THE ELLIPTIC MODULAR SURFACE OF LEVEL 4 AND ITS REDUCTION MODULO 3

ICHIRO SHIMADA

ABSTRACT. The elliptic modular surface of level 4 is a complex $K3$ surface with Picard number 20. This surface has a model over a number field such that its reduction modulo 3 yields a surface isomorphic to the Fermat quartic surface in characteristic 3, which is supersingular. The specialization induces an embedding of the Néron-Severi lattices. Using this embedding, we determine the automorphism group of this $K3$ surface over a discrete valuation ring of mixed characteristic whose residue field is of characteristic 3.

The elliptic modular surface of level 4 has a fixed-point free involution that gives rise to the Enriques surface of type IV in Nikulin-Kondo-Martin’s classification of Enriques surfaces with finite automorphism group. We investigate the specialization of this involution to characteristic 3.

1. Introduction

Let $R$ be a discrete valuation ring, and let $\mathcal{X} \rightarrow \text{Spec } R$ be a smooth proper family of varieties over $R$. We denote by $X_\bar{\eta}$ the geometric generic fiber, and by $X_\bar{s}$ the geometric special fiber. Let $\text{Aut}(\mathcal{X}/R)$ denote the group of automorphisms of $\mathcal{X}$ over $\text{Spec } R$. Then we have natural homomorphisms

$$\text{Aut}(X_\bar{\eta}) \leftarrow \text{Aut}(\mathcal{X}/R) \rightarrow \text{Aut}(X_\bar{s}).$$

In this paper, we calculate the group $\text{Aut}(\mathcal{X}/R)$ in a case where $X_\bar{\eta}$ is the elliptic modular surface of level 4 and $X_\bar{s}$ is its reduction modulo 3. In this case, the surfaces $X_\bar{\eta}$ and $X_\bar{s}$ are $K3$ surfaces, and their automorphism groups have been calculated in [11] and [14], respectively, by Borcherds’ method [2, 3].

1.1. Elliptic modular surface of level 4. The elliptic modular surface of level $N$ is a natural compactification of the total space of the universal family over $\Gamma(N) \backslash \mathbb{H}$ of complex elliptic curves with level $N$ structure, where $\mathbb{H} \subset \mathbb{C}$ is the upper-half plane and $\Gamma(N) \subset \text{PSL}_2(\mathbb{Z})$ is the congruence subgroup of level $N$. This important class of surfaces was introduced and studied by Shioda [33].

The elliptic modular surface of level 4 is a $K3$ surface birational to the surface defined by the Weierstrass equation

$$Y^2 = X(X-1) \left( X - \frac{1}{2} \left( \sigma + \frac{1}{\sigma} \right)^2 \right),$$

where $\sigma$ is an affine parameter of the base curve $\mathbb{P}^1 = \Gamma(4) \backslash \mathbb{H}$. Shioda [33, 34] studied the reduction of this surface in odd characteristics. On the other hand,
Keum and Kondo \cite{KeumKondo} calculated the automorphism group of the elliptic modular surface of level 4.

To describe the results of Shioda \cite{Shioda, Shioda2} and Keum-Kondo \cite{KeumKondo}, we prepare some notation. A lattice is a free \(\mathbb{Z}\)-module \(L\) of finite rank with a nondegenerate symmetric bilinear form \(\langle \cdot, \cdot \rangle: L \times L \to \mathbb{Z}\). The group of isometries of a lattice \(L\) is denoted by \(O(L)\), which we let act on \(L\) from the right. A lattice \(L\) of rank \(n\) is said to be hyperbolic (resp. negative-definite) if the signature of \(L \otimes \mathbb{R}\) is \((1, n-1)\) (resp. \((0, n)\)). For a hyperbolic lattice \(L\), we denote by \(O^+(L)\) the stabilizer subgroup of a connected component of \(\{x \in L \otimes \mathbb{R} \mid \langle x, x \rangle > 0\}\) in \(O(L)\). Let \(Z\) be a smooth projective surface defined over an algebraically closed field. We denote by \(S_Z\) the lattice of numerical equivalence classes \([D]\) of divisors \(D\) of \(Z\), and call it the Néron-Severi lattice of \(Z\). Then \(S_Z\) is hyperbolic by Hodge index theorem. We denote by \(P_Z\) the connected component of \(\{x \in S_Z \otimes \mathbb{R} \mid \langle x, x \rangle > 0\}\) that contains an ample class. We then put \(N_Z := \{x \in P_Z \mid \langle x, [C] \rangle \geq 0 \text{ for all curves } C \text{ on } Z\}\).

We let the automorphism group \(\text{Aut}(Z)\) of \(Z\) act on \(S_Z\) from the right by the pull-back of divisors. Then we have a natural homomorphism

\[\text{Aut}(Z) \to \text{Aut}(N_Z) := \{g \in O^+(S_Z) \mid N_Z^g = N_Z\}\.\]

For an ample class \(h \in S_Z\), we put

\[\text{Aut}(Z, h) := \{g \in \text{Aut}(Z) \mid h^g = h\}\,\]

and call it the projective automorphism group of the polarized surface \((Z, h)\).

Let \(k_p\) be an algebraically closed field of characteristic \(p \geq 0\). From now on, we assume that \(p \neq 2\). Let \(\sigma: X_p \to \mathbb{P}^1\) be the smooth minimal elliptic surface defined over \(k_p\) by \(1.1\). Then \(X_p\) is a K3 surface. For simplicity, we use the following notation throughout this paper:

\[S_p := S_{X_p}, \quad P_p := P_{X_p}, \quad N_p := N_{X_p}\.\]

Shioda \cite{Shioda, Shioda2} proved the following:

**Theorem 1.1** (Shioda \cite{Shioda, Shioda2}). Suppose that \(p \neq 2\).

1. The elliptic surface \(\sigma: X_p \to \mathbb{P}^1\) has exactly 6 singular fibers. These singular fibers are located over \(\sigma = 0, \pm 1, \pm i, \infty\), and each of them is of type \(I_4\). The torsion part of the Mordell-Weil group of \(\sigma: X_p \to \mathbb{P}^1\) is isomorphic to \((\mathbb{Z}/4\mathbb{Z})^2\).

2. The Picard number \(\text{rank}(S_p)\) of \(X_p\) is

\[
\begin{cases}
20 & \text{if } p = 0 \text{ or } p \equiv 1 \text{ mod } 4, \\
22 & \text{if } p = 3 \text{ mod } 4. 
\end{cases}
\]

3. If \(k_0 = \mathbb{C}\), the transcendental lattice of the complex K3 surface \(X_0\) is

\[
\begin{pmatrix}
4 & 0 \\
0 & 4
\end{pmatrix}.
\]

4. The K3 surface \(X_3\) is isomorphic to the Fermat quartic surface

\[F_3 : x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0\]

in characteristic 3.
It follows from Theorem 1.1 (3) and the theorem of Shioda-Inose 32 that, over the complex number field, $X_0$ is isomorphic to the Kummer surface associated with $E \rtimes E$, where $E$ is the elliptic curve $\mathbb{C}/(\mathbb{Z} \oplus 2\mathbb{Z})$. Therefore the result of Keum-Kondo 11 contains the calculation of $\text{Aut}(X_0)$.

**Definition 1.2.** Let $Z$ be a K3 surface defined over $k_p$. A **double-plane polarization** is a vector $b = [H] \in N_Z \cap S_Z$ with $\langle b, b \rangle = 2$ such that the corresponding complete linear system $|H|$ is base-point free, so that $|H|$ induces a surjective morphism $\Phi_b : Z \to \mathbb{P}^2$. Let $b$ be a double-plane polarization, and let $Z \to Z_b \to \mathbb{P}^2$ be the Stein factorization of $\Phi_b$. Then we have a **double-plane involution** $g(b) \in \text{Aut}(Z)$ associated with the finite double covering $Z_b \to \mathbb{P}^2$. Let $\text{Sing}(b)$ denote the singularities of the normal K3 surface $Z_b$. Since $Z_b$ has only rational double points as its singularities, we have the $\text{ADE}$-type of $\text{Sing}(b)$.

**Remark 1.3.** Suppose that an ample class $a \in S_Z$ and a vector $b \in S_Z$ with $\langle b, b \rangle = 2$ are given. Then we can determine whether $b$ is a double-plane polarization or not, and if $b$ is a double-plane polarization, we can calculate the set of classes of smooth rational curves contracted by $\Phi_b : Z \to \mathbb{P}^2$, and compute the matrix representation of the double-plane involution $g(b) : Z \to Z$ on $S_Z$. These algorithms are described in detail in 27 (and also in 30). They are the key tools of this paper.

We re-calculated $\text{Aut}(X_0)$ by using these algorithms, and obtained a generating set of $\text{Aut}(X_0)$ different from the one given in 11.

**Theorem 1.4** (Keum–Kondo 11). There exist an ample class $h_0 \in S_0$ and four double-plane polarizations $b_{80}, b_{121}, b_{296}, b_{688} \in S_0$ such that $\text{Aut}(X_0)$ is generated by the projective automorphism group $\text{Aut}(X_0, h_0) \cong (\mathbb{Z}/2\mathbb{Z})^5 : S_5$ and the double-plane involutions $g(b_{80}), g(b_{112}), g(b_{296}), g(b_{688})$.

See Table 1.1 for the properties of the double-plane polarizations $b_d$. See Proposition 4.2 for the geometric meaning of these generators of $\text{Aut}(X_0)$ with respect to the action of $\text{Aut}(X_0)$ on $N_0$. In Section 4.3 we also give a detailed description of the finite group $\text{Aut}(X_0, h_0)$ in terms of a certain graph $\mathcal{L}_{40}$.

**Remark 1.5.** In 11, the automorphism group $\text{Aut}(X_3) \cong \text{Aut}(F_3)$ of the Fermat quartic surface in characteristic 3 was calculated (see Theorem 4.4). This calculation also plays an important role in the proof of our main results.

1.2. **Main results.** In 11 and 14, the following was proved, and hence, from now on, we regard $\text{Aut}(X_0)$ as a subgroup of $\text{O}^+(S_0)$ and $\text{Aut}(X_3)$ as a subgroup of $\text{O}^+(S_3)$.

**Proposition 1.6.** In each case of $X_0$ and $X_3$, the action of the automorphism group on the Néron-Severi lattice is faithful.
Let $R$ be a discrete valuation ring whose fraction field $K$ is of characteristic 0 and whose residue field $k$ is of characteristic 3. Suppose that $\sqrt{-1} \in R$. Then there exists a smooth family of $K$3 surfaces $X \to \text{Spec } R$ over $R$ such that the geometric generic fiber $X \otimes_R \overline{K}$ is isomorphic to $X_0$ and the geometric special fiber $X \otimes_R k$ is isomorphic to $X_3$ (see Section 2.5). By Proposition 3.3 of Maulik and Poonen [18], the specialization from $X \otimes_R K$ to $X \otimes_R k$ gives rise to a homomorphism $\rho: S_0 \to S_3$.

In Section 2.3, we give an explicit description of $\rho$. It turns out that $\rho$ is a primitive embedding of lattices. We regard $S_0$ as a sublattice of $S_3$ by $\rho$, and put $O^+(S_3, S_0) := \{ g \in O^+(S_3) \mid S_g^0 = S_0 \}$.

Then we have a natural restriction homomorphism $\tilde{\rho}: O^+(S_3, S_0) \to O^+(S_0)$.

The main results of this paper are as follows:

**Theorem 1.7.** The restriction of $\tilde{\rho}$ to $O^+(S_3, S_0) \cap \text{Aut}(X_3)$ induces an injective homomorphism $\tilde{\rho}|_{\text{Aut}}: O^+(S_3, S_0) \cap \text{Aut}(X_3) \hookrightarrow \text{Aut}(X_0)$.

The image of $\tilde{\rho}|_{\text{Aut}}$ is generated by the finite subgroup $\text{Aut}(X_0, h_0)$ and the two double-plane involutions $g(b_{112}), g(b_{80})$. The other double-plane involutions $g(b_{220})$ and $g(b_{296})$ do not belong to the image of $\tilde{\rho}|_{\text{Aut}}$.

Let $R'$ be a finite extension of $R$, and let $X' := X \otimes_R R' \to \text{Spec } R'$ be the pull-back of $X \to \text{Spec } R$. We have a natural embedding $\text{Aut}(X'/R') \hookrightarrow \text{Aut}(X'/R')$. We put $\text{Aut}(X/R) := \text{colim}_{R'} \text{Aut}(X'/R')$.

Let $\text{res}_3: \text{Aut}(X/R) \to \text{Aut}(X_3)$ and $\text{res}_0: \text{Aut}(X/R) \to \text{Aut}(X_0)$ denote the restriction homomorphisms. It is obvious that $\text{res}_0$ is injective, and that the following diagram commutes.

$$
\begin{array}{ccc}
O^+(S_3, S_0) \cap \text{Aut}(X_3) & \xrightarrow{\rho|_{\text{Aut}}} & \text{Aut}(X_0) \\
\text{res}_3 & \xleftarrow{\text{res}_0} & \text{res}_3 \\
\end{array}
$$

**Theorem 1.8.** The image of $\text{res}_0$ is equal to the image of $\rho|_{\text{Aut}}$.

Thus we have obtained a set of generators of $\text{Aut}(X/R)$.

1.3. *Enriques surfaces.* By Nikulin [20] and Kondo [12], the complex Enriques surfaces with finite automorphism group are classified, and this classification is extended to Enriques surfaces in odd characteristics by Martin [16]. The Enriques surfaces in characteristic $\neq 2$ with finite automorphism group are divided into seven classes I-VII. In this paper, we concentrate on the Enriques surface of type IV.

**Definition 1.9.** A fixed-point free involution of a $K3$ surface in characteristic $\neq 2$ is called an *Enriques involution*. An Enriques surface $Y$ in characteristic $\neq 2$ is of type IV if $\text{Aut}(Y)$ is of order 320. An Enriques involution of a $K3$ surface is of type IV if the quotient Enriques surface is of type IV.
Proposition 1.10 (Kondo [12], Martin [16]). In each characteristic $\neq 2$, an Enriques surface of type IV exists and is unique up to isomorphism. There exist exactly 20 smooth rational curves on an Enriques surface of type IV.

Let $Y_{IV,p}$ denote an Enriques surface of type IV in characteristic $p \neq 2$. Kondo [12] showed that the covering $K3$ surface of $Y_{IV,0}$ is isomorphic to $X_0$.

Proposition 1.11. There exist exactly 6 Enriques involutions in the projective automorphism group $\text{Aut}(X_0,h_0)$, and each of them is of type IV.

By Theorem 1.7, these six Enriques involutions in $\text{Aut}(X_0,h_0)$ are specialized to involutions of $X_3$.

Theorem 1.12. Let $\varepsilon_3 \in \text{Aut}(X_3)$ be an involution that is mapped to an Enriques involution in $\text{Aut}(X_0,h_0)$ by $\tilde{\rho}|_{\text{Aut}}$. Then $\varepsilon_3$ is an Enriques involution of type IV, and the pull-backs of the 20 smooth rational curves on $X_3/\langle \varepsilon_3 \rangle \cong Y_{IV,3}$ by the quotient morphism $X_3 \to X_3/\langle \varepsilon_3 \rangle$ are lines of the Fermat quartic surface $F_3 \cong X_3$.

During the investigation, we have come to notice that the geometry of $X_p$ and $Y_{IV,p}$ is closely related to the Petersen graph (Figure 1.1). See Section 2 for this relation. As a by-product, we see that the dual graph of the 20 smooth rational curves on $Y_{IV,p}$ is as Figure 1.2. Compare Figure 1.2 with the picturesque but complicated figure of Kondo (Figure 4.4 of [12]).

It has been observed that the Petersen graph is related to various $K3$/Enriques surfaces. See, for example, Vinberg [36] for the relation with the singular $K3$ surface with the transcendental lattice of discriminant 4. See also Dolgachev-Keum [5] and Dolgachev [6] for the relation with Hessian quartic surfaces and associated Enriques surfaces.

1.4. Plan of the paper. In Section 2 we present a precise description of the embedding $\rho: S_0 \hookrightarrow S_3$. First we introduce the notion of QP-graphs. Then, using an isomorphism $X_3 \cong F_3$ given by Shioda [34], we show that $S_0$ is a lattice obtained from a QP-graph, and write the embedding $\rho: S_0 \hookrightarrow S_3$ explicitly. An elliptic modular surface of level 4 over a discrete valuation ring is constructed, and the relation with the Petersen graph is explained geometrically. In Section 3 we review the method of Borcherds [2, 3] to calculate the orthogonal group of an even
hyperbolic lattice, and fix terminologies about chambers. The application of this method to K3 surfaces is also explained. In Section 4 we review the results of [11] for Aut($X_0$) and of [14] for Aut($X_3$). Using the chamber tessellations of $N_0$ and $N_3$ obtained in these works, we give a proof of Theorems 1.7 and 1.8 in Section 5.

In Section 6, we investigate Enriques involutions of $X_0$ and $X_3$.

In this paper, we fix bases of lattices and reduce proofs of our results to simple computations of vectors and matrices. Unfortunately, these vectors and matrices are too large to be presented in the paper. We refer the reader to the author’s web site [31] for these data. In the computation, we used GAP [9].

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2. The lattices $S_0$ and $S_3$

2.1. Graphs and lattices. First we fix terminologies and notation about graphs and lattices.

A graph (or more precisely, a weighted graph) is a pair $(V, \eta)$, where $V$ is a set of vertices and $\eta$ is a map from the set $V^2 \setminus \{(v,v)\}$ of non-ordered pairs of distinct elements of $V$ to $\mathbb{Z}_{\geq 0}$. When the image of $\eta$ is contained in $\{0, 1\}$, we say that $(V, \eta)$ is simple, and denote it by $(V,E)$, where $E = \eta^{-1}(1)$ is the set of edges. Let $\Gamma = (V,E)$ and $\Gamma' = (V',E')$ be simple graphs. A map $\gamma : \Gamma \rightarrow \Gamma'$ of simple graphs is a pair of maps $\gamma_V : V \rightarrow V'$ and $\gamma_E : E \rightarrow E'$ such that, for all $\{v,v'\} \in E$, we have $\gamma_E(\{v,v'\}) = \{\gamma_V(v),\gamma_V(v')\} \in E'$. A graph is depicted by indicating each vertex by $\bullet$ and $\eta(\{v,v'\})$ by the number of line segments connecting $v$ and $v'$. The Petersen graph $\mathcal{P} = (V_P,E_P)$ is the simple graph given by Figure 1.1. It is well-known that the automorphism group Aut($\mathcal{P}$) of $\mathcal{P}$ is isomorphic to the symmetric group $S_5$.

A submodule $M$ of a free $\mathbb{Z}$-module $L$ is primitive if $L/M$ is torsion free. A nonzero vector $v$ of $L$ is primitive if $\mathbb{Z}v \subset L$ is primitive.

Let $L$ be a lattice. We say that $L$ is even if $\langle x,x \rangle \in 2\mathbb{Z}$ for all $x \in L$. The dual lattice of $L$ is the free $\mathbb{Z}$-module $L' := \text{Hom}(L,\mathbb{Z})$, into which $L$ is embedded by
Let \( \ker \langle \, \rangle \subset \mathbb{Z}^V \) denote the submodule \( \{ x \in \mathbb{Z}^V \mid \langle x, y \rangle = 0 \text{ for all } y \in \mathbb{Z}^V \} \). Then \( \langle \Gamma \rangle := \mathbb{Z}^V / \ker \langle \, \rangle \) has a natural structure of the even lattice.

Suppose that \( \mathbb{Z} \) is a \( K3 \) surface or an Enriques surface defined over an algebraically closed field. Let \( \mathcal{L} \) be a set of smooth rational curves on \( Z \). Then the mapping \( C \mapsto [C] \) embeds \( \mathcal{L} \) into the Néron-Severi lattice \( S_Z \) of \( Z \). The dual graph of \( \mathcal{L} \) is the graph \( (\mathcal{L}, \eta) \), where \( \eta(\{C_1, C_2\}) \) is the intersection number of distinct two curves \( C_1, C_2 \in \mathcal{L} \). By abuse of notation, we sometimes use \( \mathcal{L} \) to denote the dual graph \( (\mathcal{L}, \eta) \) or the image of the embedding \( \mathcal{L} \hookrightarrow S_Z \). Then the even lattice \( \langle \mathcal{L} \rangle \) constructed from the dual graph of \( \mathcal{L} \) is canonically identified with the sublattice of \( S_Z \) generated by \( \mathcal{L} \subset S_Z \), because every smooth rational curve on \( Z \) has self-intersection number \(-2\).

**Example 2.1.** Let \( \Gamma \) be the graph given by Figure 1.2. Then \( \langle \Gamma \rangle \) is an even hyperbolic lattice of rank 10 with \( A(\langle \Gamma \rangle) \cong (\mathbb{Z}/2\mathbb{Z})^2 \). Since the Néron-Severi lattice of an Enriques surface is unimodular of rank 10, the classes of 20 smooth rational curves on \( Y_{IV, p} \) generate a sublattice of index 2 in the Néron-Severi lattice.

2.2. QP-graph. We introduce the notion of QP-graphs, where QP stands for a quadruple covering of the Petersen graph. In the following, a quadrangle means the simple graph \( \square \).

**Definition 2.2.** A QP-graph is a pair \( (Q, \gamma) \) of a simple graph \( Q = (V_Q, E_Q) \) and a map \( \gamma : Q \rightarrow P \) to the Petersen graph with the following properties.

(i) The map \( \gamma_V : V_Q \rightarrow V_P \) is surjective, and every fiber of \( \gamma_V \) is of size 4.

(ii) For any edge \( e \) of \( P \), the subgraph \( (\gamma^{-1}_V(\{e\}), \gamma^{-1}_E(\{e\})) \) of \( Q \) is isomorphic to the disjoint union of two quadrangles.

(iii) Any two distinct quadrangles in \( Q \) have at most one common vertex.

A map \( \gamma : Q \rightarrow P \) satisfying conditions (i)-(iii) is called a QP-covering map. Two QP-graphs \( (Q, \gamma) \) and \( (Q', \gamma') \) are said to be isomorphic if there exists an isomorphism \( h : Q \rightarrow Q' \) such that \( \gamma' \circ h = \gamma \).

**Proposition 2.3.** Up to isomorphism, there exist exactly two QP-graphs \( (Q_0, \gamma_0) \) and \( (Q_1, \gamma_1) \). The even lattices \( \langle Q_0 \rangle \) and \( \langle Q_1 \rangle \) are hyperbolic of rank 20. The discriminant group \( A(\langle Q_0 \rangle) \) of \( \langle Q_0 \rangle \) is isomorphic to \( (\mathbb{Z}/2\mathbb{Z})^2 \), whereas \( A(\langle Q_1 \rangle) \) is isomorphic to \( (\mathbb{Z}/4\mathbb{Z})^2 \).
Proof: We enumerate all isomorphism classes of QP-graphs. Let $\Delta$ be the set of ordered pairs $\{\{i_1, i_2\}, \{i_3, i_4\}\}$ of non-ordered pairs of elements of $\{1, 2, 3, 4\}$ such that $\{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}$. We have $|\Delta| = 6$. Let $T(\Delta)$ be the set of ordered triples $[\delta_1, \delta_2, \delta_3]$ of elements of $\Delta$ such that, if $\mu \neq \nu$, then $\delta_\mu = [[i_1, i_2], \{i_3, i_4\}]$ and $\delta_\nu = [[i_1', i_2'], \{i_3', i_4'\}]$ satisfy $|\{i_1, i_2\} \cap \{i_1', i_2'\}| = 1$. Then we have $|T(\Delta)| = 48$. The following facts can be easily verified.

(a) The natural action on $T(\Delta)$ of the full permutation group $S_4$ of $\{1, 2, 3, 4\}$ decomposes $T(\Delta)$ into two orbits $o_1$ and $o_2$ of size 24.

(b) For any triple $[\delta_1, \delta_2, \delta_3] \in T(\Delta)$ and any permutation $\mu, \nu, \rho$ of $1, 2, 3$, the triple $[\delta_\mu, \delta_\nu, \delta_\rho]$ belongs to the same orbit as $[\delta_1, \delta_2, \delta_3]$.

(c) For $\delta = [[i_1, i_2], \{i_3, i_4\}] \in \Delta$, we put $\bar{\delta} := [[i_3, i_4], \{i_1, i_2\}] \in \Delta$. Then $[\delta_1, \delta_2, \delta_3] \in T(\Delta)$ and $[\delta_1, \delta_2, \delta_3] \in T(\Delta)$ belong to different orbits.

Let $\psi$ be a map from the set $V_P$ of vertices of $P$ to the set $\{o_1, o_2\}$ of the orbits. We construct a QP-graph $(Q_\psi, \gamma_\psi)$ with the set of vertices

$$V_Q := V_P \times \{1, 2, 3, 4\}$$

as follows. For each vertex $v \in V_P$, we choose an element $[\delta_1, \delta_2, \delta_3]$ from the orbit $\psi(v)$, choose an ordering $e_1, e_2, e_3$ on the three edges of $P$ emitting from $v$, and assign $\delta_i$ to the pair $(v, e_i)$ for $i = 1, 2, 3$. Let $e = \{v, v'\}$ be an edge of $P$. Suppose that $\delta = [[i_1, i_2], \{i_3, i_4\}]$ is assigned to $(v, e)$ and $\delta' = [[i_1', i_2'], \{i_3', i_4'\}]$ is assigned to $(v', e)$. Then the edges of $Q_\psi$ lying over the edge $e$ of $P$ are the following 8 edges.

\[
\begin{array}{c}
(v, i_1) & (v', i_1') \\
(v, i_2) & (v', i_2') \\
(v, i_3) & (v', i_3') \\
(v, i_4) & (v', i_4')
\end{array}
\]

Let $\gamma_\psi : Q_\psi \to P$ be obtained from the first projection $V_Q \to V_P$. Then $(Q_\psi, \gamma_\psi)$ is a QP-graph. The isomorphism class of $(Q_\psi, \gamma_\psi)$ is independent of the choice of a representative $[\delta_1, \delta_2, \delta_3]$ of each orbit $\psi(v)$ and the choice of the ordering of the edges emitting from each vertex of $P$. Indeed, changing these choices merely amounts to relabeling the vertices in each fiber of the first projection $V_Q \to V_P$ (see fact (b)). It is also obvious that every QP-graph is isomorphic to $(Q_\psi, \gamma_\psi)$ for some $\psi : V_P \to \{o_1, o_2\}$.

For an orbit $o \in \{o_1, o_2\}$, let $\bar{o}$ denote the other orbit; $\{o_1, o_2\} = \{o, \bar{o}\}$. Let $\psi : V_P \to \{o_1, o_2\}$ be a map, and let $e = \{v, v'\}$ be an edge of $P$. We define $\psi' : V_P \to \{o_1, o_2\}$ by $\psi'(v) := \psi(v), \psi'(v') := \psi(v')$ and $\psi'(v'') := \psi(v'')$ for all $v'' \in V_P \setminus \{v, v'\}$. Then $(Q_\psi, \gamma_\psi)$ and $(Q_{\psi'}, \gamma_{\psi'})$ are isomorphic. (See the picture below and fact (c).)

\[
\begin{array}{c}
\delta_1 & \delta_2 & \delta_3 & \delta_4 \\
\delta_1' & \delta_2' & \delta_3' & \delta_4'
\end{array}
\]

Hence the isomorphism class of $(Q_\psi, \gamma_\psi)$ depends only on $|\psi^{-1}(o_1)| \mod 2$. We denote by $(Q_0, \gamma_0)$ the QP-graph $(Q_\psi, \gamma_\psi)$ with $|\psi^{-1}(o_1)| \equiv 0 \mod 2$, and by $(Q_1, \gamma_1)$ the QP-graph $(Q_\psi, \gamma_\psi)$ with $|\psi^{-1}(o_1)| \equiv 1 \mod 2$. Since we have constructed $Q_0$
and \( Q_1 \) explicitly, the assertions on \( \langle Q_0 \rangle \) and \( \langle Q_1 \rangle \) can be proved by direct computation.

**Proposition 2.4.** Let \((Q, \gamma)\) be a QP-graph. Each automorphism \( g \in \text{Aut}(Q) \) maps every fiber of \( \gamma_V: V_Q \to V_P \) to a fiber of \( \gamma_V \), and hence induces \( \tilde{g} \in \text{Aut}(P) \) such that \( \tilde{g} \circ \gamma = \gamma \circ g \). The mapping \( g \mapsto \tilde{g} \) gives a surjective homomorphism

\[
\text{Aut}(Q) \to \text{Aut}(P) \cong \mathbb{S}_5,
\]

and its kernel is isomorphic to \((\mathbb{Z}/2\mathbb{Z})^6\).

**Proof.** Since \( P \) does not contain a quadrangle, every quadrangle of \( Q \) is mapped to an edge of \( P \) by \( \gamma \). Hence two distinct vertices \( v, v' \) of \( Q \) are mapped to the same vertex of \( P \) by \( \gamma \) if and only if \( \{v, v'\} \) is not an edge of \( Q \) and there exists a quadrangle of \( Q \) containing \( v \) and \( v' \). Thus the first assertion follows. We make the complete list of elements of \( \text{Aut}(Q) \) by computer, and verify the assertion on \( \text{Aut}(Q) \to \text{Aut}(P) \).

**Corollary 2.5.** A QP-covering map \( \gamma: Q \to P \) from the graph \( Q \) is unique up to \( \text{Aut}(P) \).

2.3. **The configurations \( L_{40} \) and \( L_{112} \).** In this section, we describe the Néron-Severi lattices \( S_0 \) of \( X_0 \) and \( S_3 \) of \( X_3 \), and investigate the embedding \( \rho: S_0 \hookrightarrow S_3 \) induced by the specialization of \( X_0 \) to \( X_3 \).

By Theorem 1.1 (1), we have a distinguished set of

\[
6 \times 4 + 4^2 = 40
\]

smooth rational curves on \( X_p \), where the \( 6 \times 4 \) curves are the irreducible components of the 6 singular fibers of \( \sigma: X_p \to \mathbb{P}^1 \) and the \( 4^2 \) curves are the torsion sections of the Mordell-Weil group. We denote the configuration of these smooth rational curves by \( L_{40} \).

The set of lines on the Fermat quartic surface \( F_3 \) in characteristic 3 has been studied classically by Segre [24]. The surface \( F_3 \subset \mathbb{P}^3 \) contains exactly 112 lines, and every line on \( F_3 \) is defined over the finite field \( \mathbb{F}_9 \). We denote by \( L_{112} \) the set of these lines. We can easily make the list of defining equations of all lines on \( F_3 \), and calculate the dual graph of \( L_{112} \). It is also known ([1]) that the classes of 22 lines appropriately chosen from \( L_{112} \) form a basis of \( S_{F_3} \cong S_3 \). Fixing a basis of \( S_3 \), we can express all classes of lines as integer vectors of length 22 (see [31]).

We show that the specialization of \( X_0 \) to \( X_3 \equiv F_3 \) induces an embedding

\[
\rho_{F_3}: L_{40} \hookrightarrow L_{112}
\]

of configurations. We recall the construction of the isomorphism \( X_3 \cong F_3 \) by Shioda [34]. Let \( \sigma_F: F_3 \to \mathbb{P}^1 \) be the morphism defined by

\[
\sigma_F: [x_1 : x_2 : x_3 : x_4] \mapsto [x_3^2 - ix_4^2 : x_4^3 + ix_3^2] = [-x_1^3 + ix_2^3 : x_3^2 + ix_4^2],
\]

where \( i = \sqrt{-1} \in \mathbb{F}_9 \). The generic fiber of \( \sigma_F \) is a curve of genus 1, and \( \sigma_F \) has a section. Hence the generic fiber of \( \sigma_F \) is isomorphic to its Jacobian, which is defined by the equation (1) by the result of Bašmakov and Faddeev [1]. Therefore \( \sigma_F: F_3 \to \mathbb{P}^1 \) is isomorphic to \( \sigma: X_3 \to \mathbb{P}^1 \) over \( \mathbb{F}_3 \).

Using the defining equations of lines and the vector representations of their classes, we confirm the following facts. There exist exactly \( 6 \times 4 \) lines on \( F_3 \) that are contracted to points by \( \sigma_F \). These 24 lines form, of course, a configuration of 6 disjoint quadrangles. Moreover, there exist exactly 64 lines on \( F_3 \) that are mapped...
to $\mathbb{P}^1$ isomorphically by $\sigma_F$. Let $z_F \in L_{112}$ be one of these 64 sections of $\sigma_F$. To be explicit, we choose the following line as $z_F$. (See Remark in Section 4 of [34]):

$$x_1 + i x_3 - x_4 = x_2 + x_3 - i x_4 = 0.$$

Let $\text{MW}(\sigma_F, z_F)$ denote the Mordell-Weil group of $\sigma_F : F_3 \rightarrow \mathbb{P}^1$ with the zero section $z_F$, and let $\text{Triv}(\sigma_F, z_F)$ be the sublattice of $S_3$ generated by the classes of the zero section $z_F$ and the 24 lines in the singular fibers of $\sigma_F$. (This lattice is called the trivial sublattice of the Jacobian fibration $(\sigma_F, z_F)$ in the theory of Mordell-Weil lattices [35].) Let $\text{Triv}^-(\sigma_F, z_F)$ denote the primitive closure of $\text{Triv}(\sigma_F, z_F)$ in $S_3$.

By [35], we have a canonical isomorphism

$$(2.3) \quad \text{Triv}^-(\sigma_F, z_F)/\text{Triv}(\sigma_F, z_F) \cong \text{the torsion part of } \text{MW}(\sigma_F, z_F).$$

Therefore a section $s : \mathbb{P}^1 \rightarrow F_3$ of $\sigma_F$ is a torsion of $\text{MW}(\sigma_F, z_F)$ if the class of $s$ belongs to $\text{Triv}^-(\sigma_F, z_F)$. By this criterion, we find 16 lines among the 64 sections of $\sigma_F$ that form the torsion part of $\text{MW}(\sigma_F, z_F)$. Thus we obtain the configuration $L_{112}$ as a sub-configuration of $L_{40}$. It is obvious from the construction that this embedding $\rho_L : L_{40} \hookrightarrow L_{112}$ is obtained by the specialization of $X_9$ to $X_3$.

The dual graph of $L_{40}$ is now calculated explicitly. Hence we can prove the following by a direct computation.

**Proposition 2.6.** The dual graph of $L_{40}$ is isomorphic to the QP-graph $Q_1$. □

Comparing the ranks and the discriminants of $\langle L_{40} \rangle \cong \langle Q_1 \rangle$ and $S_0$, we obtain the following:

**Corollary 2.7.** The lattice $S_0$ is generated by the classes of curves in $L_{40}$. □

**Corollary 2.8.** The embedding $\rho_L : L_{40} \hookrightarrow L_{112}$ induces the embedding $\rho : S_0 \hookrightarrow S_3$ induced by the specialization of $X_9$ to $X_3$. □

**Remark 2.9.** By [26], we know that $X_3$ is a supersingular $K3$ surface with Artin invariant 1, and hence is isomorphic to $F_3$ by the uniqueness of a supersingular $K3$ surface with Artin invariant 1.

### 2.4. All embeddings of $L_{40}$ into $L_{112}$.

The embedding $\rho_L : L_{40} \hookrightarrow L_{112}$ constructed in the preceding section depends on the choice of $\sigma_F$ and $z_F$. In this section, we make the complete list of all embeddings $L_{40} \hookrightarrow L_{112}$.

Let $a \mapsto \bar{a} := a^3$ denote the Frobenius automorphism of the base field $k_3$. Then the projective automorphism group of $F_3 \subset \mathbb{P}^3$ is equal to

$$\text{PGU}_4(F_9) := \{ g \in \text{GL}_4(k_3) \mid \bar{g} \cdot g \text{ is a scalar matrix } \}/k_3^\times,$$

which is of order 13063680. We can calculate the action of $\text{PGU}_4(F_9)$ on $L_{112}$ and on $S_3 = \langle L_{112} \rangle$. Let $\mathcal{A}$ denote the set of all ordered 5-tuples $\{z, \ell_0, \ldots, \ell_3\}$ of lines on $F_3$ that form the configuration whose dual graph is as follows.

$$\begin{array}{c}
z \\
\ell_0 \\
\ell_1 \\
\ell_2 \\
\ell_3 \\
\end{array}$$

Note that $\text{PGU}_4(F_9)$ acts on $\mathcal{A}$ naturally. We have the following:

**Proposition 2.10.** The action of $\text{PGU}_4(F_9)$ on $\mathcal{A}$ is simple-transitive.

**Proof.** By [25], we have the following facts.
(1) Since every line on $F_3$ is defined over $\mathbb{F}_9$, the intersection points of $\ell \in \mathcal{L}_{112}$ with other lines in $\mathcal{L}_{112}$ are $\mathbb{F}_9$-rational. For each $\mathbb{F}_9$-rational point $P$ of $\ell$, there exist exactly three lines in $\mathcal{L}_{112} \setminus \{\ell\}$ that intersect $\ell$ at $P$. Hence there exist exactly $112 - 3 \times 10 - 1 = 81$ lines in $\mathcal{L}_{112}$ that are disjoint from $\ell$. The group $\mathbf{P} \mathbf{G} \mathbf{U}_4(\mathbb{F}_9)$ acts on the set of ordered pairs of disjoint lines in $\mathcal{L}_{112}$.

(2) If $\ell_1, \ell_2, \ell_3 \in \mathcal{L}_{112}$ satisfy $(\ell_1, \ell_2) = (\ell_2, \ell_3) = (\ell_3, \ell_1) = 1$, then there exist a plane $\Pi \subset \mathbb{P}^3$ containing $\ell_1, \ell_2, \ell_3$ and a point $P \in \Pi$ contained in $\ell_1, \ell_2, \ell_3$. The residual line $\ell_4 = (F_3 \cap \Pi) - (\ell_1 + \ell_2 + \ell_3)$ also passes through $P$.

(3) Let $[\ell_1, \ell_2]$ be an ordered pair of disjoint lines in $\mathcal{L}_{112}$. Then there exist exactly 10 lines that intersect both $\ell_1$ and $\ell_2$. Let $\text{Stab}([\ell_1, \ell_2])$ denote the stabilizer subgroup of $[\ell_1, \ell_2]$ in $\mathbf{P} \mathbf{G} \mathbf{U}_4(\mathbb{F}_9)$. Then the restriction homomorphism

$$\text{res}_r : \text{Stab}([\ell_1, \ell_2]) \to \text{PGL}(\ell_1, \mathbb{F}_9)$$



to the group of linear automorphisms of $\ell_1 \cong \mathbb{P}^1$ over $\mathbb{F}_9$ is surjective, and its kernel is of order 2. Let $P$ be an $\mathbb{F}_9$-rational point of $\ell_1$, and let $m_P, m'_P \in \mathcal{L}_{112}$ be the lines that intersect $\ell_1$ at $P$ but are disjoint from $\ell_2$. Then the nontrivial element of $\text{Ker} (\text{res}_r)$ exchanges $m_P$ and $m'_P$.

The transitivity of the action of $\mathbf{P} \mathbf{G} \mathbf{U}_4(\mathbb{F}_9)$ on $\mathcal{A}$ follows from these facts. Moreover we have

$$|\mathcal{A}| = 112 \cdot 81 \cdot 10 \cdot 9 \cdot 16 = 13063680 = |\mathbf{P} \mathbf{G} \mathbf{U}_4(\mathbb{F}_9)|,$$

where the factor 112 is the number of choices of $\ell_0$ in $[z, \ell_0, \ldots, \ell_3] \in \mathcal{A}$, the factor 81 is the number of choices of $\ell_2$ when $\ell_0$ is given, the factor 10·9 is the number of choices of $\ell_1$ and $\ell_3$ when $\ell_0$ and $\ell_2$ are given, and the factor 16 is the number of choices of $z$ for a given quadrangle $[\ell_0, \ldots, \ell_3]$. Therefore the action of $\mathbf{P} \mathbf{G} \mathbf{U}_4(\mathbb{F}_9)$ on $\mathcal{A}$ is simple-transitive.

Let $\mathcal{F}$ denote the set of sub-configurations of $\mathcal{L}_{112}$ isomorphic to $\mathcal{L}_{40}$. Let $\alpha = [z_\alpha, \ell_0, \ldots, \ell_3]$ be an element of $\mathcal{A}$. Then there exists a unique Jacobian fibration

$$\sigma_\alpha : F_3 \to \mathbb{P}^1$$

with the zero-section $z_\alpha$ such that $\ell_0 + \ell_1 + \ell_2 + \ell_3$ is a singular fiber of $\sigma_\alpha$. The Jacobian fibration $(\sigma_\alpha, z_\alpha)$ that was used in the construction of $\rho_\mathcal{L}$ is obtained as one of $(\sigma_\alpha, z_\alpha)$. By Proposition 2.10 all Jacobian fibrations $(\sigma_\alpha, z_\alpha)$ are conjugate under the action of $\mathbf{P} \mathbf{G} \mathbf{U}_4(\mathbb{F}_9)$. Therefore $(\sigma_\alpha, z_\alpha)$ yields a sub-configuration $\mathcal{L}_\alpha$ of $\mathcal{L}_{112}$ isomorphic to $\mathcal{L}_{40}$, and the map $\alpha \mapsto \mathcal{L}_\alpha$ gives a surjection $\lambda : \mathcal{A} \to \mathcal{F}$ compatible with the action of $\mathbf{P} \mathbf{G} \mathbf{U}_4(\mathbb{F}_9)$. The size of a fiber of $\lambda$ over $\mathcal{L}' \in \mathcal{F}$ is

$$30 \times 2 \times 16 = 960,$$

where the factor 30 is the number of quadrangle in $\mathcal{L}' \cong \mathcal{L}_{40}$, the factor 2 counts the flipping $\ell_1 \leftrightarrow \ell_3$, and the factor 16 is the number of choices of the zero-section $z_\alpha$. Thus we obtain the following:

**Corollary 2.11.** The number of sub-configurations of $\mathcal{L}_{112}$ isomorphic to $\mathcal{L}_{40}$ is $|\mathbf{P} \mathbf{G} \mathbf{U}_4(\mathbb{F}_9)|/960 = 13608$, and $\mathbf{P} \mathbf{G} \mathbf{U}_4(\mathbb{F}_9)$ acts on the set of these sub-configurations transitively. \(\square\)

2.5. **An elliptic modular surface of level 4 over a discrete valuation ring.**

Let $R$ be a discrete valuation ring such that $2 \in R^\times$ and $i = \sqrt{-1} \in R$. We construct a model of the elliptic modular surface of level 4 over $R$, that is, we perform over $R$ the resolution of the completion of the affine surface defined by \(\mathbb{Q}^4\). This construction explains the isomorphism $\mathcal{L}_{40} \cong \mathbb{Q}^4$ of graphs geometrically.
In this paragraph, all schemes and morphisms are defined over \( R \). We consider the complete quadrangle on \( \mathbb{P}^2 \) (Figure 2.1) such that each of the triple points \( t_1, \ldots, t_4 \) is an \( R \)-valued point. Let \( M \to \mathbb{P}^2 \) be the blow-up of \( \mathbb{P}^2 \) at \( t_1, \ldots, t_4 \). Let \( \tilde{l}_1, \ldots, \tilde{l}_6 \) be the strict transforms of the lines \( l_1, \ldots, l_6 \), and let \( \tilde{t}_1, \ldots, \tilde{t}_4 \) be the exceptional divisors over \( t_1, \ldots, t_4 \). It is well-known that these 6 + 4 = 10 smooth rational curves on \( M \) form a configuration whose dual graph is the Petersen graph \( \mathcal{P} \). Let

\[
\varphi_M : M \to \mathbb{P}^1
\]

be the fibration induced by the pencil of lines on \( \mathbb{P}^2 \) passing through \( t_1 \). (The dependence of the construction on the choice of this \( \mathbb{P}^1 \)-fibration \( \varphi_M \) will be discussed in Section 4.3. See Remark 4.5.) Then \( \varphi_M \) has exactly three singular fibers \( \tilde{l}_1 + \tilde{l}_4, \tilde{l}_2 + \tilde{l}_3, \tilde{l}_3 + \tilde{l}_2 \), and four sections \( \tilde{t}_1, \tilde{l}_4, \tilde{l}_5, \tilde{l}_6 \). Let \( \tilde{M} \to M \) be the blow-up at the nodes on \( \tilde{l}_1 + \tilde{l}_4, \tilde{l}_2 + \tilde{l}_3, \tilde{l}_3 + \tilde{l}_2 \), and let \( \varphi'_M : M' \to \mathbb{P}^1 \) be the composite of \( \varphi_M \) and \( M' \to M \). We choose an affine parameter \( \lambda \) on the base curve \( \mathbb{P}^1 \) of \( \varphi'_M \) such that the singular fibers are located over \( \lambda = 0, 1, \infty \). Let \( \tilde{M}' \to \mathbb{P}^1 \) be the pull-back of \( \varphi'_M : M' \to \mathbb{P}^1 \) by \( \tilde{M} \to \mathbb{P}^1 \) given by

\[
\sigma \mapsto \lambda = (\sigma + \sigma^{-1})/2,
\]

and let \( \bar{M} \to \tilde{M}' \) be the normalization of \( \tilde{M}' \). Then \( \bar{M} \) is smooth over \( R \), and the natural morphism \( \bar{\varphi}_M : \bar{M} \to \mathbb{P}^1 \) to the \( \sigma \)-line has exactly six singular fibers over \( \sigma = 0, \pm 1, \pm i, \infty \). Each singular fiber is a union of three smooth rational curves forming the configuration \( \circ \circ \circ \), the middle of which is with multiplicity 2. Let \( \tilde{t}_1, \tilde{l}_4, \tilde{l}_5, \tilde{l}_6 \) be the pull-backs of the sections \( \tilde{t}_1, \tilde{l}_4, \tilde{l}_5, \tilde{l}_6 \) of \( \varphi_M \) by \( \bar{M} \to M \). Note that, by \( \bar{M} \to M \), a fiber \( F \) of \( \varphi_M : M \to \mathbb{P}^1 \) is pulled back to a sum of two fibers of \( \bar{\varphi}_M : \bar{M} \to \mathbb{P}^1 \), and hence the line bundle on \( \bar{M} \) corresponding to the pull-back of \( F \) is divisible by 2 in the Picard group \( \text{Pic} \bar{M} \). Therefore the line bundle on \( \bar{M} \) corresponding to the divisor \( B := \tilde{t}_1 + \tilde{l}_4 + \tilde{l}_5 + \tilde{l}_6 \) is divisible by 2 in \( \text{Pic} \bar{M} \), and we can construct the double covering \( \bar{X} \to \bar{M} \) branched along \( B \). Then \( \bar{X} \) is a model of the elliptic modular surface of level 4 over \( R \), and the Jacobian fibration \( \sigma : \bar{X} \to \mathbb{P}^1 \) is obtained as the composite of the double covering \( \bar{X} \to \bar{M} \) and \( \bar{\varphi}_M : \bar{M} \to \mathbb{P}^1 \).
The QP-covering map \( \mathcal{L}_{40} \to \mathcal{P} \) (see Corollary 2.13) is constructed as follows. We consider an \( F \)-valued point of \( \text{Spec} \, R \), where \( F \) is a field. We put \( X_F := \mathcal{X} \otimes_R F \), and \( M_F := \tilde{M} \otimes_R F \), \( M_F := M \otimes_R F \). Let \( \mathcal{E}_F \) be the generic fiber of \( \sigma \otimes F \), \( X_F \to \mathbb{P}^1_F \), which is an elliptic curve over the function field \( F(\sigma) \) defined by \( (1.1) \). Let \( m_2 : X_F \to X_F \) be the rational map induced by the multiplication by 2 on \( \mathcal{E}_F \). Then the rational map
\[
(2.7) \quad \mu_F : X_F \cdot m_2 \to X_F \quad \mathcal{M}_F \to M_F
\]
gives a map from \( \mathcal{L}_{40} \) to the Petersen graph \( \mathcal{P} \) formed by \( \{\bar{t}_1, \ldots, \bar{t}_4, \bar{t}_1, \ldots, \bar{t}_6\} \).

Proposition 2.12. The rational map \( \mu_F \) induces a Galois extension of the function fields. Its Galois group \( \text{Gal}(\mu) \) is isomorphic to \((\mathbb{Z}/2\mathbb{Z})^5 \) and is generated by the inversion \( \iota : (X, Y, \sigma) \mapsto (X, -Y, \sigma) \) of the elliptic curve \( \mathcal{E}_F \), two involutions
\[
(2.8) \quad (X, Y, \sigma) \mapsto (X, Y, -\sigma), \quad (X, Y, \sigma) \mapsto (X, Y, 1/\sigma),
\]
and the translations by the 2-torsion points of \( \mathcal{E}_F \).

Proof. The inversion \( \iota \) and the involutions in \( (2.8) \) fix each 2-torsion point of \( \mathcal{E}_F \). Hence the involutions in the statement of Proposition 2.12 generate a group isomorphic to \((\mathbb{Z}/2\mathbb{Z})^5 \). By (2.6), the function field \( F(\sigma) \) is a Galois extension of \( F(\lambda) \) with Galois group generated by \( \sigma \mapsto -\sigma \) and \( \sigma \mapsto 1/\sigma \). Hence the covering \( \tilde{M}_F \to M_F \) in (2.7) is the quotient by the involutions in (2.8). The covering \( X_F \to \tilde{M}_F \) in (2.7) is the quotient by \( \iota \), and the map \( m_2 \) is the quotient by the group of translations by the 2-torsion points of \( \mathcal{E}_F \). Thus the proof is completed.

Remark 2.13. The lattice \( \langle Q_0 \rangle \) is isomorphic to the Néron-Severi lattice of the singular \( K3 \) surface with the transcendental lattice of discriminant 4 studied by Vinberg [36]. However, we do not have a configuration \( Q_0 \) of smooth rational curves on this surface.

Remark 2.14. The QP-graph \( Q_0 \) is obtained algebro-geometrically as follows. Let \( M_C \) be the complex surface obtained by blowing-up \( \mathbb{P}^2_C \) at the triple points of the complete quadrangle on \( \mathbb{P}^2_C \), and let \( M_C^\circ \) be the complement of the ten \((-1)\)-curves on \( M_C \). Since \( H_1(M_C^\circ, \mathbb{Z}) \cong \mathbb{Z}^5 \), we have a canonical surjective homomorphism \( \pi_1(M_C^\circ) \to (\mathbb{Z}/2\mathbb{Z})^5 \). The corresponding étale covering \( W^\circ \to M_C^\circ \) extends to a finite morphism \( W \to M_C \) from a smooth surface \( W \). The boundary \( W \setminus W^\circ \) is a union of 40 smooth rational curves, and their dual graph is \( Q_0 \). Since these curves are with self-intersection number \(-1 \), the lattice \( \langle Q_0 \rangle \) is not isomorphic to the Néron-Severi lattice of \( W \).

3. Borcherds’ method

3.1. Chambers. We fix notions about tessellation of a positive cone of an even hyperbolic lattice by chambers.

Let \( L \) be an even lattice. A vector \( r \in L \) is called a root if \( \langle r, r \rangle = -2 \). The set of roots of \( L \) is denoted by \( \mathcal{R}(L) \).

Let \( L \) be an even hyperbolic lattice. Let \( \mathcal{P}(L) \) be one of the two connected components of \( \{ x \in L \otimes \mathbb{R} \mid \langle x, x \rangle > 0 \} \). Then \( O^+(L) \) acts on \( \mathcal{P}(L) \). For \( v \in L \otimes \mathbb{Q} \) with \( \langle v, v \rangle < 0 \), let \( (v)^- \) denote the hyperplane of \( \mathcal{P}(L) \) defined by \( \langle x, v \rangle = 0 \). Let \( \mathcal{V} \) be a set of vectors of \( L \otimes \mathbb{Q} \) such that \( \langle v, v \rangle < 0 \) for all \( v \in \mathcal{V} \). We assume that
the family \( \{ (v)^{\perp} \mid v \in V \} \) of hyperplanes is locally finite in \( \mathcal{P}(L) \). A \( \mathcal{V} \)-chamber is the closure in \( \mathcal{P}(L) \) of a connected component of

\[
\mathcal{P}(L) \setminus \bigcup_{v \in V} (v)^{\perp}.
\]

A typical example is \( \mathcal{R}(L) \)-chambers defined by the set \( \mathcal{R}(L) \) of roots of \( L \).

**Definition 3.1.** Let \( N \) be a closed subset of \( \mathcal{P}(L) \). We say that \( N \) is tessellated by \( \mathcal{V} \)-chambers if \( N \) is a union of \( \mathcal{V} \)-chambers. Suppose that \( N \) is tessellated by \( \mathcal{V} \)-chambers, and let \( H \) be a subgroup of \( O^+(S) \) that preserves \( N \). We say that \( H \) preserves the tessellation of \( N \) by \( \mathcal{V} \)-chambers if any \( g \in H \) maps each \( \mathcal{V} \)-chamber in \( N \) to a \( \mathcal{V} \)-chamber. Suppose that this is the case. We say that the tessellation of \( N \) is \( H \)-transitive if \( H \) acts transitively on the set of \( \mathcal{V} \)-chambers in \( N \).

**Remark 3.2.** Let \( U \) be a subset of \( V \) such that the closed subset

\[
N_U := \{ x \in \mathcal{P}(L) \mid \langle x, v \rangle \geq 0 \text{ for all } v \in U \}
\]

of \( \mathcal{P}(L) \) contains an interior point. Then \( N_U \) is tessellated by \( \mathcal{V} \)-chambers. In particular, if \( V' \) is a subset of \( V \), then each \( V' \)-chamber is tessellated by \( \mathcal{V} \)-chambers.

Let \( D \) be a \( \mathcal{V} \)-chamber. We put

\[
\text{Aut}(D) := \{ g \in O^+(L) \mid D^g = D \}.
\]

A wall of \( D \) is a closed subset of \( D \) of the form \((v)^{\perp} \cap D\) such that the hyperplane \((v)^{\perp} \) of \( \mathcal{P}(L) \) is disjoint from the interior of \( D \) and \((v)^{\perp} \cap D\) contains a non-empty open subset of \((v)^{\perp} \). We say that a hyperplane \((v)^{\perp} \) of \( \mathcal{P}(L) \) defines a wall of \( D \) if \((v)^{\perp} \cap D\) is a wall of \( D \). We say that a vector \( v \in L \otimes \mathbb{Q} \) with \( \langle v, v \rangle < 0 \) defines a wall of \( D \) if \((v)^{\perp} \) defines a wall of \( D \) and \( \langle v, x \rangle \geq 0 \) for all \( x \in D \). Note that, for each wall of \( D \), there exists a unique primitive vector \( x \in V \) defining the wall. Let \((v)^{\perp} \cap D \) be a wall of \( D \). Then there exists a unique \( \mathcal{V} \)-chamber \( D' \) such that the interiors of \( D \) and \( D' \) are disjoint and that \((v)^{\perp} \cap D \) is equal to \((v)^{\perp} \cap D' \). (Hence \((v)^{\perp} \cap D' \) is a wall of \( D' \).) We say that \( D' \) is a \( \mathcal{V} \)-chamber adjacent to \( D \) across the wall \((v)^{\perp} \cap D \). A face of \( D \) is a closed subset of \( D \) of the form \( F \cap D \) such that

\[
F = (v_1)^{\perp} \cap \cdots \cap (v_n)^{\perp}, \quad \text{where} \quad (v_1)^{\perp}, \ldots, (v_n)^{\perp} \text{ define walls of } D,
\]

and that \( F \cap D \) contains a non-empty open subset of \( F \).

**Example 3.3.** We consider the tessellation of \( \mathcal{P}(L) \) by \( \mathcal{R}(L) \)-chambers. Each root \( r \) of \( L \) defines a reflection \( s_r \in O^+(L) \) by \( x \mapsto x + \langle x, r \rangle r \). Let \( W(L) \) denote the subgroup of \( O^+(L) \) generated by all the reflections with respect to the roots. Then the tessellation of \( \mathcal{P}(L) \) by \( \mathcal{R}(L) \)-chambers is \( W(L) \)-transitive. An \( \mathcal{R}(L) \)-chamber \( N \) is a fundamental domain of the action of \( W(L) \) on \( \mathcal{P}(L) \), and \( O^+(L) \) is equal to \( \text{Aut}(N) \rtimes W(L) \). Moreover, \( W(L) \) is generated by the reflections \( s_r \) associated with the roots \( r \) of \( L \) defining the walls of \( N \), and the faces of codimension 2 of \( N \) give the defining relations of \( W(L) \) with respect to this set of generators.

Let \( L_{26} \) be an even unimodular hyperbolic lattice of rank 26, which is unique up to isomorphism. The shape of an \( \mathcal{R}(L_{26}) \)-chamber was determined by Conway \[4\], and hence we call an \( \mathcal{R}(L_{26}) \)-chamber a Conway chamber. Let \( w \) be a non-zero primitive vector of \( L_{26} \) with \( \langle w, w \rangle = 0 \) such that \( w \) is contained in the closure of \( \mathcal{P}(L_{26}) \) in \( L_{26} \otimes \mathbb{R} \). We say that \( w \) is a Weyl vector if the lattice \( \langle w \rangle^{\perp} / \langle w \rangle \) is isomorphic to the negative-definite Leech lattice, where \( \langle w \rangle^{\perp} \) is the orthogonal
complement in \( L_{26} \) of \( \langle w \rangle := Zw \subset L_{26} \). Let \( w \in L_{26} \) be a Weyl vector. Then a root \( r \) of \( L_{26} \) is called a Leech root with respect to \( w \) if \( \langle w, r \rangle = 1 \). We put
\[
C(w) := \{ x \in P(L_{26}) \mid \langle x, r \rangle \geq 0 \text{ for all Leech roots } r \text{ with respect to } w \}.
\]

**Theorem 3.4** (Conway [4]). The mapping \( w \mapsto C(w) \) gives a bijection from the set of Weyl vectors to the set of Conway chambers.

### 3.2. Borcherds’ method

Borcherds [2, 3] developed a method to analyze \( \mathcal{R}(S) \)-chambers of an even hyperbolic lattice \( S \) by means of Conway chambers. We briefly review this method, and fix some terminologies. See [28] for details of the algorithms.

Let \( S \) be an even hyperbolic lattice. Suppose that we have a primitive embedding \( i : S \hookrightarrow L_{26} \) such that the orthogonal complement \( R \) of \( S \) in \( L_{26} \) satisfies the following condition:

(3.1) \( R \) cannot be embedded into the negative-definite Leech lattice.

(This condition is fulfilled, for example, if \( R \) contains a root.) We choose \( P(S) \) so that the embedding \( i : S \hookrightarrow L_{26} \) induces an embedding \( \iota_P : P(S) \hookrightarrow P(L_{26}) \).

\[
\text{pr}_S : L_{26} \otimes \mathbb{Q} \to S \otimes \mathbb{Q}
\]
denote the orthogonal projection. A hyperplane \( (v)^\perp \) of \( P(L_{26}) \) intersects \( P(S) \) in a hyperplane if and only if \( \langle \text{pr}_S(v), \text{pr}_S(v) \rangle < 0 \), and, if this is the case, we have \( P(S) \cap (v)^\perp = (\text{pr}_S(v))^\perp \). We put

(3.2) \( V(i) := \{ \text{pr}_S(r) \mid r \in \mathcal{R}(L_{26}), \langle \text{pr}_S(r), \text{pr}_S(r) \rangle < 0 \} \).

The tessellation of \( P(L_{26}) \) by Conway chambers induces a tessellation of \( P(S) \) by \( V(i) \)-chambers. Each \( V(i) \)-chamber is of the form \( i_P^{-1}(C(w)) \). It is easily seen (see [28]) that the assumption (3.1) implies that each \( V(i) \)-chamber has only a finite number of walls. The defining vectors of walls of a \( V(i) \)-chamber \( i_P^{-1}(C(w)) \) can be calculated from the Weyl vector \( w \in L_{26} \) of the Conway chamber \( C(w) \). From this set of walls of \( i_P^{-1}(C(w)) \), we can calculate the finite group \( \text{Aut}(i_P^{-1}(C(w))) \subset O^+(S) \). Moreover, for each wall \( (v)^\perp \cap i_P^{-1}(C(w)) \) of a \( V(i) \)-chamber \( i_P^{-1}(C(w)) \), we can calculate a Weyl vector \( w' \) such that \( i_P^{-1}(C(w')) \) is the \( V(i) \)-chamber adjacent to \( i_P^{-1}(C(w)) \) across the wall \( (v)^\perp \cap i_P^{-1}(C(w)) \).

Since \( \mathcal{R}(S) \subset V(i) \), Remark 3.2 implies the following:

**Proposition 3.5.** An \( \mathcal{R}(S) \)-chamber is tessellated by \( V(i) \)-chambers. \( \square \)

### 3.3. Discriminant forms

For the application of Borcherds’ method to \( K3 \) surfaces, we need the notion of discriminant forms due to Nikulin [19].

Let \( q : A \to \mathbb{Q}/2\mathbb{Z} \) be a nondegenerate quadratic form with values in \( \mathbb{Q}/2\mathbb{Z} \) on a finite abelian group \( A \). We denote by \( O(q) \) the automorphism group of \( (A, q) \). For a prime \( p \), we denote by \( A_p \) the \( p \)-part of \( A \) and by \( q_p : A_p \to \mathbb{Q}/2\mathbb{Z} \) the restriction of \( q \) to \( A_p \). Then we have a canonical orthogonal direct-sum decomposition

\[
(A, q) = \bigoplus (A_p, q_p).
\]

Hence \( O(q) \) is canonically isomorphic to the direct product of \( O(q_p) \).

Let \( L \) be an even lattice, and let \( A(L) = L^\vee/L \) denote the discriminant group of \( L \). We define the **discriminant form of \( L \)**

\[
q(L) : A(L) \to \mathbb{Q}/2\mathbb{Z}
\]
by \(q(L)(\bar{x}) := \langle x, x \rangle \mod 2\mathbb{Z}\), where \(x \mapsto \bar{x}\) is the natural projection \(L^\vee \to A(L)\). Then we have a natural homomorphism
\[
\eta_L : O(L) \to O(q(L)).
\]

Let \(M\) be a primitive sublattice of an even lattice \(L\), and \(N\) the orthogonal complement of \(M\) in \(L\). Let \(O(L, M)\) denote the subgroup \(\{ g \in O(L) \mid M^g = M \}\) of \(O(L)\). Then we have a canonical embedding \(O(L, M) \to O(M) \times O(N)\). The submodule \(L \subset M^N \oplus N^\vee\) defines a subgroup \(\Gamma_L := L/(M \oplus N) \subset A(M) \times A(N)\).

By Nikulin [19], we have the following:

**Proposition 3.6.** Let \(p\) be a prime that does not divide \(|A(M)|\). Then \(N \hookrightarrow L\) induces an isomorphism \(q(L)_P \cong q(N)_P\), which is compatible with the actions of \(O(L, M)\) on \(L\) and on \(N\). \(\square\)

**Proposition 3.7.** Let \(p\) be a prime that does not divide \(|A(L)|\). Then the \(p\)-part of \(\Gamma_L\) is the graph of an isomorphism \(q(M)_P \cong -q(N)_P\), which is compatible with the actions of \(O(L, M)\) on \(M\) and on \(N\). \(\square\)

**Proposition 3.8.** Suppose that \(L\) is unimodular, and let \(\gamma_L : q(M) \cong -q(N)\) be the isomorphism with the graph \(\Gamma_L\). Let \(H\) be a subgroup of \(O(N)\). Then \(g \in O(M)\) extends to \(\tilde{g} \in O(L, M)\) with \(\tilde{g}|N \in H\) if and only if the isomorphism \(O(q(M)) \cong O(q(N))\) induced by \(\gamma_L\) maps \(\eta_M(g) \in O(q(M))\) into \(\eta_N(H) \subset O(q(N))\). \(\square\)

**3.4. Geometric application of Borcherds’ method.** Let \(Z\) be a \(K3\) surface defined over an algebraically closed field. We use the notation \(S_Z, \mathcal{P}_Z\) and \(N_Z\) defined in Section [11]. The following is well-known.

**Proposition 3.9.** The closed subset \(N_Z\) of \(\mathcal{P}_Z\) is an \(\mathcal{R}(S_Z)\)-chamber. The mapping \(C \mapsto ([C])^\perp \cap N_Z\) gives a one-to-one correspondence between the set of smooth rational curves on \(Z\) and the set of walls of \(N_Z\). \(\square\)

Since the action of \(O^+(S_Z)\) on \(\mathcal{P}_Z\) preserves the tessellation by \(\mathcal{R}(S_Z)\)-chambers and an ample class is an interior point of \(N_Z \subset \mathcal{P}_Z\), we obtain the following.

**Corollary 3.10.** Let \(a \in S_Z\) be an ample class. Then the following three conditions on \(g \in O^+(S_Z)\) are equivalent: (i) \(N_Z = N_Z^a\), (ii) \(N_Z \cap N_Z^a\) contains an interior point of \(N_Z\). (iii) There exist no roots \(r\) of \(S_Z\) such that \(\langle r, a \rangle\) and \(\langle r, a^3 \rangle\) have different signs. \(\square\)

Let \(Z\) be a complex \(K3\) surface. Let \(T_Z\) denote the orthogonal complement of \(S_Z = H^2(Z, \mathbb{Z}) \cap H^{1,1}(Z)\) in the even unimodular lattice \(H^2(Z, \mathbb{Z})\) with the cup-product. Then \(T_Z \otimes \mathbb{C}\) contains a one-dimensional subspace \(H^{2,0}(Z) = \mathbb{C}\omega\), where \(\omega\) is a non-zero holomorphic \(2\)-form on \(Z\). We put
\[
O(T_Z, \omega) := \{ g \in O(T_Z) \mid \mathbb{C}\omega^g = \mathbb{C}\omega \}.
\]
Recall that we have a natural homomorphism \(\eta_{T_Z} : O(T_Z) \to O(q(T_Z))\). We put
\[
O(q(T_Z), \omega) := \text{the image of } O(T_Z, \omega) \text{ by } \eta_{T_Z}.
\]

The even unimodular overlattice \(H^2(Z, \mathbb{Z})\) of \(S_Z \oplus T_Z\) induces an isomorphism \(\gamma_H\) between \(q(S_Z)\) and \(-q(T_Z)\). Let \(O(q(S_Z), \omega)\) denote the subgroup of \(O(q(S_Z))\) corresponding to \(O(q(T_Z), \omega)\) by the isomorphism \(O(q(T_Z)) \cong O(q(S_Z))\) induced by \(\gamma_H\). By Proposition [KS], an isometry \(g \in O(S_Z)\) extends to an isometry \(\tilde{g}\) of \(H^2(Z, \mathbb{Z})\) that preserves \(H^{2,0}(Z)\) if and only if \(\eta_{S_Z}(g) \in O(q(S_Z), \omega)\).
Let $Z$ be a supersingular $K3$ surface defined over an algebraically closed field $k_p$ of odd characteristic $p$. Then $A(SZ)$ is an $F_p$-vector space, and we have a *period* of $Z$, which is a subspace of $A(SZ) \otimes k_p$. (See Ogus [21, 22].) Let $O(q(SZ), \omega)$ denote the subgroup of $O(q(SZ))$ consisting of automorphisms that preserve the period.

In the two cases where $Z$ is defined over $\mathbb{C}$ or supersingular in odd characteristic, we call the condition

\begin{equation}
\eta_{SZ}(g) \in O(q(SZ), \omega)
\end{equation}

on $g \in O^+(SZ)$ the *period condition*. In these two cases, we have the Torelli theorem. (See Piatetski-Shapiro and Shafarevich [23] and Ogus [21, 22].) By virtue of this theorem, we have the following:

**Theorem 3.11.** Let $Z$ be a complex $K3$ surface or a supersingular $K3$ surface in odd characteristic, and let $\psi_Z: \text{Aut}(Z) \to O^+(SZ)$ be the natural representation of $\text{Aut}(Z)$ on $SZ$. Then an isometry $g \in O^+(SZ)$ belongs to the image of $\psi_Z$ if and only if $g$ preserves $N_Z$ and satisfies the period condition \((3.3)\).

We explain the procedure of Borcherds' method in the simplest case. See [28] for more general cases. In the following, we assume that $Z$ is a complex $K3$ surface or a supersingular $K3$ surface in odd characteristic. We also assume that $\psi_Z$ is injective, and regard $\text{Aut}(Z)$ as a subgroup of $O^+(SZ)$. We search for a primitive embedding $i: SZ \to \mathbb{L}_{26}$ inducing $i_P: \mathbb{P}_Z \hookrightarrow \mathbb{P}(\mathbb{L}_{26})$ and a Weyl vector $w_0 \in \mathbb{L}_{26}$ with the following properties, and look at the tessellation of the $R(SZ)$-chamber $N_Z$ by $\mathcal{V}(i)$-chambers, where $\mathcal{V}(i)$ is defined by \((3.2)\).

(I) Let $R$ denote the orthogonal complement of $SZ$ in $\mathbb{L}_{26}$. We require that $R$ satisfies \((3.1)\), so that each $\mathcal{V}(i)$-chamber has only a finite number of walls. We also require that $\eta_R: O(R) \to O(q(R))$ is surjective. By Proposition \(3.8\) every isometry $g \in O^+(SZ)$ extends to an isometry of $\mathbb{L}_{26}$. Hence the action of $O^+(SZ)$ preserves the tessellation of $\mathbb{P}_Z$ by $\mathcal{V}(i)$-chambers. In particular, the action of $\text{Aut}(Z)$ on $N_Z$ preserves the tessellation of $N_Z$ by $\mathcal{V}(i)$-chambers.

(II) Let $D$ be the closed subset $i_P^{-1}(\mathcal{C}(w_0))$ of $\mathbb{P}_Z$. We require that $D$ contains an ample class in its interior. Then $D$ is a $\mathcal{V}(i)$-chamber contained in $N_Z$.

**Definition 3.12.** The $\mathcal{V}(i)$-chamber $D$ is called the *initial chamber* of this procedure. A wall $(v)^{\perp} \cap D$ of $D$ is called an *outer-wall* if $(v)^{\perp}$ defines a wall of the $R(SZ)$-chamber $N_Z$, that is, if there exists a root $r$ of $SZ$ such that $(v)^{\perp} = (r)^{\perp}$. We call the wall $(v)^{\perp} \cap D$ an *inner-wall* otherwise. Let $\mathcal{W}_{\text{out}}(D)$ and $\mathcal{W}_{\text{inn}}(D)$ denote the set of outer-walls and inner-walls, respectively.

We calculate the set of walls of the initial chamber $D$. Since each outer-wall corresponds to a smooth rational curve on $Z$ by Proposition \(3.9\), we obtain a configuration of smooth rational curves on $Z$ from $\mathcal{W}_{\text{out}}(D)$.

(III) We calculate $\text{Aut}(D) := \{ g \in O^+(SZ) \mid D^g = D \}$. By Corollary \(3.10\) any element of $\text{Aut}(D)$ preserves $N_Z$. Therefore the group

\begin{equation}
\text{Aut}(Z, D) := \{ g \in \text{Aut}(D) \mid g \text{ satisfies the period condition } (3.3) \}
\end{equation}

is contained in $\text{Aut}(Z)$. We find an ample class $h$ in the interior of $D$ such that $h^g = h$ for all $g \in \text{Aut}(Z, D)$. Then $\text{Aut}(Z, D)$ is equal to the projective automorphism group $\text{Aut}(Z, h)$. 
(IV) Note that $\text{Aut}(Z, D) = \text{Aut}(Z, h)$ acts on $W_{\text{out}}(D)$ and $W_{\text{inn}}(D)$. We decompose $W_{\text{inn}}(D)$ into the orbits under the action of $\text{Aut}(Z, h)$:

$$W_{\text{inn}}(D) = O_1 \cup \cdots \cup O_J.$$  

From each orbit $O_j$, we choose a wall $(v_j)^{-}\cap D$, and calculate a Weyl vector $w_j \in L_{26}$ such that $D_j := i_{\beta}^{-1}(C(w_j))$ is the $\mathcal{V}(i)$-chamber adjacent to $D$ across $(v_j)^{-}\cap D$. Since $(v_j)^{-}\cap N_Z$ is not a wall of $N_Z$, the $\mathcal{V}(i)$-chamber $D_j$ is contained in $N_Z$. For each $j = 1, \ldots, J$, we find an isometry $g_j$ of $O^+(S_Z)$ that satisfies the period condition (3.3) and $D_j^g = D_j$. Note that each $g_j$ preserves $N_Z$ by Corollary 3.10 and hence $g_j \in \text{Aut}(Z)$. Note also that, for each inner-wall $(v')^{-}\cap D \in O_j$, there exists a conjugate $g' \in \text{Aut}(Z)$ of $g_j$ by $\text{Aut}(Z, h)$ that maps $D$ to the $\mathcal{V}(i)$-chamber adjacent to $D$ across the wall $(v')^{-}\cap D$.

(V) Under the assumptions given in (I)-(IV), the group $\text{Aut}(Z)$ is generated by $\text{Aut}(Z, h)$ and the automorphisms $g_1, \ldots, g_J$. Moreover, the tessellation of $N_Z$ by $\mathcal{V}(i)$-chambers is $\text{Aut}(Z)$-transitive, and the mappings $g \mapsto h^g$ and $g \mapsto D^g$ give one-to-one correspondences between the following sets:

- The set of cosets $\text{Aut}(Z, h)/\text{Aut}(Z)$.
- The set of $\mathcal{V}(i)$-chambers contained in $N_Z$.
- The subset $\{h^g \mid g \in \text{Aut}(Z)\}$ of $S_Z$.

Moreover, considering the reflections with respect to the roots $r$ defining the outer-walls $(r)^{-}\cap D$ of $D$, we see that, under the assumptions given in (I)-(IV), the tessellation of $P_Z$ by $\mathcal{V}(i)$-chambers is $O^+(S_Z)$-transitive.

The method described in this section was applied by Kondo [13] to the calculation of the automorphism group of a generic Jacobian Kummer surface, and since then, many studies have been done on the automorphism groups of various K3 surfaces (see the references of [28]). This method was also applied to the study of automorphism group of an Enriques surface in [29].

4. BORCHERDS’ METHOD FOR $X_0$ AND $X_3$

Recall from Section 1.1 that we use the following notation:

$$S_3 := S_{X_3}, \quad P_3 := P_{X_3}, \quad N_3 := N_{X_3}, \quad S_0 := S_{X_0}, \quad P_0 := P_{X_0}, \quad N_0 := N_{X_0}.$$  

4.1. Borcherds’ method for $X_3$. We identify $X_3$ and $F_3$ by Shioda’s isomorphism explained in Section 2.3. Hence $S_3$ is the Néron-Severi lattice of $F_3$. In [14], we have obtained a generating set of $\text{Aut}(X_3)$ by finding a primitive embedding $i_3: S_3 \hookrightarrow L_{26}$ inducing $i_3, P_3 \hookrightarrow P(L_{26})$ and a Weyl vector $w_0 \in L_{26}$ that satisfy the requirements in Section 3.3. The result is as follows. See [31] or [14] for the explicit descriptions of $i_3$, $w_0$, and other computational data.

We have $A(S_3) \cong (\mathbb{Z}/3\mathbb{Z})^2$. The group $O(q(S_3))$ is a dihedral group of order 8, and $O(q(S_3), \omega)$ is a cyclic subgroup of order 4. The orthogonal complement $R_3$ of

| orbit | $(v, v)$ | $(v, h)$ | $(h, b'_d)$ | $\text{Sing}(b'_d)$ | $d = (h_3, 3h'(b'_d))$ |
|-------|---------|---------|----------|----------------|----------------|
| $O'_{648}$ | $-4/3$ | 2 | 6 | $4A_2 + 6A_1$ | 10 |
| $O'_{5184}$ | $-2/3$ | 3 | 9 | $4A_3 + 6A_1$ | 31 |

**Table 4.1. Inner-walls of $D_3$**
S3 in L26 is a negative-definite root lattice of type 2A2. The order of O(R3) is 288, the order of O(q(R3)) is 8, and the natural homomorphism O(R3) → O(q(R3)) is surjective. We put

\[ D_3 := i_{3, p}^{-1}(C(w_0)). \]

Then D3 contains the class h3 ∈ S3 of a hyperplane section of X3 = F3 ⊂ P3 in its interior. Hence D3 is a \( \mathcal{V}(i_3) \)-chamber. The group \( \text{Aut}(X_3, D_3) \) defined by (3.4) is of order 13063680. The set \( \mathcal{W}_{\text{Out}}(D_3) \) of outer-walls of the initial chamber D3 is equal to \( \{(\ell)^{\perp} \cap D_3 \mid \ell \in L_{112}\} \). Because

\[ h_3 = \frac{1}{28} \sum_{\ell \in L_{112}} [\ell], \]

we have \( \text{Aut}(X_3, D_3) = \text{Aut}(X_3, h_3) \), which is equal to the projective automorphism group \( \{g \in \text{PGL}_4(k_3) \mid g(F_3) = F_3\} = \text{PGU}_4(F_9) \) of \( F_3 \subset P^3 \). The class h3 is in fact the image of w0 by the orthogonal projection \( L_{26} \otimes \mathbb{Q} \to S_3 \otimes \mathbb{Q} \). Under the action of \( \text{Aut}(X_3, h_3) = \text{PGU}_4(F_9) \), the set \( \mathcal{W}_{\text{Inn}}(D_3) \) of inner-walls of D3 is decomposed into two orbits \( O_{648}' \) and \( O_{5184}' \) of size 648 and 5184, respectively. Each inner-wall \( (v)^{\perp} \cap D_3 \) in the orbit \( O_{648}' \) is defined by a primitive vector v of \( S_3^2 \) with the properties given in Table 4.1 and there exists a double-plane polarization \( b_0^v \in S_3 \) such that the corresponding double-plane involution \( q(b_0^v) \in \text{Aut}(X_3) \) maps D3 to the \( \mathcal{V}(i_3) \)-chamber adjacent to D3 across the wall \( (v)^{\perp} \cap D_3 \). These results prove the following:

**Theorem 4.1 (Kondo-Shimada [14]).** The automorphism group \( \text{Aut}(X_3) \) is generated by the projective automorphism group \( \text{Aut}(X_3, h_3) = \text{PGU}_4(F_9) \) and two double-plane involutions \( g(b_{10}^v), g(b_{11}^v) \) corresponding to the orbits \( O_{648}', O_{5184}' \) of the action of \( \text{PGU}_4(F_9) \) on the set \( \mathcal{W}_{\text{Inn}}(D_3) \) of inner-walls of the initial chamber D3.

4.2. **Borcherds’ method for X0.** We define an embedding \( i_0 : S_0 \to L_{26} \) by

\[ i_0 := i_3 \circ \rho, \]

where \( i_3: S_3 \to L_{26} \) is the embedding used in Section 4.1 and \( \rho: S_0 \to S_3 \) is the embedding given by the specialization of X0 to X3. The key observation of this article is that \( i_0 \) is equal to the embedding used by Keum-Kondo [11] for the calculation of \( \text{Aut}(X_0) \).

We have \( A(S_0) \cong (\mathbb{Z}/4\mathbb{Z})^2 \). The group \( O(q(S_0)) \) is isomorphic to the dihedral group of order 8, and the subgroup \( O(q(S_0), \omega) \) is cyclic of order 4. The embedding \( i_0 \) is primitive and induces \( i_{0, \rho} : \mathcal{P}_0 \to \mathcal{P}(L_{26}) \). The orthogonal complement \( R_0 \) of \( S_0 \) in \( L_{26} \) is a negative-definite root lattice of type 2A3. The order of \( O(R_0) \) is 4608,
the order of $O(q(R_0))$ is 8, and the natural homomorphism $O(R_0) \to O(q(R_0))$ is surjective. The vector
\begin{equation}
(4.2) \quad h_0 := \frac{1}{2} \sum_{\ell \in \mathcal{L}_{40}} [\ell] \in S_0 \otimes \mathbb{Q}
\end{equation}
is in fact in $S_0$, and we have $\langle h_0, h_0 \rangle = 40$. Since $\langle h_0, \ell \rangle = 2$ for all $\ell \in \mathcal{L}_{40}$, the class $h_0$ is nef. Since there exist no roots $r$ of $S_0$ such that $h_0 \in (r)^\perp$, the class $h_0$ is ample. Let $w_0 \in L_{26}$ be the same Weyl vector used in Section 4.2. The orthogonal projection of $w_0$ to $S_0 \otimes \mathbb{Q}$ is equal to $h_0/2$. (In [11], the vector $h_0/2$ is used instead of $h_0$.) We put
\[D_0 := i_{0, 26}^{-1}(C(w_0)).\]
Then $D_0$ contains $h_0$ in its interior, and hence $D_0$ is a $V(i_0)$-chamber. The set $W_{\text{out}}(D_0)$ of outer-walls of the initial chamber $D_0$ is equal to $\{ (\ell)^\perp \cap D_0 \mid \ell \in \mathcal{L}_{40} \}$. We have
\[\text{Aut}(X_0, D_0) = \text{Aut}(X_0, h_0),\]
which is of order 3840 and acts on $W_{\text{out}}(D_0)$ transitively. By the algorithms in Remark 1.3, we search for double-plane polarizations in $S_0$, and obtain the following proposition, which proves Theorem 1.4.

**Proposition 4.2.** The action of $\text{Aut}(X_0, h_0)$ decomposes the set $W_{\text{in}}(D_0)$ of inner-walls of the initial chamber $D_0$ into four orbits $O_{44}$, $O_{49}$, $O_{160}$, $O_{120}$, where $|O_{s}| = s$. For each inner-wall $(v)^\perp \cap D_0 \in O_s$, there exists a double-plane polarization $h_0 \in S_0$ such that the corresponding double-plane involution $g(h_0) \in \text{Aut}(X_0)$ maps $D_0$ to the $V(i_0)$-chamber adjacent to $D_0$ across the wall $(v)^\perp \cap D_0$. \qed

Each inner-wall $(v)^\perp \cap D_0 \in O_s$ is defined by a primitive vector $v \in S_0^\perp$ with the properties given in Table 1.2. See [3] for the matrix representations of double-plane involutions $g(h_0)$.

**4.3. The group $\text{Aut}(X_0, h_0)$**. We investigate the finite group $\text{Aut}(X_0, h_0)$ more closely. There exists a natural identification between $W_{\text{out}}(D_0)$ and $\mathcal{L}_{40}$. Therefore, by (4.3), the group $\text{Aut}(X_0, h_0)$ acts on $\mathcal{L}_{40}$ faithfully, and hence $\text{Aut}(X_0, h_0)$ is embedded into the automorphism group $\text{Aut}^+(\mathcal{L}_{40})$ of the dual graph of $\mathcal{L}_{40}$. On the other hand, since $(\mathcal{L}_{40}) = S_0$ (Corollary 2.7), we have an embedding $\text{Aut}(\mathcal{L}_{40}) \hookrightarrow O^+(S_0)$. In fact, we confirm by direct calculation the following:
\[\text{Aut}(X_0, h_0) = \left\{ g \in \text{Aut}(\mathcal{L}_{40}) \left| \begin{array}{c} g, \text{as an element of } O^+(S_0), \text{satisfies} \hfill \\
\text{the period condition \[3.3\]} \end{array} \right. \right\},\]
and $\text{Aut}(X_0, h_0)$ is of index 2 in $\text{Aut}(\mathcal{L}_{40})$. By Propositions 2.3 and 2.6 we have a natural homomorphism $\text{Aut}(\mathcal{L}_{40}) \to \text{Aut}(\mathcal{P})$ to the automorphism group of the Petersen graph $\mathcal{P}$. Recall that, in Section 2.5, we have constructed a rational map $\mu: X_0 \cdot \cdot \cdot \to M \otimes k_0$ that induces the QP-covering map $\mathcal{L}_{40} \to \mathcal{P}$, and calculated the Galois group $\text{Gal}(\mu)$ in Proposition 2.12.

**Proposition 4.3.** The homomorphism
\begin{equation}
(4.4) \quad \text{Aut}(X_0, h_0) \hookrightarrow \text{Aut}(\mathcal{L}_{40}) \to \text{Aut}(\mathcal{P})
\end{equation}
is surjective, and its kernel is equal to the Galois group $\text{Gal}(\mu) \cong (\mathbb{Z}/2\mathbb{Z})^5$. 


Proof. By the list of elements of $\text{Aut}(X_0, h_0)$ (see [31]), we see that the homomorphism $\varphi_{\mu}$ is surjective, and its kernel is of order 32. Each generator of $\text{Gal}(\mu)$ given in Proposition 2.12 preserves $L_{40}$, and hence $\text{Gal}(\mu)$ is contained in $\text{Aut}(X_0, h_0)$. Since $\mu$ induces the QP-covering map $L_{40} \to P$, it follows that $\text{Gal}(\mu)$ is contained in the kernel of $\varphi_{\mu}$. Comparing the order, we complete the proof. \hfill $\Box$

For $v \in S_0$, we put

$$\text{Aut}(X_0, v) := \{ g \in \text{Aut}(X_0) \mid v^g = v \}. $$

Let $f \in S_0$ be the class of a fiber of the Jacobian fibration $\sigma: X_0 \to \mathbb{P}^1$ defined by (1.1). For each element $g$ of $\text{Aut}(X_0, f)$, there exists an automorphism $\bar{g} \in \text{Aut}(\mathbb{P}^1)$ such that the diagram

$$\begin{array}{ccc}
X_0 & \xrightarrow{g} & X_0 \\
\sigma & \downarrow & \downarrow \sigma \\
\mathbb{P}^1 & \xrightarrow{\bar{g}} & \mathbb{P}^1
\end{array}
$$

(4.5)

commutes, and hence $g$ preserves $L_{40}$. Therefore $\text{Aut}(X_0, f)$ is contained in $\text{Aut}(X_0, h_0)$, and we have a homomorphism

$$\beta: \text{Aut}(X_0, f) \to \text{Stab}(\text{Cr}(\sigma)), $$

where $\text{Cr}(\sigma) := \{0, \infty, \pm 1, \pm i\}$ is the set of critical values of $\sigma$, and $\text{Stab}(\text{Cr}(\sigma))$ is the stabilizer subgroup of $\text{Cr}(\sigma)$ in $\text{Aut}(\mathbb{P}^1)$.

We have the inversion $\iota_\sigma: X_0 \to X_0$ of the Jacobian fibration $\sigma$. We also have a subgroup $T_\sigma$ of $\text{Aut}(X_0, f)$ consisting of translations by the 16 sections of $\sigma$.

Proposition 4.4. The order of $\text{Aut}(X_0, f)$ is 768. The image of $\beta$ is isomorphic to $S_4$, and the kernel of $\beta$ is equal to the subgroup $T_\sigma \times \langle \iota_\sigma \rangle$ of $\text{Aut}(X_0, f)$.

Proof. By means of $\rho_C: L_{40} \to L_{112}$ and (2.3), we can calculate the quadrangle $F_c$ in $L_{40}$ consisting of the classes of irreducible components of the singular fiber $\sigma^{-1}(c)$ for each $c \in \text{Cr}(\sigma)$. Then $f$ is the sum of vectors in one of these $F_c$, and hence we can calculate $\text{Aut}(X_0, f)$ from the list of elements of $\text{Aut}(X_0, h_0)$. Looking at the action of $\text{Aut}(X_0, f)$ on the set of the quadrangles $F_c$, we see that the image of $\beta$ is isomorphic to $S_4$ generated by permutations $(0, -1, -i)(\infty, 1, i)$ and $(0, -i)(\infty, i)(1, -1)$ of $\text{Cr}(\sigma)$. Therefore the kernel is of order 32. Since $T_\sigma \times \langle \iota_{\sigma, z} \rangle$ is of order 32 and contained in the kernel, we complete the proof. \hfill $\Box$

Remark 4.5. Since $|\text{Aut}(X_0, h_0)|/|\text{Aut}(X_0, f)| = 5$, the orbit of $f$ by $\text{Aut}(X_0, h_0)$ consists of 5 elements $f = f^{(1)}, f^{(2)}, \ldots, f^{(5)}$. We can easily confirm that $\text{Gal}(\mu)$ is the intersection of these 5 subgroups $\text{Aut}(X_0, f^{(\nu)})$ of $\text{Aut}(X_0, h_0)$. The 5 classes $f^{(\nu)}$ give rise to 5 elliptic fibrations $\sigma^{(\nu)}: X_0 \to \mathbb{P}^1$. These elliptic fibrations correspond to the choices of the $\mathbb{P}^1$-fibration $\varphi_M: M \to \mathbb{P}^1$ in (2.3): the ones induced by the pencil of lines passing through the triple point $t_{\nu}$ for $\nu = 1, \ldots, 4$, and the one induced by the pencil of conics passing through all the triple points.

5. Proof of Theorems 1.7 and 1.8

We use the same notation as in Section 4. The following fact has been established.
Proposition 5.1. (1) The tessellation of $N_3$ by $V(i_3)$-chambers is $\text{Aut}(X_3)$-transitive, and the tessellation of $P_3$ by $V(i_3)$-chambers is $O^+(S_3)$-transitive.

(2) The tessellation of $N_0$ by $V(i_0)$-chambers is $\text{Aut}(X_0)$-transitive, and the tessellation of $P_0$ by $V(i_0)$-chambers is $O^+(S_3)$-transitive. \hfill \Box

From now on, we consider $S_0$ as a subspace of $S_3$ by $\rho: S_0 \hookrightarrow S_3$ and $P_0$ as a subspace of $P_3$. For example, we use notation such as $h_0 \in S_3$, $D_0 \subset P_3$, $P_0 \subset P_3$, ... By the definition (4.1) of $i_0$, we have the following:

Proposition 5.2. The tessellation of $P_0$ by $V(i_0)$-chambers is obtained as the restriction to $P_0$ of the tessellation of $P_3$ by $V(i_3)$-chambers. \hfill \Box

5.1. Proof of Theorem 3.7 First, we show that the restriction homomorphism $\tilde{\rho}$ from $O^+(S_3, S_0)$ to $O^+(S_0)$ maps $O^+(S_3, S_0) \cap \text{Aut}(X_3)$ to $\text{Aut}(X_0)$. By Theorem 3.11 it suffices to show that, for each $g \in O^+(S_3, S_0) \cap \text{Aut}(X_3)$, the restriction $g|_{S_0} \in O^+(S_0)$ satisfies the period condition (3.3) and preserves $N_0$.

Lemma 5.3. If $g \in O^+(S_3, S_0)$ satisfies the period condition $\eta_{S_0}(g) \in O(q(S_3), \omega)$ for $X_3$, then $g|_{S_0} \in O^+(S_0)$ satisfies the period condition $\eta_{S_0}(g|_{S_0}) \in O(q(S_0), \omega)$ for $X_0$.

Proof. Let $Q$ denote the orthogonal complement of $S_0$ in $S_3$. Then $Q$ is a negative-definite lattice of rank 2 with a Gram matrix

$$
\begin{pmatrix}
-12 & 0 \\
0 & -12
\end{pmatrix}.
$$

We consider the commutative diagram in Figure 5.1. The two isomorphisms in the bottom line of this diagram are derived from the isomorphism $q(S_3) \cong q(Q)_3$ given by Proposition 5.6 and the isomorphism $q(Q)_2 \cong q(S_0)$ given by Proposition 3.7. It is easy to verify that $O(Q)$ is a dihedral group of order 8, and the composites $p_3 \circ \eta_Q: O(Q) \to O(q(Q)_3)$ and $p_2 \circ \eta_Q: O(Q) \to O(q(Q)_2)$ are isomorphisms, where $p_2$ and $p_3$ are projections to the 2-part and 3-part, respectively. Using the image of $\eta_Q: O(Q) \to O(q(Q))$ as the graph of an isomorphism between $O(q(Q)_3)$ and $O(q(Q)_2)$, we obtain an isomorphism $O(q(S_3)) \cong O(q(S_0))$ that is compatible with the homomorphisms from $O^+(S_3, S_0)$. Recall that $O(q(S_3), \omega)$ and $O(q(S_0), \omega)$ are cyclic of order 4. Since the cyclic subgroup of order 4 is a characteristic subgroup of the dihedral group of order 8, the isomorphism $O(q(S_3)) \cong O(q(S_0))$ maps $O(q(S_3), \omega)$ to $O(q(S_0), \omega)$. \hfill \Box
Since we have calculated the embedding $\rho: S_0 \hookrightarrow S_3$ in the form of a matrix and the set $\mathcal{W}_{\text{out}}(D_3) \cup \mathcal{W}_{\text{inn}}(D_3)$ of walls of the initial chamber $D_3$ for $X_3$ in the form of a list of vectors (see [31]), we can easily prove the following:

**Lemma 5.4.** (1) The ample class $h_0$ of $X_0$ is contained in $D_3$, and no outer-walls of $D_3$ pass through $h_0$. In particular, $h_0$ belongs to the interior of $N_3$ and hence is ample for $X_3$.

(2) Among the walls $(v)^\perp \cap D_3$ of $D_3$, there exist exactly two walls such that the hyperplane $(v)^\perp$ of $\mathcal{P}_3$ contains $\mathcal{P}_0$. These two walls $(v_1)^\perp \cap D_3$ and $(v_2)^\perp \cap D_3$ belong to the orbit $O'_{648} \subset \mathcal{W}_{\text{inn}}(D_3)$. Moreover, we have $\langle v_1, v_2 \rangle = 0$. □

Combining Lemma 5.4 with Propositions 5.1 and 5.2, we obtain the following:

**Corollary 5.5.** (1) We have $\mathcal{P}_0 = (v_1)^\perp \cap (v_2)^\perp$, where $(v_1)^\perp$ and $(v_2)^\perp$ are the hyperplanes of $\mathcal{P}_3$ given in Lemma 5.4.

(2) For each $\mathcal{V}(i_0)$-chamber $D'_0 \subset \mathcal{P}_0$, there exist exactly four $\mathcal{V}(i_3)$-chambers that contains $D'_0$.

(3) The initial chamber $D_0$ for $X_0$ is a face $(v_1)^\perp \cap (v_2)^\perp \cap D_3$ of the initial chamber $D_3$ for $X_3$, and the interior of $D_0 \subset \mathcal{P}_0$ is contained in the interior of $N_3 \subset \mathcal{P}_3$.

(4) The four $\mathcal{V}(i_3)$-chambers containing $D_0$ are contained in $N_3$. In particular, we have $\gamma_1, \gamma_2, \varepsilon \in \text{Aut}(X_3)$ such that the four $\mathcal{V}(i_3)$-chambers containing $D_0$ are $D_3$ and $D_3^{\gamma_1}, D_3^{\gamma_2}, D_3^{\varepsilon}$. See Figure 5.2. □

**Remark 5.6.** The automorphisms $\gamma_1$ and $\gamma_2$ of $X_3$ in Corollary 5.5(4) can be obtained as conjugates of the double-plane involution $g(b'_{10})$ by PGU$_4(\mathbb{F}_9)$. Let $(v'\cap D_3)$ be the wall of $D_3$ that is mapped to the wall $(v_2)^\perp \cap D_3^{\gamma_1}$ of $D_3^{\gamma_1}$ by $\gamma_1$. Then $(v''\cap D_3)$ is an inner-wall belonging to $O'_{648}$, and hence we have a conjugate $\gamma''$ of $g(b'_{10})$ by PGU$_4(\mathbb{F}_9)$ that maps $D_3$ to the $\mathcal{V}(i_3)$-chamber adjacent to $D_3$ across $(v'\cap D_3)$. Then, as the automorphism $\varepsilon$, we can take $\gamma''\gamma_1$. See Section 6.2 for another construction of $\varepsilon$.

Let $\text{pr}_3: L_{26} \otimes \mathbb{Q} \to S_3 \otimes \mathbb{Q}$, $\text{pr}_0: L_{26} \otimes \mathbb{Q} \to S_0 \otimes \mathbb{Q}$ and $\text{pr}_{30}: S_3 \otimes \mathbb{Q} \to S_0 \otimes \mathbb{Q}$ be the orthogonal projections. Then we have $\text{pr}_{30} \circ \text{pr}_3 = \text{pr}_0$. We put

$$\mathcal{V}(\rho) := \{ \text{pr}_{30}(r) \mid r \in \mathcal{R}(S_3), \quad \langle \text{pr}_{30}(r), \text{pr}_{30}(r) \rangle < 0 \}.$$

The restriction to $\mathcal{P}_0$ of the tessellation of $\mathcal{P}_3$ by $\mathcal{R}(S_3)$-chambers is the tessellation of $\mathcal{P}_0$ by $\mathcal{V}(\rho)$-chambers. The closed subset

$$N_{30} := N_3 \cap \mathcal{P}_0$$

![Figure 5.2. \(\mathcal{V}(i_3)\)-chambers containing \(D_0\)](image-url)
of $P_0$ contains $D_0$ by Corollary 5.5(3), and hence its interior is non-empty. Therefore $N_{30}$ is a $V(\rho)$-chamber. We have
\[ R(S_0) \subset V(\rho) \subset V(i_0), \]
where the second inclusion follows from $R(S_3) \subset R(L_{26})$ and $pr_{30} \circ pr_3 = pr_0$. It follows from Remark 3.2 that
\[ (5.1) \quad D_0 \subset N_{30} \subset N, \]
and that the $V(\rho)$-chamber $N_{30}$ is tessellated by $V(i_0)$-chambers. If $g \in O^+(S_3, S_0)$ preserves $N_3$, then $g|_{S_0} \in O^+(S_0)$ preserves $N_{30}$, and hence preserves $N_0$ by Corollary 5.3. Combining this fact with Lemma 5.4, we conclude that every element of the image of $\tilde{\rho}|_{\text{Aut}}$ belongs to $\text{Aut}(X_0)$.

Next we calculate a generating set of the image of $\tilde{\rho}|_{\text{Aut}}$.

**Lemma 5.7.** The group $\text{PGU}_4(F_9) = \text{Aut}(X_3, h_3)$ acts transitively on the set of non-ordered pairs $\{(v)\perp, (v')\perp\}$ of hyperplanes of $P_3$ such that $(v)\perp \cap D_3$ and $(v')\perp \cap D_3$ are inner-walls of $D_3$ belonging to $O_{648}$ and that $(v, v') = 0$.

**Proof.** As can be seen from the list $[51]$ of walls of $D_3$, for each inner-wall $(v)\perp \cap D_3$ in $O_{648}$, the number of inner-walls $(v')\perp \cap D_3$ in $O_{648}$ satisfying $(v, v') = 0$ is 42. Comparing $42 \times 648/2 = 13608$ with Corollary 2.11, we obtain the proof. \[ \square \]

**Corollary 5.8.** Let $g$ be an element of $\text{Aut}(X_3)$ such that $D_0' := P_0 \cap D_3^g$ is a $V(i_0)$-chamber, that is, $D_0'$ has an interior point as a subset of $P_0$. Then there exists an element $\gamma \in \text{PGU}_4(F_9)$ such that $\gamma g \in \text{Aut}(X_3)$ maps the face $D_0$ of $D_3$ to the face $D_0'$ of $D_3^g = D_3^g$.

**Proof.** We put $v_1' := v_1^g$ and $v_2' := v_2^g$, where $v_1$ and $v_2$ are given in Lemma 5.4. Then $D_3^g = P_0^g \cap D_3 = (v_1')\perp \cap (v_2')\perp \cap D_3$ is a face of $D_3$, which is the intersection of two perpendicular inner-walls $(v_1')\perp \cap D_3$ and $(v_2')\perp \cap D_3$ in $O_{648}$. Hence the existence of $\gamma \in \text{PGU}_4(F_9)$ follows from Lemma 5.7. \[ \square \]

We put
\[ (5.2) \quad \text{Aut}(X_3, D_0) := \{ g \in \text{Aut}(X_3) \mid D_0^g = D_0 \}, \]
and compare it with $\text{Aut}(X_0, D_0) = \text{Aut}(X_0, h_0)$. Note that $\text{Aut}(X_3, D_0)$ is a subgroup of $O^+(S_3, S_0) \cap \text{Aut}(X_3)$ including the kernel of $\tilde{\rho}|_{\text{Aut}}$.

**Lemma 5.9.** The homomorphism $\tilde{\rho}|_{\text{Aut}}$ maps $\text{Aut}(X_3, D_0)$ to $\text{Aut}(X_0, h_0)$ isomorphically. In particular, the homomorphism $\tilde{\rho}|_{\text{Aut}}$ is injective, and the image of $\tilde{\rho}|_{\text{Aut}}$ contains $\text{Aut}(X_0, h_0)$.

**Proof.** By Corollary 5.5(4), the subgroup $\text{Aut}(X_3, D_0)$ of $\text{Aut}(X_3)$ is contained in the finite subset
\[ (5.3) \quad \text{PGU}_4(F_9) \sqcup \text{PGU}_4(F_9) \cdot \gamma_1 \sqcup \text{PGU}_4(F_9) \cdot \gamma_2 \sqcup \text{PGU}_4(F_9) \cdot \varepsilon \]
of $\text{Aut}(X_3)$. For each element $g$ of this subset, we determine whether $g$ preserves $P_0$ or not. We see that, in each coset $\text{PGU}_4(F_9) \cdot \gamma$ in (5.3), exactly 960 elements $g$ satisfy $P_0^g = P_0$, and that the set of restrictions $g|_{S_0}$ of these 960 $\times 4 = 3840$ elements $g$ is equal to $\text{Aut}(X_0, h_0)$. \[ \square \]
Let \( D_0 \) or orbits \( N_30 \) of codimension 3. Let \( F \) denote the hyperplane \( (v) \perp \) of \( \mathcal{P}_0 \) considered as a linear subspace of \( \mathcal{P}_3 \) of codimension 3. Let \( D'_0 \) be one of the four \( \mathcal{V}(i_3) \)-chambers such that \( D'_0 = \mathcal{P}_0 \cap D'_3 \). (See Corollary 5.10 (2).) We have \( F \cap D_0 = F \cap D'_0 = F \cap D_3 = F \cap D'_3 \), and this set contains a non-empty open subset of \( F \). Lemma 5.10 implies that there exists no...
root $r$ of $S_3$ such that the hyperplane $(r)^\perp$ of $P_3$ contains $F$. Since $F \cap D_3 = F \cap D_3'$, we see that $D_3$ and $D_3'$ are on the same side of $(r)^\perp$ for any root $r$ of $S_3$, and hence $D_3'$ is contained in $N_3$. Therefore we have an element $g'$ of $\text{Aut}(X_3)$ such that $D_3'^g = D_3'$. By Lemma 5.8 there exists an element $\gamma$ of $\text{PGU}_4(F_3)$ such that $\gamma g'$ maps the face $D_0$ of $D_3$ to the face $D_0' = D_3'^\gamma$. Since both $D_0$ and $D_0'$ are open in $P_0$, we see that $\gamma g' \in \text{Aut}(X_3)$ belongs to $O^+(S_3, S_0)$. Then $\gamma g'|_{S_0}$ maps $D_0$ to $D_0'$.

Lemmas 5.9 and 5.10 implies that $g(b_{112})$ and $g(b_{688})$ are in the image of $\tilde{\rho}|_{\text{Aut}}$. Let $G$ be the subgroup of $\text{Aut}(X_0)$ generated by $\text{Aut}(X_0, h_0)$ and $g(b_{112})$ and $g(b_{688})$. Since $G$ is contained in the image of $\tilde{\rho}|_{\text{Aut}}$, each $g \in G$ preserves $N_{30}$.

**Lemma 5.14.** If a $\mathcal{V}(i_0)$-chamber $D'$ is contained in $N_{30}$, then there exists an element $g \in G$ such that $D' = D_0^g$.

**Proof.** Since $N_{30}$ is tessellated by $\mathcal{V}(i_0)$-chambers, there exists a sequence

$$D^{(0)} = D_0, \ D^{(1)}, \ldots, \ D^{(N)} = D'$$

of $\mathcal{V}(i_0)$-chambers such that each $D^{(\nu)}$ is contained in $N_{30}$ and that $D^{(\nu)}$ is adjacent to $D^{(\nu-1)}$ for $\nu = 1, \ldots, N$. We prove the existence of $g \in G$ by induction on $N$. The case $N = 0$ is trivial. Suppose that $N > 0$, and let $g' \in G$ be an element such that $D_0^g = D^{(N-1)}$. Note that $g'$ preserves $N_{30}$. The $\mathcal{V}(i_0)$-chambers $D_0$ and $D^{(\nu)}$ are adjacent, and both are contained in $N_{30}$. Hence, by Lemma 5.10, the wall of $D_0$ across which $D^{(\nu-1)}$ is adjacent to $D_0$ is either in $O_{40}$ or in $O_{320}$. Therefore we have an element $g'' \in G$ (a conjugate of $g(b_{112})$ or $g(b_{688})$ by $\text{Aut}(X_0, h_0)$) such that $D^{(\nu-1)} = D_0^g$. Then $g''g' \in G$ maps $D_0$ to $D'$.

Let $g$ be an arbitrary element of the image of $\tilde{\rho}|_{\text{Aut}}$. Since $g$ preserves $N_{30}$, there exists an element $g' \in G$ such that $D_0^g = D_0^{g'}$. Then $g'g^{-1} \in \text{Aut}(X_0, h_0)$, and hence $g \in G$. Thus the proof of Theorem 1.7 is completed.

**5.2. Proof of Theorem 1.8** By the commutativity of the diagram (1.2) and Theorem 1.7, it suffices to prove that the image of $\text{res}_0 : \text{Aut}(\mathcal{X}/R) \rightarrow \text{Aut}(X_0)$ contains $\text{Aut}(X_0, h_0)$ and the double-plane involutions $g(b_{112})$ and $g(b_{688})$. Let $\pi : \mathcal{X} \rightarrow \text{Spec} \ R$ be the elliptic modular surface of level 4 over a discrete valuation ring $R$ of mixed characteristic with the residue field $k$ of characteristic 3. Let $K$ be the fraction field of $R$. We put $X_K := \mathcal{X} \otimes_R K$ and $X_k := \mathcal{X} \otimes_R k$, and identify $X_0$ with $X_K \otimes_K K$ and $X_3$ with $X \otimes_k k$, where $K$ and $k$ are algebraic closures of $K$ and $k$, respectively.

Replacing $R$ by a finite extension of $R$, we can assume that $h_0$ is the class of a line bundle $L_K$ of $X_K$, and that every element of $\text{Aut}(X_0, h_0)$ is defined over $K$. We can extend $L_K$ to a line bundle $L$ of $X$ by (21.6.11) of EGA, IV [3]. Then the class of the line bundle $L_k := L|_{X_k}$ of $X_k$ is $\rho(h_0) \in S_3$. Hence $L_k$ is ample by Lemma 5.4. Therefore $L$ is ample relative to $\text{Spec} \ R$ by (4.7.1) of EGA, III [7]. We choose $n > 0$ such that $L^{\otimes n}$ is very ample relative to $\text{Spec} \ R$, embed $\mathcal{X}$ into a projective space $\mathbb{P}^N_R$ over $\text{Spec} \ R$ by $L^{\otimes n}$, and regard $\text{Aut}(X_0, h_0)$ as the group of projective automorphisms of $X_K \subset \mathbb{P}^N_K$. Since $X_3$ is not birationally uniruled, we can apply the theorem of Matsusaka-Mumford [17] and conclude that every element of $\text{Aut}(X_0, h_0)$ has a lift in $\text{Aut}(\mathcal{X}/R)$. 




Remark 5.15. The argument in the preceding paragraph is a special case of Theorem 2.1 of Lieblich and Maulik [15].

Let $b$ be either $b_{112}$ or $b_{688}$. Replacing $R$ by a finite extension of $R$, we can assume that $b$ is the class of a line bundle $M_K$ of $X_K$, and that each smooth rational curve contracted by $Φ_b: X_K \to \mathbb{P}^2_K$ is defined over $K$. Let $Σ(b) \subset S_0$ be the set of classes of smooth rational curves contracted by $Φ_b$. We extend $M_K$ to a line bundle $\mathcal{M}$ of $X$. Then the class of the line bundle $M_k := \mathcal{M}|X_k$ of $X_k$ is $ρ(b) \in S_3$. By the algorithms in Remark 1.3, we can verify that $b$ is a double-plane polarization of $X$, and calculate the set $Σ(ρ(b)) \subset S_3$ of classes of smooth rational curves contracted by $Φ_{ρ(b)}: X_k \to \mathbb{P}^2_k$. Then we have the following equality:

$$(5.4) \quad Σ(ρ(b)) = ρ(Σ(b)).$$

Since the complete linear systems $|M_K|$ and $|M_k|$ are of dimension 2 and fixed-point free, we see that $π_*\mathcal{M}$ is free of rank 3 over $R$ and defines a morphism

$$\Phi: X \to \mathbb{P}^2_R$$

over $R$. We execute, over $R$, Horikawa’s canonical resolution for double covering branched along a curve with only $ADE$-singularities (see Section 2 of [10]). Let $C_1, K, \ldots, C_3, K$ be the smooth rational curves on $X_K$ contracted by $Φ_b$, where $μ$ is the total Milnor number of the singularities of the branch curve of $Φ_b$ (and hence of $Φ_{ρ(b)}$). It follows from (5.4) that the closure $C_j$ of each $C_{j,i}$ in $X$ is a smooth family of rational curves over $\text{Spec} \; R$, that $\Phi$ contracts $C_j$ to an $R$-valued point $q_{0j}$ of $\mathbb{P}^2_R$ (that is, a section of the structure morphism $\mathbb{P}^2_R \to \text{Spec} \; R$), and that $\Phi$ is finite of degree 2 over the complement of $\{q_{01}, \ldots, q_{0μ}\}$ in $\mathbb{P}^2_R$. We put $J_0 := \{1, \ldots, μ\}$, $P_0 := \mathbb{P}^2_{R_0}$, and let $β_0: P_0 \to \mathbb{P}^2_R$ be the identity. Suppose that we have a morphism $β_i: P_i \to \mathbb{P}^2_R$ over $R$ from a smooth $R$-scheme $P_i$ and a subset $J_i \subset J_0$ such that

(i) $\Phi$ factors as

$$X \xrightarrow{α_i} P_i \xrightarrow{β_i} \mathbb{P}^2_R,$$

(ii) $α_i$ contracts $C_j$ to an $R$-valued point $q_{ij}$ of $P_i$ for each $j \in J_i$, and

(iii) $α_i$ is finite of degree 2 over the complement of $\{q_{ij} \mid j \in J_i\}$ in $P_i$.

Suppose that $J_i$ is non-empty. We choose an index $j_0 \in J_i$, and let $β_i: P_{i+1} \to P_i$ be the blow-up at the $R$-valued point $q_{ij_0}$. Let $β_{i+1}: P_{i+1} \to \mathbb{P}^2_R$ be the composite of $β'$ and $β_i$. Then properties (i)-(iii) are satisfied with $i$ replaced by $i+1$ for some $J_{i+1} \subset J_i$ with $J_{i+1} \neq J_i$. Indeed, $α_{i+1}$ induces a finite morphism from at least one of $C_j$ with $j \in J_i$ to the exceptional divisor of $β'$. Therefore, after finite number of this procedure, we obtain a finite double covering $X \to P$ that factors through $Φ$, where $P$ is obtained from $\mathbb{P}^2_R$ by finite number of blow-ups at $R$-valued points. Then the deck-transformation of $X \to P$ gives a lift of the double-plane involution $g(b) \in \text{Aut} \; (X_0)$ to $\text{Aut} \; (X/R)$. \hfill \-box

Remark 5.16. The double-plane polarizations $ρ(b_{112}), ρ(b_{688}) \in S_3$ have the following properties with respect to $h_3$:

$$\langle h_3, ρ(b_{112}) \rangle = 9, \quad \langle h_3, h_3, g(ρ(b_{112})) \rangle = 34,$$

$$\langle h_3, ρ(b_{688}) \rangle = 19, \quad \langle h_3, h_3, g(ρ(b_{688})) \rangle = 178.$$
6. ENRIQUES SURFACE OF TYPE IV

Let $Z$ be a $K3$ surface defined over an algebraically closed field of characteristic $\neq 2$. For an element $g \in O^+(S_Z)$ of order 2, we put

$$S_Z^g := \{ v \in S_Z \mid v^g = v \}, \quad S_Z^{-g} := \{ v \in S_Z \mid v^g = -v \}.$$ 

Suppose that $\varepsilon : Z \to Z$ is an Enriques involution, and let $\pi : Z \to Y := Z/\langle \varepsilon \rangle$ be the quotient morphism. Note that the lattice $S_Y$ of numerical equivalence classes of divisors of the Enriques surface $Y$ is an even unimodular hyperbolic lattice of rank 10, which is unique up to isomorphism. Then the pull-back homomorphism $\pi^* : S_Y \to S_Z$ is the lattice obtained from $\pi$ unimodular lattice. Moreover, since $\varepsilon$ is an even unimodular lattice of rank 10, and (ii) if $M$ is a Gram matrix of $S_Z^{-\varepsilon}$, then $(1/2)M$ is an integer matrix that defines an even unimodular lattice. Moreover, since $\pi$ is étale, we have that (iii) the orthogonal complement $S_Z^{-\varepsilon}$ of $S_Z^{\varepsilon}$ in $S_Z$ contains no roots.

6.1. Proof of Proposition 1.11. We check conditions (i), (ii), (iii) for all involutions in the finite group $\text{Aut}(X_0, h_0)$. It turns out that there exist exactly 6 involutions $\varepsilon^{(1)}, \ldots, \varepsilon^{(6)}$ satisfying these conditions. They are conjugate to each other, and they belong to the subgroup $\text{Gal}(\mu)$ of $\text{Aut}(X_0, h_0)$ (see Proposition 4.3). We show that these involutions are Enriques involutions of type IV.

Let $\varepsilon_0 \in G$ be one of $\varepsilon^{(1)}, \ldots, \varepsilon^{(6)}$. Recall that $\sigma : X_0 \to \mathbb{P}^1$ is the Jacobian fibration defined by (1.1), and let $f \in S_0$ be the class of a fiber of $\sigma$. Since $\varepsilon_0 \in \text{Gal}(\mu)$, we have $\varepsilon_0 \in \text{Aut}(X_0, f)$ by Remark 1.5. Let $F_c \subset L_{40}$ be the set of classes of irreducible components of the singular fiber $\sigma^{-1}(c)$ over $c \in \text{Cr}(\sigma)$. Looking at the action of $\varepsilon_0$ on these 6 quadrangles $F_c$, we see that the element $\varepsilon_0 \in \text{Stab}(\text{Cr}(\sigma))$ defined by the diagram (1.5) is of order 2 and fixes exactly 2 points of $\text{Cr}(\sigma)$. Suppose that $F_c$ is fixed by $\varepsilon_0$. Then $\varepsilon_0$ acts on $F_c$ as $\ell_0 \leftrightarrow \ell_2$ and $\ell_1 \leftrightarrow \ell_3$, where $\ell_0, \ldots, \ell_3$ are labelled as in (2.4). Therefore $\varepsilon_0$ is fixed-point free, and $Y_0 := X_0/\langle \varepsilon_0 \rangle$ is an Enriques surface.

Let $\pi_0 : X_0 \to Y_0$ be the quotient morphism. We put

$$S_Y := S_{Y_0}, \quad P_Y := P_{Y_0}, \quad N_Y := N_{Y_0}.$$

By $\pi_0^* : S_Y \to S_0$, we regard $P_Y$ as a subspace of $P_0$. Then we have

$$P_Y = \{ x \in P_0 \mid x^{\varepsilon_0} = x \},$$

and $N_Y = N_0 \cap P_Y$. We consider the tessellation of $N_Y$ induced by the tessellation of $N_0$ by $\mathcal{V}(i_0)$-chambers. We put

$$D_Y := D_0 \cap P_Y.$$ 

Note that the interior point $h_0$ of $D_0$ is contained in $D_Y$, and hence $D_Y$ is a chamber in $P_Y$. Let $\text{pr}_Y : S_0 \otimes \mathbb{Q} \to S_Y \otimes \mathbb{Q}$ be the orthogonal projection. Then, by definition, we have

$$D_Y = \left\{ y \in P_Y \bigg| \langle y, \text{pr}_Y(v) \rangle \geq 0 \text{ for every primitive vector } v \in S_0^0 \text{ defining a wall of } D_0 \text{ and satisfying } \langle \text{pr}_Y(v), \text{pr}_Y(v) \rangle < 0 \right\}. $$

If $v^0 \cap D_0$ is an inner-wall of $D_0$, then $\langle \text{pr}_Y(v), \text{pr}_Y(v) \rangle \geq 0$, that is, the hyperplane $H^0$ of $P_0$ does not intersect $P_Y$. Hence we have

$$D_Y = \{ y \in P_Y \mid \langle y, \text{pr}_Y(\ell) \rangle \geq 0 \text{ for all } \ell \in L_{40} \}.$$
Therefore $D_Y$ is equal to $N_Y$, and the set of smooth rational curves on $Y_0$ is the image of $L_{40}$ by $\pi_0$. Let $Z_{\text{Aut}(X_0)}(\varepsilon_0)$ be the centralizer of $\varepsilon_0$ in $\text{Aut}(X_0)$. Then any $g \in Z_{\text{Aut}(X_0)}(\varepsilon_0)$ preserves $P_Y \subset P_0$, and we have a canonical identification

$$\text{Aut}(Y_0) = Z_{\text{Aut}(X_0)}(\varepsilon_0)/\langle \varepsilon_0 \rangle.$$  

Since $h_0 \in D_9 \cap P_Y = D_Y = N_Y = N_9 \cap P_Y$, any $g \in Z_{\text{Aut}(X_0)}(\varepsilon_0)$ preserves $D_9$ and therefore $Z_{\text{Aut}(X_0)}(\varepsilon_0)$ is a subgroup of $\text{Aut}(X_0, h_0)$. Since we have calculated all the elements of $\text{Aut}(X_0, h_0)$, we can readily compute $Z_{\text{Aut}(X_0)}(\varepsilon_0)$, and see that $Z_{\text{Aut}(X_0)}(\varepsilon_0)$ is of order 640. Therefore $Y_0$ is an Enriques surface of type IV. \hfill \Box

Since $\varepsilon^{(v)}$ is an element of $\text{Gal}(\mu)$, we have the following:

**Corollary 6.1.** The rational map $\mu : X_0 \cdots \to M \otimes_R k_0$ factors through the covering morphism of the Enriques surface $X_0/\langle \varepsilon^{(v)} \rangle$. \hfill \Box

### 6.2. Proof of Theorem 1.12

Let $\varepsilon_0 \in \text{Aut}(X_0, h_0)$ be the image of $\varepsilon_3$ by $\tilde{p}|_{\text{Aut}}$, which is one of $\varepsilon^{(i)}_0, \ldots, \varepsilon^{(s)}_0$. Since $\varepsilon_0 \in \text{Aut}(X_0, h_0)$, the involution $\varepsilon_3$ preserves the face $D_0 = P_9 \cap D_3$ of $D_3$. Therefore $\varepsilon_3$ belongs to the finite group $\text{Aut}(X_3, D_0)$ defined by (5.2). We check all involutions in $\text{Aut}(X_3, D_0)$ and find $\varepsilon_3$ in the form of a matrix acting on $S_3$. We have $(h_0, h_0^{\tau}) = 16$. Indeed, the $Y(\varepsilon_3)$-chamber $D_3^{\varepsilon_3}$ is the chamber $D_3$ in Figure 5.2. The action of $\varepsilon_3$ on the fibers of the Jacobian fibration $\sigma : X_3 \to \mathbb{P}^1$ defined by (1.1) is exactly the same as $\varepsilon_0$. Hence $\varepsilon_3$ is fixed-point free. Let $\pi_3 : X_3 \to Y_3 := X_3/\langle \varepsilon_3 \rangle$ be the quotient morphism. The centralizer $Z_{\text{Aut}(X_3)}(\varepsilon_3)$ of $\varepsilon_3$ in $\text{Aut}(X_3)$ is contained in the finite group $\text{Aut}(X_3, D_0)$. Hence we can easily verify that $Y_3$ of type IV. By 6.1, the set of pull-backs of the smooth rational curves on $Y_3$ by $\pi_3$ is $L_{40} \subset L_{112}$. Hence they are lines on $F_3$. \hfill \Box

**Remark 6.2.** In [12], Kondo showed that the covering $K3$ surface of the complex Enriques surface of type III is also isomorphic to $X_0$. It seems to be an interesting problem to find an Enriques involution on the Fermat quartic surface $F_3$ that gives rise to the Enriques surface of type III in characteristic 3.

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