The relationship between skew group algebras and orbifold theory

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Abstract

Let $V$ be a simple vertex operator algebra and let $G$ be a finite automorphism group of $V$. In [DY], it was shown that any irreducible $V$-module is a completely reducible $V^G$-module where $V^G$ is the $G$-invariant sub-vertex operator algebra of $V$. In this paper, we give an alternative proof of this fact using the theory of skew group algebras. We also extend this result to any irreducible $g$-twisted $V$-module when $g$ is in the center of $G$ and $V$ is a $g$-rational vertex operator algebra.

1 Introduction

In orbifold theory, one studies a simple vertex operator algebra (VOA) $V$, a finite automorphism group $G$ and the $G$-invariant sub-vertex operator algebra $V^G$. One of the important problems in this area is to determine the module category for $V^G$. In considering this question, the twisted $V$-modules become important because they restrict to $V^G$-modules. Roughly speaking, for $g \in G$, a $g$-twisted $V$-module $M$ is a $\mathbb{C}$-graded infinite-dimensional vector space such that each element $v \in V$ acts on $M$ by $Y_M(v,z) = \sum_{n \in \mathbb{Q}} v_n z^{-n-1}$, where $z$ is a formal variable and $v_n \in \text{End}M$. If $g$ is the identity element, then $g$-twisted $V$-modules are called $V$-modules.

A vertex operator algebra $V$ is called rational if any $V$-module is completely reducible. The main conjecture in orbifold theory is the following: If $V$ is rational then $V^G$ is rational. Moreover, every irreducible $V^G$-module is contained in some irreducible $g$-twisted $V$-module for some $g \in G$ (cf. [DVVV]). This suggests that for each $g \in G$ one must study each irreducible $g$-twisted $V$-module as a $V^G$-module. In order to approach this conjecture, it is natural to start with the case when the irreducible $g$-twisted $V$-module is the simple vertex operator algebra $V$, itself. This case was done by Dong, Li and Mason in [DLM1]. For a given simple VOA $V$ and a finite-dimensional compact-Lie group $G$ such that the action of $G$ on $V$ is continuous, they showed that $V$ is a completely reducible $V^G$-module. To be more precise, they showed that every simple $G$-module occurs in $V$ and the multiplicity space of each simple $G$-module in $V$ is an irreducible $V^G$-module. Furthermore, they showed that inequivalent irreducible $G$-modules produce inequivalent $V^G$-modules. In [DY], Dong and the author extended the results in [DLM1] to any irreducible $V$-module $M$ when $G$ is a finite automorphism group. Several duality

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theorems of Schur-Weyl type were also obtained. In particular, all inequivalent irreducible $V^G$-modules which occur as $V^G$-submodules of $M$ were classified. Moreover, we established relationships between the irreducible $V^G$-submodules of $M$ and the irreducible $V^G$-submodules of another irreducible $V$-module $N$. Indeed, the results in [DY] hold for any irreducible $g$-twisted $V$-modules when $g$ is in the center of $G$ (cf. [Y]).

In this paper, we use the theory of skew group algebras to provide an alternative way of showing that any irreducible $g$-twisted $V$-module is a completely reducible $V^G$-module when $g$ is in the center of $G$ and $V$ is a $g$-rational simple vertex operator algebra. Let $A$ be a finite-dimensional algebra over $\mathbb{C}$ and $G$ be a finite automorphism group of $A$. To these objects we associate the skew group algebra $A \rtimes G$ as defined in section 2. We then consider a finite-dimensional simple $A$-module $N$ and its inertia subgroup $G_N = \{ h \in G \mid hN \cong N \text{ as } A\text{-modules} \}$. Here, $hN$ is a new simple $A$-module associated to $N$ and $h$. The group $G_N$ acts projectively on $N$. We let $\alpha$ be the corresponding 2-cocycle whose values are roots of unity and we denote by $I$ the set of inequivalent simple $\mathbb{C}^\alpha[G_N]$-modules occurring in $N$. For each $W \in I$, we let $N_W$ be the multiplicity space of $W$ in $N$. The space $N_W$ is, in fact, an $A^G$-module (see Lemma 3.1) where, $A^G$ is the $G$-invariant sub-algebra of $A$. By using the invariant theory of skew group algebras, we prove the following result.

**Theorem 1.1** If $A$ is semisimple, then the space $N_W$ is a simple $A^G$-module for all $W \in I$.

In [DLM4], Dong, Li, and Mason constructed a series of associative algebras $A_{g,n}(V)$ from a given vertex operator algebra $V$. These associative algebras are quotient spaces of $V$ associated with an element $g \in G$ and $n \in \frac{1}{T}\mathbb{Z}_+$ where $T$ is the order of $g$ and $\mathbb{Z}_+$ is the set of nonnegative integers. These algebras were first introduce by Zhu for the case when $g$ is the identity element and $n = 0$ (cf. [Z]). The important properties of the $A_{g,n}(V)$ that are used in this paper are the following: let $M = \bigoplus_{n \in \frac{1}{T}\mathbb{Z}_+} M(n)$ be an admissible $g$-twisted $V$-module with $M(0) \neq 0$. Then

1) $M$ is an irreducible $g$-twisted $V$-module if and only if each $M(n)$ is a simple $A_{g,n}(V)$-module for all $n \in \frac{1}{T}\mathbb{Z}_+$.

2) $M^1$ and $M^2$ are isomorphic as irreducible $g$-twisted $V$-modules if and only if $M^1(n)$ and $M^2(n)$ are isomorphic as nonzero simple $A_{g,n}(V)$-modules.

3) If $V$ is a $g$-rational VOA, then $A_{g,n}(V)$ is a finite-dimensional semisimple associative algebra.

This key results allow us to reduce an infinite dimensional problem to a finite dimensional one.

Now we assume that $g$ is the element in the center of $G$ and that $V$ is a $g$-rational simple vertex operator algebra. Set $G_M = \{ h \in G \mid M \circ h \cong M \text{ as } g\text{-twisted } V\text{-modules} \}$. Here, $M \circ h$ is a new irreducible $g$-twisted $V$-module associated with $g$ and $M$. The group $G_M$ acts projectively on $M$. We let $\alpha_M$ be the corresponding 2-cocycle whose values are roots of unity and denote by $\Lambda_M$ the set of all inequivalent simple $\mathbb{C}^\alpha[M]$-modules. Hence $M = \bigoplus_{W \in \Lambda_M} M^W$ where $M^W$ is the sum of all simple $\mathbb{C}^\alpha[M]$-submodules of $M$ isomorphic to $W$. Furthermore,
\[ M \cong \bigoplus_{W \in \Lambda} W \otimes M_W \] as \( C^\alpha M \otimes V^G \)-modules where \( M_W \) be the multiplicity space of \( W \) in \( M \). By using the properties of the associative algebras \( A_{g,n}(V) \) combined with Theorem 1.1, we obtain the following.

**Theorem 1.2** The space \( M_W \) is an irreducible \( V^G \)-module.

This immediately implies that

**Theorem 1.3** If \( g \) is in the center of \( G \) and \( V \) is a \( g \)-rational simple VOA, then every irreducible \( g \)-twisted \( V \)-module is a completely reducible \( V^G \)-module.

This paper is organized as follows. In the second section, we recall the definition of a skew group algebra \( A \rtimes G \). We also discuss the relation between the modules of \( A \rtimes G \) and \( A \)-modules, and we review several important theorems of \( A^G \)-modules. In the third section, we prove Theorem 1.2. In the fourth section, we recall the definitions of vertex operator algebras and their twisted modules, and we discuss the representation theory of VOAs. We also review the construction of the series of associative algebra \( A_{g,n}(V) \) and their important properties. In the last section, Theorem 1.1 is used to prove Theorem 1.2 and Theorem 1.3.

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## 2 Skew Group Algebras

Let \( A \) be an algebra and let \( G \) be a finite automorphism group of \( A \). We give the definition of the skew group algebra \( A \rtimes G \). We also discuss the relations between modules of \( A \rtimes G \) and \( A \)-modules. Then, we study the \( G \)-invariant sub-algebra of \( A \) and its modules.

**Definition 2.1** [BR, Mo] Let \( A \) be an algebra over a field \( F \). Let \( G \) be a finite automorphism group of \( A \). The skew group algebra \( A \rtimes G \) is the set \( \{ \sum_{g \in G} a_g g \mid a_g \in A \} \) of all finite \( A \)-linear combinations of elements of \( G \) with multiplication that is the linear extension of

\[ a g b h = a g(h) b \]

for \( a, b \in A \) and \( g, h \in G \).

For an \( A \)-module \( M \) and \( g \in G \), we set \( ^g M = M \) as vector spaces over \( F \). We define the \( A \)-action on \( ^g M \) in the following way: for \( a \in A \), \( m \in ^g M \),

\[ a * _g m = g^{-1}(a)m. \quad (2.1) \]

Now assume that \( M \) is a finite-dimensional simple \( A \)-module and the field \( F \) is algebraically closed. The *inertia subgroup* of \( M \) is \( G_M = \{ h \in G \mid ^h M \cong M \text{ as } A\text{-modules} \} \). By Schur’s
Lemma, we conclude that $\text{Hom}_{A}(h^{}M, M)$ is one-dimensional. For each $h \in G_M$, we fix an isomorphism $\phi(h) : h^{}M \rightarrow M$. Here, $\phi(h)$ satisfies $\phi(h)(h^{-1}\alpha m) = \phi(h)(a * h\ m) = a \phi(h)m$. This implies that

$$\phi(h)a = h(a)\phi(h). \quad (2.2)$$

Any linear transformation $\psi$ on $M$ satisfying $\psi a = h(a)\psi$ for all $a \in A$ must belong to $\text{Hom}_{A}(h^{}M, M)$. Hence, $\psi$ is a multiple of $\phi(h)$. Since $\phi(h)\phi(k)a = (hk)(a)\phi(h)\phi(k)$, we see that $\phi(h)\phi(k) \in \text{Hom}_{A}(h^{}k^{}M, M)$. Furthermore, there is $\alpha(h, k) \in F^\times$ such that

$$\phi(h)\phi(k) = \alpha(h, k)\phi(h^{}k).$$

By associativity of composition, we have

$$\alpha(h, k)\alpha(h^{}k, l) = \alpha(h, kl)\alpha(k, l) \quad \text{for all} \quad h, k, l \in G_M.$$ 

Note that one can choose $\phi(1) = 1_M$ in order to obtain $\alpha(1, h) = \alpha(h, 1) = 1$ for $h \in G$. Thus $F[G_M]$ acts projectively on $M$ and $\alpha$ is a 2-cocycle. By using this $\alpha$ we define a twisted group algebra $F^{\alpha^{-1}}[G_M]$ over $F$ with basis $c_h, h \in G_M$, and with multiplication defined by

$$c_h c_k = \alpha(h, k)^{-1}c_{hk} \quad h, k \in G_M.$$ 

**Proposition 2.2** [BR] Let $M$ be a simple $A$-module. For any $F^{\alpha^{-1}}[G_M]$-module $V$, there is an $(A \rtimes G_M)$-module action on $M \otimes V$ defined by

$$(ah)(m \otimes v) = a\phi(h)(m) \otimes c_h v. \quad (2.3)$$

**Proof:** Let $a, b \in A$ and $h, k \in G_M$. We have

$$(ah)((bk)(m \otimes v)) = (ah)(b\phi(k)(m) \otimes c_k v) = a\phi(h)(b\phi(k)(m)) \otimes c_h c_k v = ah(b)\phi(h)\phi(k)(m) \otimes \alpha(h, k)^{-1}c_{hk} v = ah(b)\alpha(h, k)\phi(hk) \otimes \alpha(h, k)^{-1}c_{hk} v = ah(b)\phi(hk)(m) \otimes c_{hk} v = ah(b)hk(m \otimes v) = (ahbk)(m \otimes v). \quad \square$$

Next we discuss the modules of the algebra $A \rtimes G$ and their relation to $A$-modules. In fact, we recall that simple $A \rtimes G$-modules can be obtained from the simple $A$-modules and certain twisted group algebras.
Theorem 2.3 [BR] Assume $N$ is a finite-dimensional simple $A \times G$-module. Let $A^\lambda$ be a simple $A$-submodule of $N$, and let $H$ denotes the inertia subgroup of $A^\lambda$. Suppose that the isomorphisms $hA^\lambda \cong A^\lambda, h \in H$, determine the cocycle $\alpha$ by $\phi(h)\phi(k) = \alpha(h,k)\phi(hk)$. Then there is a simple $F^{\alpha^{-1}}[H]$-module $H'$ so that $$N \cong \text{Ind}_{A \times H}^{A \times G}(A^\lambda \otimes H').$$

Corollary 2.4 In fact, $H' = \text{Hom}_A(A^\lambda, P)$ where $P = \sum_{h \in H} hA^\lambda$.

Next, we discuss the $G$-invariants, $A^G = \{a \in A \mid g(a) = a \text{ for all } g \in G\}$. We assume that $|G|^{-1} \in F$. Set $$e = |G|^{-1} \sum_{g \in G} g \in A \times G.$$ Clearly, $e$ is an idempotent of $A \times G$. Furthermore, $ea = ae$ and $eh = he$ for all $a \in A^G, h \in G$. If $a$ is an arbitrary element of $A$, then $\sum_{g \in G} g(a) \in A^G$. Moreover, we obtain the following.

Proposition 2.5 [BR]

i) The map $\Phi : A^G \to e(A \times G)e, \ a \mapsto ae$ is an algebra isomorphism.

ii) The map $\Psi : A \to (A \times G)e, \ a \mapsto ae$, is an isomorphism of $(A \times G, A^G)$-bimodules.

The following theorem is the key theorem of this paper. This theorem can be found in [BR].

Theorem 2.6 (Invariant Theory) Let $M$ be a simple module for $A \times G$. Then $eM$ is a simple module for $e(A \times G)e$. Furthermore, every simple $e(A \times G)e$-module is obtained in this way.

3 The Main Theorem

For the rest of this paper, we set $F = \mathbb{C}$. We also assume that $A$ is semisimple. Let $M$ be a finite-dimensional simple $A$-module and let $G_M$ be its inertia subgroup. By the previous section, we know that $G_M$ acts projectively on $M$ and we let $\alpha_M$ be the corresponding 2-cocycle whose values are roots of unity. Since $C^{\alpha_M}[G_M]$ is semisimple, $M$ is a semisimple $C^{\alpha_M}[G_M]$-module. Let $J_{M,\alpha_M}$ be the set of irreducible $\alpha_M$-characters $\lambda$ of $C^{\alpha_M}[G_M]$ such that the corresponding simple-modules $W_\lambda$ occur in $M$. We write $M = \bigoplus_{\lambda \in J_{M,\alpha_M}} M^\lambda$ where $M^\lambda$ is the sum of simple $C^{\alpha_M}[G_M]$-submodules of $M$ isomorphic to $W_\lambda$. Let $M_\lambda$ be the multiplicity space of $W_\lambda$ in $M$ (ie. $M_\lambda = \text{Hom}_{C^{\alpha_M}[G_M]}(W_\lambda, M)$). We can realize $M_\lambda$ as a subspace of $M$ in the following way. Let $w \in W_\lambda$ be a fixed nonzero vector. We identify $\text{Hom}_{C^{\alpha_M}[G_M]}(W_\lambda, M)$ with the subspace $\{f(w) \mid f \in \text{Hom}_{C^{\alpha_M}[G_M]}(W_\lambda, M)\}$ of $M^\lambda$.

Lemma 3.1 $M_\lambda$ is an $A^G$-module.

Proof: Since $\phi(h)a = a\phi(h)$ for all $a \in A^G$ (cf. [2, 3]), hence the actions of $C^{\alpha_M}[G_M]$ and $A^G$ on $M$ are commutative and the lemma follows immediately. □
Remark 3.2 $M_\lambda$ is also an $A^{G_M}$-module.

Observe that $M^\lambda$ and $M_\lambda \otimes W_\lambda$ are isomorphic as both $\mathbb{C}^{aM}[G_M]$-modules and $A^G$-modules. Here, $A^G$ acts on the first tensor factor of $M_\lambda \otimes W_\lambda$ and $\mathbb{C}^{aM}[G_M]$ acts on the second tensor factor. We now identify $M^\lambda$ with $M_\lambda \otimes W_\lambda$ as $A^G \otimes \mathbb{C}^{aM}[G_M]$-modules.

Let $\{\bar{g}|g \in G\}$ be the basis for $\mathbb{C}^a[G]$-module. Recall that for a $\mathbb{C}[G]$-module $V$, $V^* = \text{Hom}_\mathbb{C}(V, \mathbb{C})$ becomes a $\mathbb{C}^{a-1}[G]$-module via

$$(c_g \psi)(v) = \psi(\bar{g}^{-1} v)$$

for all $g \in G, v \in V, \psi \in V^*$. Here, $\{c_g|g \in G\}$ is a basis of $\mathbb{C}^{a-1}[G]$. The dual space $V^*$ is called the contragredient module of $V$.

For a fixed $\gamma \in J_{M,a_M}$, we let $\gamma^*$ be the $\alpha_{M}^{-1}$-character of $\mathbb{C}^{aM-1}[G_M]$ dual to $\gamma$. The corresponding $\mathbb{C}^{aM-1}[G_M]$-module is $W_\gamma^*$. By Proposition (2.3), we conclude that $M \otimes W_\gamma^*$ is an $A \times G_M$-module.

Theorem 3.3 If $A$ is a semisimple associative algebra, then $\text{Ind}_{A \times G_M}^{A \times G} M \otimes W_\gamma^*$ is a simple $A \times G$-module.

Proof: For any nonzero $w \in W_\gamma^*$, we let $M_h = M \otimes c_h w$ for $h \in G_M$. Then we have $M \otimes W_\gamma^* = \bigoplus_{h \in G_M} M_h$. Note that for $h \in G_M$, $M_h$ is isomorphic to $M$ as $A$-modules. Let $G = \bigcup_{i=1}^k g_i G$ be a left coset decomposition. Hence,

$$\text{Ind}_{A \times G_M}^{A \times G} M \otimes W_\gamma^* = \bigoplus_{i=1}^k g_i \otimes \sum_{h \in G_M} M_h = \bigoplus_{i=1}^k \sum_{h \in G_M} g_i \otimes M_h.$$

The space $g_i \otimes M_h$ is an $A$-module under the following action: for $a \in A, g_i \otimes m \otimes c_h w \in g_i \otimes M_h$

$$a \cdot (g_i \otimes m \otimes c_h w) = g_i \otimes g_i^{-1}(a)m \otimes c_h w.$$ 

Indeed, $g_i \otimes M_h$ is isomorphic to $g_i M$ as $A$-modules. Hence, $g_i \otimes M_h$ is a simple $A$-module.

Finally, we will show that $\text{Ind}_{A \times G_M}^{A \times G} M \otimes W_\gamma^*$ is a simple $A \times G$-module. We let $N$ be an $A \times G$-submodule of $\text{Ind}_{A \times G_M}^{A \times G} M \otimes W_\gamma^*$. Consequently, $N$ is an $A$-module. Since $A$ is semisimple, $N$ contains a simple $A$-submodule of $\bigoplus_{i=1}^k \sum_{h \in G_M} g_i \otimes M_h$. Hence, there exist $j \in \{1, ..., k\}$ and $h' \in G_M$ such that $g_j \otimes M_{h'}$ is contained in $N$. Then $1 \otimes M_1 = (g_j h')^{-1} \cdot g_j \otimes M_{h'} \subset N$. These imply that $\bigoplus_{i=1}^k \sum_{h \in G_M} g_i \otimes M_h \subset N$. Therefore, $\text{Ind}_{A \times G_M}^{A \times G} M \otimes W_\gamma^*$ is a simple $A \times G$-module. □

For any finite group $G$, and any $G$-module $W$, we denote the $G$-invariant submodule of $W$ by $\text{Inv}_G(W)$ (i.e., $\text{Inv}_G(W) = \{x \in W \mid gx = x \text{ for all } g \in G\}$).

Proposition 3.4 [K] For any finite dimensional $\mathbb{C}^a[G]$-modules $N$, $M$, we have the natural isomorphisms

$$\text{Hom}_{\mathbb{C}^a[G]}(M, N) \cong \text{Hom}_{\mathbb{C}[G]}(\mathbb{C}, M^* \otimes N) \cong \text{Inv}_G(M^* \otimes N).$$

Moreover, $\text{Inv}_G(M^* \otimes N) = \left(\sum_{g \in G} g\right) \cdot (M^* \otimes N)$. 
Set
\[ \mathcal{M} = \text{Ind}_{A \ltimes G}^{A \ltimes G_M} M \otimes W_{\gamma}^*. \]

Then
\[ \mathcal{M} = \bigoplus_{\lambda \in J_{M,\alpha M}} \bigoplus_{i=1}^k g_i \otimes M_\lambda \otimes W_\lambda \otimes W_{\gamma}^* \]

where \( G = \bigcup_{i=1}^k g_i G_M \) is a left coset decomposition. Note that for any \( \lambda \in J_{M,\alpha M} \), \( W_\lambda \otimes W_{\gamma}^* \) is a \( \mathbb{C}[G_M] \)-module.

**Theorem 3.5 (The Main Theorem)** For any \( \gamma \in J_{M,\alpha M} \), the space \( M_{\gamma} \) is a simple \( A \ltimes G \)-module.

**Proof:** Let \( \sum_{\lambda \in J_{M,\alpha M}} \sum_{i=1}^k g_i \otimes a_i^\lambda \otimes b_i^\lambda \otimes w_i^\gamma \in \mathcal{M} \). Then
\[
e( \sum_{\lambda \in J_{M,\alpha M}} \sum_{i=1}^k g_i \otimes a_i^\lambda \otimes b_i^\lambda \otimes w_i^\gamma ) = \sum_{\lambda \in J_{M,\alpha M}} \sum_{i=1}^k e \otimes a_i^\lambda \otimes b_i^\lambda \otimes w_i^\gamma
\]
\[= \sum_{\lambda \in J_{M,\alpha M}} \sum_{i=1}^k e^2 \otimes a_i^\lambda \otimes b_i^\lambda \otimes w_i^\gamma
\]
\[= \sum_{\lambda \in J_{M,\alpha M}} \sum_{i=1}^k e(\sum_{j=1}^k g_j)(\sum_{h \in G_M} h)(a_i^\lambda \otimes b_i^\lambda \otimes w_i^\gamma)
\]
\[= \sum_{\lambda \in J_{M,\alpha M}} \sum_{i=1}^k e(\sum_{j=1}^k g_j)(a_i^\lambda \otimes \sum_{h \in G_M} h)(b_i^\lambda \otimes w_i^\gamma)
\]
\[= \sum_{i=1}^k e \otimes a_i^\gamma \otimes \frac{1}{|G_M|} \sum_{h \in G_M} h)(b_i^\gamma \otimes w_i^\gamma) \quad \text{(by Proposition 3.4)}
\]

We have \( e(\mathcal{M}) \subset e \otimes M_{\gamma} \otimes \text{Inv}_G(W_{\gamma} \otimes W_{\gamma}^*) \). In fact, \( e(\mathcal{M}) = e \otimes M_{\gamma} \otimes \text{Inv}_G(W_{\gamma} \otimes W_{\gamma}^*) \). Since \( \mathcal{M} \) is a simple \( A \ltimes G \)-module, Theorem 2.6 implies that \( e \otimes M_{\gamma} \otimes \text{Inv}_G(W_{\gamma} \otimes W_{\gamma}^*) \) is a simple \( A \ltimes G \)-module. Hence \( M_{\gamma} \) is a simple \( A \ltimes G \)-module. \( \square \)

**Corollary 3.6** \( M \) is a completely reducible \( A \ltimes G \)-module.

### 4 Vertex operator algebras and related results

We begin by reviewing the definitions of vertex operator algebras and their twisted modules. We also recall some relevant results about the representation theory of vertex operator algebras. Furthermore, for a fixed VOA \( V \) we review the construction of a series of associative algebra \( A_{g,n}(V) \) and some of their main properties which play an important role in the next section.
For a vector space \( W \), we let \( W[[z, z^{-1}]] \) be the space of \( W \)-valued formal series in arbitrary integral powers of \( z \).

**Definition 4.1 [FLM, FHL]** A vertex operator algebra is a \( \mathbb{Z} \)-graded vector space

\[
V = \oplus_{n \in \mathbb{Z}} V_n;
\]

such that

\[
\dim V_n < \infty \quad \text{and} \quad V_n = 0 \quad \text{if} \quad n \text{ is sufficiently small.} \tag{4.4}
\]

Moreover, there is a linear map

\[
V \to (\text{End} V)[[z, z^{-1}]])
\]

\[
v \mapsto Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \quad (v_n \in \text{End} V) \tag{4.6}
\]

and two distinguished vectors \( 1 \in V_0, \omega \in V_2 \) satisfying the following conditions for all \( u, v \in V \):

\[
u_n v = 0 \quad \text{for} \quad n \text{ sufficiently large;} \tag{4.7}
\]

\[
Y(1, z) = 1_V; \tag{4.8}
\]

\[
Y(v, z)1 \in V[[z]] \quad \text{and} \quad \lim_{z \to 0} Y(v, z)1 = v; \tag{4.9}
\]

\[
z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u, z_1)Y(v, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{z_0}\right) Y(v, z_2)Y(u, z_1)
\]

\[
= z_2^{-1}\delta\left(\frac{z_1 - z_2}{z_2}\right) Y(Y(u, z_0)v, z_2) \tag{4.10}
\]

(Jacobi identity) where \( \delta(z) = \sum_{n \in \mathbb{Z}} z^n \) is the algebraic formulation of the \( \delta \)-function at \( 1 \), and all binomial expressions are to be expanded in nonnegative integral powers of the second variable;

\[
[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}(\text{rank} V) \tag{4.11}
\]

for \( m, n \in \mathbb{Z} \), where

\[
L(n) = \omega_{n+1} \quad \text{for} \quad n \in \mathbb{Z}, \quad \text{i.e.,} \quad Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2} \tag{4.12}
\]

and

\[
\text{rank} V \in \mathbb{Q}; \tag{4.13}
\]

\[
L(0)v = nv = (\text{wt} v)v \quad \text{for} \quad v \in V_n \quad (n \in \mathbb{Z}); \tag{4.14}
\]

\[
\frac{d}{dz} Y(v, z) = Y(L(-1)v, z). \tag{4.15}
\]

We denote the vertex operator algebra just defined by \( (V, Y, 1, \omega) \) (or briefly, by \( V \)). The series \( Y(v, z) \) are called vertex operators.

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Remark 4.2 The operators \( L(n) \) generate a copy of the Virasoro algebra represented on \( V \) with the central charge rank \( V \).

Definition 4.3 An automorphism of \( (V, Y, 1, \omega) \) is a linear isomorphism \( g: V \to V \) satisfying

\[
gY(v, z)g^{-1} = Y(gv, z), \quad v \in V,\]
\[
g1 = 1,\]
\[
g\omega = \omega.
\]

Let \( \text{Aut}(V) \) denote the group of all automorphisms of \( V \).

Remark 4.4 For \( g \in \text{Aut}(V) \), \( g \) commutes with the component operators \( L(n) \) of \( \omega \), and in particular, \( g \) preserves the homogeneous spaces \( V_n \) which are the eigenspaces for \( L(0) \). Consequently, each \( V_n \) is a module for \( \text{Aut}(V) \).

For \( g, \) an automorphism of the VOA \( V \) of order \( T \), we denote the decomposition of \( V \) into eigenspaces with respect to the action of \( g \) as \( V = \bigoplus_{r=0}^{T-1} V^r \) where \( V^r = \{ v \in V | gv = e^{2\pi i r/T} v \} \). For a vector space \( W \), we denote the space of \( W \)-valued formal series in arbitrary complex powers of \( z \) by \( W \{ z \} \).

Definition 4.5 A weak \( g \)-twisted \( V \)-module \( M \) is a vector space equipped with a linear map

\[
v \mapsto Y_M(v, z) = \sum_{n \in \mathbb{Q}} v_n z^{-n-1} \quad (v_n \in \text{End} M)
\]

satisfying axioms analogous to (4.7)-(4.8) and (4.10). To describe these, we let \( u \in V^r, v \in V \) and \( w \in M \). Then

\[
Y_M(u, z) = \sum_{n \in r/T + \mathbb{Z}} u_n z^{-n-1};
\]

\[
u_n w = 0 \quad \text{for} \quad n \quad \text{sufficiently large};
\]

\[
Y_M(1, z) = 1_M;
\]

\[
z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_M(u, z_1) Y_M(v, z_2) - z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y_M(v, z_2) Y_M(u, z_1)
\]

\[
= z_2^{-1} \left( \frac{z_1 - z_0}{z_2} \right)^{-r/T} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y_M(Y(u, z_0)v, z_2).
\]

We denote this module by \( (M, Y_M) \), or briefly by \( M \). Equation (4.20) is called the twisted Jacobi identity. If \( g \) is the identity element, this reduces to the definition of a weak \( V \)-module and (4.20) is the untwisted Jacobi identity.
Definition 4.6 Suppose that $(M_i, Y_i)$ are two weak $g$-twisted $V$-modules, $i=1,2$. A homomorphism from $M_1$ to $M_2$ is a linear map $f : M_1 \to M_2$ which satisfies

$$fY_{M_1}(v, z) = Y_{M_2}(v, z)f$$

for all $v \in V$. We call $f$ an isomorphism if $f$ is also a linear isomorphism.

Definition 4.7 An (ordinary) $g$-twisted $V$-module is a weak $g$-twisted $V$-module $M$ which carries a $\mathbb{C}$-grading induced by the spectrum of $L(0)$. Then

$$M = \bigoplus_{\lambda \in \mathbb{C}} M_{\lambda}$$

where $M_{\lambda} = \{w \in M | L(0)w = \lambda w\}$, $\dim M_{\lambda} < \infty$. Moreover, for fixed $\lambda$, $M_{\lambda+\frac{n}{t}} = 0$ for all small enough integers $n$.

Lemma 4.8 If $M$ is a simple (ordinary) $g$-twisted $V$-module, then $M = \bigoplus_{n=0}^{\infty} M_{\lambda+\frac{n}{t}}$ for some $\lambda \in \mathbb{C}$ such that $M_\lambda \neq 0$ and $M_{\lambda+\frac{n}{t}} = 0$ for $n < 0$.

Proof: Let $v \in V^r, m \in \frac{1}{t} \mathbb{Z} + \mathbb{Z}$. We recall that, $[L(0), v_m] = (w_m - m - 1)v_m$ on $M$. By using this relation, we can show that $M(\beta) = \sum_{n \in \mathbb{Z}} M_{\beta+\frac{n}{t}}$ is a $g$-twisted $V$-submodule of $M$. Here, $\beta \in \mathbb{C}$. Hence, $M = M(\beta)$. By using the fact that $M_{\beta+\frac{n}{t}} = 0$ for all small enough integer $n$. We can choose $\lambda \in \mathbb{C}$ such that $M = \bigoplus_{n=0}^{\infty} M_{\lambda+\frac{n}{t}}$, $M_\lambda \neq 0$, and $M_{\lambda+\frac{n}{t}} = 0$ for $n < 0$. □

Definition 4.9 An admissible $g$-twisted $V$-module is a weak $g$-twisted $V$-module $M$ which carries a $\frac{1}{t} \mathbb{Z}+$ grading $M = \bigoplus_{n \in \frac{1}{t} \mathbb{Z}+} M(n)$ satisfying the following condition:

$$v_m M(n) \subset M(n + wt - m - 1)$$

for homogeneous $v \in V$. Here, $\frac{1}{t} \mathbb{Z}+$ is the set of nonnegative integers.

The notion of admissible $g$-twisted $V$-module here is equivalent to the notion of a module in $[\mathbb{Z}]$ when $g$ is the identity element. Using a grading shift we can always arrange the grading on $M$ so that $M(0) \neq 0$. This shift is important in the study of the algebra $A_{g,n}(V)$ below.

Lemma 4.10 [DLM2] Any $g$-twisted $V$-module is an admissible $g$-twisted $V$-module.

So, there is a natural identification of the category of $g$-twisted $V$-modules with a sub-category of the category of admissible $g$-twisted $V$-modules.

Definition 4.11 $V$ is called $g$-rational if every admissible $g$-twisted $V$-module is a direct sum of irreducible admissible $g$-twisted $V$-modules.
**Theorem 4.12** [DLM2] If $V$ is $g$-rational then there are only finitely many inequivalent irreducible admissible $g$-twisted $V$-modules. Moreover, each irreducible admissible $g$-twisted $V$-module is an ordinary $g$-twisted $V$-module.

**Proposition 4.13** [L, DM1] If $M$ is a simple weak $g$-twisted $V$-module then $M$ is spanned by \( \{u_n|u \in V, n \in \mathbb{Q}\} \) where $m \in M$ is a fixed nonzero vector.

Next, we recall the construction of the associative algebra $A_{g,n}(V)$ and some related results. This algebra was first introduced by Zhu for the case $g$ is the identity element and $n$ is zero. It was later done for the general case by Dong, Li, and Mason.

Let $V = (V, Y, 1, \omega)$ be a vertex operator algebra and $g$ be an automorphism of $V$ of order $T$. Then $V$ is a direct sum of eigenspaces of $g$:
\[
V = \bigoplus_{r=0}^{T-1} V^r \quad \text{where} \quad V^r = \{v \in V | gv = e^{2\pi ir/T}v\}.
\]
Fix $n = l + \frac{m}{T} \in \mathbb{Z}$ with $l$ a nonnegative integer and $0 \leq i \leq T - 1$. For $0 \leq r \leq T - 1$ we define
\[
\delta_i(r) = \begin{cases} 
1 & \text{if } i \geq r \\
0 & \text{if } i < r.
\end{cases}
\]
We also set $\delta_i(T) = 1$. Let
\[
O_{g,n}(V) = \{u \circ_{g,n} v, \quad L(-1)u + L(0)u \mid \text{homogeneous } u \in V^r, \text{ and } v \in V \}.
\]
Here, $u \circ_{g,n} v = \text{Res}_z Y(u, z)v^{(1+2)^{\pi i u-1+\delta_i(r)l+r/T}}z^{-2l+\delta_i(r)l+\delta_i(T-r)}$.

Define $A_{g,n}(V) = V/O_{g,n}(V)$. Then $A_{g,n}(V)$ is the untwisted associative algebra $A_n(V)$ as defined in [DLM3] if $g$ is the identity element and is $A_g(V)$ in [DLM2] if $n = 0$. We also define a second product $*_{g,n}$ on $V$ for $u$ and $v$ as above:
\[
u *_{g,n} v = \begin{cases} 
\sum_{m=0}^{T-1} (-1)^m (m+1) \text{Res}_z Y(u, z) v^{(1+2)^{\pi i u-1+\delta_i(r)l+r/T}}z^{2l+\delta_i(T-r)} & \text{if } r = 0 \\
0 & \text{if } r > 0.
\end{cases}
\]

Extend this linearly to obtain a bilinear product on $V$.

Let $M = \bigoplus_{n \in \frac{1}{T} \mathbb{Z}+} M(n)$ be an admissible $g$-twisted $V$-module such that $M(0) \neq 0$. Following [Z] we define weight zero operator $o_M(v) = v_{\text{wt} - 1}$ on $M$ for homogeneous $v$ and extend $o_M(v)$ to all $v$ by linearity. It is clear from the definition that $o_M(v)M(n) \subset M(n)$ for all $n$.

**Theorem 4.14** Let $V$ be a vertex operator algebra and $M$ an admissible $g$-twisted $V$-module. Then

1) The product $*_{g,n}$ induces the structure of an associative algebra on $A_{g,n}(V)$ with identity $1 + O_{g,n}(V)$. Moreover, $\omega + O_{g,n}(V)$ is a central element of $A_{g,n}(V)$.

2) For $0 \leq m \leq n$, the map $\psi_n : v + O_{g,n}(V) \mapsto o_M(v)$ from $A_{g,n}(V)$ to $\text{End}M(m)$ makes $M(m)$ an $A_{g,n}(V)$-module.

3) $M$ is irreducible $g$-twisted $V$-module if and only if $M(n)$ is a simple $A_{g,n}(V)$-module for all $n \in \frac{1}{T} \mathbb{Z}+$. 

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Parts 1), and 2) are proved in [DLM4], and 3) is proved in [DM2].

**Theorem 4.15** [DLM4] There is a bijection map between the isomorphism classes of the irreducible admissible $g$-twisted $V$-modules and the isomorphism classes of the simple $A_{g,n}(V)$-modules which can not factor through $A_{g,n-1/T}(V)$.

**Theorem 4.16** [DLM4] Suppose $V$ is a $g$-rational. Then

1) $A_{g,n}(V)$ is a finite-dimensional semisimple associative for all $n \in \frac{1}{g} \mathbb{Z}_+$. 

2) There is a bijection map between the category of finite-dimensional $A_{g,n}(V)$-modules whose irreducible components cannot factor though $A_{g,n-1/T}(V)$ and the category of ordinary $g$-twisted $V$-modules.

5 An application of skew group algebras in orbifold theory

Let $V$ be a simple vertex operator algebra. Let $G$ be a finite automorphism group of $V$. In this section we will use the theory of skew group algebras to show the complete reducibility of any irreducible $g$-twisted $V$-module as a $V^G$-module when $g$ is in the center of $G$ and $V$ is a $g$-rational vertex operator algebra.

We begin with the following setting. For $g \in G$, we let $(M,Y_M)$ be an irreducible $g$-twisted $V$-module. For $h \in G$, we set

$$(M,Y_M) \circ h = (M \circ h, Y_M \circ h).$$

Here, $M \circ h = M$ as vector spaces and

$$Y_M(v,z) = Y_M(hv,z) \text{ for } v \in V.$$ (5.1)

In fact, the space $M \circ h$ is an irreducible $h^{-1}gh$-twisted $V$-module. Thus, $M \circ h$ is an irreducible $g$-twisted $V$-module if and only if $h$ is in the centralizer of $g$.

We set $G_M = \{ h \in G \mid (M,Y_M) \cong (M \circ h, Y_M \circ h) \text{ as } g\text{-twisted } V\text{-modules} \}$. It was proved in [DM3] that $g \in G_M$. To be more precise, we write $M = \bigoplus_{\lambda,\mu} M_{\lambda,\mu}$ where $T$ is the order of $g$. Define $\phi(g) : M \to M$ by $\phi(g)|_{M_{\lambda,\mu}} = e^{-2\pi i \frac{\mu}{\lambda}}$. Then $\phi(g)$ is an automorphism of $(M,Y_M)$.

For $h \in G_M$, there is a linear isomorphism $\phi(h) : M \to M$ satisfying

$$\phi(h)Y_M(v,z)\phi(h)^{-1} = Y_{M \circ h}(v,z) = Y_M(hv,z)$$ (5.2)

for $v \in V$. The simplicity of $M$ together with Schur’s lemma shows that $h \to \phi(h)$ is a projective representation of $G_M$ on $M$. Note that we can choose $\phi(1) = 1_M$. Let $\alpha_M \in Z^2(G_M, \mathbb{C}^*)$ be the corresponding 2-cocycle whose values are roots of unity. Then $M$ is a module for $\mathbb{C}^{\alpha_M}[G_M]$. 

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Remark 5.1 Since each \( \phi(h) \) commutes with \( L(n) \) for all \( n \in \mathbb{Z} \), \( \phi(h) \) preserves the homogeneous subspaces \( M(m) \), and \( M(m) \) is a \( C^\alpha M[G_M] \)-module.

Remark 5.2

1) If \( M = V \), we can take \( \phi(h) = h \) and \( C^\alpha M[G_M] = C[G] \).

2) Let \( G = \langle h \rangle \) be a cyclic group of prime order \( p \). For \( 1 \leq i \leq p - 1 \), let \( M \) be an irreducible \( h^i \)-twisted \( V \)-module. Then \( G_M = \langle h \rangle \) and the 2-cocycle \( \alpha_M \) can be taken to be trivial.

Assume that \( g \) is an element in the center of \( G \) and \( V \) is \( g \)-rational. Let \( M \) be an irreducible \( g \)-twisted \( V \)-module. Then \( M(n) \) is a simple \( A_{g,n}(V) \) for all \( n \in \frac{1}{T} \mathbb{Z}_+ \). Moreover, by equations (2.1), (5.1), we have

\[
hM(n) = M \circ h^{-1}(n) \quad \text{for} \quad h \in G, n \in \frac{1}{T} \mathbb{Z}_+.
\]

We let

\[
G_{M(n)} = \{ h \in G \mid hM(n) \cong M(n) \quad \text{as} \quad A_{g,n}(V)\text{-modules} \}
\]

be the inertia subgroup of \( M(n) \).

Lemma 5.3 For \( n \in \frac{1}{T} \mathbb{Z}_+ \), if \( M(n) \neq 0 \), we have \( G_{M(n)} = G_M \).

Proof: Recall that \( M \) and \( M \circ h \) are irreducible \( g \)-twisted \( V \)-modules. By Theorem 4.15, we conclude that \( M \cong M \circ h \) as \( g \)-twisted \( V \)-modules if and only if \( M(n) \cong M \circ h(n) \) as \( A_{g,n}(V) \)-modules. Thus \( G_{M(n)} = G_M \) if \( M(n) \neq 0 \). □

Let \( \Lambda_{G_M, \alpha_M} \) be the set of all irreducible characters \( \lambda \) of \( C^\alpha M[G_M] \). For each \( \lambda \in \Lambda_{G_M, \alpha_M} \), we denote the corresponding simple module for \( \lambda \) by \( W_\lambda \) and we let \( M^\lambda \) be the sum of simple \( C^\alpha M[G_M] \)-submodules of \( M \) isomorphic to \( W_\lambda \). Since \( M^\lambda \) is nonzero for any \( \lambda \in \Lambda_{G_M, \alpha_M} \) (cf. [DY]), we see that

\[
M = \bigoplus_{\lambda \in \Lambda_{G_M, \alpha_M}} M^\lambda.
\]

As in section 3, we let \( M_\lambda = \{ f(w) \mid f \in \text{Hom}_{C^\alpha M[G_M]}(W_\lambda, M) \} \) for a fixed nonzero \( w \in W_\lambda \).

Theorem 5.4 For any \( \gamma \in \Lambda_{G_M, \alpha_M} \), \( M_\gamma \) is a \( V^G \)-module.

Proof: By equation (5.2), we have \( \phi(h)Y_M(v, z) = Y_M(v, z)\phi(h) \) for all \( v \in V^G \). Thus the action of \( C^\alpha M[G_M] \) and \( V^G \) are commute on \( M \). Hence, \( M_\gamma \) is a \( V^G \)-module. □

For any \( \lambda \in \Lambda_{G_M, \alpha_M} \), we identify \( M^\lambda \) with \( M_\lambda \otimes W_\lambda \) as \( V^G \otimes C^\alpha M[G_M] \)-modules. Thus

\[
M = \bigoplus_{\lambda \in \Lambda_{G_M, \alpha_M}} W_\lambda \otimes M_\lambda.
\]

Theorem 5.5 For any \( \gamma \in \Lambda_{G_M, \alpha_M} \), \( M_\gamma \) is an irreducible \( V^G \)-module.
Proof: For \( n \in \frac{1}{2}\mathbb{Z}_+ \), we set \( M^\gamma(n) = M^\gamma \cap M(n) \) and \( M_\gamma(n) = M_\gamma \cap M(n) \). In order to show that \( M_\gamma \) is an irreducible \( V^G \)-module, it is enough to show that for any \( n \in \frac{1}{2}\mathbb{Z}_+ \), if \( M^\gamma(n) \) is nonzero then \( M_\gamma(n) \) is a simple \( A_{g,n}(V^G) \)-module (cf. Theorem 4.14 3). We now take \( n \in \frac{1}{2}\mathbb{Z}_+ \) such that \( M^\gamma(n) \neq 0 \). Since \( V \) is \( g \)-rational, \( A_{g,n}(V) \) is a finite-dimensional semisimple associative algebra (cf. Theorem 4.16). Since \( M(n) \) is a simple \( A_{g,n}(V) \)-module, Theorem 3.3 implies that \( M_\gamma(n) \) is a simple \( A_{g,n}(V^G) \)-module. Since the identity on \( V^G \) induces an algebra epimorphism from \( A_{g,n}(V^G) \) to \( A_{g,n}(V^G) \), we can conclude \( M_\gamma(n) \) is a simple \( A_{g,n}(V^G) \)-module. Consequently, \( M_\gamma \) is an irreducible \( V^G \)-module. □

Corollary 5.6 \( M \) is a completely reducible \( V^G \)-module.

Corollary 5.7 If \( V \) is a rational vertex operator algebra then every irreducible \( V \)-module is a completely reducible \( V^G \)-module.

Corollary 5.8

1) Assume \( g \) is in the center of \( G \). Let \( M \) be an irreducible \( g \)-twisted \( V \)-module. If \( G_M \) is a cyclic group, says \( \langle h \rangle \), then, in fact, the eigenspaces of \( M \) with respect to the action of \( h \) are irreducible \( V^G \)-modules.

2) Suppose \( G \) is a cyclic group of prime order \( p \). Let \( M \) be an irreducible \( g^i \)-twisted \( V \)-module where \( 1 \leq i \leq p - 1 \). Hence the eigenspaces of \( M \) are irreducible \( V^G \)-modules.

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