A systematic method for constructing time discretizations of integrable lattice systems: local equations of motion

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Abstract

We propose a new method for discretizing the time variable in integrable lattice systems while maintaining the locality of the equations of motion. The method is based on the zero-curvature (Lax pair) representation and the lowest-order ‘conservation laws’. In contrast to the pioneering work of Ablowitz and Ladik, our method allows the auxiliary dependent variables appearing in the stage of time discretization to be expressed locally in terms of the original dependent variables. The time-discretized lattice systems have the same set of conserved quantities and the same structures of the solutions as the continuous-time lattice systems; only the time evolution of the parameters in the solutions that correspond to the angle variables is discretized. The effectiveness of our method is illustrated using examples such as the Toda lattice, the Volterra lattice, the modified Volterra lattice, the Ablowitz–Ladik lattice (an integrable semi-discrete nonlinear Schrödinger system) and the lattice Heisenberg ferromagnet model. For the modified Volterra lattice, we also present its ultradiscrete analogue.

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1. Introduction

The quest for a finite-difference analogue of a given differential equation can be justified for several sound reasons. A suitable discretization can reproduce most of the important properties of the differential equation in the small-value range of the difference interval and can be considered a ‘generalization’ of the original continuous equation. Such a discretization facilitates a better and more intuitive understanding of the differential equation without using a limiting procedure, which is needed to define differentiation, and is ideal for performing numerical experiments.
The suitable discretization of a completely integrable system is usually required to retain the integrability; if this is satisfied, it is called an integrable discretization. An integrable system often admits more than one integrable discretization; in such a case, we can consider the properties of each integrable discretization other than integrability and discuss which one is the most favourable for our purpose. The problem of integrable discretization has been sporadically studied since the mid-1970s, i.e. the dawn of the modern theory of integrable systems. For more than 30 years, various techniques have been developed to obtain integrable discretizations of continuous systems. Readers interested in the history of integrable discretizations are referred to the preface of Suris’s book [1].

Partial differential equations (PDEs) involve more than one independent variable. The discretization of an integrable nonlinear PDE is generally performed in two steps; in the first step, we discretize the spatial variable(s) and in the second step, the time variable is discretized. Of course, for some PDEs such as the sine-Gordon equation $u_{xt} = \sin u$, the roles of the space and time variables can be swapped and it is not meaningful to discuss the order of discretization. However, for an integrable nonlinear PDE wherein the roles of individual variables are essentially different and not interchangeable, the order in which the independent variables may be discretized appears to be unique. Thus, we can consider that the time variable is always discretized last, after the spatial variable(s) have been discretized.

In this paper, we focus on the problem of the integrable full discretization of differential-difference equations in 1+1 dimensions. The continuous independent variable to be discretized is regarded as time, as noted above. Most of the differential-difference equations considered reduce to integrable PDEs in a proper continuous limit; however, this is not necessary and we can also start with integrable lattice systems that have no continuous counterpart. The problem of time discretization is, by its nature, distinct from the problem of space discretization. In fact, the former problem has its own peculiarities and difficulties that the latter does not have. This point was uncovered by Ablowitz and Ladik [2, 3] in their attempt to fully discretize the space-discretized nonlinear Schrödinger (semi-discrete NLS) equation [4]. It turned out that unexpected nonlocality emerges in the stage of time discretization; the fully discrete NLS equation involves infinite sums and/or infinite products (in the case of an infinite chain) with respect to the discrete spatial variable and is thus a global-in-space scheme. The fully discrete NLS equation can superficially be written in a local form using additional dependent variables called auxiliary variables, but it does not provide any essential resolution of the nonlocality problem. The subsequent paper by Taha and Ablowitz [5] reinforces the impression that the appearance of global terms is a general feature of the problem of time discretization for integrable lattice systems.

The pioneering work of Ablowitz and coworkers [2, 3, 5] is based on the zero-curvature (Lax pair) representation; the guiding principle is that the time discretization does not change the spatial part of the Lax pair for an integrable lattice system. This automatically guarantees the major advantages of the full discretization; that is, the time-discretized lattice systems have the same integrals of motion and the same structures of the solutions as the original continuous-time lattice systems. In more modern terms, each of their time discretizations belongs to the same integrable hierarchy as the underlying continuous-time system [6, 7].

Despite the elegance of this result, the appearance of infinite sums/products is a shortcoming not acceptable to everybody, and new ideas are needed to remove the nonlocality. In this regard, Suris recently introduced the notion of localizing changes of variables, applied it to a large number of integrable lattice systems and obtained their time discretizations written in local equations of motion [1]. Note that some of his discretizations coincide with earlier results obtained using Hirota’s bilinear method [8], as described in the bibliographical remarks in his book [1]. Suris’s idea was to find, by guessing, a change of variables such that the equations
of motion as well as the auxiliary variables can be expressed locally in terms of the new dependent variables; the change of variables involves the step size of time as a parameter and is considered as a discrete Miura transformation giving a one-parameter deformation of the original lattice hierarchy. Thus, the integrals of motion and the solution formulas for the time-discretized lattice system are deformed accordingly. Moreover, although Suris’s approach has successfully provided many interesting examples, its applicability is rather limited. In particular, it is not applicable to the time discretization of NLS-type lattices, such as the semi-discrete NLS equation (also called the Ablowitz–Ladik lattice) [4], wherein the two dependent variables can be related by a complex conjugacy reduction.

The main objective of this paper is to propose a systematic method for constructing time discretizations of integrable lattice systems written as local equations of motion. In contrast to the other known methods, our method generally requires no ad hoc treatment on a case-by-case basis and appears to have no serious limitations in its applicability; it can be applied to possibly all lattice systems in $1 + 1$ dimensions possessing a Lax pair representation. In particular, it can be used to obtain local full discretizations of NLS-type lattices, including the Ablowitz–Ladik lattice [4]. Actually, our method can be considered as a completed version of the Ablowitz–Ladik approach [2, 3]; it both refines and extends their work in an essential way. A decisive breakthrough has been made by considering the lowest-order ‘conservation laws’, derived from the zero-curvature condition written in the matrix form. In the process, a critical role is played by an arbitrary parameter in the Lax pair called the spectral parameter. The requirement that all the fluxes corresponding to the same conserved density have to essentially coincide results in an ‘ultralocal’ algebraic system for the auxiliary variables; the simpler case where the conserved density is trivially a constant can be treated in a similar manner. Thus, by solving this algebraic system, we can restore the locality of the equations, that is, the global terms appearing in the stage of time discretization can be replaced by local expressions in terms of the original dependent variables.

This paper is organized as follows. In section 2, we describe the general method for discretizing the time variable, using the Toda lattice in Flaschka–Manakov coordinates [9–11] as an example. In section 3, we construct time discretizations of the Ablowitz–Ladik lattice, the Volterra lattice, the modified Volterra lattice and the lattice Heisenberg ferromagnet model; as a spin-off, we obtain an ultradiscrete analogue [12, 13] of the modified Volterra lattice; an ultradiscrete analogue of the Volterra lattice is presented in [14]. In addition, we uncover unexpected relationships with the work of Nijhoff, Quispel, Capel and others [15–18]. Section 4 is devoted to concluding remarks.

2. Method for time discretization

In this section, we discuss the problem of time discretization for a given integrable lattice system. We start with a Lax pair formulation in the continuous-time case and then proceed to discretize the time variable. A set of auxiliary variables is introduced to express the time-discretized lattice as a closed system of equations. Using the lowest-order ‘conservation laws’ derived from the Lax pair, we can obtain local expressions for the auxiliary variables in terms of the original variables.

2.1. The Toda lattice in Flaschka–Manakov coordinates

To make the description of our method easier to understand, in the following subsections, we use one illustrative example, i.e. the Toda lattice written in Flaschka–Manakov coordinates.
Here, \( n \) is the discrete spatial variable and the subscript \( t \) denotes differentiation with respect to the continuous time variable \( t \). The parametrization
\[
\begin{align*}
  u_n &= e^{x_n-x_{n-1}}, \\
  v_n &= u_{n,t},
\end{align*}
\]
(2.2)
enables the system (2.1) to be rewritten as the Newtonian equations of motion for the Toda lattice,
\[
\begin{align*}
  x_{n,tt} &= e^{x_{n+1}-x_n} - e^{x_n-x_{n-1}}.
\end{align*}
\]

2.2. Lax pair and a conservation law in the continuous-time case

The Lax pair formulation in a semi-discrete spacetime comprises a pair of linear equations
\[
\Psi_{n+1} = L_n(\lambda)\Psi_n, \quad \Psi_{n,t} = M_n(\lambda)\Psi_n.
\]
(2.3)
Here, \( \Psi_n \) is a column-vector function. The square matrices \( L_n \) and \( M_n \) depend on the spectral parameter \( \lambda \), which is an arbitrary constant independent of \( n \) and \( t \). The compatibility condition of the overdetermined system (2.3) is given by [2, 4, 6, 7, 19, 20]
\[
L_{n,t} + L_n M_n - M_{n+1} L_n = O,
\]
(2.4)
which is (a semi-discrete version of) the zero-curvature condition. The symbol \( O \) is used to stress that this is a matrix equation. If we specify the \( \lambda \)-dependent matrices \( L_n \) and \( M_n \) appropriately, (2.4) results in a closed differential-difference system for some \( \lambda \)-independent quantities in \( L_n \) and \( M_n \). In such a case, the pair of matrices \( L_n \) and \( M_n \) is called a Lax pair.

**Example.** The Lax pair for the Toda lattice (2.1) in Flaschka–Manakov coordinates is given by [1]
\[
\begin{align*}
  L_n &= \begin{bmatrix} \lambda + v_n & u_n \\ -1 & 0 \end{bmatrix}, \quad \Psi_n = \begin{bmatrix} \lambda & u_n \\ 1 & v_n \end{bmatrix}. \\
  M_n &= \begin{bmatrix} 0 & -u_n \\ 1 & \lambda + v_{n-1} \end{bmatrix}.
\end{align*}
\]
(2.5)

Note that one can add any \( n \)-independent scalar matrix to \( M_n \) (cf (2.4)).

We assume that the \( \lambda \)-dependence of \( L_n \) is essential, so that it cannot be removed by any gauge transformation. Thus, the integrals of motion can be generated from the trace of the monodromy matrix, the ordered product of \( L_n \) (cf [20, 21]). The matrix \( L_n \) is usually ultralocal in the dependent variables, that is, if \( L_n \) involves some variable, say \( u_n \), then it does not involve shifted variables such as \( u_{n-1} \) and \( u_{n+1} \). In addition, the determinant of \( L_n \) is required to be nonzero for generic \( \lambda \) so that the spectral problem is well-posed on the entire infinite chain.

The zero-curvature condition (2.4) generates a conservation law of the following form:
\[
\frac{\partial}{\partial t} \log(\det L_n) = \Delta_n^+(\text{tr } M_n).
\]
(2.6)
Here, \( \Delta_n^+ \) is the forward difference operator in the spatial direction, i.e.
\[
\Delta_n^+ f_n := f_{n+1} - f_n.
\]
(2.7)
For a proper integrable lattice system, \( \det L_n(\lambda) \) has to be either a time-independent function of \( \lambda \) or the exponential of a \( \lambda \)-independent conserved density multiplied by a \( t \)-independent factor. Indeed, if this was not satisfied, e.g. \( \det L_n(\lambda) = 1 + \lambda u_n \), then the expansion of \( \log \det L_n(\lambda) \) with respect to \( \lambda \) would yield an infinite number of (almost) ultralocal conserved densities, say \( u_0, u_1, \ldots \), and thus the lattice system would be trivial in some sense.
2.3. Lax pair in the discrete-time case and auxiliary variables

Now, we discuss how to construct the time discretization of a given lattice system having the Lax pair $L_n$ and $M_n$. The natural discrete-time analogue of the linear system (2.3) is given by

$$
\Psi_{n+1} = L_n(\lambda)\Psi_n, \quad \widetilde{\Psi}_n = V_n(\lambda)\Psi_n,
$$

where the tilde denotes the forward shift ($m \to m + 1$) in the discrete-time coordinate $m \in \mathbb{Z}$. Here and hereafter, the dependence on $m$ is suppressed unless it is shifted. The compatibility condition of this overdetermined linear system is given by [2, 3, 5–7, 22]

$$
\widetilde{L}_n V_n = V_{n+1} L_n,
$$

which is (a fully discretized version of) the zero-curvature condition.

Following the work of Ablowitz and coworkers [2, 3, 5], the matrix $L_n(\lambda)$ is assumed to be the same as that in the semi-discrete case. Then, we look for a $V_n(\lambda)$ such that the zero-curvature condition (2.9) results in a closed system of partial difference equations providing a discrete-time analogue of the semi-discrete system. For this purpose, we assume that the matrix $V_n$ has asymptotic behaviour

$$
V_n = I + h [M_n + O(h)],
$$

where $h$ is a sufficiently small (but nonzero) parameter independent of $\lambda$ and is considered the difference interval of time (cf (2.4) and (2.9)). Moreover, we assume that $V_n(\lambda)$ has essentially the same $\lambda$-dependence as $I + hM_n(\lambda)$. Note, however, that $M_n(\lambda)$ can only be determined up to the addition of an $n$-independent scalar matrix. Thus, this arbitrariness has to be taken into account in determining $V_n(\lambda)$; this corresponds to the freedom (nonuniqueness) of choosing the linear part of the time-discretized lattice system that determines the dispersion relation [3]. Moreover, some $n$-independent quantities (usually set as constants) in $M_n$ translate into $n$-dependent quantities in $V_n$, which typically constitute, up to the reformulation of the dependent variables, new auxiliary variables.

**Example.** For the Toda lattice in Flaschka–Manakov coordinates, the simplest ansatz for $V_n$ is (cf (2.5))

$$
V_n = I + h \left\{ \lambda \begin{bmatrix} 0 & 0 \\ 0 & \Lambda_n \end{bmatrix} + V_n^{(0)} \right\},
$$

where $\Lambda_n$ is an auxiliary variable and $V_n^{(0)}$ is a $\lambda$-independent matrix to be determined later.

Let us decompose $L_n(\lambda)$ into a sum of terms, each of which is the product of an $(n, m)$-independent scalar function of $\lambda$ and a $\lambda$-independent matrix, i.e.

$$
L_n(\lambda) = \sum_{i=\text{min}}^{i=\text{max}} f_i(\lambda) L_n^{(i)}.
$$

The scalar functions $f_i(\lambda)$ ($i_{\text{min}} \leq i \leq i_{\text{max}}$) are linearly independent; typically, they are powers of $\lambda$, e.g. $f_i(\lambda) = \lambda^i$. The nonzero elements of the matrices $L_n^{(i)}$ are classified into two types, that is, constants and dynamical variables depending on both independent variables. We express the entire set of functionally independent dynamical variables in $L_n^{(i)}$ ($i_{\text{min}} \leq i \leq i_{\text{max}}$) as $\{L_i\}$. The set of dynamical variables $\{v_n\}$ is defined from $V_n(\lambda)$ in exactly the same way. Then, the zero-curvature condition (2.9), which is an identity in $\lambda$, provides a (typically bilinear algebraic) system for $\{\widetilde{L}_n, L_n\}$ and $\{v_n, v_{n+1}\}$. In particular, this system contains a useful subsystem, that is, an ultralocal and linear system in $\{v_n\}$, wherein the coefficients involve $\{L_i\}$ and its shifts. We solve this subsystem to express a subset of $\{v_n\}$ in terms of...
the remaining \( \{v_n\} \) as well as \( \{l_n\} \) and its shifts such as \( \{\tilde{l}_n\} \) and \( \{l_{n-1}\} \). Thus, we can reduce the number of independent dynamical variables in \( V_n \) while maintaining the ultralocality of \( V_n \) with respect to \( \{v_n\} \).

**Example.** Substituting (2.5a) and (2.11) into (2.9) and noting that it is an identity in \( \lambda \), we can express \( V_{n}(0) \) in terms of \( u_n, v_n \) and \( \Lambda_n \). Thus, up to an overall factor, \( V_n \) can be written as

\[
V_n = I + h \Lambda_n \begin{bmatrix} 0 & -\tilde{u}_n \\ 1 & \lambda + v_{n-1} \end{bmatrix}.
\]

(2.12)

Then, the zero-curvature condition (2.9) for the Lax pair (2.5a) and (2.12) is equivalent to the following system of partial difference equations (cf (3.6) in [23]):

\[
\begin{align*}
\frac{1}{h} (\tilde{u}_n - u_n) &= \Lambda_{n+1} u_n (\tilde{v}_n - v_{n-1}), \\
\frac{1}{h} (\tilde{v}_n - v_n) &= \Lambda_{n+1} (\tilde{u}_{n+1} - u_n), \\
\tilde{u}_n \Lambda_n &= u_n \Lambda_{n+1}.
\end{align*}
\]

(2.13)

Note that using the third equation, we can rewrite the first and second equations in many different forms.

The fully discrete system arising from the zero-curvature condition (2.9) should define a consistent and unique time evolution under appropriate boundary conditions for the dynamical variables. In particular, we assume suitable boundary conditions for \( L_n \) as \( n \to \pm \infty \) and a constant boundary condition for \( V_n \)

\[
\lim_{n \to -\infty} V_n = \text{const.} \quad \text{or} \quad \lim_{n \to +\infty} V_n = \text{const.}
\]

(2.14)

It is redundant to impose the boundary conditions on \( V_n \) at both spatial ends as in the existing literature [1–3, 6, 7] and it is nontrivial that the redundant boundary conditions are compatible. In general, it can be shown that the two limits, \( \lim_{n \to -\infty} V_n \) and \( \lim_{n \to +\infty} V_n \), indeed coincide.

**Example.** In view of (2.2) and (2.14), we impose the following boundary conditions for \( u_n, v_n \) and \( \Lambda_n \) in (2.13):

\[
\lim_{n \to \pm \infty} u_n = 1, \quad \lim_{n \to \pm \infty} v_n = 0, \quad \lim_{n \to -\infty} \Lambda_n = 1 \quad \text{or} \quad \lim_{n \to +\infty} \Lambda_n = 1.
\]

(2.15)

Thus, the auxiliary variable \( \Lambda_n \) can be written explicitly (but globally) as

\[
\Lambda_n = \prod_{j=-\infty}^{n-1} \tilde{u}_j \quad \text{or} \quad \prod_{j=n}^{+\infty} \tilde{u}_j.
\]

In the continuum limit of time \( h \to 0 \), the auxiliary variable \( \Lambda_n \) approaches 1 and the time discretization (2.13) reduces to the Toda lattice (2.1).

**2.4. Lowest-order 'conservation laws' in the discrete-time case**

We consider the determinant of both sides of (2.9) to obtain the equality

\[
(\det \tilde{L}_n)(\det V_n) = (\det L_n)(\det V_{n+1}).
\]

(2.16)

This relation can be written more explicitly in the form of a discrete conservation law:

\[
\Delta^+_{m} \log(\det L_n) = \Delta^+_{m} \log(\det V_n).
\]

Here, \( \Delta^+_{m} \) is the forward difference operator in the time direction, i.e. \( \Delta^+_{m} f_n := \tilde{f}_n - f_n \). This is the discrete-time version of (2.6). Note that after cancelling the \( m \)-independent factor of \( \det L_n \), relation (2.16) should reduce to either

\[
\det V_n = \det V_{n+1}
\]

(2.17)
or

\[ \exp(\tilde{\rho}_n) \det V_n = \exp(\rho_n) \det V_{n+1}, \]  

(2.18)

where \( \rho_n \) is a nontrivial conserved density. We employ a simplified but still ultralocal (with respect to \( \{v_n\} \)) form of \( V_n \), compute its determinant and expand it with respect to \( \lambda \) in the summed form

\[ \det V_n(\lambda) = \sum_{j=\min}^{\max} g_j(\lambda) a_n^{(j)}. \]  

(2.19)

Here, \( g_j(\lambda) (j_{\min} \leq j \leq j_{\max}) \) are linearly independent functions of \( \lambda \), e.g. \( g_j(\lambda) = \lambda^j \), and their coefficients \( a_n^{(j)} \) are \( \lambda \)-independent functions of \( \{v_n\} \), \( \{l_n\} \) and the space/time shifts of \( \{l_n\} \). In the first case, substituting (2.19) into (2.17), we obtain the \( n \)-independence of the coefficients of \( g_j(\lambda) \) for all \( j \), i.e.

\[ a_n^{(j)} = \lim_{n \to \pm \infty} a_n^{(j)}, \quad j_{\min} \leq j \leq j_{\max}. \]  

(2.20)

In the second case (2.18), which is more common than (2.17), the substitution of (2.19) gives the set of relations

\[ \exp(\tilde{\rho}_n) a_n^{(j)} = \exp(\rho_n) a_{n+1}^{(j)}, \quad j_{\min} \leq j \leq j_{\max}. \]

Thus, there exist seemingly more than one flux \( (-) \log a_n^{(j)} \) associated with the same conserved density \( \rho_n \), but the \( a_n^{(j)} \) should coincide up to trivial proportionality factors. Indeed, calculating the ratio of the above equality for different values of \( j \) on both sides, we obtain

\[ \frac{a_n^{(j_1)}}{a_n^{(j_2)}} = \lim_{n \to \pm \infty} \frac{a_n^{(j_1)}}{a_n^{(j_2)}}, \quad j_{\min} \leq j_1 \neq j_2 \leq j_{\max}. \]  

(2.21)

That is, the ratio \( a_n^{(j_{\max})} : \cdots : a_n^{(j_{\min})} \) is independent of \( n \). In both the above cases, the right-hand side of (2.20) or (2.21) is determined by the boundary condition for \( V_n \) (2.14). This results in an ‘ultralocal’ algebraic system for a subset of \( \{v_n\} \) that essentially constitutes the auxiliary variables; the number of independent unknowns is usually equal to that of the independent equalities, so that this algebraic system is neither overdetermined nor underdetermined.

**Example.** The determinant of the Lax matrix \( L_n \) (2.5a) can be immediately computed as

\[ \det V_n(\lambda) = \lambda h \Lambda_n + 1 + hv_{n-1} \Lambda_n + h^2 \tilde{u}_n \Lambda_n. \]

Therefore, equality (2.16) implies the set of relations

\[
\begin{align*}
\tilde{u}_n \Lambda_n &= u_n \Lambda_{n+1}, \\
\tilde{u}_n \left( 1 + hv_{n-1} \Lambda_n + h^2 \tilde{u}_n \Lambda_n^2 \right) &= u_n \left( 1 + hv_n \Lambda_{n+1} + h^2 \tilde{u}_{n+1} \Lambda_{n+1}^2 \right).
\end{align*}
\]

The first relation already appeared in (2.13); the second relation is new and highly nontrivial, although it is certainly a consequence of (2.13). Comparing these two relations, we obtain

\[ \frac{1 + hv_{n-1} \Lambda_n + h^2 \tilde{u}_n \Lambda_n^2}{\Lambda_n} = \frac{1 + hv_n \Lambda_{n+1} + h^2 \tilde{u}_{n+1} \Lambda_{n+1}^2}{\Lambda_{n+1}}. \]

Because this value is \( n \)-independent, the boundary conditions (2.15) can determine it as \( 1 + h^2 \). Thus, we arrive at a quadratic equation in \( \Lambda_n \):

\[ h^2 \tilde{u}_n \Lambda_n^2 - (1 + h^2 - hv_{n-1}) \Lambda_n + 1 = 0. \]  

(2.22)
2.5. Algebraic system for the auxiliary variables and local equations of motion

We solve the obtained algebraic system for the auxiliary variables appearing in \( V_n \) (and not in \( M_n \)). By eliminating all but one of the auxiliary variables, this system becomes a scalar algebraic equation in the remaining auxiliary variable; the other auxiliary variables are expressible in terms of the solution of this equation. The degree and complexity of this algebraic equation depend on the boundary condition for \( V_n \), which determines the dispersion relation of the time-discretized lattice system. In this paper, we mainly consider the case for which the equation is solvable by the quadratic formula. For sufficiently small \( h \) (\( 0 < |h| < 1 \)), referring to the prescribed behaviour (2.10) of \( V_n \), we can discard one of the two solutions as improper and obtain a unique proper solution of the quadratic equation; note that \( (V_n - I)/h \) does not involve \( O(1/h) \) terms. In general, the larger the number of grid points defining the lattice system as well as the dispersion relation, the higher the degree of the algebraic equation determining the auxiliary variables [14]. The higher-degree case can be interpreted as a composition of lower-degree cases, that is, the time evolution in the higher-degree case can be factorized into sequential applications of more elementary time evolutions. This point is illustrated in subsection 3.2 of [14]. Once all the auxiliary variables have been expressed locally in terms of the original dependent variables that have already appeared in the zero-curvature condition (2.9). Because of the use of the ‘conservation laws’ (cf (2.17) or (2.18)), not all of the equations arising from (2.9) are independent and necessary any longer. We choose a minimal subset of these equations so that the substitution of the local expressions in the auxiliary variables produces the unique discrete-time evolution of the lattice system.

**Example.** The time discretization (2.13) of the Toda lattice (2.1) involves only one auxiliary variable \( \Lambda_n \). The asymptotic behaviour (2.10) of the Lax matrix \( V_n \) (2.12) implies that the proper solution of (2.22) is given by

\[
(1 + h^2) \Lambda_n = \frac{2}{1 - \epsilon \nu_{n-1} + \sqrt{(1 - \epsilon \nu_{n-1})^2 - 4\epsilon^2 \tilde{u}_n}},
\]

where \( \epsilon := h/(1 + h^2) \). Thus, if \( h \in \mathbb{R} \), then \(-1/2 \leq \epsilon \leq 1/2 \). In this case, the local expression (2.23) is valid only if \(-1 < h < 1 \), which covers the range \(-1/2 < \epsilon < 1/2 \). The borderline cases \( h = \pm 1 \) are excluded from our consideration. If \( h^2 > 1 \), (2.23) is inconsistent with the boundary conditions, and the other solution of (2.22) should be adopted. In any case, the boundary conditions for \( u_n \) and \( v_n \) in (2.15) imply that \( \lim_{n \to -\infty} \Lambda_n = \lim_{n \to +\infty} \Lambda_n = 1 \).

Substituting (2.23) into the first and second equations in (2.13), we obtain a time discretization of the Toda lattice (2.1) in the local form

\[
\begin{align*}
\tilde{u}_n - u_n &= \frac{2u_n (\tilde{u}_n - \nu_{n-1})}{1 - \epsilon \nu_n + \sqrt{(1 - \epsilon \nu_n)^2 - 4\epsilon \tilde{u}_{n+1}}}, \\
\tilde{v}_n - v_n &= \frac{2 (\tilde{u}_{n+1} - u_n)}{1 - \epsilon \nu_n + \sqrt{(1 - \epsilon \nu_n)^2 - 4\epsilon \tilde{u}_{n+1}}}.
\end{align*}
\]

Under the boundary conditions \( \lim_{n \to -\infty} u_n = 1 \) and \( \lim_{n \to +\infty} \nu_n = 0 \), \( \tilde{u}_n \) is determined from \( u_n, \nu_n \) and \( \tilde{u}_{n+1} \), and subsequently, \( \tilde{v}_n \) is determined from \( \nu_{n-1}, u_n, \nu_n, \tilde{u}_n \) and \( \tilde{u}_{n+1} \). If \( h \in \mathbb{R} \), the \( u_n \) are nonzero and real-valued, and the \( v_n \) are real-valued at the initial time, then (2.13) implies that the auxiliary variable \( \Lambda_n \) is always real-valued. Consequently, the discriminant of the quadratic equation (2.22) must be nonnegative. Thus, as long as the amplitudes of \( u_n - 1 \) and \( v_n \) are sufficiently small and their effects can be regarded as perturbations, the
real-valuedness of \( u_n \) and \( v_n \) is preserved under the time evolution of the discrete-time Toda lattice (2.24) for \(-1/2 < \epsilon < 1/2\).

To obtain the backward time evolution explicitly, we only have to replace \( \Lambda_{n+1} \) in the first and second equations of (2.13) with \( \Lambda_n \tilde{u}_n/u_n \) and then substitute the local expression (2.23).

### 3. Examples

In this section, we apply the general method for time discretization in section 2 to four examples other than the Toda lattice: the Ablowitz–Ladik lattice (semi-discrete NLS), the Volterra lattice, the modified Volterra lattice and the lattice Heisenberg ferromagnet model. The parameter \( h \) is generally assumed to be nonzero; alternatively, one can allow the case \( h = 0 \) as the trivial identity mapping \( \tilde{l}_n = l_n \).

#### 3.1. The Ablowitz–Ladik lattice

In this subsection, we discuss the time discretization of the Ablowitz–Ladik lattice, which we consider to be the most instructive example to illustrate our general method. The (nonreduced form of the) Ablowitz–Ladik lattice is [4]

\[
\begin{align*}
q_{n,t} + c(q_{n+1} - q_{n-1}) + (\beta - \alpha)q_n - c^{n+1}r_nq_n + d^{n+1}q_nq_{n-1} &= 0, \\
r_{n,t} + dr_{n+1} - cr_{n-1} + (\alpha - \beta)r_n - dr_{n+1} + cr_{n+1} - cr_{n-1} &= 0,
\end{align*}
\]

and its Lax pair (cf (2.3) and (2.4)) is given by

\[
\begin{align*}
L_n &= \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} q_n \\ 0 \end{bmatrix} + \frac{1}{\lambda} \begin{bmatrix} 0 & 0 \\ r_n & 1 \end{bmatrix} = \begin{bmatrix} \lambda & q_n \\ r_n & 1 \end{bmatrix}, \\
M_n &= \begin{bmatrix} c + \frac{d}{\lambda} & -\lambda c q_n - \frac{d}{\lambda} q_{n-1} \\ -\lambda c r_{n-1} - \frac{d}{\lambda} r_n & \lambda^2 c + \frac{d}{\lambda} r_{n+1} + \beta \end{bmatrix}.
\end{align*}
\]

Here, \( c, d, \alpha \) and \( \beta \) are free parameters. The familiar form of the Ablowitz–Ladik lattice is obtained from (3.1) through the reduction of the complex conjugate \( d = c^*, \beta = \alpha^* \), \( r_n = \sigma q_n^n \), where \( \sigma \) is a real constant. The case where \( c \) and \( d \) are purely imaginary leads to an integrable semi-discretization of the NLS equation. Taking into account the \( \lambda \)-dependence of the Lax matrix \( M_n \) (3.2b), we look for a Lax matrix \( V_n \) with the following dependence on \( \lambda \):

\[
V_n(\lambda) = I + \frac{h}{\lambda} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \frac{\lambda}{\lambda} V_n^{(1)} + \frac{\lambda^{n+1}}{\lambda} V_n^{(0)} + \frac{\lambda^{n+1}}{\lambda} V_n^{(1)} + \frac{\lambda^{n+1}}{\lambda} V_n^{(0)}
\]

Here, \( c_n \) and \( d_n \) are auxiliary variables, \( V_n^{(1)} \) and \( V_n^{(0)} \) are off-diagonal matrices and \( V_n^{(0)} \) is a diagonal matrix. Substituting (3.2a) and (3.3) into (2.9), we find that the matrix \( V_n \) should assume the form

\[
V_n = I + \frac{h}{\lambda} \begin{bmatrix} \tilde{q}_n c_n r_{n-1} + \alpha + \frac{1}{\lambda} d_n & -\lambda \tilde{q}_n c_n - \frac{1}{\lambda} d_n q_{n-1} \\ -\lambda c_n r_{n-1} - \frac{1}{\lambda} \tilde{r}_n d_n & \lambda^2 c_n + \tilde{r}_n d_n q_{n-1} + \beta \end{bmatrix}.
\]

Then, the zero-curvature condition (2.9) is equivalent to the following system of partial difference equations (cf [3]):

\[
\begin{align*}
\frac{1}{\lambda} (\tilde{q}_n - q_n) + \tilde{q}_{n+1} c_{n+1} (1 - r_n q_n) - d_{n+1} (1 - q_n r_n) q_{n-1} - \alpha q_n + \tilde{q}_n \beta &= 0, \\
\frac{1}{\lambda} (\tilde{r}_n - r_n) + \tilde{r}_{n+1} d_{n+1} (1 - q_n r_n) - c_{n+1} (1 - r_n q_n) r_{n-1} - \beta r_n + \tilde{r}_n \alpha &= 0, \\
(1 - \tilde{r}_n \tilde{q}_n) c_n &= c_{n+1} (1 - r_n q_n), \\
(1 - \tilde{q}_n r_n) d_n &= d_{n+1} (1 - q_n r_n).
\end{align*}
\]
We impose the following boundary conditions for \( q_n, r_n, c_n \) and \( d_n \):

\[
\lim_{n \to \pm \infty} q_n = \lim_{n \to \pm \infty} r_n = 0, \quad \lim_{n \to -\infty} (c_n, d_n) = (c, d) \quad \text{or} \quad \lim_{n \to +\infty} (c_n, d_n) = (c, d).
\] (3.6)

Because \( h \) can be absorbed by \( \alpha, \beta, c_n \) and \( d_n \), it is possible to set \( h \) as unity. However, we prefer to leave it as a small parameter, usually \( 0 < |h| \ll 1 \). In the continuum limit of time \( h \to 0 \), we obtain \( c_n \to c, d_n \to d \) and the time discretization (3.5) reduces to the nonreduced Ablowitz–Ladik lattice (3.1).

The determinant of the \( 2 \times 2 \) Lax matrix \( \Lambda_n \) (3.2a) can be immediately computed as

\[
\det \Lambda_n = \frac{\lambda^2 h c_n (1 + h \alpha) + h^2 c_n c_{n+1}}{\lambda^2 h d_n (1 + h \beta) + h^2 d_n d_{n+1}}
\]

Thus, equality (2.16) combined with the boundary conditions (3.6) implies the set of relations

\[
\begin{aligned}
\hat{c}_n &= c \Lambda_n, \\
\hat{d}_n &= d \Lambda_n, \\
h^2 \hat{c}_n c_{n+1} + h \hat{q}_n r_n c_n + h \alpha (1 + h q_{n-1} r_n d_n + h \beta) \\
&- h^2 \hat{q}_n c_n r_n d_n - h^2 q_{n-1} d_n r_n c_n = [h^2 c_d + (1 + h \alpha)(1 + h \beta)] \Lambda_n.
\end{aligned}
\] (3.7)

Here, \( \Lambda_n \) is the quantity that satisfies the recurrence formula

\[
(1 - \hat{q}_n \hat{r}_n) \Lambda_n = (1 - q_n r_n) \Lambda_{n+1}
\] (3.8)

and has the normalized boundary value, \( \lim_{n \to -\infty} \Lambda_n = 1 \) or \( \lim_{n \to +\infty} \Lambda_n = 1 \), in accordance with the boundary conditions for \( c_n \) and \( d_n \). It can be shown that \( \Lambda_n \) has the same limit value for \( n \to -\infty \) and \( n \to +\infty \). Thus, \( \Lambda_n \) can be written explicitly as

\[
\Lambda_n = \prod_{j=-\infty}^{n-1} \frac{1 - \hat{q}_j \hat{r}_j}{1 - q_j r_j} = \prod_{j=n}^{+\infty} \frac{1 - \hat{q}_j \hat{r}_j}{1 - q_j r_j}.
\] (3.9)

Note that in (3.7), only the third relation is new and nontrivial (cf (3.5)). The algebraic system (3.7) results in

\[
\begin{aligned}
h^2 c_d (1 - \hat{q}_n \hat{r}_n)(1 - q_{n-1} r_{n-1}) \Lambda_n^2 &= [(1 + h \alpha)(1 + h \beta) + h^2 c_d - (1 + h \alpha) h d q_{n-1} r_n] \\
&- (1 + h \beta) h c \hat{q}_n r_{n-1} \Lambda_n + (1 + h \alpha)(1 + h \beta) = 0.
\end{aligned}
\] (3.10)

In terms of the normalized parameters

\[
\hat{c} := \frac{hc}{1 + h \alpha}, \quad \hat{d} := \frac{hd}{1 + h \beta},
\] (3.11)

(3.10) can be rewritten as

\[
\hat{c} \hat{d} (1 - \hat{q}_n \hat{r}_n)(1 - q_{n-1} r_{n-1}) \Lambda_n^2 - [1 + \hat{c} \hat{d} - \hat{d} q_{n-1} \hat{r}_n - \hat{c} \hat{q}_n r_{n-1}] \Lambda_n + 1 = 0.
\] (3.12)

When \( cd \neq 0 \) (and thus \( \hat{c} \hat{d} \neq 0 \)), this is a quadratic equation in \( \Lambda_n \), which has two solutions. The simplest way to reject the improper solution is to recall the asymptotic behaviour of the Lax matrix \( \Lambda_n \) for small \( h \) (2.10), but here we take a different route. Let us first consider the ‘trivial’ case where \( q_n \) and \( r_n \) are zero for all \( n \); if this is satisfied at some instant \( m = m_0 \), then it holds true identically for any time \( m \). Thus, the two solutions of (3.12) are given by \( \Lambda_n = 1, 1/(\hat{c} \hat{d}) \). The recurrence formula (3.8) for \( \Lambda_n \) implies that these two solutions are unconnected, i.e. \( \Lambda_n \) takes the same value for all \( n \). The solution \( \Lambda_n = 1/(\hat{c} \hat{d}) \) can be discarded if \( \hat{c} \hat{d} \neq 1 \), because it is inconsistent with the boundary condition for \( \Lambda_n \). Next, in
the general case where \( q_n \) and \( r_n \) are not identically zero, we assume that their amplitudes are always so small that their effects can be regarded as weak perturbations of the identically zero case. In particular, the value of \( \Lambda_n \) is restricted to a neighbourhood of unity. Thus, we obtain the proper solution of the quadratic equation (3.12) as

\[
\Lambda_n = \frac{2}{1 + C_n + \sqrt{(1 + C_n)^2 - 4D_n}},
\]

with

\[
C_n := \hat{c} \hat{d} - \hat{d} q_{n-1} \hat{r}_n - \hat{e} \hat{q}_n r_{n-1},
\]

\[
D_n := \hat{c} \hat{d} (1 - \hat{q}_n \hat{r}_n) (1 - q_{n-1} r_{n-1}).
\]

Note that this solution is also valid for the linear case \( \hat{c} \hat{d} = 0 \). The decaying boundary conditions for \( q_n \) and \( r_n \) in (3.6) imply that \( \lim_{n \to -\infty} \Lambda_n = \lim_{n \to +\infty} \Lambda_n = 1 \). When \( \hat{c} \hat{d} \in \mathbb{R} \), the local expression (3.13) is valid only if \( \hat{c} \hat{d} \leq 1 \). If \( \hat{c} \hat{d} > 1 \), the other solution of (3.12) should be adopted; alternatively, one can understand the right-hand side of (3.13) as being defined by a Taylor series for small \( \hat{c} \) and \( \hat{d} \), and its analytic continuation. At the ‘threshold’ value \( \hat{c} \hat{d} = 1 \), the discrete-time system (3.5) has the trivial solution \( \tilde{q}_n = d q_{n-1}, \tilde{r}_n = \hat{e} r_{n-1}, \)

\[
c_n = c (1 - q_{n-1} r_{n-1}),
\]

\[d_n = \hat{d} (1 - q_{n-1} r_{n-1}).\]

Thus, the discriminant of the quadratic equation (3.12) vanishes at \( \hat{c} \hat{d} = 1 \), and the two solutions indeed intersect. Unless \( \hat{c} \hat{d} = 1 \), a unified expression for \( \Lambda_n \),

\[
\Lambda_n = \frac{2}{1 + C_n + \sqrt{(1 + C_n)^2 - 4D_n}},
\]

can resolve the sign problem of the square root. In addition, the above square root allows the Taylor expansion with respect to \( \{\tilde{q}_n, \tilde{r}_n, q_{n-1}, r_{n-1}\} \). However, we do not use this unwieldy formula.

The first and second equations in (3.5) are linear in \( \tilde{q}_n \) and \( \tilde{r}_n \), respectively. Thus, the forward time evolution can be expressed as follows (cf (3.7) and (3.11)):

\[
\tilde{q}_n = \frac{1 + h (1 + \alpha \hat{d} \hat{r}_n)}{1 + \alpha \hat{d} \hat{r}_n} q_n + (1 - q_n r_n) \Lambda_{n+1} (-\frac{1 + h (1 + \alpha \hat{d} \hat{r}_n)}{1 + \alpha \hat{d} \hat{r}_n} \tilde{q}_{n+1} + \hat{d} q_{n-1}),
\]

\[
\tilde{r}_n = \frac{1 + h (1 + \alpha \hat{d} \hat{r}_n)}{1 + \alpha \hat{d} \hat{r}_n} r_n + (1 - q_n r_n) \Lambda_{n+1} (-\frac{1 + h (1 + \alpha \hat{d} \hat{r}_n)}{1 + \alpha \hat{d} \hat{r}_n} \tilde{r}_{n+1} + \hat{e} r_{n-1}),
\]

where \( \Lambda_n \) is given by (3.13) with (3.14). In the simplest case of \( \hat{e} = \hat{c} = 0 \) or \( \hat{d} = \hat{d} = 0 \), the forward time evolution (3.15) is given by a simple rational mapping. This fact was disclosed by Suris [7] (also see [24]). To obtain the backward time evolution, we only have to replace \( (1 - q_n r_n) \Lambda_{n+1} \) with \( (1 - \tilde{q}_n \tilde{r}_n) \Lambda_n \) (cf (3.8)).

When \( h \) is real, \( \beta = \alpha^* \) and \( \sigma = \gamma \), we can impose the complex conjugacy reduction \( r_n = \sigma q_n^* \) with a real constant \( \sigma \). In particular, setting \( \alpha = -\beta = -i \gamma / \Delta^2, c = -d = -1 / \Delta^2 \) and \( r_n = -\Delta^2 q_n^* \), we obtain the fully discretized NLS equation

\[
\frac{i}{h} (\tilde{q}_n - q_n) + \frac{2(1 + \Delta^2 |q_n|^2)}{1 + C_{n+1} + \sqrt{(1 + C_{n+1})^2 - 4D_{n+1}}} \tilde{q}_{n+1} + q_{n-1} = \frac{\gamma}{\Delta^2} (\tilde{q}_n + q_n) = 0,
\]

where

\[
C_{n+1} = \frac{\hbar^2}{(\Delta^2 + i \gamma h)(\Delta^2 - i \gamma h)} + \frac{i \Delta^2 h}{\Delta^2 + i \gamma h} q_n \tilde{q}_{n+1} + \frac{i \Delta^2 h}{\Delta^2 - i \gamma h} \tilde{q}_{n+1} q_n^*,
\]

\[
D_{n+1} = \frac{\hbar^2}{(\Delta^2 + i \gamma h)(\Delta^2 - i \gamma h)} (1 + \Delta^2 |\tilde{q}_{n+1}|^2)(1 + \Delta^2 |q_n|^2).
\]

The choice of \( \gamma = 1 \) for the real parameter \( \gamma \) is the most natural when taking the continuum limit, while the choice of \( \gamma = 0 \) simplifies the equation considerably. The aforementioned
condition $\hat{c} \hat{d} \leq 1$ implies that $(1 - \gamma^2)h^2 \leq \Delta^2$; this is automatically satisfied if $\gamma^2 \geq 1$. At first glance, it is far from evident that the quantity in the square root is nonnegative: $(1 + C_{n+1})^2 - 4D_{n+1} \geq 0$. However, this inequality should hold true because the left-hand side represents the discriminant of the quadratic equation in $\Lambda_{n+1}$ (cf (3.12)). In fact, the reduction $r_n = -\Delta^2 q_n^\ast$ with $\Delta^2 > 0$ guarantees the auxiliary variable $\Lambda_n$ to be positive (cf (3.9)). Thus, we also have the inequality $1 + C_{n+1} > 0$. To summarize, $\tilde{q}_{n+1}$ and $q_n$ are not fully independent and satisfy the inequality $1 + C_{n+1} \geq 2\sqrt{D_{n+1}}$. Note that (3.16) with $n \to n + 1$ determines $\tilde{q}_{n+1}$ from $q_{n+1}$, $\tilde{q}_{n+2}$ and $q_n$.

Similarly to the continuous NLS equation, this full discretization is homogeneous under the following weighting scheme: weight($\Delta$) = $-1$, weight($h$) = $-2$, weight($q_n$) = $1$. It might be aesthetically pleasing to eliminate the first constant term of $C_{n+1}$ by a suitable redefinition of the parameters. For example, in the simple case of $\gamma = 0$ and $\Delta^2 = 1$, (3.16) can be rewritten as

$$\frac{i}{\delta} (\tilde{q}_n - q_n) + \frac{2(1 + \vert q_n \vert^2) (\tilde{q}_{n+1} + q_{n-1})}{1 + i\delta C_n + \sqrt{(1 + i\delta C_n)^2 - 4\Delta^2 D_n}} = 0,$$

with

$$C_n := q_0 q_{n-1}^* - \tilde{q}_0 \tilde{q}_{n-1}^*, \quad D_n := (1 + \vert \tilde{q}_{n+1} \vert^2)(1 + \vert q_n \vert^2).$$

Here, $\delta := h/(1 + h^2)$ is a new parameter. Because $-1 \leq h \leq 1$, the range of $\delta$ is $-1/2 \leq \delta \leq 1/2$; at the endpoints $\delta = \pm 1/2$, the time evolution is trivial because $\hat{c} \hat{d} = 1$.

In the complex conjugacy reduction wherein $h$ is real, $\beta = \alpha^*$ and $d = c^*$, we can ‘normalize’ the scaling of the dependent variable by setting $r_n = -q_n^*$. In this case, we rewrite the first equation for $q_n$ in (3.5) as

$$(1 + \alpha h^*) \tilde{q}_n - (1 + \alpha h) q_n + (1 - \tilde{q}_n^2) \Lambda_n \left(h c \tilde{q}_n + h^* c^* q_{n-1}\right) = 0.$$  \hspace{1cm} (3.18)

Moreover, (3.10) can be rewritten as

$$\vert 1 + \alpha h^* \vert^2 h^2 \alpha^2 + (1 + \alpha h c^* q_{n-1} \tilde{q}_n^*) + (1 + \alpha h c^*) h c \tilde{q}_n + q_{n-1}^{*} \Lambda_n = \vert 1 + \alpha h^* \vert^2 \frac{1}{\Lambda_n} + h^2 \alpha^2 (1 + \tilde{q}_n^2)(1 + \vert q_{n-1} \vert^2) \Lambda_n.$$

Using (3.18), we can replace $1/\Lambda_n$ and $\Lambda_n$ in (3.19) with rational expressions in $q_n$ as well as its shifts and complex conjugate. Thus, we obtain a rational form of the fully discrete NLS equation

$$\vert 1 + \alpha h^* \vert^2 h^2 \alpha^2 + 2 \Re[(1 + \alpha h c^* q_{n-1} \tilde{q}_n^*)]$$

$$\vert 1 + \alpha h^* \vert^2 (1 + \tilde{q}_n^2) \frac{h c \tilde{q}_{n+1} - h c^* q_{n-1}}{\left(1 + \alpha h^* \right) q_n - \left(1 + \alpha h^* \right) \tilde{q}_n} + h^2 \alpha^2 (1 + \vert q_{n-1} \vert^2) \frac{1 - (1 + \alpha h^*) \tilde{q}_n}{h c \tilde{q}_{n+1} - h c^* q_{n-1}}.$$  \hspace{1cm} (3.20)

Surprisingly, this coincides with the double-discrete NLS equation proposed by Quispel, Nijhoff, Capel and van der Linden [16, 18], up to a minor change of coordinates and parameters; despite its ‘elegance’, this rational version has the drawback that the forward/backward time evolution cannot be uniquely determined. In addition, one cannot immediately recognize that (3.20) reduces to the NLS equation in a continuous limit. Thus, we prefer our version, which is seemingly less elegant because of the existence of the square root but can define the unique time evolution properly and allow an easy-to-follow continuous limit.

Actually, using a coordinate transformation, it is possible to ‘identify’ our time discretization ((3.15) with $\hat{d} = \hat{c}^*$, $\alpha = \beta = 0$ and $r_n = -q_n^*$) with the auto-Bäcklund transformation of the Ablowitz–Ladik hierarchy derived by Nijhoff, Quispel and Capel [15].
However, their expression (see (19) in [15]) involves the indefinite sign \pm in front of the square root and it is not clear how to understand and determine it.

3.2. The Volterra lattice

The Volterra lattice

\[ u_{n,t} = u_n(u_{n+1} - u_{n-1}) \]  

(3.21)

is obtained from the Ablowitz–Ladik lattice (3.1) through the reduction \( c = d = -1, \alpha = \beta = 0, q_n = u_n - 1 \) and \( r_n = -1 \). Thus, the time discretization of the Volterra lattice can be obtained from the discrete-time Ablowitz–Ladik lattice in subsection 3.1 through the reduction \( q_n = u_n - 1 \) and \( r_n = -1 \) together with \( c_n = d_n \) and \( \alpha = \beta = 0 \). The discrepancy between this reduction and the decaying boundary conditions in (3.6) is nonessential in this regard. However, in the following, we prefer to consider the Volterra lattice independently as an interesting example. The Lax pair for the continuous-time Volterra lattice (3.21) is given by [1, 25]

\[
L_n = \begin{bmatrix}
\lambda & u_n \\
-1 & 0
\end{bmatrix},
\]

(3.22a)

\[
M_n = \begin{bmatrix}
u_n & \lambda u_n \\
-\lambda & -\lambda^2 + u_{n-1}
\end{bmatrix}.
\]

(3.22b)

Indeed, the substitution of (3.22) into the zero-curvature condition (2.4) results in (3.21).

In the discrete-time case, we assume the Lax matrix \( V_n \) of the following form:

\[
V_n = I + h \left[ \lambda^2 \begin{bmatrix} 0 & 0 \\ 0 & -\Lambda_n \end{bmatrix} + \lambda V_n^{(1)} + V_n^{(0)} \right].
\]

(3.23)

Here, \( \Lambda_n \) is an auxiliary variable; \( V_n^{(1)} \) and \( V_n^{(0)} \) are \( \lambda \)-independent off-diagonal and diagonal matrices, respectively. Substituting (3.22a) and (3.23) into (2.9), we can express \( V_n^{(1)} \) and \( V_n^{(0)} \) in terms of \( u_n \) and \( \Lambda_n \). Thus, up to an overall factor, \( V_n \) can be written as

\[
V_n = I + h \left[ \tilde{u}_n \Lambda_n + \lambda \tilde{u}_n \Lambda_n \right].
\]

(3.24)

The zero-curvature condition (2.9) for the Lax pair (3.22a) and (3.24) is equivalent to the following system of partial difference equations (cf [5]):

\[
\begin{align*}
\frac{1}{h} (\tilde{u}_n - u_n) &= \Lambda_{n+1} u_n (\tilde{u}_{n+1} - u_{n+1}), \\
\tilde{u}_n \Lambda_n &= u_n \Lambda_{n+1}.
\end{align*}
\]

(3.25)

In view of (2.14), we impose the following boundary conditions for \( u_n \) and \( \Lambda_n \):

\[
\lim_{n \to -\infty} u_n = 1, \quad \lim_{n \to -\infty} \Lambda_n = 1 \quad \text{or} \quad \lim_{n \to +\infty} \Lambda_n = 1.
\]

(3.26)

The boundary values of \( u_n \) and \( \Lambda_n \) are normalized by scaling \( h \). Note that in the case \( h = -1 \), (3.25) has the trivial solution \( \tilde{u}_n = u_{n-1}, \Lambda_n = 1/u_{n-1} \). In the continuum limit of time \( h \to 0 \), we obtain \( \Lambda_n \to 1 \). Thus, in this limit, the time discretization (3.25) reduces to the Volterra lattice (3.21). The determinant of the \( 2 \times 2 \) Lax matrix \( L_n \) (3.22a) can be immediately computed as \( \det L_n = u_n \). Using the recurrence formula for \( \Lambda_n \) in (3.25), we can rewrite the Lax matrix \( V_n \) (3.24) in an ultralocal form with respect to the auxiliary variable \( \Lambda_n \). Thus, its \( 2 \times 2 \) determinant is computed as

\[
\det V_n(\lambda) = (1 + h\tilde{u}_n \Lambda_n)[1 + h(-\lambda^2 \Lambda_n + u_{n-1} \Lambda_n)] + h^2 \lambda^2 \tilde{u}_n \Lambda_n^2
\]

\[
= -\lambda^2 h \Lambda_n + (1 + h\tilde{u}_n \Lambda_n)(1 + h u_{n-1} \Lambda_n).
\]
Therefore, equality (2.16) combined with the boundary conditions (3.26) implies a quadratic equation in $\Lambda_n$
\[
(1 + h\bar{a}_n\Lambda_n)(1 + hu_{n-1}\Lambda_n) = (1 + h)^2\Lambda_n,
\]
or equivalently,
\[
h^2\bar{a}_n u_{n-1} \Lambda_n^2 - [(1 + h)^2 - h(\bar{a}_n + u_{n-1})] \Lambda_n + 1 = 0. \tag{3.27}
\]
The asymptotic behaviour (2.10) of the Lax matrix $V_n$ implies that the proper solution of (3.27) is given by
\[
(1 + h)^2\Lambda_n = \frac{2}{1 - \epsilon(\bar{a}_n + u_{n-1}) + \sqrt{[1 - \epsilon(\bar{a}_n + u_{n-1})]^2 - 4\epsilon^2\bar{a}_nu_{n-1}}}, \tag{3.28}
\]
where $\epsilon := h/(1 + h)^2$. Thus, if $h \in \mathbb{R}$, then $\epsilon \leq 1/4$. In this case, the local expression (3.28) is valid only if $-1 < h < 1$, which covers the range $\epsilon < 1/4$. The case $h = 1$, corresponding to $\epsilon = 1/4$, involves a subtle sign problem of the square root, which we do not discuss here. If $h^2 > 1$, (3.28) is inconsistent with the boundary conditions, and the other solution of (3.27) should be adopted. In any case, the boundary conditions for $u_n$ imply that $\lim_{n \to -\infty} \Lambda_n = \lim_{n \to +\infty} \Lambda_n = 1$.

Substituting (3.28) into the first equation in (3.25), we obtain a time discretization of the Volterra lattice (3.21) in the local form
\[
\bar{a}_n - u_n = \frac{2u_n(\bar{a}_{n+1} - u_{n-1})}{1 - \epsilon(\bar{a}_{n+1} + u_n) + \sqrt{[1 - \epsilon(\bar{a}_{n+1} + u_n)]^2 - 4\epsilon^2\bar{a}_nu_{n-1}}}, \tag{3.29}
\]
Thus, the value of $\bar{a}_n$ is uniquely determined by $u_n$, $\bar{a}_{n+1}$ and $u_{n-1}$. If $h \in \mathbb{R}$ and the $u_n$ are nonzero and real-valued at the initial time, then (3.25) implies that the auxiliary variable $\Lambda_n$ is always real-valued. Consequently, the discriminant of the quadratic equation (3.27) must be nonnegative. Thus, if $\epsilon < 1/4$, $u_n$ approaches 1 sufficiently smoothly and fast as $n \to \pm\infty$, and the $u_n$ are close to 1 at the initial time, then the real-valuedness of $u_n$ is preserved under the time evolution of the discrete-time Volterra lattice (3.29).

To express the backward time evolution explicitly, we only have to replace $\Lambda_{n+1}u_n$ in the first equation of (3.25) with $\Lambda_n\bar{a}_n$ and then substitute the local expression (3.28).

Ultradiscretization [12, 13]. We can derive an ultradiscrete analogue of the Volterra lattice. The derived ultradiscrete equation given in [14] involves one arbitrary parameter $L$. We conjecture that the ultradiscrete flows defined for distinct values of $L$ are commutative and comprise an ultradiscrete analogue of the KdV hierarchy (see [14] for details).

3.3. The modified Volterra lattice

The modified Volterra lattice
\[
q_{n,t} = (1 + q_n^2)(q_{n+1} - q_{n-1}), \tag{3.30}
\]
where $q_n$ is a scalar dependent variable, was introduced by Hirota [26] (also see [4, 27, 28]); it is obtained from the Ablowitz–Ladik system (3.1) through the reduction $c = d = -1$, $\alpha = \beta = 0$ and $r_n = -q_n$. The Lax pair for the continuous-time modified Volterra lattice (3.30) is given by
\[
L_n = \begin{bmatrix}
\lambda & q_n \\
-q_n & \frac{1}{\lambda}
\end{bmatrix}, \tag{3.31a}
\]
\[
M_n = \begin{bmatrix}
-\frac{1}{\lambda^2} + q_nq_{n-1} & \lambda q_n + \frac{1}{\lambda}q_{n-1} \\
-\lambda q_{n-1} - \frac{1}{\lambda}q_n & -\lambda^2 + q_nq_{n-1}
\end{bmatrix}. \tag{3.31b}
\]
Indeed, the substitution of (3.31) into the zero-curvature condition (2.4) results in (3.30).

In the discrete-time case, we consider the reduction $c_n = d_n = -\Lambda_n$, $\alpha = \beta = 0$ and $r_n = -q_n$ of the Lax matrix $V_n$ (3.4). Thus, we obtain

$$V_n = I + h\Lambda_n\left[\frac{q_n}{\lambda}q_{n+1} - \frac{1}{\lambda^2} \frac{\lambda q_n + \frac{1}{\lambda}q_{n-1}}{-\lambda q_{n-1} - \frac{1}{\lambda}q_n - \frac{\lambda^2}{2}q_{n-1}}\right],$$

(3.32)

where $\Lambda_n$ is the auxiliary variable. The zero-curvature condition (2.9) for the Lax pair (3.31a) and (3.32) amounts to the following system of partial difference equations:

$$\begin{align*}
\frac{1}{\lambda} (\tilde{q}_n - q_n) &= (1 + q_n^2)\Lambda_{n+1} (\tilde{q}_{n+1} - q_{n-1}), \\
(1 + \tilde{q}_n^2)\Lambda_n &= (1 + q_n^2)\Lambda_{n+1}.
\end{align*}$$

(3.33)

We assume the boundary conditions

$$\lim_{n \to \pm\infty} q_n = 0, \quad \lim_{n \to -\infty} \Lambda_n = 1 \quad \text{or} \quad \lim_{n \to +\infty} \Lambda_n = 1.$$  

(3.34)

The boundary value of $\Lambda_n$ is set as 1 without any loss of generality; it can be changed to any nonzero value by rescaling the step-size parameter $h$.

The determinant of the Lax matrix $L_n$ (3.31a) is given by $\det L_n = 1 + q_n^2$; the determinant of the Lax matrix $V_n$ (3.32) is computed as

$$\det V_n(\lambda) = -\left(\frac{\lambda^2}{1 - \lambda^2}\right) h\Lambda_n + h^2 (1 + \tilde{q}_n^2) (1 + q_n^2)\Lambda_n^2 + 2h\tilde{q}_n q_{n-1}\Lambda_n + 1.$$  

Thus, equality (2.16) combined with the boundary conditions (3.34) leads to the quadratic equation in $\Lambda_n$:

$$h^2 (1 + \tilde{q}_n^2) (1 + q_n^2)\Lambda_n^2 - (1 + h^2 - 2h\tilde{q}_n q_{n-1})\Lambda_n + 1 = 0.$$  

(3.35)

By recalling the prescribed asymptotic behaviour (2.10) of the Lax matrix $V_n$ for small $h$, we obtain the proper solution of (3.35) as

$$\Lambda_n = \frac{2}{1 + h^2 - 2h\tilde{q}_n q_{n-1} + \sqrt{(1 - h^2)^2 - 4h (\tilde{q}_n + hq_{n-1}) (h\tilde{q}_n + q_{n-1})}}.$$  

(3.36)

When $h \in \mathbb{R}$, the local expression (3.36) is valid only if $-1 \leq h \leq 1$. If $h^2 > 1$, the other solution of (3.35) should be used. In any case, the decaying boundary conditions for $q_n$ imply that $\lim_{n \to -\infty} \Lambda_n = \lim_{n \to +\infty} \Lambda_n = 1$.

Substituting (3.36) into the first equation in (3.33), we obtain a time discretization of the modified Volterra lattice (3.30) in the local form:

$$\frac{1}{h} (\tilde{q}_{n+1} - q_n) = \frac{2(1 + q_n^2) (\tilde{q}_{n+1} - q_{n-1})}{1 + h^2 - 2h\tilde{q}_{n+1} q_n + \sqrt{(1 - h^2)^2 - 4h (\tilde{q}_{n+1} + hq_n) (h\tilde{q}_{n+1} + q_n)}}.$$  

(3.37)

Note that the right-hand side of (3.37) does not involve $\tilde{q}_n$. If $-1 \leq h \leq 1$, the real-valuedness of $q_n$ is preserved under the discrete-time evolution (3.37) (cf. (3.33)); the discriminant in the square root is nonnegative as long as we start with sufficiently small real-valued $q_n$ at the initial time. To express the backward time evolution explicitly, we only have to replace $(1 + q_n^2)\Lambda_n$ in the first equation of (3.33) with $(1 + \tilde{q}_n^2)\Lambda_n$ and then substitute the local expression (3.36).

At the ‘threshold’ values of $h$, $h = \pm 1$, we can extract the square root in (3.37) to obtain a rational mapping. Note that in the cases $h = \pm 1$, the discrete-time system (3.33) under the boundary conditions (3.34) has the trivial solution $\tilde{q}_n = \mp q_{n-1}$, $\Lambda_n = 1/(1 + q_n^2)$ (cf section 4.6 in [1]). Thus, the discriminant of the quadratic equation (3.35) vanishes at
\( h = \pm 1 \), and the two solutions intersect. To obtain a nontrivial mapping from this observation, we set \( h = +1 \) in (3.33) and replace \( q_n \) with \( i w_n \), namely

\[
\begin{align*}
\tilde{w}_n - w_n &= (1 - w_n^2) \Lambda_{n+1} (\tilde{w}_{n+1} - w_{n-1}), \\
(1 - \tilde{w}_n^2) \Lambda_n &= (1 - w_n^2) \Lambda_{n+1}.
\end{align*}
\]

(3.38)

Moreover, we generalize the boundary conditions (3.34) as

\[
\lim_{n \to \pm \infty} w_n = \gamma,
\]

\[
\lim_{n \to -\infty} \Lambda_n = \frac{1}{(1 + \gamma)^2}, \quad \lim_{n \to +\infty} \Lambda_n = \frac{1}{(1 + \gamma)^2}, \quad \gamma \neq -1,
\]

and assume that \( |w_n - \gamma| \) is sufficiently small. Thus, the additional condition \( \gamma \neq 0 \) excludes the trivial time evolution \( \tilde{w}_n = -w_{n-1} \). Following the same procedure as above, we obtain the quadratic equation in \( \Lambda_n \) that can be factorized as

\[
[(1 + \tilde{w}_n) (1 + w_{n+1}) \Lambda_n - 1] [(1 - \tilde{w}_n) (1 - w_{n-1}) \Lambda_n - 1] = 0.
\]

Substituting its proper solution \( \Lambda_n = 1/[(1 + \tilde{w}_n) (1 + w_{n+1})] \) into (3.38), we obtain a single equation (\( \tilde{w}_n - 1) (\tilde{w}_{n+1} + 1) = (w_n - 1) (w_{n+1} - 1) \)). By a trivial change of the dependent variable \( w_n := 1 + 2 \nu y_n (\nu \neq 0) \), we obtain the well-known ‘discrete Volterra equation’ \([18, 29]\)

\[
\tilde{y}_n (1 + \nu \tilde{y}_{n+1}) = y_n (1 + \nu y_{n+1}).
\]

(3.39)

Therefore, the discrete-time equation (3.39) belongs to the modified Volterra hierarchy and not the original Volterra hierarchy [1]; this corresponds to the special case where the quadratic equation for the auxiliary variable \( \Lambda_n \) is factorized to provide a rational solution.

**Ultradiscretization** [12, 13]. We can derive an ultradiscrete analogue of the modified Volterra lattice given by [14]

\[
\tilde{Q}_n = Q_n + \max \left( 0, Q_n + Q_{n-1} - L, \frac{\tilde{Q}_{n+1} + Q_n - L}{2} \right) - \max(0, \tilde{Q}_{n+1} + Q_n - L).
\]

(3.40)

Here, \( L \) is a positive parameter and the boundary conditions \( \lim_{n \to \pm \infty} Q_n = 0 \) are assumed. It is hoped that the issue of the *commutativity* of the ultradiscrete flows defined by (3.40) for distinct values of \( L \) will be investigated. In this regard, we expect that (3.40) and its time reversal will comprise an ultradiscrete analogue of the modified KdV hierarchy, wherein the parameter \( L \) labels each flow in the hierarchy. Fortunately, for the specific examples that we considered, (3.40) allows the ‘stable’ propagation of solitons and their elastic (but nontrivial) collisions; thus, (at least some of) the integrability properties appear to be retained in this ultradiscretization.

### 3.4. The lattice Heisenberg ferromagnet model

The lattice Heisenberg ferromagnet model was proposed in 1982 by several different authors [21, 30–32]. The equation of motion for this semi-discrete model can be derived almost systematically using either Ishimori’s approach [31] based on a gauge transformation from the Ablowitz–Ladik lattice or the \( r \)-matrix formalism based on the fundamental Poisson bracket relation [21, 30]. However, for a concise and easy-to-understand derivation, we use a more heuristic approach based on the zero-curvature representation.

We start with the Lax pair of the following form in the semi-discrete case:

\[
L_n = I + \lambda S_n,
\]

(3.41a)

\[
M_n = \frac{\lambda}{1 - \lambda^2} A_n L_n,
\]

(3.41b)
where the conditions \((S_n)^2 = I\) and \(A_n S_n = S_{n-1} A_n\) are assumed. The latter condition guarantees the useful relation \(A_n L_n = L_{n-1} A_n\). Thus, substituting the Lax pair (3.41) into the zero-curvature condition (2.4), we obtain

\[
(L_n^{-1})_t + \frac{\lambda}{1 - \lambda^2} (A_{n+1} - A_n) = 0.
\]

Noting the identity \((I + \lambda S_n)(I - \lambda S_n) = (1 - \lambda^2)I\), this results in

\[
S_{n,t} = A_{n+1} - A_n.
\]

Because \((S_n)^2 = I\) and \(A_n S_n = S_{n-1} A_n\), (3.42) implies the relation

\[
A_{n+1}(S_n + S_{n-1}) = (S_n + S_{n-1}) A_n.
\]

Thus, we choose \(A_n\) as

\[
A_n = 2iaS_n(s_n + S_{n-1})^{-1} + 2b(S_n + S_{n-1})^{-1}
\]

so that the above relation is automatically satisfied. Here, \(a\) and \(b\) are free parameters.

Substituting this expression for \(A_n\) into (3.42), we obtain

\[
S_{n,t} = \Delta_n^a [2iaS_n(s_n + S_{n-1})^{-1} + 2b(S_n + S_{n-1})^{-1}]
\]

where \(\Delta_n^a\) is the forward difference operator in the discrete space (cf (2.7)). Note that the evolution equation (3.43) is consistent with the condition \((S_n)^2 = I\) for a general \(l \times l\) matrix \(S_n\). We now consider the simplest nontrivial case of \(l = 2\) and express \(S_n\) in terms of the Pauli matrices as

\[
S_n = \sum_{j=1}^{3} S_n^{(j)} \sigma_j = \begin{bmatrix} S_n^{(1)} & iS_n^{(2)} \\ -iS_n^{(2)} & S_n^{(3)} \end{bmatrix}.
\]

Here, \(S_n := (S_n^{(1)}, S_n^{(2)}, S_n^{(3)})\) is a unit vector, i.e. \((S_n, S_n) = 1\). Because \(2(S_n + S_{n-1})^{-1} = (S_n + S_{n-1})/(1 + (S_n, S_{n-1}))\) [31], (3.43) reduces to a single vector equation involving both the scalar product and the vector product:

\[
S_{n,t} = \Delta_n^a \begin{bmatrix} a \frac{S_n \times S_{n-1}}{1 + (S_n, S_{n-1})} + b \frac{S_n + S_{n-1}}{1 + (S_n, S_{n-1})} \end{bmatrix}, \quad (S_n, S_n) = 1.
\]

The case \(b = 0\) gives the lattice Heisenberg ferromagnet model [31, 32], while the case \(a = 0\) corresponds to its simplest higher symmetry [33].

Let us examine the discrete-time case. We mainly consider the time discretization of the reduced system (3.45), and not the general matrix system (3.43), because the latter problem is expected to be too complicated. We start with the Lax matrix \(V_n\) of the following form:

\[
V_n = I + \hbar \frac{\lambda}{1 - \lambda^2} A_n L_n,
\]

where the conditions \((S_n)^2 = I\) and \(A_n S_n = \tilde{S}_{n-1} A_n\) are assumed. The latter condition guarantees the useful relation \(A_n L_n = L_{n-1} A_n\). Thus, substituting the Lax pair, (3.41a) and (3.46), into the zero-curvature condition (2.9), we obtain

\[
\frac{1}{\hbar} (\tilde{L}^{-1}_n - L^{-1}_n) + \frac{\lambda}{1 - \lambda^2} (A_{n+1} - A_n) = 0
\]

or equivalently,

\[
\frac{1}{\hbar} (\tilde{S} - S_n) = A_{n+1} - A_n.
\]
Under the condition \((S_n)^2 = I\), the relation \(A_n S_n = \tilde{S}_{n-1} A_n\) is automatically satisfied if \(A_n\) takes the general form \(A_n = B_n + \tilde{S}_{n-1}^{-1} B_n\). However, in analogy with the semi-discrete case, we employ a more specific form of \(A_n\):

\[
A_n = 2i a_n \tilde{S}_{n-1} (S_n + \tilde{S}_{n-1})^{-1} + 2b_n (S_n + \tilde{S}_{n-1})^{-1},
\]

which also satisfies the relation \(A_n S_n = \tilde{S}_{n-1} A_n\). Here, the scalar unknowns \(a_n\) and \(b_n\) are auxiliary variables. Substituting this expression for \(A_n\) into (3.47), we obtain

\[
\frac{1}{\hbar} (\tilde{S}_n - S_n) = \Delta_n \left[ 2i a_n \tilde{S}_{n-1} (S_n + \tilde{S}_{n-1})^{-1} + 2b_n (S_n + \tilde{S}_{n-1})^{-1} \right].
\]

It only remains necessary to fix the auxiliary variables \(a_n\) and \(b_n\). To this end, we recall that the time evolution determined by (3.48) has to be consistent with the condition \((S_n)^2 = I\). In the following, we focus on the case of the \(2 \times 2\) matrix \(S_n\) given by (3.44). The requirement \(\text{tr} S_n = 0\) results in the \(n\)-independence of \(a_n\); thus, we set \(a_n = a\). Therefore, (3.48) reduces to

\[
\frac{1}{\hbar} (\tilde{S}_n - S_n) = \Delta_n \left[ a \frac{S_n \times \tilde{S}_{n-1}}{1 + (S_n, \tilde{S}_{n-1})} + b_n \frac{S_n + \tilde{S}_{n-1}}{1 + (S_n, \tilde{S}_{n-1})} \right].
\]

To fix \(b_n\), we invoke the procedure presented in section 2; we assume the boundary conditions

\[
\lim_{n \to -\infty} (S_n, \tilde{S}_{n-1}) = 1, \quad \lim_{n \to -\infty} b_n = b.
\]

Because \((S_n, S_n) = 1\), we have \(\det L_n = 1 - \lambda^2\) (cf (3.41a) and (3.44)). Thus, the equality (2.17) derived from (2.16) implies that the determinant of \(V_n\):

\[
\det V_n = \det \left( I + \hbar \frac{\lambda}{1 - \lambda^2} A_n L_n \right)
\]

\[
= \det \left( I - \lambda S_n + \hbar A_n \right) \det \left( \frac{1}{1 - \lambda^2} L_n \right),
\]

is an \(n\)-independent quantity. Therefore, both the trace and the determinant of \(S_n - \hbar A_n\) must be \(n\)-independent. The \(n\)-independence of \(\text{tr} (S_n - \hbar A_n)\) is already satisfied by setting \(a_n = a\). The determinant of \(S_n - \hbar A_n\) can be computed as

\[
\det(S_n - \hbar A_n) = \det(S_n - 2iha \tilde{S}_{n-1} (S_n + \tilde{S}_{n-1})^{-1} - 2hb_n (S_n + \tilde{S}_{n-1})^{-1})
\]

\[
= \det((1 - 2hb_n)I + S_n \tilde{S}_{n-1} - 2iha \tilde{S}_{n-1})
\]

\[
= \frac{\det(S_n + \tilde{S}_{n-1})}{(1 + (S_n, \tilde{S}_{n-1}) - 2hb_n)^2 + 1 + 4(ha)^2 - (S_n, \tilde{S}_{n-1})^2}
\]

\[
= -2(1 + (S_n, \tilde{S}_{n-1}))
\]

which coincides with its boundary value determined by (3.50). This results in a quadratic equation in \(hb_n\):

\[
2(hb_n)^2 - 2(1 + (S_n, \tilde{S}_{n-1}))hb_n + 2(ha)^2 + [hb(2 - hb) - (ha)^2](1 + (S_n, \tilde{S}_{n-1})) = 0.
\]

For sufficiently small \(h\), the proper solution of this equation is given by

\[
hb_n = \frac{2(ha)^2 + [hb(2 - hb) - (ha)^2](1 + g_n)}{1 + g_n + \sqrt{(1 + g_n)^2 - 4(ha)^2 - 2[hb(2 - hb) - (ha)^2](1 + g_n)}}
\]

where \(g_n := (S_n, \tilde{S}_{n-1})\). Substituting this local expression for \(hb_n\) into (3.49), we obtain a time discretization of (3.45); this time discretization is essentially equivalent to (B.18) in [16]
We write the equation of motion for three important cases: the case $b = 0$, 

$$
\tilde{S}_n - S_n = \Delta^+_n \left\{ \frac{\hbar}{1 + \langle S_n, S_{n-1} \rangle} S_n \times \tilde{S}_{n-1} \right\} + \left[ 1 - \sqrt{1 - \left( \frac{(\hbar a)^2}{1 + \langle S_n, S_{n-1} \rangle} \right)^2} \right] \left( \frac{S_n + \tilde{S}_{n-1}}{2} \right), \quad \langle S_n, S_n \rangle = 1;
$$

the case $h b (2 - h b) = (\hbar a)^2$, 

$$
\tilde{S}_n - S_n = \Delta^+_n \left\{ \frac{\hbar}{1 + \langle S_n, S_{n-1} \rangle} S_n \times \tilde{S}_{n-1} \right\} + \left[ 1 - \sqrt{1 - \frac{4(\hbar a)^2}{(1 + \langle S_n, S_{n-1} \rangle)^2}} \right] \left( \frac{S_n + \tilde{S}_{n-1}}{2} \right), \quad \langle S_n, S_n \rangle = 1;
$$

and the case $a = 0$, 

$$
\frac{1}{\delta} (\tilde{S}_n - S_n) = \Delta^+_n \left[ \frac{S_n + \tilde{S}_{n-1}}{1 + \langle S_n, \tilde{S}_{n-1} \rangle + \sqrt{(1 + \langle S_n, \tilde{S}_{n-1} \rangle)^2 - 2\delta (1 + \langle S_n, \tilde{S}_{n-1} \rangle)}} \right], \quad \langle S_n, S_n \rangle = 1.
$$

The first and second cases provide time discretizations of the lattice Heisenberg ferromagnet model (3.45) with $b = 0$, while the third case gives a discrete-time analogue of its simplest higher symmetry (3.45) with $a = 0$. Note that the first case $b = 0$ can be identified with (6.8) in [16].

4. Concluding remarks

In this paper, we have developed an effective method for obtaining proper time discretizations of integrable lattice systems in $1 + 1$ dimensions. This method, which is based on the zero-curvature condition (2.9), allows us to obtain local equations of motion that can determine the time evolution uniquely. Using this method, we constructed new time discretizations of the Toda lattice, the Ablowitz–Ladik lattice, the Volterra lattice and the modified Volterra lattice, while we obtained the same time discretizations of the lattice Heisenberg ferromagnet model and its symmetry as those in [16–18]. It should be stressed that this is a systematic method and also applies to other integrable lattice systems. As a bonus, we were able to derive ultradiscrete analogues of the Volterra lattice (see [14]) and the modified Volterra lattice involving one arbitrary parameter $L$. Each of these ultradiscrete equations, as well as its time reversal, appears to form a hierarchy of mutually commuting flows labelled by the parameter $L$.

A notable feature of our method is that the time discretization does not modify the integrable hierarchy to which the original lattice system belongs. Thus, the integrals of motion and the functional form of the solutions remain invariant; only the time dependence of certain parameters corresponding to the angle variables in the solutions is changed (cf [2, 3, 19]). Such a time discretization can be identified with the spatial part of an auto-Bäcklund transformation of the continuous-time lattice system or, from a more unified point of view, an auto-Bäcklund transformation of the whole hierarchy of commuting flows (cf [22, 34–36]). Therefore, any time discretization obtained by our method preserves the
qualitative properties of the original integrable hierarchy and thus is expected to serve as an excellent scheme for numerical integration of the continuous-time lattice system. This is in contrast with other known methods that generally modify the integrable hierarchy to find a time discretization in the local form.

In our approach, a local-in-space time discretization is always derived from a global-in-space time discretization involving ‘nonlocal’ auxiliary variables. The obtained local equations of motion generally determine the forward/backward time evolution uniquely under the specified boundary conditions; thus, they indeed give a discrete-time analogue of evolutionary differential-difference equations. One might consider that the original nonlocal time discretization no longer has any use once the local discretization has been derived from it; however, this is often not the case. In actually solving an initial-value problem, the nonlocal time discretizations often provide crucial information on the attributes of the solution, such as the real-valuedness, positivity and rationality with respect to the initial data and the parameters; it appears that the local discretizations are not useful for establishing such properties directly.

Let us illustrate this point with one instructive example, namely the discrete-time Ablowitz–Ladik lattice studied in subsection 3.1. The global scheme (3.5) (cf (3.7) and (3.9)) is free from irrational functions; consequently, under the decaying boundary conditions (cf (3.6)), the discrete-time updates of the dependent variables are given by a rational mapping. On the other hand, the local scheme (3.15) with (3.13) involves a square root; thus, it is extremely difficult to foresee that the time evolution can be described by a rational mapping. However, these two schemes naturally define the same time evolution. Therefore, the quantity inside the square root in (3.13) is always equal to the square of a rational function of the dependent variables at the previous moment and the parameters; typically, one encounters the form

$$\sqrt{(1-\hat{c}\hat{d})^2(1+f_n)^2},$$

where $\hat{c}$ and $\hat{d}$ depend on the step-size parameter $h$ through (3.11). The rational function $f_n$ involves the dependent variables and the parameters appearing in (3.15) and is of the order $O(h)$. In addition, it vanishes when the dependent variables are identically zero. For sufficiently small $h$, this square root is extracted as $(1-\hat{c}\hat{d})(1+f_n)$. However, for relatively large $|h|$, this is not obvious; even when $\hat{c}\hat{d} < 1$, the values of the dependent variables have to satisfy rather restrictive conditions. One simple way to bypass this sign problem is to consider the square root as being defined by its Taylor expansion for small $h$ or small amplitudes of the dependent variables and then to extend the domain of the definition by analytic continuation; this is briefly explained in subsection 3.1. Thus, this example also illustrates the difficulty of the sign problem in computing updates of the dependent variables using the local equations of motion.

In this paper, we mainly considered the constant boundary conditions at spatial infinity and assumed simple boundary values of the dependent variables. Note, however, that our method is not sensitive to the boundary values of the dependent variables and is also applicable to other boundary conditions, including periodic, nonvanishing or nonconstant boundary conditions. Indeed, as is clear from the description in subsection 2.4, one can freely modify the boundary condition for $V_n$ (cf (2.14)) as long as it determines a definite value for the right-hand side of (2.20) or (2.21). Alternatively, one can first set each value of (2.20) or (2.21) as a constant free parameter and then elicit the corresponding boundary conditions for the dependent variables. We can also understand in a more intuitive way that the time discretizations derived in this paper are integrable under other suitable boundary conditions; note that the conservation laws should be derived from the local equations of motion using only local operations, i.e. without referring to the boundary conditions.
Very recently, Adler, Bobenko and Suris [37, 38] successfully classified discrete integrable systems on quad-graphs using the notion of three-dimensional consistency [39]. In particular, under some assumptions, they presented a short but complete list of one-field integrable equations defined by polynomial relations of degree 1 in each of the four arguments. Their results have been attracting a lot of interest from researchers. The time discretizations of polynomial lattice systems obtained by our method generally contain irrational nonlinearity; the nonlinear terms are determined through the proper solution of an ‘ultralocal’ algebraic equation of degree higher than 1. Thus, such time discretizations essentially lie outside the class considered by Adler et al [37, 38], although their three-dimensional consistency can, in principle, be investigated using the approach outlined in subsection 2.5 of [14]. This probably partially explains why their list of one-field integrable equations is short.

Another interesting topic closely related to the three-dimensional consistency is nonautonomous extensions of fully discrete systems involving arbitrary functions of the independent variables. To save space and enhance readability, we were unable to discuss the topic in this paper. Interested readers are again referred to [14], wherein we propose an intriguing procedure for constructing such nonautonomous extensions.

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