Uniqueness for contagious McKean–Vlasov systems in the weak feedback regime

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Abstract

We present a simple uniqueness argument for a collection of McKean–Vlasov problems that have seen recent interest. Our first result shows that, in the weak feedback regime, there is global uniqueness for a very general class of random drivers. By weak feedback we mean the case where the contagion parameters are small enough to prevent blow-ups in solutions. Next, we specialise to a Brownian driver and show how the same techniques can be extended to give short-time uniqueness after blow-ups, regardless of the feedback strength. The heart of our approach is a surprisingly simple probabilistic comparison argument that is robust in the sense that it does not ask for any regularity of the solutions.

1 Introduction

In this paper we study uniqueness of the McKean–Vlasov problem

\[
\begin{aligned}
X_t &= X_0 + Z_t - \alpha f(L_t) \\
\tau &= \inf\{t \geq 0 : X_t \leq 0\} \\
L_t &= \mathbb{P}(\tau \leq t),
\end{aligned}
\]

where \(X_0 \in (0, \infty)\) is a random start point, \(Z\) is an independent continuous stochastic process, \(\alpha \in \mathbb{R}\) is a feedback parameter, and \(f : [0,1] \to \mathbb{R}\) is a continuous function. By a solution to this problem we mean an increasing càdlàg function \(L : [0,\infty) \to [0,1]\) that satisfies (MV) and is initially zero. The law of \(X_0\) is denoted \(\nu_0\) with density \(V_0\).

Recently, variants of (MV) have been studied in [15, 16, 20, 22, 23] motivated by the study of contagion in large financial markets, and in [7, 8, 17] motivated by nonlinear integrate-and-fire models for large networks of electrically coupled neurons. In this respect, our main interest is the case \(Z_t = \rho B_t + \sqrt{1 - \rho^2} \beta_t\), where \(B_t\) is a Brownian
motion and \( \beta_t \) is a fixed Brownian path: uniqueness of the deterministic \( L \) in \( \text{(MV)} \) would then give pathwise uniqueness of the stochastic \( L \) in the conditional McKean–Vlasov problem

\[
\begin{align*}
X_t &= X_0 + \rho B_t + \sqrt{1 - \rho^2} B_0^t - \alpha L_t \\
\tau &= \inf\{t \geq 0 : X_t \leq 0\} \\
L_t &= \mathbb{P}(\tau \leq t \mid B_0)
\end{align*}
\] (CMV)

where \( B \) and \( B_0 \) are independent Brownian motions. In \cite{20}, ‘relaxed’ solutions to \( \text{(CMV)} \), for which the adaptedness of \( L \) to \( B_0 \) is relaxed, are obtained as limit points of the following finite particle system: \( N \) particles move according to Brownian motions correlated through \( B_0 \), except that when a particle hits the origin, it is absorbed, and this then has a contagious feedback effect that causes all the other particles to jump down by \( \alpha/N \) with \( \alpha > 0 \) — possibly leading to more particles being absorbed and hence further rounds of downward jumps.

If, as in \cite{20}, each particle measures the ‘distance-to-default’ of a financial entity, and absorption at zero corresponds to default, then such a feedback loop could model a cascade of bankruptcies caused by the interplay between default contagion \( (\alpha > 0) \) and common exposures \( (\rho > 0) \). Similarly to \cite{16}, each particle could also model the log leverage ratio of a bank — defined as the log of capital over assets — for which a minimum value is enforced by regulation. When reaching some lower threshold above this minimum, the distressed banks must sell assets to increase their leverage ratios, and if part of these sales pertain to common illiquid assets \( (\rho > 0) \), then it will depress the price of these, hence causing a drop in the leverage ratios of all the other banks \( (\alpha > 0) \). Note, however, that a bank reaching the threshold should now be reset to some higher leverage ratio (after the selling of assets) instead of defaulting, but the main mathematical difficulty is still the positive feedback from hitting a boundary.

By a simple sign change, this latter system can be rephrased as a model for the ‘spiking’ of electrically coupled neurons. In this case, each particle models the membrane potential of a neuron, and when this potential reaches an upper threshold the neuron is said to ‘spike’: that is, it emits an electrical signal which causes all the other potentials to increase by \( \alpha/N \) and the spiking neuron itself is reset to a predetermined value. This neuronal system is the one studied in \cite{7,8}, only without a common noise.

**Blow-ups and the physical jump condition**

As a result of the positive feedback loops, for large enough \( \alpha > 0 \), the solutions to \( \text{(CMV)} \) can develop jump discontinuities — which we call blow-ups as in \cite{15,20}. In order to study uniqueness in these cases, it is necessary to resolve ambiguity at the blow-ups (see \cite{15} Prop. 1.2 & Sect. 2) for illustrations and discussion). This is achieved by specifying that the appertaining jump sizes must satisfy the physical jump condition

\[
\Delta L_t = \inf\{x > 0 : \nu_\tau([0, \alpha x]) < x\},
\] (PJC)

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with probability 1, where \( \nu_t(S) := \mathbb{P}(X_t \in S, t < \tau \mid B^0) \). In short, (PJC) is the correct specification of jump sizes for two reasons: First, in the case of (CMV), the condition gives the minimal jump sizes that are necessary for \( L \) to be càdlàg \([20, \text{Prop. 3.3}]\). Secondly, the solutions constructed in \([8, 20]\) are obtained as limit points of finite particle systems (as described above), for which the corresponding discrete version of (PJC) gives the only sensible physical description of the jump behaviour, and this discrete condition then yields (PJC) in the limit. It should also be noted that generic (possibly non-physical) solutions to (MV) can be constructed directly from a generalised Schauder fixed point theorem as proved by Nadtochiy and Shkolnikov \([23]\) under suitable conditions on \( Z \) and \( f \). For (MV), the condition (PJC) should be understood in terms of \( \nu_t = \mathbb{P}(X_t \in \cdot, t < \tau) \), corresponding to a pathwise realisation of \( B^0 \) in (CMV).

With the physical jump condition in place, it is then natural to only consider initial densities that do not want to jump immediately (in alignment with \( L \) being càdlàg and initially zero). Indeed, it is natural to restrict solely to states that could be reached by the evolving system and, recalling that (PJC) is a rule for the left limit \( \nu_t - \), this translates as those initial conditions \( \nu_0 \) satisfying

\[
\inf\{x > 0 : \nu_0([0, \alpha x]) < x\} = 0, \tag{1.1}
\]

in the case \( f(x) = x \) that is. While this condition is necessary for càdlàg solutions under Brownian drivers as in (CMV), it is possible to construct càdlàg solutions to (MV) violating this condition for more general \( Z \) (see \([23, (3.15)]\)). Specifically, one can take \( Z_t := B_t + A_t \) with \( A_t := \alpha \mathbb{P}(X_0 + \inf_{s < t} B_s \leq 0) \) and \( f(x) = x \). If \( X_0 \) is such that (1.1) is positive, and it is not imposed that positivity in (1.1) should cause \( L \) to jump immediately, then \( L := A \) gives a continuous solution to (MV) despite violating (1.1).

In this paper, the only mathematical contributions concerning jumps will be for the case when \( Z \) is a standard Brownian motion and \( f(x) = x \) (see Section 3). If considering a generic feedback function \( f \), the condition corresponding to (PJC) is simply

\[
\Delta L_t = \inf\{x > 0 : \nu_t - ([0, \alpha \cdot (f(x + L_t) - f(L_t -))] < x\}.
\]

### 1.1 PDE viewpoint in the Brownian case

We now turn to a brief review of existing PDE approaches to the fundamental setting of \( Z_t = B_t \) and \( f(x) = x \), where \( B \) is a standard Brownian motion. If we let \( V_t \) denote the density of \( \nu_t \), i.e. the law of \( X_t \) absorbed at the origin, then we arrive (at least formally) at the PDE

\[
\partial_t V_t(x) = \frac{1}{2} \partial_{xx} V_t(x) + \alpha L'_t \partial_x V_t(x), \quad L'_t = \frac{1}{2} \partial_x V_t(0), \quad V_t(0) = 0, \tag{1.2}
\]

for \( x \in (0, \infty) \). Here the contagious feedback emerges as a transport term that pushes mass towards the origin at a rate proportional to the current flux across the boundary.

Setting \( v(t, x) := -\alpha V_t(x - \alpha L_t) \), for \( \alpha > 0 \), the equations for \( V \) and \( L \) become

\[
\partial_t v = \frac{1}{2} \partial_{xx} v \quad \text{on} \quad (\alpha L_t, \infty), \quad \partial_x v(t, \alpha L_t) = -\frac{1}{2} \alpha L'_t, \quad v(t, \alpha L_t) = 0. \tag{1.3}
\]
This is a Stefan problem modelling the freezing of a supercooled liquid occupying the semi-infinite strip \((\alpha L_t, \infty)\): the liquid is initially supercooled to a temperature \(v(0, x) = -\alpha V_0(x)\) that is below the freezing point \(v = 0\), and the evolving ‘freezing front’ is given by \(x = \alpha L_t\). Forgetting for a moment that \(V_0\) is a probability density, if \(v(0, \cdot) = -c\), then the well-posedness of this system displays a clear dichotomy: for \(c < 1\) it admits an explicit similarity solution, while no solution can exist for \(c \geq 1\). This situation motivates the recent analysis of Dembo and Tsai [10], which investigates the critical case \(c = 1\) via the scaling behaviour of a discretised particle approximation.

A related line of study is [2, 3, 21], which considers the PDE

\[
\partial_t u(t, x) = \frac{1}{2} \partial_{xx} u(t, x) + u(t, 0) \partial_x u(t, x), \quad \partial_x u(t, 0) = -u(t, 0)^2, \quad (1.4)
\]

for \(x \in (0, \infty)\), as a model for cell polarisation: \(u\) is the density of molecular markers on a cell, identified with the positive half-line, and the time evolution of the density is coupled with the concentration of markers on the cell membrane at \(x = 0\). If we set \(\tilde{V}_t(x) := \alpha^{-1} \int_0^x u(t, y) dy\), then we get back (1.2) with initial condition \(\tilde{V}_0(x) = \alpha^{-1} \int_0^x u_0(y) dy\). Calvez et al. [2, 3] show that (1.4) admits a global weak solution if \(\int_0^\infty u_0(y) dy \leq 1\), while \(u\) explodes to infinity in finite time if \(\int_0^\infty u_0(y) dy > 1\) with \(u_0\) non-increasing (see also [21] for some refinements).

Note that in both of the above examples, the solvability depends critically on how the initial condition compares to \(\alpha^{-1}\). This same relationship will play an important role in the results of the next sections. Returning for now to the supercooled Stefan problem, there is an early literature on its well-posedness for slight variations of (1.3) on a finite strip. This goes back to Fasano and Primicerio [12, 13], who give conditions on the initial datum under which the system is uniquely solvable in the class of classical solutions, either for all times or up to some finite time where \(L'\) explodes.

If a source term \(L' \delta_0(x - c)\) is added to (1.2), for some \(c > 0\), where \(\delta_0\) is the delta function, then the mass of the system is preserved. Recalling the finite particle systems described earlier, this corresponds to the case where particles are instantly reset to a predetermined value \(c\) upon reaching the boundary (instead of being absorbed). By reducing the analysis to that of a Stefan-like problem, Carillo et al. [4] show that, for \(\alpha \leq 0\), a unique classical solution exists for all times, while, for \(\alpha > 0\), classical solutions exist up to a possibly finite explosion time (for initial conditions that are \(C^1\) up to the boundary and vanish there). For further background, see also [1, 5].

At this point, it is important to emphasise that the solutions to (MV) and (CMV) from [3, 20, 23] are global in time: they do not cease to exist at some explosion time, even despite the fact that there must be blow-ups in the form of jump discontinuities for large enough \(\alpha > 0\) [15, Thm. 1.1].

### 1.2 Recent history of the problem

Aside from the earlier mentioned existence results [20, 23], the literature on (MV) is centred around the case where \(Z\) is a standard Brownian motion, \(B\), up to various
forms of drift and volatility. For clarity, we thus focus on $Z_t = B_t$ and $f(x) = x$ in this subsection.

Motivated by the incomplete results of the PDE viewpoint, when $\alpha > 0$, Delarue et al. [7, 8] introduced what is essentially a generalised probabilistic notion of solution, namely $(\text{MV})$. Recently, the study of uniqueness and regularity for this problem has been continued through independent efforts of Nadtochyi and Shkolnikov [22] and the authors of this paper together with Hambly [15]. Here is a brief overview of the results:

- Let $X_0 = x_0 > 0$. By [7] there is an $\alpha_0 \in (0, 1]$ such that, for any $\alpha \in (0, \alpha_0)$, $(\text{MV})$ has a $C^1$ solution $L$ on any time-interval, and the solution is unique in this class. The result is formulated for the (neuronal) version where particles are reset upon hitting the boundary.

- In the same setting as [7], for any $\alpha > 0$, [8] obtains global solutions to $(\text{MV})$ as limit points of the (neuronal) particle system described earlier. Moreover, there is propagation of chaos provided there is uniqueness among the limit points.

- Let $V_0 \in H^1_0(0, \infty)$. For any $\alpha > 0$, [22] gives a solution $L$ to $(\text{MV})$ up to an explosion time $t_* > 0$ such that $L' \in L^2(0, t)$ for $t < t_*$ and $\|L'\|_{L^2(0,t)}$ explodes as $t \uparrow t_*$. Moreover, there is uniqueness up to $t_*$ in this class of solutions.

- Let $V_0 \in L^\infty$ and $V_0(x) \leq Cx^\beta$ for some $C, \beta > 0$. For any $\alpha > 0$, [15] gives a solution $L$ to $(\text{MV})$ up to an explosion time $t_* > 0$ such that $L' \in L^2(0, t)$ for $t < t_*$ and $\|L'\|_{L^2(0,t)}$ explodes as $t \uparrow t_*$. Moreover, there is uniqueness up to $t_*$ among all candidate solutions, and $|L'_t| \leq K_0 t^{-\frac{1-\beta}{2}}$ on $[0, t_0]$ for any $t_0 < t_*$. Note that the final result can be combined with that of the second bullet point to give propagation of chaos for $(\text{CMV}, \rho = 0)$ up to the explosion time (or globally for small $\alpha > 0$). Based on the above, numerical schemes for $(\text{CMV}, \rho = 0)$ have been proposed and analysed by Kaushansky and Reisinger [19] and Kaushansky, Lipton and Reisinger [18], up to the explosion time.

1.3 Overview of the paper

In Section 2 we prove global uniqueness of $(\text{MV})$ under a smallness condition on $\alpha > 0$, i.e. (2.1), giving the weak feedback regime. The result applies with any continuous stochastic process, $Z$, as the driver. In Section 3 we specialise to $(\text{MV})$ with $Z_t = B_t$ and $f(x) = x$, but for a general $\alpha > 0$, and we show that the comparison argument from Section 2 can be extended to give short-time uniqueness after a blow-up.

2 Uniqueness for weak feedback

In this section we prove uniqueness of solutions to $(\text{MV})$ in the weak feedback regime, that is, when the feedback parameter, $\alpha > 0$, is sufficiently small relative to the initial condition and the feedback function $f$ (Theorem 2.2).
Our main method of proof is the simple comparison argument in Lemma 2.1. It is important to recognise that this method is of ‘zeroth order’, in the sense that it makes no use of differential or analytic properties of the solutions, $L$, as a function of time or of the initial density, $V_0$, as a function of space. This is very natural probabilistically, as the McKean–Vlasov formulation (MV) does not involve any derivatives and always makes sense globally, in contrast to the PDE point of view discussed in Section 1.1.

The value of a zeroth order approach becomes clear when there is a rough drift in the driving noise, $Z$, as the analytical tools developed in [7,15,22] cannot be applied to such a setting. For example, path by path, the solutions to (CMV) will be non-differentiable when $\rho > 0$, yet the uniqueness argument presented below is robust enough to include (CMV) in the weak feedback regime (Theorem 2.3). In fact, this is the first uniqueness result for any version of (MV) with a rough drift. Moreover, the robustness of our approach will turn out to be very useful even without a rough drift, when attacking the question of uniqueness after a blow-up in Section 3.

With a view to what follows, it is convenient to introduce the notation

$$
\|\ell\|_t := \sup_{s \leq t} |\ell(s)| \quad \text{and} \quad \|f\|_{Lip(x)} := \sup_{y \neq z \in [0,x]} \frac{|f(y) - f(z)|}{|y - z|}.
$$

For now, our first task is the essential comparison argument alluded to above. A visualisation of its proof is provided by Figure 2.1

**Lemma 2.1 (Comparison).** Suppose $L$ and $\bar{L}$ are two solutions to (MV) with the same initial condition $\nu_0$. Then

$$
|L_t - \bar{L}_t| \leq \mathbb{E}[\nu_0(\sup_{s \leq t} \{\alpha f(L_s) - Z_s\}, \sup_{s \leq t} \{\alpha f(\bar{L}_s) - Z_s\})]
\vee \mathbb{E}[\nu_0(\sup_{s \leq t} \{\alpha f(L_s) - Z_s\}, \sup_{s \leq t} \{\alpha f(\bar{L}_s) - Z_s\})].
$$

**Proof.** Couple the solutions via the Brownian motion and random start point:

$$
\begin{cases}
X_t = X_0 + Z_t - \alpha f(L_t) \\
\tau = \inf\{t \geq 0 : X_t \leq 0\} \\
L_t = \mathbb{P}(\tau \leq t),
\end{cases}
\quad
\begin{cases}
X_t = X_0 + Z_t - \alpha f(\bar{L}_t) \\
\bar{\tau} = \inf\{t \geq 0 : X_t \leq 0\} \\
\bar{L}_t = \mathbb{P}(\bar{\tau} \leq t),
\end{cases}
$$

then we have

$$
L_t - \bar{L}_t \leq \mathbb{P}(\inf_{s \leq t} X_s \leq 0, \inf_{s \leq t} X_s > 0)
\quad
= \mathbb{P}(\inf_{s \leq t} \{X_0 + Z_s - \alpha f(L_s)\} \leq 0, \inf_{s \leq t} \{X_0 + Z_s - \alpha f(\bar{L}_s)\} > 0)
\quad
= \mathbb{P}(\sup_{s \leq t} \{\alpha f(L_s) - Z_s\} \leq X_0 < \sup_{s \leq t} \{\alpha f(\bar{L}_s) - Z_s\}).
$$

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Figure 2.1: A visual proof of Lemma 2.1: The red curve is $X^1$ and the blue curve is $X^2$, coupled through the same initial value and driving noise. Writing $L_t^1 - L_t^2 = \mathbb{P}(\tau_1 \leq t < \tau_2)$, we see that, as $x_0$ decreases, we only get contributions to this probability from when the red curve first hits the $x$-axis until the blue curve first hits it. Hence the probability equals $\nu_0(0.59, 0.72)$. Note that the difference between the curves is bounded by $\|L_t^1 - L_t^2\|_t$.

Conditioning on $X_0$ gives

$$L_t - \bar{L}_t \leq \mathbb{E} \int_0^\infty 1\{\sup_{s \leq t} \{\alpha f(L_s) - Z_s\} \leq x_0 < \sup_{s \leq t} \{\alpha f(\bar{L}_s) - Z_s\}\} \nu_0(dx_0)$$

$$= \mathbb{E}[\nu_0(\sup_{s \leq t} \{\alpha f(L_s) - Z_s\}, \sup_{s \leq t} \{\alpha f(\bar{L}_s) - Z_s\})].$$

Notice that the same inequality with $L$ and $\bar{L}$ interchanged also holds, by symmetry, therefore we have the result. \hfill \Box

From here the main uniqueness result is almost immediate. Note that we refer to the weak feedback regime, as the uniqueness is guaranteed through a smallness condition on $\alpha \geq 0$, namely (2.1) below.

**Theorem 2.2** (Uniqueness in the weak feedback regime). Suppose that $\nu_0$ has a density satisfying $V_0 : (0, \infty) \rightarrow (0, \infty)$ and let $L$ and $\bar{L}$ be two solutions to (MV). If

$$\alpha \cdot \|V_0\|_\infty \cdot \|f\|_{\text{Lip}(L_t, \bar{L}_t)} < 1, \quad (2.1)$$

then $L = \bar{L}$ on $[0,t]$.

**Proof.** Let $L$ and $\bar{L}$ be two solutions to (MV). Applying the inequality $\bar{L}_s = L_s + \bar{L}_s - L_s \geq L_s - \|L - \bar{L}\|_s$, for $s \leq r$, to Lemma 2.1 gives

$$|L_r - \bar{L}_r| \leq \mathbb{E}[\nu_0(-\alpha\|f(L) - f(\bar{L})\|_r + \sup_{s \leq r} \{\alpha f(L_s) - Z_s\}, \sup_{s \leq r} \{\alpha f(\bar{L}_s) - Z_s\})]$$

$$\vee \mathbb{E}[\nu_0(-\alpha\|f(L) - f(\bar{L})\|_r + \sup_{s \leq r} \{\alpha f(L_s) - Z_s\}, \sup_{s \leq r} \{\alpha f(\bar{L}_s) - Z_s\})].$$

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Writing \( Z^*_r := \sup_{s \leq r} \{ \alpha f(L_s) - Z_s \} \) and \( \bar{Z}^* \) for the same supremum corresponding to \( \bar{L} \), the above becomes
\[
|L_r - \bar{L}_r| \leq \mathbb{E} \left[ \int_{Z^*_r}^{Z^*} V_0(x)dx \right] \vee \mathbb{E} \left[ \int_{Z^*_r}^{Z^*} V_0(x)dx \right],
\]
and so it is immediate that, for all \( r \geq 0 \),
\[
|L_r - \bar{L}_r| \leq \alpha \| V_0 \|_\infty \| f(L) - f(\bar{L}) \|_r.
\]
Using the Lipschitz property of \( f \) and taking a supremum over \( r \in [0, t] \) gives
\[
\| L - \bar{L} \|_t \leq \alpha \| V_0 \|_\infty \| f \|_{\text{Lip}(L \lor \bar{L})} \| L - \bar{L} \|_t.
\]
Therefore the hypothesis in the statement forces \( \| L - \bar{L} \|_t = 0 \), as required. \( \square \)

Recall the notation \( \nu_t = \mathbb{P}(X_t \in \cdot, t < \tau) \). If the driver \( Z_t \) has a density \( p_t \), then \( \nu_t \) has a density \( V_t \) with \( \| V_t \|_\infty \leq \| V_0 \|_\infty \). Indeed, ignoring the absorption at the origin,
\[
\nu_t(S) \leq \int_0^\infty \int_S p_t(x_0 + z - \alpha f(L_t)) V_0(x_0)dzdx_0 \leq \| V_0 \|_\infty \cdot |S|.
\]
Hence the smallness condition (2.1) gives \( \Delta L_t = 0 \) by virtue of the physical jump condition, so the solution remains continuous for all time in the weak feedback regime.

As immediate corollaries of Theorem 2.2 we obtain the following two results which are the first contributions to the question of uniqueness for the conditional McKean–Vlasov problem \((CMV)\) from \([20]\) and for a generic continuous driving noise in \((MV)\) as considered in \([23]\).

**Theorem 2.3** (Uniqueness of CMV in the weak feedback regime). The solutions to the conditional McKean–Vlasov problem \((CMV)\) are pathwise unique when \( \| V_0 \|_\infty < \alpha^{-1} \).

Furthermore, this ensures that the ‘relaxed’ solutions constructed in \([20\), Thm. 3.2\] are in fact adapted to the systemic driver \( B^0 \), and thus it implies full convergence in law of the finite particle system from \([20]\) to the unique solution of \((CMV)\).

Compared with Section 1.1, \((CMV)\) gives rise to a stochastic PDE, namely
\[
dV_t(x) = \frac{1}{2} \partial_{xx} V_t(x)dt + \alpha \partial_x V_t(x)dL_t - \rho \partial_x V_t(x)dB^0_t
\]
with \( L_t = 1 - \int_0^\infty V_t(x)dx \), understood in the weak sense for test functions \( \phi \in C^2(0, \infty) \) with \( \phi(0) = 0 \) \([20\), Rmk. 3.13\]. In the weak feedback regime this SPDE makes sense globally and, by virtue of Theorem 2.3, the solution is adapted to the driving noise.

**Theorem 2.4** (Small-time uniqueness in \([23]\)). Suppose \( \| V_0 \|_\infty < \alpha^{-1} \). The McKean–Vlasov system \((MV)\) with \( f(x) = -\log(1 - x) \), as defined in \([23\), Thm. 2.2\], has a unique solution for all \( t \geq 0 \) such that \( L_t < 1 - \alpha \| V_0 \|_\infty \).

We should note that Theorem 2.2 is the most straightforward application of Lemma 2.1. Indeed, it is possible to obtain stronger results with harder work and a more specific setting. The next section gives one such example.
3 Short-time uniqueness after a blow-up

For the remaining part of this paper we return to the setting \( Z_t = B_t \) and \( f(x) = x \). In this case, there is full uniqueness of \((MV)\) up to an explosion time \( t^* > 0 \) [15, Thm. 1.8], but while solutions continue to exist there are currently no results on whether the system restarts in a unique way.

More specifically, the current approaches to uniqueness break down, as the system may be restarting from a density \( V_t \), with mass at the boundary where, in principle, all we know is that \((PJC)\) imposes

\[
\inf\{x > 0 : \int_0^x V_t(y) \, dy < x\} = 0.\tag{3.1}
\]

The problem here is that (3.1) implies little about the regularity of \( V_t \) near zero. Without further information, we cannot rule out pathological cases like those in Figure 3.1, so it seems difficult to gain sufficient control to prove that uniqueness can be propagated after a blow-up. In practise, however, we do not expect these edge cases to arise, and indeed we are able to show here that we have at least polynomial control on the density after a jump time of \( L \) (Proposition 3.2 and Theorem 3.3). As the next result shows, this will be sufficient to give small time uniqueness after the blow-up. The idea is to use the methods from the previous section, but to ensure that insufficient time has passed for a large amount of mass to reach the boundary.

**Theorem 3.1** (Short-time uniqueness). Suppose \( v_0 \) has a density, \( V_0 \), for which there exists \( c > 0 \), \( x_0 > 0 \) and \( n \in \mathbb{N} \) such that

\[
V_0(x) \leq \alpha^{-1} - cx^n, \quad \text{for all } x < x_0.\tag{3.2}
\]

Let \( L \) and \( \bar{L} \) be two solution to \((MV)\) with \( Z_t = B_t \) and \( f(x) = x \). Then there exists \( t_0 > 0 \) such that \( L_t = \bar{L}_t \) for all \( t \in [0,t_0] \).

**Proof.** For shorthand let \( Z_t^* := \sup_{s \leq t} \{\alpha L_s - B_s\} \) and

\[
F(t) := \mathbb{E}v_0[(-\alpha\|L - \bar{L}\|_t + Z_t^*, Z_t^*)] \leq E(t) \cdot \alpha\|L - \bar{L}\|_t,
\]

where \( E(t) := \mathbb{E}[\sup_{x \in [-\alpha\|L - \bar{L}\|_t + Z_t^*, Z_t^*]} V_0(x)] \). We decompose this expectation into the three regions:

\[
Z_t^* \in [0, \alpha\|L - \bar{L}\|_t + t), \quad Z_t^* \in [\alpha\|L - \bar{L}\|_t + t, x_0), \quad Z_t^* \in [x_0, \infty).
\]

Since \( L \) and \( \bar{L} \) are càdlàg, we can take \( t_1 > 0 \) sufficiently small so that for all \( t \leq t_1 \) we have \( \alpha\|L - \bar{L}\|_t + t \leq x_0/2 \) and \( \alpha L_t, \alpha \bar{L}_t \leq x_0/4 \). Therefore

\[
E(t) \leq \alpha^{-1}\mathbb{P}(Z_t^* \in [0, \alpha\|L - \bar{L}\|_t + t]) + (\alpha^{-1} - ct^n)\mathbb{P}(Z_t^* \in [\alpha\|L - \bar{L}\|_t + t, x_0]) + \|V_0\|_\infty\mathbb{P}(Z_t^* \in [x_0, \infty))
\]

\[
\leq \alpha^{-1} - ct^n\mathbb{P}(Z_t \in [x_0/2, x_0]) + \|V_0\|_\infty\mathbb{P}(Z_t^* \in [x_0, \infty))
\]

\[
\leq \alpha^{-1} - ct^n\mathbb{P}(\sup_{s \leq t} B_s \in [\frac{1}{2}x_0, \frac{3}{4}x_0]) + \|V_0\|_\infty\mathbb{P}(\sup_{s \leq t} B_s \in [\frac{3}{4}x_0, \infty))
\]

\[
= \alpha^{-1} - ct^n(\Phi(-x_0/2t^{1/2}) - \Phi(-3x_0/4t^{1/2})) + \|V_0\|_\infty\Phi(-3x_0/4t^{1/2}),
\]

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Figure 3.1: Two pathological initial densities that satisfy (3.1) with $\alpha = \frac{5}{3}$. The blue curve takes infinitely many values above the critical value of $\frac{3}{5}$ near zero. The red curve remains strictly below $\frac{3}{5}$, but all derivatives vanish at the origin, so control of the form in (3.2) is not possible.

where $\Phi$ is the normal cdf. Using the asymptotic bounds

$$\Phi(-cx) \asymp x^{-1}e^{-c^2x^2/2}, \quad \text{as } x \to \infty,$$

we can find $t_2 \in (0, t_1]$ sufficiently small so that $E(t) < \alpha^{-1}$ for all $t \leq t_2$, which guarantees that $F(t) < \|L - \bar{L}\|_t$. By symmetry and Lemma 2.1, we conclude that

$$|L_t - \bar{L}_t| < \|L - \bar{L}\|_t, \quad \text{for every } t \in (0, t_2]. \quad (3.3)$$

Let $t_0 > 0$ be the smaller of the first jump times of $L$ and $\bar{L}$ and $t_2$. By continuity there exists $t_* \in [0, t_0]$ for which the supremum is attained:

$$|L_{t_*} - \bar{L}_{t_*}| = \|L - \bar{L}\|_{t_*}.$$

If $t_* = 0$, then $\|L - \bar{L}\|_{t_0} = 0$ and we are done. Otherwise $t_* \in (0, t_0]$, but this contradicts (3.3) at $t = t_*$. \hfill \Box

The next result allows us to propagate Theorem 3.1 to blow-up times. The main idea is that, although we cannot control the solution density near zero at arbitrary times, we can prove that the density is analytic in the interior of the half-line. At a blow-up this is then sufficient to give control at zero, since the new point at the origin was in the interior of the density before the jump discontinuity. Our proof of analyticity rely on kernel smoothing and energy estimate techniques from [14, 16], and is presented in the next section.
Proposition 3.2 (Analyticity). Suppose \( \nu_0 \) has a density \( V_0 \in L^2(0, \infty) \) and \( V_t \) is the corresponding density process for a solution of \( \text{(MV)} \). Then for all \( t > 0 \) and \( x > 0 \), \( y \mapsto V_t(y) \) is analytic at the point \( x \).

Theorem 3.3 (Short-time uniqueness after a blow-up). Suppose \( \nu_0 \) has a density \( V_0 \in L^2(0, \infty) \). Let \( t_* \) be the first blow-up time of the solution. Then the system can be restarted at time \( t_* \) and the restarted solution is the unique solution on an additional non-zero time interval, amongst all possible càdlàg solutions on this period.

Proof. Note that, at the first blow-up time \( t_* \), we have
\[
V_{t_*}(x) = V_{t_*} - (x + \alpha \Delta L_{t_*}),
\]
for \( x \geq 0 \). So although \( V_{t_*} \) is not analytic at \( x = 0 \), because \( \Delta L_{t_*} > 0 \) and \( V_{t_*} \) is analytic in the interior, \( V_{t_*} \) is analytic at \( x = 0 \). Therefore we have a series expansion
\[
V_{t_*}(x) = V_{t_*}(0) + \sum_{n \geq 1} c_n x^n, \quad \text{for every } x \in [0, x_0],
\]
for some \( x_0 > 0 \). If \( V_{t_*}(0) < \alpha^{-1} \) we have the required condition on \( V_{t_*} \) by taking \( x_0 \) sufficiently small.

If \( V_{t_*}(0) > \alpha^{-1} \), then, since the physical jump condition \( \text{(PJC)} \) on \( \Delta L \) ensures that \( V_t \) satisfies \( \text{(3.1)} \), by taking \( x \) sufficiently small we have a contradiction. Therefore suppose \( V_{t_*}(0) = \alpha^{-1} \), then by the last case we cannot have \( c_n = 0 \) for all \( n \geq 1 \), so let \( n_0 := \min\{n : c_n \neq 0\} \). For \( x_1 > 0 \) sufficiently small we have
\[
V_{t_*}(x) \geq \alpha^{-1} + (c_{n_0} + \varepsilon)x^{n_0}, \quad \text{for every } x \in [0, x_1],
\]
where \( |\varepsilon| \leq \frac{1}{2}|c_{n_0}| \). Again if \( c_{n_0} > 0 \) then we contradict \( \text{(3.1)} \), hence \( c_{n_0} < 0 \) and so we have that \( V_{t_*} \) satisfies the condition \( \text{(3.2)} \) in Theorem 3.1. Consequently, Theorem 3.1 can be applied up to a small time after the first blow-up time \( t_* \).

Although we get small-time uniqueness after a blow-up, it is important to note that global uniqueness is out of reach of the techniques presented in this paper, as they do not allow for bootstrapping at continuity times. Simply put, we pay a price for the robustness of our comparison arguments, by way of not obtaining any regularity. This translates into a lack of control over the boundary behaviour of the densities, and hence we cannot rule out the appearance of pathological states like those in Figure 3.1 at continuity times.

Starting from an initial density satisfying \( \text{(3.2)} \), and simply ignoring the absorption at the origin, it is easy to establish a zeroth order estimate \( \nu_t(x) \leq \alpha^{-1}x - Cx^k \) for some \( C > 0 \) and \( k \in \mathbb{N} \) near \( x = 0 \) on a small time interval. However, this of course does not imply control on the density of the form \( \text{(3.2)} \). As is already evident from \([7, 22, 15]\), there is a fine relationship between absorption and regularity at the boundary, which one could hope to exploit more carefully. We understand that Delarue, Nadtochiy and Shkolnikov are close to finalising results that could clear up this picture in the present case of \( Z_t = B_t \) and \( f(x) = x \) \([9]\).
3.1 Analyticity of the density in the interior

In this final subsection we present a proof of Proposition 3.2. As already mentioned, it goes via kernel smoothing, so for any $\delta > 0$ and any measure $\mu$ on $(0, \infty)$, we define

$$T_\delta \mu(x) := \int_0^\infty G_\delta(x_0, x) \mu(dx_0) \quad \text{and} \quad T_\delta^r \mu(x) := \int_0^\infty G_\delta^r(x_0, x) \mu(dx_0),$$

for $x \geq 0$, where $G_\delta$ and $G_\delta^r$ are, respectively, the absorbing and reflecting Gaussian kernels on the positive half-line, given by

$$G_\delta(x, y) := \frac{1}{\sqrt{2\pi \delta}} \left\{ e^{-\frac{(x-y)^2}{2\delta}} - e^{-\frac{(x+y)^2}{2\delta}} \right\}, \quad G_\delta^r(x, y) := \frac{1}{\sqrt{2\pi \delta}} \left\{ e^{-\frac{(x-y)^2}{2\delta}} + e^{-\frac{(x+y)^2}{2\delta}} \right\}.$$

We will sometimes abuse notation and apply $T_\delta$ and $T_\delta^r$ to $\phi \in C^\infty(0, \infty)$.

**Lemma 3.4** (Existence of weak derivatives). Suppose $\liminf_{\delta \to 0} \| \partial^n_\delta T_\delta \mu \|_2 < \infty$, then $\mu$ has an $n$th weak derivative $\partial^n_\mu \mu \in L^2(0, \infty)$ and $\| \partial^n_\delta T_\delta \mu \|_2 \to \| \partial^n_\mu \mu \|_2$ as $\delta \to 0$.

From [15] Prop. 2.1, we know that regardless of the initial condition, for all times $t > 0$, $\nu_t$ has a density function $V_t : (0, \infty) \to (0, \infty)$. A similar argument shows that:

**Lemma 3.5.** If $\nu_0$ has a density $V_0 \in L^2(0, \infty)$, then $\| V_t \|_2 \leq \| V_0 \|_2$ for every $t \geq 0$.

Let $\langle \nu_t, \phi \rangle := \int \phi d\nu_t = \mathbb{E}[\phi(X_t)1_{t<\tau}]$ and let $t_*$ be the first jump time of $L$. Strictly before $t_* > 0$, Itô’s formula gives the weak PDE

$$\langle \nu_t, \phi \rangle = \langle \nu_0, \phi \rangle + \frac{1}{2} \int_0^t \langle \nu_s, \partial_{xx} \phi \rangle ds - \alpha \int_0^t \langle \nu_s, \partial_x \phi \rangle dL_s,$$

for $\phi \in C^2(0, \infty)$ with $\phi(0) = 0$. Now take $\phi = T_\delta \psi$ for an arbitrary $\psi \in C^\infty_0(0, \infty)$. Integrating by parts, and differentiating $n$ times, we can deduce that

$$d \partial^n_t T_\delta \nu_t(x) = \frac{1}{2} \partial^{n+2}_x T_\delta \nu_t(x) dt + \alpha \partial^{n+1}_x T_\delta \nu_t(x) dL_t,$$

for $x \geq 0$ (a.e.). Using $d(\partial^n_t T_\delta \nu_t)^2 = 2 \partial^n_t T_\delta \nu_t(x) d\partial^n_t T_\delta \nu_t(x)$, and rearranging, we get

$$d \partial^n_x T_\delta \nu_t(x) = \partial^n_x T_\delta \nu_t(x) \partial^{n+2}_x T_\delta \nu_t(x) dt + 2\alpha \partial^n_x T_\delta \nu_t(x) \partial^{n+1}_x T_\delta \nu_t(x) dL_t \quad (3.4)$$

$$+ 4\alpha \partial^n_x T_\delta \nu_t(x) \partial^{n+1}_x R_\delta \nu_t(x) dL_t,$$

where we have introduced the remainder term

$$R_\delta \nu_t(x) := \int_0^\infty \frac{1}{\sqrt{2\pi \delta}} e^{-\frac{(x+x_0)^2}{2\delta}} \nu_t(dx_0).$$

Fix any two open sets $U \Subset W \Subset (0, \infty)$, where ‘$\Subset$’ denotes compact containment. Let $\zeta$ be a smooth cut-off function with $\zeta = 1$ on $U$, $\zeta \in (0, 1)$ on $W \setminus U$, and $\zeta = 0$ otherwise. Note that $|\partial_x \zeta| + |\partial_{xx} \zeta| \leq C 1_{W \setminus U}$, where $C$ only depends on $W$ and $U$. 

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Lemma 3.6 (Smoothed energy estimate). For all integers $a \geq 2$ and $b \geq 1$ we have

$$t^b \| \zeta^\frac{a}{2} \partial_x^n \partial_T \nu_t \|_2 \leq \left( \frac{1}{2} \right)^{a-1} \| \partial_x \zeta \partial_x \partial_T \nu_t \|_2^2 dt \left( a-1 \right) \| \partial_x \partial_T \nu_t \|_2^2 dt + 2 \alpha \| \partial_x \partial_T \nu_t \|_2^2 \eta^{-1} \| \partial_x \partial_T \nu_t \|_2^2 dt + 4 \alpha \left( \int_0^t \zeta(x) \partial_x \partial_T \nu_t(x) \partial_x \partial_T \nu_t(x) dx \right) dt,$$

where we have used $a \zeta^{(a-1)/2} \leq a(a-1) \zeta^{(a-2)/2}$. Differentiating $t \mapsto t^b \| \zeta^\frac{a}{2} \partial_x^n \partial_T \nu_t \|_2$ with respect to $t$ and taking the range of integration up to $t_*$ gives

$$t^b \| \zeta^\frac{a}{2} \partial_x^n \partial_T \nu_t \|_2 \leq \left( \frac{1}{2} \right)^{a-1} \| \partial_x \zeta \partial_x \partial_T \nu_t \|_2^2 dt \left( a-1 \right) \| \partial_x \partial_T \nu_t \|_2^2 dt + 4 \alpha \| \partial_x \partial_T \nu_t \|_2^2 \eta^{-1} \| \partial_x \partial_T \nu_t \|_2^2 dt + 4 \alpha \left( \int_0^t \zeta(x) \partial_x \partial_T \nu_t(x) \partial_x \partial_T \nu_t(x) dx \right) dt.$$

It remains to show the final term above vanishes as $\delta \to 0$. Writing $p_{\delta}(\cdot)$ for the standard Gaussian transition kernel, it follows from the definition of $R_\delta$ that

$$| \partial_x \partial_T \nu_t(x) | \leq | \partial_x \partial_T \nu_t(y) | \left| \int_0^\infty p_{\delta}(y) \nu_t(dy) \right| \leq \int_0^\infty | Q(\delta^{-\frac{1}{2}} y, \delta^{-\frac{1}{2}} x) | p_{\delta}(x+y) \nu_t(dy),$$

for a two-variable polynomial $Q$. Note $p_{\delta}(x+y) \leq p_{\delta}(y) \cdot e^{-x^2/2\delta}$. Applying Lemma 3.4 and Cauchy–Schwarz, we conclude $| \partial_x^{n+1} R_\delta \nu_t(x) | = O(e^{-x^2/4\delta})$ uniformly in $t$. Since $\zeta$ is supported on $W$,

$$\| \zeta^\frac{a}{2} \partial_x \partial_T \nu_t \|_2 = O(e^{-w^2/2\delta})$$

uniformly in $t$, where $w := \inf W > 0$. This is sufficient to complete the proof. □
As Lemma 3.6 gives control over the \((n+1)\)th spatial derivative in terms of the \(n\)th derivative, we can use it to inductively prove that \(V_{t,-}\) has weak derivatives of all orders, and is therefore smooth in the interior of \((0, \infty)\).

**Corollary 3.7 (Smoothness).** If \(\nu_0\) has a density \(V_0 \in L^2(0, \infty)\), then \(V_{t,-} \in C^\infty(0, \infty)\). Furthermore \(V_t \in C^\infty(0, \infty)\) for almost all \(t \in (0, t_*)\) and

\[
t^b \| \zeta \frac{\partial^n}{\partial \nu_t^2} V_{t,-} \|_2^2 + \int_0^{t_\ast} t^b \| \zeta \frac{\partial^n}{\partial \nu_t^2} V_t \|_2^2 dt \\
\leq c_1 a(a-1) \int_0^{t_\ast} t^b \| \zeta \frac{\partial^n}{\partial \nu_t^2} V_t \|_2^2 dt + b \int_0^{t_\ast} t^{b-1} \| \zeta \frac{\partial^n}{\partial \nu_t^2} V_t \|_2^2 dt.
\]

**Proof.** Fix \(n \geq 0\). Suppose that for all \(U \subseteq W \subseteq (0, \infty)\), \(a \geq 2n\), and \(b \geq n\) we have

\[
\liminf_{\delta \to 0} \int_0^{t_\ast} t^b \| \zeta \frac{\partial^n}{\partial \nu_t^2} T_\delta \nu_t \|_2^2 dt < \infty. \tag{3.5}
\]

Then Fatou’s Lemma and Lemma 3.4 imply that \(\partial^n_V\) exists and is in \(L^2(U)\) for every \(U \subseteq (0, \infty)\) and almost all \(t \in (0, t_*)\), with

\[
\int_0^{t_\ast} t^b \| \zeta \frac{\partial^n}{\partial \nu_t^2} T_\delta \nu_t \|_2^2 dt \to \int_0^{t_\ast} t^b \| \zeta \frac{\partial^n}{\partial \nu_t^2} V_t \|_2^2 dt < \infty.
\]

Therefore taking a \(\liminf\) over \(\delta\) in Lemma 3.6 gives that (3.5) holds for \(n+1\), for any \(a \geq 2(n+1)\), \(b \geq n+1\). Since (3.5) holds for \(n = 0\) and all \(a, b \geq 0\), by Lemma 3.4 using induction we conclude that the statement holds for all \(n \geq 0\). Returning to Lemma 3.6 once more, we deduce that

\[
\liminf_{\delta \to 0} \| \partial^n_{\nu_t} T_\delta \nu_{t,-} \|_2^2(U) \leq \liminf_{\delta \to 0} \| \zeta \frac{\partial^n}{\partial \nu_t^2} T_\delta \nu_{t,-} \|_2^2 < \infty,
\]

for every \(n \geq 0\). Therefore \(V_{t,-}\) has weak derivatives of all orders, so is smooth. Finally, sending \(\delta \to 0\) in Lemma 3.6 gives the inequality. \(\square\)

Finally we are in a position to complete the proof of Proposition 3.2 by using the inequality in Corollary 3.7 to show that \(x \mapsto V_{t,-}(x)\) is analytic for all \(x \in (0, \infty)\).

**Proof of Proposition 3.2.** Introduce the short-hand notation

\[
I(n, a, b) := \int_0^{t_\ast} t^b \| \zeta \frac{\partial^n}{\partial \nu_t^2} V_t \|_2^2 dt,
\]

where we recall the cut-off \(\zeta\) is defined for fixed \(U \subseteq W \subseteq (0, \infty)\). Corollary 3.7 implies

\[
I(n+1, a, b) \leq c_1 a(a-1) I(n, a-2, b) + bI(n, a, b-1) \leq (c_1 t_* a(a-1) + b) I(n, a-2, b-1).
\]

Iterating the argument gives

\[
I(n, 2n, n) \leq I(0, 0, 0) \prod_{1 \leq i \leq n} (c_1 t_* 2i(2i-1) + i) \leq C^n \cdot (2n)!,
\]
for $C > 0$ a constant depending on $V_0$ and $\zeta$. Returning to Corollary 3.7 we have

$$t^b_* \| \zeta^a \partial^a_x V_{t_*} \|_2^2 \leq (c_1 t_* a(a - 1) + b) I(n, a - 2, b - 1).$$

Therefore setting $a = 2n + 2$ and $b = n + 1$ gives

$$\| \partial^n_x V_{t_*} \|_{L^2(U)} \leq t^{-(n+1)}_* (c_1 t_* (2n + 2)(2n + 1) + n + 1) \cdot C^n \cdot (2n)! = (C')^n \cdot (2n)! ,$$

for a further constant $C' > 0$ also depending on $t_*$. Since $(2n)! \leq 4^n (n!)^2$, we conclude that

$$\| \partial^n_x V_{t_*} \|_{L^2(U)} \leq (C'')^n \cdot n! ,$$

for every $n \geq 0$. (3.6)

Morrey’s inequality [11, Sect. 5.6, Thm. 4] gives a constant $c_2 > 0$ such that

$$\| \partial^n_x V_{t_*} \|_{L^\infty(U)} \leq c_2 \| \partial^n_x V_{t_*} \|_{H^1(U)} \leq (C''')^n \cdot n! ,$$

where we have applied (3.6), for $C''' > 0$ a further constant independent of $n$. This inequality guarantees that $V_{t_*}$ is analytic in the interior of $U$, and therefore, since $U \subset (0, \infty)$ was arbitrary, $V_{t_*}$ is analytic at all strictly positive spatial points.

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