Quantum magnetic monopole condensate

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Despite the decades-long efforts, magnetic monopoles were never found as elementary particles. Monopoles and associated currents were directly measured in experiments and identified as topological quasiparticle excitations in emergent condensed matter systems. These monopoles and the related electric-magnetic symmetry were restricted to classical electrodynamics, with monopoles behaving as classical particles. Here we show that the electric-magnetic symmetry is most fundamental and extends to full quantum behavior. We demonstrate that at low temperatures magnetic monopoles can form a quantum Bose condensate dual to the charge Cooper pair condensate in superconductors. The monopole Bose condensate manifests as a superinsulating state with infinite resistance, dual to superconductivity. The monopole supercurrents result in the electric analog of the Meissner effect and lead to linear confinement of the Cooper pairs by Polyakov electric strings in analogy to quarks in hadrons.

INTRODUCTION

Magnetic monopoles [1], while elusive as elementary particles [2], exist in many materials in the form of emergent quasiparticle excitations [3, 4]. Magnetic monopoles and associated currents were directly measured in experiments [5, 6], confirming the predicted symmetry between electricity and magnetism [1]. So far, these monopoles and the related electric-magnetic symmetry were restricted to classical electrodynamics, with monopoles behaving as classical particles. Here we show that the electric-magnetic symmetry is most fundamental and extends to full quantum behavior. We demonstrate that at low temperatures magnetic monopoles can form a quantum Bose condensate dual to the charge condensate in superconductors. The monopole Bose condensate manifests as a superinsulating state with infinite resistance, dual to superconductivity [7, 8]. The monopole supercurrents result in the electric analog of the Meissner effect and lead to linear confinement of the Cooper pairs by Polyakov electric strings in analogy to quarks in hadrons [9–11].

Maxwell’s equations in vacuum are symmetric under the duality transformation $E \rightarrow B$ and $B \rightarrow -E$ (we use natural units $c = 1$, $\hbar = 1$, $\varepsilon_0 = 0$). Duality is preserved, provided that both electric and magnetic sources (magnetic monopoles and magnetic currents) are included [1]. The existence of monopoles requires that gauge fields are compact, implying, in turn, the quantization of charge and Dirac strings [12] or a core with additional degrees of freedom to regularize the singularities of the vector potential [13, 14].

The monopoles observed so far are massive classical particles, so that at low temperatures they are strongly suppressed by the large Boltzmann factor. Here we show that quantum magnetic monopoles can arise in certain bosonic insulators, where they Bose condense at low temperatures. This creates a new state of the system, in which Cooper pairs are linearly bound by electric fields squeezed into strings by the monopole condensate, in analogy to quarks within hadrons [15]. This state is dual to superconductivity and is called superinsulator [7, 8]. As a dual mirror to the effect of the Cooper pair condensate, which mediates an infinite conductance, the monopole condensate results in an infinite resistance at finite temperatures. The monopole condensate realizes therefore a 3D version of superinsulators, that have been observed in 2D superconducting films, and result from quantum tunneling events, or instantons [7–11].

RESULTS

Universal procedure

Magnetic monopoles housed by bosonic insulators are bosons themselves in simple insulators and are fermions in topological insulators [16]. We focus here on the former case. Simple bosonic insulators have an effective action [17] that can be written down in terms of two fictitious gauge fields, a vector field $a_{\mu}$ and an antisymmetric pseudotensor field...
\[ b_{\mu} [18]: \]
\[ \mathcal{L}_{BI} = \frac{1}{4\pi} b_{\mu\nu} \epsilon^{\mu\nu\alpha\beta} \partial_{\alpha} a_{\beta} + \frac{1}{2} b_{\mu\nu} m^{\mu\nu}. \]  

This so-called BF model [18] is topological since it is metric-independent. It is invariant under the usual gauge transformations \( a_{\mu} \rightarrow a_{\mu} + \partial_\mu \xi \) and under the gauge transformation of the second kind, \( b_{\mu\nu} \rightarrow b_{\mu\nu} + \partial_\mu a_\nu - \partial_\nu a_\mu \). The BF action for a model defined on a compact space endowed with non-trivial topology yields a ground state with degeneracy reflecting, one-to-one, this topology and hence referred to as topological order [19]. The choice of the coefficient \( 1/4\pi \sigma \) of the first term in Eq. (1) ensures that the system does not have such a topological order [20]. In turn, the topological coupling between a vector and a pseudotensor in 3D ensures the parity \((\mathbb{Z}^3)\) and time-reversal \((\mathbb{Z}^3)\) symmetries of the model. The field strength associated to \( a_{\mu} \) is \( f_{\mu\nu} = \partial_{\mu} a_{\nu} - \partial_{\nu} a_{\mu} \) whereas \( b_{\mu\nu} \) has associated 3-tensor field strength \( h_{\mu\nu\alpha} = \partial_{\mu} b_{\nu\alpha} + \partial_{\nu} b_{\alpha\mu} + \partial_{\alpha} b_{\mu\nu} \). The dual field strengths \( \tilde{p}^\mu = (1/2\pi) \tilde{\phi}^\mu = (1/4\pi) \epsilon^{\mu\nu\alpha\beta} \partial_\alpha b_{\beta\nu} \) and \( \phi^{\mu\nu} = (1/2\pi) f^{\mu\nu} = (1/2\pi) \epsilon^{\mu\nu\alpha\beta} a_{\alpha\beta} \) describe topologically conserved charge and vortex currents. When the gauge symmetries are compact \( U(1) \), these expressions admit singularities describing point charges with current \( q^\mu \) and line vortices with current \( m^{\mu\nu} \). If we choose the radius of the two \( U(1) \) groups as 1, the charge and vortex numbers are integer units. If the bosons are Cooper pairs, charges are measured in integer units of 2e and vortices in integer units of \( 2\pi/2e = \pi/e \). Magnetic monopoles are the endpoints of open vortices, playing the role of Dirac strings: their current is given by \( m^{\nu} = \partial_{\nu} m^{\mu\nu} \). This is easiest to recognize by noting that \( m_{\mu\nu} \) encodes the singularities in the dual field strength \( f^{\mu\nu} \) of the fictitious gauge field \( a_{\mu} \). These are exactly fictitious magnetic monopoles with their Dirac strings [1]. As we show below, they become real magnetic monopoles upon coupling to the real electromagnetic gauge field \( A_{\mu} \). It is important to stress that these monopoles are not classical solitons, representing static saddle-points of the Hamiltonian but, rather, full quantum mechanical degrees of freedom summed over in the quantum partition function of the model. As we now show, quantum corrections to their mass due to interactions with charges can make them light enough to Bose condense.

**Phase transitions and phase diagram**

In the bosonic insulator, magnetic monopoles are gapped excitations. Therefore, their density is suppressed at low temperatures. Let us, however, consider a granular system (irrespective to whether the granularity is the self-induced electronic granularity [8] or is of the structural origin, such as, e.g. granular diamond [21]), characterized by the length scale \( \ell \) playing the role of the granule size, and examine the various phases that can emerge. To that end, we define an action (1) on a lattice of spacing \( \ell \) and we add all possible local gauge invariant terms. Rotating to Euclidean space-time we arrive at the action

\[ S = \sum_i \frac{\ell^4}{4\pi^2} f_{\mu\nu} f_{\mu\nu}^{\ddagger} - i \frac{\ell^4}{4\pi^2} a_{\mu\nu} k_{\mu\nu\alpha\beta} a_{\alpha\beta}^{\ddagger} - \frac{\ell^4}{12g^2} h_{\mu\nu\alpha\beta} h_{\mu\nu\alpha\beta}^{\ddagger} + i \ell a_{\mu\nu} + \ell^2 \frac{1}{2} b_{\mu\nu} m^{\mu\nu}, \]

where \( k_{\mu\nu\alpha\beta} \) is the lattice BF term [7], see Methods, \( f \) is a dimensionless coupling, and \( g \) has the canonical dimension of mass \((msec)\). To describe materials with the relative electric permittivity \( \varepsilon \) and relative magnetic permeability \( \mu \), we incorporate the velocity of light \( v = 1/\sqrt{\varepsilon\mu} < 1 \) by defining the Euclidean time lattice spacing as \( \ell_0 = \ell/v \) and by rescaling all time derivatives, currents, and zero-components of gauge fields by the factor \( 1/v \). As a consequence, both gauge fields acquire a dispersion relation \( E = \sqrt{m^2 v^4 + p^2} \) with the topological mass \([22]\) \( m = fg/Nv \). The dimensionless parameter \( f = O(\varepsilon) \) encodes the effective Coulomb interaction strength in the material, \( g \) is the magnetic scale \( g = O(1/\lambda_c) \), where \( \lambda_c \) is the London penetration depth of the superconducting phase.

To analyze how the additional interactions can drive quantum phase transitions taking the system out of the bosonic insulator, we integrate over the fictitious gauge fields to obtain a Euclidean action \( S_{cv} \) for point charges and line vortices alone. As we show in Methods, this is proportional to the length of the charge world-lines and to the area of vortex world-surfaces, exactly as their configurational entropy. Charges and vortices can thus be assigned an effective action (equivalent to a quantum “free energy” in this Euclidean field theory context)

\[ F_{cv} = (S_c - h_c) N + (s_c - h_c) A, \]

where \( N \) and \( A \) are the length of word-lines in number of lattice links and the area of world-surfaces in number of lattice plaquettes, respectively and \( s_c \) and \( h_c \) denote the action and entropy contributions (per length and area) of charges and vortices, respectively. The possible phases that are realized are determined by the relation between the values of the Coulomb and magnetic scales and by materials parameters determining whether the coefficients of \( N \) and \( A \) are positive or negative. For positive coefficients, long world-lines and large world-surfaces are suppressed. If either of the parenthesis becomes negative, then either a charge or a monopole condensate forms. In the case of charges, the proliferation of long-world lines is the geometric picture of Bose condensation first put forth by Onsager [23] and elaborated by Feynman [24], see [25] for a recent discussion. In the case of vortices, the string between magnetic monopole endpoints becomes loose and assumes the role of an unobservable Dirac string. In this case, the monopoles are characterized by long world-lines describing their Bose condensate phase, see Fig.1. The details of this vortex transition have been discussed in [26, 27]. The resulting phase diagram is determined by the value of the parameter \( \eta = \pi (mv/\ell^3)G/\sqrt{\mu_0\mu}c \) encoding the strength of quantum fluctuations and the material properties of the system [11] and tuning parameter, \( \gamma = (f/\ell^2g)\sqrt{\mu_0/\mu_c} \), taking the
system across superconductor-insulator transition (SIT). Here $G = O(G(mv\ell))$, where $G(mv\ell)$ is the diagonal element of the lattice kernel $G(x-y)$ representing the inverse of the operator $\ell^2 \left( (mv^2 - \nabla^2) \right)$, and $\mu_v$ and $\mu_s$ are the entropy per unit length of the world line and per unit area of the world surface, respectively. The phase structure at $T = 0$ and the domains of different phases in the critical vicinity of the SIT are defined by the relations, see Methods for details:

$$\eta < 1 \rightarrow \begin{cases} \gamma < 1, \text{charge Bose condensate}, \\ \gamma > 1, \text{monopole Bose condensate}, \end{cases}$$

$$\eta > 1 \rightarrow \begin{cases} \gamma < \frac{1}{\eta}, \text{charge Bose condensate}, \\ \frac{1}{\gamma} < \gamma < \eta, \text{bosonic insulator}, \\ \gamma > \eta, \text{monopole Bose condensate}. \end{cases}$$

and are shown in Fig. 2. The finite-temperature decay of the condensates, corresponding to the deconfinement transitions into the bosonic insulator, is described by the same approach [28]. One sees that at $\ell g / f < O(1)$, a superconducting phase is realized, as observed in granular diamond [21]. If $\ell g / f < O(1)$, there is a dual superinsulating phase governed by the magnetic monopole condensate.

**Electromagnetic response and the electric string tension**

To reveal the nature of the superinsulating phase, we examine its electromagnetic response. To that end we minimally couple the electric current $j_{\mu}$ to the electromagnetic gauge field $A_{\mu}$ and compute its effective action (see Methods). Taking the limit $mv\ell \gg 1$, we find

$$\exp(-S_{\text{e.m.}}) = \sum_{(p, q)} e^{-\frac{1}{\ell^2} \sum_{\nu} \left( p_{\nu}^2 - 2p_{\nu}q_{\nu} \right)^2}. \tag{5}$$

The monopole condensation for strong $f$ renders the real electromagnetic field a compact variable, defined on the interval $[-\pi, +\pi]$ and the electromagnetic response is given by Polyakov’s compact QED action [29, 30]. This changes drastically the Coulomb interaction. To see that, take two external probe charges $\pm q_{\text{ext}}$ and find the expectation value for the corresponding Wilson loop operator $W(C)$, where $C$ is the closed loop in 4D Euclidean space-time (the factor $\ell$ is absorbed into the gauge field $A_{\mu}$ to make it dimensionless)

$$\langle W(C) \rangle = \frac{1}{Z_{A_{\mu}p}} \sum_{(p, q)} \int_{-\pi}^{\pi} \mathcal{D}A_{\mu} e^{-\frac{1}{\ell^2} \sum_{\nu} \left( p_{\nu}^2 - 2p_{\nu}q_{\nu} \right)^2} e^{i q_{\text{ext}} \sum_{C} A_{\nu}}. \tag{6}$$

When the loop $C$ is restricted to the plane formed by the Euclidean time and one of the space coordinates, $\langle W(C) \rangle$ measures the interaction energy between charges $\pm q_{\text{ext}}$. A perimeter law indicates a short-range potential, while an area-law is tantamount to a linear interaction between them [30]. For Cooper pairs, $q_{\text{ext}} = 1$, see Methods, $\langle W(C) \rangle = \exp(-\sigma A)$ where $A$ is the area of the surface $S$ enclosed by the loop

**FIG. 2. Cooper pairs–quantum magnetic monopoles phase diagram at $T = 0$.** The inter-phases lines are $\eta = 1 / \gamma$ ($\gamma > 1$), $\gamma = 1$ ($\eta < 1$), and $\eta = \gamma$ ($\eta > 1$).

C. This yields a linear potential between probe Cooper pairs, with the string tension

$$\sigma = \frac{64 f}{\sqrt{2\pi^2 \epsilon \mu}} \frac{1}{\ell^2} \exp \left( -\frac{\pi G(0)}{16 f^2} \right), \tag{7}$$

where $G(0)$ is the value of the 4D lattice Coulomb potential at coinciding points. The monopole condensate, thus, generates a string binding together charges and preventing charge transport in systems of sufficient spatial size. A magnetic monopole condensate is a 3D superinsulator, characterized by an infinite resistance at finite temperatures [7–11]. The critical value of the effective Coulomb interaction strength for the transition to the superinsulating phase is $f_{\text{crit}} = O(\ell / \lambda)$.

**DISCUSSION**

Superinsulation has been observed in 2D, where magnetic monopoles are instantons rather than particles [8, 9, 11]. The signature of 3D superinsulation [31], however, has been detected in InO films [32, 33], which can thus be considered the first material to host a magnetic monopole condensate. Strongly type II granular superconductors with a fine, inherent or self-induced texture are other most plausible candidates to house 3D superinsulators. Finally, another class of candidates are layered materials. Vortex lines in such materials can be regarded as a stack of pancake vortices, with one pancake vortex in each layer [34]. The pancakes at two outer layers behave as magnetic monopoles. If ‘strings’ connecting pancakes are loose, these pancakes behave as independent monopoles (albeit extremely anisotropic ones) and condense into a superinsulator phase.

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APPENDIX

Lattice BF term

To formulate the gauge-invariant lattice BF-term, we follow [7] and introduce the lattice BF operators

\[ k_{\mu \nu} \equiv S_{\rho} \varepsilon_{\rho \mu \nu \lambda} \tilde{d}_\lambda , \]

\[ \hat{k}_{\mu \nu} \equiv \varepsilon_{\mu \nu \rho} \tilde{d}_\rho S_{\mu} , \]  

(8)

where

\[ d_\mu f(x) \equiv f(x + \ell \hat{\mu}) - f(x) , \quad S_{\mu} f(x) \equiv f(x + \ell \hat{\mu}) , \]

\[ \hat{d}_\mu f(x) \equiv f(x) - f(x - \ell \hat{\mu}) , \quad \hat{S}_{\mu} f(x) \equiv f(x - \ell \hat{\mu}) , \]  

(9)

are the forward and backward lattice derivative and shift operators, respectively. Summation by parts on the lattice interchanges both the two derivatives (with a minus sign) and the two shift operators; gauge transformations are defined using the forward lattice derivative. The two lattice BF operators are interchanged (no minus sign) upon summation by parts on the lattice and are gauge invariant so that:

\[ k_{\mu \nu} d_\lambda = k_{\mu \nu} \tilde{d}_\lambda = 0 , \]

\[ \hat{k}_{\mu \nu} \tilde{d}_\lambda = \hat{k}_{\mu \nu} \tilde{d}_\lambda = 0 . \]  

(10)

And satisfy the equations

\[ \hat{k}_{\mu \nu} k_{\rho \lambda} = - \left( \delta_{\mu \rho} \delta_{\nu \lambda} - \delta_{\mu \lambda} \delta_{\nu \rho} \right) V^2 \]

\[ + \left( \delta_{\mu \lambda} \tilde{d}_\rho + \delta_{\nu \lambda} \tilde{d}_\rho - \delta_{\rho \lambda} \tilde{d}_\mu - \delta_{\rho \lambda} \tilde{d}_\nu \right) , \]

\[ \hat{k}_{\mu \nu} \hat{k}_{\rho \lambda} = \hat{k}_{\mu \nu} \hat{k}_{\rho \lambda} = 2 \left( \delta_{\mu \nu} V^2 - \delta_{\mu \lambda} \tilde{d}_\lambda \right) , \]  

(11)

where \( V^2 = \tilde{d}_\mu \tilde{d}_\mu \) is the lattice Laplacian. We use the notation \( \Delta_\mu \) and \( \Delta_\mu \) for the forward and backwards finite difference operators.

Phases of monopoles

To find the topological action for monopoles, we start from eq. (2) and integrate out fictitious gauge fields \( a_\mu \) and \( b_{\mu \nu} \)

\[ S_{\text{top}} = \sum_x \frac{f^2}{2 \ell^2} q_\mu \frac{\delta_{\mu \nu}}{(m_\nu)^2 - V^2} q_\nu + \frac{g^2}{8} m_{\mu \nu} \delta_{\mu \nu} \delta_{\rho \lambda} - \frac{\delta_{\mu \nu} \delta_{\rho \lambda} - \delta_{\rho \lambda} \delta_{\mu \nu}}{(m_\nu)^2 - V^2} m_{\mu \nu} \]

\[ + \frac{\pi (m_\nu)^2}{2 \ell} q_\mu \frac{\delta_{\mu \nu}}{(m_\nu)^2 - V^2} m_{\mu \nu} \]

The last term represents the Aharonov-Bohm phases of charged particles around vortices of width \( \lambda_c \). On scales much larger than \( \lambda_c \), where the denominator reduces to \( (m_\nu)^2 V^2 \), this term becomes \( i 2 \pi \) integer, as can be easily recognized by expressing \( q_\mu = \ell k_{\mu \alpha \beta} g_{\alpha \beta} \). This reflects the absence of Aharonov-Bohm phases between charges \( ne \) and magnetic flux \( 2 \pi / ne \). Accordingly, we shall henceforth neglect this term.

The important consequence of the topological interactions is that they induce self-energies in form of the mass of Cooper pairs and tension for vortices between magnetic monopoles. These self-energies are encoded in the short-range kernels in the action (12), which we approximate by a constant. Worldlines and world-surfaces are thus assigned “energies” (formally Euclidean actions in the present statistical field theory setting and thus dimensionless in our units) proportional to their length \( N \) and area \( A \) (measured in numbers of links and plaquettes),

\[ S_N = \pi (m_\nu)^2 G \frac{f \ell}{g} Q^2 N , \]

\[ S_A = \pi (m_\nu)^2 G \frac{g \ell}{f} M^2 A , \]  

(13)

where \( G = O(G(m_\nu)) \), with \( G(m_\nu) \) the diagonal element of the lattice kernel \( G(x - y) \) representing the inverse of the operator \( \ell^2 (m_\nu)^2 - V^2 \), and \( Q \) and \( M \) are the integer quantum numbers carried by the two kinds of topological defects. However, also the entropy of link strings and plaquette surfaces is proportional to their length and area [35], \( \mu_s N \) and \( \mu_A A \). Both coefficients \( \mu \) are non-universal: \( \mu \approx \ln(7) \) since at each step the non-backtracking string can choose among 7 possible directions on how to continue, while \( \mu_s \) does not have such a simple interpretation but can be estimated numerically. This gives for both types of topological defects a “free energy” proportional to their dimension and with coefficients that can be positive or negative depending on the parameters of the theory. The total free energy is

\[ \frac{F}{\pi (m_\nu)^2 G} = \left[ \left( \frac{f \ell}{g} Q^2 - \frac{1}{\eta_\ell} \right) N + \left( \frac{g \ell}{f} M^2 - \frac{1}{\eta_m} \right) A \right] , \]

where we have defined

\[ \eta_\ell = \frac{\pi (m_\nu)^2 G}{\mu_s} , \quad \eta_m = \frac{\pi (m_\nu)^2 G}{\mu_A} . \]  

(14)

If the coefficients are positive, the self-energy dominates and large string/surface configurations are suppressed in the partition function. In this regime Cooper pairs and/or vortices are gapped excitations, suppressed by their large action. If the coefficients, instead are negative, the entropy dominates and large configurations are favoured in the “free energy” (effective action). The phase in which long world-lines of Cooper pairs dominate the Euclidean partition function is a charge Bose condensate, as discussed originally by Onsager [23] and Feynmann [24] (for a recent discussion see [25]). This phase is the Bose condensate of magnetic monopoles. For vortices, proliferation of large world-surfaces means that the strings binding monopoles and antimonopoles into neutral pairs become loose. We will show below that in this case the long real monopole world-lines dominate the electromagnetic response.
The combined energy-entropy balance equations are best viewed as defining the interior of an ellipse on a 2D integer lattice of electric and magnetic quantum numbers,

\[ \frac{Q^2}{r_0^2} + \frac{M^2}{r_\text{ext}^2} < 1, \]  

(15)

where the semi-axes are given by

\[ r_\text{ext}^2 = \frac{\ell g}{\eta} \quad \text{and} \quad r_0^2 = \frac{\ell g}{\eta} \sqrt{\frac{\mu_s}{\mu_L}} \eta, \]  

(16)

with

\[ \eta = \sqrt{\eta_L \eta_s} = \pi (me^2) G / \sqrt{\mu_s} \mu_L. \]  

Of course, configurations with \( Q \neq 0 \) and \( M \neq 0 \) must be excluded since the two types of excitations are different, only pairs \([0,M]\) or \([Q,0]\) have to be considered. The phase diagram is found by establishing which integer charges lie within the ellipse when the semi-axes are varied. This yields Eq. (4) in the main text.

Electromagnetic response in the magnetic monopole condensate

To establish the electromagnetic response of the monopole condensate we add the minimal coupling of the charge current \( j^\mu \) to the electromagnetic field,

\[ \mathcal{L} \rightarrow \mathcal{L} + i \sum_x \ell^4 A_\mu \dot{j}_\mu = \mathcal{L} + i \sum_x \ell^4 \frac{1}{4\pi} A_\mu b_{\mu\nu} b_{\nu\lambda}, \]  

(18)

and we compute its effective action by integrating over the fictitious gauge fields \( a_\mu \) and \( b_{\mu\nu} \). This requires no new computation since, by a summation by parts, the above coupling amounts only to a shift

\[ m_{\mu\nu} \rightarrow m_{\mu\nu} + \frac{1}{2\pi} \ell^2 \dot{k}_{\mu\nu \alpha} A_\alpha, \]  

(19)

in (12). Setting \( q_\mu = 0 \) for the phase with gapped Cooper pairs gives Eq. (5) in the main text.

Computation of the string tension

The starting point is equation (6) in the main text. For large values of the coupling \( f \), the action is peaked around the values \( F_{\mu\nu} = 2\pi t_{\mu\nu} \), allowing for the saddle-point approximation to compute the Wilson loop. Using the lattice Stoke’s theorem, one rewrites Eq. (6) as

\[ \langle W(C) \rangle = \frac{1}{Z_{A_\mu, m_{\mu\nu}}} \int_{-\pi}^{\pi} DA_\mu e^{-\frac{\ell^2}{2\pi} \Sigma_{x} (F_{\mu\nu} - 2\pi m_{\mu\nu})^2} e^{\frac{\pi}{2\eta} \Sigma_{x} S_{\mu
u}(F_{\mu\nu} - 2\pi m_{\mu\nu})}, \]  

(20)

where the quantities \( S_{\mu\nu} \) are unit surface elements perpendicular (in 4D) to the plaquettes forming the surface \( S \) encircled by the loop \( C \) and vanish on all other plaquettes. We have also multiplied the Wilson loop operator by 1 in the form \( \exp(-i\pi q_{\text{ext}} \sum_{x} S_{\mu\nu} m_{\mu\nu}) \). Following Polyakov \( [30] \), we decompose \( m_{\mu\nu} \) into transverse and longitudinal components,

\[ m_{\mu\nu} = m_{\mu\nu}^T + m_{\mu\nu}^L, \]

(21)

where \( \{n_\mu\} \) are integers and we adopt the gauge choice \( \Delta_\mu A_\mu = 0 \), so that \( \nabla^2 A_\mu = \Delta_\mu A_\mu = m_\mu \), with \( m_\mu \in \mathbb{Z} \) describe the world-lines of the magnetic monopoles on the lattice. The set of 6 integers \( \{m_{\mu}\} \) has thus been traded for 3 integers \( \{n_\mu\} \) and 3 integers \( \{n_\mu\} \) representing the magnetic monopoles. The former are then used to shift the integration domain for the gauge field \( A_\mu \) to \([−\infty, +\infty]\). The real variables \( \{\xi_\mu\} \) can then also be absorbed into the gauge field. The integral over the now non-compact gauge field \( A_\mu \) gives the Gaussian fluctuations around the saddle points \( m_\mu \). Gaussian fluctuations contribute the usual Coulomb potential \( 1/|x| \) in 3D. We shall henceforth focus only on the magnetic monopoles.

\[ \langle W(C) \rangle = \frac{1}{Z_{m_{\mu\nu}}} \sum_{\{n_\mu\}} e^{-\frac{\pi}{2\eta} \Sigma_{x} m_{\mu} \sum_{\mu} e^{2\pi \eta_\mu} \sum_{\mu} m_{\mu} \delta S_{\mu}}, \]  

(22)

Following \( [36] \) we introduce a dual gauge field \( \chi_\mu \) with field strength \( g_{\mu\nu} = \Delta_\mu A_\nu - \Delta_\nu A_\mu \) and we rewrite (22) as

\[ \langle W(C) \rangle = \frac{1}{Z_{m_{\mu\nu}}} \int D\chi_\mu e^{-\frac{\ell^2}{\pi^2} \sum_{\mu} g_{\mu\nu}^2} \sum_{\mu} \frac{e}{N} \sum_{n_{\mu}} \sum_{n_{\mu} = 1} g_{\mu\nu} m_{\mu}(\chi_\mu + q_{\text{ext}} n_{\mu}) \]  

(23)

where the angle \( \eta_\mu = 2\pi \delta S_{\mu\nu} / (−\nabla^2) \) represents a dipole sheet on the Wilson surface \( S \) and the monopole fugacity \( \zeta \) is determined by the self-interaction as

\[ \zeta = e^{-\frac{\pi}{\eta} G(0)}, \]  

(24)

with \( G(0) \) being the inverse of the Laplacian at coinciding arguments. We also used the dilute gas approximation, valid at large \( f \), in which one takes into account only single monopoles \( m_\mu = \pm 1 \). The sum can now be explicitly performed \( [36] \), with the result

\[ \langle W(C) \rangle = \frac{1}{Z_{\chi_\mu}} \int D\chi_\mu e^{-\frac{\ell^2}{\pi^2} \sum_{\mu} g_{\mu\nu}^2 + \frac{\pi}{\eta} (1 − \cos(\chi_\mu + q_{\text{ext}} n_{\mu}))}, \]  

(25)

By shifting the gauge field \( \chi_\mu \) by \( −q_{\text{ext}} n_{\mu} \) and introducing \( M^2 = (\pi^2/2f^2) \zeta \), we can rewrite this as

\[ \langle W(C) \rangle = \frac{1}{Z_{\chi_\mu}} \int D\chi_\mu e^{-\frac{\ell^2}{\pi^2} \sum_{\mu} g_{\mu\nu}^2 + \frac{\pi}{\eta} (1 − \cos(\chi_\mu))}, \]  

(26)

where \( g_{\mu\nu} = g_{\mu\nu}(\chi_\mu + q_{\text{ext}} n_{\mu}) \). For large \( f \), this integral is dominated by the classical solution to the equation of motion

\[ \hat{\Delta}_\mu g_{\mu\nu} = −2\pi q_{\text{ext}} \hat{\Delta}_\mu S_{\mu\nu} + M^2 \sin^2 \chi_\mu. \]  

(27)
Let us assume that the Wilson loop lies in the (0-3) plane formed by the Euclidean time direction 0 and the z axis. In this case, there are non-trivial solutions only for the 1- and 2-components of the gauge field, while $\chi^3 = 0$. With the Ansatz $\chi^1_1 = \chi^1_{cl}(x_2)$, $\chi^1_2 = \chi^1_{ext}(x_1)$, we are left with two one-dimensional equations in the region far from the boundaries of the Wilson surface $S$,

$$\dot{\chi}^1_1 \chi^2_1 = -2\pi q_{ext} \dot{\chi}^1_1 \chi^2_1 = M^2 \sin^2 \chi^2_1,$$
$$\dot{\chi}^1_2 \chi^2_1 = -2\pi q_{ext} \dot{\chi}^1_2 \chi^2_1 = M^2 \sin^2 \chi^2_1.\quad (28)$$

Following [30], we solve these equations in the continuum limit,

$$\partial_1 \partial_1 \chi^2_1 = 2\pi q_{ext} \partial_1 \dot{\chi}^1_1 + M^2 \sin^2 \chi^2_1,$$
$$\partial_2 \partial_2 \chi^2_1 = 2\pi q_{ext} \partial_1 \dot{\chi}^1_1 + M^2 \sin^2 \chi^2_1.\quad (29)$$

For $q_{ext} = 1$ (corresponding to Cooper pairs in our case), the classical solutions with the boundary conditions $\chi^1_{cl} \to 0$ for $|x_1, x_2| \to \infty$ are

$$\chi^1_{cl} = \text{sign}(x_2) 4 \arctan e^{-M|x_2|},$$
$$\chi^2_{cl} = \text{sign}(x_1) 4 \arctan e^{-M|x_1|}.\quad (30)$$

Inserting this back in (26) we get formula (7) in the main text.

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