D-branes in the Lorentzian Melvin Geometry

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We consider string theory on the Lorentzian Melvin geometry, which is obtained by analytically continuing the two-parameter Euclidean Melvin background. Because this model provides a solvable conformal field theory that describes time-dependent twisted string dynamics, we study the string one-loop partition function and the D-brane spectrum. We found that both the wrapping D2-brane and the codimension-one D-string emit winding strings, and this behavior can be traced to the modified open string Hamiltonian on these probe D-branes.

§1. Introduction and summary

In recent years, the string theory approach has provided insights into the study of time-dependent and cosmological backgrounds. Among the theoretical constructs provided by this approach, time-dependent string orbifolds are of interest because they are solutions of the string equations of motion to all orders in $\alpha'$ and exactly solvable.\textsuperscript{2)} In this paper, we consider an analytically continued two-parameter Melvin background which, before the Kaluza-Klein reduction, has the metric

$$\begin{align*}
&ds^2 = -dt^2 + \frac{t^2}{(1 + q^2 t^2)(1 + p^2 t^2)} d\theta^2 + \frac{1 + q^2 t^2}{1 + p^2 t^2} (dy + A_{\theta} d\theta)^2, \\
&B_{y\theta} = \frac{pt^2}{1 + p^2 t^2}, \quad A_{\theta} = \frac{qt^2}{1 + q^2 t^2}, \quad e^{2(\phi - \phi_0)} = \frac{1}{1 + p^2 t^2}
\end{align*}$$

where we have $-\infty < t, \theta < \infty$, and $y \sim y + 2\pi R$. Note that via the Wick rotations $t \rightarrow ir, \theta \rightarrow i\varphi, p \rightarrow ip$ and $q \rightarrow iq$, we can recover the usual two-parameter Euclidean Melvin metric.\textsuperscript{42)} Now we see that the metric (1.1) reduces to an orbifold as follows. The sigma model associated with closed strings in the above background (1.1) is

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\textsuperscript{2)} There are many works studying time-dependent string backgrounds, e.g. Refs. 1) - 17). Aspects of string (and D-brane) dynamics are explored in Refs. 4), 18) - 27). Other issues regarding curvature singularity resolution and the matrix model are investigated in Refs. 28) - 39).
written as
\[ S = \frac{1}{\pi \alpha'} \int d^2 z \left[ -\partial t \partial t + \partial y \partial y + \frac{t^2}{1 + p^2 t^2} \left( \partial \theta + (q - p) \partial y \right) \left( \partial \theta + (q + p) \partial y \right) \right], \]
(1.2)

with the convention \( \partial = \frac{1}{2} (\partial_{\sigma} + \partial_{\tau}) \) and \( \bar{\partial} = \frac{1}{2} (\partial_{\sigma} - \partial_{\tau}) \). Closely following the treatment presented in Ref. 42), if one first compactifies \( \theta \) to some finite radius and then T-dualizes it to \( \bar{\theta} \), next shifts \( y \) as \( y \to y' = y + b \bar{\theta} \), subsequently T-dualizes \( \bar{\theta} \) to \( \theta' \), and finally decompactifies \( \theta \), the action \( (1.2) \) becomes that of a free sigma model in terms of \( x'^{\pm} = \frac{1}{\sqrt{2}} t e^{\pm \theta'} \) and \( y' \). Here, \( (x'^{\pm}, y') \) satisfies the periodicity conditions
\[ x'^{\pm}(\sigma + 2\pi, \tau) = e^{\pm 2\pi q R w + \alpha' p (\frac{\pi}{2} - q J)} x^{\pm}(\sigma, \tau), \quad y'(\sigma + 2\pi, \tau) = y'(\sigma, \tau) + 2\pi R w - 2\pi \alpha' p J. \]
(1.3)

Note that \((n, w) \in \mathbb{Z}\) and \(J\) is the boost generator of the \( x'^{\pm}\)-plane. Another feature of the above action is that in the limit of vanishing \( p \), \( (1.3) \) reduces to the shifted-boost orbifold proposed in Ref. 3).

In Ref. 21), it is shown that the probe D-brane emits twisted closed strings in Misner space\(*\) which is the simplest time-dependent orbifold. Inspired by that, we set out to determine if there is similar behavior in our case when using the above free field representation to study the D-brane spectrum. We observe that the winding string coupling to D-branes is suggested by their classical profiles. On the other hand, by explicitly constructing their boundary states, this property becomes evident and we are also able to compute the winding string emission rate.

In addition, we note that the behavior of winding string emission can be traced to the open string Hamiltonian \( H_o \) on these probe D-branes. This is because the open string annulus amplitude,
\[ \mathcal{A}(t) = \int_0^\infty \frac{dt}{t} \text{Tr} \ e^{-2\pi t H_o}, \]
upon the moduli transformation \( t \to s = \frac{1}{t} \), has a form similar to the one-loop partition function \( \mathcal{A}_{\text{elec}}(t) \) of open strings subject to a constant electric field. As is widely known, in the latter case, worldsheet fields acquire twisted periodicity via a doubling trick, which is responsible for string pair creation.\(47)\) In our case, after moduli integration, \( \mathcal{A}(s = \frac{1}{t}) \) (the closed-channel cylinder amplitude) also acquires

\(*\) Though \( (1.2) \) is generated from one half of \( \mathbb{R}^{1,3} \), it is possible to extend to the entire \( \mathbb{R}^{1,3} \) using the same manipulation.

\(**\) In static backgrounds, e.g. the near-horizon region of a stack of NS5-branes, closed string emission from D-branes can take place.\(46)\)
an imaginary part, and this, along with the optical theorem, accounts for the winding string emission.

The outline of this paper is as follows. In §2, we examine the worldvolume theory of D-branes and construct their boundary states. Next, we explicitly compute the winding string emission rate. In the appendix, we calculate the torus amplitude. We find that when the B-field parameter $p$ is taken to zero, this amplitude is identical to that of the shifted-boost orbifold derived in Ref. 3).

§2. D-branes in the Lorentzian Melvin geometry

In order to study the D-brane spectrum, we classify D-branes according to the free field representation ($x'^{\pm} = \frac{1}{\sqrt{2}}te^{\pm \theta'}, y'$) appearing in (1.3). There are four kinds of boundary conditions we can impose on ($\theta', y'$), i.e.

(i) : (N, N), (ii) : (N, D), (iii) : (D, D), (iv) : (D, N),

(2.1)

where $N$ ($D$) stands for the Neumann (Dirichlet) boundary condition. We restrict our attention to only those which carry the twisted sector, i.e. types (i) and (ii).

2.1. Classification of D-branes

Type (i): D2-branes

In the following DBI analysis, the relevant coordinates are those of the curved metric (1.1). The DBI action which describes the dynamics of slowly varying fields on a D2-brane is

$$S_{D2} = -\tau_2 \int dt d\theta dy e^{-\phi} \sqrt{-\det(G + B)},$$

(2.2)

where the Dp-brane tension is expressed as $\tau_p = g_s^{-1}(2\pi)^{-p} \alpha'^{-1(p+1)/2}$. The energy-momentum tensor $T^{\mu\nu}$ and the NS-source $S^{\mu\nu}$ can be derived by infinitesimally varying the action w.r.t. $G^{\mu\nu}$ and $B^{\mu\nu}$ as

$$\delta S_{D2} \delta (G + B)_{\mu\nu} = -\frac{\tau_2}{2} e^{-\phi} \sqrt{-\det(G + B)}(G + B)^{\mu\nu} = (T + S)^{\mu\nu}. \quad (2.3)$$

Plugging the metric (1.1) into this relation (with $\mu, \nu = t, \theta, y$), we have

$$T^{\mu\nu} = \frac{-\tau_2 |t|}{2} \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1+q^2t^2}{t^2} & -q \\ 0 & -q & 1 \end{pmatrix}, \quad S^{\mu\nu} = \frac{-\tau_2 |t|}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & -p & 0 \end{pmatrix}. \quad (2.4)$$

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* See Ref. 44 for the Euclidean counterparts.

** Note that the D-particle of type (iii) and its $y'$ direction T-dual counterpart, namely, that of type (ii), cannot couple to the twisted sector.
Because there appears a non-trivial worldvolume B-field, the non-commutative Yang-Mills theory on the D-brane is now described by a new set of open string parameters, namely, the open string metric $G_{\mu\nu}$, the open string coupling $G_s$, and the non-commutativity parameter $\Theta$. Making use of the Seiberg-Witten map, i.e.

$$(G + B)^{\mu\nu} = G^{\mu\nu} + \frac{\Theta^{\mu\nu}}{2\pi\alpha'}, \quad G_{\mu\nu} = (G - BG^{-1}B)_{\mu\nu},$$

and

$$G_s = e^{\Phi}g_s \sqrt{-\det G},$$

we find

$$G_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & t^2 & qt^2 \\ 0 & qt^2 & 1 + q^2t^2 \end{pmatrix}, \quad \Theta^{\theta y} = 2\pi\alpha'p, \quad G_s = g_s.$$

We can comment on the holographical dual of this open string theory for $l_s \to 0$ in large 't Hooft coupling.

Recall that the open (closed) string spectrum is truncated at massless modes when $l_s \to 0$, meanwhile bulk closed strings decouple from D-branes because the Newton constant $G_1 \sim l_s^8g_s^2$ goes to zero. Also, according to the usual AdS/CFT correspondence, the gauge theory on a stack of $N$ ($N \gg 1$) D3-branes is dual to the near-horizon geometry of a classical supergravity solution whose characteristic scale is about $l_s(g_sN)^{1/4}$. The near-horizon geometry of a stack of $N$ D3-branes extending over $(x^0, x^1, x^2, x^3) = (t \cosh \phi, t \sinh \phi, y, z)$ is

$$\frac{ds^2}{\alpha'} = \frac{U^2}{\sqrt{4\pi g_s N}}(-dt^2 + t^2d\phi^2 + dy^2 + dz^2) + \sqrt{4\pi g_s N}\left(\frac{dU^2}{U^2} + d\Omega_5^2\right).$$

Through the decoupling limit,\(^2)\(,\)\(^3)\(,\)\(^4)\(,\)\(^5)\)

$$\lim_{\alpha' \to 0} \left. U \right|_{\alpha' = 0} = \text{finite}, \quad \lim_{\alpha' \to 0} \Theta^{\mu\nu} = 2\pi\eta = \text{finite}, \quad \frac{p}{\alpha'} = \frac{\eta}{\alpha'}.$$

the metric part of the gravity solution can thus be constructed as

$$\frac{ds^2}{\alpha'} = \frac{U^2}{\sqrt{\lambda}}(-dt^2 + dz^2) + \frac{U^2}{\lambda + \eta^2t^2U^4}(t^2(d\theta + qdy)^2 + dy^2) + \sqrt{\lambda}\left(\frac{dU^2}{U^2} + d\Omega_5^2\right),$$

where $\lambda = g_s^2N$ is the 't Hooft coupling. It is seen that the second term on the RHS of (2.10) differs from that in the flat case, due to the presence of a non-zero

\(^*)\text{We are grateful to A. Hashimoto for discussions regarding this point.}\)
TYPE (II): D1-BRANES

According to (1.3), it is seen that the momentum conjugate to $y'$ is

$$\frac{1}{2}(p'_L + p'_R) = \left(\frac{n}{R} - qJ\right), \quad \frac{1}{2}(p'_L - p'_R) = \left(\frac{Rw}{\alpha'} - pJ\right).$$

(2.11)

This implies that the parameters $p$ and $q$ in (1.3) become exchanged through $T$-dualizing $y'$. Accordingly, the metric part in (1.1), after $T$-dualizing $y'$, should be modified in the corresponding manner via $q \leftrightarrow p$ as

$$ds^2 = -dt^2 + \frac{1}{1 + q^2 t^2}(t^2d\Theta^2 + dy^2), \quad \Theta = (\theta + py), \quad \begin{pmatrix} \Theta \\ y \end{pmatrix} \sim \begin{pmatrix} \Theta + 2\pi p R' \\ y + 2\pi R' \end{pmatrix}, \quad R' = \frac{\alpha'}{R}.$$

(2.12)

Note that the equations of motion of the sigma model on the metric (2.12) are related to those of the aforementioned free field one as

$$(1 + q^2 t^2)\partial_\sigma \theta' = \partial_\sigma (\theta + py) + q\partial_\tau y,$$

$$(1 + q^2 t^2)\partial_\tau y' = \partial_\tau y - qt^2 \partial_\sigma (\theta + py),$$

(2.13)

which are obtained from the relation between two sigma models connected by $T$-duality. Based on this, the above type (ii) boundary condition can be transformed into

$$\partial_{\sigma} t = \partial_\sigma (\theta + py) = \partial_\tau y = 0.$$

(2.14)

Employing the static gauge ($\xi^0 = t$, $\xi^1 = \Theta$), where $\xi^0$ and $\xi^1$ parameterize the D-string worldvolume, we observe that its classical profile is $y = C$ ($C$: constant).

Due to the fundamental region $S^1 \times \mathbb{R}$ (a cylinder), which comes from the identification in (2.12), all images of this D-string, i.e. $y = C + 2\pi m R'$ ($m \in \mathbb{Z}$), should be summed over. Hence, the D-string takes the form of a spiral that makes an angle $\vartheta = \arctan(1/pt)$ with $S^1$ and couples to winding strings. In the case $\vartheta \to \frac{\pi}{2}$, i.e. the background B-field parameter $p \to 0$, the D-string becomes parallel to the cylinder axis. This suggests that winding mode coupling is no more possible. In other words, the non-vanishing coupling to winding modes can be understood as being induced by a non-zero $p$. This point is revisited below in the context of the boundary state.

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*) We consider physics in the case of $t \neq 0$. 
2.2. Boundary state

To include the $\alpha'$ corrections, we next study the boundary state formalism. For simplicity, we work out only the bosonic part, without loss of generality. Note that the relevant coordinates for the following analysis are those used in defining the free field representation (1.3). Subsequently, these results are utilized to explicitly derive the winding string emission rate.

Boundary state

We first consider the case of a D2-brane that wraps the entire $(x'^\pm, y')$. From (2.11), the Neumann boundary condition along $y'$ implies

$$\partial_\tau y |B\rangle_{D2} = 0 \implies \left( p'_L + p'_R \right) |B\rangle_{D2} = \left( \frac{n}{R} - qJ \right) |B\rangle_{D2} = 0 ;$$

(2.15)

that is, $J |B, n = 0\rangle_{D2} = 0$. Therefore, according to (1.3), closed strings which couple to the D2-brane satisfy the periodicity condition

$$x'^\pm (\tau, \sigma + 2\pi) = e^{\pm 2\pi \gamma w} x'^\pm (\tau, \sigma) , \quad \gamma = qR ,$$

(2.16)

and the mode expansion of $x'^\pm$ reads

$$x'^\pm (\tau, \sigma) = i \sqrt{\frac{\alpha'}{2}} \sum_{m \in \mathbb{Z}} \left[ \frac{\alpha'^+_m}{m + i\gamma w} e^{-i(m \pm i\gamma w)(\tau + \sigma)} + \frac{\tilde{\alpha}'^+_m}{m \mp i\gamma w} e^{-i(m \mp i\gamma w)(\tau - \sigma)} \right].$$

(2.17)

(2.15), together with the Neumann condition $\partial_\tau x'^\pm |B\rangle_{D2} = 0$, dictates that the boundary state be of the form

$$|B\rangle_{D2} = \sum_w \mathcal{N}_w \exp \left( \sum_{m \geq 1} \frac{\alpha'^+_m \tilde{\alpha}'^-_{-m}}{m + i\gamma w} + \sum_{m \geq 0} \frac{\alpha'^-_m \tilde{\alpha}'^+_{-m}}{m - i\gamma w} - \sum_{l \geq 1} \frac{\alpha'^+_l \tilde{\alpha}'^-_{l}}{l} \right) |n = 0, w\rangle \otimes |X_\perp\rangle \otimes |GH\rangle .$$

(2.18)

Here, the transverse components, $|X_\perp\rangle$, and the ghost part, $|GH\rangle$, are the usual flat

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*) The commutation relations are defined to be $[\alpha^+_m, \alpha^-_n] = (-m - i\nu) \delta_{m+n}$, $[\tilde{\alpha}'^+_m, \tilde{\alpha}'^-_n] = (-m + i\nu) \delta_{m+n}$. For $\nu < 0$, $\alpha^+_0$ and $\tilde{\alpha}'^+_0$ ($\alpha^-_0$ and $\tilde{\alpha}'^-_0$) act as creation (annihilation) operators, and vice versa for $\nu > 0$. 

space ones. To evaluate \( N_w \) explicitly, we compute the annulus amplitude as

\[
Z(t) = \text{Tr} \ e^{2\pi i \gamma_J t + 2\pi i R w y'} e^{-2\pi t L_0^{\text{flat}}},
\]

\[
\int_0^\infty \frac{dt}{t} Z(t) = \int_0^\infty \frac{dt}{t} \sum_{w \neq 0} \frac{i e^{-2\pi \alpha'^{-2} \gamma_J t}}{2 \sinh(\pi \gamma_J w)} \partial_1(\gamma_J w | it) \eta(it) \tag{2.19}
\]

\[
\rightarrow \int_0^\infty ds \sum_{w \neq 0} \frac{s^{-\frac{23}{2}} e^{-\pi \gamma_J w^2 s - \frac{\pi \gamma_J w^2}{2}}}{2 \sinh(\pi \gamma_J w)} \partial_1(\gamma_J w s | is) \eta(is) \tag{2.20}
\]

where the insertion in the first line reflects the twisted periodicity conditions. Also, we have carried out a moduli transformation in the third line. The Cardy condition then requires \( N_w = \sqrt{\frac{8\pi^2 \alpha'}{2 \sinh(\pi \gamma_J w)}} N_2 \), where \( N_2 \) denotes the usual flat space normalization. Note that the \( y' \) part of the above partition function can be recast into the form

\[
Z_{D2}^{y'}(t) = \text{Tr} \ e^{-2\pi t \alpha' (\frac{R^2}{\alpha'} - q^2 J)^2 + N_{y'} - \frac{1}{24}}, \quad a \in \mathbb{Z},
\]

which results from integrating out \( p_{y'} \) and applying the Poisson resummation w.r.t. \( w \).

Next, we construct the above D-string boundary state. In contrast to (2.15), \(|B\rangle_{D1}\), which wraps only \( x'^{\pm} \), satisfies

\[
\partial_\sigma y' |B\rangle_{D1} = 0 \rightarrow \left( \frac{R w}{\alpha'} - p J \right) |B\rangle_{D1} = 0, \tag{2.21}
\]

i.e. \( J |B, w = 0\rangle_{D1} = 0 \). Together with \( \partial_\sigma x'^{\pm} |B\rangle_{D1} = 0 \), we find that the twisted condition on closed strings coupled to the D-string is \( x'^{\pm} (\sigma + 2\pi) = e^{\pm 2\pi i \alpha' n} x'^{\pm} (\sigma) \).

Based on these, the open string annulus amplitude is computed as

\[
\text{Tr} \ e^{2\pi p_{y'} \alpha' n J + \frac{R w}{\alpha'} y'} e^{-2\pi t L_0^{\text{flat}}}, \quad n \in \mathbb{Z}. \tag{2.22}
\]

By integrating out \( y' \) and applying the Poisson resummation w.r.t. \( n \), it is straightforward to extract the \( y' \) part of the partition function,

\[
Z_{D1}^{y'}(t) = \text{Tr} \ e^{-2\pi t \alpha' (\frac{R^2}{\alpha'} - p J)^2 + N_{y'} - \frac{1}{24}}. \tag{2.23}
\]

The above expression elucidates the T-dual relation between (2.20) and (2.23). The corresponding boundary state is of the form

\[
|B\rangle_{D1} = \sum_{n} \mathcal{N}_n \exp \left( \sum_{m \geq 1} \frac{\alpha^+ - \alpha^-}{m + i \xi n} + \sum_{m \geq 0} \frac{\alpha^- - \alpha^+}{m - i \xi n} + \sum_{l \geq 1} \frac{\alpha'^+_l}{\alpha'^- - 1} \right) |n, w = 0\rangle \otimes |X_\perp\rangle \otimes |\text{GH}\rangle, \tag{2.24}
\]

\(^{\circledast}\) Here, an overall volume factor is omitted and the untwisted sector is ignored.
with $\xi = p\frac{\alpha'}{R}$ and $\mathcal{N}_n = \sqrt{\frac{8\pi^2\alpha'}{2\sinh(\pi\xi|n|)}}\mathcal{N}_1$.

**TWISTED STRING EMISSION**

Because the integrand in the third line of (2.19) contains poles at $s = \frac{l}{\gamma|w|}$, $l = 1, 2, 3, \ldots$, an imaginary part can be gained after performing the moduli integral along a slightly deformed route over the complex plane. As asserted in Ref. 21), this imaginary part is responsible for the twisted string emission in accordance with the optical theorem in usual field theory. The emission rate is determined to be

$$-2\text{Im}\left[\frac{\alpha'\pi}{2} \int ds_{D2} \langle \langle B|e^{-\pi s(L_0+\tilde{L}_0)}|B\rangle \rangle_{D2} \right]$$

$$= \sum_{\nu \neq 0} \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{2\gamma|w| \sinh(\pi\gamma|w|)(8\pi^2\alpha')^{\frac{3}{2}} \sum \text{states}} \sum_{|n| \neq 0} e^{-\frac{2\pi l}{\xi|n|}(\frac{\alpha'^2}{4R^2}+N-1)-\pi l \gamma|w|}.$$

(2.25)

Correspondingly, that of the codimension-one D-string is

$$-2\text{Im}\left[\frac{\alpha'\pi}{2} \int ds_{D1} \langle \langle B|e^{-\pi s(L_0+\tilde{L}_0)}|B\rangle \rangle_{D1} \right]$$

$$= \left(\frac{\mathcal{N}_1^2}{\mathcal{N}_2^2}\right) \sum_{\nu \neq 0} \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{2\xi|n| \sinh(\pi\xi|n|)(8\pi^2\alpha')^{\frac{3}{2}} \sum \text{states}} \sum_{|n| \neq 0} e^{-\frac{2\pi l}{\xi|n|}(\frac{\alpha'^2}{4R^2}+N-1)-\pi l \xi|n|}.$$

(2.26)

where the prefactor is $\mathcal{N}_1^2/\mathcal{N}_2^2 = 4\pi^2\alpha'$.

The reason that the D-string sources winding strings can be traced to the open string Hamiltonian in (2.23), where the term $(\frac{mR}{\alpha'} - pJ)^2$ rules this key dynamics. In summary, the mechanism closely resembles that of open string pair creation in a constant electric field. Let us investigate this point.

In fact, due to a non-zero $p$, we see that a projection in (2.22) is encoded, i.e. only states invariant under $e^{2\pi i J}$ survive after traveling around a loop in spacetime. Using the worldsheet open-closed duality to exchange $(\tau, \sigma)$, we can treat this projection as a twisted periodicity condition imposed on worldsheet fields. This situation strongly resembles the case of open strings subject to a constant electric field $E$. In the latter case, worldsheet fields acquire twisted periodicity through a doubling trick. That is, the lightcone coordinates of open strings satisfy the relation $x^\pm(\sigma + 2\pi) = e^{\pm 2\pi \beta}x^\pm(\sigma)$, where $E = \tanh \pi \beta > 0$, as is shown in Ref. 47). Hence, the one-loop partition function for the $x^\pm$ part reads

$$Z_E^\pm(s') \sim \frac{\eta(is')}{\vartheta_1(\beta s'|is')} e^{-\pi s'^2 \beta^2},$$

(2.27)
where \( s' \) is the annulus modulus. In our case, the moduli-transformed annulus amplitude, for example, in \((2.19)\) is

\[
Z^\pm_{\text{closed}}(s) \sim \frac{\eta(is) e^{-\pi s^2 \nu}}{2 \sinh(\pi \nu) \vartheta_1(s \nu | is)},
\]

where \( s \) is the cylinder modulus, and \( \nu \) is the twist parameter. Just as in the case of \((2.27)\), which, after the moduli integral, leads to open string pair creation,\(^{47}\) \( Z^\pm_{\text{closed}}(s) \), together with the optical theorem, accounts for the winding string emission, after \( s \) is integrated out.

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**Appendix A**

**Torus amplitude**

Here, we compute the closed string torus amplitude based on \((1.3)\). Note that the orbifold \((1.3)\) can be deformed smoothly to a shifted-boost orbifold by taking \( p \to 0 \). It is non-trivial to determine whether, as all oscillating modes are considered, the one-loop partition function is identical to that of the shifted-boost orbifold first computed in Ref. 3.

Adding seven flat spectator directions, we embed the Lorentzian Melvin geometry into 10-dimensional superstring theory. The partition function in the operator formalism can be written as

\[
Z = \left[ \text{Tr} \left. NS \frac{1}{2} (1 + (-1)^F) q^{L_0} - \text{Tr} \left. R \frac{1}{2} (1 + (-1)^F) q^{L_0} \right] \right. \times \left[ \text{Tr} \left. NS \frac{1}{2} (1 + (-1)^F) q^{\bar{L}_0} - \text{Tr} \left. R \frac{1}{2} (1 + (-1)^F) q^{\bar{L}_0} \right] \right. ,
\]

where \( q = e^{2 \pi i \tau}, \tau = \tau_1 + i \tau_2 \), and \(- (+)\) is assigned to the type IIA (IIB) theory. In terms of the free field representation \((x'^\pm, y')\), the Virasoro generators are

\[
L_0 = c + \frac{\alpha'}{4} p_L^2 + \frac{\alpha'}{4} b_7^2 + \nu J_L + N, \quad \bar{L}_0 = \bar{c} + \frac{\alpha'}{4} p_R^2 + \frac{\alpha'}{4} b_7^2 - \nu J_R + \bar{N},
\]

where \( \nu = q R w + p a' (\frac{4}{R} - q J) \), and \( c, \bar{c} \) are the usual untwisted ordering constants.
To deal with the second line, we apply the Poisson resummation

\[ 1 = \int d^2 j \delta(J_L - j)\delta(J_R - j) = \int d^2 \chi e^{2\pi i \chi (J_L - j) + 2\pi i \bar{\chi} (J_R - j)} , \]

(A.3)

we can rewrite the partition function as \((J = j + \bar{j})\)

\[ Z(\tau)_{\text{super}} = \frac{V_T}{(2\pi)^3} (\alpha' \tau_2)^{-\frac{3}{2}} \sum_{n,w \in \mathbb{Z}} \int d^2 \chi d^2 j \left[ \text{Tr} \ e^{2\pi i \chi J_L q_{N-c}^L} \left[ \text{Tr} e^{2\pi i \bar{\chi} J_R q_{N-c}^R} \right] \right. \]

\[ \times e^{2\pi i \tau_1 (nw - \nu J)} e^{-\pi \tau_2 \alpha' \left( \left( \frac{\nu}{R} - q J \right)^2 + \frac{\delta w}{\alpha'} - p J^2 \right)} \exp \left[ -2\pi i \chi j - 2\pi i \bar{\chi} \bar{j} + 2\pi i \tau \nu j + 2\pi i \tau \bar{\nu} \bar{j} \right] . \]

(A.4)

To deal with the second line, we apply the Poisson resummation\(^3\) w.r.t. \(n\) and obtain

\[ R(\alpha' \tau_2)^{-\frac{7}{8}} \sum_{m,w \in \mathbb{Z}} \exp \left[ -\frac{\pi}{\alpha' \tau_2} \left( K + 2i\mu \tau_2 j \right) \left( \bar{K} - 2i\bar{\mu} \tau_2 \bar{j} \right) \right] \]

\[ \times e^{-2\pi i q R (m - \tau_1 w) J - 2\pi q R \tau_2 w (j - \bar{j})} e^{-2\pi i \chi j - 2\pi i \bar{\chi} \bar{j}} , \]

(A.5)

where \(K = m - \tau w\) and \(\mu = \frac{\alpha' p}{R}\). Then, by inserting the identity

\[ 1 = \frac{1}{\alpha' \tau_2} \int d\Xi d\bar{\Xi} \exp \left[ -\frac{\pi}{\alpha' \tau_2} (\Xi + RK + 2i\alpha' \tau_2 j) (\bar{\Xi} - R\bar{K} + 2i\alpha' \tau_2 \bar{j}) \right] , \]

(A.6)

\[ R(\alpha' \tau_2)^{-\frac{7}{8}} \int d\Xi d\bar{\Xi} \sum_{m,w \in \mathbb{Z}} \exp \left[ -\frac{\pi}{\alpha' \tau_2} (\Xi \bar{\Xi} - \Xi \bar{R}\bar{K} + \bar{\Xi} RK) - 2\pi i (\chi + \Theta) j - 2\pi i (\bar{\chi} + \bar{\Theta}) \bar{j} \right] , \]

(A.7)

where \(\Theta = p \Xi + q R K\). Finally, evaluating the trace in (A.4) and integrating out \(j, \bar{j}, \chi\) and \(\bar{\chi}\), we obtain\(^4\)

\[ Z(\tau)_{\text{super}} = \frac{V_T R}{(2\pi)^7 (\alpha' \tau_2)^{-5}} \sum_{m,w \in \mathbb{Z}} \int d\Xi d\bar{\Xi} \left| \vartheta_3(\Theta | \tau) \vartheta_3(0 | \tau)^3 - \vartheta_4(\Theta | \tau) \vartheta_4(0 | \tau)^3 - \vartheta_2(\Theta | \tau) \vartheta_2(0 | \tau)^3 \right|^2 \]

\[ \times \frac{1}{4|\eta(\tau)|^{18}} \left| \vartheta_1(\Theta | \tau) \right|^2 \exp \left[ -\frac{\pi}{\alpha' \tau_2} (\Xi \bar{\Xi} - \Xi \bar{R}\bar{K} + \bar{\Xi} RK) \right] . \]

(A.8)

\(^3\) \(\sum_n \exp(-\pi an^2 + 2\pi ibn) = \frac{1}{\sqrt{\pi}} \sum_m \exp \left( -\frac{\pi(m - b)^2}{a} \right)\)

\(^4\) That no negative norm states propagate in this kind of Lorentzian torus amplitude is proven in Ref. 21).
Next, using in (A.8) the following identity

\[ |\vartheta_3(\Theta|\tau)\vartheta_3(0|\tau)^3 - \vartheta_4(\Theta|\tau)\vartheta_4(0|\tau)^3 - \vartheta_2(\Theta|\tau)\vartheta_2(0|\tau)^3|^2 = 4|\vartheta_1(\Theta|\tau)|^4 |^2, \quad (A.9) \]

we know that the supersymmetry is broken. Further, the modular invariance of \[ \int \frac{d^2\tau}{\tau_2} Z(\tau) \] can also be seen under the transformation \( \tau \rightarrow -\frac{1}{\tau}, \ (m, w) \rightarrow (w, -m) \) and \( (\Xi, \bar{\Xi}) \rightarrow (\Xi, -\bar{\Xi}) \).

By taking the limit of vanishing B-field \( (p \rightarrow 0) \), the bosonic part of (A.8) involving only \( (x'\pm, y') \) is extracted to be

\[ \lim_{p \rightarrow 0} Z_3 = R(\alpha'\tau_2)^\frac{1}{2} \sum_{m, w \in \mathbb{Z}} \exp\left(-\frac{\pi R^2 K\bar{K}}{\alpha'\tau_2} - 2\pi\tau_2 q^2 R^2 w^2\right) |\vartheta_1(iqRK|\tau)|^{-2}. \]

(A.10)

The result is just identical to that of the shifted-boost orbifold previously derived in Ref. 3).

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