Existence and Uniqueness of Solutions for Some Basic Stochastic Differential Equations

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Abstract. Stochastic differential equation (SDE) has plenty of applications on economic and physics. However, how to find the explicit solution usually remains a serious trouble. In this paper, we aim to study the uniqueness and existence of the stochastic differential equations. To this end, we used the Ito formula and the Lipschitz condition. As a result, we found that the Lipschitz condition is a sufficient condition for a stochastic differential equation to have a unique solution. If the Lipschitz condition can be proved by calculating the index, then it is possible to find the numerical solution.

Research on the stochastic differential equation has gained much attention after the finding of the Ito formula. The Ito formula illustrates the chain rule of a semi-martingale. Due to the difficulty in obtaining the explicit solution, one primary problem is whether there exists a unique solution. One approach to solve this problem is the use of Lipschitz condition, which is used in the ordinary differential equations.

1. Introduction
Stochastic differential equations (SDE) is an important mean to construct stochastic processes. It plays an important role in the study of economy, biology and physics. For example, in economics, option pricing can be calculated by stochastic differential equation. In the market sales, we can determine the stochastic variables by analyzing large amount of sample data and building the mathematical model from the SDE.

Since Ito (1944) introduced the definition of the stochastic integral and Ito’s formula, lots of researchers has paid much attention in this topic. There are many monographs on SDEs. One can refer to [1-20] and references therein for more details.

In the past, people focus mainly on the solvable SDE which have an obvious structure. However, they are almost meaningless in the daily application. In this article, we present a result on the existence and uniqueness for general SDEs with Lipschitz condition.

2. Method
Let \{W_t\} \geq 0 be a standard Brownian motion on a probability space (\Omega, F, P) with an admissible filtration F = \{F_t\} \geq 0. A strong solution of the stochastic differential equation with initial condition x_0 \in R is an adapted process X_t = X_t^x with continuous paths such that for all t \geq 0,

\[ X_t = x_0 + \int_0^t \mu(X_s) ds + \int_0^t \sigma(X_s) dB_s \] (1)

(1)
It seems that this stochastic differential equation has little information by comparison with the common ordinary differential equation. However, there are some hidden points behind.

First, the existence of the integrals requires some degree of regularity on $X_t$ and the following two inequalities have a probability one.

$$\int_0^t \|\mu(X_s)\| \, ds < \infty \text{ And } \int_0^t \sigma^2(X_s) < \infty$$

Second, the solution is required to exist for all $t < \infty$ with a probability one.

Here, there are some properties are required for the integral process $W(t) = \int_0^t \sigma(X_s) \, dB_s$ as well, where the $W(t)$ is called the Brownian motion. First, $W(0)$ should be equal to 0. Second, the increments are independent. Third, the increments should be stationary, that is, $W(t_1 + t) - W(t_2 + t) = W(t_1) - W(t_2)$. And last but not the least, the expectation $E [W(t)]$ should be 0.

In order to find the explicit solution of the SDE, the Ito formula helps us in applying coefficient matching technique. The Ito formula is as follow

$$dX_t = \left(\frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s, x) + \frac{\partial f}{\partial s}(s, x)\right) \, dt + \frac{\partial f}{\partial x}(s, x) \, dB_t$$

A strong solution of the stochastic differential equation with initial condition $x$ is a continuous stochastic process $X(t)$ defined on some probability space $(\Omega, F, P)$ such that for a given Wiener process $B(t)$ and admissible filtration $F$, the process $X(t)$ is adapted and satisfies the stochastic integral equation. In this paper, we will focus on the strong solution only.

### 3. Results and Discussion

In this section, we start to study the uniqueness and existence of solution for a stochastic differential equation and its relative comparison with the normal differential equation.

For some kinds of sde, it is possible for us to find the solution by applying the coefficient matching technique according to the Ito formula.

Consider the sde $dX_t = \mu X_t \, dt + \sigma X_t dB_t$ which is the simplest, after applying Ito formula, we have the following two equations.

$$\frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s, x) + \frac{\partial f}{\partial s}(s, x) = 1$$

$$\frac{\partial f}{\partial x}(s, x) = \mu$$

$$f(s, x) = x + g(s)$$

$$f(s, x) = s + h(x)$$

$$f(s, x) = x_0 + s + x$$

Now, consider a more complicated equation $dX_t = \mu X_t dt + \sigma X_t dB_t$ using the coefficient matching method as well,

$$\frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s, x) + \frac{\partial f}{\partial s}(s, x) = \mu f$$

$$\frac{\partial f}{\partial x}(s, x) = \sigma f$$
\begin{equation}
 f(s, x) = e^{\sigma x + \theta(s)}
\end{equation}
(11)

\begin{equation}
 \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s, x) = \frac{1}{2} \sigma^2 f
\end{equation}
(12)

\begin{equation}
 f(s, x) = x_0 e^{\sigma x + \left(\frac{\mu - \frac{1}{2} \sigma^2}{\sigma^2}\right)s}
\end{equation}
(13)

However, for most of SDE, it is impossible for us to find the solution. In such an occasion, the uniqueness and existence of solution becomes important. In this paper, we will focus on the strong solution only.

Theorem: For an SDE with equation

\begin{equation}
 dX = b(t, x) dt + B(t, x) dW_t
\end{equation}

(14)

These are called the global Lipschitz and linear growth conditions.

For every \(0 \leq t < \infty, x, y \in \mathbb{R}^d\), on some probability space, let \(X_0\) be an \(\mathbb{R}^d\)-valued random vector, independent of the Brownian motion \(W\), and with finite second moment \(E\|X_0\|^2 < \infty\). Then there exists a continuous, adapted process \(X\) which is a strong solution of the equation relative to \(W\), with an initial condition \(X_0\). Moreover, this process is square-integrable: for every \(T > 0\), there exists a constant \(C\), depending only on \(K\) and \(T\), such that

\begin{equation}
 E\|X_t\|^2 \leq C(1 + E\|X_0\|^2)e^{\epsilon t} \quad 0 \leq t \leq T
\end{equation}
(15)

First, we prove the uniqueness of the solution, in order to achieve this, we assume that there are two different solutions, namely \(X\) and \(\tilde{X}\), with the same initial value \(X_0\).

\begin{equation}
 X_t - \tilde{X}_t = \int_0^t \mu(u, X_u) - \mu(u, \tilde{X}_u) \, du + \int_0^t \sigma(u, X_u) - \sigma(u, \tilde{X}_u) \, dB_u
\end{equation}
(16)

\begin{equation}
 E(X_t - \tilde{X}_t)^2 \leq 2E\left(\int_0^t (\mu(u, X_u) - \mu(u, \tilde{X}_u)) \, du + \int_0^t \sigma(u, X_u) - \sigma(u, \tilde{X}_u) \, dB_u\right)^2
\end{equation}
(17)

Using Ito isometry

\begin{equation}
 E(X_t - \tilde{X}_t)^2 \leq 2t E\left(\int_0^t (\mu(u, X_u) - \mu(u, \tilde{X}_u))^2 \, du + 2\int_0^t (\sigma(u, X_u) - \sigma(u, \tilde{X}_u))^2 \, dB_u\right)
\end{equation}
(18)

\begin{equation}
 E(X_t - \tilde{X}_t)^2 \leq 2(t + 1)K^2 \int_0^t E(X_u - \tilde{X}_u)^2 \, du
\end{equation}
(19)

Suppose that the continuous function \(g\) satisfies \(0 \leq g(t) \leq \alpha(t) + \beta \int_0^t g(s) \, ds\) with \(\beta \geq 0\) and \(\alpha: [0, t] \rightarrow \mathbb{R}\) is integrable, then \(g(t) \leq \alpha(t) + \beta \int_0^t \alpha(s)e^{\beta(t-s)} \, ds\)

Here, see the \(E(X_t - \tilde{X}_t)^2\) as \(g(t)\) and \(2(t + 1)K^2\) as \(\beta\). The coefficient matching method obtains that the \(\alpha(t)\) is a zero function. Therefore, \(0 \leq g(t) \leq \alpha(t) + \beta \int_0^t \alpha(s)e^{\beta(t-s)} \, ds = 0\) and \(0\).

Also, as \((X_t - \tilde{X}_t)^2\) is non-negative, \(E(X_t - \tilde{X}_t)^2 = 0\) implies \((X_t - \tilde{X}_t) = 0\)

Next, we prove the existence of solution.

\begin{equation}
 X_{k+1}^t = \xi + \int_0^t \mu(s, X_k^s) \, ds + \int_0^t \sigma(s, X_k^s) \, dB_s
\end{equation}
(20)

We have \(X_{k+1}^t - X_k^t = B_t + M_t\)
Where \( B_t = \int_0^t \mu(s, X^k_s) \, ds \) and 
\[
M_t = \int_0^t \sigma(s, X^k_s) \, ds
\]  

\[ E[(M_s)^2] \leq E \int_0^t (\sigma(s, X^k_s) - \sigma(s, X^{k-1}_s))^2 \, dB_s \leq K^2 E \int_0^t (X^k_s - X^{k-1}_s)^2 \, ds \]  

\[ E[(B_s)^2] \leq K^2 t E \int_0^t (X^k_s - X^{k-1}_s)^2 \, ds \]  

\[ E[(X^{k+1}_t - X^k_t)^2] \leq L \int_0^t E(X^k_s - X^{k-1}_s)^2 \, ds \]  

Now, we have an iterated relationship about \( E[(X^{k+1}_t - X^k_t)^2] \), by repeating this process, we have the following inequality
\[ E \left[ (X^{k+1}_t - X^k_t)^2 \right] \leq L \int ... \int (X^k_s - \xi)^2 \, ds = \frac{(Lt)^2}{k!} E(X^k_s - \xi)^2 \]  

\[ \lim_{k \to \infty} \frac{(Lt)^2}{k!} E(X^k_s - \xi)^2 = 0 \]  

\[ \lim_{k \to \infty} X^{k+1}_t - X^k_t = 0 \]  

Which is the solution of the SDE

Another sensible technique is the fixed-point theorem in topology. Define a mapping \( (LX)_t = X_0 + \int_0^t \mu(X_s) \, ds + \int_0^t \sigma(X_s) \, dB_s \), if it is a contraction mapping, there exists one and only one point \( x \) such that \( X = X_0 + \int_0^t \mu(X_s) \, ds + \int_0^t \sigma(X_s) \, dB_s \) which is the solution of the SDE.

First, prove \( E \left[ \sup_{t \in [0,T]} (LX)_t \right]^2 < \infty \)
\[ E \left[ \sup_{t \in [0,T]} (LX)_t \right]^2 = E \left[ \sup_{t \in [0,T]} X_0 + \int_0^t \mu(X_s) \, ds + \int_0^t \sigma(X_s) \, dB_s \right]^2 \leq E \left[ \sup_{t \in [0,T]} 3X_0^2 + 3 \int_0^T |\mu(X_s)|^2 \, ds + 3 \int_0^T |\sigma(X_s)|^2 \, dB_s \right]^2 \leq 3x^2 + 3E \left[ \int_0^T |\mu(X_s) - \mu(X_0)| \, ds \right]^2 + 3 \int_0^T |\sigma(X_s)|^2 \, dB_s \leq 3x^2 + 3E \left[ \int_0^T KX_s \, ds + T|\mu(X_0)| \right]^2 + 3 \int_0^T |\sigma(X_s)|^2 \, ds \leq 3x^2 + K' \int_0^T E|X|^2 + C + 3 \int_0^T |\sigma(X_s)|^2 \, ds \text{ same process as before } \]  

Second, prove the mapping is a contraction mapping i.e. \( (LX)_s - (LY)_s < X_s - Y_s \)
\[ E \left[ \sup_{s \in [0,T]} (LX)_s - (LY)_s \right]^2 \leq 2 \sup_{s \in [0,T]} \left[ \int_0^s |\mu(X_s) - \mu(Y_s)| \, ds \right]^2 + 2 \sup_{s \in [0,T]} \left[ \int_0^s |\sigma(X_s) - \sigma(Y_s)| \, dB_s \right]^2 = l_1 + l_2 \]  

\[ l_1 \leq 2E \int_0^T |\mu(X_s) - \mu(Y_s)|^2 \, ds \leq 2E \int_0^T K|X_s - Y_s|^2 \, ds \leq 2EK^2 \int_0^T |X_s - Y_s|^2 \, ds = 2K^2E \int_0^T |X_s - Y_s|^2 \, ds \]  

\[ l_2 \leq 2E \int_0^T |\sigma(X_s) - \sigma(Y_s)|^2 \, ds \leq 2E \int_0^T K|X_s - Y_s|^2 \, ds \leq 2EK^2 \int_0^T |X_s - Y_s|^2 \, ds = 2K^2E \int_0^T |X_s - Y_s|^2 \, ds \]  

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\[ I_1 + I_2 \leq 2K^2(T^2 + T) \text{Es} \| X_s - Y_s \|_2^2 \quad 2K^2(T^2 + T) < 1 \quad (32) \]

So, it is a contraction mapping.

Moving on to the ordinary differential equation, the Lipschitz condition holds as well. By comparison with the Lipschitz condition for sde above, it is obvious that they are very similar, if without considering the Brownian motion; however, it is worth to mention that the Lipschitz condition is not a sufficient and necessary condition for the uniqueness and existence of solutions for both sde and ode.

4. Conclusion

Based on the calculations, we have concluded that the Lipschitz condition provides a nearly confirmative evidence of the existence of the solution. However, there are some new criteria ensuring the uniqueness and existence of the solution as well. In the next study, we will focus on the properties of those SDE that does not satisfy the Lipschitz condition.

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