KÄHLER METRICS ON $G^C$

ROGER BIELAWSKI

Abstract. We study $G$-invariant Kähler metrics on $G^C$ from the Hamiltonian point of view. As an application we show that there exist $G \times G$-invariant Ricci-flat Kähler metrics on $G^C$ for any compact semisimple Lie group $G$.

1. Introduction

We study $G$-invariant Kähler metrics on $G^C$. We give Hamiltonian ansätze for such metrics, in the spirit of [14, 17] but for non-abelian Lie groups. We are particularly interested in such metrics being Ricci-flat. We give sufficient conditions for Ricci-flatness in terms of our Hamiltonian ansatz. For example, when $G$ is a Heisenberg group, we construct explicitly $G$-invariant Ricci-flat Kähler metrics on a neighbourhood of $G$ in $G^C$. For compact $G$ we can do better:

Theorem 1. Let $G$ be a compact simple Lie group and let $\gamma$ be a real closed $G \times G$-invariant $(1,1)$-form on $G^C$. Then there exists a $G \times G$-invariant Kähler metric on $G^C$ such that its Ricci form is $\gamma$. In particular, there exists a $G \times G$-invariant Ricci-flat Kähler metric on $G^C$.

This result has been previously known for $G = SU(2)$ [7, 18, 15, 9]. In this case it is known that the Ricci-flat metric is complete and we expect this to be true in the general case, although we have been unable to prove it. Similarly the question of uniqueness is left open.

The proof of Theorem 1 relies on existence and regularity of entire solutions to the (real) Monge-Ampère equation

$$f(\nabla u) \det D_{ij} u = g$$

where $f, g$ are certain smooth functions defined on $\mathbb{R}^n$. The $\mathbb{R}^n$ in question is a Cartan subalgebra of Lie($G$) and all the functions are invariant under the Weyl group $W$. There are very few results about entire solutions to (1.1). In fact, it is usually assumed that either $g \in L^1(\mathbb{R}^n)$ or that $f = 1$ and $g, 1/g$ are bounded [8]. None of these conditions applies in our situation and so we prove:

Theorem 2. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be two nonnegative locally bounded functions invariant under a finite reflection group $W \subset O(n)$ which acts irreducibly. If $\int_{\mathbb{R}^n} f(x)dx = +\infty$, then there exists an entire convex $W$-invariant weak solution $u : \mathbb{R}^n \to \mathbb{R}$ of (1.1). If, in addition, $f$ and $g$ are strictly positive and of class $C^{p,\alpha}$, then $u$ is of class $C^{p+2,\alpha}$.

Theorem 1 can be viewed in a more general context. Given a real-analytic Riemannian manifold $(M,g)$ one asks whether the metric $g$ can be extended to a Ricci-flat Kähler metric $\bar{g}$ on a complex thickening $M^C$ of $M$, perhaps satisfying

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some additional conditions. In this direction Bryant [2] showed that if \( \dim M = 3 \), then such an extension exists with the original manifold \( M \) being a special Lagrangian submanifold. As Bryant indicates, such a result should also hold for higher-dimensional manifolds with trivial tangent bundles. A similar result is true when \( (M, g) \) is Kähler [11, 13]. A stronger condition on \( g \) is that there exists an anti-holomorphic and isometric involution on \( M^C \) fixing \( M \) (which is true in [2, 11, 13]).

Our Theorem 1 (for \( \gamma = 0 \)) can be viewed as an example of such an extension when \( M = G \) with the bi-invariant metric. The main point, however, is that our extension is global. In this direction, one should ask a) when does there exist a geodesically complete extension; b) what is the relation with the global existence of the adapted global structure of Lempert and Szöke [16] and of Guillemin and Stenzel [12]. Recall that these authors have shown that a real-analytic Riemannian manifold \( (M, g) \) admits a canonical complex complex structure, called adapted, on \( TM \) or part thereof, characterised by the condition that the geodesic foliation is holomorphic. Thus one could ask whether there exists a Ricci-flat Kähler metric on the maximal domain of definition of the adapted complex structure. The only previous cases where we know of some sort global Ricci-flat Kähler extension are compact homogeneous Kähler manifolds, where the Ricci-flat Kähler extensions are actually hyperkähler, and for compact symmetric spaces of rank 1 [18].

Although Theorem 1 for \( G = SU(2) \) has been proved by Candelas and de la Ossa [7] and by Stenzel [18], our approach is somewhat different and it expresses the Kähler potential directly as an invariant function on \( SL(2, \mathbb{C}) \).

**Theorem 3.** There exists a unique (up to homothety) complete \( SU(2) \times SU(2) \) invariant Ricci-flat Kähler metric on \( SL(2, \mathbb{C}) \). It is given by the Kähler potential

\[
K(u) = \int_0^R \sqrt{\sinh(2t) - 2t} \, dt,
\]

where \( R = |h| \) and \( h \) is an element of \( su(2) \) such that \( u = g \exp ih \) with \( g \in SU(2) \).

This metric is of cohomogeneity one. Ricci-flat Kähler metrics of cohomogeneity one were studied by Dancer and Wang [10] under the assumption (generically satisfied) that the isotropy representation of the principal orbit is multiplicity-free. The above metric is an example of cohomogeneity one metric for which this assumption does not hold.

### 2. Notation and conventions

All \( G \)-actions are on the left. If \( G \) acts on a smooth manifold \( M \) and \( \rho \) is an element of the Lie algebra \( \mathfrak{g} \), then \( \rho^* \) denotes the fundamental vector field generated by \( \rho \). As the action is on the left, the map \( \rho \mapsto \rho^* \) is an antihomomorphism (i.e. \( [\rho^*, \tilde{\rho}^*] = -[\rho, \tilde{\rho}]^* \)).

If \( E \) is a representation of \( G \), then by an \( E \)-valued \( p \)-form we mean a \( G \)-equivariant map from \( \Lambda^p(TM) \) to \( E \), linear on fibers. If \( E \) is equipped with a \( G \)-invariant bilinear form \( \langle \cdot, \cdot \rangle \) and \( \phi, \psi \) are \( E \)-valued 1-forms on \( M \), then we define a real-valued 2-form \( \phi \wedge \psi \) by

\[
\phi \wedge \psi(X, Y) = \langle \phi(X), \psi(Y) \rangle - \langle \phi(Y), \psi(X) \rangle.
\]

In general, we adopt the convention that \( v \wedge w = v \otimes w - w \otimes v \). Because of this and the fact that we consider left \( G \)-actions, the structure equation for a connection
1-form $\theta$ on a principal $G$-bundle $P$ takes the form
\[ d\theta = [\theta, \theta] + \Omega \quad (2.2) \]
where $\Omega$ is the curvature of $\theta$.

### 3. Complex structures on $G \times g$

Let $P$ be a (trivial) principal $G$-bundle over $g$ and suppose that we are given a $G$-invariant complex structure on $P$ (in other words we have a $G$-invariant map $G^C \to g$). Any tangent vector on $P$ can be written uniquely as $\rho^* + \tilde{\rho}^*$ for some $\rho, \tilde{\rho} \in g$. Since $I$ is $G$-invariant, we have a connection 1-form $\theta$ on $P$ defined by
\[ \theta(\rho^* + \tilde{\rho}^*) = \rho. \quad (3.1) \]

We also have a horizontal (i.e. ad $G$-valued and vanishing on vertical vector fields) 1-form $L$ defined by:
\[ L(\rho^* + \tilde{\rho}^*) = \tilde{\rho}. \quad (3.2) \]

Conversely, given a connection 1-form $\theta$ and a non-degenerate horizontal 1-form $L$ we can define an almost complex structure $I$ by:
\[ I\rho^* = \text{unique horizontal } Y \text{ such that } L(Y) = \rho. \quad (3.3) \]

We have

**Proposition 3.1.** Let $P$ be a principal $G$–bundle over $g$. A connection 1-form $\theta$ and a non-degenerate horizontal 1-form $L$ on $P$ define an integrable $G$-invariant complex structure on $P$ if and only if
\[ \begin{cases} \Omega = -[L, L] \\ DL = 0. \end{cases} \quad (3.4) \]

**Remark 3.2.** The minus sign is the consequence of $G$ acting on the left. For $G$ semisimple, the assumption of horizontality of $L$ is unnecessary as it follows from the first equation (and the non-degeneracy).

**Proof.** Since $I$ is $G$-invariant by definition, it satisfies $[\rho^*, I_X] = I[\rho^*, X]$, for any vector field $X$ and any $\rho^* \in g$. It follows then from the formula for the Nijenhuis tensor that $I$ is integrable if and only if $[I\rho^*, I\tilde{\rho}^*] = [\rho, \tilde{\rho}]^*$ for any $\rho, \tilde{\rho} \in g$. From the definition of $I$ we have:
\[ DL(I\rho^*, I\tilde{\rho}^*) = dL(I\rho^*, I\tilde{\rho}^*) = -L([I\rho^*, I\tilde{\rho}^*]), \]
where we have used the fact, that for any 1-form $\phi$ and any vector fields $X, Y$:
\[ d\phi(X, Y) = X(\phi(Y)) - Y(\phi(X)) - \phi([X, Y]). \]
Therefore $[I\rho^*, I\tilde{\rho}^*]$ is vertical for all $\rho, \tilde{\rho}$ if and only if $DL = 0$. On the other hand, from the properties of curvature,
\[ \Omega(I\rho^*, I\tilde{\rho}^*) = -\theta([I\rho^*, I\tilde{\rho}^*]) \]
and so the vertical part of $[I\rho^*, I\tilde{\rho}^*]$ is equal to $[\rho, \tilde{\rho}]^*$ precisely when $\Omega = -[L, L]$. □
As an example of a solution to (3.4) consider a compact $G$ with the $G$-equivariant diffeomorphism between $G \times g$ and $G^C$ given by:

$$(g, h) \mapsto g \exp(ih),$$

i.e. by the polar decomposition. We shall describe $\theta$ and $L$ given by this diffeomorphism. As everything is equivariant it is enough to describe $\theta$ and $L$ at points of $\{1\} \times g$.

For the exponential map of any Lie group we have the following formula:

$$(d\exp)_u \exp(-u) = \frac{\exp(\text{ad} u) - 1}{\text{ad} u},$$

Applying this to $\exp ih$ and separating into the real and imaginary parts we obtain:

$$\theta(\rho, v) = \rho + \cos(\text{ad} h) - 1\text{ad} h(v), \quad (3.5)$$

$$L(\rho, v) = \sin(\text{ad} h)\text{ad} h(v) \quad (3.6)$$

where $(\rho, v)$ is tangent to $G \times g$ at the point $(1, h)$.

4. Kähler metrics on $G^C$

We now consider the following problem. Let $\mu : G \times g \to g^*$ be a regular $G$-equivariant map. For a given $G$-invariant complex structure on $P = G \times g$ defined by $\theta$ and $L$ we wish to describe Kähler metrics on $P$ for which $\mu$ is the moment map.

The covariant derivative $D\mu = d\mu + [\mu, \theta]$ vanishes on vertical vector fields. This is also true for $L$ and since both $D\mu$ and $L$ are non-degenerate, there exists an invertible $\text{Hom(ad}^* G, \text{ad} G)$-valued function $\Phi$ on $P$ such that

$$L = \Phi(D\mu).$$

A map $\Phi : g^* \to g$ can be viewed as giving a bilinear form on $g^*$: $\langle x, \Phi(y) \rangle$. We can therefore speak of $\Phi$ being symmetric, positive-definite etc. Let us introduce the following notation. If $\mu$ is a map from a manifold into $g^*$, then $\text{ad} \mu$ denotes the map into $\text{Hom}(g, g^*)$ defined by $\langle \text{ad} \mu(m)(x), y \rangle = \langle \mu(m), [x, y] \rangle$. Before stating the next result, let us explain the notation used there. For any $m \in G \times g$, $\text{ad} \mu \circ \Phi$ is a map from $g^*$ to itself and we can talk about its square. We have

**Theorem 4.1.** A regular equivariant map $\mu : P \to g^*$ on $P = G \times g$ is a moment map for a $G$-invariant Kähler metric on $P$ if and only if the function $\Phi$ defined by (4.1) is symmetric and if both $\Phi$ and $\Phi + \Phi \circ (\text{ad} \mu \circ \Phi)^2$ are positive-definite. If this is the case then the Kähler metric is given by

$$g = \langle \Phi(D\mu), D\mu \rangle + \langle \Phi^{-1}(\theta), \theta \rangle + \langle \mu, [\Phi(D\mu), \theta] - [\theta, \Phi(D\mu)] \rangle$$

and its Kähler form $\omega$ by

$$\omega = -d(\mu, \theta).$$

**Proof.** Suppose that $\mu$ is a moment map for a Kähler metric $g$. Then, for any $\rho, \tilde{\rho} \in g$,

$$g(\rho^*, \tilde{\rho}^*) = \langle d\mu(I\rho^*), \tilde{\rho} \rangle = \langle D\mu(I\rho^*), \tilde{\rho} \rangle = \langle \Phi^{-1}(\rho), \tilde{\rho} \rangle \quad (4.4)$$
which shows that $\Phi$ is symmetric. Moreover
\[ g(\rho^*, I\tilde{\rho}^*) = \omega(\tilde{\rho}^*, \rho^*) = \langle d\mu(\rho^*), \tilde{\rho} \rangle = \langle [\rho, \mu], \tilde{\rho} \rangle = -\langle [\rho, \mu], \tilde{\rho} \rangle \]
which shows that the metric has the form (4.2) (as $I\theta = -L$).

To find out the conditions for this $g$ to be positive-definite we rewrite $g$ as a metric on a Riemannian submersion. Let $\sigma$ be the connection form of the Riemannian submersion defined by $g$. In other words $\sigma(X) = \rho$ where $X - \rho^*$ is orthogonal to all vertical vector fields. Since $\langle d\mu(I\rho), \rho \rangle = g(X, \rho^*) = g(\sigma(X)^*, \rho^*)$, we have from (4.4)
\[ Id\mu = \Phi^{-1}(\sigma). \quad (4.5) \]
It follows that
\[ \sigma = I\Phi(\mu) = I\Phi(D\mu - [\mu, \theta]) = \theta + \Phi([\mu, L]). \quad (4.6) \]
Using this formula, we can write metric (4.2) as
\[ g = \langle \Phi(D\mu), D\mu \rangle - \langle \Phi([\mu, \Phi(D\mu)]), [\mu, \Phi(D\mu)] \rangle + \langle \Phi^{-1}(\sigma), \sigma \rangle. \quad (4.7) \]
Computing the metric separately on vertical and horizontal (with respect to $\sigma$) vectors, we see that both $\Phi$ and $\Phi + \Phi \circ (\text{ad} \mu \circ \Phi)^2$ must be positive-definite.

It remains to show that $\omega = g(I, \cdot)$ has the form (4.3). We compute (using symmetry of $\Phi$ and (3.4))
\[ g(I, \cdot) = -\langle \Phi^{-1}(L), \theta \rangle + \langle \Phi^{-1}(\theta), L \rangle + \langle [\mu, L], L + [\theta, \theta] \rangle = \theta \wedge D\mu + [\mu, -d\theta + 2[\theta, \theta]] = \theta \wedge D\mu - [\mu, d\theta] = -d(\mu, \theta). \]
Here $\phi \wedge \psi$ for a $\mathfrak{g}$-valued 1-form $\phi$ and a $\mathfrak{g}^*$-valued 1-form $\psi$ denotes the 2-form $\phi \wedge \psi(X, Y) = \langle \psi(Y), \phi(X) \rangle - \langle \phi(Y), \psi(X) \rangle$. This proves the theorem. \qed

If we are interested only in pseudo-Kähler metrics, then the relevant condition is much simpler.

**Proposition 4.2.** A regular equivariant map $\mu : P \to \mathfrak{g}^*$ on $P = G \times \mathfrak{g}$ is a moment map for a $G$-invariant pseudo-Kähler metric on $P$ if and only if the 1-form $\langle \mu, L \rangle$ is closed.

**Proof.** From the proof of the above theorem, $\mu$ defines a pseudo-Kähler metric if and only if $\Phi$ is symmetric ($\Phi$ is non-degenerate, since $\mu$ is regular). Computing $d\langle \mu, L \rangle$ and using (3.4) shows that $d\langle \mu, L \rangle = D\mu \wedge L$. As $L = \Phi(D\mu)$, this last expression vanishes precisely when $\Phi$ is symmetric. \qed

**Example 4.3.** Let $G$ be compact. There is a canonical $G \times G$ invariant Kähler metric on $G^C$ given by the Kähler potential $K(u) = \frac{1}{2} |h|^2$, where $u = g \exp \imath h$, $g \in G$, $h \in \mathfrak{g}$. The moment map is given at points of $\{1\} \times \mathfrak{g}$ by $\mu(h) = h$. The Kähler form, from (3.3) and (3.5), is
\[ \omega = -d\langle h, \rho \rangle \]
where $\rho$ is the canonical flat connection on $G$. In other words $\omega$ is just the canonical Kähler form of $T^*G = G \times \mathfrak{g}$. We also have
\[ D\mu = \text{cos}(\text{ad} h), \]
\[ \Phi^{-1} = \text{cos}(\text{ad} h) \frac{\text{ad} h}{\sin(\text{ad} h)}. \]
Using (4.7) one can easily compute the metric on $G \backslash G^C/G$, i.e. on a Weyl chamber, which turns out to be the standard Euclidean metric.

5. Ricci-flat metrics and proof of Theorem 1

As the Ricci curvature of a Kähler metric $g$ is given by $-i\partial\bar{\partial}\ln \det g$, we wish to write the Kähler form in terms of a holomorphic frame, namely
\[ \omega = -d\mu \wedge \theta - \langle \mu, d\theta \rangle = -(d\mu \wedge \theta + \langle \mu, [\theta, \theta] - [L, L] \rangle) = \]
\[ -\frac{i}{2} (\Phi^{-1}(\theta + iL) \wedge (\theta - iL) + i[\mu, \theta + iL] \wedge (\theta - iL)) = \]
\[ -\frac{i}{2} ((\Phi^{-1} + i \text{ad} \mu)(\theta + iL)) \wedge (\theta - iL). \]

Thus $g$ is Ricci-flat if the determinant of the operator $\Phi^{-1} + i \text{ad} \mu$ is constant for some basis of $\mathfrak{g}$ and the dual basis.

Example 5.1. We shall find invariant Ricci-flat Kähler metrics on the complexification of a Heisenberg group. Let $(V, \omega)$ be a symplectic vector space and let $\mathfrak{h}(V) = V \times \mathbb{R}$ denote the corresponding Heisenberg algebra. Thus $[(v, r), (w, s)] = (0, \omega(v, w))$. Let $H(V)$ denote the corresponding (simply-connected) Lie group. To define a Kähler metric on $H(V)^C$ we need $\mu, \theta$ and $L$. Choose a symplectic basis $p_1, q_1, \ldots, p_n, q_n$ of $V$, so that
\[ \omega = \sum p_i^* \wedge q_i^*. \]

Then $p_1, q_1, \ldots, p_n, q_n, 1$ is a basis of $\mathfrak{h}(V)$ and we define $\mu$ to be the map sending each vector of this basis to the corresponding vector of the dual basis. We shall look for a $\Phi$ which, in the above basis, is of the form
\[ \Phi_{(v, t)} = \text{diag}(1, \ldots, 1, f(t)) \]
for a positive function $f$. It follows easily that
\[ \det(\Phi^{-1} + i \text{ad} \mu)_{(v, t)} = \frac{(1 - t^2)^n}{f(t)}. \]

The connection 1-form $\theta$ will be given at a point $(p_i, q_i, t)$ by $(0, 0, \sum p_i dq^i)$. Then $D\mu = d\mu$ and $L = \Phi(d\mu)$ is given at a point $(p_i, q_i, t)$ by $(dp_i, dq_i, f(t)dt)$. Since $[L, \theta] = 0$ and $dL = 0$, the equations (3.4) are satisfied. Thus we obtain an $H(V)$-invariant Kähler metric on $H(V)^C$ for any positive function $f(t)$. The equation (5.1) implies that the metric is Ricci-flat if $f(t) = c(1 - t^2)^n$ for some constant $c$. These metrics are defined only on an open subset of $H(V)^C$ and are incomplete there.

We shall now consider in detail the case of a compact semisimple $G$ and a metric $g$ which is also invariant with respect to the right $G$-action. First of all we have:

Proposition 5.2. Let $G$ be compact semisimple and let $\gamma$ be a real closed $G \times G$-invariant $(1, 1)$-form on $G^C$. Then there exists a unique (up to an additive constant) real $G \times G$-invariant function $u$ such that $\gamma = -i\partial\bar{\partial}u$.

Proof. The existence of a function $u$ satisfying $\gamma = -i\partial\bar{\partial}u$ follows from the fact that $G^C$ is Stein (and its second Betti number is zero). As $G$ is compact we can average over $G \times G$ and obtain a $G \times G$-invariant $u$. To show uniqueness we have to prove that a $G \times G$ invariant pluriharmonic ($\partial\bar{\partial}f = 0$) function $f$ on $G^C$ is
constant. We use the complex structure given by \((\theta, L)\) in (3.5) and (3.6). Under this diffeomorphism \(G^\mathbb{C} \simeq G \times \mathfrak{g}\), the right \(G\)-action on \(G^\mathbb{C}\) becomes the adjoint action on \(\mathfrak{g}\). Let \(p_1, \ldots, p_n\) be a basis of invariant polynomials on \(\mathfrak{g}\). A \(G \times G\) invariant function on \(G \times \mathfrak{g}\) can be written as \(f = f(p_1, \ldots, p_n)\). We shall compute \(dIdf\).

Let us write \(df = \langle F, dh \rangle\) for a map \(F : \mathfrak{g} \to \mathfrak{g}\). Since \(dp_i([\rho, h]) = 0\) for any \(\rho\) and at any point \(h \in \mathfrak{g}\), \([F(h), h] = 0\) at any regular \(h\). Therefore, by (3.5) and (3.6), \(dIdf = \langle F, dgg^{-1}\rangle\) and

\[
dIdf = dF \wedge dgg^{-1} + \langle F, [dgg^{-1}, dgg^{-1}]\rangle.
\]

Since \(F\) is invariant for the left \(G\)-action, \(dF(\rho^*)\) vanishes for any \(\rho \in \mathfrak{g}\) and so \(\langle F, [\rho, \rho]\rangle = 0\) for all \(\rho, \tilde{\rho} \in \mathfrak{g}\). Since \(\tilde{\mathfrak{g}}\) is semisimple, \(F = 0\) and \(f\) is constant. \(\square\)

Therefore finding a Kähler metric with prescribed Ricci form is equivalent to finding a Kähler metric of the form (4.2) with prescribed (positive) determinant of the hermitian operator \(\Phi^{-1} + i \text{ad } \mu\).

Now, notice that

\[
(\Phi^{-1} + i \text{ad } \mu) \circ L = D\mu + i[\mu, L] = d\mu + [\mu, \theta + iL].
\]

Therefore, using \((\theta, L)\) given by (3.5) and (3.6) we require finding a map \(\mu : \mathfrak{g} \to \mathfrak{g}\) such that, at every point \(h \in \mathfrak{g}\), the determinant of the operator

\[
d\mu + \left[\mu, \frac{\exp(i\text{ad } h) - 1}{\text{ad } h}\right]
\]

is equal to \(e^u\)-multiple of the determinant of the operator (3.6).

Since we look for \(G \times G\) invariant metrics on \(G^\mathbb{C}\), \(\mu\) must have a special form. As the right action of \(G\) becomes the adjoint action on \(\mathfrak{g}\) under the polar decomposition and the two actions commute, the moment map \(\mu\) for the left action must be \(\text{ad } G\)-invariant. This means that \(\mu\) maps any Cartan subalgebra \(\mathfrak{h}\) to itself. Since, for any \(x \in \mathfrak{h}\) and any \(\rho \in \mathfrak{g}\), \(d\mu([\rho, x]) = [\rho, \mu(x)]\), we easily compute the determinant of (5.2) as \((\text{det } d\mu|_\mathfrak{h}) \prod a(\mu(x)) a(\alpha(x))\), where the product is taken over all roots \(\alpha\). On the other hand, the operator \(L\) given by (3.6) has determinant \(\prod \frac{\sinh a(x)}{a(x)}\).

Thus, to prove Theorem 1 we have to find a \(W\)-equivariant \((W\) being the Weyl group\) map \(\mu : \mathfrak{h} \to \mathfrak{h}\) satisfying the equation

\[
\left(\prod a(\mu(x))\right) \det d\mu = e^{\tilde{u}(x)} \prod \sinh a(x)
\]

for an arbitrary \(W\)-invariant function \(\tilde{u}\). Moreover, from Proposition 4.2, \(\langle \mu, L \rangle\) must be a closed 1-form on \(\mathfrak{g}\). As \(\mu(x)\) commutes with \(x\), this form is just \(\langle \mu(x), dx \rangle\).

Therefore \(\mu\) is the gradient of a \(G\)-invariant function \(K\) defined on \(\mathfrak{g}\). \(K\) is of course the Kähler potential of the metric. By restricting \(K\) to \(\mathfrak{h}\) we obtain:

**Proposition 5.3.** For a compact semisimple Lie group \(G\), the \(G \times G\)-invariant Kähler metrics on \(G^\mathbb{C}\) with Ricci form \(\gamma = -i\partial\bar{\partial}u\) are (up to homothety) in one-to-one correspondence with smooth convex \(W\)-invariant solutions \(\tilde{K}\) to the Monge-Ampère equation

\[
\left(\prod \alpha(\nabla \tilde{K})\right) \det D_{ij} \tilde{K} = e^{\tilde{u}(x)} \prod \sinh a(x)
\]

defined on an entire Cartan subalgebra \(\mathfrak{h}\), where the product is taken over all roots \(\alpha\) and \(\tilde{u} = u|_{\mathfrak{h}}\).
Equivalently, such metrics are in one-to-one correspondence with smooth convex $G$-invariant solutions $K$ to the Monge-Ampère equation
\[
\det D_{ij}K = e^{u(x)} \prod \sinh \alpha(x) \prod \alpha(x) 
\]
defined on all of $\mathfrak{g}$ (where the right-hand side is viewed as an invariant function on $\mathfrak{g}$).

Proof. We have to show that $\Phi$ and $\Phi \circ (\text{ad} \mu \circ \Phi)^2$ are positive definite. As $D\mu$ is given by the real part of (5.2), it follows that $L$ and $D\mu$ commute. As they are both self-adjoint, so is $\Phi = L \circ (D\mu)^{-1}$. $L$ is positive definite and $D\mu$ is positive definite on vectors tangent to $\mathfrak{h}$, where it is given by the matrix of second derivatives of $\tilde{K}$. On the other hand $D\mu$ is diagonal on the root spaces and its eigenvalues are $\frac{\alpha(\nabla \tilde{K})}{\alpha(x)} \cosh \alpha(x)$. Since $\tilde{K}$ is convex, these eigenvalues are positive. Therefore $\Phi$ is positive definite. To check that $\Phi + \Phi \circ (\text{ad} \mu \circ \Phi)^2$ is positive-definite note that $\Phi$ and $\text{ad} \mu$ commute. Therefore we have
\[
\Phi + \Phi \circ (\text{ad} \mu \circ \Phi)^2 = \Phi^2(\Phi^{-1} + i \text{ad} \mu)(\Phi^{-1} - i \text{ad} \mu)
\]
which is positive definite as $\Phi^{-1} + i \text{ad} \mu$ is hermitian (it is nondegenerate as $\det(\Phi^{-1} + i \text{ad} \mu)$ is positive).

Remark 5.4. For $u = 0$ (i.e. the Ricci-flat case), the Monge-Ampère equation (5.4) has the following interpretation. We seek a map $\mu : \mathfrak{g} \to \mathfrak{g}$ with a convex potential (i.e. $\mu = \nabla K$ for a convex function $K$) such that the pullback under $\mu$ of the canonical volume form of $\mathfrak{g}$ (given by the Killing metric) is equal to the volume form $\hat{\omega}$ which is the pullback of the canonical volume form on the symmetric space $G \backslash G^C$ via the exponential map $\exp : \mathfrak{g} \to G \backslash G^C$.

Corollary 5.5. Let $G$ be a compact simple Lie group and let $\gamma$ be a real closed $G \times G$-invariant $(1,1)$-form on $G^C$. Then there exists a $G \times G$-invariant Kähler metric on $G^C$ whose Ricci form is $\gamma$.

Proof. Theorem A.1 in the appendix shows the existence of a weak solution $\tilde{K}$ to (5.3). By Proposition A.2, $\tilde{K}$ is proper. Since a weak solution is equivalent to a solution a.e. (every convex function has first and second derivatives a.e.), $\tilde{K}$ gives rise to a proper (convex) weak solution $K$ of (5.4) on $\mathfrak{g}$. To show that $K$ is smooth apply Theorem 2 in [4] to $u(x) = K - c$, $c \in \mathbb{R}$ and the convex set $\Omega_c = \{x; K(x) \leq c\}$.

We have been unable to show that the Ricci-flat Kähler metrics obtained from Theorem 1 are complete. From (4.7), it follows easily

Proposition 5.6. Let $G$ be a compact semisimple Lie group of rank $n$. A $G \times G$-invariant Kähler metric on $G^C$ given by a $W$-invariant convex solution $\tilde{K}$ to (5.3) is complete if and only if
\[
\sum \frac{\partial^2 \tilde{K}}{\partial x_i \partial x_j} dx_i \otimes dx_j
\]
is a complete metric on $\mathbb{R}^n$. 

\[ \square \]
6. Ricci-flat Kähler metrics on $SU(2)$

We will compute explicitly $SU(2) \times SU(2)$-invariant metrics on $SL(2, \mathbb{C})$. By the arguments of the previous section such a metric is given by a moment map of the form $\mu(h) = f(|h|)h$ for some real function $f$ (at points of $\{1\} \times su(2)$). We compute the determinant of the operator (5.2). It can be rewritten as

$$f(|h|)1 + \dot{f}(|h|)d(|h|)h + f(|h|)(\cos(\text{ad} \ h) - 1 + i \sin(\text{ad} \ h)).$$

Now, on $su(2)$, $(\text{ad} \ h)^2$ has the eigenvalue 0 corresponding to the eigenvector $h$ and the eigenvalue $-|h|^2$ with the eigenspace corresponding to vectors orthogonal to $h$ (we fix here an ad $SU(2)$-invariant metric on $su(2)$, so that the norm of the Pauli matrices is 1). For the operator (6.1) $h$ is also an eigenvector and its eigenvalue is:

$$\lambda_0 = f(|h|) + \dot{f}(|h|)|h|.$$ 

We can also easily compute the other two eigenvalues as being

$$\lambda_{\pm} = f(|h|)(\cosh |h| \pm \sinh |h|) = f(|h|)e^{\pm|h|}.$$ 

On the other hand, the operator $L$ given by (3.6) has eigenvalues 1 and (twice) $\sinh(|h|)/|h|$. Therefore the metric will be Ricci flat if and only if the function $f = f(t)$ satisfies

$$f^2(f + tf) = c \left( \frac{\sinh t}{t} \right)^2$$

where $c$ is a positive constant. The constant $c$ amounts to changing the metric by a homothety and so we can assume that $c = 1$. Let us denote the function on the right by $G(t)$ and let us also write $u = f^3/3$. Then the last equation becomes

$$t \frac{du}{dt} + 3u = G(t),$$ 

whose general solution is

$$u(t) = \left( a + \int_0^t s^2 G(s) ds \right) t^{-3}.$$ 

As $u$ needs to be smooth at the origin, $a = 0$. Therefore

$$u(t) = t^{-3} \int_0^t (\sinh \tau)^2 d\tau = \frac{\sinh(2t) - 2t}{4t^3}$$

To check that we obtained a Kähler metric we have to, according to Theorem 3.1, check that $\Phi$ is self-adjoint and that both $\Phi$ and $\Phi + \Phi \circ (\text{ad} \ \mu) \Phi$ are positive-definite. This is done in the proof of Proposition 5.3.

Thus we have obtained an $SU(2) \times SU(2)$ invariant Ricci-flat Kähler metric on $SL(2, \mathbb{C})$. It follows from the arguments that any other such a metric is homothetic to this one.

Let us show that this metric is complete. As it is $SU(2) \times SU(2)$ invariant, it is of cohomogeneity one and it is enough to show that the metric on the quotient $\mathbb{R}_{\geq 0}$ is complete. This means computing the first two terms in (4.7) in the radial direction $h/|h|$. The second term becomes zero. The first one is $\langle L, D\mu \rangle$, and since $L(h/|h|) = h/|h|$ and $D\mu(h/|h|) = (f(|h|) + \dot{f}(|h|)|h|h/|h|$ (from (6.1) and the subsequent paragraph), the quotient metric is just

$$(t\dot{f} + f)dt^2.$$
From (6.2) \( t\dot{f} + f = G(t)(3u)^{-2/3} \) and it follows from (6.3) that \( t\dot{f} + f \) has an exponential growth. Therefore the metric is complete.

Finally, let us compute its Kähler potential, i.e. a real-valued function \( K \) such that \( \omega = -i\partial\bar{\partial}K \). By Theorem 3.1, \( \omega = -d\langle \mu, \theta \rangle \). Therefore a Kähler potential will be a function \( K \) such that \( 2\langle \mu, \theta \rangle = \text{Id}K \). i.e. \( dK = 2\langle \mu, L \rangle \). In our case the right-hand side becomes \( 2\langle f(\cdot|h|), dh \rangle \), which means that \( K = K(\cdot|h|) \) with \( K(t) \) satisfying \( \frac{dK}{dt} = tf \). This shows that the Kähler potential has the form stated in Theorem 3 (up to a constant multiple).

Appendix A. Entire \( W \)-invariant solutions of Monge-Ampère equations

We wish to show existence and regularity of entire solutions to a class of Monge-Ampère equations

\[
f(\nabla \phi) \det D_{ij}\phi = g(x)
\]

where \( f \) and \( g \) are nonnegative functions on \( \mathbb{R}^n \). We recall first the concept of a weak solution of (A.1). Let \( \phi \) be a convex function. Then \( \nabla \phi \) is a well-defined multi-valued mapping: \( (\nabla \phi)(x) \) is the set of slopes of all supporting hyperplanes to the graph of \( \phi \) at \( (x, \phi(x)) \). If \( B \) is a subset of \( \mathbb{R}^n \), let \( \nabla \phi(B) \) be its image in the multi-valued sense. Then \( \phi \) is a weak solution of (A.1) if

\[
\int_B g(x)dx = \int_{\nabla \phi(B)} f(y)dy
\]

for every Borel set \( B \). Let us denote the right-hand side by \( \omega(B, \phi, f) \). It can be shown that it is a Borel measure on \( \mathbb{R}^n \). A basic result is that if \( u_k \to u \) compactly and \( f_k \to f \) uniformly, then \( \omega(\cdot, u_k, f_k) \) converges to \( \omega(\cdot, u, f) \) weakly, i.e. as functionals on the space of compactly supported continuous functions.

After these preliminaries we are going to prove:

**Theorem A.1.** Let \( W \subset O(n) \) be a finite reflection group acting irreducibly on \( \mathbb{R}^n \) and let \( f, g \) be two nonnegative \( W \)-invariant functions on \( \mathbb{R}^n \). Furthermore, assume that \( f \) and \( g \) are locally bounded and that

\[
\int_{\mathbb{R}^n} f = +\infty.
\]

Then there exists a (weak) \( W \)-invariant solution \( \phi : \mathbb{R}^n \to \mathbb{R} \) of the Monge-Ampère equation (A.1). Moreover \( \phi \) is convex and Lipschitz continuous.

**Proof.** Put \( f_k = f + 1/k \) and \( g_k = g + 1/k, k \in \mathbb{N}_+ \). Let \( B_r \) denote the ball of radius \( r \) centred at the origin. Let \( R_k \) be a number defined by

\[
\int_{B_{R_k}} f_k = \int_{B_k} g_k.
\]

According to [1, 6] there exists a unique (up to a constant) strictly convex solution \( \phi_k \) of

\[
f_k(\nabla \phi_k) \det D_{ij}\phi_k = g_k(x)
\]

which is of class \( C^{1,\beta} \), for some \( \beta > 0 \), and such that \( \nabla \phi_k \) maps \( B_k \) onto \( B_{R_k} \). Moreover, as \( W \subset O(n) \) and all the data is \( W \)-invariant, \( \phi_k \) is \( W \)-invariant (this follows from uniqueness, since \( \phi_k \circ w \) is also a solution of (A.2)). To prove the existence of a weak solution to (A.1), defined on all of \( \mathbb{R}^n \), it is enough to show
that the functions $\phi_k$ are uniformly (in $k$) bounded on any ball $B_R$, $k \geq R$. Indeed, a bounded sequence of convex functions on a bounded open convex domain has a convergent subsequence. This follows from the elementary fact, which we will use repeatedly, that the slopes of supporting hyperplanes of a convex function, bounded by $R$ on a domain $G$, are bounded by $2R/\delta$ on any subdomain $G'$ such that $\text{dist}(G', \partial G) \geq \delta$.

Let $\Delta$ be a minimal set of reflections generating $W$ and consider a subgroup $W'$ of $W$ generated by $n - 1$ elements of $\Delta$. Let $L$ be the one-dimensional subspace of $\mathbb{R}^n$ fixed by $W'$. Since $\phi_k$ is $W$-invariant, $\nabla \phi_k$ maps $L$ to itself. We claim that $\nabla \phi_k$ are uniformly bounded on $L \cap B_R$. Suppose that they are not. Then there exists a subsequence $k_j$ such that $\nabla \phi_{k_j}$ maps $L \cap B_R$ onto $L \cap B_{s_j}$ ($\phi_k$, being $W$-invariant, satisfy $\nabla \phi_k(0) = 0$, where $s_j \to +\infty$. Now consider the unique solution $\psi_k$, $\psi_k(0) = 0$, to the equation

$$g_k(\nabla \psi_k) \det D_{ij} \psi_k = f_k(y)$$

mapping $B_{R_{k_j}}$ onto $B_{k_j}$. According to [1], $\nabla \psi_k$ is the inverse of $\nabla \phi_k$. Thus $\nabla \psi_{k_j}$ maps $L \cap B_{s_j}$ onto $L \cap B_R$. It follows that the functions $\psi_{k_j}$ are uniformly bounded (by $Rr$) on any $L \cap B_r$ (for sufficiently large $j$). As the $\psi_{k_j}$ are $W$-invariant and convex, they are bounded by $Rr$ on the convex hull of the set $W(L \cap B_r)$. Since $W$ acts irreducibly, $W(L \cap B_r)$ contains $\mathbb{R}^n$ and so the sets $\text{conv}(W(L \cap B_r))$ cover $\mathbb{R}^n$. In fact, there is a number $\delta \in (0,1)$, such that $B_{4R} \subset \text{conv}(W(L \cap B_r))$ for any $r$. It follows that there is a subsequence of $\psi_{k_j}$ convergent to a convex solution $\psi : \mathbb{R}^n \to \mathbb{R}$ of

$$g(\nabla \psi) \det D_{ij} \psi = f.$$ 

The function $\psi$ is bounded by $Rr/\delta$ on any $B_r$. Since $\psi$ is convex, $\nabla \psi$ is bounded by $4R/\delta$ on $B_{r/2}$. Thus $U = \nabla \psi(\mathbb{R}^n)$ is contained in $B_{4R/\delta}$. However

$$+\infty = \int_{\mathbb{R}^n} f = \int_U g,$$

which leads to a contradiction.

We have shown so far that $\nabla \phi_k$ are uniformly bounded on $L \cap B_R$. Therefore the $\phi_k$ are bounded on $L \cap B_R$ (we assume $\phi_k(0) = 0$). The argument applied before to $\psi_k$ can now be used for the $\phi_k$: since $W(L \cap B_r)$ covers $\mathbb{R}^n$ and $\phi_k$ are convex and $W$-invariant, $\phi_k$ are bounded on every compact subset of $\mathbb{R}^n$. Thus there exists a Lipschitz continuous limit of some subsequence. \hfill $\blacksquare$

**Proposition A.2.** Suppose, in addition, that $\int_{\mathbb{R}^n} g > 0$. Then any entire $W$-invariant convex solution $\phi$ of (A.1) is a proper function.

**Proof.** First of all, the assumption on $g$ implies that $\phi$ cannot be a bounded function. Indeed, otherwise $\phi$, being convex and defined on all of $\mathbb{R}^n$, is constant.

Let $C_W$ denote the polytopal complex of $W$, i.e. all hyperplanes fixed by reflections in $W$ and all their intersections. We proceed by induction on $\text{dim} L$, $L \in C_W \cup \{\mathbb{R}^n\}$, to show that $\phi$ is proper on all elements of $C_W \cup \{\mathbb{R}^n\}$. Let $L$ be a 1-dimensional element of $C_W$. Then $\phi$ is proper on $L$, since otherwise $\phi$ would be bounded (as $\text{conv}(W(L))$ is all of $\mathbb{R}^n$). Now assume that $\phi$ is proper on all $p$-dimensional elements of $C_W$. Let $L \in C_W$, $\text{dim} L = p + 1$. We claim that $\phi$ is actually proper on the vector space $V_L$ spanned by $L$. Indeed, otherwise there is a number $K$ and a set of points $x_k \to \infty$ such that $\phi(x_k) \leq K$. For sufficiently large $k$, the points $x_k$ do not
lie on $p$-dimensional elements of $C_W$ contained in $L$. There is a reflection subgroup $W' \subset W$ acting on $V_L$. By taking the intervals joining points of $W'x_k$, for each $k$, we obtain a contradiction, as these intervals intersect the $p$-dimensional elements of $C_W$ contained in $L$ and $\phi$ is convex. □

**Corollary A.3.** In the situation of Theorem A.1 suppose, in addition, that $f$ and $g$ are strictly positive and of class $C^{p,\alpha}$. Then any entire $W$-invariant convex solution $\phi$ of (A.1) is $C^{p+2,\alpha}$.

**Remark.** It is well-known that the conclusion of this corollary does not hold for solutions which are not $W$-invariant.

**Proof.** We can apply Corollary 2 in [3] to the convex (and bounded by the last proposition) set $\Omega_c = \{x; \phi(x) \leq c\}$. Doing this for every $c$ shows that $\phi$ is strictly convex. Now the main result of [5] implies that $\phi$ is of class $C^{1,\beta}$. Therefore $\nabla \phi$ is Hölder continuous and $\phi$ is a solution of $\det D_{ij} \phi = \tilde{g}$, where $\tilde{g} = g/f(\nabla \phi)$ is of class $C^{0,\beta}$. We can apply Theorem 2 in [4] to $u(x) = \phi(x) - c$, $c \in \mathbb{R}$ and the convex and bounded set $\Omega_c = \{x; u(x) \leq 0\}$ to conclude that $\phi$ is of class $C^{2,\beta}$. Higher regularity is standard. □

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Department of Mathematics, University of Glasgow, Glasgow G12 8QW, UK
E-mail address: R.Bielawski@maths.gla.ac.uk