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Published in:
Studia Mathematica

DOI:
10.4064/sm8762-5-2017

Publication date:
2018

Document version:
Accepted manuscript

Document license:
Unspecified

Citation for published version (APA):
Kwaśniewski, B. K., & Meyer, R. (2018). Aperiodicity, topological freeness and pure outerness: From group actions to Fell bundles. Studia Mathematica, 241(3), 257-302. https://doi.org/10.4064/sm8762-5-2017

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APERIODICITY, TOPOLOGICAL FREENESS AND PURE OUTERNESS: FROM GROUP ACTIONS TO FELL BUNDLES

BARTOSZ KOSMA KWAŚNIEWSKI AND RALF MEYER

Abstract. We generalise various non-triviality conditions for group actions to Fell bundles over discrete groups and prove several implications between them. We also study sufficient criteria for the reduced section $C^*_r(B)$ of a Fell bundle $B = (B_g)_{g \in G}$ to be strongly purely infinite. If the unit fibre $A := B_e$ contains an essential ideal that is separable or of Type I, then $B$ is aperiodic if and only if $B$ is topologically free. If, in addition, $G = \mathbb{Z}$ or $G = \mathbb{Z}/p$ for a square-free number $p$, then these equivalent conditions are satisfied if and only if $A$ detects ideals in $C^*_r(B)$, if and only if $A^+ \setminus \{0\}$ supports $C^*_r(B)^+ \setminus \{0\}$ in the Cuntz sense. For $G$ as above and arbitrary $A$, $C^*_r(B)$ is simple if and only if $B$ is minimal and pointwise outer. In general, $B$ is aperiodic if and only if each of its non-trivial fibres has a non-trivial Connes spectrum. If $G$ is finite or if $A$ contains an essential ideal that is of Type I or simple, then aperiodicity is equivalent to pointwise pure outerness.

1. Introduction

Several deep results on the relationship between various non-triviality conditions for group actions and the simplicity of reduced crossed products were proved around 1980 by Olesen and Pedersen [40–42], Kishimoto [26–28], and Rieffel [52]. Their powerful results are used, for instance, in [20,24,43,53,56] to study of the ideal structure and pure infiniteness of crossed products for actions of discrete groups.

The main point of this article is to generalise this theory to Fell bundles over discrete groups. This contains (twisted) crossed products for partial group actions as a special case, see [14,15] or Example 3.11 below. In fact, Fell bundles over $G$ model all $C^*$-algebras graded by $G$, and many important $C^*$-algebras come with such gradings. For instance, the relative Cuntz–Pimsner algebras of a $C^*$-correspondence and Doplicher–Roberts algebras are naturally $\mathbb{Z}$-graded, and these gradings are well understood, see [2,30,55]. The Cuntz–Nica–Toeplitz algebra of a product system over a quasi-lattice order is graded by the ambient group, and this grading is exploited, for instance, in [9]. The Cuntz–Pimsner algebras of product systems over Ore semigroups are naturally graded by the group completion of the Ore semigroup, with well understood fibres, see [4,32]. Thus an extension of the classical theory to Fell bundles allows to study the ideal structure and pure infiniteness properties for large classes of $C^*$-algebras.

Let $\alpha: G \to \text{Aut}(A)$ be an action of a discrete group $G$ on a $C^*$-algebra $A$. Popular among the non-triviality conditions for such actions are proper outerness (see [12]) and topological freeness (see [5]). However, proper outerness is only useful

2010 Mathematics Subject Classification. Primary 46L55; Secondary 46L40.

Key words and phrases. group action; Fell bundle; aperiodic; topologically free; purely outer; Connes spectrum; purely infinite; strongly purely infinite; filling family.
if $A$ is separable. Then it is equivalent to topological freeness and to *aperiodicity* by the work of Olesen and Pedersen. Aperiodicity requires a seemingly technical condition for $\alpha_g$ for all $g \in G \setminus \{e\}$ (see Definition 2.8 below), which we call *Kishimoto’s condition* because its role was first highlighted by Kishimoto [27]. The term “aperiodicity” was coined in [37], where single $C^*$-correspondences were considered, and carried over to Fell bundles over discrete groups in [33]. By a result of Kishimoto [28], the aperiodic group actions are the same as the *pointwise spectrally non-trivial* actions of Pasnicu and Philips [43]. All conditions above spectrally non-trivial.

The results about with exactness imply that the property that $A$ detects ideals in $B := A \rtimes_{\alpha, r} G$, that is, that $I \cap A \neq 0$ whenever $I$ is a non-zero ideal in $B$. Their residual versions together with exactness imply that $A$ separates ideals in $B$, that is, $I \cap A \neq J \cap A$ whenever $I, J < B$ are ideals with $I \neq J$, see [56]. Furthermore, aperiodicity implies that elements of $A^+ \setminus \{0\}$ support the non-zero positive elements in the reduced crossed product $A \rtimes_{\alpha, r} G$ in the Cuntz sense; this is exactly the property one uses to detect pure infiniteness, see [31], Subsection 2.8 and [33]. By arguments in [24], residual aperiodicity implies that $A^+$ is a *filling family* for $A \rtimes_{\alpha, r} G$. Filling families are introduced in [25] to detect strong pure infiniteness.

Aperiodic and topologically free Fell bundles over discrete groups have been defined already in [1,32,33]. Several other non-triviality conditions for group actions generalise readily to Fell bundles (see Definition 4.5 below). This includes pointwise pure outerness or pointwise pure universal weak outerness, but apparently not proper outerness. To establish relationships among these pointwise conditions, it suffices to study a single Hilbert bimodule.

We highlight our main achievements. Let $B = (B_g)_{g \in G}$ be a Fell bundle with unit fibre $A := B_e$. If $A$ contains an essential ideal that is separable or of Type I, then $B$ is aperiodic if and only if $B$ is topologically free, if and only if $B$ is pointwise purely universally weakly outer (Theorems 9.8 and 9.1). If, in addition, $G = \mathbb{Z}$ or $G = \mathbb{Z}/p$ for a square-free number $p$, then this is further equivalent to the condition that $A$ detects ideals in $C^*_r(B) = C^*(B)$ (Theorem 9.12). Under the latter assumptions, $B$ is residually aperiodic if and only if $A^+$ is a filling family for $C^*(B)$, if and only if $A$ separates ideals in $C^*(B)$ (Theorem 9.13). We generalise the Connes spectrum and use it to characterise aperiodicity of Fell bundles (Theorem 9.9), and the property that $A$ detects ideals in $C^*(B)$ when $G$ is Abelian (Proposition 9.5). If $G$ is finite or $A$ contains an essential ideal that is simple or of Type I, then $B$ is aperiodic if and only if $B$ is pointwise purely outer (Theorem 9.9). If $G = \mathbb{Z}$ or $G = \mathbb{Z}/p$ for a square-free number $p$, then $C^*(B)$ is simple if and only if $B$ is pointwise purely outer and minimal (Theorem 9.15).

For group actions by automorphisms, most of our results are already contained in the classic articles by Olesen–Pedersen, Kishimoto and Rieffel mentioned above. The results about $G = \mathbb{Z}/p$ with square-free $p$ are not stated explicitly there, but the proofs for $\mathbb{Z}$ evidently cover this case as well. There are many other important results in these articles. The culmination of the work of Olesen and Pedersen in [42, Theorem 6.6] shows that eleven non-triviality properties are equivalent for an automorphism of a separable $C^*$-algebra. The work of Kishimoto adds a few more equivalent properties. Some of the properties of automorphisms that are used in the proofs do not generalise to Hilbert bimodules. Therefore, it seems difficult to prove our main results directly. We proceed differently. We reduce our main results to the special case of group actions by automorphisms. We use that every
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Fell bundle is Morita equivalent to a Fell bundle that comes from a partial action that globalises to an action by automorphisms. Such a \textit{Morita globalisation} for Fell bundles over discrete groups is constructed by Abadie \cite{Abadie} and by Quigg \cite{Quigg} using Takai Duality (see Section \ref{sec:Morita}). In order to show that this construction preserves the non-triviality properties we are interested in, we introduce the concepts of a \textit{Morita covering} and a \textit{dense Morita covering} (see Section \ref{sec:Morita_coverings}). We show that these preserve Kishimoto’s condition and a number of other properties of Hilbert bimodules and Fell bundles (see Proposition \ref{prop:Kishimoto}) and that Abadie’s Morita globalisation is also a Morita covering. This allows us to reduce our main theorems to the special case treated in the classical theory. Instead of separability or simplicity of \( A \), it suffices to assume that \( A \) contains an essential ideal with that property or an essential ideal of Type I.

It is not clear whether the conditions that are relevant for pure infiniteness and strong pure infiniteness are invariant under Morita globalisations. So we study the relationships between them directly. In particular, in Section \ref{sec:strong_pure_infiniteness} we formulate strong pure infiniteness criteria for reduced Fell bundle \( C^* \)-algebras that generalise those in \cite{Kishimoto, Rordam}. The paper is organised as follows. Section \ref{sec:group_actions} collects the classical results about automorphisms and group actions by automorphisms in a way suitable for our later generalisations, with a few small additions. Section \ref{sec:preliminaries} recalls preliminaries on Hilbert bimodules, Fell bundles and their crossed products. Section \ref{sec:non_triviality_conditions} introduces some of the non-triviality conditions for Hilbert bimodules and Fell bundles, as well as notions of residually supporting and filling subalgebras. Section \ref{sec:Morita} contains general criteria for strong pure infiniteness of Fell bundle \( C^* \)-algebras. Section \ref{sec:permanence} studies permanence of non-triviality conditions for single Hilbert bimodules under Morita restrictions and dense Morita coverings. Section \ref{sec:Morita_globalisation} studies a canonical Morita globalisation for a Fell bundle that goes back to Abadie \cite{Abadie} and Quigg \cite{Quigg}. In Section \ref{sec:strong_pure_infiniteness}, we use Morita globalisations and Morita coverings to generalise and extend relationships between non-triviality conditions from single automorphisms to single Hilbert bimodules. In Section \ref{sec:Connes_spectra}, we define the Connes and strong Connes spectra for Fell bundles over Abelian discrete groups and for single Hilbert bimodules. We relate these notions to aperiodicity or pure outerness of Fell bundles, and to detection of ideals in \( C^*_r(B) \). We show that a number of conditions are equivalent if \( G = \mathbb{Z} \) or \( G = \mathbb{Z}/p \) for a square-free number \( p \). We also prove residual versions of our results, which are related to separation of ideals in \( C^*_r(B) \).

\section{1.1. Acknowledgements.} The research leading to these results has been funded by the European Union’s 7th Framework Programme (FP7/2007–2013) under grant agreement number 621724. The first-named author was partially supported by the NCN (National Centre of Science) grant 2014/14/E/ST1/00525. This work was finished while he participated in the Simons Semester at IMPAN – Fundation grant 346300 and the Polish Government MNiSW 2015–2019 matching fund. He also would like to express his gratitude to Suliman Albandik for inviting him to the Mathematisches Institut in Göttingen, where this project started.

\section{2. Group actions by automorphisms}

Throughout this article, \( G \) is a discrete group and \( e \) denotes its unit element. Let \( \alpha : G \to \text{ Aut}(A) \) be an action of \( G \) on a \( C^* \)-algebra \( A \). Let \( \mathcal{H}(A) \) be the set of
non-zero, hereditary subalgebras of $A$, and let
\[ \mathcal{H}^{\alpha}(A) := \{ D \in \mathcal{H}(A) \mid \alpha_g(D) = D \text{ for all } g \in G \}. \]

Let $\mathcal{I}(A)$ be the ideal lattice of $A$, and $\mathcal{I}^{\alpha}(A)$ the sublattice of $\alpha$-invariant ideals.

For the time being, we assume that $G$ is Abelian, so that the Pontryagin dual $\hat{G}$ and various spectra for group actions are defined. The Arveson spectrum of $\alpha$ is the set $\text{Sp}(\alpha)$ of all $\chi \in \hat{G}$ for which there is a net $(a_{\lambda})$ in $A$ with $\|a_{\lambda}\| = 1$ for all $\lambda$ and $\lim \|\alpha_g(a_{\lambda}) - \chi(g)a_{\lambda}\| = 0$ for all $g \in G$, compare \[45\] Proposition 8.1.9(iii)]. The Connes and Borchers spectra were introduced by Olesen \[26\], see also \[45\] Chapter 8]. The Connes spectrum of $\alpha$ is
\[ \Gamma(\alpha) := \bigcap_{D \in \mathcal{H}^{\alpha}(A)} \text{Sp}(\alpha|_{D}). \]

It is a closed subgroup of $\hat{G}$. The Borchers spectrum $\Gamma_{\text{Bor}}(\alpha)$ of $\alpha$ is a similar intersection taken over those $D \in \mathcal{H}^{\alpha}(A)$ for which the ideal $\overline{AD}A$ is essential in $A$. In general, it is the closure of a union of subgroups of $\hat{G}$, see \[45\] Propositions 8.8.4 and 8.8.5]. The strong Connes spectrum $\hat{\Gamma}(\alpha)$ is introduced by Kishimoto \[26\] and used by him in \[27\], \[28\]. Since the group $G$ is discrete, $\hat{\Gamma}(\alpha)$ is just a residual version of $\Gamma(\alpha)$. Namely, \[28\] Proposition 4.1] identifies
\[ (2.1) \quad \hat{\Gamma}(\alpha) = \bigcap_{I \in \mathcal{I}^{\alpha}(A), I \neq A} \Gamma(\alpha|_{A/I}), \]

where $\alpha|_{A/I}$ denotes the induced action on $A/I$. By (2.1), $\hat{\Gamma}(\alpha)$ is a closed subgroup of $\hat{G}$.

We are going to describe the Connes spectrum and the strong Connes spectrum in ways that generalise immediately to Fell bundles. Then we use these spectra to characterise when $A$ detects or separates ideals in $A \rtimes_\alpha G$.

**Proposition 2.1.** Let $\alpha : G \to \text{Aut}(A)$ be an action of a discrete Abelian group, and let $\beta : \hat{G} \to \text{Aut}(A \rtimes_\alpha G)$ be the dual action. Let $\chi \in \hat{G}$.

1. $\chi \in \Gamma(\alpha)$ if and only if $I \cap \beta_{\chi}(I) \neq 0$ for each non-zero ideal $I$ in $A \rtimes_\alpha G$.
2. $\chi \in \hat{\Gamma}(\alpha)$ if and only if $\beta_{\chi}(I) = I$ for any ideal $I$ in $A \rtimes_\alpha G$.

**Proof.** Part (1) is \[40\] Proposition 5.4 and part (2) is almost \[26\] Lemma 3.4]; we may replace $\beta_{\chi}(I) \subseteq I$ by $\beta_{\chi}(I) = I$ because $\hat{\Gamma}(\alpha)$ is a subgroup. \[ \square \]

**Definition 2.2.** Let $A$ be a C*-subalgebra of a C*-algebra $B$. We say that $A$ detects ideals in $B$ if $J \cap A = 0$ implies $J = 0$ for each ideal $J$ in $B$. It separates ideals if $I \cap A = 0$ for two ideals $I, J \subset B$ only happens if already $I = J$.

**Remark 2.3.** A C*-subalgebra $A \subseteq B$ separates ideals in $B$ if and only if $A$ residually detects ideals in $B$, that is, for each ideal $J$ in $B$ the image of $A$ detects ideals in the quotient $B/J$.

**Theorem 2.4** (\[41\] Theorem 2.5]). Let $\alpha : G \to \text{Aut}(A)$ be an action of a discrete Abelian group $G$. Then $A$ detects ideals in $A \rtimes_\alpha G$ if and only if $\Gamma(\alpha) = \hat{G}$.

An action is called minimal if 0 and $A$ are the only invariant ideals in $A$. This is necessary for the crossed product to be simple. We denote by $\mathcal{M}(A)$ the multiplier algebra of $A$, and by $\mathcal{UM}(A)$ the group of its unitaries.
Theorem 2.5. Let \( \alpha : G \to \text{Aut}(A) \) be a minimal action of a discrete Abelian group \( G \). Consider the following conditions:

(2.5.1) \( \alpha \) is simple;
(2.5.2) \( \Gamma(\alpha) = \hat{G} \);
(2.5.3) \( \mathcal{M}(A \rtimes \alpha G) \) has trivial centre;
(2.5.4) there are no \( g \in G \setminus \{e\} \) and \( u \in \mathcal{UM}(A) \) with \( \alpha_g = \text{Ad}_{u} \) and \( \alpha_h(u) = u \) for all \( h \in G \);
(2.5.5) there are no \( g \in G \setminus \{e\} \) and \( u \in \mathcal{UM}(A) \) with \( \alpha_g = \text{Ad}_{u} \).

Then \((2.5.1) \Rightarrow (2.5.2) \Rightarrow (2.5.3) \Rightarrow (2.5.4) \Rightarrow (2.5.5)\). If the group \( G \) is cyclic or finite, then \((2.5.1) \Rightarrow (2.5.4)\) are equivalent. If \( G \) is \( \mathbb{Z} \) or \( \mathbb{Z}/p \) for a square-free number \( p \), then \((2.5.1) \Rightarrow (2.5.5)\) are equivalent.

Proof. The implication \((2.5.1) \Rightarrow (2.5.2)\) is \([11, \text{Theorem 3.1}]\). All central multipliers of a simple \( C^\ast \)-algebra are scalar multiples of 1. Thus \((2.5.1) \Rightarrow (2.5.3)\). The implication \((2.5.5) \Rightarrow (2.5.4)\) is trivial.

We prove by contradiction that \((2.5.3) \Rightarrow (2.5.4)\). So assume that \( \alpha_u = \text{Ad}_u \) for some \( g \in G \setminus \{e\} \) and some \( G \)-invariant \( u \in \mathcal{UM}(A) \). Let \( G \rhd h \mapsto \lambda_h \) denote the canonical homomorphism from \( G \) to the group of unitary multipliers of \( A \rtimes \alpha G \). The unitary \( u^* \lambda_g \in \mathcal{UM}(A \rtimes \alpha G) \) commutes with \( A \subseteq \mathcal{M}(A \rtimes \alpha G) \) because \( u^* \lambda_g a(u^* \lambda_g)^\ast = u^\ast \alpha_g(a)u = a \) for all \( a \in A \), and it commutes with all \( \lambda_h \) because \( \lambda_h(u^* \lambda_g)\lambda_h = \alpha_h(u)\lambda_h \lambda_{gh}^{-1} = u^* \lambda_g \); here we use that \( u \) is \( \lambda_h \)-invariant and \( G \) Abelian. Thus \( u^* \lambda_g \) is a central unitary multiplier of \( A \rtimes \alpha G \). It is not a scalar multiple of 1 because \( g \neq e \). Thus \( \mathcal{M}(A \rtimes \alpha G) \) has non-trivial centre.

Suppose \( G = \mathbb{Z} \) or that \( G \) is finite. We prove by contradiction that \((2.5.4) \Rightarrow (2.5.5)\) if \( G \) is \( \mathbb{Z} \) or \( \mathbb{Z}/p \) with square-free \( p \). Suppose \( \alpha_g = \text{Ad}_u \) for some \( g \in G \setminus \{e\} \) and \( u \in \mathcal{UM}(A) \). Then \( \alpha_g(u) = \text{Ad}_u(u) = u \).

We argue as in the proof of \([11, \text{Theorem 4.6}]\) to show that \( \alpha_u = \text{Ad}_u \) for an \( \alpha \)-invariant unitary multiplier \( w \in \mathcal{UM}(A) \). If \( G = \mathbb{Z}/p \), we lift \( g \) to \( \hat{g} \in \mathbb{Z} \). Since \( p \) does not divide \( \hat{g} \) and \( p \) is square-free, it does not divide \( \hat{g}^2 \) either, so \( \hat{g}^2 \neq e \) in \( G \). Define

\[
 w := u\alpha_1(u)\alpha_2(u)\cdots\alpha_{g-1}(u).
\]

Then \( \text{Ad}_w = \text{Ad}_u \circ \text{Ad}_{\alpha_1(u)} \circ \cdots \circ \text{Ad}_{\alpha_{g-1}(u)}(u) = \text{Ad}_u^{2} = \alpha_{\hat{g}-\hat{g}} \) because \( \text{Ad}_u \circ \text{Ad}_v = \text{Ad}_{uv} \) and \( \text{Ad}_{\alpha(u)} = \alpha \circ \text{Ad}_u \circ \alpha^{-1} \) for any automorphism \( \alpha \) and any unitaries \( u, v \). And \( w \) is invariant under \( \alpha_1 \) and hence under \( \alpha_h \) for all \( h \in G \) because

\[
 \alpha_1(w) = \alpha_1(u)\alpha_2(u)\alpha_3(u)\cdots\alpha_{g}(u) = u^* w\alpha_{\hat{g}}(u) = u^* wu = \alpha_{g}^{-1}(w) = w;
\]

the last step uses \( \alpha_g(u) = u \).

\[ \square \]

Remark 2.6. Let \( G = \mathbb{Z}/4 \) and let A be the CAR algebra, that is, the UHF algebra of type \( 2^\infty \). Phillips has constructed an action \( \alpha : G \to \text{Aut}(A) \) such that \( \alpha_2 \) is inner and \( A \rtimes \alpha \mathbb{Z}/4 \) is simple, see \([46, \text{Example 9.3.9 and Remark 9.3.10}]\). Thus \((2.5.5)\) is strictly weaker than \((2.5.1) \Rightarrow (2.5.4)\) for the finite cyclic group \( \mathbb{Z}/4 \).

Next we consider some properties of a single automorphism \( \alpha \in \text{Aut}(A) \). We will later apply these to group actions by requiring them for all \( g \in G \setminus \{e\} \). Let
\( \Gamma(\alpha), \Gamma_{\text{Bor}}(\alpha) \) and \( \tilde{\Gamma}(\alpha) \) be the corresponding spectra for the \( \mathbb{Z} \)-action generated by \( \alpha \).

**Lemma 2.7.** Let \( \alpha : G \to \text{Aut}(A) \) be an action of a cyclic group \( G \) with generator \( g \in G \). Then

\[
\begin{align*}
(2.2) & \quad \Gamma(\alpha) = \Gamma(\alpha_g), \quad \Gamma_{\text{Bor}}(\alpha) = \Gamma_{\text{Bor}}(\alpha_g), \quad \tilde{\Gamma}(\alpha) = \tilde{\Gamma}(\alpha_g), \\
(2.3) & \quad \Gamma_{\text{Bor}}(\alpha) = \bigcup_{D \in H(\alpha)} \Gamma(\alpha|_D).
\end{align*}
\]

**Proof.** The Arveson spectrum \( \text{Sp}(\alpha) \) of \( \alpha \) coincides with the spectrum \( \text{Sp}(\alpha_g) \) of \( \alpha_g \) treated as an operator on \( A \). A subset of \( A \) is invariant for the action \( \alpha \) if and only if it is invariant under the automorphism \( \alpha_g \). Thus the spectra \( \Gamma(\alpha) \) and \( \Gamma_{\text{Bor}}(\alpha) \) depend only on \( \alpha_g \) and, in view of (2.1), the same applies to \( \tilde{\Gamma}(\alpha) \). This implies (2.2). Equation (2.3) follows from [42, Theorem 3.9]. □

**Definition 2.8** (see [27, Lemma 1.1]). An automorphism \( \alpha \in \text{Aut}(A) \) satisfies Kishimoto’s condition if, for all \( D \in H(A), b \in A \),

\[
(2.4) \quad \inf \{ \|ab\alpha(a)\| \mid a \in D^+, \|a\| = 1 \} = 0.
\]

A similar condition was first used by Connes in the von Neumann algebraic setting and later by Elliott [12] to prove that reduced crossed products for outer group actions on AF-algebras are simple. Kishimoto identified (2.4) as a key step to generalise Elliott’s result to group actions on arbitrary simple \( \mathbb{C}^* \)-algebras.

The following elementary lemma will be used several times:

**Lemma 2.9.** Let \( b \in A^+ \) with \( \|b\| = 1 \) and let \( \varepsilon > 0 \). There is \( d \in \overline{bAb}^+ \) with \( \|d\| = 1 \), such that

\[
(2.5) \quad D_0 := \{ x \in A \mid dx = x = xd \} \in H(\overline{bAb}) \subseteq H(A)
\]

and \( \|bx - x\| < \varepsilon\|x\| \) and \( \|bx\| \geq (1 - \varepsilon)\|x\| \) for all \( x \in D_0 \).

**Proof.** We may assume \( \varepsilon \in (0, 1) \). Given \( \delta \in (0, 1) \), we define

\[
(2.6) \quad f_\delta : [0, 1] \to [0, 1], \quad t \mapsto \begin{cases} 
0 & \text{if } 0 \leq t < 1 - \delta, \\
\frac{\delta}{2}(t - 1 + \delta) & \text{if } 1 - \delta \leq t < 1 - \frac{\delta}{2}, \\
1 & \text{if } 1 - \frac{\delta}{2} \leq t \leq 1.
\end{cases}
\]

This function is continuous, and \( \|f_\delta(b)\| = 1 \) because \( \|b\| = 1 \). Put \( d := f_\varepsilon(b) \in \overline{bAb}^+ \). Then \( D_0 \) in (2.5) is a hereditary subalgebra of \( \overline{bAb} \). It is non-zero because it contains \( f_{\varepsilon/2}(b) \). Let \( x \in D_0 \). Then \( \|bx - x\| = \|bdx - dx\| \leq \|bd - d\|\|x\| \leq \varepsilon\|x\| \) because \( \|(t - 1)f_\varepsilon(t)\| \leq \varepsilon \) for all \( t \in [0, 1] \). Hence \( \|x\| \leq \|bx - x\| + \|bx\| \leq \varepsilon\|x\| + \|bx\| \), that is, \( \|bx\| \geq (1 - \varepsilon)\|x\| \). □

**Theorem 2.10.** For any \( \alpha \in \text{Aut}(A) \) the following conditions are equivalent:

\[
\begin{align*}
(2.10.1) & \quad \alpha \text{ satisfies Kishimoto’s condition: } (2.4) \text{ holds for all } D \in H(A), b \in A; \\
(2.10.2) & \quad (2.4) \text{ holds for all } D \in H(A) \text{ and } b \in M(A); \\
(2.10.3) & \quad (2.4) \text{ holds for all } D \in H(A) \text{ and } b = 1; \\
(2.10.4) & \quad \Gamma_{\text{Bor}}(\alpha|_I) \neq \{1\} \text{ for all non-zero } I \in \mathcal{I}^a(A); \\
(2.10.5) & \quad \text{for } I \in \mathcal{I}^a(A) \text{ with } I \neq 0 \text{ there is } D \in H^a(I) \text{ with } \Gamma(\alpha|_D) \neq \{1\}; \\
(2.10.6) & \quad \sup \{ \|x - \alpha(x)\| \mid x \in D^+, \|x\| = 1 \} = 1 \text{ for all } D \in H(A).
\end{align*}
\]
Proof. The conditions [2.10.2] (2.10.4) are equivalent by [28, Theorem 2.1]. Conditions [2.10.3] and [2.10.6] are equivalent by [42, Theorem 5.1], and [2.10.4] and [2.10.5] are equivalent by [2.3]. We check that [2.10.1] implies [2.10.2] which is rather implicit in [27,28]. Let \( a \in D^+ \) with \( \|a\| = 1 \) and \( \varepsilon > 0 \). Pick \( d \in aAa^+ \) and \( D_0 \in \mathcal{H}(D) \) as in Lemma 2.9. Kishimoto’s condition for \( D_0 \) and \( db \in A \) gives \( x \in D^+ \) with \( xd = x \), \( \|x\| = 1 \), and \( \|xdb(x)\| = \|xdba(x)\| < \varepsilon \).

**Remark 2.11.** Kishimoto [28] calls automorphisms satisfying [2.10.4] freely acting. Pasnicu and Phillips [43] call such automorphisms spectrally non-trivial. We only mentioned the Borchers spectrum to show that the conditions used in [28,43] are equivalent to what we call Kishimoto’s condition. We prefer the formulation using (2.4) because it generalises to Hilbert bimodules.

The following definition names several triviality and non-triviality conditions for automorphisms:

**Definition 2.12.** We call \( \alpha \in \text{Aut}(A) \) inner or universally weakly inner if there are unitaries \( u \in \mathcal{M}(A) \) or in the bidual W*-algebra \( A^{**} \) with \( \alpha = \text{Ad}_u \), respectively. We call \( \alpha \) partly inner or partly universally weakly inner if there are \( 0 \neq I \in \mathcal{T}^\alpha(A) \) and a unitary \( u \in \mathcal{UM}(I) \) or \( I^{**} \), respectively, such that \( \alpha|_I = \text{Ad}_u \). We call \( \alpha \in \text{Aut}(A) \) outer, purely outer or purely universally weakly outer if it is not inner, not partly inner or not partly universally weakly inner, respectively, see [52, Section 1].

We call \( \alpha \) properly outer if \( \|\alpha|_I - \text{Ad}_u\| = 2 \) for any \( 0 \neq I \in \mathcal{T}^\alpha(A) \) and any unitary multiplier \( u \in \mathcal{UM}(I) \) (see [12, Definition 2.1]).

Let \( \hat{A} \) be the spectrum of \( A \). We call \( \alpha \) topologically non-trivial if the set of \( [\pi] \in \hat{A} \) with \( \hat{\alpha}([\pi]) \neq [\pi] \) is dense or, equivalently, \( \{[\pi] \in \hat{A} : [\pi \circ \alpha] = [\pi] \} \) has empty interior in \( \hat{A} \).

**Theorem 2.13.** Consider the following conditions for \( \alpha \in \text{Aut}(A) \):

- (2.13.1) \( \alpha \) satisfies Kishimoto’s condition;
- (2.13.2) \( \alpha \) is purely universally weakly outer;
- (2.13.3) \( \alpha \) is topologically non-trivial;
- (2.13.4) \( \alpha \) is properly outer;
- (2.13.5) \( \alpha \) is purely outer.

Then \( (2.13.2) \Rightarrow (2.13.3) \Rightarrow (2.13.4) \Rightarrow (2.13.5) \Rightarrow (2.13.1) \). If \( A \) is separable then \( (2.13.1) \Rightarrow (2.13.3) \Rightarrow (2.13.4) \Rightarrow (2.13.5) \). If \( A \) is simple then \( (2.13.1) \Rightarrow (2.13.3) \Rightarrow (2.13.4) \Rightarrow (2.13.5) \).

Proof. The implications \( (2.13.2) \Rightarrow (2.13.3) \Rightarrow (2.13.4) \) follow from [41, Lemma 4.3] and the proof of [5, Proposition 1], respectively. The implication \( (2.13.4) \Rightarrow (2.13.5) \) is obvious. To see \( (2.13.5) \Rightarrow (2.13.1) \), assume that \( \alpha|_I = \text{Ad}_u \) for some \( 0 \neq I \in \mathcal{T}^\alpha(A) \) and a unitary \( u \in \mathcal{UM}(I) \). Let \( b = u^* \). Then \( \|aba_\gamma(a)\| = \|aa\| \) for all \( a \in I^+ \) with \( \|a\| = 1 \), which contradicts (2.10.2).

If \( A \) is separable, then \( (2.13.1) \Rightarrow (2.13.4) \) are equivalent by [42, Theorem 6.6]. Indeed, Kishimoto’s condition is equivalent to [42, Theorem 6.6.(ii)] by Theorem 2.10 and conditions [42, Theorem 6.6.(iv)] are (2.13.4) and (2.13.2) respectively.

If \( A \) is simple and \( \alpha \) is outer, then \( \Gamma(\alpha) \neq \{1\} \) by [45, Corollary 8.9.10]. This implies \( (2.10.5) \) which is equivalent to Kishimoto’s condition by Theorem 2.10. Thus \( (2.13.1) \) and \( (2.13.5) \) are equivalent. □
In general, pure outerness does not imply proper outerness, even for separable C*-algebras. We thank George Elliott for explaining the following counterexample.

**Example 2.14.** Let $\mathbb{N}^\infty$ be the set of all sequences $(n_k)_{k \in \mathbb{N}}$ with $n_k \in \mathbb{N}$ for all $k \in \mathbb{N}$ and $n_k = 0$ for all but finitely many $k$. Let $\mathcal{S} = \ell^2(\mathbb{N}^\infty)$, viewed as the infinite tensor product of copies of $\ell^2(\mathbb{N})$. Let $A_n \subseteq \mathcal{B}(\mathcal{S})$ be the unital C*-subalgebra of $\mathcal{B}(\mathcal{S})$ spanned by operators of the form $x \otimes 1$ with $x \in \mathbb{K}(\ell^2(\mathbb{N}^m))$ for $m = 0, 1, \ldots, n$. That is, $x \otimes 1$ acts by $x$ on the tensor product of the first $m$ tensor factors $\ell^2(\mathbb{N})$, and identically on the remaining tensor factors $\ell^2(\mathbb{N})$. We have $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$. Let $A$ be the closure of $\bigcup A_n$. The C*-algebra $A$ is isomorphic to the fixed-point algebra $\mathcal{O}_\infty^T$ for the standard gauge action of $T$ on the Cuntz algebra $\mathcal{O}_\infty$. It is an AF-algebra. Let $I_n \triangleleft A$ be the ideal generated by $\mathbb{K}(\ell^2(\mathbb{N}^n)) \otimes 1$. These ideals form a decreasing chain $A = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ with $\bigcap I_n = 0$. Any non-zero ideal of $A$ is of the form $I_n$ for some $n \in \mathbb{N}$.

Let $P \in \mathcal{B}(\ell^2(\mathbb{N}))$ be the projection onto $\ell^2(2\mathbb{N})$, the even numbers. Let $P_m \in \mathcal{B}(\mathcal{S})$ denote the operator that acts by $P$ on the $m$th tensor factor and identically on the other factors. Let $h := \sum_{m=1}^{\infty} 2^{-m-1} P_m \in \mathcal{B}(\mathcal{S})$. If $a \in \mathbb{K}(\ell^2(N^k)) \otimes 1$, then $[P_m, a]$ vanishes for $k < m$, and $[P_m, a] \in \mathbb{K}(\ell^2(N^k)) \otimes 1$ if $k \geq m$. Thus $[P_m, A] \subseteq A$ for all $m \in \mathbb{N}$ and hence $[h, A] \subseteq A$. The *-derivation $\delta(A) := i[h, A]$ of $A$ generates an automorphism $\alpha := \exp(2\pi i \delta) = \text{Ad}_u$ with $u := \exp(2\pi i h)$.

The automorphism $\alpha$ is universally weakly inner by construction. If $x \in \mathbb{K}(\ell^2(\mathbb{N}^n))$ is non-zero, then $h \cdot (x \otimes 1)$ is not in $A$ because it contains $2^{-m-1} P$ as its $m$th tensor factor for $m > n$, which has distance $2^{-m-1}$ from $\mathbb{K}(\ell^2(\mathbb{N}))$. Hence $h$ is not a multiplier of $I_n$ for any $n \in \mathbb{N}$. Since $\|h\| \leq 1/2$, we get $h$ from $u$ by $h = \log(u)$. So $u$ is not a multiplier of $I_n$ either. If there were another unitary $v \in \text{UM}(I_n)$ with $\text{Ad}_v = \text{Ad}_u$, then $v^* u$ would commute with the image of $I_n$ in $\mathcal{B}(\mathcal{S})$. Since $I_n$ acts irreducibly on $\mathcal{S}$, the unitary $v^* u$ would have to be a scalar multiple of 1, contradicting $u \notin \mathcal{M}(I_n)$. So $\alpha$ is not partly inner.

All the properties of a single automorphism defined above generalise to group actions by requiring them pointwise. Actions that satisfy Kishimoto’s condition pointwise are called aperiodic in \cite{33,37}. Pasnicu and Phillips \cite{43} call such actions pointwise spectrally non-trivial, compare Remark 2.11.

**Definition 2.15.** Let $\alpha : G \to \text{Aut}(A)$ be an action of a discrete group $G$. We call $\alpha$ pointwise outer, pointwise purely outer or pointwise purely universally weakly outer if, for each $g \in G \setminus \{e\}$, the automorphism $\alpha_g$ is outer, purely outer or purely universally weakly outer, respectively. We call $\alpha$ aperiodic if $\alpha_g$ satisfies Kishimoto’s condition for all $g \in G \setminus \{e\}$.

**Definition 2.16.** Let $\alpha : G \to \text{Aut}(A)$ be an action of a discrete group $G$. We call $\alpha$ pointwise topologically non-trivial if $\alpha_g$ is topologically non-trivial for each $g \in G \setminus \{e\}$, that is, for each $g \in G \setminus \{e\}$, the set $F_g := \{[\pi] \in \hat{A} \mid [\pi \circ \alpha] = [\pi]\}$ has empty interior in $\hat{A}$. We call $\alpha$ topologically free if the dual topological action $\hat{\alpha} : G \to \text{Homeo}(\hat{A})$ is topologically free, that is, the union $F_{g_1} \cup F_{g_2} \cup \cdots \cup F_{g_n}$ has empty interior in $\hat{A}$ for any $g_1, \ldots, g_n \in G \setminus \{e\}$ (see \cite[Definition 1]{3})).

The following lemma shows that topological freeness and pointwise topological non-triviality are equivalent in the separable case.

**Lemma 2.17.** Let $\alpha : G \to \text{Aut}(A)$ be an action of a discrete group $G$ on a separable C*-algebra $A$. The following are equivalent:
(2.17.1) $\alpha$ is pointwise topologically non-trivial;
(2.17.2) $\alpha$ is topologically free.

Moreover, if $G$ is countable, then the above conditions are equivalent to:

(2.17.3) $\hat{\alpha}$ is free on a dense subset of $\hat{A}$.

Proof. The implications $\{2.17.3\} \Rightarrow \{2.17.2\} \Rightarrow \{2.17.1\}$ are obvious. Conversely, if $G$ is countable then $\{2.17.1\}$ implies $\{2.17.3\}$ by the proof of [41, Proposition 4.4]. Since $\{2.17.2\}$ holds for $G$ if and only if it holds for all countable subgroups of $G$, the equivalence $\{2.17.1\} \Leftrightarrow \{2.17.2\}$ for countable $G$ implies the same for all $G$. $\Box$

The following theorem highlights the role of aperiodicity and topological freeness.

Theorem 2.18. Let $A$ be a C$^*$-algebra and $G$ a discrete group. If $\alpha: G \to \text{Aut}(A)$ is an aperiodic or topologically free group action, then $A$ detects ideals in the reduced crossed product $A \rtimes_{\alpha, r} G$.

Proof. The statement for topological freeness is [5, Theorem 1]. The statement using aperiodicity is contained, for instance, in [18, Theorem 3.12]. $\Box$

The next theorem asserts that aperiodicity and topological freeness are also necessary for $A$ to detect ideals in $A \rtimes_{\alpha, r} G$ if $A$ is separable and $G$ is cyclic of infinite or square-free order. This fails for most groups $G$, even if $A$ is separable and simple, so that topological non-triviality and aperiodicity are equivalent to various pointwise outerness conditions by Theorem 2.13 and Lemma 2.17. For the group $\mathbb{Z}/4$, the example mentioned in Remark 2.6 is a counterexample: the crossed product is simple although $2 \in \mathbb{Z}/4 \setminus \{0\}$ acts by an inner automorphism. The noncommutative torus $A_\theta$ is stably isomorphic to $\mathbb{K}(\ell^2(\mathbb{Z})) \rtimes \mathbb{Z}^2$ for a $\mathbb{Z}^2$-action on $\mathbb{K}(\ell^2(\mathbb{Z}))$. It is simple for irrational $\theta$ although all automorphisms of $\mathbb{K}(\ell^2(\mathbb{Z}))$ are inner.

Theorem 2.19. Let $G = \mathbb{Z}$ or $G = \mathbb{Z}/p$ for a square-free number $p$, let $A$ be a separable C$^*$-algebra and let $\alpha: G \to \text{Aut}(A)$ be a group action. The following are equivalent:

(2.19.1) $A$ detects ideals in $A \rtimes_{\alpha} G = A \rtimes_{\alpha, r} G$;
(2.19.2) $\alpha$ is aperiodic;
(2.19.3) $\alpha$ is topologically free.

Proof of Theorem 2.19. The statements (2.19.2) and (2.19.3) are equivalent by Theorem 2.13 and Lemma 2.17. If $G = \mathbb{Z}$, then the equivalence of (2.19.1) and (2.19.2) is contained in [42, Theorem 10.4]. We claim that the proof of [42, Theorem 10.4] still works for $G = \mathbb{Z}/p$ if $p$ is square-free and shows that (2.19.1) and (2.19.2) are equivalent. Since (2.19.2) always implies (2.19.1) by Theorem 2.18 we only have to look at how the implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii) in [42, Theorem 10.4] are proved. Let $n \in \mathbb{Z}$ be such that $\alpha^n \in \text{Aut}(A)$ is not properly outer and $p$ does not divide $n$. Then $\alpha^n$ is not purely universally weakly outer by Theorem 2.13. The proof in [42, Proposition 4.2] shows that $n^2 \perp \Gamma(\alpha)$. Since $p$ is square-free and does not divide $n$, it does not divide $n^2$. So $\Gamma(\alpha) \neq \hat{G}$, which is equivalent to (2.19.1) by Theorem 2.4. $\Box$

For each $n > 0$, let $T_n := \{z \in \mathbb{T} \mid z^n = 1\}$ be the roots of unity of order $n$. We also put $T_\infty := \mathbb{T}$. Let $\text{ord}(g)$ be the order of an element $g$ of a group $G$. 
Theorem 2.20. Let $A$ be a $C^*$-algebra, $G$ a discrete group and $\alpha : G \to \text{Aut}(A)$ a group action. The following are equivalent:

1. $\alpha$ is aperiodic;
2. $\Gamma(\alpha_g) = T_{\text{ord}(g)}$ for all $g \in G$;
3. $\Gamma(\alpha_g) \neq \{1\}$ for all $g \in G \setminus \{e\}$;

The above equivalent conditions imply the following:

4. $\alpha$ is pointwise purely outer.

If $G$ is finite or $A$ is simple, then all the conditions (2.20.1) and (2.20.4) are equivalent.

Proof. The implication (2.20.1) $\Rightarrow$ (2.20.3) follows from Theorem 2.10 (condition (2.10.5) holds because $\Gamma(\alpha_g) \subseteq \Gamma(\alpha_g|_D)$ for any restriction $\alpha_g|_D$ of $\alpha_g$). Theorem 2.13 shows that (2.20.1) $\Rightarrow$ (2.20.2). To prove the first part of the theorem, it remains to show that (2.20.1) implies (2.20.2). So assume (2.20.1) and pick $g \in G$. The automorphism $\alpha_g$ generates an action $\beta : \langle g \rangle \to \text{Aut}(A)$ of the subgroup $\langle g \rangle \cong \mathbb{Z}/\text{ord}(g)$ of $G$ generated by $g$, which is aperiodic by (2.20.1). Hence $A$ detects ideals in $A \rtimes_{\beta} \langle g \rangle$ by Theorem 2.18. Thus Theorem 2.4 and Lemma 2.7 imply $\Gamma(\alpha_g) = T_{\text{ord}(g)}$.

If $A$ is simple, then conditions (2.20.1) and (2.20.4) are equivalent by Theorem 2.13. Thus it remains to show that (2.20.4) implies (2.20.2) if $G$ is finite. Let $g \in G \setminus \{e\}$ and let $\beta$ be the action of $\langle g \rangle$ generated by $\alpha_g$ as above. By assumption, $\beta$ is pointwise purely outer. Therefore, $A$ detects ideals in $A \rtimes_{\beta} \langle g \rangle$ by [52, Theorem 1.1]. Hence $\Gamma(\alpha_g) = \Gamma(\beta) = T_{\text{ord}(g)}$ by Theorem 2.4 and Lemma 2.7.

Example 2.21. Let $\bar{\alpha} : \mathbb{Z} \to \text{Aut}(A)$ be the $\mathbb{Z}$-action generated by the automorphism $\alpha$ of $A = O_{\infty,n}$ built in Example 2.14. Since $\alpha$ is universally weakly inner, so are all its powers $\bar{\alpha}_n$ for $n \in \mathbb{Z}$. On the one hand, the action $\bar{\alpha}$ is far from being aperiodic: $\bar{\alpha}_n$ cannot satisfy Kishimoto’s condition for any $n \in \mathbb{Z} \setminus \{0\}$ by Theorem 2.13. On the other hand, $\bar{\alpha}_n$ is purely outer for all $n \in \mathbb{Z} \setminus \{0\}$ by the same argument as for $\alpha$. Thus the last statement in Theorem 2.20 may fail if $G$ is infinite and $A$ is not simple.

3. Hilbert bimodules, Fell bundles, and their crossed products

We recall some basic definitions and results about Hilbert bimodules and Fell bundles. See [15, 51] for more details on Hilbert bimodules, Morita equivalence, and Fell bundles, and also the section on preliminaries in [33].
We denote this Hilbert $b$-module structure and the right Hilbert $B$-module structure $a \cdot b := a\varphi(b)$, $\langle a_1|a_2\rangle_B := \varphi^{-1}(a_1^* \cdot a_2)$ for $a, a_1, a_2 \in A$, $b \in B$. This is an equivalence bimodule.

We view a Hilbert $A, B$-bimodule as an arrow from $B$ to $A$, that is, the source is on the right and the range on the left. This convention is helpful in connection with groupoid $C^*$-algebras and with Fell bundles over groups and inverse semigroups, see \cite{7}. In this article, the direction convention does not matter much. It ensures that the isomorphisms in Example \ref{example3.9} below preserve the natural $G$-gradings.

**Remark 3.3.** The Hilbert bimodule $A\varphi$ is isomorphic through $\varphi^{-1}$ to $B$ with the obvious right Hilbert $B$-module structure and the left Hilbert $A$-module structure by $a \cdot b := \varphi^{-1}(a)b$ and $\langle a_1|b_1|b_2\rangle := \varphi(b_1 b_2^*)$ for $a \in A$ and $b, b_1, b_2 \in B$. We denote this Hilbert $A, B$-bimodule by $\varphi_{-1}B$. The construction $\varphi B$ (without inverse) still gives a $C^*$-correspondence for any morphism $\varphi: A \to B$. This is why the conventions for $\varphi B$ are used, for instance, in \cite{8}. So the conventions in \cite{8} would associate the equivalence bimodule $A\varphi$ to $\varphi^{-1}$ and not to $\varphi$.

We may compose Hilbert bimodules by the interior tensor product, see \cite{35}. The bimodule structure and the inner products induce isomorphisms of Hilbert bimodules

\begin{align}
(3.1) \quad A \otimes_A \mathcal{E} &\xrightarrow{\sim} \mathcal{E}, \\
(3.2) \quad \mathcal{E} \otimes_B B &\xrightarrow{\sim} \mathcal{E}, \\
(3.3) \quad \mathcal{E} \otimes_B \mathcal{E}^* &\xrightarrow{\sim} A\langle \mathcal{E}|\mathcal{E}\rangle \triangleleft A, \\
(3.4) \quad \mathcal{E}^* \otimes_A \mathcal{E} &\xrightarrow{\sim} \langle \mathcal{E}|\mathcal{E}\rangle_B \triangleleft B.
\end{align}

These are part of a bicategory structure with Hilbert bimodules as arrows, see \cite{7}. More generally, if $I \triangleleft A$ and $J \triangleleft B$ are ideals, the multiplication maps restrict to isomorphisms

\begin{align}
(3.5) \quad I \otimes_A \mathcal{E} &\xrightarrow{\sim} I \cdot \mathcal{E} \subseteq \mathcal{E}, \\
\mathcal{E} \otimes_B J &\xrightarrow{\sim} \mathcal{E} \cdot J \subseteq \mathcal{E}.
\end{align}

A Hilbert $A, B$-bimodule $\mathcal{E}$ is an $A\langle \mathcal{E}|\mathcal{E}\rangle, \langle \mathcal{E}|\mathcal{E}\rangle_B$-equivalence bimodule and thus an equivalence between ideals in $A$ and $B$. Conversely, any equivalence between ideals in $A$ and $B$ comes from a unique Hilbert $A, B$-bimodule. Thus Hilbert $A, B$-bimodules are the same as equivalences between ideals in $A$ and $B$. Hilbert bimodules are interpreted in \cite{7} as partial equivalences of $C^*$-algebras because a partial isomorphism is defined as an isomorphism between ideals in $A$ and $B$.

A Hilbert $A, B$-bimodule $X$ induces a dual partial homeomorphism

$$\widehat{X}: \hat{B} \supseteq \langle X|X\rangle_B \xrightarrow{\sim} A\langle \hat{X}|\hat{X}\rangle \subseteq \hat{A}, \quad [\pi] \mapsto [X{-}\text{Ind}_B^A(\pi)],$$

where $X{-}\text{Ind}_B^A(\pi): A \to \mathbb{B}(X \otimes_{\pi} \delta_\pi)$ is the Hilbert space $X \otimes_{\pi} \delta_\pi$ with the obvious representation of $A$. The partial homeomorphism of $\hat{A}$ associated to the identity Hilbert $A$-bimodule $A$ is the identity on $\hat{A}$, and the partial homeomorphism associated to $X \otimes_B Y$ for two composable Hilbert bimodules is the product $\hat{X} \odot \hat{Y}$ of partial homeomorphisms. Hence $\hat{X}^*$ is the partial inverse $\hat{X}^{-1}$ of $\hat{X}$ (see \cite{1,33}).

**Example 3.4.** Let $\varphi: J \xrightarrow{\sim} I$ be an isomorphism between two ideals $I \triangleleft A$ and $J \triangleleft B$. Consider the equivalence $I, J$-bimodule $I_\varphi$ as a Hilbert $A, B$-bimodule.
Then $A$ acts on the left on $I_g$ by multiplication and $B$ acts on the right by $x \cdot b := \varphi(\varphi^{-1}(x)b)$ for $x \in I, b \in B$. The partial homeomorphism $\tilde{\varphi}_g: \tilde{B} \supseteq \tilde{J} \rightrightarrows \tilde{I} \subseteq \tilde{A}$ is the partial inverse to $\varphi: \tilde{A} \supseteq \tilde{I} \rightrightarrows \tilde{J} \subseteq \tilde{B}$. In particular, $\tilde{A}_\alpha = \alpha^{-1}$ for an automorphism $\alpha$.

**Definition 3.5.** Let $X$ be a Hilbert $A$-bimodule. An ideal $I \triangleleft A$ is $X$-invariant if $IX = XI$. Let $\mathcal{I}^X(A)$ be the lattice of $X$-invariant ideals. We call $X$ minimal if $\mathcal{I}^X(A) = \{0, A\}$. If $I \in \mathcal{I}^X(A)$, then $XI$ is naturally a Hilbert $I$/-$I$-bimodule, called the restriction of $X$ to $I$, and the quotient $X/XI$ is naturally a Hilbert $A/I$-bimodule, called the restriction of $X$ to $A/I$.

**Remark 3.6.** Let $f: M \supseteq \Delta \rightrightarrows f(\Delta) \subseteq M$ be a partial bijection of a set $M$. We call a subset $U \subseteq M$ $f$-invariant if $f(U \cap \Delta) = U \cap f(\Delta)$. For any such $U$, let $f|_U$ be the restriction of $f$ to a partial bijection $U \supseteq U \cap \Delta \rightrightarrows U \cap f(\Delta) \subseteq U \cap f(\Delta)$ if $X$ is a Hilbert $A$-bimodule, there is a bijection between $X$-invariant ideals in $A$ and $\tilde{X}$-invariant open subsets of $\tilde{A}$. Moreover, if $I \in \mathcal{I}^X(A)$, then [33, Lemma 2.2] gives $\tilde{X}I = \tilde{X}|_I$ and $\tilde{X}/XI = \tilde{X}|_{\tilde{A}\setminus I}$.

**Definition 3.7** ([13, Definition 16.1]). Let $G$ be a discrete group. A Fell bundle $\mathcal{B}$ over $G$ consists of Banach spaces $(B_g)_{g \in G}$ with bilinear, associative multiplication maps and conjugate-linear, antimultiplicative involutions

$$\cdot: B_g \times B_h \rightarrow B_{gh}, \quad \ast: B_g \rightarrow B_{g^{-1}}$$

for $g, h \in G$, such that $\|ab\| \leq \|a\|\|b\|$ and $\|a^*a\| = \|a\|^2$ for all $a \in B_g$, $b \in B_h$, $g, h \in G$, and for each $a \in B_g$ there is $c \in B_e$ with $a^*a = c^*c$. Then $A := B_e$ is a $C^*$-algebra and $a^*a \geq 0$ in $A$ for all $a \in B_g$, $g \in G$. So each $B_g$ is a Hilbert $A$-bimodule with inner products $A(x|y) := x \cdot y^*$ and $\langle x|y\rangle_A := x^* \cdot y$ for $x, y \in B_g$, $g \in G$. Linear extension of the multiplication and involution defines a $^\ast$-algebra structure on $\bigoplus_{g \in G} B_g$. The (full) cross-section $C^\ast$-algebra $C^\ast_{\mathcal{B}}(\mathcal{B})$ of $\mathcal{B}$ is the $C^\ast$-completion of the $^\ast$-algebra $\bigoplus_{g \in G} B_g$ for its maximal $C^\ast$-norm. The reduced cross-section $C^\ast$-algebra $C^\ast_{\mathcal{B}}(\mathcal{B})_{\text{red}}$ of $\mathcal{B}$ is the $C^\ast$-completion of the $^\ast$-algebra $\bigoplus_{g \in G} B_g$ in the minimal $C^\ast$-norm $\|\cdot\|_r$ that satisfies

$$\|b_e\| \leq \left\| \sum_{g \in G} b_g \right\|_r \quad \text{for all} \quad \sum_{g \in G} b_g \in \bigoplus_{g \in G} B_g. \quad (3.6)$$

The coordinate projection $\bigoplus_{g \in G} B_g \rightarrow B_e = A$ extends to a faithful conditional expectation $E: C^\ast_{\mathcal{B}}(\mathcal{B}) \rightarrow A$. Exel calls a Fell bundle amenable if $A: C^\ast(\mathcal{B}) \rightarrow C^\ast_{\mathcal{B}}(\mathcal{B})$ is an isomorphism. This always happens if the underlying group $G$ is amenable and, in particular, if $G$ is Abelian, see [13, Theorem 20.7].

**Remark 3.8.** Let $\mathcal{B} = (B_g)_{g \in G}$ be a Fell bundle over a discrete Abelian group $G$. There is a dual action $\beta$ of $\hat{G}$ on $C^\ast_{\mathcal{B}}(\mathcal{B})$ defined by $\beta_\chi(x) = \chi(g) \cdot x$ for all $\chi \in \hat{G}$, $g \in G$, $x \in B_g \subseteq C^\ast(\mathcal{B})$. Thus $\bigoplus_{g \in G} B_g$ is the decomposition of $C^\ast(\mathcal{B})$ into its homogeneous subspaces for the $\hat{G}$-action. Conversely, if $B$ is any $C^\ast$-algebra with a continuous $\hat{G}$-action $\beta$, then the homogeneous subspaces $B_g := \{b \in B \mid \beta_\chi(b) = \chi(g) \cdot b \text{ for all } \chi \in \hat{G}\}$ with the multiplication and involution from a Fell bundle $\mathcal{B}$ form a Fell bundle. Moreover, $B \cong C^\ast(\mathcal{B})$ by the obvious $\hat{G}$-equivariant isomorphism. Thus Fell bundles over $G$
are the same as spectral decompositions for $\hat{G}$-actions on $C^*$-algebras. This is a Fell bundle variant of Takai duality.

**Example 3.9** (Crossed products). Let $\alpha: G \to \text{Aut}(A)$ be a group action. Then $A_\alpha := (A_a)_{a \in G}$ with the multiplication maps $(a, g) \cdot (b, h) := (a \alpha(g)(b), g \cdot h)$ and involutions $(a, g)^* := (a^{-1}(a^*), g^{-1})$ for all $a, b \in A$, $g, h \in G$ is a Fell bundle over $G$. We have natural isomorphisms

$$C^*(A_\alpha) \cong A \rtimes_\alpha G \quad C^*_r(A_\alpha) \cong A \rtimes_{\alpha_r} G.$$  

One could also associate to $\alpha$ the Fell bundle $A := (\alpha^{-1}_a A)_{a \in G}$. The isomorphisms $A_{\alpha_a} \cong \alpha_a^{-1} A$ of Hilbert bimodules in Remark 3.3 combine to an isomorphism of Fell bundles $A_\alpha \cong \alpha A$.

**Example 3.10** (Crossed products by partial actions). Example 3.9 generalises naturally to partial actions. Let $\alpha = (\alpha_g)_{g \in G}$ be a partial action of $G$ on a $C^*$-algebra $A$, that is, for each $g \in G$, $\alpha_g: D_{g^{-1}} \to D_g$ is an isomorphism between ideals of $A$ such that $\alpha_e = \text{id}_A$ and $\alpha_{gh} = \alpha_g \circ \alpha_h$ for all $g, h \in G$. Then $A_\alpha := ((D_a)_{a \in G})_{g \in G}$ with involutions as above and the multiplication maps $(a, g) \cdot (b, h) := (g(a\alpha_g^{-1}(a) \cdot b), g \cdot h)$ for all $a \in D_g$, $b \in D_h$, $g, h \in G$ is a Fell bundle over $G$. The crossed products $A \rtimes_\alpha G$ and $A \rtimes_{\alpha_r} G$ may be defined as $C^*(A_\alpha)$ and $C^*_r(A_\alpha)$, respectively, see [15] Proposition 16.28. Fell bundles may be interpreted as partial group actions by Hilbert bimodules, compare [7]. The full and reduced section $C^*$-algebras of a Fell bundle play the role of the full and reduced crossed products for a partial group action.

**Example 3.11** (Twisted cross section algebras). Let $B = (B_g)_{g \in G}$ be a Fell bundle over a discrete group $G$ and let $\omega: G \times G \to \mathbb{T}$ be a 2-cocycle, that is,

$$\omega(g, h)\omega(gh, k) = \omega(g, hk)\omega(h, k) \quad \text{and} \quad \omega(e, g) = \omega(g, e) = 1$$

for all $g, h, k \in G$. Then we deform the multiplication and involution in $B$ by the formulas $b \cdot_\omega c := \omega(g, h)bc$, $b^{\omega} = \omega(g, g^{-1})b^*$ for $b \in B_g$, $c \in B_h$. This gives another Fell bundle $B_\omega = (B_g)_{g \in G}$, see [50] Proposition 3.3. We call $C^*(B_\omega)$ and $C^*_r(B_\omega)$ the full and reduced cross section $C^*$-algebra of $B$ twisted by $\omega$, respectively. If $\alpha = (\alpha_g)_{g \in G}$ is a partial action of $G$ on a $C^*$-algebra $A$, then

$$A \rtimes_{\alpha}^\omega G = C^*(A_{\alpha}^\omega), \quad A \rtimes_{\alpha_r}^\omega G = C^*_r(A_{\alpha}^\omega)$$

are the twisted full and reduced partial crossed products, compare [14].

**Example 3.12** (Crossed products by Hilbert bimodules). Let $X$ be a Hilbert $A$-bimodule. Let $X_0 := A$, $X_n := X^{\otimes_A n}$, and $X_{-n} := (X^*)^{\otimes_A n}$ for $n \in \mathbb{N}$. This becomes a Fell bundle for the obvious involutions and the obvious multiplication maps between $X_n$ and $X_m$ if $n, m$ have the same sign, and more complicated multiplication maps that use the inner product maps (3.3) and (3.4) if the signs are different. The *Hilbert bimodule crossed product* $A \times_X Z$ is the cross-section $C^*$-algebra $C^*(X_{n} \in \mathbb{Z})$ of this Fell bundle $(X_n)_{n \in \mathbb{Z}}$, see [2]. A Fell bundle $(B_n)_{n \in \mathbb{Z}}$ is of this form for some Hilbert bimodule $X$ if and only if $(B_1)^{\otimes_A n} \cong B_n$ for $n > 0$, and then $A = B_0$ and $X = B_1$. Such Fell bundles are called semi-saturated in [13] Definition 4.1.

Let $B = (B_g)_{g \in G}$ be a Fell bundle and let $A := B_0$ be its unit fibre. An ideal $I \triangleleft A$ is $B$-invariant if it is $B_g$-invariant for all $g \in G$, that is, $IB_g = B_gI$ for all
Let $g \in G$. Let $\mathcal{I}^B(A) \subseteq \mathcal{I}(A)$ be the lattice of $\mathcal{B}$-invariant ideals. We call $\mathcal{B}$ minimal if $\mathcal{I}^B(A) = \{0, A\}$. If $I \in \mathcal{I}^B(A)$, then the family of restrictions $\mathcal{B}|_I = (B_g I)_{g \in G}$ forms a Fell subbundle (even an ideal) of $\mathcal{B}$, which we call the restriction of $\mathcal{B}$ to $I$. We restrict the Fell bundle to $A/I$ by taking $\mathcal{B}|_{A/I} := \mathcal{B}/I = (B_g/I)_{g \in G}$ with the induced multiplication maps and involutions.

An ideal $J$ in $C^*_r(\mathcal{B})$ is called graded if $\bigoplus (B_g \cap J)$ is dense in $J$. Let $\mathcal{I}^G(C^*_r(\mathcal{B}))$ denote the lattice of graded ideals. If $G$ is Abelian, then an ideal $J$ in $\mathcal{C}^*(\mathcal{B}) = C^*_r(\mathcal{B})$ is graded if and only if $J$ is invariant under the dual action $\beta: \hat{G} \to \text{Aut}(C^*(\mathcal{B}))$.

**Proposition 3.13.** Let $\mathcal{B}$ be a Fell bundle over $G$. The map

$$\mathcal{I}(C^*_r(\mathcal{B})) \to \mathcal{I}(A), \quad I \mapsto I \cap A,$$

preserves intersections, and its image is $\mathcal{I}^B(A)$, the lattice of $\mathcal{B}$-invariant ideals in $A$. It restricts to a lattice isomorphism $\mathcal{I}^G(C^*_r(\mathcal{B})) \to\to \mathcal{I}^B(A)$ on the sublattice $\mathcal{I}^G(C^*_r(\mathcal{B}))$ of graded ideals. The inverse isomorphism maps $I \in \mathcal{I}^B(A)$ to the graded ideal $C^*_r(\mathcal{B}|_I)$ in $C^*_r(\mathcal{B})$.

**Proof.** This follows from [33 Proposition 3.2 and Corollary 3.3]. \hfill \square

**Corollary 3.14.** The unit fibre $A = B_e$ detects ideals in $C^*_r(\mathcal{B})$ if and only if each non-zero ideal in $C^*_r(\mathcal{B})$ contains a non-zero graded ideal.

**Corollary 3.15.** The unit fibre $A = B_e$ separates ideals in $C^*_r(\mathcal{B})$ if and only if each ideal in $C^*_r(\mathcal{B})$ is graded.

**Definition 3.16.** A Fell bundle $\mathcal{B}$ is called exact if the canonical maps

$$C^*_r(\mathcal{B}) / C^*_r(\mathcal{B}|_I) \to C^*_r(\mathcal{B}|_{A/I})$$

are isomorphisms for all $I \in \mathcal{I}^B(A)$ (see [1] Definition 3.14] or [33 Definition 3.4]). This holds automatically if the group $G$ is exact.

**Proposition 3.17.** The unit fibre $A = B_e$ separates ideals in $C^*_r(\mathcal{B})$ if and only if $\mathcal{B}$ is exact and $A/I$ detects ideals in $C^*_r(\mathcal{B}|_{A/I})$ for each $I \in \mathcal{I}^B(A)$ with $I \neq A$.

**Proof.** Use Corollary 3.15 and [33 Theorem 3.12]. \hfill \square

The Hilbert $A$-bimodules $B_g$ in a Fell bundle induce partial homeomorphisms $\hat{B}_g$ of $\hat{A}$ as above. The range of $\hat{B}_g$ for $g \in G$ is the open subset of $\hat{A}$ corresponding to the ideal $D_g := \cap (B_g B_g) = B_g B_g^{-1}$.

**Lemma 3.18** ([1 Proposition 2.2]). The family $\hat{\mathcal{B}} := (\hat{B}_g)_{g \in G}$ forms a partial action of $G$ on the space $\hat{A}$, that is, $\hat{B}_g : \hat{D}_g^{-1} \to \hat{D}_g$ are homeomorphisms between open subsets of $\hat{A}$ such that $\hat{B}_e = \text{id}_\hat{A}$ and $\hat{B}_{gh}$ extends $\hat{B}_g \circ \hat{B}_h$ for $g, h \in G$.

**Remark 3.19.** Remark 3.6 generalises to Fell bundles in the obvious way: the open subset $\hat{I}$ of $\hat{A}$ is invariant for the partial action $\hat{\mathcal{B}} = (\hat{B}_g)_{g \in G}$ if and only if $I$ is $\mathcal{B}$-invariant, and the partial actions $\hat{\mathcal{B}}|_I$ and $\hat{\mathcal{B}}|_{A/I}$ dual to the restrictions of $\mathcal{B}$ to $I$ and $A/I$ agree with the restrictions $(\hat{B}_g|_I)_{g \in G}$ and $(\hat{B}_g|_{A/I})_{g \in G}$ of the partial action $\hat{\mathcal{B}} = (\hat{B}_g)_{g \in G}$ to $\hat{I}$ and $\hat{A}/I$, respectively.
4. Non-triviality conditions for Hilbert bimodules and Fell bundles

We are going to generalise various non-triviality conditions from automorphisms and group actions to Hilbert bimodules and to Fell bundles over discrete groups. It will become useful in a future project to formulate the following definition and lemma for arbitrary bimodules, without inner products.

**Definition 4.1.** Let $A$ be a $C^*$-algebra and $X$ an $A$-bimodule. Let $\text{Kish}(X) \subseteq X$ be the set of all $x \in X$ with

\[
(4.1) \quad \inf \{ \|axa\| \mid a \in D^+, \|a\| = 1 \} = 0 \quad \text{for all } D \in \mathcal{H}(A).
\]

We say that $x \in X$ satisfies Kishimoto’s condition if $x \in \text{Kish}(X)$, and that $X$ satisfies Kishimoto’s condition if $\text{Kish}(X) = X$.

**Lemma 4.2.** For any $A$-bimodule $X$, the subset $\text{Kish}(X)$ is a closed linear subspace of $X$.

**Proof.** The subset $\text{Kish}(X)$ is clearly closed under limits and under multiplication by scalars. We show that it is closed under addition. Let $x, y \in \text{Kish}(X)$. We want to prove that $x + y \in \text{Kish}(X)$. We may assume $\|x\| = 1$. Let $D \in \mathcal{H}(A)$ and $\varepsilon > 0$. Since $x \in \text{Kish}(X)$, there is $b \in D^+$ with $\|b\| = 1$ and $\|bxb\| < \varepsilon$. Lemma 2.9 gives $d \in (bDb)^+$ such that $D_0 := \{ a \in D \mid da = a = ad \}$ is a non-zero hereditary subalgebra of $D$ and hence of $A$, and $\|a - ba\| = \|a - ab\| < \varepsilon \|a\|$ for all $a \in D_0^+$. Therefore, if $a \in D_0^+$ and $\|a\| \leq 1$, then

$$
\|axa\| \leq \|(a - ab)x\| + \|abx(a - ba)\| + \|abxba\| < 3\varepsilon.
$$

Since $y \in \text{Kish}(X)$ and $D_0 \in \mathcal{H}(A)$, there is $a \in D_0^+ \subseteq D^+$ with $\|a\| = 1$ and $\|aya\| < \varepsilon$. Hence $\|a(x + y)a\| < 4\varepsilon$. This proves that $x + y \in \text{Kish}(X)$. \qed

From now on, $X$ will be a Hilbert bimodule over a $C^*$-algebra $A$. It may be weakly completed to a Hilbert bimodule $X^{**}$ over the bidual $W^*$-algebra $A^{**}$, see \[3\] .

**Definition 4.3.** A Hilbert $A$-bimodule $X$ is topologically non-trivial if the subset $\{[\pi] \in \hat{A} \mid \hat{X}([\pi]) = [\pi] \}$ in $\hat{A}$ has empty interior. Equivalently, any open subset of $(X|X)_A \subseteq \hat{A}$ contains $[\pi]$ with $\hat{X}([\pi]) \neq [\pi]$.

A Hilbert $A$-bimodule $X$ is inner if it is isomorphic to $A$ as a Hilbert bimodule. It is universally weakly inner if $X^{**} \cong A^{**}$ as a Hilbert $A^{**}$-bimodule. It is partly inner if there is $0 \neq I \in \mathcal{I}(A)$ so that $X \cdot I$ is isomorphic to $I$ as a Hilbert $A, I$-bimodule. It is partly universally weakly inner if there is $I \in \mathcal{I}(A)$ so that $X^{**} \cdot I^{**} \cong I^{**}$ as a Hilbert $A^{**}, I^{**}$-bimodule. In both cases, $I$ is $X$-invariant automatically. We call $X$ purely outer or purely universally weakly outer if $X$ is not partly inner or not partly universally weakly inner, respectively.

**Lemma 4.4.** An automorphism $\alpha \in \text{Aut}(A)$ satisfies Kishimoto’s condition if and only if $A_{\alpha}$ does; and it is topologically non-trivial, inner, partly inner, or partly universally weakly inner, respectively, if and only if $A_{\alpha}$ is so.

**Proof.** The assertion for Kishimoto’s condition is trivial. The assertion for topological non-triviality follows from Example 3.4. Let $0 \neq I \in T^*(A)$. Then $IA_{\alpha} = A_{\alpha}I = I_{0\alpha \upharpoonright}$. An isomorphism of Hilbert bimodules $I \cong I_{0\alpha \upharpoonright}$ is, in particular, a unitary operator of left Hilbert $I$-modules. So it is of the form $x \mapsto x \cdot u$
for a unitary $u$ in $\mathbb{B}(I) = \mathcal{M}(I)$. This map is an isomorphism of right Hilbert modules as well if and only if $x \cdot u \cdot \alpha(y) = x \cdot y \cdot u$ for all $x, y \in I$. Equivalently, $\alpha|_I = \text{Ad}_{u \cdot \alpha}$. Thus $\alpha$ is partly inner if and only if $A_\alpha$ is. The case $A = I$ shows that $\alpha$ is inner if and only if $A_\alpha$ is. The Hilbert $A^\ast\ast$-bimodule $(A_\alpha)^{\ast\ast}$ associated to $A_\alpha$ is equal to $(A^{\ast\ast})_{\alpha^{\ast\ast}}$. Hence the same argument shows that $\alpha$ is (partly) universally weakly inner if and only if $A_\alpha$ is. $\square$

All the properties of Hilbert bimodules defined above translate naturally to Fell bundles by considering them pointwise (fibrewise). Aperiodicity for Fell bundles was introduced in [33, Definition 4.1]. Topological freeness, which is not a pointwise condition, was considered for systems dual to saturated Fell bundles in [32, Corollary 6.5], and for general Fell bundles in [1, Section 3]. We are not aware of any source that uses other conditions in the general context of Fell bundles.

**Definition 4.5.** Let $\mathcal{B} = (B_g)_{g \in G}$ be a Fell bundle over a discrete group $G$. Let $A := B_e$ be its unit fibre. We call $\mathcal{B}$ pointwise outer, pointwise purely outer or pointwise purely universally weakly outer if, for each $g \in G \setminus \{e\}$, the Hilbert $A$-bimodule $B_g$ is outer, purely outer or purely universally weakly outer, respectively. We call $\mathcal{B}$ aperiodic if $B_g$ satisfies Kishimoto’s condition for all $g \in G \setminus \{e\}$. We call $\mathcal{B}$ pointwise topologically non-trivial if $B_g$ is topologically non-trivial for all $g \in G \setminus \{e\}$, that is, for each $g \in G \setminus \{e\}$ the subset

$$F_g := \{[\pi] \in \hat{A} \mid \tilde{B}_g([\pi]) = [\pi]\}$$

in $\hat{A}$ has empty interior; here $\tilde{B} = (\tilde{B}_g)_{g \in G}$ is the dual partial action, see Lemma 3.18. We call $\mathcal{B}$ topologically free if $\tilde{B}$ is topologically free, that is, for any $g_1, \ldots, g_n \in G \setminus \{e\}$ the union $F_{g_1} \cup F_{g_2} \cup \cdots \cup F_{g_n}$ has empty interior in $\hat{A}$ (see [36]). This only depends on the partial $G$-action on $\hat{A}$ induced by the Fell bundle.

**Remark 4.6.** Let $\mathcal{B}^\omega = (B_g)_{g \in G}$ be a deformation of $\mathcal{B} = (B_g)_{g \in G}$ by a 2-cocycle $\omega: G \times G \to \mathbb{T}$, see Example 5.11. Then the fibres $B_g, g \in G$, of the bundles $\mathcal{B}$ and $\mathcal{B}^\omega$ coincide not only as Banach spaces, but also as Hilbert bimodules over $A := B_e$. Hence $\mathcal{B}$ has any of the properties in Definition 4.5 if and only if $\mathcal{B}^\omega$ has this property.

**Proposition 4.7.** Let $\mathcal{B} = (B_g)_{g \in G}$ be a Fell bundle over a discrete group $G$. If $\mathcal{B}$ is aperiodic or topologically free, then $A := B_e$ detects ideals in $C^*_r(\mathcal{B})$.

**Proof.** The two assertions are [33, Corollary 4.3] and [1, Corollary 3.4]. $\square$

**Definition 4.8.** For $a, b \in B^+$, we write $a \preceq b$ and say that $a$ supports $b$ if there is a sequence $(x_k)_{k \in \mathbb{N}}$ in $B$ with $\lim x_k^* x_k = a$ (see [10]). We call $a, b \in A^+$ Cuntz equivalent if $a \preceq b$ and $b \preceq a$. Let $A \subseteq B$ be a $C^*$-subalgebra. We say that $A$ supports $B$ if for each $b \in B^+ \setminus \{0\}$ there is $a \in A^+ \setminus \{0\}$ with $a \preceq b$. We say that $A$ residually supports $B$ if for each ideal $I$ in $B$ the image of $A$ in the quotient $B/I$ supports $B/I$ (see [31, Definition 2.39]).

**Lemma 4.9.** If $A$ supports $B$, then $A$ detects ideals in $B$. If $A$ residually supports $B$, then $A$ separates ideals in $B$.

**Proof.** A $C^*$-subalgebra $A$ supports $B$ if and only if for each $b \in B^+ \setminus \{0\}$ there is $z \in B$ such that $0 \neq zb^* \in A$, see [43, Proposition 3.9]. This implies the first assertion. The second one follows from this and Remark 2.3. $\square$
Proposition 4.10 ([33 Corollary 4.4]). If $\mathcal{B}$ is aperiodic, then $A := B_e$ supports ideals in $C^*_r(\mathcal{B})$.

We define residual versions of the non-triviality conditions for Fell bundles:

Definition 4.11. A Fell bundle $\mathcal{B} = (B_g)_{g \in G}$ with unit fibre $A := B_e$ is residually pointwise purely outer, residually pointwise purely universally weakly outer, residually aperiodic, residually pointwise topologically non-trivial or residually topologically free if, for each $\mathcal{B}$-invariant ideal $I \triangleleft A$, the restriction $\mathcal{B}|_{A/I}$ is pointwise purely outer, pointwise purely universally weakly outer, aperiodic, pointwise topologically non-trivial or topologically free, respectively (compare [33 Definition 4.7] and [18 Definition 3.4]).

Remark 4.12. By Remark 3.19 a Fell bundle $\mathcal{B}$ is residually topologically free if and only if each restriction of the dual partial action $\overline{\mathcal{B}}$ to a closed invariant subset is topologically free. Moreover, if $G$ is Abelian and $\overline{\mathcal{A}}$ is Hausdorff, then both residual topological freeness and residual pointwise topological non-triviality coincide with freeness, compare the proof of [43 Proposition 1.10].

Remark 4.13. Let $\mathcal{B}^\omega = (B_g)_{g \in G}$ be a deformation of $\mathcal{B} = (B_g)_{g \in G}$ by a 2-cocycle $\omega$, see Example 3.11. Then $\mathcal{I}^B(A) = \mathcal{I}^{B^\omega}(A)$ and for any $I \in \mathcal{I}^B(A)$ and $g \in G$ we have $B_g|_{A/I} = B_g^\omega|_{A/I}$, see Remark 4.6. Hence $\mathcal{B}$ has any of the properties in Definition 4.11 if and only if $\mathcal{B}^\omega$ has this property.

We add one more separation condition introduced recently by Kirchberg and Sierakowski in [25] to detect strong pure infiniteness.

Definition 4.14 ([25 Definition 4.2]). Let $B$ be a $C^*$-algebra. A subset $\mathcal{F} \subseteq B^+$ is a filling family for $B$ if, for each $D \in \mathcal{H}(B)$ and each $I \in \mathcal{I}(B)$ with $D \not\subseteq I$, there is $z \in B \setminus I$ such that $z^*z \in D$ and $zz^* \in \mathcal{F}$.

It suffices to consider only primitive ideals $I$ in the above definition.

Lemma 4.15. If $A$ is a $C^*$-subalgebra of $B$ and $A^+$ is a filling family for $B$, then $A$ residually supports $B$.

Proof. Let $I \in \mathcal{I}(B)$ and let $q : B \to B/I$ be the quotient map. Let $b \in q(B)^+ \setminus \{0\}$. There is $d \in B^+ \setminus I$ with $q(d) = b$. Let $D := dBd$. Then $q(D) = bq(B)b$ and $D \not\subseteq I$. Hence there is $z \in B \setminus I$ with $z^*z \in D$ and $zz^* \in A^+$. Thus $q(z) \not= 0$ in $B/I$, $q(z)^*q(z) \in bq(B)b$ and $q(z)q(z)^* \in q(A)^+ \setminus \{0\}$. Hence $q(A)$ supports $q(B)$ by [43 Proposition 3.9].

In a future work, we shall prove the converse implication in the above lemma.

Theorem 4.16. Let $\mathcal{B} = (B_g)_{g \in G}$ be an exact Fell bundle over a discrete group $G$ and let $A := B_e$. Consider the following conditions:

1. $\mathcal{A}$ separates ideals of $C^*_r(\mathcal{B})$;
2. each ideal in $\mathcal{B}$ is of the form $C^*_r(\mathcal{B}|_I)$ for a $\mathcal{B}$-invariant ideal $I \triangleleft A$;
3. $\mathcal{B}$ residually supports $C^*_r(\mathcal{B})$;
4. $A^+$ is a filling family for $C^*_r(\mathcal{B})$;
5. $\mathcal{B}$ is residually aperiodic;
6. $\mathcal{B}$ is residually topologically free.

Then $(4.16.6) \Rightarrow (4.16.1) \Leftarrow (4.16.2) \Leftarrow (4.16.3) \Leftarrow (4.16.4) \Leftarrow (4.16.5)$. 
Proof. The equivalence between (4.16.1) and (4.16.2) follows from [33, Theorem 3.12], compare also Propositions 3.13 and 3.17. That (4.16.6) implies (4.16.1) follows from [1, Corollary 3.20], from [33, Corollary 3.23], or by combining Propositions 4.7 and 3.17. Condition (4.16.3) implies (4.16.1) by Lemma 4.9. The implication (4.16.4) → (4.16.3) is Lemma 4.13. Thus we must only prove that (4.16.5) implies (4.16.4). We mimic the proof of [24, Theorem 3.8].

Assume that $\mathcal{B}$ is residually aperiodic. Pick $D \in \mathcal{H}(C^*_r(\mathcal{B}))$ and $J \in \mathcal{I}(C^*_r(\mathcal{B}))$ with $D \not\subseteq J$. We need $z \in A$ with $z^*z \in D$ and $zz^* \in A^+$ \ J. Propositions 4.7 and 3.17 show that $A \subseteq C^*_r(\mathcal{B})$ separates ideals. Hence $I(A) C^*_r(\mathcal{B})) \cong I^\mathcal{B}(A)$ by Proposition 3.13. Since $B$ is exact, we may identify $C^*_r(\mathcal{B}) / J$ with $C^*_r(\mathcal{B} | A/I)$, where $I := A \cap J$. In particular, there is a faithful conditional expectation $E : C^*_r(\mathcal{B}) / J \rightarrow A/I$. Let $q : C^*_r(\mathcal{B}) \rightarrow C^*_r(\mathcal{B}) / J$ be the quotient map. Let $d \in D^+ \setminus J$. Define $b := q(d)$ and $\varepsilon := \frac{1}{3} \|E(b)\| > 0$. Since $\mathcal{B}|_{A/I}$ is aperiodic, [33, Lemma 4.2] gives $x \in (A/I)^+$ satisfying

$$\|x\| = 1, \quad \|bx - xE(b)x\| < \varepsilon, \quad \|xE(b)x\| > \|E(b)\| - \varepsilon = 3\varepsilon.$$  

Now [23, Lemma 2.2] gives a contraction $y \in C^*_r(\mathcal{B}) / J$ with

$$y^*(xbx)y = (xE(b)x - \varepsilon)_+ \in (A/I)^+.$$  

Moreover, $y^*xby \neq 0$ because

$$\|(xE(b)x - \varepsilon)_+\| \geq \|xE(b)x\| - \varepsilon > 2\varepsilon > 0.$$  

There are $c \in A^+$ and a contraction $w \in C^*_r(\mathcal{B})$ with $q(c) = (xE(b)x - \varepsilon)_+$ and $q(w) = xy$. Then $q(c) = y^*xby = q(w^*dw)$. So $c = w^*dw + v$ for some $v \in J$.

Since an approximate unit in $I$ is also one for $J$, there is a contraction $f \in I^+$ with $\|v - f^*v\| < \varepsilon$. Let $1$ denote the formal unit in the unitisations of $A$ or $C^*_r(\mathcal{B})$ and let $g := 1 - f \in A^+$. Then $\|g\| \leq 1$ and

$$\|gw^*dvw - gcg\| = \|gcg\| \leq \|v - f^*v\| < \varepsilon.$$  

Now [23, Lemma 2.2] gives a contraction $h \in C^*_r(\mathcal{B})$ with

$$h^*(gw^*dvw)h = (gcg - \varepsilon)_+ \in A^+.$$  

Let $z := (d^\frac{1}{2}wgh)^*$. Then $z^*z \in D$ and $zz^* = (gcg - \varepsilon)_+ \in A^+$. Moreover, since $q(gc) = q(c + fcf - c - f) = q(c)$, we get

$$\|q(zz^*)\| = \|q((gcg - \varepsilon)_+)\| = (\|q(gc)\| - \varepsilon)_+ = (\|q(c)\| - \varepsilon)_+$$  

$$= (\|xE(b)x - \varepsilon\| + \|E(b)\| - \varepsilon)_+ \neq \|xE(b)x\| - 2\varepsilon > \varepsilon > 0.$$  

Hence $zz^* \notin J$.  

5. STRONG PURE INFINITESIMAL OF REDUCED SECTION C*-ALGEBRAS

We generalise the main results from [24] to Fell bundles and slightly improve the main result of [33]. This shows why residually supporting and filling families are important.

If $a, b \in A$ are elements of a C*-algebra $A$ and $\varepsilon > 0$, we write $a \approx_\varepsilon b$ if $\|a - b\| < \varepsilon$. Infinite and properly infinite elements in $A^+$ are defined in [22]. We recall their equivalent description in [33, Lemma 2.1]. We also introduce the notion of separated pairs of elements in $A^+$.  

**Definition 5.1.** Let $A$ be a C*-algebra and let $a \in A^+ \setminus \{0\}$.  

(1) We call \( a \in A^+ \) infinite in \( A \) if there is \( b \in A^+ \setminus \{0\} \) such that for all \( \varepsilon > 0 \) there are \( x, y \in aA \) with \( x^*x \approx_\varepsilon a \), \( y^*y \approx_\varepsilon b \) and \( x^*y \approx_\varepsilon 0 \).
(2) We call \( a^+ \setminus \{0\} \) properly infinite if for all \( \varepsilon > 0 \) there are \( x, y \in aA \) with \( x^*x \approx_\varepsilon a \), \( y^*y \approx_\varepsilon a \) and \( x^*y \approx_\varepsilon 0 \).
(3) We call \( a, b \in A^+ \) separated in \( A \) if for all \( \varepsilon > 0 \) there are \( x \in aA \) and \( y \in bA \) with \( x^*x \approx_\varepsilon a \), \( y^*y \approx_\varepsilon b \) and \( x^*y \approx_\varepsilon 0 \).

Any properly infinite element is infinite. An element \( a \in A^+ \setminus \{0\} \) is infinite if and only if it is separated from some other element \( b \in A^+ \setminus \{0\} \), and properly infinite if and only if it is separated from itself. Moreover, \( a \in A^+ \setminus \{0\} \) is properly infinite if and only if \( a + I \) in \( A/I \) is either zero or infinite for each ideal \( I \) in \( A \), see [22, Proposition 3.14]. Thus proper infiniteness is residual infiniteness. For the above approximate equalities, we may assume \( x = \sqrt{a}d_1 \) and \( y = \sqrt{b}d_2 \) for some \( d_1, d_2 \in A \); thus we may reformulate the condition of being separated as follows: for all \( \varepsilon > 0 \) there are \( d_1, d_2 \in A \) with \( d_1^*a d_1 \approx_\varepsilon a \), \( d_2^*b d_2 \approx_\varepsilon b \) and \( d_1^*a^{1/2}\sqrt{b}\sqrt{d_2} \approx_\varepsilon 0 \).

The following definition uses characterizations of purely infinite and strongly purely infinite \( C^* \)-algebras in [22, Theorem 4.16] and [23, Remark 5.10].

**Definition 5.2.** A \( C^* \)-algebra \( A \) is purely infinite if each element \( a \in A^+ \setminus \{0\} \) is properly infinite, and strongly purely infinite if each pair of elements \( a, b \in A^+ \setminus \{0\} \) is separated in \( A \).

We present two general criteria that allow to prove that a \( C^* \)-algebra \( B \) is strongly purely infinite by analysing a \( C^* \)-subalgebra \( A \subseteq B \). The first one, established in [25], requires no structural assumptions on the \( C^* \)-algebras, but the conditions on \( A \) are quite strong.

**Proposition 5.3.** Let \( A \) be a \( C^* \)-subalgebra of \( B \) such that \( A^+ \) is a filling family for \( B \). Then \( B \) is strongly purely infinite if and only if each pair of elements \( a, b \in A^+ \) has the matrix diagonalisation property in \( B \), that is, for each \( x \in B \) with \( \begin{pmatrix} a & x \\ x^* & b \end{pmatrix} \in M_2(B)^+ \) and each \( \varepsilon > 0 \) there are \( d_1 \in B \) and \( d_2 \in B \) such that

\[
d_1^*a d_1 \approx_\varepsilon a, \quad d_2^*b d_2 \approx_\varepsilon b, \quad d_1^*x d_2 \approx_\varepsilon 0.\]

**Proof.** The subset \( A^+ \) is closed under \( \varepsilon \)-cut-downs, so that [25, Lemma 5.4] applies here. Therefore, it suffices to check the “matrix diagonalisation property” needed in [25, Theorem 1.1] for pairs of elements. Hence the claim follows from [25, Theorem 1.1].

The matrix diagonalisation described in the above proposition is a strong version of the separation of elements introduced in Definition 5.1. It is hard to check it in practice. However, if “the ratio of ideals in \( B \) to projections in \( A \) is not large” we may characterise when \( B \) is strongly purely infinite using much weaker conditions. The ideal property is one of the conditions that guarantee that the ideals and projections in a \( C^* \)-algebra are well-balanced. We recall that a \( C^* \)-algebra \( A \) has the ideal property if the set of projections in \( A \) separates ideals in \( A \). A \( C^* \)-algebra with the ideal property is purely infinite if and only if it is strongly purely infinite by [44, Proposition 2.14].

The proof of the following proposition uses known arguments, see, for instance, the proofs of [31, Proposition 2.46] and [33, Theorem 4.10].

**Proposition 5.4.** Assume that \( A \subseteq B \) residually supports \( B \). Let \( \mathcal{I}^B(A) := \{ J \cap A \mid J \triangleleft B \} \) and assume that \( \mathcal{I}^B(A) \) is finite or that projections in \( A \) separate
the ideals in \( \mathcal{I}^B(A) \). Then \( B \) is purely infinite if and only if each element in \( A^+ \setminus \{0\} \) is properly infinite in \( B \). Moreover, if \( B \) is purely infinite, then it has the ideal property, and so \( B \) is strongly purely infinite.

**Proof.** We may assume that each element in \( A^+ \setminus \{0\} \) is properly infinite in \( B \). It suffices to prove that \( B \) is purely infinite and has the ideal property. Then it is strongly purely infinite by \cite{4} Proposition 2.14. By Lemma 4.9, \( A \) separates ideals in \( B \). In particular, restriction to \( A \) is a lattice isomorphism \( \mathcal{I}(B) \cong \mathcal{I}^B(A) \).

Suppose first that \( \mathcal{I}^B(A) \cong \mathcal{I}(B) \) is finite. Then there is a chain \( 0 = J_0 \subset J_1 \subset \cdots \subset J_n = B \) of ideals in \( B \) such that \( J_i/J_{i-1} \) is simple for \( i = 1, \ldots, n \). Fix \( i \) and let \( q: B \to B/J_i \) be the quotient map. For any \( b \in q(J_i)^+ \setminus \{0\} \) there is \( a \in q(A)^+ \setminus \{0\} \) such that \( a \lesssim b \) in \( B/J_i \). Thus \( a \in q(J_i \cap A)^+ \setminus \{0\} \). Since \( a \) is the image under \( q \) of a properly infinite element in \( A^+ \), it is properly infinite. Since \( q(J_i) \) is simple, \( b \lesssim a \) by \cite{22} Proposition 3.5(ii)]. Hence \( b \) is Cuntz equivalent to \( a \). Thus \( b \) is properly infinite in \( q(J_i) \). It follows that \( q(J_i) \) is pure infinite. Since pure infiniteness is closed under extensions by \cite{22} Theorem 4.19, \( B \) is purely infinite. Since \( B \) has finite ideal structure, \( B \) has the ideal property by \cite{33} Lemma 4.9.

Now suppose that projections in \( A \) separate the ideals in \( \mathcal{I}^B(A) \cong \mathcal{I}(B) \). Then \( B \) has the ideal property. Let \( b \in B^+ \setminus \{0\} \) and let \( J \) be an ideal in \( B \) with \( b \notin J \). By \cite{22} Proposition 3.14, it suffices to show that the image of \( b \) under the quotient map \( q: B \to B/J \) is infinite in \( q(B) \). We may find \( a \in A^+ \setminus J \) such that \( q(a) \lesssim q(b) \). By \cite{22} Proposition 3.14, \( q(a) \) is properly infinite in \( q(B) \). By our assumption, we may find a non-zero projection \( p \) that belongs to \( A \cap B/J \) but not to \( I := A \cap J \). Since \( q(a) \) is properly infinite and \( q(p) \) belongs to the ideal \( q(B)q(a)q(B) \), we get \( q(p) \lesssim q(a) \) by \cite{22} Proposition 3.5(ii)]. Hence \( q(p) \lesssim q(b) \). Since \( q(p) \) is a properly infinite projection, \( b \) is infinite by \cite{22} Lemma 3.17, see also \cite{22} Lemma 3.12(iv)].

The analogues of (properly and residually) infinite elements for Fell bundles are defined in \cite{33} Definition 5.1. We add strongly separated pairs to this list:

**Definition 5.5.** Let \( B = \{B_g\}_{g \in G} \) be a Fell bundle and let \( A := B_e \).

1. We call \( a \in A^+ \setminus \{0\} \) **\( B \)-infinite** if there is \( b \in A^+ \setminus \{0\} \) such that for each \( \varepsilon > 0 \) there are \( n, m \in \mathbb{N} \) and \( t_i \in G \), \( a_i \in aB_{t_i} \) for \( i = 1, \ldots, n + m \) such that

\[
a \approx_{\varepsilon} \sum_{i=1}^{n} a_i^* a_i, \quad b \approx_{\varepsilon} \sum_{i=n+1}^{n+m} a_i^* a_i \quad \text{and} \quad a_i^* a_j \approx_{\varepsilon/\max\{n^2, m^2\}} 0 \quad \text{for} \ i \neq j.
\]

We call \( a \in A^+ \setminus \{0\} \) **residually \( B \)-infinite** if \( a + I \) is \( B|_{A/I} \)-infinite in \( A/I \) for all \( I \in \mathcal{I}^B(A) \) with \( \alpha \notin I \).

2. We call \( a \in A^+ \setminus \{0\} \) **properly \( B \)-infinite** (or \( B \)-paradoxical) if for each \( \varepsilon > 0 \) there are \( n, m \in \mathbb{N} \) and \( a_i \in aB_{t_i} \) for \( i = 1, \ldots, n + m \) such that

\[
a \approx_{\varepsilon} \sum_{i=1}^{n} a_i^* a_i, \quad a \approx_{\varepsilon} \sum_{i=n+1}^{n+m} a_i^* a_i \quad \text{and} \quad a_i^* a_j \approx_{\varepsilon/\max\{n^2, m^2\}} 0 \quad \text{for} \ i \neq j.
\]

3. We call \( a, b \in A^+ \setminus \{0\} \) **strongly \( B \)-separated** if for each \( c \in B_g \), \( g \in G \), and each \( \varepsilon > 0 \) there are \( n, m \in \mathbb{N} \) and \( s_i \in G \), \( a_i \in aB_{s_i} \) for \( i = 1, \ldots, n \) and
$t_j \in G, b_j \in bBt_j$ for $j = 1, \ldots, m$ such that 

$$a \approx \varepsilon \sum_{i=1}^{n} a_i^* a_i, \quad b \approx \varepsilon \sum_{j=1}^{m} b_j^* b_j \quad \text{and} \quad a_i^* c b_j \approx \varepsilon/\varepsilon_{nm} 0 \quad \text{for all } i, j$$

and $a_i^* a_j \approx \varepsilon/\varepsilon_{i,j}$ and $b_i^* b_j \approx \varepsilon/\varepsilon_{i,j}$ for all $i, j$ with $i \neq j$.

**Remark 5.6.** Let $\alpha: G \to Aut(A)$ be a group action and let $\mathcal{B} := A_\alpha$ be the associated Fell bundle. By [24, Remark 5.4], the action $\alpha: G \to Aut(A)$ is $G$-separating in the sense of [24, Definition 5.1] if and only each pair $a, b \in A^+ \setminus \{0\}$ is strongly $\mathcal{B}$-separated with $n = m = 1$. Allowing arbitrary $n$ and $m$ gives a weaker condition.

**Lemma 5.7.** Let $\mathcal{B} = \{B_g\}_{g \in G}$ be a Fell bundle. Put $A := B_e$ and let $B = \bigoplus_{g \in G} B_g$ be any $C^*$-completion. If each pair of elements $a, b \in A^+ \setminus \{0\}$ is strongly $\mathcal{B}$-separated, then each pair of elements $a, b \in A^+ \setminus \{0\}$ has the matrix diagonalisation in $B$ described in Proposition 5.3.

**Proof.** Let $C := \bigcup_{g \in G} B_g$ and $S := \bigoplus_{g \in G} B_g$. Since each pair of elements in $A^+ \setminus \{0\}$ is strongly $\mathcal{B}$-separated, each pair of elements $a, b \in A^+ \setminus \{0\}$ has the matrix diagonalisation property with respect to $C$ and $S$ as introduced in [25, Definition 4.6]. Indeed, let $x \in B_g$ be such that $(a^* x^* b) \in M_2(B)^+$ and let $\varepsilon > 0$. Let $a_i \in aB_e$ and $b_j \in bB_e$ satisfy the conditions described in Definition 5.3 with $c := a^{1/2} x b^{1/2}$. We may assume $a_i = a^{1/2} x_i$ and $b_j = b^{1/2} y_j$ for some $x_i, y_j$. Let

$$d_1 := \sum_{i=1}^{n} x_i, \quad d_2 := \sum_{j=1}^{n} y_j.$$

The estimates in Definition 5.3 imply

$$d_1^* a d_1 \approx_{2\varepsilon} a, \quad d_2^* b d_2 \approx_{2\varepsilon} b, \quad d_1^* x d_2 \approx_{\varepsilon} 0.$$

This proves our claim.

Clearly, $S$ is a multiplicative subsemigroup of $B$, $S^*CS \subseteq C$, $ASA \subseteq S$, and $\text{span}\{C\} = B$. Thus [25, Lemma 5.6], implies that each pair of elements $a, b \in A^+ \setminus \{0\}$ has the matrix diagonalisation property in $B$. \hfill $\Box$

**Theorem 5.8.** Let $\mathcal{B} = \{B_g\}_{g \in G}$ be a residually aperiodic, exact Fell bundle. Let $A := B_e$. Then $C_\alpha^*(\mathcal{B})$ is strongly purely infinite if one of the following conditions holds:

1. (5.8.1) each pair of elements in $A^+ \setminus \{0\}$ is strongly $\mathcal{B}$-separated;
2. (5.8.2) $\mathcal{T}^B(A)$ is finite and each element in $A^+ \setminus \{0\}$ is Cuntz equivalent in $C_\alpha^*(\mathcal{B})$ to a residually $\mathcal{B}$-infinite element;
3. (5.8.3) projections in $A$ separate ideals in $\mathcal{T}^B(A)$ and each element in $A^+ \setminus \{0\}$ is Cuntz equivalent in $C_\alpha^*(\mathcal{B})$ to a residually $\mathcal{B}$-infinite element;
4. (5.8.4) $A$ is of real rank zero and each non-zero projection in $A$ is Cuntz equivalent in $C_\alpha^*(\mathcal{B})$ to one which is residually $\mathcal{B}$-infinite.

**Proof.** By Theorem 4.16, $A^+$ is a filling family for $C_\alpha^*(\mathcal{B})$, and $A$ residually supports $C_\alpha^*(\mathcal{B})$. Therefore, (5.8.1) implies that $C_\alpha^*(\mathcal{B})$ is strongly purely infinite by Lemma 5.7 and Proposition 5.3. If we assume (5.8.2) or (5.8.3) then $C_\alpha^*(\mathcal{B})$ is strongly purely infinite by Proposition 5.4. the assumption that $a \in A^+ \setminus \{0\}$ is Cuntz equivalent to a residually $\mathcal{B}$-infinite element implies that $a$ is properly...
infinite in $C^*_r(B)$. By [31, Lemma 2.44], condition (5.8.4) implies that each element in $A^+ \setminus \{0\}$ is properly infinite in $C^*_r(B)$. Hence $C^*_r(B)$ is strongly purely infinite, again by Proposition 5.3.

**Remark 5.9.** Let $B^\omega = (B_g)_{g \in G}$ be a deformation of a Fell bundle $B = (B_g)_{g \in G}$ by a 2-cocycle $\omega$. For any $b \in B_g$, we have $b^* \omega \cdot b = b^*b \in A := B_e$. Thus every property of $a \in A^+ \setminus \{0\}$ described in Definition 5.5 holds in $B$ if and only if it holds in $B^\omega$. Accordingly, by Remark 4.13 if $G$ is exact or $B$ is minimal, then the assumptions and conditions in Theorem 5.8 hold for $B$ if and only if they hold for $B^\omega$. The pure infiniteness criteria in Theorem 5.8 when applied to crossed products are weaker than those in [18,20,24,34,53], compare [33, Corollary 5.14]. In particular, the pure infiniteness criteria in the above sources remain valid also in the twisted case. Moreover, twisted $k$-graph $C^*$-algebras [29, Definition 7.4] are modelled in a natural way by twisted Fell bundles, see [50, Corollary 4.9]. Therefore, Theorem 5.8 can be used to analyse when such algebras are purely infinite, compare [33, Theorem 7.9].

6. Morita restrictions and coverings of Hilbert bimodules

First, we prove that Kishimoto’s condition is invariant under Morita equivalence and preserved by restriction to possibly non-invariant ideals. We combine both processes in one step, which we call Morita restriction.

**Proposition 6.1.** Let $Y$ be a Hilbert $B$-bimodule and $E$ a Hilbert $A,B$-bimodule that is full over $A$. Then $X := E \otimes_B Y \otimes_B E^*$ is a Hilbert $A$-bimodule. If $Y$ satisfies Kishimoto’s condition, then so does $X$.

**Proof.** Let $x := e \otimes_B y_0 \otimes_B f^* \in E \otimes_B Y \otimes_B E^*$ with $e, f \in E$, $y_0 \in Y$. Elements of this form are linearly dense in $X$ by definition. Hence, by Lemma 4.2, it suffices to check Kishimoto’s condition for $x$ of this form. Fix $D \in \mathcal{H}(A)$ and $\varepsilon > 0$. We are going to find $a \in D^+$ with $\|a\| = 1$ and $\|axa\| < \varepsilon$.

Let $\mathcal{F} := D \cdot E = \{d \cdot x \mid d \in D, x \in E\}$. This is a Hilbert $B$-submodule of $E$. The left action of $A$ on $E$ gives an isomorphism $A \cong \mathbb{K}(E)$ because $E$ is full over $A$. We claim that this isomorphism maps $D$ onto $\mathbb{K}(\mathcal{F})$. Since $D = DAD$, it maps $D$ onto $D\mathbb{K}(E)D$, which is the closed linear span of $d_1|x\rangle\langle y|d_2^* = |d_1x\rangle\langle d_2y|$ for $d_1, d_2 \in D$, $x, y \in E$. This is the same as the closed linear span of $|x\rangle|y\rangle$ for $x, y \in \mathcal{F}$, which is $\mathbb{K}(\mathcal{F})$, viewed as a subalgebra of $\mathbb{K}(E)$.

So $\mathcal{F} \neq \{0\}$ and there is $\eta \in \mathcal{F}$ with $\|\eta\| = 1$. Lemma 2.9 for $|\eta| := \sqrt{\langle \eta, \eta \rangle}_E \in B$ gives $D_0 \in \mathcal{H}([\eta]|B|\eta])$ with $\|\eta|b\| \geq (1 - \varepsilon)\|b\|$ for all $b \in D_0$. If $b \in D_0$, then

\begin{equation}
\|\eta b\|^2 = \|\langle \eta|b\rangle e\|_E = \|b^*|\eta|2b\| = \|\|\eta|b\|^2 \geq (1 - \varepsilon)^2\|b\|^2.
\end{equation}

Put $y := \langle \eta|e\rangle_B \cdot y_0 \cdot (f|\eta\rangle)_B \in Y$.

Kishimoto’s condition for $Y$ gives $b \in D_0^+$ with $\|b\| = 1$ and $\|byb\| < \varepsilon(1 - \varepsilon)^2$. Let $b_0 := b/\|\eta b\| \in D_0^+$, so that $\|\eta b_0\| = 1$. Equation (6.1) implies $\|b_0y_0b_0\| < \varepsilon$. The rank-one operator $|\eta b_0\rangle\langle \eta b_0|$ belongs to $\mathbb{K}([\eta]|B|\eta])^+$ and has norm $\|\eta b_0\|^2 = \|\eta b_0\|^2 = 1$. The isomorphism $\mathbb{K}(\mathcal{F}) \cong D$ maps it to an element $a \in D^+$ with
\[ \|a\| = 1 \text{ and} \]
\[ \|axa\| = \|\eta b_0 \langle \eta b_0 | e \otimes y_0 \otimes f^* \eta b_0 \rangle \eta b_0 \| \]
\[ = \|\eta b_0 \cdot \langle \eta b_0 | e \rangle_B \otimes y_0 \otimes (\eta b_0 \cdot \langle \eta b_0 | f \rangle_B)^* \| \]
\[ = \|\eta b_0 \otimes b_0^* \langle \eta | e \rangle_B \cdot y_0 \cdot (f| \eta)_{Bb_0} \otimes (\eta b_0)^* \| \]
\[ = \|\eta b_0 \otimes b_0 b_0^* \eta b_0 \otimes (\eta b_0)^* \| < \varepsilon. \]

Hence \( x \in \text{Kish}(X) \).

**Definition 6.2.** In the situation of Proposition 6.1, we call \( X \) a *Morita restriction* of \( Y \). If, in addition, \( \mathcal{E} \) is an equivalence bimodule, we call \( X \) and \( Y \) *Morita equivalent*.

**Corollary 6.3.** If \( X \) and \( Y \) are Morita equivalent Hilbert bimodules, then \( X \) satisfies Kishimoto’s condition if and only if \( Y \) does.

Our Morita equivalences between Hilbert bimodules are the same as in [2]:

**Lemma 6.4.** An equivalence \( A-B \)-bimodule \( \mathcal{E} \) witnesses an equivalence between \( X \) and \( Y \) if and only if \( X \otimes_A \mathcal{E} \cong \mathcal{E} \otimes_B Y \).

**Proof.** On the one hand, \( X \cong \mathcal{E} \otimes_B Y \otimes_B \mathcal{E}^* \) implies
\[ X \otimes_A \mathcal{E} \cong \mathcal{E} \otimes_B Y \otimes_B \mathcal{E}^* \otimes_A \mathcal{E} \cong \mathcal{E} \otimes_B Y \otimes_B B \cong \mathcal{E} \otimes_B Y \]
by (3.4) and (3.2). On the other hand, \( X \otimes_A \mathcal{E} \cong \mathcal{E} \otimes_B Y \) and (3.3) and (3.2) imply
\[ \mathcal{E} \otimes_B Y \otimes_B \mathcal{E}^* \cong X \otimes_A \mathcal{E} \otimes_B \mathcal{E}^* \cong X \otimes_A A \cong X. \]

**Remark 6.5.** If \( \mathcal{E} \) is only full over \( A \), then it is an equivalence between \( A \) and the ideal \( J := \{ \mathcal{E}| \mathcal{E} \}_{B} \triangleleft B \). Then \( \mathcal{E} = \mathcal{E} \cdot J \cong \mathcal{E} \otimes_B J \) by (3.5), and hence
\[ X := \mathcal{E} \otimes_B Y \otimes_B \mathcal{E}^* \cong \mathcal{E} \otimes_B J \otimes_B Y \otimes_B J \otimes_B \mathcal{E}^* \cong \mathcal{E} \otimes_B (J \cdot Y \cdot J) \otimes_B \mathcal{E}^*. \]

Thus \( \mathcal{E} \) witnesses a Morita equivalence between \( X \) and the Hilbert \( J \)-module \( J Y J \).

We may split a Morita restriction into two steps. First, we restrict the Hilbert \( B \)-module \( Y \) to a Hilbert \( J \)-module \( J Y J \) for an ideal \( J \) in \( B \); secondly, we replace this by a Morita equivalent Hilbert bimodule. Restriction to any ideal is a special case of Morita restriction.

**Remark 6.6.** Let \( C = (C_g)_{g \in G} \) be a Fell bundle with unit fibre \( C := C_e \). Let \( \mathcal{E} \) be a Hilbert \( A,C \)-bimodule that is full over \( A \). We transfer the Fell bundle \( C \) to a Fell bundle \( \mathcal{E} \mathcal{C}^* := (\mathcal{E} \otimes_C C_g \otimes_C \mathcal{E}^*)_{g \in G} \), which has the unit fibre \( \mathcal{E} \otimes_C C \cong \mathcal{E} \otimes_C \mathcal{E}^* \cong \mathcal{E} \otimes_C \mathcal{E}^* \cong \mathcal{E} \) by (3.3); it is equipped with the multiplication maps and involutions defined by
\[ (x_1 \otimes (c_1, g_1) \otimes y_1^*) \cdot (x_2 \otimes (c_2, g_2) \otimes y_2^*) := x_1 \otimes (c_1, g_1) \cdot ((y_1| x_2)_C \cdot c_2, g_2) \otimes y_2^*, \]
\[ (x_1 \otimes (c_1, g_1) \otimes y_1^*)^* := y_1 \otimes (c_1, g_1)^* \otimes x_1^* \]
for \( x_1, x_2, y_1, y_2 \in \mathcal{E}, c_1, c_2 \in C, g_1, g_2 \in G \). Here \( (c_i, g_i) \) denotes \( c_i \) viewed as an element of the fibre \( C_{g_i} \) of our Fell bundle at \( g_i \), and \( y_i^* \) denotes \( y_i \) viewed as an element of the complex conjugate Banach space \( \mathcal{E}^* \). If \( C := (C_{g_f})_{g \in G} \) is the Fell
bundle associated to a group action $\gamma : G \to \text{Aut}(C)$, then the multiplication and involution on $\mathcal{E}_i \mathcal{E}_i^*$ become

$$(x_1 \otimes (c_1, g_1) \otimes y_1^*) \cdot (x_2 \otimes (c_2, g_2) \otimes y_2^*) = x_1 \otimes (c_1 \gamma_{g_1}((y_1|x_2) C \cdot c_2), g_1 g_2),$$

$$(x_1 \otimes (c_1, g_1) \otimes y_1^*)^* = y_1 \otimes (\gamma_{g_1^{-1}}(c_1^*), g_1^{-1}) \otimes x_1^*.$$

Our next goal is Proposition 6.8, which allows, roughly speaking, to detect various non-triviality conditions for a Hilbert bimodule $X$ by looking at other Hilbert bimodules that are Morita equivalent to pieces of $X$. It combines various permanence properties of Kishimoto’s condition. The first property is Morita invariance. The second is invariance in both directions under passing to essential ideals. The third is a locality condition: if $A$ is a sum of ideals, then $X$ satisfies Kishimoto’s condition if and only if its restrictions to all these ideals do so. The proofs of our main theorems will show the usefulness of the last technical property. The second property is related to the equivalence of (2.10.1) and (2.10.2) because $B \subseteq \mathcal{M}(I)$ if $I$ is an essential ideal in $B$.

**Definition 6.7.** Let $A$ be a C*-algebra and $X$ a Hilbert $A$-bimodule. Let $S$ be a set. For each $i \in S$, let $B_i$ be a C*-algebra and $\mathcal{E}_i$ a Hilbert $B_i, A$-bimodule. Define $I := \langle (A\langle X \rangle X) + (X \langle X \rangle A) \rangle \triangleleft A$, $K_i := \langle \mathcal{E}_i \mathcal{E}_i^* \rangle \triangleleft A$ and $Y_i := \mathcal{E}_i \otimes_A X \otimes_A \mathcal{E}_i^*$ for $i \in S$. Let $K \in \mathcal{I}(X)(A)$ be the smallest $X$-invariant ideal that contains $K_i$ for all $i \in S$. We say that $(Y_i)_{i \in S}$ essentially covers $X$ (up to Morita equivalence) if and only if $I \cap K$ is an essential ideal in $K$. We say that $(Y_i)_{i \in S}$ covers $X$ (up to Morita equivalence) and call it a Morita covering for $X$ if $I \subseteq K$.

In this definition, $I$ is the smallest ideal for which $X$ is a Hilbert $I$-bimodule. The Hilbert $B_i$-bimodule $Y_i$ is Morita equivalent through $\mathcal{E}_i$ to the restriction $K_i \cdot X \cdot K_i$ of $X$ to the ideal $K_i$. An ideal $J$ in $A$ corresponds to an open subset $J$ in $\hat{A}$, and $J$ is $X$-invariant if and only if $J$ is $\hat{X}$-invariant. The ideal $I \cap K$ is essential in $I$ if and only if $I \cap \hat{K}$ is dense in $\hat{I}$. So the condition for $(Y_i)_{i \in S}$ to essentially cover $X$ says that the $\hat{X}$-orbit of $\bigcup \hat{K}_i \cap \hat{I}$ is dense in $\hat{I}$; equivalently, any $\hat{X}$-invariant open subset of $\hat{I}$ meets $\bigcup \hat{K}_i$. And $(Y_i)_{i \in S}$ covers $X$ if the $\hat{X}$-orbit of $\bigcup \hat{K}_i$ contains $\hat{I}$.

**Proposition 6.8.** Let $(Y_i)_{i \in S}$ essentially cover $X$ up to Morita equivalence.

1. $X$ satisfies Kishimoto’s condition if and only if $Y_i$ does so for all $i \in S$;
2. $X$ is topologically non-trivial if and only if $Y_i$ is so for all $i \in S$;
3. $X$ is purely outer if and only if $Y_i$ is so for all $i \in S$;
4. $X$ is purely universally weakly outer if and only if $Y_i$ is so for all $i \in S$.

**Proof.** First we improve our Morita covering by enlarging $S$. Let $(X_n)_{n \in \mathbb{Z}}$ be the Fell bundle generated by $X$ and let $\mathcal{E}_{i,n} := \mathcal{E}_i \otimes_A X_{-n}$, $K_{i,n} := \langle \mathcal{E}_{i,n} | \mathcal{E}_{i,n} \rangle_A$ for $i \in S$, $n \in \mathbb{Z}$. Then

$$(Y_{i,n} := \mathcal{E}_{i,n} \otimes_A X \otimes_A \mathcal{E}_{i,n}^*) = \mathcal{E}_i \otimes_A X_{-n} \otimes_A X \otimes_A X_{n} \otimes_A \mathcal{E}_i^* \subseteq \mathcal{E}_i \otimes_A X \otimes_A \mathcal{E}_i^* = Y_i$$

because $(X_n)_{n \in \mathbb{Z}}$ is a Fell bundle. Kishimoto’s condition is inherited by submodules, and so are the properties in (2)–(4). Thus we may replace $Y_i$ by the family $(Y_{i,n})_{i \in S, n \in \mathbb{Z}}$. 

The tensor product of Hilbert bimodules corresponds to the composition of partial maps. Thus the ideal \( \langle \mathcal{E}_{i,n} | \mathcal{E}_{i,n} \rangle_A \) corresponds to the domain of the partial map \( \hat{A} \to \hat{B}_i \) associated to \( \hat{E}_i \circ \hat{X}_n^{-1} = \hat{X}_n \) applied to the domain of \( \hat{E}_i \), which is \( \hat{K}_i \). Thus \( \hat{K}_{i,n} = \hat{X}_n(\hat{K}_i) \). So \( \hat{K} := \bigcup \hat{K}_{i,n} \) is the \( \hat{X} \)-orbit of \( \bigcup \hat{K}_i \). The ideal \( K \) corresponding to \( \hat{K} \) is \( X \)-invariant. By assumption, each open subset of \( \hat{I} \) meets \( \hat{K} \). Equivalently, \( I \cap K \) is essential in \( I \).

Now we prove [1]. First assume that \( X \) satisfies Kishimoto’s condition. By definition, \( Y_i \) for \( i \in S \) is a Morita restriction of \( X \). Thus it satisfies Kishimoto’s condition by Proposition [6.1]. Conversely, assume that \( Y_i \) satisfies Kishimoto’s condition for all \( i \in S \). We want to prove Kishimoto’s condition for \( X \). So we fix \( x \in X \) and \( D \in \mathcal{H}(A) \). If \( D \cap I = \emptyset \), then \( X \cdot D = 0 \) and hence any \( x \in D^+ \) with \( \|x\| = 1 \) will do. So we may assume that \( D \cap I \neq \emptyset \). This implies \( D \cap K \neq \emptyset \) because \( I \cap K \) is essential in \( I \).

The ideal \( K \) is the closed linear span of the ideals \( K_{i,n} := \langle \mathcal{E}_{i,n} | \mathcal{E}_{i,n} \rangle_A \) with \( \mathcal{E}_{i,n} \) for \( i \in S, n \in \mathbb{Z} \) as above. Hence there are \( i \in S, n \in \mathbb{Z} \) with \( D \cap K_{i,n} \neq \emptyset \). Kishimoto’s condition for \( x \) and \( D \) is weaker than the same condition for \( x \) and \( D \cap K_{i,n} \). Hence we may assume without loss of generality that \( D \in \mathcal{H}(K_{i,n}) \). The Hilbert bimodule \( \mathcal{E}_{i,n} \) witnesses that the Hilbert \( K_{i,n} \)-bimodule \( K_{i,n} \cdot \mathcal{E}_{i,n} \) is a Morita restriction of \( Y_i \):

\[
K_{i,n} \cdot \mathcal{E}_{i,n} = \mathcal{E}_{i,n} \otimes_{B_i} \mathcal{E}_{i,n} \otimes_A X \otimes_A \mathcal{E}_{i,n} \otimes_{B_i} \mathcal{E}_{i,n} = \mathcal{E}^*_{i,n} \otimes_{B_i} Y_i \otimes_{B_i} \mathcal{E}_{i,n}.
\]

Since \( Y_i \) satisfies Kishimoto’s condition, so does \( K_{i,n} \cdot \mathcal{E}_{i,n} \) by Proposition [6.1]. Lemma [2.9] gives \( d \in D^+ \) with \( \|d\| = 1 \) such that \( D_0 := \{ a \in A \mid da = ad = a \} \) is in \( \mathcal{H}(D) \). Since \( dxd \in DXD \subseteq K_{i,n} \cdot \mathcal{E}_{i,n} \), we may apply Kishimoto’s condition for \( K_{i,n} \cdot \mathcal{E}_{i,n} \) to the pair \( (dxd, D_0) \). This gives \( a_\varepsilon \in D^+ \) with \( \|a_\varepsilon\| = 1 \) and \( \|a_\varepsilon dxd\| < \varepsilon \) for all \( \varepsilon > 0 \). Since \( D_0 \subseteq D \) and \( ad = a = ad \) for \( a \in D_0 \), the elements \( a_\varepsilon \) witness Kishimoto’s condition for \( (x, D) \). Hence \( X \) satisfies Kishimoto’s condition.

We prove [2]. Assume first that \( Y_i \) is not topologically non-trivial for some \( i \in S \). That is, there are \( i \in S \) and an open subset \( U \) of \( \hat{B}_i \) on which \( \hat{Y}_i \) acts identically. The Hilbert bimodule \( \mathcal{E}_i \) induces a homeomorphism \( \hat{E}^*_i \) from \( \hat{B}_i \) to the open subset \( \hat{K}_i \) in \( \hat{A} \). Since the action of Hilbert bimodules on representations is functorial with respect to the composition of Hilbert bimodules, \( \hat{X} \) acts identically on \( \hat{E}^*_i(U) \), which is open in \( \hat{A} \). Conversely, assume that \( X \) is not topologically non-trivial. So there is a nonempty open subset \( U \) of \( \hat{I} \) with \( \hat{X}|_U = \text{id}_U \). The subset \( U \) is \( \hat{X} \)-invariant. It intersects the image \( \hat{K}_i \) of \( \hat{E}^*_i \) non-trivially for some \( i \in S \) because \( (Y_i)_{i \in S} \) essentially covers \( X \). Then \( V := \hat{E}_i(U) \) is a nonempty open subset of \( \hat{B}_i \) with \( \hat{Y}_i|_V = \text{id}_V \). So \( Y_i \) is not topologically non-trivial.

We prove [3]. The same argument for the bidual \( W^* \)-algebras shows [4]. Assume first that \( Y_i \) is partly inner for some \( i \in S \). Then there is a non-zero invariant ideal \( L \subseteq B \) such that \( \hat{L} \cdot Y_i = Y_i \cdot L \) is isomorphic to \( L \) with the obvious Hilbert bimodule structure. For any ideal \( L \) in \( B_i \), \( \mathcal{E}^*_i \otimes_{B_i} Y_i \cdot L \otimes_{B_i} \mathcal{E}_i \) is a Hilbert subbimodule in \( \mathcal{E}^*_i \otimes_{B_i} Y_i \otimes_{B_i} \mathcal{E}_i \), which is, in turn, a Hilbert subbimodule in \( \mathcal{E}_i \). Thus \( \mathcal{E}^*_i \otimes_{B_i} Y_i \cdot L \otimes_{B_i} \mathcal{E}_i = X \cdot L' = L' \cdot X \), where \( L' := \langle L, \mathcal{E}_i | L, \mathcal{E}_i \rangle_A \neq \emptyset \) is the ideal in \( A \) corresponding to \( L \) under the Rieffel correspondence for \( \mathcal{E}_i \). Since \( Y_i \cdot L \cong L \), the restriction \( L' \cdot X \) is isomorphic to \( \mathcal{E}^*_i \otimes_{B_i} Y_i \cdot L \otimes_{B_i} \mathcal{E}_i \cong \mathcal{E}^*_i \otimes_{B_i} L \otimes_{B_i} \mathcal{E}_i \cong \langle \mathcal{E}_i \cdot L | \mathcal{E}_i \cdot L \rangle_A = L' \). Thus \( X \) is partly inner.
Conversely, if $X$ is partly inner, then there is a non-zero ideal $L \triangleleft A$ such that $X \cdot L \cong L$ as Hilbert bimodules. We have $L \subseteq I$ because $X = X \cdot I$. Since $K \cap I$ is essential in $I$, the intersection $L \cap K$ is non-zero. Then $L \cap K \neq \emptyset$ for some $i \in S$, $n \in \mathbb{Z}$ because $L \cap \bigcup K_{i,n} \neq \emptyset$. Then $Y_{i,n} := E_{i,n} \otimes A X \otimes A E^*_{i,n}$ contains the non-zero Hilbert subbimodule $E_{i,n} \otimes A (X \cdot L) \otimes A E^*_{i,n} \cong E_{i,n} \otimes A L \otimes A E^*_{i,n} \cong B_i (E_{i,n} L | E_{i,n} L)$, which is isomorphic to the ideal of $B_i$ corresponding to $L \cap K_{i,n}$ under the Rieffel correspondence. This is a non-zero ideal $L'$ with $Y_{i,n} L' \cong L'$. Thus $Y_{i,n}$ is partly inner.

**Corollary 6.9.** Let $K$ be an essential ideal in $(A \langle X | X \rangle + \langle X | X \rangle_A)$. The Hilbert $A$-bimodule $X$ has one of the properties mentioned in Proposition 6.8 if and only if the restricted Hilbert $K$-bimodule $K X K$ has that property.

**Proof.** Let $E := K$ with the Hilbert $K, A$-bimodule structure inherited from $A$. It establishes that $K X K$ essentially covers $X$ because $E \otimes_A X \otimes_A E = K \otimes_A X \otimes_A K \cong K X K$. \hfill $\square$

7. A convenient Morita globalisation

Let $B = (B_g)_{g \in G}$ be a Fell bundle over a discrete group $G$ with unit fibre $A := B_e$. A Morita globalisation of $B$ consists of a $C^*$-algebra $C$, a group action $\gamma: G \to \text{Aut}(C)$, a Hilbert $A, C$-bimodule $E$ that is full over $A$, and an isomorphism between $B$ and the Fell bundle $E C \otimes C^*$ constructed in Remark 6.6. This Fell bundle isomorphism consists of Hilbert bimodule isomorphisms $B_g \cong E \otimes_C C_{\gamma g} \otimes_C E^*$ for $g \in G$ that are compatible with the multiplication maps and involutions.

We are going to construct a canonical Morita globalisation using a variant of Takesaki–Takai duality. This construction is already studied by Quigg [49] in the language of coactions. If $B$ comes from a partial action $\alpha$, then this Morita globalisation agrees with the Morita enveloping action of $\alpha$ analysed in [3].

Following [49] Definition 2.13, we call a property $P$ of $C^*$-algebras ideal if it is invariant under Morita equivalence and inherited by ideals and if every $C^*$-algebra has a largest ideal with this property. Examples of such properties are: being liminal, antiliminal, Type I, Type I or nuclear (see [49]). We warn the reader that the “ideal property” is not “ideal” in this sense.

**Proposition 7.1.** Let $B = (B_g)_{g \in G}$ be a Fell bundle over a discrete group $G$ with unit fibre $A$. Let $C^*_r(B)$ be its reduced section $C^*$-algebra, equipped with the dual $G$-coaction $\delta_G$, and let $C := C^*(B) \rtimes_{r,\delta_G} \hat{G}$ be the corresponding reduced crossed product. Let $\gamma: G \to \text{Aut}(C)$ be the $G$-action on $C$ dual to $\delta_G$. Let $P$ be a property of $C^*$-algebras that is ideal. Then

1. $\gamma: G \to \text{Aut}(C)$ is a Morita globalisation of $B$;
2. for each $g \in G$, the Hilbert bimodules $(B_{hgh^{-1}})_{h \in G}$ cover $C_{\gamma g}$ up to Morita equivalence;
3. $C^*_r(B)$ and $C \rtimes_{r,\gamma} G$ are Morita–Rieffel equivalent, and the induced lattice isomorphism $\mathcal{I}(C^*_r(B)) \cong \mathcal{I}(C \rtimes_{r,\gamma} G)$ restricts to an isomorphism between the lattices of graded ideals, $\mathcal{I}^G(C^*_r(B)) \cong \mathcal{I}^G(C \rtimes_{r,\gamma} G)$;
4. there is lattice isomorphism $\mathcal{I}^G(A) \cong \mathcal{I}^G(C)$, $I \mapsto C^*(B | I) \rtimes_{r,\delta_G} \hat{G}$;
5. the Hilbert bimodule $E$ that witnesses that $\gamma: G \to \text{Aut}(C)$ is a Morita globalisation of $B$ is an equivalence bimodule if and only if $B$ is saturated, that is, the fibres $B_g$, $g \in G$, are equivalence bimodules over $A$;
(6) A has the property \( \mathcal{P} \) if and only if \( C \) has that property;
(7) if \( G \) is countable, then \( A \) is separable if and only if \( C \) is separable;
(8) \( C \) is simple if and only if \( A \) is simple and \( B_g \neq 0 \) for all \( g \in G \).

Proof. The double crossed product \( C \rtimes_{\gamma, r} G \cong (C^*(\mathcal{B}) \rtimes \delta_G, \hat{G}) \rtimes_{\gamma, r} G \) is “\( \hat{G} \)-equivariantly” isomorphic to \( C^*(\mathcal{B}) \otimes \mathbb{K}(L^2(G)) \) with the diagonal coaction by Katayama’s version of Takesaki duality \([21]\). Thus \( C^*_r(\mathcal{B}) \) and \( C \rtimes_{\gamma, r} G \) are \( \hat{G} \)-equivariantly Morita–Rieffel equivalent. This implies an isomorphism between the ideal lattices that preservers the sublattices of graded ideals, proving \([3]\) This implies \([4]\) by Proposition \(3.13\).

Next we prove \([1]\). We first recall a more concrete description of \( C \) from \([3]\) Proposition \(8.1\). The finitely supported functions \( k: G \times G \to \mathcal{B} \) with \( k(r, s) \in B_{rs^{-1}} \) form a *-algebra with the product and involution

\[
k_1 \ast k_2(r, s) := \sum_{t \in G} k_1(r, t)k_2(t, s), \quad k^*(r, s) := k(s, r^*).
\]

This is a normed *-algebra for the “Hilbert–Schmidt” norm

\[
\|k\|_2 := \left( \sum_{r, s \in G} \|k(r, s)\|^2 \right)^{1/2}.
\]

The group \( G \) acts on it by *-automorphisms:

\[
\gamma_t(k)(r, s) := k(rt, st), \quad r, s, t \in G.
\]

The enveloping C*-algebra of this normed *-algebra is identified in \([3]\) Proposition \(8.1\) with the crossed product \( C \) with its dual \( G \)-action.

Let \( b 1_{(s, r)} \) for \( r, s \in G \), \( b \in B_{rs^{-1}} \) be the section \( G \times G \to \mathcal{B} \) with \( b 1_{(s, r)}(s, r) = b \) and \( b 1_{(s, r)}(t, u) = 0 \) for \( (s, r) \neq (t, u) \). These elements span \( C \), and they satisfy

\[
(b 1_{(s, r)})^* (c 1_{(t, u)}) = \delta_{r,t} (b \ast c) 1_{(s, u)}, \quad (b 1_{(s, r)})^* = b^* 1_{(r,s)}, \quad \text{and } \gamma_t(b 1_{(s, r)}) = b 1_{(st^{-1}, rt^{-1})}.
\]

When \( r = s \) we extend this notation by considering also \( b = 1 \), a multiplier of \( B_e \). Hence \( p_r := 1_{rr} \) is a multiplier of \( C \). This is an orthogonal projection with \( \gamma_t(p_r) = p_{rt^{-1}} \). (If \( C \) is treated as a C*-subalgebra of adjointable operators on the Fock module \( L^2(\mathcal{B}) \), then \( p_g \) corresponds to the projection onto the summand \( B_g \), see \([3]\) Proposition \(5.6\).) The convolution formula implies

\[
C \ast p_e = \overline{\text{span}} \{ b 1_{(g, e)} \mid b \in B_g, \ g \in G \},
\]

\[
p_g \ast C \ast p_e = \{ b_g 1_{(g, e)} \mid b \in B_g \}
\]

for all \( g \in G \). In particular, there are linear isomorphisms

\[
B_g \xrightarrow{\psi_g} p_g \ast C \ast p_e, \quad b_g \mapsto b_g 1_{(g, e)}, \quad \forall g \in G.
\]

The subspace \( p_g \ast C \ast p_e \subseteq C \) is a Hilbert \( p_g \ast C \ast p_g \ast p_e \) -bimodule with operations inherited from \( C \). The group action restricts to isomorphisms \( \gamma_{g^{-1}}: p_e \ast C \ast p_e \to p_g \ast C \ast p_g \) for \( g \in G \). Composing the left Hilbert \( p_g \ast C \ast p_g \)-module structure on \( p_g \ast C \ast p_e \) with this isomorphism gives a Hilbert bimodule \( \gamma_g^{-1}(p_g \ast C \ast p_e) \) over \( p_e \ast C \ast p_e \). Its Hilbert bimodule structure is given by

\[
\gamma_g^{-1}(a 1_{(e, e)}) \ast (b 1_{(g, e)}) = (ab) 1_{(g, e)}, \quad \gamma_g^{-1}(a 1_{(e, e)}) \ast (c 1_{(g, e)})^* = (bc^*) 1_{(e, e)},
\]

\[
(b 1_{(g, e)}) \ast (a 1_{(e, e)}) = (ba) 1_{(g, e)}, \quad (b 1_{(g, e)})^* \ast (c 1_{(g, e)}) = (b^* c) 1_{(e, e)}
\]
for $a \in A$, $b, c \in B_g$. Thus the pair $(\psi_e, \psi_g)$ of isomorphisms $p_e \ast C \ast p_e \cong B_e = A$ and $p_g \ast C \ast p_e \cong B_g$ in (7.1) is an isomorphism of Hilbert bimodules $B_g \cong \gamma_g^{-1}(p_g \ast C \ast p_e)$. In particular, the maps $\psi_g$ for $g \in G$ are isometric.

We turn the group action $\gamma$ on $C$ into a Fell bundle over $G$ as in Example 3.9.

We shall use the isomorphic variant $\gamma_g^{-1} \ast C$ instead of $C_{\gamma_g}$. The right ideal $E := p_e \ast C$ is a Hilbert $p_e \ast C \ast p_e, C$-bimodule full over $p_e \ast C \ast p_e$. Identifying $p_e \ast C \ast p_e$ with $A$ by $\psi_e$, we view $E$ as a Hilbert $A, C$-bimodule. The multiplication isomorphisms (3.1) and (3.2) and the isomorphisms $\psi_g$ give Hilbert bimodule isomorphisms

$$E \otimes C_{\gamma_g} \otimes_C E^* \cong \varepsilon \otimes C_{\gamma_g^{-1}} C \otimes_C e \cong p_e \ast C \otimes C_{\gamma_g^{-1}} C \otimes C \ast p_e \cong \gamma_g^{-1}(p_e) \ast C \ast p_e \cong \gamma_g^{-1}(p_g \ast C \ast p_e) \xrightarrow{\psi_g} B_g.$$ 

All isomorphisms are explicit, and it is easy to check that they give an isomorphism of Fell bundles $B \cong \varepsilon \otimes C \ast e \ast e$ for the canonical Fell bundle structure on $\varepsilon \otimes C \ast e \ast e = (\varepsilon \otimes C_{\gamma_g} \otimes_C e)^{\ast}$ described in Remark 6.6. This finishes the proof of (1).

Now we prove (2). Fix $g \in G$. Let $A_h := A$ and $E_h := \varepsilon \otimes C_{\gamma_h}$ for $h \in G$. Then

$$Y_h := \varepsilon \otimes C_{\gamma_h} \otimes_C e^*; := (\varepsilon \otimes C \gamma_h) \otimes_C C_{\gamma_h} \otimes_C (\varepsilon \otimes C \gamma_h) \ast \cong \varepsilon \otimes C \gamma_h \otimes_C C_{\gamma_h} \otimes_C C_{\gamma_h^{-1}} \otimes_C e \ast \cong \varepsilon \otimes C \gamma_h \ast \gamma_h^{-1} \otimes C e \ast \cong B_{h^*}.$$ 

We claim that the family of Hilbert $A$-bimodules $(B_{h^*})_{h \in G}$ covers $C_{\gamma_g}$ up to Morita equivalence, witnessed by the Hilbert bimodules $(E_h)_{h \in G}$. The Hilbert $A, C$-bimodules $E_h := \varepsilon \otimes C_{\gamma_h}$ for $h \in G$ are full over $A$. The Hilbert bimodule $E = p_e \ast C$ is an equivalence bimodule for $A$ and the ideal $I := C \ast p_e \ast C \subset C$ generated by $p_e$. Thus

$$E_h \otimes_A E_h \cong C_{\gamma_h} \otimes_C I \otimes_C C_{\gamma_h} \cong \langle I \cdot C_{\gamma_h} | I \cdot C_{\gamma_h} \rangle_C = \gamma_h^{-1}(I) = C \ast \gamma_h^{-1}(p_e) \ast C = C \ast p_h \ast C.$$ 

The series $\sum_{h \in G} p_h$ converges to 1 in the strict topology on $\mathcal{M}(C^*(\hat{G})) \subseteq \mathcal{M}(C)$. Hence $\sum C \ast p_h \ast C = C$. Thus $(B_{h^*})_{h \in G}$ covers $C_{\gamma_g}$ up to Morita equivalence through the Hilbert $A, C$-bimodules $E_h$ for $h \in G$.

Statement (5) is [49, Corollary 2.7]. We include a proof in our notation. Let $g \in G$. The isomorphism $E \otimes C_{\gamma_g} \otimes_C e \ast$ implies that the Hilbert $A$-bimodule $B_g$ is Morita equivalent to the restriction $IC_{\gamma_g} \ast I$ of $C_{\gamma_g} \ast I$ to $I := C \ast p_e \ast C$, see Remark 6.3. Clearly, $IC_{\gamma_g} \ast I = I_{\alpha_g}$, where $\alpha_g$ is the restriction of $\gamma_g$ to a partial isomorphism with domain $D_{\gamma_g^{-1}} := I \cap \gamma_g^{-1}(I)$. Hence $B$ is saturated if and only if the Hilbert bimodules $I_{\alpha_g}$ for $g \in G$ are full (both on the left and right), if and only if $I$ is $\gamma$-invariant. Since $C = \sum_{g \in G} \gamma_g(I)$, this happens only if $I = C$.

Statement (6) is [49, Theorem 2.15] or [3, Corollary 5.2].

We prove (7). If $A$ is separable, then each $B_g$ is separable by Lemma 7.2 below. Then $C$ is separable because $G$ is countable. Conversely, if $C$ is separable, then so is $p_e \ast C \ast p_e \cong A$.

Statement (8) is [49, Theorem 2.10]. We include a proof. Let $A$ be simple and $B_g \neq 0$ for all $g \in G$. Then $B_g$ is full for all $g \in G$, so that $B$ is saturated. Hence $A$ and $C$ are Morita–Rieffel equivalent by (5). So $C$ is simple. Conversely, let $C$
be simple. Then $I = C$, that is, $A$ is Morita–Rieffel equivalent to $C$. Then $A$ is simple. And $B$ is saturated by $[5]$.

**Lemma 7.2.** If $A$ is separable, then any Hilbert $A$-bimodule $X$ is separable.

**Proof.** Since $\mathbb{K}(X)$ is isomorphic to an ideal in $A$ (see, for instance, [30, Proposition 1.11]), $\mathbb{K}(X)$ is separable. Thus $\mathbb{K}(X)$ has a countable approximate unit $(u_n)$. Approximating each $u_n$ by a sequence of finite-rank operators, we see that $X$ is countably generated as a right Hilbert $A$-module. Let $(x_n)_{n \in \mathbb{N}} \subseteq X$ be a sequence generating $X$ as a right Hilbert bimodule and let $(a_n)_{n \in \mathbb{N}} \subseteq A$ be a dense sequence. The linear span of $\{x_n a_m | n, m \in \mathbb{N}\}$ is dense in $X$. □

8. Application of Morita coverings to Hilbert bimodules

In this section, we use Morita coverings to generalise Theorem 2.13 to Hilbert bimodules. Moreover, we show that separability is not needed in the Type I case or, more generally, if there is an essential ideal of Type I.

**Theorem 8.1.** Let $X$ be a Hilbert $A$-bimodule $X$. Consider the following conditions:

1. $X$ satisfies Kishimoto’s condition;
2. $X$ is topologically non-trivial;
3. $X$ is purely universally weakly outer;
4. $X$ is purely outer.

Then \(8.1.2 \Rightarrow 8.1.3 \Rightarrow 8.1.4 \Leftarrow 8.1.1\). If $A$ contains a separable, essential ideal, then the conditions \(8.1.1\) and \(8.1.3\) are equivalent. If $A$ contains a simple, essential ideal, then \(8.1.1\) is equivalent to \(8.1.4\). If $A$ contains an essential ideal of Type I, then all the conditions \(8.1.1\)–\(8.1.4\) are equivalent.

**Proof.** Extend $X$ to a Fell bundle $(X_n)_{n \in \mathbb{N}}$ over $\mathbb{Z}$ as in Example 3.12. Proposition 7.1 gives a canonical Morita globalisation $\gamma: \mathbb{Z} \rightarrow \mathrm{Aut}(C)$ for this Fell bundle such that a countable number of copies of $X$ cover the Hilbert $C$-bimodule $C_\gamma$. By Proposition 6.8, $X$ has one of the four properties in our theorem if and only if $C_\gamma$ has it. Hence the implications \(8.1.2 \Rightarrow 8.1.3 \Rightarrow 8.1.4 \Leftarrow 8.1.1\) follow from the corresponding implications in Theorem 2.13. Furthermore, by Corollary 6.9, if $K$ is an essential ideal in $A$, then $X$ has one of the four properties in our theorem if and only if its restriction $KXK$ has it. Therefore, to prove the remaining part of the assertion we may assume that the essential ideal in question is equal to $A$. If $A$ is separable, then $C$ is separable by Proposition 7.1, and the equivalence \(8.1.1\) and \(8.1.3\) follows from Theorem 2.13. If $A$ is simple and $X \neq 0$, then $C$ is simple by Proposition 7.1(8). By Theorem 2.13, \(8.1.1\) and \(8.1.4\) are equivalent for $\gamma_1$ and hence for $X$. The case $X = 0$ is trivial because, by convention, 0 satisfies Kishimoto’s condition and is purely outer.

Finally, assume that $A$ contains an essential ideal of Type I. Now a different proof is needed because Theorem 2.13 does not apply. We reduce this case in two steps to the case of Hilbert bimodules over commutative $C^*$-algebras coming from partial homeomorphisms. Let $Z$ be a locally compact space and let $\theta$ be a partial homeomorphism of $Z$. Let $\theta^*$ be the induced partial isomorphism of $C_0(Z)$. Since $C_0(Z)^*$ is commutative, $\theta^*$ is partly inner if and only if it is partly universally weakly inner, if and only if $\theta$ restricts to the identity map on some open subset. This is equivalent to not being topologically non-trivial because the space of irreducible
representations of $C_0(Z)$ is just $Z$, and it is equivalent to Kishimoto’s condition by an elementary argument, see also [18] Proposition A.7. Now assume that $A$ contains an essential ideal of Type I. We are going to construct a dense Morita covering of $X$ where each piece is of the form $\theta^*$ for some partial isomorphism. We have just seen that our four properties are equivalent for these pieces $\theta^*$. By Proposition [6.8], $X$ has one of our four properties if and only if each of the pieces of a dense Morita covering has it. Thus the construction of the dense Morita covering will finish the proof.

The essential Type I ideal in $A$ contains an essential ideal $K$ with continuous trace by [45] Theorem 6.2.11. This ideal is still essential in $A$. There is a set of ideals $(K_j)_{j \in S}$ in $K$ with $K = \sum K_j$ and such that each $K_j$ is Morita–Rieffel equivalent to a commutative $C^*$-algebra $C_0(Z_j)$ for a locally compact space $Z_j$, compare [19] Theorem 3.3. Let $E_j$ be the equivalence $C_0(Z_j), K_j$-bimodule. Let $Y_j := E_j \otimes_A X \otimes_A E_j^*$. The family of Hilbert $C_0(Z_j)$-bimodules $(Y_j)_{j \in S}$ essentially covers $X$ up to Morita equivalence because $\sum K_j = K$ is essential in $A$. Thus we have reduced the case where $A$ contains an essential ideal of Type I to the case where $A$ is commutative. So let $A = C_0(Z)$ and let $X$ be a Hilbert $C_0(Z)$-bimodule.

The structure of a Hilbert bimodule over $C_0(Z)$ is well known. Namely, let $\theta$ be the partial homeomorphism on the spectrum $Z$ of $C_0(Z)$ induced by $X$. Then there is a line bundle $L$ over the domain of $\theta$ such that $X$ is isomorphic to the space of sections of $L$, with $C_0(Z)$ acting by pointwise multiplication on the right and by pointwise multiplication combined with $\theta^*$ on the left. We could now treat this case directly. We prefer, however, to remove the line bundle by another dense Morita covering. We construct this without reference to the structure of Hilbert bimodules over $C_0(Z)$.

Let $U \subseteq Z$ be the open subset that corresponds to the ideal $(X|X)_{C_0(Z)} \subset C_0(Z)$. For each $z \in U$, there is $x_z \in X$ with $\langle x_z|x_z \rangle(z) \neq 0$. Let $U_z := \{ z \in U \mid \langle x_z|x_z \rangle(z) \neq 0 \}$ and let $U_\infty := Z \setminus U$. Then $U_\infty \cup \bigcup_{z \in U} U_z = Z \setminus \partial U$ is dense in $Z$.

So the Hilbert $C_0(U_z)$-bimodules $X_z := C_0(U_z) \cdot X \cdot C_0(U_z)$ for $z \in U$ and $z = \infty$ essentially cover $X$, compare Corollary [6.9]. Thus it suffices to prove the theorem for each of these restrictions of $X$. The case $X_\infty = 0$ is trivial. So it remains to consider the Hilbert $C_0(U_z)$-bimodules $X_z$ for $z \in U$. We are going to prove that each $X_z$ is associated to a partial homeomorphism of $U_z$.

First, we claim that $X_z := X \cdot C_0(U_z)$ is isomorphic to $C_0(U_z)$ as a right Hilbert $C_0(U_z)$-module. We have $\eta \in X_z$ if and only if $(\eta|\eta) \in C_0(U_z)$. Hence $X_z$ contains $x_z$. Since $X_z$ is a Hilbert $C_0(Z), C_0(U_z)$-bimodule, the compact operators on $X_z$ are isomorphic to an ideal in $C_0(Z)$ and hence commutative. Therefore, $|\eta| \langle x_z|x_z \rangle \langle x_z| \rangle = \langle x_z \rangle \langle x_z| \rangle |\eta|$ for any $\eta \in X_z$. This implies $\eta \in \eta \cdot C_0(U_z) \subseteq x_z \cdot C_0(U_z)$ because $\langle x_z|x_z \rangle$ is strictly positive in $C_0(U_z)$. Hence the rank-1 operator $C_0(U_z) \to X_z, f \mapsto x_z f$, has dense range. Its adjoint also has dense range because $\langle x_z|x_z \rangle$ is strictly positive. Polar decomposition now gives the required unitary $X'_z \cong C_0(U_z)$.

Since $X'_z$ is a Hilbert bimodule, the left $C_0(Z)$-module structure on $X'_z$ must map some ideal in $Z$ isomorphically onto $\mathbb{K}(X'_z) \cong C_0(U_z)$. This isomorphism gives a homeomorphism between $U_z$ and some open subset of $Z$. By definition, $X_z := C_0(U_z) \cdot X \cdot C_0(U_z) = C_0(U_z) \cdot X'_z$. Thus $X_z$ is the Hilbert $C_0(U_z)$-bimodule associated to a partial homeomorphism of $U_z$, as required. This finishes the proof in case $A$ contains an essential ideal of Type I.
9. The Connes spectrum and the main results for Fell bundles

**Definition 9.1.** Let $\mathcal{B}$ be a Fell bundle over an Abelian group $G$. Let $\beta$ be the dual $\hat{G}$-action on $C^*(\mathcal{B})$. The Connes spectrum of $\mathcal{B}$ is
\[
\Gamma(\mathcal{B}) := \{ z \in \hat{G} \mid I \cap \beta_z(I) \neq 0 \text{ for each non-zero ideal } I \text{ in } C^*(\mathcal{B}) \}.
\]
The strong Connes spectrum of $\mathcal{B}$ is
\[
\hat{\Gamma}(\mathcal{B}) := \{ z \in \hat{G} \mid \beta_z(I) \subseteq I \text{ for any ideal } I \text{ in } C^*(\mathcal{B}) \}.
\]

If $X$ is a Hilbert $A$-bimodule, let $(X_n)_{n \in \mathbb{Z}}$ be the Fell bundle generated by $X$ as in Example 3.12 and define $\Gamma(X) := \Gamma((X_n)_{n \in \mathbb{Z}})$ and $\hat{\Gamma}(X) := \hat{\Gamma}((X_n)_{n \in \mathbb{Z}})$.

**Remark 9.2.** Proposition 2.1 says that these definitions give the usual notions for $G$-actions by automorphisms, viewed as Fell bundles as in Example 3.9. In general, Takai Duality tells us that the double crossed product $C^*(\mathcal{B}) \rtimes_{\gamma} G \rtimes_{\beta} G$ for the action $\gamma: G \to \text{Aut}(C^*(\mathcal{B}) \rtimes_{\beta} \hat{G})$ that is dual to $\beta$ is $\hat{G}$-equivariantly isomorphic to $C^*(\mathcal{B}) \otimes \mathbb{K}(L^2(\hat{G}))$, compare the proof of Proposition 7.1. Thus $C^*(\mathcal{B})$ and $C \rtimes_{\gamma} G$ are $\hat{G}$-equivariantly Morita–Rieffel equivalent. This induces a $\hat{G}$-equivariant lattice isomorphism between the ideal lattices of $C^*(\mathcal{B})$ and $C^*(\mathcal{B}) \rtimes_{\beta} \hat{G} \rtimes_{\gamma} G$. The questions whether each ideal in a particular $\hat{G}$-$C^*$-algebra contains a $\hat{G}$-invariant ideal or is $\hat{G}$-invariant are invariant under $\hat{G}$-equivariant Morita equivalence. Hence
\[
(9.1) \quad \Gamma(\mathcal{B}) = \Gamma(\gamma) \quad \text{and} \quad \hat{\Gamma}(\mathcal{B}) = \hat{\Gamma}(\gamma).
\]

We could use (9.1) to define the spectra for Fell bundles. Schweizer 54 defines $\Gamma(X)$ for an equivalence bimodule $X$ in this fashion, as the Connes spectrum of the automorphism that generates the $\mathbb{Z}$-action on $(A \rtimes_{\chi} \mathbb{Z}) \rtimes_{\beta} T$. So our definition generalises Schweizer’s.

As in the case of group actions, the strong Connes spectrum for Fell bundles is a residual version of the (ordinary) Connes spectrum.

**Proposition 9.3.** Let $\mathcal{B}$ be a Fell bundle over a discrete, Abelian group $G$ with unit fibre $A$. Then
\[
\hat{\Gamma}(\mathcal{B}) = \bigcup_{I \in \mathcal{I}^{\mathbb{Z}}(A)} \Gamma(\mathcal{B}|_{A/I}),
\]
where we use the restrictions of $\mathcal{B}$ to the $\mathcal{B}$-invariant quotients $A/I$.

**Proof.** Put $C := C^*(\mathcal{B}) \rtimes_{\beta} \hat{G}$ and let $\gamma: G \to \text{Aut}(C)$ be the action dual to $\beta$. Equation (2.1) gives
\[
(9.2) \quad \hat{\Gamma}(\gamma) = \bigcap_{I \in \mathcal{I}^{\mathbb{Z}}(C)} \Gamma(\gamma|_{C/I}),
\]
where $\gamma|_{C/I}$ denotes the $G$-action on $C/I$ induced by $\gamma$ (we may add $I = C$ to the intersection because $C/C = \{0\}$ and $\Gamma(\gamma|_{\{0\}}) = \hat{G}$). Every $I \in \mathcal{I}^{\mathbb{Z}}(C)$ is of the form $I = J \rtimes_{\beta} \hat{G}$ for some $J \in \mathcal{I}^{\mathbb{Z}}(C^*(\mathcal{B}))$. In turn, this is of the form $J = C^*(\mathcal{B})|_I$ for some $I \in \mathcal{I}^{\mathbb{Z}}(A)$ by Proposition 3.13. Equip $C^*(\mathcal{B})/J$ with the $\hat{G}$-action $\beta|_{C^*(\mathcal{B})/J}$ induced by $\beta$, and let $\beta|_{A/I}$ be the dual $\hat{G}$-action on $C^*(\mathcal{B}|_{A/I})$. We have $\hat{G}$-equivariant isomorphisms
\[
C^*(\mathcal{B}|_{A/I}) \cong C^*(\mathcal{B})/C^*(\mathcal{B})|_I = C^*(\mathcal{B})/J,
\]
\[
(C^*(\mathcal{B})/J) \rtimes_{\beta|_{C^*(\mathcal{B})/J}} \hat{G} \cong (C^*(\mathcal{B}) \rtimes_{\beta} \hat{G}) / (J \rtimes_{\beta|_{J}} \hat{G}) = C/I.
\]
These induce a $G$-equivariant isomorphism $C^*(\mathcal{B}|_{A/I}) \cong \mathcal{G}$. Therefore,
\[ \Gamma(\mathcal{B}|_{A/I}) = \Gamma(\gamma|_{C/I}), \]
see Remark 9.2. Since $\hat{\Gamma}(\mathcal{B}) = \hat{\Gamma}(\gamma)$, (9.2) becomes the desired formula.

**Corollary 9.4.** If $\mathcal{B}$ is a minimal Fell bundle over an Abelian group, then $\hat{\Gamma}(\mathcal{B}) = \Gamma(\mathcal{B})$.

**Proposition 9.5.** Let $\mathcal{B}$ be a Fell bundle over an Abelian group $G$ and $A := B_e$.

1. $A$ detects ideals in $C^*(\mathcal{B})$ if and only if $\Gamma(\mathcal{B}) = \hat{G}$;
2. $A$ separates ideals in $C^*(\mathcal{B})$ if and only if each ideal in $C^*(\mathcal{B})$ is graded.

**Proof.** Remark 9.2 reduces (1) to the case of automorphisms, which is Theorem 2.4. (In fact, our definition of $\Gamma(\mathcal{B})$ allows for a more elementary proof.) By Corollary 3.15, $A$ separates ideals in $C^*(\mathcal{B})$ if and only if each ideal in $C^*(\mathcal{B})$ is graded. With our definition of $\hat{\Gamma}(\mathcal{B})$, this is clearly equivalent $\hat{\Gamma}(\mathcal{B}) = \hat{G}$.

**Lemma 9.6.** Let $\mathcal{B}$ be a Fell bundle over a cyclic group $G$ generated by an element $g \in G$. Then $\Gamma(\mathcal{B}) = \Gamma(B_g)$ and $\hat{\Gamma}(\mathcal{B}) = \hat{\Gamma}(B_g)$.

**Proof.** This follows from (9.1) and Lemma 2.7 applied to the Morita globalisation $\gamma$ of $\mathcal{B}$.

We generalise and extend Lemma 2.17.

**Proposition 9.7.** Let $\mathcal{B} = (B_g)_{g \in G}$ be a Fell bundle over a discrete group $G$ such that the unit fibre $A := B_e$ contains an essential ideal that is separable or whose spectrum is Hausdorff. The following are equivalent:

1. $\mathcal{B}$ is topologically free;
2. $\mathcal{B}$ is pointwise topologically non-trivial.

If, in addition, $G$ is countable, the above conditions are equivalent to

3. $\hat{\mathcal{B}}$ is free on a dense subset of $\hat{A}$.

**Proof.** As in the proof of Lemma 2.17, it suffices to assume that $G$ is countable and prove that (9.7.2) implies (9.7.3). So assume (9.7.2). Let $K \triangleleft A$ be an essential ideal. Let $K_g := K B_g K$ for each $g \in G$. Then $(K_g)_{g \in G}$ is naturally a Fell bundle with $K_e = K$. The dual partial homeomorphisms $\tilde{K}_g$ for $g \in G$ are the restrictions of the partial homeomorphisms $\tilde{B}_g$ to the open dense set $\tilde{K}$. More precisely, $\tilde{K}_g$ is the restriction of $\tilde{B}_g: \tilde{D}_{g^{-1}} \to \tilde{D}_g$ to $\tilde{D}_{g^{-1}} \cap \tilde{K} \cap \tilde{B}_{g^{-1}}(\tilde{D}_g \cap \tilde{K})$. Thus the Fell bundle $(K_g)_{g \in G}$ is pointwise topologically nontrivial, and (9.7.3) holds if and only if $(\tilde{K}_g)_{g \in G}$ is free on a dense subset of $\tilde{K}$. This replaces $A$ by an essential ideal $K$. So it suffices to prove the proposition if $A$ itself is separable or has Hausdorff spectrum.

Suppose first that $A$ is separable. Let $\gamma: G \to \text{Aut}(C)$ be the Morita globalisation of $\tilde{\mathcal{B}}$ described in Proposition 7.1. The $C^*$-algebra $C$ is separable by Proposition 7.1(7) and the family of Hilbert bimodules $(B_{hgh^{-1}})_{h \in G}$ covers $C_{\gamma_g}$ up to Morita equivalence. Hence $\gamma_g$ is topologically non-trivial for all $g \in G \setminus \{e\}$ by Proposition 6.8(2). Lemma 2.17 gives a dense subset of $\tilde{C}$ on which $\tilde{\gamma}$ is free. Since $\gamma: G \to \text{Aut}(C)$ is a Morita globalisation of $\mathcal{B}$, the partial action $\tilde{\mathcal{B}}$ may be
identified with a restriction of \( \hat{\gamma} \) to an open subset. Hence there is a dense subset of \( \hat{A} \) on which \( \hat{B} \) is free.

Suppose now that the spectrum \( \hat{A} \) of \( A := B_e \) contains an essential ideal that is separable or of Type I. Then \( B \) is aperiodic if and only if \( B \) is topologically free.

Theorem 9.8. Let \( B = (B_g)_{g \in G} \) be a Fell bundle whose unit fibre \( A := B_e \) contains an essential ideal that is separable or of Type I. Then \( B \) is aperiodic if and only if \( B \) is topologically free.

Proof. If \( A \) has an essential ideal of Type I, then it also has an essential ideal with Hausdorff spectrum by [45, Theorem 6.2.11]. Thus the assertion follows from Proposition 9.7 and Theorem 8.1.

Theorem 9.9. Let \( B \) be a Fell bundle over a discrete group \( G \). The following are equivalent:

1. \( B \) is aperiodic;
2. \( \Gamma(B_g) = T_{\text{ord}(g)} \) for all \( g \in G \);
3. \( \Gamma(B_g) \neq \{1\} \) for all \( g \in G \setminus \{e\} \).

These equivalent conditions imply

4. \( B \) is pointwise purely outer.

If \( G \) is finite or \( A \) contains an essential ideal that is simple or of Type I, then (9.9.1)–(9.9.4) are equivalent.

Proof. Propositions 7.1 and 6.8 and (9.1) reduce the theorem to the case of automorphisms, which is mostly done in Theorem 2.20. If \( A \) contains an essential ideal which is simple or of Type I, then (9.9.1) ⇔ (9.9.4) by Theorem 8.1.

Theorem 9.10. Let \( B \) be a Fell bundle over a discrete group \( G \). Consider the following conditions:

1. \( \Gamma(B_g) = T_{\text{ord}(g)} \) for all \( g \in G \);
2. \( \Gamma(B_g) \neq \{1\} \) for all \( g \in G \setminus \{e\} \);
3. \( B \) is residually aperiodic;
4. \( B \) is residually pointwise purely outer;
5. for any \( g \in G \setminus \{e\} \) and any two \( B_g \)-invariant ideals \( I \subset J \subset A \), the restriction \( B_g|_{J/I} \) is outer.

Then [9.10.1] ⇒ [9.10.2] ⇒ [9.10.3] ⇒ [9.10.4] ⇒ [9.10.5]. If \( G \) is finite, all conditions [9.10.1]–[9.10.5] are equivalent.

Proof. Most claims are just residual versions of assertions in Theorem 9.9 by Proposition 9.3, the strong Connes spectrum is the residual version of the Connes spectrum. This gives the implications [9.10.1] ⇒ [9.10.2] ⇒ [9.10.3] ⇒ [9.10.4] for all \( G \) and the converse implication [9.10.4] ⇒ [9.10.3] for finite \( G \). Beware that [9.10.2] is a priori stronger than [9.10.3] because the former involves all ideals invariant under the single automorphism \( \alpha_g \), whereas the latter involves only those ideals invariant under the whole action \( \alpha \). Condition [9.10.5] implies [9.10.4] because it requires \( B_g|_{J/I} \) to be outer for more subquotients. For finite \( G \) and an action by automorphisms, the converse implication [9.10.4] ⇒ [9.10.5] is asserted in [43, Lemma 1.15]. Propositions 7.1 and 6.8 allow to generalise this implication.
from actions by automorphisms to Fell bundles. Thus it suffices to show that (9.10.5) implies (9.10.1) when $G$ is finite.

To this end, note that (9.10.5) is equivalent to the condition: for any $g \in G \setminus \{e\}$ and $I \in \mathcal{I}^{B_g}(A)$, the Hilbert $A/I$-bimodule $B_g/B_gI$ is purely outer. For finite $G$, Theorem 9.10 shows that (9.10.5) is equivalent to the condition: $\Gamma(B_g/B_gI) = T_{\ord(g)}$ for all $g \in G$ and $I \in \mathcal{I}^{B_g}(A)$. The latter condition is equivalent to (9.10.1) by Proposition 9.3.

**Remark 9.11.** Phillips calls a group action by automorphisms $\alpha: G \to \text{Aut}(A)$ strongly pointwise outer if it satisfies (9.10.5), see [47, Definition 4.11] or [43, Definition 1.1]. In particular, for actions of finite groups by automorphisms, the equivalence between (9.10.2), (9.10.3) and (9.10.5) is [43, Theorem 1.16].

**Theorem 9.12.** Let $G = \mathbb{Z}$ or $G = \mathbb{Z}/p$ for a square-free number $p > 0$. Let $\mathcal{B} = (B_g)_{g \in G}$ be a Fell bundle over $G$. Suppose that $A := B_0$ contains an essential, ideal that is separable, simple, or of Type I. The following are equivalent:

(9.12.1) $\Gamma(\mathcal{B}) = \hat{G}$;
(9.12.2) $A$ detects ideals in $C^*(\mathcal{B})$;
(9.12.3) each non-zero ideal in $C^*(\mathcal{B})$ contains a non-zero gauge-invariant ideal;
(9.12.4) $A$ supports $C^*(\mathcal{B})$;
(9.12.5) $\mathcal{B}$ is aperiodic;
(9.12.6) $\Gamma(B_g) \neq \{1\}$ for all $g \in G \setminus \{0\}$;
(9.12.7) $\mathcal{B}$ is pointwise topologically nontrivial;
(9.12.8) $\mathcal{B}$ is topologically free;
(9.12.9) $\hat{\mathcal{B}}$ is free on a dense subset of $\hat{A}$;
(9.12.10) $\mathcal{B}$ is pointwise purely universally weakly outer.

If $G$ is finite or if $A$ contains an essential ideal of Type I, then these conditions are also equivalent to

(9.12.11) $\mathcal{B}$ is pointwise purely outer.

**Proof.** Conditions (9.12.1) and (9.12.3) are equivalent by Proposition 9.5 and Corollary 3.14. We have (9.12.5) $\implies$ (9.12.6) $\implies$ (9.12.4) $\implies$ (9.12.2) by Theorem 9.9. Proposition 9.4.10 and Lemma 4.9. Conditions (9.12.7) and (9.12.9) are equivalent by Proposition 9.7. Thus (9.12.5) and (9.12.10) are equivalent by Theorem 8.1 and if $A$ contains an essential ideal of Type I, then they are also equivalent to (9.12.11). If $G$ is finite, then (9.12.5), (9.12.10) are equivalent to (9.12.11) by Theorem 9.10. Thus it only remains to show that (9.12.2) implies (9.12.7). Thus assume (9.12.2).

Let $K < A$ be an essential ideal. Put $K_g := KB_gK$ for each $g \in G$. Then $\mathcal{K} := (K_g)_{g \in G}$ is a hereditary sub-Fell bundle of $\mathcal{B}$, see [15, Definition 21.10]. Hence $C^*(\mathcal{K})$ is a hereditary subalgebra of $C^*(\mathcal{B})$, see [15, Theorem 21.12]. If $J < C^*(\mathcal{K})$ then $C^*(\mathcal{B})JC^*(\mathcal{B}) \cap A \neq 0$ by (9.12.2). Since $C^*(\mathcal{K})$ is hereditary in $C^*(\mathcal{B})$ and $K$ is essential in $A$, we get $J \cap K = C^*(\mathcal{B})JC^*(\mathcal{B}) \cap K \neq 0$. Thus $K_e = K$ detects ideals in $C^*(\mathcal{K})$. Furthermore, by Corollary 6.9 $\mathcal{B}$ is pointwise topologically nontrivial if and only if $\mathcal{K}$ is. Therefore, to prove the remaining part of the assertion, we may assume that $A$ itself is separable, of Type I.

Now, let $\gamma: G \to \text{Aut}(C)$ be the Morita globalisation of $\mathcal{B}$ described in Proposition 7.4. It preserves both (9.12.2) and (9.12.7). If $A$ is separable, then $C$ is separable by Proposition 7.1.(7). Hence (9.12.2) $\implies$ (9.12.7) follows from Theorem 2.19. If $A$ is of Type I, then so is $C$ by Proposition 7.1.(6). Thus (9.12.2)
implies [9.12.7] by the proof of [41] Theorem 4.6. It is remarked in [41, Remark 4.7] that the implication we care about does not need separability. And the proof works both for \( \mathbb{Z} \) and for \( \mathbb{Z}/p \) with square-free \( p \), compare the proof of Theorem 2.5.

\[ \square \]

**Theorem 9.13.** Let \( G = \mathbb{Z} \) or \( G = \mathbb{Z}/p \) with square-free \( p \). Let \( \mathcal{B} \) be a Fell bundle over \( G \). Suppose that \( A := B_0 \) is separable or of Type I. The following are equivalent:

1. (9.13.1) the strong Connes spectrum \( \tilde{\Gamma}(\mathcal{B}) \) is \( \hat{G} \);
2. (9.13.2) \( A \) separates ideals of \( C^*(\mathcal{B}) \);
3. (9.13.3) each ideal in \( C^*(\mathcal{B}) \) is of the form \( C^*(\mathcal{B}|_I) \) for some \( I \in \mathcal{I}(\hat{G}, A) \);
4. (9.13.4) \( A \) residually supports \( C^*(\mathcal{B}) \);
5. (9.13.5) \( A^+ \) is a filling family for \( C^*(\mathcal{B}) \);
6. (9.13.6) \( \mathcal{B} \) is residually aperiodic;
7. (9.13.7) \( \mathcal{B} \) is residually pointwise topologically nontrivial;
8. (9.13.8) \( \mathcal{B} \) is residually topologically free;
9. (9.13.9) \( \mathcal{B} \) is residually pointwise purely universally weakly outer.

Moreover, if \( G = \mathbb{Z}/p \), these conditions are further equivalent to

10. (9.13.10) \( \mathcal{B} \) is residually pointwise purely outer;
11. (9.13.11) \( \tilde{\Gamma}(B_g) \neq \{1\} \) for all \( g \in G \setminus \{e\} \);
12. (9.13.12) \( \tilde{\Gamma}(B_g) = \text{Top}_\text{ord}(g) \) for all \( g \in G \).

If \( G = \mathbb{Z} \) and \( A \) is of Type I, then conditions (9.13.1)–(9.13.10) are equivalent.

**Proof.** Theorem 4.16 and Proposition 9.5 give the implications

\[ (9.13.1) \equiv (9.13.2) \equiv (9.13.3) \equiv (9.13.4) \equiv (9.13.5) \equiv (9.13.6) \]

Since separation of ideals is the residual version of detection of ideals (see Remark 2.3) and being separable or of Type I passes to quotients, Theorem 9.12 implies that all conditions (9.13.1)–(9.13.9) are equivalent. Theorem 9.12 also shows that these conditions are equivalent to (9.13.10) if \( G \) is finite or \( A \) is simple or of Type I. If \( G \) is finite, then independently of \( A \) these conditions are equivalent to (9.13.11)–(9.13.12) by Theorem 9.10. \[ \square \]

**Remark 9.14.** The above theorem proves the conjecture stated in [33, Remark 7.4]. Namely, if \( E \) is a countable directed graph, then the graph \( C^\ast \)-algebra \( C^*(E) \) can be viewed as a crossed product \( A \rtimes X \mathbb{Z} \), where \( A \) is the core subalgebra of \( C^*(E) \) and \( X \) is the first spectral subspace of \( C^*(E) \) with respect to the associated gauge action. Since \( A \) is separable, all conditions (i)–(iv) in [33, Proposition 7.3] are equivalent without finiteness assumptions on \( E \).

For minimal actions of the above groups, several conditions are equivalent without any assumptions on \( A \):

**Theorem 9.15.** Let \( G = \mathbb{Z} \) or \( \mathbb{Z}/p \) for a square-free number \( p \). Assume that \( \mathcal{B} \) is a minimal Fell bundle over \( G \). Then the following are equivalent:

1. (9.15.1) \( C^*(\mathcal{B}) \) is simple;
2. (9.15.2) \( \Gamma(\mathcal{B}) = \hat{G} \);
3. (9.15.3) \( \mathcal{M}(C^*(\mathcal{B})) \) has trivial centre;
4. (9.15.4) \( \mathcal{B} \) is pointwise outer.
If $G = \mathbb{Z}/p$ or $A := B_0$ is simple, then these conditions are further equivalent to

(9.15.5) $A$ supports $C^*(B)$;
(9.15.6) $A^+$ is a filling family for $C^*(B)$;
(9.15.7) $B$ is aperiodic.

Proof. Conditions ([9.15.1] and [9.15.2]) are equivalent by Proposition 9.5 because $B$ is minimal. The implications ([9.15.7] $\Rightarrow$ [9.15.6] $\Rightarrow$ [9.15.5] $\Rightarrow$ [9.15.1]) follow from Theorem 4.16. If either $G = \mathbb{Z}/p$ or $A$ is simple, then [9.15.7] and [9.15.4] are equivalent by Theorem 9.9.

It remains to show the equivalence of [9.15.1], [9.15.3] and [9.15.4]. For actions of $G$ by automorphisms, this is contained in Theorem 2.5. Proposition 7.1 gives $C^*(B)$ is equivalent by Theorem 9.9. Since $\mathcal{T}^+(C) \cong \mathcal{T}^+(C \rtimes \gamma G)$ is a filling family for $\mathcal{G}$ and both have isomorphic centre, say, by the Dauns–Hofmann Theorem. Since $\mathcal{T}^+(C) \cong \mathcal{T}^+(C \rtimes \gamma G)$ is simple if and only if $C \rtimes \gamma G$ is; and both have isomorphic centre, say, by the Dauns–Hofmann Theorem. Since $\mathcal{T}^+(C) \cong \mathcal{T}^+(C \rtimes \gamma G)$ is simple if and only if $C \rtimes \gamma G$ is; and both have isomorphic centre, say, by the Dauns–Hofmann Theorem. Since $[9.15.1]$ is minimal, being outer and purely outer are equivalent. Hence Proposition 6.8(3) shows that $B$ is pointwise outer if and only if $\gamma$ is. \qed

Corollary 9.16. If $X$ is a Hilbert $A$-bimodule which is not an equivalence bimodule, then the crossed product $A \rtimes_X \mathbb{Z}$ is simple if and only if $X$ is minimal.

Proof. For any $n > 0$, $\langle X^{\otimes_A n} | X^{\otimes_A n} \rangle_A \subseteq \langle X | X \rangle_A$ and $A \langle X^{\otimes_A n} X^{\otimes_A n} \rangle \subseteq A \langle X | X \rangle$. Since $X$ is not an equivalence bimodule, this implies $X^{\otimes_A n} \not\sim A$ for all $n > 0$. So the Fell bundle $(X_n)_{n \in \mathbb{N}}$ is pointwise outer. \qed

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