Open Mathematics

Research Article

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\( L^p \) estimates for maximal functions along surfaces of revolution on product spaces

https://doi.org/10.1515/math-2019-0118
Received June 10, 2018; accepted October 18, 2019

Abstract: This paper is concerned with establishing \( L^p \) estimates for a class of maximal operators associated to surfaces of revolution with kernels in \( L^q(S^{n-1} \times S^{m-1}) \), \( q > 1 \). These estimates are used in extrapolation to obtain the \( L^p \) boundedness of the maximal operators and the related singular integral operators when their kernels are in the \( L((\log L)^{\kappa}(S^{n-1} \times S^{m-1})) \) or in the block space \( B_q^{0, \kappa-1}(S^{n-1} \times S^{m-1}) \). Our results substantially improve and extend some known results.

Keywords: maximal functions, \( L^p \) boundedness, Rough kernels, surfaces of revolution, extrapolation

MSC: Primary 42B20; Secondary 40B25, 47G10

1 Introduction and main results

Let \( n, m \geq 2 \), and let \( \mathbb{R}^N \) (\( N = n \) or \( m \)) be the \( N \)-dimensional Euclidean space. Let \( S^{n-1} \) be the unit sphere in \( \mathbb{R}^n \) equipped with the normalized Lebesgue surface measure \( d\sigma = d\sigma(z) \). Also, let \( x' = x/|x| \) for \( x \in \mathbb{R}^n \setminus \{0\} \), \( y' = y/|y| \) for \( y \in \mathbb{R}^m \setminus \{0\} \).

Let \( K_{\Omega,h}(x,y) = \Omega(x',y')|x'|^{-n}|y'|^{-m}h(|x|, |y|) \), where \( h \) is a measurable function on \( \mathbb{R}^+ \times \mathbb{R}^+ \) and \( \Omega \) is an integrable function on \( S^{n-1} \times S^{m-1} \) that satisfies

\[
\int_{S^{n-1}} \Omega(x', y) d\sigma(x') = \int_{S^{m-1}} \Omega(\cdot, y') d\sigma(y') = 0 \quad \text{and} \quad (1.1)
\]

\[
\Omega(rx, ty) = \Omega(x, y) \quad \text{for all } r, t > 0. \quad (1.2)
\]

For suitable mappings \( \phi, \psi : \mathbb{R}^+ \to \mathbb{R} \), consider the singular integral operator \( T^{P_1, P_2}_{\Omega, h, \phi, \psi} \) defined, initially for \( C_0^\infty \) functions on \( \mathbb{R}^{n+1} \times \mathbb{R}^{m+1} \), by

\[
T^{P_1, P_2}_{\Omega, h, \phi, \psi}(f)(\xi, \eta) = p.v. \int_{\mathbb{R}^{n+2}} e^{iP_1(u) + iP_2(v)} \times f(x - u, x_{n+1} - \phi(u), y - v, y_{m+1} - \psi(|v|))K_{\Omega,h}(u, v)du dv,
\]

where \((\xi, \eta) = (x_{n+1}, y_{m+1}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{m+1} \) and \( P_1 : \mathbb{R}^n \to \mathbb{R}, P_2 : \mathbb{R}^m \to \mathbb{R} \) are two real-valued polynomials.

When \( P_1(u) = 0 \) and \( P_2(v) = 0 \), we denote \( T^{P_1, P_2}_{\Omega, h, \phi, \psi} \) by \( T_{\Omega, h, \phi, \psi} \). Also, when \( \phi(t) = \psi(t) = t \), then \( T_{\Omega, h, \phi, \psi} \) (denoted by \( T_{\Omega, h} \)) is just the classical singular integral operator introduced by Fefferman in [1] in which he

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obtained the $L^p$ boundedness of $T_{\Omega,h}$ for all $1 < p < \infty$ whenever $\Omega$ satisfies some regularity conditions and $h \equiv 1$. As a matter of fact, the systematic study of such operator began by Fefferman in [1], and then it was elaborated very much by Fefferman and Stein in [2]. Subsequently, the investigation of the $L^p$ boundedness of $T_{\Omega,h}$ under very various conditions on $\Omega$ and $h$ has attracted the attention of many authors. For example, it was proved in [3] that $T_{\Omega,h}$ is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $1 < p < \infty$ whenever $\Omega \in L(\log L)^2(S^{n-1} \times S^{m-1})$ and $h$ satisfies certain integrability-size condition. Furthermore, the authors of [3] established the optimality of the condition in the sense that the space $L(\log L)^2(S^{n-1} \times S^{m-1})$ cannot be replaced by $L(\log L)^{2-\varepsilon}(S^{n-1} \times S^{m-1})$ for any $0 < \varepsilon < 2$. For more information about the importance and the recent advances on the study of such operators, the readers are referred (for example to [1–5], and the references therein).

On the other side, the study of the singular integrals on product spaces along surfaces of revolution has been started. For example, if $\phi$ and $\psi$ are in $C^2([0, \infty))$, convex and increasing functions with $\phi(0) = \psi(0) = 0$, then Al-Salman in [4] showed that $T_{\Omega,1,\phi,\psi}$ is bounded on $L^p(\mathbb{R}^{n+1} \times \mathbb{R}^{m+1})$ $(1 < p < \infty)$ provided that $\Omega \in L(\log L)^2(S^{n-1} \times S^{m-1})$. Recently, Al-Salman improved this result in [6]. In fact, when $\phi$, $\psi$ are given as in [4], he verified the $L^p$ boundedness of $T_{\Omega,h,\phi,\psi}$ for all $p \in (1, \infty)$ under the conditions $\Omega \in L(\log L)(S^{n-1} \times S^{m-1})$ and $h \in L^2(\mathbb{R}^n \times \mathbb{R}^m, \frac{d\mu}{r})$ with $\|h\|_{L^2(\mathbb{R}^n \times \mathbb{R}^m, \frac{d\mu}{r})} \leq 1$.

The maximal operator that relates to our singular integral operator is $M^{\psi(1,\phi,\psi)}_{\Omega,\phi,\psi}$ that given by

$$M^{\psi(1,\phi,\psi)}_{\Omega,\phi,\psi}(f)(\xi,\eta) = \sup_{h \in U} \|T^{\psi(1,\phi,\psi)}_{\Omega,\phi,\psi}(f)(\xi,\eta)\|_{L^1(\mathbb{R}^n \times \mathbb{R}^m, \frac{d\mu}{r})},$$

where $U = \{h \in L^2(\mathbb{R}^n \times \mathbb{R}^m, \frac{d\mu}{r}); \|h\|_{L^2(\mathbb{R}^n \times \mathbb{R}^m, \frac{d\mu}{r})} \leq 1\}$.

Again, when $P_1(u) = 0$ and $P_2(v) = 0$, we denote $M^{\psi(1,\phi,\psi)}_{\Omega,\phi,\psi}$ by $M_{\Omega,\phi,\psi}$. Also, when $\phi(t) = \psi(t) = t$, then $M_{\Omega,\phi,\psi}$ reduces to the classical maximal operator denoted by $M_\Omega$. Historically, the operator $M_\Omega$ was introduced by Ding in [7] in which he proved the $L^2$ boundedness of $M_\Omega$ whenever $\Omega \in L(\log L)^2(S^{n-1} \times S^{m-1})$. This result was improved independently by Al-Qassem and Pan in [8] and by Al-Salman in [9]. Precisely, they showed that $M_\Omega$ is of type $(p, p)$ for all $p \geq 2$ provided that $\Omega \in L(\log L)(S^{n-1} \times S^{m-1})$. Moreover, they pointed out that the condition $\Omega \in L(\log L)(S^{n-1} \times S^{m-1})$ is optimal in the sense that the exponent 1 in $L(\log L)(S^{n-1} \times S^{m-1})$ cannot be replaced by any smaller positive number $\varepsilon < 1$ so that $M_\Omega$ is bounded on $L^2(\mathbb{R}^{n+1} \times \mathbb{R}^{m+1})$. Also, an improvement of the result in [7] was obtained by Al-Qassem in [10]. Indeed, Al-Qassem established the $L^p(2 \leq p < \infty)$ estimates for the class $M_\Omega$ whenever $\Omega$ belongs to the block space $B_q^{0,0}((S^{n-1} \times S^{m-1})$ for some $q > 1$. Furthermore, he proved that the condition $\Omega \in B_q^{0,0}((S^{n-1} \times S^{m-1})$ is nearly optimal in the sense that the operator $M_\Omega$ may lose the $L^2$ boundedness if $\Omega$ is assumed to be in the space $B_q^{0,0}((S^{n-1} \times S^{m-1})$ for some $-1 < \varepsilon < 0$. Recently, it was found in [6] that the maximal operator $M_{\Omega,\phi,\psi}$ is bounded on $L^p(\mathbb{R}^{n+1} \times \mathbb{R}^{m+1})$ for any $p \geq 2$ if $\Omega \in L(\log L)(S^{n-1} \times S^{m-1})$, and $\phi$, $\psi$ are in $C^2([0, \infty))$, convex and increasing functions with $\phi(0) = \psi(0) = 0$. Very recently, when $\phi(t) = \psi(t) = t$, Al-Dolat and al. found in [11] that the $L^p$ $(p \geq 2)$ boundedness of $M_{\Omega,\phi,\psi}$ is obtained under the condition $\Omega \in L(\log L)(S^{n-1} \times S^{m-1}) \cup B_q^{0,0}((S^{n-1} \times S^{m-1})$ with $q > 1$. Subsequently, the investigation of the $L^p$ boundedness of $M_{\Omega,\phi,\psi}$ under weak conditions has received much attentions from many mathematicians. For the significance of considering the integral operators $M_{\Omega,\phi,\psi}$, we refer the readers to consult [8] and [11–13], among others.

The main result of this work is formulated as follows:

**Theorem 1.1.** Let $\Omega \in L^q(S^{n-1} \times S^{m-1})$, $q > 1$ and satisfy the conditions (1.1)-(1.2) with $\|\Omega\|_{L^q(S^{n-1} \times S^{m-1})} \leq 1$, and let $\mu = \mu_q(\Omega) = \log(e + \|\Omega\|_{L^q(S^{n-1} \times S^{m-1})})$. Assume that $\phi$, $\psi$ are in $C^2([0, \infty))$, convex and increasing functions with $\phi(0) = \psi(0) = 0$. Let $P_1 : \mathbb{R}^n \to \mathbb{R}$ and $P_2 : \mathbb{R}^m \to \mathbb{R}^m$ be two real-valued polynomials of degrees $d_1$, $d_2$, respectively. Then there exists a constant $C_{p,q} > 0$ such that

$$\|M_{\Omega,\phi,\psi}(f)\|_{L^p(\mathbb{R}^{n+1} \times \mathbb{R}^{m+1})} \leq C_{p,q}(1 + \mu) \|f\|_{L^p(\mathbb{R}^{n+1} \times \mathbb{R}^{m+1})} \tag{1.3}$$

for all $p \geq 2$, where $C_{p,q} = \left(\frac{2\lambda_q}{2\lambda_q-1}\right)^2 C_p$ and $C_p$ is a positive constant that may depend on the degrees of the polynomials $P_1$, $P_2$ but it is independent on $\Omega$, $\phi$, $\psi$, $q$, and the coefficients of the polynomials $P_1$, $P_2$.  


We remark that by the result in Theorem 1.1 and using an extrapolation argument, we get that \( N_{\Omega, \phi, \psi} \) is bounded on \( L^p(\mathbb{R}^{n+1} \times \mathbb{R}^{m+1}) \) for \( 2 \leq p < \infty \) if \( \Omega \in L(\log L)(S^{n-1} \times S^{m-1}) \cup B_q^{(0,0)}(S^{n-1} \times S^{m-1}) \) for some \( q > 1 \).

Here and henceforth, the letter \( C \) denotes a bounded positive constant that may vary at each occurrence but independent of the essential variables.

## 2 Preliminary lemmas

In this section, we present and prove some lemmas used in the sequel. The first lemma can be derived by applying the same technique that Al-Qassam and Pan used in [14, pp. 64-65].

**Lemma 2.1.** Let \( \Omega \in L^q(S^{n-1}), q > 1 \) be a homogeneous function of degree zero on \( \mathbb{R}^n \) with \( \| \Omega \|_{L^q(S^{n-1})} \leq 1 \), and let \( \phi: \mathbb{R}^n \to \mathbb{R} \) be a \( C^2([0, \infty)) \), convex and increasing function with \( \phi(0) = 0 \). Consider the maximal function \( N_{\Omega, \phi} \) given by

\[
N_{\Omega, \phi}(x) = \sup_{y \in \mathbb{R}^n} \int_{|z| \leq 2} |f(z - y, x + \phi(y)) - \phi(y)| dy.
\]

Then for \( p > 1 \) and \( f \in L^p(\mathbb{R}^{n+1}) \) there exists a positive number \( C_p \) such that

\[
\| N_{\Omega, \phi}(f) \|_p \leq C_p \| f \|_p.
\]

**Lemma 2.2.** Assume that \( \phi, \psi \) are \( C^2([0, \infty)) \), convex and increasing functions with \( \phi(0) = \psi(0) = 0 \). Let \( \Omega \in L^q(S^{n-1} \times S^{m-1}), q > 1 \) and satisfy the conditions (1.1)-(1.2) with \( \| \Omega \|_{L^q(S^{n-1} \times S^{m-1})} \leq 1 \). Then for all \( f \in L^p(\mathbb{R}^{n+1} \times \mathbb{R}^{m+1}) \) and \( p > 1 \), the maximal function

\[
N_{\Omega, \phi, \psi}(f, y) = \sup_{x, \xi, \eta \in \mathbb{R}^n} \int_{\mathbb{R}^n} |f(x - u, \xi, u + \phi(u), y - v, \eta, v + \psi(v)) - \phi(u) - \psi(v)| \frac{\| \Omega(x, y) \|_{\mathbb{R}^{n+m}}}{{|u|^{n/|y|}}{|v|^{m/|y|}}} du dv
\]

satisfies

\[
\| N_{\Omega, \phi, \psi}(f) \|_p \leq C_p \| f \|_p,
\]

where \( A_{ij} = \left\{ (u, v) \in \mathbb{R}^n \times \mathbb{R}^m : 2^j \leq |u| \leq 2^{j+1}, 2^j \leq |v| \leq 2^{j+1} \} \) and the positive constant \( C_p \) is independent of the functions \( \phi, \psi \) and \( \Omega \).

It is easy to prove the above lemma by using Lemma 2.1 and the inequality \( N_{\Omega, \phi, \psi}(f, y) \leq N_{\Omega, \phi} \circ N_{\Omega, \psi}(f, y) \), where \( N_{\Omega, \phi, \psi}(f, y) = N_{\Omega, \phi}(f, \cdot)(y), N_{\Omega, \phi}(f, \cdot)(y) = N_{\Omega, \phi}(f, \cdot)(y), \) and \( \circ \) denotes the composition of operators.

A significant step toward proving Theorem 1.1 is to estimate the following Fourier transform:

**Lemma 2.3.** Let \( \Omega \in L^q(S^{n-1} \times S^{m-1}), q > 1 \) and satisfy the conditions (1.1)-(1.2) with \( \| \Omega \|_{L^q(S^{n-1} \times S^{m-1})} \leq 1 \), and let \( \mu = \mu_q(\Omega) = \log(e + \| \Omega \|_{L^q(S^{n-1} \times S^{m-1})}) \). Assume that \( \phi, \psi \) are arbitrary functions on \( \mathbb{R}^n \), and assume also that

\[
P_1 = \sum_{|\alpha| \leq d_1} a_{\alpha} x^\alpha \text{ is a polynomial of degree } d_1 \geq 1 \text{ such that } |x|^{d_1} \text{ is not one of its terms and } \sum_{|\alpha| = d_1} |a_{\alpha}| = 1;
\]

and

\[
P_2 = \sum_{|\beta| \leq d_2} b_{\beta} y^\beta \text{ is a polynomial of degree } d_2 \geq 1 \text{ such that } |y|^{d_2} \text{ is not one of its terms and } \sum_{|\beta| = d_2} |b_{\beta}| = 1.
\]

For \( i, j \in \mathbb{Z}, \) define \( \tilde{j}_{ij}(x, y) : \mathbb{R}^{n+1} \times \mathbb{R}^{m+1} \to \mathbb{R} \) by

\[
\tilde{j}_{ij}(x, y) = \int_{1}^{2^{j+1}} \int_{1}^{2^{j+1}} \Omega(u, y) A_{ij}(u, \xi, u + \phi(u), y - v, \eta, v + \psi(v)) d\mu(u) d\mu(v)
\]

where

\[
A_{ij}(u, \xi, u + \phi(u), y - v, \eta, v + \psi(v)) = e^{-i \int [P_1(x^{d_1} \xi u + (2^{(d_1+1)u})^\ast \phi((2^{d_1+1}u)^\ast) \xi u)]} \quad \text{and}
\]

\[
B_{ij}(u, v, \eta, v + \psi(v)) = e^{-i \int [P_2((2^{d_2}u)^\ast \eta v + (2^{(d_2+1)u})^\ast \psi((2^{d_2+1}u)^\ast) \eta v)]}.
\]
Then, a positive constant $C$ exists such that

$$\sup_{(\xi, \eta) \in \mathbb{R}^{n+1}} |\mathcal{A}_{i,j,\Omega,\psi}(\xi, \eta)| \leq C\mu^2 2^{(i+j)/2q'}.$$

**Proof.** On one hand, it is trivial to get that

$$\mathcal{A}_{i,j,\Omega,\psi}(r, u, \xi, u, \xi_{n+1}) \leq C\mu.$$ 

Thus, using Hölder’s inequality leads to

$$\mathcal{B}_{j,\Omega,\psi}(t, v, \eta, v, \eta_{m+1}) \leq C\mu.$$ 

Also, it is easy to see that

$$\mathcal{B}_{j,\Omega,\psi}(t, v, \eta, v, \eta_{m+1}) \leq C\mu.$$ 

With Van der-Corput Lemma, we obtain

$$\mathcal{B}_{j,\Omega,\psi}(t, v, \eta, v, \eta_{m+1}) \leq C\mu.$$ 

Then, a positive constant $C$ exists such that

$$\sup_{(\xi, \eta) \in \mathbb{R}^{n+1}} |\mathcal{A}_{i,j,\Omega,\psi}(\xi, \eta)| \leq C\mu^2 2^{(i+j)/2q'}.$$ 

Proof. On one hand, it is trivial to get that

$$\mathcal{A}_{i,j,\Omega,\psi}(r, u, \xi, u, \xi_{n+1}) \leq C\mu.$$ 

Also, it is easy to see that

$$\mathcal{B}_{j,\Omega,\psi}(t, v, \eta, v, \eta_{m+1}) \leq C\mu.$$ 

Thus, using Hölder’s inequality leads to

$$\mathcal{B}_{j,\Omega,\psi}(t, v, \eta, v, \eta_{m+1}) \leq C\mu.$$ 

Since $\sum_{|\alpha|=d_1} |a_\alpha| = \sum_{|\beta|=d_2} |b_\beta| = 1$, then by taking $\theta = 1/4\mu q'$, we have

$$\mathcal{A}_{i,j,\Omega,\psi}(\xi, \eta) \leq C\mu^2 2^{(i+j)/2q'}.$$ 

□
We shall need the following Lemma which can be acquired by using the arguments employed in the proof of [6, Theorem 4.1] as well as [15, Theorem 1.6].

**Lemma 2.4.** Let $\Omega \in L^q(S^{n-1} \times S^{m-1})$, $q > 1$ and satisfy the conditions (1.1)-(1.2) with $\|\Omega\|_{L^1(S^{n-1} \times S^{m-1})} \leq 1$. Assume that $\phi$, $\psi$ and $\mu$ are given as in Theorem 1.1. Then there exists a constant $C_{p,q} > 0$ such that

$$
\|M_{\Omega,\phi,\psi}(f)\|_{L^p(\mathbb{R}^{n+1} \times \mathbb{R}^{m+1})} \leq C_{p,q} (1 + \mu) \|f\|_{L^p(\mathbb{R}^{n+1} \times \mathbb{R}^{m+1})}
$$

(2.4)

for $2 \leq p < \infty$. 

**Proof.** Choose collections of functions $\{\Phi_i\}_{i \in \mathbb{Z}}$ and $\{\Psi_j\}_{j \in \mathbb{Z}}$ defined on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively with the following properties:

(i) $\widehat{\Phi}_i$ is supported in $\{\xi \in \mathbb{R}^n : |\xi| \in 2^i, 2^{-i}\}$; 
(ii) $\widehat{\Psi}_j$ is supported in $\{\eta \in \mathbb{R}^m : |\eta| \in 2^j, 2^{-j}\}$; 
(iii) $0 \leq \Phi_i, \Psi_j \leq 1$; 
(iv) $\sum_{i \in \mathbb{Z}} (\Phi_i)^2(\xi) = \sum_{j \in \mathbb{Z}} (\Psi_j)^2(\eta) = 1$.

Define the multiplier operators $S_{j,i}$ in $\mathbb{R}^{n+1} \times \mathbb{R}^{m+1}$ via the Fourier transform given by

$$
\widehat{S_{j,i}}(\xi, \eta) = \widehat{\Phi}_i(\xi) \widehat{\Psi}_j(\eta).
$$

Hence, for any $f \in C_c(\mathbb{R}^{n+1} \times \mathbb{R}^{m+1})$, we have

$$
M_{\Omega,\phi,\psi}(f)(x, \bar{y}) \leq \sum_{j,i \in \mathbb{Z}} T_{\Omega,\phi,\psi,j,i}(f)(x, \bar{y}),
$$

(2.5)

where

$$
T_{\Omega,\phi,\psi,j,i}(f)(x, \bar{y}) = \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} |\mathcal{W}_{\Omega,\phi,\psi,j,i}(f)(x, y)|^2 \frac{dr dt}{rt} \right)^{1/2},
$$

(\mathcal{W}_{\Omega,\phi,\psi,j,i}(f))(x, y) = \sum_{s, l \in \mathbb{Z}} \int_{\mathbb{R}^{n+1}} \int_{\mathbb{R}^{m+1}} S_{j+i, l+s}(f)(x - ru, x_{n+1} - \phi(r), y - tv, y_{m+1} - \psi(t)) \chi_{\varepsilon_1, \varepsilon_2; \varepsilon_1, \varepsilon_2}(\Omega(u, v)) d\sigma(u) d\sigma(v).
$$

Therefore, by using [6, Theorem 4.1], we get

$$
\|T_{\Omega,\phi,\psi,j,i}(f)\|_p \leq C_{p,q,\mu} 2^{-\varepsilon_1|j|} 2^{-\varepsilon_2|l|} \|f\|_p
$$

(2.6)

for some constants $0 < \varepsilon_1, \varepsilon_2 < 1$ and for all $2 \leq p < \infty$. Consequently, the inequality (2.4) follows by using (2.5) and (2.6). 

\[\square\]

## 3 Proof of Theorem 1.1

The proof of Theorem 1.1 mainly depends on the approaches employed in the proof of [11, Theorem 1.1], which have their roots in [16]. Precisely, we argue the mathematical induction on the degrees of the polynomials $P_1$ and $P_2$.

If $d_1 = d_2 = 0$, then by Lemma 2.4, we directly attain

$$
\|M_{\Omega,\phi,\psi}^{P_1, P_2}(f)\|_p \leq C_{p,q} (1 + \mu) \|f\|_p
$$

(3.1)

for all $p \geq 2$. Also, if $d_1 = 0$ or $d_2 = 0$, then by [17, Theorem 1.1], it is easy to satisfy the inequality (1.3) for all $p \geq 2$. 

\[\square\]
Now, assume that (1.3) is true for any polynomial $P_1$ of degree less than or equal to $d_1$ and for any polynomial $P_2$ of degree $d_2$. We need to show that (1.3) is still true if $\text{degree}(P_1) = d_1 + 1$, and $\text{degree}(P_2) = d_2$. Without loss of generality, we may assume $P_1(x) = \sum |a_{\alpha}|d_{\alpha} + 1$ is a polynomial of degree $d_1 + 1$ such that $\sum |a_{\alpha}| = 1$ and does not contain $|x|^{d_1+1}$ as one of its terms. Also, we may assume $P_2(y) = \sum b_{\beta}y^\beta$ is a given polynomial of degree $d_2$ such that $\sum |b_{\beta}| = 1$ and does not contain $|y|^{d_2}$ as one of its terms. By duality and a simple change of variables, we have

$$M_{\Omega,\phi,\psi,0}^{P_1,P_2}(f)(\bar{x}, \bar{y}) = \left( \int \int \int_{\mathbb{R}^n \times \mathbb{R}^n} |G_{P_1,P_2,\phi,\psi,0}(f)(\bar{x}, \bar{y}, r, t)|^2 \frac{d \bar{r} d \bar{t}}{rt} \right)^{1/2},$$

where

$$G_{P_1,P_2,\phi,\psi,0}(f)(\bar{x}, \bar{y}, r, t) = \int_{S_{n-1} \times S_{n-1}} e^{ip_1(ru) + ip_2(tv)} f(x - ru, x_{n+1} - \phi(r), y - tv, y_{m+1} - \psi(t)) Q(u, v) d\sigma(u) d\sigma(v).$$

Choose two collections of $c^\infty$ functions $\{Y_i\}_{i \in \mathbb{Z}}$ and $\{\Gamma_j\}_{j \in \mathbb{Z}}$ on $(0, \infty)$, that satisfying the following conditions:

$$\text{supp } Y_i \subseteq \Gamma_{i,\mu} = \left[ 2^{-(i+1)\mu}, 2^{-(i-1)\mu} \right]; \supp \Gamma_j \subseteq \Omega_{j,\mu}; \quad 0 \leq Y_i, \Gamma_j \leq 1; \quad \text{and } \sum_{i \in \mathbb{Z}} Y_i(u) = \sum_{j \in \mathbb{Z}} \Gamma_j(v) = 1.$$

Define the multiplier operators $S_{j,i}$ in $\mathbb{R}^{n+1} \times \mathbb{R}^{m+1}$ by

$$S_{j,i}(f)(\bar{\xi}, \bar{\eta}) = Y_i(\bar{\xi}) \Gamma_j(\bar{\eta}) f(\bar{\xi}, \bar{\eta}) \text{ for } (\bar{\xi}, \bar{\eta}) = (\xi, \xi_{n+1}, \eta, \eta_{m+1}) \in R^{n+1} \times R^{m+1}.$$

Set

$$A_\infty(u) = \sum_{i=\infty}^0 Y_i(u), \quad A_0(u) = \sum_{i=0}^\infty Y_i(u), \quad B_\infty(v) = \sum_{j=\infty}^0 \Gamma_j(v), \quad \text{and } B_0(v) = \sum_{j=0}^\infty \Gamma_j(v).$$

Thanks to Minkowski’s inequality, we have

$$M_{\Omega,\phi,\psi,\infty}^{P_1,P_2}(f)(\bar{x}, \bar{y}) \leq M_{\Omega,\phi,\psi,\infty}^{P_1,P_2}(f)(\bar{x}, \bar{y}) + M_{\Omega,\phi,\psi,\infty}^{P_1,P_2}(f)(\bar{x}, \bar{y}) + M_{\Omega,\phi,\psi,\infty}^{P_1,P_2}(f)(\bar{x}, \bar{y}), \quad (3.2)$$

where

$$M_{\Omega,\phi,\psi,\infty}^{P_1,P_2}(f)(\bar{x}, \bar{y}) = \left( \int \int \int_{2^{-\mu} \times 2^{-\mu}} A_\infty(r)B_\infty(t) G_{P_1,P_2,\phi,\psi,0}(f)(\bar{x}, \bar{y}, r, t)|^2 \frac{d \bar{r} d \bar{t}}{rt} \right)^{1/2},$$

$$M_{\Omega,\phi,\psi,\infty}^{P_1,P_2}(f)(\bar{x}, \bar{y}) = \left( \int \int \int_{2^{-\mu} \times 2^{-\mu}} A_\infty(r)B_0(t) G_{P_1,P_2,\phi,\psi,0}(f)(\bar{x}, \bar{y}, r, t)|^2 \frac{d \bar{r} d \bar{t}}{rt} \right)^{1/2},$$

$$M_{\Omega,\phi,\psi,\infty}^{P_1,P_2}(f)(\bar{x}, \bar{y}) = \left( \int \int \int_{2^{-\mu} \times 2^{-\mu}} A_0(r)B_\infty(t) G_{P_1,P_2,\phi,\psi,0}(f)(\bar{x}, \bar{y}, r, t)|^2 \frac{d \bar{r} d \bar{t}}{rt} \right)^{1/2},$$

and

$$M_{\Omega,\phi,\psi,\infty}^{P_1,P_2}(f)(\bar{x}, \bar{y}) = \left( \int \int \int_{2^{-\mu} \times 2^{-\mu}} A_0(r)B_0(t) G_{P_1,P_2,\phi,\psi,0}(f)(\bar{x}, \bar{y}, r, t)|^2 \frac{d \bar{r} d \bar{t}}{rt} \right)^{1/2}. $$
Let us first estimate the $L^p$-norm of $M^{P_1, P_2}_{Ω, φ, ψ, ∞, j}(f)$. Define

$$M^{P_1, P_2}_{Ω, φ, ψ, ∞, j}(f)(x, y) = \left( \int_{2^{-j-1}r}^{2^{-j}r} \int_{2^{-j-1}r}^{2^{-j}r} \left| g_{P_1, P_2, Ω}(f)(x, y, r, t) \right|^2 \frac{dr dt}{rt} \right)^{1/2}.$$ 

Hence, by generalized Minkowski's inequality, it is easy to reach

$$M^{P_1, P_2}_{Ω, φ, ψ, ∞, j}(f)(x, y) \leq \sum_{j=0}^{∞} M^{P_1, P_2}_{Ω, φ, ψ, ∞, j}(f)(x, y). \quad (3.3)$$

If $p = 2$, then by a simple change of variables, Plancherel’s theorem, Fubini’s theorem, and Lemma 2.3, we get that

$$\left\| M^{P_1, P_2}_{Ω, φ, ψ, ∞, j}(f) \right\|_2 = \left( \int_{R^{n+1} \times R^{n+1}} \left| \hat{g}(ξ, η) \right|^2 |Ω(ξ, η)| dξ dη \right)^{1/2} \leq C^2 \left\| f \right\|_2.$$ 

(3.4)

However, if $p > 2$, then by the duality, there exists $b \in L^{p/2'}(R^{n+1} \times R^{n+1})$ with $\|b\|_{L^{p/2'}(R^{n+1} \times R^{n+1})} = 1$ such that

$$\left\| M^{P_1, P_2}_{Ω, φ, ψ, ∞, j}(f) \right\|_p \leq \int_{R^{n+1} \times R^{n+1}} |Ω(ξ, η)| dξ dη \left| b(z, 2^{-j}w) \right|^2 \left| f(z, w) \right|^2 \frac{dz dw}{(p/2)\|Ω\|_1},$$

So, by Hölder’s inequality and Lemma 2.2, we conclude that

$$\left\| M^{P_1, P_2}_{Ω, φ, ψ, ∞, j}(f) \right\|_p \leq C \left| b(z, 2^{-j}w) \right|^2 \left| f(z, w) \right|^2 \frac{dz dw}{(p/2)\|Ω\|_1},$$

$$\left\| M^{P_1, P_2}_{Ω, φ, ψ, ∞, j}(f) \right\|_p \leq C_p (1 + \mu)^2 \left\| f \right\|_p \left\| b \right\|_{L^{p/2'}} \left\| Ω \right\|_1,$$

where $\tilde{b}(z, w) = b(-z, -w)$. Thus,

$$\left\| M^{P_1, P_2}_{Ω, φ, ψ, ∞, j}(f) \right\|_p \leq C_p (1 + \mu) \left\| f \right\|_p,$$

which when Combined with (3.4) gives that there is $ε \in (0, 1)$ so that

$$\left\| M^{P_1, P_2}_{Ω, φ, ψ, ∞, j}(f) \right\|_p \leq C_p 2^{\frac{ε(p+1)}{p}} (1 + \mu) \left\| f \right\|_p \quad (3.5)$$

for all $p \geq 2$. Therefore, by (3.3) and (3.5), we obtain

$$\left\| M^{P_1, P_2}_{Ω, φ, ψ, ∞, j}(f) \right\|_p \leq C_{p, q} (1 + \mu) \left\| f \right\|_p \quad (3.6)$$
for all \( p \geq 2 \). Now, let us estimate the \( L^p \)-norm of \( \mathcal{N}_{\Omega, \phi, \psi, 0, 0}^{p_1, p_2}(f) \). Take \( Q_1(x) = \sum_{|\alpha| \leq d_1} a_\alpha x^\alpha \), and define 
\[ \mathcal{M}_{\Omega, \phi, \psi, 0, 0}^{Q_1, p_2}(f) \] and \( \mathcal{M}_{\Omega, \phi, \psi, 0, 0}^{p_1, p_2, Q}(f) \) by

\[
\mathcal{M}_{\Omega, \phi, \psi, 0, 0}^{Q_1, p_2}(f)(x, y) = \left( \int_0^1 \int_0^1 |\mathcal{J}_{Q_1, p_2, \phi, \psi, 0, 0}(f)(x, y, r, t)|^2 \frac{dr dt}{rt} \right)^{1/2},
\]

and

\[
\mathcal{M}_{\Omega, \phi, \psi, 0, 0}^{p_1, p_2, Q}(f)(x, y) = \left( \int_0^1 \int_0^1 |\mathcal{J}_{Q_1, p_2, \phi, \psi, 0, 0}(f)(x, y, r, t)|^2 \frac{dr dt}{rt} \right)^{1/2},
\]

where

\[
\mathcal{J}_{Q_1, p_2, \phi, \psi, 0, 0}(f)(x, y, r, t) = \int_{S^{n-1} \times S^{n-1}} \left( e^{ip_2(ru + tv)} - e^{ip_2(ru)} \right) f(x - ru, x_{n+1} - \phi(r), y - tv, y_{m+1} - \psi(t)) \Omega(u, v) du dv.
\]

Thus, by Minkowski’s inequality, we deduce

\[
\mathcal{M}_{\Omega, \phi, \psi, 0, 0}^{p_1, p_2, Q}(f)(x, y) \leq \mathcal{M}_{\Omega, \phi, \psi, 0, 0}^{Q_1, p_2}(f)(x, y) + \mathcal{M}_{\Omega, \phi, \psi, 0, 0}^{p_1, p_2, Q}(f)(x, y).
\]

On one hand, since \( \deg(Q_1) \leq d_1 \), then by induction step we have

\[
\left\| \mathcal{M}_{\Omega, \phi, \psi, 0, 0}^{Q_1, p_2}(f) \right\|_p \leq C_{p, q} (1 + \mu) \left\| f \right\|_p
\]

for all \( p \geq 2 \). On the other hand, it is easy to check that

\[
\left| e^{ip_2(ru)} - e^{ip_1(ru)} \right| \leq r^{d_1+1} \left| \sum_{|\alpha| = d_1+1} a_\alpha u^\alpha \right| \leq r^{d_1+1}.
\]

So, by following a similar argument as in [18] and by Cauchy-Schwartz inequality, we have that

\[
\mathcal{M}_{\Omega, \phi, \psi, 0, 0}^{p_1, p_2, Q}(f)(x, y) \leq C \left( \int_0^1 \int_0^1 \int_{S^{n-1} \times S^{n-1}} e^{ip_2(tv)} f(x - ru, x_{n+1} - \phi(r), y - tv, y_{m+1} - \psi(t)) \Omega(u, v) du dv \left| \frac{dr dt}{rt} \right| \right)^{1/2}
\]

\[
\times \Omega(u, v) du dv \left| \frac{dr dt}{rt} \right|^{1/2}
\]

\[
\leq C \left( \int_0^1 \int_0^1 \int_{S^{n-1} \times S^{n-1}} e^{ip_2(tv)} f(x - ru, x_{n+1} - \phi(r), y - tv, y_{m+1} - \psi(t)) \Omega(u, v) du dv \left| \frac{dr dt}{rt} \right| \right)^{1/2}
\]

\[
\times \Omega(u, v) du dv \left| \frac{dr dt}{rt} \right|^{1/2}
\]

\[
\leq C \left( \sum_{j=1}^{\infty} \left( 2^{-jd_1} \right)^{2^{j+1}} \int_{2^j S^{n-1}} \int_{2^j S^{n-1}} e^{ip_2(tv)} f(x - ru, x_{n+1} - \phi(r), y - tv, y_{m+1} - \psi(t)) \Omega(u, v) du dv \left| \frac{dr dt}{rt} \right| \right)^{1/2}
\]

\[
\times \Omega(u, v) du dv \left| \frac{dr dt}{rt} \right|^{1/2}
\]

\[
\leq C \left( \sum_{j=1}^{\infty} \left( 2^{-jd_1} \right)^{2^{j+1}} \int_{2^j S^{n-1}} \left( \mathcal{M}_{\Omega, \phi, \psi, 0, 0}^{p_1, p_2, Q}(f)(x, y) \right)^2 du dv \right)^{1/2}
\]
Assume that

\[ \sup_{\Omega, \phi, \psi} \int_{\Omega} (D_{\phi, \psi} f_r(x, y))^2 \, dx \, dy \leq C \left( \int_{\Omega} \left( \sum_{l=1}^{n} (D_{\phi, \psi} f_r(x, y))^2 \right)^{1/2} \, dx \, dy \right)^2 \]

where \( \circ \) denotes the composition of operators, \( N_{\Omega, \phi, \psi} f_r(x, y) = N_{\Omega, \phi, \psi} f_r^r(x, y) \) is the maximal function defined as in Lemma 2.1, and \( M_{\Omega, \phi, \psi} f_r(x, y) = M_{\Omega, \phi, \psi} (f_r(x, y))^r \) is the maximal operator in the one parameter setting defined as in [17, Eq. (1.2)]. Hence, by following a similar argument as in [18, p. 607] together with [17] and Lemma 2.1, we get

\[ \left\| M_{\Omega, \phi, \psi} f_r(x, y) \right\|_p \leq C_{p, q} (1 + \mu) \| f \|_p \]

for all \( p \geq 2 \). Therefore, by (3.7)-(3.9), we obtain that for all \( p \geq 2 \),

\[ \left\| M_{\Omega, \phi, \psi} f_r(x, y) \right\|_p \leq C_{p, q} (1 + \mu) \| f \|_p \]  

(3.10)

In the same manner, we can derive that

\[ \left\| M_{\Omega, \phi, \psi} f_r(x, y) \right\|_p \leq C_{p, q} (1 + \mu) \| f \|_p \]  

(3.11)

and

\[ \left\| M_{\Omega, \phi, \psi} f_r(x, y) \right\|_p \leq C_{p, q} (1 + \mu) \| f \|_p \]  

(3.12)

for all \( p \geq 2 \). Consequently, by (3.2), (3.6) and (3.10)-(3.12), we satisfy the inequality (1.3) for any polynomial \( P_1 \) of degree \( d_1 + 1 \) and for any polynomial \( P_2 \) of degree \( d_2 \). Similarly, we can show that the inequality (1.3) holds for any polynomial \( P_2 \) of degree \( d_2 + 1 \) and for any polynomial \( P_1 \) of degree \( d_1 \). This completes the proof of Theorem 1.1.

### 4 Further results

For \( \gamma > 1 \), define \( A_{\gamma} (R^n \times R^n) \) to be the set of all measurable functions \( h \) on \( R^n \times R^n \) satisfying the condition

\[ \sup_{R_1, R_2 > 0} \left( \frac{1}{R_1 R_2} \int_0^{R_1} \int_0^{R_2} |h(t, r)|^{\gamma} \, dr \, dt \right)^{1/\gamma} < \infty \]

and define \( L_{\gamma} (R^n \times R^n) \) as \( L_{\gamma} (R^n \times R^n) = A_{\gamma} (R^n \times R^n) \). Also, for \( 1 \leq \gamma < \infty \), define \( L_{\gamma} (R^n \times R^n) \) to be the set of all measurable functions \( h : R^n \times R^n \rightarrow R \) that satisfy the condition \( \| h \|_{L_{\gamma}(R^n \times R^n)} = \left( \int_0^\infty \int_0^\infty |h(t, r)|^{\gamma} \, dt \, dr \right)^{1/\gamma} \leq 1 \) and define \( L_{\gamma}(R^n \times R^n) \) as \( L_{\gamma}(R^n \times R^n) = A_{\gamma} (R^n \times R^n) \).

It is obvious that \( L_{\gamma}(R^n \times R^n) \) is a subset of \( L_{\gamma}(R^n \times R^n) \) for \( 1 < \gamma < \infty \), \( A_{\gamma}(R^n \times R^n) \subseteq A_{\gamma}(R^n \times R^n) \) for \( \gamma > 2 \), and \( L_{\gamma}(R^n \times R^n) = L_{\gamma}(R^n \times R^n) \).

The purpose of this section is to study the \( L^p \) boundedness of the singular integral operator \( T_{\Omega, \phi, \psi}(f)(x, y) \) and the maximal operator \( M(f)(x, y) \) under weaker conditions, where \( M(f)(x, y) \) is defined, initially for \( f \in C_0^\infty(R^{n+1} \times R^{m+1}) \), by

\[ M(f)(x, y) = \sup_{h \in L_{\gamma}(R^n \times R^n)} \left| T_{\Omega, \phi, \psi}(f)(x, y) \right| . \]

The first result of this section is the following:

**Theorem 4.1.** Suppose that \( \Omega \in L^q(S^n \times S^m) \), \( q > 1 \) and satisfy the conditions (1.1)-(1.2) with \( \| \Omega \|_1 \leq 1 \). Assume that \( \phi, \psi, \mu, P_1, \) and \( P_2 \) are given as in Theorem 1.1. Then there exists a constant \( C_{p, q} > 0 \) such that

\[ \left\| M(f)(x, y) \right\|_p \leq C_{p, q} (1 + \mu)^{2/\gamma} \| f \|_p \]

(4.1)
for all \( \gamma' \leq p < \infty \) with \( 1 < \gamma \leq 2 \); and
\[
\left\| M_{p_1, p_2}^{P_1, P_2, \gamma} (f) \right\|_\infty \leq C \| f \|_\infty. \tag{4.2}
\]

**Proof.** It is clear that if \( \gamma = 2 \), then we have \( M_{p_1, p_2}^{P_1, P_2, \gamma} = M_{p_1, p_2}^{P_1, P_2} \). So, by Theorem 1.1, the inequality (4.1) holds for all \( p > 2 \). However, if \( \gamma = 1 \); we assume that \( h \in L^1(\mathbb{R}^n \times \mathbb{R}^m, \frac{dr dt}{rt}) \) and \( f \in L^\infty(\mathbb{R}^{n+1} \times \mathbb{R}^{m+1}) \). Then for all \( (\bar{x}, \bar{y}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{m+1} \), we have
\[
\left| \int\int_{\mathbb{R}^n \times \mathbb{R}^m} h(r, t) \mathcal{S}_{P_1, P_2, \phi, \psi, D}(f)(\bar{x}, \bar{y}, r, t) \frac{dr dt}{rt} \right| \leq C \| f \|_\infty \| h \|_1.
\]
Hence, by taking the supremum on both sides over all \( h \) with \( \| h \|_1 \leq 1 \), we reach
\[
\mathcal{M}_{p_1, p_2}^{P_1, P_2, \gamma}(f)(\bar{x}, \bar{y}) \leq C \| f \|_\infty
\]
for almost every where \( (\bar{x}, \bar{y}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{m+1} \), which leads to
\[
\left\| \mathcal{M}_{p_1, p_2}^{P_1, P_2, \gamma}(f) \right\|_\infty \leq C \| f \|_\infty \tag{4.3}
\]
Finally, if \( 1 < \gamma \leq 2 \). We follow a similar approach as in [15]. By duality, we get
\[
\mathcal{M}_{p_1, p_2}^{P_1, P_2, \gamma}(f)(\bar{x}, \bar{y}) = \left( \int\int_{\mathbb{R}^n \times \mathbb{R}^m} \left| \mathcal{S}_{P_1, P_2, \phi, \psi, D}(f)(\bar{x}, \bar{y}, r, t) \right|^\gamma \frac{dr dt}{rt} \right)^{1/\gamma'} \tag{4.4}
\]
which gives
\[
\left\| \mathcal{M}_{p_1, p_2}^{P_1, P_2, \gamma}(f) \right\|_p = \left\| \mathcal{S}_{P_1, P_2, \phi, \psi, D}(f) \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m, \frac{dr dt}{rt})} \tag{4.5}
\]
Therefore, by applying the interpolation theorem for the Lebesgue mixed normed spaces to the inequalities (1.3) and (4.3), we directly obtain
\[
\left\| \mathcal{M}_{p_1, p_2}^{P_1, P_2, \gamma}(f) \right\|_p \leq C_{p, q} (1 + \mu)^{2/\gamma'} \| f \|_p \tag{4.5}
\]
for \( \gamma' \leq p < \infty \) with \( 1 < \gamma \leq 2 \); and \( \left\| \mathcal{M}_{p_1, p_2}^{P_1, P_2, \gamma}(f) \right\|_\infty \leq C \| f \|_\infty \). This completes the proof. \( \square \)

It is worth mentioning that when \( \phi(t) = \psi(t) = t \) and \( P_1(u) = P_2(v) = 0 \), Al-Qassem and Pan in [8] extended the results of Theorem 4.1. In fact, they established the \( L^p \) boundedness of \( \mathcal{M}_{p_1, p_2}^{P_1, P_2, \gamma} \) provided that \( \Omega \in L(\log L)^{2/\gamma'}(\mathbb{R}^{n+1} \times \mathbb{R}^{m+1}) \) for \( \gamma' \leq p < \infty \) with \( 1 < \gamma \leq 2 \).

By the conclusion in Theorem 4.1 and applying an extrapolation argument (see [16, 19, 20]), we shall improve and extend the corresponding results in [4, 6, 8, 11, 13]. Precisely, we obtain the following:

**Theorem 4.2.** Suppose that \( P_1, P_2, \phi, \psi \) are given as in Theorem 1.1. Assume that \( \Omega \in L(\log L)^{2/\gamma'}(\mathbb{R}^{n+1} \times \mathbb{R}^{m+1}) \cup B^{0,2/\gamma'-1}(\mathbb{R}^{n+1} \times \mathbb{R}^{m+1}) \) with \( q > 1 \). Then \( \mathcal{M}_{\Omega, p_1, p_2}^{P_1, P_2, \gamma}(f) \) is bounded on \( L^p(\mathbb{R}^{n+1} \times \mathbb{R}^{m+1}) \) for \( \gamma' \leq p < \infty \) with \( 1 < \gamma \leq 2 \); and it is bounded on \( L^{\infty}(\mathbb{R}^{n+1} \times \mathbb{R}^{m+1}) \) for \( \gamma = 1 \).

**Proof.** The idea of proving Theorem 4.2 is taken form [17], which has its roots in [16] as well as in [19]. When \( \Omega \in L(\log L)^{2/\gamma'}(\mathbb{R}^{n+1} \times \mathbb{R}^{m+1}) \) with \( 1 < \gamma \leq 2 \) and \( \Omega \) satisfies the conditions (1.1)-(1.2), then \( \Omega \) can be decomposed as a sum of functions in \( L^2(\mathbb{R}^{n+1} \times \mathbb{R}^{m+1}) \) (see [21]). In fact, we have
\[
\Omega = \sum_{k = 0}^{\infty} \Omega_k, \tag{4.6}
\]
where

\[
\int_{\mathbb{S}^{n-1}} \Omega_k(x',)d\sigma(x') = \int_{\mathbb{S}^{n-1}} \Omega_k(\cdot, y')d\sigma(y') = 0,
\]

\[
\Omega_0 \in L^2(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}), \quad \|\Omega_k\|_{\infty} \leq C 2^k, \quad \|\Omega_k\|_1 \leq C,
\]

and

\[
\sum_{i=1}^{\infty} k^{2/\gamma'} \|\Omega_k\|_1 \leq C \|\Omega\|_{L(\log L)^{2/\gamma'}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})} \text{ for } k = 0, 1, 2, \ldots.
\]

Hence, it is easy to see that

\[
\mathcal{M}_{\Omega, \phi, \psi}^{(P_1, P_2, \gamma)}(f)(x, y) \leq \mathcal{M}_{\Omega, \phi, \psi}^{(P_1, P_2, \gamma)}(f)(x, y) + \sum_{k=1}^{\infty} \|\Omega_k\|_1 \mathcal{M}_{\tilde{\Omega}, \tilde{\phi}, \tilde{\psi}}^{(P_1, P_2, \gamma)}(f)(x, y)
\]

and

\[
(1 + \log^{2/\gamma'}(e + \|\Omega_k\|_{\infty})) \leq (1 + \log^{2/\gamma'}(e + C 2^k)) \leq C k^{2/\gamma'}.
\]

As \(\Omega_0 \in L^2(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})\), then by Theorem 4.1 we get

\[
\left\|\mathcal{M}_{\Omega_0, \phi, \psi}^{(P_1, P_2, \gamma)}(f)\right\|_p \leq C_p \left(1 + \log^{2/\gamma'}(e + \|\Omega_0\|_p)\right) \|f\|_p
\]

for \(\gamma' \leq p < \infty\). Therefore, by Minkowskii’s inequality and (4.7)-(4.9), we obtain that

\[
\left\|\mathcal{M}_{\Omega, \phi, \psi}^{(P_1, P_2, \gamma)}(f)\right\|_p \leq \left\|\mathcal{M}_{\Omega_0, \phi, \psi}^{(P_1, P_2, \gamma)}(f)\right\|_p + \sum_{k=1}^{\infty} \|\Omega_k\|_1 \left\|\mathcal{M}_{\tilde{\Omega}, \tilde{\phi}, \tilde{\psi}}^{(P_1, P_2, \gamma)}(f)\right\|_p
\]

\[
\leq C_p \left(1 + \sum_{k=1}^{\infty} \|\Omega_k\|_1 k^{2/\gamma'}\right) \|f\|_p
\]

\[
\leq C_p \|\Omega\|_{L(\log L)^{2/\gamma'}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})} \|f\|_p \leq C_p \|f\|_p.
\]

However, when \(\Omega \in B_q^{(0, 2/\gamma' - 1)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})\) with \(q > 1, 1 < \gamma \leq 2\) and \(\Omega\) satisfies the conditions (1.1)-(1.2), then \(\Omega\) can be written as

\[
\Omega = \sum_{\mu=1}^{\infty} c_{\mu} b_{\mu},
\]

where each \(c_{\mu}\) is a complex number, each \(b_{\mu}\) is a \(q\)-block supported in an interval \(I_{\mu}\) on \((\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})\) and

\[
M_q^{(0, 2/\gamma' - 1)}(\{c_{\mu}\}) = \sum_{\mu=1}^{\infty} |c_{\mu}| \left(1 + \log^{2/\gamma'}(|I_{\mu}|^{-1})\right) < \infty.
\]

For each \(\mu\), define the blocklike function \(\tilde{b}_{\mu}\) by

\[
\tilde{b}_{\mu}(x, y) = b_{\mu}(x, y) - \int_{\mathbb{S}^{n-1}} b_{\mu}(u, y)d\sigma(u) - \int_{\mathbb{S}^{n-1}} b_{\mu}(x, v)d\sigma(v) + \int_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} b_{\mu}(u, v)d\sigma(u)d\sigma(v).
\]

It is clear that each \(\tilde{b}_{\mu}(x, y)\) satisfies the following:

\[
\int_{\mathbb{S}^{n-1}} \tilde{b}_{\mu}(x, \cdot) d\sigma(u) = \int_{\mathbb{S}^{n-1}} \tilde{b}_{\mu}(\cdot, v) d\sigma(v) = 0,
\]

\[
\left\|\tilde{b}_{\mu}\right\|_q \leq C |I_{\mu}|^{-1/q'}, \quad \text{and} \quad \left\|\tilde{b}_{\mu}\right\|_1 \leq C.
\]
Without loss of generality, we may assume that \(|I_\mu| < 1\). Therefore, by Minkowski’s inequality, Theorem 4.1 and (4.10)-(4.13), we obtain that
\[
\left\| M_{\Omega,\phi\psi}^{p_1,p_2}(\gamma)(f) \right\|_p \leq \sum_{\mu=1}^{\infty} |c_\mu| \left\| M_{\Omega,\phi\psi}^{p_1,p_2}(\gamma)(f) \right\|_p \\
\leq C_{p,q} \sum_{\mu=1}^{\infty} |c_\mu| \left( 1 + \log^{2\gamma}(e + |I_\mu|^{-1}) \right) \|f\|_p \\
\leq C_{p,q} \|f\|_p
\]
for all \(p \geq \gamma'\).

We point out that under the assumptions \(\Omega\) belongs to the block space \(B_q^{0,1}(S^{n-1} \times S^{m-1}), h \in A_\gamma (R^+ \times R^+)\) for some \(q, \gamma > 1\), and when \(\phi, \psi\) are \(C^\infty([0,\infty))\), convex increasing functions with \(\phi(0) = \psi(0) = 0\), the author of [22] proved that for every \(p\) satisfying \(|1/p - 1/2| < \min \{1/2, 1/\gamma'\}\), there exists a constant \(C_p\) such that
\[
\|T_{\Omega,\phi\psi}^{P_1,P_2}(f)\|_p \leq C_p \|f\|_p
\]
for every \(f \in L^p(R^{n+1} \times R^{m+1})\). By this result, it is clear that the range of \(p\) is the full range \((1, \infty)\) whenever \(h \in \mathcal{L}^\gamma(R^+ \times R^+)\) with \(\gamma \geq 2\). But what is about the \(L^p\) boundedness of \(T_{\Omega,\phi\psi}^{P_1,P_2}\) when \(h \in \mathcal{L}^\gamma(R^+ \times R^+)\) for \(1 < \gamma < 2\) ? We shall obtain an answer to this question in the affirmative as described in the following theorem.

**Theorem 4.3.** Assume that \(\Omega \in L(\log L)^{(2/\gamma')} (S^{n-1} \times S^{m-1}) \cup B_q^{0,2/\gamma'-1}(S^{n-1} \times S^{m-1}), q > 1\), and satisfying the conditions (1.1)-(1.2). Let \(h \in \mathcal{L}^\gamma(R^+ \times R^+)\) for some \(1 < \gamma < 2\), and let \(\phi, \psi\) be given as in Theorem 1.1. Then the singular integral operator \(T_{\Omega,\phi\psi}^{P_1,P_2}(f)(x, y)\) is bounded on \(L^p(R^{n+1} \times R^{m+1})\) for all \(1 < p < \infty\).

**Proof.** As a direct consequence of Theorem 4.2 and the statement that
\[
\left| T_{\Omega,\phi\psi}^{P_1,P_2}(f)(x, y) \right| \leq \|h\|_{\mathcal{L}^\gamma(R^+ \times R^+, \frac{d\mu}{d\gamma})} \gamma_{\Omega,\phi\psi}^{P_1,P_2}(\gamma)(f)(x, y),
\]
we achieve that \(T_{\Omega,\phi\psi}^{P_1,P_2}\) is bounded on \(L^p(R^{n+1} \times R^{m+1})\) for \(\gamma' < p < \infty\) with \(1 < \gamma < 2\). Moreover, by a standard duality argument, we can show that \(T_{\Omega,\phi\psi}^{P_1,P_2}\) is bounded on \(L^p\) for \(1 < p < \gamma'\) with \(1 < \gamma < 2\). So, if \(\gamma = 2\), then we are done. However, if \(1 < \gamma < 2\), then we apply the real interpolation theorem to acquire the \(L^p\) boundedness of \(T_{\Omega,\phi\psi}^{P_1,P_2}\) for \((\gamma < p < \gamma')\). This completes the proof.

**Acknowledgement:** The authors would like to thank the referees for their valuable comments and suggestions.

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