ON THE RANK OF $\pi_1(\text{Ham})$

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Abstract. We show that for any positive integer $k$ there exists a closed symplectic 4-manifold, such that the rank of the fundamental group of the group of Hamiltonian diffeomorphisms is at least $k$.

1. Introduction

The problem of determining the homotopy type of the group of Hamiltonian diffeomorphisms for a closed symplectic manifold is nowadays a far reaching problem in symplectic topology. In order to have an idea of the limited knowledge in the subject, absolutely nothing is known about the homotopy type of the group $\text{Ham}(\mathbb{T}^4,\omega)$ where $(\mathbb{T}^4,\omega)$ is the 4-dimensional torus with the standard symplectic form. Note that we stated an example of a 4-dimensional manifold, since for 2-dimensional symplectic manifolds the problem is partially understood due to the fact that in two dimensions symplectic geometry agrees with area and orientation preserving geometry. See for instance [11, Sec. 7.2]. In this direction, $\text{Ham}(S^2,\omega) = \text{Symp}_0(S^2,\omega)$ has the same homotopy type as $SO(3)$ [13]; and for the surface of genus $g \geq 1$, $\text{Ham}(M_g,\omega)$ is simply connected.

In higher dimensions there are some cases where the homotopy type of $\text{Ham}(M,\omega)$ is completely understood, due the techniques of holomorphic curves. For instance, $\text{Ham}(\mathbb{C}P^2,\omega_{FS})$ has the homotopy type of $PU(3)$; $\text{Ham}(\mathbb{C}P^1 \times \mathbb{C}P^1,\omega_{FS} + \omega_{FS})$ has the homotopy type of $SO(3) \times SO(3)$. These results are due to M. Gromov [4]. The rational homotopy type for the case of one-point blow up of $(\mathbb{C}P^2,\omega_{FS})$ was settled by M. Abreu and D. McDuff in [1]. For more examples, see the work of F. Lalonde and M. Pinsonnault [6]; and J. Evans [3].

Leave behind the problem of determining the homotopy type and focus on first stage of the problem: the fundamental group of $\text{Ham}(M,\omega)$. Recall that the fundamental group of a topological group is an abelian group. Hence is natural to ask: Given any positive integer $k$, does there exists a symplectic manifold such that the free part of $\pi_1(\text{Ham}(M,\omega))$ is isomorphic to $\mathbb{Z}^k$. Using cartesian products of symplectic manifolds together with Seidel’s representation [12] or Weinstein’s morphism [14] if possible to provide a weak answer to the problem. Namely, is possible to construct a symplectic manifold $(M,\omega)$.
such that the rank of $\pi_1(\text{Ham}(M, \omega))$ is at least $k$. See for instance [9], where Seidel’s morphism on cartesian products is studied. In this note we arrive to the same conclusion but on 4-dimensional symplectic manifolds. That is, in the smallest possible dimension.

**Theorem 1.1.** Given a positive integer $k$, there exists a closed, connected and simply connected symplectic 4-manifold $(M, \omega)$ such that

$$\text{rank } \pi_1(\text{Ham}(M, \omega)) \geq k.$$

The proof that we provide is a hands-on proof. The symplectic 4-manifold of the theorem turns out to be the blow up of $(\mathbb{C}P^2, \omega_{FS})$ at $k$ points of distinct weights. The techniques used throughout this note are soft techniques of symplectic topology, where Weinstein’s morphism plays a key role.

If $\text{Symp}_0(M, \omega)$ stands for the connected component of $\text{Symp}(M, \omega)$ that contains the identity map, then the inclusion $\text{Ham}(M, \omega) \subset \text{Symp}_0(M, \omega)$ induces an injective map

$$\pi_1(\text{Ham}(M, \omega)) \rightarrow \pi_1(\text{Symp}_0(M, \omega))$$

due to the Flux morphism, [8, Ch. 10]. Therefore, Theorem 1.1 also holds if the group $\text{Ham}(M, \omega)$ is replaced by $\text{Symp}_0(M, \omega)$.

Unlike the group of Hamiltonian diffeomorphisms of a closed symplectic manifold, the group of symplectic diffeomorphisms is not necessarily connected. Therefore, following the same line of ideas is natural to ask if given a positive number $k$ does there exists a closed connected symplectic manifold $(M, \omega)$ such that the number of connected components of $\text{Symp}(M, \omega)$ is equal to $k$. Recently, D. Aroux and I. Smith solved this problem in [2, Thm 1.3] via Floer-theoretic arguments.

As a byproduct of the arguments used to prove the main result, we are also able to show that Calabi’s morphism on the one-point blow up of $(\mathbb{R}^4, \omega_0)$ is non trivial. The first examples of open manifolds whose Calabi’s morphism is non trivial are due to A. Kislev [5]. Alongside we prove that the rank of the fundamental group of $\text{Ham}(\tilde{M}, \tilde{\omega}_\kappa)$ is positive where $(\tilde{M}, \tilde{\omega}_\kappa)$ is the one-point blow of weight $\kappa$ of $(M, \omega)$. Hence our result improves the one obtained by D. McDuff [7], since information about $\pi_1(M)$ and $\pi_2(M)$ is irrelevant in our arguments.

**Theorem 1.2.** Let $(\tilde{M}, \tilde{\omega}_\kappa)$ be the one-point blow up of weight $\kappa$ of the closed manifold $(M, \omega)$. Then for infinitely many values of $\kappa$, the rank of $\pi_1(\text{Ham}(\tilde{M}, \tilde{\omega}_\kappa))$ is positive.

It is worth mentioning the case $M = \mathbb{T}^4$ with the standard symplectic form. Therefore, $\pi_1(\text{Ham}(\tilde{\mathbb{T}}^4, \tilde{\omega}_\kappa))$ has positive rank. However, nothing is known about the group $\pi_1(\text{Ham}(\mathbb{T}^4, \omega))$.

We are grateful to L. Polterovich from bringing [3] to our attention.
2. Preliminary Computations

2.1. A compactly supported path of Hamiltonian diffeomorphisms on \((\mathbb{R}^4, \omega_0)\). Consider 1-periodic smooth functions \(a_1, a_2, a_3, a_4 : \mathbb{R} \to \mathbb{R}\) such that \(a_1(0) = a_4(0) = 1\) and \(a_2(0) = a_3(0) = 0\). Let \(\alpha : \mathbb{R} \to \mathbb{R}\) be also a smooth function such that \(\alpha(0) \in \mathbb{Z}\). Furthermore, the functions \(a_j\) are also subject to the condition that

\[
(1) \quad A_t := \begin{pmatrix} a_1(t)e^{2\pi i \alpha(t)} & a_2(t)e^{-2\pi i t} \\ a_3(t)e^{2\pi i \alpha(t)} & a_4(t)e^{-2\pi i t} \end{pmatrix}, \quad t \in [0, 1]
\]

is a \(2 \times 2\) unitary matrix. Notice that the only constraint on \(\alpha\) is \(\alpha(0) \in \mathbb{Z}\), it does not have to be periodic unlike the functions \(a_j\). Thus \(\{A_t\}_{0 \leq t \leq 1}\) is a path in \(U(2)\) that stars at the identity. Let \(\psi_t^a : (\mathbb{C}^2, \omega_0) \to (\mathbb{C}^2, \omega_0)\) be the path of Hamiltonian diffeomorphism induced by \(\{A_t\}\). A direct computation yields the Hamiltonian \(H_t^a\) and time-dependent vector field \(X_t^a\) induced by \(\{\psi_t^a\}\).

**Lemma 2.3.** The path \(\{\psi_t^a\}_{0 \leq t \leq 1}\) induced by the path of unitary matrices \(\{A_t\}\) induces in \((\mathbb{R}^4, \omega_0)\) the time-dependent vector field

\[
X_t^a = \left\{ 2\pi y_1(a_2^2 - a_1^2 \alpha') + x_2(a_1' a_3 + a_2' a_4) + 2\pi y_2(a_2 a_4 - a_1 a_3 \alpha') \right\} \frac{\partial}{\partial x_1}
\]

\[
+ \left\{ 2\pi x_1(a_1^2 \alpha' - a_2^2) + 2\pi x_2(a_1 a_3 \alpha' - a_2 a_4) + y_2(a_1' a_3 + a_2' a_4) \right\} \frac{\partial}{\partial y_1}
\]

\[
+ \left\{ x_1(a_1 a_3' + a_2 a_4') + 2\pi y_1(a_1 a_4 - a_1 a_3 \alpha') + 2\pi y_2(a_2^2 - a_3^2 \alpha') \right\} \frac{\partial}{\partial x_2}
\]

\[
+ \left\{ 2\pi x_1(a_1 a_3 \alpha' - a_2 a_4) - y_1(a_1 a_3' + a_2 a_4') + 2\pi x_2(a_3^2 \alpha' - a_4^2) \right\} \frac{\partial}{\partial y_2},
\]

and Hamiltonian function

\[
H_t^a(x_1, y_1, x_2, y_2) = \pi(-a_1^2 \alpha' + a_2^2)(x_1^2 + y_1^2) + \pi(-a_2^2 \alpha' + a_3^2)(x_2^2 + y_2^2)
\]

\[
+ 2\pi(a_2 a_4 - a_1 a_3 \alpha')(x_1 x_2 + y_1 y_2)
\]

\[
+ (a_1 a_3' + a_2 a_4')(x_1 y_2 - x_2 y_1).
\]

For \(r > 0\) denote by \(B_r\) the open ball of radius \(r\) in \(\mathbb{R}^4\) centered at the origin. Using the fact that the functions \(a_j\) define the unitary matrix \(\mathbb{I}\), it follows that the integral of \(H_t^a\) over the ball \(B_r\) only depends on \(\alpha\) and \(r\).

**Lemma 2.4.** If \(H_t^a\) is the Hamiltonian function of Lemma 2.3, then

\[
\int_0^1 \int_{B_r} H_t^a \omega_0^2 \, dt = \left( \frac{\pi^3 r^6}{6} \right)(1 + \alpha(0) - \alpha(1)).
\]
Proof. The integral over $B_r$ of the last two terms of $H^a_t$ are zero. Now since the functions $a_j$ are the entries of a unitary matrix, it follows that

$$\int_0^1 \int_{B_r} H^a_t \omega_0 \, dt = \pi \int_{B_r} (x^2_1 + y^2_1) \omega_0 \cdot \int_0^1 -a_1^2 \alpha' + a_2^2 - a_3^2 \alpha' + a_4^2 \, dt$$

$$= \pi \left( \frac{\pi^2 r^6}{6} \right) \int_0^1 1 - \alpha' \, dt$$

$$= \pi \left( \frac{\pi^2 r^6}{6} \right) (1 + \alpha(0) - \alpha(1)).$$

□

We are interested in the case when the above integral is non zero. Therefore, by the last result we neglect the functions $a_j$ and focus our attention on $\alpha$.

Next we used a bump function to obtain a compactly supported Hamiltonian path. This is the reason that we considered a Hamiltonian path instead of a Hamiltonian loop. If a Hamiltonian function induces a loop, then multiplying it by a bump function will no longer induce a loop. To that end fix $r_0, R_0 \in \mathbb{R}$ such that $0 < r_0 < R_0$ and also fix a bump function $\rho : \mathbb{R}^4 \to \mathbb{R}$ such that $\rho \equiv 1$ on $B_{r_0}$ and $\rho \equiv 0$ on $\mathbb{R}^4 \setminus B_{R_0}$. Consider the compactly supported smooth function $H^{a,\rho}_t : (\mathbb{R}^4, \omega_0) \to \mathbb{R}$ defined as

$$H^{a,\rho}_t := \rho \cdot H^a_t$$

and let $\{\psi^{a,\rho}_t\}_{0 \leq t \leq 1}$ be the induced Hamiltonian path. Observe that $\psi^{a,\rho}_t \in \text{Ham}^c(\mathbb{R}^4, \omega_0)$ and that on the ball $B_{r_0}$ we have that $\psi^{a,\rho}_t = \psi^a_t$ and $H^{a,\rho}_t = H^a_t$ for all $t \in [0, 1]$.

2.2. A compactly supported loop of Hamiltonian diffeomorphisms on $(\mathbb{R}^4, \omega_0)$. Next we define a loop in $\text{Ham}^c(\mathbb{R}^4, \omega_0)$ based at the identity by concatenating two paths of the kind defined above. Fix $a_1, \ldots, a_4, b_1, \ldots, b_4$ smooth 1-periodic functions and $\alpha$ and $\beta$ also smooth functions, such that all functions are subject to the conditions previously imposed. Further, we also impose the condition that the two path of matrices $\{A_t\}$ and $\{B_t\}$ agree in a neighborhood of $t = 1$. Therefore, the exponent functions satisfy $\alpha(1) - \beta(1) \in \mathbb{Z}$.

Define the Hamiltonian loop $\psi = \{\psi_t\}_{0 \leq t \leq 2}$ in $\text{Ham}^c(\mathbb{R}^4, \omega_0)$ as

$$\psi_t := \begin{cases} 
\psi^{a,\rho}_t & t \in [0, 1] \\
\psi^{b,\rho}_{1-t} & t \in [1, 2].
\end{cases}$$

Thus, its compactly supported Hamiltonian function is given by

$$H_t := \begin{cases} 
H^{a,\rho}_t & t \in [0, 1] \\
H^{b,\rho}_{1-t} & t \in [1, 2].
\end{cases}$$
The above observations on the paths \( \{ \psi^a_t, \rho_t \} \) and \( \{ \psi^b_t, \rho_t \} \) and the computation of Lemma 2.4 imply the following facts about the Hamiltonian loop \( \psi \).

**Proposition 2.5.** Given \( r_0, R_0 \in \mathbb{R} \) such that \( R_0 > r_0 > 0 \) there is a Hamiltonian loop \( \psi = \{ \psi_t \}_{t \leq 2} \) defined as in (2) such that it is supported in \( B_{R_0} \) and on \( B_{r_0} \) it agrees with a loop of unitary matrices. Moreover, its Hamiltonian function \( H_t \) satisfies

\[
\int_0^2 \int_{B_{r_0}} H_t \omega_0^2 \, dt = \left( \frac{\pi^3 r_0^6}{6} \right) (\alpha(0) - \alpha(1) - \beta(0) + \beta(1)).
\]  

The condition that the paths \( \{ \psi^a_t, \rho_t \} \) and \( \{ \psi^b_t, \rho_t \} \) agree in a neighborhood of 1, imply that \( \alpha(0) - \alpha(1) - \beta(0) + \beta(1) \) is a integer. Thus Eq. (4) is a kind of winding number for the loop of unitary matrices. Furthermore, it is possible to choose the functions \( \alpha \) and \( \beta \) so that the paths agree in a neighborhood of 1 and \( \alpha(0) - \alpha(1) - \beta(0) + \beta(1) \) is non zero. From now on we fix the Hamiltonian loop \( \psi \) in \( \text{Ham}^c(\mathbb{R}^4, \omega_0) \) defined in (2) so that \( \alpha(0) - \alpha(1) - \beta(0) + \beta(1) \) is equal to 1. The actual value is not important, what is relevant at this point is that it is a non zero integer.

On an open manifold \( (M, \omega) \) the Calabi morphism \( \text{Cal} : \pi_1(\text{Ham}^c(M, \omega)) \to \mathbb{R} \) is defined as

\[
\text{Cal}(\phi) := \int_0^1 \int_M F_t \omega^n \, dt,
\]

where the loop \( \phi \) is generated by the compactly supported Hamiltonian \( F_t \). Moreover, if \( (M, \omega) \) is exact then Calabi’s morphism is identically zero. Consequently, for the the loop \( \psi \) defined in (2) we have that

\[
0 = \int_0^2 \int_{\mathbb{R}^4} H_t \omega_0^2 \, dt,
\]

in contrast with the integral over \( B_{r_0} \) that is non zero.

Equation (5) will be useful when we consider the loop \( \psi \) in a Darboux chart on a closed manifold. For, in this case the normalization condition corresponds to zero mean.

### 3. A LOOP OF HAMILTONIAN DIFFEOMORPHISMS IN THE ONE-POINT BLOW UP OF INFINITE ORDER

In [10] it is proved that if a closed symplectic manifold admits a Hamiltonian circle action, then after blowing up one point the fundamental group of the group of Hamiltonian diffeomorphisms has positive rank. In this section we prove that the above result always holds, namely we prove Theorem 1.2. Henceforth, the hypothesis about the Hamiltonian circle action is no longer required. We restrict to the four-dimensional case in view of the main result of
this paper; however the result presented in this section holds on any symplectic manifold of dimension greater than or equal than four.

This section can be considered as the first step of the proof of the main theorem. Recall that for any \( R_0 > 0 \), the loop \( \psi = \{ \psi_t \}_{0 \leq t \leq 2} \) defined in (2) is supported in the open ball \( B_{R_0} \subset (\mathbb{R}^4, \omega_0) \). Let \((M, \omega)\) be a closed rational symplectic 4-manifold. For \( R_0 > 0 \) small enough, by Darboux’s theorem the loop \( \psi \in \text{Ham}^c(\mathbb{R}^4, \omega_0) \) can be regarded in \( \text{Ham}(M, \omega) \).

Since we are in a closed symplectic manifold \((M, \omega)\), we must normalize the Hamiltonian function \( H_t : (M, \omega) \to \mathbb{R} \). Thus, for \( t \in [0, 1] \) define

\[
c_t := \frac{1}{\text{Vol}(M, \omega^2)} \int_M H_t \omega^2.
\]

Afterwards define \( H^N_t : M \to \mathbb{R} \) as \( H^N_t := H_t - c_t \). Thus, \( H^N_t \) is the normalized Hamiltonian function that induces the same loop \( \psi \) in \( \text{Ham}(M, \omega) \).

Call \( \iota B_{r_0} \subset M \) the image of \( B_{r_0} \subset \mathbb{R}^4 \) under the Darboux embedding. Then the loop \( \psi = \{ \psi_t \} \) satisfies the following:

\[
\begin{align*}
\bullet & \quad \psi_t(\iota(0)) = \iota(0) \text{ for all } t \in [0, 2] \\
\bullet & \quad \psi_t \text{ behaves like a unitary matrix on } \iota B_{r_0} \text{ for all } t \in [0, 2].
\end{align*}
\]

Using the embedded ball \( \iota B_{r_0} \subset M \), define \((\widetilde{M}, \widetilde{\omega}_{r_0})\) to be the one-point blow up at \( \iota(0) \) of \((M, \omega)\) of weight \( r_0 \). From the above remarks on the loop \( \psi \), it follows from [10, Sec. 3] that \( \psi \) induces a Hamiltonian loop \( \widetilde{\psi} = \{ \widetilde{\psi}_t \}_{0 \leq t \leq 2} \) in \( \text{Ham}(\widetilde{M}, \widetilde{\omega}_{r_0}) \). For appropriate values of \( r_0 \), we claim that \( \widetilde{\psi} \) has infinite order in \( \pi_1(\text{Ham}(\widetilde{M}, \widetilde{\omega}_{r_0})) \). We prove this using Weinstein’s morphism

\[
\mathcal{A} : \pi_1(\text{Ham}(\widetilde{M}, \widetilde{\omega}_{r_0})) \to \mathbb{R}/\mathcal{P}(\widetilde{M}, \widetilde{\omega}_{r_0}),
\]

where \( \mathcal{P}(\widetilde{M}, \widetilde{\omega}_{r_0}) \) is the period group.

Next we prove Theorem 1.2 that was stated at the Introduction. We give a more precise statement of the theorem in terms of the loop of Hamiltonian diffeomorphisms \( \psi \) defined above and the weight of the blow up. Keep in mind that we state this result for four-dimensional symplectic manifolds, but the same argument works in higher dimensions.

**Theorem 3.6.** Let \( r_0 > 0 \) such that \( \pi r_0^2 \) is a transcendental number. If \((M, \omega)\) is a rational symplectic 4-manifold, then the induced loop \([\widetilde{\psi}]\) has infinite order in \( \pi_1(\text{Ham}(\widetilde{M}, \widetilde{\omega}_{r_0})) \).

**Proof.** Since \((M, \omega)\) is rational, there are \( q_1, \ldots, q_s \in \mathbb{Q} \) such that \( \mathcal{P}(M, \omega) = \mathbb{Z}(q_1, \ldots, q_s) \). In fact, \( \mathcal{P}(\widetilde{M}, \widetilde{\omega}_{r_0}) = \mathbb{Z}(q_1, \ldots, q_s, \pi r_0^2) \) since the area of the line in the exceptional divisor is \( \pi r_0^2 \).
From [10, Thm. 1.1], it is possible to compute $A(\tilde{\psi})$ in terms solely of the loop $\psi$. Namely,

$$A(\tilde{\psi}) = \left[ A(\psi) + \frac{1}{\text{Vol}(M, \tilde{\omega}^2)} \int_0^2 \int_{tB_{r_0}} H_t^N \omega^2 dt \right].$$

Since $\{\psi_t\}$ is null homotopic, it follows that $A(\psi) = 0$ in $\mathbb{R}/\mathcal{P}(M, \omega)$. Observe that the integral of $H_t^N$ over $tB_{r_0} \subset M$ is the same as the integral over $B_{r_0} \subset \mathbb{R}^4$. Therefore, from Proposition 2.5 we have that

$$\int_0^2 \int_{tB_{r_0}} H_t^N \omega^2 dt = \int_0^2 \int_{tB_{r_0}} H_t - c_t \omega^2 dt = \left( \frac{\pi^3 r_0^6}{6} \right) \cdot 1 - \text{Vol}(B_{r_0}, \omega^2) \int_0^2 c_t dt.$$

The functions $H_t$ are supported $B_{r_0} \subset \mathbb{R}^4$, thus they can also be considered as a functions on $M$. Hence using Eq. 5 it follows that

$$\int_0^2 c_t dt = \int_0^2 \frac{1}{\text{Vol}(M, \omega^2)} \int_M H_t \omega^2 dt = \frac{1}{\text{Vol}(M, \omega^2)} \int_0^2 \int_M H_t \omega^2 dt = 0.$$

If $V$ stands for $\text{Vol}(M, \omega^2)$, then $\text{Vol}(\tilde{M}, \tilde{\omega}^2) = V - \pi^2 r_0^4/2$. Substituting the above computations, we have

$$A(\tilde{\psi}) = \left[ \frac{1}{V - \pi^2 r_0^4/2} \left( \frac{\pi^3 r_0^6}{6} \right) \right] \in \mathbb{R}/\mathbb{Z}(q_1, \ldots, q_s, \pi r_0^2).$$

Then the equation $A(\tilde{\psi}^m) = 0$, for $m \in \mathbb{N}$, is equivalent to a polynomial equation on $\pi r_0^2$ with rational coefficients. Recall that $V \in \mathbb{Q}$. Since $\pi r_0^2$ is transcendental, the Hamiltonian loop $\tilde{\psi} = [\{\tilde{\psi}_t\}_{0 \leq t \leq 2}]$ has infinite order in $\pi_1(\text{Ham}(\tilde{M}, \tilde{\omega}_{r_0}))$. □

4. Proof of the Main Theorem

The proof of Theorem 3.6 gives the blueprint that we follow in order to prove the main result. The proof of the main theorem deals with $k$ distinct Hamiltonian loops supported in $k$ mutually disjoint balls. Thereby, the hypothesis in Theorem 3.6 about $\pi r_0^2$ being transcendental, is replaced by the following lemma.

**Lemma 4.7.** Given $k \in \mathbb{N}$ there exist $k$ distinct real numbers $y_1, \ldots, y_k$ such that for any $j, s \in \{1, \ldots, k\}$ the equation

$$(a + q_1 y_1 + \cdots + q_k y_k)(b - (y_1^2 + \cdots + y_k^2)) + cy_j^3 + y_s^2 = 0$$

Then the equation $A(\tilde{\psi}^m) = 0$, for $m \in \mathbb{N}$, is equivalent to a polynomial equation on $\pi r_0^2$ with rational coefficients. Recall that $V \in \mathbb{Q}$. Since $\pi r_0^2$ is transcendental, the Hamiltonian loop $\tilde{\psi} = [\{\tilde{\psi}_t\}_{0 \leq t \leq 2}]$ has infinite order in $\pi_1(\text{Ham}(\tilde{M}, \tilde{\omega}_{r_0}))$. □
has no solution for any \( q_1, \ldots, q_k, a, b, c \in \mathbb{Q} \).

The proof of the lemma is a consequence of the fact that \( \mathbb{R} \) is an infinite dimensional \( \mathbb{Q} \)-vector space. Next we provide the proof of the main theorem built on the ideas of the proof of Theorem 3.6.

For the proof of the main theorem is important to consider the complex 2-dimensional projective space \((\mathbb{C}P^2, \omega_{FS})\) endowed with its standard Fubini-Study symplectic form. If \( \omega_{FS} \) is normalized so that \((\mathbb{C}P^2 \setminus \mathbb{C}P^1, \omega_{FS})\) is symplectomorphic to the open ball of radius \( R \) in \((\mathbb{R}^4, \omega_0)\), then the area of a complex line is \( \pi R^2 \). Therefore, the period group of \((\mathbb{C}P^2, \omega_{FS})\) is \( \mathbb{Z}(\pi R^2) \) and \( \text{Vol}(\mathbb{C}P^2, \omega_{FS}^2) = (\pi R^2)^2/2 \).

**Proof of Theorem 1.1**. Given \( k > 0 \), it follows from Lemma 1.7 that there are \( k \) distinct numbers \( r_1, \ldots, r_k \in \mathbb{R}_{>0} \) where \( y_j := \pi r_j^2 \) for \( j \in \{1, \ldots, n\} \).

Let \( R_1, \ldots, R_k \in \mathbb{R} \) be any numbers such that \( R_j > r_j \) for all \( j \in \{1, \ldots, k\} \). On \( \mathbb{R}^4 \), fix \( k \) mutually disjoint balls \( B_1, \ldots, B_k \) centered at \( p_1, \ldots, p_k \) and of radii \( R_1, \ldots, R_k \) respectively. Inside each ball \( B_j \) fix a a smaller ball \( B_{r_j} \) of radius \( r_j \) centered at \( p_j \). Next, consider the complex 2-dimensional projective space \((\mathbb{C}P^2, \omega_{FS})\) such that the symplectic form is normalized so that \((\mathbb{C}P^2 \setminus \mathbb{C}P^1, \omega_{FS})\) is symplectomorphic to an open ball of radius \( R \) in \((\mathbb{R}^4, \omega_0)\) that contains all the balls \( B_1, \ldots, B_k \) and \( \pi R^2 \) is a rational number. Therefore the period group of \((\mathbb{C}P^2, \omega_{FS})\) is \( \mathbb{Z}(\pi R^2) \). Call \( \iota_j B_{r_j} \subset \mathbb{C}P^2 \) the image of the fixed ball \( B_{r_j} \subset \mathbb{R}^4 \). Hence in \((\mathbb{C}P^2, \omega_{FS})\) there are \( k \) mutually disjoint embedded balls \( \iota_1 B_{r_1}, \ldots, \iota_k B_{r_k} \). Denote by \((\mathbb{C}P^2 \#_k \overline{\mathbb{C}P}^2, \tilde{\omega}_r)\) the symplectic manifold that is obtained by blowing up the \( k \) points \( \iota_1(p_1), \ldots, \iota_k(p_k) \) in \((\mathbb{C}P^2, \omega_{FS})\) where the weight at \( \iota_j(p_j) \) is \( r_j \). That is, the embedded ball \( \iota_j B_{r_j} \) is removed from the torus for \( j \in \{1, \ldots, k\} \).

\((\mathbb{C}P^2 \#_k \overline{\mathbb{C}P}^2, \tilde{\omega}_r)\) is the desired closed symplectic 4-manifold of the main theorem. Notice that it is simply connected.

Next we define \( k \) Hamiltonian loops. From Proposition 2.5 for each \( j \in \{1, \ldots, k\} \) there is a loop \( \psi^{(j)} = \{\psi^{(j)} \}_0 \leq t \leq 2 \) in \( \text{Ham}^c(\mathbb{R}^4, \omega_0) \) supported in the ball of radius \( R_j \) such that inside the ball of radius \( r_j \) it agrees with a loop of unitary matrices. Therefore, the \( k \) loops in \( \text{Ham}^c(\mathbb{R}^4, \omega_0) \) induced \( k \) loops \( \text{Ham}(\mathbb{C}P^2, \omega_{FS}) \) that we also denoted by \( \psi^{(1)}, \ldots, \psi^{(k)} \).

For every \( j \in \{1, \ldots, k\} \), the loop \( \psi^{(j)} \) on \((\mathbb{C}P^2, \omega_{FS})\) behaves as a loop of unitary matrices on each of the embedded ball \( \iota_s B_{r_s} \). Moreover, if \( s \neq j \) it is the constant loop on \( \iota_s B_{r_s} \). In any case, the loop \( \psi^{(j)} \) fixes the points \( \iota_1(p_1), \ldots, \iota_k(p_k) \). Therefore, from [10] Sec. 3 it follows that \( \psi^{(j)} \) induces a loop \( \tilde{\psi}^{(j)} \) in \( \text{Ham}(\mathbb{C}P^2 \#_k \overline{\mathbb{C}P}^2, \tilde{\omega}_r) \). We claim that the loops \( \tilde{\psi}^{(1)}, \ldots, \tilde{\psi}^{(k)} \) generate a subgroup isomorphic to \( \mathbb{Z}^k \) in \( \pi_1(\text{Ham}(\mathbb{C}P^2 \#_k \overline{\mathbb{C}P}^2, \tilde{\omega}_r)) \).
Fix $j \in \{1, \ldots, k\}$. The corresponding Hamiltonian $H^{(j)}_t$ of $\psi^{(j)}$ is compactly supported, as before let $H^{(j),N}_t : (\mathbb{C}P^2, \omega_{FS}) \to \mathbb{R}$ be its normalization. As in the proof of Theorem 3.6 we have that

$$\int_0^2 \int_{t_j B_{r_j}} H^{(j),N}_t \omega^2 dt = \left(\frac{\pi^3 r_j^6}{6}\right) \cdot 1$$

and

$$\mathcal{A}(\tilde{\psi}^{(j)}) = \left[\frac{1}{V - \pi^2 (r_1^4 + \cdots + r_k^4)/2} \left(\frac{\pi^3 r_j^6}{6}\right)\right] \in \mathbb{R}/\mathbb{Z}(\pi R^2, \pi r_1^2, \ldots, \pi r_k^2).$$

where $V := \text{Vol}(\mathbb{C}P^2, \omega_{FS}) \in \mathbb{Q}$.

Since $\pi R^2 \in \mathbb{Q}$ and the numbers $\pi r_1^2, \ldots, \pi r_k^2$ were chosen according to Lemma 4.7 it follows that for any $m \in \mathbb{Z}_{>0}$ the equation $\mathcal{A}((\tilde{\psi}^{(j)})^m) = 0$ does not hold. Therefore $[\tilde{\psi}^{(j)}]$ has infinite order in $\pi_1(\text{Ham}(\mathbb{C}P^2 \#_k \mathbb{C}P^2, \tilde{\omega}_r))$. The same reasoning implies that $\mathcal{A}((\tilde{\psi}^{(j)})^m) \neq \mathcal{A}((\tilde{\psi}^{(s)})^n)$ for distinct $j, s \in \{1, \ldots, k\}$ and any $m, n \in \mathbb{Z}_{>0}$. Henceforth the rank of $\pi_1(\text{Ham}(\mathbb{C}P^2 \#_k \mathbb{C}P^2, \tilde{\omega}_r))$ is at least $k$.

5. Calabi’s morphism

As noted before, Calabi’s morphism on $\pi_1(\text{Ham}^c(\mathbb{R}^4, \omega_0))$ is trivial. However, the arguments used before show that Calabi’s morphism on the one-point blow up of $(\mathbb{R}^4, \omega_0)$ is non trivial.

Fix $r > 0$, let $(\mathbb{R}^4, \tilde{\omega}_r)$ be the blowup of the origin in $(\mathbb{R}^4, \omega_0)$ of weight $r$. As we have seen the loop $\psi$ in $\text{Ham}^c(\mathbb{R}^4, \omega_0)$ induces the loop $\tilde{\psi}$ in $\text{Ham}^c(\mathbb{R}^4, \tilde{\omega}_r)$.

**Proposition 5.8.** For any $r > 0$, the Calabi morphism on $\text{Ham}^c(\mathbb{R}^4, \tilde{\omega}_r)$ does not vanish.

**Proof.** Let $\psi$ be the Hamiltonian loop in $\text{Ham}^c(\mathbb{R}^4, \omega_0)$ defined in Eq. 2 and let $\tilde{\psi}$ be the induced loop in $\text{Ham}^c(\mathbb{R}^4, \tilde{\omega}_r)$. According to [10, Sec. 1], $\text{Cal}(\tilde{\psi})$ and $\text{Cal}(\psi)$ are related as

$$(6) \quad \text{Cal}(\tilde{\psi}) = \text{Cal}(\psi) - \frac{1}{2} \int_0^2 \int_{B_r} H_t^\omega \omega^2 dt$$

where $H_t : (\mathbb{R}^4, \omega) \to \mathbb{R}$ is the compactly supported Hamiltonian function of the loop $\psi$ defined in Eq. 3. Thus from Proposition 2.5,

$$\text{Cal}(\tilde{\psi}) = -\frac{\pi^3 r^6}{12} \cdot 1.$$

□
Remark. The above result is true for any dimension; Calabi’s morphism on $\text{Ham}^c(\mathbb{R}^{2n}, \tilde{\omega}_r)$ does not vanish for $n \geq 2$ and $r > 0$.

Finally, note that since Calabi’s morphism does not vanish on $\pi_1(\text{Ham}^c(\mathbb{R}^4, \tilde{\omega}_r))$ it does not descend to $\text{Ham}^c(\mathbb{R}^4, \tilde{\omega}_r)$. Furthermore, for the Hamiltonian loop $\tilde{\psi}$ in $\text{Ham}^c(\mathbb{R}^4, \tilde{\omega}_r)$ that appears in the previous proof we have that

$$\ell(\tilde{\psi}) \geq \frac{\pi^3 r^6}{12}.$$  

Here $\ell(\cdot)$ stands for the Hofer length of the class in $\pi_1(\text{Ham}(\cdot))$. For the precise definition of $\ell(\cdot)$, see [11, Sec. 7.3].

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