Numerical Linear Algebra in the Sliding Window Model

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Abstract

We initiate the study of numerical linear algebra in the sliding window model, where only the most recent \( W \) updates in the data stream form the underlying set. Although most existing work in the sliding window model uses the smooth histogram framework, most interesting linear-algebraic problems are not smooth; we show that the spectral norm, vector induced matrix norms, generalized regression, and low-rank approximation are not amenable for the smooth histogram framework.

To overcome this challenge, we first give a deterministic algorithm that achieves spectral approximation in the sliding window model that can be viewed as a generalization of smooth histograms, using the Loewner ordering of positive semidefinite matrices. We then give algorithms for both spectral approximation and low-rank approximation that are space-optimal up to polylogarithmic factors. Our algorithms are based on a new notion of “reverse online” leverage scores that account for both how unique and how recent a row is. We show that by sampling rows based on their reverse online leverage scores and repeatedly downsampling each time a new row arrives, we can both oversample rows with respect to their true leverage scores, and also bound the total number of rows stored. The downsampling procedure can be streamlined so that both our spectral approximation algorithm and our low-rank approximation algorithm run in input sparsity runtime, up to lower order factors.

We show that our techniques have a number of applications to linear-algebraic problems in other settings. Specifically, we show that our analysis immediately implies an algorithm for low-rank approximation in the online setting that is space-optimal up to logarithmic factors, as well as nearly input sparsity time. We then show our deterministic spectral approximation algorithm can be used to handle \( \ell_1 \) spectral approximation in the sliding window model under.

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a certain assumption on the bit complexity of the entries. Finally, we show that our downsampling framework can be applied to the problem of approximate matrix multiplication and provide upper and lower bounds that are tight up to $\log \log W$ factors.

1 Introduction

The advent of big data has reinforced efforts to design and analyze algorithms that process data streams, i.e., datasets that are presented as a sequence of items and can be examined in only one or a few passes. In such settings, one of the main research focuses is the streaming model, where data arrives sequentially, can be observed only once, and the proposed algorithms are allowed to use space that is typically sublinear in the size of the input. However, this model does not fully address settings where the data is time-sensitive: such applications include network monitoring [CM05, CG08, Cor13], event detection in social media [OMM+14], etc. In such settings recent data is considered more accurate and important than data that arrived prior to a certain time window.

To model such settings, Datar et al. [DGIM02] introduced the sliding window model, which is parameterized by the size $W$ of the window. The parameter $W$ represents the size of the “most recent” data, called the active data. It is precisely the active data, in contrast to the so-called “expired data”, that one wishes to concentrate on in subsequent analyses. The objective is to design and analyze algorithms to compute statistics only on the active data using memory that is sublinear in the window size $W$. Since its introduction, much work has studied statistics that can be computed in the sliding window model [AM04, BDM02, BGO14, BO07, BGL+18, BOZ12, CLLTT12, DGIM02, DM07, FKZ05, LT06a, LT06b, TXB06, ZG08]. The analysis of all the above algorithms uses one of two frameworks: the exponential histogram [DGIM02] or the smooth histogram [BO07]. Unfortunately, despite being quite general, these frameworks cannot be applied to many interesting problems that have been extensively studied in the streaming model, such as clustering, submodular maximization, as well as most interesting numerical linear algebra problems.

While these frameworks can be used for some elementary linear-algebraic problems, such as estimating $\ell_p$ norms, we show that many more complicated numerical linear algebra problems, such as spectral approximation, principal component analysis, and generalized linear regression are not amenable to these frameworks. Recently, numerical linear algebra has found applications in many areas of computer science and there is a rich literature on what can be achieved without the sliding window constraints (see [Woo14, DM17] for a survey of such approaches). A natural question thus arises:

Are there efficient algorithms for numerical linear algebra in the sliding window model?

In this paper, we initiate the study of numerical linear algebra in the sliding window model. The complexity of approximating various linear-algebraic functions in the streaming model (including determinants, inverses, low-rank approximation, matrix multiplication, and singular value computation) was asked by [Mut05] and has been followed by a large body of work and extensive literature presenting tight upper and lower bounds on the achievable guarantees for many numerical linear algebra problems. In this work, we present the first algorithms for spectral approximation, low rank approximation, and approximate matrix multiplication in the sliding window model.
1.1 Our Contributions

Even though a few elementary linear-algebraic problems, such as estimating the Frobenius norm, can be addressed in the sliding window model using the smooth histogram framework [BO07], most of the “interesting” problems are not smooth. We show in Appendix B that various linear-algebraic functions are not smooth according to the definitions of [BO07] and therefore cannot be used in the smooth histogram framework. Namely, we show that the spectral norm, vector induced matrix norms, generalized regression, and low-rank approximation are not amenable to the smooth histogram framework. This motivates the need for new frameworks for problems of linear algebra in the sliding window model.

Our first contribution is a deterministic, space and time-efficient algorithm for spectral approximation in the sliding window model under the assumption that the matrix \( A \) in the current window has bounded spectrum, i.e., \( \frac{1}{\text{poly}(W)} \leq \sigma_{\min}(A) \leq \sigma_{\max}(A) \leq \text{poly}(W) \), where \( \sigma_{\min}(A) \) and \( \sigma_{\max}(A) \) are the smallest and largest nonzero singular values respectively. (The reader might want to first consult Appendix A.1 for basic background on the sliding window model and Appendix A for details on notation.) For matrices \( A \in \mathbb{R}^{m \times n} \) and \( B \in \mathbb{R}^{d \times n} \), we use the notation \( A \circ B \) to represent the matrix \( \begin{pmatrix} A \\ B \end{pmatrix} \). Similarly, we use \( A \circ r_t \) to represent the matrix \( \begin{pmatrix} A \\ r_t \end{pmatrix} \), where \( A \in \mathbb{R}^{m \times n} \) is a matrix and \( r_t \in \mathbb{R}^{1 \times n} \) is a row vector.

**Theorem 1.1** (Deterministic Spectral Approximation). Let \( r_1, \ldots, r_t \in \mathbb{R}^{1 \times n} \) be a stream of rows and \( W < n^p \) be the size of the sliding window for some constant \( p > 0 \). Let \( A = r_{t-W+1} \circ \ldots \circ r_t \) be the matrix consisting of the \( W \) most recent rows and suppose \( \frac{1}{\text{poly}(W)} \leq \sigma_{\min}(A) \leq \sigma_{\max}(A) \leq \text{poly}(W) \). Then given a parameter \( \epsilon > 0 \), there exists an algorithm that outputs a matrix \( \mathbf{M}^\top \mathbf{M} \) such that

\[
(1 - \epsilon)\mathbf{M}^\top \mathbf{M} \preceq \mathbf{A}^\top \mathbf{A} \preceq (1 + \epsilon)\mathbf{M}^\top \mathbf{M}.
\]

The algorithm uses \( O\left( \frac{n^3}{\epsilon^2} \log n \right) \) space and has \( O(n^3) \) amortized update time.

We then give a randomized algorithm for spectral approximation that is space-optimal, up to logarithmic factors.

**Theorem 1.2** (Space Efficient Spectral Approximation). Let \( r_1, \ldots, r_t \in \mathbb{R}^{1 \times n} \) be a stream of rows and \( W < n^p \) be the size of the sliding window for some constant \( p > 0 \). Let \( A = r_{t-W+1} \circ \ldots \circ r_t \) be the matrix consisting of the \( W \) most recent rows and suppose \( \frac{1}{\text{poly}(W)} \leq \sigma_{\min}(R) \leq \sigma_{\max}(R) \leq \text{poly}(W) \) across all matrices \( R \) formed by consecutive rows in the stream. Then given a parameter \( \epsilon > 0 \), there exists an algorithm that outputs a matrix \( \mathbf{M} \) with a subset of (rescaled) rows of \( \mathbf{A} \) such that

\[
(1 - \epsilon)\mathbf{A}^\top \mathbf{A} \preceq \mathbf{M}^\top \mathbf{M} \preceq (1 + \epsilon)\mathbf{A}^\top \mathbf{A}
\]

and uses \( O\left( \frac{n^2}{\epsilon^2} \log n \log \frac{n}{\epsilon} \right) \) space, with probability \( 1 - \frac{1}{\text{poly}(W)} \). The total running time of the algorithm is \( O\left( \left| S \right| \log^2 \frac{n}{\epsilon} + \log \frac{n}{\epsilon} \cdot \text{nnz} \right) \), where \( |S| \) is the length of the stream and \( \text{nnz} \) is the input sparsity of the stream. (See Theorem 4.5.)

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1Throughout the paper we use the notation \( \mathbf{X} \preceq \mathbf{Y} \) to indicate that \( \mathbf{X} - \mathbf{Y} \in \mathbb{R}^{n \times n} \) is a positive semidefinite (PSD) matrix, i.e., for all \( \mathbf{x} \in \mathbb{R}^n \), \( \mathbf{x}^\top \mathbf{X} \mathbf{x} \leq \mathbf{x}^\top \mathbf{Y} \mathbf{x} \).
We remark that this result has applications to covariance estimation, generalized linear regression, row subset selection, and Schatten $p$-norm estimation.

We also give an algorithm for low rank approximation that is optimal in space up to polylogarithmic factors [CW09]. In fact, we give the stronger result of outputting a rank $k$ projection-cost preserving sample, which is defined as follows:

**Definition 1.3 (Rank $k$ Projection-Cost Preserving Sample [CEM+15]).** For $d_1 < m$ (resp. $d_2 < n$), a matrix $M_1 \in \mathbb{R}^{d_1 \times n}$ of rescaled rows of $A \in \mathbb{R}^{m \times n}$ (resp. a matrix $M_2 \in \mathbb{R}^{m \times d_2}$ of rescaled columns of $A$) is a $(1 + \epsilon)$ projection-cost preserving sample if, for all rank $k$ orthogonal projection matrices $P_1 \in \mathbb{R}^{n \times n}$ (resp. $P_2 \in \mathbb{R}^{m \times m}$),

$$
(1 - \epsilon) \|A - AP_1\|_F^2 \leq \|M_1 - M_1P_1\|_F^2 \leq (1 + \epsilon) \|A - AP_1\|_F^2
$$

(resp. $(1 - \epsilon) \|A - P_2A\|_F^2 \leq \|M_2 - P_2M_2\|_F^2 \leq (1 + \epsilon) \|A - P_2A\|_F^2$).

Note that a rank $k$ projection-cost preserving sample provides a solution for low-rank approximation of a matrix $A$ by choosing the appropriate rank $k$ orthogonal projection matrix $P$ that minimizes $\|A - PA\|_F^2$.

**Theorem 1.4 (Space Efficient Rank $k$ Projection-Cost Preserving Sample).** Let $r_1, \ldots, r_t \in \mathbb{R}^{1 \times n}$ be a stream of rows and $W < n^p$ be the size of the sliding window for some constant $p > 0$. Let $A = r_t \circ \ldots \circ r_1$ be the matrix consisting of the $W$ most recent rows and suppose $rac{1}{\text{poly}(W)} \leq \sigma_{\min}(R) \leq \sigma_{\max}(R) \leq \text{poly}(W)$ across all matrices $R$ formed by consecutive rows in the stream. Then given a parameter $\epsilon > 0$, there exists an algorithm that outputs a matrix $M$ that is a $(1 + \epsilon)$ rank $k$ projection-cost preserving sample of $A$, using $\tilde{O}\left(\frac{n}{\epsilon^2} \right)$ space, with probability at least $1 - \frac{1}{\text{poly}(W)}$. The total running time of the algorithm is $\tilde{O}\left(\frac{|S|}{\epsilon^2} + \log \frac{n}{\epsilon} \cdot \text{nnz}\right)$, where $|S|$ is the length of the stream and $\text{nnz}$ is the input sparsity of the stream. (See Theorem 5.8)

We show that our techniques have broad applications. We show that our analysis for the sum of the reverse online $\lambda$-ridge leverage scores [Lemma 5.5] immediately gives an online algorithm for low-rank approximation that uses optimal space up to polylogarithmic factors. In this online setting, the rows of a matrix $A \in \mathbb{R}^{W \times n}$ arrive sequentially and an algorithm must irrevocably choose at each time whether to store or discard the most recently arrived row. We emphasize that here $W$ is the length of the stream rather than the size of the sliding window.

**Theorem 1.5 (Online Rank $k$ Projection-Cost Preserving Sample).** There exists an online algorithm that, given a stream of rows as input, returns a matrix $\hat{A}$ that is a $(1 + \epsilon)$ rank $k$ projection-cost preserving sample of $A$ and uses $\mathcal{O}\left(\frac{n\epsilon^2}{\epsilon^2} \log^2 \|A\|_F \log n\right)$ space. (See Theorem 6.4)

We then show that our deterministic spectral approximation algorithm can be used as a key ingredient for an algorithm that preserves $\|Ax\|_1$ in the sliding window model up to relative and additive errors, assuming certain conditions about the entries of $A$ and $x$.

**Theorem 1.6 ($\ell_1$ Spectral Approximation).** Let $r_1, \ldots, r_t \in \mathbb{R}^{1 \times n}$ be a stream of rows and suppose the entries of each row are integers that are bounded by some polynomial in $n$. Let $A = r_t \circ \ldots \circ r_1$ be

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2We use the convention that $A \in \mathbb{R}^{n \times d}$ in Section 6.1 in convention with standard notation for online algorithms.
Table 1: Space bounds for various numerical linear algebra problems, where lower order terms are omitted for ease of readability. Matrices have dimension $W \times n$, $W \gg n$ and $\varepsilon$ is the approximation parameter. We also obtain $O\left(\text{poly}\left(n, \frac{1}{\varepsilon}\right)\right)$ space for $\ell_1$ spectral approximation in the sliding window model under a bit complexity assumption (see Theorem 6.14).

Table 1:
| Problem                                    | Space           | Reference       |
|--------------------------------------------|-----------------|-----------------|
| Deterministic Spectral Approximation       | $\tilde{O}\left(\frac{m}{\varepsilon}\right)$ | Theorem 1.1     |
| Spectral Approximation                     | $\tilde{O}\left(\frac{n^2}{\varepsilon^2}\right)$ | Theorem 4.5     |
| Rank $k$ Approximation                     | $\tilde{O}\left(\frac{n^2 k}{\varepsilon^2}\right)$ | Theorem 5.8     |
| Online Rank $k$ Approximation               | $\tilde{O}\left(\frac{n^2 k}{\varepsilon^2}\right)$ | Theorem 6.4     |
| Covariance Matrix Approximation            | $\Theta\left(\frac{n^2}{\varepsilon^2}\right)$ | Theorem 6.20, Theorem 6.24 |

the matrix consisting of the $W$ most recent rows. Given a parameter $\varepsilon > 0$, there exists an algorithm that outputs a matrix $M$ such that for any vector $x \in \mathbb{R}^{n \times 1}$ with polynomially bounded entries,

$$\|Mx\|_1 - \|Ax\|_1 \leq \varepsilon \|Ax\|_1$$

with high probability. The algorithm uses $O\left(\text{poly}\left(n, \frac{1}{\varepsilon}\right)\right)$ space. (See Theorem 6.14)

If the entries of the input matrix are not polynomially bounded, we can also obtain additive error (see Theorem 6.15).

We also use our downsampling techniques to give the first upper bound for covariance matrix approximation in the sliding window model. In this problem, the rows of a matrix $A \in \mathbb{R}^{W \times n}$ arrive sequentially, and the goal is to output a matrix $B$ such that $||A^\top A - B^\top B||_F \leq \varepsilon \|A\|_F^2$.

**Theorem 1.7 (Covariance Matrix Approximation).** There exists an algorithm that performs covariance matrix approximation in the sliding window model, using $O\left(\frac{n^2}{\varepsilon^2} \log n \left(\log \log n + \log \frac{1}{\varepsilon}\right)\right)$ bits of space. (See Theorem 6.20)

We complement our covariance matrix approximation algorithm with a lower bound that is tight up to $\log \log n + \log \frac{1}{\varepsilon}$ factors.

**Theorem 1.8.** Any sliding window algorithm that performs covariance matrix approximation with probability at least $\frac{8}{9}$ requires $\Omega\left(\frac{n^2}{\varepsilon^2} \log n\right)$ bits of space. (See Theorem 6.24)

**Remark 1.9.** Our techniques for covariance matrix approximation can be easily adapted to the problem of approximate matrix multiplication of matrices $A \in \mathbb{R}^{m \times W}$ and $B \in \mathbb{R}^{W \times p}$ when the columns of $A$ arrive in sync with the rows of $B$, and the goal is to approximate $AB$.

Our results can be summarized in Table 1.

### 1.2 Overview of Our Techniques

Smooth histograms and exponential histograms give a framework for sliding window algorithms for smooth functions (see Definition A.1). Informally, a function $f$ is smooth if, given substreams
Consider real-valued functions \( A, B \) where \( B \) is a suffix of \( A \) and \( f(B) \) is a “good” approximation to \( f(A) \), then \( f(B \cup C) \) will also be a “good” approximation to \( f(A \cup C) \) for all substreams \( C \) that arrive afterwards. A first attempt would be to apply these frameworks to various linear-algebraic quantities by showing smoothness properties for these functions. Unfortunately, we show in Appendix \( B \) that many interesting linear-algebraic functions, including low-rank approximation, generalized regression, spectral norms, and vector-induced matrix norms, are not smooth. Therefore, we require a new approach altogether.

**Deterministic Spectral Approximation.** Our first observation is that many problems in numerical linear algebra can be solved if we can maintain a spectral approximation of the covariance matrix in the sliding window model, i.e., compute \( M \) such that \((1 - \varepsilon)M \preceq A^\top A \preceq (1 + \varepsilon)M^\top M\). The covariance matrix is nice because the Loewner order interacts nicely with row updates in the following sense: If \( A \) and \( B \) are substreams, where \( B \) is a suffix of \( A \), corresponding to matrices \( A \) and \( B \) and \((1 - \varepsilon)A^\top A \preceq B^\top B \preceq A^\top A \), then

\[
(1 - \varepsilon)(A^\top A + C^\top C) \preceq B^\top B + C^\top C \preceq A^\top A + C^\top C
\]

for any matrix \( C \) that represents a substream \( C \) that arrives right after the substream \( A \). Although the similarity between the above consequence and the definition of smooth functions is notable, we cannot immediately apply existing data structures in the sliding window model because they consider real-valued functions.

We now describe a deterministic algorithm to show that spectral approximation is possible in the sliding window model, as well as build a template for our more space-efficient algorithms. To gain additional intuition behind our algorithms, first consider the case when we do not have any space constraints. Then we can have the following naïve data structure \( D \) that, at time \( t \), stores \( W \) matrices, \( X_1^\top X_1, \ldots, X_W^\top X_W \) and all the time-stamps \( t, t - 1, \ldots, t - W + 1 \), such that

\[
X_1^\top X_1 \preceq X_2^\top X_2 \preceq \ldots \preceq X_W^\top X_W,
\]

where \( X_i^\top X_i = \sum_{j=-i+1}^i r_j^\top r_j \) is the covariance matrix formed from the updates \( t - i + 1 \) to \( t \).

The space required by this naïve algorithm is \( WN^2 \), which is prohibitively large. The key to our algorithm is the observation that many matrices in the naïve data structure have their spectrum quite “close” to each other and can be approximated by a single matrix. That is, we show that we need to only store \( O \left( \frac{W}{\varepsilon} \log W \right) \) matrices and delete the rest of them.

The main point here is that, since we want a \((1 + \varepsilon)\)-spectral approximation of the matrix formed by the sliding window, we only need to store those matrices whose spectrum are \((1 + \varepsilon)\)-close to the one that we would delete. We first describe a purely linear-algebraic data structure that achieves this task in Section 2. We then augment this data structure with the necessary components in Section 2.1 and Section 3 to maintain a spectral approximation in the sliding window model. In particular, the data structure stores a subset of \( X_1^\top X_1, \ldots, X_W^\top X_W \) (let this subset be \( S_1, \ldots, S_s \)) with the invariant that \((1 - \varepsilon)S_i \preceq S_{i+2} \) for all \( 1 \leq i \leq s - 2 \). We show that this invariant not only suffices to give a \((1 + \varepsilon)\)-spectral approximation, but that it also provides a good upper bound on the number of matrices \( S_i \) that we store.

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3Positive semidefinite ordering is a partial order; therefore, \( A \not\preceq B \) does not imply that \( B \preceq A \). On the other hand, \( f(x) \not\preceq f(y) \) implies that \( f(x) \geq f(y) \) – this fact is used in a non-trivial manner in the existing results.
We can then break our algorithm into three main subroutines. Upon the arrival of each new row $r_t$, the procedure \textsc{Update} adds $r_t^\top r_t$ to each $S_i$. We then use a procedure \textsc{Compress} to maintain the above invariant. Finally, we use \textsc{Expire} to remove any matrix $S_1$ that is sufficiently older than the sliding window.

**Space Efficient Spectral Approximation.** Observe that the previous deterministic algorithm maintains separate instances $X_j^\top X_i$ and also returns an approximation to $A^\top A$, where $A \in \mathbb{R}^{W \times n}$ is the matrix consisting of the $W$ most recent rows. Thus, we have two potential directions to improve upon the deterministic algorithm. First, we would like to produce a spectral approximation to the underlying matrix $A$, rather than $A^\top A$. Secondly, we would like to improve upon the $n^3$ dependence to something more comparable to the $n^2$ dependent lower bounds that also hold for the easier, insertion-only streaming model.

The challenge for space efficient spectral approximation in the sliding window model results from two conflicting forces. Suppose we have a good approximation $\tilde{M}$ to the matrix $M$ consisting of the rows that have already arrived in the stream. When a new row $r$ arrives, we would like to sample the row with high probability if $r$ is “important” based on the rows in $\tilde{M}$. Namely, if $r$ has high magnitude or a different direction than the rows of $\tilde{M}$, we would like to capture that information by sampling $r$. On the other hand, if $r$ has low importance based on the existing rows of $\tilde{M}$, then it seems like we should not sample $r$.

However, the sliding window model also places great importance on the recent rows. Suppose the rows that follow $r$ all only contain zeroes and that all rows of $M$, i.e., all rows before $r$ are expired at the time of query. If we did not sample $r$, then we would be left with no information about the underlying matrix. Hence, it is obvious that we must always store the most recent row, and similarly place greater emphasis on more recent rows. Although the leverage score of a row is a good indicator of how “unique” (and hence, important) the row is with respect to the previous rows, there is no measure that combines both uniqueness and recency.

On a positive note, it is possible to get a good approximation $\tilde{X}_i$ to each $X_i$ maintained by the deterministic algorithm using a much smaller number of rows through any streaming algorithm for spectral approximation. We can then set $S_i = \tilde{X}_i^\top \tilde{X}_i$ rather than $X_i^\top X_i$ and use the same maintenance rules to ensure that our algorithm only stores a subset of the matrices $S_i$. However, $\tilde{X}_i$ can be expressed using roughly $\tilde{O}(n^2)$ space whereas storing $X_i$ can require $O(Wn)$ space, we can store each $\tilde{X}_i$ explicitly and output $\tilde{X}_1$ at the end of the stream, as a good approximation to $M$.

This gives a randomized algorithm that outputs a spectral approximation to $A$, rather than $A^\top A$ as by the deterministic algorithm, but actually uses space worse than the deterministic algorithm.

Nevertheless, this randomized algorithm crucially gives a template for a more space efficient spectral approximation. Observe that the rows of $X_j$ are a subset of the rows of $X_i$ for any $j > i$, so $\tilde{X}_i$ might contain a lot of the rows in $\tilde{X}_j$, which suggests a more optimal form of row sampling. As previously discussed, recent rows are more important in the sliding window model compared to the streaming model, regardless of the values of previous rows. Intuitively, if the rows are roughly uniform, then the “importance” of each row is inversely proportional to how recent the row has appeared in the stream. Since the rows will likely not be uniform, a natural approach is to adapt squared norm sampling, where each row is sampled with probability proportional to its squared norm, and inversely proportional to the sum of the squared norms of the rows that have appeared afterwards. However, squared norm sampling does not give relative error, which suggests using
a sampling approach using leverage scores.

To that effect, we introduce the concept of reverse online leverage scores (see Definition 4.1). Given a matrix $\mathbf{R} = \mathbf{r}_1 \circ \ldots \circ \mathbf{r}_t$, we say the $i^{th}$ reverse online leverage score is the leverage score of $\mathbf{r}_i$ with respect to the matrix $\mathbf{r}_1 \circ \ldots \circ \mathbf{r}_i$. The reverse online leverage scores value both the recency of a row as well as the uniqueness. However, we cannot track the reverse online leverage scores of each row in the stream without storing the entire stream. Instead, we use the rows that we have sampled as a proxy to $\mathbf{R}$ in computing approximations to the reverse online leverage scores. Specifically, we can use the rows of $\tilde{\mathbf{X}}_j$ while iteratively filtering the rows of $\tilde{\mathbf{X}}_i$ for $j > i$.

In summary, our algorithm stores a number of (rescaled) rows $\mathbf{r}_{t_1}, \mathbf{r}_{t_2}, \ldots, \mathbf{r}_{t_s}$ of the stream, where $t_1 < t_2 < \ldots < t_s$. Each time a new row $\mathbf{r}_i$ arrives in the data stream, our algorithm first performs downsampling to see whether $\mathbf{r}_i$ should be retained, based on the approximate reverse online leverage scores. Our algorithm then iteratively, from $i = s - 1$ to $i = 1$, performs downsampling to see whether each row $\mathbf{r}_i$ should be retained, based on the rows of $\mathbf{r}_{t_1}, \ldots, \mathbf{r}_i$ that have been retained. Each time a row is retained, the appropriate rescaling of the row is performed. Thus the algorithm can reconstruct each $\tilde{\mathbf{X}}_i$ by considering the matrix of all rows that have been sampled since time $t - W + i$.

To show correctness, we prove an invariant [Lemma 4.2] and subsequent corollaries, that says that each row $\mathbf{r}_i$ is oversampled, i.e., sampled with probability sufficiently larger than the reverse online leverage score. Hence, the rows that have been sampled since time $i$ form a good spectral approximation to the matrix $\mathbf{r}_1 \circ \ldots \circ \mathbf{r}_i$ that contains all the rows that have arrived since time $i$ in the data stream. This invariant holds for all times $i > t - W$ and in particular, it holds for time $t - W + 1$ and thus the sliding window.

To bound the total number of rows our algorithm samples, we upper bound the probability that each row is sampled and show it is at most a factor $c = O\left(\frac{\log n}{\varepsilon^2}\right)$ multiple of the reverse online leverage score. The sum of the reverse online scores turns out to equal the sum of the online leverage scores. Since the sum of the online leverage scores is roughly $\tilde{O}(n \log \|\mathbf{A}\|_2)$, it follows that our algorithm will sample $\tilde{O}(cn \log \|\mathbf{A}\|_2)$ rows with high probability.

Because our space efficient algorithm employs row sampling, it further enjoys the property that the total runtime is dependent on the input sparsity, up to polylogarithmic factors. By batching until a certain number of rows have arrived before we run an instance of the downsampling procedure, we can amortize the runtime needed to compute the reverse online leverage scores for each row. Again it suffices to approximate each reverse online leverage score within some constant factor, since we can just oversample with respect to the approximation guarantee. We can then use standard projection techniques to embed each of the sampled rows into a $O\left(\log \frac{n}{\varepsilon}\right)$ dimensional subspace by applying a Johnson-Lindenstrauss transform, and computing the leverage scores of the rows with respect to the matrix of the transformed rows. For more details, see Section 4.

**Space Efficient Projection Cost Preserving Sample.** Our space efficient spectral approximation algorithm outputs a matrix $\mathbf{M}$ whose singular values are within $(1 + \varepsilon)$ of $\mathbf{A}$, where $\mathbf{A} \in \mathbb{R}^{W \times n}$ is the matrix consisting of the $W$ most recent rows. However, certain tasks such as rank $k$ approximation can be achieved by sampling roughly $O\left(\frac{n}{\varepsilon^2} \log n\right)$ rows with the appropriate regularization [CMM17]. Thus it seems reasonable that these tasks can be performed in the sliding window model also by sampling a number of rows linear in $k$ rather than $n$.

Suppose we know the value of $\|\mathbf{A} - \mathbf{A}_k\|_F^2$ in advance, where we use $\mathbf{A}_k$ to denote the best
rank $k$ approximation $A$. Then we can use the same downsampling procedure as before with reverse online ridge leverage scores and regularization parameter $\lambda = \|A - A_k\|_F^2$. Using the same invariant as before, the procedure oversamples each row $r_i$ relative to the reverse online ridge leverage score. This suffices to show that the rows that have been sampled since time $i$ form a projection cost preserving sample for the matrix that contains all the rows that have arrived since time $i$ in the data stream. That is, the algorithm maintains a good low rank approximation for the matrix in the window.

The total number of rows our algorithm samples is again bounded by some multiple of the sum of the reverse online ridge leverage scores, which equals the sum of the online ridge leverage scores. [CMP16] previously showed that the sum is bounded by $O\left(n \log \frac{\|A\|_F^2}{\lambda}\right)$. However, this does not suffice for our purpose. Thus, we provide a tighter analysis when $\lambda = \|A - A_k\|_F^2$, showing that the sum of the online ridge leverage scores is then $O(k + k \log \|A\|_2)$. Hence under additional assumptions about the underlying singular values and suppressing $\epsilon$ and log factors, the total space used by our algorithm is $\tilde{O}(nk)$, when $\|A - A_k\|_F^2$ is known in advance and in fact, any constant factor underestimation of $\|A - A_k\|_F^2$ suffices to achieve an algorithm with the same asymptotic space.

We now describe a procedure $\text{ESTIMATE}$ to obtain this constant factor underestimation, which we surprisingly achieve by using another projection-cost preserving sample. Cohen et al. [CEM+15] show that $AJ$ is a projection-cost preserving sketch for $A$ when $J$ is a sufficiently large dense Johnson-Lindenstrauss matrix. So if we obtain $AJ$, we can approximate $\|A - A_k\|_F^2$. Note that we cannot use $J$ directly to obtain a projection-cost preserving sample, since $J$ does not preserve the rows of $A$. Moreover, while $J$ has $d = O(k + \log n)$ columns, $AJ$ still has $W$ rows, so we can only afford to sketch a spectral approximation for $AJ$. Fortunately, our space efficient spectral approximation $\text{META-SPETRAL}$ does exactly that when we input the rows $r_1J, r_2J, \ldots, r_iJ \in \mathbb{R}^{1 \times d}$. Hence, we can use the spectral approximation of $AJ$ to repeatedly update the constant factor underestimation of $\|A - A_k\|_F^2$. Similar to our spectral approximation algorithm, our low-rank approximation algorithm has total runtime dependent on the input sparsity, up to lower order factors. We describe the algorithm in full in Section 5.

Applications of our Techniques. We show that our techniques have applications to linear-algebraic functions in a number of interesting other models. We start by using our analysis for the sum of the reverse online ridge leverage scores to give a nearly space optimal algorithm for low-rank approximation in the online setting. In this problem, the rows of an underlying matrix $A$ arrive sequentially in a stream and an algorithm must irrevocably either store or discard the row, to output a good rank $k$ approximation of $A$. If we set $\lambda = \frac{\|A - A_k\|_F^2}{k}$, where $A_k$ is the best rank $k$ approximation of $A$, then it is known that by sampling each row with probability at least the $\lambda$-ridge leverage score, we can obtain a good rank $k$ approximation of $A$ [CM17]. Moreover, the online $\lambda$-ridge leverage score of a row is at least the $\lambda$-ridge leverage score, so it suffices to sample each row with probability at least the online $\lambda$-ridge leverage score. Thus, if we knew the value of $\frac{\|A - A_k\|_F^2}{k}$, then our tighter bounds for the sum of the online $\lambda$-ridge leverage scores would imply a nearly space optimal algorithm for the online low-rank approximation problem. Hence, we simultaneously run an algorithm to estimate the value of $\frac{\|A - A_k\|_F^2}{k}$, similar to the $\text{ESTIMATE}$ procedure described previously.
We then show that our deterministic spectral approximation algorithm can be applied as a crucial subroutine for an algorithm to preserve $\|Ax\|_1$ in the sliding window model up to relative and additive errors, assuming the entries of $A$ and $x$ are integers bounded in absolute value by $\text{poly}(n)$. We would like to form an $\ell_1$ spectral approximation in the sliding window model. However, it does not seem evident how to bound “online” versions of the Lewis weights [CP15] or the $\ell_p$ leverage scores [DDH+08]. While our analysis to bound the sum of the online ridge leverage scores relates the change in volume of the parallelepiped spanned by the columns of the stream to the online ridge leverage score of the new row, a similar geometric argument using (online) Lewis weights or (online) $\ell_p$ leverage scores does not seem obvious. A primary reason for this is that, unlike for $\ell_2$, the unit ball $\{x \text{ such that } \|Ax\|_1 \leq 1\}$ is not an ellipsoid, and, it is not clear how the John ellipsoid or the Lewis weights of this polytope change when a new row is added.

Instead, we observe that if $R_i = r_1 \circ \ldots \circ r_i$, the entries of $A$ and $x$ are bounded by some polynomials in $n$, then each time there is a relative increase from $\|Rx\|_1$ to $\|Rx\|_2^2$ for $j > i$, there must also be a significant increase from $\|Rx\|_1^2$ to $\|Rx\|_2^2$. Therefore, we can use our deterministic spectral approximation algorithm to track the times in which this relative increase might happen. At each of these times, we run a separate streaming algorithm for $\ell_1$ spectral approximation and we output the algorithm corresponding to the sliding window at the end of the stream. This approach is highly reminiscent of the smooth histogram data structure and further illustrates why our deterministic spectral approximation algorithm may be viewed with independent interest as a framework with possible applications to other linear-algebraic functions in the sliding window model.

Finally, we show that our downsampling techniques can be used to perform covariance matrix approximation in the sliding window model. It is known that squared row norm sampling suffices for approximate matrix multiplication in the offline setting. In the sliding window model, each time a new row arrives, the Frobenius norm of the underlying matrix changes; therefore, all previous rows must be downsampled accordingly. Although we do not know the Frobenius norm of the underlying matrix, we certainly have the squared row norm when the row arrives. This can be used to obtain an estimate of the Frobenius norm using smooth histograms. Hence, we can emulate squared row norm sampling by tracking the probability that each row stored by our algorithm has been sampled with, and downsampling accordingly.

Organization of the rest of the paper. As a warm-up, we describe in Section 2 a data structure that serves as a generalization of the smooth histogram framework from real-valued functions to matrix-valued functions. We then describe in Section 3 how this data structure can be used to obtain our deterministic algorithm for spectral approximation in the sliding window model. We improve upon both the time and space complexities by giving a sampling based algorithm for spectral approximation in Section 4. We show in Section 5 that we can further optimize the space complexity if we are interested in low-rank approximation, while also giving a tighter analysis on the sum of the online ridge leverage scores. In Section 6, we give applications of our techniques, first showing that this tighter analysis implies a nearly space optimal algorithm for online low-rank approximation. We then discuss $\ell_1$ spectral approximation in Section 6.2 and covariance matrix approximation in Section 6.3.

We give a brief overview of the background of the sliding window model in Appendix A. We show in Appendix B that various linear-algebraic functions cannot be used in the smooth histogram framework.
2 A General Data Structure

In this section, we describe a general data structure with the relevant numerical linear algebra properties that we shall eventually use in the sliding window model.

Definition 2.1. Given a parameter $0 < \epsilon < 1$, we define a spectrogram to be a data structure $D$ that consists of a sequence of PSD matrices $S_1 \succeq S_2 \succeq \ldots \succeq S_s \in \mathbb{R}^{n \times n}$ such that

$$(1 - \epsilon)S_i \not\preceq S_{i+2}, \text{ for each } 1 \leq i \leq s - 2.$$  \hspace{1cm} (Invariant ⋆)

Further, $D$ supports the following operations:

- **UPDATE**$(D, v)$: For a vector $v \in \mathbb{R}^n$, operation **UPDATE**$(D, v)$ adds $v^\top v$ to each matrix $S_i \in D$, appends the matrix $v^\top v$ to $D$.

- **COMPRESS**$(D, \epsilon)$: Operation **COMPRESS**$(D, \epsilon)$ enforces the invariant $(1 - \epsilon)S_i \not\preceq S_{i+2}$ for each matrix in the sequence, by deleting a number of matrices in $D$ and reordering the indices.

- **EXPRIE**$(D)$: Operation **EXPRIE**$(D)$ erases $S_1$ and reorders the indices.

We define the operations **UPDATE**, **COMPRESS**, and **EXPRIE** formally as follows.

**Algorithm 1** **UPDATE**$(D, v)$

**Input:** A data structure $D$ that consists of a sequence of PSD matrices $S_1 \succeq \ldots \succeq S_s \in \mathbb{R}^{n \times n}$, and a row vector $v \in \mathbb{R}^n$.

**Output:** An updated data structure $D$ where $v^\top v$ is added to each matrix and also appended at the end.

1: Set $S_{s+1} \leftarrow v^\top v$.
2: for $i = 1, \ldots, s$ do
3: \hspace{1cm} $S_i \leftarrow S_i + v^\top v$.
4: \hspace{1cm} end for
5: $s \leftarrow s + 1$.
6: return $D$.

**Observation 2.2.** The procedure **UPDATE**$(D, v)$ increases each matrix in $D$ by $v^\top v$ and then deletes a number of matrices so that $(1 - \epsilon)S_i \not\preceq S_{i+2}$ for each matrix $S_i$ that remains in $D$.

**Observation 2.3.** The procedure **COMPRESS**$(D, \epsilon)$ enforces $(1 - \epsilon)S_i \not\preceq S_{i+2}$ for each matrix $S_i$ that remains in $D$, enforcing Invariant ⋆.

**Observation 2.4.** The procedure **EXPRIE**$(D)$ deletes $S_1$ and reorders the indices.

We now show that the number of matrices in a spectrogram can be bounded, given certain restrictions on the eigenvalues of the matrices in the data structure.

**Lemma 2.5.** Suppose $D$ is a spectrogram that contains matrices $S_1 \succeq \ldots \succeq S_s \in \mathbb{R}^{n \times n}$ satisfying Invariant ⋆. If there exist parameters $\alpha, \beta > 0$ such that for each $1 \leq i \leq s$ and $1 \leq k \leq n$, either $\sigma_k(S_i) = 0$ or $\beta \leq \sigma_k(S_i) \leq \alpha$, then $s = O\left(\frac{n}{\epsilon} \log \frac{n}{\beta}\right)$.
Algorithm 2 \textsc{Compress}(\mathcal{D}, \epsilon)

\textbf{Input:} A data structure \(\mathcal{D}\) that consists of a sequence of PSD matrices \(S_1 \succeq \ldots \succeq S_s \in \mathbb{R}^{n \times n}\) and approximation parameter \(\epsilon > 0\).

\textbf{Output:} An updated data structure \(\mathcal{D}\) that enforces Invariant \(\star\).

\begin{algorithmic}[1]
\FOR \text{\(i = 1, \ldots, s - 2\)}
\STATE Let \(j = \max\{k : S_k \succeq (1 - \epsilon)S_i\}\).
\IF {\(j > i + 1\)}
\STATE Delete \(S_{i+1}, \ldots, S_{j-1}\).
\FOR \text{\(k = i + 1\) to \(s - j + i + 2\)}
\STATE \(S_k \leftarrow S_{j-i+k-1}\).
\ENDFOR
\STATE \(s \leftarrow s - j + i + 1\).
\ENDIF
\ENDFOR
\RETURN \(\mathcal{D}\).
\end{algorithmic}

Proof. First, observe that there cannot exist matrices \(S_i\) and \(S_j\) with \(j > i + 1\) such that for all \(1 \leq k \leq n\), \((1 - \epsilon)\sigma_k(S_i) \leq \sigma_k(S_j)\) by Invariant \(\star\). In other words, each pair of matrices must have some eigenvalues that differ by a factor of at least \(1 - \epsilon\). By assumption, the eigenvalues of each \(S_i\) satisfy \(\sigma_k(S_i) = 0\), or \(\beta \leq \sigma_k(S_i) \leq \alpha\). Thus for a fixed \(k\) for which \(\sigma_k(S_1) \neq 0\), there can be at most \(O\left(\frac{\alpha}{\beta} \log \frac{\alpha}{\beta}\right)\) indices \(i\) such that \((1 - \epsilon)\sigma_k(S_i) > \sigma_k(S_{i+2})\). Each matrix has dimension \(n \times n\), and so there are \(n\) eigenvalues, which implies there can be at most \(O\left(\frac{n}{\epsilon} \log \frac{n}{\epsilon}\right)\) indices in \(\mathcal{D}\). \(\square\)

2.1 Data Structure Augmented with Timestamps

We now define a data structure by augmenting each of the matrices \(S_i\) in a spectrogram data structure with a corresponding timestamp \(t_i\).

Definition 2.6. We define a data structure \(\mathcal{H}\) to be a spectral histogram on a data stream of rows \(r_1, \ldots, r_t \in \mathbb{R}^n\) and a sliding window parameter \(W\) if it contains a sequence of PSD matrices \(S_1 \succeq \ldots \succeq S_s \in \mathbb{R}^{n \times n}\), along with corresponding timestamps \(t_1, \ldots, t_s\) that satisfy the following properties:

1. Property \([\mathcal{H}1]\) For each \(1 \leq i \leq s\), \(S_i = \sum_{k=t_i}^{t} r_k^\top r_k\).

2. Property \([\mathcal{H}2]\) For each \(1 \leq i \leq s - 1\), one of the following holds:
(a) Property 2a: \( t_{i+1} = t_i + 1 \) and \((1 - \varepsilon)S_i \not\preceq S_{i+1} \).
(b) Property 2b: \((1 - \varepsilon)S_i \preceq S_{i+1}\) and if \( i + 1 \leq s \), then \((1 - \varepsilon)S_i \not\preceq S_{i+2} \).

(c) Property 3: \( t_1 \leq t - W + 1 < t_2 \).

Observation 2.7. Property 2 is equivalent to Invariant \( \star \).

Property 1 and Property 2 together should be considered a timestamp augmented analogue of Invariant \( \star \), while Property 3 should be considered a timestamp augmented analogue of procedure EXPIRE.

Thus we define AUGUPDATE, AUGCOMPRESS, and AUGEXPRESS to be time-sensitive analogues of UPDATE, COMPRESS, and EXPIRE.

\[\text{Algorithm 4 AUGUPDATE}(\mathcal{H}, r_t, t)\]
\[\text{Input: Data structure (spectral histogram) } \mathcal{H} \text{ consisting of PSD matrices } S_1 \succeq \ldots \succeq S_s \in \mathbb{R}^{n \times n} \text{ and corresponding timestamps } t_1, \ldots, t_s, \text{ row vector } r_t \in \mathbb{R}^n, \text{ and current timestamp } t.\]
\[\text{Output: Updated data structure } \mathcal{H}.\]

1. Set \( S_{s+1} = r_t^\top r_t \) and \( t_{s+1} = t.\)
2. \textbf{for} \( i = 1, \ldots, s \) \textbf{do}
3. \hspace{1em} \( S_i \leftarrow S_i + r_t^\top r_t.\)
4. \textbf{end for}
5. \( s \leftarrow s + 1.\)
6. \textbf{return } \mathcal{H}.

\[\text{Algorithm 5 AUGCOMPRESS}(\mathcal{H}, \varepsilon)\]
\[\text{Input: Data structure (spectral histogram) } \mathcal{H} \text{ consisting of PSD matrices } S_1 \succeq \ldots \succeq S_s \in \mathbb{R}^{n \times n} \text{ with corresponding timestamps } t_1, \ldots, t_s, \text{ and approximation parameter } \varepsilon.\]
\[\text{Output: Updated data structure } \mathcal{H}.\]

1. \textbf{for} \( i = 1, \ldots, s - 2 \) \textbf{do}
2. \hspace{1em} Compute \( j = \max\{k : S_k \succeq (1 - \varepsilon)S_i\}.\) \hspace{1em} \(\triangleright\text{This implies } S_{k+1} \not\preceq (1 - \varepsilon)S_i.\)
3. \hspace{1em} \textbf{if} \( j > i + 1 \) \textbf{then}
4. \hspace{2em} Delete \( S_{i+1}, \ldots, S_{j-1} \) and \( t_{i+1}, \ldots, t_{j-1}.\)
5. \hspace{2em} \textbf{for} \( k = i + 1 \) \text{ to } s - j + i + 2 \textbf{ do}
6. \hspace{3em} \( S_k \leftarrow S_{j-i+k-1}.\)
7. \hspace{3em} \( t_k \leftarrow t_{j-i+k-1}.\)
8. \hspace{3em} \textbf{end for}
9. \hspace{2em} \( s \leftarrow s - j + i + 1.\)
10. \hspace{2em} \textbf{end if}
11. \textbf{end for}
12. \textbf{return } \mathcal{H}.

3 Spectral Approximation in the Sliding Window Model

We now show how a spectral histogram can be used to maintain a spectral approximation in the sliding window model.
Algorithm 6 \textsc{AugExpire}(\mathcal{H}, t, W)

\textbf{Input:} Data structure (spectral histogram) $\mathcal{H}$ consisting of PSD matrices $S_1 \succeq \ldots \succeq S_s \in \mathbb{R}^{n \times n}$ and timestamps $t_1 < \ldots < t_s$, current time $t$, and window size $W$.

\textbf{Output:} Updated data structure $\mathcal{H}$.

1: \textbf{if} $t_2 \leq t - W + 1$ \textbf{then} \hspace{1cm} $\triangleright$ If first two timestamps are both expired.
2: \hspace{1cm} Delete $S_1$ and $t_1$.
3: \hspace{1cm} for $k = 1$ to $s$ \textbf{do}
4: \hspace{2cm} $S_k \leftarrow S_{k+1}$ \hspace{1cm} $\triangleright$ Reorder indices.
5: \hspace{1cm} $t_k \leftarrow t_{k+1}$
6: \hspace{1cm} \textbf{end for}
7: \hspace{1cm} $s \leftarrow s - 1$.
8: \hspace{1cm} \textbf{end if}
9: \textbf{return} $\mathcal{H}$.

Algorithm 7 Deterministic Algorithm for Spectral Approximation

\textbf{Input:} A stream of rows $r_1, \ldots, \in \mathbb{R}^{1 \times n}$, a parameter $W$ for the size of the sliding window, and an accuracy parameter $\epsilon$.

\textbf{Output:} A $(1 + \epsilon)$-spectral approximation in the sliding window model.

1: $S_1 = 0^{n \times n}, t_1 = 0$.
2: $\mathcal{H} \leftarrow \{\{S_1\}, \{t_1\}\}$.
3: for each row $r_t$ \textbf{do}
4: \hspace{1cm} $\mathcal{H} \leftarrow \textsc{AugUpdate}(\mathcal{H}, r_t, t)$
5: \hspace{1cm} $\mathcal{H} \leftarrow \textsc{AugCompress}(\mathcal{H}, \epsilon)$
6: \hspace{1cm} $\mathcal{H} \leftarrow \textsc{AugExpire}(\mathcal{H}, t, W)$
7: \hspace{1cm} \textbf{end for}
8: \textbf{return} $S_1$.

Lemma 3.1. \textit{Algorithm 7} maintains a spectral histogram, i.e., Definition 2.6.

Upon receiving each row $r_t$ in the stream, \textit{Algorithm 7} runs three subroutines: \textsc{AugUpdate}, \textsc{AugCompress}, and \textsc{AugExpire}. We show that these subroutines enforce the three important properties in Definition 2.6. The proof of Lemma 3.1 follows immediately from Lemma 3.2, Lemma 3.3, and Lemma 3.7.

Intuitively, the \textsc{AugUpdate} subroutine updates each of the matrices to maintain spectral approximations, incorporating the newly arrived row, so that Property [1] holds (see Lemma 3.2). The \textsc{AugCompress} subroutine ensures that the number of matrices remains small, by removing any three consecutive matrices whose spectra are within a factor of $(1 - \epsilon)$, and maintains Property [2] (see Lemma 3.3). The \textsc{AugExpire} subroutine also ensures that the number of matrices remains small, by removing any matrix that is “too old” for the sliding window so that Property [3] holds (see Lemma 3.7).

We first show that Property [1] is satisfied after running the \textsc{AugUpdate} subroutine.

Lemma 3.2. Upon receiving row $r_t$, the data structure $\mathcal{H}$ maintained by \textit{Algorithm 7} after the \textsc{AugUpdate} subroutine satisfies Property [1].

Proof. By the \textsc{AugUpdate} subroutine, $S_i = \sum_{k=1}^t r_k^\top r_k$, enforcing Property [1]. Since the \textsc{AugCom-
PRESS and AUGEXPRESS subroutines only delete and reorder indices, the invariant is maintained by the algorithm.

We next show that Property 2 is satisfied after running the AUGCOMPRESS subroutine. The following lemma says that the spectra of two adjacent matrices are either “far” (Property 2a) or “close” (Property 2b), but in the latter case, there cannot be three adjacent matrices whose spectra are within a $(1 + \varepsilon)$ factor.

**Lemma 3.3.** Upon receiving row $a_i$, the data structure $H$ maintained by Algorithm 7 after the AUGCOMPRESS subroutine consists of matrices $S_1, \ldots, S_s$ that satisfy Property 2. That is, for each $1 \leq i \leq s - 2$, one of the following holds:

1. Property 2a $t_{i+1} = t_i + 1$ and $(1 - \varepsilon)S_i \not\preceq S_{i+1}$.
2. Property 2b $(1 - \varepsilon)S_i \preceq S_{i+1}$ and if $i + 2 \leq s$, then $(1 - \varepsilon)S_i \not\preceq S_{i+2}$.

**Proof.** Let $J = \{u_1, \ldots, u_s\}$ be the timestamps maintained by the data structure after receiving row $B_i - 1$ and $I = \{t_1, \ldots, t_s\}$ be the timestamps maintained by the data structure after receiving row $B_i$. We fix an index $i$ and observe that if $t_i \in I$, then $t_i$ cannot have been deleted in a previous step. Thus, $t_i \in J$, so that $t_i = u_j$ for some index $j$. Let $S_1, \ldots, S_s$ be the corresponding matrices after receiving row $B_i$. We say that $t_{i+1}$ is the successor of $t_i$ in $I$ and $u_{j+1}$ is the successor of $u_j$ (and thus the successor of $t_i$) in $J$.

We first consider the case when $u_{j+1} \notin I$, i.e., whether the successor of $t_i$ in $J$ was deleted subsequently.

**Claim 3.4.** If $u_{j+1} \notin I$, then Property 2b holds.

**Proof.** If $u_{j+1} \notin I$, then the AUGCOMPRESS subroutine must have deleted timestamp $u_{j+1}$ but not timestamp $t_{i+1}$ (since $t_{i+1} \in I$). By the deletion condition of the AUGCOMPRESS subroutine, it follows that $(1 - \varepsilon)S_i \preceq S_{i+1}$ and also if $i + 2 \leq s$, then $(1 - \varepsilon)S_i \not\preceq S_{i+2}$. Thus, Property 2b holds and the claim follows.

We now perform a case analysis on whether $u_{j+1} \in I$, i.e., whether the successor of $t_i$ in $I$ was also the successor of $t_i$ in $J$. For ease of analysis, we break the case into two parts, depending on whether $u_{j+1} = u_{j+1}$ or not.

**Claim 3.5.** If $u_{j+1} \in I$ and $u_{j+1} > u_{j+1}$, then Property 2b holds.

**Proof.** If $u_{j+1} \in I$ and $u_{j+1} > u_{j+1}$, i.e., the successor of $u_j$ is not $u_{j+1}$, then consider the first time $\tau$ at which point $u_{j+1}$ became the successor of $u_j$. That is, $\tau$ is the first timestamp where all timestamps between $u_j$ and $u_{j+1}$ have been removed by the AUGCOMPRESS subroutine.

By the deletion condition of the AUGCOMPRESS subroutine, it follows that $T_{j+1} \geq (1 - \varepsilon)T_j$, where $T_j = \sum_{k=u_j}^{\tau} r_k^T r_k$ (and similarly for $T_{j+1}$). Therefore,

$$T_{j+1} + \sum_{k=\tau+1}^{t} r_k^T r_k \geq (1 - \varepsilon) \left( T_j + \sum_{k=\tau+1}^{t} r_k^T r_k \right).$$

We remark that this property is somewhat analogous to the smoothness property required for smooth histograms (see Definition A.1), but generalized to PSD matrices.
In other words, \( S_{j+1} \geq (1 - \varepsilon)S_j \), since \( S_{j+1} = T_{j+1} + \sum_{k=t+1}^{j} r_k^T r_k \) and \( S_j = T_j + \sum_{k=t+1}^{j} r_k^T r_k \). Moreover, if \( i + 2 \leq s \), then \( (1 - \varepsilon)S_i \not\preceq S_{i+2} \) holds by the deletion condition of the AUGCOMPRESS subroutine, the observation that \( \mathcal{H} \) has a PSD ordering, and the fact that \( t_{i+2} \in I \), i.e., the index was not deleted. Thus, Property 2b holds and the claim follows.

**Claim 3.6.** If \( u_{j+1} \in I \) and \( u_{j+1} = u_j + 1 \), then either Property 2a or Property 2b holds.

**Proof.** Finally, if \( u_{j+1} \in I \) and \( u_{j+1} = u_j + 1 \), i.e., the successor of \( u_j \) is \( u_j + 1 \), then \( t_{j+1} = t_i + 1 = u_j + 1 = u_{j+1} \), so the successor of \( t_i \) is also \( t_i + 1 \). Then either \( (1 - \varepsilon)S_i \not\preceq S_{i+1} \) so Property 2a holds or \( (1 - \varepsilon)S_i \preceq S_{i+1} \). In the second case, Property 2b holds, since if \( i + 2 \leq s \), then \( (1 - \varepsilon)S_i \not\preceq S_{i+2} \) holds by the deletion condition of the AUGCOMPRESS subroutine, the observation that \( \mathcal{H} \) has a PSD ordering, and the fact that \( t_{i+2} \in I \), i.e., the index was not deleted, and the claim follows.

The proof of Lemma 3.3 then follows from Claim 3.4, Claim 3.5, and Claim 3.6.

Finally, we show that Property 3 is satisfied after running the AUGEXPRISE subroutine.

**Lemma 3.7.** Upon receiving row \( r_i \), the data structure \( \mathcal{H} \) maintained by Algorithm 7 after the AUGEXPRISE subroutine consists of timestamps that satisfy \( t_1 \leq t - W + 1 < t_2 \).

**Proof.** By the AUGEXPRISE subroutine, \( t_1 \) is deleted if \( t_2 \leq t - W + 1 \). The inequality follows since \( t_1 + 1 \leq t_2 \). Since the AUGUPDATE and AUGCOMPRESS subroutines cannot delete \( S_1 \) and \( t_1 \), then the invariant is maintained by the algorithm.

We now show that Algorithm 7 maintains a spectral approximation in the sliding window model.

**Theorem 3.8.** For any time \( t \), let \( A = \sum_{k=t-W+1}^{t} r_k^T r_k \). Moreover, suppose that any nonzero eigenvalue \( \sigma \) of any matrix maintained by Algorithm 7 satisfies \( \beta \leq \sigma \leq \alpha \). Then Algorithm 7 outputs a \((1 + \varepsilon)\) spectral approximation to \( A \), using \( O \left( \frac{n^2}{\varepsilon^2} \log \frac{\alpha}{\beta} \right) \) space.

**Proof.** Let \( \mathcal{H} \) consist of matrices \( S_1, \ldots, S_t \) and corresponding timestamps \( t_1, \ldots, t_s \). By Property 3 in Lemma 3.1, \( t_1 \leq t - W + 1 < t_2 \), so that \( S_1 \succeq A \succeq S_2 \). If \( t_2 = t_1 + 1 \), then \( t_1 = t - W + 1 \) and so \( S_1 = A \) by Property 1. Otherwise, if \( t_2 > t_1 + 1 \), then by Property 2, \( (1 - \varepsilon)S_1 \preceq S_2 \) and hence, \( S_1 \succeq A \succeq (1 - \varepsilon)S_1 \), and so \( S_1 \) is a \((1 + \varepsilon)\) spectral approximation of \( A \).

To analyze the space complexity, observe that Algorithm 7 only maintains data structure \( \mathcal{H} \), which consists of a spectogram data structure consisting matrices \( S_1, \ldots, S_t \), augmented with corresponding timestamps \( t_1, \ldots, t_s \). Since each \( S_i \in \mathbb{R}^{n \times n} \) and \( s = O \left( \frac{n^2}{\varepsilon^2} \log \frac{\alpha}{\beta} \right) \) by Lemma 2.5, the total space is \( O \left( \frac{n^2}{\varepsilon^2} \log \frac{\alpha}{\beta} \right) \).

To complete the proof of the space bounds of Theorem 1.1, we use the assumption that \( \frac{1}{\text{poly}(W)} \leq \sigma_{\min}(A) \leq \sigma_{\max}(A) \leq \text{poly}(W) \).

Finally, we remark that the amortized running time per arriving row can be improved by running each of the subroutines AUGUPDATE, AUGCOMPRESS, and AUGEXPRISE only after receiving \( \frac{n^2}{\varepsilon^2} \log n \) rows since the last iteration of the subroutines. Since Algorithm 7 stores \( O \left( \frac{n^2}{\varepsilon^2} \log n \right) \) rows, the asymptotic space used by the algorithm will remain the same. It takes \( O \left( n^2 \right) \) time to
update each $S_{i-1}$ through an outer product and $O(n^3)$ time to perform the singular value decomposition. Thus if the subroutines are only performed after receiving $\frac{n^2}{\epsilon^3} \log n$ rows since the previous instance, then the amortized running time per arriving row is $O(n^3)$.

4 Space Efficient Algorithm

In this section, we give a space efficient algorithm for spectral approximation in the sliding window model. We first recall the definition of $\lambda$-ridge leverage scores, as well as define one of our key contributions, the reverse online $\lambda$-ridge leverage scores.

Given a matrix $\mathbf{R}$, the $\lambda$-ridge leverage score $[\text{AM15}]$ of the $i$th row is defined to be

$$r_i(\mathbf{R}^\top \mathbf{R} + \lambda \mathbf{I})^+ r_i^\top.$$

We use leverage score to refer to the case where $\lambda = 0$.

**Definition 4.1 (Reverse Online $\lambda$-Ridge Leverage Scores).** We define the reverse online $\lambda$-ridge leverage score of the $i$th row of a matrix $\mathbf{R} \in \mathbb{R}^{m \times n}$ to be

$$\min \left( r_i(M_i^\top M_i + \lambda \mathbf{I})^+ r_i^\top, 1 \right),$$

where $\mathbf{M}$ is the matrix $\mathbf{r}_m \circ \ldots \circ \mathbf{r}_1$ whose rows are in the reverse order as $\mathbf{R} = \mathbf{r}_1 \circ \ldots \circ \mathbf{r}_m$ and $M_i$ is the matrix consisting of the first $i$ rows of $\mathbf{M}$. The reverse online leverage scores are the reverse online $\lambda$-ridge leverage score for $\lambda = 0$.

Observe that the sum of the reverse online $\lambda$-ridge leverage scores must also be $O \left( n \log \frac{\|\mathbf{R}\|_2}{\lambda} \right)$, since the singular values of $\mathbf{R}$ are invariant when the rows are reversed, i.e., $\mathbf{Rv} = \lambda \mathbf{v}$ implies $\mathbf{R}'\mathbf{v}' = \lambda \mathbf{v}'$, where $\mathbf{R}'$ is the matrix whose rows are in the reverse order as $\mathbf{R}$ and $\mathbf{v}'$ is the vector whose entries are in the reverse order as $\mathbf{v}$.

4.1 Algorithm

Our main algorithm is **META-SPECTRAL** [Algorithm 8], that uses three subroutines **OptUpdate**, **DownSample**, and **OptExpire**. Here, the first and third subroutines are analogous to **Update** and **Expire** described in Section 2.1.

The **DownSample** procedure handles the previously described downsampling for each row and replaces the role of the **Compress** procedure in limiting the total number of rows. Intuitively, our algorithm runs an instance of online row sampling in reverse each time a new row arrives, with respect to the rows that have been sampled (and rescaled). [CMT16] uses matrix martingales to bound the total number of rows sampled by their algorithm. However, this line of analysis does not work for our algorithm, since we must restart a separate new instance each time a new row arrives. Hence, we prove an invariance that says: if our current sketch gives a good estimate of all matrices formed by tails of the stream, then our next sketch will give a good estimate of all matrices formed by tails of the stream after the arrival of the next row, with high probability.

Instead of keeping entire matrices $\mathbf{S}_i$ associated with each timestamp $t_i$, as in [Algorithm 7] **META-SPECTRAL** keeps a number of rows $\mathbf{a}_1, \ldots, \mathbf{a}_s \in \mathbb{R}^{1 \times n}$ along with their timestamps $t_1, \ldots, t_s$. 

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Algorithm 8 META-SPECTRAL: Algorithm for spectral approximation in the sliding window model

**Input:** A stream of rows \( r_1, r_2, \ldots \in \mathbb{R}^n \), a parameter \( W \leq n^p \) for the size of the sliding window, and an accuracy parameter \( \varepsilon \)

**Output:** A \((1 + \varepsilon)\)-spectral approximation in the sliding window model.

1. \( \mathcal{H} \leftarrow \{\emptyset, \emptyset\} \)
2. for each row \( r_t \) do
3. \( \mathcal{H} \leftarrow \text{OptUpdate}(\mathcal{H}, r_t, t) \)
4. Let \( \mathcal{H} \) contain rows \( a_1, \ldots, a_{s+1} \).
5. for each \( j = s \) down to \( j = 1 \) do
6. \( \text{Let } \tilde{A}_{j+1} = a_{j+1} \circ \ldots \circ a_{s+1} \).
7. \( M_j \leftarrow \text{Downsample} \left( a_j, \tilde{A}_{j+1}, \varepsilon, 0 \right) \)
8. end for
9. \( \mathcal{H} \leftarrow \text{Optexpire}(\mathcal{H}, t, W) \)
10. end for

Thus if \( r_1, \ldots, r_t \in \mathbb{R}^{1 \times n} \) are the rows of the stream, then the algorithm maintains the following invariant: \( \tilde{A}_i = a_1 \circ \ldots \circ a_i \) is a good spectral approximation of \( A_i = r_t \circ \ldots \circ r_1 \) for all \( 1 \leq i \leq s \).

To maintain this invariant, every time a new row arrives, the algorithm iteratively uses the procedure Downsample (Algorithm 11) to test whether each row \( a_i \) from \( i = s \) down to \( j = 1 \) is...
retained. This is done by computing the approximate reverse online leverage score, \( \tilde{l}_i \), of \( a_i \) with respect to the rows \( a_{i+1}, \ldots, a_s \) that have been retained, and choosing to keep \( a_i \) with probability that is some constant multiple of this score.

**Algorithm 11** **DOWNSAMPLE** \((M, \hat{A}, \epsilon, \delta)\)

**Input:** Matrices \( M \in \mathbb{R}^{m \times n}, \hat{A} \in \mathbb{R}^{d \times n} \), approximation parameter \( \epsilon \), and parameter \( \delta \).

1. \( \hat{M} \leftarrow \emptyset, \alpha \leftarrow 12p, c \leftarrow \frac{\alpha \log n}{\epsilon^2}, \lambda = \frac{\delta}{\epsilon} \).
2. for \( i = 1 \) to \( i = m \) do \( \triangleright m = 1 \) when **DOWNSAMPLE** is called by **META-SPECTRAL**
   3. \( \triangleright \mathcal{M} \) has multiple rows for time efficient algorithm
   4. Let \( \tilde{l}_i = \min(m_i(M^\top M + \hat{A}^\top \hat{A} + \lambda I)^\top m_i^\top, 1) \).
   5. Let \( p_i = \min\left(2\tilde{l}_i, 1\right) \).
   6. with probability \( p_i \) do
   7. \( \hat{M} \leftarrow \hat{M} \odot \frac{m_i}{\sqrt{p_i}} \).
5. end with probability
9. end for
10. return \( \hat{M} \).

Our main technical lemma is **Lemma 4.2** which is effectively another invariant that offers insight as to why a \((1 + \epsilon)\)-spectral approximation for each tail of the substream suffices to obtain a \((1 + \epsilon)\) approximation. Suppose that \( X \) is a \((1 + \epsilon)\) spectral approximation of \( X \). Then the reverse online leverage score of any row \( r_i \) of \( X \) with respect to \( X \) is within a \((1 + \epsilon)\) factor of the leverage score of \( r_i \) with respect to \( X \). In other words, \( X \) provides approximate reverse online leverage scores, so we can still oversample with respect to the true reverse online leverage scores and obtain a \((1 + \epsilon)\) spectral approximation after a new row arrives. Therefore, if we obtain \( \hat{X} \) by oversampling the rows of \( X \) with respect to the reverse online leverage scores and we have a matrix \( \hat{Y} \circ \hat{Z} \) that is a good spectral approximation of \( Y \circ Z \), then with high probability, **DOWNSAMPLE** \((X, \hat{Y} \circ \hat{Z})\) will still oversample the rows of \( X \) compared to the reverse online leverage scores, but at most by a constant factor.

We will use this lemma to both show that each \( a_i \circ \ldots \circ a_s \) is a good spectral approximation of \( a_i \circ \ldots \circ a_t \) (in **Corollary 4.3**). We next state and prove our first main technical lemma in this section.

**Lemma 4.2** (Sampling probabilities are bounded). Let \( \lambda \geq 0, 0 < \epsilon < \frac{1}{2}, \alpha = 12p \) and \( c = \frac{a \log n}{\epsilon^2} \). Suppose \( X, Y, Z, \tilde{X}, \tilde{Y}, \tilde{Z} \) are matrices with:

\[
\begin{align*}
1. & (1 - \epsilon)(Y^\top Y + \lambda I) \preceq \tilde{Y}^\top \tilde{Y} + \lambda I \preceq (1 + \epsilon)(Y^\top Y + \lambda I) \\
2. & (1 - \epsilon)(X^\top X + Y^\top Y + \lambda I) \preceq \tilde{X}^\top \tilde{X} + \tilde{Y}^\top \tilde{Y} + \lambda I \preceq (1 + \epsilon)(X^\top X + Y^\top Y + \lambda I) \\
3. & (1 - \epsilon)(Y^\top Y + Z^\top Z + \lambda I) \preceq \tilde{Y}^\top \tilde{Y} + \tilde{Z}^\top \tilde{Z} + \lambda I \preceq (1 + \epsilon)(Y^\top Y + Z^\top Z + \lambda I),
\end{align*}
\]

and furthermore, \( \tilde{X} \) is obtained by sampling each row \( x_i \) in \( X \) with probability at least

\[
p_i \geq \min(1, c \cdot x_i^\top (X^\top X + Y^\top Y + \lambda I)^\top x_i)
\]

and rescaling by \( \frac{1}{\sqrt{p_i}} \). Then each row \( x_i \) in \( X \) is sampled by **DOWNSAMPLE** \((\tilde{X}, \tilde{Y} \circ \tilde{Z}, \epsilon, \epsilon \lambda)\) with probability \( q_i \), such that

\[
\min(1, c \cdot x_i(D^\top D + \lambda I)^\top x_i^\top) \leq q_i \leq \min(1, 2(1 + 6\epsilon)c \cdot x_i(D^\top D + \lambda I)^\top x_i^\top),
\]

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where \( D^\top D := X^\top X + Y^\top Y + Z^\top Z \)

Before we prove this lemma, we make an important remark. Lemma 4.2 holds even when \( \tilde{X} = \emptyset \). That is, it shows a stronger result that \( a_1 \circ \ldots \circ a_s \) is a good spectral approximation of \( r_1 \circ \ldots \circ r_t \) for all \( t_{i-1} < j \leq t_i \), where \( t_0 := t - W \). Moreover, Lemma 4.2 can be formulated to show that each row is sampled with probability proportional to its reverse online leverage score. We will thus also use Lemma 4.2 to bound the total number of rows maintained by the algorithm (in Corollary 4.4) by bounding the sum of the reverse online leverage scores. Although Meta-Spectral uses \( \lambda = 0 \), we consider the full generality so that we can also apply this lemma for other regularization parameters.

Proof of Lemma 4.2. First, note that by Lemma A.3 sampling each row \( x_i \) in \( X \) with probability at least
\[
p_i \geq \min(1, c \cdot x_i^\top (X^\top X + Y^\top Y + \lambda I)^\dag x_i)
\]
suffices to form \( \tilde{X} \) such that \((1 - \epsilon)(X^\top X + \lambda I) \preceq \tilde{X}^\top \tilde{X} + \lambda \tilde{I} \preceq (1 + \epsilon)(X^\top X + \lambda I)\). Hence, there exist matrices \( \tilde{Y} \) and \( \tilde{Z} \) that satisfy the conditions.

Let \( D^\top D := X^\top X + Y^\top Y + Z^\top Z \). First, observe that subtracting condition 1 from condition 2, we get
\[
-2\epsilon (Y^\top Y + \lambda I) + (1 - \epsilon)X^\top X \preceq \tilde{X}^\top \tilde{X} \preceq 2\epsilon (Y^\top Y + \lambda I) + (1 + \epsilon)X^\top X
\]
Now adding condition 3 to the above implies that
\[
(1 - 3\epsilon)(D^\top D + \lambda I) \preceq \tilde{X}^\top \tilde{X} + \tilde{Y}^\top \tilde{Y} + \tilde{Z}^\top \tilde{Z} + \lambda I \preceq (1 + 3\epsilon)(D^\top D + \lambda I).
\]

Fact A.7 (inverse of positive semidefinite matrices) then implies that
\[
(1 + 3\epsilon)^{-1}(D^\top D + \lambda I)^\dag \preceq (\tilde{X}^\top \tilde{X} + \tilde{Y}^\top \tilde{Y} + \tilde{Z}^\top \tilde{Z} + \lambda I)^\dag.
\]
Hence for any row \( x_i \) of \( X \),
\[
x_i(D^\top D + \lambda I)^\dag x_i^\top \leq (1 + 3\epsilon)x_i(\tilde{X}^\top \tilde{X} + \tilde{Y}^\top \tilde{Y} + \tilde{Z}^\top \tilde{Z} + \lambda I)^\dag x_i^\top.
\]
Now any row \( x_i \) of \( X \) remains in \( \tilde{X} \) with probability
\[
p_i \geq \min(1, c \cdot x_i(\tilde{X}^\top \tilde{X} + \tilde{Y}^\top \tilde{Y} + \lambda I)^\dag x_i^\top).
\]
By construction, any row \( x_i \) that is sampled with probability \( p_i \) is scaled to \( \tilde{x}_i / p_i \). Furthermore, any row \( x_i \) of \( X \) that remains in \( \tilde{X} \) is sampled by DOWNSAMPLE \((\tilde{X}, \tilde{Y}, \tilde{Z}, \epsilon, \epsilon, \lambda)\) with probability \( \min \{ 1, \tilde{x}_i(\tilde{X}^\top \tilde{X} + \tilde{Y}^\top \tilde{Y} + \tilde{Z}^\top \tilde{Z} + \lambda I)^\dag x_i^\top \} \). Therefore, the probability that any row \( x_i \) of \( X \) that is present in \( \tilde{X} \) is also sampled by DOWNSAMPLE \((\tilde{X}, \tilde{Y}, \tilde{Z}, \epsilon, \epsilon, \lambda)\) is
\[
q_i = \min(p_i, 2c \cdot x_i(\tilde{X}^\top \tilde{X} + \tilde{Y}^\top \tilde{Y} + \tilde{Z}^\top \tilde{Z} + \lambda I)^\dag x_i^\top).
\]
Then for \( \epsilon < \frac{1}{3} \),
\[
c \cdot x_i(D^\top D + \lambda I)^\dag x_i^\top \leq 2c \cdot x_i(\tilde{X}^\top \tilde{X} + \tilde{Y}^\top \tilde{Y} + \tilde{Z}^\top \tilde{Z} + \lambda I)^\dag x_i^\top.
\]
Now the monotonicity of ridge-leverage scores (Lemma A.4) on the rows of \( Z \) gives us
\[
p_i \geq c \cdot x_i(D^\top D + \lambda I)^\dag x_i^\top.
\]
Hence, each row $x_i$ in $X$ is sampled in $\text{DOWNSAMPLE}(\tilde{X}, \tilde{Y} \circ \tilde{Z}, \varepsilon, \varepsilon \lambda)$ with probability $q_i$ at least $c \cdot x_i(D^\top D + \lambda I)^\dagger x_i^\top$.

For the other half of the inequality, observe that Fact $A.7$ also implies that

$$(1 - 3\varepsilon)^{-1}(D^\top D + \lambda I)^\dagger \succeq (\tilde{X}^\top \tilde{X} + \tilde{Y}^\top \tilde{Y} + \tilde{Z}^\top \tilde{Z} + \lambda I)^\dagger.$$ 

For $\varepsilon < \frac{1}{6}$, $(1 + 6\varepsilon)(1 - 3\varepsilon)^{-1} > 1$ and hence for any row $x_i$ of $X$,

$$(1 + 6\varepsilon)x_i(D^\top D + \lambda I)^\dagger x_i^\top \geq x_i(\tilde{X}^\top \tilde{X} + \tilde{Y}^\top \tilde{Y} + \tilde{Z}^\top \tilde{Z} + \lambda I)^\dagger x_i^\top.$$ 

Since $q_i \leq 2c \cdot x_i(\tilde{X}^\top \tilde{X} + \tilde{Y}^\top \tilde{Y} + \tilde{Z}^\top \tilde{Z} + \lambda I)^\dagger x_i^\top$, then it follows that each row $x_i$ in $X$ is sampled in $\text{DOWNSAMPLE}(\tilde{X}, \tilde{Y} \circ \tilde{Z}, \varepsilon, \varepsilon \lambda)$ with probability $q_i$ at most $2(1 + 6\varepsilon)c \cdot x_i(D^\top D + \lambda I)^\dagger x_i^\top$ as required.

We next show that under certain conditions on $\tilde{Y}$ and $\tilde{Z}$, each row of $X$, $Y$, and $Z$ are oversampled by $\text{DOWNSAMPLE}(\tilde{X}, \tilde{Y} \circ \tilde{Z}, \varepsilon, \varepsilon \lambda) \circ \tilde{Y} \circ \tilde{Z}$. This is done in Corollary 4.3. We then prove that the space is bounded using Corollary 4.4.

**Corollary 4.3** (Spectral approximation guarantee). Let $\alpha = 12p$, $0 < \varepsilon < \frac{1}{7}$, $c = \frac{\alpha \log n}{\varepsilon}$, and $\lambda \geq 0$. Given matrices $Y$ and $Z$, suppose $\tilde{Y}$ (resp. $Z$) is obtained by sampling each row $r_i$ in $Y$ (resp. $Z$) with probability at least

$$p_i \geq \min(1, c \cdot r_i(Y^\top Y + Z^\top Z + \lambda I)^\dagger r_i^\top)$$

and rescaling by $\frac{1}{\sqrt{p_i}}$. Let matrices $D$, $X$ and $\tilde{X}$ satisfy the conditions of Lemma 4.2.

1. If $M = \text{DOWNSAMPLE}(\tilde{X}, \tilde{Y} \circ \tilde{Z}, \varepsilon, \varepsilon \lambda) \circ \tilde{Y} \circ \tilde{Z}$, then each row $r_i$ in $X$, $Y$, and $Z$ is sampled by $M$ with probability at least $\min(1, c \cdot r_i(D^\top D + \lambda I)^\dagger r_i^\top)$.

2. If $M = \text{DOWNSAMPLE}(\tilde{X}, \tilde{Y} \circ \tilde{Z}, \varepsilon, 0) \circ \tilde{Y} \circ Z$, then with probability at least $1 - n^{-a/3}$,

$$(1 - \varepsilon)D^\top D \preceq M^\top M \preceq (1 + \varepsilon)(X + Y + Z)^\top (X + Y + Z).$$

**Proof.** Observe that by monotonicity of ridge-leverage scores (Lemma A.4) and Lemma 4.2 each row $r_i$ in $X$, $Y$, and $Z$ is sampled by $\text{DOWNSAMPLE}(\tilde{X}, \tilde{Y} \circ \tilde{Z}, \varepsilon, \varepsilon \lambda)$ with probability at least $\min(1, c \cdot r_i(D^\top D + \lambda I)^\dagger r_i^\top)$.

Now, when $M = \text{DOWNSAMPLE}(\tilde{X}, \tilde{Y} \circ \tilde{Z}, \varepsilon, 0) \circ \tilde{Y} \circ Z$, we have that each row $r_i$ in $X$, $Y$, and $Z$ is sampled by $\text{DOWNSAMPLE}(\tilde{X}, \tilde{Y} \circ \tilde{Z}, \varepsilon, 0)$ with probability at least $\min(1, c \cdot r_i(D^\top D)^\dagger r_i^\top)$. Thus by oversampling with respect to the leverage scores, i.e. Lemma A.3,

$$(1 - \varepsilon)(X + Y + Z)^\top (X + Y + Z) \preceq M^\top M \preceq (1 + \varepsilon)(X + Y + Z)^\top (X + Y + Z)$$

with probability at least $1 - n^{-a/3}$. 

Similarly, we can show that Lemma 4.2 bounds the number of rows output by $\text{DOWNSAMPLE}$.

**Corollary 4.4** (Number of rows are bounded). Let $X$, $Y$, $Z$, $\tilde{X}$, $\tilde{Y}$, and $\tilde{Z}$ be matrices that satisfy Corollary 4.3 and let $M = \text{DOWNSAMPLE}(\tilde{X}, \tilde{Y} \circ \tilde{Z}, \varepsilon, 0) \circ \tilde{Y} \circ Z$. If $W < n^p$ for some constant $p > 0$ and $\gamma \leq \varepsilon c \min(R)^2$ for any matrix $R$ by any number of consecutive rows of $X \circ Y \circ Z$, then $M$ has $O\left(\frac{\alpha n}{\varepsilon^2} \log n \log \|R\|_F^2\right)$ rows with probability at least $1 - \frac{1}{W}$. 

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Proof. By [Lemma 4.2] each row \( r_i \) in \( X, Y, \) and \( Z \) is sampled by \( \text{DOWNSAMPLE}(\overline{X}, \overline{Y} \circ \overline{Z}, \epsilon, 0) \) with probability \( p_i \) such that
\[
p_i \leq \min(1, 2(1 + 6\epsilon)c \cdot r_i(D^\top D)^\top r_i^\top),
\]
where \( D^\top D := X^\top X + Y^\top Y + Z^\top Z \). Hence, each row in \( r_i \) is sampled by \( \text{DOWNSAMPLE}(\overline{X}, \overline{Y} \circ \overline{Z}, \epsilon, 0) \) with probability \( p_i \leq 2(1 + 6\epsilon)(1 + \epsilon)c \cdot l_i \), where \( l_i \) is the leverage score of the \( i^{th} \) row with respect to \( W := X \circ Y \circ Z \). Moreover, observe that \( \sigma_i(W) \neq 0 \) and \( \gamma = \varepsilon \sigma_{\min}(W)^2 \) imply
\[
\sigma_i(W)^2 \leq \sigma_i(W)^2 + \gamma \leq (1 + \varepsilon)\sigma_i(W)^2,
\]
so for the purpose of analysis, we can relate the sum of the reverse online leverage scores with the sum of the online \( \lambda \)-ridge leverage scores for \( \lambda = \gamma \). Since the sum of the online \( \lambda \)-ridge leverage scores is known to be \( \mathcal{O}\left(n \log \frac{\|R\|_2^2}{\gamma}\right) \) (Lemma A.5), then it follows that
\[
\sum_{i=1}^{m} p_i = \mathcal{O}\left(cn \log \frac{\|R\|_2^2}{\gamma}\right),
\]
where \( m \) is the number of rows in \( X \circ Y \circ Z \). Then by a simple Chernoff bound and the observation that \( c = \mathcal{O}\left(\frac{\log n}{\varepsilon^2}\right) \), it follows that \( M \) has \( \mathcal{O}\left(\frac{n^2}{\varepsilon^2} \log n \log \frac{\|R\|_2^2}{\gamma}\right) \) rows with probability \( 1 - \frac{1}{W} \). \( \square \)

As noted earlier, Corollary 4.3 and Corollary 4.4 apply when \( \overline{X} = a_i \) or even \( \overline{X} = \emptyset \) and \( X = r_j \circ \ldots \circ r_i \) for all \( t_{i-1} < j \leq t_i \) (where \( t_0 := t - W \)), showing that \( \overline{A}_i = a_i \circ \ldots \circ a_s \) is a good spectral approximation of \( r_j \circ \ldots \circ r_i \). Intuitively, the rows we have sampled since time \( j \) form a good spectral approximation for the matrix formed by all the rows since time \( j \). In particular since \( t - W + 1 \leq t_i \), then \( \overline{A}_i \) is a good approximation to \( A \), the matrix represented by the sliding window. We now formalize the correctness and space complexity of Algorithm 8.

**Theorem 4.5.** Let \( r_1, \ldots, r_t \in \mathbb{R}^{1 \times n} \) be a stream of rows and \( W < n^p \) be the size of the sliding window for some constant \( p > 0 \). Let \( A = r_{t-W+1} \circ \ldots \circ r_t \) be the matrix consisting of the \( W \) most recent rows and suppose \( \frac{1}{\text{poly}(W)} \leq \sigma_{\min}(R) \leq \sigma_{\max}(R) \leq \text{poly}(W) \) across all matrices \( R \) formed by consecutive rows in the stream. Then given a parameter \( \epsilon > 0 \), there exists an algorithm that outputs a matrix \( M \) with a subset of (rescaled) rows of \( A \) such that
\[
(1 - \epsilon)A^\top A \preceq M^\top M \preceq (1 + \epsilon)A^\top A
\]
and uses \( \mathcal{O}\left(\frac{n^2}{\varepsilon^2} \log n \log \frac{n}{\varepsilon} \right) \) space, with probability \( 1 - \frac{1}{\text{poly}(W)} \). The total running time of the algorithm is \( \mathcal{O}\left(|S| \log^2 \frac{n}{\varepsilon} + \log \frac{n}{\varepsilon} \cdot \text{nnz} \right) \), where \( |S| \) is the length of the stream and \( \text{nnz} \) is the input sparsity of the stream.

**Proof.** Let \( t_{i}^{(j)} \) denote the time \( t_i \) for a matrix \( \overline{A}_i \) maintained by Algorithm 8 at time \( j \). Observe that it suffices to assume \( \epsilon < \frac{1}{8} \). Then by setting \( \lambda = 0 \) in Corollary 4.3 and a union bound for all \( 1 \leq i \leq s \) and \( 1 \leq j \leq t \), \( \overline{A}_i \) at time \( j \) is a \( (1 + \epsilon) \) spectral approximation of \( a_k \circ \ldots \circ a_j \) for any \( t_{i-1} < k \leq t_i \) (where \( t_0 := t - W \)), with probability at least
\[
1 - \frac{W^2}{n^{2/3}} = 1 - \frac{W^2}{n^{3p}} \geq 1 - \frac{1}{W^2}.
\]
Since $t - W + 1 \leq t_1$, then

$$(1 - \epsilon) A^T A \preceq \tilde{A}_1^T \tilde{A}_1 \preceq (1 + \epsilon) A^T A$$

at the end of the stream. By Corollary 4.4, $\tilde{A}_1$ contains $O \left( \frac{n^2}{\epsilon^2} \log n \log \frac{\|A\|_f^2}{\gamma} \right)$ rows, where $\gamma := \epsilon \sigma_{\min}(R)^2$ across all matrices $R$ formed by consecutive rows in the stream, with probability at least $1 - \frac{1}{W^2}$.

By construction any row maintained by Algorithm 8 is contained in $\tilde{A}_1$, so it follows that the algorithm uses $O \left( \frac{n^2}{\epsilon^2} \log n \log \frac{n}{\epsilon} \right)$ space, since $\frac{1}{\text{poly}(W)} \leq \sigma_{\min}(R) \leq \sigma_{\max}(R) \leq \text{poly}(W)$ and $\log W = O(\log n)$.

Finally, we remark that the amortized running time per arriving row can be improved by batching, i.e. running DOWNSAMPLE only after receiving $\frac{n}{\epsilon^2} \log n \log \frac{n}{\epsilon}$ rows since the last iteration of DOWNSAMPLE. Since Algorithm 8 stores $O \left( \frac{n^2}{\epsilon^2} \log n \log \frac{n}{\epsilon} \right)$ rows, the asymptotic space used by the algorithm will remain the same. Observe that it suffices to obtain some constant factor of the reverse online leverage score, since we can just modify DOWNSAMPLE by oversampling with that constant factor. To compute a constant factor approximation of the reverse online leverage score of any sampled row $a_t$, we use standard projection tricks [SS11, CMI16] embedding each of the $O \left( \frac{n^2}{\epsilon^2} \log n \log \frac{n}{\epsilon} \right)$ rows into a $O \left( \frac{n}{\epsilon^2} \right)$ dimension subspace by applying a Johnson-Lindenstrauss transform. Applying this subspace embedding to each of the rows takes $O \left( \log \frac{n}{\epsilon^2} \cdot \text{nnz} \right)$ time, and then computing the pseudoinverses for each submatrix takes $O \left( \frac{n^2}{\epsilon^2} \log n \log \frac{n}{\epsilon} \cdot \log^2 \frac{n}{\epsilon} \right)$ time, where $\text{nnz}$ is the input sparsity of the batch. Finally, it takes $O \left( \log \frac{n}{\epsilon^2} \cdot \text{nnz} \right)$ additional time to compute the approximate reverse online leverage scores from the pseudoinverses. Hence, the update time for each batch of $O \left( \frac{n^2}{\epsilon^2} \log n \log \frac{n}{\epsilon} \right)$ rows is

$O \left( \frac{n^2}{\epsilon^2} \log n \log \frac{n}{\epsilon} \cdot \log^2 \frac{n}{\epsilon} + \log \frac{n}{\epsilon} \cdot \text{nnz} \right)$,

so the total runtime is $O \left( |S| \log^2 \frac{n}{\epsilon} + \log \frac{n}{\epsilon} \cdot \text{nnz} \right)$, where $|S|$ is the length of the stream and nnz is the input sparsity of the stream.

We emphasize that Algorithm 8 actually provides a spectral approximation of not only the matrix $A = r_{-W+1} \cdots r_1$, represented by the sliding window at the end of the stream via $\tilde{A}_1$, but also a spectral approximation of any matrix $r_k \cdots r_t$ with $k \geq t - W + 1$ through $\tilde{A}_j$ where $j$ is the smallest index such that $k \leq t_j$.

## 5 Low-Rank Approximation

We first remark that low rank approximation can be achieved through the $O \left( \frac{n^2}{\epsilon^2} \log n \log \frac{n}{\epsilon} \right)$ space algorithm in Algorithm 8. In this section, we show a more space efficient algorithm. In fact, we describe an algorithm for low rank approximation in the sliding window algorithm that uses optimal space up to polylogarithmic factors.

Recall the definition of a projection-cost preserving sample, defined as follows:

**Definition 5.1 (Rank $k$ Projection-Cost Preserving Sample [CEM+15]).** For $d_1 < m$ (resp. $d_2 < n$), a matrix $M_1 \in \mathbb{R}^{d_1 \times n}$ of rescaled rows of $A \in \mathbb{R}^{m \times n}$ (resp. a matrix $M_2 \in \mathbb{R}^{m \times d_2}$ of rescaled columns of $A$) is a $(1 + \eta)$ projection-cost preserving sample if, for all rank $k$ orthogonal projection matrices $P_1 \in \mathbb{R}^{n \times n}$.
(resp. $P_2 \in \mathbb{R}^{m \times m}$),
\[
(1 - \eta) \|A - AP_1\|_F^2 \leq \|M_1 - M_1P_1\|_F^2 \leq (1 + \eta) \|A - AP_1\|_F^2
\]
(resp. $(1 - \eta) \|A - P_2A\|_F^2 \leq \|M_2 - P_2M_2\|_F^2 \leq (1 + \eta) \|A - P_2A\|_F^2$).

In this section, we use $A_k$ to denote the best rank $k$ approximation to a matrix $A$:
\[
A_k := \arg\min_{\|X\|_F^2 \leq k} \|A - X\|_F^2.
\]

We first show in Section 5.1 how to obtain a constant factor approximation to $\frac{\|A - A_k\|_F^2}{k}$ at all times in the sliding window using a projection-cost preserving sketch to reduce the dimension of each row, and then apply Algorithm 8 to the vectors of reduced dimension. We then show in Section 5.2 that calling DOWNSAMPLE with a different regularization, setting $\lambda$ to be some constant factor approximation to $\frac{\|A - A_k\|_F^2}{k}$, yields a good projection-cost preserving sample, and hence a good low-rank approximation, with space $O\left(\frac{nk^3}{\log^3 n}\right)$.

5.1 Constant Factor Approximation to Regularization

The challenge in obtaining the correct regularization factor is not to approximate $\|A - A_k\|_F^2$ for the matrix $A$ represented by the sliding window, but rather to approximate $\|X - X_k\|_F^2$ for any matrix $X = r_1 \circ \ldots \circ r_i$, formed by consecutive rows of the stream beginning at some time $i$. In order to approximate $\|X - X_k\|_F^2$, we obtain a projection-cost preserving sketch of $X$. We first recall a method for obtaining a projection-cost preserving sample in the streaming model, where rows of a matrix $A$ arrive one-by-one.

**Theorem 5.2** (Theorem 12 in [CEM+15]). Suppose $J$ is a random dense Johnson-Lindenstrauss matrix with $n$ rows and $O\left(\frac{k + \log \left(\frac{1}{\delta}\right)}{\delta}\right)$ columns and $O\left(\log(k/\delta)\right)$-wise independent elements [CW09]. Then for any matrix $A \in \mathbb{R}^{W \times n}$, with probability at least $1 - \delta$,
\[
\forall P \in \Pi, \quad (1 - \eta) \|A - AP\|_F^2 \leq \|AJ(I - P)\|_F^2 \leq (1 + \eta) \|A(I - P)\|_F^2,
\]
where $\Pi$ denotes the set of rank-$k$ projection matrices.

However, the sketch $AJ$ still contains $W$ rows, so the size of the resulting sketch is still too large. We thus use META-SPECTRAL to give a constant factor spectral approximation to $AJ$. Using the notation $B_k$ to denote the best rank-$k$ approximation to the matrix $B$, we query the data structure ESTIMATE to give a constant factor approximation to $\|X_i - (X_i)_k\|_F^2$ for any $X_i = r_{t-W+i} \circ \ldots \circ r_i$ with $i \leq W$ is fully described in Algorithm 12. We now prove the correctness and space complexity of ESTIMATE.

**Lemma 5.3.** Given a stream of rows $r_1, \ldots, r_i \in \mathbb{R}^{1 \times n}$, let $X_i = r_{t-W+i} \circ \ldots \circ r_i$ and $W < n^p$ for some constant $p > 0$. Suppose $\frac{1}{\text{poly}(W)} \leq \sigma_{\min}(R) \leq \sigma_{\max}(R) \leq \text{poly}(W)$ across all matrices $R$ formed by consecutive rows in the stream. With probability at least $1 - \frac{1}{W^p}$, ESTIMATE uses $O\left(d^2 \log^2 d \log n\right)$ space, where $d = k + \log n$ and for each $1 \leq i \leq W$, outputs $\zeta_i$ that satisfies
\[
\zeta_i \leq \|X_i - (X_i)_k\|_F^2 \leq 8\zeta_i.
\]

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Algorithm 12 ESTIMATE: Data structure for constant factor approximation of $\|A - A_k\|_F^2$.

**Input:** A stream of rows $r_1, \ldots, r_t \in \mathbb{R}^{1 \times n}$, a parameter $W$ for the size of the sliding window.

**Output:** 8-approximation to $\|X_i - (X_i)_k\|_F^2$ for any $X_i = r_i - W + i \odot \cdots \odot r_i$ with $i \leq W$.

1. $\delta = \frac{1}{W^2}, d = \mathcal{O}(k + \log \frac{k}{\delta})$
2. Let $J$ be a random dense $n \times d$ Johnson-Lindenstrauss matrix. \hspace{1cm} $\blacksquare$ **Theorem 5.2**
3. Run META-SPECTRAL with inputs $r_1, \ldots, r_t$ and accuracy parameter $\eta = \frac{1}{8}$. \hspace{1cm} $\blacksquare$ **Algorithm 8**
4. Suppose META-SPECTRAL maintains rows $a_1, \ldots, a_s$ and timestamps $t_1, \ldots, t_s$.
5. **for** $1 \leq i \leq W$ **do**
6. Let $X_i = r_i - W + i \odot \cdots \odot r_i$.
7. Let $j := \max \{k : t_k \leq t - W + i\}$.
8. Let $\tilde{A}_j = a_j \odot \cdots \odot a_s$.
9. Use $\zeta_i = \frac{1}{8} \|\tilde{A}_j - (\tilde{A}_j)_k\|_F^2$ as a constant factor underestimate of $\|X_i - (X_i)_k\|_F^2$.
10. **end for**
11. On query $i$, return $\zeta_i$.

**Proof.** Since $X_iJ$ is a projection-cost preserving sketch for $X_i$ with error $\frac{1}{2}$, then for any rank $k$ orthogonal projection matrix

$$\frac{1}{2} \|X_i - X_iP\|_F^2 \leq \|X_iJ - X_iJP\|_F^2 \leq \frac{3}{2} \|X_i - X_iP\|_F^2 \leq 2 \|X_i - X_iP\|_F^2.$$ 

Let $\Pi$ be a set of rank-$k$ orthonormal projection matrices. Define

$$U_1 := \arg\min_{U \in \Pi} \|X_i(I - U)\|_F, \quad U_2 := \arg\min_{U \in \Pi} \|X_iJ(I - U)\|_F,$$

Then using the projection cost preserving property of $J$ and the minimality condition on $U_1$ and $U_2$, we have

$$2 \|X_i(I - U_1)\|_F \geq \|X_iJ(I - U_1)\|_F^2 \geq \|X_iJ(I - U_2)\|_F^2 \geq \frac{1}{2} \|X_i(I - U_2)\|_F^2 \geq \frac{1}{2} \|X_i(I - U_1)\|_F^2.$$

Let

$$U_3 := \arg\min_{U \in \Pi} \|\tilde{A}_j - \tilde{A}_jU\|_F,$$

where $\tilde{A}_j$ is given in ESTIMATE and is a 2 factor spectral approximation of $X_iJ$. Then we have

$$\frac{1}{4} \|X_iJ(I - U_1)\|_F^2 \leq \|\tilde{A}_j(I - U_3)\|_F^2 \leq 4 \|X_iJ(I - U_1)\|_F^2.$$ 

Because $\zeta_i = \frac{1}{8} \|\tilde{A}_j(I - U_3)\|_F^2$, it follows that

$$\zeta_i \leq \|X_i(I - U_1)\|_F^2 \leq 8 \zeta_i.$$

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Correctness then follows from the definition of $U_1$, so that $(X_i)_k = X_i U_1$. Since META-SPECTRAL is called with vectors $r_i J \in \mathbb{R}^{1 \times d}$ of size $d = O(k + \log n)$ and $\varepsilon = 1$, the algorithm uses $O\left(d^2 \log^2 d\right)$ space by [Theorem 4.5]. It also requires $md$ random bits to form the matrix $J$, so the overall space complexity is $O\left(d^2 \log^2 d \log n\right)$. Since $\delta = \frac{1}{\sqrt{m}}$, then by a union bound over all $W$ updates, the probability of correctness is at least $1 - \frac{1}{W^2}$.

5.2 Known Regularization

Since ESTIMATE gives a constant factor approximation to $\|X_i - (X_i)_k\|_F^2$ for any $X_i = r_{i-W+i} \circ \ldots \circ r_i$ with $i \leq W$. We condition on a good estimate of $\|X_i - (X_i)_k\|_F^2$. Recall that sampling with overestimates of the $\lambda$-ridge leverage scores yields a good projection-cost preserving sample and hence, a good low-rank approximation (Lemma A.6).

Hence, it should intuitively suffice to set $\delta = \frac{1}{\sqrt{m}} \cdot \|A - A_k\|_F^2$ in the DOWNSAMPLE procedure of Algorithm 8 to obtain a good projection-cost preserving sketch of the matrix $A$. In this section, we give a tighter analysis of the sum of the online $\lambda$-ridge leverage scores $l_i$ for $\lambda = \frac{\|A - A_k\|_F^2}{k}$. We require an upper bound from [CMP16] on the sum of the online $\lambda$-ridge leverage scores, proven using the matrix determinant lemma.

Lemma 5.4 ([CMP16]). For any matrix $A \in \mathbb{R}^{W \times n}$,

$$
\det(A^\top A + \lambda I) \geq \lambda^n e^{\sum l_i / 2},
$$

where $l_i$ is the $i^{th}$ online $\lambda$-ridge leverage score.

We now give a tighter analysis of the sum of the online $\lambda$-ridge leverage scores $l_i$ for $\lambda = \frac{\|A - A_k\|_F^2}{k}$.

Lemma 5.5 (Sum of online ridge leverage scores is bounded). Let $\beta \geq 1$ and $k \geq 1$ be some constants.

For $\lambda = \frac{\|A - A_k\|_F^2}{k}$,

$$
\sum_{i=1}^n l_i = O(k + k \log \|A\|_2).
$$

Proof. Let $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq \lambda$ be the singular values of $A^\top A + \lambda I$. Then,

$$
\det(A^\top A + \lambda I) = \prod_{i=1}^n \sigma_i.
$$

Observe that $\sigma_{k+1} + \ldots + \sigma_n = \|A - A_k\|_F^2 + (n-k)\lambda$. By the AM-GM inequality, we have

$$
\prod_{i=k+1}^n \sigma_i \leq \left(\frac{\|A - A_k\|_F^2 + (n-k)\lambda}{n-k}\right)^{n-k}.
$$

Combining with the fact that $\sigma_i \leq \|A\|_2^2 + \lambda$ for $1 \leq i \leq k$, we have

$$
\det(A^\top A + \lambda I) = \prod_{i=1}^n \sigma_i \leq (\|A\|_2^2 + \lambda)^k \left(\frac{\|A - A_k\|_F^2}{n-k} + \lambda\right)^{n-k}.
$$
**Algorithm 13** SLIDING-PCP: Projection-cost preserving sample on sliding windows

**Input:** A stream of rows $r_1, \ldots, r_t \in \mathbb{R}^{1 \times n}$, a parameter $W$ for the size of the sliding window, an accuracy parameter $\epsilon$.

**Output:** $(1 + \epsilon)$ projection-cost preserving sample in the sliding window model.

1. $M_1 \leftarrow \emptyset$, $t_1 \leftarrow 0$
2. $\mathcal{H} \leftarrow \{\{M_1\}, \{t_1\}\}$
3. Initialize an instance of $\text{ESTIMATE}$.
4. for each row $r_t$ do
5. Update $\text{ESTIMATE}$ with $r_t$.
6. $\mathcal{H} \leftarrow \text{OPTUPDATE}(\mathcal{H}, r_t, t)$
7. Let $\mathcal{H}$ contain matrices $M_1, \ldots, M_{s+1}$.
8. for each $j = s$ down to $j = 1$ do
9. Query $\text{ESTIMATE}$ at index $W - (t - t_j)$ to obtain $\zeta_j$.
10. $\tilde{A}_{j+1} \leftarrow M_{j+1} \circ \ldots \circ M_{s+1}$
11. $M_j \leftarrow \text{DOWNSAMPLE}(M_j, \tilde{A}_{j+1}, \zeta_j \epsilon)$
12. end for
13. $\mathcal{H} \leftarrow \text{OPTEXPIRE}(\mathcal{H}, t, W)$
14. end for
15. return $\tilde{A}_1 = M_1 \circ \ldots \circ M_{s+1}$.

For $\lambda = \frac{\|A - A_k\|^2}{\beta k}$, 

$$\det(A^\top A + \lambda I) \leq (\|A\|^2_2 + \lambda)^k \lambda^{n-k} \left(\frac{\beta k}{n-k} + 1\right)^{n-k}$$

$$\leq (\|A\|^2_2 + \lambda)^k \lambda^{n-k} e^{\beta k}$$

$$\leq 2^k (\|A\|^2_2 + \lambda^k) \lambda^{n-k} e^{\beta k}.$$ 

Combining with [Lemma 5.4](#) and taking logarithms, it follows that 

$$n \log \lambda + \sum \frac{l_i}{2} \leq k + 2k \log \|A\|_2 + k \log \lambda + (n - k) \log \lambda + \beta k$$

$$\leq k + 2k \log \|A\|_2 + k \log \lambda + n \log \lambda + \beta k$$

or equivalently $\sum l_i \leq 2(1 + \beta)k + 4k \log \|A\|_2$. [Lemma 5.5](#) follows.

Again, we remark that the sum of the online $\lambda$-ridge leverage scores equals the sum of the reverse online $\lambda$-ridge leverage scores.

We give our algorithm for efficient space low-rank approximation in the sliding window model in [Algorithm 13](#). Note the similarity to [Algorithm 8](#) with the exception of the regularization parameter input to $\text{DOWNSAMPLE}$.

We now show that [Algorithm 13](#) gives a relative error low-rank approximation with efficient space usage.

**Theorem 5.6** (Relative error low-rank approximation). *Let $r_1, \ldots, r_t \in \mathbb{R}^{1 \times n}$ be a stream of rows and $W = O(n^p)$ be the size of the sliding window for some constant $p > 0$. Let $A = r_{t-W+1} \circ \ldots \circ r_t$ be*
the matrix consisting of the $W$ most recent rows and suppose $\frac{1}{\text{poly}(W)} \leq \sigma_{\min}(R) \leq \sigma_{\max}(R) \leq \text{poly}(W)$ across all matrices $R$ formed by consecutive rows in the stream. Then given a parameter $\epsilon > 0$, there exists an algorithm that outputs a matrix $M$ that is a $(1+\epsilon)$ rank $k$ projection-cost preserving sample of $A$, using $O\left(\frac{\|W\|_2^2 \log^2 n}{\epsilon^2}\right)$ space, with probability at least $1 - \frac{1}{\text{poly}(W)}$.

Proof. For any $X_i = r_{t-W+1} \ldots \circ r_t$ with $i \leq W$, let $\lambda_i = \|X_i - (X_i)_k\|_F^2$. By Lemma 5.3, each row $X_i$ is sampled into $\tilde{A}_i$ with probability at least

$$\min(1, c \cdot a_i(W^T W + \lambda_i I)^T a_i^T),$$

where $c = \frac{\alpha \log n}{\epsilon^2}$, for some constant $\alpha > 0$. Let $\Pi$ be the set of all rank-$k$ projection matrices. Since oversampling rows with probability proportional to their ridge leverage scores is known to give a good estimate of any rank $k$ orthogonal projection (Lemma A.6), we have that

$$\forall P \in \Pi, \quad (1 - \epsilon) \|A - AP\|_F^2 \leq \|M_1 - M_1 P\|_F^2 \leq (1 + \epsilon) \|A - AP\|_F^2$$

We now turn to bounding the space required by our algorithm by bounding the number of rows maintained by our data structure. By ridge leverage score monotonicity (Lemma A.4) and Lemma 4.2, each row $r_i$ in $A$ for $i \geq t - W + 1$ is sampled into $\tilde{A}_i$ with probability at least

$$\min(1, 2(1 + 6\epsilon)c \cdot r_i(A^T A + \lambda_i I)_{r_i} r_i^T).$$

Hence, each row in $r_i$ is sampled with probability

$$p_i \leq 2(1 + 6\epsilon)(1 + \epsilon) \cdot l_i,$$

where $l_i$ is the online $\lambda_i$-ridge leverage score of the $i^{th}$ row with $\lambda_i = \|X_i - (X_i)_k\|_F^2$.

For the purposes of analysis, set $u_i = t - W + 1$ and for each $i > 1$, let $u_i$ be the first time $j$ such that $\lambda_j \leq \frac{1}{2} \lambda_{u_i}$. This breaks the stream into points at which $\lambda_j$ decreases by a factor of $\frac{1}{2}$. Let $\tau_j$ be the online $\lambda$-ridge leverage score for the $j^{th}$ row with $\lambda = \|X_{u_i} - (X_{u_i})_k\|_F^2$. Then for each $i$,
From our assumptions that
\[ \frac{1}{\text{poly} (W)} \leq \sigma_{\min} (R) \leq \sigma_{\max} (R) \leq \text{poly} (W) \quad \text{and} \quad W = \mathcal{O} (n^p) \]
for some constant \( p > 0 \), it follows that
\[ \log \| X_u \|_2 \leq \log \| X_u \|_F = \mathcal{O} (\log n) \]
for each \( i \). Let \( \ell = \log \| A \|_F^2 \), so that \( \ell = \mathcal{O} (\log n) \), since all entries of \( A \) are polynomial in \( n \). Hence,
\[ \sum_{i=1}^{W} l_i \leq \sum_{i=1}^{\ell} \sum_{j=u_i}^{u_i+1-1} l_j \leq \sum_{i=1}^{\ell} \sum_{j=u_i}^{u_i+1-1} \tau_j \leq \sum_{i=1}^{\ell} \mathcal{O} (k \log n) = \mathcal{O} (k \log^2 n). \]

Then by standard Chernoff bounds and the observation that \( c = \mathcal{O} \left( \frac{\log n}{\epsilon^2} \right) \), it follows that \( \tilde{A}_1 \) has \( \mathcal{O} \left( \frac{k}{\epsilon^2 \log^3 n} \right) \) rows, each with dimension \( 1 \times n \), with probability at least \( 1 - \frac{1}{W^p} \). By construction, any row maintained by SLIDING-PCP is contained in \( \tilde{A}_1 \). Finally, ESTIMATE uses \( \mathcal{O} (d^2 \log^2 d \log n) \), where \( d = k + \log n \).

To optimize the running time of our low-rank approximation algorithm, we use a distribution of sparse Johnson-Lindenstrauss matrices in the ESTIMATE procedure (at the expense of polylogarithmic factors in the number of columns).

**Theorem 5.7** (Theorem 12 in [CEM+15]). Suppose \( J \) is an OSNAP sparse subspace embedding matrix with \( n \) rows and \( \mathcal{O} \left( \frac{1}{\eta^2} k \log^8 \frac{k}{\eta^2} \right) \) columns [NN13]. Then for any matrix \( A \in \mathbb{R}^{W \times n} \), with probability at least \( 1 - \mathcal{O} (\delta) \),
\[ \forall P \in \Pi, \quad (1 - \eta) \| A - AP \|_F^2 \leq \| AJ (I - P) \|_F^2 \leq (1 + \eta) \| A (I - P) \|_F^2, \]
where \( \Pi \) denotes the set of rank-\( k \) projection matrices.

We claim replacing the dense Johnson-Lindenstrauss matrix in the ESTIMATE procedure by an OSNAP matrix suffices to obtain input sparsity time, up to polylogarithmic factors, setting \( \eta = \frac{1}{\ell} \) and \( \delta = \frac{1}{W^p} \) as before.

**Theorem 5.8** (Relative error low-rank approximation). Let \( r_1, \ldots, r_t \in \mathbb{R}^{1 \times n} \) be a stream of rows and \( W < n^p \) be the size of the sliding window for some constant \( p > 0 \). Let \( A = r_{1-W+1} \circ \ldots \circ r_t \) be the matrix consisting of the \( W \) most recent rows and suppose \( \frac{1}{\text{poly} (W)} \leq \sigma_{\min} (R) \leq \sigma_{\max} (R) \leq \text{poly} (W) \) across all matrices \( R \) formed by consecutive rows in the stream. Then given a parameter \( \epsilon > 0 \), there exists an algorithm that outputs a matrix \( M \) that is a \( (1 + \epsilon) \) rank-\( k \) projection-cost preserving sample of \( A \), using \( \tilde{O} \left( \frac{n k}{\epsilon^2} \right) \) space, with probability at least \( 1 - \frac{1}{\text{poly} (W)} \). The total running time of the algorithm is \( \tilde{O} \left( |S| \frac{k}{\epsilon^2} + \log \frac{n}{\epsilon} \cdot \text{nnz} \right) \), where \( |S| \) is the length of the stream and \( \text{nnz} \) is the input sparsity of the stream.

**Proof.** Since **Theorem 5.7** offers the same correctness guarantees as **Theorem 5.2**, then the ESTIMATE procedure offers the same correctness guarantees, so correctness of **Algorithm 13** with replacing the dense Johnson-Lindenstrauss matrix in the ESTIMATE procedure by an OSNAP matrix follows. Similarly, the space complexity follows because the vectors output by the OSNAP
matrix transformation have polylogarithmic dimension greater than the vectors output by dense Johnson-Lindenstrauss matrix transform. Hence, the algorithm uses space polylogarithmic factors greater than the space complexity claimed by Theorem 5.6.

As before, we can batch the computations after the arrival of every $O \left( \frac{k}{\varepsilon} \log^3 n \right)$ rows. To analyze the runtime, we first note that the OSNAP matrix has $s = \frac{\text{polylog}(n)}{\varepsilon}$ entries per column, and so it takes $O \left( \frac{\text{polylog}(n)}{\varepsilon} \cdot \text{nnz} \right)$ time to transform all of the input rows, where nnz is the input sparsity of the stream. To compute the estimation of the ridge $\zeta_i$, ESTIMATE can run an SVD in time $\tilde{O} \left( \frac{k^3}{\varepsilon^2} \right)$, since $\tilde{O} \left( \frac{k}{\varepsilon} \right)$ transformed rows are stored and each row has dimension $\tilde{O} \left( k \right)$.

Once the estimates of the ridges are complete, we can use the same tricks as in Section 4 to compute a constant factor approximation of the reverse online leverage score of any sampled row $a_i$, applying a subspace embedding to each of the $O \left( \frac{k^3}{\varepsilon^2} \log^3 n \right)$ rows into a $O \left( \log n \cdot \text{nnz} \right)$ dimension subspace through a Johnson-Lindenstrauss transform. Applying this subspace embedding to each of the rows takes $O \left( \log n \cdot \text{nnz} \right)$ time. It then takes $O \left( \log n \cdot \text{nnz} \right)$ additional time to compute the approximate reverse online leverage scores from the pseudo inverses. Hence, the total runtime is $\tilde{O} \left( \frac{k^3}{\varepsilon^2} |S| + \log n \cdot \text{nnz} \right)$, where $|S|$ is the length of the stream and nnz is the input sparsity of the stream.

6 Applications of Techniques

In this section, we describe several applications of our techniques to algorithms for important linear algebraic in other settings.

6.1 Online Low-Rank Approximation

We first demonstrate how our previous analysis can be applied to online algorithms for low-rank approximation. Under this model, the rows of a matrix $A$ arrive sequentially, but upon the arrival of each row, an algorithm must choose to irrevocably store or discard the row, without the ability to change these decisions. Our algorithm for online low-rank approximation samples each row with probability equal to the online $\lambda$-ridge leverage scores, similar to [CMP16]. To choose the regularization parameter $\lambda$ at each time $i$, we use an algorithm that gives a $2$-approximation to $\|A - B_i\|_F^2$, where $A_i = a_1 \circ \ldots \circ a_i$, and $B_i$ is the best rank $k$ approximation to $A_i$. We show that these sampling probabilities suffice to obtain a low-rank approximation to $A$ and observe that our tighter analysis of the sum of the online $\lambda$-ridge leverage scores in Lemma 5.5 provides an upper bound on the number of rows sampled.

We use the following formulation of Theorem 1.3 in [AN13].

**Theorem 6.1** ([AN13]). Given rows $a_1, \ldots, a_n \in \mathbb{R}^{1 \times d}$ and a parameter $k$, there is an algorithm RESIDUAL that gives a 2-approximation to $\frac{\|A - A_i\|_F^2}{k}$, where $A = a_1 \circ \ldots \circ a_n$, with probability $1 - \frac{1}{n}$ and uses $O \left( k^2 \log n \right)$ space.

**Lemma 6.2.** Given a stream $a_1, \ldots, a_n \in \mathbb{R}^{1 \times d}$ of rows, let $A_n = a_1 \circ \ldots \circ a_n$, $\lambda_n = \frac{\|A_n - A_i\|_F^2}{k}$, and $\varepsilon \leq \frac{1}{2}$. Then ONLINE-LRA, described in Algorithm 14, outputs a matrix $\tilde{A}_n$ such that with probability at least $1 - \frac{1}{n^2}$.
Algorithm 14 ONLINE-LRA

Input: Stream of rows \( a_1, \ldots, a_n \in \mathbb{R}^{1 \times d} \), accuracy \( \epsilon \leq \frac{1}{2} \), and parameter \( k \).
Output: Rank \( k \) projection-cost preserving sample of \( A := a_1 \circ \ldots \circ a_n \).
1: Let \( a \) be a constant with \( n \leq d^a \).
2: \( c \leftarrow \frac{6a \log d}{\epsilon^2} \).
3: \( \tilde{\lambda}_0 \leftarrow 0 \).
4: Let \( \tilde{A}_0 \) be a \( 0 \times d \) matrix.
5: Initialize an instance of RESIDUAL \( \rho \), as in Theorem 6.1.
6: for each row \( a_i \) do
7: \( \tilde{\lambda}_i \leftarrow \frac{1}{2} \cdot \text{RESIDUAL} \).
8: Let \( p_i = \min(2c \cdot a_i(\tilde{A}_{t-1} \tilde{A}_{t-1}^\top + a_i^\top a_i + \tilde{\lambda}_i I)^\dagger a_i^\top, 1) \).
9: with probability \( p_i \):
10: \( A_t \leftarrow A_{t-1} \circ \frac{r}{\sqrt{p_t}} \).
12: otherwise:
13: \( \tilde{A}_t \leftarrow \tilde{A}_{t-1} \).
14: end for

(1) Property \( \square \) \( (1 - \epsilon)(A_n^\top A_n + \lambda_n I) \preceq \tilde{A}_{n-1}^\top \tilde{A}_{n-1} + \lambda_n I \preceq (1 + \epsilon)(A_n^\top A_n + \lambda_n I) \)

(2) Property \( \square \) Each row \( a_n \) is sampled with probability \( p_i \), where
\[
\min(1, c \cdot a_n(A_n^\top A_n + \lambda_n)^\dagger a_n^\top) \leq p_i \leq \min(1, 16c \cdot a_n(A_n^\top A_n + \lambda_n)^\dagger a_n^\top).
\]

Proof. We prove the statement by induction on \( n \). The base case holds for \( n = 1 \) since ONLINE-LRA samples \( a_1 \) with probability \( p_1 = 1 \) so that \( \tilde{A}_1 = A_1 \). Also, \( \tilde{\lambda}_1 = \lambda_1 = 0 \), so \( a_n(A_n^\top A_n + \lambda_n)^\dagger a_n^\top = 1 \) for \( n = 1 \).

Now suppose that at some time \( n - 1 \), the rows stored by ONLINE-LRA form a matrix \( \tilde{A}_{n-1} \) such that
\[
(1 - \epsilon)(A_{n-1}^\top A_{n-1} + \lambda_{n-1} I) \preceq \tilde{A}_{n-1}^\top \tilde{A}_{n-1} + \lambda_{n-1} I \preceq (1 + \epsilon)(A_{n-1}^\top A_{n-1} + \lambda_{n-1} I).
\]
Then row \( a_n \) is sampled with probability \( p_n = \min(1, 2c \cdot a_n(\tilde{A}_{n-1}^\top \tilde{A}_{n-1} + a_n^\top a_n + \tilde{\lambda}_n)^\dagger a_n^\top) \). By Theorem 6.1 \( \frac{1}{2}\lambda_n \leq \tilde{\lambda}_n \leq \lambda_n \) and \( \lambda_n \geq \lambda_{n-1} \), so
\[
\tilde{A}_{n-1}^\top \tilde{A}_{n-1} + a_n^\top a_n + \tilde{\lambda}_n I \preceq \tilde{A}_{n-1}^\top \tilde{A}_{n-1} + a_n^\top a_n + \lambda_n I \preceq (1 + \epsilon)(A_{n-1}^\top A_{n-1} + a_n^\top a_n + \lambda_n I).
\]
Thus the first half of Property \( \square \) follows from Fact A.7 that for \( \epsilon < 1 \),
\[
2c \cdot a_n(\tilde{A}_{n-1}^\top \tilde{A}_{n-1} + a_n^\top a_n + \tilde{\lambda}_n)^\dagger a_n^\top \geq c \cdot a_n(A_n^\top A_n + \lambda_n)^\dagger a_n^\top.
\]
On the other hand, \( (1 - \epsilon)(A_{n-1}^\top A_{n-1} + a_n^\top a_n + \lambda_n I) \preceq \tilde{A}_{n-1}^\top \tilde{A}_{n-1} + a_n^\top a_n + \lambda_n I \) and \( \frac{1}{2}\lambda_n \leq \tilde{\lambda}_n \) imply that
\[
\frac{1 - \epsilon}{4}(A_{n-1}^\top A_{n-1} + a_n^\top a_n + \lambda_n I) \preceq \frac{1}{4}(\tilde{A}_{n-1}^\top \tilde{A}_{n-1} + a_n^\top a_n + \lambda_n I)
\]
\[
\preceq \tilde{A}_{n-1}^\top \tilde{A}_{n-1} + a_n^\top a_n + \tilde{\lambda}_n I.
\]
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Then for $\varepsilon \leq \frac{1}{2}$,
\[
2c \cdot a_n (A_{n-1}^T \bar{A}_{n-1} + a_n^T a_n + \bar{\lambda}_n) a_n^T \leq 16c \cdot a_n (A_n^T A_n + \lambda_n) a_n^T,
\]
thus giving the second half of Property 2.

Since each row $a_i$ is sampled with probability at least $c \cdot a_i (A_i^T A_i + \lambda_i) a_i^T$, then Property 1 results from oversampling with respect to ridge leverage scores (Lemma A.3), with probability at least $1 - \frac{1}{n^2}$. Taking a union bound over all $n$ updates completes the induction. \hfill \Box

**Lemma 6.3.** For $O (\log n) = O (\log d)$, ONLINELRA stores $O \left( \frac{k}{\varepsilon^2} \log^2 \| A_n \|_F \log d \right)$ rows with probability at least $1 - \frac{1}{n^3}$.

**Proof.** Let $\tau_i = a_i \cdot (A_i^T A_i + \lambda_i) a_i^T$. For the purpose of analysis, set $t_1 = 1$ and for each $i > 1$, let $t_i$ be the smallest index $j$ such that $\lambda_j \leq \frac{1}{2} \lambda_{t_i}$, which partitions the stream into break points at which $\lambda_i$ decreases by a factor of $\frac{1}{2}$. For each $i$, $\sum_{j=t_i}^{t_{i+1}-1} \tau_j \leq 2 \sum_{i=t_i}^{t_{i+1}-1} \alpha_j$, where $\alpha_j$ is the online $\lambda$-ridge leverage score for the $j$th row with regularization $\lambda_i$.

By Lemma 5.5, $\sum_{j=t_i}^{t_{i+1}-1} \tau_j = O (k + k \log \| A_n \|_2)$. Therefore,
\[
\sum_{i=1}^{t} \tau_i \leq \sum_{i=1}^{t} \sum_{j=t_i}^{t_{i+1}-1} \tau_j \\
\leq \sum_{i=1}^{t} \sum_{j=t_i}^{t_{i+1}-1} \alpha_j \\
\leq \sum_{i=1}^{t} O \left( k + k \log \| A_n \|_2 \right) \\
= O \left( k \log^2 \| A_n \|_F \right)
\]

Applying a Chernoff bound and using the fact that $c = O \left( \frac{\log d}{\varepsilon^2} \right)$, it follows that the number of rows $\bar{A}_n$ stores is $O \left( \frac{k}{\varepsilon^2} \log^2 \| A_n \|_F \log d \right)$ with probability at least $1 - \frac{1}{n^3}$, since $O (\log d) = O (\log n)$. \hfill \Box

**Theorem 6.4.** With probability $1 - \frac{1}{\text{poly}(n)}$, ONLINELRA returns a matrix $\bar{A}_n$ that is a $(1 + \varepsilon)$ rank $k$ projection-cost preserving sample of $A_n$, with $O \left( \frac{k}{\varepsilon^2} \log^2 \| A_n \|_F \log d \right)$ rows.

**Proof.** By Property 2 from Lemma 6.2 and Lemma A.6 we have that $\bar{A}_n$ is a projection-cost preserving sample of $A_n$. By Lemma 6.3, ONLINELRA stores $O \left( \frac{k}{\varepsilon^2} \log^2 \| A_n \|_F \log d \right)$ rows. The probability of correctness results from a union bound over the correctness of Lemma 6.3 and Lemma 6.2. \hfill \Box

**Corollary 6.5.** Given a $A \in \mathbb{R}^{n \times d}$, whose rows $a_1, \ldots, a_n$ arrive sequentially, there exists an online algorithm that returns a matrix $B$ containing $O \left( \frac{k}{\varepsilon^2} \log^2 \| A \|_F \log d \right)$ (rescaled) rows of $A$ such that with high probability,
\[
(1 - \varepsilon) \| A - A_k \|_F^2 \leq \| B - B_k \|_F^2 \leq (1 + \varepsilon) \| A - A_k \|_F^2.
\]
Proof. Let $U_1$ be the rank $k$ orthogonal projection matrix that minimizes $\|A - AU_1\|_F$ and $U_2$ be the rank $k$ orthogonal projection matrix that minimizes $\|B - BU_2\|_F$, i.e., $AU_1 = A_k$ and $BU_2 = B_k$. By Theorem 6.4, $B$ is a $(1 + \varepsilon)$ rank $k$ projection-cost preserving sketch of $A$. Hence,

$$(1 - \varepsilon) \|A - AU_1\|_F^2 \leq (1 - \varepsilon) \|A - AU_2\|_F^2 \leq \|B - BU_2\|_F^2 \leq \|B - BU_1\|_F^2 \leq (1 + \varepsilon) \|A - AU_1\|_F^2.$$ 

\[ \Box \]

6.2 $\ell_1$ Spectral Approximation

In this section, we show that our deterministic spectral approximation can be applied as a subroutine for an algorithm to preserve $\|Ax\|_1$ in the sliding window model up to relative and additive errors, given certain assumptions about the entries of $A$ and $x$. We require the use of well-conditioned basis and $\ell_1$ leverage scores, which we shall introduce with the assumption that $A \in \mathbb{R}^{W \times n}$ has full rank for ease of presentation. The previous results also hold for $A$ without full rank, though the dimensions of some matrices and vectors will depend on $\text{rank}(A)$ rather than $n$.

**Definition 6.6** (Well-conditioned basis). Let $A \in \mathbb{R}^{W \times n}$. Then $U \in \mathbb{R}^{W \times n}$ is an $(\alpha, \beta, 1)$-well conditioned basis for $A$ if:

1. The column space of $U$ is equal to that of $A$.
2. $\|U\|_1 \leq \alpha$.
3. For all $x \in \mathbb{R}^n$, $\|x\|_\infty \leq \beta \|Ux\|_1$.
4. $\alpha$ and $\beta$ are independent of $W$.

**Theorem 6.7** (Theorem 4 in [DDH+08]). Let $A \in \mathbb{R}^{W \times n}$. Then there exists an $(\alpha, \beta, 1)$-well conditioned basis $U$ for the column space of $A$ with $\alpha = n^{3/2}$ and $\beta = 1$. $U$ can be computed in deterministic time $O(Wn^2 + Wn^3 \log n)$.

Specifically, a well-conditioned basis for $A \in \mathbb{R}^{W \times n}$ can be computed by taking the QR factorization $A = QR$, where $Q \in \mathbb{R}^{W \times n}$ is an orthonormal basis for span($A$) and $R \in \mathbb{R}^{n \times n}$. It can be shown that $\|x\|_{Q,1} := \|Qx\|_1$ is a norm. Let $C = \{x \in \mathbb{R}^n : \|x\|_{Q,1} \leq 1\}$ be the unit ball of the norm $\|\cdot\|_{Q,1}$ and $F \in \mathbb{R}^{n \times n}$ so that $\mathcal{E}_1 = \{x \in \mathbb{R}^n : x^\top Fx \leq 1\}$ be the Loewner-John ellipsoid of $C$. Then the $(\alpha, \beta, 1)$ well-conditioned basis is $U = QG^{-1}$, where $G$ is full rank and upper triangular with $G^\top G = F$. Moreover, we have

$$\|x\|_{LJ} \leq \|x\|_{Q,1} \leq \sqrt{n} \|x\|_{LJ},$$

where $\|x\|_{LJ}^2 = x^\top Fx$. By setting $S = GR$, we obtain $QR = UGR = US = A$.

**Definition 6.8.** Let $e_i$ denote the $i^{th}$ elementary row vector and $S$ be a change of basis matrix such that $U = AS^{-1}$ is a well-conditioned basis for the column space of $A$. The $\ell_1$ leverage scores are $\omega_i = \|e_i U\|_1$.

**Definition 6.9.** Let $A$ be a matrix and $B$ be a subset of the rows of $A$. The local $\ell_1$ leverage scores of $B$ with respect to $A$ are the leverage scores of $B$ found by computing a well-conditioned basis for $B$ rather than $A$. 

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Lemma 6.10 (Lemma A.1 in [CDW18]). Let $A \in \mathbb{R}^{W \times n}$ and let $w_i$ be the $i^{th}$ $\ell_1$ leverage score of $A$, and the corresponding local leverage score in a subset $B$ of the rows of $A$ be denoted $\tilde{w}_k$. Then $\frac{1}{\max} w_i \leq \tilde{w}_k$.

Theorem 6.11 (Theorem 5 in [DDH+08]). Let $\epsilon > 0$ and $U$ be a $(\alpha, \beta, 1)$-well-conditioned basis for the column space of $A \in \mathbb{R}^{W \times n}$. Let $e_i$ denote the $i^{th}$ elementary row vector. Suppose each row $a_i$ of $A = US$ is sampled with probability

$$p_i \geq \min \left( 1, \frac{\|e_iU\|_1}{\|e_iU\|_1} r \right),$$

where $r \geq \frac{32(\alpha \beta)}{\epsilon^2} \left( n \log \frac{12}{\epsilon} + \log \frac{2}{\delta} \right)$, and rescaled by $\frac{1}{p_i}$ to obtain a matrix $\tilde{A}$. Then with probability at least $1 - \delta$,

$$\left\| \tilde{A}x \right\|_1 - \left\| Ax \right\|_1 \leq \epsilon \left\| Ax \right\|_1,$$

for all $x \in \mathbb{R}^n$.

**Algorithm 15 L1STREAM**

**Input:** Rows $r_1, r_2, \ldots, r_m$ of a matrix $R \in \mathbb{R}^{m \times n}$, approximation parameter $\epsilon$.

**Output:** Matrix $M$ such that for any vector $x \in \mathbb{R}^{n \times 1}$, $(1 - \epsilon) \left\| Rx \right\|_1 \leq \left\| Mx \right\|_1 \leq (1 + \epsilon) \left\| Rx \right\|_1$.

1: $M \leftarrow \emptyset$
2: for each row $r_i$ do
3: \hspace{1em} $M \leftarrow \text{L1DOWNSAMPLE}(\tilde{M}, r_i, \epsilon, \frac{1}{p_i})$
4: end for
5: $M \leftarrow \emptyset$
6: for each row $m_i \in \tilde{M}$ do
7: \hspace{1em} $M \leftarrow \frac{1}{p_i} m_i$
8: end for
9: return $M$.

Lemma 6.12. Let $R \in \mathbb{R}^{m \times n}$ and $\epsilon > 0$. Then each row $r_i$ of $R$ is sampled by L1STREAM with probability at least

$$\min \left( 1, \frac{\|e_iU\|_1}{\|U\|_1} r_i \right) \leq q_i \leq \min \left( 1, \frac{\|e_iU\|_1}{\|U\|_1} r_1 r_2 \right).$$

Thus with probability at least $1 - \delta$, L1STREAM returns a matrix $M$ such that for any vector $x \in \mathbb{R}^{n \times 1}$,

$$(1 - \epsilon) \left\| Rx \right\|_1 \leq \left\| Mx \right\|_1 \leq (1 + \epsilon) \left\| Rx \right\|_1.$$ 

Moreover, with probability at least $1 - \frac{1}{\text{poly}(n)}$, L1STREAM stores $O \left( \frac{\epsilon}{\epsilon^2} n^3 \left( n \log \frac{1}{\epsilon} + \log \frac{1}{\delta} \right) \right)$ rows.

**Proof:** Let $U$ be a well-conditioned basis for $R$. We show that any row $r_i$ of $R$ is sampled by L1STREAM with probability at least

$$p_i \geq \min \left( r_1 \frac{u_i}{\|U\|_1}, 1 \right).$$

At each time $t$, any row $r_i$ that remains in $\tilde{M}$ is sampled by L1DOWNSAMPLE with probability $q_i = \min \left( \frac{\tau_i}{\max}, 1 \right)$, where $\tau_i$ is an $r_1 r_2$ multiple of the local $\ell_1$ leverage score of the rows $\tilde{M} \circ r_i$. 

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Algorithumen 16 L1DOWNSAMPLE(M, r_t, ε, δ)

Input: Matrix \( \tilde{M} \in \mathbb{R}^{d \times n} \), row \( r_t \in \mathbb{R}^{1 \times n} \), approximation parameter \( \epsilon \), failure parameter \( \delta \). Each row \( m_i \) in \( M \) has a corresponding sampling probability \( p_i \).

1: \( \alpha \leftarrow n^{3/2}, \beta \leftarrow 1 \)
2: \( \tilde{M} \leftarrow \emptyset \)
3: \( U \leftarrow \) well-conditioned basis for \( \tilde{M} \circ r_t \)
4: \( r_1 \leftarrow 32n\beta \left( n \log \frac{12}{\epsilon} + \log \frac{2}{\delta} \right) \)
5: \( r_2 \leftarrow n\alpha \beta \)
6: \( p_{d+1} \leftarrow 1 \)
7: for \( i = 1 \) to \( i = d + 1 \) do
8: Let \( \tilde{M} \) be the \( i \)th row in \( \tilde{M} \circ r_t \)
9: \( \tilde{w}_i \leftarrow \frac{u_i}{\|U\|_1} \)
10: Let \( \tau_i = \min (r_1r_2\tilde{w}_i, 1) \).
11: Let \( q_i = \min \left( \frac{\tau_i}{r_i}, 1 \right) \).
12: with probability \( q_i \), do
13: \( \tilde{M} \leftarrow \tilde{M} \circ r \)
14: \( p_i \leftarrow q_ip_i \)
15: end with probability
16: end for
17: return \( \tilde{M} \) and associated probabilities with each row

Therefore, the probability that any row \( x_i \) of \( X \) is sampled by L1DOWNSAMPLE(\( \tilde{X}, \tilde{Y}, \epsilon, \delta \)) is at most \( \max(p_i, r_1r_2\tau_i) \) and at least \( \min(p_i, r_1r_2\tau_i) \).

By Lemma 6.10, \( \tau_ir_1r_2 \geq \frac{\|u_i\|_1}{\|U\|_1}r_1 \). Hence,

\[
\min \left( 1, \frac{\|u_i\|_1}{\|U\|_1}r_1 \right) \leq q_i \leq \min \left( 1, \frac{\|u_i\|_1}{\|U\|_1}r_1r_2 \right) .
\]

By Theorem 6.11, L1STREAM returns a matrix \( M \) such that for any vector \( x \in \mathbb{R}^{n \times 1} \), \((1 - \epsilon) \|Rx\|_1 \leq \|Mx\|_1 \leq (1 + \epsilon) \|Rx\|_1 \).

Since \( r_1r_2 = O \left( \frac{1}{\epsilon}n^3 \left( n \log \frac{1}{\epsilon} + \log \frac{1}{\delta} \right) \right) \), then by a Chernoff bound, the total number of rows sampled is \( O \left( \frac{1}{\epsilon}n^3 \left( n \log \frac{1}{\epsilon} + \log \frac{1}{\delta} \right) \right) \). \( \square \)

6.2.1 Algorithm

To obtain an \( \ell_1 \) subspace embedding, we recall that Meta-Spectral provides an \( \ell_2 \) subspace embedding with approximation parameter \( \epsilon \). Let \( \mathcal{I} \) be the set of integers \( \{ -n^c, -n^c + 1, \ldots, n^c - 1, n^c \} \) for some constant \( c > 0 \). We observe that if \( x \in \mathcal{I}^{n \times 1} \) and \( A \in \mathcal{I}^{W \times n} \), then \( \|Ax\|_2^2 \) must increase by a relative \( \frac{\epsilon}{\epsilon} \) amount if \( \|Ax\|_1 \) increases by a relative \( \epsilon \) amount as more rows of \( A \) arrive. We thus run Meta-Spectral with error parameter \( \frac{\sqrt{\mathcal{I}}}{\text{poly}(n)} \) and augment each row \( r_i \) with a data structure L1STREAM that begins at time \( t_i \), where \( t_i \) is the time that row \( r_i \) arrives.
Algorithm 17 L1SLIDING: Algorithm for \( \ell_1 \) Spectral Approximation

**Input:** A stream of rows \( r_1, r_2, \ldots \in \mathbb{I}^n \) for \( I = \{-n^c, -n^c + 1, \ldots, n^c - 1, n^c\} \) for some constant \( c \), a parameter \( W \leq n^p \) for the size of the sliding window, and an accuracy parameter \( \varepsilon \).

**Output:** A subspace embedding \( \tilde{A} \) in the sliding window model.

1. \( \mathcal{H} \leftarrow \emptyset \)
2. \( \varepsilon' \leftarrow \frac{\varepsilon}{3^n} \)
3. for each row \( r_t \) do
   4. \( \mathcal{H} \leftarrow \text{OPTUPDATE}(\mathcal{H}, r_t, t) \)
   5. Start a new instance \( A_{s+1} \) of L1STREAM with approximation \( \frac{\varepsilon}{3} \) at time \( t \), associated with \( r_t \).
   6. For each \( s \) do
      7. \( \mathcal{H} \leftarrow \text{OPTEXPRISE}(\mathcal{H}, t, W) \)
     8. Update each instance \( A_i \) of L1STREAM with \( r_t \).
3. end for
10. for each \( j = s \) down to \( j = 1 \) do
11. \( \tilde{A}_{j+1} = a_{j+1} \circ \ldots \circ a_1 \)
12. \( M_j \leftarrow \text{DOWNSAMPLE}(a_j, \tilde{A}_{j+1}, \varepsilon', 0) \)
13. end for
14. : Algorithm 17 for \( \ell_1 \) Spectral Approximation

**Lemma 6.13.** Let \( I \) be the set of integers \( \{-n^c, -n^c + 1, \ldots, n^c - 1, n^c\} \) for some constant \( c > 0 \). Let \( A \in \mathbb{I}^{m \times n} \) and \( B \in \mathbb{I}^{d \times n} \). Let \( \varepsilon > 0 \), \( x \in \mathbb{I}_n \) and \( \eta = \frac{\varepsilon}{3^n} \). If \( \|Bx\|_1 \geq \varepsilon \|Ax\|_1 \), then

\[
\|Bx\|_2 \geq \eta \|Ax\|_2.
\]

**Proof.** Since \( A \in \mathbb{I}^{m \times n} \), \( x \in \mathbb{I}_n \) and \( I = \{-n^c, -n^c + 1, \ldots, n^c - 1, n^c\} \), then each entry of \( Ax \) is some integer that is at most \( n^2 \) in magnitude. Let \( y = Ax \) have entries \( y_1, \ldots, y_n \). Then for all \( 1 \leq i \leq n \), \( |y_i| \leq y_i^2 \leq n^{2c} |y_i| \). Hence,

\[
\|Bx\|_2^2 \geq \|Bx\|_1 \geq \varepsilon \|Ax\|_1 \geq \frac{\varepsilon}{n^{2c}} \|Ax\|_2^2,
\]

from which it follows that

\[
\|Bx\|_2 \geq \eta \|Ax\|_2.
\]

**Theorem 6.14.** Let \( I \) be the set of integers \( \{-n^c, -n^c + 1, \ldots, n^c - 1, n^c\} \) for some constant \( c > 0 \). Let \( r_1, \ldots, r_t \in \mathbb{I}_n \) be a stream of rows and \( W < n^p \) be the size of the sliding window for some constant \( p > 0 \). Let \( A = r_{t-W+1} \circ \ldots \circ r_1 \) be the matrix consisting of the \( W \) most recent rows. Given a parameter \( \varepsilon > 0 \), there exists an algorithm that outputs a matrix \( M \) with a subset of (rescaled) rows of \( A \) such that for all vectors \( x \in \mathbb{I}_n \),

\[
\|Mx\|_1 - \|Ax\|_1 \leq \varepsilon \|Ax\|_1
\]

with probability at least \( 1 - \frac{1}{\text{poly}(W)} \). The algorithm uses \( O\left(\frac{1}{\varepsilon^2} n^{4+2c} (n \log \frac{1}{\varepsilon} + \log n) \log n \right) \) space.
Proof. Suppose \( \mathcal{H} \) contains rows \( a_1, \ldots, a_s \) at the end of the stream, with corresponding times \( t_1, \ldots, t_s \) and algorithms \( A_1, \ldots, A_s \). Let \( A_1 = r_{t_1} \circ r_{t_1+1} \circ \cdots \circ r_t \). By Lemma 6.13 and the choice of \( \varepsilon' \) in Algorithm 17, \( \| A_1 x \|_1 \leq (1 + \frac{\varepsilon}{3}) \| A x \|_1 \) for any vector \( x \in \mathcal{I}^n \). By Lemma 6.12, L1Stream returns a matrix \( M \) such that with probability at least \( 1 - \delta \),

\[
(1 - \frac{\varepsilon}{3}) \| A_1 x \|_1 \leq \| M x \|_1 \leq (1 + \frac{\varepsilon}{3}) \| A_1 x \|_1 .
\]

Hence for sufficiently small \( \varepsilon \),

\[
(1 - \varepsilon) \| A x \|_p \leq \| M x \|_1 \leq (1 + \varepsilon) \| A x \|_1 .
\]

By Lemma 6.12 each instance of L1Stream stores \( O \left( \frac{1}{\varepsilon} n^3 (n \log \frac{1}{\varepsilon} + \log \frac{1}{\delta}) \right) \) rows with probability \( 1 - \frac{1}{\text{poly}(n)} \). Since \( \delta = \frac{1}{\text{poly}(W)} \) and \( W \leq n^p \) for some constant \( p \), then \( \log \frac{1}{\delta} = \log n \). Then by Theorem 4.5 and the choice of \( \varepsilon' = \frac{\sqrt{\varepsilon}}{3n} \), the algorithm uses space \( O \left( \frac{1}{\varepsilon^{2/3}} n^{4+2c} \left( n \log \frac{1}{\varepsilon} + \log n \right) \log n \right) \) in words.

In particular, note that Theorem 6.14 implies our algorithm preserves vectors from the hypercube \( \{-1, +1\}^n \) up to relative error despite the fact that the condition number can be some much larger polynomial in \( n \). For example, let \( x \) be a vector from the unit hypercube and \( A \in \{-1, +1\}^{W \times n} \) be a matrix that consists of the same row \( a_1 \) repeated \( W \) times, with \( \langle x, a_1 \rangle \neq 0 \). Then note that \( \| A x \|_1 \geq W \) while \( \| x \|_1 = n \) and \( W \gg n \). Moreover, since each row is repeated \( W \) times, then the leverage scores can be as large as 1 and as small as \( \frac{1}{W} \).

When the entries of \( A \) are not bounded, we can still provide an additive error guarantee to \( \| A x \|_1 \) given the same previous restrictions on the entries of \( x \).

Theorem 6.15. Let \( \mathcal{I} = \{-n^c, -n^c + 1, \ldots, n^c - 1, n^c\}^n \) for some constant \( c > 0 \). Let \( r_1, \ldots, r_t \in \mathbb{R}^{1 \times n} \) be a stream of rows and \( W < n^p \) be the size of the sliding window for some constant \( p > 0 \). Let \( A = r_{t-W+1} \circ \cdots \circ r_t \) be the matrix consisting of the \( W \) most recent rows. Given a parameter \( \varepsilon > 0 \), there exists an algorithm that outputs a matrix \( M \) with a subset of (rescaled) rows of \( A \) such that for all vectors \( x \in \mathcal{I}^n \),

\[
\| M x \|_1 - \| A x \|_1 \leq \varepsilon \| A x \|_1 + \frac{\| A \|_{\infty} \| x \|_1}{n^{c}}
\]

with probability at least \( 1 - \frac{1}{\text{poly}(W)} \). The algorithm uses space \( O \left( \frac{1}{\varepsilon^{2/3}} n^{4+2c} \left( n \log \frac{1}{\varepsilon} + \log n \right) \log n \right) \) space.

Proof. Note that when the entries of \( A \) are not bounded by some polynomial \( n^c \), then it is possible to obtain a matrix \( B \) whose entries are bounded by \( n^c \) by dividing each entry in \( A \) by \( \| A \|_{\infty} \). Furthermore, a matrix \( X \) can be formed by rounding each entry in \( B \) to the nearest integer, so that \( B = X + Y \) for some matrix \( Y \) with \( \| Y \|_{\infty} < 1 \). By Theorem 6.14 there exists an algorithm that outputs a matrix \( M \) such that with probability at least \( 1 - \frac{1}{W} \),

\[
\| M v \|_1 - \| X v \|_1 \leq \varepsilon \| X v \|_1 ,
\]

for all \( v \in \mathcal{I}^n \). By triangle inequality,

\[
\| M v \|_1 - \| B v \|_1 \leq \| Y v \|_1 \leq \varepsilon \| X v \|_1 + \| Y \|_{\infty} \| v \|_1 \leq \varepsilon \| B v \|_1 + \| v \|_1 .
\]
Scaling the entries of $M$ by $\|A\|_\infty$ to form a matrix $\tilde{A}$, it follows that
\[
\|\tilde{A}v\|_1 - \|Av\|_1 \leq \varepsilon \frac{\|A\|_\infty}{n^c} \cdot \|Bv\|_1 + \frac{\|A\|_\infty \|v\|_1}{n^c}.
\]

### 6.3 Covariance Matrix Approximation

In this section, we describe how our downsampling procedure can be applied as a subroutine to perform covariance matrix approximation in the sliding window model. Given a data stream of elements $r_1, r_2, \ldots, r_t \in \mathbb{R}^{1 \times n}$ be a stream of rows and $W = O(n^p)$ be the size of the sliding window for some constant $p > 0$. Let $A = r_{t-W+1} \circ \ldots \circ r_t$ be the matrix consisting of the $W$ most recent rows. Then the goal is to estimate $A^\top A$. We also point out that a similar algorithm and analysis handles approximate matrix multiplication in the sliding window model, where the columns of $A \in \mathbb{R}^{m \times W}$ and the rows of $B \in \mathbb{R}^{W \times p}$ arrive in sync, and the goal is to estimate $AB$.

#### 6.3.1 Preliminaries

In the classical offline model, $A^\top A$ can be estimated by sampling rows of $A$ to obtain a matrix $B$ with
\[
\mathbb{E} \left[ \left\| A^\top A - B^\top B \right\|_F^2 \right] \leq \varepsilon \|A\|_F^2.
\]
The probability of sampling each row of $A = a_1 \circ \ldots \circ a_n$ is proportional to
\[
p_k = \frac{\|a_k\|^2}{\|A\|_F^2}.
\]
The following result essentially follows the same proof as Theorem 22 in [DM17].

**Lemma 6.16.** If each row $a_k$ of $A$ is sampled with some probability $p_k$, where $p_k \geq \frac{c\|a_k\|^2}{\|A\|_F^2}$ for some fixed constant $c$ and all $1 \leq k \leq n$, and then rescaled by $\frac{1}{\sqrt{p_k}}$ to obtain matrix $B$, then
\[
\mathbb{E} \left[ \left\| A^\top A - B^\top B \right\|_F^2 \right] \leq \frac{1}{c} \|A\|_F^4.
\]

**Proof.** Observe that
\[
\mathbb{E} \left[ (B^\top B)_{i,j} \right] = \sum_{k=1}^n p_k \left( \frac{1}{p_k} a_k^\top a_k \right)_{i,j} = (A^\top A)_{i,j}.
\]

Moreover,
\[
\text{var} \left( (B^\top B)_{i,j} \right) \leq \sum_{k=1}^n \frac{1}{p_k} (a_k^\top a_k)_{i,j}^2 = \sum_{k=1}^n \frac{1}{p_k} (A_{k,i}A_{k,j})^2.
\]
Since $B^\top B$ is an unbiased estimator of $A^\top A$, then

$$
E \left[ \left\| B^\top B - A^\top A \right\|_F^2 \right] = \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{E} \left[ (A^\top A - B^\top B)_{i,j}^2 \right]
$$

$$
= \sum_{i=1}^{n} \sum_{j=1}^{n} \text{var} \left( (A^\top A - B^\top B)_{i,j} \right)
$$

$$
\leq \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{1}{p_k} (A_{k,i} A_{k,j})^2
$$

$$
= \sum_{k=1}^{n} \frac{1}{p_k} \left( \sum_{i=1}^{n} A_{k,i}^2 \right) \left( \sum_{j=1}^{n} A_{k,j}^2 \right)
$$

$$
= \sum_{k=1}^{n} \frac{1}{p_k} \|a_k\|_2^4.
$$

Hence for $p_k \geq \frac{c \|a_k\|_2^2}{\|A\|_F^2}$,

$$
E \left[ \left\| B^\top B - A^\top A \right\|_F^2 \right] \leq \frac{1}{c} \left( \sum_{j=1}^{n} \|a_j\|_F^2 \right) \cdot \|A\|_F^2 = \frac{1}{c} \|A\|_F^4.
$$

We cannot maintain $\|A\|_F^2$ in the sliding window model without using prohibitive space. However, using the smoothness of the Frobenius norm, we can instead maintain a 2-approximation of $\|A\|_F^2$ so that we can maintain $\hat{p}_i$ to be a 2-approximation of $p_i$.

**Lemma 6.17** (Lemma 5 in [BO07]). The Frobenius norm is $(\varepsilon, \varepsilon^2)$-smooth.

We refer definition, discussion, and properties of smooth functions to Section A.2.

### 6.3.2 Algorithm

Each time an update arrives, we add the new rows to our samples. We then downsample each existing sampled row so that at the end of the stream, the probability we keep row $i$ is proportional to $\frac{18 \|r_i\|_F^2}{\varepsilon^2 f}$, where $f$ is a 2-approximation to the Frobenius norm of the matrix represented by the sliding window. We give the algorithm in full in COVMatrix (Algorithm 18).

We first show that each row $r_i$ in $A$ is sampled with probability roughly $\frac{\|r_i\|_F^2}{\|A\|_F^2}$, where $A$ is the matrix represented by the sliding window. Intuitively, upon the arrival of each new row, each previous row $r_j$ is downsampled in MULTUpdate with probability that equals the relative change of the Frobenius norm. Specifically, if $g_j$ is the Frobenius norm of the rows $r_j \circ \ldots \circ r_t$, then $r_j$ is retained with probability $\frac{g_j}{g_j - \|r_j\|_F^2}$, which is the relative change of the Frobenius norm. Similarly, when two sketches are compressed in MULTCOMPRESS, $r_i$ is downsampled with probability that equals the ratio of the Frobenius norm of the matrices represented by the sketches. That is, when sketch $k$ is merged with sketch $j$, then a row $r_i$ in sketch $k$ is retained with probability $\frac{g_j}{g_i}$, which is
Algorithm 18 COVMATRIX: Covariance matrix approximation on sliding windows

**Input:** A stream of rows $r_1, \ldots, r_t \in \mathbb{R}^{1 \times n}$, a parameter $W$ for the size of the sliding window, an accuracy parameter $\varepsilon$.

**Output:** $(1 + \varepsilon)$-covariance matrix approximation in the sliding window model.

1. $M_1 = 0^{1 \times n}, t_1 = 0, g_1 = 0$.
2. $H \leftarrow \{\{M_1\}, \{t_1\}, \{g_1\}\}$.
3. **for** each row $r_t$ **do**
   4. $H \leftarrow$ MULTUPDATE($H, r_t, t, \varepsilon$)
   5. $H \leftarrow$ MULTCOMPRESS($H$)
   6. $H \leftarrow$ MULTEXPIRE($H, t, W$)
4. **end for**
5. $M \leftarrow \emptyset$
6. **for** $j = 1$ to $j = s$, where $s$ is the number of matrices in $H$ do
   7. **for** each row $r_i \in M_j$ **do**
      8. $M \leftarrow M \odot \frac{1}{\sqrt{p_i}} r_i$  \hspace{1cm} \triangleright Rescale output matrix by sampling probabilities
   9. **end for**
10. **end for**
11. return $M$.

Algorithm 19 MULTUPDATE($H, r_t, t, \varepsilon$)

**Input:** Data structure $H$ consisting of matrices $M_1, \ldots, M_s$ with corresponding timestamps $t_1, \ldots, t_s$ and values $g_1, \ldots, g_s$ such that $g_i = \sum_{j=t_i}^{t} \|r_j\|^2_2$, row vector $r_t \in \mathbb{R}^{1 \times n}$, current timestamp $t$, and accuracy parameter $\varepsilon$. Each matrix has some stored row(s) $r_i \in \mathbb{R}^{1 \times n}$, each with an associated value $p_i$ and timestamp $i$.

**Output:** Updated data structure $H$.

1. $c \leftarrow \frac{1}{2\varepsilon^2}, p_t \leftarrow c$
2. $M_{s+1} \leftarrow r_t, t_{s+1} \leftarrow t, g_{s+1} \leftarrow \|r_t\|^2_2$ \hspace{1cm} \triangleright Also store timestamp $t$ and associated probability $p_t$
3. **for** $j = 1$ to $j = s$ do
   4. $g_j \leftarrow g_j + g_{s+1}$ \hspace{1cm} \triangleright Update Frobenius norm
   5. **for** each row $r_k$ in $M_j$ **do**
      6. $q_k = \frac{g_j - g_{s+1}}{g_j} \cdot p_k$ \hspace{1cm} \triangleright Update downsampling probability
      7. if $q_k < 1$ then
         8. with probability $1 - \frac{q_k}{\min(p_k, 1)}$ do
            9. Delete $r_k$ from $M_j$ \hspace{1cm} \triangleright Downsample
         10. **end with probability**
      11. **end if**
      12. $p_k \leftarrow q_k$
   13. **end for**
   14. **end for**
15. $s \leftarrow s + 1$.
16. return $H$. 

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Algorithm 20 \textsc{MultCompress}(\mathcal{H})

\textbf{Input:} Data structure \mathcal{H} consisting of matrices \(M_1, \ldots, M_s\) with corresponding timestamps 
\(t_1, \ldots, t_s\) and values \(g_1, \ldots, g_s\) such that \(g_i = \sum_{j=t_i}^{t_{i+1}} \|r_j\|_F^2\) row vector \(r_i \in \mathbb{R}^{1 \times n}\), and current 
timestamp \(t\). Each matrix has some stored row(s) \(r_i \in \mathbb{R}^{1 \times n}\), each with an associated value \(p_i\) and 
timestamp \(i\).

\textbf{Output:} Updated data structure \(\mathcal{H}\).

1: \textbf{for} \(i = 1, \ldots, s - 2\) \textbf{do}
2: \hspace{1em} Compute \(j = \max\{k : g_k \geq \frac{1}{2}g_i\}\).
3: \hspace{1em} \textbf{if} \(j > i + 1\) \textbf{then}
4: \hspace{2em} \textbf{for} \(k = i + 1\) to \(s - j + i + 2\) \textbf{do}
5: \hspace{3em} \textbf{for each} row \(r_{\ell}\) in \(M_k\) \textbf{do}
6: \hspace{4em} \(q_{\ell} = \frac{g_k}{g_i} \cdot p_{\ell}\)
7: \hspace{4em} \textbf{if} \(q_{\ell} < 1\) \textbf{then}
8: \hspace{5em} with probability \(\frac{q_{\ell}}{\min(p_{\ell}, t)}\) \textbf{do}
9: \hspace{6em} \(M_j \leftarrow M_j \odot r_{\ell}\)
10: \hspace{6em} \textbf{end with probability}
11: \hspace{5em} \(p_{\ell} \leftarrow q_{\ell}\)
12: \hspace{4em} \textbf{end if}
13: \hspace{3em} \textbf{end for}
14: \hspace{2em} \textbf{end for}
15: \hspace{1em} \textbf{end if}
16: \hspace{1em} \textbf{end for}
17: \hspace{1em} \textbf{Delete} \(M_{i+1}, \ldots, M_{j-1}\) and \(t_{i+1}, \ldots, t_{j-1}\).
18: \hspace{1em} \textbf{for} \(k = i + 1\) to \(s - j + i + 2\) \textbf{do}
19: \hspace{2em} \(M_k \leftarrow M_{j-i+k-1}\)
20: \hspace{2em} \(t_k \leftarrow t_{j-i+k-1}\)
21: \hspace{2em} \textbf{end for}
22: \hspace{1em} \(s \leftarrow s - j + i + 1\).
23: \hspace{1em} \textbf{end if}
24: \textbf{end for}
25: \textbf{return} \(\mathcal{H}\).

the ratio of the Frobenius norm of the matrices represented by the sketches. Hence, the probability
that \(r_j\) is retained at the end of the stream is the ratio \(\|r_j\|_F^2\) (scaled by \(c\)) to the Frobenius norm of
the final sketch, \(p_j = \frac{18}{c^2} \|r_j\|_2^2\).

\textbf{Lemma 6.18.} For a row \(r_j\) where \(j \geq t - W + 1\), let \(t_i \leq j < t_{i+1}\) and \(A_i = r_i \circ \ldots \circ r_t\). Then each row
\(r_j\) is sampled in \textbf{Algorithm 18} with probability \(p_j = \frac{18}{c^2} \|r_j\|_2^2\).

\textbf{Proof.} We give a proof by induction, observing for the base case that the newest row is always
sampled by \textsc{MultUpdate}. Prior to the arrival of row \(r_t\), let the rows stored by \textbf{Algorithm 18} be
\(a_1, \ldots, a_s\) and suppose \(t_k \leq j < t_{k+1}\). Let \(A_k = r_k \circ \ldots \circ r_{t-1}\) and suppose the probability that row
\(r_j\) is sampled is \(\frac{18}{c^2} \|r_j\|_2^2\). Upon the arrival of row \(r_t\), note that \(r_j\) is downsamples in \textsc{MultUpdate}.
Algorithm 21 \textsc{MultExpire}(\mathcal{H}, t, W)

\textbf{Input:} Data structure \(\mathcal{H}\) consisting of matrices \(M_1, \ldots, M_s\) with corresponding timestamps \(t_1, \ldots, t_s\) and values \(g_1, \ldots, g_s\) such that \(g_i = \sum_{j=t_i}^{t_i+W-1} \|r_j\|_2\), row vector \(r_t \in \mathbb{R}^{1 \times n}\), current timestamp \(t\), and window size \(W\).

\textbf{Output:} Updated data structure \(\mathcal{H}\).

\begin{enumerate}
  \item \textbf{for} row \(r_j\) in \(M_1\) \textbf{do}
  \item \textbf{if} \(j \leq t - W + 1\) \textbf{then} \hfill \(\triangleright\) If row is expired.
  \item Delete \(r_j\).
  \item \textbf{end if}
  \item \textbf{end for}
  \item \textbf{if} \(t_2 \leq t - W + 1\) \textbf{then} \hfill \(\triangleright\) If second timestamp is expired.
  \item Delete \(M_1\) and \(t_1\).
  \item \textbf{for} \(k = 1\) to \(s\) \textbf{do}
  \item \(a_k \leftarrow a_{k+1}\) \hfill \(\triangleright\) Reorder indices.
  \item \(t_k \leftarrow t_{k+1}\)
  \item \textbf{end for}
  \item \(s \leftarrow s - 1\).
  \item \textbf{end if}
  \item \textbf{return} \(\mathcal{H}\).
\end{enumerate}

With probability \(\frac{g_{s+1} - g_s}{g_k} = \frac{\|A_k\|_F^2}{\|A_{k+W-1}\|_F^2 + r_t^2}\). Then row \(r_j\) is sampled with probability \(\frac{\|r_j\|_2^2}{\|A_k\|_F^2 + r_t^2}\). However if \(A_k\) is not deleted by \textsc{MultCompress}, then \(\|A_k\|_F^2 + r_t^2\) is just the definition of \(\|A_i\|_F^2\) after the arrival of row \(r_t\) in the statement of \textbf{Lemma 6.13} and so the induction is complete.

On the other hand, if \(A_k\) is deleted by \textsc{MultCompress}, then \(r_j\) is downsampled by \textsc{MultCompress} with probability \(\frac{g_{s+1} - g_s}{g_k} = \frac{\|A_{k+W-1}\|_F^2}{\|A_k\|_F^2 + r_t^2}\), where \(A_i\) is the updated Frobenius norm after the arrival of row \(r_t\). Hence, row \(r_j\) is sampled with probability \(\frac{\|r_j\|_2^2}{\|A_i\|_F^2}\), and the induction is complete. \hfill \(\square\)

We now justify the correctness and space complexity of \textbf{Algorithm 18}.

**Theorem 6.19.** Let \(r_1, \ldots, r_t \in \mathbb{R}^{1 \times n}\) be a stream of rows and \(W = O(n^p)\) be the size of the sliding window for some constant \(p > 0\). Let \(A = r_{t-W+1} \circ \ldots \circ r_t\) be the matrix consisting of the \(W\) most recent rows. Then given a parameter \(\varepsilon > 0\), there exists an algorithm that, with probability at least \(\frac{2}{3}\), outputs a matrix \(B\) with a subset of (rescaled) rows of \(A\) such that

\[ \left\| A^\top A - B^\top B \right\|_F \leq \varepsilon \left\| A \right\|_F^2. \]

With high probability, the algorithm uses \(O\left(\frac{n}{\varepsilon^2} \log^2 n\right)\) space.

**Proof.** Let \(t_1, \ldots, t_s\) be the timestamps maintained by \textbf{Algorithm 18} and \(A_1 = r_t \circ \ldots \circ r_t\). By construction, the sequence \(g_1, \ldots, g_s\) corresponds exactly to a smooth histogram that computes a 2-approximation to the Frobenius norm (see Section A.2 for reference). Thus by \textbf{Lemma 6.17} and
the properties of the smooth histogram, \( \|A_1\|_F^2 \leq 2 \|A\|_F^2 \). Hence by Lemma 6.16 each row \( a_i \) in \( A \) is sampled with probability at least \( \frac{\epsilon}{\|a_i\|_F^2} \). From Lemma 6.16, it follows that

\[
E \left[ \|A^\top A - B^\top B\|_F^2 \right] \leq \frac{\epsilon^2}{9} \|A\|_F^4.
\]

Then by Jensen’s inequality,

\[
E \left[ \|A^\top A - B^\top B\|_F \right] \leq \sqrt{E \left[ \|A^\top A - B^\top B\|_F^2 \right]} \leq \frac{\epsilon}{3} \|A\|_F^2.
\]

A standard application of Markov’s inequality gives the desired result for correctness.

To analyze the space complexity, first observe that \( s = O(\log n) \) since we only require a 2-approximation to the Frobenius norm, and a separate matrix is associated with each instance in the smooth histogram. Then by a simple Chernoff bound and the observation that \( c = O\left(\frac{1}{\epsilon^2}\right) \), it follows that \( \text{COVMatrix} \) stores \( O\left(\frac{1}{\epsilon^2} \log n\right) \) rows, with high probability, each with dimension \( 1 \times n \). Therefore, \( \text{COVMatrix} \) uses \( O\left(\frac{1}{\epsilon^2} \log^2 n\right) \) bits of space.

We observe that by rounding the entries in each stored row after the sampling probabilities are computed, we can avoid using \( O(n \log n) \) space per row.

**Theorem 6.20.** Let \( r_1, \ldots, r_t \in \mathbb{R}^{1 \times n} \) be a stream of rows and \( W = O(n^p) \) be the size of the sliding window for some constant \( p > 0 \). Let \( A = r_t - W + 1 \circ \ldots \circ r_t \) be the matrix consisting of the \( W \) most recent rows. Then given a parameter \( \epsilon > 0 \), there exists an algorithm that, with probability at least \( \frac{2}{3} \), outputs a matrix \( B \) with a subset of (rescaled) rows of \( A \) such that

\[
\|A^\top A - B^\top B\|_F \leq \epsilon \|A\|_F^2.
\]

With high probability, the algorithm uses \( O\left(\frac{n^2}{\epsilon^2} \log n \left(\log \log n + \log \frac{1}{\epsilon}\right)\right) \) space.

**Proof.** Given a matrix \( A \), we round each nonzero entry of \( A \) to the closest power of \( (1 + \frac{\epsilon}{3}) \) to form a matrix \( C \). Then \( (1 - \frac{\epsilon}{6}) A_{i,j} \leq C_{i,j} \leq (1 + \frac{\epsilon}{6}) A_{i,j} \). Hence, we can write \( A = C + D \) for some matrix \( D \) where \( |D_{i,j}| \leq \frac{\epsilon}{6} |A_{i,j}| \). Thus,

\[
\|A^\top A - C^\top C\|_F = \|C^\top D + D^\top C + D^\top D\|_F \leq \|C^\top D\|_F + \|D^\top C\|_F + \|D^\top D\|_F \\
\leq 2 \|C\|_F \|D\|_F + \|D\|_F^2 \\
\leq (2 + \frac{\epsilon}{3} + \frac{\epsilon}{6}) \left(\frac{\epsilon}{6}\right) \|A\|_F^2 \\
< \frac{\epsilon}{2} \|A\|_F^2,
\]

for \( \epsilon < 1 \).

By sending the rows of \( C \) into \( \text{COVMatrix} \) with approximation parameter \( \frac{\epsilon}{6} \), the output matrix \( B \) satisfies

\[
\|C^\top C - B^\top B\| \leq \frac{\epsilon}{8} \|C\|_F^2.
\]
For \( \varepsilon < 1 \), it follows that \( \| C \|^2 \leq 4 \| A \|^2 \), so that \( \| C^\top C - B^\top B \| \leq \frac{\varepsilon}{4} \| A \|^2 \). Hence, \( \| A^\top A - B^\top B \| \leq \varepsilon \| A \|^2 \).

If all nonzero entries of \( A \) are bounded by some polynomial in \( n \), then each entry of \( C \) can be written as \( (1 + \frac{\varepsilon}{4})^m \) for some \( m = \mathcal{O} \left( \frac{1}{\varepsilon} \log n \right) \). Since storing the value of \( m \) requires \( \log \log n + \log \frac{1}{\varepsilon} \) bits, then each entry of \( C \) can be succinctly expressed in \( \log \log n + \log \frac{1}{\varepsilon} \) bits. \( \text{COVMATRIX} \) stores \( \mathcal{O} \left( \frac{1}{\varepsilon^2} \log n \right) \) rows, with high probability, each with dimension \( 1 \times n \), so the total bit complexity is \( \mathcal{O} \left( \frac{\varepsilon^2}{n^2} \log n \left( \log \log n + \log \frac{1}{\varepsilon} \right) \right) \).

### 6.3.3 Lower Bounds

To show a lower bound for the covariance matrix approximation problem in the sliding window model, we consider the following variant of the Index problem. For any positive integer \( n \), suppose the first player Alice is given a string \( S \in \{0,1\}^n \) chosen uniformly at random, while the second player Bob is given an index \( m \in [n] \) chosen uniformly at random. Alice sends a message to Bob, who must output \( S[m] \) with probability at least \( \frac{2}{3} \). We also use \( R_{\mu,\delta}^+(\text{Index}) \) to denote the minimum communication of a protocol that sends a single message from Alice to Bob for solving Index with probability at least \( 1 - \delta \), where the probability is taken over both the input distribution \( \mu \) and the randomness of the protocol.

**Lemma 6.21.** \([\text{MNSW}95]\) Let \( \mu \) be the uniform distribution on \( \{0,1\}^n \times [n] \). Then \( R_{\mu,\delta}^+(\text{Index}) = \Omega(n) \).

We first give a lower bound for the covariance matrix approximation problem in the streaming model to build intuition. In this setting, Alice will convert her string into a matrix \( A = [M|E] \). The submatrix \( M \) will encode the characters of her string while the submatrix \( E \) will consist of a number of elementary columns. Hence, \( A^\top A \) will consist of many cross-products between \( M \) and \( E^\top \) that will inform Bob about the characters of the string. In other words, if Bob obtains a good approximation of \( A^\top A \), Bob can determine a large fraction of the string.

**Theorem 6.22.** Let \( A \) be a \( W \times n \) matrix and \( \varepsilon > 0 \). Any one-pass streaming algorithm that returns a matrix \( B \) such that \( \| A^\top A - B^\top B \|_F \leq \varepsilon \| A \|^2 \) with probability at least \( \frac{8}{9} \) requires \( \Omega \left( \frac{W^2}{\varepsilon^2} \right) \) bits of space.

**Proof.** Suppose Alice receives a string \( S \in \{0,1\}^{n/324\varepsilon^2} \) chosen uniformly at random and creates a \( \frac{1}{324\varepsilon^2} \times n \) matrix \( M \) by setting each entry

\[
M_{ij} = \begin{cases} 
18\varepsilon & \text{if } S[(i-1)n+j] = 1 \\
-18\varepsilon & \text{if } S[(i-1)n+j] = 0.
\end{cases}
\]

Let \( e_i \) be the elementary column vector with a one in the \( i^{th} \) position and zeroes elsewhere. Alice creates a matrix \( E \) that has \( 324\varepsilon^2 n \) instances of \( e_{11} \), followed by \( 324\varepsilon^2 n \) instances of \( e_{22} \), and so forth, for each of the \( \frac{1}{324\varepsilon^2} \) elementary column vectors. She then creates a \( \frac{1}{324\varepsilon^2} \times 2n \) matrix \( A = [M \ E] \). Intuitively, the entries of \( M^\top E \) and \( E^\top M \) in \( A^\top A \) reveal the entries of \( M \), in which Alice has encoded characters of \( S \). Alice sends each row of \( A \) to the covariance matrix approximation algorithm and then passes the state of the algorithm to Bob.

Bob uses the state of the algorithm to compute a matrix \( B \). Given an index \( m \in \left[ \frac{n}{324\varepsilon^2} \right] \) chosen uniformly at random, Bob sets integers \( (y-1)n + z = m \), where \( 1 \leq y \leq \frac{1}{324\varepsilon^2} \) and \( 1 \leq z \leq n \), and deduces that

\[
S[m] = 1 \text{ if } A_{yz} = 18\varepsilon \quad \text{or} \quad S[m] = 0 \text{ if } A_{yz} = -18\varepsilon.
\]
To compute $A_{yz}$, Bob computes $B^\top B$. He then looks at the entries 

$$(B^\top B)_{zj} \text{ for all } j = n + (y - 1)(324\varepsilon^2 n) + b,$$

where $1 \leq b \leq 324\varepsilon^2 n$, and outputs $S[m] = 1$ if the majority of these entries are positive and $S[m] = 0$ if the majority of these entries are negative.

**Claim 6.23.** Let $A$, $B$, and $S$ be as defined above. Then if $\|A^\top A - B^\top B\| \leq \varepsilon \|A\|_F^2$, then Bob can identify a large fraction of the indices $S[i]$. 

**Proof.** First note that each column in $A$ is a unit vector so $\|A\|_F^2 = 2n$. Let $c_i$ represent the $i^{th}$ column of $A$ so that $(A^\top A)_{ij} = \langle c_i, c_j \rangle$.

In particular, if $1 \leq z \leq n$ and $j = n + (y - 1)(324\varepsilon^2 n) + b$ for some integers $1 \leq y \leq \frac{1}{324\varepsilon^2}$ and $1 \leq b \leq 324\varepsilon^2 n$, then 

$$(A^\top A)_{zj} = (A^\top A)_{zj} = \langle c_z, c_j \rangle = A_{yz} = \pm 18\varepsilon.$$

Since there are $n$ total such indices $j$, then the total contribution of these entries $(A^\top A)_{zj}$ to the Frobenius norm of $A^\top A$ is $18\sqrt{2}\varepsilon n$. Hence, for $\|A^\top A - B^\top B\| \leq \varepsilon \|A\|_F^2 \leq 2\varepsilon n$, more than $\frac{8}{9}$ of the entries in $(B^\top B)_{zj}$ and $(B^\top B)_{zj}$ must have the same sign as $(A^\top A)_{zj}$ and $(A^\top A)_{zj}$ for $1 \leq z \leq n$ and $j = n + (y - 1)(324\varepsilon^2 n) + b$, where $1 \leq y \leq \frac{1}{324\varepsilon^2}$ and $1 \leq b \leq 324\varepsilon^2 n$. Intuitively, there can only be a small number of positions in $B^\top B$ that differ in sign with the corresponding positions of $M^\top E$ or $E^\top M$. This allows Bob to recover a large fraction of the indices of $S$ and we get the claim. \qed 

We call an entry in $B^\top B$ **bad** if it does not have the same sign as the corresponding entry in $A^\top A$, so that at most $\frac{1}{9}n^2$ entries in $(B^\top B)_{ij}$ can be bad, for $1 \leq i \leq n$ and $j = n + (y - 1)(324\varepsilon^2 n) + b$ for some integers $1 \leq y \leq \frac{1}{324\varepsilon^2}$ and $1 \leq b \leq 324\varepsilon^2 n$. Since each column vector is repeated $324\varepsilon^2 n$ times, the majority of the entries in the set $(B^\top B)_{zj}$ is bad only if at least $162\varepsilon^2 n$ are bad across $j = n + (y - 1)(324\varepsilon^2 n) + b$, where $1 \leq b \leq 324\varepsilon^2 n$. Thus, Bob can err on at most $\frac{\frac{1}{2}n^2}{162\varepsilon^2 n} = \frac{n}{1458\varepsilon^2}$ of the $\frac{n}{324\varepsilon^2}$ entries in $A$. The probability that Bob successfully computes $S[m]$ is at least $\frac{324^2}{1458\varepsilon^2} = \frac{2}{9}$, conditioned on the algorithm correctly giving a matrix $B$ such that $\|A^\top A - B^\top B\| \leq \varepsilon \|A\|_F^2$. Hence, the overall probability that Bob succeeds is at least $\frac{8}{9} - \frac{2}{9} = \frac{2}{9}$, so by **Lemma 6.21** any covariance matrix approximation algorithm that succeeds with probability at least $\frac{8}{9}$ must use $\Omega\left(\frac{n}{\varepsilon^2}\right)$ bits of space. \qed

We now give a lower bound for the covariance matrix approximation problem in the sliding window model. The argument follows similarly from **Theorem 6.22**, but we show Alice and Bob would be able to communicate a longer string if they were able to do covariance matrix approximation in the sliding window model. Alice breaks the input string $S$ into $\log n$ substrings $S_1, \ldots, S_{\log n}$. She uses $S_i$ to create a matrix $A_i = |M_i|E_i| as before. However, the magnitude of the entries in each $A_i$ are geometrically decreasing as $i$ increases. Hence, Bob can deduce the sign of any entry in $A_k$ of interest by expiring the rows of all matrices $A_j$ with $j < k$ so that the entries of $A_k$ dominates the remaining entries. The rest of the argument follows as before, showing that $A_k^\top A_k$ encodes information about a large fraction of the characters of $S_k$. Since he can do this for any $k$ of interest, then Bob can determine a large fraction of the entire string $S$. 

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Theorem 6.24. Let $\varepsilon > 0$ and $A$ be a $W \times n$ matrix for $W = \frac{1}{72^2 \varepsilon^2} \log n$. Any sliding window algorithm that returns a matrix $B$ such that $\|A^\top A - B^\top B\|_F \leq \varepsilon \|A\|_F^2$ with probability at least $\frac{8}{7}$ requires $\Omega\left(\frac{n}{\varepsilon^2} \log n\right)$ bits of space.

Proof. Suppose Alice receives a random string $S \in \{0, 1\}^{\frac{n \log n}{72^2 \varepsilon^2}}$ chosen uniformly at random and creates matrices $A^{(1)}, A^{(2)}, \ldots, A^{(\log n)}$ in the following manner. For each $1 \leq k \leq \log n$, she initially creates the $\frac{1}{72^2 \varepsilon^2} \times n$ matrix $M^{(k)}$ by setting each entry

$$M^{(k)}_{ij} = \begin{cases} 2^{-k+\log n} \cdot 72\varepsilon & \text{if } S \left[72^2 \varepsilon^{-2} (k - 1) + (i - 1)n + j\right] = 1 \\ -2^{-k+\log n} \cdot 72\varepsilon & \text{if } S \left[72^2 \varepsilon^{-2} (k - 1) + (i - 1)n + j\right] = 0. \end{cases}$$

Let $e_i$ be the elementary column vector with a one in the $i$th position and zeroes elsewhere. Alice creates a matrix $E^{(k)}$ that has $72^2 \varepsilon^2 n$ instances of $2^{-k+\log n} e_1$, followed by $72^2 \varepsilon^2 n$ instances of $2^{-k+\log n} e_2$, and so forth, for each of the $\frac{1}{72^2 \varepsilon^2}$ elementary column vectors. She then creates a $\frac{1}{72^2 \varepsilon^2} \times 2n$ matrix $A^{(k)} = [M^{(k)} \ E^{(k)}]$. Alice then sets

$$M = \begin{pmatrix} A^{(1)} \\ A^{(2)} \\ \vdots \\ A^{(\log n)} \end{pmatrix} \in \{0, 1\}^{\frac{n \log n}{72^2 \varepsilon^2} \times n}.$$

She sends each row of $M$ to the covariance matrix approximation algorithm and passes the state of the algorithm to Bob.

For an index $m \in \left[\frac{n \log n}{72^2 \varepsilon^2}\right]$ chosen uniformly at random, let $1 \leq k \leq \log n$ be the integer such that

$$\frac{n(k - 1)}{72^2 \varepsilon^2} < m \leq \frac{nk}{72^2 \varepsilon^2}.$$ 

Bob expires the rows of all matrices $A^{(i)}$ for $i < k$ by inserting $\frac{1}{72^2 \varepsilon^2} (k - 1)$ rows of all zeroes so that the matrix represented by the sliding window is

$$A := \begin{pmatrix} A^{(k)} \\ A^{(k+1)} \\ \vdots \\ A^{(\log n)} \\ 0 \end{pmatrix},$$

where $0$ represents a matrix of dimension $\ell \times n$ such that $\ell := \frac{1}{72^2 \varepsilon^2} (-k + 1 + \log n)$.

Bob then uses the state of the algorithm to compute a matrix $B$. Recall that the entries of $M^{(k)}$ are bigger than the sum of the corresponding entries in $M^{(j)}$ for $j = k + 1$ to $j = \log n$, since each $M^{(j)}$ has entries that are geometrically decreasing as $j$ increases. Similarly, the entries of $E^{(k)}$ are larger than the sum of the corresponding entries in $E^{(j)}$ for $j = k + 1$ to $j = \log n$. Thus, the entries of $A^\top A$ that correspond to $\sum_k^{\log n} (M^{(j)})^\top E^{(j)}$ or $\sum_k^{\log n} (E^{(j)})^\top M^{(j)}$ are dominated by the values of $(M^{(k)})^\top E^{(k)}$ or $(E^{(k)})^\top M^{(k)}$ respectively, so Bob can use the sign of each entry in $B^\top B$ to deduce values in $S$, similar to the proof of Claim 6.23.
Hence, the overall probability that Bob succeeds is at least 
\[ 8 \lesssim B \] conditioned on the algorithm correctly giving a matrix 
\[ \left( \begin{array}{c}
\end{array} \right) \]
Note that each entry are negative. Since the number of bad entries in 
\[ B \leq 72^2 \varepsilon^2 n(y - 1) + b \] for some integers \( 1 \leq y \leq \frac{1}{72 \varepsilon^2} \) and \( 1 \leq b \leq 72^2 \varepsilon^2 n \), then 
\[ (A_k^T A_k)_{j,z} = \left( (A^{(k)})^T A^{(k)} \right)_{z,j} = \pm 72 n^2 \varepsilon. \]
Note that each entry \( (A^T A)_{j,z} \) has the same sign as the corresponding entry \( ((A^{(k)})^T A^{(k)})_{j,z} \) since 
\[ \left| \sum_{i=k+1}^{n} \left( (A^{(k)})^T A^{(i)} \right)_{j,z} \right| \leq \frac{36 n^2}{2^{2k}} \varepsilon. \]
We call an entry \((j, z)\) in \( B^T B \), with \( 1 \leq y \leq \frac{1}{72 \varepsilon^2} \) and \( 1 \leq b \leq 72^2 \varepsilon^2 n \), bad if \( |(B^T B)_{j,z} - (A^T A)_{j,z}| \geq \frac{36 n^2}{2^{2k}} \varepsilon \). Since \( \|A\|_F^2 \leq \frac{4n^2}{2^{2k}} \), any covariance matrix approximation algorithm that produces \( B \) with \( \|A^T A - B^T B\|_F \leq \varepsilon \|A\|_F^2 \) can have at most \( \frac{1}{2} n^2 \) bad entries.

Given this, Bob uses \( m \) to set integers \( y, z \) so that \( m = \left( \frac{n}{72 \varepsilon^2} \right) (k - 1) + (y - 1)n + z \), where \( 1 \leq y \leq \frac{1}{72 \varepsilon^2} \) and \( 1 \leq z \leq n \), and deduces that \( S[m] = 1 \) if \( A_{y,z} > 0 \) or \( S[m] = 0 \) if \( A_{y,z} < 0 \). Bob can look at the entries \( (B^T B)_{j,z} \) for all \( j = n + (72 \varepsilon^2) n (y - 1) + b \), where \( 1 \leq b \leq (72 \varepsilon^2) n \), and output \( S[m] = 1 \) if the majority of these entries are positive and \( S[m] = 0 \) if the majority of these entries are negative. Since the number of bad entries in \( B^T B \) is bounded by \( \frac{1}{2} n^2 \), by similar reasoning to the proof of Theorem 6.22, the probability that Bob successfully computes \( S[m] \) is at least \( \frac{2}{7} \), conditioned on the algorithm correctly giving a matrix \( B \) such that \( \|A^T A - B^T B\| \leq \varepsilon \|A\|_F^2 \). Hence, the overall probability that Bob succeeds is at least \( \frac{8}{9} - \frac{4}{7} = \frac{2}{7} \), so by Lemma 6.21 any covariance matrix approximation algorithm that succeeds with probability at least \( \frac{2}{7} \) must use \( \Omega \left( \frac{n}{\varepsilon^2} \log n \right) \) bits of space.

\[ \square \]

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A Notations and Preliminaries

Given a matrix $A \in \mathbb{R}^{m \times n}$, the *singular value decomposition* is $A = U \Sigma V^T$, where $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ are orthogonal matrices that contain the left and right singular vectors of $A$, respectively. Moreover, $\Sigma \in \mathbb{R}^{m \times n}$ is a diagonal matrix containing the *singular values* of $A$, in decreasing order. We use $\sigma_i$ for $0 \leq i \leq \min(m, n)$ to denote the singular values: $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\min(m,n)} \geq 0$. 

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Given a matrix $A \in \mathbb{R}^{m \times n}$, the Moore-Penrose pseudoinverse is the matrix $A^\dagger$ such that: (i) $AA^\dagger A = A$, (ii) $A^\dagger AA^\dagger = A^\dagger$, (iii) $(AA^\dagger)^\top = AA^\dagger$, and (iv) $(A^\dagger A)^\top = A^\dagger A$ If $A$ has full rank, then $A^\dagger = A^{-1}$.

A.1 Background on the Sliding Window Model

An initial framework for approaching problems in the sliding window model is the exponential histogram data structure, introduced by Datar et al. [DGIM02]. Given a function $f$ to approximate in the sliding window model, the exponential histogram partitions the data stream into “buckets”, time intervals for which the evaluation of $f$ on the data in each partition is exponentially increasing. For example, suppose we are given a data stream of integers and we want to approximate the number of ones in the sliding window within a factor of 2. In the exponential histogram data structure, the smallest bucket consists of all elements in the data stream from the most recent element to the most recent element whose value is one. The next bucket would consist of all previous elements until two elements whose value is one are seen. Similarly, the $i^{th}$ bucket consists of the previous elements until $2^i$ instances of ones are seen. The key observation is that because the buckets are exponentially increasing by powers of two, the starting point of the sliding window falls inside some bucket, and it will provide a 2-approximation to the number of ones seen in the sliding window even though it does not know exactly where the starting point is. Datar and Motwani [DM07] show that the exponential histogram framework is applicable to the class of “weakly additive” functions, with some properties defined next. Let $A$ and $B$ be adjacent buckets and $A \cup B$ represent the concatenation of $A$ and $B$ such that if $0 \leq f(A) \leq \text{poly}(N)$, where $N$ is the length of the data stream, and $f(A) + f(B) \leq f(A \cup B) \leq C_f (f(A) + f(B))$ for some fixed constant $C_f \geq 1$, then there exist a sketch of $f$, as well as a “composition” function that computes the sketch of $f(A \cup B)$ from the sketches of $f(A)$ and $f(B)$.

Since the buckets in the exponential histogram data structure consist of disjoint elements, a crucial underlying requirement is that an approximation of $f$ must be deducible from the merger of the information from these buckets. For example, it is not clear how to maintain buckets for the goal of approximating the geometric mean of a sliding window. To that effect, Braverman and Ostrovsky [BO07] define the notion of a smooth function, and provide the smooth histogram data structure as a framework for approximating smooth functions of sliding windows. They also show that the class of smooth functions contains the class of weakly additive functions, as well as a number of other functions, such as the geometric mean.

**Smooth histogram.** Given adjacent buckets $A$, $B$, and $C$, a smooth function demands that if $(1 - \beta) f(A \cup B) \leq f(B)$, then $(1 - \alpha) f(A \cup B \cup C) \leq f(B \cup C)$ for some constants $0 < \beta \leq \alpha < 1$. Informally, a smooth function has the property that once a suffix of a data stream becomes a good approximation, then it always remains a good approximation, even with the arrival of new elements in the stream. With this definition of smooth function in mind, the smooth histogram data structure maintains a number of “checkpoints” throughout the data stream. Each checkpoint corresponds to a sketch of all the elements seen from the checkpoint until the most recently arrived element. Unlike the exponential histogram, the most recently arrived element impacts all sketches in the smooth histogram. A checkpoint is created with the arrival of each new element and checkpoints are discarded when their corresponding sketches get “too close” to the next checkpoint. That is, when the corresponding sketches of two checkpoints produce values that are
within \((1 - \beta)\) of each other, the later checkpoint is discarded, since by the property of smooth functions, the two checkpoints would thereafter always produce values that are within \((1 - \alpha)\) of each other. This implies that, if the function is polynomially bounded, then the smooth histogram data structure only needs a logarithmic number of checkpoints. Moreover, Braverman and Ostrovsky \([BO07]\) extend their results to the case where the sketch only provides an approximation to the evaluation of the function.

Unfortunately, a number of useful functions are either not smooth or do not provide strong enough parameters \((\alpha, \beta)\) to yield non-trivial results in the case that the sketching function is an approximation. The \(k\)-means clustering is an example of a function that is not smooth \([BLLM15]\), while there exists no known algorithm for submodular maximization that provides better than a \(1/5\)-approximation guarantee \([Fei98]\), and so submodular maximization does not provide meaningful guarantees under the smooth histogram framework. Nevertheless, \([BLLM16]\) provide a sliding window algorithm for \(k\)-means clustering, while \([CNZ16]\) and \([ELVZ17]\) provide sliding window algorithms for submodular maximization under cardinality constraints. Each of these algorithms focuses heavily on the specific structure of the problem, and so their techniques are not obviously generalizable.

### A.2 Smooth Functions

We use the notation \(B \subseteq A\) if the stream of elements indexed by \(B\) are suffix of stream of elements indexed by \(A\) (see Figure 1 for a diagrammatic representation).

**Definition A.1** (Braverman and Ostrovsky \([BO07]\)). A function \(f \geq 1\) is \((\alpha, \beta)\)-smooth if it has the following properties:

**Monotonicity** \(f(A) \geq f(B)\) for \(B \subseteq A\) (\(B\) is a suffix of \(A\)).

**Polynomial boundedness** There exists \(c > 0\) such that \(f(A) \leq n^c\).

**Smoothness** There exists \(\alpha \in (0, 1), \beta \in (0, \alpha]\) so that if \(B \subseteq A\) and \((1 - \beta)f(A) \leq f(B)\), then \((1 - \alpha)f(A \cup C) \leq f(B \cup C)\) for any adjacent \(C\).

The smooth histogram data structure estimates smooth functions in the sliding window model.

**Definition A.2** (Smooth Histogram \([BO07]\)). Let \(g\) be a function that maintains a \((1 + \epsilon)\)-approximation of an \((\alpha, \beta)\)-smooth function \(f\) that takes as input a starting index and ending index in the data stream.
The approximate smooth histogram is a structure that consists of an increasing set of indices $X_N = \{x_1, \ldots, x_s = N \}$ and $s$ instances of an algorithm $\Lambda$, namely $\Lambda_1, \ldots, \Lambda_s$ with the following properties:

1. $x_1$ corresponds to either the beginning of the data stream, or an expired point.
2. $x_2$ corresponds to an active point.
3. For all $i < s$, one of the following holds:
   - $x_{i+1} = x_i + 1$ and $g(x_{i+1}, N) < \left(1 - \frac{\beta}{2}\right) g(x_i, N)$.
   - $(1 - \alpha) g(x_i, N) \leq g(x_{i+1}, N)$ and if $i + 2 \leq s$ then $g(x_{i+2}, N) < \left(1 - \frac{\beta}{2}\right) g(x_i, N)$.
4. $\Lambda_i = \Lambda(x_i, N)$ maintains $g(x_i, N)$.

### A.3 Auxiliary Lemmas

We use several results proven by previous works. In this section, we give a brief overview of those results.

Our algorithm heavily uses importance sampling via leverage score estimates. It is well-known (see Lemma 4 in [CLM$^{+}$15] or Theorem 2.1 in [CMP16] for example) that sampling each row with a probability that is an overestimate of the $\lambda$-ridge leverage score suffices to produce a good spectral approximation:

**Lemma A.3 ([CLM$^{+}$15, CMP16])**. Given $R \in \mathbb{R}^{m \times n}$ with rows $r_1, \ldots, r_m \in \mathbb{R}^{1 \times n}$, set $0 < \varepsilon < 1$, $\delta \geq 0$, $\lambda = \frac{\delta}{\varepsilon}$, and $c = \frac{a \log n}{\varepsilon^2}$ for any constant $a > 0$. Suppose for all $1 \leq i \leq m$, we have

$$l_i \geq r_i^\top (R^\top R + \lambda I)^\dagger r_i.$$

For $1 \leq i \leq m$, define $p_i = \min(c l_i, 1)$. Let $\tilde{R}$ be a matrix that is formed by independently sampling each row of $R$ with probability $p_i$ and rescaling the row by a factor of $\frac{1}{\sqrt{p_i}}$ if it is included in the sample. Then with probability at least $1 - n^{-a/3}$,

$$(1 - \varepsilon) R^\top R - \delta I \preceq \tilde{R}^\top \tilde{R} \preceq (1 + \varepsilon) R^\top R + \delta I.$$

It follows that the $\lambda$-ridge leverage score of a row cannot increase as more rows are added to the matrix (see Theorem 11 in [CMM17], for example).

**Lemma A.4 (Ridge Leverage Score Monotonicity [CMM17])**. For any $R \in \mathbb{R}^{m \times n}$ and row vector $r \in \mathbb{R}^{1 \times n}$ and $1 \leq i \leq m$:

$$r_i^\top (R^\top R + \lambda I)^\dagger r_i \geq r_i^\top (R^\top R + r^\top r + \lambda I)^\dagger r_i.$$

For each $0 \leq i \leq m$, let $R_i$ denote the matrix consisting of the first $i$ rows of $R$. Then [CMP16] defines the online $\lambda$-ridge leverage score of the $i^{th}$ row to be

$$\min \left( r_i^\top (R_{i-1}^\top R_{i-1} + \lambda I)^\dagger r_i^\top, 1 \right).$$

Since $R_i^\top R_i \preceq R^\top R$ for all $i$, then the online $\lambda$-ridge leverage score is an overestimate of the $\lambda$-ridge leverage score.
It is easy to bound the sum of leverage scores using standard trace argument. This in turn allows us to bound the number of rows sampled when we sample rows based on its leverage scores. However, it is not a priori clear if the sum of online ridge leverage scores are also bounded. Cohen et al. [CMP16] showed that it is indeed the case:

**Lemma A.5** (Theorem 2.2 in [CMP16]). Let $R$ be an $m \times n$ matrix with rows $r_1, \ldots, r_m$ and let $l_i$ be the online $\lambda$-ridge leverage score of the $i^{th}$ row, with $\lambda > 0$. Then

$$\sum_{i=1}^{m} l_i = O \left( n \log \frac{\|R\|^2}{\lambda} \right).$$

An important consequence of the ridge-leverage score based sampling is that it also gives a projection cost preserving sketch. In particular, Cohen et al. [CMM17] showed the following:

**Lemma A.6** (Theorem 6 in [CMM17]). Given $R \in \mathbb{R}^{m \times n}$ with rows $r_1, r_2, \ldots \in \mathbb{R}^{1 \times n}$, set $0 < \epsilon < 1$, $\delta \geq 0$, $\lambda = \frac{\|R - R_k\|^2}{k}$, and $c = \frac{\alpha \log n}{\epsilon}$ for any constant $\alpha > 0$. Suppose for all $1 \leq i \leq m$, we have

$$l_i \geq r_i (R^\top R + \lambda I)^\dagger r_i^\top.$$

For $1 \leq i \leq m$, define $p_i = \min\{cl_i, 1\}$. Let $\tilde{R}$ be a matrix that is formed by independently sampling each row of $R$ with probability $p_i$ and rescaling the row by a factor of $\frac{1}{\sqrt{p_i}}$ if it is included in the sample. Then with probability at least $1 - n^{-a/3}$, for any rank $k$ orthogonal projection $P$,

$$(1 - \epsilon) \|R - RP\|^2_F \leq \|\tilde{R} - \tilde{R}P\|^2_F \leq (1 + \epsilon) \|R + RP\|^2_F.$$

**Linear algebra.** We use use several facts about positive semidefinite matrices. Positive semidefinite matrices are the class of matrices whose eigenvalues are all non-negative, i.e., a symmetric matrix $A \in \mathbb{R}^{n \times n}$ is a positive semidefinite if

$$\forall x \in \mathbb{R}^n, x^\top Ax \geq 0.$$

We use the notation $A \succeq 0$ to denote a positive semidefinite matrix. We use the following fact regarding positive semidefinite matrices:

**Fact A.7.** Suppose $A$ and $B$ are positive semidefinite matrices with $0 \preceq B \preceq A$. Then $0 \preceq A^\dagger \preceq B^\dagger$.

**Smooth histograms.** Braverman and Ostrovsky [BO07] describe the maintenance of the approximate smooth histogram using the procedure described in Algorithm 22.

**Theorem A.8** (Theorem 3 in [BO07]). Let $f$ be a $(\alpha, \beta)$-smooth function. If there exists an algorithm $A$ that maintains an $(\epsilon, \delta)$-approximation of $f$ on $D$, using space $g(\epsilon, \delta)$ and performing $h(\epsilon, \delta)$ operations per stream element, then there exists an algorithm $A'$ that maintains an $(\alpha + \epsilon)$-approximation of $f$ on sliding windows and uses $O \left( \frac{1}{p} \left( g\left( \epsilon, \frac{\delta}{n} \right) + \log n \right) \log n \right)$ bits and $O \left( \frac{1}{p} h\left( \epsilon, \frac{\delta}{\log n} \right) \log n \right)$ operations per element.

We use the following result shown by Braverman and Ostrovsky [BO07].

**Theorem A.9.** Frobenius norm of a matrix is $(\epsilon, \frac{\epsilon^2}{2})$-smooth.
Algorithm 22 Smooth Histogram Construction of Braverman and Ostrovsky [BO07]

**Input:** An underlying data set in the sliding window model.

**Output:** A smooth histogram data structure.

1: **Initialization.** For $N = 1$, put $x_1 = s = 1$ and initiate $\Lambda$ with the element corresponding to $x_1$.
2: for each new element at time $N + 1$ do
3: Sketches For all $i$, calculate $g(x_i, N + 1)$ using $\Lambda_i = \Lambda(x_i, N)$ and the element corresponding to time $N + 1$.
4: Increment $s = s + 1$ and $x_s = N + 1$, and initiate a new instance $\Lambda(N + 1, N + 1)$.
5: for $i = 1, \ldots, s - 2$ do
6: Find the largest $j > 1$ such that $g(x_j, N + 1) \geq \left(1 - \frac{\beta}{2}\right) g(x_i, N + 1)$.
7: Delete all $x_t$ for $i < t < j$ and all instances $\Lambda(x_t, N)$, and shift the list accordingly.
8: end for
9: Find the smallest $i$ such that $x_i$ is expired and $x_{i+1}$ is active. Delete all $x_j$ with $j < i$ and $\Lambda_j$ data structures, and change the enumeration accordingly.
10: end for

### B Lack of $(\alpha, \beta)$-smoothness of RandNLA Functions

In this section, we show that the spectral norm, vector induced matrix norms, linear and generalized regression, and low-rank approximation are not amenable for the smooth histogram framework.

We first prove that low-rank approximation is not smooth for any meaningful parameters $(\alpha, \beta)$ in [Definition A.1] even when the best low-rank approximations are nonzero.

**Lemma B.1.** Low-rank approximation is not smooth for any meaningful parameters $(\alpha, \beta)$ in [Definition A.1] that gives us a constant factor approximation.

**Proof.** Our proof constructs a matrix and a vector explicitly. Let $e_i$ be the elementary row vector with entry one in the $i$th position and zero elsewhere. Given $0 < \alpha < 1$, let $d = \frac{2}{\alpha}$. Let $S_A$ be the data stream whose first element is $2d e_1$, second element is $e_2$, followed by data stream $S_B$, which is a suffix of $S_A$, i.e., $S_B \subseteq S_A$. Then suppose $B$ consists of the elements $2d e_3$, followed by $e_{i+3}$ for $1 \leq i \leq d$. Finally, let $S_C$ be the data stream consisting of the single element $2d e_j$, where $j = 4 + d$. Then the corresponding matrices appear as below:

\[
\begin{bmatrix}
2d & 1 \\
2d & 1 \\
\vdots & \vdots \\
1 & 2d \\
0 & \vdots
\end{bmatrix}
\]
where we have the following matrices

\[ \mathbf{A} = \begin{bmatrix} 2d & 1 & 2d & 1 & \ldots & 1 \\ 2d & 1 & 1 & 2d & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 2d & 1 & 1 & 1 & \ldots & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2d & 1 & \ldots & 1 \\ 2d & 1 & \ldots \\ \vdots & \vdots & \ddots & \vdots \\ 2d & 1 & \ldots \end{bmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 2d & 0 & \ldots \\ 2d & 0 & \ldots \end{bmatrix}. \]

Then for \( k = 2 \), the best rank \( k \) approximation of \( \mathbf{A} \) consists of two rows containing \( 2d \) so that

\[
\min_{\substack{X \in \mathbb{R}^{N \times n} \\ \text{rank}(X) = 2}} \| \mathbf{A} - X \|_F = \sqrt{d + 1}.
\]

The best rank \( k \) approximation of matrix formed by \( S_B \) consists of the row containing \( 2d \) and any other elementary row in \( \mathbf{B} \) so that

\[
\min_{\substack{Y \in \mathbb{R}^{N \times n} \\ \text{rank}(Y) = 2}} \| \mathbf{B} - Y \|_F = \sqrt{d - 1}.
\]

Hence, the ratio of the best low-rank approximation of \( \mathbf{B} \) to the best low-rank approximation of \( \mathbf{A} \) is

\[
\frac{\sqrt{d - 1}}{\sqrt{d + 1}} > \frac{d - 1}{d + 1} = 1 - \frac{2}{d + 1} > 1 - \alpha.
\]

Now, let \( \mathbf{C} \) represent the matrix corresponding to stream \( S_{AUC} \) and let \( \mathbf{D} \) represent the matrix corresponding to stream \( S_{BUC} \). Then the best rank \( k \) approximation of \( S_{AUC} \) consists of two rows containing \( 2d \) so that

\[
\min_{\substack{X \in \mathbb{R}^{N \times n} \\ \text{rank}(X) = 2}} \| \mathbf{C} - X \|_F = \sqrt{4d^2 + d + 1},
\]

while the best rank \( k \) approximation of \( S_{BUC} \) consists of two rows containing \( 2d \) so that

\[
\min_{\substack{Y \in \mathbb{R}^{N \times n} \\ \text{rank}(Y) = 2}} \| \mathbf{D} - Y \|_F = \sqrt{d}.
\]

Thus, the ratio of the best low-rank approximation of \( S_{BUC} \) to the best low-rank approximation of \( S_{AUC} \) is at least 2, i.e., \( \beta \leq -1 \). Therefore, low-rank approximation is not smooth. \( \square \)

We now show that \( \ell_p \) regression is not smooth as per the smooth histogram framework. Recall that for \( \mathbf{A} \in \mathbb{R}^{N \times n} \) and \( \mathbf{B} \in \mathbb{R}^{N \times d} \), the generalized \( \ell_p \) regression problem is the minimization problem

\[
\min_{X \in \mathbb{R}^{n \times d}} \| \mathbf{A} X - \mathbf{B} \|_p,
\]

where \( \| \cdot \|_p \) denotes the entrywise \( \ell_p \) norm, that is, \( \| X \|_p = \left( \sum_{i,j} |X_{ij}|^p \right)^{1/p} \).
In our setting, each update to the data stream consists of a row vector $a_i \in \mathbb{R}^d$ and an element $b_i$, i.e., the underlying matrix $A$ represented by a sliding window of size $W$ consists of the rows

$$A = \begin{bmatrix} a_{t-W+1} & a_{t-W+2} & \cdots & a_t \end{bmatrix}^\top$$

$$b = \begin{bmatrix} b_{t-W+1} & b_{t-W+2} & \cdots & b_t \end{bmatrix}^\top.$$

We next show that $\ell_p$ regression is not smooth as per Definition A.1 for any reasonable parameters $(\alpha, \beta)$ (see Lemma B.2).

Lemma B.2. $\ell_p$ regression is not smooth as per Definition A.1 for any reasonable parameters $(\alpha, \beta)$ that gives us a constant factor approximation.

Proof. Let $0 < \alpha < 1$. Let $A$ be the data stream whose first element is the row $\{a_1 = \{100, 0, 0, 0, 0\}, b_1 = 100\}$, second element is the row $\{a_2 = \{0, \alpha, 0, 0, 0\}, b_2 = 0\}$, followed by data stream $B$, which is a suffix of $A$. Then suppose $B$ consists of the elements $\{a_3 = \{0, 0, 1, 0, 0\}, b_3 = 1\}$, followed by $\{a_4 = \{0, 0, 0, 1, 0\}, b_4 = 0\}$. Finally, let $C$ be the data stream consisting of the single element $\{a_5 = \{0, 0, 0, 0, 1000\}, b_5 = 2000\}$ Then the corresponding matrices appear as below:

$$A = \begin{bmatrix} 100 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1000 \end{bmatrix}, \quad b = \begin{bmatrix} 100 \\ 0 \\ 1 \\ 0 \\ 2000 \end{bmatrix},$$

where

$$A_1 := \begin{bmatrix} 100 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \end{bmatrix}, A_2 := \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

and $A_3 := \begin{bmatrix} 0 & 0 & 0 & 0 & 1000 \end{bmatrix}$ and $a_1, a_2, a_3$ are the corresponding rows of the vector $b$. Let the matrix $A_1$ and vector $b_1$ represent data stream $S_A$, $A_2$ and $b_2$ represent data stream $S_B$, $A_3$ and $b_3$ represent $A \cup C$, and $A_4$ and $b_4$ represent $S_B \cup S_C$. Finally, let $Z_i = \arg\min_{x \in \mathbb{R}^n} \|Ax - b_i\|_p$ for $1 \leq i \leq 4$. Then one can verify that $Z_1 = 1 + \alpha$ and $Z_2 = 1$. On the other hand, $Z_3 > 100$ but $Z_4 = \sqrt[2p]{2p} + 1$. Thus, $\ell_p$ regression is not smooth as per Definition A.1 for any reasonable parameters $(\alpha, \beta)$ that gives us a constant factor approximation.

Recall that the vector induced $\ell_p$ matrix norm is defined as

$$\|A\|_p = \max_{\|x\|_p = 1} \|Ax\|_p,$$

and for $p = 2$, it is the same as the spectral norm (Schatten-$\infty$ norm). We now show that vector induced norms are not smooth for all values of $p$.

Theorem B.3. The vector induced matrix norm $\|\cdot\|_p$ is not a smooth function as per Definition A.1 for a meaningful parameters $(\alpha, \beta)$ for constant factor approximation.
Proof. We give explicit construction of streams to show that there exists a stream for which the vector induced matrix norm is not smooth as per Definition A.1. Let $\mathbf{e}_1, \cdots, \mathbf{e}_4$ be the standard basis of $\mathbb{R}^n$. Let $A$ be the data stream consisting of $\{\mathbf{e}_1\}$, followed by the suffix $B$, the data stream consisting of $\{\mathbf{e}_2\}$. Let $C$ be the data stream consisting of $\{\mathbf{e}_1\}$. Define $A$ to be the matrix representing $A$ and $B$ to be the matrix representing $B$. Let $R$ be the matrix representing $A \cup C$ and $S$ be the matrix representing $B \cup C$. Thus,

$$
W := A^\top A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, X := B^\top B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
$$

$$
Y := R^\top R = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, Z := S^\top S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
$$

Observe that

$$
\|W\|_p = \max_{\|x\|_p = 1} \|Wx\|_p
$$

Thus, $\|W\|_p = 1$. Similarly, $\|X\|_p = 1$ so that $(1 - \alpha) \|W\|_p \leq \|X\|_p$ for any $0 < \alpha < 1$. On the other hand, $\|Y\|_p = 4$ and $\|Z\|_p = 1$. Hence, the vector induced matrix norm $\|\cdot\|_p$ is not a smooth function as per Definition A.1. \[\square\]