An Algebraic Perspective on Multivariate Tight Wavelet Frames with Rational Masks

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Abstract

The ideas of sums of squares representations for polynomials and trigonometric polynomials are classical, but new results and applications for them continue to be discovered. Recently, new tight wavelet frame constructions have been found that use sums of squares representations for certain nonnegative trigonometric polynomials to obtain wavelet masks generating a tight wavelet frame for $L^2(\mathbb{R}^n)$. While there has been some effort to extend these results to obtain tight wavelet frames with higher vanishing moments, which have better approximation properties, this has been done under restrictive assumptions. In this paper, we generalize the previous constructions using sums of squares to allow for the inclusion of a vanishing moment recovery function. This results in rational wavelet masks generating tight wavelet frames for $L^2(\mathbb{R}^n)$, with lower bounds on their numbers of vanishing moments. In particular, our construction may be used to create tight wavelet frames with better vanishing moments in a wide variety of new settings. Our construction is made possible by a new theoretical result, which states that nonnegative multivariate trigonometric polynomials always have representations as sums of squares of rational functions. This theorem settles a long-standing question in the area of sums of squares representations for trigonometric polynomials.

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1. Introduction

A sum of squares (sos) representation for a nonnegative polynomial $f$ is a collection $\{g_j\}_{j=1}^J$ of polynomials such that $f(x) = \sum_{j=1}^J g_j(x)^2$ for all $x \in \mathbb{R}^n$, where $n \geq 1$ is the spatial dimension. Similar representations are possible in a variety of settings, such as trigonometric polynomials (see \cite{9} and references therein), matrices with polynomial or trigonometric polynomial entries \cite{11, 13, 19}, and several others (see \cite{3} for a survey of these). While research in this area has focused on the case where the $g_j$ are polynomials, when they are allowed to be quotients of two polynomials, we call this a sum of rational squares (sors) representation. It was known to Hilbert that not all polynomials have sos representations, and his famous 17th problem asks whether they all have sors representations. This was later proved to be true by Artin \cite{1}.

In the case of trigonometric polynomials, the situation has been less clear. In one variable, all nonnegative trigonometric polynomials have sos representations with a single generator, a result known as the Fejér-Riesz lemma (see \cite{6}). In two variables, it has been shown that all nonnegative trigonometric polynomials have sors representations \cite{2}, and later, that they all have sos representations \cite{20} (see also \cite{4} Theorem 2.4). Efforts have been made to extend the Fejér-Riesz lemma to the case of two dimensions for stable polynomials,

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i.e., those which have no zeroes in the closed unit bidisk [10]. In the case of three or more variables, it is known that there are nonnegative trigonometric polynomials with no sos representation [4, Theorem 2.5], but also that positive trigonometric polynomials have sos representations [8]. On the other hand, the existence of sos representations in this setting has not been confirmed, which is a break between the polynomial and trigonometric polynomial settings. One of the main results of this paper settles this question: In Theorem [1], we prove that a rational trigonometric polynomial in n variables, which is nonnegative everywhere it is defined, is a sum of hermitian squares of at most 2^n rational trigonometric polynomials.

Meanwhile, in the area of tight wavelet frame construction, recent work [4, 16] has shown how sos representations for certain nonnegative trigonometric polynomials may be used to create highpass masks generating a tight wavelet frame for L^2(R^n). This construction makes use of the unitary extension principle (UEP) conditions on a collection of trigonometric polynomials, which are sufficient for the highpass masks to generate a tight wavelet frame [7, 12]. In this setting, we call trigonometric polynomials masks, which are lowpass when equal to one at ω = 0, and are highpass or wavelet when they are equal to zero there. Considering the case of dyadic dilation for now, when we are given a lowpass mask τ and collection of highpass masks \( \{q_\ell\}_{\ell=1}^r \), we say that these satisfy the UEP conditions when

\[
\tau(\omega)\tau(\omega + \gamma) + \sum_{\ell=1}^r q_\ell(\omega)q_\ell(\omega + \gamma) = \begin{cases} 
1 & \text{if } \gamma = 0, \\
0 & \text{if } \gamma \in \{0, \pi\}^n \setminus \{0\},
\end{cases}
\]

for all \( \omega \in \mathbb{T}^n := [-\pi, \pi]^n \). These conditions necessitate that \( f(\tau; \omega) = 1 - \sum_{\gamma \in \{0, \pi\}^n} |\tau(\omega + \gamma)|^2 \geq 0 \) for all \( \omega \in \mathbb{T}^n \), which is called the sub-QMF condition, and when equality holds for all \( \omega \in \mathbb{T}^n \), then \( \tau \) is said to satisfy the QMF condition.

In fact, the UEP conditions imply that \( f(\tau; \cdot) \) is a sum of squares, and the work in [4, 16] shows the converse. When \( f(\tau; \cdot) \) has a sum of hermitian squares representation \( \sum_{j=1}^J |g_j(2\omega)|^2 \) with trigonometric polynomials \( g_j, 1 \leq j \leq J \), they construct highpass masks satisfying the UEP conditions with \( \tau \). Their construction proceeds as follows [4]. Construct the column vectors \( H(\omega) = [\tau(\omega + \gamma)]_{\gamma \in \{0, \pi\}^n} \) and \( G(\omega) = [g_j(\omega)]_{j=1}^J \), and the Fourier transform matrix \( X(\omega) = 2^{-n/2}[e^{i(\omega + \gamma)v}]_{\gamma \in \{0, \pi\}^n, v \in \{0, 1\}^n} \). Then using block matrix notation, we have

\[
|H(\omega)\ H(\omega)G(2\omega)^* (I - H(\omega)H(\omega)^*)X(\omega)| = I,
\]

which are just another way of writing the UEP conditions, with the highpass masks \( q_{1,j}(\omega) = g_j(2\omega)\tau(\omega), 1 \leq j \leq J \), and \( q_{2,\nu}(\omega) = 2^{-n/2}[e^{i(\omega + \gamma)v} - \tau(\omega)\sum_{\gamma \in \{0, \pi\}^n} \tau(\omega + \gamma)e^{i(\omega + \gamma)v}, \nu \in \{0, 1\}^n \), which we can read off from the first row of the left-hand matrix. This means that the wavelet system generated by \( (\tau, \{q_{1,j}\}, \{q_{2,\nu}\}) \) is a tight wavelet frame.

Rewriting this matrix product and using the relationship \( G(2\omega)^*G(2\omega) = 1 - H(\omega)^*H(\omega) \), we see that this could be written as

\[
|H(\omega)(I - H(\omega)H(\omega)^*)X(\omega)| = \begin{bmatrix} 1 + (1 - H(\omega)^*H(\omega)) & 0 \\
0 & I \end{bmatrix} \begin{bmatrix} H(\omega)^* \\
X(\omega)^*(I - H(\omega)H(\omega)^*) \end{bmatrix}.
\]

In [15], this is interpreted as a scaling of the Laplacian pyramid matrix \( |H(\omega)(I - H(\omega)H(\omega)^*)X(\omega)| \), and it was shown that this scaling matrix is the unique diagonal matrix which makes this product equal to the identity. There, the scaling matrix was factored under the assumption that \( 2 - H(\omega)^*H(\omega) = \|g(2\omega)\|^2 \) for some trigonometric polynomial \( g \), giving rise to a modified lowpass mask \( g^2(\cdot) \), but the above construction instead assumes that \( 2 - H(\omega)^*H(\omega) \) factorizes as \( 1 + \sum_{j=1}^J |g_j(2\omega)|^2 \). Combining these ideas, if there is a

\[\text{In [4], they consider a construction with a general dilation matrix, but for the purposes of explaining the construction here, we continue to assume the special case of dyadic dilation.}\]
sum of squares representation for \(2 - H(\omega)^*H(\omega)\) as \(\sum_{j=0}^J |g_j(2\omega)|^2\) where \(g_0(0) = 1\), then modifying the lowpass mask to be \(g_0(2)\tau(\cdot)\) and constructing the highpass masks \(q_{1,j}\) as above leads to a tight wavelet frame. Moreover, these constructions which modify the lowpass mask do not require the original lowpass mask to satisfy the sub-QMF condition, but after modifying, the new lowpass mask will satisfy this condition. We give more details about such constructions in a more general context in Section 5.

One downside to these constructions is that they rely on the UEP, which may result in highpass masks having suboptimal vanishing moments. A highpass mask’s number of vanishing moments is just the order of its vanishing at zero, but is related to approximation rates for the corresponding wavelet system [7]. A method for correcting this introduces a vanishing moment recovery (vmr) function, and uses highpass masks \(\{q_\ell\}_{\ell=1}^\infty\) satisfying the oblique extension principle (OEP) conditions with the lowpass mask \(\tau\) and vmr function \(S\) to generate a tight wavelet frame [7]:

\[
S\infty\tau(\omega)\tau(\omega + \gamma) + \sum_{\ell=1}^\infty q_\ell(\omega)q_\ell(\omega + \gamma) = \begin{cases} S(\omega) & \text{if } \gamma = 0, \\ 0 & \text{if } \gamma \in \{0, \pi\}^n \setminus \{0\}, \end{cases}
\]

where \(S\infty\tau\) is a box spline which do not satisfy the sub-OEP condition, but after modifying, the new lowpass mask will satisfy this condition.

As before, when equality holds for all \(\omega \in \mathbb{T}^n\), this is the oblique QMF condition. We will show that this is in fact equivalent to the existence of wavelet masks satisfying the UEP conditions for the given \(S\) and \(\tau\), under the condition that \(S\) is a rational trigonometric polynomial. Moreover, if we allow rational trigonometric polynomial highpass masks, we are able to eliminate the assumption about the existence of a sum of squares representation for \(f(S, \tau; \cdot)\), because we have proved that nonnegative rational trigonometric polynomials have sors representations in Theorem 1. This stands in contrast to the UEP constructions considered above, in particular Theorem 2.5, they discuss a lowpass mask \(\tau\) for which there is no sos representation for \(f(\tau; \cdot)\), meaning that the construction there fails.

In Examples 2 and 3, we consider this same lowpass mask, showing that \(f(\tau; \cdot)\) has an sors representation, and then finding rational trigonometric polynomial highpass masks satisfying the UEP conditions with \(\tau\), which generate a tight wavelet frame. One way of interpreting this is that when \(f(\tau; \cdot)\) is nonnegative, but fails to have an sos representation, then we are still able to show that there is a tight wavelet frame based in the multiresolution analysis generated by the refinable function \(\phi\) associated with \(\tau\).

In Section 5, we show that we may once again interpret these results in terms of scaling an “oblique Laplacian pyramid” matrix, clearly demonstrating how different assumptions on the factorization of \(S(2\omega) + S(2\omega)^2f(S, \tau; \omega)\) lead to constructions with just a modified lowpass mask, as in [15]: the original lowpass mask, and a collection of highpass masks corresponding to these sums of squares generators, as in [4][16]; or a combination of the two, as discussed above. Moreover, this process of modifying the lowpass mask turns lowpass masks which do not satisfy the oblique sub-QMF condition with the given \(S\) into new lowpass masks which do satisfy this condition.

These new theorems allow us to construct tight wavelet frames with better or maximum vanishing moments in a wide variety of new settings. We focus on the case of box spline lowpass masks, and demonstrate that if a number of univariate trigonometric polynomials equal to the number of distinct directions in the box spline can be constructed satisfying certain properties, we can find a vanishing moment recovery function leading to a tight wavelet frame with nearly maximum vanishing moments. The reason why one might consider doing this is that it allows for a much simpler form of \(S\) than ones that achieve maximum vanishing moments, as discussed in [16]. The ability to trade off different criteria in this way while still obtaining a tight wavelet frame is another benefit of the flexibility afforded by our result. Our construction in this case...
is similar in some ways to the one in [3], and when the necessary univariate trigonometric polynomials exist,
we have the same number of highpass masks as appear there. However, their method relies on the UEP, and
results in tight wavelet frames with one vanishing moment.

1.1. Contributions and Organization of the Paper

We begin by describing the major contributions of the paper, and then describe the overall organization.

- In Theorem 1, we prove that a rational trigonometric polynomial in \( n \) variables, which is nonnegative
everywhere it is defined, is a sum of hermitian squares of at most \( 2^n \) rational trigonometric polynomials.

We note that this first theorem is of independent interest beyond the setting of wavelet construction.

- In Theorem 2, for a lowpass mask \( \tau \) and rational trigonometric polynomial \( \text{vmr} \) function \( S \), we establish
the equivalence between the oblique sub-QMF condition and the existence of rational trigonometric
polynomial highpass masks satisfying the OEP conditions with this \( S \) and \( \tau \).

- In Theorem 3, we show that under the additional assumptions that \( S \) is continuous at 0 with \( S(0) = 1 \), and
that \( S \) and \( 1/S \) belong to \( L^\infty(\mathbb{T}^n) \), the wavelet system generated by the rational trigonometric
polynomial highpass masks in Theorem 2 is a tight wavelet frame.

In Section 2, we review the definitions and some properties of the spaces of polynomials and rational poly-
nomials with different coefficient fields, and relate these to the corresponding spaces of rational trigonometric
polynomials. We also review sums of squares representations and some related literature.

In Section 3, we state and prove Theorem 1, along with several necessary lemmata for establishing this
result. We end with two examples where we find sos representations, one of which is known not to have an
sos representation.

In Section 4, after reviewing the oblique extension principle, we prove Theorems 2 and 3. Then we extend
parts of these results in Propositions 2 and 3 to the setting where \( S \) is not assumed to be a rational trigono-
metric polynomial. Under some assumptions on \( S \), when \( \tau \) and \( q_j \) are rational trigonometric polynomials
satisfying the OEP conditions, \( \tau \) and \( S \) satisfy the oblique sub-QMF condition on a suitably large set, and
this allows us to show that the resulting wavelet system is still a tight wavelet frame.

In Section 5, we discuss the scaling matrix interpretation of our OEP construction, and show how different
factorizations of the scaling matrix lead to different tight wavelet frame constructions, as we did in the
introduction for the UEP setting.

In Section 6, we apply the constructions of the previous section in the case that the lowpass mask
corresponds to a box spline refinable function, obtaining tight wavelet frames with maximum vanishing
moments for any such lowpass mask, provided that certain univariate trigonometric polynomials may be
constructed.

Finally, in the Appendix, we present the proof of a technical lemma, Lemma 7.

2. Preliminaries

We denote by \( \mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_n] \) the ring of polynomials in the variables \( z_1, \ldots, z_n \) with coefficients in
\( \mathbb{C} \), and similarly for \( \mathbb{R}[z] \). We denote by \( \mathbb{C}(z) = \mathbb{C}(z_1, \ldots, z_n) \), or the field of fractions of \( \mathbb{C}[z] \), which is the
space of rational polynomials \( f = p/q \) where \( p, q \in \mathbb{C}[z] \) and \( q \neq 0 \), and similarly for \( \mathbb{R}(z) \). We will also
refer to Laurent polynomials, which are the elements of \( \mathbb{C}(z) \) of the form \( z_1^{k_1} \cdots z_n^{k_n} p \), where \( p \in \mathbb{C}[z] \) and
\( k_1, \ldots, k_n \in \mathbb{Z} \). We will denote the set of Laurent polynomials as \( \mathbb{C}[z^{\pm 1}] \), or \( \mathbb{R}[z^{\pm 1}] \) when the coefficient field
is \( \mathbb{R} \). The next remark discusses and clarifies some of these spaces.

Remark 1. For the sake of clarity, we point out that the space of rational trigonometric polynomials is
equivalent to the restriction of functions in \( \mathbb{C}(z) \) to evaluation at \( z \in (\partial \mathbb{D})^n \), where \( \mathbb{D} \) is the complex unit
disk. That is, \( (\partial \mathbb{D})^n = \{ (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| = 1 \text{ for all } j \} \). This is also equivalent to the same restric-
tion for functions in the field of quotients of Laurent polynomials with complex coefficients. In either
case, the trigonometric polynomial $g$ associated with the polynomial or Laurent polynomial $f$ is defined by

\[ g(\omega) = f(e^{i\omega_1}, \ldots, e^{i\omega_n}) \]

for all $\omega \in \mathbb{T}^n := [-\pi, \pi]^n$, and similarly for a rational trigonometric polynomial $g$. In particular, a trigonometric polynomial is any function which is a finite sum of the form: $\sum_{i=1}^{n} c_k e^{-ik \omega}$, where $k$ ranges over some finite subset of $\mathbb{Z}^n$, and $k \cdot \omega$ denotes the ordinary Euclidean inner product in $\mathbb{R}^n$.

Recall that irreducible polynomials (see, e.g., [17]) $p, q \in \mathbb{C}[z]$ are called associated when $p = wq$, for $w \in \mathbb{C} \setminus \{0\}$. For any $p \in \mathbb{C}[z] \setminus \{0\}$, $p = up_1 \cdots p_r$, where $u \in \mathbb{C} \setminus \{0\}$, and $p_1, \ldots, p_r$ are irreducible polynomials. Moreover, this representation is unique in the sense that if it also holds that $p = wq_1 \cdots q_s$, with $w \in \mathbb{C}$ and $q_1, \ldots, q_s$ irreducible polynomials, then $s = r$, and for all $1 \leq i \leq r$, $q_i$ is associated with some $p_j$, $1 \leq j \leq r$. This means that if $f \in \mathbb{C}(z) \setminus \{0\}$, it has a representation as $\frac{u p_1 \cdots p_r}{v q_1 \cdots q_s}$, where $u \in \mathbb{C} \setminus \{0\}$, and $p_i, 1 \leq i \leq r$, $q_j, 1 \leq j \leq s$ are irreducible polynomials, such that for all $i$ and $j$, it is not the case that $p_i$ and $q_j$ are associated. Let $p$ be the numerator of the previous expression, and let $q$ be the denominator. So if $f = v/t$, where $v, t \in \mathbb{C}[z]$, since $p/q = f = v/t$, clearly $pt = vq$. Since every $p_i$ must divide $vq$, but $p_i$ is not associated with any $q_j$, $p_i | v$ for all $1 \leq i \leq r$. Similarly, $q_j | t$ for all $1 \leq j \leq s$, and any further irreducible factors of either side of this equation are common to both $v$ and $t$ (up to association).

Thus, if $f = p/q$ is such that $p$ and $q$ are as in the previous paragraph, and $f = v/t$ is another representation of $f$, then there exists some polynomial $m$ such that $v = pm$, $t = qm$. We will say that $f = p/q$ is in lowest terms when this holds.

In the sequel, we will use $x$ to denote an element of $\mathbb{R}^n$ and $z$ to denote an element of $\mathbb{C}^n$ (which may also happen to be real). This will be useful when we consider changing domains between $(\partial \mathbb{D})^n$ and $\mathbb{R}^n$. When we discuss the set of points where $f \in \mathbb{C}(z)$ is defined, we mean the set of points $z \in \mathbb{C}^n$ for which $q(z) \neq 0$, where $f = p/q$ is in lowest terms (noting that this set is invariant under association of the irreducible factors of $q$).

**Definition 1.** We say that a polynomial or Laurent polynomial $f \in \mathbb{C}[z^{\pm 1}]$ has a sum of hermitian squares representation (or sos representation) on $\Omega \subseteq \mathbb{C}^n$ with $\{g_j\}_{j=1}^J \subset \mathbb{C}[z]$, $J < +\infty$, when

\[ f(z) = \sum_{j=1}^J |g_j(z)|^2 \quad \forall z \in \Omega. \quad (1) \]

We may abbreviate this when the set $\Omega$ is clear (we will typically consider $\mathbb{R}^n$ or $(\partial \mathbb{D})^n$), or just say that $f$ has an sos representation or is an sos on $\Omega$ if there is some finite collection $\{g_j\}_{j=1}^J \subset \mathbb{C}[z]$ for which this equation holds for all $z \in \Omega$.

Similarly, we say that $f \in \mathbb{C}(z)$ has a sum of rational hermitian squares representation (or sors representation) on $\Omega \subseteq \mathbb{C}^n$ with $\{g_j\}_{j=1}^J \subset \mathbb{C}(z)$, if Equation (1) holds for all $z \in \Omega$ at which $f(z)$ is defined.

Using the identifications described in Remark [1], we may also apply these definitions in the case that $f$ and the $g_j$ are trigonometric polynomials, or rational trigonometric polynomials, where in this case, the natural domain to consider will be $\Omega = \mathbb{T}^n = [-\pi, \pi]^n$.

We now cite several related results from the literature, many of which we will be using in the sequel. This first collection of results concern trigonometric polynomials.

**Result 1.** (a) (Fejér-Riesz [2]) Let $f \in \mathbb{C}[z]^{\pm 1}$ be such that $f(z) \geq 0$ for all $z \in \partial \mathbb{D}$. Then $f(z) = |p(z)|^2$ for all $z \in \partial \mathbb{D}$, where $p \in \mathbb{C}[z_1]$.

(b) (Scheiderer [27]) Let $f \in \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}]$ be such that $f(z) \geq 0$ for all $z \in (\partial \mathbb{D})^2$. Then $f$ has an sos representation on $(\partial \mathbb{D})^2$.

(c) (Charina et al. [3]) Let $n \geq 3$. There is a nonnegative trigonometric polynomial in $n$ variables with no sos representation.

(d) (Dritschel [8]) Let $n \geq 2$, and $f \in \mathbb{C}[z^{\pm 1}]$ be such that $f(z) > 0$ for all $z \in (\partial \mathbb{D})^n$. Then $f$ has an sos representation on $(\partial \mathbb{D})^n$. 


Proof. (a) is proved for a special case in [6]. (b) may be found as Corollary 3.4 in [20] (see also [4, Theorem 2.4]). (c) is shown in [4]. (d) comes from [8], but the statement here comes from [9].

Moving away from the trigonometric polynomial case to that of ordinary polynomials with real coefficients, we have the following results, the first of which comes from Artin in 1927 [11] (see also [3]), but translated and adapted to our notation. The second comes from Pfister in 1967 [18] (see also [3]), and gives a bound on the number of squares in the representation guaranteed by Artin’s Theorem. We will use these theorems and the following corollary in the proof of our main theorem.

**Result 2.** (a) (Artin [11]) Let \( f \in \mathbb{R}[x] \) be such that \( f(x) \geq 0 \) for all \( x \in \mathbb{R}^n \). Then \( f \) is an sors on \( \mathbb{R}^n \) of functions in \( \mathbb{R}(x) \), i.e., there exists \( \{g_j\}_{j=1}^J \subset \mathbb{R}(x), J < +\infty \) such that for all \( x \in \mathbb{R}^n \), \( f(x) = \sum_{j=1}^J g_j(x)^2 \).

(b) (Pfister [18]) Let \( f \in \mathbb{R}[x] \) be such that \( f(x) \geq 0 \) for all \( x \in \mathbb{R}^n \). Then \( f \) is an sors on \( \mathbb{R}^n \) of at most \( 2^n \) functions in \( \mathbb{R}(x) \), i.e., there exist \( \{g_j\}_{j=1}^J \) satisfying the conclusion of part (a) with \( J \leq 2^n \).

The following corollary combines parts (a) and (b) of the result above and extends to the case where \( f \in \mathbb{R}(x) \). The respective parts of this result were known to Artin and Pfister, but we show the proof because we will use this technique later.

**Corollary 1.** Let \( f \in \mathbb{R}(x) \), \( f(x) \geq 0 \) for all \( x \in \mathbb{R}^n \) at which it is defined. Then \( f \) is an sors on \( \mathbb{R}^n \) of at most \( 2^n \) functions in \( \mathbb{R}(x) \).

Proof. Let \( f = p/q \) be in lowest terms. Then \( f = pq/q^2 \), and since \( q^2(x) \geq 0 \) for all \( x \in \mathbb{R}^n \), \( (pq)(x) = f(x)q^2(x) \geq 0 \) for all \( x \in \mathbb{R}^n \). Then by Result 2(a) and (b), \( pq = \sum_{j=1}^J g_j^2 \), where \( g_j \in \mathbb{R}(x) \) for all \( 1 \leq j \leq J \leq 2^n \), and thus \( f = \sum_{j=1}^J (g_j/q)^2 \), which is an sors representation with at most \( 2^n \) squares.

3. Sors Representations for Nonnegative Rational Polynomials on \((\partial \mathbb{D})^n\)

In this section, our goal is to prove one of our main theorems, which says that a nonnegative multivariate trigonometric polynomial has an sors representation.

The proof of this theorem will proceed in three major steps. First, we will define a particular Möbius transformation \( \mu \in \mathbb{C}(z) \) which maps \( \partial \mathbb{D} \) to \( \mathbb{R} \), and which has an inverse also in \( \mathbb{C}(z) \). We use this map \( \mu \) to define the induced map \( M : \mathbb{C}(z) \to \mathbb{C}(z) \) by \( M(f)(z) = f(\mu(z_1), \ldots, \mu(z_n)) \), along with its inverse, and we then prove several properties of \( \mu \) and the induced map, as well as their inverses. Finally, we will show that for \( f \in \mathbb{C}[z^{\pm 1}] \) satisfying the assumptions of the theorem, \( M(f) \) belongs to \( \mathbb{R}(z) \) and is nonnegative on \( \mathbb{R}^n \), so that it has an sors representation of functions in \( \mathbb{R}(z) \) by Corollary 1. We will then apply the inverse of \( M \) to each of these generators, and show that this leads to an sors representation of \( f \) on \((\partial \mathbb{D})^n\).

**Theorem 1.** Let \( f \in \mathbb{C}(z) \) be such that \( f(z) \geq 0 \) for all \( z \in (\partial \mathbb{D})^n \) at which it is defined. Then \( f \) is an sors on \((\partial \mathbb{D})^n\) of at most \( 2^n \) functions in \( \mathbb{C}(z) \), i.e., there exist \( \{g_j\}_{j=1}^J \subset \mathbb{C}(z) \) with \( J \leq 2^n \) such that

\[
    f(z) = \sum_{j=1}^J |g_j(z)|^2, \quad \text{for all } z \in (\partial \mathbb{D})^n \text{ where } f(z) \text{ is defined.}
\]

In the proof of this theorem, we will use the standard multiindex notation \( z^\alpha = z_1^{\alpha_1}z_2^{\alpha_2} \cdots z_n^{\alpha_n} \), where \( \alpha \in \mathbb{Z}^n \). For multiindices \( \alpha, \beta \), we will write \( \alpha \geq \beta \) to denote the lexicographical order, and similarly for \( \alpha > \beta \). We will write \( \alpha \geq \beta i \) to say that \( \alpha_i \geq \beta_i \) for all \( 1 \leq i \leq n \) (the “entrywise” comparison).

When \( \alpha \geq 0 \), we write \( |\alpha| = \sum_{j=1}^n \alpha_j \), and for \( \alpha \geq 0 \), \( \beta \geq 0 \), we let \( (\alpha) = \prod_{j=1}^n (\alpha_j) \). For \( a, b \in \mathbb{C}^n \) and \( \alpha \geq 0 \), this gives the formula

\[
    (a + b)^\alpha = \sum_{0 \leq k \leq \alpha} \binom{\alpha}{k} a^k b^{\alpha - k}, \quad (2)
\]
which is the result of applying the binomial theorem to each factor \((a_j + b_j)^{\alpha_j}\) and expanding.

When we write the expanded form of a polynomial or Laurent polynomial \(f \in \mathbb{C}[z^{\pm 1}]\), we will frequently use the shorthand notations \(f(z) = \sum_{\alpha} f_{\alpha} z^{\alpha}\), \(\sum_{\alpha > 0} f_{\alpha} z^{\alpha}\), or \(\sum_{\alpha \geq 0} f_{\alpha} z^{\alpha}\), which in each case refers to a sum over an appropriate finite subset of \(\mathbb{Z}^n\).

We introduce a notation for the rational polynomial obtained by inverting each variable below, since this is an operation we will consider several times in what follows.

**Definition 2.** Given \(f \in \mathbb{C}(z)\), we define \(\tilde{f}(z_1, \ldots, z_n) := f(z_1^{-1}, \ldots, z_n^{-1})\), and denote by \(\mathbb{R}(z)_{\sim}\) the subfield of \(\mathbb{R}(z)\) of all elements \(f\) with \(\tilde{f} = f\).

For \(f \in \mathbb{C}(z)\), it is clear that \(\tilde{f}(z) = f(z)\) for all \(z \in (\partial \mathbb{D})^n\). When the coefficients are real, this gives \(\tilde{f}(z) = f(z)\) for all \(z \in (\partial \mathbb{D})^n\). Thus, for \(f \in \mathbb{R}(z)\), \(f \in \mathbb{R}(z)_{\sim}\) if and only if \(f\) is real-valued on \((\partial \mathbb{D})^n\). In the next lemma, we give an alternative characterization for \(\mathbb{R}(z)_{\sim}\).

**Lemma 1.** If \(f \in \mathbb{R}(z)\), \(f = p/q\), where \(p, q \in \mathbb{R}[z]\), \(q \neq 0\), then \(f \in \mathbb{R}(z)_{\sim}\) if and only if \(pq \in \mathbb{R}(z)_{\sim}\). If \(f \in \mathbb{R}[z^{\pm 1}]\), \(f = \sum_{\alpha \in \mathbb{Z}^n} f_{\alpha} z^{\alpha}\), then \(f \in \mathbb{R}(z)_{\sim}\) if and only if \(f_{\alpha} = f_{-\alpha}\) for all \(\alpha \in \mathbb{Z}^n\).

**Proof.** In the first case, \(f = (pq)/(qq)\), and since \(qq \in \mathbb{R}(z)_{\sim}\), \(f = \tilde{f}\) if and only if \(pq = (\tilde{q}q)\). In the second case, \(\tilde{f} = \sum_{\alpha} f_{\alpha} z^{-\alpha} = \sum_{\alpha} f_{-\alpha} z^{\alpha}\), so \(f - \tilde{f} = \sum_{\alpha} (f_{\alpha} - f_{-\alpha}) z^{\alpha}\) equals 0 if and only if \(f_{\alpha} = f_{-\alpha}\) for all \(\alpha \in \mathbb{Z}^n\).

The following lemma tells us more about the sets on which a nonzero polynomial or rational polynomial can vanish.

**Lemma 2.** Let \(S_1, \ldots, S_n \subseteq \mathbb{C}\) such that for each \(1 \leq j \leq n\), the cardinality of \(S_j\) is infinite. If \(f \in \mathbb{C}[z]\) is such that \(f(z) = 0\) for all \(z \in S_1 \times \cdots \times S_n\), \(f = 0\). If \(f \in \mathbb{C}(z)\) is such that \(f(z)\) is defined and equal to 0 for all \(z \in S_1 \times \cdots \times S_n\), \(f = 0\).

**Proof.** The first case is just [17] Ch. IV.1, Corollary 1.6. If \(f = p/q \in \mathbb{C}(z)\) is in lowest terms, then the assumptions on \(f\) imply that \(p(z) = 0\) for all \(z \in S_1 \times \cdots \times S_n\), so \(p = 0\), and thus \(f = 0\).

**Definition 3.** We define the forward M"obius transformation \(\mu \in \mathbb{C}(z_1)\) by

\[
\mu(z_1) = \frac{z_1 + i}{i(z_1 - i)} = \frac{2 - i(z_1 - z_1^{-1})}{z_1 + z_1^{-1}}.
\]

We call \(\tilde{\mu}\) the reverse M"obius transformation, since it is the inverse of \(\mu\) (see Lemma [3a]). For \(f \in \mathbb{C}(z)\), let \(M(f)(z) = f(\mu(z_1), \ldots, \mu(z_n))\), which we call the induced forward M"obius transformation, and \(\tilde{M}(f)(z) = f(\tilde{\mu}(z_1), \ldots, \tilde{\mu}(z_n))\) the induced reverse M"obius transformation.

We note that similar maps are used in [5] and elsewhere, but the map \(\mu\) that we have chosen has several additional symmetries which are important for proving our results. In the next lemma, we describe some of the properties of \(\mu\) and \(\tilde{\mu}\), showing in particular that they are inverses of each other.

**Lemma 3.** Let \(\mu\) be the forward M"obius transformation.

(a) \(\mu\) maps \(\mathbb{C} \setminus \{i\}\) continuously and bijectively to \(\mathbb{C} \setminus \{-i\}\) with continuous inverse \(\tilde{\mu}\) on this set. \(\mu\) has a simple pole at \(i\), and \(\tilde{\mu}\) has a simple pole at \(-i\).

(b) \(\mu\tilde{\mu} = 1\).

(c) For all \(z \in \mathbb{C} \setminus \{i\}\), \(\overline{\mu(z)} = \tilde{\mu}(\bar{z})\).

(d) For \(x \in \mathbb{R}\), \(\mu(x), \tilde{\mu}(x) \in \partial \mathbb{D}\).

(e) For \(z \in \partial \mathbb{D} \setminus \{i\}\), \(\mu(z) \in \mathbb{R}\). For \(z \in \partial \mathbb{D} \setminus \{-i\}\), \(\tilde{\mu}(z) \in \mathbb{R}\).
Proof. For \(z_1 \in \mathbb{C}\), using the definition of \(\tilde{\gamma}\), \(\tilde{\mu}(z_1) = \frac{z_1^{-1} + i}{i(z_1^{-1} - i)} = \frac{1 + iz_1}{i(1 - iz_1)} = \frac{-z_1^{-1}i}{z_1^{-1} - i} = \frac{2 + i(z_1 - z_1^{-1})}{z_1 + z_1^{-1}}\).

(a) We have
\[
\mu(\tilde{\mu}(z_1)) = \frac{\tilde{\mu}(z_1) + i}{i(\tilde{\mu}(z_1) - i)} = \frac{z_1 - i + i(-i)(z_1 + i)}{i(z_1 - i - i(-i)(z_1 + i))} = \frac{2z_1}{2} = z_1,
\]
where the first formula is valid for evaluation whenever \(z \in \mathbb{C} \setminus \{-i\}\) and the second is valid whenever \(z \in \mathbb{C} \setminus \{i\}\), but it is clear that the singularity is removable in either case. This proves that the image of \(\mu\) is \(\mathbb{C} \setminus \{i\}\), and the image of \(\tilde{\mu}\) is \(\mathbb{C} \setminus \{i\}\), so that \(\mu\) maps \(\mathbb{C} \setminus \{i\}\) bijectively to \(\mathbb{C} \setminus \{-i\}\), the set on which \(\tilde{\mu}\) is defined, and \(\tilde{\mu}\) gives the inverse.

(b) We have \(\mu(z_1)\tilde{\mu}(z_1) = \frac{(z_1 + i)(z_1^{-1} - i)}{i(z_1^{-1} - i)(z_1 + i)} = 1\) (with removable singularities at \(\pm i\)).

(c) From the definition and computation of \(\tilde{\mu}\) above, we see that \(\tilde{\mu}(\bar{z}) = \overline{\tilde{\mu}(z)}\) for all \(z \in \mathbb{C} \setminus \{i\}\).

(d) Note that whenever \(x \in \mathbb{R}\), \(\overline{\mu(x)} = \overline{\tilde{\mu}(x)} = \tilde{\mu}(x)\), so \(1 = \mu(x)\tilde{\mu}(x) = |\mu(x)|^2\), which proves that \(\mu(x), \tilde{\mu}(x) \in \partial \mathbb{D}\).

(e) Whenever \(z \in (\partial \mathbb{D} \setminus \{i\})\), \(\overline{\mu(z)} = \overline{\tilde{\mu}(z)} = \tilde{\mu}(z^{-1}) = \mu(z)\), so \(\mu(z) \in \mathbb{R}\). Similarly, for \(z \in (\partial \mathbb{D} \setminus \{-i\})\), \(\tilde{\mu}(z) \in \mathbb{R}\).

In the following lemma, we describe some of the properties of the induced maps \(M\) and \(\tilde{M}\).

**Lemma 4.** Let \(f \in \mathbb{C}(\mathbb{Z})\), and let \(M, \tilde{M}\) be the induced forward and reverse Möbius transformations, respectively.

(a) \(M(f), \tilde{M}(f) \in \mathbb{C}(\mathbb{Z})\);

(b) \(M(\tilde{M}(f)) = \tilde{M}(M(f)) = f\);

(c) \(\{z \in (\mathbb{C} \setminus \{i\})^n : M(f)(z) = 0\} = \{(\tilde{\mu}(z_1), \ldots, \tilde{\mu}(z_n)) : z \in (\mathbb{C} \setminus \{-i\})^n, f(z) = 0\}\);

(d) \(\{z \in \mathbb{C}^n : M(f)(z) \text{ is undefined}\} \subseteq \{z \in \mathbb{C}^n : z_j = i \text{ for some } 1 \leq j \leq n\} \cup \{(\tilde{\mu}(z_1), \ldots, \tilde{\mu}(z_n)) : z \in (\mathbb{C} \setminus \{-i\})^n, f(z) \text{ is undefined}\}\);

(e) \(M(\tilde{f}) = \tilde{M}(f) = \overline{\tilde{M}(f)}\);

(f) Suppose \(f \in \mathbb{R}(\mathbb{Z})\). For \(z \in \mathbb{C}^n\), \(\overline{M(f)(z)} = \tilde{M}(f)(\bar{z})\) whenever \(M(f)(z)\) is defined. In particular, for \(x \in \mathbb{R}^n\), \(M(f)(x) = \tilde{M}(f)(x)\) whenever \(M(f)(x)\) is defined, and for \(z \in (\partial \mathbb{D})^n\), \(M(f)(z) = M(f)(z)\) and \(\tilde{M}(f)(z) = \tilde{M}(f)(z)\), whenever \(M(f)(z)\) and \(\tilde{M}(f)(z)\) are respectively defined;

(g) If \(f \in \mathbb{R}(\mathbb{Z})\), then \(M(f) = \tilde{M}(f) \in \mathbb{R}(\mathbb{Z})\);

(h) If \(f \in \mathbb{R}(\mathbb{Z})\), \((\pm i)M(f - \bar{f}) \in \mathbb{R}(\mathbb{Z})\); and

(i) If \(f(z) \geq 0\) for all \(z \in (\partial \mathbb{D})^n\) at which it is defined, then \(M(f)(x) \geq 0\) for all \(x \in \mathbb{R}^n\) at which it is defined.
Proof. (a) If \( f \in \mathbb{C}[z] \), \( f = \sum_{\alpha \geq 0} f_\alpha z^\alpha \), \( f_\alpha \in \mathbb{C} \) for all \( \alpha \in \mathbb{Z}^n \), then

\[
M(f) = \sum_{\alpha \geq 0} f_\alpha \prod_{j=1}^n \left( \frac{z_j + i}{i(z_j - i)} \right)^{\alpha_j}
\]

clearly belongs to \( \mathbb{C}(z) \). If \( M(f) = 0 \), then \( 0 = M(f)(x) = f(\mu(x_1), \ldots, \mu(x_n)) \) for all \( x \in \mathbb{R}^n \), which means that \( f(z) = 0 \) for all \( z \in (\partial \mathbb{D} \setminus \{-i\})^n \) by Lemma 3(a) and (d). By Lemma 2, with \( S_j = \partial \mathbb{D} \setminus \{-i\} \) for all \( j \), this means that \( f = 0 \). Then if \( f = p/q \in \mathbb{C}(z) \) is in lowest terms, \( q \neq 0 \), so \( M(f) = M(p)/M(q) \in \mathbb{C}(z) \), since we have just seen that \( M(q) \neq 0 \). The argument for \( \tilde{M} \) is similar.

(b) \( M(\tilde{M}(f)) = M(\tilde{f}(\tilde{\mu}(z_1), \ldots, \tilde{\mu}(z_n))) = f(\tilde{\mu}(\mu(z_1)), \ldots, \tilde{\mu}(\mu(z_n))) = f \), by the calculation in the proof of Lemma 3(a). The case of \( M(M(f)) = f \) is similar, using the other calculation at the beginning of that proof.

(c) By Lemma 3(a), \((\mathbb{C} \setminus \{i\})^n = (\tilde{\mu}(\mathbb{C} \setminus \{i\}))^n \). By (b), for \( z \in (\mathbb{C} \setminus \{i\})^n \), \( f(z) = \tilde{M}(M(f))(z) = M(f)(\tilde{\mu}(z_1), \ldots, \tilde{\mu}(z_n)) \). Then given a point \( z \in (\mathbb{C} \setminus \{i\})^n \), the \( f(z) = 0 \), \( z = (\tilde{\mu}(z_1), \ldots, \tilde{\mu}(z_n)) \) for some \( z' \in (\mathbb{C} \setminus \{i\})^n \), and \( f(z') = 0 \); conversely, for \( z' \in (\mathbb{C} \setminus \{i\})^n \) with \( f(z') = 0 \), \( z = (\tilde{\mu}(z_1), \ldots, \tilde{\mu}(z_n)) \) satisfies \( M(f)(z) = 0 \).

(d) Since \( \mu \) is not defined at \( i \), \( M(f)(z) \) may not be defined if \( z_j = i \) for some \( 1 \leq j \leq n \). Otherwise, letting \( f = \mu(p)/q \) be in lowest terms, we can apply (c) to the set where \( M(q)(z) = 0 \), since \( M(f) = M(p)/M(q) \).

(e) \( M(\tilde{f}(\tilde{\mu}(z_1), \ldots, \tilde{\mu}(z_n))) = f(\mu(z_1)^{-1}, \ldots, \mu(z_n)^{-1}) = \tilde{M}(f), \) by Lemma 3(b).\( M(f) = M(1/z_1^{-1}, \ldots, z_n^{-1}) = f(\mu(z_1)^{-1}, \ldots, \mu(z_n)^{-1}) = \tilde{M}(f) \).

(f) Let \( f \in \mathbb{R}[z] \), \( f = \sum_{\alpha \geq 0} f_\alpha z^\alpha \), \( f_\alpha \in \mathbb{R} \) for all \( \alpha \in \mathbb{Z}^n \). Then

\[
M(f) = \sum_{\alpha \geq 0} f_\alpha \prod_{j=1}^n \left( \frac{z_j + i}{i(z_j - i)} \right)^{\alpha_j},
\]

so

\[
\tilde{M}(f)(\bar{z}) = \sum_{\alpha \geq 0} f_\alpha \prod_{j=1}^n \left( \frac{\bar{z}_j - i}{(-i)(z_j + i)} \right)^{\alpha_j} = \tilde{M}(f)(\bar{z}),
\]

using the computation of \( \tilde{\mu} \) at the beginning of the proof of Lemma 3. Now if \( f = p/q \) is in lowest terms, \( \tilde{M}(f)(\bar{z}) = \tilde{M}(p)(\bar{z})/\tilde{M}(q)(\bar{z}) = \tilde{M}(p)(\bar{z})/\tilde{M}(q)(\bar{z}) = \tilde{M}(f)(\bar{z}). \)

So when \( x \in \mathbb{R}^n \), \( M(f)(x) = \tilde{M}(f)(\bar{x}) = \tilde{M}(f)(x) \), and when \( z \in (\partial \mathbb{D})^n \), \( M(f)(z) = \tilde{M}(f)(\bar{z}) = \tilde{M}(f)(z_1^{-1}, \ldots, z_n^{-1}) \), which equals \( M(f)(z) = M(f)(z) \) by (e). The proof is similar for the case with \( \tilde{M} \).

(g) By (e), since \( f \in \mathbb{R}[z] \), \( M(f) = \tilde{M}(f) \), so \( M(f) \in \mathbb{R}(z)_{\infty} \) if it is in \( \mathbb{R}(z) \), which we now verify. From Lemma 3, if \( f = p/q \in \mathbb{R}(z)_{\infty} \), then \( p\bar{q} \in \mathbb{R}(z)_{\infty} \), so if we can show that \( M(p\bar{q}) \), \( M(q\bar{q}) \) are in \( \mathbb{R}(z)_{\infty} \), \( M(f) = M(p\bar{q})/M(q\bar{q}) \in \mathbb{R}(z)_{\infty} \) also. Thus, it suffices to consider the case where \( f \in \mathbb{R}[z^{\pm 1}] \). Again from Lemma 3, this means that \( f_\alpha = f_{-\alpha} \) for all \( \alpha \in \mathbb{Z}^n \), so \( f = f_0 + \sum_{\alpha > 0} f_\alpha(z^\alpha + z^{-\alpha}) \), and \( M(f) = f_0 + \sum_{\alpha > 0} f_\alpha((\mu(z_j)z_j)^\alpha + (\mu(z_j))_j^{-\alpha} \alpha) \). Thus, it suffices to show that \( (\mu(z_j))^\alpha + (\mu(z_j))_j^{-\alpha} \) is in \( \mathbb{R}(z) \) whenever \( \alpha > 0 \).

For \( \alpha > 0 \), let \( \varepsilon \in \{-1, 1\}^n \) be such that \( \varepsilon_j = 1 \) if \( \alpha_j \geq 0 \) and is \(-1 \) otherwise. By Lemma 3(b), we have \( \mu(z_j)^{\alpha_j} = \tilde{\mu}(z_j)^{-\alpha_j} \), so using the definition of \( \mu \), Lemma 3(b), and the calculation of \( \tilde{\mu} \) at the beginning
of the proof of Lemma \[\square\] and letting $\beta = (|\alpha_j|)_j$:

$$(\mu(z_j))^{\alpha} + ((\mu(z_j))_j)^{-\alpha} = \prod_{j=1}^{n} \left( \frac{2 - \varepsilon_j i (z_j - z_j^{-1})}{z_j + z_j^{-1}} \right)^{\beta_j} + \prod_{j=1}^{n} \left( \frac{2 + \varepsilon_j i (z_j - z_j^{-1})}{z_j + z_j^{-1}} \right)^{\beta_j}

= ((z_j + z_j^{-1})_{j})^{-\beta} \left( \prod_{j=1}^{n} (2 - \varepsilon_j i (z_j - z_j^{-1}))^{\beta_j} + \prod_{j=1}^{n} (2 + \varepsilon_j i (z_j - z_j^{-1}))^{\beta_j} \right).$$

Now we apply Equation (\[\square\]) to both of these products, obtaining for the right factor of the above expression

$$\sum_{0 \leq \varepsilon \leq \beta} \binom{\beta}{k} 2^{\beta - |k|} ((\varepsilon_j (z_j - z_j^{-1}))_{j})^{k} ((-i)^{|k|} + i^{|k|}).$$

Since $(-i)^{|k|} + i^{|k|}$ is always in $\mathbb{R}$, we see that this is in $\mathbb{R}(z)$, which completes the proof, since the factor $((z_j + z_j^{-1})_{j})^{-\beta} \in \mathbb{R}(z)$.

(h) For $f = p/q \in \mathbb{R}(z)$, $f - \tilde{f} = (pq - \tilde{p}q)/(pqq)$, so $M(f - \tilde{f}) = M(pq - \tilde{pq})/M(qq)$, and since $pq \in \mathbb{R}(z)_\infty$, by (g), $M(qq) \in \mathbb{R}(z)_\infty$, so it suffices to show that $(\pm i) M(f - \tilde{f}) \in \mathbb{R}(z)$ when $f \in \mathbb{R}(\mathbb{Z}^\infty)$.

In this case, $f - \tilde{f} = \sum_{\alpha>0}(f_{-\alpha} - f_{-\alpha})z^\alpha = \sum_{\alpha>0}c_\alpha(z^\alpha - z^{-\alpha})$, where $c_\alpha \in \mathbb{R}$ for all $\alpha \in \mathbb{Z}^\infty$, $\alpha \geq 0$. This gives $M(f - \tilde{f}) = \sum_{\alpha>0}c_\alpha((\mu(z_j))_j)^{\alpha} - ((\mu(z_j))_j)^{-\alpha}$, so we need to show that for all $\alpha \in \mathbb{Z}^\infty$, $\alpha \geq 0$, $(\pm i)((\mu(z_j))_j)^{\alpha} - ((\mu(z_j))_j)^{-\alpha}) \in \mathbb{R}(z)$.

For $\alpha > 0$, we proceed as in the proof of (g) to see that

$$(\mu(z_j))_j^{\alpha} - ((\mu(z_j))_j)^{-\alpha} = ((z_j + z_j^{-1})_{j})^{-\beta} \sum_{0 \leq \varepsilon \leq \beta} \binom{\beta}{k} 2^{\beta - |k|} ((\varepsilon_j (z_j - z_j^{-1}))_{j})^{k} ((-i)^{|k|} - i^{|k|}).$$

Since $(-i)^{|k|} - i^{|k|} = 0$ whenever $|k|$ is even, the right factor equals

$$i \left( \sum_{0 \leq \varepsilon \leq \beta, |k| \text{ odd}} \binom{\beta}{k} 2^{\beta - |k| + 1} ((\varepsilon_j (z_j - z_j^{-1}))_{j})^{k} (-1)^{|k|+1}/2 \right),$$

which is clearly in $\mathbb{R}(z)$ when multiplied by $\pm i$. Since $((z_j + z_j^{-1})_{j})^{-\beta} \in \mathbb{R}(z)$, we see that $(\pm i)((\mu(z_j))_j)^{\alpha} - ((\mu(z_j))_j)^{-\alpha} \in \mathbb{R}(z)$, which completes the proof.

(i) By Lemma \[\square\](d), for $x \in \mathbb{R}^n$, $(\mu(x_1), \ldots, \mu(x_n)) \in (\partial \mathbb{D})^n$. Then using the definition of $M(f)$, when $x \in \mathbb{R}^n$, $M(f)(x) = f(\mu(x_1), \ldots, \mu(x_n)) \geq 0$.

\[\square\]
Let \( f \in \mathbb{C}[z^{\pm 1}] \) such that \( f(z) \geq 0 \) for all \( z \in (\partial \mathbb{D})^n \). Then \( 0 = f(z) - \overline{f}(z) = \sum_\alpha (f_\alpha - \overline{f_\alpha}) z^\alpha \) for all \( z \in (\partial \mathbb{D})^n \), so \( f_\alpha = \overline{f_\alpha} \) for all \( \alpha \in \mathbb{Z}^n \). Then \( f_\alpha = \frac{1}{2}(f_\alpha + \overline{f_\alpha}) + i \left( \frac{1}{2i}(f_\alpha - \overline{f_\alpha}) \right) \), so

\[
f(z) = f_0 + \sum_{\alpha>0} f_\alpha z^\alpha + f_{-\alpha} z^{-\alpha}
= f_0 + \sum_{\alpha>0} \frac{1}{2}(f_\alpha + \overline{f_\alpha})(z^\alpha + z^{-\alpha}) + i \left( \frac{1}{2i}(f_\alpha - \overline{f_\alpha}) \right) (z^\alpha - z^{-\alpha})
= g_1 + i(g_2 - \overline{g_2}),
\]
where \( g_1, g_2 \in \mathbb{R}(z) \), since \( g_1 = f_0 + \sum_{\alpha>0} \frac{1}{2}(f_\alpha + \overline{f_\alpha})(z^\alpha + z^{-\alpha}) \), which is actually in \( \mathbb{R}(z)_\infty \), and \( g_2 = \sum_{\alpha>0} \frac{1}{2i}(f_\alpha - \overline{f_\alpha})z^\alpha \), which is clearly in \( \mathbb{R}(z) \).

Let \( M \) be the induced forward Möbius transformation, and let \( \widetilde{M} \) be the induced reverse Möbius transformation. Now from Lemma 3(g) and (h), we see that \( M(f) \in \mathbb{R}(z) \), since \( M(g_1) \in \mathbb{R}(z) \), \( M(i(g_2 - \overline{g_2})) = iM(g_2 - \overline{g_2}) \in \mathbb{R}(z) \). By Lemma 3(i), \( M(f)(x) \geq 0 \) for all \( x \in \mathbb{R}^n \). Then by Corollary 1 \( M(f) = \sum_{j=1}^{\infty} r_j^2 \), where \( r_j \in \mathbb{R}(z) \) for \( 1 \leq j \leq J \leq 2^n \). Now \( \widetilde{M}(r_j) \in \mathbb{C}(z) \), and by Lemma 3(f), for \( z \in (\partial \mathbb{D})^n \), \( \widetilde{M}(r_j)(z) \in \mathbb{R} \). By Lemma 3(b), \( f = \widetilde{M}(M(f)) = \widetilde{M}(\sum_{j=1}^{\infty} r_j^2) = \sum_{j=1}^{\infty} \widetilde{M}(r_j)^2 \). We just showed that \( \widetilde{M}(r_j) \) are real-valued on \( (\partial \mathbb{D})^n \), so \( f(z) = \sum_{j=1}^{\infty} |\widetilde{M}(r_j)(z)|^2 \) for all \( z \in (\partial \mathbb{D})^n \). This completes the proof.

In the following example, we give a first application of our result and the method in its proof.

**Example 1.** Consider the trigonometric polynomial \( 1 - \cos^2(\omega_1) \cos^2(\omega_2) \), which is nonnegative for all \( \omega_1, \omega_2 \in \mathbb{R} \). This corresponds (via \( z_j = e^{i\omega_j}, \ j = 1, 2 \)) to the Laurent polynomial \( f \in \mathbb{R}[z_1^{\pm 1}, z_2^{\pm 1}] \),

\[
f(z_1, z_2) := 1 - \frac{1}{16} (z_1 + z_1^{-1})(z_2 + z_2^{-1})^2.
\]

Setting \( x_j \in \mathbb{R} \) for \( j = 1, 2 \) so that \( z_j = \mu(x_j) \), we compute \( M(f)(x) = f(\mu(x_1), \mu(x_2)) \):

\[
M(f)(x) = 1 - \frac{4}{(x_1 + x_1^{-1})^2} - \frac{4}{(x_2 + x_2^{-1})^2}.
\]

For each \( x_1, x_2 \in \mathbb{R}, (\mu(x_1), \mu(x_2)) \) is a point in \( (\partial \mathbb{D} \setminus \{-i\})^2 \), so \( M(f)(x) \geq 0 \) for all \( x \in \mathbb{R}^2 \). By Corollary 1 this has an sors representation on \( \mathbb{R}^2 \), and indeed,

\[
M(f)(x) = \left( x_1 - x_1^{-1} \right)^2 + \frac{4}{(x_1 + x_1^{-1})^2} - \frac{4}{(x_2 + x_2^{-1})^2}.
\]

Applying \( \widetilde{M} \) to each of these squares, we get

\[
f(z) = \left( \frac{z_1 - z_1^{-1}}{2i} \right)^2 + \left( \frac{z_1 + z_1^{-1}}{2} \right)^2 - \frac{4}{(z_2 + z_2^{-1})^2} - \frac{4}{(z_2 + z_2^{-1})^2}.
\]

and translating these back into trigonometric polynomials, we get

\[
1 - \cos^2(\omega_1) \cos^2(\omega_2) = \sin^2(\omega_1) + \cos^2(\omega_1) \sin^2(\omega_2).
\]

In the next example, we consider a lowpass mask \( \tau \) satisfying the sub-QMF condition such that \( f(\tau; \cdot) \) does not have an sors representation, as shown in [4] Th. 2.5. From Theorem 1 \( f(\tau; \cdot) \) must have an sors representation, but we can actually show this independently for this example, showing a different method for obtaining an sors representation from the one in the proof. In Example 3 we extend this example to a tight wavelet frame with rational highpass masks.
Example 2. We consider an example of a nonnegative trigonometric polynomial for which there is no sos representation, from [4 Th. 2.5]. We begin by considering $\tau(\omega) = (1 - c \cdot r(\omega)) a(\omega), \omega \in \mathbb{T}^3$ where $0 < c \leq 1/3$, $r(\omega) = \sin^4(\omega_1) \sin^2(\omega_2) + \sin^2(\omega_1) \sin^4(\omega_2) - 3 \sin^2(\omega_2) \sin^2(\omega_3)$ is the Motzkin polynomial evaluated at $(\sin(\omega_1), \sin(\omega_2), \sin(\omega_3))$, and we choose $a(\omega) = m(\omega_1)m(\omega_2)m(\omega_3)$, where $m$ is a lowpass mask with accuracy number at least 8 (i.e., $D^3 m(\pi) = 0$ for all $0 \leq j < 8$), and satisfies the sub-QMF condition $t(\omega) := |m(\omega)|^2 + |m(\omega + \pi)|^2 \leq 1$. Using the $\pi \mathbb{Z}^3$-periodicity of the first factor of $\tau$, we see that

$$f(\omega) := 1 - \sum_{\gamma \in \{0, \pi\}^3} |\tau(\omega + \gamma)|^2 = 1 - (1 - cr(\omega))^2 \prod_{j=1}^3 t(\omega_j),$$

where $f$ is known not to have an sos representation from [4 Th. 2.5]. This equals

$$1 - (1 - cr(\omega))^2 + (1 - cr(\omega))^2(1 - t(\omega_1)) + (1 - cr(\omega))^2t(\omega_1)(1 - t(\omega_2)) + (1 - cr(\omega))^2t(\omega_1)t(\omega_2)(1 - t(\omega_3)).$$

The latter 3 terms are all squares, since $t$ and $1 - t$ have sos representations with a single generator by Result 1(a). For the first two terms, $1 - (1 - cr(\omega))^2 = 2cr(\omega)(1 - (c/2)r(\omega))$, the right factor of which has an sos representation by Result 1(d) since $r(\omega) \leq 3$, which means that $1 - (c/2)r(\omega) \geq 1/2 > 0$ for all $\omega \in \mathbb{T}^3$. The remaining factor $2cr$ does not have an sos representation, but since

$$r(\omega) = \frac{(\sin^2(\omega_1) \sin(\omega_2) + \sin(\omega_1) \sin^2(\omega_2) - 2 \sin(\omega_1) \sin(\omega_2) \sin^2(\omega_3)) (\sin^2(\omega_1) + \sin^2(\omega_2) + \sin^2(\omega_3))}{(\sin^2(\omega_1) + \sin^2(\omega_2))^2},$$

we see that $f$ has an sos representation. 

4. OEP Tight Wavelet Frames from Sors Representations

Having established Theorem 1, we now turn to a setting where the existence of these sors representations may be fruitfully applied, namely wavelet construction with the oblique extension principle. We begin by reviewing the key ideas of this theorem, before proceeding to our construction.

Let $\mathcal{M} \in M_n(\mathbb{Z})$ be a dilation matrix, so that the set of eigenvalues of $\mathcal{M}$ lie in $\{z \in \mathbb{C} : |z| > 1\}$. Let $Q = |\det(\mathcal{M})|$. Let $\Gamma$ be a complete set of distinct coset representatives of $\mathbb{Z}^n / \mathcal{M} \mathbb{Z}^n$ containing 0, and let $\Gamma^*$ be a complete set of distinct coset representatives of $\langle 2\pi \mathcal{M}^{-T} \mathbb{Z}^n \rangle / \langle 2\pi \mathbb{Z}^n \rangle$ containing 0. In the introduction, we considered only the case of $\mathcal{M} = 2I$, and chose $\Gamma = \{0,1\}^n$, $\Gamma^* = \{0,\pi\}^n$, but we now consider the general case.

Definition 4. Given a lowpass mask $\tau$, which is a trigonometric polynomial satisfying $\tau(0) = 1$, the refinable function $\phi$ associated with $\tau$ satisfies $\phi(\mathcal{M}^T \omega) = \tau(\omega)\phi(\omega)$ for all $\omega \in \mathbb{R}^n$. It may be defined by its Fourier transform as $\hat{\phi}(\omega) = \prod_{j=1}^\infty \tau((\mathcal{M}^{-T})^j \omega)$ for all $\omega \in \mathbb{R}^n$. Given a collection of highpass masks $q_\ell$, $1 \leq \ell \leq r$ which are rational trigonometric polynomials satisfying $q_\ell(0) = 0$, we define $\mathbf{\tau} = (\tau, q_1, \ldots, q_r)$ as the combined MRA mask, and the wavelet system defined by $\mathbf{\tau}$ is then the set $\{\psi_{j,k}^{(\ell)} : j \in \mathbb{Z}, k \in \mathbb{Z}^n, 1 \leq \ell \leq r\}$, where $\psi_{j,k}^{(\ell)} = Q^{j/2} \psi(\mathcal{M}^j \cdot -k)$, and $\psi^{(\ell)}$ is defined by its Fourier transform as $\hat{\psi}^{(\ell)}(\mathcal{M}^T \omega) = q_\ell(\omega)\hat{\phi}(\omega)$ for all $\omega \in \mathbb{R}^n$.

When a wavelet system has the property that $\sum_{j,k,\ell} \langle f, \psi_{j,k}^{(\ell)} \rangle^2 = \|f\|^2_{L^2(\mathbb{R}^n)}$ for all $f \in L^2(\mathbb{R}^n)$, we say that it is a tight wavelet frame (TWF), and in this case we have the representation $f = \sum_{j,k,\ell} \langle f, \psi_{j,k}^{(\ell)} \rangle \psi_{j,k}^{(\ell)}$, which holds in the sense of $L^2(\mathbb{R}^n)$. This expansion is similar to the one given by an orthonormal basis for $L^2(\mathbb{R}^n)$, but the wavelets $\psi_{j,k}^{(\ell)}$ may not be orthogonal in the case that they only form a TWF. When in addition $\{\psi_{j,k}^{(\ell)}\}$ is an orthonormal set, we say that the wavelet system is an orthonormal wavelet basis. For more details on these ideas, see [6].
The statement of the OEP is given below, from [7]. This uses the notation (for \( u = (u_0, \ldots, u_r), \) and \( v = (v_0, \ldots, v_r) \)) \( \langle u, v \rangle_a = au_0\bar{v}_0 + \sum_{i=1}^r u_i\bar{v}_i \) and \( \sigma(\omega) := \{ \omega \in \mathbb{T}^n : \hat{\phi}(\omega + 2\pi k) \neq 0, \text{ for some } k \in \mathbb{Z}^n \} \), as well as \( \delta : \Gamma^* \rightarrow \{0, 1\} \), which always takes the value zero except for \( \delta(0) = 1 \). We have adapted the notation of this theorem to our setting.

**Result 3.** Let \( \tau, q_1, \ldots, q_r \) be 2\( \pi \)-periodic functions, and let \( \tau = (\tau, q_1, \ldots, q_r) \) be the combined MRA mask. Suppose that

(a) Each mask \( \tau, q_j \) in the combined MRA mask \( \tau \) belongs to \( L^\infty(\mathbb{T}^n) \).

(b) The refinable function \( \phi \) satisfies \( \lim_{\omega \rightarrow 0} \hat{\phi}(\omega) = 1 \).

(c) The function \( \hat{\phi}, \hat{\phi} := \sum_{k \in \mathbb{Z}^n} |\hat{\phi}(\cdot + 2\pi k)|^2 \) belongs to \( L^\infty(\mathbb{T}^n) \).

Suppose there exists a 2\( \pi \)-periodic function \( S \) that satisfies the following:

(i) \( S \in L^\infty(\mathbb{T}^n) \) is nonnegative, continuous at the origin, and \( S(0) = 1 \).

(ii) If \( \omega \in \sigma(\phi) \), and if \( \gamma \in \Gamma^* \) is such that \( \omega + \gamma \in \sigma(\phi) \), then \( \langle \tau(\omega), \tau(\omega + \gamma) \rangle_S(\mathcal{M}^T\omega) = S(\omega)\delta(\gamma) \).

Then the wavelet system defined by \( \tau \) is a tight wavelet frame for \( L^2(\mathbb{R}^n) \).

In what follows, we will always be considering \( \tau \) as a trigonometric polynomial lowpass mask, which corresponds to a compactly supported refinable function. In this case, \( \sigma(\tau) = \mathbb{T}^n \) (see [7]). We also note that when \( \phi \) satisfies condition (c), it necessarily belongs to \( L^2(\mathbb{R}^n) \). The function \( S \) is called the vanishing moment recovery (vmr) function, since the additional flexibility it brings may be used to construct highpass masks with better vanishing moments than in the unitary extension principle setting, which forces \( S \equiv 1 \).

### 4.1. Construction

Given a trigonometric polynomial lowpass mask \( \tau \) and rational trigonometric polynomial vmr function \( S \), we now give a condition on this pair which guarantees the existence of rational trigonometric polynomial highpass masks \( q_\ell \) satisfying the OEP conditions (specifically (ii) above), and we show how to find these masks constructively. The condition we impose might be thought of as an oblique extension of the well-known sub-QMF condition, and indeed when \( S \equiv 1 \), the “oblique sub-QMF condition” reduces to the sub-QMF condition, which is necessary for constructing a tight wavelet frame with the UEP [4]. Analogously, the OEP conditions will necessitate our oblique version. We begin by stating the main theorems, which are split into an algebraic part and an analytical part. The proofs of these theorems then follow, along with a few requisite definitions and lemmata.

**Theorem 2.** Let \( S \) be a nonzero rational trigonometric polynomial which is nonnegative on \( \mathbb{T}^n \), and let \( \tau \) be a trigonometric polynomial lowpass mask. The following are equivalent:

(A) The Oblique sub-QMF condition holds:

\[
\sum_{\gamma \in \Gamma^*} |\tau(\omega + \gamma)|^2 S(\omega + \gamma) \leq \frac{1}{S(\mathcal{M}^T\omega)} \quad \text{for all } \omega \in \mathbb{T}^n \text{ at which both sides are defined,} \tag{3}
\]

(B) There exist rational trigonometric polynomials \( \{q_\ell\}_{\ell=1}^r \) such that for all \( \gamma \in \Gamma^* \) and \( \omega \in \mathbb{T}^n \) at which both sides are defined:

\[
S(\mathcal{M}^T\omega)\tau(\omega)\tau(\omega + \gamma) + \sum_{\ell=1}^r q_\ell(\omega)q_\ell(\omega + \gamma) = \begin{cases} 
S(\omega) & \text{if } \gamma = 0 \\
0 & \text{otherwise.}
\end{cases} \tag{4}
\]
Moreover, provided that either (A) or (B) holds, there exist (a potentially different set of) rational trigonometric polynomials \( \{q_i\}_{i=1} \) such that for all \( \gamma \in \Gamma^* \) and \( \omega \in \mathbb{T}^n \) at which both sides are defined, Equation (4) holds, where \( r \leq 2^n(1 + Q) \).

The next theorem combines these conditions with an additional assumption on the rational trigonometric polynomial \( S \), guaranteeing that these masks generate a TWF. This might be seen as an extension of [12, Lemma 2.1].

**Theorem 3.** Assume the setting of the previous theorem. Suppose, in addition to satisfying one of (A) or (B), that \( S \) is continuous at 0 with \( S(0) = 1 \), and that \( S \) and \( 1/S \) belong to \( L^\infty(\mathbb{T}^n) \). Then the wavelet system defined by the combined MRA mask \((\tau, q_1, \ldots, q_r)\) is a tight wavelet frame.

The assumptions that \( S, 1/S \in L^\infty(\mathbb{T}^n) \) guarantee that \( S \) and \( 1/S \) are pole-free, so in this setting, Equations (3) and (4) hold for all \( \omega \in \mathbb{T}^n \). Now, we turn to the proofs of these theorems.

We begin with some definitions and basic lemmas. Recall that \( M \in M_n(\mathbb{Z}) \) is a dilation matrix.

**Definition 5.** We define the group action of \( G = (2\pi \mathcal{M}^{-T}\mathbb{Z}^n)/(2\pi \mathbb{Z}^n) \) on a rational trigonometric polynomial in the following way: for \( \gamma \in \Gamma^* \) and \( f \) a rational trigonometric polynomial, \( \gamma : f \mapsto f^\gamma(\cdot) = f(\cdot + \gamma) \). Note that this group action is independent of the set \( \Gamma^* \) of coset representatives for \( G \).

We say that a rational trigonometric polynomial \( f \) is \( G \)-invariant if for all \( \gamma \in \Gamma^* \), \( f^\gamma = f \).

We say that \( H(\omega) \) is a \( G \)-vector for the rational trigonometric polynomial \( h \) if \( H(\omega) = [h^\gamma(\omega)]_{\gamma \in \Gamma^*} \). We also call a vector a \( G \)-vector if it is of this form for some rational trigonometric polynomial. \( \square \)

**Definition 6.** For \( f \) a rational trigonometric polynomial, we define its polyphase components \( f_\nu, \nu \in \Gamma \), by

\[
f_\nu(M^T\omega) = Q^{-1/2} \sum_{\gamma \in \Gamma^*} f^\gamma(\omega) e^{-i(\omega + \gamma) \cdot \nu}.
\]

We will also use the following dual relation, which is easy to show using the definition above:

\[
f(\omega) = Q^{-1/2} \sum_{\nu \in \Gamma} f_\nu(M^T\omega) e^{i\omega \cdot \nu}.
\]

Note that if \( f \) is a rational trigonometric polynomial, then \( f(M^T\cdot) \) is \( G \)-invariant, since \( (f(M^T\cdot))^\gamma(\omega) = f(M^T(\omega + \gamma)) = f(M^T\omega) \), because \( M^T\gamma \in 2\pi \mathbb{Z}^n \). If \( f \) is \( G \)-invariant, then \( f(\omega) = Q^{-1} \sum_{\gamma \in \Gamma^*} f^\gamma(\omega) = Q^{-1/2} f_0(M^T\omega) \), so \( f \) is a rational trigonometric polynomial in \( M^T\omega \).

**Definition 7.** Let \( f \) be a rational trigonometric polynomial with an sos or sors of functions \( g_j \), \( 1 \leq j \leq J \). We say that \( f \) has a \( G \)-invariant so(r)s if its so(r)s representation has the property that \( g_j = g_j^\gamma \) for all \( \gamma \in \Gamma^* \), for all \( 1 \leq j \leq J \). \( \square \)

We now prove a few lemmas regarding \( G \)-invariance and sums of squares representations. Similar results appear in [11, Lemma 2.1] under more restrictive assumptions.

**Lemma 5.** Let \( f \) be a rational trigonometric polynomial.

(a) \( \sum_{\gamma \in \Gamma^*} |f^\gamma|^2 = \sum_{\nu \in \Gamma} |f_\nu(M^T\cdot)|^2 \).

(b) If \( f \) is \( G \)-invariant, then it is an so(r)s if and only if it is a \( G \)-invariant so(r)s.

**Proof.** (a) We use Equation (6) to compute, for \( \omega \in \mathbb{T}^n \) such that the left hand side is defined:

\[
\sum_{\gamma \in \Gamma^*} |f^\gamma(\omega)|^2 = Q^{-1} \sum_{\gamma \in \Gamma} \sum_{\nu, \nu' \in \Gamma} f_\nu(M^T\omega) \overline{f_{\nu'}(M^T\omega)} e^{i(\omega + \gamma) \cdot (\nu - \nu')} = \sum_{\nu \in \Gamma} |f_\nu(M^T\omega)|^2,
\]

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using the fact that for \( k \in \mathbb{Z}^n \), \( \sum_{\gamma \in \Gamma^*} e^{i\gamma \cdot k} \) is equal to \( Q \) when \( k \equiv 0 \pmod{M\mathbb{Z}^n} \) and is 0 otherwise.

(b) The converse is obvious, so if \( f = \sum_{j=1}^J |g_j|^2 \), then for all \( \omega \in \mathbb{T}^n \) where it is defined,

\[
f(\omega) = Q^{-1} \sum_{\gamma \in \Gamma^*} f(\omega) = Q^{-1} \sum_{j=1}^J \sum_{\gamma \in \Gamma^*} |g_j(\omega)|^2 = Q^{-1} \sum_{j=1}^J \sum_{\nu} |(g_j)_{\nu}(M^T\omega)|^2,
\]

the last equality following from part (a) which was just proved. The last expression is clearly a sum of \( G \)-invariant squares, which completes the proof. \( \square \)

From the proof of (b), it may appear that the number of sos generators will increase by a factor of \( Q \) when we require them to be \( G \)-invariant. However, if \( f \) is \( G \)-invariant and an sos, then \( f = g(M^T \cdot) \), where \( g \) is certainly nonnegative. By Theorem 1, \( g \) has an sos representation which may require at most \( Q \times M \) times as many generators as the original sos. Moreover, from the argument after Lemma 5, for \( (\omega,j) \in \Gamma^* \times \{1, \ldots, J\} \), we see that \( A(\omega) = a_{(\nu,j)}(\omega + \gamma) \), so \( A(\omega) \) has \( G \)-vector columns. Then for all \( \gamma, \gamma' \in \Gamma^* \), letting \( \delta : 2\pi M^{-T} \mathbb{Z}^n \to \{0,1\} \) always take value zero except \( \delta(0) = 1 \),

\[
(A(\omega)A(\omega)^*)_{\gamma,\gamma'} = Q^{-1} \sum_{j=1}^J |s_j(\omega + \gamma)|^2 \sum_{\nu} e^{i\nu(\gamma - \gamma')}
\]

\[
= \delta(\gamma - \gamma') \sum_{j=1}^J |s_j(\omega + \gamma)|^2
\]

\[
= \delta(\gamma - \gamma') \frac{1}{S(\omega + \gamma)} = \Sigma^{-1}(\omega)_{\gamma,\gamma'}.
\]

This clearly holds wherever \( \Sigma^{-1}(\omega) \) is defined. \( \square \)

We are now ready to prove Theorem 2.

Proof of Theorem 2. To avoid excessive verbiage, throughout this proof, all equalities should be taken to hold wherever both sides are defined, which because we are considering a finite collection of rational trigonometric polynomials, is an open, dense set with full measure.

(i) Proof that \( A \) implies \( B \): Suppose Statement A of the theorem. We observe that by Theorem 1 and Lemma 5(b), for \( \omega \in \mathbb{T}^n \),

\[
\frac{1}{S(M^T \omega)} - \sum_{\gamma \in \Gamma^*} \frac{|\tau(\omega + \gamma)|^2}{S(\omega + \gamma)} = \sum_{j=1}^J |g_j(M^T \omega)|^2, \quad (7)
\]

where \( g_j \) are rational trigonometric polynomials, since the left hand side is nonnegative by Equation 3 and is clearly \( G \)-invariant. Moreover, from the argument after Lemma 5, \( J \) in Equation 7 is no greater than \( 2^n \). Let
$H(\omega) = [\tau(\omega + \gamma)]_{\gamma \in \Gamma^*}$ be a column vector, and let $G(\omega) = [g_j(\omega)]_{j=1}^J$. Recall that $\Sigma(\omega) = \text{diag}(S(\omega + \gamma))_{\gamma \in \Gamma^*}$. We observe that

$$[H(\omega) S(M^T\omega) H(\omega)]_{\gamma \in \Gamma^*} = S(M^T\omega) H(\omega) S(M^T\omega)^* \Sigma(\omega) S(M^T\omega) H(\omega) H(\omega)^*$$

$$= S(M^T\omega) H(\omega) H(\omega)^* + \Sigma(\omega) - 2S(M^T\omega) H(\omega) H(\omega)^* + \Sigma(\omega) - S(M^T\omega) H(\omega) H(\omega)^*$$

$$= \Sigma(\omega),$$

where the last equation follows by Equation (7). The columns of $S(M^T\omega) H(\omega) G(M^T\omega)^*$ are $G$-vectors, since $H(\omega)$ is and the other factors are $G$-invariant. Now we use Lemma 6 to see that $\Sigma(\omega)^{-1} = A(\omega) A(\omega)^*$, where the columns of $A$ are $G$-vectors. We observe that for any rational trigonometric polynomial $g$:

$$(\Sigma(\omega) - S(M^T\omega) H(\omega) H(\omega)^*) [g^\gamma(\omega)]_{\gamma \in \Gamma^*} = \left[S^\gamma(\omega) g^\gamma(\omega) - S(M^T\omega) \tau^\gamma(\omega) \sum_{\gamma \in \Gamma^*} \tau^{\gamma}(\omega) g^{\gamma}(\omega)\right]_{\gamma \in \Gamma^*}$$

So we see that the columns of $(\Sigma(\omega) - S(M^T\omega) H(\omega) H(\omega)^*) A(\omega)$ are $G$-vectors, and $A$ has $M \leq 2^n Q$ columns. Then the following rational trigonometric polynomials are defined and satisfy Equation (4) with $\tau$:

$$q_{1,j}(\omega) = S(M^T\omega) \tau(\omega) g_j(M^T\omega)$$

$$q_{2,m}(\omega) = S(\omega) a_m(\omega) - S(M^T\omega) \tau(\omega) \sum_{\gamma \in \Gamma^*} \tau(\omega + \gamma) a_m(\omega + \gamma)$$

$$1 \leq j \leq J \leq 2^n,$$

$$1 \leq m \leq M \leq 2^n Q,$$

where the $n$th column of $A(\omega)$ is a $G$-vector for the rational trigonometric polynomial $a_m(\omega), 1 \leq m \leq M$. This construction also gives the bound $r \leq 2^n (1 + Q)$.

(ii) Proof that $B$ implies $A$: Suppose Statement B of the theorem. Then for $\omega \in \mathbb{T}^n$,

$$\Sigma(\omega) - S(M^T\omega) H(\omega) H(\omega)^* = Q(\omega) Q(\omega)^*,$$

where $Q(\omega)$ is a $Q \times r$ matrix with rational trigonometric polynomial entries, and columns of the form $[q(\omega + \gamma)]_{\gamma \in \Gamma^*}$. Taking the determinant on both sides of the previous equation, the left hand gives

$$\det(\Sigma(\omega) - S(M^T\omega) H(\omega) H(\omega)^*) = \det(\Sigma^{1/2}(\omega)(I - S(M^T\omega) \Sigma^{-1/2}(\omega) H(\omega) H(\omega)^* \Sigma^{-1/2}(\omega)) \Sigma^{1/2}(\omega))$$

$$= \det(\Sigma(\omega))(1 - S(M^T\omega) H(\omega)^* \Sigma^{-1}(\omega) H(\omega)).$$

Since $Q(\omega) Q(\omega)^*$ is positive semidefinite for all $\omega \in \mathbb{T}^n$, its determinant is nonnegative. $S$ is nonnegative, so for $\omega \in \mathbb{T}^n$, $1/\det(\Sigma(\omega)) \geq 0$, and $1/S(M^T\omega) \geq 0$. Then Equation (3) follows, since

$$\frac{1}{S(M^T\omega)} - \sum_{\gamma \in \Gamma^*} \frac{\tau(\omega + \gamma)^2}{S(\omega + \gamma)} = \frac{\det(Q(\omega) Q(\omega)^*)}{\det(\Sigma(\omega)) S(M^T\omega)}.$$

Now we turn to the analytical part of the construction, with the proof of Theorem 3. We will see that the additional conditions on the vnr function $S$ are used in order to guarantee the ess. boundedness of the constructed highpass masks, as well as that of $[\hat{\phi}, \hat{\varphi}]$, as required in Result 3(a) and (c).
Proof of Theorem 3. We seek to apply Result 3. It is clear from the assumptions that (i) and (ii) hold, since the conditions on $S$ mean that (4) holds for all $\omega \in \mathbb{T}^n$. Since $\tau$ is a trigonometric polynomial, it is continuous and therefore bounded; because its corresponding filter has finite support, $\phi$ is compactly supported, so $\phi$ is continuous, and (b) also holds. Rearranging Equation (4) and looking at the case $\gamma = 0$, we see that $\sum_{\ell=1}^{\infty} |q_{\ell}(\omega)|^2 = S(\omega) - S(MT\omega) |\tau(\omega)|^2$, so the ess. boundedness of the right hand side implies the ess. boundedness of $q_\ell$ for all $1 \leq \ell \leq r$. This proves (a), so it remains to show (c).

We argue as in the proof of the first half of [12 Lemma 2.1]. Recall that $\hat{\phi}(\omega) := \prod_{j=1}^{\infty} \tau((M^{-T})^j \omega)$ for all $\omega \in \mathbb{R}^n$. Let
\[
f_0(\omega) := \chi_{[-\pi,\pi]^n}(\omega)(S(\omega))^{-1/2}, \quad \text{and for all } j \geq 1, \text{ let }
f_j(\omega) := \tau(M^{-T}\omega) f_{j-1}(M^{-T}\omega) = \chi_{(M^{-T})([-\pi,\pi]^n)}(\omega)(S((M^{-T})^j \omega))^{-1/2} \prod_{\ell=1}^{j} \tau((M^{-T})^\ell \omega).
\]
We now prove by induction that $[f_j, f_j](\omega) = \sum_{k \in \mathbb{Z}^n} |f_j(\omega + 2\pi k)|^2 \leq 1/S(\omega)$. Clearly, $[f_0, f_0](\omega) = 1/S(\omega)$, so suppose by way of induction that for some $j - 1 \geq 0$, $[f_{j-1}, f_{j-1}](\omega) \leq 1/S(\omega)$ for all $\omega \in \mathbb{R}^n$. Then
\[
[f_j, f_j](\omega) = \sum_{k \in \mathbb{Z}^n} |\tau(M^{-T}(\omega + 2\pi k))|^2 |f_{j-1}(M^{-T}(\omega + 2\pi k))|^2
= \sum_{\gamma \in \Gamma^*} \sum_{k \in \mathbb{Z}^n} |\tau(M^{-T}\omega + \gamma + 2\pi k)|^2 |f_{j-1}(M^{-T}\omega + \gamma + 2\pi k)|^2
= \sum_{\gamma \in \Gamma^*} |\tau(M^{-T}\omega + \gamma)|^2 |f_{j-1} + f_{j-1}|(M^{-T}\omega + \gamma)
\leq \sum_{\gamma \in \Gamma^*} |\tau(M^{-T}\omega + \gamma)|^2 \frac{1}{S(M^{-T}\omega + \gamma)}
\leq \frac{1}{S(M^{-T}\omega)}
= \frac{1}{S(\omega)}
\]
where we applied Equation (3), which holds for all $\omega \in \mathbb{T}^n$ because of the assumptions on $S$, to obtain the last inequality. We note that as $j \to \infty$, using the continuity of $S$ at 0, $f_j(\omega) \to \hat{\phi}(\omega)$. Applying Fatou’s Lemma with the counting measure, since $|f_j(\omega)|^2 \geq 0$ for all $\omega \in \mathbb{R}^n$ and $j \geq 0$, we see that
\[
[f, f](\omega) = \sum_{k \in \mathbb{Z}^n} |\hat{\phi}(\omega + 2\pi k)|^2 = \sum_{k \in \mathbb{Z}^n} \lim_{j \to \infty} |f_j(\omega + 2\pi k)|^2 \leq \liminf_{j \to \infty} \sum_{k \in \mathbb{Z}^n} |f_j(\omega + 2\pi k)|^2 \leq \frac{1}{S(\omega)} \quad \text{for all } \omega \in \mathbb{T}^n,
\]
whence applying the ess. boundedness assumption on $1/S$ yields (c), which completes the proof. □

It is easy to see that under the weaker assumption that $1/S$ is integrable over $[-\pi,\pi]^n$, we may not have that all of the masks $S, q_\ell$ are ess. bounded, so that (i) or (a) may not hold, but the argument for (c) shows that
\[
\|\hat{\phi}\|^2 = \int_{[-\pi,\pi]^n} |\hat{\phi}(\omega)|^2 d\omega \leq \int_{[-\pi,\pi]^n} \frac{1}{S(\omega)} d\omega < +\infty,
\]
which shows that $\hat{\phi}$, and therefore also $\phi$, belong to $L^2(\mathbb{R}^n)$.

Combining Theorems 2 and 3 yields the following corollary, which applies in the setting that $\tau$ satisfies the ordinary sub-QMF condition.

Corollary 2. Let $\tau$ be a trigonometric polynomial lowpass mask. The following are equivalent:

(A) The sub-QMF condition holds:
\[
\sum_{\gamma \in \Gamma^*} |\tau(\omega + \gamma)|^2 \leq 1 \quad \text{for all } \omega \in \mathbb{T}^n,
\]
(B) There exist rational trigonometric polynomials \( \{q_{\ell}\}_{\ell=1}^{r} \) such that for all \( \gamma \in \Gamma^* \) and \( \omega \in \mathbb{T}^{n} \):

\[
\tau(\omega)\tau(\omega + \gamma) + \sum_{\ell=1}^{r} q_{\ell}(\omega)q_{\ell}(\omega + \gamma) = \begin{cases} 
1 & \text{if } \gamma = 0 \\
0 & \text{otherwise.} 
\end{cases}
\]  

(11)

Moreover, provided that either (A) or (B) holds, there exist (a potentially different set of) rational trigonometric polynomials \( \{q_{\ell}\}_{\ell=1}^{r} \) such that for all \( \gamma \in \Gamma^* \) and \( \omega \in \mathbb{T}^{n} \), Equation (11) holds, with \( r \leq 2^{n}(1 + Q) \). When one of (A) or (B) holds, the wavelet system defined by the combined MRA mask \((\tau,q_{1},\ldots,q_{r})\) is a tight wavelet frame.

This corollary is quite similar to the result [4, Thm. 2.2], but both statements here are weaker than the ones that appear in that theorem. In particular, the analogous statement for (A) in [4] requires the existence of an sos representation for \( 1 - \sum_{\gamma} |\tau\gamma|^2 \), but their result guarantees the existence of \( q_{\ell} \) which are trigonometric polynomials in (B).

We apply this corollary to construct a tight wavelet frame using the lowpass mask considered in Example 2, for which \( 1 - \sum_{\gamma} |\tau\gamma|^2 \) has no sos representation.

Example 3. We extend Example 2 to construct a tight wavelet frame. Since \( f(\tau;\cdot) \) has an sos representation, by Result 2(b), it has such a representation with at most \( 2^{3} = 8 \) generators. By Lemma 3(b) and the discussion after that lemma, \( f \) has a \( G \)-invariant sos representation with at most 8 generators. Let \( g_{1},\ldots,g_{8} \) be these rational trigonometric polynomials. Then the wavelet system defined by the combined MRA mask \((\tau,q_{1,1},\ldots,q_{1,8},q_{2,(0,0,0)},\ldots,q_{2,(1,1,1)})\) is a tight wavelet frame, with highpass masks

\[
q_{1,j}(\omega) = \tau(\omega)g_{j}(2\omega) \quad 1 \leq j \leq 8, \\
q_{2,\nu}(\omega) = 2^{-3/2}e^{i\omega\nu} - \tau(\omega)\tau_{\nu}(2\omega) \quad \nu \in \{0,1\}^{3},
\]

where \( \tau_{\nu} \) are the polyphase components of \( \tau \), as usual.

Let us recall a few basic definitions.

Definition 8. For a lowpass mask \( \tau \), the accuracy number is defined to be the order of vanishing of \( \tau \) at \( \omega = \gamma \in \Gamma^* \setminus \{0\} \). For a highpass mask \( g \), the vanishing moments are the order of vanishing of \( g \) at \( \omega = 0 \).

Inspecting the OEP conditions, we see that we can give a lower bound on the number of vanishing moments of the constructed wavelet system in terms of the accuracy number of \( \tau \) and the vanishing moments of

\[
f(S,\tau;\cdot) = \frac{1}{S(M^{T}\tau)} - \sum_{\gamma \in \Gamma^*} \frac{|\tau\gamma|^2}{S^{\gamma}}.
\]

A similar discussion for approximation orders is given in [4], without the formulation involving \( f(S,\tau;\cdot) \). The following proposition describes this lower bound.

Proposition 1. In Theorem 3 let \( \tau \) have accuracy number \( a > 0 \), \( f(S,\tau;\cdot) \) have vanishing moments \( m \), and \( S - S(M^{T}\cdot)|\tau|^2 \) have vanishing moments \( j \). If (A) or (B) holds in that theorem, then the highpass masks \( q_{\ell}, 1 \leq \ell \leq r \) in (B) have at least \( \lfloor j/2 \rfloor \geq \lfloor \min\{a,m/2\} \rfloor \) vanishing moments.

Proof. Rearranging the OEP conditions [4] with \( \gamma = 0 \), we get

\[
S(\omega) - S(M^{T}\omega)|\tau(\omega)|^2 = \sum_{\ell=1}^{r} |q_{\ell}(\omega)|^2,
\]

so if the left-hand side is \( O(|\omega|^j) \) for \( \omega \approx 0 \), then \( q_{\ell} = O(|\omega|^{j/2}) \) there, for all \( 1 \leq \ell \leq r \). If \( f(S,\tau;\omega) = O(|\omega|^m) \) for \( \omega \approx 0 \), then

\[
\frac{1}{S(M^{T}\omega)} - \frac{|\tau(\omega)|^2}{S(\omega)} = O(|\omega|^m) + \sum_{\gamma \in \Gamma^* \setminus \{0\}} \frac{|\tau(\omega + \gamma)|^2}{S(\omega + \gamma)},
\]

\[
18
\]
which is \( O(|\omega|^{\min\{m,2a\}}) \) for \( \omega \approx 0 \). Then

\[
S(\omega) - S(\mathcal{M}^T\omega)|\tau(\omega)|^2 = S(\omega)S(\mathcal{M}^T\omega) \left( \frac{1}{S(\mathcal{M}^T\omega)} - \frac{|\tau(\omega)|^2}{S(\omega)} \right) \\
= S(\omega)S(\mathcal{M}^T\omega)O(|\omega|^{\min\{m,2a\}}) \text{ for } \omega \approx 0,
\]

which means that \( j \geq \min\{m,2a\} \). This completes the proof. \( \square \)

4.2. Extending the Setting

If we move away from the rational trigonometric polynomial setting in Theorem 2, we observe that the argument that the OEP conditions imply the oblique sub-QMF condition is still valid, provided that we restrict to a set where the functions are all defined. We state some generalizations of this implication in the proposition below, and its proof makes clear the relationship between the set on which Equation (3) is guaranteed to hold, given the sets on which \( S \) is nonzero and Equation (4) holds.

**Proposition 2.** Let \( S \) be a \( 2\pi \)-periodic function which is nonnegative on \( \mathbb{T}^n \), and let \( \mathcal{N} = \{ \omega \in \mathbb{T}^n : S(\omega) \neq 0 \} \). Let \( \tau, q_\ell, 1 \leq \ell \leq r \) be \( 2\pi \)-periodic functions, and let \( \mathcal{O} \) be the set of all points \( \omega \in \mathbb{T}^n \) such that Equation (4) holds for all \( \gamma \in \Gamma^* \). Then the oblique sub-QMF condition, the inequality in (3), holds for all \( \omega \in \mathcal{S} \), where \( \mathcal{S} \) is a subset of \( \mathbb{T}^n \) with the following properties:

(a) If \( \mathcal{N}, \mathcal{O} \) are open and dense, then so is \( \mathcal{S} \).

(b) If \( \mathcal{N}^c, \mathcal{O}^c \) have measure zero, then so does \( \mathcal{S}^c \).

(c) If \( \omega \in \mathcal{S} \), and \( \omega' \in \mathbb{T}^n \) is such that \( \omega' \equiv \omega + \gamma \mod 2\pi \mathbb{Z}^n \) for some \( \gamma \in \Gamma^* \), then \( \omega' \in \mathcal{S} \).

**Proof.** For a \( 2\pi \)-periodic function \( f \), let \( \mathcal{N}(f) := \{ \omega \in \mathbb{T}^n : f(\omega) \neq 0 \} \), and let \( \mathcal{N}^* := \mathcal{N}(S(\mathcal{M}^T \cdot)) \cap \bigcap_{\gamma \in \Gamma^*} \mathcal{N}(\mathcal{S}^\gamma) \). If Equation (4) holds on \( \mathcal{O} \), then by part (ii) of the proof of Theorem 2, we see that the inequality of (3) holds for all \( \omega \in \mathcal{S} = \mathcal{N}^* \cap \mathcal{O} \). Since \( \mathcal{S} \) is a finite intersection of sets, the various assumptions on \( \mathcal{N} \) and \( \mathcal{O} \) now clearly lead to the same properties for \( \mathcal{S} \), and in particular, \( \mathcal{S} \) is nonempty under either of the sets of assumptions in (a) or (b). The last property follows since both sides of the inequality in (3) are \( \mathcal{G} \)-invariant. \( \square \)

We similarly observe that the argument for Theorem 3 does not explicitly require that \( S \) is a rational trigonometric polynomial, besides ensuring the existence of the \( q_\ell \) with this property from Theorem 2. However, the argument by induction is complicated in this setting, because the sets on which inequalities (9) and (10) hold may no longer be all of \( \mathbb{R}^n \). After addressing these technical considerations, we have the following proposition.

**Proposition 3.** Let \( \tau \) be a trigonometric polynomial lowpass mask, let \( q_\ell, 1 \leq \ell \leq r \) be rational trigonometric polynomial masks, and let \( S \in L^\infty(\mathbb{T}^n) \) be a \( 2\pi \)-periodic, nonnegative function which is continuous at 0, with \( S(0) = 1 \). If \( 1/S \) also belongs to \( L^\infty(\mathbb{T}^n) \), and Equation (4) holds for all \( \gamma \in \Gamma^* \) and \( \omega \in \mathbb{T}^n \), then the wavelet system defined by the combined MRA mask \( (\tau, q_1, \ldots, q_r) \) is a tight wavelet frame.

We first state a technical lemma which is used in the proof of this proposition.

**Lemma 7.** Suppose \( \tau \) is a trigonometric polynomial lowpass mask, and let \( S \) be a \( 2\pi \)-periodic, nonnegative, measurable function, such that \( \text{meas}(S^c) = 0 \), where \( S \) is the set on which Equation (4) holds. Let \( f_j \) be defined as in Equation (5) for all \( j \geq 0 \). Then there exists \( \mathcal{D} \subseteq \mathbb{T}^n \) with \( \text{meas}(\mathcal{D}^c) = 0 \) such that for all \( j \geq 0 \), \( |f_j(\omega)| \leq 1/S(\omega) \) for all \( \omega \in \mathcal{D} \).

This lemma is proved in the appendix. Now we give the proof of the proposition.

**Proof of Proposition 3.** As in the proof of Theorem 3, we want to apply Result 3. We see that we have assumed (i) and (ii), and we prove (a) and (b) as we did in the proof of Theorem 3, since these proofs do
not make use of the assumption that $S$ is a rational trigonometric polynomial. Thus, the main change is in the proof of (c). Since $1/S$ is ess. bounded, we see that $N^c = \{ \omega \in \mathbb{T}^n : S(\omega) = 0 \}$ has measure zero. Then by Proposition 2(b), using $O = \mathbb{T}^n$, we see that Equation (3) holds for all $\omega$ in some set $S \subseteq \mathbb{T}^n$ with $\text{meas}(S^c) = 0$. Define $f_j$ as in Equation (3) for all $j \geq 0$. Applying Lemma 7, we see that there is some set $D \subseteq \mathbb{T}^n$ such that $[f_j, f_j](\omega) = 1/S(\omega)$ for all $\omega \in D$, where $\text{meas}(D^c) = 0$.

Applying Fatou’s Lemma as in the proof of Theorem 5, we see that $[\hat{\phi}, \hat{\phi}](\omega) \leq \liminf_{j \to \infty} [f_j, f_j](\omega)$ for all $\omega \in \mathbb{R}^n$, so

$$\hat{\phi}, \hat{\phi}(\omega) \leq 1/S(\omega) \text{ for all } \omega \in D.$$  (12)

In particular, this bound holds almost everywhere, so $[\hat{\phi}, \hat{\phi}] \in L^\infty(\mathbb{T}^n)$ from the assumption on $1/S$. □

5. Scaling Oblique Laplacian Pyramids

We now consider another perspective on Theorem 2, which is similar to the one described in the introduction for the UEP-based construction. This approach follows the one taken in [15], but rather than working with the Laplacian pyramid (LP) matrix coming from the polyphase components as was done there, in keeping with the current work, we consider the LP matrix with $G$-vector columns. Given a lowpass mask $\tau$, let $H(\omega) = [\tau(\omega + \gamma)]_{\gamma \in \Gamma^*}$. Then we define the Laplacian pyramid matrix $\Phi_\tau(\omega) = [H(\omega) (I - H(\omega)H(\omega)^*) X(\omega)]$, which is a $Q \times (Q + 1)$ matrix with trigonometric polynomial entries and $G$-vector columns, and $X(\omega) = [Q^{-1/2} e^{i(\omega + \gamma) / 2}]_{\gamma \in \Gamma^*}$. If $\tau$ is the Fourier transform matrix, then we see that $\Phi_\tau(\omega) \Phi_\tau(\omega)^* = I$, since by inspecting this entry matrix product, we see that $\tau$ satisfies the UEP conditions with the highpass masks $q_\nu(\omega) = Q^{-1/2} e^{-i\omega / 2} - \tau(\omega) \tau_g(M^T \omega)$. However, this requires $\tau$ to satisfy the restrictive QMF condition. Even when this condition does not hold, however, we can see that

$$\Phi_\tau(\omega) \begin{bmatrix} H(\omega)^* \\ X(\omega)^* \end{bmatrix} = [H(\omega) (I - H(\omega)H(\omega)^*) X(\omega)] \begin{bmatrix} H(\omega)^* \\ X(\omega)^* \end{bmatrix} = I,$$

so $\Phi_\tau$ has a right-inverse. But this right-inverse does not have the structure of a wavelet filter bank: the first row is a $G$-vector for the lowpass mask $\tau$, and the remaining rows are $G$-vectors for some trigonometric polynomials $q_\nu, \nu \in \Gamma$. Then $q_\nu(\omega) = Q^{-1/2} e^{i\omega / 2}$, so $q_\nu(0) = Q^{-1/2} \neq 0$, and therefore $q_\nu$ are not wavelet masks. To correct this, we try scaling the matrix $\Phi_\tau$ to get a new filter bank satisfying the UEP conditions. We want a matrix $D(\omega)$ with trigonometric polynomial entries such that $\Phi_\tau(\omega) D(\omega) \Phi_\tau(\omega)^* = I$. If $D(\omega)$ has a factorization as $B(M^T \omega) B(M^T \omega)^*$ for some $(Q + 1) \times (r + 1)$ matrix $B(\omega)$ with trigonometric polynomial entries, the product $\Phi'(\omega) = \Phi_\tau(\omega) B(M^T \omega)$ will have $G$-vector columns and satisfy $\Phi'(\omega) \Phi'(\omega)^* = I$.

Provided $\Phi'$ still has a lowpass mask generating its first column and highpass masks for the remaining columns, we will have a new collection of masks satisfying the UEP conditions, and an associated tight wavelet frame generated by these. In the introduction, we discussed some of the possible factorizations for $D(\omega)$ and the different constructions to which these lead. Now, we translate these ideas to the case of the OEP.

Suppose that $S$ is a nonnegative rational trigonometric polynomial satisfying $S(0) = 1$, and let $\tau$ be a lowpass mask satisfying the oblique sub-QMF condition with $S$. Recalling that $\Sigma(\omega) = \text{diag}(S(\omega + \gamma))_{\gamma \in \Gamma^*}$, we define the $Q \times (Q + 1)$ oblique Laplacian pyramid (OLP) matrix $\Phi_{S,\tau}(\omega) = [H(\omega) (\Sigma(\omega) - S(M^T \omega)H(\omega)H(\omega)^*) X(\omega)]$, where $X(\omega)$ is the Fourier transform matrix as above. Then

$$\Phi_{S,\tau}(\omega) \begin{bmatrix} S(M^T \omega)H(\omega)^* \\ X(\omega)^* \end{bmatrix} = \Sigma(\omega),$$

so inverting $\Sigma(\omega)$, the matrix $\Phi_{S,\tau}$ has a right-inverse almost everywhere, in particular, wherever $S(\omega + \gamma) \neq 0$ for all $\gamma \in \Gamma^*$. However, the second matrix in this product is once again not a wavelet filter bank, so as

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4To translate between these, consider the Fourier transform matrix $X(\omega) = [Q^{-1/2} e^{i(\omega + \gamma) / 2}]_{\gamma \in \Gamma^*}$. For a mask $g(\omega)$, if $G(\omega) = [g(\omega + \gamma)]_{\gamma \in \Gamma^*}$, then $[g(\omega + \gamma)]_{\gamma \in \Gamma} = X(\omega)^* G(\omega)$ (c.f. Equation 3).
before, we will attempt to correct this by scaling the masks in $\Phi_{S,\tau}$ in order to get a new collection of masks satisfying the OEP conditions.

Let $a(M^T \omega) = S(M^T \omega)(2 - S(M^T \omega)H(\omega)^*\Sigma(\omega)^{-1}H(\omega))$. Then

$$\Phi_{S,\tau}(\omega) = a(M^T \omega) 0 X(\omega)^*\Sigma(\omega)^{-1}X(\omega) \Phi_{S,\tau}(\omega)^*$$

$$= a(M^T \omega)H(\omega)H(\omega)^* + [\Sigma(\omega) - S(M^T \omega)H(\omega)^*\Sigma(\omega)]\Sigma(\omega) - S(M^T \omega)H(\omega)H(\omega)^*$$

$$= a(M^T \omega)H(\omega)H(\omega)^* + \Sigma(\omega) - 2S(M^T \omega)H(\omega)^* + S(M^T \omega)^2H(\omega)^*\Sigma(\omega)^{-1}H(\omega)H(\omega)^*$$

$$= \Sigma(\omega) + [a(M^T \omega) + S(M^T \omega)(-2 + S(M^T \omega)(H(\omega)^*\Sigma(\omega)^{-1}H(\omega)))\Sigma(\omega)\Sigma(\omega)^{-1}H(\omega)]H(\omega)H(\omega)^*$$

$$= \Sigma(\omega).$$

Note that $\Sigma(\omega)^{-1}$ is always well-defined as a rational trigonometric polynomial matrix so long as $S \neq 0$, but if $S$ has zeroes, then $\Sigma(\omega)^{-1}$ will have poles. The assumptions of Theorem 2 on $S$ also ensure that $\Sigma(\omega)^{-1}$ has no poles.

Then to obtain masks satisfying the OEP conditions, we want a factorization of the scaling matrix as

$$a(M^T \omega) 0 X(\omega)^*\Sigma(\omega)^{-1}X(\omega) = B(M^T \omega) \begin{bmatrix} S(M^T \omega) & 0 \\ 0 & I \end{bmatrix} B(M^T \omega)^*,$$

in which case $\Phi'(\omega) = \Phi_{S,\tau}(\omega)B(M^T \omega)$ will have $G$-vector columns, and provided its first column is generated by a lowpass mask and the rest are highpass, we will have a new collection of masks satisfying the OEP conditions, since

$$\Phi'(\omega) \begin{bmatrix} S(M^T \omega) & 0 \\ 0 & I \end{bmatrix} \Phi'(\omega)^* = \Phi_{S,\tau}(\omega) \begin{bmatrix} a(M^T \omega) & 0 \\ 0 & X(\omega)^*\Sigma(\omega)^{-1}X(\omega) \end{bmatrix} \Phi_{S,\tau}(\omega)^* = \Sigma(\omega).$$

Now we see that different factorizations of the scaling matrix lead to different constructions. In the proof of Theorem 2, we do not modify the lowpass mask, and add highpass masks corresponding to the sors generators for $a(M^T \omega) - S(M^T \omega) = S(M^T \omega)^2(1/S(M^T \omega) - H(\omega)^*\Sigma(\omega)^{-1}H(\omega))$. Then we might write the factorization of Equation 13 with

$$B(M^T \omega) = \begin{bmatrix} 1 & S(M^T \omega)G(M^T \omega)^* \\ 0 & 0 \end{bmatrix} X(\omega)^*A(\omega),$$

and $G(\omega), A(\omega)$ are as in the proof of the theorem. Since the columns of $A(\omega)$ are $G$-vectors, $X(\omega)^*A(\omega) = [(a_m)_e(M^T \omega)]_{e \in G, 1 \leq m \leq M}$, where $a_m$ is the rational trigonometric polynomial generating the $m$th column of $A(\omega)$.

If instead there is a square root for $a(\omega)/S(\omega)$, so that $a(\omega)/S(\omega) = |g(\omega)|^2$ for all $\omega \in \mathbb{T}^n$, then we might write the factorization of Equation 13 with

$$B(M^T \omega) = \begin{bmatrix} g(M^T \omega) & 0 \\ 0 & X(\omega)^*A(\omega) \end{bmatrix}.$$

This corresponds to modifying the lowpass mask to obtain $g(M^T \omega)\tau$.

A third possibility combines these two ideas, requiring a representation for $a(\omega)$ as $S(\omega)|g_0(\omega)|^2 + S(\omega)^2\sum_{j=1}^{J} |g_j(\omega)|^2$, where $g_0(0) = 1$. Then we might write the factorization of Equation 13 with

$$B(M^T \omega) = \begin{bmatrix} g_0(M^T \omega) & S(M^T \omega)G(M^T \omega)^* \\ 0 & 0 \end{bmatrix} X(\omega)^*A(\omega),$$

obtaining a modified lowpass mask $g_0(M^T \tau)$, as well as new highpass masks corresponding to the generators $g_1, \ldots, g_J$.

As such, depending on the kinds of sors representations available for $a(\omega)$, and the criteria of the tight wavelet filter bank designer (such as whether or not the lowpass mask should be modified), some of these constructions may be preferable to others. Further investigation of possible factorizations of this scaling matrix may also lead to entirely new constructions.
6. Examples

In this section, we consider the case of lowpass masks associated with box spline refinable functions. The first is a simple example which has been well-studied, for example, in [14, Example 1], [10, Example 5.2]. This is the piecewise linear box spline in two dimensions.

**Example 4.** Let \( \tau(\omega) = 8^{-1}(1 + e^{i\omega})(1 + e^{i\omega})(1 + e^{i(\omega_1 + \omega_2)}) \) be the lowpass mask associated with the piecewise linear box spline refinable function in dimension 2 with dyadic dilation. Note that \( \tau \) has accuracy and flatness numbers 2. Suppose \( S(\omega) = 1/B(\omega) \), where \( B(\omega) = a + b(\cos(\omega_1) + \cos(\omega_2)) + c\cos(\omega_1 + \omega_2) \).

We will see that there are ranges of values for \( a, b, c \) for which Theorem 2 applies, which demonstrates the flexibility afforded by the theorem above compared to that of [16], for which there is a single choice of \( S \) given.

We compute
\[
\frac{1}{S(2\omega)} - \sum_{\gamma \in \mathbb{Z}^n} |\tau(\omega + \gamma)|^2 \leq \frac{3a - 6b - 3c}{8} - \left( \frac{a - 4b + c}{8} \right)(\cos(2\omega_1) + \cos(2\omega_2))
\]

This has an sos representation with two sos generators, making it equal to
\[
\frac{\sqrt{3a - 12b + 3c}}{4} (\sqrt{2/3} - \sqrt{1/6}(e^{-2i\omega_1} + e^{2i\omega_2}))^2 + \left| \frac{\sqrt{3a - 9c}}{4\sqrt{2}} (1 - e^{2i(\omega_1 + \omega_2)}) \right|^2,
\]

which is valid whenever \( a \geq \max\{4b - c, 3c\} \). This ensures that \( 3a - 12b + 3c \geq 0 \) and \( 3a - 9c \geq 0 \). In fact, since we assume that \( S(0) = 1 \), which gives \( a = 1 - 2b - c \), these conditions become \( 6b \leq 1 \), and \( 2b + 4c \leq 1 \), which are together sufficient to guarantee that \( 1/S(\omega) \) has an sos representation as
\[
\frac{\sqrt{1/3 - 3b/2}(\sqrt{2/3} - \sqrt{1/6}(e^{-i\omega_1} + e^{i\omega_2}))}{2} + \left| \frac{\sqrt{1/3 - b/2 - c}}{\sqrt{2}} (1 - e^{i(\omega_1 + \omega_2)}) \right|^2 + \left| \frac{1}{3} (1 + e^{i\omega_1} + e^{-i\omega_2}) \right|^2.
\]

That is, \( 1/3 - 3b/2 \geq 0 \) and \( 1/3 - b/2 - c \geq 0 \) hold whenever \( 6b \leq 1 \) and \( 2b + 4c \leq 1 \).

But now we observe that when these are nonzero, \( g_1(\omega) = \frac{\sqrt{3a - 12b + 3c}}{4} (\sqrt{2/3} - \sqrt{1/6}(e^{-i\omega_1} + e^{i\omega_2})) \) has exactly 1 vanishing moment, and \( g_2(\omega) = \frac{\sqrt{3a - 9c}}{4\sqrt{2}} (1 - e^{i(\omega_1 + \omega_2)}) \) does also. In [16], a tight wavelet frame for this same lowpass mask is constructed with 2 vanishing moments for all wavelet masks, but there is only one possible choice of \( S \) considered there (with \( a = 1/2, b = c = 1/6 \)). If we choose these same values of \( a, b, \) and \( c \), we see that Equation (14) has right hand side equal to 0, and changing the sos representation for \( 1/S(\omega) \) using these values leads to the same tight wavelet frame as in [16]. In [14], this example is studied using the UEP, giving 7 wavelet masks, three of which have 1 vanishing moment, and the other four having 2 vanishing moments, as part of a construction which works for any dimension. We extend this example to arbitrary dimension in Example 4.

In [16], they discuss one method for finding a vmr function \( S \) such that the oblique QMF condition is satisfied with \( \tau \), which leads to a construction with maximum vanishing moments. To illustrate some of the flexibility of our method, we consider another approach for finding a vmr function, where \( S \) is just a product of univariate trigonometric polynomials aligned in the same directions as the factors of \( \tau \). What is perhaps surprising is that this simple approach to constructing the vmr function \( S \) can still result in constructions with nearly optimal vanishing moments. In particular, we will see that giving up one vanishing moment allows us to use this simple form of \( S \), in Examples 4 and 5.

**Example 5.** Let \( \mathcal{M} = 2I \), where \( I \) is the \( n \times n \) identity matrix for spatial dimension \( n \), and let \( \Xi \) be an integer matrix with \( n \) rows. Let \( \tau(\omega) = \prod_{k=1}^{n} (2^{-1}(1 + e^{i\omega})^{m_k}) \), where \( \{\xi_1, \ldots, \xi_d\} \) are the distinct...
columns of $\Xi$, so that $\Xi$ has $\sum_{j=1}^{d} m_j$ columns, where $\xi_k$ is repeated $m_k$ times for $1 \leq k \leq d$. We suppose that $\Xi = [\Xi_0, \Xi_1]$, where $\Xi_0$ is square and invertible mod 2. In particular, this means that $d \geq n$. Let $\Xi_0^{-1}$ be a square matrix such that $\Xi_0^{-1} \Xi_0 \equiv I \pmod{2^{2n \times n}}$, and with slight abuse of notation, let the columns of $\Xi_0^{-1}$ be $\xi_k^{-1}, 1 \leq k \leq n$.

We will find a simple vnr function $S$ for which $f(S, \tau; \cdot)$ has $2\mu = 2 \min\{m_k : 1 \leq k \leq n\}$ vanishing moments. If we write $\tau = \tau_0 \tau_1$, where $\tau_0$ is the box spline lowpass mask associated with $\Xi_0$ and all multiples of these columns in $\Xi$, and $\tau_1$ is the product of the remaining factors, then $\mu$ is the accuracy number of $\tau_0$. This is typically less than the accuracy number of $\tau$, but in light of Proposition\[\text{[1]}\] we see that the highpass masks satisfying the OEP conditions with this $\tau$ and $S$ will all have at least $\mu$ vanishing moments.

Let $\ell \geq \mu$ be an integer. Now we suppose that there exists a collection of univariate trigonometric polynomials $s_k, 1 \leq k \leq d$, such that for $\omega \in \mathbb{T}$,

\begin{align*}
s_k(\omega) > 0 & \quad \text{for all } 1 \leq k \leq d, \\
s_k(2\omega) - \sum_{\gamma \in \{0, \pi\}} \cos^{2m_k}((\omega + \gamma)/2)s_k(\omega + \gamma) \geq 0 & \quad \text{for } 1 \leq k \leq n, \\
s_k(2\omega) - \sum_{\gamma \in \{0, \pi\}} \cos^{2m_k}((\omega + \gamma)/2)s_k(\omega + \gamma) = O(|\omega|^{2\ell}) & \quad \text{for } \omega \approx 0 \text{ for } 1 \leq k \leq n, \\
s_k(2\omega) - \cos^{2m_k}(\omega/2)s_k(\omega) \geq 0 & \quad \text{for } n+1 \leq k \leq d, \\
s_k(2\omega) - \cos^{2m_k}(\omega/2)s_k(\omega) = O(|\omega|^{2\ell}) & \quad \text{for } \omega \approx 0 \text{ for } n+1 \leq k \leq d.
\end{align*}

In Table\[\text{[1]}\] we give the coefficients for trigonometric polynomials satisfying all of these conditions with some values of $m_k$ and $\ell$. In that table, we write $s_{1,m,\ell}$ for the trigonometric polynomials $s_k$ satisfying the second and third lines here (setting $m$ in the table to the desired $m_k$), and $s_{2,m,\ell}$ for the trigonometric polynomials $s_k$ satisfying the last two equations here. All of the trigonometric polynomials in the table are positive for all $\omega \in \mathbb{T}$.

Let

$$\begin{align*}
S(\omega) = \left( \prod_{k=1}^{d} s_k(\omega \cdot \xi_k) \right)^{-1}.
\end{align*}$$

For $\omega \in \mathbb{T}^n, \gamma = [\gamma_i]_{i=1}^n \in \{0, \pi\}^n$, and $1 \leq i \leq d$, let $t_i(\omega, \gamma) = \cos^{2m_i}((\omega + \Xi_0^{-T} \gamma) \cdot \xi_i) s_i((\omega + \Xi_0^{-T} \gamma) \cdot \xi_i)$. Note that for $1 \leq i \leq n$, using $\Xi_0^{-1} \Xi_0 \equiv I \pmod{2^{2n \times n}}$, we have that $(\omega + \Xi_0^{-T} \gamma) \cdot \xi_i \equiv \omega \cdot \xi_i + \gamma_i \pmod{2\pi}$, so for $1 \leq i \leq n$, using $2\pi$-periodicity of $s_i$ and $\pi$-periodicity of $\cos^{2m_i}()$, we may write $t_i(\omega \cdot \xi_i, \gamma_i) = \cos^{2m_i}((\omega + \Xi_0^{-T} \gamma) \cdot \xi_i)$.
cos^{2m_i}((\omega \cdot \xi_i + \gamma_i)/2)s_{i}(\omega \cdot \xi_i + \gamma_i), since these t_i do not depend on the full vectors \omega and \gamma. In the computation below, we only write t_i in this way when we want to emphasize this fact.

Changing our set of representatives for \((\pi \mathbb{Z}^n/2\pi \mathbb{Z}^n)\) from \(\{0, \pi\}^n\) to \(\Xi_0^{-T}\{0, \pi\}^n\), and using the \(2\pi \mathbb{Z}^n\)-periodicity of these functions, we have

\[
\frac{1}{S(2\omega)} - \sum_{\gamma \in \{0, \pi\}^n} \frac{|\tau(\omega + \gamma)|^2}{S(\omega + \gamma)} = \frac{1}{S(2\omega)} - \sum_{\gamma \in \{0, \pi\}^n} \frac{|\tau(\omega + \Xi_0^{-T}\gamma)|^2}{S(\omega + \Xi_0^{-T}\gamma)}
\]

\[
= \prod_{k=1}^{d} s_k(2\omega \cdot \xi_k) - \sum_{\gamma \in \{0, \pi\}^n} \left( \prod_{i=1}^{n} t_i(\omega \cdot \xi_i, \gamma_i) \right) \left( \prod_{j=n+1}^{d} t_j(\omega, \gamma) \right)
\]

\[
= \prod_{k=1}^{d} s_k(2\omega \cdot \xi_k) - \left( \prod_{i=1}^{n} \sum_{\gamma \in \{0, \pi\}^n} t_i(\omega \cdot \xi_i, \gamma_i) \right) \left( \prod_{k=n+1}^{d} s_k(2\omega \cdot \xi_k) \right) + \left( \sum_{\gamma \in \{0, \pi\}^n} \left( \prod_{i=1}^{n} t_i(\omega, \gamma) \right) \left( \prod_{k=n+1}^{d} s_k(2\omega \cdot \xi_k) \right) - \sum_{\gamma \in \{0, \pi\}^n} \left( \prod_{i=1}^{d} t_i(\omega, \gamma) \right) \right)
\]

(16)

where the middle two quantities are equal, so we have added 0 to get the last equation. Expanding Line (16) as a telescoping series, we get

\[
\sum_{k=1}^{n} \left( \prod_{j=k+1}^{d} s_j(2\omega \cdot \xi_j) \right) \left( \prod_{i=1}^{k-1} \sum_{\gamma \in \{0, \pi\}^n} t_i(\omega \cdot \xi_i, \gamma_i) \right) \left( s_k(2\omega \cdot \xi_k) - \sum_{\gamma \in \{0, \pi\}^n} t_k(\omega \cdot \xi_k, \gamma_k) \right)
\]

(17)

For Line (17), since both terms are being summed over \(\gamma \in \{0, \pi\}^n\), and \(s_k(2\cdot)\) is \(G\)-invariant, telescoping again gives

\[
\sum_{\gamma \in \{0, \pi\}^n} \left[ \prod_{k=n+1}^{d} s_j(2\omega \cdot \xi_j) \right] \left( \prod_{i=1}^{k-1} t_i(\omega, \gamma) \right) \left( s_k(2\omega \cdot \xi_k) - \sum_{\gamma \in \{0, \pi\}^n} t_k(\omega, \gamma) \right)
\]

(19)

By the nonnegativity assumptions on the \(s_k\) in Equation (15), all of the factors in every term of (18) and (19) are nonnegative. Each factor in every term of (18) and (19) is a nonnegative univariate trigonometric polynomial, so applying the Fejér-Riesz Lemma several times, we have an sos representation for \(f(S, \tau; \cdot)\) with \(n\) terms from (18) and \(2^d(d - n)\) terms from (19), for a total of \(n + 2^n(d - n)\) sos generators. In fact, the terms from (18) are products of nonnegative univariate trigonometric polynomials of the form \(g(2\omega \cdot \xi_i)\) for some \(1 \leq i \leq d\), so applying the Fejér-Riesz Lemma to \(g = |p|^2\), we get that \(g(2\omega \cdot \xi) = |p(2\omega \cdot \xi)|^2\), which means these sos generators are \(G\)-invariant. On the other hand, in (19), these factors are not all functions of \(2\omega\), but since our application of the Fejér-Riesz Lemma gave us trigonometric polynomials \(g_k\) for which (19) equals (switching the order of summation) \(\sum_{k=n+1}^{d} \sum_{\gamma \in \{0, \pi\}^n} |g_k(\omega)|^2\), applying Lemma 5(a) yields a representation for (19) as

\[
\sum_{k=n+1}^{d} \sum_{\nu \in \{0, 1\}^n} |(g_k)_\nu(2\omega)|^2,
\]

which is a \(G\)-invariant sos with \(2^n(d - n)\) terms.

If we apply the method in the proof of Theorem 2, we get a highpass mask \(q_{1,j}\) for every sos generator of \(f(S, \tau; \cdot)\), plus an additional highpass mask \(q_{2,m}\) for every column of \(A(\omega)\). We just showed that \(f(S, \tau; \cdot)\) has a \(G\)-invariant sos representation with \(n + 2^n(d - n)\) generators, and since \(1/S\) is a product of positive univariate
trigonometric polynomials, applying the Fejér-Riesz Lemma \( d \) times gives \( 1/S = |q|^2 \) for a trigonometric polynomial \( q \). Using the method of proof in Lemma 6 for writing \( A(\omega) \), \( 1/S \) has \( J = 1 \) sos generators, so we get a matrix \( A(\omega) \) with \( Q = 2^n \) columns. This leads to a tight wavelet frame with \( n + 2^n(d - n + 1) \) wavelet masks, using the method of Theorem 2.

Moreover, using the conditions in Equation (15), we observe that in (18), the factors where subtraction is taking place are \( O(|\omega|^{2\ell}) \) for \( \omega \approx 0 \), and this is true for the \( \gamma = 0 \) term in (19) as well. For \( \gamma \neq 0 \), the factors in Equation (19) of the form \( \ell_k(\omega, \gamma) \) can be equal to 1 or 0 at \( \omega = 0 \) depending on \( \Xi_0 \). The \( \gamma \)-vanishing moments have to come from the factors \( \ell_i(\omega, \gamma) \), \( 1 \leq i \leq k - 1 \). Since \( k \geq n + 1 \), this includes all of the factors \( \cos^{2m_i}((\omega \cdot \xi_i + \gamma_i)/2) \), \( 1 \leq i \leq n \). Then for any \( \gamma \in \{0, \pi\}^n \setminus \{0\} \), some \( \gamma_i = \pi \), so the vanishing moments have to come from the factors \( \ell_i(\omega, \gamma) \), \( 1 \leq i \leq k - 1 \). Since \( k \geq n + 1 \), this includes all of the factors \( \cos^{2m_i}((\omega \cdot \xi_i + \gamma_i)/2) \), \( 1 \leq i \leq n \). Then for any \( \gamma \in \{0, \pi\}^n \setminus \{0\} \), some \( \gamma_i = \pi \), and the factor of \( \cos^{2m_i}((\omega \cdot \xi_i + \gamma_i)/2) = \sin^{2m_i}((\omega \cdot \xi_i)/2) \) means this term must be \( O(|\omega|^{2m_i}) \leq O(|\omega|^{2\mu}) \) for \( \omega \approx 0 \), where \( \mu = \min\{m_k: 1 \leq k \leq n\} \) as above. Since \( f(S, \tau; \cdot) \) is the sum of (18) and (19), and \( \ell \geq \mu \), we see that \( f(S, \tau; \cdot) \) has at least \( 2\mu \) vanishing moments. Applying Proposition 1, the highpass masks in the wavelet system constructed in Theorem 2 will all have at least \( \mu \) vanishing moments, since \( \tau \) has accuracy number at least \( \mu \).

In [5], a construction based on the UEP obtained the same number of highpass masks generating a tight wavelet frame with any box spline refinable function, but their construction always has some masks having only one vanishing moment, whereas ours typically gives more. The next few examples use the procedure just described to obtain tight wavelet frames with near-optimal vanishing moments in a few particular cases.

In the next three examples, we apply this general construction to get tight wavelet frames in arbitrary dimension with near-optimal vanishing moments. We are able to describe the highpass masks for any dimension for box splines having these particular forms because of the simple form of \( S \) that we are considering, but we still obtain one fewer vanishing moment than optimal in these cases. We start by generalizing the piecewise linear box spline example to any dimension.

Example 6. Let \( \Xi = [I \ e] \), which corresponds to the piecewise linear box spline refinable function in \( n \) dimensions. Using Table 1 entries \((a, m, \ell) = (1, 1, 2) \) and \((2, 1, 2) \), we get \( S(\omega) = 6/(5 + \cos(\omega \cdot e)) \), since \( s_{1,1,2} = 1 \) and \( s_{2,1,2} = (5 + \cos(\omega))/6 \). Since

\[
1 - \cos^2(\omega/2) - \sin^2(\omega/2) = 0, \quad \text{and} \quad 5 + \cos(2\omega)/6 - \cos^2(\omega/2)(5 + \cos(\omega))/6 = \sin^4(\omega/2),
\]

the calculation in Example 5 yields

\[
f(S, \tau; \omega) = \sum_{\gamma \in \{0, \pi\}^n} \left( \prod_{i=1}^{n} \cos^2((\omega_i + \gamma_i)/2) \right) \sin^4((\omega + \gamma) \cdot e)/2.
\]

Then if we let \( g(\omega) = \left( \prod_{j=1}^{n} (1 + e^{i\omega_j})/2 \right) \sin^2(\omega \cdot e)/2 \), by Lemma 5(a),

\[
f(S, \tau; \cdot) = \sum_{\gamma \in \{0, \pi\}^n} |g^{(\gamma)}|^2 = \sum_{\nu \in \{0, 1\}^n} |g_\nu(2\omega)|^2,
\]

which is a \( \mathcal{G} \)-invariant sos representation. Since \( 1/S(\omega) = |((2 + \sqrt{6}) + (-2 + \sqrt{6})e^{i\omega - \epsilon})(2\sqrt{6})|^2 \), following the proof of Theorem 2 we obtain the \( 2^{n+1} \) highpass masks, for \( \nu \in \{0, 1\}^n \):

\[
q_{1,\nu}(\omega) = S(2\omega)\tau(\omega)g_\nu(2\omega),
\]

\[
q_{2,\nu}(\omega) = S(\omega)e^{i\omega \nu}(2 + \sqrt{6}) + (-2 + \sqrt{6})e^{i\omega \nu})
\]

\[
- S(2\omega)\tau(\omega) \sum_{\gamma \in \{0, \pi\}^n} \tau(\omega + \gamma)e^{i(\omega + \gamma) \cdot \nu} 2\sqrt{6}((2 + \sqrt{6}) + (-2 + \sqrt{6})e^{i(\omega + \gamma) - \epsilon}).
\]
If we denote \( r = \tau((2 + \sqrt{6}) + (-2 + \sqrt{6})e^{-i\omega \cdot e})/(2\sqrt{6}) \), then the sum in the equation for \( q_{2,\nu} \) is just \( 2^{n/2}r_{\nu}(2\omega) \). Moreover, if we want to write \( S \) as a series, so that we may find the filter coefficients for these highpass masks, it is not difficult to show that

\[
\frac{6}{5 + \cos(\omega)} = \frac{\sqrt{6}}{2} \left( 1 + \frac{2}{\omega} \sum_{n=1}^{\infty} \cos(n\omega \cdot e)(2\sqrt{6} - 5)^n \right),
\]

which converges for all \( \omega \in \mathbb{T} \) since \( 5 - 2\sqrt{6} \approx 0.101 \).

In Figure 2(a), we give the picture of the refinable function for this example, with \( n = 2 \). In Figure 1 we draw the the wavelets \( \psi^{(1,\nu)}, \psi^{(2,\nu)} \) associated with the corresponding wavelet masks, for \( n = 2 \). In this case, the four squares \( |g_{\nu}|^2 \) have two equal pairs, which we combine, so we have 6 total mother wavelets. Note that the wavelets technically have infinite support, since the factor \( S \) is a rational trigonometric polynomial with nonconstant denominator, but looking at the series representation for \( S \), we see that the coefficients are decaying exponentially \( ((5 - \sqrt{6})^n \approx e^{-2.3n}) \), so the picture is changed imperceptibly by the terms with \( n \geq 4 \).

In the next example, we consider a collection of lowpass masks with accuracy number 3, and we obtain a construction which works for any dimension with at least 2 vanishing moments for all highpass masks.

**Example 7.** Let \( \Xi = [I \ I \ e] \), which yields a lowpass mask \( \tau \) having accuracy number 3. Using Table 1 entries \((a, m, \ell) = (1, 2, 2)\) and \((2, 1, 2)\), we get

\[
S(\omega) = \left[ \prod_{j=1}^{n} (2 + \cos(\omega_j))/3 \right] (5 + \cos(\omega \cdot e))/6.\]

Since

\[
(2 + \cos(2\omega))/3 - \cos^4(\omega/2)(2 + \cos(\omega))/3 - \sin^4(\omega/2)(2 - \cos(\omega))/3 = 0, \text{ and }
(5 + \cos(2\omega))/6 - \cos^2(\omega/2)(5 + \cos(\omega))/6 = \sin^4(\omega/2),
\]

the computation in Example 3 yields

\[
f(S, \tau; \omega) = \sum_{\gamma \in \{0, \pi\}^n} \left( \prod_{j=1}^{n} \cos^4((\omega_j + \gamma_j)/2)(2 + \cos(\omega_j + \gamma_j))/3 \right) \sin^4((\omega + \gamma) \cdot e)/2.
\]

When \( \gamma = 0 \), we see that \( \sin^4(\omega \cdot e)/2 \) has 4 vanishing moments, and when \( \gamma \neq 0 \), some \( \gamma_j = \pi \), in which case \( \cos^4((\omega_j + \gamma_j)/2) = \sin^4(\omega_j/2) \) has 4 vanishing moments. Letting

\[
g(\omega) = \left( \prod_{j=1}^{n} \cos^2(\omega_j/2)((1 + \sqrt{3}) + (-1 + \sqrt{3})e^{i\omega_j})/(2\sqrt{3}) \right) \sin^2(\omega \cdot e)/2,
\]

we see that

\[
f(S, \tau; \omega) = \sum_{\gamma \in \{0, \pi\}^n} |g^\gamma(\omega)|^2 = \sum_{\nu \in \{0, 1\}^n} |g_{\nu}(2\omega)|^2,
\]

which is a \( G \)-invariant sos representation with \( 2^n \) generators. This leads to a collection of \( 2^{n+1} \) highpass masks satisfying the OEP conditions with this \( S \) and \( \tau \), using the method of proof in Theorem 2 and all of these highpass masks have at least 2 vanishing moments, by Proposition 1.

In Figure 2(b), the refinable function associated with \( \tau \) in this example is given for \( n = 2 \). As in Example 6 the wavelet functions have the same regularity as the refinable function, so in particular, they are continuously differentiable and piecewise quadratic.
Figure 1: Wavelet Functions for Example $n = 2$

(a) Wavelet $\psi^{(1,0)}$
(b) Wavelet $\psi^{(1,e_1)}$
(c) Wavelet $\psi^{(2,0)}$
(d) Wavelet $\psi^{(2,e_1)}$
(e) Wavelet $\psi^{(2,e_2)}$
(f) Wavelet $\psi^{(2,e)}$

Figure 2: Refinable Functions for Examples 6, 7, and 8, $n = 2$

(a) $\phi$ for Example 6
(b) $\phi$ for Example 7
(c) $\phi$ for Example 8

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In the following example, we consider a collection of lowpass masks with accuracy number 4, and we obtain a construction which works for any dimension with at least 3 vanishing moments for all highpass masks.

**Example 8.** Let \( \Xi = [I \ I \ I \ e] \), which yields a lowpass mask \( \tau \) having accuracy number 4. Using Table I entries \((a, m, \ell) = (1, 3, 3) \) and \((2, 1, 3) \), we get

\[
S(\omega) = \left( \prod_{j=1}^{n} (33 + 26 \cos(\omega_j) + \cos(2\omega_j))/60 \right) (97 + 24 \cos(\omega \cdot e) - \cos(2\omega \cdot e))/120 \right)^{-1}.
\]

Since

\[
s_{1,3,3}(2\omega) - \cos^6(\omega/2)s_{1,3,3}(\omega) - \sin^6(\omega/2)s_{1,3,3}(\omega + \pi) = 0, \quad \text{and} \quad s_{2,1,3}(2\omega) - \cos^2(\omega/2)s_{2,1,3}(\omega) = \sin^6(\omega/2)(23 + 8 \cos(\omega))/15,
\]
calling the right hand side of the second line \( r(\omega) \), the computation in Example 8 yields

\[
f(S, \tau; \omega) = \sum_{\gamma \in \{0, \pi\}^n} \left( \prod_{j=1}^{n} \cos^6((\omega_j + \gamma_j)/2)s_{1,3,3}(\omega_j + \gamma_j) \right) r((\omega + \gamma) \cdot e).
\]

When \( \gamma = 0 \), the factor of \( \sin^6(\omega \cdot e/2) \) in \( r(\omega \cdot e) \) has 6 vanishing moments, and when \( \gamma_j = \pi, \cos^6((\omega_j + \gamma_j)/2) = \sin^6(\omega_j/2) \) has 6 vanishing moments, so \( f(S, \tau; \cdot) \) has 6 vanishing moments. As in the previous examples, we may define \( g \) using the relation \( f(S, \tau; \cdot) = \sum_{\gamma} \left| g(\gamma) \right|^2 \), and using the polyphase components of \( g \), we obtain a \( G \)-invariant sos representation for \( f(S, \tau; \cdot) \). This may be used to obtain \( 2^n+1 \) highpass masks satisfying the OEP conditions with \( S \) and \( \tau \), using the method of proof in Theorem 1 and all of these have at least 3 vanishing moments, by Proposition 1.

In Figure 3(c), the refinable function associated with \( \tau \) in this example is given for \( n = 2 \). Since the wavelet functions have the same regularity as the refinable function, they are continuously twice-differentiable and piecewise cubic.

The following example leaves the setting of dyadic dilation, and shows how a tight wavelet frame with maximum vanishing moments may be obtained for the lowpass mask associated with the cubic B-spline refinable function in one dimension with dilation factor 3. In this example, \( S \) and \( \tau \) satisfy the oblique QMF condition.

**Example 9.** Consider the lowpass mask for the univariate cubic B-spline with dilation 3, \( \tau(\omega) = \left( \frac{1}{4}(1 + 2 \cos(\omega)) \right)^3 \), which has accuracy number 3 and flatness number 1. Let \( S(\omega) = 120/(66 + 52 \cos(\omega) + 2 \cos(2\omega)) \). Then

\[
\frac{1}{S(3\omega)} - \sum_{\gamma \in \{0,2\pi/3,4\pi/3\}} \frac{|\tau(\omega + \gamma)|^2}{S(\omega + \gamma)} = 0,
\]

and since \( 1/S(\omega) = |p(\omega)|^2 \) with

\[
p(\omega) = \frac{1}{4} \left( 2(1 - 2/\sqrt{30}) + 2(1 + 2/\sqrt{30}) \cos(\omega) - 2i \sqrt{4 + 12(1 + 2\sqrt{30} \sin(\omega))} \right),
\]

we get a tight wavelet frame generated by

\[
q_{2,3}(\omega) = 3^{-1/2} e^{3i\omega} p(\omega) S(\omega) - 3^{-1/2} S(3\omega) \tau(\omega) \sum_{\gamma \in \{0,2\pi/3,4\pi/3\}} \tau(\omega + \gamma) e^{i(j(\omega + \gamma)} p(\omega + \gamma), \quad j \in \{-1,0,1\}.
\]

Moreover, all three of these highpass masks have 3 vanishing moments.
If we want to write \( S \) as a series, then letting \( \alpha = \frac{1}{2} \left( -13 + \sqrt{105} + \sqrt{2(135 - 13\sqrt{105})} \right) \approx -0.4306, \beta = \frac{1}{2} \left( -13 - \sqrt{105} + \sqrt{2(135 + 13\sqrt{105})} \right) \approx -0.0431 \), we have

\[
S(\omega) = \frac{120\alpha\beta}{(1 - \alpha^2)(1 - \beta^2)} \left[ \left( 1 + \alpha\beta \right) \left( 1 + 2 \sum_{k=1}^{\infty} \cos(k\omega)(\alpha^k + \beta^k) \right) + 2\alpha\beta \sum_{k=2}^{\infty} \cos(k\omega) \left( \frac{\alpha^{k-1} - \beta^{k-1}}{\alpha - \beta} \right) \right],
\]

where the coefficients of \( \cos(k\omega) \) in \( S \) are once again decaying exponentially in magnitude as \( k \) increases. In Figure 3 we show the refinable function and central portions of the wavelet functions \( \psi(j), j \in \{-1, 0, 1\} \), since the wavelets have infinite support. Here, the exponential decay of the coefficients means that we can neglect the terms \( \cos(k\omega) \) with \( k > 10 \), because their effect on the picture is imperceptible.

\[
\Box
\]

Appendix A. Proof of Lemma 7

Since the proof is technical, we begin with a sketch of the argument. We would like to argue by induction as in Theorem 3, but see that this will be complicated by Equations (9) and (10), which no longer hold on all of \( \mathbb{T}^n \). If the desired set \( D \) exists, and \( \omega \in D \), then Equation (9) will hold whenever \( \mathcal{M}^{-T}\omega + \gamma \in D \), and by definition, Equation (10) will hold whenever \( \mathcal{M}^{-T}\omega \in S \), so we seek a set \( D \) such that for \( \omega \in D \), these two properties will hold. The second condition suggests using \( D = \mathcal{M}^T S \), but if, from the first condition, \( \mathcal{M}^{-T}\omega + \gamma \in \mathcal{M}^T S \), we see that \( \omega \in (\mathcal{M}^T)^2(S - \mathcal{M}^{-T}\gamma) \) (once these sets are appropriately defined), so we might take \( D \) to be their intersection. Continuing with this idea gives the following set:

\[
D = (\mathcal{M}^T S) \cap \bigcap_{k=1}^{\infty} (\mathcal{M}^T)^{k+1} \left[ \bigcap_{\gamma_j \in \Gamma^*} \left( S - \sum_{j=1}^{k} (\mathcal{M}^{-T})^j \gamma_j \right) \right].
\]

Now we make things precise.

Proof of Lemma 7. In this equation for \( D \), we use the following definitions: For \( C \subseteq \mathbb{T}^n, c \in \mathbb{R}^n, A \in M_n(\mathbb{R}) \), let \( C + c := \{ \omega \in \mathbb{T}^n : \exists \omega' \in C \text{ such that } \omega \equiv \omega' + c \text{ (mod } 2\pi\mathbb{Z}^n) \}, AC := \{ \omega \in \mathbb{T}^n : \exists \omega' \in C \text{ such that } \omega \equiv A\omega' \text{ (mod } 2\pi\mathbb{Z}^n) \} \).

We want to repeat the induction in the proof of Theorem 3. Since the quantities in the inequalities (9) and (10) are \( 2\pi \)-periodic, we want to show that \( D \) above has the following properties:

1. Whenever \( \omega \in D \) and \( \gamma \in \Gamma^* \), there is some \( \omega' \in D \) such that \( \mathcal{M}^{-T}\omega + \gamma \equiv \omega' \text{ (mod } 2\pi\mathbb{Z}^n) \); and
2. Whenever $\omega \in D$, there is some $\omega' \in S$ such that $M^{-T}\omega \equiv \omega' \pmod{2\pi \mathbb{Z}^n}$.

**Proof of Property 2:** For $\omega \in D$, $\omega \in M^T S$, so by definition, there is some $\omega' \in S$, $\ell \in \mathbb{Z}^n$ such that $\omega = M^T\omega' + 2\pi \ell$. Then $M^{-T}\omega = \omega' + 2\pi M^{-T}\ell \equiv \omega' + \gamma \pmod{2\pi \mathbb{Z}^n}$ for some $\gamma \in \Gamma$. Using Proposition [2] (or just the $2\pi$-periodicity of both sides of the inequality [3]), $M^{-T}\omega \equiv \omega'' \pmod{2\pi \mathbb{Z}^n}$ for some $\omega'' \in S$. This proves property 2.

**Proof of Property 1:** Let $\omega \in D$, and suppose for some $\gamma \in \Gamma^*$ that we can show that there is some $\omega_0 \in H_0 := M^T S$, and for all $k \geq 1$, $\omega_k \in H_k := (M^T)^{k+1} \left[ S - \sum_{j=1}^{k} (M^{-T})^j \gamma_j \right]$, such that $M^{-T}\omega + \gamma \equiv \omega_k \pmod{2\pi \mathbb{Z}^n}$ for all $k \geq 0$. Then if we let $\omega' \in \mathbb{T}^n$ be such that $M^{-T}\omega + \gamma \equiv \omega'$ (mod $2\pi \mathbb{Z}^n$), we see that $\omega' \equiv \omega_k \pmod{2\pi \mathbb{Z}^n}$ for all $k \geq 0$, which implies that $\omega' \in H_k$ for all $k \geq 0$, using the definition of $AC$ above. This means that $\omega' \in D$ as desired. So to prove property 1, it is sufficient for us to show that for $\omega \in D$, $\gamma \in \Gamma^*$, and each $k \geq 0$, there is some $\omega_k \in H_k$ such that $M^{-T}\omega + \gamma \equiv \omega_k$ (mod $2\pi \mathbb{Z}^n$).

For $\omega \in D$ and $k \geq 2$, $\omega \in H_k$, so there is an $\omega' \in H_k' := \bigcap_{\omega \in H_k} \left( S - \sum_{j=1}^{k} (M^{-T})^j \gamma_j \right)$ such that $\omega = (M^T)^{k+1} \omega' + 2\pi \ell$, $\ell \in \mathbb{Z}^n$. Then for any $\gamma \in \Gamma^*$, $M^{-T}\omega + \gamma = (M^T)^k \omega' + 2\pi M^{-T} \ell + \gamma = (M^T)^k \omega' + \gamma' + 2\pi \ell'$, for some $\gamma' \in \Gamma^*$ and $\ell' \in \mathbb{Z}^n$. Provided that for any $\gamma' \in \Gamma^*$, $\omega' + (M^{-T})^k \gamma' \equiv \omega''$ (mod $2\pi \mathbb{Z}^n$), for all $\omega'' \in H_{k-1}$, this means that $M^{-T}\omega + \gamma \equiv (M^T)^k (\omega' + (M^{-T})^k \gamma') \equiv (M^T)^k \omega'' \equiv \omega_{k-1}$ (mod $2\pi \mathbb{Z}^n$) for some $\omega_{k-1} \in H_{k-1}$. Then $\omega_{k-1} \in H_{k-1}$.

Now we want to show that $\omega' \in H_k'$ means that for any $\gamma' \in \Gamma^*$, $\omega' + (M^{-T})^k \gamma' \equiv \omega''$ (mod $2\pi \mathbb{Z}^n$), for some $\omega'' \in H_{k-1}$. By definition of $H_k'$, for any $\gamma_1, \ldots, \gamma_k \in \Gamma^*$, there is some $\eta \in S$ such that $\omega' \equiv \eta - \sum_{j=1}^{k} (M^{-T})^j \gamma_j \pmod{2\pi \mathbb{Z}^n}$. But if $\gamma_k = \gamma'$, $\omega' + (M^{-T})^k \gamma' \equiv \eta - \sum_{j=1}^{k-1} (M^{-T})^j \gamma_j$. Since for all $\gamma_1, \ldots, \gamma_{k-1} \in \Gamma^*$, this congruence holds for some $\eta \in S$, if we let $\omega'' \in \mathbb{T}^n$ be such that $\omega'' \equiv \omega' + (M^{-T})^k \gamma' \pmod{2\pi \mathbb{Z}^n}$, $\omega'' \in H_{k-1}$. This shows that $M^{-T}\omega + \gamma \equiv \omega_{k-1}$ (mod $2\pi \mathbb{Z}^n$) for some $\omega_{k-1} \in H_{k-1}$, for all $k \geq 2$.

It is also the case that $\omega \in H_1$, and essentially the same argument in this case shows that $M^{-T}\omega + \gamma \equiv \omega_0 \pmod{2\pi \mathbb{Z}^n}$ for $\omega_0 \in M^T S = H_0$. This proves that for all $\gamma \in \Gamma^*$, there exist $\omega_k \in H_k$ for all $k \geq 0$ such that $M^{-T}\omega + \gamma \equiv \omega_k$ (mod $2\pi \mathbb{Z}^n$), which completes the proof of property 1.

Since $D$ is a countable intersection of sets, and $\operatorname{meas}(H_k') = 0$ for all $k \geq 0$, $\operatorname{meas}(D^c) = 0$ also. Now, arguing by induction as in the proof of Theorem [3] we see that for $\omega \in D$, by property 1, $M^{-T}\omega + \gamma \equiv \omega'$ (mod $2\pi \mathbb{Z}^n$) for some $\omega' \in D$, so by $2\pi$-periodicity and the induction hypothesis, Equation [9] holds. By property 2, $2\pi$-periodicity, and the definition of $S$, Equation [10] holds. Then the argument of the induction is correct for $\omega$ restricted to the set $D$, which completes the proof. □

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