The light-cone theorem

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Abstract
We prove that the area of cross-sections of light cones, in spacetimes satisfying suitable energy conditions, is smaller than or equal to that of the corresponding cross-sections in Minkowski, or de Sitter, or anti-de Sitter spacetime. The equality holds if and only if the metric coincides with the corresponding model in the domain of dependence of the light cone.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

It is a well-known fact in general relativity that gravitation tends to focus on null geodesics; this fact lies at the heart of, e.g., the singularity theorems of Hawking and Penrose [1]. In this work we wish to point out a simple and striking illustration of this fact, which seems to have been overlooked in the literature, concerning the area of cross-sections of light cones: we prove that such cross-sections, in spacetimes satisfying the Einstein equations with vanishing cosmological constant $\Lambda$, and with the energy–momentum satisfying the dominant energy condition, are smaller than the corresponding areas of cross-sections of light cones in Minkowski spacetime. Moreover, under supplementary restrictions on the energy–momentum tensor, equality of areas for a cross-section $S$ implies that the spacetime is Minkowskian in the domain of dependence of that part of the light cone which lies between the vertex and the cross-section $S$. A similar result holds when $\Lambda \neq 0$: in the statement just given one needs to replace the Minkowski spacetime by the de Sitter or anti-de Sitter spacetime. The precise statements can be found in section 2.

The idea of the argument is to show, using the dominant energy condition, that the expansion of the light cone is smaller than that of the model space; this implies the area inequality. The rigidity part of our statement is based on an analysis, closely following that in [2], of the associated characteristic Cauchy problem; see also [3–8] and references therein.
2. The theorem

Consider an \((n + 1)\)-dimensional spacetime \((\mathcal{M}, g)\), \(n \geq 2\), satisfying the dominant energy condition

\[
T_{\mu\nu}X^\mu Y^\nu \geq 0 \text{ for all future oriented timelike vectors } X \text{ and } Y. \tag{2.1}
\]

This will be the only condition needed for our comparison result. However, to obtain rigidity, more conditions will be needed. We shall say that the rigid dominant energy condition holds at \(q \in \mathcal{M}\) if (2.1) holds, together with the implication

\[
T_{\mu\nu}X^\mu X^\nu = 0 \text{ for some causal vector } X \text{ at } q \implies T_{\mu\nu}X^\nu = 0 \text{ at } q. \tag{2.2}
\]

(It is well known that the implication is always true for timelike vectors by (2.1) (compare appendix B), so this is only a restriction for null vectors.) We note a related condition used by Galloway and Solis [9] (see condition (C) in section 4 of that last reference), also in a null rigidity context.

General relativistic fluids with timelike flow vector \(u^\mu\), with \(0 \leq |p| \leq \rho\), and with an equation of state which excludes the possibility \(p = -\rho\) except when \(\rho = 0\), provide energy–momentum tensors satisfying (2.2) everywhere. Another example is provided by the energy–momentum \(T_{\mu\nu} = \rho \ell_\mu \ell_\nu\), where \(\rho \geq 0\) and \(\ell_\mu\) is null.

Examples of the energy–momentum tensor satisfying the dominant energy condition and which do not satisfy (2.2) are given by \(T_{\mu\nu} = -\rho g_{\mu\nu}\), \(\rho \geq 0\), or by massless scalar fields, or by the Maxwell energy–momentum tensor, as discussed in appendix A.

There is, however, a version of (2.2) which applies to both massless scalar fields and Maxwell fields; see propositions A.2 and A.3 below; we emphasize that the argument there is non-local (as it requires integration) and non-algebraic (as it makes use of the field equations): to define this, let \(\ell\) be a field of null tangents to a null hypersurface \(\mathcal{N}\). We shall say that the rigid dominant energy condition holds on \(\mathcal{N}\) if (2.1) holds together with the implication

\[
T_{\mu\nu}\ell^\mu \ell^\nu = 0 \text{ on } \mathcal{N} \implies T_{\mu\nu}\ell^\nu = 0 \text{ on } \mathcal{N}. \tag{2.3}
\]

Let \(p \in \mathcal{M}\) and let \(\mathcal{C}^+_p\) be the future light cone emanating from \(p\). Let \(T\) be any unit timelike vector at \(p\), and normalize all null vectors \(\ell\) at \(p\) by requiring that \(g(\ell, T) = -1\). This defines an affine parameter, denoted by \(s\), on the future null geodesics \(s \mapsto \gamma_\ell(s)\) with \(\gamma_\ell(0) = p\) and with initial tangent \(\ell\). Let \(\mathcal{A}(s)\) denote the \((n - 1)\)-dimensional surface reached by these geodesics after affine time \(s\)

\[
\mathcal{A}(s) = \{\gamma_\ell(s)\} \subset \mathcal{C}^+_p, \tag{2.4}
\]

where the vectors \(\ell\) run over all null future vectors at \(p\) normalized as above; see figure 1. We denote by \(\mathcal{C}(t)\) the subset of the light cone covered by all the geodesics up to affine time \(t\)

\[
\mathcal{C}(t) = \bigcup_{0 \leq s \leq t} \mathcal{A}(s). \tag{2.5}
\]

Note that \(\gamma_\ell(s)\) might not be defined for all \(s\). Further, \(\mathcal{A}(s)\) might not be a smooth surface. However, for every point \(p\) there exists a maximal \(s_0 > 0\) such that \(\mathcal{A}(s)\) is defined and smooth for all \(s < s_0\). We restrict ourselves to \(s < s_0\), though it is rather clear that this can be relaxed using the methods of [10]; we have, however, not attempted to verify all details of that.

Let \(|\mathcal{A}(s)|\eta\) denote the area of \(\mathcal{A}(s)\). So for the Minkowski metric, which we denote by \(\eta\), we have

\[
|\mathcal{A}(s)|\eta = \omega_{n-1} s^{n-1},
\]

where \(\omega_{n-1}\) is the area of the unit round sphere in \(\mathbb{R}^n\).

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6 Our signature is \((-,+ \ldots ,+)\).
Figure 1. The cross-section \( A(s) \) of the light cone \( \mathcal{C}_p^+ \); \( \mathcal{C}(s) \) is the shaded blue region. Two generators \( \gamma_1 \) and \( \gamma_2 \) are also shown.

We consider metrics satisfying the Einstein equations with cosmological constant \( \Lambda \in \mathbb{R} \) and sources. We assume smoothness of the metric for simplicity, though our result can be proved under weaker differentiability conditions.

**Theorem 2.1.** Let \((\mathcal{M}, g)\) be a smooth globally hyperbolic spacetime, solution of the Einstein equations with the energy–momentum tensor satisfying the dominant energy condition. We restrict our attention to \( s \) such that \( \mathcal{C}(s) \) lies within the domain of injectivity of the exponential map at \( p \). Then

(i) The area \(|A(s)|_g\) satisfies the inequality

\[
|A(s)|_g \leq |A(s)|_\eta. \tag{2.6}
\]

(ii) Let equality be attained at some \( s = s_2 \). If either

(a) the rigid dominant energy holds at \( \mathcal{C}(s_2) \), or

(b) the energy–momentum tensor is traceless,

then the domain of dependence of \( \mathcal{C}(s_2) \) is isometric to the corresponding domain of dependence in Minkowski or (anti) de Sitter spacetime.

**Proof.** Let \( \theta \) denote the rate of change of area along the null geodesic generators of \( \mathcal{C}_p^+ \), and let \( \sigma \) denote the shear of \( \mathcal{C}_p^+ \) (see, e.g., [11]). Let \( \gamma \) be such a generator, and recall the Raychaudhuri equation in spacetime dimension \( n + 1 \) [11] (note that the rotation term vanishes because our family of null geodesics forms a hypersurface)

\[
\frac{d\theta}{ds} = -\sigma^{AB}\sigma_{AB} - \frac{1}{n-1}\theta^{2} - R_{\sigma \mu \gamma \nu} \dot{\gamma}^{\mu} \dot{\gamma}^{\nu}. \tag{2.7}
\]

Here \( s \) is an affine parameter along the generators: \( \nabla_{\gamma} \dot{\gamma} = 0 \).

For the proof of rigidity we will be using a coordinate system \((u, r, x^A)\), where \( s = r \), with a wave-map condition imposed on the extension of the coordinates away from the light cone. However, no such condition is needed for the comparison argument.
Before giving a detailed proof, it might be useful to present an outline: let \( \theta_0 \) denote the expansion of a light cone in Minkowski spacetime

\[
\theta_0 := \frac{n - 1}{s},
\]

then \( \theta_0 \) satisfies (2.7) with vanishing Ricci tensor and \( \sigma \). Since \( \theta \) approaches \( (n - 1)/s \) as the tip of the light cone is approached, a comparison argument using (2.7) shows that \( \theta \) is smaller than its Minkowskian value. This, subsequently, implies the area inequality. Equality holds on \( C(s_2) \) if and only if \( \sigma \) and \( R_{\mu\nu}\dot{\gamma}^\mu \dot{\gamma}^\nu \) vanish along all geodesic generators of \( \mathcal{C}^+ \) until these generators reach \( \mathcal{C}(s_2) \), i.e., on \( C(s_2) \). When the rigid dominant energy condition holds (in either its local or nonlocal form), the usual energy calculation implies that the metric is vacuum in the domain of dependence of \( C(s_2) \). Under the traceless condition a more detailed analysis is necessary. This, together with the vanishing of \( \sigma \) on \( C(s) \), is used to show that the metric tensor takes the model-metric values on \( C(s) \), and the result follows by uniqueness of solutions of the characteristic initial value problem.

Let us pass now to the details of the above. Since \( \theta_0 \) satisfies the equation

\[
\frac{d\theta_0}{ds} = -\frac{\theta_0^2}{n - 1},
\]

from (2.7) we have

\[
\frac{d(\theta - \theta_0)}{ds} = \frac{\theta_0^2 - \theta^2}{n - 1} - \sigma_{AB}\sigma^{AB} - R_{\mu\nu}\dot{\gamma}^\mu \dot{\gamma}^\nu
\]

\[
= -\frac{(\theta - \theta_0)^2}{n - 1} - \frac{2}{s}(\theta - \theta_0) - \sigma_{AB}\sigma^{AB} - R_{\mu\nu}\dot{\gamma}^\mu \dot{\gamma}^\nu
\]

\[
\leq -\frac{2}{s}(\theta - \theta_0) - \sigma_{AB}\sigma^{AB} - R_{\mu\nu}\dot{\gamma}^\mu \dot{\gamma}^\nu.
\]

Hence, for \( s > s_1 > 0 \),

\[
s^2(\theta - \theta_0)(s) \leq s_1^2(\theta - \theta_0)(s_1) - \int_{s_1}^{s}(\sigma_{AB}\sigma^{AB} + R_{\mu\nu}\dot{\gamma}^\mu \dot{\gamma}^\nu)s^2 ds.
\]

(2.9)

Now, for a smooth metric we have

\[
\theta = \frac{(n - 1) + o(1)}{s}
\]

(2.10)

for small \( s \), so we can pass to the limit \( s_1 \to 0 \) to obtain

\[
(\theta - \theta_0)(s) \leq 1 \int_{0}^{s}(\sigma_{AB}\sigma^{AB} + R_{\mu\nu}\dot{\gamma}^\mu \dot{\gamma}^\nu)s^2 ds.
\]

(2.11)

Since the dominant energy condition has been assumed to hold, the right-hand-side of (2.11) is non-positive and we conclude that

\[
\theta(s) \leq \frac{n - 1}{s}
\]

(2.12)

as long as the geodesic exists. Furthermore, equality holds for some \( s_2 > 0 \) if and only if

\[
\forall s \text{ satisfying } 0 < s < s_2, \quad \sigma_{AB} = 0 = R_{\mu\nu}\dot{\gamma}^\mu \dot{\gamma}^\nu.
\]

(2.13)

The area inequality follows from (2.12) in a standard way, we give the details for completeness. In a coordinate system adapted to the light cone we can write the metric on the cone in the form

\[
g = -\alpha du^2 + 2\nu A dx^A du - 2e^{2\beta} du dr + g_{AB} dx^A dx^B,
\]

(2.14)
so that $\mathcal{C}_p^+ = \{ q \in \mathcal{M} : u(q) = 0 \}$, where $r$ is an affine parameter along the generators of $\mathcal{C}_p^+$, vanishing at the vertex, denoted by $s$ in the previous equations. A calculation shows

$$
\theta = \frac{1}{\sqrt{\det g_{AB}}} \partial_r (\sqrt{\det g_{AB}}).
$$

(2.15)

Let us denote by $\hat{g}_{AB} \, dx^A \, dx^B$ the $(n - 1)$-dimensional corresponding metric arising on a light cone in the $(n + 1)$-dimensional Minkowski spacetime. Then our analysis so far shows that

$$
\theta \equiv \partial_r \log \sqrt{\det g_{AB}} \leq \theta_0 \equiv \partial_r \log \sqrt{\det \hat{g}_{AB}}.
$$

Thus $\log(\det g_{AB}/\det \hat{g}_{AB})$ is decreasing. By elementary considerations the quotient $\det g_{AB}/\det \hat{g}_{AB}$ tends to one as $r$ tends to zero, and we conclude

$$
\log \sqrt{\det g_{AB}} \leq \log \sqrt{\det \hat{g}_{AB}}, \quad \text{hence } \sqrt{\det g_{AB}} \leq \sqrt{\det \hat{g}_{AB}}.
$$

The areas $|\mathcal{A}'(r)|_g$ and $|\mathcal{A}'(r)|_g$ are

$$
|\mathcal{A}'(r)|_g = \int_{S_{n-1}} \sqrt{\det g_{AB}} \, dx^2 \cdots dx^n, \quad |\mathcal{A}'(r)|_g = \int_{S_{n-1}} \sqrt{\det \hat{g}_{AB}} \, dx^2 \cdots dx^n,
$$

therefore

$$
|\mathcal{A}'(r)|_g \leq |\mathcal{A}'(r)|_g,
$$

which establishes part 1 of the theorem.

Assume, now, that equality in this last equation holds at $s = s_2$. Equation (2.13) implies the vanishing of $T_{\mu\nu} \hat{\gamma}^\mu \hat{\gamma}^\nu$ on $\mathcal{C}(s_2)$.

If we assume that the energy–momentum tensor $T$ satisfies the rigid dominant energy condition, as in (2.2), or the rigid dominant energy condition on $\mathcal{C}(s_2)$, as in (2.3), we can conclude that $T_{\mu\nu} \hat{\gamma}^\nu$ vanishes on $\mathcal{C}(s_2)$. The proof that the metric is vacuum in the domain of dependence of $\mathcal{C}(s_2)$ is then standard, and proceeds as follows:

Consider the manifold

$$\mathcal{M} := \mathcal{M} \setminus J^+(\mathcal{A}'(s_2)),
$$

with the metric obtained from $g$ by restriction, still denoted by $g$. Then $(\mathcal{M}, g)$ is globally hyperbolic, with

$$
\mathcal{D}^+(\mathcal{C}(s_2), \mathcal{M}) = \mathcal{D}^+(\mathcal{C}(s_2), \mathcal{M}),
$$

(2.16)

where $\mathcal{D}^+(\Omega, \mathcal{M})$ denotes the domain of dependence of an achronal set $\Omega$ within a spacetime $(\mathcal{M}, g)$. The equality in (2.16) means that the manifolds, equipped with the obvious metrics, are isometric.

Let $t$ be any Cauchy time function on $\mathcal{M}$, i.e., a time function ranging over $\mathbb{R}$, the level sets of which are Cauchy surfaces. Replacing $t$ by $t - t(p)$, we can without loss of generality assume that $t(p) = 0$.

Let

$$
E(s) = - \int_{\mathcal{D}^+(\mathcal{C}(s_2)) \cap \{ t = s \}} T^\mu \, n^\nu \, dS_\mu,
$$

(2.17)

where $n^\mu$ is the field of future directed unit normals to the level sets of $t$; $E$ is positive in our signature $(−, +, \cdots, +)$. The divergence identity on the set bounded by $\mathcal{C}(s_2) \cap \{ t \leq s \}$ and $\mathcal{D}^+(\mathcal{C}(s_2)) \cap \{ t = s \}$ (compare (D.13) and lemma B.1) shows that, for any time interval $[0, T]$, there exists a constant $C = C(T)$ such that

$$
E(s) \leq C \int_0^s E(t) \, dt - \int_{\mathcal{D}^+(\mathcal{C}(s_2)) \cap \{ t = s \} \cap \{ t = 0 \}} T^\mu \, n^\nu \, dS_\mu,
$$

(2.18)

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where the boundary integrand vanishes by the rigid dominant energy condition, being proportional to $T^{\mu\nu}n^\mu \dot{\gamma}^\nu$. Since $E(s)$ approaches zero as $s$ tends to zero, from Gronwall’s lemma we obtain

$$E(s) = 0 \quad \text{for} \quad 0 < s < s_2.$$  

Positivity of the integrand implies

$$T^{\mu\nu}n^\mu n^\nu = 0 \quad \text{on} \quad D^+(\mathcal{C}(s_2)). \tag{2.19}$$

From (2.19) and lemma B.1, we conclude that an energy–momentum tensor satisfying the rigid dominant energy condition must vanish on every level set of $t$ within the domain of dependence of $\mathcal{C}(s_2)$. As $D^+(\mathcal{C}(s_2))$ is covered by these level sets, the vanishing of $T^{\mu\nu}$ on $D^+(\mathcal{C}(s_2))$ follows.

The proof of (2.19) for tensors that do not satisfy the rigid dominant energy condition requires more care. In view of (2.13), at this stage of the analysis we can only conclude that

$$R_{rr} = 8\pi T_{rr} = 8\pi T_{rA} = R_{Ar} \tag{2.20}$$

on $\mathcal{C}(s_2)$. Indeed, to see the vanishing of $T_{Ar}$, set $\ell = \partial_r$. Then, by the dominant energy condition, the vector field $T^{\mu\nu} \ell^\mu \partial_\nu$ is causal, and has vanishing scalar product with $\ell$, hence is proportional to $\ell$. So $T^{\mu\nu} \ell^\mu$ is proportional to $\partial_r$; subsequently

$$T_{Ar} = g_{Am} \left. \frac{T^{\mu}_{\ r}}{0 \text{ unless } \mu = r} \right|_{0}^0 = 0,$$

as desired\(^8\).

Let $x^\mu$ denote normal coordinates centered at $p$, let $R > 0$ denote the largest number so that the exponential map at $p$ is a diffeomorphism from a truncated solid cone $\Omega(R) \subset T_p \mathcal{M}$, defined as

$$\Omega(R) := \left\{ 0 \leq x^0 \leq R, r := \sqrt{\sum (x^i)^2} < x^0 \right\},$$

to its image in $\mathcal{M}$. Note that this image is included in $D^+(\mathcal{C}_p(R))$ when the level sets of $x^0$ are timelike within $\Omega(R)$.

If $\Lambda = 0$, we let the functions $y^\mu$ be solutions of the following characteristic Cauchy problem:

$$\square g y^\mu = 0, \tag{2.21}$$

$$y^\mu |_{\mathcal{C}(R)} = x^\mu. \tag{2.22}$$

For nonzero $\Lambda$, we impose again the boundary conditions (2.22), but we require instead that the map $x^\mu \mapsto y^\mu(x^\alpha)$ satisfies the wave-map equation, with the (anti)-de Sitter metric in the target

$$\hat{g} = -\left(1 - \frac{2\Lambda}{n(n-1)}r^2\right) dr^2 + \frac{dr^2}{1 - \frac{2\Lambda}{n(n-1)}r^2} + r^2 \hat{h}_{AB} \, dx^A \, dx^B, \tag{2.23}$$

\(^8\) Actually, we can further show that a traceless $T_{\mu\nu}$ must vanish at the vertex of the light cone: for this, by continuity and (2.13) we find that $T_{\mu\nu} \ell^\mu \ell^\nu = 0$ at $p$ for every null vector $\ell \in T_p \mathcal{M}$. By [12, lemma 2.8], $T_{\mu\nu}$ is proportional to the metric at $p$, and tracelessness implies the claim. But this fact does not seem to be useful in the analysis that follows.
where $\hat{h}_{AB} \, dx^A \, dx^B$ is the round unit metric on $S^{n-1}$. Thus, in both cases, the functions $y^\mu$ satisfy the set of equations (see, e.g., [13, page 162])

$$
g^{\alpha\beta} \left( \partial_\alpha \partial_\beta y^\mu - \hat{\Gamma}^\mu_{\alpha\beta} \frac{\partial y^\nu}{\partial x^\alpha} + \hat{\Gamma}^\mu_{\nu\alpha} \frac{\partial y^\nu}{\partial x^\beta} \frac{\partial y^\alpha}{\partial x^\beta} \right) = 0,  \tag{2.24}$$

where the $\hat{\Gamma}^\mu_{\alpha\beta}$'s are the Christoffel symbols of $\hat{g}$, except that (2.24) is linear when $\Lambda = 0$, and thus the solutions exist globally on the domain of dependence of the smooth part of the light cone, while for $\Lambda \neq 0$ the solutions might exist only for some neighborhood of the tip of the light cone.

By [14, theorem 5.4.2] (compare [15]) the functions $y^\mu$ are smooth up-to-boundary on $D^+(C^p(R))$. Decreasing $R$ if necessary, the functions $y^\mu$ form a smooth coordinate system on $D^+(C^p(R))$. Let $g_{\mu\nu}$ denote the components of the metric in the coordinates $y^\mu$, then the $g_{\mu\nu}$'s are smooth up-to-boundary on $D^+(C^p(R))$. If we pass to a coordinate system so that $u := y^0 - |\vec{y}|$, and $r = |\vec{y}|$, and where $x^A$'s are local coordinates on $S^{n-1}$, then the cone is given by the equation $u = 0$, and the metric on $C^p(R)$ takes the form (2.14)

$$
g = -\alpha \, du^2 + 2\nu_A \, dx^A \, du - 2 e^{-\beta} \, du \, dr + h_{AB} \, dx^A \, dx^B.  \tag{2.25}$$

We emphasize that we do not assume that the metric takes the form (2.25) away from $\{u = 0\}$, so care must be taken when $\partial_u$-derivatives are taken.

By definition [11], $\sigma_{AB}$ is the trace-free part of $g(\nabla A \partial_r, \partial_B) = g_{BC} \hat{\Gamma}^C_{A\alpha} = \frac{1}{2} \partial_\alpha g_{AB}$, so from the vanishing of $\sigma_{AB}$, and from the explicit formula for $\theta = \theta_0$ we obtain

$$
\partial_r h_{AB} = \frac{2}{r} h_{AB} \iff \partial_r (r^{-2} h_{AB}) = 0.  \tag{2.26}
$$

Since $r^{-2} h_{AB}$ tends to the unit round metric $\hat{h}_{AB}$ on $S^{n-1}$ as $r$ tends to zero, we conclude that $h_{AB} = r^2 \hat{h}_{AB}$.

We continue by showing that $\beta = 0$. For this note that, by definition of normal coordinates, $r$ is an affine parameter along the geodesics generators of $C^p_+$. So $\nabla_\alpha \partial_r = 0$, which is equivalent to $0 = \Gamma^\alpha_r$. But

$$
\Gamma^\mu_r = \delta^\mu_r \left( 2\partial_r \beta + \frac{1}{2} e^{-2\beta} \partial_\alpha g_{rr} \right), \quad \text{and we conclude that} \quad e^{-2\beta} \partial_\alpha g_{rr} = -4 \partial_r \beta.  \tag{2.27}
$$

We set

$$
\lambda^\mu := -g^{\alpha\beta} \hat{\Gamma}^\mu_{\alpha\beta},  \tag{2.28}
$$

$$
\hat{\lambda}^\mu := -g^{\alpha\beta} \hat{\Gamma}^\mu_{\alpha\beta}.  \tag{2.29}
$$

The wave-map condition $\lambda_r = g_{\tau\mu} \hat{\lambda}^\mu$ can be shown to read

$$
\frac{1}{2} \hat{h}^{AB} \partial_r h_{AB} + e^{-2\beta} \partial_\alpha g_{rr} \equiv \lambda_r = g_{\tau\mu} \hat{\lambda}^\mu \equiv \frac{n-1}{r} e^{2\beta}.  \tag{2.27}
$$

Writing $y = e^{2\beta}$, this is the same as

$$
\partial_r y = \frac{n-1}{2r} y(1-y).$$
Integrating, we obtain either $y \equiv 1$, or

$$y = \frac{C(x^A)r^{(n-1)/2}}{1 + C(x^A)r^{(n-1)/2}},$$

for some function $C(x^A)$. But, in normal coordinates, $\beta$ approaches zero as $r$ goes to zero, and we conclude that $y \equiv 1$; equivalently, $\beta \equiv 0$.

In appendix C we show that the vanishing of $R_A$ is equivalent to

$$0 = \frac{(n-2)(n-3)}{2} v_A + \frac{3n-5}{2r} \partial_r v_A + \partial_r \partial_r v_A = \left( \frac{1}{r^{n-1}} \partial_r \left[ r^{n-1} \left( \partial_r v_A + \frac{n-3}{2r} v_A \right) \right] \right).$$  \tag{2.30}

Integrating (2.30) in $r$ once we obtain, for some smooth functions $\hat{v}_A = \hat{v}_A(x^B)$,

$$\hat{v}_A r^{1-n} = \partial_r v_A + \frac{n-3}{2r} v_A = r^{\frac{n-1}{2}} \partial_r \left( r^{\frac{n-1}{2}} v_A \right).$$

Integrating again, we conclude that there exist smooth functions $\bar{v}_A(x^B)$ such that, for $n > 1$,

$$v_A(r, x^B) = r^{\frac{n-1}{2}} \bar{v}_A(x^B) - \frac{2}{(n-1)} r^{2-n} \bar{v}_A(x^B).$$  \tag{2.31}

But from the definition of our coordinate system it is elementary to show that $v_A$ approaches zero as $r \to 0$, which implies that $\bar{v}_A \equiv 0$.

We are ready now to establish (2.19) for traceless energy–momentum tensors. For this let

$$\Omega(s_\ast) := J^\ast(p) \cap \{ t < s_\ast \},$$  \tag{2.32}

where $t = x^0$ is a normal coordinate. We define $s_\ast \leq R$ to be the largest number smaller than or equal to $s_2$ such that $\Omega(s_\ast)$ lies within the domain of definition of normal coordinates. Moreover, we assume that $\partial_\mu$ and $\nabla t$ are timelike on $\Omega(s_\ast)$, and that the functions $y^\mu$, defined as solutions of (2.24), form a coordinate system on $\Omega(s_\ast)$. The proof of the vanishing of $T_{\mu\nu}$, to be found in appendix D, is again an energy calculation, using instead the energy functional defined as

$$E(s) = -\int_{\mathcal{D}^\ast(\mathcal{C}(s_\ast)) \cap \{ t = s \}} T^\mu \nu X^\nu dS_\mu,$$  \tag{2.33}

where the normal–coordinates components of $X = X^\mu \partial_\mu$ are, very roughly, of the form

$$X^\mu = x^\mu.$$  \tag{2.34}

This choice of $X^\mu$ ensures the vanishing of the boundary term that arises on $\mathcal{C}(s_\ast)$ in the divergence identity (D.13). However, this leads to a difficulty because $X^\mu$ is null at $\mathcal{C}(s)$, which implies that the integrand of (2.33) does not control uniformly the energy as the boundary $\mathcal{C}(s_\ast)$ of $\Omega(s_\ast)$ is approached. Thus, the standard energy argument requires a careful reinspection. The price to pay is the need to impose tracelessness of $T_{\mu\nu}$. Moreover the argument does not guarantee that the metric is vacuum throughout $\mathcal{D}^\ast(\mathcal{C}(s_2))$, but only on $\mathcal{D}^\ast(\mathcal{C}(s_\ast))$, and we will return to this issue at the end of the proof.

We let $s_\ast$ be the number defined in the paragraph after (2.32) when $T_{\mu\nu}$ is traceless, and we set $s = s_2$ if the rigid dominant energy condition holds on $\mathcal{C}(s_2)$. Since the metric is now

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9 Strictly speaking, the argument presented in appendix D only proves that the metric is vacuum in $\Omega(s_\ast)$. But $\{ t = s_\ast \} \cap J^\ast(\mathcal{C}(s_\ast))$ is a Cauchy surface for $\mathcal{D}^\ast(\mathcal{C}(s_\ast))$, so a standard argument proves then that the metric is vacuum in $\mathcal{D}^\ast(\mathcal{C}(s_\ast))$. 

8
vacuum on $\mathcal{C}^+(s_*)$, we have

$$g^{AB}R_{AB} = 2\Lambda$$

(2.35)

there. We shall use (2.35) to prove that $\alpha = 1 - 2\Lambda r^2/n(n - 1)$ on $\mathcal{C}(s_*)$.

Recall that, at this stage, on $\mathcal{C}(s_*)$ the metric takes the form

$$g = -\alpha du^2 - 2du dr + r^2 h_{AB} dx^A dx^B.$$ (2.36)

In appendix E we show that

$$g^{AB}R_{AB} = \frac{4\Lambda n + 1}{n - 1} + 2\partial_r \partial_r \alpha + \frac{3(n - 1)}{r} \partial_r \alpha + \frac{(n - 1)(n - 2)}{r^2} (\alpha - 1).$$ (2.37)

This, together with (2.35), provides a Fuchsian ODE for $\alpha - 1$, with characteristic exponents $\lambda$ whose solve the equation

$$2\lambda(\lambda - 1) + 3(n - 1)\lambda + (n - 1)(n - 2) = 0,$$

and thus the solutions are

$$\alpha = 1 - \frac{2\Lambda}{n(n - 1)} r^2 + \alpha_+ (x^A) r^{\lambda_+} + \alpha_- (x^A) r^{\lambda_-},$$

where $\alpha_{\pm}$ are smooth functions on $S^{n-1}$, and

$$\lambda_{\pm} \in \left\{ \frac{1 - n}{2}, 2 - n \right\}.$$ (2.38)

Since both characteristic exponents are negative, the only regular solution is $\alpha \equiv 1 - \frac{2\Lambda}{n(n - 1)} r^2$.

We have therefore shown that $g_{\mu\nu}$ takes the Minkowski, or (anti)-de Sitter form on $\mathcal{C}(s_*)$. Note that the energy argument above can be used to prove uniqueness of solutions of the reduced Einstein equations, with the components of the metric in the wave-map gauge prescribed on the light cone, in the usual way (compare [3, 6, 8] and references therein).

It follows that $g_{\mu\nu}$ equals the corresponding reference metric on the domain of dependence of $\mathcal{C}(s_*)$.

So, we have that $x^\mu = y^\mu$ on $\Omega(s_*)$, with $g_{\mu\nu} = \delta_{\mu\nu}$ there. If $s_* < s_2$, then one can repeat the argument of appendix D to obtain the above conclusions on $\Omega(s_*)$, for some $s_*$ satisfying $s_* < \hat{s}_* \leq s_2$. Using this observation, an easy open-closed argument shows that $s_* = s_2$, which had to be established.

For further reference we note the following result, which follows immediately from (2.8) and (2.11):

**Proposition 2.2.** The expansion $\theta(s)$ will become negative along a generator $\gamma$ of $\mathcal{C}^+_{p}$ at some value of $s$ strictly smaller than $s_2$ whenever

$$\int_{0}^{s_2} (\sigma_{AB} \sigma^{AB} + R_{\mu\nu} y^\mu y^\nu) s^2 ds \geq (n - 1)s_2.$$ (2.39)

Once $\theta(s)$ has become negative, standard arguments imply that $\theta$ will diverge in finite time, so that either $\gamma$ will be incomplete, or will leave $J^+(p)$ in finite time.

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Appendix A. The rigid dominant energy condition on the null cone: Maxwell and scalar fields

We start by verifying:

**Proposition A.1.** Both the Maxwell energy–momentum tensor and the massless scalar field energy–momentum tensors satisfy the dominant energy condition.

**Proof.** It suffices to show that if \( n^\mu \) is unit and timelike, then \( P_\mu = T_\mu^\nu n^\nu \) is causal. Now, in an orthonormal frame \( e_\mu \) with \( n^\mu \partial_\mu = x_\mathbf{0} \) we have, for the massless scalar field

\[
T_{00} = \frac{1}{2} (e_0(\phi))^2 + \frac{1}{2} \sum_i e_i(\phi))^2, \quad T_{0i} = e_0(\phi)e_i(\phi),
\]

and the causal character of \( P_\mu = T_{0\mu} \) follows from \( \sqrt{a^2 + |\vec{b}|^2} \leq \frac{1}{2} (a^2 + |\vec{b}|^2) \).

For the Maxwell field, further rotating the frame so that \( F_{01} \sim \delta^1_1 \), it holds that

\[
T_{00} = \frac{1}{2} \sum_j F_{0j}^2 + \frac{1}{4} \sum_{i,j}^2 F_{ij}^2 = \frac{1}{2} F_{01}^2 + \frac{1}{2} \sum_j F_{1j}^2 + \frac{1}{4} \sum_{i,j \neq 1} F_{ij}^2, \quad T_{0i} = F_{01} F_{i1},
\]

and the result follows as for the scalar field. \( \square \)

Now we show that the scalar and Maxwell fields do not obey the rigid dominant energy condition in its local form (2.2) at a point \( q \). For a scalar field \( \phi \), define \( k_\mu \equiv \partial_\mu \phi|_q \). Then the energy–momentum tensor at \( q \in \mathcal{M} \) can be expressed as

\[
T_{\mu\nu} = k_\mu k_\nu - \frac{1}{2} |k|^2 g_{\mu\nu}.
\]

For spacelike \( k_\mu \) the associated tensor \( T^\mu_\nu \) has null eigenvectors (which are orthogonal to \( k_\mu \)) with nonzero eigenvalue \( -\frac{1}{2} |k|^2 \), which implies that \( T_{\mu\nu} \) does not obey the rigid dominant energy condition (2.2).

The Maxwell stress–energy tensor of an electromagnetic field is, whatever the space dimension \( n \geq 2 \),

\[
T_{\alpha\beta} = F^\lambda_\alpha F_{\lambda\beta} - \frac{1}{4} \epsilon_{\alpha\beta\gamma\delta} F^{\gamma\mu} F_{\lambda\mu}.
\]

At any point at which \( F_{\mu\nu} \) is of the form \( Y_{[\mu} Z_{\nu]} \), for some spacelike vectors \( Y \) and \( Z \), there exist null vectors \( l^\mu \) for which \( F_{\mu\nu} l^\nu = 0 \). Such vectors are eigenvectors of \( T^\alpha_\beta \) with nonzero eigenvalue. This implies that the Maxwell field does not obey the rigid dominant energy condition (2.2).

Next, let \( \mathcal{C}(s_2) \) be the subset of the future null cone \( \mathcal{C}_+ \) defined by (2.4)–(2.5). We have:

**Proposition A.2.** In spacetimes satisfying the Einstein–Maxwell field equations, the rigid dominant energy condition holds on \( \mathcal{C}(s_2) \).
Proof. We use a coordinate system in which the metric takes the form (2.25). The condition $T_{\mu\nu} \ell^\mu \ell^\nu = 0$ in (2.3) reads in those coordinates

$$T_{rr} = F^A_r F_{Ar} = 0,$$

with, by antisymmetry, $F_{rr} = F^{Ar} = 0$. Hence

$$T_{rr} = g^{AB} F_{Ar} F_{Br} = 0,$$

which implies

$$F_{Ar} = 0, \quad \text{hence also} \quad F^{Ar} = 0.$$ 

Keeping in mind $g_{rr} = g_{rA} = 0$, we obtain a direct, alternative justification of (2.20)

$$T_A = F^A_r F_{rA} = 0.$$

The Maxwell equation $dF = 0$ shows that

$$\partial^r F_{AB} = 0.$$

Because of the polar character of the coordinates $x^A$, regularity of $F$ at the vertex gives the vanishing of $F_{AB} = 0$ there, and hence everywhere.

The Maxwell equation

$$\partial_\mu (\sqrt{|\det g_{\alpha\beta}|} F^\mu u) = 0$$

reduces in our coordinates to

$$\partial_r (e^{-2\beta} \sqrt{\det h_{AB}} F_{ru}) = 0.$$

Since $e^{-2\beta} \sqrt{\det h_{AB}} F_{ru}$ tends to zero as $r \to 0$, we conclude that $F_{ur} \equiv 0$. Now (recall that $g_{ur} = -e^{2\beta}$ and $F_{Ar} = F^{Ar} = 0$),

$$T_{ur} = F^u_r F_{ru} + e^{2\beta} 2 F_{ur} F_{ur} + \frac{e^{2\beta}}{4} F_{AB} F_{AB},$$

and so

$$T_{ur} = 0.$$

Hence $T_{\mu\nu} \ell^\mu$ vanishes on $\mathcal{C}(s_2)$, as desired. □

Similarly, we have

**Proposition A.3.** In spacetimes satisfying the Einstein-massless scalar field equations, the rigid dominant energy condition holds on $\mathcal{C}(s_2)$.

Proof. In this case

$$T_{a\beta} = \partial_a \phi \partial_\beta \phi - \frac{1}{2} g_{a\beta} \partial_\lambda \phi \nabla^\lambda \phi.$$

Hence

$$T_{rr} = (\partial_r \phi)^2,$$

and $T_{rr} = 0$ implies $\partial_r \phi = 0$. So $\phi$ is constant on $\mathcal{C}(s_2)$; uniqueness of solutions of the wave equation implies that $\phi$ is constant in the domain of dependence of $\mathcal{C}(s_2)$, and so $T_{\mu\nu}$ vanishes there. □
Appendix B. The dominant energy condition and its consequences

Let us denote \( \sqrt{|g(Z, Z)|} \) by \(|Z|_g\). Given a timelike vector \( n \), let us denote by \(|Z|_{g,n}\) the square root of
\[
g(Z, Z) + 2\frac{g(Z, n)^2}{|g(n, n)|} \geq 0. \tag{B.1}
\]
Note that \(|n|_g = |n|_{g,n}\), and also \(|Z|_g = |Z|_{g,n}\) when \( Z \) is orthogonal to \( n \).

We recall a well-known result, which we prove for completeness.

**Lemma B.1.** Suppose that a symmetric two-covariant tensor \( T \) satisfies the dominant energy condition (2.1), and let \( n \) be a timelike vector.\(^\text{10}\) Then for any vectors \( W, Z \) we have
\[
|T(W, Z)| \leq \frac{|W|_{g,n}^2 + |Z|_{g,n}^2}{|n|_g^2} T(n, n). \tag{B.2}
\]
Furthermore, for any causal vector \( X \) we also have
\[
T(X, X) \leq \frac{2|X|_{g,n}}{|n|_g} T(X, n). \tag{B.3}
\]

**Remark B.2.** Denoting by \(|T|_{g,n}\) the norm of \( T \) with respect to the Riemannian metric associated with the quadratic form (B.1), (B.2) implies
\[
|T|_{g,n} \leq \frac{2}{|n|_g^2} T(n, n). \tag{B.4}
\]

**Proof.** Let, first \( W \) be orthogonal to \( n \). As \(|W|_{g,n} = |W|_g\), the vectors \( W_{\pm} := |W|_{g,n} n \pm |n|_g W \) are null consistently time oriented, thus
\[
0 \leq T(W_+, W_-) = |W|_{g,n}^2 T(n, n) - |n|_g^2 T(W, W),
\]
giving, for \( W \perp n \),
\[
T(W, W) \leq \frac{|W|_{g,n}^2}{|n|_g^2} T(n, n). \tag{B.5}
\]
Adding the two equations obtained by writing explicitly \( T(W_+, W_+) \geq 0 \) and \( T(W_-, W_-) \geq 0 \) gives
\[
T(W, W) \geq -\frac{|W|_{g,n}^2}{|n|_g^2} T(n, n) \quad \implies \quad |T(W, W)| \leq \frac{|W|_{g,n}^2}{|n|_g^2} T(n, n). \tag{B.6}
\]
We also have,
\[
0 \leq T(n, W_{\pm}) = T(n, |W|_g n) \pm T(n, |n|_g W),
\]
giving, again for \( W \perp n \),
\[
|T(W, n)| \leq \frac{|W|_{g,n} |n|_g}{|n|_g^2} T(n, n). \tag{B.7}
\]

\(^\text{10}\) We hope that the clash of notation with the space dimension \( n \), as used elsewhere in this paper, will not lead to confusion.
Next, if both $W$ and $Z$ are orthogonal to $n$, using (B.6) we find

$$|T(W, Z)| = \frac{1}{4} |T(W + Z, W + Z) - T(W - Z, W - Z)|$$

$$\leq \frac{|W + Z|_{g,n}^2 + |W - Z|_{g,n}^2}{4|n|_g^2} T(n, n)$$

$$= \frac{|W|_{g,n}^2 + |Z|_{g,n}^2}{2|n|_g^2} T(n, n).$$

(B.8)

Finally, for general vectors $W$ and $Z$ we can write

$$W = w \frac{n}{|n|_g} + W^\perp, \quad Z = z \frac{n}{|n|_g} + Z^\perp,$$

with both $W^\perp$ and $Z^\perp$ orthogonal to $n$. Then

$$|W|_{g,n}^2 = w^2 + |W^\perp|_{g,n}^2, \quad |Z|_{g,n}^2 = z^2 + |Z^\perp|_{g,n}^2,$$

and, from what has been said so far,

$$|T(W, Z)| = \frac{|wz| T(n, n) + \frac{w}{|n|_g} T(n, Z^\perp) + \frac{z}{|n|_g} T(n, W^\perp) + T(W^\perp, Z^\perp)}{|n|_g^2}$$

$$\leq \frac{|wz| + |wZ^\perp|_{g,n} + |zW^\perp|_{g,n} + \frac{1}{2}(|W^\perp|_{g,n}^2 + |Z^\perp|_{g,n}^2)}{|n|_g^2} T(n, n)$$

$$\leq \frac{w^2 + z^2 + |W^\perp|_{g,n}^2 + |Z^\perp|_{g,n}^2}{|n|_g^2} T(n, n)$$

$$= \frac{|W|_{g,n}^2 + |Z|_{g,n}^2}{|n|_g^2} T(n, n).$$

This proves (B.2).

For (B.3), set $Z^\mu = -T^\mu_\nu X^\nu$; the dominant energy condition implies that $Z^\mu$ is causal future directed. Let $e_\alpha, \alpha \in \{0, \ldots, n\}$, be any orthonormal frame such that $n = |n|_g e_0$, and let $X^\alpha$ denote the components of $X$ in this frame, thus $X = X^\alpha e_\alpha$, similarly for $Z^\alpha$. Then (B.3) is equivalent to

$$-g(Z, X) \leq \frac{2|X|_{g,n}}{|n|_g} (-g(Z, n)).$$

(B.9)

Now, since both $Z$ and $X$ are causal and future directed we have $|\sum X^i Z^i| \leq Z^0 X^0$, so

$$-g(Z, X) = Z^0 X^0 - \sum_i X^i Z^i \leq 2Z^0 X^0 = 2 \frac{X^0}{n^0} (-g(Z, n))$$

$$= 2 \frac{X^0}{|n|_g} (-g(Z, n)),$$

and (B.3) follows. \qed

We note that the constant in the lemma B.1 is optimal, with the inequality becoming an equality when $Z$ is null, when $T_{\mu\nu} = Z_{(\mu} n_{\nu)}$, and when $X^\alpha = Z^0, X^i = -Z^i$. 

Appendix C. $R_{rA}$

In this appendix we calculate the components $R_{rA}$ of the Ricci tensor of a metric which on a null hypersurface $\mathcal{N} = \{ u = 0 \}$ takes the form

$$ g = -\alpha \, du^2 + 2v_A \, dx^A \, du + 2\varepsilon \, du \, dr + r^2 \hat{h}_{AB} \, dx^A \, dx^B. \quad (C.1) $$

Here we allow $\varepsilon = \pm 1$, according to whether a future ($\varepsilon = -1$) or a past ($\varepsilon = 1$) light cone is considered. We emphasize that the above form of the metric is only assumed at $\{ u = 0 \}$, so all $g_{uv}$’s are allowed a priori to be nonzero away from $\mathcal{N}$; similarly for their derivatives.

The equations in this appendix, and in appendix E, have been checked with the xAct system for tensor computer algebra [16].

Writing $g^\sharp$ for the inverse metric, we have

$$ g^\sharp = \psi \, \partial_r^2 + 2\mu^A \partial_r \partial_A + 2\varepsilon \partial_u \partial_r + \frac{1}{r^2} \hat{h}^{AB} \partial_A \partial_B, \quad (C.2) $$

$$ g'^A \equiv \mu^A = -\varepsilon \frac{1}{r^2} \hat{h}^{AB} \nu_B, \quad g'^r \equiv \psi = \alpha + \frac{1}{r^2} \hat{h}^{AB} v_A v_B. \quad (C.3) $$

We reserve the symbols $\nu_A, \mu^A, \alpha, \psi$ and $\hat{h}_{AB}$ for objects defined on $\{ u = 0 \}$, so that, e.g. $\partial_u \nu_A$ does not make sense (but $\partial_u g_{uu}$ does, and might a priori be nonzero).

The Lévi-Civitá connection of the metric $\hat{h}_{AB}$ will be denoted as $D_A$ and will have Christoffel symbols $\gamma^C_{AB}$ with respect to the derivative $\partial_A$.

All the equations that follow are on $\mathcal{N}$.

We have the following Christoffel symbols (the remaining ones can be obtained by symmetry):

$$ \Gamma^u_{uu} = \frac{\varepsilon}{2} (\partial_u \alpha + 2\partial_u g_{ur}), \quad (C.4) $$

$$ \Gamma^u_{ur} = \frac{\varepsilon}{2} \partial_u g_{rr}, \quad (C.5) $$

$$ \Gamma^u_{rr} = 0, \quad (C.6) $$

$$ \Gamma^r_{uu} = \frac{1}{2} \mu^A \partial_u \alpha + \frac{1}{2} \psi (\partial_u \alpha + 2\partial_u g_{ur}) + \mu^A \partial_u \nu_A + \frac{\varepsilon}{2} \partial_u g_{uu}, \quad (C.7) $$

$$ \Gamma^r_{ur} = -\frac{1}{2} \partial_r \alpha + \frac{1}{2} \mu^A \partial_r \nu_A + \frac{1}{2} \mu^A \partial_u g_{rA} + \frac{1}{2} \psi \partial_u g_{rr}, \quad (C.8) $$

$$ \Gamma^r_{rr} = -\frac{\varepsilon}{2} \partial_r g_{rr}, \quad (C.9) $$

$$ \Gamma^u_{Au} = \frac{\varepsilon}{2} (\partial_u g_{rA} - \partial_r \nu_A), \quad (C.10) $$

$$ \Gamma^u_{Ar} = 0, \quad (C.11) $$

$$ \Gamma^r_{Au} = -\frac{\varepsilon}{2} \partial_A \alpha + \frac{1}{2} \mu^B (D_A v_B - D_B v_A + \partial_u g_{AB}) + \frac{1}{2} \psi (\partial_u g_{rA} - \partial_r \nu_A), \quad (C.12) $$

$$ \Gamma^r_{Ar} = \frac{\varepsilon}{2} (\partial_u g_{rA} - \partial_r \nu_A) + \frac{1}{2} \mu^B \partial_r h_{AB}, \quad (C.13) $$

$$ \Gamma^u_{AB} = -\frac{\varepsilon}{2} \partial_r h_{AB}, \quad (C.14) $$
\[
\Gamma_{AB}^C = \frac{\varepsilon}{2} (D_A v_B + D_B v_A - \partial_u g_{AB}) - \frac{1}{2} \psi \partial_r h_{AB},
\]
(C.15)

\[
\Gamma_{\alpha\alpha}^C = \frac{1}{2} h^{CA} \partial_\alpha \alpha + \frac{1}{2} \mu^C \partial_\alpha \alpha + h^{CA} \partial_\alpha g_{AA} + \mu^C \partial_\alpha g_{rr},
\]
(C.16)

\[
\Gamma_{\alpha r}^C = \frac{1}{2} h^{CA} (\partial_\alpha g_{rA} + \partial_r v_A) + \frac{1}{2} \mu^C \partial_\alpha g_{rr},
\]
(C.17)

\[
\Gamma_{rr}^C = 0,
\]
(C.18)

\[
\Gamma_{A\alpha}^C = \frac{1}{2} h^{BC} (D_A v_B - D_B v_A + \partial_u g_{AB}),
\]
(C.19)

\[
\Gamma_{A\alpha}^C = \gamma_{A\alpha}^C - \frac{1}{2} \mu^C \partial_u h_{AB},
\]
(C.20)

The traces of the Christoffel symbols read

\[
\Gamma_{\mu \mu}^C = \varepsilon \partial_\mu g_{rr} + \frac{1}{2} \psi \partial_u g_{rr} + \mu^A \partial_\mu g_{rA} + \frac{1}{2} h^{AB} \partial_\mu g_{AB},
\]
(C.22)

\[
\Gamma_{\mu r}^C = \frac{1}{2} h^{AB} \partial_r h_{AB},
\]
(C.23)

\[
\Gamma_{\mu A}^C = \frac{1}{2} h^{BC} \partial_\mu h_{BC}.
\]
(C.24)

Let \( \lambda^C \) be defined by (2.28), we have

\[
\lambda_u^C = -\partial_u g_{rr} + \frac{\varepsilon}{2} h^{AB} \partial_r h_{AB},
\]
(C.25)

\[
\lambda^C = -\partial_\mu g_{rr} + \frac{\varepsilon}{2} h^{AB} \partial_r h_{AB}
\]
- \( 2\varepsilon \mu^A \partial_\mu v_A + \frac{\varepsilon}{2} h^{AB} \partial_\mu g_{AB} - \frac{\varepsilon}{2} \psi \partial_u g_{rr} \)
- \( \frac{\varepsilon}{2} h^{AB} \partial_r h_{AB} - \frac{\varepsilon}{2} \psi \partial_u g_{rr}, \)
(C.26)

\[
\lambda^A = -\varepsilon h^{AB} D_B v_A + \partial_\alpha \alpha = \mu^B \partial_r h_{AB} + \frac{1}{2} h^{AB} \psi \partial_r h_{AB}
\]
(C.27)

\[
\lambda_{\alpha\alpha} = -\varepsilon h^{AB} D_B v_A + \partial_\alpha \alpha = \mu^B \partial_r h_{AB} + \frac{1}{2} h^{AB} \psi \partial_u g_{rr},
\]
(C.28)

\[
\lambda_A = -\mu^B h_{AB} v_B + \varepsilon \partial_\alpha v_A - \mu^A \partial_\alpha g_{AB} + \mu^A \partial_\alpha g_{rA} + \frac{1}{2} \psi \partial_u g_{rr},
\]
(C.29)

\[
\lambda_{\alpha r} = \frac{1}{2} h^{AB} \partial_r h_{AB} - \varepsilon \partial_u g_{rr},
\]
(C.30)

\[
\lambda_A = -h^{BC} h_{AD} \gamma_{D\alpha}^C - \mu^B \partial_r h_{AB} - \varepsilon (\partial_\alpha v_A + \partial_\alpha g_{rA}).
\]
(C.31)

We choose the metric (2.23) as model metric, expressed in the following coordinate system:

\[
\hat{g} = -\left(1 - \frac{2\Lambda}{n(n-1)} r^2\right) \frac{dr^2}{(\alpha^2 - \frac{2\Lambda}{n(n-1)} r^2 + r^2 h_{AB}) dx^A dx^B}
\]
- \( \alpha = \hat{\alpha} + \varepsilon \partial_u u + r^2 h_{AB} dx^A dx^B, \)

(C.32)
Its non-vanishing Christoffel symbols are, up to symmetry,
\[ \Gamma^u_{uv} = -\frac{2\varepsilon\Lambda r}{n(n-1)} \hat{u}, \quad \Gamma^v_{uu} = -\varepsilon r \hat{h}_{BC}, \quad \Gamma^u_{uv} = -\frac{2\Lambda r}{n(n-1)} \hat{u}, \tag{C.33} \]
\[ \Gamma^v_{ar} = \frac{2\varepsilon\Lambda r}{n(n-1)}, \quad \Gamma^v_{BC} = -r \hat{h}_{BC} \hat{a}, \quad \Gamma^A_{Br} = \frac{1}{r} \delta^A_B, \quad \Gamma^A_{BC} = \hat{\gamma}^A_{BC}. \]

We shall shortly assume that the metric \( g \) satisfies the wave-map conditions (see, e.g., [13, chapter VI])
\[ \lambda^\mu = \dot{\lambda}^\mu, \]
with \( \dot{\lambda}^\mu \) defined in (2.29). We find
\[ \dot{\lambda}^u = -g^{uv} \Gamma^u_{v\nu} = r \varepsilon g^{AB} \hat{h}_{AB} = \varepsilon \frac{n-1}{r}, \tag{C.34} \]
\[ \dot{\lambda}^r = -g^{uv} \Gamma^r_{v\nu} = \frac{n-1}{r} - \frac{2(n+1)\Lambda r}{n(n-1)}, \tag{C.35} \]
\[ \dot{\lambda}^A = -g^{uv} \Gamma^A_{v\nu} = -2g^{AB} \hat{g}_{AB} - g^{BC} \hat{g}_{BC} = -\frac{2}{r} \hat{h}^{BC} \hat{\gamma}^A_{BC} = -\frac{2}{r^2} \hat{h}_{BC} \hat{\gamma}^A_{BC} \]
\[ = -\frac{1}{r} \mu^A + \frac{1}{r^2 \sqrt{\det \hat{h}_{EF}} \partial_B (\sqrt{\det \hat{h}_{EF}} \hat{h}^{AB})}. \tag{C.36} \]

Using \( \lambda^u = \dot{\lambda}^u \), from (C.25) and (C.34) we obtain
\[ \partial_u g_{rr} = 0, \quad \text{hence also} \quad \partial_u g^{uu} = 0. \]
From \( \lambda^A = \dot{\lambda}^A \) we deduce that
\[ \partial_u g_{Cr} = -\frac{(n-1)}{r} v_C - \partial_r v_C, \]
and finally \( \dot{\lambda}^r = \dot{\lambda}^r \) gives
\[ \frac{1}{2} \hat{h}^{AB} \partial_u g_{AB} = \hat{h}^{AB} D_A v_B = \varepsilon \frac{(n-1)^{-1} \psi}{r} + (n-1)\varepsilon r - \frac{2(n+1)\Lambda r^3}{n(n-1)}. \]

Now,
\[ R_{Ar} = \partial_r \Gamma^r_{Ar} - \partial_r \Gamma^r_{A\gamma} + \Gamma^r_{A\gamma} \Gamma^\gamma_{Ar} - \Gamma^\gamma_{A\gamma} \Gamma^r_{Ar}, \]
and from what has been said so far, in particular using the harmonicity conditions, we obtain
\[ R_{Ar} = \partial_u \Gamma^u_{Ar} + \partial_r \Gamma^r_{Ar} + \partial_B \Gamma^B_{Ar} - \partial_r \Gamma^r_{AB} + \Gamma^\nu_{Ar} \Gamma^\nu_{A\gamma} + \Gamma^\mu_{Ar} \Gamma^\mu_{A\gamma} + \Gamma^\nu_{By} \Gamma^\nu_{Ar} + \Gamma^\gamma_{By} \Gamma^\gamma_{Ar} \]
\[ = \partial_u \Gamma^u_{Ar} + \partial_r \Gamma^r_{Ar} - \partial_r \Gamma^r_{AB} + \partial_B \Gamma^B_{Ar} + \Gamma^\nu_{By} \Gamma^\nu_{Ar} + \Gamma^\gamma_{By} \Gamma^\gamma_{Ar} \]
\[ = \Gamma^B_{ar} \Gamma^r_{AB} - \Gamma^r_{Br} \Gamma^B_{Ar} + \Gamma^r_{B\gamma} \Gamma^r_{AC}. \]

We have
\[ \partial_u g^{uB} = \frac{1}{2} \partial_u (g^{uA} (\partial_A g_{\mu\nu} + \partial_r g_{A\gamma} - \partial_r g_{A\gamma})) g_{rr} = 0 + \frac{1}{2} \partial_u g^{uB} \partial_u g_{BA}. \]

On the null surface it holds that
\[ \partial_u g^{uB} = -\varepsilon \hat{h}^{BA} \partial_u g_{BA} + \varepsilon \mu^B \partial_u g_{rr}. \tag{C.37} \]
so, using the harmonicity conditions, we are led to
\[ \partial_u g^{uB} = -\frac{n-1}{r} \mu^B + \epsilon h^{AB} \partial_r v_A. \]  
(C.38)

Hence
\[ \partial_u \Gamma^u_{Ar} = \frac{1}{r^2} ((n-1)v_A + r \partial_r v_A). \]  
(C.39)

With some work, using the formulae derived so far, one similarly obtains
\[ \partial_B \Gamma^B_{rA} - \partial_r \Gamma^B_{AB} = -\frac{\epsilon}{r} \partial_r v_A + \frac{\epsilon}{r^2} v_A, \]  
(C.40)

\[ -\partial_r \Gamma^r_{Ar} = -\frac{\epsilon}{2} \partial_r (\partial_u g_{rA} - \partial_r v_A), \]  
(C.41)

\[ \Gamma^r_{rA} \Gamma^B_{rB} = \frac{\epsilon}{r} \left( \frac{1}{2} (\partial_r v_A + \partial_u g_{rA}) - \frac{1}{r^2} v_A \right), \]  
(C.42)

\[ \Gamma^B_{rA} (\Gamma^r_{Bu} + \Gamma^C_{Br} + \Gamma^C_{BC}) = \frac{1}{r} \gamma^C_{AC}, \]  
(C.43)

\[ -\Gamma^B_{ru} \Gamma^u_{AB} = \frac{\epsilon}{2r} (\partial_r v_A + \partial_u g_{rA}), \]  
(C.44)

\[ -\Gamma^C_{rB} \Gamma^B_{rA} = \frac{\epsilon}{2r} (\partial_r v_A - \partial_u g_{rA}) - \frac{1}{r} \gamma^B_{AB}. \]  
(C.45)

Adding, we are led to
\[ \epsilon R_{Ar} = \frac{(n-2)(n-3)}{2r^2} v_A + \frac{3n-5}{2r} \partial_r v_A + \partial_r v_A \]  
\[ = \frac{1}{r^{n-1}} \partial_r \left[ r^{n-1} \left( \partial_r v_A + \frac{n-3}{2r} v_A \right) \right]. \]  
(C.46)

**Appendix D. An energy inequality for traceless \( T_{\mu\nu} \)**

We let \( x^\mu \), with \( x^0 \equiv t \), denote normal coordinates centered on the vertex \( p \) of the future light cone; we restrict consideration to the region where those are well defined. Passing to a subset of the domain of normal coordinates if necessary, we can and will, assume that \( \partial_t \) and \( \nabla t \) are timelike. We will only consider metrics which behave as in the proof of theorem 2.1: thus, we assume existence of a set of coordinates \( y^\mu \) which are required to coincide with the normal coordinates \( x^\mu \) on the light cone, and we assume that the map \( x^\mu \mapsto y^\mu \) is a smooth diffeomorphism in a neighborhood of the future light cone of \( p \). We let \( u \equiv y^0 = |\vec{y}| \), \( r \equiv \frac{|\vec{y}|}{r} \), and we denote by \( z^A \) angular coordinates parameterizing the unit vector \( \frac{\vec{y}}{r} \). We denote by \( (z^\mu) \equiv (u, r, z^A) \) these coordinates; by definition, \( \{u = 0\} \) is \( \mathcal{G}(s_*) \). Furthermore we assume that, on \( \mathcal{G}(s_*) \), the metric takes the form (2.36)
\[ g = -\alpha \, du^2 - 2 \, du \, dr + r^2 \hat{h}_{AB} \, dz^A dz^B. \]  
(D.1)

Note that we write here \( z^A \) for what is denoted by \( x^A \) elsewhere in the paper since, to avoid confusion, in the considerations below we reserve the symbol \( x^\mu \) for normal coordinates.

We will also need the hypothesis that \( g_{uu} < 0 \) and that the \( u \)-derivatives of the metric at the light cone satisfy
\[ \partial_u g_{rr} = 0. \]
As already pointed out, all those conditions will be the satisfied by the wave-map coordinates from the main body of the paper at the current stage of the argument. But we emphasize that we do not need to assume that the coordinates $y^\mu$ satisfy more conditions than those just listed.

Consider a vector field $X$ which, near the light cone, equals

$$X = u \partial_u + r \partial_r.$$  \hspace{1cm} (D.2)

So, wherever $X$ takes this form,

$$g(X, X) = g_{uu} u^2 + 2 g_{ur} u r + g_{rr} r^2.$$  

On the light cone this vanishes, so that $X$ is null there.

Keeping in mind that $g_{uu} < 0$ and $g_{ur} = -\frac{\epsilon}{u}$ at $u = 0$, for every $R > 0$ there exists $u_0 > 0$ and $\epsilon > 0$ such that for $0 \leq r \leq R$ and $0 < u \leq u_0$ we have $g_{ur} < 0$ and $g_{ur} < -\epsilon$.

Since $\partial_u g_{rr} = 0$ at $u = 0$ we further have, in the same ranges of $u$ and $r$, $|g_{rr}| \leq C u^2$.

So the second term is negative, while there exists a (possibly small) $r_0 > 0$ such that for $0 < u < u_0$ and $0 < r \leq r_0$ the first term dominates the third one, which shows that $X$ is timelike in that region.

At $u = 0$ we have

$$\partial_u (g(X, X))|_{u=0} = (-2 + r \partial_u g_{rr}) r.$$  

This shows that, reducing $u_0$ if necessary, $X$ is again timelike in the range $0 < r < R$ and $0 < u < u_0$.

Since the set of future directed timelike vectors is convex, for every $R > 0$ one can interpolate in the region $u_0 / 2 \leq u \leq u_0$ between $X$ as given by (D.2) and some vector field timelike everywhere to obtain a vector field, still denoted by $X$, which is timelike in the timelike future of $p$ and which takes the form (D.2) for $0 < r < R$ and $0 < u < u_0 / 2$.

As $X$ is null at the light cone, the integrand of (2.33) does not control all components of $T_{\mu\nu}$ uniformly as one approaches the light cone, and we need to quantify that. So we start by showing that, for any $T \in [0, \infty)$ there exists a constant $C > 0$ such that for $0 < r < t \leq T$ we have

$$T_{\mu\nu} n^\mu X^\nu \geq C^{-1} (t - r) T_{00} n^0.$$  \hspace{1cm} (D.3)

Note that for $u \geq u_0$, the inequality follows immediately from the fact that all three vectors $X^\mu$, $\partial_0$ and $n^\mu$ are uniformly timelike there, and from (B.3). So it remains to consider points for which $0 < u < u_0 / 2$. For this, we let $Z^\mu$ be a future directed null vector which, near the light cone, takes the form $Z^\mu = a \partial_u + r \partial_r$. Then

$$T_{\mu\nu} X^\mu n^\nu = (u T_{uv} + r T_{rv}) n^v = ((u - a) T_{uv} + a T_{uv} + r T_{rv}) n^v = (u - a) T_{00} n^v + T_{\mu\nu} Z^\mu n^\nu.$$  

The last term is non-negative by the dominant energy condition. Equation (D.3) will follow if

$$(u - a) \geq \frac{1}{2} (t - r) \iff a \leq \frac{1}{2} (t - r).$$  \hspace{1cm} (D.4)

Now, the condition that $Z^\mu$ is null reads

$$0 = g_{\mu\nu} Z^\mu Z^\nu = g_{uu} a^2 + 2 g_{ur} ar + r^2 g_{rr}.$$  \hspace{1cm} (D.5)

Keeping in mind that $g_{ur}$ is negative, we choose the solution

$$a = \frac{r}{g_{uu}} (-g_{ur} - \sqrt{g_{ur}^2 - g_{uu} g_{rr}}) = \frac{rg_{rr}}{-g_{ur} + \sqrt{g_{ur}^2 - g_{uu} g_{rr}}}.$$
As already seen, since $g_{rr} = 0$ at $u = 0$, the hypothesis $\partial_u g_{rr}|_{u=0} = 0$ implies $|g_{rr}| \leq C u^2$ on any bounded domain of $u$ and $r$, from which it easily follows that, reducing $u_0$ if necessary, for $0 \leq u \leq u_0$ the inequality (D.4) holds.

We continue with

**Proposition D.1.** On any bounded interval of $t$, say $0 \leq t \leq T$, and assuming as before that we are working within the domain of definition of normal coordinates, there exists a constant $C$ such that, for $0 < r \leq t \leq T$,

$$\left| \mathcal{L}_X g_{\mu\nu} - \left(2g_{\mu\nu} - \frac{\partial_r \alpha}{r} X_\mu X_\nu \right) \right|_b \leq C(t - r), \quad (D.6)$$

where the norm $| \cdot |_b$ is taken with some arbitrarily chosen Riemannian metric $b$.

**Proof.** For $u \geq u_0/2$ the estimate is clear, so it remains to consider the region $0 \leq u \leq u_0/2$, where $X$ takes the form (D.2). By definition of Lie derivative,

$$\mathcal{L}_X dz^\mu|_{u=0} = \delta^\mu_r \, dr + \delta^\mu_u \, du. \quad (D.7)$$

Writing the metric along the light cone as $g = \eta + (1 - \alpha) \, du^2$, with $\eta = -\, du^2 - 2 \, du \, dr + r^2 \delta_{AB} \, dx^A \, dx^B$, one obtains

$$\mathcal{L}_X \eta = 2\eta, \quad \mathcal{L}_X [(1 - \alpha) \, du^2] = -r \partial_r \alpha \, du^2 + 2(1 - \alpha) \, du^2, \quad (D.8)$$

and, still at $\{u = 0\}$, one finds

$$\mathcal{L}_X g = 2g - r \partial_r \alpha \, du^2. \quad (D.9)$$

We note the estimate

$$|r \partial_r \alpha| \leq Ct^2 \text{ for } 0 < r \leq t \leq T. \quad (D.10)$$

On the light cone we have

$$du = -\frac{1}{r} X_\mu \, dx^\mu.$$

So we can rewrite (D.9) as

$$\left| \mathcal{L}_X g_{\mu\nu}|_{u=0} = 2g_{\mu\nu} - \frac{1}{r} \partial_r \alpha X_\mu X_\nu. \right. \quad (D.11)$$

A Taylor expansion at $u = 0$ gives (D.6).}$

Let $E(s)$ be defined as in (2.33), except that $t$ there is taken now to be a normal coordinate $x^0$ within its domain of definition. Recall that

$$\Omega(s) := J^*(p) \cap \{t < s\}. \quad (D.12)$$

We consider the divergence identity on $\Omega(s)$

$$E(s) + \int_{E(s)} T^\mu\nu X^\nu \, dS_\mu = -\int_{\partial \Omega(s)} T^\mu\nu X^\nu \, dS_\mu = -\int_{\Omega(s)} \nabla_\mu (T^\mu\nu X^\nu)$$

$$= -\int_{\Omega(s)} \frac{1}{2} T^\mu\nu \mathcal{L}_X g_{\mu\nu}. \quad (D.13)$$

Since $T^\mu\nu$ is traceless by hypothesis, from (D.6) and from (B.2) we obtain

$$\left| T^\mu\nu \left( \mathcal{L}_X g_{\mu\nu} - \frac{\partial_r \alpha}{r} X_\mu X_\nu \right) \right|_b \leq C(t - r) T^\mu\nu n^\mu n^\nu. \quad (D.14)$$

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As \( \partial / r \alpha / r \) is bounded, (D.14) together with (B.3) imply

\[
|T^{\mu\nu} \mathcal{L}_X g_{\mu\nu}| \leq C(T_{\mu\nu} X^\mu X^\nu + (t - r) T_{\mu\nu} n^\mu n^\nu)
\]

\[
\leq C'(T_{\mu\nu} n^\mu X^\nu + (t - r) T_{\mu\nu} n^\nu).
\]  

(D.15)

By (D.3) the right-hand-side is bounded by a multiple of \( T_{\mu\nu} n^\mu X^\nu \), and we can conclude that

\[
E(s) \leq C \int_0^s E(t) \, dt - \int_{\mathcal{C}(s)} T_{\mu\nu} X^\mu n^\nu \, dS^\mu,
\]

where the vanishing of the last integral follows from the fact that \( X^\nu \) is tangent to the generators of \( \mathcal{C} \), hence null there, and from (2.13). Since \( E(s) \) approaches zero as \( s \) tends to zero, from Gronwall’s lemma we obtain

\[
E(s) = 0 \quad \text{for} \quad 0 < s < s_*.
\]

Positivity of the integrand implies

\[
T_{\mu\nu} X^\mu n^\nu = 0 \quad \text{on} \quad \Omega(s_*).
\]  

(D.16)

Since \( X \) is timelike on the interior of \( \Omega(s_*), \) from (B.3) we conclude that the spacetime is vacuum in \( \Omega(s_*) \).

**Appendix E.** \( g^{AB} R_{AB} \)

In this appendix we continue our analysis for a metric which, in addition to the hypotheses of appendix C, satisfies further \( \nu_A = 0 \) at \( \{ u = 0 \} \); thus, there we have

\[
g = -\alpha u^2 + 2\varepsilon du \, dr + h_{AB} dx^A dx^B.
\]  

(E.1)

with

\[
g^\sharp = \alpha \partial_u^2 + 2\varepsilon \partial_u \partial_r + h_{AB} \partial_A \partial_B.
\]  

(E.2)

As in appendix C, all calculations are done on the null hypersurface \( \{ u = 0 \} \).

In addition to the previous list of vanishing Christoffel symbols,

\[
\Gamma^u_{rr} = \Gamma^r_{ur} = \Gamma^r_{ru} = 0,
\]  

(E.3)

we now also have, due to the wave-map conditions and the vanishing of \( \nu_A \),

\[
\partial_u g_{rr} = \partial_u g_{ru} = \Gamma^u_{ur} = \Gamma^r_{ru} = \Gamma^r_{rr} = \Gamma^r_{ru} = \Gamma^r_{ru} = 0.
\]  

(E.4)

The remaining Christoffel symbols can be obtained from those listed in appendix C by setting \( \nu_A = 0 \) there.

We will need the following traces:

\[
\Gamma^u_{uu} = \frac{1}{2r^2} \hat{h} h^{AB} \partial_u g_{AB} + \varepsilon \partial_u g_{ru},
\]  

(E.5)

\[
\Gamma^u_{ru} = \frac{n - 1}{r},
\]  

(E.6)

\[
\Gamma^u_{ru} = \frac{1}{2} \hat{h} h^{BC} \partial_A \hat{h}_{BC}.
\]  

(E.7)

In view of (C.35),

\[
\lambda^r = \frac{\varepsilon}{2r^2} \hat{h} h^{AB} \partial_u g_{AB} + \frac{n - 1}{r} \alpha + \partial_u \alpha,
\]

\[
\delta^r = \frac{n - 1}{r} - 2\Lambda r - \frac{n + 1}{n(n - 1)}.
\]  

(E.8)
and of the wave-map condition $\lambda' = \dot{\lambda}'$, we conclude that
\[
\frac{\varepsilon}{r^2} \hat{h}^{AB} \partial_\theta g_{AB} = \frac{n - 1}{r} \left( (1 - \alpha) - 2 \Lambda r \frac{n + 1}{n(n - 1)} - \partial_\alpha. \right. \tag{E.9}
\]

We want, next, to calculate
\[
g^{AB} R_{AB} = g^{AB} \left( \partial_u \Gamma^u_{AB} - \nabla_A \Gamma^u_{Bu} + \nabla_B \Gamma^u_{Au} - \nabla^u \Gamma^u_{AB} \right)
\]
\[
= g^{AB} \left( \partial_u \Gamma^u_{AB} + \partial_A \Gamma^u_{Bu} + \partial_B \Gamma^u_{Au} - \partial^u \Gamma^u_{AB} \right) + \Gamma^u_{CA} \Gamma^C_{AB} - \Gamma^D_{CB} \Gamma^C_{AB} - 2 \Gamma^u_{CB} \Gamma^C_{Ar} - 2 \Gamma^u_{CB} \Gamma^C_{Au}. \tag{E.10}
\]

We will calculate separately various terms above, in random order, starting with the last two
\[
-2h^{AB} \Gamma^u_{aA} \Gamma^u_{BC} = \frac{\varepsilon}{r^3} \hat{h}^{AB} \partial_\theta g_{AB}, \tag{E.11}
\]
\[
-2h^{AB} \Gamma^u_{rA} \Gamma^u_{BC} = \frac{\varepsilon}{r^3} \hat{h}^{AB} \partial_\theta g_{AB} + \frac{2(n - 1)}{r^2} \alpha, \tag{E.12}
\]
\[
h^{AB} \left( \partial_u \Gamma^u_{AB} + \partial_A \Gamma^u_{Bu} \right) = -\frac{\varepsilon}{r^2} \hat{h}^{AB} \partial_\theta \partial_u g_{AB} - \frac{n - 1}{r^2} \partial_\alpha (r \alpha) + \frac{n - 1}{r} \partial_\theta g_{sr}, \tag{E.13}
\]
\[
h^{AB} \left( \Gamma^u_{au} + \Gamma^u_{ua} \right) - \Gamma^u_{ra} \Gamma^u_{BC} = -\frac{n - 1}{r^3} \hat{h}^{AB} \partial_\theta g_{AB} - \frac{(n - 1)^2}{r^2} \alpha - \frac{n - 1}{r} \partial_\theta g_{sr}. \tag{E.13}
\]

The remaining terms are
\[
g^{AB} \left( \partial_u \Gamma^C_{AB} - \nabla_B \Gamma^u_{Au} + \nabla_A \Gamma^u_{Bu} - \nabla^u \Gamma^C_{AB} \right) = \hat{h}^{AB} R_{AB},
\]
where $R$ is the Ricci tensor of the metric $h_{AB}$. Adding, one is led to the simple identity
\[
g^{AB} R_{AB} = -\frac{\varepsilon}{r^2} \hat{h}^{AB} \partial_\theta \partial_u g_{AB} - \frac{\varepsilon}{r^3} \hat{h}^{AB} \partial_\theta g_{AB}
\]
\[
- \frac{(n - 1)(n - 3)}{r^2} \alpha - \frac{n - 1}{r^3} \partial_\alpha (r \alpha) + \hat{h}^{AB} R_{AB}. \tag{E.14}
\]

But, in view of the wave-map condition (E.9),
\[
\frac{\varepsilon}{r^2} \hat{h}^{AB} \partial_\theta \partial_u g_{AB} = \frac{\varepsilon}{r^3} \hat{h}^{AB} \partial_\theta g_{AB}
\]
\[
= 4 \Lambda \frac{n + 1}{n - 1} + 2 \partial_\alpha \partial_\alpha + \frac{4(n - 1)}{r} \partial_\alpha \frac{2(n - 1)(n - 2)}{r^2} (\alpha - 1). \tag{E.15}
\]

For the model metrics (2.23) we have $\hat{h}^{AB} R_{AB} = (n - 1)(n - 2)/r^2$, so
\[
g^{AB} R_{AB} = 4 \Lambda \frac{n + 1}{n - 1} + 2 \partial_\alpha \partial_\alpha + \frac{3(n - 1)}{r} \partial_\alpha \frac{(n - 1)(n - 2)}{r^2} (\alpha - 1). \tag{E.16}
\]

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