Polynomial stability of piezoelectric beams with magnetic effect and tip body

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In this paper, we consider a dissipative system of one-dimensional piezoelectric beam with magnetic effect and a tip load at the free end of the beam, which is modeled as a special form of double boundary dissipation. Our main aim is to study the well-posedness and asymptotic behavior of this system. By introducing two functions defined on the right boundary, we first transform the original problem into a new abstract form, so as to show the well-posedness of the system by using Lumer–Phillips theorem. We then divide the original system into a conservative system and an auxiliary system, and show that the auxiliary problem generates a compact operator. With the help of Weyl’s theorem, we obtain that the system is not exponentially stable. Moreover, we prove the polynomial stability of the system by using a result of Borichev and Tomilov (Math. Ann. 347 (2010), 455–478).

1 | INTRODUCTION

Piezoelectric materials are materials that can exchange mechanical energy, electrical energy and nuclear energy in motion. Their structures are generally composed of beams or slabs. Due to the advantages like small size, high power density, fast response time, large mechanical force and high resolution, they are widely applied in many fields, such as the latest cutting-edge applications: cardiac pacemaker [7], course changing bullet, structural health monitoring [6], nano locator [16], ultrasonic imaging device, ultrasonic welding and cleaning device, energy collection [10]. The piezoelectric effect usually is shown as two types. One is to generate charge in the interior by applying mechanical force, which is called direct piezoelectric effect [19, 26]. Another is from the external electric field through its internal mechanical stress, which is called reverse piezoelectric effect. Due to the asymmetry of crystals, the above two effects have the same origin [14]. In the piezoelectric beam, which constitutes the electronic device, the mechanical disturbance responds in the form of electricity. When piezoelectric materials are integrated into components of electronic circuits, the mechanical effects on structures are also very important as they are interfered by electrical, magnetic or electromagnetic properties. There are three main ways to drive piezoelectric materials in such electronic devices: to supply voltage, current or charge to the electrodes. Therefore, it is significant to describe the interaction of these three effects (mechanical, electrical and magnetic) for understanding the stability conditions of these systems (see refs. [2, 8, 38]). The equation of piezoelectric beam with magnetic effects
is based on the description of electromagnetic coupling by Maxwell equation and the mechanical behavior of beam by Mindlin–Timoshenko theory (see refs. [2, 8]).

Let us refer to several previous works on the stability results for the piezoelectric models. In ref. [28, 29], Morris and Özer considered the effects of three effects (mechanical, electrical and magnetic) for the first time. They showed that system with only one boundary control was not exponentially stable. In ref. [33], Ramos et al. studied a one-dimensional system of piezoelectric beams with magnetic effects, the system is shown as

\[
\begin{align*}
\rho v_{tt} - \alpha v_{xx} + \gamma \beta p_{xx} &= 0, \quad (x, t) \in (0, L) \times (0, T), \\
\mu p_{tt} - \beta p_{xx} + \gamma \beta v_{xx} &= 0, \quad (x, t) \in (0, L) \times (0, T),
\end{align*}
\]

(1.1)

with boundary conditions

\[
\begin{align*}
v(0, t) &= \alpha v_x(L, t) - \gamma \beta p_x(L, t) + \frac{\xi_1}{h} v(L, t) = 0, \quad t \in (0, T), \\
p(0, t) &= \beta p_x(L, t) - \gamma \beta v_x(L, t) + \frac{\xi_2}{h} p(L, t) = 0, \quad t \in (0, T),
\end{align*}
\]

(1.2)

where \(v(x, t)\) and \(p(x, t)\) represent respectively the longitudinal vibrations of the centerline of the beam and the total charge accumulated at the electrodes of the piezoelectric beam of length \(L\), and \(\rho, \mu, \alpha, \beta, \gamma\) and \(\xi_i, i = 1, 2\) denote respectively the mass density per unit volume, the magnetic permeability, the elastic stiffness, the beam coefficient of impermeability, the piezoelectric coefficient and positive constant feedback gains. The relationship between \(\alpha, \beta\) and \(\gamma\) is given as \(\alpha = \alpha_1 + \gamma^2 \beta\), where \(\alpha_1 > 0\) represents the elastic stiffness of the model derived from the electrostatic and quasi-static methods of Euler Bernoulli small displacement (see example in ref. [29]). By using multiplier method, the authors proved that the system is exponentially stable, and obtained that the exponential stability is equivalent to the exact observability at the boundary.

Feng and Özer [13] studied the fully-dynamic models

\[
\begin{align*}
\rho v_{tt} - \alpha v_{xx} + \gamma \beta p_{xx} + c_1 v_t + a_1 v_t(t - \tau) &= 0, \quad (x, t) \in (0, L) \times \mathbb{R}^+, \\
\mu p_{tt} - \beta p_{xx} + \gamma \beta v_{xx} + c_2 p_t + a_2 p_t(t - \tau) &= 0, \quad (x, t) \in (0, L) \times \mathbb{R}^+, 
\end{align*}
\]

(1.3)

with clamped-free boundary conditions

\[
\begin{align*}
v(0, t) = p(0, t) &= 0, \quad t \in \mathbb{R}^+, \\
(\alpha v_x - \gamma \beta p_x)(L, t) &= -b_1 v(L, t) - a_1 v(L, t - \tau), \quad t \in \mathbb{R}^+, \\
(p_x - \gamma v_x)(L, t) &= -b_2 p(L, t) - a_2 p(L, t - \tau), \quad t \in \mathbb{R}^+, 
\end{align*}
\]

(1.4)

where the terms \(c_1 v_t(x, t), c_2 p_t(x, t)\), and \(b_1 v(L, t)\) and \(b_2 p(L, t)\) with the gains \(c_1, c_2, b_1, b_2 > 0\) are the distributed and boundary controller terms. By using the Lyapunov theory, they proved that the exponential stability of system (1.3)–(1.4) is retained if the coefficients of the delayed damping terms and the boundary feedback controllers satisfy explicit conditions.

Recently, some researchers have studied Timoshenko system with tip body and hybrid system with tip load damped, see [21, 30, 34, 36]. In industry, many piezoelectric beam devices are in the form of a boundary with a tip body, such as the electrostatic energy harvester mentioned in reference [11, 35]. The tip body has mass, so its appearance will bring tip inertia, which will affect the stability of the system. For example, Timoshenko system is polynomially stable or exponentially stable with or without tip body, respectively (see refs. [21, 30]). Therefore, it is an interesting question whether piezoelectric beam systems with tip bodies have different stability than (1.1)–(1.2).

As a piezoelectric beam with a tip load, the beam is clamped at \(x = 0\), and the tip is fixed at \(x = L\). The center of mass of the tip is the connection point between the tip body and the piezoelectric beam plate. By using the feedback boundary force control to the displacement velocity at \(x = L\), dissipation is introduced into the piezoelectric system. Then the coupling
model is given by
\[
\begin{align*}
\rho V_{tt} - \alpha V_{xx} + \gamma \beta P_{xx} &= 0, \quad (x, t) \in (0, L) \times \mathbb{R}^+, \\
\mu P_{tt} - \beta P_{xx} + \gamma \beta V_{xx} &= 0, \quad (x, t) \in (0, L) \times \mathbb{R}^+,
\end{align*}
\]
with the double boundary conditions
\[
\begin{align*}
V(0, t) = P(0, t) &= 0, \quad t \in \mathbb{R}^+, \\
\alpha V_x(L, t) - \gamma \beta P_x(L, t) + \xi_1 V_t(L, t) + m_1 V_{tt}(L, t) &= 0, \quad t \in \mathbb{R}^+, \\
\beta P_x(L, t) - \gamma \beta V_x(L, t) + \xi_2 P_t(L, t) + m_2 P_{tt}(L, t) &= 0, \quad t \in \mathbb{R}^+,
\end{align*}
\]
and the initial conditions
\[
(V(x, 0), V_t(x, 0), P(x, 0), P_t(x, 0)) = (V_0(x), V_1(x), P_0(x), P_1(x)), x \in (0, L),
\]
where \(V(x, t)\) and \(P(x, t)\) represent respectively the longitudinal vibrations of the centerline of the beam and the total charge accumulated at the electrodes of the piezoelectric beam of length \(L\), \(\xi_1, \xi_2 > 0\) are boundary controller terms, \(m_1\) represents mass of tip load, and \(m_2\) is the inverse of coil inductance.

We assume that the beam interacts with the tip body, and the force of the vibrating beam moves to the end load according to Newton’s law. From \([28, 29]\), \(\alpha V_x(L, t) - \gamma \beta P_x(L, t)\) in boundary condition (1.6)\(2\) represent strains, and \(\beta P_x(L, t) - \gamma \beta V_x(L, t)\) in the boundary condition (1.6)\(3\) refer to voltages. By using the analysis method similar to ref. \([30]\), equation (1.6)\(2\) can be obtained by the force balance at the end \(x = L\). On the other hand, the relationship between voltage and current of inductive circuit is connected by magnetic flux, which is defined as \(U = N \frac{d\Phi}{dt}\) with \(N\Phi = I\). And, \(U\) is voltage, \(I\) is electric current, \(N\) is the number of turns of the coil, \(l\) is the inductance of the coil, and it is a constant for the fixed coil. The above two formulas can deduce \(U = L \frac{dI}{dt}\). Since \(P(L, t)\) is the total charge at the end \(x = L\), we can obtain that \(P_t(L, t) = I(L, t)\) and \(P_{tt}(L, t) = \frac{dI(L, t)}{dt} = \frac{1}{L}U(L, t)\). By combining these results with the fact that \(\beta P_x(L, t) - \gamma \beta V_x(L, t)\) is voltage, (1.6)\(3\) can be obtained by the voltage balance at the end \(x = L\).

We first show the well-posedness of system (1.5)–(1.7) by using the classical Lumer–Phillips theorem. Dealing with the resolvent equation of system (1.5)–(1.7), we obtain an observable inequality. Combining with Borichev and Tomilov theorem \([4]\), we can prove that the system is polynomial stable. The difficulty of stability analysis lies in how to obtain that the system is lack of exponential stability. For this purpose, we divide the original system into a conservation system and an auxiliary system, and show that auxiliary system generates a compact operator. With the help of Wely’s theorem \([37]\), we obtain that the growth bound of original system is zero as in the conservation system. Some research on this type of problem can be found in reference \([1, 3, 5, 12, 17, 18, 20, 22–25, 27, 31, 32]\).

The structure of this paper is as follows. In the next section, we will give the well-posedness of system (1.5)–(1.7). In Section 3, we will show the lack of exponential stability. Finally, we will get the polynomial stability of the system in Section 4.

## 2 WELL-POSEDNESS

In this section, we give a well-posedness result for problem (1.5)–(1.7) by using a semigroup approach.

To define the semigroup associated with (1.5)–(1.7), we introduce two new functions which are defined by
\[
\begin{align*}
\eta(t) &= P_t(L, t), \\
u(t) &= V_t(L, t)
\end{align*}
\]
respectively, with
\[
\begin{align*}
u(0) &= V_1(L) = u_0 \\
\eta(0) &= P_1(L) = \eta_0.
\end{align*}
\]
By using the definition of \(u, \eta\), we can change system (1.5)–(1.7) to
\[
\rho V_{tt} - \alpha V_{xx} + \gamma \beta P_{xx} = 0, \quad (x, t) \in (0, L) \times \mathbb{R}^+, \tag{2.3}
\]
\[
\mu P_{tt} - \beta P_{xx} + \gamma \beta V_{xx} = 0, \quad (x, t) \in (0, L) \times \mathbb{R}^+, \tag{2.4}
\]
with the boundary conditions
\[
V(0, t) = \alpha V_x(L, t) - \gamma \beta P_x(L, t) + \xi_1 u(t) + m_1 u_t(t) = 0, \quad t \in \mathbb{R}^+, \tag{2.5}
\]
\[
P(0, t) = \beta P_x(L, t) - \gamma \beta V_x(L, t) + \xi_2 \eta(t) + m_2 \eta_t(t) = 0, \quad t \in \mathbb{R}^+, \tag{2.6}
\]
and the initial conditions
\[
(V(x, 0), V_t(x, 0), P(x, 0), P_t(x, 0), u(0), \eta(0)) = (V_0, V_1, P_0, P_1, u_0, \eta_0) \quad x \in (0, L). \tag{2.7}
\]

The energy of system (2.3)–(2.7) is given by
\[
E(t) = \frac{1}{2} \int_0^L \left[ \rho |V_t|^2 + \alpha_1 |V_x|^2 + \mu |P_t|^2 + \beta |\gamma V_x - P_x|^2 \right] dx + \frac{m_1}{2} |u|^2 + \frac{m_2}{2} |\eta|^2, \quad t \geq 0. \tag{2.8}
\]

Multiplying (2.3) and (2.4) by \(V_t\) and \(P_t\) respectively, and using the boundary conditions (2.5)–(2.6), we get
\[
\frac{d}{dt} E(t) = -\xi_1 |V_t(L, t)|^2 - \xi_2 |P_t(L, t)|^2, \quad t \geq 0.
\]

Let us define the space \(\mathcal{H}\) as
\[
\mathcal{H} := H^1_0(0, L) \times L^2(0, L) \times H^1_0(0, L) \times L^2(0, L) \times \mathbb{C} \times \mathbb{C},
\]
for \(H^1_0(0, L) = \{ f \in H^1(0, L) : f(0) = 0 \}\), equipped with the inner product
\[
\langle U_1, U_2 \rangle_{\mathcal{H}} = \int_0^L \left[ \rho \Phi_1 \Phi_2 + \mu \Theta_1 \Theta_2 + \alpha_1 V_{1,x} V_{2,x} + \beta (\gamma V_{1,x} - P_{1,x}) (\gamma V_{2,x} - P_{2,x}) \right] dx + m_1 u_1 u_2 + m_2 \eta_1 \eta_2,
\]
where \(U_i = (V_i, \Phi_i, P_i, \Theta_i, u_i, \eta_i) \in \mathcal{H}, i = 1, 2\). Set the vector function \(U = (V, V_t, P, P_t, u, \eta)^T\), then system (2.3)–(2.7) can be written as
\[
\begin{cases}
U_t = AU \\
U(0) = U_0
\end{cases}
\tag{2.9}
\]

where \(U_0 = (V_0, V_1, P_0, P_1, u_0, \eta_0)^T\) and \(A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}\) is given by
\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
\frac{\alpha}{\rho} \partial_{xx} & 0 & -\frac{\gamma \beta}{\rho} \partial_{xx} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
-\frac{\gamma \beta}{\mu} \partial_{xx} & \frac{\beta}{\mu} \partial_{xx} & 0 & 0 & 0 \\
-\frac{\alpha}{m_1} \xi & \frac{\gamma \beta}{m_1} \xi & 0 & -\xi_1 I & 0 \\
\frac{\gamma \beta}{m_2} \xi & -\frac{\beta}{m_2} \xi & 0 & 0 & -\xi_2 I
\end{bmatrix},
\]

with \(\zeta \varphi = \varphi_x(L)\). The domain of the operator \(A\) is given by
\[
D(A) : = \{ U \in \mathcal{H}; \; AU \in \mathcal{H}, \; \Phi(L) = u, \; \Theta(L) = \eta \},
\]
with $U = (V, \Phi, P, \Theta, u, \eta)$. It is not difficult to see that $D(A)$ is densely in the phase space $H$.

We now show that operator $A$ generates a $C_0$-semigroup $\{S(t)\}_{t \geq 0}$ of contractions in the space $H$. For this purpose, we need the following two lemmas.

**Lemma 2.1.** The operator $A$ is dissipative and satisfies that for any $U \in D(A)$,

$$
\text{Re}(AU, U)_H = -\xi_1 |u|^2 - \xi_2 |\eta|^2 \leq 0.
$$

**Proof.** For any $U \in D(A)$, relation (2.10) can be easily verified by using the inner product in $H$ and integration by parts.

**Lemma 2.2.** The operator $A$ is bijective and $0 \in \varphi(A)$, where $\varphi(A)$ is the resolvent set of $A$.

**Proof.** We need to prove that for any $F = (f_1, f_2, f_3, f_4, f_5, f_6) \in H$, there exists a $U = (V, \Phi, P, \Theta, u, \eta) \in D(A)$ such that

$$
AU = F.
$$

Equivalently, we shall consider the existence of unique solution of the system

$$
\begin{align*}
\Phi &= f_1 \quad \text{in} \quad H^1_1(0, L), \\
\frac{\alpha}{\rho} V_{xx} - \frac{\gamma \beta}{\rho} P_{xx} &= f_2 \quad \text{in} \quad L^2_2(0, L), \\
\Theta &= f_3 \quad \text{in} \quad H^1_1(0, L), \\
\frac{\beta}{\mu} P_{xx} - \frac{\gamma \beta}{\mu} V_{xx} &= f_4 \quad \text{in} \quad L^2_2(0, L), \\
-\frac{\alpha}{m_1} V_x(L) + \frac{\gamma \beta}{m_1} P_x(L) - \frac{\xi_1}{m_1} u &= f_5, \\
-\frac{\beta}{m_2} P_x(L) + \frac{\gamma \beta}{m_2} V_x(L) - \frac{\xi_2}{m_2} \eta &= f_6.
\end{align*}
$$

That is, since

$$
\Phi = f_1, \quad \Theta = f_3, \\
u = \Phi(L) = f_1(L), \quad \eta = \Theta(L) = f_3(L),
$$

we need to prove the existence of unique solution of the system

$$
\begin{align*}
\alpha V_{xx} - \gamma \beta P_{xx} &= \rho f_2, \\
\beta P_{xx} - \gamma \beta V_{xx} &= \mu f_4, \\
-\alpha V_x(L) + \gamma \beta P_x(L) &= m_1 f_5 + \xi_1 f_1(L), \\
-\beta P_x(L) + \gamma \beta V_x(L) &= m_2 f_6 + \xi_2 f_3(L).
\end{align*}
$$
After a simple arrangement of system (2.11), we can get that the above system is equivalent to
\[
\begin{align*}
V_{xx} &= \frac{\rho}{\alpha_1} f_2 + \frac{\gamma \mu}{\alpha_1} f_4 \in L^2(0,L), \\
P_{xx} &= \frac{\gamma \rho}{\alpha_1} f_2 + \frac{\alpha \mu}{\alpha_1 \beta} f_4 \in L^2(0,L),
\end{align*}
\]
with the boundary conditions
\[
\begin{align*}
V_x(L) &= -\frac{m_1}{\alpha_1} f_5 - \frac{\xi_1}{\alpha_1} f_1(L) - \frac{\gamma m_2}{\alpha_1} f_6 - \frac{\gamma \xi_2}{\alpha_1} f_3(L) \in \mathbb{C}, \\
P_x(L) &= \frac{m_1 \gamma}{\alpha_1} f_5 - \frac{\gamma \xi_1}{\alpha_1} f_1(L) - \frac{\alpha m_2}{\alpha_1 \beta} f_6 - \frac{\alpha \xi_2}{\alpha_1 \beta} f_3(L) \in \mathbb{C}, \\
V(0) &= P(0) = 0.
\end{align*}
\]
where \( V(0) = P(0) = 0 \) comes from the space \( \mathcal{H} \) itself.

By calculation, we can know that the above problem is well-posed, which means that the operator \( \mathcal{A} \) is a bijection between \( D(\mathcal{A}) \) and the space \( \mathcal{H} \). Since \( \mathcal{A} \) is closed, using the Closed Graph Theorem, we can obtain that \( 0 \in \varphi(\mathcal{A}) \).

Hence, using Lemma 2.1 and Lemma 2.2, we conclude that the operator \( \mathcal{A} \) generates a \( C_0 \)-semigroup \( \{S_A(t)\}_{t \geq 0} \) of contractions on the space \( \mathcal{H} \) by Lumer–Phillips theorem [21]. And we obtain the well-posedness result.

**Theorem 2.1.** Let \( U_0 \in D(\mathcal{A}) \), there exists a unique solution \( U(t) = S(t)U_0 \) of (2.9) such that
\[U \in C([0, \infty); D(\mathcal{A})) \cap C^1([0, \infty); \mathcal{H}).\]

### 3 | THE LACK OF EXPONENTIAL STABILITY

In this section, we are interested in studying the lack of exponential stability of the solution of problem (2.3)–(2.7). Before the proof begins, let us recall some useful definitions and theorems.

The type of a semigroup \( e^{At} \) is defined as (see ref. [9])
\[
\omega(\mathcal{A}) := \lim_{t \to \infty} \frac{\ln \|e^{AT}\|}{t} = \inf_{t > 0} \frac{\ln \|e^{AT}\|}{t}.
\]
Pay attention to that \( \omega(\mathcal{A}) = 0 \) implies \( \|e^{AT}\| = 1 \). Therefore, the conclusion that \( \omega(\mathcal{A}) = 0 \) can be indicated the lack of exponential stability. And note that the spectral radius of the semigroup \( R_\sigma(e^{AT}) = e^{\omega(\mathcal{A})t} \).

**Definition 3.1** [15]. Let \( \sigma_d(S) \) be the set of isolated eigenvalues of \( S \) with finite algebraic multiplicity then
\[
r_{\text{ess}}(S) = \inf \{R > 0; \lambda \in \sigma(S), |\lambda| > R \Rightarrow \lambda \in \sigma_d(S)\}
\]
is called the essential spectrum radius of \( S \).

It is well known that (see ref. [15]) the radius of the essential spectrum is invariant by compact perturbations
\[
r_{\text{ess}}(S) = r_{\text{ess}}(S + \mathcal{K}),
\]
where \( \mathcal{K} \) is a compact operator.
Theorem 3.1 ([9]). Let \( S(t) = e^{At} \) be a \( C_0 \)-semigroup on Hilbert space. Then

\[
\omega(S) = \max\{\omega_{ess}(S), s(A)\},
\]

where \( \omega(S) \) is the growth bound, \( \omega_{ess}(S) \) is the essential growth bound, and \( s(A) \) is the spectral bound of the infinitesimal generator \( A \) of \( S(t) \).

Since system (1.5)–(1.7) is dissipative, we have that \( s(A) \leq 0 \). That is, we only need to show that \( \omega_{ess}(S) = 0 \), which implies that \( \omega(S) = 0 \).

Theorem 3.2 ([37, Weyl’s Theorem]). If the difference of the two operator is compact, then the essential spectrum radius are the same.

Lemma 3.1. Let us fix \( \alpha, \beta, \gamma, \rho, \mu \) and the finite interval \([0, L]\). Assume that there exists a weak solution to equation

\[
\rho V_{tt} - \alpha V_{xx} + \gamma \beta P_{xx} = 0, \quad (x, t) \in (0, L) \times \mathbb{R}^+,
\]

(3.1)

\[
\mu P_{tt} - \beta P_{xx} + \gamma \beta V_{xx} = 0, \quad (x, t) \in (0, L) \times \mathbb{R}^+.
\]

(3.2)

If \( q(x) = mx + n, (m, n \in \mathbb{R}) \) and the functions

\[
E_1(t) = \int_0^L \left( \rho |V_t|^2 + \alpha_1 |V_x|^2 + \mu |P_t|^2 + \beta |\gamma V_x - P_x|^2 \right) dx, \quad t \geq 0,
\]

\[
I(x, t) = \rho |V_t(x, t)|^2 + \alpha_1 |V_x(x, t)|^2 + \mu |P_t(x, t)|^2 + \beta |\gamma V_x - P_x(x, t)|^2, \quad 0 \leq x \leq L, t \geq 0,
\]

then, for \( T \) large enough, there exists a non-negative constant \( M \) satisfying

\[
\left| \int_0^T (q(L)I(L, t) - q(0)I(0, t)) dt - \int_0^T mE_1(t) dt \right| \leq M(E_1(T) + E_1(0)).
\]

Proof. Multiplying (3.1) by \( q(x)\overline{V}_x \), and integrating over \([0, L]\), we have

\[
\int_0^L \rho V_{tt}q(x)\overline{V}_x dx - \frac{\alpha_1}{2} \int_0^L q(x) \frac{d}{dx} |V_x|^2 dx - \int_0^L \gamma \beta q(x)(\gamma V_x - P_x)\overline{V}_x dx = 0.
\]

(3.3)

Multiplying (3.2) by \( q(x)\overline{P}_x \), and integrating over \([0, L]\), we get

\[
\int_0^L \mu P_{tt}q(x)\overline{P}_x dx - \int_0^L \beta q(x)(\gamma V_x - P_x)(-\overline{P}_x) dx = 0.
\]

(3.4)

Adding (3.3) and (3.4), we obtain

\[
\int_0^L \left( \rho V_{tt}q(x)\overline{V}_x + \mu P_{tt}q(x)\overline{P}_x \right) dx - \frac{\alpha_1}{2} \int_0^L q(x) \frac{d}{dx} |V_x|^2 dx - \frac{\beta}{2} \int_0^L q(x) \frac{d}{dx} |\gamma V_x - P_x|^2 dx = 0.
\]

Integrating by parts over \([0, L]\), and using the fact of \( q'(x) = m \), we can show

\[
\int_0^L \left( \rho V_{tt}q(x)\overline{V}_x + \mu P_{tt}q(x)\overline{P}_x \right) dx - \frac{1}{2} \left[ \alpha_1 q(x)|V_x|^2 + \beta q(x)|\gamma V_x - P_x|^2 \right]_0^L
\]

\[
= -\frac{1}{2} \int_0^L \alpha_1 m|V_x|^2 + \beta m|\gamma V_x - P_x|^2 dx.
\]

(3.5)
Integrating (3.5) over $[0, T]$, integrating by parts and applying the Fubini theorem, we obtain

\[
\left| \int_0^T (q(L)I(L,t) - q(0)I(0,t))dt - \int_0^T mE_1(t)dt \right| = 2 \left| \operatorname{Re} \int_0^T q(x) \left[ \mu P \nabla_x \bar{P}_x \right]_0^L dx \right|.
\]

Let $f(x, t) = 2(\rho V_x \nabla_t + \mu P \nabla_x)$. With the help of Young’s inequality, we have

\[
\int_0^L |f(t, x)|dx \leq \int_0^L \left[ |\rho^2 V_t|^2 + |\rho V_x|^2 + \mu^2 |P_t|^2 + |P_x - \gamma V_x + \gamma V_x|^2 \right]dx \\
\leq \int_0^L \left[ |\rho^2 V_t|^2 + (1 + 2\gamma^2)|V_x|^2 + \mu^2 |P_t|^2 + 2 |\gamma V_x - P_x|^2 \right]dx \leq m_0 E_1(t),
\]

where $m_0 = \max\{\rho, (1 + 2\gamma^2)/\alpha_1, \mu, 2/\beta\}$. By using the above inequality, and the fact that $|f(T, x) - f(0, x)| \leq |f(T, x)| + |f(0, x)|$, we can obtain

\[
\left| \int_0^T (q(L)I(L,t) - q(0)I(0,t))dt - \int_0^T mE_1(t)dt \right| \leq \|q\|_{\infty} \left| \operatorname{Re} \int_0^L \left[ f(t, x) \right]_0^L dx \right| \\
\leq \|q\|_{\infty} \int_0^L \left| f(t, x) \right|_0^L dx \leq M(E_1(T) + E_1(0)),
\]

where $M = \|q\|_{\infty} \max\{\rho, (1 + 2\gamma^2)/\alpha_1, \mu, 2/\beta\}$. The conclusion follows immediately. \hfill \Box

Then we consider the undamped piezoelectric beams with tip body

\[
\rho \ddot{V}_x - \alpha \nabla_{xx} \dot{V}_x + \gamma \beta \nabla_{xx} \dot{P}_x = 0, \quad (x, t) \in (0, L) \times \mathbb{R}^+,
\]

\[
\mu \ddot{P}_x - \beta \nabla_{xx} \dot{P}_x + \gamma \beta \nabla_{xx} \dot{V}_x = 0, \quad (x, t) \in (0, L) \times \mathbb{R}^+,
\]

and the boundary conditions

\[
\begin{align*}
\bar{V}(0, t) &= \alpha \bar{V}_x(L, t) - \gamma \beta \bar{P}_x(L, t) + m_1 \ddot{V}_x(L, t) = 0, \quad t \in \mathbb{R}^+, \\
\bar{P}(0, t) &= \beta \bar{P}_x(L, t) - \gamma \beta \bar{V}_x(L, t) + m_2 \ddot{P}_x(L, t) = 0, \quad t \in \mathbb{R}^+,
\end{align*}
\]

with the same initial condition as in (2.7)

\[
(\bar{V}(x, 0), \bar{V}_x(x, 0), \bar{P}(x, 0), \bar{P}_x(x, 0)) = (V_0, V_1, P_0, P_1), \quad x \in (0, L).
\]

Multiplying (3.6), (3.7) by $\bar{V}_t, \bar{P}_t$ respectively, integrating by parts over $[0, L]$, and using the boundary conditions (3.8), we can obtain

\[
\frac{d\bar{E}(t)}{dt} = 0,
\]

where

\[
\bar{E}(t) = \frac{1}{2} \int_0^L \left( \rho |\nabla V|^2 + \alpha_1 |\nabla V_x|^2 + \mu |\nabla P|^2 + \beta |\nabla P_x|^2 \right)dx + \frac{m_1}{2} |\nabla V_x(L, t)|^2 + \frac{m_2}{2} |\nabla P_x(L, t)|^2.
\]

Denote by $S_0$ the semigroup defined by system (3.6)–(3.9).

Remark 3.1. It is straightforward to see that $S_0$ is unitary and $\omega_{\text{ess}}(S_0) = 0$. The detailed proof content can be found in references [9, 36].
Lemma 3.2. \( S(t) - S_0(t) \) is a compact operator, where \( S(t) \) is the semigroup associated to system (2.3)--(2.7), \( S_0(t) \) is the semigroup associated to system (3.6)--(3.9).

Proof. In order to prove this theorem, we need to first give the definition of auxiliary problems related to \( S(t) - S_0(t) \) and prove that \( S(t) - S_0(t) \) is a compact operator by combining the correlation results of compact operators.

**Step 1.** Give the definition of auxiliary problem associated with \( S(t) - S_0(t) \). Solving for \( V(L) \) and \( P(L) \), we can rewrite the boundary conditions (1.6)_2--(1.6)_3 as

\[
V_t(L, t) = e^{-\frac{\lambda_1}{m_1} t} V_1(L) + \frac{1}{m_1} \int_0^t e^{-\frac{\lambda_1}{m_1} (t-s)} (\alpha V_x(L, s) - \gamma \beta P_x(L, s)) ds,
\]
\[
P_t(L, t) = e^{-\frac{\lambda_2}{m_2} t} P_1(L) + \frac{1}{m_2} \int_0^t e^{-\frac{\lambda_2}{m_2} (t-s)} (\beta P_x(L, s) - \gamma \beta V_x(L, s)) ds.
\]

Denote by

\[
D_1(t) = e^{-\frac{\lambda_1}{m_1} t} V_1(L), \quad K_1(t) = \frac{1}{m_1} \int_0^t e^{-\frac{\lambda_1}{m_1} (t-s)} (\alpha V_x(L, s) - \gamma \beta P_x(L, s)) ds,
\]
\[
D_2(t) = e^{-\frac{\lambda_2}{m_2} t} P_1(L), \quad K_2(t) = \frac{1}{m_2} \int_0^t e^{-\frac{\lambda_2}{m_2} (t-s)} (\beta P_x(L, s) - \gamma \beta V_x(L, s)) ds.
\]

Similarly, the boundary condition (3.8) can be rewritten as

\[
\tilde{V}(0, t) = \tilde{P}(0, t) = 0,
\]

and

\[
\tilde{V}_t(L, t) = \tilde{V}_t(L, 0) + \frac{1}{m_1} \int_0^t (\alpha \tilde{V}_x(L, s) - \gamma \tilde{P}_x(L, s)) ds,
\]
\[
\tilde{P}_t(L, t) = \tilde{P}_t(L, 0) + \frac{1}{m_2} \int_0^t (\beta \tilde{P}_x(L, s) - \gamma \tilde{V}_x(L, s)) ds.
\]

Denote by

\[
D_3 = V_1(L), \quad K_3(t) = \frac{1}{m_1} \int_0^t (\alpha \tilde{V}_x(L, s) - \gamma \tilde{P}_x(L, s)) ds,
\]
\[
D_4 = P_1(L), \quad K_4(t) = \frac{1}{m_2} \int_0^t (\beta \tilde{P}_x(L, s) - \gamma \tilde{V}_x(L, s)) ds.
\]

Let \( \hat{V} = V - \tilde{V} \), \( \hat{P} = P - \tilde{P} \). It is easy to see that \( \hat{V} \) and \( \hat{P} \) verify the system

\[
\rho \hat{V}_{tt} - \alpha \hat{V}_{xx} + \gamma \beta \hat{P}_{xx} = 0, \quad (x, t) \in (0, L) \times \mathbb{R}^+,
\]
\[
\mu \hat{P}_{tt} - \beta \hat{P}_{xx} + \gamma \beta \hat{V}_{xx} = 0, \quad (x, t) \in (0, L) \times \mathbb{R}^+,
\]

with the boundary conditions

\[
\hat{V}(0, t) = \hat{P}(0, t) = 0, \quad t \in \mathbb{R}^+,
\]
\[
\hat{V}_t(L, t) = D_1(t) - D_3 + K_1(t) - K_3(t), \quad t \in \mathbb{R}^+,
\]
\[
\hat{P}_t(L, t) = D_2(t) - D_4 + K_2(t) - K_4(t), \quad t \in \mathbb{R}^+.
\]
and the initial conditions
\[
(\hat{\mathbf{V}}(x,0), \hat{\mathbf{P}}(x,0), \hat{\mathbf{V}}_t(x,0), \hat{\mathbf{P}}_t(x,0)) = (0, 0, 0, 0), \quad x \in (0, L).
\]

**Step 2.** Show that the operators \( D_i : \mathcal{H} \to L^2(0,T) \), \( U_0 \mapsto D_i(t) \) and \( K_i : \mathcal{H} \to L^2(0,T) \), \( U_0 \mapsto K_i(t) \), \( i = 1, \ldots, 4 \) are compact operators, where \( D_i, K_i \) are given by (3.10) and (3.11), \( V, P \) are solutions of system (1.5)–(1.7), and \( \hat{\mathbf{V}}, \hat{\mathbf{P}} \) are solutions of system (3.6)–(3.9). Since the energy identity and Lemma 3.1, we have that
\[
t \mapsto \alpha \mathbf{V}_x(L, t) - \gamma \beta \mathbf{P}_x(L, t), \quad t \mapsto \beta \mathbf{P}_x(L, t) - \gamma \beta \mathbf{V}_x(L, t)
\]
are bounded in \( L^2(0, T) \), which implies that \( t \mapsto K_i(U_0, t) \) maps bounded sets of \( \mathcal{H} \) to bounded set of \( H^1(0,T) \). Therefore, \( t \mapsto K_i(U_0, t) \) is a compact application from \( \mathcal{H} \) to \( L^2(0,T) \). Similar proof method can be found in ref. [30].

**Step 3.** Prove that \( S - S_0 \) is compact. Multiplying (3.12), (3.13) by \( \hat{\mathbf{V}}_t, \hat{\mathbf{P}}_t \) respectively, integrating by parts over \([0, L]\), and using the boundary conditions (3.14), we can obtain
\[
\frac{d}{dt} \hat{E}(t) = (\alpha \mathbf{V}_x(L, t) - \gamma \beta \mathbf{P}_x(L, t))(D_1 - D_3 + K_1 - K_3) + (\beta \mathbf{P}_x(L, t) - \gamma \beta \mathbf{V}_x(L, t))(D_2 - D_4 + K_2 - K_4).
\]
Integrating the above equation over \([0, t]\) and using the fact that \( \hat{E}(0) = 0 \), we have
\[
\hat{E}(t) = \int_0^t \left[ (\alpha \mathbf{V}_x(L, s) - \gamma \beta \mathbf{P}_x(L, s))(D_1 - D_3 + K_1 - K_3) + (\beta \mathbf{P}_x(L, s) - \gamma \beta \mathbf{V}_x(L, s))(D_2 - D_4 + K_2 - K_4) \right] ds.
\]
Using Lemma 3.1, we know that \( \alpha \mathbf{V}_x(L, s) - \gamma \beta \mathbf{P}_x(L, s), \beta \mathbf{P}_x(L, s) - \gamma \beta \mathbf{V}_x(L, s) \) are bounded in \( L^2(0, T) \). From the result of Step 2, we have that if \( U_0^\mu \) is a bounded set, then \( D_i(U_0^\mu) \) and \( K_i(U_0^\mu) \) are compact sets in \( L^2(0,T) \). This means that for any bounded sequence of initial data \( U_0^\mu \) in \( \mathcal{H} \), there exists a subsequence such that \( D_i(U_0^\mu) \) and \( K_i(U_0^\mu) \) converges strongly in \( L^2(0,T) \). Therefore, the right-hand side of the equality (3.16) converges strongly. Since the left-hand side of (3.16) is a norm, we know that the solutions of auxiliary system \( \hat{\mathbf{V}}^\mu, \hat{\mathbf{P}}^\mu \) converges in norm, which means that \( (S - S_0)(U_0^\mu) \) is also strongly convergent. So the strong convergence follows which means that \( S(t) - S_0(t) \) is compact.

**Theorem 3.3.** The piezoelectric beam system with magnetic effects and tip body is not exponentially stable.

**Proof.** From Lemma 3.2, we have show that \( S(t) - S_0(t) \) is a compact operator, where \( S(t) \) is the semigroup associated to system (1.5)–(1.7) and \( S_0(t) \) is the semigroup associated to system (3.6)–(3.9).

Since \( S_0(t) \) is unitary, then \( \omega_{ess}(S_0) = 0 \). With the help of Weyl’s Theorem in ref. [37], we have \( \omega_{ess}(S) = \omega_{ess}(S_0) = 0 \). That is, using the Theorem 3.1 in ref. [9], we have that \( \omega(S) = 0 \). Therefore, we conclude that \( S(t) \) is not exponentially stable.

\[\square\]

4 | POLYNOMIAL STABILITY

In the previous section, we have shown that the piezoelectric beam system (2.3)–(2.7) is not exponentially stable. In this section, we will state and prove the polynomial stability of our system. It will be achieved by using the following result of Borichev and Tomilov and two lemmas.

**Theorem 4.1** [4]. Assume that \( \{S(t)\}_{t \geq 0} \) be a bounded \( C_0 \)-semigroup on Hilbert space \( H \). Let \( A \) be the infinitesimal generator of \( \{S(t)\}_{t \geq 0} \) such that \( i \mathbb{R} \subset \rho(A) \). Then, for any \( k > 0 \), the following conditions are equivalent:

1. \( \| (iA-I)^{-1} \|_{\mathcal{L}(H)} = O(|\lambda|^{-k}), \lambda \to \infty; \)
2. \( \| S(t)A^{-1} \|_{\mathcal{L}(H)} = O(t^{-\frac{k}{2}}), t \to \infty. \)
The spectral equation is given by

\[ i\lambda U - AU = F. \]  \hspace{1cm} (4.1)

Rewriting (4.1) in term of its components, we have

\[
\begin{align*}
&\begin{cases}
  i\lambda V - \Phi = f_1 & \text{in } H^1_0(0,L), \\
  i\lambda \rho \Phi - \alpha V_{xx} + \gamma \beta P_{xx} = \rho f_2 & \text{in } L^2(0,L), \\
  i\lambda P - \Theta = f_3 & \text{in } H^1_0(0,L), \\
  i\lambda \mu \Theta - \beta P_{xx} + \gamma \beta V_{xx} = \mu f_4 & \text{in } L^2(0,L), \\
  i\lambda m_1 u + \alpha V_x(L) - \gamma \beta P_x(L) + \xi_1 u = m_1 f_5, \\
  i\lambda m_2 \eta + \beta P_x(L) - \gamma \beta V_x(L) + \xi_2 \eta = m_2 f_6,
\end{cases}
\end{align*}
\]  \hspace{1cm} (4.2)

where \( \lambda \in \mathbb{R}, F = (f_1, f_2, f_3, f_4, f_5, f_6) \in \mathcal{H} \) and

\[ \Phi(L) = u, \quad \Theta(L) = \eta. \]  \hspace{1cm} (4.3)

Taking (4.2) inner product with \( U \) on \( \mathcal{H} \), we have

\[ i\lambda \|U\|^2 - \langle AU, U \rangle_H = \langle F, U \rangle_H. \]

Taking the real part, we obtain that

\[ -\text{Re} \langle AU, U \rangle_H = \text{Re} \langle F, U \rangle_H. \]

From (2.10) and (4.3), we have

\[ \xi_1 |u|^2 + \xi_2 |\eta|^2 = \xi_1 |\Phi(L)|^2 + \xi_2 |\Theta(L)|^2 = -\text{Re} \langle AU, U \rangle_H = \text{Re} \langle F, U \rangle_H \leq C \|U\|_H \|F\|_H. \]  \hspace{1cm} (4.4)

For further proof, we introduce the following functionals and notions.

\[ I_V = \rho q(L)|\Phi(L)|^2 + \alpha_1 q(L)|V_x(L, t)|^2; \]
\[ I_P = \mu q(L)|\Theta(L)|^2 + \beta q(L)|(\gamma V_x - P_x)(L, t)|^2; \]
\[ \mathcal{N}^2 = \int_0^L \rho |\Phi|^2 dx + \int_0^L \mu |\Theta|^2 dx + \int_0^L \alpha_1 |V_x|^2 dx + \int_0^L \beta |\gamma V_x - P_x|^2 dx. \]

**Lemma 4.1.** Let us consider \( F = (f_1, f_2, f_3, f_4, f_5, f_6) \in \mathcal{H}, \lambda \in \mathbb{R}, \) and \( U = (V, \Phi, P, \Theta, u, \eta) \in D(A) \) such that \((i\lambda U - AU) = F\). For \( q \in C^2([0,L]), q(0) = 0, \) we have

\[ I_V + I_P - \int_0^L \rho q_x |\Phi|^2 dx - \int_0^L \mu q_x |\Theta|^2 dx - \int_0^L \alpha_1 q_x |V_x|^2 dx - \int_0^L \beta q_x |\gamma V_x - P_x|^2 dx \]

\[ = - R_1 - R_2, \]

where

\[ R_1 = \text{Re} \int_0^L \left( 2\mu q f_4 \overline{P_x} + 2\mu q \overline{f_{3,x}} \Theta \right) dx, \]
\[ R_2 = \text{Re} \int_0^L \left( 2\rho q f_2 \overline{V_x} - 2\rho q \Phi \overline{f_{1,x}} \right) dx. \]
Proof. Multiplying (4.2) by $q\overline{V}_x$ and integrating on $[0, L]$, we get

$$
\int_0^L \left(-i\lambda \rho \Phi \overline{V}_x + \alpha q V_{xx} \overline{V}_x - \gamma \beta q P_{xx} \overline{V}_x\right) dx = -\int_0^L \rho q f_2 \overline{V}_x dx. \tag{4.5}
$$

Using (4.2)\textsubscript{1}, we can rewrite the first term in (4.5) as

$$
\int_0^L -i\lambda \rho \Phi \overline{V}_x dx = \int_0^L (i\lambda V_x) \rho \Phi dx = \int_0^L \rho q \Phi (\Phi_x + f_{1,x}) dx. \tag{4.6}
$$

Multiplying (4.2)\textsubscript{4} by $q\overline{P}_x$ and integrating on $[0, L]$, we have

$$
\int_0^L \left(-i\lambda \mu \Theta \overline{P}_x + \beta q P_{xx} \overline{P}_x - \gamma \beta q V_{xx} \overline{P}_x\right) dx = -\int_0^L \mu q f_4 \overline{P}_x dx. \tag{4.7}
$$

From (4.2)\textsubscript{3}, we can rewrite the first term in (4.7) as

$$
\int_0^L -i\lambda \mu \Theta \overline{P}_x dx = \int_0^L (i\lambda P_x) \mu \Theta dx = \int_0^L \mu q \Theta (\Theta_x + f_{3,x}) dx. \tag{4.8}
$$

By combining (4.5) with (4.7), and employing (4.6) and (4.8) into it, we conclude that

$$
\int_0^L \rho q \frac{d}{dx}|\Phi|^2 dx + \int_0^L \alpha q \frac{d}{dx}|V_x|^2 dx + \int_0^L \mu q \frac{d}{dx}|\Theta|^2 dx + \int_0^L \beta q \frac{d}{dx}|\gamma V_x - P_x|^2 dx
$$

$$
= \text{Re} \int_0^L \left(-2\mu q f_4 \overline{P}_x - 2\mu q f_{3,x} \Theta - 2\rho q f_2 \overline{V}_x - 2\rho q \Phi f_{1,x} \right) dx.
$$

Then, integrating by part, we obtain that the relation in Lemma 4.1 is correct. \hfill \Box

**Lemma 4.2.** Let $\mathcal{N}, I_V, I_P$ be functionals defined above. Then, we have

$$
\mathcal{N}^2 \leq C \left(I_V + I_P + \|F\|^2_H\right), \tag{4.9}
$$

where $C$ is a constant.

**Proof.** Let $q(x) = x, x \in [0, L]$. From the result of Lemma 4.1, we have

$$
\mathcal{N}^2 = (I_V + I_P) + R_1 + R_2. \tag{4.10}
$$

By the definition of $R_1, R_2$, we conclude that

$$
|R_1| \leq C \mathcal{N} \|F\|_H, \quad |R_2| \leq C \mathcal{N} \|F\|_H. \tag{4.11}
$$

Thanks to the estimate (4.11) and Cauchy–Schwartz inequality, it is straightforward to verify that the relation (4.9) is valid. \hfill \Box

**Theorem 4.2.** $i\mathbb{R} \in \rho(A)$, where $\rho(A)$ is the resolvent set of the operator $A$.

**Proof.** Since the fact of $0 \in \rho(A)$ which we have proved in Section 2, we have that the set

$$
\mathcal{M} = \{\beta > 0 : (-i\beta, i\beta) \subset \rho(A)\} \neq \emptyset.
$$

If $\sup \mathcal{M} = \infty$, then the conclusion clearly holds. Next, we will consider $\sup \mathcal{M} < \infty$ by using reduction to absurdity. Assume that there exists $\lambda > 0$ such that $\sup \mathcal{M} = \lambda < \infty$. Clearly $\lambda \notin \mathcal{M}$. Therefore, there exist $\lambda_n \in \mathcal{M}$ and $\overline{F}_n \in H$.
with \( \|\bar{F}_n\| = 1 \) such that
\[
\left\| (i\lambda_n I - A)^{-1}\bar{F}_n \right\|_H \to \infty.
\]
Let us define \( \bar{U}_n = (i\lambda_n I - A)^{-1}\bar{F}_n \). Then we have that \( i\lambda_n \bar{U}_n - A\bar{U}_n = \bar{F}_n \). Denoting \( U_n = \frac{\bar{U}_n}{\| (i\lambda_n - A)^{-1}\bar{F}_n \|_H} \). Clearly, \( U_n \) satisfies
\[
i\lambda_n U_n - AU_n = F_n,
\]
where \( F_n = \frac{\bar{F}_n}{\| (i\lambda_n - A)^{-1}\bar{F}_n \|_H} \). Since \( \|\bar{F}_n\| = 1 \) and \( \| (i\lambda_n - A)^{-1}\bar{F}_n \|_H \to \infty \), we have \( F_n \to 0 \). Taking inner product with \( U_n \) on \( H \), we obtain
\[
i\lambda_n \|U_n\|^2 - \langle AU_n, U_n \rangle_H = \langle F_n, U_n \rangle_H.
\]
By taking the real part and the fact of \( F_n \to 0 \), we have that
\[-\text{Re}\langle AU_n, U_n \rangle_H = \text{Re}\langle F_n, U_n \rangle_H \to 0,
\]
which implies that
\[
\xi_1 |u_n|^2 + \xi_2 |\eta_n|^2 \to 0.
\]
Thanks to (2.1), we obtain \( \Phi_n(L), \Theta_n(L) \to 0 \). Using \( u_n, \eta_n \to 0 \), \( \lambda_n \in \mathcal{M} \) and \( \sup \mathcal{M} < \infty \) in (4.2), we have \( V_{x_n}(L), P_{x_n}(L) \to 0 \) which implies that \( I_V + I_P \to 0 \). By using Lemma 4.2 and the fact of \( F_n, u_n, \eta_n \to 0 \), we conclude that \( U_n \to 0 \).

This relation contracts with \( \|U_n\| = 1 \). Therefore, by using reduction to absurdity, we have completed the proof of this theorem. \(\square\)

Next, by recalling the fact of Borichev and Tomilov theorem and Lemma 4.2, we prove our result of polynomial stability.

**Theorem 4.3.** The piezoelectric system (2.3)–(2.7) with tip body decays polynomially as
\[
\|U(t)\|_H \leq \frac{C}{\sqrt{t}} \|U_0\|_D(A).
\]

**Proof.** Thanks to (4.2)\(_1\), (4.2)\(_3\), and (4.10), we arrive at
\[
I_V + I_P \leq C(1 + |\lambda|^2)\|U\|_H\|F\|_H + C\|F\|^2_H,
\]
and by using (4.9), we can obtain
\[
N^2 \leq C(1 + |\lambda|^2)\|U\|_H\|F\|_H + C\|F\|^2_H.
\]
Then, from relations (4.4) together with the definition of norm in \( H \), we get
\[
\|U\|^2_H = N^2 + \frac{m_1}{2}|u|^2 + \frac{m_2}{2}|\eta|^2 \leq C(1 + |\lambda|^2)\|U\|_H\|F\|_H + C\|F\|^2_H.
\]
By using the Young inequality, we have
\[
\|U\|^2_H \leq C(1 + |\lambda|^2)^2\|F\|^2_H.
\]
For \( |\lambda| > 1 \) large enough, we have
\[
\|U\|^2_H \leq C|\lambda|^4\|F\|^2_H.
\]
That is,
\[ \|U\|_H \leq C|\lambda|^2\|F\|_H. \]

Finally, we can get the result of polynomial stability by using Theorem 4.1.

\[ \square \]

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