Mixed state on a sparsely encoded associative memory model

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Abstract

In the present paper, we analyze symmetric mixed states corresponding to the so-called concept formation on a sparsely encoded associative memory model with $0-1$ neurons. Three types of mixed states, OR, AND and a majority decision mixed state are described as typical examples. Each element of the OR mixed state is composed of corresponding memory pattern elements by means of the OR-operation. The other two types are similarly defined. By analyzing their stabilities through the SCSNA and the computer simulation, we found that the storage capacity of the OR mixed state diverges in the sparse limit, but that the other states do not diverge. In addition, we found that the optimal threshold values, which maximize the storage capacity, for the memory pattern and the OR mixed state coincide with each other in the spare limit. Thus, we conclude that the OR mixed state is a reasonable representative of the mixed state in the spare limit. Finally, the paper examines the relationship between our results and recently reported physiological findings regarding face-responsive neurons in the inferior temporal cortex.

I. INTRODUCTION

In this paper, we analyze symmetric mixed states (hereafter “mixed state”) on a sparsely encoded associative memory model. When an associative memory model is made to store memory patterns as a result of correlation learning, a pattern, which is a nonlinear superposition of the stored memory patterns using a neuron output function, automatically becomes the equilibrium state of the model. This is called the mixed state. It is not appropriate to think that this mixed state is a side-effect and/or that is unnecessary for information processing. Amari has discussed a “concept formation” using the stability of mixed state [1]. The correlated attractor proposed in [2,3], which is a model of the Miyashita attractor [2], is considered to be a (not symmetric) mixed state in a broad sense. Recently, Parga and Rolls have used the mixed state as a mechanism of invariant recognition under a coordinate
transformation \[3\]. On the other hand, the sparse coding scheme is believed to be used in
the brain according to some physiological findings \[2\] and the theoretical viewpoints \[6–10\].
Thus, the properties of the mixed states on the sparsely encoded associative memory model
should be discussed. This is the main purpose of the present paper.

\[ \text{sgn}(\sum_{\mu=1}^{s} \xi_{\mu} + h) \] is the mixed state in a model storing memory pattern \( \xi^{\mu} \), with the
output function \( \text{sgn}(\cdot) \) and threshold value \( h \). Therefore \( s \) types of mixed state can be
composed of \( s \) memory patterns by changing the threshold value. \( \text{sgn}(\sum_{\mu=1}^{s} \xi_{\mu}^{\mu}) \) among the
\( s \) types is regarded as the mixed state in a model with the output function \( \text{sgn}(\cdot) \) that is
made to learn memory patterns of a 50\% firing rate \[11\]. In other words, a “typical” mixed
state is a state of the majority decision for the firing or non-firing elements among the \( s \)
types of the mixed state. A natural question then arises: what kind of mixed state among
the \( s \) types should be considered typical in the sparse limit of \( f \rightarrow 0 \) ? (\( f \) is the firing rate
of the memory pattern). In the sparsely encoded model, storage capacity strongly depends on
the threshold value, so that an appropriate setting of the threshold value makes the storage
capacity diverge at \( 1/|f \log f| \) in the sparse limit \[6–10\]. Thus, in this study, “typical” mixed
state is defined as the mixed state with similar properties to the memory pattern for the
storage capacity and the threshold value. The purpose of this paper is to discuss which
mixed state in the sparse limit should be regarded as a typical mixed state.

First, three out of \( s \) types of the mixed state are considered: the OR mixed state, the
majority decision mixed state, and the AND mixed state. Each element of the OR mixed
state is taken from the OR operation in the Boolean sense for each element of memory
patterns composing of the mixed state. The other two types are similarly defined. The
examination of the stability of each mixed state by the self-consistent signal-to-noise analysis
(SCSNA) proposed by Shiino and Fukai \[12\] and the computer simulation shows that the
storage capacity of the OR mixed state diverges as \( 1/|f \log f| \), but the storage capacities
of the other two types do not diverge. This can be explained as follows: the overlap of
the majority decision mixed state with the stored memory pattern and the overlap of the
AND mixed state with the stored memory pattern converge at 0 in the sparse limit, whereas
the overlap of the OR mixed state with one converges at 1. Likewise, while the optimum
threshold value that maximizes the storage capacity of the memory pattern or the OR mixed
state settles at the same value, the optimum threshold value of the majority decision mixed
state and the AND mixed state does not agree with the optimum threshold value of the
memory pattern in the sparse limit. Therefore, the important factors to decide the storage
capacity and the optimum threshold value in the sparse limit are asymptotes for the overlaps
of the \( s \) types of the mixed state with the memory patterns. Besides three typical mixed
states, we also evaluate the overlaps of the \( s \) types of the mixed states to show that all the
overlaps of the \( s - 1 \) kinds of mixed states except for the OR mixed states converge to 0 in
the sparse limit. Thus, we conclude that a consideration of the OR mixed state as a typical
mixed state in the sparse coding is appropriate.

At the end of this paper, we discuss the recently reported physiological findings of Sugase
et al. \[13\] on the face-responsive neurons in the inferior temporal cortex. They have reported
that some face-responsive neurons in the IT cortex code rough classification information in
the first half of the firing and then code detailed classification information in the latter half
of the firing. Their findings suggest that the OR mixed state emerges in the first half of
the neuron’s dynamics, and then the network state converges in the memory state. We will
infer a relationship between the physiological findings and our present results.

II. MODEL

Shown below are the equilibrium properties of a recurrent neural network composed of $N$ neurons with an output function $\Theta(\cdot)$,

$$ x_i = \Theta(u_i), \quad \text{(1)} $$

$$ u_i = \sum_{j \neq i}^N J_{ij} x_j + h, \quad \text{(2)} $$

$$ \Theta(u) = \begin{cases} 1 & u \geq 0 \\ 0 & u < 0 \end{cases}, \quad \text{(3)} $$

where $x_i$ denotes an output of the $i$-th neuron in the equilibrium state, and $J_{ij}$ denotes a synaptic weight from the $j$-th neuron to the $i$-th neuron. The threshold value $h$ of the neuron is assumed to not depend on the serial number $i$ of a neuron. Its concrete value is described later. Each element $\eta^\mu_i$ of the $\mu$-th memory pattern $\eta^\mu$, which is stored in the present model, is independently generated by the probability,

$$ \text{Prob}[\eta^\mu_i = 1] = 1 - \text{Prob}[\eta^\mu_i = 0] = f, \quad \text{(4)} $$

where $\text{E}[\eta^\mu_i] = f$, and $f$ stands for the firing rate of memory pattern $\eta^\mu$. The pattern with a small firing rate $f$ is called a sparse pattern, and the use of a sparse pattern for the memory pattern is called a sparse encoding. The synaptic weight $J_{ij}$ is decided by the covariance learning method,

$$ J_{ij} = \frac{1}{N f(1-f)} \sum_{\mu=1}^{\alpha N} (\eta^\mu_i - f)(\eta^\mu_j - f), \quad \text{(5)} $$

where $\alpha N$ is a number of the stored memory patterns, and $\alpha$ is defined as the loading rate.

Let us define the mixed state $\gamma^{(s,k)}$ composed of $s$ memory patterns. The $i$-th element of the mixed state $\gamma^{(s,k)}$ is set to '1' if the number of the firing state ('1') is $k$ or more in the $i$-th elements of the $s$ memory patterns that compose the mixed state. Otherwise, it becomes '0'. The $s$ types of the mixed states exist according to this definition because $1 \leq k \leq s$. In particular, among $s$ types of mixed state, mixed state $\gamma^{(s,1)}$ is considered to be the OR mixed state, where its $i$-th element is given by the OR operation through the $i$-th elements of $s$ memory patterns. Following the definition of the OR mixed state, $\gamma^{(s,s)}$ corresponds to the AND mixed state where its $i$-th element is obtained by the AND operation. Moreover, $\gamma^{(s,s/2)}$ is the majority decision mixed state that regards the one with more numbers of 0 or 1 for the $i$-th elements of $s$ memory patterns.

The threshold value $h$ in Eq.(2) is decided as follows. The threshold value $h$ should be appropriately chosen in the sparsely encoded model [6–10]. In this paper, the threshold value is calculated by using the mean firing rate of the retrieval pattern. The threshold value obtained by this method approximately coincides with the optimum threshold value
by which the storage capacity is maximized [10]. Therefore, we employ a method with a constant firing rate as an approximate method of obtaining the optimum threshold value in the present paper. Since the mean firing rate of the memory pattern is \( f \), when the memory pattern is retrieved, threshold value \( h \) is decided from the following equation,

\[
f = \frac{1}{N} \sum_{i=1}^{N} \Theta \left( \sum_{j \neq i} J_{ij} x_j + h \right).
\] (6)

Moreover, the mean firing rate \( f^{(s,k)} \) of mixed state \( \gamma^{(s,k)} \) replaces \( f \) in Eq. (6) to recall the mixed state \( \gamma^{(s,k)} \), and threshold value \( h \) is decided,

\[
f^{(s,k)} = E[\gamma_i^{(s,k)}] = \sum_{i=k}^{s} C_v f^v (1 - f)^{s-v}.
\] (7)

Here, \( C \) is a number of the combination. The overlap of the equilibrium state \( x \) with the \( \mu \)-th memory pattern \( \eta^\mu \) is defined,

\[
m_\mu = \frac{1}{N f (1 - f)} \sum_{i=1}^{N} (\eta^\mu_i - f)x_i.
\] (8)

If the equilibrium state \( x \) is completely equal to \( \eta^\mu \), then \( m_\mu = 1 \). The overlap \( M^{(s,k)} \) of the equilibrium state \( x \) with the mixed state \( \gamma^{(s,k)} \) is defined in a manner similar to the Eq. (8),

\[
M^{(s,k)} = \frac{1}{N f^{(s,k)} (1 - f^{(s,k)})} \sum_{i=1}^{N} (\gamma^{(s,k)}_i - f^{(s,k)})x_i.
\] (9)

If the equilibrium state \( x \) is \( \gamma^{(s,k)} \), then \( M^{(s,k)} = 1 \).

III. RESULTS

A. SCSNA analysis

The SCSNA [12] and a computer simulation are used to analyze stabilities of three typical types of mixed state, the OR mixed state \( \gamma^{(s,1)} \), the majority decision mixed state \( \gamma^{(s,s/2)} \), and the AND mixed state \( \gamma^{(s,s)} \). The order parameter equations of the SCSNA are derived as shown in the Appendix.

First, the OR mixed state \( \gamma^{(s,1)} \) is described. Fig. 1 shows the loading rate \( \alpha \) dependency of overlap \( M^{(s,1)} \) on the OR mixed state \( \gamma^{(3,1)} \) composed of three memory patterns (\( s = 3 \)). The three lines indicate the results of the SCSNA and the data points and the error bars are the results of the computer simulation for the mean firing rate of the memory pattern, \( f = 0.5, 0.2 \) and \( 0.1 \). In the computer simulation, the neuron number is set as \( N \geq 10,000 \), and the simulation is performed 11 times for each parameter. The data point shows the median, and both ends of the error bar show the 1/4 deviation and the 3/4 deviations, respectively. The horizontal axis is the loading rate \( \alpha \), and the vertical axis is the overlap \( M^{(s,1)} \) of the OR mixed state \( \gamma^{(3,1)} \) with the equilibrium state \( x \). The results of the SCSNA and the computer simulation correspond well with each other for the loading
rate \( \alpha \) dependency of overlap \( M^{(3,1)} \) and the storage capacity \( \alpha_C \), which is the loading rate when the equilibrium state becomes unstable. The properties of the three types of the mixed states are examined using the SCSNA results. This is because the SCSNA results explain the computer simulation results fairly well, as shown in Fig. 1. Fig. 2 shows the mean firing rate \( f \) dependency of the storage capacity \( \alpha_C \) on \( \gamma^{(s,1)} \) \((s = 3, 5, 7)\) of the OR mixed state. The solid line shows the storage capacity of memory pattern \((s = 1)\). The higher the mixed number \( s \), the smaller the storage capacity is with the same mean firing rate \( f \). However, the lower the mean firing rate \( f \), the larger the storage capacity for all three states of \( s \). Then, we examine the asymptotes for \( f \to 0 \). The asymptotes for the all values of \( s \) shown in Fig. 3 are the same \( 1/|f \log f| \) as the case for the memory patterns. In addition, the storage capacity of the OR mixed state \( \gamma^{(s,1)} \) with the \( s \) memory patterns with firing rate \( f \) is the same as that of the memory pattern with firing rate \( sf \) in the sparse limit. The reason is explained in the next section \( \S \) III B. Fig. 4 shows that the mean firing rate \( f \) dependency of threshold value \( h_c \) at the loading rate \( \alpha \) is set to the storage capacity, i.e., \( \alpha = \alpha_C \). The figure shows the threshold value of the memory pattern with a solid line for comparison. In the limit of \( f \to 0 \), threshold value \( h_c \) converges to the same value without depending on the \( s \) that includes the memory pattern \( s = 1 \). The reason is also explained in the next section \( \S \) III B. These findings imply that both of the memory patterns \( \eta^s \) and the OR mixed state \( \gamma^{(s,1)} \) easily coexist as stable in the sparse limit \( f \to 0 \).

Next, we examine the majority decision mixed state \( \gamma^{(s,s/2)} \) and the AND mixed state \( \gamma^{(s,s)} \). Fig. 5 and 6 show the storage capacity for the majority decision mixed state \( \gamma^{(s,s/2)} \), \((s = 3, 5, 7)\) and the AND mixed state \( \gamma^{(s,s)} \), \((s = 3, 5, 7)\), respectively. Fig. 5 shows that the storage capacity gradually increases in \( s = 3 \) as the mean firing rate \( f \) decreases, and the storage capacity decreases again without diverging in the sparse limit. It is about \( \alpha_c = 0.065 \) even at the maximum. If the number \( s \) of memory patterns is increased to \( s = 5, 7 \), the maximum storage capacity is reduced further, and it goes toward 0 in the sparse limit. Fig. 6 shows that the storage capacity \( \alpha_c \) of the AND mixed state \( \gamma^{(s,s)} \) decreases as the mean firing rate becomes small in each number \( s \) of the mixed state, and it goes toward 0 in the sparse limit. These results show that storage capacity \( \alpha_C \) does not diverge in the sparse limit except in the memory pattern and the OR mixed state \( \gamma^{(s,1)} \). The reason is qualitatively explained in \( \S \) III B. Fig. 7 and 8 show the mean firing rate \( f \) dependency of threshold value \( h_c \) at the storage capacity \( \alpha_c \) for the majority decision mixed state \( \gamma^{(s,s/2)} \), \((s = 3, 5, 7)\) and the AND mixed state \( \gamma^{(s,s)} \), \((s = 3, 5, 7)\). The figure shows the threshold value of the memory pattern with a solid line for comparison. The threshold values of these types of mixed state do not correspond to the threshold value of the memory pattern in the sparse limit, while that of the OR mixed state \( \gamma^{(s,1)} \) is the same as that of the memory pattern. This means that the memory pattern and these types of mixed state cannot coexist in the sparse limit as stable. On the other hand, the threshold value of the majority decision mixed state \( \gamma^{(s,s/2)} \) crosses the threshold value of the memory pattern at the mean firing rate of \( f = 0.5 \). As mentioned chapter \( \S \), this implies that the majority decision mixed state \( \gamma^{(s,s/2)} \) can be considered as the typical mixed state in \( f = 0.5 \).
B. Qualitative evaluation of mixed state by the naive S/N analysis

In this section, we explain the analytical results of the SCS NA by using the naive S/N analysis \[9,10\]. First, the reason why only the storage capacity of the OR mixed state \(\gamma(s,k)\) diverges in the sparse limit is discussed. Assuming that output \(x_i\) of each neuron is equal to the mixed state \(\gamma(s,k)\), the internal potential \(u_i\) is rewritten with the synaptic weight \(J_{ij}\) of Eq. (5). Without loss of generality, the mixed state \(\gamma(s,k)\) is composed of \(\eta^\mu, \quad (1 \leq \mu \leq s)\), because the synaptic weight \(J_{ij}\) in Eq. (5) is invariant for the replacement of the order of \(\mu\). The \(u_i\) in Eq. (2) is decomposed into \(1 \leq \mu \leq s\) and other parts,

\[
\begin{align*}
    u_i &= \sum_{j \neq i}^{N} J_{ij} \gamma_j^{(s,k)} + h \\
    &= \sum_{\mu=1}^{s} (\eta^\mu_i - f) m_{\mu}^{(s,k)} + h + \bar{z}, \\
    m_{\mu}^{(s,k)} &= \frac{1}{Nf(1-f)} \sum_{i=1}^{N} (\eta^\mu_i - f) \gamma_i^{(s,k)}, \\
    \bar{z}_i &= \frac{1}{Nf(1-f)} \sum_{\mu=s+1}^{\alpha N} \sum_{j \neq i}^{N} (\eta^\mu_i - f)(\eta^\mu_j - f) \gamma_j^{(s,k)}. \\
\end{align*}
\]

The first term of Eq. (10), which is composed of finite number of overlap \(m_{\mu}^{(s,k)}\), is a signal term to retrieve. The second term is the threshold value, and \(\bar{z}_i\) the third term is a cross-talk noise, which prevents the state \(\gamma(s,k)\) from being stable. In this case, \(\bar{z}_i\) obeys the normal distribution \(N(0, \alpha f^{(s,k)})\) in the limit of \(N \to \infty\). Let us evaluate \(m_{\mu}^{(s,k)}\) in the sparse limit. However, we examine \(k \geq 1\) because all elements of the mixed state with \(k = 0\) become 1. The overlap \(m_{\mu}^{(s,k)}\) of the mixed state \(\gamma(s,k)\) with the memory pattern \(\eta^\mu\) can be calculated by using the expectation value in memory pattern \(\eta^\mu\) generated with the probability in Eq. (4),

\[
\begin{align*}
    m_{\mu}^{(s,k)} &= \frac{1}{f(1-f)} E[(\eta^\mu_i - f) \gamma_i^{(s,k)}] \\
    &= \sum_{v=0}^{s-1} C_v f^v (1-f)^{s-v-1} \\
    &\quad \times [\Theta(1+v-k) - \Theta(v-k)]. \\
\end{align*}
\]

Here \(C\) is a number of the combination. Note that \(\Theta(0) = 1\) as in Eq. (3). In the sparse limit of \(f \to 0\), \(f^v\) in Eq. (14) becomes 1 in \(v = 0\) and 0 in \(v \geq 1\). Consequently, the summation for \(v\) of Eq. (14) has to be taken into account only when \(v = 0\). Thus, \((1-f)^{s-v-1} \to 1\) for \(f \to 0\), and \(s-1C_v = 1\) for \(v = 0\). \(\Theta(0-k) = 0\) as a result of taking \(k \geq 1\) into consideration. Therefore, the Eq. (14) becomes the following in the sparse limit,

\[
\begin{align*}
    m_{\mu}^{(s,k)} &= \Theta(1-k). \\
\end{align*}
\]
This expression explains that $m^{(s,k)}_{\mu} = 1$ only when $k = 1$, and $m^{(s,k)}_{\mu} = 0$ in $k \geq 2$ regardless of the number of $s$ in the sparse limit. Fig. 9 shows the mean firing rate $f$ dependency of the overlap of $m^{(s,k)}_{\mu}$. This figure also shows that only when $k = 1$, $m^{(s,k)}_{\mu} \rightarrow 1$ in the sparse limit, and $m^{(s,k)}_{\mu} \rightarrow 0$ in $k \geq 2$.

If we use Eq. (7), the variance $\alpha f^{(s,k)}$ of cross-talk noise $\bar{z}_i$ in Eq. (12) becomes,

$$\alpha f^{(s,k)} = \alpha \sum_{v=k}^{s} C_v f^v (1 - f)^{s-v}. \quad (16)$$

The mean firing rate $f^{(s,1)}$ of the $\gamma^{(s,1)}$ converges on $f^{(s,1)} \rightarrow sf$ in the sparse limit. Therefore, in the sparse limit, the variance of cross-talk noise when the memory pattern with the $sf$ mean firing rate is recalled, is $\alpha sf$. That is the case of the mixed state composed a memory pattern with $f$ mean firing rate is $\alpha sf$. Thus, they are the same as each other in the sparse limit. This is the reason why the storage capacity of the OR mixed state $\gamma^{(s,1)}$, which is composed of $s$ memory patterns with the mean firing rate $f$, is the same as that of the memory pattern with the mean firing rate $sf$. In the same way, when the mixed state $\gamma^{(s,k)}$, which is $k \geq 2$, is evaluated on the variance of the cross-talk noise in Eq. (16), it becomes 0 in the sparse limit. Thus, both $m^{(s,k)}_{\mu}$ constituting the signal term and variance for the cross-talk noise term become 0 in the sparse limit. When the S/N ratio at that time is calculated for $k \geq 2$, it converges to 0. Therefore, the study showed that the storage capacity did not become 0 only in the OR mixed state $\gamma^{(s,1)}$ of $k = 1$ in the sparse limit.

Here we use the naive S/N analysis to discuss why the threshold value of only OR mixed state $\gamma^{(s,1)}$ converges at the value excluding 0, while the threshold value converges at 0 for $k \geq 2$ in the sparse limit. Threshold value $h$ is a boundary value where the neuron state $x_i$ with the internal potential $u_i$ is assigned to either 1 or 0 such as $x_i = \Theta(u_i + h)$. Thus, if the absolute value of the signal term approaches 0, the threshold value, which is the boundary value, goes toward 0. This is why the threshold value converges at 0 on the mixed state with $k \geq 2$ where the absolute value of the signal term goes toward 0 in the sparse limit. On the other hand, $m^{(s,1)}_{\mu} \rightarrow 1$ on the OR mixed state $\gamma^{(s,1)}$, and the overlap $m^{(1,1)}_{\mu}$ on the memory pattern is always $m^{(1,1)}_{\mu} = 1$. Therefore, the threshold value that stabilizes the OR mixed state $\gamma^{(s,1)}$ and the threshold value that stabilizes the memory pattern coincide with each other at $h = -0.5$ in the naive S/N analysis. The threshold value obtained from the SCSNA as shown in Fig. 4 becomes $h_C = -0.7$, and it differs somewhat from the result $h = -0.5$. However, this is the qualitative reason for the optimum threshold value of the OR mixed state $\gamma^{(s,1)}$ that corresponds to the threshold value of the memory pattern in the sparse limit $f \rightarrow 0$.

### IV. SUMMARY AND DISCUSSION

The various types of symmetric mixed states on the sparsely encoded associative memory model composed of $0-1$ neurons were investigated using the SCSNA and the computer simulation. The results showed that the storage capacity of the OR mixed state diverges in manner similar to the memory pattern in the sparse limit with $1/|f \log f|$, but that the storage capacities of the other types of the mixed states do not diverge. In order to clarify this reason, we evaluated the overlaps of $s$ types of the mixed states with the memory
pattern. The overlap of the OR mixed state converged to 1 in the sparse limit, while the overlaps of all of the other $s - 1$ types of the mixed states converged to 0. Moreover, the evaluation of the variance of the cross-talk noise on the naive S/N analysis provided the above qualitative causes. The threshold value of the OR mixed state corresponded to the threshold value of the memory pattern in the sparse limit. This was understood from an investigation of the overlap mentioned above. We conclude that the OR mixed state is the appropriate mixed state in the sparse encoding scheme.

Finally, let us infer a relationship between the present theoretical results and recently reported physiological findings of Sugase et al. [13] for the face-responsive cell in the IT cortex. Sugase et al. performed a single unit recording in the IT cortex of two macaque monkeys by showing them various visual stimuli, such as monkey- and human-faces with various facial expressions and simple geometric shapes. They measured the temporal change of the information carried by the firing of neurons for the classification of visual stimulus sets [14]. They found that the initial transient firing correlated well with rough categorizations (e.g. face vs. nonface stimuli), and the subsequent sustained firing represented more detailed information. Their results suggest that the neuron firing pattern is initially a superposition of patterns representing different faces or expressions, but it then converges to a single pattern representing a specific face or expression. To reinterpret this transient phenomenon by the "words" discussed in this paper, we might be able to say that the OR mixed state appears in the initial part of dynamics of neurons, and the network state finally converges on the memory pattern. To our regret, such transient phenomenon did not occur in this model.

Recently, Toya et al. and Okada et al. have discussed an associative memory model with hierarchically correlated memory patterns [15,16]. The hierarchically correlated patterns employed in [15] may capture the qualitative properties of the visual stimuli used by Sugase et al. Okada et al. reported the following transient phenomena that were similar to the results of Sugase et al.. The network state approached a mixed state at the first stage of the retrieval process. After that it diverged away from the mixed state and finally converged with the memory pattern that was closest to the input pattern. However, in their model, the mixed state was not the OR mixed state but the majority decision mixed state. It was thought to be the majority decision mixed state because the memory pattern was made to be 50% of the firing rate in [15]. As mentioned before, the typical mixed state for the 50% firing rate is the majority decision mixed state. Considering our results and the results of [15], we conjecture that the associative memory model stores the sparsely encoded memory patterns with a hierarchical structure, and the state of the model is not directly drawn into any memory pattern, but drawn into the memory patterns after being drawn once into the OR mixed state. This prediction agrees with the behavior of IT cortex neurons suggested by Sugase et al.

**APPENDIX A:**

The order parameter equation of the SCSNA is given as follows [12],

$$Y = F \left( \sum_{\mu=1}^{s} (p_{\mu} - f)m_{\mu} + h + \Gamma Y + \sigma z \right),$$  \hspace{1cm} (A1)
\[ m' = \int_{-\infty}^{\infty} Dz < \eta' - f > Y, \quad \text{(A2)} \]
\[ q = \int_{-\infty}^{\infty} Dz < Y^2 >, \quad \text{(A3)} \]
\[ U = \frac{1}{\sigma} \int_{-\infty}^{\infty} Dz z < Y >, \quad \text{(A4)} \]
\[ \sigma^2 = \frac{\alpha q}{(1 - U)^2}, \quad \text{(A5)} \]
\[ \Gamma = \frac{\alpha U}{1 - U}. \quad \text{(A6)} \]

where,
\[ Dz = \frac{dz}{\sqrt{2\pi}} \exp(-z^2/2). \quad \text{(A7)} \]

Here < > stands for the average over the elements of retrieved memory pattern \( \eta^\mu \) (1 \( \leq \mu \leq s \)) that obeys Eq. (A1). Since the output function is \( F(\cdot) = \Theta(\cdot) \), we apply the Maxwell rule [12,10], and obtain the following expression,
\[ Y(\eta, \alpha, z) = \Theta \left( \sum_{\mu=1}^{s} (\eta^\mu - f)m_\mu + h + \frac{\Gamma}{2} + \sigma z \right). \quad \text{(A8)} \]

We also discuss the symmetric mixed state in this paper, where it is possible to put \( m^\mu = m \) (1 \( \leq \mu \leq s \)). By substituting these conditions into Eq. (A1) to (A7), we obtain the following Eq. (A9) to (A11),
\[ m = \frac{1}{2} \sum_{v=0}^{s-1} C_v f^v (1 - f)^{s-v-1} \times \left[ \text{erf} \left( \frac{m(v - sf + 1) + h + \frac{\Gamma}{2}}{\sqrt{2\sigma}} \right) - \text{erf} \left( \frac{m(v - sf) + h + \frac{\Gamma}{2}}{\sqrt{2\sigma}} \right) \right], \quad \text{(A9)} \]
\[ q = \frac{1}{2} + \frac{1}{2} \sum_{v=0}^{s} C_v f^v (1 - f)^{s-v} \times \text{erf} \left( \frac{m(v - sf) + h + \frac{\Gamma}{2}}{\sqrt{2\sigma}} \right), \quad \text{(A10)} \]
\[ U = \frac{1}{\sqrt{2\pi}\sigma} \sum_{v=0}^{s} C_v f^v (1 - f)^{s-v} \times \exp \left( - \left( \frac{m(v - sf) + h + \frac{\Gamma}{2}}{\sqrt{2\sigma}} \right)^2 \right). \quad \text{(A11)} \]

Eq. (B) decides the threshold value \( h \) so that,
\[ f = \frac{1}{2} + \frac{1}{2} \sum_{v=0}^{s} C_v f^v (1 - f)^{s-v} \times \text{erf} \left( \frac{m(v - sf) + h + \frac{\Gamma}{2}}{\sqrt{2\sigma}} \right). \quad \text{(A12)} \]
For the mixed state $\gamma^{(s,k)}$, $f$ in the left term of Eq. (A12) only has to be changed into $f^{(s,k)}$. Eq. (7) gives the mean firing rate $f^{(s,k)}$. The overlap $M^{(s,k)}$ of an equilibrium state with the mixed state $\gamma^{(s,k)}$ in Eq. (4) is shown as,

$$M^{(s,k)} = \sum_{v=0}^{s} \frac{\Theta(v - k) - f^{(s,k)}}{2f^{(s,k)}(1 - f^{(s,k)})} s C_v f^v (1 - f)^{s-v}$$

$$\times \text{erf}(\frac{m(v - sf) + h + \frac{\Gamma}{2}}{\sqrt{2\sigma}}).$$

(A13)
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FIGURES

FIG. 1. Storage rate $\alpha$ dependency of overlap $M^{(3,1)}$ on OR mixed state $\gamma^{(3,1)}$ of $s = 3$

FIG. 2. Storage capacity $\alpha_C$ on OR mixed state $\gamma^{(s,1)}$

FIG. 3. Asymptotic characteristic of storage capacity $\alpha_C$ on OR mixed state $\gamma^{(s,1)}$.

FIG. 4. Mean firing rate $f$ dependency of optimum threshold value $h_c$ on OR mixed state $\gamma^{(s,1)}$

FIG. 5. Mean firing rate $f$ dependency of storage capacity on majority decision mixed state $\gamma^{(s,s/2)}$

FIG. 6. Mean firing rate $f$ dependency of storage capacity on AND mixed state $\gamma^{(s,s)}$

FIG. 7. Mean firing rate $f$ dependency of threshold value $h_c$ on majority decision mixed state $\gamma^{(s,s/2)}$

FIG. 8. Mean firing rate $f$ dependency of the threshold value $h_c$ on AND mixed state $\gamma^{(s,s)}$

FIG. 9. Mean firing rate $f$ dependency of overlap $m_{(5,k)}$ as recalled state is mixed state $\gamma^{(5,k)}$
Figure 1: T. Kimoto and M. Okada

Figure 2: T. Kimoto and M. Okada
Figure 3: T. Kimoto and M. Okada

Figure 4: T. Kimoto and M. Okada
Figure 5: T. Kimoto and M. Okada

Figure 6: T. Kimoto and M. Okada
Figure 7: T. Kimoto and M. Okada

Figure 8: T. Kimoto and M. Okada
Figure 9: T. Kimoto and M. Okada