It is known that the complex spin group $\text{Spin}(n, \mathbb{C})$ is the universal covering group of complex orthogonal group $\text{SO}(n, \mathbb{C})$. In this work we construct a new kind of spinors on some classes of Kahler–Norden manifolds. The structure group of such a Kahler–Norden manifold is $\text{SO}(n, \mathbb{C})$ and has a lifting to $\text{Spin}(n, \mathbb{C})$. We prove that the Levi-Civita connection on $M$ is an $\text{SO}(n, \mathbb{C})$-connection. By using the spinor representation of the group $\text{Spin}(n, \mathbb{C})$, we define the spinor bundle $S$ on $M$. Then we define covariant derivative operator $\nabla$ on $S$ and study some properties of $\nabla$.

Lastly we define Dirac operator on $S$.

**Keywords**: Spinor; Norden metric; anti-Kahler; complex orthogonal group; spin structure; complex spin group.

1. Introduction

Manifolds equipped with additional geometric structures occur in many cases in differential geometry. For example Riemannian, semi-Riemannian, almost complex structures and spin structures are important geometric structures on manifolds. On an $n$-dimensional manifold $M$, the existence of additional structures is related to the reduction of structure group from $\text{GL}(n, \mathbb{R})$ to a subgroup $G \subset \text{GL}(n, \mathbb{R})$. The most important structure groups are $\text{O}(n), \text{SO}(n), \text{O}(p,q), \text{SO}(p,q), \text{GL}(n, \mathbb{C}), U(n), \text{SU}(n)$. The groups $\text{O}(n, \mathbb{C})$ and $\text{SO}(n, \mathbb{C})$ are subgroups of $\text{GL}(2n, \mathbb{R})$ and they have escaped from our attention in general.

A Kahler manifold can be defined as a triple $(M, J, g)$ where $M$ is a smooth manifold, $J$ an almost complex structure on $M$, $g$ a Riemannian metric on $M$ with the hermitian property $g(JX, JY) = g(X, Y)$ for any $X, Y \in \mathcal{X}(M)$ and $J$ is parallel with respect to the Levi-Civita connection $\nabla$, that is $\nabla J = 0$. Kahler manifolds are being studied widely in differential geometry ([9]). In this work we consider slightly different family of almost complex manifolds, namely, Kahler–Norden manifolds. By a Kahler–Norden manifold we mean a triple $(M, J, g)$ which consists of a smooth manifold $M$, an almost complex structure $J$ on $M$, and a semi-Riemannian metric $g$ on $M$ with the anti-hermitian property $g(JX, JY) = -g(X, Y)$ for any $X, Y \in \mathcal{X}(M)$ and $J$ is parallel with respect to the Levi-Civita connection $\nabla$, that is $\nabla J = 0$. Note that a Kahler–Norden manifold $M$ must be
2. The Spinor Representation

For more detailed explanation of the following facts see e.g. [2, 4, 10]. Spinors are geometric objects on manifolds like tensors and have various applications in mathematics and mathematical physics. In the classical theory, for the construction of spinor one uses the spinor representation of the spin group \( \text{Spin}(n) \). In this work we construct similar objects on a certain class of Kahler–Norden manifolds. To achieve this we use the spinor representation of the complex spin group \( \text{Spin}(n, \mathbb{C}) \) which is comes from the representation of complex Clifford algebra \( \mathcal{C}_n \). The complex Clifford algebra \( \mathcal{C}_n \) and its representation are well described in literature [10]. If \( n = 2k \) is even then \( \mathcal{C}_{2k} \cong \mathbb{C}^{2^k} \), if \( n = 2k + 1 \) is odd then \( \mathcal{C}_{2k+1} \cong \mathbb{C}^{2^k} \oplus \mathbb{C}^{2^k} \). When \( n = 2k \) or \( n = 2k + 1 \) the vector space \( \mathbb{C}^2 \) is called vector space of complex \( n \)-spinors and denoted by \( \Delta_n \). Using this notation we can write \( \mathcal{C}_{2k} \cong \text{End}(\Delta_k) \) and \( \mathcal{C}_{2k+1} \cong \text{End}(\Delta_{k+1}) \otimes \text{End}(\Delta_k) \).

Denote by \( \kappa_n \) the so-called spinor representation of the Clifford algebra \( \mathcal{C}_n \). In case of even dimension \( n = 2k \), \( \kappa_n \) is the isomorphism from \( \mathcal{C}_n \) to \( \text{End}(\Delta_k) \). If \( n = 2k + 1 \) is odd, then \( \kappa_n \) is the composition of the isomorphism from \( \mathcal{C}_n \) to \( \text{End}(\Delta_k) \oplus \text{End}(\Delta_{k+1}) \) with the projection from \( \text{End}(\Delta_k) \oplus \text{End}(\Delta_{k+1}) \) onto first component \( \text{End}(\Delta_k) \). Thus \( \Delta_k \) become a module over the complex Clifford algebra \( \mathcal{C}_n \).

It is clear that \( \text{SO}(n) \subset \text{SO}(n, \mathbb{C}) \) and \( \text{Spin}(n) \subset \text{Spin}(n, \mathbb{C}) \). It is known that all of these groups are connected. The fundamental groups of the orthogonal groups \( \text{SO}(n) \) and \( \text{SO}(n, \mathbb{C}) \) are the same, namely \( \pi_1(\text{SO}(n)) = \pi_1(\text{SO}(n, \mathbb{C})) = \mathbb{Z}_2 \) (\( n \geq 2 \)) and the fundamental groups of both real and complex spin groups is trivial. The maps \( \text{Ad} : \text{Spin}(n) \to \text{SO}(n) \) by \( \text{Ad}_g(v) = gvg^* \) for \( g \in \text{Spin}(n) \), \( v \in \mathbb{C}^n \) and \( \text{Ad} : \text{Spin}(n, \mathbb{C}) \to \text{SO}(n, \mathbb{C}) \) by \( \text{Ad}_g(v) = gvg^* \) for \( g \in \text{Spin}(n, \mathbb{C}) \), \( v \in \mathbb{C}^n \) are onto group homomorphisms with kernel \( \{ \pm 1 \} \). Thus \( \text{Spin}(n) \) is the universal covering group of \( \text{SO}(n) \) and \( \text{Spin}(n, \mathbb{C}) \) is the universal covering group of \( \text{SO}(n, \mathbb{C}) \). In this work we mainly deal with the groups \( \text{SO}(n, \mathbb{C}) \) and \( \text{Spin}(n, \mathbb{C}) \).

The restriction of \( \kappa_n \) to \( \text{Spin}(n) \subset \mathcal{C}_n \) gives a group homomorphism \( \kappa = \kappa_n : \text{Spin}(n) \to \text{Aut}(\Delta_k) \), called spinor representation of \( \text{Spin}(n) \). Similarly the restriction of \( \kappa_n \) to \( \text{Spin}(n, \mathbb{C}) \subset \mathcal{C}_n \) gives a group homomorphism \( \kappa = \kappa_n : \text{Spin}(n, \mathbb{C}) \to \text{Aut}(\Delta_k) \), called spinor representation of \( \text{Spin}(n, \mathbb{C}) \). Some properties of the spinor representation of \( \text{Spin}(n, \mathbb{C}) \) are as follows: (see [4])

(i) If \( n = 2k + 1 \) is odd then \( \kappa \) is irreducible.

(ii) If \( n = 2k \) is even then the spinor space \( \Delta_{2k} \) decomposes into two subspaces \( \Delta_{2k} = \Delta_{2k+} \oplus \Delta_{2k-} \) and \( \dim \Delta_{2k+} = \dim \Delta_{2k-} = 2^{k-1} \). From this decomposition one get new representations \( \kappa^+ : \text{Spin}(2k, \mathbb{C}) \to \text{End}(\Delta_{2k+}) \) and \( \kappa^- : \text{Spin}(2k, \mathbb{C}) \to \text{End}(\Delta_{2k-}) \). Both of these representations are irreducible.

(iii) If \( n = 2k + 1 \) is odd and \( k \equiv 0, 3 \pmod{4} \) then there is a non-degenerate symmetric bilinear form on \( \Delta_{2k+1} \) and the spinor representation \( \kappa \) takes value in the complex orthogonal group \( \text{SO}(2^k, \mathbb{C}) \) that is \( \kappa : \text{Spin}(2k+1, \mathbb{C}) \to \text{SO}(2^k, \mathbb{C}) \).
(iv) If \( n = 2k + 1 \) is odd and \( k \equiv 1, 2 \pmod{4} \) then there is a non-degenerate skew-symmetric bilinear form on \( \Delta_{2k+1} \) and the spinor representation \( \kappa \) takes value in the complex symplectic group \( \text{Sp}(2k, \mathbb{C}) \) that is \( \kappa : \text{Spin}(2k+1, \mathbb{C}) \to \text{Sp}(2k, \mathbb{C}) \).

(v) If \( n = 2k \) is even and \( k \equiv 0 \pmod{4} \) then there is a non-degenerate symmetric bilinear form on \( \Delta_{2k} \) and the spinor representation \( \kappa^+ \) takes value in the complex orthogonal group \( \text{SO}(2k-1, \mathbb{C}) \) that is \( \kappa : \text{Spin}(2k, \mathbb{C}) \to \text{SO}(2k-1, \mathbb{C}) \).

(vi) If \( n = 2k \) is even and \( k \equiv 2 \pmod{4} \) then there is a non-degenerate skew-symmetric bilinear form on \( \Delta_{2k} \) and the spinor representation \( \kappa^+ \) takes value in the complex symplectic group \( \text{Sp}(2k-1, \mathbb{C}) \) that is \( \kappa : \text{Spin}(2k, \mathbb{C}) \to \text{Sp}(2k-1, \mathbb{C}) \).

The Lie algebra \( \text{spin}(n, \mathbb{C}) \) of the complex spin group \( \text{Spin}(n, \mathbb{C}) \) lives in \( \mathbb{C} \mathfrak{n} \) and it is very similar to the Lie algebra of the real spin group \( \text{Spin}(n) \), and given by

\[
\text{spin}(n, \mathbb{C}) = \text{Lin}(e_i e_j : 1 \leq i < j \leq n).
\]

Then the differential of the map \( \text{Ad} : \text{Spin}(n, \mathbb{C}) \to \text{SO}(n, \mathbb{C}) \) is a map \( \text{Ad}_* : \text{spin}(n, \mathbb{C}) \to \mathfrak{so}(n, \mathbb{C}) \) defined by \( \text{Ad}_*(e_i e_j) = 2E_{ij} \), where \( E_{ij} \) are basis for \( \mathfrak{so}(n, \mathbb{C}) \).

### 3. Kahler–Norden Spin Manifolds

In this work we consider 2\( n \)-dimensional manifold \( M \) with structure group \( \text{SO}(n, \mathbb{C}) \subset \text{GL}(2n, \mathbb{R}) \). Since \( \text{SO}(n, \mathbb{C}) \subset \text{O}(n, \mathbb{C}) = \text{GL}(n, \mathbb{C}) \cap \text{O}(n, n) \), the manifold \( M \) can be endowed with a complex structure \( J \) and a semi-Riemannian metric \( g \) with signature \((n, n)\). It can be also checked that

\[
g(JX, JY) = -g(X, Y)
\]

for all vector fields \( X, Y \) on \( M \). Additionally if the condition \( \nabla^g J = 0 \) holds, that is \( J \) is parallel with respect to Levi-Civita connection \( \nabla \), then \( M \) is a Kahler–Norden manifold and we denote it by \((M, J, g)\). More informations about these manifolds can be found in ([11,12]).

**Theorem 1.** Let \((M, J, g)\) be a Kahler–Norden manifold. The Levi-Civita connection \( \nabla^g \) is an \( \text{SO}(n, \mathbb{C}) \)-connection. That is, the local connection forms \( \omega_a \) take their values in the Lie algebra \( \mathfrak{so}(n, \mathbb{C}) \).

Spin manifolds constitute an important class of manifolds ([10]). In the present paper we consider a similar but different class of manifolds. Let \( M \) be a 2\( n \)-dimensional differentiable manifold with structure group \( \text{SO}(n, \mathbb{C}) \), then there is an open covering \( \{U_a\}_{a \in A} \) of \( M \) and transition functions \( g_{ab} : V_a \cap V_b \to \text{SO}(n, \mathbb{C}) \) for \( TM \). If there exists another collection of transition functions

\[
g_{ab} : U_a \cap U_b \to \text{Spin}(n, \mathbb{C})
\]

such that following diagram commutes

\[
\begin{array}{ccc}
\text{Spin}(n, \mathbb{C}) & \xrightarrow{g_{ab}} & \text{SO}(n, \mathbb{C}) \\
\text{Ad}_{2i} \downarrow & & \downarrow \\
U_a \cap U_b & \xrightarrow{g_{ab}} & \text{SO}(n, \mathbb{C})
\end{array}
\]
that is \( \text{Ad} \circ \tilde{g}_{ab} = g_{ab} \) and the cocycle condition
\( \tilde{g}_{ab} \tilde{g}_{bc} = \tilde{g}_{ac} \) on \( U_a \cap U_b \cap U_c \) is satisfied
then \( M \) is called a Kahler–Norden spin manifold. Then one can construct a principal
\( \text{Spin}(n, \mathbb{C}) \)-bundle \( P_{\text{Spin}(n, \mathbb{C})} \) on \( M \) and a \( 2 - 1 \) bundle map \( \Lambda : P_{\text{Spin}(n, \mathbb{C})} \to P_{SO(n, \mathbb{C})} \).

We stated in Theorem 1 that the connection form \( \omega \) of \( \nabla g \) takes value in the Lie algebra
\( \mathfrak{so}(n, \mathbb{C}) \). If \( U_\beta \) is another coordinate trivializing neighborhood for
\( TM \) with \( U_\alpha \cap U_\beta \neq \emptyset \) then following relation holds between the connection forms
\( \omega \alpha \) and \( \omega \beta \):

\[
\omega \beta = g^{-1}_{\alpha \beta} \omega \alpha g_{\alpha \beta} + g^{-1}_{\alpha \beta} g_{\alpha \beta} d g_{\alpha \beta}
\]

where \( g_{ab} : U_a \cap U_b \to SO(n, \mathbb{C}) \) is transition function. These
\( \mathfrak{so}(n, \mathbb{C}) \)-valued 1-forms determine a connection 1-form \( \omega \) on the principal bundle
\( P_{SO(n, \mathbb{C})} \) with values in \( \mathfrak{so}(n, \mathbb{C}) \).

Now we define a connection 1-form \( Z \) on the principal bundle
\( P_{\text{Spin}(n, \mathbb{C})} \) with values in the Lie algebra \( \mathfrak{spin}(n, \mathbb{C}) \) by using following diagram

\[
\begin{array}{ccc}
TP_{\text{Spin}(n, \mathbb{C})} & \xrightarrow{Z} & \mathfrak{spin}(n, \mathbb{C}) \\
\downarrow \Lambda & & \downarrow \lambda \\
TP_{SO(n, \mathbb{C})} & \xrightarrow{\lambda} & \mathfrak{so}(n, \mathbb{C})
\end{array}
\]

Note that the equality
\( \omega \circ \Lambda = \text{Ad} \circ Z \)

holds.

4. Spinor Bundle on Kahler–Norden Manifolds

The spinor bundle \( S \) on a 2\( n \)-dimensional Kahler–Norden spin manifold \( M \) is defined as the
associated vector bundle

\[
S = P_{\text{Spin}(n, \mathbb{C})} \times_\kappa \Delta_n
\]

where \( \kappa : \text{Spin}(n, \mathbb{C}) \to \text{Aut}(\Delta_n) \) is the spinor representation of \( \text{Spin}(n, \mathbb{C}) \). In case \( n = 2k \),
this vector bundle splits into the sum of two subbundles \( S^+ \), \( S^- \):

\[
S = S^+ \oplus S^-, \quad S^\pm = P_{\text{Spin}(n, \mathbb{C})} \times_\kappa \Delta_n^\pm
\]

The composite map \( \rho \circ \text{Ad} : \text{Spin}(n, \mathbb{C}) \to \text{Aut}(\mathbb{R}^{2n}) \) is a representation of \( \text{Spin}(n, \mathbb{C}) \) on
\( \mathbb{R}^{2n} \) and gives

\[
P_{\text{Spin}(n, \mathbb{C})} \times_\rho \text{Ad} \mathbb{R}^{2n} \simeq TM.
\]

Such interpretations of tangent bundle enable us to product the elements of spinor bundle
with tangent vectors by the formula

\[
[p, v] \cdot [p, \phi] = [p, \kappa(v) \phi]
\]

where \( p \in P_{\text{Spin}(n, \mathbb{C})}, \; v \in \mathbb{R}^{2n}, \phi \in \Delta_n \). Since the spinor representation is \( \text{Spin}(n, \mathbb{C}) \)-
equivariant, the definition of product is independent from the representatives. This product
is bilinear, so we extend it the tensor product space

\[ TM \otimes S \rightarrow S, \]

\[ [p, v] \otimes [p, \phi] = [p, \kappa(v)\phi], \]

we denote it as a map \( \mu : TM \otimes S \rightarrow S \) and call it Clifford multiplication.

We want to define a covariant derivative operator on the spinor bundle \( S \). A section \( \Phi \in \Gamma(S) \) is called a spinor field on \( M \). Since \( S \) is an associated vector bundle, any spinor field \( \Phi \) can be identified with the mapping \( \hat{\Phi} : P_{\text{Spin}(n, C)} \rightarrow \Delta_n \) obeying the transformation rule

\[ \hat{\Phi}(pg) = \kappa(g^{-1})\hat{\Phi}(p). \]

Such maps are called equivariant.

The connection 1-form \( Z \) on the principal bundle \( P_{\text{Spin}(n, C)} \) determines a covariant derivative operator \( \nabla \) on the spinor bundle \( S = P_{\text{Spin}(n, C)} \times_\kappa \Delta_n \). Let \( X \in \chi(M) \). The operator

\[ \nabla_X : \Gamma(S) \rightarrow \Gamma(S), \]

given by \( (\nabla_X \hat{\Phi})(p) = (\hat{\Phi}_p)(X_p) \) is a covariant derivative on the spinor bundle, where \( \hat{\Phi} : P_{\text{Spin}(n, C)} \rightarrow \Delta_n \) is an equivariant map associated to the spinor field \( \Phi \in \Gamma(S) \), \( p \in P_{\text{Spin}(n, C)} \) and \( X_p \) is the horizontal lift of \( X_p \) in \( H\sigma_{\Phi}(P_{\text{Spin}(n, C)}) \) (see \([3,8]\)). It can be expressed as

\[ \nabla_X \hat{\Phi} = \hat{\Phi} X + \kappa_\lambda(Z(X')) \hat{\Phi} \]

where \( \kappa_\lambda : \text{spin}(n, C) \rightarrow \text{End}(\Delta_n) \) is the derivative of \( \kappa \) at identity \( 1 \in \text{Spin}(n, C) \). It can be also shown that \( \kappa_\lambda(e_i e_j) = \kappa(e_i e_j) \).

We can write the covariant derivative operator \( \nabla \) locally as follows: Let \( s : U \rightarrow P_{\text{Spin}(n, C)} \) be a local section of the frame bundle \( P_{\text{Spin}(n, C)} \). \( s \) consists of orthonormal frame \( s = \{ e_1, e_2, \ldots, e_n, f, f_1, f_2, \ldots, f_n \} \) of vector fields defined on the open set \( U \subset M \). We know that the local connection form \( \omega_{\sigma} \) is given by the formula

\[ \omega_{\sigma}(X) = \sum_{i<j}(w_{ij}(X) - i\tilde{w}_{ij}(X))E_{ij} \]

where \( w_{ij} \) and \( \tilde{w}_{ij} \) denote the forms defining the Levi-Civita connection, \( w_{ij}(X) = g(\nabla_{X}e_i, e_j) = \frac{1}{2}g(\nabla_{X}e_i, e_j) \), \( \tilde{w}_{ij}(X) = -g(\nabla_{X}f, e_i) \), and \( E_{ij} \in \mathfrak{so}(n, C) \) are the standard basis matrices of the Lie algebra \( \mathfrak{so}(n, C) \). Let \( s : U \rightarrow P_{\text{Spin}(n, C)} \) be a lift of \( s \) to the 2-fold covering \( \Lambda : P_{\text{Spin}(n, C)} \rightarrow P_{\text{SO}(n, C)} \). Then the local connection forms of \( Z \) are given by

\[ Z_\sigma(X) = \frac{1}{2} \sum_{i<j}(w_{ij}(X) - i\tilde{w}_{ij}(X))e_i e_j \]

and

\[ \kappa_\lambda(Z_\sigma(X)) = \frac{1}{2} \sum_{i<j}(w_{ij}(X) - i\tilde{w}_{ij}(X))\kappa_\lambda(e_i e_j) \]

\[ \kappa_\lambda(Z_\sigma(X)) = \frac{1}{2} \sum_{i<j}(w_{ij}(X) - i\tilde{w}_{ij}(X))\kappa(e_i e_j). \]

Let \( \Phi \in \Gamma(S) \) be a spinor field and \( \hat{\Phi} : P_{\text{Spin}(n, C)} \rightarrow \Delta_n \) be associated equivariant map, and consider the composition \( \Phi_s = \hat{\Phi} \circ \tilde{s} : U \rightarrow \Delta_n \), then we can write \( \Phi(x) = [\tilde{x}(x), \Phi_s(x)] \) for
each $x \in U_a$. Hence we can express the covariant derivative of spinors by the formula

$$\nabla_X \Phi = \left[ \tilde{s}_d \Phi(X) + \frac{1}{2} \sum_{i<j} (w_{ij}(X) - i\tilde{w}_{ij}(X))\epsilon(e_i,e_j)\Phi_s \right]$$

such expression will be useful for our computations (see [7]).

In the classical theory of spin manifolds, there is a hermitian inner product on the spinor bundle, but in our case there is no such inner product. Instead, one can define some special forms on the spinor bundle $S$.

**Theorem 2.** Let $M$ be an $2n$-dimensional Kahler–Norden spin manifold and $S$ be spinor bundle on $M$.

(i) If $n = 2k + 1$ is odd and $k \equiv 1, 2 \pmod{4}$, then there is a non-degenerate skew-symmetric bilinear form $F$, so called symplectic form, on the spinor bundle $S$ with values in $\mathbb{C}$.

(ii) If $n = 2k$ is even and $k \equiv 0 \pmod{4}$, then there is a non-degenerate symmetric bilinear form $B$ on the spinor bundle $S^+$ with values in $\mathbb{C}$.

(iii) If $n = 2k$ is even and $k \equiv 2 \pmod{4}$, then there is a non-degenerate skew-symmetric bilinear form $F$, so called symplectic form, on $S^+$ with values in $\mathbb{C}$.

**Proof.** Recall $S = P_{\text{Spin}(n,\mathbb{C})} \times \Delta_n$ and $S^+ = P_{\text{Spin}(n,\mathbb{C})} \times_{\kappa^+} \Delta^+_n$.

(i) For $[p, \psi_1], [p, \psi_2] \in S$, we set

$$F([p, \psi_1], [p, \psi_2]) = \epsilon(\psi_1, \psi_2)$$

where $\epsilon$ is the standard symplectic form on $\Delta_n$. Since the spinor representation $\kappa$ of $\text{Spin}(n,\mathbb{C})$ is symplectic, above equation defines a symplectic form on $S$.

(ii) For $[p, \psi_1], [p, \psi_2] \in S^+$, we set

$$B([p, \psi_1], [p, \psi_2]) = b(\psi_1, \psi_2)$$

where $b$ is the standard symmetric bilinear form on $\Delta_n$. Since the spinor representation $\kappa^+$ of $\text{Spin}(n,\mathbb{C})$ is orthogonal, above equation defines a symmetric bilinear form on $S^+$.

(iii) Similar to the case (i).

The following theorem states that the forms $F$ and $B$ are compatible with the connection $\nabla$.

**Theorem 3.**

(i) For any $X \in \Gamma(TM)$ and $\Phi, \Psi \in \Gamma(S)$,

$$X(F(\Phi, \Psi)) = F(\nabla_X \Phi, \Psi) + F(\Phi, \nabla_X \Psi).$$

(ii) For any $X \in \Gamma(TM)$ and $\Phi, \Psi \in \Gamma(S^+)$,

$$X(B(\Phi, \Psi)) = B(\nabla_X \Phi, \Psi) + B(\Phi, \nabla_X \Psi).$$
Proof. (i) With respect to a local section \( s : U_\alpha \rightarrow P_{\text{Spin}(n,\mathbb{C})} \), we have
\[
F(\nabla X \Phi, \Psi) + F(\Phi, \nabla X \Psi) = \varepsilon \left( \Phi \cdot d\Phi_s(X) + \frac{1}{2} \sum_{i<j} (\omega_{ij}(X) - i\tilde{\omega}_{ij}(X)) \kappa(e_i e_j) \Phi_s, \Phi_s \right) \\
+ \varepsilon \left( \Phi \cdot d\Phi_s(X) + \frac{1}{2} \sum_{i<j} (\omega_{ij}(X) - i\tilde{\omega}_{ij}(X)) \kappa(e_i e_j) \Psi_s, \Psi_s \right) \\
= \varepsilon (\Phi_s(X), \Psi_s) + \frac{1}{2} \sum_{i<j} (\omega_{ij}(X) - i\tilde{\omega}_{ij}(X)) \kappa(e_i e_j) (\Phi_s, \Psi_s) \\
+ \varepsilon (\Phi_s(X), \Psi_s) + \frac{1}{2} \sum_{i<j} (\omega_{ij}(X) - i\tilde{\omega}_{ij}(X)) \kappa(e_i e_j) (\Phi_s, \Psi_s) \\
= \varepsilon (\Phi_s(X), \Psi_s) + \varepsilon (\Phi_s(X)) = X\varepsilon(\Phi_s, \Psi_s).
\]
(ii) With respect to a local section \( s : U_\alpha \rightarrow P_{\text{Spin}(n,\mathbb{C})} \), we have
\[
B(\nabla X \Phi, \Psi) + B(\Phi, \nabla X \Psi) = b \left( \Phi \cdot d\Phi_s(X) + \frac{1}{2} \sum_{i<j} (\omega_{ij}(X) - i\tilde{\omega}_{ij}(X)) \kappa(e_i e_j) \Phi_s, \Phi_s \right) \\
+ b \left( \Phi \cdot d\Phi_s(X) + \frac{1}{2} \sum_{i<j} (\omega_{ij}(X) - i\tilde{\omega}_{ij}(X)) \kappa(e_i e_j) \Psi_s, \Psi_s \right) \\
= b (\Phi_s(X), \Psi_s) + \frac{1}{2} \sum_{i<j} (\omega_{ij}(X) - i\tilde{\omega}_{ij}(X)) b(\kappa(e_i e_j) \Phi_s, \Phi_s) \\
+ b (\Phi_s(X), \Psi_s) + \frac{1}{2} \sum_{i<j} (\omega_{ij}(X) - i\tilde{\omega}_{ij}(X)) b(\kappa(e_i e_j) \Psi_s, \Psi_s) \\
= b (\Phi_s(X), \Psi_s) + b (\Phi_s, d\Phi_s(X)) \\
= X b(\Phi_s, \Psi_s).
\]

Now we want to define the Dirac operator on the spinor bundle \( S \). The connection \( \nabla \) on \( S \) can be thought as a map linear map
\[
\nabla : \Gamma(S) \rightarrow \Gamma(T^*M \otimes S)
\]
satisfies the following Leibnitz rule:
\[
\nabla (f \Phi) = (df) \otimes \Phi + f \nabla \Phi.
\]
In a local orthonormal frame \{e_1, e_2, \ldots, e_n, Je_1, Je_2, \ldots, Je_n\} it can be written in the following form:

\[
\nabla \Phi = \sum_{i=1}^{n} \left( e_i^* \otimes \nabla_{e_i} \Phi + (Je_i)^* \otimes \nabla_{Je_i} \Phi \right).
\]

Note that the Clifford multiplication \( \mu : TM \otimes S \rightarrow S \) induces a map \( \mu : \Gamma(TM \otimes S) \rightarrow \Gamma(S) \) this means we can product a spinor field with a vector field.

**Definition 1 (Dirac Operator).** The composition

\[
D = \mu \circ \nabla : \Gamma(S) \xrightarrow{\nabla} \Gamma(TM \otimes S) \xrightarrow{\mu} \Gamma(S)
\]

is called Dirac operator on Kahler–Norden spin manifold \( M \).

Obviously \( D : \Gamma(S) \rightarrow \Gamma(S) \) is first order differential operator. With respect to local orthonormal frame \( \{e_1, e_2, \ldots, e_n, Je_1, Je_2, \ldots, Je_n\} \)

\[
D \Phi = \sum_{i=1}^{n} (e_i \cdot \nabla_{e_i} \Phi - (Je_i) \cdot \nabla_{Je_i} \Phi).
\]

The investigations of the main properties of such a Dirac operator will be a subject of another paper.

**References**

[1] A. Borowiec, M. Francaviglia and I. Volovich, Anti-Kahlerian manifolds, *Diff. Geom. Appl.* 12 (2000) 281–289.

[2] C. Chevalley, *The Algebraic Theory of Spinors and Clifford Algebras* (Springer-Verlag, Berlin, 1997).

[3] T. Friedrich, Dirac operators in Riemannian geometry, *Amer. Math. Soc.* (2000).

[4] W. Fulton and J. Harris, *Representation theory: A First Course* (Springer-Verlag, New York, 1991).

[5] G. Ganchev and S. Ivanov, Connections and curvatures on complex Riemannian manifold, *Internal I.C.T P.-Trieste* (1991).

[6] G. Ganchev, K. Gribachev and V. Mihova, B-connections and their conformal invariants on conformally Kahler manifold with B-metric, *Publ. Inst. Math. N. S.* 42 (1987) 107–121.

[7] K. Habermann and L. Habermann, *Introduction to Symplectic Dirac Operators* (Springer-Verlag, 2006).

[8] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Intersience, New York, Vol. I (1963).

[9] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Intersience, New York, Vol. II (1969).

[10] H. B. Lawson and M. L. Michelsohn, Spin Geometry Princeton Univ. (1989).

[11] D. Mekerov, M. Manev and K. Gribachev, Quasi-Kahler Manifolds with a pair of Norden Metrics, Results in Mathematics 49(1-2) (2006) 161–170.

[12] K. Sluka, On the Curvature of Kahler-Norden manifolds, *J. Geom. Phys.* 54 (2005) 131–145.