Asymmetric, helical and mirror-symmetric travelling waves in pipe flow

Chris Pringle and Rich R. Kerswell

Department of Mathematics, University of Bristol, University Walk, Bristol BS8 1TW, United Kingdom

(Dated: September 25, 2018)

New families of three-dimensional nonlinear travelling waves are discovered in pipe flow. In contrast to known waves (Faisst & Eckhardt Phys. Rev. Lett. 91, 224502 (2003), Wedin & Kerswell, J. Fluid Mech. 508, 333 (2004)), they possess no rotational symmetry and exist at much lower Reynolds numbers. Particularly striking is an ‘asymmetric mode’ which has one slow streak sandwiched between two fast streaks located preferentially to one side of the pipe. This family originates in a pitchfork bifurcation from a mirror-symmetric travelling wave which can be traced down to a Reynolds number of 773. Helical and non-helical rotating waves are also found emphasizing the richness of phase space even at these very low Reynolds numbers. The delay in Reynolds number from when the laminar state ceases to be a global attractor to turbulent transition is then even larger than previously thought.

PACS numbers: 47.20.Ft,47.27.Cn,47.27.nf

Wall-bounded shear flows are of tremendous practical importance yet their transition to turbulence is still poorly understood. The oldest and most famous example is the stability of flow along a straight pipe of circular cross-section first studied over 120 years ago [1]. A steady, unidirectional, laminar solution - Hagen-Poiseuille flow [2, 3] - always exists but is only realised experimentally for lower flow rates (measured by the Reynolds number $Re = UD/\nu$, where $U$ is the mean axial flow speed, $D$ is the pipe diameter and $\nu$ is the fluid’s kinematic viscosity). At higher $Re$, the fluid selects a state which is the immediate spatially and temporally complex rather than adopting a sequence of intermediate states of gradually decreasing symmetry. The exact transition Reynolds number $Re_t$ depends sensitively on the shape and amplitude of the disturbance present and therefore varies across experiments with quoted values typically ranging from 2300 down to a more recent estimate of 1750 [4, 5, 6, 7, 8, 9, 10, 11, 12]. A new direction in rationalising this abrupt transition revolves around identifying alternative solutions (beyond the laminar state) to the governing Navier-Stokes equations. These have only recently been found in the form of travelling waves (TWs) [13, 14] and all seem to be saddle points in phase space. They appear though saddle node bifurcations with the lowest found at $Re = 1251$. The delay before transition occurs ($Re_t \geq 1750$) is attributed to the need for phase space to become sufficiently complicated (through the entanglement of stable and unstable manifolds of an increasing number of saddle points) to support turbulent trajectories.

In this Letter, we present four new families of ‘asymmetric’, ‘mirror-symmetric’, helical and non-helical rotating TWs which have different structure to known solutions and exist at lower Reynolds numbers. The asymmetric family, which have one slow streak sandwiched between two fast streaks located preferentially to one side of the pipe, are particularly significant as they are the missing family of rotationally-asymmetric waves not found in [13, 14] and have the structure preferred by the linear transient growth mechanism [15]. They bifurcate from a new mirror-symmetric family which can be traced down to a saddle node bifurcation at $Re = 773$. This figure substantially lowers the current best estimate of 1251 for $Re_g$ - the Reynolds number at which the laminar state stops being a global attractor. The relative sizes of this new $Re_g$ and $Re_t$ for pipe flow are then more in line with plane Couette flow ($Re_t = 323$ [16] and $Re_g = 1277$ [14, 18]: $Re$ based on half the velocity difference and half the channel width) than plane Poiseuille flow ($Re_t \approx 1300$ [19, 20, 21, 22, 23] and $Re_g = 860$ [18]: $Re$ based on the mean flow rate and the channel width). Beyond suggesting that $Re_g$ in plane Poiseuille flow can be significantly lowered, these latest discoveries highlight the substantial delay in $Re$ between new solutions appearing in phase space and the emergence of sustained turbulent trajectories.

The new solutions were captured by inserting a fully 3-dimensional spectral representation (Chebychev in $s$, Fourier in $\phi$ and $z$ where $(s, \phi, z)$ are the usual cylindrical coordinates aligned with the pipe) of the velocity and pressure field into the governing Navier-Stokes equations as viewed from an appropriately rotating and translating reference frame in which the TW is steady [14]. The resultant nonlinear algebraic system was solved using the Newton-Raphson algorithm [30]. To start the procedure off, an artificial body force was added to the Navier-Stokes equations (see [14]) designed to give streamwise-independent vortices and streaks of a finite amplitude. The size of the forcing was then adjusted to find a bifurcation point at which the translational flow symmetry along the pipe is broken. New finite-amplitude solutions to pipe flow were found if this solution branch could be continued back to the zero-forcing limit.

The TWs previously isolated [13, 14] were induced us-
FIG. 1: Velocity fields for the asymmetric mode at Re = 2900 (top) and the mirror-symmetric mode at Re = 1344 (bottom) (both at α = 0.75). An instantaneous state is shown on the left and a streamwise-averaged state on the right. The coloring indicates the downstream velocity relative to the parabolic laminar profile: red (dark) through white (light) represents slow through fast (with zero corresponding to the shading outside the pipe). In-plane velocity components are shown by vectors. The maximum and minimum streamwise velocities (with the laminar flow subtracted) and maximum in-plane speed for the asymmetric mode are 0.33, −0.42 and 0.03 respectively while for the mirror-symmetric mode they are 0.31, −0.43 and 0.08 (all in units of U).

FIG. 2: Instantaneous axial vorticity along one wavelength of the pipe for the asymmetric mode (top) and the mirror-symmetric mode (bottom) at the same Re and α as Fig 1. Contours are at +/-60% of the maximum absolute value (green-light/red-dark).

FIG. 3: Phase velocity C in units of U as a function of α for the mirror-symmetric modes (solid lines) and asymmetric modes (dashed) at 4 values of Re near the saddle node bifurcation at Re = 773.
The helicity in the range for non-rotating, axially-aligned streaks. Interestingly, correspond to an isola in the (fixed mirror-symmetric branches (see Fig. 5). These points face passes through zero twice in going between the two O the mirror-symmetric modes except for a slight twist in helical and non-helical rotating waves look very similar to reflect symmetric nor have any rotational symmetry. The of rotating non-helical modes which are neither shift-&-reflect symmetric nor have any rotational symmetry. The TWs should exist with these fast streaks inclined to the flow, and a smaller (typically by an order of magnitude) 3-dimensional wave field. By continuity, helical TWs should exist with these fast streaks inclined to the flow direction and indeed a surface of such solutions can be found connecting the upper and lower branches of the mirror-symmetric TWs (see Fig. 5). These helical TWs take the form \( u(s, \phi, z, t) = u(s, \phi - \beta[z - Ct] - \omega t, z - Ct) \) with \( \beta \) measuring the helicity in the Galilean frame moving at \( C \hat{z} \) and \( \omega \) being an azimuthal phase speed relative to the Galilean frame. Helicity destroys S-symmetry but a modified form of O-symmetry \( (O_\beta) \) is preserved where the rotation transformation is now \( \phi \to \phi + (1 - \frac{\beta}{\alpha}) \pi \). The helicity \( \beta \) and rotational speed \( \omega \) never rise above \( O(10^{-2}) \) for \( Re \leq 1500 \) confirming the flow preference for non-rotating, axially-aligned streaks. Interestingly, in the range \( Re = 1165 - 1330 \), the helicity \( \beta \) on this surface passes through zero twice in going between the two mirror-symmetric branches (see Fig. 5). These points correspond to an isola in the (fixed \( \alpha \)) C vs \( Re \) plane of rotating non-helical modes which are neither shift-&-reflect symmetric nor have any rotational symmetry. The helical and non-helical rotating waves look very similar to the mirror-symmetric modes except for a slight twist in the streak structure along the pipe (see Figs 6 and 7). Helical modes continued off the asymmetric modes have no symmetry at all and originate in a symmetry-breaking bifurcation off the \( O_\beta \)-symmetric helical solutions extended from the mirror-symmetric waves: see Fig. 5.

The asymmetric, mirror-symmetric and helical TWs all represent saddle points in phase space with a very low-dimensional unstable manifolds (e.g. 2 for the asymmetric mode at \( (\alpha, Re) = (0.75, 1820) \) and 4 for the mirror-symmetric mode at \( (\alpha, Re) = (0.75, 1184) \)). Their presence indicates the richness of phase space even at
Reynolds numbers approaching 773. The delay of transition until $Re \geq 1750$ suggests that the establishment of a ‘turbulence-bearing’ scaffold constituted of all their stable and unstable manifolds is far from immediate. The clear implication is that while the emergence of alter-

able and unstable manifolds is far from immediate. The ’turbulence-bearing’ scaffold constituted of all their sta-

ble manifolds) sit on the separatrix between which is emerging that lower branch TWs (and therefore computations [28, 29], thereby corroborating the picture rationalise some interesting results from recent numerical computations [28, 29], using a shooting technique to converge onto this dividing surface appear to have already found that the asymmetric wave sits there too (compare Fig 4 to Fig. 8 of [28] and Fig. 1 of [29]). The fact that this wave bears some resemblance to $m = 1$ ‘optimal’ disturbances which emerge from linear transient growth analyses also suggests an enticing opportunity to bridge the gap between linear and nonlinear approaches.

In summary, we have presented a series of new traveling wave solutions to the pipe flow problem which have different structure to existing solutions and which exist at far lower Reynolds numbers. One type - the asymmetric modes - represents the missing $m = 1$ family from the waves found initially [13, 14]. These waves also appear to rationalise some interesting results from recent numerical computations [28, 29], thereby corroborating the picture which is emerging that lower branch TWs (and therefore also their stable manifolds) sit on the separatrix between laminar and turbulent states.

We acknowledge encouraging discussions with Fabian Waleffe and the support of EPSRC.

* Electronic address: Chris.Pringle@bris.ac.uk
† Electronic address: R.R.Kerswell@bris.ac.uk

[1] O. Reynolds, Proc. R. Soc. Lond. 35, 84 (1883).
[2] G. H. L. Hagen, Poggendorfs Annalen der Physik und Chemie 16, 423 (1839).
[3] J. L. M. Poiseuille, Comptes Rendus de l’Académie des Sciences 11, 961,1041 (1840).
[4] A. M. Binnie and J. S. Fowler, Proc. Roy. Soc. Lond. A 192, 32 (1947).
[5] E. R. Lingren, Arkiv för Physik 12, 1 (1958).
[6] R. J. Leite, J. Fluid Mech. 5, 81 (1959).
[7] I. J. Wygnanski and F. H. Champagne, J. Fluid Mech. 59, 281 (1973).
[8] A. G. Darbyshire and T. Mullin, J. Fluid Mech. 289, 83 (1995).
[9] B. Hof, A. Juel, and T. Mullin, Phys. Rev. Lett. 91, 244502 (2003).
[10] J. Feixinho and T. Mullin, Proc. IUTAM Symp. on Laminar-Turbulent Transition (eds Govindarajan, R. and Narasimha, R.), pp. 45–55 (2005).
[11] J. Feixinho and T. Mullin, Phys. Rev. Lett. 96, 094501 (2006).
[12] A. P. Willis and R. R. Kerswell, Phys. Rev. Lett. 98, 014501 (2007).
[13] H. Faisst and B. Eckhardt, Phys. Rev. Lett. 91, 224502 (2003).
[14] H. Wedin and R. R. Kerswell, J. Fluid Mech. 508, 333 (2004).
[15] P. J. Schmid and D. S. Henningson, J. Fluid Mech. 277, 197 (1994).
[16] S. Bottin, O. Dauchot, F. Daviaud, and P. Manneville, Phys. Fluids 10, 2597 (1998).
[17] M. Nagata, J. Fluid Mech. 217, 519 (1990).
[18] F. Waleffe, Phys. Fluids 15, 1517 (2003).
[19] S. J. Davies and C. M. White, Proc. Roy. Soc. A 119, 92 (1928).
[20] T. W. Kao and C. Park, J. Fluid Mech. 43, 145 (1970).
[21] V. C. Patel and M. R. Head, J. Fluid Mech. 38, 181 (1969).
[22] S. A. Orszag and L. C. Kells, J. Fluid Mech. 96, 159 (1980).
[23] D. R. Carlson, S. E. Widnall, and M. F. Peeters, J. Fluid Mech. 121, 487 (1982).
[24] B. Hof and et al, Science 305, 1594 (2004).
[25] B. Hof, C. W. H. van Doorne, J. Westerweel, and F. T. M. Nieuwstadt, Phys. Rev. Lett. 95, 214502 (2005).
[26] R. R. Kerswell and O. R. Tutty, J. Fluid Mech. in press (arXiv.org/physics/0611009) (2007).
[27] T. M. Schneider, B. Eckhardt, and J. Vollmer, Preprint (arXiv.org/physics/0611020) (2007).
[28] T. M. Schneider and B. Eckhardt, Chaos 16, 041103 (2006).
[29] B. Eckhardt, T. M. Schneider, B. Hof, and J. Westerweel, Annual Review of Fluid Mechanics 39, 447 (2007).
[30] In the nomenclature of [14], typical resolutions used to represent the modes were (15, 25, 5) representing about 20,000 degrees of freedom.

FIG. 7: The four fast streaks of the helical mode shown in Fig. 6 plotted over one $\beta$ wavelength $\approx 170 D$. 