Explicit solutions of the kinetic and potential matching conditions of the energy shaping method

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Abstract

In this paper we present a procedure to integrate, up to quadratures, the matching conditions of the energy shaping method. We do that in the context of underactuated Hamiltonian systems defined by simple Hamiltonian functions. For such systems, the matching conditions split into two decoupled subsets of equations: the kinetic and potential equations. First, assuming that a solution of the kinetic equation is given, we find integrability and positivity conditions for the potential equation (because positive-definite solutions are the interesting ones), and we find an explicit solution of the latter. Then, in the case of systems with one degree of underactuation, we find in addition a concrete formula for the general solution of the kinetic equation. An example is included to illustrate our results.

1 Introduction

The energy shaping method is a technique for achieving (asymptotic) stabilization of underactuated Lagrangian and Hamiltonian systems. See Ref. [10] for a review on the subject and [9,19] for more recent works. In this paper, we shall concentrate on Hamiltonian systems only. We shall represent underactuated Hamiltonian systems by pairs \((H,W)\), where \(H : T^*Q \rightarrow \mathbb{R}\) is a Hamiltonian
function on a finite-dimensional smooth manifold $Q$, and $W$ is a (proper) sub-bundle of the vertical bundle of $T^*Q$, containing the actuation directions.

The energy shaping method is based on the idea of feedback equivalence \cite{10], and its purpose is to construct, for a given pair $(H, W)$, a state feedback controller and a Lyapunov function $\hat{H} : T^*Q \to \mathbb{R}$ for the resulting closed-loop system. To do that, a set of partial differential equations (PDEs), known as matching conditions, must be solved. Such PDEs have the pair $(H, W)$ as datum and the aforementioned Lyapunov function $\hat{H}$ as its unknown.

In the Chang version of the method \cite{6, 7, 8, 9}, on which we shall focus, only simple functions $H$ and $\hat{H}$ are considered (i.e. functions with the kinetic plus potential energy form), and the subbundle $W$ is assumed to be the vertical lift of a subbundle $W \subseteq T^*Q$. In such a case, the matching conditions decompose into two subsets: the kinetic and potential matching conditions, or simply, the kinetic and potential equations. In canonical coordinates $(q, p)$, if we write

\begin{equation}
H(q, p) = \frac{1}{2} p_i \mathbb{H}^{ij}(q) p_j + h(q)
\end{equation}

and

\begin{equation}
\hat{H}(q, p) = \frac{1}{2} p_i \mathbb{\hat{H}}^{ij}(q) p_j + \hat{h}(q),
\end{equation}

such equations read [see Ref. \cite{13], Eqs. (56) and (57)]

\begin{equation}
\left( \frac{\partial \mathbb{\hat{H}}^{ij}(q)}{\partial q^k} \mathbb{\hat{H}}^{kl}(q) - \frac{\partial \mathbb{H}^{ij}(q)}{\partial q^k} \mathbb{\hat{H}}^{kl}(q) \right) p_i p_j p_l = 0
\end{equation}

(the kinetic equation) and

\begin{equation}
\left( \frac{\partial \hat{h}(q)}{\partial q^k} \mathbb{\hat{H}}^{kl}(q) - \frac{\partial h(q)}{\partial q^k} \mathbb{\hat{H}}^{kl}(q) \right) p_l = 0
\end{equation}

(the potential equation), and they must hold for all $p \in W^\perp_q$ (the orthogonal must be calculated with respect to the metric defined by $\mathbb{\hat{H}}$). Once a solution $(\hat{h}, h)$ of these equations is found, the method gives a prescription to construct a state feedback controller. So, we can say that the core of the method consists of solving above PDEs.

Let us mention that one looks for a solution $\hat{h}$ of the potential equation which is positive-definite around some critical point of $h$. This insures that the function $\hat{H}$ is a Lyapunov function for the resulting closed-loop system and the given critical point.

In Reference \cite{16], necessary and sufficient conditions were given for the existence of a solution $\hat{h}$ of the potential equation, once a solution $\mathbb{\hat{H}}$ of the kinetic equation is given (note that $\mathbb{\hat{H}}$ can be seen as a datum for the potential equation). This was done within the framework of the Goldschmidt’s integrability theory for linear partial differential equations \cite{12], which only works in the analytic
category. Nevertheless, no conditions have been presented in order to ensure the existence of positive-definite solutions. Also, no general recipe to construct an explicit solution \( \hat{h} \) has been developed. The first goal of the present paper is three-fold:

- to extend the results of [16] to the \( C^\infty \) category,
- to include positivity conditions,
- and to present a systematic procedure to integrate the potential equation up to quadratures.

Regarding the kinetic equation, few general results are known about the existence of solutions. In the case of underactuated systems with one degree of underactuation, the problem was completely solved in References [6, 7, 14, 11]. However, a general prescription for finding explicit solutions is still lacking. The second goal of this paper is, for one degree of underactuation, to give such a prescription.

The paper is organized as follows. In Section §2 we re-define the unknown \( (\hat{H}, \hat{h}) \) of the matching conditions. This gives rise to a new set of equations, in terms of which we shall study a particular subclass of solutions of the kinetic equation. Given \( \hat{H} \) inside the mentioned subclass, in Section §3 we find sufficient conditions for the existence of solutions \( \hat{h} \) of the potential equation. Also, sufficient conditions to ensure positive-definiteness of \( \hat{h} \) are given. All of these conditions together give rise to a procedure that enable us to find an explicit expression for local solutions of the potential equation (up to quadratures). Finally, we devote Section §4 to extend the procedure to integrate up to quadratures the kinetic equation also, but in the particular subclass of underactuated Hamiltonian systems with only one degree of underactuation. To conclude, we apply such a procedure to the inverted double pendulum.

We assume that the reader is familiar with basic concepts of Differential Geometry [1, 15, 18], Hamiltonian systems in the context of Geometric Mechanics [1, 2, 17] and Control Theory in a geometric language [3, 5].

**Basic notation and definitions.** Along all of the paper, \( Q \) will denote a smooth connected manifold of dimension \( n \) and \( (TQ, \tau, Q) \) and \( (T^*Q, \pi, Q) \) the tangent bundle and its dual bundle, respectively. As it is customary, we denote by \( \langle \cdot, \cdot \rangle \) the natural pairing between \( T_qQ \) and \( T^*_qQ \) at every \( q \in Q \) and by \( \mathfrak{X}(Q) \) and \( \Omega^1(Q) \) the sheaves of sections of \( \tau \) and \( \pi \), respectively. If \( F: Q \to P \) is a smooth function between differentiable manifolds, we denote by \( F^* \) and \( F^\ast \) the push-forward map and its transpose, respectively.

Consider a local chart \( (U, \varphi) \) of \( Q \), with \( \varphi: U \to \mathbb{R}^n \). Given \( q \in U \), we write \( \varphi(q) = (q^1, \ldots, q^n) = q \). For the induced local chart \( (T^*U, (\varphi^*)^{-1}) \) on \( T^*Q \), i.e. the canonical coordinates of the cotangent bundle, we write, for all \( \alpha \in T^*U \),

\[
(\varphi^*)^{-1}(\alpha) = (q^1, \ldots, q^n, p_1, \ldots, p_n) = (q, p) ,
\]
or simply
\[ (\varphi^*_q)^{-1} (\alpha) = p. \]

By a quadratic form on a subbundle \( V \subseteq T^*Q \) we shall understand a function \( \mathcal{H} : V \to \mathbb{R} \) given by the formula
\[ \mathcal{H}(\alpha) = \frac{1}{2} \langle \alpha, \rho^\sharp(\alpha) \rangle, \quad \forall \alpha \in V, \tag{5} \]
where \( \rho \) is a fibered inner product on \( V^* \), \( \rho^\sharp : V^* \to V \) is given by
\[ \langle \rho^\sharp(u), v \rangle = \rho(u, v), \]
and \( \rho^\sharp : V \to V^* \) is the inverse of \( \rho^\flat \). Given a subbundle \( W \subseteq V \), by \( W^\perp \) we shall denote the orthogonal complement w.r.t. \( \rho \) and by \( W^\sharp \) the subbundle \( \rho^\sharp(W) \subseteq V^* \). It is easy to see that
\[ W^\perp = \rho^\sharp(W^\circ) = (W^\circ)^\sharp. \tag{6} \]
Here, by \( W^\circ \subseteq V^* \) we are denoting the annihilator of \( W \). We shall also say that \( W^\perp \) is the orthogonal complement of \( W \) with respect to \( \mathcal{H} \).

The vertical subbundle associated to any vector bundle is given by the kernel of the push-forward of the bundle projection, and every element (resp. smooth section) of this subbundle is called vertical (resp. vertical vector field). In particular, for the cotangent bundle, it is given by \( \ker \pi^* \subseteq T^TQ \). If \( \alpha \in T^*_qQ \), there is a canonical way to identify \( \ker \pi^* \alpha \) and \( T^*_qQ \). This can be done through the vertical lift map \( \text{vlift}^\pi : T^*_qQ \to \ker \pi^* \alpha \), defined by
\[ \text{vlift}^\pi_\alpha(\beta) = \left. \frac{d}{dt} \alpha(t\beta) \right|_{t=0}. \tag{7} \]
This map is in fact an isomorphism of vector spaces for every \( \alpha \in T^*Q \).

Given a smooth function \( f : T^*Q \to \mathbb{R} \), we define the fiber derivative of \( f \) as the vector bundle morphism \( \mathbb{F} f : T^*Q \to TQ \) such that, for every \( \alpha, \beta \in T^*_qQ \),
\[ \langle \beta, \mathbb{F} f(\alpha) \rangle = \left. \frac{d}{dt} f(\alpha + t\beta) \right|_{t=0} = \langle df(\alpha), \text{vlift}^\pi_\alpha(\beta) \rangle. \tag{8} \]
In canonical coordinates,
\[ \langle dq_i^\sharp, \mathbb{F} f(q^\flat dq_i^\flat) \rangle = \left. \frac{\partial f \circ (\varphi^*)^{-1}}{\partial p_i} \right|_{q, p}(q, p). \tag{9} \]
For a quadratic form \( \mathcal{H} : T^*Q \to \mathbb{R} \) [see Eq. (5)], it can be shown that
\[ \mathbb{F} \mathcal{H} = \rho^\sharp, \tag{10} \]
and consequently
\[ \mathcal{H}(\alpha) = \frac{1}{2} \langle \alpha, \mathbb{F} \mathcal{H}(\alpha) \rangle, \quad \forall \alpha \in T^*Q. \tag{11} \]
Denote by $\omega$ the canonical symplectic 2-form on $T^*Q$. Given two functions $f, g : T^*Q \to \mathbb{R}$, its canonical Poisson bracket is the function
\[
\{f, g\} := \langle d f, \omega^\#(d g) \rangle,
\]where $\omega^\#$ is the inverse of $\omega^\flat : TQ \to T^*Q$ and the latter is given by the equation
\[
\langle \omega^\flat(u), v \rangle = \omega(u, v).
\]
In canonical coordinates, omitting $(\varphi^*)^{-1}$ for simplicity,
\[
\{f, g\}(q, p) = \left( \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p_i} \right)(q, p).
\]Throughout all of the paper, we will use the following convention for indices
\[
\begin{cases}
\text{latin indices} & i, j, k, l = 1, \ldots, n; \\
\text{latin indices} & a, b, c, d = 1, \ldots, m; \\
\text{greek indices} & \mu, \nu, \sigma, \rho = 1, \ldots, n - m.
\end{cases}
\]

2 Re-writing the matching conditions

Suppose that we have an underactuated Hamiltonian system $(H, W)$ with
\[
H = \hat{H} + h \circ \pi,
\]where $\hat{H} : T^*Q \to \mathbb{R}$ is a quadratic form [see Eq. (5)] and $h : Q \to \mathbb{R}$ is an arbitrary smooth function. [In canonical coordinates, this means that $H$ is given by Eq. (1)]. In other words, we are assuming that $H$ is simple. Note that $\mathcal{F}H = \mathcal{F}\hat{H} = \rho^\circ$ [see Eqs. (8) and (10)]. Suppose also\footnote{This will be the general setting and notation from now on.} that there exists a subbundle $W$ of $T^*Q$ of rank $m$ such that [see (7)]
\[
\mathcal{W}_\alpha = \text{vlift}_\alpha^\circ(W_{\pi(\alpha)}), \quad \forall \alpha \in T^*Q.
\]

Remark 1. Note that $(H, W)$ can be described by the triple $(\hat{H}, h, W)$.

In such a case, according to Ref. [13], the matching conditions of the Chang’s version of the energy shaping method, for a simple unknown $\hat{H} = \hat{H} + h \circ \pi$ [see Eq. (2) for a local expression], are given by
\[
\left\{ \hat{H}, \hat{H} \right\} \circ \mathcal{F}\hat{H}^{-1}(v) = 0, \quad \forall v \in W^\circ,
\]the kinetic equation, and
\[
\left( dh \circ \mathcal{F}\hat{H} - dh \circ \mathcal{F}\hat{H} \right) \circ \mathcal{F}\hat{H}^{-1}(v) = 0, \quad \forall v \in W^\circ,
\]the potential equation [see Eqs. (73) and (74) and Remark 18 of Ref. [13]].
Remark 2. In order to compare (16) and (17) with Eqs. (73) and (74) of Ref. [13], we must use that \( F^\hat{H}(W^\circ) = W^\perp \) [see Eqs. (6) and (10)].

Here \( \{\cdot, \cdot\} \) denotes the canonical Poisson bracket on \( T^*Q \) [see Eq. (12)]. Equations above are the intrinsic counterpart of Eqs. (3) and (4). Their data are given by the triple \((\hat{H}, \hat{h}, W)\), and their unknown by the pair \((\hat{H}, \hat{h})\). In the following, we are going to re-write (16) and (17) by redefining the unknown.

2.1 Intrinsic version

Lemma 1. Given a subbundle \( W \subseteq T^*Q \), the set of quadratic forms on \( T^*Q \) are in bijection with the triples \((\hat{W}, \hat{R}, \hat{L})\), where \( \hat{W} \) is a complement of \( W \), \( \hat{R} \) is a quadratic form on \( \hat{W} \), and \( \hat{L} \) is a quadratic form on \( W \).

Proof. To any quadratic form \( \hat{H} : T^*Q \to \mathbb{R} \), we can assign a triple \((\hat{W}, \hat{R}, \hat{L})\) with
\[
\hat{W} := W^\perp, \quad \hat{R} := \hat{H}\big|_{\hat{W}} \quad \text{and} \quad \hat{L} := \hat{H}\big|_W.
\]
(18)

Here, \( W^\perp \) means the orthogonal of \( W \) w.r.t. \( \hat{H} \). Reciprocally, to any triple \((\hat{W}, \hat{R}, \hat{L})\) as described in the lemma, we can assign the quadratic form
\[
\hat{H} := \hat{R} \circ \hat{p} + \hat{L} \circ p,
\]
(19)

where
\[
\hat{p} : T^*Q \to \hat{W} \quad \text{and} \quad p : T^*Q \to W
\]
(20)

are the maps defined by Eq. (18) is the inverse of the map given by (19). \( \square \)

Remark 3. If \( \hat{H} \) and \((\hat{W}, \hat{R}, \hat{L})\) are related as in the previous proof, then \( \hat{W} \) is always the orthogonal complement of \( W \) with respect to \( \hat{H} \).

Proposition 1. Fix a subbundle \( W \subseteq T^*Q \) and a quadratic form \( \hat{H} : T^*Q \to \mathbb{R} \). If a second quadratic form \( \hat{H} : T^*Q \to \mathbb{R} \) is a solution of (16), then \( \hat{R} := \hat{H}\big|_{\hat{W}} \) is a solution of [see Eq. (20)]
\[
\{\hat{R} \circ \hat{p}, \hat{H}\}(\sigma) = 0, \quad \forall \sigma \in \hat{W}.
\]
(21)

where \( \hat{W} := W^\perp \) (the orthogonal complement of \( W \) w.r.t. \( \hat{H} \)). On the other hand, given a complement \( \hat{W} \) of \( W \) and its related projections \( \hat{p} \) and \( p \) [see Eq. (20) again], if a quadratic form \( \hat{R} : \hat{W} \to \mathbb{R} \) satisfies (21), then, for every quadratic form \( \hat{L} : W \to \mathbb{R} \), the function \( \hat{H} \) given by (19) satisfies (16).

Proof. Given both a quadratic form \( \hat{H} \) and a triple \((\hat{W}, \hat{R}, \hat{L})\) related as in the previous lemma, let us show that, for all \( \sigma \in \hat{W} \),
\[
\{\hat{L} \circ p, \hat{H}\}(\sigma) = 0.
\]
(22)
To do that, fix a coordinate chart \((U, (q^1, \ldots, q^n))\) of \(Q\) and a local basis \(\{\xi_1, \ldots, \xi_m\} \subseteq \Omega^1(U)\) of the subbundle \(W\). Define
\[
H^{ij}(q) := \left( dq^i \big|_q, F \left( dq^j \big|_q \right) \right), \quad q \in U,
\]
and write
\[
p \left( dq^k \big|_q \right) = P^{ka}(q) \xi_a(q).
\]
Note that, using (11),
\[
H \left( p_k dq^k \big|_q \right) = \frac{1}{2} \left( p_k dq^k \big|_q, F \left( p_l dq^l \big|_q \right) \right) = \frac{1}{2} p_k p_l H^{kl}(q).
\]
On the other hand, if \(\lambda\) is the fibered inner product defining \(\mathcal{L}\), consider the matrix
\[
\mathbb{L}_{ab}(q) := \langle \xi_a(q), \lambda \circ \xi_b(q) \rangle, \quad q \in U.
\]
Then, omitting the dependence on \(q\), just for simplicity, we have that
\[
\mathcal{L} \circ p \left( p_k dq^k \right) = \frac{1}{2} \left( \mathcal{L} \circ p(p_k dq^k), \lambda^2 \left( p(p_l dq^l) \right) \right) = \frac{1}{2} p_k p_l \mathbb{H}^{kl} \mathbb{L}_{ab},
\]
and consequently [using (13)]
\[
\left\{ \mathcal{H} \circ p, \mathbb{H} \right\} \left( p_k dq^k \right) = \left( \frac{\partial}{\partial q^k} (P^{ia} P^{jb} \mathbb{L}_{ab}) \mathbb{H}^{ij} - \frac{\partial \mathbb{H}^{ij}}{\partial q^k} P^{ka} P^{lb} \mathbb{L}_{ab} \right) p_i p_j p_l.
\]
In addition, since \(p_k dq^k \in \hat{W}\) if and only if
\[
0 = p \left( p_k dq^k \right) = p_k \mathbb{P}^{ka} \xi_a,
\]
which in turn is equivalent to \(p_k \mathbb{P}^{ka} = 0\) for all \(a\), the Eq. (22) is immediate. To end the proof, it is enough to note that (19) and (22) imply the equality
\[
\left\{ \hat{\mathcal{H}}, \hat{\mathcal{H}} \right\} (\sigma) = \left\{ \mathcal{H} \circ p, \mathcal{H} \right\} (\sigma), \quad \forall \sigma \in \hat{W}.
\]
from which the proposition easily follows.

So, we can replace the kinetic equation by Eq. (21), whose unknown is a pair \((\mathcal{H}, \hat{W})\): \(\hat{W}\) is a complement of \(W\) and \(\mathcal{H} : \hat{W} \rightarrow \mathbb{R}\) is a quadratic form.

Now, let us study the potential equation. Given \(\beta \in T^*Q\) and \(\sigma \in \hat{W}\), on the same fiber, it follows that
\[
\langle \beta, F(\mathcal{L} \circ p)(\sigma) \rangle = \frac{d}{dt} \mathcal{L} \circ p(\sigma + t \beta) \bigg|_{t=0} = \frac{d}{dt} \mathcal{L} \circ p(t \beta) \bigg|_{t=0} = \mathcal{L} \circ p(\beta) \left( \frac{d}{dt} t \right)^2 \bigg|_{t=0} = 0,
\]
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because \( p(\sigma) = 0 \) and \( L \) is a quadratic form. So, \( F(\mathcal{L} \circ p)(\sigma) = 0 \) for all \( \sigma \in \hat{W} \), and the potential equation (17) can be written

\[
\left( d\hat{h} \circ F \circ \mathfrak{R} - d\hat{h} \circ F \circ \mathfrak{R} \circ \hat{p} \right)(\sigma) = 0, \quad \forall \sigma \in \hat{W},
\]

(25)

where only \( \hat{W} \) and \( \mathfrak{R} \) are involved (the quadratic form \( \mathcal{L} \) plays no role in either of the two matching conditions). As a consequence, instead of (16) and (17), we can think of the matching conditions as the Eqs. (21) and (25) for the unknown \((\mathfrak{R}, \hat{h}, \hat{W})\), and we shall do it from now on.

2.2 Local expressions

For reasons that will be clear later, we shall concentrate on those solutions \((\mathfrak{R}, \hat{W})\) of the kinetic equations for which \( \hat{W}^\sharp = F\mathfrak{R}(\hat{W}) \) is integrable. The fact that this is always possible, unless locally, is proved in the next lemma.

**Lemma 2.** Given a subbundle \( W \subseteq T^*Q \) of rank \( m \) and a quadratic form \( \mathfrak{R} : T^*Q \to \mathbb{R} \), around every point \( q_0 \in Q \) we can construct, by using only algebraic manipulations, a local coordinate chart \((U, (q^1, \ldots, q^n))\) such that

\[
\hat{W} := F\mathfrak{R}^{-1}(\text{span}\{ \partial/\partial q^1, \ldots, \partial/\partial q^{n-m}\})
\]

(26)

is a complement of \( W \) along \( U \). In particular, \( \hat{W}^\sharp \) is an integrable subbundle.

**Proof.** Given \( q_0 \in Q \), consider a coordinate neighborhood \((V, (q^1, \ldots, q^n))\) and a local basis \( \{X_1, \ldots, X_m\} \subseteq T(V) \) of \( W^\dagger \) around \( q_0 \). It is clear that

\[
X_i(q) = \sum_{j=1}^{n} C_{ij}(q) \frac{\partial}{\partial q^j} \bigg|_q, \quad q \in V,
\]

being \( C_{ij}(q) \) the coefficients of an \( m \times n \) matrix \( C(q) \) of maximal rank. Reordering the coordinate functions \( q^i \)'s, if necessary, we can ensure that the \( m \times m \) sub-matrix \( S(q_0) \), given by the last \( m \) columns of \( C(q_0) \), is non-singular. Then, the vectors

\[
\left\{ \frac{\partial}{\partial q^1} \bigg|_{q_0}, \ldots, \frac{\partial}{\partial q^{n-m}} \bigg|_{q_0} \right\}
\]

define a complement of \( W^\sharp_{q_0} \) and, by continuity, the first \( n - m \) coordinate vector fields \( \{ \partial/\partial q^1, \ldots, \partial/\partial q^{n-m} \} \) span a complement of \( W^\sharp \) along the open neighborhood \( U \subseteq V \) of \( q_0 \) where \( S(q) \) is non-singular. As a consequence, the subbundle \( \hat{W} \) given by (26) is a complement of \( W \) along \( U \).

**Remark 4.** By Frobenius theorem, if \( \hat{W}^\sharp \) is an integrable subbundle, then \( \hat{W} \) is locally given by (26) for some coordinate chart.
Now, let us fix a complement $\hat{W}$ of $W$ such that $\hat{W}^J = P\mathfrak{h}(\hat{W})$ is integrable and, given $q_0 \in Q$, consider a coordinate chart $(U, \varphi = (q^1, \ldots, q^n))$ around $q_0$ where $\hat{W}$ is locally given by (26) (as in the last lemma). We want to find the form that the matching conditions adopt in such coordinates. Consider the matrix with entries $H_{ij}$ given by (23) and define

$$H_{ij} := \langle \hat{\mathfrak{h}}^{-1} \left( \frac{\partial}{\partial q^i} \right), \frac{\partial}{\partial q^j} \rangle.$$  

(27)

We are omitting the dependence on $q$, just for simplicity. Clearly,

$$H_{ij} H_{jk} = \delta_{ik}.$$  

(28)

Also, the co-vectors

$$\sigma_\mu := \hat{\mathfrak{h}}^{-1} \left( \frac{\partial}{\partial q^\mu} \right) = \mathbb{H}_{\mu k} dq^k \in \hat{W}, \ \mu = 1, \ldots, n - m,$$

give a basis for $\hat{W}$ and we can write

$$\hat{\mathfrak{p}} (dq^k) = \hat{p}^{k\mu} \sigma_\mu.$$  

(29)

Note that, since $\hat{\mathfrak{p}} (\sigma_\mu) = \sigma_\mu,$

$$\mathbb{H}_{\mu k} \hat{p}^{k\mu} = \delta^\mu_\nu.$$  

(30)

**Remark 5.** If the (local) forms $\xi_a := \vartheta_{ai} dq^i$, for $a = 1, \ldots, m$, give a (local) basis for $W$, since $\ker \hat{\mathfrak{p}} = W$, we also have the identity

$$\vartheta_{ai} \hat{p}^{b\mu} = 0.$$  

(31)

As a consequence, the matrix $\hat{\mathfrak{p}}$ is univocally determined by the (30) and (31).

On the other hand, if $\kappa$ denotes the fibered inner product on $\hat{W}$ defining $\hat{R}$, consider the matrix

$$\mathbb{K}_{\mu \nu} := \langle \sigma_\mu, \kappa^\tau (\sigma_\nu) \rangle.$$  

With this notation, we have that

$$\hat{R} \circ \hat{\mathfrak{p}} (p_k dq^k) = \frac{1}{2} \langle \hat{\mathfrak{p}} (p_k dq^k), \kappa^2 (\hat{\mathfrak{p}} (p_l dq^l)) \rangle = \frac{1}{2} \hat{p}_k p_l \hat{p}^{k\mu} \hat{p}^{l\nu} \mathbb{K}_{\mu \nu}.$$  

(32)

Thus, using $\{13\}$, $\{24\}$ and $\{32\}$, the Eq. (21) translates to

$$\left( \frac{\partial}{\partial q^\tau} \left( \hat{\mathfrak{p}}^{\mu \nu} \hat{p}_{\mu \nu} \mathbb{K}_{\mu \nu} \right) \right) H^{k l} - \frac{\partial \mathbb{H}^{ij}}{\partial q^\tau} \hat{\mathfrak{p}}^{k \mu} \hat{p}^{l \nu} \mathbb{K}_{\mu \nu} p_i p_j p_l = 0.$$  

In addition, since a generic element of $\hat{W}$ has the form $a^\tau \sigma_\tau = a^\tau \mathbb{H}_{\tau k} dq^k$, i.e. $p_k = a^\tau \mathbb{H}_{\tau k}$, using the identities (28) and (30) we have that the kinetic equation reads

$$\left( \frac{\partial \mathbb{K}}{\partial q^\tau_1} + \mathbb{G}_{\tau_1 \tau_2 \tau_3}^{\mu \nu} \mathbb{K}_{\mu \nu} \right) a^\tau_1 a^\tau_2 a^\tau_3 = 0.$$  

(33)
with
\[ G_{\tau_1 \tau_2 \tau_3} := \hat{P}^k \delta_{\tau_1} \frac{\partial H_{\tau_2}}{\partial q^k} + \frac{\partial (\hat{P}^\mu \hat{P}^\nu)}{\partial q^{\tau_1}} H_{\tau_2}^j H_{\tau_3}^j. \] (34)

Now, let us study the potential equation in the above coordinates. From (23) we know that
\[ F(\hat{h} \circ \hat{\pi}) (p_k \, dq^k) = p_k \hat{P}^k \hat{P}^\mu \hat{\pi}_\mu \partial \partial q^k \] and, if \( p_k = a^\tau H_{\tau k} \) [see (28)],
\[ F(\hat{h} \circ \hat{\pi}) (a^\tau H_{\tau k} \, dq^k) = a^\tau \partial \partial q^k. \] (35)

On the other hand, using Eqs. (9) and (32) we have that
\[ F(K \circ \hat{\pi}) (p_k \, dq^k) = p_k \hat{P}^k \hat{P}^\mu \hat{\pi}_\mu \partial \partial q^k \] and, again, if \( p_k = a^\tau H_{\tau k} \),
\[ F(K \circ \hat{\pi}) (a^\tau H_{\tau k} \, dq^k) = a^\tau \hat{P}^\mu H_{\tau \mu \tau}. \] (36)

Thus, from (35) and (36), the potential equation (25) reads
\[ \left( \frac{\partial h}{\partial q^\mu} - \frac{\partial h}{\partial q^k} \hat{P}^k \hat{\pi}_\mu \hat{\pi}_\tau \right) a^\mu = 0, \] which is equivalent to
\[ \frac{\partial h}{\partial q^\mu} - \frac{\partial h}{\partial q^k} \hat{P}^k \hat{\pi}_\mu \hat{\pi}_\tau = 0, \quad \mu = 1, \ldots, n - m. \] (37)

[For simplicity, we are identifying \( h \) (resp. \( \hat{h} \)) with its local representative \( h \circ \varphi^{-1} \) (resp. \( \hat{h} \circ \varphi^{-1} \)).]

Summarizing the results of the entire section, we have the next two theorems.

**Theorem 1.** Consider an underactuated Hamiltonian system satisfying (14) and (15), i.e. one defined by a triple \( (\mathfrak{H}, h, W) \) (see Remark 7). Then, every (simple) solution \( \hat{H} = \mathfrak{H} + \hat{h} \circ \pi \) of the matching conditions (16) and (17) is univocally described by: \( \text{(i)} \) a subbundle \( \hat{W} \) complementary to \( W \), \( \text{(ii)} \) a quadratic form \( \mathfrak{H} : \hat{W} \to \mathbb{R} \) solving (21), \( \text{(iii)} \) a solution \( \hat{h} \) of (25), and \( \text{(vi)} \) a quadratic form \( \mathcal{L} : W \to \mathbb{R} \).

**Theorem 2.** Let \( \hat{W} \) be a complement of \( W \) such that \( \hat{W} = F(\mathfrak{H}(\hat{W})) \) is integrable. Then, in a coordinate chart \( (U, (q^1, \ldots, q^n)) \) of \( Q \) satisfying (26), the Equations (21) and (25) translate to (33) and (37), respectively.

### 3 Solving the potential equation after solving the kinetic one

In this section, given a solution of the kinetic equation (21), we shall study under which conditions a positive solution \( \hat{h} \) of the potential equation (25) does exist.
In fact, we will develop a systematic procedure to find (unless locally) an explicit solution of this equation (up to quadratures). In addition, we will provide necessary and sufficient conditions to ensure the positivity of the solution.

### 3.1 An integrability condition

Suppose that the conditions in Theorems 1 and 2 hold, and that a solution \( \mathfrak{K} \) of (21) is given. We want to find a (local) solution \( \hat{h} \) of the potential equation (25) where \( \hat{p} \) and \( \mathfrak{K} \) are considered as datum. Given \( q_0 \in Q \), denote by \((U, \varphi = (q', \ldots, q^n)) \) the coordinate chart in which \( \hat{W} \) is given by (26). In such coordinates, according to Theorem 2, Eq. (25) translates to Eq. (37). As it is well-known, the necessary and sufficient conditions to integrate these equations are

\[
\frac{\partial}{\partial q^\mu} \left( \frac{\partial \hat{h}}{\partial q^\nu} \hat{p}^{k\tau} \mathbb{K}_{\tau\nu} \right) = \frac{\partial}{\partial q^\nu} \left( \frac{\partial \hat{h}}{\partial q^\mu} \hat{p}^{k\tau} \mathbb{K}_{\tau\mu} \right),
\]

(38)

for all \( \mu, \nu \leq n - m \). In global terms they say that

\[
d \left( d\hat{h} \circ \mathfrak{F} (\mathfrak{K} \circ \hat{p}) \circ \mathfrak{F} \right)^{-1} \circ (u, v) = 0, \quad \forall u, v \in \hat{W}^\sharp.
\]

(39)

In such a case, the solution not only exists but, furthermore, it can be computed up to quadratures. We shall give a formula for \( \hat{h} \) at the end of this section [see (48)].

**Remark 6.** If \( n - m = 1 \), i.e. if the degree of underactuation is one, the sub-bundle \( \hat{W}^\sharp \) is always integrable because of dimensional reasons. In addition, the condition (38) reduces to a single equation for \( \mu = \nu = 1 \), that immediately holds. Therefore, if we find a solution \( \mathfrak{K} \) of the kinetic equation for a system with one degree of underactuation, not only there exists a solution of the single potential equation (as it is already known in the literature), but even more, we can construct such a solution (in an appropriate coordinate chart) up to quadratures.

### 3.2 A positivity condition

Suppose that \( q_0 \) is a critical point of \( h \) and, for simplicity, assume that the above given coordinate chart \((U, \varphi)\) is centered at \( q_0 \), i.e. \( \varphi(q_0) = (0, \ldots, 0) =: 0 \). As we have mentioned in the Introduction, one is actually interested in a solution \( \hat{h} \) of (37) which is positive-definite around \( q_0 \). Identifying \( \hat{h} \) with its local representative \( \hat{h} \circ \varphi^{-1} \), this is the same as saying that:

1. \( 0 \) is a critical point of \( \hat{h} \) and
2. the Hessian matrix of \( \hat{h} \) at \( 0 \) is positive-definite.

If (38) holds, then a solution \( \hat{h} \) of (37) exists and we can use the Method of Characteristics to construct it. To do that, we must impose a boundary condition, for instance, along the subset

\[
S := \varphi(U) \cap \{(0, \ldots, 0) \times \mathbb{R}^m\} \subseteq \mathbb{R}^n.
\]
So, let us impose on $S$ the condition

$$\hat{h}(0, \ldots, 0, s^1, \ldots, s^m) = \frac{\varpi}{2} \sum_{a=1}^{m} (s^a)^2,$$

(40)

for some constant $\varpi > 0$. It follows from (37), the boundary condition above and the fact that $0$ is a critical point of $h$ that $\partial \hat{h}/\partial q^i(0) = 0$ for all $i = 1, \ldots, n$. Then, $0$ is a critical point of $\hat{h}$. On the other hand, the Hessian of $\hat{h}$ at $0$,

$$\text{Hess}(\hat{h})_{ij}(0) = \frac{\partial^2 \hat{h}}{\partial q^i \partial q^j}(0),$$

(41)

can be written

$$\text{Hess}(\hat{h})(0) = \begin{bmatrix} M & A \\ A^t & \varpi I_m \end{bmatrix},$$

where:

- $I_m$ is the $m \times m$ identity matrix;
- $A$ is the $((n - m) \times m)$-matrix with entries

$$A_{\mu a} := \left( \text{Hess}(h)_{n-m+a,k} \hat{P}^{k \tau} K_{\tau \mu} \right)(0) = \frac{\partial}{\partial q^{n-m+a}} \left( \frac{\partial \hat{h}}{\partial q^k} \hat{P}^{k \tau} K_{\tau \mu} \right)(0), \quad \mu \leq n - m \text{ and } a \leq m$$

(42)

and

- $M$ is the square matrix of dimension $n - m$ with

$$M_{\mu \nu} := \left( \text{Hess}(h)_{\mu k} \hat{P}^{k \tau} K_{\tau \nu} \right)(0) = \frac{\partial}{\partial q^k} \left( \frac{\partial \hat{h}}{\partial q^\mu} \hat{P}^{k \tau} K_{\tau \nu} \right)(0).$$

(43)

**Remark 7.** The entries of the matrices $A$ and $M$ are obtained just by differentiating the potential equation (37) and using Eqs. (40) and (41) and the criticality of $0$ for $h$.

**Proposition 2.** Consider a local solution $\hat{h}$ of (37). If $\text{Hess}(\hat{h})(0)$ is positive-definite, then the matrix $M$, given by (43), is positive-definite. Now, suppose that $\hat{h}$ satisfies (40). If $M$ is positive-definite, then there exists a constant $\varpi$ such that $\text{Hess}(\hat{h})(0)$ is positive-definite too.

**Proof.** The first part of the proposition easily follows from the fact that $M$ is an upper-left corner square sub-matrix of $\text{Hess}(\hat{h})(0)$. So, let us prove the second part. Assume that $M$ is positive-definite and take $u \in \mathbb{R}^{n-m}$ and $w \in \mathbb{R}^m$. Then

$$\begin{pmatrix} u \\ w \end{pmatrix} \begin{bmatrix} M & A \\ A^t & \varpi I_m \end{bmatrix} \begin{bmatrix} u^t \\ w^t \end{bmatrix} = u^t M u + 2 u^t A w + \varpi \| w \|^2$$

$$= \| u \|^2_M + 2 u^t A w + \varpi \| w \|^2$$

$$\geq \| u \|^2_M - 2 \| u \| \| A \| \| w \| + \varpi \| w \|^2,$$
where $\| \cdot \|$ denotes the Euclidean norm in the corresponding vector space and $\| \cdot \|_M$ is the norm associated with $M$. In particular [recall Eq. (42)],

$$\|A\|^2 = \sum_{\mu=1}^{n-m} \sum_{a=1}^{m} |A_{\mu a}|^2 = \sum_{\mu=1}^{n-m} \sum_{a=1}^{m} \left| \frac{\partial}{\partial q_{n-m+a}} \left( \frac{\partial h}{\partial q^k} \hat{p}^{k} \right) \right|^2 (0).$$

(44)

Since every norm on a finite-dimensional vector space is equivalent to the Euclidean norm, there exists a positive constant $\alpha$ such that

$$\|u\|_M \geq \alpha \|u\|, \quad \forall u \in \mathbb{R}^{n-m}.$$ 

The constant $\alpha$ may be computed as

$$\alpha = \min_{\|u\|=1} \sqrt{u^T M u} = \sqrt{\lambda_{\min}^M},$$

where $\lambda_{\min}^M$ is the least eigenvalue of $M$. Then,

$$(u, w) \begin{bmatrix} M & A^t & \hat{A} \\ \hat{A} & \varpi & I \end{bmatrix} \begin{bmatrix} u^t \\ w^t \end{bmatrix} \geq \alpha^2 \|u\|^2 - 2 \|u\| \|A\| \|w\| + \varpi \|w\|^2$$

$$= \alpha^2 \left( \|u\|^2 - 2 \|u\| \|A\| \frac{\|w\|}{\alpha^2} + \varpi \frac{\|w\|^2}{\alpha^2} \right).$$

If $A = 0$, this expression is clearly nonnegative and vanishes only when $u$ and $w$ vanish. Suppose now that $A \neq 0$. Defining $\beta = \frac{\|A\|}{\alpha}$, we have

$$(u, w) \begin{bmatrix} M & A^t & \hat{A} \\ \hat{A} & \varpi & I \end{bmatrix} \begin{bmatrix} u^t \\ w^t \end{bmatrix} \geq \alpha^2 \beta^2 \left( \frac{\|u\|^2}{\beta^2} - 2 \frac{\|w\|}{\alpha} \frac{\|u\|}{\beta} + \varpi \frac{\|w\|^2}{\alpha^2} \right)$$

$$= \alpha^2 \beta^2 \left[ \left( \frac{\varpi}{\beta^2} - 1 \right) \left( \frac{\|w\|}{\alpha} \right)^2 + \left( \frac{\|w\|}{\alpha} - \frac{\|u\|}{\beta} \right)^2 \right].$$

Hence, if we choose $\varpi > \beta^2$, i.e.

$$\varpi > \left( \frac{\|A\|}{\alpha} \right)^2,$$

it follows that Hess$(\hat{h})(0)$ is positive-definite.

Remark 8. Since $q_0$ is a critical point of the potential term $h$, we can find a coordinate-free expression of the matrix $M$ using the covariant Hessian tensor $\nabla \nabla h$, given by

$$\nabla \nabla h(X, Y) = X(Y h) - \langle dh, \nabla_X Y \rangle,$$
whose matrix representation at $q_0$ is exactly Hessian matrix of $h$. Indeed, it is easy to see that

$$M_{\mu\nu} = M \left( \left. \frac{\partial}{\partial q_\mu} \right|_{q_0}, \left. \frac{\partial}{\partial q_\nu} \right|_{q_0} \right),$$

where

$$M(u, v) = \nabla\nabla h \left( \mathcal{F} (\hat{F} \circ \hat{p}) \circ \mathcal{F} \mathcal{F}^{-1}(u), v \right). \quad (47)$$

### 3.3 The integration procedure

We shall now condense the results of the previous subsections in the following theorem. Consider again an underactuated system defined by a triple $(\mathcal{H}, h, W)$.

**Theorem 3.** Let $q_0$ be a critical point of $h$ and $(U, \varphi)$ a coordinate neighborhood of $q_0$ such that the subbundle $\hat{W}$ given by (26) is a complement of $W$. Let $\hat{\mathcal{H}}$ be a solution of the kinetic equation (21) for $\hat{W}$. Then, a solution $\hat{h}$ of the potential equation (25) exists around $q_0$ if and only if the condition

$$d \left( d h \circ (\hat{\mathcal{H}} \circ \hat{p}) \circ \mathcal{F} \mathcal{F}^{-1} \right) \bigg|_{\hat{W}^\# \times \hat{W}^\#} = 0,$$

holds [see Eq. (39)]. Moreover, in such a case, $\hat{h}$ can be found up to quadratures. On the other hand, $\hat{h}$ can be chosen positive-definite around $q_0$ if and only if the bilinear form [see Eq. (47)]

$$M = \nabla\nabla h \circ \left( \mathcal{F} (\hat{\mathcal{H}} \circ \hat{p}) \circ \mathcal{F} \mathcal{F}^{-1} \times id_{\mathcal{T}Q} \right) \bigg|_{\hat{W}^\# \times \hat{W}^\#}$$

is positive-definite at that point.

Gathering all the results we have presented so far, we can state a procedure to explicitly construct local solutions of the potential equation that are positive-definite around $q_0$, provided a solution of the kinetic equation is given. We must proceed as follows:

1. find coordinates $(q^1, \ldots, q^n)$ centered at $q_0$ such that

$$\hat{W} := \mathcal{F} \mathcal{F}^{-1} \left( \operatorname{span} \left\{ \frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^n} \right\} \right)$$

is a complement of $W$ (which can be done just reordering an arbitrary coordinate system centered at $q_0$, as mentioned in the proof of Lemma 2);

2. consider a (local) solution $\hat{\mathcal{H}}$ of the kinetic equation (21) for $\hat{W}$;

3. in the coordinates of the step 1, define the functions $u_\mu := \frac{\partial h}{\partial q^\mu} \hat{p}^{k \tau} i_{\mathcal{T}Q}^\tau, \mu$, for $\mu = 1, \ldots, n - m$;

4. verify that $\frac{\partial u_\nu}{\partial q^\mu} = \frac{\partial u_\mu}{\partial q^\nu}$ for all $\mu, \nu \leq n - m$ [see Eq. (38)];
5. define \( \hat{h} \) as

\[
\hat{h}(q^1, \ldots, q^n) := \sum_{\mu=1}^{n-m} \int_0^{q^\mu} u_\mu(0, \ldots, 0, t, q^{\mu+1}, \ldots, q^n) \, dt + \frac{\varpi}{2} \sum_{a=1}^m (q^{n-m+a})^2
\]  

(48)

for some constant \( \varpi \);

6. check that the matrix \( M \) [recall Eq. (43)] is positive-definite, i.e. check that \( \lambda_{M, \text{min}} > 0 \),

7. choose \( \varpi \) such that [recall Eqs. (44), (45) and (46)]

\[
\varpi > \sum_{\mu=1}^{n-m} \sum_{a=1}^m \left( \frac{\partial u_\mu}{\partial q^{n-m+a}}(0) \right)^2 \lambda_{M, \text{min}}^{-1}
\]

The idea of studying the potential equation assuming we have a solution of the kinetic equation appeared also in Reference [16]. In such work, the author finds integrability conditions for the potential matching conditions using Goldschmidt’s integrability theory (see [12]). Then, assuming such conditions hold and supposing all the objects involved belong to the category \( C^\infty \), it is proved that there exists indeed a solution of these equations. However, the positivity of such solutions is not analyzed. Here, although our integrability conditions are similar to those found in [16], our approach is valid in the category \( C^\infty \). Moreover, we give necessary and sufficient conditions in order to ensure that the solution is positive-definite and, in addition, we show how to build such solution by computing ordinary integrals in an appropriate coordinate chart.

4 Solving both the kinetic and the potential equations

In this section we are going to apply the steps above to a particular subclass of underactuated systems.

4.1 One degree of underactuation

Assume that \((H, W)\) has one degree of underactuation, i.e. \( W \) is given by a vector subbundle \( W \subset T^*Q \) of rank \( m = n - 1 \). Suppose that the step 1 was already performed. In such a case, we can explicitly find a solution \( \hat{K} \) of the local kinetic equation (33) (corresponding to the step 2). Let us see that. To simplify the calculations, we will write

\[
q^1 = x, \quad q^{1+a} = y^a, \quad a = 1, \ldots, n-1,
\]

and

\[
\hat{K}_{11} = K, \quad \hat{G}^{11}_{111} = G, \quad \hat{P}^{k1} = \hat{P}^k, \quad k = 1, \ldots, n.
\]
Note that, according to (34),

$$G = \frac{\partial H}{\partial x} \hat{P}^1 + \frac{\partial H}{\partial y^a} \hat{P}^{1+a} + \frac{\partial (\hat{P}^a \hat{P})}{\partial x} \hat{H}_{1i} \hat{H}_{1j}. \quad (49)$$

Under this notation, Eq. (33) reads

$$\frac{\partial K}{\partial x} + GK = 0,$$

whose general solution is

$$K(x, y) = \xi(y) e^{-\int_0^x G(t, y) dt}. \quad (50)$$

In order for $K$ to define a quadratic form, we must ask $\xi$ to be a positive function.

Following with the step 3, define

$$u := u_1 = \left( \frac{\partial h}{\partial x} \hat{P}^1 + \frac{\partial h}{\partial y^a} \hat{P}^{1+a} \right) K. \quad (51)$$

It is clear that the step 4 is trivial in this case. According to step 5, we have

$$\hat{h}(x, y) = \int_0^x u(t, y) dt + \sum_{a=1}^{n-1} (y^a)^2. \quad (52)$$

The step 6 reduces to check that the number

$$M_{11} = \lambda_{\text{min}}^M = \frac{\partial u}{\partial x}(0)$$

is positive. Finally, step 7 says that we must take

$$\varpi > \frac{\sum_{a=1}^{n-1} (\partial u / \partial y^a(0))^2}{\partial u / \partial x(0)}. \quad (53)$$

### 4.2 The planar inverted double pendulum

Let us end our work with a concrete example. Consider the manifold $Q = S^1 \times S^1$ and let $(U, (\psi, \varphi))$ be a system of angular coordinates. Consider on $Q$ the simple Hamiltonian function $H$ with

$$H(\psi, \varphi, p_\psi, p_\varphi) = \frac{1}{2m} (p_\psi, p_\varphi) \begin{pmatrix} C & -B \cos(\psi - \varphi) \\ -B \cos(\psi - \varphi) & A \end{pmatrix} \begin{pmatrix} p_\psi \\ p_\varphi \end{pmatrix}$$

and

$$h(\psi, \varphi) = D_1 \cos(\psi) + D_2 \cos(\varphi),$$

where $m := AC - B^2 \cos^2(\psi - \varphi)$ and $A, B, C, D_1$ and $D_2$ are positive constants. This Hamiltonian corresponds to the system depicted in Figure [1] the (planar) inverted double pendulum, for appropriate values of the constants $A, B, C, D_1$
Consider in addition the subbundle $W \subseteq T^*Q$ generated by the 1-form $d\phi$. The latter, together with $H$, define an underactuated system with one actuator, which produces a torque around the coordinate $\phi$. To find a solution of the matching conditions for $(\delta, h, W)$, let us follow the steps above.

1. Since $W = \langle d\phi \rangle$ along $U$, then

$$W^\sharp = \mathbb{F}\delta (\langle d\phi \rangle) = \left\langle -B \cos(\psi - \varphi) \frac{\partial}{\partial \psi} + A \frac{\partial}{\partial \varphi} \right\rangle$$

there. Accordingly, the subbundle $\langle \frac{\partial}{\partial \psi} - \gamma \frac{\partial}{\partial \varphi} \rangle$ is complementary to $W^\sharp$ (shrinking $U$ around $\psi = \varphi = 0$, if needed) if we choose the constant $\gamma \neq \frac{A}{B}$. In what follows we will assume that $\gamma$ is a generic constant which, in the end, we will choose to fulfill our requirements.

Define the coordinates

$$x := \psi \quad \text{and} \quad y := \varphi + \gamma \psi,$$

we have that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \psi} - \gamma \frac{\partial}{\partial \varphi}, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial \varphi},$$

and consequently

$$\hat{W} := \mathbb{F}\delta^{-1}\left( \begin{pmatrix} \frac{\partial}{\partial x} \end{pmatrix} \right)$$

---

In fact, if we consider massless bars of lengths $L_1$ and $L_2$ with particles of masses $m_1$ and $m_2$ attached to the ends, the values of these constants are $A = m_1 L_1^2 + m_2 L_2^2$, $B = m_1 L_1$, $C = m_2 L_2^2$, $D_1 = m_2 g L_2$ and $D_2 = (m_1 + m_2) g L_1$, where $g$ is the acceleration of gravity.
is complementary to $W$ (along $U$). In this way, the first step is done. Let us mention that the matrix $\mathbb{H}^{-1}$ representing $\mathbb{F}\mathcal{J}^{-1}$ [see Eq. (27)] in the new coordinates $(x, y)$ is

$$\mathbb{H}^{-1} = \begin{bmatrix} A - 2b\gamma + C\gamma^2 & b - \gamma C \\ b - \gamma C & C \end{bmatrix},$$  

where

$$b := B \cos((1 + \gamma)x - y);$$  

while the matrix $\hat{P}$ of the projection $\hat{p}$, using $\sigma := \mathbb{F}\mathcal{J}^{-1} \left( \frac{\partial}{\partial x} \right)$ as a basis for $\hat{W}$ [see Eq. (29)], is given by the column vector

$$\hat{P} = \begin{bmatrix} 1 \\ A - \gamma b \end{bmatrix}.$$  

Also, the potential energy $h$ in these coordinates reads

$$h(x, y) = D_1 \cos(x) + D_2 \cos(\gamma x - y).$$  

2. The general solution of the kinetic equation is given by (50), where, according to (49), (54), (55) and (56),

$$G(x, y) = -\frac{2\gamma^2 b_x}{(1 + \gamma)(A - b\gamma)},$$

where we have considered $\gamma \neq -1$. Concretely,

$$K(x, y) = \xi(y) (A - b\gamma)^{-\frac{2\gamma}{1 + \gamma}},$$

being $\xi$ a positive function.

3. In this case, the function $u$ defined by (51) is given by [see (56) and (57)]

$$u(x, y) = -\frac{D_1 \sin(x)}{A - b\gamma} K(x, y).$$

4. Nothing to do.

5. According to (52),

$$\hat{h}(x, y) = -\int_0^x \frac{D_1 \sin(t)}{A - \gamma B \cos((1 + \gamma)t - y)} K(t, y) \, dt + \frac{\phi}{2} y^2.$$

6. Since

$$M = \frac{\partial u}{\partial x} (0, 0) = \frac{-D_1}{A - \gamma B} K(0, 0)$$

and $K(0, 0)$ and $D_1$ are positive, in order for $M$ to be positive, we must choose $\gamma$ such that

$$A - \gamma B < 0.$$
This is true if and only if
\[ \gamma > \frac{A}{B} > 0. \]

In particular, observe that the value \( \gamma \neq -1 \) we discarded earlier would end up yielding non-positive solutions.

7. Finally, since
\[ \frac{\partial u}{\partial y}(0,0) = 0, \]
we must take [see Eq. (53)] \( \varepsilon > 0. \)

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