Quantum Cosmology and the value of $\Lambda$

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Abstract
We analyse a simple quantum cosmological model with just a $\Lambda$ term present. Differing results are obtained depending on the boundary conditions applied. In the Euclidean regime the Hartle-Hawking boundary condition gives the factor $\exp(1/\Lambda)$ but, in agreement with Rubakov et al.\cite{10}, is badly behaved for negative $\Lambda$. Tunneling boundary conditions suggest an initially large value for $\Lambda$.

If only Lorentzian regions are considered all boundary conditions suggest an initially large value of $\Lambda$ for spatial curvature $k = 1$. This differs from the previously obtained result of Strominger \cite{11} for such models.
Introduction

One possible solution to the cosmological constant $\Lambda$ problem that has attracted a lot of interest is due to the idea that wormhole solutions can lead $\Lambda$ to become a dynamical variable with a distribution function $P(\Lambda)$ [1] - for a review of this proposal see eg. ref.[2]. It is suggested that this function is peaked, due to De Sitter instantons, with the Baum-Hawking factor $P(\Lambda) \sim \exp(1/\Lambda)$ [3,4] so predicting $\Lambda \to 0$ [1].

Wormholes are used in two distinct ways in such arguments, firstly they are used to justify why $\Lambda$ should be treated as a dynamical quantum variable instead of a usual classical variable. This is the most important aspect, as it allows one to make predictions of the possible values of $\Lambda$. Wormholes have further been used in connecting many universes together which produces a further exponentiation [1] i.e.

$$P(\Lambda) \sim \exp (\exp (1/\Lambda))$$

This is only useful if the first factor $\sim 1/\Lambda$ is correct. In other words, this aspect of wormholes only exaggerates any underlying behaviour. In this letter we wish to analyse a quantum cosmological model to see if the Baum-Hawking factors are present or justified. Although this simple model has been studied by many authors there are widely differing predictions for the expected value of $\Lambda$.

Cosmological constant model

When only a cosmological constant is present the Wheeler-DeWitt (WDW) equation takes the form (ignoring factor ordering corrections) see eg.[5-7]

$$\left( \frac{d}{da^2} - U \right) \Psi(a) = 0$$

(2)

Where the WDW potential $U$ for a closed $k = 1$ universe is given by

$$U = a^2 - \Lambda a^4$$

(3)

The potential is sketched in Fig.(1). This has been studied by many authors especially as the case of quantum tunnelling to a Lorentzian universe [8,9]. We follow particularly the analysis of Rubakov et.al. [10] and Strominger [11]. The WKB solutions have the form cf.[10]

$$\Psi = \frac{1}{\sqrt{|U(a)|}} \exp \left( \pm \int \sqrt{U(a)} da \right)$$

(4)
where the ‘action’ $S = - \int \sqrt{U} da$ is given by

$$- \int a(1 - \Lambda a^2)^{1/2} da = \frac{(1 - \Lambda a^2)^{3/2}}{3\Lambda}.$$  (5)

Taking the limits between $a = \Lambda^{-1/2}$ and $a = 0$ gives the solutions

$$\Psi_{\pm} \sim \exp(\pm 1/\Lambda).$$  (6)

The (+) sign corresponds to the Hartle-Hawking (HH)[12] boundary condition $\exp(-S)$ and the (-) sign to the tunnelling one $\exp(-|S|)$- see for example [5-7].

If we assume that the probability of having a specific $\Lambda$ is $P(\Lambda) \sim \Psi^2 \sim \exp(\pm 2/\Lambda)$ then the two approaches predict $\Lambda \to 0$ and $\Lambda \to \infty$ respectively. For the HH case we appear to get the suppression of $\Lambda$ but the opposite for the tunnelling case. However the tunnelling occurs through the barrier to $U = 0$ where $a^2 \sim 1/\Lambda$. When the barrier is small i.e. when $\Lambda$ is large tunnelling is enhanced: this is somewhat analogous to the application of an electric potential to an atom which allows electron to tunnel away. In this case a large value of $\Lambda$ is enhancing the possibility of the universe tunnelling into existence.

According to Strominger [11] because the scale factor $a$ today is very large the value of $\Lambda \sim a^{-2}$ is very small as required to fit observation. This does not however agree with the usual interpretation of quantum cosmology which is that of predicting initial conditions. As quantum tunneling is expected to occur to an initial size of roughly Planck dimensions the initial value of $\Lambda$ is correspondingly big $\sim 1$ in Planck units. If instead the initial scale factor was large (and so the initial value of $\Lambda \sim$ small) it would mean that the Euclidean domain would extend to large sizes. It would then be inconsistent with the present structure of space-time which appears Lorentzian down to at least sizes of $\sim 10^{-20}$ meters. To restate this point: quantum cosmology should not compute conditional probabilities for classical epochs of the universe long after the quantum era is over. So in contrast to ref.[11], the question “given that the scale factor is $\sim 10^{60}$ what is the probability distribution for $\Lambda$ ” is not the correct question to ask of the WDW equation. Rather we should be asking: what is the distribution function for $\Lambda$ when the classical epoch of the universe first started?
We now consider a problem that has arisen for the case of a $-ve$ cosmological constant \cite{10}. There seem a number of unnecessary complications in this analysis, such as 3rd quantization and addition of matter fields, that can be neglected, but with the problem (of $\Lambda \rightarrow -\infty$) still remaining. We therefore repeat their arguments in a more simplified and transparent manner.

The equivalent expression to eq. (5) for negative $\Lambda$ is

$$S = -\frac{(1 + |\Lambda|a^2)^{3/2}}{3|\Lambda|}$$  \hspace{1cm} (7)

It is no longer clear what integration limits have to be placed on $a$. Choosing them from $a = 0$ to $a$ gives the solutions

$$\Psi \sim \exp \pm \left[ \frac{(1 + |\Lambda|a^2)^{3/2}}{3|\Lambda|} - \frac{1}{3|\Lambda|} \right]$$  \hspace{1cm} (8)

If we keep track of the signs then the $(+)$ one corresponds to the HH case and will be dominated by large

$$\sim \left[ \frac{(1 + |\Lambda|a^2)^{3/2}}{3|\Lambda|} - \frac{1}{3|\Lambda|} \right]$$  \hspace{1cm} (9)

i.e. by $\sqrt{|\Lambda|}a^3$ large\footnote{1}. This is the problem that the HH boundary condition predicts $\Lambda \rightarrow -\infty$ found by Rubakov et. al.\cite{10}\footnote{2}. It is uncertain that this makes any sense and is rather an artifact of the HH wavefunction being peaked around the exponentially increasing solution. cf. Fig.(2) in Ref.\cite{9}.

There is another reason to discount this solution. If the cosmological constant was absent the wave function would be

$$\Psi \sim \exp \left( \pm a^2 / 2 \right)$$  \hspace{1cm} (10)

If we choose the $(+)$ sign there is a contradiction with our notions of classical behaviour since the universe would apparently prefer to have large size. Rather the other sign is more correctly peaked around $a = 0$. This point has recently been made by Vilenkin \cite{13} in criticism of the “generic”

\footnote{1} If we had not subtracted the part corresponding to $a = 0$ we would also find a divergence when $|\Lambda| \rightarrow 0$

\footnote{2} It was not necessary, as done in ref.\cite{10} to include matter fields or to consider a third quantized theory to obtain this dominant $\sqrt{|\Lambda|}a^3$ factor.
boundary conditions, but which is also valid against the HH ones when \( \Lambda \) is negative.

The other \((-\) sign solution in eq.(8) formally appears to predict \( \Lambda \to 0 \) if \( a \neq 0 \). But since there is no barrier to tunnel through the tunneling condition will simply imply that the universe stays at the origin \( a = 0 \) and \( \Lambda \) is left undefined.

One can seemingly obtain large or small \( |\Lambda| \) depending on how one applies the boundary condition (the limits of integration in eq.(4) ). When considering a -ve \( \Lambda \) it seems that we should conclude that the universe will wish to stay at the origin and no predictions about \( \Lambda \) should be drawn from the factor \( \exp \sqrt{|\Lambda| a^3} \).

We generally return to the case of positive values of \( \Lambda \) again from now; but we keep in mind that the HH boundary condition which gives the “wanted” factor of \( \exp(1/\Lambda) \) for positive \( \Lambda \) also apparently gives the prediction of \( \Lambda \to -\infty \) when \( \Lambda \) is negative.

In this Euclidean region we have found that the possible values of \( \Lambda \) depend upon the choice of boundary conditions. This point has also been mentioned by Kiefer [14] in the context of wave packet solutions to the WDW equation. Because the choice of boundary conditions is not known a priori, it seems that to simply choose the boundary condition that solves the cosmological constant problem is merely to pass the problem down the line. What is required is a measure of solutions to the WDW equation which give either large or small \( \Lambda \). Fortunately any possible wave function

\[
\Psi \sim \alpha \exp(1/\Lambda) + \beta \exp(-1/\Lambda)
\]

(\( \alpha, \beta \) arbitrary complex coefficients) will have the critical behaviour at \( \Lambda = 0 \), even for \( \alpha \) small. It also would make \( D \to 0 \) (defined in ref.[15]) and according to the measure given in ref.[15] a typical wavefunction has \( D < 1 \). This lends credence to the claim of Coleman that the mechanism is somewhat immune from the choice of boundary conditions. Although this wavefunction (11) will work alright for +ve \( \Lambda \) its modified form cf. eq(8) for negative \( \Lambda \) will be peaked at \( \Lambda \to -\infty \). The tunneling boundary condition predicts \( \Lambda \) large for positive \( \Lambda \), and also gives the preferable prediction that \( \Lambda \to 0 \) when it is negative.

We consider next what happens when the universe starts in a Lorentzian region where the WKB wave functions have the oscillating behaviour \( \Psi \sim \)
exp(±iS). The exponents in the terms exp(±iS) no longer have any critical influence, but instead the pre-factor contains any dominant behaviour. We do however have to exclude the Euclidean regime from the expression for the action cf. eq.(5) i.e. the lower limit in the integral is taken to be \( a = \Lambda^{-1/2} \). Otherwise we would simply introduce the factors exp(±1/Λ) again and reach similar conclusions.

Typically the wavefunction has the form

\[
\Psi \sim \frac{1}{a \sqrt{a^2 \Lambda - 1}} \left( e^{iS} + e^{-iS} \right).
\]

(12)

There is a similar peak around \( a^2 \Lambda \sim 1 \) as the WDW potential is zero. For \( a \) fixed and \( a^2 \Lambda >> 1 \) then \( \Psi^2 \sim 1/\Lambda \) so that larger values of \( \Lambda \) are suppressed inversely. These are again initial conditions to be followed by classical evolution, and it would appear correct to assume the quantum behavior made predictions for an initially small universe. The initial value of \( \Lambda \) would therefore appear large which would produce an inflationary phase.

This prediction occurs for both HH and tunneling boundary conditions since they only determine which combination of \( \exp(iS) \) and \( \exp(-iS) \) to take. There is a heuristic reason to see this: in the Lorentzian region the HH and tunneling solutions look almost alike (damped oscillations) and so should not differ much in their predictions. Contrast this with their behaviour in the Euclidean regime - see eg. Fig.(11.2) in ref. [7].

Similar predictions could be made if we considered a spatially open \( k = -1 \) model together with a -ve \( \Lambda \). This has Lorentzian behaviour for small \( a \) beyond which is a Euclidean regime. The peak would be around \( a^2 |\Lambda| \sim -k \). Notice how the spatial curvature is crucial for any predictions about \( \Lambda \). If we set \( k = 0 \) then \( P(\Lambda) \sim 1/(a^4 \Lambda) \) and we would obtain the prediction that \( \Lambda \rightarrow 0 \), although without the exponential peak.

It might appear that this property \( a^2 \Lambda \sim 1 \) is an artifact of using WKB solutions which are simply blowing up at the turning point \( U \equiv a^2 - \Lambda a^4 = -k \) for the same reason these factors appear when you wish to normalize the wavefunction as \( a \rightarrow 0 \) -see e.g.[16].

4 We should perhaps be careful and say the prediction is not strongly dependent on boundary conditions since there might be more unusual ways of imposing them cf. Ref.[17]. Note the slight discrepancy with Cline [17] who using a Lorentzian path integral approach concluded that \( P(\Lambda) \sim 1 \) so that any \( \Lambda \) is equally likely. In the Euclidean region he found that only special boundary conditions gave the \( \exp(1/\Lambda) \) factor.
0. However, exact solutions of the WDW equation can be found and this behaviour remains. For example, the equation [18]

\[
\left( \frac{d^2}{da^2} + \frac{p}{a} \frac{d}{da} - U \right) \Psi(a) = 0 \tag{13}
\]

with \( p \) a factor ordering correction, has solutions

\[
\Psi \sim a^{1/2} \left\{ J_{(1-p)/4}(\sqrt{-U}a) + Y_{(1-p)/4}(\sqrt{-U}a) \right\} \tag{14}
\]

Since both Bessel functions \( J_\nu(x) \) and \( Y_\nu(x) \) both decrease for increasing \( x \), they both take their maximum value when \( x = 0 \) and as we require \( a > 0 \) this occurs for \( U = 0 \), so again when \( \Lambda \sim 1/a^2 \). For \( p \neq 1 \) this is slightly modified \( \sqrt{-U}a \sim \text{small} \).

The addition of additional matter fields is likely to round off this spike at \( \Lambda \sim 1/a^2 \) cf. ref.[19]. We see that the initial value of \( \Lambda \) is expected to be large in the Lorentzian regime provided \( a \) is not large in Planck units. If the initial size of the universe is 'big' \( \sim 1 \text{ cm} \) the probable value of \( \Lambda \) is smaller but still huge compared to its present value cf. Ref.[11].

Let us finally try to understand the Euclidean regime results (when \( U \geq 0 \)) in terms of the solutions eq.(13). For factor ordering \( p = 1 \) the solutions simplify to

\[
\Psi \sim K_0(aU^{1/2}) + I_0(aU^{1/2}) \tag{15}
\]

Since \( K_0(x) \rightarrow \infty \) as \( x \rightarrow 0 \) it picks out the \( U = 0 \) or \( \Lambda \sim 1/a^2 \) case. The other Bessel function \( I \) increases with increasing \( aU^{1/2} \equiv a^2(1 - \Lambda a^2)^{1/2} \) so is maximized for \( \Lambda = 0 \), or negative \( \Lambda \) if we allow it. This is as expected since the tunneling boundary condition is the decaying solution \( K \) and the HH one a mixture of both \( I \) and \( K \) - see eg.ref.[20].

Recently a WDW equation corresponding to a classical signature change has been obtained [21]. Such an approach always has an oscillating wavefunction and so does not have the possibility of having Baum-Hawking factors \( \exp(1/\Lambda) \). It seems only consistent with tunneling boundary conditions [21] and for this reason we suspect it would also predict \( \Lambda \) large when \( \Lambda \) is a dynamical variable.

**Conclusions**

In the Euclidean regime the initial value of \( \Lambda \) is expected large if tunneling boundary conditions or tunneling like behaviour is correct. This would not be
suitable for setting $\Lambda$ small but would allow for an Inflation regime to proceed. If we consider HH boundary conditions then you can get the factor $\exp(1/\Lambda)$ (or if $\Lambda$ is -ve the anomalous $\exp(\sqrt{|\Lambda|a^3}$ factor). The two behaviours are complimentary: we could not have Inflation together with $\Lambda \rightarrow 0$ unless some other dynamical mechanism could give the two mechanisms differing time scales cf. Ref.[22].

In fact, if the tunneling boundary condition is correct and $\Lambda$ correspondingly large, we would not want the wormhole mechanism to take place. This would have the effect of transferring the large value of $\Lambda$ in the average “quantum foam of universes” to any universe, so giving a large cosmological constant in our universe.

We then considered the purely Lorentzian regime and found that the predictions in this case are not dependent on the boundary conditions. We found an initial value of $\Lambda \sim k/a^2$ and since we expect the universe to start small, due to quantum gravity processes, the corresponding value of $\Lambda$ is large. We are still left with the problem of why $\Lambda$ should be a dynamical variable with a distribution function. The 3-form (axion) field might still work in this regard even if its wormhole solution cannot be invoked cf.ref.[2].

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Figures
Fig. 1
The Wheeler-DeWitt potential $U$. The Euclidean regime has $U \geq 0$ beyond which it is Lorentzian. The tunneling boundary condition describes the decay from the origin to $a = \Lambda^{-1/2}$.