Gauge Invariant Exact Renormalization Group and Perfect Actions in the Open Bosonic String Theory.

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**Abstract**

The exact renormalization group is applied to the world sheet theory describing bosonic open string backgrounds to obtain the equations of motion for the fields of the open string. Using loop variable techniques the equations can be constructed to be gauge invariant. Furthermore they are valid off the (free) mass shell. This requires keeping a finite cutoff. Thus we have the interesting situation of a scale invariant world sheet theory with a finite world sheet cutoff. This is possible because there are an infinite number of operators whose coefficients can be tuned. This is in the same sense that "perfect actions" or "improved actions" have been proposed in lattice gauge theory to reproduce the continuum results even while keeping a finite lattice spacing.
1 Introduction

The renormalization group has been applied to the world sheet action for a string propagating in non-trivial backgrounds to obtain equations of motion. [[1]-[14]]. As a generalization of this technique, Loop Variable techniques have been used to write down gauge invariant equations of motion for both open and closed strings [15, 16, 17]. These are essentially equations that set to zero the change in coupling constants of the two dimensional world sheet field under scale transformations, i.e. these are conditions for a fixed point under a renormalization group (RG) transformation. There are a couple of noteworthy features: One is that gauge invariance (in space time) necessitates including all the modes of the string. Another is that in order to deal with non-marginal vertex operators, i.e. for space time fields that do not obey the mass shell constraint, it is necessary to keep the world sheet cutoff finite, at least in the intermediate stages of the calculation. Some aspects of the finite cutoff theory has been discussed in [7, 18, 19] where it was shown that if one keeps a finite cutoff, the proper time equation for the tachyon (which in this situation is related to the RG equation), become quadratic. This is as expected both from string field theory and also from the exact renormalization group [[20]-[23]]. In [19] it was also shown that one can make precise contact with light cone string field theory by keeping a finite cutoff ¹ In [18] it was also shown that if one wants to maintain gauge invariance while maintaining a finite cutoff one needs to include all the massive modes in the proper time equation. In this sense string field theory [25, 24, 26, 27] can be thought of as a way of keeping a finite cutoff while maintaining gauge invariance.

Another approach to off shell string theory is the background independent approach pioneered in [28] and further developed in [29, 30, 31]. The connection with the RG approach is discussed in [30, 31].

Implicit in the RG approach is the interesting fact that one is maintaining a finite cutoff while discussing a scale invariant theory. This is possible because one has an action with all possible operators and thus it is conceivable that with an infinite number of fine tunings one can satisfy the fixed point conditions and attain scale invariance even when the cutoff is non zero.

The main aim of this work is to write the gauge invariant loop variable equations in the form of an exact RG (ERG) equation (with finite

¹In light cone string field theory there is also a quartic term, which is a subtlety that is not addressed here.
cutoff). (See [23, 32, 33, 34, 35] for a discussion of many conceptual issues encountered in the ERG.) The gauge fixed version is quadratic in fields and therefore is similar to string field theory. But in the absence of a world sheet symmetry principle there could be a lot of arbitrariness in the scheme and one is not sure if the result is equivalent to string theory, in particular whether it can be made gauge invariant. In the loop variable approach, space-time gauge invariance is built in. This ensures that in the critical dimension negative norm states decouple. Having space time gauge invariance built into it at the outset is thus reassuring. The fact that we do not rely on world sheet reparametrization (or BRS) for space time gauge invariance is an advantage in that there is no clash at any time between world sheet regularization and spacetime gauge invariance. This gives a lot of freedom in choice of regularization. This was exploited recently to construct a (free) higher spin action in AdS space-time [36]. This is not so straightforward in string field theory. However unlike string field theory the equations of motion are not quadratic - they involve higher order terms.

The phenomenon of scale invariance at finite cutoff is interesting in its own right. This is related to the idea of "improved actions" [37] or "perfect actions" [38] introduced in the context of lattice gauge theory. (See also [32, 33].) The basic idea is that if one is exactly on the RG trajectory connecting the UV and IR fixed points then one is infinitely far from the continuum (i.e. one has a finite lattice spacing) and yet it is physically equivalent to it - because it is on the same RG trajectory. One can take this one step further and say that if one starts at the fixed point itself with a finite cutoff then after an infinite number of steps, when one has reached the continuum, we still have the same action! Thus the fixed point action is scale invariant with either finite or zero cutoff.

In string theory this perfect action also describes the precise values of all the infinite number of massive modes in a background that is a solution to the classical equations of motion. The extension to the quantum theory (i.e loop corrections) is an open question.

This paper is organized as follows: In Section 2 we derive the ERG in position space. In Section 3 we apply it to gauge fixed backgrounds - as one would in the "old covariant formulation" of string theory. In section 4 we derive the gauge invariant version using loop variables. Section 5 contains some conclusions and speculations.
2 RG in Position Space

In this section we derive the exact RG in position space. This is essentially a repetition of Wilson’s original derivation [20]. We include it here only because usual discussions use momentum space rather than position space. We start with point particle quantum mechanics:

2.1 Quantum Mechanics

We start with the Schrodinger equation
\[ \frac{i}{\hbar} \frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial y^2} \] (2.1.1)

for which the Green’s function is
\[ \frac{1}{\sqrt{2\pi(t_2-t_1)}} e^{\frac{(y_2-y_1)^2}{2(t_2-t_1)}} \], and change variables
\[ y = xe^\tau, t = e^{2\tau} \] and \[ \psi' = e^\tau \psi \] to get the differential equation
\[ \frac{\partial \psi'}{\partial \tau} = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} + x \right) \psi' \] (2.1.2)

with Green’s function
\[ G(x_2, \tau_2; x_1, 0) = \frac{1}{\sqrt{2\pi(1-e^{-2\tau_2})}} e^{\frac{(x_2-x_1 e^{-\tau_2})^2}{2(1-e^{-2\tau_2})}} \] (2.1.3)

Thus as \( \tau_2 \to \infty \) it goes over to \( \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2} \). As \( \tau_2 \to 0 \) it goes to \( \delta(x_1-x_2) \).

\[ \psi(x_2, \tau_2) = \int dx_1 G(x_2, \tau_2; x_1, 0) \psi(x_1, 0) \]

So \( \psi(x_2, \tau_2) \) goes from being unintegrated \( \psi(x_1) \) to completely integrated
\[ \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2} \int dx_1 \psi(x_1) \]. Thus consider
\[ \frac{\partial}{\partial \tau} \psi(x_2, \tau) = \frac{\partial}{\partial x_2} \left( \frac{\partial}{\partial x_2} + x_2 \right) \psi(x_2, \tau) \] (2.1.4)

with initial condition \( \psi(x, 0) \) Thus if we define \( Z(\tau) = \int dx_2 \psi(x_2, \tau) \), where \( \psi \) obeys the above equation, we see that \( \frac{d}{d\tau} Z = 0 \). Also for \( \tau = 0 \) \( \psi \) is the unintegrated \( \psi(x, 0) \). At \( \tau = \infty \) it is proportional to the integrated object \( \int dx \psi(x, 0) \). \( Z(\tau) \) has the same value. Thus as \( \tau \) increases the integrand in \( Z \) is more completely integrated.
We need to repeat this for the case where the initial wave function is replaced by $e^{i\bar{S}[x]}$ where $x$ denotes the space-time coordinates. Then for $\tau = \infty$ $\psi \approx \int Dxe^{iS[x]}$ the integrated partition function. At $\tau = 0$ it is the unintegrated $e^{iS[x]}$. $Z(\tau)$ is the fully integrated partition function for all $\tau$. We shall also split the action into a kinetic term and interaction term as in [23]. Thus in the quantum mechanical case discussed above we write

$$\psi = e^{-\frac{1}{2}x^2f(\tau)+L(x)}$$

By choosing $a, b, B$ suitably ($b = 2af, B = \frac{f}{bf}$) in

$$\frac{\partial \psi}{\partial \tau} = B\frac{\partial}{\partial x}(a\frac{\partial}{\partial x} + bx)\psi(x, \tau)$$

we get

$$\frac{\partial L}{\partial \tau} = \dot{f} \left[ \frac{\partial^2 L}{\partial x^2} + (\frac{\partial L}{\partial x})^2 \right]$$

(2.1.5)

Note that if $f = G^{-1}$ ($G$ is like the propagator) then $\frac{\dot{f}}{f^2} = -\dot{G}$

### 2.2 Field Theory

We now apply this to a Euclidean field theory.

$$\psi = e^{-\frac{1}{2} \int dz \int dz'X(z)G^{-1}(z,z')X(z')} + \int dzL[X(z)]$$

(2.2.6)

We apply the operator

$$\int dz \int dz' B(z, z') \delta \delta X(z') \delta X(z) + \int b(z, z'')X(z'')$$

(2.2.7)

to $\psi$ and require, as before, that this should be equal to $\frac{\partial \psi}{\partial \tau}$.

We get the following five terms (all multiplied by $B$):

$$(b - G^{-1})(z, z')$$

$$+ \left[ \frac{\partial^2 L}{\partial X(z)\partial X(z')} \delta(z - z') + \frac{\partial L}{\partial X(z)} \frac{\partial L}{\partial X(z')} \right]$$

$$+ \frac{\partial L}{\partial X(z)}( - \int G^{-1}(z', z'')X(z'')dz'')$$

$$+ \frac{\partial L}{\partial X(z')}( \int (b - G^{-1})(z, z'')X(z'')dz'')$$

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\[-[(b - G^{-1})X](z)[G^{-1}X](z')\]  \hspace{1cm} (2.2.8)

The first term is independent of \(X\) and is therefore an unimportant overall constant. If we choose \(b = 2G^{-1}\), the third and fourth terms add up to zero. Thus the second term becomes

\[
\int dz \int dz' B(z, z')[\frac{\partial^2 L}{\partial X(z)\partial X(z')} \delta(z - z') + \frac{\partial L}{\partial X(z)} \frac{\partial L}{\partial X(z')}]
\]

and the last term becomes:

\[
-\int dz \int dz' B(z, z') dz'' dz''' G^{-1}(z, z'') X(z'') G^{-1}(z', z''') X(z''') \psi
\]

We can set (2.2.7) equal to

\[
\frac{\partial \psi}{\partial \tau} = -\frac{1}{2} \int dz \int dz' X(z) \frac{G^{-1}}{\partial \tau}(z, z') X(z') \psi + \int dz \frac{\partial L}{\partial \tau} \psi.
\]

This ensures that \(Z = \int D\psi\) satisfies \(\frac{\partial Z}{\partial \tau} = 0\). If we now set \(B = -\frac{1}{2} G^{-1}(z, z')\) the equation for \(\psi\) reduces to:

\[
\int dz \frac{\partial L}{\partial \tau} = -\int dz \int dz' \frac{1}{2} \hat{G}(z, z')[\frac{\partial^2 L}{\partial X(z)\partial X(z')} \delta(z - z') + \frac{\partial L}{\partial X(z)} \frac{\partial L}{\partial X(z')}]
\]

If we now interpret \(\tau\) as \(\ln a\) this becomes easy to interpret as an RG equation diagrammatically as done in [23]: the first term in the RHS represents contractions of fields at the same point - self contractions within an operator, and the second one represents contractions between fields at two different points - between two different operators.

3 \hspace{1cm} \textbf{ERG in the Old Covariant Formalism}

We can assume that there is an infrared cutoff in all the integrals i.e. \(\int_R dz\) otherwise in a conformal field theory there could be infrared divergences in the integrals. When we integrate modes above a value \(\Lambda = \frac{1}{a}\) analyticity would demand that we also partially integrate some of the low energy modes. This is a potential source of IR divergences and could bring in dependences on the parameter \(\frac{R}{a}\). However if the cutoff is sharp enough (consistent with analyticity) one can safely take the limit \(R \to \infty\). It is also possible to have a cutoff so sharp that even for finite \(R\) the ERG equations have no dependence on \(\frac{R}{a}\). However such a cutoff would not be consistent with
analyticity. Analyticity is important in the present case because we will be making essential use of the OPE to reexpress non-local products of operators as higher dimensional local terms in the action. Since our starting point is an action that contains all the open string modes as backgrounds, this is a perfectly reasonable thing to do. Thus we can assume that the action is

\[ S = \int_{0}^{R} dzL[X(z)] = \int_{0}^{R} dz \int dk \left[ \frac{\phi(k)}{a} e^{ikX(z)} + A_{\mu}(k)\partial_{z}X^{\mu}e^{ikX(z)} \right. \]

\[ \left. + \frac{1}{2}aS_{2}(k)\partial_{z}^{2}X^{\mu}e^{ikX(z)} + aS_{\mu\nu}(k)\partial_{X}^{\mu}\partial_{X}^{\nu}e^{ikX(z)} + ... \right] \quad (3.0.12) \]

Before we implement the ERG we need a specific form for \( \dot{G}(z,z') = \dot{G}(z - z') \). As mentioned above we need \( \dot{G}(u) \) to be short ranged, otherwise the dimensionless ratio \( R \) is bound to enter in the equations.

One can make use of functions of the form \( e^{-\frac{1}{x^2}} \theta(x) \) that vanishes at \( x \leq 0 \) along with all derivatives and yet is continuous at \( x = 0 \) along with all derivatives. The precise form is not very important - although it will fix the various numerical constants in the RHS of the ERG. The main property is that it should vanish for \( |u| > a \). Thus it could be \( e^{-\frac{1}{(u-a)^{2}}} e^{-\frac{1}{(u+a)^{2}}} e^{\frac{2}{a^{2}}} \) for \( |u| \leq a \). And it has the usual form - \( G(u) = \ln u \) for \( |u| > a \) so that \( \dot{G}(u) = 0 \). With this function it is easy to see that even for finite \( R \) the equations do not depend on \( \frac{R}{a} \). However being non-analytic one cannot perform an OPE - at least not in the usual way that involves Taylor expansions.

We will use a different cutoff Green’s function:

\[ G(u) = \int \frac{d^2k}{(2\pi)^2} \frac{e^{iku}e^{-a^2k^2}}{k^2} \quad (3.0.13) \]

This has a cutoff at short distances of \( O(a) \) and at long distances reduces to the usual propagator. We now apply the ERG (2.2.11) to the action \( S \) (3.0.12).

The LHS gives

\[ \int dz \int dk \left[ \beta_{\phi}(k) - \frac{\phi(k)}{a} e^{ikX(z)} + \beta_{A_{\mu}}(k)\partial_{z}X^{\mu}(z)e^{ikX(z)} + ... \right] \quad (3.0.14) \]

where \( \beta_{\phi} \equiv \dot{g} \). The first term of the RHS gives

\[ \int dz \int dk \frac{1}{2}(-k^2) e^{ikX(z)} \quad (3.0.15) \]
The second term gives

\[ \int dk_1 \int dk_2 \frac{\phi(k_1)\phi(k_2)}{a^2} \left( \frac{-k_1 \cdot k_2}{2} \right) \int_{-R}^{+R} du \dot{G}(u) e^{ik_1 \cdot X(z)} e^{ik_2 \cdot X(z+u)} \]  

(3.0.16)

One can do an OPE for the product of exponentials to get

\[ e^{i(k_1 + k_2) \cdot X(z)} + ik_1 [u \partial_{\xi} X + \frac{u^2}{2} \partial^2 X + ...] \]

This gives

\[ e^{i(k_1 + k_2) \cdot X(z)} \int_{-R}^{+R} du \dot{G}(u) + i k_1 \partial X e^{i(k_1 + k_2) \cdot X(z)} \int_{-R}^{+R} du u \dot{G}(u) \]

\[ + ik_1 \frac{\partial^2 X}{2} e^{i(k_1 + k_2) \cdot X(z)} \int_{-R}^{+R} du u^2 \dot{G}(u) + \frac{ik_\mu ik_\nu}{2} \partial X^\mu \partial X^\nu e^{i(k_1 + k_2) \cdot X(z)} \int_{-R}^{+R} du u^2 \dot{G}(u) \]

It is easy to see that the first term of the OPE contributes to the tachyon equation:

\[ \beta_{\phi(k)} - \phi(k) = \phi(k) \left( \frac{-k^2}{2} \right) - \frac{1}{2} \int dk_1 \phi(k_1) \phi(k-k_1) \frac{k_1 \cdot (k-k_1)}{2a} \int_{-R}^{+R} du \dot{G}(u) \]  

(3.0.17)

Similarly the second term of the OPE contributes to the photon equation:

\[ \beta_{A^\mu(k)} = \int dk_1 \frac{\phi(k_1)\phi(k-k_1)}{a^2} \left( \frac{-k_1 \cdot (k-k_1)}{2} \right) ik_1^\mu \int_{-R}^{+R} du u \dot{G}(u) \]  

(3.0.18)

We have thus obtained the contribution of the tachyon field to the beta functions of the tachyon and photon. Similarly there are contributions to the beta functions of the higher spin massive fields \( S^{\mu}, S^{\mu \nu} \) etc. Note that the dimensionless number \( R/a \) does appear as expected because of the analytic nature of the cutoff. However since there are no infrared divergences one can take the limit \( R \to 0 \) without any problem. \( \dot{G}(u) = \frac{1}{\pi} e^{-\frac{R^2}{4a^2}} \). Thus integrals \( \int_{-R}^{R} du \dot{G}(u) u^n \) all have \( R/a \)-dependent pieces that contain the factor \( e^{-\frac{R^2}{4a^2}} \). So in the \( R \to \infty \) limit, all \( R/a \) dependence disappears, and the equations become completely independent of \( a \). Thus the conditions for the fixed point do not depend on \( a \), i.e. as mentioned in the introduction, there is scale invariance even though the lattice spacing \( a \) is non zero. This is the kind of

\[ ^2 \text{This can be understood as an example of finite size scaling, which has been much studied [40].} \]
situation envisaged in [37, 38] where the coupling constants of the irrelevant operators are all tuned so that the physical quantities calculated with this action do not depend on the cutoff $a$, and thus they have the same values as in the continuum. These are the "improved" actions [37] or "perfect" actions [38]. If on top of that, the background fields are tuned to satisfy the fixed point condition, then we have a scale invariant theory, even while the lattice spacing is non-zero.

One can also include the contribution due to the photon field in the RHS as shown below:

$$\frac{\delta}{\delta X^\nu(z)} \int dz'' \int dk A_\nu(k) \partial_{z''} X^\nu(z'') e^{ikX(z'')}$$

$$= \int dz'' \int dk A_\nu(k) [\delta^{\nu\sigma} \partial_{z''} \delta(z - z'') e^{ikX(z'')} + \partial_{z''} X^\nu(z'') i k^\mu \delta(z - z'') e^{ikX(z)}]$$

$$= \int dz'' \int dk A_\nu(k) [-\delta^{\nu\sigma} \delta(z - z'') i k^\mu \partial_{z''} X^\nu(z'') + \partial_{z''} X^\nu(z'') i k^\mu \delta(z - z'') e^{ikX(z)}]$$

$$= \int dz'' \int dk [-A^\mu(k) i k^\nu + i k^\mu A^\nu] \partial_{z''} X^\nu(z'') \delta(z - z'') e^{ikX(z)}$$

$$+ \partial_{z''} X^\nu[-i k^\nu A^\mu + i k^\mu A^\nu] i k^\mu \delta(z'' - z')$$

(3.0.19)

The second term $\frac{\partial L}{\partial X^\nu(z)} \frac{\partial L}{\partial X^\nu(z)}$ becomes

$$\int dz'' \int dk [-A^\mu(k) i k^\nu + i k^\mu A^\nu] \partial_{z''} X^\nu(z'') \delta(z - z'') e^{ikX(z'')}$$

$$\int dz'''' \int dk [-A^\mu(k) i k^\nu + i k^\mu A^\nu] \partial_{z'''} X^\nu(z''') \delta(z' - z''') e^{ikX(z''')}$$

Thus putting everything together we get

$$\int dz \int dz' \delta(z - z') \{ \int dk [-A^\mu(k) i k^\nu + i k^\mu A^\nu] i k^\mu \partial_{z'} X^\nu(z') \delta(z - z') e^{ikX(z)}$$

$$+ \int dk \int dk' [-i k^\rho A^\mu] [-i k'^{\sigma} A^\nu] \partial_{z} X^\rho(z) e^{ikX(z)} \partial_{z'} X^\nu(z') e^{ik'X(z')})$$

(3.0.20)

The first term is the usual Maxwell equation of motion that one should obtain at the linearized level in the beta function. On performing the OPE
in the second term (this assumes analyticity of $\hat{G}(z - z')$) we can reexpress as a sum of vertex operators for the various modes, exactly as in the case of the tachyon, above.

Notice that both terms are gauge invariant. From our experience with loop variables we see immediately that this is because of the integral over $z, z'$ which allows integration by parts. We also see that for the higher modes one need not expect full gauge invariance. In this formalism the gauge invariance due to $L_{-1}$ is usually present - this is simply the freedom to add total divergences in $z$, which we have - as in the case of the photon. For the higher gauge invariances due to $L_{-2}, L_{-3}$... we need some additional variables. So in this formalism as it stands, the equations are not invariant under the higher gauge transformations. In the next section we will use the loop variable approach to address this problem of making the ERG invariant under the full set of gauge transformations.

4 Gauge Invariant ERG

The gauge invariant construction involves writing the action in terms of the covariantized loop variable [15, 16]

$$\int [Dk_n(t)dx_n]e^{it\sum_{n\geq 0}\int dt k_n(t)Y_n(t)}\Psi[k_n(t)]$$

(4.0.21)

It is therefore useful to first redo the analysis of the previous section in the loop variable formalism where we will use the loop variable without covariantizing. This should reproduce the results of the previous section.

$$\int [Dk_n(t)dx_n]e^{it\sum_{n\geq 0}\int dt' k_n(t')\tilde{Y}_n(t')}\Psi[k_n(t)]$$

(4.0.22)

Here $\tilde{Y}_n = \frac{1}{(n-1)!}\frac{\partial^n X}{\partial t^n}$.

4.1 ERG in the Loop Variable Formalism - Gauge Fixed Case

Let us consider for concreteness the vector field (level 1). We assume that

$$\int \prod_{n=1,2...} dk_n[k^\mu_1(t_1)\Psi[k_n] = A^\mu(k_0(t_1))$$

We also assume that

$$\int \prod_{n=1,2...} dk_n[k^\mu_1(t_1)k^\nu_1(t_2)\Psi[k_n] = A^\mu(k_0(t_1))A^\nu(k_0(t_2))$$

(4.1.23)
and similarly for all products of $k_i^\mu(t_i)$. Furthermore

$$\int \prod_{n=1,2\ldots} dk_n \int k_i(t_1)k_j(t_2)\Psi[k_n] = 0 \quad \forall i,j > 1$$

which is equivalent to saying that we are setting the higher spin massive fields to zero.

In this notation we can write

$$e^{i\int dt L[X(t)]} \equiv e^{i\int dt \int dk \ A_\mu(k)e^{ikX}\partial_\mu X(t)} = \int \mathcal{D}k_n(t)e^{i\sum_{n=0}^{\infty} \int dt' k_n(t')\tilde{Y}_n(t')}\Psi[k_n(t)]$$

To verify the correctness of this equation expand the LHS in powers of $A_\mu$. The linear term is just $\int dt L[X(t)]$. On the RHS the first term gives

$$\int \mathcal{D}k_n(t)i\int dt_1k_1^\mu(t_1)\tilde{Y}_1(t_1)e^{i\int dt' k_0(t')X(t')}\Psi[k_n(t)]$$

Using

$$\langle k_1^\mu(t_1)k_0^\nu(t_2) \rangle = \delta(t_1-t_2)A^\mu(k_0(t_1))k_0^\nu(t_1) \quad (4.1.24)$$

we see that the RHS becomes

$$i\int dt_1\int dt_0 A^\mu(k_0(t_1))\tilde{Y}_1(t_1)e^{ik_0(t_1)X(t_1)}$$

which is the same as the LHS.

Let us go to the next order. LHS gives:

$$\frac{1}{2}\int dt_1\int dt_2\int dk_1 A^\mu(k_1)\partial_1 X^\mu(t_1)e^{ik_1 X(t_1)}\int dk_2 A^\nu(k_2)\partial_2 X^\nu(t_2)e^{ik_2 X(t_2)}$$

In the loop variable expression we consider:

$$\frac{1}{2}\int \mathcal{D}k_n(t)i\int dt_1 \ k_1^\mu(t_1)\tilde{Y}_1^\mu(t_1)\int dt_2 \ k_2^\nu(t_2)\tilde{Y}_1^\nu(t_2)e^{i\int dt' k_0(t')X(t')}\Psi[k_n(t)]$$

Using (4.1.23) and (4.1.24) we see that

$$\frac{1}{2}\int dk_0(t_1)\int dk_0(t_2)\int dt_1 A^\mu(k_0(t_1))\tilde{Y}_1^\mu(t_1)e^{ik_0(t_1)}\int dt_2 A^\nu(k_0(t_2))\tilde{Y}_1^\nu(t_2)e^{ik_0(t_2)}$$

which agrees with the LHS.

Let us now work out the ERG in this formalism:

$$\frac{\delta^2}{\delta X^\mu(t')}\delta X^\mu(t) e^{i\int dt_1 [k_1(t_1)\partial_1 X(t_1)+k_0(t_1)X(t_1)]}$$
\[= i \int dt_1 \delta^{\mu\nu}[k_1^\mu(t_1)\partial_t \delta(t'-t_1)+k_0^\mu(t_1)\delta(t'-t_1)]i \int dt_2 [k_1^\nu(t_2)\partial_t \delta(t-t_2)+k_0^\nu(t_2)\delta(t-t_2)]e^i \int dt_3[k_1^\mu(t_3)\partial_{t_3}X(t_3)+k_0(t_3)X(t_3)]\]

Let us concentrate on the part that is linear in \(k_1^\mu\) (and therefore \(A^\mu\)).

\[
\delta_{\mu\nu}2i \int dt_1 k_1^\mu(t_1)\partial_{t_1} \delta(t'-t_1) \int dt_2 k_0^\nu(t_2)\delta(t-t_2)e^i \int dt_3 k_1^\mu(t_3)X(t_3)\]

\[+\delta_{\mu\nu}i \int dt_1 i \int dt_2 k_0^\mu(t_1)k_0^\nu(t_2)\delta(t'-t_1)\delta(t-t_2)i \int dt_3 k_1^\rho(t_3)\partial_{t_3}X^\rho(t_3)e^i \int dt_4 k_0(t_4)X(t_4)\]

\[= \delta_{\mu\nu}2i \int dt_1 k_1^\mu(t_1)k_0^\nu(t_1)\partial_{t_1} \delta(t-t_1)\delta(t'-t_1)e^i k_0(t_1)X(t_1)\]

\[+i \int dt_3 k_0^\mu(t_3)k_0^\nu(t_3)\delta(t-t_3)\delta(t'-t_3)k_1^\rho(t_3)\partial_{t_3}X^\rho(t_3)e^i k_0(t_3)X(t_3)\]

Noting that \(\partial_{t_1} \delta(t-t_1)\delta(t'-t_1) = \delta(t-t_1)\partial_{t_1} \delta(t'-t_1) = \frac{1}{2}\partial_{t_1} (\delta(t-t_1)\delta(t'-t_1))\) and integrating by parts,

\[= \delta_{\mu\nu}i \int dt_1 [-k_1^\mu k_0^\nu ik_0^\mu(t_1)+k_0^\mu k_0^\nu k_1^\nu(t_1)]\partial_{t_1}X^\rho(t_1)e^i k_0X(t_1)\]

This is just Maxwell’s equation.

The part that has two \(k_1\)'s has three terms:

(a) \(\delta_{\mu\nu}i \int dt_1 k_1^\mu(t_1)\partial_{t_1} \delta(t-t_1) i \int dt_2 k_1^\nu(t_2)\partial_{t_2} \delta(t'-t_2)e^i \int dt_3 k_0(t_3)X(t_3)\)

(b) \(2\delta_{\mu\nu}i \int dt_1 k_1^\mu(t_1)\partial_{t_1} \delta(t-t_1)i \int dt_2 k_0^\nu(t_2)\partial_{t_2} \delta(t'-t_2)i \int dt_3 k_1^\rho(t_3)\partial_{t_3}X^\rho(t_3)e^i \int dt_4 k_0(t_4)X(t_4)\)

(c) \(\delta_{\mu\nu}i \int dt_1 k_0^\mu(t_1)\delta(t-t_1)i \int dt_2 k_0^\nu(t_2)\delta(t-t_2)k_1^\rho(t_2)\partial_{t_2}X^\rho(t_2)e^i \int dt_3 k_0(t_3)X(t_3)\)

Integrating by parts on \(t_1\) and \(t_2\) in (a)

\[\delta_{\mu\nu}ik_1^\mu(t)k_0^\nu(t)\partial_t X^\rho(t)ik_1^\nu(t')k_0^\rho(t')\partial_{t'} X^\sigma(t')e^i \int dt_4 k_0(t_4)X(t_4)\]

Similarly one can simplify (b) and (c) to get finally:

\[\delta_{\mu\nu}k_1^\mu(t)k_0^\nu(t)\partial_t X^\rho k_1^\nu(t')k_0^\rho(t')\partial_{t'} X^\sigma e^{i[k_0(t)X(t)]+k_0(t')X(t')}\]

One also sees that more than two \(k_1\)'s cannot contribute to the equation: there is one \(k_1(t)\) and one \(k_1(t')\). Additional \(k_1(t)\) would introduce massive fields which we have set to zero. The rest of the terms in the exponential
\[ e^{i \int dt k(t) \partial X(t)} \] constitute an overall multiplicative factor (involving \( A^\mu(X) \)) in the ERG equation and we need not worry about it. This is exactly as happens in the conventional field theory formalism.

These results can now be compared with (3.0.20) and are seen (on using (4.1.23)) to agree.

### 4.2 ERG - Gauge Invariant Loop Variable Formalism

We can proceed to do the above calculation in the gauge-invariant formalism. The main difference is that instead of integrating by parts on \( t \) we integrate by parts on an infinite number of variables, \( x_n \) - but these are global i.e. not \( x_n(t) \). This implies that vertex operators at all locations participate in the ERG equations, which therefore, are no longer quadratic in fields. Another important difference is that right from the beginning, all vertex operators are Taylor expanded about one point in the world sheet. This is equivalent to Taylor expanding \( \dot{G}(u) \). The Gaussian fall off at distances of order the cutoff, \( a \), is thus not seen in this power series expansion. The upshot is that one needs an IR cutoff \( R \) and the dimensionless ratio \( R/a \) enters in all the equations.

Our starting point in (4.0.21). We act with 

\[ \int dt \int dt' \dot{G}(t-t') \frac{\partial^2}{\delta X(0) \delta X(t')} \]

on \( \int [Dk_n(t) dx_n] e^{i \sum_{n \geq 0} \int dt k_n(t) Y_n(t)} \Psi[k_n(t)] \). We use [15, 16]

\[ Y(t) = X(t) + \alpha_1 \partial X(t) + \alpha_2 \partial^2 X(t) + \frac{1}{2} \alpha_3 \partial^3 X(t) + \ldots + \frac{\partial^n X(t)}{(n-1)!} + \ldots \]

where

\[ \sum_n \alpha_n t^{-n} = e^{\sum_n x_n t^{-n}} \]

and

\[ \frac{\delta Y(t)}{\delta X(t')} = \sum_{n=0}^\infty \frac{\alpha_n \partial^n \delta(t-t')}{(n-1)!} \]

The loop variable is however rewritten after first Taylor expanding about the point \( t = 0 \) and then covariantising, as \( e^{i \sum_n \int dt k_n(t) Y_n(0)} \). Thus we get

\[
\int dt \int dt' \dot{G}(t-t') \prod_{r \geq 1} [Dk_r(t) dx_r] i \sum_n \int dt_1 k_n(t_1) \frac{\partial}{\partial x_n} \left[ \sum_p \alpha_p \partial^p t_{2} \delta(t-t_2) \frac{1}{(p-1)!} \right]_{t_2=0} \]

\[
i \sum_m \int dt_3 k_m(t_3) \frac{\partial}{\partial x_m} \left[ \sum_q \alpha_q \partial^q t_{4} \delta(t'-t_4) \frac{1}{(q-1)!} \right]_{t_4=0} e^{i \sum_n \int dt k_n(t) Y_n(0)} \Psi[k_n(t)]
\]

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Rewrite $\frac{\partial}{\partial t}$ as $-\frac{\partial}{\partial t}$ and integrate by parts on $t$ to act on $\dot{G}(t-t')$, and do the same for $t_4$ (rewrite in terms of $t'$). We get:

$$\int \prod_{s \geq 1}[Dk_s(t)dx_s] \int dt \int dt' \sum_{n,m \geq 0} \left\{ \frac{\partial}{\partial x_n} \left[ \sum_p \frac{\alpha_p \partial_p^p}{(p-1)!} \right] \frac{\partial}{\partial x_m} \left[ \sum_q \frac{\alpha_q \partial_q^q}{(q-1)!} \right] \dot{G}(t-t') \right\}$$

$$\int dt_1 \int dt_3 \tilde{k}_n(t_1)\tilde{k}_m(t_3)\delta(t)\delta(t')e^{i \sum_r \int dt_5 \tilde{k}_r(t_5)Y_r(0)} \Psi[k_n(t)]$$

$$= \int \prod_{s \geq 1}[Dk_s(t)dx_s] \sum_{n,m \geq 0} \left\{ \frac{\partial}{\partial x_n} \left[ \sum_p \frac{\alpha_p \partial_p^p}{(p-1)!} \right] \frac{\partial}{\partial x_m} \left[ \sum_q \frac{\alpha_q \partial_q^q}{(q-1)!} \right] \dot{G}(t-t') |_{t=t'=0} \right\}$$

$$\int dt_1 \int dt_3 \tilde{k}_n(t_1)\tilde{k}_m(t_3)e^{i \sum_r \int dt_5 \tilde{k}_r(t_5)Y_r(0)} \Psi[k_n(t)]$$

We assume that when $n = 0$ $\frac{\partial}{\partial x_n} = 1$. Implementing the delta functions in $t, t'$ we see the Taylor expansion of $\dot{G}(u)$ mentioned above. The object $\Sigma$ can be identified with the generalized Liouville mode introduced in earlier papers on loop variables. There the EOM were obtained by varying w.r.t. $\Sigma$. Here we have two options. $\Sigma$ is something that can be evaluated (in principle) as a function of $x_n$. We can then take this equation as it stands, and evaluate it at say, $x_n = 0$. This will then reproduce the gauge fixed equation derived in the last section. In this form the equation will be quadratic in fields, because the higher order terms just factorize. We saw this in the example where we had only $A^\mu$. In the more general case, massive modes will be involved, but the factorization will still be true. However this equation is not gauge invariant. The second option is to integrate by parts on $x_n$ so that there are no derivatives on $\Sigma$. The coefficient of $\Sigma$ is gauge invariant. We can use this as the EOM. This is what is done in the gauge invariant loop variable formalism. However the equations no longer factorize. They reduce to the sum of two terms, each of which factorizes. This can be seen as follows:

Schematically the exponential $e^{i \sum_{n \geq 0} \int dt k_n(t)Y_n(t)}$ can be written as $E(T)E(T')$ where $T$ corresponds to either $t_1$ or $t_3$ in the above expression and $T'$ represents any other value of $t$. Thus in terms of fields there will be a factorization: $\langle E(T') \rangle$ is an overall multiplication factor. $E(T)$ introduces higher modes into the equation, but as there are only two points $t_1$ and $t_3$, it is always a product of two fields and so the equations are quadratic. Now consider the effect of integration by parts: We get terms of the form
\[ \frac{\partial}{\partial x_n} (E(t)E'(T')) = (\frac{\partial}{\partial x_n} E(T)) E(T') + E(T) \frac{\partial}{\partial x_n} E(T'). \] Each term can be seen to factorize, but the sum clearly will not be factorizable. Since it is only the sum that is gauge invariant, we no longer get a quadratic equation.

The RHS of the gauge invariant equation obtained is (LHS is just the beta function of each field):

\[ \int \prod_{s \geq 1} [Dk_s(t)dx_s] \sum_{n,m \geq 0} \int dt_1 \int dt_3 \tilde{k}_n(t_1), \tilde{k}_m(t_3) \]

\[ \frac{1}{2} \left( \frac{\partial^2}{\partial x_n \partial x_m} + \frac{\partial}{\partial x_{n+m}} \right) \left[ e^i \sum_n \int dt \tilde{k}_n(t)Y_n(0) \right] \Psi[k_n(t)] \]

After differentiation, one can set \( x_n = 0 \) to evaluate the expressions. The LHS is an expansion in \( Y_n \) and so is the RHS. Thus matching coefficients we get an infinite number of equations - each equation defines a beta function. If we set LHS=RHS=0 we get the condition for the fixed point and this is the EOM for the fields of the string.

Let us work out the electromagnetic case worked out earlier:

\[ [\Sigma(0)k_0.k_0 + \frac{\partial}{\partial x_1} \Sigma(0)k_1.k_0] e^i \sum_{n \geq 0} \int dt k_n(t)Y_n(t) \]

\[ = \Sigma(0)(k_0.k_0 - k_0.k_0) \frac{\partial}{\partial x_1} e^i \sum_{n \geq 0} \int dt k_n(t)Y_n(t) = \Sigma(0)(k_0.k_0i\epsilon k_1.k_1k_0 - k_1.k_0ik_0)Y_1^\mu e^{ik_0.Y} + ... \]

where the three dots indicate terms involving other vertex operators. On the LHS is \( \langle i\epsilon k_1^\mu Y_1^\mu e^{ik_0.Y} + ... \rangle \).

Gauge invariance of the equations follow exactly as in the usual loop variable formalism. The generalized tracelessness constraint

\[ \langle \int dt \int dt_1 \int dt_2 \lambda_p(t) \tilde{k}_n(t_1), \tilde{k}_m(t_2) \rangle = 0 \quad \forall n, m > 0 \]

is also required (as before).

The dimensional reduction with mass has to be done exactly as in the usual case. Thus \( q_n(t) \) is the generalized loop variable momentum in the 27th dimension. \( q_0^2 \) is to be set equal to the engineering dimension of the operator. We do not need \( \tilde{q}_n(t) \) because it is assumed that the long distance part of the Green’s function is zero for the 27th coordinate - so \( X^{26} \) has no \( t \)-dependence, so \( \tilde{q}_n(t) = q_n(t) \).
5 Conclusion

We have written down an exact renormalization group equation for the world sheet theory describing a general open string background. These equations are valid for finite cutoff. Indeed in the limit $R \to \infty$, the cutoff parameter $a$ does not enter the ERG. This means the finite cutoff RG equations - or the theory on a lattice with finite spacing - is the same as with $a = 0$. An action with this property has been described as a "perfect" action [38] - and also earlier similar ideas were introduced under the name of "improved" action [37]. Furthermore if we tune the parameters so that the beta functions are set to zero, then the resulting action is conformally invariant even if $a \neq 0$!

Furthermore using the loop variable formalism it is possible to make these equations gauge invariant. Gauge invariance usually ensures that the space-time theory is consistent. It is therefore a good check on the procedure. This takes the place of the usual checks such as BRST. Since there is a lot of freedom in the world sheet action, there is the possibility that this procedure can have a certain background independence, in that all backgrounds are on an equivalent footing. This was exploited in [36] to get a gauge invariant space time action for massive higher spin modes in AdS space time. One should be able to do this in a more general way using the ERG. This remains to be investigated.

However there is one new feature - at least in the way gauge invariance is achieved here, the parameter $R/a$ enters the equation. This can be traced to the Taylor expansion of all vertex operators about one common point. Of course the value of the parameter is arbitrary. If one were to fix gauge, one could resum the series and take the limit $R \to \infty$. In the expanded form it is not possible to take $R \to \infty$. The limit $R/a \to 0$ seems well defined - although it may not be physically reasonable. It may be that critical information is lost in this limit, thus invalidating it. This remains to be explored. It is worth pointing out that string field theory (BRST, light cone) also has some similar parameter, though for a given formulation it is a fixed number.

Finally on a speculative level, the role of a finite cutoff in space-time (as against world sheet) was discussed in [15]. The speculation was that string theory effectively imposes a finite space-time cutoff, and the large gauge symmetry - a generalized RG - of string theory makes the details of the cutoff unimportant, i.e. physically unobservable. Thus string theory should then be an example of a "perfect" (in the RG sense) space-time action.
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