THE CRITICAL ONE-DIMENSIONAL MULTI-PARTICLE DLA

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ABSTRACT. We study one-dimensional multi-particle Diffusion Limited Aggregation (MDLA) at its critical density $\lambda = 1$. Previous works have verified that the size of the aggregate $X_t$ at time $t$ is $t^{1/2}$ in the subcritical regime and linear in the supercritical regime. This paper establishes the conjecture that the growth rate at criticality is $t^{2/3}$. Moreover, we derive the scaling limit proving that

$$\{t^{-2/3}X_{st}\}_{s \geq 0} \overset{d}{\to} \left\{ \int_0^s Z_u du \right\}_{s \geq 0},$$

where the speed process $\{Z_t\}$ is a $(-1/3)$-self-similar diffusion given by $Z_t = (3V_t)^{-2/3}$, where $V_t$ is the $\frac{5}{2}$-Bessel process.

The proof shows that locally the speed process can be well approximated by a stochastic integral representation which itself can be approximated by a critical branching process with continuous edge lengths. From these representations, we determine its infinitesimal drift and variance to show that the speed asymptotically satisfies the SDE $dZ_t = 2Z_t^{5/2} dB_t$. To make these approximations, regularity properties of the process are established inductively via a multiscale argument.

CONTENTS

1. Introduction 2
2. Approximations of the speed and proof outline 6
3. An a priori upper bound and the induction base 17
4. Fixed rate process and its perturbation 21
5. The age-dependent critical branching process 39
6. Regularity of the speed 50
7. The inductive analysis on the speed: Part 1 59
8. The inductive analysis on the speed: Part 2 74
9. The second order contributions and the moments of the increment 94
10. Multi-scale analysis and the scaling limit 105
Acknowledgements 118
References 118
Appendix A. The renewal process, Fourier transform and singularity analysis 120
Appendix B. Martingale concentration lemmas 131
Appendix C. Negligibility of the third order contributions 132

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1. Introduction

We study a variant of the DLA model called the multi-particle Diffusion Limited Aggregation (MDLA), where a cloud of particles diffuse simultaneously before adhering to a growing aggregate. This model was first studied in the physics literature [30] and later in the mathematics literature [26, 25, 18, 27, 8, 17]. The model in dimension $d \geq 1$ and density $\lambda > 0$ is defined as follows. For each time $t > 0$ the aggregate is a set of vertices $A_t \subseteq \mathbb{Z}^d$. Initially, $A_0 = \{0\}$ and on each vertex $v \neq 0$ there is a random number of particles distributed as Poisson($\lambda$), independently of the other vertices. At time $t = 0$ each particle starts to move according to a simple, continuous time random walk, independently of other particles. The aggregate $A_t$ grows according to the following rule: If at time $t$ one of the particles at $v \in A_{t^-}$ attempts to jump into the aggregate, it freezes in place, together with all the other particles at $v$ and the aggregate grows by $A_t = A_{t^-} \cup \{v\}$. Frozen particles do not move for the rest of the process and cannot make the aggregate grow.

Perhaps surprisingly at first sight, only the one dimensional model is expected to exhibit a phase transition, while in higher dimensions the aggregate is expected to grow linearly for all initial densities $\lambda > 0$ which has been confirmed only for large enough $\lambda$ by Sidoravicius and Stauffer [26] and also in [27]. From now on we focus only on the one dimensional case. In this case the aggregate is simply a line segment and the processes on the positive and negative axes are independent so we simply restrict our attention to the rightmost position of the aggregate at time $t$ which we denote $X_t$. If at time $t$ a particle at $X_t + 1$ attempts to take a step to the left, the aggregate grows by one: $X_t = X_{t^-} + 1$.

Kesten and Sidoravicious [18] proved that in the subcritical regime when $\lambda < 1$, $X_t = \Theta(\sqrt{t})$ with high probability. Moreover, using the results of Dembo and Tsai [8], one can show that in fact $X_t = (c_-(\lambda) + o(1))\sqrt{t}$ for an explicit $c_-(\lambda) > 0$. In the supercritical regime, the third author proved the existence of a phase transition [22] by showing that when $\lambda > 1$, $X_t$ grows linearly and moreover that $t^{-1}X_t \to c_+(\lambda) > 0$ almost surely.

It was widely conjectured that in the critical case, when $\lambda = 1$, the aggregate grows like $t^{2/3}$. Sidoravicious and Rath [25] gave a heuristic explanation for this prediction via PDEs. In [8], Dembo and Tsai proved an upper bound of $O(t^{2/3})$ by studying a modified “frictionless” variant of the model which stochastically dominates the aggregate (see Section 1.1 for details). We verify the conjectured $t^{2/3}$ growth in the following theorem and determine the scaling limit.

**Theorem 1.** Let $V_t$ be the Bessel process with dimension $\frac{8}{3}$ given by

$$dV_t = \frac{5}{6} \frac{dt}{V_t} + dB_t, \quad V_0 = 0,$$

and let $Z_t = (3V_t)^{-2/3}$. Then,

$$\left\{ t^{-\frac{2}{3}} X_{st} \right\}_{s > 0} \xrightarrow{d} \left\{ \int_0^s Z_x dx \right\}_{s > 0}, \quad t \to \infty.$$

In particular $t^{-2/3}X_t$ has a limiting distribution which is positive almost surely. The limiting speed process $Z_t$ is a local martingale which is $(-1/3)$-self-similar which means $\{Z_{st}\}_{s \geq 0} \stackrel{d}{=} \{t^{1/3} Z_s\}$. While this representation is perhaps surprising, any $(-1/3)$-self-similar diffusion must be a $(-2/3)$ power of a Bessel process. An alternative representation, which can easily be verified by Itô’s Formula, is that

$$dZ_t = 2Z_t^{5/2} dB_t,$$

with $Z_0 = \infty$. The initial condition $Z_0 = \infty$ can be understood by the limit of the diffusion satisfying (1.1) with finite initial condition $Z_0$ sent to infinity. See Sections 2.4.6 and 10 for details.
Figure 1. We obtained the following graphs from a computer simulation of the critical 1-d MDLA. The first picture shows the size as a function of time of 5 independent aggregates running for time 300,000. The second picture shows the speed of growth of the blue aggregate in the first picture. Theorem \[ dZ_t = 2Z_t^{5/2} dB_t \] states that the speed converges to the solution of \[ dB_t \] with initial condition \( Z_0 = \infty \).
1.1. Related models. The MDLA model originated from the study of Diffusion Limited Aggregation (DLA). In the DLA model, particles come from infinity one by one by a diffusion and adhere to the aggregate. The fundamental problem is determining the asymptotic growth and shape of the aggregate. The model was introduced by Witten and Sanders [32], and describes physical phenomena such as mineral growths and electrodeposition. Although it has been studied extensively (e.g., [33, 20, 29, 12, 13, 14, 3]), there has been limited progress in obtaining rigorous results.

The frictionless model, which is a variant of MDLA, was introduced and successfully studied by Dembo and Tsai [8]. The model is defined in one dimension as follows. Like the MDLA, there is a growing aggregate on the set of positive integers and a cloud of particles that perform a continuous time random walk but the aggregate grows by the number of particles it absorbs rather than 1. In the supercritical case it explodes in finite time. The size of the aggregate is exactly equal to the number of particles absorbed, which is called in [8] the “flux condition” allowed Dembo and Tsai to derive the asymptotic behavior of the model based on PDE techniques, deriving a discontinuous scaling limit. We note that the flux condition does not hold in the MDLA model but the front stochastically dominates it and thus gives an upper bound.

Another way to interpret the MDLA is to view it as a special case of a two-type particle system (so-called the A-B model), which is sometimes used to model the spread of a rumor or a disease in a network. In the A-B model, there are two types of particles, the A-type (“healthy individuals”) and the B-type (“infected”). All the particles perform a continuous time random walk on \( \mathbb{Z}^d \) in which the A particles jump with rate \( D_A \geq 0 \) and the B particles jump with rate \( D_B > 0 \). When A and B particles coincide, the A particle turns into a B particle. Theses models were studied extensively and we refer to [15, 4, 7, 19, 24, 16].

In the case \( D_B = 0 \) and \( D_A > 0 \) (that is, only the A particles move) the transition rule is modified such that when an A particle tries to jump into a vertex with a B particle, the jump is suppressed and the A particle, together with all other A particles on the same vertex turn into B particles. It is clear that when \( D_A = 1 \) and when initially there is one B particle in the origin this corresponds to the MDLA.

The opposite case \( D_A = 0 \) and \( D_B > 0 \) is sometimes referred to in the physics literature as the Stochastic combustion process and it is used to model the burning of propellant material [23, 6]. In the mathematics literature it is sometimes called the frog model [1, 2].

1.2. Further directions and open problems.

1.2.1. Near critical behavior. For the one-dimensional MDLA with initial density \( \lambda \) very close to 1, Sidoravicius and Rath [25] asked to determine the critical exponent of the aggregate size as \( \lambda \to 1 \). To be precise, when \( \lambda > 1 \), they predicted that the constant \( c_+(\lambda) := \lim_{t \to \infty} t^{-1} X_t \) is linear in \( \lambda - 1 \) as \( \lambda \searrow 1 \), and conjectured that \( c_+(\lambda) \sim \frac{1}{2}(\lambda - 1) \) from a heuristic argument counting the sites in the aggregate with more than one particle. On the other hand, for the subcritical case, they expected that the constant \( c_-(\lambda) := \lim_{t \to \infty} t^{-1/2} X_t \) satisfies \( c_-(\lambda) \sim (1 - \lambda)^{-1/2} \).

In a forthcoming companion paper [9] building on the techniques we develop here, we give an affirmative answer to the latter conjecture, but disprove the first one by establishing

\[
\lim_{\lambda \to 1} c_+(\lambda) = \frac{1}{3}, \quad \lim_{\lambda \to 1} (1 - \lambda)^{1/2} c_-(\lambda) = 1. \tag{1.2}
\]

The heuristic calculation from [25] for the slightly supercritical regime leads to a wrong answer as it assumed that the speed of the growth is concentrated as \( t \to \infty \). Instead, it converges to a nondegenerate diffusion upon appropriate rescaling as \( \lambda \searrow 1 \). We remark that the second identity of (1.2) can be obtained by the results of [8] from a completely different approach.

When \( \lambda \) is very close to 1, it is intuitively clear that when \( t \) is not very large, the aggregate will look the same as the critical case \( \lambda = 1 \). This leads to the question of how large must \( t \) be (in terms of \( \lambda \)) in order for the model to “feel” that it is slightly supercritical or subcritical, and is related to
the rescaling of the process mentioned in the previous paragraph. In [9], we show that the answer to this question is $t \sim |\lambda - 1|^{-3}$, and give a more precise description on the scaling limit of $X_t$ as follows: Let $U_t$ and $R_t$ be the solutions to the SDEs

$$dU_t = 2U_t^3dt + 2U_t^{5/2}dB_t, \quad dR_t = -2R_t^3dt + 2R_t^{5/2}dB_t,$$

with initial conditions $U_0 = R_0 = \infty$. The methods from the current paper can be extended to prove that the aggregate size $X_t = X_t(\lambda)$ satisfies

$$\{(\lambda - 1)^2X_{s(\lambda-1)^{-3}}\}_{s>0} \xrightarrow{d} \left\{ \int_0^s U_x dx \right\}_{s>0}, \quad \lambda \approx 1.$$

Similarly, in the slightly subcritical case we have

$$\{(1 - \lambda)^2X_{s(1-\lambda)^{-3}}\}_{s>0} \xrightarrow{d} \left\{ \int_0^s R_s dx \right\}_{s>0}, \quad \lambda \approx 1.$$

One can see that $U_t$ converges to a nondegenerate stationary process as $t \rightarrow \infty$, and a standard calculation tells us that its stationary measure is an inverse gamma distribution whose mean is $\frac{1}{3}$, implying (1.2). On the other hand, in the subcritical case, we have almost surely that $2\sqrt{t}R(t) \rightarrow 1$ as $t \rightarrow \infty$, since the diffusive term of the SDE becomes negligible compared to the drift as $t$ increases. This leads to the second equation of (1.2).

1.2.2. Different initial distribution. One can consider the one-dimensional MDLA with an initial distribution that is not Poisson. Suppose that at time 0, the number of particles on vertex $i$ is $Z_i$ where

$$Z_1, Z_2, \ldots \text{ are i.i.d., } \mathbb{E}Z_1 = \lambda, \quad \text{Var}(Z_1) = \sigma^2. \quad (1.3)$$

In case where all the particles just perform independent random walks on $\mathbb{Z}$ without a growing aggregate, it is clear that for all $\lambda > 0$, i.i.d. Poisson($\lambda$) on each point of $\mathbb{Z}$ is a stationary distribution for the dynamics. Thus, one might expect that the aggregate with the initial condition (1.3) grows exactly like the original MDLA with the Poisson initial profile. However, we expect that in the critical case, the aggregate grows faster than the time it takes for large intervals to approach stationarity. Therefore, the scaling limit in the critical case is expected to depend on the initial distribution of particles. More precisely, we conjecture formulate the following conjecture.

**Conjecture 1.** Let $X_t$ be the size of the aggregate with initial condition as in (1.3). Then, with high probability

1. If $\lambda < 1$, then $X_t = \Theta(\sqrt{t})$.
2. If $\lambda > 1$, then $X_t = \Theta(t)$.
3. If $\lambda = 1$, then $X_t = \Theta(t^{2/3})$. Morover, we have

$$\left\{ t^{-\frac{2}{3}}X_{st} \right\}_{s>0} \xrightarrow{d} \left\{ \int_0^s Z_x dx \right\}_{s>0}, \quad t \rightarrow \infty,$$

where $Z_t$ is the solution to the equation

$$dZ_t = (4\sigma^2 - 4)Z_t^4 dt + 2\sigma Z_t^{5/2}dB_t, \quad (1.4)$$

with $Z_0 = \infty$.

**Remark 1.1.** We elaborate several aspects of the conjecture as follows.

1. **Theorem 1** is compatible with this conjecture. Indeed, in the Poisson case with $\lambda = 1$, we have that $\sigma^2 = 1$ and therefore the drift term vanishes.

2. For the deterministic initial condition where each vertex has exactly one particle in the beginning, we see that the diffusive term vanishes and we get a deterministic scaling limit of the form $X_t = (c_0 + o(1))t^{2/3}$ with high probability.
(3) Note that $Z_t$ from \cite{1,4} is $(-\frac{1}{2})$-self-similar and hence is a power of a Bessel process. Also, recall that the Bessel process with dimension $n \leq 2$ is recurrent. Applying Itô’s formula, it turns out that the corresponding Bessel process for $Z_t$ has dimension $(\frac{3}{4} + \frac{1}{3\sigma^2})$. Therefore, $Z_t$ explodes in finite time if $\sigma^2 \geq 2$ while it tends to 0 if $\sigma^2 < 2$. Thus, we expect that the scaling limit of $X_t$ is differentiable when $\sigma^2 < 2$ and is not differentiable when $\sigma^2 \geq 2$.

2. Approximations of the speed and proof outline

In this section, we define the speed of the process, which is our main object of study. As the speed is a complicated process by itself, we give several approximations of the speed by more tractable processes. Based on those expressions, we also give an overview of the proof of Theorem 1.

**Definition 2.1.** The speed $S(t)$ of the aggregate at time $t$ is $\frac{1}{2}$ times the conditional expectation of the number of particles at position $X_t + 1$, given $\{X_t\}_{0 \leq t \leq T}$.

This corresponds to the infinitesimal jump rate of $X_t$; note that the particles at position $X_t + 1$ are equally likely to jump to $X_t$ and $X_t + 2$, hence the multiplication by $\frac{1}{2}$. To give a more explicit expression of $S(t)$, we follow the approach of \cite{27} and define

$$Y_t(s) := \begin{cases} X_t - X_{t-s} & \text{if } 0 \leq s \leq t; \\ \infty & \text{if } s > t. \end{cases} \quad (2.1)$$

The following observation was first made in \cite{27}, and this is the starting point of our work as well. Here, we consider a general MDLA with initial density $\lambda > 0$, i.e., having i.i.d. Poisson($\lambda$) number of particles at each position in the beginning.

**Proposition 2.2.** Let $\{X_t\}_{t \geq 0}$ be the 1D MDLA with density $\lambda > 0$ and let $W(s)$ denote an independent continuous-time simple random walk starting at 0. Then, we have

$$S(t) = \frac{\lambda}{2} \mathbb{P} \left( \sup_{s \geq 0} \left\{ W(s) - Y_t(s) \right\} \leq 0 \right| Y_t \right). \quad (2.2)$$

Moreover, $\{X_t\}_{t \geq 0}$ is a Poisson process with rate $\{S(t)\}_{t \geq 0}$.

In \cite{27}, (2.2) was used to prove that $X_t$ grows linearly in $t$ if $\lambda > 1$. The approach was to show that the speed has the property of mean reversion. If the growth rate of $X_t$ becomes too small, say, $\alpha$, for a period of time, then $S(t) \approx \alpha t$, corresponding to a faster growth rate. However, at criticality, the same analysis would give $S(t) \approx \alpha(1 + o(1))$, and thus understanding the lower order terms becomes necessary. Indeed, our ultimate goal is to deduce the scaling limit of the smoothed speed by conducting a more refined and quantitative analysis on such error terms.

In Section 2.1, we introduce the notion of the *speed process* which generalizes (2.2) and explain how they are approximated by the objects we can quantitatively deal with. Then, we see how the speed of the critical aggregate is described in Section 2.2. Furthermore, in Section 2.3, we give an overview on how to obtain its scaling limit by using a smoothed version. In Section 2.4, we describe an outline of the proof. In the final subsection, Section 2.5, we illustrate the necessary estimates on the important quantities that appear frequently in the article.

2.1. The speed process and its approximation. In the remainder of this section, $Y$ denotes a *unit-step function* which is defined as follows.

**Definition 2.3** (Unit-step function and its speed process). $Y : \mathbb{R}_{\geq 0} \to \mathbb{N} \cup \{\infty\}$ is called a unit-step function if there exist either $x_1 < x_2 < \ldots$ such that

$$Y(s) = \sum_{i \geq 1} I\{x_i < s\},$$

where $I\{x_i < s\}$ is the indicator function for the event that $x_i < s$. Note that $Y$ is a non-decreasing sequence of step functions.
or \( n \in \mathbb{N} \) and \( x_1 < x_2 < \ldots < x_n < x \) such that
\[
Y(s) = \begin{cases} 
\sum_{i=1}^{n} I\{x_i < s\} & \text{if } s \leq x; \\
\infty & \text{if } s > x.
\end{cases}
\]

Note that in both cases \( Y(0) = 0 \), \( Y \) is increasing and left-continuous. Furthermore, the speed process of \( Y \) is defined as
\[
S(Y) := \frac{1}{2} \mathbb{P} \left( \sup_{s \geq 0} \{W(s) - Y(s)\} \leq 0 \bigg| Y \right),
\]
where \( W(s) \) denotes the continuous-time simple random walk starting at 0.

We let \( \mathbb{P}_\alpha \) denote the probability with respect to \( Y \) being a rate-\( \alpha \) Poisson process. We will proceed by considering the formula \( S(Y) \) under \( \mathbb{P}_\alpha \). In doing so we obtain a formula for \( S(Y) \) described in Proposition 2.4 below which enables us to conduct a refined quantitative study on the speed process. While our starting point is a Poisson \( Y \), the formula will hold for all unit-step functions \( Y \).

We define the stopping time \( T \) as
\[
T = T(Y) := \inf\{s > 0 : W(s) > Y(s)\},
\]
and set
\[
U(s) := Y(s) - W(s) + 1.
\]
which is a continuous time random walk with drift under \( \mathbb{P}_\alpha \). Setting \( \xi = (1 + 2\alpha)^{-1} \) we have that \( \xi^{U_s} \) is a martingale, and hence
\[
\xi^{U_s} \frac{\ln^2 s}{s} I\{T < \infty\}. \tag{2.5}
\]
Thus, an application of the optimal stopping theorem gives
\[
\mathbb{P}_\alpha(T < \infty) = \xi.
\]
Thus, we obtain that
\[
\mathbb{E}_\alpha[S(Y)] = \frac{\alpha}{1 + 2\alpha}.
\]

Coming back to the general case, let \( Y \) be a given unit-step function, let \( \alpha > 0 \) and set \( \xi = \frac{1}{1 + 2\alpha} \). We define \( D_t \) to be
\[
D_t := \mathbb{E}_\alpha \left[ \xi^{U_{s,t}} I\{Y_{\leq s} \} \right],
\]
where \( \mathbb{E}_\alpha[\cdot | Y_{\leq t}] \) denotes the expectation over \( W \) and the unit-step function obtained by the following procedure: take the profile of \( Y \) in \([0, t]\) and generate the rest beyond \( t \) by a rate-\( \alpha \) Poisson process. In such a case, \( \xi^{U_{s,t}} \) forms a martingale in \( t \) which converges to \( I\{T < \infty\} \) as \( t' \to \infty \). Thus, we also know that
\[
D_t = \mathbb{P}_\alpha(T < \infty | Y_{\leq t}). \tag{2.6}
\]
The proposition below tells us a way to understand \( D_t \) from its infinitesimal differences. In what follows, \( dY \) denote the additive combination of the Dirac measures at the points of discontinuity, that is, under the notation of Definition 2.3, we define
\[
dY(s) := \sum x_i \delta_{x_i}(s).
\]

**Proposition 2.4.** Let \( \alpha, t > 0 \), and \( Y \) be a unit-step function that is finite on \([0, t]\). Denoting \( d\tilde{Y}(s) := dY(s) - \alpha ds \), we have
\[
D_t = \frac{1}{1 + 2\alpha} - \frac{2\alpha}{1 + 2\alpha} \int_0^t H_s \, d\tilde{Y}(s), \tag{2.7}
\]
where \( H_s = H_s(Y_{s\leq}, \alpha) := \mathbb{P}_\alpha(s < T < \infty \mid Y_{s\leq}). \)

**Proof.** Set \( \xi = \frac{1}{1+2\alpha} \) as before. For \( Y \) which is a unit-step function, we write
\[
\frac{dY(s)}{ds} = Y(s+) - Y(s).
\]

Then, observe that
\[
\lim_{\Delta \searrow 0} \frac{1}{\Delta} \left[ \mathbb{E}[D_{s+\Delta} \mathcal{F}_s] - \mathbb{E}[D_{s} \mathcal{F}_s] \right] = \mathbb{E}_\alpha \left[ \xi^{U_{s,T}} I\{s < T\} \left( (\xi - 1) \frac{dY(s)}{ds} + \left( \frac{\xi + \xi^{-1}}{2} - 1 \right) \right) \right] \mid Y_{s\leq}
\]
\[
= - \frac{2\alpha}{1 + 2\alpha} \left( \frac{dY(s)}{ds} - \alpha \right) \mathbb{E}_\alpha \left[ \xi^{U_{s,T}} I\{s < T\} \mid Y_{s\leq} \right]
\]
\[
= - \frac{2\alpha}{1 + 2\alpha} \left( \frac{dY(s)}{ds} - \alpha \right) \mathbb{P}_\alpha(s < T < \infty \mid Y_{s\leq}),
\]

where the last identity can be deduced similarly as (2.6): for a fixed \( s > 0 \),
\[
\left\{ \xi^{U_{s,T}} I\{s < T\} \right\}_{s \geq s}
\]
is a martingale in \( s' \), considering \( Y \) as a rate-\( \alpha \) Poisson process on \([s, \infty)\). Since this converges to \( I\{s < T < \infty \} \) as \( s' \to \infty \), the optimal stopping theorem tells us that
\[
\mathbb{E}_\alpha \left[ \xi^{U_{s,T}} I\{s < T\} \mid Y_{s\leq} \right] = \mathbb{P}_\alpha(s < T < \infty \mid Y_{s\leq}).
\]

Then, we can conclude the proof by observing that \( D_0 = \xi \). \( \square \)

From Proposition 2.4, we can also deduce the value of \( \int_0^\infty \mathbb{E}_\alpha H_s ds \) as follows.

**Corollary 2.5.** Let \( \alpha > 0 \). For \( H_s = H_s(Y_{s\leq}, \alpha) \) defined as above, we have
\[
\int_0^\infty \mathbb{E}_\alpha[H_s] ds = \frac{1}{\alpha(1 + 2\alpha)}.
\]

**Proof.** Let \( \alpha' > 0 \) (different from \( \alpha \)), and let \( Y \) be the rate-\( \alpha' \) Poisson process. Recall the expression for \( D_\infty(Y) \) from Proposition 2.4 and take the expectation over \( Y \):
\[
\frac{1}{1 + 2\alpha'} = \mathbb{E}_\alpha' D_\infty(Y) = \frac{1}{1 + 2\alpha} - \frac{2\alpha}{1 + 2\alpha} \mathbb{E}_\alpha' \left[ \int_0^\infty H_s(Y_{s\leq}, \alpha) d\bar{Y}(s) \right].
\]

Here, note that \( d\bar{Y}(s) = dY(s) - \alpha ds \) does not depend on \( \alpha' \). Thus, differentiating each side with \( \alpha' \) and then plugging in \( \alpha' = \alpha \) gives
\[
-\frac{2}{(1 + 2\alpha)^2} = - \frac{2\alpha}{1 + 2\alpha} \int_0^\infty \mathbb{E}_\alpha[H_s(Y_{s\leq}, \alpha)] ds,
\]
which gives the conclusion after rearranging. \( \square \)

The above corollary motivates us to define the *branching density* \( K_\alpha \), which is one of the main objects that we need to understand very precisely.

**Definition 2.6** (The branching density). For each \( \alpha > 0 \), the function \( K_\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \), which we call the branching density, is defined as
\[
K_\alpha(s) := \alpha(1 + 2\alpha) \mathbb{E}_\alpha[H_s] = \alpha(1 + 2\alpha) \mathbb{P}_\alpha(s < T < \infty).
\]
Corollary \ref{cor:K} tells us that \( \int_0^\infty K_\alpha(s)\,ds = 1 \), and hence \( K_\alpha \) can be understood as a probability density function. It turns out that \( K_\alpha \) describes the density of the branch lengths of a certain branching process which locally approximates \( X_t \) (see Remark \ref{rem:K}), and hence we call it the branching density.

Next, we introduce a similar decomposition as \eqref{eq:decomp} that works for \( H_s \), in order to approximate \( H_s \) by \( \mathbb{E}_\alpha[H_s] \), or \( K_\alpha(s) \). Here, we stress that the expectation \( \mathbb{E}_\alpha[H_s] = \mathbb{E}_\alpha[H_s(Y_{\leq s}, \alpha)] \) is taken over the randomness of \( Y_{\leq s} \) as the rate-\( \alpha \) Poisson process. For each \( u > 0 \), define the stopping time \( T_u \) to be

\[
T_u = T_u(Y) := \inf\{ s > 0 : Y(s) - W(s) + I\{ s > u \} < 0 \}. \tag{2.8}
\]

In other words, it is the first time when \( W \) exceeds \( Y \), with an additional unit jump added to \( Y \) at \( u \). Then, the decomposition for \( H_s \) is given by the following proposition.

**Proposition 2.7.** Suppose that \( Y \) is finite in \([0, t]\) and let \( d\tilde{Y}, H_s \) and \( T_u \) be as above. Define

\[
J_{u,s} = J_{u,s}(Y; \alpha) := \mathbb{P}_\alpha(s < T_u \leq \infty \mid Y_{\leq u}) - \mathbb{P}_\alpha(s < T \leq \infty \mid Y_{\leq u}). \tag{2.9}
\]

Then, we have

\[
H_s = \mathbb{E}_\alpha[H_s] + \int_0^s J_{u,s} \,d\tilde{Y}(u).
\]

In particular, we can rewrite the formula \eqref{eq:decomp} as

\[
D_t = \frac{1}{1 + 2\alpha} - \frac{1}{(1 + 2\alpha)^2} \int_0^t K_\alpha(s) \,d\tilde{Y}(s) - \frac{2\alpha}{1 + 2\alpha} \int_0^t \int_0^s J_{u,s} \,d\tilde{Y}(u)\,d\tilde{Y}(s).
\]

**Remark 2.8.** We use bold alphabet \( J_{u,s} \) to define \eqref{eq:J} to emphasize its stochastic nature, since it depends on \( Y_{\leq u} \) which is going to be random. This will prevent confusion from its “averaged” form introduced in \eqref{eq:Jt}.

**Proof of Proposition 2.7.** For \( u < s \), observe that we have the following expression for an infinitesimal difference of \( H_s \):

\[
\lim_{\Delta \to 0} \frac{1}{\Delta} \left( \mathbb{E}_\alpha[H_s \mid Y_{\leq u + \Delta}] - \mathbb{E}_\alpha[H_s \mid Y_{\leq u}] \right) = J_{u,s} \left( \frac{dY(u)}{du} - \alpha \right).
\]

Thus, we directly obtain the conclusion by noticing that \( H_s = \mathbb{E}_\alpha[H_s \mid Y_{ \leq s}] \).

Combining \eqref{eq:ST}, Corollary \ref{cor:K} and Proposition \ref{prop:decomp} we obtain the following expression for the speed process:

\[
\mathbb{E}_\alpha[S(Y) \mid Y_{\leq s}] = \mathbb{P}_\alpha(T = \infty \mid Y_{\leq s})
\]

\[
= \frac{\alpha}{1 + 2\alpha} + \frac{1}{(1 + 2\alpha)^2} \int_0^t \int_0^s J_{u,s} \,d\tilde{Y}(u)\,d\tilde{Y}(s)
\]

\[
= \frac{2\alpha^2}{1 + 2\alpha} + \frac{1}{(1 + 2\alpha)^2} \int_0^t K_\alpha(s) \,dY(s) + \frac{\alpha}{1 + 2\alpha} \int_0^t \int_0^s J_{u,s} \,d\tilde{Y}(u)\,d\tilde{Y}(s). \tag{2.10}
\]

This formula is one of the two primary formulas which lies at the heart of our work, and sometimes we refer to this as the first-order expansion of the speed process. In the next subsection, we discuss in more detail that how we express the actual speed of the aggregate in the form of \eqref{eq:ST}.

**Remark 2.9.** We will later see that \( \int_0^t K_\alpha(s)\,dY(s) \) is the leading order in the expression \eqref{eq:ST}, which is of order \( \alpha \) while other terms are of smaller order. Moreover, this term can be understood as a branching process, where each jump in \( Y \) at distance \( s \) results in a new jump to branch out at rate \( K_\alpha(s) \). More detailed illustrations are given in the following subsections (see, e.g., \eqref{eq:K} and the discussion below).
The second main formula gives an additional order of approximation of the speed process. In (2.10), it turns out that the double integral of $J_{u,s}$ has a non-negligible order in the derivation of the scaling limit of (the smoothed version of) the speed. Thus, we need another layer of approximation to control its size more accurately. To this end, we first define the stopping time $T_{v,u}$ for each $u > v > 0$ similarly as (2.8).

$$T_{v,u} = T_{v,u}(Y) := \inf\{s > 0 : Y(s) - W(s) + I\{s > v\} + I\{s > u\} < 0\}.$$  

In other words, it is the first time when $W$ exceeds $Y$, with two additional unit jumps added to $Y$ at $v$ and $u$.

**Proposition 2.10.** Suppose that $Y$ is finite on $[0,t]$ and let $d\bar{Y}$, $J_{u,s}$, $T_{v,u}$ and $T_u$ be as above. Define

$$Q_{v,u,s} = Q_{v,u,s}(Y;\alpha) := \mathbb{P}_\alpha(s < T_{v,u} < \infty \mid Y_{\leq v}) - \mathbb{P}_\alpha(s < T_v < \infty \mid Y_{\leq v})$$

$$- \mathbb{P}_\alpha(s < T_u < \infty \mid Y_{\leq v}) + \mathbb{P}_\alpha(s < T < \infty \mid Y_{\leq v}).$$  

(2.11)

Then, we have

$$J_{u,s} = \int_0^u Q_{v,u,s} \, d\bar{Y}(v).$$  

In particular, the speed process admits the following expression:

$$\mathbb{E}_\alpha[S(Y) \mid Y_{\leq t}] = \frac{2\alpha^2}{(1 + 2\alpha)^2} + \frac{1}{(1 + 2\alpha)^2} \int_0^t K_\alpha(s) \, dY(s)$$

$$+ \frac{\alpha}{1 + 2\alpha} \int_0^t \int_0^s J^{(\alpha)}_{u,s} \, d\bar{Y}(u) \, d\bar{Y}(s) + \frac{\alpha}{1 + 2\alpha} Q_t,$$

where $J^{(\alpha)}_{u,s}$ is defined as

$$J^{(\alpha)}_{u,s} := \mathbb{E}_\alpha[J_{u,s}],$$  

(2.13)

and $Q_t$ denotes

$$Q_t = Q_t(Y;\alpha) := \int_0^t \int_0^u \int_0^s Q_{v,u,s} \, d\bar{Y}(v) \, d\bar{Y}(u) \, d\bar{Y}(s).$$

**Proof.** Proof follows analogously as Proposition 2.7 by observing that

$$\lim_{\Delta \to 0} \frac{1}{d\Delta} \mathbb{E}_\alpha[J_{u,s} \mid Y_{\leq v + \Delta}] = Q_{v,u,s} \left( \frac{d\bar{Y}(v)}{dv} - \alpha \right).$$  

□

We sometimes refer to the formula (2.12) as the second-order expansion of the speed process. Later, it turns out that the triple integral of $Q_{v,u,s}$ is negligible for the purpose of obtaining the scaling limit. Moreover, the terms $K_\alpha(s)$ and $J^{(\alpha)}_{u,s}$ no longer possess dependence on $Y$ and in fact they can be studied very explicitly. These facts eventually allow us to analyze (2.12) very precisely, which leads to the derivation of its scaling limit.
2.2. The speed of the aggregate. The goal of this section is to see how the actual speed of the aggregate is described in terms of the first- and second-order expansion. Recall the definition of $Y_t$ [2.1] and the formula [2.2] for $S(t)$. In the aggregate, a new particle is added at rate $S(t)$. In other words, if we let $\Pi$ be the Poisson point process on the plane with the standard Lebesgue intensity, we can write $X_t = |\Pi_s(0,t)|$, where

$$\Pi_S[0,t] = \{x \in [0,t]: (x,y) \in \Pi \text{ for some } y \in [0, S(x)]\}.$$

Based on this notation, we can write $dY_t(s)$ as

$$dY_t(s) = d\Pi_S(t-s),$$

which is the notation we mainly use throughout the rest of the paper. From this identity, we can recover $\{Y_t(s)\}_{s \geq 0}$ from $\Pi_S[0,t]$ as

$$Y_t(s) = |\Pi_S[t-s,t]|.$$ (2.14)

In principle, $S(t)$ can be written as [2.10] and [2.12] by plugging in $Y_t$ in the place of $Y$. However, there are several aspects that we need to keep in mind when making such a substitution.

(a) $Y_t$ reads off the aggregate backwards in time, and hence as $t$ increases, the jump is added at the origin as a particle sticks to the aggregate.

(b) $Y_t(s)$ is infinite for $s > t$, and on this regime Propositions 2.4, 2.7 and 2.10 are not applicable.

Furthermore, in order to give valid approximations on the speed, we introduce several technical assumptions on the aggregate that will be justified in the proof section.

(A1) Let $t_0 > 0$ be a large enough number. In some time interval $[t_0, t_0 + \Delta]$, $S(t)$ stays “close” to some small enough constant $\alpha > 0$. Here, one may understand $\alpha$ as the average of $S(t)$ over $t \in [t_0, t_0 + \Delta]$.

(A2) There exists $0 < t_0 < t_0$ such that $S(t)$ is “sufficiently close” to

$$S'(t) = S'(t; t_0, \alpha) := \mathbb{E}_\alpha[S(Y_t) \mid (Y_t)_{\leq t-t_0}],$$

uniformly on $t \in [t_0, t_0 + \Delta]$, where we denote $(Y_t)_{t \leq t_0} = \{Y_t(s)\}_{s \leq t-t_0}$ as before.

Note that we can avoid the issue raised in (b) if we work with $S'(t)$ defined in (A2). Moreover, (A1) will tell us that the first- and second-order approximations on $S'(t)$ are valid. Both assumptions (A1) and (A2) will be a part of the notion that we call regularity of the speed. This will be discussed in more detail in Section 2.4.

Having the issue (a) in mind, we can express $S'(t)$ as [2.10] and [2.12] by plugging in $Y_t$, or $\Pi_S$, in the place of $Y$. We first define

$$S_1(t) = S_1(t; t_0, \alpha) := \int_{t_0}^t K_\alpha(t-s) d\Pi_S(s),$$

(2.16)

Then, the first-order expansion of the speed for $t \in [t_0, t_0 + \Delta]$ is given as

$$S'(t) = \frac{2\alpha^2}{(1+2\alpha)^2} + \frac{S_1(t)}{(1+2\alpha)^2} + \frac{\alpha}{1+2\alpha} \int_{t_0}^t \int_{t_0}^s d\Pi_S(u) d\Pi_S(s),$$

(2.17)

where $d\Pi_S(s) := d\Pi_S(s) - \alpha ds$. We also call $S_1$ [2.16] the first-order approximation of the speed. One important change we want to emphasize is the definition of $J_{\Pi_S}^{t-s,t-u,t}$ which is slightly different from [2.9]: due to the time-reversed nature of $Y_t$, it is defined as

$$J_{\Pi_S}^{t-s,t-u,t} = J_{\Pi_S}^{t-s,t-u,t}(\alpha) := \mathbb{P}_\alpha(t-u < T_{[s,t]}) - \mathbb{P}_\alpha(t-u < T_s \mid \Pi_S[s,t]).$$

(2.18)

From [2.14], note that conditioning on $\Pi_S(s,t)$ is equivalent to conditioning on $\{Y_t(s')\}_{0 \leq s' \leq t-s}$. 

THE CRITICAL ONE-DIMENSIONAL MDLA 11
Further, the second order expansion is written as

\[ S'_2(t) = \frac{2\alpha^2}{(1 + 2\alpha)^2} + \frac{S_1(t)}{(1 + 2\alpha)^2} + \frac{\alpha}{1 + 2\alpha} \int_{t_0}^t \int_{t_0}^s J^{(\alpha)}_{t-s,t-u} \, d\Pi_S(u) \, d\Pi_S(s) + \frac{\alpha}{1 + 2\alpha} Q_t, \]  

(2.19)

where we define \( J^{(\alpha)}_{t-s,t-u} \) as \([2.13]\) and \( Q_t \) as

\[ Q_t = Q_t(t_0, \alpha) := \int_{t_0}^t \int_{t_0}^s Q_{t-s,t-u,t-v;\alpha} \, d\Pi_S(v) \, d\Pi_S(u) \, d\Pi_S(s), \]  

(2.20)

with \( Q_{t-s,t-u,t-v;\alpha} \) given by

\[ Q_{t-s,t-u,t-v;\alpha} := \mathbb{P}_\alpha(t - v < T_{t-s,t-u} < \infty \mid \Pi_S[s, t]) - \mathbb{P}_\alpha(t - v < T_{t-s} < \infty \mid \Pi_S[s, t]) - \mathbb{P}_\alpha(t - v < T_{t-u} < \infty \mid \Pi_S[s, t]) + \mathbb{P}_\alpha(t - v < \infty \mid \Pi_S[s, t]). \]  

(2.21)

Further, we define the second-order approximation of the speed as

\[ S_2(t) = S_2(t; t_0, \alpha) := \frac{2\alpha^2}{(1 + 2\alpha)^2} + \frac{S_1(t)}{(1 + 2\alpha)^2} + \frac{\alpha}{1 + 2\alpha} \int_{t_0}^t \int_{t_0}^s J^{(\alpha)}_{t-s,t-u} \, d\Pi_S(u) \, d\Pi_S(s). \]  

(2.22)

**Remark 2.11.** We need both the first- and second-order approximations \([2.16], (2.22)\) in order to understand the speed precisely enough for our purpose. Although the first-order approximation often turns out to be enough in many cases, the second-order approximation is essential in the derivation of the scaling limit of the speed. See Sections \([2.3, 2.4.4, 2.4.5]\) for more discussion.

Suppose that we define a process \( R(t) \) by

\[ R(t) = R(t; t_0, t_0, \alpha) := \begin{cases} \alpha & \text{if } t < t_0, \\ \int_{t_0}^t K_\alpha(t - s) \, d\Pi_R(s) & \text{if } t \geq t_0. \end{cases} \]  

(2.23)

We can interpret \( R(t) \) as an age-dependent critical branching process: the average number of offspring that a particle at \( t > t_0 \) produces is \( \int_0^\infty K_\alpha(t' - t) \, dt = 1 \). In this branching process, a particle at \( t > t_0 \) generates \( \text{Pois}(1) \) number of offspring, and each of them will be located at \( t + s \) with probability \( K_\alpha(s) \, ds \) independently.

The critical branching process \( R(t) \) is easier to understand than \( S(t) \), and our study on \( S(t) \) largely depends on comprehending \( R(t) \). A more detailed outline of the plan is discussed in Section \([2.4]\).

### 2.3. Smoothing the speed and the scaling limit

In this subsection, we give the heuristics on how the speed converges to its scaling limit. Recall the critical branching process \( R(t) \) defined in the previous subsection. Since its branching is age-dependent, \( R(t) \) itself is not a martingale in \( t \geq t_0 \) (unlike the discrete time critical Galton-Watson branching process where the number of particles at each generation forms a martingale). To build a martingale based on \( R(t) \), we first define the following renewal process of \( K_\alpha^* \):

\[ K_\alpha^*(s) = \sum_{j \geq 1} K_\alpha^{*j}(s), \]  

(2.24)

where \( K_\alpha^{*j} \) is the convolution of \( K_\alpha \) with itself taken \( j \) times. At the moment, assume that \( K_\alpha^* \) is well-defined, and moreover that the limit

\[ K_\alpha^* = \lim_{s \to \infty} K_\alpha^*(s) \]
exists. These facts are explained in Section 2.5. Then, we consider a “smoothed” version of the speed, which is defined for \( t \geq t_0 \) as

\[
L(t) = L(t; t_0, \alpha) := \int_{t_0}^{t} \int_{t}^{\infty} K_\alpha^* \cdot K_\alpha(x - s) dx d\Pi_S(s). \tag{2.25}
\]

In words, \( L(t) \) is the expected branching rate at infinity from time \( t \) with initial points given by \( \Pi_S(t_0, t) \). For details on this process, see Section 5.

We choose to study the scaling limit of \( L(t) \) rather than \( S(t) \), since it has a more tractable structure described as a martingale added with some perturbation. On the other hand, the behavior of \( S(t) \) has a very spiky nature: \( S(t) \) jumps significantly whenever a new point arrives at \( \Pi_S \). Furthermore, our ultimate interest is to obtain \( X_t \), which is closely related with the integral of \( S(t) \). From this perspective, we would not lose anything from studying \( L(t) \) since it is essentially a smoothed version of \( S(t) \).

In order to obtain the scaling limit of \( L(t) \), we need to understand the mean and variance of the increment \( L(t_0 + \Delta) - L(t_0) \) (conditioned on \( \mathcal{F}_{t_0} \)). Observe that

\[
dL(t) = K_\alpha^* d\Pi_S(t) - \int_{t_0}^{t} K_\alpha^* \cdot K_\alpha(t - s) d\Pi_S(s) dt = K_\alpha^* \{d\Pi_S(t) - S_1(t) dt\}. \tag{2.26}
\]

which gives

\[
L(t_0 + \Delta) - L(t_0) = K_\alpha^* \int_{t_0}^{t_0 + \Delta} (d\Pi_S(t) - S(t) dt) + K_\alpha^* \int_{t_0}^{t_0 + \Delta} (S(t) - S_1(t)) dt. \tag{2.27}
\]

As seen in the formula, we can decompose this increment into a “martingale part” and a “drift part”. Computations of its mean and variance is the most important ingredient in establishing the main theorem. To this end, we need to assume that the processes \( S(t) \) and \( S_1(t) \) behaves nicely enough, otherwise the terms such as \( S(t) - S_1(t) \) might become too large and the increment would not scale as expected. At the moment, we state this assumption on regularity informally as follows.

(A3) In addition to (A1) from Section 2.2, the processes \( S(t) \) and \( S_1(t) \) both stay “close” to \( \alpha \) and to each other in the interval \([t_0, t_0 + \Delta] \).

Formulating (A1), (A2) and (A3) into a rigorous, quantitative condition of regularity and proving it are two of the core endeavours of this article, and the methods are outlined in the next subsection. Under these assumptions, we can state the estimates on the increment informally as follows.

**Theorem 2.12 (Informal).** Suppose that (A1), (A2) and (A3) are satisfied in the interval \([t_0, t_0 + \Delta] \) for a sufficiently small \( \alpha > 0 \). Then, we have

\[
\begin{align*}
\mathbb{E}[L(t_0 + \Delta) - L(t_0) \mid \mathcal{F}_{t_0}] &= o(\alpha^4) \Delta; \\
\text{Var}[L(t_0 + \Delta) - L(t_0) \mid \mathcal{F}_{t_0}] &= (1 + o(1))4\alpha^5 \Delta.
\end{align*}
\]

**Sketch of proof.** Recall the expression (2.27). We can first see that

\[
\mathbb{E}[L(t_0 + \Delta) - L(t_0) \mid \mathcal{F}_{t_0}] = \mathbb{E}
\left[
K_\alpha^* \int_{t_0}^{t_0 + \Delta} (S(t) - S_1(t)) dt \mid \mathcal{F}_{t_0}
\right].
\]

Later, we will see that \( \int_{t_0}^{t_0 + \Delta} |S(t) - S_1(t)| dt = o(\alpha^2 \Delta) \) when \( S(t) \) is regular. Moreover, it turns out that (see Section 2.5)

\[
K_\alpha^* = \left( \int_{0}^{\infty} xK_\alpha(x) dx \right)^{-1} = \frac{2\alpha^2}{1 + 2\alpha}.
\]
Thus we obtain the first equation of (2.28). For the second one, we will show that

\[
\text{Var}[L(t_0 + \Delta) - L(t_0) \mid \mathcal{F}_{t_0}] \approx \text{Var}\left[ K_\alpha^+ \int_{t_0}^{t_0 + \Delta} (d\Pi_S(t) - S(t)dt) \bigg| \mathcal{F}_{t_0}\right]
\]

\[
= (K_\alpha^+)^2 \mathbb{E}\left[ \int_{t_0}^{t_0 + \Delta} S(t)dt \bigg| \mathcal{F}_{t_0}\right] = (1 + o(1))4\alpha^5 \Delta.
\]

Note that the last equality will follow from the assumption (A1).

To obtain the scaling limit of \( L(t) \) from the above, we need to rescale \( L(t) \) by \( \alpha^{-1} \), and then we can see that the time \( \Delta \) should be rescaled by \( \alpha^{-3} \). In fact, roughly speaking, we will later see that

\[
\left\{ M^{1/3} L(t_0 + Ms) \right\}_{s \geq 0} \xrightarrow{M \to \infty} \{ Z_s \}_{s \geq 0},
\]

where \( \{ Z_s \}_{s \geq 0} \) satisfies the following stochastic differential equation:

\[
dZ_s = 2Z_s^{5/2}dB_s. \tag{2.29}
\]

Here, \( t_0 \) can be understood as a large time parameter where we start to see \( S(t) \) being regular with high enough probability. A more detailed description is given in Section 2.4.6.

Note that \( Z_t \) given by (2.29) is \((-1/2)\)-self-similar: \( Z_t^s := s^{1/3} Z_{st} \) has the same law as \( Z_t \) for any \( s > 0 \). This suggests that the decay of \( Z_t \) (and hence, \( S(t) \)) in a larger time scale is asymptotically \( t^{-1/3} \), and consequently the asymptotic growth of \( X_t \) should be of order \( t^{2/3} \).

**Remark 2.13.** Since the only \( \frac{1}{2} \)-self-similar processes are Bessel processes, \( (Z_t)^{-3/2} \) should be a constant multiple of a Bessel process. One can see by applying Itô’s formula that \( Z_t = (3V_t)^{-2/3} \), where \( V_t \) is the \( \frac{8}{3} \)-Bessel process.

2.4. **Proof outline.** Before delving into the actual proof, we devote this subsection to outlining the proof step by step. The following describes the main steps that we go through in order.

- \( O(t^{3/4 + \epsilon}) \) upper bound and reduction to the fixed rate processes.
- Analysis of the fixed rate process and its perturbation.
- The age-dependent critical branching process.
- Regularity of the speed.
- Refined computation of the lower order terms.
- Scaling limit of the speed.

In the following subsections, we explain the purpose of each step and the main ideas to achieve it.

2.4.1. \( O(t^{3/4 + \epsilon}) \)-upper bound and reduction to the fixed rate processes. All the discussions from the previous subsections assumes \( \alpha > 0 \) to be small enough. In order to apply such arguments, we at least need to know that the speed tends arbitrarily close to 0 in finite time.

We obtain this by a simpler but weaker argument that gives an \( O(t^{3/4 + \epsilon}) \)-upper bound on \( X_t \). This is done by counting the number of excess particles, which is the difference between the total number of particles attached to the aggregate up to time \( t \) and the size \( X_t \) of the aggregate.

Then, we relate \( S(t) \) with the fixed rate processes as follows: since \( S(t) \) drops to an arbitrary low scale in finite time, we show that \( S(t) \) can be bounded above and below by arbitrary small constants for a while. Thus, our later analysis focuses on the study of the speed process generated by those fixed constants, which will converge to the same scaling limit.
2.4.2. Analysis of the fixed rate process and its perturbation. In studying (2.17) and (2.19), bounding the sizes of double and triple integrals in the expressions turns out to be essential, because they are the error terms of the first- and second-order approximations (equations (2.16) and (2.22)), respectively. However, these integrals are in terms of \( d\Pi_S \), which is difficult to understand since we do not have a good understanding on \( S(t) \). Furthermore, for instance the term \( J_{t-a,t-s}^{1/2} \) in (2.17) is only measurable with respect to \( \mathcal{F}_t \), meaning that the double integral in (2.17) would not have a martingale structure. Instead, we choose to work with the integrals over the fixed rate Poisson process, namely,

\[
\int_0^t \int_0^s J_{u,s} \, d\Pi_\alpha(u) \, d\Pi_\alpha(s),
\]

upon reversing the time axis, and the corresponding analogue for \( Q_t \). Here, \( \Pi_\alpha \) denotes the rate-\( \alpha \) Poisson process. A precise form of (2.30) will be given in Section 4. These integrals can be understood using Azuma-type concentration inequalities for martingales. In addition, we also need an appropriate estimate on \( J_{u,s} \) and \( Q_{\tau,v,u,s} \). Recalling its definition in (2.9), we couple the events \( \{ s < T_u < \infty \} \) and \( \{ s < T < \infty \} \) and rely on fundamental hitting time estimates of random walk. The same but more complicated coupling argument can be carried out for \( Q_{\tau,v,u,s} \) as well.

To translate the estimates for the fixed rate process to those for \( S(t) \), we analyze the Radon-Nykodym derivative of \( d\Pi_S \) with respect to \( d\Pi_\alpha \). We show that it does not become too big with high probability under an appropriately mild assumption on \( S(t) \) which is included in the definition of regularity. On the other hand, the double integral over \( d\Pi_S(u) \, d\Pi_S(s) \) would stay small with high probability if the Radon-Nykodym derivative is smaller compared to the reciprocal of the probability that (2.30) becomes too large. Details will be discussed in Section 4.

2.4.3. The age-dependent critical branching process. As discussed in Section 2.2, studying the speed \( S(t) \) directly is rather complicated. Thus, we work with the age-dependent critical branching process \( R(t) \) in Section 3 as an intermediate step to understand the speed, which is particularly useful in understanding the evolution of the fixed rate process from Section 2.4.1.

The main property of \( R(t) \) we deduce here is that it stays close to \( \alpha \) by the right distance, roughly of order \( \alpha^2 \). We obtain this by applications of martingale concentration inequalities and relating it with the standard Galton-Watson critical branching processes.

2.4.4. Regularity of the speed. As discussed in the previous subsection, a notion of regularity is introduced to deduce Theorem 2.12 and thus to obtain the scaling limit of the speed. Its definition provides a quantitative description of the assumptions (A1), (A2) and (A3) from Sections 2.2 and 2.3 and is given by various stopping times: Each stopping time will denote the first time when a certain desired property is violated, and \( S(t) \) will be called regular at time \( t \) if the minimum of all those stopping times is larger than \( t \) by a certain amount. For instance, the first times when \( S(t) \), \( |S(t) - S_i(t)| \) become larger than they are supposed to be are included in the definition. In Section 6 we give a thorough overview on the notion of regularity and explain why such conditions need to be understood.

The main purpose of defining the regularity using stopping times is to effectively understand how all the conditions we impose are intertwined with each other. If \( \tau = \min \{ \tau_i : \tau_i < t' \} \), then there should exist a \( j \) such that \( \tau_j = \tau < t' \), meaning that the property described by \( \tau_j \) is violated for the first time among all others. Thus, we can show that \( \tau \geq t' \), that is, the speed is regular with high probability, by proving \( \tau_i = \tau < t' \) happens with very small probability for every \( i \). In Section 7 we explain the details of this argument.

Another major issue is the variation of the frame of reference \( \alpha \), to complete building the inductive framework of the argument. After we prove that the speed is regular from time \( t_0 \) to \( t_1 = t_0 + \Delta \) with respect to \( \alpha \), we want to argue that the speed continues to be regular in \( [t_1, t_1 + \Delta'] \) with high
probability, with respect to a different value \( \alpha' \) to take account of the change of \( L(t) \) during the previous time interval. This means that we generate the processes \( S_1(t) \) and \( S_2(t) \) with \( \alpha' \), and show that they still satisfy the conditions of regularity as long as the change from \( \alpha \) to \( \alpha' \) is small. To this end, we require a refined understanding on analytic properties of \( K_\alpha(s) \), \( K_\alpha'(s) \) and \( J_{\alpha,t}' \), as well as a combination of various tools mentioned in the previous subsections.

The computations to establish the regularity turn out to be long and complicated, and hence we divide their details into three sections, Sections 2.4.1–2.4.5.

2.4.5. Refined computation of the lower order terms. In the computation of the mean of the increment in Theorem 2.12, we need to understand the mean of \( S(t) - S_1(t) \), which can be expressed, by using (2.19), as

\[
S(t) - S_1(t) = \frac{2\alpha^2}{(1+2\alpha)^2} + \frac{4(\alpha + \alpha^2)}{(1+2\alpha)^2} S_1(t) + \frac{\alpha}{1+2\alpha} \int_t^s \int_{t_0}^t J_{\alpha, t-t-u} \, d\Pi_S(u) d\Pi_S(s) + \frac{\alpha Q_2}{1+2\alpha}.
\]

Although we have derived an appropriate bound on the double integral part which is essentially sharp, it is an estimate that holds with high probability that does not give the precise information on the mean

\[
E \left[ \int_t^s \int_{t_0}^t J_{\alpha, t-t-u} \, d\Pi_S(u) d\Pi_S(s) \right]. \tag{2.31}
\]

To precisely calculate (2.31), we switch the integral over \( d\Pi_S(u) d\Pi_S(s) \) with simpler ones that only involves \( K_\alpha'(x) \) and \( \Pi_{\alpha} \), and show that the error arising from the change is of negligible compared to the whole integral. After such modification, we carry out an analytic study to compute the precise expected value.

Combining with the previous analysis on regularity, we also discuss the formal version of Theorem 2.12

2.4.6. Obtaining the scaling limit. Given all the works mentioned in Sections 2.4.1–2.4.5, we are almost ready to deduce the scaling limit of \( L(t) \) based on Theorem 2.12. However, when we consider the process \( M^{1/3} L(t_0 + sM) \), the initial condition at \( s = 0 \) blows up as \( M \) tends to infinity, if \( t_0 \) is a fixed value. The classical results on the convergence to the limiting diffusion (which we mention in the next subsection) would not be applicable in such a case.

Thus, our choice is to make \( t_0 \) be a random time \( \tau(M) \) depending on \( M \), which is roughly

\[
\tau(M) := \inf\{ t > 0 : L(t) \leq CM^{-1/3} \text{ and } L(t) \text{ is regular} \},
\]

where \( C \) is a fixed, large enough constant. Then, we need to show that \( \tau(M) \leq \epsilon M \) with high probability, with a constant \( \epsilon > 0 \) that can be arbitrarily small by setting \( C \) to be large, and this is equivalent to show that \( X_t = O(t^{2/3}) \) with high probability.

This is done by a multi-scale analysis on \( L(t) \) based on Theorem 2.12. To ensure it is regular, we first drop \( L(t) \) to a small enough scale using the results from Section 2.4.1. From then, we show that \( L(t) \) is more likely to halve than to double its size. Further, we argue that either event is going to happen after an appropriate amount of time that makes \( L(t) \) scale like \( t^{-1/3} \). The main technical tools we use to achieve this goal are the martingale concentration inequalities such as the Azuma-Bernstein inequality.
When the above issue on the initial regularity is resolved, we can appeal to the classical result of Helland [11] and Theorem 2.12 to show that
\[
\left\{ M^{1/3}L(\tau(M) + sM) \right\}_{s \geq 0} \to \left\{ A_s \right\}_{s \geq 0}, \tag{2.32}
\]
where \( \{A_s\}_{s \geq 0} \) is the solution of
\[
dA_s = 2A_s^{5/2}dB_s, \quad A_0 = C.
\]
The convergence (2.32) is in Stone topology, which is good enough to ensure the convergence of the integrated process
\[
\left\{ M^{-2/3} \int_0^{Mt} L(\tau(M) + s)ds \right\}_{t \geq 0} \to \left\{ \int_0^t A_s ds \right\}.
\]
Finally, the main result is obtained by increasing the level of \( C \) to infinity, and then observing that \( \int_0^{Mt} L(s)ds \) is close enough to \( X_{Mt} \) based on the conditions of regularity.

2.5. Estimates on fundamental quantities. As mentioned before, understanding the analytic properties of the branching density \( K_\alpha(s) \) and the renewal process \( K_\alpha^*(s) \) will be crucial throughout the paper. In this subsection, we review the main estimates we need, while their proofs are deferred to Appendix A. They can be obtained by studying the moment generating function of \( T \) via Fourier analysis.

To begin with, the branching density \( K_\alpha(s) \) (Definition 2.6) satisfies the following property.

**Lemma 2.14.** There exist absolute constants \( c, C > 0 \) such that for any \( t > 0 \) and \( 0 < \alpha < \frac{1}{2} \),
\[
K(t) \leq \frac{C\alpha}{\sqrt{t+1}} e^{-c\alpha^2 t}.
\]
Furthermore, its derivative in \( t \) satisfies
\[
|K'_\alpha(t)| \leq C\alpha(t+1)^{-\frac{3}{2}} e^{-c\alpha^2 t}.
\]

Moreover, the properties of \( K_\alpha^*(s) \) (2.24) are summarized by the following lemma.

**Lemma 2.15.** Let \( K_\alpha^* := \frac{2\alpha^2}{1+2\alpha} \). There exist absolute constants \( c, C > 0 \) such that for any \( t > 0 \) and \( 0 < \alpha < \frac{1}{2} \),
\[
|K_\alpha^*(t) - K_\alpha^*| \leq \frac{C\alpha}{\sqrt{t+1}} e^{-c\alpha^2 t}.
\]
Furthermore, its derivative in \( t \) satisfies
\[
|K'_\alpha(t)| \leq C\alpha(t+1)^{-\frac{3}{2}} e^{-c\alpha^2 t}.
\]

Finally, we state a similar bound on \( J_{s,t}^{(\alpha)} \) (2.13).

**Lemma 2.16.** There exist absolute constants \( c, C > 0 \) such that for any \( t > s > 0 \) and \( 0 < \alpha < \frac{1}{2} \),
\[
\left| J_{s,t}^{(\alpha)} \right| \leq \frac{Ce^{-c\alpha^2 t}}{\sqrt{(s+1)(t+1)}}.
\]

3. An a priori upper bound and the induction base

The first step of our proof is establishing that the speed drops down to arbitrary small level in finite time. In Section 3.1 we achieve this by showing an \( t^{3/2} \) polylog\((t)\)-bound on \( X_t \) which follows from counting the excessive particles in the aggregate. Based on this result, we prove in Sections 3.2, 3.3 that when \( t \) gets sufficiently large (but finite), the speed stays between arbitrary small constants for a long enough time. Such a sandwiching argument enables us to analyze the speed process that evolves from particles given by a fixed rate process in the later sections.
3.1. Simple upper bound of $t^{\frac{3}{2}}$.

**Lemma 3.1.** For any $t > 0$ we have that

$$\mathbb{P}(X_t \geq t^{\frac{3}{2}} \log^2 t) \leq Ct^{-10}$$

**Proof.** Denote by $N_t$ the total number of particles that were absorbed into the aggregate by time $t$. We start by bounding $N_t$. To this end, note that the event $A := \{X_t \leq t\}$ satisfies $\mathbb{P}(A) \geq 1 - e^{-ct}$. Indeed, we have with probability one that $S(x) \leq 1/2$ for all $x > 0$ and therefore $X_t \leq \text{Poisson}(t/2)$.

Next, for any $l \geq 0$, define the random variable $Y_l$ to be the number of particles that at some point, before time $t$, were inside the interval $[0, l]$. Define the event $B_l := \{N_t \leq l + \sqrt{l} \log^2 t\}$. It is clear that on the event

$$\mathcal{B} := A \cap \bigcap_{l=1}^{|t|} B_l$$

we have that

$$N_t \leq X_t + \sqrt{l} \log^2 t. \quad (3.1)$$

Indeed, a particle that was absorbed into the aggregate before time $t$ was, at some point, inside the interval $[0, X_t]$.

We turn to show that $\mathcal{B}$ happens with high probability. For any $l \leq t$, let $\mathcal{D}_l$ be the event that any particle which was initially outside the interval $[0, l + \frac{1}{2} \sqrt{l} \log^2 t]$ did not reach the interval $[0, l]$ and let $\mathcal{E}_l$ be the event that the number of particles initially in the interval $[0, l + \frac{1}{2} \sqrt{l} \log^2 t]$ is at most $l + \sqrt{l} \log^2 t$. It is clear that $\mathcal{D}_l \cap \mathcal{E}_l \subseteq \mathcal{B}_l$ and that $\mathbb{P}(\mathcal{D}_l) \geq 1 - Ce^{-c \log^2 t}$. Since the number of particles that were initially in $[0, l + \frac{1}{2} \sqrt{l} \log^2 t]$ is distributed by Poisson($l + \frac{1}{2} \sqrt{l} \log^2 t$) we have for any $l \leq t$ that $\mathbb{P}(\mathcal{E}_l) \geq 1 - Ce^{-c \log^2 t}$. Thus we have that $\mathbb{P}(\mathcal{B}_l) \geq 1 - Ce^{-c \log^2 t}$ and therefore, from a union bound we get $\mathbb{P}(\mathcal{B}) \geq 1 - Ce^{-c \log^2 t}$.

Next, let $Z_t$ be the number of particles at $X_t + 1$ at time $t$, and recall that the conditional distribution of $Z_t$ given $\{X_s, s \leq t\}$ is Poisson($2S(t)$).

The aggregate size, $X_t$, increases by 1 with rate $S(t)$ and therefore $X_t - \int_0^t S(x) \, ds$ is a martingale. Similarly, for each $j \geq 1$, $N_t$ increases by $j$ with rate $\frac{1}{2} j \cdot \mathbb{P}(Z_t = j \mid X_s, s \leq t)$ and therefore

$$N_t - \frac{1}{2} \int_0^t \sum_{j=1}^{\infty} j^2 \mathbb{P}(Z_x = j \mid X_s, s \leq x) \, dx = N_t - \frac{1}{2} \int_0^t \mathbb{E}[Z_x^2 \mid X_s, s \leq x] \, dx = N_t - \int_0^t S(x) + 2S(x)^2 \, dx$$

is a martingale. By Corollary 4.6 with $M = t$ and $f = 1$ (using the bounds $S, S + 2S^2 \leq 1$) we have probability at least $1 - Ce^{-c \log^2 t}$ that

$$\left| X_t - \int_0^t S(x) \, dx \right| \leq \sqrt{t} \log^2 t, \quad \left| N_t - \int_0^t S(x) + 2S(x)^2 \, dx \right| \leq \sqrt{t} \log^2 t. \quad (3.2)$$

Denote by $\mathcal{C}$ the event where these inequalities hold. On $\mathcal{B} \cap \mathcal{C}$ we have that

$$2 \int_0^t S(x)^2 \, dx \leq -X_t + \sqrt{t} \log^2 t + \int_0^t S(x) + 2S(x)^2 \, dx \leq N_t - X_t + 2\sqrt{t} \log^2 t \leq 3\sqrt{t} \log^2 t,$$

where in the first two inequalities we used (3.2) and in the last inequality we used (3.1). Thus, on $\mathcal{B} \cap \mathcal{C}$, by Cauchy Schwartz inequality we have

$$X_t - \sqrt{t} \log^2 t \leq \int_0^t S(x) \, dx \leq \sqrt{t} \cdot \left( \int_0^t S(x)^2 \, dx \right)^{\frac{1}{2}} \leq 2t^{\frac{3}{2}} \log t.$$

This finishes the proof of the lemma. \qed
The next corollary follows immediately from Lemma 3.1.

**Corollary 3.2.** We have that

1. \( \mathbb{P}(X_t = O(t^{3/2} \log^2 t)) = 1. \)
2. For all \( t \geq 2, \mathbb{E}X_t \leq Ct^{3/2} \log^2 t. \)

*Proof.* The first part follows immediately from Lemma 3.1 and the Borel-Cantelli lemma. We turn to prove the second statement. By Lemma 3.1 and the fact that \( X_t \) is Poisson\((t/2)\) we have for sufficiently large \( t \),

\[
\mathbb{E}X_t \leq t^{3/2} \log^2 t + t \cdot \mathbb{P}(t^{3/2} \log^2 t \leq X_t) \leq t^{3/2} \log^2 t + \sum_{k=1}^{\infty} (k+1)t \cdot \mathbb{P}(X_t \geq kt) \leq Ct^{3/2} \log^2 t,
\]
as needed. \( \square \)

### 3.2. Stochastic domination and bound on rate

The process \( Y_t \) is itself a function valued Markov chain and it will be useful at time to initialize it from different starting states. We define the generalized aggregate with initial condition \( Y_0 \) for some \( t_0 \geq 0 \).

**Definition 3.3.** Let \( Y_0 : \mathbb{R}_+ \to \mathbb{N} \cup \{\infty\} \) be a monotone random step function and let \( \Pi \) be an independent 2-D Poisson process. The rate of the aggregate with initial condition \((Y_0, t_0)\) is defined by

\[
S(t) := \frac{1}{2} \mathbb{P}(W(s) \leq Y_t(s) \mid Y_t), \quad t \geq t_0,
\]

where \( Y_t \) is defined by

\[
Y_t(s) = \begin{cases} 
X_t - X_{t-s} & s \leq t - t_0 \\
Y_t(t-t_0) + Y_0(s-(t-t_0)) & s > t - t_0
\end{cases}
\]

As before, the size of the generalised aggregate will be \( X_t := \Pi_s[t_0, t] \). We say that \((Y_t, S(t), X_t)\) is the aggregate with initial condition \((Y_0, t_0)\) driven by the Poisson process \( \Pi \).

The following claim follows immediately from the above definition.

**Claim 3.4.** Let \( Y_0^{(1)}, Y_0^{(2)} : \mathbb{R}_+ \to \mathbb{N} \cup \{\infty\} \) be monotone random step functions and let \( \Pi \) be an independent Poisson process. Let \((Y_t^{(1)}, S^{(1)}(t), X_t^{(1)})\) and \((Y_t^{(2)}, S^{(2)}(t), X_t^{(2)})\) be the aggregates with initial conditions \((Y_0^{(1)}, 0)\) and \((Y_0^{(2)}, 0)\) respectively, both driven by \( \Pi \). We also let \((Y_t, S(t), X_t)\) be the usual aggregate.

1. Suppose that \( Y_0^{(1)} = \infty \) with probability one. Then
   \[
   \{ S^{(1)}(t) \}_{t \geq 0} \overset{d}{=} \{ S(t) \}_{t \geq 0}, \quad \{ X_t^{(1)} \}_{t \geq 0} \overset{d}{=} \{ X_t \}_{t \geq 0}.
   \]
2. Let \( t_0 > 0 \) and suppose that \( Y_0^{(1)} \overset{d}{=} Y_{t_0} \). Then
   \[
   \{ S^{(1)}(t) \}_{t \geq 0} \overset{d}{=} \{ S(t + t_0) \}_{t \geq 0}, \quad \{ X_t^{(1)} \}_{t \geq 0} \overset{d}{=} \{ X_{t+t_0} \}_{t \geq 0}.
   \]
3. On the event \( \{ Y_0^{(1)}(s) \leq Y_0^{(2)}(s) \text{ for all } s \} \) we have for all \( t > 0 \)
   \[
   S^{(1)}(t) \leq S^{(2)}(t), \quad X_t^{(1)} \leq X_t^{(2)}, \quad \forall s > 0, \ Y_t^{(1)}(s) \leq Y_t^{(2)}(s).
   \]

**Corollary 3.5.** We have that \( S(t) \) is stochastically decreasing in the sense that, if \( t_1 \leq t_2 \) then there is a coupling of the processes \( \{ S(t + t_1) \}_{t \geq 0} \) and \( \{ S(t + t_2) \}_{t \geq 0} \) such that the first process is larger than the second.
Proof. Let \( Y_0 \) and \( Y_1 \) be independent Poisson processes. By parts (1) and (2) of Claim 3.4 we have that
\[
\{ S^{(1)}(t) \}_{t \geq 0} \overset{d}{=} \{ S(t) \}_{t \geq 0} \quad \text{and} \quad \{ S^{(2)}(t) \}_{t \geq 0} \overset{d}{=} \{ S( t + t_1 - t ) \}_{t \geq 0}.
\]
Since \( Y_0(s) \leq Y_1(s) \) for all \( s \) almost surely, we have by part (2) of Claim 3.4 that \( S^{(1)}(t) \leq S^{(2)}(t) \) for all \( t > 0 \). Thus, the processes \( \{ S^{(1)}(t + t_1) \}_{t \geq 0} \) and \( \{ S^{(2)}(t + t_1) \}_{t \geq 0} \) give the desired coupling between \( \{ S(t + t_1) \}_{t \geq 0} \) and \( \{ S( t + t_2 ) \}_{t \geq 0} \). \( \square \)

In the following lemma we show that the rate of the aggregate gets arbitrarily close to 0 for long periods of time.

Lemma 3.6. For any \( \alpha > 0 \) sufficiently small the following holds. For all \( t \geq \alpha^{-20} \) we have that
\[
\mathbb{P} \left( \sup_{t - \alpha^{-3} \leq s \leq t} S(s) \leq \alpha \right) \geq 1 - \alpha.
\]

Proof. Let \( t \geq \alpha^{-20} \) and let \( I_k := [t - k \alpha^{-3}, t - (k - 1) \alpha^{-3}] \) for all \( 1 \leq k \leq \lfloor \alpha^3 t \rfloor \).

Define the random sets
\[
A_1 := \left\{ 1 \leq k \leq \lfloor \alpha^3 t \rfloor - 1 : \text{the aggregate grew in the time interval } I_{k+1} \cup I_k \right\},
\]
and
\[
A_2 := \left\{ 1 \leq k \leq \lfloor \alpha^3 t \rfloor - 1 : \text{the aggregate grew in the time interval } I_{k+1} \cup I_k \right\},
\]
and
\[
B := \left\{ 1 \leq k \leq \lfloor \alpha^3 t \rfloor - 1 : \sup_{s \in I_k} S(s) > \alpha \right\}.
\]

It is clear that \( |A_1| \leq X_t \) and \( |A_2| \leq X_t \). Next, we claim that \( B \subseteq A_1 \cup A_2 \). Indeed suppose that \( k \notin A_1 \cup A_2 \) and let \( s \in I_k \). By the definition of \( A_1, A_2 \), the aggregate did not grow in the time interval \( [s - \alpha^{-3}, s] \) and therefore
\[
S(s) = \frac{1}{2} \mathbb{P} \left( \forall x > 0, W(x) \leq Y_s(x) \mid Y_s \right) \leq \mathbb{P} \left( \forall x \leq \alpha^{-3}, W(x) \leq 0 \right) \leq C \alpha^3 \leq \alpha,
\]
where the last two inequalities hold for sufficiently small \( \alpha \). Since the last bound holds for all \( s \in I_k \) we get that \( k \notin B \).

Finally, by Corollary 3.5 we have that \( \mathbb{P}(k \in B) \geq \mathbb{P}(1 \in B) \) and therefore using Lemma 3.1 we obtain
\[
(|\alpha^3 t| - 1) \mathbb{P}(1 \in B) \leq \sum_{k=1}^{\lfloor \alpha^3 t \rfloor - 1} \mathbb{P}(k \in B) = \mathbb{E}|B| \leq \mathbb{E}|A_1| + \mathbb{E}|A_2| \leq 2\mathbb{E}X_t \leq C t^2 \log^5 t.
\]

Thus, using that \( t \geq \alpha^{-20} \) and that \( \alpha \) is sufficiently small we get
\[
\mathbb{P} \left( \sup_{t - \alpha^{-3} \leq s \leq t} S(s) \geq \alpha \right) = \mathbb{P}(1 \in B) \leq \alpha
\]
as needed. \( \square \)

3.3. Bounding with fixed rate processes. In this section we show that the aggregate can be bounded from above and below by regular processes. Recall that \( \Pi \) is the \( 2 - D \) Poisson process that is driving the aggregate.

For \( \alpha, t > 0 \) and define the random functions
\[
Y_{t, \alpha}(s) := \Pi_{\alpha}[t - s, t], \quad \overline{Y}_{t, \alpha}(s) := \begin{cases} \Pi_{\alpha}[t - s, t], & s \leq \alpha^{-3} \\ \infty, & s > \alpha^{-3} \end{cases}
\]  

(3.3)
Moreover, for convenience, we denote instance, \( x \) analogue for \( Q \) process on the plane with the standard Lebesgue intensity, and let \( \Pi \) order approximations of the speed (2.16), (2.22). Throughout this section, \( g \) we define random function which represents the rate of the Poisson process at time \( s \) both when \( g \) and abbreviate the products among them by (2.20). In this section, our objective is to understand the multiple integral aggregate by Lemma 3.7 then conclude that the same results hold for the usual aggregate as these processes sandwich the some regularity properties which will allow us to apply the inductive argument on them. We then conclude that the same results hold for the usual aggregate as these processes sandwich the aggregate by Lemma 3.7.

4. Fixed rate process and its perturbation

Recall the first- and second-order expansions of speed (2.17), (2.19) and the definitions (2.18), (2.20). In this section, our objective is to understand the multiple integral

\[
\int_0^t \int_0^s J_{t-s,t-u:t}^H d\Pi_g(u)d\Pi_g(s),
\]

both when \( g \) is a fixed rate \( g \equiv \alpha \) and when it is slightly perturbed, and also the corresponding analogue for \( Q_t \). Consequently, we further derive estimates on the error of the first- and second-order approximations of the speed (2.16), (2.22). Throughout this section, \( g(s) > 0 \) denotes a random function which represents the rate of the Poisson process at time \( s \).

To begin with, we introduce some notations as follows. As before, \( \Pi \) denotes the Poisson point process on the plane with the standard Lebesgue intensity, and let \( \Pi_g \) be the collection of the \( x \)-coordinate locations of points lying below \( g \), that is,

\[
\Pi_g[0,t] = \{ x \in [0,t] : (x,y) \in \Pi \text{ for some } y \in [0,g(x)] \},
\]

we define \( \pi_i(t;g) \) to be the distance in \( x \)-axis from \( t \) to the \( i \)-th closest point to \( t \) in \( \Pi_g[0,t] \). For instance,

\[
\pi_1(t;g) = \pi_1(t;\Pi_g) = t - \max \{ x : x \in \Pi_g[0,t] \}.
\]

Moreover, for convenience, we denote

\[
\sigma_i(t;g) := (\pi_i(t;g) + 1)^{-1/2},
\]

and abbreviate the products among them by

\[
\sigma_1\sigma_2(t;g) := \sigma_1(t;g)\sigma_2(t;g), \quad \text{and} \quad \sigma_1\sigma_2\sigma_3(t;g) := \sigma_1(t;g)\sigma_2(t;g)\sigma_3(t;g).
\]

Further, \( \alpha, C > 0 \) denote small enough and large enough constants, respectively, and we set

\[
\hat{h} = \hat{h}(\alpha, C) := \alpha^{-2}\log^C(1/\alpha).
\]

Then, the goal of this section is to establish the following statements.
Proposition 4.1. Let \( \epsilon > 0 \) be arbitrary, \( \alpha > 0 \) be a sufficiently small constant depending on \( \epsilon, C \), and recall the definition of \( J \) [2.18]. Let \( \tau \) be a stopping time, and \( \{g(s)\}_{s \geq 0} \) be a positive stochastic process progressively measurable with respect to \( \Pi_g \), and suppose that it satisfies the following three conditions almost surely:

\[
\int_0^{\hat{\lambda}_T} (g(s) - \alpha)^2 \, ds \leq \alpha^{-1} \frac{\sigma_1}{\sigma_2}, \\
\int_0^{\hat{\lambda}_T} (g(s) - \alpha)^2 g(s) \, ds \leq \alpha^{-2} \frac{\sigma_1}{\sigma_2}, \\
\sup_{s \leq \hat{\lambda}_T} g(s) \leq \alpha^{-1} \frac{\sigma_1}{\sigma_2}.
\]

(4.4)

Furthermore, denoting \( d\Pi_g(s) = d\Pi_g(s) - \alpha ds \) as before, we define for \( t' \leq t \) that

\[
\mathcal{J}[t; \Pi_g] := \int_0^t \int_{s,t,t-u:s} d\Pi_g(u) d\Pi_g(s).
\]

(4.5)

Then, there exist \( c_\epsilon, \alpha_0(\epsilon, C) > 0 \) such that for any \( \alpha \in (0, \alpha_0) \), we have

\[
\mathbb{P}\left(-\alpha^{\frac{1}{2} - \epsilon} \sigma_1(t; g) \leq \mathcal{J}[t; \Pi_g] \leq \alpha^{-\epsilon} \sigma_1 \sigma_2(t; g), \forall t \leq \hat{h}(\alpha, C) \wedge \tau\right) \geq 1 - \exp(-\alpha^{-c_\epsilon}).
\]

Proposition 4.2. Recall the definition of \( Q \) [2.21], and define \( Q[t; \Pi_g] \) as

\[
Q[t; \Pi_g] := \int_0^t \int_0^s Q^{\Pi_g}_{t-s,t-u:s} d\Pi_g(u) d\Pi_g(s).
\]

(4.6)

Then, under the same setting as Proposition 4.1, we have

\[
\mathbb{P}\left(|Q[t; \Pi_g]| \leq \alpha^{-\epsilon} \sigma_1 \sigma_2 \sigma_3(t; g), \forall t \leq \hat{h}(\alpha, C) \wedge \tau\right) \geq 1 - \exp(-\alpha^{-c_\epsilon}).
\]

Based on these results, we discuss the error estimates for the first- and second-order approximations. Formal statements and details will be given in Section 4.5.

Remark 4.3. The assumption (4.4) will be revisited in Section 6 when we discuss regularity. In fact, it will be shown that the speed \( S(t) \) of the aggregate actually satisfies (4.4) with high probability.

Remark 4.4. In Proposition 4.1, observe that the lower bound of \( \mathcal{J}[t; \Pi_g] \) is stronger than the upper bound. This comes from the nature of \( J \) who can be (positively) larger compared to the (negative) lower bound, which will be clear in Section 4.2. Having a stronger lower bound on \( \mathcal{J} \) will play an important role in the discussion of regularity, in Sections 6.3 and 7.

As mentioned in Section 2.4.2, the proofs consist of four major steps which are discussed one by one in detail in the following subsections:

- The Azuma-type martingale concentration lemmas: Section 4.1
- Controlling the sizes of \( J_{u,s} \) and \( Q_{v,u,s} \): Section 4.2
- Proving the propositions with respect to the fixed rate process \( \Pi_\alpha \): Section 4.3
- Converting the results into a general form: Section 4.4
- The error estimate on the first- and second-order approximations of the speed: Section 4.5

4.1. The martingale concentration lemmas. We begin with developing lemmas on martingale concentration used in the proof of Propositions 4.1 and 4.2. Moreover, the lemmas we announce here will appear again frequently in Sections 5 and 8.

For the Poisson point process \( \Pi \) with standard Lebesgue intensity, let \( \mathcal{F}_t \) denote the \( \sigma \)-algebra generated by \( \Pi \) up to time \( t \) (including \( t \)). For any positive stochastic process \( g(t) \) which is predictable with respect to \( \mathcal{F}_t \), we write \( \Pi_g \) to denote the points in \( \Pi[0,t] \) below the function \( g \) as before. In particular, for a constant \( \alpha > 0 \), \( \Pi_\alpha \) refers to the rate-\( \alpha \) Poisson process. We note that for any such function \( g \) the process \( \tilde{\Pi}_g(t) := |\Pi_g[0,t]| - \int_0^t g(s) ds \) is a martingale.

We begin with the simplest form of concentration lemma for martingales. The statement includes a stopping time which is introduced for a later purpose.
Lemma 4.5. Let \( a, M, \lambda > 0 \), let \( g(t) > 0 \) and \( f(t) \) be stochastic processes predictable with respect to \( \mathcal{F}_t \), and let \( \tau \) be a stopping time. Suppose that
\[
\lambda |f(x \wedge \tau)| \leq 1, \quad \text{and} \quad \int_0^{t \wedge \tau} f(x)^2 g(x) dx \leq M, \quad \text{a.s.} \quad (4.7)
\]
Then
\[
\mathbb{P} \left( \sup_{s \leq t \wedge \tau} \int_0^s f(x) d\bar{\Pi}_g(x) \geq a\sqrt{M} \right) \leq Ce^{\lambda^2 M - a\lambda \sqrt{M}}.
\]

Proof. For \( 0 \leq s \leq t \) let
\[
M_s := \exp \left( \lambda \int_0^s f(x) d\Pi_g(x) - \int_0^s \left( e^{\lambda f(x)} - 1 \right) g(x) dx \right)
\] \[ L_s := \exp \left( \int_0^s \left( e^{\lambda f(x)} - 1 \right) g(x) - \lambda f(x) g(x) dx \right).
\]
Note that \( M_s \) is a martingale and that, using the inequality \( e^x \leq 1 + x + x^2 \) for \( |x| \leq 1 \) we get
\[
L_{s \wedge \tau} \leq \exp \left( \lambda^2 \int_0^{s \wedge \tau} f(x)^2 g(x) dx \right) \leq \exp \left( \lambda^2 \int_0^{t \wedge \tau} f(x)^2 g(x) dx \right) \leq e^{\lambda^2 M}.
\]
Define the stopping time
\[
\tau_1 := \inf \left\{ s > 0 : \int_0^s f(x) d\bar{\Pi}_g(x) \geq a\sqrt{M} \right\}, \quad \tau_2 := \inf \left\{ s > 0 : \int_0^s f(x) d\bar{\Pi}_g(x) \leq -a\sqrt{M} \right\}
\]
and let \( \tau' = \tau \wedge \tau_1 \). We have
\[
e^{\lambda a \sqrt{M}} \mathbb{P} (\tau_1 \leq \tau) \leq \mathbb{E} \left[ \lambda \exp \left( \int_0^{t \wedge \tau'} f(x) d\bar{\Pi}_g(x) \right) \right] = \mathbb{E} [M_{t \wedge \tau'} \cdot L_{t \wedge \tau'}] \leq e^{\lambda^2 M} \mathbb{E} [M_{t \wedge \tau'}] = e^{\lambda^2 M},
\]
which gives the desired bound for \( \tau_1 \). In order to get the same bound for \( \tau_2 \) we switch \( f \) by \(-f\). \( \square \)

In particular, taking \( \lambda = 1/\sqrt{M} \) in Lemma 4.5 gives the following corollary.

Corollary 4.6. Let \( a, M > 0 \), let \( g(t) > 0 \) and \( f(t) \) be predictable processes with respect to \( \mathcal{F}_t \) and let \( \tau \) be a stopping time. Suppose that
\[
|f(x \wedge \tau)| \leq \sqrt{M} \quad \text{and} \quad \int_0^{t \wedge \tau} f(x)^2 g(x) dx \leq M, \quad \text{a.s.}
\]
Then
\[
\mathbb{P} \left( \sup_{s \leq t \wedge \tau} \int_0^s f(x) d\bar{\Pi}_g(x) \geq a\sqrt{M} \right) \leq Ce^{-a}.
\]

The following corollary deals with the case \( f \equiv 1 \), which is useful when estimating the size of the Poisson process itself.

Corollary 4.7. Let \( a, h, \Delta > 0 \), \( M \geq 1 \), let \( g(t) > 0 \) a process with respect to \( \mathcal{F}_t \) and let \( \tau \) be a stopping time. Suppose that
\[
\int_{(t-\Delta) \wedge \tau}^{t \wedge \tau} g(x) dx \leq M, \quad \text{a.s.,}
\]
and define the stopping time
\[
\tau' = \inf \{ t > 0 : |\Pi_g(t - \Delta) \vee 0, t] \geq 2M + 2a\sqrt{M} + 2 \} \land h.
\]
Then, we have \( \mathbb{P}(\tau' \leq \tau) \leq \left( \frac{h}{A} \right) e^{-a}. \)

**Proof.** For each \( t \in [0, h] \), we apply Corollary 4.6 to the quantity
\[
P(t) := |\Pi_g(t \wedge \tau - \Delta, t \wedge \tau)|,
\]
and obtain that
\[
\mathbb{P}(P(t) \geq M + a\sqrt{M}) \leq e^{-a}.
\]
Then, we take a union bound over all \( t \in [0, h] \) of the form \( t = k\Delta, k \in \mathbb{Z} \):
\[
\mathbb{P}(P(\Delta) \leq M + a\sqrt{M}, \forall k\Delta \in [0, h], k \in \mathbb{Z}) \geq 1 - \left( \frac{h}{\Delta} \right) e^{-a}.
\]
Under the event described inside the above probability, the intervals \([t \wedge \tau - \Delta, t \wedge \tau]\) should contain at most \( 2M + 2a\sqrt{M} \) points for all \( t \in [0, h] \), implying that \( \tau' > \tau \). \( \square \)

Unfortunately, Lemma 4.5 often turns out to be insufficient due to several reasons, and we require more involved versions of it. To motivate the formulation of the more complicated lemmas below, we briefly explain the setting. Suppose that \( f_t(x), g(x) \) are predictable processes with respect to \( \mathcal{F}_x \), where \( f_t \) is defined for each \( t \in [0, h] \). We want to control the size of
\[
\int_{\tau_-}^{t \wedge \tau} f_t(x) d\Pi_g(x),
\]
where \( \tau_-, \tau \geq 0 \) are two given stopping times. Our desired estimate should take account of the following traits, which causes additional complication compared to the previous lemma.

- We want our bound to hold for all \( t \in [0, h] \), which would require a union bound followed by a continuity argument.
- We need to deal with the cases where \( |f_t(x)| \) is not bounded uniformly in \( x \) as in the first condition of (4.7).
- The integral starts from \( \tau_- \), which potentially causes the second bound of (4.7) to be \( \tau_- \)-measurable (which is random).

Having these aspects in mind, the device we need is stated as follows. Due to its complicatedness, we suggest the reader to skip this lemma for the moment and come back later when it is actually applied (e.g., in Section 4.3).

**Lemma 4.8.** Let \( h, \Delta, D > 0 \) be given constants, and let \( g(x) > 0 \) and \( f_t(x) \) be stochastic processes predictable with respect to \( \mathcal{F}_x \), where \( f_t : [0, t] \to \mathbb{R} \) is defined for each \( t \in [0, h] \). Suppose that there exist stopping times \( \tau_-, \tau \), and random variables \( M, A > 0 \) that satisfy the following conditions:

- \( M, A \) are \( \tau_- \)-measurable.
- \( f_t \) and \( g \) satisfy
\[
|f_t(x)| \leq \sqrt{M} \forall x \geq \Delta, \int_0^\Delta |f_t(x)| g(x) dx \leq A, \int_0^{t \wedge \tau} f_t(x)^2 g(x) dx \leq M, \text{ a.s. } \forall t \in [0, h]. \tag{4.8}
\]

- With probability one, we have \( |\partial_x f_t(x)| \vee |\partial_t f_t(x)| \leq D \) for all \( t \in [0, h], x \leq t \).
- We define \( f_t(x) \geq 0 \) and \( f_t(x) \leq 0 \) to be random functions that are decreasing and increasing, respectively, and satisfy the following conditions:
  - For all \( x \leq t \), \( f_t(x) \) and \( f_t(x) \) are \( \tau_- \)-measurable.
  - For all \( x \leq t \leq \tau \), \( f_t(x) \geq \sup_{y \geq x} f_t(y) \) and \( f_t(x) \leq \inf_{y \geq x} f_t(y) \) a.s.
Furthermore, for given constants $N, \eta > 0$, define stopping times
\[ \tau' := \inf\{ t > \Delta : |\Pi_g[t - \Delta, t]| \geq N \}, \]
\[ \tau'' := \inf\{ t > 0 : |g(t)| \geq \eta \}, \]
\[ \tau_0 := \tau \land \tau' \land \tau'', \]
and let $\delta > 0$ be a constant satisfying a.s. that
\[ \delta \leq \frac{A}{D(2N\Delta + TN + T\eta \Delta)}. \hspace{1cm} (4.10) \]

Then, denoting the closest point to 0 in $\Pi_g[0, t]$ by $p_1$, we have for all $a > 0$ that
\[ \mathbb{P}\left( \sup_{0 \leq s \leq t \land T} \int_{\tau_0}^s f_t(x) d\Pi_g(x) \leq 2Nf_t(p_1) + 3A + a\sqrt{M}, \ \forall t \in [0, h] \right) \geq 1 - \left( \frac{Ch}{\delta \Delta} \right) e^{-a}; \]
\[ \mathbb{P}\left( \inf_{0 \leq s \leq t \land T} \int_{\tau_0}^s f_t(x) d\Pi_g(x) \geq 2Nf_t(p_1) - 3A - a\sqrt{M}, \ \forall t \in [0, h] \right) \geq 1 - \left( \frac{Ch}{\delta \Delta} \right) e^{-a}. \]

with an absolute constant $C > 0$. In the integral, we regard $\int_a^b f = 0$ if $a \geq b$.

Its proof is based on a union bound applied to Lemma 4.5 to cover a discretized points in the interval $[0, h]$ and a continuity argument to cover the whole interval. Distinction of upper and lower bounds on the quantity is necessary to deduce Proposition 4.1. Due to its technicality, the proof is deferred to Appendix B.

The following corollary will not be used in Section 4, but they will be applied frequently later in Sections 5–8. It is an analogue of Lemma 4.8, but with deterministic $f_t(x)$ which is increasing in $x$, and also simpler due to the absence of $\tau$, allowing $M, A$ above to be deterministic.

**Corollary 4.9.** Let $h, M, \Delta, D, A > 0$ be given, let $g(x) > 0$ be a predictable process with respect to $\mathcal{F}_x$ and let $\tau$ be a stopping time. For each $t \in [0, h]$, let $f_t : (-\infty, t] \rightarrow \mathbb{R}_{\geq 0}$ be a deterministic, increasing function that satisfies
\[ |f_t(x)| \leq \sqrt{M} \quad \forall x \leq t - \Delta, \quad \int_{t-\Delta}^t |f_t(x)| g(x) dx \leq A, \quad |\partial_x f_t(x)| \vee |\partial_t f_t(x)| \leq D \quad \forall 0 \leq t \leq h, \ x < t. \]

Suppose that for each $t \in [0, h],$
\[ \int_0^{t \land T} f_t(x)^2 g(x) dx \leq M, \quad a.s. \]

Furthermore, for given constants $N, \eta > 0$, let the stopping times $\tau', \tau'', \tau_0$ and the constant $\delta > 0$ be as (4.9), (4.10). Then, denoting the closest point to $t$ in $\Pi_g[0, t]$ by $p_1(t)$, we have
\[ \mathbb{P}\left( \int_0^{t \land T_0} f_t(x) d\Pi_g(x) \leq 2Nf_t(p_1(t)) + 3A + a\sqrt{M}, \ \forall t \in [0, h] \right) \geq 1 - \left( \frac{Ch}{\delta \Delta} \right) e^{-a}; \]
\[ \mathbb{P}\left( \int_0^{t \land T_0} f_t(x) d\Pi_g(x) \geq -3A - a\sqrt{M}, \ \forall t \in [0, h] \right) \geq 1 - \left( \frac{Ch}{\delta \Delta} \right) e^{-a}. \]

Note that the absence of the term $Nf_t(p_1(t))$ in the lower bound comes from the assumption that $f_t$ is nonnegative. It turns out that this can be proven analogously as Lemma 4.8 and we omit the details (see Appendix B).
4.2. Estimating $J_{u,s}$ and $Q_{v,u,s}$. Recall the definitions of $J_{u,s}$ [2.9] and $Q_{v,u,s}$ [2.11]. We study its quenched version $J_{u,s}$ and show that it satisfies a similar but weaker bound than $J_{u,s}^{(a)}$ from Section [2.5]. Moreover, we derive an analogous result for $Q_{v,u,s}$. The goal is to establish the following estimates on $J_{u,s}$ and $Q_{v,u,s}$.

Throughout this section we fix $C_0 > 0$ and let $\alpha > 0$ sufficiently small depending on $\epsilon$ and $C_0$. We also let $\hat{h} := \alpha^{-2} \log C_0(1/\alpha)$.

**Proposition 4.10.** For all $\epsilon > 0$ there exists $c_\epsilon > 0$ such that
\[
\mathbb{P}_\alpha \left( -\frac{\alpha^{-\epsilon}}{\sqrt{s+1}} \leq J_{u,s} \leq \frac{\alpha^{-\epsilon}}{\sqrt{t(t+1)}}(s+1), \forall 0 \leq u < s \leq \hat{h} \right) \geq 1 - \exp(-\alpha^{-c_\epsilon}).
\]

**Proposition 4.11.** For all $\epsilon > 0$ there exists $c_\epsilon > 0$ such that
\[
\mathbb{P}_\alpha \left( |Q_{v,u,s}| \leq \frac{\alpha^{-\epsilon}}{\sqrt{(u+1)(u+1)}}(s+1), \forall 0 \leq s \leq \hat{h} \right) \geq 1 - \exp(-\alpha^{-c_\epsilon}).
\]

Proposition 4.10 can be established by a coupling argument of the events $\{s < T_u < \infty\}$ and $\{s < T < \infty\}$, and using hitting time estimates of random walks. Proof of Proposition 4.11 is based on the same idea, but the coupling for the events defining $Q_{v,u,s}$ is more technical and we defer the proof to Appendix [C].

As the first step in verifying Proposition 4.10, we address the following elementary lemma on simple random walk.

**Lemma 4.12.** Let $W_t$ be a continuous time random walk with $W_0 = 0$. Define the maximum process $M_t := \max_{s \leq t} W_s$. For any $t > 0$ we have:

1. for any $A \geq 1$
\[
\mathbb{P}(M_t \leq A) \leq C \frac{A}{\sqrt{t+1}}
\]
2. For any integer $k \geq 0$
\[
\mathbb{P}(M_t \leq 0, W_t = -k) \leq C \frac{k}{t+1}
\]

**Proof.** By the reflection principle we have
\[
\mathbb{P}(M_t \leq A) \leq \mathbb{P}(|W_t| \leq A) \leq C \frac{A}{\sqrt{t+1}}
\]
where in the last inequality we used that $\mathbb{P}(W_t = k) \leq C/\sqrt{t+1}$ for any $k \in \mathbb{Z}$ which follows from the local central limit theorem.

The second part of the claim also follows from the reflection principle. 

Based on the above observation, we deduce the following estimate on $T$ defined in [2.4].

**Lemma 4.13.** For all $\epsilon > 0$ there exists $\delta > 0$ such that on the event
\[
\mathcal{C} = \mathcal{C}_\delta := \{\forall x > 0, \ Y_x \leq \alpha^{-\delta} + 2\alpha x\}
\]
we have for all $s \leq \hat{h}$
\[
\mathbb{P}(T \geq s \mid Y) \leq \frac{\alpha^{-\epsilon}}{\sqrt{s+1}}.
\]

In particular there exist $c_\epsilon > 0$ such that
\[
\mathbb{P}_\alpha \left[ \mathbb{P}(T \geq s \mid Y) \leq \frac{\alpha^{-\epsilon}}{\sqrt{s+1}}, \forall 0 \leq s \leq \hat{h} \right] \geq 1 - \exp(-\alpha^{-c_\epsilon}).
\]
Note that the outer probability $\mathbb{P}_\alpha$ considers the probability over the rate-$\alpha$ Poisson process $Y$, and the inner probability $\mathbb{P}$ is taken over the simple random walk conditioned on $Y$.

**Proof.** Let $\delta > 0$ sufficiently small depending on $\epsilon$, and $\alpha > 0$ sufficiently small depending on all other parameters. It is easy to check that $\mathbb{P}(C) \geq 1 - \exp(-\alpha^{-\epsilon})$ and therefore the second statement in the lemma follows from the first one.

On $C$ we have

$$\mathbb{P}(T \geq s \mid Y) = \mathbb{P}(\forall x \leq s, \, W_x \leq Y_x \mid Y) \leq \mathbb{P}(\forall x \leq s, \, W_x \leq \alpha^{-\delta} + 2\alpha x)$$

Denote the last probability by $p_s$. If $s \leq \alpha^{-1-\delta}$ by the first part of Lemma 4.12 we have

$$p_s \leq \mathbb{P}(M_s \leq 3\alpha^{-\delta}) \leq C\frac{\alpha^{-\delta}}{\sqrt{s + 1}}.$$

If $\alpha^{-1-\delta} \leq s \leq \alpha^{-\frac{3}{2}-\delta}$ we have

$$p_s \leq \mathbb{P}(M_{\alpha^{-1}} \leq 2\alpha^{-\delta}, \, M_s \leq 2\alpha^{-\frac{1}{2}-\delta})$$

$$\leq \mathbb{P}(M_{\alpha^{-1}} \leq 2\alpha^{-\delta}, \, W_{\alpha^{-1}} \geq -\alpha^{-\frac{1}{2}-\delta}, \, \max_{\alpha^{-1} \leq x \leq s} W_x - W_{\alpha^{-1}} \leq 3\alpha^{-\frac{1}{2}-\delta}, \, W_{\alpha^{-1}} \leq -\alpha^{-\frac{1}{2}-\delta}) + \mathbb{P}(W_{\alpha^{-1}} \leq -\alpha^{-\frac{1}{2}-\delta})$$

$$\leq \mathbb{P}(M_{\alpha^{-1}} \leq 2\alpha^{-\delta}) \mathbb{P}(M_{s-\alpha^{-1}} \leq 3\alpha^{-\frac{1}{2}-\delta}) + Ce^{-\alpha^{-\delta}} \leq C\frac{\alpha^{-2\delta}}{\sqrt{s + 1}}$$

if $\alpha^{-\frac{3}{2}-\delta} \leq s \leq \alpha^{-\frac{7}{4}-\delta}$ by the same arguments we have $p_s \leq C_\delta\alpha^{-3\delta}/\sqrt{s + 1}$. We can repeat this process $C\log(1/\delta)$ times to obtain the bound $p_s \leq C_\delta\alpha^{-C\delta\log(1/\delta)}/\sqrt{s + 1}$ for any $s \leq \alpha^{-2-\frac{\epsilon}{2}}$. Taking $\delta > 0$ sufficiently small depending on $\epsilon$ we get $p_s \leq \alpha^{-\epsilon}\sqrt{s + 1}$ for any $s \leq \alpha^{-2-\frac{\epsilon}{2}}$. \hfill $\Box$

Let $f_Y$ be the probability density of $T$ given $Y$. That is

$$f_Y(s) := \frac{d}{ds} \mathbb{P}(T \leq s \mid Y).$$

In the following lemma we give a bound on $f$ that folds with high probability.

**Lemma 4.14.** For all $\epsilon > 0$ there exists $c_\epsilon > 0$ such that

$$\mathbb{P}_\alpha \left[ f_Y(s) \leq \frac{\alpha^{-\epsilon}}{(s + 1)^{\frac{3}{2}}}, \, \forall 0 \leq s \leq \hat{h} \right] \geq 1 - \exp(-\alpha^{-c_\epsilon}). \quad (4.13)$$

**Proof.** Let $\delta > 0$ sufficiently small such that Lemma 4.13 holds and recall the definition of $C = C_\delta$ in (4.11). It is clear that $f_Y$ is given by

$$f_Y(s) = \frac{1}{2} \cdot \mathbb{P}(\forall x \leq s, \, W_x \leq Y_x, \, W_s = Y_s \mid Y).$$
Thus, on $\mathcal{C}$ we have
\[
\begin{align*}
f_Y(s) &= \frac{1}{2} \sum_{k \leq Y(s/2)} \mathbb{P}(\forall x \leq s, \ W_x \leq Y_x, \ W(s/2) = k, \ W_s = Y_s \mid Y) \\
&\leq \sum_{k \leq Y(s/2)} \mathbb{P}(\forall x \leq s/2, \ W_x \leq Y_x \mid Y) \mathbb{P}(W(s/2) = k \mid \forall x \leq s/2, \ W_x \leq Y_x, \ Y) \\
&\quad \cdot \mathbb{P}\left(\forall s/2 \leq x \leq s, \ W_x \leq Y_s, \ W_s = Y_s \mid W(s/2) = k, \ Y\right) \\
&\leq \frac{C\alpha^{-\epsilon}}{\sqrt{s+1}} \sum_{k \leq Y(s/2)} \mathbb{P}(W(s/2) = k \mid \forall x \leq s/2, \ W_x \leq Y_x, \ Y) \\
&\quad \cdot \mathbb{P}\left(\forall x \leq s/2, \ W_x \leq Y_s, \ W(s/2) = k \mid W_0 = Y_s, \ Y\right) \\
&= \frac{C\alpha^{-\epsilon}}{\sqrt{s+1}} \sum_{k \leq Y(s/2)} \mathbb{P}(W(s/2) = k \mid \forall x \leq s/2, \ W_x \leq Y_x) \\
&\quad \cdot \mathbb{P}\left(M_{\frac{s}{2}} = 0, \ W(s/2) = k - Y_s \mid W_0 = 0, Y\right) \\
&\leq \frac{C\alpha^{-\epsilon}}{(s+1)^{3/2}} \sum_{k \leq Y(s/2)} \mathbb{P}(W(s/2) = k \mid \forall x \leq s/2, \ W_x \leq Y_x) \leq \frac{C\alpha^{-\epsilon}}{(s+1)^{3/2}} \leq \frac{\alpha^{-2\epsilon}}{(s+1)^{3}}.
\end{align*}
\]
where in the second inequality we used Lemma 4.13 in order to bound the first factor and we shifted and inverted time in the third factor. In the third inequality we used Lemma 4.12. The last bound finishes the proof as $\epsilon$ is arbitrary.

Let $f_u$ be the density of $T$ conditioned on $\mathcal{F}_u := \sigma(Y_{\leq u}) = \sigma(Y_x, \ x \leq u)$. That is
\[
f_u(s) := \mathbb{E}[f_Y(s) \mid \mathcal{F}_u] = \frac{d}{ds} \mathbb{P}(T \leq s \mid \mathcal{F}_u).
\]

In the following corollary we show, using Lemma 4.14 and Doob’s martingale inequality that a bound similar to (4.13) holds for $f_u$ as well.

**Corollary 4.15.** For all $\epsilon > 0$ there exists $c_\epsilon > 0$ such that
\[
\mathbb{P}_\alpha\left[f_u(s) \leq \frac{\alpha^{-\epsilon}}{(s+1)^{3/2}}, \ \forall 0 \leq u, s \leq \hat{h}\right] \geq 1 - \exp(-\alpha^{-c_\epsilon}).
\]

**Proof.** Let $\mathcal{D}$ be the event inside the probability in (4.13). $\mathbb{P}(\mathcal{D}^c \mid \mathcal{F}_u)$ is a martingale and therefore by Doob’s martingale inequality we have
\[
\mathbb{P}\left(\sup_{u \leq \hat{h}} \mathbb{P}(\mathcal{D}^c \mid \mathcal{F}_u) \geq \alpha^3\right) \leq \alpha^{-3} \mathbb{P}(\mathcal{D}^c) \leq \exp(-\alpha^{-c_\epsilon}).
\]

Next we claim that $f_Y(s) \leq 1$. Indeed, on the event $\{s < T \leq s + \delta\}$ we have that $\{W_{s+\delta} \neq W_s\}$ which happens with probability $1 - e^{-\delta} \leq \delta$ and therefore $f_Y(s) = \frac{d}{ds} \mathbb{P}(T \leq s \mid Y) \leq 1$. Thus, on the event $\{\sup_{u \leq s} \mathbb{P}(\mathcal{D}^c \mid \mathcal{F}_u) \leq \alpha^3\}$ we have for all $u, s \leq \hat{h}$
\[
f_u(s) = \mathbb{E}[f_Y(s) \mid \mathcal{F}_u] = \mathbb{E}[f_Y(s) \cdot 1_{\mathcal{D}} \mid \mathcal{F}_u] + \mathbb{E}[f_Y(s) \cdot 1_{\mathcal{D}^c} \mid \mathcal{F}_u]
\]
\[
\leq \frac{\alpha^{-\epsilon}}{\sqrt{s+1}} + \mathbb{P}(\mathcal{D}^c \mid \mathcal{F}_u) \leq \frac{\alpha^{-\epsilon}}{\sqrt{s+1}} + \frac{\alpha^3}{\sqrt{s+1}} \leq \frac{\alpha^{-3\epsilon}}{\sqrt{s+1}},
\]
where in the first inequality we used the definition of $\mathcal{D}$ and the fact that $f_Y(s) \leq 1$. This finishes the proof as $\epsilon$ is arbitrary.

Now we are ready to conclude the proof of Proposition 4.10.
Proof of Proposition 4.10. Let \( u < s < \hat{h} \). We have
\[
\mathbf{J}_{u,s} = \mathbb{P}_\alpha(s < T_u < \infty \mid Y_{\leq u}) - \mathbb{P}_\alpha(s < T < \infty \mid Y_{\leq u}) = \mathbb{P}(A_{u,s} \mid Y_{\leq u}) - \mathbb{P}(B_s \mid Y_{\leq u}) \tag{4.14}
\]
where
\[
A_{u,s} := \{ u \leq T < s, s \leq T_u < \infty \}, \quad B_s := \{ s \leq T < \infty, T_u = \infty \}.
\]

Let \( \delta > 0 \) sufficiently small such that Lemma 4.13 holds and let \( \mathcal{D} \) be the event inside the probability in Lemma 4.14. Define
\[
\mathcal{E} := \mathcal{D} \cap \{ v \leq \alpha^{-3}, \forall x > u, Y_x - Y_u \leq \alpha^{-\delta} + 2\alpha(x - u) \}
\]

From Lemma 4.14 it is clear that \( \mathbb{P}(\mathcal{E}) \geq 1 - \exp(-\alpha^{-c_\alpha}) \). We start by bounding the first term on the right hand side of (4.14). To this end we first condition of all of \( Y \) and then use the same arguments as in Corollary 4.15 to get the same bound when conditioning only on \( Y_{\leq u} \). By integrating over the different values \( x \) that \( T \) can take we get that on \( \mathcal{E} \),
\[
\begin{align*}
\mathbb{P}(A_{u,s} \mid Y) & \leq \int_{u}^{s} f_Y(x) \cdot \mathbb{P}(\forall x \leq y \leq s, W_y \leq Y_y + 1 \mid W_x = Y_x + 1, Y) \, dx \\
& \leq \int_{u}^{s} \frac{\alpha^{-\epsilon}}{(x + 1)^{\frac{3}{2}}} \mathbb{P}(\forall y \leq s - x, W_y \leq Y_{y_x} - Y_x \mid W_0 = 0, Y) \, dx \tag{4.15} \\
& \leq \int_{u}^{s} \frac{\alpha^{-\epsilon}}{(x + 1)^{\frac{3}{2}}} \frac{\alpha^{-\epsilon}}{\sqrt{s - x + 1}} \, dx \leq \frac{C\alpha^{-2\epsilon}}{\sqrt{s + 1}/\sqrt{u + 1}},
\end{align*}
\]

where the second inequality is by the definition of \( \mathcal{D} \) and the third inequality is by Lemma 4.13. Indeed, on \( \mathcal{E} \) the process \( Y'_y := Y_{y+x} - Y_x \) satisfies \( Y'_y \leq \alpha^{-\delta} + 2\alpha y \) and therefore by Lemma 4.13 the bound (4.12) holds when we replace \( Y \) by \( Y' \). The last inequality is by Claim A.14.

Thus, as \( \epsilon \) is arbitrary we get
\[
\mathbb{P}(A_{u,s} \mid Y) \geq \frac{\alpha^{-\epsilon}}{\sqrt{s + 1}/\sqrt{u + 1}}, \quad \forall 0 \leq u < s \leq \hat{h} \geq 1 - \exp(-\alpha^{-c_\alpha}).
\]

By the same arguments as in Corollary 4.15 we get that
\[
\mathbb{P}(A_{u,s} \mid Y_{\leq u}) \geq \frac{\alpha^{-\epsilon}}{\sqrt{s + 1}/\sqrt{u + 1}} \geq 1 - \exp(-\alpha^{-c_\alpha}). \tag{4.16}
\]

We turn to bound the second term in (4.14). By the same arguments as in (4.15) we have on \( \mathcal{E} \),
\[
\begin{align*}
\mathbb{P}(B_s \mid Y) & \leq \int_{s}^{\infty} f_Y(x) \cdot \mathbb{P}(\forall y \geq x, W_y \leq Y_y + 1 \mid W_x = Y_x + 1, Y) \, dx \\
& \leq \int_{s}^{\infty} \frac{\alpha^{-\epsilon}}{(x + 1)^{\frac{3}{2}}} \mathbb{P}(\forall y \leq \hat{h}, W_y \leq Y_{y_x} - Y_x \mid W_0 = 0, Y) \, dx \\
& \leq \int_{s}^{\infty} \frac{\alpha^{1-2\epsilon}}{(x + 1)^{\frac{3}{2}}} \frac{\alpha^{1-3\epsilon}}{\sqrt{s + 1}} \, dx \leq \frac{\alpha^{1-\epsilon}}{\sqrt{s + 1}},
\end{align*}
\]

where in the third inequality we used Lemma 4.13. Since \( \epsilon \) is arbitrary and by the same arguments as in Corollary 4.15 we get that
\[
\mathbb{P}(B_s \mid Y_{\leq u}) \geq \frac{\alpha^{1-\epsilon}}{\sqrt{s + 1}}, \quad \forall 0 \leq s \leq \hat{h} \geq 1 - \exp(-\alpha^{-c_\alpha}). \tag{4.17}
\]

Substituting (4.16) and (4.17) into (4.14) finishes the proof of the proposition.
4.3. **Multiple integrals over the fixed rate process.** Based on the results we obtained in the previous subsections, we prove Propositions 4.1 and 4.2 in the case of $\Pi_\alpha$, the fixed rate Poisson process. Switching $\Pi_g$ into $\Pi_\alpha$ not only makes the underlying process be simpler, but more importantly, enables us to interpret the integral as martingales. Previously, this was impossible since $g$ is progressively measurable with respect to $\Pi$ and $J_{t-s,t-u,t}$ is measurable only in terms of $\Pi_g[s,t]$, which essentially requires to revealing the entire information of $\Pi_g$.

Let $\Pi_\alpha^{(t)}$ denote the backwards-time point process of $\Pi_\alpha$ with respect to $t$, defined as

$$\Pi_\alpha^{(t)}[0,s] := \Pi_\alpha[t-s,t], \quad \text{with } d\Pi_\alpha^{(t)}(s) = d\Pi_\alpha^{(t)}(s) - \alpha ds.$$  

Then, we can see that $\mathcal{J}$ defined in (4.5) becomes

$$\mathcal{J}[t;\Pi_\alpha] = \int_0^t \int_0^s J_{u,s}^{\Pi_\alpha} d\Pi_\alpha^{(t)}(u) d\Pi_\alpha^{(t)}(s),$$

where $J_{u,s}^{\Pi_\alpha} := \mathbb{P}_\alpha(s < T_u < \infty \mid \Pi_\alpha^{(t)}[0,u]) - \mathbb{P}_\alpha(s < T < \infty \mid \Pi_\alpha^{(t)}[0,u])$ follows the previous definition (2.9), with respect to $Y$ generated by $\Pi_\alpha^{(t)}$. Similarly, we can write $Q$ in (4.6) by

$$Q[t;\Pi_\alpha] := \int_0^t \int_0^s Q_{v,u,s}^{\Pi_\alpha} (v) d\Pi_\alpha^{(t)}(u) d\Pi_\alpha^{(t)}(s),$$

where $Q_{v,u,s}^{\Pi_\alpha}$ is defined as (2.11), namely,

$$Q_{v,u,s}^{\Pi_\alpha} := \mathbb{P}_\alpha(s < T_{v,u} < \infty \mid \Pi_\alpha^{(t)}[0,v]) - \mathbb{P}_\alpha(s < T_{v} < \infty \mid \Pi_\alpha^{(t)}[0,v]) - \mathbb{P}_\alpha(s < T_{v,u} < \infty \mid \Pi_\alpha^{(t)}[0,v]) + \mathbb{P}_\alpha(s < T < \infty \mid \Pi_\alpha^{(t)}[0,v]).$$

Furthermore, the points $\pi_i(t;\alpha)$ from (4.1) can now be interpreted as the $i$-th closest point from the origin in $\Pi_\alpha^{(t)}$. Based on these identities, our goal is to establish the following estimates.

**Proposition 4.16.** Let $\epsilon, C > 0$ be given, and set $\hat{h}$ as (4.3). Then, there exists $\alpha_0 = \alpha_0(\epsilon,C) > 0$ such that for all $0 < \alpha < \alpha_0$, we have

$$\mathbb{P}(-\alpha^{2/\epsilon} \sigma_1(t) \leq \mathcal{J}[t;\Pi_\alpha] \leq \alpha^{-\epsilon} \sigma_1 \sigma_2(t), \forall t \leq \hat{h}) \geq 1 - \exp(-\alpha^{-\epsilon \sigma_3 t});$$

$$\mathbb{P}(|Q[t;\Pi_\alpha]| \leq \alpha^{-\epsilon} \sigma_1 \sigma_2 \sigma_3(t), \forall t \leq \hat{h}) \geq 1 - \exp(-\alpha^{-\epsilon \sigma_3 t}).$$

Since everything dealt in this subsection is within the fixed rate Poisson process $\Pi_\alpha$, we use the simplified notation $\pi_i(t) = \pi_i(t;\Pi_\alpha)$ and $\sigma_i(t) = \sigma_i(t;\Pi_\alpha)$.

We discuss the proof only for the second one, since the first one can be derived by analogous but simpler arguments; Although there is a difference that the first one has distinct upper and lower bound while the other is not, they can be obtained from the same proof as we will see in Remark 4.22. The main idea is applying results from Section 4.11 to control the triple integral (4.18) based on the bound we got from Proposition 4.11. However, there are a couple of major differences we need to keep in mind:

1. We want to estimate each integral in (4.18) one by one, starting from the inner one. This entails understanding an appropriate continuity property (in $u,s,t',t$) of the inner integrals, which we will obtain using Lemma 4.17.

2. $Q_{v,u,s}^{\Pi_\alpha}$ is not small enough for small $v,u,s$, and we need to control such cases relying on Lemma 4.9 not on Lemma 4.5. This is why an upper bound of $\alpha^{-\epsilon} \sigma_1 \sigma_2 \sigma_3$ is necessary.

The following estimate on the derivative of $Q_{v,u,s}$ gives the desired continuity property later.
Lemma 4.17. Let \(Q^{\Pi(t)}_{v,u,s}\) be defined as (4.19). With probability 1, we have
\[
\left| \frac{\partial}{\partial u} Q^{\Pi(t)}_{v,u,s} \right| \vee \left| \frac{\partial}{\partial s} Q^{\Pi(t)}_{v,u,s} \right| \leq 4
\]
for all \(0 \leq v < u < s \leq t\).

Proof. Let \(\delta > 0\) be a small number such that \(s + \delta < t\) and recall the definition (4.19). We investigate each of the four terms in (4.19) separately. To study the partial derivative of the first term, we observe that
\[
\left| P_\alpha(s \leq T_{v,u+\delta} < \infty | \Pi^{(t)}_\alpha(0,v)) - P_\alpha(s < T_{v,u} < \infty | \Pi^{(t)}_\alpha(0,v)) \right|
\]
\[
= P(u \leq T_{v,u+\delta} < u + \delta, t \leq T_{v,u} < \infty | \Pi^{(t)}_\alpha(0,v)),
\]
since the only possible way to have \(T_{v,u+\delta} = T_{v,u}\) is when \(u \leq T_{v,u+\delta} < u + \delta \leq T_{v,u}\). Then, we can write down a crude bound such that
\[
P(s \leq T_{v,u+\delta} < u + \delta, t \leq T_{v,u} < \infty | F_v) \leq \delta,
\]
since \(s \leq T_{u,s+\delta} < s + \delta\) implies that the random walk \(W\) has performed a jump between times \(s\) and \(s + \delta\). The same argument holds for \(\delta < 0\) as well, and hence we obtain that
\[
\left| \frac{\partial}{\partial u} P_\alpha(s \leq T_{v,u} < \infty | F_v) \right| \leq 1.
\]
Applying this argument to the three other terms of (4.19), we get the desired bound for the \(s\)-partial derivative of \(Q_{v,u,s}\).

The derivative with respect to \(s\) can be estimated analogously, by noticing that (for \(\delta > 0\))
\[
\left| P_\alpha(s + \delta \leq T_{v,u} < \infty | F_v) - P_\alpha(s < T_{v,u} < \infty | F_v) \right| = P(s \leq T_{v,u} < s + \delta | F_v).
\]
\[\Box\]

A straight-forward generalization gives the analogous estimate on \(\partial_t J(s,t|F_v)\), which we record in the following corollary.

Corollary 4.18. With probability 1, we have
\[
\left| \frac{\partial}{\partial s} J^{\Pi(t)}_{v,u,s} \right| \leq 2,
\]
for all \(0 \leq u < s \leq t\).

Next, we introduce stopping times that provides a fundamental advantage on dealing with the issue (2) mentioned above.

\[
\tau_{\Pi}^{rev}(t) = \tau_{\Pi}^{rev}(t,\alpha) := \inf \left\{ s > \alpha^{-1} : |\Pi^{(t)}_\alpha(s - \alpha^{-1}, s) - \alpha^{-\frac{\delta}{3\gamma}}| \right\};
\]
\[
\tau_{\Pi}^{rev}(t) = \tau_{\Pi}^{rev}(t,\alpha) := \inf \left\{ s > \alpha^{-1} - \frac{\delta}{3\gamma} : |\Pi^{(t)}_\alpha(s - \alpha^{-1} - \frac{\delta}{3\gamma}, s) - 0| \right\},
\]
\[
\tau_{3}^{rev}(t) = \tau_{3}^{rev}(t,\alpha) := \inf \left\{ s > 0 : \sup_{v,u \leq t} \left\{ Q^{\Pi(t)}_{v,u,s} - \frac{\alpha^{-\frac{\delta}{3\gamma}}}{\sqrt{(v+1)(u+1)(s+1)}} \right\} > 0 \right\},
\]
\[
\tau_{1}^{rev}(t) = \tau_{1}^{rev}(t,\alpha) := \tau_{1}^{rev}(t) \wedge \tau_{2}^{rev}(t) \wedge \tau_{3}^{rev}(t).
\]

In the proof of Proposition 4.16, \(\tau_{\Pi}^{rev}\) provides fundamental control on the size of the Poisson process, and \(\tau_{2}^{rev}\) is needed to ensure that \(\pi_1, \pi_2, \pi_3 \leq 4\alpha^{-1} - \frac{\delta}{3\gamma}\). The purpose of \(\tau_{3}^{rev}\) is obvious in controlling the integral (4.18).
Lemma 4.19. Let \( \hat{h} \) be as (4.3). Under the above definition, there exists \( a_0 = a_0(\epsilon, C) \) such that for all \( 0 < \alpha < a_0 \),

\[
P\left( \inf_{t \leq \hat{h}} \{ \tau^{rev}_t(t, \alpha) \} \geq \hat{h} \right) \geq 1 - \exp(-\alpha^{-3/10}).
\]

Proof. To estimate \( \tau^{rev}_{\Omega 1} \), we divide the interval \([0, \hat{h}]\) into subintervals of length \( \alpha^{-1} \). Each interval \([k\alpha^{-1}, (k+1)\alpha^{-1}]\) contains more than \( \alpha^{-5/4}/2 \) points with probability less than \( \exp(-\alpha^{-5/4}) \), and we take union bound over \( k = 0, 1, \ldots, \alpha^{-2} \). Note that if each interval \([k\alpha^{-1}, (k+1)\alpha^{-1}]\) contains at most \( \alpha^{-5/4}/2 \) points, then every interval \([t - \alpha^{-1}, t]\) has at most \( \alpha^{-5/4} \) points.

Controlling \( \tau^{rev}_{\Omega 2} \) is similar, where we divide \([0, \hat{h}]\) into subintervals of length \( \alpha^{-1} \) \( \alpha^{-5/4}/4 \). Each of these subintervals contains at least one point with probability \( 1 - \exp(-\alpha^{-5/4}) \).

Finally, we will see that \( \tau^{rev}_{\Omega 3} \geq \hat{h} \) with very high probability from Proposition 4.11. \( \square \)

Lemma 4.20. Let \( \hat{h} \) be as (4.3), and for each \( t \in [0, \hat{h}] \) let \( \tau^{rev}_t(t) \) be as (4.21). Then, we have

\[
P \left( \sup_{u' \leq u \leq \tau^{rev}_t(t)} \left| \int_0^{u'} Q^{(t)}_{v,u',s} d\Pi^{(t)}_{\alpha} (v) \right| \leq \frac{5\alpha^{-7/4} \sigma_1(t)}{\sqrt{(u+1)(s+1)}}, \forall 0 < u < s \leq t \leq \hat{h} \right) \geq 1 - \exp(-\alpha^{-5/4}). \tag{4.22}
\]

Remark 4.21. Another way to phrase Lemma 4.20 is the following: for each \( t \in [0, \hat{h}] \), define the stopping time \( \tau^{inf}_1(t) \) as

\[
\tau^{inf}_1(t) := \inf \{ s > 0 : \exists u \leq s \text{ s.t. } \sup_{u' \leq u} \left| \int_0^{u'} Q^{(t)}_{v,u',s} d\Pi^{(t)}_{\alpha} (v) \right| \geq \frac{5\alpha^{-7/4} \sigma_1(t)}{\sqrt{(u+1)(s+1)}} \}.
\]

Then, Lemmas 4.17 and 4.20 imply that

\[
P_{\alpha} \left( \bigcap_{t \leq \hat{h}} \{ \tau^{inf}_1(t) \land \tau^{rev}_t(t) \geq t \} \right) \geq 1 - 2 \exp(-\alpha^{-7/5}).
\]

Proof of Lemma 4.20. We first establish (4.22) for fixed \( s \) and \( t \), then extend the result to the form of (4.22).

Let \( s < t \leq \hat{h} \) be fixed, and set

\[
f_u(x) := Q^{(t)}_{x,u,s}, \quad \tilde{f}_u(x) := \frac{\alpha^{-5/4}}{\sqrt{(x+1)(u+1)(s+1)}}, \quad g(x) \equiv \alpha.
\]

We apply Lemma 4.8 to these functions, setting the parameters in the lemma as follows:

- We set \( \tau_\pm = 0 \) be deterministic, and let \( \tau = \tau(t) = \tau^{rev}_t(t) \).
- Set \( h = \hat{h}, \eta = 2\alpha, \Delta = \alpha^{-1}, \) and \( N = \alpha^{-5/4} \). Under these values, we let \( \tau' = \tau^{rev}_t(t) \), and \( \tau'' = \infty \).
- Letting

\[
D = 4, \quad M = \frac{\alpha^{1-5/8}}{(u+1)(s+1)}, \quad A = \frac{3\alpha^{-1/2} \cdot 5/4}{\sqrt{(u+1)(s+1)}},
\]

(where we set \( M \) to be deterministic as \( \tau_\pm = 0 \)) we see that the conditions given in (4.8) are satisfied.
- Set \( \delta = \alpha^{10} \) which satisfies (4.10), and let \( a = \alpha^{-7/10} \).
Applying Lemma 4.8 under this setting gives

\[
\mathbb{P}\left( \sup_{u' \leq u \wedge \tau_1^{rev}(t)} \int_0^{u'} Q_{v,u,s}^{(t)} d\tilde{\Pi}_\alpha(t)(v) \leq \alpha^{-\frac{\gamma}{2}} \frac{\sigma_1(t)}{\sqrt{(u+1)(s+1)}}, \ \forall u \in [0, s] \right) \\
\geq 1 - \exp\left(-\alpha^{-\frac{\gamma}{20}}\right),
\]

and this holds for all small enough \( \alpha > 0 \). Note that we used \( \pi_1(t) \leq \alpha^{-1-\frac{\gamma}{20}} \), which comes from \( \tau_1^{rev}(t) \), to simplify the bound inside the probability.

What remains is to extend this bound to hold for all \( s \) and \( t \). To this end, we first take a union bound over \( s < t \) in a discretized interval such that

\[
s, t \in \mathcal{T} := \{ x \in [0, \hat{h}] : x = k\delta, \ k \in \mathbb{Z} \},
\]

where \( \delta = \alpha^{10} \). This gives

\[
\mathbb{P}\left( \sup_{u' \leq u \wedge \tau_1^{rev}(t)} \int_0^{u'} Q_{v,u,s}^{(t)} d\tilde{\Pi}_\alpha(t)(v) \leq \alpha^{-\frac{\gamma}{2}} \frac{\sigma_1(t)}{\sqrt{(\pi_1(t) + 1)(u+1)(s+1)}}, \ \forall u \in [0, s], \ s, t \in \mathcal{T}, \ s \leq t \right) \\
\geq 1 - \exp\left(-\alpha^{-\frac{\gamma}{20}}\right).
\]

We first extend this to all \( s < t \) and \( t \in \mathcal{T} \). For any \( s < t \) with \( t \in \mathcal{T} \), let \( s_\delta \geq s \) such that \( s_\delta \in \mathcal{T} \) and \( s_\delta \leq s + \delta \). Using the estimate

\[
\left| Q_{v,u,s}^{(t)} - Q_{v,u,s_\delta}^{(t)} \right| \leq 4\delta = 4\alpha^{10},
\]

and the fact that \( \int_0^{u \wedge \tau_1^{rev}(t)} d\tilde{\Pi}_\alpha(t)(v) \leq \alpha^{-1-\frac{\gamma}{20}} \) which comes from the definition of \( \tau_1^{rev}(t) \), we obtain that

\[
\mathbb{P}\left( \sup_{u' \leq u \wedge \tau_1^{rev}(t)} \int_0^{u'} Q_{v,u,s}^{(t)} d\tilde{\Pi}_\alpha(t)(v) \leq \alpha^{-\frac{\gamma}{2}} \frac{2\sigma_1(t)}{\sqrt{(\pi_1(t) + 1)(u+1)(s+1)}}, \ \forall u < s \leq t, \ t \in \mathcal{T} \right) \\
\geq 1 - \exp\left(-\alpha^{-\frac{\gamma}{20}}\right).
\]

Finally, we extend this result to hold for all \( t \leq \hat{h} \). Let \( t_\delta \geq t \) be \( t_\delta \in \mathcal{T} \) and \( \delta' := t_\delta - t \leq \delta \). Here, we need to consider the difference between measure \( \Pi_\alpha^{(t)} \) and \( \Pi_\alpha^{(t_\delta)} \). Note that for any \( u' \leq u < s \leq t \) such that \( u' \leq \tau_1^{rev}(t) \),

\[
\int_0^{u'} Q_{v,u,s}^{(t)} d\tilde{\Pi}_\alpha(t)(v) = \int_{\delta'}^{u'+\delta'} Q_{v,u,s}^{(t_\delta)} d\tilde{\Pi}_\alpha(t_\delta)(v).
\]

We can then proceed by switching \( Q_{v,u,s}^{(t_\delta)} \) to \( Q_{v,u,s}^{(t)} \), and noting that on the event \( \inf_{t \leq \hat{h}} \tau_1^{rev}(t) \geq \hat{h} \) which holds with high probability from Lemma 4.19, we have

\[
\left| \int_{\delta'}^{u'+\delta'} Q_{v,u,s}^{(t_\delta)} d\tilde{\Pi}_\alpha(t_\delta)(v) + \int_0^{u'+\delta'} Q_{v,u,s}^{(t)} d\tilde{\Pi}_\alpha(t)(v) \right| \\
\leq \left| \int_0^{(u'+\delta') \wedge u} Q_{v,u,s}^{(t)} d\tilde{\Pi}_\alpha(t)(v) \right| \left| \int_{\delta'}^{u'+\delta'} Q_{v,u,s}^{(t)} d\tilde{\Pi}_\alpha(t)(v) \right| + \alpha^{-\frac{\gamma}{20}} \hat{f}_u(\pi_1(t)). \tag{4.23}
\]
Thus, we obtain
\[
\mathbb{P}\left( \sup_{u \leq u \wedge \tau_1^{\text{rev}}(t)} \int_0^{u'} Q_{u, s, t}^{\alpha}(v) \leq \frac{5\alpha^{-\frac{s}{2}}}{\sqrt{(\pi_1(t) + 1)(u + 1)(s + 1)}}, \forall u < s \leq t \leq \hat{h} \right) \geq 1 - \exp\left(-\alpha^{-\frac{s}{160}}\right),
\]
concluding the proof.

**Remark 4.22.** We stress that the same method can be applied to deduce the corresponding bound for the integral of \( J \). Namely, changing the definition of \( \tau_3^{\text{rev}} \) into the one that has the bound from Lemma 4.10 instead of 4.11, we obtain
\[
\mathbb{P}\left( \sup_{s \leq \tau_{31}^{\text{rev}}(t)} \int_0^{s'} J_{u, s}^{\alpha}(u) \leq \frac{\alpha^{-\frac{s}{2}}\sigma_1(t)}{\sqrt{s + 1}}, \forall 0 < s \leq t \leq \hat{h} \right) \geq 1 - \exp\left(-\alpha^{-\frac{s}{160}}\right);
\]
\[
\mathbb{P}\left( \inf_{s \leq \tau_{31}^{\text{rev}}(t)} \int_0^{s'} J_{u, s}^{\alpha}(u) \geq -\alpha^{-\frac{s}{2}}\sigma_1(t) - \frac{\alpha^{-\frac{s}{2}}\sigma_1(t)}{\sqrt{s + 1}}, \forall 0 < s \leq t \leq \hat{h} \right) \geq 1 - \exp\left(-\alpha^{-\frac{s}{160}}\right).
\]
(Note that unlike the upper bound, the two terms in the lower bound cannot be unified together since one is not always larger than the other.) Based on this observation, we similarly obtain the analogue of Corollary 4.23 for \( J \) to deduce the first conclusion of Proposition 4.16.

Next step, we derive an analogous result after integrating once more. For convenience we define
\[
F_1(u, s, t) := \int_0^u Q_{u, s, t}^{\alpha}(v).
\]

**Corollary 4.23.** For each \( t \), let \( \bar{z}_1^{\text{int}}(t) := \tau_1^{\text{int}}(t) \wedge \tau_1^{\text{rev}}(t) \) with \( \tau_1^{\text{int}}(t) \) given in Remark 4.21. Then, we have
\[
\mathbb{P}\left( \sup_{s \leq \tau_1^{\text{int}}(t)} \int_0^{s'} F_1(u, s, t) d\Pi_{\alpha}^{(i)}(u) \leq \frac{\alpha^{-\frac{s}{2}}\sigma_1(t)}{\sqrt{s + 1}}, \forall 0 \leq s \leq t \leq \hat{h} \right) \geq 1 - \exp\left(-\alpha^{-\frac{s}{160}}\right).
\]

**Remark 4.24.** Similarly as Remark 4.21, define
\[
\bar{z}_2^{\text{int}}(t) := \inf\left\{ s > 0 : \sup_{s \leq s} \int_0^{s'} F_1(u, s, t) d\Pi_{\alpha}^{(i)}(u) \geq \frac{\alpha^{-s}\sigma_1(t)}{\sqrt{s + 1}} \right\}.
\]
Then, Lemma 4.20 and Corollary 4.23 imply
\[
\mathbb{P}\left( \bigcap_{t \leq \hat{h}} \left\{ \tau_2^{\text{int}}(t) \wedge \bar{z}_1^{\text{int}}(t) \geq t \right\} \right) \geq 1 - 3\exp\left(-\alpha^{-\frac{s}{160}}\right).
\]

**Proof of Corollary 4.23.** The proof goes similar to that of Lemma 4.20 and hence we describe the argument more concisely. We first work with fixed \( t \), and then extend the result to the desired conclusion. For each \( s \leq t \leq \hat{h} \), define
\[
f_s(x) := F_1(x, s, t), \quad \tilde{f}_s(x) := \frac{5\alpha^{-\frac{s}{2}}\sigma_1(t)}{\sqrt{(x + 1)(s + 1)}}, \quad g(x) \equiv \alpha.
\]
We again apply Lemma 4.8 to these functions, with the other parameters set to be as follows.
- Let \( \tau_- = \tau_-(t) := \bar{z}_1^{\text{int}}(t) \), and let \( \tau = \tau(t) = \bar{z}_1^{\text{int}}(t) \).
• Set $h = \tilde{h}$, $\eta = 2\alpha$, $\Delta = \alpha^{-1}$, $N = \alpha^{-\frac{\Delta}{2}}$, $\tau' = \tau_1^{\text{rev}}(t)$, and $\tau'' = \infty$ as in the proof of Lemma 4.20.

• Similarly before, the parameters

$$D = \alpha^{-2}, \quad M = \frac{\alpha^{1-\frac{\Delta}{2}}\sigma_1(t)^2}{s + 1}, \quad A = \frac{\alpha^{1-\frac{\Delta}{2}}\sigma_1(t)}{\sqrt{s + 1}},$$

satisfy the conditions in (4.8). Note that $D$, the parameter for the derivative can be controlled deterministically by

$$|\partial_u F_1(u, s, t)| \leq Q_{u,u,s}^{\Pi(t)} \cdot (1 + \alpha) + \int_0^u |\partial_u Q_{u,u,s}^{\Pi(t)}| \cdot |d\Pi_\alpha(t)| \leq \alpha^{-2},$$

for all $s \leq \tau \leq \tau_1^{\text{rev}}$. $|\partial_s F_1(u, s, t)|$ can be estimated in the same but simpler way, since the regime of integral is independent of $s$.

• Set $\delta = \alpha^{-10}$ as before which satisfies (4.10), and let $a = \alpha^{\frac{\Delta}{10}}$.

Then, Lemma 4.8 implies that for each $t \in [0, \tilde{h}]$,

$$\mathbb{P}\left( \sup_{s' \leq s \leq \alpha^{\frac{\Delta}{10}}} \left| \int_{\tau_-}^{s'} F_1(u, s, t) d\Pi_\alpha(t)(u) \right| \leq \frac{\alpha^{-\frac{\Delta}{2}}\sigma_1\sigma_2(t)}{\sqrt{s + 1}}, \quad \forall s \in [0, t] \right) \geq 1 - \exp\left( -\alpha^{-\frac{\Delta}{10}} \right),$$

which holds for all small enough $\alpha > 0$. Note that we used $\pi_2(t) \leq 3\alpha^{-1-\frac{\Delta}{2}}$, which comes from $\tau_1^{\text{rev}}(t)$, to simplify the bound inside the probability.

Moreover, note that the definition $\tau_- = \tau_1(t)$ gives that with probability one,

$$\left| \int_0^{\tau_2} F_1(u, s, t) d\Pi_\alpha(t)(u) \right| \leq \alpha^{-\frac{\Delta}{2}}\sigma_1\sigma_2(t) \int_0^u \frac{\alpha^{-\frac{\Delta}{2}}\sigma_1\sigma_2(t)}{\sqrt{(v + 1)(u + 1)(s + 1)}} dv du$$

$$= \frac{2\alpha^{-\frac{\Delta}{2}}\sigma_1\sigma_2(t)}{\sqrt{s + 1}} \leq \frac{\alpha^{-\frac{\Delta}{2}}\sigma_1\sigma_2(t)}{\sqrt{s + 1}},$$

where the first and the last inequality hold if $s \leq \tau_1^{\text{rev}}(t)$. Thus, we combine the two to obtain

$$\mathbb{P}\left( \sup_{s' \leq s \leq \alpha^{\frac{\Delta}{10}}} \left| \int_0^{s'} F_1(u, s, t) d\Pi_\alpha(t)(u) \right| \leq \frac{2\alpha^{-\frac{\Delta}{2}}\sigma_1\sigma_2(t)}{\sqrt{s + 1}}, \quad \forall s \in [0, t] \right) \geq 1 - \exp\left( -\alpha^{-\frac{\Delta}{10}} \right),$$

Extending this to the desired result is analogous as the argument in the proof of Lemma 4.20. Namely we take a union bound over $t$ in the discretized interval $T$, and deduce an estimate corresponding to (4.23) appealing to the fact that the derivatives of $F_1$ are bounded. The details are omitted and left for interested readers. \(\square\)

Now we are only left with the outer integral. The corollary below follows analogously as the proof of Lemma 4.20 and Corollary 4.23. Thus, we state the result and omit the details of its proof.

**Corollary 4.25.** Let $\tau_2^{\text{int}}(t) := \tau_2^{\text{int}}(t) \wedge \tau_1^{\text{int}}(t)$ with $\tau_2^{\text{int}}(t), \tau_1^{\text{int}}(t)$ given in Remark 4.24, Lemma 4.23 respectively, and define

$$F_2(s, t) := \int_0^s F_1(u, s, t) d\Pi_\alpha(t)(u).$$
Then, we have

\[
\mathbb{P}\left( \sup_{t' \leq \alpha^{-\epsilon} t} \left| \int_0^{t'} F_2(s, t) d\Pi^{(t)}_\alpha(s) \right| \leq \alpha^{-\epsilon} \sigma_1 \sigma_2 \sigma_3(t), \ \forall t \leq \hat{h} \right) \geq 1 - \exp\left(-\alpha^{-\frac{\epsilon}{20}}\right).
\]

We conclude Section 4.3 by establishing Proposition 4.16.

Proof of Proposition 4.16. Combining Remark 4.24 and Corollary 4.25 implies that

\[
\mathbb{P}\left( \left| \mathcal{Q}[t; \Pi_\alpha] \right| \leq \alpha^{-\epsilon} \sigma_1 \sigma_2 \sigma_3(t), \ \forall t \leq \hat{h} \right) \geq 1 - \exp\left(-\alpha^{-\frac{\epsilon}{20}}\right),
\]

proving the second statement of (4.20) in Proposition 4.16. The first inequality in (4.20) can be obtained analogously by developing Lemma 4.20, Corollary 4.23, Remarks 4.21 and 4.24, based on the bound on \( J^{(t)}_{u,s} \) given in Proposition 4.10. We omit the details which are left for interested readers.

To conclude this subsection, we introduce the following result which controls the integral of the error bound given in Proposition 4.16.

Lemma 4.26. Let \( \epsilon, C > 0 \) be given, and set \( \hat{h} \) as (4.3). Then, there exists \( \alpha_0 = \alpha_0(\epsilon, C) > 0 \) such that for all \( 0 < \alpha < \alpha_0 \), we have

\[
\mathbb{P}_\alpha\left( \left| \int_0^{\hat{h}} \sigma_1 \sigma_2 \sigma_3(t) dt \right| \leq \alpha^{-1-\epsilon} \right) \geq 1 - \exp\left(-\alpha^{-\frac{\epsilon}{20}}\right);
\]

\[
\mathbb{P}_\alpha\left( \left| \int_0^{\hat{h}} \sigma_1 \sigma_2 \sigma_3(t) dt \right| \leq \alpha^{-\frac{1}{2} - \epsilon} \right) \geq 1 - \exp\left(-\alpha^{-\frac{\epsilon}{20}}\right).
\]

Proof. Let \( N := \Pi_\alpha[0, \hat{h}] \) and let \( \pi_1 \leq \pi_2 \leq \cdots \) be the points of the process. For convenience we let \( \pi_0 \equiv 0 \) and \( \pi_{N+1} = T \). Note that with very high probability \( \alpha^{-1-\epsilon} N \leq \alpha^{-1-\epsilon} \) and for any \( 0 \leq i \leq N \), \( \pi_{i+1} - \pi_i \leq \alpha^{-1-\epsilon} \). We start with the first inequality. We have with very high probability

\[
\int_0^{T} \frac{dt}{\sqrt{(\pi_1(t)+1)(\pi_2(t)+1)}} \leq \int_0^{T} \frac{dt}{\pi_1(t)+1} = \sum_{i=0}^{N} \pi_{i+1} \int_{\pi_i}^{T} \frac{dt}{t-\pi_i+1} \leq CN\log(1/\alpha) \leq \alpha^{-1-\epsilon}
\]

We turn to prove the second inequality. For \( k \leq \log_2(1/\alpha) \) and \( \alpha^{-1-\epsilon} \leq n \leq \alpha^{-1-\epsilon} \) define the sets

\( I_k^n := \{ i \leq n | \pi_i - \pi_{i-1} \in [2^k, 2^{k+1}] \} \)

and \( I_k := I_k^N \). We have that \( \mathbb{P}(i \in I_k^n) \leq \alpha 2^k \) and therefore with very high probability for all \( \alpha^{-1+\epsilon} \leq n \leq \alpha^{-1-\epsilon} \) we have \( |I_k^n| \leq \alpha^{-1+\epsilon} n 2^k \leq \alpha^{-2\epsilon} 2^k \). Thus with very high probability \( |I_k| \leq \alpha^{-2\epsilon} 2^k \). Thus, if we let \( I' := [N] \setminus \cup_k I_k \) we have

\[
\int_0^{T} \frac{dt}{\sqrt{(\pi_1(t)+1)(\pi_2(t)+1)(\pi_3(t)+1)}} \leq \int_0^{T} \frac{dt}{(\pi_1(t)+1)(\pi_2(t)+1)} = \sum_{i=0}^{N} \sum_{i \in I'} \frac{\pi_{i+1}}{\sqrt{(t-\pi_i+1)(\pi_2(t)+1)}} \leq C \alpha \sum_{i \in I'} \frac{1}{\sqrt{(t-\pi_i+1)}} + C \sum_{k=1}^{\log_2(1/\alpha)} 2^{-k} \sum_{i \in I_k} \frac{1}{\sqrt{(t-\pi_i+1)}}
\]

\[
\leq C \alpha^{-\frac{1}{2} - \epsilon} |I'| + C \alpha^{-\frac{1}{2} - \epsilon} \sum_{k=1}^{\log_2(1/\alpha)} 2^{-k} |I_k| \leq CN \alpha^{-\frac{1}{2} - \epsilon} + C \alpha^{-\frac{1}{2} - 3\epsilon} \log(1/\alpha) \leq C \alpha^{-\frac{1}{2} - 4\epsilon}. \]

\( \square \)
4.4. Perturbation of the underlying Poisson processes. In this subsection, we establish Propositions 4.1 and 4.2. As mentioned before, our method is to derive an estimate on the Radon-Nykodym derivative of $\Pi_g$ with respect to $\Pi_\alpha$ and use Proposition 4.16.

Let $P^\tau_g$ denote the law of $\Pi_g[0,t]$. Using this notation, we will write, for instance,

$$\mathbb{P}(\mathcal{J}[t;\Pi_g] \in \mathcal{E}) = \int_{\mathbb{R}} I\{\mathcal{J}[t;\Pi] \in \mathcal{E}\} P^\tau_g(d\Pi).$$

Similarly, we let $P^\tau_\alpha$ to be the law of $\Pi_\alpha[0,t]$. The main observation to establish Propositions 4.1 and 4.2 is the following: for a stopping time $\tau$ in terms of $g$ and $\Pi$, the Radon-Nykodym derivative of $P^\tau_g$ with respect to $P^\tau_\alpha$ can be written as

$$r^\tau_g(\Pi) := \frac{dP^\tau_g}{dP^\tau_\alpha}(\Pi) = \exp\left(-\int_0^\tau (g(y) - \alpha)dy\right) \prod_{x \in \Pi[0,\tau]} \frac{g(x)}{\alpha}.$$  \hspace{1cm} (4.24)

This comes from the fact that $g$ is predictable with respect to $\Pi$.

From now on, let $\tau$ be the stopping time given in Proposition 4.1 and for convenience we set $\hat{\tau} := \hat{h} \wedge \tau$.

The next lemma shows how to control the size of $r^\tau_g(\Pi)$, when $\Pi$ is given by $\Pi_g \sim P^\tau_g$.

Lemma 4.27. Under the setting of Proposition 4.1, we have

$$\mathbb{P}_{\Pi-P^\tau_g}(r^\hat{\tau}_g(\Pi) \geq \exp(\alpha^{-\frac{250}{\lambda}})) \leq \exp(-\alpha^{-\frac{2000}{\lambda}}).$$

Proof. From (4.24), we use $1 + x \leq e^x$ to obtain that

$$r^\hat{\tau}_g(\Pi) \leq \exp\left(\alpha^{-1} \int_0^{\hat{\tau}} (g(x) - \alpha) d\Pi(x) - \int_0^{\hat{\tau}} (g(x) - \alpha) dx\right).$$

Thus, for $\lambda := \alpha^{-\frac{250}{\lambda}}$ we have $r^\tau_g(\Pi)^\lambda \leq M_\tau \cdot L_\tau$ where

$$M_\tau := \exp\left(\lambda \alpha^{-1} \int_0^{\hat{\tau}} (g(x) - \alpha) d\Pi_g(x) - \int_0^{\hat{\tau}} g(x)(e^{\alpha^{-1}g(x)} - 1) dx\right),$$

$$L_\tau := \exp\left(\int_0^{\hat{\tau}} g(x)(e^{\alpha^{-1}g(x)} - 1) - \lambda(g(x) - \alpha) dx\right).$$

We start by bounding $L_\tau$. We have that

$$L_\tau \leq \exp\left(\int_0^{\hat{\tau}} \{g(x)\left(\lambda \alpha^{-1}(g(x) - \alpha) + \lambda^2 \alpha^{-2}(g(x) - \alpha)^2\right) - \lambda(g(x) - \alpha)\} dx\right)$$

$$\leq \exp\left(\int_0^{\hat{\tau}} \{\lambda \alpha^{-1}(g(x) - \alpha) + \lambda^2 \alpha^{-2}(g(x) - \alpha)^2\} g(x) dx\right) \leq 2$$

where in the first inequality we used $\lambda \alpha^{-1} |g(x) - \alpha| \leq \lambda \alpha^{-1} (g(x) + \alpha) \leq 1$ from the third assumption of (4.4), along with the fact that $e^y \leq 1 + y + y^2$ for $y \leq 1$. In the last inequality we used the first two assumptions of (4.4). Thus, we obtain that

$$\exp(\lambda \alpha^{-\frac{250}{\lambda}}) \cdot \mathbb{P}_{\Pi-P^\tau_g}(r^\hat{\tau}_g(\Pi) \geq \exp(\alpha^{-\frac{250}{\lambda}})) \leq \mathbb{E}_{\Pi-P^\tau_g}[r^\hat{\tau}_g(\Pi)^\lambda]$$

$$\leq \mathbb{E}[M_\tau \cdot L_\tau] \leq 2\mathbb{E}[M_\tau] = 2,$$

where in the last equality we used that $M_\tau$ and therefore $M_\tau \wedge \tau$ are martingales and that $M_0 = 1$. This concludes the proof of the lemma. □
Now, We are ready to establish Propositions 4.1 and 4.2.

Proof of Propositions 4.1 and 4.2. Define the events
\[ \mathcal{A}_1(t) := \{ \Pi : -\alpha^{-\frac{1}{2}} \sigma_1(t) \leq J[t; \Pi] \leq \alpha^{-\epsilon} \sigma_2(t) \} , \quad \mathcal{A}_2(t) := \{ \Pi : |Q[t; \Pi]| \leq \alpha^{-\epsilon} \sigma_1 \sigma_2 \sigma_3(t) \} , \]
\[ \mathcal{A}(t) := \mathcal{A}_1(t) \cap \mathcal{A}_2(t) , \quad B := \{ \Pi : r_{\hat{g}}^s(\Pi) \geq \exp(\alpha^{-\frac{r_{\hat{g}}}{t} \sigma}) \} . \]
Moreover, we set \( \mathcal{A}_{\leq t} := \cap_{s \leq t} \mathcal{A}(s) \). Then, we have
\[ \mathbb{P}_{\Pi - P_{\hat{g}}} ((A_{\hat{g}}) \cap B) \leq \mathbb{P}_{\Pi - P_{\hat{g}}} ((A_{\hat{g}}) \cap B) + \mathbb{P}_{\Pi - P_{\hat{g}}} (B) . \tag{4.27} \]
Lemma 4.27 tells us that the second term in the RHS is bounded by \( \exp(-\alpha^{-\frac{r_{\hat{g}}}{t} \sigma}) \), and the first term can be estimated by
\[ \begin{align*}
\mathbb{P}_{\Pi - P_{\hat{g}}} ((A_{\hat{g}}) \cap B) & \leq \int \Pi \{ (A_{\hat{g}}) \cap B \} r_{\hat{g}}(\Pi) \mathbb{P}_\alpha^h (d\Pi) \\
& \leq \exp\left(\alpha^{-\frac{r_{\hat{g}}}{t} \sigma}\right) \int \Pi \{ (A_{\hat{g}}) \cap B \} \mathbb{P}_\alpha^h (d\Pi) \\
& \leq \exp\left(-\alpha^{-\frac{r_{\hat{g}}}{t} \sigma}\right),
\end{align*} \]
where the last inequality comes from Proposition 4.16. Plugging the two bounds into (4.27) deduces the desired results.

The same proof as above can be applied to generalize Lemma 4.26. Due to its similarity, we omit the proof of the following result.

Lemma 4.28. Let \( \epsilon > 0 \) be arbitrary, \( \alpha > 0 \) be a sufficiently small constant depending on \( \epsilon \), and \( \hat{h} \) be as (4.3). Let \( \tau \) be a stopping time, and \( \{ g(s) \}_{s \geq 0} \) be a positive stochastic process progressively measurable with respect to \( \Pi_g \), and suppose that they satisfy (4.1). Then, we have
\[ \begin{align*}
\mathbb{P} \left( \left\| \int_0^\tau \sigma_1 \sigma_2 (t; g) dt \right\| \leq \alpha^{-1-\epsilon} \right) & \geq 1 - \exp\left(-\alpha^{-\frac{r_{\hat{g}}}{t} \sigma}\right) ; \\
\mathbb{P} \left( \left\| \int_0^\tau \sigma_1 \sigma_2 \sigma_3 (t; g) dt \right\| \leq \alpha^{-\frac{1}{2}-\epsilon} \right) & \geq 1 - \exp\left(-\alpha^{-\frac{r_{\hat{g}}}{t} \sigma}\right).
\end{align*} \]

4.5. The error estimates of the speed. In this subsection, we translate the main estimates into the error estimates of the first- and second-order approximations (2.16) and (2.22). The results are direct consequences of Propositions 4.1 and 4.2, and we can state them as follows.

Proposition 4.29. Let \( \epsilon, C > 0 \) be arbitrary, \( \alpha > 0 \) be a sufficiently small constant depending on \( \epsilon, C \), and let \( t, \hat{t} \) be \( \hat{t} - t \leq \alpha^{-2} \log^C(1/\alpha) \). Define \( S_1(s) = S_1(s; t, \alpha) \) and \( S_2(s) = S_2(s; t, \alpha) \) as (2.16) and (2.22), respectively, and recall the notation \( \sigma_1(s; S) \) from (4.2). Suppose that there is a stopping time \( \tau \) satisfying the following conditions almost surely:
\[ \int_{t-}^{t+} \left| S(s) - \alpha \right|^2 ds \leq \alpha^{-1-\epsilon} \]
\[ \int_{t-}^{t+} \left| S(s) - \alpha \right|^2 S(s) ds \leq \alpha^{-2-\epsilon} \]
\[ \sup_{t \leq s \leq t+} \{ S(s) + S_1(s) \} \leq \alpha^{-\frac{r_{\hat{g}}}{t} \sigma}. \tag{4.28} \]
Further, let \( S'(s) := S'(s; t, \alpha) \) be defined as (2.15). Then, we have that
\[ \mathbb{P} \left( -\alpha^{-\frac{1}{2}-\epsilon} \sigma_1 (s; S) \leq S'(s) - S_1(s) \leq \alpha^{-\epsilon} \sigma_1 \sigma_2 (s; S), \forall s \in [t, \hat{t} \wedge \tau] \left| \mathcal{F}_{\hat{t}} \right) \right. \geq 1 - 2 \exp\left(-\alpha^{-\frac{r_{\hat{g}}}{t} \sigma}\right) ; \]
\[ \mathbb{P} \left( |S'(s) - S_2(s)| \leq \alpha^{-\epsilon} \sigma_1 \sigma_2 \sigma_3 (s; S), \forall s \in [t, \hat{t} \wedge \tau] \left| \mathcal{F}_{\hat{t}} \right) \right. \geq 1 - 2 \exp\left(-\alpha^{-\frac{r_{\hat{g}}}{t} \sigma}\right) , \]
where this holds for any given $\mathcal{F}_{t^-}$, the sigma-algebra generated by $\Pi[0,t^-]$ and $\{S(s)\}_{s \leq t^-}$.

Proof. Note that the second statement follows directly from the assumptions, Proposition 4.2 and the formula 2.19. The first inequality follows similarly: we first claim that

$$\mathbb{P}\left(\pi_2(s;S) \leq \alpha^{-1-\frac{\theta}{100}} \right), \forall s \in [t^-,\hat{t} + \tau]) \geq 1 - \exp\left(-\alpha^{-\frac{\theta}{200}}\right).$$

This follows from the analogous argument as Lemma 4.19 used to establish bounds on $\tau^{\text{rev}}$, in our case relying on the assumption $\sup_{t^- \leq s \leq t^+} S_1(s) \leq \alpha^{-\frac{\theta}{200}}$. This implies

$$\mathbb{P}\left(\frac{4\alpha + 4\alpha^2}{(1 + 2\alpha)^2} S_1(s) \leq \frac{1}{2} \alpha^{\frac{3}{2}-\epsilon} \sigma_1(s;S) \leq \frac{1}{2} \alpha^{1-\epsilon} \sigma_1(s;S), \forall s \in [t^-,\hat{t} + \tau]) \geq 1 - \exp\left(-\alpha^{-\frac{\theta}{200}}\right),$$

and combining this with Proposition 4.1 and the formula (2.17) gives the conclusion. \qed

We remark that the assumptions (4.28) are actually satisfied by the speed with high probability with appropriate parameters $t^-, t$ and the stopping time $\tau$. The formal definitions and their proofs will be discussed in Sections 6–8.

5. THE AGE-DEPENDENT CRITICAL BRANCHING PROCESS

In Section 3 we saw that the speed drops down to an arbitrarily low size and can be sandwiched by two fixed rate processes. To continue our investigation, we need to understand the evolution of a speed process which begins with points given by a fixed rate such as (3.3). In this section, we study an age-dependent critical branching process which is closely related with the actual speed process. In the first (or second) order approximation of the speed, it will turn out that its main term stays close to a certain critical branching process, and hence understanding the latter process will be crucial in our analysis.

For two parameters $t^-_0 < t_0$, we define the age-dependent critical branching process starting from a configuration of points $\Pi_0[t^-_0,t_0]$ in the interval $[t^-_0,t_0]$. From now on, we use the following notation to emphasize the prescribed initial points: For $t \geq t_0$, we define

$$R(t) = R_b(t;\Pi_0[t^-_0,t_0],\alpha) := \int_{t_0}^{t} K_\alpha(t-x) d\Pi_0(x) + \int_{t_0}^{t} K_\alpha(t-x) d\Pi_R(x), \quad (5.1)$$

and set $R(t) = \alpha$ for $t < t_0$. In words, the process $R(t)$ can be described as follows. The points $\Pi_0[t^-_0,t_0]$ are given initially, acting as roots of the branching process. Each particle at $x \in [t^-_0,t_0]$ independently gives births to its children on $[t_0,\infty)$, at position $y \geq t_0$ at rate $K_\alpha(y-x) dy$. Each child of a root then gives birth to another generation with the same rule, and the branching continues.

Since $\int_0^{\infty} K_\alpha(x) dx = 1$, all particles except the roots ($\Pi_0[t^-_0,t_0]$) perform critical branching process (i.e., average number of offspring is 1). Moreover, the distance between a child and a parent is determined by the density $K_\alpha$, and hence it is an age-dependent branching process.

Throughout the section (and the rest of the paper), $\theta > 0$ denotes a large absolute constant, say, $\theta = 10000$ and $C_\alpha = 50$ is a fixed constant that is significantly smaller than $\theta$. For given positive number $\alpha$ and $t_0$, we denote

$$\beta := \log(1/\alpha).$$

Further, let $t^-_0$ be an arbitrary fixed number satisfying

$$t_0 - 2\alpha^{-2}\beta^\theta < t^-_0 < t_0 - \frac{1}{2}\alpha^{-2}\beta^\theta,$$

and set $\hat{t}_0 := t_0 + 3\alpha^{-2}\beta^\theta$. 
In what follows, we illustrate a way of interpreting $R(t)$ as a perturbation of the fixed rate process. Consider the process $R_0(t)$ defined as

$$R_0(t) := \begin{cases} \alpha & \text{for } t_0 \leq t < t_0; \\ \int_{t_0}^{t} K_\alpha(t-x) d\Pi_\alpha(x) & \text{for } t \geq t_0, \end{cases} \quad (5.2)$$

For $t \geq t_0$, observe that $R(t) = R_0(t)$ until we see the first point in $\Pi_{R_0 \triangle \alpha}$. This motivates us to define the following two processes $R^-_0(t)$ and $R^+_0(t)$ as follows: For all $t_0 \leq t < t_0$, we set $R^-_0(t) = R^+_0(t) = \alpha$. For $t \geq t_0$, let

$$R^-_0(t) := \int_{t_0}^{t} K_\alpha(t-x) d\Pi_{R^-_0-(R_0-\alpha)_-}(x);$$

$$R^+_0(t) := \int_{t_0}^{t} K_\alpha(t-x) d\Pi_{R^+_0+(R_0-\alpha)_+}(x),$$

where $(s)_+ := s \vee 0$ and $(s)_- := (-s) \vee 0$. Later in Proposition 5.7, we will see that $R^-_0$ (resp. $R^+_0$) lower (resp. upper) bounds $R(t)$.

To state the main result of this section, let

$$\hat{R}^-_0(t) := (R^+_0(t) + |R_0(t) - \alpha|) \vee \alpha.\quad (5.4)$$

Recalling the definition of $\sigma_1(t;g)$ (4.2), define

$$\tau_{B1} = \tau_{B1}(\alpha,t_0) := \inf \{ t \geq t_0 : (R(t) - \alpha) \notin (-\alpha^{\frac{3}{2}} \beta^{5\theta}, \alpha^{\beta^{5\theta}} \sigma_1(t;\hat{R}^-_0(t)) + \alpha^{\frac{3}{2}} \beta^{5\theta}) \};$$

$$\tau_{B2} = \tau_{B2}(\alpha,t_0) := \inf \{ t \geq t_0 : \Pi_{\hat{R}^-_0}[(t - \alpha^{-1}) \vee t_0, t] \geq \beta^{C_0} \};$$

$$\tau_B = \tau_B(\alpha,t_0) := \tau_{B1} \wedge \tau_{B2}.\quad (5.5)$$

Then, we have the following result for $R(t)$.

**Theorem 5.1.** Under the above setting, for all sufficiently small $\alpha > 0$, we have

$$\mathbb{P}(\tau_B(\alpha,t_0) > t_0) \geq 1 - \exp(-\beta^2).$$

The theorem will be essential later, serving as the induction base of our inductive argument in settling the regularity (Section 6; Theorem 6.6). Establishing the theorem consists of the four main steps as follows.

- **Section 5.1**: We develop further theory on representing the branching process using martingales. Not only will this be helpful in studying $\tau_{B1}$, but also serve as a fundamental tool in the later sections.
- **Section 5.2**: We study the critical branching process that starts from a single initial particle. The analysis provides a useful tool to understand the gap between $R_0(t)$ and $R(t)$.
- **Section 5.3**: We investigate the processes $R_0(t)$, $R^-_0(t)$ and $R^+_0(t)$, exploiting the martingale concentration lemmas from Section 4.1.
- **Section 5.4**: Combining the above analysis, we deduce the bound on $\tau_{B1}$ and $\tau_{B2}$.

Before moving on, we record a simple lemma that will be used throughout the rest of the paper.

**Lemma 5.2.** Let $g : [0,h] \rightarrow (0,\infty)$ be a positive function which defines a (deterministic) set of points $\Pi_g[0,h]$, and set $n = |\Pi_g[0,h]|$. Recalling the definition $\pi_1(t;g)$ (4.1), there exists an absolute
constant $C > 0$ such that
\[
\int_0^h \frac{dt}{\pi_1(t; g) + 1} \leq (n + 1) \log(h + 1);
\]

\[
\int_0^h \frac{dt}{\sqrt{\pi_1(t; g) + 1}} \leq C\sqrt{(n + 1)h}.
\]

Proof. Let $\Pi_g[0, h] := \{p_1 < p_2 < \ldots < p_n\}$ with $p_0 = 0, p_{n+1} = h$. For the first one, we have
\[
\int_0^h \frac{dt}{\pi_1(t; g) + 1} = \sum_{i=1}^{n+1} \int_0^{p_i - p_{i-1}} \frac{dx}{x + 1} \leq (n + 1) \log(h + 1).
\]

The second one follows by
\[
\int_0^h \frac{dt}{\sqrt{\pi_1(t; g) + 1}} = \sum_{i=1}^{n+1} \int_0^{p_i - p_{i-1}} \frac{dx}{\sqrt{x + 1}} \leq \sum_{i=1}^{n+1} 2\sqrt{p_i - p_{i-1}} \leq 2\sqrt{(n + 1)h},
\]
where we used Cauchy-Schwarz inequality to obtain the last inequality. \hfill \square

5.1. **Connection to martingales.** Recall the definition $\mathcal{R}_b(t; \Pi_0[t_0, t], \alpha)$ (5.1). For a given point process $\Pi_Q$ with respect to $Q(t) > 0$, we introduce a similar notation given by
\[
\mathcal{R}_c(s, t; \Pi_Q[t_0, s], \alpha) := \int_{t_0}^s K_{\alpha_0}(t - x) d\Pi_Q(x) + \int_s^t K_{\alpha}^*(t - u) K_{\alpha_0}(u - x) dud\Pi_Q(x). \tag{5.6}
\]

In words, it is a conditional expectation of the rate at time $t$ of the critical branching process starting from the points $\Pi_Q[t_0, s]$, conditioned on $\Pi_Q[t_0, s]$. In particular, $\mathcal{R}_c(t, t; \Pi_Q[t_0, t], \alpha) = \mathcal{R}_b(t; \Pi_Q[t_0, t], \alpha)$.

Abbreviating the notation by $Q_1(s, t) := \mathcal{R}_c(s, t; \Pi_Q[t_0, s], \alpha)$, the main advantage of studying this process comes from the following observation, which is an analogue of (2.26):
\[
\frac{\partial}{\partial s} Q_1(s, t) = K_{\alpha}(t - s) + \int_s^t K_{\alpha}^*(t - u) K_{\alpha}(u - s) du \left( \frac{d\Pi_Q(s)}{ds} - Q_1(s, s) \right) = K_{\alpha}^*(t - s) \left( \frac{d\Pi_Q(s)}{ds} - Q_1(s, s) \right),
\]
where the second line comes from the fact that $K_{\alpha_0}^* = K_{\alpha_0} + K_{\alpha_0}^* K_{\alpha_0}^*$. This gives
\[
Q_1(s_1, t) - Q_1(s_0, t) = \int_{s_0}^{s_1} K_{\alpha}^*(t - s) \left\{ d\Pi_Q(s) - Q_1(s, s) ds \right\}
\]
\[
= \int_{s_0}^{s_1} K_{\alpha}^*(t - s) \left\{ d\Pi_Q(s) + [Q(s) - Q_1(s, s)] ds \right\},
\]
\[
(5.7)
\]
where we wrote $d\Pi_Q(s) := d\Pi_Q(s) - Q(s) ds$. This decomposes the identity to the “martingale part” and the “drift part.”

In the definition of $\mathcal{R}_c$, the case $t = \infty$ will play important roles, and we specify this case with another notation:
\[
\mathcal{L}(t; \Pi_Q[t_0, t], \alpha) := \int_{t_0}^{t} \int_t^{\infty} K_{\alpha}^* \cdot K_{\alpha}(x - s) dxd\Pi_Q(s). \tag{5.8}
\]
where \( K_\alpha^* = \lim_{t \to \infty} K_\alpha^*(t) = \frac{2\alpha^2}{1+2\alpha} \) (Lemma 2.15). Note that the definition is an analogue of (2.25).

5.2. The critical branching from a single particle. In this subsection, we study the age-dependent critical branching process that starts from a single initial particle. The results in this subsection will be useful tools in the next subsection.

We let \( \{r(t)\}_{t \geq 0} \) denote the rate of the critical branching process starting from a single point at the origin \((t = 0)\). That is, we define \( r(t) \) by

\[
r(t) = r(t, \alpha) := K_\alpha(t) + \int_0^t K_\alpha(t-x)d\Pi_r(x).
\]

Also, we set \( \hat{\theta} := 2\alpha^{-2}\beta^0 \). Recall the definitions of \( \pi_i(t; g) \) (4.1) and \( \sigma_i(t; g) \) (4.2), and consider the stopping times

\[
\begin{align*}
\tau_{SB1} &:= \inf \left\{ t \geq 0 : |\Pi_r[0, t]| \geq \beta^{\frac{5}{2}\theta} \right\} ; \\
\tau_{SB2} &:= \inf \left\{ t \geq 0 : r(t) \geq \alpha^2 \sigma_1(t; r) + \alpha^2 \right\} ; \\
\tau_{SB} &:= \tau_{SB1} \land \tau_{SB2}.
\end{align*}
\]

Then, our main estimate can be described as follows.

**Proposition 5.3.** Under the above setting, we have

\[
\mathbb{P} \left( \tau_{SB} \leq \hat{\theta} \right) \leq \exp \left( -\beta^4 \right).
\]

Furthermore, we will deduce a bound on the probability that \( \Pi_r \) contains more than certain number of points in a short interval. For instance, for any \( t \in [k\alpha^{-\frac{3}{2}}, (k+1)\alpha^{-\frac{3}{2}}] \) with \( k \in \mathbb{N} \), we can write

\[
\mathbb{P} \left( |\Pi_r[t - \alpha^{-\frac{3}{2}}, t]| \geq 1 \right) \leq \mathbb{E} |\Pi_r[t - \alpha^{-\frac{3}{2}}, t]| = \int_{t - \alpha^{-\frac{3}{2}}}^t K_\alpha^*(x)dx \leq C \left\{ k^{-\frac{1}{2}}\alpha^\frac{1}{2} \lor \alpha^\frac{3}{2} \right\},
\]

for some constant \( C > 0 \). Note that the last inequality is from the estimate on \( K_\alpha^*(x) \) (Lemma 2.15). Our goal is to deduce a similar inequality as follows.

**Proposition 5.4.** Under the above setting, for all \( \alpha^{-\frac{3}{2}} \leq t \leq \hat{\theta} \) we have

\[
\mathbb{P} \left( |\Pi_r[(t - \alpha^{-\frac{3}{2}}) \land \tau_{SB}, t \land \tau_{SB}]| \geq 3 \right) \leq \alpha^\frac{3}{2}\beta^{10\theta}.
\]

The reason why we are interested in the intervals of length \( \alpha^{-3/2} \) will become more clear in the next subsection. To explain it briefly, we interpret \( R(t) \) as a perturbation of the rate-\( \alpha \) process, and it turns out that \( \alpha^{-3/2} \) is the length of an interval where we start to see particles between \( R(t) \) and \( \alpha \). The proof of Proposition 5.4 follows as a consequence of Proposition 5.3 and we discuss it at the end of this subsection.

Now we establish Proposition 5.3, beginning with the control on \( \tau_{SB1} \).

**Lemma 5.5.** For any sufficiently small \( \alpha > 0 \), we have \( \mathbb{P} \left( \tau_{SB1} \leq \hat{\theta} \right) = \exp \left( -\beta^\frac{1}{30\theta} \right) \). In other words,

\[
\mathbb{P} \left( |\Pi_r[0, \hat{\theta}]| \geq \beta^{\frac{5}{2}\theta} \right) \leq \exp \left( -\beta^\frac{1}{30\theta} \right).
\]

To establish Lemma 5.5 we start with verifying a similar property for the critical Galton-Watson branching process with Poisson offspring.

**Lemma 5.6.** Let \( Z_n \) be a critical branching process with offspring distribution \( \text{Poisson}(1) \) and \( Z_0 = 1 \). Then

\[
\mathbb{P} \left( \sum_{k=1}^n Z_k \geq n^3 \right) \leq e^{-cn}
\]
Proof. Let $\varphi_n$ be the probability generating function of $X_n$. For any $u > 0$ we have

$$\varphi_n(u) = \mathbb{E} u^{Z_n} = \mathbb{E} \left[ u^{Z_n} | Z_{n-1} \right] = \mathbb{E} \left[ \left( \mathbb{E} u^{X_1} \right)^{X_{n-1}} \right] = \varphi_{n-1}(e^{u-1}) = \psi^{(n)}(u),$$

where $\psi(u) := e^{u-1}$ and where $\psi^{(n)}(u)$ is the $n$-fold composition of $\psi$ with itself. Let $u := 1 + 1/5n$. We prove using induction on $k \leq n$ that $\varphi_k(u) \leq 1 + \frac{1}{5n} + \frac{k}{16n^2}$. In fact,

$$\varphi_{k+1}(u) = \exp(\varphi_k(u) - 1) \leq \varphi_k(u) + (\varphi_k(u) - 1)^2 \leq 1 + \frac{1}{5n} + \frac{k + 1}{16n^2}$$

where in the last inequality we used that $\varphi_k(u) \leq 1 + 1/4n$ which follows from the induction hypothesis when $k \leq n$.

Finally, using Markov’s inequality we get for any $k \leq n$

$$\mathbb{P}(Z_k \geq u^2) = \mathbb{P} \left( u^{Z_k} \geq u^{n^2} \right) \leq \varphi_k(u) \cdot u^{-n^2} \leq e^{-cn}.$$

The result now follows from a union bound. □

Proof of Lemma 5.5 Let $X_i$ be i.i.d with density $K_\alpha$. We have $\mathbb{P}(X_1 \leq \alpha^{-2}) \leq 1 - c$ for some $c > 0$. Thus

$$\mathbb{P} \left( \sum_{k=1}^{\lceil \frac{n}{10} \rceil} X_k \leq \hat{h} \right) \leq \mathbb{P} \left( \sum_{k=1}^{\lceil \frac{n}{10} \rceil} 1_{(X_k \leq \alpha^{-2})} \geq \left( 1 - \frac{c}{2} \right) \beta^{10} \right) \leq \exp \left( -\beta \frac{1}{100} \right),$$

(5.11)

where in the last inequality we used Azuma. Now, if there are more than $\beta \frac{n}{10} \theta$ particles in the branching before time $\hat{h}$ then either there are more than $\beta^{40}$ particles up to the $\beta^{10} \theta$ generation or there are less particles than that up to the $\beta^{10} \theta$ generation but one of the particles in this generation came before time $\hat{h}$. The first event happens with very small probability by Lemma 5.6 and the second one happens with small probability by union bound and (5.11). □

We conclude the proof of Proposition 5.3.

Proof of Proposition 5.3. For all $t \leq \tau_{SB1}$, observe that

$$r(t) = K_\alpha(t) + \int_{0^+}^t K_\alpha(t-x) d\Pi_r(x)$$

$$\leq K_\alpha(\tau_1(t,r) \wedge \alpha^{-2} \beta^{40}) \cdot |\Pi_r[0,t]| \leq \alpha^{3/2} \sigma_1(t,r) + \alpha^2,$$

where the last inequality follows from the estimate on $K_\alpha(x)$ (Lemma 2.14). Thus, we obtain the desired result by combining with Lemma 5.5. □

We conclude this subsection by verifying Proposition 5.4.

Proof of Proposition 5.4. For any $\alpha^{-3/2} \leq t \leq \hat{h}$, denote $s_1 := t - \alpha^{-3/2} \wedge \tau_{SB}$ and $s_2 := t \wedge \tau_{SB}$. First, based on the definition of $\tau_{SB2}$, applying Lemma 5.2 with $n = \beta \frac{n}{10} \theta$ gives that

$$\int_{s_1}^{s_2} r(x) dx \leq \alpha^{3/2} \cdot \alpha^{-3/4} \sqrt{n} \leq \alpha^2 \beta^{6\theta}.$$

Thus, we can write

$$\mathbb{P} \left( |\Pi_r[s_1, s_2]| \geq 3 \right) \leq \int_{s_1 \leq x < y < z \leq s_2} r(x) r(y) r(z) dz dy dx \leq \alpha^2 \beta^{18\theta},$$

noting that any addition of three extra points in $\Pi_r[s_1, s_2]$ does not change the bound given in the definition of $\tau_{SB2}$. Thus, we conclude the proof by combining the above with Proposition 5.3. □
5.3. Understanding the fixed rate process and its perturbations. In this subsection, we study the processes $R_0(t), R_0^-(t),$ and $R_0^+(t)$ introduced in (5.2), (5.3). We begin with observing that $R_0^-$ (resp. $R_0^+$) lower (resp. upper) bounds both $R_0$ and $\tilde{R}$.

**Proposition 5.7.** Under the above setting, $R_0^-(t) \leq R(t), R_0^+(t) \leq R_0(t)$ for all $t \geq t_0$.

*Proof.* Observe first that at $t = t_0$, $R^+(t_0) = R^-(t_0) = R(t_0) = R_0(t_0)$, and for $t \in [t_0, t_0)$ they are all identical to $\alpha$. Recalling the definition of $R(t)$, which can be written as

$$R(t) = \int_{t_0}^{t} K_\alpha(t-x)d\Pi_\alpha(x) + \int_{t_0}^{t} K_\alpha(t-x)d\Pi_R(x),$$

we observe that $R_0^-(t) \leq R(t) \leq R_0^+(t)$: They maintain the same value until time $t_0$, and then after that $R_0^-(t)$ picks up more points than $R(t)$, since it absorbs additional points from $(R_0 - \alpha)_+$ in addition to all the points taken by $R(t)$. The same reasoning for opposite direction works for $R_0^+(t)$.

To see that the conclusion holds for $R_0$, we observe that $R_0^+(t) \geq R_0(t)$ for $t \geq t_0$ inductively: Initially, we have $R_0^+(t_0) = R_0(t_0)$. Furthermore, as long as we have $R_0^+(s) \geq R_0(s)$ for all $s < t$, it holds that

$$R_0^+(t) + (R_0(t) - \alpha)_+ \geq R_0(t) + (\alpha - R_0(t)) \geq \alpha,$$

meaning that the new points picked up by $R_0(t)$ will also be included in $R_0^+(t)$. Thus, it will continue to hold that $R_0^+(t) \geq R_0(t)$. The other inequality, $R_0^+ \leq R_0$, can be obtained analogously. \(\square\)

We move on to the study of $R_0(t)$. Define the notation

$$\Pi_{R_0 + \Delta}[[x_0, x_1]] := \Pi_{R_0}[x_0, x_1] \triangle \Pi_{\alpha}[x_0, x_1],$$

that is, the collection of points lying between $R_0$ and $\alpha$. Then, define

$$\tau_1 := \inf \{t \geq t_0 : (R_0(t) - \alpha) \notin (-\alpha^{\frac{3}{2}}\beta^{C_6}, \alpha^{\frac{3}{2}}\beta^{C_6})\};$$

$$\tau_2 := \inf \{t \geq t_0 : \Pi_{\alpha}[(t - \alpha^{-1}) \vee t_0, t] \geq \beta^{s_1}\};$$

$$\tau_3 := \inf \{t \geq t_0 : \Pi_{R_0 + \Delta}[(t - \alpha^{-\frac{3}{2}}) \vee t_0, t] \geq \beta^{2C_6}\};$$

$$\tau_4 := \inf \{t \geq t_0 : R_0(t) \geq \alpha\beta^{C_6}\};$$

$$\tau_t := \tau_1 \wedge \tau_2 \wedge \tau_3 \wedge \tau_4.$$  \hspace{1cm} (5.12)

The goal of this subsection is establishing the following lemma.

**Lemma 5.8.** We have

$$\mathbb{P}(\tau \leq \hat{t}_0) \leq \exp(-\beta^5).$$

*Proof.* By an elementary estimate on Poisson processes, we have

$$\mathbb{P}(\tau_2 \leq \hat{t}_0) \leq \exp(-\beta^6).$$  \hspace{1cm} (5.13)

To obtain appropriate control on $\tau_1$, we apply Corollary 4.9 to $R_0(t)$. Recalling the estimate Lemma 2.14 on $K_\alpha$, we may set $M = \alpha^{\frac{3}{2}}\beta^2$. Writing $d\Pi_\alpha(x) := d\Pi_{\alpha}(x) - \alpha dx$, this gives that

$$\mathbb{P}\left(\int_{t_0}^{t \wedge \tau_2} K_\alpha(t - x)d\Pi_\alpha(x) \leq \alpha^{\frac{3}{2}}\beta^{C_6}(t; \alpha) + \frac{1}{2} \alpha^{\frac{3}{2}}\beta^{C_6}, \forall t \in [t_0, \hat{t}_0]\right) \geq 1 - \exp(-\beta^6);$$

$$\mathbb{P}\left(\int_{t_0}^{t \wedge \tau_2} K_\alpha(t - x)d\Pi_\alpha(x) \geq -\frac{1}{2} \alpha^{\frac{3}{2}}\beta^{C_6}, \forall t \in [t_0, \hat{t}_0]\right) \geq 1 - \exp(-\beta^6).$$  \hspace{1cm} (5.14)
Moreover, we have for all \( t \in [t_0, \hat{t}_1] \) that

\[
R_0(t) - \alpha = \int_{t_0}^{t} K_\alpha(t - x)d\Pi_\alpha(x) + \int_{t_0}^{t} K_\alpha(t - x)dx - \alpha
\]

\[
= \int_{t_0}^{t} K_\alpha(t - x)d\Pi_\alpha(x) + O(\alpha^{100}),
\]

where the last identity follows again from the estimate of \( K_\alpha(s) \) (Lemma 2.14) applied to \( s \geq t_0 - \hat{t}_0 \). Thus, combining this with (5.13) and (5.14), we obtain that

\[
\mathbb{P}(\tau_{\Pi} \leq \hat{t}_0) \leq 2 \exp\left(-\beta^6\right).
\]

Moving on to \( \tau_3 \), write \( \tilde{\tau}_1 := \tau_{\Pi} \wedge \tau_2 \), and observe that from applying Lemma 5.2 with \( n = \alpha^{-\frac{3}{2}}\beta^{C_0} \), we get

\[
\int_{(t-\alpha^{-\frac{3}{2}})\wedge \tilde{\tau}_1}^{t \wedge \tilde{\tau}_1} |R_0(t) - \alpha| \leq 2\beta_0^\frac{3}{2}C_0+1.
\]

Thus, applying Corollary 4.7 to \( \Pi_{R_0 \Delta \alpha} \) gives that

\[
\mathbb{P}(\tau_{\Pi} \leq \hat{t}_0 \wedge \tilde{\tau}_1) \leq \exp\left(-\beta^6\right).
\]

Lastly, we note that \( \tau_{\Pi} \geq \tau_1 \) deterministically from their definitions. Thus, along with (5.13), (5.15), and (5.17), we obtain the desired conclusion.

5.4. **Proximity to \( \alpha \) for small \( t \).** In this subsection, we conclude the proof of Theorem 5.1 combining the results from the previous subsections. We begin with investigating \( \tau_{B1} \). Thanks to Proposition 5.7 we can write

\[
|R(t) - \alpha| \leq |R_0(t) - \alpha| + \{R_0^\ast(t) - R_0(t)\}.
\]

Since the bound on \( |R_0(t) - \alpha| \) has already been given by \( \tau_{11} \) in Lemma 5.8 we focus on the term \( \{R_0^\ast(t) - R_0(t)\} \).

By the definition of \( R_0^\ast \) and \( R_0 \) (5.3), the process \( \delta R_0(t) := R_0^\ast(t) - R_0(t) \) satisfies

\[
\delta R_0(t) = \int_{t_0}^{t} K_\alpha(t - x)d\Pi_{\{R_0^\ast + (R_0 - \alpha), \Delta (R_0 - (R_0 - \alpha)_-)(x)\}},
\]

for all \( t \geq t_0 \), starting with \( \delta R_0(t_0) = 0 \). To lighten up our notation, we symbolically define

\[
\Delta R_0 := (R_0^\ast + (R_0 - \alpha), \Delta (R_0 - (R_0 - \alpha)_-)).
\]

**Remark 5.9.** In (5.19), we can interpret \( \delta R_0(t) \) in the following way: \( \delta R_0 \) starts as an empty process at time \( t_0 \). As time goes on, new particles are added to \( \delta R_0 \) at rate \( |R_0 - \alpha| \). We call these particles the external additions. Then, each particle from external addition performs the critical branching process.

To study the process \( \delta R_0(t) \), we introduce the following stopping time.

\[
\tau_{B1} = \tau_{B1}(\alpha, t_0) := \inf \left\{ t \geq t_0 : \left| \Pi_{\Delta R_0} \left[ (t - \alpha^{-\frac{3}{2}}) \vee t_0, t \right] \right| \geq \beta^{|t|}\right\}.
\]

Our first goal is to show the following lemma on \( \tau_{B1} \).

**Lemma 5.10.** Under the above setting, we have

\[
\mathbb{P}(\tau_{B1} \leq \hat{t}_0) \leq \exp\left(-\beta^3\right).
\]
Proof. We interpret $\delta R(t)$ as mentioned in Remark 5.9 recalling the notion of external additions. For each $j \in \mathbb{N}$, $0 \leq j \leq 2\alpha^{-\frac{3}{2}}\beta^\theta$, we write $\{p_{j,1}, \ldots, p_{j,n_j}\}$ to be the external additions at $\delta R_0(t)$ during time $[j\alpha^{-\frac{3}{2}}, (j+1)\alpha^{-\frac{3}{2}}]$. For each point $p_{j,l}$ and the critical branching process $r_{j,l}(t)$ that starts from a single point at $p_{j,l}$, we define the stopping time $\tau_{SB}(j,l)$ analogously as (5.9):

$$
\tau_{SB1}(j,l) := \inf \left\{ t \geq p_{j,l} : \Pi_{r_{j,l}}[0,t] \geq \beta^\frac{3}{2} \right\};
$$

$$
\tau_{SB2}(j,l) := \inf \left\{ t \geq p_{j,l} : r_{j,l} \geq \alpha \beta^{4\theta} \sigma_1(t;r_{j,l}) + \alpha^2 \right\};
$$

$$
\tau_{SB}(j,l) := \tau_{SB1}(j,l) \wedge \tau_{SB2}(j,l);
$$

and the critical branching process

$$
\tau := \tau \wedge \bigwedge_{j=0}^n N_{SB}(j,l).
$$

(Recall the definition of $\tau_i$ in (5.12)) Moreover, for $k, j, l \in \mathbb{N}$ with $0 \leq k, j \leq 2\alpha^{-\frac{3}{2}}\beta^\theta$ and $1 \leq l \leq n_j$, we define

$$
N^{(k)}_{j,l} := \text{the number of points in } \Pi_{\Delta R_0}(k \alpha^{-\frac{3}{2}} \wedge \tau_{SB}, (k+1)\alpha^{-\frac{3}{2}} \wedge \tau_{SB})
$$

that are descendants of $p_{j,l}$.

Note that for each $k$, $\{N^{(k)}_{j,l}\}_{j,l}$ forms a collection of independent random variables. Furthermore, due to the definition of $\tau_3$, we can consider $l \leq \beta^{2C_0}$ only.

For each $k, j, l$ as above, define $\bar{N}^{(k)}_{j,l}$ to be the random variable as follows:

- $\bar{N}^{(k)}_{j,l} \equiv 0$ for any $j > k$.
- Otherwise, we define

$$
\bar{N}^{(k)}_{j,l} = \begin{cases} 
2, & \text{with probability } C \left\{ (k - j + 1)^{-\frac{1}{2}} \alpha^{\frac{1}{2}} \lor \alpha^{\frac{1}{2}} \right\}; \\
\beta^\frac{3}{2}\theta, & \text{with probability } \alpha^2 \beta^{19\theta}; \\
0, & \text{otherwise.}
\end{cases}
$$

- For each $k$, $\{\bar{N}^{(k)}_{j,l}\}_{j,l}$ is a collection of independent random variables.

Then, (5.10) and Proposition 5.4 tell us that $\{\bar{N}^{(k)}_{j,l}\}_{j,l}$ stochastically dominates $\{N^{(k)}_{j,l}\}_{j,l}$. Now, we estimate

$$
\left| \Pi_{\Delta R_0}(k \alpha^{-\frac{3}{2}} \wedge \tau_{SB}, (k+1)\alpha^{-\frac{3}{2}} \wedge \tau_{SB}) \right| \leq \sum_{j,k} \sum_{l=1}^{n_j} \bar{N}^{(k)}_{j,l}.
$$

First, we note that each $\bar{N}^{(k)}_{j,l}$ is bounded by $\beta^\frac{3}{2}\theta$ and also observe that

$$
\sum_j \mathbb{E}\left[\bar{N}^{(k)}_{j,l}\right] \leq 2\alpha^\frac{1}{2}\beta^\theta \beta^{2C_0} \sum_j \sum_{l=1}^{n_j} \mathbb{E} \left\{ (j - \frac{1}{2})^\frac{1}{2} \alpha^\frac{1}{2} \lor \alpha^\frac{1}{2} \right\} \leq \beta^{2\theta}
$$

and

$$
\sum_j \mathbb{E}\left[\left(\bar{N}^{(k)}_{j,l}\right)^2\right] \leq 2\alpha^\frac{1}{2}\beta^\theta \beta^{2C_0} \sum_j \sum_{l=1}^{n_j} \mathbb{E} \left\{ j^{-\frac{1}{2}} \alpha^{\frac{1}{2}} \lor \alpha^{\frac{1}{2}} \right\} \leq \beta^{2\theta}.
$$

Thus, Bernstein’s inequality (Lemma 5.11 below) tells us that for each $k$,

$$
\mathbb{P}\left(\left| \Pi_{\Delta R_0}(k \alpha^{-\frac{3}{2}} \wedge \tau_{SB}, (k+1)\alpha^{-\frac{3}{2}} \wedge \tau_{SB}) \right| \geq \beta^{4\theta} \right) \leq \exp \left( -\beta^{\theta/3} \right).
$$

Thus, we can conclude the proof by taking a union bound over $k$, and by noting that

$$
\mathbb{P}\left( \tau_{SB} > t_0 \right) \geq 1 - \exp \left( -\beta^3 \right),
$$
Recall the decomposition (5.18), which can be rewritten as

\[ \text{Proof of Theorem 5.1.} \]

Noting that \( \delta R \) which comes from Proposition 5.3 and Lemma 5.8 followed by a union bound.

The Bernstein’s inequality used in the above proof can be stated as below.

**Lemma 5.11 (Theorem 3.6, [2]).** Let \( X_1, \ldots, X_n \) be independent random variables satisfying \( |X_i| \leq M \) almost surely for all \( i \). Then, we have

\[
\mathbb{P}\left( \sum_{i=1}^{n} (X_i - \mathbb{E}X_i) \geq x \right) \leq \exp\left( -\frac{x^2}{2 \sum_{i=1}^{n} \mathbb{E}[X_i^2] + \frac{2}{3} Mx} \right).
\]

The following corollary translates Lemma 5.10 into the form that we can apply to our analysis.

**Corollary 5.12.** Recall the definition of \( \hat{\Delta}R_0(t) \) (5.20), and define the stopping times

\[
\tau_{\delta B2} = \tau_{\delta B2} := \inf \left\{ t \geq t_0 : \delta R_0(t) \leq \alpha \beta^{4\theta+1} \sigma_1(t; \hat{\Delta}R_0) + \alpha \beta^{4\theta+1} \right\};
\]

\[
\tau_{\delta B3} = \tau_{\delta B3} := \inf \left\{ t \geq t_0 : \delta R_0(t) \leq 2 \alpha \beta^{4\theta+1} \right\}.
\]

Then, we have

\[
\mathbb{P}\left( \tau_{\delta B3} > \tilde{t}_0 \right) \geq \mathbb{P}\left( \tau_{\delta B2} > \tilde{t}_0 \right) \geq 1 - 2 \exp\left( -\beta^3 \right).
\]

**Proof.** The first inequality is immediate from definition. To establish the second inequality, recall the definition of \( \tau_{\delta 3} \) from (5.12). For \( t \leq \tau_{\delta 3} \land \tau_{\delta B1} \), note that

\[
\delta R_0(t) = \int_{t_0}^{t} K_{\alpha_0}(t - x) d\Pi_{\hat{\Delta}R_0}(x)
\]

\[
\leq \left| \Pi_{\hat{\Delta}R_0}\left[ t - \alpha^{-\frac{3}{2}}, t \right] \right| \cdot C\sigma_1(t; \hat{\Delta}R_0) + \sum_{k \geq 1} \left| \Pi_{\hat{\Delta}R_0}\left[ t - (k + 1)\alpha^{-\frac{3}{2}}, t - k\alpha^{-\frac{3}{2}} \right] \right| \cdot \frac{C \alpha \sigma_{\delta k}^2}{\sqrt{k\alpha^{-2}}} \]

\[
\leq \alpha \beta^{4\theta+1} \sigma_1(t; \hat{\Delta}R_0) + \alpha \beta^{4\theta+1}.
\]

Thus, the proof follows from Lemmas 5.8 and 5.10.

We now are ready to establish Theorem 5.1.

**Proof of Theorem 5.1.** Recall the decomposition (5.18), which can be rewritten as

\[
R_0^-(t) - \alpha \leq R(t) - \alpha \leq (R_0(t) - \alpha) + \delta R_0(t).
\]

From the upper bound, the results on \( \tau_{\delta 3} \) (Lemma 5.8) and \( \tau_{\delta B2} \) give the control

\[
R(t) - \alpha \leq \alpha \beta^{5\theta} \sigma_1(t; \hat{\Delta}R_0^+) + \alpha \beta^{5\theta}.
\]

For the lower bound, we study \( R_0^-(t) \) instead. Let \( t_0^k \) be another parameter such that \( t_0^k := \frac{1}{2}(t_0 + t_0) \). In particular, it satisfies

\[
t_0 - t_0^k = t_0^k - t_0 \geq \frac{1}{4} \alpha^{-2} \beta^3.
\]

Noting that \( R_0^-(t) = R_0(t) = \alpha \) for \( t < t_0 \), applying (5.7) to \( R_0^-(t) \) gives the following:

\[
R_0^-(t) - \mathcal{R}_\alpha(t_0^k - t_0^k, t; \Pi_{\alpha}[t_0^k, t_0^k], \alpha) = \int_{t_0^k}^{t} K_{\alpha}^+(t - s) \left\{ \Pi_{\hat{\Delta}R_0^-}((R_0^- - \alpha)^-) \right\} (s) \cdot (R_0(s) - \alpha) ds,
\]

(5.22)
The first integral can be estimated using Corollary 4.9. Since \( R_0(t) - (R_0(t) - \alpha)_- \leq R_0(t) \) for \( t \geq t_0 \), the definition of \( \tau_1 \) and Lemma 2.15 imply that

\[
\int_{t_0}^{t_0 + \tau_1} (K_\alpha^*(t - s))^2 c_\alpha ds + \int_{t_0}^{t \wedge \tau_1} (K_\alpha^*(t - s))^2 R_0(s)ds \\
\leq \int_{t_0}^{t} C \left( \frac{\alpha^2}{t - s + 1} \right) \alpha \beta C \alpha ds = \alpha^3 \beta \theta^2 C \alpha,
\]

for all \( t \leq t_0 \). Thus, Corollary 4.9 and Lemma 5.8 tell us that

\[
P \left( \int_{t_0}^{t} K_\alpha^*(t - s) \overline{R}_0(R_0(s) - \alpha) ds \geq -\alpha^2 \beta^3, \forall t \in [t_0, t_1] \right) \geq 1 - \exp(-\beta^5). \tag{5.23}
\]

On the other hand, for \( t \leq \tau_1 \) we can write (note that when \( s \leq t_0 \), \( R_0(s) - \alpha = 0 \))

\[
\left| \int_{t_0}^{t} K_\alpha^*(t - s)(R_0(s) - \alpha)ds \right| \leq \int_{t_0}^{t} C \left( \frac{\alpha}{\sqrt{t - s + 1}} \right) \alpha \beta C \alpha \sigma_1(s; \alpha) + \alpha^2 \beta \theta C \alpha ds. \tag{5.24}
\]

Then, the RHS can be bounded using Lemmas 5.2 and 5.13 below, by setting the parameters in Lemma 5.13 as \( \Delta_0 = \Delta_1 = \alpha^{-1}, K = \alpha^{-1} \beta^3 \), and \( N = \beta^6 \). Whenever there is an empty interval of length \( \Delta_1 \), we can add a point to justify the choice of \( \Delta_1 \) which will only increase the value of the integral. This implies that the RHS is smaller than \( \alpha^2 \beta^2 C \alpha \).

Lastly, we need to understand difference between the term \( \mathcal{R}_e(t^b_0, t; \Pi_\alpha [t_0^b, t] \Pi_\alpha, \alpha) \) and \( \alpha \). Recalling the definition of \( \mathcal{L} \) \( [5.8] \), we first observe that

\[
\left| \mathcal{R}_e(t^b_0, t; \Pi_\alpha [t_0^b, t] \Pi_\alpha, \alpha) - \mathcal{L}(t^b_0; \Pi_\alpha [t_0^b, t] \Pi_\alpha, \alpha) \right| \\
\leq \int_{t_0}^{t} K_\alpha(t - x) d\Pi_\alpha(x) + \int_{t_0}^{t} \int_{t}^{\infty} K_\alpha^* \cdot K_\alpha(u - x) du d\Pi_\alpha(x) \\
+ \int_{t_0}^{t} \int_{t_0}^{t} \left| K_\alpha^* - K_\alpha^*(t - u) \right| \cdot K_\alpha(u - x) du d\Pi_\alpha(x). \tag{5.25}
\]

For any \( t \in [t_0, t_2] \), we see that the RHS is at most \( \alpha^{100} \) since \( t - t^b_0 \geq \alpha^{2}\beta^{12} \). The first two integrals are small due to the decay of \( K_\alpha \) (Lemma 2.14), and in the last integral, either \( |K_\alpha^* - K_\alpha^*(t - u)| \) or \( K_\alpha(u - x) \) is small depending on the size of \( u \) (see Lemma 2.15). Moreover, letting \( F(x) = \int_{x}^{\infty} K_\alpha(y)dy \), we also have

\[
\mathcal{L}(t^b_0; \Pi_\alpha [t_0^b, t] \Pi_\alpha, \alpha) = K_\alpha^{*} \int_{t_0}^{t} F(t^b_0 - x) d\Pi_\alpha(x); \tag{5.26}
\]

\[
= K_\alpha^{*} \int_{t_0}^{t} F(t^b_0 - x) d\Pi_\alpha(x) + K_\alpha^{*} \int_{t_0}^{t} F(t^b_0 - x) x dx.
\]
The first term in the second line can be bounded by Lemma 4.5 using $|F(x)| \leq 1$, and the last term is dealt with integration by parts which is

$$
\int_{t_0}^{t_0} F(x)dx = \int_{0}^{\infty} F(x)dx + O(\alpha^{50}) = \int_{0}^{\infty} xK_\alpha(x)dx + O(\alpha^{50}) = (K_\alpha^*)^{-1} + O(\alpha^{50}).
$$

(5.27)

Thus, we get

$$
|\mathcal{L}(t_0^\dagger; \Pi_\alpha[t_0^\dagger, t_0], \alpha) - \alpha| \leq \alpha^{\frac{3}{2}} \beta^C,
$$

(5.28)

with probability at least $1 - e^{-\beta^3}$.

Therefore, combining (5.22), (5.23), (5.24), (5.25) and (5.27), we obtain

$$
\mathbb{P} \left( \tau_{B1} > t_0 \right) \geq 1 - 3 \exp \left( -\beta^3 \right).
$$

(5.29)

The remaining task is to control $\tau_{B2}$. First, for $t \leq \tau_1 \wedge \tau_{B3}$, observe that

$$
\hat{R}_0^+(t) := R_0^+(t) + |R_0(t) - \alpha| \leq R_0(t) + \delta R_0(t) + |R_0(t) - \alpha| \\
\leq \alpha + 2|R_0(t) - \alpha| + \delta R_0(t) \leq \alpha^C + 1 + 2\alpha^4 + 1 \leq 3\alpha^4 + 1,
$$

Thus, we have

$$
\tau_{B3} := \inf \left\{ t \geq t_0 : |\Pi_{R_0}(t) - \alpha| \geq \beta^{50} \right\}; \\
\mathbb{P} \left( \tau_{B3} > t_0^\dagger \right) \geq 1 - 3 \exp \left( -\beta^3 \right),
$$

from Corollary 4.7 based on Lemma 5.8 and Corollary 5.12. We also note that $\tau_{B3}$ is a weaker version of $\tau_{B2}$.

The decomposition (5.21) works similarly for $\hat{R}_0^+(t)$ as follows.

$$
|\hat{R}_0^+(t) - \alpha| \leq |R_0(t) + |R_0(t) - \alpha| - \alpha| \leq 2|R_0(t) - \alpha| + \delta R_0(t).
$$

Let $\tau_{S3} := \tau_{S1} \wedge \tau_{S2} \wedge \tau_1 \wedge \tau_{B3}$. By following similar computation as (5.16) and what follows afterwards,

$$
\int_{(t-\alpha^{-1})\wedge \tau_{S3}}^{t \wedge \tau_{S3}} |\hat{R}_0^+(s) - \alpha| ds \leq \int_{(t-\alpha^{-1})\wedge \tau_{S3}}^{t \wedge \tau_{S3}} 2|R_0(s) - \alpha| ds + \int_{(t-\alpha^{-1})\wedge \tau_{S3}}^{t \wedge \tau_{S3}} \delta R_0(s) ds \\
\leq \int_{(t-\alpha^{-1})\wedge \tau_{S3}}^{t \wedge \tau_{S3}} \left( 2\alpha^C + \frac{1}{2}\sigma_1(s; \hat{R}_0^+) + \alpha^4 + 1 \sigma_1(s; \hat{R}_0^+) + 2\alpha^3 \beta^4 \right) ds \leq 1,
$$

where the second inequality follows from the definitions of $\tau_{S1}$ and $\tau_{S2}$, and the last one comes from Lemma 5.2 along with the bounds from $\tau_{B3}$. Then, applying Corollary 4.7 implies that

$$
\mathbb{P} \left( \tau_{B2} > t_0^\dagger \right) \geq 1 - 7 \exp \left( -\beta^3 \right).
$$

(5.30)

Thus, we conclude the proof from (5.29) and (5.30).

To conclude the section, we record a technical but simple lemma used above, and it will be useful for the rest of the paper.

**Lemma 5.13.** Let $g : [0, h] \to (0, \infty)$ be a positive function which defines a (deterministic) set of points $\Pi_g[0, h]$. Furthermore, assume additionally that $K, N_0, \Delta_0, \Delta_1 > 0$ satisfy

- $\sup \{|\Pi_g(t - \Delta_0, t) : t \in [\Delta_0, h]| \leq N_0$;
- $\Delta_1 \geq \Delta_0$ and $\inf \{|\Pi_g(t - \Delta_1, t) : t \in [\Delta_1, h]| \geq 1$;
- $h = K \Delta_0$.


Then, there exists an absolute constant \( C > 0 \) such that the following hold true:

\[
\int_0^h \frac{dt}{\sqrt{(h-t+1)(\pi_1(t;g) + 1)}} \leq C \left( N_0 + \sqrt{K \Delta_1 N_0} \right);
\]

\[
\int_0^h \frac{dt}{\sqrt{h-t+1}(\pi_1(t;g) + 1)} \leq C N_0 \left( 1 + \frac{\sqrt{K \log \Delta_1}}{\sqrt{\Delta_0}} \right);
\]

\[
\int_0^h \frac{dt}{(h-t+1)\sqrt{\pi_1(t;g) + 1}} \leq C \left( \frac{N_0 \log \Delta_1}{\sqrt{\pi_1(h;g) + 1}} + \frac{N_0 \sqrt{\Delta_1}}{\pi_1(h;g) + 1} + \log K \sqrt{\frac{\Delta_1 N_0}{\Delta_0}} \right);
\]

\[
\int_0^h \frac{dt}{(h-t+1)(\pi_1(t;g) + 1)} \leq C N_0 \log \Delta_1 \left( \frac{1}{\pi_1(h;g) + 1} + \frac{\log K}{\Delta_0} \right).
\]

Proof. Let \( \Pi_{\{0, h]\}} := \{ p_1 < p_2 < \ldots < p_n \} \) with \( p_0 = 0, p_{n+1} = h \). To establish the first inequality, we let \( \Gamma_k := \{ \alpha \} \) \((K-k)\Delta_0, (K-k)\Delta_0\). In the first integral, note that the integrals from \( p_i \) to \( p_{i+1} \) is bounded by an absolute constant, for \( p_i \in \Gamma_0 \). Hence,

\[
\int_0^h \frac{dt}{\sqrt{(h-t+1)(\pi_1(x;g) + 1)}} \leq C N_0 + \sum_{k=1}^{K-1} \sum_{p_i \in \Gamma_{k+1}} \int_{p_i}^{p_{i+1}} \frac{dx}{\sqrt{k \Delta_0(x-p_i)}}
\]

\[
\leq C N_0 + \sum_{k=1}^{K-1} \frac{\Delta_1 N_0}{\sqrt{k \Delta_0}} \leq C N_0 + C \sqrt{\frac{K \Delta_1 N_0}{\Delta_0}},
\]

where the second inequality came from Lemma 5.2. The other inequalities can be obtained similarly, and we omit the details. For the last two inequalities, we just keep in mind that the contribution from the regime \([h-2\Delta_0, h]\) are dealt separately. \( \square \)

6. Regularity of the speed

In this section, we formulate the notion of regularity, and build up an inductive analysis to study the regularity property of the speed. To motivate the works done in Sections \( \ref{sec:6} \) \( \ref{sec:8} \) we briefly review our future goal to explain why establishing the regularity is essential.

Recall the definition of \( L \) given in \((\ref{eq:2.25})\) (see also \((\ref{eq:5.3})\)). The main goal of this paper is to establish the scaling limit of \( L \) driven by the speed process. To this end, we need to study the mean and the variance of its increment at each interval, which can be conceptually described by

\[
\mathcal{L}(t_1; \Pi_S[t_1, t_0], \alpha_1) - \mathcal{L}(t_0; \Pi_S[t_0, t_0], \alpha_0), \tag{6.1}
\]

conditioned on the information up to time \( t_0 \). Here, \( t_0 < t_0 < t_1 < t_1 \), and \( \alpha_1, \alpha_0 \) are the frame of reference in each interval which are essentially the constants that approximates the speed. That is, on the interval \([t_0, t_1]\), the speed \( S(t) \) behaves roughly like \( \alpha_0 \), with a smaller order fluctuation which will indeed turn out to be essentially \( \alpha_0^{3/2} \text{polylog}(\alpha_0) \).

The primary difficulty of computing the mean and variance of \((6.1)\) stems from the complicated nature of the process \( S(t) \). As we have seen in \((2.27)\) and \((5.7)\), the increment \((6.1)\) can be described as an integral involving \( d\Pi_S \), and if \( S(t) \) does not behave nicely enough we may not be able to estimate such an integral accurately. This motivates us to demonstrate regularity properties of \( S(t) \), which is going to tell us that \( S(t) \) stays close to \( \alpha_0 \) during the interval \([t_0, t_1]\) in an appropriate sense, so that we can calculate the first and second moment of \((6.1)\) conditioned on \( \mathcal{F}_{t_0} \).
With this goal in mind, our argument is based on an inductive study of the speed. We will begin with defining the regularity property, which is a fairly complicated mixture of all the properties that \( S(t) \) and \( \Pi_S \) should satisfy. Then, we will show that

If \( S(t) \) is regular at time \( t_0 \) with respect to \( \alpha_0 \),

then \( S(t) \) is regular at time \( t_1 \) with respect to \( \alpha_1 \), with high probability.

Before moving on to the definition of regularity, we briefly explain how we define the parameters used in the section. Throughout the rest of the paper, we fix \( \epsilon = \frac{1}{10000} \) to be a small constant, and \( C_0 = 50, \theta = 10000 \) be large constants (where \( \theta \) needs to be larger depending on \( C_0 \)) as in Sections 4 and 5. For given \( \alpha_0 > 0 \) and \( t_0 > 0 \), we set \( \beta_0 := \log(1/\alpha_0) \), and the time parameters \( t_0^-, t_0^+, \hat{t}_0, \) and \( \tilde{t}_0 \) are defined as follows:

- \( t_0^-, t_0^- \) are arbitrary numbers satisfying
  \[
  t_0 - 2\alpha_0^{-2} \beta_0^\theta < t_0^- < t_0 - \frac{1}{2} \alpha_0^{-2} \beta_0^\theta, \quad t_0 + \frac{1}{2} \alpha_0^{-2} \beta_0^\theta < t_0^+ < t_0 + 2\alpha_0^{-2} \beta_0^\theta. \tag{6.2}
  \]

- \( \hat{t}_0, \tilde{t}_0 \) are fixed numbers set to be
  \[
  \hat{t}_0 := t_0 + 3\alpha_0^{-2} \beta_0^\theta, \quad \tilde{t}_0 := t_0 + 2\alpha_0^{-2} \beta_0^{10\theta}. \tag{6.3}
  \]

Note that the definitions of \( t_0^- \) and \( \hat{t}_0 \) are consistent with those from Section 5. We also stress that

\[
(\hat{t}_0 - t_0^-) \wedge (t_0^+ - t_0^-) \wedge (t_0 - \hat{t}_0) \geq \frac{1}{2} \alpha_0^{-2} \beta_0^\theta. \tag{6.3}
\]

For the speed \( S(t) \), its first-order approximation \( S_1(t) \) \((2.16)\) is set to be

\[
S_1(t) = S_1(t; t_0^-, \alpha_0) := \int_{t_0^-}^{t} K_{\alpha_0}(t - s) d\Pi_S(s). \tag{6.4}
\]

We also recall the definitions of \( \pi_i(t; S) \) and \( \sigma_i(t; S) \) from \((4.1)\) and \((4.2)\).

The rest of the section is organized as follows. We begin with giving a formal definition of regularity in Section 6.1 which will be studied exhaustively in Sections 7 and 8. In Section 6.2, we state the main theorem and give a more involved overview on the induction argument. We also discuss some important consequences of regularity in Section 6.3. Finally, in Section 6.4, we introduce a preliminary analysis on the speed that makes it possible to ignore the history of the far past, enabling us to study \( S'(t) \) \((2.15)\) instead of \( S(t) \).

6.1. **Formal definition of regularity.** Let \( \mathcal{F}_{t_0} \) be the \( \sigma \)-algebra generated by \( (S(s))_{s \leq t_0} \) and \( \Pi_S[0, t_0] \). Since the process \( (S(s))_{s \leq t_0} \) can completely be recovered from the given configuration of points \( \Pi_S[0, t_0] \), we can also identify \( \mathcal{F}_{t_0} \) with \( \Pi_S[0, t_0] \). Our idea is to define \( \Pi_S[0, t_0] \) to be regular if

\[
\mathbb{P} \left( S \text{ behaves nicely until time } \hat{t}_0 \mid \Pi_S[0, t_0] \right) \approx 1.
\]

To introduce the **nice** traits that \( S \) should satisfy, we will investigate the process from the following perspectives.

1. Control on the size of the speed;
2. Control on the size of the aggregate;
3. Refined control of the first order approximation;
4. Regularity of the history until time \( t_0 \).

In the following subsections, we detail the formal definitions of regularity for each category, and explain their purpose. Each criterion will be described in the language of stopping times, as we saw in Section 3. In the rest of the section and throughout Sections 7 and 8 \( \kappa > 0 \) denotes a constant which is either \( \frac{1}{2} \) or 2. This constant is the parameter used to distinguish the “stronger” \((\kappa = \frac{1}{2})\) regularity from the “weaker” \((\kappa = 2)\) one. We show in Section 7 that the initial “weaker” regularity leads to the “stronger” regularity at later times.
6.1.1. Control on the size of the speed. Maintaining an appropriate control on $S(t)$ is the primary property we want from the regularity. We begin with giving the formal definitions of the stopping times of interest. For simplicity, we write $\sigma_1 \sigma_2(t) := \sigma_1(t; S) \sigma_2(t; S)$. Also, recall the definition of $S_1(t)$ from (6.4)

\[
\tau_1(\alpha_0, t_0, \kappa) := \inf \{ t \geq t_0 : S(t) \geq \kappa \alpha_0 \beta_0^{C_0} \};
\]
\[
\tau_2(\alpha_0, t_0, \kappa) := \inf \{ t \geq t_0 : S_1(t) \geq \kappa \alpha_0 \beta_0^{C_0} \};
\]
\[
\tau_3(\alpha_0, t_0) := \inf \left\{ t \geq t_0 : S(t) - S_1(t) \notin \left( -\alpha_0^{3 + \epsilon} \sigma_1(t), \alpha_0^{1 - \epsilon} \sigma_1 \sigma_2(t) \right) \right\};
\]
\[
\tau_4(\alpha_0, t_0, \kappa) := \inf \left\{ t \geq t_0 : \int_{t_0}^t (S(s) - \alpha_0)^2 ds \geq \kappa \alpha_0 \beta_0^{25 \theta} \right\};
\]
\[
\tau_5(\alpha_0, t_0, \kappa) := \inf \left\{ t \geq t_0 : \int_{t_0}^t (S(s) - \alpha_0)^2 S(s) ds \geq \kappa \alpha_0 \beta_0^{25 \theta} \right\}.
\]

We briefly explain the purpose of each stopping time as follows:

- $\tau_1$ not only gives a fundamental understanding on $S$, but also useful in controlling other stopping times such as $\tau_4$ and $\tau_5$. In particular, it is heavily used in bounding the quadratic variations of various martingales, such as the second criterion in (4.7).
- $\tau_3$ provides a crucial estimate in bounding $|S(t) - \alpha_0| \leq |S(t) - S_1(t)| + |S_1(t) - \alpha_0|$, which is important in studying various different quantities including $\tau_4$ and $\tau_5$. Moreover, the lower bound on $S(t) - S_1(t)$ will lead us to obtaining the lower bound on $S(t)$: See Proposition 6.11.
- $\tau_2$, $\tau_4$ and $\tau_5$ verify the main assumptions of Proposition 4.29 enabling us to utilize the first- and second-order approximations.

6.1.2. Size of the aggregate. Note that the number of points in $\Pi_S[0, t]$ describes the size of the aggregate, i.e., $X_t = |\Pi_S[0, t]|$. The stopping times that deals with the size of the aggregate have similar role as $\tau_{32}$ in Section 5 but we need several of them for different purposes.

\[
\tau_6(\alpha_0, t_0, \kappa) := \inf \left\{ t \geq t_0 : |\Pi_S[t_0^-, t]| \leq \frac{1}{100 \kappa} \alpha_0(t - t_0) \right\};
\]
\[
\tau_7(\alpha_0, t_0, \kappa) := \inf \left\{ t \geq t_0 : |\Pi_S[(t - \alpha_0^{-1}) \vee t_0, t]| \geq \kappa \beta_0^4 \right\};
\]
\[
\tau_8(\alpha_0, t_0, \kappa) := \inf \left\{ t \geq t_0 + \kappa \alpha_0^{-1} \beta_0^{C_0} : |\Pi_S[t - \kappa \alpha_0^{-1} \beta_0^{C_0}, t]| = 0 \right\}.
\]

- $\tau_6$ provides useful estimates needed in Section 6.4. If there are reasonably many particles in the interval $[t_0^-, t]$, then it turns out we can eliminate the effect of information before $t_0$.
- $\tau_7$ plays a similar role as $\tau_{32}$ from Section 5, such as helping to control $\tau_2$ and $\tau_3$.
- $\tau_8$ makes it easier to control quantities involving $\sigma_1$ and $\sigma_2$, since the all neighboring points before $\tau_8$ cannot be too far from each other.

6.1.3. Refined control of the first order approximation. We introduce a stopping time which is a strengthened versions of $\tau_1$ and $\tau_3$. Instead of $\tau_1$, we describe a sharper control on the average of $|S(t) - \alpha_0|$. On the other hand, in $\tau_3$, the bound on $|S(t) - S_1(t)|$ has the term $\alpha_0^{-\epsilon}$ which is not strong enough in the later analysis (see Theorem 9.8). Thus, we seek for a better bound, which is
of $\beta_0^\gamma$ rather than $\alpha_0^{-\epsilon}$. We define the stopping times $\tau_1$ and $\tau_3$ to be

$$
\tau_1 = \tau_1'(\alpha_0, t_0) := \inf \left\{ t \geq t_0 : \int_{t_0}^t |S(s) - \alpha_0| ds \geq \alpha_0^{-\frac{1}{2}\epsilon} \right\};
$$

$$
\tau_3 = \tau_3'(\alpha_0, t_0) := \inf \left\{ t \geq t_0 : \int_{t_0}^t |S(s) - S_1(s)| ds \geq \beta_0^{4\theta} \right\}.
$$

It is clear that $\tau_1$ does not imply $\tau_1'$, and also is not difficult to see that $\tau_1'$ is stronger than $\tau_3$; since integrating $|S(s) - S_1(s)|$ based on the bound in $\tau_3$ (along with $\tau_7$, $\tau_8$, and Lemma 5.2) results in $\alpha_0^{-\epsilon}$, rather than $\beta_0^{4\theta}$-bound given in $\tau_3$.

We remark that in $\tau_3'$, having a smaller exponent $4\theta$ of $\beta_0$ than $10\theta$ (which is the exponent in $\hat{t}_0 - t_0$) plays a very important role in this analysis. See Section 6.2.3 for more discussion; details will be presented in Section 8.

6.1.4. Regularity of the history. We introduce several additional events which are measurable with respect to $\mathcal{F}_{t_0}$. Recalling the definition of $\mathcal{L}$ (5.8), let

$$
\alpha_0' = \alpha_0'(\alpha_0, t_0, t_0) := \mathcal{L}(t_0; \Pi S[t_0', t_0], \alpha_0),
$$

where $\mathcal{L}$ is defined in (5.8). We also set $S(s) = \infty$ for $s < 0$. Define

$$
\mathcal{A}_1(\alpha_0, t_0) := \left\{ |\Pi S[s, t_0']| \geq \sqrt{t_0' - s + \tilde{C}_0 \log^2(t_0' - s + \tilde{C}_0)} - \alpha_0^{-\beta_0^\theta}, \forall 0 \leq s \leq t_0 \right\};
$$

$$
\mathcal{A}_2(\alpha_0, t_0) := \left\{ \frac{1}{4} \leq \frac{|\Pi S[t_0', t_0]|}{\alpha_0^{-\beta_0^\theta}} \leq 4 \right\};
$$

$$
\mathcal{A}_3(\alpha_0, t_0) := \left\{ |\alpha_0 - \alpha_0'| \leq \alpha_0^{-\beta_0^\theta} \right\}.
$$

- $\mathcal{A}_1$ is the key event used in Section 6.4. By setting $|\Pi S[s, t_0']|$ to be of strictly bigger order than $\sqrt{t_0' - s}$ (which must actually be linear in $(t_0' - s)$), the contribution from the history before $t_0'$ becomes negligible from the representation (2.2), since the probability of the simple random walk to hit the aggregate before time $t_0'$ is small enough.
- $\mathcal{A}_2$ provides a basic estimate on the size of the aggregate in the past, and is used in investigating $\tau_6$.
- $\mathcal{A}_3$, is introduced to ensure that the choice of $\alpha$ is appropriate. Later, our choice of $\alpha_1$ will satisfy $|\alpha_1 - \alpha_0| = \alpha_0^{3/2} \beta_0^{O(1)}$, which is the same order as the bound in $\mathcal{A}_3$.

6.1.5. Definition of regularity. We give the formal definition of regularity combining the notions introduced in Sections 6.1.1, 6.1.4. We let

$$
\tau(\alpha_0, t_0, \kappa) := \hat{t}_0 \wedge \min \{ \tau_i(\alpha_0, t_0, \kappa) : 1 \leq i \leq 8 \};
$$

$$
\tau^A(\alpha_0, t_0, \kappa) := \tau(\alpha_0, t_0, \kappa) \wedge \tau_1'(\alpha_0, t_0) \wedge \tau_3'(\alpha_0, t_0),
$$

where we define $\tau_3(\alpha_0, t_0, \kappa) := \tau_3(\alpha_0, t_0)$ which does not have dependence on $\kappa$. Note that $\tau$ is increasing in $\kappa$. Moreover, we set

$$
\mathcal{A}(\alpha_0, t_0) := \mathcal{A}_1(\alpha_0, t_0) \cap \mathcal{A}_2(\alpha_0, t_0) \cap \mathcal{A}_3(\alpha_0, t_0).
$$

Note that these events are $\mathcal{F}_{t_0}$-measurable. We now introduce the two notions of regularity.

**Remark 6.1.** In the following definitions and throughout the paper, the notation

$$
\mathbb{P}(\cdot | \Pi S(-\infty, t_0))
$$

denotes the probability with respect to the law of the aggregate starting from time $t_0$ under the initial points given by $\Pi S(-\infty, t_0)$, which was defined in Definition 3.3. Moreover, note that if $S(s) = \infty$
on \( s < t^b \) for some \( t^b \), then \( \Pi_S(-\infty, t^b) \) is not well defined (for instance, \( \overline{S}(t) \) from (3.4)). Even in this case when there exists such an \( t^b \leq t_0 \), we still use the notation \( \Pi_S(-\infty, t_0) \) for convenience, which will actually refer to

\[
\Pi_S(-\infty, t_0) := \text{the points in } [t^b, t_0] \text{ are given by } \Pi_S[t^b, t_0], \text{ and } S(s) = \infty \text{ for all } s \leq t^b.
\]

**Definition 6.2 (Regularity).** Let \( t_0, t_0^*, r > 0 \) denote \( [t_0] := (t_0, t_0^*) \) and set \( \hat{t}_0 \) as (6.3). We say \( \Pi_0(-\infty, t_0) \) is \((\alpha_0, r; [t_0])\)-regular if it satisfies the following two conditions:

\[
\begin{align*}
\varepsilon\left(\tau(\alpha_0, t_0, \kappa = 2) \leq \hat{t}_0 \mid \Pi_S(-\infty, t_0) = \Pi_0(-\infty, t_0) \right) & \leq r; \\
\varepsilon(\alpha_0, t_0) & \in A(\alpha_0, t_0).
\end{align*}
\]

We also write \( \Pi_0(-\infty, t_0) \in R(\alpha_0, r; [t_0]) \) to denote that \( \Pi_0(-\infty, t_0) \) is \((\alpha_0, r; [t_0])\)-regular. Note that the dependence on \( t_0^* \) comes from the definitions of \( S_1 \) and \( A \).

**Definition 6.3 (Sharp regularity).** For \( \alpha_0, t_0, t_0^*, r > 0 \) with \( [t_0] := (t_0, t_0^*) \), and set \( \hat{t}_0 \) as (6.3). We say \( \Pi_0(-\infty, t_0) \) is \((\alpha_0, r; [t_0])\)-sharp-regular if it satisfies the following two conditions:

\[
\begin{align*}
\varepsilon(\alpha_0, t_0, \kappa = 2) \left(\Pi_S(-\infty, t_0) = \Pi_0(-\infty, t_0) \right) & \leq r; \\
\varepsilon(\alpha_0, t_0) & \in A(\alpha_0, t_0).
\end{align*}
\]

We also write \( \Pi_0(-\infty, t_0) \in R(\alpha_0, r; [t_0]) \) to denote that \( \Pi_0(-\infty, t_0) \) is \((\alpha_0, r; [t_0])\)-sharp-regular.

The reason for separating two concepts of regularity will be clear when we state the main result in the next subsection (see also Remark 6.7). There, we also briefly explain the main ideas of proof.

### 6.2. Overview of the argument.

Let \( t_1 \) be an arbitrary number that satisfies

\[
t_0 + \alpha_0^2 \varepsilon_0^6 \leq t_1 \leq t_0 + \alpha_0^2 \varepsilon_0^{10\theta}.
\]

(6.9)

We note that the upper bound of \( t_1 \) is substantially smaller than \( \hat{t}_0 \), namely, \( t_1 \leq t_0 + \alpha_0^2 \varepsilon_0^{10\theta} = \hat{t}_0 - \alpha_0^2 \varepsilon_0^{10\theta} \). Then, the goal of Sections 6–8 is to establish the following theorems.

**Theorem 6.4.** Let \( \alpha_0, t_0, r > 0 \), let \( t_0^* \) be any number satisfying (6.2), set \( \hat{t}_0 \) as in (6.3), and let \( t_1 \) be an arbitrary fixed number selected within (6.9). Set \( t_1^* := t_1 - \alpha_0^{-2} \varepsilon_0^6 \), and define

\[
\alpha_1 := L(t_1^*; \Pi_S(t_1^*, t_1^*), \alpha_0), \quad \beta_1 := \log(1/\alpha_1).
\]

(6.10)

Then, the following hold true for all sufficiently small \( \alpha_0 \) and for any \( \Pi_S(-\infty, t_0) \in R(\alpha_0, r; [t_0]) \):

1. \( \varepsilon(\Pi_S(-\infty, t_1) \in R(\alpha_1, e^{-\beta_1}; [t_1]) \mid \Pi_S(-\infty, t_0) \leq r + e^{-\beta_1} \).
2. \( \varepsilon(\alpha_1 - \alpha_0 \geq \beta_0^2 \varepsilon_0^{10\theta} \mid \Pi_S(-\infty, t_0) \leq r + e^{-\beta_1} \).

Note that the second statement directly implies

\[
\varepsilon(t_1 \text{ satisfies (6.2) in terms of } t_1, \alpha_1 \mid \Pi_S(-\infty, t_0) \geq 1 - r - e^{-\beta_1/2},
\]

which means that w.h.p., we can reapply the theorem for \( \Pi_S(-\infty, t_1) \) and \( [t_1] \) to obtain \( \Pi_S(-\infty, t_2) \in R(\alpha_2, e^{-\beta_2}; [t_2]) \), and so on.

**Remark 6.5.** In the interval starting from \( t_1 \), the first-order approximation \( S_1 \) is defined as \( S_1(t) = S_1(t; t_1^*, \alpha_1) \). Thus, by setting \( \alpha_1 \) to be \( F_{t_1^*}^0 \)-measurable as above, the points in \( \Pi_{S_1}[t_1, t_1^*] \) before time \( t_1 \) do not affect the frame of reference \( \alpha_1 \) of \( S_1(t) \).

In addition to the above theorem, we need the regularity of the fixed rate process obtained in Section 3.3 which serves as the induction base of our argument. Recall the definitions of \( Y_{t_0, \alpha_0} \) and \( Y_{t_0, \alpha_0} \), and let \( \overline{S}(t) = \overline{S}_{t_0, \alpha_0}(t) \) and \( \overline{S}(t) = \overline{S}_{t_0, \alpha_0}(t) \) be the corresponding speed processes (3.4), respectively. For \( t_0 \) and \( t_0^* \) as above, we define

\[
\overline{S}_1(t) = \overline{S}_1(t; t_0^*, \alpha) := \int_{t_0}^{t_0^*} \kappa_{\alpha_0}(t - x)d\Pi_{\alpha_0}(x) + \int_{t_0}^{t} \kappa_{\alpha_0}(t - x)d\Pi_{\alpha_0}(x)
\]
Thus, using these processes, we can define the stopping times \( \{ \tau_i(\alpha_0, t_0, \kappa) \}_{1 \leq i \leq S} \) (Sections 6.1-6.1.2), and the event \( \mathcal{A}(\alpha_0, t_0) \) (6.8), analogously as before. Then, they define the regularity (Definition 6.2) of \( \overline{S}(t) \). Everything can be done in the same way for \( \underline{S}(t) \). The main theorem on the regularity of \( S(t) \) and \( \overline{S}(t) \) is stated as follows.

**Theorem 6.6.** Let \( \alpha_0, t_0 > 0 \), set \( \hat{t}_0 \) as (6.3), and set \( t_0^- := t_0 - \alpha_0^{-2} \beta^0 \). Then, under the above setting, we have

\[
P \left( \Pi_{\overline{S}}(\infty, t_0), \Pi_{\underline{S}}(\infty, t_0) \in \mathcal{R}(\alpha_0, e^{\beta^0/2}; [t_0]) \right) \geq 1 - e^{-\beta^0/2}.
\]

**Remark 6.7.** After some large constant time since the critical aggregate starts growing, we require the speed to be very small and the process to be sharp-regular. However, due to Theorem 6.4, a regular (does not have to be sharp-regular) interval beginning will result in sharp-regular intervals afterwards w.h.p., and hence Theorem 6.6 will be enough for our purpose.

The rest of the section and Sections 7, 8 are mostly devoted to the proof of Theorem 6.4 which consists of three major steps as follows.

1. **Approximating** \( S(t) \) by \( S'(t) \) (2.15) to validate the first and second order approximations;
2. **Continuation of regularity** (i.e., the stopping times) until time \( \hat{t}_0 \), which will be larger than \( \hat{t}_1 = t_1 + 2\alpha_1^{-2} \beta^0 \);
3. **Regularity** at time \( t_1 \), with new the frame of reference \( \alpha_1 \).

In the remaining subsection, we briefly explain the main purposes and ideas of (1), (2) and (3).

6.2.1. *The first and second order approximation of the speed.* To study \( S(t) \), we attempt to use the first- and second-order approximations given in Section 2.2. However, the formulas (2.17), (2.19) give expansions on \( S'(t) \) defined in (2.15), not \( S(t) \). Thus, we need to justify that \( S(t) \) is sufficiently close to \( S'(t) \).

This analysis is conducted in the next subsection by studying the original formula (2.2) of \( S(t) \). We show that under the regularity condition, the hitting probability of the random walk affected by conditioning on \( \mathcal{F}_{t_0^-} \) is negligible, since \( t_0^- \) is far away from \( t_0 \).

6.2.2. *Continuation of regularity.* The next step is to show that if \( \tau(\alpha_0, t_0, \kappa = 2) \geq \hat{t}_0 \), then it is unlikely to be smaller than \( t_0 \). The purpose of such an analysis is fairly obvious: In order to establish regularity at time \( t_1 \) in terms of \( \alpha_1 \), we would like to argue that (1) the process at time \( t_1 \) maintains the desired regularity properties in terms of \( \alpha_0 \), and (2) changing \( \alpha_0 \) to \( \alpha_1 \) does not have a significant effect. For (2), see the discussion in the following subsection.

To establish (1), we essentially argue that \( \{ \tau_i = \tau \} \) happens with small enough probability for all \( i \). In the following, we discuss several major ideas needed to process the argument.

- We study \( \tau_i \) via the original formula (2.2), by upper bounding \( Y_i \) by an appropriate choice of \( Y_i' \) that achieves the desired estimate. The construction of \( Y_i' \) is done based on the control of \( X_i \) which is given by \( \tau_i \).
- To obtain estimates on \( \tau_3 \), we appeal to the results we obtained in Section 4.5 where \( \tau_1, \tau_2, \tau_4 \) and \( \tau_5 \) provide the appropriate assumptions we need for utilizing Proposition 4.29.
- To control the rest of the stopping times, we first express

\[
|S(t) - \alpha_0| \leq |S(t) - S_1(t)| + |S_1(t) - \alpha_0|.
\]

The first term in the RHS are controlled by \( \tau_3 \), and the remaining task is to control \( |S_1(t) - \alpha_0| \). To this end, we observe that \( S_1(t) = \mathcal{R}_c(t, t; \Pi S[t_0^-, t], \alpha_0) \) (see (5.6) for its definition) and interpret it via (5.7). After we understand the gap \( |S(t) - \alpha| \), the results will follow mostly by applying the lemmas from Section 4.1.
6.2.3. Regularity in the next time step. The remaining step is to establish (2) in the first paragraph of Section 6.2.2. We note several major issues in achieving this goal:

- Our argument is mostly based on analyzing what happens when we change $\alpha_0$ to $\alpha_1$, and hence the control on $|\alpha_1 - \alpha_0|$ is essential. This is done by expressing the difference in the integral form which is similar to (5.7) and studying it based on martingale techniques such as Lemma 4.5 and the analytic properties of $K_\alpha(s)$ from Section A.

- The most crucial task is to understand $\tau_0^2(\alpha_1, t_1)$. We need to avoid having $\alpha^{-\epsilon}$ term, which is present in $\tau_3$. To this end, we need to study the gap between $S(t)$ and its second-order approximation $S_2(t)$ (2.22), rather than $|S(t) - S_1(t)|$. Then, however, it turns out that the double integral term of $J_{t-s,t-a}$ in the formula of $S_2(t)$ cannot be studied in the same way as Section 4 since the perturbation argument in Section 4 is too costly to guarantee a sharp control like $\tau_3^2$. To overcome this issue, we study this double integral rather directly, appealing to the proximity of $S(t)$ to $\alpha_0$ obtained from the previous time step.

One interesting issue we emphasize is that $\tau_3^2(\alpha_1, t_1)$ is typically much smaller than $t_2 = t_1 + \alpha_1^{-2}\beta_1^{100}$. As $t$ gets closer to $t_2$, the variance accumulated from the double integral becomes so large that we need a larger exponent on $\beta_1$ for a correct bound.

6.3. Consequences of regularity. In this subsection, we highlight some properties that a regular aggregate must have. These will all be needed in Section 9 where we study the formal version of Theorem 2.12.

We begin with the control on the number of particles in $\Pi_S[t_0, t]$ when the process is regular. Its proof will be discussed in Section 7.5.

**Proposition 6.8.** Let $\alpha_0, t_0, r > 0$, set $t_0, t_0$ as (6.3), and let $t_0'$ be any number satisfying (6.2). For any $\Pi_S(-\infty, t_0) \in \mathcal{R}(\alpha_0, r; [t_0])$, we have

$$\mathbb{P}\left( |\Pi_S[t_0, t] - \alpha_0(t - t_0)| \leq \alpha_0^{-1} - 10e, \forall t \in [t_0, t_0'] \right) \Pi_S(-\infty, t_0) \geq 1 - r - 2\exp(-\beta_0^{1.9}).$$

Next, the following describes an important property of how the frame of reference changes.

**Proposition 6.9.** Let $\alpha_0, t_0, r > 0$, and let $t_0, t_0, t_1, t_1'$ be as Theorem 6.4. Furthermore, we define $\alpha_1' := \mathcal{L}(t_1; \Pi_S[t_1, t_1], \alpha_1)$ analogously as (6.5), and set

$$\tilde{\alpha}_1 := \mathcal{L}(t_1; \Pi_S[t_1, t_1], \alpha_0).$$

Then, the following holds true for all sufficiently small $\alpha_0$ and for any $\Pi_S(-\infty, t_0) \in \mathcal{R}(\alpha_0, r; [t_0])$:

$$\mathbb{P}\left( |\alpha_1' - \tilde{\alpha}_1| \geq 2\alpha_0^2 \beta_0^{8g+1} | \Pi_S(-\infty, t_0) \right) \leq r + \exp(-\beta_0^2).$$

In fact, we will later derive scaling limit of $\alpha'$ (see Sections 9 and 10). To this end, we study the increment $\alpha_1' - \alpha_0'$ (defined in (6.5)) in Theorem 9.8. Expressing that

$$\alpha_1' - \alpha_0' = (\alpha_1' - \tilde{\alpha}_1) + (\tilde{\alpha}_1 - \alpha_0'),$$

the estimate in the proposition gives an essentially sharp estimate on $(\alpha_1' - \tilde{\alpha}_1)$. Also, note that it is strictly stronger than the error bound in $\mathcal{A}_3$ (Section 6.1.4). The proof of Proposition 6.9 will be discussed in Section 8.1.

Furthermore, we record estimates on the integrals of the errors such as $(S(t) - \alpha_0)$ in the following lemma.

**Lemma 6.10.** Let $\alpha_0, t_0, r > 0$, and let $\tilde{t}_0, \tilde{t}_0, t_1, t_1'$ be as Theorem 6.4. Furthermore, let $S_1(t) = S_1(t; \tilde{t}_0, \alpha_0)$ and $S_2(t) = S_2(t; \tilde{t}_0, \alpha_0)$ be the first- and second-order approximations of the speed
For any \( \Pi_S(-\infty, t_0] \in \mathcal{R}(\alpha_0, r; [t_0]) \), we have

\[
\begin{align*}
(1) & \quad \mathbb{P} \left( \int_{t_0}^{t_1} |S(t) - \alpha_0| dt \geq \alpha_0^{-\frac{1}{2}-\epsilon} \right| \Pi_S(-\infty, t_0] \right) \leq r + \exp \left(-\beta_0^{1.9}\right) ; \\
(2) & \quad \mathbb{P} \left( \int_{t_0}^{t_1} |S_1(t) - \alpha_0| dt \geq \alpha_0^{-\frac{1}{2}-\epsilon} \right| \Pi_S(-\infty, t_0] \right) \leq r + \exp \left(-\beta_0^{1.9}\right) ; \\
(3) & \quad \mathbb{P} \left( \int_{t_0}^{t_1} |S(t) - S_1(t)| dt \geq \alpha_0^{-\epsilon} \right| \Pi_S(-\infty, t_0] \right) \leq r + \exp \left(-\beta_0^{1.9}\right) ; \\
(4) & \quad \mathbb{P} \left( \int_{t_0}^{t_1} |S(t) - S_2(t)| dt \geq \alpha_0^{-\frac{1}{2}-\epsilon} \right| \Pi_S(-\infty, t_0] \right) \leq r + \exp \left(-\beta_0^{1.9}\right).
\end{align*}
\]

Note that all the regimes of the integrals are \([t_0, t_1]\), except for (3) which is \([t_0, t_1]\). One can essentially prove the same for the integral over \([t_0, t_1]\), but the above is enough for our purpose as well and is simpler to establish. The proof is discussed later:

- (1) and (2) are proven in Corollary 7.17
- (3) is established in Lemma 7.4
- (4) is verified in Section 7.1

Lastly, we state a lower bound on \(S(t)\). Although the bounds given in \(\tau_1\) implies that \(S(t)\) can get pretty high compared to \(\alpha_0\), it turns out that \(S(t)\) cannot stay too low from \(\alpha_0\).

**Proposition 6.11.** Let \(\alpha_0, t_0, r > 0\), and let \(t_0, t_0^+, t_0^\ast, \tilde{t}_0\) be as \((6.2)\) and \((6.3)\). Define the stopping time \(\tau_{lo}\) by

\[
\tau_{lo} = \tau_{lo}(\alpha_0, t_0, t_0^\ast) := \inf \left\{ t \geq t_0^\ast : S(t) \leq \alpha_0 - \alpha_0^{\frac{3}{2}-\epsilon} \right\}.
\]

Then, for any \(\Pi_S(-\infty, t_0] \in \mathcal{R}(\alpha_0, r; [t_0])\), we have

\[
\mathbb{P}(\tau_{lo} \leq \tilde{t}_0 | \Pi_S(-\infty, t_0]) \leq r + \exp \left(-\beta_0^2\right).
\]

This proposition will be useful later in Section 9 and its proof is presented in Section 7.1

### 6.4. Neglecting the far away history.

In this section, we resolve the issue addressed in Section 6.2.1. Recall the formula \((2.2)\):

\[
S(t) = \frac{1}{2} \mathbb{P}(W(s) \leq Y_t(s), \forall s \geq 0 | Y_t).
\]

(6.15)

Our goal is to show that its dependence on \(\mathcal{F}_{t_0}^\ast = \{Y_s(s)\}_{s \geq t_0^\ast}\) is negligible. To this end, we introduce an approximate version of speed and study its first and second order approximations. Let \(\Pi'\) be the rate-1 Poisson point process on \(\mathbb{R}^2\) independent from \(\Pi\), and define \(Y'_t(s)\) as

\[
Y'_t(s) := \begin{cases} 
Y_t(s), & \text{for } s \leq t - t_0^\ast; \\
Y(t_0^\ast) + \Pi'_0[t - s, t_0^\ast], & \text{for } s > t - t_0^\ast.
\end{cases}
\]

Then, we define the *quenched* approximated speed \(S'_q(t)\) as

\[
S'_q(t) = S'_q(t; t_0^\ast, \alpha_0) := \frac{1}{2} \mathbb{P}(W(s) \leq Y'_t(s), \forall s \geq 0 | Y'_t).
\]

(6.16)

We recall the previous definition of \(S'(t)\) in \((2.15)\), and observe that

\[
\mathbb{E}_{\Pi'_0} [S'_q(t; t_0^\ast, \alpha_0)] = S'(t; t_0^\ast, \alpha_0).
\]

Recalling the definitions of \(\tau_6\) and \(A\) from Sections 6.1.2 and 6.1.4 respectively, the main purpose of this subsection is to establish the following property.
Proposition 6.12. Suppose that we have \( \Pi_S[t_0, t_0] \in \mathcal{A}(\alpha_0, t_0) \). Then, for all sufficiently small \( \alpha_0 > 0 \), we have
\[
|S(t) - S'(t; t_0, \alpha_0)| \leq \alpha_0^{100},
\]
for all \( t \in [t_0, \tau_0(\alpha_0, t_0, \kappa = 2)] \). Furthermore, for any \( t \in [\tau_0, \tau_0 \wedge \tau_0(\alpha_0, t_0, \kappa = 2)] \), we have
\[
|S(t) - S'(t; t_0, \alpha_0)| \leq \alpha_0^{100}.
\]

It turns out that the same proof as Proposition 6.12 implies the following corollary, which will be used to study the induction base.

Corollary 6.13. Under the setting of Theorem 6.6 and the discussions above, define \( \overline{\mathcal{S}}'(t; t_0, \alpha_0) \) analogously as above in terms of \( \overline{\mathcal{S}}(t) \), and suppose that \( \Pi_{\overline{\mathcal{S}}}(\infty, t_0) \in \overline{\mathcal{A}}(\alpha_0, t_0) \). Then, for all sufficiently small \( \alpha_0 > 0 \), we have
\[
|\overline{\mathcal{S}}(t) - \overline{\mathcal{S}}'(t; t_0, \alpha_0)| \leq \alpha_0^{100},
\]
for all \( t \in [t_0, \overline{\tau}_0(\alpha_0, t_0, 2)] \). The same thing holds for \( \overline{\mathcal{S}}(t) \) and \( \overline{\mathcal{S}}'(t; t_0, \alpha_0) \).

Proof of Proposition 6.12. We prove the first statement of the proposition, and then the second one follows analogously. Let \( t \in [t_0, \tau_0(\alpha_0, t_0, 2)] \), and for simplicity we write \( S'(t) = S'(t; t_0, \alpha_0) \). Moreover, we write \( P(\cdot) \) to denote the probability over both the random walk \( W \) and the independent point process \( \Pi_{\alpha_0} \) together. The main idea is to compare the formulas (6.15) and (6.16) by coupling the random walk \( W \) together. This gives that
\[
|S(t) - S'(t)| \leq P_{\Pi_{\alpha_0}} \left( W(s) \leq Y_t(s) \forall s \leq t - t_0^0, \text{ and } \exists s' \geq t - t_0^0 : W(s') \geq Y_t(s') \mid Y_t \right) + \forall s' \geq t - t_0 \mid W(s') \geq Y_t(s') \mid Y_t\right) =: P_1 + P_2.
\]

We begin with bounding \( P_2 \). Since \( t \leq \tau_0(\alpha_0, t_0, 2) \),
\[
Y_t(t_0^0) \geq \frac{\alpha_0}{200}(t - t_0^0).
\]

Thus, we can write
\[
P_2 \leq P \left( W(t - t_0^0) \geq \frac{\alpha_0}{400}(t - t_0^0) \right) + P \left( \exists s \geq t_0^0 : W(s) \geq |\Pi_{\alpha_0}[t - s, t - t_0^0]| + \frac{\alpha_0}{400}(t - t_0^0) \right).
\]

Note that since \( t - t_0^0 \geq \alpha_0^{-2} \beta_0^\theta \),
\[
\frac{\alpha_0}{400}(t - t_0^0) \geq \sqrt{t - t_0^0} \log^{\theta/3}(t - t_0^0).
\]

Plugging this into the first term of (6.19), we can see that
\[
P \left( W(t - t_0^0) \geq \frac{\alpha_0}{400}(t - t_0^0) \right) \leq \alpha_0^{101}.
\]

On the other hand, to study the second term of (6.19), we define
\[
U'(s) = |\Pi_{\alpha_0}[0, s]| - W(s) + \frac{\alpha_0}{400}(t - t_0^0),
\]
and define the stopping time \( T' := \inf\{s > 0 : U'(s) \leq 0\} \). Then, we can write
\[
P \left( \exists s \geq t_0^0 : W(s) \geq |\Pi_{\alpha_0}[t - s, t - t_0^0]| + \frac{\alpha_0}{400}(t - t_0^0) \right) = P(T' < \infty)
= \lim_{s \to \infty} \mathbb{E} \left[ \left( \frac{1}{1 + 2\alpha_0} \right)^{U'(s) \wedge T'} \right].
\]
As we saw in \([2.5]\), \((1 + 2\alpha_0)^{-U'(s)}\) is a martingale, and hence the Optimal Stopping Theorem tells us that
\[
P(T' < \infty) = \mathbb{E}\left[\left(\frac{1}{1+2\alpha_0}\right)^{U'(0)}\right] \leq \left(\frac{1}{1+2\alpha_0}\right)^{-\frac{300}{200}} \leq \exp\left(-\frac{\beta_0^2}{400}\right) \leq \alpha_0^{101}. \tag{6.21}
\]
Combining \((6.20)\) and \((6.21)\) tells us that \(P_2 \leq 2\alpha_0^{101}\).

To control \(P_1\), we first note that \(\mathcal{A}_1(\alpha_0, t_0)\) and \(t \leq \tau(\alpha_0, t_0, 2)\) implies the following for all \(u \leq t_0\):
\[
X_t - X_u = X_t - X_{t_0} + X_{t_0} - X_u \\
\geq \frac{\alpha_0}{200} (t - t_0) + \sqrt{t_0 - u + C_0 \log^2(t_0 - u + C_0) - \alpha_0^{-1} \beta_0^{1/2}} \\
\geq \frac{\alpha_0}{300} (t - t_0) + \sqrt{t_0 - u + C_0 \log^2(t_0 - u + C_0)},
\]
where the last line is from \(t - t_0 \geq \alpha_0^{-2} \beta_0^2\). We can also see that
\[
\frac{\alpha_0}{300} (t - t_0) \geq \sqrt{t - t_0 + C_0 \log^2(t - t_0 + C_0)},
\]
which gives
\[
X_t - X_u \geq \sqrt{t - t_0 + C_0 \log^2(t - t_0 + C_0)} + \sqrt{t_0 - u + C_0 \log^2(t_0 - u + C_0)} \\
\geq \sqrt{t - u + C_0 \log^2(t - u + C_0)},
\]
where the last inequality follows from the fact that the function \(\sqrt{x + 20} \log^2(x + 20)\) is increasing and concave. From this estimate, we can see that
\[
P_1 \leq \mathbb{P}\left(\exists s' \geq t - t_0 : W(s') \geq \sqrt{s'} \log^2(s')\right) \leq e^{-\beta_0^2} \leq \alpha_0^{101}, \tag{6.22}
\]
where the second inequality is a standard estimate on a simple random walk. We conclude the proof by combining \((6.17)\), \((6.19)\) and \((6.22)\).

To obtain the second statement of the proposition, we apply the same argument with
\[
Y_1(t_0) \geq \frac{\alpha_0}{2\beta_0 C_0} (t - t_0),
\]
instead of \((6.18)\). This estimate is obtained from the definition of \(\tau_8\). \(\square\)

As a concluding remark of the section, we record a simple lemma on \(\mathfrak{S}(t)\) and \(\overline{\mathcal{S}}(t)\) as a step towards establishing Theorem 6.6.

**Lemma 6.14.** Under the setting of Theorem 6.6 and the discussions above, we have
\[
P\left(\Pi_{\mathfrak{S}}(-\infty, t_0) \in \overline{\mathcal{A}}(\alpha_0, t_0)\right) \geq 1 - \exp\left(-\beta_0^4\right).
\]
The same holds for \(\overline{\mathcal{S}}(t)\).

**Proof.** The estimate on the event \(\{\Pi_{\mathfrak{S}}(-\infty, t_0) \in \overline{\mathcal{A}}_1(\alpha_0, t_0) \cap \overline{\mathcal{A}}_2(\alpha_0, t_0)\}\) follows straight-forwardly from the properties of a fixed rate Poisson process, and we omit the details. The bound on the event \(\overline{\mathcal{A}}_3(\alpha_0, t_0)\) comes from the same argument as \((5.26)\) and \((5.28)\), replacing \(t_0'\) by \(t_0\). \(\square\)

### 7. The inductive analysis on the speed: Part 1

The purpose of this section is to formulate the argument discussed in Section 6.2.2. Before we state the main theorem, we briefly explain a technical device required to understand the procedure of the induction argument. In the definition of \(\tau\) from \((6.7)\), recall that there was a parameter \(\kappa \in \{\frac{1}{2}, 2\}\) that provides a constant multiplicative room in controlling the quantities inside \(\tau_8\)'s. One of the main observations in the induction argument is that even if we have \(\{\tau(\alpha_0, t_0, \kappa) > t_0\}\) with a weaker (i.e., larger) \(\kappa\), the control on the regularity will bootstrap as the time passes by and at a larger time we still have \(\{\tau(\alpha, t_0', \kappa') > t_0\}\) but with a stronger (i.e., smaller) \(\kappa'\), with \(t_0' > t_0\).
Note that we can only have this bootstrapped estimate with \( t'_0 > t_0 \), not with \( t_0 \), since the initial assumption is made under a weaker \( \kappa \). By doing this bootstrapping, we acquire enough room to cover the error that comes from changing \( \alpha_0 \) to \( \alpha_1 \) in the next time step.

Having this idea in mind, we introduce another notation before stating the main objective. We define \( \tau^+ \) to be

\[
\tau^+(\kappa) := \tau(\alpha_0, t'_0, \kappa),
\]

except that we use the same \( S_1(t) = S_1(t; t'_0, \alpha_0) \) when defining \( \tau(\alpha_0, t'_0, \kappa) \). That is, for instance, we define

\[
\begin{align*}
\tau^+_2(\kappa) &:= \inf\{ t \geq t'_0 : S_1(t; t'_0, \alpha_0) \geq \kappa \alpha_0 \beta'_0 \}; \\
\tau^+_4(\kappa) &:= \inf\left\{ t \geq t'_0 : \int_{t'_0}^{t} \kappa \alpha_0 \beta'_0 \right\}; \\
\tau^+_6(\kappa) &:= \inf\left\{ t \geq t'_0 : |\Pi_S[t'_0, t] - \alpha_0(0) + \beta_0(0) + \beta_0(9)| \leq \frac{1}{100}\kappa \alpha_0(t - t'_0) \right\},
\end{align*}
\]

and corresponding analogue for all other \( \tau^+_i(\kappa), i = 1, 3, 5, 7, 8 \), which are all defined in a straightforward way unlike the three mentioned above. Then, set \( \tau^+(\kappa) \) to be the minimum of \( \tau^+_i(\kappa) \)’s. The main result for this section can be stated as follows.

**Theorem 7.1.** Recall the definitions of \( \tau \) and \( A \) from (6.7) and (6.8), respectively. Write \( \tau(\kappa) = \tau(\alpha_0, t_0, \kappa), A = A(\alpha_0, t_0) \), and set \( \tau^+(\kappa) \) as above. For all sufficiently small \( \alpha_0 > 0 \) and any \( t_0 > 0 \), we have

\[
\mathbb{P}\left( \tau^+(1/2) < t_0 \mid A \right) \leq \mathbb{P}\left( \tau(2) \leq t_0 \mid A \right) + \exp\left( -\beta_0^{1.9} \right).
\]

**Remark 7.2.** The equations involving the notation \( \mathbb{P}( \pm | A ) \) are understood as follows throughout Sections 7 and 8. The equation holds for any \( \Pi_S(\infty, t_0) \) such that \( \Pi_S(\infty, t_0) \in A \).

As done in Section 5, our idea is to show that each stopping time \( \tau^+_i(1/2) \) is likely to be larger than \( \tau(2) \) if \( \tau(2) \geq t_0 \). In the following sections, we conduct this task through three major steps as follows.

(1) In Section 7.1, we study \( |S(t) - S_1(t)| \) to control \( \tau^+_3 \), done by utilizing the result we saw in Section 4.5.

(2) In Section 7.2, we estimate the gap \( |S_1(t) - \alpha_0| \) based on ideas from Section 5.1.

(3) In Section 7.3, we control all stopping times except \( \tau^+_3 \), with the estimate

\[
|S(t) - \alpha_0| \leq |S(t) - S_1(t)| + |S_1(t) - \alpha_0|,
\]

obtained from Sections 7.1, 7.2.

In each of these subsections, we add brief explanations on how the result is generalized to the case of induction base, Theorem 6.6. The only major difference compared to the proof of Theorem 7.1 is the availability of the assumption \( \{ \tau(2) > t_0 \} \). Although we can only be interested in \( t \geq t'_0 \) when establishing Theorem 7.1, we need to consider all \( t \geq t_0 \) for Theorem 6.6. This is the place where Theorem 5.1 comes in to play.

As a consequence of the arguments used in Theorem 7.1, we establish

(4) Theorem 6.6 in Section 7.4.

(5) Proposition 6.8 in Section 7.5.

For convenience, we restate Theorem 6.6 to shape it into a more convenient form to work with. For its proof, we only discuss the case of \( \overline{S}(t) \), since the corresponding result for \( S(t) \) follows analogously.
Proposition 7.3. Assume the setting of Theorem 6.6 and the discussions above, recall the definition of \( \tau_B = \tau_B(\alpha_0, t_0) \) from (5.5), we define
\[
\tau = \tau(\alpha_0, t_0) := \tau_B \wedge \min \{ \tau_i(\alpha_0, t_0, \kappa = 2) : 1 \leq i \leq 8 \}.
\]
(Note that \( \tau_3(\alpha_0, t_0) \) does not depend on \( \kappa \) although we wrote as above for convenience.) Then,
\[
P(\tau(\alpha_0, t_0) > t_0, \Pi_S(\infty, t_0) \in \overline{A}(\alpha_0, t_0)) \geq 1 - \exp(-\beta_0^{1.9}).
\]
The same result holds for \( \Pi_S \) and \( \tau(\alpha_0, t_0) \).

Before moving on, we establish the following lemma which is not only a useful tool in the later section, but also leads to the proof of (3) of Lemma 6.10.

**Lemma 7.4.** Under the setting of Theorem 7.1, we have that
\[
\int_{t_0}^{\tau(2)} |S(t) - S_1(t)| dt \leq \alpha_0^{-2}.
\]

**Proof.** From the definition of \( \tau_3 \) (Section 6.1) and Lemma 5.2, we have
\[
\int_{t_0}^{\tau(2)} |S(t) - S_1(t)| dt \leq \int_{t_0}^{\tau(2)} \frac{\alpha_0^{-2} dt}{\tau_1(t; S)} + 1 \leq \alpha_0^{-2} \Pi_S[t_0, \tau(2)] \leq \alpha_0^{-2},
\]
where we obtained the last inequality from the definition of \( \tau_7 \) (Section 6.1.2).

**Proof of Lemma 6.10-(3).** Recall in Theorem 7.1 that \( \tau(1/2) \leq \tau(2) \) if \( \tau(2) > t_0 \). Thus, combining Theorem 7.1 and Lemma 7.4 directly implies (3) of Lemma 6.10.

**7.1. The error from the first order approximation.** As the first step towards establishing Theorem 7.1, we begin with studying \( \tau_3 \), the error of the first-order approximation.

**Lemma 7.5.** Under the setting of Theorem 7.1, we have
\[
P(\tau_0^+ = \tau(2) \wedge \hat{t}_0, \tau(2) > \hat{t}_0 | A) \leq \exp(-\beta_0^3).
\]

(Note that \( \tau_0^+ \geq \tau(2) \) by definition.)

**Proof.** Let \( t \in [\hat{t}_0, \hat{t}_0] \), and we begin with expressing that
\[
|S(t) - S_1(t; t_0, \alpha_0)| \leq |S(t) - S'(t; t_0, \alpha_0)| + |S'(t; t_0, \alpha_0) - S_1(t; t_0, \alpha_0)| + |S_1(t; \alpha_0) - S_1(t; t_0, \alpha_0)|.
\]
and note that the first term in the RHS is bounded by \( \alpha_0^{100} \) for any \( \Pi_S(\infty, t_0) \in A \), which is from Proposition 6.12. Moreover, for any \( t \in [\hat{t}_0, \tau(2) \wedge \hat{t}_0] \), Lemma 2.14 tells us that
\[
|S_1(t; \alpha_0) - S_1(t; t_0, \alpha_0)| = \int_{t_0}^{t_0} K_{\alpha_0}(t - x) d\Pi_S(x) \leq K_{\alpha_0}(\hat{t}_0 - t_0) \cdot \Pi_S(\hat{t}_0, t_0) \leq \alpha_0^{100}.
\]

The rest of the proof follows as a direct consequence of Proposition 4.29. In fact, by setting \( \alpha, t^-, h, \tau \) in Proposition 4.29 to be
\[
\alpha = \alpha_0, \ t^- = t_0, \ \hat{h} = \hat{t}_0, \ \tau = \tau(2),
\]
we see that the assumptions in the proposition are all satisfied. Thus, we obtain that
\[
P\left( |S(t) - S_1(t; t_0, \alpha_0)| \leq 4\alpha_0^{-1} \sigma_1 \sigma_2(t; S), \ \forall t \in [\hat{t}_0, \hat{t}_0 \wedge \tau(2)] | A \right) \geq 1 - 4 \exp\left(-\frac{\alpha_0^{-1}}{\beta_0^3}\right).
\]
The lower bound on \( S(t) - S_1(t) \) is obtained analogously, from the expression
\[
S(t) - S_1(t; t_0, \alpha_0) \geq (S'(t; t_0, \alpha_0) - S_1(t; t_0, \alpha_0)) - |S(t) - S'(t; t_0, \alpha_0)| - |S_1(t; t_0^-, \alpha_0) - S_1(t; t_0, \alpha)|,
\]
and Proposition 4.29.

Note that the corresponding analogue of (7.6) holds the same for the second-order approximation \( S_2(t) = S_2(t; t_0^-, \alpha_0) \), and we record this result in the following corollary.

**Corollary 7.6.** Under the setting of Theorem 7.1, there exists a constant \( c_* > 0 \) such that
\[
\mathbb{P}\left( \{|S(t) - S_2(t)| \leq 4\alpha_0^{1 - \epsilon} \sigma_1 \sigma_2 \sigma_3(t; S), \forall t \in [t_0, \tau(2) \wedge t_0^+]|A\right) \\
\geq 1 - 4 \exp\left(-\alpha_0^{-c_*}\right).
\]

**Proof.** The only difference in the proof compared to the previous lemma is the way we control \( S_2(t; t_0, \alpha) - S_2(t; t_0^-, \alpha) \). The difference is bounded by
\[
\frac{1}{(1 + 2\alpha)^2} \int_{t_0}^{t_0^+} K_{\alpha_0}(t - x) d\Pi_S(x) + \frac{\alpha}{1 + 2\alpha} \int_{t_0}^{t_0^+} \int_{t_0}^{t_0^+} J^{(\alpha_0)}_{\tau, \beta, \lambda, \mu} d\Pi_S(u) d\Pi_S(s),
\]
which is also smaller than \( \alpha_0^{100} \) for all \( t \in [t_0, \tau(2) \wedge t_0^+] \) due to the decay properties of \( K \) and \( J \) (Lemma 2.16).

Along with Lemma 4.28, this leads to the proof of (4) of Lemma 6.10.

**Proof of Lemma 6.10-(4).** The result is obtained by integrating the bound on \( |S(t) - S_2(t)| \) in Corollary 7.6 using Lemma 4.28 and relying on the estimate on \( \tau(2) \) from Theorem 7.1.

We conclude this subsection by presenting the analogue of Lemma 7.5 for the induction base.

**Corollary 7.7.** Under the setting of Proposition 7.3, we have
\[
\mathbb{P}\left( \tau_3 = \tau \wedge t_0, \tau > t_0 \right) \leq \exp\left(-\beta_0^3\right).
\]

**Proof.** Relying on the same decomposition (7.4), we can obtain the same estimate on \( \overline{S}(t) - \overline{S}_1(t; t_0^-, \alpha_0) \) from Lemma 6.14 and Corollary 6.13 along with Lemma 7.5.

### 7.2. Connection to the critical branching process

In this subsection, we estimate \( |S_1(t) - \alpha_0| \). By understanding the size of this quantity, we will eventually be able to control \( |S(t) - \alpha_0| \) in the next subsection. Define the stopping time \( \tau_0 \) as
\[
\tau_0 = \tau_0(\alpha_0, t_0, t_0^+) := \inf \left\{ t \geq t_0^+: (S_1(t) - \alpha_0) \notin \left(-2\alpha_0^{\frac{3}{2}} \beta_0^{6\theta}, \alpha_0^{\frac{3}{2}} \beta_0^{6\theta}\right) \right\}.
\]

Our goal is to establish the following Lemma:

**Lemma 7.8.** Under the setting of Theorem 7.1, we have
\[
\mathbb{P}\left( \tau_0 < t_0, \tau(2) > t_0 | A \right) \leq \exp\left(-\beta_0^3\right).
\]

To prove this lemma, we rely on the methods in Section 5.1. Recall the definition (5.6), and for \( t \geq s \geq t_0 \), define
\[
R(s, t) = R_c(s, t; \Pi_S[t_0^-, s], \alpha_0).
\]
We note that $R(t, t) = S_1(t)$ and for $t \geq t_0^*$ that

$$|R(t_0, t) - \alpha_0^*| \leq \int_{t_0}^{t_0^*} K_{\alpha_0}(t-x) d\Pi_S(x) + \int_{t_0}^{t_0^*} \int_{t_0}^{t} K_{\alpha_0}(u-x) du d\Pi_S(x)$$

$$\quad + \int_{t_0}^{t} \int_{t_0}^{t} |K_{\alpha_0}^* - K_{\alpha_0}(t-u)| K_{\alpha_0}(u-x) du d\Pi_S(x)$$

$$\leq \alpha_0^{100},$$

for all $t \in [t_0, \tau(2) \wedge t_0]$, similarly as (5.25). Then, (5.7) gives

$$S_1(t) - \alpha_0^* + O(\alpha_0^{100}) = R(t, t) - R(t_0, t)$$

$$= \int_{t_0}^{t} K_{\alpha_0}^*(t-x) \left\{ d\Pi_S(x) + (S(x) - S_1(x)) dx \right\},$$

where we wrote $d\Pi_S(x) := d\Pi_S(x) - S(x) dx$. This decomposes $S_1(t) - \alpha_0^*$ into a martingale part and a drift part.

**Proof of Lemma 7.8.** We begin with estimating the martingale part of $S_1(t) - \alpha_0^*$ in (7.8). We apply Corollary 4.9 in the following setting:

- Set $\tau = \tau(2)$, $f_t(x) = K_{\alpha_0}^*(t-x)$, $D = 1$ (such a choice of $D$ is justified by Corollary A.7).
- From the definition of $\tau_3$ and the bound in Lemma 2.15, we can set

$$M := \alpha_0^{3 \beta_0^{12 \theta}} \geq \int_{t_0}^{\tau(2)} (K_{\alpha_0}^*(t-x))^2 S(x) dx. \quad (7.9)$$

- $\Delta = \alpha_0^{-1}$, $A = C \alpha_0^{3 \beta_0^{12 \theta}}$, $\eta = \alpha_0 \beta_0^{C_0}$, $N = \beta_0^{5}$ and $\delta = \alpha_0^{10}$ (see definitions of $\tau_1$ and $\tau_7$).

Then Corollary 4.9 gives

$$\mathbb{P}\left( \int_{t_0}^{\tau(2)} K_{\alpha_0}^*(t-x) d\Pi_S(x) \leq \alpha_0 \beta_0^5 \sigma_1(t; S) + \alpha_0^{3 \beta_0^{6 \theta}}, \forall t \in [t_0, \tilde{t}_0] \big| A \right) \geq 1 - e^{-\beta_0^{C_0}};$$

$$\mathbb{P}\left( \int_{t_0}^{\tau(2)} K_{\alpha_0}^*(t-x) d\Pi_S(x) \geq -\alpha_0^{3 \beta_0^{6 \theta}}, \forall t \in [t_0, \tilde{t}_0] \big| A \right) \geq 1 - e^{-\beta_0^{C_0}}.$$

To control the drift term of (7.8), we use the bound given by $\tau_3$. Express that

$$\int_{t_0}^{\tau(2)} K_{\alpha_0}^*(t-x) |S(x) - S_1(x)| dx \leq \int_{t_0}^{\tau(2)} C \left( \frac{\alpha_0}{\sqrt{t-x}} \vee \alpha_0^2 \right) \frac{\alpha_0^{1-\epsilon} dx}{\pi_1(x; S) + 1} \leq \alpha_0^{2-2\epsilon},$$

where the last inequality is from Lemma 5.13 with parameters $\Delta_0 = \alpha_0^{-1}$, $\Delta_1 = \alpha_0^{-1} \beta_0^{C_0}$, $K = \alpha_0^{-1} \beta_0^{11 \theta}$ and $N_0 = \beta_0^{5}$. Thus, the conclusion follows by combining the above two bounds, along with the condition $|\alpha_0' - \alpha_0| \leq \alpha_0^{3 \beta_0^{6 \theta}}$ from $A_3(\alpha_0, t_0)$.

We record a direct consequence of our analysis, for a future usage in Section 9.1.
Corollary 7.9. Under the setting of Theorem 7.1, define the stopping time \( \tilde{\tau}_b \) as
\[
\tilde{\tau}_b := \inf \left\{ t \geq t_0^* : \int_{t_0^*}^t K_{\alpha_0}^* (t-x) d\Pi_{\alpha_0}(x) \geq \alpha_0^2 \beta_0^{6\theta} \right\}.
\]
Then, we have \( \mathbb{P}(\tilde{\tau}_b < \hat{t}_0, \tau(2) > \hat{t}_0 | A) \leq \exp(-\beta_0^5) \).

Note that the exponent \( 6\theta \) of the error term \( \alpha_0^2 \beta_0^{6\theta} \) is related with the length of the interval \([\hat{t}_0, \hat{t}_0] \), as seen in (7.9). If we are interested in a shorter interval, we can obtain the following stronger bound, which will be essential in the analysis of \( \tau_3 \) in Section 8.3. Its proof is omitted since it is identical to that of Lemma 7.8.

Corollary 7.10. Under the setting of Theorem 7.1, define the stopping time
\[
\tau'_b = \tau'_b(\alpha_0, t_0, t_0^*) := \inf \left\{ t \geq t_0^* : |S_1(t) - \alpha_0'| \leq \alpha_0^2 \beta_0^{6\theta} \sigma_1(t; S) + \alpha_0^2 \beta_0^{3\theta} \right\},
\]
with \( \alpha_0' = \mathcal{L}(t_0; \Pi_S[t_0, t_0], \alpha_0) \) as before. Then, we have
\[
\mathbb{P}(\tau'_b \leq \hat{t}_0, \tau(2) > \hat{t}_0 | A) \leq \exp(-\beta_0^5).
\]

We stress that the dependence on \( t_0^* \) in the definition (7.10) comes from \( \alpha_0' \) and \( S_1(t) = S_1(t, t_0, \alpha_0) \). We include \( \hat{t}_0 \) in the notation (unlike the case of \( \tau_b \)) to prevent confusion when it is used later in Section 8.3.

We continue this subsection by deriving an analogue for the process \( \tilde{S}(t) \) (and \( \tilde{S}(t) \)). To be more concrete, we set \( \tilde{S}_1(t) := \int_{t_0}^t K_\alpha(t-x) d\tilde{\Pi}_\alpha(x) \), and define the processes \( \tilde{R}_b(t) \) and \( \tilde{R}_b(t) \) as in (5.3) and (5.4), replacing \( \alpha \) with \( \alpha_0 \). Define \( \tilde{S}^+(t) = \tilde{S}(t) \lor \tilde{R}_b^+(t) \), and let
\[
\tilde{\tau}_b = \tilde{\tau}_b(\alpha_0, t_0) := \inf \left\{ t \geq t_0 : (\tilde{S}_1(t) - \alpha_0) \notin \left\{ -2\alpha_0^2 \beta_0^{6\theta}, \alpha_0^2 \beta_0^{5\theta} \sigma_1(t; \tilde{S}^+) + 2\alpha_0^2 \beta_0^{3\theta} \right\} \right\}.
\]

The major difference in the definition is that the stopping time reads the process starting from \( t_0 \), not from \( t_0^* \), where we entail Theorem 5.1. Then, the following corollary holds true:

Corollary 7.11. Under the setting of Proposition 7.3, we have
\[
\mathbb{P}(\tilde{\tau}_b = \tau \land \hat{t}_0, \tau > t_0) \leq \exp(-\beta_0^4).
\]

Proof. The proof is analogous to Lemma 7.8 using the decomposition (7.8) corresponding to \( \tilde{S}_1(t) \). This gives
\[
\tilde{S}_1(t) - \tilde{R}(t_0, t) \in \left( -\alpha_0^2 \beta_0^{6\theta}, \alpha_0^2 \beta_0^{5\theta} \sigma_1(t; \tilde{S}) + 2\alpha_0^2 \beta_0^{3\theta} \right)
\]
for all \( t \in [t_0, \tau(2) \land \hat{t}_0] \), which holds with high probability.

The only difference is that we are not guaranteed with the bound on \( \lvert \tilde{R}(t_0, t) - \alpha_0 \rvert \) as before, since we need to cover \( t \geq t_0 \), not just \( t \geq t_0^* \). To study \( \lvert \tilde{R}(t_0, t) - \alpha_0 \rvert \), consider the process \( R(t) = \tilde{R}_b(t; \Pi_{\alpha_0}[t_0, t_0], \alpha_0) \) (5.1). Then, \( \tau_{B1} \) from Theorem 5.1 gives the conclusion, since \( \tilde{R}(t_0, t) \) is an averaged version of \( R(t) \).

7.3. Proximity of the speed from the base rate. The goal of this subsection is to control all stopping times introduced in Sections 6.1.1 and 6.1.2 except \( \tau_3 \). Furthermore, we also establish Proposition 6.11. To this end, we exploit the bound on \( |S(t) - \alpha_0| \) based on (7.3).

Let \( F(\alpha_0, t) \) be the following random function which is essentially the sum of error bounds from \( \tau_3 \) (Section 6.1.1), and \( \tau_b \) (7.7):
\[
F(\alpha_0, t) := 4\alpha_0^{1-\epsilon} \sigma_1 \sigma_2(t; S) + \alpha_0^2 \beta_0^{C_\alpha+1} \sigma_1(t; S) + 3\alpha_0^{3/2} \beta_0^{6\theta}.
\]
Define
\[ \tau_0^+ := \inf\{t \geq t_0^* : |S(t) - \alpha_0| \geq F(\alpha_0, t)\}. \tag{7.12} \]

**Corollary 7.12.** Under the setting of Theorem 7.1, we have
\[ \mathbb{P}\left( \tau_0^+ \leq \tau(2) \land \tau(2) > t_0^* \mid A \right) \leq 3 \exp(-\beta_0^3). \]

**Proof.** This is an immediate consequence of Lemmas 7.5 and 7.8. \qed

Recall the definition of \( \tau_1^+(1/2) \) (7.2). In the following subsections, we show that each stopping time \( \tau_i^+ \) is unlikely to be smaller than or equal to \( \bar{\tau}(2) := \tau(2) \land \tau_0^+ \).

- In Section 7.3.1, we control \( \tau_1^+(1/2), \tau_5^+(1/2) \), the stopping times related with the square integral of \( |S(t) - \alpha_0| \).
- In Section 7.3.2, we study \( \tau_0^+(1/2), \tau_7^+(1/2), \tau_8^+(1/2) \), which describe the size of the aggregate.
- In Section 7.3.3, we work with \( \tau_1^+(1/2), \tau_2^+(1/2) \).
- In Section 7.3.4, we show how the analysis from Sections 7.3.1–7.3.3 can be done in the setting of Proposition 7.3.
- In Section 7.3.5, we establish several consequences of our analysis which will be useful later in Section 9.

### 7.3.1. The square integrals.

The main goal of this subsection is to deduce the following lemma.

**Lemma 7.13.** Under the setting of Theorem 7.1, let \( \bar{\tau}(2) \) be as (7.13). We have
\[ \mathbb{P}\left( \tau_1^+(1/2) \land \tau_5^+(1/2) \leq \bar{\tau}(2) \land \tau(2) > t_0^* \mid A \right) \leq \exp(-\beta_0^3). \]

**Proof.** The proof is based on estimating the integrals in the definition of \( \tau_1^+, \tau_5^+ \) directly from the bound we have from \( \tau_9^+ \). We first observe that
\[
\frac{1}{9} \int_{t_0^*}^{\bar{\tau}(2)} (S(t) - \alpha_0)^2 dt \leq \int_{t_0^*}^{\bar{\tau}(2)} \frac{\alpha_0^{2-2\epsilon}}{\pi_1(t; S) + 1} \left( \frac{\alpha_0^{2-2\epsilon}}{\pi_1(t; S) + 1} + \frac{\alpha_0^{2}}{\pi_1(t; S) + 1} \right) dt. \tag{7.14} \]

We control the integral of each term in the RHS separately.

The integral over the constant \( \alpha_0^3 \beta_0^{12\theta} \) turns out to give the leading order and satisfies
\[ \int_{t_0^*}^{\bar{\tau}(2)} \alpha_0^3 \beta_0^{12\theta} dt \leq \alpha_0 \beta_0^{23\theta}. \tag{7.15} \]

To control the first term in the RHS of (7.14), we write
\[ \Pi_{S[t_0^*, \bar{\tau}(2)]} = \{p_1 < p_2 < \ldots < p_N\}. \]

Setting \( p_0 = p_{-1} = t_0^*, p_{N+1} = \bar{\tau}(2) \), we express
\[
\int_{t_0^*}^{\bar{\tau}(2)} \frac{\alpha_0^{2-2\epsilon} dt}{\pi_1(t; S) + 1} \left( \frac{\alpha_0^{2-2\epsilon}}{\pi_1(t; S) + 1} + \frac{\alpha_0^{2}}{\pi_1(t; S) + 1} \right) \leq 2 \int_{t_0^*}^{\bar{\tau}(2)} \frac{\alpha_0^{2-2\epsilon} dt}{\pi_1(t; S) + 1} \left( \frac{\alpha_0^{2-2\epsilon}}{\pi_1(t; S) + 1} + \frac{\alpha_0^{2}}{\pi_1(t; S) + 1} \right) \leq 2 \sum_{i=0}^{N} \int_{0}^{p_{i+1} - p_i} \frac{dx}{(x + 1)(x + 2 + p_i - p_{i-1})}. \]

THE CRITICAL ONE-DIMENSIONAL MDLA

65
We can bound the summand at $i = 0$ by 1, and the rest can be written as
\[
\sum_{i=1}^{N} \int_0^{p_{i+1} - p_i} \frac{dx}{(x + 1)(x + 2 + p_i - p_{i-1})} = \sum_{i=1}^{N} \int_0^{p_{i+1} - p_i} \frac{dx}{p_i - p_{i-1} + 1} \left( \frac{1}{x + 1} - \frac{1}{x + 2 + p_i - p_{i-1}} \right) \leq \sum_{i=1}^{N} \frac{\beta_0^2}{p_i - p_{i-1} + 1},
\]
where we used $\log(t_0 - t_0') \leq \beta_0^2$ to obtain the last inequality. The following statement gives a control on the quantity in the RHS, and its proof is given after finishing the proof of Lemma 7.13.

**Claim 7.14.** Under the above setting,
\[
\mathbb{P} \left( \sum_{i=1}^{N} \frac{1}{p_i - p_{i-1} + 1} \geq \alpha_0^{-\frac{1}{2} - \epsilon} \bigg| \mathcal{A} \right) \leq \exp \left( -\beta_0^3 \right).
\]

Using the claim, we obtain
\[
\mathbb{P} \left( \int_{t_0'}^{\hat{\tau}(2)} \frac{\alpha_0^{2-2\epsilon} dt}{(\pi_1(t; S) + 1)(\pi_2(t; S) + 1)} \geq \alpha_0^{\frac{3}{2} - 4\epsilon} \bigg| \mathcal{A} \right) \leq \exp \left( -\beta_0^3 \right). \tag{7.16}
\]

We move on to the second term in the RHS of (7.14). Using Lemma 5.2
\[
\int_{t_0'}^{\hat{\tau}(2)} \frac{\alpha_0^2 \beta_0^{2C_s+2} dt}{\pi_1(t; S) + 1} \leq \alpha_0^2 \beta_0^{2C_s+4} |\Pi_0[t_0^+, \hat{\tau}(2)]| \leq \alpha_0 \beta_0^{11\theta}, \tag{7.17}
\]
Note that we bounded $\log(t_0 - t_0') \leq \beta_0^2$.

Thus, we obtain the conclusion for $\tau_4^*(1/2)$ by combining (7.14), (7.15), (7.16), and (7.17), which gives that
\[
\mathbb{P} \left( \int_{t_0'}^{\hat{\tau}(2)} (S(t) - \alpha_0)^2 dt \geq \alpha_0 \beta_0^{24\theta} \bigg| \mathcal{A} \right) \leq \exp \left( -\beta_0^3 \right).
\]

Note that the result for $\tau_5^*(1/2)$ follows as well, by writing
\[
\int_{t_0'}^{\hat{\tau}(2)} (S(t) - \alpha_0)^2 dt \leq 2\alpha_0 \beta_0^{C_s} \int_{t_0'}^{\hat{\tau}(2)} (S(t) - \alpha_0)^2 dt,
\]
from the definition of $\tau_1(\kappa = 2)$ (Section 6.1.1).

**Proof of Claim 7.14.** Let $\hat{\alpha}_0 := 2\alpha_0 \beta_0^{C_s}$, and let
\[
\Pi_{\hat{\alpha}_0}[t_0^+, \hat{\tau}] = \{p'_1 < p'_2 < \ldots < p'_{N'}\}.
\]
Due to the definition of $\tau_1(\kappa = 2)$ (Section 6.1.1), it suffices to establish the main inequality in terms of
\[
\sum_{i=1}^{N} \frac{1}{p'_i - p'_{i-1} + 1},
\]
where we set $p'_0 = t_0^+$ as before.

Our idea is to count the number of neighboring pairs of points with distance less than $\alpha_0^{-1/2}$. To this end, we make the following simple observation:

- If $p'_i - p'_{i-1} \leq \alpha_0^{-1/2}$, then there exists $k \in \mathbb{Z}$ such that $p'_i, p'_{i-1} \in t_0^+ + \alpha_0^{-1/2}[k, k + 2]$. 

Note that for each $k \in \mathbb{Z}$,
\[
\Pr\left( \left| \Pi_{\tilde{\alpha}_0} \left[ t_0^+ + k\alpha_0^{-\frac{1}{2}}, t_0^+ + (k + 2)\alpha_0^{-\frac{1}{2}} \right] \right| \geq 2 \right) \leq C\alpha_0^2\beta_0^2,
\]
where $C > 0$ is an absolute constant. Further, for all even $k$, the above events are independent (and same for odd $k$). Thus, we apply a Chernoff bound for even $k$ and odd $k$ separately, and then use a union bound over the two to deduce that
\[
\Pr\left\{ \left\{ k \in \mathbb{Z}, k \leq \alpha_0^{-\frac{1}{2}}\beta_0^{100} : \left| \Pi_{\tilde{\alpha}_0} \left[ t_0^+ + k\alpha_0^{-\frac{1}{2}}, t_0^+ + (k + 2)\alpha_0^{-\frac{1}{2}} \right] \right| \geq 2 \right\} \right\} \leq \alpha_0^{-\frac{1}{2}}\beta_0^{110} \leq \exp \left( -\beta_0^3 \right).
\]
Furthermore, in each interval $[t_0^+ + k\alpha_0^{-1}, t_0^+ + (k + 1)\alpha_0^{-1}]$, there can be more than $2\beta_0^{10}$ particles with probability at most $\exp \left( -\beta_0^3 \right)$. Thus, we take a union bound over $0 \leq k' \leq \alpha_0^{-1}\beta_0^{100}$ and obtain that
\[
\Pr \left( \left| \Pi_{\tilde{\alpha}_0} \left[ t - \alpha_0^{-1}, t \right] \right| \leq 4\beta_0^{10}, \quad \forall t \in [t_0^+, t_0] \right) \geq 1 - \exp \left( -\beta_0^3 \right).
\]
This implies that at each interval $[t_0^+ + k\alpha_0^{-1}, (k + 2)\alpha_0^{-1}]$ with at least two points, there cannot be more than $4\beta_0^{10}$ points. Also, we have that there are at most $\alpha_0^{-1}\beta_0^{110}$ points in the entire interval.

Combining the above information, we have
\[
\sum_{i=1}^{N'} \frac{1}{p_i^2 - p_{i-1}^2 + 1} \leq \alpha_0^{-\frac{1}{2}}\beta_0^{110} \cdot 4\beta_0^{10} + \alpha_0^{-\frac{1}{2}}\beta_0^{110} \leq \alpha_0^{-\frac{1}{2}}\epsilon,
\]
with probability at least $1 - \exp \left( -\beta_0^3 \right)$.

7.3.2. The size of the aggregate. The goal of this subsection is to study $\tau_0^\ast(1/2), \tau_1^\ast(1/2), \tau_8^\ast(1/2)$ (Section 6.1.2).

Lemma 7.15. Under the setting of Theorem 7.1, let $\tilde{\tau}(2)$ be as (7.13), and set $\hat{\tau}^\ast(\kappa) = \min \{ \tau_i^\ast(\kappa) : i = 6, 7, 8 \}$. Then, we have
\[
\Pr \left( \hat{\tau}^\ast(1/2) \leq \tilde{\tau}(2), \ \tau(2) > t_0 \ | \ \mathcal{A} \right) \leq \exp \left( -\beta_0^3 \right).
\]

A key quantity in establishing the lemma is the following. For each $\Delta > \alpha_0^{-1}$, we define
\[
\tau_{10}^\ast(\Delta) := \inf \left\{ t \geq t_0^+ : \left| \int_{t-\Delta}^{t} S(s) - \alpha_0 |ds| \right| \geq \alpha_0^{-\frac{3}{2}}\theta \Delta \right\}.
\]
In fact, if we have a good control on $\tau_{10}^\ast$, then we can estimate the number of points in an interval of size $\Delta$ using Corollary 4.7. Based on Corollary 7.12 we begin with showing the following result.

Lemma 7.16. Under the setting of Theorem 7.1, let $\tilde{\tau}(2)$ be as (7.13). Then, we have for all $\Delta > \alpha_0^{-1}$ that
\[
\Pr \left( \tau_{10}^\ast(\Delta) \leq \tilde{\tau}(2), \ \tau(2) > t_0 \ | \ \mathcal{A} \right) \leq \exp \left( -\beta_0^3 \right).
\]

Proof. Based on the definition of $\tilde{\tau}(2)$ and $\tau_0$ (7.12), (7.13), we have for $t \in [t_0^+ + \Delta, \tilde{\tau}(2)]$ that
\[
\int_{t-\Delta}^{t} |S(t) - \alpha_0| dt \leq \int_{t-\Delta}^{t} \left\{ \frac{3\alpha_0^{1-\epsilon}}{\pi_1(t, S)} + 1 + \frac{\alpha_0\beta_0^5}{\sqrt{\pi_1(t, S)} + 1} + \frac{3\alpha_0^{3/2}\beta_0^6}{\beta_0^6} \right\} dt.
\]
Using Lemma 5.2 the RHS is upper bounded by
\[
\alpha_0^{-1}\beta_0^5 |\Pi_S[t - \Delta, t]| + 2\alpha_0\beta_0^5 \sqrt{\Delta |\Pi_S[t - \Delta, t]|} + 3\Delta\alpha_0^{3/2}\beta_0^6.
\]
Among these terms, we control $|\Pi_S[t-\Delta, t]|$ from Corollary 4.7. The definition of $\tau_1$ gives that
\[
\int_{(t-\Delta) \land \tau(2)}^{t \land \tau(2)} S(x) \, dx \leq 2\Delta \alpha_0 \beta_0^{C_0},
\]
and the RHS is at least $\beta_0^{C_0} \geq 1$ since $\Delta \geq \alpha_0^{-1}$. Therefore, Corollary 4.7 tells us that
\[
P\left( |\Pi_S[(t - \Delta) \land \tau(2), t \land \tau(2)]| \leq 5\Delta \alpha_0 \beta_0^{C_0}, \ \forall t \in [t_0^+, \Delta + \hat{t}_0] \right) \geq 1 - \exp\left(-\beta_0^{C_0/3}\right).
\]
When the above event holds, (7.19) is upper bounded by
\[
\alpha_0^{2-2\epsilon} \Delta + \alpha_0^{1-\epsilon} \sqrt{\Delta} + \alpha_0^{3-\epsilon} \Delta \leq \alpha_0^{3-\epsilon} \Delta,
\]
where the inequality followed from $\alpha_0^{1/2} \Delta \geq \sqrt{\Delta}$. Thus, combining the error probabilities in the three events, we deduce conclusion.

We can conduct an analogous but simpler investigation on $|S(t)-\alpha_0|$ and $|S_1(t)-\alpha_0|$, leading to (1) and (2) of Lemma 6.10. We restate them in the following corollary without spelling out the details of the proof due to similarity.

**Corollary 7.17.** Under the setting of Theorem 7.1, we have that
\[
P\left( \int_{t_0^+}^{t} |S(s) - \alpha_0| \, ds \leq \alpha_0^{1-\epsilon}, \ \forall t \in [t_0^+, \tau(2)] \ \mid \mathcal{A} \right) \leq \exp\left(-\beta_0^{C_0/3}\right); \quad (7.20)
\]
\[
P\left( \int_{t_0^+}^{t} |S_1(s) - \alpha_0| \, ds \leq \alpha_0^{1-\epsilon}, \ \forall t \in [t_0^+, \tau(2)] \ \mid \mathcal{A} \right) \leq \exp\left(-\beta_0^{C_0/3}\right).
\]

We conclude the subsection by establishing Lemma 7.15.

**Proof of Lemma 7.15.** Set $\tilde{\tau}'(\Delta) := \tilde{\tau}(2) \land \tau_0^+(\Delta) = \tau(2) \land \tau_0^+ \land \tau_0^1(\Delta)$. Having Lemma 7.16 on hand, Lemma 7.15 follows from applying Corollary 4.7 for different values of $\Delta$. Indeed, we can choose $\Delta = \alpha_0^{-1}$ for $\tau_0^1(1/2)$ and $\Delta = \alpha_0^{-1} \beta_0^{C_0}$ for $\tau_0^1(1/2)$ to see that
\[
P\left( \tau_0^1(1/2) \leq \tilde{\tau}'(\alpha_0^{-1}), \ \tau(2) > \hat{t}_0 \ \mid \mathcal{A} \right) \leq \exp\left(-\beta_0^{3}\right); \quad (7.20)
\]
\[
P\left( \tau_0^1(1/2) \leq \tilde{\tau}'(\alpha_0^{-1} \beta_0^{C_0}), \ \tau(2) > \hat{t}_0 \ \mid \mathcal{A} \right) \leq \exp\left(-\beta_0^{C_0/3}\right).
\]

The remaining task is to study $\tau_0^+(1/2)$. On the event $\mathcal{A}_2 \supset \mathcal{A}$ (Section 6.1.4), we have that
\[
\hat{t}_0^+ := \inf \left\{ t \geq t_0^+: X_t - X_{\hat{t}_0} \leq \frac{\alpha_0}{50} (t - t_0) \right\} \leq \tau_0^+(1/2). \quad (7.21)
\]
Letting $\Delta_0 = \alpha_0^{-3/2}$, we have for all $t \in [t_0^+, \hat{t}_0^+]$, $t \leq \tilde{\tau}'(\Delta_0)$ that
\[
\int_{t-\Delta_0}^{t} S(t) \, dt \in \left[ \frac{\alpha_0 \Delta_0}{2}, 2\alpha_0 \Delta_0 \right].
\]
Choose $M = 2\Delta_0$ to apply Corollary 4.6 and union bound over all $t \in \mathcal{T}_{\Delta_0}$, where $\mathcal{T}_{\Delta_0} := \{ t \in (t_0^+, \hat{t}_0^+) : t = t_0^+ + k\Delta_0, k \in \mathbb{Z} \}$. This gives that
\[
P\left( |\Pi_S[t-\Delta_0, t]| \geq \frac{\alpha_0 \Delta_0}{3}, \ \forall t \in \mathcal{T}_{\Delta_0}, \ t \leq \tilde{\tau}'(\Delta_0) \right) \geq 1 - \exp\left(-\alpha_0^{-\epsilon}\right).
We can also see that if this event holds, then \( \tilde{\tau}_0^+ \geq \tilde{\tau}' \). Therefore, combining this with (7.21),

\[
\mathbb{P}\left( \tau_0^* (1/2) \leq \tilde{\tau}' \left( \alpha_0^{-3/2} \right), \tau(2) > \hat{t}_0 \mid A \right) \leq \exp (-\alpha_0^{\delta}), \tag{7.22}
\]

Thus, we obtain conclusion from Lemma 7.16 equations (7.20) and (7.22).

7.3.3. The magnitude of the speed. We conclude this section by establishing the estimates on \( \tau_1 \) and \( \tau_2 \), and their corresponding analogue for \( \overline{S}(t) \). Further, we conclude the proof Theorem 7.1 by combining all results obtained throughout this section. We begin with controlling \( \tau_2 \).

Lemma 7.18. Under the setting of Theorem 7.1, write \( \tau_2^* (\kappa) = \tau_2(\alpha_0, t_0^*, \kappa) \). Conditioned on \( A \) and \( \{ \tau(2) > \hat{t}_0 \} \), we have \( \tau_2^*(1/2) > \tau(2) \) almost surely.

Proof. Recall the definition of \( S_1(t) \) given by (2.16):

\[
S_1(t) = \int_{t_0}^t K_{\alpha_0}(t-x) d\Pi_S(x).
\]

Based on the definitions of \( \tau_7 \) (Section 6.1.2) and \( A_2 \) (Section 6.1.4), we express that

\[
\int_{t_0}^t K_{\alpha_0}(t-x) d\Pi_S(x) \leq \sum_{k: 0 \leq \kappa^{-1} t_0^{-1} \leq t-t_0} |\Pi_S([t-(k+1)\alpha^{-1}, t-k\alpha^{-1}])| K_{\alpha_0}(k\alpha^{-1}) \leq \sum_{k: 0 \leq \kappa^{-1} t_0^{-1} \leq t-t_0} \frac{C_\alpha \beta_0 C_\kappa}{\sqrt{k\alpha^{-1}+1}} e^{-\alpha_0 k} + \alpha_0^{-1} \beta_0^0 K_{\alpha_0}(t-t_0) \leq \frac{1}{2} \alpha_0 \beta_0 C_\kappa + 1,
\]

where we used Lemma 2.14 to obtain the second inequality.

Corollary 7.19. Under the setting of Proposition 7.3, we have

\[
\mathbb{P}(\overline{\tau}_2 = \tau \wedge \hat{t}_0, \tau > \hat{t}_0) \leq \exp (-\beta_0^3).
\]

Proof. The proof is identical to the previous lemma, except that we are not conditioning on \( A_2 \). Instead, we use Lemma 6.14 to conclude the proof.

Lemma 7.20. Under the setting of Theorem 7.1, write \( \tau_1^* (\kappa) = \tau_1(\alpha_0, t_0^*, \kappa) \). Conditioned on \( A \) and \( \{ \tau(2) > \hat{t}_0 \} \), we have \( \tau_1(1/2) > \tau(2) \) almost surely.

We establish Lemma 7.20 by studying the original formula (2.2) of \( S(t) \). To this end, we first introduce the following lemma.

Lemma 7.21. Let \( \alpha > 0 \) sufficiently small and \( 0 < x \leq \alpha^{-1} \). Let \( W(s) \) be a continuous time simple random walk. We have

\[
\mathbb{P}(\forall s > 0, W(s) < x + \alpha s) \leq 3 \alpha x
\]

Proof. Let \( \xi_{\alpha} \) be the unique solution in \((0,1)\) to the equation

\[
\left( \frac{\xi + \xi^{-1} - 1}{2} \right) + \alpha \log \xi = 0. \tag{7.23}
\]

We claim that \( \xi_{\alpha} = 1 - 2\alpha + O(\alpha^2) \). Indeed, if \( \xi = 1 - \delta \) then

\[
\xi^{-1} = 1 + \delta + \delta^2 + O(\delta^3), \quad \log \xi = -\delta + O(\delta^2).
\]

Substituting these estimates in (7.23) we get the equation \( \frac{x^2}{2} + O(\delta^3) - \alpha \delta + O(\alpha \delta^2) = 0 \) and therefore \( \delta = 2\alpha + O(\alpha^2) \). We have that

\[
\mathbb{E}^{\alpha s-W(s)} = \xi_{\alpha}^{\alpha s} \sum_{k=1}^{\infty} \frac{e^{-s} s^k}{k!} \left( \frac{\xi + \xi^{-1}}{2} \right)^k = \exp \left( \alpha s \log \xi + \left( \frac{\xi + \xi^{-1}}{2} - 1 \right) s \right).
\]
Thus, by the definition of $\xi_s$ the process $M_s := \xi_{\alpha s - W(s)}$ is a martingale. Define the stopping time $T_0 := \inf\{s > 0 : W(s) \geq x + \alpha s\}$.

Almost surely we have $M_{t_{\alpha T_0}} \to 1$ as $T_0 \to \infty$. On $\{T_0 < \infty\}$ we have $-x \leq \alpha T_0 - W(T_0) \leq -x + 1$ and therefore, by the bounded convergence theorem we have

$$1 = \mathbb{E}M_0 = \mathbb{E}\left[1 \{T_0 < \infty\} \xi_{\alpha T_0 - W(T_0)}\right] \leq \theta^{-x} \mathbb{P}(T_0 < \infty).$$

Thus $\mathbb{P}(T_0 < \infty) \geq \xi^x \geq (1 - 3\alpha)^x \geq 1 - 3ax$. This finishes the proof of the lemma.

**Proof of Lemma 7.20.** Throughout the proof, we condition on $A$ and $\{\tau(2) > \hat{t}_0\}$, and let $t \in [\hat{t}_0, \tau(2)]$. Because of $\tau_\gamma$ and $A_2$, we have for all $s \in [\hat{t}_0, t]$ that

$$|\Pi_S[t - s, t]| \leq 2\beta_0^3 + 2\alpha_0\beta_0^3 s =: Y'(s). \quad (7.24)$$

From Lemma 7.21 we have

$$S(Y') \leq 12\alpha_0\beta_0^3,$$

where $S(Y')$ is the speed process $\{2.3\}$ in terms of $Y'$. Let us define $Y''$ as

$$Y''_t(s) := \begin{cases} |\Pi_S[t - s, t]| & \text{for } s \leq t - \hat{t}_0; \\ |\Pi_S[t_0, t]| + \alpha_0\beta_0^3(s - t + \hat{t}_0) & \text{for } s \geq t - \hat{t}_0. \end{cases}$$

Then, an analogous argument as Proposition 6.12 implies that under the event $A$,

$$|S(t) - S(Y'')| \leq \alpha_0^{100},$$

for any $t \geq \hat{t}_0$. Thus, since $S(Y') \geq S(Y'')$, we obtain $S(t) \leq 12\alpha_0\beta_0^3 + \alpha_0^{100} \leq \frac{1}{2}\alpha_0\beta_0^3$ which holds deterministically. This tells us that $\tau_1'(1/2) > \tau(2)$ almost surely.

It turns out that the proof of Lemma 7.20 applies similarly to the case of $\overline{S}(t)$, and we record the result below.

**Corollary 7.22.** Under the setting of Proposition 7.3, we have

$$\mathbb{P}\left(\tau_1 = \tau \wedge \hat{t}_0 \right) \leq \exp\left(-\beta_0^3\right).$$

**Proof.** The proof of Lemma 7.20 applies analogously except for the following issues:

- The statement we have does not condition on $A$, in contrast to Lemma 7.20.
- Proof of Lemma 7.20 deals only with $t \geq t_0^\circ$, whereas we need to cover all $t \geq t_0$ in our case.

The first item is resolved using Lemma 6.14. For the second one, observe that the reason we wanted $t \geq t_0^\circ$ in the previous proof was to deduce $\left(7.24\right)$; since we did not have control on $|\Pi_S[t - \alpha_0^{-1}t|$, for $t \leq t_0$, we started looking at $t_0^\circ$ to ensure such a linear bound on $Y_t(s)$.

In our case, since $S(t) = \alpha_0$ on $t \in [\hat{t}_0, t_0)$, we do not have such an issue. In fact, we can obtain $\left(7.24\right)$ for $S(t)$, $t \geq t_0$ from $\left(5.12\right)$ and Lemma 5.8 along with $\tau$. One advantage of this is that we also have

$$\mathbb{P}(\tau_1 = t_0) \leq \exp\left(-\beta_0^3\right), \quad (7.25)$$

which means that we do not need to include the event $\{\tau > t_0\}$ as before. Thus, having this discussion in mind, following the proof of Lemma 7.20 concludes the proof.

We conclude this subsection by giving the proof of Theorem 7.1.

**Proof of Theorem 7.1.** The proof follows by combining the lemmas discussed in this section. Namely,

- Control on $\tau_3^\circ$: Lemma 7.5
- Control on $\tau_1'(1/2), \tau_2'(1/2)$: Corollary 7.12 and Lemma 7.13
- Control on $\tau_6'(1/2), \tau_7'(1/2)$, $\tau_4'(1/2)$: Lemma 7.15 (with Corollary 7.12).
- Control on $\tau_1'(1/2)$ and $\tau_4'(1/2)$: Lemmas 7.20 and 7.18

Combining all the aforementioned results gives the conclusion.
7.3.4. **Proximity to \( \alpha \) of the induction base.** In this section, we explain how to deduce the corresponding analogues of Lemmas 7.13 and 7.15 for \( S(t) \). The only difference is that the definition of \( \tau_b \ (7.11) \) compared to \( \tau_0 \ (7.7) \), and hence the corresponding change needs to be made in \( \tau_0^* \ (7.12) \).

Recall the definition of \( S^* \ (7.11) \). Let us define

\[
F(\alpha_0, t) := 4\alpha_0^{1-\epsilon} \sigma_1 \sigma_2(t; S) + \alpha_0 \beta_0^{5\epsilon} \sigma_1(t; S^*) + 3\alpha_0^{3/2} \beta_0^{6\epsilon}.
\]

Then, the analogue of \( \tau_0^+ \) is given by

\[
\tau_9 := \inf \{ t \geq t_0 : |S(t) - \alpha_0| \geq F(\alpha_0, t) \}.
\]

Furthermore, we set \( \tau_{10}(\Delta) \) to be the analogue of (7.18), which is

\[
\tau_{10}(\Delta) := \inf \{ t \geq t_0 : \int_{t - \Delta \vee t_0}^{t} \{S(s) - \alpha_0\} \, ds \geq \alpha_0^{\frac{3}{2} - 7\epsilon} \Delta \}.
\]

Note the important difference from the previous definitions: they read off the process starting from \( t_0 \), not \( \tau_0^* \). Then, we have the following as corresponding counterparts of Corollary 7.12 and Lemma 7.16

\[
P(\tau_9 \leq \tau \wedge \hat{t}_0, \tau > t_0) \leq \exp(-\beta_0^2);
\]

\[
P(\tau_{10}(\Delta) \leq \tau \wedge \tau_9 \wedge \hat{t}_0, \tau > t_0) \leq 5 \exp(-\beta_0^3),
\]

where the second one holds for all \( \Delta \geq \alpha_0^{-1} \). In fact, the first one is immediate from the definitions of \( \tau_3 \) and \( \tau_b \), and the second one follows from the same proof as Lemma 7.16 relying on \( \tau_7 \) and \( \tau_{b2} \) (Theorem 5.1) when estimating \( |\Pi_{S^*}[t - \Delta, t]| \). Thus, the following corollary can be obtained analogously as Lemmas 7.13 and 7.15 whose details are omitted thanks to similarity.

**Corollary 7.23.** **Under the setting of Proposition 7.3,** let \( \tau' = \min\{\tau_i : 4 \leq i \leq 8\} \). Then, we have

\[
P(\tau' \leq \tau \wedge \hat{t}_0, \tau > t_0) \leq 2 \exp(-\beta_0^2).
\]

7.3.5. **The martingale generated by the gap \( S - \alpha_0 \).** In this subsection, we conduct further analysis based on \( \tau_{10}(\Delta) \ (7.18) \) which will be useful later in Section 9. Define the gap process

\[
\Pi_{S\Delta\alpha_0}[a, b] := \Pi_S[a, b] \triangle \Pi_{\alpha_0}[a, b], \tag{7.26}
\]

and write \( d\Pi_{S\Delta\alpha_0}(x) := d\Pi_S(x) - d\Pi_{\alpha_0}(x) \). We begin with stating a direct consequence of Theorem 7.1 and Lemma 7.16

**Corollary 7.24.** **Let** \( \alpha_0, t_0, r > 0 \), **set** \( t_0^*, \hat{t}_0 \) **as before, and let** \( \Delta_0 := \alpha_0^{-\epsilon} 7\epsilon \). **Define**

\[
\tau_{\text{gap}}^{(1)} := \inf \{ t \geq t_0^* + \Delta_0 : |\Pi_{S\Delta\alpha_0}|[(t - \Delta_0), t] \geq \beta_0^5 \}.
\]

If \( \Pi_{S(-\infty, t_0]} \in \mathcal{R}(\alpha_0, r; [t_0]) \), **then we have**

\[
P(\tau_{\text{gap}}^{(1)} < \hat{t}_0 | \mathcal{F}_{t_0}) \leq 2 \exp(-\beta_0^2) + r.
\]

**Proof.** Proof follows directly from applying Corollary 4.7 based on Lemma 7.16, and then combining with Theorem 7.1.

The next object of interest is the martingale defined as

\[
G(t) := \int_{t_0^*}^{t} K_{t_0^*}(t - x) \{d\Pi_{S\Delta\alpha_0}(x) - (S^*(t) - \alpha_0)dx\},
\]

which satisfies the following property.
Corollary 7.25. Define the stopping time
\[ \tau_{\text{gap}}^{(2)} := \left\{ t \geq t_0^* : |G(t)| \geq \alpha_0 \beta_0^{C_0} \sigma_1(t; S \Delta \alpha_0) + \alpha_0^{\frac{3}{2} - \epsilon} \right\}. \]
Then, under the setting of Corollary 7.24, we have
\[ \mathbb{P} \left( \tau_{\text{gap}}^{(2)} < t_0^* \bigg| \mathcal{F}_{t_0} \right) \leq \exp \left( -\beta_0^{2} \right) + r. \]

Proof. Recall the definition of \( \tau(2) \) (Theorem 7.1), \( F(\alpha_0, t) \) and \( \tau_0^+ \) (7.12) and write
\[ t \wedge \tau(2) \wedge \tau_0^+ = \int_{t_0^*}^{t} (K_{\alpha_0}^*(t - x))^2 |S(x) - \alpha_0| dx \]
\[ \leq \int \left[ C \left( \frac{\alpha_0^2}{t - x + 1} + \alpha_0^4 \right) (\alpha_0^{1 - \epsilon} \sigma_1(t; S)^2 + \alpha_0 \beta_0^{C_0 + 1} \sigma_1(t; S) + \alpha_0^{\frac{3}{2} \beta_0^{6}}) \right] dx. \]
Each terms in the integral can be estimated using Lemmas 5.2 and 5.13 with parameters \( \Delta_0 = \alpha_0^{-1}, \Delta_1 = \alpha_0^2 \beta_0^{C_0}, K = \alpha_0^{1 - \epsilon} \) and \( N_0 = \beta_0^{5} \). This gives that for \( t \leq t_0, \)
\[ \int_{t_0^*}^{t} (K_{\alpha_0}^*(t - x))^2 |S(x) - \alpha_0| dx \leq \alpha_0^{\frac{7}{2} - \epsilon}, \]
and hence we can apply Corollary 4.9 with parameters
\[ A^2 = M = \alpha_0^{\frac{7}{2} - \epsilon}, \Delta = \alpha_0^{\frac{3}{2} - \epsilon}, D = 1, \eta = \alpha_0 \beta_0^{C_0}, N = \beta_0^{5}, \delta = \alpha_0^{10}. \]
Combining the result with Theorem 7.1 concludes the proof. \( \square \)

The same proof applies to establishing the next corollary, whose details are omitted due to similarity.

Corollary 7.26. Let \( \alpha_0 := \alpha_0 - \alpha_0^{\frac{3}{2} - \epsilon}, d\Pi_{\Delta \alpha_0}(x) := d\Pi_{\alpha_0}(x) - d\Pi_{\alpha_0}(x) - (\alpha_0 - \alpha_0) dx, \) and define the stopping time
\[ \tau_{\text{gap}}^{(3)} := \left\{ t \geq t_0^* : \int_{t_0^*}^{t} K_{\alpha_0}^*(t - x) d\Pi_{\Delta \alpha_0}(x) \geq \alpha_0 \beta_0^{C_0} \sigma_1(t; \Delta \alpha_0) + \alpha_0^{\frac{3}{2} - \epsilon} \right\}. \]
Under the setting of Corollary 7.24, we have
\[ \mathbb{P} \left( \tau_{\text{gap}}^{(3)} < t_0^* \bigg| \mathcal{F}_{t_0} \right) \leq \exp \left( -\beta_0^{2} \right) + r. \]

7.4. The induction base: regularity of the fixed rate process. We conclude the proof of Proposition 7.3 and Theorem 6.6. Since we collected most of the ingredients from the previous subsections, we need the last piece of argument that tells us \( \tau \) is not likely to be trivial, i.e., \( \tau > t_0. \)

Lemma 7.27. Under the setting of Proposition 7.3, we have
\[ \mathbb{P} (\tau > t_0) \geq 1 - 3 \exp \left( -\beta_0^{2} \right). \]

Proof. From their definitions, it is clear that \( \tau_i > t_0 \) for all \( 4 \leq i \leq 8. \) Furthermore, we saw from (7.25) in Corollary 7.22 that \( \tau_1 > t_0 \) w.h.p. Thus, what remain to investigate are \( \tau_2 \) and \( \tau_3. \)

For \( \tau_2, \) recall the definition of \( R_0(t) \) from (5.2) and note that \( R_0(t_0) = S(t_0). \) Thus, Lemma 5.8 implies the desired estimate on \( \tau_2. \)
To understand \( \tau_3, \) we write
\[ \left| \left[ \bar{S}(t_0) - \bar{S}_1(t_0) \right] - \left[ \bar{S}'(t_0; t_0, \alpha_0) - \bar{S}_1(t_0) \right] \right| \leq |\bar{S}(t_0) - \bar{S}'(t_0; t_0, \alpha_0)|. \]
The RHS is \( O(\alpha^{50}) \) w.h.p. due to Corollary 6.13 and Lemma 6.14 and the second term of the LHS can be controlled from Proposition 4.29. Since we are only interested in estimating the above at point \( t_0 \) (not for entire \( t \)), it is not difficult to see that the third assumption of (4.28) can be weakened into

\[
\sup_{t_0 \leq t < t_0} \overline{S}(t) \leq \alpha_0^{1-\frac{1}{20}}, \quad \overline{S}_1(t_0) \leq \alpha_0^{1-\frac{1}{20}}.
\]

We also have the first two assumptions since \( \overline{S}(t) = \alpha_0 \) for \( t \in [t_0, t_0) \). Combining all the above discussions concludes the proof. \( \square \)

**Proof of Proposition 7.3.** We can obtain Proposition 7.3 by linking all the ingredients we observed so far: Building upon Lemmas 6.14 and 7.27 the estimates on \( \{\overline{S}_i\}_{i \leq 5} \) (Corollaries 7.19, 7.22, and \( \{\overline{S}_i\}_{4 \leq i \leq 8} \) (Corollary 7.23) deduce the desired result. The proof of the corresponding result for \( \overline{S}(t) \) follows analogously. \( \square \)

We conclude this subsection by verifying Theorem 6.6 based on Proposition 7.3.

**Proof of Theorem 6.6.** From Markov’s inequality, Proposition 7.3 implies that

\[
\mathbb{P}\left[ \mathbb{P}\left( \tau \leq t_0 \mid \Pi_S(-\infty, t_0) \right) \geq e^{-\beta_0^{3/2}} \right] \leq e^{-\beta_0^{1.9}},
\]

and it also tells us that

\[
\mathbb{P}\left( \Pi_S(-\infty, t_0) \in \mathcal{A}(\alpha_0, t_0) \right) \geq 1 - \exp\left( -\beta_0^{1.9} \right).
\]

This concludes Theorem 6.6 for \( \Pi_S(-\infty, t_0) \), and the result for \( \Pi_S(-\infty, t_0) \) follows analogously. \( \square \)

### 7.5. The growth of a regular aggregate

This section, we verify Proposition 6.8 which follows as a consequence of Theorem 7.1 and Lemma 7.16.

**Proof of Proposition 6.8.** Recall the definition of \( \tau(\kappa) \) and \( \tau^+() \) from Theorem 7.1 and \( \tau_{10}^+(\Delta) \) from (7.18). In the proof, we let

\[
\tilde{\tau} := \tau(2) \wedge \tau^+(1/2) \wedge \tau_{10}^+(\alpha_0^{-1}).
\]

Then, Theorem 7.1 and Lemma 7.16 tell us that

\[
\mathbb{P}\left( \tilde{\tau} \leq t_0 \mid \Pi_S(-\infty, t_0) \in \mathcal{A}(\alpha_0, r; [t_0]) \right) \geq 1 - r - \exp\left( -\beta_0^{1.9} \right).
\]

(Note that \( \tau(2) \geq \tau^+(1/2) \) if \( \tau(2) > t_0 \), and also that by definition \( \tilde{\tau} \leq t_0 \).)

Now, we apply Lemma 4.3 under the following setting:

\[
f \equiv 1, \ g(t) = S(t), \ \tau = \tilde{\tau}, \ M = \alpha_0^{-1} \beta_0^{12g}, \ \lambda = M^{-\frac{1}{2}}, \ \text{and} \ a = \beta_0^g.
\]

The definition of \( \tau_1 \) justifies the assumption (4.7). Hence, from the lemma we obtain that

\[
\mathbb{P}\left[ \sup_{t_0 \leq t \leq \tilde{\tau}} \left| \Pi_S[t_0, t] - \int_{t_0}^{t} S(s)ds\right| \geq \alpha_0^{-\frac{1}{2}} \beta_0^{\theta g} \mid \Pi_S(-\infty, t_0) \right] \leq \exp\left( -\frac{1}{2} \beta_0^g \right),
\]

and this holds for any given \( \Pi_S(-\infty, t_0) \).

Lastly, we observe that the definition of \( \tau_{10}^+(\alpha_0^{-1}) \) gives that

\[
\sup_{t_0 \leq t \leq \tau_{10}^+(\alpha_0^{-1})} \left| \int_{t_0}^{t} S(s)ds - \alpha_0(t - t_0) \right| \leq \alpha_0^{-\frac{1}{2}-8\epsilon}.
\]

Thus, we obtain the conclusion by combining (7.27), (7.28), and (7.29). \( \square \)
8. The inductive analysis on the speed: Part 2

In this section, we present the proof of Theorem 6.4, completing the inductive argument on regularity. Recall the notations \( t_0, t_1, t_0^\star, t_1^\star, t_1 \) (6.2 and (6.3)), and the definitions of \( \tau^t, \mathcal{A}, \) and \( \tau^t(\kappa) \) given in Section 6.1 ((6.7), (6.8)) and in (7.1). To emphasize the time step and the frame of reference where \( \tau^t(\kappa) \) is defined, we write \( \tau^t(\alpha_0, t_0, \kappa) = \tau^t(\kappa) \) (Hence, \( \tau^t(\alpha_0, t_0, \kappa) = \tau(\alpha_0, t_0^\star, \kappa) \)). We also set \( t_1 \) to be any number that satisfies (6.9), and let

\[
t_1^\star := t_1 - \alpha_0^{-2} \beta_0^2, \quad t_1^\star := t_1 + \alpha_0^{-2} \beta_0^2, \quad t_1 := t_1 + 4 \alpha_0^{-2} \beta_0^2.
\]

Further, let \( \alpha_1 \) be given as (6.10) and let \( \beta_1 := \log(1/\alpha_1) \).

**Theorem 8.1.** Let \( \alpha_0, t_0 > 0 \), let \( t_0^\star, t_1^\star \) be as above, and let \( t_1 \) be any number satisfying (6.9). Under the above notations, let \( \tau^t := \tau(\alpha_0, t_0, 2) \land \tau^t(\alpha_0, t_0, 1/2) \). Then, for all sufficiently small \( \alpha_0 \), the following hold true:

1. \( \mathbb{P}(\tau^t(\alpha_1, t_1, 2) \leq t_1^\star \mid \mathcal{A}(\alpha_0, t_0)) \leq \mathbb{P}(\tau^t < t_0 \mid \mathcal{A}(\alpha_0, t_0)) + 4 \exp(-\beta_0^2) \);
2. \( \mathbb{P}(\mathcal{A}(\alpha_1, t_1)^c \mid \mathcal{A}(\alpha_0, t_0)) \leq \mathbb{P}(\tau^t < t_0 \mid \mathcal{A}(\alpha_0, t_0)) + \exp(-\beta_0^2) \).

Our program is to follow the description given in Section 6.2.3.

- In Section 8.1, we study \(|\alpha_1 - \alpha_0|\), the change of the frame of reference. Moreover, we establish that \( \mathcal{A}(\alpha_1, t_1) \) holds w.h.p. at time \( t_1 \), proving Theorem 8.1-(2). As a byproduct of our analysis, we obtain control on \( \tau_1^\star \) and establish Proposition 6.9 as well.
- In Section 8.2, we study all the stopping times except \( \tau_1^\star \) and \( \tau_3^\star \).
- In Section 8.3, we establish control on \( \tau_3^\star \).
- In Section 8.4, we combine all the analysis done in Sections 8.1 and 8.3 and complete the proofs of Theorems 8.1 and 6.4.

Before moving on to the proofs, we clarify the definitions of \( \alpha_0', \alpha_1', \alpha_1': \) given \( \alpha_0, t_0 \), we have

\[
\alpha_0' = \mathcal{L}(t_0; \Pi_S[t_0^\star, t_0], \alpha_0) = \int_{t_0^\star}^{t_0} \int_{t_0^\star}^{\infty} K_{\alpha_0}^\star \cdot K_{\alpha_0}(u-x) dud\Pi_S(x);
\]

\[
\alpha_1 = \mathcal{L}(t_1^\star; \Pi_S[t_0^\star, t_1^\star], \alpha_0) = \int_{t_1^\star}^{t_1} \int_{t_1^\star}^{\infty} K_{\alpha_0}^\star \cdot K_{\alpha_0}(u-x) dud\Pi_S(x);
\]

\[
\alpha_1' = \mathcal{L}(t_1; \Pi_S[t_1^\star, t_1], \alpha_1) = \int_{t_1^\star}^{t_1} \int_{t_1^\star}^{\infty} K_{\alpha_1}^\star \cdot K_{\alpha_1}(u-x) dud\Pi_S(x),
\]

where \( K_{\alpha}^\star = \frac{2\alpha^2}{1+2\alpha} \).

8.1. Change of rates for critical branching. Our goal is to establish that \( \mathcal{A}(\alpha_1, t_1) \) (6.8) happens with high probability, conditioned on the regularity in the previous step. As a consequence of our analysis, we prove Proposition 6.9.

We begin with studying \( \mathcal{A}_3(\alpha_1, t_1) \). The other events \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are investigated in Sections 8.1.2 and 8.1.3 respectively.

**Lemma 8.2.** Under the setting of Theorem 8.1, we have

\[
\mathbb{P}(\mathcal{A}_3(\alpha_1, t_1)^c, \tau^t \geq t_0 \mid \mathcal{A}(\alpha_0, t_0)) \leq 10 \exp(-\beta_0^2) .
\]

Since the base rates in the definition of \( \alpha_1 \) and \( \alpha_1' \) are \( \alpha_0 \) and \( \alpha_1 \), respectively, we may expect that understanding the size of \(|\alpha_1 - \alpha_0|\) is important, which is also what we want in Theorem 6.4-(2). We define the following \( \mathcal{F}_{t_1} \)-measurable event

\[
\mathcal{A}_4 = \mathcal{A}_4(\alpha_0, t_0) := \left\{|\alpha_1 - \alpha_0| \leq 2 \alpha_0^{3/2} \beta_0^6\right\}, \quad (8.1)
\]
and begin with showing that $A_4$ happens with high probability.

**Lemma 8.3.** Under the setting of Theorem 8.1, we have

$$P\left(A_4', \tau' \geq \tilde{t}_0 \mid A(\alpha_0, t_0)\right) \leq 2 \exp\left(-\beta_0^3\right).$$

**Proof.** In the proof, set $A = A(\alpha_0, t_0)$ for convenience.

Since the event $A_3(\alpha_0, t_0) \supset A$ is given, we can focus on deriving $|\alpha_1 - \alpha'_0| \leq \alpha_0^{3/2} \beta_0^\theta$. Define $S_1(t) = S_1(t; t_0^\gamma, \alpha_0)$. From equation (5.7), we can write

$$\alpha_1 - \alpha'_0 = \mathcal{L}(t_1^\gamma; \Pi_S[t_0^\gamma, t_1^0], \alpha_0) - \mathcal{L}(t_0; \Pi_S[t_0^\gamma, t_0], \alpha_0)$$

$$= \int_{t_0}^{t_1^\gamma} K_{\alpha_0}^* d\tilde{\Pi}_S(x) + \int_{t_0}^{t_1^\gamma} K_{\alpha_0}^*(S(x) - S_1(x)) dx,$$

(8.2)

where we wrote $d\tilde{\Pi}_S(x) = d\Pi_S(x) - S(x) dx$.

To study the first term, we note that

$$\int_{t_0}^{t_1^\gamma \wedge \tau'} (K_{\alpha_0}^*)^2 S(x) dx \leq \alpha_0^3 \beta_0^{10\theta + 2C_0},$$

(8.3)

due to the definition of $\tau_1$ (Section 6.1.1). Since $K_{\alpha_0}^* \leq C \alpha_0^2 \leq \alpha_0^{3/2}$, we apply Corollary 4.6 to deduce that

$$P\left(\left|\int_{t_0}^{t_1^\gamma \wedge \tau'} K_{\alpha_0}^* d\tilde{\Pi}_S(x)\right| \geq \alpha_0^{3/2} \beta_0^{5\theta + 2C_0} \mid A\right) \leq \exp\left(-\beta_0^{C_\theta/3}\right).$$

(8.4)

The second term of (8.2) follows directly from Lemma 7.4 (note that $\tau' \leq \tau(2)$ for $\tau(2)$ in Lemma 7.4):

$$P\left(\left|\int_{t_0}^{t_1^\gamma \wedge \tau'} K_{\alpha_0}^*(S(x) - S_1(x)) dt\right| \geq \alpha_0^{2-3\epsilon} \mid A\right) \leq \exp\left(-\beta_0^{C_{\theta}/3}\right).$$

(8.5)

Combining (8.2), (8.4), (8.5) and the condition from $A_3(\alpha_0, t_0)$, we obtain conclusion. \hfill \Box

Before moving on, we establish the following lemma, which gives a desired control on the stopping time $\tau_1^\gamma$ (Section 6.1.3).

**Lemma 8.4.** Under the setting of Theorem 8.1, we have

$$P\left(\tau_1^\gamma(\alpha_1, t_1) \leq \tilde{t}_1^\gamma, \tau' \geq \tilde{t}_0 \mid A(\alpha_0, t_0)\right) \leq \exp\left(-\beta_0^4\right).$$

**Proof.** This comes as a direct consequence of Lemmas 7.16 and 8.3. For $t' = \tilde{t}_1^\gamma \wedge \tau_{10}^\gamma(\alpha_0^{-1}) \wedge \tau'$ and on the event $A_4$, note that

$$\int_{\tilde{t}_1^\gamma}^{t'} |S(s) - \alpha_1| ds \leq \int_{\tilde{t}_1^\gamma}^{t'} |S(s) - \alpha_0| ds + \int_{\tilde{t}_1^\gamma}^{t'} |\alpha_0 - \alpha_1| ds \leq \alpha_0^{-\frac{1}{2} - \epsilon}. $$

\hfill \Box

To establish Lemma 8.2 we set

$$t_1^- := t_1^\gamma - \alpha_0^{-2} \beta_0^\theta = t_1 - 2\alpha_0^{-2} \beta_0^\theta,$$

(8.6)
and introduce auxiliary parameters $\alpha_1^{(1)}, \alpha_1^{(2)}$ and $\alpha_1^{(3)}$ defined as

$$
\alpha_1^{(1)} := \mathcal{L}(t_1; \Pi_S[t_1^-, t_1], \alpha_0) = \int_{t_1}^{t_1} \int_{t_1^+}^{t_1} K_{\alpha_0}^* \cdot K_{\alpha_0}(u-x) dud\Pi_S(x);
$$

$$
\alpha_1^{(2)} := \mathcal{L}(t_1; \Pi_S[t_1^-, t_1], \alpha_0) = \int_{t_1}^{t_1} \int_{t_1^+}^{t_1} K_{\alpha_0}^* \cdot K_{\alpha_0}(u-x) dud\Pi_S(x);
$$

$$
\alpha_1^{(3)} := \mathcal{L}(t_1; \Pi_S[t_1^-, t_1], \alpha_0) = \int_{t_1}^{t_1} \int_{t_1^+}^{t_1} K_{\alpha_0}^* \cdot K_{\alpha_0}(u-x) dud\Pi_S(x). \quad (8.7)
$$

From these notations, we write

$$
|\alpha_1 - \alpha_1'| \leq |\alpha_1 - \alpha_1^{(3)}| + |\alpha_1^{(2)} - \alpha_1^{(3)}| + |\alpha_1^{(1)} - \alpha_1^{(2)}| + |\alpha_1' - \alpha_1^{(1)}|. \quad (8.8)
$$

Recalling that $\alpha_1 := \mathcal{L}(t_1^-, \Pi_S[t_0^-, t_1], \alpha_0)$, we can see that $|\alpha_1^{(2)} - \alpha_1^{(3)}|$ can be investigated by the same method as Lemma 8.3, leading us to the following corollary.

**Corollary 8.5.** Under the setting of Theorem 8.1 and the above notations, we have

$$
\mathbb{P}\left( |\alpha_1^{(2)} - \alpha_1^{(3)}| \geq \frac{1}{2} \alpha_0^\frac{3}{2} \beta_0^\theta, \tau' \geq \hat{t}_0 \mid A(\alpha_0, t_0) \right) \leq 2 \exp\left(-\beta_0^5\right).
$$

**Proof.** From the proof of Lemma 8.3, our task is to control

$$
\int_{t_1^+}^{t_1} K_{\alpha_0}^* \{d\Pi_S(x) + (S(x) - S_1(x; t_1^-; \alpha_0)) dx\}.
$$

The rest of the proof is analogous to Lemma 8.3 except the following two things:

- Since the regime of integration $[t_1^+; t_1]$ is much shorter than $[t_0^-, t_1]$ from the previous lemma, the RHS 8.3 can be improved to $\alpha_0^3 \beta_0^\theta + 2Cp$, resulting in a corresponding enhancement in (8.4).
- For any $t_1^- \leq t \leq t_1 \wedge \tau'$, we have

$$
|S_1(t; t_0^-, \alpha_0) - S_1(t; t_1^-; \alpha_0)| = \int_{t_0^+}^{t_1} K_{\alpha_0}(t-x) d\Pi_S(x) \leq \alpha_0^{100},
$$

due to the decay property of $K_{\alpha_0}(s)$. Thus, we can obtain (8.5) for our case as well.

□

Next, we observe that the first and the third terms in the RHS are negligible, which comes as a simple consequence of regularity.

**Lemma 8.6.** Under the setting of Theorem 8.1 and (8.7), we have

$$
\mathbb{P}\left( |\alpha_1 - \alpha_1^{(3)}| \geq \alpha_0^{100}, \tau' \geq \hat{t}_0 \mid A(\alpha_0, t_0) \right) \leq 2 \exp\left(-\beta_0^5\right);
$$

$$
\mathbb{P}\left( |\alpha_1^{(1)} - \alpha_1^{(2)}| \geq \alpha_0^{100}, \tau' \geq \hat{t}_0 \mid A(\alpha_0, t_0) \right) \leq 2 \exp\left(-\beta_0^5\right).
$$

**Proof.** We start with investigating the first inequality. We observe that

$$
\alpha_1 - \alpha_1^{(3)} = \int_{t_0}^{t_1^-} \int_{t_1^-}^{t_1} K_{\alpha_0}^* \cdot K_{\alpha_0}(u-x) dud\Pi_S(x) \leq Ce^{-c\beta_0^\theta} |\Pi_S[t_0^-, t_1^-]|, \quad (8.9)
$$

...
where the last inequality came from the estimate Lemma 2.14, noting that $t_1^{-} - t_1^{-} = \alpha_0^{-2}\beta_0^\theta$.

Moreover, from the definition of $\tau_{t_0}^0(\alpha_0, t_0, 1/2)$ and $A_2(\alpha_0, t_0)$ (Sections 6.1.2 6.1.4), we have $|\Pi_S[t_0, t']| \leq \alpha_0^{-1}\beta_0^\theta$. Plugging this estimate to (8.9), we obtain the first estimate of Lemma 8.6. For the second one, observe that

$$
\alpha_1^{(2)} - \alpha_1^{(1)} = \int_{t_1^{-}}^{t_1} \int_{t_1^+}^{\infty} K_{\alpha_0}^* \cdot K_{\alpha_0}(u-x)dud\Pi_S(x) \leq Ce^{-\alpha_0^3\beta_0^\theta}|\Pi_S[t_1^+, t_1^-]|,
$$

analogously as (8.9). Thus, the same reasoning as above gives the conclusion. 

In (8.8), what remains to understand is $|\alpha'_1 - \alpha_1^{(1)}|$. The two terms differ only in the parameter $\alpha_1$ and $\alpha_0$ in the integrand, and hence we can expect to study the difference $|\alpha'_1 - \alpha_1^{(1)}|$ based on analytic properties of $K_{\alpha}^*$ and $K_{\alpha}$. Building upon such an intuition, we prove the following result.

Lemma 8.7. Under the setting of Theorem 8.2 and (8.7), we have

$$
P \left( |\alpha'_1 - \alpha_1^{(1)}| \geq \alpha_0^2\beta_0^{8\theta + 1}, \tau' \geq t_0 | A(\alpha_0, t_0) \right) \leq \exp \left( -\beta_0^5 \right).
$$

Proof of Lemma 8.7. is given in the following subsection. Before delving into the proof, we establish Lemma 8.2.

Proof of Lemma 8.2. Lemma 8.2 follows directly from a union bound over the four probabilities in Corollary 8.3, Lemmas 8.6 and 8.7. Note that by the four estimates along with Lemma 8.3, we get

$$
|\alpha_1 - \alpha'_1| \leq \frac{2}{3} \alpha_0^2\beta_0^\theta \leq \alpha_1^3\beta_1^\theta,
$$

with high probability, which is much stronger than $A_3(\alpha_1, t_1)$ in terms of the exponent of $\beta_1$. 

Before moving on, we also conclude the proof of Proposition 6.9

Proof of Proposition 6.9. Recall the definition of $\tilde{\alpha}_1$, and we write

$$
|\alpha'_1 - \tilde{\alpha}_1| \leq |\alpha'_1 - \alpha_1^{(1)}| + |\tilde{\alpha}_1 - \alpha_1^{(1)}|,
$$

where $\alpha_1^{(1)}$ is given as (8.7). Note that we can write

$$
\tilde{\alpha}_1 - \alpha_1^{(1)} = \int_{t_0}^{t_1} \int_{t_1}^{\infty} K_{\alpha_0}^* \cdot K_{\alpha_0}(u-x)dud\Pi_S(x),
$$

which can be bounded in the same way as Lemma 8.6. Thus, combining Theorem 7.1, Lemmas 8.6 and 8.7 concludes the proof.

8.1.1. Proof of Lemma 8.7: stability under change of rate. Let $Y_\alpha(s)$ be a rate $\alpha$ Poisson process and let $W(s)$ be a continuous time random walk. Define the stopping time $T_\alpha := \inf\{t > 0 : W(s) > Y_\alpha(s)\}$ and the function $h_\alpha(t) := P(t \leq T_\alpha < \infty)$. Recall that $K_\alpha(t) = \alpha(1 + 2\alpha)h_\alpha(t)$ (Definition 2.6). In the following claim we bound the derivative of $h_\alpha(t)$ with respect to $\alpha$.

Claim 8.8. We have that

$$
\left| \frac{d}{d\alpha} h_\alpha(t) \right| \leq Ce^{-\alpha^2t}
$$

Proof. Throughout the proof we write $h'_\alpha(x)$ for the derivative of $h$ with respect to $x$ and write explicitly $\frac{d}{d\alpha} h_\alpha(x)$ when we differentiate with respect to $\alpha$. Let $\delta \leq \alpha^10$ and let $Y_\alpha, Y_\delta$ be independent Poisson processes with rates $\alpha$ and $\delta$ respectively. Let $Y_{\alpha+\delta} := Y_\alpha + Y_\delta$ and note that $T_\alpha \leq T_{\alpha+\delta}$.
Recall that $\mathbb{P}(T_\alpha < \infty) = h_\alpha(0) = 1/(1 + 2\alpha)$. Integrating over the values $x$ that $T_\alpha$ can take we obtain

$$|h_{\alpha+\delta}(t) - h_\alpha(t)| \leq \mathbb{P}(T_\alpha < t, t \leq T_{\alpha+\delta} < \infty) + \mathbb{P}(t \leq T_\alpha < \infty, T_{\alpha+\delta} = \infty)$$

$$\leq \int_0^t h'_\alpha(x) \mathbb{P}(Y_\delta(x) = 1) h_{\alpha+\delta}(t-x) \frac{2(\alpha + \delta)}{1 + 2(\alpha + \delta)} \int_t^\infty h'_\alpha(x) \mathbb{P}(Y_\delta(x) = 1) - \int_0^t h'_\alpha(x) \mathbb{P}(Y_\delta(x) \geq 2)$$

$$\leq C \int_0^t x^{-\frac{3}{2}} e^{-c\alpha^2 x} \cdot \delta x \cdot (t-x)^{-\frac{1}{2}} e^{-c\alpha^2(t-x)} dx + C \alpha \int_0^t x^{-\frac{3}{2}} e^{-c\alpha^2 x} \cdot \delta x + C \int_0^\infty x^{-\frac{3}{2}} e^{-c\alpha^2 x} dx^2$$

$$\leq C \delta e^{-c\alpha^2 t} \int_0^t x^{-\frac{1}{2}} (t-x)^{-\frac{1}{2}} + C \delta \alpha e^{-c\alpha^2 t} \int_0^\infty x^{-\frac{3}{2}} e^{-c\alpha^2 x} dx + C \alpha^{-3} \delta^2 \leq C \delta e^{-c\alpha^2 t} + C \alpha^2,$$

where in the third inequality we used Corollary A.7 in order to bound $h'_\alpha$. The claim follows from the last bound.

We claim that we can approximate the speed by a fixed value in the time interval $[t_1, t_1]$.

**Claim 8.9.** Under the setting of Theorem 6.4 there exists a random variable $\alpha_2 \in \mathcal{F}_{t_1}$ such that

$$\mathbb{P} \left( \left\| S(t) - \alpha_2 \right\| dt \geq \alpha_0^{-\frac{1}{2}} \beta_0^2 \theta, \tau' \geq \hat{t}_0 \right) \leq \exp(-\beta_0^5) .$$

**Proof.** Recalling the definition of $\alpha_1(3)$ from (8.7), we show that the choice $\alpha_2 = \alpha_1(3)$ is enough for our purpose. To this end, we split the integrand into

$$|S(t) - \alpha_1| \leq |S(t) - S_1(t; t_1^{-}, \alpha_0)| + |S_1(t; t_1^{-}, \alpha_0) - \alpha_1(3)|,$$

where $t_1^{-}$ is given as (8.6).

Note that if $t_1^{-} \leq t \leq t_1 \wedge \tau'$, then $|S_1(t; t_1^{-}, \alpha_0) - S_1(t; t_1^{-}, \alpha_0)| \leq \alpha_0^{100}$, by the same argument as (7.5). Thus, the definition of $\tau_3$ (Section 6.1.1) gives

$$|S(t) - S_1(t; t_1^{-}, \alpha_0)| \leq 2\alpha_0^{-\gamma} \sigma_1 \sigma_2(t; S) . (8.10)$$

On the other hand, the same expression as (7.8) gives

$$S_1(t; t_1^{-}, \alpha_0) - \alpha_1(3) = \int_{t_1}^t K_{t_1}^* (t-x) \left\{ d \Pi_t(x) - S_1(x; t_1^{-}, \alpha_0) dx \right\} .$$

Using the bound (8.10), we can control this term analogously as the proof of Lemma 7.8 which gives that

$$\mathbb{P} \left( \bigcap_{\tau' \geq \hat{t}_0} \left\{ |S_1(t; t_1^{-}, \alpha_0) - \alpha_1(3)| \leq \alpha_0 \beta_0^2 \sigma_1(t; S) + \alpha_0^3 \beta_0^2 \theta \right\}, \tau' \geq \hat{t}_0 \right) \leq \exp(-\beta_0^5) . (8.11)$$

Note that the exponent $\theta$ in the term $\alpha_0^3 \beta_0^2 \theta$ is smaller than that of Lemma 7.8 since the length of the interval we are looking at is of scale $t_1^{-} - t_1^{-} = \alpha_0^{-2} \beta_0^2 \theta$, not $\alpha_0^{-2} \beta_0^{100}$. 
When we have both (8.10) and (8.11), we can control the integral of $|S(t) - \alpha_1^{(3)}|$ by Lemma 5.2. This tells us that
\[
\int_{t_1}^{t_2} |S(t) - \alpha_1^{(3)}| dt \leq \int_{t_1}^{t_2} \left( \frac{\alpha_0^{1-\epsilon}}{\pi_1(t; S) + 1} + \frac{\alpha_0^{1-\epsilon}}{\sqrt{\pi_1(t; S) + 1}} + \alpha_0^3 \beta_0^6 \right) dt \leq 2\alpha_0^{-\frac{1}{2}} \beta_0^2,
\]
color{ concluing the proof. □
}

We are now ready to prove Lemma 8.7

Proof of Lemma 8.7. Let $\alpha_2$ be the random variable from Claim 8.9. Recall that
\[
\alpha_1^{(1)} = \frac{2\alpha_0^2}{1 + 2\alpha_0} \int_{t_1}^{t_2} \int_{t_1}^{t_2} K_{\alpha_0}(y - x) dy d\Pi_S(x) = \frac{2\alpha_0^2}{1 + 2\alpha_0} \int_{t_1}^{t_2} \int_{t_1}^{t_2} K_{\alpha_0}(z) dz d\Pi_S(x)
\]

Next, we break the measure $d\Pi_S$ into $(d\Pi_S(x) - \alpha_2 dx) + \alpha_2 dx$ and start by estimating the integral with respect to the Lebesgue measure. We have
\[
\frac{2\alpha_0^2\alpha_2}{1 + 2\alpha_0} \int_{t_1}^{t_2} \int_{t_1}^{t_2} K_{\alpha_0}(z) dz dx = \frac{2\alpha_0^2\alpha_2}{1 + 2\alpha_0} \int_{t_1}^{t_2} \int_{t_1}^{t_2} K_{\alpha_0}(z) dx dz + \frac{2\alpha_0^2\alpha_2}{1 + 2\alpha_0} \int_{t_1}^{t_2} \int_{t_1}^{t_2} K_{\alpha_0}(z) dx dz
\]

\[
= \frac{2\alpha_0^2\alpha_2}{1 + 2\alpha_0} \int_{t_1}^{t_2} z K_{\alpha_0}(z) dz + \frac{(t_1 - t_1)(2\alpha_0^2\alpha_2)}{1 + 2\alpha_0} \int_{t_1}^{t_2} K_{\alpha_0}(z) dz = \alpha_2 + O(\alpha_0^{10}),
\]

where in the last equality we used Corollary A.2 and Lemma 2.14 and the fact that $t_1 - t_1 \geq \alpha_0^{-2} \beta_0^2$. Thus,
\[
\alpha_1 = O(\alpha_0^{10}) + \alpha_2 + \frac{2\alpha_0^2}{1 + 2\alpha_0} \int_{t_1}^{t_2} K_{\alpha_0}(z) dz (d\Pi_S(x) - \alpha_2 dx).
\]

Using the same arguments for $\alpha_1'$ we get
\[
\alpha_1' - \alpha_1^{(1)} = O(\alpha_0^{10}) + \int_{t_1}^{t_2} F(x) (d\Pi_S(x) - \alpha_2 dx),
\]

where
\[
F(x) := \int_{t_1}^{t_2} \frac{2\alpha_0^2}{1 + 2\alpha_0} K_{\alpha_1}(z) - \frac{2\alpha_0^2}{1 + 2\alpha_0} K_{\alpha_0}(z) dz = 2 \int_{t_1}^{t_2} \alpha_1^3 h_{\alpha_1}(z) - \alpha_0^3 h_{\alpha_0}(z) dz.
\]

We turn to bound the function $F$. We have
\[
\left| \frac{d}{d\alpha} (\alpha^3 h_{\alpha}(z)) \right| \leq 3\alpha^2 h_{\alpha}(z) + \alpha^3 \frac{d}{d\alpha} h_{\alpha}(z) \leq C \alpha^2 z^{-\frac{1}{2}} e^{-c_0 z^2} + C \alpha^3 e^{-c_0 z^2} \leq C \alpha^2 z^{-\frac{1}{2}} e^{-c_0 z^2},
\]

where in the second inequality we used Lemma 2.14 and Claim 8.8 and in the third inequality we changed the values of $C$ and $c$ slightly. Thus,
\[
|F(x)| \leq C \alpha_1 - \alpha_0 \int_{t_1}^{t_2} z^{-\frac{1}{2}} e^{-c_0 z^2} dz = C \alpha_0 \alpha_1 - \alpha_0 \int_{t_1}^{t_2} y^{-\frac{1}{2}} e^{-c y} dy \leq C \alpha_0 |\alpha_1 - \alpha_0|.
\]
Next, we use this bound in order to bound the integral on the right hand side of (8.12). To this end, define
\[ I_1 := \int_{t_1}^{t_2} F(x)(d\Pi S(x) - S(x)dx), \quad I_2 := \int_{t_1}^{t_2} F(x)(S(x) - \alpha_2)dx. \]

We start by bounding \( I_1 \) using Corollary 4.6. We compute the quadratic variation
\[ M := \int_{t_1}^{t_2} F(x)^2 S(x)dx \leq \alpha_0^2|\alpha_1 - \alpha_0|^2 \int_{t_1}^{t_2} S(x)dx \leq C\alpha_0^4\beta_0^{13+\theta}C_0. \]

where the last inequality holds on the event \( A_4 \cap \{\tau' \geq \hat{t}_0\} \) using the definition of \( \tau_1 \). Thus, by Lemma 8.3 and Corollary 4.6 we have
\[ \mathbb{P}(|I_1| \geq \alpha_0^2\beta_0^{6.5\theta}C_0, \tau' \geq \hat{t}_0 | A(\alpha_0, t_0)) \leq \exp(-\beta_0^5). \]

We turn to bound \( I_2 \). On the event in Claim 8.9 we have
\[ |I_2| \leq C\alpha_0|\alpha_1 - \alpha_0| \int_{t_1}^{t_2} |S(x) - \alpha_2|dx \leq C\alpha_0^2\beta_0^{6\theta}. \]

Thus, using Claim 8.9 and substituting the bounds on \( I_1 \) and \( I_2 \) into (8.12) finishes the proof of the lemma.

We also record a generalized version of Lemma 8.7 which comes from a simple union bound.

**Corollary 8.10.** Under the setting of Theorem 8.1, let \( L(\alpha) \) denote
\[ L(\alpha) := \mathcal{L}(t_1^{-}; \Pi S[t_1^{-}, t_1], \alpha). \]

Then, we have
\[ \mathbb{P}\left( \sup_{\alpha:|\alpha - \alpha_0| \leq \alpha_0^3\beta_0^{6\theta}} |L(\alpha) - L(\alpha_0)| \geq 2\alpha_0^2\beta_0^{7\theta}, \tau' \geq \hat{t}_0 | \mathcal{A}(\alpha_0, t_0) \right) \leq \exp(-\beta_0^4). \]

**Proof.** Let \( \delta = \alpha_0^{10} \), and define \( \mathcal{D}_{\alpha_0} := \left\{ \alpha : |\alpha - \alpha_0| \leq \alpha_0^3\beta_0^{6\theta}, \alpha - \alpha_0 = k\delta \text{ for some } k \in \mathbb{Z} \right\} \). Note that Lemma 8.7 controls the difference \(|L(\alpha_1) - L(\alpha_0)|\), and we can apply a union bound over the choice of \( \alpha_1 \), extending the result into
\[ \mathbb{P}\left( \sup_{\alpha \in \mathcal{D}_{\alpha_0}} |L(\alpha) - L(\alpha_0)| \geq 2\alpha_0^2\beta_0^{7\theta}, \tau' \geq \hat{t}_0 | \mathcal{A}(\alpha_0, t_0) \right) \leq \exp(-\beta_0^4). \]

Then, note that if \(|\alpha - \hat{\alpha}| \leq \delta \) for some \( \hat{\alpha} \in \mathcal{D}_{\alpha_0} \) and \( t_1 \leq \tau' \), then \(|L(\alpha) - L(\hat{\alpha})| \leq \alpha_0^5 \), by crudely bounding (8.12), concluding the proof.

8.1.2. Estimating the event \( \mathcal{A}_4 \). Recall the definition of \( \mathcal{A}_4(\alpha_1, t_1) \) in Section 6.1.4. Our goal in this subsection is to establish the following lemma.

**Lemma 8.11.** Under the setting of Theorem 8.1, we have
\[ \mathbb{P}(\mathcal{A}_4(\alpha_1, t_1), \tau' \geq \hat{t}_0 | A(\alpha_0, t_0)) \leq \exp(-\beta_0^3). \]

To establish this lemma, we study the size of \( X_{t_1} - X_s \) in two separate regime: for \( s > \hat{t}_0 \) and \( s \leq \hat{t}_0 \). To analyze the first case, we consider the following event \( \mathcal{A}_{4,1} \):
\[ \mathcal{A}_{4,1} = \mathcal{A}_{4,1}(\alpha_0, t_0) := \left\{ \forall \Delta \in [\alpha_0^{-2}\beta_0^{2\theta/2}, t_1^{-}, \hat{t}_0], X_{t_1} - X_{t_1-\Delta} \geq \frac{\alpha_0\Delta}{3} \right\}. \]
Lemma 8.12. Under the setting of Theorem 8.1, we have
\[ \Pr \left( A_{1,1}^c, \; \tau' \geq t_0 \mid A(\alpha_0, t_0) \right) \leq \exp \left( -\beta_0^3 \right). \]

Proof. This comes as a consequence of Lemma 7.16 and Corollary 4.6, applied to
\[ |\Pi_s[s, t_1]| - \int_{s}^{t_1} S(x) dx. \]

The argument goes analogously as the proof of Lemma 7.15 and the details are omitted due to similarity. \qed

Proof of Lemma 8.11. Suppose that \( s \in [t_1 - \alpha_0^{-2} \beta_0^{\theta/2}, t_1] \). Then, given the event \( A_4 \) from (8.1), the condition for \( A_1(\alpha_1, t_1) \) is automatically satisfied, since
\[ \alpha_1^{-1} \beta_1^\theta \geq \frac{1}{2} \alpha_0^{-1} \beta_0^\theta \geq \sqrt{\alpha_0^{-2} \beta_0^{\theta/2} \log^2(\alpha_0^{-2} \beta_0^{\theta/2})} \geq \sqrt{t_1 - s + C_0 \log^2(t_1 - s + C_0)}. \] 

Now consider the case \( s \in [t_0, t_1 - \alpha_0^{-2} \beta_0^{\theta/2}] \). Conditioned on the event \( A_{1,1} \) from Lemma 8.12 we have
\[ X_{t_1} - X_s \geq \frac{1}{3} \alpha_0(t_1 - s) \geq \sqrt{t_1 - s + C_0 \log^2(t_1 - s + C_0)}, \]
which satisfies the criterion for \( A_1(\alpha_1, t_1) \).

Finally, for the case \( s \leq t_0 \) conditioned on \( A(\alpha_0, t_0) \), we observe that
\[ X_{t_1} - X_s = X_{t_1} - X_{t_0} + X_{t_0} - X_s \]
\[ \geq \frac{\alpha_0}{3}(t_1 - t_0) + \sqrt{t_0 - s + C_0 \log^2(t_0 - s + C_0)} - \alpha_0^{-1} \beta_0^{\theta/2} \]
\[ \geq \frac{\alpha_0}{4}(t_1 - t_0) + \sqrt{t_0 - s + C_0 \log^2(t_0 - s + C_0)} \]
\[ \geq \sqrt{t_1 - t_0 + C_0 \log^2(t_1 - t_0 + C_0)} \]
\[ \geq \sqrt{t_1 - s + C_0 \log^2(t_1 - s + C_0)}, \]
where the last step follows from the fact that the function \( f(x) = \sqrt{x + 50 \log^2(x + 50)} \) is positive, increasing and concave.

Thus, from (8.13), (8.14) and (8.15), we see that \( A_1(\alpha_1, t_1) \) holds if \( A(\alpha_0, t_0) \), \( A_{1,1} \) and \( A_4 \) are given. Thus, due to Lemmas 8.3 and 8.12, we obtain the conclusion. \qed

8.1.3. Estimating the event \( A_2 \). Recall the definition of \( A_2(\alpha_1, t_1) \) in Section 8.1.4. In this subsection, we show that \( A_2(\alpha_1, t_1) \) occurs with high probability.

Lemma 8.13. Under the setting of Theorem 8.1, we have
\[ \Pr \left( A_2(\alpha_1, t_1)^c, \; \tau' \geq t_0 \mid A(\alpha_0, t_0) \right) \leq \exp \left( -\beta_0^3 \right). \]

Proof. This is a direct consequence of Lemma 7.16 and Corollary 4.6, following the same argument discussed in Lemma 7.15 (see also Lemma 8.12). We omit its details due to similarity to the previous proofs. \qed

Wrapping up the discussion in Section 8.1 we finish the proof of Theorem 8.1(2).

Proof of Theorem 8.1(2). Combining Lemmas 8.2, 8.11 and 8.13, we obtain that
\[ \Pr \left( A(\alpha_1, t_1)^c, \; \tau' \geq t_0 \mid A(\alpha_0, t_0) \right) \leq \exp \left( -\beta_0^2 \right), \]
which directly implies Theorem 8.1. \qed
8.2. Regularity of speed in the next step. Recall the definition of \( \tau(\alpha_1, t_1, 2) \), \( \tau^*(\alpha_0, t_0, 1/2) \), \( A(\alpha_1, t_1) \) and \( A(\alpha_0, t_0) \) given in (6.7) and (6.8). Our goal in this subsection is to deduce the desired control on

\[
\tilde{\tau} := \tau(\alpha_1, t_1, 2)
\]

(excluding \( \tau^*_3(\alpha_1, t_1) \)), which is given by the following proposition.

Proposition 8.14. Under the setting of Theorem 8.1, the following holds true:

\[
\mathbb{P}(\tilde{\tau} \leq \tau', \ \tau' \geq \hat{t}_0 \mid A(\alpha_0, t_0)) \leq \exp(-\beta_0^4).
\]

Recall the event \( A_4 \) in [8.1]. On the event \( \{\tau' \geq \hat{t}_0\} \cap A_4 \), observe that the following hold true with probability 1:

- \( \tau_1(\alpha_1, t_1, 2) \geq \tau^*_1(\alpha_0, t_0, 1/2) \geq \tau' \), since \( \frac{1}{2}\alpha_0\beta_0^{C_0} \leq 2\alpha_1\beta_1^{C_0} \).
- \( \tau_7(\alpha_1, t_1, 2) \geq \tau^*_7(\alpha_0, t_0, 1/2) \geq \tau' \), since for any \( t \leq \tau^*_7(\alpha_0, t_0, 1/2) \),
  \[
  X_t - X_{t - \alpha_1^{-1}} \leq X_t - X_{t - 2\alpha_0^{-1}} \leq 2 \cdot \frac{1}{2}\beta_0^{4} \leq 2\beta_1^{4}.
  \]
- \( \tau_8(\alpha_1, t_1, 2) \geq \tau^*_8(\alpha_0, t_0, 1/2) \geq \tau' \), since for any \( t \leq \tau^*_8(\alpha_0, t_0, 1/2) \), we have
  \[
  X_t - X_{t - 2\alpha_0^{-1}} \geq X_t - X_{t - \frac{1}{2}\alpha_0^{-1}} > 0.
  \]

Moreover, conditioned on \( \{\tau' \geq \hat{t}_0\} \cap A_4 \), with probability 1 we have

\[
\tau_3(\alpha_1, t_1, 2) \land \tau_5(\alpha_1, t_1, 2) \geq \tau^*_4(\alpha_0, t_0, 1/2) \land \tau_5^*(\alpha_0, t_0, 1/2) \geq \tau',
\]

since for any \( t \leq \tau^*_1(\alpha_0, t_0, 1/2) \tau^*_4(\alpha_0, t_0, 1/2) \land \tau^*_5(\alpha_0, t_0, 1/2) \),

\[
\int_{t_1}^{t} (S(t) - \alpha_1)^2 dt \leq \int_{t_1}^{t} 2(S(t) - \alpha_0)^2 dt + 2(\hat{t}_0 - t_1)(\alpha_1 - \alpha_0)^2
\]

\[
\leq 2 \cdot \frac{1}{2}\alpha_0\beta_0^{25\theta} + 2 \cdot 2\alpha_0^{-2} \beta_0^{10\theta} \cdot 4\alpha_0^{3}\beta_0^{12\theta} \leq 2\alpha_1\beta_1^{\theta},
\]

where the last inequality followed from the definitions of \( \tau^*_4(\alpha_0, t_0, 1/2) \) and \( A_4 \). The integral in the definition of \( \tau_5(\alpha_1, t_1, 2) \) follows similarly, using the definition of \( \tau^*_4(\alpha_0, t_0, 1/2) \) together. Wrapping up the above discussion, \( \tilde{\tau}(1) := \tau_1 \land \tau_4 \land \tau_5 \land \tau_7 \land \tau_8(\alpha_1, t_1, 2) \), and obtain that

\[
\mathbb{P}(\tilde{\tau}(1) \leq \tau', \ \tau' \geq \hat{t}_0 \mid A(\alpha_0, t_0)) \leq \exp(-\beta_0^{4}).
\]  

(8.16)

Thus, the remaining task is to control \( \tau_2, \tau_3, \) and \( \tau_6 \). We conduct this separately in Lemmas 8.15 and 8.17 below. We begin with studying \( \tau_2 \).

Lemma 8.15. Under the setting of Theorem 8.1, we have

\[
\mathbb{P}(\tau_2(\alpha_1, t_1, 2) \leq \tau', \ \tau' \geq \hat{t}_0 \mid A(\alpha_0, t_0)) \leq \exp(-\beta_0^{4}).
\]

Proof. In the proof, we set

\[
S_1(t) = S_1(t, \hat{t}_1, \alpha_1) := \mathcal{R}_\delta(t; \Pi_S[t_1^-, t], \alpha_1).
\]

We can bound this quantity very similarly as in Lemma 7.18 as follows. We use the definition of \( \tau^*_7(\alpha_0, t_0, 1/2) \) to see that for all \( t \in [t_1, \tau'] \),

\[
S_1(t) = \int_{t_1}^{t} K_{\alpha_1}(t - x)d\Pi_S(x)
\]

\[
\leq \sum_{k_0 \leq k_0\alpha_0^{-1} \leq 2\alpha^{-2}\beta_0^{10\theta}} \frac{\beta_0^{4}}{2} \cdot \frac{C\alpha_1}{\sqrt{k\alpha_0^{-1} + 1}} e^{-c\alpha_0^{2}(k\alpha_0^{-1})} \leq \alpha_1\beta_1^{C_0},
\]
where the last inequality holds if the event \( A_4 \) is given. Thus, thanks to Lemma 8.3, we obtain the conclusion.

**Lemma 8.16.** Under the setting of Theorem 8.1, we have

\[
\mathbb{P}\left( \tau_3(\alpha_1, t_1, 2) \leq \tau', \tau' \geq \hat{t}_0 \mid A(\alpha_0, t_0) \right) \leq 2 \exp(-\beta_0^2).
\]

*Proof.* The proof is analogous as Lemma 7.5 using the definitions of \( \tau_4(\alpha_1, t_1, 2) \) and \( \tau_5(\alpha_1, t_1, 2) \), and the estimate (8.16). □

**Lemma 8.17.** Under the setting of Theorem 8.1, we have

\[
\mathbb{P}\left( \tau_6(\alpha_1, t_1, 2) \leq \tau', \tau' \geq \hat{t}_0 \mid A(\alpha_0, t_0) \right) \leq \exp(-\beta_0^3).
\]

*Proof.* This comes as a consequence of Lemma 7.16 and Corollary 4.6 applied to

\[
\left| \Pi_S[t_1, t] - \int_{t_1}^t S(x) \, dx \right|
\]

We also rely on \( A_4 \) to switch \( \alpha_0 \) to \( \alpha_1 \). The rest of the proof is analogous to that of Lemma 7.15, and we omit the details due to similarity (see also Lemmas 8.12 and 8.13). □

We conclude Section 8.2 by proving Proposition 8.14.

**Proof of Proposition 8.14.** It follows from combining (8.16), Lemmas 8.15, 8.16, and 8.17. □

8.3. **Refined control on the first order approximation.** In this subsection, we provide a refined analysis of the first order approximation, deducing an appropriate control on \( \tau_3^\prime \) (Section 6.1.3). The main result we aim to establish is the following.

**Lemma 8.18.** Under the setting of Theorem 8.1 and Proposition 8.14, we have

\[
\mathbb{P}\left( \tau_3^\prime(\alpha_1, t_1) \leq t_1^\prime, \tau \wedge \tau' \geq \hat{t}_0 \mid A(\alpha_0, t_0) \right) \leq 10 \exp(-\beta_0^2).
\]

Having this on hand, we can deduce Theorem 8.1 by combining the results obtained so far.

**Proof of Theorem 8.1** Theorem 8.1 (1) follows directly from combining Proposition 8.14, Lemmas 8.4 and 8.19. The second part of the theorem was already established in Section 8.1 □

\( \tau_3^\prime \) studies the integral of \( |S(t) - S_1(t)| \), and our approach is to first estimate \( |S(t) - S_1(t)| \) itself. To this end, we define the stopping time \( \tau_3^\prime \) as

\[
\tau_3^\prime(\alpha_0, t_0) := \inf\left\{ t \geq t_0 : |S(t) - S_1(t)| \geq \alpha_0 \beta_0^\theta \sigma_1(t; S)^2 + \alpha_0^{1-\eta} \sigma_1 \sigma_2 \sigma_3(t; S) \right\}.
\]

**Lemma 8.19.** Under the setting of Theorem 8.1 and Proposition 8.14, we have

\[
\mathbb{P}\left( \tau_3^\prime(\alpha_1, t_1) \leq t_1^\prime, \tau \wedge \tau' \geq \hat{t}_0 \mid A(\alpha_0, t_0) \right) \leq 10 \exp(-\beta_0^2).
\]

Assuming Lemma 8.19, we conclude the proof of Lemma 8.18.

**Proof of Lemma 8.18** We set \( \tau_{\text{int}} \) to be

\[
\tau_{\text{int}} := \inf\left\{ t \geq t_1 : \int_{t_1}^t \sigma_1 \sigma_2 \sigma_3(t; S) \, dt \geq \alpha^{-\frac{1}{2}} \right\}.
\]

We also recall the definition of \( \tau' \) in Theorem 8.1 and set

\[
\tau_3^\prime := \tau_3^\prime \wedge \tau' \wedge \tau_{\text{int}}.
\]

Then, Lemmas 4.28, 8.19 and Proposition 8.14 tell us that

\[
\mathbb{P}\left( \tau_3^\prime \leq t_1^\prime, \tau' \geq \hat{t}_0 \mid A(\alpha_0, t_0) \right) \leq \exp(-\beta_0^2).
\]
The proof of the desired statement follows by observing that
\[
\int_{t_1}^{t_2} |S(t) - S_1(t)| dt \leq \int_{t_1}^{t_2} \left\{ \frac{\alpha_0 \beta_0^3 \theta}{\pi_1(t; S) + 1} + \alpha_0^{1-\epsilon} \sigma_2 \sigma_3(t; S) \right\} dt \lesssim \beta_1^\theta,
\]
where we used Lemma 5.2 and \( \tau_7 \) to deduce the last inequality, relying on \( A_4 \) (8.1) to show that \( \beta_0 \) and \( \beta_1 \) are close.

The rest of this subsection is devoted to the proof of Lemma 8.19. Unlike when we studied \( \tau_3^+ \) in Lemmas 7.5 and 8.16, we can no longer work with the first-order approximation of the speed. Thus, the second-order approximation (2.22) is required. However, for technical reasons (which will be clarified in Remark 8.21), we will read the process from time \( t_1^- \) given by
\[
t_1^- := t_1 - 2\alpha_0^{-2} \beta_0 = t_1 - \alpha_0^{-2} \beta_0, \quad t_1^+ := t_1 - \frac{1}{3} \alpha_0^{-2} \beta_0,
\]
with respect to the same frame of reference \( \alpha_1 \). Recall that \( \alpha_1 \) is measurable with respect to \( F_{t_1^-} \).

In what follows, we first introduce the setting we use in this subsection, which the framework in terms of \( \alpha_1 \) and \( t_1^- \).

To begin with, the second-order approximation that we work with in this subsection is defined as
\[
S_2(t) = S_2(t; t_1^-, \alpha_1) := \frac{2\alpha_1^2}{(1 + 2\alpha_1)^2} + \frac{S_1(t)}{(1 + 2\alpha_1)^2} + \frac{\alpha_1}{(1 + 2\alpha_1)} \int_{t_1^-}^{t} \int_{t_1^-}^{s} J_{u,s,t}^{(\alpha_1)}(u) d\Pi_S(u) d\Pi_S(s),
\]
where \( S_1(t) \) in this subsection is given by
\[
S_1(t) = S_1(t, t_1^-, \alpha_1) = \int_{t_1^-}^{t} K_{\alpha_1}(t-x) d\Pi_S(x),
\]
and \( J_{u,s,t}^{(\alpha_1)} \) is as (2.9) and (2.13). We also remark that \( d\Pi_S(x) = d\Pi_S(x) - \alpha_1 dx \).

Recall that Lemma 8.3 implies that the event \( A_4 := \{|\alpha_1 - \alpha_0| \leq 2\alpha_0^{3/2} \beta_0^2\} \) happens with high probability if \( \Pi_S(-\infty, t_0] \) was regular, and it is \( F_{t_1^-} \)-measurable. In such a case, \( (t_1^-)^- := t_1^- \), \( (t_1^+)^+ := t_1^+ \) satisfies (6.2) in terms of \( t_1^- \) and \( \alpha_1 \), and thus we can redefine \( \tau(\alpha_1, t_1^-, \kappa) (6.7) \) and \( A(\alpha_1, t_1^-) (6.8) \) in terms of \( (t_1^-)^-, t_1^- \) and \( S_1(t) \). Furthermore, the stopping time \( \tau_0'(\alpha_1, t_1^-, t_1^+) \) from (7.10) can be defined in the same way, and it will play an important role in this subsection. Since the choice of \( t_1 \) is flexible in Theorem 8.1, the results from Theorem 8.1-(2), Proposition 8.14 and Corollary 7.10 tell us the following.

**Corollary 8.20.** Under the setting of Theorem 8.1, let \( \alpha_1, t_1^- (t_1^-) = t_1^- (t_1^+) = t_1, \tau(\alpha_1, t_1^-) \), \( \tau_0'(\alpha_1, t_1^-, t_1^+) \) and \( A(\alpha_1, t_1^-) \) be as above. Then, we have

1. \( \mathbb{P}(\tau(\alpha_1, t_1^-, 2) < \hat{t}_0 \mid A(\alpha_0, t_0)) \leq \mathbb{P}(\tau' < \hat{t}_0 \mid A(\alpha_0, t_0)) + 4 \exp(-\beta_0^2); \)
2. \( \mathbb{P}(\tau_0'(\alpha_1, t_1^-, t_1^+) < \hat{t}_0 \mid A(\alpha_0, t_0)) \leq \mathbb{P}(\tau' < \hat{t}_0 \mid A(\alpha_0, t_0)) + 4 \exp(-\beta_0^2); \)
3. \( \mathbb{P}(A(\alpha_1, t_1^-) \mid A(\alpha_0, t_0)) \leq \mathbb{P}(\tau' < \hat{t}_0 \mid A(\alpha_0, t_0)) + \exp(-\beta_0^2). \)

Before moving on, we clarify the role of \( \tau_0'(\alpha_1, t_1^-, t_1^+) \). It tells us the bound on \( |S_1(t) - \alpha_1''| \), where we denoted
\[
\alpha_1'' := L(t_1^-; \Pi_S[t_1^-; t_1^+], \alpha_1).
\]
On the event $\mathcal{A}_4$ where $|\alpha_1 - \alpha_0| \leq \frac{3}{2} \beta_0^\theta$, we have with high probability that

\[
|\alpha''_1 - \alpha_1| \leq |\mathcal{L}(t_i^-; \Pi_S[t_i^-], \alpha_1) - \mathcal{L}(t_i^-; \Pi_S[t_i^-], \alpha_0)| + |\mathcal{L}(t_i^-; \Pi_S[t_i^-], \alpha_0) - \mathcal{L}(t_i^-; \Pi_S[t_i^-], \alpha_0)| \\
\leq \alpha_0^2 \beta_0^\theta,
\]

where we bounded two terms using Lemma 8.6 and Corollary 8.10. Note that Corollary 8.10 is necessary (instead of Lemma 8.7) since $\alpha_1$ is not $\mathcal{F}_{t_i^-}$-measurable. Thus, combining with the estimates on $\tau''_0$ of $(\alpha_1, t_i^-, t_i^\dagger)$, we have control on the stopping time $\tau''_0$ given by

\[
\tau''_0 := \inf \left\{ t \geq t_0^1 : |S_1(t) - \alpha_1| \leq \alpha_0 \beta_0^C \sigma_1(t; S) + 2 \alpha_0^2 \beta_0^\theta \right\},
\]

in such a way that

\[
P(\tau''_0 \leq t_0^1 | \mathcal{A}(\alpha_0, t_0)) \leq P(\tau'' \leq t_0^1 | \mathcal{A}(\alpha_0, t_0)) + \exp(-\beta_0^2).
\]

Moving on to estimating $|S(t) - S_1(t; t_i^{-}, \alpha_1)|$ in $\tau_{3,1}^1 (\alpha_1, t_1)$, we write

\[
|S(t) - S_1(t; t_i^-; \alpha_1)| \leq |S(t) - S'(t; t_i^-; \alpha_1)| + |S'(t; t_i^-; \alpha_1) - S_2(t)| \\\n+ |S_1(t) - S_2(t)| + |S_1(t; t_i^-; \alpha_1) - S_1(t)|
\]

(8.19)

If $\Pi_S(-\infty, t_i^-) \in \mathcal{A}(\alpha_1, t_i^-)$ (which happens with high probability due to (3) of Corollary 8.20), Proposition 6.12 tells us that

\[
|S(t) - S'(t; t_i^-; \alpha_1)| \leq \alpha_1^{100}.
\]

Also, if $\mathcal{A}_4$ is given, then for any $t_0^1 \leq t \leq t'$ we have

\[
|S_1(t; t_i^-; \alpha_1) - S_1(t)| = \int_{t_i^-}^{t_i^\dagger} K_{\alpha_1}(t - x) d\Pi_S(x) \leq e^{-\beta_0^\theta} |\Pi_S[t_i^-; t_i^\dagger]| \leq \alpha_1^{100},
\]

due to the decay property of $K_\alpha(s)$. Moreover, note that the assumptions of Proposition 4.29 are satisfied with $t^- = t_i^-$, $\hat{t} = t_0$, $\alpha = \alpha_1$, and $\tau = \tau(\alpha_1, t_i^-, 2)$. Hence, it tells us that

\[
P \left( |S'(t; t_i^-; \alpha_1) - S_2(t)| \leq 2 \alpha_1^{1/2} \sigma_1 \sigma_2 \sigma_3(t; S), \ \forall t \in [t_i^-, \hat{t} \wedge \tau(\alpha_1, t_i^-; 2)] \bigg| \mathcal{A}(\alpha_0, t_0) \right) \geq 1 - \exp(-\beta_0^2).
\]

Note that we again used $\mathcal{A}_4$ (Lemma 8.3) to claim that $\alpha_1$ is close enough to $\alpha_0$, enabling us to exchange $\alpha_1$ with $\alpha_0$ in RHS.

Therefore, among the terms in (8.19), what remains is estimating $|S_2(t) - S_1(t)|$. It can written as

\[
S_2(t) - S_1(t) = \frac{2 \alpha_1^2}{(1 + 2 \alpha_1)^2} - \frac{4 \alpha_1 + 4 \alpha_2^2}{(1 + 2 \alpha_1)^2} S_1(t) + \frac{\alpha_1}{(1 + 2 \alpha_1)} \int_{t_i^-}^{\hat{t}} \int_{t_i^-}^{t} J_{\alpha_1}(t_s, t_u) d\Pi_S(u) d\Pi_S(s).
\]

Here, the first two terms $\frac{2 \alpha_1^2}{(1 + 2 \alpha_1)^2}$ and $\frac{4 \alpha_1 + 4 \alpha_2^2}{(1 + 2 \alpha_1)^2} S_1(t)$ are going to absorbed by $\alpha_1 \beta_0^\theta \sigma_1(t; S)^2$ in $\tau_{3,1}^1$, due to the definitions of $\tau_2(\alpha_1, t_i^-; 2)$, $\tau_3(\alpha_1, t_i^-; 2)$, and $\mathcal{A}_4$ (8.1). Thus, our main duty now is to investigate the double integral, the third term in RHS.

To this end, we recall that $t_0^1 = t_1 - \frac{1}{3} \alpha_0^2 \beta_0^\theta$, and we attempt to switch $t_i^-$ in the double integral to $t_0^1$, which will give us a significant technical advantage (see Remark 8.21). We claim that if we
have $|\alpha_1 - \alpha_0| \leq 2\alpha_0^{\frac{5}{2}}\beta_0^6$, then for any $t_1 \leq t \leq \tau'$

$$
\left| \int_{t_1}^{t} \int_{t_1}^{s} J_{t-s,t-u}^{(\alpha_1)} d\Pi_S(u) d\Pi_S(s) - \int_{t_1}^{t} \int_{t_1}^{s} J_{t-s,t-u}^{(\alpha_1)} d\Pi_S(u) d\Pi_S(s) \right| \\
= \left| \int_{t_1}^{t} \int_{t_1}^{s} J_{t-s,t-u}^{(\alpha_1)} d\Pi_S(u) d\Pi_S(s) \right| \leq \alpha_1^{50}.
$$

(8.20)

This is because for any $u, s, t$ such that $t_1 \leq u \leq t_1$, and $t \geq t_1$, we have $t - u \geq \frac{1}{3}\alpha_0^{-2}\beta_0^6$, and hence the estimate on $J$ (Lemma 2.16) implies

$$
|J_{t-s,t-u}^{(\alpha_1)}| \leq \alpha_1^{100}.
$$

Combined with the definitions of $\tau_1, \tau_7$, we obtain (8.20).

Thus, it suffices to consider the following stopping time:

$$
\tau_{3,2}^{1} = \tau_{3,2}^{1}(\alpha_1, t_1; \alpha_0) \\
:= \inf \left\{ t \geq t_1 : \int_{t_1}^{t} \int_{t_1}^{s} J_{t-s,t-u}^{(\alpha_1)} d\Pi_S(u) d\Pi_S(s) \geq \beta_0^{2\theta+4C_{\theta}^1}\sigma_1(t; S) \right\}.
$$

Remark 8.21. In the definition of $\tau_{3,2}^{1}$, note that it reads the process starting from $t_1$, not $t_1'$. In controlling the double integral in $\tau_{3,2}^{1}$, we will rely on the bound on $|S_1(t) - \alpha_1|$, which works for $t \geq t_1$, due to the definition of $\tau_0''$. This is the main reason why we shift the interval of interest to $[t_1, t]$ from $[t_1', t]$ and consider an estimate such as (8.20).

Proof of Lemma 8.19. Summing up the discussions from (8.19) to (8.20), the proof follows directly from Lemma 8.22 below.

Lemma 8.22. Under the setting of Theorem 8.1 and Proposition 8.14, we have

$$
\mathbb{P}\left( \tau_{3,2}^{1} \leq t_1', \ \hat{x} \land \tau' \geq \hat{t}_0 \mid A(\alpha_0, t_0) \right) \leq 3 \exp\left(-\beta_0^2\right).
$$

Proof of Lemma 8.22 turns out to be more involved than the methods used in Section 4 since we now do not allow the error to be of size $\alpha_1^\epsilon$. We rather analyze the double integral directly, relying on the fact that the length scale of $t_1' - t_1$, which is roughly $\alpha_0^{-2}\beta_0^6$, is much smaller than $\alpha_0^{-2}\beta_0^{10\theta}$.

To begin with, we first study the inner integral. Define the stopping time $\tau_{3,3}^{1}$ as

$$
\tau_{3,3}^{1} := \inf \left\{ t \geq t_1 : \exists s \in [t_1, t] \text{ s.t. } \int_{t_1}^{s} J_{t-s,t-u}^{(\alpha_1)} d\Pi_S(u) \geq \frac{2\beta_1^{2\theta+3}}{\sqrt{t-s+1}} + \frac{\beta_1 C_{\theta}^1}{t-s+1} \right\}.
$$

(8.21)

Lemma 8.23. Under the setting of Theorem 8.1 and Proposition 8.14, we have

$$
\mathbb{P}\left( \tau_{3,3}^{1} \leq t_1', \ \hat{x} \land \tau' \geq \hat{t}_0 \mid A(\alpha_0, t_0) \right) \leq 3 \exp\left(-\beta_0^2\right).
$$

Recall the definitions $S_1(t)$ and $R(t)$. In order to study the integrals over $d\Pi_S(x)$ which appear in Lemmas 8.22 and 8.23, we decompose it as follows:

$$
d\Pi_S(x) = d\Pi_S(x) + [S(x) - S_1(x)] + [S_1(x) - \alpha_1],
$$

(8.22)

where we remind that $d\Pi_S(x) = d\Pi_S(x) - S(x)dx$. We will rely on this formula when establishing both Lemmas 8.22 and 8.23.
In the remaining of Section 8.3, we first prove Lemma 8.23 in Section 8.3.1 and then establish Lemma 8.22 in Section 8.3.2.

8.3.1. Proof of Lemma 8.23. We decompose the integral given in the definition of \( \tau_{3,3}^4 \) into four parts using the decomposition (8.22), and then we study the following lemmas in order to show Lemma 8.23. Recall the definition of the event \( \mathcal{A}_4 \) given in (8.1), and set

\[ \mathcal{A}' := \mathcal{A}(\alpha_0, t_0) \cap \mathcal{A}_4. \]

Moreover, recall the definitions of \( \tau''_0 \) (8.17) and set

\[ \tilde{\tau} := \tau \wedge \tau''_0. \]

We note that Lemmas 8.24 and 8.25 below hold the same with the weaker stopping time \( \tilde{\tau} \) instead of \( \tau \). However, we state them in terms of \( \tau \) to be consistent with Lemma 8.26.

**Lemma 8.24.** Assume the setting of Theorem 8.1. For each \( t \in [t_1, t_1^+] \) and \( s \in [t_1^-, t] \), define the event

\[ B_1(s, t) := \left\{ \int_{t_1^-, t}^{s \wedge \tilde{\tau}} J_{t-s-t-u}^{(\alpha_1)} d\Pi_S(u) = \left( \frac{\beta^0 \alpha_0}{t-s+1} + \frac{\alpha_0^2 \beta^0}{\sqrt{t-s+1}} \right) e^{-\alpha_0^2 (t-s)} \right\}. \]

Then, we have

\[ \mathbb{P} \left( \bigcap_{t \in [t_1, t_1^+]} \bigcap_{s \in [t_1^-, t]} B_1(s, t) \bigg| \mathcal{A}' \right) \geq 1 - \exp \left( -\beta^0_0 \right). \]

**Lemma 8.25.** Under the setting of Theorem 8.1, suppose that we condition on the event \( \mathcal{A}_4 \). Then, we have

\[ \int_{t_1^-, t}^{u \wedge \tilde{\tau}} J_{t-s-t-u}^{(\alpha_1)} (S(u) - S_1(u)) du \leq \frac{\alpha_0^{1-\epsilon} \beta^0_0}{\sqrt{t-s+1}} e^{-\alpha_0^2 (t-s)}, \]

for all \( t \in [t_1, t_1^+] \) and \( s \in [t_1^-, t] \).

**Lemma 8.26.** Under the setting of Theorem 8.1, suppose that we condition on the event \( \mathcal{A}_4 \). For all \( t \in [t_1, t_1^+] \) and \( s \in [t_1^-, t] \), we have

\[ \left| \int_{t_1^-, t}^{s \wedge \tilde{\tau}} J_{t-s-t-u}^{(\alpha_1)} (S_1(u) - \alpha_1) du \right| \leq \frac{\alpha_0^{\frac{\alpha_0}{2}} \beta^{\theta_0+2}}{\sqrt{t-s+1}} e^{-\alpha_0^2 (t-s)}. \]

Before we delve into the details, we address a simple fact that is useful to keep in mind for the rest of the subsection. Note that (8.18) and Theorem 8.1 (2) shown in Section 8.1 tells us that

\[ \mathbb{P} \left( \tilde{\tau} \leq t_1^+, \tilde{\tau} \wedge \tau' \geq \hat{t}_0 \bigg| \mathcal{A}(\alpha_0, t_0) \right) \leq 2 \exp \left( -\beta^0_0 \right). \]

Furthermore, we see from Lemma 8.3 and Theorem 8.1 (2) that

\[ \mathbb{P} \left( (\mathcal{A}_4)^c \cup \mathcal{A}(\alpha_1, t_1^+)^c, \tilde{\tau} \wedge \tau' \geq \hat{t}_0 \bigg| \mathcal{A}(\alpha_0, t_0) \right) \leq 2 \exp \left( -\beta^0_0 \right). \]

This tells us that along with the event \( \{ \tilde{\tau} \wedge \tau' \geq \hat{t}_0 \} \), conditioning on \( \mathcal{A}' \) or on \( \mathcal{A}(\alpha_1, t_1^+) \) instead of \( \mathcal{A}(\alpha_0, t_0) \) produces only a small additional error probability. Moreover, we stress that \( \mathcal{A}_4 \) and \( \mathcal{A}(\alpha_1, t_1^+) \) are \( \mathcal{F}_{t_1^+} \)-measurable.
Lemma 2.16 which is

Note that we can condition on $B$ to apply Lemma 4.5 and use (8.28) to obtain that

$$\int_{t^+_1}^{u^+_1} J_{t-s,t-u}^{(s_1)} d\Pi_S(u) \leq \left( \frac{\alpha_0^4 \beta_0^{s+3}}{\sqrt{t-s+1}} + \frac{\beta_0^{s+1}}{t-s+1} \right) e^{-\alpha_0^2 (t-s)},$$

and hence we obtain the conclusion. \hfill \Box

In the remaining of this subsection, we prove Lemmas 8.24–8.26. In the proof, we repeatedly use Lemma 2.16 which is

$$J_{x,y} \leq \frac{Ce^{-2cy \alpha_0^2}}{\sqrt{(x+1)(y+1)}} \leq \frac{Ce^{-c(x+y) \alpha_0^2}}{\sqrt{(x+1)(y+1)}}, \quad \text{(8.27)}$$

where the first inequality holds upon the event $A_4$.

Proof of Lemma 8.24 Define the set

$$\mathcal{T} := \{ \alpha_0^{10} k \in [t^+_1, t^+_1] : k \in \mathbb{Z} \}.$$

For any $t, u \in \mathcal{T}$ with $u \leq t$, we first observe from (8.27) that on $A_4$,

$$\int_{t^+_1}^{u^+_1} \left| \int_{s-\alpha_0^2}^{s} J_{t-s,t-u}^{(s_1)} d\Pi_S(u) \right| \leq \frac{\alpha_0^5 \beta_0^2 e^{-\alpha_0^2 (t-s)}}{t-s+1} + \frac{\alpha_0^5 \beta_0^2 e^{-\alpha_0^2 (t-s)}}{t-s+1} \int_{s-\alpha_0^2}^{s} \frac{\alpha_0^5 \beta_0^2 \alpha_0^2 \beta_0^2 du}{\sqrt{t-u+1}} \leq \frac{2C \beta_0^2 e^{-\alpha_0^2 (t-s)}}{t-s+1}. \quad \text{(8.28)}$$

To take care of the integral from $t^+_1$ to $(s - \alpha_0^2)$, note that we have

$$\int_{t^+_1}^{(s - \alpha_0^2)} \left( J_{t-s,t-u}^{(s_1)} \right)^2 S(u) du \leq \frac{C e^{-2c \alpha_0^2 (t-s)}}{t-s+1} \int_{t^+_1}^{s} \frac{\alpha_0^5 \beta_0^2 \alpha_0^2 \beta_0^2 du}{\sqrt{t-u+1}} \leq \frac{\alpha_0^5 \beta_0^2 e^{-2\alpha_0^2 (t-s)}}{t-s+1}.$$

Thus, we apply Lemma 4.5 to obtain that

$$\mathbb{P} \left( \left| \int_{t^+_1}^{u^+_1} J_{t-s,t-u}^{(s_1)} d\Pi_S(u) \right| \geq \left( \frac{2C \beta_0^2 e^{-\alpha_0^2 (t-s)}}{t-s+1} + \frac{\alpha_0^5 \beta_0^2 \alpha_0^2 \beta_0^2}{\sqrt{t-s+1}} \right) e^{-\alpha_0^2 (t-s)} \right| A' \right) \leq \exp \left( -\beta_0^4 \right).$$

Note that we can condition on $A'$ when applying Lemma 4.5 since it is $F_{t^+_1}$-measurable. Moreover, the event $A_4$ is used to ensure that $\alpha_1$ and $\alpha_0$ are close enough.

Now define

$$B'_1(s, t) := \left\{ \left| \int_{t^+_1}^{u^+_1} J_{t-s,t-u}^{(s_1)} d\Pi_S(u) \right| \leq \left( \frac{2C \beta_0^2 e^{-\alpha_0^2 (t-s)}}{t-s+1} + \frac{\alpha_0^5 \beta_0^2 \alpha_0^2 \beta_0^2}{\sqrt{t-s+1}} \right) e^{-\alpha_0^2 (t-s)} \right\}.$$

Note that its bound is slightly stronger than $B_1(s, t)$, to leave some room to take a union bound and then conduct a continuity argument. Then, we use a union bound to deduce that

$$\mathbb{P} \left( \bigcap_{t \in \mathcal{T}} \bigcap_{s \geq t} B'_1(s, t) \bigg| A' \right) \geq 1 - \exp \left( -\beta_0^4 \right).$$
The remaining duty is to improve this bound to work on general \( t, s \) which are not necessarily in \( \mathcal{T} \). This is done by the exact same argument presented in Lemmas \ref{lem:4.20} and \ref{lem:4.9} and the details of the proof are left to the reader.

**Proof of Lemma \ref{lem:8.25}** In the proof, we assume that we have \( \mathcal{A}_4 \). Using (8.27) and the definition of \( \tau_3(\alpha_1, t_1) \) (Section 6.1.1), we express that

\[
\int_{t_1}^{s} |J_{t-s, t-u}^{(\alpha_1)}| |S(u) - S_1(u)| \, du \leq \int_{t_1}^{s} \frac{C e^{-\alpha_0^2(t-s)} e^{-\alpha_0^2(t-u)}}{\sqrt{(t-u+1)(t-s+1)}} \cdot \frac{\alpha_0^{1-e}}{\pi_1(u; S) + 1} \, ds
\]

\[
= \frac{C \alpha_0^{-e} e^{-\alpha_0^2(t-u)}}{\sqrt{t-u+1}} \int_{t_1}^{s} \frac{e^{-\alpha_0^2(t-u)}}{\sqrt{t-u+1}} \cdot \frac{ds}{\pi_1(u; S) + 1}.
\]

To simplify the RHS, we recall Lemma \ref{lem:5.13} with parameters \( \Delta_0 = \alpha_0^{-1}, \Delta_1 = \alpha_0^{-1} \beta_0^0 \) and \( N_0 = \beta_0^5 \). This gives

\[
\int_{t_1}^{s} \frac{C e^{-\alpha_0^2(t-u)}}{\sqrt{t-u+1}} \cdot \frac{ds}{\pi_1(u; S) + 1} \leq \beta_0^0.
\]

Note that although we need to set \( K \geq \alpha_0^{-1} \beta_0^5 \) in Lemma \ref{lem:5.13}, the contribution from \( k \geq \alpha_0^{-1} \beta_0^5 \) is negligible due to the extra exponential decay. This concludes the proof of Lemma \ref{lem:8.25}. \( \square \)

**Proof of Lemma \ref{lem:8.26}** Suppose that we are on the event \( \mathcal{A}_4 \). Using the bound (8.27) and the definition of \( \tau_b'' \) from (8.17), we have

\[
\int_{t_1}^{s} |J_{t-s, t-u}^{(\alpha_1)}| |S_1(u) - \alpha_1| \, du \leq \int_{t_1}^{s} \frac{C e^{-\alpha_0^2(t-s)}}{\sqrt{t-s+1}} \cdot \frac{e^{-\alpha_0^2(t-u)}}{\sqrt{t-u+1}} \left( \frac{\alpha_0 \beta_0^0}{\pi_1(u; S) + 1} + \frac{3}{2} \beta_0^0 \right) \, ds.
\]

We first note that

\[
\int_{t_1}^{s} \frac{\alpha_0^{3/2} \beta_0^0 e^{-\alpha_0^2(t-u)}}{\sqrt{t-u+1}} \, ds \leq \frac{1}{2} \beta_0^0.
\]

Furthermore, from Lemma \ref{lem:5.13} with \( \Delta_0 = \alpha_0^{-1}, \Delta_1 = \alpha_0^{-1} \beta_0^0 \) and \( N_0 = \beta_0^5 \),

\[
\int_{t_1}^{s} \frac{\alpha_0 \beta_0^0 e^{-\alpha_0^2(t-u)} \, du}{\sqrt{(t-u+1)(\pi_1(u; S) + 1)}} \leq \frac{1}{2} \beta_0^2 \beta_0^0.
\]

Note that the contribution from \( k \geq \alpha_0^{-1} \beta_0^5 \) is negligible similarly as in Lemma \ref{lem:8.25}. Combining the two estimates, we obtain conclusion. \( \square \)

8.3.2. **Proof of Lemma \ref{lem:8.22}** To finish the proof of Lemma \ref{lem:8.22} we define

\[
Z(s, t) := \int_{t_1}^{s} J_{t-s, t-u}^{(\alpha_1)} d\Pi_S(u),
\]

which we have studied in the previous subsection. Moreover, for \( \hat{\tau} \) and \( \tau_{3,3}^\parallel \) given in (8.24) and (8.21), respectively, we set

\[
\hat{\tau}' := \hat{\tau} \wedge \tau_{3,3}^\parallel.
\]

Splitting \( d\Pi_S(s) \) into four parts as described in (8.22) and recalling the definition of \( \mathcal{A}' \) from (8.23), we prove the following lemmas.
Lemma 8.27. Assume the setting of Theorem 8.1. For each \( t \in [t^1, t^1] \), define the event
\[
\mathcal{C}_1(t) := \left\{ \left| \int_{t^1}^{t} Z(s, t) d\Pi_S(s) \right| \leq \frac{\beta_0^{2\theta+3\epsilon}}{\pi_1(t; S) + 1} \right\}.
\]
Then, we have
\[
\mathbb{P} \left( \bigcap_{t \in [t^1, t^1]} \mathcal{C}_1(t) \bigg| \mathcal{A}' \right) \geq 1 - \exp(-\beta_0^3) .
\]

Lemma 8.28. Under the setting of Theorem 8.1, suppose that we condition on \( \mathcal{A}' \). Then, we have
\[
\left| \int_{t^1}^{t} Z(s, t)(S(s) - S_1(s)) ds \right| \leq \frac{\alpha_0^{1-2\epsilon}}{\pi_1(t; S) + 1} + \alpha_0^{\frac{3}{2} - 2\epsilon},
\]
for all \( t \in [t^1, t^1] \).

Lemma 8.29. Under the setting of Theorem 8.1, suppose that we condition on \( \mathcal{A}' \). Then, we have
\[
\left| \int_{t^1}^{t} Z(s, t)(S_1(s) - \alpha_1) ds \right| \leq \alpha_0^{6\theta+5},
\]
for all \( t \in [t^1, t^1] \).

Proofs of Lemmas 8.27–8.29 are similar to those of Lemmas 8.24–8.26. Before delving into those, we first conclude the proof of Lemma 8.22.

Proof of Lemma 8.22. As in the proof of Lemma 8.23 (8.25) and (8.26) tell us that along with the event \( \tau_1 \wedge \tau' \geq t_0 \), conditioning on \( \mathcal{A}' \) or on \( \mathcal{A}(\alpha_1, t^1_1) \) instead of \( \mathcal{A}(\alpha_0, t_0) \) produces only a small additional error probability. Thus, combining Lemmas 8.27–8.29, we have with probability at least \( 1 - 3\epsilon - \beta_0^3 \) conditioned on \( \mathcal{A}(\alpha_0, t_0) \) that
\[
\left| \int_{\hat{t}}^{t^1} Z(s, t) d\Pi_S(s) \right| \leq \frac{2\beta_0^{2\theta+3\epsilon}}{\pi_1(t; S) + 1} + 2\alpha_0^6 \beta_0^{2\theta+5} \leq \frac{\beta_0^{2\theta+4+3\epsilon}}{\pi_1(t; S) + 1},
\]
where the last inequality is from \( \pi_1(t; S) \leq \alpha_0^{-1} \beta_0^{C_0} \) (see the definition of \( \tau_8 \) in Section 6.1.2). Then, we obtain the conclusion using Theorem 8.1 (2) (proven in Section 8.1) and Lemma 8.23. \( \square \)

In the remaining of this subsection, we prove Lemmas 8.27–8.29.

Proof of Lemma 8.27. The proof goes analogously as Lemma 8.24, by splitting the integral into two parts, from \( t - \alpha_0^3 \) to \( t^1 \) and \( t^1 \) to \( t - \alpha_0^3 \), which is to apply Lemma 4.8. From the definition of \( \tau^1_{1,3} \), we see that for all \( t \leq \hat{t}' \),
\[
\left| \int_{t - \alpha_0^3}^{t} Z(s, t) d\Pi_S(s) \right| \leq \frac{\beta_0^{2\theta+C_0}}{\pi_1(t; S) + 1} + \frac{\alpha_0^3 \beta_0^{\theta+C_0} + 3}{\sqrt{\pi_1(t; S) + 1}} + \alpha_0^2 \beta_0^{\theta+C_0 + 4},
\]
where the second inequality follows from \( \pi_1(t \wedge \hat{t}', S) \leq 2\alpha_0^{-1} \beta_0^{C_0} \).
The other integral can be estimated using Lemma 4.8 by observing that
\[
\int_{t_1^1 \wedge t_1^0}^{(t - \alpha^{-1} t_1^0) \wedge t_1^0} Z(s,t)^2 S(s) ds \leq \int_{t_1^1}^{t} \left( \frac{2\beta_0^2 C_0 + 2 + 2\alpha_0^2 \beta_0^{\theta + 6}}{(t - s + 1)^2} \right) 2\alpha_0^{2 \theta + 2} C_0^2 ds \leq \alpha_0^{2 \theta + 2} C_0^2,
\]
and hence combining with (8.29) concludes the proof, since \( \hat{\tau} \)
\[
\sup \{ |Z(s,t)| : s \in [t_1^1 \wedge t_1^0, (t - \alpha^{-1}) \wedge t_1^0] \} \leq 2\alpha_0^{\theta + 3}.
\]
Thus, applying the lemma gives that
\[
\mathbb{P} \left( \int_{t_1^1 \wedge t_1^0}^{(t - \alpha^{-1} t_1^0) \wedge t_1^0} Z(s,t) d\Pi_2(s) \right) \leq \alpha_0^{\theta + 2} C_0^2, \quad \forall t \in [t_1^1, t_1^0], \quad \mathcal{A}' \right) \geq 1 - \exp \left( -\beta_0^3 \right),
\]
and hence combining with (8.29) concludes the proof, since \( \hat{\tau}' \geq t_1^1 \) with high probability (Proposition 8.14, equation (8.18) and Lemma 8.23).

Proof of Lemma 8.28 Recalling the definition of \( \tau_3(\alpha_1, t_1^1) \) (Section 6.1.1), we have
\[
\int_{t_1^1 \wedge t_1^0}^{t \wedge t_1^0} |Z(s,t)(S(s) - S_1(s))| ds \leq \int_{t_1^1}^{t} \left( \frac{\beta_0^{\theta + 1}}{t - s + 1} + \frac{1}{\alpha_0^2 \beta_0^{\theta + 3}} \frac{\alpha_0^4 \beta_0^{\theta + 3}}{\sqrt{t - s + 1}} \right) \frac{\alpha_0^{1-\epsilon} - \alpha_0^0 (t-s)}{\pi_1(s;S) + 1} ds.
\]
Then, applying Lemma 5.13 with \( \Delta_0 = \alpha_0^{-1}, \Delta_1 = \alpha_0^{-1} \beta_0^{C_0}, \) and \( N_0 = \beta_0^5 \) bounds the RHS by
\[
\frac{\alpha_0^{1-2\epsilon}}{\pi_1(s;S) + 1} + \frac{3}{2} \alpha_0^{2-2\epsilon},
\]
recalling that the contribution from \( k \geq \alpha_0^{-1} \beta_0^2 \) is negligible similarly as in Lemma 8.25.

Proof of Lemma 8.29 Having the definition of \( \tau_4'' \) (8.17) in mind, we express that
\[
\int_{t_1^1 \wedge t_1^0}^{t \wedge t_1^0} |Z(s,t)(S_1(s) - \alpha_1)| ds \leq \int_{t_1^1 \wedge t_1^0}^{t \wedge t_1^0} e^{-\alpha_0^2 (t-s)} \left( \frac{\beta_0^{C_0+1}}{t - s + 1} + \frac{1}{\alpha_0^2 \beta_0^{\theta + 3}} \frac{\alpha_0^4 \beta_0^{\theta + 3}}{\sqrt{t - s + 1}} \right) \frac{\alpha_0^{1-\epsilon} - \alpha_0^0 (t-s)}{\pi_1(s;S) + 1} ds.
\]
We begin with observing that
\[
\int_{t_1^1 \wedge t_1^0}^{t \wedge t_1^0} e^{-\alpha_0^2 (t-s)} \left( \frac{\beta_0^{C_0+1}}{t - s + 1} + \frac{1}{\alpha_0^2 \beta_0^{\theta + 3}} \frac{\alpha_0^4 \beta_0^{\theta + 3}}{\sqrt{t - s + 1}} \right) 2\alpha_0^3 \beta_0^{\theta} ds \leq \alpha_0^3 \beta_0^{\theta + 4} + \alpha_0 \beta_0^{2 \theta}.
\]
On the other hand, the remaining integral can be bounded using Lemma 5.13 as in the proof of Lemma 8.28. This gives
\[
\int_{t_1^1 \wedge t_1^0}^{t \wedge t_1^0} e^{-\alpha_0^2 (t-s)} \left( \frac{\beta_0^{C_0+1}}{t - s + 1} + \frac{1}{\alpha_0^2 \beta_0^{\theta + 3}} \frac{\alpha_0^4 \beta_0^{\theta + 3}}{\sqrt{t - s + 1}} \right) \frac{\alpha_0^{1-\epsilon} - \alpha_0^0 (t-s)}{\pi_1(s;S) + 1} ds \leq \alpha_0 \beta_0^{2 \theta},
\]
concluding the proof.
8.3.3. Some consequences of the integral calculation. Before moving on, we record several direct consequences of the integral calculations we have seen above. These results will be useful later in Section 9 when we encounter similar formulas under a slightly different setting.

We begin with stating the analogue of Lemmas 8.24 and 8.25 noting that the previous analysis works analogously in another interval instead of \([\hat{t}_0^3, \hat{t}_0^1]\). In this subsection, \(S_1(t)\) is defined to be \(S_1(t) = S_1(t; t_0, \alpha_0)\).

**Corollary 8.30.** Let \(\alpha_0, t_0, r > 0\), set \(t_0^3, t_0^1, \hat{t}_0^1\) as (6.2), (6.3), and let

\[
t_0^1 := t_0 + \frac{5}{2} \alpha_0^{-2} \beta_0^\theta - \hat{t}_0 - \frac{1}{2} \alpha_0^{-2} \beta_0^\theta.
\]

Define the stopping times

\[
\begin{align*}
\hat{\tau}_{3,3}^{(1)} &:= \inf \left\{ t \geq \hat{t}_0 : \exists s \in [t_0^1, t) \text{ s.t. } \int_{t_0^1}^s J^{(\alpha_0)}_{t-s, t-u} d\Pi_S(u) \geq C_{\alpha_0} + 1 \sqrt{t-s+1} \right\}; \\
\hat{\tau}_{3,3}^{(2)} &:= \inf \left\{ t \geq \hat{t}_0 : \exists s \in [t_0^1, t) \text{ s.t. } \int_{t_0^1}^s J^{(\alpha_0)}_{t-s, t-u} (S(u) - S_1(u)) du \geq C_{\alpha_0} \right\}.
\end{align*}
\]

Suppose that \(\Pi_S(-\infty, t_0) \notin \mathcal{R}(\alpha_0, r; [t_0])\). Then, we have

\[
P \left( \hat{\tau}_{3,3}^{(1)} \land \hat{\tau}_{3,3}^{(2)} < \hat{t}_0 \mid \mathcal{F}_{t_0} \right) \leq \exp(-\beta_0^3 + r).
\]

To be precise, we remark that the conclusion can be obtained from combining Theorem 7.1 and the argument from Lemma 8.23.

We also derive a similar estimate on the integral of \(K_{\alpha_0}^*\) instead of \(J^{(\alpha_0)}_{t-s, t-u}\).

**Corollary 8.31.** Under the setting of Corollary 8.30, define the stopping time

\[
\tau_{3,3}^{(3)} := \inf \left\{ t \geq \hat{t}_0 : \exists s \in [t_0^3, t) \text{ s.t. } \int_{t_0^3}^s K_{\alpha_0}^*(s-u)(S(u) - S_1(u)) du \geq \alpha_0^{-2} \right\}.
\]

Then, we have

\[
P \left( \tau_{3,3}^{(3)} < \hat{t}_0 \mid \mathcal{F}_{t_0} \right) \leq \exp(-\beta_0^3 + r).
\]

**Proof.** The conclusion can be obtained from the same method as Lemma 8.25 since \(K_{\alpha_0}^*(x)\) satisfies the estimate from Lemma 2.15 which is very similar to 8.27.

The following analogue of Lemma 8.23 can be obtained similarly.

**Corollary 8.32.** Under the setting of Corollary 8.30, define the stopping time

\[
\begin{align*}
\hat{\tau}_{3,3}^4 &:= \inf \left\{ t \geq \hat{t}_0 : \exists s \in [t_0^4, t) \text{ s.t. } \int_{t_0^4}^s J^{(\alpha_0)}_{t-s, t-u} d\Pi_S(u) \geq C_{\alpha_0} + 1 \right\}; \\
\hat{\tau}_{3,3}^{(4)} &:= \inf \left\{ t \geq \hat{t}_0 : \exists s \in [t_0^4, t) \text{ s.t. } \int_{t_0^4}^s J^{(\alpha_0)}_{t-s, t-u} (S(u) - S_1(u)) du \geq C_{\alpha_0} \right\}.
\end{align*}
\]

Then, we have

\[
P \left( \hat{\tau}_{3,3}^4 < \hat{t}_0 \mid \mathcal{F}_{t_0} \right) \leq 3 \exp(-\beta_0^2 + r).
\]

Note that in the term \(\frac{5}{2} \alpha_0^{-2} \beta_0^\theta \sqrt{t-s+1}\), we have a larger exponent \(6\theta\) than in \(\tau_{3,3}^4\), since we deal with a longer interval \([t_0, \hat{t}_0]\).
The other lemmas in the previous subsections can be extended similarly. Letting
\[ Z_0(s, t) := \int_{t_0}^{s} f^{(\alpha_0)}_{-s, t-u} d\Pi_S(u), \]
we have the analogue of Lemma 8.28, which can be deduced from using Corollary 8.32 instead of Lemma 8.23.

**Corollary 8.33.** Under the setting of Corollary 8.30, define the stopping time
\[ \tilde{\tau}_{3,1}^{(1)} := \inf \left\{ t \geq \hat{t}_0 : \int_{t_0}^{t} Z_0(s, t)(S(s) - S_1(s)) ds \geq \frac{\alpha_0^{-2\tau}}{\pi_1(t; S) + 1} + \alpha_0^{3-2\tau} \right\}. \]
Then, we have
\[ P \left( \tilde{\tau}_{3,1}^{(1)} < \hat{t}_0 \mid \mathcal{F}_{t_0} \right) \leq \exp \left( -\beta_0^3 \right) + r. \]
Due to their similarity, proofs of Corollaries 8.30, 8.32 and 8.33 are omitted.

### 8.4. Proof of Theorem 6.4
In this final section, we present the proof of Theorem 6.4. Throughout the proof, we assume that \( \mathcal{F}_{t_0} = \Pi_S(-\infty, t_0] \) is \((\alpha_0, r; [t_0])\)-regular (see Definition 6.2).

We begin with verifying the second item of Theorem 6.4. Let \( \tau := \tau(\alpha_0, t_0, 2) \) \((8.31)\), \( \tau^+ := \tau^+(\alpha_0, t_0, 1/2) \) \((7.1)\), and \( \tau' := \tau^+ \wedge \tau \). Observe that
\[ P \left( |\alpha_1 - \alpha_0| \geq 2\alpha_0^{3/2} \beta_0^{6\theta} \mid \mathcal{F}_{t_0} \right) \leq P \left( \tau \leq \hat{t}_0 \mid \mathcal{F}_{t_0} \right) + P \left( \tau' < \hat{t}_0, \tau > \hat{t}_0 \mid \mathcal{F}_{t_0} \right) + P \left( |\alpha_1 - \alpha_0| \geq 2\alpha_0^{3/2} \beta_0^{6\theta}, \tau' \geq \hat{t}_0 \mid \mathcal{F}_{t_0} \right) \]
\[ \leq r + \exp \left( -\beta_0^{3/2} \right). \]
where the last line follows from the definition of regularity, Theorem 7.1 and Lemma 8.3.

To study the event \( \{\mathcal{F}_{t_1} \) is \((\alpha_1, e^{-\beta_0^{3/2}}; [t_1])\)-regular\}, we verify the two conditions of regularity separately. The second condition follows from Theorems 7.1 and 8.1(2). Namely, we write
\[ P \left( A(\alpha_1, t_1) \cap \mathcal{F}_{t_0} \right) \leq P \left( \tau' < \hat{t}_0 \mid \mathcal{F}_{t_0} \right) + \exp \left( -\beta_0^{3/2} \right) \]
\[ \leq P \left( \tau' < \hat{t}_0, \tau > \hat{t}_0 \mid \mathcal{F}_{t_0} \right) + P \left( \tau \leq \hat{t}_0 \mid \mathcal{F}_{t_0} \right) \]
\[ \leq \exp \left( -\beta_0^{3/2} \right) + r + \exp \left( -\beta_0^{3/2} \right) \leq r + \exp \left( -\beta_0^{3/2} \right). \]

To verify the first condition of regularity, recall the definition of \( A_4 \) \((8.1)\), and \( t_0^4 = t_1 + 4\alpha_0^{-2} \beta_0^{\theta} \) which is larger than \( \hat{t}_1 = t_1 + 4\alpha_0^{-2} \beta_0^{\theta} \) on \( A_4 \). Observe that
\[ P \left( P \left( \tau^{(4)}(\alpha_1, t_1, 2) \leq \hat{t}_1 \mid \mathcal{F}_{t_1} \right) > e^{-\beta_0^{3/2}}, \tau > \hat{t}_0, A_4 \mid \mathcal{F}_{t_0} \right) \]
\[ \leq e^{2\beta_0^{3/2}} P \left( \tau^{(4)}(\alpha_1, t_1, 2) \leq \hat{t}_1, \tau > \hat{t}_0 \mid \mathcal{F}_{t_0} \right), \]
by Markov’s inequality. Note that \( A_4 \) ensures \( \beta_0^{3/2} \leq 2\beta_0^{3/2} \). Furthermore, since
\[ t_1^4 = t_1 + 4\alpha_0^{-2} \beta_0^{\theta} \leq t_0 + \alpha_0^{-2} \beta_0^{10\theta} + 4\alpha_0^{-2} \beta_0^{\theta} < t_0 + 2\alpha_0^{-2} \beta_0^{10\theta} = \hat{t}_0, \]
combining Theorem 7.1, Proposition 8.14 and Lemma 8.19 gives that
\[ P \left( \tau^{(4)}(\alpha_1, t_1, 2) \leq t_1^4, \tau > \hat{t}_0 \mid \mathcal{F}_{t_0} \right) \leq 3 \exp \left( -\beta_0^{2} \right). \]
We also know from Theorem 7.1 and Lemma 8.3 that
\[ P \left( A_4^c, \tau > \hat{t}_0 \mid \mathcal{F}_{t_0} \right) \leq 2 \exp \left( -\beta_0^{2} \right). \]
Hence, from now we will be interested in investigating the double integral starting from
\( \Pi \) since \( \alpha \).

An important relation we stress here is that to study the mean of the speed. We aim to state and proving the formal version of Theorem 2.12, which will be done in Section 9.2. To this end, an essential step is to compute the needed in deriving the scaling limit of the speed. We aim to state and proving the formal version
\[ \int_{0}^{t} \int_{t_0}^{s} J_{-s,t-u} d\Pi_{S}(u) d\Pi_{S}(s). \]

Thus, we conclude the proof of Theorem 6.4 from (8.30), (8.31) and (8.35).

9. The second order contributions and the moments of the increment

Building upon the analysis of regularity, the goal of this section is establishing the main estimate needed in deriving the scaling limit of the speed. We aim to state and proving the formal version of Theorem 2.12 which will be done in Section 9.2. To this end, an essential step is to compute the expectation of the double integral term in \( S_{2}(t) \), namely,

\[ \mathcal{J}(t) = \mathcal{J}(t; \alpha, t_0) := \frac{\alpha}{1 + 2\alpha} \int_{t_0}^{t} \int_{s}^{t} J_{t-s,t-u} d\Pi_{S}(u) d\Pi_{S}(v). \]

The following theorem gives an appropriate control on the mean of \( \mathcal{J}(t) \), and it is established in Section 9.1.

**Theorem 9.1.** Let \( \alpha, t_0 > 0 \), set \( t_0, t_0', t_0 \) and \( \hat{t}_0 \) as (6.2) and (6.3), and let \( r = e^{-\beta^2/2} \) for \( \beta := \log(1/\alpha) \). Suppose that \( \Pi_{S}(-\infty, t_0) \) is \((\alpha, r; [t_0])\)-regular. Then, for all \( t \in [t_0, \hat{t}_0] \), we have

\[ \mathbb{E}[\mathcal{J}(t) | \Pi_{S}(-\infty, t_0)] = 2\alpha^2 \left( 1 + o(\alpha^{1/2}) \right). \]

We note that the error bound \( \alpha^{1/2} \) is not essential: something better than \( O(\beta^{-2}) \) will suffice for our purpose.

9.1. The contribution of the double integral. From the analysis in the previous sections, we can deduce that \( \mathcal{J}(t) \) is typically of order \( \alpha^{2-\epsilon} \). Then, the main difficulty in establishing Theorem 9.1 is narrowing down its size to \( 2\alpha^2 \) plus a smaller order error. Here, we require more refined tools to study the mean of \( \mathcal{J}(t) \) accurately.

Let \( t_0, t_0, t_0 \) and \( \hat{t}_0 \) be as in Theorem 9.1, we introduce another parameter \( t_0^{\dagger} \) defined as

\[ t_0^{\dagger} := t_0 + \frac{5}{2} \alpha^{-2} \beta^\theta. \]

An important relation we stress here is that

\[ (t_0 - t_0^{\dagger}) \wedge (t_0^{\dagger} - t_0) \geq \frac{1}{2} \alpha^{-2} \beta^\theta. \]

Since \( \Pi_{S}(-\infty, t_0) \) is regular and we consider \( t \geq \hat{t}_0 \), we can ignore the contributions to the integral from the regime \( [t_0, t_0^{\dagger}] \), analogously as we have seen in (8.20). Namely, with probability \( 1 - e^{-\beta^2} \),

\[ \left| \mathcal{J}(t) - \int_{t_0^{\dagger}}^{t} \int_{t_0}^{s} J_{t-s,t-u} d\Pi_{S}(u) d\Pi_{S}(s) \right| \leq \alpha_0^{50}, \quad \text{for all } t \in [t_0, \hat{t}_0]. \]

Hence, from now we will be interested in investigating the double integral starting from \( t_0^{\dagger} \). The reason for our choice of such \( t_0^{\dagger} \) is explained in Remark 9.2. Controlling the contribution from the error event where the above does not hold will be discussed in the proof of Lemma 9.5 below, and it will also be bounded by \( O(\alpha^{50}) \).
Writing $F_t := \Pi_S(-\infty,t_0]$ as before, we begin with observing that

$$
\mathbb{E}[J(t) \mid F_{t_0}] = \frac{\alpha}{1 + 2\alpha} \mathbb{E} \left[ \int_{t_0}^{t} \int_{t_0}^{s} J_{t-s,t-u}^{(\alpha)} d\tilde{\Pi}_S(u)(S(s) - \alpha) ds \right] F_{t_0} + O(\alpha^{5/2}), \tag{9.1}
$$

since the outer integral with respect to $d\tilde{\Pi}_S(s) = d\Pi_S(s) - S(s)ds$ is a martingale and thus has mean zero. Suppose that we can switch $(S(s) - \alpha)ds$ into $(S_1(s) - \alpha)ds$ with a negligible error, recalling the definition $S_1(s) = S_1(s; t_0, \alpha) = R_c(s, s; \Pi_S[t_0, s], \alpha)$. Then, from (9.2), we can write

$$(S_1(s) - \alpha) = \int_{t_0}^{s} K^{(\alpha)}(s-x)(d\Pi_S(x) - S_1(x)dx) + (R_c(t_0^*, s; \Pi_S[t_0^*, \hat{t}], \alpha) - \alpha). \tag{9.2}$$

**Remark 9.2.** Note that this integral starts from $t_0^*$, not $t_0$, in order to keep our control on $R_c(t_0^*, s; \Pi_S[t_0^*, \hat{t}], \alpha)$ for all $s \in [t_0^*, \hat{t}]$ (see [9.4] below). The integral (9.2) needs to be from $t_0^*$, not $t_0$, since we have control on $|S_1(u) - \alpha|$ only on $t \geq t_0^*$ (Lemma 7.8).

From (9.2), we again attempt to approximate $(S_1(s) - \alpha)ds$ by

$$(S_1(s) - \alpha)ds \approx \int_{t_0}^{s} K^{(\alpha)}(s-x) d\tilde{\Pi}_S(x) + (\alpha'' - \alpha)ds,$$

with $\alpha'' := \mathcal{L}(t_0^*, \Pi_S[t_0^*, \hat{t}], \alpha)$. If we make these two steps of approximations rigorous and repeat a similar procedure to the inner integral, then we arrive at the following proposition.

**Proposition 9.3.** Define the integrals $I_1$ and $I_2$ by

$$I_1(t) := \mathbb{E} \left[ \int_{t_0^*}^{t} \int_{t_0^*}^{s} J_{t-s,t-u}^{(\alpha)} d\tilde{\Pi}_S(u) \cdot \int_{t_0^*}^{s} K^{(\alpha)}(s-u) d\tilde{\Pi}_S(u) ds \right];$$

$$I_2(t) := \mathbb{E} \left[ \int_{t_0^*}^{t} \int_{t_0^*}^{s} J_{t-s,t-u}^{(\alpha)} K^{(\alpha)}(u-v) d\tilde{\Pi}_S(v) du \cdot \int_{t_0^*}^{s} K^{(\alpha)}(s-x) d\tilde{\Pi}_S(x) ds \right].$$

Then, under the setting of Theorem 9.1, for all $\hat{t}_0 \leq t \leq \hat{t}_0$ we have

$$\mathbb{E}[J(t) \mid F_{t_0}] = \frac{\alpha}{1 + 2\alpha} \mathbb{E}[I_1(t) + I_2(t)] + O(\alpha^{21/2}).$$

Note that the integrals $I_1$ and $I_2$ are independent of $F_{t_0}$.

Before establishing the proposition rigorously, we first deduce Theorem 9.1 from it.

**Proof of Theorem 9.1.** Observe that for any deterministic functions $f, g$ and any numbers $a_1 \leq a_2$, $b_1 \leq b_2$, we have

$$\mathbb{E}\left[ \int_{a_1}^{a_2} f(x) d\Pi_\alpha(x) \int_{b_1}^{b_2} g(y) d\Pi_\alpha(y) \right] = \alpha \int_{a_1 \land b_1}^{a_2 \land b_2} f(x)g(x) dx,$$

since the only nontrivial correlation comes from the cases when both $x$ and $y$ are at the same point in the point process. Thus, we obtain that

$$\mathbb{E}[I_1(t)] = \alpha \int_{t_0^*}^{t} \int_{t_0^*}^{s} J_{t-s,t-u}^{(\alpha)} K^{(\alpha)}(s-u) du ds = \alpha \int_{0}^{t-t_0^*} \int_{0}^{s} J_{u,s}^{(\alpha)} K^{(\alpha)}(s-u) du ds.$$
Similarly, we can see that

\[
\mathbb{E}[J_2(t)] = \int_{t_0}^{t} \int_{t_0}^{s} \mathbb{E} \left[ \int_{t_0}^{u} J_{t-s,t-u}^{(\alpha)} K_\alpha^*(u-v) d\Xi_\alpha(v) \int_{t_0}^{s} K_\alpha^*(s-x) d\Xi_\alpha(x) \right] duds
\]

\[
= \alpha \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{u} J_{t-s,t-u}^{(\alpha)} K_\alpha^*(u-v) K_\alpha^*(s-v) dvduds
\]

\[
= \alpha \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{u} J_{t-s,t-u}^{(\alpha)} K_\alpha^*(s-u) K_\alpha^*(s-v) dvduds + O(\alpha^{100}),
\]

where in the third line, the integral over \([t_0^1, t_0^2]\) of \(dv\) has a negligible \(O(\alpha^{100})\)-order contribution due to the decay property of \(J\), since \(t \geq t_0\). The two deterministic integrals above are computed via a precise understanding of the quantities \(K_\alpha^*\) and \(J^{(\alpha)}\). This is done using analytical methods based on Fourier analysis, and can be found in Lemma A.11 in the appendix. Applying the results from the lemma concludes the proof. \(\square\)

The rest of the subsection is devoted to the proof of Proposition 9.3. The analysis is highly technical, and relies on a similar approach as that in Section 8.3 based on the decomposition

\[
d\Xi_S(x) = d\Xi_S(x) + (S(x) - S_1(x)) dx + (S_1(x) - \alpha) dx.
\]

\((d\Xi_S(x) := d\Xi_S(x) - S(x) dx as before.\)

9.1.1. Reshaping the outer integral. We begin with modifying the outer integral of \(J(t)\). Since it suffices to study the integral in the RHS of (9.1), define

\[
J_{\text{out}}^{(1)}(t) := \int_{t_0^1}^{t} \int_{t_0^1}^{s} J_{t-s,t-u}^{(\alpha)} d\Xi_S(u)(S(s) - S_1(s)) ds;
\]

\[
\tau_{\text{out}}^{(1)} := \inf \left\{ t \geq t_0 : \left| J_{\text{out}}^{(1)}(t) \right| \geq \alpha^{t - \epsilon} + \frac{\alpha^{1-\epsilon}}{\pi_1(t; S) + 1} \right\}.
\]

Moreover, recalling (9.2), the remaining outer integral is decomposed by the following formula:

\[
S_1(s) - \alpha = \int_{t_0^1}^{s} K_\alpha^*(s-x) d\Xi_\alpha(x) + \int_{t_0^1}^{s} K_\alpha^*(s-x) d\Xi_{S-\alpha}(x)
\]

\[
+ \int_{t_0^1}^{s} K_\alpha^*(s-x)(S(x) - S_1(x)) dx + [R_c(t_0^1, s, \Pi_S[t_0^1, t_0^1], \alpha) - \alpha''] + [\alpha'' - \alpha],
\]

where we defined \(d\Xi_{S-\alpha}(x) := d\Xi_S(x) - d\Xi_\alpha(x) - (S(x) - \alpha) dx\) and \(\alpha'' = \mathcal{L}(t_0^1; \Pi_S[t_0^1, t_0^1], \alpha)\). We show that the contributions to the double integral coming from the terms other than the first and
the last in the RHS are negligible. To this end, recall the notation $\Pi_{S,\alpha}$ (7.26) and define

$$I_{\text{out}}^{(2)}(t) := \int_{t_0}^{t} \int_{t_0}^{s} J_{t-s,t-u}^{(\alpha)}(u) d\Pi_{S}(u) \int_{t_0}^{s} K_{\alpha}^{\ast}(s-x) d\Pi_{S-\alpha}(x) ds;$$
$$\tau_{\text{out}}^{(2)} := \inf \{ t \geq \hat{t}_0 : |I_{\text{out}}^{(2)}(t)| \geq \alpha^{-1} \sigma_1(t; S \Delta \alpha) + \alpha^{\frac{3}{4} - \epsilon_1} \sigma_1(t; S \Delta \alpha)^2 + \alpha^{\frac{3}{4} - \epsilon} \};$$
$$I_{\text{out}}^{(3)}(t) := \int_{t_0}^{t} \int_{t_0}^{s} J_{t-s,t-u}^{(\alpha)}(u) d\Pi_{S}(u) \int_{t_0}^{s} K_{\alpha}^{\ast}(s-x)(S(x) - S_1(x)) dx ds;$$
$$\tau_{\text{out}}^{(3)} := \inf \{ t \geq \hat{t}_0 : |I_{\text{out}}^{(3)}(t)| \geq \alpha^{\frac{3}{4} - 3\epsilon} \}. $$

We remark that the contribution from the fourth term is negligible due to Lemma 8.6 which tells us that with probability at least $1 - e^{-\beta^3}$,

$$|R_{\epsilon}(t_0^{\ast}, s; \Pi_{S}[t_0^{\ast}, t_0^{\ast}], \alpha) - \alpha''| \leq \alpha^{100}, \quad (9.4)$$

for all $t \in [t_0^{\ast}, \hat{t}_0]$.

**Lemma 9.4.** Under the setting of Theorem 9.1, we have

$$\mathbb{P} \left( \tau_{\text{out}}^{(1)} \wedge \tau_{\text{out}}^{(2)} \wedge \tau_{\text{out}}^{(3)} < \hat{t}_0 \mid \mathcal{F}_{\hat{t}_0} \right) \leq \exp \left( -\beta^3 \right) + r.$$

From this lemma, deduce the conclusion of this subsection which can be written as follows.

**Lemma 9.5.** Define the integral $I_{\text{out}}(t)$ as

$$I_{\text{out}}(t) := \int_{t_0}^{t} \int_{t_0}^{s} J_{t-s,t-u}^{(\alpha)}(u) d\Pi_{S}(u) \int_{t_0}^{s} K_{\alpha}^{\ast}(s-u) d\Pi_{\alpha}(u) + \left[ \alpha'' - \alpha \right] ds.$$  

Under the setting of Theorem 9.1, we have for all $t \in [t_0^{\ast}, \hat{t}_0]$ that

$$\mathbb{E}[\mathcal{J}(t) \mid \mathcal{F}_{\hat{t}_0}] = \frac{\alpha}{1 + 2\alpha} \mathbb{E}[I_{\text{out}}(t) \mid \mathcal{F}_{\hat{t}_0}] + o(\alpha^{2+\epsilon}).$$

**Proof.** Let $\tau_{\text{out}} := \tau_{\text{out}}^{(1)} \wedge \tau_{\text{out}}^{(2)} \wedge \tau_{\text{out}}^{(3)}$. We first show that

$$\mathbb{E}[\mathcal{J}(t) 1_{\{ \tau_{\text{out}} < \hat{t}_0 \}} \mid \mathcal{F}_{\hat{t}_0}] = o(\alpha^4) = \mathbb{E}[I_{\text{out}}(t) 1_{\{ \tau_{\text{out}} < \hat{t}_0 \}} \mid \mathcal{F}_{\hat{t}_0}]. \quad (9.5)$$

To establish the left estimate, we use the fact that $S(t) \leq \frac{1}{2}$ and $|J_{u,s}^{(\alpha)}| \leq 2$ from Proposition 2.2 and (4.14). Namely, we can express that

$$|\mathcal{J}(t)| \leq \frac{\alpha}{1 + 2\alpha} \left[ \left| \Pi_{1/2}[t_0^{\ast}, t_0^{\ast}] \right|^2 + \int_{t_0}^{t} \int_{t_0}^{s} 2 \cdot \frac{1}{4} duds \right]. \quad (9.6)$$

Since the event $\{ \tau_{\text{out}} < \hat{t}_0 \}$ happens with probability less than $2e^{-\beta^3/2}$, we can couple this event with the tail events of the rate-$\frac{1}{2}$ Poisson process $\Pi_{1/2}[t_0^{\ast}, \hat{t}_0]$ and deduce the first equality of (9.5). The second estimate can be obtained analogously.

What remains to show is that the contributions from $I_{\text{out}}^{(1)}(t)$, $I_{\text{out}}^{(2)}(t)$ and $I_{\text{out}}^{(3)}(t)$ are small under the event $\{ \tau_{\text{out}} \geq \hat{t}_0 \}$. Note that we already know that the contribution from $I_{\text{out}}^{(3)}(t)$ is small, namely,

$$\frac{\alpha}{1 + 2\alpha} \mathbb{E}[I_{\text{out}}^{(3)}(t) 1_{\{ \tau_{\text{out}} \geq \hat{t}_0 \}}] = o(\alpha^{2+\epsilon}). \quad (9.7)$$
Moving on to the investigation of $I_{\text{out}}^{(2)}(t)$, let $\delta_0 = \alpha^{-\frac{3}{2}}+8\epsilon$. From Lemma 7.16, the event
\[
\bigcap_{t \in [t_0^1+\delta_0, t_0]} \left\{ \int_t^{t_0} |S(s) - \alpha|\,ds \leq \alpha^4 \right\}
\]
happens with probability at least $1 - e^{-\beta^3} - \epsilon$.

Conditioned on this event, the probability that $\pi_1(t; S \Delta \alpha) < \delta_0$ is smaller than $\alpha^4$ for each fixed $t$. Thus, we have
\[
\begin{align*}
\mathbb{E}[\sigma_1(t; S \Delta \alpha) \mathbbm{1}_\{\tau_{\text{out}} \geq \hat{t}_0\} | \mathcal{F}_{t_0}] &\leq \alpha^{\frac{3}{2}}-5\epsilon, \\
\mathbb{E}[\sigma_1(t; S \Delta \alpha)^2 \mathbbm{1}_\{\tau_{\text{out}} \geq \hat{t}_0\} | \mathcal{F}_{t_0}] &\leq \alpha^{\frac{3}{2}}-9\epsilon.
\end{align*}
\]

Combining with the definition of $\tau_{\text{out}}^{(2)}$ and Lemma 9.4, this gives that
\[
\frac{\alpha}{1 + 2\alpha} \mathbb{E}[I_{\text{out}}^{(2)}(t) \mathbbm{1}_\{\tau_{\text{out}} \geq \hat{t}_0\} | \mathcal{F}_{t_0}] = o\left(\alpha^{2+\frac{1}{2}}\right). \tag{9.8}
\]

Similarly, we have
\[
\mathbb{E}[(\sigma_1(t; S))^2 \mathbbm{1}_\{\tau_{\text{out}} \geq \hat{t}_0\} | \mathcal{F}_{t_0}] \leq \alpha^{1-\epsilon},
\]
and hence from the definition of $\tau_{\text{out}}^{(1)}$, we get
\[
\frac{\alpha}{1 + 2\alpha} \mathbb{E}[I_{\text{out}}^{(1)}(t) \mathbbm{1}_\{\tau_{\text{out}} \geq \hat{t}_0\} | \mathcal{F}_{t_0}] = o\left(\alpha^{2+\frac{1}{2}}\right). \tag{9.9}
\]

To conclude the proof, recall that
\[
\mathcal{J}(t) = \frac{\alpha}{1 + 2\alpha} \left( I_{\text{out}}^{(2)}(t) + I_{\text{out}}^{(1)}(t) + I_{\text{out}}^{(2)}(t) + I_{\text{out}}^{(3)}(t) \right),
\]
up to an $O(\alpha^{50})$-error that comes from (9.4), whose contribution can be controlled analogously as (9.6), and combine (9.7), (9.8) and (9.9).

**Proof of Lemma 9.4.** We begin with noting that $\tau_{\text{out}}^{(1)} \geq \hat{t}_0$ with high probability, due to Corollary 8.33. To establish the control on $\tau_{\text{out}}^{(2)}$ and $\tau_{\text{out}}^{(3)}$, we recall the definitions of $\tau_{\text{gap}}, \tau_{\text{gap}}^{(2)}$ (Corollaries 7.24, 7.25), $\tau_{3,3}^{(3)}$ and $\tau_{\text{gap}}^{(3)}$ (Corollaries 8.31, 8.32). On the event $\{\tau_{\text{gap}}^{(2)} \wedge \tau_{\text{gap}}^{(3)} \wedge \tau_{3,3}^{(3)} \geq \hat{t}_0\}$, we can write
\[
I_{\text{out}}^{(2)}(t) \leq \alpha \beta^6 \beta^6 \left( \frac{1}{\sqrt{t-s+1}} + \frac{\beta^{s+1}}{t-s+1} \right) \left( \frac{\alpha \beta^6}{\sqrt{\pi_1(t; s; S \Delta \alpha)}} + \frac{\alpha^7}{\epsilon} \right) \, ds.
\]

Then, the RHS is bounded using Lemma 5.13 with parameters $\Delta_0 = \alpha^{-\frac{3}{2}} + 7\epsilon$, $N_0 = \beta^{\frac{5}{2}}, K = \alpha^{-\frac{3}{2}}$ and $\Delta_1 = \alpha^{-\frac{3}{2}}$. This gives
\[
\int_{t_0}^{t} \left( \frac{\alpha \beta^6}{\sqrt{t-s+1}} + \frac{\beta^{s+1}}{t-s+1} \right) \left( \frac{\alpha \beta^6}{\sqrt{\pi_1(t; s; S \Delta \alpha)}} + \frac{\alpha^7}{\epsilon} \right) \, ds 
\leq \frac{\alpha^{-\epsilon}}{\sqrt{\pi_1(t; S \Delta \alpha) + 1}} + \frac{\alpha^7}{\pi_1(t; S \Delta \alpha) + 1} + \alpha^{7/2} - \epsilon.
\]

Note that if there were intervals of length $\Delta_1$ without any points, then we can just artificially add points to justify the choice of $\Delta_1$; this operation will only increase the value of the whole integral.

Continuing our work on the same event as above, we express that
\[
I_{\text{out}}^{(3)}(t) \leq \int_{t_0}^{t} \left( \frac{\alpha \beta^6}{\sqrt{t-s+1}} + \frac{\beta^{s+1}}{t-s+1} \right) \alpha^{2-2\epsilon} \, ds \leq \alpha^{\frac{3}{2}-3\epsilon},
\]
concluding the proof of Lemma 9.4. \qed
9.1.2. Reshaping the inner integral. Building upon Lemma \[\text{9.5}\] we investigate the inner integral of \(I_{\text{out}}(t)\) and establish Proposition \[\text{9.3}\]. Define
\[
F(s) := \int_{t_0}^{s} K_\alpha^*(s-u) d\overline{\Pi}_\alpha(u) + [\alpha'' - \alpha];
\]
\[
\tau_{\text{in}}^{(1)} := \inf \left\{ s \geq t_0^+: |F(s)| \geq \alpha^3 \beta \sigma_1(s; \alpha) + 3\alpha \beta^3 \beta^6 \right\}.
\]
Moreover, note that \(\alpha'' - \alpha = (\alpha'' - \alpha') + (\alpha' - \alpha)\), and (9.7) gives
\[
\alpha'' - \alpha' = \int_{t_0}^{t_0^+} K_\alpha^*[d\Pi_S(x) - S_1(x)dx].
\]
Then, the same argument as Lemma \[\text{8.3}\] tells us that
\[
P \left( |\alpha'' - \alpha| \geq \alpha^3 \beta^2 \right| F_{t_0} \right) \leq \exp (-\beta^5).
\]
Thus, Theorem \[\text{7.1}\], Corollary \[\text{7.9}\], and the definition of \(A_3(\alpha, t_0)\) (Section \[\text{6.1.4}\]) tell us that
\[
P \left( \tau_{\text{in}}^{(1)} < t_0^+ \right| F_{t_0} \right) \leq \exp (-\beta^4) + r.
\]
We also decompose \(d\overline{\Pi}_S(u)\) in \(I_{\text{out}}(t)\) as
\[
d\overline{\Pi}_S(u) = d\overline{\Pi}_\alpha(u) + d\overline{\Pi}_{S-\alpha}(u) + (S(u) - S_1(u))du + (S_1(u) - \alpha)du,
\]
and then writing \((S_1(u) - \alpha)\) as \[\text{9.3}\]. Define the stopping times
\[
\tau_{\text{in}}^{(2)} := \inf \left\{ t \geq t_0^+: \left| \int_{t_0}^{t} \int_{t_0^+}^{s} J_{t-s, t-u}^{(\alpha)} (S(u) - S_1(u)) F(s) du ds \right| \geq \alpha^{\frac{3}{2}} \right\};
\]
\[
\tau_{\text{in}}^{(3)} := \inf \left\{ t \geq t_0^+: \left| \int_{t_0}^{t} \int_{t_0}^{s} J_{t-s, t-u}^{(\alpha)} F(s) \int_{u}^{s} K_\alpha^*(u-v) d\Pi_{S-\alpha}(v) dv ds \right| \geq \alpha^{\frac{3}{4}} \right\};
\]
\[
\tau_{\text{in}}^{(4)} := \inf \left\{ t \geq t_0^+: \left| \int_{t_0}^{t} \int_{t_0}^{s} J_{t-s, t-u}^{(\alpha)} F(s) \int_{u}^{s} K_\alpha^*(u-v) (S(u) - S_1(v)) dv ds \right| \geq \alpha^{\frac{3}{2}} \right\}.
\]
We again stress that the integrals of \(dv\) are over \([t_0^+, u]\), not \([t_0^+, u]\). Among the rest of the integrals rather than the ones stated in the above definitions, \(d\overline{\Pi}_\alpha(u)\) and \(\int_{t_0}^{u} K_\alpha^*(u-v) d\overline{\Pi}_\alpha(v) du\) form the leading order \(I_1(t)\) and \(I_2(t)\). The other contributions coming from \(d\overline{\Pi}_{S-\alpha}(u)\) and \((\alpha'' - \alpha)du\) will be studied later. Note that in the above we consider \(u \geq t_0^+\), and hence the contribution from \([\mathcal{R}_c(t_0^+, u; \Pi_S[t_0^+, t_0^+], \alpha) - \alpha'']du\) is negligible from (9.4) by the argument in Lemma \[\text{9.5}\].

**Lemma 9.6.** Under the setting of Theorem \[\text{9.1}\] we have
\[
P \left( \tau_{\text{in}}^{(2)} \wedge \tau_{\text{in}}^{(3)} \wedge \tau_{\text{in}}^{(4)} < t_0^+ \right| F_{t_0} \right) \leq \exp (-\beta_0^2) + r.
\]

**Proof.** The proof is very similar to that of Lemma \[\text{9.4}\], and hence we concisely explain which lemmas from the previous sections are used to establish the conclusion.

For \(\tau_{\text{in}}^{(2)}\), we rely on Corollary \[\text{8.30}\] to estimate the inner integral over \(du\), and then combine with the bound from \(\tau_{\text{in}}^{(1)}\), using Lemma \[\text{5.13}\] to perform integration over \(ds\).
For $\tau_{in}^{(3)}$, recall the definition of $r_{gap}^{(1)}$, $r_{gap}^{(2)}$ from Corollaries 7.24, 7.25 and note that on $\{r_{gap}^{(1)} \wedge r_{gap}^{(2)} \geq \hat{t}_0\}$ we have

$$
\begin{align*}
&\left| \int_{t_0}^{s} J_{t-s,t-u}^{(a)} \int_{t_0}^{u} K_{\alpha}^{*}(u-v) d\bar{\Pi}_{S-\alpha}(v) du \right| \\
&\leq \frac{e^{-c\alpha^2(t-s)}}{\sqrt{t-s+1}} \int_{t_0}^{s} \frac{1}{\sqrt{t-u+1}} \left( \frac{\alpha\beta C_{\epsilon}}{\sqrt{\pi_1(u; S \triangle \alpha)}} + \frac{\alpha^{3-\epsilon}}{\sqrt{t-s+1}} \right) du \leq \frac{\alpha^{3-\epsilon} e^{-c\alpha^2(t-s)}}{\sqrt{t-s+1}},
\end{align*}
$$

where the last inequality follows from applying Lemma 5.13 as before in Lemma 9.4. Then the control on $\tau_{in}^{(3)}$ is obtained by performing the outer integral over $ds$, using the bound on $F(s)$ and Lemma 5.13.

Lastly, for $\tau_{in}^{(4)}$, we rely on $\tilde{\tau}_{3,3}$ from Corollary 8.31 where on $\{\tilde{\tau}_{3,3} \geq \hat{t}_0\}$ we have

$$
\begin{align*}
&\left| \int_{t_0}^{s} J_{t-s,t-u}^{(a)} \int_{t_0}^{u} K_{\alpha}^{*}(u-v)(S(u) - S_{1}(v)) dv du \right| \\
&\leq \int_{t_0}^{s} \left| J_{t-s,t-u}^{(a)} \right| \alpha^{2-\epsilon} du \leq \alpha^{2-\epsilon}.
\end{align*}
$$

Then, conducting the outer integral over $ds$ similarly as above gives the estimate on $\tau_{in}^{(4)}$, concluding the proof. \[\square\]

As mentioned right before Lemma 9.6, we now study the remaining integrals, beginning with

$$
I_{in}^{(5)}(t) := \int_{t_0}^{t} \int_{t_0}^{s} J_{t-s,t-u}^{(a)} d\bar{\Pi}_{S-\alpha}(u) \cdot \int_{t_0}^{u} K_{\alpha}^{*}(s-x) d\Pi_{\alpha}(x) ds;
$$

$$
I_{in}^{(6)}(t) := \int_{t_0}^{t} \int_{t_0}^{s} J_{t-s,t-u}^{(a)} d\bar{\Pi}_{S-\alpha}(u) \cdot [\alpha'' - \alpha] ds.
$$

**Lemma 9.7.** Under the setting of Theorem 9.1 we have for all $t \in [\hat{t}_0, \tilde{t}_0]$ that

$$
\mathbb{E} \left[ I_{in}^{(5)}(t) \bigg| \mathcal{F}_{\hat{t}_0} \right] = O \left( \alpha^{3-\epsilon} \right) = \mathbb{E} \left[ I_{in}^{(6)}(t) \bigg| \mathcal{F}_{\hat{t}_0} \right].
$$

Although these integrals should have a negligible size compared to the leading order, one can see that the same argument as the previous analysis does not give the correct estimate on $I_{in}^{(5)}$. To overcome this issue, we remind ourselves that our actual interest is to control the expected value of this quantity.

**Proof of Lemma 9.7.** We begin with investigating $I_{in}^{(5)}$. Define $\alpha := \alpha - \frac{3}{2} - \epsilon$, and recall from Proposition 6.11 that $\tau_{0}(\alpha, t_0) \geq \hat{t}_0$ with high probability, that is, $S(t)$ is likely to stay above from $\underline{\alpha}$. Thus, writing $\tau_{00} = \tau_{00}(\alpha, t_0)$, observe that

$$
\mathbb{E} \left[ \int_{t_0}^{t_{\tau_{00}}} \int_{t_0}^{s} J_{t-s,t-u}^{(a)} d\bar{\Pi}_{S-\alpha}(u) \cdot \int_{t_0}^{u} K_{\alpha}^{*}(s-x) d\Pi_{\alpha}(x) ds \bigg| \mathcal{F}_{\hat{t}_0} \right] = 0,
$$

since $d\bar{\Pi}_{S-\alpha}$ and $d\Pi_{\alpha}$ define martingales and depend on disjoint set of points. Thus, under extra conditioning on $\Pi_{\underline{\alpha}}[\hat{t}_0, \tilde{t}_0]$, we can see that the above will always be zero. Therefore, relying on
Proposition 6.11 and the argument from Lemma 9.5 gives that for all \( t \in [\tilde{t}_0, \tilde{t}_0] \),

\[
\mathbb{E} \left[ \int_{\tilde{t}_0}^{\tilde{t}_0} \left( \int_{\tilde{t}_0}^{\tilde{t}_0} J_{t-s,t-u}^{(a)} d\tilde{\Pi}_{S - \alpha}(u) \cdot \int_{\tilde{t}_0}^{\tilde{t}_0} K_\alpha^*(s - x) d\tilde{\Pi}_\alpha(x) \right) ds \bigg| \mathcal{F}_{\tilde{t}_0} \right] = O(\alpha^{50}).
\]

Writing \( d\tilde{\Pi}_{\Delta \alpha} = d\tilde{\Pi}_\alpha - d\tilde{\Pi}_{2\alpha} \), define the stopping time

\[
\tau_{\text{in}}^{(5)} := \inf \left\{ t \geq \tilde{t}_0 : \int_{\tilde{t}_0}^{t} \left( \int_{\tilde{t}_0}^{\tilde{t}_0} J_{t-s,t-u}^{(a)} d\tilde{\Pi}_{S - \alpha}(u) \cdot \int_{\tilde{t}_0}^{\tilde{t}_0} K_\alpha^*(s - x) d\tilde{\Pi}_\alpha(x) \right) ds \geq \alpha^{\frac{5}{2} - \epsilon} \right\}.
\]

To conclude the proof, it suffices to show that

\[
\mathbb{P} \left( \tau_{\text{in}}^{(5)} < \tilde{t}_0 \bigg| \mathcal{F}_{\tilde{t}_0} \right) \leq \exp \left( -\beta_0^2 \right) + r.
\]

Since this estimate follows similarly as the proof of Lemmas 9.4 and 9.6 we give a brief explanation on its proof. Since \( J_{t-s,t-u}^{(a)} \) satisfies a similar bound as \( K_\alpha^* \) (Lemma 2.15), analogous argument as Corollary 7.25 gives an estimate the first part of the integral, written as

\[
\tau_{\text{in}}^{(6)} := \inf \left\{ t \geq \tilde{t}_0 : \exists s \in [t_0, t] \text{ s.t. } \int_{t_0}^{s} J_{t-s,t-u}^{(a)} d\tilde{\Pi}_{S - \alpha}(u) \geq \frac{\beta_0^C}{t - s + 1} + \frac{\alpha^{\frac{3}{2} - \epsilon}}{t - s + 1} \right\};
\]

\[
\mathbb{P} \left( \tau_{\text{in}}^{(6)} < \tilde{t}_0 \bigg| \mathcal{F}_{\tilde{t}_0} \right) \leq \exp \left( -\beta_0^3 \right) + r.
\]

(Note that \( \tau_1(t; S \Delta \alpha) \) in Corollary 7.25 becomes \((t - s + \tau_1(s) + 1)^{-1/2} \leq (t - s + 1)^{-1/2}\) in this case.) The second part of the integral is controlled by Corollary 7.26 and hence combining the two bounds with an application of Lemma 5.13 concludes the proof, similarly as the previous lemmas.

Finally, we note that \( I_{\text{in}}^{(6)}(t) \) is a martingale (note that \( \alpha'' \) is \( \mathcal{F}_{t_0} \)-measurable while the integral starts from \( t_0^\dagger \)) and hence its expectation is zero.

Now we are left with the two final integrals, which are

\[
I_{\text{in}}^{(7)}(t) := \int_{t_0}^{t} \left( \int_{t_0}^{s} J_{t-s,t-u}^{(a)} du \cdot \int_{t_0}^{s} K_\alpha^*(s - x) d\tilde{\Pi}_\alpha(x) \right) (\alpha'' - \alpha) ds;
\]

\[
I_{\text{in}}^{(8)}(t) := \int_{t_0}^{t} \int_{t_0}^{s} J_{t-s,t-u}^{(a)} (\alpha'' - \alpha)^2 du ds = \int_{t_0}^{t} \int_{t_0}^{s} J_{u,s}^{(a)} (\alpha'' - \alpha)^2 du ds.
\]

Note that \( I_{\text{in}}^{(7)}(t) \) is a martingale, since \( \alpha'' \) is \( \mathcal{F}_{t_0} \)-measurable. Thus, we have \( \mathbb{E}[I_{\text{in}}^{(7)}(t) \big| \mathcal{F}_{t_0}] = 0 \). On the other hand, \( I_{\text{in}}^{(8)}(t) \), which is a deterministic integral, is bounded from (A.10): \( I_{\text{in}}^{(8)}(t) = O(\alpha^{2-\epsilon}) \) for any \( t \in [\tilde{t}_0, \tilde{t}_0] \).

Combining all the analysis done in this subsection, we conclude the proof of Proposition 9.3.

Proof of Proposition 9.3 In (9.10) and the discussions below, we have seen that \( I_{\text{out}}(t) \) in Lemma 9.5 can be decomposed into

\[
I_1(t), I_2(t), \left\{ \text{integrals in } \tau_{\text{in}}^{(2)}, \tau_{\text{in}}^{(3)}, \tau_{\text{in}}^{(4)} \right\}, I_{\text{in}}^{(5)}(t), I_{\text{in}}^{(6)}(t), I_{\text{in}}^{(7)}(t), \text{ and } I_{\text{in}}^{(8)}(t),
\]

THE CRITICAL ONE-DIMENSIONAL MDLA 101
up to an $O(\alpha^{50})$ error coming from (9.4). Then, Lemmas 9.6, 9.7 and 9.11 tell us that the conditional expectations given $\mathcal{F}_t$ of all of the above integrals except $I_1(t)$ and $I_2(t)$ are of order $O(\alpha^{\frac{5}{2}-\epsilon})$. This implies
\[
\mathbb{E}[I_{\text{out}}(t) | \mathcal{F}_t] = \mathbb{E}[I_1(t) + I_2(t) | \mathcal{F}_t] + O\left(\alpha^{\frac{5}{2}-\epsilon}\right),
\]
which concludes the proof combined with Lemma 9.5.
\[\square\]

9.2. The analysis on the increment. We give a formal statement and proof of Theorem 2.12, thus verifying the assumptions of Theorem 4.1 are indeed accurate.

**Theorem 9.8.** Let $\alpha_0,t_0 > 0$, set $t^*_0,t^*_1,t_0$ and $\tilde{t}_0$ as (6.2) and (6.3) in terms of $\alpha_0$, and let $r = e^{-\beta_0^{3/2}}$. Also, let $t_1$ be any number satisfying (6.9), set $t^*_1 := t_1 - \alpha_0^{-2}\beta_0'$. Furthermore, define
\[
\alpha_1 := \mathcal{L}(t^*_1; \Pi_S[t^*_0,t^*_1], \alpha_0),
\]
and $\alpha'_0, \alpha'_1$ as (6.5). Then, for any $\Pi_S(-\infty,t_0)$ that is $(\alpha_0,r;[t_0])$-sharp-regular, we have
\[
\mathbb{E}[\alpha'_1 - \alpha'_0 | \Pi_S(-\infty,t_0)] = o\left(\alpha_0^4 \beta_0^{-3}\right) \cdot (t_1 - t_0);
\]
\[
\text{Var}[\alpha'_1 - \alpha'_0 | \Pi_S(-\infty,t_0)] = (1 + o(\alpha_0^{-1}))4\alpha_0^5(t_1 - t_0).
\]

**Proof.** Let $\tau = \alpha(t_0,0,2)$, $\tau^s = \tau^s(\alpha_0,0,2)$ be as (6.7), and recall the definition of $\tilde{\alpha}_1$ from Proposition 6.9. We define the event
\[
\mathcal{B}_1 := \left\{\tau \geq \tilde{t}_0 \right\} \cap \left\{\tau^s \geq \tilde{t}_0 \right\} \cap \left\{\alpha'_1 - \alpha'_1 \leq 2\alpha_0^2 \beta_0^{3/2}\right\}.
\]
\[
\mathcal{B}_2 := \left\{\int_{t_0}^{t_1} S(t)dt = \alpha_0(t_1 - \tilde{t}_0) + O\left(\alpha_0^{-\frac{1}{2}}\right) = \int_{\tilde{t}_0}^{t_1} S_1(t)dt \right\};
\]
\[
\mathcal{B}_3 := \left\{\int_{t_0}^{t_1} |S(t) - S_1(t)|dt \leq \alpha_0^{-2}\right\} \cap \left\{\int_{t_0}^{t_1} |S(t) - S_2(t)|dt \leq \alpha_0^{\frac{1}{2}}\right\};
\]
\[
\mathcal{B}_4 := \left\{\int_{t_0}^{t_1} S(t)dt = \alpha_0(t_1 - t_0) + O\left(\alpha_0^{-\frac{1}{2}}\right) \right\};
\]
\[
\mathcal{B}_5 := \left\{\int_{t_0}^{t_1} \sigma_1 \sigma_2 \sigma_3(t;S)dt \leq \alpha_0^{-\frac{1}{2}}\right\};
\]
\[
\mathcal{B} := \mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3 \cap \mathcal{B}_4 \cap \mathcal{B}_5,
\]
where $S_1(t) = S_1(t;\tilde{t}_0,\alpha_0)$, $S_2(t) = S_2(t;\tilde{t}_0,\alpha_0)$ are given as (2.16), (2.22). Conditioned on $\Pi_S(-\infty,t_0)$, $\mathcal{B}$ happens with probability at least $1 - 2e^{-\beta_0^{3/2}}$:

- $\mathcal{B}_1$ is obtained from the definition of sharp-regularity, Proposition 6.9, and Theorem 7.1.
- $\mathcal{B}_2, \mathcal{B}_3$ come from Lemma 6.10 and the sharp-regularity along with $\mathcal{B}_2$ implies $\mathcal{B}_4$.
- $\mathcal{B}_5$ is from Lemma 4.28 (whose assumptions are satisfied by $\tau_1, \tau_4$, and $\tau_5$).

Hence the contributions to the mean and variance of increments on $\mathcal{B}^c$ are negligible (since $\|S\| \leq \frac{1}{2}$). We write $\alpha'_1 - \alpha'_0 = (\alpha'_1 - \tilde{\alpha}_1) + (\tilde{\alpha}_1 - \alpha'_0)$ and similarly as (2.26) and (8.2), we express that
\[
\tilde{\alpha}_1 - \alpha'_0 = \int_{t_0}^{t_1} K_{\alpha_0}^*(d\Pi_S(t) - S_1(t)dt).
\]
From these formulas, we begin with estimating the mean of the increment. Note that from Lemma 8.1
\[
\mathbb{E}[|\alpha'_1 - \alpha_1|; \mathcal{F}_0] = o(\alpha_0^4 \beta_0^{-3}) \cdot (t_1 - t_0).
\] (9.12)

Furthermore, we have
\[
\mathbb{E}[\alpha_1 - \alpha_0'; \mathcal{F}_0] = K_{\alpha_0}^* \cdot \mathbb{E} \left[ \int_{t_0}^{t_1} \{S(t) - S_1(t)\} dt \bigg| \mathcal{F}_0 \right]
\] (9.13)

We decompose this integral into two parts, from \( t_0 \) to \( t_0' \) and from \( t_0' \) to \( t_1 \). For the first one, observe that the definition of \( B \) gives
\[
\left| \mathbb{E} \left[ \int_{t_0}^{t_0'} \{S(t) - S_1(t)\} dt ; \mathcal{F}_0 \right] \right| \leq \beta_0^{4\theta} = o(\alpha_0^2 \beta_0^{-3})(t_1 - t_0).
\] (9.14)

To study the integral from \( t_0' \) to \( t_1 \) of (9.13), recall the formula (2.22) to see that
\[
S(t) - S_1(t) = S(t) - S_2(t) + \frac{2\alpha_0^2}{(1 + 2\alpha_0)^2} - \frac{4\alpha_0 + 4\alpha_0^2}{(1 + 2\alpha_0)^2} S_1(t) + J(t),
\]
where \( J(t) \) is defined in the beginning of this section. Thus, we can see that for all \( t \in [t_0', t_1] \),
\[
\mathbb{E} \left[ \frac{2\alpha_0^2}{(1 + 2\alpha_0)^2} + J(t) ; \mathcal{F}_0 \right] = (1 + O(\alpha^\epsilon))4\alpha_0^2.
\]

Moreover, the event \( B \) gives
\[
\mathbb{E} \left[ \int_{t_0}^{t_1} S_1(t) dt ; \mathcal{F}_0 \right] = \alpha_0(t_1 - t_0) + O\left(\alpha_0^{-\frac{1}{2}-\epsilon}\right).
\]

In addition, the integral of \(|S(t) - S_2(t)|\) is bounded from the definition of \( B \), which is smaller than
\[
\alpha_0^\frac{1}{2} - 2\alpha_0^2 = o(\alpha_0^4 \beta_0^{-1}) \cdot (t_1 - t_0).
\]
Combining these three estimates, we observe that the terms of order \( \alpha_0^2(t_1 - t_0) \) cancel out, and hence deduce that
\[
\mathbb{E} \left[ \int_{t_0}^{t_1} \{S(t) - S_1(t)\} dt ; \mathcal{F}_0 \right] = o(\alpha_0^2 \beta_0^{-3}) \cdot (t_1 - t_0).
\] (9.15)

Therefore, combining (9.12), (9.13), (9.14) and (9.15) concludes the proof of the first statement.

To control the variance, we first write \( \alpha'_1 - \alpha_0' = K_{\alpha_0}^* \cdot \mathcal{M} + \mathcal{D} \), where
\[
\mathcal{M} := \int_{t_0}^{t_1} \{d\Pi S(t) - S(t)\} dt, \quad \mathcal{D} := (\alpha'_1 - \alpha_1) + K_{\alpha_0}^* \cdot \int_{t_0}^{t_1} \{S(t) - S_1(t)\} dt,
\]
and also define \( \mathcal{D} := \mathbb{E}[\mathcal{D}|\mathcal{F}_0] \). Then, we express that
\[
|\text{Var} [\alpha'_1 - \alpha_0'| \mathcal{F}_0] - \text{Var} [K_{\alpha_0}^* \cdot \mathcal{M} | \mathcal{F}_0]| \leq \text{Var}[\mathcal{D}|\mathcal{F}_0] + 2K_{\alpha_0}^* \cdot \left\{ \mathbb{E}[\mathcal{M}^2 | \mathcal{F}_0] \cdot \text{Var}[\mathcal{D}|\mathcal{F}_0] \right\}^{1/2},
\] (9.16)
Thus, we can control the size of \( \alpha \) we have
\[
\mathbb{E}[\mathcal{M}^2 | \mathcal{F}_0] = \mathbb{E} \left[ \int_{t_0}^{t_1} S(t) dt \bigg| \mathcal{F}_0 \right] = o(\alpha_0^{10}) + \mathbb{E} \left[ \int_{t_0}^{t_1} S(t) dt \bigg| \mathcal{B} \right] + \mathbb{E} \left[ \int_{t_0}^{t_1} S(t) dt \bigg| \mathcal{F}_0 \right]
\]
\[
= \alpha_0(t_1 - t_0) + O\left(\alpha_0^{-\frac{3}{2} - \epsilon}\right) = \left(1 + o(\alpha_0^{\frac{3}{2}})\right) \alpha_0(t_1 - t_0).
\]
(9.17)

Moreover, on the event \( \mathcal{B} \), we have \( \mathcal{D} = O(\alpha_0^{2-2\epsilon}) \). Thus, we see that \( \text{Var}[\mathcal{D} | \mathcal{F}_0] = O(\alpha_0^{4-4\epsilon}) \), and hence
\[
\text{Var}[\mathcal{D} | \mathcal{F}_0] = o(\alpha_0^{1-5\epsilon})(K_0^*)^2 \mathbb{E}[\mathcal{M}^2 | \mathcal{F}_0].
\]

Therefore, we combine the above computations (9.16) and (9.17) to deduce the second statement which finishes the proof.

To conclude this section, we establish a variant of Theorem 9.8 which will be useful when applying to a slightly different setting in the next section.

**Corollary 9.9.** Under the setting of Theorem 9.8, let \( \mathcal{E} \) be an event that satisfies \( \mathbb{P}(\mathcal{E} | \mathcal{F}_{t_0}) \geq 1 - \alpha_0^5 \), for any \( \mathcal{F}_{t_0} = \Pi_S(-\infty, t_0] \) that is \( (\alpha_0, r; [t_0]) \)-sharp-regular. Then, we have
\[
\text{Var}\left[ \left( \alpha_1' - \alpha_0' \right) \mathbbm{1}\{\mathcal{E}^c|\mathcal{F}_0\} \right] = (1 + o(\alpha_0^{\frac{3}{2}})) 4\alpha_0^5(t_1 - t_0).
\]

**Proof.** For convenience, we write \( \mathcal{F}_0 = \mathcal{F}_{t_0} = \Pi_S(-\infty, t_0] \). Due to the analogous decomposition as (9.16), it suffices to show that
\[
\text{Var}\left[ \left( \alpha_1' - \alpha_0' \right) \mathbbm{1}\{\mathcal{E}^c|\mathcal{F}_0\} \right] = o(\alpha_0^4),
\]
since we have \( \text{Var}[\alpha_1' - \alpha_0'|\mathcal{F}_0] = \Theta(\alpha_0^{3.108}) \). The proof goes analogously as that of Lemma 9.5, based on the same idea as (9.5). Observe from (8.2) that
\[
\alpha_1 - \alpha_0' = \mathcal{L}(t_1; \Pi_S[t_0, t_1], \alpha_0) = K^*_\alpha \int_{t_0}^{t_1} d\Pi_S(x) - S_1(x) dx
\]
\[
= K^*_\alpha |\Pi_S[t_0, t_1]| - K^*_\alpha \int_{t_0}^{t_1} \int_{y} K_0(x-y) dx d\Pi_S(y).
\]
Since \( \int_{y}^{t_1} K_0(x-y) dx \in [0,1] \), we can see that \( |\alpha_1 - \alpha_0'| \) is stochastically dominated by
\[
|\alpha_1 - \alpha_0'| \leq K^*_\alpha |\Pi_{1/2}[t_0, t_1]|,
\]
due to the fact that \( S \leq \frac{1}{2} \) (Proposition 2.2). Moreover, we also have
\[
\alpha_1' = K^*_\alpha \int_{t_1}^{\infty} \int_{t_1}^{t_1} K_0(x-y) dx d\Pi_S(y) dx \leq K^*_\alpha |\Pi_{1/2}[t_1, t_1]|.
\]

Thus, we can control the size of \( \alpha_1 \) and \( \alpha_1' \) on the event \( \mathcal{E}^c \) by coupling with the tail events of the Poisson process, which easily gives us the desired estimate. (Note that \( \alpha_0' \) is already close to \( \alpha_0 \) since the speed is \( (\alpha_0, r; [t_0]) \)-sharp-regular.)
10. Multi-scale analysis and the scaling limit

In this section we repeatedly use the induction step theorem in order to prove the main results of this paper. The proof goes as follows. Let $t > 0$ be sufficiently large and recall that Theorem 10.1 says that $t^{-\frac{1}{3}}S(st)$ should behave roughly like the solution of the SDE (1.1). The proof has the following parts:

1. We show that in time $\approx \delta t$ the aggregate is regular and the speed is of order $\approx \delta^{-\frac{1}{3}}t^{-\frac{1}{3}}$.
2. We show that $t^{\frac{1}{3}}S(\delta t + st)$ is close to the solution of (1.1) with initial condition $Z(0) = \delta^{-\frac{1}{3}}$.

Theorem 10.1 follows from part (2) by taking $\delta \to 0$ and integrating the speed to obtain the aggregate size. The proof of the first part is given in Subsection 10.1 and it includes analysis of $\log t$ many scales of the aggregate speed. We show that in each scale the aggregate speed is likely to decrease and that the time it takes for this to happen is not too long. The second part is given in Subsection 10.2 and it is a direct application of a result by Helland [11]. In [11] it is shown that, under some assumptions, if the increments of a sequence of processes have the right conditional expectation and variance then the sequence of processes converges to the solution of the corresponding SDE.

Throughout this section it is convenient to work with a slightly different definition of regular. Recall the definition of $\mathcal{R}(\alpha_0, r, [t_0])$ in Section 6.1. As usual, we identify a set of points $\Pi$ with the cumulative function $Y(\delta) := |\Pi[\delta t + s, t]|$ and denote by $\Pi_Y$ the set of points corresponding to the step function $Y$. We also write $Y \in \mathcal{R}(\alpha_0, r, [t_0])$ when $\Pi_Y \in \mathcal{R}(\alpha_0, r, [t_0])$. For convenience, in the following definition and throughout this section we also switch the roles of $\alpha$ and $\alpha'$ from Section 6 and equation (6.5).

**Definition 10.1.** Let $\alpha > 0$ sufficiently small and $t > 0$. We say that a deterministic step function $Y_0: \mathbb{R}_+ \to \mathbb{N} \cup \{\infty\}$ satisfies $Y_0 \in \mathcal{R}'(\alpha, t)$ if there exists $t^-$ with

$$t - 2\alpha^{-2}\log^\theta(1/\alpha) \leq t^- \leq t - \frac{1}{2}\alpha_0^{-2}\log^\theta(1/\alpha)$$

and $\alpha' > 0$ such that $Y_0 \in \mathcal{R}(\alpha', \alpha_0^6; [t])$ and

$$\alpha = \mathcal{L}(t; \Pi_Y[t^-, t], \alpha') := \frac{2(\alpha')^2}{1 + 2\alpha'} \int_{t^-}^{t} \int_{t^-}^{t} K_{\alpha'}(z - x)dzd\Pi_Y(x).$$

Using this definition we state the main results of the induction step in the following theorem.

**Theorem 10.2.** Let $\alpha > 0$ sufficiently small, $t > 0$ and $\tilde{t} > 0$ such that

$$\alpha^{-2}\log^{106-4}(1/\alpha) \leq \tilde{t} - t \leq \alpha^{-2}\log^{106}(1/\alpha).$$

Let $Y_\tilde{t} \in \mathcal{R}'(\alpha, t)$ and let $(Y_x, S(x), X_x)$ be the aggregate with initial condition $(Y_\tilde{t}, t)$. Then, there is a random variable $\hat{\alpha} \in \mathcal{F}_\tilde{t}$ such that the following holds:

1. $|\hat{\alpha} - \alpha| \leq \sqrt{\alpha^5(\tilde{t} - t)\log^2(1/\alpha)}$

2. $\mathbb{P}(Y_\tilde{t} \in \mathcal{R}'(\hat{\alpha}, \tilde{t})) \geq 1 - \alpha^5$

3. $|\mathbb{E}(\hat{\alpha} - \alpha)| \leq \frac{\alpha^4}{\log^3(1/\alpha)}(\tilde{t} - t)$.

4. $\mathbb{E}(\hat{\alpha} - \alpha)^2 = 4\alpha^5(\tilde{t} - t)\left(1 + O(\alpha^{\frac{1}{3}})\right)$. 
\[ (5) \]

\[ \mathbb{P}\left( |X_{\tilde{t}} - X_t - \alpha(\tilde{t} - t)| \leq \alpha^{1.4}(\tilde{t} - t) \right) \geq 1 - \alpha^5 \]

**Proof.** By Definition 10.1 there exists \( \alpha' \) such that \( Y_t \in N(\alpha', \alpha^6, [t]) \) which in particular means by the definition of \( A_2 \) in (6.6) that \( |\alpha - \alpha'| \leq (\alpha')^{3/2} \log^2(1/\alpha') \leq 2(\alpha')^{3/2} \log^2(1/\alpha') \). Thus by Theorem 6.4 we have some \( \tilde{\alpha}' \) such that

\[ \mathbb{P}(Y_{t_1} \in N(\tilde{\alpha}', (\tilde{\alpha}')^{3/2}, [\tilde{t}])) \geq 1 - 2\alpha^6 \quad \text{and} \quad \mathbb{P}(|\alpha' - \tilde{\alpha}'| \leq 2(\alpha')^{3/2} \log^2(1/\alpha')) \geq 1 - 2\alpha^6. \]

Let

\[ \tilde{\alpha} := \mathcal{L}(\tilde{t}; \Pi_{\tilde{t}}[\tilde{t} - \tilde{t}], \tilde{\alpha}'). \]

Using the definition of \( A_2 \) again we have \( |\alpha - \tilde{\alpha}'| \leq (\tilde{\alpha}')^{3/2} \log^2(1/\alpha') \leq 2\alpha_0^{3/2} \log^2(1/\alpha) \) with probability at least \( 1 - 4\alpha^6 \). Thus we get

\[ \mathbb{P}(|\alpha - \tilde{\alpha}| \leq \sqrt{\alpha^{5}(\tilde{t} - t) \log^2(1/\alpha')} \geq 1 - 4\alpha^6. \]

We also get that \( \mathbb{P}(Y_{t_1} \in N(\tilde{\alpha}', \alpha^6, [\tilde{t}])) \geq 1 - 6\alpha^6 \) and therefore by Definition 10.1 \( \mathbb{P}(Y_t \in N(\tilde{\alpha}, \tilde{t})) \geq 1 - 6\alpha^6 \). This finishes the proof of the second part of the theorem. The third and fourth part of the theorem follows from Theorem 9.8. The last part of the theorem follows from Proposition 6.8.

This finishes the proof of all parts of the theorem except for part one which holds with probability \( 1 - 4\alpha^6 \) instead of deterministically. In order to fix this issue we change the definition of \( \tilde{\alpha} \) slightly. On the event where part (1) doesn’t hold we change \( \tilde{\alpha} \) to be \( \alpha \). It is clear that parts (2), (3) and (5) still hold. Part (4) of the theorem still holds by Corollary 9.9. \( \square \)

10.1. **Stitching intervals.** In this section we use Theorem 10.2 repeatedly to show that at time \( t \) the aggregate speed is \( \approx t^{-\frac{1}{3}} \). To this end, we stitch a lot of short intervals of time to obtain the right growth in a medium interval of time. Then we stitch a lot of medium intervals of time to obtain the right growth in a long interval of time. Finally, we stitch a lot of long intervals of time to obtain the right growth in the time interval \([0, t]\). See Subsections 10.1.1, 10.1.2, and 10.1.3.

10.1.1. **Stitching short intervals.** In this section we build inductively a process that approximates the speed of the aggregate and study the time it takes for this process to get smaller or larger by a factor of 2.

For the proof of the main lemma of this subsection we’ll need the following martingale concentration result due to Freedman [10]. See also Theorem 18 in [5] for a short proof.

**Claim 10.3.** Let \( X_0, X_1, \ldots, X_n \) be a martingale with filtration \( \mathcal{F}_0, \ldots, \mathcal{F}_n \). Suppose that almost surely for all \( 1 \leq i \leq n \) we have:

\[ \text{Var}(X_i \mid \mathcal{F}_{i-1}) \leq \sigma_i^2 \quad \text{and} \quad |X_i - X_{i-1}| \leq M \]

Then,

\[ \mathbb{P}( |X_n - X_0| \geq \lambda) \leq 2 \exp\left( -\frac{\lambda^2}{\sum_{i=1}^{n} \sigma_i^2 + \frac{1}{3} M \lambda} \right). \]

Let \( t_0 > 0 \) and let \( \delta_0 > 0 \) sufficiently small. Let \( \alpha_0 > 0 \) such that \( |\alpha_0 - \delta_0| \leq \delta_0^{1.1} \). Here \( \delta_0 \) should be understood as the scale of the speed and \( \alpha_0 \) as the initial speed. Let \( Y_{t_0} \in N(\alpha_0, t_0) \) and let \( \Pi_S \) be the aggregate with initial condition \( Y_{t_0} \). Let \( t_1 > 0 \) such that

\[ \delta_0^{-2} \log^{10^{6}-3}(1/\delta_0) \leq t_1 - t_0 \leq \delta_0^{-2} \log^{10^{6}-1}(1/\delta_0). \]
and $t_i := t_0 + i(t_1 - t_0)$ for any $i \geq 2$. We define the sequence $\alpha_i$ inductively. Suppose that $\alpha_1, \ldots, \alpha_i$ were already defined. Define the stopping times
\[
\begin{align*}
\zeta_1 &:= \min \{ t_i \geq t_0 : \alpha_i < \delta_0/2 \} \\
\zeta_2 &:= \min \{ t_i \geq t_0 : \alpha_i > 2\delta_0 \} \\
\zeta_4 &:= \min \{ t_i \geq t_0 : Y_{t_i} \notin \mathcal{R}(\alpha_i, t_i) \} \\
\zeta_5 &:= \min \{ t_{i+1} \geq t_i : |X_{t_{i+1}} - X_{t_i} - \alpha_i(t_1 - t_0)| \geq \alpha_i^{1.4}(t_1 - t_0) \},
\end{align*}
\]
$\zeta_3 = \zeta_4 \wedge \zeta_5$ and $\zeta := \zeta_1 \wedge \zeta_2 \wedge \zeta_3$. We note that so far $\zeta$ is not well defined because $\alpha_j$ for $j > i$ was not defined but the event $\{ \zeta > t_i \}$ is well defined. On the event $\{ \zeta > t_i \}$ we have that $Y_{t_i} \notin \mathcal{R}(\alpha_i, t_i)$ and therefore we can define $\alpha_{i+1} \in \mathcal{T}_{t_i}$ to be the random variable $\tilde{\alpha}$ from Theorem 10.2 with $t_i$ and $t_{i+1}$ as $t$ and $\tilde{t}$ respectively and with $\alpha_i$ as $\alpha$. On the event $\{ \zeta \leq t_i \}$ we just let $\alpha_{i+1} := \alpha_i$. Using Theorem 10.2 we have almost surely on the event $\{ \zeta > t_n \}$ that $\mathbb{P}(\zeta = t_{n+1} | \mathcal{F}_{t_n}) \leq 2\alpha_n^5 \leq C\delta_0^5$ and moreover
\[
|\mathbb{E}(\alpha_{n+1} - \alpha_n | \mathcal{F}_{t_n})| \leq \frac{(t_1 - t_0)\alpha_n^4}{\log^3(1/\alpha_n)} \leq C(t_1 - t_0)\delta_0^3 \log^3(1/\delta_0).
\]

Lemma 10.4. The stopping time $\zeta$ satisfies the following properties
\begin{enumerate}
\item $\mathbb{P}(\zeta = \zeta_3) \leq C\delta_0^3$
\item $\mathbb{P}(\zeta = \zeta_1) = 2/3 + O(\log^{-1}(1/\delta_0))$, $\mathbb{P}(\zeta = \zeta_2) = 1/3 + O(\log^{-1}(1/\delta_0))$
\item For all $k \geq 1$, $\mathbb{P}(\zeta \geq t_0 + k\delta_0^{-3}) \leq Ce^{-ck}$
\item For all $\epsilon < 1$, $\mathbb{P}(\zeta \leq t_0 + \epsilon\delta_0^{-3}) \leq C\delta_0^4 + Ce^{-c(1-\epsilon)}$
\end{enumerate}

Proof. We start by studying the Doob decomposition of $\alpha_i$. Consider the martingale
\[
\beta_n := \alpha_0 + \sum_{i=1}^{n} \alpha_i - \mathbb{E}[\alpha_i | \mathcal{F}_{t_{i-1}}] = \alpha_n - \sum_{i=1}^{n} \mathbb{E}[\alpha_i | \mathcal{F}_{t_{i-1}}] - \alpha_{i-1}.
\]
We show that the increments of $\beta_n$ are close to those of $\alpha_n$. We have
\[
|\beta_{n+1} - \beta_n| - (\alpha_{n+1} - \alpha_n)| \leq 1 \{ \zeta > t_n \} |\mathbb{E}[\alpha_{n+1} | \mathcal{F}_{t_n}] - \alpha_n| \leq C1 \{ \zeta > t_n \} \delta_0^3(t_1 - t_0) \leq C1 \{ \zeta > t_n \} \delta_0^3(t_1 - t_0) \log^3(1/\delta_0). \tag{10.1}
\]
Thus, for any $n \leq 2\delta_0^{-3} \log^2(1/\delta_0)/(t_1 - t_0)$ we have that
\[
|\beta_n - \alpha_n| \leq Cn \delta_0^3(t_1 - t_0) \leq C \frac{\delta_0}{\log(1/\delta_0)}. \tag{10.2}
\]
and in particular $0.4\delta_0 < \beta_n < 2.1\delta_0$ for such $n$. By (10.1) we also have
\[
|\beta_{n+1} - \beta_n| \leq |\alpha_{n+1} - \alpha_n| + C \delta_0^3(t_1 - t_0) \log^2(1/\delta_0) \leq C \delta_0^3(t_1 - t_0) \log^3(1/\delta_0) \tag{10.3}
\]
and therefore, using (10.1) once again
\[
|\beta_{n+1} - \beta_n|^2 - (\alpha_{n+1} - \alpha_n)^2 \leq C \sqrt{\delta_0^3(t_1 - t_0) \log^2(1/\delta_0)}(\beta_{n+1} - \beta_n) - (\alpha_{n+1} - \alpha_n)| \leq C1 \{ \zeta > t_n \} \delta_0^3 \log^2(1/\delta_0)(t_1 - t_0)^{3/2} \leq C1 \{ \zeta > t_n \} \delta_0^3 \log^2(1/\delta_0).
\]
Thus
\[
\mathbb{E}\left[ (\beta_{n+1} - \beta_n)^2 \mid \mathcal{F}_{t_n} \right] = \mathbb{E}\left[ (\alpha_{n+1} - \alpha_n)^2 \mid \mathcal{F}_{t_n} \right] + \mathbb{1}\{\zeta > t_n\} \{O(\delta_0^7 \log^{2\theta}(1/\delta_0)) \}
\]
\[= \mathbb{1}\{\zeta > t_n\} 4\alpha_n^5 (t_1 - t_0) (1 + O(\delta_0^{0.4})).
\]
(10.4)

It follows that \(\mathbb{P}( (\beta_{n+1} - \beta_n)^2 ) \geq 0.1\delta_0^5 (t_1 - t_0) \mathbb{P}(\zeta > t_n)\). Now, letting \(n_0 := \lceil 100\delta_0^{-3}/(t_1 - t_0) \rceil\) we have that
\[
5\delta_0^2 \geq \mathbb{E}(\beta_{n+1} - \beta_n)^2 \geq \text{Var}(\beta_{n+1}) = \sum_{n=0}^{n_0} \mathbb{E}(\beta_{n+1} - \beta_n)^2 
\]
\[\geq 0.1n_0\delta_0^5 (t_1 - t_0) \mathbb{P}(\zeta \geq t_n) \geq 10\delta_0^2 \cdot \mathbb{P}(\zeta \geq t_n).
\]
We get that \(\mathbb{P}(\zeta > t_n) \leq 1/2\). Let \(k \geq 1\). By repeating the above arguments on the event \(\{\zeta > t_{(k-1)n_0}\}\) with the initial condition \(Y_{t_{(k-1)n_0}}\) instead of \(Y_0\) and the random variable \(\alpha_{(k-1)n_0}\) instead of \(a_0\) we get that \(\mathbb{1}\{\zeta > t_{(k-1)n_0}\} \cdot \mathbb{P}(\zeta > t_{(k-1)n_0}) \leq 1/2\) almost surely (we note that we might not have \(|\alpha_{kn_0} - \delta_0| \leq \delta_0^{1.1}\) but we did not use the fact that \(|\alpha_0 - \delta_0| \leq \delta_0^{1.1}\) in the proof of \(\mathbb{P}(\zeta > t_{n_0}) \leq 1/2\). Thus, inductively we have that
\[
\mathbb{P}(\zeta > t_{kn_0}) \leq \mathbb{1}\{\zeta \geq t_{(k-1)n_0}\} \cdot \mathbb{P}(\zeta > t_{kn_0} \mid \mathcal{F}_{(k-1)n_0}) \leq \frac{1}{2} \mathbb{P}(\zeta > t_{(k-1)n_0}) \leq \cdots \leq 2^{-k}.
\]
(10.5)

Finally, by the definition of \(n_0\), for \(c_1 = 0.001\) and \(k\) sufficiently large we have that \(t_{[c_1k]n_0} = t_0 + [c_1k]n_0(t_1 - t_0) \leq t_0 + k\delta_0^{-3}\) and therefore
\[
\mathbb{P}(\zeta > t_0 + k\delta_0^{-3}) \leq \mathbb{P}(\zeta > t_{[c_1k]n_0}) \leq 2^{-[c_1k]} \leq C e^{-ck}.
\]

This finishes the proof of the third part.

We turn to prove the first part. For any \(n \geq 1\)
\[
\mathbb{P}(\zeta_3 = \zeta \leq t_n) = \mathbb{P}\left( \bigcup_{i=1}^{n} \{\zeta_3 = t_i \cap \zeta \geq t_{i-1}\} \right) \leq \sum_{i=1}^{n} \mathbb{P}(\zeta > t_{i-1} \text{ and } \zeta_3 = i)
\]
\[= \sum_{i=1}^{n} \mathbb{E}[ \mathbb{1}\{\zeta > t_{i-1}\} \cdot \mathbb{P}(\zeta_3 = t_i \mid \mathcal{F}_{t_{i-1}})] \leq C n\delta_0^5.
\]
(10.6)

Thus, if we let \(n_1 := \lceil \delta_0^{-1}\rceil n_0 \leq \delta_0^{-2}\) we get by (10.5) that
\[
\mathbb{P}(\zeta_3 = \zeta) \leq \mathbb{P}(\zeta_3 = \zeta \leq t_{n_1}) + \mathbb{P}(\zeta > t_{n_1}) \leq C \delta_0^3 + C e^{-c\delta_0^{-1}} \leq C \delta_0^3.
\]

We turn to prove the fourth part of the lemma. Let \(\log^{-2}(1/\delta_0) \leq \epsilon < 1\) and let \(n_3 := \lceil \epsilon \delta_0^{-3}/(t_1 - t_0) \rceil \leq \delta_0^{-1}\). Define
\[
M := C \sqrt{\delta_0^5 (t_1 - t_0) \log^{2\theta}(1/\delta_0)} \leq \delta_0^3 \log^{C\theta}(1/\delta_0), \quad \sigma := C \sqrt{\delta_0^5 (t_1 - t_0)}.
\]

The martingale \(\beta_n\) satisfy the assumptions of Theorem 10.3 with this \(M\) and \(\sigma_i = \sigma\) by equations (10.3) and (10.4). Thus, taking \(\lambda = \delta_0/4\) we get
\[
\mathbb{P}(\{\beta_{n_3} - \alpha_{n_3} \cap \alpha_{n_3} - \alpha_0 \geq \lambda\}) \leq 2 \exp\left( -\frac{\lambda^2}{n_3 \sigma^2 / 2 M \lambda} \right) < 2 \exp\left( -\frac{c_0^2}{2\epsilon_0^2 + c_0^2 \log^{C}(1/\delta_0)} \right) \leq C e^{-c\epsilon^{-1}}.
\]

Thus, using that \(|\alpha_0 - \delta_0| \leq \delta_0^{1.1}\) we get
\[
\mathbb{P}(\zeta \leq t_0 + \epsilon \delta_0^{-3}) \leq \mathbb{P}(\zeta_3 = \zeta \leq t_{n_3}) + \mathbb{P}(\zeta_1 = \zeta \land \zeta_2 = \zeta) = \leq C n_3 \delta_0^5 + \mathbb{P}(\{\alpha_{n_3} - \alpha_0 \geq 4\delta_0/10\})
\]
\[\leq C \delta_0^4 \mathbb{P}(\{\alpha_{n_3} - \beta_{n_3} > \delta_0/10\}) + \mathbb{P}(\{\beta_{n_3} - \alpha_0 \geq \delta_0/4\}) \leq C \delta_0^4 + C e^{-c\epsilon^{-1}}.
\]
where in the third inequality we used \([10.6]\) and in the fifth inequality we used \([10.2]\). This finishes the proof of the fourth part of the lemma when \(\epsilon \geq \log^{-2}(1/\delta_0)\). It is clear that the same inequality holds for \(\epsilon < \log^{-2}(1/\delta_0)\) as well (since the \(e^{-\epsilon^{-1}} < \delta_0^4\) in this case).

Finally, we prove the second part of the lemma. Let \(n_4 := [\log^2(1/\delta_0)\delta_0^2/(t_1 - t_0)] \leq \delta_0^{-1}\) and
\[
A := \{ \zeta = \zeta_1 \leq t_{n_4} \}, \quad B := \{ \zeta = \zeta_2 \leq t_{n_4} \}, \quad C := \{ \zeta_1 \wedge \zeta_2 > \zeta \text{ and } \zeta \leq t_{n_4} \} \cup \{ \zeta > t_{n_4} \}.
\]
We have
\[
\delta_0 + O(\delta_0^{1.1}) = \alpha_0 = \mathbb{E}[\beta_{n_4}] = \mathbb{E}[1_A\beta_{n_4}] + \mathbb{E}[1_B\beta_{n_4}] + \mathbb{E}[1_C\beta_{n_4}]. \tag{10.7}
\]
Next, we estimate each one of the terms separately. We have
\[
\mathbb{P}(C) \leq \mathbb{P}(\zeta = \zeta_3 \leq t_{n_4}) + \mathbb{P}(\zeta \geq t_{n_4}) \leq Cn_4\delta_0^5 + Ce^{-\log^2(1/\delta_0)} \leq C\delta_0^4; \tag{10.8}
\]
where in the second inequality we used \([10.6]\) and \([10.5]\). Thus,
\[
\mathbb{E}[1_C\beta_{n_4}] \leq C\delta_0^4. \tag{10.9}
\]

On the event \(A\) we have
\[
|\beta_{n_4} - \delta_0/2| \leq |\beta_{n_4} - \alpha_{n_4}| + |\alpha_{n_4} - \delta_0/2| \leq C\frac{\delta_0}{\log(1/\delta_0)} + C\delta_0^3 \log(1/\delta_0) \leq C\frac{\delta_0}{\log(1/\delta_0)},
\]
where in the second inequality we bound the first term using \([10.2]\) and bound the second term using the definition of \(\zeta_1\) and the fact that the increments of \(\alpha_n\) are small. Thus
\[
\mathbb{E}[1_A\beta_{n_4}] = \frac{\delta_0}{2} \cdot \mathbb{P}(A) + O\left(\frac{\delta_0}{\log(1/\delta_0)}\right). \tag{10.10}
\]

Using the same arguments we have
\[
\mathbb{E}[1_B\beta_{n_4}] = 2\delta_0 \cdot \mathbb{P}(B) + O\left(\frac{\delta_0}{\log(1/\delta_0)}\right) = 2\delta_0 \left(1 - \mathbb{P}(A) + O(\delta_0^4)\right) + O\left(\frac{\delta_0}{\log(1/\delta_0)}\right)
= 2\delta_0 - 2\delta_0 \cdot \mathbb{P}(A) + O\left(\frac{\delta_0}{\log(1/\delta_0)}\right), \tag{10.11}
\]
where in the second equality we used \([10.8]\).

Substituting \([10.10]\), \([10.11]\) and \([10.9]\) into \([10.7]\) we get
\[
\delta_0 = \frac{\delta_0}{2} \mathbb{P}(A) + 2\delta_0 - 2\delta_0 \cdot \mathbb{P}(A) + O\left(\frac{\delta_0}{\log(1/\delta_0)}\right)
\]
which means that
\[
\mathbb{P}(A) = \frac{2}{3} + O\left(\log^{-1}(1/\delta_0)\right), \quad \mathbb{P}(B) = \frac{1}{3} + O\left(\log^{-1}(1/\delta_0)\right)
\]
The second part of the lemma follows as the event \(\{ \zeta = \zeta_1 \} \wedge A \in \{ \zeta > t_{n_4} \}\) has a small probability. \(\square\)

**Remark 10.5.** On the event \(\{ \zeta < \zeta_3 \}\) (which happens with high probability by Lemma \([10.4]\) we have that
\[
\frac{\delta_0}{3}(\zeta - t_0) \leq X_\zeta - X_{t_0} \leq 3\delta_0(\zeta - t_0)
\]
In the next subsection we are going to change \(t_0, \alpha_0\) and \(\delta_0\) and therefore we write more specifically \(\zeta(t_0, \alpha_0, \delta_0)\) for the stopping time \(\zeta\) and similarly with \(\zeta_1, \zeta_2, \zeta_3\).

We define the random variable \(\alpha' = \alpha'(t_0, \alpha_0, \delta_0) \in \mathcal{F}_\zeta\) as follows. on the event \(\{ \zeta = t_i \}\), we let \(\alpha' := \alpha_i\).
10.1.2. **Stitching medium intervals.** In this section we stitch the medium intervals of time together to create the large intervals of time in which the speed decreases with high probability. The idea is to repeatedly use the previous section to show that the speed behaves somewhat like a random walk with a negative drift on the dyadic scales. As in the previous section we let $t_0 > 0$, $\delta_0 > 0$ sufficiently small and $\alpha_0 > 0$ with $|\alpha_0 - \delta_0| \leq \delta_0^{1/2}$. We also let $M > 0$ with $2\delta_0 < M \leq \sqrt{\delta_0}$. Finally, let $Y_{t_0} \in \mathcal{X}(\alpha_0, t_0)$ and let $\Pi_S$ be the aggregate with initial condition $Y_{t_0}$. We define the processes $\delta_i, \alpha_i$ and the stopping times $t_i$ inductively (note that the $t_i$ are defined not as in the previous subsection). Suppose we defined $\delta_1, \ldots, \delta_i, \alpha_1, \ldots, \alpha_i$ and $t_1, \ldots, t_i$ (note that the definition of $t_i$ will be different this time). Define the stopping times

\[
\begin{align*}
\xi'_1 &:= \min \{ t_i \geq t_0 : \delta_i \leq \delta_0/2 \} \\
\xi'_2 &:= \inf \{ t_i \geq t_0 : \delta_i > M \} \\
\xi'_3 &:= \inf \{ t_i \geq t_1 : (t_{i-1}, \alpha_{i-1}, t_{i-1}) = \zeta_3(t_{i-1}, \alpha_{i-1}, \delta_{i-1}) \}
\end{align*}
\]

and $\xi' = \xi'_1 \wedge \xi'_2 \wedge \xi'_3$. Let $t_{i+1} := \zeta(t_i, \alpha_i, \delta_i)$. We define $\delta_{i+1} \in \mathcal{F}_{t_{i+1}}$ as follows

\[
\delta_{i+1} = \begin{cases} 
\delta_i/2, & \text{on the event } \{ \xi' > t_i \text{ and } \xi'_3 > t_{i+1} \text{ and } \zeta(t_i, \alpha_i, \delta_i) = \zeta_1(t_i, \alpha_i, \delta_i) \} \\
\delta_i, & \text{on the event } \{ \xi' \leq t_i \text{ or } \xi'_3 = t_{i+1} \} \\
2\delta_i, & \text{on the event } \{ \xi' > t_i \text{ and } \xi'_3 > t_{i+1} \text{ and } \zeta(t_i, \alpha_i, \delta_i) = \zeta_2(t_i, \alpha_i, \delta_i) \}
\end{cases}
\]

We also let $\alpha_{i+1} = \alpha'(t_i, \alpha_i, \delta_i)$ where $\alpha'$ is defined in the end of the previous section. On the event $\{ \xi' > t_i \}$ we have that $\delta_i < M$, and there exist a random variable $\alpha_i$ with $|\alpha_i - \delta_i| \leq \delta_0^{1/2}$ so that $Y_{t_i} \in \mathcal{X}(\alpha_i, t_i)$. Thus, on this event, we can apply Lemma 10.4 with $t_i, \delta_i$ and $\alpha_i$ as $t_0, \delta_0$ and $\alpha_0$ respectively. By the lemma, with probability at least $1 - C\delta_i^4$ we have that $\xi'_3 > t_{i+1}$. The reader should think of $\delta_i$ as the sequence of different dyadic scales that the speed traveled through.

**Lemma 10.6.** The stopping time $\xi'$ satisfies the following properties:

1. \[ \mathbb{P}(\xi' = \xi'_1) \leq C\delta_0 \]
2. \[ \mathbb{P}(\xi' = \xi'_2) \leq \left( \frac{\delta_0}{M} \right)^c \]
3. For all $k \geq 1$,
\[ \mathbb{P}(\xi' > t_0 + k\delta_0^3) \leq Ce^{-c\sqrt{k}} \]
4. For all $\epsilon < 1$,
\[ \mathbb{P}(\xi' < t_0 + \epsilon\delta_0^3) \leq C\delta_0^4 + Ce^{-c\epsilon^{-1}} \]

**Proof.** Define $W_i := \log_2(\delta_i/\delta_0)$. Let $n \geq 1$ and let $\xi_1, \xi_2, \xi_3, \xi$ be the stopping times from the previous subsection with $t_0, \alpha_0, \delta_0$ being $t_n, \alpha_n, \delta_n$ respectively. By Lemma 10.4 we have that

\[
\mathbb{E}(W_{n+1} - W_n \mid \mathcal{F}_{t_n}) = \mathbb{1}\{\xi' > t_n\} \left( -\mathbb{P}(\delta_{i+1} = 2\delta_i \mid \mathcal{F}_{t_n}) + \mathbb{P}(\delta_{i+1} = \delta_i/2 \mid \mathcal{F}_{t_n}) \right) \leq \frac{1}{3} + O\left( \frac{1}{\log(1/\delta_n)} \right) \leq -\frac{1}{4} \mathbb{1}\{\xi' > t_n\},
\]

where the last inequality holds when $\delta_0$ is sufficiently small and as $\delta_0 \leq 2M \leq 2\sqrt{\delta_0}$. Thus, the process $M_n := W_n + \frac{1}{4} \sum_{j=1}^{n-1} \mathbb{1}\{\xi' > t_j\}$ is a super-martingale with increments bounded in absolute value by 2. We get, by Azuma inequality

\[
\mathbb{P}(\xi' > t_n) = \mathbb{P}(W_n \geq 1, \xi' > t_n) \leq \mathbb{P}(M_n - M_0 \geq n/4) \leq Ce^{-cn}. \tag{10.12}
\]

We can now prove the second part of the lemma. By (10.12) with $n_0 := 1/2 \log_2(\delta_0^4)$ we have

\[
\mathbb{P}(\xi' = \xi'_2) \leq \mathbb{P}(\xi' \geq t_{n_0}) \leq Ce^{-cn_0} \leq \left( \frac{\delta_0}{M} \right)^c.
\]
We turn to prove the first part of the lemma. For all $n \geq 1$, by Lemma 10.4 we have that
\[ P(\zeta' = \zeta_3 = t_n) \leq E[\mathbb{1}\{\zeta' > t_{n-1}\}] \cdot P(\zeta_3 = t_n \mid \mathcal{F}_{t_{n-1}}) \leq E\left[\mathbb{1}\{\zeta' > t_{n-1}\} C \delta_{n-1}^3\right] \leq CM^3 \]
and therefore $P(\zeta' = \zeta_3 \leq t_n) \leq CnM^3$. Thus, using also (10.12) with $n_1 = \lfloor \delta_0^{-2} \rfloor$ we get
\[ P(\zeta' = \zeta_3) \leq P(\zeta' = \zeta_3 \leq t_{n_1}) + P(\zeta' > t_{n_1}) \leq C \delta_0. \]
Next, we prove part (3). The lower bound for $\zeta'$ follows immediately from Lemma 10.4. Indeed, $\zeta' \geq \zeta(t_0, \alpha_0, \delta_0)$. We turn to prove the upper bound. By (10.12) we have that
\[ P(\zeta' = t_0 + k \delta_0^{-3}) \leq P(\zeta' = t_0) + \sum_{i=1}^{n} P(\zeta' = t_i \geq t_0 + k \delta_0^{-3}) \]
\[ \leq Ce^{-cn} + \sum_{i=1}^{n} \sum_{j=1}^{i} P(\zeta' = t_i, t_j - t_{j-1} > \frac{k}{n} \delta_0^{-3}) \]
\[ \leq Ce^{-cn} + \sum_{i=1}^{n} \sum_{j=1}^{i} E\left[\mathbb{1}\{\zeta' > t_{j-1}\} P\left(t_j - t_{j-1} > \frac{k}{n} \delta_0^{-3} \mid \mathcal{F}_{t_{j-1}}\right)\right] \]
\[ \leq Ce^{-cn} + \sum_{i=1}^{n} \sum_{j=1}^{i} e^{-ck/n} \leq Ce^{-cn} + n^2 e^{-ck/n}, \]
where the fourth inequality we used Lemma 10.4. The third part of the lemma follows from the last result by substituting $n = \sqrt{k}$. 

On the event $\{\zeta' < \zeta_3\}$ we have that
\[ \frac{\delta_0}{3} (\zeta' - t_0) \leq X_{\zeta'} - X_{t_0} \leq 2M(\zeta' - t_0). \]
For $\epsilon_0 > 0$, we denote the good event by
\[ A = A(t_0, \alpha_0, \delta_0, M, \epsilon) := \{\zeta' < \zeta_3 \wedge \zeta_3' \cap \{t_0 + \epsilon_0 \delta_0^{-3} \leq \zeta' \leq t_0 + \epsilon_0^{-1} \delta_0^{-3}\}. \]
By Lemma 10.6 we have
\[ P(A^c) \leq C \delta_0 + C \left(\frac{\delta_0}{M}\right)^{\epsilon} + Ce^{-c\epsilon^{-\frac{1}{2}}} \leq C \left(\frac{\delta_0}{M}\right)^{\epsilon} + Ce^{-c\epsilon^{-\frac{1}{2}}} \] (10.13)
Define $\alpha''(t_0, \alpha_0, \delta_0, M) := \alpha'(t_{i-1}, \alpha_{i-1}, \delta_{i-1})$ on the event $\{\zeta' = t_i\}$ for $i > 0$ and $\alpha''(t_0, \alpha_0, \delta_0, M) := \alpha_0$ on $\{\zeta' = t_0\}$. Note that, on $\{\zeta' < \zeta_2 \wedge \zeta_3\}$ we have $|\alpha'' - \delta_0/2| \leq (\delta_0/2)^{1.1}$. Since all the parameters are going to change again in the next section we write more specifically $\zeta'(t_0, \alpha_0, \delta_0, M)$ instead of $\zeta'$. We do the same with the other stopping times $\zeta_1, \zeta_2, \zeta_3, \zeta_{\alpha}$. 

10.1.3. Stitching long intervals. In this section we stitch together the long intervals of time in order to prove that aggregate has the $t^{\frac{2}{5}}$ growth.

**Theorem 10.7.** Let $\epsilon, \alpha_0 > 0$ such that $\alpha_0$ is sufficiently small depending on $\epsilon$. Let $Y_{t_0} \in \mathfrak{R}'(\alpha_0, t_0)$ and let $\Pi_{s}$ be the aggregate with initial condition $Y_{t_0}$. There exist $C_{\epsilon} > 0$ such that the following holds. Let $t \geq t'(t_0, \alpha_0, \epsilon)$ and define the stopping time
\[ \zeta_t := \inf\{s > t_0 : \exists \alpha \text{ such that } |\alpha - t^{-\frac{1}{2}}| \leq t^{-\frac{2}{5}} \text{ and } Y_s \in \mathfrak{R}'(\alpha, s)\}. \]
We have that
\[ P\left(\zeta_t \leq C_{\epsilon} t \text{ and } X_{\zeta_t} \leq C_{\epsilon} t^{\frac{2}{5}}\right) \geq 1 - \epsilon \] (10.14)
Proof. Let $m \geq 1$ sufficiently large independently of $t$ and let $n = n_t := \lceil \log_2(\alpha_0^{-1} t^{\frac{1}{4}}) \rceil$. Define for all $0 \leq j \leq n$

\[
\delta_j := \alpha_0 2^{-j}, \quad M_j := \begin{cases} \sqrt{\delta_j}, & j \leq \frac{n}{2} \\ \delta_j^{m+\frac{1}{4}(n-j)}, & j > \frac{n}{2}, \end{cases} \quad \epsilon_j := (m + n - j)^{-2}.
\]

It is easy to verify that $\frac{1}{2} t^{-\frac{1}{3}} \leq \delta_j \leq t^{-\frac{1}{3}}$ and that $\delta_j \leq M_j \leq \sqrt{\delta_j}$ for all $j$ as long as $t$ is sufficiently large.

Define the sequence of stopping times $t_i$ and the sequence of random variables $\alpha_i \in \mathcal{F}_t$, inductively by

\[
t_{i+1} := \zeta'(t_i, \alpha_i, \delta_i, M_i), \quad \alpha_{i+1} := \alpha''(t_i, \alpha_i, \delta_i, M_i)
\]

Let $A_i := \mathcal{A}(t_i, \alpha_i, \delta_i, M_i, \epsilon_i)$ and $B_j := \bigcap_{i=1}^j A_i$. We start by showing that $\mathbb{P}(B_n)$ is large. Using \eqref{10.13} we get

\[
\mathbb{P}(B_n^c) \leq \mathbb{P}(A_1^c) + \sum_{j=2}^{n} \mathbb{P}(B_j \cap A_j^c) = \mathbb{P}(A_1^c) + \sum_{j=2}^{n} \mathbb{E}\left[ 1_{B_j} \mathbb{P}(A_j^c | \mathcal{F}_{t_{j-1}}) \right] 
\]

\[
\leq C \sum_{j=1}^{n} \left( \frac{\delta_j}{M_j} \right)^c + C \sum_{j=1}^{n} e^{-c \epsilon_j^{-\frac{1}{2}}}. \tag{10.15}
\]

Let $S_1, S_2$ be the corresponding first and second sums in the right hand side of \eqref{10.15}. We bound the second term using the definition of $\epsilon_j$ by

\[
S_2 \leq C e^{-cm} \sum_{j=1}^{n} e^{-c(n-j)} \leq C e^{-cm} \sum_{k=0}^{\infty} e^{-ck} \leq C e^{-cm}.
\]

The first term is bounded as follows

\[
S_1 \leq C \sum_{j=1}^{n/2} \delta_j^c + C e^{-cm} \sum_{j=[n/2]}^{n} e^{-c(n-j)} \leq C \alpha_0^c + C e^{-cm} \sum_{k=1}^{\infty} e^{-ck} \leq C \alpha_0^c + C e^{-cm}.
\]

Since $\alpha_0$ is sufficiently small and $m$ is sufficiently large we get that $\mathbb{P}(B_n) \geq 1 - \epsilon$.

Next, we show that on $B_n$ the event in \eqref{10.14} holds. To this end we bound the stopping time $t_n$ and the aggregate size $X_{t_n}$ and then show that, on $B_n$ we have that $\zeta_t \leq t_n$. We have

\[
t_n = t_0 + \sum_{j=0}^{n-1} t_{j+1} - t_j \leq t_0 + \sum_{j=0}^{n-1} \epsilon_j^{-1} \delta_j^{-3} \leq t_0 + \delta_3^{-3} \sum_{j=0}^{n} (m + n - j)^2 \leq t_0 + C t \sum_{k=0}^{\infty} (2m + k)^2 2^{-3k} \leq t_0 + C m t \leq C t \alpha_0 t.
\]

We turn to bound the aggregate size at time $t_n$. We have

\[
X_{t_n} \leq \sum_{j=0}^{n-1} X_{t_{j+1}} - X_{t_j} \leq C \sum_{j=0}^{n-1} M_j (t_{j+1} - t_j) \leq C \sum_{j=0}^{n} M_j \epsilon_j^{-1} \delta_j^{-3}
\]

\[
\leq C \sum_{j=1}^{n/2} \epsilon_j^{-1} \delta_j^{-3} + C_m \sum_{j=[n/2]}^{n} (m + n - j)^2 \delta_j^{-2} \leq C \epsilon_0^{-1} \delta_{[n/2]}^{-3} + C_m \delta_{[n/2]}^{-2} \sum_{j=[n/2]}^{n} (m + n - j)^2 \leq C_m n^2 \delta_{[n/2]}^{-3} + C_m \delta_{[n/2]}^{-2} \sum_{k=0}^{\infty} (m + k) e^{-ck}
\]

\[
\leq C_m n^2 \delta_{[n/2]}^{-3} + C_m \delta_{[n/2]}^{-2} \leq C_m \alpha_0 t \frac{1}{2} \log^2 t + C_m \alpha_0 t \frac{1}{2} \leq C_m \alpha_0 t \frac{1}{2}.
\]
Thus, it suffices to prove that on $\mathcal{B}_n$ we have $\zeta_t \leq t_n$. To this end, recall that the time interval $[t_0, t_n]$ is the union of many short intervals. On $\mathcal{B}_n$ the process is regular in the endpoints of each of these intervals with a parameter $\alpha$ that do not change by more than $\alpha^{\frac{3}{2}} \log^C(1/\alpha)$ between consecutive short intervals. Thus, by a discrete mean value theorem the process is regular at some time $t' < t_n$ with $\alpha'$ such that $|\alpha' - t' - \frac{3}{2}| \leq t' - \frac{5}{2}$. It follows that $\zeta_t \leq t_n$ on $\mathcal{B}_n$. \hfill $\square$

10.2. Convergence in distribution. In this section we show how to deduce the convergence to the diffusion from the induction step theorem. To this end we’ll use the following theorem due to Helland [11]. The theorem we state here is weaker than the result in [11] but suffices for what we need.

10.2.1. The result of Helland. We start with the basic settings and notations in the paper of Helland [11]. Let $x_0 > 0$, $\mu : [0, \infty) \to \mathbb{R}$ and $\sigma : [0, \infty] \to [0, \infty]$. Let $X(t)$ for $t > 0$ be a solution to the stochastic differential equation

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dB_t, \quad X(0) = x_0,$$

such that almost all sample paths are continuous. Suppose that there are no accessible boundaries. That is $\mathbb{P}(\forall t > 0, \ 0 < X(t) < \infty) = 1$

For any $n \geq 1$, let $X^{(n)}(t) > 0$ be a sequence of processes. Let $\Delta_n \to 0$ and for any $k \geq 0$ let $t_k = t_{k(n)} := k\Delta_n$. In what follows $\xrightarrow{p}$ denotes convergence in probability. Suppose that:

1. $X_n(0) \xrightarrow{p} x_0$
2. $X_n(t)$ is fixed on the intervals $[t_k, t_{k+1})$
3. For all $t, \epsilon > 0$ we have as $n \to \infty$

$$\sum_{k: t_k \leq t} \mathbb{P}\left( |X_n(t_{k+1}) - X_n(t_k)| \geq \epsilon \mid \mathcal{F}_{t_k} \right) \xrightarrow{p} 0$$

4. For all $t > 0$ and a compact set $K \subseteq (0, \infty)$ we have as $n \to \infty$

$$\sum_{k: t_k \leq t} \left| \mathbb{E}\left( X_n(t_{k+1}) - X_n(t_k) \mid \mathcal{F}_{t_k} \right) - \mu(X_n(t_k)) \cdot \Delta_n \right| \cdot \mathbb{I}\{X_n(t_k) \in K\} \xrightarrow{p} 0$$

5. For all $t > 0$ and a compact set $K \subseteq (0, \infty)$ we have as $n \to \infty$

$$\sum_{k: t_k \leq t} \left| \mathbb{E}\left( (X_n(t_{k+1}) - X_n(t_k))^2 \mid \mathcal{F}_{t_k} \right) - \sigma^2(X_n(t_k)) \Delta_n \right| \cdot \mathbb{I}\{X_n(t_k) \in K\} \xrightarrow{p} 0$$

Theorem 10.8 (Helland). Suppose that the assumptions (1) – (5) above hold. Then,

$$(X_n(t))_{t>0} \xrightarrow{d} (X(t))_{t>0}$$

Remark 10.9. The convergence in Theorem [10.8] is in the sense of the Stone topology defined in [28] and discussed in [11]. We will not define this topology. Instead, we just note that:

1. when $X$ is almost surely continuous, the convergence is equivalent to the following two conditions
   (a) The finite dimensional distributions of $X_n$ converge weakly to the finite dimensional distributions of $X$.
   (b) For all $\epsilon, T > 0$ we have

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{P}\left( \sup_{0 \leq s, t \leq T \atop |s-t| \leq \delta} |X_n(t) - X_n(s)| \geq \epsilon \right) = 0$$
(2) It can be shown, using Skorhod representation theorem, that if \( X \) is almost surely continuous then there is a coupling of the processes \( X_n \) and \( X \) such that for all \( T > 0 \) almost surely we have \( \sup_{t \leq T} |X_n(t) - X(t)| \to 0 \).

10.2.2. Proof of Theorem 10.7. Throughout this section \( X_t \) is the size of the usual aggregate with no initial condition. Let \( \epsilon > 0 \) sufficiently small, \( \alpha_0 > 0 \) sufficiently small depending on \( \epsilon \) and let \( t_0 := \alpha_0^{-20} \). Recall the definition of \( \bar{Y}_{t,\alpha} \) given in (3.3). Define the event

\[
\mathcal{A}_0 := \{ X_{t_0} \leq t_0 \} \cap \{ \forall s > 0, \ Y_{t_0}(s) \leq \bar{Y}_{t_0,0,0}(s) \} \cap \{ \bar{Y}_{t_0,0,0} \in \mathcal{R}(\alpha_0, \alpha_0^8, \{t_0\}) \}. \quad (10.16)
\]

The first event on the right hand side of (10.16) holds with high probability as \( S(t) \leq \frac{1}{2} \) almost surely and therefore \( X_{t_0} \leq \text{Poisson}(t_0/2) \). The second and third events hold with high probability by Lemma 3.7 and Theorem 6.6 respectively. Thus \( \mathbb{P}(\mathcal{A}_0) \geq 1 - \epsilon \) as long as \( \alpha_0 \) is sufficiently small depending on \( \epsilon \). Let \( (\bar{Y}_t, \bar{S}(t), \bar{X}_t) \) be the aggregate with initial condition \( \bar{Y}_{t_0,0,0} \). We note that, according to Definition 3.3 we have that \( \bar{X}_{t_0} = 0 \) and by Claim 3.4 we have almost surely on \( \mathcal{A}_0 \) that \( X_t \leq t_0 + \bar{X}_t \) for all \( t > t_0 \).

We turn to bootstrap the regularity by applying Theorem 6.2. Let \( t_1 := t_0 + \alpha_0^{-2} \log^{100}(1/\alpha_0) \). Define

\[
\mathcal{A}_1 := \{ \bar{X}_{t_1} \leq t_0 \} \cap \{ \exists \alpha_1 \text{ with } \bar{Y}_{t_1} \in \mathcal{R}'(\alpha_1, t_1) \}.
\]

Almost surely on \( \mathcal{A}_0 \) we have that \( \mathbb{P}(\mathcal{A}_1 \mid \mathcal{F}_{t_0}) \geq 1 - \epsilon \). Indeed, the first event holds with high probability by the same argument as above. For the second one, by Theorem 6.2, there is a random variable \( \alpha_1' \leq 2\alpha_0 \) such that almost surely on \( \mathcal{A}_0 \) we have \( \mathbb{P}(Y_{t_1} \in \mathcal{R}'(\alpha_1', \alpha_1, \{t_1\}) \mid \mathcal{F}_{t_0}) \geq \alpha_0^{-10} \) and therefore, by Definition 10.1 there is some \( \alpha_1 \leq 3\alpha_0 \) such that almost surely on \( \mathcal{A}_0 \), \( \mathbb{P}(Y_{t_1} \in \mathcal{R}'(\alpha_1, t_1) \mid \mathcal{F}_{t_0}) \). We get that

\[
\mathbb{P}(\mathcal{A}_0 \cap \mathcal{A}_1) = \mathbb{E}(1_{\mathcal{A}_0} \mathbb{P}(\mathcal{A}_1 \mid \mathcal{F}_{t_0})) \geq (1 - \epsilon)\mathbb{P}(\mathcal{A}_0) \geq (1 - \epsilon)^2 \geq 1 - 2\epsilon.
\]

Next let \( \delta > 0 \) sufficiently small such that \( \delta^2 C_e < \epsilon \) where \( C_e \) is from Theorem 10.7. Finally, let \( t > 0 \) sufficiently large (depending on all other parameters) such that \( \delta t \geq t'/(t_1, \alpha_1, \epsilon) \) where \( t' \) is from Theorem 10.7. Let \( t_2 \) and \( \alpha_2 \) be the stopping time and random variable from Theorem 10.7 with \( \delta t \) instead of \( t \). Define the events

\[
\mathcal{A}_2 := \{ t_2 \leq \delta t \} \cap \{ \bar{X}_{t_2} \leq \delta t^2 \},
\]

and \( \mathcal{A} := \mathcal{A}_0 \cap \mathcal{A}_1 \cap \mathcal{A}_2 \). The random variable \( \alpha_2 \) is defined on \( \mathcal{A}_2 \) and satisfies \( |\alpha_2 - (\delta t)^{-\frac{1}{2}}| \leq C_s \delta^{-2} \). By Theorem 10.7 and the choice of \( \delta \) almost surely on the event \( \mathcal{A}_0 \cap \mathcal{A}_1 \) we have \( \mathbb{P}(\mathcal{A}_2 \mid \mathcal{F}_{t_1}) \geq 1 - \epsilon \). Thus we get \( \mathbb{P}(\mathcal{A}) \geq 1 - 3\epsilon \).

We let \( (\bar{Y}, \bar{S}, \bar{X}) \) be the aggregate \( (\bar{Y}, \bar{S}, \bar{X}) \) condition on the event \( \mathcal{A} \). Starting with \( t_2 \) and \( \alpha_2 \), we define the sequence \( t_i \) and the random variables \( \alpha_i \) like in Section 10.1.1 with a minor difference. Instead of stopping the process when the speed changes by a factor of 2, we stop the process when the speed changes by a factor of \( \log(1/\alpha_2) \). More precisely, we let \( t_3 := t_2 + \alpha_2^{-2} \log^{100-2}(1/\alpha_2) \) and \( t_{i+1} := t_2 + (i - 1)(t_3 - t_2) \). We define the sequence \( \alpha_i \) inductively. Let

\[
\begin{align*}
\zeta_1 &:= \min\{ t_1 \geq t_2 : \alpha_i < \alpha_2 \log^{-1}(1/\alpha_2) \} \\
\zeta_2 &:= \min\{ t_2 \geq t_2 : \alpha_i > \alpha_2 \log(1/\alpha_2) \} \\
\zeta_3 &:= \min\{ t_2 \geq t_2 : \bar{Y}_{t_2} \notin \mathcal{R}'(\alpha_i, t_1) \} \\
\zeta_4 &:= \min\{ t_{i+1} \geq t_3 : |X_{t_{i+1}} - X_{t_i} - \alpha_i(t_1 - t_0)| \geq \alpha_i^{-1/4}(t_1 - t_0) \},
\end{align*}
\]

We let \( \zeta := \zeta_1 \wedge \zeta_2 \wedge \zeta_3 \wedge \zeta_4 \). On \( \{ \zeta > t_i \} \) we let \( \alpha_{i+1} \) be the random variable \( \alpha_1 \) from the Theorem 10.2 and on \( \{ \zeta \leq t_i \} \) we let \( \alpha_{i+1} := \alpha_i \).

Claim 10.10. We have that \( \mathbb{P}(\zeta \leq t_2 + \alpha_2^{-3} \log(1/\alpha_2)) \leq C \log^{-2}(1/\alpha_2) \)
Finally, we let \( K \) be the continuation of \( \alpha_i \) that is \( A(t) \) is defined by \( A(t) := \alpha_i \) for any \( t_i \leq t < t_{i+1} \).

We also define the rescaled process

\[
V_t(s) := t^{\frac{1}{3}} A(t_2 + st)
\]

**Lemma 10.11.** We have that

\[
(V_t(s))_{s>0} \overset{d}{\to} (Z_\delta(s))_{s>0}, \quad t \to \infty
\]

where \( Z_\delta \) is the solution to \( dZ = 2Z^\frac{1}{2} dB_t \) with \( Z_\delta(0) = \delta^{-\frac{1}{3}} \). The convergence is in the sense of Remark 10.9.

**Proof.** We check all the conditions of Theorem 10.8 with \( \mu = 0, \sigma(x) = 2x^\frac{5}{2} \) and \( x_0 := \delta^{-\frac{1}{3}} \). Let

\[
\Delta_t := \frac{(t_3 - t_2)}{t} \leq C_\delta t^{-\frac{1}{3}} \log^C t \to 0, \quad t \to \infty
\]

Finally, we let \( s_k := k \Delta_t \) so that \( V_t(s_k) = A(t_{k+2}) = \alpha_{k+2} \). The first condition holds since

\[
V_t(0) = t^{\frac{1}{3}} A(t_2) = t^{\frac{1}{3}} \alpha_2 \to \delta^{-\frac{1}{3}}.
\]

It is clear that the second condition holds. Next, we check the third condition. For sufficiently large \( t \) we have that

\[
P(|V_t(s_{k+1}) - V_t(s_k)| \geq \epsilon' \mid F_{t_k}) \leq P(|\alpha_{k+3} - \alpha_{k+2}| \geq \epsilon' t^{-\frac{1}{3}} \mid F_{t_k}) = 0
\]

so in particular the third condition holds. We turn to prove the fourth condition. Let \( s > 0 \) and \( K \subseteq (0, \infty) \) a compact set. We have that

\[
\sum_{k; s_k \leq s} \left| \mathbb{E} \left[ V_t(s_{k+1}) - V_t(s_k) \mid F_{t_{k+1}} \right] \right| \mathbb{1} \{ V_t(t_k) \in K \} = t^{\frac{1}{3}} \sum_{k; s_k \leq s} \left| \mathbb{E} \left[ \alpha_{k+3} - \alpha_{k+2} \mid F_{t_{k+1}} \right] \right| \mathbb{1} \{ t^{\frac{1}{3}} \alpha_{k+2} \in K \}
\]

\[
\leq C t^{\frac{2}{3}} \sum_{k; s_k \leq s} \frac{(t_{k+3} - t_{k+2}) \alpha_{k+2}^4}{\log^3(1/\alpha_{k+2})} \mathbb{1} \{ t^{\frac{1}{3}} \alpha_{k+2} \in K \} \leq C_{K, \delta} t^{\frac{1}{3}} \sum_{k; s_k \leq s} \frac{\alpha_{k+2}^4}{\log^3(1/\alpha_{k+2})} \mathbb{1} \{ t_{k+2} - t_{k+1} \} \leq \frac{C_{K, \delta}}{\log^3 t} \to 0
\]

Lastly, we check the fifth condition

\[
\sum_{k; s_k \leq s} \left| \mathbb{E} \left[ \frac{(V_t(s_{k+1}) - V_t(s_k))^2}{\mathbb{1} \{ (V_t(s_{k+1}) - V_t(s_k))^2 \} \mid F_{t_{k+1}}} - 4V_t(t_k)^5 \Delta_t \right] \right| \mathbb{1} \{ V_t(t_k) \in K \}
\]

\[
= t^{\frac{2}{3}} \sum_{k; s_k \leq s} \left| \mathbb{E} \left[ \frac{(\alpha_{k+3} - \alpha_{k+2})^2}{\mathbb{1} \{ (\alpha_{k+3} - \alpha_{k+2})^2 \} \mid F_{t_{k+2}}} - 4\alpha_{k+2}^5(t_3 - t_2) \right] \right| \mathbb{1} \{ V_t(t_k) \in K \}
\]

\[
\leq C t^{\frac{2}{3}} \sum_{k; s_k \leq s} \alpha_{k+2}^5 (t_1 - t_0) \mathbb{1} \{ V_t(t_k) \in K, \, \zeta > t_{k+2} \} + C t^{\frac{2}{3}} \sum_{k; s_k \leq s} \alpha_{k+2}^5 (t_3 - t_2) \mathbb{1} \{ V_t(t_k) \in K, \, \zeta \leq t_{k+2} \}
\]

\[
\leq C t^{\frac{2}{3}} \alpha_{k+2}^5 + C t^{\frac{2}{3}} \alpha_{k+2}^5 \mathbb{1} \{ \zeta \leq t_2 + \alpha_{k+2}^3 \log(1/\alpha_{k+2}) \} \to 0.
\]
Claim 10.13. Let \( a \) as converge as well. The statement of the corollary follows from (10.17) and (10.18).

\[ t^{-\frac{4}{3}} (\bar{X}_{t_{2} + st} - \bar{X}_{t_{2}}) \xrightarrow{d} \int_{0}^{s} Z_{\delta}(x) dx \]

Proof. On the event \( \{ \zeta > t_{2} + \alpha_{2}^{-2} \log(1/\alpha_{2}) \} \) we have for all \( k \geq 2 \) with \( t_{k} \leq st \) that

\[
|\bar{X}_{t_{k+1}} - \bar{X}_{t_{k}} - \alpha_{k} (t_{k+1} - t_{k})| \leq  \alpha_{k}^{1.4} (t_{3} - t_{2}) \leq  \alpha_{2}^{1.3} (t_{3} - t_{2}) \leq  t^{-\frac{2}{3}} (t_{3} - t_{2}).
\]

Thus

\[
\left| \bar{X}_{t_{2} + st} - \bar{X}_{t_{2}} - \int_{t_{2}}^{t_{2} + st} A(t') dt' \right| \leq C_{s} t^{\frac{3}{5}}.
\]

Next, we have that

\[
t^{-\frac{2}{3}} \int_{t_{2}}^{t_{2} + st} A(t') dt' = t^{\frac{1}{3}} \int_{0}^{s} A(t_{2} + xt) dx = \int_{0}^{s} V_{t}(x) dx \xrightarrow{d} \int_{0}^{s} Z_{\delta}(x) dx,
\]

where the convergence is in finite dimensional distributions and it follows from Lemma 10.11 and by part (2) of Remark 10.9. Indeed, there is a coupling of the processes such that the convergence is almost sure and uniform on compact subsets of \((0, \infty)\) and therefore the integrals of the processes converge as well. The statement of the corollary follows from (10.17) and (10.18).

We need one last claim for the proof of Theorem 1.

Claim 10.12. We have the following convergence in finite dimensional distributions

\[
t^{-\frac{2}{3}} (\bar{X}_{t_{2} + st} - \bar{X}_{t_{2}}) \xrightarrow{d} \int_{0}^{s} Z_{\delta}(x) dx
\]

Proof. Suppose that both \( V_{a} (x) \) and \( V(x) \) be a \( \frac{8}{3} \)-Bessel processes with initial condition \( V_{a} (0) = a \) and \( V(0) = 0 \) By Remark 2.13 it suffices to show that

\[
\left( \int_{0}^{s} V_{a}(x) dx \right)_{s \geq 0} \xrightarrow{d} \left( \int_{0}^{s} V(x) dx \right)_{s \geq 0},
\]

as \( a \to 0 \). To this end we couple the two processes. Recall that \( V_{a} \) and \( V \) are solutions to the equation

\[
dV(x) = dB_{x} + \frac{5}{6} \frac{dx}{V(x)}.
\]

Suppose that both \( V_{a} \) and \( V \) are driven by the same Brownian motion. The drift term is larger as long as the process is smaller and therefore, under this coupling, \( V_{a}(x) - V(x) \) is positive and decreasing. Thus \( |V_{a}(x) - V(x)| \leq a \) for all \( x > 0 \). Therefore, for all \( s > 0 \)

\[
\left| \int_{0}^{s} V_{a}(x) dx - \int_{0}^{s} V(x) dx \right| \leq \int_{0}^{s} |V_{a}(x) - V(x)| dx \leq as \to 0, \quad \text{a.s.,}
\]

as \( a \to 0 \). Equation (10.19) follows from this.

Proof of Theorem 1. On the event \( \mathcal{A} \) we have almost surely

\[
X_{t} \leq X_{t_{0}} + \bar{X}_{t} \leq t_{0} + \bar{X}_{t_{2}} + \bar{X}_{t_{2} + t} - \bar{X}_{t_{2}} \leq 2et^{\frac{2}{3}} + \bar{X}_{t_{2} + t} - \bar{X}_{t_{2}},
\]

(10.20)
where the last inequality holds for sufficiently large \( t \). Thus, for all \( z > 0 \) we have
\[
\liminf_{t \to \infty} \mathbb{P}(X_t \leq z t^{\frac{2}{3}}) \geq \liminf_{t \to \infty} \mathbb{P}(X_t \leq z t^{\frac{2}{3}}, A)
\]
\[
\geq \liminf_{t \to \infty} \mathbb{P}(\tilde{X}_{t_2 + t} - \tilde{X}_{t_2} \leq (z - 2\epsilon)t^{\frac{2}{3}}, A)
\]
\[
\geq \liminf_{t \to \infty} \mathbb{P}(A) \cdot \mathbb{P}(\tilde{X}_{t_2 + t} - \tilde{X}_{t_2} \leq (z - 2\epsilon)t^{\frac{2}{3}} | A)
\]
\[
\geq (1 - 2\epsilon) \cdot \liminf_{t \to \infty} \mathbb{P}(\tilde{X}_{t_2 + t} - \tilde{X}_{t_2} \leq (z - 2\epsilon)t^{\frac{2}{3}})
\]
\[
\geq (1 - 2\epsilon) \cdot \mathbb{P}\left( \int_0^1 \tilde{Z}_d(x) dx \leq (z - 2\epsilon) \right),
\]
where in the last inequality we used Corollary 10.12. Recall that the following inequality holds for all \( \epsilon, \delta \) as long as \( \delta \) is sufficiently small depending on \( \epsilon \). Taking \( \delta \) to zero and using Claim 10.13, we get
\[
\liminf_{t \to \infty} \mathbb{P}(X_t \leq z t^{\frac{2}{3}}) \geq (1 - 2\epsilon) \cdot \mathbb{P}\left( \int_0^1 \tilde{Z}_d(x) dx \leq (z - 2\epsilon) \right),
\]
where \( \tilde{Z} \) is the solution to the same SDE with \( \tilde{Z}(0) = \infty \). Next, taking \( \epsilon \) to zero we get
\[
\liminf_{t \to \infty} \mathbb{P}(X_t \leq z t^{\frac{2}{3}}) \geq \mathbb{P}\left( \int_0^1 \tilde{Z}_d(x) dx \leq z \right).
\]
By the same arguments for all \( z_1, \ldots, z_k > 0 \) we have
\[
\liminf_{t \to \infty} \mathbb{P}(X_{s_1 t} \leq z_1 t^{\frac{2}{3}}, \ldots, X_{s_k t} \leq z_k t^{\frac{2}{3}}) \geq \mathbb{P}\left( \int_0^{s_1} \tilde{Z}_d(x) dx \leq z_1, \ldots, \int_0^{s_k} \tilde{Z}_d(x) dx \leq z_k \right).
\]
We turn to prove a matching upper bound on the last probability. As the arguments are similar we will not give all the details. To this end, we change the definition of \( A_0 \). We take a sufficiently small \( \alpha_0 \) and \( t_0 = 100\alpha_0^{-2} \) and use the same definition as in (10.16) with \( Y_{t_0,J_0} \) instead of \( Y_{t_0,J_0} \). We also let \( (Y', S(t), X') \) be the aggregate with initial condition \( Y_{t_0,J_0} \) and work with \( (Y', S(t), X') \) instead of \( (Y, S(t), X) \). The rest of the definitions such as \( A_1, A_2, A \) and \( \tilde{X} \) stay the same. As in (10.20) we have on \( A \) that
\[
X_t \geq \tilde{X}_t - X_{t_2} \geq \tilde{X}_{t_2 + (1 - 2\epsilon)t} - \tilde{X}_{t_2}.
\]
Thus
\[
\limsup_{t \to \infty} \mathbb{P}(X_t \leq z t^{\frac{2}{3}}) \leq \limsup_{t \to \infty} \mathbb{P}(A^c) + \limsup_{t \to \infty} \mathbb{P}(\tilde{X}_{t_2 + (1 - 2\epsilon)t} - \tilde{X}_{t_2} \leq z t^{\frac{2}{3}}, A)
\]
\[
\leq 3\epsilon + \limsup_{t \to \infty} \mathbb{P}(\tilde{X}_{t_2 + (1 - 2\epsilon)t} - \tilde{X}_{t_2} \leq z t^{\frac{2}{3}})
\]
\[
\leq 3\epsilon + \mathbb{P}\left( \int_0^{1 - 2\epsilon} \tilde{Z}_d(x) dx \leq z \right).
\]
Taking \( \delta \to 0 \) and then \( \epsilon \to 0 \) we get
\[
\limsup_{t \to \infty} \mathbb{P}(X_t \leq z t^{\frac{2}{3}}) \leq \mathbb{P}\left( \int_0^1 \tilde{Z}_d(x) dx \leq z \right).
\]
By the same arguments
\[
\limsup_{t \to \infty} \mathbb{P}(X_{s_1 t} \leq z_1 t^{\frac{2}{3}}, \ldots, X_{s_k t} \leq z_k t^{\frac{2}{3}}) \leq \mathbb{P}\left( \int_0^{s_1} \tilde{Z}_d(x) dx \leq z_1, \ldots, \int_0^{s_k} \tilde{Z}_d(x) dx \leq z_k \right).
\]
Thus
\[
\lim_{t \to \infty} \mathbb{P}(X_{s_1 t} \leq z_1 t^{\frac{2}{3}}, \ldots, X_{s_k t} \leq z_k t^{\frac{2}{3}}) = \mathbb{P}\left( \int_0^{s_1} \tilde{Z}_d(x) dx \leq z_1, \ldots, \int_0^{s_k} \tilde{Z}_d(x) dx \leq z_k \right).
\]
This finishes the proof of Theorem 1. □
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Appendix A. The renewal process, Fourier transform and singularity analysis

In this section we study the asymptotic behavior of functions $K$ and the density of the renewal process $K^*$ as $\alpha \to 0$. We also estimate a key integral that appears in the drift term of the speed increment from Section 9. The main tools we use are Fourier transforms and complex analysis.

A.1. Fourier transform of $\tau$ and $K$. Let $0 < \alpha < 1$. Let $Y(t)$ be a rate $\alpha$ Poisson process and $W(t)$ be a continuous time random walk with $W(0) = 0$. Let $\tau := \inf\{t > 0 : W(t) > Y(t)\}$. Recall that $K = K_\alpha$ is given by $K(t) = \alpha(1 + 2\alpha)P(t < \tau < \infty)$. We also let $X$ be a random variable with density $K$.

Lemma A.1. For all $s \in \mathbb{C}$ with $\text{Re}(s) < 0$ we have

\begin{equation}
\mathbb{E}[e^{st} \mathbb{1}_{\{\tau < \infty\}}] = \frac{1 + \alpha - s - \sqrt{(1 + \alpha - s)^2 - (1 + 2\alpha)}}{1 + 2\alpha}
\end{equation}

\begin{equation}
\mathbb{E}[e^{sX}] = \int_0^\infty e^{st} K(t) dt = \frac{\alpha}{s} \left( \alpha - s - \sqrt{(1 + \alpha - s)^2 - (1 + 2\alpha)} \right).
\end{equation}

Proof. Let $U_t := Y(t) - W(t)$. Using that $Y(t) \sim \text{Poisson}(\alpha t)$ and that the number of steps taken by the random walk up to time $t$ is distributed Poisson$(t)$ we get that for any $\theta > 0$

\[
\mathbb{E}[\theta^{U_t}] = \mathbb{E}[\theta^{Y(t)}] \mathbb{E}[\theta^{W(t)}] = \left( e^{-\alpha t} \sum_{k=0}^\infty \frac{(\alpha t)^k}{k!} \theta^k \right) \left( e^{-t} \sum_{k=0}^\infty \frac{t^k}{k!} \left( \theta + \theta^{-1} \right)^k \right) = \exp \left( t \left( \alpha(\theta - 1) + \frac{\theta + \theta^{-1}}{2} - 1 \right) \right).
\]

Thus, if we let

\[ s = s(\theta) := -\alpha(\theta - 1) - \frac{\theta + \theta^{-1}}{2} + 1 \]

we get that the process $Z_t := \theta^{U_t} e^{st}$ is a martingale. Note that for $\theta_0 := 1/(1 + 2\alpha)$ we have $s(\theta_0) = 0$ and that $s(\theta) < 0$ for $\theta < \theta_0$. Thus for $0 < \theta < \theta_0$ we have $Z_t \to 0$ almost surely and therefore $Z_{t\wedge \tau} \to \theta^{-1} e^{s\tau} \mathbb{1}_{\{\tau < \infty\}}$. Using the bounded convergence theorem and inverting the function $s(\theta)$ we obtain

\[
\mathbb{E}[e^{s\tau} \mathbb{1}_{\{\tau < \infty\}}] = \theta(s) \cdot \mathbb{E}(Z_0) = \theta(s) = \frac{1 + \alpha - s - \sqrt{(1 + \alpha - s)^2 - (1 + 2\alpha)}}{1 + 2\alpha}.
\]

We proved the last equality for all $s < 0$ however, since both sides are analytic functions in $\{\text{Re}(s) < 0\}$, by the uniqueness theorem the equality holds for all $s$ in this domain.

We turn to prove the second part. Using the definition of $K$ and part (1) we obtain

\[
\mathbb{E}e^{sX} = \int_0^\infty e^{st} K(t) dt = \alpha(1 + 2\alpha) \int_0^\infty e^{st} \mathbb{P}(t \leq \tau < \infty) dt
\]

\[
= \alpha(1 + 2\alpha) \mathbb{E} \left[ \int_0^\infty e^{st} \mathbb{1}_{\{\tau < \infty\}} \mathbb{1}_{\{t \geq 0\}} dt \right] = \alpha(1 + 2\alpha) \mathbb{E} \left[ \mathbb{1}_{\{\tau < \infty\}} \int_0^\tau e^{st} dt \right]
\]

\[
= \alpha(1 + 2\alpha) \left( \mathbb{E} e^{s\tau} \mathbb{1}_{\{\tau < \infty\}} - \mathbb{P}(\tau < \infty) \right) = \frac{\alpha}{s} \left( \alpha - s - \sqrt{(1 + \alpha - s)^2 - (1 + 2\alpha)} \right)
\]

as needed. \qed

In the next corollary we compute the moments of the random variable $X$. 
Corollary A.2. We have that
\[\int_0^\infty uK(u)du = \frac{1 + 2\alpha}{2\alpha^2} = \frac{1}{2}\alpha^{-2} + O(\alpha^{-1}), \quad \int_0^\infty u^2K(u)du = \frac{(1 + \alpha)(1 + 2\alpha)}{\alpha^4} = \alpha^{-4} + O(\alpha^{-3})\]
\[\int_0^\infty u^2K'(u)du = -\frac{1 + 2\alpha}{\alpha^2} = -\alpha^{-2} + O(\alpha^{-1}), \quad \int_0^\infty uK'(u)du = -1\]

Proof. the first two identities follows from differentiating at \(s = 0\) both sides of the equation
\[\int_0^\infty e^{st}K(t)dt = \alpha s \left(\alpha - s - \sqrt{(1 + \alpha - s)^2 - (1 + 2\alpha)}\right)\]
the last two identities follows from integration by parts. \(\square\)

Next, we find the behavior of the Fourier transform in different regions of the complex plain. For \(s > 0\) define
\[\varphi(s) := \mathbb{E}e^{-sX} = -\frac{\alpha}{s} \left(\alpha + s - \sqrt{(1 + \alpha + s)^2 - (1 + 2\alpha)}\right)\]
We think of \(\varphi\) as a complex function defined on a certain domain \(\Omega \subseteq \mathbb{C}\). See part (1) of Claim A.3 for more details.

Define also
\[\bar{\varphi}(s) := -\frac{\alpha}{s}(\alpha - \sqrt{\alpha^2 + 2s})\]
We’ll see that \(\varphi\) can be approximated by \(\bar{\varphi}\) for an appropriate range of \(s\).

Claim A.3. The function \(\varphi\) satisfies the following properties
1. The functions \(\varphi\) and \(\bar{\varphi}\) extend to analytic and bounded functions in
\[\Omega := \mathbb{C} \setminus \{s \in \mathbb{R} : s < -1 - \alpha + \sqrt{1 + 2\alpha}\}\]
2. We have that \(|\varphi(s)| \leq C\alpha/s\) for sufficiently large \(|s|\) (independently of \(0 < \alpha < 1\)).
3. For any \(s \in \Omega\) with \(|s| \geq \frac{1}{5}\alpha^2\) we have
\[|\bar{\varphi}(s)|, |\varphi(s)| \leq \frac{C\alpha}{\sqrt{|s|}}\]
4. For any \(s \in \Omega\) with \(|s| \geq \frac{1}{5}\alpha^2\) and \(|\alpha^2 + 2s| \geq \frac{1}{3}|s|\) we have
\[|\varphi(s) - \bar{\varphi}(s)| \leq C\alpha\] (A.1)
5. For any \(s \in \Omega\) with \(|s| \geq \frac{1}{5}\alpha^2\) and \(|\alpha^2 + 2s| \geq \frac{1}{3}|s|\) we have
\[|1 - \varphi(s)| \geq c, \quad |1 - \bar{\varphi}(s)| \geq c\]

Proof. The proof of the first part is standard and follows from
\[\sqrt{(1 + \alpha + s)^2 - (1 + 2\alpha)} = \sqrt{(s + 1 + \alpha + \sqrt{1 + 2\alpha})\sqrt{(s + 1 + \alpha - \sqrt{1 + 2\alpha})}}\] (A.2)
where this equality holds when \(s > 0\) and the right hand side is analytic in \(\Omega\) if we let \(\sqrt{\cdot}\) be the standard square root defined on \(\mathbb{C} \setminus \mathbb{R}_{<0}\) (note that in fact \(\varphi\) is analytic in a larger domain but we will not use it). We turn to prove that \(\bar{\varphi}\) is analytic in \(\Omega\). In the definition of \(\bar{\varphi}\) we let \(\sqrt{\cdot}\) be the standard square root and so \(\bar{\varphi}\) is analytic in
\[\mathbb{C} \setminus \{s \in \mathbb{R} : s < -\frac{1}{2}\alpha^2\}\]
which contains \(\Omega\).

The second part follows as the right hand side of (A.2) behaves like \(s + O(1)\) as \(|s| \to \infty\) uniformly in \(0 < \alpha < 1\).
We turn to prove the third part. Suppose that $\frac{1}{5} \alpha^2 \leq |s| \leq C$ for some $C > 0$. We have
\[ |\varphi(s)| \leq \frac{\alpha}{|s|} (\alpha + |s| + \sqrt{\alpha^2 + 2s + 2\alpha s + s^2}) \leq \frac{\alpha}{|s|} (\alpha + |s| + C \sqrt{|s|}) \leq \frac{C \alpha}{\sqrt{|s|}}. \]
The bound for any $s \in \Omega$ follows from the last estimate together with the second part of the claim. By the same arguments we get the same bound for $\tilde{\varphi}$.

We turn to prove part (4). When $s \in \Omega$ and $|s| \geq \frac{1}{5}$, by part (3) of the claim we have that $|\tilde{\varphi}(s)|, |\varphi(s)| \leq C \alpha$ and (A.1) follows. Let $s \in \Omega$ with $\frac{1}{5} \alpha^2 \leq |s| \leq \frac{1}{5}$ and $|\alpha^2 + 2s| \geq \frac{1}{2} |s|$. In this case we have
\[ \varphi(s) = -\frac{\alpha}{s} (\alpha + s - \sqrt{\alpha^2 + 2s + 2\alpha s + s^2}), \]
where the square root is the standard square root (note that when $Re(s) \leq -1 - \alpha$, the square root is no longer the standard one). Let $I$ be the line segment connecting $\alpha^2 + 2s + 2\alpha s + s^2$ and $\alpha^2 + 2s$. For any $z \in I$ we have
\[ |z| \geq |\alpha^2 + 2s| - |2\alpha s + s^2| \geq \frac{1}{2} |s| - \frac{1}{4} |s| = \frac{1}{4} |s|. \tag{A.3} \]
Next, we claim that $I \subset \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, indeed, $|Im(2\alpha s + s^2)| \leq |Im(s)|$ and therefore if $Im(s) \neq 0$ then $I$ is contained entirely inside the upper or lower half planes. If $Im(s) = 0$ and $I \not\subset \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ then, since $s \in \Omega$ we have that $0 \in I$ contradicting (A.3). Thus, using (A.3) again we obtain
\[ \left| \sqrt{\alpha^2 + 2s + 2\alpha s + s^2} - \sqrt{\alpha^2 + 2s} \right| \leq \int_I \frac{1}{2\sqrt{|z|}} |dz| \leq C\alpha \frac{|s| + |\alpha|}{\sqrt{|s|}} \leq C(\alpha \sqrt{|s|} + |s|^\frac{3}{2}). \]
We get that
\[ |\tilde{\varphi}(s) - \varphi(s)| \leq C \frac{\alpha}{|s|} (|s| + \alpha \sqrt{|s|}) \leq C \left( \alpha + \frac{\alpha^2}{\sqrt{|s|}} \right) \leq C \alpha. \]
Finally, we prove the last part of the claim. The inequality in the case $|s| \geq C \alpha^2$ follows from part (3) of the claim. Thus suppose that $\frac{1}{5} \alpha^2 \leq |s| \leq C \alpha^2$. $z = 0$ is the unique solution to the equation $(1 - \sqrt{1 + 2z})/z = 1$ in $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ and therefore if we let $z := \alpha^{-2} s$ we have that $|z| \geq \frac{1}{5}$ and
\[ |\tilde{\varphi}(s) - 1| = \left| 1 - \frac{1 - \sqrt{1 + 2z}}{z} \right| \geq c. \]
The same inequality holds when we replace $\tilde{\varphi}$ with $\varphi$ by part (4) of the claim.

Throughout this section we’ll use the complex contours $\gamma = \gamma^{(t)} := \gamma_1 + \gamma_2 + \gamma_3$ and $\delta := \delta_1 + \delta_2 + \delta_3$ where
\[
\gamma_1(x) := x - \frac{i}{t}, \quad x \in (-\infty, 0], \quad \delta_1(x) := x - i\alpha^2, \quad x \in \left(-\infty, \frac{1}{4} \alpha^2\right] \n\gamma_2(x) := \frac{1}{t} e^{ix}, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \quad \delta_2(x) := -\frac{1}{4} \alpha^2 + ix, \quad x \in \left[-\alpha^2, \alpha^2\right] \n\gamma_3(x) := -x + \frac{i}{t}, \quad x \in [0, \infty), \quad \delta_3(x) := -x + \alpha^2 i, \quad x \in \left[\frac{1}{4} \alpha^2, \infty\right). \n\]
See Figure 2

A.2. Estimates for $K$.

Lemma A.4. For any $t > 0$ we have
\[ K(t) \leq \frac{C \alpha}{\sqrt{t + 1}} e^{-\alpha^2 t}. \]
Proof. Consider the function \( g(t) := \int_0^t K(x)dx \). It is easy to check that the Laplace transform of \( g \) is \( \varphi(s)/s \). Thus, using the Laplace inversion theorem and the fact that \( \varphi(s)/s \) is integrable on \( 1 + i\mathbb{R} \) by part (2) of Claim A.3, we get that

\[
\int_0^t K(x)dx = \frac{1}{2\pi i} \int_{1+i\mathbb{R}} e^{ts} \varphi(s)ds = \frac{1}{2\pi i} \int_{\gamma(1)} e^{ts} \varphi(s)ds,
\]

where in the last equality we deformed the contour using again the fact that the function inside the integral decays sufficiently fast by part (2) of Claim A.3.

Using the dominated convergence theorem one can differentiate under the integral sign to obtain

\[
K(t) = \frac{1}{2\pi i} \int_{\gamma(1)} e^{ts} \varphi(s)ds.
\]

When \( t \leq \alpha^{-2} \) we deform the contour again to \( \gamma^{(t)} \) and use part (3) of Claim A.3 to obtain

\[
K(t) \leq C\alpha \int_{\gamma^{(t)}} |s|^{-\frac{3}{2}} e^{tRe(s)} \leq \frac{C\alpha}{\sqrt{t}} \int_{\gamma^{(1)}} |\omega|^{-\frac{3}{2}} e^{Re(\omega)} \leq \frac{C\alpha}{\sqrt{t}} e^{-\alpha^2 t},
\]

where in the second inequality we changed variables by \( \omega = ts \).

Suppose next that \( t \geq \alpha^{-2} \). In this case we use the contour \( \delta \) instead of \( \gamma \). By part (3) of Claim A.3 and using the fact that \( Re(s) < -\alpha^2 \) on \( \delta \) we get

\[
K(t) \leq C\alpha \int_{\delta} |s|^{-\frac{3}{2}} e^{tRe(s)} \leq C\alpha e^{-\alpha^2 t} \int_{\delta} |s|^{-\frac{3}{2}} e^{\frac{3}{2}Re(s)} \leq \frac{C\alpha}{\sqrt{t}} e^{-\alpha^2 t} \int_{\delta'} |\omega|^{-\frac{3}{2}} e^{\frac{3}{2}Re(\omega)}
\]

\[
= \frac{C\alpha}{\sqrt{t}} e^{-\alpha^2 t} \int_{\delta'_1 + \delta'_2 + \delta'_3} |\omega|^{-\frac{3}{2}} e^{\frac{3}{2}Re(\omega)} + \int_{\delta'_2} |\omega|^{-\frac{3}{2}} e^{\frac{3}{2}Re(\omega)} \leq \frac{C\alpha}{\sqrt{t}} e^{-\alpha^2 t} + \sqrt{t\alpha^3} e^{-\alpha^2 t} \leq \frac{C\alpha}{\sqrt{t}} e^{-\alpha^2 t},
\]

where \( \delta' = \delta'_1 + \delta'_2 + \delta'_3 \) and where \( \delta'_1, \delta'_2, \delta'_3 \) are the image under the change of variables \( \omega = ts \) of the contours \( \delta_1, \delta_2, \delta_3 \) respectively. Note that in the last inequality we changed the value of \( c \) slightly and used that \( t \geq \alpha^{-2} \).

A.3. Estimates for the renewal process. Define the functions

\[
K^*(t) := \sum_{n=1}^\infty K^{*n}(t), \quad \tilde{K} = K* - \frac{2\alpha^2}{1+2\alpha},
\]
where $K^{*n}$ is the $n$-fold convolution of $K$ with itself. The function $K^*$ is the density of the renewal process with inter-arrival times that have density $K$.

**Lemma A.5.** For any $t > 0$ we have

$$|\tilde{K}(t)| \leq \frac{C\alpha}{\sqrt{t+1}} e^{-c\alpha^2 t}. \quad (A.4)$$

**Proof.** By the same arguments as in the proof of Lemma A.4 we have for any $n \geq 1$

$$K^{*n}(t) = \frac{1}{2\pi i} \int_{\gamma(t)} e^{ts} \varphi(s)^n ds.$$  

Thus,

$$K^*(t) = \sum_{n=1}^{\infty} K^{*n}(t) = \frac{1}{2\pi i} \int_{\gamma(t)} e^{ts} \varphi(s)^n \frac{1}{1-\varphi(s)} ds. \quad (A.5)$$

Indeed, by part (3) of Claim A.3 we have that $|\varphi(s)| \leq C\alpha \leq 1/2$ on the contour $\gamma(t)$ and therefore the sum $\sum_{n=1}^{\infty} \varphi(s)^n$ converges absolutely and we can change the order of summation and integration. Note that the function we integrate in (A.5) is meromorphic in $\Omega$ with a single pole at $s = 0$. Indeed, it is easy to see that $\varphi(s) = 1$ if and only if $s = 0$.

Suppose that $t \leq \alpha^{-2}$. Deforming $\gamma(t)$ to $\delta$ and using part (5) of Claim A.3 we obtain

$$|\tilde{K}(t)| \leq 2\alpha^2 + K^*(t) \leq 2\alpha^2 + C \int_{\gamma(t)} e^{tRe(s)}|\varphi(s)| \leq 2\alpha^2 + \frac{C\alpha}{\sqrt{t}} e^{-c\alpha^2 t} \leq \frac{C\alpha}{\sqrt{t}} e^{-c\alpha^2 t},$$

where in the third inequality we repeat the same arguments as in the proof of Lemma A.4.

Suppose next that $t \geq \alpha^{-2}$. In this case we use the contour $\delta$. In order to deform $\gamma(t)$ to $\delta$ we need to compute the residue at $s = 0$. We have

$$\text{Res} \left( \frac{e^{ts} \varphi(s)}{1-\varphi(s)}, 0 \right) = \varphi(0) \lim_{s \to 0} \frac{s}{\varphi(0) - \varphi(s)} = -\frac{1}{\varphi'(0)} = \frac{2\alpha^2}{1+2\alpha}.$$  

Thus

$$K^*(t) = \frac{2\alpha^2}{1+2\alpha} + \frac{1}{2\pi i} \int_{\delta} e^{ts} \frac{\varphi(s)}{1-\varphi(s)} ds.$$  

Therefore, using part (5) of Claim A.3 and the same bounds as in the proof of Lemma A.4 we get

$$|\tilde{K}(t)| \leq C \int_{\delta} e^{tRe(s)}|\varphi(s)| \leq \frac{C\alpha}{\sqrt{t}} e^{-c\alpha^2 t}.$$

This finishes the proof of (A.4) for all values of $t$. \hfill $\square$

The following is an immediate corollary

**Corollary A.6.** For any $t > 0$ we have that

$$K^*(t) \leq \frac{C\alpha}{\sqrt{t+1}} + 2\alpha^2$$

Repeating exactly the same arguments as in the proof of Lemma A.4 and Lemma A.5 we obtain

**Corollary A.7.** We have

$$|\tilde{K}'(t)| \leq \frac{C\alpha}{t^{\frac{3}{2}}} e^{-c\alpha^2 t}, \quad |\tilde{K}'(t)| = |(K^*)'(t)| \leq \frac{C\alpha}{t^{\frac{3}{2}}} e^{-c\alpha^2 t}$$
Remark A.8. We do not prove it but the functions $K = K^{(a)}$ and $\tilde{K} = \tilde{K}^{(a)}$ have a scaling limit of the form
\[
\alpha^{-2} K(\alpha^{-2} x) \to k_1(x), \quad \alpha^{-2} \tilde{K}(\alpha^{-2} x) \to k_2(x), \quad \alpha \to 0.
\]
Moreover the Laplace transform of $k_1$ is $\tilde{\varphi}(\alpha^2 x)$ which is independent of $\alpha$. The following lemma shows that, due to a fortuitous cancellation, the limiting functions $k_1$ and $k_2$ are equal.

Lemma A.9. For any $t > 0$ we have
\[
|\tilde{K}(t) - K(t)| \leq \frac{C_\alpha}{t + 1}.
\]

Proof. By the proof of Lemma A.4 and Lemma A.5 we have
\[
K(t) = \frac{1}{2\pi i} \int e^{ts} \psi(s) ds, \quad \tilde{K}(t) = \frac{1}{2\pi i} \int e^{ts} \frac{\tilde{\varphi}(s)}{1 - \varphi(s)} ds.
\]
In order to prove the lemma we show first that one can replace $\varphi$ with $\tilde{\varphi}$ in the last two integrals. Then we prove that
\[
\int e^{ts} \varphi(s) ds = \int e^{ts} \frac{\tilde{\varphi}(s)}{1 - \varphi(s)} ds. \tag{A.6}
\]
By part (4) of Claim A.3 we have
\[
\left| \int e^{ts} \varphi(s) ds - \int e^{ts} \tilde{\varphi}(s) ds \right| \leq \int e^{t\Re(s)} |\varphi(s) - \tilde{\varphi}(s)| ds \leq C_\alpha \int e^{t\Re(s)} ds \leq \frac{C_\alpha}{t} e^{-\alpha t^2} \int 1 + \frac{C_\alpha}{t} \int e^{\Re(\omega)} ds \leq C_\alpha \frac{e^{-\alpha t^2}}{t} + \frac{C_\alpha}{t} \leq \frac{C_\alpha}{t}.
\]
Similarly we have
\[
\left| \int e^{ts} \frac{\varphi(s)}{1 - \varphi(s)} ds - \int e^{ts} \frac{\tilde{\varphi}(s)}{1 - \varphi(s)} ds \right| \leq \int e^{t\Re(s)} \frac{|\varphi(s) - \tilde{\varphi}(s)|}{|1 - \varphi(s)||1 - \varphi(s)|} ds \leq C_\alpha \int e^{t\Re(s)} ds \leq \frac{C_\alpha}{t}.
\]
Next, we prove the identity (A.6).
\[
\left| \int e^{ts} \frac{\tilde{\varphi}(s)}{1 - \varphi(s)} ds - \int e^{ts} \tilde{\varphi}(s) ds \right| = \left| \int e^{ts} \frac{\tilde{\varphi}(s)^2}{1 - \varphi(s)} ds - \int e^{ts} \frac{\alpha^2(\alpha - \sqrt{\alpha^2 + 2s})^2}{s^2 + \alpha s(\alpha - \sqrt{\alpha^2 + 2s})} ds \right|
\]
\[
= 4\alpha^2 \int e^{\frac{y^2 - \alpha^2}{2}} \frac{(\alpha - y)^2 y}{(y^2 - \alpha^2)^2 + 2\alpha(y^2 - \alpha^2)(\alpha - y)} dy = 4\alpha^2 e^{-\frac{\alpha^2}{2}} \int e^{\frac{i\alpha^2}{2}} \left( \frac{y}{y^2 - \alpha^2} \right) dy
\]
\[
= 4\alpha^2 e^{-\frac{\alpha^2}{2}} \int e^{\frac{i\alpha^2}{2}} \frac{y}{y^2 - \alpha^2} dy = 0,
\]
where in the third equality we changed variables by $y = \sqrt{\alpha^2 + 2s}$ and also deformed the contour to the imaginary axis (this is possible as the function is analytic except for two poles at $\alpha$ and $-\alpha$). The last equality holds as we integrate an odd function.

The last lemma gives a tight bound when $t$ is large. In the following corollary we improve the estimate when $t$ is small.

Corollary A.10. For any $t > 0$ we have
\[
|\tilde{K}(t) - K(t)| \leq C \min\{\alpha^2, \frac{\alpha}{t + 1}\}.
\]
Proof. By Lemma A.4 we have

\[ K \ast K(t) = \int_0^t K(s)K(t-s)ds = 2 \int_0^{t/2} K(s)K(t-s)ds \]

\[ \leq C \int_0^{t/2} \frac{\alpha^2}{\sqrt{t-s}}ds \leq \frac{\alpha^2}{\sqrt{t}} \int_0^{t/2} \frac{1}{\sqrt{s}}ds \leq C\alpha^2. \]  

(A.7)

Thus, letting \( X_j \) be an i.i.d. sequence with density \( K \), if \( t \leq \alpha^{-2} \) we have

\[ \sum_{j=3}^{\infty} K^{*j}(t) = \sum_{j=1}^{\infty} \int_0^t K^{*j}(s)K^{*2}(t-s)ds \leq C\alpha^2 \sum_{j=1}^{\infty} \int_0^t K^{*j}(s)ds \]

\[ \leq C\alpha^2 \sum_{j=1}^{\infty} \mathbb{P}\left( \sum_{i=1}^j X_i \leq t \right) \leq C\alpha^2 \sum_{j=1}^{\infty} \mathbb{P}\left( \forall i \leq j, X_i \leq \alpha^{-2} \right) \leq C\alpha^2 \sum_{j=1}^{\infty} e^{-c_j} \leq C\alpha^2. \]

(A.8)

By (A.7) and (A.8) we have for \( t \leq \alpha^{-2} \)

\[ |\tilde{K}(t) - K(t)| \leq C\alpha^2 + K \ast K(t) + \sum_{j=3}^{\infty} K^{*j}(t) \leq C\alpha^2. \]

The corollary easily follows from the last bound and Lemma A.9.

A.4. The double integral in the drift term. In Theorem 9.1 we show that the drift term of the aggregate speed can be expressed in terms of a certain integral involving the functions \( K^* \) and \( J \). In this subsection we show how the Fourier transform estimates we developed in this section can be used in order to estimate this integral.

Lemma A.11. Let \( \alpha^{-2} \log^2(1/\alpha) \leq \frac{T}{\alpha^{-\frac{5}{2}}} \) and let

\[ I_1 := \int_0^T \int_0^u J_{s,u}K^*(u-s)dsdu \quad I_2 := \int_0^T \int_0^x J_{s,u}K^*(x-u)K^*(x-s)dsdu dx. \]

We have that \( I_1 + I_2 = 2 + O(\sqrt{\alpha}) \).

We start by showing that the function \( J \) can be expressed in terms of \( K \)

Claim A.12. We have that

\[ J_{s,u} = -\frac{1}{\alpha^2(1+2\alpha)^2} \int_s^u K'(x)K(u-x)dx - \frac{2}{(1+2\alpha)^2} K(u). \]

Proof. Let \( \tau' \) be an independent copy of \( \tau \) and recall the definition of \( J \) in (2.13). As \( \tau_s \geq \tau \) we have

\[ J_{s,u} = \mathbb{P}(u < \tau_s < \infty) - \mathbb{P}(u \leq \tau < \infty) \]

\[ = \mathbb{P}(\tau < u, u < \tau_s < \infty) - \mathbb{P}(u \leq \tau < \infty, \tau_s = \infty) \]

\[ = \mathbb{P}(s < \tau < u, u \leq \tau + \tau' < \infty) - \mathbb{P}(u \leq \tau < \infty, \tau' = \infty) \]

\[ = -\frac{1}{\alpha^2(1+2\alpha)^2} \int_s^u K'(x)K(u-x)dx - \frac{2}{(1+2\alpha)^2} K(u) \]

where in the last equality we used the definition of \( K \) and the fact that \(-\alpha^{-1}K'(x)\) is the density of \( \tau \mathbb{1}\{\tau < \infty\} \).
Corollary A.13. For all $u > s > 0$ we have

$$|J_{s,u}| \leq \frac{Ce^{-\alpha^2u}}{\sqrt{u + 1}\sqrt{s + 1}}.$$ 

For the proof of the corollary we will need the following simple bound.

Claim A.14. For all $s < u$ we have

$$\int_{s}^{u} \frac{1}{(x + 1)^{\frac{3}{2}}(u - x + 1)^{\frac{1}{2}}} \, dx \leq \frac{C}{\sqrt{s + 1}\sqrt{u + 1}}.$$ 

Proof. Suppose first that $u > 2s$. In this case

$$I_{s,u} \leq \int_{s}^{u/2} \frac{C}{(x + 1)^{\frac{3}{2}}(u + 1)^{\frac{1}{2}}} \, dx + \int_{u/2}^{u} \frac{C}{(u + 1)^{\frac{3}{2}}(u - x)^{\frac{1}{2}}} \, dx \leq \frac{C}{\sqrt{s + 1}\sqrt{u + 1}} + \frac{C}{u + 1} \leq \frac{C}{\sqrt{s + 1}\sqrt{u + 1}}.$$ 

In the case $u \leq 2s$ we have

$$I_{s,u} \leq \int_{s}^{u} \frac{1}{(s + 1)^{\frac{3}{2}}(u - x)^{\frac{1}{2}}} \, dx \leq \frac{\sqrt{u}}{(s + 1)^{\frac{3}{2}}} \leq \frac{C}{\sqrt{s + 1}\sqrt{u + 1}},$$

as needed. \hfill \square

Proof of Corollary A.13 By Claim A.12, Lemma A.4 and Corollary A.7 we have

$$|J_{s,u}| \leq \alpha^{-2} \int_{s}^{u} |K'(x)|K(u - x) \, dx + 2K(u) \leq \int_{s}^{u} \frac{C e^{-\alpha^2u}}{(x + 1)^{\frac{3}{2}}(u - x + 1)^{\frac{1}{2}}} \, dx + \frac{C e^{-\alpha^2u}}{\sqrt{u + 1}} \leq \frac{C e^{-\alpha^2u}}{\sqrt{u + 1}\sqrt{s + 1}} + \frac{C e^{-\alpha^2u}}{\sqrt{u + 1}} \leq \frac{C e^{-\alpha^2u}}{\sqrt{u + 1}\sqrt{s + 1}},$$

where the third inequality is by Claim A.14 and the last inequality follows by slightly changing the values of $c$ and $C$. \hfill \square

Proof of Lemma A.11 First, note that by Corollary A.13 one can change the upper limit $T$ in the definition of $I_1$ to $\infty$ without changing the value of $I_1$ by more that $O(\alpha^{10})$. Using this observation and the definition of $K$ we obtain

$$I_1 = O(\alpha^{10}) + \int_{0}^{\infty} \int_{0}^{u} J_{s,u} K(u - s) \, ds \, du + \frac{2\alpha^{2}}{1 + 2\alpha} \int_{0}^{\infty} \int_{0}^{u} J_{s,u} ds \, du. \quad (A.9)$$

Denote the last two terms by $I_3$ and $I_4$ respectively. By Claim A.12 we have

$$I_4 = -\frac{2}{(1 + 2\alpha)^3} \int_{0}^{\infty} \int_{0}^{u} \int_{0}^{x} K'(x)K(u - x)ds \, dx \, du - \frac{4\alpha^{2}}{(1 + 2\alpha)^3} \int_{0}^{\infty} \int_{0}^{u} K(u)ds \, du$$

$$= -\frac{2}{(1 + 2\alpha)^3} \int_{0}^{\infty} \int_{0}^{u} K'(x)K(u - x)dx \, du - \frac{4\alpha^{2}}{(1 + 2\alpha)^3} \int_{0}^{\infty} uK(u)du \quad (A.10)$$

$$= -\frac{2}{(1 + 2\alpha)^3} \int_{0}^{\infty} K(b)db \int_{0}^{\infty} aK'(a)da - 2 + O(\alpha) = O(\alpha),$$

where in the third equality we changed variables by $a = x$ and $b = u - x$ and used Corollary A.2. In the last equality we used Corollary A.2 again.
We turn to estimate $I_3$. Let $F(u) := -1 + \int_0^u K(x)\,dx$ and note that $F$ decays exponentially fast. Using Claim A.12 we get

$$
\alpha^2(1 + 2\alpha)^2 I_3 = \frac{\alpha^2}{1 + 2\alpha} \int_0^\infty \int_y^u \int_0^\infty K'(y)K(u - y)\,\tilde{K}(u - s) - 2\alpha^2 \int_0^\infty \int_0^u K(u)\,\tilde{K}(u - s) 
$$

$$
= \int_0^\infty \int_y^u \int_0^\infty K'(a + b)K(c - b)\tilde{K}(c)\,da\,db\,dc - 2\alpha^2 \int_0^\infty \int_0^u K(a + b)\tilde{K}(b)\,da\,db
$$

(A.11)

$$
= \int_0^\infty \tilde{K}(c) \int_0^\infty K(b)K(c - b) + 2\alpha^2 \int_0^\infty F(b)\tilde{K}(b) = \int_0^\infty (K + \tilde{K})\cdot \tilde{K} + 2\alpha^2 F\cdot \tilde{K},
$$

where in the second equality we made the change of variables $a = s$, $b = x - s$, $c = u - s$. Note that the right hand side of (A.11) is of order $O(\alpha^2)$ by Lemma A.5, equation (A.7) and the definition of $F$. Thus,

$$
I_3 = O(\alpha) + \int_0^\infty \alpha^{-2}(K + \tilde{K})\cdot \tilde{K} + 2F\cdot \tilde{K}.
$$

Next, we estimate the integral $I_2$. Using the definition of $\tilde{K}$ we obtain

$$
I_2 = \frac{T}{0} \int_y^u \int_0^\infty J_{s,u} \left( \tilde{K}(x - u) + \frac{2\alpha^2}{1 + 2\alpha} \right) \left( \tilde{K}(x - s) + \frac{2\alpha^2}{1 + 2\alpha} \right)
$$

$$
= \int_0^\infty \int_y^u \int_0^\infty J_{s,u} \tilde{K}(x - u)\tilde{K}(x - s) + \frac{2\alpha^2}{1 + 2\alpha} \int_0^\infty \int_y^u \int_0^\infty J_{s,u} \tilde{K}(x - u)
$$

(A.12)

$$
+ \frac{2\alpha^2}{1 + 2\alpha} \int_0^\infty \int_y^u \int_0^\infty J_{s,u} \tilde{K}(x - s) + \frac{4\alpha^4}{(1 + 2\alpha)^2} \int_0^\infty \int_y^u \int_0^\infty J_{s,u} + O(\alpha^{10}),
$$

where in the second inequality we also changed the upper limit $T$ to $\infty$ in the first three integrals using the same arguments as in (A.9). Note that we could not change the upper limit in the fourth integral as the corresponding infinite integral doesn’t converge. We denote the four terms on the right hand side of (A.12) by $I_5, I_6, I_7$ and $I_8$ respectively.

We start with $I_5$. Using Claim A.12 we get

$$
\alpha^2(1 + 2\alpha)^2 I_5 = \alpha^2(1 + 2\alpha)^2 \int_0^\infty \int_y^u \int_0^\infty J_{s,u} \tilde{K}(x - u)\tilde{K}(x - s)
$$

$$
= \int_0^\infty \int_y^u \int_0^\infty K'(y)K(u - y)\tilde{K}(x - u)\tilde{K}(x - s) - 2\alpha^2 \int_0^\infty \int_y^u \int_0^\infty K(u)\tilde{K}(x - u)\tilde{K}(x - s)
$$

$$
= \int_0^\infty \int_y^u \int_0^\infty K'(a + b)K(c - b)\tilde{K}(c)\tilde{K}(d) - 2\alpha^2 \int_0^\infty \int_y^u \int_0^\infty K(a + b)\tilde{K}(c - b)\tilde{K}(c)
$$

$$
= \int_0^\infty \int_y^u \int_0^\infty K'(d)\tilde{K}(d - c)\tilde{K}(c)\tilde{K}(c) - 2\alpha^2 \int_0^\infty \int_y^u \int_0^\infty K(a + b)\tilde{K}(c - b)\tilde{K}(c)
$$

$$
= \int_0^\infty \int_y^u \int_0^\infty \tilde{K}(d)\tilde{K}(d - c)\tilde{K}(c)\tilde{K}(c)\tilde{K}(c) + 2\alpha^2 \int_0^\infty \int_y^u \int_0^\infty F(b)\tilde{K}(c - b)
$$

$$
= \int_0^\infty \tilde{K}\cdot (\tilde{K} + \tilde{K} \cdot K) + 2\alpha^2 \tilde{K} \cdot (F \cdot \tilde{K}),
$$
where third equality we applied the change of variables $a = s$, $b = y - s$, $c = u - s$, $d = x - s$ to the first integral and the change of variables $a = s$, $b = u - s$, $c = x - s$ to the second integral. As before, the right hand side of the last equation is of order $O(\alpha^2)$ and therefore

$$I_5 = O(\alpha) + \int_0^\infty \alpha^{-2} \tilde{K} \cdot (\tilde{K} \ast K \ast K) + 2 \tilde{K} \cdot (F \ast \tilde{K}).$$

We turn to estimating $I_6$. Using Claim A.12 again we get

$$\frac{(1 + 2\alpha)^3}{2} I_6 = \alpha^2 (1 + 2\alpha)^2 \int_0^\infty \int_0^\infty \int_0^\infty J_{s,u} \tilde{K}(x - u)$$

$$= - \int_0^\infty \int_0^\infty \int_0^\infty K'(y)K(u - y)\tilde{K}(x - u) - 2\alpha^2 \int_0^\infty \int_0^\infty \int_0^\infty K(u)\tilde{K}(x - u)$$

$$= - \int_0^\infty \int_0^\infty \int_0^\infty K'(a + b)K(c)\tilde{K}(d) - 2\alpha^2 \int_0^\infty \int_0^\infty \int_0^\infty K(a + b)\tilde{K}(c)$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty K(b)K(c)\tilde{K}(d) + 2\alpha^2 \int_0^\infty \int_0^\infty F(b)\tilde{K}(c) = \int_0^\infty \tilde{K} - \int_0^\infty \tilde{K} + O(\alpha) = O(\alpha),$$

where in the third equality we changed variables by $a = s$, $b = y - s$, $c = u - y$, $d = x - u$ and in the fifth equality we used that, by Corollary A.2, we have

$$\int_0^\infty F(x)dx = - \int_0^\infty K(y)dydx = - \int_0^\infty yK(y)dy = -\frac{1}{2} \alpha^{-2} + O(\alpha^{-1}).$$

Next, we estimate $I_7$. We have

$$\frac{(1 + 2\alpha)^3}{2} I_7 = \alpha^2 (1 + 2\alpha)^2 \int_0^\infty \int_0^\infty \int_0^\infty J_{s,u} \tilde{K}(x - s)$$

$$= - \int_0^\infty \int_0^\infty \int_0^\infty K'(y)K(u - y)\tilde{K}(x - s) - 2\alpha^2 \int_0^\infty \int_0^\infty \int_0^\infty K(u)\tilde{K}(x - s)$$

$$= - \int_0^\infty \int_0^\infty \int_0^\infty K'(a + b)K(c - b)\tilde{K}(d) - 2\alpha^2 \int_0^\infty \int_0^\infty \int_0^\infty K(a + b)\tilde{K}(c)$$

$$= \int_0^\infty \tilde{K}(d) \int_0^\infty \int_0^\infty K(b)K(c - b) + 2\alpha^2 \int_0^\infty \tilde{K}(c) \int_0^\infty F(b)$$

$$= \int_0^\infty \tilde{K} \cdot (K \ast K \ast 1) + 2\alpha^2 \tilde{K} \cdot (F \ast 1),$$

where in the third equality we used the same change of variables as before. The right hand side of the last equation is of order $O(1)$ and therefore

$$I_7 = O(\alpha) + \int_0^\infty 2\tilde{K} \cdot (K \ast K \ast 1) + 4\alpha^2 \tilde{K} \cdot (F \ast 1),$$
Finally, we estimate $I_8$. We have

$$\frac{(1 + 2\alpha)^2}{4\alpha^4}I_8 = \int_0^T \int_0^u \int_0^u J_{s,u} = \int_0^T \int_0^u \int_0^u T J_{s,u} = T \int_0^T \int_0^u J_{s,u} - \int_0^T \int_0^u uJ_{s,u}.$$  

We denote the last two terms by $I_9$ and $I_{10}$ respectively. We have that $I_9 = O(\alpha^{-\frac{7}{2}})$ by the assumption on $T$ and by the bound on $I_4$ in (A.10).

We turn to estimate $I_{10}$. We have

$$\alpha^2(1 + 2\alpha)^2I_{10} = -\alpha^2(1 + 2\alpha)^2 \int_0^\infty \int_0^u uJ_{s,u} = \int_0^\infty \int_0^u \int_0^u uK'(x)K(u - x) + 2\alpha^2 \int_0^\infty \int_0^u uK(u)$$

$$= \int_0^\infty \int_0^u uK'(x)K(u - x) + 2\alpha^2 \int_0^\infty \int_0^u uK(u) = \int_0^\infty \int_0^\infty a(a + b)K'(a)K(b) + 2\alpha^2 + O(\alpha^{-1})$$

$$= \int_0^\infty \int_0^\infty a^2K'(a)da \int_0^\infty K(b)db + \int_0^\infty \int_0^\infty aK'(a)da \int_0^\infty bK(b)db + 2\alpha^2 + O(\alpha^{-1}) = \frac{1}{2}\alpha^{-2} + O(\alpha^{-1}),$$

where in the fourth and last equality we used Corollary A.2. Thus, $I_{10} = \frac{1}{2}\alpha^{-4} + O(\alpha^{-3})$ and therefore $I_8 = 2 + O(\sqrt{\alpha})$.

Adding all the estimates for $I_3, I_4, I_5, I_7$ and $I_8$ we obtain

$$I_1 + I_2 = I_3 + I_4 + I_5 + I_6 + I_7 + I_8 = 2 + O(\sqrt{\alpha}) + \int_0^\infty \alpha^{-2}(K * K) \cdot \tilde{K} + 2F \cdot \tilde{K}$$

$$+ \int_0^\infty \alpha^{-2}\tilde{K} \cdot (\tilde{K} * K * K) + 2\tilde{K} \cdot (F * \tilde{K}) + \int_0^\infty 2\tilde{K} \cdot (K * K * 1) + 4\alpha^2 \tilde{K} \cdot (F * 1).$$

Next, using that $\tilde{K} = K^* - 2\alpha^2 + O(\alpha^3)$ we see that the third integral is completely canceled by the $\tilde{K}$ inside the brackets of the second integral. Thus,

$$I_1 + I_2 = 2 + O(\sqrt{\alpha}) + \alpha^{-2} \int_0^\infty \tilde{K} \cdot (K^* * K * K + K * K * K + 2\alpha^2(F * K^* + F))$$

$$= 2 + O(\sqrt{\alpha}) + \alpha^{-2} \int_0^\infty \tilde{K} \cdot (K^* + 2\alpha^2(1 * K^* K^* - 1 * K^* + 1 * K - 1))$$

$$= 2 + O(\sqrt{\alpha}) + \alpha^{-2} \int_0^\infty \tilde{K} \cdot (K^* + 2\alpha^2) = 2 + O(\sqrt{\alpha}) + \alpha^{-2} \int_0^\infty \tilde{K}(\tilde{K} - K),$$

where in the second equality we used that $F = K * 1 - 1$, in the third equality we used that $K^* = K + K * K^*$ and in the last equality we used that $\tilde{K} = K^* - 2\alpha^2 + O(\alpha^3)$.

Next, we estimate the integral in the right hand side of (A.13). We have

$$\int_0^\infty \tilde{K}(\tilde{K} - K) \leq C\alpha^2 \int_0^{\alpha^{-1}} \tilde{K}(t)dt + C\alpha \int_0^{\alpha^{-1}} \frac{\tilde{K}(t)}{t}dt \leq C\alpha^3 \int_0^{\alpha^{-1}} \frac{1}{\sqrt{t}}dt + C\alpha^2 \int_0^{\alpha^{-1}} \frac{1}{t^2}dt \leq C\alpha^2,$$

where in the first inequality we used Corollary A.10 and in the second inequality we used Lemma A.5.

Substituting the last bound into (A.13) finishes the proof of the lemma. $\square$
APPENDIX B. MARTINGALE CONCENTRATION LEMMAS

We provide the details of the deferred proofs of martingale concentration lemmas. We begin with establishing Lemma 4.8. It will be clear from the proof that Corollary 4.9 follows from the same proof as well.

Proof of Lemma 4.8 Since we have a somewhat weakened assumption of the upper bound on \( f_t \), we split the integral by two parts and control them separately. Moreover, as the error bound suggest, we use union bound over a discretized interval of \([0, h]\), and then complete the argument by showing a certain type of continuity of the integral. Without loss of generality, we assume that \( \tau_0 \geq \tau_- \) almost surely (otherwise the integral is zero).

For each \( t \in [0, h] \), we write

\[
\int_{\tau_-}^{t \land \tau_0} f_t(x) d\tilde{\Pi}_g(x) = \int_{\tau_-}^{(\Delta \lor \tau_0) \land \tau_0} f_t(x) d\tilde{\Pi}_g(x) + \int_{(\Delta \lor \tau_0) \land \tau_0}^{t \land \tau_0} f_t(x) d\tilde{\Pi}_g(x).
\]  \hspace{1cm} (B.1)

The first integral can be controlled using the definitions of \( \tau', \tau'' \), namely,

\[
N f_t(p_1) - A \leq \int_{\tau_-} f_t(x) d\tilde{\Pi}_g(x) \leq N f_t(p_1) + A.
\]

On the other hand, the second integral can be controlled using Corollary 4.6 where we obtain

\[
\mathbb{P}\left( \left| \int_{(\Delta \lor \tau_0) \land \tau_0} f_t(x) d\tilde{\Pi}_g(x) \right| \geq a\sqrt{M} \right) \leq Ce^{-a}.
\]

Then, we define \( \mathcal{T} := \{ t \in [0, h] : t = k\delta \Delta, k \in \mathbb{Z} \} \), and combine the above two estimates to deduce

\[
\mathbb{P}\left( \left| \int_{\tau_-}^{t \land \tau_0} f_t(x) d\tilde{\Pi}_g(x) \right| \leq N f_t(p_1) + A + a\sqrt{M}, \ \forall \ t \in \mathcal{T} \right) \geq 1 - \left( \frac{Ch}{\delta \Delta} \right) e^{-a}, \hspace{1cm} (B.2)
\]

and an analogous bound holds for the lower bound.

Next, for any \( t \in [0, h] \), let \( t_\delta \) be such that \( t_\delta \leq t, t_\delta \in \mathcal{T} \). Write \( t' = (t \land \tau_0) \lor \tau_- \), \( t'_\delta = (t_\delta \land \tau_0) \lor \tau_- \), and express that

\[
\int_{\tau_-}^{t'} f_t(x) d\tilde{\Pi}_g(x) - \int_{t'_\delta}^{t'} f_t(x) d\tilde{\Pi}_g(x) = \int_{t'_\delta}^{t'} f_t(x) d\tilde{\Pi}_g(x) + \int_{\tau_-}^{t'} (f_t(x) - f_{t_\delta}(x)) d\tilde{\Pi}_g(x).
\]

The first integral can again be bounded by using the definitions of \( \tau', \tau'' \):

\[
N f_t(p_1) - A \leq \int_{t'_\delta}^{t'} f_t(x) d\tilde{\Pi}_g(x) \leq N \tilde{f}_t(p_1) + A,
\]  \hspace{1cm} (B.3)

where we used the fact that \( \tilde{f}_t(x) \) (resp. \( f_t(x) \)) is a decreasing (resp. increasing) function. On the other hand, we control the second integral by using

\[
|f_t(x) - f_{t_\delta}(x)| \leq D(t - t_\delta) \leq D\delta \Delta. \hspace{1cm} (B.4)
\]

Moreover, due to \( \tau_0 \leq \tau' \), the total number of points \([\Pi_t[0, t'_\delta]]\) is bounded by \( hN\Delta^{-1} \). Thus,

\[
\int_{\tau_-}^{t'_\delta} (f_t(x) - f_{t_\delta}(x)) d\tilde{\Pi}_g(x) \leq D\delta \Delta (hN\Delta^{-1} + h\eta) = D\delta(hN + h\Delta \eta). \hspace{1cm} (B.5)
\]
Therefore, under the event described inside \((B.2)\), we have by combining \((B.3)\) and \((B.5)\) that
\[
\int_{\tau - t'} f_t(x) d\overline{\Pi}_g(x) \leq N \overline{f}_t(p_1) + 2A + D\delta(hN + h\Delta\eta)
\]
\[
+ N \overline{f}_t(p_1) + a\sqrt{M},
\]
and also the corresponding lower bound. Using \((B.4)\), we can simplify \((B.6)\) to
\[
\int_{\tau - t'} f_t(x) d\overline{\Pi}_g(x) \leq 2N \overline{f}_t(p_1) + 2A + D\delta(2N\Delta + hN + h\Delta\eta) + a\sqrt{M}
\]
\[
\leq 2N \overline{f}_t(p_1) + 3A + a\sqrt{M},
\]
where the last inequality is due to the assumption on \(\delta\). Again, we have the corresponding bound for the lower bound as well. Therefore, we obtain conclusion by noticing that the above holds deterministically for all \(t \in [0,h]\) under the event given in \((B.2)\). \(
\]
Note that the proof of Corollary 4.9 follows from the exact same argument, except that we split the integrals by \([0, t - \Delta]\) and \([t - \Delta, t]\) instead of \((B.1)\), namely,
\[
\int_{0}^{t'} f_t(x) d\overline{\Pi}_g(x) = \int_{0}^{(t-\Delta)\wedge \tau_0} f_t(x) d\overline{\Pi}_g(x) + \int_{(t-\Delta)\wedge \tau_0}^{t\wedge \tau_0} f_t(x) d\overline{\Pi}_g(x).
\]
The details are omitted due to similarity.

Appendix C. Negligibility of the Third Order Contributions

In this section, we establish Proposition 4.11. The proof is based on a similar approach as Proposition 4.10, but requires more delicate analysis on the coupled events of \(W\) and \(Y\). Recall the definition (2.11), and let
\[
A_{v,u} := 1\{ s \leq T_{v,u} < \infty \}, \quad A_v := 1\{ s \leq T_v < \infty \},
\]
\[
A_u := 1\{ s \leq T_u < \infty \}, \quad A_0 := 1\{ s \leq T < \infty \},
\]
\[
A := A_{v,u} - A_v - A_u + A_0.
\]
Then, \(Q_{v,u,s} = \mathbb{E}_0[A | Y_{\leq v}]\).

Recalling that \(T \leq T_u \leq T_v \leq T_{v,u}\), \(A\) can take nonzero values only in the seven cases as illustrated in the Table 1 below.

| Event | \(s, \infty\) | \(s, \infty\) | = \(\infty\) | Value of \(A\) |
|-------|----------------|----------------|-----------|---------------|
| \(\mathbf{A}_1\) | none | \(T\) | \(T_u, T_v, T_{v,u}\) | +1 |
| \(\mathbf{A}_2\) | none | \(T, T_u, T_v\) | \(T_{v,u}\) | -1 |
| \(\mathbf{A}_3\) | \(T\) | \(T_u\) | \(T_v, T_{v,u}\) | -1 |
| \(\mathbf{A}_4\) | \(T\) | \(T_u, T_v\) | \(T_{v,u}\) | -2 |
| \(\mathbf{A}_5\) | \(T, T_u\) | \(T_v\) | \(T_{v,u}\) | -1 |
| \(\mathbf{A}_6\) | \(T\) | \(T_u, T_v, T_{v,u}\) | none | -1 |
| \(\mathbf{A}_7\) | \(T, T_u, T_v\) | \(T_{v,u}\) | none | +1 |

Table 1. The cases that give nonzero values of \(A\). Each row describes the regimes that \(T, T_u, T_v\), and \(T_{v,u}\) belong to. For instance, the second row denotes that \(\mathbf{A}_2 = 1\{ t \leq T \leq T_u \leq T_v < \infty = T_{v,u}\} \), and that \(A = -1\) on \(\mathbf{A}_2\).
From Table[1] we see that
\[ A = 1 \{A_1\} - 1 \{A_2\} - 1 \{A_3\} - 2 \cdot 1 \{A_4\} - 1 \{A_5\} - 1 \{A_6\} + 1 \{A_7\}. \]
Therefore, our strategy is to estimate each \( \mathbb{P}_\alpha(A_i | \mathcal{F}_v) \). This is done by the following lemmas, and we obtain the conclusion from combining them together.

The following lemmas provide estimates on \( \mathbb{P}_\alpha(A_i | \mathcal{F}_v) \). Recall the parameter \( \hat{h} \) from [4.3]. Moreover, in the following statements, “an event holds with very high probability” means that for all sufficiently small \( \alpha > 0 \), it holds with probability at least
\[ 1 - e^{-\alpha^{-c_\epsilon}}, \]
for some constant \( c_\epsilon > 0 \).

**Lemma C.1.** We have \( 0 \leq \mathbb{P}_\alpha(A_1 | \mathcal{F}_v) - \mathbb{P}_\alpha(A_2 | \mathcal{F}_v) = \frac{4\alpha^2}{(1+2\alpha)^2}\mathbb{P}_\alpha(s \leq T < \infty | \mathcal{F}_v). \) Moreover, with very high probability,
\[
\mathbb{P}_\alpha(A_1 | \mathcal{F}_v) - \mathbb{P}_\alpha(A_2 | \mathcal{F}_v) \leq \frac{\alpha^{-2\epsilon}}{\sqrt{(v+1)(u+1)(s+1)}}. \tag{C.4}
\]

**Lemma C.2.** \( A_3 \) is an empty event. In particular, \( \mathbb{P}_\alpha(A_3 | \mathcal{F}_v) = 0 \).

**Lemma C.3.** We have \( \mathbb{P}_\alpha(A_4 | \mathcal{F}_v) = \frac{2\alpha}{1+2\alpha}\mathbb{P}_\alpha(T \leq s < T_u < \infty | \mathcal{F}_v) \). Moreover, with very high probability,
\[
\mathbb{P}_\alpha(A_4 | \mathcal{F}_v) \leq \frac{\alpha^{-2\epsilon}}{\sqrt{(v+1)(u+1)(s+1)}}.
\]

**Lemma C.4.** We have \( \mathbb{P}_\alpha(A_5 | \mathcal{F}_v) \leq \frac{2\alpha}{1+2\alpha}\mathbb{P}_\alpha(T \leq s < T_v < \infty | \mathcal{F}_v) \). Moreover, with very high probability,
\[
\mathbb{P}_\alpha(A_5 | \mathcal{F}_v) \leq \frac{\alpha^{-2\epsilon}}{\sqrt{(v+1)(u+1)(s+1)}}.
\]

**Lemma C.5.** With very high probability, we have
\[
\text{for all } 0 \leq v < u \leq \hat{h}, \quad |\mathbb{P}_\alpha(A_6 | \mathcal{F}_v) - \mathbb{P}_\alpha(A_7 | \mathcal{F}_v)| \leq \frac{\alpha^{-2\epsilon}}{\sqrt{(v+1)(u+1)(s+1)}}.
\]

Although Lemmas C.1, C.3 follow from straight-forward generalizations of Proposition 4.10, the proof of Lemma C.5 requires more work. Indeed, we will later see that the probabilities \( \mathbb{P}_\alpha(A_i | \mathcal{F}_v) \), \( i \in \{1, 2, 6, 7\} \) do not satisfy the estimate (C.4) individually, but their leading orders cancel out under the setting of Lemmas C.1 and C.5. Though such cancellation is apparent in Lemma C.1 a refined analysis is needed in Lemma C.5 to observe such phenomenon.

In the remaining of this subsection, we present the proofs of Lemmas C.2, C.4, and then prove Lemmas C.1 and C.5. Also, for \( Y = \{Y(x)\} \), we define \( Y^p \) and \( Y^{p,q} \) as
\[ Y^p(x) := Y(x) + 1 \{x > p\}, \quad Y^{p,q}(x) := Y(x) + 1 \{x > p\} + 1 \{x > q\}. \]

**Proof of Lemma C.2.** Note that \( T_u > s \) implies \( T_v = T_u \), since \( Y^u(t') = Y^u(t') \) for \( t' > s \). Therefore, \( \{s < T_u < \infty = T_v\} \) described in the definition of \( A_3 \) is an invalid event. \[ \square \]

**Proof of Lemmas C.3 and C.4.** We present the proof of Lemma C.3. Proof of Lemma C.4 follows from the same argument and is left to the reader.

On \( A_4 \), \( T < s \leq T_u \) implies that the random walk \( W \) hits \( Y \) at \( T \in [u, s) \). Then, since \( s \leq T_u < \infty \), it hits \( Y^u \) at \( T_u = T_v \in [s, \infty) \). Finally, since \( T_{v,u} = \infty \), it never hits \( Y^v,u \). Therefore,
\[ \mathbb{P}_\alpha(A_4 | \mathcal{F}_v) = \mathbb{P}_\alpha(T < s < T_u < \infty | \mathcal{F}_v) \]
\[ \times \mathbb{P}_\alpha(W(t') < Y^v,u(t'), \forall t' \geq T_u | W(T_u) = Y^v,u(T_u) - 1) \]
\[ = \frac{2\alpha}{1+2\alpha}\mathbb{P}_\alpha(T < s < T_u < \infty | \mathcal{F}_v), \]
which proves the first part of the lemma.

Then, we can use Proposition 4.10 to estimate the probability in the RHS of the above. This gives that with very high probability, we have for all \(0 \leq v < u < s \leq \hat{h}\) (which gives in particular, \(v \leq \alpha^{-2}\)) that
\[
\mathbb{P}_\alpha(A_1|F_v) \leq \frac{\alpha^{1-\epsilon}}{(u+v)(s+1)} \leq \frac{\alpha^{-2\epsilon}}{(v+1)(u+1)(s+1)}.
\]

\[\square\]

Proof of Lemma C.1. On \(A_1 = \{s \leq T < \infty = T_u = T_v = T_{v,u}\}\), the random walk \(W\) hits \(Y\) at time \(T \in (s, \infty)\) but never hits \(Y^v\) after that. Thus,
\[
\mathbb{P}_\alpha(A_1|F_v) = \mathbb{P}_\alpha(s \leq T < \infty | F_v) \\
\times \mathbb{P}_\alpha(W(t') < Y^v(t'), \forall t' > T | W(T) = Y^u(T) - 1)
\]
\[(C.5)\]
On the other hand, on \(A_2 = \{s \leq T \leq T_u \leq T_v < \infty = T_{v,u}\}\), the random walk \(W\) hits \(Y^v\) as well at \(T_u \in (T, \infty)\) after hitting \(Y\) at \(T \in (s, \infty)\). (One can also see that \(T_u = T_v\), but never hits \(Y^{v,u}\). Therefore,
\[
\mathbb{P}_\alpha(A_2|F_v) = \mathbb{P}_\alpha(s \leq T < \infty | F_v) \\
\times \{\mathbb{P}_\alpha(T_u < \infty | T \in (s, \infty)) - \mathbb{P}_\alpha(T_{v,u} < \infty | T \in (s, \infty))\}
\]
\[(C.6)\]
Combining (C.5) and (C.6), we obtain that
\[
0 \leq \mathbb{P}_\alpha(A_1|F_v) - \mathbb{P}_\alpha(A_2|F_v) = \frac{4\alpha^2}{(1+2\alpha)^2} \times \mathbb{P}_\alpha(s \leq T < \infty | F_v).
\]
This proves the first part of Lemma C.1. Then, (C.4) follows from the same argument as in the proof of Lemma C.3 by using Lemma 4.13. \[\square\]

What remains is to establish Lemma C.5. To this end, it will be useful to divide \(A_7\) into two disjoint events \(A_7^1\) and \(A_7^2\) as follows:
\[
A_7^1 := \{T \geq u\} \cap A_7, \quad \text{and} \quad A_7^2 := \{T < u\} \cap A_7.
\]
Note that \(A_7^1, A_7^2\) are equivalent to the following events:
\[
A_7^1 = \{u \leq T < T_u = T_v < s \leq T_{v,u} < \infty\};
\]
\[
A_7^2 = \{v \leq T = T_u < u \leq T_v < s \leq T_{v,u} < \infty\}.
\]
Then, we bound \(\mathbb{P}_\alpha(A_6|F_v) - \mathbb{P}_\alpha(A_7^1|F_v)\) and \(\mathbb{P}_\alpha(A_7^2|F_v)\) as the follows.

Lemma C.6. With very high probability, we have for all \(0 \leq v < u < s \leq \hat{h}\) that
\[
\mathbb{P}_\alpha(A_6|F_v) - \mathbb{P}_\alpha(A_7^1|F_v) \leq \frac{\alpha^{-2\epsilon}}{(s+1)\sqrt{u+1}}.
\]

Lemma C.7. With very high probability, we have for all \(0 \leq v < u < s \leq \hat{h}\) that
\[
\mathbb{P}_\alpha(A_7^2|F_v) \leq \frac{\alpha^{-\epsilon}}{\sqrt{(v+1)(u+1)(s+1)}}.
\]

Proof of Lemma C.5. Lemma C.5 follows straight-forwardly from Lemmas C.6 and C.7. \[\square\]
Proof of Lemma C.6 Let \( f_v \) denote the probability density function of \( T \), conditioned on \( \mathcal{F}_v \). Namely,

\[
f_v(t') = -\frac{d}{dt} \mathbb{P}_\alpha(t' \leq T | \mathcal{F}_v).
\]

Under the notation of Lemma 4.14, \( f_v = \mathbb{E}_\alpha[f_Y|\mathcal{F}_v] \). Similarly, we write

\[
f(t') = -\frac{d}{dt} \mathbb{P}_\alpha(t' \leq T) = \mathbb{E}_\alpha[f_Y(t')].
\]

For simplicity, we also introduce \( h(t') = \mathbb{P}_\alpha(t' \leq T < \infty) \).

Here, note that \( f(t') = -h'(t') \).

On \( A_6 \), we have \( u \leq T < s \), since if \( T < u \), then \( T = T_u \) which is a contradiction. Also, since both \( T_u \) and \( T_v \) are at least \( s \), they are equal. Therefore, we can write

\[
\mathbb{P}_\alpha(A_6 | \mathcal{F}_v) = \int_u^s f_v(x) \mathbb{P}_\alpha(s \leq T_u < \infty | T = x) \times \mathbb{P}_\alpha(W(t') < Y^{v,u}(t'), \forall t' > T_u | W(T_u) = Y^{v,u}(T_u) - 1) \, dx \tag{C.7}
\]

\[
= \int_u^s f_v(x) h(s - x) \times \frac{2\alpha}{1 + 2\alpha} \, dx = \int_u^s f_v(x) h(s - x) h(0) \, dx,
\]

where the last equality follows from \( h(0) = \mathbb{P}_\alpha(0 \leq T < \infty) = \frac{2\alpha}{1 + 2\alpha} \).

On the other hand, \( \mathbb{P}_\alpha(A_{\frac{1}{2}} | \mathcal{F}_v) \) can be written as

\[
\mathbb{P}_\alpha(A_{\frac{1}{2}} | \mathcal{F}_v) = \int_u^s \int_x^s f_v(x) \mathbb{P}_\alpha(T_v \in [y, y + dy] | T = x) \mathbb{P}_\alpha(s \leq T_v, u < \infty | T_v = y) \, dx dy \tag{C.8}
\]

\[
= \int_u^s \int_x^s f_v(x) f(y - x) h(s - y) \, dy \, dx
\]

Using the identity

\[
\int_x^s h'(s - y) h(y - x) \, dy = \int_x^s h(s - y) h'(y - x) \, dy,
\]

note that

\[
h(s - x) h(0) - h \left( \frac{s - x}{2} \right)^2 = \int_{\frac{s - x}{2}}^s \frac{dy}{\int_x^s} \{ h(s - y) h(y - x) \} \, dy
\]

\[
= -\int_x^s h(s - y) h'(y - x) \, dy + 2 \int_{\frac{s - x}{2}}^s h(s - y) h'(y - x) \, dy.
\]

This gives that

\[
\mathbb{P}_\alpha(A_6 | \mathcal{F}_v) - \mathbb{P}_\alpha(A_{\frac{1}{2}} | \mathcal{F}_v) = \int_u^s f_v(x) h \left( \frac{s - x}{2} \right)^2 \, dx + 2 \int_u^s \int_{\frac{s - x}{2}}^s f_v(x) h(s - y) h'(y - x) \, dy
\]

\[
= I_1 + I_2.
\]
Then, our next task is to estimate $I_1$ and $I_2$. Recall Corollary 4.15 and the bounds on $h$ and $f = -h'(t')$ (Lemma 2.14 noting that $K_0(t') = \alpha(1+2\alpha)h(t')$). This gives that there exists a constant $C > 0$ such that with very high probability,

\begin{align}
I_1 &\leq \int_{u}^{s} \frac{C \alpha^{-\epsilon} \ dx}{(x+1)^{3/2}(s-x+1)}; \\
I_2 &\leq \int_{u}^{s} \int_{\frac{u+t}{2}}^{t} \frac{C \alpha^{-\epsilon} \ dy \ dx}{(x+1)^{3/2}(y-x+1)^{3/2}(s-y+1)^{1/2}},
\end{align}

(C.9)

hold for all $0 < u < s \leq \hat{h}$.

Both integrals in the RHS of (C.9) can be estimated based on the idea used in Claim A.14. Thus, the result of Lemma C.8 tells us that

\[ I_1 + I_2 \leq \frac{C \alpha^{-\epsilon} \log s}{(s+1)\sqrt{u+1}} \leq \frac{\alpha^{-2\epsilon}}{\sqrt{(v+1)(u+1)(s+1)}}, \]

where $C > 0$ may take a different value from that in (C.9) but is still an absolute constant. The last inequality follows from the assumption $s \leq \hat{h} \leq \alpha^{-2\epsilon}$.

**Lemma C.8.** There exists an absolute constant $C > 0$ such that for any $0 \leq u < s$,

\begin{align}
I_1' &:= \int_{u}^{s} \frac{dx}{(x+1)^{3/2}(s-x+1)} \leq \frac{C \log s}{(s+1)\sqrt{u+1}}; \\
I_2' &:= \int_{u}^{s} \int_{\frac{u+t}{2}}^{t} \frac{dy \ dx}{(x+1)^{3/2}(y-x+1)^{3/2}(s-y+1)^{1/2}} \leq \frac{C \log s}{(s+1)\sqrt{u+1}}.
\end{align}

**Proof.** We give a proof for the second inequality. The first inequality can be obtained by the same approach and is simpler.

We can rewrite the LHS of the second line by

\[ I_2' = \int_{\frac{u+t}{2}}^{t} \int_{u}^{s} \frac{dy \ dx}{(x+1)^{3/2}(y-x+1)^{3/2}(s-y+1)^{1/2}}. \]

(C.10)

As done in Claim A.14 we divide into two cases, when $t \leq 2s$ and $t > 2s$. In what follows, $C > 0$ denotes an absolute constant that may take different values at different places.

**Case 1.** $s \leq 2u$.

In this case, we can bound the integral in a rather straight-forward way using the expression (C.10):

\[ I_2' \leq \int_{\frac{u+t}{2}}^{t} \frac{C \ dy}{(u+1)^{3/2}(s-y+1)} \leq \frac{C \log s}{(s+1)^{3/2}} \leq \frac{C \log s}{(s+1)\sqrt{v+1}}. \]

**Case 2.** $s > 2u$. 

Noting that $\frac{u+(2y-s)}{2} = y - \frac{s-u}{2}$, we divide the inner integral into two parts, from $u$ to $y - \frac{s-u}{2}$ and from $y - \frac{s-u}{2}$ to $2y - s$. Using (C.10), the first half can be bounded as

$$\int_{x \in \frac{s}{2}}^{x \in \frac{s}{2}} \int_{y \in \frac{s}{2}}^{y \in \frac{s}{2}} dx \, dy \leq \int_{x \in \frac{s}{2}}^{x \in \frac{s}{2}} \left( (x+1)^{3/2}(y-x+1)^{3/2}(s-y+1)^{1/2} \right) dx \, dy \leq \int_{x \in \frac{s}{2}}^{x \in \frac{s}{2}} \left( (u+1)^{3/2}(s-u+2)^{3/2}(s-y+1)^{1/2} \right) dx \, dy \leq \frac{2^{3/2}dy}{(s-u+2)^{3/2}(s-y+1)^{1/2}} \leq \frac{C}{(s+1)^{3/2}}. \quad (C.11)$$

To control the second half, we first note that

$$\int_{y \in \frac{s}{2}}^{y \in \frac{s}{2}} \frac{dx}{(x+1)^{3/2}(y-x+1)^{3/2}(s-y+1)^{1/2}} \leq \frac{2}{(y-(\frac{s-u}{2}+1)^{3/2}(s-y+1)^{1/2}}. \quad (C.12)$$

Thus, splitting the right integral into two parts, from $\frac{s+u}{2}$ to $\frac{1}{2}(s + \frac{s+u}{2})$ and from $\frac{1}{2}(s + \frac{s+u}{2})$ to $s$. This gives that

$$\int_{\frac{s}{2}}^{\frac{s}{2}} \frac{dy}{(y-(\frac{s}{2}))^{3/2}(s-y+1)^{1/2}} \leq \frac{C}{(s+1)^{3/2}}; \quad (C.13)$$

and we obtain the conclusion for Case 2: $s > 2u$ by combining the equations (C.11), (C.12) and (C.13).

**Proof of Lemma C.7.** Recall the definitions of $f_v$, $f$ and $h$ introduced in the beginning of the proof of Lemma C.6. Similarly as (C.7) and (C.8), we can write $\mathbb{P}_\alpha(A_\alpha^2 | \mathcal{F}_v)$ as

$$\mathbb{P}_\alpha(A_\alpha^2 | \mathcal{F}_v) = \int_{x \in v} f_v(x) \int_{y \in f(y-x)} h(s-y) \, dy \, dx.$$ 

As in the previous proof, recall the estimates on $f_v$ (Lemma 4.14) and on $f$ and $h$ (Lemma 2.14). Then, we obtain that the following holds with very high probability:

$$\mathbb{P}_\alpha(A_\alpha^2 | \mathcal{F}_v) \leq \int_{x \in v} \int_{y \in f(y-x)} \frac{Ca^{-t} \, dy \, dx}{(x+1)^{3/2}(s-x+1)^{3/2}(s-y+1)^{1/2}},$$

for all $0 \leq v < u < s \leq \hat{h}$. Then, Lemma C.9 below gives the correct estimate on the double integral and hence we obtain the conclusion.

**Lemma C.9.** There exists an absolute constant $C > 0$ such that for any $0 \leq v < u < s \leq \hat{h}$,

$$\int_{v}^{u} \int_{u}^{s} \frac{dy \, dx}{(x+1)^{3/2}(t-x+1)^{3/2}(s-y+1)^{1/2}} \leq \frac{C \, dy \, dx}{\sqrt{(v+1)(u+1)(s+1)}}.$$


Proof. This follows from the same argument as Claim [A.14] and Lemma [C.8] Namely, one may the following four cases separately:
\[
\{u \leq 2v\} \cap \{s \leq 2u\}; \quad \{u \leq 2v\} \cap \{s < 2u\}; \quad \{u > 2v\} \cap \{s \leq 2u\}; \quad \{u > 2v\} \cap \{s > 2u\}.
\]
Then, when \(u > 2v\) (resp. \(s > 2u\)), split the integral \(\int_u^v\) (resp. \(\int_u^s\)) into two parts, \(\int_{u/2}^u\) and \(\int_{u/2}^u\) (resp. \(\int_{s/2}^s\) and \(\int_{s/2}^s\)), as done in Claim [A.14]. The rest of the argument goes the same as Lemma [C.8] and we omit the details. \(\square\)