Quantum chaos in $SU_3$ models with trapped ions

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A scheme to generate long-range spin-spin interactions between three-level ions in a chain is presented, providing a feasible experimental route to the rich physics of well-known $SU_3$ models. In particular, we demonstrate different signatures of quantum chaos which can be controlled and observed in experiments with trapped ions.

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One of the current trends in quantum physics is the quest for controllable quantum many-body systems which can be used as quantum simulators [1, 2]. In particular, there is a growing interest in simulating spin and quantum magnetism. In recent years, the focus is moving from $SU_2$ spins towards $SU_N$ [3, 4], which can be realized in earth alkalines, or mixed spin spaces [5]. Here we show an implementation of $SU_3$ physics with trapped ions which are known to provide a large degree of control from the experimental point of view.

An important feature of quantum simulators based on ions is the possibility of studying long-range interactions, which are notoriously difficult to simulate classically [6, 7]. The implementation is based on spin dependent forces on the ions [8], which have been experimentally achieved recently [9–12]. These interactions lead to new phases, like exotic forms of superfluidity [13], supersolids [14], quantum crystals, and devil’s staircase [15, 16].

We concentrate on an important aspect present in $SU_3$ models: quantum chaos [17, 18]. Quantum chaos, opposed to classical chaos which can be defined by exponentially fast growing distance of phase space trajectories, was strongly driven by the understanding of the spectral properties of quantum many-body systems [19]. The large degree of control offered by experiments with ultracold atomic gases has triggered a vast number of experiments to look for different signatures of chaos [20]. Prominent examples are the observation of dynamical tunneling phenomena [21, 22], and more recently, the implementation of the kicked-top Hamiltonian on a single atom experiment [23]. Recent proposals look for signatures of chaotic behavior in spin-orbit coupled condensates [24] or in kicked Bose-Hubbard dimers [25].

In this letter, we demonstrate that the extremely long-range character of interactions between ions can be used to mimic shell models which are paradigmatic of quantum chaos [17, 18, 26]. We calculate experimentally controllable signatures of chaos, and estimate the fidelity of the proposed simulation in the Supplementary Material.

Spin-spin interactions of ions: The main ingredient required to achieve a strong and controllable spin interaction between trapped ions is the implementation of a state-dependent force on the ions. In an early proposal by Mintert and Wunderlich [27], such force is induced by a magnetic field gradient. More flexible proposals provide a force by a Raman coupling of two spin states, which can give rise to phonon excitations. While the proposal by Porras and Cirac [8] uses standing waves in all spatial direction, and thereby allows for up to three independent spin-spin interactions, experimental implementations have provided Ising-type spin-spin interactions using a Raman coupling due to a pair of propagating waves with a wave vector difference transverse to the ion alignment [9, 10]. By choosing the frequency difference between the two lasers close to a vibrational sideband transition, one can enhance and control the phonon excitations, and thus the Ising coupling constants. In particular, single vibrational modes can be selected.

Here we generalize this scheme to three couplings in a system of three-level ions, which allows for an implementation of $SU(3)$ spin models. All spin states, for instance represented by three hyperfine states $|1\rangle \equiv |F = 0, m = 0\rangle$, $|2\rangle \equiv |F = 1, m = -1\rangle$, and $|3\rangle \equiv |F = 1, m = 1\rangle$ in the $S_{1/2}$ manifold of Yb$^{+}$, are coupled as depicted in Fig. 1(a), via off-resonant transitions to an excited state $|e\rangle \equiv |F = 0\rangle$ in the $F_{1/2}$ manifold. The lasers are arranged as shown in Fig. 1(b), such that each Raman pair has a wave vector difference $\delta k_\alpha$ along a dif-

![Figure 1](image-url)
ferent spatial direction $\alpha = x, y, z$. Interference between different couplings can be avoided by choosing different detunings $\Delta_\alpha$ from $|e_i\rangle$.

Each coupling can be equipped with beat notes $\omega_\alpha \pm \mu_\alpha$, where $\mu_\alpha$ describes the detuning from the atomic transition frequency $\omega_\alpha$, given by the hyperfine splitting and/or the Zeeman splitting. After adiabatic elimination of the excited state $|e_i\rangle$ and a rotating wave approximation, the Hamiltonian of each coupling reads:

$$h_\alpha(t) = \hbar \sum_i \Omega_\alpha^i \sin(\mu_\alpha t) \delta \mathbf{k}_\alpha \cdot \mathbf{x}^i \cdot \tau^i_\alpha. \tag{1}$$

Here, $\Omega_\alpha^i$ is the two-photon Rabi frequency of the Raman transition $\alpha$ at the position of ion $i$. The spin-flip operators are defined as $\tau^i_\alpha = |1 > < 2| + \text{H.c.}$, $\tau^i_\beta = |2 > < 3| + \text{H.c.}$, and $\tau_z = |2 > < 3| + \text{H.c.}$. Next, we rewrite the position operator in terms of normal coordinates, $\mathbf{M}$, where

$$\mathbf{M} = \text{Lamb-Dicke parameter}$$

where $\mathbf{M}$ is the transition dipole moment, and $\mathbf{M}$ is the normal coordinate in direction $\alpha$. We have introduced the Lamb-Dicke parameter $\eta_{m\alpha}$ characterizing the strength of the spin-phonon coupling. For the validity of Hamiltonian $1$, it is necessary to have $\eta_{m\alpha} \ll 1$. It is explicitly given by

$$\eta_{m\alpha} = \sqrt{\frac{1}{\omega_m^2 M_m}}. \tag{2}$$

where $c_x, c_y = 1, c_z = -2$. We have chosen “ionic” units, in which besides the electric constant $1/(4\pi\epsilon_0)$ also the ion mass $M$, the ion charge $q$, and the equilibrium distance $d$ of neighboring ions in the chain are set to unity. Frequencies are then given in units of $\omega_0 \equiv q/(d\sqrt{4\pi\epsilon_0 M})$.

It has been shown in Ref. $10$ that the time evolution under the Hamiltonian of Eq. $1$ is (to second order in the exponent) given by

$$U_\alpha(t, 0) = \exp \left[ \sum_i \varphi_\alpha^i (t) \tau^i_\alpha - \sum_{ij} \xi_\alpha^{(i,j)} (t) \tau^i_\alpha \tau^j_\beta \right], \tag{3}$$

where $\varphi_\alpha^i = \sum_m (\xi_{m\alpha}^i) t^b_{m\alpha} - \text{H.c.}$ contains a residual spin-phonon coupling, while the second term describes a spin-spin coupling. Both functions $\xi_{m\alpha}^i (t)$ and $\xi_\alpha^{(i,j)} (t)$ consist of oscillatory terms (with frequencies $\mu_\alpha$, $\mu_\alpha \pm \omega_{m\alpha}$), which are suppressed by at least one power of $\eta_{m\alpha}^i / |\mu_\alpha - \omega_{m\alpha}| \ll 1$ for sufficiently large detuning from the sideline. In this limit, the dominant contribution to the time evolution stems from a single term in $\xi_\alpha^{(i,j)} (t)$ which is linear in $t$, and thus increases constantly.

Thus, we can set $\xi_\alpha^{(i,j)} \approx 0$, and $\xi_\alpha^{(i,j)} (t) \approx iJ_{ij}^\alpha t$, with

$$J_{ij}^\alpha = \Omega_\alpha^i \Omega_\beta^j \sum_m \eta_{m\alpha}^i \eta_{m\beta}^j m_{\alpha \beta}^m \mu_\alpha^m - \omega_{m\alpha}^2 \tag{4}$$

The time evolution is thus identical to the one of a spin model with spin-spin coupling $J_{ij}^\alpha$.

In the presence of more than one couplings, since $\{\eta_c, \eta_z\} \approx 0$, the time evolution is not simply the product of all $U_\alpha$, but consists of additional terms. Up to second order in the Magnus expansion $28$, it is equal to $U \approx \sum (U_{\alpha}) (U_{\alpha \beta})$ with $U_{\alpha \beta} = \exp \left\{ \sum_{ij} \chi_{ij} (t) \tau^i_\alpha \tau^j_\beta \right\}$. The functions $\chi_{ij}$ are given by the integral

$$\chi_{ij} = \sum_{m,n} \int_0^t \int_0^t dt_1 \int_0^t dt_2 \sin \mu_\alpha t_1 \sin \mu_\beta t_2 \times \left( a_{m\alpha} e^{i\omega_{m\alpha} t_1} + \text{H.c.} \right) \times \left( a_{m\beta} e^{i\omega_{m\beta} t_2} + \text{H.c.} \right). \tag{5}$$

For $\mu_\alpha = \mu_\beta$ and $\omega_{m\alpha} = \omega_{m\beta}$, this function is similar to $\xi_\alpha^{(i,j)} (t)$ with small oscillating terms, and one dominant term linear in $t$. However, if couplings $\alpha$ and $\beta$ have different beat notes and/or the mode frequencies are different (for instance by anisotropic transverse trapping), the linear term disappears, and this contribution can be neglected.

In addition to the spin-spin coupling, a magnetic field term can be generated by a resonant carrier transition $3, 29$, leading to the effective spin Hamiltonian

$$H_{\text{spin}} = \sum_\alpha \sum_{i=1}^N B^{(i)} \sigma_{\alpha}^i + \sum_\alpha \sum_{i<j} J_{ij}^\alpha \tau^i_\alpha \tau^j_\beta, \tag{6}$$

with $\sigma_\alpha^i \equiv |1 > < 2| - |3 > < 4|$. The couplings $J_{ij}^\alpha$ given by Eq. $4$, can be tuned by the beat notes $\mu_\alpha$. In particular, by choosing $\mu_\alpha$ sufficiently close to the center-of-mass (COM) mode in each direction $J_{ij}^\alpha$ will have only weak dependence on the ion positions $i$ and $j$. Most generally, we assume an inhomogeneous magnetic field $B^{(i)}$.

**Effective SU$_3$ Shell Model:** In the limit where $J_{ij}^{(j)} = J_{ij} \approx \text{constant} < 0$, it is convenient to define spin-flip operators $S_{\sigma \sigma'} = \sum_{i=1}^N |\sigma \rangle \langle \sigma'|^{(i)}$, acting equally on all spins. Since we have $S_{11} + S_{22} + S_{33} = N$, the $S_{\sigma \sigma'}$ provide eight independent operators spanning the SU$_3$ algebra. For simplicity, we set $J_{ij} = J$ and the magnetic field homogeneous. Defining a symmetrized spin operator $S_{\sigma \sigma'} = (S_{\sigma \sigma'} + S_{\sigma' \sigma}) / \sqrt{2}$, we may re-write the spin Hamiltonian of Eq. $4$ as an ideal model Hamiltonian in terms of these SU$_3$ operators:

$$H_{\text{ideal}} = \frac{B}{\sqrt{2}} (\hat{S}_{11} - \hat{S}_{33}) + J \sum_{\sigma \sigma'} \hat{S}_{\sigma \sigma'} \hat{S}_{\sigma' \sigma'} \tag{7}$$
besides the replacement $S_{\sigma \sigma'} \rightarrow \tilde{S}_{\sigma \sigma'}$, this hamiltonian is identical to the three-level lipkin-meshkov-glick (LMG) Hamiltonian [30], a model, where particles can occupy three different shells with single-particle energies $-B, 0, B$. Two-body interactions of particles in the same shell lead to pair-tunneling into the other shells. The LMG Hamiltonian has applications in nuclear physics, and its three-level version is particularly appealing as a not fully integrable spin model in the context of quantum chaos [18, 31].

our Hamiltonian (7) differs from the LMG Hamiltonian only by an additional interaction $\sum_{\sigma \neq \sigma'} S_{\sigma \sigma'} S_{\sigma' \sigma}$. Due to its SU3 symmetry, and the fact that this additional term is a Casimir operator of SU3 [18], the Hamiltonian (7) and the LMG Hamiltonian are fully equivalent: Having a block-diagonal structure with respect to different representations of SU3, in each block the additional interaction simply reduces to a constant.

apart from particle exchange symmetry, the LMG model has a second symmetry [31]: As particles can change the spin state only pairwise, the occupation numbers of each spin state, $\langle S_{11} \rangle$, $\langle S_{22} \rangle$, and $\langle S_{33} \rangle$, can only change by two, and thus are fixed to either even ($e$) or odd ($o$) values. This gives rise to four signature classes, eee, eeo, ooe, ooo for $N$ even, or ooo, eeo, ooe, ooe for $N$ odd. This signature class symmetry is also present in Eq. (6), that is, in the model with a space-dependent coupling $J_{\alpha}^{(i,j)}$ given by Eq. (4). On the other hand, the spin exchange symmetry is lost. Still there is invariance under parity, as $J^{(i,j)} = J(N-i)(N-j)$, due to the parity invariance of $K$. For a numerical diagonalization of this Hamiltonian, it is convenient to construct the eigenbasis of parity and signature class. While the Fock states are already signature eigenstates, a combination of at most two Fock states also yields a parity eigenstate.

quantum chaos in the LMG model: In the classical limit of the three-level LMG model, its phase-space can be divided into regions of chaotic and regular motion [31]. Accordingly, also the quantum model shows signatures of both chaotic and regular behavior. While in chaotic quantum systems the spectrum features level repulsion, regular behavior is related to level clustering. These features are nicely displayed by the unfolded distribution $P_{\text{unf}}$ of the level spacings $s$ in the spectrum. A Poisson distribution, $P(s) = e^{-s}$, indicates level clustering, while chaotic Hamiltonians with time-reversal invariance follow a Wigner distribution, $P(s) = (\pi/2)s \exp[-\pi s^2/4]$. In Ref. [31], it has been shown for the LMG model that one part of the spectrum is spaced according to the Poisson distribution, while another part follows a Wigner spacing.

This results in a level spacing distribution as shown in Fig. 2(a) for $N = 10$ and a magnetic field $B = \langle J_{\alpha}^{(i,j)} \rangle / 2$. We have used the realistic Hamiltonian (6), with a trap frequency $\omega_{s} = 0.1 \omega_{s}/\delta_{s}$ at a relative detuning from the COM mode, $\delta_{s} = (\omega_{\text{COMa}} - \mu_{s})$. As is shown in the Supplementary Material, this Hamiltonian reproduces with high fidelity the physics of the ideal model, Eq. (7). We have unfolded the spectrum separately in each symmetry block of the Hamiltonian (that is for fixed parity and signature class). The level spacing distribution is found to be broader than the Wigner distribution, and has its maximum shifted towards smaller spacings. This suggests to consider the Brody distribution $P_{q}(s)$ which interpolates between the Wigner ($q = 1$) and the Poisson distribution ($q = 0$) [32, 33].

$$P_{q}(s) = \alpha(q+1)s^{q}\exp[-\alpha s^{q+1}]$$

with $\alpha = [\Gamma((q+2)/(q+1))]^{q+1}$. The value of $q$ provides a measure of the degree of chaoticity in the system. As shown in Fig. 2(a), our distribution is well represented by $q = 0.7$.

The behavior expressed by these statistics can be illustrated by representing the evolution of a few energy levels when one parameter of the Hamiltonian is changed, e.g. the magnetic field strength $B$. In each symmetry block, we find both level crossings and avoided level crossings, as already expected from the level spacing distribution. In Fig. 2(b), we illustrate, for $N = 4$, a part of the spectrum where all level crossings belonging to states of the same symmetry (parity) are avoided. Of course, the crossings between states of different parity are not avoided. In Fig. 2(c), we then show the same part of the spectrum in the presence of an additional small magnetic field gradient $\delta B = 0.2$, that is for an inhomogeneous magnetic field $B_{\text{inhom}}(x) = B + \delta B x$. This contribution breaks the parity symmetry, turning the previously symmetry allowed level crossings, Fig. 2(b), into avoided ones.

Experimental detection of quantum chaos: The signatures of quantum chaos discussed so far are hard to measure in a system of trapped ions. More easily, quantum chaos can be detected by preparing the system...
of quantum chaos mirroring the usual one in classical systems: exponential sensitivity to initial conditions. Instead, for quantum-chaotic motion it is argued that a relevant signature is provided by high sensitivity of the time evolution onto slight changes in the Hamiltonian. In our scheme, the occupation of the level \( |m = 0 \rangle \) can be measured with high precision by resonantly exciting ions from this level, and observing the subsequent fluorescence. For a system prepared in the regular state a regular pattern is expected, while an erratic pattern is a signature of chaotic motion [17]. We exemplify this in Fig. 3(b,c), showing the time evolution of \( \langle S_{11} \rangle \) for the regular and the chaotic state given above, evolved with \( B = 0.5 \) in the full SU3 Hamiltonian and in the reduced SU2 Hamiltonian. The curves for SU2 clearly show a regular pattern, while in the SU3 case the differences between the chaotic and the regular state are less obvious. We therefore perform a Fourier analysis of these curves after subtracting its average and normalizing the amplitude of the oscillation. In the Fourier spectrum, shown in Fig. 3(d), the regular evolution is dominated by only a few peaks, while the spectrum of the chaotic evolution is much more diversified.

**Summary:** We have presented a scheme to realize SU3 spin models with trapped ions. The spin-spin interaction parameter is tunable, allowing the simulation of spin models with long-range interactions. In this paper, we have focused on the situation where interactions are almost constant with respect to the position of the ions. Then the systems describes a three-level LMG model which interpolates between quasi-integrable and chaotic dynamics. Chaos on the quantum level is usually characterized in terms of the spectral statistics which demands high-precision measurements of many energy levels, hardly possible in the settings similar to the one discussed in our paper. Instead we propose a feasible scheme to detect quantum chaos in the dynamics of the model.

![Image](image-url)
Our proposal thus provides a powerful experimental tool to study the onset and signatures of chaos in quantum systems, based on state-of-the-art techniques.

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