PROPER ACTIONS ON TOPOLOGICAL GROUPS: APPLICATIONS TO QUOTIENT SPACES

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ABSTRACT. Let $X$ be a Hausdorff topological group and $G$ a locally compact subgroup of $X$. We show that the natural action of $G$ on $X$ is proper in the sense of R. Palais. This is applied to prove that there exists a closed set $F \subseteq X$ such that $FG = X$ and the restriction of the quotient projection $X \to X/G$ to $F$ is a perfect map $F \to X/G$. This is a key result to prove that many topological properties (among them, paracompactness and normality) are transferred from $X$ to $X/G$, and some others are transferred from $X/G$ to $X$. Yet another application leads to the inequality $\dim X \leq \dim X/G + \dim G$ for every paracompact group $X$ and its locally compact subgroup $G$.

1. INTRODUCTION

By a $G$-space we mean a completely regular Hausdorff space together with a fixed continuous action of a Hausdorff topological group $G$ on it.

The notion of a proper $G$-space was introduced in 1961 by R. Palais \cite{Palais} with the purpose to extend a substantial portion of the theory of compact Lie group actions to the case of noncompact ones.

A $G$-space $X$ is called proper (in the sense of Palais \cite{Palais} Definition 1.2.2), if each point of $X$ has a, so called, small neighborhood, i.e., a neighborhood $V$ such that for every point of $X$ there is a neighborhood $U$ with the property that the set $\langle U, V \rangle = \{g \in G \mid gU \cap V \neq \emptyset\}$ has compact closure in $G$. Clearly, if $G$ is compact, then every $G$-space is proper.

Many important problems in the theory of proper actions are conjugated (see \cite{19, 1, 2, 8, 7}) to the following major open problem:

**Conjecture 1.** Let $G$ be a locally compact group. Then the orbit space $X/G$ of any paracompact proper $G$-space $X$ is paracompact.

This conjecture is open even if $X$ is metrizable; in this case it is equivalent (see \cite{8}) to the following old problem going back to R. Palais \cite{Palais}:

**Conjecture 2.** Let $G$ be a locally compact group and $X$ a metrizable proper $G$-space. Then the topology of $X$ is metrizable by a $G$-invariant metric.

Due to Palais \cite{Palais}, it is known that Conjecture 2 is true for separable metrizable proper $G$-spaces provided the acting group $G$ is Lie. Other particular cases are discussed in \cite{21} and \cite{8}.

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In this paper we prove Conjecture 1 in an important particular case, namely, when $X$ is a topological group endowed with the natural action of its locally compact subgroup $G$. (see Corollary 1.5). We first show that $X$ is a proper $G$-space and then we establish a more general result (Theorem 1.2) which has many interesting applications in the theory of topological groups.

Below all topological groups are assumed to satisfy the Hausdorff separation axiom.

**Theorem 1.1.** Let $X$ be a topological group and $G$ a locally compact subgroup of $X$. Then the action of $G$ on $X$ given by the formula $g \ast x = x g^{-1}$, $g \in G$, $x \in X$, is proper.

Recall that a subset $S$ of a proper $G$-space $X$ is called $G$-fundamental, or just fundamental, if $S$ is a small set and the saturation $G(S) = \{gs \mid g \in G, s \in S\}$ coincides with $X$.

Here is the key result of the paper:

**Theorem 1.2.** Let $X$ be a topological group and $G$ a locally compact subgroup of $X$. Then there exists a closed $G$-fundamental set in $X$.

It is easy to prove (see [2, Proposition 1.4]) that in every proper $G$-space $X$, the restriction of the orbit map $p : X \to X/G$ to any closed small set is perfect (i.e., is closed and has compact fibers). In combination with Theorem 1.2 this yields:

**Corollary 1.3.** Let $X$ be a topological group, $G$ a locally compact subgroup of $X$, and $X/G$ the quotient space of all right lateral classes $xG = \{xg \mid g \in G\}$, $x \in X$. Then there exists a closed subset $F \subset X$ such that the restriction $p|_F : F \to X/G$ is a perfect surjective map.

This fact has the following immediate corollary about transfer of properties from $X$ to $X/G$:

**Corollary 1.4.** Let $\mathcal{P}$ be a topological property stable under perfect maps and also inherited by closed subsets. Assume that $X$ is a topological group with the property $\mathcal{P}$ and let $G$ be a locally compact subgroup of $X$. Then the quotient space $X/G$ also has the property $\mathcal{P}$.

Among such properties $\mathcal{P}$ we single out just some of those which provide new results in Corollary 1.4: paracompactness, countable paracompactness, weak paracompactness, normality, perfect normality, Čech-completeness, being a $k$-space (see [16, §5.1, §5.2, §5.3, §1.5, §3.9, §3.3]) and stratifiability (see [14]). Thus, we get the following positive solution of Conjecture 1 in an important special case:

**Corollary 1.5.** Let $X$ be a paracompact topological group and $G$ a locally compact subgroup of $X$. Then the quotient space $X/G$ is paracompact.

In this connection it is in order to recall the following remarkable result of A. V. Arhangel’skii [1]: every topological group is the quotient of a paracompact zero-dimensional group. Hence, the local compactness of $G$ is essential in Corollary 1.5.

We recall that a locally compact group is called almost connected, if its space of connected components is compact.
Corollary 1.6. Let $\mathcal{P}$ be a topological property stable under open perfect maps and also inherited by closed subsets. Assume that $X$ is a paracompact group with the property $\mathcal{P}$ and let $G$ be an almost connected subgroup of $X$. Then the quotient space $X/G$ also has the property $\mathcal{P}$.

Among properties stable under open perfect maps and also inherited by closed subsets we highlight strong paracompactness and realcompactness (see [16, Exercises 5.3.C(c), 5.3H(d), and Theorem 3.11.4 and Exercises 3.11.G]). Thus, we get the following

Corollary 1.7. Let $X$ be a strongly paracompact (resp., paracompact and realcompact) topological group and $G$ an almost connected subgroup of $X$. Then the quotient space $X/G$ is strongly paracompact (resp., paracompact and realcompact).

We note that the converse of the first statement in this corollary is not true. Namely, it is known that the Baire space $B(\aleph_1)$ of weight $\aleph_1$ (which is, in fact, a commutative topological group) is strongly paracompact and its product with the additive group $\mathbb{R}$ of the reals is not (see [16, Exercises 5.3F(a) and 5.3F(b)]). Hence, the direct product $X = \mathbb{R} \times B(\aleph_1)$ and its subgroup $G = \mathbb{R} \times \{0\}$ provide the desired counterexample, answering negatively a question from [11]. Furthermore, [12, Open Problem 3.2.1, p.151] asks whether every locally strongly paracompact group is strongly paracompact? The same group $\mathbb{R} \times B(\aleph_1)$ provides a negative answer to this question too. Indeed, since the product of a compact space and a strongly paracompact space is strongly paracompact (see [16, Exercise 5.3H(a)]), we infer that $\mathbb{R} \times B(\aleph_1)$ is locally strongly paracompact.

Combining our Corollary 1.5 with a result of Abels [1, Main Theorem], we obtain the following

Corollary 1.8. Let $X$ be a paracompact group, $G$ an almost connected subgroup of $X$, and $K$ a maximal compact subgroup of $G$. Then there exists a $K$-invariant subset $S \subset X$ such that $X$ is $K$-homeomorphic to the product $G/K \times S$. In particular, $X$ is homeomorphic to $\mathbb{R}^n \times S$ for some $n \geq 1$.

In [10] A. V. Arhangel’skii has studied properties which are transferred in the opposite direction, i.e., from $X/G$ to $X$. The next corollary is a unified result of this sort which implies many of those in [10] as well as provides some new ones.

Corollary 1.9. Let $\mathcal{P}$ be a topological property invariant and inverse invariant of perfect maps, and also stable under multiplication by a locally compact group. Assume that $X$ is a topological group and let $G$ be a locally compact subgroup of $X$ such that the quotient space $X/G$ has the property $\mathcal{P}$. Then the group $X$ also has the property $\mathcal{P}$.

Among such properties we mention just some: paracompactness, being a $k$-space, Čech-completeness (see [16, §5.1, §3.3, §3.9]). That paracompactness is stable under multiplication by a locally compact group follows from a result of Morita [22], since every locally compact group is paracompact (even, strongly paracompact [12, Theorem 3.1.1]).

Corollary 1.10. Let $\mathcal{P}$ be a topological property invariant and inverse invariant of open perfect maps, and also stable under multiplication by a locally compact group. Assume that $X$ is a paracompact group and let $G$ be an almost connected subgroup

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of $X$ such that the quotient space $X/G$ has the property $\mathcal{P}$. Then the group $X$ also has the property $\mathcal{P}$.

Among such properties we highlight realcompactness (see [16, Theorem 3.11.14 and Exercise 3.11.G, and also take into account that every locally compact group is realcompact]).

Corollary 1.5 is further applied to prove the following Hurewicz type formula:

**Theorem 1.11.** Let $G$ be a locally compact subgroup of a paracompact topological group $X$. Then

$$\dim X \leq \dim X/G + \dim G.$$ 

**Remark 1.12 ([23]).** If in this theorem $X$ is a locally compact group then, in fact, the equality holds:

$$\dim X = \dim X/G + \dim G.$$ 

All the proofs are given in section 3.

2. Preliminaries

Throughout the paper, unless otherwise is stated, by a *group* we shall mean a topological group $G$ satisfying the Hausdorff separation axiom; by $e$ we shall denote the unity of $G$.

All topological spaces are assumed to be Tychonoff (= completely regular and Hausdorff). The basic ideas and facts of the theory of $G$-spaces or topological transformation groups can be found in G. Bredon [14] and in R. Palais [23]. Our basic reference on proper group actions is Palais’ article [24]. Other good sources are [21], [1] and [2].

For the convenience of the reader we recall, however, some more special definitions and facts below.

By a $G$-space we mean a topological space $X$ together with a fixed continuous action $G \times X \to X$ of a topological group $G$ on $X$. By $gx$ we shall denote the image of the pair $(g, x) \in G \times X$ under the action.

If $Y$ is another $G$-space, a continuous map $f : X \to Y$ is called a $G$-map or an equivariant map, if $f(gx) = gf(x)$ for every $x \in X$ and $g \in G$.

If $X$ is a $G$-space, then for a subset $S \subset X$ and for a subgroup $H \subset G$, the $H$-hull (or $H$-saturation) of $S$ is defined as follows: $H(S) = \{hs \mid h \in H, \ s \in S\}$. If $S$ is the one point set $\{x\}$, then the $G$-hull $G(\{x\})$ usually is denoted by $G(x)$ and called the orbit of $x$. The orbit space $X/G$ is always considered in its quotient topology.

A subset $S \subset X$ is called $H$-invariant if it coincides with its $H$-hull, i.e., $S = H(S)$. By an invariant set we shall mean a $G$-invariant set.

For any $x \in X$, the subgroup $G_x = \{g \in G \mid gx = x\}$ is called the stabilizer (or stationary subgroup) at $x$.

A compatible metric $\rho$ on a metrizable $G$-space $X$ is called invariant or $G$-invariant, if $\rho(gx, gy) = \rho(x, y)$ for all $g \in G$ and $x, y \in X$. If $\rho$ is a $G$-invariant metric on any $G$-space $X$, then it is easy to verify that the formula

$$\bar{\rho}(G(x), G(y)) = \inf \{\rho(x', y') \mid x' \in G(x), \ y' \in G(y)\}$$
defines a pseudometric $\tilde{\rho}$, compatible with the quotient topology of $X/G$. If, in addition, $X$ is a proper $G$-space then $\tilde{\rho}$ is, in fact, a metric on $X/G$ [24, Theorem 4.3.4].

For a closed subgroup $H \subset G$, by $G/H$ we will denote the $G$-space of cosets $\{gH| g \in G\}$ under the action induced by left translations.

A locally compact group $G$ is called almost connected, if the space of connected components of $G$ is compact. Such a group has a maximal compact subgroup $K$, i.e., every compact subgroup of $G$ is conjugate to a subgroup of $K$ [1, Theorem A.5]. The corresponding classical theorem on Lie groups can be found in [20, Ch. XV, Theorem 3.1].

In 1961 Palais [24] introduced the very important concept of a proper action of an arbitrary locally compact group $G$ and extends a substantial part of the theory of compact Lie transformation groups to noncompact ones.

Let $X$ be a $G$-space. Two subsets $U$ and $V$ in $X$ are called thin relative to each other [24 Definition 1.1.1], if the set $\langle U,V \rangle = \{ g \in G| gU \cap V \neq \emptyset \}$, called the transporter from $U$ to $V$, has a compact closure in $G$. A subset $U$ of a $G$-space $X$ is called $G$-small, or just small, if every point in $X$ has a neighborhood thin relative to $U$. A $G$-space $X$ is called proper (in the sense of Palais), if every point in $X$ has a small neighborhood.

Clearly, if $G$ is compact, then every $G$-space is proper. Furthermore, if $G$ acts properly on a compact space, then $G$ has to be compact as well. If $G$ is discrete and $X$ is locally compact, the notion of a proper action is the same as the classical notion of a properly discontinuous action. When $G=\mathbb{R}$, the additive group of the reals, proper $G$-spaces are precisely the dispersive dynamical systems with regular orbit space (see [13 Ch. IV]).

Important examples of proper $G$-spaces are the coset spaces $G/H$ with $H$ a compact subgroup of a locally compact group $G$. Other interesting examples the reader can find in [1, 2, 5, 6] and [24].

In what follows we shall need also the definition of a twisted product $G \times_K S$, where $K$ is a closed subgroup of $G$, and $S$ a $K$-space. $G \times_K S$ is the orbit space of the $K$-space $G \times S$ on which $K$ acts by the rule: $k(g,s) = (gk^{-1},ks)$. Furthermore, there is a natural action of $G$ on $G \times_K S$ given by $g'(g,s) = [g'g,s]$, where $g' \in G$ and $[g,s]$ denotes the $K$-orbit of the point $(g,s)$ in $G \times S$. We shall identify $S$, by means of the $K$-equivariant embedding $s \mapsto [e,s], s \in S$, with the $K$-invariant subset $\{ [e,s] | s \in S \}$ of $G \times_K S$. This $K$-equivariant embedding $S \hookrightarrow G \times_K S$ induces a homeomorphism of the $K$-orbit space $S/K$ onto the $G$-orbit space $(G \times_K S)/G$ (see [15 Ch. II, Proposition 3.3]).

The twisted products are of a particular interest in the theory of transformation groups (see [15 Ch. II, § 2]). It turns out that every $G$-space locally is a twisted product. For a more precise formulation we need to recall the following well known notion of a slice (see [24 p. 305]):

**Definition 2.1.** Let $X$ be a $G$-space and $K$ a closed subgroup of $G$. A $K$-invariant subset $S \subset X$ is called a $K$-kernel if there is a $G$-equivariant map $f: G(S) \rightarrow G/K$, called the slicing map, such that $S = f^{-1}(eK)$. The saturation $G(S)$ is called a tubular set and the subgroup $K$ will be referred as the slicing subgroup.

If in addition $G(S)$ is open in $X$ then we shall call $S$ a $K$-slice in $X$.

If $G(S) = X$ then $S$ is called a global $K$-slice of $X$.
It turns out that each tubular set with a compact slicing subgroup is a twisted product. The tubular neighborhood \( G(S) \) is \( G \)-homeomorphic to the twisted product \( G \times_K S \); namely the map \( \xi : G \times_K S \to G(S) \) defined by \( \xi([g, s]) = gs \) is a \( G \)-homeomorphism (see [15, Ch. II, Theorem 4.2]). In what follows we will use this fact without a specific reference.

One of the most powerful results in the theory of topological transformation groups states (see [24, Proposition 2.3.1]) that, if \( X \) is a proper \( G \)-space with \( G \) a Lie group, then for any point \( x \in X \), there exists a \( G_x \)-slice \( S \) in \( X \) such that \( x \in S \). In general, when \( G \) is not a Lie group, it is no longer true that a slice exists at each point of \( X \) (see [4]). Generalizing the case of Lie group actions, in [7] the following approximate version of Palais’ Slice Theorem [24, Proposition 2.3.1] for non-Lie group actions was proved, which plays a key role in the proof of Theorem 1.11:

**Theorem 2.2** (Approximate Slice Theorem [7]). Let \( G \) be a locally compact group, \( X \) a proper \( G \)-space and \( x \in X \). Then for any neighborhood \( O \) of \( x \) in \( X \), there exist a compact large subgroup \( K \subset G \) with \( G_x \subset K \), and a \( K \)-slice \( S \) such that \( x \in S \subset O \).

Thus, every proper \( G \)-space is covered by invariant open subsets each of which is a twisted product.

Note that for \( G \) a compact group this theorem was proved in [4]. A version of it, without requiring \( K \) to be a large subgroup, was obtained in [2].

We recall that here a closed subgroup \( H \subset G \) is called large, if there exists a closed normal subgroup \( N \subset G \) such that \( N \subset H \) and \( G/N \) is a Lie group.

We refer to [7] for further characterizations of large subgroups.

On any group \( G \) one can define two natural (but equivalent) actions of \( G \) given by the formulas

\[ g \cdot x = gx, \quad \text{and} \quad g * x = xg^{-1}, \]

respectively, where in the right parts the group operations are used with \( g, x \in G \).

Throughout we shall consider the second action only.

By \( U(G) \) we shall denote the Banach space of all right uniformly continuous bounded functions \( f : G \to \mathbb{R} \) endowed with the supremum norm. Recall that \( f \) is called right uniformly continuous, if for every \( \varepsilon > 0 \) there exists a neighborhood \( O \) of the unity in \( G \) such that \( |f(y) - f(x)| < \varepsilon \) whenever \( yx^{-1} \in O \).

We shall consider the induced action of \( G \) on \( U(G) \), i.e.,

\[ (gf)(x) = f(xg), \quad \text{for all} \quad g, x \in G. \]

It is easy to check that this action is continuous, linear and isometric (see e.g., [3, Proposition 7]).

**Proposition 2.3.** Let \( G \) be a group. Then for every \( f \in U(G) \), the map \( f_* : G \to U(G) \) defined by \( f_*(x)g = f(xg^{-1}) \), \( x, g \in G \), is a right uniformly continuous \( G \)-map.

**Proof.** A simple verification. \( \square \)
3. Proofs

Proof of Theorem 1.4. Choose a neighborhood $U$ of the identity in $X$ such that $U = U^{-1}$ and $U^3 \cap G$ has a compact closure in $G$. We claim that for every $x \in X$ the neighborhood $xU$ is $G$-small. Indeed, let $y \in X$ be any point. Two cases are possible.

Case 1. Assume that $y \in xU^2G$. Then $y = xu_1u_2h$ with $u_1, u_2 \in U$ and $h \in G$. We claim that $xU^2h$ is a neighborhood of $y$ thin relative to $xU$. Indeed, if $g \in \langle xU, xU^2h \rangle$, then $g^{-1}h^{-1} \in U^3 \cap G$. Since $U^3 \cap G$ has a compact closure, we see that so is the set $(U^3 \cap G)h$ to which belongs $g^{-1}$. This yields that $\langle xU, xU^2h \rangle$ is contained in $h^{-1}(U^3 \cap G)^{-1}$, which also has a compact closure. Hence the transporter $\langle xU, xU^2h \rangle$ has a compact closure, as required.

Case 2. Assume that $y \notin xU^2G$. In this case $y \notin xU^2G$. Indeed, if $y \notin xU^2G$ then the neighborhood $xUu^{-1}y$ of $y$ should meet the set $xU$. Then $xux^{-1}y = xv$ for suitable elements $u, v \in U$ and $h \in G$. Then $y = xv^{-1}vh \in xU^2G$, a contradiction.

Hence the open set $V = X \setminus xU^2G$ is a $G$-invariant neighborhood of $y$, and $V$ is thin relative to $xU$ because the transporter $\langle xU, V \rangle$ is empty in this case. This completes the proof.

Proposition 3.1. Let $X$ be a group and $G$ a locally compact subgroup of $X$. Then there exists a locally finite covering of $X$ consisting of $G$-invariant open sets of the form $S_iG$, where each $S_i$ is an open $G$-small subset of $X$.

Proof. By Theorem 1.1 $X$ is a proper $G$-space, and hence, one can choose a $G$-small neighborhood $U$ of the unity in $X$. By virtue of Markov’s theorem 12, Theorem 3.3.9], there exists a right uniformly continuous function $f : X \rightarrow [0, 1]$ such that

\begin{equation}
(3.1) \quad f(e) = 0 \quad \text{and} \quad f^{-1}([0, 1]) \subset U.
\end{equation}

Then, by Proposition 2.3 $f$ induces an $X$-equivariant map $f_* : X \rightarrow U(X)$ defined by the rule:

\[ f_*(x)(g) = f(xg^{-1}), \quad x, g \in X. \]

Denote by $Z$ the image $f_*(X)$. Clearly, $Z$ is the $X$-orbit of the point $f_*(e)$ in the $X$-space $U(X)$, and the metric of $U(X)$ induces an $X$-invariant metric on $Z$. We claim that

\begin{equation}
(3.2) \quad f_*(\Gamma_{x,V}) \subset x^{-1}U, \quad \text{for every} \quad x \in X,
\end{equation}

where $V = [0, 1)$ and $\Gamma_{x,V}$ is the open subset $\{ \varphi \in U(X) \mid \varphi(x) \in V \}$ of $U$.

First we observe that $\Gamma_{x,V} = x^{-1}\Gamma_{e,V}$ and

\[ f_*^{-1}(\Gamma_{e,V}) \subset f^{-1}(V). \]

Then (3.2) follows from (3.1) and the $X$-equivariance of $f_*$. Besides, since $f_*(x) \in \Gamma_{x,V}$ for every $x \in X$, we see that the sets $\Gamma_{x,V}, x \in X$, constitute a covering of $Z$.

From now on we restrict ourselves only by the induced actions of the subgroup $G$, i.e., we will consider $X$ and $Z$ just as $G$-spaces.

Now, it follows from (3.2) and from the $G$-equivariance of $f_*$ that

\begin{equation}
(3.3) \quad f_*^{-1}(G(\Gamma_{x,V})) \subset x^{-1}UG, \quad \text{for every} \quad x \in X.
\end{equation}
Since \( f_* : X \to Z \) is \( G \)-equivariant, it induces a continuous map \( \tilde{f}_* \) of the \( G \)-orbit spaces, i.e., we have the following commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{f_*} & Z \\
\downarrow p & & \downarrow q \\
X/G & \xrightarrow{\tilde{f}_*} & Z/G
\end{array}
\]

where \( p \) and \( q \) are the \( G \)-orbit maps.

It follows from (3.3) that

\[
(3.4) \quad \tilde{f}_*^{-1}(q(\Gamma_x, V)) \subset p(x^{-1}U), \quad \text{for every } x \in X.
\]

Thus, the open covering \( \{p(xU) \mid x \in X\} \) of the \( G \)-orbit space \( X/G \) is refined by the open covering \( \{\tilde{f}_*^{-1}(q(\Gamma_x, V)) \mid x \in X\} \).

Since the metric of \( Z \) is \( G \)-invariant, the orbit space \( Z/G \) is pseudometrizable (see Preliminaries). Hence the open covering \( \{q(\Gamma_x, V) \mid x \in X\} \) of \( Z/G \) admits an open locally finite refinement, say \( \{W_i \mid i \in I\} \) (see [16, Theorem 4.4.1 and Remark 4.4.2]). Then, clearly, \( \{p^{-1}(\tilde{f}_*^{-1}(W_i)) \mid i \in I\} \) is an open locally finite refinement of \( \{xUG \mid x \in X\} \) consisting of \( G \)-invariant sets. It then follows that each set \( p^{-1}(\tilde{f}_*^{-1}(W_i)) \) is contained in some set \( xU \), \( x \in X \), which yields that

\[
p^{-1}(\tilde{f}_*^{-1}(W_i)) = S_iG,
\]

where \( S_i = p^{-1}(\tilde{f}_*^{-1}(W_i)) \cap xU \).

Now, \( S_i \), being a subset of the \( G \)-small set \( xU \), is itself \( G \)-small (see the proof of Theorem 1.1). This completes the proof. \( \square \)

**Proof of Theorem 1.2.** Let \( \{S_iG \mid i \in I\} \) be the locally finite open covering of \( X \) from Proposition 3.1. Then the union \( S = \bigcup_{i \in I} S_i \) is a \( G \)-small set (see e.g., [2, Proposition 1.2(d)]). On the other hand,

\[
SG = \left( \bigcup_{i \in I} S_i \right)G = \bigcup_{i \in I} S_iG = X,
\]

yielding that \( S \) is a \( G \)-fundamental subset of \( X \). Since the closure of a \( G \)-small set is \( G \)-small (see e.g., [2, Proposition 1.2(b)]), the closure \( \overline{S} \) is the desired closed \( G \)-fundamental set. This completes the proof.

**Proof of Corollary 1.6.** Since \( G \) is almost connected, it has a maximal compact subgroup \( K \) (see Preliminaries). Since by Corollary 1.5 the quotient \( X/G \) is paracompact, due to a result of Abels [1, Main Theorem], \( X \) admits a global \( K \)-slice \( S \), and hence, it is \( G \)-homeomorphic to the twisted product \( G \times_K S \) (see Preliminaries). Since the group \( K \) is compact, it then follows that the \( K \)-orbit map \( G \times S \to G \times_K S \cong_G X \) is open and perfect. This yields immediately that the restriction \( p|_S : S \to X/G \) of the \( G \)-orbit map \( p : X \to X/G \) is an open and perfect surjection. Now the result follows.

**Proof of Corollary 1.9.** By virtue of Corollary 1.3 \( X \) admits a closed subset \( F \subset X \) such that the restriction of the quotient projection \( p : X \to X/G \) to \( F \) is a
perfect surjection $p|_F : F \to X/G$. It then follows from the hypothesis that $F$ has the property $P$. Since a locally compact group is paracompact (even, strongly paracompact \cite{12} Theorem 3.1.1), then again by the hypothesis, the product $G \times F$ has the property $P$. Since $F$ is a closed $G$-small set, the action map $G \times F \to X$ is perfect (see Abels \cite{2} Proposition 1.4), and since $FG = X$ we see that the map $G \times F \to X$ is surjective. Then $X$ has the property $P$ by the hypothesis.

**Proof of Corollary 1.10** It is quite similar to the proof of Corollary 1.6.

**Proof of Theorem 1.11** Consider three cases.

**Case 1.** Assume that $G$ is almost connected. In this case $G$ has a maximal compact subgroup, say, $K$ (see Preliminaries). Since, by Corollary 1.5 the quotient $X/G$ is paracompact, one can apply a theorem of H. Abels \cite{1} Main Theorem, according to which $X$ admits a global $K$-slice, say $S$, and hence, $X$ is $G$-homeomorphic to the twisted product $G \times KS$. By another result of H. Abels \cite{1} Theorem 2.1, $X$ is homeomorphic (moreover, $K$-equivariantly homeomorphic) to the product $G/K \times S$.

Since $G$ is locally compact and paracompact and the natural quotient map $G \to G/K$ is open and closed, we infer that $G/K$ is also locally compact and paracompact. Further, since $S$ is paracompact, according to a theorem of Morita \cite{22} one has:

\[(3.5) \dim (G/K \times S) \leq \dim G/K + \dim S.\]

Since $K$ is compact, according to a result of V. V. Filippov \cite{18} one has the inequality:

\[(3.6) \dim S \leq \dim S/K + \dim q\]

where $q : S \to S/K$ is the $K$-orbit projection and $\dim q = \sup \{\dim g^{-1}(a) \mid a \in S/K\}$. Further, since $K$ acts freely on $S$, we see that $\dim q = \dim K$.

Consequently, combining (3.5) and (3.6) one obtains:

\[(3.7) \dim (G/K \times S) \leq \dim G/K + \dim K + \dim S/K.\]

Next, since $X/G \cong (G \times KS)/G \cong S/K$ (see Preliminaries) and $\dim G/K + \dim K = \dim G$ (see Remark 1.12), it then follows from (3.7) that

\[\dim (G/K \times S) \leq \dim X/G + \dim G,\]

as required.

**Case 2.** Let $G$ be totally disconnected. Then $G$ is zero-dimensional (see \cite{17} Theorem 1.4.5)). Hence, one has to prove that

\[\dim X \leq \dim X/G.\]

By virtue of Approximate Slice Theorem (see Theorem 2.2), $X$ is covered by a family of open tubular sets $U_i$. Assume that $f_i : U_i \to G/K_i$ is the corresponding slicing map, where each $K_i$ is a compact large subgroup of $G$.

Let $p : X \to X/G$ be the quotient map. Since, by Corollary 1.5 $X/G$ is paracompact the open covering $\{p(U_i)\}$ of $X/G$ has a locally finite closed shrinking $\{F_i\}$, i.e., $F_i \subset p(U_i)$ and $\bigcup F_i = X/G$ (see \cite{16} Remark 5.1.7). Then, putting $\Phi_i = p^{-1}(F_i)$, we get a locally finite closed shrinking $\{\Phi_i\}$ of $\{U_i\}$ consisting of $G$-invariant sets.
Thus, $X$ is covered by a locally finite family of closed tubular sets $\Phi_i$. Consequently, by virtue of the locally finite sum theorem (see [17, Theorem 3.1.10]), it suffices to show that $\dim \Phi \leq \dim X/G$, for every member $\Phi$ of the covering $\{\Phi_i\}$.

Let $\varphi : \Phi \to G/K$ be the slicing map corresponding to a tubular set $\Phi \in \{\Phi_i\}$, where $K$ is a compact large subgroup of $G$ (of course, depending upon $\Phi$). Since $K$ is a large subgroup, the quotient $G/K$ is locally connected, in fact, it is a manifold [7, Proposition 3.2]. On the other hand $G/K$ is totally disconnected, and hence, $G/K$ should be a discrete space. This yields that, if we put $Q = f^{-1}(eK)$, then each $gQ$ is closed and open in $\Phi$, and $\Phi$ is the disjoint union of the sets $gQ$ one $g$ out of every coset in $G/K$. In other words, $\Phi$ is just homeomorphic to the product $G/K \times Q$.

Next, since $G/K$ is locally compact and paracompact, and $Q$ is paracompact (being a closed subset of the paracompact space $X$), according to a theorem of Morita [22] one has:

$$\dim \Phi = \dim (G/K \times Q) \leq \dim G/K + \dim Q. \quad (3.8)$$

Since $K$ is compact, according to a result of V. V. Filippov [18] one has the inequality:

$$\dim Q \leq \dim Q/K + \dim r \quad \text{where } r : Q \to Q/K \text{ is the } K \text{-orbit projection}. \quad (3.9)$$

where $r : Q \to Q/K$ is the $K$-orbit projection. Further, since $K$ acts freely on $Q$, we see that $\dim r = \dim K = \dim G = 0$.

Consequently, combining (3.8) and (3.9), and taking into account that $\dim K = 0 = \dim G/K$ (this follows, for instance, from $\dim G = 0$ and Remark 1.12), one obtains:

$$\dim \Phi \leq \dim Q/K. \quad (3.10)$$

Further, since $\Phi$ is a tubular set with the $K$-slice $Q$, one has $\Phi/G \cong Q/K$ (see Preliminaries). Then it follows from (3.10) that

$$\dim \Phi \leq \dim \Phi/G. \quad (3.11)$$

But $\Phi/G$ is a closed subset of $X/G$, and since $X/G$ is normal, due to monotonicity of the dimension (see [17, Theorem 3.1.4]), one has $\dim \Phi/G \leq \dim X/G$. This, together with (3.11), now yields that

$$\dim \Phi \leq \dim X/G,$$

as required.

Case 3. Let $G$ be arbitrary. By case 1, we have:

$$\dim X \leq \dim X/G_0 + \dim G_0. \quad (3.12)$$

where $G_0$ is the identity component of $G$. Since $G/G_0$ is a totally disconnected locally compact group which acts naturally on the $G_0$-orbit space $X/G_0$, then according to case 2, one has:

$$\dim X/G_0 \leq \dim \frac{X/G_0}{G/G_0}. \quad (3.13)$$

Now, since $\frac{X/G_0}{G/G_0} \cong X/G$ and $\dim G_0 \leq \dim G$, it follows from (3.12) and (3.13) that

$$\dim X \leq \dim X/G + \dim G.$$

This completes the proof.
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