SYMMETRIC POLYNOMIALS IN THE VARIETY
GENERATED BY GRASSMANN ALGEBRAS

NAZAN AKDOGAN, ŞEHMUS FINDIK

Abstract. Let \( G \) denote the variety generated by infinite dimensional Grassmann algebras; i.e., the collection of all unitary associative algebras satisfying the identity \([z_1, z_2, z_3] = 0\), where \([z_i, z_j] = z_i z_j - z_j z_i\). Consider the free algebra \( F_3 \) in \( G \) generated by \( X_3 = \{x_1, x_2, x_3\} \). We call a polynomial \( p \in F_3 \) symmetric if it is preserved under the action of the symmetric group \( S_3 \) on generators, i.e., \( p(x_1, x_2, x_3) = p(x_{\xi 1}, x_{\xi 2}, x_{\xi 3}) \) for each permutation \( \xi \in S_3 \).

The set of symmetric polynomials form the subalgebra \( F_3^{S_3} \) of invariants of the group \( S_3 \) in \( F_3 \). The commutator ideal \( F_3' \) of the algebra \( F_3 \) has a natural left \( K[X_3] \)-module structure, and \((F_3')^{S_3}\) is a left \( K[X_3]^{S_3} \)-module. We give a finite free generating set for the \( K[X_3]^{S_3} \)-module \((F_3')^{S_3}\).

1. Introduction

Let \( K[X_n] \) be the commutative associative unitary polynomial algebra over a field \( K \) of characteristic zero generated by \( X_n = \{x_1, \ldots, x_n\} \). We consider the general linear group \( GL_n(K) \) which acts on \( x_1, \ldots, x_n \). This action is extended to an action of \( GL_n(K) \) on \( K[X_n] \) canonically, which is an automorphism of the polynomial algebra \( K[X_n] \). If \( G \leq GL_n(K) \), then the algebra \( K[X_n]^G \) of invariants of \( G \) is defined to be the set of polynomials \( p(X_n) \in K[X_n] \) such that \( g(p(X_n)) = p(X_n) \) for each \( g \in G \).

One may see the works of Hilbert ([13], [14]) published in the last years of 19\(^{th}\) century, dealing with the finite generation of the algebra of invariants and finite bases of syzygies, which are known as the first and the second fundamental theorems of invariant theory. At the 1900 Paris congress, Hilbert’s 14\(^{th}\) problem [15] was proposed by himself to generalize his result to any subgroup of \( GL_n(K) \). A special case of the problem is to determine whether the algebra \( K[X_n]^G \) of invariants is finitely generated for any subgroup \( G \) of \( GL_n(K) \). Some partial affirmative results were obtained in response to the problem. For example, Noether [23] showed that the algebra \( K[X_n]^G \) of invariants is finitely generated when \( G \) is a finite subgroup of \( GL_n(K) \). Another approach in the same direction was given by Shephard and Todd [24] and Chevalley [4] which considers a special case, when the finite group \( G \) is generated by reflections, and \( K[X_n]^G \) is generated by \( n \) algebraically independent homogeneous elements. Finally, Nagata [22] gave a counterexample to Hilbert’s 14\(^{th}\) problem in 1959. One can see the survey on this problem by Humphreys [16] for more information.

---

2010 Mathematics Subject Classification. 15A75, 13A50, 16R30.
Key words and phrases. Grassmann algebras, symmetric polynomials.
The problem were applied to noncommutative invariant theory, as well. Let $K\langle X_n \rangle$ denote a free associative algebra. Kharchenko [18] extended Noether’s result proving that $(K\langle X_n \rangle/T)^G$, where $T$ is a $T$-ideal, is finitely generated for every finite subgroup $G$ of $GL_n(K)$ if and only if $K\langle X_n \rangle/T$ satisfies the ascending chain condition on two-sided ideals. Furthermore, Dicks and Formanek [5] and Kharchenko [18] identified the condition on finite groups $G$ for finite generation of $K\langle X_n \rangle^G$: The algebra of invariants $K\langle X_n \rangle^G$ of a finite subgroup $G$ of $GL_n(K)$ is finitely generated if and only if $G$ is a cyclic group acting by scalar multiplication. Lane [20] and Kharchenko [17] showed that $K\langle X_n \rangle^G$ is a free algebra for any finite subgroup of $GL_n(K)$. Long after, Domokos [6] showed that $(K\langle X_n \rangle/T)^G$ is generated by algebraically independent elements for a finite group $G$ if and only if $G$ is a pseudo-reflection group and $T$ contains the polynomial $[[z_1, z_2], z_2]$. We recommend Formanek’s survey [11] for more detail on the noncommutative generalizations of the problem.

The most basic example of an algebra of invariants is defined by the symmetric polynomials. Let $G = S_n$ be the group of permutation matrices in $GL_n(K)$ acting on the algebra $K\langle X_n \rangle$ by permuting the variables; i.e.,

$$\xi \cdot p(x_1, \ldots, x_n) = p(x_{\xi 1}, \ldots, x_{\xi n}), \quad \xi \in S_n, \quad p \in K\langle X_n \rangle.$$ 

The algebra $K\langle X_n \rangle^{S_n}$ of invariants of $S_n$ is a subalgebra of symmetric polynomials in $K\langle X_n \rangle$. It is well known that $K\langle X_n \rangle^{S_n}$ is generated by the elementary symmetric polynomials $\sigma_j = \sum x_{i_1} \cdots x_{i_j}$, $i_1 < \cdots < i_j$, $j = 1, \ldots, n$ which are algebraically independent over $K$. Furthermore, $K\langle X_n \rangle^{S_n}$ is isomorphic to a polynomial algebra in $n$ variables since $\sigma_1, \ldots, \sigma_n$ are algebraically independent. Besides, the set $\{\sum_{i=1}^n x_i^k : k = 1, \ldots, n\}$ is another generating set for $K\langle X_n \rangle^{S_n}$.

When it comes to noncommutative-algebraic algebras, one may consider Lie algebras, Leibniz algebras, etc. Bryant and Papistas [2] showed that the algebra of invariants in the free Lie algebra is not finitely generated for a finite nontrivial group of automorphisms of the algebra. A similar result is obtained for free metabelian Lie algebra in the same paper. Some recent works related with the Lie and Leibniz generalizations of the problem have been carried out. Let $L_n$ and $L_n'$ be the free metabelian Lie and Leibniz algebras of rank $n$, respectively. In both cases the commutator ideals $L_n'$ and $L_n'$ of $L_n$ and $L_n$, respectively, are right $K\langle X_n \rangle$-modules. A minimal infinite generating set for the Lie algebra $L_n^{S_2}$ was given in [9], while a finite generating set for the $K\langle X_n \rangle^{S_2}$-module $(L_n)^{S_2}$ was provided by Drensky et al. [8]. Besides, characterization of the symmetric polynomials in $(L_n')^{S_2}$ as a $K\langle X_n \rangle^{S_2}$-module was given in [10].

An important class of noncommutative algebras is the variety generated by the Grassmann identity $[[z_1, z_2], z_3] = 0$. Let $F_n$ be the free algebra of rank $n$ in this variety. The symmetric polynomials in $F_n^{S_2}$ have been studied in [1]. In this paper, we consider the free associative algebra $F_3$ of rank 3 generated by $X_3$ over a field $K$ of characteristic zero. Its commutator ideal $F_3' = F_3[F_3, F_3]$ is a left $K\langle X_3 \rangle$-module. We investigate the algebra $F_3^{S_3}$ of invariants of $S_3$, and give a finite free generating set for $(F_3)^{S_3}$ as a left $K\langle X_3 \rangle^{S_3}$-module.

2. Preliminaries

Let $K$ be a field of characteristic zero and let $K\langle Y_n \rangle$ be the free associative algebra over $K$ with the generating set $Y_n = \{y_1, \ldots, y_n\}$. An element $f$ in the
algebra $K(Y_n)$ is a polynomial identity of an algebra $A$ if $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in A$. An algebra $A$ is called a polynomial identity algebra or simply a PI-algebra if it has a nontrivial polynomial identity. The set $T(A)$ of all polynomial identities of $A$, which are invariant under all endomorphisms of $K(Y_n)$, forms an ideal of $K(Y_n)$. This ideal is called the $T$-ideal of $A$. The class of all associative algebras satisfying the polynomial identities from $T(A)$ is the variety generated by the algebra $A$.

Now let $I$ be the ideal of $K(Y_n)$ generated by the elements of the form $y_i y_j + y_j y_i$, $1 \leq i, j \leq n$. The Grassmann algebra is the factor algebra $K(Y_n)/I$, generated by the elements $y_i + I$, $1 \leq i \leq n$, over the field $K$. The Grassmann algebra is a PI-algebra satisfying the polynomial identity $[[z_1, z_2], z_3] = 0$, where $[z_1, z_2] = z_1 z_2 - z_2 z_1$. Krakowski and Regev [19] proved that the $T$-ideal of the Grassmann algebra is generated by this polynomial identity. See also [12] on the identities of the Grassmann algebras when the based field is of positive characteristic, [3] for graded identities of the Grassmann algebra, and [21] on Grassmann algebras of graphs.

The variety $\mathcal{G}$ generated by the Grassmann algebra contains unitary associative algebras satisfying the polynomial identity

$$[[z_1, z_2], z_3] = [z_1, z_2]z_3 - z_3[z_1, z_2] = (z_1 z_2 - z_2 z_1)z_3 - z_3(z_1 z_2 - z_2 z_1) = 0,$$

over the field $K$. Recall that the above identity implies

$$[z_1, z_2][z_3, z_4] = -[z_1, z_3][z_2, z_4].$$

Let $F_n = F_n(\mathcal{G})$ be the free algebra in $\mathcal{G}$ of rank $n$ generated by $X_n = \{x_1, \ldots, x_n\}$ over $K$. The commutator ideal $F'_n = F_n[F_n, F_n]F_n$ of the algebra $F_n$ has the following basis via [7].

$$x_{i_1}^{a_1} \cdots x_{i_c}^{a_c} [x_{i_1}, x_{i_2}] \cdots [x_{i_{2c-1}}, x_{i_{2c}}], a_i \geq 0, i_1 > \cdots > i_{2c}, c \geq 1.$$

The identity [11] implies that $[x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}] = 0$ if some of $x_{i_j}$’s coincide, since

$$[x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}] = -[x_{i_1}, x_{i_3}][x_{i_2}, x_{i_4}] = [x_{i_1}, x_{i_4}][x_{i_2}, x_{i_3}],$$

and $[x_{i_1}, x_{i_2}] = x_{i_2}^2 - x_{i_1}^2 = 0$. This means that

$$[x_2, x_1][x_2, x_1] = [x_3, x_1][x_2, x_1] = [x_3, x_2][x_2, x_1] = 0$$

or

$$[x_2 x_1 - x_1 x_2][x_2, x_1] = (x_3 x_1 - x_1 x_3)[x_2, x_1] = (x_3 x_2 - x_2 x_3)[x_2, x_1] = 0$$

Hence we have the followings in $F'_2$ and $F'_3$, respectively, as a consequence of Grassmann identity.

$$x_1 x_2 w = x_2 x_1 w, \ w \in F'_2,$$

$$x_1 x_2 x_3 w = x_1 x_3 x_2 w = x_2 x_1 x_3 w = x_2 x_3 x_1 w = x_3 x_1 x_2 w = x_3 x_2 x_1 w, \ w \in F'_3.$$

Therefore, $F'_2$ and $F'_3$ can be considered as a left $K[X_2]$-module and $K[X_3]$-module, respectively. Note that $F''_n$ is not a left $K[X_n]$-module when $n \geq 4$, since

$$x_1 x_2 [x_3, x_4] \neq x_2 x_1 [x_3, x_4]$$
as a consequence of the fact that \([x_1, x_2][x_3, x_4] \neq 0\). The following result is a direct consequence of the basis and equalities provided above.

**Corollary 2.1.** The commutator ideal \(F'_2\) is a free left \(K[X_2]\)-module generated by \([x_2, x_1]\), and \(F'_3\) is a free left \(K[X_3]\)-module with generators \([x_2, x_1]\), \([x_3, x_1]\), and \([x_3, x_2]\).

A polynomial \(s(X_n) \in F_n\) is called symmetric if \(s(x_1, \ldots, x_n) = s(x_{\xi 1}, \ldots, x_{\xi n})\) for all \(\xi \in S_n\). Symmetric polynomials form a subalgebra \(F_{S_n}^S\) of \(F_n\) called the algebra of invariants of the symmetric group \(S_n\). Clearly the algebra \((F'_2)^{S_2}\) and \((F'_3)^{S_3}\) can be considered as a left \(K[X_2]^{S_2}\)-module and a left \(K[X_3]^{S_3}\)-module, respectively.

Let us initiate by \(n = 2\). The next theorem gives the description of the \(K[X_2]^{S_2}\)-module \((F'_2)^{S_2}\). Although stated in \([1]\), we give a short sketch of the proof for the completeness of our paper.

**Theorem 2.2.** The algebra \((F'_2)^{S_2}\) of symmetric polynomials in the commutator ideal \(F'_2\) is generated by the single element \((x_2-x_1)[x_2, x_1]\) as a left \(K[X_2]^{S_2}\)-module.

**Proof.** Let \(e(x_1, x_2) = p(x_1, x_2)[x_2, x_1] \in (F'_2)^{S_2}\), where \(p(x_1, x_2) \in K[X_2]\). Then \(e(x_1, x_2) = e(x_2, x_1)\) implies that \((p(x_1, x_2) + p(x_2, x_1))[x_2, x_1] = 0\) in the free module generated by \([x_2, x_1]\). Thus the polynomial \(p(x_1, x_2) \in K[X_2]\) can be expressed as \(q(x_1, x_2)(x_2 - x_1)\) for some \(q(x_1, x_2) \in K[X_2]^{S_2}\).

The results obtained in this paper can be considered as a next step of the work done in Theorem 2.2.

### 3. Main Results

In this section, we study the algebra \((F'_3)^{S_3}\) of symmetric polynomials in the commutator ideal \(F'_3\) of the algebra \(F_3\). We give explicit form of symmetric polynomials in the algebra \((F'_3)^{S_3}\). We also provide a finite set of free generators for \((F'_3)^{S_3}\) as a module of \(K[X_3]^{S_3}\). In order to study the algebra \((F'_3)^{S_3}\), we work in the commutator ideal \(F'_3\), which is freely generated by \([x_2, x_1]\), \([x_3, x_1]\), and \([x_3, x_2]\) as a \(K[X_3]\)-module due to Corollary 2.1.

In the next theorem, we provide the description of \((F'_3)^{S_3}\) as a left \(K[X_3]^{S_3}\)-module.

**Theorem 3.1.** A polynomial \(f \in F'_3\) is symmetric if and only if

\[
f(x_1, x_2, x_3) = p(x_1, x_2, x_3)[x_2, x_1] + p(x_1, x_3, x_2)[x_3, x_1] + p(x_2, x_3, x_1)[x_3, x_2]
\]

for some \(p \in K[X_3]\) such that \(p(x_1, x_2, x_3) = -p(x_2, x_1, x_3)\).

**Proof.** Let \(f(x_1, x_2, x_3)\) be an arbitrary element in \((F'_3)^{S_3} \subset F'_3\) of the form

\[
f(x_1, x_2, x_3) = p(x_1, x_2, x_3)[x_2, x_1] + q(x_1, x_2, x_3)[x_3, x_1] + r(x_1, x_2, x_3)[x_3, x_2],
\]

for some \(p, q, r \in K[X_3]\). The fact that

\[
f(x_1, x_2, x_3) = f(x_{\xi 1}, x_{\xi 2}, x_{\xi 3}), \ \xi \in S_3,
\]

combined by Corollary 2.1 imply the followings.

\[
p(x_1, x_2, x_3) = q(x_1, x_3, x_2) = -p(x_2, x_1, x_3) = -q(x_2, x_3, x_1)
\]

\[
r(x_3, x_1, x_2) = r(x_3, x_2, x_1).
\]
One may reduce the equalities above to
\[ p(x_1, x_2, x_3) = p(x_1, x_3, x_2), \quad r(x_1, x_2, x_3) = p(x_2, x_3, x_1) \]
such that \( p(x_1, x_2, x_3) = -p(x_2, x_1, x_3) \).

Conversely, let \( f(x_1, x_2, x_3) \) satisfy the statement of the theorem, i.e.,
\[ f(x_1, x_2, x_3) = p(x_1, x_2, x_3)[x_2, x_1] + p(x_1, x_3, x_2)[x_3, x_1] + p(x_2, x_3, x_1)[x_3, x_2] \]
for some \( p \in K[X_3] \) such that \( p(x_1, x_2, x_3) = -p(x_2, x_1, x_3) \). We have to show that
\( f(x_1, x_2, x_3) \) is preserved under the action of transpositions (12) and (13) since they
generate the symmetric group \( S_3 \). We start with the action of (12).
\[ f(x_2, x_1, x_3) = p(x_2, x_1, x_3)[x_1, x_2] + p(x_2, x_3, x_1)[x_3, x_1] + p(x_1, x_3, x_2)[x_3, x_2] \]
\[ = -p(x_1, x_2, x_3)[x_1, x_2] + p(x_1, x_3, x_2)[x_3, x_1] + p(x_2, x_3, x_1)[x_3, x_2] \]
\[ = p(x_1, x_2, x_3)[x_2, x_1] + p(x_1, x_3, x_2)[x_3, x_1] + p(x_2, x_3, x_1)[x_3, x_2] \]
Similarly, one may easily show that \( f(x_3, x_2, x_1) = f(x_1, x_2, x_3) \). \( \square \)

**Remark 3.2.** Let \( f \) be a symmetric polynomial in \((F_3')^{S_3}\) as stated in Theorem 3.1 Then one may easily verify that \( p = p_1 p_2 \) for some \( p_1 \in K[X_3]^{S_3} \) and some \( p_2 \in K[X_3] \) of the form
\[ p_2 = \sum_{0 \leq a < b, 0 \leq c} \epsilon_{abc} (x_1^a x_2^b - x_1^b x_2^a) x_3^c \]
where \( \epsilon_{abc} \in K \). Let us denote
\[ f_{a,b,c} = (x_1^a x_2^b - x_1^b x_2^a) x_3^c \]
for nonnegative integers \( a, b, c \). This immediately implies the following result.

**Corollary 3.3.** The set \( \{ f_{a,b,c} \mid 0 \leq a < b, 0 \leq c \} \) is a basis for the \( K[X_3]^{S_3} \)-module \((F_3')^{S_3} \); i.e., \( f \in (F_3')^{S_3} \) if and only if
\[ f(x_1, x_2, x_3) = \sum_{0 \leq a < b, 0 \leq c} \delta_{abc} f_{a,b,c} \]
for some uniquely determined polynomials \( \delta_{abc} \in K[X_3]^{S_3} \).

The rest of the paper is devoted to obtain a generating set for the \( K[X_3]^{S_3} \)-module \((F_3')^{S_3} \). Let us denote
\[ \sigma_1 = x_1 + x_2 + x_3, \quad \sigma_2 = x_1 x_2 + x_1 x_3 + x_2 x_3, \quad \sigma_3 = x_1 x_2 x_3, \]
which are the elementary symmetric polynomials freely generating the algebra \( K[X_3]^{S_3} \). Note that \( \{ \nu_1, \nu_2, \nu_3 \} \) is another generating set for the algebra \( K[X_3]^{S_3} \), where \( \nu_k \) is defined to be \( \nu_k = x_1^k + x_2^k + x_3^k \in K[X_3]^{S_3} \), for each \( k \geq 1 \).

The proof of the following technical lemma is straightforward.

**Lemma 3.4.**
\begin{align*}
(3) & \quad f_{0,b,c} = \nu_c f_{0,b,0} - f_{0,b+c,0} + f_{b,c,0}, \quad b, c \geq 1, \\
(4) & \quad f_{a,b,c} = -f_{b,a,c}, \quad a, b, c \geq 0, \\
(5) & \quad f_{0,b,0} = \sigma_1 f_{0,b-1,0} - \sigma_2 f_{0,b-2,0} + \sigma_3 f_{0,b-3,0}, \quad b \geq 4 \\
(6) & \quad f_{0,3,0} = \sigma_1 f_{0,2,0} - \sigma_2 f_{0,1,0},
\end{align*}
Case II. Let $a$, $b$, and $c$ such that $0 \leq a < b , 0 \leq c$ generate the $\mathbb{K}[X_3]_{^{S_3}}$-module. Depending on the relations between $a$ and $c$, we may handle the problem in three cases.

Case I. Let $a = c$, and $b = a + k$ for some positive integer $k$. Then

$$f_{a,b,c} = (x_1^a x_2^b - x_1^b x_2^a)x_3^2[x_2, x_1] + (x_1^a x_3^b - x_1^b x_3^a)x_2^2[x_3, x_1] + (x_2^a x_3^b - x_2^b x_3^a)x_1^2[x_3, x_2]$$

Case II. Let $a < c$, and $b = a + k$, $c = a + l$, for some $k, l \in \mathbb{Z}^+$. Then similarly, we get that $f_{a,b,c} = \sigma^a f_{k,l}$. 

Case III. Let $c < a$, and $a = c + k$, $b = c + l$, for some $k, l \in \mathbb{Z}^+$. Then we have $f_{a,b,c} = \sigma^a f_{k,l}$. 

Therefore, the generating set reduces to the set consisting of the elements of the form $f_{0,0,0}$, $f_{a,b,0}$, and $f_{a,b,c}$. Now by Lemma 3.1 options 3 and 4, we may eliminate the elements $f_{0,0,0}$ from the generating set, as well. Note that we use the option 4 in case $c < b$, in order to fix the notation as in Corollary 3.3.

Considering Lemma 3.4 options 3 and 4, one may conclude by induction that every element of the form $f_{0,0,0}$, $b \geq 3$, is included in the $\mathbb{K}[X_3]_{^{S_3}}$-module generated by $f_{0,1,0}$ and $f_{0,2,0}$. Similarly, making use of Lemma 3.4 options 7, 8, we get that elements of the form $f_{a,b,0}$, $b \geq 3$, are included in the $\mathbb{K}[X_3]_{^{S_3}}$-module generated by $\{f_{a,1,0}, f_{a,2,0}, f_{a,0,0}\}$, or by $\{f_{1,a,0}, f_{2,a,0}, f_{0,a,0}\}$ via Lemma 3.4 [4]. Note that $f_{1,1,0} = f_{2,2,0} = 0$, $f_{2,1,0} = -f_{1,2,0}$, and elements of the form $f_{0,a,0}$ are in the $\mathbb{K}[X_3]_{^{S_3}}$-module generated by $\{f_{0,1,0}, f_{0,2,0}\}$ when $a \geq 3$. Hence the elements $f_{1,1,0}$ and $f_{2,2,0}$, $a \geq 3$, are included in the $\mathbb{K}[X_3]_{^{S_3}}$-module generated by the elements $f_{1,2,0}$, $f_{0,1,0}$ and $f_{0,2,0}$ using Lemma 3.4 options 3, 4, and 5, inductively.

Theorem 3.6. The set \{f_{0,1,0}, f_{0,2,0}, f_{1,2,0}\} is a minimal generating set for the $\mathbb{K}[X_3]_{^{S_3}}$-module $(F'_3)^{S_3}$. Moreover, $(F'_3)^{S_3}$ is freely generated by the elements $f_{0,1,0}$, $f_{0,2,0}$ and $f_{1,2,0}$ as a $\mathbb{K}[X_3]_{^{S_3}}$-module.

Proof. Recall that $f_{0,1,0}$, $f_{0,2,0}$, and $f_{1,2,0}$ are of homogeneous degree 3, 4, and 5, in the graded algebra $F'_3$, of the form

$$f_{0,1,0} = (x_2 - x_1)[x_2, x_1] + (x_3 - x_1)[x_3, x_1] + (x_3 - x_2)[x_3, x_2]$$

$$f_{0,2,0} = (x_3^2 - x_1^2)[x_2, x_1] + (x_3^2 - x_3^2)[x_4, x_1] + (x_3^2 - x_2^2)[x_3, x_2]$$

$$f_{1,2,0} = (x_1 x_2^2 - x_2^2 x_2)[x_2, x_1] + (x_1 x_3^2 - x_2^2 x_3)[x_3, x_1] + (x_2 x_3^2 - x_2^2 x_3)[x_3, x_2]$$

Clearly the element $f_{0,1,0}$ of degree 3 is not included in the $\mathbb{K}[X_3]_{^{S_3}}$-module generated by $\{f_{0,2,0}, f_{1,2,0}\}$. On the other hand, assuming that $f_{0,2,0}$ is in the $\mathbb{K}[X_3]_{^{S_3}}$-module generated by $\{f_{0,1,0}, f_{1,2,0}\}$, we have $f_{0,2,0} = \alpha f_{0,1,0}$ for some $\alpha \in \mathbb{K}$. Now
by Corollary 2.1 considering the coefficient of $[x_2, x_1]$ in the free $K[X_3]$-module $F'_3$ generated by $[x_2, x_1]$, $[x_3, x_1]$, and $[x_3, x_2]$, we have that

$$(x_2^2 - x_1^2) = \alpha(x_1 + x_2 + x_3)(x_2 - x_1)$$

and hence $\alpha = 0$, which contradicts with $f_{0,2,0} \neq 0$. Besides, if $f_{1,2,0}$ is included in the $K[X_3]^{S_3}$-module generated by $\{f_{0,1,0}, f_{0,2,0}\}$, then

$$f_{1,2,0} = (\alpha \sigma_1^2 + \beta \sigma_2) f_{0,1,0} + \gamma \sigma_1 f_{0,2,0}$$

for some $\alpha, \beta, \gamma \in K$. Similarly, let us consider the coefficient

$$(x_1 x_2^2 - x_1^2 x_2) = (\alpha(x_1 + x_2 + x_3)^2 + \beta(x_1 x_2 + x_1 x_3 + x_2 x_3))(x_2 - x_1)
+ \gamma(x_1 + x_2 + x_3)(x_2^2 - x_1^2)$$

of $[x_2, x_1]$ in the equation. Clearly $\alpha x_2^2 x_2 = 0$ gives $\alpha = 0$, and then by coefficients of $x_1 x_2^2$ and $x_2^2 x_3$ we have $\beta + \gamma = 1$ and $\beta + \gamma = 0$, which is a contradiction. Therefore, the set $\{f_{0,1,0}, f_{0,2,0}, f_{1,2,0}\}$ is a minimal generating set.

Finally, we have to show that $f_{0,1,0}$, $f_{0,2,0}$, and $f_{1,2,0}$ are free generators. Note that $f_{0,2,0}^2 = f_{0,2,0} f_{1,2,0} = f_{1,2,0} = 0$ due to the equation (2), and thus we may assume that these elements can be taken linearly in a possible relation. Let

$$s_1 f_{0,1,0} + s_2 f_{0,2,0} + s_3 f_{1,2,0} = 0$$

for some symmetric polynomials $s_1, s_2, s_3 \in K[X_3]^{S_3}$. In the same way, by the coefficient of $[x_2, x_1]$, we obtain that

$$s_1(x_2 - x_1) + s_2(x_2^2 - x_1^2) + s_3(x_1 x_2^2 - x_1^2 x_2) = 0$$

or

$$s_1 + s_2(x_2 + x_1) + s_3 x_1 x_2 = 0$$

in the commutative polynomial algebra $K[X_3]$. This means that the polynomial

$$s_2(x_2 + x_1) + s_3 x_1 x_2$$

is symmetric, which is preserved under the action of the transposition $(23) \in S_3$. Then we have the following.

$$s_2(x_2 + x_1) + s_3 x_1 x_2 = s_2(x_3 + x_1) + s_3 x_2 x_3
s_2 x_2 + s_3 x_1 x_2 = s_2 x_3 + s_3 x_1 x_3
x_2(s_2 + s_3 x_1) = x_3(s_2 + s_3 x_1)$$

Therefore, $(x_2 - x_3)(s_2 + s_3 x_1) = 0$, implying that $s_2 + s_3 x_1 = 0$. This implies that $s_3 x_1$ is symmetric. However, $s_3 x_1 = (12) \cdot s_3 x_1 = s_3 x_2$ gives $s_3 = 0$, where (12) $\in S_3$. Hence $s_2 = s_3 x_1 = 0$, and $s_1 = -s_2(x_2 + x_1) - s_3 x_1 x_2 = 0$. □

We give the next example to illustrate how to express a symmetric polynomial in terms of free generators $f_{0,1,0}$, $f_{0,2,0}$, and $f_{1,2,0}$.

**Example 3.7.** Let us consider the symmetric polynomial

$$f_{2,4,5} = (x_1^2 x_2^4 - x_1 x_2^2 x_3^2)^5 [x_2, x_1] + (x_1^2 x_2^4 - x_1 x_3^2)^2 x_2^2 [x_3, x_1] + (x_2^2 x_3^4 - x_2 x_3^2)^2 x_1^5 [x_3, x_2].$$

Then we have that $f_{2,4,5} = \sigma_3^2 f_{0,2,3}$ by Case II of the proof of Theorem 3.6. Now by Lemma 3.4 (3),

$$f_{2,4,5} = \sigma_3^2 (\nu_3 f_{0,2,0} - f_{0,5,0} + f_{2,3,0})$$
and by options (5), (8) and (4) of Lemma 3.4 we have that
\[
f_{2,4,5} = \sigma_2^2(\nu_3 f_0,2,0 - (\sigma_1 f_0,4,0 - \sigma_2 f_0,3,0 + \sigma_3 f_0,2,0) + (\sigma_1 f_2,2,0 - \sigma_2 f_2,1,0 - \sigma_3 f_0,2,0))
\]
\[
= \sigma_2^2(\nu_3 f_0,2,0 - (\sigma_1(\sigma_1 f_0,3,0 - \sigma_2 f_0,2,0 + \sigma_3 f_0,1,0) - \sigma_2 f_0,3,0 + \sigma_3 f_0,2,0)
\]
\[
+ (\sigma_2 f_{1,2,0} - \sigma_3 f_{0,2,0}))
\]
\[
= \sigma_2^2((-\sigma_1 \sigma_3 f_{0,1,0} + \nu_3 + \sigma_1 \sigma_2 - 2 \sigma_3) f_{0,2,0} + (-\sigma_1^2 + \sigma_2) f_{0,3,0} + \sigma_2 f_{1,2,0})
\]
\[
= \sigma_2^2((-\sigma_1 \sigma_3 f_{0,1,0} + \nu_3 + \sigma_1 \sigma_2 - 2 \sigma_3) f_{0,2,0} + (-\sigma_1^2 + \sigma_2)(\sigma_1 f_{0,2,0} - \sigma_2 f_{0,1,0})
\]
\[
+ \sigma_2 f_{1,2,0})
\]
\[
= (-\sigma_1 \sigma_3^3 + \sigma_1^2 \sigma_2 \sigma_3^2 - \sigma_2^2 \sigma_3^2) f_{0,1,0}
\]
\[
+ (\nu_3 \sigma_3^2 + \sigma_1 \sigma_2 \sigma_3^2 - 2 \sigma_3^3 - \sigma_1^2 \sigma_3^2 + \sigma_1 \sigma_2 \sigma_3^2) f_{0,2,0} + \sigma_2 \sigma_3^2 f_{1,2,0}.
\]

We complete the paper by the next problem.

**Problem 3.8.** Determine symmetric polynomials in $F_n'$ for $n \geq 4$.

**REFERENCES**

[1] N. Akdoğan, On the symmetric polynomials in the variety of Grassmann algebras, Erzincan Un. J. Sci. Tech., to appear.

[2] R. Bryant, A. Papistas, On the fixed points of a finite group acting on a relatively free lie algebra, Glasgow Mathematical Journal, 42 (2000) 2, 167-181.

[3] L. Centrone, On the graded identities of the Grassmann algebra, J. Algebra Comb. Discrete Appl. 4 (2) (2016) 165-180.

[4] C. Chevalley, Invariants of finite groups generated by reflections, Amer. J. Math., 77 (1955), 778-782.

[5] W. Dicks, E. Formanek, Poincare series and a problem of S. Montgomery, Lin. Mult. Alg. 12 (1982), 21-30.

[6] M. Domokos, Relatively free invariant algebras of finite reflection groups, Trans. Amer. Math. Soc., 348 (1996), 2217-2233.

[7] V. Drensky, Free Algebras and PI-Algebras, Springer, Singapore, 1999.

[8] V. Drensky, Ş. Fındık, N.S. Öğüşlü, Symmetric polynomials in the free metabelian Lie algebras, Mediterr. J. Math., 17 (2020) 5, 1-11.

[9] Ş. Fındık, N.S. Öğüşlü, Palindromes in the free metabelian Lie algebras, Int. J. Algebra and Comput. 29 (2019) 5, 885-891.

[10] Ş. Fındık, Z. Öz Kurt, Symmetric polynomials in Leibniz algebras and their inner automorphisms, Turkish. J. Math., 44 (2020) 6, 2306-2311.

[11] E. Formanek, Noncommutative invariant theory, Contemp. Math., 43 (1985), 87-119.

[12] A. Giambruno, F. Koshlukov, On the identities of the Grassmann algebras in characteristic $p > 0$, Isr. J. Math., 122 (2001), 305-316.

[13] D. Hilbert, Ueber die Theorie der algebraischen Formen, Math. Ann., 36 (1890), 473-534.

[14] D. Hilbert, Ueber die vollen Invariantensysteme, Math. Ann., 42 (1893), 313-373.

[15] D. Hilbert, Mathematische Probleme, Göttinger Nachrichten (1900), 253-297; Arch. Math. u. Phys., 3 (1901) 1, 44-63; Translation: Bull. Amer. Math. Soc., 8 (1902) 10, 437-470.

[16] J.E. Humphreys, Hilbert’s Fourteenth Problem, Amer. Math. Monthly Vol., 85 (1978) 5, 341-353.

[17] V.K. Kharchenko, Algebras of invariants of free algebras, Algebra i Logika 17 (1978), 478-487. Translation: Algebra and Logic, 17(1978), 316-321.

[18] V.K. Kharchenko, Noncommutative invariants of finite groups and Noetherian varieties, J.Pure Appl. Alg., 31 (1984), 83-90.

[19] D. Krakovski, A. Regev, The polynomial identities of the Grassmann algebra, Trans. Amer. Math. Soc. 181 (1973), 429-438.

[20] D.R. Lane, Free algebras of rank two and their automorphisms, Ph.D. thesis, Bedford College, London, 1976.

[21] L. Makar-Limanov, On Grassmann algebras of graphs, Journal of algebra, 87 (1984), 283-289.

[22] M. Nagata, On the 14-th problem of Hilbert, Amer. J. Math., 81 (1959), 766-772.
SYMMETRIC POLYNOMIALS IN THE VARIETY GENERATED BY GRASSMANN ALGEBRAS

[23] E. Noether, Der Endlichkeitssatz der Invarianten endlicher Gruppen, Math. Ann., 77 (1916), 89-92.
[24] G.C. Shephard, J. A. Todd, Finite unitary reflection groups, Can. J. Math., 6 (1954), 274-304.

Department of Mathematics, Çukurova University, 01330 Balcalı, Adana, Turkey

Department of Mathematics, İstanbul Technical University, İstanbul, Turkey

Email address: nakdogan@itu.edu.tr

Email address: sfindik@cu.edu.tr