AN APPROXIMATE FACTORIZATION METHOD FOR INVERSE ACOUSTIC SCATTERING WITH PHASELESS NEAR-FIELD DATA

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Abstract. This paper is concerned with the inverse acoustic scattering problem with phaseless near-field data at a fixed frequency. An approximate factorization method is developed to numerically reconstruct both the location and shape of the unknown scatterer from the phaseless near-field data generated by incident plane waves at a fixed frequency and measured on the circle \(\partial B_R\) with a sufficiently large radius \(R\). The theoretical analysis of our method is based on the asymptotic property in the operator norm from \(H^{1/2}(S^1)\) to \(H^{-1/2}(S^1)\) of the phaseless near-field operator defined in terms of the phaseless near-field data measured on \(\partial B_R\) with large enough \(R\), where \(H^s(S^1)\) is a Sobolev space on the unit circle \(S^1\) for real number \(s\), together with the factorization of a modified far-field operator. The asymptotic property of the phaseless near-field operator is also established in this paper with the theory of oscillatory integrals. The unknown scatterer can be either an impenetrable obstacle of sound-soft, sound-hard or impedance type or an inhomogeneous medium with a compact support, and the proposed inversion algorithm does not need to know the boundary condition of the unknown obstacle in advance. Numerical examples are also carried out to demonstrate the effectiveness of our inversion method. To the best of our knowledge, it is the first attempt to develop a factorization type method for inverse scattering problems with phaseless data.

Key words. Inverse acoustic scattering, approximate factorization method, phaseless near-field data, asymptotic behavior of phaseless near-field operator

AMS subject classifications. 35R30, 35Q60, 65R20, 65N21, 78A46

1. Introduction. Inverse scattering with phased data (i.e., data with phase information) has been widely studied mathematically and numerically over the past decades due to its significant applications in such diverse scientific areas as radar and sonar detection, remote sensing, geophysics, medical imaging and nondestructive testing (see, e.g., [8, 15, 16, 29, 30] for a comprehensive overview). However, in many practical applications, it is much harder to obtain data with accurate phase information compared with only measuring the modulus or intensity of the data (see, e.g., [9, 10, 43, 44] and the references quoted there). Therefore, it is often desirable to study inverse scattering problems with phaseless data (i.e., data without phase information).

Many optimization and iteration algorithms have been proposed for solving inverse scattering problems with phaseless data (see, e.g., [4, 15, 16, 20, 30, 44, 55, 56]). Optimization and iteration algorithms can achieve an accurate reconstruction of the unknown scatterers. However, this type of algorithms is time-consuming and needs to know the boundary conditions of the unknown scatterers in advance. To reduce the computational cost, non-iterative algorithms have recently attracted more and more attention in inverse scattering problems (see, e.g., [8, 30, 19]). For inverse scattering problems with phaseless data, non-iterative algorithms have also been studied recently. In [35], a non-iterative algorithm was proposed to reconstruct a polyhedral sound-soft or sound-hard obstacle from a few high frequency phaseless backscattering far-field measurements associated with incident plane waves, where the exterior unit normal vector of each side/face of the obstacle is determined first.

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with suitably chosen incident directions and the location of the obstacle is then determined with a few phased far-field data. This method has been extended to the case of inverse electromagnetic scattering in [39]. In [11,13], a direct imaging method was proposed to reconstruct scattering obstacles from acoustic and electromagnetic phaseless near-field data, based on the reverse time migration technique. A direct imaging method was developed in [57] to recover scattering obstacles from acoustic phaseless far-field data corresponding to infinitely many sets of superpositions of two plane waves with a fixed frequency as the incident fields. Recently, two direct sampling algorithms were given to reconstruct acoustic obstacles in [25] and acoustic sources in [26] from phaseless far-field data generated with incident plane waves, by adding a reference point scatterer into the scattering system. In [27], two direct sampling algorithms were introduced to recover acoustic obstacles from phaseless far-field measurements corresponding to superpositions of plane waves and point sources as the incident fields, where the point sources have fixed source location with at most three different scattering strengths. In [59], a non-iterative algorithm was proposed to recover acoustic sources from multi-frequency phaseless near-field data, where the phase information of the measured data is first recovered with the reference point source technique (that is, adding certain reference point sources into the scattering system) and the acoustic sources are then reconstructed with the Fourier method. On the other hand, uniqueness and stability have also been studied for inverse scattering with phaseless data (see, e.g., [32], [34], [10], [12] for the case of near-field measurements and [24, 27, 11, 42, 17, 53, 54, 58] for the case of far-field measurements).

In this paper, we consider inverse acoustic scattering with phaseless near-field data associated with incident plane waves. For simplicity, we restrict our attention to the 2D case. Precisely, our inverse problem is to reconstruct a unknown scatterer from the phaseless near-field data \(|u(x, d)|\) for \(x \in \partial B_R\) and \(d \in S^1\), where \(\partial B_R\) is a circle of radius \(R\) and centered at the origin enclosing the unknown obstacle, \(S^1\) is the unit circle, \(d \in S^1\) is the direction of incident plane wave and \(u\) is the total field which is the sum of the incident and scattered fields. Our purpose here is to develop a numerical algorithm based on the factorization method to solve this inverse problem. The classical factorization method was first proposed by Kirsch in [28], where a necessary and sufficient criterion was established to characterize both the location and shape of the obstacle by using the spectral system of the far-field operator defined by the far-field pattern associated with the incident plane waves. Moreover, this method can be implemented as a non-iterative algorithm for the inverse scattering problem which is very fast in the computations and does not need to know the boundary conditions of the unknown obstacles in advance. Therefore, the factorization method has been widely applied to many kinds of inverse scattering problems so far. In particular, the factorization method was extended to the case of near-field measurements in [21] to reconstruct the unknown obstacle with the aid of the spectral system of the near-field operator defined by the near-field data measured over a circle or sphere, associated with the incident point sources. We refer to the monograph [30] and the references therein for a comprehensive overview of the factorization method.

Note that, in order to establish the necessary and sufficient criterion on the characterization of the unknown obstacles, the key step of the classical factorization method is to prove that the constructed far-field operator or near-field operator defined by the measured data satisfies the Range Identity (see [30] Theorem 2.15) for the original version and [24] Theorem 3.2) and [51] Theorem 1.1) for the modified version). However, for the inverse problem under consideration, it is difficult to find a suitable
operator defined by the measured phaseless near-field data which satisfies the Range Identity. Thus, in this paper, we propose a modified factorization method, which is called the approximate factorization method, to numerically reconstruct the unknown scatterer from the phaseless near-field data. In doing so, an essential role is played by the asymptotic property in the linear space $L^2(S^1)$ of the phaseless near-field operator defined in terms of the phaseless near-field data measured on the circle $\partial B_R$ with large enough $R$. This asymptotic property is established, in this paper, by making use of the asymptotic properties of the scattered field and results from the theory of oscillatory integrals (see Theorem 3.7). In particular, utilizing this asymptotic result and constructing a modified phaseless near-field operator $\tilde{N}_R^{PW}$ defined by the measured data $|u(x, d)|$ with $x \in \partial B_R$, $d \in S^1$ and a modified far-field operator $\tilde{F}$ defined by the far-field pattern $u^\infty(\hat{x}, d)$ of the scattered field with $\hat{x}, d \in S^1$ (see the formulas (4.3) and (4.4) below), we can prove that the operators $(\tilde{N}_R^{PW})_\# := |\text{Re}(\tilde{N}_R^{PW})| + |\text{Im}(\tilde{N}_R^{PW})|$ and $\tilde{F}_\# := |\text{Re}(\tilde{F})| + |\text{Im}(\tilde{F})|$ satisfy the asymptotic property

$$
\|(\tilde{N}_R^{PW})_\# - (1/\sqrt{8k\pi R})\tilde{F}_\#\|_{L^2(S^1) \to L^2(S^1)} = O(1/R^\alpha)
$$

for any fixed $\alpha \in (1/2, 1)$ as the radius $R \to +\infty$ (see Remark 4.8). This means that the leading order term of the operator $(\tilde{N}_R^{PW})_\#$ in the linear space $L^2(S^1)$ is $(1/\sqrt{8k\pi R})\tilde{F}_#$ as $R \to +\infty$. Note that $(1/\sqrt{8k\pi R})$ is a constant for fixed $R$. On the other hand, we can prove that the operator $\tilde{F}_\#$ has a factorization satisfying the Range Identity in [31, Theorem 1.1] and thus the unknown obstacle can be recovered from the spectral system of $\tilde{F}_\#$ (see Theorem 4.7 below). Thus, it is expected that the unknown obstacle can be approximately recovered from the spectral system of $(\tilde{N}_R^{PW})_\#$ if $R$ is sufficiently large. Based on this, a numerical algorithm is proposed to reconstruct both the location and shape of the unknown scatterer from the phaseless near-field data. Numerical examples are also carried out to demonstrate the effectiveness of the inversion algorithm. It should be remarked that an approximate factorization or asymptotic factorization method has also been studied for inverse scattering problems with phased data (see [3, 19, 20, 50, 51]). To the best of our knowledge, the present paper is the first attempt to employ the idea of the factorization method in inverse scattering problems with phaseless data.

The rest part of this paper is organised as follows. In Section 2, we present the forward and inverse scattering problems considered. In Section 3, we study the asymptotic property in the linear space $L^2(S^1)$ of the phaseless near-field operator defined in terms of the phaseless near-field data measured on the circle $\partial B_R$ with large enough $R$ (see Theorem 3.7 below), which plays an essential role in the theoretical analysis of the approximate factorization method given in Section 4 for the inverse problem under consideration. The numerical implementation of our method is presented in Section 5. Numerical experiments are carried out in Section 6 to illustrate the effectiveness of the inversion algorithm. Some conclusions are given in Section 7.

2. The forward and inverse scattering problems. We now present the forward and inverse scattering problems considered in this paper. For simplicity, we restrict our attention to the 2D case. However, our analysis can be easily extended to the 3D case. Let the obstacle $D$ be an open and bounded domain in $\mathbb{R}^2$ with $C^2$-boundary $\partial D$ such that the exterior $\mathbb{R}^2 \setminus \overline{D}$ of $\overline{D}$ is connected. Given the incident field $u^i$, the total field $u = u^i + u^s$ is the sum of the incident field $u^i$ and the scattered
field $u^s$. If $D$ is an impenetrable obstacle, then the scattering problem by the obstacle $D$ is modeled as follows:

$$\Delta u^s + k^2 u^s = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{T},$$

$$\mathcal{B} u^s = f \quad \text{on } \partial D,$$

$$\lim_{r \to \infty} r^{\frac{3}{2}} \left( \frac{\partial u^s}{\partial r} - i ku^s \right) = 0, \quad r = |x|,$$

where $u^s := u - u^i$ is the scattered field, the boundary value $f := -\mathcal{B} u^i$, $k = \omega/c > 0$ is the wave number, and $\omega$ and $c$ are the wave frequency and speed in the homogeneous medium in $\mathbb{R}^2 \setminus \overline{T}$. Here, the equation (2.1) is called the Helmholtz equation, and the condition (2.2) is the well-known Sommerfeld radiation condition, which ensures that the scattered field is outgoing (see, e.g., [16]). Further, (2.2) is the boundary condition imposed on $\partial D$, which depends on the physical property of the obstacle $D$:

$$\mathcal{B} u^s = u^s \quad \text{for a sound-soft obstacle } D,$$

$$\mathcal{B} u^s = \partial u^s / \partial \nu + \rho u^s \quad \text{for an impedance obstacle } D,$$

where $\nu$ is the outward unit normal vector on the boundary $\partial D$ and $\rho \in L^\infty(\partial D)$ is the impedance function on the boundary $\partial D$ which is complex-valued with $\text{Im}(\rho) \geq 0$ almost everywhere on $\partial D$. In particular, when $\rho = 0$, the impedance boundary condition is reduced to the Neumann boundary condition, which corresponds to the sound-hard obstacle.

If $D$ is filled with an inhomogeneous medium characterized by the refractive index $n$, then the scattering problem is modeled by the medium scattering problem

$$\Delta u^s + k^2 n u^s = g \quad \text{in } \mathbb{R}^2,$$

$$\lim_{r \to \infty} r^{\frac{3}{2}} \left( \frac{\partial u^s}{\partial r} - i ku^s \right) = 0, \quad r = |x|,$$

where $g := -k^2(n-1)u^i$. In this paper, we assume that the contrast function $m := n - 1$ is supported in $\overline{T}$ and $n \in L^\infty(\mathbb{R}^2)$ with $\text{Re}[n(x)] \geq c_0 > 0$ for a constant $c_0$ and $\text{Im}[n(x)] \geq 0$ for almost all $x \in D$.

By the variational method [3] or the integral equation method [14, 16], it can be shown that the obstacle scattering problem (2.1)-(2.3) and the medium scattering problem (2.5)-(2.6) have a unique solution. In particular, it is well-known that the scattered field $u^s$ has the asymptotic behavior

$$u^s(x) = \frac{\alpha x}{\sqrt{8k\pi}} \sqrt{|x|} \left\{ u^\infty(x) + O \left( \frac{1}{|x|} \right) \right\}, \quad |x| \to \infty$$

uniformly for all observation directions $\hat{x} := x/|x| \in \mathbb{S}^1$ with $\mathbb{S}^1$ denoting the unit circle in $\mathbb{R}^2$ (see [30]). Here, $u^\infty(\hat{x})$ is called the far-field pattern of the scattered field $u^s(x)$, which is an analytic function of $\hat{x} \in \mathbb{S}^1$. In this paper, we consider the incident plane wave $u^i = u^i(x,d) := e^{ikr-d}$, where $d \in \mathbb{S}^1$ is the incident direction. Accordingly, the total field, the scattered field and the far-field pattern are denoted by $u(x,d)$, $u^s(x,d)$ and $u^\infty(\hat{x}, d)$, respectively.

To give a precise description of the inverse problem considered in this paper, let $B_R$ be the circle of radius $R$ and centered at $(0,0)$. Throughout this paper, we assume that $R$ is large enough so that $\overline{T} \subset B_R$. Then we shall consider the following inverse scattering problem with the phaseless near-field data at a fixed frequency.
Inverse Problem (IP): Reconstruct the unknown scatterer \( D \) from the measured phaseless near-field data \(|u(x, d)|\) for all \( x \in \partial D_R \) and \( d \in S^1 \), where \( u(x, d) \) is the total field of the scattering problem by the impenetrable obstacle or the inhomogeneous medium, associated with the incident plane wave \( u^i(x, d) \) at a fixed frequency.

Our purpose is to develop an approximate factorization method for solving the inverse problem (IP) with the radius \( R \) large enough, based on the asymptotic property in the linear space \( L(H^{1/2}(S^1), H^{-1/2}(S^1)) \) of the phaseless near-field operator defined in terms of the phaseless near-field data measured on the circle \( \partial B_R \) with large enough \( R \), together with the factorization of a modified far-field operator. To this end, we introduce the following notations which will be used in the rest of the paper. For any \( x, d \in S^1 \), let \( \hat{x} = (\cos \theta_x, \sin \theta_x) \), \( d = (\cos \theta_d, \sin \theta_d) \) with \( \theta_x, \theta_d \in [0, 2\pi] \). Denote by \( (\cdot, \cdot)_{L^2(S^1)} \) the inner product in the Hilbert space \( L^2(S^1) \) and by \( \langle \cdot, \cdot \rangle \) the duality pairing between \( H^{-1/2}(S^1) \) and \( H^{1/2}(S^1) \) extending the inner product \( (\cdot, \cdot)_{L^2(S^1)} \) (see, e.g., [30]). Throughout this paper, the constants may be different at different places. We remark that, to the best of our knowledge, no uniqueness result is available yet, e.g., [30]).

3. The asymptotic property of the phaseless near-field operator. In this section, we study the asymptotic property in the linear space \( L(H^{1/2}(S^1), H^{-1/2}(S^1)) \) of the phaseless near-field operator defined in terms of the phaseless near-field data measured on the circle \( \partial B_R \) with large enough \( R \), which plays an essential role in the development of the approximate factorization method for the inverse problem (IP). Precisely, we introduce the phaseless near-field operator \( N_{PW}^R : H^r(S^1) \to H^s(S^1) \), \( r, s \in \mathbb{R} \), by

\[
(N_{PW}^R \varphi)(\hat{x}) := \int_{S^1} (|u(R\hat{x}, d)|^2 - 1) e^{ikR\hat{x} \cdot d} \varphi(d) ds(d), \quad \varphi \in H^r(S^1),
\]

(3.1)

where \( \hat{x} = x/R, x \in \partial B_R, d \in S^1, u(x, d) = u^i(x, d) + u^s(x, d) \) and \( u^s(x, d) \) are the total and scattered fields of the scattering problem by the impenetrable obstacle or the inhomogeneous medium, associated with incident plane waves \( u^i(x, d) \). We also introduce the far-field operator \( F : H^r(S^1) \to H^s(S^1), r, s \in \mathbb{R} \), by

\[
(F\varphi)(\hat{x}) := \int_{S^1} u^\infty(\hat{x}, d) \varphi(d) ds(d), \quad \varphi \in H^r(S^1),
\]

(3.2)

where \( u^\infty(\hat{x}, d) \) for \( \hat{x}, d \in S^1 \) is the far-field pattern of the scattered field \( u^s(x, d) \). It can be seen that \( N_{PW}^R \) and \( F \) are well defined since \(|u(x, d)|\) is analytic in \( x \in \partial B_R \) and in \( d \in S^1 \), respectively, and \( u^\infty(\hat{x}, d) \) is analytic in \( \hat{x} \in S^1 \) and in \( d \in S^1 \), respectively (see, e.g., [16]). In what follows, we will study the relationship between \( N_{PW}^R \) and \( F \) when \( R \) is sufficiently large.

We need the following result on the property of the scattered field \( u^s \).

Lemma 3.1. For any \( x \in \mathbb{R}^2 \setminus \mathcal{D} \) with \(|x| \) large enough and \( d \in S^1 \), the scattered field \( u^s(x, d) \) has the asymptotic behavior

\[
u^s(x, d) = e^{i\pi/4} \frac{e^{ik|x|}}{\sqrt{8k\pi} |x|^{1/2}} u^\infty(\hat{x}, d) + u^s_{Res}(x, d)
\]

with

\[
||u^\infty(\cdot, d)||_{C^1(S^1)} \leq C, \quad ||u^s_{Res}(x, d)|| \leq \frac{C}{|x|^{3/2}},
\]

(3.3) (3.4) (3.5)
where $C > 0$ is a constant independent of $x$ and $d$.

Proof. This lemma can be easily obtained by using the well-posedness of the obstacle scattering problem \cite{[2.1]}-\cite{[2.3]} and the medium scattering problem \cite{[2.5]}-\cite{[2.6]}, together with the asymptotic behavior \cite{[2.7]} of the scattered field $u^*$ (see, e.g., \cite{[16]}).

From \cite{[3.1]}, \cite{[3.3]} and a direct calculation it easily follows that

\[
(N_{R \varphi})^P (\hat{x}) = \frac{e^{i\pi/4} e^{ikR}}{\sqrt{8k\pi} R^{1/2}} (F \varphi) (\hat{x}) + \left( H_{PW,R \varphi}^{(1)} \right) (\hat{x}) + \left( H_{PW,R \varphi}^{(2)} \right) (\hat{x}),
\]

where

\[
\left( H_{PW,R \varphi}^{(1)} \right) (\hat{x}) := \frac{e^{-i\pi/4} e^{-ikR}}{\sqrt{8k\pi} R^{1/2}} (L_{PW,R \varphi}) (\hat{x})
\]

with

\[
(L_{PW,R \varphi}) (\hat{x}) := \int_{\mathbb{S}^1} u_{\text{Res}}^{\alpha} (\hat{x}, d) e^{iRd \cdot \hat{x}} \varphi (d) ds(d)
\]

and

\[
\left( H_{PW,R \varphi}^{(2)} \right) (\hat{x}) := \int_{\mathbb{S}^1} \left[ u_{\text{Res}}^{\alpha} (\hat{x}, d) + u_{\text{Res}}^{\alpha} (\hat{x}, d) e^{iRd \cdot \hat{x}} + |u_{\text{Res}}^{\alpha} (\hat{x}, d)|^2 e^{iRd \cdot \hat{x}} \right] \varphi (d) ds(d).
\]

To proceed further we need the following result for oscillatory integrals (see \cite{[12]}).

Lemma 3.2 (Lemma 3.9 in \cite{[12]}). For any $-\infty < a < b < \infty$ let $u \in C^2[a, b]$ be real-valued and satisfy that $|u(t)| \geq 1$ for all $t \in (a, b)$. Assume that $a = x_0 < x_1 < \cdots < x_N = b$ is a division of $(a, b)$ such that $u'$ is monotone in each interval $(x_{i-1}, x_i)$, $i = 1, \ldots, N$. Then for any function $\phi$ defined on $(a, b)$ with integrable derivative and for any $\lambda > 0$,

\[
\left| \int_a^b e^{i\lambda u(t)} \phi(t) dt \right| \leq (2N + 2) \lambda^{-1} \left[ |\phi(b)| + \int_a^b |\phi'(t)| dt \right].
\]

We can now study the property of $H_{i \varphi}^{(i)}$, $i = 1, 2$, for $R$ large enough.

Lemma 3.3. For any $\varphi \in H^1 (\mathbb{S}^1)$ and for $R > 0$ large enough we have

\[
\| H_{PW,R \varphi}^{(i)} \|_{L^2 (\mathbb{S}^1)} \leq \frac{C}{R} \| \varphi \|_{H^1 (\mathbb{S}^1)},
\]

where $C > 0$ is a constant independent of $R$.

Proof. From \cite{[3.7]} and the fact that $C^\infty(\mathbb{S}^1)$ is dense in $H^1 (\mathbb{S}^1)$, it suffices to show that for any $\varphi \in C^\infty(\mathbb{S}^1)$ and $R$ large enough,

\[
\| L_{PW,R \varphi} \|_{L^2 (\mathbb{S}^1)} \leq \frac{C}{R^{1/2}} \| \varphi \|_{H^1 (\mathbb{S}^1)}.
\]

Let $\varphi$ be arbitrarily fixed function in $C^\infty(\mathbb{S}^1)$. For $\hat{x}, d \in \mathbb{S}^1$, let $\theta_{\hat{x}}, \theta_d$ be the real numbers as defined at the end of last section. Then, by the change of variables we have

\[
(L_{PW,R \varphi}) (\hat{x}) = \int_0^{2\pi} e^{i2kR \cos(\theta_{\hat{x}} - \theta_d)} u_{\text{Res}}^{\alpha}(\hat{x}, \theta_d) \overline{\varphi(\theta_d)} d\theta_d =: \left( L_{PW,R \varphi} \right) (\theta_{\hat{x}}),
\]
where $\tilde{u}^\infty(\theta_x, \theta_d) := u^\infty(\tilde{x}, d)$ and $\tilde{\varphi}(\theta_d) := \varphi(d)$ for $\theta_x, \theta_d \in [0, 2\pi]$. Let us define $\tilde{l}(\theta_x, \theta_d) := \frac{u^\infty(\theta_x, \theta_d)}{\varphi(\theta_d)}$ for $\theta_x, \theta_d \in [0, 2\pi]$. Since $u^\infty(\tilde{x}, d)$ is analytic in $d \in S^1$ and $\varphi \in C^\infty(S^1)$, then $\tilde{l}(\theta_x, \theta_d)$ and $\tilde{\varphi}(\theta_d)$ can be extended as $C^\infty$-smooth functions on $\mathbb{R}$ and $2\pi$-periodic with respect to $\theta_d$. Then it follows by the change of variables that for $\theta_x \in [0, 2\pi]$,

$$
(\tilde{L}_{PW,R}\tilde{\varphi})(\theta_x) = \left[ \int_{\theta_x - \pi}^{\theta_x} + \int_{\theta_x}^{\theta_x + \pi/2} + \int_{\theta_x + \pi/2}^{\theta_x + \pi} \right] e^{2ikR\cos(\theta_x - \theta_d)} \tilde{u}^\infty(\theta_x, \theta_d) \tilde{\varphi}(\theta_d) d\theta_d
$$

$$
= \int_0^\pi \tilde{\varphi} e^{-2ikR\cos(\pi/2)} \tilde{l}(\theta_x, t + \theta_d - \pi) dt + \int_0^\pi \tilde{\varphi} e^{2ikR\cos(\pi/2)} \tilde{l}(\theta_x, \theta_x - t) dt
$$

$$
+ \int_0^\pi \tilde{\varphi} e^{-2ikR\cos(\pi/2)} \tilde{l}(\theta_x, t + \theta_d + \pi - t) dt =: I_{1,PW}(\theta_x) + I_{2,PW}(\theta_x) + I_{3,PW}(\theta_x) + I_{4,PW}(\theta_x).
$$

We now estimate $I_{i,PW}$, $i = 1, 2, 3, 4$. We first consider $I_{1,PW}$. Let $\delta > 0$ be small enough so that $\sin \delta \geq \delta/2$ and let $R$ be large enough. Let $\theta_x \in [0, 2\pi]$ be arbitrarily fixed. Then we have

$$
I_{1,PW}(\theta_x) = \left[ \int_0^\pi + \int_0^\pi \right] e^{-2ikR\cos(\pi/2)} \tilde{l}(\theta_x, t + \theta_d - \pi) dt =: I_{1,PW}^{(1)}(\theta_x) + I_{1,PW}^{(2)}(\theta_x).
$$

Set $f(t) = -2\cos(t)/\delta$. Then $f'(t) = 2\sin(t)/\delta$. Thus, for $t \in [\delta, \pi/2]$ we have $|f'(t)| = 2\sin(t)/\delta \geq 2\sin\delta/\delta \geq 1$ and $f'(t)$ is monotone in $[\delta, \pi/2]$. By using Lemma 3.2, the formula (3.4) and the reciprocity relation that $u^\infty(\tilde{x}, d) = u^\infty(\tilde{-d}, \tilde{-x})$ for $\tilde{x}, \tilde{d} \in S^1$ (see, e.g., [30]), it is obtained that

$$
|I_{1,PW}^{(2)}(\theta_x)| = \left| \int_0^\pi e^{2ikRf(t)} \tilde{l}(\theta_x, t + \theta_d - \pi) dt \right|
$$

$$
\leq \frac{C}{R^\delta} \left( |\tilde{l}(\theta_x, \theta_x - \pi/2)| + \int_0^\pi \left| \frac{d}{dt} \tilde{l}(\theta_x, t + \theta_x - \pi) \right| dt \right)
$$

$$
\leq \frac{C}{R^\delta} \left( |\tilde{\varphi}(\theta_x - \pi/2)| + \|\tilde{\varphi}\|_{H^1([0, 2\pi])} \right).
$$

This yields that

$$
\|I_{1,PW}^{(2)}\|_{L^2[0,2\pi]} \leq \frac{C}{R^\delta}\|\tilde{\varphi}\|_{H^1([0, 2\pi])}. \tag{3.11}
$$

Further, it follows from the formula (3.4) that

$$
|I_{1,PW}^{(1)}(\theta_x)| \leq \int_0^\pi |\tilde{l}(\theta_x, t + \theta_x - \pi)| dt
$$

$$
\leq \left( \int_0^\pi 1^2 dt \right)^{1/2} \left( \int_0^\pi |\tilde{l}(\theta_x, t + \theta_x - \pi)|^2 dt \right)^{1/2}
$$

$$
\leq \frac{C}{R^\delta} \left( \int_0^\pi |\tilde{\varphi}(t + \theta_x - \pi)|^2 dt \right)^{1/2},
$$
which implies that
\[
\| I_{1, PW}^{(1)} \|_{L^2[0, 2\pi]} \leq C\delta \| \vec{\varphi} \|_{L^2[0, 2\pi]}.
\] (3.12)

Using (3.11) and (3.12) and taking \( \delta = R^{-1/2} \) give
\[
\| I_{1, PW} \|_{L^2[0, 2\pi]} \leq C\delta \| \vec{\varphi} \|_{L^2[0, 2\pi]} + \frac{C}{R^{1/2}} \| \vec{\varphi} \|_{H^1[0, 2\pi]} \leq \frac{C}{R^{1/2}} \| \vec{\varphi} \|_{H^1[0, 2\pi]},
\]
By a similar argument we can obtain that
\[
\| I_{i, PW} \|_{L^2[0, 2\pi]} \leq \frac{C}{R^{1/2}} \| \varphi \|_{H^1[0, 2\pi]}, \quad i = 2, 3, 4.
\]
Hence it follows that
\[
\| L_{PW,R}\varphi \|_{L^2(S^1)} = \| \tilde{L}_{PW,R}\tilde{\varphi} \|_{L^2[0, 2\pi]} \leq \frac{C}{R^{1/2}} \| \tilde{\varphi} \|_{H^1[0, 2\pi]} \leq \frac{C}{R^{1/2}} \| \varphi \|_{H^1(S^1)},
\]
that is, (3.13) holds. The proof is thus complete.

**Lemma 3.4.** For any \( \varphi \in L^2(S^1) \) and for \( R > 0 \) large enough, we have
\[
\| H_{PW,R}\varphi \|_{L^2(S^1)} \leq \frac{C}{R} \| \varphi \|_{L^2(S^1)},
\]
where \( C > 0 \) is a constant independent of \( R \).

**Proof.** Lemma 3.3 gives that for any \( \hat{x}, d \in S^1 \), \( |u^s(R\hat{x}, d)| \leq CR^{-1/2} \) when \( R \) is large enough. This, together with (3.5) and (3.9), implies the required result. The proof is then completed.

By the formula (3.10) and Lemmas 3.3 and 3.4, we can obtain the following result on the relationship between the operators \( N_{PW}^R \) and \( F \) when \( R \) is sufficiently large.

**Lemma 3.5.** For \( R > 0 \) large enough, we have
\[
\left\| N_{PW}^R - \frac{e^{i\pi/4}e^{ikR}}{\sqrt{8k\pi}} R^{1/2} F \right\|_{H^1(S^1) \to L^2(S^1)} \leq \frac{C}{R},
\]
where \( C > 0 \) is a constant independent of \( R \).

Denote by \( (N_{PW}^R)^* \) and \( F^* \) the adjoint operator of the operators \( N_{PW}^R \) and \( F \), respectively. Then, by a similar argument as in the proof of Lemma 3.3, we have the following result on the relationship between the operators \( (N_{PW}^R)^* \) and \( F^* \) when \( R \) is large enough.

**Lemma 3.6.** For \( R > 0 \) large enough, we have
\[
\left\| (N_{PW}^R)^* - \frac{e^{-i\pi/4}e^{-ikR}}{\sqrt{8k\pi}} R^{1/2} F^* \right\|_{H^1(S^1) \to L^2(S^1)} \leq \frac{C}{R},
\]
where \( C > 0 \) is a constant independent of \( R \).

**Proof.** By (3.9) it follows that
\[
(N_{PW}^R)^* = \frac{e^{-i\pi/4}e^{-ikR}}{\sqrt{8k\pi}} R^{1/2} F^* + H_{PW,R}^{(1)*} + H_{PW,R}^{(2)*},
\] (3.13)
where \( H_{PW,R}^{(1)*} \) and \( H_{PW,R}^{(2)*} \) denote the adjoint operator of \( H_{PW,R}^{(1)} \) and \( H_{PW,R}^{(2)} \), respectively, and are represented as follows: for \( d \in S^1 \) and \( \psi \in H^r(S^1) \) with \( r \in \mathbb{R} \),
\[
\left( H_{PW,R}^{(1)*}\psi \right)(d) := \frac{e^{i\pi/4}e^{ikR}}{\sqrt{8k\pi}} R^{1/2} \left( L_{PW,R}\psi \right)(d)
\]
with

\[
(L_{PW,R}^{2}) (\phi) := \int_{\mathbb{S}^1} u_{\phi}(\hat{x}, d)e^{-2ikR\hat{x} \cdot \hat{d}} \psi(\hat{x})d\hat{x},
\]

and

\[
(H_{PW,R}^{(2)}) (\phi) := \int_{\mathbb{S}^1} u_{\psi}^{*}(R\hat{x}, d) + u_{\psi}^{*}(R\hat{x}, d)e^{-2ikR\hat{x} \cdot \hat{d}} + |u_{\psi}(R\hat{x}, d)|^2 e^{-ikR\hat{x} \cdot \hat{d}} \psi(\hat{x})d\hat{x}.
\]

Similarly as in the proof of Lemmas 3.3 and 3.4, we can apply Lemmas 3.1 and 3.2 to obtain that

\[
\|H_{PW,R}^{(1)}\|_{H^1(\mathbb{S}^1) \to L^2(\mathbb{S}^1)} \leq \frac{C}{R}, \quad \|H_{PW,R}^{(2)}\|_{L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)} \leq \frac{C}{R}
\]

for \( R \) large enough. The required estimate then follows from (3.13) and (3.14). The proof is thus complete.

Making use of Lemmas 3.3 and 3.4 we can prove the following theorem.

**Theorem 3.7.** For \( R > 0 \) large enough, we have

\[
\left\| N_{PW}^R - \frac{e^{i\pi/4}}{\sqrt{8k\pi} R^{1/2}} F \right\|_{H^{1/2}(\mathbb{S}^1) \to H^{-1/2}(\mathbb{S}^1)} \leq \frac{C}{R},
\]

where \( C > 0 \) is a constant independent of \( R \).

**Proof.** Let \( R \) be large enough. First, it follows from Lemma 3.4 that

\[
\left\| N_{PW}^R - \frac{e^{i\pi/4}}{\sqrt{8k\pi} R^{1/2}} F \right\|_{L^2(\mathbb{S}^1) \to L^{-2}(\mathbb{S}^1)} \leq \frac{C}{R}.
\]

This, together with Lemma 3.3 and the interpolation property of Sobolev spaces (see [35, Theorem 8.13]), gives the required estimate. The proof is thus complete.

**4. The approximate factorization method.** In this section, we make use of the asymptotic behavior of the phaseless near-field operator to develop an approximate factorization method for the inverse problem. To this end, we first introduce some auxiliary operators. For \( \hat{x} \in \mathbb{S}^1 \) define \( \varphi_{m}(\hat{x}) := 1/\sqrt{2\pi} e^{im\theta_{\hat{x}}}, m \in \mathbb{Z}, \) where \( \theta_{\hat{x}} \in [0, 2\pi] \) is defined as above. It is well known that \( \{\varphi_{m} : m \in \mathbb{Z}\} \) is a complete orthonormal system in \( L^2(\mathbb{S}^1) \). Thus, for any \( \varphi \in L^2(\mathbb{S}^1) \) we have that, in the sense of mean square convergence,

\[
\varphi(\hat{x}) = \sum_{m=-\infty}^{+\infty} a_{m} \varphi_{m}(\hat{x}), \quad a_{m} := (\varphi, \varphi_{m})_{L^2(\mathbb{S}^1)} = \int_{\mathbb{S}^1} \varphi(\hat{x}) \overline{\varphi_{m}(\hat{x})} d\hat{x}. \tag{4.1}
\]

Further, it is known that \( H^{r}(\mathbb{S}^1) \) with \( r \geq 0 \) is a Hilbert space under the norm \( \|\varphi\|_{H^{r}(\mathbb{S}^1)} := \sum_{m=-\infty}^{+\infty} (1 + m^2)^{r/2} |a_{m}|^2 \) for \( \varphi \in H^{r}(\mathbb{S}^1) \) with the coefficients \( a_{m} \) given in (4.1). For more properties of the Sobolev Space \( H^{r}(\mathbb{S}^1), r \geq 0, \) and its dual space \( H^{-r}(\mathbb{S}^1), \) the reader is referred to [8][5]. Let \( B_{1/2} \) be the operator defined by

\[
B_{1/2}\varphi := \sum_{m=-\infty}^{+\infty} (1 + m^2)^{-1/4} a_{m} \varphi_{m}.
\]
for \( \varphi \in L^2(\mathbb{S}^1) \) with the coefficients \( a_m \) given in (4.1) and let \( B_{1/2}^* \) be the adjoint of \( B_{1/2} \). Then we have the following results concerning \( B_{1/2} \) and \( B_{1/2}^* \).

**Lemma 4.1.** \( B_{1/2} \) is bijective (and so boundedly invertible) from \( L^2(\mathbb{S}^1) \) to \( H^{1/2}(\mathbb{S}^1) \). Further, \( B_{1/2}^* \) is bijective (and so boundedly invertible) from \( H^{-1/2}(\mathbb{S}^1) \) to \( L^2(\mathbb{S}^1) \) and given by

\[
B_{1/2}^* \psi = \sum_{m=-\infty}^{+\infty} (1 + m^2)^{-1/4} b_m \varphi_m, \quad b_m := \langle \psi, \varphi_m \rangle = \int_{\mathbb{S}^1} \psi(\tilde{x}) \varphi_m(\tilde{x}) d\tilde{x} \tag{4.2}
\]

for \( \psi \in H^{-1/2}(\mathbb{S}^1) \), where \( \langle \cdot, \cdot \rangle \) is the duality pair between \( H^{-1/2}(\mathbb{S}^1) \) and \( H^{1/2}(\mathbb{S}^1) \).

**Proof.** Let \( \varphi \in L^2(\mathbb{S}^1) \) with the coefficients \( a_m \) given in (4.1). Then we have

\[
\|B_{1/2}\varphi\|_{H^{1/2}(\mathbb{S}^1)}^2 = \sum_{m=-\infty}^{+\infty} (1 + m^2)^{1/2} |(1 + m^2)^{-1/4} a_m|^2 = \sum_{m=-\infty}^{+\infty} |a_m|^2 = \|\varphi\|_{L^2(\mathbb{S}^1)}^2 < +\infty.
\]

This implies that \( B_{1/2} \) is a bounded operator from \( L^2(\mathbb{S}^1) \) to \( H^{1/2}(\mathbb{S}^1) \). For \( \varphi \in L^2(\mathbb{S}^1) \) with the coefficients \( a_m \) given in (4.1) define \( B_{-1/2} \) by

\[
B_{-1/2} \varphi := \sum_{m=-\infty}^{+\infty} (1 + m^2)^{1/4} a_m \varphi_m.
\]

Similarly as above, we can deduce that \( B_{-1/2} \) is a bounded operator from \( H^{1/2}(\mathbb{S}^1) \) to \( L^2(\mathbb{S}^1) \). It is easily seen that \( B_{-1/2} B_{1/2} \varphi = \varphi \) for \( \varphi \in L^2(\mathbb{S}^1) \) and \( B_{1/2} B_{-1/2} \varphi = \varphi \) for \( \varphi \in H^{1/2}(\mathbb{S}^1) \). Then \( B_{1/2} \) is bijective (and so boundedly invertible) from \( L^2(\mathbb{S}^1) \) to \( H^{1/2}(\mathbb{S}^1) \), and thus \( B_{1/2}^* \) is also bijective (and so boundedly invertible) from \( H^{-1/2}(\mathbb{S}^1) \) to \( L^2(\mathbb{S}^1) \). Further, let \( \psi \in H^{-1/2}(\mathbb{S}^1) \) and let \( b_m \) be given in (4.2). Then we have

\[
\langle B_{1/2} \varphi, \psi \rangle = \left( \sum_{m=-\infty}^{+\infty} (1 + m^2)^{-1/4} a_m \varphi_m, \psi \right) = \sum_{m=-\infty}^{+\infty} (1 + m^2)^{-1/4} \langle \varphi_m, \psi \rangle_{L^2(\mathbb{S}^1)} = \sum_{m=-\infty}^{+\infty} (1 + m^2)^{-1/4} \langle \varphi_m, \psi \rangle_{L^2(\mathbb{S}^1)} \tilde{b}_m = \left( \varphi, \sum_{m=-\infty}^{+\infty} (1 + m^2)^{-1/4} b_m \varphi_m \right)_{L^2(\mathbb{S}^1)}.
\]

Therefore, \( B_{1/2}^* \) has the form (4.2). This completes the proof.

With these preparations, we introduce the modified phaseless near-field operator \( \widetilde{N}_R^{PW} \) and the modified far-field operator \( \widetilde{F} \) by

\[
\widetilde{N}_R^{PW} := e^{-i(kR+\pi/4)} B_{1/2}^* N_R^{PW} B_{1/2}, \tag{4.3}
\]

\[
\widetilde{F} := B_{1/2}^* F B_{1/2}, \tag{4.4}
\]
respectively. The approximate factorization method for the inverse problem will be developed with utilizing the asymptotic property of the operator $\tilde{N}_R^{PW}$ for $R$ large enough (see Remark 4.3 below).

From the property of $B_{1/2}$ and $B_{1/2}^*$, we know that $\tilde{N}_R^{PW}$ and $\tilde{F}$ are bounded operators from $L^2(S^1)$ to $L^2(S^1)$. Further, with the aid of Theorem 4.2 and Lemma 4.1, we can obtain the following theorem on the asymptotic property of $\tilde{N}_R^{PW}$ and $\tilde{F}$ for $R$ large enough.

**Theorem 4.2.** For $R > 0$ large enough we have

$$\left\|\tilde{N}_R^{PW} - \frac{1}{\sqrt{8k\pi R}} \tilde{F}\right\|_{L^2(S^1) \to L^2(S^1)} \leq \frac{C}{R},$$

where $C > 0$ is a constant independent of $R$.

**Remark 4.3.** If the far-field operator $F$ is regarded as an operator from $L^2(S^1)$ to $L^2(S^1)$, then the modified far-field operator $\tilde{F}$ can be rewritten as

$$\tilde{F} = B_{1/2}^* I_0^* F I_0 B_{1/2}.$$  \hfill (4.5)

Here, $I_0$ is the imbedding operator from $H^{1/2}(S^1)$ to $L^2(S^1)$ and its adjoint $I_0^*$ is an imbedding operator from $L^2(S^1)$ to $H^{-1/2}(S^1)$. From [35, Chapter 8] it is seen that $I_0$ is injective and compact with a dense range in $L^2(S^1)$ and $I_0^*$ is injective and compact with a dense range in $H^{-1/2}(S^1)$. Note that the formula (1.3) will be used in the study of the operator $\tilde{F}$ (see Theorem 4.2 for details).

From Theorem 1.2 it is known that the leading order term of the operator $\tilde{N}_R^{PW}$ is $(1/\sqrt{8k\pi R})\tilde{F}$ as $R \to +\infty$ in the linear space $\mathcal{L}(L^2(S^1), L^2(S^1))$ of bounded linear operators from $L^2(S^1)$ to $L^2(S^1)$. Note that the coefficient $(1/\sqrt{8k\pi R})$ is a constant for arbitrarily fixed $R$. Thus, instead of studying the operator $\tilde{N}_R^{PW}$ directly, we will investigate the property of the operator $\tilde{F}$, making use of the factorization method presented in [30, 31], where the factorization of the far-field operator $F$ has been extensively investigated for inverse obstacle scattering problems. In what follows, we will employ some useful results in [30, 31] to derive a characterization of the obstacle $D$ from the operator $\tilde{F}$.

Define the boundary integral operators $S, K, K' : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$ and $T : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ by

$$(S\varphi)(x) := \int_{\partial D} \Phi(x, y)\varphi(y)ds(y), \quad x \in \partial D,$$

$$(K\varphi)(x) := \int_{\partial D} \frac{\partial \Phi(x, y)}{\partial n(y)}\varphi(y)ds(y), \quad x \in \partial D,$$

$$(K'\varphi)(x) := \int_{\partial D} \frac{\partial \Phi(x, y)}{\partial n(y)}\varphi(y)ds(y), \quad x \in \partial D,$$

$$(T\psi)(x) := \int_{\partial D} \frac{\partial \Phi(x, y)}{\partial n(y)}\psi(y)ds(y), \quad x \in \partial D,$$

where $\Phi(x, y) := (i/4)H_0^{(1)}(k|x - y|), x, y \in \mathbb{R}^2, x \neq y$, is the fundamental solution of the Helmholtz equation $\Delta w + k^2 w = 0$ in $\mathbb{R}^2$. Here, $H_0^{(1)}$ is the Hankel function of the first kind of order 0. From [16, 30, 45] it is known that the boundary integral operators $S, K, K' : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$ and $T : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ are bounded operators.
We now collect some results in \cite{30} for the factorization of the far-field operator $F$ in the cases of a sound-soft obstacle, an impedance obstacle and an inhomogeneous medium.

**Lemma 4.4.** Let $D$ be a sound-soft obstacle. Assume that $k^2$ is not a Dirichlet eigenvalue of $-\Delta$ in $D$. If the far-field operator $F$ is regarded as the operator from $L^2(\Omega^1)$ to $L^2(\Omega^1)$, then the following statements hold.

(a) The operator $F$ has the factorization

$$F = -G_{\text{Dir}}S^*G_{\text{Dir}},$$

where $G_{\text{Dir}} : H^{1/2}(\partial D) \to L^2(\Omega^1)$ is the data-to-pattern operator given by $G_{\text{Dir}} f = w^\infty$ with $w^\infty \in L^2(\Omega^1)$ being the far-field pattern of the scattered field $w^s$ of the exterior Dirichlet problem \cite{24} - \cite{28} with boundary value $f \in H^{1/2}(\partial D)$. Here, $G_{\text{Dir}}^* : L^2(\Omega^1) \to H^{-1/2}(\partial D)$ and $S^* : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$ are the adjoint of $G_{\text{Dir}}$ and $S$, respectively.

(b) The operator $G_{\text{Dir}}$ is compact, one-to-one with dense range in $L^2(\Omega^1)$. For any $z \in \mathbb{R}^2$ define the function $\phi_z \in L^2(\Omega^1)$ by

$$\phi_z(\hat{x}) := e^{-ik\hat{x} \cdot z}, \quad \hat{x} \in \Omega^1. \quad (4.10)$$

Then $\phi_z$ belongs to the range $\mathcal{R}(G_{\text{Dir}})$ of $G_{\text{Dir}}$ if and only if $z \in D$.

(c) The operator $S$ has the following properties.

(i) The operator $S$ is an isomorphism from $H^{-1/2}(\partial D)$ into $H^{1/2}(\partial D)$.

(ii) Let $S_i$, be defined by \cite{4} with $k = i$. Then $S_i$ is self-adjoint and coercive as an operator from $H^{-1/2}(\partial D)$ into $H^{1/2}(\partial D)$, that is, there exists $c_1 > 0$ with $\langle \varphi, S_i \varphi \rangle \geq c_1 \| \varphi \|^2_{H^{-1/2}(\partial D)}$ for all $\varphi \in H^{-1/2}(\partial D)$.

(iii) $\text{Im}(\langle \varphi, S \varphi \rangle) < 0$ for all $\varphi \in H^{-1/2}(\partial D)$ with $\varphi \neq 0$.

(iv) The difference $S - S_i$ is compact from $H^{-1/2}(\partial D)$ into $H^{1/2}(\partial D)$.

**Proof.** (a) was proved in \cite{30} Theorem 1.15, (b) was proved in \cite{30} as Theorem 1.12 and Lemma 1.13, and (c) was proved in \cite{30} Lemma 1.14.

**Lemma 4.5.** Let $D$ be an obstacle with the impedance boundary condition given in \cite{24}, with $p \in L^\infty(\partial D)$ and $\text{Im}(\rho) \geq 0$ almost everywhere on $\partial D$. Assume that $k^2$ is not an eigenvalue of $-\Delta$ in $D$ with respect to the impedance boundary condition. If the far-field operator $F$ is regarded as the operator from $L^2(\Omega^1)$ to $L^2(\Omega^1)$, then the following statements hold.

(a) The operator $F$ has the factorization

$$F = -G_{\text{imp}}T_{\text{imp}}^*G_{\text{imp}}^*,$$

where $G_{\text{imp}} : H^{-1/2}(\partial D) \to L^2(\Omega^1)$ is the data-to-pattern operator given by $G_{\text{imp}} f = w^\infty$ with $w^\infty \in L^2(\Omega^1)$ being the far-field pattern of the scattered field $w^s$ of the exterior impedance problem \cite{2.1} - \cite{2.3} with boundary value $f \in H^{-1/2}(\partial D)$ and $T_{\text{imp}} : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ is given by $T_{\text{imp}} = T + i(\text{Im}(\rho)) I + K_T + pK + \rho S_T$. Here, $G_{\text{imp}}^* : L^2(\Omega^1) \to H^{1/2}(\partial D)$ and $T_{\text{imp}}^* : H^{1/2}(\partial D) \to H^{1/2}(\partial D)$ are the adjoint of $G_{\text{imp}}$ and $T_{\text{imp}}$, respectively.

(b) The operator $G_{\text{imp}}$ is compact, one-to-one with dense range in $L^2(\Omega^1)$. For any $z \in \mathbb{R}^2$, $\phi_z$ belongs to the range $\mathcal{R}(G_{\text{imp}})$ of $G_{\text{imp}}$ if and only if $z \in D$, where $\phi_z$ is the function defined in \cite{4.11}.

(c) The operator $T_{\text{imp}}$ has the following properties.

(i) The operator $T_{\text{imp}}$ is an isomorphism from $H^{1/2}(\partial D)$ into $H^{-1/2}(\partial D)$. 

(ii) Let $T_i$ be defined by (4.9) with $k = i$. Then $-T_i$ is self-adjoint and coercive as an operator from $H^{1/2}(\partial D)$ into $H^{-1/2}(\partial D)$, that is, there exists $c_1 > 0$ with $-\langle T_i \varphi, \varphi \rangle \geq c_1 \| \varphi \|_{H^{1/2}(\partial D)}^2$ for all $\varphi \in H^{1/2}(\partial D)$.

(iii) $\text{Im}(T_{\text{imp}} \varphi, \varphi) > 0$ for all $\varphi \in H^{1/2}(\partial D)$ with $\varphi \neq 0$.

(iv) The difference $T_{\text{imp}} - T_1$ is compact from $H^{1/2}(\partial D)$ into $H^{-1/2}(\partial D)$.

Proof. (a) was proved in [30, Theorem 2.6], (b) can be shown by using [30, Theorem 2.8], and (c) can be proved by using [30, Theorem 1.26], [30, Theorem 2.6] and the argument as in the proof of [30, Lemma 2.7].

**Lemma 4.6.** Let $D$ be an inhomogeneous medium, where the refractive index $n \in L^\infty(D)$ satisfies that $\text{Re}[n(x)] \geq c_0 > 0$ and $\text{Im}[n(x)] \geq 0$ for almost all $x \in D$ with some constant $c_0$ and the contract function $m = n - 1$ is compactly supported in $\overline{T}$ and satisfies that $\text{Re}[m(x)] \geq c > 0$ or $\text{Re}[m(x)] \leq -c < 0$ for almost all $x \in D$ with some constant $c$. Assume that $k^2$ is not an eigenvalue of the interior transmission problem in $D$ in the sense of [30, Definition 4.7]. If the far-field operator $F$ is regarded as the operator from $L^2(S^1)$ to $L^2(S^1)$, then the following statements hold.

(a) The operator $F$ has the factorization

$$F = H_{\text{med}}^* T_{\text{med}} H_{\text{med}},$$

where $H_{\text{med}} : L^2(S^1) \to L^2(D)$ and $H_{\text{med}}^* : L^2(D) \to L^2(S^1)$ are defined as

$$(H_{\text{med}} \psi)(x) = \sqrt{|m(x)|} \int_{S^1} e^{ikx \cdot \psi}(d) ds(d), \quad x \in D,$$

$$(H_{\text{med}}^* \varphi)(\hat{x}) = \int_{D} e^{-ikx \cdot \varphi} \sqrt{|m(y)|} \varphi(dy), \quad \hat{x} \in S^1.$$

The operator $T_{\text{med}} : L^2(D) \to L^2(D)$ is defined by $T_{\text{med}} f = k^2 \text{sign}(m)|f + \sqrt{|m| |w| D|},$ where $\text{sign}(m) := m/|m|$ and $w \in H^1_{\text{loc}}(\mathbb{R}^2)$ is the radiating solution of the equation

$$\Delta w + k^2(1 + m)w = -k^2 \frac{m}{\sqrt{|m|}} f \quad \text{in} \quad \mathbb{R}^2.$$

(b) The operator $H_{\text{med}}^*$ is compact with dense range in $L^2(S^1)$. For any $z \in \mathbb{R}^2$, $\phi_z$ belongs to the range $\mathcal{R}(H_{\text{med}}^*)$ of $H_{\text{med}}^*$ if and only if $z \in D$, where $\phi_z$ is the function defined in (4.10).

(c) The operator $T_{\text{med}}$ has the following properties.

(i) The operator $T_{\text{med}}$ can be written in the form $T_{\text{med}} = T_{\text{med}}^{(0)} + K_{\text{med}},$ where $T_{\text{med}}^{(0)}$ has the form $T_{\text{med}}^{(0)} = k^2 \text{sign}(m)f$ for $f \in L^2(D)$ and $K_{\text{med}} : L^2(D) \to L^2(D)$ is compact. For the case when $\text{Re}[m(x)] \geq c > 0$ for almost all $x \in D$, $K_{\text{med}}^{(0)}$ is self-adjoint and coercive in $L^2(D)$. For the case when $\text{Re}[m(x)] \leq -c < 0$ for almost all $x \in D$, $K_{\text{med}}^{(0)}$ is self-adjoint and coercive in $L^2(D)$.

(ii) We have $\text{Im}(T_{\text{med}} f, f)_{L^2(D)} \geq 0$ for all $f \in L^2(D)$.

(iii) $\text{Im}(H_{\text{med}}^* f, f)_{L^2(D)} > 0$ for all $f \in \mathcal{R}(H_{\text{med}})$ with $f \neq 0$.

Proof. (a) was proved in [30, Theorem 4.5], (b) can be proved by using [30, Theorem 4.6] in conjunction with the compactness and injectivity of $H_{\text{med}}$, and (c) follows from [30, Theorem 4.8].
Using formula (14.5) in conjunction with Lemmas 4.4, 4.5 and 4.6 and the Range Identity [31, Theorem 1.1], we can obtain the following theorem on the characterization of the obstacle $D$, based on the factorization of the operator $\widetilde{F}_\# := |\text{Re}(F)| + |\text{Im}(F)|$.

**Theorem 4.7.** (a) Let $D$ be a sound-soft obstacle and let us assume that the conditions in Lemma 4.4 are satisfied. Then we have

$$z \in D \iff B_{1/2}^*\phi_z \in \mathcal{R}(\widetilde{F}_\#^{1/2})$$

$$\iff W(z) := \left[ \sum_{j=1}^{\infty} \left( B_{1/2}^*\phi_z, \psi_j \right)_{L^2(\mathbb{S}^1)} \right]^2 / \lambda_j^{-1} > 0,$$

where $\phi_z$ is the function defined in (14.10) and $\{\lambda_j; \psi_j\}_{j \in \mathbb{N}}$ is an eigensystem of the self-adjoint operator $\widetilde{F}_\#$.

(b) Let $D$ be an impedance obstacle and let us assume that the conditions in Lemma 4.4 are satisfied. Then the statements (4.11) and (4.12) hold.

(c) Let $D$ be filled with an inhomogeneous medium and let us assume that the conditions in Lemma 4.4 are fulfilled. Then the statements (4.11) and (4.12) hold.

**Proof.** We only prove (c). The proof of the statements (a) and (b) is similar.

Define $\tilde{H}_{\text{med}} := H_{\text{med}}I_0B_{1/2}$ and let $\tilde{H}_{\text{med}}^*$ be the adjoint of $\tilde{H}_{\text{med}}$. Then, by Remark 4.3 and Lemmas 4.4 and 4.6 we know that $\tilde{F}$ has the factorization $\tilde{F} = \tilde{H}_{\text{med}}^*I_0\tilde{H}_{\text{med}}$ and that $\tilde{H}_{\text{med}}^* = B_{1/2}^*I_0^*\tilde{H}_{\text{med}}^*$ is bounded from $L^2(D)$ to $L^2(\mathbb{S}^1)$ and compact with dense range in $L^2(\mathbb{S}^1)$. Thus, from the Range Identity [31, Theorem 1.1] and Lemma 4.6 it follows that the operator $\tilde{F}_\#$ is positive and $\mathcal{R}(\tilde{H}_{\text{med}}^*) = \mathcal{R}(\tilde{F}_\#^{1/2})$. On the other hand, by Lemma 4.4 and Remark 4.3 we have that $B_{1/2}^*$ is bijective (and so boundedly invertible) from $H^{-1/2}(\mathbb{S}^1)$ to $L^2(\mathbb{S}^1)$ and $I_0^*$ is injective from $L^2(\mathbb{S}^1)$ to $H^{-1/2}(\mathbb{S}^1)$. Thus it is easy to deduce that for any $z \in \mathbb{R}^2$, $\phi_z \in \mathcal{R}(\tilde{H}_{\text{med}}^*)$ if and only if $B_{1/2}^*\phi_z = B_{1/2}^*I_0^*\phi_z \in \mathcal{R}(\tilde{H}_{\text{med}})$. Consequently, by the above argument and (b) of Lemma 4.6 we obtain that $z \in D$ if and only if $B_{1/2}^*\phi_z \in \mathcal{R}(\tilde{F}_\#^{1/2})$. Finally, by Picard’s theorem [29, Theorem A.54] and the fact that the operator $\tilde{F}_\#$ is positive, the statement (4.12) follows. The proof is thus complete. \[\square\]

**Remark 4.8.** Define $(\tilde{N}_R^{PW})_\# := |\text{Re}(\tilde{N}_R^{PW})| + |\text{Im}(\tilde{N}_R^{PW})|$. Then, by Theorem 4.2 and the inequality in [37, pp. 30] we obtain that for any $\alpha \in (1/2, 1)$ and $R$ large enough,

$$\| (\tilde{N}_R^{PW})_\# - \frac{1}{\sqrt{8k\pi R}} \tilde{F}_\# \|_{L^2(\mathbb{S}^1)} \leq C^{(0)}_\alpha \left| \tilde{N}_R^{PW} - \frac{1}{\sqrt{8k\pi R}} \tilde{F} \right|_{L^2(\mathbb{S}^1)}^\alpha$$

$$\leq C^{(1)}_\alpha \left( \frac{1}{R} \right) \alpha,$$

where $C^{(0)}_\alpha$ and $C^{(1)}_\alpha$ are positive constants depending on $\alpha$ but not on $R$. This implies that the leading order term of the operator $(\tilde{N}_R^{PW})_\#$ is $(1/\sqrt{8k\pi R})\tilde{F}_\#$ in the linear space $\mathcal{L}(L^2(\mathbb{S}^1), L^2(\mathbb{S}^1))$ for $R$ large enough. On the other hand, by Theorem 4.4 the obstacle $D$ can be recovered by using the factorization of the operator $\tilde{F}_\#$. Thus, it is expected that if $R$ is large enough then the location and shape of the obstacle $D$ can be approximately recovered by using the indicator function $W(z)$ given in (4.12) with $\tilde{F}_#$ replaced by $(\tilde{N}_R^{PW})_#$. Based on these discussions, a numerical algorithm
for our inverse problem will be proposed in details in the next section. Note that
the algorithm is based on the factorization method presented in Theorem 4.7 and the
approximate formula given in [4,13] and thus called the approximate factorization
method.

5. Numerical implementation of the approximate factorization method. This section is devoted to the numerical implementation of the approximate
factorization method. Note that the operator $\tilde{N}_{R}^{PW}$ in (4.3) can not be numerically calculated since the operators $B_{1/2}$ and $B_{1/2}^{*}$ are represented as infinite series.
Thus, in order to give a numerical implementation of the method, we introduce the truncated operator of $\tilde{N}_{R}^{PW}$:

$$\tilde{N}_{R,M}^{PW} := e^{-i(kR^\pi/4)}B_{1/2,M}^{*}N_{R}^{PW}B_{1/2,M},$$

(5.1)

where $M \in \mathbb{N}$, $B_{1/2,M}$ is the truncated operator of $B_{1/2}$ given by

$$B_{1/2,M}\varphi := \sum_{m=-M}^{M} (1 + m^2)^{-1/4}a_m \varphi_m$$

for $\varphi \in L^2(\mathbb{S}^1)$ with the coefficients $a_m$ given in (4.1) and $B_{1/2,M}^{*}$ is the truncated operator of $B_{1/2}^{*}$ given by

$$B_{1/2,M}^{*}\psi := \sum_{m=-M}^{M} (1 + m^2)^{-1/4}b_m \varphi_m$$

for $\psi \in H^{-1/2}(\mathbb{S}^1)$ with the coefficients $b_m$ given in (4.2). For the truncated operators $B_{1/2,M}$ and $B_{1/2,M}^{*}$ we have the following lemma.

Lemma 5.1. For $M \in \mathbb{N}$ the following assertions hold.

(a) $\left\| B_{1/2,M} \right\|_{L^2(\mathbb{S}^1) \rightarrow H^{1/2}(\mathbb{S}^1)} \leq \left\| B_{1/2} \right\|_{L^2(\mathbb{S}^1) \rightarrow H^{1/2}(\mathbb{S}^1)}$.

(b) $\left\| B_{1/2,M}^{*} \right\|_{H^{-1/2}(\mathbb{S}^1) \rightarrow L^2(\mathbb{S}^1)} \leq \left\| B_{1/2}^{*} \right\|_{H^{-1/2}(\mathbb{S}^1) \rightarrow L^2(\mathbb{S}^1)}$.

(c) For any $r \in \mathbb{N}$, $\left\| B_{1/2}^{*} - B_{1/2,M}^{*} \right\|_{H^r(\mathbb{S}^1) \rightarrow L^2(\mathbb{S}^1)} \leq (1 + M)^{-r+1/2}$.

Proof. Assertions (a) and (b) follow easily from a direct calculation. Thus we only need to prove the assertion (c). Let $\varphi \in H^r(\mathbb{S}^1)$ be of the form (4.1). Then we have

$$\left\| B_{1/2,M}^{*}\varphi - B_{1/2,M}^{*}\varphi \right\|_{L^2(\mathbb{S}^1)}^2 = \sum_{|m| \geq M+1} |(1 + m^2)^{-1/4}a_m|^2$$

$$\leq (1 + M)^{-2(r+1/2)} \sum_{|m| \geq M+1} (1 + m^2)^r |a_m|^2$$

$$\leq (1 + M)^{-2(r+1/2)} \left\| \varphi \right\|_{H^r(\mathbb{S}^1)}^2.$$

This completes the proof of the lemma.

Using Theorem 5.1 and Lemma 5.2, we can obtain the following theorem for the truncated operator $\tilde{N}_{R,M}^{PW}$ and the modified far-field operator $\tilde{F}$.

Theorem 5.2. Let $M \in \mathbb{N}$ and let $R > 0$ be large enough. Then, for any $r \in \mathbb{N}$ there exists a constant $C_r > 0$ independent of $R$ such that

$$\left\| \tilde{N}_{R,M}^{PW} - \frac{1}{\sqrt{8\pi R}} \tilde{F} \right\|_{L^2(\mathbb{S}^1) \rightarrow L^2(\mathbb{S}^1)} \leq \frac{C_r}{R^{1/2}} \left( \frac{1}{R^{1/2}} + \frac{1}{(1 + M)^{r+1/2}} \right).$$
Proof. Let \( \tilde{F}_M \) be the truncated operator of \( \tilde{F} \) defined by
\[
\tilde{F}_M := B^*_1/2, M FB_{1/2, M},
\]
where \( F \) is the far-field operator given in (5.2). Arbitrarily fix \( r \in \mathbb{N} \). Then, by Lemma 5.1 we have
\[
\| \tilde{F}_M - \tilde{F} \|_{L^2(S^1)} \leq \|B^*_1/2, M F(B_{1/2, M} - B_{1/2})\|_{L^2(S^1)} + \|B^*_1/2, M - B^*_1/2\|_{L^2(S^1)} \leq \|B^*_1/2, M - B^*_1/2\|_{H^{1/2}(S^1)} + \|B^*_1/2, M - B^*_1/2\|_{H^{1/2}(S^1)} \leq C_r^{(1)}(1 + M)^{-(r+1)/2},
\]
where \( C_r^{(1)} > 0 \) is a constant depending on \( r \) but not on \( M \) and we have made use of Lemma 5.1 and the fact that \( u^w(\hat{x}, d) \) is analytic, respectively, in \( \hat{x} \in S^1 \) and \( d \in S^1 \) to obtain the last inequality. Then, by the above inequality, Theorem 5.2 and Lemmas 5.1 and 5.2 we obtain that
\[
\| \tilde{N}_{R,M}^{PW} - (1/\sqrt{8k\pi R}) \tilde{F} \|_{L^2(S^1)} \leq \| \tilde{N}_{R,M}^{PW} - (1/\sqrt{8k\pi R}) \tilde{F}_M \|_{L^2(S^1)} + \|B^*_1/2, M \|_{L^2(S^1)} \leq C_r^{(1)}(1 + M)^{-(r+1)/2}.
\]

The proof is thus complete. \( \square \)

Remark 5.3. Define \( (\tilde{N}_{R,M}^{PW}) := |\text{Re}(\tilde{N}_{R,M}^{PW})| + |\text{Im}(\tilde{N}_{R,M}^{PW})| \). For simplicity, we will choose \( M \geq R \) in the remaining part of this paper. Then it follows from Lemma 5.1, Theorem 5.2 and the inequality on pp. 30 that for any \( \alpha \in (1/2, 1) \), \( r \in \mathbb{N} \) and \( R \) large enough,
\[
\| (\tilde{N}_{R,M}^{PW}) \|_{L^2(S^1)} \leq C_r^{(0)}(1 + M)^{-(r+1)/2}
\]
and
\[
\| B^*_1/2, M \phi_z - B^*_1/2, M \phi_z \|_{L^2(S^1)} \leq (1 + M)^{-(r+1)/2} \| \phi_z \|_{H^r(S^1)} \leq (1 + R)^{-(r+1/2)} \| \phi_z \|_{H^r(S^1)}.
\]
where $C_a^{(0)}$, $C_{a,r}$, $C_{a,r}^{(2)}$ are positive constants independent of $M$ and $R$, and $\phi_2$ is the function defined in (1.10). Based on Theorem 4.7 and the approximate formulas (5.2) and (5.3), we can define the indicator function

$$W^\text{PW}_M(z) := \left[ \sum_{j=1}^{\infty} \left( B_{1/2,M}^{*} \phi_j, \psi_j \right)_{L^2(S)}^2 \lambda_j \right]^{-1},$$

where $\{\lambda_j; \psi_j\}_{j \in \mathbb{N}}$ is an eigensystem of the self-adjoint operator $(\hat{N}^\text{PW}_{R,M})^*$. From the above discussion and Theorem 4.7 it is expected that if $M \geq R$ and $R$ is sufficiently large, then the indicator function $W^\text{PW}_M(z)$ has a very similar property as $W(z)$ defined in (4.19). Thus it is expected that the obstacle $D$ can be numerically recovered by using the discrete form of the indicator function $W^\text{PW}_M(z)$ (see the formula (5.8)). This is indeed confirmed by the numerical examples carried out in Section 6. It should be pointed out that we are currently not able to give a rigorous theoretical analysis on the property of the indicator function $W^\text{PW}_M(z)$.

We now give the detailed numerical implementation of the approximate factorization method. The measured phaseless near-field data are obtained at $|u(x_i, x_j)|$ with $x_i \in \partial B_R$ and $x_j \in S^1$, $1 \leq i, j \leq L$, where $x_i = Rx_i$ and $\hat{x}_j$ are uniformly distributed points on $S^1$. Accordingly, by the trapezoidal rule, the operators $B_{1/2,M}$ and $B_{1/2,M}^*$ can be approximated as follows:

$$(B_{1/2,M} \varphi)(\hat{x}) = (B_{1/2,M}^* \varphi)(\hat{x})
= \sum_{m=-M}^{M} \left( 1 + m^2 \right)^{-1/4} \left( \int_{S^1} \varphi(d) \hat{\varphi}_m(d) ds(d) \right) \varphi_m(\hat{x})
\approx \frac{2\pi}{L} \sum_{m=-M}^{M} \left( 1 + m^2 \right)^{-1/4} \left( \sum_{j=1}^{L} \varphi(\hat{x}_j) \hat{\varphi}_m(\hat{x}_j) \right) \varphi_m(\hat{x}).$$

By a direct calculation the approximate values of $(B_{1/2,M} \varphi)(\hat{x})$ and $(B_{1/2,M}^* \varphi)(\hat{x})$ at the points $\hat{x}_j$, $j = 1, 2, \ldots, L$, can be computed as

$$\begin{pmatrix}
(B_{1/2,M} \varphi)(\hat{x}_1) \\
(B_{1/2,M} \varphi)(\hat{x}_2) \\
\vdots \\
(B_{1/2,M} \varphi)(\hat{x}_L)
\end{pmatrix}
\approx
\begin{pmatrix}
(B_{1/2,M}^* \varphi)(\hat{x}_1) \\
(B_{1/2,M}^* \varphi)(\hat{x}_2) \\
\vdots \\
(B_{1/2,M}^* \varphi)(\hat{x}_L)
\end{pmatrix}
\approx
B_{L,M}
\begin{pmatrix}
\varphi(\hat{x}_1) \\
\varphi(\hat{x}_2) \\
\vdots \\
\varphi(\hat{x}_L)
\end{pmatrix}.
\tag{5.4}
$$

Here, $B_{L,M}$ is a complex symmetric matrix defined by $B_{L,M} := \frac{2\pi}{L} C_{L,M} D_{M} C_{L,M}^*$, where $C_{L,M} = (c_{ij})_{1 \leq i \leq L, 1 \leq j \leq 2M+1}$ with $c_{ij} = \varphi_j \cdot (M+1)(\hat{x}_i)$ and $D_{M}$ is a diagonal matrix given by $D_{M} = \text{Diag}(d_1, d_2, \ldots, d_{2M+1})$ with $d_j = [1 + (j - M - 1)^2]^{-1/4}$. In particular, the approximate values of $(B_{1/2,M}^* \varphi)(\hat{x})$ at the points $\hat{x}_j$, $j = 1, 2, \ldots, L$, can be obtained from (5.4) with $\varphi$ replaced by $\phi_2$. Similarly, by the trapezoidal rule again, the approximate values of $(N_R^\text{PW} \varphi)(\hat{x})$ at the points $\hat{x}_j$, $j = 1, 2, \ldots, L$, can be computed as

$$\begin{pmatrix}
(N_R^\text{PW} \varphi)(\hat{x}_1) \\
(N_R^\text{PW} \varphi)(\hat{x}_2) \\
\vdots \\
(N_R^\text{PW} \varphi)(\hat{x}_L)
\end{pmatrix}
\approx \frac{2\pi}{L} N_{L}
\begin{pmatrix}
\varphi(\hat{x}_1) \\
\varphi(\hat{x}_2) \\
\vdots \\
\varphi(\hat{x}_L)
\end{pmatrix}.
\tag{5.5}$$
where $N_L = (n_{ij})_{1 \leq i, j \leq L}$ with $n_{ij} = |u(R\hat{x}_i, \hat{x}_j)|^2 - 1|e^{iR\hat{x}_i, \hat{x}_j}$. Then, by the definition (5.1), the approximate formulas (5.4) and (5.5) in conjunction with a direct calculation the approximate values of $(N_{R,M}^PW)(\hat{x})$ at the points $\hat{x}_j$, $j = 1, 2, \ldots, L$, can be easily obtained:

$$
\begin{pmatrix}
(\tilde{N}_{R,M}^{PW}\varphi)(\hat{x}_1) \\
(\tilde{N}_{R,M}^{PW}\varphi)(\hat{x}_2) \\
\vdots \\
(\tilde{N}_{R,M}^{PW}\varphi)(\hat{x}_L)
\end{pmatrix}
\approx
\frac{2\pi}{L} \tilde{N}_{L,M}
\begin{pmatrix}
\varphi(\hat{x}_1) \\
\varphi(\hat{x}_2) \\
\vdots \\
\varphi(\hat{x}_L)
\end{pmatrix},
$$

(5.6)

where

$$
\tilde{N}_{L,M} := e^{-i(kR+\pi/4)}B_{L,M}N_LB_{L,M}.
$$

(5.7)

Based on (5.4), (5.6) and the indicator function $W^{PW}_M(z)$ defined in Remark 5.3 we introduce the discrete indicator function $W^{PW}_{L,M}(z)$:

$$
W^{PW}_{L,M}(z) := \left(\sum_{i=1}^{L} |\tilde{\phi}_{z,M}^{i}\psi_{l,M}|^2 \lambda_{l,M}\right)^{-1},
$$

(5.8)

where $\tilde{\phi}_{z,M} := B_{L,M}(\phi_1(\hat{x}_1), \phi_2(\hat{x}_2), \ldots, \phi_L(\hat{x}_L))^T$ and $\{\lambda_{l,M}; \psi_{l,M}\}_{l=1}^{L}$ is the eigen-system of the complex symmetric matrix $(N_{L,M})^\# := |\text{Re}(N_{L,M})| + |\text{Im}(N_{L,M})|$. Here, the real and imaginary parts of the matrix $N_{L,M}$ are complex symmetric matrices given by

$$
\text{Re}(\tilde{N}_{L,M}) := \frac{1}{2} \left(\tilde{N}_{L,M} + \tilde{N}_{L,M}^*\right),
$$

$$
\text{Im}(\tilde{N}_{L,M}) := \frac{1}{2i} \left(\tilde{N}_{L,M} - \tilde{N}_{L,M}^*\right),
$$

respectively. Moreover, the matrices $|\text{Re}(\tilde{N}_{L,M})|$ and $|\text{Im}(\tilde{N}_{L,M})|$ are the discrete form of the operators $|\text{Re}(N^{PW}_L)|$ and $|\text{Im}(N^{PW}_L)|$, respectively, which are also complex symmetric and can be computed as in [4 Section 4]. From Theorem 4.7 and the discussion in Remark 5.3 it is expected that $W^{PW}_{L,M}(z)$ is much bigger for $z \in D$ than that for $z \notin D$ if $M \geq R$ and $R$ is sufficiently large. Here, the constant $2\pi/L$ in (5.6) is not taken into account for the indicator function (5.8) since it does not make any contribution to the numerical algorithm.

The numerical algorithm of the approximate factorization method can be presented as follows.

**Algorithm 5.1.** Let $K$ be the sampling region which contains the unknown obstacle $D$.

1. Choose $T_m$ to be a mesh of $K$. Set $R$ and $M$ to be large numbers with $M \geq R$.
2. Collect the phaseless near-field data $|u(x_i, \hat{x}_j)|$ with $x_i \in \partial B_R$ and $\hat{x}_j \in S^1$, $1 \leq i, j \leq L$, generated by the incident plane waves $\psi(x, \hat{x}_j) = e^{i\kappa x \cdot \hat{x}_j}$, $1 \leq j \leq L$.
3. Compute the matrix $N_{L,M}$ by using (5.7).
4. For all sampling points $z \in T_m$, compute the indicator function $W^{PW}_{L,M}(z)$ given in (5.8).
5. Locate all those sampling points $z \in T_m$ such that $W^{PW}_{L,M}(z)$ takes a large value, which represent the obstacle $D$. 

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6. Numerical examples. In this section, we present several numerical experiments to demonstrate the effectiveness of our inversion algorithm. To generate the synthetic data, the forward scattering problem is solved by using the Nyström method [16]. Further, the noisy phaseless near-field data \(|u_\delta(x, d)|, x \in \partial B_R, d \in S^1,\) are simulated by

\[ |u_\delta(x, d)| = |u(x, d)| (1 + \delta \zeta), \]

where \(\delta\) is the noise ratio and \(\zeta\) is the uniformly distributed random number in \([-1, 1]\).

In the following examples, we choose \(k = 10, L = 400, M = 100\) and \(R = 20\). The parametrization of the test curves for the boundary \(\partial D\) are given in Table 6.1.

| Type             | Parametrization                                      |
|------------------|------------------------------------------------------|
| Kite shaped      | \(x(t) = (\cos t + 0.65 \cos(2t) - 0.65, 1.5 \sin t), \ t \in [0, 2\pi]\) |
| Peanut shaped    | \(x(t) = \sqrt{\cos^2 t + 0.25 \sin^2 t} (\cos t, \sin t), \ t \in [0, 2\pi]\) |
| Rounded square   | \(x(t) = (3/4)(\cos^3 t + \cos t, \sin^3 t + \sin t), \ t \in [0, 2\pi]\) |
| Rounded triangle | \(x(t) = (2 + 0.3 \cos(3t))(\cos t, \sin t), \ t \in [0, 2\pi]\) |

**Table 6.1**

Parametrization of the boundary curves

Example 1. We first consider a peanut-shaped, sound-soft obstacle. See Figure 6.1(a) for the physical configuration. Figure 6.1 presents the reconstruction results of the obstacle by using the phaseless near-field data from incident plane waves without noise, with 10% noise and with 20% noise, respectively.

Example 2. We now consider a rounded square-shaped, sound-hard obstacle. See Figure 6.2(a) for the physical configuration. Figure 6.2 presents the reconstruction results of the obstacle by using the phaseless near-field data from incident plane waves without noise, with 10% noise and with 20% noise, respectively.

Example 3. This example considers a kite-shaped, impedance obstacle. The impedance function is given by \(\rho(x(t)) = (5 + 5i) * (1 + 0.5 \sin t), \ t \in [0, 2\pi]\), where \(x(t)\) is the parametrization of the boundary \(\partial D\). See Figure 6.3(a) for the physical configuration. Figure 6.3 presents the reconstruction results of the obstacle by using the phaseless near-field data from incident plane waves without noise, with 10% noise and with 20% noise, respectively.

Example 4. This example considers a rounded triangle-shaped, penetrable obstacle. The refractive index in \(D\) is given by \(n(x) = 2 + 1.5i\). See Figure 6.4(a) for the physical configuration. Figure 6.4 presents the reconstructed results of the obstacle by using the phaseless near-field data from incident plane waves without noise, with 10% noise and with 20% noise, respectively.

7. Conclusion. In this paper, we considered the inverse scattering problem with phaseless near-field data at a fixed frequency, associated with incident plane waves. An approximate factorization method is proposed to reconstruct the unknown obstacles from phaseless near-field data measured on a circle of a sufficiently large radius. The theoretical analysis of our approach is based on the asymptotic property in the linear space \(C(H^{1/2}(S^1), H^{-1/2}(S^1))\) of the phaseless near-field operator defined in terms of the phaseless near-field data measured on the circle of a large enough radius, together with the factorization of the modified far-field operator. Our inversion algorithm is independent of the physical properties of the unknown obstacles. Numerical experiments indeed show that our inversion algorithm provides satisfactory
reconstruction results of the unknown obstacles. Currently, we are extending this method to the case of incident point sources. Moreover, it is interesting to study the more challenging case of inverse electromagnetic scattering problems. This will be considered as a future work.

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Fig. 6.2. Reconstruction of a rounded square-shaped, sound-hard obstacle.

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Fig. 6.3. Reconstruction of a kite-shaped, impedance obstacle with the impedance function 
\( \rho(x(t)) = (5 + 5i) \ast (1 + 0.5 \sin t), \ t \in [0, 2\pi] \), on \( \partial D \).

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Factorization method for phaseless inverse problem

(a) Physical configuration     (b) $k = 10, R = 20$, no noise

(c) $k = 10, R = 20$, 10% noise     (d) $k = 10, R = 20$, 20% noise

Fig. 6.4. Reconstruction of a rounded triangle-shaped, penetrable obstacle with the refractive index $n(x) = 2 + 1.5i$ in $D$.

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