Research Article

Coefficient Bounds for Some Families of Starlike and Convex Functions of Reciprocal Order

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The aim of the present paper is to investigate coefficient estimates, Fekete-Szegő inequality, and upper bound of third Hankel determinant for some families of starlike and convex functions of reciprocal order.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in \mathbb{U}).$$

(1)

Also let $\mathcal{S}^*_{\alpha}$ and $\mathcal{K}^*_{\alpha}$ denote the usual classes of starlike and convex functions of order $\alpha$, $0 \leq \alpha < 1$, respectively. In 1975, Silverman [1] proved that $f(z) \in \mathcal{S}^*_{\alpha}$ if it satisfies the condition

$$\frac{|zf'(z)|}{f(z)} - 1 < 1 - \alpha, \quad (z \in \mathbb{U}).$$

(2)

Geometrical meaning of inequality (2) is that $zf'(z)/f(z)$ maps $\mathbb{U}$ onto the interior of the circle with center at 1 and radius $1 - \alpha$.

By $\mathcal{S}^*_{\alpha}$ and $\mathcal{K}^*_{\alpha}$, we mean the classes of starlike and convex functions of reciprocal order $\alpha$, $0 \leq \alpha < 1$ which are defined, respectively, by

$$\mathcal{S}^*_{\alpha} = \left\{ f(z) \in \mathcal{A} : \text{Re} \frac{zf'(z)}{zf''(z)} > \alpha, \quad (z \in \mathbb{U}) \right\},$$

$$\mathcal{K}^*_{\alpha} = \left\{ f(z) \in \mathcal{A} : \text{Re} \frac{f'(z)}{zf''(z) + f'(z)} > \alpha, \quad (z \in \mathbb{U}) \right\}.$$  

(3)

Recently in 2008, Nunokawa and his coauthors [2] improved inequality (2) for the class $\mathcal{S}^*_{\alpha}$ and they proved that, for $f(z) \in \mathcal{S}^*_{\alpha}$, $0 < \alpha < 1/2$, if and only if the following inequality holds:

$$\left| \frac{zf'(z)}{f(z)} - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}, \quad (z \in \mathbb{U}).$$

(4)

In view of these results we now define the following subclass of analytic functions of reciprocal order and investigate its various properties.
Definition 1. A function \( f(z) \in \mathcal{A} \) is said to be in the class \( \mathcal{L}(\lambda, \gamma) \), with \( \gamma \in \mathbb{C} \setminus \{0\} \) and \( \lambda \in [0, 1] \), if it satisfies the inequality
\[
\text{Re} \left( 1 + \frac{1}{\gamma} \left( \frac{F_{\lambda}(z)}{zF'_{\lambda}(z)} - 1 \right) \right) > 0,
\]
where
\[
F_{\lambda}(z) = (1 - \lambda) f(z) + \lambda z f'(z).
\]

Example 2. Let us define the functions \( F_{\lambda}(z) \) by
\[
F_{\lambda}(z) = \frac{z}{(1 + (2\gamma - 1)z)^{2\gamma/(2\gamma - 1)}}.
\]
This implies that
\[
\frac{zF'_{\lambda}(z)}{F_{\lambda}(z)} = \frac{1 - z}{1 + (2\gamma - 1)z}.
\]
Hence
\[
1 + \frac{1}{\gamma} \left( \frac{F_{\lambda}(z)}{zF'_{\lambda}(z)} - 1 \right) = \frac{1 + z}{1 - z}.
\]
and this further implies that
\[
\text{Re} \left( 1 + \frac{1}{\gamma} \left( \frac{F_{\lambda}(z)}{zF'_{\lambda}(z)} - 1 \right) \right) = \text{Re} \left( \frac{1 + z}{1 - z} \right) > 0, \quad (z \in \mathbb{U}).
\]
The \( q \)th Hankel determinant \( H_q(n) \), \( q \geq 1, n \geq 1 \), for a function \( f(z) \in \mathcal{A} \) is studied by Ňoonan and Thomas [3] as
\[
H_q(n) = \begin{vmatrix}
\alpha_n & \alpha_{n+1} & \cdots & \alpha_{n+q-1} \\
\alpha_{n+1} & \alpha_{n+2} & \cdots & \alpha_{n+q} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{n+q-1} & \alpha_{n+q} & \cdots & \alpha_{n+2q-2}
\end{vmatrix}.
\]

In literature many authors have studied the determinant \( H_q(n) \). For example, Arif et al. [4, 5] studied the \( q \)th Hankel determinant for some subclasses of analytic functions. Hankel determinant of exponential polynomials is obtained by Ehrenborg in [6]. The Hankel transform of an integer sequence and some of its properties were discussed by Layman [7]. It is well known that the Fekete-Szegő functional \( |a_3 - \lambda a_2^2| \) is \( H_2(1) \). Fekete-Szegő then further generalized the estimate \( |a_3 - \lambda a_2^2| \) with \( \lambda \) real and \( f(z) \in \mathcal{A} \).

In this paper we study some useful results including coefficient estimates, Fekete-Szegő inequality, and upper bound of third Hankel determinant for the functions belonging to the class \( \mathcal{L}(\lambda, \gamma) \).

Throughout this paper we assume that \( \gamma \in \mathbb{C} \setminus \{0\} \) and \( \lambda \in [0, 1] \) unless otherwise stated.

For our results we will need the following Lemmas.

**Lemma 3** (see [11]). If \( q(z) \) is a function with \( \text{Re} q(z) > 0 \) and is of the form
\[
q(z) = 1 + c_1 z + c_2 z^2 + \cdots
\]
then
\[
|c_n| \leq 2, \quad \text{for } n \geq 1.
\]

**Lemma 4** (see [12]). If \( q(z) \) is of the form (12) with positive real part, then the following sharp estimate holds:
\[
|c_2 - \nu c_1^2| \leq 2 \max \{1, [2\nu - 1]\}, \quad \text{for all } \nu \in \mathbb{C}.
\]

**Lemma 5** (see [13]). If \( q(z) \) is of the form (12) with positive real part, then
\[
2c_2 = c_1^2 + x (4 - c_1^2),
\]
\[
4c_3 = c_1^3 + 2 (4 - c_1^2) c_1 x - c_1 (4 - c_1^2) x^2 + 2 (4 - c_1^2) (1 - |x|^2) z,
\]
for some \( x, z \) with \( |x| \leq 1 \) and \( |z| \leq 1 \).

### 2. Some Properties of the Class \( \mathcal{L}(\lambda, \gamma) \)

**Theorem 6.** Let \( f(z) \in \mathcal{L}(\lambda, \gamma) \). Then
\[
|a_2| \leq \frac{2|\gamma|}{1 + \lambda},
\]
and for all \( n = 3, 4, 5, \ldots \)
\[
|a_n| \leq \frac{2|\gamma|}{(n - 1)(1 + \lambda (n - 1))} \sum_{k=1}^{n-1} \left( 1 + 2|\gamma| k \right),
\]
\[
|a_n| \leq \frac{2|\gamma|}{(n - 1)(1 + \lambda (n - 1))} \prod_{k=2}^{n-1} \left( 1 + 2\frac{|\gamma| k}{k - 1} \right).
\]

**Proof.** Let us define the function \( q(z) \) by
\[
q(z) = 1 + \frac{1}{\gamma} \left( \frac{F_{\lambda}(z)}{zF'_{\lambda}(z)} - 1 \right),
\]
where \( F_{\lambda}(z) \) is given by (6) with
\[
F_{\lambda}(z) = z + \sum_{k=2}^{\infty} \left[ 1 + \lambda (k - 1) \right] a_k z^k,
\]
and \( q(z) \) is analytic in \( \mathbb{U} \) with \( q(0) = 1, \text{Re} \ q(z) > 0 \).

Now using (1) and (12), we have
\[
\sum_{k=2}^{\infty} A_k z^k = \left[ 1 + \gamma \left( \sum_{k=1}^{\infty} c_k z^k \right) \right] \left( z + \sum_{k=2}^{\infty} k A_k z^k \right),
\]
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\[ A_k = [1 + \lambda (k - 1)] a_k. \] (21)

Comparing coefficient of like power of \( z^n \), we obtain

\[ (1 - n) A_n = y \{ c_{n-1} + 2A_2c_{n-2} + \cdots + (n - 1) A_{n-1}c_1 \}. \] (22)

Using triangle inequality and Lemma 3, we get

\[ |(1 - n) A_n| \leq 2 |y| \left| \frac{1 + 2 |A_2| + \cdots + (n - 1) |A_{n-1}|}{} \right|. \] (23)

For \( n = 2 \) and \( n = 3 \) in (23), we easily obtain that

\[ |a_2| \leq \frac{2 |y|}{1 + \lambda}, \quad |a_3| \leq \frac{|y| (1 + 4 |y|)}{1 + 2\lambda}. \] (24)

Making \( n = 4 \) in (23), we see that

\[ 2 |A_3| \leq 2 |y| \left( 1 + 2 |A_2| \right) \leq 2 |y| \left( 1 + \frac{4 |y| (1 + 2\lambda)}{1 + \lambda} \right); \] (25)
equivalently, we have

\[ |a_4| \leq \frac{2 |y| (1 + 3 |y|) (1 + 4 |y|)}{3 (1 + 3\lambda)}. \] (26)

Using the principal of mathematical induction, we obtain

\[ |A_n| \leq \frac{2 |y|}{(n - 1) \prod_{k=2}^{n-1} \left( 1 + \frac{2 |y| k}{k - 1} \right)}. \] (27)

Now from the use of relation (21), we obtain the required result. \[ \square \]

If we take \( \lambda = 0 \) and \( y = 1 - \alpha \), we get the following result.

**Corollary 7** (see [14]). Let \( f(z) \in \mathcal{S}_*(\alpha) \). Then, for \( n = 3, 4, 5, \ldots, \) one has

\[ |a_n| \leq \frac{2 (1 - \alpha) \prod_{k=2}^{n-1} \left( 1 + \frac{2 (1 - \alpha) k}{k - 1} \right)}{n (n - 1)}, \] (28)
with \( |a_3| \leq 2 (1 - \alpha) \).

Making \( \lambda = 1 \) and \( y = 1 - \alpha \), we get the following result.

**Corollary 8** (see [14]). Let \( f(z) \in \mathcal{K}_*(\alpha) \). Then, for \( n = 3, 4, 5, \ldots, \) one has

\[ |a_n| \leq \frac{2 (1 - \alpha) \prod_{k=2}^{n-1} \left( 1 + \frac{2 (1 - \alpha) k}{k - 1} \right)}{n (n - 1)}, \] (29)
with \( |a_3| \leq (1 - \alpha) \).

**Theorem 9.** Let \( f(z) \in \mathcal{L}(\lambda, y) \) and be of the form (1). Then

\[ |a_3 - \mu a_2^2| \leq \frac{|y|}{(1 + 2\lambda)} \max \{1, |2y - 1|\}, \] (30)

where

\[ v = 2y (1 + 2\lambda) \left( \frac{1}{1 + 2\lambda} - \frac{\mu}{(1 + \lambda)^2} \right). \] (31)

**Proof.** Let \( f(z) \in \mathcal{L}(\lambda, y) \). Then from (22) we have

\[ a_2 = \frac{-y c_1}{(1 + \lambda)}, \quad a_3 = \frac{-y}{2 (1 + 2\lambda)} (c_2 - 2y c_1^2). \] (32)

We now consider

\[ |a_3 - \mu a_2^2| = \frac{|y|}{2 (1 + 2\lambda)} \left| c_2 - 2y (1 + 2\lambda) \left( \frac{1}{1 + 2\lambda} - \frac{\mu}{(1 + \lambda)^2} \right) c_1^2 \right|. \] (33)

Using Lemma 4, we obtain

\[ |a_3 - \mu a_2^2| \leq \frac{|y|}{(1 + 2\lambda)} \max \{1, |2y - 1|\}, \] (34)

where \( v \) is given by (31).

Putting \( \mu = 1 \), we obtain the following result.

**Corollary 10.** Let \( f(z) \in \mathcal{L}(\lambda, y) \). Then

\[ |a_3 - a_2^2| \leq \frac{|y|}{(1 + 2\lambda)}. \] (35)

**Theorem 11.** Let \( f(z) \in \mathcal{L}(\lambda, y) \) and be of the form (1). Then

\[ |a_3 a_4 - a_2^3| \leq \left[ 7 + 28\lambda + 25\lambda^2 + 4 \left( 1 + 4\lambda + 10\lambda^2 \right) |y| + 48\lambda^2 |y|^2 \right] \left( \frac{3 (1 + 2\lambda)^2 (1 + 4\lambda + 3\lambda^2)}{} \right) \times |y|^2. \] (36)

**Proof.** Let \( f(z) \in \mathcal{L}(\lambda, y) \). Then, from (22), we have

\[ a_2 = \frac{-y c_1}{(1 + \lambda)}, \quad a_3 = \frac{-y}{2 (1 + 2\lambda)} (c_2 - 2y c_1^2), \] (37)

\[ a_4 = \frac{-y}{3 (1 + 3\lambda)} (c_3 - 7 y c_1 c_2 + 3y^2 c_1^3). \]
Consider

\[ |a_2 a_4 - a_3^2| = \left| \frac{y^2}{12 (1 + \lambda) (1 + 2 \lambda)^2 (1 + 3 \lambda)} \times \left( 4 (1 + 2 \lambda)^2 c_1 c_3 - 2 y (1 + 4 \lambda + 10 \lambda^2) c_1^2 c_2 + 12 y^2 \lambda^3 c_1^4 - 3 (1 + 4 \lambda + 3 \lambda^2) c_2^2 \right) \right|. \]  

(38)

Now using values of \( c_2 \) and \( c_3 \) from Lemma 5, we obtain

\[ |a_2 a_4 - a_3^2| = \left| \frac{|y|^2}{12 (1 + \lambda) (1 + 2 \lambda)^2 (1 + 3 \lambda)} \times \left[ (1 + 2 \lambda)^2 x^2 \left( 1 - |x|^2 \right) \right] \times \left( 4 - c_1^2 \right)^2 \right|. \]  

(39)

Differentiating with respect to \( \rho \), we get

\[ \frac{\partial F(c, \rho)}{\partial \rho} = \left| \frac{|y|^2}{12 (1 + \lambda) (1 + 2 \lambda)^2 (1 + 3 \lambda)} \times \left[ (1 + 2 \lambda)^2 + |y| \left( 1 + 4 \lambda + 10 \lambda^2 \right) + \frac{3}{2} \left( 1 + 4 \lambda + 3 \lambda^2 \right) \right] c^4 \right| \times (4 - c^2) \rho \]  

(41)

Now since \( \frac{\partial F(c, \rho)}{\partial \rho} > 0 \) for \( c \in [0, 2] \) and \( \rho \in [0, 1] \), maximum of \( F(c, \rho) \) will exist at \( \rho = 1 \) and let \( F(c, 1) = G(c) \). Then

\[ G(c) = \left| \frac{|y|^2}{12 (1 + \lambda) (1 + 2 \lambda)^2 (1 + 3 \lambda)} \times \left[ (1 + 2 \lambda)^2 + |y| \left( 1 + 4 \lambda + 10 \lambda^2 \right) + \frac{3}{2} \left( 1 + 4 \lambda + 3 \lambda^2 \right) \right] c^4 \right| \times (4 - c^2) \]  

(42)

Now by differentiating with respect to \( c \), we obtain

\[ G' (c) = \left| \frac{|y|^2}{12 (1 + \lambda) (1 + 2 \lambda)^2 (1 + 3 \lambda)} \times \left[ (1 + 2 \lambda)^2 + |y| \left( 1 + 4 \lambda + 10 \lambda^2 \right) + \frac{3}{2} \left( 1 + 4 \lambda + 3 \lambda^2 \right) \right] c^3 \right| \times (4 - c^2) \rho^2 + 2c (1 + 2 \lambda)^2 \left( 4 - c^2 \right) (1 - \rho^2) = F(c, \rho). \]  

(40)
Applying triangle inequality and then putting \(|z| = 1, \rho = 1\), and \(c_1 = c\), we have

\[ |a_{2}a_{3} - a_{4}| \]

\[ \leq \frac{ |y| }{ 6(1 + \lambda)(1 + 2\lambda)(1 + 3\lambda) } \times \left[ \begin{array}{c} 7 + 28\lambda + 25\lambda^2 + 4\left(1 + 4\lambda + 10\lambda^2\right)|y| + 48\lambda^2 |y|^2 \\ 3(1 + 2\lambda)^2 (1 + 4\lambda + 3\lambda^2) \end{array} \right] \]

\[ \times |y|^2. \]

(44)

**Theorem 12.** Let \( f(z) \in \mathcal{L}(\lambda, \gamma) \) and be of the form (1). Then

\[ |a_{2}a_{3} - a_{4}| \]

\[ \leq \frac{ |y| }{ 6(1 + \lambda)(1 + 2\lambda)(1 + 3\lambda) } \times \left[ \begin{array}{c} 7 + 28\lambda + 25\lambda^2 + 4\left(1 + 4\lambda + 10\lambda^2\right)|y| + 48\lambda^2 |y|^2 \\ 3(1 + 2\lambda)^2 (1 + 4\lambda + 3\lambda^2) \end{array} \right] \]

\[ \times |y|^2. \]

Proof. From (37), we can write

\[ |a_{2}a_{3} - a_{4}| \]

\[ = \frac{ 12y^2\lambda c_3 - 2y(2 + 6\lambda + 7\lambda^2)c_1c_2 + 2(1 + 3\lambda + 2\lambda^2)c_3 }{ 6(1 + \lambda)(1 + 2\lambda)(1 + 3\lambda) } \]

\[ \times |y|. \]

(46)

Using Lemma 5 for the values of \( c_2 \) and \( c_3 \), we have

\[ |a_{2}a_{3} - a_{4}| \]

\[ = \frac{ |y| }{ 6(1 + \lambda)(1 + 2\lambda)(1 + 3\lambda) } \times \frac{ 1 }{ 2 } \left( 2(3y^2 + 1) \lambda^2 \right) c_3 \\
+ \left( 1 + 3\lambda + 2\lambda^2 \right) 2(2 + 6\lambda + 7\lambda^2)c_3 \\
+ 2(1 + 3\lambda + 2\lambda^2)c_3 \] + \left( 1 + 3\lambda + 2\lambda^2 \right) (4 - c_3^2) \]

\[ \times x \]

\[ - \frac{ 1 }{ 2 } \left( 1 + 3\lambda + 2\lambda^2 \right) (4 - c_3^2) \]

\[ \times x^2 \]

\[ + (1 + 3\lambda + 2\lambda^2) (4 - c_3^2) (1 - |x|^2) \] z^2. \]

(47)

Now by using the same procedure as we did in the proof of Theorem 11, we obtain the required result.

**Theorem 13.** If \( f(z) \in \mathcal{L}(\lambda, \gamma) \) and is of the form (1), then

\[ |H_3(1)| \]

\[ \leq \frac{ |y| (1 + 4|y|) }{ (1 + 2\lambda) } \times \left[ \begin{array}{c} \left( 7 + 28\lambda + 25\lambda^2 + 4\left(1 + 4\lambda + 10\lambda^2\right)|y| + 48\lambda^2 |y|^2 \right) \\
3(1 + 2\lambda)^2 (1 + 4\lambda + 3\lambda^2) \end{array} \right] \]

\[ \times |y|^2. \]

(49)
\[
\begin{align*}
&\times |\gamma|^2 + 2 |\gamma| (1 + 4 |\gamma|) (1 + 3 |\gamma|) \\
&= \left[ \frac{2 |\gamma| \left( (4 |\gamma| + 1) \left( (3 |\gamma| + 1) 2\lambda^2 + (3\lambda + 1) \right) \right)}{3 (1 + \lambda) (1 + 2\lambda) (1 + 3\lambda)} \right] \\
&\times |\gamma| (1 + 4 |\gamma|) (1 + 3 |\gamma|) (3 + 8 |\gamma|) |\gamma|^2 \\
&\times \left[ \frac{7 + 28\lambda + 25\lambda^2 + 4 (1 + 4\lambda + 10\lambda^2) |\gamma|^2 + 48\lambda^2 |\gamma|^2}{3 (1 + 2\lambda)^3 (1 + 4\lambda + 3\lambda^2)} \right] \\
&\times |\gamma|^2 (1 + 4 |\gamma|) \\
&\times \left[ \frac{4 \left( (4 |\gamma| + 1)^2 \left( (3 |\gamma| + 1) 2\lambda^2 + (3\lambda + 1) \right) \right)}{9 (1 + \lambda) (1 + 2\lambda) (1 + 3\lambda)^2} \right] \\
&\times \left[ \frac{(1 + 4 |\gamma|) (1 + 3 |\gamma|) (3 + 8 |\gamma|) |\gamma|^2}{6 (1 + 2\lambda) (1 + 4\lambda)} \right] \\
&\times \left[ \frac{7 + 28\lambda + 25\lambda^2 + 4 (1 + 4\lambda + 10\lambda^2) |\gamma|^2 + 48\lambda^2 |\gamma|^2}{3 (1 + 2\lambda)^3 (1 + 4\lambda + 3\lambda^2)} \right].
\end{align*}
\]

This completes the proof of this result. \(\blacksquare\)

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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