Quantization of function algebras on semisimple orbits in $\mathfrak{g}^*$

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Abstract

In this paper we describe a multiparameter deformation of the function algebra of a semisimple coadjoint orbit. In the first section we use the representation of the Lie algebra on a generalized Verma module to quantize the Kirillov bracket on the family of semisimple coadjoint orbits of a given orbit type. In the second section we extend this construction to define a deformation in the category of representations of the quantized enveloping algebra. In an earlier paper we used cohomological methods to prove the existence of a two parameter family quantizing a compatible pair of Poisson brackets on any symmetric coadjoint orbit. This paper gives a more explicit algebraic construction which includes more general orbit types and which we prove to be flat in all parameters.

1 Quantizing the Kirillov bracket

Assume we have the following data: a simple Lie algebra, $\mathfrak{g}$, over the complex field, $\mathbb{C}$, with corresponding Lie group, $G$, and a Cartan subalgebra, $\mathcal{H}$, together with a system of simple positive roots. Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathcal{H} \oplus \mathfrak{n}^+$, be the corresponding Cartan decomposition, where $\mathfrak{n}^+, \mathfrak{n}^-$ are the nilpotent subalgebras which are, respectively, the sum of positive root spaces and the sum of the negative root spaces. Denote by $\mathfrak{g}^*$ and $\mathcal{H}^*$ the $\mathbb{C}$ linear duals of $\mathfrak{g}$ and $\mathcal{H}$. Given $\lambda \in \mathfrak{g}^*$, let $\mathcal{O}_\lambda$ be the orbit of $\lambda$ under the coadjoint action of $G$ on $\mathfrak{g}^*$. Finally, let $S(\mathfrak{g})$ be the symmetric algebra of $\mathfrak{g}$ considered as polynomial functions on $\mathfrak{g}^*$ and $\mathcal{I}_\lambda$ the ideal of polynomials vanishing on the orbit $\mathcal{O}_\lambda$. Then $\mathcal{F}_\lambda = S(\mathfrak{g})/\mathcal{I}_\lambda$ is the algebra of functions on $\mathcal{O}_\lambda$, with Poisson structure given by the standard Kostant-Kirillov-Souriau (KKS) bracket
\[ \{f, g\}(\lambda) = \langle \lambda, [df_\lambda, dg_\lambda]\rangle. \]

Consider the $G$ orbits of semisimple elements, that is, linear functions conjugate under the coadjoint action to ones which vanishes on $\mathfrak{n}^+ \oplus \mathfrak{n}^-$. In any such orbit we can pick as origin an element, $\lambda$, which is the trivial extension to $G$ of a functional on $\mathcal{H}$. The stabilizer of such an element will be a subalgebra generated by $\mathcal{H}$ and a subset of the simple roots. Such a subalgebra is called a Levi subalgebra. The set of all $\lambda$ with stabilizer equal to a fixed Levi algebra $\mathcal{L}$ is parametrized by a subspace of $\mathcal{H}^*$ minus its intersection with a family of coordinate hyperplanes. Denote this parameter space by $\Lambda_\mathcal{L}$, and represent the elements by
$m$-tuples, $\lambda = (\lambda_1, \ldots, \lambda_m)$, where the $\lambda_i$ are the values of $\lambda$ on a fixed subset of the set of simple positive roots and all $\lambda_i$ are different from zero. (In the case of $\mathcal{G} = sl(n, \mathbb{C})$, this corresponds to a Levi algebra which is block diagonal with $m + 1$ blocks.) The algebras we are interested in quantizing are the $\mathcal{F}_\lambda$ as $\lambda$ varies in $\Lambda_L$.

Define the parabolic subalgebra, $\mathcal{P} = \mathcal{L} \oplus \mathcal{N}^+$. Let $M_\lambda = \text{Ind}^G_P 1_\lambda$ be the representation of $\mathcal{G}$ induced from the one dimensional representation of $\mathcal{P}$ defined by $\lambda$. Let $\mathcal{N}_P^-$ be the subalgebra of $\mathcal{N}^-$ which is a vector space complement to $\mathcal{P}$ in $\mathcal{G}$. As a vector spaces all the $M_\lambda$ can be identified with $M_P := U(\mathcal{N}_P^-)$ by the isomorphism

$$M_\lambda = U(\mathcal{G}) \otimes_{U(\mathcal{P})} 1_\lambda \cong U(\mathcal{N}_P^-) = M_P.$$

Henceforth we use this identification and denote by $v_0$ the element of $M_\lambda$ corresponding to $1 \in U(\mathcal{N}_P^-)$. The element $v_0$ will be the highest weight vector of $M_\lambda$ for all $\lambda$. Let $\phi_\lambda : \mathcal{G} \to \text{End}(M_\mathcal{P})$ be the Lie algebra homomorphism defining the representation $M_\lambda$, then for all $X \in \mathcal{G}$ and all $\lambda \in \Lambda_L$

$$\phi_\lambda(X)v_0 = \lambda(X)v_0 + \ldots \text{lower weight terms}.$$

We study the quantization relative to indeterminates, $h$ (the formal deformation parameter) and $\lambda_i$ (parametrizing $\Lambda_L$). Let $\text{End}(M_\mathcal{P})[\lambda, h]$ be the ring of polynomials $h, \lambda_1, \ldots, \lambda_m$ with coefficients in $\text{End}(M_\mathcal{P})$ and $T(\mathcal{G})[\lambda, h]$ the ring of polynomials in $h, \lambda_1, \ldots, \lambda_m$ with coefficients in the tensor algebra of $\mathcal{G}$. The map

$$h\phi_\lambda/h : \mathcal{G} \longrightarrow \text{End}(M_\mathcal{P})[\lambda, h]$$

extends to a homomorphism of $\mathbb{C}[\lambda, h]$ algebras:

$$\phi_{\lambda, h} : T(\mathcal{G})[\lambda, h] \longrightarrow \text{End}(M_\mathcal{P})[\lambda, h].$$

Consider $T(\mathcal{G})[\lambda, h]$ as a graded algebra with elements of $\mathcal{G}$ and the indeterminates $\lambda$ and $h$ considered as having degree one. Define a grading on the algebra $\text{End}(M_\mathcal{P})[\lambda, h]$ with the elements of $\text{End}(M_\mathcal{P})$ having degree zero and $\lambda$ and $h$ having degree one. Then let $\mathcal{A}_{\lambda, h}$ be the subalgebra of $\text{End}(M_\mathcal{P})[\lambda, h]$ generated by the image of $\phi_{\lambda, h}$. Since $\phi_{\lambda, h}$ is a map of graded algebras, the image $\mathcal{A}_{\lambda, h}$ is a graded subalgebra. As a torsion free submodule of the free $\mathbb{C}[h]$ module, $\text{End}(M_\mathcal{P})[\lambda, h]$, it is also a free $\mathbb{C}[h]$ module. We will prove the following

**Theorem 1.1** The graded algebra $\mathcal{A}_{\lambda, h}$ is a free $\mathbb{C}[\lambda, h]$ module. At all points $\lambda = \lambda_0$ of $\Lambda_L$, $\mathcal{A}_{\lambda_0, h}$ is a quantization of the Poisson algebra $\mathcal{F}_{\lambda_0}$ with KKS bracket.
Proof. In the course of proving the second part of the theorem we will show that the quotient \( A_{\lambda,h}/hA_{\lambda,h} \) is isomorphic to a certain free \( \mathbb{C}[\lambda] \) module, \( N \otimes_{\mathbb{C}} \mathbb{C}[\lambda] \), where \( N \) is an induced representation of \( \mathcal{G} \) defined independently of \( \lambda \). Then

\[
A_{\lambda,h} \cong A_{\lambda,h}/hA_{\lambda,h} \otimes_{\mathbb{C}} \mathbb{C}[h] \cong N \otimes_{\mathbb{C}} \mathbb{C}[\lambda] \otimes_{\mathbb{C}} \mathbb{C}[h] \cong N \otimes_{\mathbb{C}} \mathbb{C}[\lambda, h],
\]

which proves the first assertion of the theorem.

In order to prove the second statement we must prove the following two statements:

1. For all \( \lambda_0 \in \Lambda_{\mathcal{L}} \) the algebra \( A_{\lambda_0,h}/hA_{\lambda_0,h} \) is isomorphic to the algebra \( \mathcal{F}_{\lambda_0} \).

2. Denoting the isomorphism in 1. by \( \sigma \) and defining the Poisson bracket of two cosets, \([f],[g]\), in \( A_{\lambda,h}/hA_{\lambda,h} \), by

\[
\{[f],[g]\} = \left[ \frac{1}{h} (fg - gf) \right] \quad \text{we obtain} \quad \sigma([f],[g]) = \{\sigma(f),\sigma(g)\}.
\]

When \( \lambda \) is evaluated at a fixed vector in \( \Lambda_{\mathcal{L}} \), the algebra \( A_{\lambda,h} \) is, by definition, generated as an \( \mathbb{C}[h] \) subalgebra of \( \text{End}(M_\mathcal{P})[h] \) by the elements \( \phi_{\lambda,h}(X) = h\phi_{\lambda,h}(X) \) for \( X \in \mathcal{G} \). To simplify notation, let \( \phi := \phi_{\lambda,h} \). The Lie algebra homomorphism property \( \phi(X)\phi(Y) - \phi(Y)\phi(X) = \phi([X,Y]) \) implies that

\[
(h\phi)(X)(h\phi)(Y) = (h\phi)(Y)(h\phi)(X) \mod h \cdot \text{Im}(h\phi).
\]

Therefore the algebra \( A_\lambda = A_{\lambda,h}/hA_{\lambda,h} \) is commutative and \( \phi_{\lambda,h} \) induces a map

\[
\psi: S(\mathcal{G}) \to A_\lambda.
\]

Let \( X_i^\pm \) be a basis for \( \mathcal{N}_\pm \) and \( X_j^0 \) a basis for \( \mathcal{H} \), and, as usual, for any multi-index \( A = (a_i^-, a_j^0, a_i^+) \), let \( X^A = \Pi(X_i^-)^{\alpha_i^-} \Pi(X_j^0)^{\alpha_j^0} \Pi(X_i^+)^{\alpha_i^+} \) and \( |A| = \sum a_i^- + \sum a_j^0 + \sum a_i^+ \). We use \( X^\hat{A} \) to denote the symmetrization of the monomials \( X^A \), considered as elements of \( T(\mathcal{G}) \) and as a basis for \( S(\mathcal{G}) \). By definition

\[
\phi_{\lambda,h}(X^\hat{A}) = h^{|A|}(\phi(X))^\hat{A},
\]

where the latter expression is the symmetrization in \( \text{End}(M_\mathcal{P}) \) of the products of the linear factors \( h\phi(X_i) \). Now suppose that \( f(X) = \sum_A c_A(\lambda)X^\hat{A} \in \text{Ker}\psi \), then there exists an element of \( g(X) \in T(\mathcal{G})[\lambda, h] \) such that \( \phi_{\lambda,h}(f(X)) = h\phi_{\lambda,h}(g(X)) \). Applying the difference to the highest weight vector, we get

\[
\phi_{\lambda,h}(f(X) - hg(X))v_0 = 0.
\]
On the other hand, the coefficient of the \( v_0 \) in this expression depends only on the summands which have weight zero relative to \( H \). Furthermore any term of weight zero containing a pair of a negative root vector, \( \phi(X^-) \), and a positive root vector, \( \phi(X^+) \), will contribute to the coefficient of \( v_0 \) only via commutators \( \phi([X^-_i, X^+_j]) \). For such terms the homogeneity in \( h \) exceeds the number of factors containing \( \phi \). The same is true of all the terms in \( \phi_{\lambda,h}(h\phi(X)) \).

This means that the constant term in \( h \) in the coefficient of \( v_0 \) in \( \phi_{\lambda,h}(f(X) - h\phi(X))v_0 \) is

\[
0 = \sum_A c_A(\lambda(X))^A = f(\lambda(X)).
\]

But identifying \( S(\mathcal{G}) \) with the polynomial functions on \( \mathcal{G}^* \), \( f(\lambda(X)) \) is the \( f(\lambda)(\lambda) \), the evaluation of the function \( f(X) \) at the point \( \lambda \in \mathcal{G}^* \). This proves that any \( f(X) \in \text{Ker}\psi \) vanishes at the point \( \lambda \). To prove that \( f(X) \) vanishes on the entire orbit, we use the invariance of \( \text{Ker}\psi \) under the adjoint action of \( G \). For any \( g \in G \) we have \( f(X)(g\lambda) = (g^{-1}f(X))(\lambda) = 0 \), which completes the proof that \( \text{Ker}\psi \subset \mathcal{I}_\lambda \).

The inclusion just proved shows that \( \psi \) induces an epimorphism

\[
\pi_\lambda : \mathcal{A}_\lambda \cong S(\mathcal{G})/\text{Ker}\psi \to S(\mathcal{G})/\mathcal{I}_\lambda = \mathcal{F}_\lambda.
\] (1)

To prove that this is an isomorphism, we use the fact that \( \mathcal{F}_\lambda \) decomposes into a direct sum of finite dimensional irreducible representations of \( \mathcal{G} \). On the other hand, being a free \( \mathbb{C}[h] \) module, \( \mathcal{A}_{\lambda,h} \cong N \otimes_{\mathbb{C}} \mathbb{C}[h] \) and as \( \mathbb{C} \) vector spaces, \( \mathcal{A}_\lambda = A_{\lambda,h}/hA_{\lambda,h} \cong N \). Evaluation at the generic value of \( h \) gives an isomorphism of \( N \) with the image of the representation \( \phi_{\lambda/h} \), where, for any \( \lambda \in \Lambda_\mathcal{L} \), \( \lambda/h \) is a generic \( \mathbb{C} \) valued weight vector in \( \Lambda_\mathcal{L} \). Let

\[
N = \oplus n_\mu(\lambda/h)V_\mu \quad \text{and} \quad \mathcal{F}_\lambda = \oplus \ell_\mu(\lambda)V_\mu.
\]

The \( \mathcal{G} \) equivariance of the epimorphism \( \pi_\lambda \) implies that \( n_\mu(\lambda/h) \geq \ell_\mu(\lambda) \).

To establish the reverse inequality \( \pi_\lambda \) implies that \( n_\mu(\lambda/h) \geq \ell_\mu(\lambda) \).

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**Lemma 1.2** Let \( M_\lambda^* \) be the restricted dual of \( M_\lambda \). For generic \( \lambda \), \( M_\lambda^* \) is a generalized Verma module for the parabolic algebra \( \mathcal{L} \oplus \mathcal{N}_\mathcal{P} \) generated as a principal \( U(N_\mathcal{P}^+) \) module by the lowest weight vector \( v_0^* \).
Let $\text{Ind}_L^G \mathbf{1}$ be the representation of $G$ induced from the trivial representation of $L$ with basis vector $\mathbf{1}$. Then $\mathbf{1} \mapsto v_0 \otimes v_0^*$ induces a $G$ equivariant morphism

$$\tau_\lambda : \text{Ind}_L^G \mathbf{1} \xrightarrow{\cong} M_\lambda \otimes_{\mathbb{C}} M_\lambda^*.$$  

We have the following isomorphisms and identities,

$$\text{Ind}_L^G \mathbf{1} \cong U(N_P^+)U(N_P^-)\mathbf{1}, \quad M_\lambda \cong U(N_P^-)v_0, \quad U(N_P^-)v_0^* = 0,$$

and for generic $\lambda$

$$M_\lambda^* \cong U(N_P^+)v_0^*.$$  

A simple inductive argument using the filtration by weights in $M_\lambda$ and $M_\lambda^*$ shows that for generic $\lambda$, $\tau_\lambda$ is an isomorphism.

Dualizing this isomorphism and restricting to $G$ finite elements in $\text{End}(M_\lambda)$ relative to the conjugation action, we get an isomorphism

$$\tau_{\lambda}^* : (M_\lambda \otimes M_\lambda^*)^{G-\text{finite}} \cong (\text{Ind}_L^G \mathbf{1})^{G-\text{finite}}.$$  

An element of $M_\lambda \otimes M_\lambda^*$ defines an element of finite rank in $\text{End}(M_\lambda)$ in the obvious way. We can define a nondegenerate pairing between $A \in \text{End}(M_\lambda)$ and $B \in M_\lambda \otimes M_\lambda^*$ by $\langle A, B \rangle = \text{trace}(A \circ B)$. This induces an imbedding $j$ of $\text{End}(M_\lambda)$ into $(M_\lambda \otimes M_\lambda^*)^*$ whose restriction to the $G$ finite elements will also be denoted by $j$.

For the generic value $\lambda/h$ the composition $\tau_{\lambda/h} \circ j$ defines an imbedding

$$\tau_{\lambda} \circ j : (\text{End}(M_{\lambda/h}))^{G-\text{finite}} \hookrightarrow (\text{Ind}_L^G \mathbf{1})^{G-\text{finite}}.$$  

The representation $\text{Ind}_L^G \mathbf{1}$ is naturally identified with point distributions on the orbit $O_\lambda$ supported at the point $\lambda$. Dualizing, we get an identification of $(\text{Ind}_L^G \mathbf{1})^{G-\text{finite}}$ and $\mathcal{F}_\lambda$.

The restriction of the imbedding $\tau_{\lambda/h} \circ j$ to the subset $N \subset \text{End}(M_{\lambda/h})$ composed with the identification of $(\text{Ind}_L^G \mathbf{1})^{G-\text{finite}}$ and $\mathcal{F}_\lambda$ defines a $G$ equivariant imbedding of $N$ in $\mathcal{F}_\lambda$. This proves the reverse inequality relating multiplicities, $n_\mu(\lambda/h) \leq \ell_\mu(\lambda)$. Therefore we have equality

$$n_\mu(\lambda/h) = \ell_\mu(\lambda),$$

which implies that $\pi_\lambda$ is an isomorphism and $\ker \psi_\lambda = \mathcal{I}_\lambda$, concluding the proof that $A_{\lambda/h}/hA_{\lambda,h} \cong \mathcal{F}_\lambda$. 

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When we consider $\lambda$ as an indeterminate, the isomorphism

$$F_\lambda \cong (\text{Ind}_G^\mu 1)^{\ast f_{\text{finite}} \otimes C[\lambda]}$$

also proves that $A_{\lambda,h}/hA_{\lambda,h}$ is a free $C[\lambda]$ module.

Finally, in relation to the Poisson structures, it is enough to check condition 2. on linear elements in $A_{\lambda,h}$ and $F_\lambda$. For $X, Y \in G$ considered as functions on $\mathcal{O}_\lambda$ their Poisson bracket is the function defined by $[X, Y]$, the bracket in $G$. On the other hand the bracket of the corresponding elements in $A_{\lambda,h}/hA_{\lambda,h}$ is

$$\frac{1}{\hbar}((h\phi)(X)(h\phi)(Y) - (h\phi)(Y)(h\phi)(X)) = h\phi([X, Y]),$$

as required. ■

2 Quantization in the category of $U_q(\mathcal{G})$ modules

Given $\mathcal{G}$ as above, let $U_q(\mathcal{G})$ be the quantized enveloping algebra with generators, $E_i, F_i, K_i^\pm, 1 \leq i \leq n$, satisfying the standard relations given by Lusztig, see [Lu]. We adopt the widely used convention of using $q^{H_i}$ to represent $K_i$. Let $\mathcal{U}^-$ be the subalgebra generated by the $F_i, \mathcal{U}^0$, the (commutative) subalgebra generated by the $q^{H_i}$ which, as an abelian group relative to multiplication, is isomorphic to the lattice $\Pi$ generated by the simple roots, and $\mathcal{U}^+$ the subalgebra generated by the $E_i$. The algebra $\mathcal{U}$ decomposes into a product $\mathcal{U} = \mathcal{U}^-\mathcal{U}^0\mathcal{U}^+$. For any Levi subalgebra $\mathcal{L}$, the quantized universal enveloping algebra $U_q(\mathcal{L})$ is naturally imbedded in $U_q(\mathcal{G})$. Let $\mathcal{U}_\mathcal{L}$ denote the image of this imbedding. It is generated by $E_i, F_i$ for $i$ in a subset of the simple roots, and all $K_i^\pm$. Define the subalgebra of $\mathcal{U}$ associated to the parabolic subalgebra $\mathcal{P} \supset \mathcal{L}$ by

$$\mathcal{U}_\mathcal{P} := U_\mathcal{L}\mathcal{U}^+.$$

For any weight vector $\lambda = \sum \lambda_i \omega_i \in \Lambda_\mathcal{L}$, let $1_{q,\lambda}$ be the one dimensional representation of $\mathcal{U}_\mathcal{P}$ with basis vector $v_0$, which is the trivial representation when restricted to $\mathcal{U}_\mathcal{L}^-$ and to $\mathcal{U}^+$ and is defined for the remaining generators of $\mathcal{U}_\mathcal{P}$, that is, those $q^H \in \mathcal{U}^0$, by

$$q^H v_0 = q^{(\lambda,H)} v_0.$$

Inducing this representation up to $\mathcal{U}$ gives the quantized generalized Verma module:

$$M_{q,\lambda} = U \otimes_{\mathcal{U}_\mathcal{P}} 1_{q,\lambda} \phi_{q,\lambda}(u)(w \otimes v_0) = uw \otimes v_0, \quad u, w \in \mathcal{U}.$$
In this construction we may consider \( q \) and the components of the vector \( \lambda \) as complex parameters, or we may let \( q = e^t \) and consider all constructions over the formal power series in \( t \) and \( \lambda_i t \). As a vector space over \( \mathbb{C} \) or a module over \( \mathbb{C}[[t, \lambda_i t]] \), \( M_{q, \lambda} \) is isomorphic to 
\[ M_E := \mathcal{U}^- / \mathcal{U}_- \.
\]

The next step is to find a subrepresentation of \( \text{End}(M_{q, \lambda}) \) which is a deformation of the adjoint representation of \( \mathcal{G} \). Previously this followed trivially from the fact that the subspace \( \mathcal{G} \subset U(\mathcal{G}) \) is \( \text{ad}(U(\mathcal{G})) \) invariant. In the present context we rely on the work of Joseph and Letzter, \([JL]\), on the structure of the \( \text{ad}(U) \) finite part of \( U \), denoted \( F(U) \).

First of all, the subspace \( \text{ad}(U)q^\alpha \) is finite dimensional if and only if \( \alpha = -4\mu \) for a nonnegative integral weight \( \mu \). Let \( R \) be the set of such elements of the lattice, \( \Pi \), and \( F(\mu) = \text{ad}(U)q^{-4\mu} \) for \( \mu \in R \). Then
\[ F(U) = \bigoplus_{\mu \in R} F(\mu) \text{ and } F(\mu) \cong \text{End}(V_{q, \mu}), \]
where \( V_{q, \mu} \) is the representation of \( U \) deforming the finite dimensional representation of \( \mathcal{G} \) with highest weight \( \mu \).

Let \( \mathcal{G}_q \) denote the deformed adjoint representation, which, by elementary deformation theory, is unique up to equivalence. The representation \( \mathcal{G}_q \) occurs infinitely often in \( F(U) \) since each \( \text{End}(V_{q, \mu}) \) contains a copy of \( \mathcal{G}_q \). However, see \([JL]\), for any finite dimensional representation, \( V_{q, \mu} \), the isotypical component, \( F(U)V_{q, \mu} \), is a finitely generated module over the center, with multiplicity equal to the dimension of the zero weight space in the nonquantized representation, \( V_{\mu} \). Thus, up to multiplication by scalar operators, there are finitely many copies of \( \mathcal{G}_q \) in the image of \( F(U) \) in \( \text{End}(M_{q, \lambda}) \).

We fix a \( \mathcal{G}_q \subset F(U) \) which specializes at \( q = 1 \) to the standard imbedding of \( \mathcal{G} \) in \( U(\mathcal{G}) \), where specialization at \( q = 1 \), is carried out most simply by replacing the generators \( q^{\pm H_i} \) with \( (q^{\pm H_i} - 1)/(q^{d_i} - q^{-d_i}) \) so that the Hopf algebra structure is defined over \( \mathbb{C}[q, q^{-1}] \). Although one could construct the quantization we are seeking in terms of the parameter \( q \) it is convenient to revert to the setting of formal deformations, setting \( q = e^t \) and considering \( t \) and the components \( \lambda_i \) as indeterminates. We continue to use the symbol \( q \) but for the remainder of this section it will be understood as a formal power series in \( t \), and \( \mathcal{G}_q \) will be a free \( \mathbb{C}[[t]] \) module. The tensor algebra \( T(\mathcal{G}_q) \) is defined relative to the tensor product over \( \mathbb{C}[[t]] \).

Let \( \text{End}(M)[[t, \lambda/h, h]] = \text{End}(M_{q, \lambda/h}) \otimes \mathbb{C}[[h]] \) considered as a \( \mathbb{C}[[t, \lambda_i, h]] \) module. From
the representation of $F(U)$ on the quantized Verma module we get a map

$$\phi = h\phi_{q,\lambda/h} : G_q[\lambda, h] \to \text{End}(M)[[t, \lambda/h, h]].$$

This extends to a $C[[t, \lambda/h, h]]$ module map of the tensor algebra

$$\phi : T(G_q)[[\lambda, h]] \to \text{End}(M)[[t, \lambda/h, h]].$$

We can define a grading on $T(G_q)[[\lambda, h]]$ by setting the elements of $G_q$, and the variables $\lambda_i$ and $h$ to be of degree one and $t$ to be of degree zero. Similarly, we have a grading on $\text{End}(M)[[t, \lambda/h, h]]$ defined in the same way on the parameters and with elements of $\text{End}(M)$ considered as degree zero. Then $\phi$ is a morphism of graded modules.

Define the subalgebra $A_{t,\lambda,h} := \text{Im } \phi$. At $t = 0$ we recover the construction of section 1 quantizing the Kirillov bracket. Now, the dependence on $t$ introduces a new deformation parameter which quantizes some quadratic bracket arising from the $U_q(G)$ module structure.

**Theorem 2.1** The algebra $A_{t,\lambda,h}$ is a $U_q(G)$ module, and is free as a $C[[t, \lambda, h]]$ module. Multiplication is $U_q(G)$ equivariant.

**Proof.** The definition of the action of $u \in U$ on $A \in \text{End}(M)[[t, \lambda, h]],$

$$ad(u) \cdot A = \sum \phi(u_{(1)}) \circ A \circ \phi(S(u_{(2)})) \text{ implies } \sum ad(u_{(1)})(A)ad(u_{(2)})(B) = ad(u)(AB).$$

Since $\phi(G_q)$ is a $U$ submodule so is the span of all products of elements in $\phi(G_q)$, but this is precisely $A_{t,\lambda,h}$. Moreover the latter formula is precisely the definition of equivariance of multiplication. To prove that $A_{t,\lambda,h}$ is free we use the fact that the quotient module, $A_{t,\lambda,h}/tA_{t,\lambda,h} \cong A_{\lambda,h}$, that is, the algebra defined in section 1, is a free $C[\lambda, h]$ module. Hence the same is true of the extension to formal power series. Moreover $A_{t,\lambda,h}$ is free as a $C[[t]]$ module since it is a submodule of the free $C[[t]]$ module, $\text{End}(M)[[t, \lambda/h, h]]$. Hence

$$A \cong A/tA \otimes C[[t]] \cong N \otimes C[\lambda, h] \otimes C[[t]] \cong N \otimes C[[t, \lambda/h, h]].$$

**Remarks 1.** The algebra $A_{t,\lambda,h}$ is graded as a quotient of $T(G_q)[[\lambda, h]]$ by a graded ideal.

**2.** The bracket defined by the commutator modulo the ideal generated by $t$ and $h$ is a linear combination of the Kirillov bracket and a second bracket which is some kind of $R$-matrix bracket on the orbit. We have verified that for $G = sl(2)$, the second bracket is the $R$ matrix bracket generated by $E \wedge F$, see [DS], using the results of Lyubashenko and Sudberry.
on quantum Lie algebras, see \cite{LS}. We believe that this is the case for the general, $G$ and symmetric coadjoint orbit. It is interesting question to determine the form of the second bracket for the general semisimple orbit, since it is known that the naive definition of an $R$-matrix bracket is not in general compatible with the Kirillov bracket.

References

[D] J. Dixmier, Algèbres Enveloppantes, Gauthier Villars, (1974)

[DS] J. Donin and S. Shnider, Quantum symmetric spaces, *Journal of Pure and Applied Algebra*, \textbf{100}, (1995), 103-117

[JL] A. Joseph and G. Letzter, Separation of variable in quantized enveloping algebras, *American Journal of Mathematics* \textbf{116} (1994) 127-177.

[Lu] G. Lusztig, Quantum deformations of certain simple modules over enveloping algebras, *Advances in Mathematics* \textbf{70} (1988) 237-249.

[LS] V. Lyubashenko and A. Sudberry, Quantum Lie algebras of type $A_n$, \texttt{q-alg/9510004}