Entanglement between artificial atoms and photons of lossless cavities

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Abstract. We investigated the dynamics of atom-field entanglement for two natural or artificial two-level atoms interacting with a one-mode quantum electromagnetic field by means of multiphoton transitions in a lossless cavity. Tavis-Cummings model is used to describe the interaction of the atoms and real microwave coplanar cavity field. We carried out the mathematical modeling of the dynamics of the system under consideration for various initial states of the atomic subsystem and an intensive coherent field of the cavity. We showed that for small multiplicities, the atoms and the field, which were initially in a pure separable state, can return to this state during the evolution. We also found that for large multiplicities the atoms and the field are in the entangled atom-field state in the process of the system evolution with the exception of the initial time instant. These results can be used in the theory of quantum networks.

1. Introduction

Entanglement is the cornerstone of quantum computation and quantum communication [1]. One of the particular interest schemes in which entanglement can be created is a system containing two two-level atoms, since they can represent two qubits, the building blocks of the quantum gates that are essential to implement quantum protocols in quantum information processing. The interaction between light and atoms is one of the most fundamental and basic processes in nature, and it represents a milestone in our understanding of a broad range of physical phenomena. The recent experimental success in engineering strong interactions between photons and atoms in high-quality cavities opens up the possibility to use light-matter systems for investigation of various quantum light-matter phenomena. In particular, the appearance of entanglement in the interaction between light and matter in cavity quantum electrodynamics, as a simple way to produce the entangled states and as a candidate for the physical realization of quantum information processing, is of special interest. In this respect, the atom-field and atom-atom entangled states have been experimentally generated through the interaction of a single natural or artificial atom or arrays of atoms with a field in a high-Q microwave cavity [2],[3].

The Jaynes-Cummings model (JCM), as is well-known, is a fully quantum mechanical and exactly solvable model which gives a pattern for the description of the most basic and important interactions between light (a single-mode quantized electromagnetic field) and matter (a two-level atom) in the rotating wave approximation [4]. After introducing the JCM, a great deal of efforts have been performed in the generalization of this model [5]-[7]. For example, many atom, multiphoton transitions and the multi-mode fields are employed instead of single two-level atom interacting with single-mode cavity field via one-photon transitions. The models with intensity-dependent coupling and time-dependent coupling are also investigated instead of
a constant atom-field coupling (see references in [8]). Among them, one of the generalization takes into account the different configurations of three-level atoms instead of two-level atoms [9]. An investigation of the atom-field entanglement for JCM has been initiated by Phoenix and Knight [10] and Gea-Banacloche [11]. Gea-Banacloche has derived an asymptotic result for the JCM state vector which is valid when the field is initially in a coherent state with a large mean photon number. It is shown that the atom prepared in arbitrary initial pure atomic state is to a good approximation in a pure state in the middle of the collapse region. This has been firstly noticed by Phoenix and Knight by using the entropy concepts. An appreciable disentanglement between atom and field is found at the half-revival time, otherwise the atom and field are strongly entangled. Moreover, at the half-revival time, the cavity field represents a coherent superposition of the two macroscopically distinct states with opposite phases or so-called Schrödinger cat state. The interaction splits the initial coherent state into two parts in the phase space, which at the disentanglement times differ in phase by an angle $\pi$. The theory outlined in [11] has been generalized for one-atom degenerate two-photon JCM [12],[13], two-photon JCM with nondegenerate two-photon [14] and Raman transitions [15], one-atom degenerate Raman model [16], two-atom JCM [17],[18], two-atom one-mode Raman coupled model [19], two-atom two-mode degenerate, nondegenerate and Raman coupled model [20]-[22]. The atom-field entanglement for degenerate and nondegenerate two-atom JCM with intensity dependent coupling has also been investigated in our paper [23].

Two-photon processes are known to play a very important role in atomic systems due to high degree of correlation between emitted photons [24],[25]. An interest for investigation of the two-photon JCM is stimulated by the experimental realization of a two-photon one-atom micromaser on Rydberg transitions in a microwave cavity [26]. A nondegenerate two-photon two-mode maser, which represents a two-level Rydberg atom interacting with two different modes of a quantum electromagnetic field in a high-quality cavity through a nondegenerate two-photon transition, is an important generalization of the model of a two-photon micromaser. A possibility of the modulation, amplification and control of one mode with another mode is an important feature of the two-photon two-mode maser. JCM with nondegenerate two-photon transitions have attracted a great deal of attention [27],[28].

In this paper we analyzed an atom-field entanglement in the framework of the two-atom multi-photon JCM. We supposed that the cavity field is initially in a one-mode coherent state with a large mean photon number and qubits are in a separable or entangled initial states. We studied model by using the linear atomic entropy concepts. The main goal of this paper is to find such initial states of the atomic subsystem which don’t provide disentanglement between the atom and the field during the system evolution. This research is important in the selection of optimal modes of operation of quantum networks [29]-[31]. To describe the real quantum network we should consider the more complicated model taking additionally into account motion of atoms through resonators, multi-modes regime and the interaction of atoms with external laser fields, i.e. we should consider the so-called generalized Tavis-Cummings model. Engineering such quantum networks offers the opportunity to explore extreme regimes of light-matter interaction that are inaccessible with natural systems. The coupling strength for such systems can be increased until it is comparable with the atomic or mode frequency. Only in the strong- or ultra-strong regime one can obtain the suitable values of the transfer factor [32]. Such regimes can be realized for superconducting qubits, impurity spins and some others artificial atoms [33],[34]. But in a strong or ultra-strong regime the rotating wave approximation which usually used for description of the atom-field interaction in the framework of the generalized Tavis-Cummings model becomes unsuitable. In this case only the numerical solution of the Tavis-Cummins model is possible. An alternative approach for describing a quantum network based on the generalized Tavis-Cummings model was implemented in our paper [35], where we used the quasi-classical approximation for calculating the transmission coefficient. In the present paper we obtained the
exact dynamic of the atom-field entanglement parameter for multi-photon Tavis-Cummins model in the rotating wave-approximation and find such initial states of the atomic subsystem which don’t provide disentanglement between the atom and the field during the system evolution. We showed that such states exist for multi-photon model. We found that two-level artificial atoms (superconducting qubits, impurity spins, quantum dots etc.) with allowed multi-photon transitions have certain advantages in creating quantum networks.

2. Model description. The exact solution of Schrödinger equation for wave function

The physical system under consideration consists of two separate identical two-level natural or artificial atoms (superconducting circuits, trapped ions, impurity spins or quantum dots) resonantly interacting with common one-mode cavity coherent quantum electromagnetic field in a lossless cavity through multi-photon transitions. The interaction Hamiltonian of such a model in dipole and rotating-wave approximation has the following form

\[ H_2 = \hbar g \sum_{i=1}^{2} (\sigma_i^+ a^m + \sigma_i^- (a^+)^m). \]  

Here we use the following notation: \( a^+ (a) \) is a creation (annihilation) operator for cavity mode, \( \sigma_i^+ = |+\>_i\langle -| \) and \( \sigma_i^- = |-\>_i\langle +| \) are the atomic transition operators, while \(|-\>_i \) and \(|+\>_i \) denote the ground and excited states of the \( i \)-th two-level atom (\( i = 1, 2 \)) respectively and \( g \) is the constant of atom-field interaction. Parameter \( m \) is the transition multiplicity.

The time-dependent wave function of the total system \( |\Psi(t)\rangle \) obeys the Schrödinger equation

\[ i\hbar \frac{\delta |\Psi(t)\rangle}{\delta t} = H|\Psi(t)\rangle. \]

The equation of motion (1) is supplemented with initial conditions of the following form. The atoms are supposed to be initially prepared in arbitrary pure atomic states superposition

\[ |\Psi(0)\rangle_A = \alpha|+, +\rangle + \beta|+, -\rangle + \gamma|-\rangle + \delta|--\rangle, \]

where \( \alpha, \beta, \gamma \) and \( \delta \) are the arbitrary complex values satisfying the condition

\[ |\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1 \]

and field is supposed to be initially in one-mode coherent state

\[ |\alpha\rangle = \sum_{n=0}^{\infty} F_n |n\rangle, \]

where the coefficients \( F_n \) are

\[ F_n = \exp(-r/2) \frac{\alpha^n e^{i\varphi}}{\sqrt{n!}} \]

and \( \alpha = r^{1/2} e^{i\varphi}, r = |\alpha|^2 \) is the mean photon number and \( \varphi \) is the phase of coherent mode. The put below \( \varphi = 0 \).

The full wave function of atom-field system at initial time can be written as

\[ |\Psi(0)\rangle = |\Psi(0)\rangle_A |\alpha\rangle. \]
The exact solution of Shrödinger equation for wave function under considered initial conditions for model with Hamiltonian (1) is

\[ |\Psi(t)\rangle = \sum_{n=0}^{\infty} (X_{1n}(t)|+,+,n\rangle + X_{2n}(t)|+,-,n+m\rangle + X_{3n}(t)|-,+,n+m\rangle + X_{4n}(t)|-,+,n+2m\rangle, \tag{2} \]

Here the following notation is accepted:

\[ X_{1n}(t) = -\frac{1}{4\Omega_n^2(-1)^{3/2}}e^{-\sqrt{2}\Omega_n t} \left( 4b_n^2 i\Omega_n e^{\sqrt{2}\Omega_n t} \alpha F_n + 2a_n^2 i\Omega_n \left( 1 + e^{2\sqrt{2}\Omega_n t} \right) \alpha F_n - i\sqrt{2}a_n^3 \left( -1 + e^{2\sqrt{2}\Omega_n t} \right) (\beta + \gamma) F_{n+m} + a_n b_n \left( -i\sqrt{2}b_n \left( -1 + e^{2\sqrt{2}\Omega_n t} \right) (\beta + \gamma) F_{n+m} + 2i\Omega_n \left( -1 + e^{2\sqrt{2}\Omega_n t} \right)^2 \delta F_{n+2m} \right) \right), \]

\[ X_{2n}(t) = \frac{1}{2\Omega_n^2} \left( \Omega_n^2 (\beta - \gamma) F_{n+m} + \Omega_n^2 (\beta + \gamma) F_{n+m} \cos(\sqrt{2}\Omega_n t) + i\sqrt{2}\Omega_n \left( a_n \alpha F_n + b_n \delta F_{n+2m} \right) \sin(\sqrt{2}\Omega_n t) \right), \]

\[ X_{3n}(t) = \frac{1}{2\Omega_n^2} \left( -\left( \Omega_n^2 \right) (\beta - \gamma) F_{n+m} + \left( \Omega_n^2 \right) (\beta + \gamma) F_{n+m} \cos(\sqrt{2}\Omega_n t) + i\sqrt{2}\Omega_n \left( a_n \alpha F_n + b_n \delta F_{n+2m} \right) \sin(\sqrt{2}\Omega_n t) \right), \]

\[ X_{4n}(t) = -\frac{1}{4\Omega_n^2(-1)^{3/2}}e^{-\sqrt{2}\Omega_n t} \left( 2a_n b_n i\Omega_n \left( -1 + e^{2\sqrt{2}\Omega_n t} \right)^2 \alpha F_n + a_n^2 \left( -i\sqrt{2}b_n \left( -1 + e^{2\sqrt{2}\Omega_n t} \right) (\beta + \gamma) F_{n+m} + 4i\Omega_n e^{\sqrt{2}\Omega_n t} \delta F_{n+2m} \right) + b_n^2 \left( -i\sqrt{2}b_n \left( -1 + e^{2\sqrt{2}\Omega_n t} \right) (\beta + \gamma) F_{n+m} + 2i\Omega_n \left( 1 + e^{2\sqrt{2}\Omega_n t} \right) \delta F_{n+2m} \right) \right), \]

where

\[ a_n = \sqrt{\frac{(n+m)!}{n!}}, \quad b_n = \sqrt{\frac{(n+2m)!}{(n+m)!}}, \quad \Omega_n = \sqrt{a_n^2 + b_n^2}. \]

We can use the wave function (2) for calculations of variables. For example, we can obtain the probability of finding the both atoms in excited state \(|+,+\rangle\) as

\[ W(t) = \sum_{n=0}^{\infty} |X_{1n}(t)|^2. \]
3. The dynamics of the reduced atomic entropy for various initial atomic and field states

A linear entropy of reduced atomic (or field) density matrix can serve for entanglement degree evaluation of the systems consisting of two subsystems and being prepared in a pure state. The linear entropy of reduced atomic density matrix for considered systems has the following form

\[ S = 1 - Tr\left(\rho_{AT}^2\right), \]

where \( \rho_{AT} = Tr_F(|\Psi\rangle\langle\Psi|) \). The case when \( S = 0 \) corresponds to completely disentangled atomic and field states, and the case when a linear entropy equals to \( \frac{3}{4} \) corresponds to maximum entanglement amount. Using the expressions (2) and (3) one can obtain

\[ S = 1 - \left[ \sum_{n=0}^{\infty} |X_{1n}|^2 \right]^2 - \left[ \sum_{n=0}^{\infty} |X_{2n}|^2 \right]^2 - \left[ \sum_{n=0}^{\infty} |X_{3n}|^2 \right]^2 - \left[ \sum_{n=0}^{\infty} |X_{4n}|^2 \right]^2 \]

\[ -2 \left[ \sum_{n=0}^{\infty} X_{1,n+m}^* X_{2n} \right] \left[ \sum_{n=0}^{\infty} X_{1,n+m}^* X_{2n} \right] \]

\[ -2 \left[ \sum_{n=0}^{\infty} X_{1,n+2m} X_{4n} \right] \left[ \sum_{n=0}^{\infty} X_{1,n+2m} X_{4n} \right] - 2 \left[ \sum_{n=0}^{\infty} X_{2,n+m} X_{3n} \right] \left[ \sum_{n=0}^{\infty} X_{2,n+m} X_{3n} \right] \]

\[ -2 \left[ \sum_{n=0}^{\infty} X_{2,n+m} X_{4n} \right] \left[ \sum_{n=0}^{\infty} X_{2,n+m} X_{4n} \right] - 2 \left[ \sum_{n=0}^{\infty} X_{3,n+m} X_{4n} \right] \left[ \sum_{n=0}^{\infty} X_{3,n+m} X_{4n} \right] \]

The results of numerical calculations of time dependence of a linear atomic entropy (3) for different atomic initial states and multiplicities \( m \) are shown in Figs. 1 and 2. Let us note that the linear entropy (3) is the dimensionless quantity. Taking into account the fact that atom-field coupling constant for real quantum networks vary in a wide range from hundred Hz to tens GHz [36] we will use a scaled dimensionless time \( gt \). The case of coherent one-mode field of high intensity is under consideration. In some pictures we draw both values of linear entropy \( S \) (black curves) and atomic possibilities to find both atoms in excited state (gray curves). Fig. 1 demonstrates the time behavior of a linear atomic entropy for case when the atomic subsystem is prepared initially in entangled state \( |\Psi(0)\rangle_A = \sqrt{1/2}(|+, -\rangle + |-, +\rangle) \) (or \( |\Psi(0)\rangle_A = \sqrt{1/2}(|+, +\rangle + |-, -\rangle) \)). Fig. 2 demonstrates the time behavior of linear atomic entropy for the case when the atomic subsystem is prepared initially in a separable state \( |\Psi(0)\rangle_A = |+, +\rangle \). Let us note that for both states the initial atom-field entanglement is absent. One can see from Figs. 1 and 2 that the entanglement behavior has the regular character only for small values of multiplicity \( m = 1 \) and \( m = 2 \). Two-atom JCM model with one-photon transitions \( (m = 1) \) shows apparently the recreation of the "two atoms+field" pure state vector in the middle of the collapse region for entangled initial atoms. For separable initial atomic states two subsystems (atoms and field) remain entangled for all time instants \( t > 0 \). Two-atom JCM with two-photon transitions exhibits the disentanglement for all the initial atomic states. For entangled initial atomic states we have three sets of disentanglement times that are divisible by fractions of the revival period independently of the initial mean value of photon numbers. For separable initial atomic states, the model exhibits one set of disentanglement times that are divisible by one revival period. For two-atom JCM with multiplicity \( m > 3 \) the entanglement parameter demonstrates the irregular behavior in time. The "two atom + field" system remains entangled in all time instants \( t > 0 \). For this time instants the system characterizes by the almost maximal amount of entanglement. Thus, we really found for multiphoton Tavis-Cummings model such initial states of the atomic subsystem which don’t provide
disentanglement between the atom and the field during the system evolution. These results may be used in a future experiments with quantum networks. Let us note that for more realistic models of quantum networks based taking into account dissipation and many modes of a cavity the sharp oscillation of the entanglement parameter should be smoothed.

It is of interest to compare the results for atom-field entanglement time behavior for multi-photon two-atom JCM with these for two-atom degenerate and nondegenerate JCM with intensity-dependent coupling presented in our paper [23].

The first of the mentioned models describes two two-level atoms resonantly interacting with one-mode coherent field $|\alpha\rangle$ in lossless cavity. The interaction Hamiltonian of such a model is

$$H_{int} = \hbar g \sum_{i=1}^{2} \left( \sqrt{a_i^+ a_i^+} \sigma_i^- + \sigma_i^+ a \sqrt{a_i^+ a_i^+} \right),$$  \hspace{1cm} (4)

Here parameter $g$ with the operator $\sqrt{a_i^+ a_i^+}$ plays the role of intensity-dependent coupling constant between atoms and the cavity field.

The second model describes two two-level atoms interacting with two-mode coherent field $|\alpha_1\rangle |\alpha_2\rangle$ in lossless cavities via nondegenerate two-photon transitions under the assumption of exact two-photon resonance. The effective interaction Hamiltonian for considered model can be written in the following form

$$H_{int} = \hbar g \sum_{i=1}^{2} \left( \sqrt{a_1^+ a_1^+} \sqrt{a_2^+ a_2^+} \sigma_i^- + \sigma_i^+ a_1 \sqrt{a_1^+ a_2^+} \right),$$  \hspace{1cm} (5)

where $a_j^+$ ($a_j$) is creation (annihilation) operator for $j$-th cavity mode ($j = 1, 2$) and $g$ is the effective constant of dipole-photon interaction. In two-photon processes the Stark shift caused by the intermediate atomic level plays the role of an intensity-dependent detuning. However, if the two fields are tuned in such a way that both have reverse detuning with the intermediate atomic level, the Stark shift will not appear as has been pointed in Ref. [27]. They noticed that such a two-photon signal could be achieved by two dye lasers.

The results for atom-field entanglement time behavior for models with Hamiltonians (4) and (5) for high intensity one- or two-mode coherent field show in Figs. 3 and 4. One can easily see that for one-mode two-atom JCM with intensity-dependent coupling the entanglement behaves much like a two-photon JCM. The entanglement behavior for two-mode two-atom JCM with intensity-dependent coupling much like those of multi-photon two-atom JCM.
Figure 1. Time evolution of linear entropy for entangled atomic initial state $|\Psi(0)\rangle_A = \sqrt{\frac{1}{2}}(|+, -| + |-, +|)$. The multiplicity $m = 1$ (a) $m = 2$ (b) $m = 3$ (c) and $m = 4$ (d).

4. Conclusion

The dynamics atom-field entanglement for system consisting with two identical two-level atoms with multi-photon transitions interacting with intense coherent one-mode cavity field are considered in the paper. Atom-field entanglement is estimated on the basis of a linear entropy criterion for separable and entangled initial atomic states and different values of multiplicities. We derived that the entanglement behavior has the regular character only for small values of multiplicity $m = 1$ and $m = 2$. The disentanglement between atoms and field states is found to appear in the model with one- and two-photon transitions. One can see in Figs. 1 and 2 that the entanglement behavior has the regular character only for small values of multiplicity $m = 1$ and $m = 2$ and for some initial atomic states and large coherent field inputs. We estimates the periods of such disentanglement. For two-atom JCM with multiplicity $m > 3$ the entanglement parameter demonstrates the irregular behavior in time. For such case the "two atom + field" system remains entangled in all time instants $t > 0$. For this time instants the system characterizes by the almost maximal amount of entanglement.
Figure 2. Time evolution of linear entropy for separable atomic initial state $|\Psi(0)\rangle_A = |+, +\rangle$. The multiplicity $m = 1$ (a), $m = 2$ (b), $m = 3$ (c) and $m = 4$ (d). The initial mean photon number $n = 30$ and phase of coherent state $\varphi = 0$. Gray curves denote the possibility to find both atoms in excited state $W + 0.5$.

Figure 3. Time evolution of linear entropy for model with Hamiltonian (4) and atomic initial states: $|\Psi(0)\rangle_A = \sqrt{1/2}(|+, -\rangle + |-, +\rangle)$ (a) and $|\Psi(0)\rangle_A = |+, +\rangle$ (b). The initial mean photon number $n = 30$ and phase of coherent state $\varphi = 0$. Gray curves denote the possibility to find both atoms in excited state $W + 0.5$. 
Figure 4. Time evolution of linear entropy for model with Hamiltonian (5) and initial atomic state $|\Psi(0)\rangle_A = (1/\sqrt{2})|+, -\rangle + |-, +\rangle$ (a) and $|\Psi(0)\rangle_A = |+, +\rangle$. The mean photon numbers $\bar{n}_1 = \bar{n}_2 = 30$. Gray curves denote the possibilities $W$ to find both atoms in exited states $W$.

5. References

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