SEQUENCE SPACE REPRESENTATIONS FOR SPACES OF ENTIRE FUNCTIONS WITH RAPID DECAY ON STRIPS

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Abstract. We obtain sequence space representations for a class of Fréchet spaces of entire functions with rapid decay on horizontal strips. In particular, we show that the projective Gelfand-Shilov spaces $\Sigma^1_\nu$ and $\Sigma^\nu_1$ are isomorphic to $\Lambda^\nu_\infty(n^{1/(\nu+1)})$ for $\nu > 0$.

1. Introduction

The representation of (generalized) function spaces by sequence spaces is a classical and well-studied topic in functional analysis [3, 6, 14–20, 22, 25–29, 32]. Apart from their inherent significance, such representations are important in connection with the problem of isomorphic classification.

Langenbruch [18, 19] gave sequence space representations for a class of weighted ($LB$)-spaces of analytic germs defined on strips near $\mathbb{R}$. More precisely, let $\omega : [0, \infty) \to [0, \infty)$ be an increasing continuous function with $\omega(0) = 0$ such that $\log t = o(\omega(t))$. Moreover, assume that $\omega$ satisfies the mild assumption

\[(1.1) \quad \omega(t+1) = O(\omega(t)).\]

For $h > 0$ we write $V_h = \mathbb{R} + i(-h, h)$. Consider the ($LB$)-space

\[\mathcal{H}_\omega(\mathbb{R}) := \bigcup_{n \in \mathbb{Z}_+} \{ \varphi \in \mathcal{O}(V_{1/n}) \mid \sup_{z \in V_{1/n}} \left| \varphi(z) \right| e^{\omega(|\Re z|)/n} < \infty \} .\]

Such spaces generalize the test function space of the Fourier hyperfunctions [12]. In [19, Theorem 4.6], it is shown that $\mathcal{H}_\omega(\mathbb{R})$ is isomorphic to $\Lambda^\nu_0((\omega^*(n))_{n \in \mathbb{N}})$, where $\omega^*(t) = (s\omega^{-1}(s))^{-1}(t)$ for $t \geq 0$. This result implies that the Gelfand-Shilov spaces $\Sigma^\nu_1$ and $S^\nu_1$ [10] are isomorphic to $\Lambda^\nu_0(n^{1/(\nu+1)})$ for $\nu > 0$ [19, Example 5.4].

In the present article, we obtain sequence space representations for the Fréchet spaces

\[\mathcal{U}_\omega(\mathbb{C}) := \{ \varphi \in \mathcal{O}(\mathbb{C}) \mid \sup_{z \in V_n} |\varphi(z)| e^{\omega(|\Re z|)} < \infty, \quad \forall n \in \mathbb{N} \} ,\]

which may be considered as the projective counterparts of the spaces $\mathcal{H}_\omega(\mathbb{R})$. Such spaces were considered in [7] and generalize the test function space of the Fourier ultrahyperfunctions [23]. Under the additional standard assumption $\omega(2t) = O(\omega(t))$, we show that $\mathcal{U}_\omega(\mathbb{C})$ is isomorphic to $\Lambda_\infty(\omega^*(n))$ (Theorem 5.1). This result yields a precise isomorphic classification of the spaces $\mathcal{U}_\omega(\mathbb{C})$. Moreover, it implies that the

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projective Gelfand-Shilov spaces $\Sigma^1_{\nu}$ and $\Sigma^1_{\nu}$, considered e.g. in \cite{6,7,24}, are isomorphic to $\Lambda^{\infty}_{\frac{1}{\nu+1}}(n)$ for $\nu > 0$ (Theorem 6.4). We believe that our main result holds true under the more general assumption (1.1), provided that $U_{\omega}(C)$ is non-trivial (cf. \cite{7, Theorem 7.2}), but were unable to show this; see Problem 6.2.

The basic functional analytic tool in our considerations is the following result from the structure theory of power series of infinite type \cite{2, 28, 31}.

**Theorem 1.1.** \cite[Corollary 1.5 and Lemma 2.3]{2} Let $\Lambda^{\infty}_{\beta}$ be a nuclear stable power series space of infinite type. Let $E$ be a Fréchet space satisfying $(DN)$ and $(\Omega)$ with diametral dimension equal to $\Lambda^{\prime}_{\infty}(\beta)$. Then, $E$ is isomorphic to $\Lambda^{\infty}_{\beta}$.

We refer to Sections 4 and 5 for the definition of the notions occurring in Theorem 1.1. We show that $U_{\omega}(C)$ is isomorphic to $\Lambda^{\infty}_{\omega(n)}$ by applying Theorem 1.1 to $E = U_{\omega}(C)$ and $\beta = (\omega(n))$. We prove that $U_{\omega}(C)$ satisfies the linear topological invariants $(DN)$ and $(\Omega)$ in Section 4. The diametral dimension of $U_{\omega}(C)$ is determined in Section 5; our proofs are inspired by and based on results from \cite{19}. Some of our arguments in Sections 4 and 5 rely on the mapping properties of the short-time Fourier transform on $U_{\omega}(C)$ and an auxiliary space of Fourier hyperfunctions of fast growth. These are established in Section 3. Finally, in Section 6 we prove our main result and provide various examples.

We are much indebted to Langenbruch for his work \cite{19}. In particular, the methods and results from \cite[Section 4]{19} (see also \cite{16}) were truly inspiring to us.

2. Spaces of entire functions with rapid decay on strips

In this preliminary section, we introduce the class of weighted Fréchet spaces of entire functions we shall be concerned with in this article. We refer to [7] for more information on these spaces.

By a weight function we mean an increasing continuous function $\omega : [0, \infty) \rightarrow [0, \infty)$ with $\omega(0) = 0$ such that $\log t = o(\omega(t))$. In particular, $\omega$ is bijective on $[0, \infty)$. We extend $\omega$ to $\mathbb{R}$ as the even function $\omega(x) = \omega(|x|), x \in \mathbb{R}$. We consider the following standard condition on $\omega$ \cite{5}: (a) $\omega(2t) = O(\omega(t))$. Condition (a) implies that there are $K, C_0 \geq 1$ such that

$$\omega(x + y) \leq K(\omega(x) + \omega(y)) + \log C_0, \quad x, y \in \mathbb{R}.$$ 

We write $z = x + iy \in \mathbb{C}$ for a complex variable. Given an open set $V \subseteq \mathbb{C}$, we denote by $\mathcal{O}(V)$ the space of analytic functions on $V$. Let $\omega$ be a weight function. For $h > 0$ we write $V_h = \mathbb{R} + i(-h, h)$. Let $\lambda \in \mathbb{R}$. We denote by $\mathcal{A}_{\omega, \lambda}(V_h)$ the Banach space consisting of all $\varphi \in \mathcal{O}(V_h)$ such that

$$\|\varphi\|_{\mathcal{A}_{\omega, \lambda}(V_h)} := \sup_{z \in V_h} |\varphi(z)| e^{\lambda \omega(x)} < \infty.$$ 

We define the Fréchet space \cite{7}

$$U_{\omega}(C) := \lim_{h \rightarrow \infty} \mathcal{A}_{\omega,h}(V_h).$$
In [7, Theorem 7.2], it is shown that $U_{\omega}(\mathbb{C})$ is non-trivial if and only if
\begin{equation}
\int_0^\infty \omega(t)e^{-\mu t}dt < \infty
\end{equation}
for all $\mu > 0$. Since $(\alpha)$ implies polynomial growth, we particularly have that $U_{\omega}(\mathbb{C}) \neq \{0\}$ if $\omega$ satisfies $(\alpha)$. This also follows from a classical result of Gelfand and Shilov on the non-triviality of the spaces $S^p_\mu$ [10, Chapter 4, Section 8]. Furthermore, the assumption $\log t = o(\omega(t))$ implies that $U_{\omega}(\mathbb{C})$ is nuclear [8, Theorem 5.1]. We also need the following $(LB)$-space
\begin{equation}
A_\omega(\mathbb{R}) := \lim_{h \to \infty} A_{\omega,-h}(V_1/h).
\end{equation}
If $\omega$ satisfies $(\alpha)$, the space $A_\omega(\mathbb{R})$ coincides with the space $H_\infty(\mathbb{R})$ from [19].

3. The short-time Fourier transform

In this auxiliary section, we characterize the spaces $U_{\omega}(\mathbb{C})$ and $(A_\omega(\mathbb{R}))'_b$ by means of the short-time Fourier transform. These characterizations will be one of the fundamental tools in the rest of this article.

The short-time Fourier transform (STFT) of $f \in L^2(\mathbb{R})$ with respect to the window $\psi \in L^2(\mathbb{R})$ is given by
\begin{equation}
V_\psi f(x, \xi) := \int_{\mathbb{R}} f(t)\overline{\psi(t-x)}e^{-2\pi i \xi t}dt, \quad (x, \xi) \in \mathbb{R}^2.
\end{equation}
The adjoint of $V_\psi : L^2(\mathbb{R}^2) \to L^2(\mathbb{R})$ is given by the weak integral
\begin{equation}
V_\psi^* F(t) = \int_{\mathbb{R}^2} F(x, \xi)e^{2\pi i \xi t}\psi(t-x)dxd\xi, \quad F \in L^2(\mathbb{R}^2).
\end{equation}
Furthermore, if $\gamma \in L^2(\mathbb{R})$ is such $(\gamma, \psi)_{L^2} \neq 0$, then the reconstruction formula
\begin{equation}
\frac{1}{(\gamma, \psi)_{L^2}}V_\gamma^* \circ V_\psi = \text{id}_{L^2(\mathbb{R})}
\end{equation}
holds. We refer to [11] for more information on the STFT.

Next, we introduce some auxiliary function spaces. Let $\omega$ be a weight function. For $h, \lambda \in \mathbb{R}$ we denote by $C_{\omega,\lambda,h}(\mathbb{R}^2)$ the Banach space consisting of all $f \in C(\mathbb{R}^2)$ such that
\begin{equation}
\|f\|_{C_{\omega,\lambda,h}(\mathbb{R}^2)} := \sup_{(x,\xi)\in\mathbb{R}^2} |f(x,\xi)|e^{\lambda \omega(x) + h|\xi|} < \infty.
\end{equation}
We define the Fréchet spaces
\begin{equation}
C_\omega(\mathbb{R}^2) := \lim_{h \to \infty} C_{\omega,h,h}(\mathbb{R}^2), \quad \tilde{C}_\omega(\mathbb{R}^2) := \lim_{h \to \infty} C_{\omega,h,-1/h}(\mathbb{R}^2),
\end{equation}
and the $(LB)$-space
\begin{equation}
C_\omega^0(\mathbb{R}^2) := \lim_{h \to \infty} C_{\omega,-h,-h}(\mathbb{R}^2).
\end{equation}
We are ready to establish the mapping properties of the STFT on the space $U_{\omega}(\mathbb{C})$. 
Proposition 3.1. Let $\omega$ be a weight function satisfying $(\alpha)$. Fix a window $\psi \in \mathcal{U}_\omega(\mathbb{C})$. The linear mappings

$$V_\psi : \mathcal{U}_\omega(\mathbb{C}) \to C_\omega(\mathbb{R}^2) \quad \text{and} \quad V_\psi^* : C_\omega(\mathbb{R}^2) \to \mathcal{U}_\omega(\mathbb{C})$$

are continuous. Furthermore, if $\gamma \in \mathcal{U}_\omega(\mathbb{C})$ is such that $(\gamma, \psi)_{L^2} \neq 0$, then the reconstruction formula

$$(3.2) \quad \frac{1}{(\gamma, \psi)_{L^2}} V_\psi^* \circ V_\psi = \text{id}_{\mathcal{U}_\omega(\mathbb{C})}$$

holds.

Proof. We first consider $V_\psi$. Fix $h > 0$ and let $\varphi \in \mathcal{U}_\omega(\mathbb{C})$ be arbitrary. By Cauchy’s integral formula we have that

$$V_\psi \varphi(x, \xi) = \int_\mathbb{R} \varphi(t - i \text{sgn}(\xi) h) \psi(t - x - i \text{sgn}(\xi) h) e^{-2\pi i t(t - i \text{sgn}(\xi) h)} dt.$$

Hence,

$$|V_\psi \varphi(x, \xi)| \leq e^{-2\pi h|\xi|} \int_\mathbb{R} |\varphi(t - i \text{sgn}(\xi) h)||\psi(t - x - i \text{sgn}(\xi) h)| dt \leq C_0 \int_\mathbb{R} e^{-K h_\omega(t)} dt \|\varphi\|_{A_\omega,K_h(V_h)} \|\psi\|_{A_\omega,2 K_h(V_h)} e^{-h_\omega(x) - 2\pi h |\xi|}.$$

This shows that $V_\psi$ is continuous. Next, we treat $V_\psi^*$. Fix $h > 0$ and let $F \in C_\omega(\mathbb{R}^2)$ be arbitrary. Note that

$$V_\psi^* F(t + i u) = \iint_{\mathbb{R}^2} F(x, \xi) e^{2\pi i t (t + i u)} \psi(t + i u - x) dx d\xi, \quad t + i u \in \mathbb{C},$$

is an entire function. For all $t + i u \in V_h$

$$|V_\psi^* F(t + i u)| \leq \iint_{\mathbb{R}^2} |F(x, \xi)| e^{2\pi h |\xi|} |\psi(t + i u - x)| dx d\xi \leq C_0 \int_\mathbb{R} e^{-K h_\omega(\alpha)} d\alpha \int_\mathbb{R} e^{-\pi h |\xi|} d\xi \|\psi\|_{A_\omega,K_h(V_h)} \|F\|_{C_{\omega,2 K_h, 3 \pi h}(\mathbb{R}^2)} e^{-h_\omega(t)}.$$

This shows that $V_\psi^*$ is continuous. Finally, $(3.2)$ follows from $(3.1)$. \qed

Our next goal is to extend the STFT and its adjoint to $\mathcal{U}_\omega’(\mathbb{C}) = (\mathcal{U}_\omega(\mathbb{C}))’$. We define the STFT of $f \in \mathcal{U}_\omega’(\mathbb{C})$ with respect to the window $\psi \in \mathcal{U}_\omega(\mathbb{C})$ as

$$V_\psi f(x, \xi) := \langle f(t), \overline{\psi(t - x) e^{-2\pi i t}} \rangle, \quad (x, \xi) \in \mathbb{R}^2.$$ 

Clearly, $V_\psi f$ is a continuous function on $\mathbb{R}^2$. We define the adjoint STFT of $F \in C_\omega^*(\mathbb{R}^2)$ as

$$\langle V_\psi^* F, \varphi \rangle := \iint_{\mathbb{R}^2} F(x, \xi) \overline{V_\psi \varphi(x, -\xi)} dx d\xi, \quad \varphi \in \mathcal{U}_\omega(\mathbb{C}).$$

Note that $V_\psi^* F$ belongs to $\mathcal{U}_\omega’(\mathbb{C})$ by Proposition 3.1.
Proposition 3.2. Let $\omega$ be a weight function satisfying $\alpha$. Fix a window $\psi \in \mathcal{U}_\omega(\mathbb{C})$. The linear mappings
\[
V_\psi : \mathcal{U}_\omega'(\mathbb{C}) \to C^0(\mathbb{R}^2) \quad \text{and} \quad V_\psi^* : C^0(\mathbb{R}^2) \to \mathcal{U}_\omega'(\mathbb{C})
\]
are continuous. Furthermore, if $\gamma \in \mathcal{U}_\omega(\mathbb{C})$ is such that $(\gamma, \psi)_{L^2} \neq 0$, then the reconstruction formula
\[
\frac{1}{(\gamma, \psi)_{L^2}} V_\psi^* \circ V_\psi = \text{id}_{\mathcal{U}_\omega'(\mathbb{C})}
\]
holds.

Proof. We first consider $V_\psi$. Since the space $\mathcal{U}_\omega'(\mathbb{C})$ is bornological, it suffices to show that $V_\psi$ is bounded. Let $B$ be an arbitrary bounded subset of $\mathcal{U}_\omega'(\mathbb{C})$. By the Banach-Steinhaus theorem, there are $h, C > 0$ such that
\[
\sup_{f \in B} |\langle f, \varphi \rangle| \leq C\|\varphi\|_{\mathcal{A}_{\omega,h}(V_h)}, \quad \varphi \in \mathcal{U}_\omega(\mathbb{C}).
\]
Note that
\[
\|\psi(t-x)e^{-2\pi i xt}\|_{\mathcal{A}_{\omega,h}(V_h)} \leq C_0\|\psi\|_{\mathcal{A}_{\omega,h}(V_h)} e^{Kh\omega(x) + 2\pi h|x|}, \quad (x, \xi) \in \mathbb{R}^2,
\]
and thus $\sup_{f \in B} \|V_\psi f\|_{\mathcal{C}_{\omega,-K_h,2\pi h(\mathbb{R}^2)}} < \infty$. This shows that the set $V_\psi(B)$ is bounded. Next, the continuity of $V_\psi^*$ follows from Proposition 3.1. Finally, (3.3) is a consequence of (3.2) and the fact that $\mathcal{U}_\omega(\mathbb{C})$ is dense in $\mathcal{U}_\omega'(\mathbb{C})$.

Finally, we consider the space $\mathcal{A}_\omega'(\mathbb{R}) = (\mathcal{A}_\omega(\mathbb{R}))'_b$. Since $\mathcal{U}_\omega(\mathbb{C})$ is dense in $\mathcal{A}_\omega(\mathbb{R})$ (cf. [7, Lemma 5.11]), we may view $\mathcal{A}_\omega'(\mathbb{R})$ as a subspace of $\mathcal{U}_\omega'(\mathbb{C})$.

Proposition 3.3. Let $\omega$ be a weight function satisfying $\alpha$. Fix a window $\psi \in \mathcal{U}_\omega(\mathbb{C})$. The linear mappings
\[
V_\psi : \mathcal{A}_\omega'(\mathbb{R}) \to \tilde{C}_\omega(\mathbb{R}^2) \quad \text{and} \quad V_\psi^* : \tilde{C}_\omega(\mathbb{R}^2) \to \mathcal{A}_\omega'(\mathbb{R})
\]
are continuous. Furthermore, if $\gamma \in \mathcal{U}_\omega(\mathbb{C})$ is such that $(\gamma, \psi)_{L^2} \neq 0$, then the reconstruction formula
\[
\frac{1}{(\gamma, \psi)_{L^2}} V_\psi^* \circ V_\psi = \text{id}_{\mathcal{A}_\omega'(\mathbb{R})}
\]
holds.

Proof. We first consider $V_\psi$. Fix $h > 0$ and let $f \in \mathcal{A}_\omega'(\mathbb{R})$ be arbitrary. Note that
\[
\|V_\psi f\|_{\mathcal{C}_{\omega,h,-1/h(\mathbb{R}^2)}} = \sup_{\varphi \in B_h} |\langle f, \varphi \rangle|,
\]
where
\[
B_h = \{\psi(t-x)e^{-2\pi i xt}e^{h\omega(x) - |\xi|/h} \mid (x, \xi) \in \mathbb{R}^2\}.
\]
The set $B_h$ is bounded in $\mathcal{A}_\omega(\mathbb{R})$, as follows from the estimate
\[
\|\psi(t-x)e^{-2\pi i xt}\|_{\mathcal{A}_{\omega,h}(V_{1/(2h)})} \leq C_0\|\psi\|_{\mathcal{A}_{\omega,h}(V_{1/(2h)})} e^{-h\omega(x) + |\xi|/h}, \quad (x, \xi) \in \mathbb{R}^2.
\]
This shows that $V_\psi$ is continuous. Next, we treat $V_\psi^*$. It suffices to show that the linear mapping
\[
V_\psi : \mathcal{A}_\omega(\mathbb{R}) \to \lim_{h \to \infty} C_{\omega,h,1/h(\mathbb{R}^2)}
\]
is continuous. This can be done by using an argument similar to the one used in the first part of the proof of Proposition 3.1. Finally, (3.4) follows from (3.3). □

4. THE CONDITIONS \(DN\) AND \(\Omega\)

We now show that the Fréchet space \(U_\omega(\mathbb{C})\) satisfies the linear topological invariants \((DN)\) and \((\Omega)\).

We start with the condition \((DN)\). A Fréchet space \(E\) with a fundamental increasing sequence \(\{\| \cdot \|_n\}_{n \in \mathbb{N}}\) of seminorms is said to satisfy \((DN)\) if

\[
\exists n \in \mathbb{N} \forall m \geq n \exists k \geq m \exists C > 0 \forall e \in E : \|e\|_m^2 \leq C \|e\|_n \|e\|_k.
\]

In such a case, \(\| \cdot \|_n\) is called a dominating norm. We need the following weighted version of the three-lines theorem.

**Lemma 4.1.** Let \(k, h \in \mathbb{R}\) with \(k < h\). Let \(\sigma\) and \(\eta\) be non-decreasing non-negative functions on \([0, \infty)\) satisfying

\[
\int_0^\infty \sigma(t)e^{\frac{\pi}{2k}t}dt < \infty \quad \text{and} \quad \int_0^\infty \eta(t)e^{\frac{\pi}{2k}t}dt < \infty.
\]

Let \(\varphi\) be analytic and bounded on \(\mathbb{R} + ik(h, k)\) and continuous on \(\mathbb{R} + ik(h, k)\). Suppose that

\[
|\varphi(x + ik)| \leq Ae^{-\sigma(x)} \quad \text{and} \quad |\varphi(x + ih)| \leq Be^{-\eta(x)}, \quad x \in \mathbb{R},
\]

for some \(A, B > 0\). Then,

\[
|\varphi(x + iy)| \leq A^{\frac{k}{k-h}}B^{\frac{k-h}{k-h}}e^{-\left(\frac{k-h}{k-h}\right)\frac{\pi}{2k} - \left(\frac{k-h}{k-h}\right)\frac{\pi}{2k}}, \quad k \leq y \leq h.
\]

**Proof.** The proof is similar to the one of [7, Proposition 3.5] and therefore left to the reader. □

**Proposition 4.2.** Let \(\omega\) be a weight function. Let \(0 < h_0 < h_2\) and \(0 \leq \lambda_0 \leq \lambda_2\). Set

\[
\lambda_1 = \lambda_1(h_0, h_2, \lambda_0, \lambda_2) = \frac{h_2\lambda_0 + h_0\lambda_2}{2(h_2 + h_0)}.
\]

For all \(0 < h_1 < h_2\) it holds that

\[
\|\varphi\|_{A_{\omega, \lambda_1}(V_{h_1})} \leq \|\varphi\|_{A_{\omega, \lambda_0}(V_{h_0})} \|\varphi\|_{A_{\omega, \lambda_2}(V_{h_2})}, \quad \varphi \in U_\omega(\mathbb{C}).
\]

In particular, \(U_\omega(\mathbb{C})\) satisfies \((DN)\) and \(\| \cdot \|_{A_{\omega, \lambda_0}(V_{h_0})} = \| \cdot \|_{L^\infty(V_{h_0})}\) is a dominating norm for each \(h_0 > 0\).

**Proof.** We may assume that \(U_\omega(\mathbb{C})\) is non-trivial. Hence, by [7, Theorem 7.2], \(\omega\) satisfies \((2.1)\) for all \(\mu > 0\). Let \(\varphi \in U_\omega(\mathbb{C})\) be arbitrary. By applying Lemma 4.1 to \(k = -h_0\) and \(h = h_2\), and the weight functions \(\sigma = \lambda_0\omega\) and \(\eta = \lambda_2\omega\), we obtain that for all \(x + iy \in \mathbb{R} + [0, h_1]\)

\[
|\varphi(x + iy)| \leq \|\varphi\|_{A_{\omega, \lambda_0}(V_{h_0})} \|\varphi\|_{A_{\omega, \lambda_2}(V_{h_2})} e^{-\left(\frac{(h_0 - y)\lambda_0 + (y + h_0)\lambda_2}{2(h_2 + h_0)}\right)\omega(x)}
\]

\[
\leq \|\varphi\|_{A_{\omega, \lambda_0}(V_{h_0})} \|\varphi\|_{A_{\omega, \lambda_2}(V_{h_2})} e^{-\lambda_1\omega(x)}.
\]
Similarly, by applying Lemma 4.1 to $k = -h_2$ and $h = h_0$, and the weight functions $\sigma = \lambda_2 \omega$ and $\eta = \lambda_0 \omega$, we obtain that (4.1) holds for all $x + iy \in \mathbb{R} + [-h_1, 0]$. This implies the result. \hfill \Box

Next, we consider the condition (Ω). A Fréchet space $E$ with a fundamental decreasing sequence $(U_n)_{n \in \mathbb{N}}$ of neighbourhoods of zero is said to satisfy (Ω) [21] if
\[ \forall n \in \mathbb{N} \exists m \geq n \forall k \geq m \exists \lambda, C > 0, \forall t \geq 1 : U_m \subseteq Ct^\lambda U_k + \frac{1}{t} U_n. \]

Note that (Ω) is inherited to quotient spaces.

**Proposition 4.3.** Let $\omega$ be a weight function satisfying (α). Then, $\mathcal{U}_\omega(\mathbb{C})$ satisfies (Ω).

**Proof.** By Proposition 3.1 $\mathcal{U}_\omega(\mathbb{C})$ is isomorphic to a complemented subspace of $C_\omega(\mathbb{R}^2)$. Hence, it suffices to show that $C_\omega(\mathbb{R}^2)$ satisfies (Ω). This can be done by using a standard argument involving cut-off functions. \hfill \Box

5. **The diametral dimension**

In this section, we determine the diametral dimension of $\mathcal{U}_\omega(\mathbb{C})$.

We start by defining power series spaces of infinite type. Let $I$ be a countable index set. Let $\beta = (\beta_i)_{i \in I}$ be a multi-indexed sequence of non-negative numbers. Let $r \in \mathbb{R}$. We write $\Lambda_r^\beta(I)$ for the Banach space consisting of all $c = (c_i)_{i \in I} \in \mathbb{C}^I$ such that
\[ \|c\|_{\Lambda_r^\beta(I)} := \sup_{i \in I} |c_i| e^{r \beta_i} < \infty. \]

We define
\[ \Lambda_\infty(I; \beta) := \lim_{r \to \infty} \Lambda_r^\beta(I). \]

By an exponent sequence we mean a non-decreasing sequence $\beta = (\beta_n)_{n \in \mathbb{N}}$ of non-negative numbers such that $\beta_n \to \infty$ as $n \to \infty$. We define the power series of infinite type as
\[ \Lambda_\infty(\beta) := \Lambda_\infty(\mathbb{N}; \beta). \]

The space $\Lambda_\infty(\beta)$ is nuclear if and only if $\log n = O(\beta_n)$ [21, Proposition 29.6]. The exponent sequence $\beta$ (and also the space $\Lambda_\infty(\beta)$) is said to be stable if $\beta_{2n} = O(\beta_n)$. The strong dual of $\Lambda_\infty(\beta)$ is given by $\Lambda_\infty'(\beta) = \lim_{r \to \infty} \Lambda_r^\beta(\mathbb{N})$.

Next, we recall the notion of diametral dimension [13]. Let $E$ be a vector space. For $n \in \mathbb{N}$ we denote by $\mathcal{L}_n(E)$ the set consisting of all subspaces $L \subseteq E$ with $\dim L \leq n$. Given subsets $V \subseteq U \subseteq E$ and $n \in \mathbb{N}$, we set
\[ \delta_n(V, U) := \inf \{ \delta > 0 \mid \exists L \in \mathcal{L}_n(E) : V \subseteq \delta U + L \}. \]

For a Fréchet space $E$ with a fundamental decreasing sequence $(U_n)_{n \in \mathbb{N}}$ of neighbourhoods of zero we define the diametral dimension as the set
\[ \Delta(E) := \{ (c_n)_{n \in \mathbb{N}} \in \mathbb{C}^\mathbb{N} \mid \forall m \in \mathbb{N} \exists k \geq m : \lim_{n \to \infty} c_n \delta_n(U_k, U_m) = 0 \}. \]

This notion is a linear topological invariant. Given an exponent sequence $\beta$, we have that $\Delta(\Lambda_\infty(\beta)) = \Lambda_\infty'(\beta)$ [13, Proposition 10.6.10].
Let \( \omega \) be a weight function. We define
\[
\omega^*(t) := (s\omega^{-1}(s))^{-1}(t), \quad t \geq 0.
\]
Note that \( \omega^* \) is a weight function satisfying (\( \alpha \)) (even if \( \omega \) itself does not satisfy (\( \alpha \)). Moreover, we have that [19 Equation (3.1)]
\[
(5.1) \quad \omega^*(t) = \frac{t}{(s\omega(s))^{-1}(t)}, \quad t \geq 0.
\]

Our goal is to show that \( \Delta(\mathcal{U}_\omega(\mathbb{C})) = \Lambda'_\infty(\omega^*(n)) \). We start with \( \Lambda'_\infty(\omega^*(n)) \subseteq \Delta(\mathcal{U}_\omega(\mathbb{C})) \). We need the following elementary fact about the diametral dimension.

**Lemma 5.1.** [9 Theorem 1.6.2.4] [30 Proposition 2] Let \( E \) and \( F \) be nuclear Fréchet spaces and suppose that \( F \) is isomorphic to a closed subspace of \( E \). Then, \( \Delta(E) \subseteq \Delta(F) \).

The proof of the next result is inspired by [19 Theorem 3.2].

**Proposition 5.2.** Let \( \omega \) be a weight function satisfying (\( \alpha \)). Then, \( \Lambda'_\infty(\omega^*(n)) \subseteq \Delta(\mathcal{U}_\omega(\mathbb{C})) \).

**Proof.** We claim that \( \Lambda'_\infty((m + \omega(j))_{(m,j)\in\mathbb{N}\times\mathbb{Z}}) \) contains a closed subspace isomorphic to \( \mathcal{U}_\omega(\mathbb{C}) \). Before we prove the claim, let us show how it implies the result. Note that \( \Lambda'_\infty((m + \omega(j))_{(m,j)\in\mathbb{N}\times\mathbb{Z}}) \cong \Lambda'_\infty(\beta) \), where \( \beta \) is the increasing rearrangement of the set \( \{m + \omega(j) \mid m \in \mathbb{N}, j \in \mathbb{Z}\} \). In the proof of [19 Theorem 3.2(b)], it is shown that \( \omega^*(n) = O(\beta_n) \). In particular, as \( \log t = o(\omega^*(t)) \), the space \( \Lambda'_\infty((m + \omega(j))_{(m,j)\in\mathbb{N}\times\mathbb{Z}}) \) is nuclear. Hence, the claim and Lemma 5.1 yield that
\[
\Lambda'_\infty(\omega^*(n)) \subseteq \Lambda'_\infty(\beta) = \Delta(\Lambda'_\infty(\beta)) = \Delta(\Lambda'_\infty((m + \omega(j))_{(m,j)\in\mathbb{N}\times\mathbb{Z}})) \subseteq \Delta(\mathcal{U}_\omega(\mathbb{C})).
\]

We now show the claim. Since \( \Lambda'_\infty(m) \cong O(\mathbb{C}) \), the space \( \Lambda'_\infty((m + \omega(j))_{(m,j)\in\mathbb{N}\times\mathbb{Z}}) \) is isomorphic to the Fréchet space \( \Lambda'_\infty((\omega(j))_{j\in\mathbb{Z}};O(\mathbb{C})) \) consisting of all \( (\varphi_j)_{j\in\mathbb{Z}} \in O(\mathbb{C})^\mathbb{Z} \) such that for all \( h > 0 \)
\[
\sup_{j \in \mathbb{Z}} \sup_{|z| \leq h} |\varphi_j(z)| e^{h\omega(j)} < \infty.
\]
A straightforward computation shows that the mapping
\[
\mathcal{U}_\omega(\mathbb{C}) \to \Lambda'_\infty((\omega(j))_{j\in\mathbb{Z}};O(\mathbb{C})): \varphi \mapsto (\varphi(\cdot + j))_{j\in\mathbb{Z}}
\]
is a topological embedding. \( \square \)

**Remark 5.3.** An inspection of the proof of Proposition 5.2 shows that this result still holds if (\( \alpha \)) is relaxed to (1.1). Next, we show that \( \Delta(\mathcal{U}_\omega(\mathbb{C})) \subseteq \Lambda'_\infty(\omega^*(n)) \). To this end, we will use the generalized diametral dimension and an extension of the condition (\( \Omega \)). Both these notions were introduced in [19]. For a Fréchet space \( E \) with a fundamental decreasing sequence \( (U_n)_{n\in\mathbb{N}} \) of neighbourhoods of zero the generalized diametral dimension is defined as the set
\[
\Delta'(E) := \{(c_n)_{n\in\mathbb{N}} \in C^\mathbb{N} \mid \forall m \in \mathbb{N} \exists k \geq m \exists \lambda > 0 : \lim_{n \to \infty} c_n \delta_n(U_k, U_m)^\lambda = 0\}.
\]
Given an exponent sequence $\beta$ with $\log n = o(\beta_n)$, we have that $\tilde{\Delta}(\Lambda_\infty(\beta)) = \Lambda_\infty(\beta)$ [16 Equality (1.4)].

Let $E$ and $\tilde{E}$ be Fréchet spaces with fundamental decreasing sequences $(U_n)_{n \in \mathbb{N}}$ and $(\tilde{U}_n)_{n \in \mathbb{N}}$ of neighbourhoods of zero, respectively. Let $d : E \to \tilde{E}$ be a continuous linear mapping. The triple $(d, E, \tilde{E})$ is said to satisfy $(\Omega)$ if

$$\forall n \in \mathbb{N} \exists m \geq n \forall k \in \mathbb{N} \exists \lambda, C > 0 \forall t \geq 1 : \tilde{U}_m \subseteq C t^\lambda d(U_k) + \frac{1}{t} \tilde{U}_n.$$  

Note that $E$ satisfies $(\Omega)$ if and only if $(\id_E, E, E)$ satisfies $(\Omega)$. The next result may be interpreted as a substitute for Lemma 5.1.

Lemma 5.4. [16 Lemma 1.3(a)] Let $E$ and $\tilde{E}$ be Fréchet spaces and let $d : E \to \tilde{E}$ be a continuous linear mapping. If $(d, E, \tilde{E})$ satisfies $(\Omega)$, then $\Delta(E) \subseteq \tilde{\Delta}(\tilde{E})$.

The following simple observation generalizes the fact that condition $(\Omega)$ for Fréchet spaces is inherited to quotient spaces.

Lemma 5.5. Let $E_i$ and $\tilde{E}_i$ be Fréchet spaces and let $d_i : E_i \to \tilde{E}_i$ be a continuous linear mapping for $i = 1, 2$. Let $T : E_1 \to E_2$ and $\tilde{T} : \tilde{E}_1 \to \tilde{E}_2$ be surjective continuous linear mappings such that $\tilde{T} \circ d_1 = d_2 \circ T$. If $(d_1, E_1, \tilde{E}_1)$ satisfies $(\Omega)$, then so does $(d_2, E_2, \tilde{E}_2)$.

Proof. The proof is straightforward and therefore left to the reader. \qed

Proposition 5.6. Let $\omega$ be a weight function satisfying $(\alpha)$. Then, $\Delta(\mathcal{U}_\omega(\mathbb{C})) \subseteq \Lambda'_\infty(\omega^*(n))$.

Proof. In [19 Section 4] (see particularly the proof of [19 Theorem 4.6]), it is shown that $\tilde{\Delta}(\mathcal{A}'_\omega(\mathbb{R})) \subseteq \Lambda'_\infty(\omega^*(n))$ (recall that $\mathcal{A}_\omega(\mathbb{R})$ coincides with the space $\mathcal{H}_\infty(\mathbb{R})$ from [19] if $\omega$ satisfies $(\alpha)$). Let $d : \mathcal{U}_\omega(\mathbb{C}) \to \mathcal{A}'_\omega(\mathbb{R})$ be the inclusion mapping. By Lemma 5.4 it suffices to show that $(d, \mathcal{U}_\omega(\mathbb{C}), \mathcal{A}'_\omega(\mathbb{R}))$ satisfies $(\Omega)$. We shall achieve this by combining Lemma 5.5 with the results from Section 3. Set $E_1 = C_\omega(\mathbb{R}^2)$ and $\tilde{E}_1 = \tilde{C}_\omega(\mathbb{R}^2)$. Let $d_1 : C_\omega(\mathbb{R}^2) \to \tilde{C}_\omega(\mathbb{R}^2)$ be the inclusion mapping. We also set $E_2 = \mathcal{U}_\omega(\mathbb{C})$, $\tilde{E}_2 = \mathcal{A}'_\omega(\mathbb{R})$ and $d_2 = d$. Next, fix a non-zero window $\psi \in \mathcal{U}_\omega(\mathbb{C})$ and consider the linear mappings $T = V_\psi^* : C_\omega(\mathbb{R}^2) \to \mathcal{U}_\omega(\mathbb{C})$ and $\tilde{T} = V_\psi^* : \tilde{C}_\omega(\mathbb{R}^2) \to \mathcal{A}'_\omega(\mathbb{R})$. The mappings $T$ and $\tilde{T}$ are continuous and surjective by Proposition 3.1 and Proposition 3.3, respectively. Moreover, it is clear that $\tilde{T} \circ d_1 = d_2 \circ T$. Hence, by Lemma 5.5 it is enough to prove that $(d_1, C_\omega(\mathbb{R}^2), \tilde{C}_\omega(\mathbb{R}^2))$ satisfies $(\Omega)$. This can be done by using a standard argument involving cut-off functions. \qed

6. SEQUENCE SPACE REPRESENTATIONS

We are ready to prove the main result of this article.

Theorem 6.1. Let $\omega$ be a weight function satisfying $(\alpha)$. Then, $\mathcal{U}_\omega(\mathbb{C})$ is isomorphic to $\Lambda_\infty(\omega^*(n))$.  


Proof. We apply Theorem 1.1 to \( E = U_\omega(\mathbb{C}) \) and \( \beta = (\omega^*(n)) \). Note that \( \Lambda_\infty(\omega^*(n)) \) is stable and nuclear because \( \omega^* \) satisfies \((\alpha)\) and \( \log t = o(\omega^*(t)) \). The space \( U_\omega(\mathbb{C}) \) satisfies \((DN)\) and \((\Omega)\) by Proposition 1.2 and Proposition 1.3 respectively. Proposition 5.2 and Proposition 5.6 yield that \( \Delta(U_\omega(\mathbb{C})) = \Lambda'_\infty(\omega^*(n)) \).

We have the following open problem.

Problem 6.2. Show that Theorem 6.1 holds for all non-trivial spaces \( U_\omega(\mathbb{C}) \), where \( \omega \) is a weight function satisfying (1.1) (but not necessarily \((\alpha)\)).

Theorem 1.1 is still applicable in the more general context of Problem 6.2 (recall that \( \omega^* \) always satisfies \((\alpha)\)). Moreover, for all weight functions \( \omega \) satisfying (1.1), it holds that \( U_\omega(\mathbb{C}) \) satisfies \((DN)\) and that \( \Lambda'_\infty(\omega^*(n)) \subseteq \Delta(U_\omega(\mathbb{C})) \); see Proposition 4.2 and Proposition 5.2 (see Remark 5.3). However, we do not know how to show that \( U_\omega(\mathbb{C}) \) satisfies \((\Omega)\) (Proposition 4.3) and that \( \Delta(U_\omega(\mathbb{C})) \subseteq \Lambda'_\infty(\omega^*(n)) \) (Proposition 5.6) without assuming that \( \omega \) satisfies \((\alpha)\).

We now explicitly determine (up to equivalence) the sequence \((\omega^*(n))\) for various weight functions \( \omega \); see also [19, Section 5]. Given two sequences \( \beta \) and \( \gamma \), we write \( \beta \asymp \gamma \) to indicate that \( \beta = O(\gamma) \) and \( \gamma = O(\beta) \).

Examples 6.3.

(i) \( \omega(t) = t^{1/\nu} \), where \( \nu > 0 \). Then,

\[
(\omega^*(n)) = (n^{1/(\nu+1)}).
\]

(ii) \( \omega(t) = (\log t)^{a_1}(\log \log t)^{a_2} \) for \( t \) large enough, where \( a_1 > 1 \) and \( a_2 \in \mathbb{R} \) or \( a_1 = 1 \) and \( a_2 > 0 \). Then,

\[
(\omega^*(n)) \asymp ((\log n)^{a_1}(\log \log n)^{a_2}).
\]

(iii) \( \omega(t) = t^{1/\nu}(\log t)^a \) for \( t \) large enough, where \( \nu > 0 \) and \( a \in \mathbb{R} \). Then,

\[
(\omega^*(n)) \asymp (n^{1/(\nu+1)}(\log n)^{a\nu/(\nu+1)}).
\]

(iv) \( \omega(t) = e^{(\log t)^a} \), where \( 0 < a < 1 \). We will use some results from the theory of regularly varying functions [4] to determine \((\omega^*(n))\). By [4, Proposition 1.5.15], the inverse of the function \( t\omega(t) \) is asymptotically equivalent to \( t\omega^#(t) \), where \( \omega^# \) is the de Bruijn conjugate of \( \omega \) [4, p. 29]. Hence, by (5.1), \( \omega^* \) is asymptotically equivalent to \( 1/\omega^# \). The function \( \omega^# \) is determined in [4, p. 434, Example 3]. From this result, we obtain that for \( (m-1)/m \leq a < m/(m+1) \), where \( m \in \mathbb{Z}_+ \),

\[
(\omega^*(n)) \asymp \left( \exp \left( \sum_{j=1}^{m} \frac{(-1)^{j+1}}{j!} \left( \prod_{k=0}^{j-2} (a j - k) \right) (\log n)^{a-1)}(a^{j+1}) \right) \right),
\]

In particular,

\[
(\omega^*(n)) \asymp (e^{(\log n)^a}), \quad \text{for } a < 1/2,
\]

\[
(\omega^*(n)) \asymp e^{(\log n)^a-a(\log n)^{2a-1}}), \quad \text{for } 1/2 \leq a < 2/3,
\]

\[
(\omega^*(n)) \asymp (e^{(\log n)^a-a(\log n)^{2a-1}}+\frac{3}{(3a-1)(\log n)^{3a-2}}), \quad \text{for } 2/3 \leq a < 3/4.
\]
Finally, we apply Theorem 6.1 to obtain sequence space representations of the projective Gelfand-Shilov spaces $\Sigma^1_\nu$ and $\Sigma^\nu_1$. Given $\mu, \nu > 0$, we define $\Sigma^\mu_\nu$ as the Fréchet space consisting of all $\varphi \in C^\infty(\mathbb{R})$ such that for all $h > 0$

$$\sup_{p,q \in \mathbb{N}} \sup_{x \in \mathbb{R}} \frac{h^{p+q}|\varphi^{(p)}(x)x^q|}{p!^\mu q!^\nu} < \infty.$$ 

The spaces $\Sigma^\mu_\nu$ are the Fréchet counterparts of the classical Gelfand-Shilov spaces $\mathcal{S}^\nu_\nu$ [10, Chapter IV].

**Theorem 6.4.** The spaces $\Sigma^1_\nu$ and $\Sigma^\nu_1$ are both isomorphic to $\Lambda_\infty(n^{1/(\nu+1)})$ for $\nu > 0$.

**Proof.** Since the Fourier transform is an isomorphism between $\Sigma^1_\nu$ and $\Sigma^\nu_1$ (cf. [10, Chapter IV, Section 6]), it suffices to show that $\Sigma^1_\nu$ is isomorphic to $\Lambda_\infty(n^{1/(\nu+1)})$. We have that $\Sigma^1_\nu = \mathcal{U}_{1/\nu}(\mathbb{C})$ (cf. [10, Chapter IV, Section 2]). Hence, the result follows from Theorem 6.1 and Example 6.3(i). $\square$

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