ON HADAMARD-TYPE INEQUALITIES FOR CO–ORDINATED $r$–CONVEX FUNCTIONS

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ABSTRACT. In this paper we defined $r$–convexity on the coordinates and we established some Hadamard-Type Inequalities.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. Then the following inequality holds:

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}.$$ 

This inequality is well known in the literature as Hadamard’s inequality.

In [1], C.E.M. Pearce, J. Pecaric and V. Simic generalized this inequality to $r$–convex positive function $f$ which is defined on an interval $[a, b]$, for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$:

$$f(\lambda x + (1 - \lambda)y) \leq \begin{cases} (\lambda [f(x)]^r + (1 - \lambda) [f(y)]^r)^{\frac{1}{r}} , & \text{if } r \neq 0 \\ [f(x)]^\lambda [f(y)]^{1-\lambda} , & \text{if } r = 0 \end{cases}.$$ 

We have that 0–convex functions are simply log-convex functions and 1–convex functions are ordinary convex functions.

In [3], N.P.G. Ngoc, N.V. Vinh and P.T.T. Hien established following theorems for $r$–convex functions:

**Theorem 1.** Let $f : [a, b] \to (0, \infty)$ be $r$–convex function on $[a, b]$ with $a < b$. Then the following inequality holds for $0 < r \leq 1$:

$$\frac{1}{b - a} \int_a^b f(x) dx \leq \left( \frac{r}{r + 1} \right)^{\frac{1}{r}} ( [f(a)]^r + [f(b)]^r )^{\frac{1}{r}}.$$ 

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**Theorem 2.** Let \( f, g : [a, b] \to (0, \infty) \) be \( r \)-convex and \( s \)-convex functions respectively on \([a, b]\) with \( a < b \). Then the following inequality holds for \( 0 < r, s \leq 2 \):

\[
\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{1}{2} \left( \frac{r}{r+2} \right)^{\frac{2}{r}} (\lfloor f(a) \rfloor^r + \lfloor f(b) \rfloor^r)^{\frac{2}{r}} + \frac{1}{2} \left( \frac{s}{s+2} \right)^{\frac{2}{s}} (\lfloor g(a) \rfloor^s + \lfloor g(b) \rfloor^s)^{\frac{2}{s}}.
\]

**Theorem 3.** Let \( f, g : [a, b] \to (0, \infty) \) be \( r \)-convex and \( s \)-convex functions respectively on \([a, b]\) with \( a < b \). Then the following inequality holds if \( r > 1 \), and \( \frac{1}{r} + \frac{1}{s} = 1 \):

\[
\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \left( \frac{\lfloor f(a) \rfloor^r + \lfloor f(b) \rfloor^r}{2} \right)^{\frac{1}{r}} \left( \frac{\lfloor g(a) \rfloor^s + \lfloor g(b) \rfloor^s}{2} \right)^{\frac{1}{s}}.
\]

Similar results can be found for several kind of convexity, in [8], [9], [10] and [12].

In [5], a convex function on the co-ordinates defined by S.S. Dragomir as follow:

**Definition 1.** A function \( f : \Delta \to \mathbb{R} \) which is convex on \( \Delta \) is called co-ordinated convex on \( \Delta \) if the partial mappings \( f_y : [a, b] \to \mathbb{R}, f_y(u) = f(u, y) \) and \( f_x : [c, d] \to \mathbb{R}, f_x(v) = f(x, v) \) are convex for all \( y \in [c, d] \) and \( x \in [a, b] \).

Again in [5], Dragomir gave the following inequalities related to definition given above.

**Theorem 4.** Suppose that \( f : \Delta \to \mathbb{R} \) is co-ordinated convex on \( \Delta \). Then one has the inequalities:

\[
f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) \leq \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b f \left( x, \frac{c+d}{2} \right) dx + \frac{1}{d-c} \int_c^d f \left( \frac{a+b}{2}, y \right) dy \right]
\]

\[
\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dxdy
\]

\[
\leq \frac{1}{4} \left[ \frac{1}{b-a} \int_a^b f(x, c) dx + \frac{1}{b-a} \int_a^b f(x, d) dx + \frac{1}{d-c} \int_c^d f(a, y) dy + \frac{1}{d-c} \int_c^d f(b, y) dy \right]
\]

\[
\leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}.
\]

The above inequalities are sharp.

In [6], M. Alomari and M. Darus proved some inequalities of the Hadamard and Jensen types for co-ordinated log-convex functions. In [7], M.K. Bakula and J.
Pecaric improved several inequalities of Jensen’s type for convex functions on the coordinates. In [11], M.E. Özdemir, E. Set and M.Z. Sarıkaya established Hadamard’s type inequalities for co-ordinated $m$–convex and $(\alpha,m)$–convex functions. Similar results can be found in [8], [9], [10] and [12].

The main purpose of this present note is to give definition of $r$–convexity on the coordinates and to prove some Hadamard-type inequalities for co-ordinated $r$–convex functions.

2. MAIN RESULTS

We can define $r$–convex functions on the coordinates as follow:

**Definition 2.** A function $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}_+$ will be called $r$–convex on $\Delta$, for all $t, \lambda \in [0, 1]$ and $(x, y), (u, v) \in \Delta$, if the following inequalities hold:

$$f(tx + (1-t)y, \lambda u + (1-\lambda)v) \leq \begin{cases} [t\lambda f^\ast(x, u) + t(1-\lambda)f^\ast(y, u) + (1-t)\lambda f^\ast(y, u) + (1-t)(1-\lambda)f^\ast(y, u)]^{\frac{1}{r}}, & \text{if } r \neq 0 \\ f^{\lambda}(x, u)f^{(1-\lambda)}(y, u)f^{(1-t)\lambda}(y, u)f^{(1-t)(1-\lambda)}(y, u), & \text{if } r = 0 \end{cases}. $$

It is simply to see that if we choose $r = 0$, we have co-ordinated log-convex functions and if we choose $r = 1$, we have co-ordinated convex functions. A function $f : \Delta \to \mathbb{R}_+$ is $r$–convex on $\Delta$ is called co-ordinated $r$–convex on $\Delta$ if the partial mappings

$$f_y : [a, b] \to \mathbb{R}_+, \ f_y(u) = f(u, y)$$

and

$$f_x : [c, d] \to \mathbb{R}_+, \ f_x(v) = f(x, v)$$

are $r$–convex for all $y \in [c, d]$ and $x \in [a, b]$.

We need the following lemma for our main results.

**Lemma 1.** Every $r$–convex mapping $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}_+$ is $r$–convex on the coordinates, where $t, \lambda \in [0, 1]$.

**Proof.** Suppose that $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}_+$ is $r$–convex on $\Delta$. Consider the mapping

$$f_y : [a, b] \to \mathbb{R}_+, \ f_y(u) = f(u, y)$$

Case 1: For $r = 0$ and $u_1, u_2 \in [a, b]$, then we have:

$$f_y(tu_1 + (1-t)u_2) = f(tu_1 + (1-t)u_2, y) = f(tu_1 + (1-t)u_2, \lambda y + (1-\lambda)y) \leq f^{\lambda}(u_1, y)f^{(1-\lambda)}(u_1, y)f^{(1-t)\lambda}(u_2, y)f^{(1-t)(1-\lambda)}(u_2, y) = f^{\lambda}(u_1)f_y^{(1-\lambda)}(u_1)f_y^{(1-t)\lambda}(u_2)f_y^{(1-t)(1-\lambda)}(u_2).$$
Case 2: For \( r \neq 0 \) and \( u_1, u_2 \in [a, b] \), then we have:

\[
f_y(tu_1 + (1 - t)u_2) = f(tu_1 + (1 - t)u_2, y) \\
\leq [t\lambda f_y'(u_1, y) + t(1 - \lambda) f_y'(u_1, y)]^{\frac{1}{r}} \\
+ (1 - t) \lambda f_y'(u_2, y) + (1 - t) (1 - \lambda) f_y'(u_2, y)]^{\frac{1}{r}}.
\]

Therefore \( f_y(u) = f(u, y) \) is \( r \)-convex on \([a, b] \). By a similar argument one can see \( f_x(v) = f(x, v) \) is \( r \)-convex on \([c, d] \).

\[\text{□}\]

**Theorem 5.** Suppose that \( f : \Delta \to \mathbb{R}_+ \) be a positive co-ordinated \( r \)-convex function on \( \Delta \). If \( t, \lambda \in [0, 1] \) and \((x, y), (u, v) \in \Delta \), then one has the inequality:

\[
\frac{1}{(b - a)(d - c)} \int_{a}^{b} \int_{c}^{d} f(x, y) dxdy \\
\leq \frac{1}{2} \left( \frac{r}{r+1} \right)^{\frac{1}{r}} \left[ \int_{a}^{b} ([f_x(c)]^r + [f_x(d)]^r)^{\frac{1}{r}} dx \\
+ \int_{c}^{d} ([f_y(a)]^r + [f_y(b)]^r)^{\frac{1}{r}} dy \right]
\]

where \( 0 < r \leq 1 \).

**Proof.** Since \( f : \Delta = [a, b] \times [c, d] \to \mathbb{R}_+ \) is co-ordinated \( r \)-convex on \( \Delta \), then the partial mappings

\[ f_x : [c, d] \to \mathbb{R}_+, \ f_x(v) = f(x, v) \]

and

\[ f_y : [a, b] \to \mathbb{R}_+, \ f_y(u) = f(u, y) \]

are \( r \)-convex, by inequality (1.1), we can write:

\[
\frac{1}{d - c} \int_{c}^{d} f_x(y) dy \leq \left( \frac{r}{r+1} \right)^{\frac{1}{r}} ([f_x(c)]^r + [f_x(d)]^r)^{\frac{1}{r}}
\]

or

\[
\frac{1}{d - c} \int_{c}^{d} f(x, y) dy \leq \left( \frac{r}{r+1} \right)^{\frac{1}{r}} ([f(x, c)]^r + [f(x, d)]^r)^{\frac{1}{r}}.
\]
Dividing both side of inequality $(b - a)$ and integrating respect to $x$ on $[a, b]$, we get

\begin{equation}
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dx dy \leq \left( \frac{r}{r+1} \right)^{\frac{1}{r}} \left[ \frac{1}{(b-a)} \int_a^b \left[ (f(x, c))^r + [f(x, d)]^r \right]^{\frac{1}{r}} dx \right].
\end{equation}

By a similar argument for the mapping, we have

\begin{equation}
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dx dy \leq \left( \frac{r}{r+1} \right)^{\frac{1}{r}} \left[ \frac{1}{(d-c)} \int_c^d \left[ (f(a, y))^r + [f(b, y)]^r \right]^{\frac{1}{r}} dy \right].
\end{equation}

By addition (2.2) and (2.3), (2.1) is proved. \hfill \Box

**Corollary 1.** In (2.7), if we choose $r = 1$ we have the mid inequality of (1.4).

**Theorem 6.** Suppose that $f, g : \Delta \rightarrow \mathbb{R}_+$ be co-ordinated $r_1$-convex function and co-ordinated $r_2$-convex function on $\Delta$. Then one has the inequality:

\begin{equation}
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) g(x, y) dy dx \leq \frac{1}{4} \left( \frac{r_1}{r_1+2} \right)^{\frac{1}{r_1}} \left[ \frac{1}{(b-a)} \int_a^b \left[ (f(x, c))^r_1 + [f(x, d)]^r_1 \right]^{\frac{1}{r_1}} dx \right]
\end{equation}

\begin{equation}
+ \frac{1}{4} \left( \frac{r_2}{r_2+2} \right)^{\frac{1}{r_2}} \left[ \frac{1}{(d-c)} \int_c^d \left[ (f(a, y))^r_2 + [f(b, y)]^r_2 \right]^{\frac{1}{r_2}} dy \right]
\end{equation}

where $r_1 > 0$, $r_2 \leq 2$.

**Proof.** Since $f, g : \Delta = [a, b] \times [c, d] \rightarrow \mathbb{R}_+$ is co-ordinated $r_1$-convex and $r_2$-convex on $\Delta$. Then the partial mappings

\begin{align*}
f_x : [c, d] & \rightarrow \mathbb{R}_+, \quad f_x(v) = f(x, v) \\
& \quad \text{and} \\
f_y : [a, b] & \rightarrow \mathbb{R}_+, \quad f_y(u) = f(u, y)
\end{align*}
are \( r_1 \)-convex on \( \Delta \). On the other hand the partial mappings
\[
g_x : [c, d] \to \mathbb{R}_+, \; g_x(v) = g(x, v)
\]
and
\[
g_y : [a, b] \to \mathbb{R}_+, \; g_y(u) = g(u, y)
\]
are \( r_2 \)-convex on \( \Delta \). From (1.2), we get
\[
\frac{1}{d - c} \int_c^d f(x)g_x(y) \, dy \leq \frac{1}{2} \left( \frac{r_1}{r_1 + 2} \right)^{\frac{r_1}{r_1 + 2}} ([f_x(c)]^{r_1} + [f_x(d)]^{r_1})^{\frac{2}{r_1}}
\]
\[
+ \frac{1}{2} \left( \frac{r_2}{r_2 + 2} \right)^{\frac{r_2}{r_2 + 2}} ([g_x(c)]^{r_2} + [g_x(d)]^{r_2})^{\frac{2}{r_2}}
\]
or
\[
\frac{1}{d - c} \int_c^d f(x, y)g(x, y) \, dy \leq \frac{1}{2} \left( \frac{r_1}{r_1 + 2} \right)^{\frac{r_1}{r_1 + 2}} ([f_x(c)]^{r_1} + [f_x(d)]^{r_1})^{\frac{2}{r_1}}
\]
\[
+ \frac{1}{2} \left( \frac{r_2}{r_2 + 2} \right)^{\frac{r_2}{r_2 + 2}} ([g_x(c)]^{r_2} + [g_x(d)]^{r_2})^{\frac{2}{r_2}}.
\]
Dividing both side of inequality \((b - a)\) and integrating respect to \(x\) on \([a, b]\), we have
\[
(b - a) \left( \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y)g(x, y) \, dy \, dx \right)
\]
\[
\leq \frac{1}{2} \left( \frac{r_1}{r_1 + 2} \right)^{\frac{r_1}{r_1 + 2}} \left[ \frac{1}{(b - a)} \int_a^b ([f(x, c)]^{r_1} + [f(x, d)]^{r_1})^{\frac{2}{r_1}} \, dx \right]
\]
\[
+ \frac{1}{2} \left( \frac{r_2}{r_2 + 2} \right)^{\frac{r_2}{r_2 + 2}} \left[ \frac{1}{(b - a)} \int_a^b ([g(x, c)]^{r_2} + [g(x, d)]^{r_2})^{\frac{2}{r_2}} \, dx \right].
\]
By a similar argument, we have
\[
(2.6)
\]
\[
(b - a) \left( \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y)g(x, y) \, dy \, dx \right)
\]
\[
\leq \frac{1}{2} \left( \frac{r_1}{r_1 + 2} \right)^{\frac{r_1}{r_1 + 2}} \left[ \frac{1}{(d - c)} \int_c^d ([f(a, y)]^{r_1} + [f(b, y)]^{r_1})^{\frac{2}{r_1}} \, dy \right]
\]
\[
+ \frac{1}{2} \left( \frac{r_2}{r_2 + 2} \right)^{\frac{r_2}{r_2 + 2}} \left[ \frac{1}{(d - c)} \int_c^d ([g(a, y)]^{r_2} + [g(b, y)]^{r_2})^{\frac{2}{r_2}} \, dy \right].
\]
Addition (2.5) and (2.6), we can write

\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)g(x,y)dydx
\]

\[
\leq \frac{1}{4} \left( \frac{r_1}{r_1 + 2} \right)^{\frac{2}{r_1}} \left[ \frac{1}{(b-a)} \int_a^b \left( [f(x,c)]^{r_1} + [f(x,d)]^{r_2} \right)^{\frac{2}{r_1}} dx \right]
\]

\[
+ \frac{1}{4} \left( \frac{r_2}{r_2 + 2} \right)^{\frac{2}{r_2}} \left[ \frac{1}{(b-a)} \int_a^b \left( [g(x,c)]^{r_2} + [g(x,d)]^{r_2} \right)^{\frac{2}{r_2}} dx \right]
\]

\[
+ \frac{1}{4} \left( \frac{r_1}{r_1 + 2} \right)^{\frac{2}{r_1}} \left[ \frac{1}{(d-c)} \int_c^d \left( [f(a,y)]^{r_1} + [f(b,y)]^{r_1} \right)^{\frac{2}{r_1}} dy \right]
\]

\[
+ \frac{1}{4} \left( \frac{r_2}{r_2 + 2} \right)^{\frac{2}{r_2}} \left[ \frac{1}{(d-c)} \int_c^d \left( [g(a,y)]^{r_2} + [g(b,y)]^{r_2} \right)^{\frac{2}{r_2}} dy \right]
\]

which completes the proof. \[\square\]

**Corollary 2.** In (2.4), if we choose \(r_1 = r_2 = 2\), we have

\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)g(x,y)dydx
\]

\[
\leq \frac{1}{8(b-a)} \left( \int_a^b [f(x,c)]^2 dx + \int_a^b [f(x,d)]^2 dx + \int_a^b [g(x,c)]^2 dx + \int_a^b [g(x,d)]^2 dx \right)
\]

\[
+ \frac{1}{8(d-c)} \left( \int_c^d [f(a,y)]^2 dy + \int_c^d [f(b,y)]^2 dy + \int_c^d [g(a,y)]^2 dy + \int_c^d [g(b,y)]^2 dy \right).
\]

**Corollary 3.** In (2.3), if we choose \(r_1 = r_2 = 2\), and \(f(x,y) = g(x,y)\), we have

\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)g(x,y)dydx
\]

\[
\leq \frac{1}{4(b-a)} \left( \int_a^b [f(x,c)]^2 dx + \int_a^b [f(x,d)]^2 dx \right) + \frac{1}{4(d-c)} \left( \int_c^d [f(a,y)]^2 dy + \int_c^d [f(b,y)]^2 dy \right).
\]
Theorem 7. Suppose that $f, g : \Delta \to \mathbb{R}_+$ be co-ordinated $r_1-$convex function and co-ordinated $r_2-$convex function on $\Delta$. Then one has the inequality:

\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)g(x, y)dydx \leq \frac{1}{2} \left( \frac{1}{(b-a)} \int_a^b \left( \frac{[f(x,c)]^{\tau_1} + [f(x,d)]^{\tau_1}}{2} \right)^{\frac{1}{\tau_1}} dx \right) \\
\times \left( \frac{1}{(d-c)} \int_c^d \left( \frac{[g(c,y)]^{\tau_2} + [g(b,y)]^{\tau_2}}{2} \right)^{\frac{1}{\tau_2}} dy \right) \\
+ \frac{1}{2} \left( \frac{1}{(d-c)} \int_c^d \left( \frac{[f(a,y)]^{\tau_1} + [f(b,y)]^{\tau_1}}{2} \right)^{\frac{1}{\tau_1}} dy \right) \\
\times \left( \frac{1}{(b-a)} \int_a^b \left( \frac{[g(a,y)]^{\tau_2} + [g(b,y)]^{\tau_2}}{2} \right)^{\frac{1}{\tau_2}} dx \right)
\]

where $r_1 > 1$ and $\frac{1}{\tau_1} + \frac{1}{\tau_2} = 1$.

Proof. Since $f, g : \Delta = [a, b] \times [c, d] \to \mathbb{R}_+$ is co-ordinated $r_1-$convex and $r_2-$convex on $\Delta$. Then the partial mappings

\[
f_x : [c, d] \to \mathbb{R}_+, \quad f_x(v) = f(x, v)
\]

and

\[
f_y : [a, b] \to \mathbb{R}_+, \quad f_y(u) = f(u, y)
\]

are $r_1-$convex on $\Delta$. On the other hand the partial mappings

\[
g_x : [c, d] \to \mathbb{R}_+, \quad g_x(v) = g(x, v)
\]

and

\[
g_y : [a, b] \to \mathbb{R}_+, \quad g_y(u) = g(u, y)
\]

are $r_2-$convex on $\Delta$. From (1.3), we can write

\[
\int_c^d f(x, y)g(x, y)dy \leq \left( \frac{[f(x,a)]^{\tau_1} + [f(x,b)]^{\tau_1}}{2} \right)^{\frac{1}{\tau_1}} \left( \frac{[g(x,a)]^{\tau_2} + [g(x,b)]^{\tau_2}}{2} \right)^{\frac{1}{\tau_2}}.
\]
Integrating this inequality respect to $x$ on $[a, b]$, we get

\[(2.8) \quad \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)g(x, y)dydx \leq \left( \frac{1}{(b-a)} \int_a^b \left( \frac{[f(x,c)]^{r_1} + [f(x,d)]^{r_1}}{2} \right)^{\frac{1}{r_1}} dx \right) \times \left( \frac{1}{(d-c)} \int_c^d \left( \frac{[g(x,c)]^{r_2} + [g(x,d)]^{r_2}}{2} \right)^{\frac{1}{r_2}} dy \right). \]

Similarly, we can write

\[(2.9) \quad \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)g(x, y)dydx \leq \left( \frac{1}{(d-c)} \int_c^d \left( \frac{[f(a,y)]^{r_1} + [f(b,y)]^{r_1}}{2} \right)^{\frac{1}{r_1}} dy \right) \times \left( \frac{1}{(b-a)} \int_a^b \left( \frac{[g(a,y)]^{r_2} + [g(b,y)]^{r_2}}{2} \right)^{\frac{1}{r_2}} dx \right). \]

Adding (2.8) and (2.9), (2.7) is proved.

\[\square\]

Corollary 4. In (2.7), if we choose $r_1 = r_2 = 2$, we have

\[(2.10) \quad \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)g(x, y)dydx \leq \frac{1}{2} \sqrt{\int_a^b [f(x,c)]^2 dx + \int_a^b [f(x,d)]^2 dx} \times \frac{1}{2} \sqrt{\int_c^d [g(x,c)]^2 dy + \int_c^d [g(x,d)]^2 dy} \times \frac{1}{2} \sqrt{\int_a^b [f(a,y)]^2 dx + \int_a^b [f(b,y)]^2 dx} \times \frac{1}{2} \sqrt{\int_c^d [g(a,y)]^2 dy + \int_c^d [g(b,y)]^2 dy}. \]
Corollary 5. In (2.7), if we choose \( r_1 = r_2 = 2 \), and \( f(x, y) = g(x, y) \), we have

\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)^2 \, dy \, dx \leq \frac{1}{4(b-a)} \left[ \int_a^b [f(x, c)]^2 \, dx + \int_a^b [f(x, d)]^2 \, dx \right] + \frac{1}{4(d-c)} \left[ \int_c^d [f(a, y)]^2 \, dy + \int_c^d [f(b, y)]^2 \, dy \right].
\]