SPECTRA AND STRAINS

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Abstract. This is a blend of two informal reports on the activities of the seminar on Galois representations and mirror symmetry given at the Conference on classification problems and mirror duality at the Steklov Institute, in March 2006, and at the Seminar on Algebra, Geometry and Physics at MPI, in November 2007. We assess where we are on the issue of the spectra of Fano varieties, and state problems. We introduce higher dimensional irreducible analogues of dessins, the low ramified sheaves, and hypothesize that Fano spectra relate to their geometric conductors. We give a recipe to a physicist.

1. The Fano Spectra

1.1. Spectra and anticanonical spectra. Let \( F \) be a Fano variety of index \( d \), so that \( -K = dH \). Consider the matrix of quantum multiplication by \( H \). It has entries in \( \mathbb{Q}[q_i, q_i^{-1}] \), where \( i \)'s correspond, as usual, to the numerical classes of curves on \( F \).

One may specialize the matrix to \( M_H \) in \( \text{Mat}(\mathbb{Q}[t, t^{-1}]) \) by

\[ q_i \mapsto t^{\text{degree of curve of class } i}, \]

where the degree is taken with respect to \( H \). There is no need to do that when \( H^2(F) = \mathbb{Z} \) which we will freely assume henceforth. Let \( \text{inv} \) stand for the multiplicative inversion \( \mathbb{G}_m \to \mathbb{G}_m \).

We define [ provisionally!] the spectrum to be the inv of the closed subscheme of \( \mathbb{G}_m = \text{Spec } \mathbb{Q}[t, t^{-1}] \) given by the principal ideal generated by \( \det(M_H) \).

A concurrent notion is that of an anticanonical spectrum. This arises in a similar way when one specializes the matrix of quantum multiplication by \( -K \) to \( M_{-K} \) in \( \text{Mat}(\mathbb{Q}[t, t^{-1}]) \) by

\[ q_i \mapsto t^{\text{degree of curve of class } i}, \]

where the degree is now taken with respect to \( -K \). Up to a shift on the torus, the anticanonical spectrum is the pullback of the spectrum.
under the $d$-isogeny $\mathbb{G}_m \to \mathbb{G}_m$. One also considers the complete anticanonical spectrum that comprises all singularities of the regularized anticanonical quantum $D$-module $\mathcal{G}_0$ by adding a suitable component at infinity.

1.2. Quantum Lefschetz and stability of spectra. Givental’s Quantum Weak Lefschetz theorem implies that the spectra of Fanos are stable under hyperplane sections: if $V$ is a hyperplane section of $F$ of index $> 1$, then $\text{spec } F$ coincides with $\text{spec } V$ up to a multiplicative shift. If the index of $V$ is 1, there exists a so-called Givental constant $g$ such that $\text{spec } F - g$ coincides with $\text{spec } V$ up to a multiplicative shift, and from now on we adjust our definition of $\text{spec } F$ to be what formerly was $\text{spec } F - g$.

1.3. Strains. Two [deformation classes of] Fano varieties are said to be in the same strain if one is a hyperplane section of the other. We extend that to an equivalence relation and define the spectrum of a strain to be the spectrum of any of its members. The spectrum is well defined up to a multiplicative shift.

If $V$ is a hyperplane section of $F$, we sometimes refer to $F$ as an unsection of $V$.

1.4. Progenitors. We call a Fano variety a progenitor of its strain if any variety in the strain is its hyperplane section. In particular, progenitor have no unsections.

1.5. Problems. Given a Fano $F$, determine whether it is a progenitor of its strain. Given a strain $S$, determine whether it is finite.

Example. The strain of complete intersections in projective space is infinite and has no progenitor.

One may choose to work with “easier” Fano varieties, or their strains:

1.6. Cellular, minimal, Tate. A Fano variety $F$ is said to be minimal if its cohomology is as small as it can be ($H^{2k+1}(F) = 0, H^{2k}(F) = \mathbb{Z}$). A Fano is said to be Tate if its motive has no non-Tate constituents. A Fano $F$ is said to be cellular if $F$ is a union of affine spaces: $F = \bigcup \mathbb{A}^{i_j}(j)$. $\mathbb{A}^{i_1}_{j_1} \cap \mathbb{A}^{i_2}_{j_2} = \emptyset$ if $j_1 \neq j_2$. A strain is called cellular/minimal/Tate if it has a cellular/minimal/Tate variety in it.
Examples. The strain of complete intersections in projective spaces is cellular and minimal and Tate. Its spectrum is one point defined over \(Q\). A less trivial example is the strain of the Grassmannian \(G(2,5)\). It is again cellular and minimal (its triple hyperplane section is a minimal Fano threefold \(V_3\)) and Tate. Its spectrum consists of the two roots of \(t^2 - 11t - 1\). Hence, \(V_3\) is not a complete intersection in a projective space. Spectra of rank 1 Fano threefolds \(F\) are formed by elliptic points and cusp points on \(X_0(N)/\mathbb{Q}\) where \(N = \frac{(-K_F)^3}{2d^2}\) [Go].

1.7. Problem. Clearly, cellular implies Tate, and minimal implies Tate. Are there non-cellular minimal strains?

1.8. Problems. Find the spectra of all Fano / all cellular / all minimal strains. Find the Fano spectral field \(F \subset \overline{Q}\), is the minimal field of definition of all Fano spectra.

1.9. Theorem. [GG] The spectra of all Grassmannians are defined over \(Q^{ab}\).

Contrary to some expectations, S. Galkin and I have found that it is possible, even in a case of a classical group, to find in the spectrum of a generalized grassmannian an irreducible component (i.e. a Gal \(Q\) orbit) whose Galois group is the symmetric group \(S_n, n \geq 5\). We have also found that two generalized grassmannians of different classical groups may have a common non-trivial irreducible component in their spectra. This shows that the spectrum is a fine invariant and raises the following

1.10. Problems. To what extent does a spectrum determine its (cellular) strain? Is it possible for two different (say, cellular) strains to have the same spectrum? And, vaguely: let \(P_1\) and \(P_2\) be the progenitors of the strains \(S_1\) and \(S_2\). Let the intersection of Spec \(S_1\) and Spec \(S_2\) be non-empty and non-trivial. Does it imply that there is a natural correspondence between \(P_1\) and \(P_2\)?

2. Low ramification sheaves and LRS spectra

Is the way the cells of a cellular variety are joined together controlled by a dessin–type combinatorics? Dessins should be generalized to positive relative dimensions as “maximally Szpiro” objects, such as flat morphisms that have as few critical points as possible. We discuss this naive approach in greater detail in the next section; meanwhile,
to fix ideas, we deal with [absolutely geometrically] irreducible Galois representations of the field $\mathbb{Q}(t)$ that are low–ramified geometrically.

2.1. Geometric ramification. Let $U \subset \mathbb{P}^1/\mathbb{C}$ be a Zariski open subset, $S = \mathbb{P}^1 - U$. Let $L$ be a rank $r$ non-trivial irreducible polarized local system over $U$, i.e. a representation $\varphi : \pi_1(U^{an}) \longrightarrow O(r)/Sp(r)$, and let $L_x$ be its generic fiber. Its ramification is

$$R(L) = \sum_{s \in S(\mathbb{C})} \dim L_x/L_x^I_s.$$ 

2.2. Low ramified local systems and their conductors. A local system $L$ as above is said to be low ramified if $R(L) = 2 \text{rk} L$.

Its geometric conductor is the respective closed subscheme of $\mathbb{P}^1 | \mathbb{C}$: the union of points $s \in S(\mathbb{C})$ each taken with multiplicity $\dim L_x/L_x^I_s$.

We want to consider the cases when $L$ is “of arithmetic nature”. One may imagine the following simplistic picture. First, one wants the image of the monodromy to be in $GL(r, \mathbb{Z})$. Given a prime number $l$, one arrives at a tower of unramified Galois covers $U(l^n)$ of $U$ by considering the respective system of mod $l^n$ representations. Assume $S$ is defined over $\mathbb{Q}$, that is, $U$ is defined over $\mathbb{Q}$. The second requirement is that the covers $U(l^n)$ be defined over $\mathbb{Q}$.

One may budge a low ramified local system on the complex analytic base isomonodromically by shifting points in $S$, but that will result, in general, in a loss of definability of the system of level covers over $\mathbb{Q}$, or in fact over any fixed number field (although each $U(l^n)$ is definable individually over some number field by Weil).

A typical example of an “arithmetic” local system is an irreducible constituent in the local system of relative cohomologies in a smooth pencil over $U | \mathbb{Q}$. As is usual in this context, we allow for monodromies in the ring of algebraic integers, and for the finite base field change $K/\mathbb{Q}$:

2.3. Low ramified $l$-adic sheaves. A [necessarily tame] lisse [absolutely] geometrically irreducible $\overline{\mathbb{Q}}_l$-sheaf $\mathcal{L}$ on $U_K$ is said to be low ramified, or to be an LRS, if its geometric ramification computed as above satisfies $R(\mathcal{L}) = 2 \text{rk} \mathcal{L}$. We say that an $l$-adic sheaf on $\mathbb{P}^1 | K$ is low ramified if its restriction to its ouvert de lissité is low ramified. We define the conductors of low ramified sheaves as above.

Conductors of low ramified sheaves will also be referred to as $LRS$ spectra.
2.4. **Problem.** Find the conductors of low ramified sheaves on $\mathbb{P}^1|_Q$ (resp. $\mathbb{P}^1|_K$) of a given rank.

2.5. **Fano spectra and LRS spectra.** One may ask what the two worlds have in common. Our interest in the LRS spectra arose from the fact that the Landau-Ginsburg models of the rank 1 threefolds are, motivically, twisted Kuga-Sato families. The due generalization of such a Kuga-Sato or a modular elliptic surface over a rational base to a higher relative dimension $N - 1$ is a pencil $\pi : \mathcal{E} \rightarrow \mathbb{P}^1|_Q$ such that the “essential” constituent of $R^{N-1}\pi_1(\mathbb{Q}_{\ell})$ is a low ramified sheaf on the base. Consider now a minimal Fano $F$; its completed anticanonical spectrum is given by the symbol of the “counting equation DN” of $F$. Now, a generic equation DN has been shown \cite{GS} to be of low ramification, in the sense that the local system of its solutions is. A conjecture of mirror symmetry asserts that the counting DNs are of Picard-Fuchs type, hence, modulo the conjecture, the spectrum of a “generic” minimal Fano is also an LRS spectrum.

2.6. **Problems.** On some genericity assumption on the variety $2$ is every component (= Galois orbit) of the completed anticanonical spectrum of a Fano/cellular Fano/Tate Fano also a component of some LRS spectrum? The reverse, ‘is every component of an LRS spectrum also a part of some cellular spectrum?’, is most probably refutable as stated, but might become a real one if the premise is made a bit more specific (prescribing types of some of local monodromies, etc).

2.7. **Arithmetic conductors of geometric conductors.** We finish this section with the following observation: the fields of definition of the components of the Fano spectra tend to have small discriminants per degree. The Galois group of the Fano threefold $V_{22}$ is $S_3$. The discriminant is $-44$. The spectrum of the blowup of $\mathbb{P}^3$ along $\mathbb{P}^1$ consists of two irreducible pieces. The Galois group of each is $S_3$. The fields of definition are unramified over the respective quadratic extensions. The discriminants are $-23, -31$, exactly the two lowest levels at which there emerge weight 1 cuspforms. Can the assertion that ‘the components of the spectrum of a cellular variety are not too ramified’ be made precise? Are they close to the border allowed by the explicit formulae $3$? Rephrasing, shall we expect the combinatorics of the affine

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1. LG’s of [quantum minimal] del Pezzos have modular meaning, too.
2. A suitable quantum analogue of absence of primitive algebraic classes in the middle dimension.
3. Diaz y Diaz.
cells to possess certain optimality properties? And on the LRS side, can one show that the arithmetic conductors of geometric conductors of low ramified sheaves are small? Vague as it is, this observation, if extended, may have very practical consequences for the search for the spectra.

3. The search

In order to tabulate low ramified sheaves of low ranks, one may proceed by fieldworking for another closely related zoology.

3.1. The special Laurent polynomials zoo. Let $M$ be the standard lattice in $\mathbb{R}^N$, and let $P$ be a Calabi-Yau lattice polytope, that is, one with 1 strictly internal point. Inasmuch as it is allowed by $P$, a generic non–zero polynomial $\pi$ ‘tends’ to be Morse–Lefschetz, i.e. have simple singularities and critical values. What we need is the opposite of the generic: stratify $X$ according to how the critical values come together and single out the Artinian strata. We refer to the geometric points in these strata as the special Laurent polynomials$^4$.

3.2. LRS vs Special Laurent. One should not expect the two classifications to directly translate one into the other. Which low ramified sheaves arise then as irreducible constituents in $R^{N-1}\pi_1(\overline{\mathcal{Q}_l})$ with $\pi$ special Laurent? What are the conditions on $P$ that guarantee that the special Laurent polynomials with support in $P$ produce low ramified sheaves, or at least some of them do?

According to Batyrev’s idea of mirror symmetry for non-torics, a Fano $F$, say of Picard rank 1, may degenerate to a toric whose fan’s unit vectors are the vertices of a polytope of CY type. We may look for the weak Landau-Ginsburg model $\mathbb{P}_F$ of $F$ in the linear space $X$ of Laurent polynomials with support in $P$. One should not therefore be too surprised$^5$ to find inherent structural similarities between the [subdivisions within] classifications of Batyrev type degenerations, the low ramified sheaves and the special Laurent polynomials.

$^4$The definition of the stratification on $X$ seems to require a good deal of local to global algebra that can account for multiple singularities, non–isolated singularities and singularities at the infinity of the compactification. It is not improbable, though, that these fine contributions may add up to a practicable total.

$^5$Upcoming is S. Galkin’s thesis where some of these matters are worked out for $N = 2, 3$. 
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