Numerical analysis of an electrostatically formed membrane mirror

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Abstract. The paper presents a numerical algorithm developed to determine the deformed shape of a membrane mirror controlled by electrostatic forces. Deformable mirrors are key components that are used in combination with wavefront sensors and real-time control systems in adaptive optics. The electrostatic membrane mirror concept implies using a thin conductive reflective membrane stretched over a solid flat frame and deformed electrostatically by applying control voltages to electrostatic actuators positioned behind the membrane. The proposed algorithm implies solving a coupled structural-electrostatic problem by using finite element and boundary element methods. Small deflections of a membrane are described by Poisson’s equation. The electric charge distribution over the membrane surface having a prescribed potential is governed by a Fredholm integral equation of the first kind. The coupled problem is solved iteratively, and a criterion for terminating iterations when searching for a steady-state solution is presented. The distinctive feature of this approach is that it allows us to take into account electrical edge effects typical for conducting thin-walled structures of very small thickness. Illustrative examples are provided to show the applicability and validity of the proposed method as well as its advantage over some existing techniques.

1. Introduction
Electrostatic formation and control of membrane mirrors has received much attention since the late 1970s as a promising adaptive optics technology that benefits both Earth and space applications [1, 2]. The key feature of any adaptive optical system is that it uses optical elements that can change their shape when outside control signal is applied [3]. Figure 1 shows two typical designs of an electrostatically formed membrane mirror that have been under intensive theoretical and experimental study in recent decades. In the first design, the mirror is considered to be a two-membrane capacitor. When the electrical voltage is applied, electrostatic forces deform the membranes so that their shape is close to the shape of a parabolic reflector. The focus length can be changed by changing the voltage. In the second design, a rigid segmented electrode is used to provide the surface shape control by means of voltage variation on the segments. The typical membrane mirror is made of a thin lightweight flexible polymer film coated with a very thin layer of metal, usually aluminum, but gold and silver can also be used. To provide a satisfactory performance of the mirror, membrane materials must have high elasticity modulus, low density, small thickness, high thermal stability, low thermal expansion coefficients, and, for applications in space, strong space radiation resistance. In [4], two candidate materials for future space applications are considered. It is shown that polyimide films have many advantages over
polyester films as they have a larger modulus of elasticity, higher tensile strength, lower thermal expansion coefficient, smaller elongation at break, and stronger anti-ultraviolet radiation ability.

Figure 1. Basic designes of an electrostatically formed membrane mirror.

Figure 2. A segmented electrode.

To determine the shape of an electrostatically formed membrane mirror, the coupled structural-electrostatic problem is to be solved. The main challenge is to compute electrostatic pressure acting on a conducting membrane. In [5], the first configuration of an electrostatically formed membrane mirror was considered. Two elastic membranes with preliminary tension were fixed along their contours. To determine the membranes’ shape under electrostatic pressure, the theory of large deflections of plates [6] was applied, and the nonlinear coupled problem was solved numerically using the shooting method [7]. The electric field between the membranes was evaluated by a simple formula

\[ E = \frac{\Delta u}{H - 2w}, \]

where \( \Delta u \) is the applied electrical voltage, \( H \) is the distance between the membranes, and \( w \) is the deflection of the membrane from its normal flat shape. The main limitation of formula (1) is that it does not take into account the so-called "edge effect", which occurs near the capacitor edges and plays an important part in the electromechanical interaction [8].

In [9], an attempt was made to obtain a more elaborate solution to the electrostatic problem by applying the finite element method to solve the Laplace equation in an open domain. In [10], a strongly coupled electro-mechanical finite element was introduced to model an electrostatic actuated deformable mirror. However, since the coupled problem is nonlinear, an iterative procedure should be applied, which makes the computational cost extremely high. To reduce the model complexity and decrease the computational cost, we suggest using integral equations to determine electrical charge distribution on a conducting surface having a prescribed potential. The suggested approach has been stimulated by recent advances in integral equations [11], boundary element methods [12], and new applications of the optic technology [13, 14, 15, 16, 17].

Our objective is to analyze the influence of the geometry of the segmented electrode and electric edge effects on the deformed membrane profile by means of computer simulations. The main focus is on the numerical aspects of modelling the electrostatic formation of metallized polymer films of arbitrary shapes. The simulation tool was developed and tested using the Matlab environment. A parallel-plate capacitor with a metal-coated membrane and a rigid metal segmented electrode is considered as an example.
2. Mathematical model

The mathematical model of an electrostatically formed membrane mirror includes equations of soft-shell structures and equations of electrostatics of thin-walled conductors that must be solved simultaneously. The problem is coupled because the value of electrostatic pressure acting on a flexible conductor is determined by the electric charge distribution over its surface. When the conductor is deformed, the charge distribution changes depending on the surface curvature.

The fundamentals of the theory of large deformations of soft shells used in the stress analysis of structures made of composite materials were presented in [18, 19]. In [20], strict nonlinear equations of soft shells in an electrostatic field were deduced. These equations form the basis of numerical algorithms for computing the formation of flexible conductors of complex shapes. They can also be used to construct solutions in simpler cases.

In this study, the mathematical model includes Poisson’s equation to describe small deflections of a membrane and a Fredholm integral equation of the first kind to describe a surface charge distribution over the membrane surface [21].

2.1. Properties of a conductor in an electrostatic field

Owing to the mobility of charges, the electric field within the volume of a conductor is always zero. In all points on the surface the tangential components of the field are also equal to zero. Therefore, outside a conductor, the electric field lines are perpendicular to its surface, ending or beginning on charges on the surface. The entire volume of a conductor is equipotential because the electric field strength $E$ is the negative gradient of the electrostatic potential $\phi$

$$E = -\nabla \phi. \quad (2)$$

For a conductor in a homogeneous and isotropic dielectric, we have

$$E = \frac{\sigma}{\varepsilon \varepsilon_0}, \quad (3)$$

where $\sigma$ is the local surface charge density of a conductor, $\varepsilon_0 = 8.854 \cdot 10^{-12} \text{F/m}$ is the vacuum permittivity, and $\varepsilon$ is the relative permittivity, which is equal to unity in this study. Distribution of electric charges on the conductor surface depends on the surface curvature: the charge density is greater where the radius of curvature is smaller.

The potential of the electrostatic field created by charged surfaces (surface field potential), given that $\phi(\infty) = 0$, is equal to

$$\phi = \frac{1}{4\pi \varepsilon \varepsilon_0} \int_S \frac{\sigma dS}{r}, \quad (4)$$

where $r$ is the distance from the surface charge $\sigma dS$ to the field point in question. Integration is performed over the surfaces where the electric charges are distributed.

2.2. Pressure on the surface of a conductor due to a surface charge distribution

The electric force $dF$ acting on a surface area $dS$ is directed along the normal to the surface and is determined by the formula

$$dF = \frac{\sigma^2}{2\varepsilon \varepsilon_0} dS = \frac{\varepsilon \varepsilon_0 E^2}{2} dS. \quad (5)$$

Hence, the force per unit area or electrostatic pressure is

$$p = \frac{dF}{dS} = \frac{\sigma^2}{2\varepsilon \varepsilon_0} = \frac{\varepsilon \varepsilon_0 E^2}{2}. \quad (6)$$

This pressure tends to draw the conductor into the field, regardless of the sign of the surface charge.
2.3. Deformation of a flexible membrane

To perform the stress-strain analysis of thin-walled structures with low flexural rigidity, the theory of soft shells is used [18]. In accordance with this theory, the soft shell is momentless and unsuited for absorbing compressive forces. On the surface, where compressive forces may occur, folds appear. We consider a flexible metallized polymer membrane, which is prestressed to maintain a tensile state and to avoid folds under applied electrostatic pressure. It is assumed that the membrane is made of a homogeneous linearly elastic material. Small normal deflections $w(x, y)$ of a prestressed metallized polymer membrane are described by Poisson’s equation

$$T_x \frac{\partial^2 w}{\partial x^2} + T_y \frac{\partial^2 w}{\partial y^2} = p(x, y),$$

where $T_x$ and $T_y$ are prestressing forces per unit length acting in the $x$ and $y$ directions, and $p(x, y)$ is electrostatic pressure depending on the deflection of the membrane and defined by relationship (6).

3. Numerical algorithm

To determine the form of a metallized membrane loaded by electrostatic forces, we need to know the charge distribution over the conducting surface. The surface charge density uniquely defines electrostatic pressure acting on the membrane and thus determines its shape. The problem is coupled and nonlinear because the deformed shape of the membrane is not known in advance but depends on the surface charge distribution which in turn depends on the shape of the membrane.

All solutions shown here are based on the partitioned scheme where individual problems are solved independently and the interaction information is transferred between them at every stage of the solution process. The models and the software developed for computing the charge distribution are based on the boundary element technique considered in [22], and those for the membrane deformation are based on the finite element method described in detail in [23, 24].

3.1. Synthesis of an electric field due to a surface potential

To explain the main points of the approach, let us consider a rectangular conducting plate of negligible thickness having a prescribed potential $\phi_0$ as shown in figure 3. According to [21], [25], the electric potential synthesis problem in this case is reduced to the solution of a Fredholm integral equation of the first kind

$$\frac{1}{4\pi \varepsilon \varepsilon_0} \int_{-b}^{b} \int_{-a}^{a} \frac{\sigma(x', y') dx' dy'}{R} = \phi_0,$$

where $R = \sqrt{(x - x')^2 + (y - y')^2}$ is the distance between a source point $(x', y')$ and the field point $(x, y)$. To solve the problem, the method of subsections with collocation is implemented. According to this method, the rectangular plate is divided into $N$ rectangular elements

$$x_{i+1} - x_i = \Delta x, \quad y_{i+1} - y_i = \Delta y.$$

Collocating the charge densities $\sigma_j$ at the centers of the elements $(x_{cj}, y_{cj})$ gives the system of $N$ linear equations

$$\sum_{j=1}^{N} a_{ij} \sigma_j = \phi_i,$$

where $\phi_i = \phi_0$, and

$$a_{ij} = \frac{1}{4\pi \varepsilon \varepsilon_0} \int_{x_i}^{x_{i+1}} \int_{y_i}^{y_{i+1}} \frac{dx' dy'}{\sqrt{(x_{cj} - x')^2 + (y_{cj} - y')^2}}.$$
The diagonal elements $a_{ii}$ of the system matrix are calculated analytically, while the off-diagonal elements are computed numerically [26]. It is essential that this electrostatic problem is solvable and has only one solution. Figures 5 and 6 show the surface charge density distribution on a rectangular metal plate divided into $64 \times 32$ square elements. The plate is assumed to have the length $2a = 10m$, the width $2b = 5m$, and the surface potential $\phi_0 = 10^5V$. The results are in good agreement with those presented in [21].

This technique can be applied with relevant modifications to the calculation of the surface charge density distribution on a metal circular disk. The charge density of an infinitely thin circular disk is given analytically by

$$\sigma(r) = \frac{Q}{2\pi a} \frac{1}{\sqrt{a^2 - r^2}},$$

(12)

where $Q$ is a total charge, $a$ is the radius of the disk, and $r$ is the radial distance from the center of the disk [27]. Note the special values $\sigma(0) = \frac{Q}{2\pi a}$ and $\sigma(r \rightarrow a) = \infty$.

Figure 7 shows 2700 collocation points on the conducting circular disk of radius $a = 1m$, and figure 8 displays the surface charge density distribution on it, given the disk potential $\phi_0 = 20kV$.

The convergence of the results of computing the surface charge density on the circular disk is shown in table 1. A double-ended estimate is used to verify the solution.
Figure 7. Collocation points on a circular disk.

Figure 8. Surface charge density distribution on a circular disk.

Table 1. Convergence of the results of computing the surface charge density on a circular disk.

| Num. of FE | Num. of nodes | Total charge Q, C | $\sigma_0$, C/m$^2$ | $\frac{Q}{2\pi a^2}$, C/m$^2$ | $\delta$, % |
|-----------|---------------|-------------------|---------------------|-----------------------------|----------|
| 12        | 17            | 1.25819 \cdot 10^{-6}  | 0.24606 \cdot 10^{-6}  | 0.20025 \cdot 10^{-6}  | 18.617   |
| 48        | 57            | 1.35669 \cdot 10^{-6}  | 0.24110 \cdot 10^{-6}  | 0.21592 \cdot 10^{-6}  | 10.444   |
| 108       | 121           | 1.38253 \cdot 10^{-6}  | 0.23391 \cdot 10^{-6}  | 0.22004 \cdot 10^{-6}  | 5.930    |
| 300       | 321           | 1.39960 \cdot 10^{-6}  | 0.22963 \cdot 10^{-6}  | 0.22275 \cdot 10^{-6}  | 2.996    |
| 588       | 617           | 1.40574 \cdot 10^{-6}  | 0.22813 \cdot 10^{-6}  | 0.22373 \cdot 10^{-6}  | 1.929    |
| 1200      | 1241          | 1.40980 \cdot 10^{-6}  | 0.22715 \cdot 10^{-6}  | 0.22437 \cdot 10^{-6}  | 1.223    |
| 2700      | 2761          | 1.41258 \cdot 10^{-6}  | 0.22648 \cdot 10^{-6}  | 0.22482 \cdot 10^{-6}  | 0.733    |
| 4800      | 4881          | 1.41382 \cdot 10^{-6}  | 0.22618 \cdot 10^{-6}  | 0.22501 \cdot 10^{-6}  | 0.517    |

We can see that the computed values of the surface charge density at the center of the disk $\sigma_0$ converge as the number of finite elements (subdomains) increases. This corresponds to the upper curve in figure 9. The lower curve displays the values of $\sigma(0)$ obtained on the basis of analytical formula (12). The total charge of the disk is computed by summing up the charges of subdomains $Q = \sum \sigma_i S_i$. It is then used to determine the charge density at the center of the disk $\sigma(0)$ by formula (12). The last column of the table contains the values of the relative error $\delta = \frac{|\sigma_0 - \sigma(0)|}{\sigma_0}$.

It should be noted that in both examples adequate simulation is also provided in boundary areas where the edge effect can be easily observed. It will be shown in what follows that edge effects become especially important when computing multiple electrode structures.

3.2. Finite element analysis of the membrane deformation

The finite element method is implemented to find the deformed shape of a flexible membrane. The variational formulation for this problem has the form

$$\int_{-a}^{a} \int_{-b}^{b} T_x \frac{\partial w}{\partial x} \delta w dx dy + T_y \frac{\partial w}{\partial y} \delta w dy dx = \int_{-a}^{a} \int_{-b}^{b} p(x, y) \delta w dx dy. \quad (13)$$
A linear 4-node quadrilateral isoparametric membrane element is used to perform the finite element analysis. According to the isoparametric element concept [28], similar expressions are used to approximate the unknown function $w$ and to determine the coordinates $x$ and $y$ within the finite element

$$w = N_i w_i, \quad x = N_i x_i, \quad y = N_i y_i,$$

where $w_i (i = 1, ..., 4)$ are nodal unknowns, $(x_i, y_i)$ are nodal coordinates, and $N_i$ are shape functions defined in terms of non-dimensional coordinates $-1 \leq \xi, \eta \leq 1$ as follows:

$$N_1 = \frac{1}{4}(1 - \xi)(1 - \eta); \quad N_2 = \frac{1}{4}(1 + \xi)(1 - \eta); \quad N_3 = \frac{1}{4}(1 + \xi)(1 + \eta); \quad N_4 = \frac{1}{4}(1 - \xi)(1 + \eta).$$

The numerical integration pattern shown in figure 4 is applied to form the stiffness matrix of a finite element.

### 3.3. Iterative procedure for solving a coupled problem

The solution of this coupled set of equations requires an iterative approach. Iteration continues until a convergence criterion of the form

$$\frac{\|W^k\|}{\|W^{k-1}\|} - 1 \leq \epsilon, \quad \|W\| = (W^T W)^{1/2}$$

is satisfied for some small tolerance $\epsilon$. According to [23], the best choice is to take it equal to half machine precision $\epsilon = 10^{-8}$. Here $W^k$ and $W^{k-1}$ are deflection vectors of the membrane at two successive iteration steps.

### 4. Results and discussion

In this section, numerical results are presented for two types of membrane mirror configurations to prove the validity and efficiency of the proposed approach.

#### 4.1. A parallel-plate capacitor with two flexible square membrane electrodes

In this example, the capacitor is composed of two parallel metallized flexible square membranes to which a certain voltage is applied. The parameters of the capacitor are presented in table 2.
Figure 11. Surface charge density distribution over the upper (a) and bottom (b) electrodes of the capacitor and along the line $y = 0.5 \text{m}$ on the upper (c) and bottom (d) electrodes of the capacitor.

Table 2. Parameters of a parallel-plate capacitor with membrane electrodes.

| Parameter                  | Notation | Value   | Unit |
|----------------------------|----------|---------|------|
| Upper electrode potential  | $\phi_u$ | $2 \cdot 10^4$ | V    |
| Bottom electrode potential | $\phi_b$ | $-2 \cdot 10^4$ | V    |
| Size of electrodes         | $a \times a$ | $1 \times 1$ | m    |
| Distance between electrodes | $H$      | 0.01    | m    |
| Prestressing forces        | $T_x, T_y$ | 100     | N/m  |
| Number of computational subsections | $N$   | $50 \times 50$ |      |

The surface charge density distribution over the upper and bottom undeformed metallized square membranes and the corresponding surface charge density distribution along the line $y = 0.5\text{m}$ are displayed in figure 11. The deformed shape of the square membrane electrode under electrostatic pressure and its profile $w$ at $y = 0.5\text{m}$ are presented in figure 12 for two types of boundary conditions: all edges fixed (a, c) and two opposite edges fixed (b, d).
Figure 12. The deformed shape of a square membrane electrode and its profile \( w (y = 0.5m) \) for different types of boundary conditions: all edges fixed (a, c) and two opposite edges fixed (b, d).

To make sure that the proposed approach provides adequate simulation results and to estimate the influence of the capacitor edge effect, the membranes’ shapes in the capacitor were also computed on the basis of simple formula (1). In figure 13, the corresponding deflection of a square membrane \( w (y = 0.5m) \) is shown for two types of boundary conditions: all edges fixed (a) and two opposite edges fixed (b), respectively.

Comparing the results presented in figures 12 and 13, we notice that, for the membrane with all its edges fixed, the maximum discrepancy of the results occurs in the center of the membrane and reaches almost 22%. As for the membrane with two opposite edges fixed, the maximum discrepancy of the results is 44%, which takes place in the middle of free edges. Furthermore, in the latter case, the deformed shape of the membrane is a double curved surface if the edge effect is counted, otherwise it is a parabolic cylinder. The presented results and their analysis prove that the electric field edge effect makes a substantial contribution to the formation of a flexible membrane in an electrostatic field.
4.2. A parallel-plate capacitor with a flexible membrane electrode and a rigid segmented electrode

In this example, the parameters of the capacitor are taken from table 2. The potential of the upper flexible membrane electrode is \(-2 \times 10^4 V\). The bottom rigid square electrode is divided into two segments according to the pattern in figure 10. The potential of the central blue segment of size 0.5 m × 0.5 m is \(10^{-4} V\), while the potential of the surrounding yellow segment is \(2 \times 10^{-4} V\). Figure 14 illustrates the surface charge density distribution over the upper (a) and bottom (b) electrodes of the capacitor and along the line \(y = 0.5m\) (c) and (d), respectively. As one can see in the pictures, the step form of the potential on the segmented electrode produces the step-like charge density distribution on the membrane surface. The corresponding deformed shape of the square membrane electrode (a) and its profile \(w (y = 0.5m)\) (b) are shown in figure 15.

To verify the results and estimate the influence of the edge effect on the deformed shape of the membrane in the capacitor, we also performed calculations of the membrane deformation using formula (1) for computing the electrostatic field strength. Figure 16 illustrates the profiles of the surface charge density (a) and the square membrane deflection (b) at \(y = 0.5m\) for this case.

A number of numerical experiments has been carried out on various meshes. Table 3 contains the values of the deflection of the central point of the square membrane in the capacitor computed on 20-fold meshes. The values \(w_0\) reflect the presence of the edge effect in the model under consideration, and the values \(w_0^{(1)}\) reflect its absence according to formula (1). The last column of the table contains percentages of the relative difference \(\delta = \left| \frac{w_0 - w_0^{(1)}}{w_0} \right| \).

| Mesh size | Deflection \(w_0\), m | Deflection \(w_0^{(1)}\), m | \(\delta\), % |
|-----------|------------------|------------------|------|
| 20 × 20   | \(-0.32104 \cdot 10^{-3}\) | \(-0.22222 \cdot 10^{-3}\) | 22.221 |
| 40 × 40   | \(-0.30604 \cdot 10^{-3}\) | \(-0.22196 \cdot 10^{-3}\) | 22.195 |
| 60 × 60   | \(-0.30103 \cdot 10^{-3}\) | \(-0.22192 \cdot 10^{-3}\) | 22.191 |
| 80 × 80   | \(-0.29849 \cdot 10^{-3}\) | \(-0.22190 \cdot 10^{-3}\) | 22.189 |

The convergence of the results of computing the square membrane deflection in the capacitor
Figure 14. Surface charge density distribution over the upper (a) and bottom (b) electrodes of the capacitor and along the line \( y = 0.5 \) m (c) and (d), respectively.

Figure 15. The deformed shape of a square membrane electrode (a) and its profile \( w \) (\( y = 0.5 \) m) (b) in the capacitor with a segmented electrode.
Figure 16. The surface charge density (a) and the square membrane deflection (b) profiles at $y = 0.5m$ computed by using formula (1).

Figure 17. Convergence of the results of computing the square membrane deflection $w_0$ (a) and $w_0^{(1)}$ (b) in the capacitor with a segmented electrode.

with a segmented electrode is also demonstrated by figure 17, where the plots visualize the data collected in the table. We can see that the computed values of the square membrane deflection at the center of the membrane $w_0$ (a) and $w_0^{(1)}$ (b) converge as the mesh refines. The relative differences between these values are very close to 22% for all mesh patterns.

The analysis of the presented results allows us to make a conclusion similar to that in the previous subsection about a significant influence of the electric field edge effect on the shape of a flexible metallized membrane loaded by electrostatic forces.

5. Conclusions
The developed mathematical model and numerical algorithm have shown their efficiency in determining the deformed shape of a metal-coated membrane in the capacitor with a segmented electrode. The mathematical formulation and the solution techniques for more general cases are straightforward. It is important that the proposed approach makes it possible to perform synthesis of an electrostatic field due to a surface potential for segmented electrodes of various configurations providing desirable shape control of deformable mirrors. A number of numerical experiments have been performed to shown that the edge electric field significantly affects the
resulting deflection of the membrane. At the same time, there is still lack of sufficiently detailed information on the interaction between an electrostatic field and flexible membranes of complex configurations. The extension of the model implies including nonlinear deformation effects and structural dynamics [29]. The model will be very helpful for designing membrane elements in adaptive optics systems involving electrostatic forces. Having such a tool will make it possible to solve many practical problems of shaping metallized membranes by electrostatic forces.

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