ON RELATIVE AND OVERCONVERGENT DE RHAM-WITT COHOMOLOGY FOR LOG SCHEMES

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Abstract. We construct the relative log de Rham-Witt complex. This is a generalization of the relative de Rham-Witt complex of Langer-Zink to log schemes. We prove the comparison theorem between the hypercohomology of the log de Rham-Witt complex and the relative log crystalline cohomology in certain cases. We construct the $p$-adic weight spectral sequence for relative proper strict semistable log schemes. When the base log scheme is a log point, we show it degenerates at $E_2$ after tensoring with the fraction field of the Witt ring. We also extend the definition of the overconvergent de Rham-Witt complex of Davis-Langer-Zink to log schemes $(X, D)$ associated with smooth schemes with simple normal crossing divisor over a perfect field. Finally, we compare its hypercohomology with the rigid cohomology of $X \setminus D$.

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1. Introduction

The de Rham-Witt complex \( \{W_m^{p} \Omega_X^\bullet \}_{m \in \mathbb{N}} \) was defined by Illusie [Ill79] for a scheme \( X \) of characteristic \( p > 0 \). He defined it as the initial object of \( V \)-pro-complexes. When \( X \) is smooth over a perfect scheme, the hypercohomology of the de Rham-Witt complex computes the crystalline cohomology. Also, Illusie and Raynaud [IR83] remarked that one can also define the de Rham-Witt complex by using the crystalline cohomology sheaf in the process of definition.

Langer and Zink [LZ04] extended Illusie’s definition to relative situations. Let \( S \) be a \( \mathbb{Z}_p \)-scheme such that \( p \) is nilpotent in \( S \). They defined the log de Rham-Witt complex \( \{W_m^{p} \Omega_{X/S}^\bullet \}_{m \in \mathbb{N}} \) for a scheme \( X \) over \( S \). Their definition is close to that of Illusie: In fact, they defined it as the initial object of \( F-V \)-pro-complexes. The hypercohomology of Langer and Zink’s de Rham-Witt complex also computes the crystalline cohomology in smooth cases.

Olsson [Ols07] extended Langer-Zink’s definition to the case of algebraic stacks. He also gave another possible definition of de Rham-Witt complex via the crystalline cohomology sheaf and compared two definitions, but it seems that they do not always coincide.

It is natural to extend the definition of the de Rham-Witt complex to the case of log schemes in the sense of Fontaine-Illusie-Kato ([Kat89]), which is our main interest. Hyodo and Kato [HK94] defined the log de Rham-Witt complex for a log smooth log scheme of Cartier type over a perfect field of characteristic \( p > 0 \) by using the log crystalline cohomology sheaf. They also proved the comparison theorem to the log crystalline cohomology.

Nakajima [Nak05] introduced a theory of formal de Rham-Witt complexes as a kind of axiomatization of Hyodo-Kato’s construction. It also covers the cohomological construction of the de Rham-Witt complex for smooth schemes with simple normal crossing divisor over a perfect field of characteristic \( p > 0 \).

In this paper, we construct the log de Rham-Witt complex for a fine log scheme \( X \) over a fine log scheme \( S \) over \( \mathbb{Z}_p \). We follow the definition of Langer-Zink, and construct the log de Rham-Witt complex as the initial object of log \( F-V \)-pro-complexes. Note that we cannot apply the methods of Hyodo, Kato and Nakajima directly to our log de Rham-Witt complex because their methods seem to be applicable only to the case of perfect base log schemes and because the definition using the log crystalline cohomology sheaf seems not to be good in the case of non-perfect base log schemes. We prove the comparison theorem between the hypercohomology of the log de Rham-Witt complex and the relative log crystalline cohomology in
case of relative semistable log schemes and that of log schemes associated to smooth schemes with normal crossing divisor.

Mokrane [Mok93] used the de Rham-Witt complex of Hyodo-Kato to construct the $(p$-adic) weight spectral sequence for the crystalline cohomology of strictly semistable log schemes. He proved its $E_2$-degeneration modulo torsion when the base scheme is the spectrum of a finite field. Nakajima [Nak05] extended his result to the case where the base scheme is the spectrum of any perfect field by using the specialization argument of Illusie-Deligne ([Ill75]). In this paper, we construct the $p$-adic weight spectral sequence for the relative crystalline cohomology of a relative strictly semistable log schemes and prove its $E_2$-degeneration modulo torsion when the base scheme is the spectrum of a (not necessarily perfect) field.

Since our definition of the log de Rham-Witt complex follows that of Langer-Zink and differs from that of Hyodo-Kato, the proof of our results is similar to that of Langer-Zink and differs from that of Hyodo-Kato and Mokrane. The key ingredient is to find certain explicit basis of the log de Rham-Witt complex called the log basic Witt differentials in explicit cases, which are generalizations of the basic Witt differentials of Langer-Zink.

We also introduce the notion of the overconvergent log de Rham-Witt complex. Davis, Langer and Zink [DLZ11] introduced the notion of the overconvergent de Rham-Witt complex for smooth schemes over a perfect field of characteristic $p > 0$. They proved the comparison theorem between its hypercohomology and the Monsky-Washnitzer cohomology in the affine case. They also proved that its hypercohomology calculates the rigid cohomology in the case of smooth quasi-projective varieties using Große-Klönne’s theory of dagger spaces [GK00]. We first treat the case of smooth affine varieties with simple normal crossing divisor over a perfect field of characteristic $p > 0$ such that they admit a global coordinates and divisors are defined by the coordinates. We define the overconvergent log de Rham-Witt complex in this case and prove the comparison theorem between its hypercohomology and the log Monsky-Washnitzer cohomology of Tsuzuki [Tsu99]. More generally, we can extend the definition of the overconvergent log de Rham-Witt complex to arbitrary log schemes obtained by smooth schemes with simple normal crossing divisor. By combining the result of local cases with a result in [Tsu99], we can prove the comparison theorem with rigid cohomology.

The content of each section is as follows: In §2, we fix notations which we use in this paper and give the definition of the crystalline cohomology over non-adic base, which we need later.

In §3, we define the log version of $F$-$V$-procomplexes and the de Rham-Witt complex defined by Langer and Zink. We extend their fundamental results to our log cases.

In §4, we define the log $p$-basic elements and the log basic Witt differentials in specific cases. They are generalizations of the $p$-basic elements and the basic Witt differentials defined in [LZ04] §2.1, 2.2. We prove that any element of the log de Rham-Witt complex is written as a convergent sum of the log basic Witt differentials. The notion of the basic differentials is a powerful tool for us and it plays a role in proofs in the later sections.

In §5, we give the definition of log Witt lifts and log Frobenius lifts for log smooth log schemes. We prove that there exists a log Frobenius lift étale locally.

In §6, we construct the comparison morphism between the log crystalline cohomology and the hypercohomology of the log de Rham-Witt complex for log smooth log schemes using log Frobenius lifts.

In §7, we prove the comparison theorem for smooth schemes with normal crossing divisor and semistable log schemes.
In §8, we define the weight filtrations of the log de Rham-Witt complex and construct the $p$-adic Steenbrink complex for proper semistable log schemes over arbitrary base. The $p$-adic Steenbrink complex defines a spectral sequence, which we call the $p$-adic weight spectral sequence. When the base scheme is the spectrum of a (not necessarily perfect) field, we prove $E_2$-degeneration after tensoring with the fractional field of the Witt ring by using Nakkajima’s specialization method.

In §9, we construct the $p$-adic weight spectral sequence of proper smooth schemes with simple normal crossing divisor and prove its $E_2$-degeneration (after tensoring with the fraction field of the Witt ring) when the base scheme is the spectrum of a (not necessarily perfect) field.

In §10, we give the definition of the overconvergent log de Rham-Witt complex for a log scheme $(X, D)$ defined by a smooth scheme $X$ with simple normal crossing divisor $D$ over a perfect field $k$ of characteristic $p > 0$. We see that the overconvergent log de Rham-Witt complex coincides with the overconvergent de Rham Witt complex of Davis-Langer-Zink ([DLZ11]) when the log structure is trivial. We compare the overconvergent log de Rham-Witt cohomology with the log Monsky-Washnitzer cohomology in affine cases, and with the rigid cohomology of $X \setminus D$ in general cases.

Finally, note that there exist several other variants of de Rham-Witt complex: When $p$ is odd, Hesselholt and Madsen defined the absolute de Rham-Witt complex \( \{ W_m^\cdot \Omega_X^\wedge \}_{m \in \mathbb{N}} \) for any $\mathbb{Z}_p$-scheme $X$ ([HM03], [HM04]). Hesselholt studied the relation with the Langer and Zink’s relative de Rham-Witt complex using $K$-theoretic methods ([Hes05]). When $p$ is odd and nilpotent in $S$ and $X$ is $S$-scheme, there is a canonical surjective map \( \{ W_m^\cdot \Omega_X^\wedge \}_{m \in \mathbb{N}} \twoheadrightarrow \{ W_m^\cdot \Omega_{X/S}^\wedge \}_{m \in \mathbb{N}} \) from the absolute de Rham-Witt complex to the Langer-Zink’s relative de Rham-Witt complex. Cuntz and Deninger defined the relative de Rham-Witt complex in arbitrary truncated sets by a different approach so that the big and the $p$-isotypical theories are covered ([CD14]). It would be an interesting problem to generalize their constructions to the case of log schemes, and compare them with our construction.

We also remark that there are other studies to construct a ($p$-adic) weight filtration. Nakkajima and Shiho [NS08] construct a theory of weights on the log crystalline cohomology of a family of open smooth variety. They used the log de Rham complex of a lift to define a weight filtration. Nakkajima [Nak15] applied their method to a proper truncated simplicial SNCL (=simple normal crossing log) scheme having affine truncated simplicial open covering. Tsuji [Tsu10] used filtrations of sheaves of $D$-modules to construct a weight spectral sequence for a semistable log scheme over a complete discrete valuation ring. It is also an interesting problem to consider their situations using our de Rham-Witt complex and to compare with their results.

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Notations. We fix a prime number $p$ throughout this paper. All schemes are assumed to be defined and separated over $\mathbb{Z}_p$.

Let $R$ be a ring. For a $W(R)$-module $N$, we write $N_{[F]}$ for the $W(R)$-module whose underlying set is $N$ and its module structure is obtained by the Frobenius map $F: W(R) \to W(R)$. 
If $R$ is an $\mathbb{F}_p$-algebra and $L$ is a sheaf of $W(R)$-modules equipped with an endomorphism $\phi$ which is $F$-linear ($F$: Frobenius map on $W(R)$) and $r$ is a negative integer, the Tate twist $L(r)$ denotes a sheaf $L$ with the endomorphism $p^{-r}\phi$.

We use the convention of Nakajima about signs. ([Nak05], Conventions)

For a $\mathbb{Z}_{(p)}$-algebra $R$, the ring of Witt vectors of any length $W_m(R)$ has a canonical pd-structure on the ideal $I = V W_m(R)$ given by

$$
\gamma_n(V \xi) = \frac{p^{n-1}}{n!} V(\xi^n), \xi \in W_{m-1}(R), n \geq 1.
$$

We always consider this pd-structure on the ring of Witt vectors.

For a $\mathbb{Z}_{(p)}$-scheme $S$ and an $S$-scheme $X$, $\{W_m \Omega_{X/S}^\bullet\}_{m \in \mathbb{N}}$ denotes the de Rham-Witt complex constructed in [LZ04] §1.3.

2. Preliminaries

2.1. Logarithmic geometry. In this paper, we use freely the terminologies concerning logarithmic geometry in the sense of Fontaine-Illusie-Kato. The basic reference is [Kat89]. All log schemes are assumed to be fine and separated and defined over $\mathbb{Z}_{(p)}$. If $X$ is a log scheme, we denote by $\tilde{X}$ the underlying scheme of $X$.

**Definition 2.1.** (1) A pre-log ring is a triple $(A, P, \alpha)$ consisting of a commutative ring $A$, a commutative fine monoid $P$ and a morphism of monoids $P \to A$ where $A$ is regarded as a monoid by its multiplicative structure. We usually suppress $\alpha$ in the notation. We denote by $\{\ast\}$ the trivial monoid.

(2) If $(A, P)$ is a pre-log ring, $\text{Spec}(A, P)$ is the log scheme whose underlying scheme is $X = \text{Spec} A$ with the log structure associated to the pre-log structure $P \to \mathcal{O}_X$ induced by the structure map $\alpha : P \to A$.

(3) We say $(Y, \mathcal{N})$ is a log scheme over a pre-log ring $(A, P)$ to mean that $(Y, \mathcal{N})$ comes equipped with a morphism of log schemes $(Y, \mathcal{N}) \to \text{Spec}(A, P)$.

**Definition 2.2.** A morphism $(A, P) \to (B, Q)$ of pre-log rings is said to be log smooth (resp. log étale) if the kernel and the torsion part of the cokernel (resp. the kernel and the cokernel) of $P^\text{gp} \to Q^\text{gp}$ are finite groups of orders invertible on $B$ and the induced morphism $A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q] \to B$ is an étale ring map.

We recall the toroidal characterization of log smoothness ([Kat89] (3.5), [Kat96] Theorem 4.1):

**Theorem 2.3.** Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a morphism of fine log schemes and $Q \to \mathcal{N}$ a chart of $\mathcal{N}$. Then the following conditions are equivalent.

1. $f$ is log smooth (resp. log étale).

2. There exists étale locally a chart $(P \to \mathcal{M}, Q \to \mathcal{N}, Q \to P)$ of $f$ extending $Q \to \mathcal{N}$ such that

   a. The kernel and the torsion part of the cokernel (resp. the kernel and the cokernel) of $Q^{\text{gp}} \to P^{\text{gp}}$ are finite groups of orders invertible on $X$.

   b. The induced map $X \to Y \times_{\text{Spec} \mathbb{Z}[Q]} \text{Spec} \mathbb{Z}[P]$ of schemes is étale (in the usual sense).

We see if $(A, P) \to (B, Q)$ is a log smooth (resp. log étale) morphism of pre-log rings, the induced map $\text{Spec}(B, Q) \to \text{Spec}(A, P)$ is a log smooth (resp. log étale) morphism of log schemes.

**Definition 2.4.** (1) Let $f : X \to S$ be a smooth morphism of schemes and $D \subset X$ a reduced Cartier divisor. Let $j : U := X \setminus D \to X$ be the natural open immersion. We call $D$ a simple normal crossing divisor (SNCD) (resp. a normal crossing divisor (NCD)) if, for any point of $z$ of $D$, there exist a Zariski open neighbourhood $V$ of $z$. If $f : \text{Spec}(A, P) \to \text{Spec}(B, Q)$. Then $f$ is log smooth (resp. log étale) if the kernel and the torsion part of the cokernel (resp. the kernel and the cokernel) of $Q^{\text{gp}} \to P^{\text{gp}}$ are finite groups of orders invertible on $B$. If $f : \text{Spec}(A, P) \to \text{Spec}(B, Q)$ is a log smooth (resp. log étale) morphism of log schemes.
z in X (resp. an étale morphism \( V \to X \) such that the image of \( V \) contains \( z \)) and the following cartesian diagram of schemes

\[
\begin{array}{ccc}
D \times X \overset{c}{\longrightarrow} V \\
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A basic example is the Witt vector $A = W(R)$ of a Noetherian $\mathbb{Z}(\varpi)$-algebra $R$ in which $\varpi$ is nilpotent and the ideals $I_m = \varpi^m W(R)$ equipped with the canonical $p$-structure.

Let $X$ be a proper smooth scheme over $A_1$. We have a canonical morphism of crystalline topoi $\iota_m : (X/A_m)_{\text{crys}} \to (X/A_n)_{\text{crys}}$ for $n \geq m$. We say the system $\mathcal{E} = \{\mathcal{E}_m\}_m$ is a compatible system of locally free crystals if for each $m$, $\mathcal{E}_m$ is a locally free crystal on the crystalline site $\text{Crys}(X/A_m)$ and for each $n \geq m$, $\iota_m^* \mathcal{E}_n \cong \mathcal{E}_m$.

**Definition 2.7.** Let $R$ be a ring and $D(R)$ be the derived category of the category of complexes of $R$-modules.

1. We call $K^\bullet \in D(R)$ is perfect if $K^\bullet$ is quasi-isomorphic to a bounded above complex of finite free $R$ modules and it has finite tor dimension. This is equivalent to the condition that $K^\bullet$ is quasi-isomorphic to a bounded complex of finite projective $R$-modules.

2. We call $K^\bullet \in D(R)$ is strictly perfect if $K^\bullet$ is quasi-isomorphic to a bounded complex of finite free $R$-modules and it has finite tor dimension.

**Lemma 2.8.** Suppose given $K_m \in D(A_m)$ and a map $K_{m+1} \to K_m$ in $D(A_{m+1})$ for each $m \geq 0$. We assume

1. $K_1$ is a perfect object, and

2. The maps induce isomorphisms

$$K_{m+1} \otimes_{A_{m+1}}^L A_m \to K_m.$$

Then $K = \varprojlim K_m$ is a perfect object of $D(A)$ and $K \otimes_{A}^L A_m \to K_m$ is an isomorphism for all $m$.

**Proof.** Since $I_n/I_m$ and $I_m/I_{m+1}$ are nilpotent ideals for all $n$ and $A$ is complete for the topology defined by $\{I_m\}$, we can extend [Sta15] More on Algebra, Lemma 15.65.2 and Lemma 15.65.3 to our case.

**Proposition 2.9.** (cf. [Sta15] Crystaline Cohomology, Remark 45.24.12)

There exists a perfect object $\underline{\Gamma}^\text{crys}(X/A, \mathcal{E})$ in $D(A)$ such that

$$\underline{\Gamma}^\text{crys}(X/A, \mathcal{E}) \otimes^L_A A_m \cong \underline{\Gamma}^\text{crys}(X/A_m, \mathcal{E}_m).$$

**Proof.** Base change theorem ([BO78] Theorem 7.8) gives

$$\underline{\Gamma}^\text{crys}(X/A_m+1, \mathcal{E}_{m+1}) \otimes_{A_{m+1}}^L A_m \cong \underline{\Gamma}^\text{crys}(X/A_m, \mathcal{E}_m)$$

for every $n$. By this result and the comparison theorem ([BO78] Theorem 7.1) we obtain

$$\underline{\Gamma}^\text{crys}(X/A_1, \mathcal{E}_1) \cong \underline{\Gamma}^\text{Zar}(X, (\mathcal{E}_1)_X \otimes \Omega^\bullet_{X/A_1}).$$

We show first that $\underline{\Gamma}^\text{Zar}(X, (\mathcal{E}_1)_X \otimes \Omega^\bullet_{X/A_1})$ is perfect. By using the stupid filtration on $(\mathcal{E}_1)_X \otimes \Omega^\bullet_{X/A_1}$, we are reduced to showing that $\underline{\Gamma}^\text{Zar}(X, (\mathcal{E}_1)_X \otimes \Omega^\bullet_{X/A_1})$ is perfect by [Sta15] More On Algebra, Lemma 58.4. It follows from the fact that $(\mathcal{E}_1)_X \otimes \Omega^\bullet_{X/A_1}$ is a locally free sheaf of finite type and $X$ is proper over a Noetherian ring $A_1$. Thus we have a perfect object $D := \varprojlim \underline{\Gamma}^\text{crys}(X/A_m, \mathcal{E}_m)$ and it has the property $D \otimes_{A}^L A_m \cong \underline{\Gamma}^\text{crys}(X/A_m, \mathcal{E}_m)$.

We have the same proposition for a proper log smooth integral scheme $X$. (Use [Kat89] Theorem (6.10)).

**Definition 2.10.** Assume that $R$ is a Noetherian $\mathbb{Z}(\varpi)$-algebra in which $\varpi$ is nilpotent.

If $X$ is a proper smooth scheme over $R$ and $\mathcal{E} = \{\mathcal{E}_m\}_m$ is a compatible system of locally free crystals of $X$, we define crystalline cohomology of $X$ with coefficients
$\mathcal{E}$ over $W(R)$ by $H^{*}_{\text{crys}}(X/W(R), \mathcal{E}) := \mathbb{R}^{+}\Gamma_{\text{crys}}(X/W(R), \mathcal{E})$. We define crystalline cohomology of $X$ by $H^{*}_{\text{crys}}(X/W(R)) := H^{*}_{\text{crys}}(X/W(R), \mathcal{O}_{X/W(R)})$.

If $(R, P)$ is a pre-log ring and $X$ is a proper log smooth integral scheme over $(R, P)$, we define the log crystalline cohomology $H^{*}_{\text{log-crys}}(X/W(R, P))$ in the similar fashion.

**Theorem 2.11.** Let $R$ be a Noetherian $\mathbb{Z}(p)$-algebra in which $p$ is nilpotent, and $X$ be a proper smooth scheme over $R$. Then we have a canonical isomorphism

$$H^{*}_{\text{crys}}(X/W(R)) \to \mathbb{H}^{*}_{\text{Zar}}(X, W\Omega^{*}_{X/R}).$$

**Proof.** Let $u_{m} : (X/W_{m}(R))_{\text{crys}} \to X_{\text{Zar}}$ be the canonical morphism of topoi. Using the simplicial method (cf. [LZ04] §3.2) we may assume $X$ is embedded in a smooth affine scheme $Y$ which admits a Frobenius lift $Y_{m}$. From the naturality of the comparison morphism of crystalline cohomology and de Rham cohomology ([BO78] Theorem 7.1, Remark 7.5) we have a commutative diagram

$$\begin{array}{ccc}
\mathbb{R}u_{m*}\mathcal{O}_{X/W_{m}(R)} & \to & \Omega^{*}_{Y_{m}/W_{m}(R)} \\
\downarrow & & \downarrow \\
\mathbb{R}u_{m-1*}\mathcal{O}_{X/W_{m-1}(R)} & \to & \Omega^{*}_{Y_{m-1}/W_{m-1}(R)}.
\end{array}$$

Moreover, the Frobenius lift $Y_{m}$ makes the following diagram commutative

$$\begin{array}{ccc}
\mathcal{O}_{Y_{m}} & \xrightarrow{\Delta_{m}} & W_{m}(Y) \\
\downarrow & & \downarrow \\
\mathcal{O}_{Y_{m-1}} & \xrightarrow{\Delta_{m-1}} & W_{m-1}(Y).
\end{array}$$

Hence we see comparison isomorphisms $\mathbb{R}u_{m*}\mathcal{O}_{X/W_{m}(R)} \to W_{m}\Omega^{*}_{X/R}$ are compatible with restriction and then obtain the canonical isomorphism $\mathbb{R}\Gamma_{\text{crys}}(X/W_{*}(R)) \to \mathbb{R}\Gamma_{\text{Zar}}(X, W_{*}\Omega^{*}_{X/R})$ in $D(N_{*}(W_{m}(R)))$. Apply $\mathbb{R}\lim$ to this, then we get the isomorphism $\mathbb{R}\Gamma_{\text{crys}}(X/W(R)) \to \mathbb{R}\Gamma_{\text{Zar}}(X, W\Omega^{*}_{X/R})$ by Proposition 2.9 and [LZ04] Proposition 1.13.

### 3. Log $F$-$V$-procomplexes and Log de Rham-Witt Complex

#### 3.1. Log pd-derivations.

**Definition 3.1.** ([Ogu06] Definition 1.1.9)

Let $\theta$ be a morphism of pre-log rings

$$\begin{array}{ccc}
Q & \xrightarrow{\beta} & S \\
\phi^{\dagger} \downarrow & & \downarrow \phi^{\star} \\
P & \xrightarrow{\alpha} & R
\end{array}$$

and let $M$ be a $S$-module. Then a log-derivation of $(S, Q)/(R, P)$ with values in $M$ is a pair $(D, \delta)$, where $D : S \to M$ is a derivation of $S/R$ with values in $M$ and $\delta : Q \to M$ is a homomorphism of monoids such that the following conditions are satisfied:

1. For every $q \in Q$, $D(\beta(q)) = \beta(q)\delta(q)$.
2. For every $p \in P$, $\delta(\phi^{\star}(p)) = 0$.

**Definition 3.2.** Let $R$ be a ring and $S$ an $R$-algebra. Let $I \subset S$ be an ideal equipped with a pd-structure $\{\gamma_{n}\}_{n \in \mathbb{N}}$. Let $M$ be a $S$-module.
(1) ([LZ04] Definition 1.1) A pd-derivation of $S/R$ with values in $M$ is a derivation $D : S \to M$ of $S/R$ which satisfies

$$D(\gamma_n(b)) = \gamma_{n-1}(b)D(b)$$

for $n \geq 1$ and each $b \in I$.

(2) Let $(R, P) \to (S, Q)$ be a morphism of pre-log rings. A log derivation $(D : S \to M, \delta : Q \to M)$ is called a log pd-derivation if $D$ is a pd-derivation.

We denote by $\mathcal{D}er((R, P), (S, Q), M)$ the set of log pd-derivations. The functor $M \mapsto \mathcal{D}er((R, P), (S, Q), M)$ is representable by a universal object

$$(d : S \to \tilde{\Lambda}^1_{(S, Q)/(R, P)}, d \log : Q \to \tilde{\Lambda}^1_{(S, Q)/(R, P)}),$$

where the $S$-module $\tilde{\Lambda}^1_{(S, Q)/(R, P)}$ is obtained as the quotient module of the log differentials $\Lambda^1_{(S, Q)/(R, P)}$ by the submodule generated by all elements $d(\gamma_n(b)) - \gamma_{n-1}(b)db$ for $b \in I, n \geq 1$.

**Definition 3.3.** (1) Let $R \to S$ be a morphism of rings. A differential graded $S/R$-algebra is a unitary graded $S$-algebra

$$E^\bullet = \bigoplus_{i \geq 0} E^i$$

equipped with an $R$-linear differential $d : E^\bullet \to E^\bullet$ such that the following relations hold:

$$d(\omega\eta) = (-1)^{ij}d\omega\eta, \quad \omega \in E^i, \eta \in E^j,$$

$$d(\omega\eta) = d(\omega)\eta + (-1)^i d(\eta)\omega, \quad \omega \in E^i, \eta \in E^j,$$

$$d^2 = 0.$$

(2) Let $(R, P) \to (S, Q)$ be a morphism of pre-log rings. A log differential graded $(S, Q)/(R, P)$-algebra is a triple $(E^\bullet, d, \partial)$, where $(E^\bullet, d)$ is a differential graded $S/R$-algebra and $\partial : Q \to E^1$ is a morphism of monoids, such that $(d : S \to E^0 \to E^1, \partial)$ is a log derivation and $d\partial = 0$.

A morphism of log differential graded $(S, Q)/(R, P)$-algebras $f : (E^\bullet, d, \partial) \to (E'^\bullet, d', \partial')$ is a morphism of differential graded $S/R$-algebras $f : (E^\bullet, d) \to (E'^\bullet, d')$ that satisfies $\partial' = f \circ \partial$.

(3) A log pd-differential graded $(S, Q)/(R, P)$-algebra $(E^\bullet, d, \partial)$ is a log differential graded $(S, Q)/(R, P)$-algebra $(E^\bullet, d, \partial)$ such that $d : S \to E^0 \to E^1$ is a pd-derivation.

### 3.2. Frobenius action on log pd-derivations.

We consider a continuous (i.e. it factors through $W_l(S)$ for some $l > 0$) $W(R)$-linear pd-derivation $D : W(S) \to M$ to a discrete $W(S)$-module (i.e. it is obtained by restriction of scalars $W(S) \to W_l(S)$ for some $l > 0$). By [LZ04] §1.1, we have a map $F^D : W(S) \to M$ given by $\xi = [x] + \mathcal{V}_\rho \mapsto [x^{(p-1)}] D([x]) + D(\rho), x \in S$. Then $F^D : W(S) \to M_{\{F\}}$ is a continuous $W(R)$-linear pd-derivation. We extend this to pre-log rings.

Let $(S, Q)$ be a pre-log ring and $(D, \delta) : W(S, Q) \to M$ a $W(R)$-linear log pd-derivation. Then the pair $(F^D, \delta)$ is also a log pd-derivation. In fact, for $q \in Q$, we have

$$F^D(W(\alpha)(q)) = F^D([\alpha(q)]) = [\alpha(q)]^{(p-1)} D([\alpha(q)]) = [\alpha(q)]^{(p-1)} \cdot [\alpha(q)] \delta(q) = F^D(W(\alpha)(q)) \delta(q).$$
By the universal property of the logarithmic differential sheaf, we obtain a morphism $F : \hat{\Lambda}^1_{W_m+1(S,Q)/W_m+1(R,P)} \to (\hat{\Lambda}^1_{W_m(S,Q)/W_m(R,P)})[F]$ from 

$$(d, d \log) : (W_m(S), Q) \to \hat{\Lambda}^1_{W_m(S,Q)/W_m(R,P)}$$

and it satisfies $F \circ (d, d \log) = (F d, d \log)$. By definition, we obtain 

$$F(d\xi) = (F d)(\xi), \quad \xi \in W_{m+1}(S),$$

$$F(d \log m) = d \log m, \quad m \in Q,$$

$$F d([x]) = [x]^{p-1} d[x], \quad x \in S,$$

$$d(F \xi) = p F d\xi, \quad \xi \in W_{m+1}(S).$$

### 3.3 Log $F$-$V$-procomplexes

Let $(R, P) \to (S, Q)$ be a morphism of pre-log rings.

**Definition 3.4.** A log $F$-$V$-procomplex over $(R, P)$-algebra $(S, Q)$ is a projective system 

$$\{E^\bullet_m = (E^\bullet_m, D_m, \partial_m), \pi_m : E^\bullet_{m+1} \to E^\bullet_m\}_{m \in \mathbb{N}}$$

of a log differential graded $W_m(S, Q)/W_m(R, P)$-algebra $(E^\bullet_m, D_m, \partial_m)$, 

$$\cdots \to E^\bullet_{m+1} \xrightarrow{\pi_m} E^\bullet_m \to \cdots \to E^\bullet_1 \to E^\bullet_0 = 0.$$ 

Moreover, $\{E^\bullet_m\}$ is equipped with two sets of homomorphisms of graded abelian groups, 

$$F : E^\bullet_{m+1} \to E^\bullet_m, V : E^\bullet_m \to E^\bullet_{m+1}, m \geq 0,$$

and the following properties hold.

(i) $\partial_m$ are compatible with $\pi_m$, i.e., $\partial_m = \pi_m \circ \partial_{m+1}$ for any $m \geq 0$.

(ii) The morphisms $W_m(S) \to E^\bullet_m$ are compatible with $F$ and $V$ for any $m \geq 0$.

(iii) The restriction maps $\pi_m : E^\bullet_m \to E^\bullet_{m-1}$ are compatible with $F$ and $V$ for any $m \geq 1$.

(iv) Let $E^\bullet_m[F]$ be the graded $W_{m+1}(S)$-algebra obtained via restriction of scalars $F : W_{m+1}(S) \to W_m(S)$. Then $F$ induces a homomorphism of graded $W_{m+1}(S)$-algebras, 

$$F : E^\bullet_{m+1} \to E^\bullet_m[F].$$

(v) We have 

$$F V\omega = p\omega, \quad \omega \in E^\bullet_m, \quad n \geq 0,$$

$$F D_{m+1} V\omega = D_{m\omega},$$

$$F D_{m+1} [x] = [x^{p-1}] D_m [x], \quad x \in S,$$

$$V(\omega F\eta) = (V\omega)\eta, \quad \eta \in E^\bullet_{m+1},$$

$$F \partial_{m+1}(q) = \partial_m(q), \quad q \in Q.$$ 

A morphism of log $F$-$V$-procomplexes $f : \{E^\bullet_m = (E^\bullet_m, D_m, \partial_m), \pi\} \to \{E'^\bullet_m = (E'^\bullet_m, D'_m, \partial'_m), \pi'\}$ is a morphism of pro-log differential graded $W_*(S, Q)/W_*(R, P)$-algebras $f = \{f_m : E^\bullet_m \to E'^\bullet_m\}$ that is compatible with $F$ and $V$.

### 3.4 Construction of log de Rham-Witt complex

Let $R$ be a $\mathbb{Z}(p)$-algebra. For a pre-log ring $(S, Q)$ over $(R, P)$, we construct the log de Rham-Witt complex 

$$\{W_m \Lambda^\bullet\}_{m \in \mathbb{N}} = \{W_m \Lambda^\bullet_{(S,Q)/(R,P)}\}_{m \in \mathbb{N}}$$

as the universal log $F$-$V$-procomplex by induction on $m$. We set 

$$W_1 \Lambda^\bullet := \Lambda^\bullet_{(S,Q)/(R,P)} = \hat{\Lambda}^1_{W_1(S,Q)/W_1(R,P)}.$$ 

To define $W_{m+1} \Lambda^\bullet$ we assume that we have
\(\{W_n^\Lambda^\bullet\}_{n \leq m}\), a system of log pd-differential graded \(W_n(S, Q)/W_n(R, P)\)-algebras \(W_n^\Lambda^\bullet\),

\(\tilde{\Lambda}_{W_n(S, Q)/W_n(R, P)}^\bullet \to W_n^\Lambda^\bullet\), for \(n \leq m\), surjective morphisms of log differential graded algebras which are compatible with the restriction maps and with \(F\),

\(V : W_n^\Lambda^\bullet \to W_{n+1}^\Lambda^\bullet\), additive maps for \(1 \leq n < m\),

\(W_n^\Lambda_0 = W_n(S)\) for \(1 \leq n < m\),

\(F\) annihilates \(I\) for \(1 \leq n < m\),

\(\bar{\Lambda}_{W_n(S, Q)/W_n(R, P)}^\bullet \to \tilde{\Lambda}_{W_n(S, Q)/W_n(R, P)}^\bullet\) as follows. Consider all relations of the form

\[\sum_{l=1}^{M} \xi(l) \cdot d \log q_1 \cdot \ldots \cdot d \log q_r \cdot d\eta_{r+1} \cdot \ldots \cdot d\eta_l = 0\]

in \(W_m^\Lambda^\bullet\), where \(\xi(l), \eta_k(l) \in W_m(S), \eta_k(l) \in Q\). Then \(I \subset \bar{\Lambda}_{W_{m+1}(S, Q)/W_{m+1}(R, P)}^\bullet\) is defined to be the ideal generated by the elements

\[\sum_{l=1}^{M} V \xi(l) \cdot d \log q_1 \cdot \ldots \cdot d \log q_r \cdot d\eta_{r+1} \cdot \ldots \cdot d\eta_l,\]

\[\sum_{l=1}^{M} dV \xi(l) \cdot d \log q_1 \cdot \ldots \cdot d \log q_r \cdot d\eta_{r+1} \cdot \ldots \cdot d\eta_l.\]

Then \(I\) is stable by \(d\). Moreover,

\[F : \bar{\Lambda}_{W_{m+1}(S, Q)/W_{m+1}(R, P)}^\bullet \to \tilde{\Lambda}_{W_{m+1}(S, Q)/W_{m+1}(R, P)}^\bullet\]

annihilates \(I\) since we have

\[FV \xi = p\xi \in \bar{\Lambda}_{W_m(S, Q)/W_m(R, P)}^0 = W_m(S), \quad \xi \in W_m(S)\]

\[FdV \eta = d\eta \in \bar{\Lambda}_{W_m(S, Q)/W_m(R, P)}^1, \quad \eta \in W_m(S)\]

\[Fd \log q = d \log q \in \bar{\Lambda}_{W_m(S, Q)/W_m(R, P)}^1, \quad q \in Q.\]

Therefore \(F\) induces

\[F : \bar{\Lambda}^\bullet_{m+1} := \bar{\Lambda}_{W_{m+1}(S, Q)/W_{m+1}(R, P)}^\bullet/I \to W_m^\Lambda^\bullet.\]

On the other hand, we have a well-defined map

\[V : W_m^\Lambda^\bullet \to \tilde{\Lambda}^\bullet_{m+1},\]

\[\xi \cdot d \log q_1 \cdot \ldots \cdot d \log q_r \cdot d\eta_{r+1} \cdot \ldots \cdot d\eta_l \mapsto V \xi \cdot d \log q_1 \cdot \ldots \cdot d \log q_r \cdot d\eta_{r+1} \cdot \ldots \cdot d\eta_l.\]

We have \(FdV \omega = d\omega\). Let \(J\) be the ideal of \(\bar{\Lambda}^\bullet_{m+1}\) generated by the elements

\[V(\omega \cdot F \eta) - V \omega \cdot \eta,\]

\[d(V(\omega \cdot F \eta) - V \omega \cdot \eta),\]

where \(\omega \in W_m^\Lambda^\bullet\) and \(\eta \in \tilde{\Lambda}^\bullet_{m+1}\). We see that \(F\) annihilate \(J\). We set \(W_{m+1}^\Lambda^\bullet := \bar{\Lambda}^\bullet_{m+1}/J\). Then we have maps

\[F : W_{m+1}^\Lambda^\bullet \to W_m^\Lambda^\bullet,\]

\[V : W_m^\Lambda^\bullet \to W_{m+1}^\Lambda^\bullet.\]

We can see that all requirements of the definition of log \(F\)-\(V\)-procomplexes are satisfied. We set \(W^\Lambda^\bullet := \varprojlim W_n^\Lambda^\bullet\). By the construction, the log de Rham-Witt complex we made is a natural extension of the de Rham-Witt complex constructed in [LZ04]; i.e., \(W_m^\Lambda^\bullet(S(\{\cdot\}))/(R(\{\cdot\})) \simeq W_m^{\Omega^\bullet S/R}\).

The following proposition is clear from the definition.
Proposition 3.5. (cf. [LZ04] Proposition 1.6) Let \( \{E_m^\bullet, D_m, \partial_m\}_{m \in \mathbb{N}} \) be a log F-V-procomplex over \((S, Q)/(R, P)\). Then there is a unique morphism of log F-V-procomplexes
\[
\{W_m^\bullet(A^{\bullet}_{(S, Q)/(R, P)})\} \to \{E_m^\bullet\}
\]
over \((S, Q)/(R, P)\).

3.6. **Standard Filtration.** The differential graded ideals
\[
\text{Fil}^i E^i_m = V^i E^i_{m-s} + d V^i E^i_{m-s} \subset E^i_m
\]
gives a filtration of a log F-V-procomplex \( \{E_m^\bullet\} \) and it is called the standard filtration. Since restriction maps and \( F, V \) are compatible, we find
\[
\pi(\text{Fil}^i E^i_m) \subset \text{Fil}^i E^i_{m-1}, \quad F(\text{Fil}^i E^i_m) \subset \text{Fil}^{i-1} E^i_{m-1}, \quad V(\text{Fil}^i E^i_m) \subset \text{Fil}^{i+1} E^i_{m-1}, \quad d(\text{Fil}^i E^i_m) \subset \text{Fil}^{i+1} E^i_{m+1}.
\]

Proposition 3.6. (cf. [HM03] 3.2.4) Let \((R, P) \to (S, Q)\) be a morphism of pre-log rings and \( m, s \) positive integers satisfying \( m \geq s \). We set \( W_m^\bullet := W_m^\bullet(A^{\bullet}_{(S, Q)/(R, P)}) \).
Then we have the following exact sequence:
\[
0 \to \text{Fil}^i W_m^\bullet \to W_m^\bullet \xrightarrow{\pi^{m-s}} W_n^\bullet \to 0.
\]

Proof. For any log F-V-procomplex \( \{E_m^\bullet\} \), the composition of the two morphisms \( \text{Fil}^i E^i_m \hookrightarrow E^i_m \xrightarrow{\pi^{m-s}} E^i_s \) is zero since \( \pi \) commutes with \( F \) and \( V \), and \( \pi(m E^i_m) \subset E^i_0 = 0 \). Therefore \( \pi^{m-s} \) induces a morphism
\[
\pi^{m-s} : E^i_m / \text{Fil}^i E^i_m \to E^i_s.
\]

Fix \( r := m - s \) and set \( E_r^\bullet := E_{r+s}^\bullet / \text{Fil}^i E_{r+s}^\bullet \). Then \( \{E_r^\bullet\} \) is a log F-V-procomplex over \((S, Q)/(R, P)\) by (\( \ast \)). We show that \( \{W_n^\bullet(A^\bullet)\} \) is the universal log F-V-procomplex. Since the projection map \( \pi \) of \( W_n^\bullet(a^\bullet) \) is surjective, we have the canonical surjective morphism \( \{W_n^\bullet(A^\bullet)\} \to \{W_n^\bullet(a^\bullet)\} \) of log F-V-procomplexes. The diagram:
\[
\begin{array}{ccc}
A^{W_m}_{W_n(S, Q)/(W_m(S), R, P)} & \longrightarrow & W_{n+r}A^{\bullet} \\
A^{W_m}_{W_n(S, Q)/(W_m(S), R, P)} & \longrightarrow & W_{n+r}A^{\bullet} \\
\end{array}
\]
shows the morphism \( A^{W_m}_{W_n(S, Q)/(W_m(S), R, P)} \to W_{n}^{\bullet} \) is surjective.

Let \( \{E_r^\bullet\} \) be any log F-V-procomplex over \((S, Q)/(R, P)\). By the universal property of \( \{W_n^\bullet(A^\bullet)\} \), there is a unique morphism \( \{W_n^\bullet(A^\bullet)\} \to \{E_r^\bullet\} \) of log F-V-procomplexes. We compose it with canonical surjection \( \{W_n^\bullet(A^\bullet)\} \to \{W_n^\bullet(a^\bullet)\} \). Then we get a morphism \( \{W_n^\bullet(a^\bullet)\} \to \{E_r^\bullet\} \). This is the unique morphism of log F-V-procomplexes from \( \{W_n^\bullet(a^\bullet)\} \) to \( \{E_r^\bullet\} \) because \( A^{W_m}_{W_n(S, Q)/(W_m(S), R, P)} \to W_{n}^\bullet \) is surjective. Hence \( \{W_n^\bullet(a^\bullet)\} \) has the universal property. 

**3.6. Base change for étale morphisms.** We establish the étale base change property of log de Rham-Witt complexes. The following propositions can be shown by the same method used in [LZ04] Proposition 1.7 and 1.9.

**Proposition 3.7.** Let \( R \) be a ring such that \( R \) is \( F \)-finite (in the sense of [LZ04] Proposition A.2) or \( p \) is nilpotent in \( R \). Let \((R, P) \to (S, Q)\) be a morphism of pre-log rings and \( S \to S' \) be an étale ring map. Then the natural morphism
\[
W_m^\bullet(A^{\bullet}_{(S', Q)/(R, P)}) \to W_m^\bullet(S') \otimes_{W_m(S)} W_m^\bullet(A^{\bullet}_{(S, Q)/(R, P)})
\]
is an isomorphism.
Proposition 3.8. Let $(R, P)$ be a pre-log ring such that $R$ is $F$-finite or $p$ is nilpotent in $R$. Assume we are given an unramified ring homomorphism $R \to R'$ and a morphism $(R', \mathcal{P}) \to (S, \mathcal{Q})$ of pre-log rings. Then we have a natural isomorphism of $W$-procomplexes relative to $(S, \mathcal{Q})/(R, \mathcal{P})$: $$\{W_m \Lambda^*_{(S, \mathcal{Q})/(R, \mathcal{P})}\} \to \{W_m \Lambda^*_{(S, \mathcal{Q})/(R', \mathcal{P})}\}.$$ We define the log de Rham-Witt complex on log schemes. The following lemma immediately follows from [Ogu06] Proposition 2.2.1.

Lemma 3.9. Let $\beta : Q \to M$ be a chart for a sheaf of fine monoids $\mathcal{M}$ on a scheme $X$. Suppose that $\beta$ factors $$Q \xrightarrow{\beta'} Q' \xrightarrow{\beta''} \mathcal{M},$$ where $Q'$ is a constant sheaf of a finitely generated monoid. Then, étale locally on $X$, $\beta'$ can be factored $$Q' \xrightarrow{\beta''} Q'' \xrightarrow{\beta'} \mathcal{M},$$ where $\beta''$ a chart for $\mathcal{M}$.

Proposition-Definition 3.10. Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a morphism of fine log schemes over $\mathbb{Z}_p$. We assume that $p$ is nilpotent in $Y$. We identify the étale topology of $W_m(X)$ and that of $X$ (See [Sta15] Étale Cohomology, Proposition 44.46.3). Then there is a unique quasi-coherent sheaf $W_m \Lambda^*_{(X, \mathcal{M})/(Y, \mathcal{N})}$ on $X_{ét}$ which has the following property: If there is a commutative diagram $$U = \text{Spec } S' \xrightarrow{\gamma'} V = \text{Spec } R'$$ $$\gamma \downarrow \quad \quad \quad \quad \quad \quad \downarrow \gamma$$ $$X \xrightarrow{f} Y,$$ where $\gamma$ and $\gamma'$ are étale morphisms and there is a chart $(Q \to M|_U, P \to N|_V, P \to Q)$ of the morphism $(U, M|_U) \to (V, N|_V)$, then we have a canonical isomorphism $$\Gamma(U, W_m \Lambda^*_{(X, \mathcal{M})/(Y, \mathcal{N})}) \cong W_m \Lambda^*_{(S', \mathcal{N})/(R', P')}.$$ Proof. When $X = \text{Spec } S$ and $Y = \text{Spec } R$ are affine and $f$ has a chart $(Q \to M, P \to N, P \to Q)$, the presheaf $$X_{ét} \ni U' \mapsto (U' = \text{Spec } S' \to X) \mapsto W_m \Lambda^*_{(S', \mathcal{N})/(R', P')}$$ defines a quasi-coherent sheaf on $X_{ét}$ because of the base change property of étale morphisms (Proposition 3.7). We temporarily denote by $F_{(P, Q)}$ this sheaf. We have to show if there exists another chart $(Q' \to M, P' \to N, P' \to Q')$ of $f$, we have an isomorphism $F_{(P, Q)} \cong F_{(P', Q')}$. We denote by $$W_m(f) : (W_m(X) = \text{Spec } W_m(S), W_m(M)) \to (W_m(Y) = \text{Spec } W_m(R), W_m(N))$$ the morphism induced by $f$. Since $(Q \to M, P \to N, P \to Q)$ (resp. $(Q' \to M, P' \to N, P' \to Q')$) is a chart of $f$, we have a canonical chart $$(Q \to W_m(M), P \to W_m(N), P \to Q)$$ (resp. $(Q' \to W_m(M), P' \to W_m(N), P' \to Q')$) of $W_m(f)$. Then we have an isomorphism $$\Lambda^*_{W_m(S, \mathcal{Q})/W_m(R, P)} \cong \Lambda^*_{W_m(S, \mathcal{Q})/W_m(M, \mathcal{N})} (W_m(X)) \cong \Lambda^*_{W_m(S, \mathcal{Q})/W_m(R, P')}$$ by [Ogu06] Corollary 1.1.11. Hence it induces an isomorphism $$\tilde{\Lambda}^*_{W_m(S, \mathcal{Q})/W_m(R, P)} \cong \tilde{\Lambda}^*_{W_m(S, \mathcal{Q})/W_m(R, P')}.$$
Let
\[ I_{(P,Q)} \subseteq \tilde{\Lambda}^{\bullet}_{W_{m+1}(S,Q)/W_{m+1}(R,P)} \] (resp. \[ I_{(P',Q')} \subseteq \tilde{\Lambda}^{\bullet}_{W_{m+1}(S,Q')/W_{m+1}(R,P')} \])

and
\[ J_{(P,Q)} \subseteq \tilde{\Lambda}^{\bullet}_{W_{m+1}(S,Q)/W_{m+1}(R,P)}/I_{(P,Q)} \] (resp. \[ J_{(P',Q')} \subseteq \tilde{\Lambda}^{\bullet}_{W_{m+1}(S,Q')/W_{m+1}(R,P')}/I_{(P',Q')} \])

be ideals defined in the construction of \([W_{m}\Lambda^{\bullet}(S,Q)/(R,P)]\) (resp. \([W_{m}\Lambda^{\bullet}(S,Q')/(R,P')]\)). See §3.4.

First we assume that there is a morphism of charts
\[(Q \to \mathcal{M}, P \to \mathcal{N}, P \to Q) \to (Q' \to \mathcal{M}, P' \to \mathcal{N}, P' \to Q').\]

This morphism induces a canonical map \(\mathcal{F}_{(P,Q)} \to \mathcal{F}_{(P',Q')}\). We show that it is an isomorphism. Let \(\alpha : Q_X \to \mathcal{O}_X\) and \(\alpha' : Q_X' \to \mathcal{O}_X\) be the structure morphisms. Let \(\beta : Q \to Q'\) be a morphism of monoids induced by the morphism of charts. For any generic point \(x\) of \(X\), we have isomorphisms
\[ Q/\alpha^{-1}(\mathcal{O}_{X,x}) \xrightarrow{\beta_*} Q'/\alpha'^{-1}(\mathcal{O}_{X,x}) \xrightarrow{\sim} \mathcal{M}_x/\mathcal{O}_{X,x}. \]

Since \(Q'\) is finitely generated, by replacing \(X\) with some étale neighbourhood of \(x\), we can assume that for for any \(q' \in Q'\) there exists \(q \in Q, s \in \alpha'^{-1}(\mathcal{O}_X(X)^*)\) such that \(q' \cdot s' = \beta(q) \cdot s\). We see
\[ d \log q' = d \log(q' \cdot s') - d \log s' \]
\[ = d \log(\beta(q) \cdot s) - d \log s' \]
\[ = d \log \beta(q) + d \log s - d \log s' \]
\[ = d \log \beta(q) + \alpha'(s)^{-1}ds - \alpha'(s')^{-1}ds'. \]

Hence we see that \(I_{(P,Q)} \to I_{(P',Q')}\) is an isomorphism. This isomorphism induces
\[ \tilde{\Lambda}^{\bullet}_{W_{m+1}(S,Q)/W_{m+1}(R,P)}/I_{(P,Q)} \xrightarrow{\sim} \tilde{\Lambda}^{\bullet}_{W_{m+1}(S,Q')/W_{m+1}(R,P')}/I_{(P',Q')}\]

By the construction, we have \(J_{(P,Q)} \xrightarrow{\sim} J_{(P',Q')}\) via this morphism. So we see
\[ W_{m}\Lambda^{\bullet}(S,Q)/(R,P) \xrightarrow{\sim} W_{m}\Lambda^{\bullet}(S,Q')/(R,P'), \]

and this shows \(\mathcal{F}_{(P,Q)} \xrightarrow{\sim} \mathcal{F}_{(P',Q')}\).

We consider the general case. Let \(x\) be any geometric point of \(X\). By Lemma 3.9, there exists a commutative diagram

\[
\begin{array}{ccc}
U = \text{Spec } S' & \longrightarrow & V = \text{Spec } R' \\
\gamma' \downarrow & & \gamma \downarrow \\
X & \longrightarrow & Y,
\end{array}
\]

where \(U\) is an étale neighbourhood of \(x\), and the morphisms \(\gamma, \gamma'\) are étale, such that we admit a chart \((Q'' \to \mathcal{M}|_U, P'' \to \mathcal{N}|_V, P'' \to Q'')\) of \((U, \mathcal{M}|_U) \to (V, \mathcal{N}|_V)\) and morphisms of coherent charts
\[(Q \to \mathcal{M}|_U, P \to \mathcal{N}|_V, P \to Q) \to (Q'' \to \mathcal{M}|_U, P'' \to \mathcal{N}|_V, P'' \to Q''),\]
\[(Q' \to \mathcal{M}|_U, P' \to \mathcal{N}|_V, P' \to Q') \to (Q'' \to \mathcal{M}|_U, P'' \to \mathcal{N}|_V, P'' \to Q'').\]
Then we see that
\[
F_{(P,Q)}(U) = W_m\Lambda_{(S',Q')/(R',P')}
\]
\[
\sim F_{(P',Q')}(U) = W_m\Lambda_{(S',Q')/(R',P')}
\]
by the proof of the previous case. The collection of these maps glue to an isomorphism \(F_{(P,Q)} \simeq F_{(P',Q')}\).

3.7. Exact sequences. The log de Rham-Witt sheaves satisfy the same exact sequences as the usual Kähler differentials. The following results are generalizations of a part of [LZ05].

**Proposition 3.11.**

1. Let \(X \to Y \to S\) be morphisms of fine log schemes. Then there is the following exact sequences:
\[
W_m\Lambda^{1}_{X/S} \otimes_{W_m(O_Y)} W_m\Lambda^{1}_{X/S} \to W_m\Lambda_{X/S} \to W_m\Lambda_{X/Y} \to 0.
\]

2. Let \(X \to Y \to S\) be morphisms of fine log schemes, where \(i : X \to Y\) is an exact closed immersion defined by a quasi-coherent ideal \(a \subset O_Y\). Then there is the following exact sequences:
\[
W_m(a)/W_m(a)^2 \otimes_{W_m(O_Y)} W_m\Lambda^{1}_{X/S} \to W_m(O_X) \otimes_{W_m(O_Y)} W_m\Lambda_{X/S} \to W_m\Lambda_{X/Y} \to 0.
\]

**Proof.**

1. Since the problem is local, we can assume that morphisms of log schemes are associated to morphisms of pre-log rings \((R, P) \to (S, Q)\). The collection of these maps glue to an isomorphism \(F_{(P,Q)} \simeq F_{(P',Q')}\).

Let \(I_m \subset W_m\Lambda_{(S',Q')/(R, P)}\) be the ideal generated by the elements of the form \(ds, d\log m\) where \(s \in W_m(S), m \in W_m(Q)\). Then we see that \(I_m\) is invariant under \(F, V\) and \(d\).

The natural surjective morphism
\[
W_m\Lambda_{(S',Q')/(R, P)} \to W_m\Lambda_{(S',Q')/(S, Q)}
\]
factors \(W_m\Lambda_{(S',Q')/(R, P)}/I_m\).

Since \(I_m\) is stable by \(F, V\) and \(d\), we see \(\{W_m\Lambda_{(S',Q')/(R, P)}/I_m\}\) is a log \(F\)-\(V\)-procomplex over \((S', Q')/(S, Q)\). We obtain an isomorphism
\[
W_m\Lambda^{1}_{(S',Q')/(R, P)}/I_m \simeq W_m\Lambda_{(S',Q')/(S, Q)}
\]
and a short exact sequence
\[
0 \to I_m \to W_m\Lambda^{1}_{(S',Q')/(R, P)} \to W_m\Lambda_{(S',Q')/(S, Q)} \to 0.
\]

Thus we have the following exact sequence
\[
W_m\Lambda^{1}_{(S',Q')/(R, P)} \otimes_{W_m(S)} W_m\Lambda^{1}_{(S',Q')/(R, P)} \to W_m\Lambda^{1}_{(S',Q')/(R, P)} \to W_m\Lambda^{1}_{(S',Q')/(S, Q)} \to 0.
\]

(2) This is reduced to the case of morphism of log schemes associated to a morphism of pre-log rings \((R, P) \to (S, Q)\). Since the canonical morphism \(W_m(S, Q) \to W_m(S', Q)\) is a strict closed immersion defined by the ideal \(W_m(a)\), we have the following exact sequence ([Ogu06], Prop 2.3.2):
\[
W_m(a)/W_m(a)^2 \otimes_{W_m(S')} W_m(S, Q)^1 \otimes_{W_m(S, Q)} W_m(S', Q) W_m(R, P) \to W_m^1 W_m(S', Q)/W_m(R, P) \to 0.
\]

Then we have the following complexes:
\[
W_m(a)/W_m(a)^2 \otimes_{W_m(S)} W_m\Lambda^{1}_{(S, Q)/(R, P)} \to W_m(S) \otimes_{W_m(S)} W_m\Lambda_{(S, Q)/(R, P)} \to W_m\Lambda_{(S',Q')/(R, P)} \to 0.
\]
It remains to prove the exactness at \( W_m(S') \otimes_{W_m(S)} W_m\Lambda_{(S,Q)/(R,P)}^* \). Since

\[
W_*(a)W_*\Lambda_{(S,Q)/(R,P)}^* + dW_*(a)W_*\Lambda_{(S,Q)/(R,P)}^* - 1
\]

is stable by operators \( F, V \) and \( d \),

\[
\{W_m\omega_{(S',Q)/(R,P)}^*\} := \{W_m\Lambda_{(S,Q)/(R,P)}^*/(W_m(a)W_m\Lambda_{(S,Q)/(R,P)}^* + dW_m(a)W_m\Lambda_{(S,Q)/(R,P)}^* - 1)\}
\]

is a log \( F-V \)-procomplex over \((S', Q)/(R, P)\).

It is easy to verify that \( \{W_m\omega_{(S',Q)/(R,P)}^*\} \) satisfies the universal property, so we have an isomorphism \( \{W_m\omega_{(S',Q)/(R,P)}^*\} \cong \{W_m\Lambda_{(S',Q)/(R,P)}^*\} \). \( \square \)

### 3.8. Log phantom components

Let \( R \) be a \( \mathbb{Z}_p \)-algebra, \((R, P) \to (S, Q)\) a morphism of pre-log rings and \( M \) an \( S \)-module. We denote by \( M_w \), the \( W(S) \)-module \( M \) obtained by the restriction of scalars \( w_m : W(S) \to S \) via the Witt polynomial.

We establish the log version of phantom components defined in [LZ04] §2.4. For \( m \geq 1 \), we define a map

\[
\delta_m : W(S) \to \Lambda_{(S,Q)/(R,P)}^* \cdot \omega_m,
\]

\[
(x_0, x_1, \ldots) \mapsto \sum_{i=0}^{m-1} x_i^{m-1} d x_i.
\]

By Lemma 2.14 of [LZ04], \( \delta_m \) is a continuous \( W(R) \)-linear \( \rho \)-derivation. We can easily verify that \( (\delta_m, d \log) : (W(S), Q) \to \Lambda_{(S,Q)/(R,P)}^* \cdot \omega_m \) is a log derivation.

The log derivation \( (\delta_m, d \log) \) induces a \( W_{m+1}(R) \)-linear morphism

\[
\omega_m : \Lambda_{(S,Q)/(W_{m+1}(R), W_{m+1}(R))}^* \to \Lambda_{(S,Q)/(R,P)}^* \cdot \omega_m.
\]

It extends to the exterior powers

\[
\omega_m : \Lambda_{(S,Q)/W_{m+1}(R), W_{m+1}(R))}^* \to \Lambda_{(S,Q)/(R,P)}^* \cdot \omega_m
\]

by

\[
\omega_m(\xi d \log q_1 \cdots d \log q_r \cdot d \eta_{r+1} \cdots d \eta_k) = w_m(\xi d \log q_1 \cdots d \log q_r \cdot \delta_m(\eta_{r+1}) \cdots \delta_m(\eta_k)),
\]

where \( q_1, \ldots, q_r \in Q \) and \( \xi, \eta_{r+1}, \ldots, \eta_k \in W_{m+1}(S) \).

Define a complex \( E_m^* \) of \( W_m(S) \)-modules by

\[
E_m^* = \bigoplus_{i=0}^{m-1} \Lambda_{(S,Q)/(R,P), w_i}^*.
\]

We define \( F : E_m^* \to E_{m-1}^* \) and \( V : E_m^* \to E_{m+1}^* \) by following formulas: For \([\rho_0, \ldots, \rho_{m-1}] \in E_m^* \), \( \rho_i \in \Lambda_{(S,Q)/(R,P), w_i}^* \),

\[
F[\rho_0, \ldots, \rho_{m-1}] := [\rho_1, \ldots, \rho_{m-1}],
\]

\[
V[\rho_0, \ldots, \rho_{m-1}] := [0, \rho_0, \ldots, p \rho_{m-1}].
\]

For \( m \geq 1 \), the sum of the maps \( (\omega_0, \ldots, \omega_{m-1}) \) define a homomorphism of \( W_m(S) \)-modules

\[
\omega_m^* : \Lambda_{W_m(S), W_m(R), R}^* \to E_m^*,
\]

which is a homomorphism of \( W_m(S) \)-modules. We can prove following proposition by using the same argument of [LZ04] Proposition 2.15.
Proposition 3.12. \( \omega^m \) factors through a homomorphism of projective systems of algebras

\[
\omega^m : W_m \Lambda^\bullet_{(S, Q)}/(R, P) \to E_m^\bullet.
\]

This homomorphism commutes with \( F \) and \( V \). We have

\[
d_{\omega^m} = [1, p, p^2, \ldots] \omega^m d
\]

where \( [1, p, p^2, \ldots] \in \prod S = E_0^m \).

4. Log basic Witt differentials in special cases

4.1. SNCD case. In this subsection, we consider the log version of the \( p \)-basic elements and the basic Witt differentials in the SNCD case. In fact, we treat a slightly more generalized case which we need later.

Let \( R \) be a \( \mathbb{Z}_{(p)} \)-algebra and \( R[T] := R[T_1, \ldots, T_n] \) and \( e \) an integer such that \( 0 \leq e \leq n \) and \( f \) an nonnegative integer. We consider the log structure associated with the map \( \mathbb{N}^e \oplus \mathbb{N}^f \to R[T] \). \( \mathbb{N}^e \ni c_i \mapsto T_i (1 \leq i \leq e) \), \( \mathbb{N}^f \ni c_i \mapsto 0 (1 \leq i \leq f) \) where \( c_i \) (resp. \( c_f \)) are basis of \( \mathbb{N}^e \) (resp. \( \mathbb{N}^f \)).

We define the log \( p \)-basic differentials of \( \Lambda^\bullet_{(R[T], \mathbb{N}^e \oplus \mathbb{N}^f)/R} := \Lambda^\bullet_{(R[T], \mathbb{N}^e \oplus \mathbb{N}^f)/(R, \{\ast\})} \) as follows. Let \( p^{-\infty} \) be a symbol and we set \( p^{-\infty} : p^{-\infty} := p^{-\infty} \) and \( p^{-1} \cdot p^{-\infty} := p^{-\infty} \) and \( \text{ord}_p(p^{-\infty}) := -\infty \).

A function \( k : [1, n] \to \mathbb{Z}_{\geq 0} \cup \{p^{-\infty}\} \) is called a weight if for every \( e < i \leq n \), \( k_i = k(i) \in \mathbb{Z}_{\geq 0} \). Let \( \text{Supp} \ k := \{ i \in [1, n] \mid k_i \neq 0 \} \).

We associate a weight without poles \( k^+ \) to a weight \( k \) by

\[
(k^+)_i := \begin{cases} 0 & (k_i = p^{-\infty}), \\ k_i & (k_i \neq p^{-\infty}). \end{cases}
\]

For each weight \( k \), we fix a total order on \( \text{Supp} \ k = \{i_1, \ldots, i_r\} \) in such a way that

\[
\text{ord}_p k_{i_1} \leq \text{ord}_p k_{i_2} \leq \cdots \leq \text{ord}_p k_{i_r}, \text{ord}_p k_{i_1} = \text{ord}_p k_{i_{r+1}} \Rightarrow i_j \leq i_{j+1}.
\]

We say \( (I_{-\infty}, I_0, I_1, \ldots, I_l) \) is a partition of \( \text{Supp} \ k \) if \( I_j \) if intervals of \( \text{Supp} \ k \) and \( I_{-\infty} = \{ i \in [1, n] \mid k_i = p^{-\infty} \} \), \( \text{Supp} \ k = I_{-\infty} \sqcup I_0 \sqcup I_1 \sqcup \cdots \sqcup I_l \)), the elements of \( I_j \) are smaller than that of \( I_{j+1} \) (with respect to the fixed order) and \( I_1, \ldots, I_l \) are not empty. \( I_{-\infty} \) and \( I_0 \) can be empty. We associate the element

\[
e(k, \mathcal{P}, J) := \left( \prod_{j \in J} d \log c_i \right) \cdot \left( \prod_{i \in I_{-\infty}} d \log T_i \right) \cdot \epsilon(k^+, \mathcal{P}^\prime = (I_0, I_1, \ldots, I_l))
\]

of \( \Lambda_{(R[T], \mathbb{N}^e \oplus \mathbb{N}^f)/R} \) to the triple \( (k, \mathcal{P} = (I_{-\infty}, I_0, \ldots, I_l), J) \). Here \( k \) is a weight, \( \mathcal{P} \) is a partition of \( \text{Supp} k \), \( J \) is a subset of \( [1, f] \) and

\[
e(k^+, \mathcal{P}^\prime = (I_0, I_1, \ldots, I_l)) = T^{k_0} (p^{-\text{ord}_p k_{i_1}} d T^{k_{i_1}}) \cdots (p^{-\text{ord}_p k_{i_l}} d T^{k_{i_l}})
\]

is the \( p \)-basic element defined in [LZ04] §2.1. We call the elements of this form \( p \)-basic elements.

Lemma 4.1. The log \( p \)-basic elements form a base of the log de Rham complex \( \Lambda_{(R[T], \mathbb{N}^e \oplus \mathbb{N}^f)/R}^\bullet \) as an \( R \)-module.

Proof. The \( R \)-module \( \Lambda_{(R[T], \mathbb{N}^e \oplus \mathbb{N}^f)/R}^l \) has the following basis:

\[
d \log c_{i_1} \cdots d \log c_{i_n} \cdot d \log T_{i_1} \cdots d \log T_{i_m} \cdot \left( \prod_{j \in [1, n]} T_j^{k_j} \right) \cdot d \log T_{j_1} \cdots d \log T_{j_{l-1-m}}.
\]
where \(1 \leq h_1 < \cdots < h_s \leq f, 1 \leq i_1 < \cdots < i_m \leq e, k : [1, n] \to \mathbb{Z}_{\geq 0}, k|_{\{i_1, \ldots, i_m\}} = 0, j_1 < \cdots < j_{e-m-s} \in \text{Supp } k.

Let \(\Lambda^I(I, k, J) \subset \Lambda^I_{(R[T], N^e \oplus N^j)/R}\) be the free \(R\)-submodule spanned by all elements of the form (1) for a fixed \(I = \{i_1, \ldots, i_m\} \subset [1, e], J \subset [1, f]\) and a weight \(k : [1, n] \to \mathbb{Z}_{\geq 0}\) such that \(k|_{\{i_1, \ldots, i_m\}} = 0\). We have a decomposition as free \(R\)-modules:

\[
\Lambda^I_{(R[T], N^e \oplus N^j)/R} = \bigoplus_{I, k, J} \Lambda^I(I, k, J).
\]

The rank of \(\Lambda^I(I, k, J)\) is \(\binom{m}{s}\) where \(m = |\text{Supp } k|\) and \(s = l - |I| - |J|\). The number of \(p\)-basic elements of the form

\[
\left(\prod_{i \in I} d\log c_i\right) \cdot \left(\prod_{i \in J} d\log T_i\right) \cdot e(k, \mathcal{P})
\]

for fixed \(I\) and \(J\) and \(k\) is also \(\binom{m}{s}\). Hence it is enough to show that the \(p\)-basic elements of this form generate \(\Lambda^I(I, k, J)\) as an \(R\)-module. It follows from the proof of [LZ04] Proposition 2.1.

Next we determine the log version of the basic Witt differentials for the pre-log ring \((R[T], N^e \oplus N^j) = (R[T_1, \ldots, T_m], N^e \oplus N^j)\) over \(R = (R, \{\ast\})\). We denote by \(X_i \in W(R)\) the Teichmüller lift \([T_i]\) of \(T_i\). We consider the log de Rham-Witt complex \(WA^\bullet_{(R[T], N^e \oplus N^j)/R}\).

We call a function \(k : [1, n] \to \mathbb{Z}_{\geq 0}[1/p] \cup \{p^{-\infty}\}\) a weight if \(e < i \leq n, k_i := k(i) \in \mathbb{Z}_{\geq 0}[1/p]\). Set \(t(k_i) := -\text{ord}_p k_i\) and \(u(k_i) = \max(0, t(k_i))\). For each weight \(k\), we fix a total order on \(\text{Supp } k = \{i_1, \ldots, i_e\}\) in such a way that

\[
\text{ord}_p k_{i_1} \leq \text{ord}_p k_{i_2} \leq \cdots \leq \text{ord}_p k_{i_e}, \text{ord}_p k_{i_j} = \text{ord}_p k_{i_{j+1}} \Rightarrow i_j \leq i_{j+1}.
\]

If \(I = \{i_t, \ldots, i_{t+m}\}\) is an interval of \(\text{Supp } k\), the restriction of \(k\) to \(I\) will be given by \(k_I\). We set \(t(k_I) = t(k_{i_t}), u(k_I) = u(k_{i_t})\). If \(k\) is fixed in our discussion, we write \(t(I)\) and \(u(I)\) instead of \(t(k_I)\) and \(u(k_I)\).

We say \((I_{-\infty}, I_0, I_1, \ldots, I_l)\) is a partition of \(\text{Supp } k\) if \(I_j\) are intervals of \(\text{Supp } k\), \(I_{-\infty} = \{i \in [1, n] \mid k_i = p^{-\infty}\}\). Supp \(k = I_{-\infty} \cup I_0 \cup I_1 \cup \cdots \cup I_l\), the elements of \(I_j\) are smaller than that of \(I_{j+1}\) (with respect to the fixed order) and \(I_1, \ldots, I_l\) are not empty. \(I_{-\infty}\) and \(I_0\) can be empty.

Let \((\xi, k, \mathcal{P}, J)\) be a quadruple such that \(k\) is a weight, \(\mathcal{P} = (I_{-\infty}, I_0, \ldots, I_l)\) is a partition of \(k, \xi \in V^{|J|+\ell} W(R)\) and \(J \subset [1, f]\). We define a log basic Witt differential \(\epsilon = \epsilon(\xi, k, \mathcal{P}, J) \in WA^{|J|+l+|I_{-\infty}|}_{(R[T], N^e \oplus N^j)/R}\) by

\[
\epsilon = \left(\prod_{i \in J} d\log c_i\right) \cdot \left(\prod_{i \in I_{-\infty}} d\log X_i\right) \cdot e(\xi, k^+, (I_0, \ldots, I_l)),
\]

where \(e(\xi, k^+, (I_0, \ldots, I_l))\) is the basic Witt differential defined in [LZ04] §2.2. We call the log basic Witt differential \(\epsilon(\xi, k, \mathcal{P}, J)\) is integral if \(e(\xi, k^+, (I_0, \ldots, I_l))\) is integral, i.e., \((k^+)\epsilon \in \mathbb{Z}_{\geq 0}\) for all \(i\). The log basic Witt differential \(\epsilon(\xi, k, \mathcal{P}, J)\) is called fractional if it is not integral.

We denote by \(\epsilon_m(\xi, k, \mathcal{P}, J)\) the image of \(\epsilon(\xi, k, \mathcal{P}, J)\) in \(W_m A^\bullet_{(R[T], N^e \oplus N^j)/R}\). The element \(\epsilon_m(\xi, k, \mathcal{P}, J)\) depends only on the residue class \(\xi\) of \(\xi\) in \(W_m(R)\). We see \(\xi \in V^n W_m(R)\) because \(\xi \in V^n W(R)\) for \(u = u(k^+)\). We have \(\epsilon_m(\xi, k, \mathcal{P}, J) = 0\) if \(p^{m-1} \cdot k^+\) is not integral.
The relations

\[ Fd \log c_i = V d \log c_i = \log c_i, \quad Fd \log X_i = V d \log X_i = \log X_i, \]

\[ d(\log c_i) = 0, \quad d(\log X_i) = 0 \]

and [LZ04] Proposition 2.5 and 2.6 give the following formulas:

(1)

\[ \begin{align*}
F \epsilon(\xi, k, (I - \infty, I_0, \ldots, I_t), J) &= \begin{cases} 
\epsilon(F \xi, pk, (I - \infty, I_0, \ldots, I_t), J) & (I_0 \neq \emptyset, k^+ \text{ is integral}), \\
\epsilon(V^{-1} \xi, pk, (I - \infty, I_0, \ldots, I_t), J) & (I_0 = \emptyset, k^+ \text{ not integral}).
\end{cases}
\end{align*} \]

(2)

\[ \begin{align*}
V \epsilon(\xi, k, (I - \infty, I_0, \ldots, I_t), J) &= \begin{cases} 
\epsilon(V^{-1} \xi, \frac{1}{p}k, (I - \infty, I_0, \ldots, I_t), J) & (I_0 \neq \emptyset, \text{or } k^+ \text{ is integral and divisible by } p), \\
\epsilon(pV^{-1} \xi, \frac{1}{p}k, (I - \infty, I_0, \ldots, I_t), J) & (I_0 = \emptyset, (1/p)k^+ \text{ is not integral}).
\end{cases}
\end{align*} \]

(3) If \( I = \text{Supp } k^+ \) and \( t = t(k_I) \),

\[ \begin{align*}
de(\xi, k, (I - \infty, I_0, \ldots, I_t), J) &= \begin{cases} 
0 & (I_0 = \emptyset), \\
\epsilon(\xi, k, (I - \infty, 0, I_0, \ldots, I_t), J) & (I_0 \neq \emptyset, k^+ \text{ not integral}), \\
\epsilon^{-1}(\xi, k, (I - \infty, 0, I_0, \ldots, I_t), J) & (I_0 \neq \emptyset, k^+ \text{ integral}).
\end{cases}
\end{align*} \]

By Proposition 3.12, the map

\[ \omega_m : \tilde{\Lambda}_{W_{m+1}^R(T_1, \ldots, T_n), N^e \oplus N^f/R} \to \Lambda_{(R[T_1, \ldots, T_n], N^e \oplus N^f)/R} \wedge \omega_m \]

which we defined in §3.8 factors through

\[ \omega_m : W_{m+1}^\bullet \Lambda_{(R[T_1, \ldots, T_n], N^e \oplus N^f)/R} \to \Lambda_{(R[T_1, \ldots, T_n], N^e \oplus N^f)/R} \wedge \omega_m. \]

Let

\[ \tilde{\omega}_m : W^\bullet \Lambda_{(R[T_1, \ldots, T_n], N^e \oplus N^f)/R} \to \Lambda_{(R[T_1, \ldots, T_n], N^e \oplus N^f)/R} \wedge \omega_m, \]

be the composition of \( \omega_m \) followed by the natural projection map.

**Proposition 4.2.** Let \( \epsilon = \epsilon(\xi, k, (I - \infty, I_0, \ldots, I_t), J) \in W^\bullet \Lambda_{(R[T_1, \ldots, T_n], N^e \oplus N^f)/R} \) be

a log basis Witt differential where \( \xi = V^u \eta, u = u(k^+) \). Then

\[ \tilde{\omega}_m(\epsilon) \]

\[ = \begin{cases} 
w_m(\xi) \cdot \left( \prod_{i \in J} d \log c_i \right) \cdot \left( \prod_{i \in L - \infty} d \log T_i \right) \cdot \left( TP^{m^k_{k_1}} \left( p^{-ord p^{m^k_{k_1}}} dT^{p^{m^k_{k_1}}} \right) \cdots \left( p^{-ord p^{m^k_{k_1}}} dT^{p^{m^k_{k_1}}} \right) \right) & (if \ p^m \cdot k^+ \ not \ integral), \\
w_m(\xi) \cdot \left( \prod_{i \in J} d \log c_i \right) \cdot \left( \prod_{i \in L - \infty} d \log T_i \right) \cdot \left( TP^{m^k_{k_1}} \left( p^{-ord p^{m^k_{k_1}}} dT^{p^{m^k_{k_1}}} \right) \cdots \left( p^{-ord p^{m^k_{k_1}}} dT^{p^{m^k_{k_1}}} \right) \right) & (if \ p^m \cdot k^+ \ integral, \ I_0 \neq \emptyset \ or \ k^+ \ integral),
\end{cases} \]

\[ w_{m - u}(\eta) \cdot \left( \prod_{i \in J} d \log c_i \right) \cdot \left( \prod_{i \in L - \infty} d \log T_i \right) \cdot \left( p^{-ord p^{m^k_{k_1}}} dT^{p^{m^k_{k_1}}} \right) \cdots \left( p^{-ord p^{m^k_{k_1}}} dT^{p^{m^k_{k_1}}} \right) & (if \ p^m \cdot k^+ \ integral, \ I_0 = \emptyset). \]

**Proof.** It follows from the construction of \( w_m \), [LZ04] Proposition 2.16 and calculations of log parts. \( \square \)
Proposition 4.3. Any element of \( W^\Lambda_\bullet (R[T_1, \ldots, T_n], N^e \oplus N^f)/R \) has a unique expression as a convergent sum of log basic Witt differentials:

\[
\sum_{k, \mathcal{P}, J} \epsilon(\xi_k, \mathcal{P}, J, k, \mathcal{P}, J),
\]

where \( k \) runs over all possible weight, \( \mathcal{P} \) over all partitions and \( J \) over all subsets of \([1, f]\). A convergent sum means that for any given number \( m \), we have \( \xi_k, \mathcal{P}, J \in V^m W(R) \) for all but finitely many weights \( k \).

Proof. For \( \xi \in W(R[T]) \), we can see that \( \xi \) can be written uniquely as a convergent sum \( \xi = \sum_{k, m \geq 0} V^m ([a_{k,m}] X^k) \), where \( X = [T] \), \( a_{k,m} \in R \) and \( k \) runs all possible integral weights.

For a given nonnegative integer \( m \) and a weight \( k, \rho \leq m \) denotes the maximum nonnegative integer such that \( p^{-\rho}k \) is integral. Then we have

\[
V^m ([a_{k,m}] X^k) = V^{m-\rho} ([a_{k,m}] x^{p^{-\rho}k}).
\]

Hence \( \xi \) is written as the convergent sum \( \xi = \sum_{k: \text{weight}} V^{\epsilon(k)} (\eta_k X^{p^{\epsilon(k)}k}) \).

Since we have a canonical surjective map

\[
\Lambda^1_{W(R[T]), N^e \oplus N^f}/R, \Omega^1_{W(R[T])/W(R)} \oplus \bigoplus_{i=1}^f W(R[T]) d \log T_i \oplus \bigoplus_{i=1}^f W(R[T]) d \log c_i \rightarrow \Lambda^1_{W(R[T]), N^e \oplus N^f}/R,
\]

any element in \( W^\Lambda_\bullet (R[T_1, \ldots, T_n], N^e \oplus N^f)/R \) is written as a convergent sum of elements of the form

\[
d \log c_{i_1} \cdots d \log c_{i_s} \cdot d \log X_{j_1} \cdots d \log X_{j_t} \cdot V^u (\eta_0 X^{p^{0}k^{(0)}})^{d \log 1} \cdots V^{u_m} (\eta_m X^{p^{m}k^{(m)}})^{d \log 1},
\]

where \( 1 \leq i_1 \leq \cdots \leq i_s \leq f, 1 \leq j_1 \leq \cdots \leq j_t \leq e, k^{(0)}, \ldots, k^{(m)} \) are weights and \( u \) is the least nonnegative integer such that \( p^{u} \cdot k^{(1)} \) is integral.

We prove that all the elements of the form \((*)\) can be written as a sum of log basic Witt differentials, by dividing them into four cases.

Case 0. \( \{j_1, \ldots, j_t\} \cap (\bigcup_{i=0}^m \text{Supp} k^{(i)}) = \emptyset \).

If \( \{j_1, \ldots, j_t\} \cap (\bigcup_{i=0}^m \text{Supp} k^{(i)}) = \emptyset \), \((*)\) can be written as a sum of log basic Witt differentials by [LZ04] Theorem 2.8 and our definition of log basic Witt differentials.

Case 1. \( k^{(1)} \) are all integral, i.e., \( u_0 = u_1 = \cdots = u_m = 0 \).

\[
(*) = d \log c_{i_1} \cdots d \log c_{i_s} \cdot d \log X_{j_1} \cdots d \log X_{j_t} \cdot (\eta_0 X^{k^{(0)}})^{d \log 1} \cdots (\eta_m X^{k^{(m)}})^{d \log 1}.
\]

It can be reduced to the case 0 by following calculations.

We write \( e_i \) for \((0, \ldots, 1, \ldots, 0)\), whose \( i \)th entry is 1 and the others are 0. If \( k \) is an integral weight without poles and \( t \in \text{Supp} k \), we have

\[
d \log X_t \cdot X^k = X^{k-k_i e_i} \cdot X^{k_i e_i - 1} d X_t,
\]

\[
d \log X_t \cdot X^k = d \log X_t \cdot (X^{k_i e_i} d X^{k-k_i e_i} + X^{k-k_i e_i} d X_t^{k_i})
\]

\[
= d \log X_t \cdot X^{k_i e_i} d X^{k-k_i e_i} = X^{k_i e_i - 1} d X_t \cdot d X^{k-k_i e_i}.
\]

Case 2. \( u_0 \geq u_j \) for \( j = 1, \ldots, m \).
We can rewrite (•) as follows:

\[ V^{\omega_0}(d\log c_1 \cdots d\log c_k \cdot d\log X_{j_1} \cdots d\log X_{j_k}) = \eta_0 X^{p^{\omega_0}k(0)} \cdot (V^{\omega_0} \cdots d\eta) X^{p^{\omega_1}k(1)} \cdots (V^{\omega_n} dh_{m}) X^{p^{\omega_m}k(m)}. \]

Since \( V \) maps log basic Witt differentials to log basic Witt differentials, it follows from case 1.

Case 3. \( u_1 \geq u_j \) for \( j = 0, \ldots, m \).

We apply Leibniz rule:

\[ V^{\omega_0}(\eta_0 X^{p^{\omega_0}k(0)}) d\log X_{j_1} = d(V^{\omega_0}(\eta_0 X^{p^{\omega_0}k(0)})) \eta_0 X^{p^{\omega_1}k(1)} \]

By the Leibniz rule and the fact that \( d \) maps log basic Witt differentials to log basic Witt differentials, we can reduce it to the former three cases.

Next we prove the independence of the log basic Witt differentials. Suppose the element \( \omega = \sum e_i \in \mathbb{F} \) of the form as (2) is equal to zero. We show \( \xi \in \mathbb{F} \) for all \( k, \mathcal{P}, J \). It suffices to show that the image of \( \xi \in \mathbb{F} \) in \( W_n(R) \) is zero for all \( m \). We fix a positive integer \( m \). Let \( \xi \) be the image of \( \xi \) in \( W_n(R) \).

First we suppose \( R \) is \( p \)-torsion free. Consider the morphism

\[ \tilde{\omega} : W_{[R[T_1, \ldots, T_n], N^e \oplus N^f]} / R \to \Lambda_{[R[T_1, \ldots, T_n], N^e \oplus N^f]} / R, \]

for \( 0 \leq i \leq m - 1 \). Proposition 4.2 shows that \( \tilde{\omega}_i(\xi) = 0 \) for \( 0 \leq i \leq m - 1 \) because log \( p \)-basic elements are linearly independent by Lemma 4.1. Since we assume that \( R \) has no \( p \)-torsion, \( \xi \in \mathbb{F} \) for all \( k, \mathcal{P}, J \). Hence the proof of independence is completed if \( R \) is \( p \)-torsion free.

We consider the general case. Take a surjective ring homomorphism \( \phi : \tilde{R} \to R \) where \( \tilde{R} \) is a ring without \( p \)-torsion. Set \( \mathfrak{a} := \ker \phi \). Let \( \tilde{\mathcal{P}} \) be a pre-log ring whose pre-log structure is given by \( \tilde{\mathcal{P}} = \tilde{R}[T_1, \ldots, T_n] \), \( \tilde{\mathcal{P}} \supset \mathfrak{a} \), \( T_i \to (1 \leq i \leq e), N^f \supset \mathfrak{a} \). We denote by \( W_{[\tilde{\mathcal{P}}, \mathcal{P}, J], N^e \oplus N^f} / \tilde{R} \) the subgroup of \( W_{[\tilde{\mathcal{P}}, \mathcal{P}, J], N^e \oplus N^f} / \tilde{R} \), which consists of convergent sums of log basic Witt differential of \( \epsilon(\xi) \in \tilde{\mathcal{P}} \) with \( \xi \in \mathcal{P} \in W_{\mathfrak{a}}(\tilde{\mathcal{P}}) \). We see \( W_{([\tilde{\mathcal{P}}, \mathcal{P}, J], N^e \oplus N^f) / \tilde{R}} \) is a ideal of \( W_{\mathfrak{a} \otimes \mathcal{P}, \mathcal{P}, J], N^e \oplus N^f} / \tilde{R} \) by the first part of the proof and Proposition 2.11 of [LZ04]. Let \( W_{\mathfrak{a} \otimes \mathcal{P}, \mathcal{P}, J], N^e \oplus N^f} / \tilde{R} \) be the image of \( W_{\mathfrak{a} \otimes \mathcal{P}, \mathcal{P}, J], N^e \oplus N^f} / \tilde{R} \) in \( W_{[\mathfrak{a} \otimes \mathcal{P}, \mathcal{P}, J], N^e \oplus N^f} / \tilde{R} \).

Define a procomplex \( \{ E_m^\bullet \} \) by

\[ E_m^\bullet := W_{\mathfrak{a} \otimes \mathcal{P}, \mathcal{P}, J], N^e \oplus N^f} / \tilde{R} \]

Set \( E^\bullet := \lim_{\leftarrow m} E_m^\bullet \). Then we have \( E_0^\bullet = W_0(R) \) and

\[ E^\bullet \cong W_{\mathfrak{a} \otimes \mathcal{P}, \mathcal{P}, J], N^e \oplus N^f} / \tilde{R} \]

Since \( W_{\mathfrak{a} \otimes \mathcal{P}, \mathcal{P}, J], N^e \oplus N^f} / \tilde{R} \) is invariant under \( F, V \) and \( d \), we see \( \{ E_m^\bullet \} \) is an log \( F-V \)-procomplex over \( (R[T], N^e \oplus N^f) / R \). Hence we obtain a morphism

\[ \{ W_{[R[T], N^e \oplus N^f] / R} \} \to \{ E_m^\bullet \} \]

of log \( F-V \)-procomplexes. Then there is the following commutative diagram
By p-torsion free case, any element $\omega$ of $W \Lambda^*_{R(T_1, \ldots, T_n), N^e \oplus N^f}/R$ is uniquely written as a convergent sum as (2). The commutativity of the diagram indicated above and the fact that the composite morphism

$$W \Lambda^*_{a(R(T_1, \ldots, T_n)), N^e \oplus N^f}/R \rightarrow W \Lambda^*_{(R(T_1, \ldots, T_n))/R} \rightarrow W \Lambda^*_{(R(T_1, \ldots, T_n))/R}$$

is zero implies Proposition 4.3 holds for any $R$. □

**Corollary 4.4.** Any element $\omega$ of $W_m \Lambda^*_{(R[T_1, \ldots, T_n]), N^e \oplus N^f}/R$ can be written as a finite sum

$$\omega = \sum_{k, \mathcal{P}, J} \epsilon_m(\xi_{k, \mathcal{P}, J}, k, \mathcal{P}, J), \xi_{k, \mathcal{P}, J} \in V^{m(k)} W_{m-k}(R).$$

Here $k$ runs over all weights such that $p^{m-1} \cdot k^+$ is integral, $\mathcal{P}$ runs over all partitions and $J$ over all subsets of $[1, f]$. The coefficients $\xi_{k, \mathcal{P}, J}$ are uniquely determined by $\omega$.

### 4.2. Semistable case

We consider the log $p$-basic elements and the basic Witt differentials in specific cases, which contains the semistable case.

For positive integers $d \leq e \leq n$ and a nonnegative integer $f$, we consider the pre-log ring

$$(A = R[T_1, \ldots, T_n]/(T_1 \cdots T_d), P = N^e \oplus N^f),$$

where $e_i$ (resp. $c_i$) are basis of $N^e$ (resp. $N^f$), for later discussions in this paper. The module of (relative) log differential forms $\Lambda^*_{(A,P)/R}$ is isomorphic to a free $A$-module $\bigoplus_{i=1}^e \text{Ad} \log T_i \oplus \bigoplus_{i=1}^f \text{Ad} T_i \oplus \bigoplus_{i=1}^{f+1} \text{Ad} \log c_i$. Hence $\Lambda^*_{(A,P)/R}$ has the following elements as a basis of $R$-module:

$$T_1^{k_1} \cdots T_n^{k_n} \prod_{i \in G} d \log T_i \cdot \prod_{i \in H} d \log T_i \cdot \prod_{j \in J} d \log T_i \cdot \prod_{i \in J} d \log c_i,$$

where $G \subset [1, d], H \subset [d+1, e], J \subset [e+1, n] \cap \text{Supp} k, J \subset [1, f]$ and $\min_{1 \leq i \leq d} k_i = 0$.

We conclude that the log $p$-basic differentials $\epsilon(k, \mathcal{P}, J)$ satisfying $[1, d] \not\subset \text{Supp} k^+$ forms the basis as an $R$-module by a similar argument to that in Lemma 4.1.

Next we study the basic Witt differentials of $W \Lambda^*_{(A,P)/R}$.

**Proposition 4.5.** Any element in $W \Lambda^*_{(A,P)/R}$ has a unique expression as a convergent sum

$$\sum_{k, \mathcal{P}, J} \epsilon(\xi_{k, \mathcal{P}, J}, k, \mathcal{P}, J)$$

of log basic Witt differentials. Here $k$ runs over all possible weight such that $[1, d] \not\subset \text{Supp}^+ k$, $\mathcal{P}$ over all partitions of $\text{Supp} k$ and $J$ over all subsets of $[1, f]$. A convergent sum means that for any given number $m$, we have $\xi_{k, \mathcal{P}, J} \in V^m W(R)$ for all but finitely many weights $k$.

**Proof.** As proof of Proposition 4.3, any element $\xi$ of $W(A)$ can be written as the following convergent sum:

$$\xi = \sum_{k, \text{weight } [1, d] \not\subset \text{Supp} k} V^{s(k)}(\eta_k X^{p^{s(k)} k}).$$

Hence an element of $W \Lambda^*_{(A,P)/R}$ can be written as a convergent sum of the following form

$$d \log c_{i_1} \cdots d \log c_{i_s} \cdot d \log X_{j_1} \cdots d \log X_{j_t} \cdot V^{n_0}(\eta_0 X^{p^n} 0) \cdot V^{n_1}(\eta_1 X^{p^{n_1} k^{(1)}}) \cdots V^{n_m}(\eta_m X^{p^{n_m} k^{(m)}})$$

for $m \geq 0$, $\eta_0, \eta_1, \ldots, \eta_m$ are elements of $R$ which vanish at $0$ and $k^{(1)}, \ldots, k^{(m)}$ are integers.
with $1 \leq i_1 < \cdots < i_s \leq f, 1 \leq j_1 < \cdots < j_l \leq e$, each $k^{(i)}$ is a weight satisfying $[1, d] \not\subset \text{Supp}\ k^{(i)}$ for all $i$ and $u_i$ is the least nonnegative integer such that $p^{u_i}k^{(i)}$ is integral.

We show this is equal to zero if $[1, d] \subset \bigcup_{i=1}^m \text{Supp}\ k^{(i)}$. We can assume that all $k^{(i)}$ are integral by the proof of Proposition 4.3. If $k$ is an integral weight, $dX^k$ is divisible by $X^{k[1, d]}$. Hence if $[1, d] \subset \bigcup_{i=1}^m \text{Supp}\ k^{(i)}$, the element indicated above is zero.

We can prove that any element of $W\Lambda^*_\mathcal{A,/} \mathcal{P}/R$ can be written as the form indicated in the proposition in the same as Proposition 4.3 because the actions of $F, V, d$ on log basic Witt differentials do not change the condition $[1, d] \not\subset \text{Supp}\ k$.

We can also show that this expression is unique by a similar argument to the proof of Proposition 4.3. □

**Corollary 4.6.** Any element $\omega$ of $W_m\Lambda^*_\mathcal{A,/} \mathcal{P}/R$ can be written as a finite sum

$$\omega = \sum_{k, \mathcal{P}, \mathcal{J}} \epsilon_m(\xi, k, \mathcal{P}, \mathcal{J}), \ \xi, k, \mathcal{P}, \mathcal{J} \in V^{p(k)} \ W_{m-u(k)}(R).$$

Here $\epsilon_m$ is the image of $\epsilon$ on $W_m\Lambda^*_\mathcal{A,/} \mathcal{P}/R$, $k$ runs over all weights such that $[1, d] \not\subset \text{Supp}\ k$; $p^{m-1}k^+$ is integral, $\mathcal{P}$ runs over all partitions of $\text{Supp}\ k$ and $\mathcal{J}$ over all subsets of $[1, f]$. The coefficients $\xi, k, \mathcal{P}, \mathcal{J}$ are uniquely determined by $\omega$.

Set $d \log X := d \log X_1 + \cdots + d \log X_e + d \log c_1 + \cdots + d \log c_f$.

We define an element $\epsilon'(\xi, k, (I_\infty, I_0, \ldots, I_l), J)$ for a log basic Witt differential $\epsilon(\xi, k, (I_\infty, I_0, \ldots, I_l), J)$ by

$$\epsilon' = \begin{cases} \epsilon(\xi, k, (I_\infty, I_0, \ldots, I_l), J) \\ (\prod_{i \in J} d \log c_i) \cdot (\prod_{i \in I_\infty, i \neq c} d \log X_i) \cdot d \log X \cdot \epsilon(\xi, k^+, (I_0, \ldots, I_l)) \\ (k_e = p^{-\infty}), \end{cases}$$

where $\epsilon(\xi, k^+, (I_0, \ldots, I_l))$ is the classical basic Witt differential defined in [LZ04].

If $k_e = p^{-\infty}$, we see

$$\epsilon'(\xi, k, (I_\infty, I_0, \ldots, I_l), J) = \epsilon(\xi, k, (I_\infty, I_0, \ldots, I_l), J)$$

$$+ \left( \prod_{i \in J} d \log c_i \right) \left( \prod_{i \in I_\infty, i \neq c} d \log X_i \right) \left( \sum_{i \in [1, c]} d \log X_i \right) \epsilon(\xi, k^+, (I_0, \ldots, I_l))$$

$$+ \left( \prod_{i \in J} d \log c_i \right) \left( \prod_{i \in I_\infty, i \neq c} d \log X_i \right) \left( \sum_{i \in [1, f] \setminus J} d \log c_i \right) \epsilon(\xi, k^+, (I_0, \ldots, I_l))$$

$$= \epsilon(\xi, k, (I_\infty, I_0, \ldots, I_l), J)$$

$$+ \text{(linear combination of } \epsilon(\xi, k, (I_\infty, I_0, \ldots, I_l), J) \text{ such that } k_e \neq p^{-\infty}).$$

From this we obtain

**Proposition 4.7.** $W\Lambda^*_\mathcal{A,/} \mathcal{P}/R$ has a decomposition as $W(R)$-modules:

$$W\Lambda^*_\mathcal{A,/} \mathcal{P}/R = WC^*_\mathcal{A,/} \mathcal{P}/R \oplus WC'_\mathcal{A,/} \mathcal{P}/R,$$

where $WC^*_\mathcal{A,/} \mathcal{P}/R$ (resp. $WC'_\mathcal{A,/} \mathcal{P}/R$) consists of the elements which can be written as a convergent sum of the elements of the form $\epsilon'$ such that $k_e \neq p^{-\infty}$ (resp. $k_e = p^{-\infty}$).

Note that the decomposition we stated above is not a decomposition as complexes.
5. Log Witt Lift and Log Frobenius Lift

Let \( R \) be a \( \mathbb{Z}_p \)-algebra in which \( p \) is nilpotent and \( (R, P) \to (S, Q) \) a log smooth morphism of pre-log rings. We define the log version of Witt lifts and Frobenius lifts of [LZ04] §3.1.

**Definition 5.1.** A log Witt lift of \((S, Q)\) over \((R, P)\) is a system \(((S_n, Q_n), \delta_n : (S_n, Q_n) \to W_n(S, Q))_{n \geq 1}\) satisfying the following conditions.

(1) For each \( n \geq 1 \), \((S_n, Q_n)\) is a log smooth over \( W_n(R, P) \), and \( W_n(S, Q) \otimes_{W_{n+1}(S, Q)} (S_{n+1}, Q_{n+1}) \simeq (S_n, Q_n) \), \((S_1, Q_1) = (S, Q)\).

(2) Let \( w_0 : W_n(S, Q) \to (S, Q) \) be the morphism induced by the Witt polynomial \( w_0 : W_n(S) \to S \) and \( \text{id}_Q \). For \( n > 1 \), \( w_0 \delta_n \) is the natural map \((S_n, Q_n) \to (S, Q)\) and the following diagram commutes:

\[
\begin{array}{ccc}
(S_{n+1}, Q_{n+1}) & \overset{\delta_{n+1}}{\longrightarrow} & W_{n+1}(S, Q) \\
\downarrow & & \downarrow \\
(S_n, Q_n) & \overset{\delta_n}{\longrightarrow} & W_n(S, Q).
\end{array}
\]

**Definition 5.2.** A log Frobenius lift of \((S, Q)\) over \((R, P)\) is a system \(((S_n, Q_n), \phi_n : (S_n, Q_n) \to (S_{n-1}, Q_{n-1}), \delta_n : (S_n, Q_n) \to W_n(S, Q))_{n \geq 1}\), satisfying the following conditions:

(1) \(((S_n, Q_n), \delta_n)\) is a log Witt lift of \((S, Q)\) over \((R, P)\).

(2) For \( n \geq 1 \), \( \phi_n \) is compatible with the Frobenius on the log Witt ring \( F : W_n(R, P) \to W_{n-1}(R, P) \), the absolute Frobenius \( \text{Frob} : S/pS \to S/pS \) and \( \times p : Q \to Q \).

(3) The following diagram commutes:

\[
\begin{array}{ccc}
(S_{n+1}, Q_{n+1}) & \overset{\delta_{n+1}}{\longrightarrow} & W_{n+1}(S, Q) \\
\downarrow & & \downarrow \\
(S_n, Q_n) & \overset{\delta_n}{\longrightarrow} & W_n(S, Q).
\end{array}
\]

We also define log Witt lifts and log Frobenius lifts for a morphism \( f : (X, \mathcal{M}) \to (Y, \mathcal{N}) \) of fine log schemes.

**Definition 5.3.** A log Witt lift of \((X, \mathcal{M})\) over \((Y, \mathcal{N})\) is a system \(((X_n, \mathcal{M}_n), \Delta_n : W_n(X, \mathcal{N}) \to (X_n, \mathcal{N}_n))_{n \geq 1}\) satisfying the following conditions.

(1) For each \( n \geq 1 \), \((X_n, \mathcal{M}_n)\) is a log smooth over \( W_n(Y, \mathcal{N}) \), and \( W_n(X, \mathcal{M}) \times_{W_{n+1}(X, \mathcal{M})} (X_{n+1}, \mathcal{M}_{n+1}) \simeq (X_n, \mathcal{M}_n) \), \((X_1, \mathcal{M}_1) = (X, \mathcal{M})\).

(2) Let \( w_0 : (X, \mathcal{M}) \to W_n(X, \mathcal{M}) \) be the morphism induced by the Witt polynomial \( w_0 : X \to W_n(X) \) and \( \text{id}_\mathcal{M} \). For \( n > 1 \), \( \Delta_n w_0 \) is the natural map \((X_n, \mathcal{M}_n) \to (X_n, \mathcal{M}_n)\) and the following diagram commutes:

\[
\begin{array}{ccc}
W_n(X, \mathcal{M}) & \overset{\Delta_n}{\longrightarrow} & (X_n, \mathcal{M}_n) \\
\downarrow & & \downarrow \\
W_{n+1}(X, \mathcal{M}) & \overset{\Delta_{n+1}}{\longrightarrow} & (X_{n+1}, \mathcal{M}_{n+1}).
\end{array}
\]

**Definition 5.4.** A log Frobenius lift of \((X, \mathcal{M})\) over \((X, \mathcal{M})\) is a system \(((X_n, \mathcal{M}_n), \Phi_n : (X_{n-1}, \mathcal{M}_{n-1}) \to (X_n, \mathcal{M}_n), \Delta_n : W_n(X, \mathcal{M}) \to (X_n, \mathcal{M}_n))_{n \geq 1}\), satisfying the following conditions:
(1) \((X_n, \mathcal{M}_n), \Delta_n\) is a log Witt lift of \((X, \mathcal{M})\) over \((Y, \mathcal{N})\).

(2) For \(n \geq 1\), \(\Phi_n\) is compatible with the Frobenius on the log Witt scheme \(F : W_n-1(X, \mathcal{M}) \to W_n(X, \mathcal{M})\), the absolute Frobenius \(\text{Frob} : X \otimes \mathbb{F}_p \to X \otimes \mathbb{F}_p\) and \(\times p : \mathcal{M} \to \mathcal{M}\).

(3) The following diagram commutes:

\[
\begin{array}{ccc}
W_n(X, \mathcal{M}) & \xrightarrow{\Delta_n} & (X_n, \mathcal{M}_n) \\
\Phi_{n+1} \downarrow & & \downarrow \\
W_{n+1}(X, \mathcal{M}) & \xrightarrow{\Delta_{n+1}} & (X_{n+1}, \mathcal{M}_{n+1}).
\end{array}
\]

**Lemma 5.5.** (1) Let \((R, P) \to (S, Q)\) be a log smooth morphism of pre-log rings. Then \((S, Q)\) has a log Frobenius lift over \((R, P)\).

(2) Let \((X, \mathcal{M}) \to (Y, \mathcal{N})\) be a log smooth morphism of fine log schemes. Then \(\text{étalement} \) locally on \((X, \mathcal{M})\) has a log Frobenius lift over \((Y, \mathcal{N})\).

**Proof.** By the toroidal characterization of the log smoothness of log schemes (Theorem 2.3), (2) follows from (1). We show (1).

The morphism \((R, P) \to (S, Q)\) has a decomposition \((R, P) \to (R \otimes \mathbb{Z}[P] \mathbb{Z}[Q], Q) \to (S, Q)\). Since \((S, Q)\) is log smooth over \((R, P)\), the ring map \(R \otimes \mathbb{Z}[P] \mathbb{Z}[Q] \to S\) is \(\text{étalement}\).

First we construct a log Frobenius map on \(T := R \otimes \mathbb{Z}[P] \mathbb{Z}[Q], Q)\) over \((R, P)\). Let \(\alpha : P \to R\) be the structure morphism of the pre-log ring \((R, P)\). Set \(T_n := W_n(R) \otimes \mathbb{Z}[P] \mathbb{Z}[Q]\) where the structural morphism \(\mathbb{Z}[P] \to W_n(R)\) is induced by \(a \in P \to [\alpha(a)]\). Then \((T_n, Q_n := Q)\) will be a pre-log ring in the obvious way. In particular, \((T_n, Q_n)\) is log smooth over \(W_n(R, P)\). We extend \(F : W_n(R) \to W_{n-1}(R)\) to a morphism

\[
\phi_n : (T_n, Q_n) \to (T_{n-1}, Q_{n-1}), \quad a \otimes b \mapsto F a \otimes b^p, \quad a \in W_n(R), b \in Q.
\]

and also define \(\delta_n : (T_n, Q_n) \to W_n(T, Q)\) induced by \(T_n \to W_n(T); a \in Q \to [1 \otimes a], \text{id} : Q_n = Q \to Q\). Then \((T_n, Q_n)\) is a log Frobenius lift of \((T, Q)\).

To obtain a log Frobenius lift on \((S, Q)\), it is suffice to show that if \((S, Q) \to (S', Q)\) is a morphism of pre-log rings such that the underlying ring map \(S \to S'\) is an \(\text{étalement}\) morphism and the underlying monoid map \(Q \to Q\) is the identity map and there a log Frobenius lift \((\phi_n, \delta_n)\) of \((S, Q)\), there is a unique log Frobenius lift of the form \((\phi_n', Q_n), \psi_n, \epsilon_n)\) of \((S', Q)\) and \((S, Q) \to (S', Q)\) lifts to a homomorphism \((\phi_n, \delta_n) \to (\phi_n', Q_n), \psi_n, \epsilon_n)\). We can prove this in the same manner as the proof of [LZ04] Proposition 3.2.

\[\square\]

### 6. Comparison morphism

We construct the comparison morphism between the log crystalline cohomology and the hypercohomology of the log de Rham-Witt complex.

#### 6.1. Extension of derivations

In this subsection, we consider the log version of the discussion in [Ill79] 0, §3.1. First we recall the definition of the trivial extension of a quasi-coherent sheaf ([Ogu06] Example 2.1.6).

**Definition 6.1.** Let \(f : X \to Y\) be a morphism of fine log schemes and \(E\) a quasi-coherent sheaf of \(\mathcal{O}_X\)-modules.

The trivial \(Y\)-extension of \(X\) by \(E\) is the log scheme \(T\) defined by \(\mathcal{O}_T := \mathcal{O}_X \oplus E\) with \((a, b)(a', b') := (aa', ab' + a'b), \) with \(\mathcal{M}_T := \mathcal{M}_X \oplus E, \) and \(\alpha_T(m, e) := (a_X(m), a_X(e))\) if \(m \in \mathcal{M}_X\) and \(e \in E\). The canonical projection \(\mathcal{O}_T \to \mathcal{O}_X\) (resp. the canonical map \(\mathcal{O}_Y \to \mathcal{O}_X \to \mathcal{O}_T\)) defines a morphism of log schemes \(X \to T\) (resp. \(T \to Y\)). We also have an evident retraction \(T \to X\) over \(Y\).
Let \((Y, \mathcal{N}, \mathcal{I}, \gamma)\) be a fine log pd-scheme. Let \(i : (X, \mathcal{M}) \to (X', \mathcal{M}')\) be a closed immersion of log schemes. We assume \(\gamma\) extends to \(X\) and \(i\) has a factorization \((X, \mathcal{M}) \xrightarrow{\gamma} (Z, \mathcal{L}) \xrightarrow{j} (X', \mathcal{M}')\) with \(j\) an exact closed immersion and \(g\) log étale. We admit this kind of factorizations étale locally on \(X\) ([Kat89] (4.10) (1)).

Set \(\mathcal{J} := \ker(\mathcal{O}_Z \to j_* \mathcal{O}_X)\). Then the log pd-envelope \((D, \mathcal{M}_D, \mathcal{J}, \{\cdot\})\) of \(i\) is the usual pd-envelope \((D, \mathcal{J}, \{\cdot\})\) of \(X\) in \(Z\) with log structure \(\mathcal{M}_D\) given by the inverse image of \(\mathcal{L}\). Since \(g\) is log étale, the canonical morphism \(g^* \Lambda_{(X', \mathcal{M}')}^{1}(Y, \mathcal{N}) \to \Lambda_{(Z, \mathcal{L})}^{1}(Y, \mathcal{N})\) is an isomorphism ([Kat89] Proposition (3.12)).

**Proposition 6.2.** The log derivation \((d, d \log) : (\mathcal{O}_X, \mathcal{M}) \to \Lambda_{(X', \mathcal{M}')}^{1}(Y, \mathcal{N})\) extends uniquely to

\[
(d', d' \log) : (\mathcal{O}_D, \mathcal{M}_D) \to \mathcal{O}_D \otimes \mathcal{O}_X, \Lambda_{(X', \mathcal{M}')}^{1}(Y, \mathcal{N}) \simeq \mathcal{O}_D \otimes \mathcal{O}_X \Lambda_{(Z, \mathcal{L})}^{1}(Y, \mathcal{N})
\]

such that \(dx'^n = x^{[n-1]} \otimes dx\) for all \(x \in \mathcal{J}, n \geq 1\) and \(d' \log m = 1 \otimes d \log m\) for all \(m \in \mathcal{L}\).

**Proof.** Let \(E := \mathcal{O}_D \otimes \mathcal{O}_X, \Lambda_{(X', \mathcal{M}')}^{1}(Y, \mathcal{N})\) and \(T = \text{Spec}(\mathcal{O}_D \oplus E, \mathcal{M}_D \oplus E)\) be the trivial \(Z\)-extension of \(D\) by \(E\). We define a pd-structure on a ideal \(E \subset \mathcal{O}_T\) by \(u^n = 0\) for \(n \geq 2\).

Since \(\mathcal{O}_T\) is an augmented \(\mathcal{O}_D\)-algebra and \(E\) is an augmented ideal, there exists a unique pd-structure \(\delta\) on \(\mathcal{T} \cdot \mathcal{O}_T + E \subset \mathcal{O}_T\) which is compatible with the pd-structures on \(\mathcal{T}\) and \(E\) by ([Ber74] 11.6.5). \(\delta\) satisfies \(\delta_u(x + u) = x^{[n]} + x^{[n-1]}u\) for \(x \in \mathcal{J}, u \in E\). By the construction, \(\delta\) is compatible with \(\gamma\). Let \(\alpha : \mathcal{O}_Z \to \mathcal{O}_T = \mathcal{O}_D \oplus E\) (resp. \(\beta : \mathcal{L} \to \mathcal{M}_T\)) be a morphism defined by \(\alpha(z) = (z, 1 \otimes dz)\) (resp. \(\beta(e) = (e, 1 \otimes d \log e)\)). They define a morphism \(\eta_0 = (\alpha, \beta) : (\mathcal{O}_Z, \mathcal{L}) \to (\mathcal{O}_T, \mathcal{M}_T)\).

By the universal property of the log pd-envelope, \(\eta_0\) induces an \(\mathcal{O}_T\)-pd-morphism \(\eta : (\mathcal{O}_D, \mathcal{M}_D) \to (\mathcal{O}_T, \mathcal{M}_T)\). We see that this morphism is a section of the canonical projection map \((\mathcal{O}_T, \mathcal{M}_T) \to (\mathcal{O}_D, \mathcal{M}_D)\). The morphisms \(d' : \mathcal{O}_D \xrightarrow{\alpha} \mathcal{O}_T \xrightarrow{\beta} E\) and \(d' \log : \mathcal{M}_D \xrightarrow{\beta} \mathcal{M}_T \xrightarrow{\alpha} E\) define a log derivation

\[
(d', d' \log) : (\mathcal{O}_D, \mathcal{M}_D) \to \mathcal{O}_D \otimes \mathcal{O}_X, \Lambda_{(X', \mathcal{M}')}^{1}(Y, \mathcal{N}) \simeq \mathcal{O}_D \otimes \mathcal{O}_X \Lambda_{(Z, \mathcal{L})}^{1}(Y, \mathcal{N})
\]

such that \(dx'^{n} = x^{[n-1]} \otimes dx\) for all \(x \in \mathcal{J}\) and \(n \geq 1\) and \(d' \log m = 1 \otimes d \log m\) for all \(m \in \mathcal{L}\). Uniqueness is easy. \(\square\)

The log derivation extends to a graded algebra \(\mathcal{O}_D \otimes \mathcal{O}_X, \Lambda_{(X', \mathcal{M}')}^{*}(Y, \mathcal{N})\). We denote by \(\Lambda^{*}_{(D, \mathcal{M}_D)}(Y, \mathcal{N})\) the log pd de Rham complex of \((D, \mathcal{M}_D)\) over \((Y, \mathcal{N})\) with respect to the pd-structure \([\cdot]\) on \((D, \mathcal{J})\). The universal property of the log pd de Rham complex induces a map \(\Lambda^{*}_{(D, \mathcal{M}_D)}(Y, \mathcal{N}) \to \mathcal{O}_D \otimes \mathcal{O}_X, \Lambda_{(X', \mathcal{M}')}^{*}(Y, \mathcal{N})\) of \(\mathcal{O}_Y\)-algebras. This map is isomorphism by the same proof to ([III] 3.1.6).

**6.2. Comparison morphism.** Let \(R\) be a \(\mathbb{Z}(p)\)-algebra, in which \(p\) is nilpotent. Let \((X, \mathcal{M}) \to \text{Spec}(R, \mathcal{P})\) be a morphism of fine log schemes and we assume that the pd-structure of \(W(R)\) extends to \(X\). We have the natural morphism

\[
u_m : (((X, \mathcal{M})/W_m(R, \mathcal{P}))^{\text{log crys}} \to X_{\text{ét}}
\]

from the log crystalline topos to the étale topos. We write the structure sheaf of the log crystalline site \(\mathcal{O}_{(X, \mathcal{M})/W_m(R, \mathcal{P})}\) as \(\mathcal{O}_m\).

Define a morphism

\[
\mathcal{R} \cdot m, \mathcal{O}_m \to W_m \Lambda^*_{(X, \mathcal{M})/W_m(R, \mathcal{P})}
\]

in the derived category \(D^+(X, W_m(R))\) of sheaves of \(W_m(R)\)-modules on \(X_{\text{ét}}\) as follows:
First, we consider the case that \((X, M)\) has an embedding into a log smooth scheme \((Y, N)\) over \((R, P)\) such that \((Y, N)\) has a log Witt lift \(((Y_m, N_m), \Delta_m)\). We already know such embedding exists \(\acute{e}tale\) locally on \(X\) by \([HK94\, (2.9.2)\) and Lemma 5.5. There exists the following commutative diagram:

\[
\begin{array}{ccc}
(X, M) & \xrightarrow{w_0} & (Y, N) \\
\downarrow & & \downarrow \quad \downarrow \\
W_m(X, M) & \xrightarrow{w_0} & W_m(Y, N).
\end{array}
\]

The left vertical arrow \(w_0 : (X, M) \to W_m(X, M)\) defines a log pd-thickening relative to the canonical pd-structure on \(V W_m(R)\).

Then the morphism \(W_m(X, M) \to (Y_m, N_m)\) factors through a morphism

\[
\mu_m : W_m(X, M) \to (\overline{Y}_m, \overline{N}_m),
\]

where \((\overline{Y}_m, \overline{N}_m)\) is the log pd-envelope of the closed immersion \((X, M) \to (Y_m, N_m)\) with respect to the canonical pd-structure on \(V W_m(R)\). Then we have an isomorphism in \(D^+(X, W_m(R))\)

\[
R\text{Hom}_{m}\mathcal{O}_m \to \mathcal{O}_{\overline{Y}_m} \otimes_{\mathcal{O}_{\overline{N}_m}} \Lambda^*_{(Y_m, N_m)/W_m(R, P)}.\]

Since \(X \to \overline{Y}_m\) is a nilimmersion, we can consider the right hand side as a sheaf on \(X_{\text{et}}\).

By the discussion in §6.1, we have an isomorphism

\[
\Lambda^*_{(\overline{Y}_m, \overline{N}_m)/W_m(R, P)} \simeq \mathcal{O}_{\overline{Y}_m} \otimes_{\mathcal{O}_{\overline{N}_m}} \Lambda^*_{(Y_m, N_m)/W_m(R, P)}.\]

We define the comparison morphism as follows:

\[
\begin{array}{ccc}
\mathcal{O}_{\overline{Y}_m} \otimes_{\mathcal{O}_{\overline{N}_m}} \Lambda^*_{(Y_m, N_m)/W_m(R, P)} & \xrightarrow{\sim} & W_m \Lambda^*_{(X, M)/(R, P)} \\
\downarrow & & \downarrow \mu_m \\
\Lambda^*_{(\overline{Y}_m, \overline{N}_m)/W_m(R, P)} & \xrightarrow{\mu_m} & \Lambda^*_{W_m(X, M)/W_m(R, P)}.\end{array}
\]

One can show this comparison morphism is independent of embeddings and Witt lifts using the fibered product argument in \([III79\, II.1.1\).

Next, we treat general cases. Recall the definition of embedding system \(([HK94\, (2.9.2)\) on p.237) :

**Definition 6.3.** Let \(f : (X, M) \to (S, \mathcal{L})\) be a morphism of fine log schemes such that underlying morphism \(X \to S\) is locally of finite type, an embedding system for \(f\) is a pair of simplicial objects \((X^*, M^*)\) and \((Z^*, N^*)\) in the category of fine log schemes endowed with morphism

\[
(X^*, M^*) \to (X, M), (X^*, M^*) \to (Z^*, N^*), (Z^*, N^*) \to (S, \mathcal{L})
\]

satisfying the following conditions (i)-(iv).

(i) The diagram

\[
\begin{array}{ccc}
(X^*, M^*) & \xrightarrow{\sim} & (Z^*, N^*) \\
\downarrow & & \downarrow \\
(X, M) & \xrightarrow{\sim} & (S, \mathcal{L})
\end{array}
\]

is commutative.

(ii) The morphism \(X^* \to X\) is a hypercovering for the \(\acute{e}tale\) topology and \(M^i\) is the inverse image of \(M\) on \(X^i\) for each \(i \geq 0\).

(iii) Each \((Z^i, N^i) \to (S, \mathcal{L})\) is log smooth.

(iv) Each \((X^i, M^i) \to (Z^i, N^i)\) is closed immersion.
Let \( \{ X(i) \}_{i \in I} \) be an étale covering of \( X \) such that each \( X(i) \) can be embedded to a log smooth scheme \( Y(i) \) which has a log Witt lift \( \{ Y_m(i) \}_m \). Set
\[
X(i_1, \ldots, i_r) := X(i_1) \times_X \cdots \times_X X(i_r),
\]
\[
Y_m(i_1, \ldots, i_r) := Y_m(i_1) \times_{W_m(R, P)} \cdots \times_{W_m(R, P)} Y_m(i_r).
\]
Then \( X(i_1, \ldots, i_r) \to Y_m(i_1, \ldots, i_r) \) is closed immersion since \( X \) is separated. For \( r \in \mathbb{N} \), let
\[
X^r := \prod_{i_1, \ldots, i_r \in I} X(i_1, \ldots, i_r), \quad Y_m^r := \prod_{i_1, \ldots, i_r \in I} Y_m(i_1, \ldots, i_r).
\]
We get an embedding system \( X^\bullet \to Y_m^\bullet \). We denote by \( \overline{Y_m} \) the log pd-envelope with respect to this closed immersion. Let \( \overline{\theta : (X^\bullet)_\et \to X^\bullet_{\et}} \) be the natural augmentation morphism.

By the liftable case, we have a morphism
\[
\mathcal{O}_{\overline{Y_m}} \otimes_{\mathcal{O}_Y} \Lambda_{(Y_m^\bullet, \mathcal{N}^\bullet_m)/W_m(R, P)} \to W_m^\bullet(X^\bullet, \mathcal{N}^\bullet_{X, M^\bullet}).
\]
Applying \( R_{\overline{\theta}} \) to both sides, we get the comparison morphism
\[
R_{\overline{\theta}} \to W_m^\bullet(X^\bullet, \mathcal{N}^\bullet_{X, M^\bullet}).
\]
This is because the canonical morphism
\[
R_{\overline{\theta}} \to R_{\overline{\theta}}(\mathcal{O}_{\overline{Y_m}} \otimes_{\mathcal{O}_Y} \Lambda_{(Y_m^\bullet, \mathcal{N}^\bullet_m)/W_m(R, P)})
\]
is quasi-isomorphism by [HK94] Proposition 2.20 and we have a natural isomorphism
\[
R_{\overline{\theta}} \Lambda_m^\bullet(X^\bullet, \mathcal{N}^\bullet_{X, M^\bullet})/(R, P) \simeq W_m^\bullet(X^\bullet, \mathcal{N}^\bullet_{X, M^\bullet})/(R, P)
\]
from the étale base change property of log de Rham-Witt complexes.

We prove that the comparison morphism is compatible with the Frobenius structure (cf. [LZ04] Proposition 3.6). Frobenius morphisms and multiplications by \( p \)
\[
W_m^\bullet(\mathcal{O}_X) \xrightarrow{F} W_{m-1}^\bullet(\mathcal{O}_X) \quad M \xrightarrow{\times p} M \quad P \xrightarrow{\times p} P
\]
defines a map of log de Rham complexes
\[
\Lambda_{(W_m(X), W_m(M))}/W_m(R, P) \to \Lambda_{(W_{m-1}(X), W_{m-1}(M))}/W_{m-1}(R, P)
\]
and it factors \( F : W_m^\bullet(X, M)/(R, P) \to W_{m-1}^\bullet(X, M)/(R, P) \). We have \( F = p^mF \) on \( W_m^\bullet(X, M)/(R, P) \) because \( dF \xi = p^m d\xi \) for \( \xi \in W_m^\bullet(\mathcal{O}_X) \) and \( d \log m^\bullet = pd \log m \) for \( m \in W_m(M) \).

Let \( (X_0, \mathcal{M}_0) := (X, \mathcal{M}) \times \mathbb{F}_p \) and \( R_0 := R \otimes \mathbb{F}_p \). Consider the commutative diagram:
\[
\xymatrix{ \Spec W_{m-1}(R) & X_0 \ar[l]_{\Frob} \ar[r] & \Spec W_m(R) }
\]
It induces a map
\[
R_{\overline{\pi}} : \mathcal{O}_{(X_0, \mathcal{M}_0)/W_m(R_0, P)} \to R_{\overline{\pi}}(\mathcal{O}_{(X_0, \mathcal{M}_0)/W_{m-1}(R_0, P)}),
\]
where \( \overline{\pi} : ((X_0, \mathcal{M}_0)/W_m(R_0, P))^{\log} \to (X_0)_{\et} = X_{\et} \) is the canonical morphism of topoi. We have canonical isomorphism
\[
R_{\overline{\pi}}(\mathcal{O}_{(X_0, \mathcal{M}_0)/W_m(R_0, P)}) \to R_{\overline{\theta}}(\mathcal{O}_{X, M}/W_m(R, P)).
\]
So we obtain $\mathcal{F} : \mathcal{R}u_m \mathcal{O}_m \to \mathcal{R}u_{m-1} \mathcal{O}_{m-1}$.

Proposition 6.4. We have a commutative diagram:

$$
\begin{array}{ccc}
\mathcal{R}u_m \mathcal{O}_m & \xrightarrow{\mathcal{F}} & W_m \Lambda^\bullet_{(X,M)/(R,P)} \\
\downarrow \mathcal{F} & & \downarrow \mathcal{F} \\
\mathcal{R}u_{m-1} \mathcal{O}_{m-1} & \xrightarrow{\mathcal{F}} & W_{m-1} \Lambda^\bullet_{(X,M)/(R,P)}
\end{array}
$$

Proof. By the simplicial method as above, we can assume that $(X, M)$ is embedded in a log smooth scheme $(Y, N)$ which admits a log Frobenius lift $\{\{Y_n, N_n\}\}_n$. Let $\Phi_m : (Y_{m-1}, N_{m-1}) \to (Y_m, N_m)$ be the given lift of the absolute Frobenius. The map $\mathcal{F} : \mathcal{R}u_m \mathcal{O}_m \to \mathcal{R}u_{m-1} \mathcal{O}_{m-1}$ is represented by the map

$$
\mathcal{O}_{Y_m} \otimes_{\mathcal{O}_{Y_m}} \Lambda^\bullet_{Y_m, N_m} / W_m(R, P) \to \mathcal{O}_{Y_{m-1}} \otimes_{\mathcal{O}_{Y_{m-1}}} \Lambda^\bullet_{Y_{m-1}, N_{m-1}} / W_{m-1}(R, P)
$$

which is induced by $\Phi_m$.

By the properties of log Frobenius lifts, we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{O}_{Y_m} \otimes_{\mathcal{O}_{Y_m}} \Lambda^\bullet_{Y_m, N_m} / W_m(R, P) & \xrightarrow{\mathcal{F}} & W_m \Lambda^\bullet_{(X,M)/(R,P)} \\
\downarrow \mathcal{F} & & \downarrow \mathcal{F} \\
\mathcal{O}_{Y_{m-1}} \otimes_{\mathcal{O}_{Y_{m-1}}} \Lambda^\bullet_{Y_{m-1}, N_{m-1}} / W_{m-1}(R, P) & \xrightarrow{\mathcal{F}} & W_{m-1} \Lambda^\bullet_{(X,M)/(R,P)}
\end{array}
$$

and this is identified to the diagram in the proposition. \hfill \Box

7. Comparison theorem

Let $R$ be a $\mathbb{Z}_{(p)}$-algebra such that $p$ is nilpotent in $R$.

7.1. NCD case. Let $Y$ be a log scheme over $S = \text{Spec} R$. We assume that the structure morphism $Y \to S$ has étale locally on $Y$ a decomposition

$$
Y \xrightarrow{u} \text{Spec}(A = R[T_1, \ldots, T_n], P = \mathbb{N}^e \oplus \mathbb{N}^f) \to \text{Spec} R
$$

with $u$ exact and étale (in the usual sense), $1 \leq e \leq n, f \geq 0,$ and $(A, P)$ is the pre-log ring we discussed in §4.1. If $X$ is a smooth scheme and $D$ is a normal crossing divisor on $X$, the log scheme $(X, D)$ satisfies this condition.

First we consider the log scheme $X := \text{Spec}(A = R[T_1, \ldots, T_n], P = \mathbb{N}^e \oplus \mathbb{N}^f)$. Let $(A_m = W_m(R)[T_1, \ldots, T_n], P)$ be the pre-log ring of the type we discussed in §4.1. Then $X_m := \text{Spec}(A_m, P)$ is a lift of $X$ over $W_m(R)$. Let $\phi_m : A_{m+1} \to A_m$ be the morphism defined by $F : W_{m+1}(R) \to W_m(R)$ and $T_i \to T_i^f$ for $1 \leq i \leq n$. The morphism $\phi_m$ and the multiplication by $p$ morphism $x \cdot p : P \to P$ define a morphism of log schemes $\Phi_m : X_m \to X_{m+1}$. Let $\delta_m : A_m \to W_m(A)$ be the morphism induced by the canonical morphism $W_m(R) \to W_m(A)$ and $T_i \to [T_i]$ for $1 \leq i \leq n$. We denote by $\Delta_m : W_m(X) \to X_m$ the morphism corresponding to $\delta_m$ and the identity morphism on $P$. The pair $(X_m, \Delta_m)$ defines the comparison morphism

$$
\Lambda^\bullet_{X_m/W_m(R)} \to W_m \Lambda^\bullet_{X/R}.
$$

If $f = 0$, the family $(X_m, \Phi_m, \Delta_m)_m$ is a log Frobenius lift of $X$ and the comparison morphism coincides with the morphism induced by the log Witt lift $(X_m, \Delta_m)_m$ in §6.

Theorem 7.1. The comparison morphism

$$
\Lambda^\bullet_{X_m/W_m(R)} \to W_m \Lambda^\bullet_{X/R}
$$

is a quasi-isomorphism. This morphism is functorial.
Proof. We prove that the canonical comparison morphism of complexes 
\[ \Lambda^{\bullet}_{(A_m, P)/W_m(R)} \to W_m \Lambda^{\bullet}_{(A, P)}/R \]
is a quasi-isomorphism. We have a decomposition of complexes 
\[ W_m \Lambda^{\bullet}_{(A, P)/R} = W_m \Lambda^{\text{int}}_{(A, P)/R} \oplus W_m \Lambda^{\text{frac}}_{(A, P)/R}, \]
where \( W_m \Lambda^{\text{int}}_{(A, P)/R} \) (resp. \( W_m \Lambda^{\text{frac}}_{(A, P)/R} \)) is the integral part (resp. the fractional part). By the formula of derivation on log basic Witt differentials given in §4, we see that \( W_m \Lambda^{\text{frac}}_{(A, P)/R} \) is acyclic. The comparison morphism maps \( \Lambda^{\bullet}_{(A_m, P)/W_m(R)} \) isomorphically to the complex \( W_m \Lambda^{\text{int}}_{(A, P)/R} \) because the comparison map sends the log \( p \)-basic differential 
\[ \left( \prod_{i \in J} d \log c_i \right) \cdot \left( \prod_{i \in I_{-\infty}} d \log T_i \right) \cdot T^{k_{i_0}} \left( p^{-\text{ord}_p k_{i_1}} dT^{k_{i_1}} \right) \cdots \left( p^{-\text{ord}_p k_{i_t}} dT^{k_{i_t}} \right) \]
to the following log basic Witt differential:
\[ \left( \prod_{i \in J} d \log c_i \right) \cdot \left( \prod_{i \in I_{-\infty}} d \log X_i \right) \cdot X^{k_{i_0}} \left( F^{-t(\ell_1)} dX^{p^{(\ell_1)} k_{\ell_1}} \right) \cdots \left( F^{-t(\ell_t)} dX^{p^{(\ell_t)} k_{\ell_t}} \right). \]
\[ \square \]

**Theorem 7.2.** Let \( Y \) be a smooth scheme over \( R \) and \( D \) be a normal crossing divisor of \( Y \). Then the canonical homomorphism 
\[ \mathbb{R}i_{(Y,D)/W_m(R)} \mathcal{O}_{(Y,D)/W_m(R)} \to W_m \Lambda^{\bullet}_{(Y,D)/R} \]
is an isomorphism in \( D^+(Y, W_m(R)) \). Moreover, if \( R \) is Noetherian and \( Y \) is proper over \( R \), we have a canonical isomorphism 
\[ H^*_{\log, \text{crys}}((Y, D)/W(R)) \to \mathbb{H}^*_{\text{et}}(Y, W \Lambda^{\bullet}_{(Y,D)/R}). \]

**Proof.** Using the similar method of [LZ04] Theorem 3.5, we may assume that \((Y, D) = \text{Spec}(A = R[T_1, \ldots, T_n], P = \mathbb{N}^e)\), where the log structure is given by \( c_i \to T_i \). There is the canonical log Frobenius lift \( (\text{Spec}(W_m(R)[T_1, \ldots, T_n], P, \Phi_m, \Delta_m)_m \) of \((Y, D)\). Since the pre-log ring \((A, P)\) is log smooth over \( R \), the comparison morphism of §4.1 becomes the map \( \Lambda^{\bullet}_{(A_m, P)/W_m(R)} \to W_m \Lambda^{\bullet}_{(A, P)/R} \). Hence the first claim follows from Theorem 7.1. The proof of the second claim is similar to that of Theorem 2.11. \[ \square \]

7.2. **Semistable case.** We prove the comparison theorem for semistable log schemes. Let \((Y, \mathcal{M})\) be a log scheme over \( S = \text{Spec}(R, \mathbb{N})\) of the following type:

Étale locally on \( Y \), the structure morphism \( Y \to S \) has a decomposition 
\[ Y \to \text{Spec}(A = R[T_1, \ldots, T_n]/(T_1 \cdots T_d), P = \mathbb{N}^e \oplus \mathbb{N}^f) \] with \( u \) exact and étale (in the usual sense), \( 1 \leq d \leq e \leq n, f \geq 0, \) \((A, P)\) is the pre-log ring we discussed in §4.2, and \( \delta \) is induced by the diagonal map \( N \to N^e \oplus N^f \). Obviously, semistable log schemes over \( S \) satisfy this condition. Set
\[ W_m \tilde{\Lambda}^{\bullet} = W_m \Lambda^{\bullet}_{(R(u))}, \ W_m \Lambda^{\bullet} = W_m \Lambda^{\bullet}_{(R(N))}. \]

Let \( t_m \in W_m(\mathcal{M}) \) be the image of the base of \( N \) under the morphism \( N \to \mathcal{M} \to W_m(\mathcal{M}) \) and \( \theta_m = d \log t_m \in W_m \tilde{\Lambda}^1 \).

**Lemma 7.3.** We have the following exact sequence:
\[ 0 \to W_m(\mathcal{O}_Y) = W_m \tilde{\Lambda}^0 \xrightarrow{\lambda \theta_m} W_m \tilde{\Lambda}^1 \xrightarrow{\lambda \theta_m} W_m \tilde{\Lambda}^2 \xrightarrow{\lambda \theta_m} \cdots. \]
Proof. It is easy to see that this is a chain complex. Since the question is local and using the étale base change property, we can assume that

\[ Y = \text{Spec}(A = R[T_1, \ldots, T_n]/(T_1 \cdots T_d), P = \mathbb{N}_e \oplus \mathbb{N}_f). \]

By Proposition 4.7 there exists a decomposition

\[ W_m \Lambda^\bullet(A,P)/R = WC^\bullet(A,P)/R \oplus WC^\bullet(A,P)/R. \]

Define \( e : W_m \Lambda^\bullet(A,P)/R \to W_m \Lambda^\bullet(A,P)/R \) by \( (a, b \land d \log X) \mapsto b \). It is easy to see \((\land \theta_m) \circ e + e \circ (\land \theta_m) = \text{id}\).

\[ \square \]

**Lemma 7.4.** The canonical morphism \( W_m \tilde{\Lambda}^\bullet \to W_m \Lambda^\bullet \) induces an isomorphism

\[ W_m \tilde{\Lambda}^\bullet/(W_m \tilde{\Lambda}^\bullet - 1 \land \theta_m) \simeq W_m \Lambda^\bullet. \]

We have an exact sequence:

\[ 0 \to W_m \Lambda^\bullet - 1 \xrightarrow{\land \theta_m} W_m \tilde{\Lambda}^\bullet \to W_m \Lambda^\bullet \to 0. \]

Proof. It can be easily seen that the surjective morphism \( W_m \tilde{\Lambda}^\bullet \to W_m \Lambda^\bullet \) factors \( W_m \tilde{\Lambda}^\bullet/(W_m \tilde{\Lambda}^\bullet - 1 \land \theta_m) \) and \( \{ W_m \tilde{\Lambda}^\bullet/(W_m \tilde{\Lambda}^\bullet - 1 \land \theta_m) \} \) is an exact sequence over \((Y, \mathcal{M}_Y)/(R, \mathbb{N})\). This implies the canonical surjective map

\[ \Lambda^\bullet_{W_m}(Y, \mathcal{M}_Y)/W_m(R, \mathbb{N}) \to W_m \tilde{\Lambda}^\bullet/(W_m \tilde{\Lambda}^\bullet - 1 \land \theta_m) \]

factors \( \Lambda^\bullet_{W_m}(Y, \mathcal{M}_Y)/W_m(R, \mathbb{N}) \).

Let \( \{ E'_m \} \) be any log \( F \)-V-procomplex over \((Y, \mathcal{M})/(R, \mathbb{N})\). There is a morphism \( \{ W_m \tilde{\Lambda}^\bullet/(W_m \tilde{\Lambda}^\bullet - 1 \land \theta_m) \} \to \{ E'_m \} \) of log \( F \)-V-procomplexes obtained by the composition

\[ \{ W_m \tilde{\Lambda}^\bullet/(W_m \tilde{\Lambda}^\bullet - 1 \land \theta_m) \} \to \{ W_m \Lambda^\bullet \} \to \{ E'_m \}, \]

where the second arrow is induced by the universal property of \( \{ W_m \Lambda^\bullet \} \). Moreover, it is unique morphism that fits into the following diagram

\[ \Lambda^\bullet_{W_m}(Y, \mathcal{M})/W_m(R, \mathbb{N}) \]

\[ \{ W_m \tilde{\Lambda}^\bullet/(W_m \tilde{\Lambda}^\bullet - 1 \land \theta_m) \} \]

\[ \{ E'_m \} \]

because the top arrow is surjective. Hence we proved \( \{ W_m \tilde{\Lambda}^\bullet/(W_m \tilde{\Lambda}^\bullet - 1 \land \theta_m) \} \) has the universal property and \( \{ W_m \tilde{\Lambda}^\bullet/(W_m \tilde{\Lambda}^\bullet - 1 \land \theta_m) \} \to \{ W_m \Lambda^\bullet \} \) is an isomorphism. The second claim follows from the isomorphism and Lemma 7.3.

\[ \square \]

Let \( X = X_{d,e,n,f} := \text{Spec}(R[T_1, \ldots, T_n]/(T_1 \cdots T_d), \mathbb{N}_e \oplus \mathbb{N}_f) \) be the log scheme corresponding to the pre-log ring of \( \mathcal{O}_X \) over \( S = \text{Spec}(R, \mathbb{N}) \) for \( 1 \leq d \leq e \leq n, f \geq 0 \). Consider the closed subschemes \( Z_1 := V(T_1 \cdots T_{d-1}) \) and \( Z_2 := V(T_d) \) and \( Z = Z_1 \cap Z_2 \) of \( X \) endowed with the inverse image log structure of \( X \). We find \( Z_1 \simeq X_{d-1,e,n,f}, Z_2 \simeq X_{d-1,e-1,n-1,f+1} \). If \( I_i := \ker(O_X \to O_{Z_i}) \) \((i = 1, 2)\), we see \( I_1 + I_2 = \ker(O_X \to O_Z) \) and \( I_1 \cap I_2 = 0 \). Then we get the following exact sequence of \( O_X \)-modules:

\[ 0 \to O_X \to O_X/I_1 \oplus O_X/I_2 \to O_X/(I_1 + I_2) \to 0. \]

The log differential sheaf \( \Lambda^1_{X/S} \) is a free \( O_X \)-module because \( \Lambda^1_{X/S} \) is the quotient of

\[ \bigoplus_{i=1}^e O_X d \log T_i \oplus \bigoplus_{i=e+1}^n O_X d T_i \oplus \bigoplus_{i=1}^f O_X d \log c_i. \]
divided by the submodule generated by \(d \log T_1 + \cdots + d \log T_e + d \log c_1 + \cdots + d \log c_f\).

Hence we obtain the exact sequence
\[
0 \to \Lambda^\bullet_{X/S} \to (\mathcal{O}_X/I_1 \otimes_{\mathcal{O}_X} \Lambda^\bullet_{X/S}) \oplus (\mathcal{O}_X/I_2 \otimes_{\mathcal{O}_X} \Lambda^\bullet_{X/S}) \\
\rightarrow \mathcal{O}_X/(I_1 + I_2) \otimes_{\mathcal{O}_X} \Lambda^\bullet_{X/S} \to 0.
\]

Since closed immersions \(Z_i \hookrightarrow X\) (\(i = 1, 2\)) and \(Z \hookrightarrow X\) are exact closed immersions defined by a coherent sheaf of ideals of \(\mathcal{M}_X\), canonical morphisms \(\mathcal{O}_X/I_i \otimes_{\mathcal{O}_X} \Lambda^\bullet_{X/S} \to \Lambda^\bullet_{Z_i/S}\) (\(i = 1, 2\)) and \(\mathcal{O}_X/(I_1 + I_2) \otimes_{\mathcal{O}_X} \Lambda^\bullet_{X/S} \to \Lambda^\bullet_{Z/S}\) are isomorphisms (Corollary 2.3.3 of [Ogu06]). So we obtain the exact sequence
\[
0 \to \Lambda^\bullet_{X/S} \to \Lambda^\bullet_{Z_1/S} \oplus \Lambda^\bullet_{Z_2/S} \to \Lambda^\bullet_{Z/S} \to 0.
\]

We prove the existence of Mayer-Vietoris exact sequences for the de Rham-Witt complex in semistable cases. We write \(W_m \Lambda^\bullet_{X/(R,+)} = W_m \Lambda^\bullet_{X/S}\), and so on.

Define \(\tilde{K}_i := W_m(I_i)W_m\tilde{\Lambda}^\bullet_{X/S} + dW_m(I_i)W_m\tilde{\Lambda}^\bullet_{X/S}^{-1} \subset W_m\tilde{\Lambda}^\bullet_{X/S}\) (\(i = 1, 2\)). From the fact \(W_m(I_1) + W_m(I_2) = W_m(I_1 + I_2)\), \(\tilde{K}_1 + \tilde{K}_2\) is equal to \(W_m(I_1 + I_2)W_m\tilde{\Lambda}^\bullet_{X/S} + dW_m(I_1 + I_2)W_m\tilde{\Lambda}^\bullet_{X/S}^{-1}\).

Then from Proposition 3.11 (2) we get \(W_m\tilde{\Lambda}^\bullet_{Z_i/S} \cong W_m\tilde{\Lambda}^\bullet_{X/S}/\tilde{K}_i\) (\(i = 1, 2\)) and \(W_m\tilde{\Lambda}^\bullet_{Z/S} \cong W_m\tilde{\Lambda}^\bullet_{X/S}/(\tilde{K}_1 + \tilde{K}_2)\).

**Lemma 7.5.** The following sequence is exact:
\[
0 \to W_m\tilde{\Lambda}^\bullet_{X/S} \to W_m\tilde{\Lambda}^\bullet_{Z_1/S} \oplus W_m\tilde{\Lambda}^\bullet_{Z_2/S} \to W_m\tilde{\Lambda}^\bullet_{Z/S} \to 0.
\]

**Proof.** Since the sequence is identified to
\[
0 \to W_m\Lambda^\bullet_{X/S} \to W_m\tilde{\Lambda}^\bullet_{X/S}/\tilde{K}_1 \oplus W_m\tilde{\Lambda}^\bullet_{X/S}/\tilde{K}_2 \to W_m\tilde{\Lambda}^\bullet_{X/S}/(\tilde{K}_1 + \tilde{K}_2) \to 0,
\]
it suffices to show that the morphism \(W_m\tilde{\Lambda}^\bullet_{X/S} \to W_m\tilde{\Lambda}^\bullet_{Z_1/S} \oplus W_m\tilde{\Lambda}^\bullet_{Z_2/S}\) is injective. Let \(\omega \in W_m\Lambda^\bullet_{X/S}\) be an element of the kernel of this morphism. By Corollary 4.6, we see \(\omega\) is uniquely written as a finite sum of log basic Witt differentials \(\sum_k \epsilon_m(\xi_{k,p,j}, k, \mathcal{P}, J)\), \(\xi_{k,p,j} \in W^{r_{\mu}(k)}_{m,u(k)}(R)\) where \(k\) runs through all weights such that \([1, d] \not\subset \text{Supp}^+ k\), \(p^{m-1} \cdot k^+\) is integral and \(J\) runs through all subsets of \([1, f]\). The image of \(\omega\) in \(W_m\tilde{\Lambda}^\bullet_{Z/S}\) is the sum of log basic Witt differentials \(\epsilon_m(\xi_{k,p,j}, k, \mathcal{P}, J)\) of \(W_m\tilde{\Lambda}^\bullet_{Z_i/S}\) such that \([1, d-1] \not\subset \text{Supp}^+ k\). Similarly, the image in \(W_m\tilde{\Lambda}^\bullet_{Z/S}\) is the sum of log basic Witt differentials \(\epsilon_m(\xi_{k,p,j}, k, \mathcal{P}, J)\) of \(W_m\tilde{\Lambda}^\bullet_{Z_i/S}\) such that \(d \not\in \text{Supp}^+ k\). If we apply Corollary 4.6 again to \(Z_1\) (resp. \(Z_2\)), we get \(\xi_{k,p,j} = 0\) for \(k\) that satisfies \([1, d-1] \not\subset \text{Supp}^+ k\) (resp. \(d \not\in \text{Supp}^+ k\)). We conclude \(\omega = 0\). \(\square\)

**Proposition 7.6.** The following sequence is exact:
\[
0 \to W_m\Lambda^\bullet_{X/S} \to W_m\Lambda^\bullet_{Z_1/S} \oplus W_m\Lambda^\bullet_{Z_2/S} \to W_m\Lambda^\bullet_{Z/S} \to 0.
\]

**Proof.** We prove by induction on the degree. Since we have an exact sequence
\[
0 \to W_m(\mathcal{O}_X) \to W_m(\mathcal{O}_X/I_1) \oplus W_m(\mathcal{O}_X/I_2) \to W_m(\mathcal{O}_X/(I_1 + I_2)) \to 0,
\]
the sequence is exact on degree zero. From Lemma 7.4 and Lemma 7.5, we get the following exact commutative diagram

Using this diagram and nine-lemma, the proposition follows by induction. □

For \( X = \text{Spec}(A = R[T_1, \ldots, T_n]/(T_1 \cdots T_d), P = \mathbb{N}^e \oplus \mathbb{N}^f) \), we set \( X_m \) by

\[ X_m := \text{Spec}(A_{m} := W_m(R)[T_1, \ldots, T_n]/(T_1 \cdots T_d), P). \]

There are morphisms of log schemes \( \Phi_m : X_m \to X_{m+1} \) and \( \Delta_m : W_m(X) \to X_m \) as the case of \( \text{Spec}(R[T_1, \ldots, T_n], \mathbb{N}^e \oplus \mathbb{N}^f) \). The pair \( (X_m, \Delta_m) \) defines the comparison morphism

\[ \Lambda^*_{X_m/S_m} \to W_m\Lambda^*_X/R. \]

If \( d = e \) and \( f = 0 \), the family \( (X_m, \Phi_m, \Delta_m)_m \) is a log Frobenius lift of \( X \) and the comparison morphism coincides with the morphism induced by the log Witt lift \( (X_m, \Delta_m)_m \) in §6.

**Lemma 7.7.** Assume \( d = 1 \). Let \( X_m = \text{Spec}(A_{m} = W_m(R)[T_1, \ldots, T_n]/(T_1), P = \mathbb{N}^e \oplus \mathbb{N}^f) \) be the canonical lift of \( X \) over \( S_m = \text{Spec}(W_m(R), \mathbb{N}) \). Then the comparison morphism

\[ \Lambda^*_{X_m/S_m} \to W_m\Lambda^*_X/S \]

is a quasi-isomorphism.

**Proof.** First, we consider two log structures on a ring \( A = R[T_1, \ldots, T_n]/(T_1) \) over \( (R, \mathbb{N}) \). The one is defined by

\[ \mathbb{N}^e \oplus \mathbb{N}^f \to A : \mathbb{N}^e \ni e_1 \mapsto 0, \ e_i \mapsto T_i \ (i \neq 1), \ \mathbb{N}^f \ni c_i \mapsto 0 \]

and the diagonal morphism \( \mathbb{N} \to \mathbb{N}^e \oplus \mathbb{N}^f \).

The other one is given by a diagram

\[
\begin{array}{ccc}
Q := \mathbb{N}^e \oplus \mathbb{N}^{e-1} \oplus \mathbb{N}^f & \to & A \\
\downarrow & & \downarrow \\
1 & \to & R,
\end{array}
\]

where the upper horizontal morphism is induced by

\[
(0, e_i, 0) \mapsto T_i, \ (1, 0, 0) \mapsto 1, \ (0, 0, c_i) \mapsto 0.
\]
The morphism of monoids \( Q \to N^e \oplus N^f \) defined by
\[
\begin{align*}
(1, 0, 0) & \mapsto (1, (1, \ldots, 1), (1, \ldots, 1)), \\
(0, e_i, 0) & \mapsto (0, e_i, 0), \quad 1 \leq i \leq e - 1, \\
(0, 0, c_i) & \mapsto (0, 0, c_i), \quad 1 \leq i \leq f
\end{align*}
\]
gives a map \( (A, Q) \to (A, N^e \oplus N^f) \) of pre-log rings over \((R, N)\).

There is also another log structure on \( A \) over \((R, \{\ast\})\), which is given by \( N^{e-1} \oplus N^f \to A \) that sends \( e_i \) to \( T_i \) for \( 1 \leq i \leq e - 1 \). We get a diagram
\[
\begin{array}{ccc}
(A, N^{e-1} \oplus N^f) & \to & (A, Q) \\
 & \downarrow & \downarrow \\
(R, \{\ast\}) & \to & (R, N)
\end{array}
\]
We also have a similar diagram for \( A_m = W_m(R)[T_1, \ldots, T_n]/(T_1) \).

They induce a diagram:
\[
\begin{array}{ccc}
\Lambda^\bullet_{(A_m, N^{e-1} \oplus N^f)/(W_m(R), \{\ast\})} & \xrightarrow{\alpha_1} & \Lambda^\bullet_{(A_m, Q)/(W_m(R), N)} \\
\downarrow & & \downarrow \\
W_m\Lambda^\bullet_{(A, N^{e-1} \oplus N^f)/(R, \{\ast\})} & \xrightarrow{\beta_1} & W_m\Lambda^\bullet_{(A, Q)/(R, N)} \\
\downarrow & & \downarrow \\
W_m\Lambda^\bullet_{(A, N^e \oplus N^f)/(R, N)} & \xrightarrow{\beta_2} & W_m\Lambda^\bullet_{(A, N^e \oplus N^f)/(R, N)}
\end{array}
\]
It is easy to see that \( \alpha_1 \) is an isomorphism. \( \alpha_2 \) is also an isomorphism because the canonical morphism \( Q^{SP} \to \mathbb{Z}^e \oplus \mathbb{Z}^f \) induced by \( Q \to N^e \oplus N^f \) is an isomorphism. We also have isomorphisms
\[
\Lambda^\bullet_{(W_m(A), N^{e-1} \oplus N^f)/(W_m(R), \{\ast\})} \cong \Lambda^\bullet_{W_m(A, Q)/(W_m(R), N)} \cong \Lambda^\bullet_{W_m(A, N^e \oplus N^f)/(W_m(R), N)}
\]
by the same reason.

By the construction of the log de Rham-Witt complexes, \( \beta_1 \) and \( \beta_2 \) are also isomorphisms. Hence we only have to show
\[
\Lambda^\bullet_{(A_m, N^{e-1} \oplus N^f)/(W_m(R), \{\ast\})} \to W_m\Lambda^\bullet_{(A, N^{e-1} \oplus N^f)/(R, \{\ast\})}
\]
is a quasi-isomorphism, but this is Theorem 7.2. \( \square \)

**Theorem 7.8.** Let \( X_m = \text{Spec}(A_m = W_m(R)[T_1, \ldots, T_n]/(T_1 \cdots T_d), P = N^e \oplus N^f) \) be the canonical lift of \( X \) over \( S_m = \text{Spec}(W_m(R), N) \). Then the comparison morphism
\[
\Lambda^\bullet_{X_m/S_m} \to W_m\Lambda^\bullet_{X/S}
\]
is a quasi-isomorphism.

**Proof.** The comparison morphisms are compatible with the Mayer-Vietoris sequence, i.e., the following diagram commutes:
\[
\begin{array}{ccc}
0 & \to & \Lambda^\bullet_{X_m/S_m} \\
\downarrow & & \downarrow \\
0 & \to & W_m\Lambda^\bullet_{X_m/S}
\end{array}
\]
Consider the long exact sequences of hypercohomology and using descending induction, it suffices to show when \( d = 1 \) and it follows from Lemma 7.7. \( \square \)
Theorem 7.9. Let $(Y, \mathcal{M})$ be a semistable log scheme over the pre-log ring $(R, N)$. Then the canonical homomorphism
\[ \mathbb{R}u_{(Y, \mathcal{M})/W_m(R, N)}^\bullet \mathcal{O}_{(Y, \mathcal{M})/W_m(R, N)} \to W_m\mathcal{A}^\bullet_{(Y, \mathcal{M})/(R, N)} \]
is an isomorphism in $D^+(Y, W_m(R))$. Moreover, if $R$ is Noetherian and $Y$ is proper over $R$, we have a canonical isomorphism
\[ H^\bullet_{\log-crys}((Y, \mathcal{M})/W(R, N)) \to H^\bullet_{\acute{e}t}(Y, W\mathcal{A}^\bullet_{(Y, \mathcal{M})/(R, N)}). \]
Proof. This follows from Theorem 7.8 by the similar proof to that of Theorem 7.2. □

8. Weight spectral sequence and its degeneration for semistable schemes

In this section, we define the $p$-adic Steenbrink complex for proper strictly semistable log schemes. First we recall some facts about topology of log structures.

8.1. Topology of log structure. We recall some facts about the topology of log structures ([Shi02] §1.1, [Ols03] Appendix).

Definition 8.1. ([Shi02] Definition 1.1.1) A fine log scheme $(X, \mathcal{M})$ is said to be of Zariski type if there exists an open covering $X = \bigcup_i X_i$ with respect to the Zariski topology such that each $(X_i, \mathcal{M}|_{X_i})$ admits a chart.

Remark 8.2. If $X$ is a smooth scheme with simple normal crossing divisor $D$, the log scheme $(X, D)$ is a fine log scheme of Zariski type. A strictly semistable log scheme is also a fine log scheme of Zariski type.

Let $X$ be a scheme and $\tau : X_{\acute{e}t} \to X_{\text{Zar}}$ be the canonical morphism of topoi. For a log structure $(\mathcal{M}, \alpha)$ on $X$, the log structure $(\tau_*\mathcal{M}, \tau_*\alpha)$ with respect to the Zariski topology on $X$ is defined by
\[ \tau_*\alpha : \tau_*\mathcal{M} \xrightarrow{\tau_*\alpha} \tau_*\mathcal{O}_{X_{\acute{e}t}} = \mathcal{O}_{X_{\text{Zar}}}. \]

Conversely, for a log structure $(\mathcal{M}', \alpha')$ with respect to the Zariski topology, we define the log structure $(\tau^*\mathcal{M}', \tau^*\alpha')$ on $X$ as the associated log structure to the pre-log structure
\[ \tau^{-1}\mathcal{M}' \xrightarrow{\tau^{-1}\alpha'} \tau^{-1}\mathcal{O}_{X_{\text{Zar}}} \to \mathcal{O}_{X_{\acute{e}t}}. \]

The pair $(\tau_*, \tau^*)$ induces an equivalence of categories
\[ \left( \text{Fine log schemes of Zariski type} \right) \simeq \left( \text{Fine log schemes with respect to the Zariski topology} \right) \]
([Shi02] Corollary 1.1.11, [Ols03] Theorem A.1).

Remark 8.3. Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a morphism of fine log schemes with respect to the Zariski topology. Then for $m \geq 1$, there is a quasi-coherent sheaf $W_m\mathcal{A}^\bullet_{(X, \mathcal{M})/(Y, \mathcal{N})}$ on $X_{\text{Zar}}$ that satisfies the following condition: If there is a commutative diagram
\[ U = \text{Spec } S' \xrightarrow{f} V = \text{Spec } R' \]
\[ X \xrightarrow{f} Y, \]
where vertical arrows are open immersions and there is a chart $(Q \to \mathcal{M}|_U, P \to \mathcal{N}|_V, P \to Q)$ of the morphism $(U, \mathcal{M}|_U) \to (V, \mathcal{N}|_V)$. Then we have a canonical isomorphism
\[ \Gamma(U, W_m\mathcal{A}^\bullet_{(X, \mathcal{M})/(Y, \mathcal{N})}) \simeq W_m\mathcal{A}^\bullet_{(S', Q)/(R', P)}. \]
This follows from the similar argument to that of Proposition-Definition 3.10.

We see that $W_m\Lambda^*_{(X,M)/(Y,N)}$ is equal to the complex of étale sheaves defined by

$$U \mapsto \Gamma(U, (W_m(\pi))^*(W_m\Delta^*_{(X,M)/(Y,N)}))$$

for any object $\pi : U \to X$ in $X_{\text{ét}}$ by Proposition 3.7. Hence

$$\mathbb{H}^q_{\text{Zar}}(X, W_m\Delta^*_{(X,M)/(Y,N)}) \simeq \mathbb{H}^q_{\text{ét}}(X, W_m\Lambda^*_{(X,M)/(Y,N)})$$

by [Fu11] Proposition 5.7.5.

In §8, we consider strictly semistable log schemes as fine log schemes with respect to the Zariski topology. By abuse of notation, we write $W_m\Lambda^*_{(X,M)/(Y,N)}$ for the log de Rham-Witt complex with respect to the Zariski topology. We use the setting of §7.2.

8.2. Poincaré residue map. Let $Y$ be a proper strictly semistable log scheme over $S = \text{Spec}(R, \mathbb{N})$ such that $R$ is Noetherian and $Y_1, \ldots, Y_d$ its irreducible components. For a subset $J = \{\alpha_1, \ldots, \alpha_j\}$ of $[1, d]$, we set $Y_J := Y_{\alpha_1} \cap \cdots \cap Y_{\alpha_j}$. We give a filtration $P_j$ of $W_m\Lambda^*$ by

$$P_j W_m\Lambda^* := \text{image}(W_m\tilde{\Lambda}^j \otimes_{W_m(\mathcal{O}_Y)} W_m\Omega^*_{Y/R} \to W_m\tilde{\Lambda}^j),$$

where $W_m\Omega^*_{Y/R}$ denotes the (classical) de Rham-Witt complex defined in [LZ04]. We first define a map

$$\tilde{\rho}_J : W_m\Omega^*_{Y/R} \to \text{Gr}_j W_m\Lambda^* := P_j W_m\Lambda^*/P_{j-1} W_m\Lambda^*$$

by $\omega \mapsto \omega \wedge d \log[T_{\alpha_1}] \wedge \cdots \wedge d \log[T_{\alpha_j}]$, where $T_1, \ldots, T_n$ are local coordinates of $Y$ such that each $Y_i$ corresponds to $T_i = 0$. One can show this map is independent of the choice of local coordinates by a similar proof to [Del70] (3.5). Let $I_J$ be the ideal of $\mathcal{O}_Y$ corresponding to the closed immersion $i_J : Y_J \hookrightarrow Y$. We would like to show that $\tilde{\rho}_J$ factors through $i_J^* W_m\Omega^*_{Y/R}$. For this, it suffices to show that $\tilde{\rho}_J(\omega a) = \tilde{\rho}_J(da \wedge \omega) = 0$ for any $a \in W_m(I_J)$ and $\omega \in W_m\Omega^*_{Y/R}$. Any $a \in W_m(I_J)$ can be written as a finite sum:

$$a = \sum_{i=0}^{m-1} V^i (c_1^{(i)} T_{\alpha_1} + \cdots + c_j^{(i)} T_{\alpha_j}),$$

where $c_i^{(i)} \in \mathcal{O}_Y$. Hence we can assume $a = V^i (c_1 T_{\alpha_1} + \cdots + c_j T_{\alpha_j})$. By [DLZ12] Proposition 2.23, we have an expression

$$[c_1 T_{\alpha_1} + \cdots + c_j T_{\alpha_j}] = \sum_{k: \text{weight} | k = 1} \beta_k [T_{\alpha_1}]^{k_1} \cdots [T_{\alpha_j}]^{k_j}, \beta_k \in V^{V(k)} W_m(R),$$

where $k : [1, j] \to \mathbb{Z}_{\geq 0}[1/p]$ runs through weights such that $|k| = k_1 + \cdots + k_j = 1$. For a weight $k$, let $u(k)$ denote the least nonnegative integer such that $p^u(k) k$ is integral. If $\beta = V^{V(k)} \eta$, the expression $\beta [T_{\alpha_1}]^{k_1} \cdots [T_{\alpha_j}]^{k_j}$ means

$$V^{V(k)} (\eta [T_{\alpha_1}]^{p^u(k) k_1} \cdots [T_{\alpha_j}]^{p^u(k) k_j}).$$

Note that $p^{u(k)} k_1, \ldots, p^{u(k)} k_j \in \mathbb{Z}_{\geq 0}$.

Without loss of generality, we may assume that $a = V^i (\eta [T_{\alpha_1}]^{l_1} \cdots [T_{\alpha_j}]^{l_j})$ with $\eta \in W_m(R), t, l_1, \ldots, l_j$ nonnegative integers such that at least one of $l_1, \ldots, l_j$ is
positive. We have
\[ (V^j(\eta[T_{\alpha_1}]_{\l_1} \cdots [T_{\alpha_j}]_{\l_j}) \cdot \omega) \land d \log[T_{\alpha_1}] \land \cdots \land d \log[T_{\alpha_j}] = V^j(\eta[T_{\alpha_1}]_{\l_1} \cdots [T_{\alpha_j}]_{\l_j} \cdot F^j \omega \land d \log[T_{\alpha_1}] \land \cdots \land d \log[T_{\alpha_j}]) = 0 \]
since at least one of \( l_1, \ldots, l_j \) is positive. Hence we see \( \tilde{\rho}_j(\omega \land \omega) = 0 \). Similarly we see \( \tilde{\rho}_j(\omega \land \omega) = 0 \) because
\[ (dV^j(\eta[T_{\alpha_1}]_{\l_1} \cdots [T_{\alpha_j}]_{\l_j}) \land \omega) \land d \log[T_{\alpha_1}] \land \cdots \land d \log[T_{\alpha_j}] = dV^j(\eta[T_{\alpha_1}]_{\l_1} \cdots [T_{\alpha_j}]_{\l_j} \land F^j \omega \land d \log[T_{\alpha_1}] \land \cdots \land d \log[T_{\alpha_j}]) = 0. \]
Hence \( \tilde{\rho}_j \) induces the map \( \rho_j : i_j_* W_m \Omega^{\bullet-j}_{Y_j/R} \to Gr_j W_m \tilde{\Lambda}^\bullet \).

Let \( Y^{(j)} \) be \( \text{Spec}(R[Y_1, \ldots, Y_{n-j}]) \) and \( i^{(j)}_*: Y^{(j)} \to Y \) the canonical map. From the collection of maps \( \{\rho_j\}_{|j|=j} \) we obtain a map \( i^{(j)}_* W_m \Omega^{\bullet-j}_{Y^{(j)}/R} \to Gr_j W_m \tilde{\Lambda}^\bullet \). We sometimes drop \( i^{(j)}_* \) when there is no risk of confusion.

**Lemma 8.4.** \( i^{(j)}_* W_m \Omega^{\bullet-j}_{Y^{(j)}/R} \to Gr_j W_m \tilde{\Lambda}^\bullet \) is an isomorphism. We call the inverse isomorphism of this map Poincaré residue isomorphism \( \text{Res} : Gr_j W_m \tilde{\Lambda}^\bullet \cong i^{(j)}_* W_m \Omega^{\bullet-j}_{Y^{(j)}/R} \).

**Proof.** Without loss of generality, we can assume \( S = \text{Spec} R \) and
\[ Y = \text{Spec}(R[T_1, \ldots, T_n]/(T_1 \cdots T_d), \mathbb{N}^d). \]
In this case, we find
\[ Y_j = \text{Spec}(R[T_1, \ldots, \tilde{T}_{\alpha_1}, \ldots, \tilde{T}_{\alpha_j}, \ldots, T_n]) \]
is the spectrum of a polynomial ring. On the other hand, an element of \( Gr_j W_m \tilde{\Lambda}^\bullet \) has a unique expression as a sum of basic Witt differentials with \( |I_{-\infty}| = j \). We already know the basic Witt differentials on the left hand side ([LZ04] §2.2). Comparing the basic Witt differential on both sides, the claim follows.

### 8.3. Weight spectral sequence

We are ready to construct the weight filtration of a strictly semistable log scheme. Put \( W_m A^\Omega := W_m \tilde{\Lambda}^{i+j+1} / P_j W_m \tilde{\Lambda}^{i+j+1} \).

**Lemma 8.5.** There exists the following exact sequences:
\[ 0 \to W_m A \xrightarrow{\theta_m} W_m A^0 \xrightarrow{(-1)^i \theta_m} W_m A^i \xrightarrow{(-1)^i \theta_m} \cdots. \]

**Proof.** It suffice to show the exactness of the following sequence (cf. [Mok93] Proposition 3.15):
\[ W_m \tilde{\Lambda}^{i-1} \xrightarrow{\theta_m} W_m \tilde{\Lambda}^i \xrightarrow{(-1)^i \theta_m} W_m \tilde{\Lambda}^{i+1} / P_0 W_m \tilde{\Lambda}^{i+1} \xrightarrow{(-1)^i \theta_m} W_m \tilde{\Lambda}^{i+2} / P_1 W_m \tilde{\Lambda}^{i+2} \xrightarrow{(-1)^i \theta_m} \cdots. \]
We deduce the exactness of this complex by a similar argument to that in Lemma 7.3.

We consider \( W_m A^{\bullet\bullet} \) as a double complex by
\[ W_m A^{i,j+1} \xrightarrow{(-1)^j \theta_m} W_m A^{i,j} \xrightarrow{(-1)^i \theta_m} W_m A^{i+1,j} \]
(cf. [Nak05] (2.2.1,*)). Consider a simple complex

\[ W_{m}A^{•} := (\cdots \xrightarrow{(-1)^{j}θ_{m} \wedge} W_{m}\tilde{Λ}^{i+j+1}/P_{j}W_{m}\tilde{Λ}^{i+j+1} \xrightarrow{(-1)^{j}θ_{m} \wedge} \cdots )_{j \geq 0}, \]

and define a weight filtration on this complex by

\[ P_{k}W_{m}A^{•} := (\cdots \xrightarrow{(-1)^{j}θ_{m} \wedge} (P_{j+k+1} + P_{j})(W_{m}\tilde{Λ}^{i+j+1}/P_{j}W_{m}\tilde{Λ}^{i+j+1} \xrightarrow{(-1)^{j}θ_{m} \wedge} \cdots )_{j \geq 0}. \]

If we ignore the compatibility with the Frobenius, we obtain an isomorphism

\[ Gr_{k}W_{m}A^{•} = \bigoplus_{j \geq \max\{-k,0\}} \bigoplus_{j \geq \max\{-k,0\}} Gr_{2j+k+1}W_{m}\tilde{Λ}^{i+j+1}\{-j\}, \]

Hence we get a spectral sequence:

\[ E_{1}^{k,h,k} = \bigoplus_{j \geq \max\{-k,0\}} \bigoplus_{j \geq \max\{-k,0\}} H_{h}^{2j-k}(Y^{(2j+k+1)},W_{m}\tilde{Λ}^{i+j+1}/R) \]

\[ \Rightarrow H_{h}^{2j}(Y,W_{m}\tilde{Λ}^{i+j+1}/(R,N)). \]

We would like to construct a spectral sequence also for the non-truncated de Rham-Witt cohomology. The canonical projection map \( π : W_{m+1}\tilde{Λ}^{•} \to W_{m}\tilde{Λ}^{•} \) satisfies \( π(P_{j}W_{m+1}\tilde{Λ}^{•}) \subset P_{j}W_{m}\tilde{Λ}^{•} \). Then \( π \) induces the map

\[ π : W_{m+1}A^{ij} \to W_{m}A^{ij} \]

and there exist two commutative diagrams

\[ \xymatrix{ W_{m+1}A^{i,j+1} \ar[r]^{π} \ar[d]_{(-1)^{j}θ_{m+1} \wedge} & W_{m}A^{i,j+1} \ar[d]^{(-1)^{j}θ_{m} \wedge (-1)^{j+1}d} \ar[r]^{π} & W_{m}A^{i,j} \ar[d]^{(-1)^{j+1}d} \ar[r] & W_{m+1}A^{i,j+1} \ar[d]_{(-1)^{j}θ_{m} \wedge} \ar[r]^{π} & W_{m}A^{i,j+1} \ar[d]^{(-1)^{j+1}d} \ar[r] & W_{m+1}A^{i,j+1} \ar[r]^{π} & W_{m}A^{i,j+1} } \]

Therefore we get a morphism of double complexes

\[ π : W_{m+1}A^{••} \to W_{m}A^{••}. \]

For any nonnegative integer \( k \), the projection morphism \( π : P_{k}W_{m+1}\tilde{Λ}^{•} \to P_{k}W_{m}\tilde{Λ}^{•} \) is surjective by definition. We know \( W_{m}\tilde{Λ}^{i} \) is a coherent sheaf of \( W_{m}(O_{Y}) \)-module and there is an exact sequence

\[ 0 \to P_{k-1}W_{m}\tilde{Λ}^{i} \to P_{k}W_{m}\tilde{Λ}^{i} \xrightarrow{\text{Res}} W_{m+1}\tilde{Λ}^{i-k}_{Y(k)/R} \to 0. \]

From this one sees that \( P_{k}W_{m}\tilde{Λ}^{i} \) is a quasi-coherent sheaf of \( W_{m}(O_{Y}) \)-modules for each \( k \) by induction. Moreover, there exists the following commutative diagram with exact rows:

\[ \xymatrix{ 0 & P_{k-1}W_{m+1}\tilde{Λ}^{i} \ar[r] & P_{k}W_{m+1}\tilde{Λ}^{i} \ar[r]^{\text{Res}} & W_{m+1}\tilde{Λ}^{i-k}_{Y(k)/R} \ar[r] & 0 \\
0 & P_{k-1}W_{m}\tilde{Λ}^{i} \ar[r] & P_{k}W_{m}\tilde{Λ}^{i} \ar[r]^{\text{Res}} & W_{m+1}\tilde{Λ}^{i-k}_{Y(k)/R} \ar[r] & 0. } \]

This exact sequence and the fact that \( \{ P_{k}W_{m}\tilde{Λ}^{•}\}_{m} \) satisfy the Mittag-Leffler condition show that the sequence

\[ 0 \to P_{k}W\tilde{Λ}^{•} \to P_{k+1}W\tilde{Λ}^{•} \xrightarrow{\text{Res}} W\tilde{Λ}^{•}_{(k+1)/R}\{ -k - 1 \} \to 0 \]
is exact. The weight spectral sequence (we ignore Frobenius action)
\[ E_1^{-k,h+k} = \bigoplus_{j \geq \max\{-k,0\}} \mathbb{H}^{h-2j-k}_{\text{Zar}}(Y^{(2j+k+1)}, W\Omega^*_{Y^{(2j+k+1)}/R}) \Rightarrow \mathbb{H}^h_{\text{Zar}}(Y, W\Lambda^*_Y/(\mathbb{N})) \]
is deduced from this exact sequence.

8.4. **Frobenius compatibility.** In this subsection we discuss about the Frobenius compatibility of the spectral sequence we constructed in the last subsection.

**Lemma 8.6.** (cf. [Mok93] Proposition 4.12, [Nak05] (10.1.16)) Let \( j \) be a nonnegative integer. For \( 1 \leq q \leq j + 1 \), \( i^{(q)} : Y^{(j+1)} \rightarrow Y^{(j)} \) denotes different closed immersions, and \( \rho^{(q)}_m : i^{(j)}_*W_m\Omega^*_{Y^{(j)}/R} \rightarrow i^{(j+1)}_*W_m\Omega^*_{Y^{(j+1)}/R} \) be a morphism induced by \( i^{(q)} \). We set \( \rho_m := \sum_{q=1}^{j+1}(-1)^{q+1} \rho^{(q)}_m \). Then there is the following commutative diagram:

\[
\begin{array}{ccc}
\text{Gr}_j W_m \tilde{\Lambda}^* & \xrightarrow{\theta_m \wedge} & \text{Gr}_{j+1} W_m \tilde{\Lambda}^{*+1} \\
\downarrow \cong \text{Res} & & \downarrow \cong \text{Res} \\
W_m \Omega^*_{Y^{(j)}/R} & \xrightarrow{(-1)^{j+1-q} \rho_m} & W_m \Omega^*_{Y^{(j+1)}/R}.
\end{array}
\]

**Proof.** Since we can check the commutativity locally, we may work on \( Y_J \) for some \( J = \{ \alpha_1, \ldots, \alpha_{j+1} \} \). For \( 1 \leq q \leq j + 1 \), let \( J_q = \{ \alpha_1, \ldots, \alpha_q, \ldots, \alpha_{j+1} \} \). The claim follows from the commutativity of the following diagram:

\[
\begin{array}{ccc}
\text{Gr}_j W_m \tilde{\Lambda}^* & \xrightarrow{\theta_m \wedge} & \text{Gr}_{j+1} W_m \tilde{\Lambda}^{*+1} \\
\downarrow & & \downarrow \\
W_m \Omega^*_{Y^{(j)}/R} & \xrightarrow{(-1)^{j+1-q} \rho_m} & W_m \Omega^*_{Y^{(j+1)}/R}.
\end{array}
\]

which we can check directly from definitions. \( \square \)

We describe the Frobenius on torsion \( p \)-adic Steenbrink complexes. We mention that for a positive integer \( k \) and nonnegative integer \( j \), the multiplication \( p^k : W_{m+1}\Lambda^j \rightarrow W_{m+1}\Lambda^j \) factors \( p^k : W_m\Lambda^j \rightarrow W_{m+1}\Lambda^j \) since \( p \) annihilates \( \ker(W_{m+1}\Lambda^j \rightarrow W_m\Lambda^j) \) (cf. [Ill79] I.3.4).

**Theorem 8.7.** (cf. [Nak05] Proposition 9.8)

Let \( m, k \) be two positive integers and \( j \) a nonnegative integer.

1. \( p^k F : W_m\Lambda^j \rightarrow W_m\Lambda^j \) is a unique morphism which makes the following diagram commutative:

\[
\begin{array}{ccc}
W_{m+1}\Lambda^j & \xrightarrow{\pi} & W_m\Lambda^j \\
p^k F & & p^k F \\
W_m\Lambda^j & \xrightarrow{p^k F} & W_m\Lambda^j
\end{array}
\]

Furthermore, \( p^k F \) is compatible with \( d \) and \( \pi \), i.e., The following two diagrams commute:

\[
\begin{array}{ccc}
W_m\Lambda^j & \xrightarrow{d} & W_m\Lambda^{j+1} \\
p^k F & & p^k F \\
W_m\Lambda^j & \xrightarrow{p^k F} & W_m\Lambda^{j+1}
\end{array}
\]

\[
\begin{array}{ccc}
W_{m+1}\Lambda^j & \xrightarrow{\pi} & W_m\Lambda^j \\
p^k F & & p^k F \\
W_{m+1}\Lambda^j & \xrightarrow{p^k F} & W_m\Lambda^j
\end{array}
\]
The morphism $p^k F : W_m \Lambda^k \to W_m \Lambda^k$ is equal to the morphism induced by the absolute Frobenius morphism on $Y$.

(2) There is a unique morphism $\tilde{\Phi}_m^{(j)} : W_m A^{0j} \to W_m A^{0j}$ which makes the following diagram commutative:

$$
\begin{array}{c}
W_{m+1} A^{0j} \xrightarrow{\pi} W_m A^{0j} \\
\downarrow F \\
W_m A^{0j}
\end{array}
$$

Furthermore, $\tilde{\Phi}_m^{(j)}$ is compatible with $\theta_m \wedge$ and $\pi$:

$$
\begin{array}{c}
W_m A^{0,j+1} \xrightarrow{\tilde{\Phi}_m^{(j+1)}} W_m A^{0,j+1} \\
\downarrow \theta_m \wedge \\
W_m A^{0j}
\end{array}
\quad
\begin{array}{c}
W_{m+1} A^{0j} \xrightarrow{\tilde{\Phi}_m^{(j+1)}} W_{m+1} A^{0j} \\
\downarrow \pi \\
W_m A^{0j}
\end{array}

(3) $p^k F : W_m \tilde{\Lambda}^j \to W_m \tilde{\Lambda}^j$ is the unique morphism which makes the following diagram commutative:

$$
\begin{array}{c}
W_{m+1} \tilde{\Lambda}^j \xrightarrow{\pi} W_m \tilde{\Lambda}^j \\
\downarrow p^k F \\
W_m \tilde{\Lambda}^j
\end{array}
$$

Furthermore, $p^k F$ is compatible with $d, \theta_m \wedge,$ and $p^k F$ on $W_m \Lambda^j$. In other words, the following diagrams commute:

$$
\begin{array}{c}
W_m \tilde{\Lambda}^j \xrightarrow{d} W_m \tilde{\Lambda}^j+1 \\
\downarrow p^k F \\
W_m \tilde{\Lambda}^j+1
\end{array}
\quad
\begin{array}{c}
W_m \tilde{\Lambda}^j+1 \xrightarrow{p^k F} W_m \tilde{\Lambda}^j+1 \\
\downarrow \theta_m \wedge \\
W_m \tilde{\Lambda}^j
\end{array}
\quad
\begin{array}{c}
W_m \tilde{\Lambda}^j \xrightarrow{d} W_m \tilde{\Lambda}^j+1 \\
\downarrow p^k F \\
W_m \tilde{\Lambda}^j+1
\end{array}
\quad
\begin{array}{c}
W_m \tilde{\Lambda}^j+1 \xrightarrow{p^k F} W_m \tilde{\Lambda}^j+1 \\
\downarrow \theta_m \wedge \\
W_m \tilde{\Lambda}^j
\end{array}

(4) The morphism $p^k F$ on $W_m \tilde{\Lambda}^j$ preserves the weight filtration $P$ on $W_m \tilde{\Lambda}^j$. For an integer $i \geq 1$, $p^k F : W_m \tilde{\Lambda}^{i+j+1} \to W_m \tilde{\Lambda}^{i+j+1}$ induces an endomorphism

$$
p^k F : W_m A^{i*} \to W_m A^{i*}
$$

of complexes.
(5) Let $i$ be a positive integer. The following diagrams commute:

\[ \begin{array}{ccc}
W_{m+1}A^{ij} & \xrightarrow{\pi} & W_m A^{ij} \\
\xrightarrow{p^k F} & & \xrightarrow{p^k F} \\
W_m A^{ij} & & W_m A^{ij}
\end{array} \quad \begin{array}{ccc}
W_m A^{ij} & \xrightarrow{(-1)^{j+1}d} & W_m A^{i+1,j} \\
\xrightarrow{p^k F} & & \xrightarrow{p^k F} \\
W_m A^{i+1,j} & & W_m A^{i+1,j}
\end{array} \]

\[ \begin{array}{ccc}
W_m A^{i,j+1} & \xrightarrow{p^k F} & W_m A^{i,j+1} \\
\xrightarrow{(-1)^j \theta_m \wedge} & & \xrightarrow{(-1)^j \theta_m \wedge} \\
W_m A^{ij} & & W_m A^{ij},
\end{array} \quad \begin{array}{ccc}
W_m A^{ij} & \xrightarrow{p^k F} & W_m A^{ij} \\
\xrightarrow{\pi} & & \xrightarrow{\pi} \\
W_m A^{ij} & & W_m A^{ij}.
\end{array} \]

(6) The following diagram is commutative:

\[ \begin{array}{ccc}
W_m A^{0j} & \xrightarrow{(-1)^{j+1}d} & W_m A^{1j} \\
\xrightarrow{\Phi_m^{(j)}} & & \xrightarrow{p^k F} \\
W_m A^{0j} & & W_m A^{1j}.
\end{array} \]

(7) The following diagram is commutative:

\[ \begin{array}{ccc}
W_m \tilde{\Lambda}^j & \xrightarrow{\tilde{\omega}^k} & W_m \tilde{\Lambda}^j \\
\xrightarrow{W_m \Lambda^j} & & \xrightarrow{W_m \Lambda^j} \\
W_m A^j = W_m \tilde{\Lambda}^j / (\theta_m \wedge W_m \tilde{\Lambda}^j) & & W_m \Lambda^j = W_m \tilde{\Lambda}^j / (\theta_m \wedge W_m \tilde{\Lambda}^j-1).
\end{array} \]

Proof. (1) Uniqueness follows from the surjectivity of $\pi$. Since $p^k = p^k \circ \pi$ and $dF \omega = p^k d\omega$, the diagrams commute. We obtain the compatibility with projections by the compatibility of $\pi$ and $p^k$, $\pi$ and $F$.

(2) The Poincaré residue morphism gives an isomorphism

\[ \text{Res} : W_m A^{0j} \rightarrow W_m (O_{Y^{(j+1)}}) \]

and there is the Frobenius morphism

\[ \Phi_m^{(j)} : W_m (O_{Y^{(j+1)}}) \rightarrow W_m (O_{Y^{(j+1)}}). \]

Define

\[ \tilde{\Phi}_m^{(j)} := \text{Res}^{-1} \circ \Phi_m^{(j)} \circ \text{Res} : W_m A^{0j} \rightarrow W_m A^{0j}. \]

The commutativity of the first diagram is deduced from following commutative diagram:

\[ \begin{array}{ccc}
W_{m+1} A^{0j} & \xrightarrow{\pi} & W_m A^{0j} \\
\xrightarrow{\sim} & & \xrightarrow{\sim} \\
W_m A^{0j} & & W_m A^{0j} \\
\xrightarrow{\sim} & & \xrightarrow{\sim} \\
W_{m+1} (O_{Y^{(j+1)}}) & & W_m (O_{Y^{(j+1)}}).
\end{array} \quad \begin{array}{ccc}
W_m A^{0j} & \xrightarrow{\Phi_m^{(j)}} & W_m A^{0j} \\
\xrightarrow{F} & & \xrightarrow{F} \\
W_m A^{0j} & & W_m (O_{Y^{(j+1)}}) \\
\xrightarrow{\text{Res}} & & \xrightarrow{\text{Res}} \\
W_{m+1} (O_{Y^{(j+1)}}) & & W_m (O_{Y^{(j+1)}}).
\end{array} \]

The commutativity of second diagram follows from Lemma 8.6 and the commutativity of $\rho_m$ and $\Phi_m^{(j)}$. Third case is trivial.

(3) The proof of (3) is the same as that of (1).
(4)(5) Trivial.

(6) Since we know \(\pi\) is surjective and \(\pi\) commutes with \(d\), this follows from the following commutative diagram:

\[
\begin{array}{ccc}
W_{m+1}A^{ij} & \xrightarrow{d} & W_{m+1}A^{1j} \\
\downarrow F & & \downarrow pF \\
W_mA^{ij} & \xrightarrow{d} & W_mA^{1j}.
\end{array}
\]

(7) Trivial. \(\square\)

**Theorem 8.8.** (cf. [Nak05] Theorem 9.9) There exists a unique endomorphism \(\tilde{\Phi}_m^{(\ast\ast)} : W_mA^{\ast\ast} \to W_mA^{\ast\ast}\) of double complexes which makes the following diagram commutative:

\[
\begin{array}{ccc}
W_{m+1}A^{\ast\ast} & \xrightarrow{\pi} & W_mA^{\ast\ast} \\
\downarrow p^*F & & \downarrow \tilde{\Phi}_m^{(\ast\ast)} \\
W_mA^{\ast\ast} & \xrightarrow{\pi} & W_mA^{\ast\ast}.
\end{array}
\]

The endomorphism \(\tilde{\Phi}_m^{(\ast\ast)}\) defines an endomorphism \(\tilde{\Phi}_m : W_mA^{\ast} \to W_mA^{\ast}\) and there is the following commutative diagram:

\[
\begin{array}{ccc}
W_mA^{\ast} & \xrightarrow{\tilde{\Phi}_m} & W_mA^{\ast} \\
\downarrow \theta_m \wedge & & \downarrow \theta_m \wedge \\
W_mA^{\ast} & \xrightarrow{\Phi_m} & W_mA^{\ast},
\end{array}
\]

where \(\Phi_m\) is the endomorphism induced by the absolute Frobenius.

The Poincaré residue isomorphism \(\text{Res}\) induces an isomorphism

\[
\text{Res} : Gr_k W_mA^{\ast} \simeq \bigoplus_{j \geq \max\{-k,0\}} (W_m\Omega^i_{Y(2j+k+1)/R}(-1)^{j+1}d)\{-2j-k\}
\]

which makes the following diagram commutative:

\[
\begin{array}{ccc}
Gr_k W_mA^{\ast} & \xrightarrow{\text{Res}} & \bigoplus_{j \geq \max\{-k,0\}} (W_m\Omega^i_{Y(2j+k+1)/R}(-1)^{j+1}d)\{-2j-k\} \\
\downarrow \tilde{\Phi}_m & & \downarrow p^{j+k}\Phi_m \\
Gr_k W_mA^{\ast} & \xrightarrow{\text{Res}} & \bigoplus_{j \geq \max\{-k,0\}} (W_m\Omega^i_{Y(2j+k+1)/R}(-1)^{j+1}d)\{-2j-k\}.
\end{array}
\]

**Proof.** Define \(\tilde{\Phi}_m^{(\ast\ast)}\) by

\[
\tilde{\Phi}_m^{(\ast\ast)} = \begin{cases} \tilde{\Phi}_m^i & (\ast = 0), \\ p^*F & (\ast \neq 0). \end{cases}
\]

The commutativity of the second diagram follows from the following commutative diagram:

\[
\begin{array}{ccc}
Gr_k W_{m+1}A^{\ast\ast} & \xrightarrow{\text{Res}} & \bigoplus_{j \geq \max\{-k,0\}} W_{m+1}\Omega^{i-j-k}_{Y(2j+k+1)/R}(-j) \\
\downarrow p^*F & & \downarrow p^{j+k}(p^{j-k}F) \\
Gr_k W_mA^{\ast\ast} & \xrightarrow{\text{Res}} & \bigoplus_{j \geq \max\{-k,0\}} W_m\Omega^{i-j-k}_{Y(2j+k+1)/R}(-j),
\end{array}
\]

which immediately follows from the definition of \(\text{Res}\). \(\square\)

By the comparison theorem 7.9, we obtain the following theorem:
Theorem 8.9. (1) There exists the spectral sequences:
\[ E_1^{k,h+k} = \bigoplus_{j \geq \max\{-k,0\}} H_{\text{crys}}^{h-2j-k}(Y^{(2j+k+1)}/W_m(R))(-j-k) \]
\[ \Rightarrow H^h_{\text{log-crys}}(Y/W_m(R,N)). \]

(2) Set
\[ WA^\bullet := \lim_{m} W_m A^\bullet, \quad WA^\bullet := \lim_{m} W_m A^\bullet, \]
\[ \Phi := \lim_{m} \Phi_m : WA^\bullet \to WA^\bullet, \quad \Phi^\bullet := \lim_{m} \Phi_m^\bullet : WA^\bullet \to WA^\bullet \]

Then there exists the spectral sequence:
\[ E_1^{k,h+k} = \bigoplus_{j \geq \max\{-k,0\}} H_{\text{crys}}^{h-2j-k}(Y^{(2j+k+1)}/W(R))(-j-k) \]
\[ \Rightarrow H^h_{\text{log-crys}}(Y/W(R,N)). \]

We call this spectral sequence the p-adic weight spectral sequence.

8.5. Gysin map. In this subsection we describe Gysin maps defined on the de Rham complexes, the de Rham-Witt complexes and the crystalline cohomology, and their relation.

Let \( X \) be a smooth scheme over a scheme \( S \) and \( D \) be a smooth divisor of \( X/S \). The Gysin map of the de Rham complexes \( G^{\text{dR}}_{D/X} : \Omega^\bullet_{D/S}[-1] \to \Omega^\bullet_{X/S} \) in the derived category of sheaves on \( X \) is equal to the boundary morphism of the exact sequence (cf. [Mok93] §4.1)
\[ 0 \to \Omega^\bullet_{X/S} \to \Omega^\bullet_{(X,D)/S} \xrightarrow{\text{Res}} \Omega^\bullet_{D/S}[-1] \to 0. \]

Next, We recall the Gysin map of the crystalline cohomology. Note that the Gysin map in the crystalline cohomology is originally defined by Berthelot [Ber74], but the construction in [NS08] is convenient for our purpose. Let \((S,\gamma)\) be a pd-scheme such that \( p \) is nilpotent in \( S \). Set \( S_0 := \text{Spec}(\mathcal{O}_S/\mathcal{I}) \). Let \( X \) be a smooth scheme over \( S_0 \) and \( D \) a smooth divisor on \( X \) over \( S_0 \). We denote by \( a \) the natural closed immersion \( D \to X \). Let \( a_{\text{Zar}} : (D_{\text{Zar}},\mathcal{O}_D) \to (X_{\text{Zar}},\mathcal{O}_X) \) (resp. \( a_{\text{crys}} : ((D/S)_{\text{crys}},\mathcal{O}_{D/S}) \to ((X/S)_{\text{crys}},\mathcal{O}_{X/S}) \)) be the induced morphism of Zariski ringed topoi (resp. crystalline ringed topoi). The Gysin map of the crystalline cohomology is defined as follows.

Choose an open covering \( X = \bigcup_{i \in I_0} X^{i_0} \) such that there exist a smooth scheme \( Y^{i_0} \) with smooth divisor \( Z^{i_0} \) on \( Y^{i_0} \) over \( S \) and a cartesian diagram
\[
\begin{array}{ccc}
X^{i_0} & \to & Y^{i_0} \\
\downarrow & & \downarrow \\
D|_{X^{i_0}} & \to & Z^{i_0}.
\end{array}
\]

Fix a total order on \( I_0 \) and let \( I \) be a category whose objects are \( \underline{i} = (i_0, \ldots, i_r)'s \) \((i_0 < i_1 < \ldots < i_r, r \in \mathbb{Z}_{\geq 0})\). Set \( \{\underline{i}\} := \{i_0, \ldots, i_r\} \). For two objects \( \underline{i}, \underline{j} \in I \), a morphism from \( \underline{j} \) to \( \underline{i} \) is the inclusion \( \{j\} \hookrightarrow \{i\} \). By abuse of notation, we sometimes write simply \( \underline{i} \) instead of \( \{i\} \).

Set \( D^{\underline{i}} := D|_{X^{i_0}} \). For an object \( \underline{i} = (i_0, \ldots, i_r) \), we set \( X^{\underline{i}} := \bigcap_{s=1}^r X^{i_s}, D^{\underline{i}} := \bigcap_{s=1}^r D^{i_s} \). Then \((X^\bullet, D^\bullet)\) is a diagram of log schemes, i.e., a contravariant functor
\[ I^{\text{op}} \to \text{LogSch} \]
over \((X,D)\). By [NS08] (2.4.0.2), there exists a closed immersion \((X^\bullet, D^\bullet) \hookrightarrow (Y^\bullet, Z^\bullet)\) to a diagram of smooth schemes with smooth divisor over \( S \). Let \( a^\bullet : \]
$D^\bullet \hookrightarrow X^\bullet$ and $b^\bullet : Z^\bullet \hookrightarrow Y^\bullet$ be diagrams of the natural closed immersions. By using Poincaré residue isomorphism, there is the following exact sequence

$$0 \to \Omega_{Y/S} \to L_{\bullet} \to D_{\bullet} \to \Omega_{Z/S} \to -1 \to 0.$$

Let $L_{X/S}$ (resp. $L_{D/S}$) be the linearization functor ([BO78] Construction 6.9) with respect to the diagram $X^\bullet \hookrightarrow Y^\bullet$ (resp. $D^\bullet \hookrightarrow Z^\bullet$) of closed immersions of schemes. Let $L_{X^\bullet/D^\bullet/S}$ be the log linearization functor ([NS08] §2.2) with respect to the diagram $(X^\bullet, D^\bullet) \to (Y^\bullet, Z^\bullet)$ of closed immersions of log schemes. Let $Q_{X/S} : (\Omega_{S/crys})_c \to (\Omega_{S/crys})_c$ be the natural morphisms from the restricted crystalline topos to the crystalline topos ([Ber74] IV 2.1). Then we have morphisms

$$Q_{X/S} : ((\Omega_{S/crys})_c \to (\Omega_{S/crys})_c, O_{X/S}),$$

$$Q_{X/S} : ((\Omega_{S/crys})_c \to (\Omega_{S/crys})_c, O_{X/S}).$$

of ringed topos ([Ber74] IV (2.1)).

The following diagram is commutative by [NS08] Corollary 2.2.12:

$$\begin{array}{ccc}
\text{(HPD differential operators)} & \xrightarrow{\theta_{zar, \bullet}} & \text{(HPD differential operators)} \\
\text{(Crystals of $\mathcal{O}_{S/crys}$-modules)} \downarrow & & \downarrow \\
L_{D^\bullet/S} & & L_{X^\bullet/S}
\end{array}$$

where $Z^\bullet$ (resp. $X^\bullet$) is the pd-envelope of the closed immersion $D^\bullet \hookrightarrow Z^\bullet$ (resp. $X^\bullet \hookrightarrow Y^\bullet$) over $(S, I, \gamma)$. Hence we have the following exact sequence:

$$0 \to Q_{X/S}^\bullet \to L_{X^\bullet/S}(\Omega_{Y^\bullet/Z^\bullet/S}) \to Q_{X/S}^\bullet \to L_{D^\bullet/S}(\Omega_{Z^\bullet/S} \{-1\}) \to 0.$$

Let $\theta_{X/S,crys} : (\Omega_{X^\bullet/S})_c \to (\Omega_{X^\bullet/S})_c$ and $\theta_{X/S,crys} : (\Omega_{X^\bullet/S})_c \to (\Omega_{X^\bullet/S})_c$ be natural augmentation morphisms of topos. Similarly, we have augmentation morphisms

$$\theta_{D^\bullet,S,crys}, \theta_{D^\bullet,S,crys}, \theta_{(X,D)/S,crys}, \theta_{(X,D)/S,crys}.$$

By [NS08] Proposition 1.6.4, we have the equality of functors $Q_{X/S}^\bullet \cong \mathbb{R} \theta_{X/S,crys}$. There is an isomorphism $\mathbb{R} a_{\bullet,crys}^\bullet \cong \mathbb{R} a_{\bullet,crys}^\bullet$ by [NS08] (1.6.0.13). Since $a^\bullet$ and $a^\bullet$ are closed immersions, we see

$$\mathbb{R} a_{\bullet,crys}^\bullet = a_{\bullet,crys}^\bullet, \mathbb{R} a_{\bullet,crys}^\bullet = a_{\bullet,crys}^\bullet.$$

([Ber74] III Corollaire 2.3.2). So we have the following triangle

$$Q_{X/S}^\bullet \mathbb{R} \theta_{X/S,crys}^\bullet \xrightarrow{\mathbb{R} \theta_{X/S,crys}^\bullet} L_{X^\bullet/S}(\Omega_{Y^\bullet/Z^\bullet/S}) \xrightarrow{\mathbb{R} \theta_{D^\bullet/S,crys}^\bullet} L_{D^\bullet/S}(\Omega_{Z^\bullet/S} \{-1\}).$$

Let $\epsilon : (X, D) \to X$ and $\epsilon^\bullet : (X^\bullet, D^\bullet) \to X^\bullet$ be the canonical morphisms of log schemes. By the cohomological descent ([NS08] Lemma 1.5.1), we have the natural isomorphisms (in derived categories)

$$\mathcal{O}_{X/S} \cong \mathbb{R} \theta_{X/S,crys}^\bullet \mathcal{O}_{X^\bullet/S},$$

$$\mathcal{O}_{(X,D)/S} \cong \mathbb{R} \theta_{(X,D)/S,crys}^\bullet \mathcal{O}_{(X^\bullet,D^\bullet)/S},$$

$$\mathcal{O}_{D/S} \cong \mathbb{R} \theta_{D/S,crys}^\bullet \mathcal{O}_{D^\bullet/S}.$$
By [NS08] Proposition 2.2.7, we have isomorphisms
\[
\mathcal{O}_{X/S} \simeq L_{X/S}(\Omega^1_{V/S}),
\]
\[
\mathcal{O}_{(X,D)/S} \simeq L_{(X,D)/S}(\Lambda^{+}_{Y,Z}/S),
\]
\[
\mathcal{O}_{D/S} \simeq L_{D/S}(\Omega^*_{D}).
\]
We also have isomorphisms
\[
\mathbb{R}\theta_{X/S,\text{crys}} L_{X/S}(\Lambda^{+}_{Y,Z}/S) \simeq \mathbb{R}\theta_{X/S,\text{crys}} \mathbb{R}e^* L_{(X,D)/S}(\Lambda^{+}_{Y,Z}/S)
\]
\[
\simeq \mathbb{R}e_\ast \mathbb{R}\theta_{(X,D)/S,\text{crys}} L_{(X,D)/S}(\Lambda^{+}_{Y,Z}/S).
\]
Hence we have the following triangle
\[
Q^+_{X/S}(\mathcal{O}_{X/S}) \rightarrow Q^+_{X/S}(\mathcal{O}_{(X,D)/S}) \rightarrow Q^+_{X/S}(\mathcal{O}_{D/S}) \{-1\} \rightarrow .
\]
From this triangle, we have the following boundary morphism
\[
G^+_{D/X} : Q^+_{X/S}(\mathcal{O}_{D/S}) \{-1\} \rightarrow Q^+_{X/S}(\mathcal{O}_{X/S})[1]
\]
in \( D^+(Q^+_{X/S}(\mathcal{O}_{X/S})) \). Applying the global section functor, we obtain a morphism
\[
G^+_{D/X} : \mathbb{R}\Gamma((X/S)_{\text{crys}}, Q^+_{X/S}(\mathcal{O}_{D/S}) \{-1\}) \rightarrow \mathbb{R}\Gamma((X/S)_{\text{crys}}, Q^+_{X/S}(\mathcal{O}_{X/S}))[1].
\]
By [Ber74] V Proposition 1.3.1, (1.3.3), the left hand is identified to
\[
\mathbb{R}\Gamma((X/S)_{\text{crys}}, Q^+_{X/S}(\mathcal{O}_{D/S}) \{-1\}) \simeq \mathbb{R}\Gamma((X/S)_{\text{crys}}, a_{\mathcal{O}_{D/S}} \mathcal{O}_{D/S}) \{-1\}
\]
and the right hand is identified to
\[
\mathbb{R}\Gamma((X/S)_{\text{crys}}, Q^+_{X/S}(\mathcal{O}_{X/S})[1]) \simeq \mathbb{R}\Gamma((X/S)_{\text{crys}}, \mathcal{O}_{X/S})[1].
\]
Therefore we have the Gysin map
\[
G^+_{D/X} : \mathbb{R}\Gamma((D/S)_{\text{crys}}, \mathcal{O}_{D/S}) \{-1\} \rightarrow \mathbb{R}\Gamma((X/S)_{\text{crys}}, \mathcal{O}_{X/S})[1].
\]
The Gysin map \( G^+_{D/X} \) is independent of the choice of the open covering \( X = \bigcup_{i \in I_0} X_i \) and the diagram of embeddings \( (X^i, D^i) \hookrightarrow (Y^i, Z^i) \) ([NS08] Proposition 2.8.2).

We define the Gysin map of the de Rham-Witt complex
\[
G^+_{D/X} : W_m \Omega^+_{D/S} \{-1\} \rightarrow W_m \Omega^+_{X/S}[1]
\]
by the boundary morphism of the exact sequence
\[
0 \rightarrow W_m \Omega^+_{X/S} \rightarrow W_m \mathcal{A}_{(X,D)/S} \underset{\text{Res}}{\longrightarrow} W_m \Omega^+_{D/S} \{-1\} \rightarrow 0.
\]
Since restriction maps are surjective, we also have the exact sequence:
\[
0 \rightarrow W_0 \Omega^+_{X/S} \rightarrow W_0 \mathcal{A}_{(X,D)/S} \underset{\text{Res}}{\longrightarrow} W_0 \Omega^+_{D/S} \{-1\} \rightarrow 0.
\]
Similarly we define \( G^+_{D/X} : W_0 \Omega^+_{D/S} \{-1\} \rightarrow W_0 \Omega^+_{X/S}[1] \).

We consider the compatibility of these Gysin maps. Let \( S \) be a scheme over \( \mathbb{Z}_p \) in which \( p \) is nilpotent, \( X \) a smooth scheme over \( S \) and \( D \) a smooth divisor of \( X \) over \( S \). We imitate the method in [NS08] §2.4 to make a simplicial log Frobenius lift.

Take an affine open covering \( X = \bigcup_{i \in I_0} X^i \) of \( X \) such that there exists an étale morphism \( X^i \rightarrow \mathbb{A}^n_{\mathbb{F}_p} \), and \( D^i := D \cap X^i = \emptyset \) or \( D^i \) is defined by the image of \( T_i \in \mathcal{O}_{\mathbb{A}^n_{\mathbb{F}_p}} \) in \( \mathcal{O}_{X^i} \). Then each \( X^i \) (resp. \( D^i \)) has a canonical Frobenius lift (in the sense of [LZ04] §3.1) \( \{X^i_m\}_m \) (resp. \( \{D^i_m\}_m \)) and there is a morphism \( \{D^i_m\}_m \rightarrow \{X^i_m\}_m \) of systems which is compatible with the structure of Frobenius lifts. For \( \delta = (i_0, \ldots, i_r) \in I, \) we set \( X^\delta := \bigcap_{\alpha=0}^r X^{i_\alpha} \) and \( D^\delta := \bigcap_{\alpha=0}^r D^{i_\alpha} \). Let \( X^\delta_m \) be the open subscheme of \( X^\delta_m \) defined by the image of \( X^\delta \rightarrow X^\delta_m \). It is
easy to see that the induced morphism $X^b \to X_m^{(i_0, \underline{\omega})}$ is a closed immersion. Set $D_m^{(i_0, \underline{\omega})} := D_m \cap X_m^{(i_0, \underline{\omega})}$ and $X_m^{(i_0, \underline{\omega})} := \times_{W_m(s), \alpha = 0} X_m^{(i_0, \underline{\omega})}$. The closed immersion $X^b \hookrightarrow X_m^{(i_0, \underline{\omega})}$ induce a closed diagonal immersion $X^b \hookrightarrow X_m^{(i_0, \underline{\omega})}$. We denote by $b : X_m^{i_0} \to X_m^{(i_0, \underline{\omega})}$ by the blowing up $X_m^{i_0}$ along $D_m^{(i_0, \underline{\omega})} := \times_{W_m(s), \alpha = 0} D_m^{(i_0, \underline{\omega})}$. We consider the complement $X_m^{(i_0, \underline{\omega})}$ of the strict transform of

\[
\bigcup_{\beta=0}^r (X_m^{(i_0, \underline{\omega})} \times \cdots \times X_m^{(i_{\beta-1}, \underline{\omega})} \times D_m^{(i_{\beta}, \underline{\omega})} \times X_m^{(i_{\beta+1}, \underline{\omega})} \times \cdots \times X_m^{(i_r, \underline{\omega})})
\]

in $X_m^{i_0}$, where fibered products $\times$ mean $\times_{W_m(S)}$, fibered products over $W_m(S)$. Let $D_m^{i_0} = X_m^{i_0} \cap b^{-1}(D_m^{(i_0, \underline{\omega})})$ be the exceptional divisor on $X_m^{i_0}$. Then $D_m^{i_0}$ is a smooth divisor on $X_m^{i_0}$ by [NS08] Theorem 2.4.2. Considering the strict transform of the image of $X^b$ of the diagonal embedding in $X_m^{(i_0, \omega)}$, we have a closed immersion $X^b \hookrightarrow X_m^{(i_0, \underline{\omega})}$. Moreover, we have $D_m^{i_0} \times X^b \twoheadrightarrow D_m^{i_0}$.

We interpret [NS08] Theorem 2.4.2 in our situation. We consider the case $D_m^{(i_0, \underline{\omega})} \neq \emptyset$ for all $0 \leq \alpha \leq r$. Then the closed immersion $D_m^{(i_0, \underline{\omega})} \hookrightarrow X_m^{(i_0, \underline{\omega})}$ is defined by a global section $x_m^{(i_0, \underline{\omega})}$ of $X_m^{(i_0, \underline{\omega})}$, which corresponds to the image of $T_1$ of $A_{W_m(S)}$ under the map $X_m^{(i_0, \underline{\omega})} \to X_m^{i_0} \to A_{W_m(S)}$. Let $A_m^{i_0} := \mathbb{E}^{r}_{s=0} O_{X_m^{(i_0, \omega)}}$ be the structure sheaf of $X_m^{(i_0, \omega)}$. Then $X_m^{i_0}$ is the spectrum over $W_m(S)$ of the following sheaf of algebras

$B_m^{i_0, \alpha} := A_m^{i_0} \langle [u_{m,1}^{\pm 1}, \ldots, u_{m,r}^{\pm 1}] \rangle / (x_m^{(i_0, \omega)} - u_{m,\alpha} x_m^{(i_0, \omega)} | 1 \leq \alpha \leq r)$,

where $u_{m, \alpha}$ are independent indeterminants and $D_m^{i_0}$ corresponds to the equation $x_m^{(i_0, \omega)} = 0$. $\{X_m^{(i_0, \omega)}\}_m$ has a natural structure of Frobenius lift. In fact, the natural morphism $X_m^{i_0} \to X_m^{i_0}$ induces $X_m^{(i_0, \omega)} \to X_m^{(i_0, \omega)}$ and it maps $x_m^{(i_0, \omega)}$ to $x_m^{(i_0, \omega)}$. Hence we obtain a natural map $B_m^{i_0} \to B_m^{i_0, \omega}$ and it satisfies $W_{m-1}(O_S) \otimes_{W_m(O_S)} B_m^{i_0} \simeq B_m^{i_0, \omega}$. Since the following diagram commutes

\[
\begin{array}{ccc}
X_m^{i_0} & \hookrightarrow & X_m^{(i_0, \underline{\omega})} \\
\Phi_m^{i_0} \downarrow & & \text{Frob} \\
X_m^{i_0} & \hookrightarrow & X_m^{i_0} / p X_m^{i_0}
\end{array}
\]

and the absolute Frobenius map Frob and horizontal arrows are homeomorphisms, $\Phi_m^{i_0}$ is also a homeomorphism. It induces a map $\Phi_m^{(i_0, \underline{\omega})} : X_m^{(i_0, \omega)} \to X_m^{(i_0, \omega)}$. The family $\{\Phi_m^{(i_0, \underline{\omega})}\}_m$ defines a map $A_m^{i_0} \to A_m^{i_0, \omega}$. Since it maps $x_m^{(i_0, \omega)}$ to $\left(\frac{x_m^{(i_0, \omega)}}{p^{\alpha}}\right)$, we can extend this map to $B_m^{i_0} \to B_m^{i_0, \omega}$ by $u_{m, \alpha} \mapsto u_{m-1, \alpha}^{p^\alpha}$.

For $\Delta_m$, we use the following commutative diagram:

\[
\begin{array}{ccc}
X_m^{i_0} & \xrightarrow{w_0} & W_m(X_\Delta) \\
\downarrow & & \downarrow \\
X_m^{(i_0, \underline{\omega})} & \xleftarrow{w_0} & W_m(X_m^{(i_0, \omega)})
\end{array}
\]

Since $X_\Delta \to X_m^{i_0}$ is an open immersion, the morphism $W(X_\Delta) \to W(X_m^{i_0})$ is also an open immersion. We also know $X_\Delta$ and $W_m(X_\Delta)$ has the same underlying topological space. Since $X_m^{(i_0, \underline{\omega})} \to X_m^{i_0}$ is also open, there is a unique map $\Delta_m : W_m(X_\Delta) \to X_m^{(i_0, \omega)}$ which makes the diagram commutative. These maps define
Δ_m : W_m(X^2) → X^m_{[1]}. We can define Δ_m : W_m(X^2) → X^m_m using this map and sending u_m,α to [α_1,α].

We have the following commutative diagram:

\[
\begin{array}{ccc}
X^* & \longrightarrow & X^*_m \\
\downarrow & & \downarrow \\
D^* & \longrightarrow & D^*_m
\end{array}
\]

Let X^*_m (resp. D^*_m) be the pd-envelope of the closed immersion X^* ↪ X^*_m (resp. D^* ↪ D^*_m). By [NS08] Lemma 2.2.16 (2), the natural morphism D^*_m → X^*_m × X^*_m D^*_m is an isomorphism.

Let θ_{Zar} : D_{Zar} → X_{Zar}, be the canonical morphism of Zariski topoi and θ_{X,Zar} : X^*_Zar → X_{Zar}, θ_{D,Zar} : D^*_Zar → D_{Zar} be the augmentation morphism. Then the following commutative diagram shows the compatibility of Gysin maps:

\[
\begin{array}{ccc}
\mathbb{R}a_{Zar} \circ u_{D/W_m(S)} & \circ O_{D/W_m(S)} & \circ [1] \\
\downarrow & & \downarrow \\
\mathbb{R}a_{Zar} & \circ u_{D/W_m(S)} & \circ Q_{D/W_m(S)} & \circ O_{D/W_m(S)} & \circ [1] \\
\downarrow & & \downarrow \\
\mathbb{R}a_{Zar} & \circ θ_{D,Zar} & \circ (Ω^{•}_{D/W_m(S)}) & \circ [1] \\
\downarrow & & \downarrow \\
\mathbb{R}a_{Zar} & \circ θ_{D,Zar} & \circ (W_m Ω^{•}_{D^*/S}) & \circ [1]
\end{array}
\]

where Ω^{•}_{D/W_m(S)} := O_{D/W_m(S)} ⊗_{O_{D^*/S}} Ω^{•}_{D^*/W_m(S)} Ω^{•}_{X^*_m/W_m(S)} := O_{X^*_m_m} ⊗_{O_{X^*_m/S}} Ω^{•}_{X^*_m/W_m(S)}.

Finally, we consider the relation between the boundary map of E_1-term of the p-adic weight spectral sequence and Gysin maps. Let Y be a stricly semistable log scheme over S = Spec(R,N). We use the convention of §7.2 and §8.2. Let G^{\text{dR}}_q : W_m Ω^{•}_{Y(j+1)/R} → W_m Ω^{•}_{Y(j)/R}[1] be the Gysin map corresponding to different immersions e^{(q)} : Y(j+1) → Y(j). We set G^{\text{dR}} := \sum_{q=1}^{j+1} (-1)^{q+1} G^{\text{dR}}_q, and let d^1 be the boundary morphism of the exact sequence

0 → Gr_j W_m \tilde{Λ}^* → (P_{j+1}/P_{j-1}) W_m \tilde{Λ}^* → Gr_j W_m \tilde{Λ}^* → 0.

Proposition 8.10. (cf. [Mok93] Proposition 4.11) The following diagram is commutative:

\[
\begin{array}{ccc}
Gr_{j+1} W_m \tilde{Λ}^* & \longrightarrow & Gr_j W_m \tilde{Λ}^*[1] \\
\downarrow & & \downarrow \text{Res}[1] \\
W_m Ω^{•}_{Y(j+1)/R} & \longrightarrow & W_m Ω^{•}_{Y(j)/R}[1] \oplus_{[1]}
\end{array}
\]

Proof. Let J = \{α_1, ..., α_{j+1}\} be a subset of [1, d] and J_q = \{α_1, ..., α_q, ..., α_{j+1}\}. The residue morphism W_m Ω^{•}_{Y_{j}/R} → Gr_j W_m \tilde{Λ}^* naturally extends to a morphism W_m Α^{•}_{\tilde{Y}_j/R} → (P_{j+1}/P_{j-1}) W_m \tilde{Λ}^*. The commutativity follows from
the following commutative diagram with exact rows:

\[
\begin{array}{ccccccc}
0 & \to & W_m \Omega^*_{Y_{j_k}}(-j) & \to & W_m A^*_{(Y_{j_k}, Y_{j_j})}(-j) & \to & W_m \Omega^*_{Y_{j_j}}(-j-1) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Gr_j W_m \tilde{A}^* & \to & (P_{j+1}/P_{j-1}) W_m \tilde{A}^* & \to & Gr_{j+1} W_m \tilde{A}^* & \to & 0.
\end{array}
\]

Proposition 8.11. (cf. [Nak05] Theorem 10.1)

Let \( s \) be a positive integer or nothing. Under the residue isomorphism,

\[
d^1 : \mathbb{H}^b_{\text{Zar}}(\hat{Y}, Gr_k W_s A^*) \to \mathbb{H}^b_{\text{Zar}}(\hat{Y}, Gr_{k-1} W_s A^*)
\]

is identified with the following morphism:

\[
\sum_{j \geq \max\{-k, 0\}} \left( \begin{array}{c}
\{ (-1)^j \mathcal{G}^{\text{cris}} \{ -2j - k + 1 \} + (-1)^{j+k} \rho_s \{ -2j - k \} \} : \\
\bigoplus_{j \geq \max\{-k, 0\}} H^{h-2j-k}_{\text{cris}}(\hat{Y}^{(2j+k+1)}/W_s(R))(-j - k) \\
\bigoplus_{j \geq \max\{-k+1, 0\}} H^{h-2j-k+2}_{\text{cris}}(\hat{Y}^{(2j+k)}/W_s(R))(-j - k + 1),
\end{array} \right)
\]

where \( \rho_s \) is the morphism defined in Lemma 8.6.

Proof. We can copy the proof of [Nak05] Theorem 10.1 using Proposition 8.6 and Proposition 8.10.

\[\square\]

8.6 Degeneration of weight spectral sequence. In this section, we prove that the weight spectral sequence degenerates up to torsion if the base scheme is a spectrum of a (not necessarily perfect) field using the method of [Nak05]. Let \( Y \) be a proper strictly semistable log scheme over a field \( k \) of characteristic \( p > 0 \).

Let \( s = (\text{Spec } k, \mathbb{N} \oplus k^*) \) be a log point with structure morphism defined by \( \mathbb{N} \oplus k^* \ni (a, u) \mapsto 0 \) for \( a \neq 0 \) and \( (0, u) \mapsto u \). By [Nak00] Lemma 2.2, there is a subring \( A_1 \) which is finitely generated over \( \mathbb{F}_p \) and a proper strictly semistable log scheme \( \mathfrak{Y} \) over \( s_1 = (\text{Spec } A_1, \mathbb{N} \oplus A_1^*) \) with structure morphism defined by \( \mathbb{N} \oplus A_1^* \ni (a, u) \mapsto 0 \) for \( a \neq 0 \) and \( (0, u) \mapsto u \) such that \( \mathfrak{Y} \times_{s_1} s = Y \). We can assume \( A_1 \) is smooth over \( \mathbb{F}_p \). Lift \( A_1 \) to a \( p \)-adically complete formally smooth algebra \( A \) over \( W(\mathbb{F}_p) = \mathbb{Z}_p \). Let \( S := (\text{Spf } A, \mathbb{N} \oplus A^*) \) be a \( p \)-adically log formal scheme over \( \text{Spf}(\mathbb{Z}_p, \mathbb{Z}_p^*) \) such that log structure of \( S \) is induced by \( \mathbb{N} \oplus A^* \ni (a, u) \mapsto 0 \) for \( a \neq 0 \) and \( (0, u) \mapsto u \). \( S \) has the pd-ideal \( pO_S \) and it defines the exact closed immersion \( s_1 \hookrightarrow S \). For an affine log formal open subscheme \( T \) of \( S \), let \( T_1 := T \otimes_{\mathbb{Z}_p} \mathbb{F}_p \) be its reduction. We fix a lift of Frobenius \( F_T : T \to T_1 \). Set \( \mathfrak{Y}_T := \mathfrak{Y} \times_{S_1} T_1 \). If \( t \) is a closed point of \( T_1 \), set \( \mathfrak{Y}_T := \mathfrak{Y} \times_{S_1} T_1 \). In this situation, the canonical inclusion \( A_1 \hookrightarrow k \) factors \( \mathcal{O}_{T_1} \). Let \( \mathcal{O}_T \to W(k_l) \) (resp. \( \mathcal{O}_T \to W(k) \)) be the Teichmüller lift of the morphism \( \mathcal{O}_{T_1} \to k_l \) (resp. \( \mathcal{O}_{T_1} \to k \)) ([Ill79] (0.1.3.20)). We consider \( W(k_l) \) and \( W(k) \) as \( \mathcal{O}_T \)-algebras via these maps.

Proposition 8.12. (cf. [Nak05] Proposition 3.2) There exists an affine log formal open subscheme \( T \) of \( S \) such that the canonical morphism

\[
H^b_{\text{log-cr}}(\mathfrak{Y}_T/T) \otimes_{\mathcal{O}_T} W(k) \to H^b_{\text{log-cr}}(Y/W(k, N))
\]

is an isomorphism.

Proof. For an affine log formal open subscheme \( T \) of \( S \), we find

\[
\mathbb{R} \Gamma_{\text{log-cr}}(\mathfrak{Y}_T/T) \otimes_{\mathcal{O}_T} W_n(k) \simeq \mathbb{R} \Gamma_{\text{log-cr}}(Y/W_n(k, N))
\]
by the base change theorem ([Kat89] (6.10)).

Let \((P^\bullet, d^\bullet)\) be a strictly perfect complex (Definition 2.7 (2)) which represents \(R\Gamma_{\log-crys}(\mathcal{Y}/T)/T\). Then

\[
R\Gamma_{\log-crys}(\mathcal{Y}/T_n) \otimes \mathcal{O}_{T_n} W_n(k) \simeq R\Gamma_{\log-crys}(\mathcal{Y}/T) \otimes \mathcal{O}_T \otimes \mathcal{O}_{T_n} W_n(k) \\
\simeq P^\bullet \otimes \mathcal{O}_T W_n(k).
\]

Since \(P^\bullet \otimes \mathcal{O}_T W_n(k)\) satisfies Mittag-Leffler condition

\[
R\Gamma_{\log-crys}(Y/W(k, N)) = \varprojlim_n R\Gamma_{\log-crys}(Y/W_n(k, N)) \\
\simeq \varprojlim_n (R\Gamma_{\log-crys}(\mathcal{Y}/T_n) \otimes \mathcal{O}_{T_n} W_n(k)) \\
\simeq \varprojlim_n (P^\bullet \otimes \mathcal{O}_T W_n(k)) \\
\simeq \varprojlim_n (P^\bullet \otimes \mathcal{O}_T W_n(k)) \\
\simeq P^\bullet \otimes \mathcal{O}_T W(k).
\]

By [Nak05] Lemma 3.1, we can suppose \(\text{Tor}_{d^j}(L/T, W(k)) = 0\) for \(L = H^j(P^\bullet)\) and \(\text{Im}(d^j)\) for any \(j\) by shrinking \(T\) if necessary. Then we get

\[
H^h_{\log-crys}(\mathcal{Y}/T) \otimes \mathcal{O}_T W(k) = H^h(P^\bullet) \otimes \mathcal{O}_T W(k) \\
= H^h(P^\bullet \otimes \mathcal{O}_T W(k)) \\
= H^h_{\log-crys}(Y/W(k, N)).
\]

\[\square\]

**Theorem 8.13.** ([Nak05] Proposition 3.5, Theorem 3.6) Let \(Y\) be a semistable log scheme over any field of characteristic \(p > 0\) and \(K\) be the fraction field of \(W(k)\). The \(p\)-adic weight spectral sequence (Theorem 8.9 (2)) degenerates at \(E_2\) after tensoring with \(K\).

**Proof.** By [Nak05] Corollary 3.4 and Proposition 8.12, there exists an affine log formal scheme \(T\) of \(S\) such that for any closed point \(t \in T\) the canonical morphisms

\[
H^h_{\log-crys}(\mathcal{Y}/T) \otimes \mathcal{O}_T W(k_t) \rightarrow H^h_{\log-crys}(\mathcal{Y}/T/W(k_t, N)), \\
H^h_{\log-crys}(\mathcal{Y}/T) \otimes \mathcal{O}_T W(k) \rightarrow H^h_{\log-crys}(Y/W(k, N))
\]

are isomorphisms. By Deligne’s remark ([Ill75] (3.10)), we can assume there exists a finitely generated \(\mathbb{Z}_p\)-module \(M\) such that \(H^h_{\log-crys}(\mathcal{Y}/T) \simeq M \otimes_{\mathbb{Z}_p} \mathcal{O}_T\) by shrinking \(T\) if necessary.

By Corollary 3.4 of [Nak05] and Corollary 8.12, there exists isomorphisms

\[
H^h_{\text{cris}}(Y(j)/W(k)) \simeq H^h_{\text{cris}}(\mathcal{Y}(j)/T) \otimes \mathcal{O}_T W(k), \\
H^h_{\text{cris}}(\mathcal{Y}(j)/W(k_t)) \simeq H^h_{\text{cris}}(\mathcal{Y}(j)/T) \otimes \mathcal{O}_T W(k)
\]

for all \(j\) and for all closed points \(t\) of \(T\) by shrinking \(T\) if necessary. Set

\[
F^{-k, h+k} := \bigoplus_{j \geq \max\{k, 0\}} H^h_{\text{cris}}(\mathcal{Y}(j)/T)/T)
\]

and

\[
G^{-k, h+k} := \text{ker}(F^{-k, h+k} \rightarrow F^{-k+1, h+k})/\text{image}(F^{-k-1, h+k} \rightarrow F^{-k, h+k}),
\]

where the morphisms \(F^{-k, h+k} \rightarrow F^{-k+1, h+k}\) and \(F^{-k-1, h+k} \rightarrow F^{-k, h+k}\) are the sums of the induced morphisms of closed immersions and Gysin maps as in Proposition 8.11.
4.3.12) and by [Nak05] Lemma 3.1, we obtain

\[ E_2^{-k,h+k}(Y/W(k, N)) = G^{-k,h+k} \otimes_{O_T} W(k) \]
\[ E_2^{-k,h+k}(\mathfrak{M}_T/W(k_T, N)) = G^{-k,h+k} \otimes_{O_T} W(k_T) \]

for all \( k, h \) by shrinking \( T \) if necessary. Using Deligne's remark, we can assume that there exists a finitely generated \( \mathbb{Z}_p \)-module \( M^{-k,h+k} \) such that \( M^{-k,h+k} \otimes_{\mathbb{Z}_p} O_T \cong G^{-k,h+k} \). Let \( K_t \) be the fraction field of \( W(k_t) \). We have

\[ \dim_K(E_2^{-k,h+k}(Y/W(k, N)) \otimes_{W(k)} K) = \dim_{\mathbb{F}_p}(M^{-k,h+k} \otimes_{\mathbb{Z}_p} \mathbb{F}_p) = \dim_K(E_2^{-k,h+k}(\mathfrak{M}_T/W(k_T, N)) \otimes_{W(k_T)} K_t), \]
\[ \dim_K(H_{log-crys}^h(Y/W(k, N)) \otimes_{W(k)} K) = \dim_{\mathbb{F}_p}(M \otimes_{\mathbb{Z}_p} \mathbb{F}_p) = \dim_K(H_{log-crys}^h(\mathfrak{M}_T/W(k_T, N)) \otimes_{W(k_T)} K_t). \]

By the purity of the weight of the crystalline cohomology ([CLS98 Théorème 1.2]), this theorem is true for \( \mathfrak{M}_T/W(k_t, N) \) because \( t \) is the spectrum of a finite field. By the above calculation of dimensions, we see this theorem is true for any field of characteristic \( p > 0 \).

9. Weight spectral sequence and its degeneration for open smooth varieties

Let \( R \) be a Noetherian \( \mathbb{Z}(p) \)-algebra in which \( p \) is nilpotent. Let \( X \) be a proper smooth scheme over \( R \) and \( D \) an SNCD on \( X \) over \( R \). We consider the log scheme \( (X, D) \) with respect to the Zariski topology. By Theorem 7.2, we have a canonical isomorphism

\[ H_{log-crys}^h((X, D)/W(R)) \cong \mathbb{H}_{Zar}^h(X, W\Lambda_{(X,D)/R}). \]

Let \( D_1, \ldots, D_d \) be the irreducible components of \( X \). For a subset \( J = \{\alpha_1, \ldots, \alpha_j\} \) of \([1, d]\), let \( D_J = D_{\alpha_1} \cap \cdots \cap D_{\alpha_j} \). We set \( D^{(j)} \) by \( \bigcap_{|J|=j} D_J \) for all nonnegative number \( j \). Then we can show that the canonical morphism

\[ W\Omega_{D^{(j)}/R}^{*,-j}(-j) \to Gr_jW\Lambda_{(X,D)/R} \]

is an isomorphism as \( \S 8.2 \). We also call the map \( Gr_jW\Lambda_{(X,D)/R} \to W\Omega_{D^{(j)}/R}^{*,-j}(-j) \) the Poincaré residue map. Using this, we obtain the following spectral sequence

\[ E_1^{k,h+k} = H^{h-k}_{crys}(D^{(k)}/W(R))(-k) \Rightarrow H_{log-crys}^h((X, D)/W(R)), \]

which we also call the \( p \)-adic weight spectral sequence.

**Theorem 9.1.** When \( R = k \) is a field, the \( p \)-adic weight spectral sequence degenerates at \( E_2 \) after tensoring with the fraction field of \( W(k) \).

**Proof.** The proof is the same as that of Theorem 8.13 (cf. [Nak05] Theorem 5.2).

\[ \square \]

10. Overconvergent log de Rham-Witt complex in SNCD case

In this section, we extend the overconvergent de Rham-Witt complex of [DLZ11] to log schemes associated to schemes with simple normal crossing divisors over a perfect field. In this section, we work on the Zariski topology when we consider log structures and log de Rham-Witt complexes (See \S 8.1).
10.1. Overconvergent log de Rham-Witt complex. Let \( k \) be a perfect field of positive characteristic \( p \) and \( K = W(k)[1/p] \) its fraction field. Let \( A = k[T_1, \ldots, T_n] \) be a polynomial ring. We consider the pre-log ring \( (A, N^d), N^d \geq 1, \rightarrow T_i \in A \) for \( d \leq n \). Recall that an element \( \omega \) of \( W^* A_{(A,N^d)/k} \) is uniquely written as a convergent sum (Proposition 4.3)

\[
\omega = \sum_{k, \mathcal{P}} \epsilon(\zeta_k, p, k, \mathcal{P}).
\]

In this section, we only consider the case that \( J \) is empty. Therefore we write \( \epsilon(\zeta_k, p, k, \mathcal{P}) \) for \( \epsilon(\zeta_k, p, \emptyset, k, \mathcal{P}) \).

For a positive real number \( \epsilon \) we define the Gauss norm \( \gamma_\epsilon \) by

\[
\gamma_\epsilon(\omega) := \inf_{k, \mathcal{P}} \{ \text{ord}_P \zeta_k - \epsilon |k^+| \}.
\]

This is equal to \( \inf_{k, \mathcal{P}} \{ \text{ord}_P \zeta_k - \epsilon |k^+| \} \) (see [DLZ11] (0.3)) because \( \text{ord}_P \zeta = \text{ord}_P \xi \) for \( \zeta, \xi \in W(k) \).

If \( \gamma_\epsilon(\omega) \geq -\infty \), we say that \( \omega \) has radius of convergence \( \epsilon \). We call \( \omega \) overconvergent if there is an \( \epsilon > 0 \) such that \( \omega \) has radius of convergence \( \epsilon \). We find

\[
\gamma_\epsilon(\omega_1 + \omega_2) \geq \min(\gamma_\epsilon(\omega_1), \gamma_\epsilon(\omega_2))
\]

and overconvergent elements form a sub differential algebra \( W^* A_{(A,N^d)/k} \) of \( W^* A_{(A,N^d)/k} \) (cf. [DLZ11] pp. 5).

**Proposition 10.1.** (cf. [DLZ11] Proposition 0.7, Proposition 0.9)

Let \( \phi : (k[S_1, \ldots, S_n], N^d) \rightarrow (k[T_1, \ldots, T_m], N^{d'}) \) be a morphism of pre-log rings over \( k \). The map

\[
\phi_* : W^* A_{(k[S_1, \ldots, S_n], N^d)/k} \rightarrow W^* A_{(k[T_1, \ldots, T_m], N^{d'})/k}
\]

induces

\[
\phi^*_k : W^* A_{(k[S_1, \ldots, S_n], N^d)/k} \rightarrow W^* A_{(k[T_1, \ldots, T_m], N^{d'})/k}.
\]

Moreover, \( \phi^*_k \) is surjective when both \( k[S_1, \ldots, S_n] \rightarrow k[T_1, \ldots, T_m] \) and \( N^d \rightarrow N^{d'} \) are surjective.

**Proof.** Let \( \omega = \sum_{k, \mathcal{P}} \epsilon(\zeta_k, p, k, \mathcal{P}) \) be any element of \( W^* A_{(k[S_1, \ldots, S_n], N^d)/k} \). Since \( \omega \) is overconvergent, there are \( \epsilon > 0 \) and \( C \in \mathbb{R} \) such that \( \text{ord}_P \zeta_k - \epsilon |k^+| \geq C \) for all \( k \) and \( \mathcal{P} \). For any subset \( J \) of \( [1, d] \), we set \( \omega_J := \sum_{k, \mathcal{P}, J \subseteq J} \epsilon(\zeta_k, p, k, \mathcal{P}) \). Then we see that \( \omega_J \) can be written as a form

\[
\omega_J = \left( \prod_{i \in J} d \log X_i \right) \cdot \sum_{k, \mathcal{P},} \epsilon(\zeta_k, p, k', \mathcal{P}').
\]

Here \( X_i := [T_i], k' : [1, n] \setminus J \rightarrow \mathbb{Z}_{\geq 0}[1/p] \) runs over all weights without log poles, \( \mathcal{P}' \) runs over all subsets of \( \text{Supp} k' \), \( (\zeta_k, p, k', \mathcal{P}) \in V^{\epsilon(\zeta_k, p, k', \mathcal{P})} W(k) \) and \( e(\zeta_k, p, k', \mathcal{P}) \) is a basic Witt differential (in the sense of [LZ04]). We see that

\[
\omega = \sum_{J \subseteq [1, d]} \omega_J
\]

and that all coefficients \( \zeta_k, p, k', \mathcal{P} ' \) satisfy \( \text{ord}_P \zeta_k - \epsilon |k^+| \geq C \). Hence we obtain

\[
\omega_J := \sum_{k, \mathcal{P}, J \subseteq J} \epsilon(\zeta_k, p, k', \mathcal{P}').
\]

By [DLZ11] Proposition 0.9, we obtain \( \phi(\omega_J) \in W^1 \Omega^*_{k[T_1, \ldots, T_m]/k} \). We see that

\[
\phi(\omega_J) = \phi \left( \prod_{i \in J} d \log X_i \right) \phi(\omega_J) \in W^1 \Omega^*_{k[T_1, \ldots, T_m], N^{d'}/k}
\]

because \( \phi(\prod_{i \in J} d \log X_i) \in W^1 \Omega^*_{k[T_1, \ldots, T_m], N^{d'}/k} \) and \( W^1 \Omega^*_{k[T_1, \ldots, T_m], N^{d'}/k} \) is a ring. This shows \( \phi(\omega) \in W^1 \Omega^*_{k[T_1, \ldots, T_m], N^{d'}/k} \).
We prove the last statement. If \( \phi \) is surjective, we can construct a map
\[
\psi : (k[T_1, \ldots, T_n], \mathbb{N}^d) \rightarrow (k[S_1, \ldots, S_n], \mathbb{N}^d)
\]
of pre-log rings such that \( \phi \circ \psi = \text{id} \). Then for any \( \eta \in W^1 \Lambda^\bullet_{(k[T_1, \ldots, T_n], \mathbb{N}^d)/k} \), the element \( \psi_\ast(\eta) \) belongs to \( W^1 \Lambda^\bullet_{(k[T_1, \ldots, T_n], \mathbb{N}^d)/k} \) and it satisfies \( \phi_\ast \psi_\ast(\eta) = \eta \). \( \square \)

Let \((B, P, \alpha)\) be a pre-log ring such that \( B \) is a finitely generated \( k \)-algebra. Then we can find a commutative diagram
\[
\begin{array}{ccc}
\mathbb{N}^d & \xrightarrow{\alpha} & A = k[T_1, \ldots, T_n] \\
\downarrow & & \downarrow \\
P & \xrightarrow{\alpha} & B,
\end{array}
\]
where the top morphism is given by \( e_i \mapsto T_i \) and the both vertical morphisms are surjective. It induces a map between log de Rham-Witt complexes \( \lambda : W^1 \Lambda^\bullet_{(B, P)/k} \rightarrow W^1 \Lambda^\bullet_{(B, P)/k} \).

**Definition 10.2.** We define \( W^1 \Lambda^\bullet_{(B, P)/k} \) as the image of \( W^1 \Lambda^\bullet_{(A, \mathbb{N}^d)/k} \) under the map \( \lambda \). We call \( W^1 \Lambda^\bullet_{(B, P)/k} \) the overconvergent log de Rham-Witt complex for the pre-log ring \((B, P)\) over \( k \).

By Proposition 10.1 (cf. [DLZ11] Definition 1.1), this definition is independent of the choice of the above diagram (3) and the correspondence \((B, P) \mapsto W^1 \Lambda^\bullet_{(B, P)/k} \) is functorial. Our definition of the overconvergent log de Rham-Witt complex is an extension of the overconvergent de Rham-Witt complex of Davis-Langer-Zink, i.e., \( W^1 \Lambda^\bullet_{(B, \{\ast\}))/k} \simeq W^1 \Omega^\bullet_{B/k} \).

### 10.2. Comparison with log Monsky-Washnitzer cohomology

Let \( k \) be a perfect field of char \( p > 0 \). We consider a finitely generated, smooth algebra \( \tilde{B} \) over Witt ring \( W(k) \) and \( X := \text{Spec} \tilde{B} \). We assume there is a (global) coordinates \( \tilde{t}_1, \ldots, \tilde{t}_n \) of \( X \), i.e., the morphism \( X \rightarrow \Lambda^n_{W(k)} \) defined by \( \tilde{t}_1, \ldots, \tilde{t}_n \) is étale. Let \( B = \tilde{B} \otimes_{W(k)} k \) be the reduction of \( \tilde{B} \) to \( k \) and \( t_1, \ldots, t_n \) be images of \( \tilde{t}_1, \ldots, \tilde{t}_n \) in \( B \), and \( X = \text{Spec} B \). We denote by \( D \) the divisor of \( X \) which is defined by the equation \( t_1 \cdots t_d = 0 \) and \( D \) its reduction to \( X \). Let \( \tilde{B}^\dagger \) be the weak completion of \( B \) with respect to \((p) \subset W(k)\) (in sense of [MW68] Definition 1.1). Let \((B, \mathbb{N}^d)\) (resp. \((\tilde{B}, \mathbb{N}^d)\)) be the pre-log ring defined by \( e_i \mapsto t_i \) (resp. \( e_i \mapsto \tilde{t}_i \)).

**Definition 10.3.** An endomorphism \( \phi \) of \( \tilde{B}^\dagger \) is called Frobenius if the following three conditions are satisfied:

1. \( \phi \) is compatible with the Frobenius map \( F \) on \( W(k) \).
2. Its reduction to \( B \simeq \tilde{B}^\dagger/p\tilde{B}^\dagger \) coincides with the absolute Frobenius on \( B \).
3. \( \phi \) satisfies the relation \( \phi(\tilde{t}_i) = \tilde{t}_i^p \cdot u_i \) for \( 1 \leq i \leq d \).

In this situation, Tsuzuki defined the logarithmic Monsky-Washnitzer cohomology \( H^n_{\log-MW}((X, D)/K) \) and proved that it depends only on \( X \) and \( D \) ([Tsu99] (3.3), Proposition 3.3.1).

To prove the comparison theorem between the logarithmic overconvergent de Rham-Witt cohomology and the logarithmic Monsky-Washnitzer cohomology, we have to extend the overconvergent Witt lift of [DLZ11] §3.

By [DLZ11] Proposition 3.2, the map \( t_\phi : \tilde{B}^\dagger \rightarrow W(B) \) defined in [Ill79] (0.1.3.20) has the image in \( W^1(B) \). The map \( s_\phi : \tilde{B}^\dagger \rightarrow W(\tilde{B}^\dagger) \) defined in [Ill79] (0.1.3.16) maps \( \tilde{t}_i \) to the unique element whose ghost components are \( (t_i, \phi(\tilde{t}_i), \phi^2(\tilde{t}_i), \ldots) \). It
easily follows by induction that \( \phi^j(\tilde{t}_i) \) is written as a form \( \tilde{t}_i^j \cdot \beta_{i,j} \) where \( \beta_{i,j} \in \tilde{B}^\dagger \). Then
\[
(\tilde{t}_i, \phi(\tilde{t}_i), \phi^2(\tilde{t}_i), \ldots) = (\tilde{t}_i, \tilde{t}_i^0, \tilde{t}_i^1, \ldots) \cdot (1, \beta_{i,1}, \beta_{i,2}, \ldots).
\]
There is a unique element \( \nu_0(\tilde{t}_i) \) of \( W(\tilde{B}^\dagger) \) whose ghost components are
\[
(1, \beta_{i,1}, \beta_{i,2}, \ldots).
\]
Let \( \lambda_0(\tilde{t}_i) \in W(B) \) be the image of \( \nu_0(\tilde{t}_i) \) in \( W(B) \) via the projection map.

Take a presentation from a polynomial algebra \( \tilde{A} = W(k)[\tilde{T}_1, \ldots, \tilde{T}_N] \rightarrow B \) such that \( \tilde{T}_i \) is mapped to \( \tilde{t}_i \) for \( 1 \leq i \leq n \) and lift the Frobenius \( \phi \) on \( \tilde{B}^\dagger \) to a Frobenius \( F \) on \( \tilde{A}^\dagger \).

Following the notation used in [DLZ11] Proposition 3.1, we define a pseudovaluation (cf. [DLZ12] Definition 1.4) \( \mu_\epsilon \) on \( \tilde{A}^\dagger \) by
\[
\mu_\epsilon \left( \sum_{k \in \mathbb{N}^n} c_k \tilde{T}_1^{k_1} \cdots \tilde{T}_N^{k_N} \right) = \inf_{k, c_k \neq 0} \{ \| \nu_0(\tilde{t}_i) \| \} = \inf_{k, c_k \neq 0} \| \nu_0(\tilde{t}_i) \|,
\]
and define \( W^+(\tilde{A}^\dagger) \subset W(\tilde{A}^\dagger) \) by
\[
W^+(\tilde{A}^\dagger) := \{ (a_0, a_1, \ldots) \in W(\tilde{A}^\dagger) \mid \exists \epsilon > 0, \exists C \in \mathbb{R}, m + \mu_\epsilon(a_m) \geq C \text{ for all } m \}.
\]

**Lemma 10.4.** \( \nu_F(\tilde{T}_i) \in W(\tilde{A}^\dagger) \).

**Proof.** We find \( \mu_\epsilon(\tilde{T}_i) = -\epsilon \) by definition. By the argument of [DLZ11] Proposition 3.1, we have \( \mu_\epsilon/\mu_0(F^j(\tilde{T}_i)) \geq -\epsilon \). Let \( \alpha_{i,j} := w_j(\nu_0(\tilde{T}_i)) \subset \tilde{A}^\dagger \). Then \( F^j(\tilde{T}_i) = \tilde{T}_i^{p^j} \cdot \alpha_{i,j} \) and we find \( \mu_\epsilon/\mu_0(F^j(\tilde{T}_i)) = \mu_\epsilon/\mu_0(\alpha_{i,j}) - \epsilon/p^j \cdot \epsilon = \mu_\epsilon/\mu_0(\alpha_{i,j}) - \epsilon \). Hence we get \( \mu_\epsilon(\nu_0(\tilde{T}_i)) = \mu_\epsilon/\mu_0(\alpha_{i,j}) \geq 0 \). By the proof of [DLZ11] Proposition 3.1, it is equivalent to \( j + \mu_\epsilon(\nu_0(\tilde{T}_i)) \geq 0 \). This means \( \nu_F(\tilde{T}_i) \in W^+(\tilde{A}^\dagger) \).

**Lemma 10.5.** (cf. [DLZ11] Proposition 3.1) The projection map \( pr : W(\tilde{A}^\dagger) \rightarrow W(A) \) induces a map \( W^+(\tilde{A}^\dagger) \rightarrow W^+(A) \).

**Proof.** We define a pseudovaluation \( \mu_\epsilon \) on \( A \) by
\[
\mu_\epsilon \left( \sum_{k \in \mathbb{N}^n} c_k T_1^{k_1} \cdots T_N^{k_N} \right) = \min_{k, c_k \neq 0} \{ -\epsilon |k| \}, c_k \in k, |k| = k_1 + \cdots + k_N.
\]
Any element \( \alpha = (a_0, a_1, a_2, \ldots) \) of \( W(A) \) has a unique expression
\[
\alpha = \sum_k \xi_k X^k, \xi_k \in V^{a(k)}(W(k),
\]
where \( a(k) \) is the denominator of \( k \). The Gauss norm \( \gamma_\epsilon \) on \( W(A) \) is defined by
\[
\gamma_\epsilon(\alpha) = \inf_{k, \xi_k} \| \xi_k \| - \epsilon |k|.
\]
By [DLZ12] Proposition 2.18, \( \gamma_\epsilon(\alpha) = \inf_m \{ m + \mu_\epsilon(\alpha_m) \} \). Hence the projection map \( pr \) maps any element of \( W^+(\tilde{A}^\dagger) \) to an element of \( W^+(A) \).

One can prove when \( (1, a_1, a_2, \ldots) \) is in \( W^+(\tilde{A}^\dagger) \), \( (a_1, a_2, \ldots) \) is in \( W^+(\tilde{A}^\dagger) \) by easy calculations. We obtain
\[
W^+(\tilde{A}^\dagger) \cap (1 + V W(\tilde{A}^\dagger)) \subset 1 + V W^+(\tilde{A}^\dagger).
\]
By this argument and by Lemma 10.4, we have \( \nu_F(\tilde{T}_i) \in 1 + V W^+(\tilde{A}^\dagger) \). The element \( \lambda_F(\tilde{T}_i) \) belongs \( 1 + V W^+(\tilde{A}^\dagger) \) by Lemma 10.5. Using the functoriality,
As a result, we obtain the following diagram (cf. [Ill79] (1.3.18)):

\[ N^d \xrightarrow{(\text{id}, \lambda_i)} N^d \oplus (1 + V W^t(B)) \]

\[ \xrightarrow{t_i} \tilde{B}^t \xrightarrow{t_e} W^t(B), \]

where the right vertical arrow is induced by \( N^d \to W^t(B); e_i \mapsto [t_i] \) and the natural inclusion \( 1 + V W^t(B) \hookrightarrow W^t(B) \). One sees \( t_e(\tilde{t}_i) = [t_i] \cdot \lambda_i(\tilde{t}_i) \).

Now we can construct a map from the differential complex with logarithmic poles \( \Omega^\bullet_{B^t/W(k)}(\mathcal{D}) := \tilde{B}^t \otimes_{B} \Omega^\bullet_{B/W(k)}(\mathcal{D}) = \tilde{B}^t \otimes_{B} \Lambda^\bullet_{(B, N^d)}(W^t/k) \) defined in [Tsu99] §3 to the logarithmic overconvergent de Rham-Witt complex \( W^t \Omega^\bullet_{B/k}(D) := W^t \Lambda^\bullet_{(B, N^d)/k} \):

\[ \sigma : \Omega^\bullet_{B^t/W(k)}(\mathcal{D}) \to W^t \Omega^\bullet_{B/k}(D), \]

which is induced by the diagram (4). We see \( \sigma(d \log \tilde{t}_i) = d \log [t_i] + d \lambda(\tilde{t}_i)/\lambda(\tilde{t}_i) \) and \( \sigma(d\tilde{t}_i) = dt_e(\tilde{t}_i) \). Note that \( \lambda(\tilde{t}_i) \in 1 + V W^t(B) \subset W^t(B)^\times \).

We endow a filtration \( \{P_j\} \) on the de Rham complex and the de Rham-Witt complex by

\[ P_j(\Omega^i_{B^t/W(k)}(\mathcal{D})) := \text{image}(\Omega^i_{B^t/W(k)}(\mathcal{D}) \otimes_{B} \Omega^{i-j}_{B/W(k)} \to \Omega^i_{B^t/W(k)}(\mathcal{D})), \]

\[ P_j(\Omega^i_{B^t/W(k)}(\mathcal{D})) := \text{image}(\Omega^i_{B^t/W(k)}(\mathcal{D}) \otimes_{B} \Omega^{i-j}_{B/W(k)} \to \Omega^i_{B^t/W(k)}(\mathcal{D})), \]

\[ P_j(W^t \Omega^i_{B/k}(D)) := \text{image}(W^t \Omega^i_{B/k}(D) \otimes_{W(k)} W^t \Omega^{i-j}_{B/k} \to W^t \Omega^i_{B/k}(D)), \]

\[ P_j(W^t \Omega^i_{B/k}(D)) := P_j(W^t \Omega^i_{B/k}(D)) \cap W^t \Omega^i_{B/k}(D). \]

Since

\[ \text{image}(W^t \Omega^i_{B/k}(D) \otimes_{W^t(B)} W^t \Omega^{i-j}_{B/k} \to W^t \Omega^i_{B/k}(D)) \subset P_j(W^t \Omega^i_{B/k}(D)), \]

\( \sigma \) induces

\[ \sigma : P_j \Omega^\bullet_{B^t/W(k)}(\mathcal{D}) \to P_j W^t \Omega^\bullet_{B/k}(D). \]

The canonical morphism \( Gr_j W^t \Omega^\bullet_{B/k}(D) \to Gr_j W^t \Omega^\bullet_{B/k}(D) \) is injective. For a subset \( J = \{a_1, \ldots, a_j\} \) of \([1, d]\), put \( \tilde{B}_J := \tilde{B}/(t_{a_1}, \ldots, t_{a_j}) \).

**Lemma 10.6.** There are residue isomorphisms of de Rham complexes:

\[ \text{Res} : Gr_j(\Omega^\bullet_{B^t/W(k)}(\mathcal{D})) \to \bigoplus_{|J| = j} \Omega^{i-j}_{B^t/W(k)}, \]

\[ \text{Res} : Gr_j(\Omega^\bullet_{B^t/W(k)}(\mathcal{D})) \to \bigoplus_{|J| = j} \Omega^{i-j}_{B^t/W(k)}. \]

**Proof.** The first claim is [Del70] II Proposition 3.6. Since \( \tilde{B} \to \tilde{B}^t \) is flat ([Mer72] Proposition 3) and \( \tilde{B}^t \otimes_{\tilde{B}} \tilde{B}_J \simeq \tilde{B}_J \), one sees

\[ \tilde{B}^t \otimes_{\tilde{B}} Gr_j(\Omega^\bullet_{B^t/W(k)}(\mathcal{D})) \simeq Gr_j(\Omega^\bullet_{B^t/W(k)}(\mathcal{D})) \]

and therefore

\[ \bigoplus_{|J| = j} \Omega^{i-j}_{B^t/W(k)} \simeq \tilde{B}^t \otimes_{\tilde{B}} \left( \bigoplus_{|J| = j} \Omega^{i-j}_{B^t/W(k)} \right) \]

\[ \simeq \tilde{B}^t \otimes_{\tilde{B}} (Gr_j(\Omega^\bullet_{B^t/W(k)}(\mathcal{D}))) \]

\[ \simeq Gr_j(\Omega^\bullet_{B^t/W(k)}(\mathcal{D})). \]

\[ \square \]
We prove the de Rham-Witt version. Note that we have an isomorphism
\[ \text{Res} : Gr_r W\Omega^\bullet_{B/k}(D) \simeq \bigoplus_{|j| = j} W\Omega^\bullet_{B_j/k} \]
by a similar proof to that of Lemma 8.4.

**Lemma 10.7.** Let \( j \) be an integer such that \( 0 \leq j \leq d \). Let
\[
\begin{array}{c}
\mathbb{N}^d \\
\downarrow q \\
\mathbb{N}^d
\end{array} \rightarrow A = k[T_1, \ldots, T_N]
\]
be a presentation of the pre-log ring \((B, \mathbb{N}^d)\). We denote by \( \phi \) the natural morphism
\[ W_m \Omega^\bullet_{A/k}(D) := W_m A^{\bullet}_{(A, \mathbb{N}^d)/k} \rightarrow W_m \Omega^\bullet_{B/k}(D). \]
Let \( \omega = \sum_{k, p, |l - \omega| > j} \epsilon(\xi, p, k, p) \) be an element of \( W_m \Omega^\bullet_{A/k}(D) \).
Then we have \( \phi(\omega) = 0 \) if \( \phi(\omega) \in P_j W_m \Omega^\bullet_{B_j/k}(D) \).

**Proof.** By Proposition 3.11 (2), we have
\[ \ker \phi = W_m(I) W_m \Omega^\bullet_{A/k}(D) + dW_m(I) W_m \Omega^\bullet_{A/k}(D) \subset W_m \Omega^\bullet_{A/k}(D) \]
where \( I = \ker(q : A \rightarrow B) \).
When \( J = \{\alpha_1, \ldots, \alpha_r\} \) is a subset of \([1, d]\), we set
\[ A_J = k[T_1, \ldots, \overline{T}_{\alpha_1}, \ldots, \overline{T}_{\alpha_r}, \ldots, T_N] \]
and \( B_J := B/(t_{\alpha_1}, \ldots, t_{\alpha_r}) \). The map \( q \) induces a surjective map \( q_J : A_J \rightarrow B_J \).
We define \( I_J := \ker(q_J) \).
Let \( \omega = \sum_{k, p, |l - \omega| > j} \epsilon(\xi, p, k, p) \) be an element of \( W_m \Omega^\bullet_{A/k}(D) \) such that \( \phi(\omega) \in P_J W_m \Omega^\bullet_{B_J/k}(D) \).
By the construction of the (log) basic Witt differentials and Proposition 4.3 and [LZ04] Proposition 2.17, we obtain an isomorphism
\[ \bigoplus_{r = 0}^{d} \bigoplus_{|j| = r} W_m \Omega^\bullet_{A_j/k} \rightarrow W_m \Omega^\bullet_{A/k}(D). \]
We see \( \omega \) is uniquely written as the sum \( \sum_{d, |j| > j} d \log X_j \cdot \omega_j \) where \( \omega_j \in W_m \Omega^\bullet_{A_j/k} \).
via this isomorphism. For \( 0 \leq s \leq d \), we set \( \omega_s := \sum_{|j| = s} d \log X_j \cdot \omega_j \).
Let \( r \) be an integer such that \( j \leq r \leq d \) and \( \omega_{r+1} = \omega_{r+2} = \cdots = \omega_d = 0 \).
It follows that \( \omega = \omega_{r+1} + \cdots + \omega_r \) and \( \omega \in P_r W_m \Omega^\bullet_{A_j/k}(D) \).
We have the following commutative diagram:
\[
\begin{array}{c}
Gr_r W_m \Omega^\bullet_{A/k}(D) \rightarrow Gr_r W_m \Omega^\bullet_{B/k}(D) \\
\downarrow \text{Res} \quad \downarrow \text{Res} \\
\bigoplus_{|j| = r} W_m \Omega^\bullet_{A_j/k} \rightarrow \bigoplus_{|j| = r} W_m \Omega^\bullet_{B_j/k}. \\
\end{array}
\]
Here \( \phi_r \) is the induced morphism of \( \phi \).
We have \( \phi_r(\omega) = 0 \) because \( \phi(\omega) \in P_J W_m \Omega^\bullet_{B_J/k}(D) \) and \( r \geq j \).
The image of \( \omega \) is mapped to \( (\omega_j)_{|j| = r} \in \bigoplus_{|j| = r} W_m \Omega^\bullet_{A_j/k} \)
by the residue isomorphism.
For any \( J \) satisfying \(|J| = r \), we obtain \( \phi_J(\omega_J) = 0 \).
By [LZ05], \( \ker \phi_J \) is equal to
\[ W_m(I_J) W_m \Omega^\bullet_{A_J/k} + dW_m(I_J) W_m \Omega^\bullet_{A_J/k} \subset W_m \Omega^\bullet_{A_J/k}. \]
Then we see $d \log X_J \cdot \omega_J \in W_n \Omega^*_A(k)(D)$ belongs to  
\[ \ker \phi = W_n(I)W_n \Omega^*_A(k)(D) \]  
and  
\[ dW_n(I)W_n \Omega^*_A(k)(D) \subset W_n \Omega^*_A(k)(D). \]  
It shows $\phi(\omega_r) = \sum_{|J|=r} \phi(d \log X_J \cdot \omega_J) = 0$. If we consider $\omega' := \omega - \omega_r$ instead of $\omega$, we find that $\omega'$ has an expression of the form $\sum_{k, p, j \gg j} \varepsilon(\xi_{k, p}, k, P)$ and that $\phi(\omega') \in P_j W_n \Omega^*_B(k)(D)$. It is clear that $\omega'_1 = \omega'_2 = \cdots = \omega'_d = 0$.

By descending induction on $r$, we find $\omega$ is a sum of elements of $\ker \phi$.  

Lemma 10.8. The residue isomorphism $\text{Res}$ induces the residue isomorphism of overconvergent de Rham-Witt complexes:  
\[ \text{Res} : Gr_j W^1 \Omega^*_B(k)(D) \to \bigoplus_{|J|=j} W^1 \Omega^*_B(k). \]

Proof. Take a presentation  
\[ \mathbb{N}^d \to A = k[T_1, \ldots, T_N] \]
\[ \mathbb{N}^d \to B \]
and consider the following commutative diagram:  
\[ Gr_j W^1 \Omega^*_B(k)(D) \xrightarrow{\subset} Gr_j W^1 \Omega^*_B(k)(D) \xrightarrow{\text{Res}} \bigoplus_{|J|=j} W^1 \Omega^*_B(k). \]

We prove that the map $Gr_j W^1 \Omega^*_A(k)(D) \to Gr_j W^1 \Omega^*_B(k)(D)$ is surjective. It suffices to show that the map $P_j W^1 \Omega^*_A(k)(D) \to P_j W^1 \Omega^*_B(k)(D)$ induced by $q : W^1 \Omega^*_A(k)(D) \to W^1 \Omega^*_B(k)(D)$ is surjective.

Let $\tilde{\omega} \in P_j W^1 \Omega^*_B(k)(D)$. Since $W^1 \Omega^*_A(k)(D) \to W^1 \Omega^*_B(k)(D)$ is surjective, there exists an element $\omega = \sum_{k, p, j} \varepsilon(\xi_{k, p}, k, P) \in W^1 \Omega^*_A(k)(D)$ such that $q(\omega) = \tilde{\omega}$. Set $\omega_1 = \sum_{k, p, j \gg j} \varepsilon(\xi_{k, p}, k, P)$ and $\omega_2 = \sum_{k, p, j \gg j} \varepsilon(\xi_{k, p}, k, P)$. Then we see $\omega_1 = P_j W^1 \Omega^*_A(k)(D)$ and $\omega = \omega_1 + \omega_2$. By Lemma 10.7, we get $q(\omega_2) = 0$. Hence we have $q(\omega_1) = q(\omega) = \tilde{\omega}$. This implies $P_j W^1 \Omega^*_A(k)(D) \to P_j W^1 \Omega^*_B(k)(D)$ is surjective. Thus, we can assume $B$ is a polynomial ring $A = k[T_1, \ldots, T_N]$.

In this case, the elements $\omega$ of $Gr_j W^1 \Omega^*_A(k)(D)$ is in $Gr_j W^1 \Omega^*_A(k)(D)$ if and only if it can be written as an overconvergent sum of log basic Witt differentials $\varepsilon(\xi, k, P)$ such that $|\xi|_\infty = j$. Hence $\text{Res} : Gr_j W^1 \Omega^*_A(k)(D) \simeq \bigoplus_{|J|=j} W^1 \Omega^*_A(k)$ induces an isomorphism $\text{Res} : Gr_j W^1 \Omega^*_A(k)(D) \simeq \bigoplus_{|J|=j} W^1 \Omega^*_A(k)$.

Theorem 10.9. Let $\tau = 4d[\log_p \dim B]$. Then the kernel and cokernel of the homomorphism  
\[ \sigma_* : H^1(\Omega^*_B(k)/W(k)(D)) \to H^1(W^1 \Omega^*_B(k)(D)) \]
induced by $\tau$ are annihilated by $p^\tau$. In particular, $\sigma_*$ is an isomorphism if $\dim B < p$.

There is a rational isomorphism  
\[ H^*_\text{log-MW}(X, D)/K \simeq H^*(W^1 \Omega^*_B(k)(D) \otimes W(k) K) \]
between log Monsky-Washnizer cohomology and logarithmic overconvergent de Rham-Witt cohomology.
Proof. Consider the following commutative diagram:

\[
\begin{array}{ccc}
Gr_j(\Omega^•_{B/W(k)}(D)) & \longrightarrow & Gr_j(W^1\Omega^•_{B/k}(D)) \\
\sim & \text{Res} & \sim \text{Res} \\
\bigoplus_{|J|=j} \Omega^{•-j}_{B_j/W(k)} & \longrightarrow & \bigoplus_{|J|=j} W^1\Omega^{•-j}_{B_j/k}.
\end{array}
\]

Let \( \kappa := [\log_p \dim B] \). By [DLZ11] Proposition 3.24, the kernel and cokernel of

\[ H^i(\Omega^•_{B_j/W(k)}) \rightarrow H^i(W^1\Omega^•_{B_j/k}) \]

are annihilated by \( p^{2\kappa(J)} \) where \( \kappa(J) = [\log_p \dim B_j] \). Thus the kernel and cokernel of

\[ H^i(Gr_j(\Omega^•_{B_j/W(k)}(D))) \rightarrow H^i(Gr_j(W^1\Omega^•_{B_j/k}(D))) \]

are annihilated by \( p^{2\kappa} \). Consider the following exact sequences:

\[
\begin{array}{cccc}
0 & \longrightarrow & P_{j-1}\Omega^•_{B/W(k)}(D) & \longrightarrow & P_j\Omega^•_{B/W(k)}(D) & \longrightarrow & Gr_j(\Omega^•_{B/W(k)}(D)) & \longrightarrow & 0 \\
0 & \longrightarrow & P_{j-1}W^1\Omega^•_{B/k}(D) & \longrightarrow & P_jW^1\Omega^•_{B/k}(D) & \longrightarrow & Gr_jW^1\Omega^•_{B/k}(D) & \longrightarrow & 0.
\end{array}
\]

It induces a long exact sequences of cohomology of chain complexes:

\[
\begin{array}{cccc}
H^r(P_{j-1}\Omega^•_{B/W(k)}(D)) & \longrightarrow & H^r(P_j\Omega^•_{B/W(k)}(D)) & \longrightarrow & H^r(Gr_j(\Omega^•_{B/W(k)}(D))) & \longrightarrow & \\
H^r(P_{j-1}W^1\Omega^•_{B/k}(D)) & \longrightarrow & H^r(P_jW^1\Omega^•_{B/k}(D)) & \longrightarrow & H^r(Gr_jW^1\Omega^•_{B/k}(D)) \quad \quad \quad .
\end{array}
\]

By diagram chase and induction, we find that the kernel and cokernel of

\[ H^r(P_j\Omega^•_{B/W(k)}(D)) \rightarrow H^r(P_jW^1\Omega^•_{B/k}(D)) \]

are annihilated by \( p^{\alpha(j)} \) where \( \alpha(j) = 4j[\log_p \dim B] \). Since \( P_d\Omega^•_{B/W(k)}(D) = \Omega^•_{B/W(k)}(D) \) and \( P_dW^1\Omega^•_{B/k}(D) = W^1\Omega^•_{B/k}(D) \), we get the claim. \( \Box \)

**Proposition 10.10.** Let \( C := B \left[ \frac{1}{t_1 \cdots t_d} \right] \) and \( Y := \text{Spec} C \). Then there is canonical morphism

\[ \theta : W\Omega^•_{B/k}(D) \rightarrow W\Omega^•_{C/k}, \quad d\log[t_i] \mapsto \frac{[t_1 \cdots \hat{t}_i \cdots t_d]}{[t_1 \cdots t_d]} d[t_i]. \]

(1) \( \theta \) induces a morphism

\[ \theta^\dagger : W^1\Omega^•_{B/k}(D) \rightarrow W^1\Omega^•_{C/k}. \]

(2) \( \theta^\dagger \) induces an isomorphism of cohomology groups

\[ \theta^\dagger : H^r(W^1\Omega^•_{B/k}(D) \otimes_{W(k)} K) \rightarrow H^r(W^1\Omega^•_{C/k} \otimes_{W(k)} K). \]

**Proof.** (1) Choose a presentation

\[
\begin{array}{ccc}
\mathbb{N}^d & \longrightarrow & k[T_1, \ldots, T_N] \\
\downarrow & \quad & \downarrow \lambda \\
\mathbb{N}^d & \longrightarrow & B.
\end{array}
\]

\[ \begin{array}{ccc}
\mathbb{N}^d & \longrightarrow & k[T_1, \ldots, T_N] \\
\downarrow & \quad & \downarrow \lambda \\
\mathbb{N}^d & \longrightarrow & B.
\end{array} \]
Then we have a presentation $\lambda' : k[T_1, \ldots, T_N, S] \to C$ induced by $\lambda$ and $S \mapsto 1/(t_1 \cdots t_d)$. We obtain two surjective morphisms

$$\tau : W^*_{\ell k[T_1, \ldots, T_N, \mathbb{N}^d]/k} \to W^*_{B}/k(D),$$

$$\tau' : W^*_{\ell k[T_1, \ldots, T_N, S]/k} \to W^*_{C}/k.$$ 

Let $\omega$ be any element of $W^1\Lambda^*_{\ell k[T_1, \ldots, T_N, \mathbb{N}^d]/k}$. Then as Proposition 10.1, $\omega$ can be written as $\omega = \sum_{J \subseteq [1, d]} (\prod_{i \in J} \log X_i) \tau J$ where $\tau J \in W^1\Omega^*_{\ell k[T_1, \ldots, T_N]}/k$.

We set

$$\tilde{\omega} := \sum_{J \subseteq [1, d]} \left( \prod_{i \in J} (Y \cdot X_1 \cdots \tilde{X}_i \cdots X_d \cdot dX_i) \right) \cdot \tau J \in W^0\Omega^*_{\ell k[T_1, \ldots, T_N, S]/k}$$

where $Y = [S]$. It is easy to see that $\theta(\tau(\omega)) = \tau'(\tilde{\omega})$.

One finds $\tilde{\omega} \in W^0\Omega^*_{\ell k[T_1, \ldots, T_N, S]/k}$ because we have

$$\prod_{i \in J} (Y \cdot X_1 \cdots \tilde{X}_i \cdots X_d \cdot dX_i) \in W^0\Omega^*_{\ell k[T_1, \ldots, T_N, S]/k}, \quad \omega_J \in W^1\Omega^*_{\ell k[T_1, \ldots, T_N]/k}$$

and $W^0\Omega^*_{\ell k[T_1, \ldots, T_N, S]/k}$ is a ring. Therefore $\theta$ induces a morphism

$$\theta^1 : W^1\Omega^*_{\ell B}/k(D) \to W^1\Omega^*_{\ell C}/k.$$ 

(2) A Frobenius map on $\tilde{B}$ induces a ring map on $\tilde{C} := \tilde{B} [1/(t_1 \cdots t_d)]$ such that it is compatible with the Frobenius map $F$ on $W(k)$ and its reduction to $C$ coincides with the absolute Frobenius morphism on $C$. Hence we have a commutative diagram

$$\begin{array}{ccc}
\Omega^*_{\ell B}/W(k)(D) & \longrightarrow & W^*_{B}/k(D) \\
\downarrow & & \downarrow \\
\Omega^*_{\ell C}/W(k) & \longrightarrow & W^*_{C}/k.
\end{array}$$

By (1), it induces the following commutative diagram

$$\begin{array}{ccc}
\Omega^*_{\ell B}/W(k)(D) & \longrightarrow & W^1\Omega^*_{\ell B}/k(D) \\
\downarrow & & \downarrow \\
\Omega^*_{\ell C}/W(k) & \longrightarrow & W^1\Omega^*_{\ell C}/k.
\end{array}$$

Taking cohomology, we obtain the following commutative diagram

$$\begin{array}{ccc}
H^*_{\text{log-MW}}((X, D)/K) & \longrightarrow & H^*(W^1\Omega^*_{\ell B}/k(D) \otimes W(k) K) \\
\downarrow & & \downarrow \\
H^*_{\text{MW}}(Y/K) & \longrightarrow & H^*(W^1\Omega^*_{\ell C}/k \otimes W(k) K).
\end{array}$$

The horizontal arrows and the left vertical arrow are isomorphisms by Theorem 10.9 and [DLZ11] Corollary 3.25 and [Tsu99] Theorem 3.5.1. Hence the right vertical arrow is also an isomorphism. \qed
10.3. Sheaf of overconvergent log de Rham-Witt complex. In this subsection, we define the Zariski sheaf of overconvergent log de Rham-Witt complexes for smooth schemes with simple normal crossing divisor.

**Proposition 10.11.** (cf. [DLZ11] Proposition 1.2)

Let \( X = \text{Spec} \ B \) be a smooth affine scheme and \( D \) a simple normal crossing divisor on \( X \). We assume that there is a global chart \( \alpha : \mathbb{N}^d \to \mathcal{M}_{(X,D)} \). Then we have a pre-log structure

\[
\beta = \beta_\alpha : \mathbb{N}^d \to \mathcal{M}_{(X,D)}(X) \to \mathcal{O}_X(X) = B
\]

of \( B \). Let \( D_1, \ldots, D_j \) be the irreducible components of \( D \). We also assume that there is an étale morphism \( X \to \mathbb{A}^n_k = \text{Spec} k[T_1, \ldots, T_n] \) such that \( w(T_i) = \beta(e_i) =: t_i \in B \) and that \( D_i \) is defined by \( t_i = 0 \). We fix a nonnegative integer \( r \).

(1) We denote by \( f \in B \) an arbitrary element. Then \( \beta \) induces a pre-log structure \( \beta_f : \mathbb{N}^d \to B \to B_f \) of \( B_f \). The presheaf

\[
D(f) \mapsto W^r\Lambda(B_f, \mathbb{N}^d)/k
\]

defines a sheaf on the Zariski topology on \( X \). We denote by \( W^r\Lambda_{(X,D)}/k,\alpha \) this sheaf.

(2) The Zariski sheaf \( W^r\Lambda_{(X,D)}/k,\alpha \) is independent of the choice of charts \( \alpha \). We denote by \( W^r\Lambda_{(X,D)/k} \) this Zariski sheaf.

(3) The Zariski cohomology of the sheaf \( W^r\Lambda_{(X,D)/k} \) vanishes in degree \( j > 0 \), i.e.,

\[
H^j_{\text{Zar}}(X, W^r\Lambda_{(X,D)/k}) = 0.
\]

**Proof.** Let \( \{f_i\}_{i=1}^t \) be a finite family of elements of \( B \) such that \( f_i \) generate \( B \) as an ideal. For \( 1 \leq i_1 < \cdots < i_s \leq t \), we denote by \( \mathcal{M}_{(X,D)} \) the intersection \( D(f_{i_1}) \cap \cdots \cap D(f_{i_s}) \). For simplicity, we set \( B_{i_1 \cdots i_s} := B_{f_i \cdots f_s} \). We define a Čech complex \( C^\bullet = C^\bullet((X,D),\alpha) \) by \( C^s := \bigoplus_{1 \leq i_1 < \cdots < i_s \leq t} W^r\Lambda_{(B_{i_1 \cdots i_s}, \mathbb{N}^d)/k} \). Then we see the filtration \( \{P_j\} \) which we introduced in §10.2 induces a filtration on \( C^\bullet \):

\[
P_j C^\bullet = \bigoplus_{1 \leq i_1 < \cdots < i_s \leq l} P_j W^r\Lambda_{(B_{i_1 \cdots i_s}, \mathbb{N}^d)/k}.
\]

Set \( G_{r,j} C^\bullet := P_j C^\bullet / P_{j-1} C^\bullet \). Then one has an exact sequence of complexes

\[
0 \to G_{r,j-1} C^\bullet \to P_j C^\bullet \to G_j C^\bullet \to 0.
\]

By the Poincaré residue map, one obtain a commutative diagram

\[
\begin{array}{ccc}
\bigoplus_{1 \leq i_1 < \cdots < i_{s-1} \leq l} \bigoplus_{|J|=j} W^r\Omega_{B_{i_1 \cdots i_{s-1}, k}}^{s-j} / k & \xrightarrow{\text{Res}^{-1}} & G_{r,j} C^{s-1} \\
\downarrow & & \downarrow \\
\bigoplus_{1 \leq i_1 < \cdots < i_{s} \leq l} \bigoplus_{|J|=j} W^r\Omega_{B_{i_1 \cdots i_{s}, k}}^{s-j} / k & \xrightarrow{\text{Res}^{-1}} & G_{r,j} C^{s}.
\end{array}
\]

Here \( B_{i_1 \cdots i_j} \) denotes the localization \( B_{f_{i_1} \cdots f_{i_j}} \) of \( B \) for \( J = \{\alpha_1, \ldots, \alpha_j\} \subset [1, d] \).

We see that

\[
\bigoplus_{1 \leq i_1 < \cdots < i_s \leq l} \bigoplus_{|J|=j} W^r\Omega_{B_{i_1 \cdots i_s, k}}^{s-j} / k \cong \bigoplus_{|J|=j} \tilde{C}^j((\text{Spec} \ B J, W^r\Omega_{B_{j,k}}^{s-j})),
\]

where \( \tilde{C}^\bullet((\text{Spec} \ B J, W^r\Omega_{B_{j,k}}^{s-j})) \) is the Čech complex with degree \( s \) elements given by \( \tilde{C}^s((\text{Spec} \ B J, W^r\Omega_{B_{j,k}}^{s-j})) = \bigoplus_{1 \leq i_1 < \cdots < i_s \leq l} W^r\Omega_{B_{i_1 \cdots i_s, k}}^{s-j} / k \). Hence the Poincaré
residue map induces an isomorphism \( \text{Res} : \text{Gr} \alpha C^* \cong \bigoplus_{j\in J} \hat{C}^*(\text{Spec } B_j, W^\dagger \Omega_{B_j/k}^{\alpha-j}) \).

The boundary morphism of \( \text{Gr}_j C^* \) is identified to the direct sum of boundary morphisms of \( \{ C^*(\text{Spec } B_j, W^\dagger \Omega_{B_j/k}^{\alpha-j}) \}_{j\in J} \). It follows that \( \text{Gr}_j C^* \) is exact by the long exact sequence of cohomology and [DLZ11] Proposition 1.6. We find \( P_j C^* \) is exact by induction for all \( j \). As \( P_0 C^* = C^* \), we get (1).

We prove (2). Let \( \alpha' : \mathbb{N}^d \to \mathcal{M}(X,D) \) be another chart. We have an isomorphism \( W^\alpha C^*_\phi (\mathbb{N}^d, \beta_\alpha)_j \cong W_{\phi}(\mathbb{N}^d, \beta_\alpha)_j^\dagger \) by Proposition-Definition 3.10. Let \( \ell'_j := \beta_\alpha(\epsilon_i) \).

We set \( B'_j := B_{\ell'_1, \ldots, \ell'_j} \) for \( J = \{ \alpha_1, \ldots, \alpha_j \} \subset [1, d] \). Since \( B_j \simeq B'_j \), we see
\[
W^\dagger \Omega_{B_{\ell'_1, \ldots, \ell'_j}/k}^{\alpha-j} \cong W^\dagger \Omega_{B'_{\ell'_1, \ldots, \ell'_j}/k}^{\alpha-j}
\]
for all \( 1 \leq i_1 < \cdots < i_s \leq l \). Hence we obtain an isomorphism \( \text{Gr}_j C^*((X,D), \alpha) \cong \text{Gr}_j C^*((X,D), \alpha') \). Using the exact sequence and induction, we see that
\[
C^*((X,D), \alpha) \cong C^*((X,D), \alpha').
\]
This shows (2).

(3) is deduced from the exactness of the Čech complex. \( \square \)

**Definition 10.12.** Let \( X \) be a smooth scheme over \( k \) and \( D \) be a simple normal crossing divisor on \( X \). Then for any point \( x \) of \( X \), there is an affine neighbourhood \( U \) of \( x \) in \( X \) such that the log scheme \( (U,D|_U) \) admits a chart of the form \( \mathbb{N}^d \to \mathcal{M}(U,D|_U) \) for some \( d \) (cf. [Kat89], §8.1) and that there is an étale morphism \( U \to K_n^s \) for some \( n \).

By Proposition 10.11, the Zariski sheaves \( W^\dagger \Lambda^r_{(U,D|_U)/k} \) glue together to give a Zariski sheaf \( W^\dagger \Lambda^r_{(X,D)/k} \).

We call \( W^\dagger \Lambda^r_{(X,D)/k} \) the sheaf of overconvergent log de Rham-Witt complexes.

### 10.4. Comparison with rigid cohomology

We generalize our results to global cases. Let \( X \) be a smooth quasi-projective variety over a perfect field \( k \) and \( D \) an SNCD of \( X \) over \( k \). Let \( j : Y := X \setminus D \to X \) be the canonical open immersion. We have the overconvergent de Rham-Witt complex \( W^\dagger \Omega^*_{Y/k} \) for a smooth variety \( Y \) ([DLZ11] §1), and the overconvergent log de Rham-Witt complex \( W^\dagger \Lambda^r_{(X,D)/k} \) for a smooth variety with SNCD. The canonical morphism \( W^\dagger \Lambda^r_{(X,D)/k} \to j_! W^\dagger \Omega^*_{Y/k} \) induces the map \( W^\dagger \Lambda^r_{(X,D)/k} \to j_! W^\dagger \Omega^*_{Y/k} \). Davis-Langer-Zink defined a map from the rigid cohomology to the overconvergent de Rham-Witt cohomology
\[
\mathbb{R} \Gamma_{\text{rig}}(Y/k) \to \mathbb{R} \Gamma_{\text{Zar}}(Y, W^\dagger \Omega^*_{Y/k}) \otimes K
\]
and showed this is a quasi-isomorphism when \( Y \) is smooth and quasi-projective over \( k \) ([DLZ11] Theorem 4.40).

**Lemma 10.13.** We have a canonical morphism
\[
\mathbb{R} \Gamma_{\text{Zar}}(X, W^\dagger \Lambda^r_{(X,D)/k}) \to \mathbb{R} \Gamma_{\text{Zar}}(Y, W^\dagger \Omega^*_{Y/k}).
\]

**Proof.** The canonical morphism \( W^\dagger \Lambda^r_{(X,D)/k} \to j_! W^\dagger \Omega^*_{Y/k} \) induces a map
\[
\mathbb{R} \Gamma_{\text{Zar}}(X, W^\dagger \Lambda^r_{(X,D)/k}) \to \mathbb{R} \Gamma_{\text{Zar}}(X, j_! W^\dagger \Omega^*_{Y/k}).
\]

We know \( \mathbb{R} \Gamma_{j_*} W^\dagger \Omega^*_{Y/k} \) is the sheaf associated to the presheaf
\[
V \mapsto H^q(V \cap Y, W^\dagger \Omega^*_{Y/k})
\]
([Sta15] Cohomology of Sheaves, Lemma 7.3) and for open set \( V \) of \( X \),
\[
\mathbb{R} \Gamma_{j_*} W^\dagger \Omega^*_{Y/k}(V \cap Y) = (\mathbb{R} \Gamma_{j_*} W^\dagger \Omega^*_{Y/k})|_V.
\]
where \( j' : V \cap Y \rightarrow V \) is the restriction of \( j \). Moreover, if \( Y \) is affine, the Zariski cohomology of \( W^+\Omega^n_{Y/k} \) vanishes in degrees \( q > 0 \) ([DLZ11] Proposition 1.2 (b)). So we conclude that \( \mathbb{R}\Gamma_{Zar}(X, j_*W^+\Omega^n_{Y/k}) = 0 \) for \( q > 0 \). Hence we get

\[
\mathbb{R}\Gamma_{Zar}(X, j_*W^+\Omega^n_{Y/k}) \simeq \mathbb{R}\Gamma_{Zar}(Y, W^+\Omega^n_{Y/k})
\]

([Sta15] Cohomology of Sheaves, Lemma 14.6) and the desired map. □

By this lemma we have a diagram

\[
\mathbb{R}\Gamma_{rig}(Y/K) \longrightarrow \mathbb{R}\Gamma_{Zar}(Y, W^+\Omega^n_{Y/k}) \otimes K.
\]

We show that the vertical arrow is a quasi-isomorphism. Take an open covering \( \{X^i\}_{i \in I} \) of \( X \) by affine schemes \( X^i = \text{Spec} A_i \) which satisfy the following condition: There is an étale morphism \( X^i \rightarrow A^i \) and \( D^i := X^i \cap D \) is defined by \( t_1 \cdots t_s = 0 \) for some \( s \leq r_i \), where \( t_j \) is the image of \( t_i \) of \( A^i \) in \( A^j \). \( A^i \) has a smooth lifting \( \tilde{A}^i \) over \( W(k) \) which has an étale morphism \( \tilde{X}^i = \text{Spec} \tilde{A}^i \rightarrow \mathbb{A}^i_{W(k)} \) such that \( \tilde{D}^i \) defined by \( t_1, \ldots, t_s \) is a lifting of \( D^i \).

For a subset \( \mathcal{I} \subset I \) we set \( X^\mathcal{I} := \bigcap_{i \in \mathcal{I}} X^i \), \( D^\mathcal{I} := X^\mathcal{I} \cap D \), \( Y^\mathcal{I} := X^\mathcal{I} \setminus D^\mathcal{I} \). Since \( X \) is quasi-projective, each \( X^\mathcal{I} \) is a smooth quasi-projective affine scheme and it satisfies the condition indicated above.

Consider the following commutative diagram of simplicial schemes:

\[
\begin{array}{ccc}
Y^\bullet & \xrightarrow{\theta_Y} & Y \\
\downarrow{j^\bullet} & & \downarrow{j} \\
X^\bullet & \xrightarrow{\theta_X} & X \\
\end{array}
\]

This diagram induces the following diagram:

\[
W^+\Lambda^\bullet_{(X,D)/k} \otimes K \xrightarrow{\sim} \mathbb{R}\theta_X^*(W^+\Lambda^\bullet_{(X,D)/k}) \otimes K \xrightarrow{j_*} (\mathbb{R}\theta_Y^*W^+\Omega^\bullet_{Y/k}) \otimes K \xrightarrow{\sim} \mathbb{R}\theta_X^*((j^\bullet)_*W^+\Omega^\bullet_{Y/k}) \otimes K.
\]

The horizontal arrows are quasi-isomorphisms. By Proposition 10.10, the right vertical arrow is a quasi-isomorphism. Hence the left vertical arrow is also a quasi-isomorphism. Therefore, we get the following comparison theorem.

**Theorem 10.14.** Let \( X \) be a smooth quasi-projective variety over a perfect field \( k \) and \( D \) be a simple normal crossing divisor of \( X \). Then we have an isomorphism

\[
H^*_\text{rig}(Y/K) \simeq \mathbb{H}^*_\text{Zar}(X, W^+\Lambda^\bullet_{(X,D)/k} \otimes_{W(k)} K).
\]

**References**

[Ber74] Pierre Berthelot. *Cohomologie cristalline des schémas de caractéristique \( p > 0 \).* Lecture Notes in Mathematics, Vol. 407. Springer-Verlag, Berlin-New York, 1974.

[BO78] Pierre Berthelot and Arthur Ogus. *Notes on crystalline cohomology.* Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1978.

[CD14] Joachim Cuntz and Christopher Deninger. Witt vector rings and the relative de rham witt complex. *http://arxiv.org/abs/1410.5249,* 2014.

[CLS98] Bruno Chiarellotto and Bernard Le Stum. Sur la pureté de la cohomologie cristalline. *C. R. Acad. Sci. Paris Sér. I Math.*, 326(8):961–963, 1998.
[Del70] Pierre Deligne. *Équations différentielles à points singuliers réguliers*. Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, Berlin-New York, 1970.

[DLZ11] Christopher Davis, Andreas Langer, and Thomas Zink. Overconvergent de Rham-Witt cohomology. *Ann. Sci. Éc. Norm. Supér. (4)*, 44(2):197–262, 2011.

[DLZ12] Christopher Davis, Andreas Langer, and Thomas Zink. Overconvergent Witt vectors. *J. Reine Angew. Math.*, 668:1–34, 2012.

[Fu11] Lei Fu. *Etale cohomology theory*, volume 13 of Nankai Tracts in Mathematics. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.

[HK94] Osamu Hyodo and Kazuya Kato. Semi-stable reduction and crystalline cohomology with logarithmic poles. *Astérisque*, (223):221–268, 1994. Périodes p-adiques (Bures-sur-Yvette, 1988).

[HM03] Lars Hesselholt and Ib Madsen. On the K-theory of local fields. *Ann. of Math. (2)*, 158(1):1–113, 2003.

[HM04] Lars Hesselholt and Ib Madsen. On the De Rham-Witt complex in mixed characteristic. *Ann. Sci. École Norm. Sup. (4)*, 37(1):1–43, 2004.

[Ill75] Luc Illusie. Report on crystalline cohomology. In *Algebraic geometry (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974)*, pages 459–478. Amer. Math. Soc., Providence, R.I., 1975.

[Ill79] Luc Illusie. Complexe de de Rham-Witt et cohomologie cristalline. *Ann. Sci. École Norm. Sup. (4)*, 12(4):501–661, 1979.

[IR83] Luc Illusie and Michel Raynaud. Les suites spectrales associées au complexe de de Rham-Witt. *Inst. Hautes Études Sci. Publ. Math.*, (57):73–212, 1983.

[Kat89] Kazuya Kato. Logarithmic structures of Fontaine-Illusie. In *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, pages 191–224. Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[Kat96] Fumiharu Kato. Log smooth deformation theory. *Tohoku Math. J. (2)*, 48(3):317–354, 1996.

[LZ04] Andreas Langer and Thomas Zink. De Rham-Witt cohomology for a proper and smooth morphism. *J. Inst. Math. Jussieu*, 3(2):231–314, 2004.

[LZ05] Andreas Langer and Thomas Zink. Gauss-Manin connection via Witt-differentials. *Nagoya Math. J.*, 179:1–16, 2005.

[Mer72] David Meredith. Weak formal schemes. *Nagoya Math. J.*, 45:1–38, 1972.

[Mok93] A. Mokrane. La suite spectrale des poids en cohomologie de Hyodo-Kato. *Duke Math. J.*, 72(2):301–337, 1993.

[MW68] P. Monsky and G. Washnitzer. Formal cohomology. I. *Ann. of Math. (2)*, 88:181–217, 1968.

[Nak00] Chikara Nakayama. Degeneration of l-adic weight spectral sequences. *Amer. J. Math.*, 122(4):721–733, 2000.

[Nak05] Yukiyoshi Nakajima. p-adic weight spectral sequences of log varieties. *J. Math. Sci. Univ. Tokyo*, 12(4):513–661, 2005.

[Nak15] Yukiyoshi Nakajima. Weight filtrations on log crystalline cohomologies and weight filtrations on infinitesimal cohomologies in mixed characteristics. preprint, 2015.

[NS08] Yukiyoshi Nakajima and Atsushi Shiho. Weight filtrations on log crystalline cohomologies of families of open smooth varieties. Lecture Notes in Mathematics, Vol. 1959. Springer-Verlag, Berlin, 2008.

[Ogu06] Arthur Ogus. Lectures on logarithmic algebraic geometry. *http://math.berkeley.edu/~ogus/preprints/log_book/logbook.pdf*, 2006.

[Ols03] Martin C. Olsson. Logarithmic geometry and algebraic stacks. *Ann. Sci. École Norm. Sup. (4)*, 36(5):747–791, 2003.

[Ols07] Martin C. Olsson. Crystalline cohomology of algebraic stacks and Hyodo-Kato cohomology. *Astérisque*, (316):1–412, 2007.

[Shi02] Atsushi Shiho. Crystalline fundamental groups. II. Log convergent cohomology and rigid cohomology. *J. Math. Sci. Univ. Tokyo*, 9(1):1–163, 2002.

[Sta15] The Stacks Project Authors. *Stacks Project*. *http://stacks.math.columbia.edu*, 2015.

[Tsu99] Nobuo Tsuzuki. On the Gysin isomorphism of rigid cohomology. *Hiroshima Math. J.*, 29(3):479–527, 1999.

[Tsu10] Takeshi Tsuji. On nearby cycles and D-modules of log schemes in characteristic p > 0. *Compos. Math.*, 146(6):1552–1616, 2010.