Non-linear vibrations of a simply supported rectangular plate carrying a centric mass

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Abstract. Vibration is something that surrounds us in all areas, from everyday life to advanced industries. Among them, plates are found anywhere, so it has taken a fair share of study and analysis. Often, plates hold one or many point masses which completely changes their vibration frequencies and mode shapes. So, the main objective of this work is to analyze the effect of an added centric mass on the vibration Characteristics of simply supported rectangular plates and in both the linear and non-linear vibration regimes. The theoretical formulation adopted is based on Hamilton’s principle and spectral analysis. The expressions for the kinetic, linear and non-linear strain energies are derived taking into account the effect of the added mass on the kinetic energy and the effect of the membrane forces induced by the non-linearity on the strain energy. The generated equation has been solved numerically via an iterative procedure permitting to obtain the amplitude dependent frequencies and mode shapes. The results presented in this article are expected to be useful, both quantitatively and qualitatively, for a better understanding of the vibration of plates with an added mass.

1. Introduction
The subject of a plate carrying a concentrated mass has interested many researchers. K. H. Low [1] investigated the vibration frequencies of a plate with different boundary conditions, carrying concentrated masses at different places, using the finite element method and in experimentally. The fundamental frequency of the plate was found to decrease as the mass added was increased. K. H. Low and his co-author in 1998 [2] examined the vibration of plate a with added masses in the cases of combined clamped and simply supported conditions using the energy method. They found that the analytical model estimated well the natural frequency of the plate, compared with experimental results. Chai Gin Boay in 1995 [3] studied also the linear vibration of a rectangular isotropic plate with various boundary conditions clamped and simply supported carrying a concentrated mass using the Ritz approach. He found that the one-term solution gives a good solution if the mass is placed at the center of the plate. Chai Gin Boay in 1993 [4] investigated the natural frequencies of plates with various combinations of clamped and simply supported conditions with and without added masses. The frequency of a plate with an added mass placed away from its center was not well predicted.

In the present work, the linear and geometrically non-linear free vibration of a simply supported rectangular plate carrying a centric added mass is analyzed. The fundamental non-linear mode has been compared with the linear one and the bending stress distribution is also presented. The effect of the centric mass on the non-linearity of the simply supported plate is studied and backbone curves are presented.
2. Numerical model
Consider the transverse vibrations of the simply supported isotropic plate shown in Figure 1, carrying a centric mass \( m \), located at the center of the plate. The bending strain energy \( V_b \), the membrane strain energy \( V_a \), and the kinetic energy can be written as follows [5]:

\[
V_a = \frac{3D}{2H^2} \int_S \left[ \left( \frac{\partial W}{\partial x} \right)^2 + \left( \frac{\partial W}{\partial y} \right)^2 \right] dS \quad (1)
\]

\[
V_b = \frac{1}{2} \int_S D \left( \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} \right)^2 dS \quad (2)
\]

\[
T = \frac{1}{2} \rho H \int_S \left( \frac{\partial W}{\partial t} \right)^2 dS + \frac{1}{2} m \left( \frac{\partial W (a/2, b/2)}{\partial t} \right)^2 \quad (3)
\]

\[ W(x, y, t) = a_i w_i(x, y) \sin(\omega t) \quad (4) \]

The last term of equation 3 represents the kinetic energy for the added mass. The transverse displacement \( W \), assumed to be harmonic, is expanded in the form of a series:

The expressions for the membrane and bending strain energies and for the kinetic energy become after discretization:

\[
V_a = \frac{1}{2} a_i a_j a_k a_l b_{ijkl} \sin^4(\omega t) \quad (5)
\]

\[
V_b = \frac{1}{2} a_i a_j k_{ij} \sin^2(\omega t) \quad (6)
\]

\[
T = \frac{1}{2} \omega^2 a_i a_j m_{ij} \cos^2(\omega t) \quad (7)
\]

With \( b_{ijkl}, k_{ij} \) and \( m_{ij} \) being the geometrical non-linearity tensor, the rigidity tensor and the mass tensor respectively. Their expressions are given, in a non-dimensional form as:

\[
b_{ijkl}' = 3 \int_S \left( \alpha \left( \frac{\partial w_i^*}{\partial x^*} \frac{\partial w_j^*}{\partial x^*} + \frac{\partial w_i^*}{\partial y^*} \frac{\partial w_j^*}{\partial y^*} \right) \left( \alpha \left( \frac{\partial w_i^*}{\partial x^*} \frac{\partial w_j^*}{\partial x^*} + \frac{\partial w_i^*}{\partial y^*} \frac{\partial w_j^*}{\partial y^*} \right) \right) dx^* dy^* \quad (8)
\]
\[ k_{ij}^* = \int_S \left( \alpha^2 \frac{\partial^2 w_i^*}{\partial x^2} + \frac{\partial^2 w_i^*}{\partial y^2} \right) \left( \alpha^2 \frac{\partial^2 w_j^*}{\partial x^2} + \frac{\partial^2 w_j^*}{\partial y^2} \right) \, dx \, dy \]  
\[ (9) \]

\[ m_{ij}^* = \int_S w_i^* w_j^* dx \, dy + \eta w_i^* (0.5, 0.5) w_j^* (0.5, 0.5) \]  
\[ (10) \]

\( \eta \) represents the ratio of the added mass to the total mass of the plate \( \eta = \frac{m}{\rho H a b} \), and \( \alpha \) is the aspect ratio of the plate \( \alpha = \frac{b}{a} \). The other non-dimensional parameters are defined by:

\[ b_{ijkl} = \frac{D a H^2}{b^3} b_{ijkl}^* \]  
\[ (11) \]

\[ k_{ij} = \frac{D a H^2}{b^3} k_{ij}^* \]  
\[ (12) \]

\[ m_{ij} = \rho H a b m_{ij}^* \]  
\[ (13) \]

The plate motion is governed by Hamilton’s principle:

\[ \delta \int_0^{2\pi} (V - T) \, dt = 0 \]  
\[ (14) \]

\( \delta \) indicates the variation of the integral, \( V \) is the total strain energy. Equation 14 leads to the following set of \( n \) non-linear algebraic equations:

\[ 3a_i a_j a_k b_{ijk}^* + 2a_i k_{ir}^* - 2\omega^2 a_i m_{ir}^* = 0 \quad r = 2, \ldots, n \]  
\[ (15) \]

as mentioned in [5], \( \omega^2 \) can written as follows

\[ \omega^2 = \frac{a_i a_j b_{ijkl}^* + \frac{3}{2} a_i a_j a_k a_l b_{ijkl}^*}{a_i a_j m_{ij}^*} \]  
\[ (16) \]

With

\[ \omega^2 = \frac{D}{\rho b^2} \omega^2 \]  
\[ (17) \]

Equation 15 becomes

\[ 3a_i a_j a_k b_{ijk}^* + 2a_i k_{ir}^* - \frac{3}{2} a_i a_j a_k a_l b_{ijkl}^* a_i m_{ir}^* = 0 \quad r = 2, \ldots, n \]  
\[ (18) \]

To treat similar non-linear problems, Benamar’s method has been often used [5] [6] and [7]. It consists on fixing first the contribution of the first contribution \( a_1 \) before solving numerically the non-linear system via an iterative procedure, based on a hybrid method combining the steepest descent and Newton’s method to get the contributions of the higher functions. The non-dimensional expression for the bending stresses, as mentioned in Ref [5], is:

\[ \sigma_{xb}^* = \alpha^2 \left( \frac{\partial^2 W^*}{\partial x^2} \right) + \nu \left( \frac{\partial^2 W^*}{\partial y^2} \right) \]  
\[ (19) \]

\[ \sigma_{yb}^* = \nu \left( \frac{\partial^2 W^*}{\partial y^2} \right) + \alpha^2 \nu \left( \frac{\partial^2 W^*}{\partial x^2} \right) \]  
\[ (20) \]

The relation between non-dimensional and dimensional expressions is:

\[ \sigma = \frac{E H^2}{2(1 - \nu^2)\bar{b}^2} \sigma^* \]  
\[ (21) \]
Figure 2. Comparison of the first mode shape at large vibration amplitudes along the plate length at \( y^* = 0.5 \) of a rectangular plate \( \alpha = 1 \) carrying a centric mass \( \eta = 0.1 \). 1: Lowest amplitude. 2: Highest amplitude

3. Numerical results and discussion
After performing the necessary calculations and observing the results, this section presents the remarks made by analyzing the mode shape deformation at large vibration amplitudes in the presence of an added mass placed at the center of the rectangular plate. Also, the non-linearity effect on the plate is illustrated using the backbone curve giving the maximum amplitude of the plate \( W_{\text{max}} \) versus the ratio of the non-linear frequency to the linear frequency \( \omega_{nl}/\omega_l \). Figure 2 presents a comparison of the first mode shape at large vibration amplitudes along the plate length at \( y^* = 0.5 \) of a square plate carrying a centric mass \( \eta = 0.1 \). The mode shape between the mass position and the supports becomes more straightened at higher amplitudes. Figure 3 presents the first mode shape of a simply supported rectangular plate with an aspect ratio \( \alpha = 0.6 \) carrying a centric mass with \( \eta = 0.1 \), along the plate length at \( y^* = 0.5 \). Figure 4 shows for the same conditions along the section of the mode along the plate width at \( x^* = 0.5 \). It is clear that the deformation at large vibration amplitudes is more likely to happen in the \( y \) direction that in the \( x \) direction. The bending stress distributions associated to the nonlinear fundamental mode of a simply supported plate of an aspect ratio \( \alpha = 0.6 \) without added mass and for a simply supported plate with an added mass at the center \( \eta = 0.2 \) along the plate length at \( y^* = 0.5 \) up to one thickness of the plate \( W_{\text{max}}^* = 1 \) are compared in figure 5. It appears that the mass does not affect the bending stress at the mass location (the center of the plate). Also, the presence of the mass reduces the bending stress between the center and the supports. In Figure 6, a normalized non-dimensional bending stress \( \sigma_{x^*b}^* \) of a simply supported plate carrying a concentrated mass \( \eta = 0.2 \) at the center has been made and compared for different amplitudes. It can be seen that at higher amplitudes, the associated bending stress increases in the middle between the center and the supports.

The presence of a centric mass affects the mode shape of a simply supported plate at large vibration amplitudes as discussed before. It also changes the non-linear behavior of the plate.
Figure 3. Comparison of the first mode shape at large vibration amplitudes along the plate length at $y^* = 0.5$ of a rectangular plate $\alpha = 0.6$ carrying a centric mass $\eta = 0.1$. 1: Lowest amplitude. 2: Highest amplitude.

Figure 4. Comparison of the first mode shape at large vibration amplitudes along the plate width at $x^* = 0.5$ of a rectangular plate $\alpha = 0.6$ carrying a centric mass $\eta = 0.1$. 1: Lowest amplitude. 2: Highest amplitude.
Figure 5. The non-dimensional bending stress $\sigma_{xb}^*$ along the plate length at $y^* = 0.5$ of a rectangular plate $\alpha = 0.6$ without added mass and for a simply supported plate carrying a centric mass $\eta = 0.2$ up to an amplitude equal to once the plate thickness.

Figure 6. The normalized non-dimensional bending stress $\sigma_{xb}^*$ at large vibration amplitudes along the plate length at $y^* = 0.5$ of a rectangular plate $\alpha = 0.6$ carrying a centric mass $\eta = 0.2$.

1: Lowest amplitude. 2: Highest amplitude.
Figure 7. Backbone curves of a simply supported square plate $\alpha = 1$ carrying a centric mass.

Figure 8. Backbone curves of a simply supported rectangular plate $\alpha = 0.6$ carrying a centric mass.
Figures 7 and 8 present the backbone curves of a square plate and a rectangular plate $\alpha = 0.6$ respectively carrying mass with mass ratio values $\eta = 0, 0.1, 0.2$. The added mass is placed at the center of the plate. From the backbone curves, it appears that the presence of the added mass decreases the hardening character of the non-linearity of simply supported rectangular plate even for square plate.

4. conclusion

The effect of an added centric mass on the vibration of a simply supported rectangular plate has been discussed in this paper, covering the deformation of the first mode shape at large vibration amplitudes, the associated bending stress distributions and also the changes in the non-linear behavior. The added centric mass changes the mode shape at large vibration amplitudes by straightening the mode shape between the mass position and the simple supports. The bending stress in the out of the center decreases in the case of an added centric mass. The hardening character of the non-linearity of the simply supported plate decreases due to the presence of the centric added mass.

References

[1] Low K 1993 Journal of sound and vibration 160 111–121
[2] Low K, Chai G, Lim T and Sue S 1998 International Journal of Mechanical Sciences 40 1119–1131
[3] Boay C G 1995 Computers & structures 56 39–48
[4] Boay C G 1993 Computers & structures 48 529–533
[5] Benamar R, Bennouna M and White R 1993 Journal of sound and vibration 164 295–316
[6] Abdelali H M, El Bikri K and Benamar R 2017 Annals of Solid and Structural Mechanics 9 57–67
[7] El Kadiri M, Benamar R and White R 1999 Journal of Sound and Vibration 228 333–358