On the gauge invariant path-integral measure for the overlap Weyl fermions in $16$ of SO(10)

Yoshio Kikukawa

Institute of Physics, University of Tokyo, Tokyo 153-8902, Japan

E-mail: kikukawa@hep1.c.u-tokyo.ac.jp

Abstract: We consider the lattice formulation of SO(10) chiral gauge theory with left-handed Weyl fermions in the sixteen dimensional spinor representation ($16$) within the framework of the Overlap fermion/the Ginsparg-Wilson relation. We define a manifestly gauge-invariant path-integral measure for the left-handed Weyl field using all the components of the Dirac field, but the right-handed part of which is just saturated completely by inserting a suitable product of the SO(10)-invariant 't Hooft vertices in terms of the right-handed field. The definition of the measure applies to all possible topological sectors. The measure possesses all required transformation properties under lattice symmetries and the induced effective action is CP invariant. The global U(1) symmetry of the left-handed field is anomalous due to the non-trivial transformation of the measure, while that of the right-handed field is explicitly broken by the 't Hooft vertices. There remains the issue of locality in the gauge-field dependence of the Weyl fermion measure, but the question can be addressed in the weak gauge-coupling expansion at least by Monte Carlo methods without encountering the sign problem. We also discuss the relations of our formulation to other approaches/proposals to decouple the species-doubling/mirror degrees of freedom. Those include Eichten-Preskill model, Ginsparg-Wilson Mirror-fermion model, Domain wall fermion model with the boundary Eichten-Preskill term, 4D Topological Insulator/Superconductor with gapped boundary phase, and the recent studies on the PMS phase/“Mass without symmetry breaking”. We clarify the similarity and the difference in technical detail and show that our proposal is a well-defined testing ground for that basic question.
Contents

1 Introduction 2

2 The SO(10) chiral lattice gauge theory with overlap Weyl fermions 4
  2.1 Gauge field of SO(10) 5
  2.2 Weyl field in 16-dimensional spinor representation of SO(10) 5
  2.3 Topology of the SO(10) lattice gauge fields 7

3 Path Integration – a proposal for the gauge-invariant measure 8
  3.1 Definition of the path integration measures 8
  3.2 Chiral determinant and ’t Hooft-vertex pfaffians 10
  3.3 The Weyl field measure in terms of chiral basis 12
  3.4 Saturation of the right-handed part of the fermion measure by ’t Hooft vertices 14
  3.5 CP invariance 16
  3.6 Schwinger-Dyson equations and Correlation functions 18
  3.7 Gauge field dependence of the Weyl field measure – Locality issue remaining 19

4 More on the saturation of the fermion measure by ’t Hooft vertices 22
  4.1 Property of the functional pfaffian for the link fields in Spin(9) subgroup 22
  4.2 The case of trivial link field in the weak gauge-coupling limit 23
  4.3 The case of representative SU(2) link fields of topologically non-trivial sectors 27
  4.4 Continuity across the mass singularity: $m_0 \rightarrow +0; +0 \rightarrow -0; -0 \rightarrow -\infty$ 28
  4.5 Disorder nature of the auxiliary spin-field path integrations 29
  4.6 A summary 33

5 Other anomalous/anomaly-free chiral gauge theories 33
  5.1 Fate of the anomalous SU(2) chiral gauge theory 33
  5.2 Anomaly-free chiral gauge theories descent from SO(10) 34
  5.3 The standard model plus the right-handed neutrinos 34

6 Relations with other approaches/proposals 35
  6.1 cf. Eichten-Preskill model 37
  6.2 cf. Ginsparg-Wilson Mirror-fermion model 40
  6.3 cf. Recent studies on the PMS phase/Mass without Symmetry Breaking 42
  6.4 cf. Domain wall fermions with the boundary Eichten-Preskill term 49
  6.5 cf. Topological Insulators/Superconductors with gapped boundary phases 51

7 Conclusion 54

A Dirac gamma matrices 55

B SO(10) gamma matrices 56
1 Introduction

The experimental verification of the elementary particle spectrum of the standard model is now completed with the observation of a new particle with a mass of 125.09 GeV by the ATLAS and CMS Collaborations through LHC Run 1[1–3], for which all measurements of the properties, including its spin, CP properties, and coupling strengths to SM particles, are consistent within the uncertainties with those expected for the SM Higgs boson. On the other hand, the discoveries of neutrino mixings by the Super-Kamiokande Collaboration and the Sudbury Neutrino Observatory (SNO) Collaboration [4–6] imply the masses of neutrinos and suggest the existence of the right-handed neutrinos. The consistency of the standard model predictions with experiments in high precision implies that the standard model particles and the possible right-handed neutrinos are well-decoupled from the fundamental scale of the Plank mass and even from the possible scale of new physics beyond the standard model. And the low-lying fermionic elementary particles fit into the sixteen-dimensional irreducible representation of SO(10), 16, which is complex, but free from gauge anomaly.

Chiral gauge theories such like the SO(10) gauge theory with 16s have several interesting possibilities in their own dynamics: fermion number non-conservation due to chiral anomaly[7, 8], various realizations of the gauge symmetry and global flavor symmetry[9, 10], the existence of massless composite fermions suggested by ‘t Hooft’s anomaly matching condition[11], the classical scale invariance and the vanishing vacuum energy[12, 13] and so on. Unfortunately, little is known so far about the actual behavior of (non-supersymmetric) chiral gauge theories beyond perturbation theory. It is then desirable to develop a formulation to study the non-perturbative aspect of chiral gauge theories.

Lattice gauge theory can now provide a framework for non-perturbative formulation of chiral gauge theories, despite the well-known problem of the species doubling [14–17]. The clue to this development is the construction of local and gauge-covariant lattice Dirac operators satisfying the Ginsparg-Wilson relation[23–28].

\[ \gamma_5 D + D \gamma_5 = 0, \quad \gamma_5 = \gamma_5(1 - 2aD). \]  

An explicit example of such lattice Dirac operator is given by the overlap Dirac operator [24, 26], which was derived by Neuberger from the overlap formalism [29–59]. The overlap formula was derived from the five-dimensional approach of domain wall fermion proposed by Kaplan[60, 61]. In the vector-like formalism of domain wall fermion by Shamir and Furman[62–65], the local low energy effective action of the chiral mode is precisely given by the overlap Dirac operator [66–68]. By the Ginsparg-Wilson relation, it is possible to realize an exact chiral symmetry on the lattice[69–75] in the manner consistent with the

\[ \text{See [18–22] for the recent reviews on this subject.} \]
no-go theorem. It is also possible to introduce Weyl fermions on the lattice and this opens the possibility to formulate anomaly-free chiral gauge theories on the lattice[18, 21, 71, 76–92]. In the case of U(1) chiral gauge theories, Lüscher[76] proved rigorously that it is possible to construct the fermion path-integral measure which depends smoothly on the gauge field and fulfills the fundamental requirements such as locality, gauge-invariance and lattice symmetries. This construction was extended to the SU(2)×U(1) chiral gauge theory of the Glashow-Weinberg-Salam model[93–95] based on the pseudo reality and anomaly-free conditions of SU(2)×U(1) by Kadoh and the author[92]. For generic non-abelian chiral gauge theories, the construction in all orders of the weak gauge-coupling expansion was given by Suzuki[81, 82] and by Lüscher[83]. However, a non-perturbative construction is not obtained yet so far.

In this article, we consider the lattice formulation of SO(10) chiral gauge theory with Weyl fermions in the sixteen dimensional spinor representation \(16\) within the framework of the Overlap fermion/the Ginsparg-Wilson relation. We propose a manifestly gauge-invariant path-integral measure of the left-handed Weyl fermions, which is defined with all the components of the Dirac field, but the right-handed part of which is just saturated completely by inserting a suitable product of the SO(10)-invariant ’t Hooft vertices in terms of the right-handed field. The definition of the measure applies to all possible topological sectors. The measure possesses all required transformation properties under lattice symmetries and the induced effective action is CP invariant. The global U(1) symmetry of the left-handed field is anomalous due to the non-trivial transformation of the measure[70–75], while that of the right-handed field is explicitly broken by the ’t Hooft vertices. There remains the issue of smoothness/locality in the gauge-field dependence of the Weyl fermion measure. But the question is well-defined and can be addressed in the weak gauge-coupling limit at least through Monte Carlo simulations without encountering the sign problem.

We also discuss the relation of our formulation with other approaches/proposals to decouple the species-doublers and mirror fermions. Those include the Eichten-Preskill model [98, 99], the Mirror-fermion model using Ginsparg-Wilson fermions [100–107], Domain wall fermions with the boundary Eichten-Preskill term[108, 109], the recent studies on “Mass without symmetry breaking” [110–118]/the PMS phase [101, 119, 120] and 4D TI/TSCs with gapped boundary phases [121–127]. We clarify the similarity and the difference in technical detail, trying to show that our proposal is a well-defined testing ground for that...
basic question.\footnote{See \cite{98, 99, 128–138} for the former attempts to decouple the species-doublers/mirror-fermions by strong Yukawa, Wilson-Yukawa and multi-fermion interactions. See also \cite{139–149} for the original Mirror-fermion approach using Wilson fermions.}

It is known that a chiral gauge theory is a difficult case for numerical simulations because the effective action induced by Weyl fermions has a non-zero imaginary part. But in view of the recent studies of the simulation methods based on the complex Langevin dynamics\cite{158–193} and the complexified path-integration on Lefschetz thimbles\cite{194–237}, it would be still interesting and even useful to develop a formulation of chiral lattice gauge theories by which one can work out fermionic observables numerically as the functions of link field with exact gauge invariance.

This article is organized as follows. In section 2, we introduce our lattice formulation of SO(10) gauge theory with left-handed Weyl field in \(16\) at the classical level. In section 3, we define the path-integral measures of the left-handed Weyl field and discuss its properties. In section 4, we examine in detail the saturation of the right-handed part of the fermion measure by ‘t Hooft vertices. In section 5, we discuss the cases of other anomalous and anomaly-free chiral gauge theories. Section 6 is devoted to the discussions of the relations to other approaches/proposals. In section 7, we conclude with a summary and discussions.

## 2 The SO(10) chiral lattice gauge theory with overlap Weyl fermions

In this section, we describe a construction of the SO(10) chiral gauge theory on the lattice within the framework of chiral lattice gauge theories based on the lattice Dirac operator satisfying the Ginsparg-Wilson relation \cite{76, 77}. We assume a local, gauge-covariant lattice Dirac operator \(D\) which satisfies the Ginsparg-Wilson relation. An explicit example of such lattice Dirac operator is given by the overlap Dirac operator \cite{24, 26}, which was derived from the overlap formalism \cite{29–33, 40, 47–51}. In this case, our formulation is equivalent to the overlap formalism for chiral lattice gauge theories\footnote{The recent proposal by Grabowska and Kaplan\cite{150–157} is “orthogonal” to the approaches discussed in this paper. It is based on the original domain wall fermion by Kaplan\cite{60}, but coupled to the “five-dimensional” link field which is obtained from the dynamical four-dimensional link field at the target wall by the gradient flow toward the mirror wall. This choice of the “five-dimensional” link field makes possible a chiral gauge coupling for the target and mirror walls, while keeping the system four-dimensional and gauge-invariant. It is “orthogonal” in the sense that the authors do not try to decouple the massless-modes at the mirror wall, but interpret them as physical degrees of freedom with very soft form factor caused by the gradient flow, and that the authors do not try (do not need) to break explicitly the continuous global symmetries with “would-be gauge anomalies” in the mirror-wall sector, which would be required if one would try to decouple the mirror-modes as claimed by Eichten and Preskill and by the other authors\cite{98, 100, 124}.} or the domain wall fermion approach \cite{84, 87}.
In the followings, we consider the four-dimensional lattice \( \Lambda \) of the finite size \( L \) and choose lattice units \( a = 1 \):

\[
\Lambda = \{ x = (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid 0 \leq x_\mu < L (\mu = 0, 1, 2, 3) \}.
\]

The unit vector in the directions \( \mu(= 0, 1, 2, 3) \) are denoted as \( \hat{\mu} \).

2.1 Gauge field of SO(10)

The gauge field of SO(10) is defined as the link field on the lattice \( \Lambda \). The SO(10) link variables are at first introduced in the (reducible) spinor representation as the thirty-two dimensional special unitary matrices, \( U(x, \mu) \in \text{Spin}(10) \). The generators of Spin(10) are given by

\[
\Sigma_{ab} = -\frac{i}{4} [\Gamma^a, \Gamma^b],
\]

where \( \{ \Gamma^a \mid a = 1, 2, \cdots, 10 \} \) form the Clifford algebra, \( \Gamma^a \Gamma^b + \Gamma^b \Gamma^a = 2 \delta^{ab} \). An explicit representation for \( \{ \Gamma^a \mid a = 1, 2, \cdots, 10 \} \) is given in the appendix B. The link variables are then parametrized as

\[
U(x, \mu) = e^{i\theta_{ab}(x, \mu) \Sigma_{ab}/2} \in \text{Spin}(10).
\]

We require the admissibility condition on the gauge field,

\[
\| 1 - P(x, \mu, \nu) \| < \epsilon,
\]

for all \( x, \mu, \nu \), where the plaquette variables are defined by

\[
P(x, \mu, \nu) = U(x, \mu) U(x + \hat{\mu}, \nu) U(x + \hat{\nu}, \mu)^{-1} U(x, \nu)^{-1}.
\]

This condition ensures that the overlap Dirac operator\([24, 26]\), which is assumed to act on the fermion fields in the spinor representations of SO(10), is a smooth and local function of the gauge field if \( \epsilon < 1/30 \)[28].

To impose the admissibility condition dynamically, we adopt the following action for the gauge field:

\[
S_G = \frac{1}{g^2} \sum_{x \in \Gamma} \sum_{\mu, \nu} \text{tr} \{ 1 - \tilde{P}(x, \mu, \nu) \} \left[ 1 - \text{tr} \{ 1 - \tilde{P}(x, \mu, \nu) \} / 10 \epsilon^2 \right]^{-1},
\]

where the SO(10) link variables are represented in the defining representation as the ten-dimensional special orthogonal matrices, \( \tilde{U}(x, \mu) \in \text{SO}(10) \). The generators of SO(10) in the defining representation are given by \( \{ \tilde{\Sigma}_{ab} \}_{cd} = i(\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}) \) and the link variables are represented with the same parameters as

\[
\tilde{U}(x, \mu) = e^{i\theta_{ab}(x, \mu) \tilde{\Sigma}_{ab}/2} \in \text{SO}(10).
\]

2.2 Weyl field in 16-dimensional spinor representation of SO(10)

The left-handed Weyl field in the 16-dimensional (irreducible) spinor representation of SO(10) is defined on the lattice \( \Lambda \) based on the Ginsparg-Wilson relation. First we introduce a Dirac field on the lattice in the 16-dimensional spinor representation of SO(10),

\[
\psi(x) = P_+ \psi(x), \quad \bar{\psi}(x) = \bar{\psi}(x) P_+,
\]

where
where
\[ P_+ = \frac{1 + \Gamma^{11}}{2}, \quad \Gamma^{11} = -\Gamma^1 \Gamma^2 \cdots \Gamma^{10}. \] (2.8)

We also introduce the overlap Dirac operator \( D \) acting on \( \psi(x) \) as
\[ D = \frac{1}{2} \left( 1 + X \frac{1}{\sqrt{X}} \right), \quad X = \gamma_\mu \frac{1}{2} (\nabla_\mu - \nabla^\dagger_\mu) + \frac{1}{2} \nabla_\mu \nabla^\dagger_\mu - m_0, \] (2.9)

where \( \nabla_\mu \) is the covariant difference operator which acts on \( \psi(x) \) as \( \nabla_\mu \psi(x) = U(x, \mu) \psi(x + \mu) - \psi(x) \) and \( 0 < m_0 < 2 \). Under the admissibility condition, \( D \) is a local, gauge-covariant lattice Dirac operator. It also satisfies the Ginsparg-Wilson relation,
\[ \gamma_5 D + D \gamma_5 = 0, \] (2.10)

where
\[ \gamma_5 \equiv \gamma_5(1 - 2D), \quad (\gamma_5)^2 = \mathbb{I}. \] (2.11)

Then we define the left-handed Weyl fermions in the 16-dimensional spinor representation of SO(10) by the eigenstates of the chiral operators, \( \hat{\gamma}_5 \) for the field and \( \gamma_5 \) for the anti-fields:
\[ \psi_-(x) = \hat{P}_- \psi(x), \quad \bar{\psi}_-(x) = \bar{\psi}(x) P_+, \] (2.12)

where \( \hat{P}_\pm \) and \( P_\pm \) are the chiral projection operators given by
\[ \hat{P}_\pm = \left( \frac{1 \pm \hat{\gamma}_5}{2} \right), \quad P_\pm = \left( \frac{1 \pm \gamma_5}{2} \right). \] (2.13)

We note that \( [\hat{P}_\pm, P_\pm] = 0 \) and \( [P_\pm, P_\pm] = 0 \).

The action of the left-handed Weyl field in the 16-dimensional spinor representation of SO(10) is given by
\[ S_W[\psi_-, \bar{\psi}_-] = \sum_{x \in \Lambda} \bar{\psi}_-(x) D \psi_-(x) = \sum_{x \in \Lambda} \bar{\psi}(x) P_+ D \psi(x), \] (2.14)

where we note the relation \( P_+ D \hat{P}_- = P_+ D \). This action is manifestly invariant under the SO(10) lattice gauge transformations. It is also invariant under the global U(1) transformation of the left-handed fields,
\[ \delta_\alpha \psi_-(x) = i \alpha \psi_-(x), \quad \text{or} \quad \delta \psi(x) = i \alpha \hat{P}_- \psi(x) \bigg|_{2}; \] (2.15)
\[ \delta_\alpha \bar{\psi}_-(x) = -i \alpha \bar{\psi}_-(x), \quad \text{or} \quad \delta \bar{\psi}(x) = -i \alpha \bar{\psi}(x) P_+. \] (2.16)

This global U(1) symmetry is, as we will see below, broken due to the non-trivial transformation property of the Weyl field path-integral measure and the non-vanishing vacuum expectation values of ’t Hooft vertices,
\[ T_-(x) = \frac{1}{2} V^a(x) V^a(x), \quad V^a(x) = \frac{1}{2} \psi_-(x)^T i \gamma_5 C_D T^a \psi_-(x), \] (2.17)
\[ \bar{T}_-(x) = \frac{1}{2} \bar{V}^a(x) \bar{V}^a(x), \quad \bar{V}^a(x) = \frac{1}{2} \bar{\psi}_-(x)^T i \gamma_5 C_D T^a \bar{\psi}_-(x)^T, \] (2.18)
in the topologically nontrivial sectors of the gauge field. Here $T^a$ ($a = 1, 2, \cdots, 10$) are the operators acting on the SO(10) spinor space, $T^a = C \Gamma^a$. The explicit representations of $C$ and $\{T^a| a = 1, \cdots, 10\}$ are given in the appendix B. The action also possesses all required transformation properties under lattice symmetries: translations, rotations, reflections and charge conjugation. In particular, under $P$ (space reflections) and $C$ (charge conjugation) the action is not invariant, while under CP the action is transformed into the same form, but the definitions of the chiral projection for the fields and anti-fields are interchanged:

$$\psi_-(x) = \hat{P}_- \psi(x) \quad \Rightarrow \quad \psi_-(x) = P_- \psi(x),$$

$$\bar{\psi}_-(x) = \bar{\psi} P_+(x) \quad \Rightarrow \quad \bar{\psi}_-(x) = \bar{\psi} \{\gamma_5 \hat{P}_+ \gamma_5\}(x).$$

But the effective action of the gauge field turns out to be CP invariant. This CP transformation property of the model will be discussed below.

### 2.3 Topology of the SO(10) lattice gauge fields

The admissibility condition ensures that the overlap Dirac operator\cite{24, 26} is a smooth and local function of the gauge field \cite{28}. Moreover, the Ginsparg-Wilson relation implies the index theorem

$$\text{Index } D = \text{Tr} \gamma_5(1 - D).$$

Then, through the lattice Dirac operator $D$, it is possible to define a topological charge of the gauge fields \cite{25, 30, 31, 33, 69}: for the admissible SO(10) gauge fields, one has

$$Q = -\frac{1}{8} \text{tr} \gamma_5(1 - D) = \frac{1}{8} \sum_{x \in \Gamma} \text{tr} \{\gamma_5(1 - D)\}(x, x),$$

where $D(x, y)$ is the kernel of the lattice Dirac operator $D$. (Our convention for the gamma matrices is such that $\gamma_0 \gamma_1 \gamma_2 \gamma_3 \gamma_5 = 1$.) Then the admissible SO(10) gauge fields can be classified by the topological numbers $Q$.\footnote{Strictly speaking, the complete topological classification of the space of admissible SO(10) gauge fields is not known yet. We assume that it is classified with $Q$ as in the continuum theory.} We denote the space of the admissible SO(10) gauge fields with a given topological charge $Q$ by $U[Q]$.

The instanton solutions of SU(2) gauge field can be embedded into the mutually commuting SU(2) subgroups of the Spin(10) gauge field. In such a case, the index counts as

$$\text{Index } D = \sum_{\text{SU(2)}} m q,$$

where $q$ is the topological charge of the embedded instanton solution, and $m$ is the multiplicity of the doublets (2s) of the embeding SU(2) subgroup, which is an integer multiple of 4 for the sixteen-dimensional irreducible representation of Spin(10).
3 Path Integration – a proposal for the gauge-invariant measure

3.1 Definition of the path integration measures

The path-integral measures for the link field and the Weyl field are formulated as follows. For the link field $U(x, \mu)$, it is defined with the group-invariant Haar measure as usual:

$$D[U] = \prod_{x \in \Lambda} \prod_{\mu=0}^3 dU(x, \mu).$$

(3.1)

For the Weyl field $\psi_-(x)$, it is defined by using all the components of the original Dirac field $\psi_{\alpha s}(x) (\alpha = 1, \cdots, 4; s = 1, \cdots, 16)$ not as usual, but the right-handed part of the measure is just saturated completely by inserting a suitable product of the 't Hooft vertexes in terms of the right-handed fields,

$$T_+(x) = \frac{1}{2} V_+^a(x)V_+^a(x), \quad V_+^a(x) = \frac{1}{2} \psi_+(x) i \gamma_5 C_D T^a \psi_+(x),$$

(3.2)

$$\bar{T}_+(x) = \frac{1}{2} \bar{V}_+^a(x)\bar{V}_+^a(x), \quad \bar{V}_+^a(x) = \frac{1}{2} \bar{\psi}_+(x) i \gamma_5 C_D T^a \bar{\psi}_+(x).$$

(3.3)

Namely, the Weyl field measure is defined as

$$D[\psi_-]D[\bar{\psi}_-] = D[\psi]D[\bar{\psi}] \prod_{x \in \Lambda} F(T_+(x)) \prod_{x \in \Lambda} F(\bar{T}_+(x)),$$

(3.4)

where

$$D[\psi]D[\bar{\psi}] = \prod_{x \in \Lambda} \prod_{\alpha=1}^4 \prod_{s=1}^{16} d\psi_{\alpha s}(x) \prod_{x \in \Lambda} \prod_{\alpha=1}^4 \prod_{s=1}^{16} d\bar{\psi}_{\alpha s}(x),$$

(3.5)

and $F(w)$ is the certain function to represent the product of the 't Hooft vertexes, $T_+(x)$ and $\bar{T}_+(x)$. The Weyl field measure so defined depends on the link field $U(x, \mu)$ through the chiral projection $\hat{P}_+$ to define $T_+(x)$ in terms of the right-handed field $\bar{\psi}_+(x) = \hat{P}_+ \psi_+(x)$.

Note that we use the four-spinor notation in the definition of the 't Hooft vertexes and the factor $\hat{P}_T^T i \gamma_5 C_D T^a E^a(x) \hat{P}_+$, not $\hat{P}_T^T i \gamma_5 C_D P_+ T^a E^a(x) \hat{P}_+$, appears for the field $\psi_+(x)$, while $P_- i \gamma_5 C_D T^a \bar{E}^a(x) P_- = P_- \{ i \gamma_5 C_D P_- T^a \bar{E}^a(x) \} P_-^T$ for the anti-field $\bar{\psi}_+(x)$.

Our choice for $F(w)$ is

$$F(w) \equiv 4! \left( \frac{z}{2} \right)^{-4} I_4(z) \bigg|_{(z/2)^2 = w} = 4! \sum_{k=0}^{\infty} \frac{w^k}{k!(k+4)!},$$

(3.6)

where $I_\mu(w)$ is the modified Bessel function of the first kind. It has the integral representation as

$$F(w) \bigg|_{w = (1/2) w^\alpha w^\alpha} = \left( \frac{\pi^5}{12} \right)^{-1} \int \prod_{a=1}^{10} d\mu^a \delta(\sqrt{e^{b} c} - 1) e^{c w^e}.$$  

(3.7)

---

8This point is crucial for our proposal. If one includes the factor $P_+$ in the definition of the 't Hooft operator for the field $\psi_+(x)$, one has $\hat{P}_T^T i \gamma_5 C_D P_+ T^a E^a(x) \hat{P}_+ = (1-D)^T i \gamma_5 C_D P_+ T^a E^a(x) (1-D)$. The factor $(1-D)$ projects out the modes with the momenta $\pi_{\mu}^{(A)} (A = 1, \cdots, 15)$, where $\pi^{(1)} \equiv (\pi, 0, 0, 0), \pi^{(2)} \equiv (0, \pi, 0, 0), \cdots, \pi^{(15)} \equiv (\pi, \pi, \pi, \pi)$. This type of the operator cannot saturate the right-handed part of the measure completely. Therefore it is not acceptable for our purpose. This point will be discussed later in relation to other formulations. See section 6.
and allows us to prove the CP invariance of the effective action of the lattice model, as discussed below.\footnote{One possible choice for $F(w)$ is simply $F(w) = e^w = \sum_{k=0}^\infty \frac{w^k}{k!}$. It also has the integral representation, \[
abla \left|_{w=(1/2)\omega u^a u^a} \right. = (2\pi)^{-\frac{1}{2}} \int_{a=1}^{10} d\varepsilon^a e^{-(1/2)\varepsilon^a \varepsilon^a + \varepsilon^a u^a} \] In this case, however, we do not succeed yet in proving the CP invariance of the effective action of the lattice model.}

The partition function of our lattice model for the SO(10) chiral Gauge theory is then given as follows,

\[
Z \equiv \int \mathcal{D}[U] \ e^{-S_G[U]+\Gamma_W[U]},
\]

where $\Gamma_W[U]$ is the effective action induced by the path-integration of the Weyl field, 

\[
e^{\Gamma_W[U]} = \int \mathcal{D}[\psi_-] \mathcal{D}[\bar{\psi}_-] e^{-S_W[\psi_-,\bar{\psi}_-]}
= \int \mathcal{D}[\psi] \mathcal{D}[\psi_0] \prod_{x \in \Lambda} F(T_+(x)) \prod_{x \in \Lambda} F(T_+(x)) e^{-S_W[\psi_-,\bar{\psi}_-]}
= \int \mathcal{D}[\psi] \mathcal{D}[\psi_0] \mathcal{D}[E] \mathcal{D}[E_0] e^{-S_W[\psi_-,\bar{\psi}_-] + \sum_{x \in \Lambda} \{ E^a(x) \bar{V}_+(x) + \bar{E}^a(x) V_+(x) \}}[\psi_+,\bar{\psi}_+].
\]

In the last equation, the integral representation of $F(w)$ is used and the path-integrations over the SO(10)-vector real spin fields with unit length, $E^a(x)$ and $\bar{E}^a(x)$, are introduced:

\[
\mathcal{D}[E] = \prod_{x \in \Lambda} (\pi^5/12)^{-1} \prod_{a=1}^{10} dE^a(x) \delta(\sqrt{E^b(x)E^b(x)} - 1)
\]

\[
\mathcal{D}[\bar{E}] = \prod_{x \in \Lambda} (\pi^5/12)^{-1} \prod_{a=1}^{10} d\bar{E}^a(x) \delta(\sqrt{\bar{E}^b(x)\bar{E}^b(x)} - 1).
\]

Defined with all the components of the Dirac field $\psi(x)$, $\bar{\psi}(x)$, the Weyl field measure is manifestly invariant under the SO(10) gauge transformation. It also possesses all required transformation properties under lattice symmetries: translations, rotations, reflections and charge conjugation. As to the global U(1) fermion symmetry of the left-handed field $\psi_-(x)$, $\bar{\psi}_-(x)$, the fermionic measure transforms as

\[
\delta_\alpha \mathcal{D}[\psi_-] \mathcal{D}[\bar{\psi}_-] = -i \sum_{x \in \Gamma} \alpha(x) \text{tr}\{\hat{P}_- - \hat{P}_+\}(x, x) \times \mathcal{D}[\psi_-] \mathcal{D}[\bar{\psi}_-]
\]

with a local parameter $\alpha(x)$, and it gives rise to the non-trivial chiral anomaly in the U(1) Ward-Takahashi relation. One may consider the similar global U(1) fermion symmetry of the right-handed field $\psi_+(x)$, $\bar{\psi}_+(x)$, but it is broken explicitly by the ’t Hooft vertexes, $T_+(x)$ and $\bar{T}_+(x)$, down to $Z_4 \times Z_4$, one $Z_4$ for the field $\psi_+(x)$ and the other $Z_4$ for the
anti-field $\bar{\psi}_+(x)$. The reason for the two independent $Z_4$ is that the bilinear kinetic term of the right-handed field, $\sum_{x \in \Lambda} \bar{\psi}_+(x) D \psi_+(x)$, is not introduced here. Conversely, this $Z_4 \times Z_4$ symmetry prohibits such bilinear terms of the right-handed field to appear, as long as it is not broken spontaneously.

### 3.2 Chiral determinant and 't Hooft-vertex pfaffians

The path-integral weight for the effective action defined by eq. (3.10) consists of $S_W[\psi_-, \bar{\psi}_-]$, the action of the left-handed fields, and $\sum_{x \in \Lambda} \{ E^a(x) V^a_+(x) + \bar{E}^a(x) \bar{V}^a_+(x) \} [\psi_+, \bar{\psi}_+]$, the 't Hooft vertex terms of the right-handed fields. These two terms can be written in terms of the Dirac fields $\psi(x)$, $\bar{\psi}(x)$, as follows:

$$
S_W[\psi_-, \bar{\psi}_-] = \sum_{x \in \Lambda} \{ E^a(x) V^a_+(x) + \bar{E}^a(x) \bar{V}^a_+(x) \} [\psi_+, \bar{\psi}_+]
$$

$$
= \sum_{x \in \Lambda} \left( \psi^T \bar{\psi} \right) (x) \left( \begin{array}{cc}
-P_+^T \gamma_5 C_D T^a E^a \hat{P}_+ & -P_+^T D^T P_+^T \\
P_+ D \hat{P}_- & -P_- \gamma_5 C_D T^a \hat{E}^a P_+^T
\end{array} \right) \left( \begin{array}{c}
\psi \\
\bar{\psi}
\end{array} \right) (x).
$$

(3.14)

Then the path-integration of the fermion fields with the Dirac field measure $D[\psi]D[\bar{\psi}]$ gives rise to the pfaffian of the above gauge-covariant anti-symmetric operator,

$$
\text{pf} \left( \begin{array}{cc}
-P_+^T \gamma_5 C_D T^a E^a \hat{P}_+ & -P_+^T D^T P_+^T \\
P_+ D \hat{P}_- & -P_- \gamma_5 C_D T^a \hat{E}^a P_+^T
\end{array} \right).
$$

(3.15)

This pfaffian factorises into the chiral determinant of the left-handed fields and the 't Hooft-vertex pfaffians of the right-handed fields in the chiral bases for the field $\psi(x)$ and the anti-field $\bar{\psi}(x)$ where $\hat{\gamma}_5$ and $\gamma_5$ are diagonalized, respectively. One can introduce the four-spinor vectors of the chiral bases as

$$
P_+ \otimes \hat{P}_+ u_i(x) = u_i(x) \quad (i = 1, \cdots, n/2 - 8Q); \quad (u_i, u_j) = \delta_{ij}, \quad (3.16)
$$

$$
P_+ \otimes \hat{P}_- v_i(x) = v_i(x) \quad (i = 1, \cdots, n/2 + 8Q); \quad (v_i, v_j) = \delta_{ij}, \quad (3.17)
$$

$$
\bar{u}_k(x) P_- \otimes P_+ = \bar{u}_k(x) \quad (k = 1, \cdots, n/2); \quad (\bar{u}_k, \bar{u}_l) = \delta_{kl}, \quad (3.18)
$$

$$
\bar{v}_k(x) P_+ \otimes P_+ = \bar{v}_k(x) \quad (k = 1, \cdots, n/2); \quad (\bar{v}_k, \bar{v}_l) = \delta_{kl} \quad (3.19)
$$

in the given topological sector $\Omega[Q]$, where $n = \dim \Lambda \times 4 \times 16$. The basis vectors $u_i(x)$ and $v_i(x)$ depend on the gauge field through the chiral projectors $\hat{P}_\pm$, while the basis vectors $\bar{u}_k(x)$ and $\bar{v}_k(x)$ can be chosen so that they are independent of the gauge field. For example, $\bar{u}_k(x)_{\alpha \sigma} = \delta_{xx'} \delta_{\alpha \sigma} + 2 \delta_{st}$ for $k = \{ x' \in \Lambda; \sigma = 1, 2; t = 1, \cdots, 16 \}$ and $\bar{v}_k(x)_{\alpha \sigma} = \delta_{xx'} \delta_{\alpha \sigma} \delta_{st}$ for $k = \{ x' \in \Lambda; \sigma = 1, 2; t = 1, \cdots, 16 \}$, assuming $\gamma_5 = \text{diag}(1, 1, -1, -1)$. One can always choose the bases of the Dirac fields, $\{ u_j(x), v_j(x) \}$, and $\{ \bar{u}_j(x), \bar{v}_j(x) \}$, so that the jacobian factors, $\det(u_j(x), v_j(x))$, and $\det(\bar{u}_j(x), \bar{v}_j(x))$, are unity independent of the gauge field.
In this choice of the chiral bases, the pfaffian can be evaluated as

\[
\text{pf}
\begin{pmatrix}
-P^T \gamma_5 C_D T^a E^a \hat{P}_+ & -P^T D^T P^T \\
P_+ D \hat{P}_- & -P_+ \gamma_5 C_D T^{a\dagger} E^a P^T
\end{pmatrix}
\]

where the matrices

\[
(u^T \gamma_5 C_D T^a E^a u) \\
0 \\
0 \\
0
\]

This is due to the appearance of the chiral zero modes with a non-trivial index \(n_+ - n_-\) in the trivial sectors, the matrix

\[
(P^T D \hat{P}_-) \]

the boundary condition, in particular,

\[
\text{ln det}(\bar{v} D v) + \text{pf}(u^T \gamma_5 C_D T^a E^a u) \times \text{pf}(\bar{u} i \gamma_5 C_D T^{a\dagger} \bar{E}^a \bar{u}^T)
\]

Thus the effective action \(\Gamma_W[U]\), with our definition of the Weyl field measure eq. (3.4), has the extra factor of logarithm of the pfaffians, \(\text{pf}(u^T \gamma_5 C_D T^a E^a u)\) and \(\text{pf}(\bar{u} i \gamma_5 C_D T^{a\dagger} \bar{E}^a \bar{u}^T)\), integrated over the auxiliary spin fields in addition to the usual effective action given by the logarithm of the chiral determinant, \(\ln \text{det}(\bar{v} D v)\) \([30–32\) \([76, 77]\].

The first factor in the r.h.s. of eq. (3.25) is nothing but the chiral determinant in the overlap formalism.\([30–32\) In the weak gauge-coupling limit, the matrix \((\bar{v} D v)\) shows the massless singularity associated with the free left-handed Weyl field. With the periodic boundary condition, in particular, \((\bar{v} D v)\) is not invertible because there appear the zero modes in the eigenvalues of \(D\), which have zero index \(n_+ - n_- = 0\). In the topologically non-trivial sectors, the matrix \((\bar{v}_k D v_i)\) is not a square matrix and \(\text{det}(\bar{v} D v)\) vanishes identically. This is due to the appearance of the chiral zero modes with a non-trivial index \(n_+ - n_- = -8Q \neq 0\). These zeromodes, saturated by the insertion of the ‘t Hooft vertices in terms of the left-handed fields \(T_{\pm}(x)\), \(\bar{T}_{\pm}(x)\) given by eqs. (2.17) and (2.18), give rise to the non-vanishing vacuum expectation values of the ‘t Hooft vertices and break the global U(1)
fermion symmetry. These are the robust results of the overlap formalism/the index theorem followed from the Ginsparg-Wilson relation\textsuperscript{[30–32]}\textsuperscript{[25].10}

As to the second and third factors, on the other hand, we note that the first matrix $(u^T i\gamma_5 C_D T^a E^a u)$ changes its size as $n/2 - 8Q$ depending on the topological charge $Q$, but remains to be a square matrix, while the second one $(\bar{u} i\gamma_5 C_D T^{\dagger a} E^a \bar{u}^T)$ is the square matrix of the fixed size $n/2$. Therefore the pfaffians of these matrices do not vanish identically in general in any topological sector $\mathfrak{U}[Q]$. And the path-integration of the pfaffians over the spin fields $E^a(x)$ and $\bar{E}^a(x)$ gives a certain non-zero functional of the admissible link field $U(x, \mu)$, as we will discuss more in detail in the following sections 3.4 and 4. We refer this fact as “saturation of the right-handed part of the fermion measure by the ’t Hooft vertices”.

This result on the saturation of the right-handed part of the measure is in sharp contrast to that on the left-handed part of the measure, which gives rise to the chiral determinant. Thanks to this result, the total Weyl field measure defined by eqs. (3.4) and (3.5) reproduces the correct zero-modes associated with only the left-handed Weyl fields in any topologically non-trivial sectors $\mathfrak{U}[Q]$, while it is manifestly gauge-invariant and there is no phase-ambiguity which depends on the gauge field due to the choice of the chiral basis vectors. We hope that the above result on the saturation of the right-handed part of the measure for the anomaly-free multiplet of 16 of SO(10) is another robust property of the overlap fermion/the Ginsparg-Wilson relation. The schematic picture of our SO(10) model is shown in fig. 1 in terms of the overlap formalism\textsuperscript{[30–32]}

\subsection*{3.3 The Weyl field measure in terms of chiral basis}

In the definition of the Weyl field measure, eqs. (3.4) and (3.5), the part of the Dirac field measure, $\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}]$, may be formulated in chiral components by using the chiral bases defined with the chiral projectors $\hat{P}_\pm$ and $P_\pm$. In the given topological sector $\mathfrak{U}[Q]$, it reads

$$\mathcal{D}_+[\psi_-] \mathcal{D}_+[\bar{\psi}_-] \mathcal{D}_+[\psi_+] \mathcal{D}_+[\bar{\psi}_+] = \prod_{j=1}^{n/2+8Q} dc_j \prod_{k=1}^{n/2} d\bar{c}_k \prod_{j=1}^{n/2-8Q} db_j \prod_{k=1}^{n/2} d\bar{b}_k,$$

(3.26)

where $n = \dim \Lambda \times 4 \times 16$ and \{c_j, \bar{c}_k\} and \{b_j, \bar{b}_k\} are the Grassmann coefficients in the expansion of the chiral component fields,

$$\psi_- (x) = \sum_j v_j (x) c_j, \quad \bar{\psi}_- (x) = \sum_k \bar{c}_k \bar{v}_k (x),$$

(3.27)

\footnote{It has been known that there is a problem about the fermion number violation\textsuperscript{[238, 239]} in the gauge-fixing approach to construct chiral lattice gauge theories. (See \textsuperscript{[19]} and references there in.) In the gauge-fixing approach, to remove the species doublers, one uses non-gauge-covariant Wilson terms for the pairs of the target (left-handed) Weyl fermions and the spectator (right-handed) Weyl fermions. But the action and the path-integral measure of the Wilson fermions are invariant exactly under the vector-like fermion number symmetry, and this causes a problem in describing the process of fermion number violation\textsuperscript{[238, 239]}. One needs, as shown by Golterman and Shamir in \textsuperscript{[240]}, to introduce the explicit symmetry-breaking term such as the non-gauge-invariant Majorana mass term for the target (left-handed) fermions to reproduce the non-vanishing vacuum expectation values of ’t Hooft vertex operators. In \textsuperscript{[240]}, the case of the SO(10) chiral gauge theory was considered and the semi-classical continuum limit of the lattice model was examined for the background gauge fields with a SU(2) instanton solution embedded in all possible ways consistent with the SO(10) global symmetry.}

- 12 -
can always choose the basis of the Dirac field, \( \det(\text{original measure}) \) in terms of the chiral orthonormal bases defined by eqs. (3.16) and (3.18).\[76, 77\] Since the link field adjusting the overall constant phase factors of the Jacobian as 

\[
\psi^+_j(x) = \sum_j u_j(x)b_j, \quad \bar{\psi}^+_j(x) = \sum_k \bar{b}_k \bar{u}_k(x),
\]

in terms of the chiral orthonormal bases defined by eqs. (3.16) and (3.18).\[76, 77\] Since the original measure \( D[\psi]D[\bar{\psi}] \) does not depend on the gauge field, it follows that one can always choose the basis of the Dirac field, \( \{u_j(x), v_j(x)\} \), so that the Jacobian factor, \( \det(u_j(x), v_j(x)) \), is independent of the gauge field. For the infinitesimal variation of the link field \( \delta_\eta U(x, \mu) = i\eta_\mu(x)U(x, \mu) \), this condition is given by

\[
\sum_j (u_j, \delta_\eta u_j) + \sum_j (v_j, \delta_\eta v_j) = 0.
\]

Adjuncting the overall constant phase factors of the Jacobian as \( \det(u_j(x), v_j(x)) = 1 \), one obtains

\[
D_\ast[\psi_-]D_\ast[\bar{\psi}_-]D_\ast[\psi_+]D_\ast[\bar{\psi}_+] = D[\psi]D[\bar{\psi}]\text{.} \tag{3.30}
\]

Using this chiral decomposition of \( D[\psi]D[\bar{\psi}] \) and the integral representation of \( F(w) \), the Weyl field measure eq. (3.4) now reads in the factorized form,

\[
D[\psi_-]D[\bar{\psi}_-] = D_\ast[\psi_-]D_\ast[\bar{\psi}_-] \times D_\ast[\psi_+]D_\ast[\bar{\psi}_+] \prod_{x \in A} F(T_+(x)) \prod_{x \in A} F(\bar{T}_+(x))
\]

\[
= D_\ast[\psi_-]D_\ast[\bar{\psi}_-] \times D_\ast[\psi_+]D_\ast[\bar{\psi}_+] \int D[E]D[\bar{E}] e^{\sum_{x \in A} \{E^a(x)\bar{V}_+^a(x) + \bar{E}^a(x)\bar{V}_+^a(x)\}}[\psi_+, \bar{\psi}_+]\text{.} \tag{3.31}
\]
Moreover, using the chiral bases, ’t Hooft vertex terms of the right-handed fields \( \psi_+(x) \), \( \bar{\psi}_+(x) \) are written as follows,

\[
\sum_{x \in \Lambda} \{ E^a(x) V^a_+(x) \} [\psi_+] = \sum_{i,j} \frac{1}{2} b_i (u^T i \gamma_5 C \bar{D}^a E^a u)_{ij} b_j, \tag{3.32}
\]

\[
\sum_{x \in \Lambda} \{ \bar{E}^a(x) \bar{V}^a_+(x) \} [\bar{\psi}_+] = \sum_{k,l} \frac{1}{2} \bar{b}_k (\bar{u} i \gamma_5 C \bar{D}^a \bar{E}^a \bar{u}^T)_{kl} \bar{b}_l. \tag{3.33}
\]

Then the path-integration over the right-handed fields \( \psi_+ \), \( \bar{\psi}_+ \) can be performed explicitly as

\[
D[\psi_-] D[\bar{\psi}_-] = D_+ [\psi_-] D_+ [\bar{\psi}_-] \times \int D[E] \text{pf} (u^T i \gamma_5 C \bar{D}^a E^a u) \int D[E] \text{pf} (\bar{u} i \gamma_5 C \bar{D}^a \bar{E}^a \bar{u}^T), \tag{3.34}
\]

where the symbol \( \text{pf} \) stands for the pfaffians of these anti-symmetric matrices. Thus, in our definition of the Weyl field measure eq. (3.4), the insertion of the product of the ’t Hooft vertexes in terms of the right-handed fields, \( T_+(x) \) and \( \bar{T}_+(x) \), adds the extra factors of the pfaffians integrated over the auxiliary spin fields to the usual definition of the (left-handed) Weyl field measure, \( D_+ [\psi_-] D_+ [\bar{\psi}_-] = \prod_{j=1}^{n/2+8Q} dc_j \prod_{k=1}^{n/2} dc_k \) \cite{76, 77}.

As we mentioned before, the first matrix \( (u^T i \gamma_5 C \bar{D}^a E^a u) \) given by eq. (3.23) changes its size as \( n/2 - 8Q \) depending on the topological charge \( Q \), but remains to be a square matrix, while the second one \( (\bar{u} i \gamma_5 C \bar{D}^a \bar{E}^a \bar{u}^T) \) given by eq. (3.24) is the square matrix of the fixed size \( n/2 \). Therefore these pfaffians do not vanish identically in general in any given topological charge \( \mathfrak{U} \) \( Q \), and the path-integration of the pfaffians over the spin fields \( E^a(x) \) and \( \bar{E}^a(x) \) gives a certain non-zero functional of the admissible link field \( U(x, \mu) \), as we will argue in the following sections 3.4 and 4.

### 3.4 Saturation of the right-handed part of the fermion measure by ’t Hooft vertexes

The pfaffian of the second matrix eq. (3.24) turns out to be unity. This is because the matrix is represented as

\[
(\bar{u} i \gamma_5 C \bar{D}^a \bar{E}^a \bar{u}^T)_{kl} = i \epsilon_{\sigma \sigma'} \delta_{xx'} (\bar{T}^a \bar{P}_+)_{\mu \sigma} \bar{E}^a(x') \tag{3.35}
\]

for \( k = \{ x, \sigma, t \} \) and \( l = \{ x', \sigma', t' \} \), in the bases \( \gamma_5 = \text{diag}(1,1,-1,-1) \), \( \bar{u}_k(x)_{\alpha \sigma} = \delta_{xx'} \delta_{\alpha \sigma} + 2 \delta_{\alpha t} \) for \( k = \{ x' \in \Lambda; \sigma = 1,2; t = 1, \cdots , 16 \} \). Then the pfaffian of the matrix is evaluated as

\[
\text{pf}(\bar{u} i \gamma_5 C \bar{D}^a \bar{E}^a \bar{u}^T) = \prod_x \det (\bar{P}_+ + \bar{P}_- i \bar{T}^a \bar{E}^a(x)) = \prod_x \det (i \bar{T}^a \bar{E}^a(x)) = \prod_x \det (i \bar{C}^a [E^{10}(x) + i \bar{u}^{a'} \bar{E}^{a'}(x)]) = 1. \tag{3.36}
\]
Note that \( \det(i\hat C^\dagger) \) and \( \det((E^{10}(x) + iF^a E^a(x))) \) are both equal to +1 and the latter, in particular, is independent of \( E^a(x) \). Then the path-integration over \( E^a(x) \) simply gives

\[
\int D[E] \text{pf} \left( \bar{u} i\gamma_5 C_D T^a \bar{E}^a \bar{u}^T \right) = 1. \tag{3.37}
\]

Thus the measure of the right-handed anti-field, \( D_\star[\bar{\psi}_+] \), is indeed saturated completely by inserting the product of the \( \hat C \) vertex \( \bar{T}_+(x)[\bar{\psi}_+] \). This is actually the known result which was first shown by Eichten and Preskill in [98], where the effects of the generalized Wilson-terms were studied in the strong coupling limit. In fact, our result reads

\[
\int D_\star[\bar{\psi}_+] F(\bar{T}_+(x)[\bar{\psi}_+]) = \int \prod_{x \in \Lambda} \prod_{\alpha=3}^{4} \prod_{s=1}^{16} d\bar{\psi}_{\alpha s}(x) \prod_{x \in \Lambda} 8! \frac{1}{12!} \left\{ \frac{1}{24} \bar{\psi}(x) P_- i\gamma_5 C_D T^a \bar{\psi}(x)^T \bar{\psi}(x) P_- i\gamma_5 C_D T^a \bar{\psi}(x)^T \right\}^8 = 1, \tag{3.38}
\]

and it provides the explicit normalization for the constant in the result there[98].

The pfaffian of the first matrix eq. (3.23), on the other hand, is a complex number in general, which depends on the spin field \( E^a(x) \) as well as the link field \( U(x, \mu) \). We do not have a rigorous proof so far that the path-integration of the pfaffian over \( E^a(x) \) is non-zero for any admissible link fields. But there are typical examples of link field configurations where one can argue that it is indeed the case. Those include the case in the weak gauge-coupling limit where the link variables are set to unity, \( U(x, \mu) = 1 \), and the cases of the SU(2)(=Spin(3)) link fields with non-zero topological charges \( Q(\neq 0) \), which represent the non-trivial topological sectors \( \Omega[Q] \). These cases will be discussed further in detail in the following section 4.

In summary, the effective action \( \Gamma_W[U] \) is obtained in the chiral basis as follows.

\[
\left( \bar{D}_\star[\bar{\psi}_+] \right) \equiv \int D[\bar{\psi}_+] \prod_{x \in \Lambda} F(T_+(x)) \prod_{x \in \Lambda} F(\bar{T}_+(x)) e^{-S_W[\bar{\psi}_- \bar{\psi}_-]} = \int D_\star[\bar{\psi}_+] \prod_{x \in \Lambda} D_\star[\bar{T}_+(x)] D_\star[\bar{\psi}_+] \prod_{x \in \Lambda} F(T_+(x)) \prod_{x \in \Lambda} F(\bar{T}_+(x)) e^{-S_W[\bar{\psi}_- \bar{\psi}_-]} \times e^{-S_W[\bar{\psi}_- \bar{\psi}_-] + \sum_{x \in \Lambda} \{ E^a(x) V^a_T(x) + \bar{E}^a(x) V^a_T(x) \} [\bar{\psi}_+ \bar{\psi}_+]} = \det(\bar{v} D\bar{v}) \int D\bar{v} D\bar{E} \text{pf} (u^T i\gamma_5 C_D T^a E^a u). \tag{3.39}
\]

For later convenience, we introduce the abbreviation \( \langle \cdots \rangle_F \) for the path-integration of
only the fermion fields and the spin fields with the link field fixed as a background field:

\[
\langle \mathcal{O} \rangle_F = \int \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] e^{-S_W[\psi, \bar{\psi}]} \mathcal{O}
\]

\[
= \int \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] \prod_{x \in \Lambda} F(T_+(x)) \prod_{x \in \Lambda} F(\bar{T}_+(x)) e^{-S_W[\psi, \bar{\psi}]} \mathcal{O}
\]

\[
= \int \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] \mathcal{D}[E] \mathcal{D}[\bar{E}] e^{-S_W[\psi, \bar{\psi}] + \sum_{x \in \Lambda} \{E^a(x) \gamma^a \bar{\psi}(x) + \bar{E}^a(x) \gamma^a \psi(x)\}[\bar{\psi}, \bar{\psi}]} \mathcal{O}.
\]

(3.40)

We also use the abbreviation \( \langle \cdots \rangle_E \) and \( \langle \cdots \rangle'_E \) for the path-integration of the spin field \( E^a(x) \):

\[
\langle \mathcal{O} \rangle_E \equiv \int \mathcal{D}[E] pf(u^T i \gamma_5 C_D T^a E^a u) \mathcal{O},
\]

(3.41)

\[
\langle \mathcal{O} \rangle'_E \equiv \int \mathcal{D}[E] \mathcal{O}.
\]

(3.42)

With these abbreviations, the effective action eq. (3.39) reads

\[
e^{\Gamma_{W[U]}[U]} = \langle 1 \rangle_F[U]
\]

\[
= \det(\bar{\psi} D \psi) \langle 1 \rangle_F[U]
\]

\[
= \det(\bar{\psi} D \psi) \langle pf(u^T i \gamma_5 C_D T^a E^a u) \rangle'_F[U].
\]

(3.43)

3.5 CP invariance

We define CP transformation as

\[
U(x, \mu) \rightarrow U(x, \mu)^{CP} = \left( U(x^P, 0)^*, U(x^P - \hat{k}, k)^* \right),
\]

(3.44)

\[
\psi(x) \rightarrow \psi(x)^{CP} = +(P \gamma_0)^{-1} C_D^{-1} \bar{\psi}(x)^T,
\]

(3.45)

\[
\bar{\psi}(x) \rightarrow \bar{\psi}(x)^{CP} = -\psi(x)^T C_D P \gamma_0,
\]

(3.46)

\[
E^a(x) \rightarrow E^a(x)^{CP} = (-1)^a E^a(x^P) \quad (a = 0, 1, \cdots, 9),
\]

(3.47)

where \( P : x \rightarrow x^P \equiv (x_0, -x_k) \). Then, the overlap Dirac operator obeys the CP-conjugation relation,

\[
D[U^{CP}] = (P \gamma_0)^{-1} C_D^{-1} D[U]^T C_D P \gamma_0
\]

\[
= (P \gamma_0)^{-1} (\gamma_5 C_D)^{-1} D[U]^T \gamma_5 C_D P \gamma_0,
\]

(3.48)

(3.49)

and accordingly, the chiral projection operators satisfy the CP-conjugation relations given by

\[
\hat{P}_\pm[U^{CP}] = (P \gamma_0)^{-1} (\gamma_5 C_D)^{-1} \hat{P}_\pm[U]^T \gamma_5 C_D P \gamma_0,
\]

(3.50)

\[
P_{\pm} = (P \gamma_0)^{-1} (\gamma_5 C_D)^{-1} P_{\pm} \gamma_5 C_D P \gamma_0.
\]

(3.51)

Under the CP transformation, the action of the left-handed fields, \( \psi_-(x) \) and \( \bar{\psi}_-(x) \), is transformed as

\[
S_W = \sum_{x \in \Lambda} \bar{\psi}(x) P_+ D \psi(x) \quad \rightarrow \quad S_W = \sum_{x \in \Lambda} \bar{\psi}(x) D \bar{P}_- \psi(x),
\]

(3.52)
while the 't Hooft vertices of the right-handed fields, \( \psi_+ (x) \) and \( \tilde{\psi}_+ (x) \), are transformed as

\[
T_+ (x) = \frac{1}{2} V^a_+ (x) V^a_+ (x), \quad V^a_+ (x) = \frac{1}{2} \psi^T (x) \tilde{P}_+ i\gamma_5 C_D T^a \tilde{P}_+ \psi (x)
\]

\[\rightarrow T'_+ (x) = \frac{1}{2} V^{a'}_+ (x) V^{a'}_+ (x), \quad V^{a'}_+ (x) = (-1)^a \frac{1}{2} \bar{\psi} (x) \{ \gamma_5 \tilde{P}_- \gamma_5 \} i\gamma_5 C_D T^{a'} \{ \gamma_5 \tilde{P}_- \gamma_5 \} \bar{\psi} (x) , \quad \] (3.53)

\[
\tilde{T}_+ (x) = \frac{1}{2} \bar{V}^a_+ (x) \bar{V}^a_+ (x), \quad \bar{V}^a_+ (x) = \frac{1}{2} \bar{\psi} (x) P_- i\gamma_5 C_D T^a \tilde{E}^a P_T \bar{\psi} (x) T
\]

\[\rightarrow \tilde{T}'_+ (x) = \frac{1}{2} \bar{V}^{a'}_+ (x) \bar{V}^{a'}_+ (x), \quad \bar{V}^{a'}_+ (x) = (-1)^a \frac{1}{2} \bar{\psi}^T (x) P_+ i\gamma_5 C_D T^a P_+ \bar{\psi} (x). \quad (3.54)\]

Therefore, in our model, CP invariance is not manifest\([32, 241–243]\). Instead, the definition of the chiral projection for the fields and anti-fields are interchanged as

\[
\psi_- (x) = \tilde{P}_- \psi (x) \Rightarrow \psi_- (x) = P_- \psi (x),
\]

\[
\tilde{\psi}_- (x) = \tilde{P}_+ \psi (x) \Rightarrow \tilde{\psi}_- (x) = \tilde{\psi} \{ \gamma_5 \tilde{P}_+ \gamma_5 \} (x).
\]

\[
\psi_+ (x) = \tilde{P}_+ \psi (x) \Rightarrow \psi_+ (x) = P_+ \psi (x),
\]

\[
\tilde{\psi}_+ (x) = \tilde{P}_- \psi (x) \Rightarrow \tilde{\psi}_+ (x) = \tilde{\psi} \{ \gamma_5 \tilde{P}_- \gamma_5 \} (x).
\]

This transformation property implies immediately that the fermion expectation value \( \langle 1 \rangle_F [U] (= e^{\Gamma_W[U]}) \) is transformed in the following manner.

\[
\langle 1 \rangle_F [U] = \det (\bar{v} D v) \int \mathcal{D}[E] \text{pf} (u^T i\gamma_5 C_D T^a E^a u)
\]

\[\rightarrow \langle 1 \rangle_F [U^{CP}] = \det (u^\dagger \gamma_5 D \gamma_5 \bar{v}^\dagger) \int \mathcal{D} [\bar{E}] \text{pf} (v^T i\gamma_5 C_D T^a \bar{E}^a \gamma_5 v^*)
\]

\[= \left\{ \det (\bar{u} D u) \int \mathcal{D} [E] \text{pf} (v^T i\gamma_5 C_D T^a E^a v) \right\}^* . \quad (3.59)\]

Therefore a necessary and sufficient condition for the CP invariance of the effective action, \( \Gamma_W[U^{CP}] = \Gamma_W [U] \), is formulated by the following identity,

\[
\det (\bar{v} D v) \int \mathcal{D} [E] \text{pf} (u^T i\gamma_5 C_D T^a E^a u) = \left\{ \det (\bar{u} D u) \int \mathcal{D} [E] \text{pf} (v^T i\gamma_5 C_D T^a E^a v) \right\}^* .
\]

\[\quad (3.60)\]

Here we assume that the (background) link field is in the topologically trivial sector, where \( \langle 1 \rangle_F [U] \) is not vanishing and the effective action is well-defined.

To prove the identity eq. (3.60), we consider the two unitary matrices of the size \( n (= \dim \Lambda \times 4 \times 16) \) defined by

\[
\begin{pmatrix} (\bar{u} u) & (\bar{u} v) \\ (\bar{v} u) & (\bar{v} v) \end{pmatrix},
\]

\[\begin{pmatrix} (u^T i\gamma_5 C_D T^a E^a u) & (u^T i\gamma_5 C_D T^a E^a v) \\ (v^T i\gamma_5 C_D T^a E^a u) & (v^T i\gamma_5 C_D T^a E^a v) \end{pmatrix},
\] (3.61) (3.62)
where \( \{ u_j, v_j \} \) and \( \{ \bar{u}_j, \bar{v}_j \} \) consist the complete orthonormal bases of the Dirac fields \( \psi(x) \) and \( \bar{\psi}(x) \), respectively (cf. [32]). One can choose the bases so that the determinant of the first matrix is unity, while the determinant of the second one is unity independent of the choice. Note that \( (\bar{u}u) = (\bar{u}Du) \) and \( (\bar{v}v) = (\bar{v}Dv) \). Note also that the second matrix is unitary because of the constraint on the spin field, \( E^a(x)E^a(x) = 1 \). If a unitary matrix \( U \) has the block structure as

\[
U = \begin{pmatrix} N & O \\ P & M \end{pmatrix},
\]

(3.63)

where \( N \) and \( M \) are non-singular square matrices, it follows that

\[
\det U = \det N \times \det (M - PN^{-1}O) = \det N / \det M^\dagger.
\]

(3.64)

In the second equality, the relations \( N^{-1} = -P^\dagger(M^\dagger)^{-1}O^{-1} \) and \( MM^\dagger + PP^\dagger = 1 \) are used, which follow from the unitarity of \( U \). This result implies that

\[
\det(\bar{v}Dv) = \left\{ \det(\bar{u}Du) \right\}^*,
\]

(3.65)

\[
\text{pf}(u^T i\gamma_5 C D^T a E^a u) = \pm \left\{ \text{pf}(v^T i\gamma_5 C D^T a E^a v) \right\}^*.
\]

(3.66)

The signature in the second result is the constant independent of the link field and the spin field, and it can be fixed at \( U(x,\mu) = 1 \) and \( E^a(x) = \delta^{a10} \) to be +1. Since the path-integration over the real spin field commute with the complex conjugation, the identity eq. (3.60) now follows.

Therefore, we have

\[
\langle 1 \rangle_F[U^{CP}] = \langle 1 \rangle_F[U],
\]

(3.67)

and the effective action is indeed CP invariant[32],

\[
\Gamma_W[U^{CP}] = \Gamma_W[U].
\]

(3.68)

Thus the Weyl field measure defined by eqs. (3.4) and (3.5) respects the CP symmetry.

### 3.6 Schwinger-Dyson equations and Correlation functions

The Schwinger-Dyson equations for the link field and the Weyl field can be derived from the path-integral definition of the partition function, eqs. (3.9) and (3.10). With respect to the local variation of the link field, \( \delta_\eta U(x,\mu) = i\eta_\mu(x)U(x,\mu) \), the simplest non-trivial example is given by

\[
\left\langle \left[ -\delta_\eta S_G[U] - \sum_{x \in A} \bar{\psi}(x)P_+ \delta_\eta D\psi(x) + \sum_{x \in A} \psi^T \hat{P}_+^T \bar{\psi} \gamma_5 C_D^T T^a E^a \delta_\eta \hat{P}_+ \psi(x) \right] \right\rangle = 0,
\]

(3.69)

The operators in the bracket \([ \cdots ]\) in the l.h.s. are all the local operators with respect to the variation point \( x \) and therefore the equation of motion is local. We note that the third term comes from the link field dependence of the Weyl field measure. With respect to the
local variations of the fermion fields $\delta \psi(x)$, $\delta \tilde{\psi}(x)$ and of the spin field $\delta E^a(x)$, one can derive the following non-trivial examples.

\[
\left\langle \psi(y) \left[ \bar{\psi} P_+ D(x) - \psi^T \hat{P}_T i \gamma_5 C_D T^a E^a \hat{P}_+ (x) \right] \right\rangle_F = \delta_{xy} \langle 1 \rangle_F, \tag{3.70}
\]

\[
\left\langle \left[ P_+ D \psi(x) - P_- i \gamma_5 C_D T^a \bar{E}^a P_- T \tilde{\psi}(x) \right] \tilde{\psi}(y) \right\rangle_F = \delta_{xy} \langle 1 \rangle_F, \tag{3.71}
\]

\[
\frac{1}{2} \left\langle \psi^T \hat{P}_T i \gamma_5 C_D C [\Sigma_{bc}, \Gamma^a] E^a(x) \hat{P}_+ \psi \right\rangle_F = 0. \tag{3.72}
\]

The first two equations can be decomposed into the chiral components by noting $P_+ D = D \hat{P}_+$ and $\delta_{xy} = (P_+ + P_-) \delta_{xy} = \hat{P}_+(x, y) + \hat{P}_-(x, y)$. We finally obtain

\[
\left\langle \psi_+(x) \psi_-(y) \right\rangle_F = \hat{P}_- D^{-1} P_+(x, y) \langle 1 \rangle_F, \tag{3.73}
\]

\[
\left\langle \psi_+(y) \left[ \psi_+^T i \gamma_5 C_D T^a E^a \hat{P}_+ (x) \right] \right\rangle_F = -\hat{P}_+(y, x) \langle 1 \rangle_F, \tag{3.74}
\]

\[
\left\langle \left[ P_- i \gamma_5 C_D T^a \bar{E}^a \psi_+^T (x) \right] \psi_+(y) \right\rangle_F = -P_- \delta_{xy} \langle 1 \rangle_F, \tag{3.75}
\]

assuming that $D$ is invertible.

As long as $\langle 1 \rangle_F$ is finite and well-defined, these results imply the following facts about the particle spectrum in the channel of the 16 representation of SO(10) symmetry: the left-handed fields $\psi_-(x)$, $\tilde{\psi}_-(x)$ support the massless Weyl fermions and have long-range correlations, while the right-handed fields $\psi_+(x)$, $\tilde{\psi}_+(x)$ are decoupled each other and have short-range correlations of order the several lattice spacings with the composite operators $[\psi_+^T i \gamma_5 C_D T^a E^a \hat{P}_+(x)]$ and $[P_- i \gamma_5 C_D T^a \bar{E}^a \psi_+^T (x)]$, respectively.[28] As to the right-handed field $\psi_+(x)$, however, the information of yet another correlation function $\langle \psi_+(y) \left[ \psi_+^T i \gamma_5 C_D T^a E^a \hat{P}_-(x) \right] \rangle_F$ is also required before deducing a definite conclusion.

### 3.7 Gauge field dependence of the Weyl field measure – Locality issue remaining

The variation of the effective action $\Gamma_W[U]$ w.r.t. the link field can be derived from the path-integral definition eq. (3.10) as follows.

\[
\delta_\eta \Gamma_W[U] = \left\langle - \sum_{x \in \Lambda} \bar{\psi}(x) P_+ \delta_\eta D \psi(x) + \sum_{x \in \Lambda} \psi^T(x) \hat{P}_+^T i \gamma_5 C_D T^a E^a \delta_\eta \hat{P}_+ \psi(x) \right\rangle_F / \langle 1 \rangle_F
\]

\[
= \text{Tr} \left\{ P_+ \delta_\eta D \langle \psi_- \tilde{\psi}_- \rangle_F \right\} / \langle 1 \rangle_F - \text{Tr} \left\{ \delta_\eta \hat{P}_+ \langle \psi_+ [\psi_+^T i \gamma_5 C_D T^a E^a] \rangle_F \right\} / \langle 1 \rangle_F. \tag{3.76}
\]

The first term can be rewritten further using the result of the two-point correlation function of the left-handed fields eq. (3.73) as

\[
\text{Tr} \left\{ P_+ \delta_\eta D \langle \psi_- \tilde{\psi}_- \rangle_F \right\} / \langle 1 \rangle_F = \text{Tr} \left\{ P_+ \delta_\eta DD^{-1} \right\}. \tag{3.77}
\]

It is identified as the physical contribution of the left-handed Weyl fermions. The second term, on the other hand, represents the gauge field dependence of the Weyl field measure.
eq. (3.4) through the right-handed ‘t Hooft vertices. It replaces the measure term $-i\Sigma_\eta = \sum_j (v_j, \delta_\eta v_j)$ [76, 77]. So we denote this term with $-i\Sigma_\eta$,

$$-i\Sigma_\eta \equiv -\text{Tr}\{\delta_\eta \hat{P}_+ \langle \bar{\psi}_+ [\bar{\psi}_+^T i\gamma_5 C_D T^a E^a] \rangle_F \} / \langle 1 \rangle_F.$$  

(3.78)

Then the variation of the effective action is written as

$$\delta_\eta \Gamma_W[U] = \text{Tr}\{P_+ \delta_\eta DD^{-1}\} - i\Sigma_\eta.$$  

(3.79)

For the gauge transformation, $\delta_\eta U(x, \mu) = i\{\omega(x)U(x, \mu) - U(x, \mu)\omega(x + \mu)\}$ and $\eta_\mu(x) = \omega(x) - U(x, \mu)\omega(x + \mu)U(x, \mu)^{-1} = -D_\mu\omega(x)$, the first term gives the gauge anomaly term,

$$\text{Tr}\{P_+ \delta_\eta DD^{-1}\}|_{\eta_\mu = -D_\mu\omega} = -i\text{Tr}\{\omega_{\gamma 5} D\},$$  

(3.80)

where, in the weak gauge-coupling expansion, the leading non-trivial term is vanishing because of the anomaly cancellation condition for the 16-dimensional (irreducible) spinor representation of SO(10), $\text{Tr}\{P_+ \Sigma_{a_1 b_1} [\Sigma_{a_2 b_2} \Sigma_{a_3 b_3} + \Sigma_{a_3 b_3} \Sigma_{a_2 b_2}]\} = 0$. The second term gives

$$-i\Sigma_\eta |_{\eta_\mu = -D_\mu\omega} = -i\text{Tr}\{[\omega, \hat{P}_+ \bar{\psi}_+ [\bar{\psi}_+^T i\gamma_5 C_D T^a E^a] \hat{P}_+] \} / \langle 1 \rangle_F$$

$$= \left( -i \frac{1}{2} \text{Tr}\{\langle \bar{\psi}_+ [\bar{\psi}_+^T i\gamma_5 C_D [\Gamma^a, \omega] E^a] \rangle_F \} \right.$$  

$$+ i \text{Tr}\{\langle \bar{\psi}_+ [\bar{\psi}_+^T i\gamma_5 C_D T^a E^a \hat{P}_+] \rangle_F \} \} / \langle 1 \rangle_F$$  

$$= +i\text{Tr}\{\omega_{\gamma 5} D\},$$  

(3.81)

where the Schwinger-Dyson equations eqs. (3.72) and (3.74) are used at the last equality. Thus we can check that the effective action is gauge-invariant.

The measure term $-i\Sigma_\eta$ is required to be a smooth and local function of the link field variables, since it appears as an operator of the link field in the Schwinger-Dyson equation w.r.t. the link field, eq. (3.69). In the weak gauge-coupling expansion, in particular, the vertex functions are derived from this term as

$$-i\Sigma_\eta = \sum_{m=0}^{\infty} \frac{(ig)^{1+m}}{\sqrt{1+m}} \frac{1}{m!} \sum_{k:p_1, \ldots, p_m} \bar{\eta}_{\mu}^{ab}(-k) \epsilon_{\nu_1 \cdots \nu_m}^{abc} \omega_{\mu_1 \cdots \nu_m}^{m} \times \sum_{p_1} \ldots \sum_{p_m} \tilde{A}_{\nu_1}^{a_1 b_1} \ldots A_{\nu_m}^{a_m b_m}(p_m)$$  

(3.82)

and they are required to be analytic w.r.t. the external momenta. Since the gauge field dependence of the Weyl field measure is induced by the path-integrations of the right-handed Weyl field $\psi_+(x)$ and the spin field $E^a(x)$, it is required that these fields have short range correlations with the correlation lengths of order the lattice spacing.

A necessary and sufficient condition for this requirement is that the corrrelation function $\langle \psi_+(x) [\bar{\psi}_+^T i\gamma_5 C_D T^a E^a(y)] \rangle_F$ is the smooth function of the (background) link field variables and it satisfies the locality bound,

$$\| \langle \psi_+(x) [\bar{\psi}_+^T i\gamma_5 C_D T^a E^a(y)] \rangle_F / \langle 1 \rangle_F \| \leq C |x - y|^\sigma e^{-|x - y|/\xi}$$  

(3.83)

for certain constants $C > 0$, $\sigma > 0$ and $\xi > 0$ and the similar bounds for its variations w.r.t. the link field. (cf. [28]) We note that the above condition is satisfied by the part
\begin{equation}
\langle \psi_+(x) [\psi_+^T \gamma_5 C_D T^a E^a \hat{P}_+(y)] \rangle \text{ because of eq. (3.74) and is therefore about the part of}
\langle \psi_+(x) [\psi_+^T \gamma_5 C_D T^a E^a \hat{P}_-(y)] \rangle. \text{ The locality range } \xi \text{ then determines the effective cutoff}
\text{ scale } \Lambda \text{ of the lattice model as } \Lambda = \pi (\xi a)^{-1}.
\end{equation}

This question is of not quite dynamical but non-perturbative nature in our model,
involved in the path-integration of the spin field \( E^a(x) \) with the weight \( \text{pf}(u^T \gamma_5 C_D T^a E^a u) \)
which is complex in general. And we do not have yet a rigorous proof on the smoothness
and locality of the measure term for any admissible link fields. But this question is well-defined. It can be
addressed in the weak gauge-coupling limit at least because the pfaffian
\( \text{pf}(u^T \gamma_5 C_D T^a E^a u) \) is positive semi-definite in this case, as we will argue in the following
section, and Monte Carlo methods are applicable to evaluate the correlation functions and
the vertex functions. We leave this important and interesting question for our future study.

For later use, we express the measure term \( -i \Sigma_\eta \) in terms of the chiral basis, although
it does not actually depend on the choice of the basis. For this, we first note that the
correlation function \( \langle \psi_+(x) [\psi_+^T \gamma_5 C_D T^a E^a (y)] \rangle_F \) is written explicitly in the chiral basis as
\begin{equation}
\langle \psi_+(x) [\psi_+^T \gamma_5 C_D T^a E^a (y)] \rangle_F
= - \langle u_i(x) (u^T \gamma_5 C_D T^a E^a u)^{-1}_{ij} (u_j^T (y) \gamma_5 C_D T^a E^a (y)) \rangle_E / \langle 1 \rangle_E.
\end{equation}

Then the measure term \( -i \Sigma_\eta \) can be also expressed in terms of the chiral basis as
\begin{equation}
-i \Sigma_\eta \equiv - \text{Tr} \{ \delta_\eta \hat{P}_+ \langle \psi_+^T \gamma_5 C_D T^a E^a \rangle_F \} / \langle 1 \rangle_E
= \left\langle \text{Tr} \{ (u^T \gamma_5 C_D T^a E^a \delta_\eta \hat{P}_+ u) (u^T \gamma_5 C_D T^a E^a u)^{-1} \} \right\rangle_E / \langle 1 \rangle_E.
\end{equation}

In this respect, we find it interesting to see that such quantity like the integral of pfaffian
\( \int \mathcal{D}[E] \text{pf}(u^T \gamma_5 C_D T^a E^a u) \) can reproduce the gauge anomaly term, \( i \text{Tr} \{ \omega_5 D \} \), besides
the chiral determinant \( \text{det}(\bar{\psi}D\psi) \) when the anomaly cancellation condition is fulfilled. The variation of
the integral of pfaffian is evaluated as
\begin{equation}
\delta_\eta \ln \left\{ \int \mathcal{D}[E] \text{pf}(u^T \gamma_5 C_D T^a E^a u) \right\}
= \sum_j \langle u_j, \delta_\eta u_j \rangle + \langle \text{Tr} \{ (u^T \gamma_5 C_D T^a E^a \delta_\eta \hat{P}_+ u) (u^T \gamma_5 C_D T^a E^a u)^{-1} \} \rangle_E / \langle 1 \rangle_E
= \sum_j \langle u_j, \delta_\eta u_j \rangle - i \Sigma_\eta.
\end{equation}

The first term in the r.h.s. is followed from the property of the pfaffian as \( \text{pf} \{ Q^T A Q \} = \text{pf} A \times \text{det} Q \) where \( Q \) is unitary. It sums up with the term \( \sum_j \langle v_j, \delta_\eta v_j \rangle (=-i \Sigma_\eta) \) from the
variation of chiral determinant \( \text{det}(\bar{\psi}D\psi) \) to zero, because of the condition eq. (3.29). Thus
the gauge-variation of the integral of pfaffian leads to the result eq. (3.81).^{11}

^{11}If one makes the other choice for \( F(w) \) as \( F(w) = e^w = \sum_{k=0}^\infty \frac{w^k}{k!} \),
the integral of pfaffian \( \int \mathcal{D}[E] \text{pf}(u^T \gamma_5 C_D T^a E^a u) \) is replaced by the hyper-pfaffian, \( \text{hpf} A \), of the
rank-four complete anti-symmetric tensor \( A_{ijkl} \equiv T_{ijkl} + T_{iklj} + T_{ijlk} \) where \( T_{ijkl} = \sum_x (1/2) u^a(x) \gamma_5 C_D T^a u^b(x) u^b(x) \gamma_5 C_D T^a u^a(x) \). It can also reproduce the gauge anomaly term. We
wonder if it is possible to interpret these quantities from the point of view of topological field theory.
4 More on the saturation of the fermion measure by ’t Hooft vertices

As discussed in the previous section, the pfaffian of the matrix eq. (3.23) is in general a complex number which depends on the spin field \(E^a(x)\) as well as the link field \(U(x, \mu)\). And we do not have yet a rigorous proof that the path-integration of the pfaffian over \(E^a(x)\),

\[
\langle 1 \rangle_E = \int \mathcal{D}[E] \text{pf}(u^T i \gamma_5 C_D T^a E^a u), \tag{4.1}
\]

is non-zero for any admissible link fields. But there are typical examples where one can argue that it is indeed the case. Those include the case in the weak gauge-coupling limit where the link variables are set to unity, \(U(x, \mu) = 1\), and the cases of the SU(2) link fields with non-zero topological charges \(Q(\neq 0)\), which represent the non-trivial topological sectors \(U[Q]\). We will examine these cases in detail.

4.1 Property of the functional pfaffian for the link fields in Spin(9) subgroup

For this purpose, let us assume that the link field \(U(x, \mu)\) is in the SO(9) subgroup and commutes with \(\Gamma^{10}\),

\[
[\Gamma^{10}, U(x, \mu)] = 0 \tag{4.2}
\]

and, accordingly,

\[
\Gamma^{10} \hat{P}_+[U] \Gamma^{10} = \hat{P}_+[U]. \tag{4.3}
\]

Then the charge-conjugation relation,

\[
C^{-1} (\gamma_5 C_D)^{-1} \hat{P}_+[U]^T (\gamma_5 C_D) C = \hat{P}_+[U],
\]

implies that one can choose the basis vectors \(\{u_j(x)\}\) so that they satisfy the relation

\[
u_j(x)^T i \gamma_5 C_D C \Gamma^{10} = C_{jk} u_k(x)^T,
\]

\[
C^{-1} C^\dagger = C^T = -C. \tag{4.6}
\]

\(C\) is then given by the expression \(C_{jk} = (u^T i \gamma_5 C_D C \Gamma^{10} u)_{jk}\) and the matrix eq. (3.23) can be represented as

\[
(u^T i \gamma_5 C_D T^a E^a u) = C \times (u^\dagger \Gamma^{10} T^a E^a u)
\]

\[
= (u^\dagger \Gamma^{10} T^a E^a u)^T \times C. \tag{4.7}
\]

This implies that while the eigenvalues of \((u^T i \gamma_5 C_D T^a E^a u)\) appear in pair with the opposite signatures as \(\{(\tilde{\lambda}_i, -\tilde{\lambda}_i) \mid i = 1, \ldots, n/4 - 4Q\}\), the eigenvalues of \((u^\dagger \Gamma^{10} T^a E^a u)\) degenerate with the multiplicity two at least as \(\{(\lambda_i, \lambda_i) \mid i = 1, \ldots, n/4 - 4Q\}\).

Then the pfaffian of the matrix \((u^T i \gamma_5 C_D T^a E^a u)\) can be written as

\[
\text{pf}(u^T i \gamma_5 C_D T^a E^a u) = \text{pf}(u^T i \gamma_5 C_D C \Gamma^{10} u) \times \prod_{i=1}^{n/4-4Q} \lambda_i. \tag{4.8}
\]
Since the space of the spin field configurations, which we denote with \( V_E \), is the direct product of multiple \( S^9 \), \( V_E = S^9 \times \cdots \times S^9 \) (by \( \dim \Lambda \) times), it is pathwise connected and any configuration of the spin field \( E^a(x) \) can be reached from the constant configuration \( E^a_0(x) = \delta^{a,10} = \text{const.} \) through a continuous deformation. For the constant configuration we have \( \lambda_i = 1 \) (\( i = 1, \cdots, n/4 - 4Q \)). And this fixes the signature of the above formula (as long as the pfaffian is not vanishing).

The half product of the eigenvalues of \( (u^\dagger \Gamma^a \Gamma^b E^a u) \), \( \prod_{i=1}^{n/4-4Q} \lambda_i \) in eq. (4.8), is independent of the choice of the basis vectors. Its square or the full product can be expressed in the basis-independent manner as follows,

\[
\prod_{i=1}^{n/4-4Q} \lambda_i^2 = \det(u^\dagger \Gamma^a \Gamma^b E^a u) = \det \left( P_- + P_+ \left[ \hat{P}_- + \hat{P}_+ \Gamma^b E^b \right] \right). \tag{4.9}
\]

### 4.2 The case of trivial link field in the weak gauge-coupling limit

In the weak gauge-coupling limit where the link variables are set to unity, \( U(x, \mu) = 1 \), one can choose the basis vectors \( \{ u_j(x) \} \) as

\[
u_j(x) = \frac{1}{\sqrt{2N}} e^{ij \mu x} u_\alpha(p, \sigma) \delta_{s,t} \quad (j = \{ p, \sigma, t \}), \tag{4.10}\]

where \( \{ u_\alpha(p, \sigma) \} \) are the four-spinor eigenvectors of the free hermitian Wilson-Dirac operator \( H_w = \gamma_5 (D_w - m_0) \) (0 < \( m_0 < 2 \)) with the negative eigenvalues in the plane-wave basis given by

\[
u_\alpha(p, \sigma) = \begin{cases} 
\left( \begin{array}{c}
-c(p) \chi_\sigma \\
(\omega(p) + b(p)) \chi_\sigma 
\end{array} \right) / \sqrt{2\omega(p)(\omega(p) + b(p))} & (p \neq 0) \\
\left( \begin{array}{c}
\chi_\sigma \\
0 
\end{array} \right) & (p = 0)
\end{cases} \tag{4.11}\]

and

\[
b(p) = \sum_{\mu} (1 - \cos p_{\mu}) - m_0, \tag{4.12}\]

\[
c(p) = I \{ i \sin p_0 \} - \sum_{k} \sigma_k \sin p_k, \tag{4.13}\]

\[
\omega(p) = \sqrt{\sum_{\mu} \sin^2 (p_{\mu}) + \left\{ \sum_{\mu} (1 - \cos (p_{\mu})) - m_0 \right\}^2}. \tag{4.14}\]

The four-momentum \( p_\mu \) is given by \( p_\mu = \frac{2\pi n_\mu}{L} \) (\( n_\mu \in \mathbb{Z} \)) for the periodic boundary condition and \( p_\mu = \frac{2\pi (n_\mu + 1/2)}{L} \) (\( n_\mu \in \mathbb{Z} \)) for the anti-periodic boundary condition. The zero modes with \( p_\mu = 0 \) in eq. (4.11) exist only for the periodic boundary condition. (See the appendix C for detail.)
For the constant configuration of the spin field, \(E_0^a(x) = \delta^{a,10}\), \(u^T i \gamma_5 C_D T^a E_0^a u\)\(jjk\)(\(= C_{jjk}\)) = \(\delta_{p+p',0} \delta_{\sigma,\sigma'} i \tilde{C}_{tt'}\) \((j = \{p, \sigma, t\}, k = \{p', \sigma', t'\})\) and \(\tilde{\lambda}_i = \pm i\), while \((u^T \Gamma^{10} T^a E_0^a u\))\(jjk\) = \(\delta_{jjk}\) and \(\lambda_i = +1\). Then \(\text{pf}(u^T i \gamma_5 C_D T^a E_0^a u)\) is unity.

For randomly generated spin-field configurations with the lattice sizes up to \(L = 4\), we computed numerically the eigenvalues of the matrices \((u^T i \gamma_5 C_D T^a E_0^a u)\) and \((u^T \Gamma^{10} T^a E_0^a u)\), the pfaffian \(\text{pf}(u^T i \gamma_5 C_D T^a E_0^a u)\) and the half product \(\prod_{i=1}^{n/4} \lambda_i\). We checked that the eigenvalue spectra of the these matrices have the structures of \(\{(\tilde{\lambda}_i, -\tilde{\lambda}_i) | i = 1, \ldots, n/4 - 4Q\}\) and \(\{\lambda_i, \lambda_i \} | i = 1, \ldots, n/4 - 4Q\}\), respectively. All eigenvalues turn out to be non-zero. But there appear relatively small eigenvalues for the periodic boundary condition. We found that both the pfaffian and the half product stay real-positive. The typical examples of the eigenvalue spectra are shown in fig. 2 for \(L = 4\) with the periodic boundary condition. One can see how the half product remains real-positive: the eigenvalues of the “half set” of the eigenvalues sum up still exactly to zero mod \(2\pi\). Accordingly, the eigenvalues of \((u^T i \gamma_5 C_D T^a E_0^a u)\) appear closely but not exactly in the quartet structure \(\{\tilde{\lambda}, -\tilde{\lambda}, \tilde{\lambda}^*, -\tilde{\lambda}^*\}\) and the pfaffian is still exactly real-positive.

As to the relatively small eigenvalues observed for the trivial link field and randomly generated spin-field configurations with the periodic boundary condition, they are attributed to the zero modes with \(p_\mu = 0\) in eqs. (4.10) and (4.11) and their mixing-partners. This is because the number of these small eigenvalues always counts to 64 (= 32 \(\times\) 2) and such small eigenvalues do not appear with the anti-periodic boundary condition (for up to \(L = 4\)) as shown in fig. 3. The non-zero components of the zero modes’ vectors are right-handed as \((\chi_\sigma, 0)^T\) \((\sigma = 1, 2)\), while the never-vanishing components of the other modes’ vectors are left-handed. The relevant matrix elements of \((u^T \Gamma^{10} T^a E_0^a u)\) for the mixing then read

\[
- \chi_\sigma^T c(p') \chi_\sigma \delta_{0,p'} k \Gamma^{10} T^a E_0^a(k) / V \sqrt{2 \omega(p') \omega(p')} + b(p'),
\]

(4.15)
Figure 3. The eigenvalue spectra of the matrices \((u^T i\gamma_5 C_D T^a E^a u)\) and \((u^\dagger \Gamma^{10} \Gamma^a E^a u)\) with a randomly generated spin-field configuration for the case of the trivial link field. The lattice size is \(L = 4\) and the boundary condition for the fermion field is anti-periodic. For reference, the eigenvalue spectrum of the matrix \((\bar{v}^k D v_i)\) is also shown with green x symbol for the same boundary condition.

where \(\tilde{E}^a(k)\) is the fourier-components of \(E^a(x)\) defined by \(\tilde{E}^a(k) \equiv \sum_x e^{-ikx} E^a(x)\) with the constraints, \(\sum_{k_\mu} \tilde{E}^a(k)^* \tilde{E}^a(k) = V^2\) and \(\sum_{k_\mu} \tilde{E}^a(k)^* \tilde{E}^a(k + p) = 0\) \((p \neq 0)\).

Therefore the zero modes mix with a linear-combination of the modes \(\{p'_\mu, \sigma'_A\}\) for which
\[
\chi^T_{\sigma} c(p') \chi_{\sigma'} \delta_{p' + k, 0} \tilde{E}^a(k) \neq 0,
\]
but they decouple completely from the modes with the momenta \(p'_\mu = \pi_{(A)}^1 (A = 1, \ldots, 15)\) where \(\pi^1 \equiv (\pi, 0, 0, 0), \pi^2 \equiv (0, \pi, 0, 0), \ldots, \pi^{15} \equiv (\pi, \pi, \pi, \pi)\). This implies that the mixing of the zero modes is completely suppressed for the following class of the spin configurations,
\[
E^a(x) = \frac{1}{V} \sum_A \cos(\pi^{(A)} x) \tilde{E}^a(\pi^{(A)}),
\]
\[
\sum_A \tilde{E}^a(\pi^{(A)}) \tilde{E}^a(\pi^{(A)}) = V^2,
\]
\[
\sum_{A \neq B} \tilde{E}^a(\pi^{(A)}) \tilde{E}^a(\pi^{(A)} + \pi^{(B)}) = 0 \quad (B = 1, \ldots, 15).
\]

In this case, zero eigenvalues appear and the multiplicity of the zero eigenvalues is at least 64. This explains why the relatively small eigenvalues appear for randomly generated spin-field configurations with the periodic boundary condition. One can verify numerically the appearance of zero eigenvalues for \(E^a_*\). The example of the eigenvalue spectra of \((u^\dagger \Gamma^{10} \Gamma^a E^a_* u)\) are shown in fig. 4 for \(L = 4\) with the periodic boundary condition.

Based on the analytical results in the previous subsection and the above numerical observations, we assume that the half product \(\prod_{i=1}^{n/4 - 4Q} \lambda_i\) stays real independently of the spin field \(E^a(x)\). Then we can argue that the pfaffian is positive semi-definite for any spin-field configuration \(E^a(x)\) in the weak gauge-coupling limit:
\[
\text{pf}(u^T i\gamma_5 C_D T^a E^a u) = \prod_{i=1}^{n/4 - 4Q} \lambda_i \geq 0 \quad (Q = 0; g \to 0).
\]
We first note, as we discussed before, that the space of the SO(10)-vector spin field configurations, which we denote with \( V_E \), is the direct product of multiple \( S^9 \) and is pathwise connected. Then any configuration of the spin field \( E^a(x) \) can be reached from the constant configuration \( E^a_0(x) = \delta^{a,10} \) through a continuous deformation. Since it is unity for the constant configuration, the half product \( \prod_{i=1}^{n/4-4Q} \lambda_i \) should be positive for a given configuration \( E^a(x) \) as long as there exists a path to \( E^a(x) \) from \( E^a_0(x) (= \delta^{a,10}) \) such that the half product never vanish along the path. On the other hand, for the spin configurations with which the half product is zero, a certain subset in the eigenvalue spectrum of \( (u^\dagger \Gamma^{10} \Gamma^a E^a u) \) are zero. Along the path which goes though such a spin configuration, the eigenvalue spectrum flow and the subset of would-be zeros pass the origin in the complex plane. Then the half product can change discontinuously in its signature(phase). Since the signature(phase) stays constant as far as the half product is nonzero, this could happen if and only if the subspace of the configurations with the vanishing determinant, which we denote with \( V^0_E \), can divide the entire space of the spin configurations \( V_E \) into the subspaces which are disconnected each other. And the divided disconnected subspaces of \( V_E \setminus V^0_E \) should be classified by the values of the signature(phase) of the half product. In this respect, however, one notes that \( \pi_k(S^9) = 0 \) \((k < 9)\) and any topological obstructions and the associated topological terms are not known in the continuum limit for the SO(10)-vector spin field \( E^a(x) \) on the four-dimensional spacetime \( S^4 \) or \( T^4 \). In particular, any topologically non-trivial configurations/defects of the SO(10)-vector spin field and the associated fermionic massless excitations are not known in the continuum limit. Then it seems reasonable to assume that \( V^0_E \) consists of lattice artifacts and in particular it is given solely by the subspace of the configurations \( E^a_*(x) \), which we denote with \( V^*_E \). If one assumes that \( V^0_E = V^*_E \), the multiplicity of the zero eigenvalues are 64 and the would-be zero eigenvalues have the approximate structure \( \{ (\lambda_i, \lambda_i, \lambda_i^*, \lambda_i^*) | i = 1, \cdots, 16 \} \). Then the signature(phase) of the half product \( \prod_{i=1}^{n/4-4Q} \lambda_i \) does not change in passing \( V^0_E (= V^*_E) \). Therefore the pfaffian \( \text{pf}(u^T i\gamma_5 C_D T^a E^a u) \) is positive semi-definite.
It then follows that the path-integration of the pfaffian is real and positive in the weak gauge-coupling limit:

\[ \langle 1 \rangle_F = \int D[E^a] \det(u^T i\gamma_5 C D T^a E^a u) > 0 \quad (Q = 0; \ g \to 0). \quad (4.20) \]

4.3 The case of representative SU(2) link fields of topologically non-trivial sectors

As for the case of the SU(2) link fields with non-zero topological charges \( Q \neq 0 \), we take the following link field which gives the topological charge \( Q = 2m_0 m_23 \) \((m_0, m_23 \in \mathbb{Z})\) \([244–246]\):

\[ U(x, \mu) = e^{i\theta_{12}(x, \mu)}\Sigma^{12}, \quad (4.21) \]

where

\[ \theta_{12}(x, 0) = \begin{cases} 0 & (x_0 < L - 1) \\ -F_{01}Lx_1 & (x_0 = L - 1) \end{cases}, \quad \theta_{12}(x, 1) = F_{01}x_0, \quad (4.22) \]

\[ \theta_{12}(x, 3) = \begin{cases} 0 & (x_2 < L - 1) \\ -F_{23}Lx_3 & (x_2 = L - 1) \end{cases}, \quad \theta_{12}(x, 4) = F_{23}x_2, \quad (4.23) \]

and

\[ F_{01} = \frac{4\pi m_0}{L^2}, \quad F_{23} = \frac{4\pi m_{23}}{L^2}. \quad (4.24) \]

With this link field, the hermitian Wilson-Dirac operator is diagonalized numerically and the normalized eigenvectors with the negative eigenvalues are computed to form the chiral basis \( \{u_j(x)\} \). We checked that the number of the eigenvectors is \( n/2 - 8Q \) for \( L = 3, 4 \) with the periodic b.c. and is consistent with the index theorem.

For the constant configuration of the spin field, \( E_0^a(x) = \delta^{a,10}, \ (u^T i\gamma_5 C D T^a E_0^a u)_{jk} (= C_{jk}) \) is a unitary matrix and \( \hat{\lambda}_i \)'s are pure complex phases, while \( (u^T \Gamma^{10} T^a E_0^a u)_{jk} \) remains the unit matrix and \( \lambda_i = +1 \). Then \( \text{pf}(u^T i\gamma_5 C D T^a E_0^a u) \) is a pure complex phase.

For randomly generated spin-field configurations with the lattice sizes up to \( L = 4 \), we again checked that the eigenvalue spectra of the matrices \( (u^T i\gamma_5 C D T^a E^a u) \) and \( (u^T \Gamma^{10} T^a E^a u) \) have the structures of \( \{(\hat{\lambda}_i, -\hat{\lambda}_i) \mid i = 1, \ldots, n/4 - 4Q\} \) and \( \{(|\lambda_i|, \lambda_i) \mid i = 1, \ldots, n/4 - 4Q\} \), respectively. All eigenvalues turn out to be non-zero. But there appear again relatively small eigenvalues for \( Q < 0 \), the number of those counts to \( -8Q \). We found that the complex phase of \( \text{pf}(u^T i\gamma_5 C D T^a E^a u) \) stays constant and equal to that of \( \text{pf}(u^T i\gamma_5 C D T^a E_0^a u) \), while the half product \( \prod_{i=1}^{n/4-4Q} \lambda_i \) stays real-positive. The typical examples of the eigenvalue spectra are shown in fig. 5 for \( Q = -2 \) and \( L = 4 \) with the periodic boundary condition.

In this case, the relatively small eigenvalues can be attributed to the chiral zero modes due to the topologically non-trivial link field. This is because the number of these small eigenvalues always counts to \( -8Q \) consistently with the index theorem.

Based on the analytical results in the previous subsection and the above numerical observations, we again assume that the half product \( \prod_{i=1}^{n/4-4Q} \lambda_i \) stays real independently of
Figure 5. The eigenvalue spectra of the matrices \( (u^T i\gamma_5 C_\beta T^a E^a u) \) and \( (u^\dagger \Gamma^{10} \Gamma^a E^a u) \) with a randomly generated spin-field configuration for the case of the representative SU(2) link field of the topological sector with \( Q = -2 \). The lattice size is \( L = 4 \) and the boundary condition for the fermion field is periodic.

The spin field \( E^a(x) \). In this case the multiplicity of the zero eigenvalues should be \( | -8Q | \) and the would-be zero eigenvalues should have the approximate structure \( \{ (\lambda_i, \lambda_i^*, \lambda_i^*, \lambda_i^*) \ | \ i = 1, \cdots, | -2Q | \} \), and then the signature(\( \text{phase} \)) of the half product does not change. Therefore the complex phase of the pfaffian is stationary. It then follows that the path-integration of the pfaffian is a non-vanishing complex number for the representative \( SU(2) \) link field of topologically non-trivial sectors \( \Lambda[Q] \):

\[
\langle 1 \rangle_E = \int \mathcal{D}[E^a] \det(u^T i\gamma_5 C_\beta T^a E^a u) = c[U] \ (\not= 0) \quad (U(x, \mu) = e^{i\theta_{12}(x, \mu) \Sigma_{12}} \in \Lambda[Q] ; \ g \to 0). \quad (4.25)
\]

### 4.4 Continuity across the mass singularity: \( m_0 \to +0; \ +0 \to -0; \ -0 \to -\infty \)

Another support for the above arguments is followed from the consideration on the continuity in the mass parameter \( m_0 \) from the negative region, \( m_0 < 0 \) to the positive region, \( 0 < m_0 < 2 \). We note first that the chiral basis for the negative mass \( m_0 < 0 \) can be defined by the same formula as for the positive mass \( 0 < m_0 < 2 \) given by eqs. (4.10) and (4.11), except that the vectors of zero modes with \( p_\mu = 0 \) should be flipped in chirality to the left-handed ones \( (0, \chi_\sigma)^T \ (\sigma = 1, 2) \) from the right-handed ones \( (\chi_\sigma, 0)^T \ (\sigma = 1, 2) \). For this case, one can take the limit \( m_0 \to -\infty \) to obtain the trivial basis vectors\(^\text{12}\) as

\[
u_j(x) = \frac{1}{\sqrt{L^4}} e^{ipx} \begin{pmatrix} 0 \\ \chi_\sigma \end{pmatrix} \delta_{s,t} \quad (j = \{ p, \sigma, t \}), \quad (4.26)
\]

and one can show that the pfaffian is unity independent of the spin field \( E^a(x) \), just like the case of the anti-field \( \bar{\psi}_+(x) \). We note next that in the limits \( m_0 \to \mp 0 \), the both formula

\(^\text{12}\)Note that we define the Wilson Dirac operator \( X \) in eq. (2.9) with the negative signature in front of the mass parameter \( m_0 \). The above fact is related to the fact that one can send to the infinity the mass parameter \( +m_0 \) of the positive mass region in Kaplan’s Domain-wall fermion\(^6\). And this is why and how Kaplan’s domain-wall fermion (with periodic b.c.) is simplified to Shamir’s boundary fermion/vectorlike domain-wall fermion\(^6, 7\).
are well-defined as long as \( L \) is finite. But, actually at \( m_0 = \mp 0 \), the zero modes of both chiralities belong to the same zero-eigenvalue-sector of the massless hermitian Wilson-Dirac operator \( H_w = \gamma_5 D_w \) and degenerate. Then one can interpolate the two regions of \( m_0 \to \mp 0 \) smoothly by the one-parameter family of the basis vectors of the zero modes,

\[
u_o(0, \sigma)(\theta) = \begin{pmatrix} \sin \theta \chi_\sigma \\ \cos \theta \chi_\sigma \end{pmatrix} \quad \theta \in [0, \pi/2]. \tag{4.27}
\]

The examples of the eigenvalue spectra of \( (u^\dagger \Gamma^{10} T^a E^a u) \) in the limit \( m_0 \to \mp 0 \) are shown for randomly generated spin configurations in fig. 6 and for \( E^a_5(x) \) in fig. 7, respectively both with \( L = 4 \) and the periodic boundary condition. In both cases, the two spectra of the limit \( m_0 \to \mp 0 \) shown in the upper two panels are interpolated by varying the parameter \( \theta \) from 0 to \( \pi/2 \), as shown in the lower three panels. We found that in the course of the interpolation, the “half product” \( \prod_{i=1}^{n/4-4} \lambda_i \) remains real and positive-definite and it vanishes only at \( \theta = \pi/2 \) if \( E^a(x) = E^a_5(x) \). This result supports the picture that the pfaffian for \( 0 < m_0 < 2 \) can be zero, but is positive semi-definite, while the pfaffian for \( m_0 < 0 \) is positive definite all the way down to the limit \( m_0 \to -\infty \), where it is unity independently of the spin field \( E^a(x) \). And the positive semi-definite pfaffian for \( 0 < m_0 < 2 \) can be smoothly connected to the positive-definite one for \( m_0 < 0 \) at \( m_0 = \pm 0 \) without any singularity. And it implies that the pfaffian integrals are both positive definite for \( 0 \leq m_0 < 2 \) and \( m_0 \leq 0 \), and there is no massless singularity at the limit \( m_0 = \pm 0 \).

4.5 Disorder nature of the auxiliary spin-field path integrations

The path-integrations over the SO(10)-vector real spin fields with unit length, \( E^a(x) \) and \( \bar{E}^a(x) \), in eq. (3.34) define two kind of the four-dimensional spin models with the partition functions,

\[
\langle 1 \rangle_E = \int \mathcal{D}[E] \text{pf}(u^T i \gamma_5 C_D T^a E^a u), \tag{4.28}
\]

\[
\langle 1 \rangle_{\bar{E}} = \int \mathcal{D}[\bar{E}] \text{pf}(\bar{u} i \gamma_5 C_D T^a \bar{E}^a \bar{u}^T). \tag{4.29}
\]

It is important and useful to understand the dynamical nature of the path-integrations in these spin models.

The second model for \( \bar{E}^a(x) \) is trivial. This is because the pfaffian weight is unity, \( \text{pf}(\bar{u} i \gamma_5 C_D T^a \bar{E}^a \bar{u}^T) = 1 \). Then the two-point correlation function is given by

\[
\langle \bar{E}^a(x) \bar{E}^b(y) \rangle_{\bar{E}} = \frac{1}{10} \delta_{xy} \delta^{ab} \langle 1 \rangle_{\bar{E}} \tag{4.30}
\]

and the spin field is completely disordered.

The first model for \( E^a(x) \) is quite non-trivial. The pfaffian weight is the rather complicated (non-local) function of the spin field variables, which can be chosen to be real and positive semi-definite for the background link fields in the SO(9) subgroup, as we have argued, but is complex in general. The mass parameter \( m_0 \) is the only parameter to control the strength of the coupling of the spin field.\(^{13}\) One may regard the number of the spin models.

\(^{13}\)The kinetic term for the spin field \( E^a(x) \) such as \(-K \sum_{\tau, \mu} E^a(x) E^a(x + \hat{\mu}) \) can be added for the analysis. We omit this term for simplicity.
Figure 6. The eigenvalue spectra of $\langle u^\dagger \Gamma^{10} \Gamma^a E^a u \rangle$ in the limit $m_0 \to \mp 0$ with a randomly generated spin configuration for the trivial link field. The interpolation parameter $\theta_\alpha$ is chosen as $\theta_\alpha = 0, 3\pi/12, 4\pi/12, 5\pi/12, \pi/2$ for the top-left, bottom-left, bottom-middle, bottom-right, top-right figures, respectively. The lattice size is $L = 4$ and the boundary condition for the fermion field is periodic.

Spin components $N (= 10)$ as another parameter and consider the large $N$ method. In order to get insights into the dynamical nature of this spin model, one needs to apply the methods such as Monte Carlo simulations and the saddle point analysis in the large $N$ expansion.[247, 248]

In the following, we apply the saddle point analysis in the spirit of the large $N$ expansion to the model with the trivial link field background (in the weak gauge-coupling limit). For this purpose, we introduce the unconstraint (linearized) field $X^a(x)$ and the Lagrange-multiplier field $\lambda(x)$ to impose the constraint $X^a(x)X^a(x) = 1$, and rewrite the original path integration eq. (4.28) as follows,

$$\langle 1 \rangle_E = \int \mathcal{D}[X] \mathcal{D}[\lambda] \text{pf} \left( u^T i\gamma_5 C_D T^a X^a u \right) e^{\sum_x \lambda(x)(X^a(x)X^a(x)-1)}.$$  

(4.31)

Then the field variables are decomposed into the modes with zero-momentum and other modes of fluctuation as

$$X^a(x) = X^a_0 + \tilde{X}^a(x), \quad \sum_x \tilde{X}^a(x) = 0,$$

(4.32)

$$\lambda(x) = \lambda_0 + \tilde{\lambda}(x), \quad \sum_x \tilde{\lambda}(x) = 0,$$

(4.33)
Figure 7. The eigenvalue spectra of \((u^\dagger \Gamma_{10} \Gamma_a E^a u)\) in the limit \(m_0 \to \mp 0\) with a spin field configuration in the class \(E_a(x)\) for the trivial link field. The interpolation parameter \(\theta_\alpha\) is chosen as \(\theta_\alpha = 0, 3\pi/12, 4\pi/12, 5\pi/12, \pi/2\) for the top-left, bottom-left, bottom-middle, bottom-right, top-right figures, respectively. The lattice size is \(L = 4\) and the boundary condition for the fermion field is periodic.

and the action of \(X^a(x)\) and \(\lambda(x)\) is expanded in terms of \(\tilde{X}^a(x)\) and \(\tilde{\lambda}(x)\) up to the second order. The result reads

\[
S[X^a, \lambda] = -\frac{1}{2} \ln \det \left( u^\dagger i\gamma_5 C_D X^a u \right) - i \sum_x \lambda(x)(X^a(x)X^a(x) - 1) \quad (4.34)
\]

\[
= \left\{ -16 \ln(X_0^a X_0^a)^{1/2} - i\lambda_0(X_0^a X_0^a - 1) \right\} V - i \sum_x 2\tilde{\lambda}(x)X_0^a \tilde{X}^a(x) + \sum_{x,y} \tilde{X}^a(x) \left\{ 4D(x - y)(2X_0^a X_0^b - X_0^c X_0^c \delta^{ab})/(X_0^d X_0^d)^2 - i\lambda_0 \delta_{xy} \delta^{ab} \right\} \tilde{X}^b(x) + \cdots ,
\]

(4.35)

where \(D(x - y)\) is the kinetic operator defined through the chiral projector \(\hat{P}_{0+}(x, y)\) of the single free overlap Dirac fermion as

\[
D(x - y) = \text{tr}[\hat{P}_{0+}(x - y)\hat{P}_{0+}(y - x)] \quad (4.36)
\]

\[
= \frac{1}{V} \sum_k q^{i,k,x} \frac{1}{V} \sum_q \left\{ 1 + \frac{\sin(q + k)\omega(q) + b(q + k)b(q)}{\omega(q + k)\omega(q)} \right\}, \quad (4.37)
\]

\[
\hat{P}_{0+}(x - y) = \frac{1}{V} \sum_q q^{i,q,x} \left\{ \frac{1}{2} - \frac{i\gamma_\mu \sin q_\mu + b(q)}{\omega(q)} \right\} . \quad (4.38)
\]
The path-integration of $\tilde{\lambda}(x)$ and $\tilde{X}^a(x)$ (in this order) gives the effective action of $X^a_0$ and $\tilde{\lambda}_0$ as follows,

$$S_{\text{eff}}[X^a_0, \lambda_0] = \left\{ -16 \ln(X^a_0 \tilde{\lambda}_0)^{1/2} - i\lambda_0(X^a_0 \tilde{\lambda}_0 - 1) \right\} V + \frac{(10 - 1)}{2} \sum_{k \neq 0} \ln \left\{ -4\tilde{D}(k)/X^a_d X^e_{0} - i\lambda_0 \right\},$$  \hspace{1cm} (4.39)

where $\tilde{D}(k)$ is the fourier transform of the kinetic operator $D(x - y)$.

In eq. (6.52), we observe that the first term scales as $2^{N}$ ($= 32$) in terms of $N (= 10)$, while the second term scales as $N$. Then, we can assume that $\lambda_0$ scales as $2^{N}/N$ so that the first two terms of the classical contribution both scale as $2^{N}$. The third term of the one-loop correction scales as $N$ and is suppressed by the factor $N/2^{N} (\approx 9/32)$.

The stationary conditions for $X^a_0$ and $\lambda_0$ are given by

$$0 = 2X^a_0 \left\{ -\frac{8}{X^a_0 \tilde{\lambda}_0} - i\lambda_0 + \frac{9}{2V} \sum_{k \neq 0} \left( -\frac{1}{X^a_0 \tilde{\lambda}_0} - i\lambda_0 \frac{1}{-4\tilde{D}(k) - i\lambda_0 X^d_e X^d_{0}} \right) \right\}, \hspace{1cm} (4.40)$$

$$0 = (X^a_0 \tilde{\lambda}_0 - 1) + \frac{9}{2V} \sum_{k \neq 0} \frac{1}{-4\tilde{D}(k)/X^d_e X^d_{0} - i\lambda_0}. \hspace{1cm} (4.41)$$

Assuming $X^a_0 \neq 0$, the above conditions imply that

$$-i\lambda_0 = 8/X^a_0 \tilde{\lambda}_0 - \frac{9}{32V} \sum_{k \neq 0} \left( -16 + \frac{32}{-\tilde{D}(k) + 2} \right), \hspace{1cm} (4.42)$$

$$X^a_0 \tilde{\lambda}_0 = 1 - \frac{9}{32V} \sum_{k \neq 0} \frac{4}{-\tilde{D}(k) + 2}, \hspace{1cm} (4.43)$$

where the leading results, $-i\lambda_0 = 8/X^a_0 \tilde{\lambda}_0$ and $X^a_0 \tilde{\lambda}_0 = 1$, are substituted in the terms suppressed by the factor $N/2^{N} (\approx 9/32)$. The r.h.s. of the condition eq. (4.43) is required to be positive for $X^a_0 \neq 0$. It is plotted in fig. 8 as the function of $m_0$,

$$f(m_0) \equiv 1 - \frac{9}{32V} \sum_{k \neq 0} \frac{4}{-\tilde{D}(k) + 2}, \hspace{1cm} (4.44)$$

One can see that $f(m_0) \leq 0$ for $m_0 < 2$ and it is in contradiction with the assumption $X^a_0 \neq 0$. In this region of the mass parameter $m_0$, the fluctuation of the spin field $E^a(x)$ is too large to maintain the non-zero expectation value of the spin field $\langle E^a(x) \rangle$. The region includes the positive region $0 \leq m_0 < 2$ and it also extends to the negative region $m_0 \leq 0$ all the way down to $m_0 \rightarrow -\infty$.

The above result supports the following picture on the dynamical nature of the spin model. The spin model for $E^a(x)$ is well-defined for all values of $m_0$ in the region $[-\infty, 2)$. For $m_0 \in [-\infty, 2)$, the spin model is in the single disordered phase. In the limit $m_0 \rightarrow -\infty$, in particular, the pfaffian is unity and the spin field is completely disordered, having the vanishing correlation length, $\xi_E = 0$. The correlation length $\xi_E$ is a monotonically increasing function of $m_0$. And SO(10) global symmetry in the weak gauge-coupling limit as well as
$Z_4 \times Z_4$ discrete symmetries are not broken spontaneously in the thermodynamic limit $L \to \infty$.

For our purpose to formulate the Weyl field measure, the spin model for $E^a(x)$ should be in the positive disordered region $m_0 \in (0, 2)$, while the spin model for $\tilde{E}^a(x)$ is equivalent to the model in the limit $m_0 \to -\infty$. Thus the both spin models have the disorder nature, which are actually in the same disordered phase.

4.6 A summary

Based on the above analytical and numerical results, we argue that in these two cases of the trivial link field and of the SU(2) link fields with $Q(\neq 0)$, the path-integration of the pfaffian $\text{pf}(u^T i\gamma_5 C_D T^a E^a u)$ over the spin field $E^a(x)$ gives a non-zero result,

$$\int D[E] \text{pf}(u^T i\gamma_5 C_D T^a E^a u) = c [U(x, \mu)] \neq 0 \quad (4.45)$$

and that the measure of the right-handed field, $D_+[\psi_+]$, is indeed saturated completely by inserting the product of the 't Hooft vertex $T_+(x)[\psi_+]$, while the SO(10) symmetry does not break spontaneously in the thermodynamic limit. Furthermore, we also argue that for the case of the trivial link field, the path-integral of the pfaffian does not vanish and remains positive-definite in the course of the interpolation of the mass parameter $m_0$ from the negative region $m_0 < 0$ to the positive region $0 < m_0 < 2$.

5 Other anomalous/anomaly-free chiral gauge theories

5.1 Fate of the anomalous SU(2) chiral gauge theory

Although it is known to be inconsistent due to the global gauge anomaly[249–251], it is instructive to try to formulate the SU(2) chiral gauge theory with the Weyl field in the doublet 2 in the similar manner as the SO(10) model. This is because the doublet 2 of SU(2) is the irreducible spinor representation of SO(3) and the (pseudo) scalar bilinear of
the Weyl field transforms as the triplet of SO(3), while the coefficient of the perturbative gauge anomaly vanishes identically as \( \text{Tr}\{\tau^a (\tau^b \tau^d + \tau^e \tau^b)\} = 0 \). Then one may try to saturate the path-integral measure of the right-handed Weyl fields by the product of the following gauge-invariant quartic operators,

\[
\frac{1}{24} \left[ \psi^T(x) i \gamma_5 C_D (i \tau_2 \tau^d) \psi(x) \right]^2, \quad \frac{1}{24} \left[ \bar{\psi}_+(x) i \gamma_5 C_D (i \tau_2 \tau^d) \right]^T \bar{\psi}_+(x) T^2. \tag{5.1}
\]

However, this does not work in the topologically nontrivial sectors \( \mathbb{U}[Q] \), because the index theorem is given by \( n_+ - n_- = -Q \) in the SU(2) theory and the number of the zero modes is not necessarily a multiple of four. In particular, when the topological charge \( Q \) is an odd integer, the dimension of the anti-symmetric matrix \( (u^T_i i \gamma_5 C_D (i \tau_2 \tau^d) E^a u_k) \) is odd and its pfaffian vanishes identically. Therefore, the above operators can not always saturate the right-handed measure. Thus the SU(2) chiral gauge theory with the single Weyl field in the doublet \( 2 \) is ill-defined in our formulation, as it should be.

### 5.2 Anomaly-free chiral gauge theories descent from SO(10)

Once the lattice model for the SO(10) chiral gauge theory with the Weyl field in \( 16 \) is formulated, it is straightforward to obtain the lattice models for the SU(5) chiral gauge theory with \( \{10, 5^*\} \) (the Georgi-Glashow model), the SU(4)×SU(2)×SU(2) chiral gauge theory with \( \{(4, 2, 1), (4^*, 1, 2)\} \) (the Pati-Salam model), the SU(3)×SU(2)×U(1) chiral gauge theory with \( \{(2, 2)_1/6, (3^*, 1)_{-2/3}, (3^*, 1)_{1/3}, (1, 2)_{-1/2}, (1, 1)_1, (1, 1)_0\} \) (the standard model) by reducing the link field in SO(10) to the subgroups, SU(5), SU(4)×SU(2)×SU(2), SU(3)×SU(2)×U(1), respectively. The spin fields may be kept the SO(10) real vectors so that the SO(10) global symmetry is maintained in the weak gauge-coupling limit. The continuous global symmetries remained in these models are all “ready to be gauged” without encountering gauge anomalies, thus anomaly-matched, and therefore free from the infrared singularities in the correlation functions of the associated conserved currents due to chiral(gauge) anomalies.

### 5.3 The standard model plus the right-handed neutrinos

As to the case of the SU(3)×SU(2)×U(1) chiral gauge theory, in particular, we can incorporate the three generations of quarks and leptons plus right-handed neutrinos by simply preparing the three copies of the Weyl fields in \( 16 \) in our formulation. Then the left-handed part of the fermion measure respects the global flavor SU(3) symmetry, while the right-handed part respects only the global permutation symmetry, \( S_3 \), because of the products of the ’t Hooft vertices of each right-handed fields. It is straightforward to incorporate the Higgs boson field and its Yukawa couplings to the left-handed fields consisting the three generations of quark and leptons plus right-handed neutrinos. In doing this, one can include the chiral projector \( P_\pm \) suitably so that the Yukawa coupling terms themselves respect the CP symmetry\([241–243]\) and that the sources of the CP violation are given by the complex phases in the flavor mixing matrices of quarks (Cabibbo-Kobayashi-Maskawa matrix\([252, 253]\)), charged leptons and neutrinos (Pontecorvo-Maki-Nakagawa-Sakata matrix\([254, 255]\)) beside the theta terms. Thus we can formulate the standard model plus the right-handed
neutrinos on the lattice with the manifest exact gauge invariance and the required symmetry properties. The detail of the formulation will be discussed elsewhere.

6 Relations with other approaches/proposals

The characteristic features of our lattice model for the SO(10) chiral gauge theory with the Weyl fermions in $\mathbf{16}$ can be summarised in the formula of the fermion partition function eq. (3.10):

$$\langle 1 \rangle_F \equiv \int \mathcal{D}[\psi_-] \mathcal{D}[\bar{\psi}_-] e^{-S_W[\psi_-, \bar{\psi}_-]} = \int \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] \prod_{x \in \Lambda} F(T_+(x)) \prod_{x \in \Lambda} F(\bar{T}_+(x)) e^{-S_W[\psi_-, \bar{\psi}_-]} = \int \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] \mathcal{D}[E] \mathcal{D}[\bar{E}] e^{-S_W[\psi_-, \bar{\psi}_-]} + \sum_{x \in \Lambda} \{ E^a(x) V^a_+(x) + \bar{E}^a(x) \bar{V}^a_+(x) \} [\psi_+, \bar{\psi}_+] .$$

(6.1)

In this formula eq. (6.1), the total action of the model, including the 't Hooft vertex terms, can be defined as

$$S_{Ov}[\psi, \bar{\psi}, E^a, \bar{E}^a] = \sum_{x \in \Lambda} \bar{\psi}_-(x) D\psi_-(x) - \sum_{x \in \Lambda} \frac{1}{2} \{ E^a(x) \gamma^5 C D T^a \psi_+(x) + \bar{E}^a(x) \bar{\psi}_+(x) \gamma^5 C D T^{a\dagger} \bar{\psi}_+(x) \} .$$

(6.2)

Here the right-handed Weyl fields are introduced explicitly, trying to make the path-integral measure of the left-handed Weyl fields in $\mathbf{16}$ simplified and manifestly gauge-invariant. The SO(10) invariant 't Hooft vertex operators of the right-handed fields are used to saturate completely the right-handed part of the fermion measure. The short range correlations of order the lattice spacing are required for the the right-handed Weyl fields and the auxiliary spin fields so that they are decoupled from physical degrees of freedom, preserving the symmetries and leaving only the smooth and local terms of the link fields. These features/requirements are actually shared with other various approaches and proposals to decouple the species doubling or mirror modes of models.

An important technical difference lies on the fact that the path-integral measure of the right-handed Weyl fields, i.e. the right-handed part of the chiral decomposition of Dirac field measure, are formulated with the non-trivial chiral basis \{ $u_i(x)|P_+ \otimes \bar{P}_+ u_i = u_i, i = 1, \cdots, n/2 - 8Q$ \} \{ $\bar{u}_k(x)|\bar{u}_k P_- \otimes P_+ = u_k, k = 1, \cdots, n/2$ \}, which depends on the gauge field, as given by eq. (3.26),

$$\psi_+(x) = \sum_i u_i(x) b_i, \quad \bar{\psi}_+(x) = \sum_k \bar{b}_k \bar{u}_k(x),$$

$$\mathcal{D}_+[\psi_+] \mathcal{D}_+[\bar{\psi}_+] = \prod_{j=1}^{n/2-8Q} db_j \prod_{k=1}^{n/2} d\bar{b}_k .$$

(6.3)

(6.4)
We need to make sure the locality of this right-handed-measure contribution to the induced effective action.

Another important technical difference is that we choose the product function for the ’t Hooft vertices $F(\omega)$ as given by eq. (3.6) and therefore use the unit $SO(10)$-vector spin fields, $E^a(x)$ and $\bar{E}^a(x)$ with the constraints $E^a(x)E^a(x) = 1$ and $\bar{E}^a(x)\bar{E}^a(x) = 1$, omitting their kinetic(hopping) terms. This choice allows us to prove the CP symmetry. It is also relevant for preserving the (global) $SO(10)$ symmetry in the thermodynamic limit.

The above action $\mathcal{S}_{\text{OV}}$ can be regarded as a certain limit of the following action of the $SO(10)$-invariant chiral Yukawa model in the framework of the Ginsparg-Wilson fermion,

\[
\mathcal{S}_{\text{OV/Mi}}[\psi, \bar{\psi}, X^a, \bar{X}^a] = \sum_{x \in \Lambda} \left\{ \tilde{\psi}_-(x)D\psi_-(x) + z_+ \bar{\psi}_+(x)D\psi_+(x) \right\} \\
- \sum_{x \in \Lambda} \left\{ y X^a(x)\psi_+^T(x)i\gamma_5 C_D T^a \psi_+(x) + \bar{y} \bar{X}^a(x)\bar{\psi}_+^T(x)i\gamma_5 C_D T^{a\dagger} \bar{\psi}_+(x)^T \right\} \\
+ S_X[X^a],
\]

where

\[
S_X[X^a] = \sum_{x \in \Lambda} \left\{ -\kappa \sum_{\mu} X^a(x)X^a(x + \mu) + \frac{1}{2}X^a(x)X^a(x) + \frac{\lambda'}{2}(X^a(x)X^a(x) - \nu^2)^2 \\
-\bar{\kappa} \sum_{\mu} \bar{X}^a(x)\bar{X}^a(x + \mu) + \frac{1}{2}\bar{X}^a(x)\bar{X}^a(x) + \frac{\bar{\lambda}'}{2}(\bar{X}^a(x)\bar{X}^a(x) - \bar{\nu}^2)^2 \right\}.
\]

The limit to the original action $\mathcal{S}_{\text{OV}}$ is achieved by

\[
y = \bar{y}, \quad \frac{z_+}{\sqrt{yy}} \rightarrow 0, \quad (6.7)
\]

\[
\nu = \bar{\nu} = 1, \quad \lambda' = \bar{\lambda}' \rightarrow \infty, \quad (6.8)
\]

\[
\kappa = \bar{\kappa} \rightarrow 0. \quad (6.9)
\]

In the lattice model defined with the action, $\mathcal{S}_{\text{OV/Mi}}$, the global U(1) symmetry of the right-handed fields is broken to $Z_4$ by the Yukawa couplings $y$ and $\bar{y}$. But the proof of the CP symmetry is not successful so far.\[14\]

\[14\] In the other limit as

\[
\lambda' = \bar{\lambda}' \rightarrow 0, \quad (6.10)
\]

\[
\kappa = \bar{\kappa} \rightarrow 0, \quad (6.11)
\]

it reduces to the model with quartic interaction of the ’t Hooft vertices,

\[
\mathcal{S}_{\text{OV/BP}}[\psi, \bar{\psi}] = \sum_{x \in \Lambda} \left\{ \tilde{\psi}_-(x)D\psi_-(x) + z_+ \bar{\psi}_+(x)D\psi_+(x) \right\} \\
- \sum_{x \in \Lambda} \left\{ y^2 \frac{1}{2} [\psi_+^T(x)i\gamma_5 C_D T^a \psi_+(x)]^2 + \bar{y}^2 \frac{1}{2} [\bar{\psi}_+^T(x)i\gamma_5 C_D T^{a\dagger} \bar{\psi}_+(x)^T]^2 \right\}. \quad (6.12)
\]

This action (in the limit $z_+ \rightarrow 0$) corresponds to the other choice of the product function $F(\omega)$ as $F(\omega) = e^\omega$. 

\[– 36 –\]
In the following, we discuss the relations to Eichten-Preskill model, Ginsparg-Wilson Mirror-fermion model, Domain wall fermion model with the boundary Eichten-Preskill term, 4D Topological Insulators/Superconductors with gapped boundary phases, and the recent studies on the Paramagnetic Strong-coupling (PMS) phase/Mass without symmetry breaking, trying to clarify the similarity and the difference in technical detail and to show that our proposal is a well-defined testing ground for that basic question.

6.1 cf. Eichten-Preskill model

The SO(10) invariant interaction terms of the ’t Hooft vertex were first used by Eichten and Preskill[98] to decouple the species doublers in their formulation of chiral lattice gauge theories based on the generalized Wilson term:

\[
S_{\text{EP}} = \sum_{x \in \Lambda} \left\{ \bar{\psi}(x) \gamma_{-} \left( \frac{1}{2} \nabla_{\mu} - \frac{1}{2} \nabla_{\mu}^{\dagger} \right) \psi(x) + z_{+} \bar{\psi}(x) \gamma_{+} \left( \frac{1}{2} \nabla_{\mu} - \frac{1}{2} \nabla_{\mu}^{\dagger} \right) \psi(x) \right\} \\
- \sum_{x \in \Lambda} \left\{ \frac{\lambda}{24} \left[ \psi^{T}(x) i \gamma_{5} C D T^{a} \psi(x) \right]^{2} + \frac{\lambda}{24} \left[ \bar{\psi}(x) i \gamma_{5} C D T^{a} \bar{\psi}(x) \right]^{2} \right\} \\
- \sum_{x \in \Lambda} \left\{ \frac{r}{48} \Delta \left[ \psi^{T}(x) i \gamma_{5} C D T^{a} \psi(x) \right]^{2} + \frac{r}{48} \Delta \left[ \bar{\psi}(x) i \gamma_{5} C D T^{a} \bar{\psi}(x) \right]^{2} \right\},
\]

(6.13)

where

\[
\Delta \{ A(x) B(x) C(x) D(x) \} \equiv \frac{1}{2} \sum_{\mu} \left\{ (\nabla_{\mu} \nabla_{\mu}^{\dagger} A(x)) B(x) C(x) D(x) + A(x) (\nabla_{\mu} \nabla_{\mu}^{\dagger} B(x)) C(x) D(x) \right\} \\
+ A(x) B(x) (\nabla_{\mu} \nabla_{\mu}^{\dagger} C(x)) D(x) + A(x) B(x) C(x) (\nabla_{\mu} \nabla_{\mu}^{\dagger} D(x)) \right\}. \quad (6.14)
\]

In this action, the right(left)-handed Weyl fields are formulated by the naive chiral projectors as \( P_{+} \psi(x) \), \( \bar{\psi}(x) P_{-} \) (\( P_{-} \psi(x) \), \( \bar{\psi}(x) P_{+} \)). The global \( U(1) \) symmetry of the right-handed fields is broken to \( Z_{4} \) by the quartic couplings \( \lambda \), \( \tilde{\lambda} \) and \( r \), \( \tilde{r} \), and the CP symmetry is manifest thanks to the naive definition of chirality. On the other hand, in their analysis of the Eichten-Preskill model, Golterman, Petcher and Rivas[99] have considered the same type of chiral Yukawa model as \( S_{\text{Ov/Mi}} \), but with the naive Dirac operator, the naive chiral projectors, and the additional Wilson-Yukawa coupling with the lattice laplacian included:

\[
S_{\text{EP/WY}} = \sum_{x \in \Lambda} \left\{ \bar{\psi}(x) \gamma_{-} \left( \frac{1}{2} \nabla_{\mu} - \frac{1}{2} \nabla_{\mu}^{\dagger} \right) \psi(x) + z_{+} \bar{\psi}(x) \gamma_{+} \left( \frac{1}{2} \nabla_{\mu} - \frac{1}{2} \nabla_{\mu}^{\dagger} \right) \psi(x) \right\} \\
- \sum_{x \in \Lambda} \left\{ y X^{a}(x) \psi^{T}(x) i \gamma_{5} C D T^{a} \psi(x) + y \bar{X}^{a}(x) \bar{\psi}(x) P_{-} i \gamma_{5} C D T^{a} \bar{\psi}(x) \right\} \\
- \sum_{x \in \Lambda} \left\{ w X^{a}(x) \psi^{T}(x) i \gamma_{5} C D T^{a} (\nabla_{\mu} \nabla_{\mu}^{\dagger} / 2) \psi(x) + w \bar{X}^{a}(x) \bar{\psi}(x) P_{-} i \gamma_{5} C D T^{a} (\nabla_{\mu} \nabla_{\mu}^{\dagger} / 2) \bar{\psi}(x) \right\} \\
+ S_{X}[X^{a}] \bigg|_{\lambda = \tilde{\lambda}},
\]

(6.15)
The authors’ intention here is to consider the right-handed sector of the above models and to leave only the physical mode (with $p_\mu \simeq 0$) of the right-handed Weyl fields $P_+ \psi(x)$, $\bar{\psi}(x)P_-$ and split/gap/decouple the species doubling modes (with $p_\mu \simeq \pi^{(A)}$, $A = 1, \cdots, 15$) of the fields by a suitable choice of the couplings, $\lambda$, $r$ or $y$, $w$ just like the Wilson term does for the naive Dirac field.

As Eichten and Preskill showed, in the strong quartic-coupling limit $z_+/\sqrt{\lambda} \to 0$ and $r/\lambda \to 0$, the path-integral measure of the right-handed Weyl fields,

$$D_0[\psi_+]D_0[\bar{\psi}_+] = \prod_{x \in \Lambda} \prod_{\alpha = 1}^2 \prod_{s = 1}^{16} d\psi_{\alpha s}(x) \prod_{x \in \Lambda} \prod_{\alpha = 3}^4 \prod_{s = 1}^{16} d\bar{\psi}_{\alpha s}(x),$$

are saturated completely by the ’t Hooft vertices in terms of the right-handed fields, as reproduced in eq.(3.38):

$$\int D_0[\psi_+] \prod_{x \in \Lambda} \frac{4!}{8!12!} \left\{ \frac{1}{23} \psi(x)^T P_+ i\gamma_5 C_D^a T^a \psi(x) \psi(x)^T P_+ i\gamma_5 C_D^a T^a \psi(x) \right\}^8 = 1,$$

$$\int D_0[\bar{\psi}_+] \prod_{x \in \Lambda} \frac{4!}{8!12!} \left\{ \frac{1}{23} \bar{\psi}(x) P_- i\gamma_5 C_D^a T^a \bar{\psi}(x) \psi(x)^T i\gamma_5 C_D^a T^a \bar{\psi}(x) \right\}^8 = 1.$$ \hspace{1cm} (6.16)

Moreover, in the hopping parameter expansion w.r.t. $z_+/\sqrt{\lambda}$, $r/\lambda$, all the modes of the right-handed Weyl fields $P_+ \psi(x)$, $\bar{\psi}(x)P_-$ acquire masses without breaking the SO(10) symmetry through the mixings with the modes of the composite fields

$$B_-(x) = P_- i\gamma_5 C_D^a T^a \bar{\psi}(x)^T \bar{\psi}(x) P_- i\gamma_5 C_D^a T^a \bar{\psi}(x)^T,$$

$$\bar{B}_-(x) = \psi^T(x) i\gamma_5 C_D^a P_+ T^a \psi(x) \psi^T(x) i\gamma_5 C_D^a P_+ T^a,$$ \hspace{1cm} (6.19)

as chiral partners.

Golterman, Petcher and Rivas applied a method of large $N_f$ expansion to the model $S_{\text{EP/WY}}$, where $N_f$ is the number of copies of the right(left)-handed Weyl fields in 16. In the region of the strong Yukawa/Wilson-Yukawa couplings, where the couplings are scaled as $y = \sqrt{N_f} \tilde{y}$, $w = \sqrt{N_f} \tilde{w}$ ($N = 0$, $z_+ = 1$), the effective hopping parameter for the spin fields $X^a(x)$ is given by

$$\kappa_{\text{eff}} = \kappa + \frac{1}{8} \int_{-\pi}^{\pi} \frac{d^4p}{(2\pi)^4} \sum_{\mu} \sin^2 p_\mu \frac{\sum_{\mu} \sin^2 p_\mu}{(\tilde{y} + \tilde{w} \sum_{\mu} (1 - \cos p_\mu))^2}. $$ \hspace{1cm} (6.21)

For larger $\tilde{y}$ and $\tilde{w}$, $\kappa_{\text{eff}}$ is less than the mean-field value of the critical hopping parameter $\kappa_c = 5/2$. Then the SO(10) symmetry is not broken spontaneously and the model is in the PMS phase. For smaller $\tilde{y}$ and $\tilde{w}$, $\kappa_{\text{eff}}$ exceeds the critical value. Then the SO(10) symmetry is broken spontaneously and the model is in the FM phase. All the modes of the right-handed Weyl fields $P_+ \psi(x)$, $\bar{\psi}(x)P_-$ form massive Wilson-Dirac fermions with the modes of the composite fields

$$B'_-(x) = P_- i\gamma_5 C_D^a \bar{\psi}(x)^T T^a X^a(x),$$

$$\bar{B}'_-(x) = \psi^T(x) i\gamma_5 C_D^a P_+ T^a X^a(x).$$ \hspace{1cm} (6.22)
as chiral partners. The inverse propagator is given by

\[
S(p)^{-1} = (P_+ z^2 + P_-)i\gamma_\mu \sin p_\mu + y + w \sum_\mu (1 - \cos p_\mu),
\]

(6.24)

where \( z^2 = \frac{1}{32}\langle X^a(x)X^a(x \pm \hat{\mu}) + \bar{X}^a(x)\bar{X}^a(x \pm \hat{\mu}) \rangle \) up to \( \mathcal{O}(1/N_f) \) corrections. Through the transition from the PMS phase to the FM phase, it remains that \( z^2 \neq 0 \) and the right-handed Weyl fields keep to form the Wilson-Dirac fermions with the composite-field chiral partners.

On the other hand, in the region of the weak Yukawa/Wilson-Yukawa couplings, where the couplings are scaled as \( y = \tilde{y}/\sqrt{N_f}, w = \tilde{w}/\sqrt{N_f}, \lambda' = \tilde{\lambda}/N_f (z_+ = 1) \), the \( SO(10) \) symmetry is broken spontaneously for larger \( \tilde{y} \) and \( \tilde{w} \) and the model is in the FM phase. The non-zero vacuum expectation value, \( \langle X^a(x) \rangle = \delta^a_{10} \tilde{v}/\sqrt{N_f} \), is determined through the gap equation of the fermion self-energy,

\[
\Sigma(p) = i\gamma_5 C_D P_+ \tilde{C} \tilde{v} \left( \tilde{y} + \tilde{w} \sum_\mu (1 - \cos p_\mu) \right),
\]

(6.25)

by

\[
1 + 2\tilde{\lambda} \tilde{v}^2 - 8\kappa = 32 \int_{-\pi}^{\pi} \frac{d^4 p}{(2\pi)^4} \frac{(\tilde{y} + \tilde{w} \sum_\mu (1 - \cos p_\mu))^2}{\sum_\mu \sin^2 p_\mu + \tilde{v}^2 (\tilde{y} + \tilde{w} \sum_\mu (1 - \cos p_\mu))^2}.
\]

(6.26)

The right-handed Weyl fields acquire the non-vanishing Majorana-Wilson masses. For smaller \( \tilde{y} \) and \( \tilde{w} \), \( \tilde{v} \) vanishes and the model is in the PMW phase. The Majorana-Wilson masses vanish identically and all the modes of the right-handed Weyl fields become massless.

Thus both in the PMS and PMW phases the massless fermion spectrum consist of Dirac fermions in \( 16 \). Then it was argued by the authors that the existence of the FM phase which separates the PMS and PMW phases is the crucial ingredient for the failure of the proposal.

For our purpose, the above results in the strong coupling limits are indeed the ideal situation. Our intention here is actually to show the same situation occurs for the Weyl fields \( \tilde{P}_+, \tilde{\psi}(x), \tilde{\bar{\psi}}(x)P_- \) in the framework of the overlap fermion/the Ginsparg-Wilson relation as defined in the model \( S_{OV/MI} \) and the model \( S_{OV} \). As we mentioned above, an important technical difference here lies on the fact that the path-integral measure of the right-handed Weyl fields, i.e. the right-handed part of the chiral decomposition of Dirac field measure, are simple and gauge-invariant for the model \( S_{WY/EP} \) as given by eq. (6.16), but are non-trivial and gauge-field dependent for the models \( S_{OV/MI} \) and \( S_{OV} \) as given by eq. (3.26). This is why we need to make sure the saturation of the right-handed-measure by the 't Hooft vertices and the locality of the right-handed-measure contribution to the induced effective action for our case.

Another important technical difference is that we choose the product function \( F(\omega) \) as given by eq. (3.6) and therefore use the unit \( SO(10) \)-vector spin fields, \( E^a(x) \) and \( \bar{E}^a(x) \) with the constraints \( E^a(x)E^a(x) = 1 \) and \( \bar{E}^a(x)\bar{E}^a(x) = 1 \), omitting their kinetic(hopping) terms. This choice allows us to prove the CP symmetry. But it is also relevant for preserving
the (global) SO(10) symmetry. This corresponds to taking the limits eqs. (6.8) and (6.9) in $S_{\text{EP/\text{WY}}}$ as

$$S_{\text{EP/WY}}|_{x=1, \lambda \to \infty} = \sum_{x \in \Lambda} \left\{ \bar{\psi}(x)\gamma_{\mu}P_-(\sqrt{\nabla_{\mu} - \nabla_{\mu}^\dagger/2})\psi(x) + z_+\bar{\psi}(x)\gamma_{\mu}P_+(\sqrt{\nabla_{\mu} - \nabla_{\mu}^\dagger/2})\psi(x) \right\}$$

$$- \sum_{x \in \Lambda} \left\{ y E^a(x)\psi(x)i\gamma_5 C_D T^a P_+\psi(x) + y \bar{E}^a(x)\bar{\psi}(x)P_-i\gamma_5 C_D T^a\bar{\psi}(x)^T \right\}$$

$$- \sum_{x \in \Lambda} \left\{ w E^a(x)\psi(x)i\gamma_5 C_D T^a(\nabla_{\mu}\nabla_{\mu}^\dagger/2)P_+\psi(x) \right. + w \bar{E}^a(x)\bar{\psi}(x)P_-i\gamma_5 C_D T^a(\nabla_{\mu}\nabla_{\mu}^\dagger/2)\bar{\psi}(x)^T \right\}. \quad (6.27)$$

This region in the coupling-constant space of $S_{\text{EP/WY}}$ has not been explored by Golterman, Petcher and Rivas\[99\]. And, in fact, we find no phase transition from the PMS phase to the FM phase towards the weak-coupling limit $y/z_+, w/z_+ \to 0$ within the saddle point analysis in the spirit of the large N expansion. We will come back to this point later in relation to the discussion about the recent studies on the PMS phase/“Mass without symmetry breaking”.

6.2 cf. Ginsparg-Wilson Mirror-fermion model

The SO(10) invariant action $S_{\text{OV/MI}}$, eq. (6.5), defines a Mirror-fermion model for the SO(10) chiral gauge theory in the framework of the Ginsparg-Wilson fermion. It is formulated in the spirit of the series of works by Bhattacharya, Chen, Giedt, Poppitz and Shang\[100–107\], although any SO(10) model has not been discussed in the literature. In the action $S_{\text{OV/MI}}$, the global U(1) symmetry of the right-handed fields is broken to $Z_4$ by the Yukawa couplings $y$ and $\bar{y}$. As mentioned above, however, the proof of the CP symmetry is not successful so far.

In this respect, we note that one can prove the CP invariance of the effective action if one modifies the Yukawa couplings by the insertion of the chiral projectors $P_{\pm}$ \[241–243\] as

$$- \sum_{x \in \Lambda} \left\{ y X^a(x)\psi_+^T(x)i\gamma_5 C_D T^a P_+\psi_+(x) + \bar{y} \bar{X}^a(x)\bar{\psi}_+(x)i\gamma_5 C_D T^a\bar{\psi}_+(x)^T \right\}$$

$$= - \sum_{x \in \Lambda} \left\{ y \psi^T(1 - D)^T i\gamma_5 C_D T^a X^a P_+(1 - D)\psi(x) + \bar{y} \bar{\psi}i\gamma_5 C_D T^a\bar{X}^a P_-\bar{\psi}(x)^T \right\}, \quad (6.28)$$

and to take the following action,

$$S'_{\text{OV/MI}}[\psi, \bar{\psi}, X^a, \bar{X}^a] = \sum_{x \in \Lambda} \left\{ \bar{\psi}_-(x) D\psi_-(x) + z_+\bar{\psi}_+(x) D\psi_+(x) \right\}$$

$$- \sum_{x \in \Lambda} \left\{ y X^a(x)\psi_+^T(x)i\gamma_5 C_D T^a P_+\psi_+(x) \right. + \bar{y} \bar{X}^a(x)\bar{\psi}_+(x)P_-i\gamma_5 C_D T^a\bar{\psi}_+(x)^T \right\}$$

$$+ S_X[X^a]. \quad (6.29)$$

But this type of Yukawa coupling is singular in the large limit $z_+ \sqrt{y\bar{y}} \to 0$: the saturation of the right-handed part of the measure is incomplete, because

$$P_+\psi_+(x) = P_+ P_+\psi(x) = P_+(1 - D)\psi(x) \quad (6.30)$$
and the factor $(1 - D)$ projects out the modes with the momenta $\pi^{(A)}_\mu$ ($A = 1, \cdots, 15$).

We note that this is the common property of the mass-like terms of the Ginsparg-Wilson fermion. For the Dirac mass term, it is usually formulated as

$$S_D = \sum_{x \in \Lambda} \{ \bar{\psi}(x) D \psi(x) + m_D \bar{\psi}(1 - D) \psi(x) \},$$

(6.31)

because the scalar and pseudo scalar operators, $\bar{\psi}(1 - D) \psi(x)$ and $\bar{\psi} i \gamma_5 (1 - D) \psi(x)$, have the good transformation properties under the chiral transformation, $\delta \psi(x) = \gamma_5 (1 - 2D) \psi(x)$, $\delta \bar{\psi}(x) = \bar{\psi}(x) \gamma_5$. However, this choice makes the limit of the large mass parameter $m_D$ singular by the same reason as above. The maximal value of the mass is given at $m_D = 1$, where $D$ cancels out in the action and the simple bilinear operator $\bar{\psi}(x) \psi(x)$ saturates the path-integral measure of the Dirac fields completely. To make the limit of the large mass parameter well-defined, we should write the action as

$$S_D = \sum_{x \in \Lambda} \{ z \bar{\psi}(x) D \psi(x) + m \bar{\psi}(x) \psi(x) \},$$

(6.32)

where $z = 1 - m_D$ and $m = m_D$ and should take the limit $z/m = (1 - m_D)/m_D \to 0$.

As for the Majorana mass term, one often formulates the action as

$$S_M = \sum_{x \in \Lambda} \{ \bar{\psi}_+(x) D \psi_+(x)$$

$$+ m_M (\psi_+(x)^T C_D \psi_+(x) + \bar{\psi}_+(x) C_D \bar{\psi}_+(x)^T) \}$$

(6.33)

$$= \sum_{x \in \Lambda} \{ \bar{\psi}(x) P_- D \psi(x)$$

$$+ m_M (\psi^T \bar{P}_+^T C_D \psi(x) + \bar{\psi} P_- C_D \bar{P}_+ \psi^T(x)) \}.$$

(6.34)

But this Majorana mass term has the matrix elements as

$$u_{j}^T C_D u_k = -\delta_{p+p',0} \frac{b(p')}{\omega(p')} \epsilon_{\sigma,\sigma'} \quad (j = \{ p, \sigma \}, k = \{ p', \sigma' \}),$$

(6.35)

$$\bar{u}_j C_D \bar{u}_k^T = -\delta_{x,x'} \epsilon_{\sigma,\sigma'} \quad (j = \{ x, \sigma \}, k = \{ x', \sigma' \}),$$

(6.36)

and the pfaffian of the first matrix has the factor $\prod_{p'} \{| b(p')/\omega(p') \}$, while the second one is unity. Since $b(p')$ can vanish for $0 < m_0 < 2$, it is singular in the limit of the large mass parameter $m_M$. This type of the Majorana-Yukawa couplings are used in the formulation of the 2D “10”, “3450” models by Chen, Giedt, Poppitz and Shang. Instead, one can formulate the action as

$$S_M = \sum_{x \in \Lambda} \{ z \bar{\psi}_+(x) D \psi_+(x)$$

$$+ M (\psi_+(x)^T i \gamma_5 C_D \psi_+(x) + \bar{\psi}_+(x) i \gamma_5 C_D \bar{\psi}_+(x)^T) \}$$

(6.37)

$$= \sum_{x \in \Lambda} \{ z \bar{\psi}(x) P_- D \psi(x)$$

$$+ M (\psi^T \bar{P}_+^T i \gamma_5 C_D \psi(x) + \bar{\psi} P_- i \gamma_5 C_D \bar{P}_+ \psi^T(x)) \}.$$

(6.38)
In the chiral basis, this Majorana mass term has the matrix elements as

\[
\begin{align*}
    u_j^T i \gamma_5 C D u_k &= i \delta_{p+p',0} \epsilon_{\sigma,\sigma'} \quad (j = \{ p, \sigma \}, k = \{ p', \sigma' \}), \\
    \bar{u}_j i \gamma_5 C D \bar{u}_k^T &= i \delta_{x,x'} \epsilon_{\sigma,\sigma'} \quad (j = \{ x, \sigma \}, k = \{ x', \sigma' \}),
\end{align*}
\]

and the pfaffians of these matrices are both unity. Then the limit \( z/M \to 0 \) is well-defined and the right-handed measure is indeed saturated completely. The Majorana-Yukawa couplings in \( SOv/Mi \) and \( SOv \) have precisely the latter structure.

As argued in sections 3 and 4, in the model \( SOv \) the functional pfaffian is real positive semi-definite in the weak gauge-coupling limit, where the link variables are set to unity, \( U(x,\mu) = 1 \), and the pfaffian path-integration over the spin fields is non-vanishing. Moreover, the correlation functions of the right-handed fields \( \psi_+(x), \bar{\psi}_+(x) \) are short-ranged, and the spin fields \( E^a(x), \bar{E}^a(x) \) are in the disordered phase. Therefore, the limit of large Majorana-Yukawa couplings, eqs. (6.7), (6.8) and (6.9), is indeed well-defined and there exists the PMS phase in that region of the coupling-constant space of \( SOv/Mi \).

### 6.3 cf. Recent studies on the PMS phase/Mass without Symmetry Breaking

As to the possible phase transitions from the PMS phase to the FM and PMW phases, the recent lattice studies on the PMS phase/“Mass without Symmetry Breaking” by Ayyar and Chandrasekharan and by Catterall, Schaich and Butt are interesting and suggestive[110, 113–118]. As to the four-dimensional case, in particular, the authors consider the reduced staggered fermion model with a certain quartic interaction term, where there exist \( SU(4)/SO(4) \) and \( Z_4 \) symmetries and any quadratic mass terms are forbidden due to the symmetries. In the classical continuum limit within the weak coupling phase, the reduced staggered fermion model describes sixteen Majorana fermions (= sixteen Weyl fermions) interacting through \( SU(4) \times Z_4 \) symmetric quartic (quadratic Yukawa) interaction. In this model, the strong-coupling limit is well-defined because the fermion measure is indeed saturated by the quartic interaction term completely, and the strong coupling expansion can be formulated. One can show in the strong coupling regime that the fermion field and the auxiliary boson field are both massive without any symmetry breaking and the model is indeed in the PMS phase. On the other hand, the path-integral weight (fermion pfaffian) can be managed to be real positive and Monte Carlo methods are applicable. Their numerical simulations have confirmed the PMS phase. Moreover, the authors have found the numerical evidences for the very narrow FM intermediate phase between the PMS and PMW phases in the four-dimensional model.

For the purpose to decouple the mirror (Overlap/Ginsparg-Wilson) fermions, the interest lies in the behavior of the model deep inside the PMS phase off the phase transition. It is still important and useful to study the nature of the possible phase transitions from the PMS phase to the FM, PMW phases in our case of the \( SO(10) \) theories, because it gives us the understanding of the relation between the phase of the massless left-handed (target) Weyl fields and the phase of the massive/gapped right-handed mirror fields.

For this purpose, let us consider the following model, as the simplest possible extension of \( SOv \), which is obtained from \( SOv/Mi \) by taking the limit (6.8) and (6.9) and by further
reducing the degrees of freedom of the spin fields through the identification \( E^a(x) = \tilde{E}^a \), \( y = \bar{y} \):

\[
\hat{S}_{\text{OV}}[\psi, \bar{\psi}, E^a] = \sum_{x \in \Lambda} \bar{\psi}_-(x) D\psi_-(x) + \sum_{x \in \Lambda} z_+ \bar{\psi}_+(x) D\psi_+(x)
\]

\[
-\sum_{x \in \Lambda} y E^a(x) \{ \bar{\psi}_+^T(x) i\gamma_5 C_D T^a \psi_+(x) + \bar{\psi}_+(x) i\gamma_5 C_D T^a \psi_+(x)^T \}. \tag{6.41}
\]

We note that although the degrees of freedom of the spin fields are halved, this model still exactly reduces to \( S_{\text{OV}} \) in the limit \( z_+/y \to 0 \) because the functional pfaffian factorizes into those of the field \( \psi_+(x) \) and the anti-fields \( \bar{\psi}_+(x) \) in this limit and the latter is unity, \( \text{pf}(\bar{u} i\gamma_5 C_D T^a E^a \bar{u}^T) = 1 \), independently of the spin field \( E^a(x) \).

The partition function of the model \( \hat{S}_{\text{OV}} \) is obtained as follows.

\[
\tilde{Z}_{\text{OV}} = \det(\bar{u} D u) \langle \text{pf}(u^T i\gamma_5 C_D T^a E^a [1 + (z_+/2y)^2 P_-] u) \rangle'_E. \tag{6.42}
\]

This is because the path-integration of the right-handed anti-field \( \bar{\psi}_+(x) \) can be performed explicitly and the effective action reads

\[
\hat{S}_{\text{OV}} = \sum_{x \in \Lambda} \bar{\psi}_-(x) D\psi_-(x)
\]

\[
-\sum_{x \in \Lambda} y E^a(x) \{ \psi_+^T(x) i\gamma_5 C_D T^a [1 + (z_+/2y)^2 P_-] \psi_+(x) \}, \tag{6.43}
\]

which has the same form as \( S_{\text{OV}} \) except for the factor \( [1 + (z_+/2y)^2 P_-] \).

Then, by the similar reasoning given in section 4, we can argue that the pfaffian \( \text{pf}(u^T i\gamma_5 C_D T^a E^a [1 + (z_+/2y)^2 P_-] u) \) is real and positive semi-definite. Therefore, the path-integration of the pfaffian over the spin field is positive definite. This holds true as long as \( 0 \leq (z_+/2y) < \infty \):

\[
\langle \text{pf}(u^T i\gamma_5 C_D T^a E^a [1 + (z_+/2y)^2 P_-] u) \rangle'_E > 0 \quad (0 \leq z_+/2y < \infty). \tag{6.44}
\]

Only at the limit of the weak Majorana-Yukawa coupling, \( z_+/2y = \infty \) or \( y/z_+ = 0 \), the partition function can show the massless singularity because the pfaffian is evaluated (formally) as

\[
\text{pf}\left(u^T i\gamma_5 C_D T^a E^a [P_-] u\right) = \text{pf}\left\{ (\bar{u} u)^T (\bar{u} i\gamma_5 C_D T^a E^a \bar{u}^T) (\bar{u} u) \right\}
\]

\[
= \det(\bar{u} u) \text{pf}(\bar{u} i\gamma_5 C_D T^a E^a \bar{u}^T)
\]

\[
= \det(\bar{u} D u). \tag{6.45}
\]

This result implies that the PMS phase extends all the way to the limit of the weak Majorana-Yukawa coupling \( y/z_+ = 0 \) and the FM phase is absent within the coupling-constant space of \( \hat{S}_{\text{OV}} \).

We can confirm the above result by applying the saddle point analysis in the spirit of the large \( N \) expansion to this model, just as discussed in section 4.5. The effective action of
the spin field \( X^a(x) = X^a_0 \) and the Lagrange multiplier field \( \lambda(x) = \lambda^a_0 \) in this case is given by

\[
S[X^a, \lambda] = - \frac{1}{2} \ln \det \left( \begin{pmatrix} u^T i \gamma_5 C_D T^a X^a u & (z+/2y)(u^T \tilde{u}) \\ -(z+/2y)(\tilde{u} u) & (u i \gamma_5 C_D T^a \tilde{u}) \end{pmatrix} \right) - i \sum_x \lambda(x) (X^a(x) X^a(x) - 1) 
\]

(6.46)

\[
= -32 \sum_p \ln \left( X^a_0 X^a_0 + (z+/2y)^2 \Delta(p) \right)^{1/2} - \sum_{x,y} \bar{X}^a(x) \left\{ 4B(x-y)(2X^a_0 X^b_0 - X^a_0 X^b_0 \delta^{ab} \right) - 4A(x-y)(z+/2y)^2 \delta^{ab} - i \lambda_0 \delta_{xy} \delta^{ab} \} \bar{X}^b(x) + \cdots 
\]

(6.47)

where \( B(x-y) \) and \( A(x-y) \) are the kinetic operators defined through the chiral projector \( \hat{P}_{0+}(x,y) \) of the single free overlap Dirac fermion as

\[
B(x-y) = \text{tr} \left[ \hat{P}_{0+} X^a_0 X^a_0 + (z+/2y)^2 \Delta \right] \hat{P}_{0+} \left[ X^a_0 X^a_0 + (z+/2y)^2 \Delta \right] \]

\[
+ \text{tr} \left[ \hat{P}_{0+} X^a_0 X^a_0 + (z+/2y)^2 \Delta \right] \hat{P}_{0+} \left[ X^a_0 X^a_0 + (z+/2y)^2 \Delta \right] \]

(6.48)

\[
A(x-y) = \text{tr} \left[ \hat{P}_{0+} X^a_0 X^a_0 + (z+/2y)^2 \Delta \right] \hat{P}_{0+} \left[ X^a_0 X^a_0 + (z+/2y)^2 \Delta \right] \]

(6.49)

and

\[
\hat{P}_{0+}(x-y) = \frac{1}{V} \sum_q e^{iq(x-y)} \left\{ \frac{1}{2} - \frac{1}{2} \gamma_5 \sin q_\mu + b(q) \right\}, \]

(6.50)

\[
\Delta(x-y) = \frac{1}{V} \sum_q e^{iq(x-y)} \left\{ \frac{\omega(q) + b(q)}{2 \omega(q)} \right\}. \]

(6.51)
The path-integration of the fluctuations $\tilde{\lambda}(x)$ and $\tilde{X}^a(x)$ (in this order) gives the effective action of $X_0^a$ and $\lambda_0$ as follows,

$$S_{\text{eff}}[X_0^a, \lambda_0] = \left\{ -32 \sum_p \ln (X_0^a X_0^a + (z/2)^2 \Delta(p))^{1/2} - i\lambda_0 (X_0^a X_0^a - 1) \right\} V$$

$$+ \left( \frac{10 - 1}{2} \right) \sum_{k \neq 0} \ln \left\{ -4 \tilde{D}(k; X_0) - i\lambda_0 \right\},$$

(6.52)

where $\tilde{D}(k; X_0)$ is the fourier transform of the kinetic operator,

$$D(x - y; X_0) = X_0^a X_0^a B(x - y) + (z+/2y)^2 A(x - y).$$

(6.53)

The stationary conditions for $X_0^a$ and $\lambda_0$ are given by

$$0 = 2 X_0^a \left\{ -16 \frac{1}{V} \sum_p X_0^a X_0^a + \frac{1}{(z+/2y)^2 \Delta(p)} - i\lambda_0$$

$$+ \left( \frac{9}{2} \right) \sum_{k \neq 0} \left\{ -4 \frac{\partial}{\partial X_0^a} \tilde{D}(k; X_0) \right\} + \frac{1}{16 \tilde{D}(k; X_0) + D(0; X_0)} \right\} \right|_{X_0^a X_0^a = 1},$$

(6.54)

$$0 = (X_0^a X_0^a - 1) + \left( \frac{9}{2} \right) \sum_{k \neq 0} \frac{1}{16 \tilde{D}(k; X_0) - i\lambda_0}.$$  

(6.55)

Assuming $X_0^a \neq 0$, the above conditions imply that

$$-i\lambda_0 = 16 \frac{1}{V} \sum_p X_0^a X_0^a + \frac{1}{(z+/2y)^2 \Delta(p)}$$

$$- \frac{9}{32} \sum_{k \neq 0} \left\{ -4 \frac{\partial}{\partial X_0^a} \tilde{D}(k; X_0) \right\} + \frac{16}{16 \tilde{D}(k; X_0) + D(0; X_0)} \right|_{X_0^a X_0^a = 1},$$

(6.56)

$$X_0^c X_0^c = 1 - \frac{9}{32} \sum_{k \neq 0} \left\{ -4 \frac{\partial}{\partial X_0^a} \tilde{D}(k; X_0) + D(0; X_0) \right\} \right|_{X_0^a X_0^a = 1},$$

(6.57)

where the leading results, $-i\lambda_0 = 16 \frac{1}{V} \sum_p X_0^a X_0^a + \frac{1}{(z+/2y)^2 \Delta(p)}$ and $X_0^a X_0^a = 1$, are substituted in the terms suppressed by the factor $N/2^2 \pi$ ($\simeq 9/32$). The r.h.s. of the condition eq. (6.57) is required to be positive for $X_0^a \neq 0$. It is plotted in fig. 9 as the function of $z_+$ for $y = 1$ and $m_0 = 1$,

$$f(m_0, z_+, y) \equiv 1 - \frac{9}{32} \sum_{k \neq 0} \left\{ -4 \frac{\partial}{\partial X_0^a} \tilde{D}(k; X_0) + D(0; X_0) \right\} \right|_{X_0^a X_0^a = 1}.$$  

(6.58)

One can see that $f(m_0, z_+, y) < 0$ for $z_+ \geq 0$ up to $z_+ \simeq 10$ $(y = 1, m_0 = 1)$ and it is in contradiction with the assumption $X_0^a \neq 0$. In this region of the coupling $z_+$, the fluctuation of the spin field $E^a(x)$ is too large to maintain the non-zero expectation value of the spin field.
Figure 9. $f(m_0, z_+, y)$ vs. $z_+$ ($y = 1$): The consistency condition for the SO(10) symmetry breaking in the effective spin model of $\tilde{S}_{OV}$ within the saddle point analysis in the spirit of the large N expansion.

Figure 10. [left]: $\tilde{\Gamma}(k)$ as the function of $k^2/(2\pi/L)^2$ for $z_+ = 0$ (green) and $z_+ = 10.0$ (blue) ($y = 1$) in comparison with $(1/16) \sum_\mu 4 \sin^2(k_\mu/2)$ (purple). [right]: $\tilde{\Gamma}(k)|_{k^2=(2\pi/L)^2}$ in logarithmic scale as the function of $z_+ / y$ ($y = 1$).

$\langle E^a(x) \rangle$, and we do not see any evidence for the FM phase. In fig. 10, the kinetic term of the fluctuation modes $\tilde{X}^a(x)$, given by $\tilde{\Gamma}(k) \equiv (1/4) \left[ -\tilde{D}(k; X_0) + \tilde{D}(0; X_0) \right] |_{X_0^2=1}$, is shown as the function of $k^2/(2\pi/L)^2$ in comparison with $(1/16) \sum_\mu 4 \sin^2(k_\mu/2)$. One can see that the kinetic term looks like the canonical form of the free theory and the normalization of the kinetic term, defined by the value of $\tilde{\Gamma}(k)$ at $k^2 = (2\pi/L)^2$, is decreasing monotonically and exponentially in $z_+ / y$ ($y = 1$). These results support the picture that the PMS phase extends all the way to the limit of the weak Majorana-Yukawa coupling $y/z_+ = 0$ and that the FM and PMW phases are absent within the coupling space of $\tilde{S}_{OV}$.

Given the result that the FM phase is absent in the model $\tilde{S}_{OV}$, it is instructive to compare the above result to the situation in the original Eichten-Preskill models $S_{EP}$ and $S_{EP/WY}$[98, 99], where it was argued by Golterman, Petcher and Rivas[99] that the crucial ingredient for the failure of the proposal is the existence of the FM phase which separates the PMS and PMW phases. In fact, as mentioned before, there is an important technical
difference in our models, which is relevant for preserving the (global) \text{SO}(10) symmetry: we choose the product function $F(\omega)$ as given by eq. (3.6) and therefore use the unit \text{SO}(10)-vector spin fields, omitting their kinetic(hopping) terms. This corresponds to taking the limits eqs. (6.8) and (6.9) in $S_{\text{EP}/WY}$ as given in eq. (6.27). This region in the coupling-constant space of $S_{\text{EP}/WY}$ has not been explored by Golterman, Petcher and Rivas. In applying the saddle point analysis in the spirit of the large $N$ expansion, we further reduce the degrees of freedom of the spin fields through the identification $E^{a}(x) = \tilde{E}^{a}$, $y = \tilde{y}$, $w = \tilde{w}$ and consider the following model.

\[
\tilde{S}_{\text{EP}/WY} = \sum_{x \in \Lambda} \left\{ \tilde{\psi}(x) \gamma_{\mu} P_{-} \left( [\nabla_{\mu} - \nabla_{\mu}^{\dagger}] / 2 \right) \psi(x) + z_{+} \tilde{\psi}(x) \gamma_{\mu} P_{+} \left( [\nabla_{\mu} - \nabla_{\mu}^{\dagger}] / 2 \right) \psi(x) \right\} \\
- \sum_{x \in \Lambda} y E^{a}(x) \left\{ \psi^{T}(x) i \gamma_{5} C_{D} T^{a} P_{+} \tilde{\psi}(x) + \tilde{\psi}(x) P_{-} i \gamma_{5} C_{D} T^{a} \right\} \\
- \sum_{x \in \Lambda} w E^{a}(x) \left\{ \psi^{T}(x) i \gamma_{5} C_{D} T^{a} (\nabla_{\mu} \nabla_{\mu}^{\dagger} / 2) P_{+} \psi(x) + \tilde{\psi}(x) P_{-} i \gamma_{5} C_{D} T^{a} (\nabla_{\mu} \nabla_{\mu}^{\dagger} / 2) \tilde{\psi}(x)^{T} \right\} .
\] (6.59)

In this model, the consistency condition for $\langle X^{a}(x) \rangle \neq 0$ is given by

\[
f(m_{0}, z_{+}, y, w) \equiv 1 - \frac{9}{32} \frac{1}{V} \sum_{k \neq 0} \frac{4}{-D'(k^{0}; X_{0}) + \tilde{D}'(k^{0}; X_{0})} \bigg|_{X_{0}^{\prime} X_{0}^{\prime} = 1} > 0,
\] (6.60)

where $k_{\mu}^{0} = 0$ or $\pi_{\mu}^{(15)}$ depending on the value of the couplings $z_{+}, y, w$. $\tilde{D}(k^{0}; X_{0})$ is the fourier transform of the kinetic operator,

\[
D'(x - y; X_{0}) = X_{0}^{\prime} X_{0}^{\prime} B'(x - y) + (z_{+} / 2)^{2} A'(x - y),
\] (6.61)

where $B'(x - y)$ and $A'(x - y)$ are defined by

\[
B'(x - y) = \frac{1}{V} \sum_{k} e^{i k x} \frac{1}{V} \sum_{q} \left\{ \tilde{W}(q + k)^{2} + \tilde{W}(q)^{2} + 2 \tilde{W}(q + k) \tilde{W}(q) \right\} \times \frac{\tilde{W}(q + k)}{X_{0}^{\prime} X_{0}^{\prime} \tilde{W}(q + k)^{2} + (z_{+} / 2)^{2} \sin^{2}(q + k) / 2} \times \frac{\tilde{W}(q)}{X_{0}^{\prime} X_{0}^{\prime} \tilde{W}(q)^{2} + (z_{+} / 2)^{2} \sin^{2}(q)} ,
\] (6.62)

\[
A'(x - y) = \frac{1}{V} \sum_{k} e^{i k x} \frac{1}{V} \sum_{q} \left\{ \sin(q + k) \mu + \tilde{W}(q)^{2} + 2 \tilde{W}(q + k) \tilde{W}(q) \right\} \times \frac{\sin(q + k) \mu}{X_{0}^{\prime} X_{0}^{\prime} \tilde{W}(q + k)^{2} + (z_{+} / 2)^{2} \sin^{2}(q + k) / 2} \times \frac{\sin q_{\mu}}{X_{0}^{\prime} X_{0}^{\prime} \tilde{W}(q)^{2} + (z_{+} / 2)^{2} \sin^{2}(q)} ;
\] (6.63)

and $W = y + (w / 2) \nabla_{\mu} \nabla_{\mu}$. In fig. 11, $f(m_{0}, z_{+}, y, w)$ is plotted as the function of $z_{+}$ for $y = w = 1$ and $m_{0} = 1$. The singular behavior of the plots around $z_{+} \simeq 1.4$ indicates the fact that at a certain critical value $z_{+} = z_{+}^{c} (\simeq 1.4)$ the kinetic operator degenerates: $\tilde{D}'(0) = \tilde{D}'(k) = \tilde{D}'(\pi^{(15)})$, where $\tilde{D}'(k) \equiv \tilde{D}'(k; X_{0}) \big|_{X_{0}^{\prime} X_{0}^{\prime} = 1}$. For $z_{+} < z_{+}^{c}$, $\tilde{D}'(k) \leq \tilde{D}'(\pi^{(15)})$ and
the saddle point is assumed to be Anti-Ferromagnetic, \( \langle X^a(x) \rangle = X^a_0 \langle -1 \rangle^{x_μ} \). For \( z_+ > z'_+ \), \( \tilde{D}'(k) \leq \tilde{D}'(0) \) and the saddle point is assumed to be Ferromagnetic, \( \langle X^a(x) \rangle = X^a_0 \). In both cases, \( f(m_0, z_+, y, w) < 0 \) and the fluctuation of the spin field \( E^a(x) \) is too large to maintain the non-zero expectation value of the spin field \( \langle E^a(x) \rangle \). Thus the model is in the PMS phase in the entire region of the coupling \( z_+ (y = 1) \) up to \( z_+ \approx 15 \).

In the case of the model \( \tilde{S}_{Ov} \), we found that \( \tilde{D}(k) \leq \tilde{D}(0) \) for the entire region \( z_+ \geq 0 \), where \( \tilde{D}(k) \equiv \tilde{D}(k; X_0) \big|_{X^a_2 = 1} \). And the saddle point is assumed to be Ferromagnetic, \( \langle X^a(x) \rangle = X^a_0 \). Therefore the coupling-constant space of the model \( \tilde{S}_{Ov} \) should correspond to the region of the weaker Majorana-Yukawa coupling, \( z_+ > z'_+ \), within the coupling-constant space of the model \( \tilde{S}_{EP/WY} \). This fact is also supporting the picture that the PMS phase extends all the way to the limit of the weak Majorana-Yukawa coupling \( y/z_+ = 0 \) in our \( SO(10) \) model \( \tilde{S}_{Ov} \).

One may study these models, \( \tilde{S}_{Ov} \) and \( \tilde{S}_{EP/WY} \), as the counter part of the reduced staggered fermion model with the \( SU(4)/SO(4) \) and \( Z_4 \) symmetries by reducing \( SO(10) \) symmetry to \( SO(6)/\text{Spin}(6) = SU(4) \) and \( SO(4)/SU(2)^+ \times SU(2)^- \) (or \( SO(3)/SU(2)^+ \)).\footnote{For the models with the symmetries reduced from \( SO(10) \), one may reformulate the model \( S_{Ov} \) more simply in terms of overlap Majorana fields as
\[
\tilde{S}_{Ov/M}[\psi, E^{a'}] = \sum_{x \in \Lambda} \{ \bar{\psi}(x)^T C D \psi(x) - y E^{a'}(x) \psi(x)^T C D E^{a'} \psi(x) \}. \tag{6.64}
\]}

Figure 11. \( f(m_0, z_+, y, w) \) vs. \( z_+ (y = w = 1) \): The consistency condition for the \( SO(10) \) symmetry breaking in the effective spin model of \( \tilde{S}_{EP/WY} \) within the saddle point analysis in the spirit of the large \( N \) expansion.
in the consistency condition eq. (6.60). But the effect still remains rather large. Then the full quantum fluctuations can reduce the region of the coupling-constant \( z_+ \) where the FM phase appears, restoring the broken SO(6) symmetry. Thus our results here seems quite consistent with the observations and arguments made by these authors about the reduced staggered fermion model with the quartic interaction term which respects the SU(4)/SO(4) and \( Z_4 \) symmetries, and about "Mass without Symmetry Breaking".

### 6.4 cf. Domain wall fermions with the boundary Eichten-Preskill term

In the proposal by Creutz, Tytgat, Rebbi, Xue[108] to formulate the standard model plus the right-handed neutrinos by the domain wall fermion, the authors have considered the quartic term with the symmetry SU(4) \( \times \) SU(2)_L \( \times \) SU(2)_R as boundary interaction terms. In fact, this type of the boundary interaction term can be obtained from the SO(10) interaction term by reducing the symmetry to SO(6) \( \times \) SO(4) (\( = \) SU(4) \( \times \) SU(2)_L \( \times \) SU(2)_R). Then, it is straightforward to lift their proposal to the SO(10) chiral gauge theory.

In fact, we can show that the following action defines such a domain wall fermion model for the SO(10) chiral gauge theory:

\[
S_{DW/Mi} = \sum_{i=1}^{L_5} \sum_{x \in \Lambda} \bar{\psi}(x, t) \left\{ [1 + a'_5(D_{4w} - m_0)]\delta_{t't'} - P_- \delta_{t+1,t'} - P_+ \delta_{t,t'+1} \right\} \psi(x, t') \\
+ \sum_{x \in \Lambda} (z_+ - 1) \bar{\psi}(x, L_5) P_- [1 + a'_5(D_{4w} - m_0)] \psi(x, L_5) \\
- \sum_{x \in \Lambda} \left\{ y X^a(x) \psi^T(x, L_5) i\gamma_5 C_D T^a \psi(x, L_5) \\
+ \bar{y} \bar{X}^a(x) \bar{\psi}(x, L_5) P_- i\gamma_5 C_D T^a \bar{\psi}(x, L_5)^T \right\} \\
+ S_X[X^a],
\]

(6.65)
where the Dirichlet b.c. is assumed,
\[ P_+ \psi(x, 0) = 0, \quad P_- \psi(x, 0) = 0 ; \quad P_- \psi(x, L_5 + 1) = 0, \quad P_+ \psi(x, L_5 + 1) = 0, \]
(6.66)

and \(a_5'(= a_5/a)\) is the lattice spacing of extra dimension in the lattice unit. In this action, the second term in the r.h.s. is introduced so that all the terms which involve the field \(\tilde{\psi}(x, L_5)P_-(= \tilde{q}_+ (x))\) in the original action of the domain wall fermion (the five-dimensional Wilson fermion) are rescaled by the factor \(z_+\) and made vanished in the limit \(z_+ \to 0\). Then the forth term with the Yukawa coupling \(y\) is required so that it saturates the path-integral measure of that field \(\tilde{\psi}(x, L_5)P_-\). On the other hand, the field \(\psi(x, L_5)\) is related to the (truncated) overlap fermion field \(\psi(x)\) by the relation \(\psi(x, L_5) = (-\gamma_5)(1 + e^{a_5L_5H})^{-1} \psi(x)\) in the subtraction scheme using the five-dimensional Wilson fermion subject to the anti-periodic b.c., and is projected to the right-handed Weyl field \(\psi_+(x) = P_+ \psi(x)\) in the limit \(L_5 \to \infty\) (plus \(a_5 \to 0\))[68]. Thus it ends up with the overlap fermion model with \(S_{Ov/Mi}\) in the limit (6.8) and (6.9), which is the similar model with \(\tilde{S}_{Ov}\), but before reducing the degrees of freedom of the spin fields through the identification \(E^n(x) = \hat{E}^n\), \(y = \hat{y}\). The global U(1) symmetry of the five-dimensional fermion fields is broken to \(Z_4\) by the boundary Yukawa couplings. The CR5, P, and CPR5 symmetries are all broken by the boundary Yukawa couplings and the term with the coupling \((1 - z_+)\).

Taking the limit eqs. (6.7), (6.8) and (6.9) first, the model provides the five-dimensional \(\phi^4\) theory in the limit (6.68), (6.69), and (6.70). Then, what we have argued in the previous sections about the four-
dimensional model $S_{Ov}$ implies that the domain wall fermion path-integral measure is properly saturated at around the right-handed boundary with the fields, $\psi(x, L_5)$, $\bar{\psi}(x, L_5)P_-$, even when the spin fields $E^a(x)$, $\bar{E}^a(x)$ have the disordered nature. Moreover, the CP symmetry is restored in the limit $L_5 \to \infty$.

Thus the five-dimensional domain wall fermion model defined by the action eq. (6.67) provides a very explicit and well-defined implementation of the proposal by Creutz, Tytgat, Rebhi, Xue for the (more general) case of the SO(10) chiral gauge theory. And our four-dimensional lattice model defined with the path-integration measure for the left-handed Weyl field eq. (6.1) is nothing but the low energy effective theory of the five-dimensional domain wall model in the limit $L_5 \to \infty$ ($a'_5 \to 0$).

In this respect, we note that one may define the action of such a SO(10) domain wall fermion model simply by

$$S'_{DW/Mi} = \sum_{x=1}^{L_5} \sum_{x \in \Lambda} \bar{\psi}(x, t) \left\{ [1 + a'_5(D_{4w} - m_0)]\delta_{t't} - P_-\delta_{t+1,t'} - P_+\delta_{t,t+1} \right\} \psi(x, t')$$

$$- \sum_{x \in \Lambda} \left\{ y X^a(x)q_{1,5}^T(x)i\gamma_5 C_D T^a P_+ q_+(x) + \bar{y} \bar{X}^a(x)\bar{q_+}(x) P_- i\gamma_5 C_D T^a \bar{q_+}(x)^T \right\} + S_X[X^a].$$

Note here that the boundary interaction terms are formulated solely with the boundary field variables, $q(x) = \psi_+(x, 1) + \psi_+(x, L_5)$, $\bar{q}(x) = \bar{\psi}_-(x, 1) + \bar{\psi}_+(x, L_5)$, which are first introduced by Shamir and Furman. In this action, the global U(1) symmetry of the five-dimensional Wilson fermion fields is broken to $Z_4$ by the boundary Yukawa couplings. The CR$_5$ and P symmetries are also broken to the CPR$_5$ symmetry in the same manner.

We note, however, that this model ends up with the overlap fermion model $S'_{Mi/Ov}$ with the Yukawa couplings eq. (6.28) in the limit $L_5 \to \infty$ in the same subtraction scheme. Therefore, this type of the Majorana-Yukawa couplings at the boundary are singular in the large limit.

6.5 cf. Topological Insulators/Superconductors with gapped boundary phases

It has been proposed by Wen, by You, BenTov and Xu, and by You and Xu to use the 4D Topological Insulators(TIs)/Superconductors(TSCs) with the gapped boundary phases in order to formulate the 3+1D chiral gauge theories in the Hamiltonian formalism. These authors have considered the same 4D TI with the time-reversal symmetry defined by the following quantum Hamiltonian,

$$\hat{H}_{4DTI} = \sum_{i=1}^{\nu} \sum_p \hat{a}_i(p)\hat{a}_i(p)\hat{a}_i(p)^\dagger + \beta \left( \left\{ \sum_{k=1}^{4} \cos(p_k) - 4 \right\} + m \right) \hat{a}_i(p),$$

where $\hat{a}_i(p)$ and $\hat{a}_i(p)^\dagger$ are fermionic annihilation-creation operators in momentum space, satisfying the canonical anti-commutation relations, $\hat{a}_i(p)\hat{a}_j(p') + \hat{a}_j(p')\hat{a}_i(p) = \delta_{p,p'}\delta_{i,j}$. The alpha and beta matrices are chosen here as $\alpha_k = \sigma_3 \otimes \sigma_k$ ($k = 1, 2, 3$), $\alpha_4 = \sigma_2 \otimes I$, and $\beta = -\sigma_1 \otimes I$. The generator of the time-reversal symmetry transformation is given as
\[ \mathcal{T} = K (iI \otimes \sigma_2), \]  where \( K \) stands for complex conjugation. This 4D quantum lattice fermion model is nothing but the Hamiltonian formulation of Kaplan’s 5-dim. domain wall fermion defined with the Wilson term.\[60, 61\] It was first examined by Creutz and Horvath[121] to study the chiral property of the massless lattice fermions realized as Shockley surface states, and later by X.-L. Qi, Hughes and S.H. Zhang[122] as a 4D extension of the 2D Integer Quantum Hall Effect (IQHE).

The insulator is in topological phase for \( m > 0 \) and in trivial phase for \( m < 0 \). On the 3D boundary of the domain wall due to the change of the mass parameter from \( m > 0 \) to \( m < 0 \), there appear \( n \in \mathbb{Z} \) copies of two-component (right-handed) Weyl fermions at low energy \( |p| \ll 0 \) (\( l = 1, 2, 3 \)) assuming the thermodynamic limit of the 4D space. These Weyl fermions are protected from acquiring mass by the topological index defined by the second Chern character of the U(1) bundle associated with the connection \( \sum_k \psi_k^\dagger \delta \psi_k \) and the time reversal symmetry. This gapless boundary phase can be described by the low energy effective Hamiltonian,

\[
\hat{H}_{3D}^{(bd)} = \sum_{i=1}^{\nu} \int d^3x \hat{\psi}_i(x) \{ \sum_{l=1}^{3} (-i) \sigma_l \partial_l \} \hat{\psi}_i(x), \tag{6.72}
\]

The generator of the time-reversal symmetry transformation acting the effective Hamiltonian is given as \( \mathcal{T} = K (i \sigma_2) \).

For the case \( \nu = 16 \), the authors have proposed the boundary interaction terms to fully gap the boundary phase with the sixteen massless Weyl fermions, or the bulk interaction terms to be able to interpolate between the topological and trivial phases without closing the mass gap nor breaking the symmetries. In fact, the boundary/bulk interaction terms introduced in these works are the SO(10)-invariant quartic (or Yukawa) term

\[
\hat{O}(x) = \frac{1}{2} [\hat{\psi}(x)^T C_D T^a \hat{\psi}(x)]^2 + \frac{1}{2} [\hat{\psi}(x)^T C_D T^a \phi^a(x)]^2 \tag{6.73}
\]

assuming that the sixteen massless Weyl fermions are in the 16 of SO(10) and its descendants, SO(7) \times SO(3) and SO(6) \times SO(4) (\( = SU(4) \times SU(2) \times SU(2) \)). It is quite interesting to see that these are essentially identical to the SO(10)-invariant quartic terms of the ’t Hooft vertices, \( T_+(x), T_+(x) \),

\[
\hat{O}_T(x) = T_+(x) + T_+(x),
\]

\[
= \frac{1}{2} [\psi^T(x) C_D T^a \psi(x)]^2 + \frac{1}{2} [\hat{\psi}(x)^T C_D T^a \phi^a(x)]^2, \tag{6.74}
\]

and their descendants.

Wen, in particular, have considered the SO(10) chiral gauge theory as a target theory[123]. The author have proposed to use the following SO(10)-invariant boundary interaction terms,

\[
\hat{H}_{3D,10} = \int d^3x \left\{ \hat{\psi}(x)^T i \sigma_2 T^a \phi^a(x) \hat{\psi}(x) - \hat{\psi}(x) i \sigma_2 T^a \phi^a(x) \hat{\psi}(x)^T + \mathcal{H}[\phi^a(x)] \right\}, \tag{6.75}
\]
where the Weyl field \( \hat{\psi}(x) = \{\hat{\psi}_a(x)\}(s = 1, \cdots, 16) \) is assumed to form the irreducible spinor representation 16 of SO(10), and \( \mathcal{H}[\phi^a(x)] \) stands for the kinetic and potential terms of the SO(10) vector real scalar field \( \phi^a(x) \). It is assumed that \( \mathcal{H}[\phi^a(x)] \) is chosen to make \( \phi^a(x)\phi^{a}(x) = M^2 \neq 0 \) without breaking the SO(10) symmetry, \( \langle \phi^a(x) \rangle = 0 \). It is also assumed that the correlation length of the field \( \phi^a(x) \), \( \xi_\phi \), is much larger than the lattice spacing so that one can then expect the Yukawa coupling to generate a (Majorana-type) mass for all the sixteen Weyl fermions. The author then discusses that topological defects with \( \phi^a(x) = 0 \) at some points or in some regions of the 3+1D space-time, which can give rise to massless (gapless) fermionic excitations, do not exist for the SO(10) vector real scalar field \( \phi^a \) with mass for all the sixteen Weyl fermions. The author then discusses that topological defects do not exist for the SO(10) vector real scalar field \( \phi \), a \( \pi \) hedgehog solitons, \( \pi \) against point-defects like the instantons, \( \pi_2(S^9) = 0 \) against line-defect like hedgehog solitons, \( \pi_1(S^9) = 0 \) against membrane-defect like vortex lines, and \( \pi_0(S^9) = 0 \) against 3-brane-defect like domain walls. The author also points out that the WZW term does not exist, because \( \pi_3(S^9) = 0 \). Based on these assumptions and considerations, the author have argued that the boundary interaction \( \hat{\mathcal{H}}_{3D,10} \) can make the boundary phase with the sixteen massless Weyl fermions fully gapped without breaking the SO(10) and time-reversal symmetries.

As mentioned above, the 4D TI, eq. (6.71), with the boundary phases of the \( \nu \) massless Weyl fermions is nothing but the Hamiltonian formulation of Kaplan’s 5-dim. domain wall fermion defined with the Wilson term. Then the 4D TI(TSC) can be formulated in the framework of 4+1D Euclidean path-integral quantization using the five-dimensional lattice domain wall fermion including suitable boundary/bulk interaction terms. Using this 5 dim. lattice formulation of the 4D TI(TSC), one can study the effect of the boundary/bulk multi-fermion (Yukawa) interactions on the properties of the 4D TI(TSC), in particular, the behaviors of the proposed 3D gapped boundary phases of the 4D TI(TSC) of \( \nu = 16 \), by using the various perturbative/non-perturbative methods in the framework of lattice field theory.

The domain wall fermion models \( S_{DW/Mi}^\nu \) and \( S_{DW/Ov}^\nu \) discussed in the previous sec. 6.4 provide such a formulation. In fact, the partition function of the 4D TI(TSC) of \( \nu = 16 \) with the SO(10)-invariant boundary interaction terms can be defined precisely by the domain wall fermion model \( S_{DW/Mi}^\nu \):

\[
Z_{4DTI/\nu=16} = \int \prod_{t=-L_5+1}^{L_5} \mathcal{D}[\psi(t)]\mathcal{D}[\bar{\psi}(t)]\mathcal{D}[E]\mathcal{D}[\bar{E}] \ e^{-S_{DW/Mi}[\psi,\bar{\psi},E,\bar{E}]_{\text{Dir}}} |_{\text{Dir}} = \det a_5'(D_{5w} - m_0)|_{\text{AP}} Z_{Ov/Mi} \quad [L_5 \to \infty (a_5' \to 0)].
\] (6.77)

In this 4+1D lattice model, one can fix the radii of the SO(10) spin fields to unity as

\[
E^a(x)\bar{E}^a(x) = 1, \quad \bar{E}^a(x)\bar{E}^a(x) = 1
\] (6.78)

from the beginning by taking the limit eqs. (6.7), (6.8) and (6.9). Moreover, one can take the limit of the large Majorana-Yukawa couplings,

\[
y = \bar{y}, \quad \frac{z_+}{\sqrt{yy}} \to 0.
\] (6.79)
Then one ends up with the domain wall fermion model $S_{\text{DW/Ov}}$ and the four-dimensional lattice model $S_{\text{Ov}}$ as a low energy effective lattice theory for the edge modes at the boundaries. The partition function of the 4D TI(TSC) of $\nu = 16$ then reads

$$Z_{\text{4DTI/}}/\nu = 16 = \det \alpha'_{5}(D_{5w} - m_{0})|_{\text{AP}} Z_{\text{Ov}} \left[ L_{5} \rightarrow \infty (\alpha'_{5} \rightarrow 0) \right]$$

Thus our four-dimensional lattice model of the SO(10) chiral gauge theory, defined with the path-integration measure for the left-handed Weyl field eq. (6.1), gives a direct and well-defined description of the $\nu = 16$ 3D gapped/gapless boundary phases proposed by Wen[123] in the framework of 3+1D local lattice theory. In sections 3 and 4, we have argued that the pfaffian $\text{pf}(u^{T} i \gamma_{5} C_{D} T^{a} E^{a} u) \text{f}$ is real positive semi-definite and its path-integration over $E^{a}(x)$ ($a = 1, \cdots, 10$) is non-vanishing for the trivial link field $U(x, \mu) = 1$ in the weak-gauge coupling limit. This implies that the partition function of the boundary phase is real-positive,

$$\langle \text{pf}(u^{T} i \gamma_{5} C_{D} T^{a} E^{a} u) \text{f} \rangle_{E} > 0 \quad (U(x, \mu) = 1), \quad (6.81)$$

and the massless singularity associated with the sixteen right-handed Weyl fermions is absent. We have also argued that the auxiliary spin fields show the disordered nature for $m_{0} < 2$ and the correlation functions of the right-handed fermions and the auxiliary spin fields are short-ranged (except the unknown one, $\langle \psi_{+} \left[ \psi_{+}^{T} i \gamma_{5} C_{D} T^{a} E^{a} \hat{P}_{-} \right] \rangle_{E}$). These results provide analytical evidences that the $\nu = 16$ 3D boundary phase is fully gapped indeed and respects the SO(10) symmetry.

We will discuss the details of the description of the gapped boundary phases of 1-4D TI/TSC in terms of overlap fermions elsewhere[256].

7 Conclusion

In this paper, we formulated the SO(10) chiral gauge theory with Weyl fermions in the sixteen dimensional spinor representation 16 within the framework of the Overlap fermion/the Ginsparg-Wilson relation. We defined the path-integral measure of the left-handed Weyl fermions with all the components of the Dirac field, but the right-handed part of which is saturated completely by inserting a suitable product of the SO(10)-invariant 't Hooft vertices in terms of the right-handed field. The definition of the measure applies to all possible topological sectors. In fact, we examined in detail the two cases of the trivial link field and of the SU(2) link fields with non-vanishing topological charge $Q(\neq 0)$. We argued that the path-integration of the pfaffian $\text{pf}(u^{T} i \gamma_{5} C_{D} T^{a} E^{a} u) \text{o}$ over the auxiliary spin field $E^{a}(x)$ gives a non-zero result,

$$\int D[E] \text{pf}(u^{T} i \gamma_{5} C_{D} T^{a} E^{a} u) = c [U(x, \mu)] \neq 0$$

and that the measure of the right-handed field, $D_{\psi}[\psi_{+}]$, is indeed saturated completely by inserting the product of the 't Hooft vertex $T_{+}(x)[\psi_{+}]$, while the SO(10) symmetry
does not break spontaneously in the thermodynamic limit. The measure possesses all required transformation properties under lattice symmetries and we gave a proof of the CP-invariance of the induced effective action. The global U(1) symmetry of the left-handed field is anomalous due to the non-trivial trasformation of the measure, while that of the right-handed field is explicitly broken by the 't Hooft vertices.

There remains the issue of smoothness/locality in the gauge-field dependence of the Weyl fermion measure term,

$$-i \Sigma_\eta \equiv - \text{Tr} \{ \delta \hat{P}_+ \langle \psi_+^T i \gamma_5 C_D T^a E^a \rangle_F \} / \langle 1 \rangle_F.$$ (7.1)

This question is of not quite dynamical but non-perturbative nature in our model, involving the path-integration of the spin field $E^a(x)$ with the weight of the pfaffian $\text{pf}(u^T i \gamma_5 C_D T^a E^a u)$, which is complex in general. But the question is well-defined and it is highly desirable to establish these properties rigorously, if possible. It can be addressed in the weak gauge-coupling expansion at least because the pfaffian is positive semi-definite and Monte Carlo methods are applicable to evaluate the correlation functions and the vertex functions. We leave this important and interesting question for our future study.\(^{16}\)

We discussed the relations of our formulation to other approaches/proposals to decouple the species-doublers and mirror fermions such as the Eichten-Preskill model\[^{98, 99}\], the Mirror-fermion model using Ginsparg-Wilson fermions\[^{100–107}\], Domain wall fermions with the boundary Eichten-Preskill term\[^{108}\], the recent studies on the PMS phase/“Mass without symmetry breaking”\[^{110, 113–118}\] and 4D TI/TSCs with gapped boundary phases\[^{123–127}\]. These discussions, we hope, clarified the similarity and the difference in technical detail and showed that our proposal is a well-defined testing ground for that basic question.

A Dirac gamma matrices

Dirac gamma matrices (The Clifford algebra of $2^{[5/2]}$ dimensions):

$$\gamma_0 = \begin{pmatrix} 0 I \\ I 0 \end{pmatrix}, \gamma_1 = \begin{pmatrix} 0 & i \sigma_1 \\ -i \sigma_1 & 0 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 & i \sigma_2 \\ -i \sigma_2 & 0 \end{pmatrix}, \gamma_3 = \begin{pmatrix} 0 & i \sigma_3 \\ -i \sigma_3 & 0 \end{pmatrix},$$ (A.1)

$$\gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$ (A.2)

$$C_D = \begin{pmatrix} i \sigma_2 & 0 \\ 0 & -i \sigma_2 \end{pmatrix}.$$ (A.3)

\(^{16}\) Two-dimensional abelian chiral gauge theories can be formulated in the similar manner. Those include the $1^4(-1)^4$ axial gauge model and the $21(-1)^3$ chiral gauge model. In these models, the two-point vertex function of the U(1) gauge field in the mirror sector was computed through Monte Carlo simulations. The simulation results indeed showed a numerical evidence that the two-point vertex function is regular and that the induced measure term $-i \Sigma_\eta$ is a local functional of the link field. See \[^{257}\] for detail.
\{\gamma_{\mu}, \gamma_{\nu}\} = 2\delta_{\mu\nu} ; \quad \gamma_{\mu}^\dagger = \gamma_{\mu} \quad (\mu = 0, 1, 2, 3), \quad (A.4)
\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3, \quad \{\gamma_5, \gamma_{\nu}\} = 0 \quad (\mu = 0, 1, 2, 3), \quad (A.5)
C_D = \gamma_2\gamma_0, \quad (A.6)
C_D\gamma_\mu C_D^{-1} = -\gamma_\mu^T, \quad C_D\gamma_5 C_D^{-1} = \gamma_5, \quad C_D^T = C_D^{-1} = C_D = -C_D. \quad (A.7)

B \quad SO(10) gamma matrices

SO(10) gamma matrices (The Clifford algebra of \(2^{11/2}\) dimensions):

\begin{align*}
\Gamma^1 &= \tau_1 \times \tau_1 \times \tau_1 \times \tau_1 \times \tau_1, \quad (B.1) \\
\Gamma^2 &= \tau_2 \times \tau_1 \times \tau_1 \times \tau_1 \times \tau_1, \quad (B.2) \\
\Gamma^3 &= \tau_3 \times \tau_1 \times \tau_1 \times \tau_1 \times \tau_1, \quad (B.3) \\
\Gamma^4 &= I \times \tau_2 \times \tau_1 \times \tau_1 \times \tau_1, \quad (B.4) \\
\Gamma^5 &= I \times \tau_3 \times \tau_1 \times \tau_1 \times \tau_1, \quad (B.5) \\
\Gamma^6 &= I \times I \times \tau_2 \times \tau_1 \times \tau_1, \quad (B.6) \\
\Gamma^7 &= I \times I \times \tau_3 \times \tau_1 \times \tau_1, \quad (B.7) \\
\Gamma^8 &= I \times I \times I \times \tau_2 \times \tau_1, \quad (B.8) \\
\Gamma^9 &= I \times I \times I \times \tau_3 \times \tau_1, \quad (B.9) \\
\Gamma^{10} &= I \times I \times I \times I \times \tau_2, \quad (B.10) \\
\Gamma^{11} &= I \times I \times I \times I \times \tau_3, \quad (B.11) \\
C &= i\tau_2 \times \tau_3 \times \tau_2 \times \tau_3 \times \tau_2. \quad (B.12)
\end{align*}

\begin{align*}
\Gamma^a \Gamma^b + \Gamma^b \Gamma^a &= 2\delta^{ab}; \quad \Gamma^a \dagger = \Gamma^a \quad (a = 1, \cdots, 10), \quad (B.13) \\
\Gamma^{11} &= -i\Gamma^1 \Gamma^2 \cdots \Gamma^{10}, \quad \{\Gamma^{11}, \Gamma^a\} = 0 \quad (a = 1, \cdots, 10), \quad (B.14) \\
C \Gamma^a C^{-1} &= -\{\Gamma^a\}^T, \quad C \Gamma^{11} C^{-1} = -\Gamma_{11}, \quad C^T = C^{-1} = C^{-\dagger} = -C. \quad (B.15)
\end{align*}

The T matrices

\begin{align*}
T^a &= C \Gamma^a; \quad T^a \dagger = T^a \quad (B.16)
\end{align*}
\[ T^1 = i(-i)(+i)(-i)(+i)\tau_3 \times \tau_2 \times \tau_3 \times \tau_2 \times \tau_3, \]
\[ T^2 = i(+1)(+i)(-i)(+i)I \times \tau_2 \times \tau_3 \times \tau_2 \times \tau_3, \]
\[ T^3 = i(+i)(+i)(-i)(+i)\tau_1 \times \tau_2 \times \tau_3 \times \tau_2 \times \tau_3, \]
\[ T^4 = i(+1)(-i)(-i)(+i)\tau_2 \times \tau_1 \times \tau_3 \times \tau_2 \times \tau_3, \]
\[ T^5 = i(+1)(+1)(-i)(+i)\tau_2 \times \tau_3 \times \tau_3 \times \tau_2 \times \tau_3, \]
\[ T^6 = i(+1)(+1)(+1)(-i)\tau_2 \times \tau_3 \times I \times \tau_2 \times \tau_3, \]
\[ T^7 = i(+1)(+1)(+i)(+i)\tau_2 \times \tau_3 \times \tau_1 \times \tau_2 \times \tau_3, \]
\[ T^8 = i(+1)(+1)(+1)(-i)\tau_2 \times \tau_3 \times \tau_2 \times \tau_1 \times \tau_3, \]
\[ T^9 = i(+1)(+1)(+1)(+1)\tau_2 \times \tau_3 \times \tau_2 \times I \times \tau_3, \]
\[ T^{10} = i(+1)(+1)(+1)(+1)\tau_2 \times \tau_3 \times \tau_2 \times \tau_3 \times I. \]

The reduced Clifford algebra of \(2^{[9/2]}\)

\[ \Gamma^{a} = \tilde{\Gamma}^{a'} \times \tau_1 \quad (a' = 1, \cdots, 9), \quad \text{(B.17)} \]
\[ C = \tilde{C} \times \tau_2. \quad \text{(B.18)} \]

The reduced \(T\) matrices

\[ T^{a'} = \tilde{T}^{a'} \times \tau_3, \quad \text{(B.19)} \]
\[ T^{10} = \tilde{T}^{10} \times I = \tilde{C} \times I. \quad \text{(B.20)} \]

\[ T^{10 \dagger} \Gamma^{a'} = \Gamma^{10} \Gamma^{a'} = -i \tilde{T}^{a'} \times \tau_3. \quad \text{(B.21)} \]

C Chiral basis in the weak coupling limit

\[ H = \gamma_5 (D_w - m_0) = \frac{1}{L^4} \sum_p e^{i p (x - y)} \begin{pmatrix} b(p) I & c(p) \\ c^\dagger(p) & -b(p) I \end{pmatrix}, \quad \text{(C.1)} \]

where

\[ b(p) = \{ \sum_\mu (1 - \cos p_\mu) - m_0 \}, \quad \text{(C.2)} \]
\[ c(p) = I \{ i \sin p_0 \} - \sum_k \sigma_k \sin p_k. \quad \text{(C.3)} \]
Orthonormal chiral basis: $j = (p, s)$

$$u_j(x) = e^{ipx} u(p, s); \quad u(0, s) = \begin{pmatrix} \chi_s \\ 0 \end{pmatrix}, \quad u(p, s) = \left( \frac{c\chi_s}{-(\omega + b)\chi_s} \right) / \sqrt{2\omega(\omega + b)} \quad (p \neq 0),$$

$$v_j(x) = e^{ipx} v(p, s); \quad v(0, s) = \begin{pmatrix} 0 \\ \chi_s \end{pmatrix}, \quad v(p, s) = \left( \frac{+(\omega + b)\chi_s}{c^\dagger\chi_s} \right) / \sqrt{2\omega(\omega + b)} \quad (p \neq 0),$$

where

$$\omega(p) = \sqrt{\sum \sin^2(p_\mu) + \left\{ \sum \mu \right\} (1 - \cos(p_\mu)) - m_0}^2.$$

Majorana-mass-type inner products:

$$\langle u_j^T, \gamma_5 C_D u_j' \rangle = \delta_{p+p',0} u_j^T(-p', s)\gamma_5 C_D u(p', s')$$

$$= \delta_{p+p',0} \chi_s^{T} \sigma_2 \chi_{s'}$$

$$= \delta_{p+p',0} \epsilon_{ss'},$$

Acknowledgments

The author would like to thank M. Sato, H. Fujii, M. Kato, Y. Okawa, T. Okuda for enlightening discussions. This work is supported in part by JSPS KAKENHI Grant Numbers 24540253, 16K05313 and 25287049.

References

[1] G. Aad et al. [ATLAS Collaboration], Phys. Lett. B 716, 1 (2012) doi:10.1016/j.physletb.2012.08.020 [arXiv:1207.7214 [hep-ex]].

[2] S. Chatrchyan et al. [CMS Collaboration], Phys. Lett. B 716, 30 (2012) doi:10.1016/j.physletb.2012.08.021 [arXiv:1207.7235 [hep-ex]].

[3] G. Aad et al. [ATLAS and CMS Collaborations], Phys. Rev. Lett. 114, 191803 (2015) doi:10.1103/PhysRevLett.114.191803 [arXiv:1503.07589 [hep-ex]].

[4] Y. Fukuda et al. [Super-Kamiokande Collaboration], Phys. Rev. Lett. 81, 1562 (1998) doi:10.1103/PhysRevLett.81.1562 [hep-ex/9807003].

[5] Q. R. Ahmad et al. [SNO Collaboration], Phys. Rev. Lett. 87, 071301 (2001) doi:10.1103/PhysRevLett.87.071301 [nucl-ex/0106015].

[6] Q. R. Ahmad et al. [SNO Collaboration], Phys. Rev. Lett. 89, 011301 (2002) doi:10.1103/PhysRevLett.89.011301 [nucl-ex/0204008].

[7] G. ’t Hooft, Phys. Rev. Lett. 37, 8 (1976).

[8] G. ’t Hooft, Phys. Rev. D 14, 3432 (1976) [Erratum-ibid. D 18, 2199 (1978)].
[9] S. Raby, S. Dimopoulos and L. Susskind, Nucl. Phys. B 169, 373 (1980).
[10] S. Dimopoulos, S. Raby and L. Susskind, Nucl. Phys. B 173, 208 (1980).
[11] G. ’t Hooft, PRINT-80-0083 (UTRECHT) Lecture given at Cargese Summer Inst., Cargese, France, Aug 26 - Sep 8, 1979
[12] B. Holdom, New J. Phys. 10, 053040 (2008) doi:10.1088/1367-2630/10/5/053040 [arXiv:0708.1057 [hep-ph]].
[13] B. Holdom, Phys. Lett. B 681, 287 (2009) doi:10.1016/j.physletb.2009.10.021 [arXiv:0907.0009 [hep-ph]].
[14] L. H. Karsten and J. Smit, Nucl. Phys. B 183, 103 (1981).
[15] H. B. Nielsen and M. Ninomiya, Nucl. Phys. B 185, 20 (1981) [Erratum-ibid. B 195, 541 (1982)].
[16] H. B. Nielsen and M. Ninomiya, Nucl. Phys. B 193, 173 (1981).
[17] D. Friedan, Commun. Math. Phys. 85, 481 (1982).
[18] M. Luscher, Nucl. Phys. Proc. Suppl. 83, 34 (2000) doi:10.1016/S0920-5632(00)91593-7 [hep-lat/9909150].
[19] M. Golterman, Nucl. Phys. Proc. Suppl. 94, 189 (2001) doi:10.1016/S0920-5632(01)00953-7 [hep-lat/0011027].
[20] H. Neuberger, doi:10.1142/9789812811455-0009 [hep-lat/0011035].
[21] M. Luscher, Subnucl. Ser. 38, 41 (2002) doi:10.1142/9789812778253-0002 [hep-th/0102028].
[22] D. B. Kaplan, arXiv:0912.2560 [hep-lat].
[23] P. H. Ginsparg and K. G. Wilson, Phys. Rev. D 25, 2649 (1982).
[24] H. Neuberger, Phys. Lett. B 417, 141 (1998) [arXiv:hep-lat/9707022].
[25] P. Hasenfratz, V. Laliena and F. Niedermayer, Phys. Lett. B 427, 125 (1998) [arXiv:hep-lat/9801021].
[26] H. Neuberger, Phys. Lett. B 427, 353 (1998) [arXiv:hep-lat/9801031].
[27] P. Hasenfratz, Nucl. Phys. B 525, 401 (1998) [arXiv:hep-lat/9802007].
[28] P. Hernandez, K. Jansen and M. Lüscher, Nucl. Phys. B 552, 363 (1999) [arXiv:hep-lat/9808010].
[29] R. Narayanan and H. Neuberger, Phys. Lett. B 302, 62 (1993) [arXiv:hep-lat/9212019].
[30] R. Narayanan and H. Neuberger, Nucl. Phys. B 412, 574 (1994) [arXiv:hep-lat/9307006].
[31] R. Narayanan and H. Neuberger, Phys. Rev. Lett. 71, 3251 (1993) [arXiv:hep-lat/9308011].
[32] R. Narayanan and H. Neuberger, Nucl. Phys. B 443, 305 (1995) [arXiv:hep-th/9411108].
[33] R. Narayanan, Nucl. Phys. Proc. Suppl. 34, 95 (1994) [arXiv:hep-lat/9311014].
[34] R. Narayanan and H. Neuberger, Nucl. Phys. Proc. Suppl. 34, 587 (1994) doi:10.1016/0920-5632(94)90453-7 [hep-lat/9311015].
[35] H. Neuberger, Found. Phys. 27, 93 (1997). doi:10.1007/BF02550158
[36] R. Narayanan, Phys. Rev. D 58, 097501 (1998) doi:10.1103/PhysRevD.58.097501 [hep-lat/9802018].
[63] V. Furman and Y. Shamir, Nucl. Phys. B 439, 54 (1995) [arXiv:hep-lat/9405004].
[64] T. Blum and A. Soni, Phys. Rev. D 56, 174 (1997) [arXiv:hep-lat/9611030].
[65] T. Blum and A. Soni, Phys. Rev. Lett. 79, 3595 (1997) [arXiv:hep-lat/9706023].
[66] P. M. Vranas, Phys. Rev. D 57, 1415 (1998) [arXiv:hep-lat/9705023].
[67] H. Neuberger, Phys. Rev. D 57, 5417 (1998) [arXiv:hep-lat/9710089].
[68] Y. Kikukawa and T. Noguchi, Nucl. Phys. Proc. Suppl. 83, 630 (2000) doi:10.1016/S0920-5632(00)91758-4 [hep-lat/9902022].
[69] M. Lüscher, Phys. Lett. B 428, 342 (1998) [arXiv:hep-lat/9802011].
[70] Y. Kikukawa and A. Yamada, Phys. Lett. B 448, 265 (1999) [arXiv:hep-lat/9806013].
[71] M. Lüscher, Nucl. Phys. B 538, 515 (1999) [arXiv:hep-lat/9808021].
[72] K. Fujikawa, Nucl. Phys. B 546, 480 (1999) [arXiv:hep-th/9811235].
[73] D. H. Adams, Annals Phys. 296, 131 (2002) [arXiv:hep-lat/9812003].
[74] H. Suzuki, Prog. Theor. Phys. 102, 141 (1999) [arXiv:hep-th/9812019].
[75] T. W. Chiou, Phys. Lett. B 445, 371 (1999) [arXiv:hep-lat/9809013].
[76] M. Lüscher, Nucl. Phys. B 549, 295 (1999) [arXiv:hep-lat/9811032].
[77] M. Lüscher, Nucl. Phys. B 568, 162 (2000) [arXiv:hep-lat/9904009].
[78] H. Suzuki, Prog. Theor. Phys. 101, 1147 (1999) [arXiv:hep-lat/9901012].
[79] H. Neuberger, Phys. Rev. D 63, 014503 (2001) [arXiv:hep-lat/0002032].
[80] D. H. Adams, Nucl. Phys. B 589, 633 (2000) [arXiv:hep-lat/0004015].
[81] H. Suzuki, Nucl. Phys. B 585, 471 (2000) [arXiv:hep-lat/0002009].
[82] H. Igarashi, K. Okuyama and H. Suzuki, arXiv:hep-lat/0012018.
[83] M. Lüscher, JHEP 0006, 028 (2000) [arXiv:hep-lat/0006014].
[84] T. Aoyama and Y. Kikukawa, arXiv:hep-lat/9905003.
[85] Y. Kikukawa and Y. Nakayama, Nucl. Phys. B 597, 519 (2001) [arXiv:hep-lat/0005015].
[86] Y. Kikukawa, Y. Nakayama and H. Suzuki, Nucl. Phys. Proc. Suppl. 106, 763 (2002) [arXiv:hep-lat/0111036].
[87] Y. Kikukawa, Phys. Rev. D 65, 074504 (2002) [arXiv:hep-lat/0105032].
[88] D. Kadoh, Y. Kikukawa and Y. Nakayama, JHEP 0412, 006 (2004) [arXiv:hep-lat/0309022].
[89] D. Kadoh and Y. Kikukawa, JHEP 0501, 024 (2005) [arXiv:hep-lat/0401025].
[90] Y. Kikukawa, lectures in ILFTN workshop on “Perspectives in Lattice QCD”; Nara, Oct. 31 – Nov. 11, 2005. The lecture notes are available from http://www2.ccs.tsukuba.ac.jp/workshop/iltf05/.
[91] D. Kadoh and Y. Kikukawa, JHEP 0802, 063 (2008) doi:10.1088/1126-6708/2008/02/063 [arXiv:0709.3656 [hep-lat]].
[92] D. Kadoh and Y. Kikukawa, JHEP 0805, 095 (2008) Erratum: [JHEP 1103, 095 (2011)] doi:10.1088/1126-6708/2008/05/095, 10.1007/JHEP03(2011)095 [arXiv:0709.3658 [hep-lat]].
[93] S. L. Glashow, Nucl. Phys. 22, 579 (1961).
[94] S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967).
[95] A. Salam, Originally printed in "Svartholm: Elementary Particle Theory, Proceedings Of The Nobel Symposium Held 1968 At Lerum, Sweden", Stockholm 1968, 367-377
[96] S. A. Frolov and A. A. Slavnov, Phys. Lett. B 309, 344 (1993).
doi:10.1016/0370-2693(93)90943-C
[97] S. Aoki and Y. Kikukawa, Mod. Phys. Lett. A 8, 3517 (1993)
doi:10.1142/S0217732393002257 [hep-th/9306067].
[98] E. Eichten and J. Preskill, Nucl. Phys. B 268, 179 (1986).
doi:10.1016/0550-3213(86)90207-5
[99] M. F. L. Golterman, D. N. Petcher and E. Rivas, Nucl. Phys. B 395, 596 (1993)
doi:10.1016/0550-3213(93)90049-U [hep-lat/9206010].
[100] T. Bhattacharya, M. R. Martin and E. Poppitz, Phys. Rev. D 74 (2006) 085028
doi:10.1103/PhysRevD.74.085028 [hep-lat/0605003].
[101] J. Giedt and E. Poppitz, JHEP 0710, 076 (2007) doi:10.1088/1126-6708/2007/10/076
[hep-lat/0701004].
[102] E. Poppitz and Y. Shang, JHEP 0708, 081 (2007) doi:10.1088/1126-6708/2007/08/081
[arXiv:0706.1043 [hep-th]].
[103] E. Poppitz and Y. Shang, Int. J. Mod. Phys. A 23, 4545 (2008)
doi:10.1142/S0217751X08041281 [arXiv:0801.0587 [hep-lat]].
[104] E. Poppitz and Y. Shang, JHEP 0903, 103 (2009) doi:10.1088/1126-6708/2009/03/103
[arXiv:0901.3402 [hep-lat]].
[105] E. Poppitz and Y. Shang, Int. J. Mod. Phys. A 25, 2761 (2010)
doi:10.1142/S0217751X10049852 [arXiv:1003.5896 [hep-lat]].
[106] C. Chen, J. Giedt and E. Poppitz, JHEP 1304, 131 (2013) doi:10.1007/JHEP04(2013)131
[arXiv:1211.6947 [hep-lat]].
[107] J. Giedt, C. Chen and E. Poppitz, PoS LATTICE 2013, 131 (2014) [arXiv:1403.5146 [hep-lat]].
[108] M. Creutz, M. Tytgat, C. Rebbi and S. S. Xue, Phys. Lett. B 402, 341 (1997)
[arXiv:hep-lat/9612017].
[109] H. Neuberger, Phys. Lett. B 413, 387 (1997) doi:10.1016/S0370-2693(97)01132-5
[hep-lat/9705022].
[110] V. Ayyar and S. Chandrasekharan, Phys. Rev. D 91, no. 6, 065035 (2015)
doi:10.1103/PhysRevD.91.065035 [arXiv:1410.6474 [hep-lat]].
[111] Y. BenTov, JHEP 1507, 034 (2015) doi:10.1007/JHEP07(2015)034 [arXiv:1412.0154
[cond-mat.str-el]].
[112] Y. BenTov and A. Zee, Phys. Rev. D 93, no. 6, 065036 (2016)
doi:10.1103/PhysRevD.93.065036 [arXiv:1505.04312 [hep-th]].
[113] V. Ayyar and S. Chandrasekharan, Phys. Rev. D 93, no. 8, 081701 (2016)
doi:10.1103/PhysRevD.93.081701 [arXiv:1511.09071 [hep-lat]].
[114] V. Ayyar and S. Chandrasekharan, JHEP 1610, 058 (2016) doi:10.1007/JHEP10(2016)058 [arXiv:1606.06312 [hep-lat]].
[115] S. Catterall, JHEP 1601, 121 (2016) doi:10.1007/JHEP01(2016)121 [arXiv:1510.04153 [hep-lat]].
[116] S. Catterall and D. Schaich, arXiv:1609.08541 [hep-lat].
[117] S. Catterall and N. Butt, arXiv:1708.06715 [hep-lat].
[118] D. Schaich and S. Catterall, arXiv:1710.08137 [hep-lat].
[119] P. Gerhold and K. Jansen, JHEP 0709, 041 (2007) [arXiv:0705.2539 [hep-lat]].
[120] P. Gerhold and K. Jansen, JHEP 0710, 001 (2007) [arXiv:0707.3849 [hep-lat]].
[121] M. Creutz and I. Horvath, Phys. Rev. D 50, 2297 (1994) doi:10.1103/PhysRevD.50.2297 [hep-lat/9402013].
[122] X. L. Qi, T. Hughes and S. C. Zhang, Phys. Rev. B 78, 195424 (2008) doi:10.1103/PhysRevB.78.195424 [arXiv:0802.3537 [cond-mat.mes-hall]].
[123] X. G. Wen, Chin. Phys. Lett. 30, 111101 (2013) doi:10.1088/0256-307X/30/11/111101 [arXiv:1305.1045 [hep-lat]].
[124] J. Wang and X. G. Wen, arXiv:1307.7480 [hep-lat].
[125] Y. You, Y. BenTov and C. Xu, arXiv:1402.4151 [cond-mat.str-el].
[126] Y. Z. You and C. Xu, Phys. Rev. B 91, no. 12, 125147 (2015) doi:10.1103/PhysRevB.91.125147 [arXiv:1412.4784 [cond-mat.str-el]].
[127] M. DeMarco and X. G. Wen, arXiv:1706.04648 [hep-lat].
[128] P. V. D. Swift, Phys. Lett. 145B, 256 (1984). doi:10.1016/0370-2693(84)90350-2
[129] J. Smit, Acta Phys. Polon. B 17, 531 (1986).
[130] S. Aoki, Phys. Rev. Lett. 60, 2109 (1988). doi:10.1103/PhysRevLett.60.2109
[131] S. Aoki, Phys. Rev. D 38, 618 (1988). doi:10.1103/PhysRevD.38.618
[132] K. Funakubo and T. Kashiwa, Phys. Rev. Lett. 60, 2113 (1988). doi:10.1103/PhysRevLett.60.2113
[133] K. Funakubo and T. Kashiwa, Phys. Rev. D 38, 2602 (1988). doi:10.1103/PhysRevD.38.2602
[134] M. F. L. Golterman and D. N. Petcher, Nucl. Phys. B 359 (1991) 91. doi:10.1016/0550-3213(91)90294-8
[135] M. F. L. Golterman, D. N. Petcher and J. Smit, Nucl. Phys. B 370, 51 (1992). doi:10.1016/0550-3213(92)90344-B
[136] W. Bock, A. K. De and J. Smit, Nucl. Phys. B 388, 243 (1992). doi:10.1016/0550-3213(92)90551-L
[137] W. Bock, A. K. De, E. Focht and J. Smit, Nucl. Phys. B 401, 481 (1993) doi:10.1016/0550-3213(93)90311-C [hep-lat/9210022].
[138] S. Aoki, I. H. Lee and R. E. Shrock, Phys. Rev. D 45, 13 (1992). doi:10.1103/PhysRevD.45.13
[139] I. Montvay, Phys. Lett. B 199, 89 (1987). doi:10.1016/0370-2693(87)91468-7
[140] I. Montvay, Phys. Lett. B 205, 315 (1988). doi:10.1016/0370-2693(88)91671-1
[141] K. Farakos, G. Koutsoumbas, L. Lin, J. P. Ma, I. Montvay and G. Munster, Nucl. Phys. B 350, 474 (1991). doi:10.1016/0550-3213(91)90268-3
[142] L. Lin, I. Montvay, H. Wittig and G. Munster, Nucl. Phys. B 355, 511 (1991). doi:10.1016/0550-3213(91)90124-G
[143] L. Lin, I. Montvay, H. Wittig and G. Munster, Nucl. Phys. Proc. Suppl. 20, 601 (1991). doi:10.1016/0920-5632(91)90312-G
[144] G. Munster, L. Lin, M. Plagge, I. Montvay and H. Wittig, Nucl. Phys. Proc. Suppl. 26, 489 (1992). doi:10.1016/0920-5632(92)90124-G
[145] L. Lin, I. Montvay and H. Wittig, Phys. Lett. B 264, 407 (1991). doi:10.1016/0370-2693(91)90369-2
[146] I. Montvay, Nucl. Phys. Proc. Suppl. 26, 57 (1992). doi:10.1016/0920-5632(92)90229-L
[147] I. Montvay, Nucl. Phys. Proc. Suppl. 29BC, 159 (1992) doi:10.1016/0920-5632(92)90017-M [hep-lat/9205023].
[148] L. Lin, G. Munster, M. Plagge, I. Montvay, H. Wittig, C. Frick and T. Trappenberg, Nucl. Phys. Proc. Suppl. 30, 647 (1993) doi:10.1016/0920-5632(93)90294-G [hep-lat/9212015].
[149] L. Lin, I. Montvay, G. Munster, M. Plagge and H. Wittig, Phys. Lett. B 317, 143 (1993) doi:10.1016/0370-2693(93)91584-A [hep-lat/9303012].
[150] D. M. Grabowska and D. B. Kaplan, Phys. Rev. Lett. 116, no. 21, 211602 (2016) doi:10.1103/PhysRevLett.116.211602 [arXiv:1511.03649 [hep-lat]].
[151] D. M. Grabowska and D. B. Kaplan, Phys. Rev. D 94, no. 11, 114504 (2016) doi:10.1103/PhysRevD.94.114504 [arXiv:1610.02151 [hep-lat]].
[152] D. B. Kaplan and D. M. Grabowska, PoS LATTICE 2016, 018 (2016).
[153] H. Fukaya, T. Onogi, S. Yamamoto and R. Yamamura, PTEP 2017, no. 3, 033B06 (2017) doi:10.1093/ptep/ptx017 [arXiv:1607.06174 [hep-th]].
[154] K. i. Okumura and H. Suzuki, PTEP 2016, no. 12, 123B07 (2016) doi:10.1093/ptep/ptw167 [arXiv:1608.02217 [hep-lat]].
[155] H. Makino and O. Morikawa, PTEP 2016, no. 12, 123B06 (2016) doi:10.1093/ptep/ptw183 [arXiv:1609.08376 [hep-lat]].
[156] H. Makino, O. Morikawa and H. Suzuki, PTEP 2017, no. 6, 063B08 (2017) doi:10.1093/ptep/ptx085 [arXiv:1704.04862 [hep-lat]].
[157] Y. Hamada and H. Kawai, PTEP 2017, no. 6, 063B09 (2017) doi:10.1093/ptep/ptx086 [arXiv:1705.01317 [hep-lat]].
[158] G. Parisi, Phys. Lett. B 131, 393 (1983).
[159] J. R. Klauder, J. Phys. A 16, L317 (1983).
[160] J. R. Klauder, Phys. Rev. A 29, 2036 (1984).
[161] J. Ambjorn and S. K. Yang, Phys. Lett. B 165, 140 (1985).
[162] J. Ambjorn, M. Flensburg and C. Peterson, Nucl. Phys. B 275, 375 (1986).
[163] J. Berges, S. Borsanyi, D. Sexty and I. -O. Stamatescu, Phys. Rev. D 75, 045007 (2007) [hep-lat/0609058].
[164] J. Berges and D. Sexty, Nucl. Phys. B 799, 306 (2008) [arXiv:0708.0779 [hep-lat]].
[165] G. Aarts and I.-O. Stamatescu, JHEP 0809, 018 (2008) [arXiv:0807.1597 [hep-lat]].
[166] G. Aarts, Phys. Rev. Lett. 102, 131601 (2009) [arXiv:0810.2089 [hep-lat]].
[167] G. Aarts, F. A. James, E. Seiler and I.-O. Stamatescu, Phys. Lett. B 687, 154 (2010) [arXiv:0912.0617 [hep-lat]].
[168] G. Aarts, E. Seiler and I.-O. Stamatescu, Phys. Rev. D 81, 054508 (2010) [arXiv:0912.3360 [hep-lat]].
[169] G. Aarts and F. A. James, JHEP 1008, 020 (2010) [arXiv:1005.3468 [hep-lat]].
[170] G. Aarts, F. A. James, E. Seiler and I.-O. Stamatescu, Eur. Phys. J. C 71, 1756 (2011) [arXiv:1101.3270 [hep-lat]].
[171] G. Aarts and F. A. James, JHEP 1201, 118 (2012) [arXiv:1112.4655 [hep-lat]].
[172] E. Seiler, D. Sexty and I.-O. Stamatescu, Phys. Lett. B 723, 213 (2013) [arXiv:1211.3709 [hep-lat]].
[173] J. M. Pawlowski and C. Zielinski, Phys. Rev. D 87, 094503 (2013) [arXiv:1302.1622 [hep-lat]].
[174] J. M. Pawlowski and C. Zielinski, Phys. Rev. D 87, 094509 (2013) [arXiv:1302.2249 [hep-lat]].
[175] G. Aarts, L. Bongiovanni, E. Seiler, D. Sexty and I.-O. Stamatescu, arXiv:1303.6425 [hep-lat].
[176] D. Sexty, “Simulating full QCD at nonzero density using the complex Langevin equation,” arXiv:1307.7748 [hep-lat].
[177] G. Aarts, “Lefschetz thimbles and stochastic quantisation: Complex actions in the complex plane,” arXiv:1308.4811 [hep-lat].
[178] P. Giudice, G. Aarts and E. Seiler, arXiv:1309.3191 [hep-lat].
[179] A. Mollgaard and K. Splittorff, Phys. Rev. D 88, no. 11, 116007 (2013) doi:10.1103/PhysRevD.88.116007 [arXiv:1309.4335 [hep-lat]].
[180] D. Sexty, Nucl. Phys. A 931, 856 (2014) doi:10.1016/j.nuclphysa.2014.09.029 [arXiv:1408.6767 [hep-lat]].
[181] T. Hayata and A. Yamamoto, Phys. Rev. A 92, no. 4, 043628 (2015) doi:10.1103/PhysRevA.92.043628 [arXiv:1411.5195 [cond-mat.quant-gas]].
[182] K. Splittorff, Phys. Rev. D 91, no. 3, 034507 (2015) doi:10.1103/PhysRevD.91.034507 [arXiv:1412.0502 [hep-lat]].
[183] G. Aarts, E. Seiler, D. Sexty and I. O. Stamatescu, PoS CPOD 2014, 060 (2015) [arXiv:1503.08813 [hep-lat]].
[184] Z. Fodor, S. D. Katz, D. Sexty and C. Tőrő, Phys. Rev. D 92, no. 9, 094516 (2015) doi:10.1103/PhysRevD.92.094516 [arXiv:1508.05260 [hep-lat]].
[185] L. L. Salcedo, Phys. Rev. D 94, no. 7, 074503 (2016) doi:10.1103/PhysRevD.94.074503 [arXiv:1510.09064 [hep-lat]].
[186] T. Hayata, Y. Hidaka and Y. Tanizaki, Nucl. Phys. B 911, 94 (2016) doi:10.1016/j.nuclphysb.2016.07.031 [arXiv:1511.02437 [hep-lat]].
[187] D. Li, arXiv:1605.04623 [hep-lat].
[188] G. Aarts, F. Attanasio, B. Jünger and D. Sexty, JHEP 1609, 087 (2016) doi:10.1007/JHEP09(2016)087 [arXiv:1606.05561 [hep-lat]].
[189] Y. Abe and K. Fukushima, Phys. Rev. D 94, no. 9, 094506 (2016) doi:10.1103/PhysRevD.94.094506 [arXiv:1607.05436 [hep-lat]].
[190] Y. Ito and J. Nishimura, JHEP 1612, 009 (2016) doi:10.1007/JHEP12(2016)009 [arXiv:1609.04501 [hep-lat]].
[191] L. L. Salcedo, Phys. Rev. D 94, no. 11, 114505 (2016) doi:10.1103/PhysRevD.94.114505 [arXiv:1611.06390 [hep-lat]].
[192] G. Aarts, E. Seiler, D. Sexty and I. O. Stamatescu, JHEP 1705, 044 (2017) doi:10.1007/JHEP05(2017)044 [arXiv:1701.02322 [hep-lat]].
[193] H. Fujii, S. Kamata and Y. Kikukawa, arXiv:1710.08524 [hep-lat].
[194] F. Pham, “Vanishing homologies and the n variable saddlepoint method”, in Proc. Symp. Pure Math, vol. 40.2, pp. 319Â–333. AMS, 1983
[195] D. Kaminski, “Exponentially improved stationary phase approximations for double integrals”, Methods and Appl. of Analysis 1 (1994) 44Â±56.
[196] C. J. Howls, “Hyperasymptotics for multidimensional integrals, exact remainder terms and the global connection problem”, Proc. R. Soc. A 453 no. 1966, (1997) 2271Â±2294.
[197] E. Witten, “Analytic Continuation Of Chern-Simons Theory,” arXiv:1001.2933 [hep-th].
[198] M. Cristoforetti et al. [AuroraScience Collaboration], Phys. Rev. D 86, 074505 (2012) [arXiv:1205.3996 [hep-lat]].
[199] M. Cristoforetti, L. Scorzato and F. Di Renzo, arXiv:1210.8026 [hep-lat].
[200] M. Cristoforetti, F. Di Renzo, A. Mukherjee and L. Scorzato, arXiv:1303.7204 [hep-lat].
[201] A. Mukherjee, M. Cristoforetti and L. Scorzato, arXiv:1308.0233 [physics.comp-ph].
[202] H. Fujii, D. Honda, M. Kato, Y. Kikukawa, S. Komatsu and T. Sano, JHEP 1310, 147 (2013) doi:10.1007/JHEP10(2013)147 [arXiv:1309.4371 [hep-lat]].
[203] A. Cherman, D. Dorion and M. Ünsal, JHEP 1510, 056 (2015) doi:10.1007/JHEP10(2015)056 [arXiv:1403.1277 [hep-th]].
[204] M. Cristoforetti, F. Di Renzo, G. Eruzzi, A. Mukherjee, C. Schmidt, L. Scorzato and C. Torrero, Phys. Rev. D 89, no. 11, 114505 (2014) doi:10.1103/PhysRevD.89.114505 [arXiv:1403.5637 [hep-lat]].
[205] A. Mukherjee and M. Cristoforetti, Phys. Rev. B 90, no. 3, 035134 (2014) doi:10.1103/PhysRevB.90.035134 [arXiv:1403.5680 [cond-mat.str-el]].
[206] G. Aarts, L. Bongiovanni, E. Seiler and D. Sexty, JHEP 1410, 159 (2014) doi:10.1007/JHEP10(2014)159 [arXiv:1407.2090 [hep-lat]].
[207] Y. Tanizaki and T. Koike, Annals Phys. 351, 250 (2014) doi:10.1016/j.aop.2014.09.003 [arXiv:1406.2386 [math-ph]].
[208] H. Nishimura, M. C. Ogilvie and K. Pangeni, Phys. Rev. D 91, no. 5, 054004 (2015) doi:10.1103/PhysRevD.91.054004 [arXiv:1411.4959 [hep-ph]].
[209] Y. Tanizaki, Phys. Rev. D 91, no. 3, 036002 (2015) doi:10.1103/PhysRevD.91.036002 [arXiv:1412.1891 [hep-th]].

[210] T. Kanazawa and Y. Tanizaki, JHEP 1503, 044 (2015) doi:10.1007/JHEP03(2015)044 [arXiv:1412.2802 [hep-th]].

[211] A. Behtash, T. Sulejmanpasic, T. Schäfer and M. Ünsal, Phys. Rev. Lett. 115, no. 4, 041601 (2015) doi:10.1103/PhysRevLett.115.041601 [arXiv:1502.06624 [hep-th]].

[212] Y. Tanizaki, H. Nishimura and K. Kashiwa, Phys. Rev. D 91, no. 10, 101701 (2015) doi:10.1103/PhysRevD.91.101701 [arXiv:1504.02979 [hep-th]].

[213] F. Di Renzo and G. Eruzzi, Phys. Rev. D 92, no. 8, 085030 (2015) doi:10.1103/PhysRevD.92.085030 [arXiv:1507.03858 [hep-lat]].

[214] A. Behtash, E. Poppitz, T. Sulejmanpasic and M. Ünsal, JHEP 1511, 175 (2015) doi:10.1007/JHEP11(2015)175 [arXiv:1507.04063 [hep-th]].

[215] K. Fukushima and Y. Tanizaki, PTEP 2015, no. 11, 111A01 (2015) doi:10.1093/ptep/ptv152 [arXiv:1507.07351 [hep-th]].

[216] Y. Tanizaki, Y. Hidaka and T. Hayata, New J. Phys. 18, no. 3, 033002 (2016) doi:10.1088/1367-2630/18/3/033002 [arXiv:1509.07146 [hep-th]].

[217] H. Fujii, S. Kamata and Y. Kikukawa, JHEP 1511, 078 (2015) Erratum: [JHEP 1602, 036 (2016)] doi:10.1007/JHEP02(2016)036, 10.1007/JHEP11(2015)078 [arXiv:1509.08176 [hep-lat]].

[218] H. Fujii, S. Kamata and Y. Kikukawa, JHEP 1512, 125 (2015) Erratum: [JHEP 1609, 172 (2016)] doi:10.1007/JHEP12(2015)125, 10.1007/JHEP09(2016)172 [arXiv:1509.09141 [hep-lat]].

[219] A. Behtash, G. V. Dunne, T. Schäfer, T. Sulejmanpasic and M. Ünsal, Phys. Rev. Lett. 116, no. 1, 011601 (2016) doi:10.1103/PhysRevLett.116.011601 [arXiv:1510.00978 [hep-th]].

[220] A. Alexandru, G. Basar and P. Bedaque, Phys. Rev. D 93, no. 1, 014504 (2016) doi:10.1103/PhysRevD.93.014504 [arXiv:1510.03258 [hep-lat]].

[221] A. Behtash, G. V. Dunne, T. Schäfer, T. Sulejmanpasic and M. Ünsal, Annals of Mathematical Sciences and Applications Volume 2, No. 1 (2017) doi:10.4310/AMSA.2017.v2.n1.a3 [arXiv:1510.03435 [hep-th]].

[222] L. Scorzato, PoS LATTICE 2015, 016 (2016) [arXiv:1512.08039 [hep-lat]].

[223] A. Alexandru, G. Basar, P. F. Bedaque, G. W. Ridgway and N. C. Warrington, JHEP 1605, 053 (2016) doi:10.1007/JHEP05(2016)053 [arXiv:1512.08764 [hep-lat]].

[224] A. Alexandru, G. Basar, P. F. Bedaque, G. W. Ridgway and N. C. Warrington, Phys. Rev. D 93, no. 9, 094514 (2016) doi:10.1103/PhysRevD.93.094514 [arXiv:1604.00956 [hep-lat]].

[225] A. Alexandru, G. Basar, P. F. Bedaque, S. Vartak and N. C. Warrington, Phys. Rev. Lett. 117, no. 8, 081602 (2016) doi:10.1103/PhysRevLett.117.081602 [arXiv:1605.08040 [hep-lat]].

[226] A. Alexandru, G. Basar, P. Bedaque, G. W. Ridgway and N. C. Warrington, Phys. Rev. D 94, no. 4, 045017 (2016) doi:10.1103/PhysRevD.94.045017 [arXiv:1606.02742 [hep-lat]].

[227] T. Fujimori, S. Kamata, T. Misumi, M. Nitta and N. Sakai, Phys. Rev. D 94, no. 10, 105002 (2016) doi:10.1103/PhysRevD.94.105002 [arXiv:1607.04205 [hep-th]].
[228] A. Alexandru, G. Basar, P. F. Bedaque, G. W. Ridgway and N. C. Warrington, Phys. Rev. D 95, no. 1, 014502 (2017) doi:10.1103/PhysRevD.95.014502 [arXiv:1609.01730 [hep-lat]].
[229] Y. Tanizaki and M. Tachibana, JHEP 1702, 081 (2017) doi:10.1007/JHEP02(2017)081 [arXiv:1612.06529 [hep-th]].
[230] T. Fujimori, S. Kamata, T. Misumi, M. Nitta and N. Sakai, Phys. Rev. D 95, no. 10, 105001 (2017) doi:10.1103/PhysRevD.95.105001 [arXiv:1702.00589 [hep-th]].
[231] M. Fukuma and N. Umeda, PTEP 2017, no. 7, 073B01 (2017) doi:10.1093/ptep/ptx081 [arXiv:1703.00861 [hep-th]].
[232] A. Alexandru, G. Basar, P. F. Bedaque and N. C. Warrington, arXiv:1703.02414 [hep-lat].
[233] J. Nishimura and S. Shimasaki, JHEP 1706, 023 (2017) doi:10.1007/JHEP06(2017)023 [arXiv:1703.09409 [hep-th]].
[234] Y. Mori, K. Kashiwa and A. Ohnishi, arXiv:1705.03646 [hep-lat].
[235] T. Fujimori, S. Kamata, T. Misumi, M. Nitta and N. Sakai, PTEP 2017, no. 8, 083B02 (2017) doi:10.1093/ptep/ptx101 [arXiv:1705.10483 [hep-th]].
[236] Y. Tanizaki, H. Nishimura and J. J. M. Verbaarschot, arXiv:1706.03822 [hep-lat].
[237] P. F. Bedaque, arXiv:1711.05868 [hep-lat].
[238] T. Banks, Phys. Lett. B 272, 75 (1991). doi:10.1016/0370-2693(91)91015-N
[239] T. Banks and A. Dabholkar, Phys. Rev. D 46, 4016 (1992) doi:10.1103/PhysRevD.46.4016 [hep-lat/9204017].
[240] M. Golterman and Y. Shamir, Phys. Rev. D 67, 014501 (2003) doi:10.1103/PhysRevD.67.014501 [hep-th/0202162].
[241] H. Suzuki, JHEP 0010, 039 (2000) [arXiv:hep-lat/0009036].
[242] K. Fujikawa, M. Ishibashi and H. Suzuki, JHEP 0204, 046 (2002) [arXiv:hep-lat/0203016].
[243] K. Fujikawa and H. Suzuki, Phys. Rev. D 67, 034506 (2003) [arXiv:hep-lat/0210013].
[244] J. Smit and J. C. Vink, Nucl. Phys. B 286, 485 (1987). doi:10.1016/0550-3213(87)90451-2
[245] A. Gonzalez-Arroyo, In *Peniscola 1997, Advanced school on non-perturbative quantum field physics* 57-91 [hep-th/9807108].
[246] M. Hamanaka and H. Kajiura, Phys. Lett. B 551, 360 (2003) doi:10.1016/S0370-2693(02)03073-3 [hep-th/0208059].
[247] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Oxford University Press, 2nd edition, 1993.
[248] C. Itzykson and J.-M. Drouffe, *Statistical field theory* vol. 1, Cambridge University Press, 1989.
[249] H. Neuberger, Phys. Lett. B 437, 117 (1998) [arXiv:hep-lat/9805027].
[250] O. Bar, Nucl. Phys. B 650, 522 (2003) [arXiv:hep-lat/0209098].
[251] O. Bar and I. Campos, Nucl. Phys. B 581, 499 (2000) [arXiv:hep-lat/0001025].
[252] N. Cabibbo, Phys. Rev. Lett. 10, 531 (1963). doi:10.1103/PhysRevLett.10.531
[253] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. 49, 652 (1973).
[254] B. Pontecorvo, Sov. Phys. JETP 7, 172 (1958) [Zh. Eksp. Teor. Fiz. 34, 247 (1957)].
[255] Z. Maki, M. Nakagawa and S. Sakata, Prog. Theor. Phys. 28, 870 (1962). doi:10.1143/PTP.28.870

[256] Y. Kikukawa, “Gapped boundary phases of 1-4D TSCs in terms of Overlap/Domain wall fermions”, in preparation.

[257] Y. Kikukawa, “Why is the mission impossible? – Decoupling the mirror Ginsparg-Wilson fermions in the lattice models for two-dimensional abelian chiral gauge theories,” arXiv:1710.11101 [hep-lat].