On the Local Lipschitz Robustness of Bayesian Inverse Problems

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Abstract

In this note we consider the robustness of posterior measures occurring in Bayesian inference w.r.t. perturbations of the prior measure and the log-likelihood function. This extends the well-posedness analysis of Bayesian inverse problems. In particular, we prove a general local Lipschitz continuous dependence of the posterior on the prior and the log-likelihood w.r.t. various common distances of probability measures. These include the Hellinger and Wasserstein distance and the Kullback–Leibler divergence. We only assume the boundedness of the likelihoods and measure their perturbations in an $L^p$-norm w.r.t. the prior. The obtained robustness yields under mild assumptions the well-posedness of Bayesian inverse problems, in particular, a well-posedness w.r.t. the Wasserstein distance which is missing in the existing literature. Moreover, our results indicate an increasing sensitivity of Bayesian inference as the posterior becomes more concentrated, e.g., due to more or more accurate data. This confirms and extends previous observations made in the sensitivity analysis of Bayesian inference.

Keywords: Bayesian inference, robust statistics, inverse problems, well-posedness, Hellinger distance, Wasserstein distance, Kullback–Leibler divergence

Mathematics Subject Classification: 60B10, 62C10, 62F15, 62G35, 65N21

1 Introduction

In recent years, Bayesian inference has become a popular approach to model and solve inverse problems in various fields of applications, see, e.g., [7, 26] for a comprehensive introduction. Here, noisy observations are used to update the knowledge of unknown parameters from a given prior distribution to a resulting posterior distribution. The relation between the parameters and the observable quantities are given by a measurable forward map which, in combination with an assumed error distribution, determines the likelihood function for the data given the parameter. The Bayesian approach is quite appealing and, in particular, yields a well-posed inverse problem [7, 21, 22, 29, 38, 40], i.e., its unique solution—the posterior distribution—depends (locally Lipschitz) continuously on the observational data and behaves also robustly w.r.t. (numerical) approximations of the forward map. However, besides the observed data and the employed likelihood model, the subjective choice of the prior distribution significantly affects the outcome of the Bayesian inference, too. In order to account for that a robust Bayesian analysis has emerged, where a class of suitable priors or likelihoods is considered and the range of all resulting posterior quantities or statistics is computed or estimated, see, e.g., [2, 25] for an introduction. Moreover, the well-known Bernstein–von Mises theorem [41] establishes a kind of “asymptotic robustness” at least in finite-dimensional spaces. This theorem tells us that, under suitable assumptions, the posterior measure concentrates around the true parameter, which generates the observations, as more and more data is observed. This convergence to the truth is called consistency and it is independent of
the chosen prior measure as long as the true parameter belongs to its support. However, for posterior measures on infinite-dimensional spaces the situation is far more delicate, and positive as well as negative results for consistency exist, see, e.g., [4, 5, 14, 30]. Furthermore, in [32, 33, 34] the authors show an extreme non-robustness of Bayesian inference—called Bayesian brittleness—w.r.t. small perturbations of the likelihood model as well as w.r.t. classes of priors based on only finitely many pieces of information. In particular, the range of attainable posterior quantities (e.g., expectations or probabilities) over a class of allowed priors and likelihoods covers the essential (prior) range of the quantity of interest. This brittleness occurs for arbitrarily many data and arbitrarily small perturbations of the likelihood model. However, the distance used to measure the size of the perturbations plays a crucial role here as we will discuss later on.

In this paper we take a slightly different approach than the classical robust Bayesian analysis: Instead of bounding the resulting posterior range of certain quantities or statistics of interest for a given class of admissible priors or likelihood models, we rather study whether the distance between the posterior measures themselves can be bounded uniformly by a constant multiplied with the distance of the corresponding prior measures or log-likelihood functions. Thus, the goal is to establish a (local) Lipschitz continuity of the posterior w.r.t. the prior or the log-likelihood with explicit bounds on the local Lipschitz constant. To this end, we employ the following common distances and divergences for (prior and posterior) probability measures: the total variation, Hellinger, and Wasserstein distance as well as the Kullback–Leibler divergence. Perturbations of the log-likelihood function are measured by suitable $L^p$-norms w.r.t. the prior measure. Indeed, under rather mild assumptions we can state the local Lipschitz continuity of posteriors on general Polish spaces w.r.t. the prior and the log-likelihood for all distances and divergences listed above. On the other hand, our estimates show that the sensitivity of the posterior to perturbations of prior or log-likelihood increases as the posterior concentrates—an observation also made in [11, 19]. We discuss this issue and its relation to the Bernstein–von Mises theorem in Section 3 and 5 in detail.

As mentioned at the beginning, a local Lipschitz dependence of the posterior measure w.r.t. the observational data and approximations of the forward map has been proven in [7, 38] for Gaussian and Besov priors and the Hellinger distance. These results have been generalized to heavy-tailed prior measures in [21, 22, 40] and a continuous dependence in Hellinger distance was recently shown under substantially relaxed conditions in [29]. In the latter work a continuous dependence on the data was also established concerning the Kullback–Leibler divergence between the resulting posteriors. Moreover, in [31] it is shown that converging approximations of the forward map yield the convergence of the perturbed posteriors to the true posterior in terms of their Kullback–Leibler divergence. These previous results relate, of course, to our robustness statements for perturbed log-likelihoods.

However, the focus of this note is rather on the general structure of the local Lipschitz dependence on the log-likelihood and the prior measure. In fact, robustness w.r.t. the data or approximations of the forward map follows—under suitable assumptions—from our general results. Besides that, the local Lipschitz dependence of the posterior on the prior has not been established in the literature to the best of our knowledge. Furthermore, a robustness or well-posedness analysis of Bayesian inverse problems in Wasserstein distance (i.e., perturbations of posterior and prior are measured in Wasserstein distance) has also been missing. This distance is of particular interest for studying the robustness w.r.t. perturbations of the prior measure on infinite-dimensional spaces such as function spaces. The reason behind is that the total variation and Hellinger distance as well as the Kullback–Leibler divergence obtain their maximum value for mutually singular measures and probability measures on infinite-dimensional spaces tend to be singular—cf., the necessary conditions for Gaussian measures on Hilbert spaces to be equivalent [3, 28]. Thus, these distances and divergences may be of little practical use for prior robustness whereas the Wasserstein distance of perturbed priors does not rely on their equivalence. Besides that, the Wasserstein distance has been proven quite flexible and useful for various topics in probability theory such as convergence of Markov processes [20] and perturbation theory for Markov chains [37]. We establish a first robustness analysis of Bayesian inverse problems w.r.t. the Wasserstein distance and show how the general robustness result yields a well-posedness of Bayesian inverse
problems in Wasserstein distance. For the latter we use the same basic assumptions stated in [7, 29, 38] for the well-posedness in Hellinger distance.

In summary, this paper contributes to the robustness analysis of Bayesian inference and provides positive statements in a quite general setting. Our results, although quantitative, are rather qualitative nature establishing a local Lipschitz robustness and, on the other hand, illustrating the increasing sensitivity of the posterior to perturbations in prior or log-likelihood for an increasingly informative likelihood. Our setting considers bounded likelihoods which excludes, e.g., the case of infinite-dimensional observations (cf. [7, Section 3.3]).

The outline of the paper is as follows: in the next section we introduce the general setting and the common form of our main results. We also discuss their relation to classical robust Bayesian analysis and Bayesian brittleness. In Section 3 to 5 we provide the exact statements and proofs for robustness in the Hellinger (and total variation) distance, Kullback–Leibler divergence, and Wasserstein distance. In particular, we establish in Section 5 the well-posedness of Bayesian inverse problems in Wasserstein distance. Furthermore, the relation of the obtained robustness statements to the existing literature and results on robust Bayesian analysis is discussed in Section 6. The Appendix includes some more detailed explanations and calculations on the relation between Bayesian brittleness and robustness as well as an explicit computation for the Hellinger distance of Gaussian measures on separable Hilbert spaces.

2 Setting and Main Results

Throughout this paper let \((E, d_E)\) be a complete and separable metric space with Borel \(\sigma\)-algebra \(\mathcal{E}\) and let \(\mathcal{P}(E)\) denote the set of all probability measures \(\mu\) on \((E, \mathcal{E})\). We also consider the special case of a separable Hilbert space \(H\) with norm \(\| \cdot \|_H\).

In this paper we focus on posterior probability measures \(\mu_\Phi \in \mathcal{P}(E)\) of the form

\[
\mu_\Phi(dx) := \frac{1}{Z} \exp(-\Phi(x)) \mu(dx),
\]

resulting from a prior measure \(\mu \in \mathcal{P}(E)\) and a measurable negative log-likelihood \(\Phi : E \to \mathbb{R}_+ := [0, \infty)\). The constant \(Z := \int_E e^{-\Phi(x)} \mu(dx) \in (0, \infty)\) denotes the normalization constant, sometimes called evidence. The assumption that \(\Phi(x) \geq 0\) is convenient and not very restrictive, since any \(\mu_\Phi\) of the form (1) with \(\Phi : E \to \mathbb{R}\) being bounded from below, i.e., \(\inf_{x \in E} \Phi(x) > -\infty\), can be rewritten as \(\mu_\Phi(dx) = \exp(-\Phi(x) - \inf \Phi)) / (Z \exp(-\inf \Phi)) \mu(dx)\).

Posterior measures as in (1) occur, for example, when we consider the Bayesian approach to inverse problems such as reconstructing an unknown \(x^\dagger \in E\) based on noisy data

\[
y = G(x^\dagger) + \epsilon,
\]

of a measurable forward map \(G : E \to \mathbb{R}^d\) with additive noise \(\epsilon \in \mathbb{R}^d\). Inverse problems are typically ill-posed and require some kind of regularization. In the Bayesian approach we employ a probabilistic regularization, i.e., we incorporate a-priori knowledge about \(x^\dagger\) by a prior probability measure \(\mu \in \mathcal{P}(E)\), and assume a random noise \(\epsilon \sim \nu_\epsilon \in \mathcal{P}(\mathbb{R}^d)\), i.e., \(\epsilon\) in the equation above is viewed as a realization of the random variable \(\epsilon\). The Bayesian approach then consists of conditioning the prior measure \(\mu\) on observing the data \(y \in \mathbb{R}^d\) as a realization of the random variable

\[
Y := G(X) + \epsilon, \quad X \sim \mu, \ \epsilon \sim \nu_\epsilon \quad \text{stochastically independent}.
\]

This results in a conditional or posterior probability measure on \(E\). If the noise distribution allows for a bounded Lebesgue density \(\nu_\epsilon(\epsilon) \propto \exp(-\ell(\epsilon))d\epsilon\), with \(\ell : \mathbb{R}^d \to \mathbb{R}\) being bounded from below, \(\inf_{\epsilon \in \mathbb{R}^d} \ell(\epsilon) > -\infty\), then the posterior measure of \(X \sim \mu\) given that \(Y = y\) is of the form (1) with \(\Phi(x) = \Phi(x, y) :=
\]
\(\ell(y - G(x))\), see, e.g., [7, 29, 38]. A common noise model is a mean-zero Gaussian noise, i.e., \(\nu_x = N(0, \Sigma)\), where \(\Sigma \in \mathbb{R}^{d \times d}\) is symmetric and positive definite, which yields \(\Phi(x) = \Phi(x, y) = \frac{1}{2}\|\Sigma^{-1/2}(y - G(x))\|^2\) where \(\|\cdot\|\) denotes the Euclidean norm on \(\mathbb{R}^d\). For clarity, we omit the dependence of the observational data \(y\) in \(\Phi\) most of the time. The main focus of this paper is to study the robustness of the posterior \(\mu_\Phi\) w.r.t. perturbations of the log-likelihood model \(\Phi\) as well as the chosen prior measure \(\mu\).

An interesting and important special case in practice are Bayesian inverse problems in function spaces, i.e., where \(E\) is a separable Hilbert space such as \(\mathcal{H} = L^2(D)\) with \(D \subset \mathbb{R}^n\) denoting a spatial domain. In such situations Gaussian measures \(\mu = N(m, C)\) on \(\mathcal{H}\) are a convenient class of prior measures, see, e.g., [38, 7, 36, 12]. Often the mean \(m \in \mathcal{H}\) and the trace-class covariance operator \(C : \mathcal{H} \rightarrow \mathcal{H}\) are chosen themselves from parametric classes. For instance, we may suppose a linear model for the mean \(m = \sum_{j=1}^J \theta_j \phi_j\), where \(\theta_j, \phi_j \in \mathcal{H},\) with parameter \(\theta = (\theta_1, \ldots, \theta_J) \in \mathbb{R}^J\). Or, furthermore, we may consider Gaussian prior measures on suitable function spaces \(\mathcal{H} \subseteq L^2(D)\) with covariance operators belonging to the Matérn class, i.e., \(C = C_{\alpha, \beta, \gamma} = \beta(I + \gamma^2 \Delta)^{-\alpha}\) with parameters \(\alpha, \beta, \gamma > 0\) and \(\Delta\) denoting the Laplace operator, see, e.g., [36, 12]. Since in practice the so-called hyperparameters \(\theta, \alpha, \beta, \gamma\) are often estimated by statistical procedures, a robustness of the resulting posterior measure w.r.t. slightly different means, covariances, or hyperparameters of the corresponding Gaussian prior measures seems highly desirable. Therefore, we remark in each of the following sections on particular bounds for posteriors resulting from perturbed Gaussian priors.

**Main Results.** We are interested in the stability of the posterior measure \(\mu_\Phi \in \mathcal{P}(E)\) w.r.t. perturbations of the negative log-likelihood function \(\Phi : E \rightarrow [0, \infty)\) and the prior measure \(\mu \in \mathcal{P}(E)\). The former includes, for instance, for \(\Phi(x) = \ell(y - G(x))\) perturbations of the observed data \(y \in \mathbb{R}^d\) or the forward map \(G\), e.g., due to numerical approximations of \(G\), cf. Remark 1 below. We then bound the difference between the original posterior \(\mu_\Phi\) and two kinds of perturbed posteriors:

1. perturbed posteriors \(\tilde{\mu}_\Phi \in \mathcal{P}(E)\) resulting from perturbed log-likelihood functions \(\tilde{\Phi} : E \rightarrow \mathbb{R}\), i.e.,
   \[
   \tilde{\mu}_\Phi(dx) := \frac{1}{Z} e^{-\tilde{\Phi}(x)} \, \mu(dx), \quad \tilde{Z} := \int_E e^{-\tilde{\Phi}(x)} \, \mu(dx),
   \]

2. perturbed posteriors \(\bar{\mu}_\Phi \in \mathcal{P}(E)\) resulting from perturbed prior measures \(\bar{\mu} \in \mathcal{P}(E)\), i.e.,
   \[
   \bar{\mu}_\Phi(dx) := \frac{1}{Z} e^{-\Phi(x)} \, \bar{\mu}(dx), \quad \bar{Z} := \int_E e^{-\Phi(x)} \, \bar{\mu}(dx).
   \]

In particular, we prove the local Lipschitz continuity of \(\mu_\Phi \in \mathcal{P}(E)\) w.r.t. the log-likelihood \(\Phi \in L^p_\mu(\mathbb{R}_+)\) and the prior \(\mu \in \mathcal{P}(E)\) in several common distances and divergences for probability measures \(d : \mathcal{P}(E) \times \mathcal{P}(E) \rightarrow [0, \infty]\). Here, \(L^p_\mu(\mathbb{R}_+)\) denotes the set of non-negative functions which are \(p\)-integrable w.r.t. the prior measure \(\mu\). For \(d\) we consider the total variation distance, the Hellinger distance, the Kullback–Leibler divergence, and the Wasserstein distance. Given suitable assumptions our results are then of the following form:

1. For a given prior \(\mu \in \mathcal{P}(E)\) and a log-likelihood \(\Phi \in L^p_\mu(\mathbb{R}_+)\) with suitable \(p \in \mathbb{N}\), there exists a radius \(r = r_{\mu, \Phi} > 0\) and a constant \(C_{\mu, \Phi, r} < \infty\) such that
   \[
   d(\mu_\Phi, \tilde{\mu}_\Phi) \leq C_{\mu, \Phi, r} \|\Phi - \tilde{\Phi}\|_{L^p_\mu} \quad \forall \tilde{\Phi} \in L^p_\mu(\mathbb{R}_+): \|\Phi - \tilde{\Phi}\|_{L^p_\mu} < r.
   \]

2. Similarly, there exists a radius \(r = r_{\mu, \Phi} > 0\) and a constant \(C_{\mu, \Phi, r} < \infty\) such that
   \[
   d(\mu_\Phi, \bar{\mu}_\Phi) \leq C_{\mu, \Phi, r} d(\mu, \bar{\mu}) \quad \forall \bar{\mu} \in \mathcal{P}(E): d(\mu, \bar{\mu}) < r.
   \]
3. In both cases the estimated local Lipschitz constant \( C_{\mu^*,\varphi, r} \) is proportional to \( Z^{-1} \), i.e., it increases if the normalization constant \( Z \) of \( \mu_\Phi \) decreases due to, for instance, a higher concentration or localization of the posterior measure.

Thus, besides positive robustness results our bounds suggest an increasing sensitivity of the posterior w.r.t. perturbations of the prior or log-likelihood for increasingly informative \( \Phi \), e.g., due to more or more precise observations employed in the Bayesian inference.

**Remark 1.** The bound \( d(\mu_\Phi, \mu_\Phi') \leq C_{\mu, \varphi, r} \| \Phi - \tilde{\Phi} \|_{L^p_\mu} \) can usually be used to prove local Lipschitz continuous dependence of the posterior on the data or show robustness w.r.t. numerical approximations, say \( G_h \), of the forward map \( G : E \to \mathbb{R}^d \). For example, for an additive Gaussian noise \( \varepsilon \sim N(0, \Sigma) \) we can set \( \tilde{\Phi}(x) := \frac{1}{2} |y - G_h(x)|^2 \) and obtain

\[
|\Phi(x) - \tilde{\Phi}(x)| \leq (2|y|)_{\Sigma^{-1}} + |G(x)|_{\Sigma^{-1}} + |G_h(x)|_{\Sigma^{-1}} |G(x) - G_h(x)|.
\]

If \( G, G_h \in L^2_{\mu_\Phi}(\mathbb{R}^d) \), then the Cauchy–Schwarz inequality yields

\[
d(\mu_\Phi, \mu_\Phi') \leq C_{\mu, \varphi, r} C_{\Sigma^{-1}} \left( |y| + \|G\|_{L^2_{\mu_\Phi}} + \|G_h\|_{L^2_{\mu_\Phi}} \right) \|G - G_h\|_{L^2_{\mu_\Phi}}.
\]

Analogous expressions can be obtained for the case of perturbed data \( \tilde{y} \in \mathbb{R}^d \), i.e., for \( \tilde{\Phi}(x) := \frac{1}{2} |\tilde{y} - G(x)|^2 \).

### 3 Robustness in Hellinger Distance

First, we study the continuity of the posterior measure \( \mu_\Phi \in \mathcal{P}(E) \) w.r.t. the log-likelihood function \( \Phi : E \to (0, \infty) \) and the prior measure \( \mu \in \mathcal{P}(E) \) in the Hellinger distance

\[
d^2_H(\mu, \tilde{\mu}) := \int_E \left( \sqrt{\frac{d\mu}{d\nu}(x)} - \sqrt{\frac{d\tilde{\mu}}{d\nu}(x)} \right)^2 \nu(dx), \quad \mu, \tilde{\mu} \in \mathcal{P}(E).
\]

Here, \( \nu \) denotes an arbitrary (probability) measure on \( E \) dominating \( \mu \) and \( \tilde{\mu} \), e.g., \( \nu = \frac{1}{2} \mu + \frac{1}{2} \tilde{\mu} \). The Hellinger distance is equivalent to the total variation distance

\[
d_{TV}(\mu, \tilde{\mu}) := \sup_{A \in \mathcal{E}} |\mu(A) - \tilde{\mu}(A)| = \frac{1}{2} \int_E \left| \frac{d\mu}{d\nu}(x) - \frac{d\tilde{\mu}}{d\nu}(x) \right| \nu(dx),
\]

due to

\[
\frac{1}{2} d^2_H(\mu, \tilde{\mu}) \leq d_{TV}(\mu, \tilde{\mu}) \leq d_H(\mu, \tilde{\mu}), \quad \mu, \tilde{\mu} \in \mathcal{P}(E),
\]

see, e.g., [17], but it yields also continuity of the moments of square-integrable functions, see [7, Theorem 21]. We investigate now the robustness of the posterior \( \mu_\Phi \) w.r.t. \( \Phi \) in Hellinger distance. The related issue of robustness w.r.t. the data \( y \) and numerical approximations \( \Phi_h \) of \( \Phi \), cf. Remark 1, was already stated for the Bayesian inference with additive Gaussian noise and a Gaussian prior \( \mu \) by Stuart [38] and recently extended by Dashti and Stuart [7] and Latz [29] to a more general setting. Moreover, in [39, Section 4] we can already find a similar result under slightly different assumptions. Nonetheless, we state the theorem and proof for completeness.

**Theorem 2.** Let \( \mu \in \mathcal{P}(E) \) and \( \Phi, \tilde{\Phi} : E \to \mathbb{R} \) belong to \( L^2_{\mu}(\mathbb{R}) \) with \( \text{ess inf}_\mu \Phi = 0 \). Then, for the two probability measures \( \mu_\Phi, \mu_\Phi' \in \mathcal{P}(E) \) given by (1) and (2), respectively,

\[
d_H(\mu_\Phi, \mu_\Phi') \leq \frac{\exp(-[\text{ess inf}_\mu \tilde{\Phi}]_\mu)}{\min(Z, \tilde{Z})} \| \Phi - \tilde{\Phi} \|_{L^2_{\mu}},
\]

where \( [t]_\mu := \min(0, t) \), \( t \in \mathbb{R} \), and \( |Z - \tilde{Z}| \leq \exp(-[\text{ess inf}_\mu \tilde{\Phi}]_\mu) \| \Phi - \tilde{\Phi} \|_{L^1_{\mu}} \).
Proof. Analogously to the proof of [38, Theorem 4.6] or [39, Theorem 4.2] we start with

\[
\begin{align*}
    d_H(\mu_\Phi, \bar{\mu}_\Phi)^2 &= \int_E \left( \frac{e^{-\Phi(x)/2}}{\sqrt{Z}} - \frac{e^{-\bar{\Phi}(x)/2}}{\sqrt{Z}} \right)^2 \mu(dx) \\
    &\leq 2 \int_E \left( \frac{e^{-\Phi(x)/2}}{\sqrt{Z}} - \frac{e^{-\bar{\Phi}(x)/2}}{\sqrt{Z}} \right)^2 + \left( \frac{e^{-\Phi(x)/2}}{\sqrt{Z}} - \frac{e^{-\bar{\Phi}(x)/2}}{\sqrt{Z}} \right)^2 \mu(dx) = I_1 + I_2,
\end{align*}
\]

where

\[
I_1 := \frac{2}{Z} \int_E \left( e^{-\Phi(x)/2} - e^{-\bar{\Phi}(x)/2} \right)^2 \mu(dx), \quad I_2 := \frac{2}{Z} \left( \frac{1}{\sqrt{Z}} - \frac{1}{\sqrt{Z}} \right)^2 = \frac{2}{Z} \left( \sqrt{Z} - \sqrt{Z} \right)^2.
\]

Since \(|e^{-t} - e^{-s}| = e^{-\min(t,s)} |1 - e^{-|t-s|}| \leq e^{-\min(t,s)} |t - s|\) for any \(t, s \in \mathbb{R}\), we obtain

\[
I_1 \leq \frac{2 \exp(-\text{ess inf}_\Phi)}{Z} \int_E \frac{|\Phi(x) - \bar{\Phi}(x)|^2}{4} \mu(dx) = \exp(-\text{ess inf}_\Phi) \|\Phi - \bar{\Phi}\|_{L_\mu}^2
\]

and since \(|t^{1/2} - s^{1/2}| \leq \frac{1}{2} \min(t, s)^{-1/2} |t - s|\) for \(t, s > 0\) we have

\[
I_2 \leq \frac{1}{2Z \min(Z, \bar{Z})} |Z - \bar{Z}|^2.
\]

Now, as for \(I_1\) we obtain

\[
|Z - \bar{Z}| \leq \int_E |e^{-\Phi(x)} - e^{-\bar{\Phi}(x)}| \mu(dx) \leq \exp(-\text{ess inf}_\Phi) \|\Phi - \bar{\Phi}\|_{L_\mu}^2
\]

and due to \(Z \leq 1\) we have

\[
\frac{1}{2Z} + \frac{1}{2Z \min(Z, \bar{Z})} \leq \frac{1}{2 \min(Z, \bar{Z})^2} + \frac{1}{2 \min(Z, \bar{Z})^2} = \frac{1}{\min(Z, \bar{Z})^2}
\]

which concludes the proof. \(\square\)

The statement of Theorem 2 implies that for a given prior \(\mu \in \mathcal{P}(E)\) there exists for each \(\Phi \in L_\mu^2(\mathbb{R}_+)\) a radius \(r_{\Phi,\mu} := \frac{1}{2} Z = \frac{1}{2} \int_E e^{-\Phi} d\mu > 0\) and a constant \(C_{\Phi,\mu} := 2/Z < \infty\) such that

\[
d_H(\mu_\Phi, \bar{\mu}_\Phi) \leq C_{\Phi,\mu} \|\Phi - \bar{\Phi}\|_{L_\mu^2} \quad \forall \bar{\Phi} \in L_\mu^2(\mathbb{R}_+) : \|\bar{\Phi} - \Phi\|_{L_\mu^2} < r_{\Phi,\mu}.
\]

This states the local Lipschitz continuity of the mapping \(L_\mu^2(\mathbb{R}_+) \ni \Phi \mapsto \mu_\Phi \in (\mathcal{P}(E), d_H)\). All results in the constant remainder of the paper will be of the form as in Theorem 2: we bound the distance of the posteriors by a constant times the distance of the log-likelihoods or priors where the constant may also depend on the perturbation but such that it can be bounded uniformly for all sufficiently small perturbations.

For robustness w.r.t. different priors we get the following result.

**Theorem 3.** Let \(\mu, \bar{\mu} \in \mathcal{P}(E)\) and \(\Phi : E \to [0, \infty)\) be measurable. Then, for \(\mu_\Phi, \bar{\mu}_\Phi \in \mathcal{P}(E)\) as in (1) and (3), respectively, we have

\[
d_H(\mu_\Phi, \bar{\mu}_\Phi) \leq \frac{2}{\min(Z, \bar{Z})} d_H(\mu, \bar{\mu}), \quad |Z - \bar{Z}| \leq d_H(\mu, \bar{\mu}).
\]
Proof. Let $\rho(x) := \frac{d\mu}{d\nu}(x)$ and $\tilde{\rho}(x) := \frac{d\tilde{\mu}}{d\nu}(x)$ denote the densities of $\mu$ and $\tilde{\mu}$ w.r.t. a dominating $\nu \in \mathcal{P}(E)$. Then, we have

$$
\frac{d\mu}{d\nu}(x) = \frac{d\mu}{d\nu}(x) \frac{d\mu}{d\nu}(x) = \frac{e^{-\Phi(x)}}{Z} \rho(x), \quad \frac{d\tilde{\mu}}{d\nu}(x) = \frac{d\tilde{\mu}}{d\nu}(x) \frac{d\tilde{\mu}}{d\nu}(x) = \frac{e^{-\Phi(x)}}{Z} \tilde{\rho}(x),
$$

where $Z = \int_E e^{-\Phi(x)} \rho(x) \nu(dx)$ and $\tilde{Z} = \int_E e^{-\Phi(x)} \tilde{\rho}(x) \nu(dx)$. We obtain analogously to Theorem 2

$$
d_H^2(\mu, \tilde{\mu}) = \int_E \left( \sqrt{\frac{d\mu}{d\nu}(x)} - \sqrt{\frac{d\tilde{\mu}}{d\nu}(x)} \right)^2 \nu(dx)
$$

$$
= \int_E \left( e^{-\Phi(x)/2} \sqrt{\frac{\rho(x)}{Z}} - e^{-\Phi(x)/2} \sqrt{\frac{\tilde{\rho}(x)}{Z}} \right)^2 \nu(dx)
$$

$$
\leq 2 \int_E e^{-\Phi(x)} \left( \frac{\sqrt{\rho(x)}}{\sqrt{Z}} - \frac{\sqrt{\tilde{\rho}(x)}}{\sqrt{Z}} \right)^2 + \frac{\nu(dx)}{\sqrt{Z}} \left( \frac{\sqrt{\rho(x)}}{\sqrt{Z}} - \frac{\sqrt{\tilde{\rho}(x)}}{\sqrt{Z}} \right)^2 \nu(dx)
$$

$$
= I_1 + I_2
$$

where

$$
I_1 := \frac{2}{Z} \int_E e^{-\Phi(x)} \left( \sqrt{\rho(x)} - \sqrt{\tilde{\rho}(x)} \right)^2 \nu(dx), \quad I_2 := \frac{2}{Z} \left( \sqrt{\frac{\tilde{Z}}{Z}} - \sqrt{\frac{Z}{Z}} \right)^2.
$$

Due to $\Phi(x) \geq 0$ we get

$$
I_1 \leq \frac{2}{Z} \int_E \left( \sqrt{\rho(x)} - \sqrt{\tilde{\rho}(x)} \right)^2 \nu(dx) = \frac{2}{Z} \int_E \left( \sqrt{\frac{d\mu}{d\nu}(x)} - \sqrt{\frac{d\tilde{\mu}}{d\nu}(x)} \right)^2 \nu(dx) = \frac{2}{Z} d_H^2(\mu, \tilde{\mu}).
$$

Moreover, as in the proof of Theorem 2, we have that $I_2 \leq \frac{1}{2Z \min(Z, Z)} |Z - \tilde{Z}|^2$ and due to (5)

$$
|Z - \tilde{Z}| \leq \int_E e^{-\Phi(x)} |\rho(x) - \tilde{\rho}(x)| \nu(dx) \leq 2d_{TV}(\mu, \tilde{\mu}) \leq 2d_H(\mu, \tilde{\mu}).
$$

Hence, since $Z, \tilde{Z} \leq 1$ we obtain

$$
d_H^2(\mu, \tilde{\mu}) \leq I_1 + I_2 \leq \left( \frac{2}{Z} + \frac{2}{Z \min(Z, Z)} \right) d_H^2(\mu, \tilde{\mu}) \leq \frac{4d_H^2(\mu, \tilde{\mu})}{\min(Z, Z)^2}.
$$

Remark 4 (Robustness w.r.t. total variation). For the total variation distance we obtain similar to Theorem 2 and Theorem 3 that for $\mu, \tilde{\mu} \in \mathcal{P}(E)$ and $\Phi, \tilde{\Phi} : E \to \mathbb{R}$ with $\inf_x \Phi(x) = 0$ we have for $\mu, \tilde{\mu},$ and $\tilde{\mu}$ as in (1), (2), and (3), respectively, that

$$
d_{TV}(\mu, \tilde{\mu}) \leq \frac{\exp(-\left[\inf_x \Phi(x)\right]_+)}{Z} \left\| \Phi - \tilde{\Phi} \right\|_{L^1(\mu)} \leq \frac{2}{Z} d_{TV}(\mu, \tilde{\mu}) \leq \frac{2}{Z} d_{TV}(\mu, \tilde{\mu}).
$$

Thus, for robustness in total variation norm, we only need that $\Phi, \tilde{\Phi} \in L^1(\mathbb{R})$ instead of $\Phi, \tilde{\Phi} \in L^2(\mathbb{R}).$

Remark 5 (Increasing sensitivity). The bounds established in Theorem 2 and 3 as well as in Remark 4 involve the inverse of the normalization constant $Z$ of $\mu$. This suggests that Bayesian inference becomes increasingly sensitive to perturbations of the log-likelihood or prior as the posterior $\mu$ concentrates due to more or more accurate data. This may seem counterintuitive given the well-known Bernstein–von Mises theorem [41, 27] in
asymptotic Bayesian statistics: under suitable conditions the posterior measure concentrates around the true, data-generating $x^\dagger \in E$ in the large data limit. This statement holds independently of the particular prior $\mu$ as long as $x^\dagger$ belongs to the support of the measure $\mu \in \mathcal{P}(E)$, i.e., as long as $x^\dagger \in \text{supp} \, \mu$. However, the latter resolves the alleged contradiction: given a suitable infinite space $E$ and a non-atomic prior $\mu \in \mathcal{P}(E)$—i.e., for each $x \in E$ we have $\lim_{r \to 0} \mu(B_r(x)) = 0$—we can construct for any $\epsilon < 0$ a perturbed prior $\tilde{\mu}$ with $d_{TV}(\mu, \tilde{\mu}) \leq \epsilon$ but $\tilde{\mu}(B_r(x^\dagger)) = 0$ for a sufficiently small radius $r = r(\epsilon)$; then $\mu_\Phi$ concentrates around $x^\dagger$ and $\tilde{\mu}_\Phi$ around another $x^\ast \in \text{supp} \, \tilde{\mu}$, see [27], and their total variation distance will tend to 1 since $x^\dagger \neq x^\ast$.

Similar arguments also apply to perturbations of the likelihood function, since $x^\dagger$ is typically the minimizer of the log-likelihood $\Phi$ on $\text{supp} \, \mu$, and, therefore, we can construct perturbed $\tilde{\Phi}$ with a different minimizer $x^\ast \neq x^\dagger$ but with arbitrarily small $L^1$-distance $\|\Phi - \tilde{\Phi}\|_{L^1}$. Thus, in general, it is indeed the case that Bayesian inference becomes more sensitive w.r.t. perturbations of the likelihood or the prior as the amount of data or its accuracy increases. As we will see this also holds for other divergences and distances than the total variation or Hellinger distance.

Remark 6 (Robustness w.r.t. Gaussian priors). Concerning Gaussian priors $\mu = N(m, C)$ and $\tilde{\mu} = N(\tilde{m}, \tilde{C})$ on a separable Hilbert space $\mathcal{H}$ with norm $\| \cdot \|_\mathcal{H}$ we can bound the Hellinger distance of the resulting posteriors by Theorem 3. In order to obtain a non-trivial bound, we require that $\mu$ and $\tilde{\mu}$ are absolutely continuous w.r.t. each other, i.e., that $m - \tilde{m} \in \text{rg} \, C^{1/2} = \text{rg} \, \tilde{C}^{1/2}$, where $\text{rg} \, A \subseteq \mathcal{H}$ denotes the range of an operator $A: \mathcal{H} \to \mathcal{H}$, and $C^{-1/2} \tilde{C} C^{-1/2} - I$ is a Hilbert–Schmidt operator on $\mathcal{H}$, see, e.g., [3, Corollary 6.4.11] or [28, Section II.3]. Assuming furthermore that $T := C^{-1/2} \tilde{C} C^{-1/2}$ is a positive definite operator on $\mathcal{H}$, we can then use the exact expressions for the Hellinger distance of equivalent Gaussian measures:

$$
\begin{align*}
d_H^2(N(m, C), N(\tilde{m}, C)) &= 2 - 2 \exp \left( -\frac{1}{8} \| C^{-1/2} (m - \tilde{m}) \|_\mathcal{H}^2 \right), \\
d_H^2(N(m, C), N(\tilde{m}, \tilde{C})) &= 2 - 2 \left[ \det \left( \frac{1}{2} \sqrt{T} + \frac{1}{2} \sqrt{T^{-1}} \right) \right]^{-1/2}.
\end{align*}
$$

We provide a detailed derivation of these expressions in B and only make the following remarks here: (a) the inverse $T^{-1}$ exists and is bounded on $\mathcal{H}$, since $T$ is positive definite and $T - I$ is Hilbert–Schmidt, i.e., the smallest eigenvalue of $T$ is bounded away from zero; (b) the determinant $\det \left( \frac{1}{2} \sqrt{T} + \frac{1}{2} \sqrt{T^{-1}} \right)$ is well-defined as a Fredholm determinant, since $I - \left( \frac{1}{2} \sqrt{T} + \frac{1}{2} \sqrt{T^{-1}} \right)$ is trace-class, see B; and (c) we have $\det \left( \frac{1}{2} \sqrt{T} + \frac{1}{2} \sqrt{T^{-1}} \right) \geq 1$ due to $\sqrt{T} + \frac{1}{2} \sqrt{T^{-1}} \geq 1$ for $T > 0$.

If, moreover, $C^{-1/2} \tilde{C} C^{-1/2} - I$ is trace class we can bound

$$
\begin{align*}
d_{TV}^2(\mu, \tilde{\mu}) &\leq \frac{1}{4} \left( \text{tr} \left( C^{-1} \tilde{C} - I \right) + \| C^{-1/2} (m - \tilde{m}) \|_\mathcal{H}^2 - \log \det (C^{-1} \tilde{C}) \right) \\
&\leq \frac{3}{2} \| C^{-1/2} \tilde{C} - I \|_{HS} + \frac{1}{2} \| C^{-1/2} (m - \tilde{m}) \|_\mathcal{H}
\end{align*}
$$

where $\|A\|_{HS} := \sqrt{\text{tr} (A^* A)}$ denotes the Hilbert–Schmidt norm of Hilbert–Schmidt operators $A: \mathcal{H} \to \mathcal{H}$.

The first inequality for the total variation distance follows by Pinsker’s inequality and the Kullback–Leibler divergence of equivalent Gaussian measures, see next section, and the second one by means of [9]. Thus, using Remark 4 we can bound the total variation distance of posteriors resulting from Gaussian priors with different mean or covariance. However, we would like to remark that Gaussian priors on function spaces are often singular w.r.t. each other. For example, Gaussian priors with Matérn covariance operator $C = C_{\alpha,\beta,\gamma} = \beta (I + \gamma^2 \Delta)^{-\alpha}$ [36, 12] are singular for different values of $\alpha > 0$ or $\beta > 0$. We refer to [12] for a further discussion and for a particular subclass of equivalent Gaussian priors with Matérn covariance.

\*Similar requirements were essential for the brittleness results in [33, 34].
4 Robustness in Kullback–Leibler Divergence

A common way to compare the relative information between two probability measures $\mu, \tilde{\mu} \in \mathcal{P}(E)$ is to compute the Kullback–Leibler divergence (KLD) between them, which in case of existence of $\frac{d\mu}{d\tilde{\mu}}$ is

$$d_{KL}(\mu || \tilde{\mu}) := \int_E \log \left( \frac{d\mu}{d\tilde{\mu}}(x) \right) \mu(dx).$$

If $\mu$ is not absolutely continuous w.r.t. $\tilde{\mu}$, then $d_{KL}(\mu || \tilde{\mu}) := +\infty$. The KLD is not a metric for probability measures due to the lack of symmetry and triangle inequality\(^2\) but nonetheless an important quantity in information theory and optimal experimental design. Moreover, the total variation and Hellinger distance measures due to the lack of symmetry and triangle inequality.

**Theorem 7.** Let $\mu\in\mathcal{P}(E)$ and $\Phi, \tilde{\Phi} : E \to \mathbb{R}$ belong to $L^1_{\Phi}(\mathbb{R})$ with $\text{ess inf}_{\mu} \Phi = 0$. Then, for the two probability measures $\mu\Phi, \tilde{\mu}\tilde{\Phi} \in \mathcal{P}(E)$ given in (1) and (2), respectively,

$$d_{KL}(\mu || \tilde{\mu}) \leq \frac{2 \exp(-[\text{ess inf}_{\mu} \tilde{\Phi}]_{-})}{\min(Z, \tilde{Z})} \| \Phi - \tilde{\Phi} \|_{L^1_{\mu}},$$

and also $|Z - \tilde{Z}| \leq \| \Phi - \tilde{\Phi} \|_{L^1_{\mu}}$.

**Proof.** We have $\frac{d\mu_{\Phi}}{d\sqrt{\mu_{\tilde{\Phi}}}}(x) = \frac{d\mu_{\Phi}}{d\mu_{\tilde{\Phi}}}(x) = \frac{Z}{\tilde{Z}} e^{\tilde{\Phi}(x) - \Phi(x)}$ and, thus,

$$d_{KL}(\mu_{\Phi} || \mu_{\tilde{\Phi}}) = \int_E \log \left( \frac{Z}{\tilde{Z}} e^{\tilde{\Phi}(x) - \Phi(x)} \right) \mu_{\Phi}(dx) \leq |\log(Z) - \log(\tilde{Z})| + \int_E |\tilde{\Phi}(x) - \Phi(x)| \frac{e^{-\Phi(x)}}{Z} \mu(dx).$$

We further obtain

$$\int_E |\tilde{\Phi}(x) - \Phi(x)| \frac{e^{-\Phi(x)}}{Z} \mu(dx) \leq \frac{1}{Z} \int_E |\tilde{\Phi}(x) - \Phi(x)| \mu(dx) = \frac{\| \Phi - \tilde{\Phi} \|_{L^1_{\mu}}}{Z}$$

as well as due to $|\log t - \log s| \leq \frac{1}{\min(t,s)} |t - s|$ for $t, s > 0$ and $|e^{-t} - e^{-s}| \leq e^{-\min(t,s)} |t - s|$ for $t, s \in \mathbb{R}$ that

$$|\log(\tilde{Z}) - \log(Z)| \leq \frac{1}{\min(Z, \tilde{Z})} |Z - \tilde{Z}| \leq \frac{1}{\min(Z, \tilde{Z})} \int_E e^{-\Phi(x)} - e^{-\tilde{\Phi}(x)} \mu(dx) \leq \frac{\exp(-[\text{ess inf}_{\mu} \tilde{\Phi}]_{-})}{\min(Z, \tilde{Z})} \int_E |\Phi(x) - \tilde{\Phi}(x)| \mu(dx) = \frac{\exp(-[\text{ess inf}_{\mu} \tilde{\Phi}]_{-})}{\min(Z, \tilde{Z})} \| \Phi - \tilde{\Phi} \|_{L^1_{\mu}}.$$
Hence, we end up with
\[
d_{\text{KL}}(\mu_\Phi \| \mu_{\tilde{\Phi}}) \leq \frac{\exp(-[\text{ess inf}_\mu \Phi]-) \| \Phi - \tilde{\Phi} \|_{L_\mu^1}}{\min(Z, \bar{Z})} + \frac{\| \Phi - \tilde{\Phi} \|_{L_{\bar{\mu}}^1}}{Z} \]
\[
\leq \frac{2\exp(-[\text{ess inf}_\mu \Phi]-) \| \Phi - \tilde{\Phi} \|_{L_\mu^1}}{\min(Z, \bar{Z})}
\]
and \(|Z - \bar{Z}| \leq \| \Phi - \tilde{\Phi} \|_{L_{\bar{\mu}}^1}\) was already shown in Theorem 2.

An analogous proof to Theorem 7 also yields
\[
d_{\text{KL}}(\mu_\Phi \| \mu_{\tilde{\Phi}}) \leq \frac{2\exp(-[\text{ess inf}_\mu \Phi]-) \| \Phi - \tilde{\Phi} \|_{L_\mu^1}}{\min(Z, \bar{Z})}.
\] (9)

Concerning the following robustness statement w.r.t. perturbed priors \(\tilde{\mu} \in \mathcal{P}(E)\) we restrict ourselves to \(\tilde{\mu}\) which are equivalent to \(\mu\). Note that whenever \(\mu\) is not absolutely continuous to \(\tilde{\mu}\) then \(d_{\text{KL}}(\mu \| \tilde{\mu})\) as well as \(d_{\text{KL}}(\mu_\Phi \| \mu_{\tilde{\Phi}})\) is infinite. Thus, assuming that \(\mu\) and \(\tilde{\mu}\) are equivalent seems to be a rather mild restriction.

**Theorem 8.** Let \(\mu, \tilde{\mu} \in \mathcal{P}(E)\) be equivalent and \(\Phi : E \to [0, \infty)\) be measurable. Then, for \(\mu_\Phi, \tilde{\mu}_\Phi \in \mathcal{P}(E)\) given in (1) and (3), respectively,
\[
d_{\text{KL}}(\mu_\Phi \| \tilde{\mu}_\Phi) \leq \frac{1}{\min(Z, \bar{Z})} (d_{\text{KL}}(\mu \| \tilde{\mu}) + d_{\text{KL}}(\mu \| \mu_\Phi))
\]
as well as \(|Z - \bar{Z}| \leq \sqrt{\frac{1}{2}d_{\text{KL}}(\mu \| \tilde{\mu})}\).

**Proof.** Let \(\rho(x) := \frac{d\mu}{d\bar{\mu}}(x)\), then we have \(\frac{d\mu_\Phi}{d\tilde{\mu}_\Phi}(x) = \frac{Z}{\bar{Z}} \rho(x)\) and obtain
\[
d_{\text{KL}}(\mu_\Phi \| \tilde{\mu}_\Phi) = \int_E \log \left( \frac{\bar{Z}}{Z} \rho(x) \right) \mu_\Phi(dx) = \int_E \log(\rho(x)) \mu_\Phi(dx) - \log \left( \frac{Z}{\bar{Z}} \right)
\]
where the first term can be bounded as follows:
\[
\int_E \log(\rho(x)) \mu_\Phi(dx) = \int_E \log(\rho(x)) \frac{\exp(-\Phi(x))}{Z} \mu(dx) \leq \frac{1}{Z} d_{\text{KL}}(\mu \| \tilde{\mu}).
\]
Concerning the second term we first note that
\[
\frac{Z}{\bar{Z}} = \frac{1}{Z} \int_E e^{-\Phi(x)} \mu(dx) = \frac{1}{Z} \int_E e^{-\Phi(x)} \rho(x) \tilde{\mu}(dx) = \int_E \rho(x) \tilde{\mu}_\Phi(dx)
\]
and then apply Jensen’s inequality for the convex function \(t \mapsto -\log(t), t > 0\), to obtain
\[
-\log \left( \frac{Z}{\bar{Z}} \right) \leq \int_E -\log(\rho(x)) \tilde{\mu}_\Phi(dx) = \int_E \log \left( \frac{1}{\rho(x)} \right) e^{-\Phi(x)} \frac{Z}{\bar{Z}} \tilde{\mu}(dx) \leq \frac{1}{Z} \int_E \log \left( \frac{d\tilde{\mu}}{d\mu}(x) \right) \tilde{\mu}(dx),
\]
where we used that \(\frac{d\tilde{\mu}}{d\mu}(x) = \frac{1}{\rho(x)}\). Hence, we end up with
\[
d_{\text{KL}}(\mu_\Phi \| \tilde{\mu}_\Phi) \leq \frac{1}{Z} d_{\text{KL}}(\mu \| \tilde{\mu}) + \frac{1}{2} d_{\text{KL}}(\mu \| \mu)
\]
which yields the first statement. The second statement is a direct implication of Pinsker’s inequality (7) and \(|Z - \bar{Z}| \leq d_{\text{TV}}(\mu, \tilde{\mu})\).
Again, Theorem 8 also implies a bound for the alternative KLD
\[
d_{KL}(\tilde{\mu}_\Phi\|\mu_\Phi) \leq \frac{1}{\min(Z, \bar{Z})} \left( d_{KL}(\mu\|\bar{\mu}) + d_{KL}(\bar{\mu}\|\mu) \right). \tag{10}
\]

**Remark 9** (Kullback–Leibler divergence of Gaussian priors). Considering again Gaussian priors \( \mu = N(m, C) \) and \( \bar{\mu} = N(\bar{m}, \bar{C}) \) on separable Hilbert spaces \( \mathcal{H} \) with norm \( \| \cdot \|_\mathcal{H} \) there is an exact formula for their KLD [35] given both measures are equivalent and \( C^{-1/2}\bar{C}C^{-1/2} - I \) is a trace-class operator on \( \mathcal{H} \):
\[
d_{KL}(\bar{\mu}\|\mu) = \frac{1}{2} \left( \text{tr}(C^{-1}\bar{C} - I) + \|C^{-1/2}(m - \bar{m})\|_\mathcal{H}^2 - \log \det(C^{-1}\bar{C}) \right).
\]
Again, this can be used in combination with Theorem 8 in order to bound the KLD of posterior measures resulting from (equivalent) Gaussian priors in terms of the perturbations in the mean and covariance (parameters).

## 5 Robustness in Wasserstein Distance

In this section we focus on measuring perturbations of posterior and prior distributions in the Wasserstein distance. The main advantage of this metric is that it does not rely on the absolute continuity of distributions. Therefore, also for singular measures such as Dirac measures \( \delta_x, \delta_{\bar{x}} \in \mathcal{P}(E) \) at \( x \neq \bar{x} \in E \) the Wasserstein distance yields a sensible value. Besides that, the Wasserstein distance is based on the metric of the underlying space \( E \) which allows some flexibility in the application by employing a suitable metric.

We introduce the following spaces of probability measures on a complete and separable metric space \((E, d_E)\) given a \( q \geq 1 \):
\[
\mathcal{P}_q(E) := \{ \mu \in \mathcal{P}(E) : |\mu|_{P_q} < \infty \}, \quad |\mu|_{P_q} := \inf_{x_0 \in E} \left( \int_E d_E^q(x, x_0) \mu(dx) \right)^{1/q}.
\]
Note that \(|\delta_x|_{P_q} = 0 \) and for \((E, \| \cdot \|_E)\) being a linear space one could also set \(|\mu|_{P_q}^q := \int_E \|x\|_E^q \mu(dx)\). For measures \( \mu, \bar{\mu} \in \mathcal{P}_q(E) \) we can now define the \( q \)-Wasserstein distance by
\[
W_q(\mu, \bar{\mu}) := \inf_{\pi \in \Pi(\mu, \bar{\mu})} \left( \int_{E \times E} d_E^q(x, y) \pi(dx, dy) \right)^{1/q},
\]
where \( \Pi(\mu, \bar{\mu}) \) denotes the set of all couplings \( \pi \in \mathcal{P}(E \times E) \) of \( \mu \) and \( \bar{\mu} \), i.e., \( \pi(A \times E) = \mu(A) \) and \( \pi(E \times A) = \bar{\mu}(A) \) for each \( A \in \mathcal{E} \). We note that \( (\mathcal{P}_q, W_q) \) is again a complete and separable metric space, see, e.g., [42].

We focus on the 1-Wasserstein distance \( W_1 \) subsequently. The advantage of this particular distance is its dual representation also known as Kantorovich–Rubinstein duality [42]:
\[
W_1(\mu, \bar{\mu}) = \sup_{f : E \to \mathbb{R}, \text{Lip}(f) \leq 1} \left| \int_E f(x) \mu(dx) - \int_E f(x) \bar{\mu}(dx) \right|,
\]
where \( \text{Lip}(f) := \sup_{x \neq y \in E} \frac{|f(x) - f(y)|}{d_E(x, y)} \) denotes the global Lipschitz constant of \( f \) w.r.t. the metric \( d_E \) on \( E \).

Our first result considers robustness in Wasserstein distance w.r.t. perturbations of the log-likelihood function.

**Theorem 10.** Let \( \mu \in \mathcal{P}_2(E) \) and assume \( \Phi, \bar{\Phi} : E \to \mathbb{R} \) belong to \( L_1^2(\mathbb{R}) \) with \( \text{ess inf}_\mu \Phi = 0 \). Then, for the two probability measures \( \mu_\Phi, \mu_{\bar{\Phi}} \in \mathcal{P}(E) \) given in (1) and (2), respectively, we have
\[
W_1(\mu_\Phi, \mu_{\bar{\Phi}}) \leq \exp(-\text{ess inf}_\mu \Phi) \frac{1}{Z} \left( |\mu_\Phi|_{P_1} \| \Phi - \bar{\Phi} \|_{L_1^\Phi} + |\mu|_{P_2} \| \Phi - \bar{\Phi} \|_{L_1^\Phi} \right)
\]
and also \( |Z - \bar{Z}| \leq \| \Phi - \bar{\Phi} \|_{L_1^\Phi} \).
Proof. Let $x_0 \in E$ be arbitrary. We start with the dual representation

$$W_1(\mu_\Phi, \mu_{\tilde{\Phi}}) = \sup_{\text{Lip}(f)\leq 1, \, f(x_0)=0} \left| \int_E f(x) (\mu_\Phi(dx) - \mu_{\tilde{\Phi}}(dx)) \right|$$

where we can take the supremum also w.l.o.g. w.r.t. all Lipschitz continuous functions $f: E \to \mathbb{R}$ with $\text{Lip}(f) = \sup_{x \neq y \in E} \frac{|f(x) - f(y)|}{d_E(x,y)} \leq 1$ and $f(x_0) = 0$. The latter two conditions imply $|f(x)| \leq d_E(x, x_0)$. Furthermore, we have that

$$\left| \int_E f(x) (\mu_\Phi(dx) - \mu_{\tilde{\Phi}}(dx)) \right| = \left| \int_E f(x) \left( \frac{e^{-\Phi(x)}}{Z} - \frac{e^{-\tilde{\Phi}(x)}}{\tilde{Z}} \right) \mu(dx) \right| \leq I_1(f) + I_2(f)$$

where

$$I_1(f) := \left| \frac{1}{Z} - \frac{1}{\tilde{Z}} \right| \left| \int_E f(x) e^{-\Phi(x)} \mu(dx) \right|, \quad I_2(f) := \left| \frac{1}{Z} \int_E f(x) \left( e^{-\Phi(x)} - e^{-\tilde{\Phi}(x)} \right) \mu(dx) \right|.$$

We can bound $I_2$ as follows using $|e^{-t} - e^{-s}| \leq e^{-\min(t,s)} |t - s|$ for $t, s \in \mathbb{R}$ and the Cauchy–Schwarz inequality:

$$\sup_{\text{Lip}(f)\leq 1, \, f(x_0)=0} I_2(f) \leq \frac{\exp(-[\text{ess inf}_\mu \tilde{\Phi}]_-)}{\tilde{Z}} \int_E d_E(x, x_0) |\Phi(x) - \tilde{\Phi}(x)| \mu(dx)
\leq \frac{\exp(-[\text{ess inf}_\mu \tilde{\Phi}]_-)}{\tilde{Z}} \left( \int_E d_E^2(x, x_0) \mu(dx) \right)^{1/2} \| \Phi - \tilde{\Phi} \|_{L^2_\mu}.$$

Moreover, due to $\left| \frac{1}{Z} - \frac{1}{\tilde{Z}} \right| = \frac{|Z - \tilde{Z}|}{Z \tilde{Z}}$ and $|Z - \tilde{Z}| \leq e^{-[\text{ess inf}_\mu \tilde{\Phi}]_-} \| \Phi - \tilde{\Phi} \|_{L^1_\mu}$, see Theorem 2, we have

$$\sup_{\text{Lip}(f)\leq 1, \, f(x_0)=0} I_1(f) \leq \frac{\exp(-[\text{ess inf}_\mu \tilde{\Phi}]_-)}{Z} \| \Phi - \tilde{\Phi} \|_{L^1_\mu} \sup_{\text{Lip}(f)\leq 1, \, f(x_0)=0} \left| \int_E f(x) \mu_\Phi(dx) \right|
\leq \frac{\exp(-[\text{ess inf}_\mu \tilde{\Phi}]_-)}{Z} \| \Phi - \tilde{\Phi} \|_{L^1_\mu} \int_E d_E(x, x_0) \mu_\Phi(dx).$$

Since $x_0 \in E$ was chosen arbitrarily we obtain the statement. \hfill \square

If one prefers an estimate without $|\mu_\Phi|_{\mathcal{P}_1}$, then we can bound $W_1(\mu_\Phi, \mu_{\tilde{\Phi}})$ also by

$$W_1(\mu_\Phi, \mu_{\tilde{\Phi}}) \leq \frac{e^{-[\text{ess inf}_\mu \tilde{\Phi}]_-} |\mu|_{\mathcal{P}_1} \| \Phi - \tilde{\Phi} \|_{L^1_\mu}}{\min(Z, \tilde{Z})^2} + \frac{e^{-[\text{ess inf}_\mu \tilde{\Phi}]_-} |\mu|_{\mathcal{P}_2} \| \Phi - \tilde{\Phi} \|_{L^2_\mu}}{\min(Z, \tilde{Z})}
\leq \frac{2 e^{-[\text{ess inf}_\mu \tilde{\Phi}]_-} |\mu|_{\mathcal{P}_2} \| \Phi - \tilde{\Phi} \|_{L^2_\mu}}{\min(Z, \tilde{Z})^2},$$

where we used Jensen’s inequality and $\min(Z, \tilde{Z}) \leq Z \leq 1$ for the second inequality.

As outlined in Remark 1, we can use Theorem 10 to show a (local Lipschitz) continuous dependence of the posterior measure w.r.t. the observed data $y \in \mathbb{R}^d$ in Wasserstein distance, i.e., establishing a Wasserstein well-posedness of Bayesian inverse problems. This is done in detail at the end of this section under similar conditions as for Hellinger well-posedness, cf. [7, 29].

Studying the stability w.r.t. the prior in Wasserstein distance is unfortunately more delicate than in the previous sections and the following result requires some restrictive assumptions which we discuss afterwards.
**Theorem 11.** Let $E$ be bounded w.r.t. the metric $d_E$, i.e.,
$$
\sup_{x,y \in E} d_E(x, y) \leq D < \infty,
$$
and let $e^{-\Phi} : E \to [0, 1]$ be Lipschitz w.r.t. $d_E$, i.e., $\text{Lip}(e^{-\Phi}) < \infty$. Then, for the two probability measures $\mu_\Phi, \tilde{\mu}_\Phi \in \mathcal{P}(E)$ given in (1) and (3), respectively, we have
$$
W_1(\mu_\Phi, \tilde{\mu}_\Phi) \leq \frac{1 + D \text{Lip}(e^{-\Phi})}{Z} \left( 1 + \text{Lip}(e^{-\Phi}) \frac{|\mu|_1}{Z} \right) W_1(\mu, \tilde{\mu})
$$
as well as $|Z - \tilde{Z}| \leq \text{Lip}(e^{-\Phi}) W_1(\mu, \tilde{\mu})$.

**Proof.** Again, let $x_0 \in E$ be arbitrary. By the duality of $W_1$ we have
$$
W_1(\mu_\Phi, \tilde{\mu}_\Phi) = \sup_{\text{Lip}(f) \leq 1, f(x_0) = 0} \left| \int_E f(x) e^{-\Phi(x)} \left( \frac{\mu(dx)}{Z} - \frac{\tilde{\mu}(dx)}{Z} \right) \right|.
$$
For any $f : E \to \mathbb{R}$ with $\text{Lip}(f) \leq 1$ and $f(x_0) = 0$ we get that $g(x) := f(x) e^{-\Phi(x)}$ satisfies $g(x_0) = 0$ as well as
$$
|g(x) - g(y)| \leq |f(x)| |e^{-\Phi(x)} - e^{-\Phi(y)}| + |e^{-\Phi(y)}| |f(x) - f(y)|
$$
$$
\leq |f(x)| \text{Lip}(e^{-\Phi}) d_E(x, y) + d_E(x, y)
$$
$$
\leq (1 + D \text{Lip}(e^{-\Phi})) d_E(x, y)
$$
since $|f(x)| \leq |f(x_0)| + |f(x) - f(x_0)| \leq d_E(x, x_0) \leq D$. Hence, we obtain
$$
W_1(\mu_\Phi, \tilde{\mu}_\Phi) \leq (1 + D \text{Lip}(e^{-\Phi})) \sup_{\text{Lip}(g) \leq 1, g(x_0) = 0} \left| \int_E g(x) \left( \frac{\mu(dx)}{Z} - \frac{\tilde{\mu}(dx)}{Z} \right) \right|
$$
and by the triangle inequality
$$
\frac{W_1(\mu_\Phi, \tilde{\mu}_\Phi)}{1 + D \text{Lip}(e^{-\Phi})} \leq \sup_{\text{Lip}(g) \leq 1, g(x_0) = 0} \left[ \left( \frac{1}{Z} - \frac{1}{Z} \right) \left| \int_E g(x) \mu(dx) \right| + \frac{1}{Z} \left| \int_E g(x) (\mu(dx) - \tilde{\mu}(dx)) \right| \right]
$$
$$
\leq \frac{|Z - \tilde{Z}|}{ZZ} \int_E d_E(x, x_0) \mu(dx) + \frac{1}{Z} W_1(\mu, \tilde{\mu}).
$$
Since $x_0 \in E$ was chosen arbitrarily and
$$
|Z - \tilde{Z}| = \left| \int_E e^{-\Phi(x)} (\mu(dx) - \tilde{\mu}(dx)) \right| \leq \text{Lip}(e^{-\Phi}) W_1(\mu, \tilde{\mu}),
$$
we obtain the statement. 

The assumption on the boundedness of $d_E$ on $E$ is not that restrictive, since we can always consider a bounded version $\tilde{d}_E(x, y) := \min(D, d_E(x, y))$, $D > 0$, of a metric $d_E$ on $E$ and, thereby, obtain a bounded metric space $(E, \tilde{d}_E)$. However, a crucial restriction in Theorem 11 is the Lipschitz condition $\text{Lip}(e^{-\Phi}) < \infty$ w.r.t. a bounded metric on $E$. For example, for Euclidean spaces $E = \mathbb{R}^n$ equipped with the bounded metric $d_E(x, y) := \min(D, |x - y|)$, $D > 0$, and a sufficiently smooth $\Phi \in C^1(\mathbb{R}^n, [0, \infty))$ the condition $\text{Lip}(e^{-\Phi}) < \infty$ would require that
$$
\sup_{x \in E} \|\nabla e^\Phi(x)\| = \sup_{x \in E} \|\nabla \Phi(x)\| < \infty,
$$
where $\nabla$ denotes the gradient w.r.t. the usual Euclidean norm $|\cdot|$ on $E = \mathbb{R}^n$. We close the discussion on Wasserstein robustness with a few remarks on the results we have obtained.
Remark 12 (Wasserstein distance of Gaussian priors). Concerning the 1-Wasserstein robustness w.r.t. Gaussian priors we remark that there exists an exact formula for the 2-Wasserstein distance of two Gaussian measures \( \mu = N(m, C) \) and \( \tilde{\mu} = N(\tilde{m}, \tilde{C}) \) on a separable Hilbert space \( \mathcal{H} \) with norm \( \| \cdot \|_H \):

\[
W_2(\mu, \tilde{\mu}) = \sqrt{\| m - \tilde{m} \|_H^2 + \text{tr}(C) + \text{tr}(\tilde{C}) - 2\text{tr} \left( \sqrt{C^{1/2}C^{1/2}} \right)},
\]

see [15, Theorem 3.5]. This can be used to bound the 1-Wasserstein distance of Gaussian priors \( W_1(\mu, \tilde{\mu}) \leq W_2(\mu, \tilde{\mu}) \) due to Jensen’s inequality. Besides that we have for Gaussian measures \( \mu = N(m, C) \in \mathcal{P}^2(\mathcal{H}) \) and \( |\mu|_{P^2} \leq |\mu|_{P^2} = \sqrt{\text{tr}(C)} \) where the latter statement holds also for arbitrary \( \mu \in \mathcal{P}^2(\mathcal{H}) \). We highlight, that we do not require the equivalence of Gaussian priors \( \mu = N(m, C) \), \( \tilde{\mu} = N(\tilde{m}, \tilde{C}) \) in order to evaluate \( W_2(\mu, \tilde{\mu}) \) or bound \( W_1(\mu, \tilde{\mu}) \).

Remark 13 (Proofs via couplings). An alternative, and maybe more powerful, approach to establish Wasserstein robustness is to construct suitable couplings \( \pi \) between \( \mu_\Phi \) and \( \tilde{\mu}_\Phi \) and use these for deriving upper bounds for \( W_q(\mu_\Phi, \tilde{\mu}_\Phi) \) or \( W_q(\mu_\Phi, \tilde{\mu}_\Phi) \) in terms of \( \| \Phi - \tilde{\Phi} \|_{L^p} \) and \( W_q(\mu, \tilde{\mu}) \), respectively. For robustness w.r.t. the log-likelihood——i.e., proving \( W_q(\mu_\Phi, \tilde{\mu}_\Phi) \leq C \| \Phi - \tilde{\Phi} \|_{L^p} \) and \( W_q(\mu, \tilde{\mu}) \) ——this could be feasible. However, for establishing robustness w.r.t. the prior——i.e., proving \( W_q(\mu_\Phi, \tilde{\mu}_\Phi) \leq C W_q(\mu, \tilde{\mu}) \)—the optimal coupling \( \pi \in \Pi(\mu_\Phi, \tilde{\mu}_\Phi) \) would have to have at least a density w.r.t. the optimal coupling \( \pi^* \in \Pi(\mu, \tilde{\mu}) \) attaining the infimum \( W_q(\mu, \tilde{\mu}) \). This seems rather difficult to guarantee.

Remark 14 (Increasing sensitivity). The bounds established in Theorem 10 and 11 suggests again an increasing sensitivity of the posterior—but this time in Wasserstein distance—w.r.t. the log-likelihood and the prior as the posterior becomes increasingly concentrated. In Remark 5 we have outlined why such an increasing sensitivity is quite natural in the topology induced by the total variation or Hellinger distance. Moreover, the same reasoning as in Remark 5 applies when perturbations are measured by the Kullback–Leibler divergence, since the KLD also relies on the equivalence of (perturbed posterior and prior) measures. We now argue why this increasing sensitivity is also natural in the topology induced by the Wasserstein distance. To this end, let us first assume a sequence of increasingly concentrated posterior measures \( \mu_\Phi^{(k)}(dx) := Z_k^{-1} e^{-k\Phi(x)} \mu(dx) \) for \( k \in \mathbb{N} \) with \( Z_k := \int_E e^{-k\Phi(x)} \mu(dx) \). Let \( S_0 := \text{supp}(\mu) \) denote the support of \( \mu \) and assume that \( x_* := \text{argmin}_{x \in S_0} \Phi(x) \) exists and is unique. Then, under mild assumptions, \( \mu_\Phi^{(k)} \) converges weakly to \( \delta_{x_*} \), see, e.g., [24]. Given a perturbed prior \( \tilde{\mu} \) we set \( \tilde{\mu}_\Phi^{(k)}(dx) := Z_k^{-1} e^{-k\Phi(x)} \tilde{\mu}(dx) \), \( k \in \mathbb{N} \), with \( \tilde{Z}_k := \int_E e^{-k\Phi(x)} \tilde{\mu}(dx) \). Given that \( \tilde{x}_* := \text{argmin}_{x \in \tilde{S}_0} \tilde{\Phi}(x) \) exists and is unique, where \( \tilde{S}_0 := \text{supp}(\tilde{\mu}) \), the \( \tilde{\mu}_\Phi^{(k)} \) converge weakly to \( \delta_{\tilde{x}_*} \) under the assumptions in [24]. Thus, in order to have non-exploiting local Lipschitz constants w.r.t. the Wasserstein distance of the mappings \( \mu \mapsto \mu_\Phi^{(k)} \) as \( k \to \infty \), we require that there exists a radius \( r > 0 \) and a constant \( C < \infty \) such that

\[
\lim_{k \to \infty} \frac{W_1(\mu_\Phi^{(k)}, \tilde{\mu}_\Phi^{(k)})}{W_1(\mu, \tilde{\mu})} \leq C \quad \forall \tilde{\mu} \in \mathcal{P}^1(E) : W_1(\mu, \tilde{\mu}) \leq r.
\]

In the following, we assume that the metric \( d_E \) of the complete and separable space \( (E, d_E) \) is bounded. This yields, given the weak convergence of \( \mu_\Phi^{(k)} \) to \( \delta_{x_*} \) and of \( \tilde{\mu}_\Phi^{(k)} \) to \( \delta_{\tilde{x}_*} \), that

\[
\lim_{k \to \infty} \frac{W_1(\mu_\Phi^{(k)}, \tilde{\mu}_\Phi^{(k)})}{W_1(\mu, \tilde{\mu})} = \frac{W_1(\delta_{x_*}, \delta_{\tilde{x}_*})}{W_1(\mu, \tilde{\mu})} = \frac{d_E(x_*, \tilde{x}_*)}{W_1(\mu, \tilde{\mu})},
\]

see [42]. Next, we construct a sequence of perturbed priors \( \tilde{\mu}_\Phi^{(c)} \in \mathcal{P}^1(E) \) with \( W_1(\mu, \tilde{\mu}_\Phi^{(c)}) < \epsilon, \epsilon > 0 \), for which the ratio \( d_E(x_*, \tilde{x}_*)/W_1(\mu, \tilde{\mu}_\Phi^{(c)}) \) deteriorates to infinity as \( \epsilon \to 0 \). To this end, we consider a
ball of radius \( \epsilon > 0 \) around \( x_* \), i.e., \( B_\epsilon(x_*) := \{ x \in E : d_E(x, x_*) \leq \epsilon \} \), and set for an arbitrarily chosen \( x_\epsilon \in \partial B_\epsilon(x_*) \)
\[
\tilde{\mu}_0^{(\epsilon)}(A) := \mu(A) - \mu(A \cap B_\epsilon(x_*)) + \mu(B_\epsilon(x_*)) \delta_{x_\epsilon}(A), \quad A \in \mathcal{E},
\]
i.e., outside the ball \( B_\epsilon(x_*) \) the measure \( \tilde{\mu}_0^{(\epsilon)} \) coincides with \( \mu \) but all the probability mass \( \mu(B_\epsilon(x_*)) \) inside the ball \( B_\epsilon(x_*) \) is now concentrated at the single point \( x_\epsilon \). Assuming that \( \Phi \) is continuous and \( \epsilon \) sufficiently small we have \( \tilde{x}_\epsilon^{(\epsilon)} : = \arg\min_{x \in \text{supp} \tilde{\mu}_0^{(\epsilon)}} \Phi(x) \) \( \in \partial B_\epsilon(x_0) \). Thus, \( d_E(x_\epsilon, \tilde{x}_\epsilon^{(\epsilon)}) = \epsilon \). On the other hand,
\[
W_1(\mu, \tilde{\mu}_0^{(\epsilon)}) = \int_{B_\epsilon(x_0)} d_E(x_\epsilon, x) \mu(dx) \leq 2\epsilon \mu(B_\epsilon(x_0)).
\]
Hence, for suitable infinite spaces \( E \) and non-atomic priors \( \mu \) such that \( \lim_{\epsilon \to 0} \mu(B_\epsilon(x)) = 0 \) for any \( x \in E^3 \) we have
\[
\lim_{\epsilon \to 0} \frac{d_E(x_\epsilon, \tilde{x}_\epsilon^{(\epsilon)})}{W_1(\mu, \tilde{\mu}_0^{(\epsilon)})} \geq \lim_{\epsilon \to 0} \frac{1}{2\mu(B_\epsilon(x_0))} = \infty.
\]
This shows that also in the Wasserstein topology, the posterior depends increasingly sensitively on perturbations of the prior as the likelihood becomes more informative. A similar reasoning can be employed to show also the increasing sensitivity w.r.t. perturbations of the likelihood measured in \( L^p \)-norms.

**Wasserstein Well-posedness.** In the following we show how general robustness results as in Theorem 10 can be used to establish well-posedness of Bayesian inverse problems (BIP). The well-posedness of BIP w.r.t. Hellinger distance has been studied in a number of works [7, 21, 22, 29, 38, 40] and has been recently extended to the Kullback–Leibler divergence [29]. We now extend it to the Wasserstein distance employing Theorem 10.

Since the unique solution of a Bayesian inverse problem is the posterior measure which exists under quite mild assumptions, see, e.g., [29], the remaining condition for well-posedness is the continuous dependence of the posterior on the observed data. To this end, we briefly recall the basic setting of Bayesian inverse problems from Section 2: we infer an unknown \( x^1 \in \mathcal{H} \) belonging to a separable Hilbert space \( \mathcal{H} \) based on (i) a-priori information given by a prior measure \( \mu \in \mathcal{P}(\mathcal{H}) \) and (ii) noisy observations \( y \in \mathbb{R}^d \) of a measurable forward map \( G : \mathcal{H} \to \mathbb{R}^d \). The noise is assumed as additive and viewed as random with a given distribution \( \nu_\epsilon \propto \exp(-\ell(\epsilon)) \) \( \nu_\epsilon \) with measurable \( \ell : \mathbb{R}^d \to \mathbb{R} \). We condition the prior \( \mu \) on the event that we observed \( Y = y \) for the random observable \( Y := G(X) + \epsilon \), where \( X \sim \mu \) and \( \epsilon \sim \nu_\epsilon \) are independent, and obtain a posterior measure of the form (1) with \( \Phi(x) = \Phi(x, y) := \ell(y - G(x)) \). In the following we assume that \( \ell : \mathbb{R}^d \to \mathbb{R} \) is bounded from below—here, w.l.o.g., by zero, i.e., \( \ell : \mathbb{R}^d \to [0, \infty) \).

We now show that by Theorem 10 we obtain well-posedness w.r.t. the Wasserstein distance under the same basic assumption on \( \Phi \) or \( \ell \), respectively, stated in [7, 38] as well as slightly modified in [21, 22, 40] for establishing well-posedness w.r.t. the Hellinger distance.

**Corollary 15.** Let \( \mu_\Phi \in \mathcal{P}(\mathcal{H}) \) be given as in (1) with \( \Phi(x) = \ell(y - G(x)) \) and assume that \( G : \mathcal{H} \to \mathbb{R}^d \) and \( \ell : \mathbb{R}^d \to [0, \infty) \) are measurable. If there exists a monotonic and non-decreasing function \( M : [0, \infty) \times \mathbb{R} \to [0, \infty) \) such that for any \( y, \tilde{y} \in \mathbb{R}^d \) with \( |y|, |	ilde{y}| \leq r < \infty, r > 0 \), we have
\[
|\ell(y - G(x)) - \ell(\tilde{y} - G(x))| \leq M(r, ||x||) |y - \tilde{y}| \quad \forall x \in \mathcal{H}
\]
as well as \( M(r, ||\cdot||) \in L^2_\mu(\mathbb{R}) \) and if there exists a bounded set \( A \subset \mathcal{H} \) with \( \mu(A) > 0 \), then for any \( r > 0 \) there exists a constant \( C_r < \infty \) such that for each \( |y|, |	ilde{y}| \leq r \) we have
\[
W_1(\mu_\Phi, \mu_{\tilde{\Phi}}) \leq C_r |y - \tilde{y}|
\]
where \( \mu_{\tilde{\Phi}} \) is as in (2) with \( \tilde{\Phi}(x) = \ell(\tilde{y} - G(x)) \).

\(^3\)Again, this is the same requirement as in Remark 5 and the works on Bayesian brittleness [33, 34].
Proof. By construction, we have \( \text{ess inf}_\mu \tilde{\Phi} \geq 0 \) and obtain by means of Theorem 10
\[
W_1(\mu_\Phi, \mu_{\tilde{\Phi}}) \leq \frac{2\|\mu\|_p^2}{\min(Z, \tilde{Z})^2} \|\Phi - \tilde{\Phi}\|_{L^1_\mu} \leq \frac{2\|\mu\|_p^2}{\min(Z, \tilde{Z})^2} \|M(r, \|\cdot\|)\|_{L^2_\mu} |y - \tilde{y}|,
\]
where the last inequality followed by our assumption. It remains to bound \( \min(Z, \tilde{Z}) \) uniformly (w.r.t. \(|y|, |\tilde{y}|\)) from below. To this end, we again use the assumption and obtain
\[
\min (\ell(y - G(x)), \ell(\tilde{y} - G(x))) \geq \ell(0 - G(x)) - rM(r, \|x\|) \geq -rM(r, \|x\|).
\]
This implies, in particular, by the assumption on \( M \) there exists an \( R < \infty \) with
\[
\inf_{x \in A} \min (\ell(y - G(x)), \ell(\tilde{y} - G(x))) \geq -r \sup_{x \in A} M(r, \|x\|) = -rR > -\infty
\]
which yields
\[
\min(Z, \tilde{Z}) \geq \int_A \exp(-rR)\mu(dx) > 0
\]
and concludes the proof. \( \square \)

By similar arguments and appropriate assumptions, cf. [7, Section 4.2], continuity in Wasserstein distance for converging approximation \( G_h \approx G \), i.e., \( \lim_{h \to 0} G_h(x) = G(x) \), can be shown.

We close this section with a result on the continuous dependence on the data under much weaker assumptions than in Corollary 15 following the approach in [29].

Corollary 16. Let \( \mu_\Phi \in \mathcal{P}(\mathcal{H}) \) be given as in (1) with \( \Phi(x) = \ell(y - G(x)) \) and assume that \( \mu \in \mathcal{P}_2(\mathcal{H}) \), \( G: \mathcal{H} \to \mathbb{R}^d \) is measurable and \( \ell: \mathbb{R}^d \to [0, \infty) \) is continuous. Then, \( \mu_\Phi \) depends continuously w.r.t. the Wasserstein distance \( W_1 \) on the data \( y \in \mathbb{R}^d \), i.e., for perturbed data \( \tilde{y}_k \in \mathbb{R}^d \) such that \( \lim_{k \to \infty} \tilde{y}_k = y \), we have for
\[
\mu_{\tilde{\Phi}}(dx) := \frac{1}{Z_k} e^{-\tilde{\Phi}_k(x)} \mu(dx), \quad \tilde{\Phi}_k(x) := \ell(\tilde{y}_k - G(x))
\]
that \( \lim_{k \to \infty} W_1(\mu_\Phi, \mu_{\tilde{\Phi}}) = 0 \).

Proof. First, we slightly modify the proof of Theorem 10 for general perturbed \( \Phi \) with \( \inf_x \tilde{\Phi}(x) = \inf_x \Phi(x) = 0 \). In particular, we obtain
\[
W_1(\mu_\Phi, \mu_{\tilde{\Phi}}) = \sup_{\text{Lip}(f) \leq 1, f(x_0) = 0} \leq I_1(f) + I_2(f)
\]
where for \( f: \mathcal{H} \to \mathbb{R} \) with \( \text{Lip}(f) \leq 1, f(x_0) = 0 \)
\[
I_1(f) := \left| \frac{1}{Z} - \frac{1}{\tilde{Z}} \right| \left| \int_{\mathcal{H}} f(x) e^{-\Phi(x)} \mu(dx) \right| \leq \frac{|Z - \tilde{Z}|}{\min(Z, \tilde{Z})^2} \int_{\mathcal{H}} \|x - x_0\| \mu(dx)
\]
\[
\leq \frac{\int_{\mathcal{H}} \|x - x_0\| \mu(dx)}{\min(Z, \tilde{Z})^2} \|e^{-\Phi} - e^{-\tilde{\Phi}}\|_{L^1_\mu},
\]
and
\[
I_2(f) := \left| \frac{1}{Z} \int_{\mathcal{H}} f(x) \left( e^{-\Phi(x)} - e^{-\tilde{\Phi}(x)} \right) \mu(dx) \right| \leq \frac{1}{Z} \left( \int_{\mathcal{H}} \|x - x_0\|^2 \mu(dx) \right)^{1/2} \|e^{-\Phi} - e^{-\tilde{\Phi}}\|_{L^2_\mu}.
\]
Thus, in summary, since \(x_0\) can be chosen arbitrarily, we have by Jensen’s inequality and \(Z, \tilde{Z} \leq 1\)

\[
W_1(\mu_\Phi, \mu_{\tilde{\Phi}}) \leq \frac{\|\mu\|_{p_1} + \|\mu\|_{p_2}}{\min(Z, \tilde{Z})^2} \|e^{\Phi} - e^{\tilde{\Phi}}\|_{L^2_{\mu}}.
\]

We can now use Lebesgue’s dominated convergence theorem, as it was done in \([29]\), in order to prove that for \(\tilde{\Phi}_k(x) := \ell(\tilde{y}_k - G(x))\) with \(\lim_{k \to \infty} \tilde{y}_k = y\) we have \(W_1(\mu_\Phi, \mu_{\tilde{\Phi}_k}) \to 0\) as \(k \to \infty\). The argumentation here is as follows: first we have by the continuity of \(\ell\) and \(e^{\Phi}\) that

\[
\lim_{k \to \infty} \exp(-2\tilde{\Phi}_k(x)) = \exp(-2\Phi(x)) \quad \forall x \in \mathcal{H}.
\]

Moreover, by the assumption on \(\ell\) we know that \(\exp(-2\Phi), \exp(-2\tilde{\Phi}_k) \leq 1\) for all \(k \in \mathbb{N}\). Since \(\int_{\mathcal{H}} 1 \mu(dx) = 1\) we can use \(g \equiv 1\) as an integrable dominating function and obtain by Lebesgue’s dominated convergence that

\[
\lim_{k \to \infty} \|\exp(-\Phi) - \exp(-\tilde{\Phi}_k)\|_{L^2_{\mu}} = 0.
\]

Since this implies also \(Z_k \to Z\) as \(k \to \infty\), we can conclude that

\[
\lim_{k \to \infty} W_1(\mu_\Phi, \mu_{\tilde{\Phi}_k}) \leq \frac{\|\mu\|_{p_1} + \|\mu\|_{p_2}}{Z^2} \lim_{k \to \infty} \|\exp(-\Phi) - \exp(-\tilde{\Phi}_k)\|_{L^2_{\mu}} = 0.
\]

\(\square\)

6 Discussion of Related Literature

Besides the rather recent well-posedness studies of Bayesian inverse problems, the idea of a robust Bayesian analysis and the question about the sensitivity of the posterior w.r.t. the prior measure (or the likelihood function) have a long history in Bayesian statistics. Some of the early references are \([8, 18, 23]\) and convenient overviews of many existing approaches and (positive) results are given in \([1, 2, 25]\). A common approach in robust Bayesian analysis is to consider a class of possible and sensible priors \(\Gamma \subset \mathcal{P}(E)\), or likelihood functions, and to study and bound the range of a functional of interest \(f: \mathcal{P}(E) \to \mathbb{R}\) over the set of resulting posterior measures, i.e., to estimate \(\inf_{\mu \in \Gamma} f(\mu_\Phi)\) and \(\sup_{\mu \in \Gamma} f(\mu_\Phi)\). These bounds can then be used for robust decision making accounting for a variation of the prior, or likelihood. Typical functionals of interest are, for instance, probabilities of certain events, e.g., \(f(\mu) = \mu(A), A \in \mathcal{E}\), the (Fréchet) mean of \(\mu\) or the covariance of \(\mu\) if \(E\) is a linear space. There exist several common types of classes of priors with corresponding bounds on the range of various functionals \(f\). We refer to the literature above and focus only on a particular, appealing type of class—the \(\epsilon\)-contamination class—later on.

Moreover, in the described setting of robust Bayesian analysis also a notion of non-robustness of Bayesian inference has been established, called the dilation phenomenon \([43]\). This occurs if

\[
\inf_{\mu \in \Gamma} f(\mu_\Phi) \leq \inf_{\mu \in \Gamma} f(\mu) \leq \sup_{\mu \in \Gamma} f(\mu) \leq \sup_{\mu \in \Gamma} f(\mu_\Phi)
\]

with one of the outer inequalities being strict. Thus, dilation means that the posterior range of \(f\) is larger than the prior range of \(f\) over the class \(\Gamma\). Recently, an extreme kind of dilation, called Bayesian brittleness, was established in \([32, 33, 34]\) w.r.t. (a) arbitrarily small perturbations of the likelihood and (b) classes of priors \(\Gamma_k \subset \mathcal{P}(E)\) specified only by \(k \in \mathbb{N}\) moments or other functionals.

Another approach to robust Bayesian analysis, starting with \([11]\), considers the Fréchet and Gâteaux derivative of the posterior measure \(\mu_\Phi\) w.r.t. perturbations of the prior measure \(\mu + \rho\) where \(\rho\) denotes a suitable signed measure of mass zero. This leads to a derivative-based sensitivity analysis of Bayesian inference, see,
e.g., [10, 16, 19]. Already in these works, particularly [11, 19], the increasing sensitivity of the posterior measure in case of an increasing amount of observational data was noticed.

In the following, we discuss in more detail the relation of our robustness results to the classical robust Bayesian analysis for $\epsilon$-contamination classes of prior measures as well as to the derivative-based sensitivity analysis of posterior measures, and, moreover, explain why our results do not contradict Bayesian brittleness.

**Robustness for $\epsilon$-contamination classes.** A commonly used class of admissible priors in robust Bayesian analysis are $\epsilon$-contamination classes: Given a reference prior $\mu \in \mathcal{P}(E)$ and a set of suitable perturbing probability measures $Q \subset \mathcal{P}(E)$, we consider the class

$$
\Gamma_{\epsilon,Q}(\mu) := \{(1 - \epsilon)\mu + \nu : \nu \in Q\} \subset \mathcal{P}(E), \quad \epsilon > 0.
$$

Common choices for $Q$ are simply $Q = \mathcal{P}(E)$, all symmetric and unimodal distributions on $E$, or all distributions such that $(1 - \epsilon)\mu + \epsilon \nu$ is unimodal if $\mu$ is. The choice $Q = \mathcal{P}(E)$ is, of course, the most conservative and comes closest to our setting. For brevity we denote $\Gamma_{\epsilon}(\mu) := \Gamma_{\epsilon,\mathcal{P}(E)}(\mu)$ in the following. If we consider now balls $B^\text{TV}_\epsilon$ of radius $\epsilon > 0$ in $\mathcal{P}(E)$ w.r.t. total variation distance $d^\text{TV}$, we have

$$
\Gamma_{\epsilon}(\mu) \subset B^\text{TV}_\epsilon(\mu) := \{\tilde{\mu} \in \mathcal{P}(E) : d^\text{TV}(\mu, \tilde{\mu}) \leq \epsilon\},
$$

since $d^\text{TV}((1 - \epsilon)\mu + \epsilon \nu, \mu) \leq \epsilon$. However, the $\epsilon$-contamination class $\Gamma_{\epsilon}(\mu)$ is in general a strict subset of the ball $B^\text{TV}_\epsilon(\mu)$, because $\sup \mu \subseteq \sup (1 - \epsilon)\mu + \epsilon \nu$ whereas there exist probability measures $\tilde{\mu}$ with $d^\text{TV}(\mu, \tilde{\mu}) \leq \epsilon$ but $\sup \mu \not\subseteq \sup \tilde{\mu}$. Thus, our prior robustness results are, in general, w.r.t. a larger class of perturbed prior measures than $\epsilon$-contamination classes.

Furthermore, we establish a local Lipschitz continuous dependence of the posterior measure on the prior w.r.t. particular probability distances such as the total variation distance. This is, in general, a different concept than bounding the posterior range of functionals of interest. Of course, for certain cases we can find relations. For example, concerning probabilities, i.e., functionals $f_A(\mu) := \mu(A)$ where $A \in \mathcal{E}$, a local Lipschitz continuity in terms of the total variation distance as in Remark 4 implies also bounds on the posterior range of $f_A$ over $\Gamma_{\epsilon}(\mu)$. In particular, we obtain with the results of Section 3 that for all $A \in \mathcal{E}$

$$
\inf_{\tilde{\mu} \in \Gamma_{\epsilon}(\mu)} \tilde{\mu}_\Phi(A) \geq \mu_\Phi(A) - \frac{2\epsilon}{Z}, \quad \sup_{\tilde{\mu} \in \Gamma_{\epsilon}(\mu)} \tilde{\mu}_\Phi(A) \leq \mu_\Phi(A) + \frac{2\epsilon}{Z}.
$$

However, in [23] we find explicit expressions for the range of posterior probabilities for an $A \in \mathcal{E}$ over the class $\Gamma_{\epsilon}(\mu)$:

$$
\begin{align*}
\inf_{\tilde{\mu} \in \Gamma_{\epsilon}(\mu)} \tilde{\mu}_\Phi(A) &= \mu_\Phi(A) \left(1 + \epsilon \sup_{x \not\in A} \exp(-\Phi(x)) \right)^{-1}, \\
\sup_{\tilde{\mu} \in \Gamma_{\epsilon}(\mu)} \tilde{\mu}_\Phi(A) &= \frac{(1 - \epsilon)Z \mu_\Phi(A) + \epsilon \sup_{x \in A} \exp(-\Phi(x))}{(1 - \epsilon)Z + \epsilon \sup_{x \in A} \exp(-\Phi(x))}.
\end{align*}
$$

On the other hand, these exact bounds do not allow the derivation of local Lipschitz continuity w.r.t. the total variation distance on $\Gamma_{\epsilon}(\mu)$, because they do not imply a bound for $|\tilde{\mu}_\Phi(A) - \mu_\Phi(A)|$ by a constant times $d^\text{TV}(\mu, \tilde{\mu})$. Nonetheless, these exact ranges can be used to study lower bounds for the total variation distance of perturbed posteriors:

$$
\sup_{\tilde{\mu} \in B^\text{TV}_\epsilon(\mu)} d^\text{TV}(\mu_\Phi, \tilde{\mu}_\Phi) \geq \sup_{\tilde{\mu} \in \Gamma_{\epsilon}(\mu)} d^\text{TV}(\mu_\Phi, \tilde{\mu}_\Phi) = \sup_{A \in \mathcal{E}} \sup_{\tilde{\mu} \in \Gamma_{\epsilon}(\mu)} |\mu_\Phi(A) - \tilde{\mu}_\Phi(A)| = \sup_{A \in \mathcal{E}} \max \left\{\mu_\Phi(A) - \inf_{\tilde{\mu} \in \Gamma_{\epsilon}(\mu)} \tilde{\mu}_\Phi(A), \sup_{\tilde{\mu} \in \Gamma_{\epsilon}(\mu)} \tilde{\mu}_\Phi(A) - \mu_\Phi(A)\right\}.
$$
Bayesian brittleness. In [32, 33, 34] the authors establish several results concerning an extreme non-robustness of Bayesian inference w.r.t. (a) small perturbations of the likelihood function and (b) w.r.t. a class of priors specified only by finitely many “generalized” moments. They call this non-robustness brittleness and state it w.r.t. the posterior range of functionals $^4 f: E \to \mathbb{R}$.

Their brittleness result concerning perturbed likelihood models is that for arbitrary small perturbations the resulting range of posterior expectations of $f$ is the same as the (essential) range of $f$ over the support of the prior $\mu$. This result is in contradiction to the local Lipschitz robustness shown in this paper. The crucial difference between both results, brittleness and robustness, is the way how perturbations of the likelihood are measured: In [33, 34] the likelihood function $L$ is considered as a function of the parameter $x \in E$ and the data $y \in \mathbb{R}^d$—i.e., $L(x, y) \propto \exp(\Phi(x, y))$—and a perturbed likelihood $\tilde{L}(x, y)$ is considered close to $L$ if for all $x \in E$ the resulting data distribution on $\mathbb{R}^d$ with Lebesgue density $\tilde{L}(x, \cdot)$ is close to the distribution with Lebesgue density $L(x, \cdot)$. For instance, employing the total variation distance for the induced data distributions on $\mathbb{R}^d$ we would consider $\tilde{L}$ close to $L$ if $d_L(L, \tilde{L}) := \sup_{x \in X} \|L(x, \cdot) - \tilde{L}(x, \cdot)\|_{L^1}$ is small—here, the $L^1$-norm is taken w.r.t. the Lebesgue measure on $\mathbb{R}^d$. Thus, closeness of likelihood functionals is considered in an average sense w.r.t. the data $y$ but then uniformly w.r.t. $x \in E$. In this paper, on the other hand, we assume fixed data $y \in \mathbb{R}^d$ and consider the negative log-likelihoods $\Phi(\cdot) = -\log L(\cdot, y)$ and $\tilde{\Phi}(\cdot) = -\log \tilde{L}(\cdot, y)$ close to each other if $\|\Phi - \tilde{\Phi}\|_{L^1_{\mu}}$ is small. Thus, in our case closeness of log-likelihoods is considered in an average sense w.r.t. the parameter $x \in E$ and for the fixed observed data $y \in \mathbb{R}^d$. In [A] we discuss in greater detail (i) why brittleness w.r.t. the likelihood is natural if perturbations are measured by the distance $d_L$, as above, and (ii) how robustness can again be obtained if we employ the alternative distance $\tilde{d}_L(L, \tilde{L}) := \sup_{y \in \mathbb{R}^d} \|\tilde{L}(\cdot, y) - L(\cdot, y)\|_{L^1_{\mu}}$. Note that the latter distance implies bounds on the perturbed marginal likelihood or evidence $\tilde{Z} = \int_E \tilde{L}(x, y) \mu(dx)$ whereas the first distance $d_L$ does not. This fact yields the difference between robustness and brittleness, see [A].

The second brittleness result in [33, 34] is stated for classes of priors on $E$ defined only by a set of finitely many functionals $^5 \Psi_k: E \to \mathbb{R}$, $k = 1, \ldots, K$. In particular, given a measure $\nu_0 \in \mathcal{P}(\mathbb{R}^K)$ we consider the class $\Gamma := \{\mu \in \mathcal{P}(E): \Psi_k \mu = \nu_0\}$ of priors where $\Psi(x) := (\Psi_1(x), \ldots, \Psi_K(x))$ and $\Psi_k \mu$ denotes the pushforward measure. This construction accounts for the fact that in practice only finitely many information are available in order to derive or choose a prior measure. In [33, 34] it is then shown under mild assumptions that the range of posterior expectations of an $f: E \to \mathbb{R}$ resulting from priors $\tilde{\mu} \in \Gamma$ coincides with the range of $f$ on $E$. Again, this is not a contradiction to the local Lipschitz robustness w.r.t. the prior established in this paper, since the class $\Gamma$ is, in general, quite different from balls $B_r(\mu) \subset \mathcal{P}(E)$ with radius $r > 0$ around a reference prior $\mu$ in Hellinger or Wasserstein distance.

Derivative of the posterior and local sensitivity diagnostics. Besides the rather global perturbation estimates derived in the robust Bayesian analysis for, e.g., contamination classes of prior measures, several authors studied the local sensitivity of the posterior measure w.r.t. the prior. As a first result we mention the derivative of the posterior $\mu_\Phi$ w.r.t. the prior $\mu$ in the total variation topology introduced by [11] as follows. Let $T_\Phi : \mathcal{P}(E) \to \mathcal{P}(E)$ denote the map from prior $\mu$ to posterior $T_\Phi(\mu) := \mu_\Phi(dx) = \frac{1}{Z} \exp(-\Phi(x)) \mu(dx)$. In order to define the derivative of $T_\Phi$ we consider the set $\mathcal{S}_0(E)$ of signed measures $\rho : E \to \mathbb{R}$ on $E$ with zero mass $\rho(E) = 0$ for modelling perturbations of probability measures, i.e., perturbed priors $\tilde{\mu} = \mu + \rho$. We introduce the set of all admissible perturbations $^6 P_\mu := \{\rho \in \mathcal{S}_0(E): \rho + \mu \in \mathcal{P}(E)\}$ of a prior $\mu \in \mathcal{P}(E)$

$^4$Actually, in [32, 33, 34] the functionals $f$ are functionals of the data distribution, i.e., $f: \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$. However, in the parametric setting considered here, e.g., the distribution of the data or observable $Y$ given $x \in E$ is $N(G(x), \Sigma)$ for the Gaussian noise model, these functionals can be understood as functionals acting on $x \in E$, i.e., $f: E \to \mathbb{R}$.

$^5$Again, in [33, 34] the functionals $\Psi_k$ are actually functionals of the data distribution associated with $x \in E$, i.e., $\Psi_k : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$.

$^6$In [11] the authors allow for any perturbation $\rho \in \mathcal{S}_0(E)$ extending the application of $T_\Phi$ also to signed measures.
and notice that $P_\mu$ is star-shaped with center $\rho_0 = 0$. Then the derivative $\partial T_\Phi(\mu)$ of $T_\Phi$ at a prior $\mu \in \mathcal{P}(E)$ is defined as the linear map from $P_\mu$ to $\mathcal{S}_0(E)$ satisfying
\[
\lim_{\|\rho\|_{TV} \to 0} \frac{\|T_\Phi(\mu + \rho) - T_\Phi(\mu) - \partial T_\Phi(\mu)\rho\|_{TV}}{\|\rho\|_{TV}} = 0,
\]
where $\|\rho\|_{TV} := \int_E \frac{d\rho}{d\nu} \, d\nu$ denotes the total variation norm of a (signed) measure $\rho$ with $\rho \ll \nu$ for a $\sigma$-finite measure $\nu$. In [11, Theorem 4] it is then shown that
\[
\partial T_\Phi(\mu)\rho = \frac{1}{Z} e^{-\Phi}\left(\rho - \int_E e^{-\Phi} \frac{d\rho}{Z} \mu\right) \in \mathcal{S}_0(E).
\]
Moreover, [11, Theorem 4] states the following bounds for the norm $\|\partial T_\Phi(\mu)\| := \sup_{\|\rho\|_{TV} = 1} \|\partial T_\Phi(\mu)\rho\|_{TV}$ of the derivative $\partial T_\Phi(\mu)$:
\[
\frac{1}{Z} \sup_{x \in E: \mu(x) = 0} e^{-\Phi(x)} \leq \|\partial T_\Phi(\mu)\| \leq \frac{1}{Z} \sup_{x \in E} e^{-\Phi(x)},
\]
i.e., for non-atomic priors $\mu$ we have $\|\partial T_\Phi(\mu)\| = \frac{1}{Z}$ given our standing assumption $\inf_x \Phi(x) = 0$. This already implies an increasing sensitivity of the posterior w.r.t. perturbations of the prior for increasingly informative likelihoods, i.e., a decreasing normalization constant $Z$.

Based on the Fréchet derivative $\partial T_\Phi(\mu)$ at $\mu$ other authors studied the sensitivities of $T_\Phi$ w.r.t. a given class of possible perturbations, see, e.g., [10, 16, 19]. For instance, given an $\epsilon$-contamination class $\Gamma_{\epsilon, Q}(\mu)$ as above the authors of [19] study the sensitivity $s(\mu, Q; \Phi) := \sup_{\nu \in Q} s(\mu, \nu; \Phi)$ with local sensitivities
\[
s(\mu, \nu; \Phi) := \lim_{\epsilon \to 0} \frac{d_{TV}(T_\Phi(\mu), T_\Phi((1 - \epsilon)\mu + \epsilon\nu))}{d_{TV}((1 - \epsilon)\mu + \epsilon\nu, (1 - \epsilon)\mu + \epsilon\nu)}.
\]
Since $(1 - \epsilon)\mu + \epsilon\nu = \mu + \epsilon(\nu - \mu)$ and $d_{TV}(\mu, (1 - \epsilon)\mu + \epsilon\nu) = \epsilon\|\nu - \mu\|_{TV}$, this local sensitivity coincides with the norm of the Gâteaux derivative of $T_\Phi$ at $\mu$ in the direction $\rho = \nu - \mu \in \mathcal{S}_0(E)$, i.e.,
\[
s(\mu, \nu; \Phi) = \|T_\Phi(\mu)(\nu - \mu)\|_{TV}.\]
In [19] the authors consider furthermore geometric perturbations of the prior such as $\tilde{\mu}(dx) \propto \left(\frac{d\nu}{d\mu}\right)^{\epsilon} \mu(dx)$, $\epsilon > 0$, and local sensitivities based on divergences rather than total variation distance, see also [10, 16] employing the Kullback–Leibler divergence. Again, they derive an increasing sensitivity $s(\mu, Q; \Phi) \to \infty$ for various classes $Q$ as the likelihood $e^{-\Phi}$ becomes more informative due to more observations in their case. In particular, they derive explicit growth rates of $s(\mu, Q; \Phi_N)$ w.r.t. $N \in \mathbb{N}$ where $N$ denotes the number of i.i.d. observations employed for Bayesian inference and $\Phi_N$ the corresponding log-likelihood.

These results on Fréchet or Gâteaux derivatives w.r.t. the prior measure are quite close to our approach establishing explicit bounds on the local Lipschitz constant. In particular, the constant $C_{\mu, \Phi, r}$ in the corresponding result (4) can be seen as an upper bound on the norm of the derivative $\|\partial T_\Phi(\mu)\|$ for all perturbed priors $\tilde{\mu} \in B_r(\mu)$ belonging to the $r$-ball around $\mu$ in $\mathcal{P}(E)$—cf. Remark 4 stating that $d_{TV}(\mu_\Phi, \mu_\Phi) \leq \frac{2}{Z} d_{TV}(\mu, \tilde{\mu})$. Compared to the studies in [10, 16, 19] we allow for arbitrary perturbed priors not restricted to (geometric) $\epsilon$-contamination classes and, moreover, we consider different topologies on $\mathcal{P}(E)$ induced by Hellinger distance, Kullback–Leibler divergence and Wasserstein distance.

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A  On Brittleness and Robustness w.r.t. Perturbed Likelihoods

In this appendix we discuss in more detail the phenomenon of Bayesian brittleness for perturbed likelihoods as stated in [33, Theorem 6.4]. Moreover, we reveal the mathematical reason behind the brittleness and show how one can obtain robustness by modifying the distance for the likelihood functions.

Setting. We first recall the setting in [33, 34]. We assume a fixed prior measure \( \mu \in \mathcal{P}(E) \) and for simplicity only consider the parametric case where the distribution of the observable data on \( \mathbb{R}^d \) depends only on \( x \in E \). I.e., consider a prior distributed random variable \( X \sim \mu \) on \( E \) and an observable random variable \( Y \) on \( \mathbb{R}^d \) such that the conditional distribution of \( Y \) given \( X = x \) is given by \( \nu_x \in \mathcal{P}(\mathbb{R}^d) \) with \( \nu_x(dy) = L(x, y)dy \) for a positive Lebesgue density \( L(x, \cdot) : \mathbb{R}^d \to (0, \infty) \). Thus, \( L(x, \cdot) \in L^1(\mathbb{R}^d) \) for all \( x \in E \) and we suppose that \( L : E \times \mathbb{R}^d \to (0, \infty) \) is jointly measurable. Moreover, rather than observing a precise realization \( y \in \mathbb{R}^d \) of \( Y \) we suppose that we observe the event \( Y \in B_\delta(y) \subset \mathbb{R}^d \), i.e., we account for a finite resolution of the data described by the radius \( \delta > 0 \) of the ball \( B_\delta(y) = \{ y' \in \mathbb{R}^d : |y - y'| \leq \delta \} \). Conditioning \( X \sim \mu \) on the observation \( Y \in B_\delta(y) \) yields a posterior probability measure on \( E \) depending on \( L \) which we denote by

\[
\mu_L(dx | B_\delta(y)) := \frac{1}{Z_L} \exp(-\Phi_L(x)) \mu(dx), \quad \Phi_L(x) := -\int_{B_\delta(y)} L(x, y') dy',
\]

where \( Z_L := \int_E \exp(-\Phi_L(x)) \mu(dx) \).

Bayesian brittleness. Let us now consider a perturbed likelihood model, namely, another jointly measurable \( \bar{L} : E \times \mathbb{R}^d \to (0, \infty) \) such that \( \int_{\mathbb{R}^d} \bar{L}(x, y) dy = 1 \) for all \( x \in E \). This model yields a perturbed posterior measure which we denote by

\[
\mu_{\bar{L}}(dx | B_\delta(y)) := \frac{1}{Z_{\bar{L}}} \exp(-\Phi_{\bar{L}}(x)) \mu(dx), \quad \Phi_{\bar{L}}(x) := -\int_{B_\delta(y)} \bar{L}(x, y') dy',
\]

and \( Z_{\bar{L}} := \int_E \exp(-\Phi_{\bar{L}}(x)) \mu(dx) \). We can then ask for stability of the mapping \( L \mapsto \mu_L \). To this end, we measure the distance between the two likelihood models \( L, \bar{L} \) by the following distance:

\[
d_L(L, \bar{L}) := \sup_{x \in E} \| L(x, \cdot) - \bar{L}(x, \cdot) \|_{L^1} = 2 \sup_{x \in E} d_{TV}(\nu_x, \bar{\nu}_x)
\]

where \( \bar{\nu}_x \in \mathcal{P}(\mathbb{R}^d) \) denotes the probability measure on \( \mathbb{R}^d \) induced by \( \bar{L}(x, \cdot) \). Although, this distance seems natural for comparing parametrized models for data distributions it leads to non-robustness, or brittleness, as stated in [33, Theorem 6.4]: Let \( f : E \to \mathbb{R} \) be a measurable quantity of interest and consider the posterior expectation of \( f \) which we simply denoted by

\[
\mu_L(f | B_\delta(y)) := \int_E f(x) \mu_L(dx | B_\delta(y));
\]

then for each \( \epsilon > 0 \) there exists a \( \delta(\epsilon) > 0 \) such that

\[
\sup_{L : d_L(L, \bar{L}) \leq \epsilon} \mu_L(f | B_\delta(y)) \geq \text{ess sup} \mu \quad \forall 0 < \delta < \delta(\epsilon) \, \forall y \in \mathbb{R}^d,
\]

with an analogous statement for the infimum. Thus, in other words, the range of all (perturbed) posterior expectations of \( f \) resulting from all perturbed likelihood models \( \bar{L} \) within an \( \epsilon \)-ball around \( L \) w.r.t. \( d_L \) covers the range of all (essential) prior values of \( f \)—as long as the observation is sufficiently accurate, i.e., \( \delta < \delta(\epsilon) \).
An explanation for brittleness. We explain the Bayesian brittleness and the mathematical reason behind in terms of the total variation distance of the posterior measures:

\[ d_{TV}(\mu_L(\cdot \mid B_\delta(y)), \mu_L(\cdot \mid B_\delta(y))) = \frac{1}{2} \int_E \left| \frac{1}{Z_L} \exp(-\Phi_L(x)) - \frac{1}{Z_L} \exp(-\Phi_{\tilde{L}}(x)) \right| \mu(dx). \]

Similarly to Remark 4 we have

\[ d_{TV}(\mu_L(\cdot \mid B_\delta(y)), \mu_L(\cdot \mid B_\delta(y))) \leq \frac{1}{Z_L} \int_E |\exp(-\Phi_L(x)) - \exp(-\Phi_{\tilde{L}}(x))| \mu(dx). \]

Thus, using the definition of \( \Phi_L \) and \( \Phi_{\tilde{L}} \), we can further bound

\[ d_{TV}(\mu_L(\cdot \mid B_\delta(y)), \mu_L(\cdot \mid B_\delta(y))) \leq \frac{1}{Z_L} \int_E \int_{B_\delta(y)} |L(x, y') - \tilde{L}(x, y')| \, dy' \, \mu(dx). \tag{11} \]

Hence, for robustness we need to control the \( L^1 \)-difference of \(|L(x, \cdot) - \tilde{L}(x, \cdot)|\) over the observed event, the ball \( B_\delta(y) \). However, the bound \( d_{TV}(\mu_L, \tilde{L}) < \epsilon \) only implies that

\[ \int_{B_\delta(y)} \left| L(x, y') - \tilde{L}(x, y') \right| \, dy' \leq \left( \frac{\epsilon}{|B_\delta(y)|} \right), \quad \forall x \in E, \]

where \(|B_\delta(y)|\) denotes the Lebesgue measure of \( B_\delta(y_0) \subset \mathbb{R}^d \). Thus, for any \( \epsilon \) we can take a sufficiently small \( \delta \) and then \( \epsilon/|B_\delta(y_0)| \) becomes arbitrarily large. Now, of course, these are just discussions about controlling upper bounds for the total variation distance between the posteriors, but it should be clear that we can easily construct sufficiently “bad” perturbed likelihoods \( \tilde{L} \) with \( d_{TV}(\mu_L(\cdot \mid B_\delta(y)), \mu_L(\cdot \mid B_\delta(y))) \approx 1 \), see the following example which is a slight modification of the illustrative example in [34].

**Example 17.** We consider \( E = [0, 1] \) as parameter space and \( \mathbb{R} \) as data space. We suppose a uniform prior \( \mu = \mathcal{U}(E) \) and use \( \nu_x = N(x, 1/16) \) as the data distribution model. i.e., we have the likelihood

\[ L(x, y) = (\pi/8)^{-1/2} \exp(-8(y - x)^2). \]

Now, suppose we observe the event \( Y \in B_\delta(0) \). Then, \( \int_{B_\delta(0)} L(x, y) \, dy = F(4(\delta - x)) - F(4(-\delta - x)) \) where \( F \) denotes the cumulative distribution function of \( N(0, 1) \) and the resulting posterior measure is then

\[ \mu_L(dx \mid B_\delta(0)) \propto \left[ F(10(-x + \delta)) - F(10(-x - \delta)) \right] dx. \]

Thus, for sufficiently small \( \delta \) most of the posterior mass will be located close to 0, i.e., for \( \delta \leq 0.05 \) we have \( \mu_L(A \mid B_\delta(0)) \leq 10^{-3} \) for the interval \( A = [0.9, 1] \)—see also the left panel in Figure 1 for an illustration.

Now, we consider a slight perturbation \( \tilde{L}_r \) of \( L \), namely, for \( x \geq 0.9 \) we set \( \tilde{L}_r(x, \cdot) = L(x, \cdot) \) and for \( x < 0.9 \) we choose

\[ \tilde{L}_r(x, y) := 10^{-20}1_{B_r(0)}(y) + c_r(x) \, L(x, y)1_{B_r(0)}(y), \]

i.e., in comparison to \( L \) we simply lower the likelihood to \( 10^{-20} \) for \(|y| \leq r \) and otherwise \( \tilde{L}_r(x, y) \propto L(x, y) \) where

\[ c_r(x) := \frac{1 - 2r \cdot 10^{-20}}{1 - (F(10(r - x)) - F(10(-r - x)))} \]

is chosen such that \( \int_{\mathbb{R}} \tilde{L}_r(x, y) \, dy = 1 \)—see the middle panel in Figure 1 for an illustration of \( \tilde{L}_r \) for \( r = 0.05 \). Here, \( B_r(0) := E \setminus B_r(0) \) denotes the complement set of the ball \( B_r(0) \). Since \( \tilde{L}_r(x, y) \geq (\pi/8)^{-1/2} \exp(-8) \geq 10^{-5} \) for \( y \in B_r(0) \) and \( x \geq 0.9 \) most of the mass of the perturbed posterior
\( \mu_{\tilde{L}}(\cdot \mid B_\delta(0)) \) will be in the interval \( A = [0.9, 1] \) if \( \delta \leq r \)—see also the right panel in Figure 1 for an illustration. In fact, we have for \( \delta \leq r \) that
\[
\mu_{\tilde{L}}([0.9, 1] \mid B_\delta(0)) \geq \frac{10^{-5}}{0.9 \cdot 10^{-20} + 0.1 \cdot 10^{-5}} \approx 1.
\]
Now, we can crudely bound the distance
\[
\|L(x, \cdot) - \tilde{L}_r(x, \cdot)\|_{L^1} \leq \frac{2r}{\sqrt{\pi}/8} + |c_r(x) - 1| \quad \text{for } x \in [0.9, 1],
\]
whereas \( \|L(x, \cdot) - \tilde{L}_r(x, \cdot)\|_{L^1} = 0 \) for \( x < 0.9 \). Moreover, we notice that for sufficiently small \( r \) we have \( c_r(x) \) is sufficiently close to one for all \( x \in [0.9, 1] \). Thus, for any \( \epsilon > 0 \) we can find an \( r = r(\epsilon) \) such that \( \|L_r(x, \cdot) - L(x, \cdot)\|_{L^1} \leq \epsilon \) for each \( x \in E \) but for all \( 0 < \delta < r(\epsilon) < 0.05 \) we have
\[
d_{TV}(\mu_L (\cdot \mid B_\delta(0)), \mu_{\tilde{L}}_r (\cdot \mid B_\delta(0))) \geq \left| \mu_L([0.9, 1] \mid B_\delta(0)) - \mu_{\tilde{L}}_r([0.9, 1] \mid B_\delta(0)) \right| \\
\geq \frac{10^{-5}}{0.9 \cdot 10^{-20} + 0.1 \cdot 10^{-5}} - 10^{-3} \approx 0.999.
\]

Figure 1: Densities of the posterior \( \mu_L(\cdot \mid B_\delta(0)) \) from Example 17 for various values of \( \delta \) (left), the perturbed likelihood \( \tilde{L}_r \) from Example 17 (middle), and the densities of the perturbed posterior \( \mu_{\tilde{L}}_r(\cdot \mid B_\delta(0)) \) from Example 17 for various values of \( \delta \) and \( r = 0.05 \) (right).

**Obtaining robustness.** The above estimate (11) suggests that robustness w.r.t. perturbed likelihoods can only be obtained in a distance for likelihoods \( L \) and \( \tilde{L} \) which allows to control \( |L(x, y) - \tilde{L}(x, y)| \) uniformly w.r.t. \( y \). Thus, if we employ the following alternative distance given the fixed prior \( \mu \)
\[
\widehat{d}_L(L, \tilde{L}) := \sup_{y \in \mathbb{R}^d} \|L(\cdot, y) - \tilde{L}(\cdot, y)\|_{L^1},
\]
then we get by Fubini’s theorem that
\[
d_{TV}(\mu_L(\cdot \mid B_\delta(y)), \mu_{\tilde{L}}(\cdot \mid B_\delta(y))) \leq \frac{1}{Z_L} \int_E \int_{B_\delta(y)} |L(x, y') - \tilde{L}(x, y')| \, dy' \, \mu(dx) \\
= \frac{1}{Z_L} \int_{B_\delta(y)} \int_E |L(x, y') - \tilde{L}(x, y')| \, \mu(dx) \, dy' \\
= \frac{1}{Z_L} \widehat{d}_L(L, \tilde{L}),
\]
i.e., a local Lipschitz robustness. We remark that using the distance \( \widehat{d}_L \) implies that we bound the range of the possible likelihoods for the observed event. As discussed before and illustrated in Example 17 such a control is crucial for a stability w.r.t. perturbed likelihood models.
B  Hellinger Distance of Gaussian Measures on Separable Hilbert Spaces

We provide a proof of the explicit expressions for the Hellinger distance of Gaussian measures on a separable Hilbert space \( \mathcal{H} \) stated in Remark 6, since this is missing so far in the literature up to our knowledge.

**Different means, same covariance.** We start with proving that if \( \tilde{m} - m \in \text{rg } C^{1/2} \), then
\[
d_H^2(N(m, C), N(\tilde{m}, C)) = 2 - 2 \exp\left(-\frac{1}{8} \| C^{-1/2}(m - \tilde{m}) \|_{\mathcal{H}}^2 \right).
\]

To this end, we require the well-known Cameron–Martin formula for the density of \( \tilde{\mu} := N(\tilde{m}, C) \) w.r.t. \( \mu := N(m, C) \). This density is, given that \( h := \tilde{m} - m \in \text{rg } C^{1/2} \),
\[
\frac{d\tilde{\mu}}{d\mu}(x) = \exp\left(-\frac{1}{2} \| C^{-1/2}h \|_{\mathcal{H}}^2 + \langle C^{-1}h, x - m \rangle \right), \quad x \in \mathcal{H},
\]
where \( \langle C^{-1}h, \cdot - m \rangle : \mathcal{H} \to \mathbb{R} \) is well-defined as a random variable in \( L^2_H(\mathbb{R}) \), see, e.g., [6, Chapter 1]. We then use that
\[
d_H^2(\mu, \tilde{\mu}) = 2 - 2 \int_{\mathcal{H}} \sqrt{\frac{d\tilde{\mu}}{d\mu}(x)} \, \mu(dx) \quad \text{since} \quad \tilde{\mu} \ll \mu,
\]
which can be verified easily, and that for any \( x' \in \mathcal{H} \) and \( \mu = N(m, C) \)
\[
\int_{\mathcal{H}} \exp\left(\langle C^{-1/2}x', x - m \rangle \right) \, \mu(dx) = \exp\left(\frac{1}{2} \| x' \|_{\mathcal{H}}^2 \right),
\]
see [6, Proposition 1.2.7], in order to derive that for \( \mu = N(m, C), \tilde{\mu} = N(\tilde{m}, C) \) with \( h = \tilde{m} - m \in \text{rg } C^{1/2} \)
\[
d_H^2(\mu, \tilde{\mu}) = 2 - 2 \exp\left(-\frac{1}{4} \| C^{-1/2}h \|_{\mathcal{H}}^2\right) \int_{\mathcal{H}} \exp\left(\frac{1}{2} \langle C^{-1}h, x - m \rangle \right) \, \mu(dx)
\]
\[
= 2 - 2 \exp\left(-\frac{1}{4} \| C^{-1/2}h \|_{\mathcal{H}}^2\right) \exp\left(\frac{1}{8} \| C^{-1/2}h \|_{\mathcal{H}}^2\right)
\]
\[
= 2 - 2 \exp\left(-\frac{1}{8} \| C^{-1/2}(m - \tilde{m}) \|_{\mathcal{H}}^2\right).
\]

**Same mean, different covariances.** Now, we show that, given \( \text{rg } C^{1/2} = \text{rg } \tilde{C}^{1/2} \), \( T := C^{-1/2}\tilde{C}C^{-1/2} \) being positive definite on \( \mathcal{H} \) and \( T - I \) being Hilbert–Schmidt on \( \mathcal{H} \), we have
\[
d_H^2(N(m, C), N(m, \tilde{C})) = 2 - 2 \left[ \det\left(\frac{1}{2} \sqrt{T} + \frac{1}{2} \sqrt{T^{-1}}\right) \right]^{-1/2}.
\]
W.l.o.g. we assume \( m = 0 \) in the following and use [28, Theorem 3.3] which states that for \( \mu := N(0, C) \) and \( \tilde{\mu} := N(0, \tilde{C}) \) and given the assumptions above, we have
\[
\frac{d\tilde{\mu}}{d\mu}(\psi(\xi)) = \rho(\xi) := \prod_{k=1}^{\infty} \frac{1}{\sqrt{t_k}} \exp\left(\frac{t_k - 1}{2t_k} \xi_k^2 \right), \quad \xi = (\xi_1, \xi_2, \ldots) \in \mathbb{R}^\mathbb{N},
\]
where the \( t_k > 0, k \in \mathbb{N} \), denote the eigenvalues of \( T \) and the measurable mapping \( \psi : \mathbb{R}^\mathbb{N} \to \mathcal{H} \) is specified in the proof of [28, Theorem 3.3]. We do not require the explicit definition of \( \psi \), only the following relation which is also stated in the proof of [28, Theorem 3.3]: with \( \nu := \bigotimes_{k=1}^{\infty} N(0, 1) \) we have \( \mu = \psi_* \nu \), i.e.,
\( \mu = N(0, C) \) is the pushforward of the product measure \( \nu \) under the mapping \( \psi \), see [28] for details. We use these facts in combination with (12) to obtain that for \( \mu := N(0, C) \) and \( \tilde{\mu} := N(0, \tilde{C}) \)

\[
d^2_H(\mu, \tilde{\mu}) = 2 - 2 \int_{\mathcal{H}} \sqrt{\rho(\psi^{-1}(x))} \psi_* \nu(dx) = 2 - 2 \int_{\mathcal{H}} \sqrt{\rho(\xi)} \nu(d\xi)
\]

\[
= 2 - 2 \prod_{k=1}^{\infty} \int_{\mathbb{R}} \frac{1}{\sqrt{t_k}} \exp \left( \frac{t_k - 1}{4t_k} \xi_k^2 \right) \exp \left( -\frac{1}{2} \xi_k^2 \right) \frac{d\xi_k}{\sqrt{2\pi}}.
\]

A straightforward calculation yields

\[
\int_{\mathbb{R}} \exp \left( \frac{t_k - 1}{4t_k} \xi_k^2 \right) \exp \left( -\frac{1}{2} \xi_k^2 \right) d\xi_k = \int_{\mathbb{R}} \exp \left( -\frac{1}{2} \xi_k^2 \right) d\xi_k = \sqrt{2\pi} \sqrt{\frac{2t_k}{1+t_k}}
\]

and, thus,

\[
d^2_H(\mu, \tilde{\mu}) = 2 - 2 \prod_{k=1}^{\infty} \sqrt{\frac{2t_k}{1+t_k}} = 2 - 2 \left[ \prod_{k=1}^{\infty} \sqrt{\frac{1+t_k}{2\sqrt{t_k}}} \right]^{-1/2} = 2 - 2 \left[ \prod_{k=1}^{\infty} \left( \frac{\sqrt{t_k}}{2} + \frac{1}{2\sqrt{t_k}} \right) \right]^{-1/2},
\]

where we assumed for the moment that the infinite products converge. Note, that the infinite product on the righthand side coincides with \( \text{det} \left( \frac{1}{2} \sqrt{T} + \frac{1}{2} \sqrt{T^{-1}} \right) \) given that this (Fredholm) determinant is finite, i.e., given that

\[
I - \left( \frac{1}{2} \sqrt{T} + \frac{1}{2} \sqrt{T^{-1}} \right)
\]

is a trace-class operator. Thus, if we can show that

\[
\sum_{k=1}^{\infty} \left( 1 - \frac{\sqrt{t_k}}{2} + \frac{1}{2\sqrt{t_k}} \right) = \sum_{k=1}^{\infty} \left( 1 - \frac{1+t_k}{2\sqrt{t_k}} \right) < \infty,
\]

then the above formula for \( d^2_H(N(m, C), N(\tilde{m}, \tilde{C})) \) is verified. To this end, we define the function \( f(t) := \frac{1+t}{2\sqrt{t}} \) for \( t > 0 \) and compute its first and second derivative \( f'(t) = \frac{t^{1/2} - t^{-1/2}}{4t} \) and \( f''(t) = \frac{3t^{-1/2} - t^{1/2}}{8t^2} \), respectively. We notice that \( f(1) = 1 \) and \( f'(1) = 0 \), hence,

\[
\left| 1 - \frac{1+t_k}{2\sqrt{t_k}} \right| = |f(1) - f(t_k)| \leq \max_{t \in [1,t_k]} |f''(t)| \left| 1 - t_k \right|^2.
\]

Moreover, we have that \( t_k - 1 \to 0 \) as \( k \to \infty \), since \( T - I \) is Hilbert–Schmidt on \( \mathcal{H} \). Thus, there exists a \( k_0 \in \mathbb{N} \) such that \( |1 - t_k| \leq \frac{1}{2} \) for \( k \geq k_0 \). We obtain by setting \( c := \max_{t \in [\frac{1}{2}, 1]} |f''(t)| \left| 1 - t_k \right|^2 \) that

\[
\left| 1 - \frac{1+t_k}{2\sqrt{t_k}} \right| \leq c \left| 1 - t_k \right|^2 \quad \forall k \geq k_0,
\]

which yields, since \( T - I \) is Hilbert–Schmidt, that

\[
\sum_{k=1}^{\infty} \left( 1 - \frac{1+t_k}{2\sqrt{t_k}} \right) \leq \sum_{k=1}^{k_0} \left( 1 - \frac{1+t_k}{2\sqrt{t_k}} \right) + c \sum_{k=k_0}^{\infty} (t_k - 1)^2 < \infty.
\]
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