A new approach to the Pontryagin maximum principle for nonlinear fractional optimal control problems

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1. Introduction

Fractional optimal control problems (FOCPs) can be regarded as a generalization of classic optimal control problems (OCPs) for which the dynamics of the control system are described by fractional differential equations (FDEs) and might involve a performance index given by fractional integration operator. The reason to formulate and solve FOCPs relies in the fact that there are a significant number of instances in which FDEs describe the behavior of the control systems of interest more accurately than the more common integer differential equations. These instances arise, for example, in diffusion processes, process control, dynamic control systems encompassing signal processing, etc., [1].

Fractional calculus is a field of mathematics that deals with integrals and derivatives whose order may be given by an arbitrary real or complex number, thus generalizing the more common integer-order differentiation and integration. It started more than 300 years ago when the notation for differentiation of noninteger order \( \frac{d^r}{dx^r} \) was discussed between Leibniz and L'Hôpital. Since then, fractional calculus has been developed gradually, being now a very active research area of Mathematical Analysis as attested by the vast number of publications ([2–5]). There are several different ways of defining fractional derivatives, and, consequently, different types of FOCPs. However, the ones in the sense the Riemann–Liouville and the Caputo have been more widely used. In most of the works that have been published on FOCPs, the state variable is obtained by the Riemann–Liouville or the Caputo fractional integration of the dynamics, but so far, only integer order integral performance indexes have been considered, although, this type of performance indexes has been adopted in the considerably different context of calculus of variations, [6]. It also should be noted that several specific numerical techniques have been developed to solve FOCPs. For more details, see [7–9].

In this paper, we consider FOCPs for which the performance index is given by an integral of fractional order, and the dynamics are mapping specifying the Caputo fractional derivative of the state variable with respect to time. The reason to choose the Caputo fractional derivative is because it is the most popular one among physicists and scientists, being the fact that the fractional derivative of constants are zero. Moreover, the assumptions that we impose on the data of the problem enables a novel approach to the proof based on a generalization of Taylor's expansion and a fractional mean value theorem. Another contribution of the paper consists on an analytic method to solve the fractional differential equation. This is illustrated by an example based on a generalization of the Mittag–Leffler function and \( \alpha \) exponential function.
Our approach has two key advantages relatively to the one often adopted that consists into converting the FCOP into an equivalent OCP and, then, decoding the obtained necessary conditions of optimality into the data of the original problem: (i) more precise insight inherent to the use of variational methods in the original modeling framework; and (ii) more direct approximating computational procedures guided by the maximum principle conditions.

It should be remarked that our result differs substantially from the one presented in [10] where, by using a quite different approach, necessary conditions of optimality are derived for a different OCP that requires the velocity set (i.e., the set of time derivatives of the state variable) to be convex. This is a very strong assumption and constitutes a key difference from our result, which covers dynamic control systems whose velocity sets might be a mere discrete set of points. Moreover, our approach is much more in line with the celebrated classic work of Pontryagin et al., [11].

This paper is organized as follows. In the next Section, we present a brief review of fractional integral and fractional derivative concepts and some basic notions specifically pertinent to this work. In Section 3, we state, discuss, and prove necessary conditions of optimality for a different OCP that requires the velocity set (i.e., the set of time derivatives of the state variable) to be convex. This is a very strong assumption and constitutes a key difference from our result, which covers dynamic control systems whose velocity sets might be a mere discrete set of points. Moreover, our approach is much more in line with the celebrated classic work of Pontryagin et al., [11].

There are several definitions of a fractional derivative. In this section, we present a review of some definitions and preliminary facts, which are particularly relevant for the results of this article [12–14].

Definition 2.1
Let \( f : (a, b) \to \mathbb{R} \) be a locally integrable function in interval \([a, b]\). For \( t \in [a, b] \) and \( \alpha > 0 \), the left and right Riemann–Liouville fractional integrals are, respectively, defined by

\[
\mathcal{I}_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha - 1} f(\tau) d\tau,
\]

and

\[
\mathcal{I}_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau - t)^{\alpha - 1} f(\tau) d\tau,
\]

where \( \Gamma(\cdot) \) is the Euler gamma function.

Definition 2.2
Let \( f : (a, b) \to \mathbb{R} \) be an absolutely continuous function in the interval \([a, b]\). For \( t \in [a, b] \) and \( \alpha > 0 \), the left and right Riemann–Liouville fractional derivatives are, respectively, defined by

\[
\mathcal{D}_a^\alpha f(t) = \frac{d^n}{dt^n} \left( \mathcal{I}_a^{n-\alpha} f(t) \right) = \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{dt} \right)^n \int_a^t (t - \tau)^{n-\alpha-1} f(\tau) d\tau,
\]

and

\[
\mathcal{D}_b^\alpha f(t) = \left( -\frac{d}{dt} \right)^n \left( \mathcal{I}_b^{n-\alpha} f(t) \right) = \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{dt} \right)^n \int_t^b (\tau - t)^{n-\alpha-1} f(\tau) d\tau,
\]

where \( n \in \mathbb{N} \) is such that \( n - 1 < \alpha \leq n \), and \( \Gamma(\cdot) \) is as in Definition 2.1.

Definition 2.3
Let \( f : AC^n[a, b] \to \mathbb{R} \) be an absolutely continuous function in the interval \([a, b]\). For \( t \in [a, b] \) and \( \alpha > 0 \), the left and right Caputo fractional derivatives are, respectively, defined by

\[
\mathcal{D}_a^\alpha f(t) = \frac{d^n}{dt^n} \left( \mathcal{I}_a^{n-\alpha} f(t) \right) = \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{dt} \right)^n \int_a^t (t - \tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau,
\]

and

\[
\mathcal{D}_b^\alpha f(t) = \mathcal{I}_b^{n-\alpha} \left( \frac{d}{dt} \right)^n f(t) = \frac{(-1)^n}{\Gamma(n - \alpha)} \int_t^b (\tau - t)^{n-\alpha-1} f^{(n)}(\tau) d\tau,
\]

where \( n \in \mathbb{N} \) is such that \( n - 1 < \alpha \leq n \).

Remark 2.1
If \( \alpha = n \in \mathbb{N}_0 \), then the Caputo and Riemann–Liouville fractional derivative coincide with the ordinary derivative \( \frac{d^nf(t)}{dt^n} \).

Remark 2.2
The Caputo fractional derivative of a constant is always equal to zero. This is not the case with the Riemann–Liouville fractional derivative.
Theorem 2.1 ([14])
Let \( f(\cdot) \) be a differentiable function in \( [a, b] \). Then, for \( \alpha > 0 \),
\[
\frac{\mathcal{D}_t^\alpha}{\mathcal{D}_a^\alpha} f(t) = f(t), \quad \frac{\mathcal{D}_t^-^\alpha}{\mathcal{D}_a^-^\alpha} f(t) = f(t),
\]
where
\[
\frac{\mathcal{D}_a^\alpha}{\mathcal{D}_t^\alpha} f(t) = f(t), \quad \frac{\mathcal{D}_a^-^\alpha}{\mathcal{D}_t^-^\alpha} f(t) = f(t),
\]
and for \( 0 < \alpha < 1 \),
\[
\frac{\mathcal{D}_a^\alpha}{\mathcal{D}_a^-^\alpha} f(t) = f(b) - f(a), \quad \frac{\mathcal{D}_a^-^\alpha}{\mathcal{D}_a^\alpha} f(t) = f(a) - f(b).
\]

Theorem 2.2
Fractional integration by parts [15].
Let \( 0 < \alpha < 1 \), and \( f(\cdot) \) be a differentiable function in interval \( [a, b] \) and \( g(\cdot) \in L^1([a, b]) \).
Then, the following integration by parts formula holds
\[
\int_a^b g(t) \frac{\mathcal{D}_t^\alpha}{\mathcal{D}_a^\alpha} f(t) dt = \int_a^b f(t) \frac{\mathcal{D}_t^-^\alpha}{\mathcal{D}_a^-^\alpha} g(t) dt + \left[ g(t) \frac{\mathcal{D}_t^\alpha}{\mathcal{D}_a^\alpha} f(t) \right]_a^b,
\]
and
\[
\int_a^b g(t) \frac{\mathcal{D}_a^\alpha}{\mathcal{D}_t^\alpha} f(t) dt = \int_a^b f(t) \frac{\mathcal{D}_a^-^\alpha}{\mathcal{D}_t^-^\alpha} g(t) dt - \left[ g(t) \frac{\mathcal{D}_a^\alpha}{\mathcal{D}_t^\alpha} f(t) \right]_a^b.
\]

Another important auxiliary result required to prove our maximum principle is the generalization of the Bellman–Gronwall lemma for fractional differential systems. Here, we will consider the following /epsintegral extracted from [16].

Theorem 2.3
Generalized Bellman–Gronwall inequality.
Suppose that \( \alpha > 0 \), \( t \in [0, T] \) and the functions \( a(t) \), \( b(t) \) and \( u(t) \) are non-negative and continuous on \( 0 \leq t < T \) with
\[
u(t) \leq a(t) + b(t) \int_0^t (t-s)^{\alpha-1} u(s) ds,
\]
where \( b(t) \) is a bounded and monotonic increasing function on \( [0, T] \), then
\[
u(t) \leq a(t) + \int_0^t \sum_{n=1}^{\infty} \frac{(b(t) \Gamma(n \alpha))}{\Gamma(n \alpha)} (t-s)^{n \alpha-1} a(s) ds, \quad t \in [0, T).
\]

Theorem 2.4
Generalized Taylor’s formula (cf. [17, 18]).
Let \( 0 < \alpha \leq 1 \), \( n \in \mathbb{N} \), \( f(\cdot) \) be a continuous function in \( [a, b] \), \( \mathcal{D}_a^{\alpha-k} f(\cdot) \in C[a, b] \) for all \( k \in \{1, \ldots, n\} \), and \( \mathcal{D}_a^{(n+1)\alpha} f(\cdot) \) is continuous on \( [a, b] \). Then for all \( x \in [a, b] \), the generalized Taylor’s formula for Caputo fractional derivatives is defined by
\[
f(x) = \sum_{k=0}^{n} \frac{(x-a)^k \alpha}{\Gamma(k \alpha + 1)} \mathcal{D}_a^{k \alpha} f(a) + R_n(x, a),
\]
where \( R_n(x, a) = \frac{\mathcal{D}_a^{(n+1)\alpha} f(\xi)}{\Gamma((n+1) \alpha + 1)} (x-a)^{n+1} \alpha \)
being, for each \( x \in [a, b] \), \( a \leq \xi \leq x \), and denoting the Caputo fractional derivative of order \( \alpha \) by \( \mathcal{D}_a^\alpha \).

Notice that, if \( \alpha = 1 \), the generalized Taylor’s formula reduces to the classical Taylor’s formula.

Lemma 2.1 ([19])
Let \( f \in C[a, b] \), and \( \alpha > 0 \). Then, there exists some \( \xi \in (a, b) \) such that
\[
f_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt = f(\xi) \frac{(x-a)^\alpha}{\Gamma(1+\alpha)},
\]
where \( \xi \) is the fractional intermediate value. Remark that there might exist more than one \( \xi \) satisfying this property.

Definition 2.4
The two-parameter Mittag–Leffler function defined by the power series in the form:
\[
E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma[n \alpha + \beta]},
\]
where \( \alpha \) and \( \beta \) are positive parameters.
If $\beta = 1$, this function is often denoted simply by $E_{\alpha}(\cdot)$. We observe that $E_{0,1}(z) = 1/(1-z)$, $E_{1,1}(z) = \exp z$, $E_{1,2}(z) = (\exp z - 1)/z$, and $E_{1,0}(z) = z \exp z$.

Definition 2.5

Let $A \in \mathbb{R}^{n \times n}$. Then, the generalization of the two-parameter Mittag–Leffler function becomes

$$E_{\alpha,\beta}(At^\alpha) = \sum_{n=0}^{\infty} A^n \frac{t^{n\alpha}}{\Gamma(n\alpha + \beta)},$$

and being the $\alpha$ exponential matrix function defined by using Mittag–Leffler function as follows:

$$e_{\alpha}(A, t) = t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha) = \sum_{n=0}^{\infty} A^n \frac{t^{n\alpha}}{\Gamma(n + 1\alpha)}.$$ (1)

The Mittag–Leffler function has several interesting properties. For details, see [20–23].

In what follows, the definition of fractional state transition matrix for a linear time-varying vector valued fractional differential equation of the type

$$\frac{\partial^\alpha x(t)}{\partial t^\alpha} = A(t)x(t), \quad x(0) = x_0,$$

(2)

where $A(\cdot) : \mathbb{R} \to \mathbb{R}^{n \times n}$, and $x(\cdot) \in \mathbb{R}^n$ is required.

Definition 2.6

The matrix valued map $\Phi_{\alpha} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n \times n}$ is the fractional state transition matrix to the linear fractional differential equation (2) on the interval $[a, t]$, if it satisfies

$$\frac{\partial^\alpha}{\partial t^\alpha} \Phi_{\alpha}(t, s) = A(t)\Phi_{\alpha}(t, s), \quad \Phi_{\alpha}(t, t) = I_n, \text{ and } \Phi_{\alpha}(t, s) = 0_n \text{ if } t < s,$$

(3)

where $I_n$ and $0_n$ are, respectively, the identity and zero matrices of order $n$. It is a simple exercise to conclude that $x(t) = \Phi_{\alpha}(t, a)x_a$ is solution to (2).

3. The fractional optimal control problems statement and its maximum principle

In this section, we discuss the FOCP considered in this article, state the associated necessary conditions of optimality, and present its proof, which uses an approach that differs from the ones usually adopted in the literature for this class of optimal control problems.

Let us consider the simple general problem as follows

$$\begin{align*}
(\hat{P}) \text{ Minimize } & \int_{t_0}^{t_f} L(t, \bar{x}(t), u(t)) \\
\text{subject to } & \frac{\partial^\alpha}{\partial t^\alpha} \bar{x}(t) = \bar{f}(t, \bar{x}(t), u(t)), \quad [t_0, t_f] \mathcal{L} - \text{a.e.} \\
& \bar{x}(t_0) = \bar{x}_0 \in \mathbb{R}^n \\
& u(t) \in \mathcal{U},
\end{align*}$$

(4)

where $\mathcal{U} = \{u : [t_0, t_f] \to \mathbb{R}^m | u(t) \in \Omega(t) \}$, $\Omega : [t_0, t_f] \to \mathbb{R}^m$ is a given set valued mapping, the functions $L : [t_0, t_f] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $\bar{f} : [t_0, t_f] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are given functions defining, respectively, the running cost (or Lagrangian) functional, and the fractional dynamics. $\frac{\partial^\alpha}{\partial t^\alpha}$ is the Riemann–Liouville fractional integral, and $\frac{\partial^\alpha}{\partial t^\alpha}$ is the left Caputo fractional derivative of order $0 < \alpha < 1$ of the state variable with respect to time.

It is not hard to see that a simple transformation allows us to convert the problem $(\hat{P})$ into an equivalent one, simply by defining an additional state variable component $y$ by

$$\frac{\partial^\alpha}{\partial t^\alpha} y(t) = L(t, \bar{x}(t), u(t)),$$

satisfying the initial condition $y(t_0) = 0$. Then, we conclude that problem $(\hat{P})$ is equivalent to the one as follows:

$$\begin{align*}
(P) \text{ Minimize } & g(x(t_f)) \\
\text{subject to } & \frac{\partial^\alpha}{\partial t^\alpha} x(t) = f(t, x(t), u(t)), \quad [t_0, t_f] \mathcal{L} - \text{a.e.} \\
& x(t_0) = x_0 \in \mathbb{R}^n \\
& u(t) \in \mathcal{U},
\end{align*}$$

(7)

where now $g(x(t_f)) = y(t_f)$, the state variable $x = \text{col}(y, \bar{x})$, that is, it includes $y$ as a first component with initial value at 0, and the mapping $f = \text{col}(L, \bar{f})$, that is, it has $L$ as first component.
Theorem 3.1

Let \( x \in \mathbb{R}^n \) be a solution to the optimal control problem. Then, there exists a function \( p : [t_0, t_f] \rightarrow \mathbb{R}^n \) satisfying

- the adjoint equation
  \[
  D_t^\alpha p(t) = D_t f(t, x^*(t), u^*(t)), \tag{10}
  \]
  with the transversality condition
  \[
  p(t_f) = \nabla g(x^*(t_f)), \tag{11}
  \]
  where the operator \( D_t^\alpha \) is right Riemann–Liouville fractional derivative, and
- \( u^* : [t_0, t_f] \rightarrow \mathbb{R}^m \) is a control strategy such that \( u^*(t) \) maximizes \([t_0, t_f] \setminus \{t_f\}\)-a.e.

Consider

\[
H(t, x, p, u) := p^T f(t, x, u),
\]

with \( p \in \mathbb{R}^n \), to be the Pontryagin function associated to problem \((P)\).

Definition 3.1

A pair \((\bar{x}, u^*)\) is an optimal control process for problem \((P)\), if it yields a cost lower than that associated with any other feasible control process.

Obviously, given the equivalence between \((P)\) and \((\bar{P})\), the same definition holds for \((x, u)\) with respect to \((P)\).

Theorem 3.1

Let \((x^*, u^*)\) be optimal control process for \((P)\).

Then, there exists a function \( p : [t_0, t_f] \rightarrow \mathbb{R}^n \) satisfying

- the adjoint equation
  \[
  D_t^\alpha p(t) = D_t f(t, x^*(t), u^*(t)), \tag{10}
  \]
  with the transversality condition
  \[
  p(t_f) = \nabla g(x^*(t_f)), \tag{11}
  \]
  where the operator \( D_t^\alpha \) is right Riemann–Liouville fractional derivative, and
- \( u^* : [t_0, t_f] \rightarrow \mathbb{R}^m \) is a control strategy such that \( u^*(t) \) maximizes \([t_0, t_f] \setminus \{t_f\}\)-a.e.

Proof of Theorem 3.1

The first key idea is that any perturbation of the optimal control \( u^* \) that affects the final value of the state trajectory may increase the cost. Thus, the proof relies on the comparison between the optimal trajectory \( x^* \) and trajectories \( x \), which are obtained by perturbing the optimal control \( u^* \).

Let \( \tau \) be a Lebesgue point in \([t_0, t_f]\), and \( \varepsilon > 0 \) sufficiently small so that \( \tau - \varepsilon \geq t_0 \). The Lebesgue point in the fractional context is defined next.

Definition 3.2

Let \( 0 < \alpha \leq 1 \). An \( \alpha \)-Lebesgue point of an integrable function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a point \( t_0 \in \mathbb{R} \) satisfying

\[
\lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_{t_0 - \varepsilon}^{t_0 + \varepsilon} |f(t) - f(t_0)|^{2-\alpha} \, dt = 0.
\]

By directly using the definition \( \alpha \)-fractional integral it is easy to conclude that the set of Lebesgue points includes those of the integer order integral, 25. It is important to point out the well-known fact that the subset of Lebesgue points of an integrable function \( f \) constitutes a full Lebesgue measure subset (see, e.g., 26).
Moreover, consider the perturbed control strategy $u_{t,e}$ defined by

$$u_{t,e}(t) = \begin{cases} \tilde{u} & \text{if } t \in [\tau - e, \tau) \\ u^*(t) & \text{if } t \in [t_0, t] \setminus [\tau - e, \tau), \end{cases}$$

(12)

where $\tilde{u} \in \Omega(t)$ for all $t \in [\tau - e, \tau)$, being $\tau$ a Lebesgue point of the reference optimal control strategy. Note that, there is no loss of generality of the choice of $\tau$ because the set Lebesgue points is of full Lebesgue measure.

Let $x_{t,e}$ be the trajectory associated with $u_{t,e}$, and with $x_{t,e}(t_0) = x_0$. Clearly, by definition of optimality of $(x^*, u^*)$,

$$\begin{cases} 0 \leq g(x_{t,e}(t)) - g(x^*(t)) \\ = \nabla_x g(x^*(t))[x_{t,e}(t) - x^*(t)] + o(\varepsilon), \end{cases}$$

(13)

where $\nabla_x g(\cdot)$ is the gradient of $g(\cdot)$, and $o(\varepsilon)$ is some positive number satisfying $\lim_{\varepsilon \to 0} \frac{o(\varepsilon)}{\varepsilon} = 0$.

Observe that $x_{t,e}(t) = x^*(t)$ for all $t \in [t_0, \tau - e)$.

Moreover, it is clear that, for all $t \in [\tau - e, \tau]$, we have,

$$\begin{align*}
|x_{t,e}(t) - x^*(t)| & \leq \tau - \varepsilon f^p_k |f(s, x_{t,e}(s), \tilde{u}) - f(s, x^*(s), u^*(s))| \\
& \leq \tau - \varepsilon f^p_1 K_1 |x_{t,e}(s) - x^*(s)| + 2M \varepsilon^{\alpha} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} \\
& \leq \tilde{M} \varepsilon^{\alpha}
\end{align*}$$

(14)

where

$$\tilde{M} = 2M \left(1 + K_1 \sum_{n=1}^{\infty} \frac{\Gamma(\alpha)^n}{\Gamma(\alpha + 1)} \varepsilon^{\alpha} \right).$$

It is not difficult to show that this series converges, and thus, $\tilde{M}$ is some finite positive number. The last inequality was obtained by applying Theorem 2.3, and, in particular, holds for $t = \tau$.

Before proceeding with the proof, we need the following auxiliary result.

In what follows, let $\Phi_{u}(\cdot, \cdot)$ denote the state transition matrix for the linear fractional differential system

$$\zeta D_{\alpha}^q \xi(t) = D_x f(t, x^*(t), u^*(t)) \xi(t).$$

Lemma 3.1

Consider the general time interval $[a, b]$ and define the function $F(t, x) = f(t, x, u(t))$, where $u(t)$ is a general feasible control function. Moreover, consider $\tilde{x}(\cdot)$, $y(\cdot)$, and $\tilde{x}_v(\cdot)$ to be, respectively, the solutions to the following fractional differential systems defined on the interval $[a, b]$:

- $\zeta D_{\alpha}^q \tilde{x}(t) = F(t, \tilde{x}(t))$ with $\tilde{x}(a) = x_0$,
- $\zeta D_{\alpha}^q y(t) = D_v F(t, \tilde{x}(t)) y(t)$ with $y(a) = \tilde{y}$, and
- $\zeta D_{\alpha}^q \tilde{x}_v(t) = F(t, \tilde{x}_v(t))$ with $\tilde{x}_v(a) \in x_0 + v^{\alpha} \tilde{y} + o(v^{\alpha}) B_1(0)$.

Then, for all $v$ positive and sufficiently small real number, we have that $\tilde{x}_v(\cdot)$ satisfies $[a, b]$,

$$\tilde{x}_v(t) \in \tilde{x}(t) + v^{\alpha} y(t) + o(v^{\alpha}) B_1(0).$$

Here, $B_1(0)$ denotes the closed unit ball of $\mathbb{R}^n$ centered at 0.

Proof of Lemma 3.1

Let us consider the first-order expansion of the map $x \to F(t, \cdot)$ around $\tilde{x}(t)$. We have

$$F(t, \tilde{x}_v(t)) - F(t, \tilde{x}(t)) = D_v F(t, \tilde{x}(t)) (\tilde{x}_v(t) - \tilde{x}(t)) \in o(||\tilde{x}_v(t) - \tilde{x}(t)||),$$

for all $t \in [a, b]$. Since

$$\zeta D_{\alpha}^q y(t) = D_v F(t, \tilde{x}(t)) y(t),$$

with $v^{\alpha} y(a) \in \tilde{x}_v(a) - \tilde{x}(a) + o(v^{\alpha}) B_1(0)$, we have that

$$\zeta D_{\alpha}^q \tilde{x}_v(t) - \zeta D_{\alpha}^q \tilde{x}(t) - v^{\alpha} \zeta D_{\alpha}^q y(t) = \zeta(t),$$

for some $\zeta \in L^1$ satisfying $\zeta(t) \in o(v^{\alpha}) B_1(0)$ in $L^1$. By integrating, we have
\[ x_\alpha(t) - x(t) - v^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} \left( F(\tau, \tilde{x}_\alpha(\tau)) - F(\tau, \tilde{x}(\tau)) - v^\alpha D_\alpha F(\tau, \tilde{x}(\tau)) y(\tau) \right) d\tau + \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} \xi(\tau) d\tau. \]

Before continuing, we should observe that it is a simple exercise to conclude that the assumption (H2) implies that \( \|D_\alpha F(x(t))\| \leq K_r \). Note also that it is not difficult to see that \( \beta(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} \xi(\tau) d\tau \in o(v^\alpha) \), for all \( t \in [a, b] \).

Now, by putting \( z(t) = x_\alpha(t) - x(t) - v^\alpha y(t) \), using the above observations and the assumption (H2), we obtain the inequality

\[ \|z(t)\| \leq \beta(t) + K_r \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} \|z(\tau)\| d\tau. \]

Theorem 2.3 of the previous section (Generalized Bellman–Gronwall inequality) yields

\[ \|z(t)\| \leq \beta(t) + \frac{K_r}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} \|z(\tau)\| d\tau. \]

Let

\[ \bar{\beta} = \sup_{t \in [a,b]} \|\beta(t)\|. \]

Obviously, we have that \( \bar{\beta} \in o(v^\alpha) \). By performing the integral in the right hand side of the previous inequality, we have that, for all \( t \in [a, b] \),

\[ \|z(t)\| \leq \bar{\beta} \left( 1 + \frac{\sum_{n=1}^{\infty} K_r^n \Gamma(n) (t - a)^n}{\Gamma(na + 1)} \right). \]

which, by using the one parameter Mittag–Leffler function, can be expressed by

\[ \|z(t)\| \leq \bar{\beta} (1 + E_{\alpha,1}(K_r \Gamma(a)(t - a)^\alpha)). \]

Thus, Lemma 3.1 is proved, that is, for all \( t \in [a, b] \),

\[ x_\alpha(t) \in x(t) + v^\alpha y(t) + o(v^\alpha)B_\alpha^1(0). \]

(15)

Now, by considering (13) and applying Lemma 3.1 on the interval \([t, \tau]\) with \( F(t, x) = f(t, x, u_{t,x}(t)), v^\alpha = \varepsilon, a = \tau, t = t_\tau, \tilde{x}_\alpha = x_{t_\tau, \varepsilon}, \) and \( \tilde{x} = x^* \), we conclude that

\[
\begin{aligned}
0 &\leq \nabla_x g(x^*(t)) [x_{t_\tau, \varepsilon}(t) - x^*(t)] + o(\varepsilon) \\
&\leq \varepsilon \nabla_x g(x^*(t)) y(t) + o(\varepsilon) \\
&= \varepsilon \nabla_x g(x^*(t)) \Phi_\alpha(t, \tau) y(\tau) + o(\varepsilon)
\end{aligned}
\]

where \( \Phi_\alpha(t, \tau) \) is the fractional state transition matrix associated with the linear system

\[ \frac{D_\alpha^\alpha y(t)}{D_\alpha y(t)} = D_\alpha F(t, x^*(t)) y(t), \]

in the interval \([t, \tau]\). By letting

\[ -p^T(t_\tau) = \nabla_x g(x^*(t_\tau)), \]

and

\[ p^T(t) = p^T(t_\tau) \Phi_\alpha(t_\tau, t), \]

we conclude that the adjoint variable \( p(\cdot) \) satisfies the right Riemann–Liouville fractional linear equation

\[ iD_\alpha^\alpha p^\alpha(t) = p^\alpha(t) D_\alpha F(t, x^*(t)), \]

that is, \( p(\cdot) \) satisfies the adjoint equation of our maximum principle as well as the associated transversality condition.

Finally, by putting together (16) and (17) and by choosing

\[ y(\tau) = f(t, x^*(\tau), \bar{u}) - f(t, x^*(\tau), u^*(\tau)), \]

we obtain

\[ 0 \geq \varepsilon p^T(\tau) [f(t, x^*(\tau), \bar{u}) - f(t, x^*(\tau), u^*(\tau))] + o(\varepsilon). \]

By dividing both sides of this inequality by \( \varepsilon > 0 \) and by taking the limit \( \varepsilon \to 0^+ \), we conclude the inequality

\[ 0 \geq p^T(\tau) [f(t, x^*(\tau), \bar{u}) - f(t, x^*(\tau), u^*(\tau))]. \]
which, from the arbitrariness of \( \tilde{u} \in \Omega \), yields the maximum condition at time \( t = \tau \),

\[
H(\tau, x^*(\tau), p(\tau), u^*(\tau)) \geq H(\tau, x^*(\tau), p(\tau), \tilde{u}).
\]

The fact that \( \tau \) is an arbitrary Lebesgue point in \([t_0, T]\) implies that the maximum condition of our main result holds, that is, \( u^*(t) \) maximizes on \( \Omega(t) \), the map \( u \rightarrow H(t, x^*(t), p(t), u), [t_0, T] \) \(-a.e.-\).

Our main result is proved.

4. Illustrative example

The Pontryagin maximum principle proved in the previous section is now applied to solve a simple problem of resources management that involves minimizing a certain fractional integral subject to given-controlled FDEs.

We consider the following problem

Minimize \( J(u) \) \hspace{1cm} (18)

subject to \( \frac{D^\alpha_t}{t} x(t) = u(t)x(t), \quad t \in [0, T], \) \hspace{1cm} (19)

\[
x(0) = x_0, \hspace{1cm} (20)
\]

\[
u(t) \in [0, 1], \hspace{1cm} (21)
\]

where \( J(u) = \frac{D^\alpha_0}{t} (1 - u(t))x(t), \) with \( 0 < \alpha < 1 \) and \( T > \Gamma(\alpha + 1)|\alpha|^{-1}. \) Here, \( \frac{D^\alpha_0}{t} \) is fractional integral, and \( \frac{D^\alpha_t}{t} \) is left Caputo fractional derivative.

The variable \( x \) represents a natural resource that takes positive values (note that \( x_0 > 0 \) necessarily) 'grows' according to the law (19), where the function \( u \), designated by control, represents the fraction of the available resource that is used to promote further growth.

The overall goal is to find the control strategy that maximizes the amount of accumulated resource over the time interval \([0, T]\) given by the fractional integral (18).

First, we consider an additional state variable component \( y \) satisfying

\[
\frac{D^\alpha_t}{t} y(t) = (1 - u(t))x(t), \hspace{1cm} y(0) = 0,
\]

in order to obtain the canonic problem statement in the form considered in our main result, that is,

Minimize \( y(T) \)

subject to \( \frac{D^\alpha_t}{t} x(t) = u(t)x(t), \quad x(0) = x_0, \)

\[
\frac{D^\alpha_t}{t} y(t) = (1 - u(t))x(t), \hspace{1cm} y(0) = 0, \hspace{1cm} (20)
\]

\[
u(t) \in [0, 1]. \hspace{1cm} (21)
\]

From Theorem 3.1, the adjoint equation (10) and the transversality condition (11) for this problem are

\[
\frac{D^\alpha_t}{t} p_1(t) = [p_1 u^*(t) + p_2 (1 - u^*(t))], \quad p_1(T) = 0, \hspace{1cm} (22)
\]

\[
\frac{D^\alpha_t}{t} p_2(t) = 0, \hspace{1cm} p_2(T) = 1, \hspace{1cm} (23)
\]

where \( \frac{D^\alpha_t}{t} \) is right Riemann–Liouville fractional derivative of order \( \alpha \). Thus, we have that \( p_2(t) \equiv p_2(T) = 1 \), and equation (22) becomes

\[
\frac{D^\alpha_t}{t} p_1(t) = [(p_1(t) - 1)u^*(t) + 1]. \hspace{1cm} (24)
\]

From the maximum condition, we know that \( u^*(t) \) maximizes, \( L -a.e. \) in \([0, 1] \), the mapping

\[
\nu \rightarrow p^T(t)f(t, x^*(t), y^*(t), \nu) = [p_1(t)\nu + p_2(t)(1 - \nu)]x^*(t).
\]

Because \( p_2 = 1 \) and \( x^*(t) > 0 \) for all \( t \in [0, T] \) (this is to conclude from the fact that \( x_0 > 0 \), the mapping to be maximized can be simplified to \( \nu \rightarrow (p_1(t) - 1)\nu \). Thus, given that the system is time invariant, we have that

\[
u^*(t) = \begin{cases} 1 & \text{if } p_1(t) > 1 \\ 0 & \text{if } p_1(t) < 1. \end{cases}
\]

Thus, \( p_2(t) = 1 \), and \( p_1(\cdot) \) is continuous, \( \exists b > 0 \) such that \( u^*(t) = 0 \), for all \( t \in [T - b, T] \). Thus, from (22), we have \( \frac{D^\alpha_t}{t} p_1(t) = 1 \) and, by backward integration, we obtain

\[
\frac{p_1(t)}{T - t} = \frac{(T - t)\alpha}{\Gamma(\alpha + 1)}. \hspace{1cm} (25)
\]
Obviously that, for \( t^* = T - (\Gamma(\alpha + 1))^{\frac{1}{\alpha + 1}} \), we obtain \( p_1(t^*) = 1 \). Now, let us determine the optimal control for \( t < t^* \). Because, independently of the control, \( p_1(t) \) remains monotonically decreasing, we have for \( t < t^* \), \( u^*(t) = 1 \), and, thus,

\[
\dot{D}_C^\alpha p_1(t) = p_1(t). \tag{26}
\]

The solution of this linear fractional differential equation (26) is given by

\[
p(t) = p(t^*)^\Phi_\alpha(t^*, t),
\]

where \( p(t^*) = 1 \), and \( \Phi_\alpha(t^*, t) \) is the fractional state transition matrix (in fact, scalar-valued) that can be computed by the Mittag–Leffler function defined in the previous section.

By setting \( \beta = \alpha \), \( A = [1] \), and by replacing \( \tau = t^* - t = T - \Gamma(\alpha + 1)^{\alpha - 1} - t \), we conclude that

\[
p_1(t) = e_\alpha(1, t^* - t) = (t^* - t)^{\alpha - 1} E_\alpha,\alpha ((t^* - t)^\alpha)
\]

\[
 = (t^* - t)^{\alpha - 1} \sum_{k=0}^{\infty} \frac{(t^* - t)^{k\alpha}}{\Gamma((k + 1)\alpha)}.
\]

Note that if \( \alpha = 1 \), then we have classical solution \( e^{t^* - t - 1} \).

Because we have the optimal control \( u^* \), we can easily compute the optimal trajectory, which satisfies \( x^*(0) = x_0 \), and

\[
\begin{cases} 
\dot{D}_C^\alpha x^*(t) = x^*(t) & \text{if } t \in [0, t^*] \\
0 & \text{if } t \in [t^*, T]. 
\end{cases}
\]

We can compute the optimal trajectory \( x^* \) by the generalization Mittag–Leffler function. For all \( t \in [0, t^*] \), and \( x^*(0) = x_0 \), we conclude that

\[
x^*(t) = E_\alpha(\alpha t^*) = x_0 E_\alpha(\alpha t^*) = x_0 \sum_{k=0}^{\infty} \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)}.
\]

Note that if \( \alpha = 1 \), then we have classical solution \( x_0 e^t \).

Now, we compute the optimal trajectory \( x^* \) in the interval \([t^*, T]\), which \( u^* = 0 \), and \( x^*(t) = x^*(T) \). We conclude that

\[
x^*(t) = x^*(t^*) = x_0 E_\alpha((t^*)^\alpha) = x_0 \sum_{k=0}^{\infty} \frac{(T - (\Gamma(\alpha + 1)^{\frac{1}{\alpha + 1}}))^k}{\Gamma(k\alpha + 1)}.
\]

Once again, note that if \( \alpha = 1 \), then we have classical solution \( x_0 e^{t^* - t - 1} \).

5. Conclusion

This article concerns the derivation of necessary conditions of optimality in the form of Pontryagin maximum principle for a nonlinear fractional optimal control problem whose differential equation involves the Caputo derivative of the state variable with respect to time. Under mild assumptions on the data of the problem, the proof involved the direct application of variational arguments, thus avoiding the often used argument of converting the optimal control problem into a conventional one and, then, express the optimality conditions for this auxiliary problem back in the fractional derivative context. Another interesting novelty consists in the fact that, unlike in most fractional optimal control problem formulations, we consider the cost functional given by a fractional integral of Riemann–Liouville type.

A simple example illustrating the application of our maximum principle was presented. The optimal control strategy was computed analytically being the fractional differential adjoint equation solved by using a technique based on a generalization Mittag–Leffler function.

A natural sequel of this article concerns the weakening of the assumptions on the data of the problem, notably the mere measurability dependence of the dynamics with respect to time and to the control variables. This will certainly require more sophisticated variational arguments and the use of methods and results of nonsmooth analysis. Another direction of research consists in increasing the structure of the fractional optimal control problem by considering additional state endpoint constraints and state and/or mixed constraints in its formulation. In this case, additional regularity assumptions will be needed to ensure that the obtained necessary conditions of optimality do not degenerate.
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