OPEs and 3-point correlators of protected operators

in $N = 4$ SYM

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Abstract

Two- and three-point correlation functions of arbitrary protected operators are constructed in $N=4$ SYM using analytic superspace methods. The OPEs of two chiral primary multiplets are given. It is shown that the $n$-point functions of protected operators for $n \leq 4$ are invariant under $U(1)_Y$ and it is argued that this implies that the two- and three-point functions are not renormalised. It is shown explicitly how unprotected operators can be accommodated in the analytic superspace formalism in a way which is fully compatible with analyticity. Some new extremal correlators are exhibited.
1 Introduction

There has been a resurgence of interest in four-dimensional superconformal field theories over the past few years largely due to the impact of the Maldacena conjecture [1] and this has led to the discovery of many new and interesting results. Most of these results have concerned properties of short (series C) operators and their correlation functions derived both directly in field theory and from supergravity via the AdS/CFT correspondence. Some recent reviews and lists of references can be found in [2, 3, 4, 5]. These operators are protected from renormalisation in the sense that their shortness implies that they cannot develop anomalous dimensions because the representations under which they transform determine these dimensions uniquely. Recently it has become apparent from various calculations that certain series A operators also have vanishing anomalous dimensions [6, 7, 8, 9, 10, 11]. In a recent note [12] the current authors have argued that the reason for this is fundamentally the same as for the series B and C operators - the operators concerned are short and this shortness is preserved in the interacting quantum theory provided that the constraints satisfied by the superfields (in super Minkowski space) are respected by gauge invariance (see also [13] and comments on this result in [14]). One way of saying this is to note that all composite operators can be realised as superfields on analytic superspace and the protected ones are those that are realised as analytic tensor superfields in the interacting quantum theory. The non-protected operators, which do acquire anomalous dimensions, have super Dynkin labels which include positive real numbers and it is this fact which stops them from being realisable as normal tensor fields.

The idea of using harmonic superspace methods [15] to study four-dimensional superconformal field theories was advocated in a series of papers [16, 17, 18] and was motivated by the realisation that the on-shell N=4 super Yang-Mills field strength superfield can be described as a (covariantly) analytic field on a certain harmonic superspace [19] (see also [20]). It was realised that there is a family of gauge-invariant scalar multiplets which can be written as analytic superfields and which therefore seemed to be a natural set of objects to study in field theory. It was later observed that this set of multiplets is actually in one-to-one correspondence with the Kaluza-Klein multiplets of IIB supergravity on $AdS_5 \times S^5$ [21]. It has subsequently been shown that various other short multiplets in the theory can be written as harmonic superfields on various harmonic superspaces [22, 23, 24]. However, until recently, not much attention has been paid to analytic superfields which carry superindices. It is the case, however, that all representations in four-dimensional SCFT with extended supersymmetry are carried by analytic superfields of various types [25]. Moreover, in [5] it was noted that the Konishi multiplet in the free theory can be realised as an analytic tensor superfield.

In this paper we take the study of analytic operators a step further by constructing the 3-point functions of analytic tensor superfields and by identifying the leading contributions of analytic tensor operators to the OPEs of two other such operators. In the interacting theory this analysis is therefore relevant to the protected operators - all series B and C operators in the theory and a subset of series A operators. There is a formal resemblance between the N=4 analytic superspace we shall use and Minkowski space which is due to the fact that they can both be presented as subsets of Grassmannians and which allows us to adapt the techniques of [26] in a reasonably straightforward manner. Using these methods we derive the complete expressions for the three-
point functions and the related OPEs. We also show that the correlation functions of protected operators for \( n \leq 4 \) points are invariant under \( U(1)_Y \) \cite{27, 28} and argue that the two- and three-point functions of such operators are not renormalised. In addition, we discuss how unprotected operators can be accommodated in the analytic superspace formalism by means of an \( N = 2 \) example. It turns out that such operators, which we call quasi-tensors, are perfectly compatible with analyticity in the internal space, but at the same time allow the introduction of non-integral powers of \( x^2 \) reflecting the occurrence of anomalous dimensions. These results suggest that the formulae we have given for the OPEs and three-point functions may also be valid, when interpreted appropriately, for arbitrary operators and not just the protected ones. In a recent paper, Eden and Sokatchev \cite{13} have studied some of these three-point functions and OPEs in harmonic superspaces but in a somewhat different approach to that adopted here. In the following we confirm some of their results, obtained by studying the constraints that analyticity imposes on three-point functions, in our formalism. These results can also be obtained directly from the OPE. We also argue that there are extremal correlators involving protected operators other than chiral primaries \(^1\) and discuss an example following the ideas of \cite{13}.

\section{Composite operators}

We begin by recalling a few facts about composite operators in four-dimensional SCFT. The representations of the superconformal group are well-known \cite{29} and their realisations on superfields have been studied by many authors, see for example \cite{30, 31, 19, 22, 23, 24}. The quantum numbers specifying a representation of the \( \mathcal{N} = 4 \) superconformal group are \((L, J_1, J_2, a_1, a_2, a_3)\) where \( L \) is the dilation weight, \( J_1 \) and \( J_2 \) are spin labels and \((a_1, a_2, a_3)\) are \( SU(4) \) Dynkin labels.

The unitarity bounds for the three series of operators are

\begin{align}
\text{Series A :} & \quad L \geq 2 + 2J_1 + 2m_1 - \frac{m}{2} \quad L \geq 2 + 2J_2 + \frac{m}{2} \\
\text{Series B :} & \quad L = \frac{m}{2}, \quad L \geq 1 + m_1 + J_1, \quad J_2 = 0 \quad \text{or} \\
& \quad L = 2m_1 - \frac{m}{2}, \quad L \geq 1 + m_1 + J_2, \quad J_1 = 0 \\
\text{Series C :} & \quad L = m_1 = \frac{m}{2} \quad J_1 = J_2 = 0
\end{align}

where \( m \) is the total number of boxes in the Young tableau of the \( SU(4) \) representation and \( m_1 \) the number of boxes in the first row.

It will be useful later on to be able to write these representations in terms of super Dynkin diagrams. For the (complexified) superconformal group \( SL(4|N) \) acting on \( \mathbb{C}^{4|N} \), the Dynkin diagram depends on the choice of basis. If the basis is ordered in the standard fashion, \( 4 \) even - \( N \) odd, we have the distinguished basis with one odd root, but we shall use a different basis, which we shall refer to as physical, in which the basis has the ordering, \( 2 \) even - \( N \) odd - \( 2 \) even.

\(^1\)We use the term chiral primary to mean a supermultiplet whose leading component is a trace of \( p \) factors of the gauge multiplet scalars in the representation \([0p0]\) of \( SU(4) \). There are other multi-trace multiplets which transform under the same representation of the superconformal group and which have the same properties as far as the results of this paper are concerned.
The physical basis has two odd roots so that the Dynkin diagram is

\[
\begin{array}{c}
\circ \quad \cdots \quad \circ \\
N-1
\end{array}
\]

Any representation can be specified by giving labels associated to each node of the Dynkin diagram. The labels associated with the two external even (black) nodes are determined by the spin quantum numbers \((J_1, J_2)\) and the \((N - 1)\) internal even labels are fixed by the Dynkin labels of \(SL(N)\). The two odd (white) labels are then determined by the dilation \((L)\) and the \(R\)-symmetry \((R)\) quantum numbers. All the Dynkin labels should be non-negative integers except for the odd ones which can be positive real numbers. These continuous labels are directly related to anomalous dimensions of operators.

In order to have unitary representations (of the real superconformal group \(SU(2, 2|N)\)) the Dynkin labels on the odd nodes must exceed those of the adjacent external nodes by at least one unless one or both pairs of these adjacent nodes are zero. This gives three series of unitary bounds. We label the nodes from the left \(n_1 \ldots n_{N+3}\) so that the two odd nodes are \(n_2\) and \(n_{N+2}\) and the adjacent external nodes are \(n_1\) and \(n_{N+3}\) respectively. For series A we have \(n_2 \geq n_1 + 1\) and \(n_{N+3} \geq n_{N+2} + 1\). For series B we have either \(n_1 = n_2 = 0\) and \(n_{N+3} \geq n_{N+2} + 1\) or we have \(n_2 \geq n_1 + 1\) and \(n_{N+3} = n_{N+2} = 0\). Finally series C requires that \(n_1 = n_2 = n_{N+3} = n_{N+2} = 0\).

For general \(N\) we have

\[
\begin{align*}
\frac{1}{2}(L - R) + J_1 + \frac{m}{N} - m_1 \\
\frac{1}{2}(L + R) + J_2 - \frac{m}{N}
\end{align*}
\]

where \(m\) is the total number of boxes in the internal Young tableau determined by the \(SU(N)\) Dynkin labels \((a_1, \ldots a_{N-1}) = (n_3, \ldots n_{N+1})\) and \(m_1\) is the number of boxes in the first row. The external black labels are \((n_1, n_{N+3}) = (2J_1, 2J_2)\). For \(N = 4\) we need to impose \(R = 0\) in order to have representations of \(PSU(2, 2|4)\). This implies that

\[
n_3 + 2n_2 - n_1 = n_5 + 2n_6 - n_7
\]

The super Dynkin diagram can also be used to represent coset spaces determined by parabolic subgroups. With respect to a given basis the Borel subalgebra consists of lower triangular matrices, and a parabolic subalgebra (which by definition is one which contains the Borel subalgebra) consists of lower block triangular matrices. The size of these blocks is determined by a set of at most \(N + 3\) positive integers \(k_1 < k_2 \ldots\) and can be represented on the Dynkin diagram by placing crosses through the \(k_i\)th nodes (starting from the left). For example, super Minkowski space is represented by

\[
\begin{array}{c}
\otimes \quad \cdots \quad \otimes
\end{array}
\]

Chiral superspaces have a single cross through one of the odd nodes, harmonic superspaces have crosses through both odd nodes and some internal nodes, and analytic superspaces have
crosses only through internal nodes. The superspace of most interest to us in this paper is \((N, p, q) = (4, 2, 2)\) analytic superspace; the Dynkin diagram is:

\[
\begin{array}{ccc}
\bullet & \| & \bullet \\
& \times & \\
\bullet & \| & \bullet
\end{array}
\]  

(6)

This space is a subset of a super Grassmannian with local coordinates

\[
X^{AA'} = \begin{pmatrix}
 x^{\alpha\dot{\alpha}} & \lambda^{\alpha\alpha'} \\
 \pi^{\dot{\alpha}} & y^{aa'}
\end{pmatrix}.
\]

(7)

where \(x\) denotes the spacetime coordinates, \(\lambda, \pi\) are odd coordinates and \(y\) are coordinates for the internal manifold. The indices \((\alpha, \dot{\alpha})\) are 2-component spacetime spinor indices while \((a, a')\) are 2-component spinor indices for the internal space which is (locally) the same as spacetime in the complexified case. The capital indices span both spacetime and internal indices, \(A = (\alpha, a), \ A' = (\dot{\alpha}, a')\), and we use the convention that \((\alpha, \dot{\alpha})\) are even indices while \((a, a')\) are odd. An important feature of analytic superspace is that superfields carrying irreducible representations are completely specified by the super Dynkin labels and analyticity; no further constraints need to be imposed.

The crosses on a super Dynkin diagram factorise the diagram into sub-(super)-Dynkin diagrams corresponding to the semi-simple subalgebra of the Levi subalgebra (the diagonal blocks in the parabolic), while the Dynkin labels above the crosses correspond to charges under internal \(U(1)\)'s or dilatation and \(R\) weights. In general the Levi subalgebra will be a superalgebra and so the fields can carry superindices. Only in cases where both odd nodes have crosses through (such as for super Minkowski space and harmonic superspaces) does the Levi subalgebra contain no superalgebra.

In the free theory the Maxwell field strength superfield, corresponding to the representation with \(n_4 = 1\) and all other Dynkin labels zero, is a single component analytic superfield \(W\). In the interacting case \(W\) is covariantly analytic and so is not a superfield on analytic superspace. However, gauge-invariant products of \(W\) are. The operators \(A_p := \text{tr}(W^p) \ p = 2, 3, \ldots\) which transform under the representations which have only the central Dynkin label non-zero are in one-to-one correspondence with the Kaluza-Klein supermultiplets of IIB supergravity on \(AdS_5 \times S^5\) [32, 21, 16]. The operator \(A_2 := T\) is special; it is the supercurrent multiplet. \(A_p\) is a scalar under \(\mathfrak{sl}(2|2) \oplus \mathfrak{sl}(2|2)\) and has charge \(p\) under the \(U(1)\) corresponding to the central node of the super Dynkin diagram. All other representations transform non-trivially under the sub-algebra \(\mathfrak{sl}(2|2) \oplus \mathfrak{sl}(2|2)\). The series B superfields must transform under the totally (generalised) antisymmetric tensor representation (or the trivial representation) of one of the \(\mathfrak{sl}(2|2)\) subalgebras and the series C superfields must transform under the totally antisymmetric representation of both \(\mathfrak{sl}(2|2)\) subalgebras (trivially in the KK case). For a general representation the highest weight state is obtained from the tensor component which has the most number of internal \((a\ \text{or} \ a')\) indices. The short representations are the series B and C operators and the series A operators which saturate at least one of the unitarity bounds and which can be written in terms of derivatives \(\partial_{A'\ A}\) acting on products of \(A_p\)s on analytic superspace.

Let us now consider an arbitrary analytic tensor operator \(\mathcal{O}\) specified by a set of super Dynkin labels. The 3 leftmost labels determine the representation of the left \(\mathfrak{sl}(2|2)\) under which it
transforms and similarly for the 3 rightmost labels. The left (right) sets of 3 labels are associated with tensors which carry $A(A')$ indices. The left super Dynkin labels $[n_1n_2n_3]$ correspond to a Young tableau of the form $<n_3+n_2-n_1,n_2-n_1,n_1>$, where the first entry gives the number of boxes in the leftmost column, the second the number of boxes in the second column and third gives the number of additional columns all of which have only one box.\footnote{This is a different convention to that used in [12].} In a similar manner the 3 right super Dynkin labels $[n_5,n_6,n_7]$ correspond to the Young tableau $<n_5+n_6-n_7,n_6-n_7,n_7>$. We shall denote these representations by $\mathcal{R}$ and $\mathcal{R}'$ respectively.

The total number of indices of representation $\mathcal{R}$ is the same as the number of boxes in the Young tableau and is given by $S := n_3+2n_2-n_1$. This number is the same for both $\mathcal{R}$ and $\mathcal{R}'$ because we are concerned only with representations for which $R = 0$. These have the same number of primed and unprimed indices.

We shall denote a general operator by $O^Q_{\mathcal{R}\mathcal{R}'}$ where $Q = L - (J_1 + J_2)$. The transformation rule for such an operator under an infinitesimal superconformal transformation is

$$\delta O^Q_{\mathcal{R}\mathcal{R}'} = V O^Q_{\mathcal{R}\mathcal{R}'} + \mathcal{R}(A(X))O^Q_{\mathcal{R}\mathcal{R}'} + \mathcal{R}'(D(X))O^Q_{\mathcal{R}\mathcal{R}'} + Q \Delta O^Q_{\mathcal{R}\mathcal{R}'} \quad (8)$$

where $V$ is the vector field generating the transformation, $A(X)$ and $D(X)$ are $X$-dependent parameters for the left and right $\mathfrak{gl}(2|2)$ algebras, $\mathcal{R}(A(X))$ and $\mathcal{R}'(D(X))$ extend the representations $\mathcal{R}, \mathcal{R}'$ to $\mathfrak{gl}(2|2)$ in a natural way, and $\Delta = \text{str}(A + XC)$. Here,

$$V X = B + AX + XD + XCX \quad (9)$$
$$A(X) = A + XC \quad (10)$$
$$D(X) = D + CX \quad (11)$$

and $A, B, C, D$ are super-matrices parametrising an infinitesimal $\mathfrak{sl}(4|4)$ matrix $\delta g$,

$$\delta g = \left( \begin{array}{cc} -A^A_B & B^{AB'} \\ -C^A_B & D^A_{B'} \end{array} \right) \quad (12)$$

Note that this defines a transformation of $\mathfrak{psl}(4|4)$ on analytic superspace because the element proportional to the unit matrix does not act.

### 3 Two-point functions

The conjugate representation is given by reversing the super Dynkin labels of a given representation, and to get non-vanishing two-point functions we have to pair an operator and its conjugate. Thus we have

$$< O^Q_{\mathcal{R}\mathcal{R}'}(1) O^Q_{\mathcal{R}'\mathcal{R}}(2) > \sim (g_{12})^Q \mathcal{R}(X^{-1}_{12}) \mathcal{R}'(X^{-1}_{12}) \quad (13)$$
In this expression \( X_{12} := X_1 - X_2 \) as usual, and the propagator \( g_{12} \) is defined by

\[
g_{12} := \text{sdet} X_{12}^{-1} = \frac{y_{12}^2}{x_{12}^2} = \frac{y_{12}^2}{\hat{x}_{12}^2}
\]

where

\[
\hat{x}_{12} = x_{12} - \lambda_{12} y_{12}^{-1} \pi_{12} \quad (15)
\]

\[
\hat{y}_{12} = y_{12} - \pi_{12} x_{12}^{-1} \lambda_{12} \quad (16)
\]

In these expressions matrix multiplication is understood, the inverses having downstairs indices \((x^{-1})_{\alpha \alpha}, (y^{-1})_{a' a}\). The meaning of the \( R \) symbols is as follows: at point 1 the unprimed indices on the operator accord with the representation \( R \), so one takes \( S \) factors of \( X_{12}^{-1} \) (which has downstairs indices) with the unprimed indices in that representation. The primed indices then have to be in the same representation, and then similarly for \( R' \). For example, consider the series C operators. These have zero for the external 2 nodes on either side and so are completely specified by their \( SU(4) \) Dynkin labels which have to be of the form \([qpq]\). If we take \( q = 1 \) for simplicity then the corresponding operator is a covector \( O_{AA'} \) on analytic superspace with dilation weight \( L = p + 2 = Q \). In this case we get

\[
< O_{AA'}^{p+2}(1) O_{BB'}^{p+2}(2) > \sim (g_{12})^{p+2} (X_{12}^{-1})_A^B (X_{12}^{-1})_B^{A'}
\]

The reason this formula works is because the variation of \( X_{12} \) can be written

\[
\delta X_{12} = A_1 X_{12} + X_{12} D_2 = A_2 X_{12} + X_{12} D_1
\]

where \( A_1 := A(X_1) \), etc. Thus, applying a superconformal transformation to the above two-point function we find that it is invariant because the propagator factors take care of the dilations and the “rotation” factors \((A(X), D(X))\) are absorbed by the \( X \)s.

As the inverses of coordinate matrices \( X \) appear repeatedly in the correlation functions it is worthwhile giving the exact form,

\[
X^{-1} = \begin{pmatrix}
\hat{x}^{-1} & -x^{-1} \lambda \hat{y}^{-1} \\
-y^{-1} \pi \hat{x}^{-1} & \hat{y}^{-1}
\end{pmatrix}
\]

In order to study possible singularities in the internal coordinates it is more useful to express \( X^{-1} \) in terms of \( x \) and \( \hat{y} \), and to write the propagator in terms of the same variables. We find

\[
\hat{x} = x (1 + x^{-1} \lambda \hat{y}^{-1} \pi)^{-1}
\]

\[
y^{-1} \pi \hat{x}^{-1} = \hat{y}^{-1} \pi x^{-1}
\]
so that each term in $X^{-1}$ behaves at worst like $\hat{y}^{-1}$, although there are nilpotent factors in all of the singular terms except for the $y$ entry.

4 Three-point functions

4.1 The general case

We now turn to three-point functions. We can adapt the Minkowski spacetime formalism of [26] more or less straightforwardly to analytic superspace. The idea is to use factors of $X_{12}^{-1}$ and $X_{13}^{-1}$ to translate points 2 and 3 to point 1, that is, we write

$$< O_{Q_1}^{R_1} O_{Q_2}^{R_2} O_{Q_3}^{R_3} > \sim (g_{12})^{Q_{12}} (g_{23})^{Q_{23}} (g_{31})^{Q_{31}} \times$$

$$R_2 (X_{12}^{-1}) R_2 (X_{12}^{-1}) R_3 (X_{13}^{-1}) R_3 (X_{13}^{-1}) \times t$$

where

$$Q_{ij} = \frac{1}{2} (Q_i + Q_j - Q_k), \quad k \neq i, j$$

and where $t$ is a tensor that transforms under the representations $R_1, R_1'$ and the contragredient representations of $R_2, R_2', R_3, R_3'$, that is, the representations with the same symmetry properties but with the indices upstairs. However, due to the factors of $X_{12}^{-1}$ and $X_{13}^{-1}$ that have been introduced, the tensor $t$ transforms under rotations only at point 1. In addition, the propagator factors take care of the dilation weights. We shall obtain a solution to the Ward identities if $t$ is a tensorial function (of the desired type) of $X_{12}^{-1} - X_{13}^{-1}$ as this combination has the property that it transforms only by rotations at point 1. This is easy to see by noting that it can be rewritten as $-X_{123}^{-1}$ where

$$X_{123} = X_{12} X_{23}^{-1} X_{31}$$

Therefore, the tensor $t$ is a monomial in $X_{123}$ and $X_{123}^{-1}$ with its indices arranged to fall into the correct representations as described above. Roughly speaking, $t \sim (X_{123})^{2S_2 + 2S_3} (X_{123}^{-1})^{2S_1}$. We note that there will in general be more than one solution for a given choice of operators, although in some cases there may be no solutions if the representations do not match properly. In addition it may be possible to have contractions between upper and lower indices on $t$ leading to smaller powers of both $X_{123}$ and its inverse. Finally, it may be the case that there are solutions but that these solutions exhibit singularities in the internal $y$ variables. In such cases the coefficients must be zero.

4.2 Chiral primaries

In order to clarify the above we shall give a few examples. Firstly, consider three chiral primaries $< A_{p_1} A_{p_2} A_{p_3} >$. In this case the formula reduces to the one given before in [16],
\[ <A_{p1}A_{p2}A_{p3}> \sim (g_{12})^{p_{12}}(g_{13})^{p_{13}}(g_{23})^{p_{13}} \]  

(27)

where

\[ p_{ij} = \frac{1}{2}(p_i + p_j - p_k), \quad k \neq i, j \]  

(28)

Analyticity implies that the sum of the charges must be even and that the sum of any two of the charges must be greater or equal to the third otherwise there will be singularities in the y’s.

Next consider the three-point function of two chiral primaries \( A_p, A_q \) and one arbitrary operator \( O \) (at point 1). From the above general formula it can be seen that the factors of \( X_{12}^{-1} \) and \( X_{13}^{-1} \) are not required, and so we simply get a product of \( X_{123}^{-1} \)’s. In order for this to be non-zero the operator \( O \) must be self-conjugate. We have

\[ <O_Q^{\hat{R}R'}(1)A_p(2)A_q(3)> \sim R( (X_{123}^{-1})(g_{12})^{\frac{1}{2}(Q+p-q)}(g_{13})^{\frac{1}{2}(Q+q-p)}(g_{23})^{\frac{1}{2}(p+q-Q)} \]  

(29)

where we have used the facts that the representations \( R \) and \( R' \) are the same and that the \( Q \) charge of \( A_p \) is \( p \). Now we need to analyse which operators are allowed by analyticity. We recall that the representation \( R \) can be represented by the Young tableau \( \langle b + c, b, a \rangle \) where \( a = n_1, b = n_2 - n_1, c = n_3 \). (For self-conjugate representations we take the super Dynkin labels to be \[ [a(a + b)cda(a + b)a] \). Consider the expression \( R(X^{-1}) \); its inverse is \( R(X) \). Now the highest power of \( y \) in \( R(X) \) resides in the component which has the maximum number of internal indices. This is achieved by filling up the first two columns with internal indices (recall that they are odd, so graded symmetrisation implies actual antisymmetrisation for them). This gives us \( (y)^{2b+c} \), but actually, again because of antisymmetry, we get \( b \) contractions, so we find a term of the form \( (y^2)^by^c \), the indices on the final \( c \) factors being totally symmetric (both primed and unprimed). There are also nilpotent terms with the same index structure but they have the effect of amending \( y \) to \( \hat{y} \). The leading singularity for the inverse is then given by \( (\hat{y}^2)^{-(b+c)} \).

Since \( X_{123} = X_{12}X_{23}^{-1}X_{31} \) we find a leading singularity structure of \( R(X_{123}^{-1}) \) of this type for both the (12) and (13) channels. This implies

\[ Q + p - q - 2(b + c) \geq 0 \]  

(30)

\[ Q + q - p - 2(b + c) \geq 0 \]  

(31)

If we assume that \( p \geq q \) we therefore have

\[ Q \geq 2(b + c) + p - q \]  

(32)

In the (23) channel, on the other hand, the \( R(X_{123}^{-1}) \) factor is regular and actually softens the singularity due to the propagators. This means that we need to look for the factor with the smallest number of \( y \)’s, and this is given by the term with the largest number of spacetime indices. The latter is \( a + 4 \), provided that \( b \geq 2 \). The number of \( y \)’s is \( 2(b - 2) + c \) and the
lowest term in $y_{23}$ is therefore $(y_{23}^2)^{(b-2)}(y_{23})^c$. There are also nilpotent contributions with the same index structure which again have the effect of changing the $y$s to $\hat{y}$s. The factor $(y_{23})^c$, which is symmetric in its $c$ primed and unprimed indices, does not soften singularities of the form $(y_{23}^2)^{-1}$ and so we obtain the analyticity bound

$$Q \leq p + q - 4 + 2b, \quad \text{for } b \geq 2 \quad (33)$$

If $b = 0, 1$, on the other hand, then the $\mathcal{R}$ factor does not affect the (23) singularities at all and so we simply get

$$Q \leq p + q, \quad \text{for } b = 0, 1 \quad (34)$$

The analyticity bounds are therefore (32) together with (33) for $b \geq 2$ or (34) for $b = 0, 1$. If $b = 0$ then necessarily $a = 0$ and so the operator is series C and protected. For such operators we have $Q = 2c + d + 2$ and the bounds imply that $d \geq p - q$ while $Q = p + q - 2k$ with $k = 0, 1, \ldots q$ and $k + c \leq q$.

If $b = 1$ we again have (34), but now $Q = 2c + d + 2$, and now the possible values of $Q$ are $Q = p + q - 2k$ for $k = 0, 1, \ldots (q - 1)$ with $k + c \leq (q - 1)$ and $d \geq p - q$. These operators correspond to saturated series A representations, and may or may not be protected in the interacting theory.

If $b \geq 2$ one has the bound (33) with $Q = 2b + 2c + d$, $c \leq q - 2$ and $d \geq p - q$. These operators are unsaturated and are thus expected to acquire anomalous dimensions in the interacting case.

These results are in accord with those of [13] but our interpretation of them is slightly different. The cases $b = 0, b \geq 2$ are not problematic; for $b = 0$ the operator at point 1 is series C and protected, while for $b \geq 2$ the operator is unsaturated and can acquire an anomalous dimension in the interacting theory. In fact, for fixed values of $a$ and $c$ all of these $b \geq 2$ representations have the same number of components. It therefore makes sense to allow $b$ to be non-integral. As we shall see in section 6 this can be done explicitly in analytic superspace and we can also consider values of $b$ less than 2 down to $b = 1$ at which point these representations become reducible. This point is relevant to the question of whether a $b = 1$ operator is protected or whether it acquires an anomalous dimension. For the latter to occur, there must be an available representation from $b \geq 2$ category which can be continuously deformed into the $b = 1$ representation under consideration.

Using the bounds on the representations that can arise it is easy to show that the only $b = 1$ operators that are guaranteed to be protected are those with the maximum possible value of $Q$, i.e. $Q = p + q$. Since the maximum value of $Q = p + q - 4 + 2b$ for $b \geq 2$, we see that when we continue this formula to $b = 1$ we arrive at $Q = p + q - 2$. So the $b = 1, Q = p + q$ representations must be protected; there is no allowed representation with $b = 1$ replaced by $b = 1 + \gamma$. Note also that these operators have the right quantum numbers to be constructible explicitly from analytic superspace derivatives acting on a product of $A_p$ and $A_q$, so that the constraints they would obey as Minkowski space superfields are consistent with interactions.
If we now consider sub-maximal values of $Q$ such as $Q = p + q - 2$, for $b = 1$, then we see that there are representations from the $b \geq 2$ series which coincide with them when $b$ is continued down to $b = 1$. Furthermore, it is straightforward to check that the $SU(4)$ quantum numbers fall into the same representations. It therefore seems to us that it is not possible to say whether a given operator with these quantum numbers is protected or not purely on the basis of representations and analyticity; one needs to know whether the operator concerned can be written as an analytic tensor in the interacting theory or not.

To illustrate this point let us consider $p = 4, q = 2$. For the $b = 1$ operators with $Q = p + q - 2 = 4$, the only possible values of the $SU(4)$ Dynkin labels are $c = 0, d = 2$. For $b \geq 2$, we again find $c = 0, d = 2$ and $Q = 2 + 2b$. Continuing this down to $b = 1$ we find that $Q = 4$ and so the two representations match. If we take $a = 0$ then the super Dynkin labels of this operator are $[002000]$. For $b = 1$ there are three possibilities in the free theory: $T^2$, $K \times T$, where $K$ is the Konishi multiplet, and $W^4$, all with leading components which are scalars in the 20$'$ representation of $SU(4)$. However, in the interacting theory only the first of these is protected as the other two become long and indeed acquire anomalous dimensions. In the context of the three-point functions involving $A_4$ and $A_2(= T)$, however, any of the three is allowed to appear.

### 4.3 Series C operators

Now let us consider three-point functions of series C operators, that is operators which have super Dynkin labels $[00cdc00]$. These include both the single-trace KK multiplets (CPOs) and multi-trace multiplets which can be either in the same $SU(4)$ representations as the CPOs (i.e. $[0d0])$ or more general ones of the form $[cdc], c \neq 0$. These more general superfields can be represented as analytic superfields on $(4, 1, 1)$ superspace, but on $(4, 2, 2)$ superspace they become analytic tensors with superindices. Such operators have the form $O^{Q}_{A_4 \ldots A_c A'_4 \ldots A'_c}$ with $c$ antisymmetric unprimed superindices and $c$ antisymmetric primed superindices; they have $Q$-charge equal to $2c + d$ which is equal to the number of boxes, $m_1$, in the first row of the $SU(4)$ Dynkin diagram $[cdc]$.

The general formula for series C three-point functions is

$$
\langle O^{Q_1}_{A_1 \ldots A_c A'_1 \ldots A'_1} O^{Q_2}_{B_1 \ldots B_c B'_1 \ldots B'_1} O^{Q_3}_{C_1 \ldots C_c C'_1 \ldots C'_1} \rangle
= (g_{12})^{Q_{12}} (g_{13})^{Q_{13}} (g_{23})^{Q_{23}} \times
$$

$$
(X^{-1})^{a_{12}}_{A'B'} (X^{-1})^{a_{21}}_{A'C'} (X^{-1})^{a_{31}}_{A''C''} (X^{-1})^{a_{13}}_{B'C'} (X^{-1})^{a_{23}}_{B''C''} (X^{-1})^{a_{32}}_{C'C'} \times
$$

$$
(X^{-1})^{a_{12}}_{A'A} (X^{-1})^{a_{21}}_{B'B} (X^{-1})^{a_{31}}_{C'C} \times
$$

$$
(X^{-1})^{a_{13}}_{A''A} (X^{-1})^{a_{23}}_{B''B} (X^{-1})^{a_{32}}_{C''C}
$$

where $(X^{-1})^{a_{12}}_{A'B'} := X^{-1}_{A'B_1} \ldots X^{-1}_{A'_1 B_1}$ with the indices antisymmetrised, $X_{ijk} := X_{ij} X^{-1}_{j} k X_{ki}$ and $\{a_{ij}\}$ is a set of nine non-negative integers such that

$$
\sum_{j=1}^{3} a_{ij} = c_i, \quad \sum_{i=1}^{3} a_{ij} = c_j
$$
The simplest solution is \( a_{ii} = c_i \), with all other \( a_{ij} \) vanishing, which corresponds to the three-point function

\[
\langle O_1 O_2 O_3 \rangle = (g_{12})^{Q_{12}}(g_{13})^{Q_{13}}(g_{23})^{Q_{23}}(X_{123}^{-1})_{A'A}(X_{231}^{-1})_{B'B}(X_{312}^{-1})_{C'C}
\]

(37)

In fact the general solution is given in terms of \( a_{12}, a_{13}, \) and \( a_{21} \) by

\[
\begin{pmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{pmatrix} =
\begin{pmatrix}
    c_1 - a_{12} - a_{13} & a_{12} & a_{13} \\
    a_{12} & c_2 - a_{12} - a_{23} & a_{23} \\
    a_{13} & a_{23} & c_3 - a_{13} - a_{23}
\end{pmatrix}.
\]

(38)

with

\[
\begin{align*}
    a_{12} + a_{13} & \leq c_1 \\
    a_{12} + a_{23} & \leq c_2 \\
    a_{13} + a_{23} & \leq c_3.
\end{align*}
\]

(39)

We now turn to the question of analyticity. In order to not have any poles in the internal coordinates \( y_{12} \) we require that for each term involving \( X_{12}^{-1} \) or \( X_{123}^{-1} = X_{13}^{-1} - X_{12}^{-1} \) or \( X_{231}^{-1} = X_{21}^{-1} - X_{23}^{-1} \) there is a corresponding power of \( (g_{12}) \) to cancel the poles, and similarly for the coordinates \( y_{13} \) and \( y_{23} \). In other words we need

\[
Q_{ij} \geq a_{ij} + a_{ji} + a_{ii} + a_{jj}
\]

(40)

but it is easy to see that \( Q_{ij} = \frac{1}{2}(d_i + d_j - d_k) + a_{ij} + a_{ji} + a_{ii} + a_{jj} - a_{kk} \) and thus analyticity gives the three relations

\[
d_i + d_j - d_k \geq 2a_{kk}.
\]

(41)

## 5 OPEs

There have been several studies of the OPE in \( N = 4 \) SYM, see for example [33, 34, 28]. In [35] a discussion of the OPE of two supercurrents in analytic superspace was given, but analytic tensor superfields were not taken into account and this vitiates somewhat the conclusions of that article. However, it has been observed that the Konishi multiplet can be accommodated in the OPE as an analytic tensor superfield, at least in the free theory [5]. In a more recent paper, Eden and Sokatchev [13] have analysed which operators can be expected to appear in the OPE of two \( T \)'s on the basis of studying the constraints that analyticity imposes on three-point functions with two \( T \)'s and an arbitrary third operator. Here we give the complete OPE of two \( T \)'s, at least as far as protected operators are concerned (and ignoring descendants). Moreover, we show that analyticity of the OPE itself is sufficient to derive the constraints on the operators that can appear. This is also true for general CPOs (and presumably for other operators).

The OPE of two \( T \)'s has the following form:
\[ T(1)T(2) \sim \sum C_R \mathcal{R}(X_{12})(g_{12})^{\frac{1}{2}(4-Q)} \mathcal{O}_{RR'}^Q(2) + \ldots \tag{42} \]

The operators allowed on the right-hand side have to be in self-conjugate representations so that the indices on \( \mathcal{R}(X_{12}) \) can hook up with the indices on the operator concerned. The \( C_R \)'s are numerical coefficients, and the dots indicate the contributions of descendants. The set of operators includes the unit operator which contributes the central charge term of the OPE.

To analyse the restrictions due to analyticity let us suppose that an operator on the RHS is in the representation specified by the Young tableau \(< b + c, b, a >\) for the unprimed \( \mathfrak{gl}(2|2) \) (and therefore also for the primed one). The leading singularity will occur when \( \mathcal{R}(X_{12}) \) has the fewest number of \( y \)'s allowed. This will occur when as many boxes as possible of the Young tableau are filled with \( \alpha \) indices. We can distinguish three cases, \( b \geq 2, b = 1, b = 0. \)

For \( b \geq 2 \) we can fill up the first two rows of the Young tableau with \( \alpha \)'s and so there can be \( 2(b-2) + c \) powers of \( y \). Explicitly, this gives \( (y^2)^{b-2}y^c \). In fact, there are nilpotent contributions with fewer \( y \)'s but these simply have the effect of changing \( y \) to \( \hat{y} \). The term \( y^c \), which is symmetric on both primed and unprimed indices, cannot help with singularities if \( y^2 \), so we obtain the inequality

\[ Q \leq 2b \tag{43} \]

Since \( Q = 2(b+c) + n_4 \) this implies that \( c = d = 0 \) and \( Q = 2b \). These operators are unsaturated series A and have trivial internal labels.

For \( b = 1 \) the lowest power of \( y \) we can obtain is \( y^c \) and does not affect the singularity. So in this case we find \( Q \leq 4 \). Since all the super Dynkin labels are integers in this case, we can have only \( Q = 4, 2 \). If \( Q = 4 \) we can have either \( d = 2, c = 0 \) or \( d = 0, c = 1 \). If \( Q = 2 \) we must have \( c = d = 0 \).

For \( b = 0 \) we have \( n_1 = n_2 = 0 \) so that the operator is series C. Again we find that \( Q \leq 4 \) so we can have \( Q = 0, 2, 4 \), and we also have \( Q = 2c + d \). So for \( Q = 4 \) we can have \( d = 4, c = 0 \) or \( d = 2, c = 1 \) or \( d = 0, c = 2 \). For \( Q = 2 \) we can have \( d = 2, c = 0 \) or \( d = 0, c = 1 \). Finally, if \( Q = 0 \) we must have \( d = c = 0 \); this is just the unit operator.

These results are in agreement with those we obtained previously from the three-point functions and also with those of [13]. We now comment on the interpretation. Firstly, for \( b = 0, Q = 2 \), if the internal Dynkin labels are \([020] \) then this operator is \( T \). If the labels are \([101] \) we get an operator which doesn’t exist in the interacting theory. For \( b = 0, Q = 4 \) the operator with \( SU(4) \) labels \([040] \) is \( T^2 \), the operator \([202] \) can be constructed from two \( T \)s and two derivatives, while the operator \([121] \) would require one derivative and two \( T \)s and so does not exist as a primary field in the interacting theory.

For \( b = 1, c = 0, d = 2 \) we have the family of operators with super Dynkin labels \([a(a + 1)020(a + 1)a] \). These are protected operators which can be constructed from two \( T \)s and \( a + 2 \) derivatives symmetrised with respect to both sets of indices. The lowest member of this family, with \( a = 0 \), is the operator whose leading component is \( T^2 \) in the \( 20' \) representation.
of \(SU(4)\). If \(b = 1, c = 1, d = 0\) we get another family of operators with super Dynkin labels \([a(a + 1)101(a + 1)a]\). These can be constructed from two \(T\)s and \((a + 3)\) derivatives with both sets of indices in the representation \(<2, 1, a>\). Note that all of these operators exist in the interacting theory and can be constructed from derivatives and \(T\)s so that the constraints they would obey as superfields on Minkowski superspace are consistent with gauge invariance.

For \(b = 1, Q = 2\) we have a family of operators with super Dynkin labels \([a(a + 1)000(a + 1)a]\). In the free theory such operators can be constructed from two \(W\)’s and \(a + 1\) derivatives. However, in the interacting theory, \(W\) carries group indices so that the derivatives would have to be gauge-covariant. Since the gauge potential itself is not a field on analytic superspace, it follows that such operators cannot be constructed in the interacting theory as analytic tensor superfields.

Finally, if \(b \geq 2\) we have the family of operators with \(Q = 2b, c = d = 0\). These have super Dynkin labels \([a(a + b)000(a + b)a]\). Such representations make sense for non-integral \(b\) and in fact they all have the same number of components for a given \(a\). We shall argue below that it makes sense to consider \(R(X_{12})\) for these representations as long as \(b > 1\). Hence, in the interacting quantum theory the \(b = 1, Q = 2\) series operators acquire anomalous dimensions to become unsaturated operators of the same type as the \(b \geq 2, Q = 2b\) operators. As an example, consider the Konishi operator, \(K = \text{tr}(W_IW_I)\) (\(I, J SO(6)\) vector indices) as a superfield on Minkowski superspace. In the free theory this has super Dynkin labels \([0100010]\); it is saturated and shortened and can be explicitly written in terms of \(W\) as

\[
K_{AB,A'B'} = \partial_{(AA')}W\partial_{B'B}W - \frac{1}{6}\partial_{(AA')\partial_{B'B}}W^2
\]  

(44)

In the interacting theory it becomes an unconstrained scalar superfield on Minkowski superspace and acquires an anomalous dimension \([34, 36]\); in analytic superspace it can be represented as a quasi-tensor superfield (see below) with super Dynkin labels \([0(1 + \gamma)000(1 + \gamma)0]\) where \(\gamma > 0\) is the anomalous dimension.

The above can be generalised straightforwardly to the case of two CPOs, \(A_p, A_q\). The OPE is

\[
A_p(1)A_q(2) \sim \sum C_{R}R(X_{12})(g_{12})^{\frac{1}{2}(p+q-Q)}O^{Q}_{R,R'}(2) + \ldots
\]  

(45)

where the notation is the same as above. Again the operators on the RHS must be in self-conjugate representations. The constraints imposed by analyticity in \(y_{12}\) are found in the same way as for two \(T\)s, but in this case this is not the complete set. To obtain the other conditions it is convenient to take point 2, say, to be at the origin, so that the OPE is written entirely in terms of \(X\), the coordinate of \(A_p\). Now \(A_p\) has a finite expansion in terms of \(y\) which has the highest power \((y^2)^p\). Therefore there is a bound on the highest power of \(y\) that can appear on the right-hand side. When one takes this into account (for both \(A_p\) and \(A_q\) taken to be at point 1) one obtains the same constraints as from the three-point analysis. In the case of two \(T\)s this procedure doesn’t give any new information.
6 Quasi-tensor superfields

In this section we shall discuss the notion of quasi-tensor superfields which are analytic fields which correspond to representations with non-integral Dynkin labels over the white nodes. To make the discussion simpler we shall work in $N = 2$ in $(2,1,1)$ analytic superspace. The super Dynkin diagram for this space is

$$
\bullet \circ \times \circ \bullet
$$

(46)

It is a super Grassmannian with local coordinates

$$X^{AA'} = \begin{pmatrix} x^{\alpha \dot{\alpha}} & \lambda^{\alpha} \\ \pi^{\dot{\alpha}} & y \end{pmatrix},$$

(47)

where the $x$s are the spacetime coordinates, $\lambda, \pi$ are odd coordinates and $y$ is the standard local coordinate for the internal space $\mathbb{C}P^1$. We shall put $A = (\alpha,3)$ and $A' = (\dot{\alpha},3)$. The Levi subalgebra for this superspace is $\mathfrak{sl}(2|1) \oplus \mathfrak{sl}(2|1) \oplus \mathbb{C}$. The aim is to consider representations of this algebra which can be extended to non-integral values of the Dynkin labels over the white nodes.

The simplest example which illustrates this is given by antisymmetric tensors. Consider a tensor $O_{ABC_1...C_p}$ which is (generalised) antisymmetric on all $n = p + 2$ indices; it is not difficult to see that such tensors all have the same number of components for $p = 0,1,2,...$. Tensors of this type carry the antisymmetric representation of $\mathfrak{sl}(2|1)$ in the obvious way. We now define $O_{AB}[p]$ to be the object with components

$$
O_{33}[p] = O_{3333}...
$$

(48)

$$
O_{a3}[p] = O_{a333}...
$$

(49)

$$
O_{a\beta}[p] = O_{a\beta33}...
$$

(50)

All of the other components of $O$ vanish by antisymmetry of the even indices. The action of $\mathfrak{sl}(2|1)$ is easy to find; if we take $\mathfrak{sl}(2|1)$ to act from the left, we find

$$
\delta O_{33}[p] = (p + 2)A_3^aO_{a3}[p] + (p + 2)A_3^3O[p]
$$

(51)

$$
\delta O_{a3}[p] = \hat{A}_a^\beta O_{\beta3}[p] + (p + 3)A_3^aO_{a3}[p] + A_3^3O_{33}[p] + (p + 1)A_3^\beta O_{a\beta}[p]
$$

(52)

$$
\delta O_{a\beta}[p] = (p + 1)A_3^3O_{a\beta}[p] - 2A_3^aO_{a\beta3}[p]
$$

(53)

where $A_A^B \in \mathfrak{sl}(2|1)$ and $\hat{A}_a^\beta$ is traceless. Now this formula makes sense for arbitrary values of $p$ (even complex) and it is straightforward to check that one still has a representation of the algebra. In the context of the $N = 2$ superconformal group, unitarity requires $p$ to be real and $p \geq -1$. (Note that $p$’s being less than zero is not a problem for $O_{a\beta}[p]$: we could,
antisymmetric on both pairs of indices. The components of this matrix are non-vanishing components in a \(4|4\) matrix. Again has the property that most of the components vanish. We can therefore arrange these components so that there is a common factor of \(\hat{y}^{n-2}\) coming from the propagator factor in the OPE so that the non-analyticity in \(\hat{y}\) will cancel out. On the other hand, there will be non-integral powers of \(x^2\) so that we can have anomalous dimensions in a way which is perfectly compatible with analyticity.

\[\begin{pmatrix}
(\hat{y} + \pi x^{-1} \lambda)^n & \pi^{\dot{A}}(\hat{y} + \pi x^{-1} \lambda)^{n-1} \\
\lambda^{\dot{A}}(\hat{y} + \pi x^{-1} \lambda)^{n-1} & \frac{\hat{y}^{n-2}}{n} \left( x^{\alpha\dot{\alpha}} (\hat{y} + (n-1)\pi x^{-1} \lambda) + (n-1)\lambda^{\alpha} \pi^{\dot{\alpha}} \right) \\
\lambda^{\alpha} \lambda^{\beta} \hat{y}^{n-2} & 2\frac{n}{n(n-1)} x^{[\alpha\dot{\alpha}] \lambda \beta} \hat{y}^{n-2}
\end{pmatrix}\]

(54)

where the index arrangement is

\[[AB][A'B'] = \left(\begin{array}{cccc}
33, 33 & 33, \dot{3} & 33, \dot{3} & 33, \dot{3} \\
\alpha 3, 33 & \alpha \dot{3} & \alpha \dot{3} & \alpha \dot{3} \\
\alpha 3, 33 & \alpha \dot{3} & \alpha \dot{3} & \alpha \dot{3} \\
\alpha \beta, \dot{3} & \alpha \beta, \dot{3} & \alpha \beta, \dot{3} & \alpha \beta, \dot{3}
\end{array}\right)\]

(55)

We observe that there is a common factor of \(\hat{y}^{n-2} = \hat{y}^p\). When this is factored out the remaining matrix is analytic in \(\hat{y}\) and depends on \(p\) only through numerical coefficients. The matrix can now be defined for arbitrary values of \(p\) and can appear on the RHS of the OPE of two CPOs contracted with the operator discussed above. The key point is that there will be a factor of \(\hat{y}^{-p}\) coming from the propagator factor in the OPE so that the non-analyticity in \(\hat{y}\) will cancel out. On the other hand, there will be non-integral powers of \(x^2\) so that we can have anomalous dimensions in a way which is perfectly compatible with analyticity.

7 \(U(1)_Y\)

In [27] it was argued that there might be a bonus \(U(1)\) symmetry of some \(N = 4\) SYM correlation functions whose origin could be traced to IIB supergravity. This idea was extended in [28] where it was conjectured that the OPE involving at least two short operators might be \(U(1)_Y\) invariant and that this would imply the non-renormalisation theorems for two- and three-point functions
of CPOs. We recall that these non-renormalisation theorems had been found on the AdS side [39] and checked in certain, mainly perturbative, field theory calculations [40, 17, 36]. It was further noted in [27] that the analytic superconformal invariants listed in [37] were invariant under this additional symmetry. It was realised in [38] that, using the reduction formula introduced in the superconformal context in [27], one could construct a proof of the non-renormalisation theorem for two- and three-point functions of CPOs which can be interpreted in terms of $U(1)_Y$. As shown in [38, 41, 5] there are additional, nilpotent invariants which do not need to be invariant under this symmetry which were omitted from the invariants listed in [37]. However, these invariants occur at five or more points, so that the $n$-point functions of CPOs for $n \leq 4$ are indeed invariant under this additional symmetry. In this section we extend this discussion to arbitrary protected operators in $N = 4$ SYM. The conclusions are the same as for CPOs: the two-, three- and four-point functions are $U(1)_Y$ invariant and the two- and three-point functions should be non-renormalised.

To see how this symmetry arises, we note that $(4, 2, 2)$ analytic superspace can be regarded as a coset space of $GL(4|4)$. However, when one works out the action of the group on the coordinates $X$ one finds that (infinitesimal) transformations proportional to the unit matrix do not act, so that one naturally obtains an action of $PGL(4|4)$. Explicitly, if the constraint (16) is dropped, we extend the transformation group in precisely this way. If we denote the diagonal elements of the super matrices $A$ and $D$ by

$$A \sim \frac{1}{2} \begin{pmatrix} a_0 I_2 & 0 \\ 0 & a_1 I_2 \end{pmatrix}$$

and similarly for $D$ we find the transformations

$$\delta x = sx \quad (57)$$
$$\delta \lambda = \frac{1}{2}(s + s' + t)\lambda \quad (58)$$
$$\delta \pi = \frac{1}{2}(s + s' - t)\lambda \quad (59)$$
$$\delta y = s'y \quad (60)$$

where

$$s = \frac{1}{2}(a_0 + d_0) \quad (61)$$
$$s' = \frac{1}{2}(a_1 + d_1) \quad (62)$$
$$t = \frac{1}{2}(a_0 - a_1 - (d_0 - d_1)) = -\frac{1}{2}\text{str}(\delta g) \quad (63)$$

The parameters $s$ and $s'$ correspond to dilations and internal dilations respectively, and we can identify $t$ as the $U(1)_Y$ parameter which acts only on the odd variables. If we enlarge the group in this way we find that
The basic propagator $g_{12}$, which can be interpreted as the two-point function of two $W$s in the free theory, will be invariant under the enlarged symmetry if we assign $U(1)_Y$ charge $-1$ to $W$. This simply has the effect of cancelling the $t$ term from $\Delta$, so that invariance follows because $g_{12}$ depends on the odd variables only as a product $\lambda \pi$.

Now consider an arbitrary protected analytic tensor operator $O^Q_{R|R'}$. If we assign it $U(1)_Y$ charge $-Q$ then the parameter of this transformation will only appear via the matrices $A(X)$ and $D(X)$ in (8). Therefore the Ward identities for two- and three-point functions of such operators will still be satisfied for this larger symmetry group because verifying that this is the case is essentially the same as for $PSL(4|4)$.

The reduction formula states that the derivative of an $n$-point correlation function with respect to the complex coupling constant $\tau$ is given by an $(n+1)$-point function which includes an integrated insertion of the on-shell action [27]. In analytic superspace this formula can be written as [38]

$$\frac{\partial}{\partial \tau} < O_1 \ldots O_n > \sim \int d\mu_o < T_o O_1 \ldots O_n >$$

where $d\mu = d^4x d^4y d^4\lambda$. This measure is easier to interpret in harmonic superspace, but for the following argument the key point is that the integrand will need a factor of $\lambda_o^4$ in order to obtain a non-zero integral. $T_o$ is the supercurrent inserted at point 0.\footnote{The fact that the on-shell action can be written as integral over a subspace of $N = 4$ superspace was noted some time ago [42].} If we apply this formula to a two- or three-point function of protected analytic operators the integrand on the RHS will involve a three- or four-point function of protected operators with the same tensorial structure as the left-hand side since $T_o$ has no superindices. Now invariance of the left-hand side for a given component will involve given powers of the $\lambda$s and $\pi$s as they are the only coordinates which carry $U(1)_Y$ charge. These powers must be the same on the right-hand side because the tensorial structure is the same. But to obtain a non-zero integral we need an extra four powers of $\lambda_o$, and this is clearly not possible. Thus the integrals must be zero (there may be a sum of terms on the right) and we conclude that the two- or three-point function on the left-hand-side must be non-renormalised.

The above argument is predicated on the fact that the integrands are $U(1)_Y$ invariant. This is true for the three-point functions by construction, but we need to show that it is also true for four-point functions of protected operators. For $n$-point functions of tensor operators we can proceed as for two- or three-points, namely we start by translating all the points to point 1, say, by means of $X_{1i}^{-1}, i = 2, \ldots n$. We can then take care of the dilation weights by multiplying by appropriate factors of the propagators. We shall then obtain solutions to the Ward identities by multiplying by tensors formed from monomials of the coordinate functions $X_{12j} := X_{12}X_{2j}^1X_{j1}, j = 2, \ldots n$ and their inverses. There may be many independent solutions of this type and we can multiply each one of them by a function of the invariants. By construction, these tensorial functions will
be $U(1)_Y$ invariant, so to prove the non-renormalisation theorem for three points we only need to observe that the four-point invariants are themselves invariant under $U(1)_Y$. The argument breaks down for four-point functions because there are nilpotent five-point invariants which are not $U(1)_Y$ invariant, as we remarked above.

8 Extremal correlators

We recall that extremal correlators are by definition correlation functions of CPOs such that the charge (central super Dynkin label) at one point equals the sum of charges at all of the other points. These were shown to have a free-field functional form and to be non-renormalised on the AdS side in [43], and this was checked on the field theory side in perturbation theory [44] and non-perturbatively [18]. There are also next-to-extremal correlators which exhibit similar behaviour [18, 45, 46, 47]. In [13, 48] this behaviour of extremals has been interpreted from the point of view of OPEs.

To be explicit consider a simple example, $\langle A_6(1)T(2)T(3)T(4) \rangle$. If one carries out OPE expansions in (12) and (34), then, using the restrictions on the OPES which are due to analyticity, it is easy to show that only short protected operators can contribute in the final two-point function. It follows from this that the functional form of the extremal correlator is a product of free propagators multiplied by a coefficient which could, in principle, depend on the coupling. However, the same argument shows that this dependence must be trivial, because the OPE coefficients and the coefficients of the two-point functions of the exchanged operators which contribute when one carries out the double OPE expansion are determined by two- or three-point functions of protected operators.

We shall now argue that this picture can be generalised to protected operators other than CPOs. To be explicit we shall consider an example of a four-point correlation function consisting of three $T$s and a series C operator with Dynkin labels [1p1]. We recall that such an operator is represented on analytic superspace by a covector operator $O_{A'A''}^{p+2}$. The claim is that such a correlation function is extremal if $p = 4$, i.e. $Q = 6$. In this case we can write

$$\langle O_{A'A''}^6(1)T(2)T(3)T(4) \rangle = (f_{123}(X_{123}^{-1})_{A'A''} + f_{124}(X_{124}^{-1})_{A'A''}) \times (g_{12g13g14})^2$$

(66)

where $f_{123}$ and $f_{124}$ are two arbitrary functions of the invariants. If we again carry out the double OPE expansion on this correlator we find as before that the only operators which can contribute in the intermediate channel are protected.

To show this we can either analyse the $TO_{A'A''}^6$ OPE or three-point functions of the form $\langle O_{R'R''}^Q O_{A'A''}^6 T \rangle$. We choose to do the latter. There are two possible solutions for this three-point function:

$$(i) \quad \langle O_{B'B''}^Q(1)O_{A'A''}^6(2)T(3) \rangle = P \times R((X_{12}^{-1})_{B'B''}(X_{12}^{-1})_{A'A''}R_1(X_{123}^{-1})_{B'B''})$$

(67)
(ii) \[ <\mathcal{O}^Q_{B'B}(1)\mathcal{O}^6_{A'A}(2)T(3) > = P \times (X_{231}^{-1})_{A'A}R(X_{123}^{-1}) \] (68)

where in both cases the propagator factor \( P \) is given by

\[ P = (g_{12})^{\frac{Q}{2}+2}(g_{13})^{\frac{Q}{2}-2}(g_{23})^{4-\frac{Q}{2}} \] (69)

The notation in (68) is as follows: \( B'(B') \) stands for all the unprimed (primed) indices in the representation \( R(R') \) while \( B_2 \) stands for one fewer index in the representation \( R_1 \) which is self-conjugate; the operation \( R \) on the RHS of (i) then puts all of the \( B'(B') \) indices within the bracket into the representations \( R(R') \).

We can now show that analyticity implies there can be no unprotected operators \( (b \geq 2) \) which can contribute to the three-point function and which can appear in the OPE of two \( T \)s. In fact, one can show that there are no solutions of type (ii) with \( b \geq 2 \) which are compatible with analyticity. For solutions of type (i) with \( b \geq 2 \) one can show that either \( d = 2 \) or \( d = 4 \) whereas the only unprotected operators in \( TT \) have \( d = 0 \). The operators which can contribute to both the three-point function and the \( TT \) OPE can also be found. The only possible contribution is type (i) with super Dynkin labels \([0004000]\), i.e. a short series \( C \) operator with \( SU(4) \) labels \([040]\).

A similar calculation shows that no unprotected operators are exchanged in the next-to-extremal case, \( <\mathcal{O}^4_{A'A}TTT > \).

The example given here can be generalised to the case of a \([1p1]\) operator with three other chiral primaries whose charges sum to \( Q = p + 2 \). Presumably there are many more extremal and next-to-extremal correlators.

9 Conclusions

In this paper we have exploited the fact that all of the unitary representations of \( PSU(4|4) \) that arise in \( N = 4 \) SYM can be realised as analytic superfields on \((4,2,2)\) superspace. Broadly speaking they divide into two categories, the protected operators, which are represented by analytic tensor superfields and which have integral super Dynkin labels, and the unprotected operators, which can be realised by what we have called quasi-tensor superfields and which have non-integral labels for at least one of the white (odd) nodes in the particular super Dynkin basis we have been using. We have given explicit formulae for the two- and three-point functions valid, in principle, for any protected operators and have derived how these operators appear in the OPEs of any two chiral primary operators. These OPE formulae can clearly be extended to more complicated protected operators without too much difficulty. We have also seen that \( n \)-point correlation functions of protected operators for \( n \leq 4 \) are invariant under an additional \( U(1)_Y \) symmetry and have used this fact to argue that the two- and three-point functions should be non-renormalised.

In section 6 we showed how unprotected operators can be accommodated in analytic superspace in an \( N = 2 \) example. This result shows how such operators can occur in the OPE of two
protected operators and leads us to believe that the two-and three-point correlators and OPE formulae given in the paper are probably also valid for the unprotected operators provided that factors such as $\mathcal{R}(X_{12})$ are interpreted appropriately. It would be interesting to use this formalism to try to prove the conjecture of [28] concerning the $U(1)_Y$ behaviour of OPES of different types of operator, but so far we have not done this.

Three-point functions with at least one unprotected operator cannot be non-renormalised, and it is not so clear that they are invariant under $U(1)_Y$ (although see comments in [28]). However, it might be possible to employ the reduction formula to obtain some information about the $\tau$-dependence of such correlators - one might imagine being able to relate the $\tau$-dependence of the coefficient of such a function to the anomalous dimension of the operator, for example.

Finally, the methods advocated in this paper may have further applications. For example, we are now in a position to analyse four- and higher point functions directly in analytic superspace using the OPE. Another application would be to superconformal field theories in other spacetime dimensions, particularly $D = 6, (2,0)$ and $D = 3, N = 8$ SCFT. In a recent paper [48] it has been shown that there are also protected series A operators in $D = 6, (2,0)$ and this has been used to give a proof of the triviality of certain extremal correlators studied on the AdS side in [49] and in analytic superspace in [50].

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