$L^2$-Density of Wild Initial Data for the Hypodissipative Navier-Stokes Equations

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Abstract

In this paper we deal with the Cauchy problem for the hypodissipative Navier-Stokes equations in the three-dimensional periodic setting. For all Laplacian exponents $\theta < 1/3$, we prove non-uniqueness of dissipative $L^2_tH^\theta_x$ weak solutions for an $L^2$-dense set of $C^\beta$ Hölder continuous wild initial data with $\theta < \beta < 1/3$. This improves previous results of non-uniqueness for infinitely many wild initial data ([8, 20]) and generalizes previous results on density of wild initial data obtained for the Euler equations ([14, 13]).

Keywords: Convex integration, hypodissipative Navier-Stokes equations, wild initial data

1 Introduction

The existence of dissipative solutions for the Euler equations

$$\begin{cases}
\partial_t v + \text{div}(v \otimes v) + \nabla p = 0 \\
\text{div} v = 0
\end{cases}$$

(1.1)

with regularity lower than $C^0_tC^2_x$ has been investigated deeply. After the pioneering works of Scheffer [33] (on the plane $\mathbb{R}^2$) and Shnirelman [34] (on the periodic torus $\mathbb{T}^2$), De Lellis and Székelyhidi, in [15], introduced the convex integration technique (first used by Nash in [28] and Kuiper in [26] in the context of isometric embeddings, and formalized in a more general setting by Gromov in [23]) in this setting, proving the existence of nontrivial compactly supported $L^\infty_tL^2_x$ weak solutions of (1.1) in $\mathbb{R}^n$ for any $n$. The subsequent paper [16] provided a proof of the non-uniqueness of weak solutions satisfying the weak energy inequality

$$\int_{\mathbb{T}^3} |v(x,t)|^2 \, dx \leq \int_{\mathbb{T}^3} |v(x,0)|^2 \, dx, \quad (1.2)$$

i.e. dissipating the total kinetic energy. We call such solutions dissipative or admissible.

Both of these papers use a Baire category argument, proving that such solutions constitute the set of continuity points of a Baire-1 map. This implies that such solutions not only exist, but are “typical” in the sense of category. These results were the first steps in the resolution of the second part of Onsager’s conjecture from [29].

Onsager’s Conjecture. Let $(v,p)$ be a weak solution of (1.1) and define the total kinetic energy as

$$E(t) := \frac{1}{2} \int_{\mathbb{T}^3} |v(x,t)|^2 \, dx$$

If $v \in C^\beta$ for $\beta > \frac{1}{3}$, then the energy is a conserved quantity, i.e. $E(t) = E(0)$.

By contrast, for any $\beta < \frac{1}{7}$, there exist $C^\beta$ weak solutions of (1.1) which do not conserve the energy.

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The first part was proved in [10], and refined for more general spaces in [6, 22].

In [17], De Lellis and Székelyhidi introduced a more constructive approach which allowed to obtain infinitely many dissipative continuous solutions (see also [18]), and then infinitely many $C^{\beta}$-Hölder solutions in [19].

In [3], Isett was able to improve the Hölder exponent for the existence of non-conservative solutions to $\frac{1}{5} - \varepsilon$ and, after the introduction of Mikado flows by Daneri and Székelyhidi in [14], he completed the proof of Onsager’s conjecture above in [25] by showing existence of infinitely many Hölder continuous dissipative solutions in the class $C_0^\beta C_t^1$, for all $\beta < \frac{1}{5}$. This result was later improved to dissipative solutions in the same class in [4].

In the class of admissible solutions, weak-strong uniqueness holds, as proved in [36]: $C^1$ solutions are unique, and moreover, if such a solution exists, any $L_t^q L_s^p$ solution with the same initial data which is admissible coincides with the $C^1$ solution.

However, cleverly adapting and improving the convex integration technique of the above-mentioned papers, the existence of infinitely many initial data giving rise to admissible solutions in the class $C^0$ for $\beta < \frac{1}{16}$ was proved in [12]. In [14] and [13], the following topologically stronger statement was proved: the set of $C^0$ initial data giving rise to admissible solutions is dense in $L^2(\mathbb{T}^3)$. It was proved in [14] for $\beta < \frac{1}{5}$, and in [13] for $\beta < \frac{1}{3}$.

Removing the admissibility condition (1.2) leads to non-uniqueness for any $C^\infty$ initial datum, as proved in [21]. For the Navier-Stokes equations

$$\begin{align*}
\partial_t v + \text{div}(v \otimes v) + \nabla p &= \Delta_v, \\
\text{div} v &= 0
\end{align*}$$

(1.3)

in [27], Leray proved the existence of global weak solutions satisfying the following energy inequality:

$$\int_{\mathbb{T}^3} |v(x,t)|^2 dx + 2 \int_0^t \int_{\mathbb{T}^3} |\nabla v(x,s)|^2 dx ds \leq \int_{\mathbb{T}^3} |v(x,0)|^2 dx \quad \text{a.e. } t > 0. \quad (1.4)$$

Such solutions are called Leray solutions or Leray-Hopf solutions, and we will call them admissible solutions of (1.3). The strategy employed in [27] can easily be adapted to prove the existence of solutions to (1.1) satisfying (1.2), which are therefore called by the same names.

Thanks to the Ladyzhenskaya-Prodi-Serrin regularity theory, weak-strong uniqueness for (1.3) holds for $L_s^q L_t^p$ solutions, where $\frac{3}{s} + \frac{2}{r} = 1$, as proved in [30] for $d = 3$ and in [31] for the general case.

The uniqueness or non-uniqueness of solutions to (1.3) satisfying (1.4) is still a long-standing open problem. The latest step in this regard is [11] where, introducing a body force in the equation, the authors exhibit two distinct admissible solutions on $\mathbb{R}^3$.

Several non-uniqueness results have been obtained for non-admissible solutions, i.e. in the absence of the energy inequality (1.4). In [5], the authors prove the existence, for any smooth energy profile $e$, of $C^0[H_s^1]$ solutions with kinetic energy $e$, i.e.

$$\int_{\mathbb{T}^3} |v(x,t)|^2 dx = e(t),$$

for some parameter $\beta$. This implies non-uniqueness for the zero initial datum. Choosing a non-increasing $e$, this also implies the existence of solutions of (1.3) satisfying (1.2).

It is known that such $\beta$ cannot be too large, since $\beta = \frac{1}{2}$ implies weak-strong uniqueness by [11]. In arbitrary dimensions, $\beta = \frac{1}{2}$ is in fact a threshold for weak-strong uniqueness. Indeed, in Terence Tao’s blog post [35], the non-uniqueness of $H^1[H_s^1]$ for any $s < \frac{1}{2}$ has been proven to hold on $\mathbb{T}^d$ where $d = d(s)$ is sufficiently large.

In the subsequent work [2], a “gluing” theorem is proved: given any two strong solutions $u_1, u_2 \in C^0[H_s^1(0,T), \mathbb{T}^3]$, there exists a weak solution $v \in C^0[H_s^1 \cap W_t^1 H_s^{1+p}]$, with a set of singular times $\Sigma$ having dimension $\dim_H \Sigma \leq 1 - \beta$, which coincides with $u_1$ on $[0, \frac{T}{2}]$ and with $u_2$ on $[\frac{T}{2}, T]$. The parameter $\beta$ is not quantified in [2], but the strategy therein allows it to reach at most a value slightly above $10^{-3}$.

Reducing the regularity in time can lead to better spatial regularity. Indeed, in [7], the authors prove an approximation result: given a smooth divergence-free field $v$, we can approximate it in $L_t^p L_s^q \cap L_t^1 W_s^{1,\infty}$ with a solution of (1.3), for any $p < 2$. The singular set of these solutions is also of low dimension. The strategy of [7] can be extended to $L_t^p L_s^q \cap L_t^1 W_s^{1,q}$, where once again $p < 2$, and we have that $s < 2$ and $q < q_{\max}(s, p)$, where $q_{\max}(s, p) \to 1$ if $s \to 2$ or $p \to 2$. 

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In this paper we are interested in studying how introducing a fractional dissipative term in (1.1) may affect the uniqueness or ill-posedness of the Cauchy problem. More specifically, we consider the hypodissipative Navier-Stokes equations

\[
\begin{cases}
\partial_t v + \text{div}(v \otimes v) + \nabla p + (-\Delta)^\theta v = 0, \\
\text{div} v = 0
\end{cases}
\]

with exponent \( \theta < \frac{1}{3} \) and admissibility condition

\[
\frac{1}{2} \int_\mathbb{T}^3 |v(x,t)|^2 \, dx + \frac{1}{2} \int_0^t \int_{\mathbb{T}^3} |(-\Delta)^{\frac{\theta}{2}} v(x,s)|^2 \, dx \, ds \leq \frac{1}{2} \int_{\mathbb{T}^3} |v(x,0)|^2 \, dx.
\]

Theorem 6.1

Theorem 6.1, we obtain the following corollary. In this case as well, admissible solutions are also known as Leray solutions

Inspired by the results in [14] and [13], which prove the existence of an \( L^2 \)-dense class of \( C^\beta \) wild initial data (namely data for which non-uniqueness holds) for admissible solutions to (1.5) in \( L^2_H^\theta \). The strategy proposed in [14] and [13] provides a quantitative criterion for non-uniqueness based on the existence of approximate solutions called adapted sub-solutions.

Here, we explore and extend that strategy to the hypodissipative Navier-Stokes equations. The main issue with respect to the Euler setting is to control the dissipative term in the energy. Our main results are the following.

**Theorem 1.1 (C^\beta weak solutions with data close to \( L^2 \) functions and time of admissibility).** Let \( \theta < \beta < \frac{1}{3} \), \( w \in L^2(\mathbb{T}^3) \). Then, for all \( \eta > 0 \), there exist a time \( T = T(\eta) > 0 \), an initial datum \( w_\eta \in C^\beta(\mathbb{T}^3) \) such that \( \| w_\eta - w \|_{L^2} < \eta \), and infinitely many weak solutions \( v_\eta \in C^0([0,T], C^\beta(\mathbb{T}^3)) \), with initial datum \( v_\eta|_{t=0} = w_\eta \), which satisfy (1.2) on \([0,T]\), but can be proved to satisfy (1.6) (i.e to be admissible) only on \([0,T(\eta)]\). Moreover

\[
\lim_{\eta \to 0} T(\eta) = 0.
\]

The fact that the admissibility condition in Theorem 1.1 cannot be guaranteed to hold for \( C^\beta \) solutions on a fixed set of times is due to the necessity of controlling the dissipation term in the energy.

**Definition 1.1 (Wild initial data).** Let \( X \) be a function space. A divergence-free vector field \( w \in L^2(\mathbb{T}^3) \) is a \( (0,X,T) \)-wild initial datum for (1.5) if there exist infinitely many weak solutions \( v : [0,T] \times \mathbb{T}^3 \to \mathbb{R}^3 \) of (1.5) such that \( v \in X, v(x,0) = w(x) \ a.e. \) in \( \mathbb{T}^3 \), and the admissibility condition (1.6) holds on \([0,T]\). The set of such data is denoted by \( W_{0,X,T} \). If \( X = L^\infty_T C^\beta_x \), we will speak of \( (0,\beta,T) \)-wild data, and of the set \( W_{0,\beta,T} \).

As a consequence of Theorem 1.1 we obtain the following corollary.

**Corollary 1.1 (Density of wild initial data – Hölder solutions).** The set \( \bigcup_T W_{0,\beta,T} \) is dense in the set of divergence-free \( L^2 \) vector fields, for all \( 0 < \beta < \frac{1}{3} \).

Moreover, by taking a solution \( v_\eta \) as given by Theorem 1.1 and continuing it with a Leray solution \( \tilde{v}_\eta : [T(\eta),\infty) \times \mathbb{T}^3 \to \mathbb{R}^3 \) with datum \( \tilde{v}_\eta(T(\eta)) = v_\eta(T(\eta)) \), as provided by Theorem 6.1 we obtain the following.

**Theorem 1.2 (Density of wild initial data – Sobolev solutions).** \( W_{0, L^2 H^\theta_x, T} \) is dense in the set of divergence-free \( L^2 \) vector fields, for all \( 0 < \frac{1}{3}, T > 0 \).

The general strategy of the paper is to define suitable relaxations of the notion of solution (the so-called “sub-solutions” of Section 3), and approximating one kind of sub-solution with another one which is closer to the notion of solution. This is done constructing sequences of sub-solutions that converge, in an appropriate
Theorem 1.1

For Lemma 2.1 (Hölder norm inequalities).

Concerning the Hölder norms, we recall the following standard inequalities.

The paper is organized as follows. Section 2 establishes some notations and contains a few useful preliminary results. Section 3 introduces three kinds of subsolutions, namely strict, strong, and adapted. Section 4 contains the approximation results. Section 5 states the approximation results. Section 6 deduces some important definitions and relations used in the following sections. Section 7 shows how one can approximate strict subsolutions with strong ones. Sections 8 and 9 contain the two substeps of each convex integration step, respectively a gluing step and a perturbation step. Sections 10 and 11 then prove the other approximation results, namely the approximation of strong subsolutions with adapted ones, and that of adapted subsolutions with weak solutions.

2 Preliminaries

Throughout the paper, we will use the following notations:

- $\mathcal{S}^{3 \times 3}$ are the symmetric 3-by-3 matrices; within this set, $\mathcal{S}^+_{3 \times 3}$ are the positive definite ones, $\mathcal{S}^0_{3 \times 3}$ are the traceless ones, and $\mathcal{S}^\geq_{3 \times 3}$ are the positive semidefinite ones.

- If $R \in \mathcal{S}^{3 \times 3}$, we decompose it as
  $$ R = \frac{1}{3} \text{tr} R \mathbb{I} + \hat{R} = \rho \mathbb{I} + \hat{R}, $$
  where $\hat{R} \in \mathcal{S}^0_{3 \times 3}$ is the traceless part of $R$.

- For scalar functions $f$, we write $\nabla f := (\partial_1 f, \partial_2 f, \partial_3 f) =: \mathbb{D} f$;

- However, for vector fields $v$, we define $\mathbb{D} v$ so that $(\mathbb{D} v)_{ij} = \partial_j v_i$, whereas $\nabla v = (\mathbb{D} v)^T$; with these choices, $(v \cdot \nabla)v = \mathbb{D} v \cdot v = v \cdot \nabla v$;

- In a similar fashion, for tensor fields $S$, $\mathbb{D} S$ is defined by $(\mathbb{D} S)_{ijk} = \partial_k S_{ij}$, whereas $\nabla S$ is defined by $(\nabla S)_{ijk} = \partial_i S_{jk}$;

- The Hölder norms are defined as follows
  $$ \|f\|_0 := \sup |f(x)|, \quad [f]_k := \max_{\beta \in \mathbb{N}^3} \sup_{|\beta| = k} |\partial^{\beta} f|, \quad [f]_{\alpha} := \sup_{x,y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}, $$
  $$ \|f\|_k := \|f\|_0 + \sum_{i=1}^{k} [f]_i, \quad \|f\|_{k+\alpha} := \|f\|_k + \max_{|\beta| = k} [\partial^{\beta} f]_{\alpha}, $$
  for $k \in \mathbb{N}$, $\alpha \in (0, 1)$.

Concerning the Hölder norms, we recall the following standard inequalities.

**Lemma 2.1 (Hölder norm inequalities).** For $0 \leq s \leq r$ and $f, g : \mathbb{T}^3 \to \mathbb{R}^d$

$$ [fg]_s \leq C(r) ([f]_r, [g]_0 + \|f\|_0 [g]_r) \quad (2.1) $$

$$ [f]_s \leq C(r,s) [f]_{1} \|f\|_0^{1-s} [f]_{1}^{s}. \quad (2.2) $$

Moreover, for $f : \mathbb{T}^3 \to \mathcal{S} \subseteq \mathbb{R}^d$ and $\Psi : \mathcal{S} \to \mathbb{R}$:

$$ [\Psi \circ f]_m \leq K(d,m)([\Psi]_1 \|\mathbb{D} f\|_{m-1} + \|\nabla \Psi\|_{m-1} \|f\|_{m-1} \|f\|_{m} \|f\|_{1}) \quad (2.3) $$

$$ [\Psi \circ f]_m \leq K(d,m)([\Psi]_1 \|\mathbb{D} f\|_{m-1} + \|\nabla \Psi\|_{m-1} [f]_1^{m}). \quad (2.4) $$


Finally, for all \( s, r \geq 0 \):
\[
\|f \ast \varphi_r\|_{r+2} \leq C(r, s) \ell^{-s} \|f\|_r \\
\|f - f \ast \varphi_r\|_{r} \leq C(r, s) \ell \|f\|_{r+1} \\
\|f - f \ast \varphi_r\|_{r} \leq C(r, s) \ell^2 \|f\|_{r+2} \\
\|(fg) \ast \varphi_t - (f \ast \varphi_r)(g \ast \varphi_t)\|_{r} \leq C(r, s) \ell^{2-s} \|f\|_1 \|g\|_1,
\]

where \( \varphi \) is a standard mollification kernel, i.e. \( \varphi \in C_0^\infty(B_1; [0, 1]) \) and \( \int \varphi = 1 \), and \( \varphi_t := \frac{1}{t} \varphi(\frac{\cdot}{t}) \).

For functions \( f, g : \mathbb{T}^3 \times [0, T] \to \mathbb{R}^d \), we denote their time-slices by \( f_t(x) := f(x, t), g_t(x) := g(x) \). The above lemma can then be applied to the time slices of time-dependent vector fields, e.g. the velocities of subsolutions, with the notation \( \|f(t, \cdot)||_{C^r}, [f(t, \cdot)]_{C^r} \) for the (semi)norms of the slices. By taking supremum norms in time, the above inequalities can be formulated with \( C_0^0 C^r \) norms.

We now introduce Mikado flows, the basic building blocks of the perturbations, and the important Stationary Phase Lemma. The proofs of the following results can be found in [14].

**Lemma 2.2 (Mikado flows).** For any compact subset \( \mathcal{N} \subset \subset \mathbb{S}^3 \) there exists a smooth vector field \( W : \mathcal{N} \times \mathbb{T}^3 \to \mathbb{R}^3 \) such that, for every \( R \in \mathcal{N} \)
\[
\begin{cases}
\text{div}_x(W(R, \xi) \otimes W(R, \xi)) = 0 \\
\text{div}_\xi W(R, \xi) = 0
\end{cases},
\]
and
\[
\int_{\mathbb{T}^3} W(R, \xi) d\xi = 0 \quad \text{and} \quad \int_{\mathbb{T}^3} W(R, \xi) \otimes W(R, \xi) d\xi = R^*
\]

Using Fourier series in \( \xi \) and the above integral and differential relations, we obtain that
\[
W(R, \xi) = \sum_{k \in \mathbb{Z}^3} a_k(R) A_k e^{ik \cdot \xi} \quad \text{and} \quad W(R, \xi) \otimes W(R, \xi) = R + \sum_{k \in \mathbb{Z}^3} C_k(R) e^{ik \cdot \xi},
\]
where the coefficients \( a_k, C_k \in C_0^\infty \), the \( A_k \) satisfy \( A_k \cdot k = 0, |A_k| = 1 \), the \( C_k \) satisfy \( C_k k = 0 \), and moreover
\[
\sup_{R \in \mathcal{N}} |\mathcal{D}_R^N a_k(R)| = \|a_k\|_{C^N(\mathcal{N})} \leq \frac{C(\mathcal{N}, N, m)}{|k|^m} \quad \text{and} \quad \sup_{R \in \mathcal{N}} |\mathcal{D}_R^N C_k(R)| = \|C_k\|_{C^N(\mathcal{N})} \leq \frac{C(\mathcal{N}, N, m)}{|k|^m}.
\]

In Section 7 we will need the fact that, if we set
\[
U(R, \xi) := \sum_k a_k(R) \frac{ik \times A_k}{|k|^2} e^{ik \cdot \xi},
\]
then we have that \( \text{curl}_x U = W \). Indeed
\[
\text{curl}_x U(R, \xi) = \sum_{k \in \mathbb{Z}^3} \epsilon_{lmn} \partial_m \left( a_k(R) \frac{ik \times A_k}{|k|} e^{ik \cdot \xi} \right) e_l = \sum_{k \in \mathbb{Z}^3} \epsilon_{lmpq} a_k(R) ik_p (A_k) q |k|^{-2} ik_m e^{ik \cdot \xi} e_l = \]
\[
= -\sum_{k \in \mathbb{Z}^3} (\delta_l p \delta_m q - \delta_l q \delta_m p) a_k(R) k_p |k|^{-2} (A_k)_q k_m e^{ik \cdot \xi} e_l = \]
\[
= \sum_k a_k(R) A_k e^{ik \cdot \xi} - \sum_{k \in \mathbb{Z}^3} a_k(R) k_p k_q |k|^{-2} (A_k)_q k_m e^{ik \cdot \xi} e_p = \]
\[
= W(R, \xi) - \sum_k a_k(R) (k \cdot A_k) \frac{k}{|k|^2} e^{ik \cdot \xi}.
\]

We now introduce a certain “anti-divergence operator” which will be used to obtain the new Reynolds stress \( R \) in the various approximation results.
Observing that $D$ see that be found e.g. in \cite{4, Proposition B.1} (transport) and \cite{20, Proposition 3.3} (transport-diffusion).

We continue by recalling some classical estimates on the transport and transport-diffusion equations, which can

\begin{equation}
\text{Definition 2.1 (Anti-divergence $R$). Define the operator $\diamond$ so that}
\begin{align*}
\{ \Delta \diamond v &= v - \int_{T^3} \nabla v \, dx, \\
\int_{T^3} \diamond v &= 0,
\end{align*}
\end{equation}

and then define

\begin{equation}
Rv := \frac{1}{4} (\mathcal{D} \mathcal{P} \diamond v + (\mathcal{D} \mathcal{P} \diamond v)^T) + \frac{3}{4} (\mathcal{D} \diamond v + (\mathcal{D} \diamond v)^T) - \frac{1}{2} (\text{div} \diamond v) \text{Id},
\end{equation}

$\mathcal{P}$ being the Leray projection onto divergence-free fields with zero average.

This operator satisfies the following properties.

\begin{equation}
\text{Theorem 2.1 (Fractional laplacian and Hölder norms).}
\end{equation}

Moreover, for every $\gamma \in (0,1), N \geq 1$. Let $a \in C^\infty(\mathbb{T}^3), \Phi \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ be smooth functions and assume

\begin{equation}
\frac{1}{K} \leq |\mathcal{D} \Phi| \leq K \quad \text{on } \mathbb{T}^3.
\end{equation}

Then

\begin{equation}
\left| \int_{T^3} a(x) e^{ik \Phi} \, dx \right| \leq C(K, N) \frac{|a|_N + |a|_0 |\Phi|_N}{|k|^N},
\end{equation}

and for the operator $R$ of (2.13) above we have that

\begin{equation}
\left| R(a(x) e^{ik \Phi}) \right|_{\alpha} \leq C(\alpha, K, N) \left( |a|_0 + |a|_{N+\alpha} + |a|_0 |\Phi|_{N+\alpha} \right).
\end{equation}

We now recall some classical estimates regarding fractional laplacians.

\begin{equation}
\text{Lemma 2.3 (Divergence and $R$). For any $C^\infty$ vector field $v$, $Rv \in \mathcal{D}^3_{0} \times 3$ is symmetric and trace-free, and moreover}
\end{equation}

\begin{equation}
\text{Lemma 2.4 (Stationary Phase Lemma). Let $\alpha \in (0,1), N \geq 1$. Let $a \in C^\infty(\mathbb{T}^3), \Phi \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ be smooth functions and assume}
\end{equation}

\begin{equation}
\frac{1}{K} \leq |\mathcal{D} \Phi| \leq K \quad \text{on } \mathbb{T}^3.
\end{equation}

Then

\begin{equation}
\left| \int_{T^3} a(x) e^{ik \Phi} \, dx \right| \leq C(K, N) \frac{|a|_N + |a|_0 |\Phi|_N}{|k|^N},
\end{equation}

and for the operator $R$ of (2.13) above we have that

\begin{equation}
\left| R(a(x) e^{ik \Phi}) \right|_{\alpha} \leq C(\alpha, K, N) \left( |a|_0 + |a|_{N+\alpha} + |a|_0 |\Phi|_{N+\alpha} \right).
\end{equation}

We now recall some elementary calculations for the reader’s convenience. With the definitions we gave for $\nabla, \mathcal{D},$ setting $D_i^{(v)} := \partial_i + v \cdot \nabla$, we have that

\begin{equation}
\nabla e^{ik \Phi} = i \nabla \Phi \cdot k e^{ik \Phi} = ie^{ik \Phi} k \cdot \nabla \Phi
\end{equation}

\begin{equation}
D_i^{(v)} (\mathcal{D} \Phi) = \mathcal{D} (D_i^{(v)} \Phi) - \mathcal{D} \Phi \cdot \nabla v.
\end{equation}

Observing that $\mathcal{D} \Phi \mathcal{D} \Phi^{-1} = \text{Id}$ and thus $0 = D_i^{(v)} (\mathcal{D} \Phi \mathcal{D} \Phi^{-1}) = D_i^{(v)} (\mathcal{D} \Phi) \cdot \mathcal{D} \Phi^{-1} + \mathcal{D} \Phi \cdot D_i^{(v)} (\mathcal{D} \Phi^{-1})$, we can see that

\begin{equation}
D_i^{(v)} \mathcal{D} \Phi^{-1} = -\mathcal{D} \Phi^{-1}D_i^{(v)} (\mathcal{D} \Phi)\mathcal{D} \Phi^{-1} = (\nabla \Phi^{-1} \nabla v)^T - \mathcal{D} \Phi^{-1} \cdot \mathcal{D} (D_i^{(v)} \Phi) \cdot \mathcal{D} \Phi^{-1}.
\end{equation}

We continue by recalling some classical estimates on the transport and transport-diffusion equations, which can be found e.g. in \cite{4, Proposition B.1} (transport) and \cite{20, Proposition 3.3} (transport-diffusion).
Proposition 2.1 (Estimates on the transport equation). Assume $|t - t_0||v|_1 \leq 1$. Then, any solution $f$ of
\[
\begin{aligned}
(\partial_t + v \cdot \nabla) f &= g \\
f(\cdot, 0) &= f_0
\end{aligned}
\]
satisfies
\[
\|f(t)\|_0 \leq \|f_0\|_0 + \int_{t_0}^{t} \|g(\cdot, \tau)\| d\tau
\]
\[
\|f(t)\|_{\alpha} \leq e^{\alpha \|f_0\|_{\alpha} + \int_{t_0}^{t} \|g(\cdot, s)\| d\alpha d\tau}
\]
for all $0 \leq \alpha \leq 1$ and, more generally, for any $N \geq 1$ and $0 \leq \alpha < 1$
\[
[f(t)]_{N+\alpha} \lesssim [f_0]_{N+\alpha} + \|v\|_{N+\alpha}[f_0]_1 + \int_{t_0}^{t} ([g(s)]_{N+\alpha} + (t-s)[v]_{N+\alpha}[g(s)]_1) ds.
\]

Define $\Phi(t, \cdot)$ to be the inverse of the flux $X$ of $v$ starting at time $t_0$ as the identity (i.e. $\frac{d}{dt} X = v(X, t)$ and $X(x, t_0) = x$). Under the same assumptions as above we have that
\[
\|\nabla \Phi(t) - 1d\|_0 \lesssim |t||v|_1
\]
\[
[\Phi(t)]_N \lesssim |t||v|_N, \quad \forall N \geq 2.
\]

Proposition 2.2 (Estimates on the transport-diffusion equation). Assume $0 \leq (t - t_0)|v|_1 \leq 1$. Then, any solution of
\[
\begin{aligned}
(\partial_t + v \cdot \nabla + (-\Delta)^0) u &= f \\
u(\cdot, t_0) &= u_0
\end{aligned}
\]
\begin{align*}
&\text{in } \mathbb{T}^3 \times (t_0, T) \\
&\text{in } \mathbb{T}^3
\end{align*}
satisfies
\[
\|u(t)\|_{\alpha} \leq e^{\alpha \|u_0\|_{\alpha} + \int_{t_0}^{t} \|f(\cdot, s)\| d\alpha d\tau}
\]
for all $0 \leq \alpha \leq 1$ and, more generally, for any $N \geq 1$ and $0 \leq \alpha < 1$
\[
[u(t)]_{N+\alpha} \lesssim [u_0]_{N+\alpha} + (t-t_0)[v]_{N+\alpha}[u_0]_1 + \int_{t_0}^{t} ([f(s)]_{N+\alpha} + (t-s)[v]_{N+\alpha}[f(s)]_1) ds,
\]
where the implicit constants depends only on $N, \alpha$.

To conclude this section, we recall classical Schauder estimates (see e.g. the book [24]), which will be used in several places in this paper.

Lemma 2.5 (Schauder estimates). For any $\alpha \in (0, 1)$ and any $m \in \mathbb{N}$, there exists a constant $C(\alpha, m)$ with the following properties. If $\varphi, \psi : \mathbb{T}^3 \to \mathbb{R}$ are the unique solutions of
\[
\begin{aligned}
\Delta \varphi &= f \\
\varphi &= 0
\end{aligned}
\]
\[
\begin{aligned}
\Delta \psi &= \text{div} F \\
\psi &= 0
\end{aligned}
\]
then
\[
\|
\begin{pmatrix}
\varphi \\
\psi
\end{pmatrix}
\|_{m+1+\alpha} \leq C(m, \alpha) \|
\begin{pmatrix}
f \\
F
\end{pmatrix}
\|_{m+\alpha}.
\]
3 Approximate solutions

For the proof of [Theorem 1.1] we begin by introducing the various notions of subsolutions needed to perform the convex integration schemes.

The first notion of subsolution is very similar to the one used in [13, 14, 17].

Definition 3.1 (Subsolutions and strict subsolutions). A subsolution is a triple \((v, p, R) : \mathbb{T}^3 \times (0, T) \rightarrow \mathbb{R}^3 \times \mathbb{R} \times \mathfrak{S}^{3\times 3}_0\) such that \(v \in L^2_{\text{loc}}, \ R \in L^1_{\text{loc}}, \ p\) is a distribution, the equations

\[
\begin{aligned}
\partial_t v + \text{div}(v \otimes v) + \nabla p + (-\Delta)^0 v &= -\text{div} R \\
\text{div} v &= 0
\end{aligned}
\]

hold in the sense of distributions in \(\mathbb{T}^3 \times (0, T), \) and moreover \(R \geq 0 \ a.e., \ i.e. \ it \ is \ positive \ semidefinite \ a.e.. \) If \(R \in \mathfrak{S}^{3\times 3}_+ \ a.e., \) then the subsolution is said to be strict.

The next notion of subsolution extends the ones of [14] and [13]. As in [13], the Reynolds stress is controlled by a power of the trace. However, the exponent \(\gamma\) will only act on the “reduced” trace \(p\Omega^{-1},\) where \(\Omega > 0\) is a constant whose role is explained in Section 4.

Definition 3.2 (Strong subsolutions). A strong subsolution with parameters \(\gamma, \Omega > 0\) is a subsolution \((v, p, R)\) such that in addition \(\text{tr} R\) is a function of \(t\) only and, if

\[
\rho(t) := \frac{1}{3}(\text{tr} R)(t) \quad \quad q(t) := \frac{\rho(t)}{\Omega},
\]

then

\[
|\hat{R}(x, t)| \leq \Omega q^{1+\gamma}(t) \quad \forall (x, t).
\]

Remark 3.1 (On strength and parameters). In our schemes \(q\) will be sufficiently small so that in particular \(q^\gamma \leq r_0,\) where \(r_0\) is the geometric constant in [14, Definition 3.2], thus leading to the conclusion that \(3.2\) implies that our strong subsolutions are also strong in the sense of [14], provided \(\Omega = O(1)\) (specifically \(\Omega q^\gamma \leq r_0)).\) Note also that, if \((v, p, R)\) is a strong subsolution for some parameters \(\gamma, \Omega > 0\) with \(q < 1,\) then it is also a strong subsolution for any \(0 < \gamma' < \gamma\) with the same \(\Omega\).

The last notion of subsolution has vanishing Reynolds stress at time \(t = 0\) and the \(C^1\)-norms blow up at certain rates as the Reynolds stress goes to zero. Such adapted subsolutions have been introduced in [14] [13]. The blow-up rate in this paper is analogous to the one of [13]. Differently from [13], the blow-up is controlled by the “reduced” trace \(q\) rather than the “full” trace \(p,\) and the estimates include a power of \(\Omega.\)

Definition 3.3 (Adapted subsolutions). Given \(\gamma, \Omega > 0, 0 < \beta < \frac{1}{3},\) and \(\nu\) satisfying

\[
\nu > \frac{1 - 3\beta}{2\beta},
\]

we call a triple \((v, p, R)\) a \(C^\beta,\) adapted subsolution on \([0, T]\) with parameters \(\gamma, \Omega, \nu\) if \((v, p, R) \in C^\infty(\mathbb{T}^3 \times (0, T)) \cap C(\mathbb{T}^3 \times [0, T])\) is a strong subsolution with parameters \(\gamma, \Omega\) with initial datum

\[
v(\cdot, 0) \in C^\beta(\mathbb{T}^3) \quad \text{and} \quad R(\cdot, 0) \equiv 0,
\]

and, setting \(\rho(t) := \frac{1}{3}\text{tr} R(x, t)\) and \(q := \rho\Omega^{-1},\) for all \(t > 0\) we have that \(\rho(t) > 0\) and there exist \(\alpha \in (0, 1)\) and \(C \geq 1\) such that

\[
||v||_{1+\alpha} \leq C \Omega^\frac{\gamma}{2} q^{-(1+\nu)},
\]

\[
|\partial \nu| \leq C \Omega^\frac{\gamma}{2} q^{-\nu}.
\]
4 Strategy of the proof

The remainder of this paper closely follows the convex integration strategy adopted by [13] in the Euler setting. Section 5 states results that allow us to approximate one kind of subsolution (as defined in the previous section) with another. One of those results uses the parameters we will introduce in this section in (4.1).

Section 6 proves the main theorem starting from the results of Section 5. Section 7 shows how to obtain a strong subsolution from a strict one. By iterating Sections 8-9, we produce with another. One of those results uses the parameters we will introduce in this section in (4.1).

In passing from one subsolution to another, the $C^0$ and $C^1$ norms of the various subsolutions are estimated in terms of parameters $(\delta_q, \lambda_q)$, where $\delta_q$ is the amplitude (in space) of $w_q := v_q - v_{q-1}$, and $\lambda_q$ is the oscillation frequency (in space) of $w_q$. The parameters, however, are partially different from those chosen in [13] and closer to the ones used in [14].

More precisely, we define

$$\lambda_q := 2\pi[a^{b\theta}], \quad \delta_q := \Lambda \varepsilon q^{2q-2}, \quad \zeta_q := \lambda_q^{-2\beta}, \quad \Lambda := \delta_q^{2\theta},$$

(4.1)

where

- $[x]$ denotes the ceiling of $x$, i.e. the smallest integer $n \geq x$;
- $\beta \in (0, \frac{1}{3})$ and $b \in (1, \frac{3}{2})$ control the Hölder exponent of the scheme and are required to satisfy

$$1 < b < \frac{1 - \beta}{2\beta},$$

(4.2)

- $a \gg 1$ is sufficiently large to absorb various $q$-independent constants in the course of the proofs.

The parameter $\Lambda$, and thus the distinction between $\delta_q$ and $\zeta_q$, were absent in [13]. They are added here to make sure $\delta_1 = \delta$, thus making (7.12) an $a$-independent estimate. Thus, in particular, we are allowed to bound $\Lambda$ from below, since such a bound will be satisfied for $a$ large enough, but not to bound it from above, which would cause $\delta$ to depend on $a$.

With this choice of parameters, we must require the conditions

$$\Lambda \geq 1,$$

(4.3)

$$\frac{1}{3} > \beta > \theta + \epsilon',$$

(4.4)

for some positive $\epsilon'$. Condition (4.3) merely requires $a$ to be sufficiently large.

The main convex integration step will consist in stating that, for a certain universal constant $M > 1$, some sufficiently small $\alpha, \gamma > 0$, and a sufficiently large $a \gg 1$, if $(v_q, p_q, R_q)$ is a strong subsolution satisfying

$$\|R_q\|_0 \leq \Lambda q^{1+\gamma},$$

(4.5)

$$\|v_q\|_{1+\alpha} \leq M \delta_q^{\frac{1}{2}q+1+\alpha},$$

(4.6)

$$\frac{3}{4} \delta_q^{q+2} \leq p_q \leq \frac{7}{2} \delta_q^{q+1},$$

(4.7)

$$|\partial_i p_q| \leq p_q \delta_q^{\frac{1}{2}q},$$

(4.8)

$$\|v_q\|_{\theta + \epsilon} \leq M \left(1 + \sum_{i=0}^{q} \lambda_i^{\theta + \epsilon - \beta}\right),$$

(4.9)

where $\rho_q := \frac{1}{q} \text{tr}R_q$, and $q_q := \Lambda^{-1} p_q$, then there exists a strong subsolution $(v_{q+1}, p_{q+1}, R_{q+1})$ satisfying the conditions (4.5)-(4.9) with $q$ replaced by $q + 1$ as well as the following additional estimate

$$\|v_{q+1} - v_q\|_0 + \lambda_{q+1} \|v_{q+1} - v_q\|_{H^{-1}} + \lambda_{q+1}^{-1-\alpha} \|v_{q+1} - v_q\|_{1+\alpha} \leq M \delta_q^{\frac{1}{2}q+1}.$$
The proof consists of three steps:

1. A mollification step, moving from \((v_q,p_q,R_q)\) to \((\bar{v}_{q,i},\bar{p}_{q,i},\bar{R}_{q,i})\), where the mollification parameter \(\ell_{q,i}\) varies on suitably chosen subintervals, as required by the different orders of the upper and lower bounds on \(p_q\) in (4.7);

2. A gluing step, which goes from \((v_{q,i},p_{q,i},R_{q,i})\) to \((\bar{v}_q,\bar{p}_q,\bar{R}_q)\);

3. A perturbation step going from \((\tau_q,\bar{p}_q,\bar{R}_q)\) to \((v_{q+1},p_{q+1},R_{q+1})\).

The change in condition (4.5) with respect to (13) was made in order to prevent the new definition of \(\delta_q\) from causing bounds of the form \(\Lambda^A \leq 1\), with \(A > 0\), to appear in the proofs. Condition (4.9) was added in order to control the new trace terms.

Section 8 proves the mollification and gluing steps, and Section 9 addresses the perturbation step. The gluing step was introduced in (25) to ensure \(\bar{R}_q\) is supported in pairwise disjoint time intervals. This allows us to construct the perturbation as \(w = \sum (w_{\alpha,i} + w_{\beta,j})\), where the \(w_{\alpha,i}\) are Mikado flows with pairwise disjoint supports and \(\text{supp} w_{\alpha,i} \subseteq \text{supp} w_{\beta,j}\), thus preventing \(w \otimes w\) from containing “mixed terms” \(w_{\alpha,i} \otimes w_{\alpha,j}\) with \(i \neq j\), which are harder to deal with.

Fixing \(\alpha > 0, \gamma > 0\), we also define

\[
\ell_q := \frac{\delta_q}{\delta_{q+2}} + \frac{\tau_q}{\delta_{q+2}} \Lambda^{\frac{1}{2}} \\
\tau_q := \frac{\delta_q}{\delta_{q+2}} \Lambda^{\frac{1}{2}} \tag{4.10}
\]

**Remark 4.1 (Homogeneity in \(\Lambda\) of \(\ell_q, \tau_q\)).** \(\ell_q\), as well as the \(\ell_{q,i}\) defined in Section 8, are 0-homogeneous in \(\Lambda\), whereas \(\tau_q\) is \(1/2\)-homogeneous. The last property allows us to cancel the \(\Lambda^{1/2}\) factors we will see appearing in the course of the proof.

We also assume

\[
\frac{\delta_{q+1} \delta_q^\alpha}{\lambda_{q+1}} \leq \delta_{q+2}, \tag{4.12}
\]

which can be achieved if \(\alpha\) is sufficiently large assuming \((15\alpha + \beta\gamma)b < (b - 1)(1 - \beta - 2b\beta)\). Moreover, we assume

\[
\lambda_{q+1}^{-\frac{1}{\alpha}} \leq \ell_q \leq \lambda_{q+1}^{-\frac{1}{\alpha}}. \tag{4.13}
\]

The right inequality in (4.13) is evident from the definition. The left inequality can be reduced to \(-b < \beta b^2(1 + \gamma) + \beta - 1 - 2\alpha\), which can easily be verified for \(\alpha = 0 = \gamma\), and thus also for \(\alpha, \gamma\) sufficiently small. We will in fact need the following sharper bound:

\[
\lambda_{q+1}^{-\frac{1}{\alpha}} \leq \tau_q^{-\frac{1}{\alpha}}. \tag{4.14}
\]

which can be achieved by imposing the following condition:

\[
\lambda_{q+1}^{-\frac{1}{\alpha}} \leq \tau_q^{-\frac{1}{\alpha}}. \tag{4.15}
\]

The above conditions can be obtained by choosing, in this order

- \(b, \beta\) as in (4.12), so that in particular \(\beta(1 + b) < 1\);
- \(0 < \alpha, \gamma\) sufficiently small depending on \(b, \beta\);
- \(\mathbb{N} \in \mathbb{N}\) sufficiently large depending on \(b, \beta, \alpha, \gamma\) so as to get (4.15).

One last notational remark: \(A \lesssim B\) (resp. \(A \gtrsim B\)) will mean \(A \leq C(b, \beta, \alpha, \gamma, M)B\) (resp. \(A \geq C(b, \beta, \alpha, \gamma, M)B\)), or \(C(N, b, \beta, \alpha, \gamma, M)\) if norms depending on \(N\) are involved (e.g. \(C^{N+1}a\) norms). \(A \sim B\) will mean \(A \lesssim B\) and \(A \gtrsim B\). Note that \(C\) does not depend on \(a \gg 1\).
5 Main iterative propositions

In this section, we state the main propositions which allow us to pass from one kind of subsolution to another one, which is closer to the notion of solution. Both the below statements use the parameters $\delta, \lambda, \Lambda$ defined in the previous section. The combination of these propositions leads to our main theorem, as illustrated in Section [6].

In the first proposition it is shown that a smooth strict subsolution can be approximated with an adapted subsolution.

**Proposition 5.1 (From strict to adapted subsolutions).** Let $(\hat{v}, \hat{\rho}, \hat{R})$ be a smooth strict subsolution on $[0, T]$. Then, for any $0 < \hat{\rho} < \frac{1}{3}, \nu > \frac{1 - \hat{\rho}}{2\hat{\rho}},$ and $\delta, \sigma > 0$, there exist $\gamma, \Omega > 0$ and a $C^\beta$-adapted subsolution $(\hat{v}, \hat{\rho}, \hat{R})$ with parameters $\gamma, \Omega, \nu$ such that $\hat{\rho} \leq \frac{5}{4}\delta$ and, for all $t \in [0, T]$

$$\int_{\mathbb{T}^3} (|\partial_t \hat{v}|^2 + \text{tr} \hat{R}) \, dx = \int_{\mathbb{T}^3} (|\partial_t \hat{v}|^2 + \text{tr} \hat{R}) \, dx$$

$$\|\hat{v} - \hat{v}_c\| \lesssim 1 + \delta^2$$

$$\|\hat{v} - \hat{v}\|_{H^{-1}} < \sigma$$

Moreover, if we define

$$\hat{T}(t) := \int_0^t \int_{\mathbb{T}^3} \left( \left| (-\Delta)^{\sigma} v \right|^2 - \left| (-\Delta)^{\sigma} v \right|^2 \right) \, dx \, ds,$$

we have the bound

$$\|\partial_t \hat{T}\| \lesssim \sum_q \Lambda \lambda_q^{0+e-\beta}.$$  (5.5)

The $q = 0$ term of this sum is the largest, and is $\delta \lambda_1^{2\beta - 1}\lambda_0^{0+e-\beta}$, which is $a$-increasing. Since we can see that $a \to \infty$ for $\delta \to 0$, for any $\eta > 0$, it can only be ensured that

$$\|\hat{T}\| (t) \leq \eta \quad t \in [0, \hat{T}(\eta, \delta, \sigma)],$$

where $\hat{T}(\eta, \delta, \sigma) \sim \eta \delta^{-1} \lambda_1^{-2\beta} \lambda_0^{-0+e} \rightarrow 0$ if $a \to \infty$ or $\eta \to 0$.

The proof will be given in Section [10].

Next, we show that adapted subsolutions can be approximated by weak solutions with the same initial data.

**Proposition 5.2 (From adapted subsolutions to weak solutions).** Let $0 < \hat{\rho} < \frac{1}{3}, \gamma > 0,$ and $\nu > 0$ with

$$\frac{1 - 3\hat{\rho}}{2\hat{\rho}} < \nu < \frac{1 - 3\hat{\rho}}{2\hat{\rho}}.$$  (5.6)

The following holds for all $\delta < 1$.

If $(\hat{v}, \hat{\rho}, \hat{R})$ is a $C^\beta$-adapted subsolution with parameters $\gamma, \Omega, \nu$ and $\hat{\rho} \leq \frac{5}{4}\delta$, then, for all $\sigma > 0$, there exists a $C^\beta$ weak solution $v$ of (1.5) with initial datum

$$v(\cdot, 0) = \hat{v}(\cdot, 0)$$  (5.7)

and such that, for all $t \in [0, T]$

$$\int_{\mathbb{T}^3} |v|^2 \, dx = \int_{\mathbb{T}^3} (|\partial_t v|^2 + \text{tr} \hat{R}) \, dx$$

$$\|v - \hat{v}\|_c \lesssim \delta^2$$

$$\|v - \hat{v}\|_{H^{-1}} < \sigma$$  (5.10)
Moreover, if we define
\[ \mathcal{T}(t) := \int_0^t \int_{\mathbb{T}^3} \left( |(\Delta)^{\frac{2}{3}}v| - |(\Delta)^{\frac{2}{3}}v| \right) dx \, dt, \]
(5.11)
then we can see that
\[ \left| \partial_t \mathcal{T} \right| \lesssim \sum \Lambda \lambda_{\eta}^{0+6-\beta}, \]
(5.12)
so that, like in the previous proposition, for any \( \eta > 0 \), it can be ensured that
\[ \left| \mathcal{T}(t) \right| \leq \eta \quad \forall t \in [0,T(\eta, \delta, a)], \]
where \( T(\eta, \delta, a) \sim \eta \delta^{-1} \lambda_1^{-2} \lambda_0^{6-\theta-\epsilon} \rightarrow 0 \) if \( \eta \rightarrow 0 \) or \( a \rightarrow \infty \).

Finally, consider the family of strong subsolutions \( (\hat{v}, \hat{p}, \hat{R} + \epsilon/3 \text{Id}) \), where \( e : [0,T] \rightarrow \mathbb{R} \) satisfies the following conditions:
\[ e(t) \leq \frac{\delta}{2} - \hat{\rho}(t) \]
(5.13)
\[ \left| \partial_t e \right| \leq \sqrt{\delta_0 \lambda_0 e}. \]
(5.14)
\[ e \geq 0. \]  
(5.15)

This family can be used to yield infinitely many distinct weak solutions with the same initial data as \( (v, p) \).

The proof will be given in Section 11.

**Proposition 5.2** allows us to prove the following wildness criterion.

**Corollary 5.1 (Wildness criterion).** Let \( 0 < \beta < \frac{1}{3} \) and \( (\hat{v}, \hat{p}, \hat{R}) \) be a \( C^\beta \)-adapted subsolution such that \( \hat{\rho} \leq \frac{\delta}{2} \delta \) for some small \( \delta > 0 \) and \( \rho^{-1} |\partial_t \rho| \leq M \delta \) for some suitably large \( M > 0 \). Assume that the following admissibility condition is satisfied for all \( t \in [0,t_a] \) for some sufficiently small \( t_a \):
\[ \frac{1}{2} \int_{\mathbb{T}^3} \left( |\hat{v}|^2(x,t) + \text{tr} \hat{R}(x,t) \right) dx + \int_0^t \int_{\mathbb{T}^3} \left( |(\Delta)^{\frac{2}{3}}\hat{v}|^2(x,s) \right) dx \, ds \leq \frac{1}{2} \int_{\mathbb{T}^3} \left( |\hat{v}|^2(x,0) + \text{tr} \hat{R}(x,0) \right) dx, \]
(5.16)
with a strict inequality for at least some \( t \in [0,t_a] \). Then \( \hat{v}(x,0) \in W_{0,\beta-\epsilon,t_a} \) for any \( \epsilon > 0 \).

The existence of infinitely many \( C^{\beta-\epsilon} \) weak solutions with \( \hat{v}(x,0) \) as their initial datum is a consequence of **Proposition 5.2** above. The admissibility of those solutions follows from (5.16) as shown in the next section, where it is also seen that the strictness of (5.16) for at least some \( t \) is vital to the admissibility of the solutions. We shall henceforth adopt the following notational convention, already applied in the statements of the propositions:

- \( (\hat{v}, \hat{p}, \hat{R}) \) will always denote strict subsolutions;
- In Sections 6-7 \( (\hat{v}, \hat{p}, \hat{R}) \) will always denote strong subsolutions; in Sections 8-9 all subsolutions will be strong, and in Sections 10-11 the subscripts will mark strong subsolutions;
- \( (\hat{v}, \hat{p}, \hat{R}) \) will always denote adapted subsolutions, and \( \hat{\beta} \) will be the Hölder regularity of adapted subsolutions;
- \( (v, p) \) will always denote (weak) solutions.
6 Proof of the existence theorem

We start by recalling the following classical result.

**Theorem 6.1 (Existence of Leray solutions).** For any \( w \in L^2(\mathbb{T}^3) \) with \( \text{div} \, w = 0 \) and every \( \theta \in (0, 1) \) there is a weak solution \( v \in L^\infty(\mathbb{R}^+, L^2(\mathbb{T}^3)) \cap L^2(\mathbb{R}^+, H^0(\mathbb{T}^3)) \) of (1.5) such that \( v(\cdot, 0) = w \) and

\[
\frac{1}{2} \int_{\mathbb{T}^3} |v|^2(x,t) \, dx + \int_0^t \int_{\mathbb{T}^3} |(-\Delta)^{\theta} v|^2(x,\tau) \, dx \, d\tau \leq \frac{1}{2} \int_{\mathbb{T}^3} |w|^2(x) \, dx \quad \forall t \geq 0. \tag{6.1}
\]

In fact, the following form of energy inequality also holds:

\[
\frac{1}{2} \int_{\mathbb{T}^3} |v|^2(x,t) \, dx + \int_s^t \int_{\mathbb{T}^3} |(-\Delta)^{\theta} v|^2(x,\tau) \, dx \, d\tau \leq \frac{1}{2} \int_{\mathbb{T}^3} |v|^2(x,s) \, dx \quad a.e. \, s, \forall t > s.
\]

Recalling the **Definition 3.1** of subsolutions and strict subsolutions, one can prove the following existence result.

**Lemma 6.1 (Existence of strict subsolutions).** Let \( w \in L^2(\mathbb{T}^3) \) with \( \text{div} \, w = 0 \). For any \( \delta > 0 \) there exists a smooth strict subsolution \((\tilde{v}, \tilde{p}, \tilde{R})\) defined on \([0, T)\) such that

\[
||\tilde{v}|_{t=0} - w||_{L^2(\mathbb{T}^3)} \leq \delta, \tag{6.2}
\]

and for all \( t \in [0, T) \)

\[
\frac{1}{2} \int_{\mathbb{T}^3} (|\tilde{v}|^2(x,t) + \text{tr} \, R(x,t)) \, dx + \int_0^t \int_{\mathbb{T}^3} (-\Delta)^{\frac{\beta}{2}} |\tilde{v}|^2(x,\tau) \, dx \, d\tau \leq \frac{1}{2} \int_{\mathbb{T}^3} |\tilde{v}|^2(x) \, dx + \delta. \tag{6.3}
\]

The proof is an adaptation of the one of [32, Lemma 6.8, p. 38], and is reported in Appendix A. The proof of the main result then follows the steps of [14, Section 4], taking care of the additional dissipation by suitably increasing the energy of the starting strict subsolution.

**Proof. (Theorem 1.1)**

We choose \( \eta > 0, \theta < \beta' < \beta, w \in L^2 \) with \( \text{div} \, w = 0 \). Using the above result, we obtain a smooth strict subsolution \((\tilde{v}', \tilde{p}', \tilde{R}')\) on \([0, T]\) such that (6.2)-(6.3) hold for some \( \delta > 0 \) which we will fix later. We now note that adding a smooth time-dependent non-negative multiple of the identity to \( \tilde{R}' \) does not change the fact that \((\tilde{v}', \tilde{p}', \tilde{R}')\) is a smooth strict subsolution. We may thus substitute our strict subsolution with \((\tilde{v}, \tilde{p}, \tilde{R}) := (\tilde{v}', \tilde{p}', \tilde{R}' + 2(3|\mathbb{T}^3|^{-1}e_K(t)) \text{Id})\), where \( K \) is a constant to be specified later in this proof, \( 0 \leq e_K(t) \leq \delta/2 - Kt \), and \( e_K(0) = \delta/2 \). Combining the choice of \( e_K \) with (6.3), we obtain the following relations for \( \delta/2 - Kt > 0 \):

\[
\frac{1}{2} \int_{\mathbb{T}^3} (|\tilde{v}|^2(x,0) + \text{tr} \, R(x,0)) \, dx = \frac{1}{2} \int_{\mathbb{T}^3} |w|^2(x) \, dx + \frac{3}{2} \delta \tag{6.4}
\]

\[
\frac{1}{2} \int_{\mathbb{T}^3} (|\tilde{v}|^2(x,t) + \text{tr} \, R(x,t)) \, dx + \int_0^t \int_{\mathbb{T}^3} (-\Delta)^{\frac{\beta}{2}} |\tilde{v}|^2(x,\tau) \, dx \, d\tau \leq \frac{1}{2} \int_{\mathbb{T}^3} |\tilde{v}|^2(x) \, dx + \frac{3}{2} \delta - Kt. \tag{6.5}
\]

Indeed, passing from \((\tilde{v}', \tilde{p}', \tilde{R}')\) to \((\tilde{v}, \tilde{p}, \tilde{R})\) adds a term \( e_K \) to the left-hand side, since \( \text{tr} \, \tilde{R} = \text{tr} \, \tilde{R}' + 2|\mathbb{T}^3|^{-1}e_K \). Now let \( \tilde{v}_0 \) be the initial datum of \( \tilde{v} \), and note that

\[
\int_{\mathbb{T}^3} \text{tr} \, R(x,0) \, dx = ||w||_{L^2}^2 - ||\tilde{v}_0||_{L^2}^2 + 3\delta \leq ||w - \tilde{v}_0||_{L^2} (||w||_{L^2} + ||\tilde{v}_0||_{L^2}) + 3\delta \leq \delta/2 ||w||_{L^2} + 3\delta \leq C(w)\delta. \tag{6.6}
\]
Using Proposition 5.1 and Proposition 5.2, we can produce a $C^\beta$-adapted subsolution $(\hat{v}, \hat{\rho}, \hat{R})$ and a $C^\beta$ weak solution $(v, p)$, satisfying the integral equalities (5.11) and (5.8) and the $H^{-1}$ estimates (5.3) and (5.10), and the functions $\hat{\mathcal{F}}, \hat{\mathcal{J}}$ of (5.4) and (5.11). Recall that we have that

$$\int_{T^3} |(\hat{v}|^2 + \text{tr} \hat{R})(x,t)\, dx = \int_{T^3} |(v|)|^2 + \text{tr} \hat{R})(x,t)\, dx = \int_{T^3} |v(x,t)|^2\, dx,$$

(6.7)

and thus

$$\|v(t)\|^2_2 - \|\hat{v}(t)\|^2_2 = \int \text{tr} \hat{R}(x,t)\, dx.$$  (6.8)

Call $v_0$ the initial datum of $v$ and of $\hat{v}$, and note that, by (5.3), (6.8) and (6.6), we have that

$$\|v_0 - \hat{v}_0\|^2_2 - \|\hat{v}_0\|^2_2 = 2 \int \hat{v}_0 \cdot (v_0 - \hat{v}_0)\, dx \leq C(w)\delta + 2\|\hat{v}_0\|_{H^1}.$$  

Thus, we first choose $\delta$ sufficiently small so that $C(w)\delta < \frac{\eta^2}{2}$ and obtain $\hat{v}$, then we fix $\sigma < \frac{\eta^2}{4\|\hat{v}_0\|_{H^1}}$ and obtain $\hat{v}$, and finally we conclude that

$$\|v_0 - \hat{v}_0\|^2_2 \leq \eta^2 \implies \|\hat{v}_0 - v_0\|_{L^2} \leq \eta.$$  

As for the admissibility condition, choosing $K$ so that $|\partial_t(\mathcal{F} + \hat{\mathcal{J}})| \leq K - 1$, as is made possible by (5.5) and (5.12), we have that

$$\frac{1}{2} \int_{T^3} |v(x,t)|^2\, dx + \int_0^t \int_{T^3} \left| (-\Delta)^{\frac{\delta}{2}} v \right|^2 (x,s)\, ds\, dx$$

$$\leq \int_{T^3} \frac{1}{2} |w|^2(x)\, dx + \frac{3}{2}\delta - t$$

$$\leq \frac{1}{2} \int_{T^3} \left( |\hat{v}_0|^2 + \text{tr} \hat{R}(x,0) \right)\, dx$$

(6.8)

$$\leq \frac{1}{2} \int_{T^3} |v|^2(x,0)\, dx,$$

where the second-last inequality is strict for all $t \neq 0$ where (6.8) is valid. This yields the energy inequality for $t$ sufficiently small. Since we can only estimate $|\partial_t(\mathcal{F} + \hat{\mathcal{J}})|$ with a quantity which is potentially unbounded as $\delta \to 0$ (as seen in (5.5) and (5.12)), and $C(w)\delta < \frac{\eta^2}{2}$ implies $\delta \to 0$ as $\eta \to 0$, we conclude that our time $T(\eta)$ of guaranteed admissibility satisfies

$$\lim_{\eta \to 0} T(\eta) = 0.$$  

So far, we have only obtained one solution for each $\eta$. Suppose that, from $(\hat{v}, \hat{\rho}, \hat{R})$, we produced the adapted subsolution $(\hat{v}, \hat{\rho}, \hat{R})$, and from there the solution $(v, p)$. As noted in Proposition 5.2 considering

$$(\hat{v}', \hat{\rho}', \hat{R}') := (\hat{v}, \hat{\rho}, \hat{R} + \frac{e}{3} \text{Id})$$

with $e$ satisfying a suitable set of conditions, we can obtain more weak solutions and ensure these solutions are admissible up to $T(\eta)$. The required conditions are listed below.

1. The first condition ensures $\hat{R}'(\cdot, 0) \equiv 0$:

   $$e(0) = 0;$$

2. The second one ensures $\text{tr}(\hat{R}') \geq 0$:

   $$e(t) \geq 0;$$

   it would be enough to require $\hat{\rho}' \geq |e|$, but we exclude $e < 0$ for convenience (cfr. Step 3 of Section 11):
3. The third one ensures the admissibility of the new solutions:

\[
\frac{1}{2}|T^3|e(t) \leq \frac{1}{2} \int_{T^3} \left(|\tilde{v}(x,0)|^2 + \text{tr} \tilde{R}(x,0) - |\tilde{v}(x,t)|^2 - \text{tr} \tilde{R}(x,t)\right) dx
\]

\[
- \int_0^1 \int_{T^3} \left|(-\Delta)^{\frac{d}{2}} \tilde{v}(x,s)\right|^2 dx ds - \mathcal{F} - K_t,
\]

\(K\) being the same constant used to find \((v,p)\); since the right-hand side of the above inequality is strictly positive for all \(t \neq 0\) where \((v,p)\) is admissible, this condition is compatible with requiring that \(e \geq 0\) as done above;

4. The last conditions are (5.13)-(5.15).

This completes the proof. \(\Diamond\)

7 From strict to strong subsolutions

We state here an analogue of [14, Proposition 3.1].

**Proposition 7.1.** Let \((\tilde{v}, \tilde{p}, \tilde{R})\) be a smooth solution of (3.1), and \(S \in C^\infty(T^3 \times [0,T]; S^3_{+})\) be a smooth positive-definite matrix field. Fix \(\alpha \in (0,1)\) and \(\varepsilon > 0\). Then for any \(\lambda > 1\) there exists a smooth solution \((\tilde{v}, \tilde{p}, \tilde{R})\) of (3.1) with

\[
(\tilde{v}, \tilde{p}, \tilde{R}) = (\tilde{v}, \tilde{p}, \tilde{R}) \quad \text{for } t \notin \text{supp } \text{tr } S,
\]

\[
\int_{T^3} |\tilde{v}(x,t)|^2 + \text{tr} \tilde{R}(x,t) dx = \int_{T^3} |\tilde{v}(x,t)|^2 + \text{tr} \tilde{R}(x,t) dx \quad \forall t \in [0,T],
\]

and the following estimates hold

\[
||\tilde{v} - \tilde{v}||_{H^{1}} \leq \frac{C}{\lambda},
\]

(7.3)

\[
||\tilde{v}||_{k} \leq C\lambda^k \quad k = 1, 2
\]

(7.4)

\[
||\tilde{R} - \tilde{R} - S||_{H^{0}} \leq \frac{C}{\lambda^{1-\alpha-N}}
\]

(7.5)

Moreover, \(\text{tr}(\tilde{R}(x,t) - \tilde{R}(x,t) - S(x,t)) = \mu(t)\) is a function of \(t\) only and satisfies

\[
|\mu'(t)| \leq C\lambda^\alpha.
\]

(7.6)

The constant \(C \geq 1\) above depends on \((v,p,R), S\) and \(\alpha\), but not on \(\lambda\). Finally, defining

\[
\tilde{\mathcal{F}}(t) := \int_0^t \int_{T^3} \left(\left|(-\Delta)^{\frac{d}{2}} \tilde{v}(x,s)\right|^2 - \left|(-\Delta)^{\frac{d}{2}} \tilde{v}(x,s)\right|^2\right) dx ds
\]

we have that

\[
|\partial_s \tilde{\mathcal{F}}(t)| \leq C\lambda^{2(0+\varepsilon)}.
\]

(7.7)

**Proof.**

Define the inverse flow of \(\tilde{v}, \Phi : T^3 \times [0,T] \rightarrow T^3\), as the solution of

\[
\begin{align*}
\frac{\partial}{\partial t} \Phi(x,t) + (\tilde{v} \cdot \nabla) \Phi(x,t) &= 0 \\
\Phi(x,0) &= x \quad x \in T^3,
\end{align*}
\]

Define
Lemma 2.2

and set

\[ \mathcal{R}(x,t) = \mathcal{D}\Phi(x,t)S(x,t)\mathcal{D}^T\Phi(x,t). \]

Observe that \( \mathcal{R} \) is defined on the compact set \( \mathbb{T}^3 \times [0,T] \) and, being continuous, has a compact image \( \mathcal{R}_0 := \mathcal{R}(\mathbb{T}^3 \times [0,T]) \subseteq \mathcal{S}^{1,3}_+ \).

By Lemma 2.2 there exists a smooth vector field \( W : \mathcal{R}_0 \times \mathbb{T}^3 \rightarrow \mathbb{T}^3 \) satisfying the differential equations (2.6) and the integral equations (2.7). Define

\[ w_o(x,t) = \mathcal{D}\Phi^{-1}W(\mathcal{R}_0,\lambda \Phi(x,t)) \]
\[ w_c(x,t) = \frac{1}{\lambda} \text{curl}(\mathcal{D}^T\Phi U(\mathcal{R}_0,\lambda \Phi(x,t))) - w_o, \]

where \( U = U(R, \xi) \) is defined as in (2.12) and thus satisfies \( \text{curl}_{\xi} U = W \). Moreover, set

\[ \bar{v} := \tilde{v} + w_o + w_c \quad \bar{p} = \tilde{p} + \mathcal{P} \quad \bar{R} := R - \mathcal{S} - \mathcal{E}^{(1)} - \mathcal{E}^{(2)}, \]

where

\[ \mathcal{P} := -\frac{1}{3}(w_c \cdot \bar{v} + w_o \cdot w_c) \]
\[ \mathcal{E}^{(1)} := \mathcal{R}(F) + (w_c \otimes \tilde{v} + w_o \otimes w_c + \mathcal{P} \text{Id}) \]
\[ F := \text{div}(w_o \otimes w_o - S) + (\partial_t + \bar{v} \cdot \nabla)w_o + \partial_t \tilde{v} + (-\Delta)^0(w_o + w_c) \]
\[ \mathcal{E}^{(2)} := \frac{1}{3} \int_{\mathbb{T}^3} \left( |\tilde{v}|^2 - |\bar{v}|^2 - \text{tr} \mathcal{S} \right) dx \cdot \text{Id}, \]

with \( \mathcal{R} \) defined as in (2.13). By construction, the relation (7.2) holds, \( \mathcal{E}^{(1)} \) is traceless, \( \mathcal{E}^{(2)} \) is only \( t \)-dependent, and \( (\bar{v}, \bar{p}, \bar{R}) \) solves (3.1). To verify this last claim, we can see that

\[ \text{div} \mathcal{E}^{(1)} = \text{div}(\tilde{v} \otimes \bar{v} - \tilde{v} \otimes \bar{v} + S + \mathcal{P} \text{Id}) + \partial_t \tilde{v} + (-\Delta)^0(w_o + w_c) \]
\[ = \partial_t \bar{v} + \text{div}(\tilde{v} \otimes \bar{v} - S + \bar{R}) + \nabla \bar{p} + (-\Delta)^0 \bar{v}. \]

We call \( w := w_o + w_c = \bar{v} - \tilde{v} \). Recall that

\[ W(R, \xi) = \sum_{k \neq 0} a_k(R)A_k e^{ik \cdot \xi} \]
\[ U(R, \xi) = \sum_k a_k(R)\frac{|k| \times A_k e^{ik \cdot \xi}}{|k|^2}, \]

which are respectively (2.8) and (2.12), with the \( a_k \) satisfying (2.10). This allows us to decompose

\[ w_o = \sum_{k \neq 0} \mathcal{D}\Phi^{-1}a_k(\mathcal{D}\Phi \mathcal{D}^T\Phi)A_k e^{ik \cdot \lambda \Phi} = \sum_{k \neq 0} b_k e^{ik \lambda \Phi} \tag{7.8} \]
\[ w_c = \frac{i}{\lambda} \sum_{k \neq 0} \nabla(a_k(\mathcal{D}\Phi \mathcal{D}^T\Phi)) \times \frac{\mathcal{D}^T\Phi(k \times A_k)}{|k|^2} e^{ik \lambda \Phi} = \sum_{k \neq 0} c_k e^{ik \lambda \Phi}. \tag{7.9} \]

The estimate (7.5) is deduced by combining arguments from [14] with estimates for \( \mathcal{E}^{(2)} \) and \( \mathcal{R}((-\Delta)^0w) \). \( \mathcal{E}^{(2)} \) is estimated in a similar fashion to how we estimate \( \mathcal{E}^{(1)} \) below. To estimate \( \mathcal{R}((-\Delta)^0w) \), using the fact that \( [\mathcal{R}, (-\Delta)^0] = 0 \) and (2.17), we see that

\[ \left\| \mathcal{R}((-\Delta)^0w) \right\|_{0} \lesssim \left\| \mathcal{R}w \right\|_{20 + \frac{\pi}{4}} \lesssim \sum_k \left( \frac{\|b_k + \frac{1}{\lambda} c_k\|_0}{|k|^{1-\frac{\alpha}{20}}} + \frac{\|b_k + \frac{1}{\lambda} c_k\|_{N+20 + \frac{\pi}{4}}}{|k|^{N-20-\frac{\pi}{4}}} \right) + \frac{\|b_k + \frac{1}{\lambda} c_k\|_{N+20 + \frac{\pi}{4}}}{|k|^{N-20-\frac{\pi}{4}}} \right) \]

\[ \lesssim \lambda^{\frac{\pi}{4} + 20 - 1} \sum_k \left( \frac{1}{|k|^{1-\frac{\alpha}{20}}} + \frac{1 + C_{\Phi}(N, \alpha, \theta)}{|k|^{N-1 - \frac{6-20}{4}}} \right). \]
where we used (2.10) to get the extra $|k|^{-6}$ in each term, and the boundedness of $\Phi$ to get the $C_\Phi(N, \alpha, \vartheta)$. Concerning (7.4), the smoothness of $\Phi, S$ combined with (2.10) gives us

$$\max \{\|c_k\|_N, \|b_k\|_N\} \lesssim |k|^{-m},$$

(7.10)

for all integers $m > 0$, where the $b_k$ and $c_k$ are as in the decompositions of $w_o, w_c$ above. This easily allows us to conclude that

$$\|w\|_N \lesssim \lambda^N,$$

since differentiating the exponential gives us a factor of $\lambda$ for each derivative. We then note that

$$\|\vec{\nu}\|_N \leq \|\vec{\nu}\|_N + \|w\|_N \lesssim 1 + \lambda^N \lesssim \lambda^N,$$

where the second step used the smoothness of $\vec{\nu}$. For $N = 1, 2$ the above reduces to (7.4).

Estimate (7.3) is proved separately for $w_c$ and $w_o$. The former is straightforward, since $w_c$ is already of order $\lambda^{-1}$ in $L^2 \to H^{-1}$ thanks to (7.9), and (7.3) is an $H^{-1}$ estimate. For the latter, the main idea is to write

$$e^{\partial_t \Phi \cdot k} = \frac{\partial_i e^{\partial_i \Phi \cdot k}}{i \lambda \partial_i \Phi \cdot k},$$

and integrating by parts. One must then make sure that $j = j(x)$ is chosen in such a way that the denominator is bounded from below. This can be done by using the fact that $\Phi$ is a diffeomorphism, which implies $|\nabla \Phi|$ is bounded below.

To continue, we note that $\mu = \text{tr} \, \bar{\epsilon}^{(2)} = \int [\vec{\nu}]^2 - |\vec{\nu}|^2 - \text{tr} S \text{d}x$. $\vec{\nu}$ and $\text{tr} S$ are both smooth, so they are bounded. In order to estimate $\int |\vec{v}(r)|^2$, note that the following energy identity for $\vec{v}$ follows from (3.1):

$$\partial_t \frac{1}{2} |\vec{v}|^2 + \text{div} \left( \vec{v} \left( \frac{|\vec{v}|^2}{2} + \bar{\rho} \right) \right) + \vec{v} \cdot (-\Delta) \vec{v} = -\vec{v} \cdot \text{div}(\hat{R} - R((-\Delta) \nu w)).$$

Moreover, with the arguments used to estimate $R_{11}$ in [14, pp. 18-20], we conclude that

$$\left\| \bar{\epsilon}^{(1)} - R((-\Delta) \nu w) \right\|_0 \lesssim \lambda^{\mp 1}.$$

These two bounds, by integrating in $x$ and using (7.4), yield

$$\left| \frac{d}{dr} \int \frac{1}{2} |\vec{v}|^2 \text{d}x \right| \lesssim \int \left| \vec{v} \right| (-\Delta) \vec{v} + |\nabla \vec{v}| \hat{R} - R((-\Delta) \nu w) \text{d}x \leq C(1 + \lambda^{\mp 1}).$$

Thus, the estimate (7.6) is proved.

The last thing left is to estimate $|\partial_i \vec{\nu}|$. Combining some simple calculations with (7.4), (2.17), (2.18), and (2.2), we obtain that

$$|\partial_i \vec{\nu}| = \int_{\mathbb{T}^d} \left| (-\Delta)^{\frac{\alpha}{2}} \vec{v} \right|^2 - \left| (-\Delta)^{\frac{\alpha}{2}} \vec{v} \right|^2 \text{d}x \text{d}s = \int_{\mathbb{T}^3} \left| 2 (-\Delta)^{\frac{\alpha}{2}} \vec{v} + (-\Delta)^{\frac{\alpha}{2}} w \cdot (-\Delta)^{\frac{\alpha}{2}} w \text{d}x \text{d}s \right|$$

$$\lesssim \int_{\mathbb{T}^3} \left| 2 (-\Delta)^{\frac{\alpha}{2}} \vec{v} \cdot (-\Delta)^{\frac{\alpha}{2}} w \text{d}x \text{d}s \right| + \left| \nabla \left( \vec{v} \right) \right|_{\text{tr}} \|w\|_{\text{tr}} + \lambda^{2(\alpha + \epsilon)}.$$

The velocity $\vec{v}$ is bounded by smoothness, so $I \leq K(\vec{v}, \theta, \epsilon) \lambda^{\alpha + \epsilon}$. Since $\lambda > 1$, this yields (7.7), thus concluding the proof. \hfill \Box

**Proposition 7.1** will be applied in the situation described by the following corollary.

**Corollary 7.1 (Strict to strong).** Let $(\vec{v}, \vec{p}, \hat{R})$ be a smooth strict subsolution on $[0, T]$. There exist $\delta, \gamma > 0$ such that the following holds.
For any $0 < \delta < \hat{\delta}$, $\alpha, \gamma > 0$ and $0 < \epsilon < \beta - \theta$ sufficiently small, there exists a smooth strong subsolution $(\bar{v}, \bar{\rho}, \bar{R})$ with $\bar{R}(x,t) = \bar{p}(t)\text{Id} + \bar{R}(x,t)$, and a “dissipative trace term” as isolated in Proposition 7.1, i.e.

$$\mathfrak{F}(t) := \int_0^t \int_{\mathbb{T}^3} \left( \left| (-\Delta)^\theta \bar{v}(x,s) \right|^2 - \left| (-\Delta)^\theta \bar{v}(x,s) \right|^2 \right) dxds,$$

such that, for all $t \in [0, T]$

$$\int_{\mathbb{T}^3} \left( |\bar{v}(x,t)|^2 + \text{tr} \bar{R}(x,t) \right) dx = \int_{\mathbb{T}^3} \left( |\bar{v}(x,t)|^2 + \text{tr} \bar{R}(x,t) \right) dx$$

(7.11)

$$\frac{3}{4} \delta \leq \bar{\rho} \leq \frac{5}{4} \delta$$

(7.12)

$$|\bar{R}| \leq \Lambda_0^{1+\gamma}$$

(7.13)

$$||\bar{v} - \bar{v}||_{H^1} \leq \delta_0^{-1}$$

(7.14)

$$||\bar{v}||_{H^{1+a}} \leq \delta_0^{1+a}$$

(7.15)

$$|\partial_t \bar{p}(t)| \leq \delta_0^{1-a}$$

(7.16)

$$|\partial_t \mathfrak{F}(t)| \leq \Lambda_0^{1+\gamma} \delta_0^{1+\epsilon}$$

(7.17)

$$||\bar{v}||_{0+\epsilon} \leq K(1 + \delta_0^{1+\epsilon})$$

(7.18)

where the constant $K$ depends on $(\bar{v}, \bar{\rho}, \bar{R})$ and $\epsilon$, the parameters $\delta_q, \lambda_q, \xi_q, \Lambda$ are defined as in (4.1) with sufficiently large $a$, and $\alpha$ is the small parameter from Section 3.

Proof.

Let

$$\hat{\delta} = \frac{1}{2} \inf \left\{ \bar{R}(x,t) \xi : |\xi| = 1, x \in \mathbb{T}^3, t \in [0, T] \right\}.$$

Since $\bar{R}$ is a smooth positive definite tensor on a compact set, $\hat{\delta} > 0$. Then $S := \bar{R} - \hat{\delta}\text{Id}$ is positive definite for any $\delta < \hat{\delta}$. We may in addition assume without loss of generality that $\delta \leq 1$. We apply Proposition 7.1 with $(\bar{v}, \bar{\rho}, \bar{R}), S,$ and $\bar{u} \in (0, 1), \epsilon > 0$ to be chosen below. This yields a smooth solution $(\bar{v}, \bar{\rho}, \bar{R})$ of (3.1) with properties (7.2), (7.3), (7.5), and (7.6). We first note that (7.11) coincides with (7.2). Next, we observe that $\bar{R} - \bar{R} + S = \bar{R} - \delta \text{Id}$, so that, since $\mu(t) = \text{tr}(\bar{R} - \bar{R} + S)$ is a function of time only, the function

$$\bar{p} = \frac{1}{3} \text{tr}(\bar{R} - \bar{R} + S) + \delta$$

(7.19)

is independent of $x$.

Let us now prove (7.12). By the above and (7.5) for $N = 0$, we have that

$$|\bar{p} - \delta| = \frac{1}{3} |\text{tr}(\bar{R} - \bar{R} + S)| \leq ||\bar{R} - \bar{R} + S||_0 \leq C\lambda^{20+\gamma-1}$$

(7.20)

We require now the following condition on $\lambda$:

$$C\lambda^{20-1+\gamma} \leq \delta_0^{1+\gamma}$$

(7.21)

Then we notice that, for $\gamma$ sufficiently small and $a$ sufficiently large, we have that

$$\delta_0^{1+\gamma} \leq \frac{1}{4} \delta_0^{1+\gamma}$$

(7.22)
Indeed, rewriting the above in terms of $\lambda_q$, it reads
\[ \Lambda^{\frac{1}{2}} \lambda_{q}^{-\beta + 2\theta + \varpi - 1} \leq \Lambda \lambda_{q}^{-2b\beta(1 + \gamma)}. \]

Since $\Lambda \geq 1$ by (4.3), this reduces to showing that
\[ -\beta + 2\theta + \varpi - 1 < -2b\beta(1 + \gamma). \]  

and taking $a$ sufficiently large. In turn, (7.23) can be proved using that, by assumption, $\theta < \beta, 2b\beta < 1 - \beta$ (see (4.2)), and taking $\bar{\gamma}, \gamma$ sufficiently small. Thus (7.22) is proved.

Now from (7.20), (7.21), and (7.22) for $q = 0$, it follows that
\[ |\check{\rho} - \delta| \leq \frac{1}{4} \delta \xi_1^2 = \frac{1}{4} \Lambda^{-\gamma} \delta^{1+\gamma} \leq \frac{1}{4} \delta, \]  

where in the last inequality we used the fact that $\delta < 1 < \Lambda$. We have thus proved (7.12).

From this estimate we can in turn deduce (7.13). Indeed, since $\check{\mathbf{R}} = \check{\mathbf{R}} - \check{\mathbf{\dot{R}}} + \check{\mathbf{S}}$, by chaining the inequalities (7.5) for $N = 0$, (7.21), and (7.22) for $q = 0$, we analogously deduce that:
\[ \left| \check{\mathbf{R}} \right| \leq \frac{1}{4} \delta \xi_1^2 = \frac{1}{4} \Lambda^{-\gamma} \delta^{1+\gamma} \leq \left( \frac{3}{4} \right)^{1+\gamma} \delta^{1+\gamma} \Lambda^{-\gamma} \leq \check{\rho}^{1+\gamma} \Lambda^{-\gamma} \leq \Lambda \check{\rho}^{1+\gamma}. \]  

The bound (7.14) follows from (7.3) together with the following condition on $\lambda$:
\[ C \lambda^{-1} \leq \delta \lambda_0^{-1}. \]  

(7.25)

To obtain (7.15), we first use standard interpolation estimates together with (7.3) to obtain that
\[ \| \check{\mathbf{v}} \|_{1+\alpha} \leq C_\varpi \| \mathbf{v} \|_{1-\alpha} \| \check{\mathbf{v}} \|_{2} \leq C_\varpi C \lambda^{1+\alpha}. \]

Therefore, (7.15) reduces to the following condition on $\lambda$:
\[ C C_\varpi \lambda^{1+\alpha} \leq \delta \lambda_0^{-1}. \]  

(7.26)

The estimate (7.16) follows from (7.6) and (7.19), giving
\[ |\partial_t \check{\mathbf{p}}| = \frac{1}{3} \left| \partial_t \text{tr}(\check{\mathbf{R}} - \check{\mathbf{\dot{R}}} + \check{\mathbf{S}}) \right| \leq \frac{C}{3} \lambda^{\varpi}. \]

Therefore, (7.16) amounts to
\[ \frac{C}{3} \lambda^{\varpi} \leq \delta \Lambda^\gamma \lambda_0^{-\beta}. \]  

(7.27)

Since by (7.7) one has that $|\partial_t \check{\mathbf{f}}| \leq C \lambda^{2(\theta + \epsilon)}$, to obtain (7.17) we require
\[ C \lambda^{2(\theta + \epsilon)} \leq \Lambda^{\frac{1}{2}} \delta \lambda_0^{\theta + \epsilon}. \]  

(7.28)

Finally, to obtain (7.18), we note that $\check{\mathbf{v}}$ is smooth and thus bounded by a constant $C_0$, so that, by interpolation and (7.4), we have that
\[ \| \check{\mathbf{v}} \|_{\theta + \epsilon} \leq C_\varpi \| \mathbf{v} \|_{1-\theta-\epsilon} \| \check{\mathbf{v}} \|_{1} \leq C_\varpi C_0^{1-\theta-\epsilon} (C \lambda)^{\theta + \epsilon}. \]

Therefore, we will require
\[ C_\varpi C_0^{1-\theta-\epsilon} (C \lambda)^{\theta + \epsilon} \leq \delta \lambda_0^{\theta + \epsilon}. \]

(7.29)

To conclude the proof of the corollary, we now show that, for suitable choices of $\check{\delta}, \gamma, \varpi$, there exists a $\lambda$ satisfying conditions (7.21), (7.25), (7.26), (7.27), (7.28), and (7.29).
In particular, for fixed constants $\overline{C}, \overline{K}$ independent of the parameters $a, \delta, b, \overline{\alpha}, \overline{\beta}$, the following conditions must be satisfied by $\lambda$:

\[
\begin{align*}
\lambda & \geq \overline{K} \beta_0^{\frac{1}{3(2b-1)+1}} \lambda_0 \quad (7.30) \\
\lambda & \geq \overline{K} \delta^{-1} \lambda_0 \quad (7.31) \\
\lambda & \leq \overline{C} \delta^{\frac{1}{\theta_0^{1+\theta}}} \lambda_0 \quad (7.32) \\
\lambda & \leq \overline{C} \delta^{\frac{1}{\theta_0^{1+\theta}}} \lambda_0 \quad (7.33) \\
\lambda & \leq \overline{C} \Lambda^{\frac{1}{\theta_0^{1+\theta}}} \lambda_0^{\frac{1}{\theta_0^{1+\theta}}} \quad (7.34) \\
\lambda & \leq \overline{C} \delta^{\frac{1}{\theta_0^{1+\theta}}} \lambda_0 \quad (7.35)
\end{align*}
\]

First we choose $\delta < 1$, and $\overline{\alpha}, \overline{\beta}$ sufficiently small, and then show that, for $a$ sufficiently large there exists a $\lambda$ satisfying all the above inequalities.

First of all, notice that, since $\delta_0 = \delta_0^{2\bar{b}(b-1)} > 1$ if $\delta$ is fixed and $a$ is sufficiently large, then $(7.31)$ implies $(7.30)$, and $(7.32)$ implies $(7.35)$ independently of the choice of $\alpha > 0$, since $\theta + \varepsilon < \beta < \frac{1}{3} < 1 + \alpha$.

Hence, we are left with showing that $(7.31)$ is compatible with $(7.32)$-$(7.34)$.

The compatibility of $(7.31)$ and $(7.32)$, independently of $\alpha > 0$, is straightforward, since $\delta_0 \gg 1$ when $a$ is sufficiently large.

Inequality $(7.33)$ does not contradict $(7.31)$ provided we choose $\overline{\alpha}$ so small that $\frac{1-\beta}{a} > 1$, and then $a$ sufficiently large.

The compatibility of $(7.31)$ with $(7.34)$ rewrites as

\[ \lambda_0^{\frac{1}{3}} \leq \frac{\overline{C}}{\overline{K}} \Lambda^{\frac{1}{\theta_0^{1+\theta}}} \delta_0^{\frac{1}{\theta_0^{1+\theta}}} \]

and, inserting the definitions of $\delta_0, \Lambda, \lambda_1$, as

\[ \lambda_0^{\frac{1}{3}} \leq \frac{\overline{C}}{\overline{K}} \delta_0^{\frac{1}{\theta_0^{1+\theta}}} \lambda_0^{\frac{1}{\theta_0^{1+\theta}}} \Lambda^{\frac{1}{\theta_0^{1+\theta}}} \theta_0^{\frac{1}{\theta_0^{1+\theta}}} \left( \frac{\varepsilon}{\theta_0^{1+\theta}} \right) \lambda_0^{\frac{1}{\theta_0^{1+\theta}}} \]

Hence the above reduces to showing that

\[ \frac{1}{2} \leq \frac{\beta(2b-1)}{2(\theta + \varepsilon)} \]

which holds since $b > 1$ and $\theta + \varepsilon < \beta$.

The proof is thus complete. \( \diamond \)

8 Localized gluing step

Definition 8.1 (Decomposing the time interval). Let $0 \leq T_1 < T_2 \leq T$ such that $T_2 - T_1 > 4\tau_q$. We define sequences of intervals $\{I_i\}, \{J_i\}$ as follows. Let

\[ t_i := i\tau_q \quad I_i := \left[ t_i + \frac{2}{3}\tau_q, t_i + \frac{1}{3}\tau_q \right] \cap [0, T], \quad (8.1) \]

and let

\[ \mathcal{I} := \left\{ \min \left\{ i : t_i + \frac{2}{3}\tau_q \geq T_1 \right\} \right\} \quad T_1 > 0 \quad \mathcal{I} := \left\{ i : t_i + \frac{2}{3}\tau_q \leq T_2 \right\} \quad (8.2) \]

Moreover, define

\[ J_i := \left( t_i - \frac{1}{3}\tau_q, t_i + \frac{1}{3}\tau_q \right) \cap [0, T], \quad \mathcal{J} := \left( t_i + \frac{2}{3}\tau_q, T \right), \quad (8.3) \]

\[ J_{n+1} := \left( 0, t_{n+1} - \frac{2}{3}\tau_q \right) \quad J_{n+1} := \left( t_{n+1} - \frac{2}{3}\tau_q, T \right) \]

\[ J_{n+1} := \left( t_{n+1} - \frac{2}{3}\tau_q, T \right) \quad (8.3) \]

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These form a pairwise disjoint decomposition of \([0, T]\):
\[
[0, T] = \bigcup_{n \geq 0} I^n_{n-1} \cup I^n_{n} \cup \ldots \cup J^n_{n+1},
\]
and
\[
t_n < T_1 + \frac{5}{3}\tau_q < T_2 - \frac{5}{3}\tau_q < t_n.
\]
Moreover, if \(T_1 > 0, n \geq 1\), otherwise we have both that \(n = 0\) and that \(J^n_{-1} \cup I^n_{-1} = \emptyset\).

Given a subsolution \((v_q, p_q, R_q)\) with \(p_q := \frac{1}{3} \text{tr} R_q\) and \(q_q := \Lambda^{-1} p_q\), we will define:
\[
(p_{q,i}, q_{q,i}, \ell_{q,i}) := \left( p_q(t_i), q_q(t_i), \frac{1 + \epsilon}{\xi^q(q_q)^{1+\alpha} q^{-\alpha}} \right),
\]
where \(\alpha, \gamma\) are the parameters of Section 4. Using (8.11) and assuming \(a \gg 1\) is sufficiently large (as in (4.13), depending on \(\alpha, \gamma, b\)), we may ensure that
\[
\lambda^{-1}_{q+1} \leq \ell_q \leq \ell_{q,i} \leq \lambda^{-1}_q.
\]
Since we will always be working with \(0 < \beta\), recalling \(\ell^{-1}_q \leq \lambda_{q+1}\), and assuming \(\epsilon \leq \alpha\), we observe that
\[
\tau_q^{\ell^{-20-\epsilon}_q} \leq \tau_q^{\ell^{-20-\epsilon}_q} \leq \ell^{3t_q \Lambda^{-1}} (q^{-1}_q)^{-1} \lambda^{20}_{q+1} \leq \lambda^{1+2b \delta} q^{-1} < 1,
\]
for \(b\) sufficiently close to 1.

\textbf{Proposition 8.1 (Gluing step).} Let \(b, \beta, \alpha, \gamma\) and \((\delta_q, \lambda_q, \Lambda, \xi_q, \ell_q, \tau_q)\) be as in Section 4 with
\[
ab < \beta
\]
\[
(8.9)
\]
\[
b^2 (1 + \gamma) < \frac{1 - \beta}{2\beta}.
\]

Let \([T_1, T_2] \subset [0, T]\) with \(|T_2 - T_1| > 4\tau_q\). Let \((v_q, p_q, R_q)\) be a strong subsolution on \([0, T]\) which on \([T_1, T_2]\) satisfies the estimates

\[
\frac{3}{4} \delta_q^{2+2} \leq p_q \leq \frac{7}{2} \delta_q^{2+1}
\]
\[
\|\tilde{R}_q\|_0 \leq \Lambda q^{1+\gamma}
\]
\[
\|v_q\|_{1+\alpha} \leq M \delta_q^{1+\alpha}
\]
\[
\|v_q\|_{\theta+\epsilon} \leq M \left( 1 + \sum_{i=0}^{q} \delta_q^{\theta+i} \right)
\]
\[
|\partial_\theta p_q| \leq p_q \xi_q^{1+\alpha}
\]
\[
(8.11)
\]
\[
(8.12)
\]
\[
(8.13)
\]
\[
(8.14)
\]
\[
(8.15)
\]
with some constant \(M > 0\), where
\[
p_q := \frac{1}{3} \text{tr} R_q \quad \quad q_q := \frac{p_q}{\Lambda}
\]

Define \((p_{q,i}, q_{q,i}, \ell_{q,i})\) as in (8.6).

Then, provided \(a \gg 1\) is sufficiently large, there exists \((v_q, p_q, R_q)\) solution of (3.1) on \([0, T]\) such that
\[
(v_q, p_q, R_q) = (v_q, p_q, R_q) \quad \text{on} \quad [0, T] \setminus [T_1, T_2],
\]
(8.16)
and on \([T_1, T_2]\) the following estimates hold:

\[
\|\bar{v}_q - v_q\|_\alpha \lesssim \Lambda^{1+\gamma} \epsilon_q^\alpha \tag{8.17}
\]
\[
\|\bar{v}_q\|_{1+\alpha} \lesssim \delta_q^{1+\alpha} \tag{8.18}
\]
\[
\|\nabla_x v_q\|_{\theta+\epsilon} \leq M \left( 1 + \sum_{i=0}^{q+1} \delta_i \lambda_i^\beta \right) \tag{8.19}
\]
\[
\|\nabla_x| v_q|\|_0 \lesssim \Lambda \epsilon_q^{1+\gamma} \tag{8.20}
\]
\[
\frac{7}{8} \rho \leq \bar{\rho} \leq \frac{9}{8} \rho \tag{8.21}
\]
\[
|\partial_t | v_q|\|_0 \lesssim \epsilon_q^{1+\gamma} \tag{8.22}
\]
\[
\|\rho_q - \bar{\rho}_q\|_0 = \frac{1}{3} \left| \int_{\mathbb{T}^3} \left( |v_q|^2 - |\nabla v_q|^2 \right) dx \right| \lesssim \Lambda \epsilon_q^{1+\gamma} \tag{8.23}
\]

Moreover, on \([t_\alpha, \tau]\) the following additional estimates hold for \(t \in I_{i-1} \cup I_i \cup I_1:\)

\[
\|\nabla v_q\|_{N+1+\alpha} \lesssim \delta_q^{1+\gamma} \epsilon_q^{-N} \tag{8.24}
\]
\[
\|\nabla \bar{R}_q\|_{N+\alpha} \lesssim \Lambda \epsilon_q^{1+\gamma} \tag{8.25}
\]
\[
\|\partial_t + \nabla v \bar{R}_q\|_{N+\alpha} \lesssim \Lambda \epsilon_q^{1+\gamma} \tag{8.26}
\]

Regarding the support of the Reynolds stress, we have that

\[
\bar{R}_q(., t) \equiv 0 \quad \forall \tau \in \bigcup_{i=2}^\tau J_i. \tag{8.27}
\]

In terms of energy, we have that

\[
\int_{\mathbb{T}^3} |v_q|^2 (x, t) + \text{tr} \bar{R}_q(x, t) \, dx = \int_{\mathbb{T}^3} |v_q|^2 (x, t) + \text{tr} R_q(x, t) \, dx, \tag{8.28}
\]

and the function

\[
\mathcal{T}_g := \int_{\mathbb{T}^3} \int_0^t \left( |(-\Delta)^{\frac{\alpha}{2}} v_q|^2 - |(-\Delta)^{\frac{\alpha}{2}} v_q|^2 \right) ds \, dx,
\]

satisfies

\[
|\partial_t \mathcal{T}_g| \lesssim \Lambda \delta_q^{\frac{\alpha}{2}} \lambda_i^{\frac{\alpha}{2}+\epsilon} \tag{8.29}
\]

and therefore

\[
|\mathcal{T}_g(t)| \lesssim t \Lambda \delta_q^{\frac{\alpha}{2}} \lambda_i^{\frac{\alpha}{2}+\epsilon} \log q \tag{8.30}
\]

Finally, if \(2\alpha < \beta_1\), then

\[
\|\nabla v_q - v_q\|_\alpha \lesssim \delta_q^{\frac{\alpha}{2}} \lambda_i^{\frac{\alpha}{2}+\epsilon}. \tag{8.31}
\]

The proof closely follows the gluing procedure of [13] Section 6], which in turn draws heavily from [13] Sections 3-4]. Recall that the solution is left unchanged outside \([T_1, T_2]\) and the gluing only happens in that interval. More precisely, recalling the decomposition (8.4):

- The gluing procedure is carried out in the interval

\[
J_\alpha \cup \ldots \cup J_\tau = \left( t_\alpha - \frac{1}{3} \tau, t_\tau + \frac{1}{3} \tau \right). \tag{8.31}
\]
• The subsolution is left unchanged in \( J_{\mathcal{L}} = J_{\mathcal{L}^{-1}} \cup J_{\mathcal{P}} \).

• The intervals \( I_{\mathcal{L}^{-1}} \) and \( I_{\mathcal{P}} \) are used as cutoff regions between the glued and unglued subsolutions.

Recall also that, since the trace \( \rho_{q} = 1/3 \text{tr} R_{q} \) has different lower and upper bounds on \([T_{1}, T_{2}]\) (respectively of order \( \delta_{q+2} \) and \( \delta_{q+1} \)), mollification with different parameters \( \ell_{q,i} \) depending on \( \rho_{q}(t_{i}) \) on intervals of size \( \tau_{q} \) around the points \( t_{i} \) is necessary.

We will also make use of the following estimates.

**Lemma 8.1 (Material derivative estimates for subsolutions and potentials).** Let \((v, p, R), (v', p', R')\) be two solutions of (3.1), and let \( z := (-\Delta)^{-1} \text{curl} v, z' := (-\Delta)^{-1} \text{curl} v'\). Then the following estimates hold for every \( N \in \mathbb{N}, \alpha \in (0, 1)\):

\[
\| (\partial_{t} + v \cdot \nabla + (-\Delta)^{0})(v - v') \|_{N+\alpha} \leq \| v - v' \|_{N+\alpha} (\| v \|_{\alpha} + \| v' \|_{\alpha}) + \| R - R' \|_{N+1+\alpha} \tag{8.32}
\]

\[
\| (\partial_{t} + v \cdot \nabla + (-\Delta)^{0})(z' - z) \|_{N+\alpha} \leq \| z' - z \|_{N+\alpha} (\| v \|_{1+\alpha} + \| z' - z \|_{\alpha}) + \| R' - R \|_{N+1+\alpha} \tag{8.33}
\]

The proof of (8.32) can be found in the proof of [20, Proposition 5.3], whereas that of (8.33) can be found within the proof of [20, Proposition 5.4]. Note that, since the proofs require the Schauder estimates of Lemma 2.5, this result does not hold for \( \alpha = 0 \).

**Proof.** (Proposition 8.1)

**Step 1: Mollification**

Let \( \varphi \) be a standard mollification kernel in space and define

\[
v_{\ell_{q,i}} := v_{q} \ast \varphi_{\ell_{q,i}},
\]

\[
p_{\ell_{q,i}} := p_{q} \ast \varphi_{\ell_{q,i}} + \frac{1}{3} \left( |v_{q}|^{2} \ast \varphi_{\ell_{q,i}} - |v_{q,i}|^{2} \right)
\]

\[
\dot{R}_{\ell_{q,i}} := \dot{R}_{q} \ast \varphi_{\ell_{q,i}} + (v_{q} \otimes v_{q}) \ast \varphi_{\ell_{q,i}} - v_{q,i} \otimes v_{q,i}.
\]

With this definition, (3.1) holds for the triple \((v_{\ell_{q,i}}, p_{\ell_{q,i}}, \dot{R}_{\ell_{q,i}})\). Using the estimates (8.12) and (8.13), together with (2.5), we deduce that

\[
\| v_{q,i} - v_{q} \|_{\alpha} \leq 2^{1+\alpha} \delta_{q}^{1+\alpha} \ell_{q,i} = \Lambda^{\frac{1+\alpha}{2}} q_{q,i}^{1+\alpha} \tag{8.34}
\]

\[
\| v_{q,i} \|_{N+1+\alpha} \leq 2^{1+\alpha} \delta_{q}^{1+\alpha} \ell_{q,i}^{-N} \tag{8.35}
\]

\[
\| v_{q,i} \|_{\theta+\varepsilon} \leq \Lambda \tag{8.36}
\]

\[
\| \dot{R}_{q,i} \|_{N+\alpha} \leq \Lambda q_{q}^{1+\gamma} \ell_{q,i}^{-N-\alpha} + \delta_{q}^{2+2\alpha} \ell_{q,i}^{2-N-\alpha} \tag{8.37}
\]

\[
\int_{\Omega} |v_{q,i}|^{2} - |v_{q,i}|^{2} \, dx \leq 2^{1+\alpha} \delta_{q}^{2+2\alpha} \ell_{q,i}^{2} = \Lambda q_{q,i}^{1+\gamma} \ell_{q,i}^{2} \tag{8.38}
\]

To obtain (8.38), we also use the trivial identity \( \int f \varphi_{\ell} = \int f = |v_{q}|^{2} \).

**Step 2: Gluing procedure**

Let \( \{I_{k}\}_{0 \leq k \leq \pi} \) be the sequence of intervals corresponding to \([T_{1}, T_{2}]\) according to Definition 8.1 above. We now fix a partition of unity on \([0, T] \)

\[
\sum_{i=0}^{\pi+1} \chi_{i} = 1
\]
subordinate to the decomposition (8.34), i.e. \([0, T] = \bigcup_{i=1}^{n} I_i \cup \bigcup_{i=n}^{\bar{n}} I_i \cup \bigcup_{i=n}^{\bar{n}} J_i \cup J_{n+1}.\) More precisely, for each \(n-1 \leq i \leq \bar{n}+1,\) the function \(\chi_i \geq 0\) satisfies

\[
\text{supp } \chi_i \subset I_{i-1} \cup I_i \cup I_{i+1}, \quad |\chi_i^{(N)}|_{\tilde{\beta}} \equiv 1, \quad |\partial_t^n \chi_i| \lesssim \tau_q^{-N} \quad \forall N \geq 0.
\]

We define

\[
\bar{\nabla}_q := \sum_{i=n}^{\bar{n}+1} \chi_i \partial_t v_i, \quad \bar{p}_q^{(1)} := \sum_{i=n}^{\bar{n}+1} \chi_i p_i,
\]

where \((v_i, p_i)\) is defined as follows. For \(n \leq i \leq \bar{n}\) we define \((v_i, p_i)\) as the solution of

\[
\begin{align*}
\partial_t v_i + \nabla (v_i \otimes v_i) + \nabla p_i + (-\Delta) v_i &= 0, \\
\nabla v_i &= 0, \\
\chi_i(t, t_i) &= v_{q_i}(t, t_i),
\end{align*}
\]

and set \((v_i, p_i) = (v_q, p_q)\) for \(i = n-1\) and \(i = \bar{n}+1.\) Thus, we note first of all that \(\nabla \bar{\nabla}_q = 0,\) and moreover

\[
(\bar{\nabla}_q, \bar{p}_q^{(1)}) = (v_q, p_q), \quad t \in [0, T] \setminus [T_i, T_{i+1}].
\]

Next, we define \(\bar{R}_q.\) We have that \(\chi_i + \chi_{i+1} = 1\) for \(t \in I_i \cup I_{i+1},\) and therefore

\[
\partial_t \bar{\nabla}_q + \nabla (\bar{\nabla}_q \otimes \bar{\nabla}_q) + \nabla \bar{p}_q^{(1)} + (-\Delta)^0 \bar{\nabla}_q = \partial_t \chi_i \cdot (v_i - v_{i+1}) \\
\chi_i \cdot (1 - \chi_i) \nabla (v_i - v_{i+1}) \otimes (v_i - v_{i+1}) \\
- \chi_i \cdot (1 - \chi_i) \nabla (v_i - v_{i+1}) \otimes \nabla (v_i - v_{i+1})
\]

for all \(n-1 \leq i \leq \bar{n},\) where we wrote \(R_i = 0\) for \(n \leq i \leq \bar{n}\) and \(R_i = R_q\) otherwise. Thus, recalling the operator \(R\) from [Definition 2.1] set

\[
\begin{align*}
\frac{\alpha^{(1)}}{R_q} := \begin{cases} \\
-\partial_t \chi_i R_i (v_i - v_{i+1}) \\
+ \chi_i \cdot (1 - \chi_i) (v_i - v_{i+1}) \otimes (v_i - v_{i+1}) \\
0
\end{cases} \quad t \in I_i \\
\frac{\alpha^{(2)}}{R_q} := \sum_{i=n}^{\bar{n}+1} \chi_i \hat{R}_i = (\chi_{n-1} + \chi_{\bar{n}+1}) \hat{R}_q,
\end{align*}
\]

and

\[
\bar{p}_q^{(2)} := \sum_{i=n}^{\bar{n}+1} \chi_i \cdot (1 - \chi_i) \left( |v_i - v_{i+1}|^2 - \int_{T^3} |v_i - v_{i+1}|^2 \, dx \right).
\]

Finally, we define

\[
\bar{R}_q = \frac{\alpha^{(1)}}{R_q} + \frac{\alpha^{(2)}}{R_q} + \bar{p}_q \text{Id}, \quad \bar{p}_q := \bar{p}_q^{(1)} + \bar{p}_q^{(2)},
\]

where

\[
\bar{p}_q := p_q + \frac{1}{3} \int_{T^3} (|v_q|^2 - |\nabla v_q|^2) \, dx.
\]

Define also

\[
\bar{F}_q(t) := \frac{1}{3} \int_{T^3} \left( |(-\Delta)^2 v_q|^2 - |(-\Delta)^2 \nabla v_q|^2 \right) \, dx.
\]

By construction, we have that

\[
\partial_t \bar{\nabla}_q + \nabla (\bar{\nabla}_q \otimes \bar{\nabla}_q) + \nabla \bar{p}_q = - \nabla \bar{R}_q,
\]

and (8.16) and (8.28) hold. Moreover

\[
\frac{\alpha}{R_q} = 0 \quad \forall t \in \bigcup_{i=n}^{\bar{n}} J_i.
\]
Step 3: Stability estimates on classical solutions

Throughout this step and the next, we will assume estimate (8.21), which will be proved in Step 5 below. This estimate will allow us to replace $\gamma_q$ with $\rho_q$ and vice versa whenever we need to do so in our estimates, since the two are of the same order.

Let us consider for the moment $n \leq i \leq n$. We recall the classical existence result for solutions of (8.40) found in [20, Proposition 3.5], by which $(v_i, p_i)$ in (8.39) above is defined at least on an interval of length $\sim \|v_{q,i}\|_{1+\alpha}^{-1}$. By (8.35) and (4.11), we have that

$$\|v_{q,i}\|_{1+\alpha} \lesssim \delta_q^{1\alpha}_{q} = \tau_q^{-1} \ell_q^{\alpha} \lesssim \tau_q^{-1}.$$  

Therefore, provided $a \gg 1$ is sufficiently large, $v_i$ is defined on $I_{n-1} \cup J_i \cup I_i$, so that $\gamma_q$ in (8.39) is well-defined.

Next, we deduce from (8.15) that $|\partial_i \log \rho_q| \lesssim \delta_q^{1/2} \lambda_q = \tau_q^{-1} \rho_q$, so that, by assuming $a \gg 1$ is sufficiently large, we may ensure that

$$p_q(t_1) \leq 4\rho_q(t_2) \quad \forall t_1, t_2 \in I_{n-1} \cup J_i \cup I_i,$$

for any $i$. In particular $\rho_q = \rho_q,i$ and $\rho_q \sim \rho_q,i$ in $I_{n-1} \cup J_i \cup I_i$. We apply Lemma 8.1 to $(v_{q,i}, p_{q,i}, R_{q,i})$ and $(v_i, p_i, 0)$, which, using (8.34), (8.35), (8.37), and (8.45), immediately yields

$$\left\| \left( \partial_t + v_{q,i} \cdot \nabla + (-\Delta)^{\alpha} \right)(v_i - v_{q,i}) \right\|_{N+\alpha} \lesssim \left\| v_i - v_{q,i} \right\|_{N+\alpha} \left( \left\| v_i \right\|_{1+\alpha} + \left\| v_{q,i} \right\|_{1+\alpha} \right).$$

The Grönwall inequality then implies that

$$\left\| v_i - v_{q,i} \right\|_{N+\alpha} \lesssim \Lambda \tau_q \sum_{j=i-1}^{i+1} \left( \left\| v_j - v_{q,i} \right\|_{\alpha} + \left\| v_{q,i} - v_q \right\|_{\alpha} \right) \lesssim \Lambda \tau_q^{1+\gamma} \ell_q^{N-1-\alpha}.$$  

The case $N = 0$ of (8.47), together with (8.34), leads to

$$\left\| v_i - v_{q,i} \right\|_{N+\alpha} \lesssim \Lambda \tau_q \sum_{j=i-1}^{i+1} \left( \left\| v_j - v_{q,i} \right\|_{\alpha} + \left\| v_{q,i} - v_q \right\|_{\alpha} \right) \lesssim \Lambda \tau_q^{1+\gamma} \ell_q.$$  

By (8.21), this is equivalent to (8.17).

The case $N = 1$ of (8.47) leads to

$$\left\| v_i - v_{q,i} \right\|_{1+\alpha} \lesssim \Lambda \tau_q^{1+\gamma} \ell_q^{-1-\alpha} \lesssim \Lambda \tau_q^{1+\gamma} \ell_q^{-1-\alpha}.  
$$

By (8.21), this is equivalent to (8.17).

Combining the above estimate with (8.13) outside the gluing region and in $I_{q-1} \cup I_q$, and with (8.35) in $J_q \cup I_q \cup \ldots \cup J_{q-1} \cup I_{q-1}$, we deduce that (8.18) is verified. More generally, as we did above for $N = 0$, we deduce from (8.35) and (4.87) that

$$\left\| v_i - v_{q,i} \right\|_{1+\alpha} \lesssim \Lambda \tau_q^{1+\gamma} \ell_q^{-1-\alpha} \lesssim \Lambda \tau_q^{1+\gamma} \ell_q^{-1-\alpha}.$$  

Combining the above estimate with (8.13) outside the gluing region and in $I_{q-1} \cup I_q$, and with (8.35) in $J_q \cup I_q \cup \ldots \cup J_{q-1} \cup I_{q-1}$, we deduce that (8.18) is verified. More generally, as we did above for $N = 0$, we deduce from (8.35) and (4.87) that

$$\left\| v_i - v_{q,i} \right\|_{1+\alpha} \lesssim \Lambda \tau_q^{1+\gamma} \ell_q^{-1-\alpha} \lesssim \Lambda \tau_q^{1+\gamma} \ell_q^{-1-\alpha}.  
$$

We have used the fact that $\ell_q,i \sim \ell_q,i+1 \sim \ell_q,i-1$. The above inequality coincides with (8.24).
We also remark the following simple interpolation of the $N = 0$ and $N = 1$ cases of (8.47), which will be used in Step 5 below.

$$
\|v_i - v_{\ell,q,i}\|_{0+e} \lesssim \|v_i - v_{\ell,q,i}\|_{0}^{1-\theta} \|v_i - v_{\ell,q,i}\|_{1+\alpha}^{\theta+e} \lesssim \Lambda\tau_q q_{\ell,q,i}^{1+\gamma \rho \theta - \theta - \epsilon - 1 - \alpha}.
$$

Further in the proof, we will need estimates for $\|v_i - v_{\ell,q,i}\|_{N+\alpha}$ and $\|\nabla (\partial_t + v_{\ell,q,i} \cdot \nabla) (v_i - v_{\ell,q,i})\|_{N+\alpha}$. Concerning the former, by applying the triangle inequality, we see that

$$
\|v_i - v_{\ell,q,i}\|_{N+\alpha} \leq \|v_i - v_{\ell,q,i}\|_{N+\alpha} + \|v_{\ell,q,i} - v_{\ell,q,i+1}\|_{N+\alpha} + \|v_{\ell,q,i+1} - v_{\ell,q,i}\|_{N+\alpha}.
$$

The first and third term are estimated by (8.47). The second one is readily shown to satisfy the same estimate, so that

$$
\|v_i - v_{\ell,q,i}\|_{N+\alpha} \lesssim \Lambda^{1+\gamma \rho \theta - 1 - N - \alpha} q_{\ell,q,i}^{-1}.
$$

As for the material derivative, we note that

$$
\|(\partial_t + v_{\ell,q,i} \cdot \nabla) (v_i - v_{\ell,q,i})\|_{N+\alpha} \lesssim \Lambda^{1+\gamma \rho \theta - 1 - N - \alpha} q_{\ell,q,i}^{-1}.
$$

Using (8.48), we can easily conclude, by interpolating in (8.47), that

$$
\|(-\Delta)^{\theta} (v_i - v_{\ell,q,i})\|_{N+\alpha} \lesssim \|v_i - v_{\ell,q,i}\|_{N+\alpha}^{1-20\epsilon} \|v_i - v_{\ell,q,i}\|_{N+\alpha}^{20\epsilon} 
\lesssim \Lambda q_{\ell,q,i}^{1+\gamma \rho \theta - 1 - N - \alpha}.
$$

so that

$$
I \leq \|(\partial_t + v_{\ell,q,i} \cdot \nabla) (v_i - v_{\ell,q,i})\|_{N+\alpha} + \|(-\Delta)^{\theta} (v_i - v_{\ell,q,i})\|_{N+\alpha} \lesssim \Lambda^{1+\gamma \rho \theta - 1 - N - \alpha} q_{\ell,q,i}^{-1}.
$$

The term $IV$ is handled similarly. Since $v_{\ell,q,i} - v_{\ell,q,i+1}$ obeys the bound (8.50), using Lemma 8.1 we conclude that $II$ also obeys the above bound. We now consider $III$, which can be estimated as follows:

$$
III = \|(v_{\ell,q,i} - v_{\ell,q,i+1}) \cdot \nabla (v_i - v_{\ell,q,i+1})\|_{N+\alpha} \lesssim \Lambda^{1+\gamma \rho \theta - 1 - N - \alpha} q_{\ell,q,i}^{-1}.
$$

in particular satisfying the bound (8.51). We thus conclude that

$$
\|(\partial_t + v_{\ell,q,i} \cdot \nabla) (v_i - v_{\ell,q,i})\|_{N+\alpha} \lesssim \Lambda^{1+\gamma \rho \theta - 1 - N - \alpha} q_{\ell,q,i}^{-1}.
$$

Step 4: Estimates on the new Reynolds stress

As is done in [4] Section 3.3, we define the vector potentials

$$
\zeta_i := (-\Delta)^{-1} \text{curl } v_i \quad \zeta_{\ell,q,i} := (-\Delta)^{-1} \text{curl } v_{\ell,q,i} \quad \zeta_q := (-\Delta)^{-1} \text{curl } v_q
$$

and, by a combination of Lemma 8.1 Proposition 2.2 and the Grönwall inequality similar to the one used to obtain (8.47) in Step 3 above, obtain that

$$
\|\zeta_i - \zeta_{\ell,q,i}\|_{N+\alpha} \lesssim \Lambda \tau_q q_{\ell,q,i}^{1+\gamma \rho \theta - 1 - N - \alpha}.
$$
Combining this with Lemma 8.1 yields
\[
\left\| (\partial_t + v_{\ell q_i}, \nabla + (-\Delta)^{\theta}) (z_i - z_{\ell q_i}) \right\|_{N+\alpha} \lesssim \Lambda q_i^{1+\gamma} e^{-\gamma} e^{-\alpha}.
\] (8.54)

By (8.8) and the Schauder estimates of Lemma 2.5, we deduce that
\[
\left\| (-\Delta)^{\theta} (z_i - z_{\ell q_i}) \right\|_{N+\alpha} \lesssim \| z_i - z_{\ell q_i} \|_{N+\alpha + 20 + \varepsilon} \lesssim \| v_i - v_{\ell q_i} \|_{N+\alpha - 1 + 20 + \varepsilon} \lesssim \Lambda e^{-\gamma} e^{-\alpha} \lesssim \Lambda q_i^{1+\gamma} e^{-\gamma} e^{-\alpha}.
\] (8.55)

By the triangle inequality, (8.55), and (8.54), we thus conclude that
\[
\left\| (\partial_t + v_{\ell q_i}, \nabla) (z_i - z_{\ell q_i}) \right\|_{N+\alpha} \lesssim \Lambda q_i^{1+\gamma} e^{-\gamma} e^{-\alpha}.
\] (8.56)

Both (8.53) and (8.56) are valid in \( I_{i-1} \cup J_i \cup I_i \) for any \( n \leq i \leq \pi \).

The sequel of this proof will require estimates on \( z_i - z_{i+1} \), which means we must now bound \( z_{\ell q_i} - z_{\ell q_{i+1}} \). We note that \( z_{\ell q_i} = q_i \phi_{q_i} \), and \( z_{\ell q_{i+1}} = q_i \phi_{q_{i+1}} \), so that, using (2.5) and Schauder estimates (Lemma 2.5), we get
\[
\left\| z_{\ell q_i} - z_{\ell q_{i+1}} \right\|_{N+\alpha} \lesssim \| z_i \|_{2+\alpha} (\ell_{q_i}^{2-N} + \ell_{q_{i+1}}^{2-N}) \lesssim \Lambda q_i^{1+\gamma} \ell_{q_i}^{2-N} e^{2N}.
\] (8.57)

The final estimate for \( z_i - z_{i+1} \) is thus
\[
\left\| z_i - z_{i+1} \right\|_{N+\alpha} \leq \left\| z_i - z_{\ell q_i} \right\|_{N+\alpha} + \left\| z_{\ell q_i} - z_{\ell q_{i+1}} \right\|_{N+\alpha} + \left\| z_{\ell q_{i+1}} - z_{i+1} \right\|_{N+\alpha} \lesssim \Lambda q_i^{1+\gamma} \ell_{q_i}^{2-N} e^{2N}.
\] (8.58)

slightly coarser than (8.53) above. As for the material derivatives, we must estimate
\[
\left\| (\partial_t + v_{\ell q_i}, \nabla) (z_i - z_{i+1}) \right\|_{N+\alpha} \lesssim \left\| (\partial_t + v_{\ell q_i}, \nabla) (z_i - z_{\ell q_i}) \right\|_{N+\alpha}
+ \left\| (\partial_t + v_{\ell q_{i+1}}, \nabla) (z_{\ell q_i} - z_{\ell q_{i+1}}) \right\|_{N+\alpha}
+ \left\| (v_{\ell q_i} - v_{\ell q_{i+1}}, \nabla) (z_{\ell q_{i+1}} - z_{i+1}) \right\|_{N+\alpha}
+ \left\| (\partial_t + v_{\ell q_{i+1}}, \nabla) (z_{\ell q_{i+1}} - z_{i+1}) \right\|_{N+\alpha}
= I + II + III + IV.
\]

The terms I and IV are estimated by (8.56). To estimate II, we apply Lemma 8.1 to \((v_{\ell q_i}, P_{\ell q_i}, R_{\ell q_i})\) and \((v_{\ell q_{i+1}}, P_{\ell q_{i+1}}, R_{\ell q_{i+1}})\) and use (8.55) and (8.57), obtaining that
\[
\left\| (\partial_t + v_{\ell q_i}, \nabla) (z_{\ell q_i} - z_{\ell q_{i+1}}) \right\|_{N+\alpha} \lesssim q_i^{1+\gamma} e^{-\gamma} e^{-\alpha} \lesssim q_i^{1+\gamma} e^{-\gamma} e^{-\alpha} \lesssim q_i^{1+\gamma} e^{-\gamma} e^{-\alpha}.
\]

We then note that, by interpolation
\[
\left\| (-\Delta)^{\theta} (z_{\ell q_i} - z_{\ell q_{i+1}}) \right\|_{N+\alpha} \lesssim \Lambda q_i^{1+\gamma} e^{-\gamma} e^{-\alpha} \lesssim q_i^{1+\gamma} e^{-\gamma} e^{-\alpha}.
\]

The above two bounds combine to yield
\[
II \lesssim \Lambda q_i^{1+\gamma} e^{-\gamma} e^{-\alpha}.
\] (8.59)

Coming to III, we estimate it by combining (8.53) with (8.50):
\[
\left\| (v_{\ell q_i} - v_{\ell q_{i+1}}, \nabla) (z_{i+1} - z_{\ell q_{i+1}}) \right\|_{N+\alpha} \lesssim \Lambda q_i^{1+\gamma} e^{-\gamma} e^{-\alpha} \lesssim q_i^{1+\gamma} e^{-\gamma} e^{-\alpha} \lesssim q_i^{1+\gamma} e^{-\gamma} e^{-\alpha}.
\] (8.60)

Combining (8.56), (8.59), and (8.60), we thus obtain that
\[
\left\| (\partial_t + v_{\ell q_i}, \nabla) (z_i - z_{i+1}) \right\|_{N+\alpha} \lesssim \Lambda q_i^{1+\gamma} e^{-\gamma} e^{-\alpha}.
\] (8.61)
Recalling the expression for $\mathbf{R}_q$ in (8.43), using (8.45), (8.58), and (8.50), we obtain, as in the proof of [4] Proposition 4.3], that
\[
\left\| \mathbf{R}_q \right\|_{N+\alpha} \lesssim \tau_q^{-1} \| z_i - z_{i+1} \|_{N+\alpha} + \| v_i - v_{i+1} \|_{N+\alpha} \| v_i - v_{i+1} \|_{\alpha}
\lesssim \Lambda q^{1+i} \ell_q^{N} \ell_q^{-2\alpha} + \Lambda q^{1+i} \ell_q^{N} \ell_q^{-2\alpha} + \Lambda q^{1+i} \ell_q^{N} \ell_q^{-2\alpha}.
\]
This, together with (8.21), gives us (8.25).
As for (8.26), we note that
\[
\left\| (\partial_t + v_q \cdot \nabla) \mathbf{R}_q \right\|_{N+\alpha} \lesssim \tau_q^{-2} \| z_i - z_{i+1} \|_{N+\alpha} + \tau_q^{-1} \| v_i - v_{i+1} \|_{N+\alpha} \| v_i - v_{i+1} \|_{\alpha}
+ \| \mathbf{R}_q \|_{N+\alpha} \| v_i - v_{i+1} \|_{N+\alpha} \| v_i - v_{i+1} \|_{\alpha}
+ \| (\partial_t + v_q \cdot \nabla)(v_i - v_{i+1}) \|_{N+\alpha} \| v_i - v_{i+1} \|_{\alpha}
+ \| (\partial_t + v_q \cdot \nabla)(v_i - v_{i+1}) \|_{N+\alpha} \| v_i - v_{i+1} \|_{\alpha}
+ \| (v_q - \nabla) \cdot \nabla \mathbf{R}_q \|_{N+\alpha}.
\]
Combining the above bound on $\mathbf{R}_q$ with (8.58), (8.61), (8.35), (8.50), (8.52), and the bound (8.48) applied to $v_q - \nabla v_q$, we obtain that
\[
\left\| (\partial_t + v_q \cdot \nabla) \mathbf{R}_q \right\|_{N+\alpha} \lesssim \Lambda q^{1+i} \ell_q^{N} \ell_q^{-2\alpha} + \Lambda q^{1+i} \ell_q^{N} \ell_q^{-2\alpha},
\]
which yields (8.26) once combined with (8.21).

**Step 5: $\nabla v_q$, $T_q$, and (8.30)**

Next, we estimate $\nabla v_q$, recalling its definition in (8.44). We wish to estimate $\nabla v_q - \rho_q$. We note that
\[
\left\| \nabla v_q - \rho_q \right\|_0 = \frac{1}{3} \int_{\mathbb{T}^3} |\nabla v_q|^2 - |v_q|^2 \, dx \leq \int_{\mathbb{T}^3} |\nabla v_q|^2 - |v_q|^2 \, dx + \int_{\mathbb{T}^3} |v_q|^2 \, dx.
\]
The second term above is already estimated by (8.38), so we proceed to estimate the first term. As in [4] Proposition 4.4], one has that
\[
|\nabla v_q|^2 - |v_q|^2 = \chi_i (|v_i|^2 - |v_q|^2) + (1 - \chi_i) ((|v_i|^2 - |v_{q,i+1}|^2) + (1 - \chi_i) (|v_{q,i+1}|^2 - |v_{q,i}|^2) - \chi_i (1 - \chi_i) |v_i - v_{i+1}|^2.
\]
Therefore
\[
\int_{\mathbb{T}^3} |\nabla v_q|^2 - |v_q|^2 \, dx \leq \int_{\mathbb{T}^3} |v_i|^2 - |v_{q,i+1}|^2 \, dx + \int_{\mathbb{T}^3} |v_i|^2 - |v_{q,i}|^2 \, dx + \int_{\mathbb{T}^3} |v_{q,i+1}|^2 \, dx + \int_{\mathbb{T}^3} |v_i - v_{i+1}|^2 \, dx.
\]
We start by estimating the fourth term as follows by using (8.50):
\[
\int_{\mathbb{T}^3} |v_i - v_{i+1}|^2 \, dx \leq \| v_i - v_{i+1} \|^2_{\alpha} \lesssim \Lambda q^{1+i} \ell_q^2 \alpha.
\]
We then proceed to estimate the third term in (8.63) by using the triangle inequality, (8.38), and the fact \( \mathbf{q}_{q,i} \sim \mathbf{q}_{q,i+1} \): 

\[
\int_{T^3} \left( |v_{q,i}|^2 - |v_{q,i+1}|^2 \right) \, dx \leq \int_{T^3} \left( |v_{q,i}|^2 - |v_q|^2 \right) \, dx + \int_{T^3} \left( |v_q|^2 - |v_{q,i+1}|^2 \right) \, dx \lesssim \Lambda \mathbf{q}_{q,i}^{1+\gamma} \epsilon^{2\alpha}. \tag{8.65}
\]

The first and second terms in (8.63) are estimated in similar ways, so we only estimate the former. To that end, we proceed in a way similar to [20 Proposition 5.5]. We start by using the fact that \( (v_i, p_i, 0) \) and \( (v_{q,i}, p_{q,i}, \hat{R}_{q,i}) \) are subsolutions, [Theorem 2.1] (8.35) and (8.37), and (8.36) and (8.49), to obtain that 

\[
\frac{d}{dt} \int_{T^3} |v_i|^2 - |v_{q,i}|^2 \, dx \leq 2 \left( \int_{T^3} |\mathcal{D}v_{q,i} - \hat{R}_{q,i}| \, dx + \int_{T^3} \left| (-\Delta)^{\theta} v_i - (-\Delta)^{\theta} v_{q,i} \right|^2 \, dx \right) \lesssim \left\| v_{q,i} \right\|_{1+\alpha} \left\| \hat{R}_{q,i} \right\|_{\alpha} + \left\| v_i + v_{q,i} \right\|_{0+\epsilon} \left\| v_i - v_{q,i} \right\|_{0+\epsilon} \\
\lesssim \delta_{q}^{1+\alpha} - \Lambda \mathbf{q}_{q,i}^{1+\gamma} \epsilon^{0-\alpha - 1 - \beta} + \Lambda^2 \cdot \Lambda \mathbf{q}_{q,i}^{1+\gamma} \epsilon^{0-\alpha - 1 - \beta} = \Lambda \mathbf{q}_{q,i}^{1+\gamma} \epsilon^{0-\alpha - 1 - \beta} (F_1 + F_2),
\]

where we write 

\[
F_1 = \delta_{q}^{1+\alpha} - \Lambda \mathbf{q}_{q,i}^{1+\gamma} \epsilon^{0-\alpha - 2\alpha} \\
F_2 = \Lambda \mathbf{q}_{q,i}^{1+\gamma} \epsilon^{0-\alpha - \beta - 2\alpha}
\]

It is easy to see that 

\[
F_1 = \epsilon^{4\alpha} \mathbf{q} \epsilon^{0-\alpha - 2\alpha} \lesssim 1.
\]

To prove that \( F_2 \) is also bounded above by a constant, we note that 

\[
F_2 = f(\alpha, \gamma, \epsilon) \Lambda^2 \frac{\mathbf{q}_{q,i}^{1+\gamma}}{\delta_{q}^{1+\alpha} - \alpha} \left( \mathbf{q}_{q,i}^{1+\gamma} \right)^{1+\theta} / \mathbf{q}_{q,i}^{1+\gamma} \mathbf{q}_{q,i}^{1+\gamma}
\]

where \( f(\alpha, \gamma, \epsilon) \sim 1 \) for \( \alpha, \gamma, \epsilon \ll 1 \). We then observe that (8.10), together with the fact that \( \theta < \frac{1}{3}, b > 1 \), implies that 

\[
\Lambda^\frac{1}{\delta_{q}^{1+\alpha} - \alpha} \left( \mathbf{q}_{q,i}^{1+\gamma} \right)^{1+\theta} \leq \mathbf{q}_{q,i}^{1+\gamma} \left( \mathbf{q}_{q,i}^{1+\gamma} \right)^{1+\theta} \leq \mathbf{q}_{q,i}^{1+\gamma} \left( \lambda_q^{1+\gamma} \right)^{1+\theta} = \lambda_q^{1+\gamma} \left( \lambda_q^{1+\gamma} \right)^{1+\theta} \leq 1.
\]

We have thus proved that 

\[
F_2 \lesssim 1,
\]

provided \( \alpha, \gamma, \epsilon \ll 1 \) are sufficiently small and \( \alpha \) is sufficiently large. This means that 

\[
\left\| \frac{d}{dt} \int_{T^3} |v_i|^2 - |v_{q,i}|^2 \, dx \right\|_{\mathbf{q}_{q,i}} \lesssim \Lambda \mathbf{q}_{q,i}^{1+\gamma} \epsilon^{2\alpha}. \tag{8.66}
\]

Combining (8.63)-(8.66), we conclude that 

\[
\int_{T^3} |v_t|^2 - |v_{q,i}|^2 \, dx \lesssim \Lambda \mathbf{q}_{q,i}^{1+\gamma} \epsilon^{2\alpha}. \tag{8.67}
\]
Estimates (8.67) and (8.38) imply
\[
\left| \int_{\mathbb{T}^3} \left| \nabla q \right|^2 - |v_q|^2 \, dx \right| \lesssim \Lambda q^{1+\gamma}_q \ell_q^{2\alpha}.
\]
This proves in particular that \( \bar{\rho}_q \sim \rho_q \) and (8.21), as well as (8.23).

Similarly, using (5.1) for \((v_q, p_q, R_q)\) and \((v_{\ell_q}, p_{\ell_q}, \bar{R}_{\ell_q})\) first, and for \((v_q, \bar{\rho}_q, \bar{R}_q)\) and \((v_{\ell_q}, p_{\ell_q}, R_{\ell_q})\) afterwards, we also deduce that
\[
\frac{d}{dt} \left( \int_{\mathbb{T}^3} \left| v_{\ell_q} \right|^2 \, dx \right) \lesssim \Lambda q^{1+\gamma}_q \ell_q^{-1} \ell_q^{2\alpha},
\]
\[
\frac{d}{dt} \left( \int_{\mathbb{T}^3} \left| \nabla v_{\ell_q} \right|^2 \, dx \right) \lesssim \Lambda q^{1+\gamma}_q \ell_q^{-1} \ell_q^{2\alpha}.
\]
Combining (8.68) and (8.69), we get
\[
\left| \partial_t \bar{\rho}_q - \partial_t \rho_q \right| \lesssim \Lambda q^{1+\gamma}_q \ell_q^{-1/2} \ell_q^{2\alpha}.
\]

To prove (8.22) note that
\[
\Lambda q^{1+\gamma}_q \ell_q^{-1} \ell_q^{2\alpha} \lesssim \rho_q \delta q_{\lambda_q} \ell_q^{1+2\alpha - 2\beta} \lesssim \rho_q \delta q_{\lambda_q},
\]
where we used the definitions of \( \tau_q \ell_{q+1} \), (8.11), the relations \( \ell_q^{-1} \leq \lambda_{q+1} \) and \( \Lambda q_{,q} = \rho_{q,1} \sim \rho_q \), and the fact that \( \alpha < a_b < \beta \), which follows from (8.9) since \( b > 1 \). Therefore, since we showed above that \( \bar{\rho}_q \sim \rho_q \), we have (8.22).

It remains to estimate \( \| \mathbf{\bar{\tau}}_q \|_0 \) on \([T_1, T_2]\) in order to verify (8.20) for the Reynolds stress. We already obtained (8.25) on \( J_{n} \cup \ldots \cup J_{\pi} \) (recall the decomposition (8.4)). Moreover, on \( J_{n-1} \cup J_{\pi+1} \) the subsolution remains unchanged, so there is nothing to prove. We are then left with the task of proving (8.20) on the cut-off regions \( I_{n-1} \) and \( I_\pi \).

To do so, we need to estimate \( \| v_i - v_{\ell_i} \|_a \) and \( \| z_i - z_{\ell_i} \|_a \). For the former, we combine estimates of \( v_i - v_{\ell_i} \) and of \( v_{\ell_i} - v_i \). For the latter, we only need to estimate \( \| z_{\ell_i} - z_i \|_a \), since we already handled \( z_i - z_{\ell_i} \), above. One has that, by (2.5), Lemma 2.5 (Schauder estimates), and (8.13)
\[
\| z_{\ell_i} - z_i \|_a \lesssim \| z_i \|_2 + \| \nabla v_i \|_a \ell_q^{2\alpha} \lesssim \| \nabla v_i \|_a \delta q_{\lambda_q} \ell_q^{1+\alpha} \lesssim \Lambda \delta q_{\lambda_q} \ell_q^{2\alpha - \lambda_q^{2\alpha} - \lambda},
\]
which gives us (8.20) as desired. We then have to verify (8.19) and (8.29). To that end, we observe that
\[
\| \mathbf{\bar{\tau}}_q \|_0 + \| v_i \|_0 \}
\]

The second term is estimated by (8.14). As for the first one, we note that
\[
\| v_q - v_{\ell_q} \|_a \lesssim \Lambda \delta q_{\lambda_q} \ell_q^{1+\alpha} \lesssim \delta q_{\lambda_q} \ell_q^{1+\alpha} \lesssim \delta q_{\lambda_q} \ell_q^{1+\alpha},
\]
where the first step is due to (8.48). We can thus estimate \( \| \mathbf{\bar{\tau}}_q - v_q \|_0 + \| v_i \|_0 + \| v_i \|_0 + \| v_i \|_0 \}
\]

Lastly
\[
\| \partial_t \bar{\mathbf{\tau}}_q \| = \int_{\mathbb{T}^3} \left( |(-\Delta)^{\alpha} \mathbf{\bar{\tau}}_q | - |(-\Delta)^{\alpha} v_q | ^2 \right) \, dx \lesssim \| \mathbf{\bar{\tau}}_q + v_i \|_0 + \| \mathbf{\bar{\tau}}_q - v_q \|_0 + \| v_i \|_0 + \| v_i \|_0 + \| v_i \|_0 \}
\]

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thus proving (8.29).
We conclude the proof by obtaining (8.30). To that end, we first note that (8.21) and (8.11) combine to give us \( \bar{\psi} \lesssim \zeta_{q+1} \). Combining this with (8.17), we conclude that
\[
\| \tau_q - v_q \| \lesssim \delta_{q+1}^{\frac{1}{2}} \zeta_{q+1}^2 \ell_q^a,
\]
thus reducing (8.30) to
\[
\lambda_{q+1}^{-\beta_0} \ell_q^{(1-\frac{2}{b})a} \lesssim 1.
\]
Since \( \ell_q^{-1} \leq \lambda_{q+1} \), the above follows from
\[
-\beta_0 + \frac{2}{b} \alpha - \frac{1}{2} \alpha < 0 \iff \alpha < \frac{2b\beta_0}{4-b}.
\]
We recall that we wish to obtain (8.30) only under the assumption \( 2\alpha < \beta_0 \). This means the above relation follows from
\[
\frac{2b}{4-b} > \frac{1}{2} \iff 5b > 4,
\]
which in turn follows from \( b > 1 \). The proof is thus complete.

\[\diamond\]

Remark 8.1 (Multi-gluing). Proposition 8.7 can easily be extended to a pairwise disjoint union of intervals \([T_1^{(i)}, T_2^{(i)}] \subset [0, T]\) with \( T_2^{(i)} - T_1^{(i)} \geq 4\tau_q\) and \( T_2^{(i)} < T_1^{(i+1)} \).

9 Perturbation step

Proposition 9.1 (Main Perturbation Step). Let \( b, \beta, \alpha, \gamma, (\delta_q, \lambda_q, \Lambda, \zeta_q, \ell_q, \tau_q) \) be as in Section 2 with
\[
\alpha < \beta \gamma.
\]
Let \([T_1, T_2] \subset [0, T]\) and let \( t_i, I_i, J_i \) be as in (8.1). Let \((v, p, R)\) be a smooth strong subsolution on \([T_1, T_2]\) satisfying
\[
\| R \|_1 \lesssim \Lambda q^{1+\gamma} \ell_q^{-2\alpha} \ell_q^{-1} q_i
\]
\[
\delta_{q+2} \lesssim \rho \lesssim \delta_{q+1}
\]
\[
\| v \|_0 \lesssim C\rho
\]
on \( K_i := [(i-1 + \frac{1}{3}) \tau_q, (i + \frac{2}{3}) \tau_q] \) for \( i-1 \leq i \leq i + 1 \), where the \( \ell_q \) are defined as in (8.6), and \( C_\rho \) is a geometric constant. Further, let \( \psi : [0, T] \to [0, 1] \) be a cutoff function and \( S_\psi \in C^\infty(\overline{T^3 \times [T_1, T_2]}; \mathbb{S}^{3 \times 3}) \) be a smooth matrix field with
\[
S_\psi(x, t) = \sigma_\psi(t) \text{Id} + \hat{S}_\psi(x, t) = \Lambda \xi_\psi(t) \text{Id} + \hat{S}_\psi(x, t),
\]
where \( \hat{S}_\psi \) is traceless, \( \sigma_\psi = \psi^2 \sigma \), and \( (\xi, \xi_\psi) = \Lambda^{-1}(\sigma, \sigma_\psi) \). Suppose \( \psi \) satisfies
\[
\| \psi \| \lesssim \delta_q^{\frac{1}{2}} \lambda_q,
\]
and \( \sigma \) satisfies
\[
0 \leq \sigma(t) \leq 4\delta_{q+1}
\]
\[
|\sigma|_{K_i} \lesssim \rho(t_i)
\]
\[
|\partial_t \sigma| \lesssim \delta_q^{\frac{1}{2}} \lambda_q
\]
Moreover, assume that for any $N \geq 0, t \in I_{i-1} \cup I_i \cup I_n \leq i \leq \Pi$

$$\left\| S_{N+1} \right\|_{N+\alpha} \lesssim \Lambda \delta^{1+\gamma \eta q \epsilon} \theta^{-2\alpha} q^{-1}$$ (9.10)

$$\left\| v \right\|_{N+1+\alpha} \lesssim \delta \delta^{1+\alpha} \epsilon q \theta^{-N} q^{-1}$$ (9.11)

$$\left\| v \right\|_{0+\epsilon} \leq M \left( 1 + \sum_{\lambda = 0}^{q+1} \Lambda \Lambda^{-q \epsilon - \beta} \right)$$ (9.12)

$$\left\| (\partial_t + v \cdot \nabla) \tilde{S}_{N+1} \right\|_{N+\alpha} \lesssim \Lambda \delta^{1+\gamma \eta q \epsilon} \theta^{-6q \epsilon} q^{-1}$$ (9.13)

Finally, assume that

$$\text{supp} \tilde{S} \subseteq T^1 \times \bigcup_i I_i.$$ (9.14)

Then, provided $a \gg 1$ is sufficiently large (depending on the implicit constants in (9.9), (9.10), (9.11), and (9.13)), there exist smooth $(\tilde{v}, \tilde{\rho}) \in C^\infty(T^1 \times T_1, T_2 ; \mathbb{R}^3 \times \mathbb{R})$ and a smooth matrix field $\mathcal{E} \in C^\infty(T^1 \times T_1, T_2 ; \mathcal{S}^{3 \times 3})$ with $\text{supp} \mathcal{E} \subseteq \text{supp} S$ such that, setting $\tilde{R}_1 := R - S - \mathcal{E}$, the triple $(\tilde{v}, \tilde{\rho}, \tilde{R})$ is a strong subsolution with

$$\int_{T^1} |\tilde{v}|^2 + \text{tr} \tilde{R} dx = \int_{T^1} |v|^2 + \text{tr} R dx \quad \forall t.$$ (9.15)

Moreover, we have the estimates

$$\left\| \bar{v} - v \right\|_{H^{-1}} \leq \frac{M}{2} \delta^{\frac{1}{q}} \delta^{\frac{1}{q}} q^{-1} q^{-1}$$ (9.16)

$$\left\| \bar{v} - v \right\|_0 \leq \frac{M}{2} \delta^{\frac{1}{q}} q^{-1} q^{-1}$$ (9.17)

$$\left\| \bar{v} - v \right\|_{1+\alpha} \leq \frac{M}{2} \delta^{\frac{1}{q}} q^{-1} q^{-1}$$ (9.18)

$$\left\| \bar{v} - v \right\|_{0+\epsilon} \leq M \delta^{\frac{1}{q}} q^{-1} q^{-1}$$ (9.19)

and the error $\mathcal{E}$ satisfies the estimates

$$\left\| \mathcal{E} \right\|_0 \leq \delta \delta^{\frac{1}{2}} q^{-6q} q^{-1} q^{-1}$$ (9.20)

$$\left| \partial_t \text{tr} \mathcal{E} \right| \leq \delta \delta^{\frac{1}{2}} q^{-6q} q^{-1} q^{-1}.$$ (9.21)

Finally, setting

$$\mathcal{T}(t) := \frac{1}{3} \int_0^t \int_{T^1} \left( \left| (-\Delta)^{\frac{1}{2}} \bar{v} \right|^2 - \left| (-\Delta)^{\frac{1}{2}} v \right|^2 \right) dx ds$$

$$= \frac{1}{3} \int_0^t \int_{T^1} \left( (-\Delta)^{\frac{1}{2}} (\bar{v} + v) \cdot (-\Delta)^{\frac{1}{2}} (\bar{v} - v) \right) dx ds,$$

we have that

$$\left| \partial_t \mathcal{T} \right| \lesssim \Lambda \theta^{-q \epsilon - \beta},$$ (9.22)

for any $\epsilon > 0$. Thus, $\mathcal{T}_p \text{Id}$ satisfies (9.21) for all $t \in [0, T]$, and (9.20) only for small times.

The proof extends [13, Section 7], which is a localization of the argument carried out in [4, Section 5]. The difference between [13] and [4] is that the latter absorbs the whole $R$ with the perturbation flow, whereas the former, as well as the proof below, aims to only absorb $S$.

Proof.
Step 1: Squiggling Stripes and the Stress Tensors $\tilde{S}_i$

As in [4, Lemma 5.3], we choose a family of smooth non-negative $\eta_i = \eta_i(x,t)$ with the following properties:

\begin{align}
\eta_i & \in C_c^\infty (T^3 \times [T_1, T_2]; [0, 1]) \tag{9.23} \\
\text{supp } \eta_i \cap \text{supp } \eta_j = \emptyset \quad & \forall i \neq j \tag{9.24} \\
T^3 \times I_i & \subset \{ (x,t) : \eta_i(x,t) = 1 \} \tag{9.25} \\
\text{supp } \eta_i & \subseteq T^3 \times (I_i \cup I_i \cup I_{i+1}) \tag{9.26}
\end{align}

\[ \exists c_0 > 0 : \sum_i \int_{T^3} \eta_i^2(x,t) dx \geq c_0 \quad \forall t \in [0, T] \tag{9.27} \]

\[ \| \partial_t^N \eta_i \|_m \leq C(N,m) \tau_q^{-N} \quad N,m \geq 0, \tag{9.28} \]

where the $c_0$ in (9.27) is a geometric constant. Define

\[ \sigma_i(x,t) := \frac{\eta_i^2(x,t)}{\sum_j \eta_j^2(y,t) dy} \sigma_q(t), \]

so that $\sum_i \int_{T^3} \sigma_i dx = \int_{T^3} \sigma_q dx$. Using the inverse flow $\Phi_t$ starting at time $t_i$

\[ \left\{ \begin{array}{l}
(\partial_t + v \cdot \nabla) \Phi_t = 0 \\
\Phi_t(x,t_i) = x
\end{array} \right., \]

set

\[ S_i := \sigma_i \text{Id} + \eta_i^2 \tilde{S}_q \]

\[ \tilde{S}_i := \frac{\mathcal{D} \Phi_t \sigma_i \mathcal{D}^T \Phi_t}{\sigma_i} = \frac{\mathcal{D} \Phi_t}{\sigma_i} \left( \frac{\sum_j \eta_j^2}{\int_{T^3} \sigma} \right) \mathcal{D}^T \Phi_t. \]

One can check from (9.27)-(9.28) and from (9.24) and (9.25) that

\[ \| \sigma_i \|_0 \leq 4 |T^3| c_0^{-1} \delta_{q+1} \tag{9.29} \]

\[ \| \sigma_i \|_N \leq \rho_i := \rho(t_i) \leq \delta_{q+1}, \tag{9.30} \]

and moreover, since by (9.14) supp $\tilde{S}_q \subseteq \{ \sum_i \eta_i^2 = 1 \}$

\[ \frac{1}{3} \text{tr} \sum_i \int_{T^3} S_i dx = \frac{1}{3} \text{tr} \tilde{S}_q, \quad \sum_i \tilde{S}_i = \tilde{S}_q \tag{9.31} \]

We next claim that for all $(x,t)$

\[ \tilde{S}_i(x,t) \in B_{\frac{1}{2}}(\text{Id}) \subseteq S^{3 \times 3}_+, \tag{9.32} \]

where $B_{\frac{1}{2}}(\text{Id})$ is the ball of radius $\frac{1}{2}$ centered at the identity $\text{Id}$ in $S^{3 \times 3}$. Indeed, by the classical estimates on transport equations reported in Proposition 2.1

\[ \| \nabla \Phi_t - \text{Id} \|_{0} \lesssim \tau_q \delta_{q}^{1+\alpha} = \rho_{q}^{1+\alpha} \lesssim c_{q}^{3\alpha} \]

(9.33)
for \( t \in J_i \cup I_i \cup J_{i+1} \), since this is an interval of length \( |J_i \cup I_i \cup J_{i+1}| \sim \tau_q \). Using (9.7), (9.10) and (4.13), we also have that, for any \( N \geq 0 \)

\[
\left\| \frac{\eta^2 S_w}{\sigma_i} \right\|_N \lesssim \left\| \frac{\eta \gamma}{\sigma_i} \right\|_N \lesssim q^{-N} \gamma^{-N} \lesssim \gamma^{-N} \lambda^{-N} q^{-1} + \lambda^{-N} q^{-1} \gamma^{-N} + \lambda^{-N} q^{-1} \gamma^{-N} \lambda^{-N} q^{-1} \gamma^{-N},
\]

(9.34)

Then, using the decomposition

\[
\tilde{S}_i - \text{Id} = \mathcal{D} \Phi_{q-1} \frac{\eta^2 S_w}{\sigma_i} \mathcal{D} \Phi^T_{q-1} + \mathcal{D} \Phi_{q-1} \left( \mathcal{D} \Phi^T_{q-1} - \text{Id} \right) + \mathcal{D} \Phi_{q-1} - \text{Id},
\]

we deduce from (9.33)-(9.34) that

\[
|\tilde{S}_i - \text{Id}| \lesssim (1 + \rho^3) \lambda_{q+1}^{-2} \rho^3 (1 + \rho^3) + 2 \rho^3 \leq \frac{1}{2},
\]

provided \( a \gg 1 \) is sufficiently large, since we assumed \( \alpha < \beta \gamma \) in (9.1). This verifies (9.32).

**Step 2: The perturbation \( w \).**

Now we can define the perturbation term as

\[
w_a := \sum_i \sqrt{\sigma_i} (\mathcal{D} \Phi_i)^{-1} W (\tilde{S}_i, \lambda_{q+1} \Phi_i) = \sum_i w_{ao},
\]

where \( W \) are the Mikado flows on the compact set \( B_\cdot (\text{Id}) \) as defined in **Lemma 2.2**. Notice that the supports of the \( w_{ao} \) are disjoint and, using the Fourier series representation of the Mikado flows

\[
w_{oi} := \sum_{k \neq 0} (\mathcal{D} \Phi_i)^{-1} b_{i,k} A_k e^{\lambda_{q+1} k \Phi_i},
\]

(9.35)

where we write

\[
b_{i,k}(x,t) := \sqrt{\sigma_i(x,t)} a_k(\tilde{S}_i(x,t)).
\]

We define \( w_c \) so that \( w := w_a + w_c \) is divergence-free:

\[
w_c := \frac{i}{\lambda_{q+1}} \sum_{k \neq 0} \mathcal{D} (b_{i,k}) \frac{\mathcal{D} \Phi^T_i (k \times A_k)}{|k|^2} e^{\lambda_{q+1} k \Phi_i} = \sum_{i,k \neq 0} c_{i,k} e^{\lambda_{q+1} k \Phi_i},
\]

(9.36)

where we write

\[
c_{i,k} = \mathcal{D} (b_{i,k}) \frac{\mathcal{D} \Phi^T_i (k \times A_k)}{|k|^2}.
\]

Define then

\[
w := w_a + w_c,
\]

\[
\bar{\nu} := \nu + w
\]

\[
\bar{p} := p - \sum_i \sigma_i
\]

\[
\bar{\varepsilon} (x,t) := \bar{\varepsilon}^{(1)} (x,t) + \bar{\varepsilon}^{(2)},
\]

where

\[
\bar{\varepsilon}^{(1)} := R (\partial_t \bar{\nu} + \text{div}(\bar{\nu} \otimes \bar{\nu}) + \nabla \bar{p} + (-\Delta)^{\frac{1}{2}} \bar{\nu} + \text{div}(R - S_w)),
\]

(9.37)

with \( R \) being the anti-divergence operator defined in **Definition 2.1** and

\[
\bar{\varepsilon}^{(2)} (t) := \frac{\text{Id}}{3} \int_{\mathbb{T}^3} \left( |\bar{\nu}|^2 - |\nu|^2 - \text{tr} S_w \right) dx.
\]

(9.38)

Equations (9.15) and (3.1) follow by construction.
Step 3: Estimates on the perturbation

The estimates on \(\tilde{v}\) follow similarly to the ones for \(v_{q+1}\) in \([3, \text{Section 5-6}]\). Obtaining those requires estimates on the coefficients \(b_{i,k}, c_{i,k}\), which in turn require estimates of \(\tilde{S}_i\) and estimates of \(\mathcal{D}\Phi_i\). The latter read as follows:

\[
\|\mathcal{D}\Phi_i - \text{Id}\|_N + \| (\mathcal{D}\Phi_i)^{-1} - \text{Id} \|_N \lesssim \ell_q^{3\alpha} \ell_{q,i}^{-N} \tag{9.39}
\]

\[
\|\mathcal{D}\Phi_i\|_N + \| (\mathcal{D}\Phi_i)^{-1} \|_N \lesssim \ell_q^{3\alpha} \ell_{q,i}^{-N} \tag{9.40}
\]

\[
\| (\partial_i + v \cdot \nabla)\mathcal{D}\Phi_i \|_N \lesssim \ell_q \ell_{q,i}^{3\alpha} \ell_{q,i}^{-N}. \tag{9.41}
\]

To obtain these, we first observe that \(\Phi_i\) is a diffeomorphism, which implies both \(\mathcal{D}\Phi_i\) and \((\mathcal{D}\Phi_i)^{-1}\) are bounded, thus yielding the \(N = 0\) case of \((9.40)\). To obtain \((9.39)\), we start by combining \((9.33)\) with the \(N = 0\) case of \((9.40)\), thus obtaining that

\[
\| (\mathcal{D}\Phi_i)^{-1} - \text{Id} \|_0 \leq \| (\mathcal{D}\Phi_i)^{-1} \|_0 \| \text{Id} - \mathcal{D}\Phi_i \|_0 \lesssim \ell_q^{3\alpha}. \tag{9.39}
\]

This yields \((9.39)\) for \(N = 0\). For \(N \geq 1\), we note that

\[
\|\mathcal{D}\Phi_i - \text{Id}\|_N + \| (\mathcal{D}\Phi_i)^{-1} - \text{Id} \|_N \lesssim \|\mathcal{D}\Phi_i - \text{Id}\|_0 + \| (\mathcal{D}\Phi_i)^{-1} - \text{Id} \|_0 + \|\mathcal{D}\Phi_i \|_N^{-1} + \|\mathcal{D}(\mathcal{D}\Phi_i)^{-1} \|_N^{-1}. \tag{9.40}
\]

The other cases of \((9.39)\) follow by combining its \(N = 0\) case with the \(N \geq 1\) cases of \((9.40)\).

The estimates for \(\|\mathcal{D}\Phi_i\|_N\) for \(N \geq 1\) follow from Proposition 2.1. By combining \((2.20)\) with Lemma 2.1, we obtain that

\[
\| (\partial_i + v \cdot \nabla)\mathcal{D}\Phi_i \|_N \lesssim \|\mathcal{D}\Phi_i \|_N \| v \|_0 + \| v \|_N \|\mathcal{D}\Phi_i \|_0. \tag{9.41}
\]

Estimates \((9.40)\) and \((9.41)\) then yield \((9.41)\).

To complete the proof of \((9.40)\), we are left with estimating \(\| (\mathcal{D}\Phi_i)^{-1} \|_N\). We note that \(\mathcal{D}^N(\Phi_i \circ \Phi_i^{-1}) = 0\) for \(N \geq 1\). Then we use the Leibniz rule and the chain rule to write

\[
\mathcal{D}^2(\Phi_i \circ \Phi_i^{-1}) = \mathcal{D}(\mathcal{D}(\Phi_i \circ \Phi_i^{-1})\mathcal{D}\Phi_i^{-1}) = (\mathcal{D}^2(\Phi_i \circ \Phi_i^{-1})(\mathcal{D}\Phi_i^{-1})^2 + (\mathcal{D}\Phi_i \circ \Phi_i^{-1})\mathcal{D}^2\Phi_i^{-1}
\]

\[
\mathcal{D}^3(\Phi_i \circ \Phi_i^{-1}) = \mathcal{D}(\mathcal{D}^2(\Phi_i \circ \Phi_i^{-1}))
\]

\[
= (\mathcal{D}^3(\Phi_i \circ \Phi_i^{-1})(\mathcal{D}\Phi_i^{-1})^3 + 3(\mathcal{D}^2(\Phi_i \circ \Phi_i^{-1})(\mathcal{D}^2\Phi_i^{-1})(\mathcal{D}\Phi_i^{-1})^2 + (\mathcal{D}\Phi_i \circ \Phi_i^{-1})\mathcal{D}^3\Phi_i^{-1}
\]

\[
\mathcal{D}^4(\Phi_i \circ \Phi_i^{-1}) = \mathcal{D}(\mathcal{D}^3(\Phi_i \circ \Phi_i^{-1}))
\]

\[
= (\mathcal{D}^4(\Phi_i \circ \Phi_i^{-1})(\mathcal{D}\Phi_i^{-1})^4 + 6(\mathcal{D}^3(\Phi_i \circ \Phi_i^{-1})(\mathcal{D}^2\Phi_i^{-1})(\mathcal{D}^2\Phi_i^{-1}) + 3(\mathcal{D}^2(\Phi_i \circ \Phi_i^{-1})(\mathcal{D}^2\Phi_i^{-1})^2
\]

\[
+ 4(\mathcal{D}^2(\Phi_i \circ \Phi_i^{-1})(\mathcal{D}^2\Phi_i^{-1})(\mathcal{D}\Phi_i^{-1}) + (\mathcal{D}\Phi_i \circ \Phi_i^{-1})\mathcal{D}^4\Phi_i^{-1}
\]

From these, we can see that

\[
\|\mathcal{D}^2\Phi_i^{-1}\|_0 \leq \|\mathcal{D}\Phi_i^{-1}\|_0 \|\mathcal{D}^2\Phi_i^{-1}\|_0 \lesssim \ell_q^{3\alpha} \ell_{q,i}^{-1}\tag{9.40}
\]

\[
\|\mathcal{D}^3\Phi_i^{-1}\|_0 \leq \|\mathcal{D}\Phi_i^{-1}\|_0 \|\mathcal{D}^3\Phi_i^{-1}\|_0 + 3\|\mathcal{D}^2\Phi_i^{-1}\|_0 \lesssim \ell_q^{3\alpha} \ell_{q,i}^{-2}\tag{9.41}
\]

\[
\|\mathcal{D}^4\Phi_i^{-1}\|_0 \leq \|\mathcal{D}\Phi_i^{-1}\|_0 \|\mathcal{D}^4\Phi_i^{-1}\|_0 + 6\|\mathcal{D}\Phi_i^{-1}\|_0 \|\mathcal{D}^3\Phi_i^{-1}\|_0 \lesssim \ell_q^{3\alpha} \ell_{q,i}^{-3}\tag{9.42}
\]

These two examples show us that

\[
\mathcal{M}(\Phi_i \circ \Phi_i^{-1}) = (\mathcal{D}\Phi_i \circ \Phi_i^{-1})\mathcal{D}^M\Phi_i^{-1} + \text{other terms},
\]

where the other terms are of the form \((\mathcal{D}^k\Phi_i \circ \Phi_i^{-1})(\mathcal{D}\Phi_i^{-1})^n(\mathcal{D}^m\Phi_i^{-1})^n\), where \(k + (m-1)n = M - 1\) and \(m < M\). If we assume \((9.40)\) for \(N < M - 1\), we see that such terms are estimated as \(\ell_{q,i}^{-(k-1)}\ell_{q,i}^{-(m-1)n} = \ell_{q,i}^{-(1-M)}\), thus so is \(\mathcal{D}^M\Phi_i^{-1} = (\mathcal{D}^{M-1}\Phi_i^{-1})\), which proves \((9.40)\) for \(N = M - 1\). Thus, by induction, the estimate \((9.40)\) is proved.
The following estimates then follow precisely as in [4, Propositions 5.7 and 5.9]:

$$
\begin{align*}
\| S_i \|_N & \lesssim \ell^{-N} \\
\| b_{i,k} \|_N & \lesssim \rho_i^{-1} |k|^{-6} \ell^{-N} \\
\| c_{i,k} \|_N & \lesssim \rho_i^{-1} |k|^{-6} \ell^{-N-1} \\
\| D_i S_i \|_N & \lesssim \tau_{q_i}^{-1} \ell^{-N} \\
\| D_i b_{i,k} \|_N & \lesssim \delta_{q_i+1} \tau_{q_i}^{-1} \ell^{-N} |k|^{-6} \\
\| D_i c_{i,k} \|_N & \lesssim \delta_{q_i+1} \tau_{q_i}^{-1} \ell^{-N-1} |k|^{-6}.
\end{align*}
$$

To obtain (9.42), observe first that, by its definition, we have that

$$
\tilde{S}_i = D_i \Phi_i \mathcal{D}^T \Phi_i + D_i \Phi_i, \quad \frac{\tilde{S}_i}{\sigma_i} = \text{Id} + f(t) \frac{\tilde{S}_i}{\sigma_i},
$$

and therefore

$$
\| \tilde{S}_i \|_{N} \lesssim \| D_i \Phi_i \|_{N} + \| D_i \Phi_i \|_{0} \left\| \frac{\tilde{S}_i}{\sigma_i} \right\|_{0} + \left\| \frac{\tilde{S}_i}{\sigma_i} \right\|_{N},
$$

where we used that \( \| D_i \Phi_i \|_{0} \lesssim 1 \). By (9.40), the first term above obeys (9.42). To estimate the remaining two terms, we use (9.10) and (9.7) to obtain that

$$
\left\| \frac{\tilde{S}_i}{\sigma_i} \right\|_{N} \lesssim \left\| \frac{\tilde{S}_i}{\sigma_i} \right\|_{N+\alpha} \lesssim \zeta q_{,i} \ell^{-N} \ell^{-2a} \lesssim \ell^{-N} \ell_{q,i}^{2a-2b}. (9.48)
$$

Estimate (9.42) then follows from (9.40) and the assumption (9.1), i.e. that \( \alpha < \beta \). The proof of (9.45) follows a similar strategy. First, we decompose \( S_i \sigma_i^{-1} \) and its material derivative as

$$
\frac{S_i}{\sigma_i} = \text{Id} + f(t) \frac{S_i}{\sigma_i} = \text{Id} + f(t) \frac{\tilde{S}_i}{\sigma_i} - f(t) \frac{\tilde{S}_i}{\sigma_i},
$$

(9.49)

We then decompose \( D_i \tilde{S}_i \) as follows:

$$
D_i \tilde{S}_i = D_i \Phi_i \frac{S_i}{\sigma_i} \mathcal{D}^T \Phi_i + D_i \Phi_i D_i \frac{S_i}{\sigma_i} \mathcal{T} \Phi_i + D_i \Phi_i \frac{S_i}{\sigma_i} \mathcal{T} \Phi_i + D_i \Phi_i \frac{S_i}{\sigma_i} \mathcal{T} \Phi_i = I + II + III.
$$

The terms \( I \) and \( III \) are estimated in similar ways, so we only estimate \( I \). To that end, we note that, using (9.48) and the fact that \( f \lesssim 1 \), we can obtain that

$$
\left\| \frac{S_i}{\sigma_i} \right\|_{N} \lesssim 1 + \left\| \frac{\tilde{S}_i}{\sigma_i} \right\|_{N} \lesssim \ell^{-N}. \quad (9.48)
$$

Using this bound together with (9.40) and (9.41), we obtain that

$$
\left\| I \right\|_{N} \lesssim \left\| D_i \Phi_i \right\|_{N} \left\| \frac{S_i}{\sigma_i} \right\|_{0} + \left\| D_i \Phi_i \right\|_{0} \left\| \frac{S_i}{\sigma_i} \right\|_{N} + \left\| \frac{S_i}{\sigma_i} \right\|_{0} \left\| \Phi_i \right\|_{0} \left\| \frac{S_i}{\sigma_i} \right\|_{N} \lesssim \delta_{q,i}^{1+1+\alpha} \ell_{q}^{3\beta \alpha N+\alpha} \ell_{q,i}^{3\beta \alpha N} \ell_{q,i}^{-N} \lesssim \delta_{q,i}^{1+1+\alpha} \ell_{q}^{3\beta \alpha N+\alpha} \ell_{q,i}^{3\beta \alpha N} \ell_{q,i}^{-N}.
$$

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Combining the above with
\[ \delta_q^2 \lambda^{1+\alpha}_q = \tau_q^{-1} \xi_q^3 \leq \tau_q^{-1}, \]  
we see that \( I \) satisfies (9.45).

Coming to \( II \), we first note that
\[ \|II\|_N \lesssim \|D \Phi_i\|_N \left[ \|D_i S_i\|_0 \right] + \|D \Phi_i\|_0^2 \left[ \|D_i S_i\|_N \right]. \]  
(9.51)

Therefore, to prove that \( II \) satisfies (9.45), we need to estimate \( D_i (\sigma_i^{-1} S_i) \). Recalling (9.49), we note that
\[ \left\| D_i \frac{S_i}{\sigma_i} \right\|_N \lesssim \left| f'(t) \right| \left\| \frac{\delta_i}{\sigma} \right\|_N + \frac{f(t)}{\sigma(t)} \left\| D_i \delta_i \right\|_N + \frac{f(t)\sigma'}{\sigma^2} \left\| \dot{S}_i \right\|_N =: P_1 + P_2 + P_3. \]  
(9.52)

To estimate \( P_1 \), we first note that, by (9.28)
\[ f'(t) = \frac{\sum \int T^3 \partial_i \eta_i(x,t) \eta_i(x,t) dx}{|T^3|} \lesssim \tau_q^{-1}. \]
Combining this with (9.48), we conclude that
\[ \|P_1\|_N \lesssim \tau_q^{-1} \xi_q^{N-1}. \]  
(9.53)

To bound \( P_2 \) from above, we recall that \( f \sim 1 \), and apply (2.13) to conclude that
\[ \|P_2\|_N \lesssim \Lambda \xi_q^{6} \xi_q^{\alpha} \delta_q \xi_q. \]

We then note that
\[ \xi_q^{\alpha} \xi_q^{\alpha} \delta_q \xi_q \lesssim \tau_q^{-1} \xi_q^{2\alpha-2\beta}, \]  
(9.54)
so that \( P_2 \) also satisfies (9.53).

Coming, finally, to \( P_3 \), we use (9.9), (49.48), and (9.50) to conclude that \( P_3 \) also satisfies (9.53). Combining the bounds on \( P_1, P_2, P_3 \) we have just obtained, we conclude that \( D_i (\sigma_i^{-1} S_i) \) also satisfies (9.53), i.e.
\[ \left\| D_i \frac{S_i}{\sigma_i} \right\|_N \lesssim \tau_q^{-1} \xi_q^{N-1}. \]

Combining this estimate with (9.40), we conclude (9.45).

To prove (9.43) and (9.46), we first prove some estimates on \( \sqrt{\sigma_i} \). Firstly, we note that, thanks to (9.27) and (9.38)
\[ \|\sqrt{\sigma_i}\|_N \lesssim \left[ \frac{\left| T^3 \right|^{1/2}}{\sqrt{C_0}} \right] \|\eta_i\|_N \lesssim \left[ \frac{\left| T^3 \right|^{1/2}}{\sqrt{C_0}} \right] \|\eta_i\|_N \lesssim \frac{\eta_i^2}{\sqrt{C_0}} \lesssim \eta_i^2. \]  
(9.55)

As for the material derivative, setting
\[ h(t) := \sum \int \eta_i^2(x,t) dx, \]
we have that
\[ \sigma_i = \frac{\left| T^3 \right| |\nabla \eta_i|^2}{h}. \]

We can thus write
\[ D_i \sqrt{\sigma_i} = \frac{1}{2} \frac{\sqrt{\sigma_i}}{2} D_i \sigma_i = \frac{\sqrt{h}}{\left| T^3 \right|^{1/2} |\nabla \eta_i| h} \left( \frac{2 \left| T^3 \right| \nabla \eta_i D_i \eta_i |\nabla \eta_i|^2}{h} + \left| T^3 \right| \nabla \eta_i \right) \]
\[ = \frac{\left| T^3 \right| |\nabla \eta_i|^2}{\sqrt{h}} D_i \eta_i + \frac{\left| T^3 \right| \nabla \eta_i}{\sqrt{h}} + \left| T^3 \right| \eta_i \frac{1}{2 \sqrt{h}} \nabla \sigma \nabla \eta_i - \left| T^3 \right| \frac{\sqrt{h}}{2h^2} \nabla \sigma \nabla h \eta_i, \]
\[ = I + II + III + IV. \]
We then note that, by (9.28) and (9.11), we have that
\[ \|D_t \eta_i\|_N \leq \|\partial_t \eta_i\|_N + \|v\|_N \|\eta_i\|_0 + \|\eta_i\|_N \lesssim \tau_q^{-1} + \delta_q^{1/2} \lambda_q^{-1} \|\eta_i\|_N. \]
Combining this with (9.8), (9.27), and the fact \( \psi \leq 1 \), we easily see that
\[ \|I\|_N \lesssim \frac{\|T^3\|_1}{c_0} \|\partial_t \eta_i\|_N \lesssim \delta_q^{1/2} \lambda_q^{-1} \|\eta_i\|_N. \]

Coming to \( II \), we estimate it by using (9.8), (9.27), (9.28), and (9.6):
\[ \|II\|_N \lesssim \frac{\|T^3\|_1}{c_0} \delta_q^{1/2} \lambda_q^{-1} \|\eta_i\|_N \lesssim \delta_q^{1/2} \lambda_q^{-1} \|\eta_i\|_N. \]

As for \( III \), we use (9.27), (9.9), (9.28), and the fact that \( \psi \leq 1 \):
\[ \|III\|_N \lesssim \frac{\|T^3\|_1}{c_0} \delta_q^{1/2} \lambda_q^{-1} \|\eta_i\|_N \lesssim \delta_q^{1/2} \lambda_q^{-1} \|\eta_i\|_N. \]

Finally, to estimate \( IV \), we first note that
\[ \|h'\|_N = \left\| \sum_j \int_T \partial_t^j \eta_j dy \right\|_N \lesssim \sum_j \int_T (\|\eta_j\|_N \|\partial_t^j \eta_j\|_0 + \|\eta_j\|_0 \|\partial_t \eta_j\|_N) dy \lesssim K \tau_q^{-1}, \]
where \( K > 0 \) is a constant. We used (9.28). The term \( IV \) is thus estimated by combining this bound with (9.27), (9.8), (9.28), and the fact \( \psi \leq 1 \):
\[ \|IV\|_N \lesssim K [T^3 + \delta_q^{-1} \lambda_q^{-1} \|\eta_i\|_N \lesssim \delta_q^{-1} \lambda_q^{-1} \|\eta_i\|_N. \]

By combining the above bounds on \( I, II, III, IV \), we conclude that
\[ \|D_t \sqrt{\sigma_t}\|_N \lesssim \delta_q^{-1} \lambda_q^{-1} \|\eta_i\|_N. \]

To prove (9.43) and (9.46), we note that
\[ \|b_{l,k}\|_N \lesssim \sqrt{\sigma_t} \|\alpha_k(\bar{S}_i)\|_0 + \sqrt{\sigma_t} \|\alpha_k(\bar{S}_i)\|_N, \]
\[ \|D_t b_{l,k}\|_N \lesssim \|D_t \sqrt{\sigma_t} \|_N \|\alpha_k(\bar{S}_i)\|_0 + \|D_t \sqrt{\sigma_t} \|_N \|\alpha_k(\bar{S}_i)\|_N + \|\sqrt{\sigma_t} \|_N \|D_t \|_N \|\alpha_k(\bar{S}_i)\|_0 + \|\sqrt{\sigma_t} \|_0 \|D_t \|_N \|\alpha_k(\bar{S}_i)\|_N. \]

The bounds (9.43) and (9.46) then readily follow by combining (9.55), (9.57), and the following applications of (9.44):
\[ \|a_k(\bar{S}_i)\|_N \lesssim \|D_t \|_N \|\bar{S}_i\|_N, \]
\[ \|D_t (a_k(\bar{S}_i))\|_N \lesssim \|D_t \|_N \|\bar{S}_i\|_N, \]
\[ \|\alpha_k(\bar{S}_i)\|_N \lesssim \|D_t \|_N \|\bar{S}_i\|_N, \]
where \( (\mathcal{D}_{2a_k})_{ij} = \partial_i a_k \) being the matrix of the first derivatives of \( a_k \) w.r.t. the components of its argument. To prove (9.44), we note that, by Leibniz rule
\[ \|c_{l,k}\|_N \lesssim \sum_{i=0}^N \|b_{l,k}\|_{i+1} \|\mathcal{D}^T \Phi_i\|_{N-i}. \]
from which (9.44) follows by (9.43) and (9.40). Coming finally to (9.47), we start by noting that
\[ D_i \nabla (b_{i,k}) = \nabla D_i (b_{i,k}) + [v \cdot \nabla, \nabla](b_{i,k}). \]
This means that
\[ ||D_i c_{i,k}||_N \leq \sum_{i=0}^{N} \left( ||\nabla D_i (b_{i,k})||_i + ||[v \cdot \nabla, \nabla](b_{i,k})||_i \right) ||D_i^T \Phi_i||_{N-i} + \sum_{i=0}^{N} ||D b_{i,k}||_i ||D_i D^T \Phi_i||_{N-i}. \]
Since, by the estimates (9.41), (9.40), (9.43), (9.46), and (9.11) on the factors here involved, we see that this scales like \( \ell_{q,i}^{-N} \), we will only need to prove the case \( N = 0 \). In that case, we obtain that
\[ ||D_i c_{i,k}||_0 \leq ||D_i (b_{i,k})||_1 ||D \Phi_i||_0 + ||[v \cdot \nabla, \nabla](b_{i,k})||_0 ||D \Phi_i||_0 + ||b_{i,k}||_1 ||D_i D \Phi_i||_0 = I + II + III. \]
By (9.46) and (9.40), we see that
\[ I \lesssim \delta_{q+1}^{\frac{1}{q}} \ell_{q,i}^{-1} |k|^{-6}. \]
By (9.43), (9.41), and (9.50), we estimate \( III \) as
\[ III \lesssim \rho_1^{\frac{1}{q}} |k|^{-6} \ell_{q,i}^{-1} \lambda_{q+1}^{\frac{1}{q}} \lesssim \delta_{q+1}^{\frac{1}{q}} \ell_{q,i}^{-1} |k|^{-6}. \]
We are then left with proving that \( II \) also satisfies this bound. To this end, we rewrite the commutator as
\[ [v \cdot \nabla, \nabla](b_{i,k}) = \sum_{j} (v_j \partial_j (\partial_i b_{i,k}) - \partial_i (v_j \partial_j b_{i,k})) e_\ell = - \sum_{j} \partial_j v_j \partial_j b_{i,k} e_\ell = \nabla v \nabla b_{i,k}. \]
It then follows from (9.40), (9.43), and (9.50) that \( II \) satisfies the same bound as \( I \) and \( III \), thus proving (9.47). In turn, the estimates on \( \tilde{v} \) in (9.17) follow from the ones just given precisely as in [4] Corollary 5.8, pp. 23-24. Indeed, once we note that
\[ ||\nabla (e^{\alpha \cdot k - 1})||_0 \leq 2 \lambda_{q+1} |k|, \]
we can deduce from the estimates above that
\[ ||w_{o,i}||_N \lesssim \sum_{i,k} ||D \Phi_i^{-1}||_N ||b_{i,k}||_0 ||e^{\alpha \cdot k - 1}||_0 + \sum_{i,k} ||D \Phi_i^{-1}||_0 ||b_{i,k}||_N ||e^{\alpha \cdot k - 1}||_0 \]
\[ + \sum_{i,k} ||D \Phi_i^{-1}||_0 ||b_{i,k}||_0 ||e^{\alpha \cdot k - 1}||_N \lesssim \delta_{q+1}^{\frac{1}{q}} \lambda_{q+1}^{N}. \]
The \( w_{o,i} \) have pairwise disjoint supports, so the sum over \( i \) always consists of a single term, which yields that the desired estimates hold for \( w_o \). The estimates on \( c_{i,k} \lambda_{q+1}^{-1} \) are always better than those on \( b_{i,k} \), meaning that any estimate that holds for \( w_o \) holds for \( w_c \) as well. Thus, (9.17) follows directly, and (9.18) and (9.19) follow by interpolation.
Coming to (9.16), the fact that \( w_c \) satisfies this bound can be easily deduced from (9.44), which tells us that
\[ \lambda_{q+1}^{-1} ||c_{i,k}||_0 \lesssim \delta_{q+1}^{\frac{1}{q}} |k|^{-6} \ell_{q,i}^{-1} \lambda_{q+1}^{-1}. \]
To estimate \( w_o \), we use a procedure similar to the one employed in Section 7 to prove (7.3), replacing (7.10) with (9.43).

**Step 4: Estimates on the new Reynolds term \( \hat{\epsilon}^{(1)} \).**

The aim of this section is to prove \( \hat{\epsilon}^{(1)} \) satisfies (9.20), namely
\[ ||\hat{\epsilon}^{(1)}||_0 \leq \delta_{q+2} \lambda_{q+1}^{-6}. \]
Drawing from [4], we decompose $\hat{\xi}^{(1)}$ as

$$
\hat{\xi}^{(1)} = R \left( \partial_t w + \text{div}(v \otimes w + w \otimes v + v \otimes w) - \sum_i \nabla \sigma_i + (-\Delta)^\theta w - \text{div} S_\psi \right)
= R \left( \partial_t w + w \cdot \nabla v + v \cdot \nabla w + \text{div}(w \otimes w) - \sum_i \nabla \sigma_i + (-\Delta)^\theta w - \text{div} S_\psi \right)
= \left[ R(\cdot \cdot) + R((\cdot \cdot) + R \left[ \text{div}(w \otimes w - \hat{S}_\psi) - \sum_i \nabla \sigma_i \right] + R((-\Delta)^\theta w) \right].
$$

We then note that, since the $w_{o,i}$ have disjoint supports and $w_o = \sum_i w_{o,i}$, by (9.31), we have that

$$
\text{div}(w \otimes w - \hat{S}_\psi) - \sum_i \nabla \sigma_i = \text{div}(w_o \otimes w_c + w_c \otimes w_o + w_c \otimes w_c) + \sum_i \left[ \text{div} \left( w_{o,i} \otimes w_{o,i} - \eta_i^2 \hat{S}_\psi - \frac{1}{3} \text{tr} S_t \right) \right]. \quad (9.59)
$$

We now rewrite the first three terms using the definition of $w_c$ (9.59) (to rewrite $\xi_O$), and the fact that $D_i e^{\beta_{q+1} k \Phi t} = 0$ (to rewrite $\xi_T$):

$$
\xi_N = \sum_{i,k} R \left( (\mathcal{D} \Phi_i^{-1} b_{i,k} A_k + \lambda_{q+1}^{-1} c_{i,k} \cdot \nabla v) e^{\beta_{q+1} k \Phi t} \right),$$

$$
\xi_T = \sum_{i,k} R \left( (D_i \mathcal{D} \Phi_i^{-1} b_{i,k} A_k + \mathcal{D} \Phi_i^{-1} D_i b_{i,k} A_k + \lambda_{q+1}^{-1} D_i c_{i,k} \cdot e^{\beta_{q+1} k \Phi t} \right),$$

$$
\xi_O = \sum_{i,k} \left[ \text{div} \left( w_{o,i} \otimes w_{o,i} - S_t \right) \right] + \sum_i \left[ \text{div} \left( w_{o,i} \otimes w_{c,i} - S_t \right) \right] = \sum_{i,k} R \left( \text{div}(\nabla \sigma_i \mathcal{D} \Phi_i^{-1} C_k(\hat{S}_\psi)) e^{\beta_{q+1} k \Phi t} \right),$$

where the $C_k$ are as defined in Lemma 2.2. We now note that the leading order terms are

$$
\xi_N^{(L)} = \sum_{i,k} R \left( (\mathcal{D} \Phi_i^{-1} b_{i,k} A_k) \cdot \nabla v e^{\beta_{q+1} k \Phi t} \right),$$

$$
\xi_T^{(L)} = \sum_{i,k} R \left( (D_i \mathcal{D} \Phi_i^{-1} b_{i,k} A_k + \mathcal{D} \Phi_i^{-1} D_i b_{i,k} A_k) e^{\beta_{q+1} k \Phi t} \right),$$

$$
\xi_O^{(L)} = \sum_{i,k} \text{div} \left( w_{o,i} \otimes w_{c,i} + w_{c,i} \otimes w_{o,i} + w_{c,i} \otimes w_{c,i} \right) + \sum_{i,k} R \left( \text{div}(\nabla \sigma_i \mathcal{D} \Phi_i^{-1} C_k(\hat{S}_\psi)) e^{\beta_{q+1} k \Phi t} \right) = \xi_O^{(L,1)} + \xi_O^{(L,2)}.
$$

We start by estimating $\xi_O^{(L,1)}$. Since $\xi_T$ is Calderón-Zygmund, we have that

$$
||R \text{div}(w_o \otimes w_c + w_c \otimes w_o)||_a \lesssim ||w_o||_q ||w_c||_0 + ||w_o||_0 ||w_c||_a.
$$

From (9.43), (9.40), and (9.44), we can conclude that

$$
||w_o||_N \lesssim \rho_i \lambda_{q+1}^N \quad ||w_c||_N \lesssim \rho_i \lambda_{q+1}^{N-1} \lambda_{q+1}^{-1}.
$$

By interpolation, this lets us conclude that

$$
||\xi_O^{(L,1)}||_a \lesssim \rho_i \lambda_{q+1}^{-1} \lambda_{q+1}^{-1}.
$$
To estimate the other leading terms, we start by using Lemma 2.4 on all three:
\[
\left\| \mathcal{E}^{(L)}_{\alpha} \right\|_N \lesssim \sum_{i,k} \left( \frac{\| \nabla \Phi_i^{-1} b_{i,k} \cdot \nabla v \|_{0} + \| \nabla \Phi_i^{-1} b_{i,k} \cdot \nabla v \|_{N + \alpha} + \| \nabla \Phi_i^{-1} b_{i,k} \cdot \nabla v \|_{0} \| \Phi_i \|_{N + \alpha} }{N_{k}^{\alpha}} \right)
\]
\[
\left\| \mathcal{E}^{(L)}_{\alpha} \right\|_T \lesssim \sum_{i,k} \left( \frac{\| D_i \nabla \Phi_i^{-1} b_{i,k} + D_i \Phi_i^{-1} D_i b_{i,k} \|_{0} }{N_{k}^{\alpha}} \right)
\]
\[
\left\| \mathcal{E}^{(L,2)}_{O} \right\|_\alpha \lesssim \sum_{i,k} \left( \frac{\| \text{div}(\sigma_i \nabla \Phi_i^{-1} C_k(\bar{S}_i) \nabla (\Phi_i^{-1})) \|_{0} }{N_{k}^{\alpha}} \right)
\]
\[
+ \sum_{i,k} \left( \frac{\| \text{div}(\sigma_i \nabla \Phi_i^{-1} C_k(\bar{S}_i) \nabla (\Phi_i^{-1})) \|_{N + \alpha} + \| \text{div}(\sigma_i \nabla \Phi_i^{-1} C_k(\bar{S}_i) \nabla (\Phi_i^{-1})) \|_{0} \| \Phi_i \|_{N + \alpha} }{N_{k}^{\alpha}} \right).
\]

To estimate \( \mathcal{E}^{(L)}_N \), we combine (9.40), (9.43), and (9.11) with a Leibniz inequality:
\[
\sum_{i,k} \left( \frac{\| \nabla \Phi_i^{-1} b_{i,k} \cdot \nabla v \|_{0} + \| \nabla \Phi_i^{-1} b_{i,k} \cdot \nabla v \|_{N + \alpha} + \| \nabla \Phi_i^{-1} b_{i,k} \cdot \nabla v \|_{0} \| \Phi_i \|_{N + \alpha} }{N_{k}^{\alpha}} \right)
\]
\[
\lesssim \sum_{i,k} \left( \frac{\| \nabla \Phi_i^{-1} b_{i,k} \|_{0} \| \nabla v \|_{0} + \| \nabla \Phi_i^{-1} b_{i,k} \|_{N + \alpha} + \| \nabla \Phi_i^{-1} b_{i,k} \|_{0} \| \nabla v \|_{N + \alpha} + \| \nabla \Phi_i^{-1} b_{i,k} \|_{N + \alpha} + \| \nabla \Phi_i^{-1} b_{i,k} \|_{N + \alpha} \| \nabla v \|_{0} }{N_{k}^{\alpha}} \right)
\]
\[
\lesssim \sum_{i,k} \left( \frac{\| \nabla \Phi_i^{-1} b_{i,k} \|_{0} \| \nabla v \|_{N + \alpha} + \| \nabla \Phi_i^{-1} b_{i,k} \|_{N + \alpha} + \| \nabla \Phi_i^{-1} b_{i,k} \|_{N + \alpha} \| \nabla v \|_{0} }{N_{k}^{\alpha}} \right) + \sum_{i,k} \left( \frac{\| \nabla \Phi_i^{-1} b_{i,k} \|_{N + \alpha} + \| \nabla \Phi_i^{-1} b_{i,k} \|_{N + \alpha} \| \nabla v \|_{0} }{N_{k}^{\alpha}} \right)
\]
\[
\lesssim \sum_{i,k} \left( \frac{\| \nabla \Phi_i^{-1} b_{i,k} \|_{0} \| \nabla v \|_{N + \alpha} + \| \nabla \Phi_i^{-1} b_{i,k} \|_{N + \alpha} + \| \nabla \Phi_i^{-1} b_{i,k} \|_{N + \alpha} \| \nabla v \|_{0} }{N_{k}^{\alpha}} \right) + \sum_{i,k} \left( \frac{\| \nabla \Phi_i^{-1} b_{i,k} \|_{N + \alpha} \| \nabla v \|_{0} }{N_{k}^{\alpha}} \right)
\]
\[
\lesssim \frac{1}{\lambda_q^{-2\alpha}} \frac{1}{\lambda_q^{-1}} + \frac{1}{\lambda_q^{-1}} + \frac{1}{\lambda_q^{-1}}.
\]

The above holds for any \( N \). If we choose \( N \) to be the \( N \) from Section 4 by (9.3) and (4.14), we conclude that
\[
\left\| \mathcal{E}^{(L)}_N \right\|_{N} \lesssim \frac{\delta_q^{\frac{1}{1 - \alpha}}}{N_{k}^{1}} \frac{\delta_q^{\frac{1}{1 - \alpha}}}{N_{k}^{1}}
\]

Notice how the leading order term here is the one that does not depend on \( N \) thanks to the \( \ell_q \) in (4.14). This is also true of \( \mathcal{E}^{(L)}_T \) and \( \mathcal{E}^{(L,2)}_O \).

\( \mathcal{E}^{(L)}_T \) is estimated in a similar manner, using (9.40), (9.41), (9.43), (9.46), and (4.14).

As for \( \mathcal{E}^{(L,1)}_O \), we first ensure that adding a derivative, whichever factor it lands on, costs at most \( \ell_q^{-1} \). This ensures that the leading term is the first one, because of that gain of \( \ell_q \) mentioned above. We then estimate the leading term.

By (9.40) and (9.43), differentiating \( b_{i,k} \) or \( \nabla \Phi_i^{-1} \) costs \( \ell_q^{-1} \), and by (9.30), differentiating \( \sigma_i \) does not cost anything, so we are left with showing that \( C_k(\bar{S}_i) \) scales like \( \ell_q^{-1} \). Thanks to (2.4), (2.11), and (9.42), we have that
\[
\| C_k(\bar{S}_i) \|_{N} \lesssim \| C_k \|_{1} \| \nabla \bar{S}_i \|_{N - 1} + \| \nabla C_k \|_{N - 1} \| \bar{S}_i \|_{1} \lesssim |k|^{-1} \ell_q^{-2}.
\]

We then use (2.7), (9.40), and the above estimate on \( C_k(\bar{S}_i) \) to estimate the leading term:
\[
\sum_{i,k} \left( \frac{\| \text{div}(\sigma_i \nabla \Phi_i^{-1} C_k(\bar{S}_i) \nabla (\Phi_i^{-1})) \|_{0} }{N_{k}^{\alpha}} \right)
\]
\[
\lesssim \sum_{i,k} \left( \frac{\| \sigma_i \|_{1} \| \nabla \Phi_i^{-1} \|_{0} \| C_k(\bar{S}_i) \|_{0} + 2 \| \sigma_i \|_{0} \| \nabla \Phi_i^{-1} \|_{1} \| C_k(\bar{S}_i) \|_{0} \| \nabla (\Phi_i^{-1}) \|_{0} + \| \sigma_i \|_{0} \| \nabla \Phi_i^{-1} \|_{1} \| C_k(\bar{S}_i) \|_{1} }{N_{k}^{\alpha}} \right)
\]
\[
\lesssim \rho_i \ell_q^{-1} \lambda_q^{-1}.
\]
We thus obtain that
\[
\|\mathcal{E}_N\|_a \lesssim \frac{\delta q_{q+1}^2 \lambda_q}{\lambda_{q+1}^{1-3a}}
\]
\[
\|\mathcal{E}_T\|_a \lesssim \frac{\delta q_{q+1}^2 \lambda_q}{\lambda_{q+1}^{1-5a}}
\]
\[
\|\mathcal{E}_O\|_a \lesssim \frac{\rho_i}{\ell_{q+1}^{1-a}}.
\]

The relation (4.12) easily yields that the above terms satisfy (9.20) for \(a \gg 1\) sufficiently large, since
\[
\frac{\delta q_{q+1}^2 \lambda_q}{\lambda_{q+1}^{1-3a}} \leq \delta q_{q+1}^2 \lambda_q \leq \delta q_{q+1}^2 \lambda_q
\]
\[
\frac{\rho_i}{\ell_{q+1}^{1-a}} \leq \frac{\Lambda \delta q_{q+1}^2 \lambda_q}{\lambda_{q+1}^{1-3a}} \leq \delta q_{q+1}^2 \lambda_q.
\]

(9.61)

Coming to \(\mathcal{E}_D\), which is not present in [4], we estimate it as follows:
\[
\left\| \mathcal{R}(\Delta \theta w) \right\|_0 \lesssim \left\| \mathcal{R} w \right\|_{20+\varepsilon} \lesssim \left\| \mathcal{R} w \right\|_{1-20-\varepsilon} \left\| \mathcal{R} w \right\|_{1+20+\varepsilon}.
\]

(9.62)

At this point, we use Lemma 2.4 to obtain that
\[
\left\| \mathcal{R}_w \right\|_0 \lesssim \left\| \mathcal{R}_w \right\|_a \lesssim \sum_{i,k} \left( \frac{\| \mathcal{D}_i \mathcal{D}_k^{-1} b_{i,k} \|_0}{|k|^{1-a}} + \frac{\| \mathcal{D}_i \mathcal{D}_k^{-1} b_{i,k} \|_{N+a}}{|k|^{N-a}} \right)
\]
\[
\lesssim \frac{1}{\lambda_{q+1}^{1-a}} + \frac{\ell_{q+1}^{N-a}}{\lambda_{q+1}^{1-a}} \lesssim \frac{\delta q_{q+1}^2 \lambda_q}{\lambda_{q+1}^{1-a}}.
\]

(9.63)

the last step being due to (4.14). We also note that
\[
\left\| \mathcal{R}_w \right\|_1 = \max_i \left\| \mathcal{R} \partial_i w \right\|_0.
\]

Proceeding on \(\mathcal{R} \partial w\) as we did on \(\mathcal{R}_w\), then yields
\[
\left\| \mathcal{R}_w \right\|_1 \lesssim \max_i \left\| \mathcal{R} \partial_i w \right\|_a \lesssim \sum_{i,k} \left( \frac{\| \mathcal{D} \mathcal{D}_i^{-1} b_{i,k} \|_1}{|k|^{1-a}} + \frac{\| \mathcal{D} \mathcal{D}_i^{-1} b_{i,k} \|_{N+a}}{|k|^{N-a}} \right)
\]
\[
\lesssim \delta q_{q+1}^2 \lambda_q \lambda_{q+1}^{1-a}.
\]

(9.64)

Such estimates analogously also hold for \(\mathcal{R}_w\), and thus for \(\mathcal{R} w\). Thus, by (9.62)–(9.64)
\[
\left\| \mathcal{R}(\Delta \theta w) \right\|_0 \lesssim \delta q_{q+1}^2 \lambda_q \lambda_{q+1}^{20+\varepsilon-1}.
\]

In particular, for \(a \gg 1\) large enough, (9.20) is satisfied if
\[
2\theta + \varepsilon - 1 + \alpha - \beta < -2b\beta - 6a \iff 7a + \varepsilon < 1 + \beta - 2\theta - 2b\beta.
\]

(9.65)

Since \(\theta < \beta\), and \(2b\beta < 1 - \beta\) by (4.2), we have that \(1 + \beta - 2\theta - 2b\beta > 0\). Thus, (9.65) above holds for \(a, \varepsilon\) sufficiently small.

**Step 5: Estimates on the new Reynolds term \(\mathcal{E}^{(2)}\).**
Now we turn to $\xi^{(2)}$. Consider the decomposition

$$
|\xi^{(2)}| = \frac{1}{3} \left| \int_{\mathbb{T}^3} |v|^2 - |v|^2 - \text{tr} S_v \right|
$$

$$
\leq \frac{1}{3} \left| \int_{\mathbb{T}^3} |w_o|^2 - \text{tr} S_v \right| + \frac{1}{3} \left| \int_{\mathbb{T}^3} 2 w \cdot v \right| + \frac{1}{3} \left| \int_{\mathbb{T}^3} 2 w_c \cdot w_o + |w_c|^2 \right|,
$$

(9.66)

and proceed as in [4] Proposition 6.2]. In the case of the first term, we will estimate the whole tensor, and therefore the trace. For the other terms, only the trace will be handled.

Concerning the first term in (9.66), thanks to (9.31), $\sum \sigma_i = \int \sigma_v$, so that two cancellations occur:

$$
\int w_o \otimes w_o - S_v \, dx = \sum_{i,k \neq 0} \int \sigma_i \mathcal{D} \Phi_i^{-1} C_k(\tilde{S}_i) \mathcal{D} \Phi_i^T e^{\beta \Phi_i} \, dx + \int \left( \sum_i \sigma_i - \sigma_v \right) \, Id \, dx
$$

$$
= \sum_{i,k \neq 0} \int Z_{i,k} e^{\beta \Phi_i} \, dx,
$$

(9.67)

where we write $Z_{i,k} := \sigma_i \mathcal{D} \Phi_i^{-1} C_k(\tilde{S}_i) \mathcal{D} \Phi_i^T$. Using (2.15), (2.11), and (4.14), we obtain that

$$
\left| \int \sum_{i,k \neq 0} Z_{i,k} e^{\beta \Phi_i} \right| \lesssim \sum_{i,k \neq 0} \left| \int Z_{i,k} e^{\beta \Phi_i} \right| \lesssim \sum_{i,k \neq 0} \frac{\|Z_{i,k}\|_{\mathcal{B}_q} + \|Z_{i,k}\|_q \|\Phi_i\|_{\mathcal{B}_q}}{|\lambda_{q+1} k|^N} \lesssim \sum_{k \neq 0} \frac{\delta_q \ell_q}{\lambda_{q+1}^N} \lesssim \delta_q \ell_q.
$$

(9.68)

The second inequality above is easily justified by using (9.30), (9.39), and (9.66) to estimate $Z_{i,k}$ as follows:

$$
\|Z_{i,k}\|_{\mathcal{B}_q} \lesssim \|\sigma_i\|_{\mathcal{B}_q} \|\mathcal{D} \Phi_i^{-1}\|_{\mathcal{B}_q} \|C_k(\tilde{S}_i)\|_{\mathcal{B}_q} + 2 \|\sigma_i\|_{\mathcal{B}_q} \|\mathcal{D} \Phi_i^{-1}\|_{\mathcal{B}_q} \|C_k(\tilde{S}_i)\|_{\mathcal{B}_q} \|\mathcal{D} \Phi_i^{-1}\|_{\mathcal{B}_q} + \|\sigma_i\|_{\mathcal{B}_q} \|\mathcal{D} \Phi_i^{-1}\|_{\mathcal{B}_q} \|C_k(\tilde{S}_i)\|_{\mathcal{B}_q}
$$

$$
\lesssim |k|^{-\delta_q \ell_q} \ell_q^{-N}.
$$

To estimate the second term in (9.66), observe that

$$
w \cdot v = \sum_{i,k} ((\mathcal{D} \Phi_i)^{-1} b_{i,k} + \lambda_{q+1}^{-1} c_{i,k}) \cdot v e^{\beta \Phi_i},
$$

so that, combining Lemma 2.4, (9.43), (9.44), (9.11), (9.39), and (4.14), we obtain that

$$
\left| \int 2 w \cdot v \, dx \right| \lesssim \sum_{i,k} \left| \int ((\mathcal{D} \Phi_i)^{-1} b_{i,k} + \lambda_{q+1}^{-1} c_{i,k}) \cdot v \right|_{\mathcal{B}_q} \lesssim \sum_{i,k} \frac{\|((\mathcal{D} \Phi_i)^{-1} b_{i,k} + \lambda_{q+1}^{-1} c_{i,k}) \cdot v \|_{\mathcal{B}_q}}{\lambda_{q+1}^N |k|^N}
$$

$$
\lesssim \delta_q \ell_q \lambda_q^{-1} \lambda_q^{-1} \lambda_q^{-1} \lesssim \delta_q \ell_q \lambda_q^{-1} \lambda_q^{-1} \lambda_q^{-1}.
$$

(9.69)

Concerning the third term in (9.66), note that the estimates on $w_c$ are always no coarser than those for $w_o$, so if we estimate $\int w_o \cdot w_c$ well, the whole term is estimated well. To this end, we observe that

$$
\int \int w_o \cdot w_c \, dx \leq \sum_{0 \neq k \neq 0} \int \sum_{i,j = 0 \neq 0} ((\mathcal{D} \Phi_i)^{-1} b_{i,k} \lambda_{q+1}^{-1} c_{i,j} - k) e^{\beta \Phi_i} \, dx
$$

$$
\lesssim \sum_{0 \neq k \neq 0} \left| \int ((\mathcal{D} \Phi_i)^{-1} b_{i,k} \lambda_{q+1}^{-1} c_{i,j} - k) \right|_{\mathcal{B}_q} \lesssim \sum_{0 \neq k \neq 0} \left| \int ((\mathcal{D} \Phi_i)^{-1} b_{i,k} \lambda_{q+1}^{-1} c_{i,j} - k) \right|_{\mathcal{B}_q}
$$

$$
= I + II,
$$

where we used Lemma 2.4 in the case $l \neq 0$, as well as the fact that the $w_o$ and $w_c$ have disjoint support so we do not have products of the form $b_{i,k} c_{j,l-k}$ for $i \neq j$. The term $I$ is easily estimated as $\delta_q \ell_q \lambda_q^{-1} \lambda_q^{-1} \lesssim \delta_q \ell_q \lambda_q^{-1} \lambda_q^{-1}$. 

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so that \( I \) satisfies the same estimate as the second term in (9.66). As for \( II \), (9.43) and (9.44) easily yield
\[
\|I\|_{2,\ell} \lesssim \rho \ell^{-1} q_{q+1}^{-1}. \tag{9.70}
\]
Combining (9.66)-(9.70) with the fact that \( \int |w_c|^2 \) also satisfies (9.70), we arrive at
\[
|\mathcal{E}|^{(2)}_{\ell} \lesssim \frac{\delta_{q+1} + \delta_{q+1} \ell^\alpha \ell_q + \rho \ell^{-1}}{q_{q+1}^{4}}.
\]
By (4.12), we thus conclude that, for \( a \gg 1 \) sufficiently large, \( \mathcal{E}^{(2)} \) satisfies (9.20). Combining with the fact (obtained in the previous step) that \( \mathcal{E}^{(1)} \) satisfies (9.20), we thus conclude that (9.20) holds.

**Step 6: Estimates on \( \partial_t \text{tr} \mathcal{E} \)**

Observe that \( \mathcal{E}^{(1)} \) is traceless, whereas \( \mathcal{E}^{(2)} \) is a function of \( t \) only. In order to estimate the time derivative of \( \mathcal{E}^{(2)} \), observe that, since \( \nu \) is solenoidal, for every \( F = F(x,t) \)
\[
\frac{d}{dt} \int_{\Omega} F = \int_{\Omega} \partial_t F,
\]
where \( \partial_t = \partial_t + \nu \cdot \nabla \). Therefore, using again the decomposition in (9.66), we have that
\[
\left| \frac{d}{dt} \int_{\Omega} |\tilde{v}|^2 - |v|^2 - \text{tr} S \right| \leq \left| \int_{\Omega} \left[ \partial_t \left( \sum_{i,k \neq 0} \sigma_i \partial \Phi_i^{-1} C_k(\tilde{S}) \partial \Phi_i^{-T} e^{\partial_{q+1} k \Phi_i} \right) \right] \right|
\]
\[
+ \left| \int_{\Omega} \partial_t (2 w_c \cdot w_o + |w_c|^2) \right| + \left| \int_{\Omega} \partial_t (2 \nu \cdot w) \right|. \tag{9.71}
\]
Let us first estimate \( \|\partial_t w_o\|_0 \). Recall from (2.21) that \( \partial_t (\partial \Phi_i)^{-1} = \partial \nu (\partial \Phi_i)^{-1} \), which, combined with the fact that \( \partial_t e^{\partial_{q+1} k \Phi_i} = 0 \), yields
\[
\partial_t w_o = \sum_{i,k \neq 0} \partial_t \left( \sqrt{\sigma_i} a_k(\tilde{S}) \right) \partial \Phi_i^{-1} A_k e^{\partial_{q+1} k \Phi_i}
\]
\[
+ \sum_{i,k \neq 0} \sqrt{\sigma_i} a_k(\tilde{S}) \partial \nu \partial \Phi_i^{-1} A_k e^{\partial_{q+1} k \Phi_i}
\]
\[
= \sum_{i,k \neq 0} \partial \Phi_i^{-1} \partial_t b_{i,k} e^{\partial_{q+1} k \Phi_i} + \sum_{i,k \neq 0} \partial \nu \partial \Phi_i^{-1} b_{i,k} e^{\partial_{q+1} k \Phi_i}.
\]
First notice that, by using (9.11), (9.39), and (9.43), we obtain that
\[
\|\partial \nu \partial \Phi_i^{-1} b_{i,k}\|_0 \lesssim \frac{\delta_{q+1}^{\frac{1}{2}} \delta_{q+1}^{\frac{1}{2}} \delta_{q+1}^{1+\alpha}}{|k|^4}.
\]
As for the coefficients \( \partial \Phi_i^{-1} \partial_t b_{i,k} \), combining (9.39) and (9.46) gives
\[
\|\partial \Phi_i^{-1} \partial_t b_{i,k}\|_0 \lesssim \tau_q^{-1} \delta_{q+1}^{\frac{1}{2}} |k|^{-6} = \frac{\delta_{q+1}^{\frac{1}{2}} \delta_{q+1}^{\frac{1}{2}} \delta_{q+1}^{1+\alpha}}{|k|^4}.
\]
Therefore
\[
\|\partial_t w_o\|_0 \lesssim \frac{\delta_{q+1}^{\frac{1}{2}} \delta_{q+1}^{\frac{1}{2}} \delta_{q+1}^{1+\alpha}}{|k|^4}.
\]
Observing that
\[
\partial_t w_c = \sum_{i,k} \lambda_{q+1}^{-1} \partial_t c_{i,k} e^{\partial_{q+1} k \Phi_i},
\]

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which follows from $D_i e^{\beta q_j + k\Phi_i} = 0$ seen above, (9.47) implies
\[
\|D_i w_c\|_0 \lesssim \delta_{q+1} \delta_{q+1} \ell_q^{-4\alpha} \lambda_{q+1}^{-1}.
\]
Combining with $\|w_o\|_0 + \|w_c\|_0 \lesssim \delta_{q+1}^7$ and using (4.11), (4.12), we obtain that
\[
\int_{Q^1} D_i (w_o \otimes w_c + w_c \otimes w_o + w_c \otimes w_c) \lesssim \|D_i w_o\|_0 \|w_c\|_0 + \|w_o\|_0 \|D_i w_c\|_0 + \|D_i w_c\|_0 \|w_c\|_0 
\leq \delta_{q+1}^7 \delta_{q+1} \ell_q^{-4\alpha} = \delta_{q+1}^7 \delta_{q+1} \ell_q^{-4\alpha} 
\lesssim \delta_{q+1}^7 \lambda_{q+1}^{1-\alpha} \lambda_{q+1}^{1-6\alpha}.
\]
The second term of (9.71) is thus estimated. We then similarly decompose the first term in (9.71) as
\[
D_i \left( \sum_{k \neq 0} \sigma_i D \Phi_j^{-1} C_k(\tilde S_i) \nabla \Phi_j^{-1} e^{\beta q_j + k\Phi_i} \right) = \sum_{i,k \neq 0} D_i \sigma_i D \Phi_j^{-1} C_k(\tilde S_i) \nabla \Phi_j^{-1} e^{\beta q_j + k\Phi_i} 
+ \sum_{i,k \neq 0} \sigma_i D v D \Phi_j^{-1} C_k(\tilde S_i) \nabla \Phi_j^{-1} e^{\beta q_j + k\Phi_i} 
+ \sum_{i,k \neq 0} \sigma_i D \Phi_j^{-1} D_i [C_k(\tilde S_i)] \nabla \Phi_j^{-1} e^{\beta q_j + k\Phi_i} 
+ \sum_{i,k \neq 0} \sigma_i D \Phi_j^{-1} C_k(\tilde S_i) \nabla \Phi_j^{-1} e^{\beta q_j + k\Phi_i}.
\]
In order to estimate this, we still need to estimate $D_i |C_k(\tilde S_i)|$ and $D_i \sigma_i$. To obtain the former, we first use (2.4):
\[
\|D_i (C_k(\tilde S_i))\|_N \leq \|D_i (C_k(\tilde S_i))\|_N \|D_i \tilde S_i\|_0 + \|D_i (C_k(\tilde S_i))\|_0 \|D_i \tilde S_i\|_0 
\lesssim \|C_k\|_{N+1} \|\tilde S_i\|_1 + \|C_k\|_2 \|\tilde S_i\|_N \|D_i \tilde S_i\|_0 + \|C_k\|_1 \|D_i \tilde S_i\|_N,
\]
We then use (2.11), (9.42), and (9.45) to conclude that
\[
\|D_i (C_k(\tilde S_i))\|_N \lesssim |k|^{-6} \tau_{q_i}^{-1} \ell_q^{-N}.
\]
Comming to $D_i \sigma_i$, we claim that
\[
\|D_i \sigma_i\|_N \lesssim \delta_{q+1} \tau_{q_i}^{-1} \ell_q^{-N}.
\]
To obtain (9.73), we set
\[
h(t) = \sum_{j} \int \eta_j^2(x,t) dx
\]
\[
D_i \sigma_i = \frac{[\mathbb{T}^3]}{h} \frac{\psi \sigma}{2} D_i \eta_i + \frac{[\mathbb{T}^3]}{h} \eta_i \frac{\rho_i}{h} \left( \frac{\psi \sigma}{h} \right) = I + II.
\]
We first estimate the term $I$. Recalling (9.28), (9.27), $\psi \leq 1$, and (9.7), we conclude that
\[
\|I\|_N \lesssim \frac{[\mathbb{T}^3]}{h} \frac{\psi \sigma}{h} \|\eta_i\|_N \|D_i \eta_i\|_0 + \|\eta_i\|_0 \|D_i \eta_i\|_N \lesssim \delta_{q+1} \tau_{q_i}^{-1} \ell_q^{-N}.
\]
As for the second term, we already see that, since the only factor depending on $x$ is $\eta_i^2$ which, by (9.28), satisfies $\|\eta_i^2\|_N \lesssim 1$ for all $N$, the estimates for $II$ will only depend on $N$ via an $a$-independent constant, thus making it sufficient to estimate $\partial_i (\psi \sigma h^{-1})$ in $C^0$. To that end, we rewrite it as
\[
\partial_i \left( \frac{\psi \sigma}{h} \right) = \frac{2 \psi \sigma}{h} + \frac{\psi^2}{h} - \frac{\psi^2 h}{R^2} = T_1 + T_2 + T_3.
\]
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To estimate $T_1$, we recall (9.7), (9.6), and (9.27):

$$
\| T_1 \|_0 \lesssim \frac{2 \delta_0^{1+\frac{1}{2}} \cdot 4 \delta_0^{q+1}}{c_0} \lesssim \tau_q^{-1} \delta_q+1.
$$

Coming to $T_2$, by (9.7), (9.9), $\psi \leq 1$, and (9.27), we obtain

$$
\| T_2 \|_0 \lesssim \frac{4 C \delta_0^{q+1} \delta_q^2}{c_0} \lesssim \delta_q+1 \tau_q^{-1},
$$

where $C$ is the implicit constant in (9.9). Finally, to estimate the term involving $R \cdot \psi$, by (9.7), (9.9), (9.56), we conclude that

$$
\| T_3 \|_0 \lesssim \frac{4 K \delta_0^{q+1} \tau_q^{-1}}{c_0} \lesssim \delta_q+1 \tau_q^{-1},
$$

where $K$ is the implicit constant in (9.56). The estimate (9.74) is thus proved. By (9.72), (9.40), (9.60), (9.30), (9.11), and (9.72), we conclude that

$$
\left\| \left[ 2 \sum_{i,k \neq 0} \delta_i \Phi_i^{-1} C_k(\bar{S}_i) \nabla \Phi_i^{-1} e^{\delta_{q+1} \Phi_i} \right] \right\|_0 \lesssim \delta_q+1 \tau_q^{-1} = \delta_q+1 \delta_0 \delta_q \tau_q^{-4a} \lesssim \delta_q+1 \delta_q \tau_q^{-4a} \tau_q^{-1} \lesssim \delta_q+1 \delta_q \tau_q^{-4a} \tau_q^{-1}.
$$

The last step being exactly as done above. Finally, to estimate the term involving $D_t(w \cdot v)$, we first note that

$$
\int D_t(v \cdot w) = \int D_t v \cdot w + \int v \cdot D_t w = - \int (\nabla p + (-\Delta)^{\alpha} v + \text{div} R) \cdot w + \int v \cdot D_t w,
$$

using (9.11) in the last step. To estimate the second term of (9.74), we write

$$
v \cdot D_t w = \sum_{i,k \neq 0} h_{i,k} e^{\delta_{q+1} \Phi_i},
$$

where

$$
h_{i,k} := v \cdot D_t[(\Phi_i)^{-1} b_{i,k} + \delta_{q+1} \hat{c}_{i,k}] = v \cdot [D_t(\Phi_i)^{-1} b_{i,k} + (\Phi_i)^{-1} D_t b_{i,k} + \delta_{q+1} D_t c_{i,k}].
$$

By Lemma 2.1 we obtain that

$$
\| h_{i,k} \|_N \lesssim \| v \|_N \left( \| D_t \Phi_i^{-1} \|_0 \| b_{i,k} \|_0 + \| \Phi_i^{-1} \|_0 \| D_t b_{i,k} \|_0 + \frac{1}{\delta_q+1} \| D_t c_{i,k} \|_0 \right)
$$

$$
+ \| v \|_0 \left( \| D_t \Phi_i^{-1} \|_N \| b_{i,k} \|_0 + \| D_t \Phi_i^{-1} \|_0 \| b_{i,k} \|_N + \| \Phi_i^{-1} \|_N \| D_t b_{i,k} \|_0 + \| \Phi_i^{-1} \|_0 \| D_t b_{i,k} \|_N + \frac{1}{\delta_q+1} \| D_t c_{i,k} \|_N \right).
$$

Thus, using (9.4), (9.11) (in the form $\| v \|_{N+\alpha} \lesssim \delta_q^{\frac{1}{2}} (1+ \frac{\alpha}{4}) \| f \|_{N-\alpha} \lesssim \tau_q^{-1} \| f \|_{N-\alpha}$), (9.43)-(9.44), (9.46)-(9.47), and (9.40)-(9.41), we conclude that

$$
\| h_{i,k} \|_N \lesssim \delta_q^{\frac{1}{2}} \tau_q^{-1} \| f \|_{N-\alpha} = \delta_q^{\frac{1}{2}} \delta_q \tau_q^{-4a-\alpha}.
$$

With Lemma 2.4 the above estimate yields that $\int v \cdot D_t w$ satisfies (9.21).

To deal with the first term of (9.74), we first note that, since $\text{div} w = 0$, $\int \nabla p \cdot w = 0$. Integrating by parts, the term $\int \text{div} R \cdot w$ can be estimated as follows:

$$
\left| \int \text{div} R \cdot w \, dx \right| \lesssim \| R \|_1 \| w \|_0 \lesssim \Lambda_q^{1+\gamma} \delta_q^{1+\alpha} \delta_q^{\frac{1}{2}} \lesssim \Lambda_q^{1+\gamma} \delta_q^{1+\alpha} \delta_q^{\frac{1}{2}} \delta_q^{\frac{1}{2}} + \Lambda_q^{1+\gamma} \delta_q^{1+\alpha} \delta_q^{\frac{1}{2}} \delta_q^{\frac{1}{2}},
$$

where we used (9.2) and (9.3). To conclude that the first term in (9.74) satisfies (9.21), we would require

$$
\frac{1+\gamma}{\delta_q+1} \lesssim \frac{1+\alpha}{\delta_q+1} \lesssim \delta_q^{\frac{1}{2}} \delta_q^{\frac{1}{2}}.
$$

For $\alpha, \gamma$ sufficiently small, this follows from

$$
-b\beta - \beta + 1 < -2b^2\beta + b \iff 1 - \beta - 2b\beta < b(1 - \beta - 2b\beta) \iff 1 - \beta - 2b\beta > 0,
$$

which in turn follows from (4.2).
Step 7: $\mathcal{T}_p$ and its derivative

The time derivative $\partial_t \mathcal{T}_p$ is readily estimated as

$$|\partial_t \mathcal{T}_p| = \left| \int_{T^3} (-\Delta)^\frac{3}{2}(2v + w) \cdot (-\Delta)^\frac{3}{2}w \, dx \right|$$

$$\leq \int_{T^3} 2(-\Delta)^\frac{3}{2}v \cdot (-\Delta)^\frac{3}{2}w \, dx + \int_{T^3} (-\Delta)^\frac{3}{2}w \, dx$$

$$\leq (2\|v\|_{\theta+\varepsilon}\|w\|_{1-\theta-\varepsilon}\|w\|_{\theta+\varepsilon} + \|w\|_1^{2-2\theta-2\varepsilon}\|w\|_1^{2+2\varepsilon}).$$

By (9.12), we have that $\|v\|_{\theta+\varepsilon} \lesssim \Lambda^\frac{3}{2}$, as for $w$, we have that $\|w\|_N \lesssim \delta_q^\frac{1}{2}\lambda_q^{N+1}$ (cfr. Step 3 above). Thus, recalling that $\theta + \varepsilon < \beta$

$$|\partial_t \mathcal{T}_p| \lesssim 2\Lambda^\frac{3}{2} \cdot \Lambda_{\theta+\varepsilon}^{0+\varepsilon-\beta} + \Lambda_{\theta+\varepsilon}^{20+2\varepsilon-2\beta} \leq \Lambda_{\theta+\varepsilon-\beta}.$$

Since this is exactly (9.22), the proposition is proved. $\Diamond$

Remark 9.1 (The fractional dissipation term). Note that (9.22) is stronger than (9.21), since

$$\Lambda_{\theta+\varepsilon-\beta} = \Lambda^\frac{1}{2} \delta_q^{\frac{1}{2}} \lambda_q^{\theta+\varepsilon} \lesssim \delta_q^\frac{1}{2} \lambda_q^{1+\beta}.$$

Indeed, this inequality follows from $\theta + \varepsilon - \beta < 1 - 6\alpha - \beta - 2b\beta$ which, for $\alpha, \varepsilon$ sufficiently small, follows from (4.2) and the fact $\theta < \beta$.

However, $\mathcal{T}_p$ is only estimated as follows:

$$|\mathcal{T}_p(t)| \lesssim t\Lambda_{\theta+\varepsilon-\beta}.$$

To ensure that this satisfies (9.20) for any $q \geq 0$, we would require

$$0 < \theta + \varepsilon - \beta < -2b\beta - 3\alpha \iff 3\alpha < \beta - \theta - \varepsilon - 2b\beta.$$

Seen as the above right-hand side is, in general, negative, we cannot require it. Thus, in general, $\mathcal{T}_p$ only satisfies (9.20) if the $q$ in the statement is sufficiently large, which is why we separated $\mathcal{T}_p$ from the other Reynolds terms. However, for $t \lesssim \Lambda^{-1} \beta^{\theta-\varepsilon-0}$, we can contrast the growth of $\Lambda_{\theta+\varepsilon-\beta}$ with the smallness of the time, meaning that $\mathcal{T}_p$ only satisfies (9.20) for a short period of time, or if $q$ is sufficiently large.

Remark 9.2 (C^0 estimate on the Reynolds stress). The requirement (9.2) is only used to obtain (9.21), meaning we only need it on $\text{supp} S$, since $S = 0 \rightarrow \varepsilon = 0$.

10 From strict to adapted subolutions

The aim of this section is to prove Proposition 5.1 (p. 11). The proof closely follows the arguments of [13] Section 8). Each stage contains a localized gluing step performed using Proposition 8.1 and a perturbation step performed using Proposition 9.1.

Proof. (Proposition 5.1)

Step 1: Setting the parameters of the scheme

Let $(\bar{v}, \bar{\beta}, \bar{R})$ be a smooth strict subolution and let $0 < \hat{\beta} < \beta < \frac{1}{2}, \nu > 0$. Choose $b > 1$ according to (4.2), furthermore let $\bar{\varepsilon} > 0$ such that:

$$b(1 + \bar{\varepsilon}) < \frac{1 - \beta}{2\beta}.$$  \hspace{1cm} (10.1)

Then, let $\delta, \gamma > 0$ be the constants given by Corollary 7.1 and choose $0 < \alpha < 1$ and $0 < \gamma < \hat{\beta} < \gamma$ so that:
Corollary 7.1

To obtain from (4.12), (4.13) are satisfied by both the pairs \((\alpha, \gamma)\) and \((\alpha, \hat{\gamma})\):

- The other conditions in Sections 8 and 9 namely (8.9)-(8.10) and consequently (9.1), (9.65), and (9.75), are satisfied by both the pairs \((\alpha, \gamma)\) and \((\alpha, \hat{\gamma})\):
- Condition (4.14) can hold for both pairs \((\alpha, \gamma)\) and \((\alpha, \hat{\gamma})\); since \(\hat{\gamma} > \gamma\), relation (4.15) reduces this to:

\[
(b - 1)(1 - \beta(b + 1)) - \hat{\gamma}b^2 - 2ab > 0;
\]

(10.2)

- The following conditions hold:

\[
\nu > \frac{1 - 3\beta + \alpha}{2\beta},
\]

(10.3)

\[
\frac{\alpha}{\beta} < b\hat{\gamma} < \frac{3\alpha}{2\beta}, \quad 0 < b\hat{\gamma} < \frac{\alpha}{\beta}, \quad 3\alpha > 2b\beta\hat{\gamma}.
\]

(10.4)

Having fixed \(b, \beta, \alpha, \gamma, \hat{\gamma}\), we may choose \(\tilde{N} \in \mathbb{N}\) so that (4.14) is also valid. For \(a \gg 1\) sufficiently large (to be determined) we then define \((\lambda_q, \delta_q)\) as in (4.11). Thus we are in the setting of Section 4.

Step 2: From strict to strong subsolution

We apply Corollary 7.1 to obtain from \((\tilde{\nu}, \tilde{\rho}, \tilde{R})\) a strong subsolution \((v_0, p_0, R_0)\) with \(\delta = \delta_1\) such that the properties from (7.12) to (7.16) hold. By (7.12)-(7.16), \((v_0, p_0, R_0)\) satisfies

\[
\begin{align*}
\frac{3}{4}\delta_1 & \leq p_0 \leq \frac{5}{4}\delta_1, \\
\|\tilde{R}_0(t)\|_0 & \leq \Lambda^{q+1}\tilde{q}, \\
\|\tilde{v}_0\|_{H^{-1}} & \leq \Lambda^{q-1}_0, \\
|\partial_t p_0| & \leq \delta_1\delta_q^{\frac{q}{2}}\|\tilde{v}_0\|_{1+\alpha}, \\
|\partial_t p_0| & \leq \delta_1\delta_q^{\frac{q}{2}}\|\tilde{v}_0\|_{1+\alpha}.
\end{align*}
\]

(10.5) (10.6) (10.7) (10.8) (10.9)

Step 3: Inductive construction of \((v_q, p_q, R_q)\)

Starting from \((v_0, p_0, R_0)\), we show how to inductively construct a sequence \(\{v_q, p_q, R_q\}_{q \in \mathbb{N}}\) of smooth strong subsolutions with:

\[
R_q(x, t) = p_q(t) Id + \tilde{R}_q(x, t)
\]

which satisfy the following properties:

- **(a)\(_q\)** For all \(t \in [0, T]\)

\[
\int_{\mathbb{T}^3} (|v_q|)^2 + \text{tr}\, R_q) \, dx = \int_{\mathbb{T}^3} (|v_0|)^2 + \text{tr}\, R_0) \, dx;
\]

- **(b)\(_q\)** For all \(t \in [0, T]\)

\[
\|\tilde{R}_q(t)\|_0 \leq \Lambda^{q+1}\tilde{q};
\]

- **(c)\(_q\)** If \(2^{-j}T < t \leq 2^{-j+1}T\) for some \(j = 1, \ldots, q\), then

\[
\frac{3}{8}\delta_{j+1} \leq p_q \leq 4\delta_j;
\]

- **(d)\(_q\)** For all \(t \leq 2^{-q}T\):

\[
\|\tilde{R}_q(t)\|_0 \leq \Lambda^{q+1}\tilde{q}, \quad \frac{3}{4}\delta_{q+1} \leq p_q \leq \frac{5}{4}\delta_{q+1};
\]

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If $2^{-j}T < t \leq 2^{-j+1}T$ for some $j = 1, \ldots, q$, then

$$
\|v_q\|_{1+\alpha} \leq M\delta_j^{\frac{1}{2}}\lambda_j^{1+\alpha}
$$

$$
|\partial_ip_q| \lesssim \delta_{j+1}\delta_j^{\frac{1}{2}}\lambda_j
$$

whereas if $t \leq 2^{-q}T$

$$
\|v_q\|_{1+\alpha} \leq M\delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}
$$

$$
|\partial_ip_q| \lesssim \delta_{q+1}\delta_q^{\frac{1}{2}}\lambda_q.
$$

For all $t \in [0,T]$ and $q \geq 1$:

$$
\|v_q - v_{q-1}\|_{H^{-1}} \leq M\delta_q^{\frac{1}{2}}(\zeta_q^{\alpha}q^{-1} + \ell^{-1}_q\lambda_q^{-1})
$$

$$
\|v_q - v_{q-1}\|_0 \leq M\delta_q^{\frac{1}{2}}.
$$

Thanks to our choice of parameters in Step 1 above, $(v_0,p_0,R_0)$ satisfies $[10.5]-[10.9]$, and thus the inductive assumptions $(a_0)-(g_0)$ (the last condition can be deduced from (7.18)). Suppose then $(v_q,p_q,R_q)$ is a smooth strong subsolution which satisfies $(a_q)-(g_q)$. The construction of $(v_{q+1},p_{q+1},R_{q+1})$ consists of two steps: first a localized gluing step performed using Proposition 8.1 to get from $(v_q,p_q,R_q)$ to a smooth strong subsolution $(v_{q+1},p_{q+1},R_{q+1})$, then a localized perturbation step done using Proposition 9.1 to get $(v_{q+1},p_{q+1},R_{q+1})$ from $(v_q,p_q,R_q)$.

We apply Proposition 8.1 with $[T_1,T_2] = [0,2^{-q}T]$.

Then $T_2 - T_1 \geq 4\pi_q$, if $a \gg 1$ is sufficiently large. Moreover, by $(d_q), (e_q),$ and $(g_q),$ $(v_q,p_q,R_q)$ fulfills the requirements of Proposition 8.1 on $[T_1,T_2]$ with parameters $\alpha, \gamma > 0$.

Then, by Proposition 8.1 we obtain a smooth strong subsolution $(v_{q},p_{q},\overline{R}_{q})$ on $[0,T]$ such that $(v_q,p_q,\overline{R}_q)$ is equal to $(v_q,p_q,R_q)$ on $[2^{-q}T,T]$, and on $[0,2^{-q}T]$ satisfies

$$
\|v_q - v_q\|_{1+\alpha} \lesssim \Lambda_q^{\alpha}v_q^{1+\alpha}
$$

$$
\|v_q\|_{1+\alpha} \lesssim \delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}
$$

$$
\|v_q\|_{0+\epsilon} \leq 1 + \sum_{i=0}^{q+1} \delta_i^{\frac{1}{2}}\lambda_i^{\epsilon+\epsilon}
$$

$$
\|\overline{R}_q\|_0 \leq \Lambda\overline{v}_q^{1+\gamma}\overline{v}_q^{-2\alpha}
$$

$$
\frac{5}{8}\delta_{q+1} \leq \overline{v}_q \leq \frac{3}{2}\delta_{q+1}
$$

$$
|\partial_i\overline{v}_q| \lesssim \delta_{q+1}\delta_q\lambda_q.
$$

Moreover, on $[0,\overline{\tau}]$ one has that

$$
\|v_q\|_{N+1+\alpha} \lesssim \delta_q^{\frac{1}{2}}v_q^{1+\alpha}\ell^{-N}
$$

$$
\|\overline{R}_q\|_{N+1+\alpha} \lesssim \Lambda\overline{v}_q^{1+\gamma}\ell^{-N-2\alpha}
$$

and

$$
(\overline{\partial}_i + v_q \cdot \nabla)\overline{R}_q \|_{N+1+\alpha} \lesssim \Lambda\overline{v}_q^{1+\gamma}\ell^{-N-6\alpha}\delta_q^{\frac{1}{2}}\lambda_q.
$$

and

$$
\overline{R}_q \equiv 0 \quad t \in \bigcup_{i=0}^{\pi} J_i,
$$

(10.12)
Recalling Definition 8.1 and (8.5) observe that
\[
\left[0, \frac{3}{4} 2^{-q} T\right] \subset [0, T],
\]
provided \(a \gg 1\) is chosen sufficiently large (e.g. so that \(\frac{5}{3} \tau_q < \frac{1}{4} 2^{-q} T\)). Then, choose a cut-off function \(\psi_q \in C_0^\infty([0, \frac{3}{4} 2^{-q} T]; [0, 1])\) such that
\[
\psi_q(t) = \begin{cases} 
1 & t \leq 2^{-(q+1)} T \\
0 & t > \frac{3}{4} 2^{-q} T
\end{cases}
\]
(10.14)
and such that \(|\psi'_q(t)| \lesssim 2^q\). By choosing \(a \gg 1\) sufficiently large, we may assume that
\[
|\psi_q(t)| \leq \frac{1}{2} \delta_q^\frac{1}{2}
\]
(10.15)
for all \(q\). Then, set
\[S_q := \psi_q^2(\overline{R}_q - \delta_{q+2} \text{Id}) = \psi_q^2 S.q\]
Using (10.15), (10.4), (10.10)-(10.13), and the easy observation that \(\overline{p}_q \lesssim \overline{p}_q - \delta_{q+2}\), we see that \(S_q\) and \((\overline{\nu}_q, \overline{p}_q, \overline{R}_q)\) satisfy the assumptions of Proposition 9.1 on the interval \([0, T]\) with parameters \(\alpha, \hat{\gamma} > 0\). We have that
\[
\sigma_q = \psi_q^2(\overline{p}_q - \delta_{q+2}) = \psi_q^2 \sigma.
\]
Recalling Remark 9.2 since \(\text{supp} S_q \subseteq [t, T]\) where (8.25) holds, we can apply Proposition 9.1 thus obtaining a new subsolution \((v_{q+1}, p_{q+1}, \overline{R}_q - S_q - \delta_{q+1})\) with
\[
\begin{align*}
\|v_{q+1} - \overline{\nu}_q\|_0 + &\ell_q \lambda_{q+1} \|v_{q+1} - \overline{\nu}_q\|_{H^1} \\
+ &\lambda_{q+1}^{-1-a} \|v_{q+1} - \overline{\nu}_q\|_{1+a} \\
+ &\lambda_{q+1}^{-1-a} \|v_{q+1} - \overline{\nu}_q\|_{1+a} \\
\int_T |v_{q+1}|^2 - tr S - \delta_{q+1} = \int_T |\overline{\nu}_q|^2 & t \in [0, T],
\end{align*}
\]
and such that the estimates (9.20) and (9.21) hold for \(\delta_{q+1}\). Let
\[R_{q+1} := \overline{R}_q - S_q - \delta_{q+1}.
\]
We claim that \((v_{q+1}, p_{q+1}, R_{q+1})\) is a smooth strong subsolution satisfying \((a_{q+1}, g_{q+1})\). Notice that \((a_{q+1})\) is satisfied by construction. Since \((v_{q+1}, p_{q+1}, R_{q+1}) = (v_q, p_q, R_q)\) for \(t \geq 2^{-q} T\), we may restrict \(t\) to \([0, 2^{-q} T]\) in the following, so that in particular (10.10) holds.
Let us now prove \((b_{q+1})\). On the one hand
\[
\|\tilde{R}_{q+1}\|_0 = \|(1 - \psi_q^2)\tilde{R}_q - \tilde{\delta}_{q+1}\|_0 \\
\leq (1 - \psi_q^2)\Lambda \tilde{q}^{1+\hat{\gamma}} e^{-2a} + \delta_{q+2} \lambda_{q+1}^{-3a} \psi_q > 0,
\]
(10.16)
on the other hand
\[
\rho_{q+1} = (1 - \psi_q^2)\Lambda \tilde{q}^{1+\hat{\gamma}} e^{-2a} + \psi_q^2 \delta_{q+2} + \frac{1}{3} tr \delta_{q+1} \\
\geq (1 - \psi_q^2)\Lambda \tilde{q}^{1+\hat{\gamma}} e^{-2a} + \psi_q^2 \delta_{q+2} - \delta_{q+2} \lambda_{q+1}^{-3a} \psi_q > 0.
\]
(10.17)
The proof of \((b_{q+1})\) thus reduces to assessing whether there exists a suitable \(\gamma\) such that
\[
(1 - \psi_q^2)\Lambda \tilde{q}^{1+\hat{\gamma}} e^{-2a} + \delta_{q+2} \lambda_{q+1}^{-3a} \psi_q > 0 \leq \Lambda^{-\gamma}(1 - \psi_q^2)\Lambda q + \psi_q^2 \delta_{q+2} - \delta_{q+2} \lambda_{q+1}^{-3a} \psi_q > 0.
\]
(10.18)
To this end set
\[
F(s) := (1 - s)\Lambda q^{1+\gamma}q^{-2\alpha} + \delta q + 2\lambda q^{-3}\alpha \\
G(s) := (1 - s)\pi q + s\delta q + 2 - \delta q + 2\lambda q^{-3}\alpha \\
H(s) := \Lambda^{-\gamma}H^{1+\gamma}(s) - F(s),
\]
and observe that (10.18) is equivalent to \(H(\psi_q^2) \geq 0\) if \(\psi_q > 0\), and follows from this inequality also in case \(\psi_q = 0\). In particular, (10.18) follows from:

(i) \(H(0) \geq 0\) and \(H(1) \geq 0\);

(ii) \(H'(0) \leq 0\) and \(H'(1) \leq 0\).

(iii) \(H''(0) \geq 0\).

We note next that, since \(2b\beta > 3\alpha\)
\[
\delta q^{2}\lambda q^{-3\alpha} \leq \Lambda q^{1+\gamma},
\]
so that we have the estimates
\[
F(0) \leq \Lambda q^{1+\gamma}q^{-2\alpha}, \quad G(0) \geq \pi q.
\]
It is also clear that \(G(s) \leq \pi q\).

It is then easy to check that the requirement \(H(0) \geq 0\), i.e. \(F(0) \leq \Lambda^{-\gamma}H^{1+\gamma}(0)\), amounts to \(\Lambda q^{1+\gamma}q^{-2\alpha} \leq \Lambda q^{1+\gamma}\), i.e. \(\pi q^{1+\gamma}q^{-2\alpha} \leq 1\). Hence, since \(q^{-1} \leq \lambda q + 1\) by (4.13) and \(q \geq \zeta q + 1\) by (d.q+1), \(H(0) \geq 0\) follows from
\[
\hat{\gamma} - \frac{\alpha}{\beta} > \gamma, \quad (10.19)
\]
provided \(a \gg 1\) is sufficiently large to absorb geometric constants. The relation (10.19) follows from (10.4) since \(b > 1\).

The next requirement, \(H(1) \geq 0\), i.e. \(\lambda q^{-3\alpha} \leq \zeta q^{-1}q^{-2\alpha}\), requires \(3\alpha > 2b\beta\gamma\) as found in (10.4), since \(1 - \lambda q^{-3\alpha} \geq \frac{1}{2}\) for a sufficiently large.

The following condition, \(H'(0) \leq 0\), can be rewritten as
\[
-\Lambda q^{1+\gamma}q^{-2\alpha} \leq (1 + \gamma)(q - \zeta q + 2\lambda q^{-3\alpha})\gamma(\delta q + 2 - \pi q) \iff \Lambda q^{1+\gamma}q^{-2\alpha} \leq (q - \zeta q + 2\lambda q^{-3\alpha})\gamma(\pi q - \delta q + 2).
\]
Noting that \(\pi q \geq \delta q + 2\) by (d.q+1), and therefore \(\pi q - \delta q + 2 \geq \frac{1}{2}\pi q\) for a sufficiently large, the above reduces to
\[
\pi q^{-\gamma}q^{-2\alpha} \leq 1 \iff \pi q^{-\gamma}q^{-2\alpha} = \pi q^{-\gamma}q^{-2\alpha} \leq 1,
\]
which follows from condition (10.19) deduced above.

We then need the condition \(H'(1) \leq 0\), which can be rewritten as
\[
-\Lambda q^{1+\gamma}q^{-2\alpha} \geq (1 + \gamma)\zeta q^{-2}(1 - \lambda q^{-3\alpha})\gamma(\delta q + 2 - \pi q),
\]
which similarly follows from (10.19).

The last condition, \(H'' \geq 0\), follows from the fact that \(F'' = 0\) and \(G'' = 0\), and thus \(H'' = \Lambda^{-\gamma}(1 + \gamma)\gamma G^{-1}G^2\) is positive.

Thus, our choice of \(\alpha, \gamma, \hat{\gamma}\) in (10.4) guarantees that (10.18) holds, which yields \((b_{q+1})\).

Consider now \((c_{q+1})\), where we only need to consider the case \(j = q + 1\), i.e. the estimate on \([2 - q^{-1}T, 2 - qT]\).

Using (10.17), the fact that \(\pi q \geq \delta q + 2\) for a large enough, and (10.10), we see that
\[
\delta q + 2(1 - \lambda q^{-3\alpha}) \leq \pi q + (\delta q + 2 - \delta q + 2\lambda q^{-3\alpha}) \leq \frac{3}{2}\pi q + \delta q + 2\lambda q^{-3\alpha}.
\]
Therefore \((c_{q+1})\) holds, provided \(a \gg 1\) is sufficiently large.
Similarly, concerning \((d_{q+1})\), observe that for \(t \leq 2^{-(q+1)}T\) we have that \(\psi_q(t) = 1\), so that

\[
\delta_{q+2} (1 - \lambda_{q+1}^{-3\alpha}) \leq \rho_{q+1} \leq \delta_{q+2} (1 + \lambda_{q+1}^{-3\alpha}).
\]

Moreover, using (10.16) and the fact that \(\psi = 1\) for \(t \leq 2^{-(q+1)}T\)

\[
\|\hat{R}_{q+1}\|_0 \leq \delta_{q+2} \lambda_{q+1}^{-3\alpha} \leq \Lambda \left( \frac{3}{4} \delta_{q+2} \right)^{1+\gamma},
\]

where we used the fact that \(2\delta_{q+2}^\gamma < 3\alpha\) and chose \(\alpha \gg 1\) sufficiently large. Therefore \((d_{q+1})\), i.e. \(\|\hat{R}_{q+1}\|_0 \lesssim \Lambda \delta_{q+1}^{1+\gamma}\), and \(\frac{3}{4}\delta_{q+2} \lesssim \rho_{q+1} \leq \frac{5}{4}\delta_{q+2}\), holds.

Concerning \((e_{q+1})\), it is once more enough to restrict to \(t \leq 2^{-q}T\), i.e. the case \(j = q + 1\). From (10.10) and (9.18) we deduce that

\[
\|v_{q+1}\|_{1+\alpha} \leq \|v_{q+1} - \nabla q\|_{1+\alpha} + \|\nabla q\|_{1+\alpha} \leq M \frac{\delta_{q+1}^\gamma}{\beta_{q+1}} \lambda_{q+1}^{1+\alpha} + C \delta_{q+1}^{\gamma} \lambda_{q+1}^{1+\alpha} \leq M \delta_{q+1}^\gamma \lambda_{q+1}^{1+\alpha},
\]

where \(C\) is the implicit constant in (10.10), which can be absorbed by choosing \(\alpha \gg 1\) sufficiently large. The estimate on \(\|\rho_{q+1}\|\) similarly follows from the trace estimate of (10.10) and (9.21). \((e_{q+1})\) is thus proved.

Finally, \((g_{q+1})\) easily follows from (9.19) and (10.10).

**Step 4: Convergence to an adapted subsolution**

We have thus obtained a sequence \((v_q, p_q, R_q)\) satisfying \((a_q)-(g_q)\).

From \((f_q)\) it follows that \((v_q, p_q)\) is a Cauchy sequence in \(C^0\). Indeed, it is clear for \(\{v_q\}\), and concerning \(\{p_q\}\) we may use (3.1) to write

\[
\Delta (p_{q+1} - p_q) = -\text{div}\text{div} (\hat{R}_{q+1} - \hat{R}_q + (v_{q+1} - v_q) \otimes v_q + v_{q+1} \otimes (v_{q+1} - v_q)),
\]

and apply Schauder estimates (Lemma 2.5). Similarly, \(\{R_q\}\) also converges in \(C^0\). Indeed, from the definition and using (8.20), (8.12), (9.10), (9.20), and \((b_q)\), we have that

\[
\|R_{q+1} - R_q\|_0 = \|R_q - R_q - S_q - \epsilon_{q+1}\|_0 \leq \|R_q\|_0 + \|S_q\|_0 + \|\epsilon_{q+1}\|_0 \lesssim \delta_{q+1}.
\]

For all \(t > 0\) there exists \(q(t) \in \mathbb{N}\) such that

\[
(v_q, p_q, R_q)(\cdot, t) = (v_{q(t)}, p_{q(t)}, R_{q(t)})(\cdot, t) \quad \forall q \geq q(t),
\]

thus \((v_q, p_q, R_q)\) converges uniformly to a strong subsolution \((\hat{v}, \hat{p}, \hat{R})\) satisfying

\[
\|\hat{R}\|_0 \leq \Lambda \delta_{1+\gamma}^\gamma,
\]

and, using (7.11) and \((a_q)\)

\[
\int_{T^3} (|\hat{v}|^2 + \text{tr} \hat{R}) \, dx = \int_{T^3} (|\hat{v}|^2 + \text{tr} \hat{R}) \, dx \quad \forall t \in [0, T].
\]
Furthermore, using (7.14) and \((f_q)\):

\[
\| \hat{v} - \tilde{v} \|_{H^{-1}} \leq \| v_0 - \tilde{v} \|_{H^{-1}} + \| v_0 - \hat{v} \|_{H^{-1}} \\
\leq \delta_1 \lambda_0^{-1} + \sum_{q=0}^{\infty} \| v_{q+1} - v_q \|_{H^{-1}} \\
\leq \delta_1 T \zeta_8^4 \beta_1^\alpha,
\]

leading to (5.3) for \(a \) sufficiently large. Using \((f_q)\) and the fact that \(\hat{v}, \tilde{v}\) are smooth and thus bounded in \(C^0\), (6.2) is proved similarly:

\[
\| \hat{v} - \tilde{v} \|_{C^0} \leq \| v_0 - \tilde{v} \|_{C^0} + \| v_0 - \hat{v} \|_{C^0} \\
\leq 1 + \sum_{q=0}^{\infty} \| v_{q+1} - v_q \|_0 \\
\leq 1 + \delta_1^2.
\]

Concerning the initial datum, from \((e_q)\) and \((f_q)\) we obtain by interpolation that \(\hat{v}(\cdot, 0) \in \dot{C}^\beta\), and from \((d_q)\) we obtain that \(\dot{R}(\cdot, 0) = 0\).

Finally, we verify conditions (3.5), and (3.6) for being a \(C^\beta\)-adapted subsolution. Let \(t > 0\). Then there exists \(q \in \mathbb{N}\) such that \(t \in [2^{-q} T, 2^{-q+1} T]\). By \((c_q)\) and \((e_q)\)

\[
\frac{3}{8} \delta_{q+1} \leq \hat{\rho} \leq 4 \delta_q
\]

\[
\| \hat{v} \|_{1+\alpha} \leq M \delta_q \lambda_q^{1+\alpha}.
\]

Therefore \(\hat{\rho}^{-1} > \frac{1}{4} \delta_q^{-1}\), and hence, using (4.1) and (10.3), we deduce that

\[
\| \hat{v} \|_{1+\alpha} \leq \Lambda \hat{q}^{1+\nu},
\]

for \(a \gg 1\) sufficiently large. Similarly, using \((e_q)\) and (10.3), we deduce that

\[
| \partial_t \hat{\rho} | \leq \delta_{q+1} \delta_q \lambda_q = \Lambda_q \lambda_q^{1-\beta} \lambda_{q+1} \sim \Lambda_q \lambda_q^{1-\beta} \lambda_{q+1}^{1-2\beta} = \Lambda_q \lambda_q^{1-\beta} \lambda_{q+1}^{1-2\beta} \leq \Lambda_q \lambda_q^{1-\beta} \lambda_{q+1}^{1-2\beta} \leq \Lambda_q \hat{q}^{1-\beta} \hat{q}^{-\nu}.
\]

Finally, a word about the term

\[
\hat{\mathcal{F}} := \sum (\mathcal{J}_s^{(q)} + \mathcal{J}_d^{(q)}),
\]

where \(\mathcal{J}_s^{(q)}\) and \(\mathcal{J}_d^{(q)}\) are the extra trace terms from the \(q\)th gluing and perturbation steps. We have that

\[
| \partial_t \mathcal{J}_s^{(q)} | + | \partial_t \mathcal{J}_d^{(q)} | \leq \Lambda q^{0+\nu},
\]

thus proving (5.5). However, adding \(\hat{\mathcal{F}}\) into \(\hat{R}\) could compromise the adaptedness of \((\hat{v}, \hat{p}, \hat{R})\) by rendering (3.5)-(3.6) invalid, which is why we keep it separated and deal with it in the final argument. The estimate (5.5) implies that

\[
| \hat{\mathcal{F}}(t) | \leq \sum t \lambda_q \lambda_q^{0+\nu}.
\]

To be able to make it as small as we desire, we must contrast the \(\alpha\)-growth of the \(q = 0\) and \(q = 1\) terms of this sum. This is easily achieved by requiring \(t \leq \lambda_q^{-\beta} \lambda_q^{0+\nu} \) for \(t\) arbitrarily small. In any case, calling \(t_s\) the maximal time where \(\hat{\mathcal{F}}\) can be estimated with small quantities, we have that

\[
\lim_{a \to +\infty} t_s = 0,
\]

since we need \(t_s \delta_0^{1/2} \lambda_0^{0+\nu}\) to be small. \(\diamondsuit\)
11 From adapted subsolutions to solutions

The aim of this section is to prove Proposition 5.2 (p. 11). The proof closely follows the arguments of [13, Section 9]. We now start from an adapted subsolution and, by a convex integration scheme, build a sequence of strong subsolutions which converge to a solution of the fractional Navier-Stokes equation. As in Proposition 5.1 the convex integration scheme needs the localized gluing and perturbation arguments of Proposition 8.1 (in the form of Remark 8.1) and Proposition 9.1. However, the choice of the cut-off functions will be, as in [14], dictated by the shape of the trace part of the Reynolds stress, and not fixed a priori as in Proposition 5.1. Before we start the proof, a remark needs to be made about starting the chain of Proposition 8.1 and Proposition 9.1 with worse estimates.

Remark 11.1 (Worse starting estimate). In Proposition 8.1 if we replace (8.12) with

$$\|\hat{R}\|_0 \leq \Lambda q^{1+\gamma} e^{2\alpha},$$

as we will need to do below, the estimates (8.37), (8.47), (8.48), (8.17), (8.18), (8.24), (8.20), (8.25), (8.23), (8.62), and (8.70) will be worsened by a factor $e^{-\alpha}$. In fact, we can gain a factor $e^{\alpha}$ in (8.20) and (8.25), and a factor $e^{\alpha}c$ in (8.23) and (8.62). To keep the inductive estimates on the velocity gap $\|v_{q+1} - v_q\|_0$ and $\|v_{q+1} - v_q\|_{H^{-1}}$, the velocity $\|v_{q+1}\|_0$, and the derivative of the trace $\partial_t \rho_q$, we will need

$$\Lambda^{1+\gamma} q^{(1-\frac{2}{3})\alpha} \lesssim \delta_{q+1}^{\frac{1}{2}}$$

$$\delta_q^{\frac{1}{2}} e^{\alpha} e^{-2\alpha} \lesssim \delta_{q+1}^{\frac{1}{2}}$$

$$Q^{\frac{3}{2}} q^{(2-\frac{2}{3})\alpha} \lesssim 1,$$

all of which can easily be deduced by assuming $2\alpha < \beta \gamma$ and $\alpha < \frac{2}{3}$. The former assumption also yields (8.30), which will allow us to bound the $H^{-1}$ norm of $v_{q+1} - v_q$ sufficiently tightly. If we then start the perturbation step of Proposition 9.1 from estimates that we can obtain from the modified output estimates mentioned above, we can get the same output estimates from Proposition 9.1.

Proof. (Proposition 5.2)

Step 1: Setting the parameters in the scheme

Let $(\hat{v}, \hat{\rho}, \hat{R})$ be a $C^{1,\alpha}$ adapted subsolution on $[0, T)$, with $\Omega = \Lambda$, satisfying the “strong” condition $|\hat{R}| \leq \Lambda q^{1+\gamma}$ for some $\gamma > 0$ and (3.5) and (3.6) for some $\alpha, \nu > 0$ as in Definition 3.3 of adapted subsolution, with

$$\frac{1 - \hat{\beta}}{2\hat{\beta}} < 1 + \nu < \frac{1 - \beta}{2\beta}.$$ 

Fix $b > 0$ so that

$$b^2 (1 + \nu) < \frac{1 - \beta}{2\beta}, \quad 2\beta (b^2 - 1) < 1.$$ 

Observe that both the strongness condition (3.2) and the adaptedness conditions (3.5)-(3.6) remain valid for any $\hat{\gamma} < \gamma$ and $\alpha' \leq \alpha$ (cfr. Remark 3.1). Then, we may assume that $\alpha, \hat{\gamma} > 0$ are sufficiently small, so that $(\hat{v}, \hat{\rho}, \hat{R})$ satisfies (3.2) for some $\hat{\gamma} > 0$ and (3.5)-(3.6) for some $\alpha, \nu > 0$, and furthermore choose $\gamma$ so that

$$2\alpha < \beta \gamma < 3\alpha \quad b\beta \hat{\gamma} < 3\alpha.$$ 

For the reasons discussed in Remark 11.1 above, and for another technical reason we will see below, we require

$$2\alpha < \hat{\beta} \gamma < 3\alpha.$$ 

Finally, having fixed $b, \hat{\beta}, \beta, \alpha, \gamma, \hat{\gamma}$, we may choose $\mathcal{N} \in \mathbb{N}$ so that (4.14) holds. For $\alpha > 1$ sufficiently large (to be determined) we then define $(\lambda_q, \delta_q)$ as in (4.1) (using $\beta$). Thus, we are in the setting of Section 4.
Step 2: Conditions on \((v_0,p_0,R_0)\) and the inductive construction of \((v_q,p_q,R_q)\)

Differently from [13, Section 9], we can take \((v_0,p_0,R_0) = (\hat{v},\hat{p},\hat{R})\), since we are assuming \(\hat{\rho} \leq \frac{5}{4}\delta_1 = \frac{5}{4}\delta\), which is \(a\)-independent. We do have some estimates to verify for \((v_0,p_0,R_0)\), namely that, wherever \(\rho_0 \geq \delta_{q+2}\)

\[
\|v_0\|_{1+\alpha} \leq \frac{1}{i+1} \delta_{q+2}^{1+\alpha}
\]

\[
|\partial_i \rho_0| \leq \rho_0 \delta_{q+2}^2
\]

(11.5)

Indeed, where \(\rho_0 \geq \delta_{q+2}\), \((11.2)\) easily yields

\[
\Lambda^\frac{i}{i-1} \mathcal{Q}_0^{1+i} \lesssim \Lambda^\frac{i}{i-1} \lambda_{q+2} \lesssim \delta_{q+2} \lambda_q,
\]

\[
\Lambda \mathcal{Q}_0^{-1} \lesssim \Lambda \lambda_{q+2} \lesssim \delta_{q+2} \delta_{q+2} \lambda_q,
\]

provided \(a \gg 1\) is sufficiently large. These two relations, combined with \((3.5)\) and \((3.6)\), yield \((11.5)\).

Start from \((v_0,p_0,R_0)\), we will inductively construct a sequence \((v_q,p_q,R_q)\) of smooth strong subsolutions for \(q = 1,2,\ldots\), with

\[
R_q(x,t) = \rho_q(t) \text{Id} + \hat{R}_q(x,t),
\]

satisfying the following properties:

(Aq) For all \(t \in [0,T]\)

\[
\int_\mathbb{T}^3 (|v_q|^2 + \text{tr} R_q) dx = \int_\mathbb{T}^3 (|v_0|^2 + \text{tr} R_0) dx;
\]

(11.6)

(Bq) For all \(t \in [0,T]\)

\[
\rho_q \leq \frac{5}{2} \delta_{q+1};
\]

(11.7)

(Cq) For all \(t \in [0,T]\)

\[
\|\hat{R}_q\|_0 \leq \begin{cases} 
\Lambda \mathcal{Q}_0^{1+i} \mathcal{Q}_q^{-i} \rho_q \geq 2 \delta_{q+2} \\
\Lambda \mathcal{Q}_q^{1+i} \rho_q \leq 2 \delta_{q+2} \\
\Lambda \mathcal{Q}_q^{1+i} \rho_q \leq 2 \delta_{q+2}
\end{cases};
\]

(11.8)

(Dq) If \(\rho_q \geq \delta_{q+2}\) for some \(j \geq q\), then

\[
\|v_q\|_{1+\alpha} \leq M \delta_{j+1} \lambda_j^{1+\alpha}
\]

(11.9)

\[
|\partial_i \rho_q| \leq \rho_q \delta_{j+1} \lambda_j;
\]

(11.10)

(Eq) For all \(t \in [0,T]\) and \(q \geq 1\)

\[
\|v_q - v_{q-1}\|_{H^{1+\alpha}} \lesssim \left( \frac{\delta_{q+2}}{i+1} \right) \|v_q - v_{q-1}\|_0 \lesssim \delta_{q+2}^3
\]

(11.11)

(Fq) \(\|v_q\|_{\theta+\epsilon} \leq M \left( 1 + \lambda_j \sum_{i=0}^q \lambda_{q+i}^{\theta+\epsilon-\beta} \right)\).

Thanks to our choice of parameters in Step 1 above, \((v_0,p_0,R_0)\) satisfies \((11.5)\), and therefore our inductive assumptions \((A_0)-(F_0)\).

Suppose now \((v_q,p_q,R_q)\) satisfies \((A_q)-(F_q)\) above. Let

\[
J_q := \{ t \in [0,T] : \rho_q(t) \geq \frac{3}{2} \delta_{q+2} \}, \quad K_q := \{ t \in [0,T] : \rho_q(t) \geq 2 \delta_{q+2} \};
\]


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Proposition 8.1

Proposition 8.1 (using Remark 11.1 above on $v$ is contained in $\mathbb{R}$).

Being (relatively) open in $[0,T]$, $J_q$ is a disjoint, possibly countable, union of (relatively) open intervals $(T_{1,i}^2, T_{2,i}^2)$. Let

$$g_q := \left\{ i : (T_{1,i}^2, T_{2,i}^2) \cap K_q \neq \emptyset \right\},$$

and let $t_0 \in (T_{1,i}^2, T_{2,i}^2) \cap K_q$ for some $i \in g_q$. Since $K_q$ is compact, we may assume that the open interval $(T_{1,i}^2, t_0)$ is contained in $J_q \setminus K_q$. Using (11.10), we then have that

$$\frac{3}{2} \delta_{q+2} = \rho_q(T_{1,i}^2) \geq \rho_q(t_0) - |T_{1,i}^2 - t_0| \sup_{g_q} |\partial_t \rho_q|$$

$$\geq 2 \delta_{q+2} - 2 \delta_{q+2} |\partial_t \rho_q| |T_{1,i}^2 - t_0|,$$

hence

$$|T_{1,i} - t_0| \geq \frac{1}{4} \left( \frac{1}{4} \delta_{q+1} \right)^{-1} \rho_q \delta_{q+1}^3 \rho_q > 4 \tau_q,$$  

(11.12)

provided $\rho_q \gg 1$ is chosen sufficiently large. A similar estimate holds for $T_{2,i}^2$. Therefore $T_{2,i}^2 - T_{1,i}^2 > 4 \tau_q$ for any $i \in g_q$, so that $g_q$ is a finite index set.

Next, we apply Proposition 8.1 (in the form of Remark 8.1), keeping Remark 11.1 in mind, to $(v_q, p_q, R_q)$ on this disjoint union of intervals $\bigcup_{i \in g_q} J_q$. Since $\rho_q > \frac{3}{2} \delta_{q+2}$, from (A_q)-(F_q) and (11.2)-(11.3) we see that the assumptions of Proposition 8.1 on $(v_q, p_q, R_q)$ hold with parameter $\tilde{\gamma}$. Then we obtain $(\overline{v}_q, \overline{p}_q, \overline{R}_q)$ such that, on $J_q$

$$\|\overline{v}_q(t) - v_q(t)\|_{t=\alpha} \lesssim \delta_{q+1}^{\frac{1}{2}} \rho_q \lesssim \delta_{q+1}^{\frac{1}{2}} \rho_q$$  

(From 8.30)

$$\|\overline{v}_q\|_{t=\alpha} \lesssim \delta_{q+1}^{\frac{1}{2}} \rho_q \lesssim \delta_{q+1}^{\frac{1}{2}} \rho_q$$  

(From 8.18)

$$\|\overline{R}_q\|_{t=\alpha} \lesssim \delta_{q+1}^{\frac{1}{2}} \rho_q \lesssim \delta_{q+1}^{\frac{1}{2}} \rho_q$$  

(From 8.20)

$$\frac{7}{8} \rho_q \leq \Lambda \delta_{q+1} \leq \frac{9}{8} \rho_q$$  

(From 8.21)

$$|\partial_t \overline{p}_q| \lesssim \delta_{q+1}^3 \rho_q.$$  

(From 8.22)

Moreover, recalling (8.32), for any $i \in g_q$ we have the following additional estimates valid for $t \in [T_{1,i}^2 + 2 \tau_q, T_{2,i}^2 - 2 \tau_q] \cap J_q$:

$$\|v_q\|_{t=\alpha} \lesssim \delta_{q+1}^{\frac{1}{2}} \rho_q \lesssim \delta_{q+1}^{\frac{1}{2}} \rho_q$$  

(11.13)

$$\|\overline{v}_q\|_{t=\alpha} \lesssim \Lambda \delta_{q+1}^{\frac{1}{2}} \rho_q \lesssim \Lambda \delta_{q+1}^{\frac{1}{2}} \rho_q$$

and

$$\text{supp} \overline{R}_q \subset T^3 \times \bigcup_i \{ \Psi_i \}_{i}.$$  

(11.14)

where $\{I_i\}$ are the intervals defined in (8.1).

Let us choose a cut-off function $\Psi_q \in C_c^\infty(J_q; [0, 1])$ such that

$$\text{supp} \Psi_q \subset \bigcup_{i \in g_q} \left( T_{1,i}^2 + 2 \tau_q, T_{2,i}^2 - 2 \tau_q \right)$$  

(11.15)

$$K_q \subset \{ \Psi_q = 1 \}$$  

(11.16)

$$|\Psi_q| \lesssim \delta_{q+2}^3 \rho_q.$$  

(11.17)

Such a choice is made possible by (11.12). We then want to apply Proposition 9.1 (using Remark 11.1 above where $\rho_q \geq 2 \delta_{q+2}$) to $(v_q, p_q, R_q)$ with $S_q := \Psi_q^2 (\overline{R}_q - \delta_{q+2} \mathbf{1}_d)$,
hence \( \sigma_q = \psi_q^2(\overline{p}_q - \delta_{q+2}) \). Using (11.17), (11.2), (11.13), (11.14), (8.21)-(8.22), and \((A_q)\)-\((F_q)\), we see that \( S \) and \((\overline{q}_q, \overline{p}_q, \overline{K}_q)\) satisfy the required assumptions on the interval \([T_1^{(i)} + 2\tau_q, T_2^{(i)} - 2\tau_q]\) with parameters \(\alpha, \gamma > 0\). In particular, (9.2) (or its worsened form discussed in Remark 11.1) follows from (11.13), since we only need it on \(\psi_q \subseteq [T_1^{(i)} + 2\tau_q, T_2^{(i)} - 2\tau_q]\).

**Proposition 9.1** gives then a new subsolution \((v_{q+1}, p_{q+1}, \overline{K}_q - S_q - \varepsilon_{q+1})\) with

\[
\begin{align*}
\|v_{q+1} - \overline{v}_q\|_0 + \|v_{q+1} - \overline{v}_q\|_{H^{-1}\lambda_{q+1}} \\
+ \lambda_q^{-1-\alpha}\|v_{q+1} - \overline{v}_q\|_{1+\alpha} \\
+ \lambda_q^{-1-\alpha}\|v_{q+1} - \overline{v}_q\|_{1+\alpha} \\
= M\delta_{q+1}^2 \\
\int_{\mathbb{T}^3} |v_{q+1}|^2 - S_q - \varepsilon_{q+1} = \int_{\mathbb{T}^3} |\overline{v}_q|^2 \quad t \in [0, T].
\end{align*}
\]

(From (9.17) and (9.18)) and such that \(\varepsilon_{q+1}\) satisfies (9.20)-(9.21). Let

\[
R_{q+1} = \overline{K}_q - S_q - \varepsilon_{q+1}.
\]

We claim that \((v_{q+1}, p_{q+1}, R_{q+1})\) is a smooth strong subsolution satisfying \((A_{q+1})\)-\((F_{q+1})\). Notice that \((A_{q+1})\) is satisfied by construction. By definition of \(S_q\), one has that

\[
\rho_{q+1} = \overline{p}_q^2(1 - \psi_q^2) + \psi_q^2\delta_{q+2} - \frac{1}{3}\text{tr}\varepsilon_{q+1}
\]

\[
\hat{R}_{q+1} = \overline{K}_q(1 - \psi_q^2) - \hat{\varepsilon}_{q+1}.
\]

For \(t \in K_q\), condition \((B_{q+1})\) follows easily from (9.20) and the fact that \(K_q \subseteq \{\psi_q = 1\}\). For \(t \notin J_q\), we have that

\[
\rho_{q+1} = \rho_q \leq \frac{3}{2}\delta_{q+2} < \frac{5}{2}\delta_{q+2},
\]

For \(t \in J_q \setminus K_q\), we have that \(\overline{p}_q \leq \frac{9}{8}\rho_q \leq \frac{9}{8}\cdot 2\delta_{q+2} = \frac{9}{4}\delta_{q+2}\), which means

\[
\rho_{q+1} \leq \frac{9}{4}\delta_{q+2}\left(1 - \frac{4}{9}\psi_q^2 + \frac{5}{9}\lambda_{q+1}^{-6\alpha}\right) \leq \frac{5}{4}\delta_{q+2}\left(\frac{9}{5} + \frac{1}{4}\lambda_{q+1}^{-6\alpha}\right),
\]

and if \(\lambda_{q+1}^{-6\alpha} \leq \frac{1}{5}\), which is a matter of choosing a large enough, we have \((B_{q+1})\).

Note that, by the construction of \(\rho_{q+1}\), we have that \(J_q \subseteq K_{q+1}\), since on the whole of \(J_q\) we have that \(\rho_{q+1} \sim \delta_{q+2} \gg \delta_{q+3}\). This is the reason why we required \(\hat{\gamma} < \gamma\) and used the larger \(\gamma\) outside of \(J_q\) in (C_q): to make sure \((C_{q+1})\) was automatically verified outside \(J_q\). This is in stark contrast to what happened in Section 10 where the perturbation regions \(P_q := [0, 2\hat{\gamma}T]\) satisfied the opposite inclusion \(P_{q+1} \subseteq P_q\) and we consequently required \(\hat{\gamma} > \gamma\) to ensure the weaker “strong condition” \((b_{q+1})\) in \(P_q \setminus P_{q+1}\), while the stronger \((d_{q+1})\) only held in \(P_{q+1}\), where \(\psi_q = 1\).

By the above paragraph, in verifying conditions \((C_{q+1})\)-\((D_{q+1})\), it suffices to restrict to the case when \(\rho_{q+1} \geq 2\delta_{q+3}\) and \(j = q + 1\), respectively.

The argument showing \((C_{q+1})\) for \(t \in K_{q+1}\) is similar to the proof of \((b_{q+1})\) in Step 3 of Proposition 5.1 above. On the one hand

\[
\|\hat{R}_{q+1}\|_0 = \|(1 - \psi_q^2)\overline{K}_q - \hat{\varepsilon}_{q+1}\|_0
\]

\[
\leq (1 - \psi_q^2)\Lambda\overline{\delta}_q^{1+\frac{1}{\hat{\gamma}}}\ell_{q-2\alpha + (1 - \frac{1}{2})\alpha^2\varepsilon_{q+1}} + \delta_{q+2}\lambda_{q+1}^{-6\alpha},
\]

on the other hand

\[
\rho_{q+1} = (1 - \psi_q^2)\Lambda\overline{\delta}_q + \psi_q^2\delta_{q+2} + \frac{1}{3}\text{tr}\varepsilon_{q+1}
\]

\[
\geq (1 - \psi_q^2)\overline{p}_q + \psi_q^2\delta_{q+2} - \delta_{q+2}\lambda_{q+1}^{-6\alpha}.
\]
So where $\psi_q = 1$ we have the condition since, for $a$ large enough, we can guarantee
\[
\delta_{q+2}\lambda_{q+1}^{-6\alpha} \leq \delta_{q+2}\lambda_{q+2}(1 - \lambda_{q+1}^{-6\alpha})^{1+\hat{\gamma}} \iff \lambda_{q+1}^{-6\alpha} \leq \delta_{q+2}(1 - \lambda_{q+1}^{-6\alpha})^{1+\hat{\gamma}},
\]  
(11.18)
since $6\alpha > 2\beta\hat{\gamma}$ is required in (11.13). If $\psi_q \neq 1$, however, we need
\[
(1 - \psi_q^2)\Lambda^{-\hat{\gamma}}p_q^{1+\hat{\gamma}}\ell_q^{-2\alpha} + \delta_{q+2}\lambda_{q+1}^{-6\alpha} + \delta_{q+2}\lambda_{q+2}(\psi_q > 0) \leq \Lambda^{-\hat{\gamma}}[(1 - \psi_q^2)p_q + \psi_q^2\delta_{q+2} - \delta_{q+2}\lambda_{q+1}^{-6\alpha}]^{1+\hat{\gamma}}\ell_{q+1}^{-2\alpha},
\]  
(11.19)
To this end set
\[
F(s) := (1 - s)\Lambda\ell_{q}^{1+\hat{\gamma}}\ell_{q}^{-2\alpha} + \delta_{q+2}\lambda_{q+1}^{-6\alpha}
G(s) := (1 - s)p_q + s\delta_{q+2} - \delta_{q+2}\lambda_{q+1}^{-6\alpha} = p_q + s(p_q - p_q) - \delta_{q+2}\lambda_{q+1}^{-6\alpha}
H(s) := \Lambda^{-\hat{\gamma}}G^{1+\hat{\gamma}}(s)\ell_{q+1}^{-2\alpha} - F(s),
\]
and, just like in Proposition 5.1 deduce that $H(\psi_q^2) \geq 0$ by proving that:

(i) $H(0) \geq 0$ and $H(1) \geq 0$;

(ii) $H'(0) \geq 0$ and $H'(1) \geq 0$.

(iii) $H''(s) \geq 0$.

To this end, we first obtain the estimates
\[
\delta_{q+2}\lambda_{q+1}^{-6\alpha} \leq \Lambda\ell_{q}^{1+\hat{\gamma}}\ell_{q}^{-2\alpha}, \quad F(0) \leq \Lambda\ell_{q}^{1+\hat{\gamma}}\ell_{q}^{-2\alpha}, \quad G(0) \leq p_q, \quad G(s) \leq p_q.
\]
The first one follows from (11.18), (8.21), and the fact we are working for $t \in J_q$. The second one follows from the first one. The fourth one is obvious, since $p_q \geq \frac{7}{8}p_q + \frac{21}{16}\delta_{q+2} > \delta_{q+2}$. For the third one, we reduce it to $\delta_{q+2}\lambda_{q+1}^{-6\alpha} \leq p_q$, and then it follows from the first estimate, since $\Lambda\ell_{q}^{1+\hat{\gamma}} \leq p_q$. We then prove (i)-(v) as follows.

- It is easy to check that the two parts of (i) amount to
  \[
  \Lambda\ell_{q}^{1+\hat{\gamma}}\ell_{q}^{-2\alpha} \leq \Lambda\ell_{q}^{1+\hat{\gamma}}\ell_{q+1}^{-2\alpha}, \quad \delta_{q+2}\lambda_{q+1}^{-6\alpha} \leq \Lambda^{-\hat{\gamma}}(\delta_{q+2}(1 - \lambda_{q+1}^{-6\alpha})^{1+\hat{\gamma}}\ell_{q+1}^{-2\alpha};
  \]
the first one follows from $\ell_{q} \sim \ell_{q+1}^{1/6}$, the second one follows from (11.13) and the following relations, which hold for $a$ sufficiently large:
\[
1 - \lambda_{q+1}^{-6\alpha} \geq \frac{1}{2} \iff \lambda_{q+1}^{-6\alpha} \leq \frac{1}{2}, \quad \lambda_{q+1}^{-6\alpha} \leq \ell_{q+1}^{-2\alpha} - 1 - \hat{\gamma};
\]

- The requirements (ii) can be rewritten as
  \[
  \ell_{q}^{1+\hat{\gamma}}\ell_{q}^{-2\alpha} \geq (1 + \hat{\gamma})(p_q - \delta_{q+2})\ell_{q+1}^{-2\alpha}\max\{p_q - \delta_{q+2}\lambda_{q+1}^{-6\alpha}\hat{\gamma}, \delta_{q+2}(1 - \lambda_{q+1}^{-6\alpha})\hat{\gamma}\},
  \]
which easily follows for sufficiently small $\hat{\gamma}$ and sufficiently large $a$, since $\ell_{q}^{-2\alpha} \sim \ell_{q+1}^{-2\alpha}$;

- Note that $G'' = 0$ because $G$ is linear in $s$, and the same is true of $F''$, meaning that (iii) is simply
  \[
  0 \leq \Lambda^{-\hat{\gamma}}(1 + \hat{\gamma})G^{2-1}(s)G^2(s)\ell_{q+1}^{-2\alpha} \leq \Lambda^{-\hat{\gamma}}(1 + \hat{\gamma})[(1 - s)p_q + \delta_{q+2}(1 - \lambda_{q+1}^{-6\alpha})\hat{\gamma} - p_q - \delta_{q+2}^2\ell_{q+1}^{-2\alpha},
  \]
which is obvious, since all those factors are positive.
We have thus obtained \((C_{q+1})\).
The velocity estimate in \((D_{q+1})\) for \(j = q + 1\) follows from (9.18) and (8.24). The trace estimate in \((D_{q+1})\) follows from (9.21) and (8.22). Finally, \((E_{q+1})\) follows precisely as \((f_{q+1})\) in the proof of Proposition 5.1 in Section 10 above, and \((F_{q+1})\) is obtained just like \((g_{q+1})\). We keep in mind Remark II.1 above for all of these. Thus, the inductive step is proved.

Finally, the convergence of \(\{v_q\}\) to a solution of the hypodissipative Navier-Stokes equations as in the statement of Proposition 5.2 (i.e. the one we are proving) follows easily from the sequence of estimates in \((A_q)-(F_q)\), analogously to Step 4 of Proposition 5.1 proved in Section 10 above.

The Navier-Stokes term \(\mathcal{F}\) will be handled in the same way as \(\hat{\mathcal{F}}\) was dealt with in Proposition 5.1, giving us once more that the maximal time \(t_*\) of “smallness” of \(\mathcal{F}\) must satisfy

\[
\lim_{\alpha \to \infty} t_* = 0.
\]

**Step 3: From one to infinitely many**

Looking at the details of the scheme, we realize that replacing \((\dot{v}, \dot{\rho}, \dot{R})\) with

\[(v_0, p_0, R_0) \equiv (\dot{v}, \dot{\rho}, \dot{R} + \epsilon/3),\]

as described in the statement of Proposition 5.2, clearly retains condition (3.4), since the initial datum is not changed. It does not necessarily preserve conditions (3.5) and (3.6). Those, however, are only needed to obtain the conditions \((D_0)\). If we then show that the conditions \((A_0)-(F_0)\) (and thus also \((D_0)\)) are maintained with such a perturbation, we need not worry about losing (3.5) and (3.6).

Conditions about the velocity are clearly preserved, and \((A_0)\) and \((E_0)\) are vacuous, so all we need is

\[
\rho'_0 \geq \frac{5}{2}\delta \quad \text{(B_0)}
\]

\[
\|\hat{R}_0\|_0 \leq \begin{cases} 
\Lambda \hat{q}_{0,1}^{1+\gamma} \Lambda_{\delta_{\frac{3}{2}}^2 \tilde{e}_q}^{2\delta_2} & \rho'_0 \geq 2\delta_2 \\
\Lambda \hat{q}_{0,1}^{1+\gamma} \Lambda_{\delta_{\frac{3}{2}}^2}^{2\delta_2} \leq \rho'_0 \leq 2\delta_2 \\
\Lambda \hat{q}_{0,1}^{1+\gamma} \rho'_0 \leq \frac{3}{2} \delta_2 
\end{cases} \quad \text{(C_0)}
\]

\[
|\partial_{t} \rho'_0| \leq \rho'_0 \delta_{\frac{3}{2}}^2 \lambda_{0}. \quad \text{(D_0, 2, i.e. (11.10))}
\]

Concerning \((B_0)\), the proposition assumes \(\hat{\rho} \leq 5/2\delta\), so that the condition is preserved by requiring (5.13). Since \(\hat{\rho}(0) = 0\), \(\epsilon\) has the possibility to vary in a neighborhood of \(t = 0\) without becoming negative.

\((D_0, 3)\) boils down to the following condition on \(\epsilon\):

\[
\left\{ \begin{array}{l}
|\partial_{t} \epsilon| \leq \epsilon \sqrt{\delta} \lambda_{0} \\
|\partial_{t} \epsilon| + |\epsilon| \sqrt{\delta} \lambda_{0} \leq \hat{\rho} \sqrt{\delta} \lambda_{0} - |\partial_{t} \hat{\rho}| \quad \text{otherwise}
\end{array} \right. \quad e > 0
\]

To keep things simple, we require (5.14) and (5.15).

Coming to \((C_0)\), we first assume \(e > 0\), which immediately yields, by the properties of \((\dot{v}, \dot{\rho}, \dot{R})\), that

\[
\|\hat{R}_0\|_0 \leq \begin{cases} 
\Lambda \hat{q}_{0,1}^{1+\gamma} \Lambda_{\delta_{\frac{3}{2}}^2}^{2\delta_2} & \hat{\rho} \geq 2\delta_2 \\
\Lambda \hat{q}_{0,1}^{1+\gamma} \frac{3}{2} \delta_2 \leq \hat{\rho} \leq 2\delta_2 \\
\Lambda \hat{q}_{0,1}^{1+\gamma} \frac{3}{2} \delta_2 & \hat{\rho} \leq \frac{3}{2} \delta_2
\end{cases}
\]

Our goal is to obtain that

\[
\|\hat{R}_0\|_0 \leq \begin{cases} 
\Lambda \hat{q}_{0,1}^{1+\gamma} \Lambda_{\delta_{\frac{3}{2}}^2}^{2\delta_2} & \hat{\rho} + e \geq 2\delta_2 \\
\Lambda \hat{q}_{0,1}^{1+\gamma} \frac{3}{2} \delta_2 \leq \hat{\rho} + e \leq 2\delta_2 \\
\Lambda \hat{q}_{0,1}^{1+\gamma} \frac{3}{2} \delta_2 & \hat{\rho} + e \leq \frac{3}{2} \delta_2
\end{cases}
\]

- We first note that \(\hat{\rho} + e \leq \frac{3}{2} \delta_2 \implies \hat{\rho} \leq \frac{3}{2} \delta_2\), so in this case we have the desired estimate for \(\|\hat{R}_0\|_0\):
• If $\hat{p} \leq \frac{3}{2}\hat{\delta}_2$ but $\frac{3}{2}\hat{\delta}_2 \leq \hat{p} + \epsilon \leq 2\hat{\delta}_2$, since $\hat{\gamma} < \gamma$ and $\|\tilde{R}_0\|_0 \leq \Lambda(\hat{\gamma} + \Lambda^{-1}\epsilon)^{1+\gamma}$, we have the desired estimate;

• If $\hat{p} \leq \frac{3}{2}\hat{\delta}_2$ but $\hat{p} + \epsilon \geq 2\hat{\delta}_2$, the desired estimate is even looser than in the previous item;

• Finally, if $\hat{p} \geq 2\hat{\delta}_2$, then so is $\hat{p} + \epsilon$, meaning again we have the desired estimate.

Thus, we need no additional conditions to obtain (C0). Summing up, the conditions we must impose on $\epsilon$ are precisely (5.13)-(5.15). The proof is complete. \(\Diamond\)

A Proof of Lemma 6.1

Proof (Lemma 6.1)

Fix $\rho \in C^\infty_c(B_1(0))$ a standard mollification kernel in space, and define:

$$\rho_\epsilon(x) := e^{-3\epsilon}\rho(xe^{-1}).$$

To ensure the regularity of the initial datum, we consider the smoothed datum

$$w_0 := w * \rho_{\eta_0},$$

where

$$\eta_0 := \max \left\{ \eta : \|w * \rho_\eta - w\|_{L^2} \leq \frac{\delta}{3} \land \int_{\mathbb{T}^3} \|[w_0]^2 - |w|^2\|(x)dx \leq \frac{2}{3}\delta \right\}. \quad (A.1)$$

By Theorem 6.1, there exists a solution $(\tilde{v}, \tilde{p})$ with initial datum $w_0$, where $\tilde{p}$ can be recovered uniquely once we impose $\int \tilde{p} = 0$.

We now fix a standard mollification kernel in time $\chi \in C^\infty_c((-1,0))$ and, with $\rho_\epsilon, \rho$ as defined above, we define

$$\chi_\epsilon(t) := e^{-1}\chi(te^{-1})$$

$$v(x,t) := \int_t^{t+\epsilon} (\tilde{v} * \rho_\epsilon)(x,s)\chi_\epsilon(t-s)ds,$$

$$p(x,t) := \int_t^{t+\epsilon} (\tilde{p} * \rho_\epsilon)(x,s)\chi_\epsilon(t-s)ds,$$

$$R(x,t) := \nabla \otimes \tilde{v} - v \otimes v,$$

where

$$\tilde{f} := \int_t^{t+\epsilon} (f * \rho_\epsilon)(x,s)\chi_\epsilon(t-s)ds.$$
Coming to the initial datum, we have that
\[ v|_{t=0} = \int_0^t (\bar{v} - w_0) \cdot \rho_e \cdot \chi(x, s) ds + w_0 \cdot \rho_e. \]

Taking the $L^2$ norm, we can easily obtain that
\[ \|v|_{t=0} - w_0\|_{L^2(B_3)} \leq \sup_{t \in [0, \delta]} \|\bar{v}(\cdot, t) - w_0\|_{L^2(B_3)} + \|w_0 \cdot \rho_e - w_0\|_{L^2(B_3)} =: \sup I_t + II_e. \]

$I_e$ can be made as small as we desire by choosing $\varepsilon$ small enough. Let $\varepsilon_0$ be the maximal parameter such that $II_e < \frac{\delta}{3}$. As for $sup I_t$, using $\bar{v}_t(x) := \bar{v}(x, t)$, we can obtain that
\[
I_t^2 = \int_{B_3} |\bar{v}_t - w_0|^2 dx = \int_{B_3} |\bar{v}_t|^2 - |w_0|^2 dx - 2 \int_0^t \langle \partial_t \bar{v}, w_0 \rangle dx ds
\leq 2 \int_0^t \langle -\varepsilon_0 \bar{v}_t + \bar{v}_t, (-\Delta)^0 w_0 \rangle dx ds
\leq 2 \int_0^t \|\bar{v}_t\|_{L^2(B_3)} \|w_0\|_{L^2} dx ds
\leq 2 \int_0^t \|\bar{v}_t\|_{L^2} \|w_0\|_{L^2} dx ds
\leq 2 \sqrt{C(1-2\theta)} \int_0^t \|\bar{v}_t\|_{L^2} \|w_0\|_{L^2} dx ds
\leq 2 \sqrt{C(1-2\theta)} \sup_{t \in [0, \delta]} \|\bar{v}(\cdot, t) - w_0\|_{L^2(B_3)} \|w_0\|_{L^2} \leq K(w) \|w_0\|_{L^2},
\]

In $\bullet$, we used the fact that $|\bar{v}_t|^2 \leq \|w_0\|^2_{L^2}$, i.e. (6.1), as well as the fact that $(\bar{v}, \bar{p})$ is a solution of (1.1c) and the fact $\text{div} w_0 = 0$. In $\bullet$, we used Theorem 2.1 choosing $\varepsilon = 1 - 2\theta$. In the last step, we used that $\|w_0\|_{L^2} \leq \|w\|_{L^2}$. This becomes arbitrarily small if we choose $t$ appropriately small, and since $t \in [0, \varepsilon]$ translates to $\varepsilon$ small enough. Since $Dw_0 = D\rho_{\varepsilon} * w = \eta_0^4 D\rho_{\varepsilon}^{-1} x \cdot w$, Hölder’s inequality yields
\[
\|Dw_0\|_{L^2} \leq \eta_0^4 \|D\rho_{\varepsilon}^{-1} x\|_{L^2(B_{\varepsilon_0})} \|w\|_{L^2} \leq \eta_0^4 \|D\rho_{\varepsilon}^{-1} x\|_{L^2(B_{\varepsilon_0})} \|w\|_{L^2} = C(w) \eta_0^{-2},
\]

so that, to ensure $sup_{t \in [0, \delta]} I_t \leq \frac{\delta}{2}$, we choose
\[
\varepsilon \leq \frac{\delta \eta_0^2}{3C(w)K(w)} =: \varepsilon_0.
\]

Choosing $\varepsilon := \min\{\varepsilon_0, \varepsilon_0, \eta_0\}$ thus yields
\[
\|\bar{v}|_{t=0} - w\|_{L^2} \leq \delta \quad \|\bar{v}|_{t=0} - w_0\|_{L^2} \leq \frac{2}{3} \delta.
\]

We have thus obtained (6.2). As for (6.3), we first notice that
\[
\int_{B_3} (\rho_e * f)(x, t) dx = \int_{B_3} f(x, t) dt,
\]

for any $t \in [0, T]$ and any function $f$. Thus, by the definition of $R$, we have that
\[
\int_{B_3} |v|^2(x, t) + \text{tr} R(x, t) dx = \int_{B_3} |\bar{v}|^2(x, t) dx = \int_{B_3} (\chi_e * |\bar{v}|^2)(x, t) dx,
\]

We have thus reduced (6.3) to the following inequality:
\[
\frac{1}{2} \int_{B_3} (\chi_e * |\bar{v}|^2)(x, t) + \int_0^t \int_{B_3} (-\Delta)^0 (\bar{v})_3^2 (x, s) dx ds \leq \int_{B_3} |v|^2(x) dx + \delta.
\]
Since \((\tilde{v}, \tilde{p})\) satisfies (6.1), we can see that

\[
\frac{1}{2} \int_{T^3} \left( \chi_{\varepsilon} \ast |\tilde{v}|^2 \right) (x, t) \, dx + \int_{0}^{t} \int_{T^3} \left| (-\Delta)^{\frac{\theta}{2}} v^2 \right| (x, s) \, dx \, ds \leq \frac{1}{2} \int_{T^3} |w|^2 \, dx + \frac{1}{2} \int_{T^3} |w_0|^2 - |w|^2 \, dx
\]

\[
+ \frac{1}{2} \int_{T^3} \left( \chi_{\varepsilon} \ast |\tilde{v}|^2 \right) \, dx + \int_{0}^{t} \int_{T^3} \left[ \left| (-\Delta)^{\frac{\theta}{2}} v \right|^2 - \left| (-\Delta)^{\frac{\theta}{2}} v \right|^2 \right] (x, s) \, dx \, ds
\]

\[
= \frac{1}{2} \int_{T^3} |w|^2 \, dx + I + II + III.
\]

Our desired estimate (6.3) will then follow from

\[
I \leq \delta \frac{3}{3}, \quad II \leq \delta \frac{3}{3}, \quad III \leq \delta \frac{3}{3}.
\]  \hspace{1cm} (A.4)

The first of these relations follows from (A.1).

The second relation in (A.4) follows, for \(\varepsilon\) sufficiently small, from the fact that, since \(\tilde{v} \in L^2 T H^0 \subseteq L^2 L^2, |\tilde{v}|^2 \in L^2 L^1\), so \(\chi_{\varepsilon} \ast |\tilde{v}|^2 \rightarrow |\tilde{v}|^2\) in \(L^2 L^1\) for \(\varepsilon \rightarrow 0\).

Coming to the third relation, we first rewrite the integral of the first integrand:

\[
\int_{0}^{t} \int_{T^3} \left| (-\Delta)^{\frac{\theta}{2}} v \right|^2 (x, s) \, dx \, ds = \int_{0}^{t} \int_{T^3} \int_{T^3} \rho_{\varepsilon}(x-y)(-\Delta)^{\frac{\theta}{2}} \tilde{v}(y, \tau) \chi_{\varepsilon}(t-\tau) \, dy \, d\tau \, dx \, ds
\]

\[
\leq \int_{0}^{t} \int_{T^3} \int_{T^3} \left| (-\Delta)^{\frac{\theta}{2}} \chi_{\varepsilon}(t-\tau) \right|^2 \, dx \, dx \, d\tau,
\]

where we used Jensen's inequality in the first step and (A.3) in the second one. Therefore, the remaining term is rewritten as:

\[
III = \int_{0}^{t} \int_{T^3} \left[ \left( \chi_{\varepsilon} \ast \left| (-\Delta)^{\frac{\theta}{2}} \tilde{v} \right|^2 \right)(x, s) - \left| (-\Delta)^{\frac{\theta}{2}} \tilde{v} \right|^2 (x, s) \right] \, dx \, ds.
\]

We now note that, since \(\tilde{v} \in L^2 T H^0\), we have that \((-\Delta)^{\frac{\theta}{2}} \tilde{v} \in L^2 L^2\), and thus \(|(-\Delta)^{\frac{\theta}{2}} \tilde{v}|^2 \in L^1 L^1\). Therefore, as before, \(\chi_{\varepsilon} \ast |(-\Delta)^{\frac{\theta}{2}} \tilde{v}|^2 \rightarrow 0 \) in \(L^1 L^1\), and the third relation of (A.4) reduces to an opportune choice of \(\varepsilon\).

Summing up, \((v, p, R)\) is a smooth solution of (3.1), which satisfies (6.3) and (6.2), and \(R \geq 0\) by (A.2).

The proof of Lemma 6.1 is thus complete. \(\diamondsuit\)

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