QUASI-ISOMETRIES, BOUNDARIES AND JSJ-DECOMPOSITIONS OF RELATIVELY HYPERBOLIC GROUPS

BRADLEY W. GROFF
THE UNIVERSITY OF ILLINOIS AT CHICAGO

Abstract. We demonstrate the quasi-isometry invariance of two important geometric structures for relatively hyperbolic groups: the coned space and the cusped space. As applications, we produce a JSJ-decomposition for relatively hyperbolic groups which is invariant under quasi-isometries and outer automorphisms, as well as a related splitting of the quasi-isometry groups of relatively hyperbolic groups.

1. Introduction

In [Gro87], Gromov introduced relatively hyperbolic groups as a generalization of δ-hyperbolic groups. The first author to elaborate on this idea was Farb, with a new definition in his thesis [Far94]. The equivalence between Gromov’s definition and Farb’s relatively hyperbolic with bounded coset penetration is established in [Bow97], and the necessity of BCP is demonstrated in [Szc98]. Following this, there have been contributions by Bowditch [Bow97] and many others, often describing these groups through new definitions [DS05, GM08, Osi06, Sis12b, Yam04]. It is important to note that the notion of relative hyperbolicity only makes sense given a chosen collection of subgroups so it is more accurate to say that \( \Gamma \) is hyperbolic relative to \( \mathcal{A} \). For instance, hyperbolic groups, which are naturally hyperbolic relative to \( \{1\} \), can be investigated through their non-trivial relatively hyperbolic structures, e.g. the free group \( F(a,b) \) is hyperbolic relative to \( \langle [a,b] \rangle \).

These groups have enough structure to make them fruitful objects of study and they naturally occur in many contexts throughout modern geometric group theory. For instance:

- \( \pi_1(M) \) for \( M \) a complete, finite volume manifold with pinched negative sectional curvature is hyperbolic relative to cusp subgroups [Far98, Bow98];
- the fundamental group of a graph of groups with finite edge groups is hyperbolic relative to vertex groups [Bow98];
- a limit group is hyperbolic relative to maximal non-cyclic abelian subgroups [Ali05, Dah03];
- a group acting geometrically on CAT(0) spaces with isolated flats is hyperbolic relative to the stabilizers of maximal flats [DS05, HK05];
- a hyperbolic group is hyperbolic relative to \( \{1\} \).

For our purposes, the Groves-Manning description of relative hyperbolicity is ideal because of the combinatorial simplicity of computing path lengths. This computation, combined with the geometric insight provided by [Dru09], is the crucial component for the proof that quasi-isometries of Cayley graphs induce quasi-isometries of cusped spaces.

**Theorem 6.3** Let \( \Gamma_1 \) and \( \Gamma_2 \) be finitely generated groups. Suppose that \( \Gamma_1 \) is hyperbolic relative to a finite collection \( \mathcal{A}_1 \) such that no \( A \in \mathcal{A} \) is properly relatively hyperbolic. Let \( q : \Gamma_1 \to \Gamma_2 \) be a quasi-isometry of groups. Then there exists \( \mathcal{A}_2 \), a collection of subgroups of \( \Gamma_2 \), such that the cusped space of \( (\Gamma_1, \mathcal{A}_1) \) is quasi-isometric to that of \( (\Gamma_2, \mathcal{A}_2) \).

We remark that there is an analogue for the coned space (Theorem 6.4). Several corollaries follow from Theorem 6.3 including Corollary 6.7 exchanging finite generating sets does not change the quasi-isometry type of the cusped space, (Definition 2.3). However, our main application of Theorem 6.3 stems from the following:

Corollary 6.5. With $(\Gamma_1, A_1)$ and $(\Gamma_2, A_2)$ as in Theorem 6.3, the cusped spaces $X(\Gamma_1, A_1)$ and $X(\Gamma_2, A_2)$ have homeomorphic boundaries.

In [Bow98a, Bow01], the actions of hyperbolic and relatively hyperbolic groups on their boundaries are analyzed to produce JSJ and peripheral splittings, respectively. The topological features of the boundary (cut pairs and cut points, Definition 3.2) correspond to the limit sets of two-ended and peripheral subgroups in the boundary, respectively.

This approach was generalized to include groups acting on continua in [PS06]. The action is inherited by an $\mathbb{R}$-tree which is constructed from the topological features of the continuum (again, cut pairs and points). We pause to note that even in the context of hyperbolic groups this strategy cannot be extended to produce different splittings from larger separating sets such as Cantor sets, [DP10]. An application for this construction to CAT(0) groups appears in [PS09]. Here, the group action on the CAT(0) boundary is analyzed via a variant of the convergence property to determine the isomorphism types of the vertex groups.

In [PS06], the authors suggest that the cut-point/cut-pair tree may be useful in the analysis of relatively hyperbolic groups. We demonstrate that this is true. On our way to this, we prove that this tree is simplicial in Theorem 4.6. Further, the splitting produced satisfies our Definition 2.16 (for more generality, see [GL10a]) of a JSJ splitting over elementary subgroups relative to peripheral subgroups, Theorem 5.1.

We also note that in Section 1 of [Bow01] a project to extend the results of [Bow98a] to relatively hyperbolic groups is suggested. Here, the terminology local cut points is used in place of our cut pairs, Definition 3.2. By using the convergence property we are able to identify the vertex groups of the combined tree. Since this is a JSJ tree and it is quasi-isometry invariant, we effectively finish this project.

Theorem 7.5. Let $\Gamma_1$ and $\Gamma_2$ be finitely generated groups. Suppose additionally that $\Gamma_1$ is one-ended and hyperbolic relative to the finite collection $A_1$ of subgroups such that no $A \in A$ is properly relatively hyperbolic or contains an infinite torsion subgroup. Let $T$ be the cut-point/cut-pair tree of $\partial (\Gamma_1, A_1)$. If $f : \Gamma_1 \to \Gamma_2$ is a quasi-isometry then

• $T$ is the cut-point/cut-pair tree for $\Gamma_2$ with respect to the peripheral structure induced by Theorem 6.3,

• if $\text{Stab}_{\Gamma_2}(v)$ is one of the following types then $\text{Stab}_{\Gamma_2}(v)$ is of the same type,
  1. hyperbolic 2-ended,
  2. peripheral,
  3. relatively QH with finite fiber.

Moreover, by [MOY12, Corollary 7.3] the set of relatively hyperbolic structures on $\Gamma$ forms a partial order and by our condition on the parabolic subgroups, this peripheral structure is the unique maximal structure. Combining this with Corollary 6.5, we see that the action of $\text{Out}(\Gamma)$ induces homeomorphisms of $\partial \Gamma$ and preserves the JSJ tree. Thus, we get the following corollary.

Corollary 7.6. The JSJ tree corresponding to the cut-point / cut-pair tree is a fixed point for the action of $\text{Out}(\Gamma)$ on the JSJ deformation space.

Lastly, we obtain some results controlling the structure of $QI(\Gamma)$ by observing that it acts on the same tree. This action (which is shown to be faithful by Theorem 8.1) can be reasonably well understood by inspecting the quasi-isometry classes of the vertex stabilizers. Using this, we can find bounds for the number of edges in the quotient graph $T/QI$ and produce a splitting of $QI(\Gamma)$ which is induced by the splitting of $\Gamma$, Theorem 8.2.

2. Preliminaries

The following section is a brief summary of the various definitions and facts which we will use directly. We also include some discussion relating our results to other research. Given this brevity, we suggest several more comprehensive resources: for coarse geometry, $\delta$-hyperbolic spaces and groups, and their boundaries, see [BH99]; for relatively hyperbolic groups, [Bow97, Dru09, Far98, GM08, Hru10, Os06]; for constructing group actions on trees from group actions
2.1. Relatively Hyperbolic Groups and $X(\Gamma, A)$.

**Definition 2.1.** ([GM08]) Let $T$ be any graph with edges of length $1$. We define the **combinatorial horoball** based at $T$, $\mathcal{H}(=\mathcal{H}(T))$ to be the following 1-complex:

- $\mathcal{H}^{(0)} = VT \times \{\{0\} \cup \mathbb{N}\}$
- $\mathcal{H}^{(1)} = \{(t, n), (t, n + 1)\} \cup \{(t_1, n), (t_2, n)\} \mid d_T(t_1, t_2) \leq 2^n \}$. We call edges of the first set **vertical** and of the second **horizontal**.

**Remark.** In [GM08], the combinatorial horoball is described as a 2-complex because they needed the complex to be simply connected. As we are not concerned with that here, we only require the 1-skeleton.

**Definition 2.2.** ([GM08]) Let $D : \mathcal{H} \rightarrow [0, \infty)$ be defined by extending the map on vertices $(t, n) \rightarrow n$ linearly across edges. We call $D$ the **depth function** for $\mathcal{H}$ and refer to vertices $v$ with $D(v) = n$ as vertices of depth $n$ or depth $n$ vertices.

Because $T \times \{0\}$ is homeomorphic to $T$, we identify $T$ with $D^{-1}(0)$.

**Definition 2.3.** ([GM08]) Let $A$ be a collection of subgroups of $\Gamma$. The **cusped space** $X(\Gamma, S, A)$ is the union of $\Gamma$ with $\mathcal{H}(gA)$ for every coset of $A \in A$, identifying $gA$ with the depth 0 subset of $\mathcal{H}(gA)$. We suppress mention of $S, A$ when they are clear from the context.

For points in $X(\Gamma)$, we do not distinguish between the depth functions of distinct horoballs because horoballs only overlap at depth 0 and so this convention is unambiguous. Thus, we can refer to the depth of any vertex in $X(\Gamma)$ without mention of the associated horoball or coset.

**Definition 2.4.** The elements of the collection of subgroups $A$ are called **parabolic subgroups** and the subgroups which are conjugate to them are called **peripheral subgroups**.

**Definition 2.5.** ([GM08]) If $X(\Gamma, S, A)$ is $\delta$-hyperbolic then we say that $\Gamma$ is **hyperbolic relative to** $A$ or that $\Gamma$ is a **relatively hyperbolic group** or that the pair $(\Gamma, A)$ is **relatively hyperbolic**.

**Remark.** As mentioned in the introduction, there are many definitions of relatively hyperbolic groups. These definitions are all equivalent, with the exception of Farb’s in which the phrase “with bounded coset penetration” needs to be appended. Equivalence of Definition 2.3 with others is established in [GM08, Theorem 3.25].

**Remark.** Substituting $S$ for some other finite generating set $S'$ changes the topology of $X(\Gamma, S, A)$ and may change the value of $\delta$, but does not affect the hyperbolicity of the cusped space for some $\delta'$. Thus we omit $S$ from the definition above. See [GM08] or [Osi06] for this result in a different context.

**Remark.** In [Bra10, Section 4], the construction of the cusped space is extended to groups which only have a finite relative generating set (ie $\Gamma = \langle S \cup H \rangle$) by declaring a proper metric on peripheral subgroups. It seems reasonable to generalize our results in this fashion.

**Definition 2.6.** ([Bow97]) Given a group $\Gamma$ hyperbolic relative to $A$, the **Bowditch boundary**, $\partial(\Gamma, S, A)$ is the Gromov boundary of the associated cusp space, $\partial X(\Gamma, S, A)$. When there is no ambiguity we simply say the boundary.

**Remark.** In [Bow97], the boundary is defined as the ideal boundary of a proper, hyperbolic space on which the group acts. This is largely motivated by the definition for relatively hyperbolic groups given in [Gro87]. Part of the appeal of the characterization of relative hyperbolicity given in [GM08] is that it satisfies the conditions of the definition of [Gro87], and so it naturally fits with the notion of boundary developed in [Bow97]. In fact, Bowditch constructs a very similar space to the cusped space in which the edges at depth $n$ are shrunk by a factor of $2^n$ instead of adding extra edges. The two spaces are quasi-isometric by the natural map on vertices, so the boundaries are the same.
Lemma 2.7. [GM08, 3.11] If \( A \) is a combinatorial horoball, then the Gromov boundary consists of a single point, denoted \( e_A \), which can be represented by a geodesic ray containing only vertical edges.

Because our proof of Theorem 6.3 can be simplified to prove an analogous result for the coned space, we include its definition.

Definition 2.8. [Bow97] Let \( G \) be a group with a finite collection of subgroups \( B \). For every coset \( gB_i \) of an element of \( B \), we add to the Cayley graph a vertex \( v_{gB_i} \) and from every element of \( gB_i \) we add an edge to \( v_{gB_i} \). We call this the coned space of \((G, B)\).

Definition 2.9. [Bow97] A graph is fine if for every \( n > 0 \) and every edge \( e \), the number of cycles of length at most \( n \) containing \( e \) is finite.

Definition 2.10 (Bow97, Alternate characterization of relative hyperbolicity). Let \( C \) be the Cayley graph of a group \( G \) generated by a finite set \( S \) and let \( A \) be a collection of subgroups of \( G \). If the coned space of \((G, A)\) is hyperbolic and fine, then \((G, A)\) is relatively hyperbolic [Bow97, p. 27].

The following three definitions appear in Chapter 4 of [Osi06].

Definition 2.11. An element \( g \) of is called hyperbolic if \( g \) is not conjugate into any \( A_i \in A \).

Definition 2.12. A subgroup \( H \) of a relatively hyperbolic group \( \Gamma = \langle S | R \rangle \) is called relatively quasi-convex if there exists \( \sigma > 0 \) such that the following condition holds. Let \( f, g \) be two elements of \( H \), and \( p \) an arbitrary geodesic path from \( f \) to \( g \) in \( \text{Cay}(\Gamma, S \cup A) \). Then for any vertex \( v \in p \), there exists a vertex \( w \in H \) such that \( \text{dist}(u, w) \leq \sigma \).

Definition 2.13. If \( H \) is relatively quasi-convex and \( H \cap A^g \) is finite for all \( g \in \Gamma \) and \( A \in A \), then \( H \) is called strongly relatively quasi-convex.

Theorem 2.14 ([Osi06], Theorem 4.19). The centralizer of a hyperbolic element in a relatively hyperbolic group is strongly relatively quasi-convex.

Definition 2.15. A subgroup of a relatively hyperbolic group is called elementary if it is either virtually cyclic or a subgroup of a peripheral subgroup.

2.2. JSJ Decompositions. JSJ decompositions were first introduced in the context of 3-manifold topology in the works of Jaco and Shalen and, independently, Johannson [JS79, Joh79]. The manifold is separated by a maximal system of disjoint, embedded tori with the goal of understanding the structure of the ambient manifold by inspecting the individual components.

This idea was ported to geometric group theory originally by Kropholler [Kro90] and then by Rips and Sela [RS97]. Several authors have expanded on these ideas, including Bow98a, DS99, FP06, Pap05, PS09, and, in a slightly different manner, SS03. In the group theoretic setting, JSJ decompositions encode a maximal amount of the information related to two-ended splittings (or, in the case of [RS97], Z-splittings) as a graph of groups.

Several of these notions have recently been unified by the results in [GL07, GL10a, GL10b]. It is the language presented here which we adopt. Since we are interested only in a particular type of JSJ-decomposition, we do not include the most general definitions.

Definition 2.16. Let \( \Gamma \) be hyperbolic relative to \( A \). An elementary JSJ splitting relative to \( A \) is a tree, \( T \), with a \( \Gamma \) action such that the following hold.

1. all edge stabilizers are elementary subgroups;
2. (universally elliptic) any edge stabilizer of \( T \) fixes a point in any other tree with property (1);
3. (maximal for domination) for any tree \( T' \) satisfying (2), every vertex stabilizer of \( T \) stabilizes a vertex of \( T' \); and
4. (relative to \( A \)) all subgroups of elements of \( A \) fix a point in \( T \).

In [GL10a], the authors promote the notion that the correct objects of study for JSJ decompositions should not be JSJ trees because they are not unique. Rather, a collection of trees is more fundamental:
Definition 2.17. A maximal collection of trees which all satisfy Definition 2.16 is called a JSJ deformation space.

In the study of JSJ decompositions, one important focus is on understanding those subgroups in which there are many mutually incompatible splittings. Such pairs of splittings are called hyperbolic-hyperbolic in [RS97] and are best understood as analogous to splittings of a surface group over two simple closed curves with an essential intersection. It is impossible to realize both splittings simultaneously with a common refinement of the graphs of groups.

The subgroups with this property have various names in different contexts, including quadratically hanging [RS97], maximal hanging Fuchsian [Bow98a], orbifold hanging vertex [Pap05]. These names all seek to describe the same central idea: many pairs of simple closed curves on surfaces intersect. Consequently, surface groups have many ℤ-splittings which cannot be simultaneously realized in a common graph of groups. The essential power of these ideas is that whenever these ‘hyperbolic-hyperbolic’ splittings occur in finitely presented groups, the situation is always very close to the surface case. There is a more general definition given in [GL10a] which happens to encompass all of the above.

Definition 2.18. Let \( \Gamma_v \) be a vertex group for a JSJ tree. If \( \Gamma_v \) is not universally elliptic, then \( \Gamma_v \) is called flexible.

For our purposes, this definition falls somewhat short. We are interested in identifying subgroups up to quasi-isometry and flexibility is not preserved under such maps. For example, compare closed hyperbolic surface groups with hyperbolic triangle groups. In the context of relatively hyperbolic groups, there is a larger class of subgroups which we can use instead of flexible subgroups and which will reflect the coarse geometry directly: relatively QH subgroups with finite fiber [GL10a].

Definition 2.19. Given a group with a JSJ tree relative to \( A \), a subgroup \( Q \) is a relatively QH-subgroup if it satisfies the following:

1. an exact sequence, with \( O \) a hyperbolic 2-orbifold, \( F \) called the fiber:
   \[
   1 \to F \to Q \to \pi_1(O) \to 1
   \]

2. the images of incident edge groups are either finite or contained in a boundary subgroup of \( \pi_1(O) \).

3. every conjugate of an element of \( A \) intersects \( Q \) with image either finite or contained in a boundary component of \( \pi_1(O) \).

Specifically, these subgroups will be detectable from the boundary of the group, and thus will be realized as vertex groups for the cut-point/cut-pair tree.

2.3. Convergence Actions. Suppose \( \Gamma \) acts on a space \( X \) by homeomorphisms.

Definition 2.20. A sequence \( (g_i) \) is a convergence sequence on \( X \) if there exists points \( x_1, x_2 \in X \) such that for any compact \( C \) not containing \( x_1, g_i(C) \to x_2 \).

Definition 2.21. If every sequence in \( G \) contains a convergence subsequence, then \( G \) acts as a convergence group on \( X \).

Both hyperbolic and relatively hyperbolic groups are characterized by types of convergence actions which they exhibit on their boundaries [Bow98b, Yam04]. This allows for the identification of stabilizers of certain topological features in these contexts. We also require the classic theorem

Theorem 2.22 ([CJ94, Gab92, Tuk88]). Let \( G \) be a subgroup of Homeo(\( S^1 \)). \( G \) acts as a convergence group on \( S^1 \) if and only if \( G \) is Fuchsian.

A Fuchsian group is a discrete subgroup of Möbius transformations of \( D^2 \). In the proof of Theorem 2.3 we will use the convergence action to identify the vertex groups of our splitting.
3. The Tree from the Boundary

Given a group acting on a continuum by homeomorphisms, \[PS06\] constructs an \( \mathbb{R} \)-tree with vertices representing the topological features of the continuum (cut points and cut pairs). The tree inherits the action of the group on the continuum. We condense their exposition in anticipation of demonstrating that this tree is simplicial and of JSJ type in the context of relative hyperbolicity in Sections 4 and 5. For the remainder of this section we assume that \( \Gamma \) is a relatively hyperbolic group.

**Definition 3.1.** A **continuum** is a compact connected metric space.

**Definition 3.2.** Given a continuum \( X \), a point \( x \in X \) is a **cut point** if \( X \setminus \{ x \} \) is not connected. If \( \{ a, b \} \subset X \) contains no cut points and \( X \setminus \{ a, b \} \) is not connected, then \( \{ a, b \} \) is a **cut pair**. A set \( Y \) is called **inseparable** if no two points of \( Y \) lie in different components of the complement of any cut pair.

These separating features occur as the fixed points of peripheral or hyperbolic two-ended subgroups over which \( \Gamma \) splits. Given that we want to also understand when there are many mutually incompatible splittings (as in Definitions 2.18 and 2.19), we have terminology reflecting interlocking cut pairs. These pairs arise in our context as the endpoints of pairs of hyperbolic two-ended subgroups over which the group splits but which admit no common refinement.

**Definition 3.3.** Let \( X \) be a continuum without cut points. A finite set \( S \) is called a **cyclic subset** if there is an ordering \( S = \{ s_1, s_2, \ldots, s_n \} \) and continua \( M_1, \ldots, M_n \) such that

1. \( M_i \cap M_{i+1} = \{ s_i \} \), subscripts mod \( n \)
2. \( M_i \cap M_j = \emptyset \) whenever \( |i - j| > 1 \)
3. \( \bigcup M_i = X \)

An infinite subset in which all finite subsets of cardinality at least 2 are cyclic is also called cyclic.

**Definition 3.4.** A maximal cyclic subset with at least 3 elements is called a **necklace**.

Cyclic subsets arise as collections of mutually separable cut pairs. We also note that an inseparable cut pair can be in the closure of more than one necklace, but if the cut pair is not inseparable then that necklace is unique.

Given a continuum \( X \), we define an equivalence relation \( \sim \) such that any cut point is equivalent only to itself and for \( x, y \) which are not cut pairs, \( x \sim y \) if and only if there is no cut point \( z \) such that \( x \) and \( y \) are in different components of \( X \setminus z \).

We would like to define a similar notion for cut pairs but the extra structure (particularly the interaction between maximal inseparable sets and necklaces) makes this difficult. Instead, we directly construct subsets of the powerset of \( X \) which reflect the topology. Let \( \mathcal{R} \) be the collection containing all inseparable cut pairs, necklaces and maximal inseparable sets of \( X \). We claim that this structure is compatible with \( \sim \), ie that \( \mathcal{R} \) is the union of sets defined similarly on each class of \( \sim \). This follows from the following lemma, that cut points do not separate cut pairs.

**Lemma 3.5.** Suppose that \( T \) is a connected topological space with cut point \( x \). If \( \{ y, z \} \) is a cut pair then \( y \) and \( z \) are in the same component of \( T \setminus x \).

**Proof.** Let \( C_1 \) be the component of \( T \setminus \{ y, z \} \) containing \( x \). Let \( w \) be a point in another component, \( C_2 \). Clearly \( x \) separates \( C_1 \) but not \( C_2 \). Thus, \( w \) and \( y \) are in the same component of \( T \setminus \{ x, z \} \) and \( w \) and \( z \) are in the same component of \( T \setminus \{ x, y \} \). Thus, \( y \) and \( z \) are in the same component of \( T \setminus x \). \( \square \)

In \[PS06\] Theorems 12, 13, 14], \( \sim \) is shown to satisfy a ‘betweenness’ property so that a process of ‘connecting the dots’ can fill it in to an \( \mathbb{R} \)-tree. \[PS06\] Corollary 31] serves the same purpose for \( \mathcal{R} \). The combination of the two (which Lemma 3.5 justifies) is discussed in Section 5 of \[PS06\], see Figure 1. Obviously a group action by homeomorphisms on a continuum is inherited by this tree, although the action is not a priori isometric. In fact, this \( \mathbb{R} \)-tree does not necessarily come equipped with a metric.
Of course, the most important question about an action of a group on an \( \mathbb{R} \)-tree is whether it can be promoted to an action on a simplicial tree, or whether the \( \mathbb{R} \)-tree itself is simplicial. This is often accomplished via the Rips machine, which can be applied in the context of a reasonably nice action.

**Definition 3.6.** Let \( \Gamma \) act on the \( \mathbb{R} \)-tree \( T \) by homeomorphisms. We say the action is **nesting** if there exists a \( g \in \Gamma \) and an interval \( I \subset T \) such that \( g(I) \) is properly contained in \( I \). Otherwise, we say the action is **non-nesting**.

A non-degenerate arc \( I \) is called **stable** if there is a non-degenerate sub-arc such that for any non-degenerate arc \( K \subset J \), \( \text{Stab}(K) = \text{Stab}(J) \). An action is called **stable** if any closed arc \( I \) of \( T \) is stable.

We remark that the cut-point tree (ie the tree produced by connecting the dots for \( \sim \)) is simplicial whenever the boundary is connected and locally connected [Bow01, Theorem 9.2]. This can be achieved by the following mild constraints on \( A \).

**Theorem 3.7** ([Bow01], Theorem 1.5). Suppose that \( \Gamma \) is relatively hyperbolic and that each peripheral subgroup is one- or two-ended and contains no infinite torsion subgroup. If \( \Gamma \) is connected then it is locally connected.

We can produce a tree by ‘blowing up’ the vertices of subcontinua with no cut points according to the cut-pair structure (\( R \)), as discussed in [PS06, Section 5]. We call this the **combined tree** or cut-point / cut-pair tree, which we will denote by \( \mathcal{T} \) (Figure 1). As mentioned above, we seek to apply the Rips machine so we must demonstrate that the action is stable and non-nesting.

### 4. \( \mathcal{T}(\partial X(\Gamma)) \) is Simplicial

Because the cut point tree is simplicial, the only intervals which can be unstable are those which contain multiple inseparable cut pairs. Thus, we restrict our attention to those. We first require some information on the variety of finite subgroups of a relatively hyperbolic group.

**Lemma 4.1.** Let \( \Gamma \) be a relatively hyperbolic group. There are finitely many conjugacy classes of finite order subgroups \( F \) such that \( F \) is contained in a hyperbolic two-ended subgroup \( H \) with \( F \) fixing the ends of \( H \).

**Proof.** The following argument is adapted from [Mos12]. For every hyperbolic two-ended subgroup \( H \), take \( A_H \subset X(\Gamma, \mathcal{A}) \) to be the set of all geodesics between the endpoints of \( H \). There is a uniform width \( W \) for all \( A_H \) which is independent of the choice of \( H \) [Hru10, Corollary 8.16], ie for a point \( p \in A_H \cap \text{Cay}(\Gamma) \subset X(\Gamma, \mathcal{A}), H \setminus N_W(p) \) is not connected.

It follows that for some \( k \in \mathbb{N} \), if \( h \in H \) has the property that \( [h.N_kW(p)] \cap N_kW(p) = \emptyset \) and \( h \) fixes the endpoints of \( H \), then \( h \) has infinite order.

For \( F \) as above, it must be that the \( F \)-translates of \( N_kW(p) \) are not disjoint from \( N_kW(p) \).

Conjugating by \( p \) sends every such \( F \) to a subgroup \( F' \subset N_kW(1) \) of the same conjugacy class, and there are only finitely many possible \( F' \). \( \square \)
Lemma 4.2. There is a uniform bound on the order of the stabilizer of an interval containing at least two vertices corresponding to cut pairs.

Proof. Let I be any interval containing at least two inseparable cut pairs, A, B. We show that Stab(I) is finite. If \{g_n\} \subset Stab(I) is an infinite sequence of elements then we may assume that \(g_n(x) \to p\) for all \(x \in \partial \Gamma\), perhaps after passing to a subsequence. However, each \(g_n\) must fix all of the points of both cut pairs. This is a contradiction.

Additionally, this finite subgroup satisfies the hypotheses of Lemma 4.1 because it is a subgroup of Stab(A). There are only finitely many conjugacy classes of such subgroups so there is a uniform bound on the order of Stab(I). It follows immediately that the action is stable. □

Corollary 4.3. The action of \(\Gamma\) on \(\mathcal{T}\) is stable.

Lemma 4.4. The action of \(\Gamma\) on \(\mathcal{T}\) is non-nesting.

Proof. Assume not. Then there exists an interval \(I \subset T\) and \(g \in \Gamma\) (replaced by \(g^2\) if necessary) such that \(g(I)\) is a proper subset of \(I\). By the Brouwer fixed point theorem, there is a fixed point of \(g\) in \(I\), call this \(A\). We may assume \(I = [A, B], g(B) \in [A, B]\). Note that \(g\) has infinite order.

By convergence, there exists \(p, q \in \partial \Gamma\) such that \(g^n(x) \to p\) for every \(x \neq q\). Clearly, \(p \in A\). Because \(g^{-n}(x) \to q\), also \(q \in A\) otherwise \(q \in C \in (A, B)\) but \(g^{-n}(C) \neq C\). However, this implies that for all \(x \neq p, g^{-n}(x) \to q \in A\). Yet \(g^{-n}(B) \notin [A, B]\) for any \(n\), a contradiction. □

Lemma 4.5. This action is also minimal.

Proof. Every cut point is stabilized by a peripheral subgroup and every cut pair is stabilized by a finite or two-ended subgroup. Since the group is one-ended, cut pair stabilizers must be two-ended, showing that a 2-dense collection of vertices of the cut point tree and of every cut pair tree have non-trivial stabilizers. Thus, no proper invariant subtree exists. □

Theorem 4.6. Let \(\Gamma\) be a finitely presented, one-ended group, hyperbolic relative to \(\mathcal{A}\) such that for every \(A \in \mathcal{A}, A\) is not properly relatively hyperbolic and \(\mathcal{A}\) contains no infinite torsion subgroup. Let \(\mathcal{T}\) be the combined tree obtained by the action of \(\Gamma\) on its Bowditch boundary. Then \(\mathcal{T}\) is simplicial.

Proof. Suppose not. Then because the action is non-nesting, by [Lev98], there is an \(\mathbb{R}\)-tree \(\mathcal{T}'\) equipped with an isometric \(\Gamma\)-action and an equivariant quotient map \(\mathcal{T} \to \mathcal{T}'\). Furthermore, stabilizers of non-simplicial segments in \(\mathcal{T}'\) stabilize segments in \(\mathcal{T}\), and so are finite of uniformly bounded order by Lemma 4.2. Therefore, as in Corollary 4.3 the \(\Gamma\)-action is stable.

In all cases of [BF95, Theorem 9.5] other than the pure surface case, one obtains a splitting over a finite group. However, \(\Gamma\) is one-ended, so we reduce to this case. By [BF95, Theorems 9.4(1) & 9.5] \(\Gamma\) admits a splitting over a two-ended group \(V\), and this two-ended group corresponds to an essential, non-boundary parallel simple closed curve in the associated orbifold. If \(g \in V\) corresponds to this curve, then since the associated lamination on the orbifold has no closed leaves \(g\) must act hyperbolically on \(\mathcal{T}'\). This implies that \(g\) also acts hyperbolically on \(\mathcal{T}\). However, a splitting of \(\Gamma\) over a two-ended group must induce a cut pair corresponding to the endpoints of the axis of \(\langle g \rangle\). This cut pair must be stabilized by \(g\), so \(g\) cannot act hyperbolically. This is a contradiction. □

In summary, the combined tree \(\mathcal{T}\) is simplicial and has one vertex for each of the following topological structures in the continuum:

1. cut points
2. inseparable cut pairs
3. necklaces
4. equivalence classes of points not separated by cut points or cut pairs

Additionally, there is an edge between two vertices if the corresponding sets in the continua have intersecting closures. We note that, by the construction of \(\mathcal{R}\), points of the continuum can be contained in multiple elements of \(\mathcal{R}\).
5. \( T(\Gamma, A) \) is a JSJ-Tree

Now that we know that \( T \) is simplicial in common situations, we are left with the task of classifying it as a JSJ tree. This effectively labels the splitting as the ideal splitting for understanding the algebraic content of the group.

**Theorem 5.1.** With \( \Gamma, A \) as in Theorem 4.6, \( T \) is a JSJ tree over elementary subgroups relative to peripheral subgroups.

**Proof.** We show that \( T \) satisfies the conditions of Definition 2.16. By the construction of the tree every edge group must be the stabilizer of either a cut point or a cut pair. Because relatively hyperbolic groups act on their boundaries with a convergence action, these stabilizers must be elementary subgroups (condition (1)). Every peripheral subgroup fixes a point in the tree because it fixes a point in the boundary (this point is \( e_A \) of Lemma 2.7), which implies condition (4). Furthermore, this tree satisfies (3) because every elementary splitting always has a topological expression in the boundary. In particular, [Bow01] implies the existence of a cut point and [Pap05] implies the existence of a cut pair whenever there is an peripheral or hyperbolic two-ended splitting, respectively. Thus, every vertex in every such tree comes from one of these structures and hence is already a vertex stabilizer in \( T \). Finally, every splitting of this group must reflect the topology of the boundary. In particular, every edge represents the intersection of topological features. Fixing these features means fixing a vertex in a tree for another splitting. In essence, the argument that this tree satisfies (3) goes in both directions. This is (2). \( \square \)

6. **Induced Quasi-Isometries of Cusped Spaces**

In this section we prove that quasi-isometries of Cayley graphs induce quasi-isometries of cusped spaces:

**Theorem 6.3.** Let \( \Gamma_1 \) and \( \Gamma_2 \) be finitely generated groups. Suppose that \( \Gamma_1 \) is hyperbolic relative to a finite collection \( A_1 \) such that no \( A \in A \) is properly relatively hyperbolic. Let \( q : \Gamma_1 \to \Gamma_2 \) be a quasi-isometry of groups. Then there exists \( A_2 \), a collection of subgroups of \( \Gamma_2 \), such that the cusped space of \( (\Gamma_1, A_1) \) is quasi-isometric to that of \( (\Gamma_2, A_2) \).

We first show that horoballs of quasi-isometric spaces are themselves quasi-isometric. To that end, we distinguish among types of geodesics which exist in horoballs. We assume that \( n_2 > n_1 \).

**Definition 6.1.** Let \( \hat{T} \) be a horoball over the graph \( T \) with \((t_1, n_1)\) and \((t_2, n_2)\) vertices of \( \hat{T} \). We say \([ (t_1, n_1), (t_2, n_2) ] \) is vertical or a vertical geodesic if \( n_2 \) is the maximal depth among vertices of \([ (t_1, n_1), (t_2, n_2) ] \). See Figure 2.

![Figure 2. A vertical geodesic (right) and a non-vertical geodesic.](image)

**Lemma 6.2.** Let \( q : T \to S \) a \((k, c)\)-quasi-isometry between graphs. There is a \((1, C)\)-quasi-isometry \( \hat{q} : H(T) \to H(S) \) between combinatorial horoballs such that \( \hat{q} \) extends \( q \). Furthermore, \( C \) depends only on \( k \) and \( c \).
Proof. Extend \( q \) to \( \hat{q} \) by defining \( \hat{q}(v, n) = (q(v), n) \) and let \( s_i = q(t_i) \). The proof proceeds by comparing lengths of geodesics; we show that the length of a segment in \( H(S) \) is less than a linear function of the length of the corresponding segment in \( H(T) \). Since we start with a quasi-isometry, this argument is also valid in the reverse direction. Moreover, the quasi-inverse has coefficients which obey the same dependencies which establishes the proof. We partition the proof into cases by which geodesics are vertical.

\[ [(s_1, n_1), (s_2, n_2)] \text{ vertical:} \quad d_{H(S)}((s_1, n_1), (s_2, n_2)) \leq 2 \log_2 [d_S(s_1, s_2)] + 3 - n_2 - n_1 \]

\[ \leq 2 \log_2 [kd_T(t_1, t_2) + c] + 3 - n_2 - n_1 \]

\[ \leq 2 \log_2 [(k + c)d_T(t_1, t_2)] + 3 - n_2 - n_1 \]

\[ \leq 2 \log_2 (k + c) + 2 \log_2 (d_T(t_1, t_2)) + 3 - n_2 - n_1 \]

\[ \leq \hat{d}_T((t_1, n_1), (t_2, n_2)) + 2 \log_2 (k + c) + 3 \]

For the remaining two cases we assume that \([ (s_1, n_1), (s_2, n_2) ] \) is not vertical.

\[ [(t_1, n_1), (t_2, n_2)] \text{ not vertical:} \]

\[ d_{H(S)}((s_1, n_1), (s_2, n_2)) \leq 2 \log_2 (d_S(s_1, s_2)) - n_2 - n_1 + 3 \]

\[ \leq 2 \log_2 (k + c) + 2 \log_2 (d_T(t_1, t_2)) - n_2 - n_1 + 3 \]

\[ \leq 2 \log_2 (k + c) + n_2 - n_1 + 3 \]

\[ \leq 2 \log_2 (k + c) + \hat{d}_T(t_1, t_2) + 3 \]

Since \([ t_1, t_2 ] \) is not descending, \( \log_2 (d_T(t_1, t_2)) \leq n_2 \), justifying (2.1).

The coarse density of the image is clear. Since \( q \) is a quasi-isometry, we can also get identical results in the reverse direction with a symmetric argument so that geodesics in \( H(T) \) are bounded by a linear function of the lengths in \( H(S) \). Thus, \( \hat{q} \) is a \((1, 2 \log_2 (k + c) + 3)\)-quasi-isometry. \( \square \)

**Theorem 6.3.** Let \( \Gamma_1 \) and \( \Gamma_2 \) be finitely generated groups. Suppose that \( \Gamma_1 \) is hyperbolic relative to a finite collection \( \mathcal{A}_1 \) such that no \( A \in \mathcal{A} \) is properly relatively hyperbolic. Let \( q : \Gamma_1 \to \Gamma_2 \) be a quasi-isometry of groups. Then there exists \( \mathcal{A}_2 \), a collection of subgroups of \( \Gamma_2 \), such that the cusped space of \((\Gamma_1, \mathcal{A}_1)\) is quasi-isometric to that of \((\Gamma_2, \mathcal{A}_2)\).

**Proof.** We extend \( q \) to a map \( Q : X(\Gamma_1) \to X(\Gamma_2) \). First, let \( Q = q \) on \( \text{Cay}(\Gamma_1) \). By the proof of Theorem 5.12 of [Dru09], \( q \) induces a quasi-isometric embedding of cosets of elements of \( \mathcal{A}_1 \) into those of \( \mathcal{A}_2 \) and we can take these to have uniform constants. We observe that this can be made coarsely surjective because no peripheral subgroup is properly relatively hyperbolic. The reason for this is that if we had a quasi-isometry which was not coarsely surjective then this would give an element of \( \mathcal{A}_2 \) that would be hyperbolic relative to the image of \( q \) of some coset of an element of \( \mathcal{A}_1 \), again by [Dru09], contrary to our hypothesis. When the map is coarsely surjective, we get that this element is hyperbolic relative to itself, which is the trivial case that we allow.

Now, \( q \) might only take a coset to within a bounded distance of the corresponding coset in \( \Gamma_2 \), rather than directly to it as in Lemma 6.2. We can still use the induced quasi-isometry on the subsets of the horoballs which have positive depth but at depth 0 we have to make an adjustment.

However, the proof of [Dru09] Theorem 5.12] shows us that there exists a bound \( T \) such that the image of \( A_i \) is at most \( T \) from the coset to which it is quasi-isometric. Thus, we have to account only for an extra additive \( T \) in the constants.

Thus we know that \( q \) induces quasi-isometries between cosets of peripheral subgroups and that the constants of these quasi-isometries do not depend on the particular cosets but only on the constants of \( q \) and on \((\Gamma_1, \mathcal{A})\). Therefore, \( Q \) restricts to a quasi-isometry on each individual horoball and on the Cayley graph. We only need to show that these can be combined across all of \( X \).
Now let \([x, y]\) be a geodesic arc between points of \(X(\Gamma_1)\). We divide this arc into several subarcs by taking the collection of maximal subarcs \(I_1\) which have every vertex of depth 0 and also take the complimentary collection of segments \(I_2\). We note that some of these arcs may be degenerate (length 0, just a singleton) and that our methods account for this. In other words, we have
\[
[x, y] = [x_0, x_1] \cup [x_1, x_2] \cup \ldots \cup [x_{n-1}, x_n] = [x_0, x_n]
\]
with \([x_{2i}, x_{2i+1}] \in I_1\) and \([x_{2i+1}, x_{2i+2}] \in I_2\). Essentially, we have divided \([x_0, x_n]\) into segments which go between two different horoballs (again, possibly with length 0) and segments which traverse individual horoballs. We should mention that the subdivision used here has a parity which suggests that the terminal points \(x_0\) and \(x_n\) must have depth 0, but that this is easily surmounted. Simply attach a vertical segment to a depth 0 vertex whenever necessary.

We have the following expression:
\[
\hat{d}_1(x_0, x_n) = \sum_{i=0}^{n-1} d(x_i, x_{i+1}) = \sum_{I \in I_1} \text{length}(I) + \sum_{I \in I_2} \text{length}(I)
\]

We construct a path in \(\Gamma_2\) which tracks the image of \([x_0, x_n]\). For each \(x_i, x_{i+1}\), we take any geodesic in \(X(\Gamma_2), [Q(x_i), Q(x_{i+1})] = [q(x_i), q(x_{i+1})]\), see Figure 3. Because \(q\) is a quasi-isometry between Cayley graphs, \(d_1 = d_\hat{1}\) for any segment in \(I_1\), and \(d_2 \geq d_\hat{2}\), we get the following estimate on the lengths of the images of endpoints of segments of \(I_1\):
\[
\sum_{i=0}^{(n-1)/2} \hat{d}_2(q(x_{2i}), q(x_{2i+1})) \leq \sum_{i=0}^{(n-1)/2} d_2(q(x_{2i}), q(x_{2i+1})) \leq \sum_{i=0}^{(n-1)/2} [kd_1(x_{2i}, x_{2i+1}) + c] = \sum_{i=0}^{(n-1)/2} [kd_\hat{1}(x_{2i}, x_{2i+1}) + c]
\]

By Lemma 6.2, we know that the horoballs paired by \(q\) are quasi-isometric and, by [Dru09], that the constants can be chosen uniformly. If we let \(\Lambda - 2T\) be the maximum among those constants and \(k, c\), we get the following length estimate:
\[
\hat{d}_2(q(x_0), q(x_n)) \leq \sum_{i=0}^{(n-1)/2} \hat{d}_2(q(x_{2i}), q(x_{2i+1})) + \sum_{i=1}^{(n-1)/2} \hat{d}_2(q(x_{2i-1}), q(x_{2i})) \leq \sum_{i=0}^{(n-1)/2} [\Lambda \hat{d}_1(x_{2i}, x_{2i+1}) + \Lambda] + \sum_{i=1}^{(n-1)/2} [\Lambda \hat{d}_1(x_{2i-1}, x_{2i}) + \Lambda]
\]

Now, since we know that the horoball-transversals have length \(\geq 1\), we can move the additive constant \(\Lambda\) from the first sum to the second sum, except for a single summand. We do this to

**Figure 3.** A typical geodesic in \(X(\Gamma_1)\) and a reconstructed piecewise geodesic in \(X(\Gamma_2)\).
account for scenarios in which a geodesic is constructed from several paths contained entirely in horoballs, making the first sum \( \sum \).

\[
\hat{d}_2(q(x_0), q(x_n)) \leq \Lambda + \sum_{i=0}^{(n-1)/2} [\Lambda \hat{d}_1(x_{2i}, x_{2i+1})] + \sum_{i=1}^{(n-1)/2} [\Lambda \hat{d}_1(x_{2i-1}, x_{2i}) + 2\Lambda] \\
\leq \Lambda + \sum_{i=0}^{(n-1)/2} [\Lambda \hat{d}_1(x_{2i}, x_{2i+1})] + \sum_{i=1}^{(n-1)/2} [3\Lambda \hat{d}_1(x_{2i-1}, x_{2i})] \\
\leq \Lambda + 3\Lambda \left[ \sum_{i=0}^{(n-1)/2} [\hat{d}_1(x_{2i}, x_{2i+1})] + \sum_{i=1}^{(n-1)/2} [\hat{d}_1(x_{2i-1}, x_{2i})] \right] \\
= 3\Lambda \hat{d}_1(x_0, x_n) + \Lambda'
\]

(3)

We still need to establish coarse density. However, this is clear for depth 0 vertices since the Cayley graphs are quasi-isometric and it is clear for the positive depth vertices since the horoballs are in bijection and this pairing is by quasi-isometries.

We also need to produce the other bound, but by a symmetric argument using the quasi-inverse \( r \),

\[
\hat{d}_1(r(x_0), r(x_n)) \leq 3\Lambda' \hat{d}_2(x_0, x_n) + \Lambda'
\]

Therefore,

\[
\hat{d}_1(r(q(x_0)), r(q(x_n))) \leq 3\Lambda' \hat{d}_2(q(x_0), q(x_n)) + \Lambda'
\]

\[
\leq 3\Lambda'[3\Lambda \hat{d}_1(x_0, x_n) + \Lambda] + \Lambda' = 9\Lambda \Lambda' \hat{d}_1(x_0, x_n) + 3\Lambda \Lambda' + \Lambda
\]

Because \( q \) and \( r \) are quasi-inverse, there exists an \( a > 0 \) such that

\[
\hat{d}_1(x_0, x_n) \leq \hat{d}_1(r(q(x_0)), r(q(x_n))) + a
\]

Combining these, we get

\[
\frac{1}{3\Lambda'} \hat{d}_1(x_0, x_n) - \frac{a + \Lambda'}{3\Lambda'} \leq \hat{d}_2(r(x_0), r(x_n)) \leq 3\Lambda' \hat{d}_1(x_0, x_n) + \Lambda'
\]

We conclude by maximizing among constants. \( \square \)

As indicated previously, our proof of Theorem 6.3 simplifies to prove an analogous result for the coned space.

**Theorem 6.4.** Let \( \Gamma_1, \Gamma_2, A, q \) be as above. Then there exists a collection of subgroups \( B \) of \( \Gamma_2 \) such that the coned spaces of \( (\Gamma_1, A) \) and \( (\Gamma_2, B) \) are quasi-isometric.

**Proof.** Adjust the proof of the previous theorem by replacing the intra-horoball arcs with arcs through the cone-points. These all have length 2 so simply change (2) to

\[
\hat{d}_2(q(x_0), q(x_n)) \leq \sum_{i=0}^{(n-1)/2} \hat{d}_2(q(x_{2i}), q(x_{2i+1})) + \sum_{i=1}^{(n-1)/2} 2
\]

\( \square \)

**Corollary 6.5.** With \( (\Gamma_1, A_1) \) and \( (\Gamma_2, A_2) \) as in Theorem 6.3, the cusped spaces \( X(\Gamma_1, A_1) \) and \( X(\Gamma_2, A_2) \) have homeomorphic boundaries.

**Corollary 6.6.** With \( (\Gamma_1, A_1) \) as in Theorem 6.3, the trees describing the maximal peripheral splitting \( \text{Bow98}[A] \) and the cut-point/cut-pair tree \( \text{PS06} \) for the boundary of the cusped space are quasi-isometry invariant.

**Corollary 6.7.** Let \( \Gamma \) be a group hyperbolic relative to \( A \) with finite compatible generating sets \( S \) and \( T \). Then \( X(\Gamma, S, A) \) and \( X(\Gamma, T, A) \) are quasi-isometric. The analogous result for the coned space also holds.
Proof. We only need to show that we can drop the condition on peripheral subgroups. Inspecting the proof of Theorem 6.3, the requirement that no peripheral subgroup is properly relatively hyperbolic is used to establish a bijection between cosets of peripheral subgroups from [Dru09]. Because peripheral cosets will be mapped to themselves under the identity map, this bijection is automatic and the hypothesis is unnecessary. \qed

We note that this corollary is also new for hyperbolic groups with a non-trivial relatively hyperbolic structure.

7. Vertex Stabilizers

The first step towards identifying the vertex stabilizers is establishing the quasi-convexity of vertex groups.

Lemma 7.1. Let Γ be finitely presented and one-ended. Additionally suppose that (Γ, A) is relatively hyperbolic with A finite such that no A ∈ A is properly relatively hyperbolic and no A contains an infinite torsion subgroup. Let T be the combined tree from the boundary. If Γv is a vertex group of T then Γv is relatively quasi-convex.

Proof. This is clearly true for vertex groups which are peripheral. It is also true for hyperbolic two-ended vertex groups by Theorem 2.14. Assume Γv is not of these types.

T is a bipartite graph in which all vertices of one color have corresponding groups which are either peripheral or hyperbolic two-ended. To see this, note that maximal inseparable sets and necklaces must only be adjacent to cut pairs or points. Furthermore, cut pairs and points must come between these larger sets. Now, let \{Z_1, \ldots, Z_n\} and \{P_1, \ldots, P_m\} be the collection of all hyperbolic two-ended subgroups and the collection of all peripheral subgroups incident to Γv, respectively.

Let γ be any geodesic between points in Γv. Decompose γ into maximal segments of length ≥ 1 which have endpoints in cosets of some Z_i, P_j or are contained completely within Γv. The segments contained in each Z_i must stay within a bounded distance of Γv because hyperbolic two-ended subgroups are strongly relatively quasi-convex (Theorem 2.14). Additionally the P_j segments stay within bounded distance of P_j [Dru09 Theorem 4.21]. Since they are peripheral and we are investigating relative quasi-convexity, we are not concerned with how far they stray from P_j ∩ Γv inside P_j. \qed

Proposition 7.2. With Γ, A, T as in Lemma 7.1, a vertex group Γ_v of T, is relatively QH with finite fiber if and only if Γ_v is the stabilizer of a necklace in T.

Proof. If Γ_v is relatively QH with finite fiber, then by Definition 2.19 there is a short exact sequence

\[ 1 \to F \to Γ_v \to π_1(O) \to 1 \]

with F finite and O a hyperbolic orbifold. We first determine that Γ_v ∉ A. Because F is finite, Γ_v ∼\_qi π_1(O) and because O is a hyperbolic 2-orbifold, Γ_v is itself hyperbolic relative to \{1\}. By our condition on peripheral subgroups, Γ_v ∉ A.

Now, by definition Γ_v is virtually Fuchsian. Let C be the set of bi-infinite curves in the universal cover \( \tilde{O} \) which are not homotopic to a boundary component of \( \tilde{O} \). Since F is finite, Γ_v is quasi-isometric to π_1(O) and so \( \partial Γ_v \simeq \partial π_1(O) \) with respect to the relatively hyperbolic structure induced by A. Call this set N. Let \( \tilde{N} \) be the image of N in \( \partial \tilde{Γ} \) induced by the inclusion of Γ_v in Γ. This map is well-defined because of relative quasi-convexity of Γ_v. Specifically, by [MMP10], relative quasi-convexity is equivalent to quasi-convexity in the cusped space and by general results on δ-hyperbolic spaces the boundary will embed.

We claim that \( \tilde{N} \) is a necklace in \( \partial \tilde{Γ} \). By definition, every edge group must be either finite or contained in a boundary component. Because Γ is one-ended, the finite case is excluded. Consequently, for any γ ∈ C each coset of an edge group is contained in a single component of \( \tilde{O} \setminus γ \). Let η ∈ C be any curve which has an essential crossing with γ. Such η exists because if not then γ must be boundary parallel, but C contains no boundary parallel curves.
Since $\eta^+$ and $\eta^-$ are in different components of $N \setminus \{\gamma^+\gamma^-,\gamma^+\gamma^-\}$ and each edge group is attached to only a single boundary component, it must be that every edge group has image contained in either the same component of $N \setminus \gamma$ as $\eta^+$ or $\eta^-$, but never both. Thus, the image of $\{\gamma^+\gamma-,\gamma^+\gamma^-\}$ also separates the image of $\{\eta^+\eta^-\}$ in $N$ and the endpoints of $\gamma$ form a cut pair in $\partial \Gamma$.

In the reverse direction, if $\Gamma_v$ stabilizes a necklace $N$ then, by the last paragraph of the proof of [PS06, Theorem 22], it has an action on $S^1$ which preserves the cyclic order. Since $\Gamma_v = \text{Stab}(\mathcal{N})$, $\Gamma_v$ inherits the convergence property of $\Gamma$ and this property must be realized on $\mathcal{N}$. Any sequence of group elements contained in the kernel of this action is not a convergence sequence, so the kernel must be finite. The fiber, $F$, is the kernel of this action. By Theorem 2.22, the quotient must be a Fuchsian group. Let $\mathcal{O}$ be the quotient of $\mathbb{H}^2$ by the action of $\Gamma_v/F$, truncating cusps so that $\mathcal{O}$ is compact. We are left with showing that edge groups are boundary parallel.

First, we show that no element of $\mathcal{C}$ (again defined as the set of non-boundary bi-infinite curves on $\mathcal{O}$ - those which cross other bi-infinite curves and interlock in $\partial N$) can be contained in the image of a peripheral edge group. Because peripheral subgroups have a unique boundary point (Lemma 2.7), such a curve would induce a cut point in $\mathcal{N}$. However, by Lemma 3.5, no cut pair can be separated by a cut point. In this context, every cut pair separates another cut pair with the only exception arising from those cut pairs which are end points of boundary curves of $\mathcal{O}$. In other words, for every cut pair $C$ of $\mathcal{N}$ there is an interlocked cut pair unless $C$ forms the endpoints in $\partial \mathcal{O}$ of a curve homotopic to a boundary component of $\mathcal{O}$.

Now we are left with only the possibility that some $\gamma \in \mathcal{C}$ is identified with a hyperbolic two-ended edge group. Let $\{x_1, x_2\}$ be any cut pair interlocked with $\{\gamma^+, \gamma^-\}$. We claim that $\{x_1, x_2\}$ is not actually a cut pair in $\partial \Gamma$. $\partial \Gamma \setminus \{\gamma^+, \gamma^-\}$ must have at least 3 components in this situation. Let $Y$ be a component which does not contain any points of $\mathcal{N}$. In particular, $Y \cap \mathcal{N} = \{\gamma^+, \gamma^-\}$. Thus, no cut pair of $\mathcal{N}$ separates $\gamma^+$ from $\gamma^-$ as both are contained in the component $Y$. However, then $\partial \mathcal{O}$ must separate $\{x_1, x_2\}$, contrary to Lemma 3.5. Thus, we must have that no such $\gamma$ exists in $\mathcal{C}$.

Lemma 7.3. If $\{x, y\}$ is an inseparable cut pair in $\partial \Gamma$ then $\text{Stab}(\{x, y\})$ is a hyperbolic two-ended subgroup of $\Gamma$.

Proof. Let $Z = \text{Stab}(\{x, y\})$. $Z$ must be infinite else $\Gamma$ would not be one-ended. As a subgroup of $\Gamma$, $Z$ acts on $\partial \Gamma$ with the convergence property. Since $Z$ fixes $\{x, y\}$, this implies that $Z$ is two-ended. □
Table 1. Subgroups which stabilize particular topological features in the boundary.

| Stabilizer                  | Topological Feature |
|-----------------------------|---------------------|
| hyperbolic 2-ended          | cut-pair            |
| peripheral                  | cut-point            |
| relatively QH with finite fiber | necklace            |

Corollary 7.4. With $\Gamma, \mathcal{A}, \mathcal{T}$ as in Lemma 7.1, there is a correspondence between vertex groups of $\mathcal{T}$ tree and the topological features of the boundary given by Table 2.

Proof. Since hyperbolic 2-ended subgroups are strongly relatively quasi-convex [Osi06], their boundaries embed [Hru10]. Similarly, [Bow01] demonstrates that cut points correspond to peripheral splittings. As vertex groups, these features must be separating. The last point is Proposition 7.2. □

With this in place, we are ready to show:

Theorem 7.5. Let $\Gamma_1$ and $\Gamma_2$ be finitely generated groups. Suppose additionally that $\Gamma_1$ is one-ended and hyperbolic relative to the finite collection $\mathcal{A}_1$ of subgroups such that no $A \in \mathcal{A}_1$ is properly relatively hyperbolic or contains an infinite torsion subgroup. Let $\mathcal{T}$ be the cut-point/cut-pair tree of $\partial(\Gamma_1, \mathcal{A}_1)$. If $f : \Gamma_1 \to \Gamma_2$ is a quasi-isometry then

- $\mathcal{T}$ is the cut-point/cut-pair tree for $\Gamma_2$ with respect to the peripheral structure induced by Theorem 6.3.
- if $\text{Stab}_{\Gamma_1}(v)$ is one of the following types then $\text{Stab}_{\Gamma_2}(v)$ is of the same type,
  1. hyperbolic 2-ended,
  2. peripheral,
  3. relatively QH with finite fiber.

Proof. By Corollary 6.5, there exists a relatively hyperbolic structure for $\Gamma_2$ such that the boundaries of the cusped spaces are homeomorphic. Since $\mathcal{T}$ depends only on the topology of this continuum, $\mathcal{T}$ is the cut-point/cut-pair tree for $\Gamma_2$.

By the correspondence given in Corollary 7.4, these vertex types depend only on the topology of the boundary. Since these topological features are preserved, the vertex group types are preserved as well. □

We conclude this section with a consequence of the fact that $\text{Out}(\Gamma)$ acts on $\partial \Gamma$ by homeomorphisms.

Corollary 7.6. Let $(\Gamma, \mathcal{A})$ be relatively hyperbolic with no $A \in \mathcal{A}$ properly relatively hyperbolic. Then, the $\text{Out}(\Gamma)$ action on the JSJ-deformation space over elementary subgroups relative to peripheral subgroups fixes $\mathcal{T}$.

Proof. The action passes to an action on the boundary by homeomorphisms which induces an action on the combined tree so that vertex groups map to vertex groups and adjacencies are preserved. Furthermore, because maximal relatively hyperbolic structures are unique [MOY12], there is no change in the choice of peripheral structure. □

As demonstrated in Theorem 6.3, the action of $QI(\Gamma)$ on the Cayley graph of $\Gamma$ induces an action on the cusped space $X(\Gamma, \mathcal{A})$ and, because the boundary is invariant under quasi-isometries, on $\partial(\Gamma)$. As stated in Corollary 6.6, this naturally yields an action on the tree $\mathcal{T}$.

8. Splitting $QI(\Gamma)$ with $\mathcal{T}$

Our last section describes how to understand $QI(\Gamma)$ from the action it inherits on $\mathcal{T}(\Gamma)$.
8.1. Faithfulness.

**Theorem 8.1.** With $\Gamma, A, T$ as in Lemma 7.1, the action of $\mathcal{QI}(\Gamma)$ on $T(\Gamma, A)$ is faithful, assuming that $T$ is not a point.

**Proof.** Given a $(k, c)$-quasi-isometry $\phi \in \mathcal{QI}(\Gamma)$ which has a trivial induced action on $T(\Gamma)$, it must be that $\phi$ coarsely fixes all peripheral subgroups over which $G$ splits. For now, assume one such subgroup exists and call this $A$. By the same citation of [Dru09] applied to Theorem 6.3, $\phi$ must map every coset $gA$ within $N_T(gA)$. However, given two adjacent cosets, $hA$ and $shA$, both must satisfy this condition. Consider adjacent points $a \in hA$, $b \in shA$. It must be that $\phi(a) \in N_{k+c}(hA)$ and $\phi(b) \in N_{k}(shA)$. Further, as $d(a,b) = 1$, $d(\phi(a), \phi(b)) \leq k + c$. Consequently, $\phi(a) \in N_{k+c}(hA)$ and $\phi(b) \in N_{k+c}(hA)$ and $\phi(a), \phi(b) \in N_{k+c}(hA) \cap N_{k+c}(shA)$. This intersection must have a finite diameter which is uniformly bounded over any chosen pair of cosets. Therefore, $d(a, \phi(a))$ is bounded by this diameter. Since every point can be found within such a neighborhood by an appropriate choice of such cosets, $\phi$ must be a finite distance from the identity map.

The two-ended case is similar. These subgroups have been shown to satisfy a similar divergence property called hyperbolically embedded [DGO12, Sis12a] and they are strongly relatively quasi-convex, so the argument is nearly identical. \[
\square
\]

![Figure 5. $\mathcal{QI}(\Gamma)$ acts faithfully on the JSJ tree.](image)

8.2. Bounding the Number of Edges of $T/QI(\Gamma)$. We have that $\Gamma$ and $\mathcal{QI}(\Gamma)$ act on the same tree so that $\mathcal{QI}$ splits whenever $\Gamma$ has a non-trivial JSJ as described here. We can also control the number of edges which $\mathcal{QI}$ admits in this induced splitting. Let $\Lambda$ be $|E(T/\Gamma)|$ and let $\text{Aut}_{qi}(T/\Gamma)$ be the group of graph automorphisms which respect the quasi-isometry type of each edge and vertex stabilizer.

**Theorem 8.2.** The graph of groups decomposition of $\mathcal{QI}(\Gamma)$ induced by the JSJ-decomposition of $\Gamma$ has at most $\Lambda$ edges and at least $\Lambda/|\text{Aut}_{qi}(T/\Gamma)|$ edges.

**Proof.** First, note that $\Gamma$ acts on itself by quasi-isometries and if $T$ is not a point then this action will have no global fixed points so obviously the same is true for the $\mathcal{QI}(\Gamma)$ action.

Now, each $g \in \Gamma$ acts by (quasi)-isometries on $X(\Gamma, A)$ which gives the upper bound, and $\mathcal{QI}(\Gamma)$ acts on $T/\Gamma$ by graph automorphisms which must preserve the quasi-isometry class of each stabilizer because the action is filtered through the action on $X(\Gamma, A)$, thus giving the lower bound. \[
\square
\]
Acknowledgements

I would like to thank Daniel Groves for his generous support and guidance. I would also like to thank Mark Feighn for pointing out an unstated hypothesis for the peripheral subgroups in Sections 3 and 5, Panos Papazoglou and Eric Swenson for some helpful remarks concerning [PS06 Section 5] and Chris Cashen for his useful comments on my first version.

References

[Ali05] Emina Alibegovi´c. A combination theorem for relatively hyperbolic groups. Bull. London Math. Soc., 37(3):459–466, 2005.

[BF95] Mladen Bestvina and Mark Feighn. Stable actions of groups on real trees. Invent. Math., 121(2):287–321, 1995.

[BH99] Martin R. Bridson and André Haefliger. Metric spaces of non-positive curvature, volume 319 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1999.

[Bow97] Brian Bowditch. Relatively hyperbolic groups. Preprint, 1997.

[Bow98a] Brian Bowditch. Cut points and canonical splittings of hyperbolic groups. Acta Math., 180(2):145–186, 1998.

[Bow98b] Brian Bowditch. A topological characterisation of hyperbolic groups. J. Amer. Math. Soc., 11(3):643–667, 1998.

[Bow01] Brian Bowditch. Peripheral splittings of groups. Trans. Amer. Math. Soc., 353(10):4057–4082 (electronic), 2001.

[CJ94] Andrew Casson and Douglas Jungreis. Convergence groups and Seifert fibered 3-manifolds. Invent. Math., 118(3):441–456, 1994.

[Dah03] François Dahmani. Combination of convergence groups. Geom. Topol., 7:933–963 (electronic), 2003.

[DGO12] François Dahmani, Vincent Guirardel, and Denis Osin. Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces. arXiv preprint, identifier: 1111.7048, 2012.

[DP10] Thomas Delzant and Panos Papasoglu. Codimension one subgroups and boundaries of hyperbolic groups. Groups Geom. Dyn., 4(3):533–548, 2010.

[Dru09] Cornelia Drutu. Relatively hyperbolic groups: geometry and quasi-isometric invariance. Comment. Math. Helv., 84(3):503–546, 2009.

[DS99] Martin Dunwoody and Michah Sageev. JSJ-splittings for finitely presented groups over slender groups. Invent. Math., 135(1):25–44, 1999.

[DS05] Cornelia Drutu and Mark Sapir. Tree-graded spaces and asymptotic cones of groups. Topology, 44(5):959–1058, 2005. With an appendix by Denis Osin and Sapir.

[Far94] Benson Farb. Relatively hyperbolic and automatic groups with applications to negatively curved manifolds. ProQuest LLC, Ann Arbor, MI, 1994. Thesis (Ph.D.)–Princeton University.

[Far98] Benson Farb. Relatively hyperbolic groups. Geom. Funct. Anal., 8(5):810–840, 1998.

[FP06] Koji Fujiwara and Panos Papasoglu. JSJ-decompositions of finitely presented groups and complexes of groups. Groups Geom. Dyn., 16(1):70–125, 2006.

[GL07] Vincent Guirardel and Gilbert Levitt. Deformation spaces of trees. Groups Geom. Dyn., 1(2):135–181, 2007.

[GL10a] Vincent Guirardel and Gilbert Levitt. JSJ decompositions: definitions, existence, uniqueness, I: The jsj deformation space. Preprint, 2010.

[GL10b] Vincent Guirardel and Gilbert Levitt. JSJ decompositions: definitions, existence, uniqueness, II. compatibility and acylindricity. Preprint, 2010.

[GMS87] F. W. Gehring and G. J. Martin. Discrete convergence groups. In Complex analysis, I (College Park, Md., 1985–86), volume 1275 of Lecture Notes in Math., pages 158–167. Springer, Berlin, 1987.

[GMS08] Daniel Groves and Jason Fox Manning. Dehn filling in relatively hyperbolic groups. Israel J. Math., 168:317–429, 2008.

[Gos87] M. Gromov. Hyperbolic groups. In Essays in group theory, volume 8 of Math. Sci. Res. Inst. Publ., pages 75–263. Springer, New York, 1987.

[HK05] G. Christopher Hruska and Bruce Kleiner. Hadamard spaces with isolated flats. Geom. Topol., 9:1501–1538, 2005. With an appendix by the authors and Mohammad Hidawi.

[Hru10] G. Christopher Hruska. Relative hyperbolicity and relative quasiconvexity for countable groups. Algebr. Geom. Topol., 10(3):1807–1856, 2010.

[JS79] William H. Jaco and Peter B. Shalen. Seifert fibered spaces in 3-manifolds. Mem. Amer. Math. Soc., 21(220):viii+192, 1979.

[Kro90] P. H. Kropholler. An analogue of the torus decomposition theorem for certain Poincaré duality groups. Proc. London Math. Soc. (3), 60(3):503–529, 1990.

[Lev98] Gilbert Levitt. Non-nesting actions on real trees. Bull. London Math. Soc., 30(1):46–54, 1998.
[MMP10] Jason Fox Manning and Eduardo Martínez-Pedroza. Separation of relatively quasiconvex subgroups. *Pacific J. Math.*, 244(2):309–334, 2010.
[Mos12] Lee Mosher. Finite subgroups of relatively hyperbolic groups - answer. *Math Overflow*, http://mathoverflow.net/questions/94689/finite-subgroups-of-relatively-hyperbolic-groups, 2012.
[MOY12] Yoshifumi Matsuda, Shin-ichi Oguni, and Saeko Yamagata. The universal relatively hyperbolic structure on a group and relative quasiconvexity for subgroups. *Preprint*, 2012.
[Osi06] Denis V. Osin. Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems. *Mem. Amer. Math. Soc.*, 179(843):vi+100, 2006.
[Pap05] Panos Papasoglu. Quasi-isometry invariance of group splittings. *Ann. of Math. (2)*, 161(2):759–830, 2005.
[PS06] Panos Papasoglu and Eric Swenson. From continua to R-trees. *Algebr. Geom. Topol.*, 6:1759–1784 (electronic), 2006.
[PS09] Panos Papasoglu and Eric Swenson. Boundaries and JSJ decompositions of CAT(0)-groups. *Geom. Funct. Anal.*, 19(2):559–590, 2009.
[RS97] E. Rips and Z. Sela. Cyclic splittings of finitely presented groups and the canonical JSJ decomposition. *Ann. of Math. (2)*, 146(1):53–109, 1997.
[Sis12a] Alessandro Sisto. On metric relative hyperbolicity. *arXiv preprint*, identifier: 1210.8081, 2012.
[Sis12b] Alessandro Sisto. Projections and relative hyperbolicity. *arXiv preprint*, identifier: 1010.4552, 2012.
[SS03] Peter Scott and Gadde A. Swarup. Regular neighbourhoods and canonical decompositions for groups. *Astérisque*, (289):vi+233, 2003.
[Szc98] Andrzej Szczepański. Relatively hyperbolic groups. *Michigan Math. J.*, 45(3):611–618, 1998.
[Tuk88] Pekka Tukia. Homeomorphic conjugates of Fuchsian groups. *J. Reine Angew. Math.*, 391:1–54, 1988.
[Yam04] Asli Yaman. A topological characterisation of relatively hyperbolic groups. *J. Reine Angew. Math.*, 566:41–89, 2004.

_E-mail address: bradleywgroff@gmail.com_