\( \alpha \)-ideals in a 0-distributive lattice

R. M. Hafizur Rahman

Department of Mathematics, Begum Rokeya University, Rangpur, Bangladesh

E-mail: salim030659@yahoo.com

https://doi.org/10.26782/jmcms.2019.08.00006

Abstract

In this paper we have studied the \( \alpha \)-ideals in a 0-distributive lattice. We have described the \( \alpha \)-ideals by two definition and proved that these are equivalent. We have given several characterizations. They have proved that a lattice \( L \) is disjunctive if and only if each ideal is an \( \alpha \)-ideals. We have also included a prime separation theorem for \( \alpha \)-ideals. At the end we have studied the \( \alpha \)-ideals in a sectionally quasi-complemented lattice.

Keywords: \( \alpha \)-ideals, 0-distributive lattice, separation theorem, quasi-complemented lattice.

I. Introduction

\( \alpha \)-ideals have been studied by many authors concluding Cornish [V] in case of distributive lattices with 0. In a non-distributive lattice \( L \) with 0, if \( I(L) \) is pseudo complemented, then concept of \( \alpha \)-ideals is possible. Thus, in particular, we can study the \( \alpha \)-ideals for a 0-distributive lattice as a lattice \( L \) with 0 is 0-distributive if and only if \( I(L) \) is pseudo complemented. A lattice \( L \) with 0 is called a 0-distributive lattice if for all \( a, b, c \in L, a \wedge b = 0 = a \wedge c \) imply \( a \wedge (b \vee c) = 0 \). 0-distributive lattices were first studied by [XII]. Then a series of lattice theorists have studied the topic including [II], [III], [IV], [VIII], [IX], [X] and [XI]. In [VII], Jayaram has given a result on Prime Separation Theorem using \( \alpha \)-ideals in a 0-distributive lattice. On the other hand, Noor, Ayub and Islam [I] have generalized the result of [V] for distributive near lattices. In this paper, we would like to discuss the \( \alpha \)-ideals of a 0-distributive lattice in a very simple way. Then we generalized several results of [V] in a 0-distributive lattice.

For an ideal \( J \) in \( L \), we define \( \alpha(J) = \{ (x)^+ : x \in J \} \) is a filter in \( A_0(L) \), where \( A_0(L) \) is the set of all annihilator ideals of the form \( (x)^+ \); \( x \in L \), which is a sublattice of lattice of annihilator ideals of \( L \). We also define \( \leftarrow \alpha(F) = \{ x \in L : (x)^+ \in F \} \) where \( F \) is any filter in \( A_0(L) \). It is easy to check that \( \leftarrow \alpha(F) \) is an ideal in \( L \).

We start with the following result which is due to [V] Problem 3.1.
Proposition 1.1. Let $L$ be a lattice with 0. The following holds:

a) For an ideal $J$ in $L$, $\alpha(J) = \{x^* : x \in J\}$ is a filter in $A_0(L)$,

b) For a filter $F$ in $A_0(L)$, $\alpha(F) = \{x \in L : (x)^* \in F\}$ is an ideal in $L$,

c) If $J_1, J_2$ are ideals in $L$ then $J_1 \subseteq J_2$ implies $\alpha(J_1) \subseteq \alpha(J_2)$ and $F_1, F_2$ are filter in $A_0(L)$ then $F_1 \subseteq F_2$ implies $\alpha(F_1) \subseteq \alpha(F_2)$,

d) The map $I \rightarrow \alpha(I) = \{(\alpha(I))\}$ is a closure operation on the lattice of ideals of $L$.

In a 0-distributive lattice $L$, an ideal $J_0$ is called an $\alpha$-ideal if $\alpha \alpha(J) = J$. Thus $\alpha$-ideals are simply the closed elements with respect to the closure operation of proposition 1.1. Thus the following result is an immediate consequence of above result.

Proposition 1.2. The $\alpha$-ideals of a lattice $L$ with 0 form a complete lattice isomorphic to the lattice of filters ordered by set inclusion of the lattice $A_0(L)$ of annulets of $L$.

The infimum of a set of $\alpha$-ideals $J_i$ is $\cap J_i$ is their set theoretic intersection. The supremum is $\alpha \alpha(v J)$ where $\vee J$ is the supremum in the lattice of ideals of $L$.

Proposition 1.3. For an ideal $J$ of a distributive lattice $L$, the following are equivalent:

a) $J$ is an $\alpha$-ideal

b) For $x, y \in L$, $(x)^* = (y)^*$ and $x \in J \Rightarrow y \in J$

c) $x \in J$ implies $(x)^* \subseteq J$.

Proof. (a) $\Leftrightarrow$ (b) Suppose a) holds, so $\alpha \alpha(J) = J$. Let $(x)^* = (y)^*$ and $x \in J$. Then $(y)^* = (x)^* \in \alpha(J)$. So by the definition, $y \in \alpha \alpha(J) = J$. This implies $(x)^* \in \alpha(J)$. Then by definition of $\alpha(J)$, $(x)^* = (p)^*$ for some $p \in J$. Hence by (b) $x \in J$, and so $\alpha \alpha(J) \subseteq J$. Since by Proposition 1.1, reverse inclusion always holds. Thus $\alpha \alpha(J) = J$ and hence $J$ is an $\alpha$-ideal.

(b) $\Leftrightarrow$ (c) Suppose (b) holds. Let $x \in J$ and $y \in (x)^*$. Then $(x)^* \subseteq (y)^*$. Thus, $(y)^* = (x)^* \cup (y)^* = ((x)^* \cap (y)^*)^* = (x \land y)^* = (x \land y)^*$. Moreover,
$x \wedge y \in J$ as $x \in J$. Hence applying (b), we have $y \in J$. This implies $(x)^* \subseteq J$ and so (c) holds. Finally suppose (c) holds. Let $x \in J$ and $(x)^* = (y)^*$, $y \in L$. By (c), $(x)^{**} \subseteq J$, and so $y \in (y)^{**} = (x)^{**} \subseteq J$ implies $y \in J$, which is (b).

Thus we can define an $\alpha$ - ideal as follows:

An ideal $I$ in a 0-distributive lattice $L$ is called an $\alpha$ - ideal if for each $x \in L$, $x \in I$ implies $(x)^{**} \subseteq I$.

We know that $A^\bot = \{x \in L : x \wedge a = 0 \text{ for all } a \in L\}$ is an ideal if $L$ is 0-distributive and if $A$ is an ideal, then $A^\bot = A^\bot$ is the annihilator ideal.

**Theorem 1.4.** For any ideal $J$ in a 0-distributive lattice $L$, the set $I^\alpha = \{x \in L : (a)^* \subseteq (x)^* \text{ for some } a \in J\}$ is the smallest $\alpha$ - ideal containing $J$ and ideal $I$ in $L$ is an $\alpha$ - ideal if and only if $I = I^\alpha$.

**Proof.** Let $x \in I^\alpha$. Then $(a)^* \subseteq (x)^*$ for some $a \in I$ and so $(x)^{**} \subseteq (a)^{**}$. Suppose $y \in (a)^{**}$. Thus $(y)^* \subseteq (a)^{**}$ and so $(a)^* \subseteq (y)^*$. This implies $x \in I^\alpha$. Therefore, $(a)^{**} \subseteq I^\alpha$ and so $(x)^{**} \subseteq I^\alpha$. It follows that $I^\alpha$ is an $\alpha$ - ideal. Now suppose $x \in I$, then by definition, $x \in I^\alpha$ and so $I \subseteq I^\alpha$. Suppose $K$ is an $\alpha$ - ideal containing $I$. Let $x \in I^\alpha$. Then $(a)^* \subseteq (x)^*$ for some $a \in I \subseteq K$. This implies $(x)^{**} \subseteq (a)^{**} \subseteq K$ as $K$ is an $\alpha$ - ideal. Thus $(x) \subseteq K$ and $x \in K$. Hence $I^\alpha \subseteq K$. That is $I^\alpha$ is the smallest $\alpha$ - ideal containing $I$.

**Theorem 1.5.** Every annihilator ideal in a 0-distributive lattice $L$ is an $\alpha$ - ideal.

**Proof.** Let $I = A^\bot$ be the annihilator ideal of $L$. Suppose $y \in I = A^\bot$. Then $y \wedge a = 0$ for all $a \in A$. Then $(y)^\wedge (a)^* = (0)^*$ and so $(y)^* \subseteq (a)^*$. Thus $(y)^{**} \subseteq (a)^{**} = (a)^*$ for all $a \in A$. Hence, $(y)^{**} \subseteq \bigcap_{a \in A} (a)^* = A^\bot = I$ and so $I$ is an $\alpha$ - ideal.

**Theorem 1.6.** Let $L$ be a 0-distributive nearlattice. $A$ be a meet semilattice of $L$. Then $A^0$ is an $\alpha$ - ideal, where $A^0 = \{x \in L : x \wedge a = 0 \text{ for some } a \in A\}$.

**Proof.** By [6; Theorem 4.2.8], $A^0$ is an ideal. Now let $x \in A^0$ and $y \in (x)^{**}$. Clearly $x \in A^0$ implies $x \wedge a = 0$ for some $a \in A$. This shows that $y \in A^0$, consequently $(x)^{**} \subseteq A^0$. Hence $A^0$ is an $\alpha$ - ideal of $L$. 

*Copyright reserved © J. Mech. Cont.& Math. Sci.*

*R. M. Hafizur Rahman*
Theorem 1.7. If a prime ideal \( P \) of a 0-distrbutive lattice \( L \) is non dense then \( P \) is an \( \alpha \) -ideal.

Proof. By assumption \( P^* \neq (0) \). Hence there exists \( x \in P^* \) such that \( x \neq 0 \).
But then \( (x)^* \supseteq P^* \) gives \( (x)^* \supset P \) as \( P \subseteq P^* \). Further if \( t \in (x)^* \), then \( x \land t = 0 \in P \). But as \( P \) is a prime ideal, so \( t \in P \) (since \( P \cap P^* = (0) \Rightarrow x \not\in P \)). This implies \( (x)^* \subseteq P \). Combining both the inclusions, we get \( P = (x)^* \). Hence \( P \) is an annihilator ideal and so by Theorem 1.5, \( P \) is an \( \alpha \) -ideal.

Corollary 1.8. Every non-dense prime ideal of a 0-distributive lattice is an annulet.

Theorem 1.9. For an \( \alpha \) -ideal \( I \) of a 0-distributive lattice \( L \), \( I = \{ y \in L : (y) \subseteq (x)^* \text{ for some } x \in I \} \).

Proof. Let \( a \in I \). Then -ideal lattice, J. C. Varlet [12] introduced the notion of 0-distributive lattices. Then \( (a) \subseteq (a)^* \) implies that \( a \in \{ y \in L : (y) \subseteq (x)^* \text{ for some } x \in I \} \).
Conversely, let \( a \in \{ y \in L : (y) \subseteq (x)^* \text{ for some } x \in I \} \). Then \( (a) \subseteq (x)^* \) for some \( x \in I \). Since \( I \) is an \( \alpha \) -ideal, so \( (x)^* \subseteq I \) and so \( (a) \subseteq I \). Hence \( a \in I \).

Now we include a prime Separation Theorem for \( \alpha \)-ideals in a 0-distributive lattice. This result has also been proved in [1]. But we claim that our proof is much better and easier.

Theorem 1.10. Let \( F \) be a filter and \( I \) be an \( \alpha \) -ideal in a 0-distributive lattice \( L \) such that \( I \cap F = \phi \). Then there exists a prime \( \alpha \) -ideal \( P \supseteq I \) such that \( P \cap F = \phi \).

Proof. Let \( \chi \) be the collection of all filters containing \( F \) and disjoint from \( I \). \( \chi \) is non-empty as \( F \in \chi \). Then by [6; Lemma 3.3.3] there exists a maximal filter \( Q \) containing \( F \) and disjoint from \( I \). Suppose \( Q \) is not prime. Then there exist \( f, g \in Q \) such that \( f \lor g \) exists and \( f \lor g \in Q \). Then by [6; Lemma 3.3.4], there exist \( a \in Q, \ b \in Q \) such that \( a \land f \in I \) and \( b \land g \in I \). Thus we have \( a \land b \land f \in I \) and \( a \land b \land g \in I \). Then \( (a \land b \land f) \subseteq (x)^* \) and \( (a \land b \land g) \subseteq (y)^* \) for some \( x, y \in I \). Thus we have \( (a \land b \land f) \land (x)^* = (0) = (a \land b \land g) \land (y)^* \). That is \( (a \land b) \land (x)^* \land (a \land b) \land (y)^* \land (g) \). Since \( I(L) \) is 0-distributive, it follows that \( (a \land b) \land (x)^* \land (y)^* \land ((f) \lor (g)) = (0) \).

\( Copyright\ reserved\ ©\ J.\ Mech.\ Cont.&\ Math.\ Sci.\ \\
R. M. Hafizur Rahman \)

69
Let \( I \) is an ideal, which is contradiction to \( F \). Moreover, by proposition 1.1, both ideals \( L \) where \( b \), such that \( P \) is a prime ideal. Thus, \( P \) is an \( \alpha \)-ideal.

For an \( \alpha \)-ideal \( I \), \( \alpha \alpha(I) = I \). Also, it is clear that for any filter \( F \) of \( A_b(L) \), \( \alpha \alpha(F) = F \). Moreover, by proposition 1.1, both \( \alpha \) and \( \alpha \) are isotone. Hence the lattice of \( \alpha \)-ideals of \( L \) is isomorphic to the lattice of filters.

**Corollary 1.11.** Let \( L \) be a 0-distributive lattice. Then the set of prime \( \alpha \)-ideals of \( L \) are isomorphic to the set of prime filters of \( A_b(L) \).

A 0-distributive lattice \( L \) is called disjunctive if for there is an element \( x \in L \) such that \( a \land x = 0 \) where \( 0 \leq a < b \). It is easy to check that is \( L \) is disjunctive if and only \( (a) = (b) \) implies \( a = b \) for any \( a, b \in L \).

**Proposition 1.12.** In a 0-distributive lattice \( L \) the following conditions are equivalent:

(i) each ideal is an \( \alpha \)-ideal.

(ii) each prime ideal is an \( \alpha \)-ideal.

(iii) \( L \) is disjunctive.

**Proof.** (i) \( \Rightarrow \) (ii); Suppose \( P \) is any prime ideal of \( L \) then by (i) \( P \) is an \( \alpha \)-ideal, that is \( \alpha \alpha(P) = P \). Let \( I \) be any ideal of \( L \) then we have \( I = \cap(P : P \supseteq I) \) implies \( \alpha \alpha(I) = \alpha \alpha(\cap(P : P \supseteq I)) = \cap(\alpha \alpha(P) : P \supseteq I) = \alpha \alpha(\cap(P : P \supseteq I)) \) implies that \( \alpha \alpha(I) = I \). So \( I \) is an \( \alpha \)-ideal.

(ii) \( \Rightarrow \) (i) is trivial.

(i) \( \Rightarrow \) (iii); For any \( x, y \in L \), \( (x) = (y) \). Since \( (x) \) is an \( \alpha \)-ideal, so by definition of \( \alpha \)-ideal, \( y \in (x) \). Therefore, \( y \leq x \). Similarly, \( x \leq y \). Hence \( L \) is disjunctive.
(iii) \(\Rightarrow\) (i); Suppose \(I\) is any ideal of \(L\). By proposition 1.1, \((x)^* \subseteq \alpha \alpha(I)\).

For the reverse inclusion, let \(x \in \alpha \alpha(I)\). Then by definition \((x)^* \in \alpha(I)\), and so \((x)^* = (y)^*\) for some \(y \in (x)^*\). This implies \(x = y\), as \(L\) is disjunctive. So \(x \in L\) and hence \(\alpha \alpha(I) = I\). Therefore \(I\) is an \(\alpha\)-ideal of \(L\).

**Lemma 1.13.** A 0-distributive lattice \(L\) is relatively complemented if and only every prime filter is an ultra filter (Proper and maximal).

**Proof.** By Theorem 2.11 in [XII] we have \(L\) is relatively complemented if and only if its prime ideals are unordered. Thus the result follows.

A lattice \(L\) with 0 is called a quasi-complemented lattice if for each \(x \in L\), there exists \(y \in L\) such that \(x \land y = 0\) and \(((x) \lor (y))^* = (x)^* \land (y)^* = (0)\).

A 0-distributive lattice \(L\) is called quasi-complemented if for each \(x \in L\), there exists \(x' \in L\) such that \(x \land x' = 0\) and \(((x) \lor (x'))^* = (0)\).

A lattice \(L\) with 0 is called sectionally quasi-complemented if each interval \([0, x], x \in L\) is quasi-complemented.

We conclude the paper with the following result.

**Theorem 1.14.** Let \(L\) be a 0-distributive lattice. Then the following conditions are equivalent:

(i) \(L\) is sectionally quasi-complemented.

(ii) each prime \(\alpha\)-ideal is a maximal prime ideal.

(iii) each \(\alpha\)-ideal is an intersection of minimal prime ideals.

Moreover, the above conditions are equivalent to \(L\) being quasi-complemented if and only if there is an element \(d \in L\) such that \((d)^* = (0)\).

**Proof.** (i) \(\Rightarrow\) (ii); Suppose \(L\) is a sectionally quasi-complemented. Then by [6; Theorem 4.3.7], \(A_q(L)\) is relatively complemented. Hence its every prime filter is an ultra filter. Then by Corollary 1.11, each prime \(\alpha\)-ideal is a minimal prime ideal.

(ii) \(\Rightarrow\) (iii); It is not hard to show that each ideal of \(L\) is an intersection of prime \(\alpha\)-ideals. This shows (ii) \(\Rightarrow\) (iii).

(iii) \(\Rightarrow\) (ii); This is obvious by the minimality property of prime \(\alpha\)-ideals.
(ii) $\Rightarrow$ (i); Suppose (ii) holds. Then by Corollary 1.11, each prime filter of $A_0(L)$ is maximal. Then by Lemma 1.13, $A_0(L)$ is relatively complemented and so by Proposition 2.7 in [5] $L$ is sectionally quasi-complemented.

**Conclusion.** This paper shows that $\alpha$-ideals can be studied in non-distributive lattices by the 0-distributive property of a lattice. Following the technique of this paper, one can generalize those results for a 0-distributive near lattice.

**References**

I. Ayub Ali, Noor, A. S. A. and Islam, A. K. M. S. *Annulets in a Distributive Nearlattice*; Annals of Pure and Applied Mathematics, Vol. 3, No. 1, (2012), 91-96.

II. Ayub Ali, R. M. Hafizur Rahman and A. S. A. Noor; *Prime Separation Theorem for $\alpha$-ideals in a 0-distributive Lattice*; Journal of Pure and Applied Science, Assam, India. 12(1) (2012), pp. 16-20.

III. Ayub Ali, R. M. Hafizur Rahman & A. S. A. Noor; *On Semi prime n-ideals in Lattices*; Annals of Pure and Applied Mathematics. Vol. 2, No.-1, Page: 10-17 (2012).

IV. Md. Ayub Ali, R. M. Hafizur Rahman, A. S. A. Noor & Jahanara Begum; *Some characterization of n-distributive lattices*; Institute of Mechanics of Continua and Mathematical Sciences, Township, Madhyamgram, Kolkata-700129, Volume-7, Number-2, Page: 1045-1055 (2013).

V. Cornish, W. H., *Annulets and $\alpha$-ideals in a distributive lattice*; J. Aust. Math. Soc. 15(1) (1975), 70-77.

VI. Jaidur Rahman, A study on 0-distributive near lattice; Ph. D Thesis, Khulna university of Engineering and Technology.

VII. Jayaram, C., *Prime $\alpha$ – ideals in a 0-distributive lattice*; Indian J. Pure Appl. Math. 173 (1986), 331-337.

VIII. Pawar, Y. S and Thakare, N. K., *0-Distributive semilattice*; Canad. Math. Bull. Vol. 21(4) (1978), 469-475.

IX. Pawar, Y. S and Thakare, N. K., *0-Distributive semilattices*; Canad. Math. Bull. Vol. 21(4) (1978), 469-475.
X. R. M. Hafizur Rahman; Annulates in a 0-distributive lattice, Annals of Pure and Applied Mathematics, Vol. 3, No. 1, (2012), 91-96.

XI. R. M. Hafizur Rahman, Md. Ayub Ali & A. S. A. Noor; On Semi prime Ideals of a Lattice; Journal Mechanics of Continua and Mathematical Sciences, Township, Madhyamgram, Kolkata-700129. Volume-7, Number-2, Page: 1094-1102 (2013).

XII. Varlet, J. C., A generalization of the notion of pseudo-complementedness; Bull. Soc. Sci. Liege, 37 (1968), 149-158.