Continuous and discontinuous transitions in geophysical turbulence

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Abstract. In geophysical turbulent flows, it is customary to have two or more attractors (typically responsible for low-frequency variability). Such a phenomenon is investigated in the dynamics of two-dimensional and quasi-geostrophic turbulence. In the inertial limit, with a time scale separation (between spin-up time and inertial time), the attractors are concentrated near a set of steady states of the inviscid equations. Statistical mechanical approaches help us determine which of these steady states are eligible. Then, we can establish phase diagrams, where phase transition lines delimitate regions corresponding to different regimes, i.e., different flow structures. The latter can be related to the different (say, two) regimes, which the Kuroshio alternatively finds itself into (bistability). Our present aim is to classify the bifurcations —given the two inertial equilibria— when tuning an external parameter. We give a general theory predicting whether the transitions are continuous (second-order) or discontinuous (first-order). The continuity or discontinuity depends on the value of some moments of the potential vorticity. We discuss applications to oceanic (Kuroshio) and experimental geophysical flows.

1. Introduction

The aim of the present work is to present analytical and numerical computations of phase diagrams for a large class of equilibrium states of two-dimensional and geophysical turbulent flows. This is carried out in the framework of the Robert–Sommeria–Miller (RSM) theory. The RSM theory is an equilibrium statistical mechanics theory, which predicts the large scales of two-dimensional turbulent flows (Miller, 1990; Robert & Sommeria, 1991). It is thus relevant to quasi-2D geophysical flows, such as oceans and atmospheres. The detailed computations for the present results can be found in a companion paper (Corvellec & Bouchet, 2011).

Recently, it was shown that ocean currents, such as the Kuroshio (in the Northern Pacific, off Japan) or the Gulf Stream, may be understood as equilibria of the inertial dynamics, in very simple ocean models (Venaille & Bouchet, 2010). Such a conservative theory does not take into account the long-term effects of forcing and dissipation, which occur in any real flow. But in a recent work, it was shown that even with forcing and dissipation acting, the inertial description in terms of equilibria is relevant (Bouchet & Simonnet, 2009).

In the inertial limit, with a time scale separation (between spin-up time and inertial time), the attractors are found to concentrate near a set of steady states of the inviscid equations. Bouchet & Simonnet (2009) report especially the existence of out-of-equilibrium transitions between close-to-equilibrium states, in a numerical simulation of the 2D stochastic Navier–
Stokes equations. This is an instance of bistability. This phenomenology is expected to be observed in quasi-geostrophic models or primitive equations, owing to their analogy to the 2D Euler equations.

2. Bistability and phase transitions

Bistability is indeed observed in geophysical flows. The Kuroshio is seen to oscillate between an intense jet-like state and a weak meandering state, on decadal time scales. In fast-rotating tank experiments, transitions between a ‘zonal’ state and a ‘blocked’ state are found (Tian et al., 2001). They are remindful of the atmospheric phenomenon known as ‘blocking’. In geophysical turbulent flows, it is customary to have two or more attractors (typically responsible for low-frequency variability).

Finding and characterizing the transitions in a system is crucial, since the possibly interesting qualitative changes lie there. In the equilibrium statistical mechanical context, we deal with phase transitions. Flows will have their structure change as they undergo phase transitions. It is thus important to know whether they are first-order (discontinuous) or second-order (continuous). One would spontaneously associate bistability with discontinuous transitions.

In a first-order phase transition, a stable equilibrium and a metastable one exchange their stabilities. In a second-order phase transition, a stable equilibrium becomes unstable, while two stable equilibria show up. This picture corresponds to the minimization of a functional (see section 3). We are interested in phase diagrams which display qualitatively (two) different states, because we wish to relate these to the two regimes given by the bistability of real and experimental flows. Thus, one may see bistability as a transition process between one minima and the other (back and forth), when the system is in the vicinity of a first-order phase transition line. The picture is then that of a double-well potential, with potential barrier of a certain height.

3. The barotropic quasi-geostrophic model

The system we consider is that of the barotropic quasi-geostrophic equations, which model the 2D dynamics of one oceanic or atmospheric layer. It is a nonlinear transport equation, where the nonlinearity is due to nonlocal and nonintegrable interactions. We refer to the latter as long-range interactions (Bouchet & Barré, 2005). Because of them, we expect statistical ensemble inequivalence and negative specific heat when computing the statistical equilibria (see next section). The advected quantity is the potential vorticity. Bidimensionality allows for a streamfunction formulation of the problem. The barotropic quasi-geostrophic equations read

$$\partial_t q + u \cdot \nabla q = 0; \quad u = e_z \times \nabla \psi; \quad q = \Delta \psi + h$$

where $u$ denotes the (two-dimensional) velocity field, $\psi$ the streamfunction (defined up to a constant), $q$ the potential vorticity (in vorticity units), and $h$ an equivalent topography. $e_z$ is just the upward vertical unit vector. We consider a domain $D = \{(x, y) \in [0, \tau^{1/2}] \times [0, \tau^{-1/2}]\}$ of area unity ($|D| = 1$), with $\tau \geq 1$. The boundary condition is $\psi = 0$ on $\partial D$. The natural scalar product for the fields at play is denoted by $\langle q_1, q_2 \rangle := \int_D q_1 q_2$.

One of the main physical features of this system is its self-organization into large-scale coherent structures (monopoles, dipoles, parallel flows). Such large-scale structures are analogous to geophysical cyclones, anticyclones, and jets in the oceans and atmospheres (Bouchet & Sommeria, 2002). The barotropic quasi-geostrophic model is also relevant to the description of experimental flows, such as the infinite–Reynolds-number (turbulent) approximation of fluid dynamics when three-dimensional motion is constrained by a strong transverse field (e.g., rotation) or takes place in geometries of small (vertical-to-horizontal) aspect ratio.
The quasi-geostrophic model is thus an appropriate model for geophysical and experimental flows on time scales much less than the dissipation time scale, but large enough for turbulent mixing to have operated as much as allowed by the constraint of energy conservation. Turbulence is known to develop complex vorticity filaments at finer and finer scales. Equilibrium statistical theories of 2D turbulent flows, predict —assuming ergodicity— the final organization of the flow at a coarse-grained level: (large scales).

The dynamical system (1) conserves the so-called Casimirs, i.e., all functionals of the potential vorticity, or equivalently, all moments of the potential vorticity (Miller, 1990; Robert & Sommeria, 1991). Besides energy, there is thus an infinity of flow invariants.

4. Computation of the statistical equilibria

According to the RSM theory, a mixing entropy (for the potential vorticity values) is to be maximized, while taking into account all the flow invariants. Then, statistical equilibria are found to verify a certain functional relationship between potential vorticity and streamfunction: 

\[ q = f(\psi) \]

From (1), it can be seen that statistical equilibria are particular steady states of the barotropic quasi-geostrophic model. Analytical computation of RSM equilibrium states would be a difficult task though: it would be about solving a variational problem involving an infinite number of constraints.

Simpler variational problems (taking into account only a few constraints) were shown to give access to some classes of RSM equilibria (Bouchet, 2008). For instance, such a class is the one for which \( q = f(\psi) \) is linear (or affine). An example of using statistical mechanics for predicting and describing real turbulent flows can be found in (Bouchet & Simonnet, 2009) and references therein. There, \( h = 0 \) and (1) reduces to the 2D Euler equations. Bifurcations between stable steady Euler solutions (seen as particular RSM equilibria) are found to occur when varying the domain shape, the nonlinearity of \( f(\psi) \), or the energy.

This suggests that a general theory of phase transitions for statistical mechanics of 2D and geophysical flows should be looked for —it is not available at the present day. Only instances of such phase transitions have been reported in the literature. Working towards this goal, we present new results on phase transitions related to the nonlinearity of the function \( f(\psi) \). Note that key results regarding statistical ensemble inequivalence, encompassing the case of a nonlinear equation \( q = f(\psi) \), were already presented by Ellis et al. (2002).

The simpler variational problem we consider is the following:

\[ C_s(E, \Gamma) = \min_q \left\{ \int_D s(q) \mid \mathcal{E}[q] = E, \Gamma[q] = \Gamma \right\}. \]

Let us refer to it as the microcanonical (energy-circulation) variational problem. Note that this variational problem corresponds to (CVP) in (Bouchet, 2008). In (2), \( s(q) \) is a strictly convex function, and the functionals \( \mathcal{E}[q] = -\frac{1}{2} \int_D \psi(q-h) \) and \( \Gamma[q] = \int_D q \) are the energy and the circulation, respectively. Note that we have taken \( \int_D h = 0 \). For simplicity, we restrict our attention to even functions \( s(q) \). For given values of the constraints \( E \) and \( \Gamma \), the \( q \) fields solving (2) are statistical equilibria. For regular enough \( q \)’s, there exists \((\beta, \gamma) \in \mathbb{R}^2\) such that these statistical equilibria are critical points of \( \mathcal{G}[q] = \int_D s(q) + \beta \mathcal{E}[q] + \gamma \Gamma[q] \). This functional is referred to as the Gibbs free energy, in analogy with usual thermodynamics. \( \beta \) and \( \gamma \) are the Lagrange multipliers associated with the energy and circulation constraints, respectively. The critical points of \( \mathcal{G} \) are the \( q \) fields for which the first-order variation of \( \mathcal{G} \) vanish, i.e.,

\[ s'(q) - \beta \psi + \gamma = 0. \]

Since \( s(q) \) is even, \( s'(q) \) is odd, so its inverse \((s')^{-1}(q)\) exists and is odd. Then, \( q = (s')^{-1}(\beta \psi - \gamma) \). This is the form of our relationship \( q = f(\psi) \). At lowest order, the nonlinearity of this
relationship will be characterized by the scalar parameter $a_4$: the Taylor expansion of $(s')^{-1}$ around 0 reads $(s')^{-1}(x) = x + a_4 x^3 + o(x^4)$. Note that $a_4$ is the weight of $-\int_D \omega^4$ in the variational problem.

The variational problem dual to (2), i.e.,

$$G(\beta, \gamma) = \min_q \mathcal{G}[q] = \min_q \left\{ \mathcal{G}[q] = \int_D s(q) + \beta \mathcal{E}[q] + \gamma \Gamma[q] \right\},$$

is referred to as the grand canonical variational problem. Because it is relaxed (unconstrained), it is more easily tractable. For all couples $(\beta, \gamma)$, minima $G(\beta, \gamma)$ are also minima $S(E(\beta, \gamma), \Gamma(\beta, \gamma))$. When varying $(\beta, \gamma)$ though, $E(\beta, \gamma)$ and $\Gamma(\beta, \gamma)$ might not span their entire accessible range ($E \in \mathbb{R}_+, \Gamma \in \mathbb{R}$). When they do not, the microcanonical ensemble and the (dual) grand canonical ensemble are said to be inequivalent.

It is natural to begin with the study of the convexity of $\mathcal{G}[q]$. Since $\Gamma[q]$ is a linear form, it is equivalent and enough to investigate the convexity of the Helmholtz free energy functional $\mathcal{F}[q] = \int_D s(q) + \beta \mathcal{E}[q]$. In statistical equilibrium theories, the classical stability criteria amount to sufficient conditions for the nondegeneracy of the minimizer Ellis et al. (2002). Here, the sufficient condition is that the second-order variation of $\mathcal{F}$ be positive-definite (Michel & Robert, 1994). Phase transitions may occur only where $\mathcal{F}[q]$ ceases to be convex, i.e., where solutions to (4) may cease to be unique or cease to exist.

Phase transitions can be found and characterized (in their continuous or discontinuous nature) very conveniently, by studying bifurcation problems for the critical points (3). The bifurcation parameter is $\beta \in \mathbb{R}$. The system is symmetric ($q \mapsto -q$) for $\gamma = 0$ in the grand canonical ensemble, and for $\Gamma = 0$ in the canonical ensemble. Then, $q = 0$ is a critical point. The point is to determine the possible bifurcations from this trivial solution, when varying $\beta$. For a bifurcation to occur, the Jacobian of the bifurcation problem has to vanish. This happens for a certain value of $\beta$, along a (nontrivial) direction, the neutral direction.

According to classical bifurcation theorems (Chow & Hale, 1982), the (infinite-dimensional) bifurcation problem reduces to an equivalent scalar problem, given by an appropriate projection of the bifurcation problem onto the neutral direction. That is the celebrated Lyapunov–Schmidt decomposition/reduction. A supercritical bifurcation corresponds to a continuous transition, while a subcritical bifurcation corresponds to a discontinuous transition. We find that the criterion for being in one case or the other is simply the sign of $a_4$.

This description is generic but valid only at low energy (in the vicinity of the bifurcation point). At high energy, other branches must come into play. Also, the situation at high values of $\gamma$ or $\Gamma$ cannot be derived from these results at $\gamma = 0$ or $\Gamma = 0$, respectively.

5. Results

Solving the grand canonical variational problem in a symmetric case, we find a tricritical point (figure 1). The latter is a point in the phase diagram, where a second-order phase transition turns into a first-order one, as the sign of $a_4$ changes. In the corresponding constrained ensemble, we find a ‘canonical’ spinodal point, when $a_4 > 0$. We refer the reader to (Bouchet & Barré, 2005) for a systematic classification of all these singularities.

Solving the canonical variational problem —energy constraint relaxed, circulation fixed at a low value, we find interesting phase transitions, where the flow structure completely changes (Figure 2). For elongated rectangular domains (aspect ratio $\tau > \tau_c$), we recover the showing up of a dipolar structure, while for square-like domains ($\tau < \tau_c$), we recover that of a central monopole with counter-circulating cells at the corners (Chavanis & Sommeria, 1996; Venaille & Bouchet, 2009). The novelty here is to distinguish between a first-order transition and a second-order one, depending on the sign of $a_4$, at zero circulation. At small but nonzero circulation,
Figure 1. The ‘canonical’ tricritical point is at \((a, b) = (0, 0)\). The curve \((4a = b^2, b < 0)\) corresponds to the appearance of three local maxima. The bold curve \((16a = 3b^2, b < 0)\) is a first-order phase transition line. The bold-dashed curve is a second-order phase transition line. Here, ‘canonical’ simply refers to the relaxed ensemble in general, with respect to a constrained one. In (Bouchet & Barré, 2005).

Figure 2. Phase diagram for \(\Gamma = 0.01\) and \(\tau = 2\). The blue curve plots \(a_4(\beta)\) so as to fit the three first-order phase transition points computed from numerical continuation. Insets show vorticity fields right near the discontinuous phase transition, on either side; color scale ranges from \(-0.5\) to \(0.5\) (from blue to red); the black contours are six iso-vorticity lines on each inset plot. For \(a_4 > 0\), as \(\beta\) decreases and crosses a certain value \(-\lambda_1'\), a large-scale dipole appears in the flow, and the energy increases in a jump-like way.
we lose the second-order phase transition to symmetry-breaking, but then we have metastable states (of which stability tends to that of the equilibrium, as the circulation tends to zero). In the square-like case, we can be in the presence of three qualitatively different states (stable or metastable).

Canonical first-order phase transitions are associated with ensemble inequivalence between the canonical and microcanonical ensembles. We have a statistical ensemble inequivalence range (a certain energy range), wherein some states have negative specific heat (Bouchet & Barré, 2005). We find no microcanonical phase transitions. Recall that the computations are carried out in the low-energy limit. We wish to emphasize that the canonical ensemble may be relevant to geophysical applications, since the two regimes of known bistable systems have different energies. The area of phase diagram near the discontinuous transition should be that of interest, when investigating stochastically induced transitions.

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