HOPF HYPERSURFACES OF SMALL HOPF PRINCIPAL CURVATURE 
IN \( \mathbb{CH}^2 \)

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Abstract. Using the methods of moving frames and exterior differential systems, we show that there exist Hopf hypersurfaces in complex hyperbolic space \( \mathbb{CH}^2 \) with any specified value of the Hopf principal curvature \( \alpha \) less than or equal to the corresponding value for the horosphere. We give a construction for all such hypersurfaces in terms of Weierstrass-type data, and also obtain a classification of pseudo-Einstein hypersurfaces in \( \mathbb{CH}^2 \).

1. Introduction

In this paper, we address two classic problems in the study of hypersurfaces in the complex space forms \( \mathbb{CP}^n \) and \( \mathbb{CH}^n \), namely

- the basic structure theory for Hopf hypersurfaces;
- the classification of pseudo-Einstein hypersurfaces.

1.1. History of the first problem. A Hopf hypersurface is one for which the structure vector \( W \) is a principal vector. (Definitions will be given in §1.3.) In 1982, Cecil and Ryan [2] gave a local characterization of Hopf hypersurfaces in \( \mathbb{CP}^n \) as tubes over complex submanifolds. In 1985, Montiel [9] established a similar characterization for Hopf hypersurfaces in \( \mathbb{CH}^n \) whose Hopf principal curvature \( \alpha \) is greater than \( 2/r \), the Hopf principal curvature of the horosphere.

In this paper, we construct the class of Hopf hypersurfaces in \( \mathbb{CH}^2 \) for which \( 0 \leq \alpha \leq 2/r \), and show that each such hypersurface (for \( \alpha < 2/r \)) can be characterized in terms of Weierstrass-type data which take the form of a pair of contact curves in \( S^3 \). (For a precise statement, see Theorem 2.) We expect that a similar approach will show how to construct new examples in higher dimensions.

1.2. History of the second problem. The complex space forms \( \mathbb{CP}^n \) and \( \mathbb{CH}^n \) do not admit Einstein hypersurfaces. However, as Kon [6] discovered, there is a nice class of hypersurfaces satisfying

\[
SX = \rho X + \sigma \langle X, W \rangle W,
\]

for all tangent vectors \( X \), where \( S \) is the Ricci tensor, and \( \rho \) and \( \sigma \) are constants. He called such hypersurfaces pseudo-Einstein. For \( n \geq 3 \), the pseudo-Einstein hypersurfaces were classified by Kon for \( \mathbb{CP}^n \) and by Montiel for \( \mathbb{CH}^n \). Each is an open subset of a homogeneous Hopf hypersurface (see Theorems 6.1 and 6.2 in [10]). For \( n = 2 \), the classification problem remained open until recently, when Kim and Ryan [5] showed that pseudo-Einstein hypersurfaces in \( \mathbb{CP}^2 \) and \( \mathbb{CH}^2 \) must be Hopf. However, in addition to the known homogeneous examples, this classification includes all Hopf hypersurfaces with \( \alpha = 0 \).

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In this paper, we complete the classification of pseudo-Einstein hypersurfaces in \( C^H \) by applying our solution to the Hopf hypersurface construction problem for \( \alpha = 0 \). For more details, see §2.4.

1.3. Hypersurfaces in Complex Space Forms: Notation and Definitions. The complex space forms \( \mathbb{C}P^n \) and \( \mathbb{C}H^n \) can both be viewed as quotients of spheres by \( S^1 \) actions. In the case of \( \mathbb{C}P^n \), the circle acts on the sphere \( S^{2n+1} \) of radius \( r \) in \( \mathbb{C}^{n+1} \) by multiplication by a unit modulus complex number. For \( \mathbb{C}H^n \), the action is the same, but the ‘sphere’ is the anti-de Sitter space \( H^{2n+1}_{1} \) defined by \( \langle z, z \rangle_C = -r^2 \) using an Hermitian inner product with signature \((n, 1)\):

\[
\langle z, w \rangle_C = -z_0\bar{w}_0 + \sum_{k=1}^{n} z_k\bar{w}_k, \quad z, w \in \mathbb{C}^{n+1}.
\]

(In what follows, \( \langle \ , \ \rangle \) without the subscript \( C \) will indicate the real part of this inner product. A nonzero vector \( z \) such that \( \langle z, z \rangle_C = 0 \) will be called a null vector. Also, as in [10], we will use \( \mathbb{H} \) as an abbreviation for \( H^{2n+1}_{1} \).) In each case, the projection \( \pi \) to the quotient is a Riemannian (respectively, semi-Riemannian) submersion, and the quotient is a complex space form, i.e., it has a positive definite Kähler metric of constant holomorphic sectional curvature, given by \( 4/r^2 \) or \( -4/r^2 \), respectively.

A real hypersurface \( M \) in \( \mathbb{C}P^n \) or \( \mathbb{C}H^n \) inherits two structures from the ambient space. First, given a unit normal \( \xi \), the structure vector field \( W \) on \( M \) is defined so that \( JW = \xi \), where \( J \) is the complex structure. This gives an orthogonal splitting of each tangent space as

\[
\text{span}\{W\} \oplus W^\perp.
\]

Second, we define on \( M \) the \((1, 1)\) tensor field \( \varphi \) which is the complex structure \( J \) followed by tangential projection, so that

\[
\varphi X = JX - \langle X, W \rangle \xi.
\]

The shape operator \( A \) is defined by

\[
AX = -\nabla_X \xi
\]

where \( \nabla \) is the Levi-Civita connection of the ambient space. The Gauss equation expresses the curvature tensor of \( M \) in terms of \( A \) and \( \varphi \) as follows:

\[
R(X, Y) = AX \wedge AY + c(X \wedge Y + \varphi X \wedge \varphi Y + 2\langle X, \varphi Y \rangle \varphi),
\]

where \( 4c = \pm 4/r^2 \) is the holomorphic sectional curvature of the complex space form, and \( \langle X \wedge Y \rangle Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y \).

The Ricci tensor is given by

\[
SX = (2n+1)cX - 3c\langle X, W \rangle W + mAX - A^2X,
\]

where \( m = \text{trace} \ A \).

Let \( \alpha = \langle AW, W \rangle \). If \( W \) is a principal vector everywhere (i.e. \( AW = \alpha W \)), we say that \( M \) is a Hopf hypersurface. Following J.K. Martins [8], we use the term Hopf principal curvature to mean the principal curvature that corresponds to the principal vector \( W \) for a Hopf hypersurface.
It is a nontrivial fact that the Hopf principal curvature $\alpha$ is constant for a (connected) Hopf hypersurface. This was proved by Y. Maeda \cite{7} for $\mathbb{CP}^n$ and by Ki and Suh \cite{4} for $\mathbb{CH}^n$. Further details on the material of this section may be found in §2 of \cite{10}.

1.4. **Our approach.** Many classification results in the study of hypersurfaces in complex space forms rely on the ingenious use of the identities that arise between the shape operator, the curvature tensor, and $\varphi$. Unfortunately, some of these arguments work only when the complex dimension $n$ of the space form is at least three. Motivated by some unresolved questions in the case $n = 2$, we began a program of using moving frames and exterior differential systems to study special hypersurfaces in $\mathbb{CP}^2$ and $\mathbb{CH}^2$. This approach has the advantage of being systematic. One does not have to choose which tensor to differentiate in which direction; instead, Cartan’s test for involutivity, alternating with prolongation, is guaranteed to tell you how many solutions there are. Nonetheless, some effort is required to set up the machinery of the frame bundle and its canonical framing, in terms of which our exterior differential systems are defined. Although the results of this paper do not make use of Cartan’s test, we used it to show existence of Hopf hypersurfaces with $0 \leq |\alpha| \leq 2/r$ after which we were able to make an explicit construction as described in Theorem 2.

In this paper, all manifolds are assumed connected and all manifolds and maps are assumed smooth ($C^\infty$) unless stated otherwise. For basic reference material on hypersurfaces, see \cite{10}. More on exterior differential systems may be found in the monograph \cite{1} or the textbook \cite{3}.

2. **Setup and Statement of Results**

In this section we state our main result (Theorem 2 below). In order to make sense of the mappings used in that theorem, we first need to define moving frames for hypersurfaces in $\mathbb{CH}^2$. In fact, we will work with lifted frames, which are sections of the frame bundle of $\mathbb{H} = H_5^1$ that are orthonormal with respect to the semi-Riemannian metric given by the real part of (1). Because this may be less familiar to some readers than the Riemannian case, and also in order to fix notation, we review the basic tools in some detail.

2.1. **Moving Frames for Hypersurfaces in $\mathbb{CH}^2$.** Let $G$ be the Lie group $U(2,1)$ of matrices whose columns form a ‘unitary’ basis for $\mathbb{C}^3$ with respect to the Hermitian inner product (1) with signature $(2,1)$. We take the convention that the first column of a matrix $u \in G$ has inner product $-1$ with itself, while the second and third columns have inner product $+1$ with themselves.

Let $\zeta : G \to \mathbb{H}$ be given by multiplying the first column of $u \in G$ by $r$, and let $\rho = \pi \circ \zeta : G \to \mathbb{CH}^2$. Let $e_2$ and $e_3$ denote the second and third columns of $u$. At the point $z = \zeta(u) \in \mathbb{H}$, vectors $e_2(u), e_3(u)$ span the complex plane that is orthogonal to $z$; thus, these vectors push forward under $\pi$ to span the tangent space of $\mathbb{CH}^2$ at $\pi(z)$. For later use, we will need to consider frames that span these spaces as real vector spaces. To that end, define additional $\mathbb{C}^3$-valued functions $e_0, e_1, e_4$ on $G$ such that

\begin{equation}
\frac{i}{r} \zeta = e_0, \quad ie_1 = e_2, \quad ie_3 = e_4.
\end{equation}

The vector $e_0(u)$ is tangent to the fiber of $\pi$ at $z$, normalized so that $\langle e_0, e_0 \rangle_C = -1$, while the remaining unit vectors $e_1(u), \ldots, e_4(u)$ push forward under $\pi$ to give a basis (over $\mathbb{R}$) for the tangent space to $\mathbb{CH}^2$ at $\pi(z)$. (These are referred to as *horizontal* vectors, as they are orthogonal to the fiber of $\pi$.)
Definition 1. Let \( M^3 \subset \mathbb{CH}^2 \) be an oriented embedded hypersurface. We say that a map \( f : M \rightarrow G \) is an adapted lift of \( M \) if \( p \circ f = \text{id}_M \), \( \pi_\ast(e_1 \circ f) \) is normal to \( M \) and agrees with the orientation. (Note that this means that \( \pi_\ast(e_1 \circ f) \) is the structure vector \( W \) of \( M \) and that \( \pi_\ast(e_1 \circ f) \) and \( \pi_\ast(e_2 \circ f) \) are also tangent to \( M \)).

The choice of adapted lift is not unique; at a given point of \( M \), we may change \( z \) by the \( S^1 \) action on \( \mathbb{H} \), and we may also multiply \( e_2 \) by a unit modulus complex number.

2.2. Gauss Maps and Characteristics. Now we will specialize to Hopf hypersurfaces \( M \subset \mathbb{CH}^2 \) for a fixed Hopf principal curvature \( \alpha \in (-2/r, 2/r) \). (We will discuss the case where \( |\alpha| = 2/r \) in \( \S \ref{sec:examples} \)). Write \( \alpha = \frac{2}{r} \sin \phi \) where \( \phi \in (-\pi/2, \pi/2) \). We define two additional vector-valued functions on \( G \),

\[
\mathbf{g}^\pm = \mathbf{e}_0 - (\sin \phi \mathbf{e}_3 \pm \cos \phi \mathbf{e}_4),
\]

and let \( \mathbf{g}_C^\pm = \pi \circ \mathbf{g}^\pm \) denote the line in \( \mathbb{C}^3 \) spanned by \( \mathbf{g}^\pm \).

For an adapted lift \( f : M \rightarrow G \), the compositions \( \mathbf{g}^\pm \circ f \) may be regarded as modified versions of the Gauss map: first, the hypersurface normal is rotated within a complex plane, through an angle determined by \( \alpha \), and then lifted to a null vector tangent to the anti-de Sitter space \( \mathbb{H} \).

Let \( \mathcal{V} \) denote the set of nonzero null vectors in \( \mathbb{C}^3 \). Then the \( \mathbf{g}^\pm \) take values in \( \mathcal{V} \), and the \( \mathbf{g}_C^\pm \) are maps from \( G \) into the projectivized null cone \( \pi(\mathcal{V}) \subset \mathbb{CP}^2 \). This space is a smooth manifold of real dimension three, and we will identify it with \( S^3 \) as follows: Given a point \( z \in \mathcal{V} \), satisfying

\[
|z_1|^2 + |z_2|^2 = |z_0|^2,
\]

we note that \( z_0 \neq 0 \), and thus \( \pi(\mathcal{V}) \) lies entirely in the domain of one of the standard coordinate charts on \( \mathbb{CP}^2 \). Letting \( w_1 = z_1/z_0 \) and \( w_2 = z_2/z_0 \), we see immediately that points in \( \pi(\mathcal{V}) \) satisfy \( |w_1|^2 + |w_2|^2 = 1 \), the equation of the unit sphere in \( \mathbb{C}^2 \).

The relationship between the geometry of the Hopf hypersurface \( M \) and the null vectors \( \mathbf{g}^\pm \) is as follows. Choose any point \( p \in M \) and an orthonormal principal basis \( (e_1, e_2, e_3) \) for \( T_pM \) satisfying \( e_2 = \varphi e_1 \) and \( e_3 = W \), so that \( Ae_1 = \lambda e_1 \), \( Ae_2 = \nu e_2 \), and \( Ae_3 = \alpha e_3 \). A fundamental identity for Hopf hypersurfaces (following from the Codazzi equation – see [10], pp. 245–246) gives

\[
\lambda \nu = \frac{\lambda + \nu}{2} \alpha + c
\]

(recall that \( c = -1/r^2 \)). This shows that \( \lambda \) and \( \nu \) are distinct, since

\[
\left( \lambda - \frac{\alpha}{2} \right) \left( \nu - \frac{\alpha}{2} \right) = c + \frac{\alpha^2}{4} = -\frac{\cos^2 \phi}{r^2}.
\]

Hence \( (e_1, e_2, e_3) \) extends to a smooth local principal orthonormal frame near \( p \) with corresponding smooth principal curvatures \( (\lambda, \nu, \alpha) \). Moreover, \( \S \ref{sec:properties} \) implies that the quadratic form corresponding to the restriction of \( A - \frac{\alpha}{2} I \) to \( W^\perp \) has signature ++. We define two vectors \( \mathbf{v}^\pm \) in \( W^\perp \) that are null for this quadratic form, i.e.

\[
\langle (A - \frac{\alpha}{2}) \mathbf{v}^+, \mathbf{v}^+ \rangle = 0, \quad \langle (A - \frac{\alpha}{2}) \mathbf{v}^-, \mathbf{v}^- \rangle = 0,
\]

given by

\[
\mathbf{v}^\pm = \pm \cos \phi \ e_1 + (r \lambda - \sin \phi) \varphi e_1.
\]
Although the values of $v^+$ and $v^-$ depend on the choice of which eigenvector is assigned to $e_1$, it is easy to check using (7) that a different choice merely multiplies each of $v^\pm$ by a nonzero factor. Thus, the respective distributions spanned by $v^+$ and $v^-$ are well-defined and $M$ is foliated in two ways by curves tangent to these distributions.

We define the characteristic distributions $\chi^+$ and $\chi^-$ on $M$ as the direct sum of the span of $v^+$ (respectively, $v^-$) and the span of the structure vector $W$. It is somewhat surprising that these two-dimensional distributions on $M$ are integrable; in fact, we have

**Proposition 1.** Let $M \subset \mathbb{C}H^2$ be a Hopf hypersurface with Hopf principal curvature $\alpha \in (-2/r, 2/r)$. Then $M$ is foliated by two-dimensional leaves of each of the distributions $\chi^+, \chi^-$. Moreover, if $f$ is an adapted lift of $M$, then the maps $g_C^+ \circ f$ and $g_C^- \circ f$ are constant along the leaves of $\chi^+, \chi^-$, respectively.

This proposition will be proved in §3.2.

2.3. Main Result.

**Theorem 2.**

1. Let $M$ be an oriented Hopf hypersurface in $\mathbb{C}H^2$ with Hopf principal curvature $\alpha \in (-2/r, 2/r)$. Then there are well-defined maps $\sigma^\pm : M \to S^3$, such that $\sigma^\pm = g_C^\pm \circ f$ for any adapted lift $f$ of $M$, and such that the images of $\sigma^\pm(M)$ are contact curves in $S^3$.

2. Conversely, given any two embedded contact curves $C_1, C_2$ in $S^3$, let $P$ be the intersection of their inverse images under $g_C^\pm$ in $G$. Then $\rho(P)$ is a Hopf hypersurface with Hopf principal curvature $\alpha$.

We will prove this theorem in §3.3. For the moment, we note how these results carry over to the borderline case, when $\alpha = \pm 2/r$. In this case, $\cos \phi = 0$, so $g^+ = g^-$. Labelling this as $g$, we note that $g \circ f$ is well-defined (in fact, it is $e_0 - e_3$, which is a just a null lift of $-W$, i.e., one of two null vectors tangent to $\mathbb{H}$ that project to $-W$). The image of the map $g_C$ is still a contact curve in $S^3$, which will be regular at points where $\lambda \neq \nu$. However, given the image contact curve, more data is necessary to reconstruct the Hopf hypersurface $M$; more details will be given in §3.4.

As for the characteristic distributions, in the borderline case (8) becomes

\[
(\lambda - \alpha/2) (\nu - \alpha/2) = 0.
\]

If $\lambda \neq \nu$, we may assume, without loss of generality, that $\lambda \neq -\alpha/2$ so that $v^+ = v^-$ and the unique characteristic distribution is spanned by $\varphi e_1$, the principal direction corresponding to $\nu = 1/r$. On the other hand, if $\lambda = \nu$ globally, then $M$ is just an open subset of a horosphere with $\lambda = 1/r$. Finally, there is the possibility that $\lambda = \nu$ at some points and $\lambda \neq \nu$ at others.

**Remark 1.** Tubes over real hyperbolic planes occur in the list of standard examples of Hopf hypersurfaces with constant principal curvatures. As seen in Theorems 3.4 and 3.12 of [10], they have Hopf principal curvature $0 < \alpha < 2/r$. According to Theorem 2, these must be constructible by our methods. It would be interesting to identify which contact curves produce these special examples.

1The contact structure here is the standard one in $S^3$, whose contact planes are orthogonal to the fibers of the Hopf fibration $S^3 \to \mathbb{C}P^1$. 

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2.4. Classification of Pseudo-Einstein Hypersurfaces. As defined by Kon [6], a hypersurface $M^{2n-1}$ in a complex space form is pseudo-Einstein if there are constants $\rho$ and $\sigma$ such that the Ricci tensor $S$ of $M$ satisfies

$$SX = \rho X + \sigma \langle X, W \rangle W,$$

for all tangent vectors $X$, where $W$ is the structure vector. When $n \geq 3$, it is known that every such hypersurface is an open subset of a homogeneous Hopf hypersurface (see Theorems 6.1 and 6.2 in [10]). When $n = 2$, we have

**Theorem 3** (Kim-Ryan [5]). Every pseudo-Einstein hypersurface $M$ in $\mathbb{C}H^2$ or $\mathbb{C}P^2$ is a Hopf hypersurface. In $\mathbb{C}P^2$, $M$ is either an open subset of a geodesic sphere, or has $\alpha = 0$ and is generically an open subset of a tube of radius $r\pi/4$ over a holomorphic curve. In $\mathbb{C}H^2$, either $\alpha = 0$ or $M$ is an open subset of a horosphere or a tube over a totally geodesic $\mathbb{C}H^1$.

Moreover, it is easy to check that every Hopf hypersurface in $\mathbb{C}H^2$ with $\alpha = 0$ is pseudo-Einstein. In fact, equation (2.2b) and Proposition 2.21 of [5] shows that for such a hypersurface, $SW = 2cW$ and $SX = 6cX$ for all $X \in W^\perp$. One can also see this from (7) and (3) in the current paper. First note that $\lambda \nu = c$ from (7) so that in (3) we have $mA_X - A^2X = cX$ for $X \in W^\perp$ and thus $SX = 6cX$. The fact that $SW = 2cW$ when $\alpha = 0$ is immediate.

Thus, the construction given earlier in this section, in the special case where $\alpha = 0$, completes the classification of pseudo-Einstein hypersurfaces in complex space forms. We have

**Theorem 4.** A pseudo-Einstein hypersurface in $\mathbb{C}H^2$ must be a Hopf hypersurface. Denoting the Hopf principal curvature by $\alpha$, which we may arrange to be nonnegative, we have the following possibilities:

1. $\alpha > 2/r$. An open subset of a geodesic sphere or tube over a totally geodesic $\mathbb{C}H^1$, as described in Theorems 3.7 and 3.8 of [10].
2. $\alpha = 2/r$. An open subset of a horosphere, as described in Theorem 3.4 of [10].
3. $\alpha = 0$. A hypersurface constructed from two arbitrary contact curves as in Theorem 2 (using $\alpha = 0$).

3. Proofs

In this section we will give proofs of Proposition 1 and Theorem 2. Because it will be convenient to compute using differential forms, we begin by deriving the structure equations for a basis for the left-invariant 1-forms on $G = U(2, 1)$.

3.1. **Structure Equations.** Recall from (2.1) that we define vector-valued functions $\zeta, e_2, e_3$ on $G$ by decomposing a matrix $u \in G$ into columns:

$$u = \begin{bmatrix} r^{-1}\zeta(u), e_2(u), e_3(u) \end{bmatrix}.$$

Let $\gamma^i_j$, for $1 \leq i, j \leq 3$, denote the components of the Maurer-Cartan form $u^{-1}du$ on $G$. Note that these are complex-valued 1-forms. Then

$$d\zeta = \gamma^1_1\zeta + r(\gamma^2_2 e_2 + \gamma^3_3 e_3),$$

$$de_2 = \frac{1}{r}\gamma^2_1 e_2 + \gamma^3_2 e_3,$$

$$de_3 = \frac{1}{r}\gamma^3_1 e_2 + \gamma^3_2 e_3.$$
Differentiating the relations $\langle \zeta, \zeta \rangle_C = -1/r^2$, $\langle \zeta, e_2 \rangle_C = 0$, $\langle e_2, e_2 \rangle_C = 1$, etc., and using (13), shows that
\[
\gamma^j_i = \begin{cases} 
\gamma^i_j & \text{if exactly one of } i, j \text{ equals } 1 \\
-\gamma^j_i & \text{otherwise.}
\end{cases}
\]
As well, the $\gamma^j_i$ satisfy the usual Maurer-Cartan equations:
\[
d\gamma^j_i = -\gamma^k_i \wedge \gamma^j_k. \tag{14}
\]
(We use the usual summation convention from now on.)

We will identify $G$ with a sub-bundle of the orthonormal frame bundle of $\mathbb{H}$, by associating to $u \in G$ the orthonormal frame $(e_0(u), \ldots, e_4(u))$ at base point $z$. (Recall that $e_0, e_1, e_4$ are related to $\zeta, e_2, e_3$ by (11).) To analyze the geometry of real hypersurfaces in $\mathbb{C}H^2$, we will need to use real-valued differential forms on the frame bundle. To that end, we first define real-valued 1-forms $\eta^0, \ldots, \eta^4$ on $G$ such that
\[
d\zeta = \eta^0 e_0 + \eta^1 e_1 + \eta^2 e_2 + \eta^3 e_3 + \eta^4 e_4. \tag{15}
\]
Comparison with (13) shows that the $\eta^i$ are linearly independent left-invariant 1-forms on $G$, and that
\[
\gamma^1_1 = \frac{i}{r} \eta^0, \quad \gamma^2_1 = \frac{1}{r} (\eta^2 - i \eta^1), \quad \gamma^3_1 = \frac{1}{r} (\eta^3 + i \eta^4). \tag{16}
\]
Moreover, (15) shows that the $\eta^i$ are the canonical forms on $G$, once we identify $G$ with a frame bundle.

The connection forms $\omega^\alpha_\beta$ on $G$, where $0 \leq \alpha, \beta \leq 4$, are uniquely characterized by satisfying the structure equations
\[
d\eta^\alpha = -\omega^\alpha_\beta \wedge \eta^\beta \tag{17}
\]
and taking value in the Lie algebra $\mathfrak{o}(1, 4)$, i.e., $\omega^0_j = \omega^4_j$ and $\omega^i_j = -\omega^j_i$. (We now take the index ranges $1 \leq j, k \leq 4$.) Thus, to calculate the connection forms we need to express the derivatives of the canonical forms in terms a basis of 1-forms on $G$. These 1-forms are in turn defined by differentiating the frame vectors, regarded as vector-valued functions on $G$.

Multiplying (15) by $i/r$, we have
\[
de e_0 = -\frac{1}{r^2} \eta^0 \zeta + \frac{1}{r} \mathcal{J}^j_k \eta^k e_j, \quad \text{where } \mathcal{J} = \begin{bmatrix} 0 & -1 \\
1 & 0 \\
0 & 1 \\
0 & 0 \end{bmatrix}.
\]
The matrix $\mathcal{J}$ represents multiplication by $i$ on the span of $\{e_1, \ldots, e_4\}$. Differentiating $\langle \zeta, e_j \rangle$ and $\langle e_0, e_j \rangle$ shows that
\[
de e_j = \eta^k_j e_k + \frac{1}{r^2} \eta^j_k \zeta + \frac{1}{r} \mathcal{J}^j_k \eta^k e_0, \tag{19}
\]
for some (real-valued) 1-forms $\eta^j_k$. By differentiating $\langle e_j, e_k \rangle$, we see that the $\eta^j_k$ are skew-symmetric. In fact, comparing with (13) shows that
\[
\eta^2_1 = \text{Im } \gamma^2_1, \quad \eta^3_1 = -\eta^4_1, \quad \eta^4_1 = \text{Re } \gamma^2_1, \quad \eta^2_2 = \text{Im } \gamma^2_2, \quad \eta^3_2 = -\eta^4_2, \quad \eta^4_2 = \text{Re } \gamma^2_2.
\]
The 1-forms $\eta^0, \ldots, \eta^4$, $\eta^1, \eta^2, \eta^3, \eta^4$ are linearly independent, and form a basis for the left-invariant 1-forms on $G$.\]
Using (16), (20) and the Maurer-Cartan equations (14), we compute that

\begin{align}
(21) & \quad d\eta^0 = -\frac{1}{r} g^k_j \eta^j \wedge \eta^k, \\
(22) & \quad d\eta^j = -(\eta^k_j - \frac{1}{r} g^k_j \eta^0) \wedge \eta^k.
\end{align}

Comparing these equations with (17) shows that the connection forms are given by \(\omega^0_k = \omega_j^k = \frac{1}{r} g^k_j \eta^j\) and

\[\omega^k_j = \eta^k_j - \frac{1}{r} g^k_j \eta^0.\]

Next, let \(F\) be the orthonormal frame bundle of \(\mathbb{CH}^2\). We define a map \(\Pi : G \to F\) which takes \(u \in G\) to an orthonormal frame at \(\pi(\zeta(u))\) given by \((\pi_* e_1, \ldots, \pi_* e_4)\). The relationship between the connection forms on \(F\) and on \(G\) is given by the following

**Lemma 5.** Let \(f_0 = (e_1, e_2, e_3, e_4)\) be an orthonormal frame on \(U \subset \mathbb{CH}^2\) such that \(J e_1 = e_2\) and \(J e_3 = e_4\), and let \(f : U \to G\) be any map such that \(\Pi \circ f = f_0\). Then for any tangent vector \(v\) on \(U\),

\begin{align}
(23) & \quad v = (v \cdot f^* \eta^k) e_k, \\
(24) & \quad \nabla_v e_j = (v \cdot f^* \omega^k_j) e_k,
\end{align}

where \(\nabla\) denotes the Levi-Civita connection on \(\mathbb{CH}^2\).

**Proof.** Let \(h = \zeta \circ f\), and let \(\hat{e}_1, \ldots, \hat{e}_4\) be the horizontal vector fields on \(h(U) \subset \mathbb{H}\) such that \((\pi_*( h(p)) \hat{e}_j(h(p)) = e_j(p)\) for all \(p \in U\). Note that \(\hat{e}_j \circ h = e_j \circ f\). We may extend these vector fields to all of \(\hat{U} = \pi^{-1}(U)\) using the \(S^1\)-action, and still have \(\pi_* \hat{e}_j = e_j\). These vector fields, together with \(\hat{e}_0(z) = \frac{1}{r} z\), define a section \(\hat{f} : \hat{U} \to G\) such that \(f = \hat{f} \circ h\).

Suppose \(v \in T_p \mathbb{CH}^2\), and let \(\hat{v} \in T_{h(p)} \mathbb{H}\) be the horizontal vector that projects to \(v\). By the defining property of the canonical forms on \(G\),

\[\hat{v} = (\hat{v} \cdot \hat{f}^* \eta^k) \hat{e}_k.\]

Then, because \(\hat{v}\) and \(h_* v\) differ by a multiple of \(\hat{e}_0\),

\[\hat{v} = ((h_* v) \cdot \hat{f}^* \eta^k) \hat{e}_k = (v \cdot f^* \eta^k) \hat{e}_k.\]

**Lemma 5** implies that the pullbacks, under \(\Pi : G \to F\), of the canonical and connection forms on \(F\) are the \(\eta^k\) and \(\omega^k_j\) on \(G\). Any computation made on \(G\) using these pullback forms will be valid on \(F\).
3.2. Characteristics.

**Lemma 6.** Let $M \subset \mathbb{C}H^2$ be a hypersurface, and let $f : M \rightarrow G$ be an adapted lift of $M$. Let $e_i = \pi_*(e_i \circ f)$ for $1 \leq i \leq 4$. Then $f^*\eta^4 = 0$ and

$$f^*\omega_j^4 = \sum_{k=1}^{3} \langle Ae_j, e_k \rangle f^*\eta^k,$$

for $1 \leq j \leq 3$.

**Proof.** Near any point $p \in M$, we can extend the domain of $f$ to an open set $U$ around $p$. (Of course, the condition that the vector $e_4 \circ f$ projects to be normal to $M$ does not apply at points not on $M$.) Then, applying (23) to vectors $v$ tangent to $M$ shows that $f^*\eta^4 = 0$.

Furthermore, for $1 \leq j, k \leq 3$, we have $(f^*\eta^k)(e_j) = \delta_j^k$ on $M$. Thus, using (24), we have for $1 \leq \ell \leq 3$,

$$\sum_{k=1}^{3} \langle Ae_j, e_k \rangle f^*\eta^k(e_\ell) = \sum_{k=1}^{3} \langle Ae_j, e_k \rangle \delta_k^\ell = \langle Ae_\ell, e_j \rangle$$

$$= -\langle \nabla_{e_\ell}e_4, e_j \rangle = -f^*\omega_4^\ell(e_\ell) = f^*\omega_j^4(e_\ell)$$

which establishes the desired result. \qed

**Proof of Proposition [7].** We will verify the statements in the proposition using an adapted lift $f$ of the Hopf hypersurface $M$. (Such adapted lifts exist locally near any point of $M$.) Let $e_1, \ldots, e_4$ be as in the proof of Lemma 6. By the Hopf condition,

$$\langle Ae_1, e_3 \rangle = \langle Ae_2, e_3 \rangle = 0, \quad \langle Ae_3, e_4 \rangle = \alpha.$$

Thus, $f^*(\omega_3^4 - \alpha \eta^3) = 0$. (For the rest of this section, differential forms will be understood to be pulled back by $f$.)

Differentiating $\eta^4 = 0$ and $\omega_3^4 - \alpha \eta^3 = 0$ (and substituting for $\eta^4$ and $\omega_3^4$ using these equations) gives

$$\omega_1^4 \wedge \eta^1 + \omega_2^4 \wedge \eta^2 = 0,$$

$$2\omega_4^4 \wedge \omega_2^4 + \alpha (\omega_2^4 \wedge \eta^1 - \omega_1^4 \wedge \eta^2) + \frac{2}{r^2} \eta^1 \wedge \eta^2 = 0.$$

Linearly combining (26) and (27) gives two equivalent equations,

$$(\omega_1^4 - \frac{1}{r} (\sin \phi \eta^1 + \cos \phi \eta^2)) \wedge (\omega_2^4 - \frac{1}{r} (\sin \phi \eta^2 - \cos \phi \eta^1)) = 0,$$

$$(\omega_1^4 - \frac{1}{r} (\sin \phi \eta^1 - \cos \phi \eta^2)) \wedge (\omega_2^4 - \frac{1}{r} (\sin \phi \eta^2 + \cos \phi \eta^1)) = 0.$$

(Recall that $\alpha = (2/r) \sin \phi$.) For the sake of convenience, we define

$$\kappa_1^+ = \omega_1^4 - \frac{1}{r} (\sin \phi \eta^1 \pm \cos \phi \eta^2),$$

$$\kappa_2^+ = \omega_2^4 - \frac{1}{r} (\sin \phi \eta^2 \mp \cos \phi \eta^1).$$

Then the equations (28) imply that $\kappa_1^+$ and $\kappa_2^+$ are linearly dependent, as are $\kappa_1^-$ and $\kappa_2^-$. (Note, however, that at least one form in each pair is nonzero at every point.)

Recall the characteristic distributions on $M$ defined in (2.2). There is no loss of generality here in assuming that our frame is principal on $M$, i.e $Ae_1 = \lambda e_1$ and $Ae_2 = \nu e_2$ for
distinct principal curvature functions $\lambda, \nu$. Using $\omega_1^4 = \lambda \eta^1$ and $\omega_2^4 = \nu \eta^2$, it is easy to check that $\kappa_1^+, \kappa_2^+$ vanish on $\chi^+$, and $\kappa_1^-, \kappa_2^-$ vanish on $\chi^-$. To show integrability, suppose that $\kappa_1^+ \neq 0$ near a point $p \in M$. Choose 1-forms $\tau_1, \tau_2$ so that $\kappa_1^+ \wedge \tau_1 \wedge \tau_2 \neq 0$. Then, $d\kappa_1^+ = \kappa_1^+ \wedge \theta + k \tau_1 \wedge \tau_2$ for some 1-form $\theta$ and some scalar $k$. Once we check that $\kappa_1^+ \wedge d\kappa_1^+ = 0$, we get $k = 0$, which establishes the integrability condition for $\chi^+$ near $p$. Using $\eta^4 = 0$ and $\omega_3^4 = \alpha \eta^3$, we calculate that

$$d\kappa_1^+ = \frac{1}{r} (\cos \phi \kappa_1^+ + \sin \phi \kappa_2^+) \wedge \eta^3 + \eta_1^2 \wedge \kappa_2^+.$$

Since $\kappa_1^+$ and $\kappa_2^+$ are linearly dependent, we must have $d\kappa_1^+ \wedge \kappa_1^+ = 0$.

A similar argument works for $\kappa_2^+ \neq 0$ and for the distribution $\chi^-$. This proves the first assertion of the proposition.

Next, we show that the maps $g_C^\pm$ are constant along the corresponding characteristic surfaces in $M$. For simplicity, we check this assertion for $g_C^+$; the argument for $g_C^-$ being similar. First, before restricting to $M$, we use the equations (18) and (19) to calculate

$$dg^+ = \frac{1}{r} (\sin \phi \eta^4 - \cos \phi \eta^3) g^+ + \frac{1}{r} (\eta^0 + \sin \phi \eta^3 \pm \cos \phi \eta^4) ig^+$$

$$- \kappa_1^+ (\sin \phi e_2 - \cos \phi e_1) + \kappa_2^+ (\sin \phi e_1 + \cos \phi e_2) - \kappa_3^+ (\sin \phi e_4 - \cos \phi e_3),$$

where, for future reference, we define

$$\kappa_3^\pm = \omega_3^4 - \frac{2}{r} (\sin \phi \eta^3 \pm \cos \phi \eta^4) = \omega_3^4 - \alpha \eta^3 \mp \frac{2}{r} \cos \phi \eta^4.$$

Once we restrict to $M$, then $\eta^4 = \kappa_3^\pm = 0$, so that

$$dg^+ = \frac{1}{r} \cos \phi \eta^3 g^+ + \frac{1}{r} (\eta^0 - \sin \phi \eta^3) ig^+ \mod \kappa_1^+, \kappa_2^+.$$

Thus, along a curve in $M$ tangent to the leaves of $\chi^+$, the derivative of $g^+$ is a complex multiple of the value of $g^+$. Therefore, the complex line spanned by $g^+$ remains fixed as we move along the leaves of $\chi^+$.

\[\square\]

3.3. Proof of Main Result. Given any point of $M$, we can construct an adapted lift $f$ on a neighborhood of that point. To prove the first part of Assertion 1 in the theorem, we need to show that the composition $g_C^\pm \circ f$ is unchanged when we modify the adapted lift. For simplicity, we will show that $g_C^+ \circ f$ is unchanged, the argument for $g_C^-$ being similar.

At a given point of $M$, any two adapted lifts differ either by moving the point $z$ along the fiber of the quotient map $\pi : H \rightarrow CH^2$, or by rotating $e_1$ and $e_2$ while keeping their span fixed. We will define vector fields on $G$ whose trajectories correspond to modifying the frame in these ways. Note that by Lemma 6, an adapted lift of a hypersurface is a 3-dimensional submanifold $\Sigma \subset G$ on which the 1-form $\eta^4$ pulls back to be zero. (As well, because the projection of this submanifold to $H$ must be transverse to the fibers of $\pi$, the 3-form $\eta^1 \wedge \eta^2 \wedge \eta^3$ must pull back to be nonzero at each point of $\Sigma$.)

First, a left-invariant vector field $X$ on $G$ that moves $z$ in the direction of $e_0$ (i.e., tangent to the fiber of $\pi$) must, by (15), have the property that $X \cdot \eta^0$ is a nonzero constant, while its interior product with $\eta^1, \ldots, \eta^4$ must be zero. In order that motion along the trajectories of $X$ take adapted lifts to adapted lifts, we need $L_X \eta^4$ to be a multiple of $\eta^4$. Because

$$d\eta^4 = -\omega_1^4 \wedge \eta^1 - \omega_2^4 \wedge \eta^2 - \omega_3^4 \wedge \eta^3,$$
the interior product of $X$ with $\omega_1^4, \omega_2^4, \omega_3^4$ must be zero. (To define $X$ uniquely, we normalize $X \cdot \eta^0 = 1$ and require also that $X \cdot \omega^2 = 0$.) Then from (30) we see that
\[ X \cdot d\mathfrak{g}^+ = \frac{i}{r} \mathfrak{g}^+. \]
Thus, this change of adapted lift only changes $\mathfrak{g}^+$ by multiplication by a unit modulus constant.

Next, note that (19) specializes to
\[ de_1 = \omega_1^2 e_2 + \omega_2^2 e_3 + \omega_3^2 e_4 + \frac{1}{r^2} \eta^1 \zeta - \frac{1}{r^2} \eta^2 e_0. \]
Therefore, a left-invariant vector field $Y$ on $G$ that corresponds to rotating $e_1$ in the direction of $e_2$ satisfies $Y \cdot \omega^2 = 0$, while the interior product of $Y$ with all the other 1-forms $\eta^0, \ldots, \eta^4, \eta^1, \eta^2$ and $\eta^3$ is zero. (To see that motion along the trajectories of $Y$ takes adapted lifts to adapted lifts, it is easy to check that $\mathcal{L}_Y \eta^4 = 0$.) Then we easily calculate that $Y \cdot d\mathfrak{g}^+ = 0$.

It follows from Proposition 1 that the images of $\mathfrak{r} = \mathfrak{g}^+_C \circ f$ are curves. (Moreover, because for each choice of sign, one of $\kappa^\pm_1, \kappa^\pm_2$ is nonzero at every point of $M$, (30) shows that the images are regular curves.) To prove the rest of Assertion 1, we need to show that these are contact curves.

Recall that the maps $\mathfrak{g}^+_C$ take value in the projectivization of $\pi(V)$ of the null cone $V \subset \mathbb{C}^3$. As explained in [2.2] this may be identified with $S^3$. We now give a characterization of the standard contact structure in $S^3$ under this identification:

**Lemma 7.** Let $\mu : I \to \pi(V)$ be a regular curve (where $I$ is an interval on the real line), and let $n : I \to V$ be any lift of $\mu$. Then the image of $\mu$ is a contact curve if and only if
\[ \left\langle \frac{dn}{dt}, \mathfrak{r}(t) \right\rangle = 0. \]

**Proof.** Note that condition (31) is independent of choice of lift $n$.

Suppose that $n(t) = (z_0(t), z_1(t), z_2(t))$ with $|z_0|^2 = |z_1|^2 + |z_2|^2$. Then
\[ \left\langle \frac{dn}{dt}, \mathfrak{r} \right\rangle = \text{Re} \left( -i \left( \overline{z_1} z_1' + \overline{z_2} z_2' - \overline{z_0} z_0' \right) \right). \]
Meanwhile, the corresponding curve in $\mathbb{C}^2$, given by $w_1 = z_1/z_0$ and $w_2 = z_2/z_0$, satisfies
\[ \left\langle \frac{dw}{dt}, iw \right\rangle = \text{Re} \left( -i \left( \frac{1}{|z_0|^2} \left( \overline{z_1} (z_0 z_1' - z_1 z_0') + \overline{z_2} (z_0 z_2' - z_2 z_0') \right) \right) \right) \]
\[ = \text{Re} \left( \frac{1}{|z_0|^2} \left( \overline{z_1} z_1' + \overline{z_2} z_2' - \overline{z_0} z_0' \right) \right), \]
where we use the usual Hermitian inner product on $\mathbb{C}^2$. (Note that at $w \in S^3$, the vector $iw$ is tangent to the fibers of the Hopf fibration.) Therefore, $\mu(t)$ is a contact curve in the space of complex null lines if and only if $w(t)$ is a contact curve in $S^3 \subset \mathbb{C}^2$.

We finish the proof of Assertion 1 with the following lemma, which will also be used in the proof of Assertion 2.

**Lemma 8.** Let $\mu : I \to S^3$ be any regular curve (where $I$ is an interval on the real line), and let $\gamma : I \to G$ be any lift such that $\mathfrak{g}^+_C \circ \gamma = \mu$. Then $\mu$ is a contact curve if and only if $\gamma^* \kappa^\pm_3 = 0$. 

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Proof. We will verify the assertion for \( g_C^+ \), the argument for \( g_C^- \) being similar. First, note that \( g^+ \circ \gamma \) is a lift of \( \mu \) into the null cone \( \mathcal{V} \). Let \( t \) be a coordinate on \( I \), and let \( g^+(t) \) stand for \( g^+(\gamma(t)) \). Then, applying Lemma 7 and using (30), we see that \( \mu \) is a contact curve if and only if

\[
0 = \left< \frac{dg^+}{dt} \cdot ig^+(t) \right> = \frac{\partial}{\partial t} \omega(\gamma^* \kappa_3^+).
\]

\( \square \)

To begin the proof of Assertion 2, note that the nine 1-forms \( \eta^0, \eta^3, \omega^1 \) and \( \kappa_1^+, \kappa_2^+, \kappa_3^+ \) comprise a coframe on \( G \). Thus, by (30), the map \( g_C^+ \) has constant rank 3 (and the same is true for \( g_C^- \)). Suppose that \( C_1 \) is an embedded contact curve in \( S^3 \), and let \( N \subset G \) be its inverse image under \( g_C^+ \). Then \( N \) is a 7-dimensional submanifold. Because \( \kappa_3^+ \) must pull back to be zero on \( N \), and \( \kappa_1^+, \kappa_2^+ \) pull back to be linearly dependent on \( N \), then the remaining forms \( \eta^0, \eta^3, \omega^1, \kappa_1^+, \kappa_2^+, \kappa_3^- \) pull back to \( N \) to be linearly independent. In particular, the restriction of \( g_C^- \) to \( N \) still has rank 3. It follows that the intersection \( P \) of the inverse images of the contact curves is a smooth submanifold of dimension 5.

Since the maps \( g_C^\pm \) are constant along the trajectories of \( X \) and \( Y \), then \( P \) is foliated by these trajectories. It follows that the image \( M = \rho(P) \) inside \( \mathbb{C}P^2 \) is three-dimensional. Moreover, any adapted lift of \( M \) lies inside \( P \). Because the 1-forms \( \kappa_3^+ \) and \( \kappa_3^- \) pull back to be zero on \( P \), the same is true for \( \omega^3 - \alpha \eta^3 \). So, for any adapted lift \( f \) of \( M \), \( f^*(\omega^3 - \alpha \eta^3) = 0 \). It then follows from Lemma 6 that \( M \) is a Hopf hypersurface with Hopf principal curvature \( \alpha \).

This finishes the proof of Theorem 2.

3.4. Borderline Case. In the borderline case, where \( \alpha = \pm 2/r \) the functions \( g^+ \) and \( g^- \) coincide. However, we can still use this modified Gauss map to associate a single contact curve to a Hopf hypersurface, and to some extent we can reconstruct the hypersurface from the curve. For simplicity, say that \( \alpha = 2/r \), whence \( \sin \phi = 1 \) and \( \cos \phi = 0 \). Let

\[
g = e_0 - e_3,
\]

and let \( g_C = \pi \circ g \) denote the projectivization of \( g \). Then (30) specializes to

\[
dg = \frac{1}{r} \eta^4 \, g + \frac{1}{r} (\eta^0 + \eta^3) ig - \kappa_1 e_2 + \kappa_2 e_1 - \kappa_3 e_4,
\]

where we define

\[
\kappa_1 = \omega_1^4 - \frac{1}{r} \eta^4, \quad \kappa_2 = \omega_2^4 - \frac{1}{r} \eta^2, \quad \kappa_3 = \omega_3^4 - \frac{2}{r} \eta^3 = \omega_3^4 - \alpha \eta^3.
\]

The proof in 3.3 for Assertion 1 may be suitably modified to show that the composition \( \sigma = g_C \circ f \), where \( f \) is an arbitrary adapted lift of a Hopf hypersurface \( M \) with Hopf principal curvature \( \alpha \), gives a well-defined map to \( S^3 \), and whose image is a contact curve. (In fact, the proof of Lemma 8 can be modified to show that a curve \( \mu : I \to S^3 \) is contact if and only if \( \gamma^* \kappa_3 = 0 \), where \( \gamma : I \to G \) is any lift such that \( g_C \circ \gamma = \mu \).) Note that (8) implies that, at each point of \( M \), one of the other principal curvatures \( \lambda, \nu \) must equal \( 1/r \). If they are not both equal to \( 1/r \), then one of \( \kappa_1, \kappa_2 \) is nonzero at each point, and rank of \( \sigma \) is one.

Thus, given an arbitrary regular contact curve \( \mathcal{C} \) in \( S^3 \), the inverse image \( N \) under \( g_C \) of this curve is a codimension two submanifold of \( G \) on which \( \kappa_3 \) pulls back to be zero and \( \kappa_1, \kappa_2 \) pull back to be linearly dependent. We wish to construct a three-dimensional submanifold \( \Sigma \subset N \) which will be an adapted lift of a Hopf hypersurface. To do this, we
only need to ensure that \( \eta^4 \) pulls back to be zero on \( \Sigma \), and the forms \( \eta^1, \eta^2, \eta^3 \) pull back to be linearly independent. (We refer to the latter as the independence condition for \( \Sigma \).) Then the vanishing of \( \kappa_3 \) will ensure that the Hopf condition holds.

It is easy to check that
\[
\text{d}\eta^4 \equiv -\kappa_1 \wedge \eta^1 - \kappa_2 \wedge \eta^2 \mod \kappa_3.
\]

Thus, because the span of \( \{ \kappa_1, \kappa_2 \} \) is one-dimensional on \( N \) (the regularity of the contact curve rules out the vanishing of both \( \kappa_1 \) and \( \kappa_2 \) at any point), the Pfaff rank of \( \eta^4 \) is one. (In other words, \( \eta^4 \wedge \text{d}\eta^4 \neq 0 \) but \( \eta^4 \wedge \text{d}\eta^4 \wedge \text{d}\eta^4 = 0 \).) It follows by Pfaff’s theorem (see [3] Chapter One) that around any point of \( N \) there is an open set in \( N \) and a local coordinate system \( x_0, \ldots, x_6 \) defined on that set, in which \( \eta^4 \) is a multiple of \( \text{d}x_1 - x_2 \text{d}x_0 \). It follows that by setting \( x_1 = \psi(x_0), \ x_2 = \psi'(x_0) \) for an arbitrary function \( \psi \), we can construct a five-dimensional submanifold \( P \) on which both \( \kappa_3 \) and \( \eta^4 \) pull back to be zero. As argued in §3.3 the trajectories of \( X \) and \( Y \) foliate \( P \), and its image under \( \rho \) is a Hopf hypersurface in \( \mathbb{C}H^2 \).

To summarize, here is the analogue of Theorem 2 in the borderline case:

**Theorem 9.**

1. Let \( M \) be an oriented Hopf hypersurface in \( \mathbb{C}H^2 \) with Hopf principal curvature \( \alpha = \pm 2/r \). Then there is a well-defined map \( \sigma : M \to S^3 \), such that \( \sigma = g_C \circ f \) for any adapted lift \( f \) of \( M \), and such that the image \( \sigma(M) \) is a contact curve in \( S^3 \). (The map \( \sigma \) is regular at points where the principal curvatures of \( M \) are distinct.)
2. Conversely, given any regular contact curve \( C \) in \( S^3 \), let \( N \subset G \) be its inverse image under \( g_C \). Given \( p \in N \), there exists an open neighborhood \( U \subset N \) containing \( p \), and a family of five-dimensional manifolds \( P \subset U \) containing \( p \), such that \( \pi(P) \) is a Hopf hypersurface with principal curvature \( \pm 2/r \). (The family is parametrized by a choice of one function of one real variable.)

4. **Afterword**

Our original approach to the construction of Hopf hypersurfaces in \( \mathbb{C}H^2 \) was to analyze the Hopf condition using the techniques of exterior differential systems (EDS). From this point of view, adapted lifts of Hopf hypersurfaces are 3-dimensional integral submanifolds of the EDS on \( G \) generated by the 1-forms \( \eta^4 \) and \( \omega^4 - \alpha \eta^3 \). Applying Cartan’s Test shows that this Pfaffian system is involutive with last nonzero characteristic \( s_1 = 2 \). This indicates that locally-defined solutions may be constructed by solving a sequence of Cauchy problems, in which the last time we have freedom to specify the initial data, that choice amounts to two functions of one variable. The existence of these solutions is guaranteed by the Cartan-Kähler theorem.

However, the Cartan-Kähler theorem does not give an explicit construction for solutions, so one has to work harder. In fact, our analysis shows that, when \( \alpha \) is strictly between \(-2/r \) and \( 2/r \), this EDS is hyperbolic, and is integrable by the method of Darboux. The construction scheme for Hopf hypersurfaces given by Theorem 2 is based on this observation. We note that the function count given by Cartan’s Test is explicitly realized by this construction, since specifying a contact curve in a three-dimensional manifold amounts to choosing an arbitrary function of one variable (because the contact form may be written as \( dy - z \, dx \) in suitable local coordinates).
References

[1] Bryant, R.L., Chern, S.-S., Gardner, R.B., Goldschmidt, H.L., Griffiths, P.A.: Exterior differential systems. MSRI Publications 18, Springer-Verlag, New York (1989)

[2] Cecil, T.E., Ryan, P.J.: Focal sets and real hypersurfaces in complex projective space. Trans. Amer. Math. Soc. 269, 481–499 (1982)

[3] Ivey, T.A., Landsberg, J.M.: Cartan for Beginners: Differential geometry via moving frames and exterior differential systems. Graduate Studies in Mathematics 61, American Mathematical Society, Providence, RI (2003)

[4] Ki, U-H., Suh, Y.J.: On real hypersurfaces of a complex space form. Math. J. Okayama Univ. 32, 207–221 (1990)

[5] Kim, H.S., Ryan, P.J.: A classification of pseudo-Einstein hypersurfaces in CP^2, Differential Geom. Appl. 26 106–112 (2008)

[6] Kon, M.: Pseudo-Einstein real hypersurfaces in complex space forms. J. Differential Geom. 14, 339–354 (1979)

[7] Maeda, Y.: On real hypersurfaces of a complex projective space. J. Math. Soc. Japan 28, 529–540 (1976)

[8] Martins, J.K.: Hopf hypersurfaces in space forms. Bull. Braz. Math. Soc. (N.S.) 35, 453–472 (2004)

[9] Montiel, S.: Real hypersurfaces of a complex hyperbolic space. J. Math. Soc. Japan 37, 515–535 (1985)

[10] Niebergall, R., Ryan, P.J.: Real hypersurfaces in complex space forms. In: Chern, S.-S., Cecil, T.E. (eds.) Tight and taut submanifolds, pp. 233–305. MSRI Publications 32, Cambridge University Press, Cambridge (1997)

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