Inflection reflection: images in mirrors whose curvature changes sign

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Abstract
Mirrors that are convex in some places and concave in others can generate images of extended objects (such as the viewer’s face) that are curiously distorted and often topologically disrupted. Understanding these images involves the caustics of the family of rays emitted by each point of the object, and the totality of all such families constituting the rays from all points of the object. The general theory is illustrated by the simplest mirror with an inflection, whose profile is a cubic function. Simulations, and observations with a flexible plastic mirror, show how the image changes as the viewer moves relative to the mirror.

Keywords: rays, caustics, geometrical optics, topology

Supplementary material for this article is available online
(Some figures may appear in colour only in the online journal)

1. Introduction
Most people have seen images distorted by reflection in curved surfaces, such as the shiny surfaces of cars, cosmetic mirrors, backs and fronts of spoons, and Christmas-tree baubles. With mirrors that are purely convex or concave, the distortions can be considerable, especially if the curvatures are not uniform, but they are not difficult to understand in terms of familiar magnification or demagnification. More extreme deformations, including topological disruption, can also occur with mirrors that are purely concave. But such effects are much more dramatic with mirrors that are convex in some regions and concave in others, that is, mirrors with lines of inflection, on which one or both principal curvatures vanish, for example fairground
‘funhouse’) mirrors. Such mirrors can generate several images, that merge and disappear as the viewer moves. Similar multiple images occur in nature: in mirages [1, 2], in the rising or setting Sun when the atmosphere possesses an inversion layer [3], in reflections from wavy water [4, 5], and in gravitational lensing [6–8]. The multiple images have been studied in detail in refraction by a cylinder filled with water [9].

The explanation of these distortions and disruptions is rarely if ever taught to students in optics courses. This is a pity, because the explanation involves important concepts commonly regarded as abstract, such as mappings and caustic singularities. My purpose here—part of a programme of revealing the arcane in the mundane [10]—is to describe and illustrate the theory of these easy-to-see and sometimes amusing reflections in mirrors, presenting the general theory and illustrating it with the simplest mirror with an inflection: cubically curved in one direction, flat in the other.

An object that is being imaged, for example one’s face when looking in a mirror, can be regarded as a collection of luminous points. Each point emits a family of rays, finitely many of which are reflected into one’s eye(s). This number changes when caustics [11, 12]—focal singularities of the ray family—cross the eye or camera. Understanding such images therefore requires the study of the ray families, and their caustics, from all the source points: families of families of rays. This theory is described in section 2.

For explicit analytical calculations, it is useful to make the paraxial approximation, involving rays with a small range of directions; this is the subject of section 3. For objects characterised by boundary curves, disruptions of the image are determined by the ‘caustic-touching theorem’ [13]; this is explained in section 4, and illustrated by the reflections of a disk seen when traversing the inflected mirror. However, many objects, including faces, are characterised by smooth distributions of intensity, whose contour curves are disrupted at different stages of the viewer’s traverse and whose reflections are more complicated; this is illustrated, also in section 4, by simulated images of my grandson’s face. Section 5 describes a simple way to create an inflected mirror, and displays a sequence of photographs of the distortions and disruptions of my wife’s face during a traverse of the inflection (the supplementary material (https://stacks.iop.org/EJP/42/065301/mmedia) contains a movie). The concluding section 6 includes a brief discussion of the not fully explored images from more complicated mirrors; these could provide a rich source of student projects. Also included is a brief description of a related unsolved problem in the wave physics of an inflected mirror.

2. General theory: rays, families of rays, families of families of rays

The shape of the mirror is described by its deviation \( h(r') \) (figure 1) from the plane \( r' = x'e_x + y'e_y \), in which \( e_x \) and \( e_y \) are unit vectors in the \( x \) and \( y \) directions. Rays from a source point (for example, part of a face looking into the mirror), in the plane \( r_0 = x_0e_x + y_0e_y \), distant \( z_0 \) from the mirror plane, are reflected from the mirror at \( r' \) and reach points \( r = xe_x + ye_y \) in an observation plane distant \( z \) from the mirror plane. Later we will set \( z = z_0 \), but for now it is convenient to keep these distances separate.

Rays are reflected according to the law of equal angles: the unit direction vector \( e_{\text{refl}} \) of the reflected ray is the direction \( e_{\text{inc}} \) of the incident ray with its component along the unit normal \( n \) at \( r' \) reversed. Thus

\[
 e_{\text{refl}} = e_{\text{inc}} - 2 (n \cdot e_{\text{inc}}) n. \tag{1}
\]

Note that \( e_{\text{refl}}, e_{\text{inc}} \) and \( n \) are unit vectors in three dimensions, and \( r', r_0 \) and \( r \) are vectors in the indicated planes with \( z \) components zero; the full three-dimensional vectors in the source
Figure 1. Ray geometry: a source at \( r_0 \) is reflected at \( r' \) and received at \( r \).

and observation planes are \( r_0 + z_0 e_z \) and \( r + z e_z \), where \( e_z \) is the unit vector in the \( z \) direction. Explicitly, the vectors \( e_{\text{inc}} \) and \( n \) are

\[
e_{\text{inc}} = N (r' - r_0 - (z_0 - h(r')) e_z), \quad n = \frac{-\nabla h + e_z}{\sqrt{1 + (\nabla h)^2}},
\]

in which \( N \) is a normalisation constant. Thus the three-dimensional vector describing the point at distance \( s \) along the reflected ray is

\[
R = R(r_0, r'; s) = r' + h(r') e_z + s e_{\text{refl}}(r_0, r').
\]

The solution of these equations gives the reflection point(s) \( r' \) of the ray(s) reaching \( r \) from \( r_0 \).

An immediate check on the formulas is a flat mirror, i.e. \( h = 0, n = e_z \), with \( z = z_0 \). The reflection point (just one, in this case) is confirmed as

\[
r' = \frac{1}{2} (r + r_0).
\]

(This explains why a plane mirror must be at least half as big as one’s head in order to see the reflected head in its entirety.)

Central to the formation of images are the caustics of the family of rays issuing from the source \( r_0 \). These are the points \( r \) determined by the focussing equation

\[
det \frac{\partial r}{\partial r'} = \frac{\partial x}{\partial x'} \frac{\partial y}{\partial y'} - \frac{\partial y}{\partial x'} \frac{\partial x}{\partial y'} = 0.
\]

A feature that is important to the images of extended objects is that each of their points is infinitely magnified as its caustic crosses the eye. This follows from the complementary interpretation of the focussing equation: a small region \( \partial r \) (pupil of the eye), corresponds to a large region of the image \( \partial r' \) as seen on the mirror. We will note this large local magnification in distorted images in sections 4 and 5; it is of central importance in interpreting images in gravitational lensing [6].
An eye at \( r \) sees each point \( r_0 \) of an object by looking along the direction of the ray (or rays), given by (3), from the corresponding point(s) \( r' \) on the mirror. On each of these rays, the image will appear to be located where that ray touches its caustic—the counterpart, in this more general geometrical optics, of the elementary point focus when aberrations are neglected. Although this feature is of fundamental importance in optical instruments, we ignore it here, because when looking at the reflections we are usually unaware of our eye’s (or eyes’) accommodation in seeing where they are located. Here it will be convenient to specify the direction of gaze to the images by their reflection positions \( r' \) on the mirror.

Another feature we will neglect, because it is not relevant to understanding the simplest distortions and disruptions we will study, is that in general rays touch their caustics at two places, not one. This is because caustics in 3D are two-sheeted surfaces, with the ray from \( r' \) touching its caustic at distances from the mirror given by the two principal radii of curvature of the reflected wavefront at \( r' \). A plane mirror is a degenerate case, in which the two sheets coincide, and the caustic consists of a single point beyond the mirror—the familiar virtual image of each object point. For mirrors curved only in one direction, as in the simplest inflected mirror to be considered in detail, the second caustic, associated with the uncurved direction (\( y \) in the following) is a degenerate line focus on the other side of the mirror. Double touching would be important in understanding the more complicated images in general doubly-curved mirrors, when observed near the ‘umbilic points’ [14, 15] at which the two sheets are connected.

To illustrate these ideas, we choose the simplest mirror with an inflection, curved only in the \( x \) direction, namely

\[
h(x') = -\frac{x'^{3}}{a^{2}}, \tag{6}\]

and calculate the rays reflected by sources at different positions \( x_0 \). Figure 2 shows three such ray families. The caustic singularities—envelopes of each ray family—are obvious; they separate regions reached by different numbers of rays, in this case one or three. An eye at \( x_0 \), looking into the mirror, would see one image in cases (a) and (c), and three images in case (b). These changes of numbers of images occur as caustics cross the eye. They are the real caustics; the virtual caustics, that is, envelopes of the rays continued to the other side of the mirror, are irrelevant to the multiple-image phenomena of concern here, so they are not shown in figure 2.

(Readers wishing to reproduce these ray patterns should note that computing them is not quite straightforward. Rays reaching points \( x' \) on the convex side that are reflected downwards

\[
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\]
will never reach the \( x \) plane, and unless these are excluded naive algorithms show them traveling backwards through the mirror. And on the concave side points \( x' \) for which \( h(x') > z_0 \) must also be excluded. Moreover, for a source close to the concave side some rays will be multiply reflected.

We consider an observer, with eye at \( r \), looking at reflections of her head, which is located in the \( r_0 \) plane with \( z_0 = z \), and specified by an intensity function \( I(r_0 - r) \), where the displacement \( r \) represents the fact that her head is not a fixed object but moves with her. The reflected image she sees in the \( r' \) plane is, up to intensity, the point-by-point mapping of her head, namely

\[
I_{\text{refl}}(r') = I(r_0 (r', r) - r) .
\]

The function \( r_0(r', r) \) represents the ray from \( r_0 \) that reaches \( r \) via \( r' \), determined by (1)–(3) with \( r_0 \) and \( r \) interchanged. We will discuss the structure of these images, illustrated by examples, in sections 4 and 5.

### 3. Paraxial approximations

The geometrical optics of the previous section is exact, enabling reflected rays and images to be computed without approximation. But to get explicit analytic formulas for caustics and images is difficult, even for the simplest mirrors such as our inflected model (6). An approximation that overcomes this difficulty while preserving the essence of the image distortions and disruptions is to restrict attention to mirrors with gently varying profiles, and rays making small angles with the \( z \) direction. This is the paraxial approximation, where

\[
|\nabla h| \ll 1, \quad \frac{|r - r_0|}{\min(z - h, z_0 - h)} \ll 1.
\]

Regarding these quantities as first-order small eliminates \((\nabla h)^2\) in the denominator in \( n \) in (2), and consistent expansion of (1)–(3) leads to the ray equation in the paraxial approximation:

\[
\text{paraxial} : r = \left( 1 + \frac{z - h (r')}{z_0 - h (r')} \right) r' - \left( \frac{z - h (r')}{z_0 - h (r')} \right) r_0 - 2 (z - h (r')) \nabla h (r') .
\]

A further approximation, commonly employed and used extensively in the following, is to also regard the height of the mirror as small, that is,

\[
h \ll \min(z, z_0) .
\]

This is equivalent to regarding the reflection as taking place at \( r' \) on the mirror plane \( z = 0 \) rather than \( z = h(r') \). The resulting ‘paraxial flat’ approximation gives the ray equation

\[
\text{paraxial flat} : r = \left( 1 + \frac{z}{z_0} \right) r' - \frac{z}{z_0} r_0 - 2 z \nabla h (r') .
\]

This gives the rays for mirrors that can be curved in both directions. The mirror (6) is curved only in the \( x \) direction, so the variables separate, and the two components of (11), including the trivial \( y \) equation, are

\[
x = \left( 1 + \frac{z}{z_0} \right) x' - \frac{z}{z_0} x_0 + \frac{6z x'^2}{a^2} , \quad y = y' + \frac{z}{z_0} (y' - y_0) .
\]
Figure 3. As figure 2, for $x_0 = 0$; (a) exact rays; (b) paraxial rays (9); (c) paraxial flat rays (11); the red dotted line is the caustic (13) in the paraxial flat approximation.

The focussing condition (5) can now be implemented explicitly, leading to the shape $x_{\text{caustic}}$ of the caustic:

$$\frac{\partial x}{\partial x'} = 0 \Rightarrow 1 + \frac{z}{z_0} + \frac{12zx'}{a^2} = 0 \Rightarrow x_{\text{caustic}} = -\frac{a^2}{24z} \left(1 + \frac{z}{z_0}\right)^2 - \frac{z}{z_0}x_0. \quad (13)$$

Figure 3 compares the exact ray pattern (a) with the paraxial (b) and flat paraxial (c) approximations. From (b), the pattern of paraxial rays is almost indistinguishable from the exact pattern. But (c) shows that the paraxial flat pattern misses one of the branches of the caustic, the cusp singularity where this branch is connected with the previous one, and one of the images of each source point. Nevertheless, the branch predicted by paraxial flatness very accurately reproduces that branch of the exact caustic. And we are here concentrating on the simplest image distortions and disruptions, associated with a single caustic, so it is justifiable to use the paraxial flat approximation for the simulations to follow.

4. Simulations

To simulate the distortions and disruptions of images, it is simplest to begin by considering objects that can be represented as boundary curves, i.e. one-dimensional sequences of source points, generating a family of families of rays.

As explained elsewhere, with many illustrations [13], topological changes of such images are described by the ‘caustic-touching theorem’. This envisages a family of false light rays from the viewing eye. When reflected into the region where the object is, these rays will form caustics: surfaces in 3D, curves in 2D. When the caustics of false light touch the boundary curve representing the object, its image will be undergoing a topological change. Central to the theorem is the symmetry of the optical path length under exchange of source and observation point: this implies that if B lies on the caustic of rays from a source at A, then A lies on the caustic of rays from a source at B. The topology changes are of two types: splitting (or reconnection) of two image curves (for example $\infty$ changing to $\circ \circ$), and shrinking and disappearance of an image loop (or its opposite).

To illustrate this process, consider an object in the form of a disk of radius $R$, centred on the viewer’s eye at $x_0$—a crude representation of her face (we choose $y_0 = 0$, but the same results follow for any value of $y_0$). Caustic-touching will occur at the two eye positions $x_0$ where the
two edges of the disk, at $x = \pm R$, coincide with the caustic, given for the cubic inflected mirror (6) by (13) with $z_0 = z$. Thus
\[ x = x_{\text{caustic}} = -\frac{a^2}{6z} - x_0 = \pm R + x_0, \quad (14) \]
whose solution is
\[ x_0 = \pm \frac{R}{2} - \frac{a^2}{12z}. \quad (15) \]
As $x_0$ decreases, the $+$ solution corresponds to image curve disconnection and the $-$ solution to disappearance of an image loop.

Figure 4 illustrates this process with deformations of the disk calculated by the mapping (7). The topological changes occur close to (a), and at (d). The stretching in (c) and (d) is a consequence of the high magnification of points within the disk whose caustic crosses the eye. Points around the original disk are mapped to scrambled sequences around its images; explicit examples have been described in detail and illustrated elsewhere [13].

For objects characterised by smooth intensity distributions, such as faces, images are more complicated than those of curves. Such objects can be represented by their set of intensity contours, generating a ‘family of families of families’ of rays (a family from each object point, a family of object points round each contour, and a family of contours). In the resulting sequence of images as the position $x_0$ varies, different features will transform at different $x_0$, as their curves touch the caustic of the false light from the eye.

Figure 5 is a series of images of a photograph, created by the mapping (7). The mapping could be applied pixel by pixel to the digitised photograph. In fact it is simpler to interpolate
Figure 5. Simulation of reflections of my grandson’s face as it passes across an inflection of a mirror. Adjacent images of one eye merge near (d), and disappear between (c) and (d); images of the other eye merge between (d) and (c), and disappear between (b) and (c); images of the nose merge and disappear between (c) and (d); images of the mouth merge between (d) and (e), and disappear between (c) and (d).

Figure 6. Inflected mirror, made from flexible polyester sheet.

between the pixels, converting the digital array to an intensity function on $-1 \leq x, y \leq 1$, and apply (7) directly (in Mathematica, the command to interpolate is ImageValue). Again the high magnification is evident, as revealed by the stretching of the images, especially in (d) and (e).

5. Observation

A mirror with an inflection can easily be created, without the need for a frame, from a flexible polyester sheet silvered on both sides, approximately 200 $\mu$m thick; this is thicker than reflecting Mylar, which crumples too easily, and thinner than acrylic sheet, which is more difficult
Figure 7. Merging and disappearance of two images of my wife’s face in the mirror of figure 6, as the head position is varied relative to the inflection; when the head moves in the other direction (not shown here), the third image, on the right, merges with the closer of the two, in a similar sequence. (See the movie, including the transformation involving the third image, in the supplementary material: inflected mirror.mp4). Why does the camera not appear in these images? I leave this as an exercise (hint: think y).

to bend. I used a sheet that is 2 m long and 0.6 m wide. When bent under itself as shown in figure 6, gravity forms an inflection naturally.

The image transformations can be seen looking down at the mirror. Of course these images are generated by reflections that are exact, not paraxial flat. Therefore the existence of the second branch of the caustic (cf, figure (2)) means that there will be three images or one, not two or zero. Figure 7 shows an example of a sequence of such images as the view traverses the mirror. As in the simulation of figure 5, different features (eyes, nose, spectacle lenses) merge and disappear at different stages of the traverse, and high magnification results in stretching. Another feature that is clear is that the middle image is parity-reversed, and the two outer images are not, for a movie, see inflected mirror.mp4 in supplementary material.

6. Concluding remarks

An aim of this paper has been to emphasise that geometrical optics is not trivial, even when the only physics involved is the ancient law of specular reflection combined with the rectilinear propagation of rays. Understanding the curiously deformed and divided images of an object, whose reflection is viewed in a mirror whose curvature changes sign, involves the multiple rays reaching the eye, caustics in the family of rays emitted by each object point, and the family of families of rays from the different object points.

The arguments presented here do not exhaust the subtle and varied deformations of images in curved mirrors, and their changes as the view is varied. For illustrative purposes, we have considered mirrors with inflections, because these provide the simplest models. But caustics, and therefore multiple images, can be generated in reflections from concave surfaces whose
curvature does not change sign, for example the insides of coffee cups, and parabolic and paraboloidal reflectors [16].

More fundamentally, we have considered only images associated with the simplest caustics, namely smooth curves in 2D and surfaces in 3D. Even in the simple cubically inflected model, with exact and paraxially approximated rays, these simplest caustics can connect at cusps, as illustrated in figures 2 and 3(b). Close to cusps, three images can collide and transform triply as well as pairwise; an example was considered in [9] (see also the supplementary movie, and [13]). More complicated caustics can occur. The geometries of those that are typical (in a well-defined sense of stability under perturbation), are classified by the mathematics of catastrophe theory [14, 17, 18]. The simplest caustic in this classification—the smooth curves or surfaces considered here—is the fold catastrophe.

An important distinction is between mirrors curved in one direction, whose profiles are described by functions $h(x')$ of a single variable (such as our inflected model (6)), whose Gaussian curvature vanishes, and mirrors with non-zero Gaussian curvature, with profiles $h(r')$ varying in both directions. The former reflect ray families with caustics in the cuspoid family: folds, cusps, swallowtails, etc, and reflections from the latter include different caustics, such as the elliptic and hyperbolic umbilic catastrophes [11, 19–22]. A start has been made in understanding distortions and disruptions of images associated with these more general caustics [23].

Notwithstanding the subtleties of caustic singularities and associated image deformations, geometrical optics is the most elementary description of light. The next deeper level is wave optics. The decoration of geometrical caustics by wave interference patterns is well understood [12]; in the simplest case, of the smooth caustics considered here, the wave decoration is described by the Airy function [22, 24]. This is not the subject of the present paper. However, for inflected mirrors there is an important problem concerning rays and waves that deserves mention because it is unsolved and of current interest. Very close to the mirror and on the concave side, rays reflect repeatedly while skipping along the surface, and form a caustic just above the surface. The associated waves are localised near the mirror while propagating along it. For closed concave surfaces, rays that skip many times correspond to ‘whispering-gallery modes’ [25]. On the convex side, there are no whispering-gallery modes; instead, there are creeping waves and rays [26–29] propagating close to the mirror while getting weaker as they shed waves tangentially into the space above. The whispering-gallery modes and the creeping waves are well understood, but the transformation of one into the other across an inflection is not [30–32].

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