DISPERSION AND ATTENUATION FOR AN ACOUSTIC WAVE EQUATION CONSISTENT WITH VISCOELASTICITY

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Abstract

An acoustic wave equation for pressure accounting for viscoelastic attenuation is derived from viscoelastic equations of motion. It is assumed that the relaxation moduli are completely monotonic. The acoustic equation differs significantly from the equations proposed by Szabo (1994) and in several other papers. Integral representations of dispersion and attenuation are derived. General properties and asymptotic behavior of attenuation and dispersion in the low and high frequency range are studied. The results are compatible with experiments. The relation between the asymptotic properties of attenuation and wavefront singularities is examined. The theory is applied to some classes of viscoelastic models and to the quasi-linear attenuation reported in seismology.

keywords: wave propagation; viscoelasticity; attenuation; completely monotonic functions; complete Bernstein functions; ultrasound

Notation.

\[ f \ast g \] convolution
\[ \mathcal{L}(f) = \tilde{f} \] Laplace transform of \( f \)
\[ [a, b] \] the set \( a < x \leq b \)
\[ f(x) \sim_a g(x) \] asymptotic equivalence \( 0 < \lim_{x \to a} f(x)/g(x) < \infty \) for \( a = 0 \) or \( \infty \).

1 Introduction.

Correct modeling of wave attenuation is an important objective for several branches of acoustics and a vast literature is dedicated to this subject. We shall be concerned here with modeling of intrinsic attenuation (i.e. excluding attenuation due to backscatter) in the context of pressure wave equations used in acoustics. In this context many attenuation models have been constructed ad
hoc to match some aspect of the experimental data without taking into account the physical nature of the wave motion involved.

In many applications the mathematical model of propagation of acoustic pulses (longitudinal waves) in bio-tissues and polymers is based on a linear wave equation for the pressure field. The pressure wave equation can be derived from the equations of motion of linear elasticity. In order to account for attenuation some authors included in the wave equation an additional term involving a fractional time derivative or a more general pseudo-differential operator acting on the time variable [1] [2] [3]. This method of accounting for attenuation is called the time causal method in [4]. In this context causality means that the pseudo-differential operator is the convolution with a causal distribution, which ensures the validity of Kramers-Kronig relations. In [5] [6] the time derivatives have been modified in order to ensure that the attenuation obeys a fixed power law with an exponent $> 1$ in the entire frequency range. It will be shown that in all these cases the resulting pressure wave equations are inconsistent with the viscoelastic equations of motion.

Introducing attenuation by modifying the time derivatives in the equations of motion is justified in the context of poroelasticity [7] [8] [9] [10] [11] [12] [13]. In poroelasticity and poroacoustics the attenuation due to viscous flow in a porous medium is represented by the viscodynamic operator, which is a pseudo-differential operator acting on the time variables [14] [15] [16]. In poroelastic equations of motion the viscodynamic operator is applied to the inertial terms. In viscoelasticity attenuation is introduced through a time convolution operator in the constitutive equations. It will be shown in Sec. 2 that in both the viscoelastic equations of motion and in the acoustic wave equation attenuation is represented by a term involving a pseudodifferential operator acting on both the time and spatial variables. This observation is not new, for example Stokes’ equation has this structure.

The attempts to explain the experimentally observed power law frequency dependence of attenuation in polymers, bio-tissues and some viscous fluids in terms of an oversimplified power law attenuation model [1] [5] [6] result in unbounded phase speeds. Experiments covering the range 1 – 250 MHz indicate that the exponent of the power law lies between 1 and 2. In the oversimplified model it is assumed that the power law extends to the high frequency range. A power law with an exponent $> 1$ in the high-frequency range entails wave propagation with unbounded phase speed. It has been shown in [17] that experimental data lie in the low frequency range represented by the low-frequency asymptotics of viscoelastic relaxation models. It is shown in [17] that viscoelastic solids (i.e. viscoelastic media in which strain under constant load does not relax to 0) have power law exponents in the range 1 – 2 in the low-frequency range. Exponents below one indicate that the material is a viscoelastic fluid - such as some bio-tissues subject to a constant shear load. These general results are corroborated by the analysis of specific relaxation models in [18] for longitudinal waves. This indicates that experimental data can be explained in terms of viscoelastic models.

Kowar, Schertzer and coworkers [19] [20] constructed lossy wave equations...
with a bounded propagation speed by applying causal attenuation operators to the elastic pressure wave equation, which ensures bounded phase speed. They found that superlinear attenuation rates exist only in the low frequency range covered by the experiments. In their model, attenuation exhibits sublinear frequency dependence in the high frequency range. This approach however amounts to a modification of the time derivatives in the pressure wave equation, which is not consistent with viscoelastic equations of motion.

If the pressure wave equation is derived from viscoelastic equations of motion but the attenuation grows at a superlinear rate in the high frequency range then the creep compliance is not concave \[21\]. This contradicted by observations. In the special case of strict power law attenuation with an exponent \(> 1\) in the entire frequency range it is easy to prove that the creep compliance is decreasing and convex.

Several researchers have avoided the problem by working directly with the equations of motion of linear viscoelasticity or thermo-viscoelasticity with an appropriate stress relaxation model \[22, 23, 24\]. We shall however present an approximate derivation of a linear pressure wave equation based on the equations of motion of viscoelasticity. It will be thus demonstrated that viscoelastic attenuation is represented in the pressure wave equation by a mixed temporal and spatial derivative or a time convolution operator acting on a Laplacian of the pressure field (Sec. 3.2). It will then be shown in Sec. 3 that the attenuation and dispersion associated with the wave equation has the same properties as the attenuation and dispersion in linear viscoelasticity examined in \[25, 17\]. The theory will be applied to some classes of viscoelastic media.

In Sec. 3.5 we shall establish the relations between the asymptotics of the attenuation function and regularity properties of Green’s function at the wavefronts. So far regularity at the wavefronts has only been studied in connection with the singularity of the relaxation modulus or its derivative.

The dispersion-attenuation theory will also be used to examine the controversial linear frequency dependence of attenuation observed in geological media \[26, 27, 28\]. Linear attenuation is incompatible with viscoelasticity and with bounded phase speed. We shall therefore look for the closest approximation to linear attenuation compatible with bounded phase speed. We investigate here attenuation models which differ from linear attenuation by a slowly varying factor.

In Sec. 2 a wave equation for pressure in a viscoelastic medium will be derived. In Sec. 3.2 the dispersion and attenuation theory will be developed for this equation. The dissipation-attenuation theory presented in Sec. 3 depends on the assumption that the viscoelastic relaxation modulus is a completely monotonic (CM) function \[29\]. The CM property is so universal in viscoelasticity that several attempts have been made to justify it by an \(a priori\) argument \[30, 31, 32\]. In particular every spring-dashpot model and every fractional generalization of such models has a completely monotonic relaxation modulus. The last statement is an easy consequence of the duality theorem proved in \[29\]. Many other models of viscoelastic relaxation (Prony series, the Havriliak-Negami model and its special cases) are also expressed in terms CM relaxation moduli.
In Sec. 3.5 it is shown that the order of the wavefront singularity in media with bounded wavefront speed depends on the asymptotic behavior of the attenuation function at infinity.

In Sec. 4 the general theory is applied to some examples involving strongly singular convolution operators (\( K \) is singular at 0 but locally integrable, the phase speed is unbounded, Sec. 4.3), weakly singular kernels (\( K \) is non-singular, but its derivative \( K' \) has a singularity at 0, Sec. 4.2) and non-singular kernels (\( K \) and \( K' \) are continuous at 0, the attenuation function is bounded). In the first case the phase speed is unbounded while in the second case the wave fields exhibit finite wavefront speed. In Section 4.4 the case of nearly linear frequency dependence of attenuation is examined.

2 Derivation of the pressure wave equation.

Consider a homogeneous isotropic compressible viscous fluid defined by the constitutive equation
\[
\sigma = -P \mathbf{I} + \lambda \text{div} \mathbf{u}_t \mathbf{I} + \mu \left[ \nabla \mathbf{u}_t + (\nabla \mathbf{u}_t)^T \right]
\]
\[
P = c_0^2 (\rho - \rho_0) \tag{2}
\]
where \( \sigma \) denotes the Cauchy stress tensor, \( P \) represents the elastic part of the pressure, \( \mathbf{u} \) is the displacement vector, \( \rho \) is the density and \( \rho_0 \) is a reference density. The parameters \( K := \lambda + 2\mu \) and \( \mu \) are dynamic viscosities of volumetric and shear deformations and \( c_0 \) is an elastic propagation speed.

Mass conservation can be expressed in the form \( \tau_t = \tau \text{div} \mathbf{u}_t \), where \( \tau := 1/\rho \). Hence
\[
\rho_t = -\rho \text{div} \mathbf{u}_t \tag{3}
\]
For further reference we note that \( \rho - \rho_0 = \rho_0 [\exp(-\text{div} \mathbf{u}) - 1] \approx -\rho_0 \text{div} \mathbf{u} \) and thus equation (2) assumes the form \( P \approx -K \text{div} \mathbf{u}, K = \rho_0 c_0^2 \), in the linear approximation.

Equation (3) implies that
\[
\text{div} \sigma = -\nabla P + (\lambda + \mu) \nabla \text{div} \mathbf{u}_t + \mu \nabla^2 \mathbf{u}_t \tag{4}
\]
The momentum balance \( \rho \mathbf{u}_{tt} = \text{div} \sigma \) and the mass balance imply the following equation for \( \xi := \text{div} \mathbf{u} \):
\[
\rho \xi_{tt} = -\nabla^2 P + (\lambda + 2\mu) \nabla^2 \xi_{t} \tag{4}
\]
Eqs (2) and (3) imply that \( c_0^{-2} P_t = -\rho \xi_t \) and \( c_0^{-2} P_{tt} = -\rho \xi_{tt} - \rho_t \xi_t = -\rho \left( \xi_{tt} - \xi_t^2 \right) \). In a linearized theory we assume that
\[
|\xi_{tt}|^2 \ll |\xi_{tt}| \tag{5}
\]

hence \( c_0^{-2} P_{tt} = -\rho \xi_{tt} \). The final pressure equation is obtained by substituting the last results in (4)
\[
c_0^{-2} P_{tt} = \nabla^2 P + \frac{\lambda + 2\mu}{\rho_0 c_0^2} \nabla^2 P_t \tag{6}
\]
If multiplication by $\lambda$ and $\mu$ in eq. (1) is replaced by convolutions with CM kernels $\lambda(t)$ and $\mu(t)$, respectively, then eq. (6) assumes the following form
\[ c_0^{-2}P_{tt} = \nabla^2 P + \left(\rho_0 c_0^2\right)^{-1}\nabla^2 K * P_t \]
for $x \in \mathbb{R}^d, d = 1$ or $3$. The convolution kernel $K(t) := \lambda(t) + 2\mu(t)$ is a CM function.

We shall consider the Cauchy problem for equation (7) with the initial conditions
\[ P(t, x) = 0 \quad \text{for } t < 0, \]
\[ \lim_{t \to 0^+} P(t, x) = P_0(x), \quad \lim_{t \to 0^+} P_t(t, x) = Q_0(x) \]

3 Dispersion and attenuation theory.

3.1 Mathematical preliminaries.

We shall recall the notion of completely monotonic functions and complete Bernstein functions (CBF) and some properties of the latter class of functions. For more details see [33] or [17].

**Definition 1** A real function $f$ defined on $]0, \infty[$ is said to be completely monotonic (CM) if it is infinitely differentiable and its derivatives satisfy the inequalities
\[ (-1)^n D^n f(t) \geq 0 \]
for $n = 0, 1, 2, \ldots$.

A CM function can have a singularity at 0. Any linear combination of CM functions with positive coefficients is obviously CM.

If $f$ is CM and it is integrable over $[0, 1]$, then $f$ is said to be locally integrable CM (LICM).

According to Bernstein’s theorem [34] a CM function can be expressed in terms of a positive Radon measure:

**Theorem 1** If $f$ is a CM function then there is a positive Radon measure $m$ such that
\[ f(x) = \int_{[0, \infty]} e^{-xy} m(dy), \quad x > 0 \]

For our purposes a Radon measure is essentially a measure with infinite mass. A CM function $f$ is locally integrable if and only if the measure $m$ satisfies the inequality
\[ \int_{[0, \infty]} \frac{m(dy)}{1 + y} < \infty \]

A function $f$ on $]0, \infty[$ is said to be a Bernstein function (BF) if it is differentiable and its derivative is LICM. A BF is thus non-negative and non-decreasing, hence it has a finite limit at 0.
Definition 2 A real function $f$ on $\mathbb{R}_+$ is a CBF if and only if there is a BF $g$ such that $f(x) = x^2 \tilde{g}(x)$.

If $f$ is locally integrable CM, then $p \tilde{f}(p)$ is a CBF, where $\tilde{f}$ denotes the Laplace transform of $f$.

The theorems on CBFs cited below can be found in the monograph [33].

Theorem 2 Every CBF $f$ has an integral representation

$$f(x) = a + bx + x \int_{[0, \infty]} \frac{\nu(dr)}{x + r}, \quad x \geq 0$$

(11)

with $a, b \geq 0$ and a positive Radon measure $\nu$ satisfying the inequality

$$\int_{[0, \infty]} \frac{\nu(dr)}{1 + r} < \infty$$

(12)

The integral in equation (11) is a decreasing function of $x$. Consequently

$$\lim_{x \to \infty} f(x)/x = b$$

(13)

while

$$\lim_{x \to 0} f(x) = a$$

(14)

Theorem 3 A non-zero function $f$ is a CBF if and only if the function $x/f(x)$ is a CBF.

Theorem 4 If $f$ is a CBF and $0 \leq \alpha \leq 1$ then $f(x)^\alpha$ is also a CBF.

Definition 3 A real function $f$ defined on $[0, \infty]$ is slowly varying at $w = 0$ or $\infty$ if for all $\lambda > 0$

$$\lim_{x \to w} f(\lambda x)/f(x) = 1.$$

The logarithm $\ln(1 + x)$ has this property.

Definition 4 A real function $f$ defined on $[0, \infty]$ is regularly varying at $w = 0$ or $\infty$ if for all $\lambda > 0$ the limit $\lim_{x \to w} f(\lambda x)/f(x)$ is finite.

If $f$ is regularly varying at $w$ then $f(x) \sim_w x^\alpha l(x)$ for some real $\alpha$ and a function $l(x)$ slowly varying at $w$, where $w$ is either 0 or infinity.

Theorem 5 (Valiron 1911)

If $f$ is an increasing function satisfying the condition $\lim_{y \to -\infty} f(y) = 0$ and $g(x)$ is given by the Stieltjes integral

$$g(x) = \int_{[0, \infty]} \frac{df(y)}{x + y}$$

$0 \leq \beta < 1$ and the function $l$ is slowly varying at infinity, then the following two statements are equivalent:
\( f(y) \sim y^\beta l(y) \) for \( y \to \infty \)

\( g(x) \sim \left[ (\pi \beta) / \sin(\pi \beta) \right] x^{\beta - 1} l(x) \)

3.2 Application to the dispersion-attenuation theory.

The attenuation-dispersion theory will be presented along the lines of [17]. Although there are some differences between (15) below and eq. (9) in op. cit., the analysis of attenuation and dispersion is very similar. We shall therefore present the main arguments in brief. Many other results obtained in [17] can be extended to the wave equation under consideration without changing the argument.

The wavenumber (or the length of the wave number vector) of a pressure field satisfying equation (7) is given by the formula

\[
\kappa(p) = \frac{p}{c_0} \left( 1 + p \tilde{K}(p)/(\rho_0 c_0^2) \right)^{-1/2}
\]

(15)

The convolution kernel \( K(t) \) is assumed to be a locally integrable completely monotonic function, hence

\[
K(t) = \int_{[0, \infty]} e^{-rt} \lambda(dr),
\]

where \( \lambda \) is a positive Radon measure satisfying the inequality

\[
\int_{[0, \infty]} \frac{\lambda(dr)}{1 + r} < \infty.
\]

Theorem 2 implies that \( p \tilde{K}(p) = p \int_{[0, \infty]} \lambda(dr)/(r + p) \) is a CBF.

By Theorem 3 and Theorem 4 the function \( \kappa \) is a CBF. Consequently the theory of attenuation and dispersion developed for linear viscoelastic media applies for the acoustic equation (7).

Since \( \kappa(0) = 0 \), Theorem 2 implies that

\[
\kappa(p) = Bp + \beta(p)
\]

(16)

where \( B \geq 0 \),

\[
\beta(p) := p \int_{[0, \infty]} \frac{\nu(dr)}{r + p}
\]

(17)

and \( \nu \) is a positive Radon measure satisfying the inequality

\[
\int_{[0, \infty]} \frac{\nu(dr)}{r + 1} < \infty
\]

(18)
The support of $\nu$ is called the attenuation spectrum. Equation (17) implies that $\beta(p)/p = o[1]$ for $p \to \infty$ uniformly in the closed right-half complex plane $[17]$. This in turn implies that the wavefield is bounded by a wavefront moving with the speed $C_0 = 1/B$ provided $B > 0$. If $B = 0$ then the phase speed is unbounded and the wavefield immediately spreads to the entire available space. The constant $B$ is given by the limit $B = \lim_{p \to \infty} \kappa(p)/p$ (see Appendix A). For some functions $K(t)$ the Radon measure $\nu$ can be calculated explicitly using the analytic properties of the function $\kappa(p)$ as given by eq. (15), cf [17].

The function $\beta$ can be split into its real part $A$ (attenuation) and imaginary part $-D$ (excess dispersion). These two quantities will be expressed as functions of the circular frequency $\omega = i p$ on the real $\omega$ axis. We then have

$$A(\omega) = \omega^2 \int_0^\infty \frac{\nu(dr)}{\omega^2 + r^2}$$  \hspace{1cm} (19)

$$D(\omega) = \omega \int_0^\infty \frac{r \nu(dr)}{\omega^2 + r^2}$$  \hspace{1cm} (20)

Equations (19–20) express the two functions $A$ and $D$ in terms of the same measure $\nu$. They can be viewed as a parametric form of the dispersion relations. It will be seen that the parametric form of the dispersion relations is often more convenient to use than the Kramers-Kronig relations.

It is clear from (20) that the function $D$ has sublinear growth in the high-frequency range, i.e. $D(\omega)/\omega \to 0$ as $\omega \to \infty$.

Since $y \to y/(r^2 + y)$ is an increasing function for every $r > 0$, equation (19) implies that the attenuation function $A(\omega)$ is increasing unless $\nu = 0$. The statement about the asymptotic behavior of $\beta(p)/p$ at infinity made previously implies that the attenuation function $A(\omega)$ is also sublinear in the high-frequency range.

If $\nu$ has a finite total mass

$$M := \int_0^\infty \nu(dr) < \infty$$

then $\lim_{\omega \to \infty} A(\omega) = M$ [17].

The phase speed $c(\omega)$ is defined as $\omega/\text{Re} \, k(\omega)$. The definition of the dispersion function $D$ implies that

$$\frac{1}{c(\omega)} = \frac{1}{C_0} + \frac{D(\omega)}{\omega}.$$  \hspace{1cm} (21)

But

$$\frac{D(\omega)}{\omega} = \int_0^\infty \frac{r \nu(dr)}{r^2 + \omega^2} \geq 0$$

hence $c(\omega) \leq C_0$. If $K_0 := \lim_{t \to 0} K(t) < \infty$ then $\lim_{p \to \infty} [p \tilde{K}(p)] = K_0$ in the right half-plane $\text{Re} \, p \geq 0$ and

$$C_0 = \lim_{p \to \infty} \frac{p}{\kappa(p)} = c_0 \left(1 + K_0/\mathcal{K}\right) \geq c_0.$$
Equation (20) implies that $D(\omega)/\omega$ is a decreasing function of frequency, hence $c(\omega)$ is an increasing function of $\omega$. Also $D(\omega)/\omega \to 0$ for $\omega \to \infty$, hence $\lim_{\omega \to \infty} c(\omega) = C_0$.

On the other hand

$$\lim_{p \to 0} \frac{k(p)}{p} = c_0^{-1} (1 + K_\infty/K)^{-1/2}$$

(22)

where $K_\infty := \lim_{t \to \infty} K(t) = \lim_{p \to 0} [p \tilde{K}(p)] \geq 0$. The limit (22) exists for arbitrary azimuths $\arg p \in [-\pi/2, \pi/2]$, hence $\lim_{\omega \to 0} c(\omega) = C_\infty := c_0 (1 + K_\infty/K)^{1/2} > 0$. We also note that in view of equation (21) for $\omega \to 0$ the ratio $D(\omega)/\omega$ tends to the finite limit $1/C_\infty - 1/C_0$. Equation (20) implies that this limit is equal to $\int_{[0,\infty]} \nu(dr)/r$. An important conclusion is that the last integral is finite:

$$\int_{[0,\infty]} \frac{\nu(dr)}{r} < \infty.$$  

(23)

Since the function $K$ is non-increasing, the inequality $K_\infty \leq K_0$ holds and thus $C_\infty \leq C_0$. Hence the phase speed increases monotonically from $C_\infty$ to $C_0$.

The parameter $K_\infty$ can be eliminated without changing the model by subtraction $K_\infty$ from $K(t)$ and adding it to $C$: $K \to \bar{K} = K + K_\infty$, $K(t) \to \bar{K}(t) = K(t) - K_\infty$. In terms of $\bar{K}$ and $\bar{K}(t)$ we have $\bar{K}_\infty = 0$, $C_\infty = c_0$.

### 3.3 Asymptotic behavior of attenuation.

If $\nu$ has regular variation at $\infty$ and $\nu([0,r]) \sim_a a r^\alpha \ln(r)$, where $l(r)$ is a function slowly varying at $\infty$ and $a, \alpha > 0$, then in view of the inequality (18) $\alpha < 1$. It is shown in [17] that $A(\omega) \sim L(\omega) \omega^{\gamma}$, where $L$ is slowly varying at $\infty$.

If the Radon measure $\nu$ is regularly varying at $0$, $\nu([0,r]) \sim_0 l_1(r) r^\gamma$, then in view of the inequality (23) the exponent $\gamma$ must be greater than 1. Theorem 1.11 in [17] implies that $A(\omega) \sim_0 \omega^{\gamma} L_1(\omega)$, where $L_1$ is a slowly varying function at $0$. This result is confirmed by a frequently reported experimental observation for longitudinal waves in polymers and bio-tissues for frequencies in the range 0–250 MHz [3]. This observation is often misinterpreted as evidence of a power law behavior of attenuation $A(\omega) = A \omega^\gamma$, $\gamma > 1$ valid for all the frequencies.

In the exceptional case $R := \int_{[0,\infty]} \nu(dr) < \infty$ the inequality $\nu(dr) < \nu(dr)/(1 + r^2/\omega^2)$ implies that $\lim_{\omega \to \infty} A(\omega) = R$.

Another frequently reported behavior $A(\omega) \sim a \omega^2$ for $\omega \to 0$ (cf [36, 37]) occurs when the support of the measure $\nu$ is contained in $[b, \infty]$, where $b > 0$. In this case

$$\int_{[0,\infty]} \frac{\nu(dr)}{r^2 + \omega^2} \to \int_{[b,\infty]} \frac{\nu(dr)}{r^2} := a < \infty$$

as $\omega \to 0$, which implies the quadratic behavior of attenuation at low frequencies. According to equation (15) this behavior is expected in the case of Newtonian viscosity ($K(t) = N \delta(t), \dot{p} \tilde{K}(p) = N p$) and, more generally, when $p \tilde{K}(p) \sim a p$ for $p \to 0$ and a constant $a$.  


3.4 Kramers-Kronig dispersion relations.

Equation (16) implies that \( \beta(p) \) is the Laplace transform of a causal distribution \( f \), where

\[
f(s) = \begin{cases} 
    \int_0^\infty e^{-sr} \nu(r) \, dr, & s > 0 \\
    0, & s \leq 0
\end{cases}
\]

Indeed, in view of the inequality (18) the function \( f \) is locally integrable. Hence its primitive \( g \) of \( f \) is continuous and the function \( f \) can be viewed as a distribution of first order. Thus \( \beta(p) = p \tilde{f}(p) = \mathcal{L}(f')(p) \) is the Laplace transform of the causal distribution \( f' \) of second order. If \( A(\omega) \sim \text{const} \times \omega^\alpha \) for \( \omega \to \infty \), where \( 0 < \alpha < 1 \), then \( A/(1 + \omega^2) \) and \( D(\omega)/(1 + \omega^2) \) are integrable. Since these functions are the real and imaginary part of a Fourier transform of a causal function, they satisfy the Kramers-Kronig dispersion relations with one subtraction \((38, \text{Sec. 1.8(f), pp. 42–43})\).

It is however more convenient to work with the parametric Kramers-Kronig relations \((19–20)\) because they do not involve singular integrals.

3.5 Regularity of Green’s function at the wavefronts.

Low-frequency asymptotics of the attenuation function can be verified by experiments. High-frequency asymptotics of \( A(\omega) \) is not accessible to such a verification but it determines the regularity at the wavefronts. The influence of the singularity of the relaxation modulus and its derivative at 0 has been examined in several papers \((39, 40, 41, 42, 43, 44, 45, 46)\). In this section we shall however link regularity at the wavefront to the asymptotic behavior of the attenuation function at infinity.

If \( C_0 = \infty \) then Green’s function does not vanish anywhere and is an analytic function of \( t \) and \( x \) \((39)\). In the remaining cases Green’s function vanishes outside the wavefront \( |x| = C_0 t \) and is an analytic function of \( t \) and \( x \) in the region of space-time defined by the inequality \( t - |x|/C_0 > 0 \). The transition between this region and the unperturbed region is determined by the discontinuities of Green’s function and its derivatives at the wavefront or lack thereof. We shall say that the order of the wavefront singularity is \( N \) if if at least one derivative \( \partial_t^n \partial_x^m P(t, x) \) with \( n + m = N \) has a jump discontinuity at the wavefront and this statement is not true for any derivatives with \( n + m < N \). Green’s function of (7) is given by the solution (36) for \( P_0(x) = 0 \), \( Q_0(x) = \delta(x) \)

\[
P(t, x) = \frac{1}{4\pi} \int_{-\infty}^{\infty} F_1(-i\omega) e^{-i\omega(t - |x|/c(\omega)) - A(\omega) r} \, d\omega
\]

with \( F_1(p) = \kappa(p)/p \) and \( r := |x| \). Note that \( |\kappa(-i\omega)/\omega|^2 = c(\omega)^{-2} + A(\omega)^2/\omega^2 \). The second term is bounded for \( \omega > \Omega \), where \( \Omega \) is an arbitrary positive number, because \( A(\omega) \) is sublinear. The first term is bounded by \( c(\Omega)^{-2} \) because \( c(\omega) \) is non-decreasing. Let \( W_0(\omega, r) := C_1 \exp(-A(\omega) r) \). The absolute value of \( F_1 \) is bounded for \( \omega > \Omega \) and the absolute value of the integrand is majorized by the function \( W_0(\omega, r) \), for \( \omega > \Omega \), where \( C_1 \) is a positive constant.
Distributional derivatives of $P(t,x)$ are given by the inverse Fourier transform

$$
\partial_t^m \partial_x^n P(t,x) = \frac{1}{4\pi} \int_{-\infty}^{\infty} F_1(-i\omega)(-i\omega)^m (-\kappa(-i\omega))^n e^{-i\omega(t-r/c(\omega)) - A(\omega) r} \, d\omega
$$

(25)

The absolute value of the integrand is majorized by a function $W_{m,n}(\omega,r) := C_2 \omega^{m+n+1} \exp(-A(\omega) r)$. Note that the attenuation function is non-negative and non-decreasing, hence it has a limit $\lim_{\omega \to \infty} A(\omega) = C$, possibly infinite, $0 \leq C \leq \infty$. If $C$ is infinite then $A(\omega)$ can be increasing in the high-frequency range at a logarithmic rate

$$
A(\omega) \sim A \ln^{1+\gamma}(\omega), \quad A > 0, \gamma > -1
$$

(26)

or according to the power law

$$
A(\omega) \sim A \omega^{\alpha}, \quad 0 < \alpha < 1
$$

(27)

The attenuation is often bounded in the case of a bounded attenuation spectrum, which is the case for the Cole-Davidson relaxation [17] as well as for the Strick and Becker creep [47].

We now assume that the wavefront speed $C_0 < \infty$.

If the attenuation function satisfies equation (27) then for every positive constant $\varepsilon$ and for $r \geq \varepsilon$ the absolute values of the integrands of (24) and (25) are majorized by the functions $W_0(\omega,\varepsilon)$ and $W_{m,n}(\omega,\varepsilon)$ are integrable over $[0,\infty[$ and and consequently $P(t,x)$ and its derivatives of arbitrary order are continuous everywhere outside the origin, in particular at the wavefront if $t > 0$. Since they vanish outside the wavefront, they tend to zero at the wavefront. Consequently the signal propagates with a delay with respect to the wavefront and is preceded by a flat ”pedestal” [48] (Fig. 1). The importance of the pedestal for seismic inversion was demonstrated in [11].

If $A(\omega)$ is bounded then the function $W_0(\omega,r)$ is not integrable and Green’s function $P(t,x)$ can be discontinuous at the wavefront. This effect is demonstrated numerically for the Strick and Becker creep in [49]. This effect is frequent if the attenuation spectrum is bounded, like in the case of relaxation models defined in terms of finite Prony sums, the Cole-Davidson relaxation [17] and the Strick-Becker creep compliance [47]. The appearance of discontinuities at the wavefronts in the last-mentioned case is demonstrated numerically in [49].

A curious intermediary situation arises if $A(\omega)$ satisfies equation (26) with $\gamma = 0$. In this case $W_0(\omega,r) \sim \infty C_1 \omega^{-A r}$ is integrable if the propagation distance is sufficiently large: $r > 1/A$, while $W_{m,n}(\omega,r) \sim \infty C_2 \omega^{m+n-\alpha} \exp(-A(\omega) r)$ is integrable if $r > (m+n+1)/A$. At the wavefront $r = c_0 t$ and thus $P(t,x)$ is continuous at the wavefront for $t > 1/(A c_0)$, while the derivatives of order $N$ are continuous at the wavefront for $t > (N+1)/(A c_0)$. The order of the wavefront singularity thus increases stepwise with time. This effect was deduced by a different argument by Desch and Grimmer [43] [44].
Figure 1: Cross-section of Green’s function near the wavefront. (a) Green’s function and all its derivatives are continuous at the wavefront (solid line); (b) Green’s function and its first-order derivatives are continuous (dashed line); (c) Green’s function has a jump discontinuity at the wavefront (dot-dashed line); (d) Green’s function is continuous at the wavefront but its derivatives are not (dotted line).

If $\gamma > 1$ then $W_0(\omega, r) = C_1 \omega^{-A r \ln^\gamma(\omega)}$ and $W_{m,n}(\omega, r) = C_2 \omega^{m+n-A r \ln^\gamma(\omega)}$ are integrable for all $r \geq \varepsilon > 0$. Consequently $P(t, x)$ and its derivatives are continuous outside the origin, in particular at the wavefront if $t > 0$. The regularity properties of Green’s function are thus the same as for power law attenuation.

If $-1 < \gamma < 0$, then $W_0(\omega) = C_1 \omega^{-A r \frac{1}{\ln^\gamma(\omega)}}$ and $W_{m,n}(\omega) = C_2 \omega^{m+n-A r \frac{1}{\ln^\gamma(\omega)}}$. In this case both $P(t, x)$ and its derivatives can be discontinuous. It follows that strictly logarithmic growth of attenuation ($\gamma = 1$) constitutes a sharp boundary between media which allow for discontinuity propagation and those which do not allow for discontinuities at the wavefronts.

Equation (37) implies that the order of the wavefront in three-dimensional space is higher by one than in one-dimensional space for the same attenuation function.

4 Examples.

4.1 Viscoelastic media with a bounded attenuation function.

A special class of viscoelastic relaxation models is characterized by the inequalities $K_0 < \infty$ and $K'_0 := K'(0) > -\infty$. A frequent representative of this class is the Dirichlet series $K(t) = \sum_{n=1}^{N} \lambda_n e^{-\lambda_n t}$, with $N \leq \infty$, $\lambda_n$, $r_n > 0$ for $n = 1, \ldots, N$, $\sum_{n=1}^{N} \lambda_n = K_0 < \infty$ and $\sum_{n=1}^{N} r_n \lambda_n = -K'_0 < \infty$. A finite Dirichlet series is known as the Prony series. This kind of relaxation mechanism is often used to model multiple relaxation due to several relaxation mechanisms such as chemical reactions, cf [22]. In seismology it was suggested by Liu et al.
In this class
\[ \int_{0, \infty} r \lambda(dr) = -K_0' < \infty \]
and therefore
\[ p \tilde{K}(p) = p \int_{0, \infty} \frac{\lambda(dr)}{1 + r/p} \approx \int_{0, \infty} \lambda(dr) \frac{1}{p} \int_{0, \infty} r \lambda(dr) + o[1/p] = K_0 + K_0'/p + o[1/p] \]
for \( p \to \infty \), \( \text{Re } p \geq 0 \), so that
\[ \rho(p) = \frac{1}{p} C_0^{-1} + \frac{R}{p} \]
with \( R = -K_0' \left[ 1 + K_0/\rho c_0^2 \right]^{-3/2} / (2 \rho c_0^3) \). Note that
\[ \frac{\rho(p)}{p} = \frac{p}{C_0} + \int_{0, \infty} \frac{\nu(dr)}{1 + r/p} \approx \frac{p}{C_0} + R \]
for large \( p \) implies that \( \int_{0, \infty} \nu(dr) = R \). As it has already been noted, this implies that the attenuation function tends to a constant as \( \omega \to \infty \).

### 4.2 Viscoelastic media with an asymptotic power law attenuation and weakly singular \( K(t) \).

The attenuation function grows at a power law rate in the Cole-Cole relaxation model \[51\], originally proposed for dielectric relaxation and subsequently applied in polymer viscoelasticity by Bagley and Torvik \[52\]:
\[ p \tilde{K}(p) = M \frac{1 + a (\tau p)^{-\alpha} - (1 + (\tau p)^{-\alpha})}{1 - a} \]
with \( M, \tau, a > 0 \), \( 0 < \alpha < 1 \). We have subtracted a constant term \( Ma \) so that \( K_\infty = \lim_{p \to 0} [p \tilde{K}(p)] = 0 \) because a non-zero static modulus is already represented by \( K \). Since \( K_0 = \lim_{p \to \infty} [p \tilde{K}(p)] = M (1 - a) \geq 0 \), the parameter \( a \) satisfies the inequality \( a \leq 1 \). The phase speed is contained between \( C_\infty > 0 \) and \( C_0 < \infty \), where \( C_0 := c_0 \left[ 1 + K_0/K \right]^{1/2} \) is the wavefront speed, the phase speed is bounded from below: \( c(\omega) \geq c_0 \), with \( c_0 \leq C_0 \). The formula
\[ f(y) = y^{\alpha - 1} / (1 + y^{\alpha}) \]
for the Laplace transform of \( f(x) = E_\alpha (-x^\alpha) \) yields the kernel \( K \):
\[ K(t) = M (1 - a) E_\alpha (-t/\tau)^\alpha, \quad t \geq 0, \]
where \( E_\alpha \) denotes the Mittag-Leffler function \[53\]. It is proved in Appendix B that \( K(t) \) in equation \[30\] is CM. This function is shown in Fig. 2. The function has been calculated numerically using equation \[39\].
The high-frequency behavior of attenuation in the above model is given by the formula

\[ A(\omega) \sim \frac{(1 - a) M_1 \sin(\alpha \pi/2)}{2c_0 \tau (1 + M_1)^{3/2}} (\tau \omega)^{1-\alpha}, \]

where \( M_1 = M/K \). For \( \omega \to 0 \) we have a different picture:

\[ A(\omega) \sim \frac{(1 - a) M_1 \sin(\alpha \pi/2)}{2c_0 \tau (1 + M_1)^{3/2}} (\tau \omega)^{1+\alpha}. \]

The exponent \( 1 + \alpha \) lies between 1 and 2, in accordance with experimental data for polymers and bio-tissues in the frequency range 0–250 MHz.

The Cole-Cole attenuation function can be calculated numerically using the formula

\[
A(\omega) = \omega \sqrt{X(\omega)^2 + Y(\omega)^2} - X(\omega) / \left( 2c_0 \sqrt{X(\omega)^2 + Y(\omega)^2} \right) \\
X(\omega) = 1 + (M \ast (1 - a)/K)(\omega \tau)^\alpha \ ( (\omega \tau)^\alpha + \cos(\pi \alpha/2) ) /  \left( 1 + (\omega \tau)^{2\alpha} + 2(\omega \tau)^\alpha \cos(\pi \alpha/2) \right) \\
Y(\omega) = - (M/K) (1 - a) (\omega \tau)^{\alpha} \sin(\pi \alpha/2) / \left( 1 + (\omega \tau)^{2\alpha} + 2(\omega \tau)^\alpha \cos(\pi \alpha/2) \right)
\]

Using these formulas the functions \( A(\omega) \) and \( c(\omega) \) were plotted in Fig. 3 for \( c_0 = 1500 \text{ m/s}, a = 0.5, M/K = 1, K_\infty/K = a, \tau = 10^{-13} \text{ s} \) and \( \alpha = 0.2, 0.5 \) and 0.8. The plot shows that the Cole-Cole attenuation function obeys two approximate power laws, a superlinear one in the low frequency range \( \omega \ll 3.27 \times 10^6 \text{ MHz} \) and a sublinear one in the high frequency range \( \omega \gg 3.27 \times 10^6 \text{ MHz} \).
4.3 A viscoelastic model with unbounded phase speed.

If \( K(t) = A(t/\tau)^{-\alpha}/\Gamma(1 - \alpha) \) with \( A, \tau > 0 \) and \( 0 < \alpha < 1 \), then the pressure equation assumes the form

\[
c_0^{-2} P_{tt} = \nabla^2 P + (A \tau^\alpha / \rho_0 c_0^2) D^\alpha \nabla^2 P
\]

(34)

where \( D^\alpha \) denotes the Caputo fractional time derivative of order \( \alpha \). This particular equation was considered in [18]. It is related to the constant-Q model in seismology [54, 55]. In this case \( K_0 = \infty \), hence \( C_0 = \infty \) and the solutions do not exhibit wavefronts. In this case \( X(\omega) = 1 + A_1 (\tau \omega)^\alpha \cos(\pi \alpha/2) \) and \( Y(\omega) = A_1 (\tau \omega)^\alpha \sin(\pi \alpha/2) \), where \( A_1 := A/K \). The attenuation function and phase speed are shown for \( A_1 = 0.5 \) in Fig. 4, which confirms that the phase speed is unbounded. This is hardly surprising because the order of the derivatives in the last term of equation (34) is \( 2 + \alpha \), higher than the orders of the other derivatives and the equation of motion is parabolic.

Asymptotic behavior of dispersion and attenuation associated with special wave equations based on fractional versions of spring-dashpot models has been examined in a series of papers by Holm, Nåsholm and Sinkus [18, 24, 56, 57]. The convolution kernels \( K \) corresponding to the fractional spring-dashpot models are CM, hence the general theory developed in Sec. 3.2 applies to their equations.
4.4 Quasi-linear attenuation.

The attenuation function is almost linear while phase speed varies very slowly in seismological applications [26, 58] as well as in marine sediments [59]. This results in an approximately constant $Q$ factor, defined by the formula $Q(\omega) := \omega/[2c(\omega)A(\omega)]$.

We shall use asymptotic considerations to investigate dispersion, attenuation and existence of wavefronts for a nearly linear attenuation function. Note that an exactly linear rate of growth of the attenuation function in the high frequency range would be inconsistent with the assumption that the origin of attenuation is purely viscoelastic and the relaxation modulus is CM. A model of a linear attenuation function and an approximately linear $D(\omega)$ was elaborated by Futterman [26]. Futterman was only concerned with finite phase speed for his strictly linear attenuation model. In contrast to Futterman we shall take into account the fact that attenuation grows at a strictly sublinear rate. Furthermore, the investigations of this section also shed some light on the relation between boundedness of the phase speed and existence of wavefronts. One might wonder whether it is possible that the phase speed has a finite upper bound $C_0$ but the wave field extends beyond the surface $|x| = C_0 t$. It will turn out that this can happen.

Note that the attenuation function in the constant-$Q$ model of the previous section is sublinear, while absence of wavefronts manifests itself through unboundedness of phase speed (Fig. 4). Sublinearity of the attenuation function alone does not however guarantee bounded phase speed. The Fourier transform of the one-dimensional Green’s function is

$$g(\omega) = F(\omega) e^{-i\omega(t-|x|/C_0)} - e^{-iD(\omega)|x|-A(\omega)|x|}$$

where $|F(\omega)|$ is bounded. The function $g$ is square integrable for $|x| > 0$ if $A(\omega)/|\ln(\omega)|$ is unbounded for $\omega \to \infty$. Suppose that this condition is satisfied. According to the Paley-Wiener theorem ([60], Theorem XII) Green’s function vanishes for $t < |x|/C_0$ if and only if

$$\int_0^\infty \frac{A(\omega)}{1 + \omega^2} d\omega < \infty$$

This is true in particular for

$$A(\omega) \sim a \omega^{1+\lambda}/|\ln(\omega)|^\gamma, \quad a > 0, \, \omega \to \infty \quad (35)$$

if the following condition is satisfied:

(*) Either $\lambda < 0$ or $\lambda = 0$ and $\gamma > 1 + \varepsilon$.

In order to check the behavior of attenuation and phase speed in these cases we have to turn to equations (19) and (20). Consider a Radon measure $\nu(dr) = h(r) dr$, whose density has the asymptotic behavior $h(r) \sim br^\lambda/|\ln r|^\gamma$ at infinity, where $b > 0$. Inequality (18) is satisfied if

$$b \int_\mathbb{N} \frac{r^\lambda dr}{(1 + r) \ln(r)^\gamma} \equiv b \int_\mathbb{N} \frac{e^{\lambda y} dy}{y^\gamma (1 + e^{-y})} < \infty$$

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for some sufficiently large \( N > 1 \), hence it is satisfied if and only if Condition \((\ast)\) is satisfied.

We shall now calculate the asymptotic behavior of the functions \( A(\omega) \) and \( D(\omega) \). In terms of the integration variable \( s = r^2 \)

\[
A(\omega) = \omega^2 \int_{0, \infty} \frac{\mu(ds)}{\omega^2 + s}
\]

where \( \mu(ds) = h(\sqrt{s}) \frac{ds}{2\sqrt{s}} \sim b \frac{s^{(\lambda-1)/2} ds}{(2|\ln(\sqrt{s})|)^\gamma} \) for \( s \to \infty \). Note that \( 1/|\ln(\sqrt{s})|^{\gamma} \) is a slowly varying function. By Theorem \( 5 \)

\[
A(\omega) \sim (b/2) \left[ \pi/\cos(\pi\lambda/2) \right] \omega^{1+\lambda/|\ln(\omega)|^\gamma}
\]

for \( \omega \to \infty \). It is thus seen that in the case under consideration inequality \((18)\)

is equivalent to the statement that \( |x| = C_0 t \) is a wavefront of Green’s function and also to Condition \((\ast)\).

By a similar argument equation \((20)\) implies that

\[
D(\omega) \sim (b/2) \left[ \pi/\cos(\pi\lambda/2) \right] \omega^{1+\lambda/|\ln(\omega)|^\gamma}
\]

and, using equation \((21)\),

\[
c(\omega) \sim C_0 \left[ 1 - b\omega^\lambda/(2|\ln(\omega)|^\gamma) \right]
\]

If \( \lambda < 0 \) or \( \lambda = 0 \) and \( \gamma > 0 \), then \( \lim_{\omega \to \infty} c(\omega) = C_0 \).

If \( \lambda = 0 \) and \( 0 < \gamma \leq 1 \) then the phase speed has a finite upper bound \( C_0 \) but the Paley-Wiener theorem implies that the wavefield is not bounded by the surface \( |x| = C_0 t \). It should however be noted that in this special case failure of inequality \((18)\) implies that the kernel \( K(t) \) cannot be locally integrable and CM.

For \( \lambda = 0 \) and \( \gamma > 0 \) the dispersion is very weak. A nearly linear attenuation and a nearly constant phase speed is observed in marine sediments as well as in the Earth’s crust and mantle.

5 Conclusions.

Consistency with linear viscoelasticity requires that attenuation in the pressure wave equation is represented by a term of the form \( K \ast \nabla^2 P \). The operator \( \nabla^2 \) is here crucial for truly viscoelastic attenuation.

If the kernel \( K \) is completely monotonic then the wavenumber function \( \kappa(p) \) is a CBF and a rich theory of dispersion-attenuation developed in \([25, 17]\) applies. The general results presented in Sec. 3.2 has a very strong predictive power even before any particular model of the medium is substituted.

It was pointed out in Section 3.2 that \( A(\omega) \) is sublinear in the high-frequency range. This is also true when phase speed is unbounded and there are no wavefronts. As the results of Sections 3.2 and 4.4 indicate, absence of wavefronts manifests itself by unboundedness of the phase speed rather than by the Paley-Wiener criterion alone. On the other hand experimental data for many
polymers, castor oil and bio-tissues indicate a power law for the attenuation function $A(\omega)$ with an exponent in the range 1–2. Such values of the exponent in the high frequency range are incompatible with viscoelasticity but they are consistent with the low frequency asymptotics of the attenuation function. They are also incompatible with bounded phase speed. Viscoelasticity does not however exclude an unbounded phase speed. Comparison of characteristic relaxation times with the range of experimental data obtained for polymers and bio-tissues (0–250 MHz) indicate that the observed data are pertinent for the low-frequency behavior of the attenuation function. Low-frequency behavior of attenuation and dispersion is examined in some more detail in the context of linear viscoelasticity in [17].

High-frequency asymptotic properties of the attenuation function are unavailable to direct measurements but they affect the singularity carried by the wavefront if $C_0 < \infty$.

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A Solution of the Cauchy problem (7–9)

Consider the initial value problem defined by equation (7) in one-dimensional space with the initial conditions $P(0,x) = P_0(x)$ and $P_1(0,x) = Q_0(x)$. The Laplace-Fourier transform

$$\hat{P}(p,k) := \int_0^\infty e^{-pt} \left[ \int_{-\infty}^{\infty} e^{-ikx} P(t,x) \, dx \right] \, dt$$

is given by the expression $\hat{P}(p,k) = F(p,k)/\left[k^2 + \kappa(p)^2\right]$, where

$$F(p,k) := c_0^{-2} \left[ \frac{p (1 + k^2/p) \hat{P}_0(k) + \hat{Q}_0(k)}{1 + (\rho_0 c_0^2)^{-1} p \hat{K}(p)} \right]$$

Hence

$$P(t,x) = \frac{1}{2\pi i} \int_{-i\infty + \varepsilon}^{i\infty + \varepsilon} e^{pt} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} F(p,k) \frac{1}{k^2 + \kappa(p)^2} \, dk \right] \, dp$$
for an arbitrary $\varepsilon > 0$.

If $x > 0/x < 0$ then the contour of the inner integral can be closed by a semicircle at infinity in the upper/lower complex $k$-half-plane, $\text{Im } k > 0/\text{Im } k < 0$. Since $\text{Re } k^{-i\omega} = A(\omega) \geq 0$, the only residue in the upper/lower $k$-half-plane is $k = i\kappa(-i\omega)/k = -i\kappa(-i\omega)$, hence

$$P(t, x) = \frac{1}{4\pi i} \int_{-i\infty + \varepsilon}^{i\infty + \varepsilon} F_1(p) e^{pt - k(p)|x|} \, dp = \frac{1}{4\pi i} \int_{-i\infty + \varepsilon}^{i\infty + \varepsilon} F_1(p) e^{(t-B|x| - \beta(p)|x|)} \, dp$$

where $F_1(p) := F(p, \kappa(p) \text{sgn } x)/\kappa(p)$.

For $t < B|x|$ the Bromwich contour can be closed in the right $p$-half-plane $\text{Re } p > 0$. The function $\kappa(p)$ does not have any singularity except for a cut along the negative $p$-semi-axis. The integrand does not have any singularity in the right-half plane, hence $P(t, x) = 0$ for $t < B|x|$. If $B > 0$ then $C_0 := 1/B$ can be identified as the wavefront speed. If $B = 0$ then $F(t, x)$ vanishes for $t < 0$, which implies causality.

Define the function $Q(t, y)$ in such a way that $P(t, x) = Q(t, |x|)$. The solution $P^{(3)}(t, x)$ of the same initial-value problem in three dimensions is then given by the formula [61]

$$P^{(3)}(t, x) = -\frac{1}{2\pi r} \frac{\partial Q(t, r)}{\partial r}$$

where $r = |x|$, and therefore $P^{(3)}(t, x)$ vanishes for $t < Br$.

**B The Cole-Cole relaxation kernel.**

The Mittag-Leffler function $E_\alpha(-x^\alpha), 0 < \alpha \leq 1$, can be calculated by applying the inverse Laplace transform to equation [29]

$$E_\alpha(-x^\alpha) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{yx} y^{\alpha-1} \, dy$$

where $\varepsilon > 0$. The Bromwich contour can be deformed to a Hankel loop running along a the straight line $L_-$ from $-\infty - i\varepsilon$ to $-i\varepsilon$, then along a semicircle of radius $\varepsilon$ centered at 0 and along the straight line $L_+$ from $i\varepsilon$ to $-\infty + i\varepsilon$. The integral over the semi-circle

$$\int_{-\pi}^{\pi} e^{\varepsilon \varphi} \exp(i\varphi) \frac{\varphi^{\alpha-1}\exp(i(\alpha - 1)\varphi)}{1 + \varepsilon^{\alpha}\exp(i\alpha \varphi)} \, d\varphi$$

tends to 0 as $\varepsilon \to 0$. The remaining two integrals tend to

$$\frac{1}{\pi} \int_{0}^{\infty} e^{-rt} \text{Im} \left[ \frac{r^{\alpha-1}\exp(-i(\alpha - 1)\pi)}{1 + r^{\alpha}\exp(-i\alpha \pi)} \right] \, dr$$

(38)
where $r \exp(\pm \pi i)$ has been substituted for $y$ on $\mathcal{L}_\pm$. Equation (38) works out to

$$E_\alpha(-x^\alpha) = \frac{\sin(\alpha \pi)}{\pi} \int_0^\infty e^{-rx} r^{\alpha-1} \left[r^{2\alpha} + 2r^\alpha \cos(\alpha \pi) + 1\right]^{-1} dr \quad (39)$$

Equation (38) shows that $E_\alpha(-x^\alpha)$ is the Laplace transform of a non-negative function. By Theorem 1, it is CM.

The Cole-Cole kernel function (30) differs from $E_\alpha(-x^\alpha)$ by a linear scaling transformation, hence it is CM.