Domain of attraction of the quasi-stationary distribution for one-dimensional diffusions

Hanjun Zhang, Guoman He

School of Mathematics and Computational Science, Xiangtan University, Hunan 411105, P.R. China.

Abstract
In this paper, we study quasi-stationarity for one-dimensional diffusions killed at 0, when 0 is a regular boundary and \(+\infty\) is an entrance boundary. We give a necessary and sufficient condition for the existence of exactly one quasi-stationary distribution, and that this distribution attracts all initial distributions. In particular, a novelty here is that we show that if the killed semigroup is intrinsically ultracontractive, then it is not only a sufficient condition ensuring the uniqueness of the quasi-stationary distribution, but also a necessary condition.

Keywords: One-dimensional diffusion; Quasi-stationary distribution; Quasi-limiting distribution; Intrinsic ultracontractivity

2000 MSC: primary 60J60; 60J70 secondary 47D07; 37A30

1. Introduction
In this paper, we consider the one-dimensional diffusions \(X\) on \([0, \infty)\) given by

\[
\frac{dX_t}{dt} = dB_t - q(X_t)dt, \quad X_0 = x > 0,
\]

where \((B_t; t \geq 0)\) is a standard one-dimensional Brownian motion and \(q \in C^1([0, \infty))\). Observe that, under the condition \(q \in C^1([0, \infty))\), \(\int_0^\infty e^{Q(y)}dy < \infty\) and \(\int_0^\infty e^{-Q(y)}dy < \infty\) for some (and, therefore, for all) \(d > 0\), which is equivalent to saying that the boundary point 0 is regular in the sense of Feller, where \(Q(y) = \int_y^\infty 2q(x)dx\).

Let \(P_x\) and \(E_x\) stand for the probability and the expectation, respectively, associated with \(X\) when initiated from \(x\). For any distribution \(\nu\) on \((0, \infty)\), we define \(P_\nu(\cdot) := \int_0^\infty P_x(\cdot)\nu(dx)\), and \(E_\nu\) denotes the expectation with respect to \(P_\nu\). Let \(\tau\) be the hitting time of 0, that is, \(\tau = \inf\{t > 0 : X_t = 0\}\).

Associated to \(X\) we consider the sub-Markovian semigroup given by \(T_t f(x) = E_x(f(X_t), \tau > t)\), with density kernel denoted by \(r(t, x, y)\). We denote by \(L\) the infinitesimal operator of \(X\), that is,

\[
L = \frac{1}{2} \frac{\partial}{\partial x} x - q \frac{\partial}{\partial x}.
\]

Let us introduce the following useful measure defined on \((0, \infty)\):

\[
\mu(dy) := e^{-Q(y)}dy.
\]

Notice that \(\mu\) is the speed measure for \(X\).

One of the fundamental problems for a killed Markov processes conditioned on long-term survival is to study its long-term asymptotic behavior. Conditional stationarity, which we call quasi-stationarity, is one of the most interesting topics in this direction. More formally, the following definition captures the main object of interest of this work.

\(^*\)Corresponding author
Email address: hgm01640163.com (Guoman He)

Preprint submitted to Elsevier

September 30, 2014
Our main results are Theorem 2 (see Section 3) and Theorem 3 (see Section 4). Let us re-state them at here.

We also say that \( \nu \) is attracted to \( \pi \), or is in the domain of attraction of \( \pi \), for the conditional evolution.

In some cases, the long time behavior of the conditional distribution can be proved that is independent of the initial state. This leads us to study the notion of Yaglom limit.

Definition 2. We say that a probability measure \( \pi \) supported on \((0, \infty)\) is a QLD, if there exists a probability measure \( \nu \) such that the following limit exists in distribution:

\[
\lim_{t \to \infty} \mathbb{P}_\nu(X_t \in \bullet|\tau > t) = \pi(\bullet).
\]

We also say that \( \nu \) is attracted to \( \pi \), or is in the domain of attraction of \( \pi \), for the conditional evolution.

Definition 3. We say that a probability measure \( \pi \) supported on \((0, \infty)\) is a Yaglom limit, if for any \( x \in (0, \infty) \)

\[
\lim_{t \to \infty} \mathbb{P}_\pi(X_t \in \bullet|\tau > t) = \pi(\bullet).
\]

It is generally believed that QSD, QLD and Yaglom limit have the following relation (see [16]):

Yaglom limit \( \Rightarrow \) QSD \( \iff \) QLD.

A complete treatment of the QSD problem for a given family of processes should accomplish the following two things (see [18]):

(i) determination of all QSD's; and

(ii) solve the domain of attraction problem, namely, characterize all laws \( \nu \) such that a given QSD \( \nu \) attracts all \( \nu \).

Although ever since the pioneering work by Mandl [13], the existence of the Yaglom limit and that of a QSD for killed one-dimensional diffusion processes have been proved by many authors (see, e.g., [14, 14, 16, 16]), it is very difficult to give a complete answer to the question of domain of attraction for initial distributions are different from the Dirac measures and the compactly supported initial distributions. In fact, details about (ii) are known only for the Brownian motion with strictly negative constant drift [15] and the Ornstein-Uhlenbeck process [12]. Under Mandl’s conditions are not satisfied, the problem of the existence, uniqueness and domain of attraction of QSDs for one-dimensional diffusions killed at 0 and whose drift is allowed to go to \(-\infty\) at 0 and the process is allowed to have an entrance boundary at \(+\infty\) are solved in a satisfactory way by Cattiaux et al. [2]. In the present paper, we will show that there is exactly one QSD for the one-dimensional diffusion \( X \) killed at 0, when 0 is a regular boundary and \(+\infty\) is an entrance boundary, and that this distribution attracts all initial distributions.

In this paper, we give a necessary and sufficient condition for the existence of exactly one QSD in terms of \( q \) (see Hypothesis (H)). If the ground state \( \eta_1 \) (eigenfunction associated to the principal eigenvalue \( \lambda_1 \)) belongs to \( L^1(\mu) \), we show that this unique QSD \( \nu_1 \) can be written by

\[
d\nu_1 = \frac{\eta_1 d\mu}{\langle \eta_1, 1 \rangle_\mu},
\]

where \( \langle f, g \rangle_\mu := \int_0^\infty f(u)g(u)\mu(du) \). In order to obtain \( \eta_1 \in L^1(\mu) \) and \( L^2(\mu) \) estimates for the heat kernel \( r(t, x, y) \), we show that the semigroup \( \{T_t\}_{t \geq 0} \) is intrinsically ultracontractive [IU]. Although [5, 17] have proved that intrinsic ultracontractivity is a sufficient condition for the measure \( \nu_1 \) being a unique QSD, it seems that, up to now, no one has pointed out that it is also a necessary condition. In this paper, we plan to fill this gap. We show that intrinsic ultracontractivity is not only a sufficient condition ensuring the uniqueness of the QSD, but also a necessary condition.

For all the results in this paper we will use the following hypothesis (H), that is,

**Hypothesis (H).**

\[
\int_0^\infty e^{\theta t} \left( \int_0^\infty e^{-\theta z} dz \right) dy < \infty.
\]

Our main results are Theorem 2 (see Section 3) and Theorem 3 (see Section 4). Let us re-state them at here.
Theorem 1. (1) The following are equivalent:
(i) (H) holds;
(ii) [IU] holds;
(iii) There exists a unique quasi-stationary distribution

\[ d\nu_1 = \frac{\eta_1 d\mu}{\langle \eta_1, 1 \rangle_{\mu}} \]

for the process \( X \).

(2) Assume (H) holds. Then the unique quasi-stationary distribution \( \nu_1 \) attracts all initial distributions \( \nu \) supported in \( (0, \infty) \), that is, for any Borel set \( A \subseteq (0, \infty) \)

\[ \lim_{t \to \infty} \mathbb{P}_x(X_t \in A | \tau > t) = \nu_1(A). \]

Moreover, we point out the following interesting fact (which is basically Proposition 5).

Proposition 1. Assume (H) holds. Then \( \eta_1 \) is bounded.

This paper is organized as follows. In Section 2, we study the spectrum of the operator \( L \). Section 3 contains the proof of the intrinsic ultracontractivity. We also show in Section 5 that there exists a unique QSD for the process \( X \). In the last section, we solve the problem of domain of attraction of this unique QSD.

2. The spectrum of \( L \)

Throughout this paper, we shall assume the process \( X \) has a finite lifetime, i.e. for \( x > 0 \)

\[ \mathbb{P}_x(\tau < \infty) = 1. \]

This is very closely related to the following function defined for \( x \in (0, \infty) \)

\[ \Lambda(x) = \int_0^x e^{Q(y)} dy. \]  

(3)

Notice that \( \Lambda \) is the scale function for \( X \), is finite and satisfies \( L \Lambda = 0, \Lambda(0) = 0, \Lambda'(0) = 1 \). In fact, as can be seen from the definition of natural scale that \( \Lambda(X_t) \) is a nonnegative local martingale, and so that \( \mathbb{P}_x(\tau < \infty) = 1 \) for \( x > 0 \) if and only if the scale function is finite at \( \infty \); That is, for \( c > 0 \), \( \int_0^\infty e^{Q(y)} dy = \infty \).

If \( \int_0^\infty e^{Q(y)} \left( \int_y^\infty e^{-Q(z)} dz \right) dy = \infty \) and hypothesis (H) holds, then \( +\infty \) is called an entrance boundary according to Feller’s classification (see, e.g., Chapter 15 in [8]). Observe that \( \int_0^\infty e^{Q(y)} \left( \int_y^\infty e^{-Q(z)} dz \right) dy = \infty \) if \( \Lambda(\infty) = \infty \). However, we know from the following lemma that hypothesis (H) holds can deduce that \( \Lambda(\infty) = \infty \). Hence, hypothesis (H) directly implies that \( +\infty \) is an entrance boundary.

Lemma 1. Assume (H) holds. Then \( \mu(0, \infty) < \infty \) and \( \Lambda(\infty) = \infty \).

Proof. For any \( x \in (0, \infty) \), we have

\[ \mu(0, \infty) = \int_0^\infty e^{-Q(z)} dz = \int_0^x e^{-Q(z)} dz + \int_x^\infty e^{-Q(z)} dz. \]

Under the condition \( q \in C^1((0, \infty)) \) we have that \( \int_0^x e^{-Q(z)} dz < \infty \), for all \( x \in (0, \infty) \). In addition, if (H) holds then for all \( x \in (0, \infty) \), \( \int_0^\infty e^{-Q(z)} dz < \infty \). We have thus proved that \( \mu(0, \infty) < \infty \). Applying the Cauchy–Schwarz inequality, we get \( x^2 = (\int_0^x e^{Q(z)/2} e^{-Q(z)/2} dz)^2 \leq \int_0^x e^{Q(z)} dz \int_0^x e^{-Q(z)} dz \), and therefore, (H) implies that \( \Lambda(\infty) = \infty \). \( \square \)

It is well known (see, e.g., [8]) that a QSD for any absorbing, continuous-time Markov chain on \( \{0\} \cup \{1, 2, \ldots\} \) can exist only if absorption at 0 is certain and the decay parameter is positive, and so that the study of QSD for such a process needs to consider the positiveness of the decay parameter. Similarly, we also need to consider the positiveness of the principal eigenvalue \( \lambda_1 \) for one-dimensional diffusions. In fact, we have the following result.
Lemma 2. Assume (H) holds. Then $\lambda_1 > 0$.

Proof. For any $x > 0$, when $0 < y \leq x$, we have

$$
\int_0^x e^Q(t)dy \int_y^\infty e^{-Q(t)}dz = \int_0^x e^Q(t)\int_y^\infty e^{-Q(t)}dzdy \\
\leq \int_0^\infty e^Q(t)\int_y^\infty e^{-Q(t)}dzdy \\
\leq \int_0^\infty e^Q(t)\int_y^\infty e^{-Q(t)}dzdy.
$$

It follows that

$$
\delta := \sup_{x>0} \int_0^x e^Q(t)dy \int_y^\infty 2e^{-Q(t)}dy \leq \int_0^\infty e^Q(t)\int_y^\infty 2e^{-Q(t)}dzdy.
$$

If (H) is satisfied, then $\delta < \infty$. In addition, we know from Lemma 1 that (H) holds imply $\mu(0, \infty) < \infty$, then from Theorem 1.1 in [3] we know that $(4\delta)^{-1} \leq \lambda_1 \leq (\delta)^{-1}$. From this estimate the result follows. □

Next we will show the discreteness of the spectrum. Although this fact has already been proved in Theorem 3.16 of [10], we give another proof. Assertion (ii) of the following proposition was proved in Theorem 3.1 of [11] for a drifted Brownian motion killed at 0, when 0 is an exit boundary and $+\infty$ is an entrance boundary, but the proof carries over without essential changes to our case.

Proposition 2. Assume (H) holds. Then we have

(i) the spectrum of $L$ is discrete.

(ii) for any nonnegative $f, g \in L^2(\mu)$,

$$
\lim_{t \to \infty} e^{\lambda t} \langle g, T_tf \rangle_\mu = \langle \eta_1, f \rangle_\mu \langle \eta_1, g \rangle_\mu. 
$$

Proof. Assertion (i): Assume (H) holds. Then it is easy to see that

$$
\lim_{t \to \infty} \mu([r, \infty)) \int_0^r e^{Q(t)}dx = 0. \tag{5}
$$

From Lemma 1 we have that $\mu$ is finite. Thus we know from Corollary 3.4.14 in [23] that (5) holds if and only if the following super-Poincaré inequality holds, which has been introduced in the work [23]:

$$
\mu(f^2) \leq cE(f, f) + \beta(c)\mu(|f|^2), \quad c > 0,
$$

where $\beta : (0, \infty) \to (0, \infty)$ is a decreasing function, $(E, D(E))$ a Dirichlet form on $L^2(\mu)$. It has been proved in Theorem 2.1 of [23] that (6) is equivalent to $\sigma_{\text{ess}}(L) = 0$, i.e. the spectrum of $L$ is discrete, where $\sigma_{\text{ess}}(L)$ denotes the essential spectrum of $L$.

Assertion (ii): It is straightforward from the $L^2$ version of the process. □

3. Existence and uniqueness of quasi-stationary distributions

In this section, we study the standard QSDs of a one-dimensional diffusion $X$ killed at 0, when 0 is a regular boundary and $+\infty$ is an entrance boundary, a typical problem for absorbing Markov processes. We first show that the semigroup $\{T_t\}_{t \geq 0}$ is intrinsically ultracontractive, which ensures the integrability of $\eta_1$ with respect to $\mu$. Before this, we need to do some preparation.

According to [2], the one-dimensional diffusion $X$ is symmetric with respect to $\mu$ and satisfies the following assumptions:

(I) (Irreducibility) If a Borel set $A$ is $T_t$-invariant, i.e. $T_t(1_Af)(x) = 1_A T_t f(x)$ $\mu$-a.e. for any $f \in L^2(\mu) \cap B_{lip}(0, \infty)$ and $t > 0$, then $A$ satisfies either $\mu(A) = 0$ or $\mu((0, \infty) \setminus A) = 0$. Here $B_{lip}(0, \infty)$ is the space of bounded Borel functions on $(0, \infty)$. 

4
(II) (Strong Feller Property) For each \( t > 0 \), \( T_t(\mathfrak{B}_b(0, \infty)) \subset C_b(0, \infty) \), where \( C_b(0, \infty) \) is the space of bounded continuous functions on \( (0, \infty) \).

(III) (Tightness) For any \( \epsilon > 0 \), there exists a compact set \( K \) such that
\[
\sup_{x \in (0, \infty)} R_t 1_K(x) \leq \epsilon.
\]

Here the resolvent \( R_t f(x) = \int_0^\infty e^{-rt} T_t f(x) dt \) and \( 1_K \) is the indicator function of the complement of the compact set \( K \).

Based on the above facts, we have the following result.

**Lemma 3.** ([7], Lemma 6.4.5) Assume that \( X \) satisfies (I)–(III). Then there exists a ground state \( \eta_1 \) of \( (E, D(E)) \) uniquely up to a sign. \( \eta_1 \) can be taken to be strictly positive on \( (0, \infty) \).

Following [17], let us recall the notion of intrinsic ultracontractivity, which was introduced by Davies and Simon [6], is a very important concept in both analysis and probability and has been studied extensively.

**Definition 4.** Assume that the ground state \( \eta_1 \) of \( (E, D(E)) \) exists and \( \eta_1 \) is strictly positive on \( (0, \infty) \). Let \( \{T_t\}_{t \geq 0} \) be the semigroup generated by \( (E, D(E)) \). \( \{T_t\}_{t \geq 0} \) is said to be intrinsically ultracontractive [IU] if, for any \( t > 0 \), there exists a constant \( c_t > 0 \) such that
\[
r(t, x, y) \leq c_t \eta_1(x) \eta_1(y) \quad \text{for } x, y \in (0, \infty).
\]  

(7)

Here we should recall (see [5], Theorem 4.2.5) that [IU] implies that for any \( t > 0 \) there exists a constant \( c'_t > 0 \) such that
\[
c'_t \eta_1(x) \eta_1(y) \leq r(t, x, y) \quad \text{for } x, y \in (0, \infty).
\]

We can see that [IU] implies the following condition [SIU] trivially.

**Definition 5.** Assume that the ground state \( \eta_1 \) of \( (E, D(E)) \) exists and \( \eta_1 \) is strictly positive on \( (0, \infty) \). Let \( \{T_t\}_{t \geq 0} \) be the semigroup generated by \( (E, D(E)) \). \( \{T_t\}_{t \geq 0} \) is said to be semi-intrinsically ultracontractive [SIU] if, for any \( t > 0 \) and any compact set \( K \) of \( (0, \infty) \), there exists positive constants \( A, B \) such that
\[
A \eta_1(x) \eta_1(y) \leq r(t, x, y) \leq B \eta_1(x) \eta_1(y) \quad \text{for } x \in K, \ y \in (0, \infty).
\]  

(8)

This notion was introduced by Bañuelos and Davis [1], where they called it one half [IU]. By integrating the left-hand inequality of (8) with respect to \( y \) over \( (0, \infty) \), we get
\[
A \eta_1(x) \int_0^\infty \eta_1(y) \mu(dy) \leq 
\int_0^\infty r(t, x, y) \mu(dy) \leq 1
\]

(9)

for \( x \in K \), namely, the [SIU] implies the \( L^1 \)-integrability of \( \eta_1 \). Similarly, by integrating the left-hand inequality of (8) with respect to \( x \) over \( K \), we also obtain \( \eta_1 \in L^\infty(\mu) \). Thus \( \eta_1 \in L^1(\mu) \cap L^\infty(\mu) \) which is included into \( L^2(\mu) \). Based on this fact, from the inequality (7), it is trivial to see that for any \( t > 0 \),
\[
\int_0^\infty r^2(t, x, y) \mu(dy) \leq c^2_t \eta_1^2(x) \int_0^\infty \eta_1^2(y) \mu(dy) < \infty,
\]

that is, \( r(t, x, y) \in L^2(\mu) \).

The following lemma, due to Tomisaki [20] who gives a sufficient condition for [IU] in terms of the speed measure, the scale function and the killing measure of a one-dimensional diffusion process, plays an important role for the proof of Proposition 3.

**Lemma 4.** ([20], Remark 2.10 and Theorem 2.11) Assume that the killing measure is equal to zero, 0 is a regular boundary and \( +\infty \) an entrance boundary. Then [IU] holds.

In [17], the author prove that the boundary point 0 is exit and \( +\infty \) is entrance for the one-dimensional logistic Feller diffusion process and the semigroup of this process has the [IU]. For our case, we have the following result.
Proposition 3. Assume (H) holds. Then [IU] holds.

Proof. The result follows from Lemma 4.

We present the following lemma for convenience of the reader to understand the proof of Theorem 2.

Lemma 5. ([17], Corollary 2.6) Assume that \([T_t]_{t \geq 0}\) has the [SIU]. Then a quasi-stationary distribution of the process uniquely exists.

Now we are ready to state the main result of this section.

Theorem 2. The following are equivalent:

(i) (H) holds;
(ii) [IU] holds;
(iii) There exists a unique quasi-stationary distribution

\[ d\nu_1 = \frac{\eta_1 d\mu}{\langle \eta_1, 1 \rangle_{\mu}} \]

for the process \(X\).

Proof. (i) \(\Rightarrow\) (ii). See Proposition 3.
(ii) \(\Rightarrow\) (iii). First, notice that [IU] holds implies \(\eta_1 \in L^1(\mu)\). Thanks to the symmetry of the semigroup, for all \(f \in L^2(\mu)\) we have

\[ \int T_t f \eta_1 d\mu = \int f T_t \eta_1 d\mu = e^{-\lambda_1 t} \int f \eta_1 d\mu. \]  \(\text{(10)}\)

The equality (10) can extend to all bounded function \(f\). In particular, we may use it with \(f = 1_A\) and with \(f = 1_{(0, \infty)}\).

Note that

\[ \int T_t 1_A \eta_1 d\mu = \mathbb{P}_{\nu_1}(X_t \in A, \tau > t)(\eta_1, 1)_\mu \]

and

\[ \int T_t (1_{(0, \infty)}) \eta_1 d\mu = \mathbb{P}_{\nu_1}(\tau > t)(\eta_1, 1)_\mu, \]

then

\[ \mathbb{P}_{\nu_1}(X_t \in A|\tau > t) = \frac{\mathbb{P}_{\nu_1}(X_t \in A, \tau > t)}{\mathbb{P}_{\nu_1}(\tau > t)} = \frac{\int T_t 1_A \eta_1 d\mu}{\int (1_{(0, \infty)})(1_{(0, \infty)}) \eta_1 d\mu} = \frac{\int 1_A T_t \eta_1 d\mu}{\int (1_{(0, \infty)})(1_{(0, \infty)}) \eta_1 d\mu} = \nu_1(A). \]

Thus, we get that \(\nu_1\) is a QSD. Moreover, we know from Lemma 5 that \(\nu_1\) is a unique QSD of \(X\).

(iii) \(\Rightarrow\) (i). If there exists a unique QSD for \(X\), then it is easy to prove that (H) holds. In fact, from Theorem 4.14 in [10] we know the following truth:

If 0 is regular, \(+\infty\) is inaccessible and \(\mathbb{P}_x(\tau < \infty) = 1\), then there exists a unique QSD if and only if for every \(a > 0\) there exists \(y_a > 0\) such that

\[ \sup_{x > y_a} \mathbb{E}_x[e^{\alpha \tau_x}] < \infty. \]

This is true if and only if infinity is an entrance boundary. \(\square\)
4. Domain of attraction of the quasi-stationary distribution

In this section, we consider the problem of the domains of attraction in our framework. We have proved that \( v_1 \) is a QSD in previous section. Next we will use the same arguments as in the proof of Theorem 5.3 of [2] to show \( v_1 \) is the Yaglom limit distribution.

**Proposition 4.** Assume \((H)\) holds. Then for any \( x > 0 \) and any Borel subset \( A \) of \((0, \infty)\),

\[
\lim_{t \to \infty} e^{t(\lambda - 1)}P_x(\tau > t) = \eta_1(x)\nu(\eta_1, 1)_\mu, \tag{11}
\]

\[
\lim_{t \to \infty} e^{t\lambda}P_x(X_t \in A, \tau > t) = v_1(A)\eta_1(x)(\eta_1, 1)_\mu. \tag{12}
\]

This implies that

\[
\lim_{t \to \infty} P_x(X_t \in A|\tau > t) = v_1(A),
\]

that is, \( v_1 \) is the Yaglom limit distribution.

**Proof.** If \((H)\) is satisfied, we have that \( \mu \) is a bounded measure from Lemma[1] For any Borel set \( A \subseteq (0, \infty) \) such that \( I_A \in L^2(\mu) \) and any \( x > 0, t > 1 \), we have

\[
P_x(X_t \in A, \tau > t) = \int P_y(X_{t-1} \in A, \tau > t-1)r(1, x, y)\mu(dy)
\]

\[
= \int T_{t-1}(I_A)(y)r(1, x, y)\mu(dy)
\]

\[
= \int I_A(y)(T_{t-1}r(1, x, \cdot))(y)\mu(dy).
\]

Since both \( I_A \) and \( r(1, x, \cdot) \) are in \( L^2(\mu) \), by using Proposition[2] we obtain

\[
\lim_{t \to \infty} e^{t(\lambda - 1)}P_x(X_t \in A, \tau > t) = (I_A, \eta_1)_\mu(r(1, x, \cdot), \eta_1)_\mu.
\]

Since

\[
\int r(1, x, y)\eta_1(y)\mu(dy) = (T_1\eta_1)(x) = e^{-\lambda t}\eta_1(x),
\]

thus we get that \( v_1 \) is the Yaglom limit. \( \square \)

In the next result we give a sharper estimate on \( \eta_1 \), which has not been mentioned by previous authors (see, e.g., [19, 2, 10, 11]). We will use the same arguments as in the proof of Theorem 5.3 of [2] to show \( e^{t(\lambda - 1)}P_x(\tau > t) \) is uniformly bounded in the variables \( t \) and \( x \).

**Proposition 5.** Assume \((H)\) holds. Then \( \eta_1 \) is bounded.

**Proof.** Let us first remark that for \( 0 < x < x_0, P_{x_0}(\tau > t) \leq P_{x_0}(\tau > t) \). Thus from the equality (11), we get that \( \eta_1(x) \leq \eta_1(x_0) \). If \((H)\) is satisfied, we know from the proof of Theorem 5.3 of [2] that \( H \) holds can deduce that there is \( x_0 > 0 \) such that \( B_1 := \sup_{u \leq x_0} E_x[\lambda^{x_0}] < \infty \). From the equality (11) again, we get that \( B_2 := \sup_{u \geq 0} e^{-\lambda t}P_{x_0}(\tau > u) < \infty \). Then for large \( x > x_0 \), we have

\[
P_x(\tau > t) = \int_0^\infty P_{x_0}(\tau > u)P_x(\tau_{x_0} \in d(t - u)) + P_x(\tau_{x_0} > t)
\]

\[\leq B_2 \int_0^\infty e^{-\lambda t}P_x(\tau_{x_0} \in d(t - u)) + P_x(\tau_{x_0} > t)
\]

\[\leq B_2e^{-\lambda t}E_x[1^{\lambda_{x_0}}] + e^{-\lambda t}E_x[0^{\lambda_{x_0}}]
\]

\[\leq e^{-\lambda t}B_1(B_2 + 1).
\]

Thus, we get that \( e^{t(\lambda - 1)}P_x(\tau > t) \) is uniformly bounded in the variables \( t \) and \( x \). By using the equality (11), it is easily seen that for large \( x > x_0 > 0, \eta_1(x) \leq \frac{B_1(x_0 + 1)}{\eta_1(x_0)} \). Hence, for any \( x > 0 \), there exists \( x_0 > 0 \) such that \( \eta_1(x) \leq \max[\eta_1(x_0), \frac{B_1(x_0 + 1)}{\eta_1(x_0)}] \). This completes the proof. \( \square \)
Although in the literature there are several articles which have studied the problem of the domains of attraction (see, e.g., [19, 10]), our result is of particular interest in the analysis of the domain of attraction of QSD for one-dimensional diffusions because we use the definition of QLD to prove the domain of attraction but not impose any condition on initial distribution. Inspired by the proof of Lemma 19 in [22], we have the following result.

**Theorem 3.** Assume (H) holds. Then the unique quasi-stationary distribution \( \nu_1 \) attracts all initial distributions \( \nu \) supported in \( (0, \infty) \), that is, for any Borel set \( A \subseteq (0, \infty) \)

\[
\lim_{t \to \infty} \mathbb{P}_\nu(X_t \in A | \tau > t) = \nu_1(A).
\]

**Proof.** Let \( \nu \) be a probability measure whose support is contained in \( (0, \infty) \). If (H) is satisfied, we know from the proof of Proposition 5 that \( e^{t \lambda} \mathbb{P}_\nu(\tau > t) \) is uniformly bounded in the variables \( t \) and \( x \), and \( \eta_1 \) is bounded. If (H) is satisfied, we also know from Lemma 1 that \( \mu(0, \infty) < \infty \). Then, by the dominated convergence theorem, one can integrate with respect to \( \nu \) under the limit in the equality (11).

\[
\lim_{t \to \infty} e^{t \lambda} \mathbb{P}_\nu(\tau > t) = \int_{0}^{\infty} \eta_1(x) \int_{0}^{\infty} \eta_1(y) \mu(dy) \nu(dx).
\]

The same holds for the equality (12).

\[
\lim_{t \to \infty} e^{t \lambda} \mathbb{P}_\nu(X_t \in A, \tau > t) = \nu_1(A) \int_{0}^{\infty} \eta_1(x) \int_{0}^{\infty} \eta_1(y) \mu(dy) \nu(dx).
\]

This implies that

\[
\lim_{t \to \infty} \mathbb{P}_\nu(X_t \in A | \tau > t) = \lim_{t \to \infty} \frac{\mathbb{P}_\nu(X_t \in A, \tau > t)}{\mathbb{P}_\nu(\tau > t)} = \nu_1(A).
\]

We complete the proof. \( \square \)

**Acknowledgements**

The first author would like to thank Prof. Servet Martínez for his kind hospitality during a visit to the Centro de Modelamiento Matemático of Universidad de Chile, where part of this work was done. The work is supported by the National Natural Science Foundation of China (Grant No.11371301) and the Science and Technology Planning Project of Hunan Provincial Government of China (Grant No. 2012FJ4093).

**References**

[1] R. Bañuelos, B. Davis, Heat kernel, eigenfunctions, and conditioned Brownian motion in planar domains, J. Funct. Anal. 84 (1989) 188–200.

[2] P. Catiaux, P. Collet, A. Lambert, S. Martínez, J. San Martín, Quasi-stationary distributions and diffusion models in population dynamics, Ann. Probab. 37 (2009) 1926–1969.

[3] M. F. Chen, Explicit bounds of the first eigenvalue, Sci. in China, Ser. A. 43 (2000) 1051–1059.

[4] P. Collet, S. Martínez, J. San Martín, Asymptotic laws for one-dimensional diffusions conditioned to nonabsorption, Ann. Probab. 23 (1995) 1300–1314.

[5] E. B. Davies, Heat Kernels and Spectral Theory, Cambridge Univ. Press, Cambridge, 1989.

[6] E. B. Davies, B. Simon, Ultracontractivity and the heat kernel for Schrödinger operators and the Dirichlet Laplacians, J. Funct. Anal. 59 (1984) 335–395.

[7] M. Fukushima, Y. Oshima, M. Takeda, Dirichlet Forms and Symmetric Markov Processes, 2nd rev. and ext. ed., Walter de Gruyter, Berlin, 2010.

[8] S. Karlin, H. M. Taylor, A Second Course in Stochastic Processes, 2nd edn, Academic Press, New York, 1981.

[9] R. Knobloch, L. Partzsch, Uniform conditional ergodicity and intrinsic ultracontractivity, Potential. Anal. 33 (2010) 107–136.

[10] M. Kolb, D. Steinsaltz, Quasiliimiting behavior for one-dimensional diffusions with killing, Ann. Probab. 40 (2012) 162–212.

[11] J. Lattin, Uniqueness of quasi-stationary distributions and discrete spectra when \( \infty \) is an entrance boundary and 0 is singular, J. Appl. Probab. 49 (2012) 719–730.

[12] M. Lladser, J. San Martín, Domain of attraction of the quasi-stationary distributions for the Ornstein-Uhlenbeck process, J. Appl. Probab. 37 (2000) 511–520.

[13] P. Mandl, Spectral theory of semi-groups connected with diffusion processes and its application, Czech. Math. J. 11 (1961) 558–569.
[14] S. Martínez, J. San Martín, Quasi-stationary distributions for a Brownian motion with drift and associated limit laws, J. Appl. Probab. 31 (1994) 911–920.
[15] S. Martínez, P. Picco, J. San Martín, Domain of attraction of quasi-stationary distributions for the Brownian motion with drift, Adv. appl. prob. 30 (1998) 385–408.
[16] S. Méléard, D. Villemonais, Quasi-stationary distributions and population processes, Probab. Surv. 9 (2012) 340–410.
[17] Y. Miura, Ultracontractivity for Markov semigroups and quasi-stationary distributions, Stoch. Anal. Appl. 32 (2014) 591–601.
[18] A. G. Pakes, Quasi-stationary laws for Markov processes: examples of an always proximate absorbing state, Adv. Appl. Prob. 27 (1995) 120–145.
[19] D. Steinsaltz, S.N. Evans, Quasi-stationary distributions for one-dimensional diffusions with killing, Trans. Amer. Math. Soc. 359 (2007) 1285–1324 (electronic).
[20] M. Tomisaki, Intrinsic ultracontractivity and small perturbation for one-dimensional generalized diffusion operators, J. Funct. Anal. 251 (2007) 289–324.
[21] E. A. van Doorn, P. K. Pollett, Quasi-stationary distributions for discrete-state models, European J. Oper. Res. 230 (2013) 1–14.
[22] D. Villemonais, Approximation of quasi-stationary distributions for 1-dimensional killed diffusions with unbounded drifts, Available at arXiv:0905.3636v1, 2009.
[23] F. Y. Wang, Functional inequalities for empty essential spectrum, J. Funct. Anal. 170 (2000) 219–245.
[24] F. Y. Wang, Functional Inequalities, Markov Semigroups and Spectral Theory, Science Press, Beijing, 2005.