THEORY AND ASTROPHYSICAL CONSEQUENCES OF A MAGNETIZED TORUS AROUND A RAPIDLY ROTATING BLACK HOLE

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ABSTRACT

We analyze the topology, lifetime, and emissions of a torus around a black hole formed in hypernovae and black hole–neutron star coalescence. The torus is ab initio uniformly magnetized, represented by two counteroriented current rings, and develops a state of suspended accretion against a "magnetic wall" around the black hole. Magnetic stability of the torus gives rise to a new fundamental limit \( \delta_a/\delta_k < 0.1 \) for the ratio of poloidal magnetic field energy to kinetic energy, corresponding to a maximum magnetic field strength \( B_\ell \simeq (10^{16} \text{ G})/(7 M_\odot/M_H)(6 M_H/R)^2(M_T/0.03 M_H)^{1/2} \). The lifetime of rapid spin of the black hole, effectively defined by the timescale of dissipation of spin energy \( E_{\text{rot}} \) in the horizon, hereby satisfies \( T \simeq (40 \text{ s}) (M_H/7 M_\odot)(R/6 M_H)^3(0.03 M_H/M_T) \) for a black hole of mass \( M_H \) surrounded by a torus of mass \( M_T \) and radius \( R \). The torus converts a major fraction \( E_{\text{gw}}/E_{\text{rot}} \simeq 10^{-3} \) into gravitational radiation through a finite number of multipole mass moments and a smaller fraction into MeV neutrinos and baryon-rich winds. At a source distance of 100 Mpc, these emissions over \( N = 2 \times 10^4 \) periods give rise to a characteristic strain amplitude \( N^{1/2} h_{\text{char}} \simeq 6 \times 10^{-21} \). We argue that torus winds create an open magnetic flux tube on the black hole, which carries a minor fraction \( E_{\ell}/E_{\text{rot}} \simeq 10^{-3} \) in baryon-poor outflows to infinity. We conjecture that these are not high-\( \sigma \) outflows, owing, in part, to magnetic reconnection in surrounding current sheets. The fraction \( E_{\ell}/E_{\text{rot}} \simeq 1/2 (M_H/R)^3 \) is standard for a universal horizon half-opening angle \( \theta_H \simeq M_H/R \) of the open flux tube. We identify this baryon-poor output of tens of seconds with gamma-ray bursts with contemporaneous and strongly correlated emissions in gravitational radiation, conceivably at multiple frequencies. Ultimately, this leaves a black hole binary surrounded by a supernova remnant.

Subject headings: black hole physics — gamma rays: bursts — gravitational waves

1. INTRODUCTION

Black holes surrounded by a magnetized torus or disk are believed to constitute the central engines that power various high-energy sources, notably active galactic nuclei, galactic microquasars, and gamma-ray bursts. The latter systems are thought to be the outcome of catastrophic events such as core collapse in massive stars and black hole–neutron star coalescence (Eichler et al. 1989; Woosley 1993; Paczyński 1991, 1998) and are of interest as potentially the most extreme and short-lived black hole–torus systems.

We describe a theory for the topology, lifetime, and emissions of black hole–torus systems as a function of three parameters: the mass \( M_H \) of an extreme Kerr black hole, the radius \( R \), and the mass \( M_T \) of the torus. We shall do so largely by studying the torus by equivalence to pulsars, in both topology and millisecond rotation periods. The energy emissions are powered by the spin energy of the black hole (Kerr 1963). Most of the black hole luminosity—the rate at which the black hole deposits energy into its surroundings in all channels—is incident on the torus, which hereby creates a major energy output of the system. A minor energy output is released in baryon-poor outflows through an open magnetic flux tube along the spin axis of the black hole. The lifetime of these black hole–torus systems is identified with the lifetime of rapid spin of the black hole in a state of suspended accretion (van Putten & Ostriker 2001) against a "magnetic wall" around the black hole (van Putten 1999).

The suspended accretion state results from a strong coupling of the torus to the spin energy of the black hole. We point out that this mechanism is based on a uniform magnetization of the torus, represented by two oppositely oriented current rings (van Putten 1999). This magnetization is a natural outcome of both black hole–neutron star coalescence and core collapse in hypernovae.

In this paper we quantify (1) the lifetime of rapid spin of the black hole in terms of a new magnetic stability criterion for the torus, (2) baryon-rich outflows from the torus at MeV temperatures, and (3) the fraction of black hole spin energy in baryon-poor outflows through an open magnetic flux tube on the black hole, created from outer layers of the inner torus magnetosphere by these torus winds.

The torus is luminous in various channels, which are strongly correlated by the properties of the torus: in gravitational radiation, winds, and thermal and MeV neutrino emissions (van Putten 2001b; van Putten & Levinson 2002). Calorimetric constraints on the torus winds hereby obtain predictions for the proposed emissions in gravitational radiation, while calorimetry on the gravitational wave emissions by upcoming gravitational wave experiments obtains a method for identifying Kerr black holes as objects in nature (van Putten & Levinson 2002).

Gravitational radiation forms a major output of the system and the dominant output of the torus (van Putten 2001b; van Putten & Levinson 2002). This is emitted by a finite number of multipole mass moments in a torus of finite...
slenderness, as a result of the Papaloizou-Pringle instability (Papaloizou & Pringle 1984; van Putten 2002) and, conceivably, other wave modes in the torus. These emissions are candidate sources for the upcoming gravitational wave experiments by laser interferometric instruments LIGO (Abramovici et al. 1992), VIRGO (Bradaschia et al. 1992), TAMA (Masaki et al. 2001), and GEO (Schultz & Papa 1999). The frequency in gravitational radiation is determined by the Keplerian frequency of the torus. The latter is strongly correlated to the output energy in torus winds. This suggests the possibility of performing calorimetry on the impact of these torus winds on the remnant stellar envelope (van Putten 2003) and on hypernova remnants, in order to constrain the expected frequency in gravitational radiation.

Baryon-poor outflows form a small fraction of the total output from the black hole through an open magnetic flux tube (van Putten & Levinson 2002). We here attribute the formation of this open flux tube to powerful baryon-rich torus winds, driven from the surface by escaping MeV neutrinos. This neutrino output provides the dominant cooling torus winds, driven from the surface by escaping MeV neutrinos. This neutrino output provides the dominant cooling.

Tentative observational evidence for a black hole luminosity incident into surrounding matter in the case of supermassive black holes is found in MCG – 6-30-15 (Wilms et al. 2001) and in the case of stellar mass black hole candidates in the galactic source XTE J1650–500 (Miller et al. 2002). Independent observational evidence of the presence of magnetic fields remains elusive. We point out, however, that in our model heating of the torus is due to viscous shear between its inner and outer faces, possibly in the form of magnetohydrodynamical turbulence, which is different from electric dissipation as envisioned in Wilms et al. (2001).

In §§ 2 and 3 we describe the topology and stability of the magnetosphere of a torus formed in black hole–neutron star coalescence and hypernovae, the suspended accretion state around rapidly rotating black holes, and the secular timescale of its rapid spin. The formation of a finite number of multipole mass moments by the Papaloizou-Pringle instability in tori of finite slenderness is summarized in § 4. Energy emissions in various channels by the torus are described in § 5. We calculate the neutrino-driven mass-loss rate from the surface of the torus in § 6. These baryon-poor torus winds have some implications for energy extraction from the black hole and the creation of open magnetic flux tubes from outer layers in the inner torus magnetosphere. In §§ 7–9 we describe the structure of the inner flux tube supported by the black hole and dissipation by magnetic reconnection in its interface with the surrounding outer flux tube. We conclude with observational consequences in § 10.

2. FORMATION AND STRUCTURE OF THE TORUS MAGNETOSPHERE

A torus formed in core collapse of a massive star or black hole–neutron star coalescence will be magnetized with a remnant of the progenitor magnetic field. An aligned poloidal magnetic field in the progenitor star provides a magnetic moment density in the torus, aligned with its axis of rotation. Equivalently, the magnetic field in the torus is produced by two concentric current loops with opposite orientation. In the case of a torus formed from the breakup of a neutron star around a black hole, these two current loops form out of a single current loop representing the magnetization of the neutron star, upon stretching the latter around the black hole followed by a reconnection (see Fig. 1). Conceivably, the magnetic field in the torus is amplified by winding or a dynamo process. The stability of this magnetic field configuration is discussed in § 3 and sets an upper limit of $B \approx 10^{16}$ G on the field strength.

Below we outline the resulting magnetosphere of the black hole–torus system in the force-free limit. For illustrative purposes we consider first the poloidal topology of the vacuum magnetic field configuration. We proceed by discussing the more realistic situation of a force-free magnetosphere and show that the inner and outer faces of the torus are each equivalent to a pulsar with, however, generally different angular velocities.

In the subsequent sections we shall consider some essential energetic aspects in greater detail. In particular, it will be shown that appreciable matter outflows are expected along open field lines to infinity, and a reconnection boundary is expected to form near the rotation axis. This may give rise to a non–force-free magnetic field in those regions.

2.1. Topology of the Vacuum Magnetic Field

A uniform magnetization of the torus is approximately described by two counteroriented current rings in the equatorial plane. A third current loop is associated with the black hole, representing its equilibrium magnetic moment in its lowest energy state (van Putten 2001a). This induced magnetic moment is oriented antiparallel to the magnetic moment of the torus, facilitating an essentially uniform and maximal horizon flux at arbitrary spin rates. In flat spacetime, the magnetic field produced by a superposition of current rings can be calculated analytically (Jackson 1975, § 5.5). The topology of this ab initio flat spacetime vacuo
The magnetic field is shown in Figure 2. As seen, at large radii (compared with the radius of the outer current ring) it quickly approaches a dipole solution. In the inner region the field lines intersect the horizon, giving rise to a strong coupling between the black hole and the inner face of the torus. This topology and flux distribution are preserved in the face of general relativistic effects, as a result of the equilibrium magnetic moment of the black hole.

2.2. Equivalence to Pulsars

By vacuum breakdown, the flux surfaces will evolve with electric charges to a largely force-free state similar to pulsars (Goldreich & Julian 1969). As a result, a magnetosphere develops that consists of conductive flux surfaces and magnetic winds. The torus hereby supports an inner and an outer torus magnetosphere. These are equivalent in poloidal topology to pulsar magnetospheres, wherein the horizon of the black hole is equivalent to a compactified infinity with nonzero angular velocity (Fig. 3). This equivalence implies similar, causal interactions by magnetic winds acting on the inner and outer faces of the torus.

In the force-free limit, the flux surfaces in the outer/inner torus magnetosphere assume rigid corotation with the outer/inner face of the torus by no-slip/no-slip boundary conditions. The last closed field line of the inner torus magnetosphere reaches the light cylinder associated with the angular velocity of the outer face of the torus, similar to the last closed field lines in pulsar magnetospheres. The last closed field line of the inner torus magnetosphere reaches the inner light surface (Znajek 1977) associated with the angular velocity of the inner face of the torus (Fig. 2 of van Putten 1999). Beyond, field lines are open and extend to infinity or to the horizon of the black hole with no-slip/slip boundary conditions. The former are created by torus winds that cross the outer light cylinder.

2.3. Suspended Accretion against a Magnetic Wall

Most of the black hole luminosity is incident on the torus (van Putten 1999). The torque exerted on the inner face of the torus by the black hole is obtained by integrating the angular momentum flux \( \mathcal{J}^\theta = F^\theta F_{\phi \theta}/4\pi \) over the section of the horizon that is threaded by magnetic field lines that are anchored to the torus:

\[
T_+ = 4\pi \int_{\theta_H}^{\pi/2} \sqrt{-g} \mathcal{J}^\theta d\theta = (\Omega_H - \Omega_+) \times \int_{\theta_H}^{\pi/2} \frac{\Sigma}{\rho} \sin \theta \left(F_{\phi \theta}\right)^2 d\theta,
\]

where equation (A18) has been used. Here \( \theta_H \) is the angle of the last field line that connects the torus and the horizon (see Fig. 3). \( \Omega_+ \) and \( \Omega_H \) are the angular velocities of the inner face of the torus and the black hole, respectively, and the metric components \( \rho \) and \( \Sigma \) are defined in the Appendix. In terms of the net poloidal magnetic flux associated with the open field lines in the torus, \( 2\pi A \), we may write

\[
T_+ = (\Omega_H - \Omega_+) f_H^2 A^2.
\]

Formally \( f_H \) is defined through equation (1). Likewise, the torque exerted on the outer face by field lines that extend to infinity can be expressed as

\[
T_- = \Omega_- f_w^2 A^2,
\]

where \( \Omega_- \) is the angular velocity of the outer face of the torus and \( f_w \) is the fraction of open field lines that extend to infinity. It is seen that when \( \Omega_H > \Omega_+ \), angular momentum is transferred from the black hole to the torus, tending to spin up the inner face, whereas in the slowly rotating case (\( \Omega_H < \Omega_+ \)) the black hole receives angular momentum from the torus (Fig. 4). The outer face always loses angular momentum via a wind to infinity. By mechanical work, the magnetic torus winds to infinity and into the
2.4. Frame Dragging and Electric Charge Distribution

The electric charge distribution can be obtained from Maxwell’s equation: \( F_{\mu\nu} = \frac{4\pi}{c} j^\mu \). Assuming the radial magnetic field, \( B_r = -F_{\theta r}/(\Sigma \sin \theta) \), to be independent of \( \theta \), we obtain from equation (A19) the Goldreich-Julian charge density near the horizon, on field lines that emanate from the inner face:

\[
\rho_e = \alpha^2 j^\theta = -\frac{(\Omega_+ + \beta) B_r \cos \theta}{2\pi},
\]

where \( \beta \) is defined below equation (A1) and equals \( -\Omega_H \) on the horizon. We find that for \( \Omega_H > \Omega_+ \) the charge density changes sign on field lines threading the horizon. The same holds true also for the inner flux tube that extends to infinity (see discussion following eq. [44]).

The charge distribution given by equation (3) represents a bifurcation from the magnetosphere around a Schwarzschild black hole. Ingoing boundary conditions and outgoing boundary conditions at infinity may hereby carry a continuous electric current. If the black hole rotates slower than the inner face of the torus, the inner light surface is absent and the sign of the charge distribution near the torus carries through to the horizon (Fig. 5a). If the black hole rotates faster than the inner face of the torus, the inner light surface becomes apparent (a bifurcation from slow rotation) and, by foregoing arguments, introduces a sign change in the charge distribution (Fig. 5b).

3. The Lifetime of Rapid Spin of the Black Hole

A magnetized torus of a few tenths of solar masses around a stellar mass black hole of about 7 \( M_\odot \) is subject to magnetic self-interaction and a stabilizing tidal interaction in the central potential well. We here derive limits on the average strength of a poloidal magnetic field that can be supported by the torus. We do not consider the problem of stability of a poloidal magnetic field itself, such as the magnetorotational (MRI) instability. A limit on the energy in
poloidal magnetic field defines a lower bound on the lifetime of rapid spin of the black hole.

In regards to wave motion within the equatorial plane, the contribution of poloidal magnetic fields is that of magnetic pressure, which is generally stabilizing on the motion of the fluid. In regards to poloidal wave motion, a poloidal magnetic field generally conspires toward instabilities. This can be calculated by partitioning the torus in a finite number of fluid elements with current loops, representing local magnetic moments. The two leading-order partitions are shown in configurations C and B1 of Figure 1, for which we derive critical magnetic field strengths. The first is subject to magnetic tilt instability between the inner and the outer face, and the second is subject to a magnetic buckling instability.

3.1. A Magnetic Tilt Instability

Following C in Figure 1, consider the magnetic interaction energy of a pair of concentric current rings, given by

$$U_\mu(\theta) = -\mu B \cos \theta . \quad (4)$$

Here $\mu$ is the magnetic moment of the inner ring, $B$ is the magnetic field produced by the outer ring, and $\theta$ denotes the angle between $\mu$ and $B$. Note that $U_\mu(\theta)$ has a period of $2\pi$ and is maximal (minimal) when $\mu$ and $B$ are antiparallel (parallel; see Fig. 6). Consider tilting a fluid element of a ring out of the equatorial plane to a height $z$ along a cylinder of radius $R$. (This is different from tilting a rigid ring, whose elements move on a sphere.) A tilt hereby changes the distance to central black hole to $\rho = (R^2 + z^2)^{1/2} \approx R(1 + z^2/2R^2)$. In the approximation of equal mass in the inner and outer faces of the torus, simultaneous tilt of one ring upward and the other ring downward is associated with the potential energy

$$U_\mu(\theta) \simeq -\frac{M_T M_H}{R} \left[ 1 - \frac{1}{4} \tan^2 \left( \frac{\theta}{2} \right) \right] , \quad (5)$$

with $\tan(\theta/2) = z/R$, where we averaged over all segments of a ring. Note that $U_\mu(\theta)$ has period $\pi$ and is minimal when $\theta = 0$. Stability is accomplished provided that the total potential energy $U(\theta) = U_\mu(\theta) + U_\phi(\theta)$ satisfies

$$\frac{d^2 U}{d\theta^2} > 0 . \quad (6)$$

The potential $U(\theta)$ is shown in Figure 7, which shows the bifurcation at $b = 1/12$ of the stable equilibrium $\theta = 0$ into an unstable equilibrium with the appearance of two neighboring stable equilibria at nonzero angles. The bifurcation point is therefore second order. Nevertheless, the torus may become nonlinearly unstable at large angles ($b \gg b^*$). We therefore consider below the physical parameters at this bifurcation point.

For two rings of radii $R_o$ with $(R_+ - R_-)/(R_+ + R_-) = O(1)$, we have $U_\mu \simeq \frac{1}{2} B^2 R^3 \cos \theta$, so that equation (6) gives $B_c^2 M_H^2 = \frac{1}{2} (M_H/R)^4 (M_T/M_H)$, or

$$B_c \simeq (10^{16} \text{ G}) \left( \frac{7 M_\odot}{M_H} \right) \left( \frac{6 M_H}{R} \right)^2 \left( \frac{M_T}{0.03 M_H} \right)^{1/2} . \quad (7)$$

The critical value of the ratio of poloidal magnetic energy ($\delta_B = f_B B^2 R^3 / 6$) to kinetic energy ($M_T M_H/2R$) in the torus becomes

$$\frac{\delta_B}{\delta_k} = \frac{f_B}{12} , \quad (8)$$

where $f_B$ denotes a factor of order unity, representing the volume of the inner torus magnetosphere as a fraction of $4\pi R^3/3$. We emphasize that the limit given by equation (8) is fundamental, independent of the mass of the black hole and the mass and radius of the torus. This result can equivalently be attributed to stable balance of Lorentz forces against tidal forces, preventing a small misalignment to cause a tilt.
The poloidal magnetic field introduces an anisotropic pressure tensor. At the critical magnetic field strength (eq. [7]), pressure components in the equatorial plane are predominantly magnetic rather than thermal at MeV temperatures. The poloidal pressure components (along the magnetic field lines) are thermal pressures. In equilibrium, these poloidal pressure components are unaffected by poloidal tidal forces if the poloidal flux surfaces assume a spherical shape inside of the torus.

The magnetic field may be generated in response to the power received from the black hole. If so, then \( d\omega_a/dt \leq L_H \), where (Thorne et al. 1986; van Putten 1999)

\[
L_H \approx \frac{1}{2} \alpha_B B^2 M_H^2
\]

(9)
denotes the black hole luminosity expressed in terms of a fraction \( \eta = \Omega_T/\Omega_H \) of the angular velocity of the torus to that of the black hole. The discrepancy \( L_H - d\omega_B/dt \) is carried away in the various channels, as described in the suspended accretion state, and will be zero in equilibrium. This gives rise to a minimum \( c \)-falling time

\[
\tau_B = \frac{3}{2} B^{-2} \left( \frac{R}{M_H} \right)^2 \approx (0.2 \text{s}) \left( \frac{\eta}{0.1} \right)^{-1} \left( \frac{R}{M_H^7/3} \right)^3 \left( \frac{M_H}{7 M_\odot} \right)
\]

(10)

A critical magnetic field of \( B_c \approx 10^{16} \text{ G} \) is hereby reached on a timescale of at least a few seconds. The suspended accretion state may thereby be intermittent on a timescale of seconds, associated with magnetic field build-up powered by the rotational energy of the black hole.

Most of the rotational energy is dissipated in the horizon, creating entropy \( S \) for a black hole temperature \( T_H \) at a maximal rate \( T_H S \approx B^2 M^2 / 32 \) (Thorne et al. 1986). The lifetime of rapid spin of the black hole becomes effectively the timescale of dissipation of black hole spin energy \( E_{\text{rot}} \approx M_H^3 / 3 \) in the horizon; i.e.,

\[
T \approx \frac{E_{\text{tot}}}{T_H S} \geq (40 \text{s}) \left( \frac{M_H}{7 M_\odot} \right) \left( \frac{R}{M_H^7/3} \right)^4 \left( \frac{0.03 M_H}{M_T} \right)
\]

(11)

where equation (7) has been employed.

The suspended accretion state, conceivably intermittent on a timescale (10), lasts for a duration similar to equation (11) until the innermost stable circular orbit reaches the torus or until \( \Omega_H \approx \Omega_T \), whichever comes first.

### 3.2. A Magnetic Buckling Instability

We partition the magnetization of the torus into \( N \) equi-
distant fluid elements with dipole moments, \( \mu_i = \mu/N = (1/2) B R^3 / N \). We consider the vertical degree of freedom of fluid elements that move to a height \( z \) above the equatorial plane. By conservation of angular momentum, this motion is restricted to a cylinder of constant radius. Their position vectors in and off the equatorial plane will be denoted by

\[
\begin{align*}
\mathbf{r}_i^* &= (R \cos \phi_i, R \sin \phi_i, 0), \\
\mathbf{r}_i &= (R \cos \phi_i, R \sin \phi_i, z_i), \quad \phi_i = \frac{2 \pi i}{N}.
\end{align*}
\]

(12)

A fluid element \( i \) assumes an energy that consists of magnetic moment–magnetic moment interactions and the tidal interaction with the central potential well. The total potential energy of the \( i \)th fluid element is given by

\[
U_i = -\frac{\mu_i B^2}{N} \sum_{j 
eq i} \left| \frac{\mathbf{r}_i^* - \mathbf{r}_j^*}{|\mathbf{r}_i^* - \mathbf{r}_j^*|} \right|^3 \cos \theta_{ij} + U_g(\theta_i),
\]

(13)

where \( B^2 = B/N^2 \) denotes the magnetic field strength of a magnetic dipole at distance \( d = 2n R/N, \theta_d \) denotes the angle between the \( i \)th magnetic moment and the local magnetic field of the \( j \)th magnetic moment, and \( U_g = -(M_T M_H/RN) (1 - z_i^2 / 2R^2) \) is the tidal interaction of the \( i \)th fluid element with the black hole. Here \( N^2 \) is a factor of order \( N \) that satisfies the normalization condition \( \sum_i U_i = -\mu B \) (in equilibrium). Upon neglecting azimuthal curvature in the interaction of neighboring magnetic moments, we have a magnetic moment–magnetic moment interaction

\[
\mu_i B^2 \left| \frac{\mathbf{r}_i^* - \mathbf{r}_j^*}{|\mathbf{r}_i^* - \mathbf{r}_j^*|^3} \right| \cos \theta_{ij} \approx \mu_i B^2 \left[ 1 - \left( 1 + \frac{3}{2 |i - j|^2} \right) \alpha_{ij}^2 \right],
\]

(14)

where \( \alpha_{ij} = (z_i - z_j) / d \),

\[
\cos \theta_{ij} = -\sqrt{1 - \frac{\alpha_{ij}^2}{1 + \alpha_{ij}^2}} \approx -(1 - \alpha_{ij}^2),
\]

(15)

and \( |\mathbf{r}_i^* - \mathbf{r}_j^*| \approx |i - j|d(1 + \alpha_{ij}^2 / 2 |i - j|^2) \).

We shall use a small-amplitude approximation, whereby \( z_i / R = \tan \theta_i \approx \theta_i \). We study the stability of this configuration to derive an upper limit for the magnetic field strength. An upper limit is obtained by taking into account only interactions between neighboring magnetic moments. (The sharpest limit is obtained by taking into account interactions between one magnetic moment and all its neighbors.) Thus, we have \( N^2 = 2N \) and consider the total potential energy

\[
U_i = \frac{\mu_i B^2}{N} \sum_{j \neq i} \left( 1 - \frac{5}{2} \alpha_{ij}^2 \right) + U_g(\theta_i),
\]

(16)

where \( \alpha_{ij} = N(\theta_i - \theta_j) / 2\pi \). The Euler-Lagrange equations of motion are

\[
\frac{M_T R \dot{\theta}_i}{N} + \frac{\partial U_i}{\partial \theta_i} = 0.
\]

(17)

This defines the system of equations for the vector \( x = (\theta_1, \theta_2, \ldots, \theta_N) \) given by

\[
\frac{M_T R^2}{N^2} x = \frac{5 \mu B}{2N} \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & -1 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ -1 & 0 & \ldots & 0 & -1 & 2 \end{pmatrix} x.
\]

(18)

The least stable eigenvector is \( x = (1, -1, 1, \ldots, -1) \) (for \( N \) even), for which the critical value of the magnetic field is

\[
B_c^2 M_H^2 = \frac{1}{5} M_T^4 M_H^4 \left( \frac{M_H}{M_T} \right)^4.
\]

(19)

This condition is very similar to equation (7) and gives a
A torus tends to develop instabilities in response to shear, which can be studied analytically in the approximation of incompressible fluid about an unperturbed angular velocity

$$\Omega = \Omega_0 \left( \frac{\delta}{\delta_0} \right)^q,$$

where $q \geq 3/2$ denotes the rotation index and $a = (R_+ + R_-)/2$ denotes the major radius of the torus. In the inviscid limit, irrotational modes in response to initially irrotational perturbations to the underlying flow (vortical if $q \neq 2$) show the Papaloizou-Pringle instability (Papaloizou & Pringle 1984) to also operate in wide tori (van Putten 2002). The neutral stability curves of the resulting buckling modes are described by a critical rotation index $q_c = q_c(\delta, m)$ as a function of the slenderness parameter $\delta = b/2a$, where $b = (R_+ - R_-)/2$ denotes the minor radius of the torus and $a = (R_+ + R_-)/2$ the mean radius (Fig. 9). Quadratic fits to the stability curves are

$$q_c(\delta, m) = \begin{cases} 
0.10 \left( \frac{\delta}{0.1} \right)^2 + 1.73 & (m = 2), \\
0.26 \left( \frac{\delta}{0.1} \right)^2 + 1.73 & (m = 3), \\
0.50 \left( \frac{\delta}{0.1} \right)^2 + 1.73 & (m = 4), \\
0.80 \left( \frac{\delta}{0.1} \right)^2 + 1.73 & (m = 5), \\
0.034 m^2 \left( \frac{\delta}{0.1} \right)^2 + 1.73 & (m > 5).
\end{cases}$$

Instability sets in above these curves, stability below. For

![Fig. 9.—Diagram showing the neutral stability curves for the nonaxisymmetric buckling modes in a torus of incompressible fluid, as an extension of the Papaloizou-Pringle instabilities to finite slenderness ratios $\delta = b/2a > 0$, where $b$ and $a$ denote the minor and major radius of the torus, respectively. Curves of critical rotation index $q_c$ are labeled with azimuthal quantum numbers $m = 1, 2, \ldots$, where instability sets in above and stability sets in below. The central pressure produced by super- and sub-Keplerian motions of the inner and outer faces of the torus is balanced by magnetic and thermal pressure at MeV temperatures. At finite slenderness $0 < b/a < 1$, this produces instability to a finite number of Papaloizou-Pringle modes (reprinted from van Putten 2002).](image-url)
m = 2, the critical value \( q_c = 2 \) is obtained for \( \delta = 0.16 \), associated with the Rayleigh stability criterion for the azimuthally symmetric wave mode \( m = 0 \). For large \( m \), we use the numerical result of critical values \( \delta = 0.28/m \) for \( q = 2 \), where \( m \) denotes the azimuthal wavenumber.

It will be appreciated that the torus is \( m = 0 \) stable for perturbations of its radius (in the mean). This is due to the frozen-in condition of magnetic flux surfaces. In terms of angular momentum transport, the horizon magnetic flux \( \propto (M/a)^2 \) hereby defines a black hole–to–torus coupling which is dominant over the coupling \( \propto \Omega_T \simeq (M/a)^{3/2} \) of the torus to infinity.

The central pressure of the torus is produced by differential rotation, as a result of a state of super-Keplerian motion of the inner face and a state of sub-Keplerian motion of the outer face. This pressure is balanced by both magnetic and thermal pressure at MeV temperatures, which gives rise to finite slenderness \( \delta > 0 \) and hence instability to a finite number of Papaloizou-Pringle modes. Thermal pressure alone gives the estimate (adapted from van Putten 2002)

\[
q \approx 1.5 + 0.2 \left( \frac{0.1}{\delta} \right)^2 \left( \frac{kT}{2 \text{ MeV}} \right).
\]

(23)

Magnetic pressure enhances this estimate to larger values of \( q \). This shows that a torus of finite slenderness (\( \delta < 0.1 \)) and a radius around \( 5M \) is unstable \( (q > \sqrt{3}) \) to the formation of a finite number of multipole mass moments by the Papaloizou-Pringle instability. A torus with multiple mass moments potentially defines a gravitational wave spectrum consisting of several lines.

5. ENERGY EMISSIONS BY THE TORUS

The suspended accretion state in the case of a symmetric flux distribution, given by equal fractions of open magnetic flux on the inner and the outer face, gives rise to remarkably simple expressions for the predicted energy output in the relevant channels. To leading order the angular velocities of the inner and outer faces are given in terms of the mean angular velocity of the torus, \( \Omega_T = (\Omega_+ + \Omega_-)/2 \), and the slenderness ratio, \( \delta \), as

\[
\Omega_{\pm} = \Omega_T (1 \pm \delta).
\]

Equation (2) implies that for a small slenderness ratio, \( \delta \ll 1 \), the resulting mass ejection is to that of the black hole. In the limit of strong magnetohydrodynamical viscosity and small slenderness ratio we then have the asymptotic results (van Putten 2003)

\[
\frac{E_{gw}}{E_{\text{rot}}} \sim \eta \gamma, \quad \frac{E_w}{E_{\text{rot}}} \sim \eta^2 \gamma, \quad \frac{E_d}{E_{\text{rot}}} \sim \eta \delta,
\]

(24)

where \( E_{gw} \), \( E_w \), and \( E_d \) are defined below.

**Gravitational radiation.—** The major energy output from the torus is in gravitational radiation,

\[
E_{gw} = (6 \times 10^{53} \text{ ergs}) \left( \frac{\eta}{0.1} \right) \left( \frac{M_H}{10 M_{\odot}} \right).
\]

(25)

A quadrupole buckling mode radiates gravitational waves at close to twice the angular frequency of the torus (van Putten 2002). These emissions are relatively powerful, representing about 10% of the rotational energy of the black hole for the lifetime of the system. The associated mass inhomogeneity \( \delta M_T \) in the torus assumes a value commensurate with the inferred luminosity in gravitational radiation. For quadrupole emissions, we have

\[
E_{gw} \simeq (32/5)(M_H/R)^4 \left( \delta M_T/M_H \right)^2 /R^{3/2},
\]

where \( \omega \approx M_H^{1/2}R^{-3/2} \) denotes the orbital angular frequency at a radius \( R \), approximating a circular motion. The estimated gravitational wave luminosity is hereby produced by \( \delta M_T \simeq 0.5\% M_H (R/5M_H)^{3/4} \). This corresponds to a mass inhomogeneity of 20% for a torus \( M_T = 0.2 M_H \) around a black hole of mass \( M_H = 7 M_H \).

**Torus winds.—** The energy output in torus winds is a factor \( \eta \) less than that in gravitational radiation, or

\[
E_w = (6 \times 10^{52} \text{ ergs}) \left( \frac{\eta}{0.1} \right)^2 \left( \frac{M}{10 M_{\odot}} \right).
\]

(26)

These powerful torus winds may produce hypernova remnants in the host environment, e.g., a shell associated with a molecular cloud. They may be baryon loaded and deposit some torus matter onto the companion star as in the Brown et al. (2000) association of hypernovae to soft X-ray transients, and they may be important in collimating baryon-poor jets produced by the black hole. Finally, these winds are potentially relevant in \( r \)-processes (Levinson & Eichler 1993). This suggests considering observational methods to determine \( E_{gw} \), from which to determine the system parameter \( \eta \), and hence the frequency of gravitational radiation. Of potential interest are model-dependent estimates of \( E_d \), from their role in collimating outflows and calorimetry on hypernova remnants. The first can be pursued using existing data on gamma-ray burst (GRB) beaming angles, which suggests a value of \( \eta \simeq 0.1 \). The second method is potentially more reliable but awaits further study on the identification and observational aspects of hypernova remnants.

**Torus temperature.—** The energy output in thermal and neutrino emissions is a factor \( \delta \) less than that in gravitational radiation, or

\[
E_d = (10^{53} \text{ ergs}) \left( \frac{\delta}{0.15} \right) \left( \frac{M_H}{10 M_{\odot}} \right).
\]

(27)

Here we refer to a fiducial value of \( \eta \simeq 0.1 \) as before, as well as a value \( \delta \simeq 0.15 \) or less. The latter is suggested by quadrupole radiation of gravitational waves, which requires \( \delta \leq 0.16 \) at the threshold of Rayleigh stability, according to equation (22). This dissipation rate corresponds to a temperature of a few MeV (see eq. [29]), which thereby produces baryonic winds. The torus winds considered here, therefore, are baryon-rich.

6. PRESSURE-DRIVEN MASS EJECTION FROM THE TORUS

It has been argued in § 5 that a fraction \( E_d/E_{\text{rot}} \) of the black hole spin energy is dissipated in the torus, thereby heating it to a temperature in excess of a few MeV. This drives a powerful wind by the pressure gradients in the surface layers of the torus. The resulting mass ejection opens magnetic field lines that pass through the outer Alfven point and folds some of those in the outer layers of the inner magnetosphere. This creates a change in poloidal topology in the form of a coaxial structure of open flux tubes with opposite magnetic orientation (toward infinity). Because the torus is rotating and magnetized, the ejection of the wind is partially anisotropic. Specifically, mass flux is generally suppressed along magnetic field lines that are inclined toward
the rotation axis and enhanced along field lines that are strongly inclined away from the axis (Blandford & Payne 1982; Romanova et al. 1997), as a result of centrifugal forces. The details of the outflow depend on the heating and cooling rate of the corona and on its structure, and analysis thereof is beyond the scope of the present discussion. In what follows, we provide a rough estimate for the mass flux expelled in the vertical direction, speculate on the implications for opening of magnetic field lines, and discuss the consequences of mass ejection in the equatorial plane for the energy extraction process.

6.1. An Estimate of the Vertical Mass Flux

A torus having a temperature in excess of a few MeV and an average density \( \rho_0 \sim 10^{11}(M_T/0.1 M_{\odot})/V_{21} \) g cm\(^{-3}\), where \( V = 10^{21}V_{21} \) cm\(^3\) is the volume of the torus, cools predominantly by neutrino emission through electron and positron capture on nucleons. The cooling rate is given by (e.g., Bethe & Wilson 1985)

\[
e_{\text{cap}} \simeq 10^{29} \rho_0 T_6^6 \text{ ergs s}^{-1} \text{ cm}^{-3},
\]

where \( T_{10} \) is the average temperature in units of 10\(^{10}\) K. For a total energy dissipation rate of \( L = 10^{52}L_{52} \) ergs, we then find an average temperature of

\[
T_{10} \simeq 2L_{52}^{1/6} \left( \frac{M_T}{0.1 M_{\odot}} \right)^{-1/6}.
\]

As will be shown below, the mass flux from the surface depends on the temperature profile in the neighborhood of the flow critical point, where the density is well below the average. The latter can be determined in principle by equating the local heating and cooling rates, provided that adiabatic cooling there can be neglected (which we find to be justified only if the temperature exceeds \( \sim 2 \times 10^{10} \) K). While the fraction of the black hole spin energy dissipated in the surface layers is not well constrained, a fraction of the energy of neutrinos escaping from the dense regions—well beneath the surface—will be deposited in the surface layers of the torus. The dominant absorption processes for the latter are neutrino capture on neutrons and protons and pair neutrino annihilation into electron-positron pairs. These processes typically dominate at lower densities and higher temperatures. In spherical geometry, the heating rate is dominated by neutrino annihilation at densities below \( \rho = 6 \times 10^6 (L_{52}/R_{96}^2)^{1/2} (R_6/r)^3 \) g cm\(^{-3}\), where \( R_6 \) is the radius of the neutrino production region (Levinson & Eichler 1993). The cooling rate due to electron-positron capture on nucleons at densities below \( \rho = 5 \times 10^6 T_{10}^3 \) g cm\(^{-3}\) (Levinson & Eichler 1993). Taking into account both processes, we find, up to a geometrical factor, that the critical point (see eq. [37]) will be maintained roughly at the average temperature given by equation (29).

Now, the torus material should be a mixture of baryons and a light fluid (photons and electron-positron pairs in equilibrium). The light and baryonic fluids will be tightly coupled as a result of the large Thomson depth. Deep beneath its surface, the torus is in a hydrostatic equilibrium where the vertical gravitational force exerted on it by the black hole is supported by the baryon pressure \( p_b = n_b kT \).

At baryon densities

\[
\rho_b < \frac{p_{\text{up}}}{kT} \simeq 10^{11} T_{10}^3 \text{ g cm}^{-3},
\]

the light fluid pressure,

\[
p_l = 2 \times 10^{25} T_{10}^4 \text{ dyn cm}^{-2},
\]

exceeds the pressure contributed by the baryons. At sufficiently shallow layers, the light fluid pressure gradient overcomes the vertical gravitational force, and the matter starts to accelerate. The transonic flow should pass through a critical point. To estimate the mass flux, we calculate below the wind density and velocity at the critical point.

To simplify the analysis, we neglect general relativistic effects. Since we are merely interested in an estimate for the mass-loss rate, we need only to consider the properties of the flow in the neighborhood of the critical point. Since the latter is located well within the light cylinder, we neglect rotation of the torus and the toroidal magnetic field. (This is not valid in general but is used only in regards to mass ejection from the upper and lower faces of the torus.) Below we find that the flow becomes mildly relativistic at the critical point. The MHD limit applies, in view of high electric conductivity in the coupled light plus baryonic fluids, whereby the streamlines of the flow are along magnetic flux surfaces. Baryon number conservation, viz., \( \nabla \cdot (n_b u_b) = 0 \), where \( u_b \) denotes the poloidal four-velocity and \( n_b \) the baryon number density, and Maxwell’s equation, \( \nabla \times E = 0 \), imply that the flux of baryons per unit magnetic flux is conserved:

\[
\left( \frac{n_b u_b}{B_p} \right)' = 0.
\]

Here \( B_p \) is the poloidal magnetic field and the prime denotes a derivative along streamlines (i.e., \( u \cdot \nabla \)). In the limit of weak gravitational field, the projection of the momentum equation, \( T_{\psi}' = 0 \), on the poloidal direction and the use of equation (32) give (Camenzind 1986; Takahashi et al. 1990)

\[
u_p (1 - M^{-2}) u_p = - \frac{\gamma'}{1 - a_e^2} \left[ a_e^2 (\ln B_p)' + \psi' \right],
\]

where \( \gamma' = 1 + u_p^2 \) is the Lorentz factor of the flow, \( \psi = GM_H/c^2r^2 \) is the gravitational potential,

\[
a_e = \left( \frac{4p_l}{12p_l + 3p_b c^2} \right)^{1/2}
\]

is the sound speed of the mixed fluid (measured in units of \( c_s \) and \( M = u_p/c_s \), with \( c_s = a_e/(1 - a_e^2)^{1/2} \) being the sound four-velocity, is the corresponding Mach number. It is seen that the flow has a critical point at \( M = 1 \). We note that, under the assumption made above that the toroidal magnetic field can be ignored, this critical point coincides essentially with the fast magnetosonic point. We now suppose that the flow passes through this critical point, which should be true in the case of a wind expelled along open magnetic field lines. There the right-hand side of equation (33) must also vanish. The latter condition can be solved to yield the sound speed at the critical point:

\[
a_e^2 = - \frac{\psi'}{(\ln B_p)'} = \frac{GM_H}{c^2 r_e^2 (\ln B_p)^7},
\]

where \( r_e \sim a \) is the radius of the critical point. It is seen that the critical sound speed depends on the geometry of the field lines in the vicinity of the critical point. To obtain an
order-of-magnitude estimate, let us assume that \( r' \sim \xi_c/r_c \) and
\( \ln(B_p) \sim \xi_c^{-1} \), where \( \xi_c < r_c \) denotes the distance from
the torus midplane to the critical point along streamlines. Then
\[
\frac{a_{\text{crit}}}{c_s} = \left( \frac{GM_H}{c^2 r_c} \right)^{1/2} \left( \frac{\xi_c}{r_c} \right) = 0.3 \left( \frac{M_H}{10 M_\odot} \right)^{1/2} \left( \frac{r_c}{\xi_c} \right)^{1/2} \left( \frac{\xi_c}{r_c} \right).
\]
(36)
The critical density can be obtained now by employing equations (29), (34), and (36):
\[
\rho_{\text{crit}} = \frac{p_f}{c^2} \frac{4 - 12 \xi_c}{a_{\text{crit}}^2}
\approx 1 \times 10^{17} \left( \frac{M_H}{10 M_\odot} \right)^{-1} \left( \frac{M_T}{0.1 M_\odot} \right)^{-2/3}
\times r_c^2 \left( \frac{\xi_c}{r_c} \right)^2 L_{52}^{2/3} \text{ g cm}^{-3}.
\]
(37)
The associated mass flux is given by
\[
\rho_{\text{crit}} c_s \approx 1 \times 10^{17} \left( \frac{M_H}{10 M_\odot} \right)^{-1/2} \left( \frac{M_T}{0.1 M_\odot} \right)^{-2/3}
\times r_c^{1/2} \left( \frac{\xi_c}{r_c} \right) L_{52}^{1/3} \text{ g cm}^{-2} \text{ s}^{-1},
\]
(38)
which corresponds to a mass-loss rate of \( \dot{M} \approx 1 \times 10^{30} \text{ g s}^{-1} \)
for a surface area of \( A = 0.1 a^2 = 10^{13} \text{ cm}^2 \). We conclude that during the \( \sim 30 \text{ s} \) suspended accretion state the torus will be partially evaporated.

Finally, we note that the Alfvén Mach number at the critical point is
\[
M_A = \left( \frac{16 \pi p_f}{3 B_p^2} \right)^{1/2} \approx 0.07 \left( \frac{M_T}{0.1 M_\odot} \right)^{-1/3} L_{52}^{1/3} B_{p15}^{-1}.
\]
(39)
We thus conclude that, for our choice of torus parameters, the outflow is sub-Alfvénic (but not highly so) at the critical point.

### 6.2. Creation of Open Field Lines

The ejection of matter from the hot torus corona will result in the opening of some magnetic field lines in the outer layers of the inner torus magnetosphere (Fig. 10). By equation (39), plasma that streams along field lines that extend to large distances quickly reaches the Alfvén point. These field lines become nearly radial several scale heights above the torus, forming an open flux tube. For field lines that converge toward the rotation axis of the black hole, the above analysis is probably inapplicable, as the centrifugal force cannot be ignored. Large pressure gradients in the corona would tend to push matter along some of the magnetic field lines in the outer layers of the inner torus magnetosphere. A combination of buoyancy and centrifugal forces may subsequently give rise to a twist of these field lines, some of which may ultimately fold and open, to form a region of oppositely directed magnetic field lines. Field lines thus created near the axis constitute the inner flux tube that extends from the horizon to infinity; those anchored to the torus now extend to infinity and constitute an outer flux tube. The inner and outer flux tubes have opposite magnetic orientation and are separated by a charge and current sheet, whenever the outer tube carries a (super-Alfvénic) wind to infinity. Reconnection may occur in the boundary layer, which is of interest in the rearrangement of the magnetosphere near the axis. This coaxial structure of open flux tubes is discussed in greater detail in the next section.

### 6.3. Implications for Energy Extraction from the Black Hole

The rate at which the torus catalyzes black hole spin energy has been calculated in § 5 under the assumption that the torus magnetosphere is force-free. However, ejection of appreciable mass flux along magnetic field lines that thread the horizon may alter the extraction process. Detailed study of ideal MHD flow in Kerr geometry is provided by Takahashi et al. (1990) and Hirota et al. (1992). They analyze the conditions under which energy extraction occurs in a flow that starts with a zero poloidal velocity at the plasma source and is pulled inward by the black hole. They show that the MHD flow can carry a negative energy flux into the horizon (which is equivalent to the condition that the energy given by eq. [A9] is negative), provided that (1) the angular velocity of magnetic field lines lies in the range \( 0 < \Omega_F < \Omega_H \), where \( \Omega_H \) is the angular velocity of the black hole (which is identical to the condition found by Blandford & Znajek 1977), and (2) the Alfvén point is located inside the ergosphere, which implies that inflow of negative electromagnetic energy exceeds inflow of positive kinetic energy. The latter condition restricts the mass flux expelled along field lines connecting the torus and the horizon. The vertical mass flux estimated above assumes maximum extraction efficiency. We argue that inward mass ejection in the equatorial plane may be suppressed by the centrifugal barrier. The details are complicated by virtue of general relativistic effects. Whether this is sufficient to allow energy extraction is not clear at present. If not, it would mean that the rate at which the spin-down energy of the hole is dissipated in the torus and, hence, its temperature must be regulated by mass ejection.
7. STRUCTURE OF THE INNER FLUX TUBE

In the preceding section we argued that mass ejection from the torus opens magnetic field lines on the outer layers of the inner torus magnetosphere. This creates an open magnetic flux tube that extends from the horizon to infinity, surrounded by an outer flux tube that is anchored to the torus. The upper sections of this structure result from a fold, stretch, and cut in poloidal topology, whereby the inner and outer tubes assume mutually antiparallel poloidal magnetic fields. The lower section remains unfolded, leaving parallel poloidal magnetic fields in the inner and outer flux tubes (see Figs. 4 and 10). This additional structure was not analyzed in van Putten & Levinson (2002). In what follows, we describe the structure of the inner tube in some detail.

7.1. Asymptotic Boundary Conditions on the Horizon and at Infinity

The inner flux tube satisfies slip/slip boundary conditions both on the horizon and at infinity. It is well known that in the limit of infinite conductivity, every magnetic surface must rotate rigidly (although the angular velocity of different flux surfaces should not be the same). Finite resistivity effects, however, may give rise to a differential rotation along magnetic flux tubes (implying $\mathbf{E} \cdot \nabla \neq 0$). To make our analysis general, we shall allow for such a differential rotation and denote the angular velocities of a given flux surface on the horizon and at infinity by $\Omega_{F+}$ and $\Omega_{F-}$, respectively.

As stated above, we anticipate the horizon half-opening angle $\theta_H$ of the inner flux tube to be sufficiently small so that the small-angle approximation applies in solving Maxwell’s equations. From equation (A20) we obtain

$$2\pi \int f' \sqrt{-\tilde{g}} \, d\theta = \frac{\Lambda}{2\rho^2} \sin \theta F_{\rho \theta} \equiv \frac{1}{2} B_T.$$  

(40)

It can be readily shown (e.g., Blandford & Znajek 1977) that in the force-free limit, viz., $F_{\rho \theta} = 0$, the Boyer-Lindquist toroidal magnetic field, $B_T$, is conserved along magnetic flux surfaces $\Psi(r, \theta)$ and that the current contained within a magnetic flux surface is given exactly by equation (40), viz., $I(\Psi) = \int_{\Psi} B_T \, d\theta$, and, hence, is also conserved. Current conservation along streamlines is not guaranteed in general, however, even in the limit of infinite conductivity, since inertial effects may give rise to cross field currents. At any rate, beyond the fast magnetosonic point of the inflow (outflow), the poloidal current becomes radial asymptotically, as it is carried purely by the inflowing (outflowing) Goldreich-Julian charges (see below). By employing the asymptotically frozen-in condition on the horizon, $F_{\rho \theta} = (\Sigma/\Lambda) (\Omega_{H} - \Omega_{F+}) F_{\rho \theta}$ (see eq. [A18] with $\beta = -\Omega_H$), we obtain the poloidal current flowing through the horizon:

$$I_H \approx \frac{r_H^2 + a^2}{2(r_H^2 + a^2 \cos^2 \theta)} \sin \theta F_{\rho \theta} (\Omega_{H} - \Omega_{F+}).$$  

(41)

Assuming the ZAMO radial magnetic field, $B_r = -F_{\rho \theta}/(\Sigma \sin \theta)$, to be independent of $\theta$ near the horizon, we can calculate the total magnetic flux through the horizon in one hemisphere: $\Psi = 2\pi A_\rho = \int B_r (|\phi| = 0) \, d\theta \, d\phi \approx \pi \Sigma \sin^2 \theta B_r = -\pi \sin \theta F_{\rho \theta}$. Equation (41) then yields to leading order

$$I_H \approx - (\Omega_{H} - \Omega_{F+}) A_\phi.$$  

(42)

Likewise, the frozen-in condition at infinity gives $F_{\rho \theta} = \Omega_{F-} F_{\rho \theta}$, leading in the small-angle approximation to

$$I_\infty = \frac{1}{2} \sin \theta F_{\rho \theta} \Omega_{F-} \approx - \Omega_{F-} A_\phi.$$  

(43)

The electric charge distribution in the inner flux tube can be obtained from Maxwell’s equation: $F_{\rho \theta}^\mu = 4\pi j^\mu$. At small angles this equation is given, to a good approximation, by equation (A19) (see van Putten 2001b). Assuming as before that the ZAMO radial magnetic field is independent of $\theta$ near the axis, we obtain from equation (A19) the Goldreich-Julian charge density:

$$\rho_e = \alpha^2 j^\rho = - \frac{(\Omega_{F} + \beta) B_r \cos \theta}{2\pi},$$  

(44)

where $\Omega_F$ denotes the local angular velocity of the flux surface at hand. Evidently, the electric charge changes along the inner flux tube from $\rho_{eH} = (\Omega_{H} - \Omega_{F+}) B_r/2\pi$ on the horizon to $\rho_{e\infty} = -\Omega_{F-} B_r/2\pi$ at infinity and vanishes at the radius at which $-\beta = \Omega_{F}$, corresponding to a locally zero angular momentum state of the flux surface. This is illustrated in Figure 4. Using equations (A20) and (44), one finds $j^\rho = j^\rho \nu^\rho$, where $\nu^\rho = 1$ at infinity and $\nu^\rho = -\alpha^2$ on the horizon (see the Appendix). This implies that the current on the horizon and at infinity is purely due to convection of Goldreich-Julian charges. A similar conclusion was drawn by Punsly & Coroniti (1990).

7.2. Two Steady State Limits

Current closure of $I_\infty$ through the inner tube over the outer tube with equal magnetic flux of opposite sign gives rise to a differentially rotating inner tube, described by

$$\Omega_{F+} = \Omega_{F-} = \frac{\Omega_{H}}{2}.$$  

(45)

(van Putten & Levinson 2002). This assumes the force-free limit, whereby current is conserved along flux surfaces and $I_H = I_\infty$.

Alternatively, current closure of $I_\infty$ through the inner tube over the outer tube and the outer torus magnetosphere allows for the limit of infinity conductivity. Combined with the force-free limit, this approach was used by Blandford & Znajek (1977) and Phinney (1983) to construct their force-free solutions and determine the efficiency of the energy extraction process. Near the axis equations (42) and (43) yield to leading order

$$\Omega_{F+} = \Omega_{F-} = \frac{\Omega_{H}}{2}.$$  

(46)

It will be appreciated that this implies a nearly maximal energy extraction rate on the inner tube.

Punsly & Coroniti (1990) argue that the assumption made by Blandford & Znajek (1977) and Phinney (1983), namely, that the magnetosphere is force-free in the entire region between the horizon and infinity, violates the principle of MHD causality. Their argument relies on the observation that the inflow must become superfast on the horizon and, therefore, cannot communicate with the plasma source region (e.g., the gap in the Blandford-Znajek model). They concluded that the use of the Znajek frozen-in condition on
the horizon to determine \( \Omega_F \) is unphysical and that \( \Omega_F \) must be determined by the dissipative process that leads to ejection of plasma on magnetic field lines. This point is of interest, as the black hole has a finite, though secular, lifetime of rapid spin as an inner engine to GRBs. In what follows we reexamine this issue.

We remark that MHD causality prohibits infinitely rigid structures (the Alfvén velocity is bounded by the velocity of light). This is true in particular over distances much larger than the system size. Nevertheless, we may examine a steady state force-free limit \( \Omega_{F+} \approx \Omega_{F-} \approx \Omega_H/2 \) as a close approximation on scales comparable to the system size, modified only by a differentially rotating gap that injects the associated current in a region where frame-dragging \( \beta \) is pronounced.

The current flow in the inner tube is created in a slightly differentially rotating gap in a neighborhood of \( \Omega_F = -\beta \) between two Alfvén surfaces. The gap size (and hence current output) is determined by the local degree of differential frame dragging in \( \beta \). As the magnetosphere around the black hole is transparent to \( \beta \), the gap remains in causal contact with the angular velocity of the black hole: a change in the asymptotic value \( -\beta = \Omega_H \) on the horizon is associated with a change in \( \beta \) throughout the surrounding spacetime. This indicates that the gap is subject to changes \( \delta \Omega_H \) under general conditions.

If \( I_H \neq I_{\infty} \), the inner tube develops a response by charge conservation: time variable or in steady state, depending on the absence or presence of cross field currents (i.e., \( J_\theta \)) between the inner and the outer tube. Conceivably, the plasma injection process maintains the required cross field currents, and there exist multiple current closure paths. In view of equations (42) and (43), note that this implies additional differential rotation in the inner tube with accompanying dissipation processes. In the absence of cross currents, there results a time-variable adjustment of the two Alfvén surfaces that delimit the gap by application of Gauss’s theorem to, respectively, the black hole plus lower section and the upper section. This causes adjustment of the current injected into the lower and the upper sections. In steady state, this recovers \( I_H = I_{\infty} \). This argument leaves open the possibility that the gap is time variable, however, especially on the light crossing timescale.

7.3. Output Power through the Inner Tube

The inner tube releases black hole spin energy at a certain rate, set by the horizon half-opening angle \( \theta_H \) shown in Figures 4 and 10. Aforementioned force-free limits (eqs. [45] and [46]) define bounds for the bipolar outflows of black hole spin energy through the inner tube, i.e.,

\[
\frac{\Omega_{F+}^2 (\Omega_H - \Omega_{F+}) A^2_{in}}{2 \pi A_{in}} \leq S_m \leq \frac{1}{4} \frac{\Omega_H^2 A_{in}}{2 \pi},
\]

where \( 2\pi A_{in} \) is the net magnetic flux on the horizon associated with the inner tube. Hence, we have a fraction

\[
\frac{E_i}{E_{rot}} = \frac{1}{4} \frac{\theta_H^4}{\Omega_H^4 H}
\]

of the rotational of the black hole (see eqs. [1] and [2]).

The observed true GRB energies cluster around \( 3 \times 10^{50} \) ergs (Frail et al. 2001). This corresponds to \( E_i = (2 \times 10^{51} \text{ ergs}) (0.16/\epsilon) \), where \( \epsilon \) denotes the efficiency of conversion of kinetic energy to gamma rays. In our model of GRBs from rotating black holes, this observational constraint introduces the small parameter

\[
\frac{E_i}{E_{rot}} = 10^{-3} \left( \frac{7 M_e}{M} \right) \left( \frac{0.16}{\epsilon} \right).
\]

This corresponds to a horizon half-opening angle \( \theta_H \geq 15^\circ \), \( \theta_H \approx 35^\circ \) associated with the output from the gap in equation (45).

We propose to identify \( \theta_H \) with the curvature in poloidal topology of the inner torus magnetosphere (van Putten 2003), \( \theta_H \approx M/R \) within a factor close to 1. In this event, we have

\[
\frac{E_i}{E_{rot}} \sim \frac{1}{4} \left( \frac{M}{R} \right)^4.
\]

A spread in torus radius by a factor of about 2 corresponds to a spread in energies in baryon-poor outflows by about 1 order of magnitude, consistent with the observed spread in true GRB energies.

The fate of this Poynting flux outflow is determined by specific mechanisms for dissipation, as these may arise downstream in the upper section of the open flux tube.

8. STRUCTURE OF THE OUTER TUBE

The open field lines that are anchored to the torus form the outer flux tube. The baryon-rich material ejected from the torus, derived from the spin energy of the black hole (see § 6), flows along those open field lines to infinity. We have estimated this to constitute a substantial mass loss. In the ideal MHD approximation the angular velocity of the outer flux tube is conserved along magnetic field lines (on the scale of the system) and, therefore, equals that of the torus, viz., \( \Omega_T = \Omega_T \). Above the Alfvén point the ratio of toroidal to poloidal field in the outflow is (see eq. [A11] with \( \Omega_T = \Omega_T \) and \( v_\theta \rightarrow r^{-1} \))

\[
\frac{F_{r\theta}}{F_{\phi\theta}} = \frac{\Omega_T}{v_r}.
\]

As argued above, the direction of the poloidal magnetic field in the outer flux tube is opposite to its direction in the inner tube. Thus, if the torus and the black hole rotate in the same direction, as envisioned here, then the toroidal magnetic field \( F_{\phi\theta} \) has opposite orientations in the inner and outer tubes. The Goldreich-Julian charge density in the outer tube is

\[
\rho_e = \frac{\Omega_T B_r \cos \theta}{2\pi}
\]

and is opposite in sign to the outflowing charges in the inner flux tube. These charges carry the current flowing along the open field lines from the torus. This region of the torus magnetosphere is equivalent to a pulsar magnetosphere.

We conclude that the outer tube forms out of the outer layers of the inner torus magnetosphere (the magnetic wall around the black hole) and joins the open field lines in the outer torus magnetosphere in creating a super-Alfvénic baryon-rich wind to infinity. It conceivably contributes to collimation of the baryon-poor outflows in the inner flux tube.

9. THE INTERFACE BETWEEN THE INNER AND THE OUTER FLUX TUBE

The creation of the open magnetic flux tubes from the closed torus magnetosphere topologically represents folding
of magnetic field lines in the upper section of the inner/outer flux tube, which gives rise to an antiparallel orientation of the poloidal magnetic field. This is accompanied by a cylindrical current and charge sheet that accounts for the jump in the electric and magnetic fields across the interface. The lower section of the inner/outer flux tube that connects to the horizon has, instead, a parallel orientation between the poloidal magnetic field. In the perfect MHD limit, the properties of the interface are described by jump conditions as follow from Maxwell’s equations, \( F^\mu_{\nu} = 4\pi J^\nu \), i.e.,

\[
4\pi\sigma_c = [\Omega r \sin \theta B_r] = r \sin \theta \Omega T B_{\theta} - r \sin \theta \left( \frac{\Omega_H}{2} \right) B_{\phi},
\]

(53)

where \( B_{\theta} \) denotes the radial magnetic field near the interface in the outer (inner) flux tube. The poloidal and toroidal interface currents are, likewise,

\[
4\pi J^r = [B_\phi] = \left[ \frac{\Omega r \sin \theta B_r}{v'} \right],
\]

\[
4\pi J^\phi = [-B_r].
\]

(54)

The poloidal current given by equation (54) results beyond the Alfvén point, where the wind transports angular momentum outward to infinity. The outflow in the inner tube is expected to be relativistic \((v' = 1)\). If the baryon-rich outflow from the torus also becomes relativistic, then the latter equation implies that \( J^r = \sigma_c \), namely, the poloidal current in the boundary layer is solely due to the outflowing surface charges. The poloidal current sheet—possibly further carrying a poloidal current—is a potential site for reconnection of magnetic field lines, which would convert magnetic energy in the inner flux tube into kinetic energy.

### 9.1. Dissipation in the Folded Upper Section of the Flux Tubes

The upper interface between the inner and the outer flux tubes is subject to antiparallel magnetic fields and strong differential rotation (whenever \( \Omega_T \neq \Omega_H / 2 \)). It is therefore a potential site for reconnection of magnetic field lines and may give rise to conversion of a fraction of the magnetic energy in the inner flux tube into kinetic energy. In order to assess the fraction of magnetic energy that is converted in the interface, a detailed reconnection model is required. Here we point out that the reconnection time is limited by the crossing time of an Alfvén wave across the inner tube, which is typically very short. If this limit applies, then an appreciable fraction of magnetic energy will be dissipated. (The reconnection rate could be much smaller.) In any case, the power extracted from the hole by the inner tube is uncertain because the angular velocity of the inner tube on the horizon is unknown. In or near the lowest energy state of the gap, the inner tube mediates a power \( S_{m} \), as given by equation (47).

The electric fields produced by the reconnection process in the interface may inject plasma, including baryons that should be present in the boundary layer, into the inner tube. This would lead to mass loading of the inner tube and additional conversion of the Poynting flux. The amount of baryonic contamination by this process is yet an open issue.

Additional dissipation may arise from a differential rotation of the inner flux tube, as discussed in van Putten & Levinson (2002). It is clear that if the system fluctuates over a timescale \( \Delta t \), then nonlinear disturbances may induce differential rotation over length scales less than \( c\Delta t \) (which corresponds to \( 10^5 \) gravitational radii for the entire lifetime of the system). Such electromagnetic fronts should be highly dissipative by virtue of the large parallel electric fields associated with the differential rotation of the magnetic flux tubes. It is anticipated that, under the conditions envisioned here, copious pair and photon productions would ensue inside the differentially rotating fronts.

### 10. OBSERVATIONAL CONSEQUENCES

We have described in some detail the topology, lifetime, and emissions of black hole–torus systems in the suspended accretion state. Figure 11 summarizes the energy transport and conversion of black hole spin energy, catalyzed by the torus. Most energy is dissipated in the horizon, while the major output is converted by the torus into gravitational radiation, winds, and neutrino emissions (as well as thermal emissions). A minor output is released in baryon-poor outflows as input to the observed GRB afterglows. The analysis is based largely on equivalence to pulsar magnetospheres. The suspended accretion state develops against a magnetic wall around rapidly rotating black holes. This is based on a uniform magnetization of the torus, represented by two oppositely oriented current rings in a torus formed in

![Diagram of energy transport and conversion](image-url)
black hole–neutron star coalescence and core-collapse supernovae. The proposed emissions in gravitational radiation are powered by the spin energy of the black hole, which renders these emission candidates for the upcoming gravitational wave experiments. This output is contemporaneous with GRBs from baryon-poor outflows and hence has the same duration as the redshift-corrected durations of long GRBs: tens of seconds, as a result of the condition of magnetic stability of the torus.

Calorimetry on the emissions in gravitational radiation provides a rigorous compactness test for Kerr black holes (van Putten 2001a), which can be pursued by upcoming gravitational wave experiments. We emphasize that our predictions on the energies and duration of emissions in gravitational radiation are robust and independent of an association with GRBs. Hypernovae could be overabundant as burst sources of gravitational radiation, with only a small fraction making successful GRBs. Calorimetry on supernova wind energies provides a method for constraining the angular velocity of the torus and, hence, its quadrupole emissions in gravitational radiation. Possible channels are those recently observed X-ray line emissions in GRB 011211 (Reeves et al. 2002), hypernova remnants, which remain to be identified, and by association with soft X-ray transients with chemically enhanced companion stars (Brown et al. 2000).

X-ray and radio shells that may result from the interaction of the baryon-rich torus winds with ambient matter, as well as the gamma-ray emission expected to be produced by the baryon-poor outflows in the inner tube, should be sought (Levinson et al. 2002).

Below we summarize our findings and draw additional conclusions:

1. Most of the black hole luminosity is incident on the torus by topological equivalence to pulsar magnetospheres, when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly. We identify a new magnetic instability, which defines a minimum lifetime when the black hole spins rapidly.

2. At MeV temperatures of the torus, a torus with a minor radius of less than about the linear size of the black hole becomes unstable to the formation of finite multipole moments by the Papaloizou-Pringle instability and, conceivably, other modes. The torus hereby converts a major fraction of black hole spin energy by horizon Maxwell stresses as proposed by Blandford & Znajek (1977). Causality in the formation of baryon-poor outflows from the open magnetic flux tube is due to transparency of the magnetosphere to the gravitational field: the associated current injection is subject to current continuity between asymptotic boundary conditions on the horizon and at infinity by Gauss’s theorem, and regulated by differential frame dragging and current closure at infinity. By transparency of the magnetosphere to frame dragging, the angular velocity of the inner tube is coupled to the angular velocity of the black hole under general conditions.

3. A minor fraction of black hole spin energy is released in baryon-poor outflows. We attribute this fraction to curvature in poloidal topology, whereby $E_j / E_{rot} \approx \frac{1}{2} \left( \frac{M}{R} \right)^2$ is standard for a universal horizon half-opening angle $\theta_H \approx M / R$ of the associated open magnetic flux tube. These outflows are probably not high-$\sigma$—the ratio of Poynting flux to kinetic energy flux—owing in part to magnetic reconnection in an interface with baryon-rich winds flowing along the surrounding outer flux tube.

4. Causality in the process of spin-up of the torus by the black hole follows by topological equivalence to pulsar magnetospheres (van Putten 1999). In response, the black hole spins down by conservation of energy and angular momentum. This establishes causality in the extraction of black hole spin energy by horizon Maxwell stresses as proposed by Blandford & Znajek (1977). Causality in the formation of baryon-poor outflows from the open magnetic flux tube is due to transparency of the magnetosphere to the gravitational field: the associated current injection is subject to current continuity between asymptotic boundary conditions on the horizon and at infinity by Gauss’s theorem, and regulated by differential frame dragging and current closure at infinity. By transparency of the magnetosphere to frame dragging, the angular velocity of the inner tube is coupled to the angular velocity of the black hole under general conditions.

5. Calorimetry on the predicted energy output in gravitational radiation by the upcoming gravitational wave experiments provides a method for identifying Kerr black holes as objects in nature when $2\pi \int_0^{E_{rot}} f_{gw} dE > 0.005$ (van Putten 2001a); where $f_{gw}$ denotes the quadrupole frequency of gravitational waves. Current experiments consist of laser interferometric detectors LIGO, VIRGO, TAMA, and GEO, bar and sphere detectors. An individual source is band limited in gravitational radiation in terms of a frequency sweep of about 10%, corresponding to emission of the first 50% of the output in gravitational radiation from a maximally spinning Kerr black hole. Collectively black hole–torus systems are conceivably luminous in multiple frequencies with broad distributions owing to a spread in black hole mass.

6. Frequencies of quadrupole gravitational radiation $f_{gw} \simeq (470 \text{ Hz}) (E_{\text{w}} / 4 \times 10^{52} \text{ ergs})^{1/2} (7 M_\odot / M_H)^{3/2}$ can be predicted by calorimetry on the torus wind energies $E_w$. Calorimetry on X-ray line emissions points toward frequencies around 500 Hz (van Putten 2003).

7. Calorimetry on $E_w$ may also be pursued by calorimetry on hypernova remnants, given the observed supernova association (e.g., Bloom et al. 2002). Our model suggests the ejection of the remnant stellar envelope as a shell with kinetic energy $0.5 \beta E_{\text{w}} \sim 3 \times 10^{51} \text{ ergs} (\beta / 0.1) (M_H / 10 M_\odot) (\eta / 0.1)$. Here $\beta$ denotes the initial radial velocity relative to the velocity of light, in response to the impact of $E_w$ from within. For a supernova association to molecular clouds, see, e.g., Chu & MacLow (1990) and Wang & Halfand (1991); a hypernova or GRB association to molecular clouds has been considered by Efremov, Elmegreen, & Hodge (1998); Wang (1999); Dunne, Points, & Chu (2001); Lai et al. (2001); Chen et al. (2002); Price et al. (2002). Determining kinetic energies of these remnants can be pursued analogously to studying supernova remnants, in the radio and X-ray. Chandra observations may hereby identify hypernova remnants in X-ray–bright (super) shells around a black hole binary, possibly in the form of a soft X-ray transient.

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APPENDIX

IDEAL MHD IN KERR GEOMETRY

We express the Kerr metric in Boyer-Lindquest coordinates with the following notation:

\[ ds^2 = -\alpha^2 dt^2 + \omega^2 (d\phi + \beta dt)^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 , \]

where \( \alpha = \rho \Delta^{1/2}/\Sigma \) is the lapse function, \( \omega = (\Sigma^2/\rho^2) \sin^2 \theta \), and \( -\beta = 2aMr/\Sigma^2 \) is the angular velocity of a ZAMO with respect to a distant observer, with \( \Delta = r^2 + a^2 - 2Mr, \rho^2 = r^2 + a^2 \cos^2 \theta \), and \( \Sigma^2 = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta \). The parameters \( M \) and \( a \) are the mass and angular momentum per unit mass of the black hole.

We denote by \( n, p, \rho, \) and \( h = (\rho + p)/n \) the proper particle density, pressure, energy density, and specific enthalpy, respectively. The stress energy tensor then takes the form

\[ T^{\alpha\beta} = h u^\alpha u^\beta - pg^{\alpha\beta} + \frac{1}{4\pi} \left( F^\alpha{}_{\beta} F^\beta{}_{\alpha} + \frac{1}{4} g^{\alpha\beta} F^2 \right) , \]

where \( u^\alpha \) is the four-velocity and \( F_{\mu\nu} \) is the electromagnetic tensor that satisfies Maxwell equations. The dynamics of the MHD flow is then governed by the equation

\[ T^\nu_{\mu\nu} = 0 , \]

subject to conservation of particle flux,

\[ (nu^\alpha)_\alpha = \frac{1}{\sqrt{-g}} \left( \sqrt{-g} u^\alpha \right)_\alpha = 0 . \]

The ideal MHD assumption (i.e., infinite conductivity) imposes the additional constraint

\[ u^\alpha F_{\alpha\beta} = 0 . \]

The stationary axisymmetric flow considered here is characterized by two Killing vectors: \( \xi^\mu = \partial_t \) and \( \chi^\mu = \partial_\phi \). By contracting with the stress energy tensor, we can construct the energy and angular momentum currents: \( \mathcal{E}^\alpha = T^\alpha{}_{\beta} \xi^\beta = T^\alpha_t \) and \( \mathcal{L}^\alpha = T^\alpha{}_{\beta} \chi^\beta = T^\alpha_\phi \), which are conserved, viz.,

\[ \mathcal{L}^\alpha_{\beta\alpha} = \mathcal{E}^\alpha_{\alpha\beta} = 0 . \]

Equations (A4), (A5), and (A6), together with the homogeneous Maxwell equations, admit four quantities that are conserved along magnetic flux surfaces, the shape of which is given by a stream function \( \Psi(r, \theta) \): the particle flux per unit magnetic flux,

\[ \eta(\Psi) = \frac{\sqrt{-g} u^\phi}{F_{\theta\phi}} = \frac{\sqrt{-g} u^\theta}{F_{\theta\phi}} , \]

the angular velocity of magnetic field lines,

\[ \Omega_F(\Psi) = \frac{F_\theta}{F_\phi} = \frac{\Omega_F}{F_{\theta\phi}} , \]

and the total energy and angular momentum per particle carried by the MHD flow,

\[ E(\Psi) = h u_t - \sqrt{-g} \frac{\Omega_F}{4\pi \eta} F^{\theta\phi} , \]

\[ L(\Psi) = -h u_\phi - \sqrt{-g} \frac{\Omega_F}{4\pi \eta} F^{t\phi} . \]

The above equations also yield the relation

\[ \frac{F_{\theta\phi}}{F_{\phi\theta}} = \frac{\Omega_F - v^\phi}{v^\theta} , \]

where \( v^\phi = u^\phi/u^t \) and \( v^\theta = u^\theta/u^t \) are the corresponding components of the three-velocity. Equations (A9) and (A10) can be
used to express \( u', v', \) and \( P' \) in terms of \( E, L, \) and the Alfvén Mach number, defined by \( M^2 = 4\pi \eta r^2 / n. \) One finds

\[
F_\theta' = \frac{4\pi \eta}{\sin \theta} (g_{tt} + g_{t\varphi} \Omega_F) L + (g_{t\varphi} + g_{\varphi\varphi} \Omega_F) E, \quad (A12)
\]

\[
h u' = \frac{h(-u + \beta u_0)}{\alpha^2} = (E - \Omega_F L) - M^2 (E + \beta L), \quad (A13)
\]

\[
v' = -\alpha^2 g_{\varphi\varphi} (E - \Omega_F L) + M^2 (g_{t\varphi} E + g_{\varphi\varphi} L)
-\alpha^2 g_{\varphi\varphi} (E - \Omega_F L) - M^2 (g_{t\varphi} E + g_{\varphi\varphi} L), \quad (A14)
\]

where \( k_0 = g_{tt} + 2g_{t\varphi} \Omega_F + g_{\varphi\varphi} \Omega_F^2. \) Finally, using the normalization condition, \( u' \mu u_\mu = -1, \) we obtain for the poloidal velocity, defined as \( u'_p = u_\mu u' \mu + u_\mu u' \mu, \) the relation

\[
u'_p = (\alpha u')^2 + \left(\frac{u_\mu}{\alpha^2}\right)^2. \quad (A15)
\]

Note that \( u_\mu \) is the poloidal four-velocity as measured by a ZAMO and \( \alpha u' \) is the corresponding \( t \) component of the four-velocity. As seen, the Lorentz factor of the inflow in the ZAMO frame approaches \( \Gamma = \alpha u' \sim \alpha^{-1} \) on the event horizon.

Consider first the behavior of the solution near the horizon. There \( \alpha \to 0 \) and so

\[
\nu' = \frac{4}{\sqrt{\alpha^2}} = -\frac{\Delta}{r^2 + a^2}. \quad (A16)
\]

The poloidal velocity is radial on the horizon, implying \( u'_p = g_{\varphi\varphi} u' \varphi \). Using equation (A15), we find

\[
u' = u' = \frac{\alpha}{\sqrt{\alpha^2}} = -\frac{\Delta}{r^2 + a^2}. \quad (A17)
\]

Substituting the above results into equation (A11), we finally obtain

\[
F_{t\theta} = -\frac{r^2 + a^2}{\Delta} (\Omega + \beta). \quad (A18)
\]

Equation (A18) gives the boundary condition on the horizon.

The Boyer-Lindquist electric charge density in the flow is determined from Maxwell’s equation: \( F_{t\mu}^{t\mu} = 4\pi \rho' \). It can be readily shown that if the poloidal component of the magnetic field is radial on the event horizon (which is the case in the force-free limit of an uncharged black hole), then near the horizon the latter equation reduces to

\[
\frac{\partial}{\partial \theta} \left[ \frac{\sin \theta}{\alpha^2} (\Omega + \beta) F_{t\theta} \right] = 4\pi \sqrt{-g} j'. \quad (A19)
\]

The poloidal current follows from the equation

\[
\frac{\partial}{\partial \theta} \left( \frac{\sin \theta}{\alpha^2} F_{t\theta} \right) = 4\pi \sqrt{-g} j'. \quad (A20)
\]

Using equations (A11), (A16), (A19), and (A20), we finally recover the result (Punsly & Coroniti 1990)

\[
j' = \frac{\rho_c}{v_p}, \quad (A21)
\]

where \( \rho_c = \alpha^2 j' \) and \( v_p = u_p / u_t \) is the three-velocity as measured by a ZAMO.

**REFERENCES**

Abramovici, A., et al. 1992, Science, 256, 325

Balbus, S. A., & Hawley, J. F. 1992, ApJ, 400, 610

Bethe, H. A., & Wilson, J. R. 1985, ApJ, 295, 14

Blandford, R. D., & Payne, D. G. 1982, MNRAS, 199, 883

Blandford, R. D., & Znajek, W. L. 1977, MNRAS, 179, 433

Bloom, J. S., et al. 2002, ApJ, 572, L45

Bradač, C., et al. 1992, Phys. Lett. A, 163, 15

Brown, G. E., Lee, C.-H., Wijers, R. A. M. J., Lee, H. K., Israeli, M., & C. H. Bethe, H. A. 2000, NewA, 5, 91

Camenzind, M. 1986, A&A, 162, 32

Chen, C.-H. R., Chu, Y.-H., Graedel, R., Lai, S.-P., & Wang, Q. D. 2002, AJ, 123, 2462

Chu, Y.-H., & MacLow, M.-M. 1990, ApJ, 365, 510

Cohen, J. M., Kegeles, L. S., & Rosenblum, A. 1975, ApJ, 201, 783

Coward, D., van Putten, M. H. P. M., & Burman, R. 2002, ApJ, 580, 1084

Dunne, B. C., Points, S. D., & Chu, Y.-H. 2001, ApJS, 136, 119

Efremov, Y. N., Elmegreen, B. G., & Hodge, P. W. 1998, ApJ, 501, L163

Eichler, D., Livio, M., Piran, T., & Schramm, D. N. 1989, Nature, 340, 126

Fratl, D. A., et al. 2001, ApJ, 562, L55

Goldreich, P. M., & Julian, W. H. 1969, ApJ, 157, 869

Hirotani, K., Takahashi, M., Nitta, S., & Tomimatsu, A. 1992, ApJ, 386, 455

Jackson, J. D. 1975, Classical Electrodynamics (New York: Wiley)

Kerr, R. P. 1963, Phys. Rev. Lett., 11, 237

Lai, S.-P., Chu, Y.-H., Chen, C.-H. R., Ciardullo, R., & Grebel, E. K. 2001, ApJ, 547, 754

Levinson, A., & Eichler, D. 1993, ApJ, 418, 386

Levinson, A., et al. 2002, ApJ, 576, 923

Masaki, A., et al. 2001, Phys. Rev. Lett., 86, 3950

Müller, J. M., et al. 2002, ApJ, 570, L69

Paczyński, B. 1991, Acta Astron., 41, 257

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