A QUANTUM APPROACH TO KELLER-SEGEL DYNAMICS VIA A DISSIPATIVE NONLINEAR SCHRÖDINGER EQUATION

José Luis López
Departamento de Matemática Aplicada and Excellence Research Unit “Modeling Nature” (MNat)
Facultad de Ciencias, Universidad de Granada
18071 Granada, Spain
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Abstract. The parabolic-parabolic Keller-Segel model of chemotaxis is shown to come up as the hydrodynamic system describing the evolution of the modulus square \( n(t,x) \) and the argument \( S(t,x) \) of a wavefunction \( \psi = \sqrt{n} e^{iS} \) that solves a cubic Schrödinger equation with focusing interaction, frictional Kostin nonlinearity and Doebner-Goldin dissipation mechanism. This connection is then exploited to construct a family of quasi-stationary solutions to the Keller-Segel system under the influence of no-flux and anti-Fick laws.

1. Introduction. The modeling of quantum dissipation and diffusion processes has been widely explored over last years [4, 9, 30, 34, 35, 36, 40] due to their impact in various fields of science. Particularly, many of the nonlinear Schrödinger equations proposed in the literature involve complex nonlinearities describing different phenomenologies such as incoherent solitons, propagation of optical pulses or damping effects in nonlinear media. In this spirit, the family of Doebner-Goldin equations

\[
\begin{align*}
    i\partial_t \psi + \frac{1}{2} \Delta_x \psi + V[\psi] & = \frac{iD}{2} \left( \frac{\Delta_x n}{n} \right) \psi + D' \mu_1 \left( \frac{\nabla_x \cdot J}{n} \right) \psi + D' \mu_3 \left( \frac{|J|^2}{n^2} \right) \psi \\
    & + D' \mu_4 \left( \frac{J \cdot \nabla_x n}{n^2} \right) \psi + D' \left( \mu_2 \frac{\Delta_x n}{n} + \mu_5 \frac{|
abla_x n|^2}{n^2} \right) \psi
\end{align*}
\]

was introduced in [20] as the most general class of Schrödinger type equations compatible with a Fokker-Planck continuity equation governing the evolution of the probability density \( n(t,x) = |\psi(t,x)|^2 \) (cf. Eq. (20) below). Here, \( D, D' \in \mathbb{R} \) are diffusion constants, \( \mu_1, \ldots, \mu_5 \in \mathbb{R} \) are the nonlinear coupling coefficients, \( V[\psi] \) is a (eventually nonlinear) interaction potential and \( J = \text{Im}(\overline{\psi} \nabla_x \psi) \) is the electric current, where \( \text{Im}(\cdot) \) denotes the imaginary part of the complex function.
between brackets. In this formulation, a units system has been adopted in which the physical constants specific to the Schrödinger picture (that is, the Planck constant and the particle mass) have been normalized to unity for simplicity. Eq. (1) will be shown below to play a fundamental role (on the microscopic picture) in the explanation of the diffusion processes occurring at the Keller-Segel (macroscopic) level of description.

The Schrödinger-Kostin equation is also of paramount importance in describing the Langevin dynamics of a quantum particle immersed in a heat bath [30]:

\[
\begin{align*}
    i \partial_t \psi + \frac{1}{2} \Delta_x \psi - \tau V_K[\psi] \psi &= 0, \\
    V_K[\psi] &= S - \langle S \rangle,
\end{align*}
\]

where

\[
\langle S \rangle(t) := \frac{\int_{\Omega} S(t,x) n(t,x) \, dx}{\int_{\Omega} n(t,x) \, dx},
\]

is the expected value of $S$ with respect to the state $\psi$, and $\tau > 0$ stands for the friction coefficient. Here, $S(t,x)$ is intended to denote an argument of the complex wavefunction $\psi(t,x)$. The main difficulty in writing Kostin’s potential is of course the a priori ambiguity induced by the multivalued nature of the function $S$, that is, associated with each wavefunction $\psi$ there exist infinitely many functions $S$ that fulfill the modulus-argument decomposition

\[
\psi(t,x) = \sqrt{n(t,x)} \, e^{iS(t,x)}. \tag{4}
\]

Precisely, the subtraction of the positional expectation value $\langle S \rangle$ in (3) is performed to circumvent this shortcoming. Indeed, given a wavefunction $\psi$ such that both $S$ and $S'$ satisfy (4), the equality $S - \langle S \rangle = S' - \langle S' \rangle$ holds, and the mapping $\psi \mapsto V_K[\psi]$ is thus uniquely determined. At last, $\langle S \rangle$ is typically removed from (2)-(3) by means of the gauge transformation [36]

\[
\psi \mapsto \psi e^{i\nu(t)}, \quad \nu(t) = -\tau \int_0^t \langle S \rangle(s) e^{\tau(t-s)} \, ds, \tag{5}
\]

so that the simplest form of Kostin’s equation is achieved:

\[
i \partial_t \psi + \frac{1}{2} \Delta_x \psi - \tau S \psi = 0. \tag{6}
\]

The physical background in which Eqs. (1) and (6) are embedded is that of a quantum open system, consisting of an ensemble of quantum particles interacting dissipatively with a local thermal environment which, in the simplest case, can be thought of as an infinite set of harmonic oscillators in thermal equilibrium whose degrees of freedom are linearly coupled to those of the main system. Then, after tracing out the reservoir degrees of freedom, a reduced density matrix description is obtained for the evolution of the particle ensemble under observation, from which the corresponding hydrodynamics is simply deduced. Finally, the Schrödinger equation governing the evolution of the wavefunction (4) is derived according to the aforementioned hydrodynamics. In the particular case of Kostin’s approach, Eq. (6) represents the Schrödinger counterpart of the well-known Caldeira-Leggett model [12] describing quantum Brownian motion.

Some recent works have provided a glimpse of a potential connection between the blow-up of solutions to the Keller-Segel model of chemotaxis [29] and the collapse of wavefunctions obeying a nonlinear Schrödinger equation with nonlinearity of the type $|\psi|^\alpha \psi$, among which we may highlight [10, 14, 21, 26, 37]. Indeed, the eventual
blow-up of the solutions depends in both cases on the size of the initial population for dimension $N = 2$ and higher (see for instance [5, 8] and references therein). What we intend to achieve here is to specify the exact Schrödinger model (that will ultimately be a peculiar combination of Eqs. (1) and (6)) from which the Keller-Segel equations of chemotaxis are deduced in the fluid regime accompanying the decomposition (4).

The parabolic-parabolic Keller-Segel system takes the form

$$\begin{align*}
\partial_t \rho &= D_1 \Delta_x \rho - \nabla_x \cdot \left( \rho \nabla_x \varphi(c) \right), \\
\partial_t c &= D_2 \Delta_x c - \tau c + \rho,
\end{align*}$$

(7)

where $\rho(t, x)$ stands for the density of a cell population, $c(t, x)$ is the concentration of some chemical substance (usually known as chemoattractant), $\varphi(c)$ is the so-called sensitivity function, $D_1, D_2 > 0$ are the corresponding diffusion coefficients, and where the parameter $\tau \geq 0$ denotes the degradation rate of the chemoattractant. In the biological domain, this model describes the process through which the cells move in the direction of the concentration of some chemical agents released by themselves with velocity $\nabla_x (\varphi(c))$. The time scale governing the diffusion process for $c$ is typically much shorter than that for $\rho$, which is why the concentration of chemoattractant is often assumed to be quasi-stationary. It is also commonly assumed that $\tau = 0$ for mathematical simplicity, so that the evolution law for $c(t, x)$ is reduced to the gravitational Poisson equation, thus yielding the simplified (parabolic-elliptic) Keller-Segel model

$$\begin{align*}
\partial_t \rho &= D_1 \Delta_x \rho - \nabla_x \cdot \left( \rho \nabla_x \varphi(c) \right), \\
-D_2 \Delta_x c &= \rho.
\end{align*}$$

(8)

This system along with some of its variants are in the focus of attention of much of the current research in biomathematics (see for instance [3, 5, 10, 38] and references therein).

Our main purpose in this paper is to derive Eq. (7) (or where applicable Eq. (8)) as the hydrodynamical system associated with a nonlinear Schrödinger equation of the type (1) with focusing cubic nonlinearity $n\psi$ (responsible for the production term of the second equation in (7)) and Kostin’s nonlinearity $S\psi$ (responsible for the degradation term of the second equation in (7)), see [30, 36, 40], so that the wavefunction can be reconstructed from the Keller-Segel laws for the pair $(n, S)$ in the sense given by (4), provided that the linear sensitivity function $\varphi(S) = S$ is chosen. In this situation it is clear that the blow-up dynamics of both systems become closely related, so much so that a manipulation in the order of the cubic nonlinearity, say changing $n\psi$ to $f(n)\psi$ with $f$ satisfying some kind of strict sublinear growth, or the addition of a reaction term of logistic type in the equation for $\rho$, might lead to blow-up prevention [16, 32, 33, 43]. Besides, the intimate relationship between both descriptions might help to find some particular classes of solutions to (7) such as steady states or solitonic profiles, as well as to derive new variants of (7) from a different point of view than usual. In this spirit, special attention should be paid to the modeling of mechanisms accounting for flux-limited nonlinear diffusion as well as for fractional diffusion, which has recently attracted the interest of a number of authors in the context of chemotactic and related systems (see for instance [2, 3, 6, 7, 11, 13, 15, 18, 22, 39]). To the author’s knowledge, these modified diffusions do not find a counterpart in the Schrödinger-Doebner-Goldin literature at present.
The paper is organized as follows: Section 2 is devoted to the derivation of the fluid system (7) ruling the time evolution of \( n \) and \( S \) in a strong sense. Before that, we argue on the well-definition and regularity of \( S \) under appropriate hypotheses, and thus on the possibility of an unambiguous polar decomposition \( \psi = \sqrt{n} e^{iS} \) and a strong formulation of the corresponding evolution equation. Finally, in Section 3 the relation between (7) and the associated Schrödinger formulation is exploited in order to construct a family of quasi-stationary solutions (namely, \( n \) is found to be time-independent while \( S \) is shown to decay exponentially in time toward an equilibrium state) to the Neumann boundary problem associated with the two-dimensional Keller-Segel system, ruled by the vanishing current condition \( J = 0 \) or by the action of anti-Fick’s law \( J = D_1 \nabla_x n \). The technical details on the existence of such solutions are postponed to an Appendix.

2. From the Schrödinger formalism to Keller-Segel dynamics. With our sights set on avoiding vacuum regions so that the singularities present in the Doebner-Goldin equations (1) make full sense, we introduce the following notations:

\[
X^T = C([0, T); H^2(\Omega)) \cap C^1([0, T); L^2(\Omega)),
\]

\[
X^T_\delta = \{ \psi \in X^T : |\psi| > \delta > 0 \text{ a.e. } x \in \Omega, \forall 0 \leq t < T \}.
\]

The main result of this section reads as follows:

**Theorem 2.1.** Let \( \Omega \subset \mathbb{R}^N \) be a simply-connected, \( C^1 \) bounded domain. Let also \( \delta > 0, T > 0 \) and \( \psi \in X^T_\delta \) be a strong solution of the following nonlinear Schrödinger-Kostin equation with Doebner-Goldin dissipation-diffusion mechanism:

\[
i \partial_t \psi + \frac{1}{2} \Delta_x \psi + (n - \tau S) \psi = \left( iD_1 \frac{\Delta_x n}{2n} - D_2 \nabla_x \cdot \left( \frac{J}{n} \right) - \frac{|J|^2}{2n^2} - Q \right) \psi,
\]

where \( n = |\psi|^2 \) is the probability density, \( J = \text{Im}(\nabla_x \psi) \) is the electric current and \( Q = -\frac{\Delta_x \sqrt{n}}{2\sqrt{n}} \) is the quantum Bohm potential. Then, for a wavefunction

\[
\psi(t, x) = \sqrt{n(t, x)} e^{iS(t, x)},
\]

the parabolic-parabolic Keller-Segel system governing the hydrodynamic evolution of \( n(t, x) \) and \( S(t, x) \) holds strongly in \([0, T)\):

\[
\partial_t n = D_1 \Delta_x n - \nabla_x \cdot (n \nabla_x S),
\]

\[
\partial_t S = D_2 \Delta_x S - \tau S + n.
\]

Before proving it, we give an overview on the mathematical machinery that makes the decomposition (10) possible from a rigorous point of view.

2.1. On the existence of a continuous argument and the viability of a polar decomposition for the wavefunction. With a view to the right definition of Eq. (2)-(3), in this section we review some of the technical issues tackled in [25, 36] regarding the rigorous existence of a continuous argument and the corresponding decomposition of the complex wavefunction in polar form. If \( \psi \) is assumed to be decomposable as in (10), then the identity

\[
\nabla_x V_K = \text{Im} \left( \frac{\nabla_x \psi}{\psi} \right) = \frac{J}{n}
\]

formally holds. The solvability of Eq. (13) requires the irrotationality of the field in the right-hand side, that can be deduced from Schwartz’s Lemma if \( \Omega \) is assumed
to be simply-connected. This hypothesis on the domain is of crucial importance for our aim, since otherwise one is obliged to admit jump discontinuities of \( S \). Under this assumption, the existence of a unique solution to Eq. (13) (up to an additive constant) can be claimed for any given \( \psi \) which is separated a minimum distance \( \delta \) from zero, that is to say, belonging to

\[
\psi \in H^2_\delta := \{ \varphi \in H^2(\Omega) : |\varphi| > \delta > 0 \text{ a.e. } x \in \Omega \},
\]

and satisfying some additional regularity. Thus, a family of countably many functions \( S_k \in H^2(\Omega), k \in \mathbb{Z} \), exists such that the decomposition (10) is fulfilled along with the identity

\[
S_k - S_l = 2\pi (k - l), \quad k, l \in \mathbb{Z},
\]

(14)

More precisely, we have:

**Lemma 2.2** ([25]). Let \( \Omega \subset \mathbb{R}^N \) be a simply-connected, Lipschitz-continuous bounded domain and \( 0 \leq k \in \mathbb{Z} \). Then, for all complex functions \( \psi \in H^k(\Omega) \) such that

\[
\nabla \psi \psi \in (H^{k-1}(\Omega))^N, \quad \frac{\nabla \psi \psi}{\psi} \nabla \psi \psi \in (H^{k-2}(\Omega))^{N^2},
\]

there exists a unique (up to an additive constant) function \( S \in H^k(\Omega) \) that solves

\[
\nabla_x S = \text{Im} \left( \frac{\nabla_x \psi}{\psi} \right).
\]

(15)

Besides, given \( \mu \in \mathbb{R} \) there exists a unique \( S \in H^k(\Omega) \) solving Eq. (15) such that \( \int_\Omega S dx = \mu \), and also a unique \( \beta_\mu \in [0, 2\pi) \) such that the family

\[
S_l := S + \beta_\mu + 2\pi l, \quad l \in \mathbb{Z},
\]

(16)

satisfies (10) and (14).

Furthermore, for any fixed \( \mu \in \mathbb{R} \), the mapping \( \psi \mapsto S \) is continuous from \( H^2_\delta \) to the subset

\[
S^\mu = \left\{ S \in H^2(\Omega) : \int_\Omega S dx = \mu \right\}.
\]

(17)

Indeed, it is this property that makes it possible to find a (unique) continuous-in-time argument for any given \( \psi \in X^T_\delta \) solving a general Schrödinger-like equation. As a matter of fact, the main result in [25] establishes that for any strong solution \( \psi \) in \([0, T]\) to the Schrödinger equation

\[
i\partial_t \psi + \frac{1}{2} \Delta_x \psi = \Theta[n, J] \psi,
\]

(18)

with \( \Theta : (H^2_\delta(\Omega))^N \to L^2(\Omega) \) being any continuous mapping of \( n \) and \( J \), there exists a family of arguments \( \{ S_k \}_{k \in \mathbb{Z}} \subset X^T \) fulfilling Eqs. (10) and (14) a.e. \( \Omega \), for all \( t \in [0, T] \). This is possible in accordance with the evolution law satisfied by \( S \), stemming from Eq. (18), which can be described in terms of the only observables \( n \) and \( J \) (see §2.2). In particular, the existence of a continuous-in-time argument in \( X^T \) follows for any solution to Eq. (1).

In the end, the following result establishes the existence of a well-defined operator that yields the correct frictional term in the Schrödinger-Kostin picture (2)-(3).
Lemma 2.3 ([36]). Let \( \Omega \subset \mathbb{R}^N \) \((1 \leq N \leq 3)\) be a simply-connected, Lipschitz-continuous bounded domain. Let also \( \delta > 0 \) and \( T > 0 \). Then, there exists a mapping \( V_K : H^2_\delta \to H^2(\Omega) \) such that for all \( \psi \in H^2_\delta \), \( V_K[\psi] \) is a solution to Eq. (19) coupled with (3). Besides, if \( S \in H^2(\Omega) \) is any argument of \( \psi \), then \( V_K[\psi] = S - \langle S \rangle \). In particular, it is fulfilled that \( V_K[\psi] = \psi(e^{it} \psi) \) for all function \( \nu = \nu(t) \).

Once the problem (2)-(3) is known to be well defined (and thus (6)), we focus our attention on a model Schrödinger-Kostin equation with the form

\[
i \partial_t \psi + \frac{1}{2} \Delta_x \psi - \tau S \psi = \Theta[n,J] \psi, \tag{19}\]

where \( \Theta : (H^2_\delta(\Omega)) \times (H^1(\Omega))^N \to L^2(\Omega) \) is meant to be a (complex) continuous mapping belonging to the Doebner-Goldin class. Local well-posedness in bounded domains of the initial-boundary value problem associated with (19) was proved in [36].

2.2. Proof of Theorem 2.1: Keller-Segel \((n,S)\)-hydrodynamics. By making the particular choices

\[D = D'\mu_4 = 1, \quad D'\mu_1 = 2D'\mu_3 = -1, \quad D'\mu_2 = -2D'\mu_5 = \frac{1}{4}\]

in Eq. (1), and then defining the continuous operator \( \Theta : (H^2_\delta(\Omega)) \times (H^1(\Omega))^N \to L^2(\Omega) \) by

\[\Theta[n,J] := iD_1 \frac{\Delta_x n}{2n} - D_2 \nabla_x \left( \frac{J}{n} \right) - \frac{|J|^2}{2n^2} - Q - n\]

for any given \( \psi \in X^T_\delta \) (cf. Eq. (19)), we obtain the Doebner-Goldin dissipative version of the cubic-Kostin Schrödinger equation stated in (9).

Notice that \( \overline{\nu} \nabla \psi \in L^2(\Omega) \) and

\[\nabla \otimes (\overline{\psi} \nabla \psi) = \nabla \overline{\psi} \otimes \nabla \psi + \overline{\psi} (\nabla \otimes \nabla) \psi \in L^2(\Omega),\]

given that \( \nabla_x \psi \in H^1(\Omega) \subset L^4(\Omega) \) by virtue of the Rellich-Kondrachov theorem, and thus \( \nabla \overline{\psi} \otimes \nabla \psi \in L^2(\Omega) \). As consequence, \( \overline{\psi} \nabla \psi \in (H^1(\Omega))^N \). Also, since the identity \( \overline{\psi} \nabla \nabla \psi = \frac{1}{2} \nabla x n + iJ \) is straightforwardly satisfied, we find that \( J, \nabla_x n \in (H^1(\Omega))^N \), hence \( n \in H^2(\Omega) \). The nonvacuum condition \( |\psi| > \delta \) then implies that \( n \in H^2_\delta \), and allows to guarantee that \( \frac{\nabla_x n}{n}, \frac{J}{n} \in (H^1(\Omega))^N \). This along with the fact that \( t \mapsto \psi(t) \) is a continuous mapping in \( H^2(\Omega) \), allows concluding that the regularity properties

\[n \in X^T_\delta, \quad \nabla_x n, J, \frac{J}{n} \in C([0,T];H^1(\Omega)) \cap C^1([0,T];H^{-1}(\Omega))\]

are fulfilled. Now, after multiplying Eq. (9) times \( -\overline{\psi} \) and taking real parts we find that the continuity equation

\[\partial_t n + \nabla_x \cdot J = D_1 \Delta_x n, \tag{20}\]

holds in a strong sense, as \( t \mapsto \Delta_x n(t) \) and \( t \mapsto \nabla_x \cdot J(t) \) are continuous from \([0,T]\) onto \( L^2(\Omega) \), and thus \( n \in C^1([0,T];L^2(\Omega)) \). Now, Eq. (20) can be equivalently reformulated as

\[\partial_t n = D_1 \Delta_x n - \nabla_x \cdot (n \nabla_x S), \tag{21}\]

by virtue of Eq. (13) and Lemma 2.3, thus yielding Eq. (11) of the Theorem.
There only remains to establish the equation governing the evolution of $S(t,x)$. To that aim, we start deriving the evolution equation satisfied by $\frac{J}{n}$. Firstly, we find that the following law for the density current

\[
\partial_t J = \text{Im}(\partial_t \bar{\psi} \nabla_x \psi + \bar{\psi} \partial_t \nabla_x \psi)
\]

\[
= \frac{1}{2} \text{Re} \left( \bar{\psi} \nabla_x \Delta_x \psi - \nabla_x \psi \Delta_x \bar{\psi} \right) - \tau n \nabla_x S + \text{Re} \left( \nabla_x \Theta[n,J] \bar{\psi} - \bar{\psi} \nabla_x (\Theta[n,J]) \right)
\]

\[
= \frac{1}{2} \text{Re} \left( \psi^2 \nabla_x \left( \frac{\Delta_x \psi}{\psi} \right) \right) - \tau n \nabla_x S - \text{Re} \left( \psi^2 \nabla_x \left( \frac{\Theta[n,J]}{\psi} \right) \right)
\]

holds in the sense of distributions. Then,

\[
\partial_t \left( \frac{J}{n} \right) = \frac{1}{2} \text{Re} \left( \frac{\psi}{\bar{\psi}} \nabla_x \left( \frac{\Delta_x \psi}{\psi} \right) \right) - \tau \nabla_x S - \text{Re} \left( \frac{\psi}{\bar{\psi}} \nabla_x \left( \frac{\Theta[n,J]}{\psi} \right) \right) + J \left( \frac{\nabla_x \cdot J}{n} - D_1 \Delta_x n \right).
\]  

(22)

A simple computation leads to $\nabla_x \psi = \left( \frac{\nabla_x n}{2n} + i \frac{J}{n} \right) \psi \in H^1(\Omega)$, from which it is deduced that

\[
\Delta \psi = \left\{ 2Q + \left( \frac{|J|^2}{n^2} - i \frac{\nabla_x \cdot J}{n} \right) \right\} \psi \in L^2(\Omega)
\]

by just taking the divergence in both sides. Therefore, the arguments of the real parts in Eq. (22) are developed as follows:

\[
\frac{\psi}{\bar{\psi}} \nabla_x \left( \frac{\Delta_x \psi}{\psi} \right) = \nabla_x \left( \frac{\Delta_x n}{2n} - \frac{[\nabla_x n]^2}{4n^2} \right) - \left\{ \nabla_x \left( \frac{|J|^2}{n^2} \right) + i \nabla_x \left( \frac{\nabla_x \cdot J}{n} \right) \right\}
\]

\[
- \frac{2i}{n^2} J \left\{ \frac{\Delta_x n}{2} - \frac{[\nabla_x n]^2}{4n} - \frac{|J|^2}{n} - i \nabla_x \cdot J \right\}.
\]

\[
\frac{\psi}{\bar{\psi}} \nabla_x \left( \frac{\Theta[n,J]}{\psi} \right) = \nabla_x (\Theta[n,J]) + \Theta[n,J] \left( \frac{\nabla_x \psi}{\psi} - \frac{\nabla_x \psi}{\psi} \right).
\]

Finally, after making some simplifications we find

\[
\partial_t \left( \frac{J}{n} \right) = -\nabla_x \left( Q + \frac{|J|^2}{2n^2} + \tau S + \text{Re}(\Theta[n,J]) \right)
\]

\[
= \nabla_x \left( D_2 \nabla_x \left( \frac{J}{n} \right) + n - \tau S \right).
\]  

(23)

To conclude, we introduce the operator $K : H^2_{\bar{\delta}} \times (H^1(\Omega))^N \rightarrow L^2(\Omega)$ defined as

\[
K[n,J] := D_2 \nabla_x \cdot \left( \frac{J}{n} \right) + n
\]

and denote $\tilde{S}(t,x) = S(t,x) + \tau \int_0^t S(s,x) ds$, so that Eq. (23) now reads $\nabla_x (\partial_t \tilde{S} - K) = 0$. Now, we can define

\[
\tilde{S}(t,x) := S_0(x) + \int_0^t K(s,x) ds - \frac{1}{|\Omega|} \int_\Omega \int_0^t K(s,y) ds dy
\]

for all $t \in [0,T)$. Here, $S_0$ is an argument of the initial datum constructed as follows. Since $\psi(0) \in H^2_\delta$, Lemma 2.2 guarantees the existence of $S_{\psi(0)} \in H^2(\Omega)$ and $\beta_0 \in [0,2\pi)$ such that

\[
\psi(0,x) = \sqrt{n(0,x)} e^{iS_{\psi(0)}(x)} \quad \text{a.e. } x \in \Omega,
\]
where we fixed \( \mu = 0 \) in (17) and denoted \( S_0 = S_{\psi(0)} + \beta_0 \). Then, after denoting \( \Phi(t) = \frac{1}{|\Omega|} \int_\Omega K(t, x) \, dx \) and using (13) we find that
\[
\partial_t \tilde{S} = D_2 \nabla_x \cdot \left( \frac{J}{n} \right) + n - \Phi(t) = D_2 \Delta_x S + n - \Phi(t)
\]
is fulfilled in a strong sense, which can be straightforwardly rewritten as
\[
\partial_t S = D_2 \Delta_x S + n - \tau S - \Phi(t).
\]
The final step consists in showing that \( \Phi \) can be chosen to vanish identically without altering the dynamics. Indeed, the wavefunction \( \varphi = \sqrt{n} e^{i S} \) satisfies
\[
i \partial_t \varphi + \frac{1}{2} \Delta_x \varphi + (n - \tau S) \varphi = \left( iD_1 \frac{\Delta_x n}{2n} - D_2 \nabla_x \cdot \left( \frac{J}{n} \right) - \frac{|J|^2}{2n^2} - Q + \Phi(t) \right) \varphi,
\]
that differs from Eq. (9) only in the term \( \Phi \varphi \). Now, it suffices to consider (cf. (5))
\[
\nu(t) := \int_0^t \Phi(s) e^{-\tau(t-s)} \, ds
\]
and define the following gauge transformation \( \psi(t, x) := e^{i \nu(t)} \varphi(t, x) \) (notice that \( |\psi|^2 = |\varphi|^2 \) and \( J_\psi = J_\varphi \)) to deduce that \( \psi \) exactly solves Eq. (9), for which the relation \( S_\psi = S_\varphi + \nu \) has been taken into account. As consequence, the fluid \((n, S)\)-system associated with Eq. (9) reproduces the parabolic-parabolic Keller-Segel system (7) in a strong sense.

Now, we are done with the proof of Theorem 2.1.

**Remark 1.** If the Kostin contribution \( \tau S \) in Eq. (9) is ignored, then the degradation term in the second equation of (7) does not appear and the parabolic-elliptic Keller-Segel system (8) is recovered in the quasi-stationary regime.

### 3. On a family of quasi-stationary solutions to the parabolic-parabolic Keller-Segel model under vanishing current and anti-Fick diffusion conditions.

In this section we are interested in studying some particular classes of quasi-stationary solutions to the Keller-Segel system (11)-(12) in dimension \( N = 2 \), stemming from the assumptions (i) vanishing current density: \( J = 0 \), or (ii) vanishing diffusion current: \( j := J - D_1 \nabla_x n = 0 \) on the associated quantum system (9), which typically generate steady state dynamics (see for instance [35], where a set of explicit wave function profiles with time-independent probability density was found for a family of dissipative Schrödinger equations of logarithmic type). In what follows, homogeneous Neumann conditions
\[
\frac{\partial u}{\partial \eta} = 0 \tag{24}
\]
are considered on \( \partial \Omega \). More precisely, the following result is proved.

**Theorem 3.1.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \) with smooth boundary. The following assertions hold true.

(i) Assume that the vanishing current condition \( J = 0 \) holds and let
\[
n_0(x) = \alpha > 0, \quad S_0(x) = \beta > 0 \quad \text{in} \ \Omega.
\]

Then, the couple
\[
n(t, x) = \alpha, \quad S(t, x) = \frac{\alpha}{\tau} + \left( \frac{\beta - \alpha}{\tau} \right) e^{-\tau t}
\]
solves the parabolic-parabolic Keller-Segel system (11)-(12) subject to (24),
and decays exponentially (for large times) toward the equilibrium state
\[(n_\infty, S_\infty) = \left(\alpha, \frac{\alpha}{\tau}\right)\].

(ii) Assume that the anti-Fick law
\[J = D_1 \nabla_x n \]
is fulfilled. Let also
\[\lambda > \tau D_1 \max \left\{1 - \log(\tau D_1), -\xi_- \right\}, \tag{26}\]
\[C_0 > \frac{\lambda}{\tau}, \text{ and } (n_0, S_0) \text{ be such that} \]
\[D \Delta_x \log(n_0) - \log(n_0) + \nu n_0 = \bar{\lambda}, \tag{27}\]
\[S_0 = D_1 \log(n_0) + C_0, \tag{28}\]
in \(\Omega\), with
\[D = \frac{D_2}{\tau}, \quad \nu = \frac{1}{\tau D_1}, \quad \bar{\lambda} = \frac{\lambda}{\tau D_1}, \tag{29}\]
where \(\xi_- > 0\) stands for the least of the two real roots of the function
\[f(\xi) = \nu e^\xi - \xi - \bar{\lambda}. \tag{30}\]

Then, the couple
\[n(t, x) = n_0(x), \quad S(t, x) = D_1 \log(n_0(x)) + \frac{\lambda}{\tau} + \left(C_0 - \frac{\lambda}{\tau}\right)e^{-\tau t} \]
solves the parabolic-parabolic Keller-Segel system (11)-(12) subject to (24),
and decays exponentially (for large times) toward
\[(n_\infty(x), S_\infty(x)) = \left(n_0(x), D_1 \log(n_0(x)) + \frac{\lambda}{\tau}\right), \tag{31}\]
which is an equilibrium state of (11)-(12) under the condition (27). Besides,
if the chemical diffusion \(D > 0\) is sufficiently small, then nonconstant positive
solutions to (11)-(12) with the form (30) do exist with \(e^{\xi_-} < n_0 \leq e^{\xi_- + \kappa e^n}\),
for some \(\kappa > 0\).

Proof. The condition \(J = 0\) becomes \(n \nabla_x S = 0\) by virtue of the polar decomposition
of the wavefunction \(\psi = \sqrt{n} e^{iS}\). Indeed,
\[J = \text{Im}(\psi \nabla_x \psi) = \text{Im}\left(\sqrt{n} \nabla_x \sqrt{n} + i n \nabla_x S\right) = n \nabla_x S. \tag{31}\]

Then, \(S\) is homogeneous in the position variable, say \(S(t, x) = \phi(t)\), and system
(11)-(12) can be rewritten as
\[\partial_t n = D_1 \Delta_x n, \tag{32}\]
\[\phi' + \tau \phi = n. \tag{33}\]

According to Eq. (33) \(n\) must be \(x\)-independent too, thus Eq. (32) entails \(n = n_0 = \alpha \in \mathbb{R}^+\). This leads to
\[\phi(t) = \left(\phi(0) - \frac{n_0}{\tau}\right)e^{-\tau t} + \frac{n_0}{\tau}, \tag{34}\]
which concludes the proof of (i).

We now prove (ii). In the situation described by (25), the identity (31) gives
\[n \nabla_x S = D_1 \nabla_x n. \tag{34}\]
Eq. (11) then reads $\partial_t n = 0$, so that on one hand only stationary density profiles $n(t, x) = n_0(x)$ are admissible. On the other hand, Eq. (12) can be rewritten as
\[ \partial_t S + \tau S = n_0 + D_1 D_2 \Delta_x \log(n_0), \tag{35} \]
where $S$ is given by (cf. formula (34))
\[ S(t, x) = D_1 \log(n_0(x)) + \varphi(t) \tag{36} \]
for an arbitrary function $\varphi : \mathbb{R}^+ \to \mathbb{R}$. By inserting the profiles (36) into the left-hand side of Eq. (35) one finds the following separation of variables relation:
\[ \varphi'(t) + \tau \varphi(t) = n_0 - \tau D_1 \log(n_0) + D_1 D_2 \Delta_x \log(n_0) = \lambda \in \mathbb{R}. \tag{37} \]
As consequence, by first solving (37) for $\varphi$ we obtain
\[ \varphi(t) = \frac{\lambda}{\tau} + \left(C_0 - \frac{\lambda}{\tau}\right)e^{-\tau t}, \]
hence
\[ S(t, x) = D_1 \log(n_0(x)) + \frac{\lambda}{\tau} + \left(C_0 - \frac{\lambda}{\tau}\right)e^{-\tau t}, \]
as stated in the Theorem. Now, making the change of unknown function
\[ u(x) := \log(n_0(x)) \tag{38} \]
in (37) we are led to
\[ D\Delta_x u - u = \bar{\lambda} - \nu e^u, \tag{39} \]
with $D, \bar{\lambda}, \nu$ as defined in (29). The rest of the proof consists in studying the existence of solutions to Eq. (39) on bounded domains $\Omega \subset \mathbb{R}^2$, subject to the homogeneous Neumann boundary conditions established in (24). Similar problems have been dealt with in various scenarios, see for instance [17, 19, 23, 28, 41, 42]. When solutions to (39)-(24) do exist, the couple of functions
\[ (n(x), S(t, x)) = \left(e^{u(x)}, D_1 u(x) + \frac{\lambda}{\tau} + \left(C_0 - \frac{\lambda}{\tau}\right)e^{-\tau t}\right) \]
solves the parabolic-parabolic Keller-Segel system (11)-(12), which coincides with (30) after undoing the change of variables (38).

Consider
\[ \bar{\lambda} > 1 + \log(\nu) \tag{40} \]
or, in terms of the original parameters, $\lambda > \tau D_1 (1 - \log(\tau D_1))$. Within this range, the function $f(\xi) = \nu e^\xi - \xi - \bar{\lambda}$ is easily checked to have two positive roots $0 < \xi_- < \xi_+$. Hence, the existence of constant solutions to (39)-(24) can be concluded.

We now search for nonconstant solutions. According to Eq. (39) and given that $f(\xi_-) = 0$ under the constraint (40), the function
\[ \omega := u - \xi_- \tag{41} \]
satisfies
\[ D\Delta_x \omega - \omega = (\xi_- + \bar{\lambda})(1 - e^\omega), \]
or equivalently
\[ D\Delta_x \omega - \sigma \omega + (1 - \sigma)(e^\omega - \omega - 1) = 0, \tag{42} \]
where $\sigma = 1 - \xi_- - \bar{\lambda}$. Since we are intended to search for positive solutions to Eq. (42), we can finally write
\[ D\Delta_x \omega - \sigma \omega + (1 - \sigma)F(\omega) = 0, \tag{43} \]
\[ w > 0 \text{ in } \Omega, \]
\[ \frac{\partial \omega}{\partial \eta} = 0 \quad \text{on } \partial \Omega, \]

where we denoted
\[ F(z) = \begin{cases} e^z - z - 1, & z > 0 \\ 0, & z \leq 0 \end{cases}. \tag{44} \]

We remark at this point that \( \sigma > 0 \), as
\[ 0 > f'(\xi_-) = \nu e^{\xi_-} - 1 = \xi_- + \lambda - 1 = -\sigma. \]

Also, if \( \xi_- + \lambda > 0 \) is assumed, then \( \sigma < 1 \). Now, it is enough to show that the functional
\[ I(\omega) = \frac{D}{2} \int_{\Omega} |\nabla \omega|^2 \, dx + \frac{\sigma}{2} \int_{\Omega} \omega^2 \, dx - (1 - \sigma) \int_{\Omega} (e^\omega - \omega - 1) \, dx \tag{45} \]

admits nontrivial critical points, since any critical point of \( I \) is a solution to Eq. (42) subject to the boundary condition (24). This follows from an application of the Mountain Pass Theorem (MPT) provided that \( D > 0 \) is chosen sufficiently small. Furthermore, it can be proved that any nonconstant critical point of (45) is positive in \( \Omega \) as consequence of the Hopf boundary point lemma (HBPL). Thus, \( u > \xi_- \) in \( \Omega \) or, in terms of the original variables, \( n_0 > e^{\xi_-} \). Finally, the boundedness of \( \omega \) also holds in \( \Omega \) as a result of a Sobolev inequality. The technical aspects related to these properties are postponed to an Appendix.

Appendix. We sketch here the application of the MPT (introduced by A. Ambrosetti and P. H. Rabinowitz in the pioneering work [1]) carried out in the proof of Theorem 3.1, as well as the arguments that allow us to conclude that the critical point given by the MPT is in our case nonconstant, positive and bounded (full details for a similar situation can be found in [27, 31]). First of all, we recall the statement of the MPT and (a simplified version of) the HBPL for the sake of a self-contained presentation.

**Theorem 3.2 (MPT).** Let \( X \) be a Banach space and \( I \in C^1(X, \mathbb{R}) \) such that

(i) There exist \( r, \alpha > 0 \) such that \( I > 0 \) in \( B_r \setminus \{0\} \) and \( I \geq \alpha \) on \( \partial B_r \);

(ii) there exists \( 0 \neq x \in X \) such that \( I(x) = 0 \);

(iii) [Palais-Smale condition] if \( \{u_n\} \subset X \) is such that \( I(u_n) > 0 \), \( I(u_n) \) is bounded and \( I'(u_n) \to 0 \) as \( n \to \infty \), then \( \{u_n\} \) admits a convergent subsequence.

Then,
\[ m = \inf_{f \in \Gamma} \max_{0 \leq z \leq 1} \{I(f(z))\} \tag{46} \]

is a critical value of \( I \) with \( 0 < \alpha \leq m < \infty \), where we denoted
\[ \Gamma = \{ f \in C([0, 1], X) : f(0) = 0, \ f(1) = x \}. \]

The proof of the following result is available for instance in [24].

**Lemma 3.3 (HBPL).** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with \( n \geq 2 \) and \( \Delta_x u \geq 0 \) in \( \Omega \). Let also \( x_0 \in \partial \Omega \) be such that

(i) \( u \) is continuous at \( x_0 \);

(ii) \( u(x_0) > u(x) \) for all \( x \in \Omega \);

(iii) [Interior sphere condition] there exists a ball \( B \subset \Omega \) with \( x_0 \in \partial B \).
Then
\[ \frac{\partial u}{\partial \eta}(x_0) > 0, \]
where \( \eta \) denotes the unit normal vector exterior to \( \Omega \).

We organize the subjects of our study in four steps.

**Step 1: I admits a nontrivial critical point.** To prove this claim we make essential use of Theorem 3.2. We first define
\[ H(\omega) := \int_0^\omega F(z) \, dz = e^\omega - \frac{1}{2} \omega^2 - \omega - 1 \]
and remark that

**Lemma 3.4.** The following assertions hold true:

(i) \( F(\omega) \leq e^\omega \).
(ii) If \( \theta > \frac{1}{3} \), then \( H(\omega) \leq \theta \omega F(\omega) \) for all \( \omega \geq 0 \).

**Proof.** (i) is an obvious consequence of the definition of \( F \). To verify (ii), we define
\[ G(\omega) := \theta \omega F(\omega) - H(\omega) = (\theta \omega - 1)F(\omega) + \frac{1}{2} \omega^2 \]
and notice that \( G(0) = 0 \) and
\[ G'(\omega) = \omega + \theta F(\omega) + (\theta \omega - 1)F'(\omega) > \frac{1}{3}(\omega + 2 + (\omega - 2)e^\omega) \geq 0, \]
for all \( \omega \geq 0 \).

Then, the Palais-Smale condition is satisfied by virtue of the following result ([1], Lemma 3.6).

**Lemma 3.5.** Let \( L \) be the second order uniformly elliptic operator given by
\[ L[u] = -\sum_{i,j=1}^{N} (a_{ij}(x)u_{x_i})_{x_j} + c(x)u \]
and consider the partial differential equation \( L[u] = f \). Assume that \( f = f(x, u) \) satisfies the following two conditions:

(i) There exists \( a > 0 \) such that \( |f(x, u)| \leq ae^k(u) \), where \( \frac{k(u)}{u} \to 0 \) as \( u \to \infty \).
(ii) There exists \( b > 0 \) and \( \theta \in [0, \frac{1}{2}] \) such that \( \int_0^u f(x, t) \, dt \leq \theta uf(x, u) \) for \( |u| \geq b \).

Then, \( I \) satisfies the Palais-Smale condition (statement (iii) in Theorem 3.2).

Furthermore, by noting that
\[ \frac{F(\omega)}{\omega} \to 0 \quad \text{as} \quad \omega \to 0, \]  
the following result (Lemma 3.3 in [1]) also applies to meet assumption (i) of Theorem 3.2.

**Lemma 3.6.** Let \( f \) fulfill the condition (i) of Lemma 3.5 as well as

(iii) \( f(x, \omega) = o(|\omega|) \) at \( \omega = 0 \) uniformly in \( x \in \Omega \).

Then, \( I \) satisfies the statement (i) of Theorem 3.2.
Therefore, there only remains to prove that assumption (ii) is satisfied in order that Theorem 3.2 can be applied to yield the existence of a nontrivial critical point of \( I \). Without loss of generality, we can assume that \( 0 \in \Omega \). Then, define

\[
H^1(\Omega) \ni \phi(x) = \begin{cases} \frac{1}{D} (1 - \frac{\delta}{\sqrt{D}}) , & |x| \leq \sqrt{D} \\ 0 , & |x| \geq \sqrt{D} \end{cases}
\]

with \( D > 0 \) as in (45), and \( h(t) := I(t\phi) \). By assumption (i) of Theorem 3.2, \( h(t) > 0 \) if \( t > 0 \) is small enough. The existence of \( 0 < t_1 := kD < t_2 \) (for some constant \( k > 0 \)) such that \( h'(t) < 0 \) if \( t > t_1 \) and \( h(t) < 0 \) if \( t > t_2 \) can also be shown (see Lemma 2.4 in [31]), so that there exists \( t_0 > 0 \) satisfying \( h(t_0) = 0 \), which makes the hypothesis (ii) of Theorem 3.2 to be fulfilled.

Now, Theorem 3.2 can be applied with \( X = H^1(\Omega) \) and \( x = t_0 \phi \) to conclude the existence of a positive critical value of \( I \) given by (46).

**Step 2: \( I \) admits a nonconstant critical point.** To prove this, we first observe that (cf. (45))

\[
I(t\phi) = \frac{\pi(6 + \sigma)}{12D} t^2 - (1 - \sigma) \int_\Omega \left( e^{t\phi(x)} - \frac{t^2}{2} \phi(x)^2 - t\phi(x) - 1 \right) dx
\]

by virtue of

\[
\int_\Omega |\phi(x)|^2 dx = \frac{\pi}{6D}, \quad \int_\Omega |\nabla \phi(x)|^2 dx = \frac{\pi}{D^2},
\]

as follows from straightforward calculations.

Let \( \delta = \frac{\pi k^2 (6 + \sigma)}{12} \). Then,

\[
\max_{t \geq 0} \{ h(t) \} = \max_{0 \leq t \leq t_1} \{ h(t) \} \leq \frac{\pi(6 + \sigma)}{12D} t_1^2 = \delta D.
\]

Consider now the set of positive constant solutions to Eq. (43), namely

\[
\Lambda = \left\{ z > 0 : e^z - z - 1 = \frac{\sigma}{1 - \sigma} \right\},
\]

and define

\[
\gamma := \inf_{z \in \Lambda} \left\{ \frac{1}{2} z^2 - (1 - \sigma)(e^z - z - 1) \right\}.
\]

We have \( \Lambda \neq \emptyset \) and \( \gamma > 0 \). Indeed, the function \( g(z) = e^z - (1 + \frac{\sigma}{1 - \sigma}) z - 1 \) satisfies \( g(0) = 0 \), decreases until reaching its minimum value at \( z_{\min} = -\log(1 - \sigma) \) and then increases unboundedly, so there exists \( z_+ > 0 \) such that \( g(z_+) = 0 \). Hence, \( z_+ \in \Lambda \). On the other hand, the function \( M(z) = \frac{1}{2} z^2 - (1 - \sigma)(e^z - z - 1) \) is rewritten as simply \( M(z) = \frac{1}{2} z^2 - \sigma z \) when acting on \( \Lambda \), which is shown to be positive for all \( z > 2\sigma \). Finally, the positivity of \( \gamma \) follows from the fact that \( g(2\sigma) < 0 \), which entails \( z_+ > 2\sigma \). To conclude, it suffices to choose \( D \) small enough so that \( \delta D < \gamma |\Omega| \). In that case, we get from (46) and (48)

\[
m \leq \max_{0 \leq t \leq t_1} \{ I(t\phi) \} < \gamma |\Omega|,
\]

which tells us that not all the critical points of \( I \) are constant, since otherwise we would have

\[
\gamma |\Omega| > m = I(z) = \left( \frac{1}{2} z^2 - (1 - \sigma)(e^z - z - 1) \right) |\Omega|,
\]

which leads to a contradiction by virtue of the definition of \( \gamma \).
Step 3: Boundedness of the nonconstant critical point. By multiplying Eq. (43) times $\omega^{2s-1}$, with $s \geq 1$, and after integrating by parts in the first term of the left-hand side, we find

$$\left(\frac{2s-1}{s^2}\right) D \int_\Omega |\nabla x\omega|^2 dx + \sigma \int_\Omega \omega^{2s} dx = (1 - \sigma) \int_\Omega \omega^{2s-1} F(\omega) dx. \quad (49)$$

In the particular case $s = 1$, the estimate

$$\int_\Omega (D|\nabla x\omega|^2 + \sigma \omega^2) dx = (1 - \sigma) \int_\Omega \omega F(\omega) dx \leq \frac{2}{(1 - 2\theta)} I[\omega]$$

holds, as follows from (45) and the statement (ii) of Lemma 3.4 with $\frac{1}{3} < \theta < \frac{1}{2}$. Also, if $\omega$ is such that $I[\omega] = m$ (cf. Step 2), then (48) applies to yield

$$\int_\Omega (D|\nabla x\omega|^2 + \sigma \omega^2) dx \leq \varepsilon D, \quad (50)$$

where we denoted $\varepsilon = \frac{2}{(1 - 2\theta)} > 0$. We will also use the following Sobolev inequality (see formula (3.2) in [27] and the references invoked there):

$$\|\omega\|_{L^p(\Omega)} \leq C \left(\frac{p}{2}\right)^{\frac{p+2}{p}} D^{\frac{2}{p}-1} \int_\Omega (D|\nabla x\omega|^2 dx + \sigma \omega^2) dx \leq C \left(\frac{p}{2}\right)^{\frac{p+2}{p}} D^{\frac{2}{p}}, \quad (51)$$

where $1 \leq p < \infty$ and where $C$ denotes different positive constants here and hereafter. Now, we can make use of the Taylor expansion of $F^2$ to estimate

$$\int_\Omega F(\omega)^2 dx = \sum_{j=2}^{\infty} \int_\Omega (2^j - 2\omega - 2) \frac{\omega^j}{j!} dx \leq CD, \quad (52)$$

by virtue of (51) as well as of the convergence of the numerical series involved. Applying the Cauchy-Schwarz inequality to the right-hand side of (49) along with (52) leads to

$$\int_\Omega \omega^{2s-1} F(\omega) dx \leq C \sqrt{D} \left(\int_\Omega \omega^{4s-2} dx\right)^{\frac{1}{2}}. \quad (53)$$

Then, the first inequality in (51) combined with (49) and (53) yields

$$\|\omega^s\|_{L^2(\Omega)} \leq CD^{\frac{s}{2}-\frac{1}{2}} \left(\int_\Omega \omega^{4s-2} dx\right)^{\frac{1}{2}}, \quad (54)$$

where we used the elementary inequality $1 \leq \frac{s^2}{2s^2-1} \leq s$ for $s \geq 1$. Finally, two sequences $s_j$ and $M_j$ are introduced such that

$$4s_0 - 2 = q, \quad 4s_{j+1} - 2 = qs_j \quad \text{for } j = 0, 1, 2...$$

which can be explicitly written as $s_j = (\frac{q}{4})^j \left(s_0 + \frac{2}{q-4}\right) - \frac{2}{q-4}$, and

$$M_0 = C^\frac{q}{4}, \quad M_{j+1} = (C s_j)^\frac{q}{2} M_j^\frac{q}{2} \quad \text{for } j = 0, 1, 2...$$

An inductive procedure allows us to deduce that

$$\int_\Omega \omega^{4s_j-2} dx \leq M_j D, \quad M_j \leq e^{a s_j-1} \text{ for some } a > 0.$$
Hence, the estimate \( \| \omega \|_{L^\infty(\Omega)} \leq e^{\frac{e}{q}} \) follows for \( q > 4 \) after letting \( j \to \infty \) in
\[
\| \omega \|_{L^{\infty}(\Omega)}^{q_j - 1} \leq CD^{1 - \frac{2}{q} \frac{q}{q_j - 1}} \left( \int_\Omega \omega^{4^{q_j - 1} - 2} \, dx \right)^{\frac{q}{q_j - 1}} \leq DM_j \leq De^{a_{q_j - 1}}. 
\]
This particularly implies that \( n_0 \leq e^{\xi + \epsilon_0} \) according to (41).

**Step 4: Positivity of the nonconstant critical point.** Given \( \omega \in H^1(\Omega) \) a critical point of \( I \), it necessarily solves Eq. (43) subject to the boundary condition (24). Since \( F \in C^1(\mathbb{R}) \), \( F(0) = 0 \) and \( \omega \) is bounded by virtue of Step 3, the chain rule for Sobolev spaces implies that \( F \circ \omega \) belongs to \( H^1(\Omega) \), thus the elliptic regularity theorem along with the Sobolev inequalities in dimension two apply in our case to give \( \omega \in C^2(\Omega) \). We now argue by contradiction. Indeed, let us assume that \( \inf_\Omega \omega < 0 \). Then, there exists \( B_0 \subset \Omega \) and \( x_0 \in \Omega \) such that \( \omega(x_0) = \inf_\Omega \omega \) with \( x_0 \in \partial B_0 \) and \( \omega < 0 \) in \( B_0 \). Hence, \( F(\omega) = 0 \) and \( \Delta_x \omega = \frac{\partial^2}{\partial x} \omega < 0 \) in \( B_0 \) according to (44) and (43), respectively. As consequence,
\[
\frac{\partial \omega}{\partial \eta}(x_0) < 0
\]
by virtue of Hopf’s Lemma 3.3. This is already a contradiction, since if \( x_0 \in \Omega \) then \( \nabla_x \omega(x_0) = 0 \), while if \( x_0 \in \partial \Omega \) then \( \frac{\partial \omega}{\partial \eta}(x_0) = 0 \). Thus, \( \omega \geq 0 \) in \( \Omega \). To check the strict positivity, let us now assume that \( \inf_\Omega \omega = 0 \). In this case, there exists \( x_1 \in \overline{\Omega} \) and \( B_1 \subset \Omega \) such that \( \omega(x_1) = 0 \) with \( x_1 \in \partial B_1 \) and \( \omega \geq 0 \) in \( B_1 \). Eq. (43) can then be rewritten as
\[
\Delta_x \omega = \frac{1}{D} \left( \sigma - (1 - \sigma) \frac{e^\omega - \omega - 1}{\omega} \right) \omega \geq 0
\]
in \( B_1 \) (recall the discussion on the behaviour of the function \( g(z) \) in Step 2). Thus, Lemma 3.3 applies again to guarantee that
\[
\frac{\partial \omega}{\partial \eta}(x_1) > 0 \quad (54)
\]
unless \( \omega = 0 \) in \( B_1 \). Repeating the same arguments as for \( x_0 \), the condition (54) is easily shown to lead to a contradiction. On the other hand, \( \omega = 0 \) in \( B_1 \) entails \( \omega = 0 \) in \( \Omega \), which contradicts the fact that \( \omega \) is nonconstant by hypothesis. Consequently \( \omega \) is positive in \( \Omega \), which means that \( n_0 > e^{\xi} \) in \( \Omega \) according to (41) and (38).

This completes the proof of Theorem 3.1.

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E-mail address: jllopez@ugr.es