Kaon Weak Decays in Chiral Theories*

M.D. Scadron

Physics Dept. Univ. of Arizona, Tucson AZ 85721, USA

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The ten nonleptonic weak decays $K \to 2\pi$, $K \to 3\pi$, $K_L \to 2\gamma$, $K_S \to 2\gamma$, $K_L \to \pi^0 2\gamma$, are predicted for a chiral pole model based on the linear sigma model theory which automatically satisfies the partial conservation of axial current (PCAC) hypothesis. These predictions, agreeing with data to the 5% level and containing no or at most one free parameter, are compared with the results of chiral perturbation theory (ChPT). The latter ChPT approach to one-loop level is known to contain at least four free parameters and then predicts a $K_L \to \pi^0 \gamma \gamma$ rate which is 60% shy of the experimental value. This suggests that ChPT is an unsatisfactory approach towards predicting kaon weak decays.

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I. INTRODUCTION

In this paper we contrast the kaon weak decay predictions of the two chiral theories based on (i) the linear sigma model (LσM) characterized here by the non-loop tree graphs of the chiral pole model (CPM); (ii) chiral perturbation theory (ChPT) involving loop diagrams. Prior studies of the CPM and its direct link with the model-independent approach of current algebra - partial conservation of axial currents (PCAC) were worked out in refs.[1], while the LσM-CPM extension was given in ref.[2], including the weak decays $K \to 2\pi$, $K \to 3\pi$, $K_L \to \gamma \gamma$, $K_S \to \gamma \gamma$ and $K_L \to \pi^0 \gamma \gamma$. At about the same time, the predictions of ChPT were summarized for $K \to 2\pi$ and $K \to 3\pi$ decays in ref.[3] and extended to $K_S \to \gamma \gamma$ and $K_L \to \pi^0 \gamma \gamma$ in ref.[4].

We shall show that the former LσM-CPM-PCAC approach predicts the above-mentioned 10 weak decay amplitudes to within 5% accuracy in terms of no or at most one free parameter. In contrast, the latter ChPT formalism based on 10 strong interaction parameters $L_1 - L_{10}$ requires at least 4 weak interaction parameters $[3] c_2, c_3, G_1, G_2$ to explain the 7 decays $K \to 2\pi$, $K \to 3\pi$ and even then the one-loop ChPT prediction of the $K_L \to \pi^0 \gamma \gamma$ rate recovers only 35% of the observed rate [4].

In Sec.II we study the LσM-CPM chiral symmetry scheme for $K \to 2\pi$ and $K \to 3\pi$ decays, predicting all 7 amplitudes in terms of tree graphs and one $\Delta I = 1/2$ scale. The latter is at first taken as the one fitted parameter in this scheme in Sec.II. Then it too will be predicted from the CPM tree approximation for $K_L \to \gamma \gamma$ in Sec.III, or from the (quark tadpole) one-loop order graph for the $\Delta I = 1/2$ $s \to d$ self energy in Sec.IV. Also in Sec.III we extend this LσM-CPM chiral symmetry approach to tree graphs for $K_S \to \gamma \gamma$ and $K_L \to \pi^0 \gamma \gamma$. Finally in Sec.V we summarize the ChPT results for the 10 weak decays and indicate where two $\Delta I = 1/2$ and two independent $\Delta I = 3/2$ fitted parameters and also one $K_L \to \pi^0 \gamma \gamma$ fitted parameter are required. We draw our conclusions in Sec.VI.

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II. LσM-CPM-PCAC APPROACH TO \( K \to 2\pi \) AND \( K \to 3\pi \) DECAYS

The strong interaction SU(2) linear \( \sigma \) model (LσM) lagrangian and its implication for chiral symmetry and partial conservation of axial currents (PCAC) are well-documented in text books [5]. The natural extension of the SU(2) LσM (for pseudoscalar \( \pi \) and scalar \( \sigma \) mesons) to weak interactions of kaons is via a chiral pole model (CPM) involving again intermediate \( \pi \) and \( \sigma \) mesons [2,6].

Specifically the dominant \( \Delta I = 1/2 \) CPM graph is depicted in Fig. 1 for parity-violating (pv) \( K_S \to \pi \pi \) decays via \( K_S^{PV} \to \sigma \to 2\pi \), with the latter \( \sigma \to \pi \pi \) transition given by the LσM vertex [5] \( \langle \pi \pi | \sigma \rangle = -m_\sigma^2/f_\pi \) for \( f_\pi \approx 93 \text{ MeV} \). The former weak vertex \( \langle \sigma | H_w^{\mu\nu} | K_S \rangle \) is given by the chiral symmetry relation

\[
\langle \sigma | H_w^{\mu\nu} | K_S \rangle = \langle \pi^o | H_w^{\mu\nu} | K_L \rangle.
\]

Since the intermediate \( \sigma \) resonance has a broad width as suggested by many experiments [7], or from the LσM theory or mended chiral symmetry [8] with \( \Gamma_\sigma = m_\sigma \approx 700 \text{ MeV} \), the \( \Delta I = 1/2 \) CPM \( K_S \to 2\pi \) amplitude in the chiral limit based on Fig. 1 is [1,2]

\[
\langle \pi \pi | H_w^{\mu\nu} | K_S \rangle = \langle \pi \pi | \sigma \rangle \frac{1}{m_K^2 - m_\sigma^2 + i m_\sigma \Gamma_\sigma} \langle \sigma | H_w^{\mu\nu} | K_L \rangle
\]

\[
\approx (i/f_\pi) \langle \pi^o | H_w^{\mu\nu} | K_L \rangle
\]

when \( m_\pi = 0 \). Here we have used (1) and dropped the small real part of (2) relative to its imaginary part since \( |m_K^2 - m_\sigma^2| << m_\sigma^2 \). This LσM-CPM result (2) also is a consequence [1] of PCAC applied to both pions (PCAC consistency) with charge commutator amplitude \( M_{CC} \):

\[
\langle \pi_1 \pi_2 | H_w | K_S \rangle = M_{CC1} + M_{CC2} + O(m_\sigma^2/m_K^2).
\]

Returning to the CPM version (2), the value \( \langle \pi^o | H_w^{\mu\nu} | K_L \rangle | \approx 3.2 \times 10^{-8} \text{ GeV}^2 \) to be found in Sec.III from the CPM version of \( K_L \to \gamma\gamma \) in turn sets the \( K_S \to 2\pi^o \ \Delta I = 1/2 \) scale from eq.(2) for \( f_\pi \approx 93 \text{ MeV} \):

\[
|\langle \pi^o \pi^o | H_w^{\mu\nu} | K_S \rangle|_{CPM} \approx |\langle \pi^o | H_w^{\mu\nu} | K_L \rangle|/f_\pi \approx 34 \times 10^{-8} \text{ GeV}.
\]

The CPM extension to \( K_S \to \pi^+ \pi^- \) includes Fig. 1 along with Fig. 2 for charged pions. These latter W emission graphs (\( W_{em} \)) have small \( \Delta I = 1/2 \) and \( \Delta I = 3/2 \) parts and can be computed using the “vacuum saturation” method [9]

\[
|\langle \pi^+ \pi^- | H_w^{\mu\nu} | K_S \rangle|_{W_{em}} = (G_F s_1 c_1/2 \sqrt{2}) |\langle \pi^+ | A_{\mu} | 0 \rangle \langle \pi^- | V^\mu | K_S \rangle| + \leftrightarrow
\]

\[
= G_F s_1 c_1 f_\pi (m_K^2 - m_\sigma^2)/\sqrt{2} \approx 4 \times 10^{-8} \text{ GeV},
\]

for V-A chiral left-handed vector currents simulating the vector \( W \). Then the total \( K_S \to \pi^+ \pi^- \) weak CPM amplitude is the sum of (4) and (5):

\[
|\langle \pi^+ \pi^- | H_w^{\mu\nu} | K_S \rangle|_{CPM} \approx (34 + 4) \times 10^{-8} \text{ GeV} = 38 \times 10^{-8} \text{ GeV}.
\]
Lastly the pure $\Delta I = 3/2 \ K^+ \to \pi^+\pi^0$ amplitude can be computed in the CPM via the analog W emission (or vacuum saturation) value [9]

$$
|\langle \pi^+ \pi^0 | H_{\pi\pi}^\alpha | K^+ \rangle |_{\text{CPM}} = (G_F s_1 c_1 / 2 \sqrt{2}) |\langle \pi^+ | A_\mu | 0 \rangle \langle \pi^0 | V^\mu | K^+ \rangle |
$$

$$
= G_F s_1 c_1 f_+(0) f_\pi (m_K^2 - m_\pi^2) / 2\sqrt{2} \approx 1.83 \times 10^{-8} \text{ GeV.} \tag{7}
$$

In (5) and (7) we invoke $f_+(0) \approx 0.96$ as the $\mathcal{O}(\varepsilon^2)$ small deviation from the nonrenormalization limit of unity as found in various quark model schemes [10].

Although the above CPM is quite simple (yet manifesting chiral symmetry), it is also very accurate as the following experimental (exp) amplitudes $M_{\pi\pi}$ indicate [11]:

$$
|M_{K^+}^\pm|_{\text{exp}} = (39.08 \pm 0.08) \times 10^{-8} \text{ GeV}
$$

$$
|M_{K^0}^{00}|_{\text{exp}} = (37.11 \pm 0.17) \times 10^{-8} \text{ GeV}
$$

$$
|M_{K^0}^{+0}|_{\text{exp}} = (1.833 \pm 0.006) \times 10^{-8} \text{ GeV.} \tag{8}
$$

The CPM predictions (4), (6), (7) are respectively within 1%, 8%, 1% of the observed $K_{2\pi}$ amplitudes in (8).

Similar 5% accuracy for these $K_{2\pi}$ $\Delta I = 1/2$ and $\Delta I = 3/2$ scales follows by invoking “PCAC consistency” [1] of eq. (3), giving

$$
a_{S}^+ = i \langle \pi^+ \pi^- | H_{\pi\pi} | K_S \rangle = \langle \pi^+ | H_{\pi\pi} | K^+ \rangle (1 - m_\pi^2 / m_K^2) / f_\pi
$$

$$
a_{S}^{00} = i \langle \pi^0 \pi^0 | H_{\pi\pi} | K_S \rangle = \langle \pi^0 | H_{\pi\pi} | K_L \rangle (1 - m_\pi^2 / m_K^2) / f_\pi
$$

$$
a_{+}^{+0} = i \langle \pi^+ \pi^- | H_{\pi\pi} | K^+ \rangle = \langle \pi^+ | H_{\pi\pi} | K^+ \rangle (1 - m_\pi^2 / m_K^2) / 2 f_\pi
$$

$$
+ \sqrt{2} \langle \pi^0 | H_{\pi\pi} | K^0 \rangle (1 - m_\pi^2 / m_K^2) / 2 f_\pi. \tag{9}
$$

Note that the $a_{S}^{00}$ equation is compatible with CPM-PCAC given by (2). Note too the explicit factors of $(1 - m_\pi^2 / m_K^2)$ occurring in eqs.(9) which force all $K_{2\pi}$ amplitudes to vanish in the strict SU(3) limit, a result originally obtained by Cabibbo and Gell-Mann [12] due to CP and SU(3) invariance. Then one models the reduced matrix elements $\langle \pi^+ | H_{\pi\pi} | K^+ \rangle$ and $\langle \pi^0 | H_{\pi\pi} | K^0 \rangle$ via the s-d quark self energy and the W-exchange graphs [2,13] or alternatively uses a pure meson loop model [1]. Lastly one can further tune the above 5% CPM discrepancy to the 2% level by accounting for final-state $\pi\pi$ interactions [1,13] given the observed $\delta_0 - \delta_2 \approx 57^0$ phase shift difference, but we shall not do so here.

Instead we accept the above CPM predictions for the three $K_{2\pi}$ amplitudes to 5% accuracy (but containing no free parameters), and extend the scheme to the four $K_{3\pi}$ amplitudes via PCAC consistency [1,13] in analogy with (2) and (9):

$$
A_{L}^{+0} = i \langle \pi^+ \pi^- \pi^0 | H_{\pi\pi} | K_L \rangle = - \langle \pi^0 \pi^0 | H_{\pi\pi} | K_S \rangle (1 - m_\pi^2 / m_K^2) / 4 f_\pi
$$
\[ A_{-}^{000} = i\langle \pi^0 \pi^0 | H_w | K^+ \rangle = \langle \pi^+ | H_w | K_S \rangle (1 - m^2_{\pi}/m_{K}^2) / 4f_\pi \]

\[ A_{+}^{++} = i\langle \pi^+ \pi^+ | H_w | K_L \rangle = \langle \pi^+ | H_w | K_S \rangle (1 - m^2_{\pi}/m_{K}^2) / 4f_\pi. \]

In the final forms of eqs.(9) we have used the \( K_{2\pi} \) sum rule \( M_{-}^{+} = 2M_{+}^{0} \) along with the PCAC consistency extension of \( K_{2\pi} \) in (3) to the \( K_{3\pi} \) version [1]

\[ \langle \pi_1 \pi_2 \pi_3 | H_w | K \rangle = \frac{1}{2} (M_{CC1} + M_{CC2} + M_{CC3}) + O(m^2_{\pi}/m_{K}^2). \] (11)

The factor of \( \frac{1}{2} \) in (11) (already occurring in (10)) accounts for the “mismatch” between Feynman amplitudes (where the pions are treated as independent) and PCAC consistency (where the PCAC procedure must be symmetrized over the final-state pions) with the decaying kaon always kept on mass shell. Just as the PCAC consistency \( K_{2\pi} \) form (3) also follows from a (tedious) analysis of rapidly varying pole terms [1], the PCAC consistency \( K_{3\pi} \) form (11) (including the factor of 1/2) likewise follows from an (even more tedious) analysis of rapidly varying pole terms [14].

Given the three \( K_{2\pi} \) L\( \sigma \)M-CPM predictions, (4), (6), (7), the PCAC consistency extension to the four \( K_{3\pi} \) amplitudes in (10) is

\[ |A_{-}^{000}|_{PCAC} \approx 0.85 \times 10^{-6} \]

\[ |A_{+}^{000}|_{PCAC} \approx 0.95 \times 10^{-6} \]

\[ |A_{+}^{++} - |PCAC \approx 1.89 \times 10^{-6} \]

\[ |A_{+}^{000}|_{PCAC} \approx 2.53 \times 10^{-6}. \] (12)

These \( K_{3\pi} \) predictions in (12) are respectively within 6%, 1%, 2%, 3% of the experimental amplitudes [11]

\[ |A_{-}^{000}|_{exp} = (0.91 \pm 0.01) \times 10^{-6} \]

\[ |A_{+}^{000}|_{exp} = (0.96 \pm 0.01) \times 10^{-6} \]

\[ |A_{+}^{++} - |exp = (1.93 \pm 0.01) \times 10^{-6} \]

\[ |A_{+}^{000}|_{exp} = (2.60 \pm 0.02) \times 10^{-6}. \] (13)

The latter amplitudes are extracted from the standard three-body phase space integral [15] with \( N \) being the Feynman statistical factor for the rate

\[ \Gamma = \frac{2}{N(8\pi M)^3} |A|^2 \int_{4\mu^2}^{(M^2 - m^2)^2} ds \left[ \frac{s - 4\mu^2}{s} \right]^{1/2} \]
\[ I = |A|^2, \]  
(14)

where \( M \) is the kaon mass, \( m \) is the odd-pion mass, and \( \mu \) is the non-odd-pion mass. The amplitudes \( A \) in (14) are taken as constant (empirically valid to within 5\% ) and the resulting integrals in (14) are \( I(+ - 0) = 1.95, I(00+) = 0.996, I(+ + -) = 0.798, I(000) = 0.397 \) in units of \( 10^{-6} \) GeV.

Suffice it to say that this LoM-CPM-PCAC approach used in Sec. II predicts all seven \( K_{2\pi} \) and \( K_{3\pi} \) weak decay amplitudes to within 5\% accuracy relative to the data - in terms of just \( \Delta I = 1/2 \) scale here

\[ |\langle \sigma|H_w|K_S \rangle| = |\langle \pi^0|H_w|K_L \rangle| \approx 3.2 \times 10^{-8} \text{ GeV}^2. \]  
(15)

### III. EXTENSION OF CPM TO \( K_L \rightarrow \gamma\gamma, K_S \rightarrow \gamma\gamma, K_L \rightarrow \pi^0\gamma\gamma \)

First we consider \( K_L \rightarrow 2\gamma \) decay with CPM \( \pi^0 \) pole graph of Fig.3 generating the amplitude

\[ \langle 2\gamma|H_{\pi^0}^w|K_L \rangle = (2\gamma|\pi^0)(m_K^2 - m_{\pi^0}^2)^{-1}\langle \pi^0|H_{\pi^0}^w|K_L \rangle = F_{K_L\gamma\gamma}\varepsilon^\mu\varepsilon^\nu\varepsilon_{\mu\nu\alpha\beta}k^\alpha k^\beta. \]  
(16)

One knows the ABJ [16] or equivalently the LoM \( \pi^0 \rightarrow \gamma\gamma \) amplitude has magnitude \( \alpha/\pi f_{\pi} \) and the analogue \( F_{K_L\gamma\gamma} \) amplitude for branching ratio [11] \( 5.9 \times 10^{-4} \) with lifetime \( \tau_{K_L} = 5.17 \times 10^{-8} \) sec. gives

\[ F_{K_L\gamma\gamma} = \left( \frac{64\pi}{m_K^2} \Gamma_{K_L\gamma\gamma} \right)^{1/2} = (3.51 \pm 0.04) \times 10^{-9} \text{ GeV}^{-1}. \]  
(17)

Then eq.(16) requires the scale

\[ |\langle \pi^0|H_w|K_L \rangle| \approx 3.2 \times 10^{-8} \text{ GeV}^2, \]  
(18)

which matches the \( \Delta I = 1/2 \) scale of (15) needed to explain all \( K_{2\pi} \) and \( K_{3\pi} \) decays by construction.

Next we apply the CPM and the \( \sigma \) pole graph of Fig 4 to compute the \( K_S \rightarrow \gamma\gamma \) decay amplitude [2]

\[ \langle 2\gamma|H_{\sigma}^w|K_L \rangle = F_{K_S\gamma\gamma}\varepsilon^\mu\varepsilon^\nu(k^\mu k^\nu - k^\rho k^\rho) \]

\[ = (2\gamma|\sigma)(m_K^2 - m_\sigma^2 + im_\sigma \Gamma_\sigma)^{-1}\langle \sigma|H_{\pi^0}^w|K_S \rangle. \]  
(19)

The scalar analogue \( \sigma \rightarrow 2\gamma \) of the LoM \( \pi^0\gamma\gamma \) amplitude in (17) receives a quark-loop \( u \) and \( d \) enhancement of 5/3 in Fig 5a:

\[ F_{qk \ loop} = N_c \left( \frac{4}{9} + \frac{1}{9} \right) \frac{\alpha}{\pi f_{\pi}} = \frac{5}{3} \frac{\alpha}{\pi f_{\pi}} \]  
(20)

for \( N_c = 3 \). But the LoM also requires the \( \pi^+ \) meson loop of Fig 5b, generating the \( \sigma \rightarrow \gamma\gamma \) amplitude [17]

\[ F_{\pi \ loop} = -\frac{2g'\alpha}{\pi m_\sigma^2} \left[ -\frac{1}{2} + \xi I(\xi) \right] = -\left[ -\frac{1}{2} + \xi I(\xi) \right] \frac{\alpha}{\pi f_{\pi}}, \]  
(21)
where we have used the LoSM coupling $g' = m_{\pi}^2/2f_{\pi}$. With $\xi \equiv m_{\pi}^2/m_{\eta}^2 \approx 0.04$ for [7,8] $\sigma(700)$, the Feynman integral $I(\xi)$ in (21) is [17]

$$I(\xi) = \int_0^1 dy \int_0^1 dx \ [\xi - xy(1 - y)]^{-1} = \frac{\pi^2}{2} - 2 \ln^2 \left[ \frac{1}{\sqrt{4\xi}} + \sqrt{\frac{1}{4\xi} - 1} \right] \approx .025.$$ (22)

Substituting (22) into (21), one notes that the pion loop amplitude of Fig 5b changes sign [18] and enhances the quark loop amplitude of (20), giving for (21)

$$F_{\pi \text{ loop}} = -(-0.50) \frac{\alpha}{\pi f_{\pi}}.$$ (23)

Then the net SU(2) LoSM $\sigma \rightarrow \gamma\gamma$ amplitude is

$$F_{\sigma \gamma\gamma}^{\text{LoSM}} \approx (1.67 + 0.50) \frac{\alpha}{\pi f_{\pi}} = 2.17 \frac{\alpha}{\pi f_{\pi}},$$ (24)

predicting a scalar $\rightarrow \gamma\gamma$ rate now compatible with data [19].

Returning to the $K_S \rightarrow \gamma\gamma$ amplitude (19) and using the same approximation $|m_{\pi}^2 - m_{\eta}^2| << m_{\eta}^2$ as in (2) we find, given the observed [11] branching ratio $B(K_S \rightarrow \gamma\gamma) = (2.4 \pm 0.9) \times 10^{-6}$ and corresponding amplitude $F_{K_S\gamma\gamma} = (5.4 \pm 1.0) \times 10^{-9}$ GeV,

$$|\langle \sigma | H_\omega | K_S \rangle| \approx m_{\sigma}^4 \frac{F_{K_S\gamma\gamma}}{F_{\sigma\gamma\gamma}^{\text{LoSM}}} = (4.9 \pm 0.9) \times 10^{-8} \text{ GeV}^2$$ (25)

assuming $m_{\sigma} \approx 700$ MeV. Actually we prefer [20] the LoSM-NJL scalar mass $m_{\sigma} = 2m_q \approx 650$ MeV, in which case (25) predicts $|\langle \sigma | H_\omega | K_S \rangle| = (4.2 \pm 0.8) \times 10^{-8}$ GeV$^2$

Although the latter estimate is within one standard deviation of the $K_L \rightarrow \gamma\gamma$ value (18) for this crucial $\Delta I = 1/2$ weak scale, the extreme sensitivity of (25) on $m_{\sigma}^2$ makes this latter successful estimate at best only plausible (but nonetheless consistent with the overall CPM picture). Stated in a more phenomenological way, a CPM picture for $K_S \rightarrow \gamma\gamma$ decay dominated by an intermediate scalar $\xi(1000)$ with observed PDG rate [11] $\Gamma_{\xi\gamma\gamma} \approx 6$ keV (as emphasized in ref.[19]) roughly predicts a $K_S \rightarrow \gamma\gamma$ rate

$$\Gamma_{K_S\gamma\gamma} \sim \Gamma_{\xi\gamma\gamma} |\langle \xi | H_\omega | K_S \rangle|^2/m_{\xi}^4 \sim 6 \times 10^{-21} \text{ GeV}$$ (26)

for our usual $\Delta I = 1/2$ weak scale $|\langle \xi | H_\omega | K_S \rangle| \sim 3.2 \times 10^{-8}$ GeV$^2$ as given by (15) or (18). For this rate (26) to be compatible with data [11],

$$\Gamma_{K_S\gamma\gamma} = (18 \pm 7) \times 10^{-21} \text{ GeV},$$ (27)

the scalar mass $\xi(1000)$ in (26) should be replaced by $\sigma(760)$, close to the theoretical value in ref.[20].

Finally we study $K_L \rightarrow \pi^0\gamma\gamma$ in the CPM. Following ref.[2] we consider only the CPM graph of Fig 6, generating the weak parity-conserving (pc) amplitude

$$\langle \pi^0\gamma_{q_1}\gamma_{q_2} | H_{\pi^0}^pc | K_L \rangle = \langle \gamma\gamma | \sigma \rangle \frac{1}{s - m_{\pi}^2 + im_{\sigma} \Gamma_{\sigma}} \langle \pi^0 | H_{\pi^0}^pc | K_L \rangle$$ (28)

where $s = (q_1 + q_2)^2$. We shall use the chiral symmetry constraint analogous to eq(1):

$$\langle \pi^0 | H_{\pi^0}^pc | K_L \rangle = \langle \pi^0 \pi^0 | H_{\pi^0}^pc | K_S \rangle$$ (29)
and scale the latter directly to $K_S \to \pi^0\pi^0$ data in eq.(8) (or equivalently the predicted CPM amplitude in eq.(4)). The corresponding weak decay rate involves the three-body phase space integral [15,21] over the square of (28):

$$\Gamma(K_L \to \pi^0\gamma\gamma) = |\langle\pi^0\sigma|H_w|K_L\rangle|^2 |F_{\pi\gamma\gamma}|^2 \frac{\pi^2}{m_K^4(4\pi)^5} \times$$

$$\int_{s_o}^{(m_K-m_\pi)^2} ds s^2 \left\{ \frac{(s - (m_K + m_\pi)^2)}{(s - (m_K - m_\pi)^2 + m_\pi^2)} \right\}^{1/2}. \tag{30}$$

The integral in (30) has the numerical value $1.7 \times 10^{-4}$ GeV$^4$ for the same lower cutoff $s_o = 0.0784$ GeV$^2$ as used by the experimental groups [22] which measured the rate of $K_L \to \pi^0\gamma\gamma$, the latter PDG average being [11]

$$\Gamma(K_L \to \pi^0\gamma\gamma)_{exp} = (2.16 \pm 0.36) \times 10^{-23} \text{ GeV}. \tag{31}$$

Using the chiral symmetry relation (29), the CPM prediction (4) (only 5% shy of the observed $K_{2\pi}$ amplitude), and the $L\sigma M \sigma \to 2\gamma$ amplitude (24) (only 10% shy of the data [19]), the predicted CPM rate in (30) becomes

$$\Gamma(K_L \to \pi^0\gamma\gamma)_{CPM} = |\langle2\pi^0|H_w|K_S\rangle|^2 (2.17 \times \frac{m_K^2}{m_K^4(4\pi)^5}) \times$$

$$(1.7 \times 10^{-4} \text{ GeV}^4) \approx 1.5 \times 10^{-23} \text{ GeV}, \tag{32}$$

within 2 standard deviations of the measured rate in (31). Moreover, the CPM invariant $\gamma\gamma$ spectrum in Fig. 6 of ref.[2] peaks in a manner compatible with data, a result also true for ChPT [4,21].

Thus the 3 weak radiative rates computed in this section III for $K_L \to \gamma\gamma$, $K_S \to 2\gamma$, $K_L \to \pi^02\gamma$ have the CPM predictions in (18), (25 or 26), (32) which are all near the data in (17), (27), (31), respectively.

### IV. SINGLE QUARK LINE PREDICTION FOR $\Delta I = 1/2$ SCALE

To complete the $L\sigma M$-CPM picture, we should reconfirm this one $\Delta I = 1/2$ scale based on the underlying quark model, where e.g. the quark loop for $\pi^0 \to 2\gamma$ or its extension to $\sigma \to 2\gamma$ do make contact with data. To this end we consider the $\Delta I = 1/2$ single quark line (SQL) transition $s \to d$ depicted in Fig.7 via the self energy effective hamiltonian $\Sigma_{sd} = b\tilde{d}\bar{\psi}(1-\gamma_5)s + h.c.$ according to the dimensionless weak scale [23]

$$-b \approx \frac{G_F s_1 c_1}{8\pi^2 \sqrt{2}} (m_c^2 - m_u^2)^2 \approx 5.6 \times 10^{-8}. \tag{33}$$

Here the GIM [24] enhancement factor $m_c^2 - m_u^2$ in (33) is big because the charmed quark mass $m_c \approx 1.6$ GeV is large relative to $m_u \approx 0.34$ GeV.

Recently it has been shown [25] that this SQL $\Delta I = 1/2$ scale (33) not only predicts $K_S \to \pi\pi$ correctly, but it also maps out hyperon $B \to B^'\pi$, $\Xi^- \to \Sigma^-\gamma$ and $\Omega^- \to \Xi^-\gamma$ weak decays. It is sometimes suggested that this SQL scale (33) can be transformed away for $K_S \to \pi\pi$ decays. While we have previously argued that this cannot be done for $K_{2\pi}$ decays [26], it most certainly cannot be extended to the above SQL hyperon decays in any case (else these hyperon decays would vanish). Thus we proceed with (33) and apply it to $K_{2\pi}$ decays.
Specifically the first-order weak axial-vector LSZ amplitude is [27]

$$M_\mu = i \int d^4 x \ e^{i q x} \langle 0 | T(H^w \ A^3_\mu(x)) | K^0 \rangle \approx ib \sqrt{2} f_K q_\mu,$$

where the weak scale $b$ multiplies the strong axial current as depicted in Fig.8. This multiplication suggests a very short-distance weak structure of (33) relative to the strong scale generating $f_K$ (because $M_w^{-1} << m_K^{-1}$). Then the soft-pion theorem predicts on the kaon mass shell

$$q^\mu M_\mu = i f_\pi \langle \pi^0 | H^w | K_L \rangle = ib \sqrt{2} f_K m_K^2,$$

$$\langle \pi^0 | H^w | K_L \rangle = 2b(f_K/f_\pi)m_K^2 \approx -3.4 \times 10^{-8} \ GeV^2.$$

for $(f_K/f_\pi) \approx 1.2$ and $b \approx -5.6 \times 10^{-8}$ from (33).

We note that this predicted $\Delta I = 1/2$ SQL scale in (36) is very close to the $3.2 \times 10^{-8}$ GeV$^2$ scale in (15) and (18) needed to properly fix the $K_L \rightarrow 2\gamma$ rate. If instead we fixed the $\langle \pi^0 | H_w | K_L \rangle$ scale in (15) and (18) to this predicted SQL-GIM-enhanced scale of (36) driven by (33), then the “worst” $K_{2\pi}$ and $K_{3\pi}$ CPM predictions in (4) for $K^0_{2\pi}$ and in (12) for $A^{+0}$ become even closer to the data, namely 1% and 2% respectively.

**V. CHIRAL PERTURBATION THEORY PREDICTIONS**

In ref.[3] it was shown that the three $K_{2\pi}$ amplitudes could be accurately predicted if two parameters, $c_2$ for $\Delta I = 1/2$ and $c_3$ for $\Delta I = 3/2$ transitions, were allowed to be fitted freely. Moreover, higher order four-derivative couplings (generating 82 terms) are needed in ChPT to explain the four $K_{3\pi}$ amplitudes to within 5%. This corresponds to fitting not only $c_2$ and $c_3$ (as in $K_{2\pi}$ decays), but also two more parameters $G_1$ and $G_2$.

Then in ref.[4] the $K_S \rightarrow 2\gamma$ and $K_L \rightarrow \pi^0 2\gamma$ decays were considered (but not $K_L \rightarrow 2\gamma$). For $K_S \rightarrow 2\gamma$ the tree-level and one-loop level ChPT theory predictions (generating 37 terms in the four-derivative Lagrangian) are in good agreement with the branching ratio $B(K_S \rightarrow \gamma \gamma) = 2.0 \times 10^{-6}$ (near the PDG value $2.4 \pm 0.9 \times 10^{-6}$) provided the parameter $G^C_A$ is freely fitted to $9.1 \times 10^{-6} \ GeV^{-2}$. Given this value of $G^C_A$, the resulting $K_L \rightarrow \pi^0 2\gamma$ rate in one-loop order ChPT has branching ratio $0.68 \times 10^{-6}$, which is only 40% of the observed $K_L \rightarrow \pi^0 2\gamma$ branching ratio [15] of $1.70 \times 10^{-6}$. However as noted before, the ChPT $\gamma \gamma$ spectrum for $K_L \rightarrow \pi^0 2\gamma$ roughly matches the data, as does the $\Sigma M$-CPM $\gamma \gamma$ spectrum.

In Table 1 we contrast the predictions of the $\Sigma M$-CPM-PCAC approach described in Secs II-IV with the one loop ChPT results summarized in Sec.V and compare them to experiment.

| Table 1: Contrasting Chiral Theories |
|--------------------------------------|
| **L$\sigma$M-CPM-PCAC** | **ChPT** |

8
| Process          | Predictions                                      | Parameters          |
|------------------|--------------------------------------------------|---------------------|
| $K \rightarrow 2\pi$ | Predicts all 3 amplitudes to within 5% of data with no free parameters | $c_2, c_3$          |
| $K \rightarrow 3\pi$ | Predicts all 4 amplitudes to within 5% of data with no free parameters | $c_2, c_3, G_1, G_2$ |
| $KL \rightarrow 2\gamma$ | Amplitude predicted to within 3% of data          | $G_{CA}^8$          |
| $KS \rightarrow 2\gamma$ | Amplitude predicted to within 15% of data         | $G_{CA}^8$          |
| $KL \rightarrow \pi^02\gamma$ | Rate predicted to within 28% of data              | Given $G_{CA}^8$ above, predicts branching ratio 40% of data |
VI. CONCLUSION

In this paper we have shown that the chiral symmetry approach of the SU(2) linear $\sigma$ model (L$\sigma$M) extended for weak interactions to the chiral pole model (CPM), involving tree-level $\pi^0$ and $\sigma$ poles, provides a very accurate description of nonleptonic weak kaon decays. Specifically if we input the one $\Delta I = 1/2$ scale derived from a single quark line (SQL) GIM-enhanced transition nonperturbatively inducing

$$-\langle \pi^0 | H_w | K_L \rangle = \frac{G_F s_t c_t}{4\pi^2 \sqrt{2}} (m_c^2 - m_u^2) (f_K/ f_\pi) m_K^2 \approx 3.4 \times 10^{-8} \text{ GeV}^2,$$

then the 8 predicted decays $K \to 2\pi$, $K \to 3\pi$, $K_L \to 2\gamma$ all match experiment to within 2% - without introducing any free parameters. Moreover the decays $K_S \to 2\gamma$, $K_L \to \pi^0 2\gamma$ are then predicted to be within 2 standard deviations of the data central values scaled to this weak SQL transition (37). At the very least, even if the SQL scale (37) is not used, then this $L\sigma$M-CPM-PCAC scheme correctly predicts these 10 decay amplitudes in terms of only one free parameter.

Since this $K_{2\pi}$ L$\sigma$M-CPM scheme reduces to standard PCAC formulæ, we have also used PCAC to obtain our $K_{3\pi}$ predictions. By way of contrast we have compared the above L$\sigma$M-CPM-PCAC results with the much more complicated and far less predictive approach of chiral perturbation theory (ChPT).

In particular, the two scales of $K_{2\pi}$ decays, for $\Delta I = 1/2$ and for $\Delta I = 3/2$ transitions, must both be assumed for ChPT (whereas they are both predicted accurately in the L$\sigma$M-CPM-PCAC scheme). Furthermore two more ChPT parameters must be assumed for $K_{3\pi}$ decays (even with the cumbersome 82 Lagrangian terms). Moreover the single $K_S \to 2\gamma$ weak scale must be assumed (even with 37 more terms in the Lagrangian), and then the $K_L \to \pi^0 2\gamma$ ChPT rate is only 40% of the data.

We therefore conclude that the former L$\sigma$M-CPM-PCAC chiral symmetry approach is far more predictive and less complicated than is ChPT. In a prior study [28] we also conclude that a L$\sigma$M approach to pion interactions occurring in strong transitions, $r_\pi$, $F_A(0)/F_V(0)$, $\alpha_{\pi+}$, $\alpha_{\pi\pi(0)}$ is also more predictive than is ChPT.

It is interesting that there has been a recent attempt [29] to merge a L$\sigma$M-type picture with $m_\sigma \sim 700$ MeV together with $K \to 2\pi$ weak decays and ChPT. While this former link is compatible with data and with refs.[1] and [2], the above analysis suggests that an extension to ChPT is quite implausible.

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VII. FIGURE CAPTIONS

Fig.1 CPM graph for $\Delta I = 1/2 \ K_S \to \pi\pi$ amplitudes.

Fig.2 W-emission extension to $\Delta I = 3/2 \ K_S \to \pi^+\pi^-$ amplitude.

Fig.3 CPM graph for $K_L \to \gamma\gamma$ decay.

Fig.4 CPM graph for $K_S \to \gamma\gamma$ decay.

Fig.5 L$\sigma$M quark loops (a) and $\pi^\pm$ loop (b) for $\sigma \to \gamma\gamma$ decay.
Fig. 6 CPM graph for $K_L \to \pi^0\gamma\gamma$ decay.

Fig. 7 W-mediated $s \to d$ loop (a) becoming $\Delta I = 1/2$ SQL transition (b)

Fig. 8 Quark $s \to d$ loop representing $K^0 \to$ vacuum matrix element of weak axial current.

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