A Group Theoretical Approach to Graviton Two-Point Function

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Abstract

From the group theoretical point of view, it is proved that the theory of linear conformal gravity should be written in terms of a tensor field of rank-3 and mixed symmetry [1]. Such field equation was obtained in de Sitter space [2]. In this paper, a proper solution to the physical sector of this field equation in de Sitter ambient space is written as a product of a generalized polarization tensor and a massless minimally coupled scalar field. Similar to the minimally coupled scalar field, the Krein space quantization has been exploited to quantize this sector. We have calculated the physical part of the linear graviton two-point function. This two-point function is simultaneously de Sitter invariant and free of any pathological large-distance behavior.

1 Introduction

Many people believe that conformal invariance may be the key to a future theory of quantum gravity. In this paper, we consider linear theories of gravitation, in which, not only the field equations but also the free field commutation relations are conformal invariant. The main input into this construction of linear gravity is to insist that the propagating modes must be a pair of massless particles with helicity $\pm 2$. It was supposed that, a natural choice for such a field is a symmetric tensor field of rank-2.

However, as Barut and Böhm [3] have shown, for the physical representation of conformal group, the value of conformal Casimir operator is 9. While, based on the work of Binegar et al [1], for tensor fields of rank-2, this value will become 8. Hence, the tensor field of rank-2 does not correspond to any unitary irreducible representation (UIR) of conformal group (physical state of group). Indeed, such physical requirement implies that the theory of linear conformal quantum gravity must be formulated in terms of a tensor field of rank-3 and mixed symmetry with conformal degree zero [1]. By mixed symmetry, we mean

$$\Psi_{abc} = -\Psi_{bac}, \quad \sum_{cyc} \Psi_{abc} = 0,$$

while a field of conformal degree zero satisfies $u^d \partial_d \Psi_{abc} = 0$.

On the other hand, according to Wigner’s theorem, a linear gravitational field should transform under the unitary irreducible representation of its space-time symmetry group. Therefore, it seems that we need a theory which remains invariant under conformal transformation (as one expected for every massless theory [4]), and this theory should also be invariant under de Sitter (dS) group as space-time symmetry group. Our choice of dS space-time is due to the recent cosmological observation. These observational data are strongly in favour of a positive acceleration of the present universe [5],

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which means, in the first approximation, our universe might currently be in a dS phase. Accordingly, a mixed symmetry tensor field of rank-3 with conformal degree zero, which transforms according to both UIRs of the conformal and de Sitter groups was achieved in Ref. [2]. In this paper, we first obtain the solution of the physical part of this conformal field equation and then the conformally invariant (CI) two-point function is calculated in such a way that de Sitter invariance is preserved and the theory is free of pathological large-distance behavior.

Here, we would like to mention that, although, the geometrical interpretation of this linear theory is not entirely clear, but it may have an interesting property linked to quantum approach to modified gravitational theories, say metric-affine theories of gravity. The advent of a rank-3 tensor field implies that, contrary to General Relativity (GR) assumptions, space-time geometry is not fully described by the metric only, and other geometrical objects which can be independent of metric, such as connections, must be taken into account. In general, the connection does carry dynamics, so that the theory presents more degrees of freedom than GR. Consequently, torsion does not remain non-propagating [6].

Actually, if we accept that quantum theory of gravity should be an effective field theory, as many do [7], we can conclude remarkable results; It is proved that, torsion is zero in vacuum and in the presence of a scalar field or the electromagnetic field, however, in the presence of a Dirac field or other vector and tensor fields it does not necessarily vanish [6]. This shows a correspondence between torsion and the presence of fields that describe particles with spin. So, though when torsion is present, the concept of a perfect fluid has to be generalized if one wants to include particles with spin, but since many cosmological and astrophysical applications are related to either vacuum or to environments where matter can more or less be accurately described as a perfect fluid, these contributions to torsion will be negligible in most cases [8]. Therefore, it seems that these dynamical degrees of freedom can be eliminated in low-energy regimes [6] and still, one can consider the dS space-time as the classical background with good accuracy. Nevertheless, we believe that in high-energy physics, where quantum corrections are important, these effects cannot be ignored.

The layout of the paper is as follows. Section (II) is devoted to a brief review of the notations; the CI “massless” spin-2 wave equations in dS space will be studied. The solution of the physical part of the field equations is considered in Section (III). It is shown that this solution can be written in terms of a polarization tensor and a massless minimally coupled scalar field. In Section (IV), CI bi-tensor two-point function has been calculated. Specially, it is discussed that obtaining a covariant two-point function without infrared divergence necessitates using Krein space field quantization. Finally, a brief conclusion and outlook has been presented in Section (V). We have supplied some useful mathematical details of calculations in the appendices.

2 De Sitter Space and Dirac’s Six-cone Formalism

2.1 de Sitter Space:

The de Sitter solution to the cosmological Einstein field equation (with positive cosmological constant \( \Lambda \)) can be viewed as a one-sheeted hyperboloid embedded in a five dimensional Minkowski space \( M^5 \):

\[
X_H = \{ x \in R^5 ; x^2 = \eta_{\alpha \beta} x^\alpha x^\beta = -H^{-2} = -\frac{3}{\Lambda} \}, \quad \alpha, \beta = 0, 1, 2, 3, 4,
\]

where \( \eta_{\alpha \beta} = \text{diag} (1, -1, -1, -1, -1) \) and \( H \) is the Hubble parameter. The dS metric is

\[
ds^2 = \eta_{\alpha \beta} dx^\alpha dx^\beta = g_{\mu \nu} dX^\mu dX^\nu, \quad \mu, \nu = 0, 1, 2, 3
\]

2. The antisymmetric part of the connection is often called the Cartan torsion tensor.

3. It is expected that at some intermediate or high energy regimes, the spin of particles might interact with the geometry [9].
where \( X^\mu \)'s are 4 space-time intrinsic coordinates of the dS hyperboloid. One can write any geometrical object in this space in terms of both four local coordinates \( X^\mu \) (intrinsic space notation) and five global coordinates \( x^\alpha \) (ambient space notation).

In what follows, the ambient space notation is used; in this notation, there exist a straightforward connection with unitary irreducible representations (UIRs) of dS group, because the Casimir operators are easily identifiable [10]. [Kinematical group of the dS space is the 10-parameter group \( SO_0(1,4) \) (connected component of the identity in \( O(1,4) \)), which is one of the two possible deformations of the Poincaré group.] There are two Casimir operators

\[
Q^{(1)} = -\frac{1}{2} L_\alpha^\beta L_\alpha^\beta, \quad Q^{(2)} = -W_\alpha W^\alpha, \tag{2.2}
\]

where \( W_\alpha = -\frac{1}{5} \epsilon_{\alpha \beta \gamma \sigma \eta} L^\beta_\gamma L^{\sigma \eta} \) (The symbol \( \epsilon_{\alpha \beta \gamma \sigma \eta} \) holds for the usual antisymmetric tensor), with 10 infinitesimal generators \( L_\alpha^\beta = M_\alpha^\beta + \sum_\alpha L_\alpha^\beta \). \( M_\alpha^\beta \) and \( \sum_\alpha L_\alpha^\beta \) are the orbital and the spinorial parts respectively and the action of them are defined by [10]

\[
M_\alpha^\beta \equiv -i(x_\alpha \partial^\beta - x^\beta \partial_\alpha) = -i(x_\alpha \tilde{\partial}^\beta - x^\beta \tilde{\partial}_\alpha),
\]

\[
\sum_\alpha \kappa_\gamma^\beta \ldots \equiv -i(\eta_\gamma \kappa_\beta^\gamma \ldots - \eta_\beta \kappa_\gamma^\beta \ldots + \eta_\alpha \kappa_\gamma^\alpha \ldots - \eta_\beta \kappa_\gamma^\beta \ldots + \ldots).
\tag{2.3}
\]

\( \tilde{\partial}_\alpha \) is the tangential (or transverse) derivative on dS space, defined by

\[
\tilde{\partial}_\alpha = \theta_{\alpha \beta} \partial^\beta = \partial_\alpha + H^2 x_\alpha x \cdot \partial, \quad \text{with} \quad x \cdot \tilde{\partial} = 0,
\tag{2.4}
\]

and \( \theta_{\alpha \beta} \) is the transverse projector (\( \theta_{\alpha \beta} = \eta_{\alpha \beta} + H^2 x_\alpha x_\beta \)).

The operator \( Q^{(1)} \) commutes with the action of the group generators, thus, it is constant in each UIR. The eigenvalues of \( Q^{(1)} \) can be used to classify the UIRs i.e.,

\[
(Q^{(1)} - \langle Q^{(1)} \rangle) \mathcal{K}(x) = 0.
\tag{2.5}
\]

Following Dixmier [11], one can get a classification scheme using a pair \((p,q)\) of parameters involved in the following possible spectral values of the Casimir operators:

\[
Q^{(1)} = (p(p+1) - (q+1)(q-2)) I_d, \quad Q^{(2)} = (p(p+1)q(q-1)) I_d.
\tag{2.6}
\]

According to the range of values of the parameters \( p \) and \( q \), there exist three distinct types of UIRs for \( SO(1,4) \) [13] [14], namely: principal, complementary and discrete series. In the case of the principal and complementary series, the flat limit compels the value of \( p \) to bear the meaning of spin. For the discrete series case, the only representation which has a physically meaningful Minkowskian counterpart is \( p = q \) case. For more mathematical details of the group contraction and the physical principles underlying the relationship between dS and Poincaré groups, one can refer to Refs. [13] [14].

The spin-2 tensor representations relevant to the present work are as follows: I) The UIRs \( U^{2,\nu} \) in the principal series where \( p = s = 2 \) and \( q = \frac{1}{2} + i \nu \) correspond to the Casimir spectral values:

\[
\langle Q^{\nu} \rangle = \nu^2 - \frac{15}{4}, \quad \nu \in \mathbb{R},
\tag{2.7}
\]

note that \( U^{2,\nu} \) and \( U^{2,-\nu} \) are equivalent.

II) The UIRs \( V^{2,q} \) in the complementary series where \( p = s = 2 \) and \( q - q^2 = \mu \), correspond to

\[
\langle Q^{\mu} \rangle = q - q^2 - 4 \equiv \mu - 4, \quad 0 < \mu < \frac{1}{4}.
\tag{2.8}
\]
III) The UIRs $\Pi_{2,1}^\pm$ in the discrete series where $p = s = 2$ correspond to

$$
\langle Q^{(1)} \rangle = -4, \; q = 1 \; (\Pi_{2,1}^\pm) ; \quad \langle Q^{(2)} \rangle = -6, \; q = 2 \; (\Pi_{2,2}^\pm).
$$

Regarding the de Sitter group, the "massless" spin-2 field is symbolized by $\Pi_{2,2}^\pm$ and $\Pi_{2,1}^\pm$ in which the sign $\pm$, stands for the helicity. In these cases, the two representations $\Pi_{2,2}^\pm$, in the discrete series with $p = q = 2$, have a Minkowskian interpretation. It is worth to mention that $p$ and $q$ do not bear the meaning of mass and spin. For discrete series in the limit $H \to 0$, $p = q = s$ are veritably none other than spin.

The compact subgroup of conformal group $SO(2,4)$ is $SO(2) \otimes SO(4)$, in which, by considering $E$ as the eigenvalues of the conformal energy generator of $SO(2)$ and $(j_1, j_2)$ as the $(2j_1 + 1)(2j_2 + 1)$ dimensional representation of $SO(4) = SU(2) \otimes SU(2)$, the mathematical symbols $C(E; j_1, j_2)$ can be used to denote the irreducible projective representation of the conformal group. The representation $\Pi_{2,2}^\pm$ has a unique extension to a direct sum of two UIRs $C(3; 2, 0)$ and $C(-3; 2, 0)$ of the conformal group, with positive and negative energies respectively \cite{13}. The latter is restricted to the massless Poincaré UIRs $P^>(0, 2)$ and $P<0, 2)$ with positive and negative energies respectively. $P^<(0, 2)$ (resp. $P^<(0, -2)$) are the massless Poincaré UIRs with positive and negative energies and positive (resp. negative) helicity. The following diagrams elucidate these connections

$$
\begin{align*}
\Pi_{2,2}^+ & \leftrightarrow \quad C(3, 2, 0) \quad \oplus \quad C(-3, 2, 0) \quad \leftarrow \quad P^>(0, 2) \\
& \leftarrow \quad H=0 \quad \oplus \quad \oplus \\
\Pi_{2,2}^- & \leftrightarrow \quad C(3, 0, 2) \quad \oplus \quad C(-3, 0, 2) \quad \leftarrow \quad P^>(0, -2)
\end{align*}
$$

$$
\begin{align*}
\Pi_{2,2}^+ & \leftrightarrow \quad C(3, 2, 0) \quad \oplus \quad C(-3, 2, 0) \quad \leftarrow \quad P<(0, 2) \\
& \leftarrow \quad H=0 \quad \oplus \quad \oplus \\
\Pi_{2,2}^- & \leftrightarrow \quad C(3, 0, 2) \quad \oplus \quad C(-3, 0, 2) \quad \leftarrow \quad P<(0, -2)
\end{align*}
$$

where the arrows $\leftrightarrow$ indicate unique extension. It is important to note that the representations $\Pi_{2,1}^\pm$ do not have corresponding flat limit \cite{13,14}.

### 2.2 Dirac’s six cone formalism and Conformal-invariant field equations:

The concept of conformal space and six-cone formalism was first used by Dirac to obtain the field equations for spinor and vector fields in $1 + 3$ dimensional space-time in CI form \cite{15}. He proposed a manifestly conformally covariant formulation in which the Minkowski space is embedded as the hyper surface $\eta_{ab}u^au^b = 0$, $(a, b = 0, 1, 2, 3, 4, 5)$, $\eta_{ab} = \text{diag}(1, -1, -1, -1, -1, 1)$ in $R^6$. Then the fields are extended by homogeneity requirements to the whole of the space of homogeneous coordinates, namely $R^6$. Reduction to four dimensions is achieved by projection, that is by fixing the degrees of homogeneity of all fields. Wave equations, subsidiary conditions, etc., must be expressed in terms of operators that are defined intrinsically on the cone. These are well-defined operators that map tensor fields to tensor fields with the same rank on the cone $u^2 = 0$. So, the resultant equations which are obtained by this method, are conformally invariant. This approach to conformal symmetry which leads to best path to exploit the physical symmetry was then developed by Mack and Salam \cite{16} and many others \cite{17}.

\footnote{Note that in dS space, concept of mass does not exist by itself as a conserved quantity, actually, is used in reference to propagation on the dS light cone (conformal invariance). The conformal invariance and the light-cone propagation, constitute the basis for constructing massless field in dS space. The term "massive" is referred to fields that in their zero curvature limit would be reduced to massive Minkowskian fields \cite{3}. Concept of light-cone propagation, however, does exist and leads to the conformal invariance.}
Considering this method in de Sitter space provides us with the opportunity to acquire the CI field equations for massless scalar, vector and tensor fields \cite{2, 18, 19}. It has been shown that in the flat limit \((H \to 0)\), these CI equations, would be reduced exactly to their counterpart in Minkowsi space, e.g., Maxwell equations are obtained from the vector field case \cite{18, 19}.

We are interested in the conformal invariance properties of massless spin-2 field in dS space, i.e. dS linear gravity. Generalizing the group theoretical approach, based on what was proposed by Binegar et al \cite{1} to the de Sitter space and using a mixed symmetry tensor field of rank-3 with conformal degree zero, which transforms according to both UIRs of the conformal and de Sitter groups, related CI wave equation in dS space is best obtained by first establishing the wave equation in Dirac’s null-cone in \(R^6\), and then followed by the projection of this equation to the dS space, as follows \cite{2}:

\[
2Q_0^{(1)}(Q_0^{(1)} - 2)(F_{\alpha\beta\gamma} - \frac{1}{4}x_\gamma A_{\alpha\beta}) + (\bar{\partial}_\alpha + 3x_\alpha)(Q_0^{(1)} - 2)(4\bar{\partial} \cdot F_{\beta\gamma} - A_{\gamma\beta} - x_\gamma \bar{\partial} \cdot A_{\alpha}) + (\bar{\partial}_\beta + 3x_\beta)(Q_0^{(1)} - 2)(4\bar{\partial} \cdot F_{\alpha\gamma} - A_{\alpha\gamma} - x_\gamma \bar{\partial} \cdot A_{\alpha}) = 0,
\]

\[(2.12)\]

in which \(Q_0^{(1)} = -\frac{1}{2}M^{\alpha\beta}M_{\alpha\beta}, \ F_{\alpha\beta\gamma}\) is the projected field and \(A_{\alpha\beta} \equiv \bar{\partial}^\gamma F_{\alpha\beta\gamma} - x_\alpha F_{\gamma\beta}^\gamma + x_\beta F_{\gamma\alpha}^\gamma\). Now, by imposing the mixed symmetry, transversality, divergenceless and traceless conditions on the tensor field \(F_{\alpha\beta\gamma}\), which are necessary for UIRs of dS and conformal groups, the CI equation \(2.12\) reduces to (See Appendix A)

\[
Q_0^{(1)}(Q_0^{(1)} - 2)F_{\alpha\beta\gamma} = 0, \quad \text{or equivalently,} \quad (Q^{(1)} + 6)(Q^{(1)} + 4)F_{\alpha\beta\gamma} = 0,
\]

\[(2.13)\]

Obviously this CI field corresponds to the two representations of discrete series, the physical representation of dS group, namely \(\Pi^+_2\) and \(\Pi^-_2\) (Eq. \(2.9\)). Accordingly the parameter \(p\) does have a physical significance. It is indeed spin. On the other side, in the Minkowski space, for every massless representation of Poincaré group there exists only one corresponding representation in the conformal group \cite{3, 20}. In the dS space, as mentioned, for the massless tensor field, only two representations in the discrete series \(\Pi^+_2\) and \(\Pi^-_2\), have a Minkowskian interpretation (The signs \(\pm\) correspond to two types of helicity for the massless tensor field). Thus, in the following, we only consider the tensor field that corresponds to the representations of \(\Pi^+_2\) which has the Minkowskian limit, i.e.

\[
(Q^{(1)} + 6)F_{\alpha\beta\gamma} = 0.
\]

\[(2.14)\]

### 3 De Sitter Field Solution

In this section, we want to obtain solution of the physical part of CI field equation. Let us start with the most generic form of \(F_{\alpha\beta\gamma}\) which can be chosen as follows

\[
F_{\alpha\beta\gamma} = (\bar{\partial}_\alpha + x_\alpha)K_{\beta\gamma} - (\bar{\partial}_\beta + x_\beta)K_{\alpha\gamma} + \bar{Z}_\alpha H_{\beta\gamma} - \bar{Z}_\beta H_{\alpha\gamma},
\]

\[(3.15)\]

where \(K_{\alpha\beta}\) and \(H_{\alpha\beta}\) are two rank two tensor fields and \(Z\) is a constant 5-dimensional vector. Bar over the vector makes it a tangential (or transverse) vector on dS space (See \(2.3\)). Imposing the mixed symmetry, transversality, divergenceless and traceless conditions on \(F_{\alpha\beta\gamma}\), which are needed in order to relate it to the physical representation, leads to

\[
K_{\alpha\beta} = K_{\beta\alpha}, \quad x \cdot K_{\cdot\beta} = x \cdot K_{\cdot\alpha} = 0,
\]

\[
H_{\alpha\beta} = H_{\beta\alpha}, \quad x \cdot H_{\cdot\beta} = x \cdot H_{\cdot\alpha} = 0, \quad \bar{\partial} \cdot H_{\cdot\beta} = \bar{\partial} \cdot H_{\cdot\alpha} = 0, \quad \mathcal{H} = 0,
\]

\[(3.16)\]

\(^4\text{Not: For simplicity from now on we take } H = 1 \text{ and use the notation } \bar{\partial}^\alpha F_{\alpha\beta\gamma} \equiv \bar{\partial} \cdot F_{\beta\gamma}\)
where $\mathcal{H}' = H^\alpha_\alpha$ is the trace of $H_{\alpha\beta}$. In addition one obtains useful relations as follows:

\[
(Q^{(1)}_0 - 2)K_{\beta\gamma} + (\bar{\partial}_\beta + 2x_\beta)\bar{\partial} \cdot K_{\gamma} - (Z \cdot \bar{\partial} + 3x \cdot Z)H_{\beta\gamma} + x_\beta Z \cdot H_{\gamma} = 0, \quad (I)
\]
\[
(\bar{\partial}_\alpha + 2x_\alpha)\bar{\partial} \cdot K_{\beta} - (\bar{\partial}_\beta + 2x_\beta)\bar{\partial} \cdot K_{\alpha} + x_\alpha Z \cdot H_{\beta} - x_\beta Z \cdot H_{\alpha} = 0, \quad (II)
\]
\[
(\bar{\partial}_\alpha + x_\alpha)K' - \bar{\partial} \cdot K_{\alpha} - Z \cdot H_{\alpha} = 0, \quad (III)
\]

$K' = K^\alpha_\alpha$ is the trace of $K_{\alpha\beta}$.

On the other hand, substituting $F_{\alpha\beta\gamma}$ in (2.14), results in [From now on, in order to get shorthand equations, we define a symmetrizer operator, i.e. $S_{\alpha\beta}K_{\alpha\beta} \equiv K_{\alpha\beta} + K_{\beta\alpha}$, and an anti-symmetrizer operator, i.e. $S_{\alpha\beta}K_{\alpha\beta} \equiv K_{\alpha\beta} - K_{\beta\alpha}$]

\[
\begin{align*}
\{ S_{\alpha\beta}[(\bar{\partial}_\alpha + 3x_\alpha)Q^{(1)}_0 - 4x_\alpha]K_{\beta\gamma} & = S_{\alpha\beta}\left((8x_\alpha + 2\bar{\partial}_\alpha)(x \cdot Z) + 2x_\alpha(Z \cdot \bar{\partial})\right)H_{\beta\gamma}, \quad (I) \\
Q^{(1)}_0H_{\beta\gamma} & = 0. \quad (II)
\end{align*}
\]

From Eq. (3.18 I) together with the conditions given in (3.16), (3.17) and after using the procedure given in Appendix B, it is proved that $K_{\beta\gamma}$ can be written in terms of $H_{\beta\gamma}$ as

\[
K_{\beta\gamma}(x) = \left(-\frac{1}{2}(x \cdot Z) + \frac{1}{8}(Z \cdot \bar{\partial})\right)H_{\beta\gamma} - \frac{1}{8}\left(x_\beta Z \cdot H_{\gamma} + x_\gamma Z \cdot H_{\beta}\right),
\]

Thus we can construct the tensor field (3.15) as follows:

\[
F_{\alpha\beta\gamma}(x) = S_{\alpha\beta}\left[(\bar{\partial}_\alpha + x_\alpha)\left(-\frac{1}{2}(x \cdot Z) + \frac{1}{8}(Z \cdot \bar{\partial})\right) + \bar{Z}_\alpha\right]H_{\beta\gamma}
\]

\[
- \frac{1}{8}S_{\alpha\beta}(\bar{\partial}_\alpha + x_\alpha)(x_\beta Z \cdot H_{\gamma} + x_\gamma Z \cdot H_{\beta}),
\]

where $H_{\beta\gamma}$ must satisfy the Eq. (3.18 II). After utilizing the similar procedure, which is given in Ref. [18], it is proved that

\[
H(x) = \left[-\frac{2}{3}\theta Z_1 \cdot + S\bar{Z}_1 + \frac{1}{3}S(\bar{\partial} - x)\left(\frac{1}{9}\bar{\partial}Z_1 \cdot + x \cdot Z_1\right)\right]\left[\bar{Z}_2 - \frac{1}{2}\bar{\partial}\left(\bar{Z}_2 \cdot \bar{\partial} + 2x \cdot Z_2\right)\phi,
\]

where $Z_2$ and $Z_3$ are another 5-dimensional constant vectors and $\phi$ is a massless minimally coupled scalar field.

### 4 Two-point function

The two-point functions in de Sitter space can be written in terms of bi-tensors [21]. These are functions of two points $(x, x')$ which behave like tensors under coordinate transformations at each point. Bi-tensors are called maximally symmetric if they respect de Sitter invariance. Furthermore, as explained in [22] and [23], the axiomatic field theory in de Sitter is based on bi-tensor Wightman two-point function. This two-point function is defined by

\[
W_{\alpha\beta\gamma'}(x, x') = \langle \Omega | F_{\alpha\beta\gamma'}(x)F_{\alpha'\beta'\gamma'}(x') | \Omega \rangle,
\]

where $x, x' \in X_H$ and $|\Omega\rangle$ is the Fock-vacuum state. The two-point function which is a solution of the Eq. (2.14) with respect to $x$ and $x'$, can be written simply in terms of the scalar two-point function. As the previous section, we consider the following possibility for the transverse two-point function

\[
W_{\alpha\beta\gamma'}(x, x') = S_{\alpha\beta}(\bar{\partial}_\alpha + x_\alpha)\left(\bar{S}_{\alpha'\beta'}(\bar{\partial}_\alpha' + x_\alpha')W^K_{\beta'\gamma'}(x, x') + \bar{S}_{\alpha\beta}\bar{S}_{\alpha'\beta'}((\theta_\alpha \cdot \theta_\alpha')W^H_{\beta'\gamma'}(x, x')\right).
\]
function has been found in [24] as follows

\[ W^K_{\gamma_\beta' \gamma'} (x, x') = S_{\alpha \beta'} (\partial_{\alpha'} + x'_{\alpha'} \partial_{\alpha'}) W^K_{\gamma_\beta' \gamma'} (x, x') = S'_{\alpha \beta'} \left( -\frac{1}{2} (x \cdot \theta'_{\alpha'}) + \frac{1}{8} (\theta'_{\alpha'} \cdot \theta') \right) W^H_{\gamma_\beta' \gamma'} (x, x') - \frac{1}{8} S_{\alpha \beta'} \left( x_{\beta'} \theta'_{\alpha'} \cdot W^H_{\gamma_\beta' \gamma'} (x, x') + x_{\gamma} \theta'_{\alpha'} \cdot W^H_{\gamma_\beta' \gamma'} (x, x') \right). \]  

(4.24)

According to Eqs. (4.23) and (4.24), it turns out that the two-point function can be written in the following form

\[ W_{\alpha \beta' \gamma' \gamma'} (x, x') = S_{\alpha \beta'} S'_{\alpha' \beta'} \left( \bar{\partial}_{\alpha} + x_{\alpha} \right) \left( -\frac{1}{2} (x \cdot \theta'_{\alpha'}) + \frac{1}{8} (\theta'_{\alpha'} \cdot \theta') \right) + (\theta_{\alpha} \cdot \theta'_{\alpha'}) \right) W^H_{\gamma_\beta' \gamma'} 

- \frac{1}{8} S_{\alpha \beta'} S'_{\alpha' \beta'} \left( \bar{\partial}_{\alpha} + x_{\alpha} \right) \left( x_{\beta'} \theta'_{\alpha'} \cdot W^H_{\gamma_\beta' \gamma'} + x_{\gamma} \theta'_{\alpha'} \cdot W^H_{\gamma_\beta' \gamma'} \right), \]  

(4.25)

where \( W^H_{\gamma_\beta' \gamma'} (x, x') \) applies in Eq. (3.18). Meanwhile, such transverse function was found in Ref. [18] as

\[ W^H_{\gamma_\beta' \gamma'} (x, x') = \left( -\frac{2}{3} S' \theta \theta' + SS' \theta \cdot \theta' \right) \left( \theta \cdot \theta' - \frac{1}{2} \bar{\partial} \theta' \cdot x \right) W_{mc} (x, x'), \]  

(4.26)

where \( W_{mc} \) is the two-point function for a minimally coupled massless scalar field in dS space. This two-point function has been found in [24] as follows

\[ W_{mc} (x, x') = \frac{1}{8 \pi^2} \left[ \frac{1}{1 - Z(x, x')} - \ln (1 - Z(x, x')) + \ln 2 + f (\eta, \eta') \right] \]  

(4.27)

it should be noted that, \( Z \) is an invariant object under the isometry group \( O(1, 4) \) which is defined for two given points on the dS hyperboloid \( x \) and \( x' \), by:

\[ Z \equiv -x \cdot x' + 1 + \frac{1}{2} (x - x')^2, \]

so that, any function of \( Z \) is dS-invariant, as well. Whereas, \( f \) is a function of the conformal time \( \eta \) that breaks the dS invariance. In addition, the term \( \ln (1 - Z(x, x')) \) at largely separated points, is responsible for the advent of infrared divergences. However, by constructing a covariant quantization of the massless minimally coupled scalar field in indefinite metric field quantization (Krein space method), we are capable of calculating the physical graviton two-point function, that is dS-invariant and free of any divergences [25, 26, 27]. Accordingly

\[ W_{mc}^{K \text{Krein}} (x, x') = \frac{i}{8 \pi^2} \epsilon (x^0 - x'^0) \left[ \delta (1 - Z(x, x')) + \bar{\partial} (Z(x, x') - 1) \right], \]  

(4.28)

Similarly, with the choice of \( x' \), the two-point function \[ W_{mc}^{K \text{Krein}} (x, x') \] satisfies Eq. (2.14) (with respect to \( x' \), see Appendix A).
where $\vartheta$ is the Heaviside step function and

$$
\epsilon (x^0 - x^0) = \begin{cases} 
1 & x^0 > x^0, \\
0 & x^0 = x^0, \\
-1 & x^0 < x^0.
\end{cases}
$$

(4.29)

Notice that this two-point function has been written in terms of $\mathcal{Z}$, therefore dS invariance is indeed preserved and it is clearly free of infrared divergence.

Pursuing this path, it is the work of a few lines to show that (4.25) in terms of $\mathcal{Z}$ becomes (Appendix A):

$$
W_{\alpha \beta \gamma \alpha' \beta' \gamma'} = \frac{\mathcal{Z}}{108(1 - \mathcal{Z}^2)^2} S_{\alpha \beta} S_{\alpha' \beta'} S_{\gamma \gamma'} \left( \theta_{\beta \gamma} \theta_{\beta' \gamma'} (\theta_{\alpha} \cdot \theta_{\alpha'}) f_1(\mathcal{Z}) + x_{\beta} \theta_{\alpha \gamma} \theta_{\beta' \gamma'} (x \cdot \theta_{\alpha'}) f_2(\mathcal{Z}) \\
+ \theta_{\beta \gamma} \theta_{\beta' \gamma'} (x \cdot \theta_{\alpha'}) f_3(\mathcal{Z}) + \theta_{\beta' \gamma'} (\theta_{\alpha} \cdot \theta_{\alpha'}) (x' \cdot \theta_{\beta}) f_4(\mathcal{Z}) \\
+ x_{\beta} \theta_{\beta' \gamma'} (x \cdot \theta_{\alpha'}) f_5(\mathcal{Z}) + (\theta_{\alpha} \cdot \theta_{\alpha'}) (\theta_{\beta} \cdot \theta_{\beta'}) f_6(\mathcal{Z}) \\
+ x_{\gamma} (x \cdot \theta_{\alpha'}) (\theta_{\beta} \cdot \theta_{\beta'}) f_7(\mathcal{Z}) + x_{\alpha \gamma} (x \cdot \theta_{\alpha'}) (\theta_{\beta} \cdot \theta_{\beta'}) f_8(\mathcal{Z}) \\
+ (x \cdot \theta_{\alpha'}) (x' \cdot \alpha) (\theta_{\beta} \cdot \theta_{\beta'}) f_9(\mathcal{Z}) + (x \cdot \theta_{\beta'}) (x' \cdot \alpha) (\theta_{\gamma} \cdot \gamma') f_{10}(\mathcal{Z}) \\
+ x_{\alpha \gamma} (x \cdot \theta_{\alpha'}) (\theta_{\beta} \cdot \theta_{\beta'}) f_{11}(\mathcal{Z}) + x_{\alpha} (x \cdot \theta_{\alpha'}) (x' \cdot \gamma) (\theta_{\beta} \cdot \theta_{\beta'}) f_{12}(\mathcal{Z}) \right) \frac{d}{dZ} \mathcal{W}_{mc}^{Krein}(\mathcal{Z}),
$$

(4.30)

in which we have $[18]

$$
\frac{d}{dZ} \mathcal{W}_{mc}^{Krein}(\mathcal{Z}) = \frac{i}{8\pi^2} \frac{\mathcal{Z} - 2}{\mathcal{Z} - 1} \epsilon (x^0 - x^0) \delta (\mathcal{Z} - 1).
$$

Eq. (4.30) is the explicit form of the two-point function in ambient space notations. This two-point function is obviously dS-invariant and free of any divergences. [Note that, using the projection rules given in Ref. [18, 28, one can easily calculate the intrinsic counterpart of (4.30).]

5 Conclusion

A group theoretical approach to quantum gravity, based on Wigner’s theorem and Dirac’s six-cone formalism, led to the field equation for the massless spin-2 field in de Sitter space which was conformally invariant [2]. In the present work, the related two-point function was calculated. It is invariant under the conformal transformation and free of any pathological large-distance behavior. Indeed, the latter is achieved by carrying out the calculation in a procedure based on Krein space structure. This two-point function may play an important role in formulating the quantum effects of gravity in the interacting cases.

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A Some useful relations

In this appendix we collect some useful relations.

I) The action of the Casimir operator \(Q^{(1)}\) on a rank-3 tensor field can be brought in the more explicit form (Regarding (2.2) and (2.3))

\[
Q^{(1)}F_{\alpha\beta\gamma} = (Q_0^{(1)} - 6)F_{\alpha\beta\gamma} + 2\left(\eta_{\alpha\beta}F_{\delta\delta\gamma} + \eta_{\beta\gamma}F_{\alpha\delta\delta} + \eta_{\gamma\alpha}F_{\delta\delta\delta}\right) + 2\left(x_\alpha\partial \cdot F_{\beta\gamma} + x_\beta\partial \cdot F_{\alpha\gamma} + x_\gamma\partial \cdot F_{\alpha\beta}\right) - 2\left(\partial_\alpha x \cdot F_{\beta\gamma} + \partial_\beta x \cdot F_{\alpha\gamma} + \partial_\gamma x \cdot F_{\alpha\beta}\right) - 2\left(F_{\beta\alpha\gamma} + F_{\gamma\beta\alpha} + F_{\alpha\gamma\beta}\right).
\]

(A.1)

It is important to note that by imposing the following conditions on the tensor field \(F_{\alpha\beta\gamma}\), which are necessary for the UIRs of dS and conformal groups,

- \(F_{\alpha\beta\gamma} = -F_{\beta\alpha\gamma}\) and \(F_{\alpha\beta\gamma} + F_{\beta\gamma\alpha} + F_{\gamma\alpha\beta} = 0\); Mixed symmetry conditions (Note: These conditions are necessary for UIRs of conformal group [1]),
- \(x \cdot F_{\beta\gamma} = x \cdot F_{\alpha\gamma} = x \cdot F_{\alpha\beta} = 0\); Transversality conditions,
- \(\partial \cdot F_{\beta\gamma} = \partial \cdot F_{\alpha\gamma} = \partial \cdot F_{\beta\alpha} = 0\); Divergenceless conditions (Note: For transverse tensors, like \(F_{\alpha\beta\gamma}\), \(\partial \cdot F_{\alpha\beta\gamma} = \partial \cdot F_{\beta\alpha\gamma}\)),
- \(F_{\alpha\delta\delta} = 0\); Traceless condition.

Eq. (A.1) reduces to

\[
(Q^{(1)} + 6)F_{\alpha\beta\gamma} = Q_0^{(1)}F_{\alpha\beta\gamma}.
\]

For more mathematical details of the action of the Casimir operators (\(Q^{(1)}\) and \(Q^{(2)}\)), the commutation rules and algebraic identities of the various operators and fields, one can refer to [18, 29].

II) The two-point function (4.25) with the choice of \(x'\) reads as:

\[
W_{\alpha\beta\gamma\alpha'\beta'\gamma'}(x, x') = \tilde{S}_{\alpha\beta}\tilde{S}_{\alpha'\beta'}\left[(\partial_{\alpha'} + x'_{\alpha'})\left(-\frac{1}{2}(x' \cdot \theta_\alpha) + \frac{1}{8}(\theta_\alpha \cdot \partial')\right) + (\theta_{\alpha'} \cdot \theta_\alpha)\right]W^{\eta}_{\beta'\gamma'\gamma''} + \frac{1}{8}\tilde{S}_{\alpha\beta}\tilde{S}_{\alpha'\beta'}(\partial'_{\alpha'} + x'_{\alpha'})\left(x'_{\beta'}\theta_\alpha \cdot W^{H}_{\beta'\gamma'\gamma''} + x'_{\gamma'}\theta_\alpha \cdot W^{H}_{\beta'\gamma'\gamma''}\right).
\]

(A.2)

To obtain the two-point function, the following identities become important

\[
\tilde{\partial}_\alpha f(Z) = -(x' \cdot \theta_\alpha)\frac{df(Z)}{dZ},
\]

(A.3)

\[
\theta^{\alpha\beta}\theta'_{\alpha'\beta'} = \theta \cdot \theta' = 3 + Z^2, \quad (x \cdot \theta_{\alpha'})\theta'_{\alpha'} = Z^2 - 1, \quad (x \cdot \theta'_{\alpha'})\theta_{\alpha'} = Z(1 - Z^2),
\]

(A.4)

\[
\tilde{\partial}_\alpha(x \cdot \theta'_{\beta'}) = \theta_\alpha \cdot \theta'_{\beta'}, \quad \tilde{\partial}_\alpha(x' \cdot \theta_\beta) = x_\beta(x' \cdot \theta_\alpha) - Z\theta_\alpha \theta_\beta,
\]

\[
\tilde{\partial}_\alpha(\theta_\beta \cdot \theta'_{\beta'}) = x_\beta(\theta_\alpha \cdot \theta'_{\beta'}) + \theta_{\alpha\beta}(x \cdot \theta'_{\beta'}) - Z(x \cdot \theta'_{\alpha'}),
\]

\[
\theta^\beta_{\alpha'}(\theta_\gamma \cdot \theta'_{\beta'}) = \theta^\beta_{\alpha'\gamma'} + (x \cdot \theta'_{\alpha})(x \cdot \theta'_{\beta}),\quad \theta^{\alpha'}_{\alpha}(\theta_\gamma \cdot \theta'_{\beta'}) = \theta^{\alpha'}_{\alpha'\gamma} + (x \cdot \theta'_{\gamma})(x \cdot \theta'_{\beta'}),
\]

(A.5)

\[
Q_0^{(1)} f(Z) = (1 - Z^2)\frac{d^2 f(Z)}{dZ^2} - 4Z\frac{df(Z)}{dZ}.
\]
B Mathematical Relations Underlying Eq. (3.19)

Generally, following form for \( K_{\beta\gamma} \) can be considered,
\[
K_{\beta\gamma} = C_1(x \cdot Z H_{\beta\gamma}) + C_2(Z \cdot \bar{\partial} H_{\beta\gamma}) + C_3(\bar{\partial}_\beta Z \cdot H_{\gamma} + \bar{\partial}_\gamma Z \cdot H_{\beta}) + C_4(x_\beta Z \cdot H_{\gamma} + x_\gamma Z \cdot H_{\beta}) + C_5(x_\gamma Z \cdot H_{\beta}) + C_6(\bar{\partial}_\beta \bar{\partial}_\gamma - x_\gamma \bar{\partial}_\beta) Z \cdot H \cdot Z, \tag{B.1}
\]
clearly \( K_{\beta\gamma} = K_{\gamma\beta} \). \( C_1, \ldots, C_6 \) are six arbitrary real numbers, which are determined by considering the following physical requirements:
The transversality conditions \( (x \cdot K_{\gamma} = x \cdot K_{\beta} = 0) \) require that
\[
C_2 + C_3 + C_4 = 0. \tag{B.2}
\]
And then the condition \((3.17-III)\) makes
\[
C_5 = -C_6, \quad \text{and} \quad C_1 + 4C_4 + 1 = 0. \tag{B.3}
\]
Regarding condition \((3.17-I \text{ and } II)\), one can obtain
\[
C_1 = -\frac{1}{2}, \quad C_4 = -\frac{1}{8}, \tag{B.4}
\]
and also a new auxiliary equation \( \bar{\partial}_\beta Z \cdot H_{\gamma} = x_\gamma Z \cdot H_{\beta} \), which states that the third and fourth terms in \((B.1)\) are not independent, so, without any damage to the generality of the solution, one can take \( C_3 = 0 \). Then we have \( C_2 = \frac{1}{8} \), and so, one can rewrite the general solution for \( K_{\beta\gamma} \) as follows,
\[
K_{\beta\gamma} = -\frac{1}{2}(x \cdot Z H_{\beta\gamma}) + \frac{1}{8}(Z \cdot \bar{\partial} H_{\beta\gamma}) - \frac{1}{8}(x_\beta Z \cdot H_{\gamma} + x_\gamma Z \cdot H_{\beta}) + C_5(\bar{\partial}_\beta x_\gamma - \bar{\partial}_\gamma x_\beta) Z \cdot H \cdot Z. \tag{B.5}
\]
Note that, a straightforward calculation shows that Eq. (3.18) does not create new constraints to be imposed on \((B.5)\). Therefore, since we’re looking for the easiest possible answer, we choose \( C_5 = 0 \).

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