text{min}} = 0.052$, while $P_A^3$ and $P_A^4$ increase from zero rapidly. When $\tau \approx 0.1$, the system arrives at a steady stage with $P_A^1 = P_A^3 \approx 3/8$ and $P_A^0 \approx 1/4$. The equality $P_A^1 = P_A^3$ implies that the system has given up all its previous polarization. Afterward, the system keeps quiet in a long duration until $\tau \approx 0.9$. Then, in the last stage of the period, the previous magnetization is completely recovered. Thus, a zone of oscillation (ZoS) followed by a quiet zone, and again a ZoS appear successively and repeated. However, there is a small turbulence occurring at the right middle of the quiet zone.

During all the time the behavior of the B-atoms match exactly with the A-atoms, namely, $P_B^1(\tau) = P_B^3(\tau)$ due to symmetry as expected. It is emphasized that the appearance of the quiet zone is a noticeable and quite popular feature of spin-evolution. This feature is in nature a quantum phenomenon of interference as shown by eqs. [9], [10], and [3]. From these formulae it is clear that the effect of interactions is simply to adjust the length of the zone (longer or shorter).

It is noted that $g_{AB} \propto \int \text{d} \tau |\phi_A|^2 |\phi_B|^2$, which is roughly proportional to $\omega^6/3N^{-3/5}$. Thus it is clear that increasing the confinement or reducing the particle number will accelerate the evolution and shorten the period $t_p$ (e.g., Numerically, when $N_A = N_B = 40$ and $\omega = 300$, we have $t_p = 558.6$ sec. If $\omega$ increases by ten times, $t_p = 35.2$ sec). In eq. [3] all factors, except the time-dependent factor, are irrelevant to $\omega$ and insensitive to $N$ (if the ratios of the $N_C$ of the initial state remain the same). Therefore the change of $\omega$ is simply equivalent to a re-scale of time, and the variation of $N$ (in the above sense) would cause only a slight change quantitatively but not qualitatively.

In order to see the effect of the asymmetry of particle numbers, the initial case $(40, 0, 0)$ is considered as shown in Fig.1a. A remarkable feature is the great separation of $P_C^1$ and $P_C^3$ curves in the quiet zone. It implies that the initial polarization of both species are partially preserved.

Experimentally, the polarization might not be perfect. An example is given in Fig.1b, where each species has four atoms in $\mu = 0$ initially. Comparing Fig.1a with Fig.1b, it is clear that the four $\mu = 0$ atoms of each species cause additionally four rounds of oscillations in each ZoS. Thus the ZoS becomes longer and the quiet zone becomes shorter accordingly. In each round of oscillation $P_C^1$ and $P_C^3$ keep to have reverse phases. In general, when the initial state is $(N_A - K, K, 0)_A(0, K, N_A - K)_B$, it was found that $K$ rounds of oscillation will emerge in each ZoS additionally.

(I) The interference of a fully polarized system with a zero-polarized system

When the initial state is $(40, 0, 0)_A(20, 0, 20)_B$, the evolution is shown in Fig.2, which is greatly different from Fig.1a. Meanwhile a half of B-atoms are up while the other half are down initially. Thus the total spin of the group of up-B-atoms and that of the down-B-atoms have the same magnitude but opposite directions. Therefore the vector sum of them must be small. Hence, those $|\alpha|$ with a large $S_B$ will lead to a very small $\langle \alpha | O \rangle$, and therefore can be omitted from the expansion of $\Psi(t)$ (refer to eq. [4]). In other words, during the evolution, $S_B$ remains to be small. Consequently, the magnetization of the B-atoms, $M_B$, remains to be small. Thus, in order to keep $M = M_A + M_B$ to be conserved, $P_A^1(\tau)$ must be always close to 1. This deduction is confirmed by Fig.2.

Since the A-atoms are always close to be fully-polarized, they are inert. One might therefore guess that the B-atoms would behave just as if the A-atoms do not exist. This suggestion is partially true. Comparing the curves of $P_B^1(\tau)$ shown in Fig.2, with those with the A-atoms removed (not plotted), they are one-to-one similar in pattern, but different in the scale of time. In fact, the former is a compressed version of the latter, namely, the evolution of the former is more rapid. For an example, the former is roughly periodic with a period $\approx 0.64 t_p$ while the latter is exactly periodic with a period $1.65 t_p$. It is mentioned that the Hamiltonian of the latter is $g_B^{\text{free}} S_B^z$, where $g_B^{\text{free}} = \pi \hbar / (4 \epsilon B^2 \int \text{d} \tau |\phi_B|^4)$. On the other hand, both $c_2^A$ and $c_2^B$ are contributed to $g_B$. It turns out that these two strengths have opposite signs. This leads to $g_B \gg g_B^{\text{free}}$ (refer to eq. [2]), and therefore a faster evolution. Incidentally, $P_B^1(\tau)$ plotted in Fig.2 overlaps nearly with $P_B^1(\tau)$. This confirms the previous deduction that $M_B$ remains small.

On the other hand, although $P_A^1(\tau)$ varies very slightly with time, it is strictly periodic. In general, when one of the species (say, A) is fully-polarized initially, its evolution is strictly periodic disregarding how B is initially. This arises because meanwhile the factor $\langle O | \alpha \rangle \langle \alpha | O \rangle$ in eq. [3] assures $S_A = S_A^\prime = N_A$, while the factor $\langle \alpha | N_C^\prime | \alpha \rangle$ assures $S_B^\prime = S_B$ (refer to the appendix). Accordingly, the time-dependent factor in [3] becomes $\cos((S^\prime(S^\prime + 1) - S(S + 1)) g_B^{\text{free}} \tau / \hbar) = \cos(I_{g_B^{\text{free}}} \tau / \hbar)$, thus the strict periodicity with the period $t_p$ is clear.

Incidentally, if A and B interchange their initial status, namely, the initial state is $(20, 0, 20)_A(40, 0, 0)_B$, then the B-atoms would be inert, while the behavior of $P_A^1$ would be similar to the $P_B^1$ of Fig.2a, in pattern but with a longer period. The reason is that both $c_2^A$ and $c_2^B$ are negative resulting in a weaker $g_A$ and therefore a slower evolution.

When the initial state of Fig.2a is slightly changed to $(40, 0, 0)_A(18, 4, 18)_B$, namely, a few B-atoms are initially in $\mu = 0$, the evolution is thereby changed dramatically as shown in Fig.2b. This suggests that a larger $S_B$ can be contained in $\Psi(t)$, so that a greater part of the total magnetization can be transferred to the B-atoms. Accordingly, $P_A^1(\tau)$ may differ from 1 explicitly, and the interference of the two species becomes stronger. In Fig.2b, the periodicity
of the A-species with the period \( t_p \) is clearly shown, and the two ZoS together with the quiet zone are also found. In the quiet zone of \( P^A_{\pi} \), we have \( P^A_{\pi} - P^A_{\pi} \approx 0.85 \). It implies that 15% of magnetization has been transferred to B. However, \( P^A_{\pi} \) remains to be very small. It is noticeable that \( P^B_\beta \) is strongly affected by \( P^A_\pi \). In particular, when the A-atoms oscillate strongly, the B-atoms also. Thus, \( P^B_\beta \) seems to be a mixture of two modes, one matches the evolution of A with the two ZoS, the other one is just the residual of its original mode.

When half of the B-atoms have \( \mu = 0 \) initially, the evolution is plotted [2], which is similar to [2]. However, \( P^B_\beta \) and \( P^A_\pi \) match more strongly with each other in reverse phase (a peak of \( P^B_\beta \) matches a dip of \( P^A_\pi \)), and the residual mode in \( P^B_\beta \) becomes very weak. Thus [2] and [2] are two typical examples to show a very weak and a very strong interferences, respectively.

(III) Effect of the inter-species interaction

Let the realistic \( V_{ij}^{AB} \) be changed to \( \beta V_{ij}^{AB} \), then we study the effect of the adjustable strength \( \beta \). When the initial state, for an example, is \( (40, 0, 0) \) in the A-polarized state, three versions with \( \beta = 0, 0.05 \) and 0.3, respectively, are plotted in Fig [3a] to c. It is reminded that, once the inter-species interaction is removed, the evolution of each species is strictly periodic with the period \( t_p^{AB} = \pi \hbar / (\beta C^{ij}) \). In particular, \( t_p^A = 3.721 t_p \) and \( t_p^B = 1.65 t_p \) (with our parameters). When \( \beta = 0 \), the A-atoms do not evolve, while the B-atoms evolve according to the law given in ref [2]. In particular, each time when \( t \) is close to \( K_t^{AB} / 4 \), where \( K = 0, 1, 2, \cdots \), the ZoS appears wherein \( P^B_\beta (\tau) \) would contain a few rounds of oscillation. Between two adjacent ZoS, it is the quiet zone.

The case with \( \beta = 0.05 \) is shown in [3b]. Two points are noticeable. (i) Since the A-atoms have been fully-polarized initially, if they can evolve via the help of the B-atoms, they should behave periodically as mentioned previously. The period should be \( t_\beta^{(3)} = \pi \hbar / |g^B_{\beta AB}| \), where \( g^B_{\beta AB} \) is similarly defined as \( g_{AB} \) but with \( c^{2AB}_B \) replaced by \( c\beta c^{AB}_B \). Since \( |g^B_{\beta AB}| \ll \beta |g_{AB}| \), \( t_\beta^{(3)} \) is very large. In fact, the range of \( \tau \) in [3b] is only a small portion of \( t_\beta^{(3)} \).

(ii) Although \( \beta V_{ij}^{AB} \) is so weak, \( P^B_\beta \) is seriously affected by the A-atoms. Consequently, \( P^B_\beta \) is a mixture of two modes, just as the previous case in Fig [2] one matches \( P^A_\pi \) in reverse phase, while the other is the residual of the original mode. Thus, both systems oscillate with a large amplitude and a very low frequency, while additional small oscillation takes place in \( P^B_\beta \) occasionally due to the second mode.

When \( \beta \) increases further, \( t_\beta^{(3)} \) becomes shorter, and the corresponding \( P^C_\beta \) is plotted in Fig [3c] which is a compression version of [3] towards left. In [3]: the range of \( t \) is close to \( 1 t_\beta^{(3)} / 3 \), and we can see the broad quiet zones.

When \( \beta = 1 \), the patterns in [3] are further compressed.

One more example is shown in Fig [4] with the initial state \((36, 4, 0) A (0, 4, 36) B \) corresponding to Fig [1]. If the inter-species interaction is removed, both species are inert as shown in [4] because they are nearly fully-polarized initially. When \( \beta = 0.05 \) as shown in [4b], the evolutions of both systems become nearly periodic with the same long period close to \( t_\beta^{(3)} = \pi \hbar / |g_{\beta AB}| \) as mentioned before. When \( \beta \) is larger, the pattern is compressed towards left as shown in [4]. A further compression of [4] leads to Fig [4].

In summary, a theoretical tool (the fractional parentage coefficients) has been introduced to study the spin-evolution of a mixture of the condensates of \( ^8 \)Rb (A) and \( ^23 \)Na (B) atoms. Making use of the single-mode-approximation, a formula has been derived to describe the evolution beyond the mean field theory. The evolution of each spin component \( P^C_\beta (\tau) \) appears as an oscillation around a background. The background is determined by the inherent symmetry and the initial state, and is not at all affected by the interactions. Based on the formula, selected cases have been studied numerically. A number of predictions on the character of evolution have been made.

When both species are (nearly) fully polarized in reverse direction initially (Fig [1]), this is a case of strong interference. The evolution of two species are closely match with each other \((P^A_\pi (\tau) = P^B_\beta (\tau))\) and evolve with the same mode (in reverse phase) and the same period \( t_p \) determined by the inter-species interaction. ZoS and quiet zone appear alternately. In the former, several rounds of oscillation will emerge in \( P^C_\beta (\tau) \), and the total magnetization of each species will undergo a great change (from being fully polarized to zero polarized, or vice versa). The appearance of quiet zones is quite popular in spin-evolution and is a quantum phenomenon of interference.

When A is fully polarized while B is zero-polarized, and if B does not have \( \mu = 0 \) atoms initially (Fig [2a]), this is a case of weak interference. However, if B does have a few \( \mu = 0 \) atoms initially, the few atoms can work as a catalyst and will cause a strong interference (comparing [2a] and [2b]). The sensitivity of the initial \( \mu = 0 \) atoms is a noticeable point.

When A is fully polarized, disregarding how B is initialized, if we introduce \( \beta \) to adjust the strength of the inter-species interaction, we found that the interference can be initiated by a very small \( \beta \) and a new mode of evolution is thereby caused (Fig [3]). Accordingly, A evolves with the new mode and B evolves with a mixture of the new mode and a residual. However, if \( \beta \) is very small, the new mode will have a very long period, therefore the interference is difficult to be observed in the early stage. Nonetheless, when \( \beta \) increases, the period of the new mode becomes shorter and causes a compression of \( P^C_\beta (\tau) \) towards left [3]. Finally, both species are dominated by the new mode.

The above predictions remain to be confirmed experimentally. Furthermore, the effectiveness of the single-
mode approximation and the particle numbers (if they are very large) deserved to be further studied.

Acknowledgments

We appreciate the support from the NSFC under the grants 10574163 and 10674182.

Appendix

1. The calculation of $\langle O | \alpha \rangle$
The initial state

$$ | O \rangle = | N_{A,1}^I, N_{A,0}^I, N_{A,-1}^I | N_{B,1}^I, N_{B,0}^I, N_{B,-1}^I \rangle \quad (11) $$

Let

$$ M_C^I = N_C^I - N_{C,-1}^I \quad (12) $$

where $C = A$ or $B$, and $M = M_A^I + M_B^I$. Thus

$$ \langle O | \alpha \rangle = C^{SM}_{S_A M_A S_B M_B} D_{N_A A_0 M_A N_0 B_0 M_B} D_{N_B S_B M_B} \quad (13) $$

where $D^{NC,SC}_{M_C^I} = \{ N_{C,1}^I, N_{C,0}^I, N_{C,-1}^I | \hat{\rho}_{NC,MC}^I \}$ is for a single species and is very useful. These coefficients can be calculated via the following recursion formulae

$$ \sqrt{(N - N_0 + M)/(2N)} D_{N,M,N_0}^{N,S} = A(N, S, M, 1) D_{M,N_0}^{N-1,S+1+1} + B(N, S, M, 1) D_{M,N_0}^{N-1,S-1} \quad (14) $$

$$ \sqrt{N_0/N} D_{M,N_0}^{N,S} = A(N, S, M, 0) D_{M,N_0}^{N-1,S+1} + B(N, S, M, 0) D_{M,N_0}^{N-1,S-1} \quad (15) $$

$$ \sqrt{(N - N_0 - M)/(2N)} D_{M,N_0}^{N,S} = A(N, S, M, -1) D_{M+1,N_0}^{N-1,S+1} + B(N, S, M, -1) D_{M+1,N_0}^{N-1,S-1} \quad (16) $$

where

$$ A(N, S, M, \mu) = \left[ \frac{1 + (-1)^{N-S}(N - S)(S + 1)}{(2N(2S + 1))} \right]^{1/2} C^{S,M}_{S+1,M-\mu, 1,\mu} \quad (17) $$

$$ B(N, S, M, \mu) = \left[ \frac{(1 + (-1)^{N-S})S(N + S + 1)}{2N(2S + 1)} \right]^{1/2} C^{S,M}_{S-1,M-\mu, 1,\mu} \quad (18) $$

The above recursion formulae are derived based on the analytical expressions of the fractional parentage coefficients given in [10] and [17].

2. The calculation of $\langle \alpha' | \hat{\rho}_A | \alpha \rangle$
Let $C = A$, then

$$ \langle \alpha' | \hat{\rho}_A | \alpha \rangle = \langle S_A', S_B', S_A' | \hat{\rho}_A | S_A, S_B, S, M \rangle \quad (19) $$

$$ = \delta_{S_A'S_B'} \sum_{M_A, M_B} C^{S',M}_{S_A'M_A, S_B'M_B} C^{S,M}_{S_A'M_A, S_B'M_B} \langle \hat{N}^A | \hat{N}^A \rangle \quad (20) $$

Where the last factor concerns only the $A$-species, we have

$$ \langle \hat{N}^A | \hat{N}^A \rangle = \delta_{M_A, M_A} N_A \quad (21) $$

$$ \{ \delta_{S_A', S_A} (A(N_A, S_A, M_A, \mu))^2 + (B(N_A, S_A, M_A, \mu))^2 \} + \delta_{S_A', S_A} 2A(N_A, S_A, 2, M_A, \mu)B(N_A, S_A, M_A, \mu) \}

The derivation of (21) is also based on the fractional parentage coefficients. For the case $C = B$, the calculation is similar.

In particular, when $\alpha' = \alpha$, $\mu = 0$, and $N_A$ is large, we have

$$ \langle \alpha | \hat{\rho}_A | \alpha \rangle \approx \frac{1}{2N_A} N_A \left[ 1 - \sum_{M_A, M_B} (C_{S_A'M_A, S_B'M_B} N_A/S_A)^2 \right] \leq N_A/2 \quad (21) $$

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It is recalled that the interactions of both the A and B species are dominated by the spin-independent part, which are both repulsive. Therefore, we can assume that $\phi_A$ and $\phi_B$ would depend on the particle number and $\omega$ in a similar way. Accordingly, $\int |d_{r}\phi_A|^2 |d_{r}\phi_B|^2$ and $\int |d_{r}\phi_A|^4$ would have similar dependences. It is well known that the latter is approximately $\propto \omega^{6/5} N^{-3/5}$. It is reasonable to suggest that this dependence holds also for the former. This suggestion has been supported by numerical results.
FIG. 1: Interference of the evolutions of two initially fully polarized condensates A(Rb) and B(Na). $P_{1A}^A(t)$ (black), $P_{-1A}^A(t)$ (red), $P_{1B}^B(t)$ (blue), and $P_{-1B}^B(t)$ (green) are plotted against $\tau = t/t_p$. The initial particle numbers of the spin-components $(N_{C1}, N_{C0}, N_{C, -1})_C$ ($C = A$ or $B$) are marked in each panel. $\omega = 300$ sec is assumed. The implication of the colors and the magnitude of $\omega$ are the same in the follows. In (a) and (c) the black curve overlaps the green, the red overlaps the blue.
FIG. 2: Interference of an initially fully polarized condensate with an initially non-polarized condensate.

FIG. 3: Interference of an initially fully polarized condensate with a half polarized condensate. The actual inter-species interaction has been reduced by a factor $\beta$ which is marked in each panel.
FIG. 4: Interference of two initially polarized condensates. The polarizations of both systems are not perfect, each contains four $\mu = 0$ atoms. The actual inter-species interaction has been reduced by a factor $\beta$ which is marked in each panel.