COMPUTATION OF THE HAUSDORFF DISTANCE 
BETWEEN TWO ELLIPSES

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Abstract. We are interested in the problem of finding the Hausdorff distance between two objects in $\mathbb{R}^2$, or in $\mathbb{R}^3$. In this paper, we develop an algorithm for computing the Hausdorff distance between two ellipses in $\mathbb{R}^3$. Our algorithm is mainly based on computing the distance between a point $u \in \mathbb{R}^3$ and a standard ellipse $E_s$, equipped with a pruning technique. This algorithm requires $O(\log M)$ operations, compared with $O(M)$ operations for a direct method, to achieve a comparable accuracy. We give an example, and observe that the computational cost needed by our algorithm is only $O(\log M)$.

1. Introduction and preliminaries

The problem of computing the Hausdorff distance between two objects in $\mathbb{R}^2$ or $\mathbb{R}^3$ has been widely used in a large variety of applications, including computer vision, CAD/CAM, pattern recognition, and approximation theory. Furthermore, it is very important to propose more efficient algorithms for solving the Hausdorff distance problem like that. In the literature, many problems already have been studied, and various numerical techniques have been given to compute the optimal Hausdorff distance. Alt et al[1] considered finite collections of simplices(e.g., line segments, triangles, or tetrahedrons), and developed polynomial-time algorithms. Belogay et al[2] discussed an algorithm for computing the Hausdorff distance between two discretized curves in $\mathbb{R}^2$. Bouts[3], Rote[6] and Scharf[7] considered planar sets of curves which have parametric representations. In Kim[4] the minimum distance problem between any two ellipses has been studied and an iterative algorithm for finding the optimal distance has been given. This problem can be
frequently found in computer-aided geometric design systems. On the other hand, in order to observe the similarity between two objects we need to use the Hausdorff distance measure as a similarity distance measure instead of the minimum Euclidean distance, which is different from any other distance functions. In this paper we are interested in the problem of finding the Hausdorff distance between two ellipses in \( \mathbb{R}^3 \).

One efficient algorithm for computing the Hausdorff distance between two ellipses will be proposed. Our algorithm is based on computing the orthogonal distance between any one point and a standard ellipse, equipped with a pruning technique.

Let \( d(a, b) \) be a distance between two points \( a \) and \( b \) in \( \mathbb{R}^3 \). Then, the distance from a point \( a \in \mathbb{R}^3 \) to a non-empty subset \( B \subset \mathbb{R}^3 \) is given by

\[
d(a, B) = \inf_{b \in B} d(a, b).
\]

(1)

The Hausdorff distance between two sets \( A \) and \( B \) in \( \mathbb{R}^3 \) is defined as

\[
H(A, B) = \max \{ h(A, B), h(B, A) \},
\]

(2)

where \( h(A, B) \) is called the one-sided Hausdorff distance from \( A \) to \( B \), given by

\[
h(A, B) = \sup_{a \in A} d(a, B).
\]

(3)

We now assume that \( d(a, b) \) is the Euclidean distance throughout this paper. Namely, for \( a = (a_1, a_2, a_3)^T \) and \( b = (b_1, b_2, b_3)^T \) in \( \mathbb{R}^3 \)

\[
d(a, b) = |a - b| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}.
\]

(4)

This distance may be replaced by any other common used appropriate distance functions as well.

Here, in \( \mathbb{R}^3 \) we can characterize an ellipse by giving the three points \( c, p, \) and \( q \), where \( c \) is its center, and \( p \) and \( q \) denote one of the ends of the major axis and one of the ends of the minor axis respectively. Further, the representation of an ellipse can also be given by using the corresponding geometric transformation of a standard ellipse lying in the \( xy \)-plane. So, our problem is reduced to the problem of computing the Hausdorff distance between both of an ellipse in \( \mathbb{R}^3 \) and a standard ellipse in the \( xy \)-plane.
The standard form of an ellipse $E_s$ centered at origin in the $xy$-plane can be given by

$$E_s : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a$ and $b$ are the semimajor and the semiminor axis respectively. Further, the parametric form of $E_s$ is expressed by

$$x = a \cos t, \quad y = b \sin t,$$

where $-\pi \leq t \leq \pi$. It follows that $E_s$ can be parameterized by a continuous function $f : [-\pi, \pi] \to \mathbb{R}^3$ such that $f(t) = (a \cos t, b \sin t, 0)^T$.

Let $\tilde{E}$ be an arbitrary ellipse given by the center $\tilde{c} = (\tilde{c}_1, \tilde{c}_2, \tilde{c}_3)^T$ and two points $\tilde{p} = (\tilde{p}_1, \tilde{p}_2, \tilde{p}_3)^T$ and $\tilde{q} = (\tilde{q}_1, \tilde{q}_2, \tilde{q}_3)^T$. Then $\tilde{E}$ can be represented by a transformation (rotation and translation) of the corresponding standard ellipse $E_s$.

Let $E_s$ be a relative standard ellipse with respect to $\tilde{E}$:

$$E_s : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

with its parametric function

$$f(t) = (a \cos t, b \sin t, 0)^T \in \mathbb{R}^3, \quad -\pi \leq t \leq \pi,$$

where

$$a = |\tilde{p} - \tilde{c}| = \sqrt{(\tilde{p}_1 - \tilde{c}_1)^2 + (\tilde{p}_2 - \tilde{c}_2)^2 + (\tilde{p}_3 - \tilde{c}_3)^2},$$

$$b = |\tilde{q} - \tilde{c}| = \sqrt{(\tilde{q}_1 - \tilde{c}_1)^2 + (\tilde{q}_2 - \tilde{c}_2)^2 + (\tilde{q}_3 - \tilde{c}_3)^2}.$$

Then, by using a $3 \times 3$ rotation matrix $R = (r_{ij})$ given by

$$(r_{11}, r_{21}, r_{31})^T = Re_1 = R(1,0,0)^T = \frac{\tilde{p} - \tilde{c}}{|\tilde{p} - \tilde{c}|},$$

$$(r_{12}, r_{22}, r_{32})^T = Re_2 = R(0,1,0)^T = \frac{\tilde{q} - \tilde{c}}{|\tilde{q} - \tilde{c}|},$$

$$(r_{13}, r_{23}, r_{33})^T = Re_1 = R(0,0,1)^T = \frac{(\tilde{p} - \tilde{c}) \times (\tilde{q} - \tilde{c})}{| (\tilde{p} - \tilde{c}) \times (\tilde{q} - \tilde{c}) |}$$

the coordinate $\tilde{w} = (\tilde{x}, \tilde{y}, \tilde{z})^T$ of a point on $\tilde{E}$ can be represented by

$$\tilde{w} = Rw + \tilde{c},$$
where \( w = (x, y, 0)^T \) is the coordinate of the corresponding point of \( E_s \) with respect to \( \tilde{w} \). Furthermore, the corresponding parametric function \( \tilde{f}(t) \) for \( \tilde{E} \) can be defined as

\[
\tilde{f}(t) = R f(t) + \tilde{c} = \left( \begin{array}{ccc} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{array} \right) \left( \begin{array}{c} a \cos t \\ a \sin t \\ 0 \end{array} \right) + \left( \begin{array}{c} \tilde{c}_1 \\ \tilde{c}_2 \\ \tilde{c}_3 \end{array} \right), \quad -\pi \leq t \leq \pi.
\]

2. Distance from a point to an ellipse in \( R^3 \)

The following distance problem between one point and a standard ellipse in \( R^3 \) can be found in Kim[4].

Suppose that \( u = (u_1, u_2, u_3)^T \) is a point in \( \mathbb{R}^3 \) and \( E_s \) is a standard ellipse in the \( xy \)-plane defined by

\[
E_s : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,
\]

with its parametric function

\[
f(t) = (a \cos t, b \sin t, 0)^T \in \mathbb{R}^3.
\]

In order to compute the distance \( d(u, E_s) \) from \( u \) to \( E_s \) we consider the problem of finding the optimal point \( v \) which lies on \( E_s \), such that \( d(u, E_s) = \inf_{w \in E_s} d(u, w) = d(u, v) \). Let \( Q(u, E_s; t) \) be an object function of a parameter \( t \) with respect to \( u \) and \( E_s \), defined as

\[
Q(u, E_s; t) = (u_1 - a \cos t)^2 + (u_2 - b \sin t)^2.
\]

Then, by finding the optimal point \( \hat{v} = (a \cos \hat{t}, b \sin \hat{t}, 0)^T \) on \( E_s \) such that

\[
m = Q(u, E_s; \hat{t}) = \inf_{-\pi \leq t \leq \pi} Q(u, E_s; t),
\]

the distance \( d(u, E_s) \) can be obtained by

\[
d(u, E_s) = \inf_{w \in E_s} d(u, w) = \inf_{-\pi \leq t \leq \pi} |u - f(t)| = \inf_{-\pi \leq t \leq \pi} \sqrt{(u_1 - a \cos t)^2 + (u_2 - b \sin t)^2 + u_3^2} = \sqrt{m + u_3^2}.
\]

Due to the necessary condition \( \frac{\partial Q}{\partial t} = 0 \) for a minimum we have the following equation:

\[
A \sin t - B \cos t + C \sin t \cos t = 0,
\]
the Hausdorff distance between two ellipses

with $A = au_1$, $B = bu_2$, and $C = (b^2 - a^2)$. In case of $C = 0$ in (19) we can easily find $t = \hat{t}$ by

$$
\hat{t} = \tan^{-1}\left(\frac{B}{A}\right).
$$

If $C = b^2 - a^2 \neq 0$, then the equation (19) induces the following two equations:

$$(21) \quad A \sin t - B \cos t + C \sin t \cos t = 0 \quad \text{for} \quad 0 \leq t \leq \pi,$$

and

$$(22) \quad A \sin t + B \cos t + C \sin t \cos t = 0 \quad \text{for} \quad -\pi \leq t \leq 0.$$  

Let $\omega = \cos t$ in (21) and (22). Then the corresponding values of $\sin t$ can be given by

$$
\sin t = \begin{cases} 
\sqrt{1 - \omega^2} & (0 \leq t \leq \pi) \\
-\sqrt{1 - \omega^2} & (-\pi \leq t \leq 0),
\end{cases}
$$

and the equation (19) leads to the following quartic equation:

$$
\omega^4 + 2 \left(\frac{A}{C}\right) \omega^3 + \left(\frac{A^2 + B^2 + C^2}{C^2}\right) \omega^2 - 2 \left(\frac{A}{C}\right) \omega - \left(\frac{A}{C}\right)^2 = 0.
$$

By solving the equation (23) for $\omega$ we find some solutions $\omega_j \quad (j = 1, 2, \ldots, n)$ such that $-1 \leq \omega_j \leq 1$ and $1 \leq n \leq 4$. Further, for each $\omega_j$ we have the value $\bar{t}_j$ such that $\cos \bar{t}_j = \omega_j \quad (0 \leq \bar{t}_j \leq \pi)$. Since $t$ has two values $\theta^1_j = \bar{t}_j$ and $\theta^2_j = -\bar{t}_j$ for each $\bar{t}_j$, the corresponding values of $\sin t$ can be obtained by

$$
\begin{cases} 
\sin \theta^1_j = \sqrt{1 - (\omega_j)^2} & (j = 1, 2, \ldots, n) \\
\sin \theta^2_j = -\sqrt{1 - (\omega_j)^2} & (j = 1, 2, \ldots, n),
\end{cases}
$$

In this case we can choose $t = \tilde{t} = \theta^m_1$ for the solution of (19) such that

$$
(u_1 - a \cos \theta^m_1)^2 + (u_2 - b \sin \theta^m_1)^2
$$

$$
(25) \quad = \min_{i=1,2} \left\{ (u_1 - a \cos \theta^i_1)^2 + (u_2 - b \sin \theta^i_1)^2 \right\}.
$$

Since the Euclidean metric is invariant under every rotation and translation, the problem of finding the distance $d(\tilde{u}, \tilde{E})$ between a point $\tilde{u} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)^T$ and an ellipse $\tilde{E}$ can be transformed into a simpler one. That is, from (12) and (13) the corresponding point $u = (u_1, u_2, u_3)^T$
with respect to $\tilde{u}$ can be given by $u = R^{-1}(\tilde{u} - \tilde{c})$. Finally, we can easily compute

$$d(\tilde{u}, E) = d(u, E_s).$$

3. The Hausdorff distance between two ellipses

Let $E^\alpha$ and $E^\beta$ be two ellipses in $\mathbb{R}^3$. Suppose that $E^\alpha$ is characterized by its center $\tilde{c}^\alpha = (\tilde{c}_1^\alpha, \tilde{c}_2^\alpha, \tilde{c}_3^\alpha)^T$ and the two points $\tilde{p}^\alpha = (p_1^\alpha, p_2^\alpha, p_3^\alpha)^T$ and $\tilde{q}^\alpha = (q_1^\alpha, q_2^\alpha, q_3^\alpha)^T$, and $E^\beta$ is given by $\tilde{c}^\beta = (\tilde{c}_1^\beta, \tilde{c}_2^\beta, \tilde{c}_3^\beta)^T$, $\tilde{p}^\beta = (p_1^\beta, p_2^\beta, p_3^\beta)^T$, and $\tilde{q}^\beta = (q_1^\beta, q_2^\beta, q_3^\beta)^T$. Then, we can define the corresponding standard ellipses $E_s^\alpha$ and $E_s^\beta$ with their parametric functions:

$$E_s^\alpha : f(t^\alpha) = (a^\alpha \cos t^\alpha, b^\alpha \sin t^\alpha, 0)^T \in R^3, \quad (-\pi \leq t^\alpha \leq \pi),$$

$$E_s^\beta : g(t^\beta) = (a^\beta \cos t^\beta, b^\beta \sin t^\beta, 0)^T \in R^3, \quad (-\pi \leq t^\beta \leq \pi),$$

where

$$a^\alpha = |\tilde{p}^\alpha - \tilde{c}^\alpha| = \sqrt{(p_1^\alpha - c_1^\alpha)^2 + (p_2^\alpha - c_2^\alpha)^2 + (p_3^\alpha - c_3^\alpha)^2},$$

$$b^\alpha = |\tilde{q}^\alpha - \tilde{c}^\alpha| = \sqrt{(q_1^\alpha - c_1^\alpha)^2 + (q_2^\alpha - c_2^\alpha)^2 + (q_3^\alpha - c_3^\alpha)^2},$$

$$a^\beta = |\tilde{p}^\beta - \tilde{c}^\beta| = \sqrt{(p_1^\beta - c_1^\beta)^2 + (p_2^\beta - c_2^\beta)^2 + (p_3^\beta - c_3^\beta)^2},$$

$$b^\beta = |\tilde{q}^\beta - \tilde{c}^\beta| = \sqrt{(q_1^\beta - c_1^\beta)^2 + (q_2^\beta - c_2^\beta)^2 + (q_3^\beta - c_3^\beta)^2}.$$

Also, the coordinate $\tilde{w}^\alpha = (\tilde{x}^\alpha, \tilde{y}^\alpha, \tilde{z}^\alpha)^T$ of a point on $E^\alpha$ and the coordinate $\tilde{w}^\beta = (\tilde{x}^\beta, \tilde{y}^\beta, \tilde{z}^\beta)^T$ of a point on $E^\beta$ can be represented by

$$E^\alpha : \tilde{w}^\alpha = R^\alpha \tilde{w}^\alpha + \tilde{c}^\alpha,$$

$$E^\beta : \tilde{w}^\beta = R^\beta \tilde{w}^\beta + \tilde{c}^\beta,$$

where $\tilde{w}^\alpha = (x^\alpha, y^\alpha, z^\alpha)^T$ and $\tilde{w}^\beta = (x^\beta, y^\beta, z^\beta)^T$ are the corresponding coordinates on $E_s^\alpha$ and $E_s^\beta$ respectively, and $R^\alpha = (r_{ij}^\alpha)$ and $R^\beta = (r_{ij}^\beta)$ are the rotation matrices with respect to $E_s^\alpha$ and $E_s^\beta$ respectively. Their rotation matrices are given by the following:
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for \( k = \alpha \) or \( \beta \)

\[
(r^k_{11}, r^k_{21}, r^k_{31})^T = R^k e_1 = R^k (1, 0, 0)^T = \frac{\tilde{p}^k - \tilde{c}^k}{|\tilde{p}^k - \tilde{c}^k|},
\]

\[
(r^k_{12}, r^k_{22}, r^k_{32})^T = R^k e_2 = R^k (0, 1, 0)^T = \frac{\tilde{q}^k - \tilde{c}^k}{|\tilde{q}^k - \tilde{c}^k|},
\]

\[
(r^k_{13}, r^k_{23}, r^k_{33})^T = R^k e_1 = R^k (0, 0, 1)^T = \frac{(\tilde{p}^k - \tilde{c}^k) \times (\tilde{q}^k - \tilde{c}^k)}{|(\tilde{p}^k - \tilde{c}^k) \times (\tilde{q}^k - \tilde{c}^k)|}.
\]

Further, the parametric functions \( \tilde{f}(t^\alpha) \) and \( \tilde{g}(t^\beta) \) of \( \tilde{E}^\alpha \) and \( \tilde{E}^\beta \) respectively are given by

\[
\tilde{f}(t^\alpha) = R^\alpha f(t^\alpha) + \tilde{c}^\alpha,
\]

\[
\tilde{g}(t^\beta) = R^\beta g(t^\beta) + \tilde{c}^\beta.
\]

In order to compute the Hausdorff distance between two ellipses \( \tilde{E}^\alpha \) and \( \tilde{E}^\beta \):

\[
H(\tilde{E}^\alpha, \tilde{E}^\beta) = \max \{ h(\tilde{E}^\alpha, \tilde{E}^\beta), h(\tilde{E}^\beta, \tilde{E}^\alpha) \}
\]

we now consider the one-sided Hausdorff distance \( h(\tilde{E}^\alpha, \tilde{E}^\beta) \) from \( \tilde{E}^\alpha \) to \( \tilde{E}^\beta \). Here, using the rotation matrix \( R^\beta \) and the center \( \tilde{c}^\beta \) we can make a new ellipse \( \bar{E}^\alpha \) with respect to \( \tilde{E}^\alpha \). The coordinate \( \bar{w}^\alpha = (\bar{x}^\alpha, \bar{y}^\alpha, \bar{z}^\alpha)^T \) of a point on \( \bar{E}^\alpha \) is defined as

\[
\bar{w}^\alpha = (R^\beta)^{-1}(\bar{w}^\alpha - \bar{c}^\beta).
\]

Due to the invariant property of Euclidean distance under every rotation and translation we have

\[
h(\bar{E}^\alpha, E^\beta) = \sup_{\bar{w}^\alpha \in \bar{E}^\alpha} d(\bar{w}^\alpha, E^\beta)
\]

\[
= \sup_{\bar{w}^\alpha \in \bar{E}^\alpha} \inf_{w^\beta \in E^\beta} d(\bar{w}^\alpha, w^\beta)
\]

\[
= \sup_{\bar{w}^\alpha \in \bar{E}^\alpha} \inf_{w^\beta \in E^\beta} d(\bar{w}^\alpha, w^\beta)
\]

\[
= h(\bar{E}^\alpha, E^\beta)
\]

From (37) the corresponding parametric function of \( \bar{E}^\alpha \) is given by

\[
\bar{E}^\alpha : h(t^\alpha) = (R^\beta)^{-1}(R^\alpha f(t^\alpha) + \bar{c}^\alpha - \bar{c}^\beta).
\]
Firstly, by subdividing the interval $[-\pi, \pi]$ with parameter values: $-\pi \leq t^0 \leq t^1 \leq \ldots \leq t^M \leq \pi$ and $-\pi \leq t^{\beta}_1 \leq t^{\beta}_2 \leq \ldots \leq t^{\beta}_N \leq \pi$, and setting $\tilde{u}_i = \tilde{f}(t^{\alpha}_i)$ and $\tilde{v}_j = \tilde{g}(t^{\beta}_j)$, we obtain the two sets of sample points $\tilde{U} = \{\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_M\}$ and $\tilde{V} = \{\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_N\}$ on $\tilde{E}^\alpha$ and $\tilde{E}^\beta$ respectively. Moreover, by using the well-known direct method we can compute an approximation to the one-sided Hausdorff distance,

$$h(\tilde{E}^\alpha, \tilde{E}^\beta) \approx h(\tilde{U}, \tilde{V}) = \max_{1 \leq i \leq M} \min_{1 \leq j \leq N} |\tilde{u}_i - \tilde{v}_j|,$$

at a cost of $O(MN)$ operations. Here, in order to reduce the computational cost it is necessary to seek a new method for approximating $h(\tilde{E}^\alpha, \tilde{E}^\beta)$ that requires fewer operations than the direct method. To do so, a so-called pruning technique may be one of the best methods suitable for that. We are willing to use the pruning technique for solving our problem. In fact, our purpose is to reduce the necessary work by pruning the subintervals of the parametric interval $[-\pi, \pi]$. In this case, if we know an algorithm for computing $d(\tilde{u}, \tilde{E}^\beta)$ exactly for $\tilde{u} \in \tilde{E}^\alpha$ like that in previous section, in other words, if each computation of the distance $d(\tilde{u}, \tilde{E}^\beta)$ counts as a single operation, then our algorithm requires only $O(\log M)$ operations, for fewer than $O(M)$ operations needed by the direct method. Whenever we add more sample points for achieving high accuracy, the following lemma provides a criterion for ignoring parts of $\tilde{E}^\alpha$, so that the number of sample points can be diminished.

**Lemma 3.1.** Let $\tilde{u}$ be a point on $\tilde{E}^\alpha$ such that $d(\tilde{u}, \tilde{E}^\beta) \leq m \leq h(\tilde{E}^\alpha, \tilde{E}^\beta)$ for a positive number $m$. And, let $w = m - d(\tilde{u}, \tilde{E}^\beta)$. Then, for every $\tilde{v} \in \tilde{E}^\alpha$ we have

$$d(\tilde{u}, \tilde{E}^\beta) \leq m \quad \text{whenever} \quad |\tilde{v} - \tilde{u}| \leq w.$$

**Proof.** If $|\tilde{v} - \tilde{u}| \leq w$, then for all $\tilde{x} \in \tilde{E}^\beta$,

$$d(\tilde{v}, \tilde{E}^\beta) \leq |\tilde{v} - \tilde{x}| \leq |\tilde{v} - \tilde{u}| + |\tilde{u} - \tilde{x}| \leq w + d(\tilde{u}, \tilde{E}^\beta) = m.$$

From the lemma above, if we know a current value $m$ of a variable $\text{maxdist}$ as an estimate for $h(\tilde{E}^\alpha, \tilde{E}^\beta)$ after some iterations, we can prune every point $\tilde{v} \in \tilde{E}^\alpha$ whenever $|\tilde{u} - \tilde{v}| \leq w$. We now propose a pruning procedure:
Consider a subinterval \([\alpha, \beta]\) of the parametric interval \([-\pi, \pi]\). Suppose that we have a current value \(m_{axdist} = m\) as an estimate for \(h(\tilde{E}^\alpha, \tilde{E}^\beta)\) such that

\[
\max\{d(\tilde{f}(\alpha), \tilde{E}^\beta), d(\tilde{f}(\beta), \tilde{E}^\beta)\} \leq m \leq h(\tilde{E}^\alpha, \tilde{E}^\beta).
\]

Let \(\mu = \frac{\alpha + \beta}{2}\), \(\tilde{\sigma} = \tilde{f}(\mu)\), \(\delta = d(\tilde{\sigma}, \tilde{E}^\beta)\). Then, firstly, in case of \(\delta > m\) we have a new estimate \(m_{axdist} = \delta\) for \(h(\tilde{E}^\alpha, \tilde{E}^\beta)\) instead of \(m\). From lemma 3.1 with \(\tilde{u} = \tilde{f}(\alpha)\) and \(w = m_{axdist} - m = \delta - m\) it follows that

\[
d(\tilde{v}, \tilde{E}^\beta) \leq m_{axdist} = \delta \quad \text{whenever} \quad |\tilde{v} - \tilde{u}| = |\tilde{v} - \tilde{f}(\alpha)| \leq w.
\]

Also, let \(\tilde{u} = \tilde{f}(\beta)\) and \(w = \delta - m\). Then, we have

\[
d(\tilde{v}, \tilde{E}^\beta) \leq \delta \quad \text{whenever} \quad |\tilde{v} - \tilde{u}| = |\tilde{v} - \tilde{f}(\beta)| \leq w.
\]

Since \(\tilde{f}\) is continuous, the following sets can be found from (35):

\[
R_\alpha = \big\{ t \in [-\pi, \pi] : |\tilde{f}(t) - \tilde{f}(\alpha)| = m \big\} = \big\{ t \in [-\pi, \pi] : |\tilde{f}(t) - f(\alpha)| = m \big\},
\]

\[
R_\beta = \big\{ t \in [-\pi, \pi] : |\tilde{f}(t) - \tilde{f}(\alpha)| = m \big\} = \big\{ t \in [-\pi, \pi] : |\tilde{f}(t) - f(\alpha)| = m \big\}.
\]

Let \(t_\alpha = \min\{t \in R_\alpha : \alpha \leq t\}\), and \(t_\beta = \max\{t \in R_\beta : t \leq \beta\}\). Then, it follows that

\[
|\tilde{f}(t) - \tilde{f}(\alpha)| < w \quad \text{for} \quad \alpha \leq t \leq t_\alpha,
\]

and

\[
|\tilde{f}(t) - \tilde{f}(\beta)| < w \quad \text{for} \quad t_\beta \leq t \leq \beta.
\]

Thus, the subinterval \([\alpha, \beta]\) can be pruned to \([t_\alpha, t_\beta]\).

Secondly, if \(\delta \leq m\), then applying lemma 3.1 with \(\tilde{u} = \tilde{\sigma} = \tilde{f}(\mu)\) and \(w = m_{axdist} - \delta = m - \delta\) we have

\[
d(\tilde{v}, \tilde{E}^\beta) \leq m_{axdist} = m \quad \text{whenever} \quad |\tilde{v} - \tilde{\sigma}| = |\tilde{v} - \tilde{f}(\mu)| \leq w.
\]

Since \(\tilde{f}\) is continuous, the following sets can be obtained:

\[
R_{\alpha,\beta} = \big\{ t \in [-\pi, \pi] : |\tilde{f}(t) - \tilde{f}(\mu)| = w \big\} = \big\{ t \in [-\pi, \pi] : |\tilde{f}(t) - f(\mu)| = w \big\},
\]

and

\[
t_\alpha = \max\{t \in R_{\alpha,\beta} : t \leq \mu\},
\]

\[
t_\beta = \min\{t \in R_{\alpha,\beta} : \mu \leq t\}.
\]

Thus, if \(w = 0\), then \(t_\alpha = t_\beta = \mu\). Otherwise,

\[
|\tilde{f}(t) - \tilde{\sigma}| < w \quad \text{for} \quad t_\alpha \leq t \leq t_\beta.
\]
So, from (43) we have
\[ d(\tilde{f}(t), E^\beta) \leq \maxdist = m \quad \text{for} \quad t_\alpha \leq t \leq t_\beta. \]
It means that for every \( t \in [t_\alpha, t_\beta] \), \( \tilde{f}(t) \) can be ignored when we add new sample points. The subinterval \([\alpha, \beta]\) can be pruned to the smaller subintervals \([\alpha, t_\alpha]\) and \([t_\beta, \beta]\).

This analysis leads to the following pruning procedure that returns a set \( \Gamma \) of 0, 1 or 2 subintervals of \([\alpha, \beta]\), whose union \( \bigcup \Gamma \) contains all values of \( t \in [\alpha, \beta] \) that might satisfy \( d(\tilde{f}(t), E^\beta) > m \). In other words, if \( \tilde{f}(t) \) is a potentially new sample point with \( t \in [\alpha, \beta] \), then \( t \) must belong to one of the subintervals in \( \Gamma \).

**procedure**: prune \(([\alpha, \beta])\)

**step1**: Set \( \text{midpt} = \tilde{f}(\frac{\alpha + \beta}{2}) \), \( \text{dist} = d(\text{dist}, E^\beta) \), and \( \Gamma = \phi \). From (26) we can compute
\[ d(\text{midpt}, E^\beta) = d(u, E^\beta) \quad \text{for} \quad u = (R^\beta)^{-1}(\text{midpt} - \tilde{c}^\beta). \]

**step2**: If \( \text{dist} > \maxdist \), then set \( w = \text{dist} - \maxdist \). And,
\[ \text{maxdist} \leftarrow \text{dist}, \]
\[ R_\alpha \leftarrow \{ t \in [\alpha, \beta] : |f(t) - f(\alpha)| = w \}, \]
\[ R_\beta \leftarrow \{ t \in [\alpha, \beta] : |f(t) - f(\beta)| = w \}, \]
\[ t_\alpha \leftarrow \min\{ t \in R_\alpha : \alpha \leq t \}, \]
\[ t_\beta \leftarrow \max\{ t \in R_\beta : t \leq \beta \}, \]
\[ \Gamma \leftarrow \Gamma \bigcup \{ [t_\alpha, t_\beta] \}. \]
If \( \text{dist} \leq \maxdist \), then
\[ w \leftarrow \maxdist - \text{dist}, \]
\[ R_{\alpha, \beta} \leftarrow \{ t \in [\alpha, \beta] : |f(t) - \text{midpt}| = w \} \]
\[ = \{ t \in [\alpha, \beta] : |f(t) - f(\frac{\alpha + \beta}{2})| = w \}, \]
\[ t_\alpha \leftarrow \max\{ t \in R_{\alpha, \beta} : t \leq \frac{\alpha + \beta}{2} \}, \]
\[ t_\beta \leftarrow \min\{ t \in R_{\alpha, \beta} : \frac{\alpha + \beta}{2} \leq t \}. \]
Further, if \( t_\alpha > \alpha \), then
\[ \Gamma \leftarrow \Gamma \bigcup \{ [\alpha, t_\alpha] \}. \]
the Hausdorff distance between two ellipses

If $t_\beta < \beta$, then

\[ \Gamma \leftarrow \Gamma \bigcup \{ [t, \beta] \} . \]

**step3:** Return $\Gamma$.

We first start from the whole interval $\Gamma = \{ [-\pi, \pi] \}$, and the pruning procedure can be applied repeatedly until arriving at a collection of subintervals $\Gamma$, where all their lengths are smaller than a specified tolerance $\epsilon$. Let us describe an algorithm for finding the one-sided Hausdorff distance $h(\tilde{E}_\alpha, \tilde{E}_\beta)$ from $\tilde{E}_\alpha$ to $\tilde{E}_\beta$. Our algorithm can be stated as follows:

**Algorithm:** Hausdorff distance $(\tilde{c}_\alpha, \tilde{p}_\alpha, \tilde{q}_\alpha, \tilde{c}_\beta, \tilde{p}_\beta, \tilde{q}_\beta)$

**step1:** Compute $a^\alpha, b^\alpha, a^\beta, b^\beta$ from (28), (29), (30), and (31). Define the rotation matrices $R^\alpha$ and $R^\beta$, and their inverses $(R^\alpha)^{-1}$ and $(R^\beta)^{-1}$ from (34). Give a tolerance $TOL = \epsilon > 0$. Initialize the following:

First, it follows from (35) and (39) that for a point $\tilde{u} = \tilde{f}(t)$ on $\tilde{E}$ we can see

\[ d(\tilde{u}, \tilde{E}_\beta) = d(\tilde{f}(t), \tilde{E}_\beta) = d(h(t), E_\beta^\alpha) = d(\tilde{u}, E_\beta^\alpha), \]

where $h(t) = \tilde{u} = (R^\alpha)^{-1}(\tilde{u} - \tilde{c}_\beta) = (R^\beta)^{-1}(\tilde{f}(t) - \tilde{c}_\beta) = (R^\beta)^{-1}(R^\alpha \tilde{f}(t) + \tilde{c}_\beta - \tilde{c}_\beta)$ is the corresponding point on $E_\alpha$ with respect to $\tilde{u}$. Thus, we have the following:

\[ \text{maxdpt} \leftarrow \max \{ d(\tilde{f}(-\pi), \tilde{E}_\beta), d(\tilde{f}(\pi), \tilde{E}_\beta) \} \]

where $h(-\pi) = (R^\beta)^{-1}(\tilde{f}(-\pi) - \tilde{c}_\beta)$ and $h(\pi) = (R^\beta)^{-1}(\tilde{f}(\pi) - \tilde{c}_\beta)$.

And,

\[ \text{N} \leftarrow 1, \]

\[ \text{interval}(1) = [\alpha_1, \beta_1] \leftarrow [-\pi, \pi], \]

\[ \text{maxsize} \leftarrow 2\pi. \]

**step2:** Repeat

\[ \Gamma \leftarrow \phi \]
for $j = 1$ to $N$ do

begin

\[
\Gamma \leftarrow \Gamma \cup \text{prune}(\text{interval}(j)) = \Gamma \cup \text{prune}([\alpha(j), \beta(j)]).
\]

end;

Reset $\alpha(j), \beta(j), N$, so that
\[
\Gamma = \{\text{interval}(j) : 1 \leq j \leq N\}.
\]

for $j = 1$ to $N$ do

begin

It follows from (26) that
\[
d(f(\frac{\alpha(j) + \beta(j)}{2}), \tilde{E}^j) = d(u, E^j).
\]

where $u = (R^3)^{-1}(\tilde{f}(\frac{\alpha(j) + \beta(j)}{2}) - \tilde{c}^j)$.

\[
dist(j) \leftarrow d(f(\frac{\alpha(j) + \beta(j)}{2}), \tilde{E}^j)
\]

\[
\text{maxdist} \leftarrow \max\{\text{dist}(j), \text{maxdist}\}
\]

end;

\[
\text{maxsize} \leftarrow \max\{\beta(j) - \alpha(j) : j = 1, 2, \ldots, N\}
\]

Until $\text{maxsize} < \text{TOL} = \epsilon$

step3: Return $\text{maxdist}$

To test our algorithm we give an example for computing the one-sided Hausdorff distance $h(\tilde{E}^\alpha, \tilde{E}^\beta)$. If we let $\text{TOL} = \epsilon = \frac{2\pi}{M}$, then we have from (40)
\[
h(\tilde{E}^\alpha, \tilde{E}^\beta) \approx h(\tilde{U}, \tilde{E}^\beta) = \max_{1 \leq i \leq M} d(\tilde{u}_i, \tilde{E}^\beta).$

The semidirect method requires the computational cost $O(M)$. The purpose of proposing our method is to reduce the computational cost from $O(M)$ to something like $O(\log M)$. The following example shows this by dint of obtaining a desired result. We can see that the computational cost needed by our method is nearly $O(\log M)$. Here, all computations are performed using MATLAB, and the running time is the average for 20 trials.

**Example:** Let $\tilde{E}^\alpha$ and $\tilde{E}^\beta$ be two ellipses in $\mathbb{R}^3$. $\tilde{E}^\alpha$ is characterized by its center $\tilde{c}^\alpha = (1, 3, 2)^T$ and the two points $\tilde{p}^\alpha = (2, 5, 1)^T$ and $\tilde{q}^\alpha = (4, 7, -3)^T$. Also, $\tilde{E}^\beta$ is given by $\tilde{c}^\beta = (2, 2, 3)^T$, $\tilde{p}^\beta = (-2, 5, -3)^T$, and $\tilde{q}^\beta = (3, 1, 4)^T$. In table 1 we can see the optimal result for computing $h(\tilde{E}^\alpha, \tilde{E}^\beta)$. Figure 1 shows the running time as a function of $M$. This indicates that the computational cost is approximately $O(\log M)$.

| $M$ | # subinterval | max. subinterval | approximation | seconds |
|-----|---------------|------------------|---------------|---------|
| 10  | 3             | 3.33540e-01      | 8.0808079096319343 | 0.1705  |
| 30  | 4             | 1.66770e-01      | 8.089373923326425  | 0.1770  |
| 50  | 4             | 8.33850e-02      | 8.0909399537777757 | 0.2012  |
| 70  | 4             | 8.33850e-02      | 8.0909399537777757 | 0.2008  |
| 100 | 4             | 4.16925e-02      | 8.0916184521529146 | 0.2338  |
| 200 | 6             | 2.08462e-02      | 8.0916447321673193 | 0.2928  |
| 300 | 6             | 2.08462e-02      | 8.0916447321673193 | 0.2945  |
| 500 | 6             | 1.04231e-02      | 8.0917229590390676 | 0.3503  |
| 700 | 8             | 5.21156e-03      | 8.091706689477028  | 0.4282  |
| 1000| 8             | 5.21156e-03      | 8.091706689477028  | 0.4295  |
| 2000| 12            | 2.57163e-03      | 8.091720405944331  | 0.5569  |
| 3000| 16            | 1.28097e-03      | 8.091723146314327  | 0.7309  |
| 5000| 22            | 6.40483e-04      | 8.0917234218317784 | 0.9618  |
| 7000| 22            | 6.40483e-04      | 8.0917234218317784 | 0.9598  |
| 8000| 22            | 6.40483e-04      | 8.0917234218317784 | 0.9602  |
| 10000| 32           | 3.20241e-04      | 8.091723394812352  | 1.2813  |

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Figure 1. Running times are shown for the example.

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