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Lyapunov spectrum of the separated flow around the NACA 0012 airfoil and its dependence on numerical discretization

P. Fernandez¹,²,*, Q. Wang¹,²,*

Abstract
We investigate the impact of numerical discretization on the Lyapunov spectrum of separated flow simulations. The two-dimensional chaotic flow around the NACA 0012 airfoil at a low Reynolds number and large angle of attack is considered to that end. Time, space and accuracy order refinement studies are performed to examine each of these effects separately. Numerical results show that the time discretization has a small impact on the dynamics of the system, whereas the spatial discretization can dramatically change them. Also, the finite-time Lyapunov exponents associated to unstable modes are shown to be positively skewed, and quasi-homoclinic tangencies are observed in the attractor of the system. The implications of these results on flow physics and sensitivity analysis of chaotic flows are discussed.

Keywords: Computational fluid dynamics, Flow instability, Lyapunov stability analysis, Nonlinear dynamics and chaos, Sensitivity analysis, Separated flows, Turbulence

1. Introduction
Lyapunov analysis is a mathematical tool to characterize the stability of dynamical systems. With the first attempts to apply these techniques to chaotic fluid flows dating back from the ‘90s [23, 37, 41, 44, 45], Lyapunov analysis currently gains attention in the flow physics community as a promising approach for flow instability, vortex dynamics and turbulence research [3, 8, 29, 30, 47, 48]. Lyapunov stability analysis, in its finite-time version, has also been applied in the field of Lagrangian coherent structures [22, 39, 40]. While the interest in flow physics lies in the Lyapunov exponents of the actual flow, numerical algorithms compute the Lyapunov spectrum of the finite-dimensional representation obtained after numerical discretization. Before these techniques become routine in the community and more publications appear, it is

*Corresponding author
Email addresses: pablof@mit.edu (P. Fernandez), qiqi@mit.edu (Q. Wang)
¹Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, USA
²Center for Computational Engineering, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, USA
critical to understand the role of the spatial and temporal discretization on the resulting dynamics: Is the spectrum of the discrete system that of the actual flow? If so, how much resolution is required to reproduce the true dynamics? That is, how much resolution is required for these studies to be reliable and trustworthy?

The Lyapunov spectrum of chaotic flow simulations also plays an important role in engineering. In particular, conventional sensitivity analysis methods break down for chaotic systems [24], and this compromises tasks such as flow control, design optimization, error estimation, data assimilation, and uncertainty quantification. While a number of sensitivity analysis methods have been proposed for chaotic systems, including Fokker-Planck methods [43], Fluctuation-Dissipation methods [25], the Ensemble Adjoint (EA) method [24], the Least Squares Shadowing (LSS) method [46] and the Non-Intrusive Least Squares Shadowing (NILSS) method [7, 33, 34], they all come at a high computational cost. This is ultimately related to the positive portion of the Lyapunov spectrum of the system, and the cost of each method is sensitive to different aspects of it. For example, the cost of NILSS is, by construction, proportional to the number of positive exponents, whereas the cost of EA has been shown to depend on the magnitude of the largest Lyapunov exponent and the rate of decay of correlations of the system [9, 10]. Understanding the Lyapunov spectrum of chaotic flow simulations, and its dependence on numerical discretization, is therefore necessary to estimate the cost and feasibility of sensitivity analysis for chaotic flows.

In this paper, we investigate the impact of numerical discretization on the Lyapunov spectrum of the two-dimensional, separated flow around the NACA 0012 airfoil at Reynolds number $Re_\infty = 2,400$, Mach number $M_\infty = 0.2$, and angle of attack $\alpha = 20$ deg. Since the simulation is two-dimensional, the flow physics are different to those of three-dimensional flows. However, the moderate computational cost of this problem allows us to perform a more comprehensive study than otherwise possible. In particular, the impact of the temporal resolution ($t$-refinement), the spatial resolution ($h$-refinement) and the accuracy order ($p$-refinement) are investigated. Also, we analyze three different notions of convergence of Lyapunov exponents. Finally, we examine other aspects of the nonlinear dynamics of this flow in order to gain insight into the mechanisms that make separated flows chaotic.

The remainder of the paper is structured as follows. In Section 2, we present an overview of Lyapunov stability analysis. Section 3 describes the methodology to discretize the Navier-Stokes equations and compute Lyapunov exponents and covariant vectors. The details of the flow around the NACA 0012 airfoil considered in this paper are discussed in Section 4. The Lyapunov spectrum, and its dependence on numerical discretization, are investigated in Section 5. In Section 6, we analyze other aspects of the nonlinear dynamics of this flow, including finite-time Lyapunov exponents, covariant Lyapunov vectors and the presence of homoclinic tangencies in the attractor. Finally, some concluding remarks and a discussion on the implications of these results on flow physics and sensitivity analysis are presented in Section 7.

2. Lyapunov stability analysis

The spatial discretization of the compressible Navier-Stokes equations for a Newtonian fluid yields an autonomous, continuous-time, first-order dynamical system of
where \( u_h = u_h(t) \) is an \( n \)-dimensional vector of state variables, such as the density, momentum and total specific energy at every grid point. We note that, for a given flow problem, different meshes and numerical schemes lead to different dimensions \( n \) and different dynamics \( f_h \).

While it has only been proved for some two-dimensional, incompressible Navier-Stokes systems [16, 17, 19, 28], physical evidence suggests that most chaotic flows are ergodic, that is, \( f_h \) has an ergodic invariant probability measure that we shall denote by \( \mu \). Roughly speaking, ergodicity means that the time-average behavior of the system is independent of the initial condition. We shall assume ergodicity hereinafter.

Under fairly minor additional assumptions, for an ergodic system of the form (1) and for \( \mu \)-almost every initial condition \( u_h,0 \), there exist scalars \( \Lambda_1^h \geq \Lambda_2^h \geq \cdots \geq \Lambda_n^h \in \mathbb{R} \) and linearly independent vectors \( \psi_1^h(u_h), \psi_2^h(u_h), \ldots, \psi_n^h(u_h) \in \mathbb{R}^n \) such that

\[
\frac{d}{dt}\left(\psi_j^h(u_h(t))\right) = \frac{\partial f_h}{\partial u_h} \bigg|_{u_h(t)} \psi_j^h(u_h(t)) - \Lambda_j^h \psi_j^h(u_h(t)), \quad u_h(t_0) = u_{h,0} \tag{2}
\]

for \( t \geq t_0 \) and \( j = 1, \ldots, n \). \( \psi_j^h(u_h) \) and \( \Lambda_j^h \) are the so-called covariant Lyapunov vectors (CLVs) and Lyapunov exponents (LEs), respectively. We note that the CLVs depend on the state \( u_h \), whereas the LEs are a property of the system independent of \( u_h \). This result is part of the Oseledecs’ Multiplicative Ergodic Theorem [36] and has been shown to hold also in several infinite-dimensional settings [2], including Hilbert spaces [38] and, more generally, Banach spaces [27]. That is, LEs and CLVs exist for the actual flow as well.

The intuitive interpretation of Lyapunov vectors and exponents is as follows: “Any infinitesimal perturbation \( \delta u_{h,0} \) in the direction \( \psi_j^h(u_h(t_0)) \) at \( t = t_0 \) will remain along \( \psi_j^h(u_h(t_0)) \) at all times \( t \geq t_0 \). Also, the magnitude of the perturbation increases or decreases at an asymptotic rate \( ||\delta u_h(t)|| \sim \exp(\Lambda_j^h(t-t_0)) \) as \( t \to \infty \).” Hence, the magnitude and sign of the Lyapunov exponents characterize how infinitesimal perturbations to the system evolve over time. Depending on the sign of the Lyapunov exponents, the dynamical system (1) can be classified into four types:

- **Stationary (or steady-state) system**: All the exponents are negative and the system converges to a stationary point (also known as fixed point) as \( t \to \infty \). This is the case for steady laminar flows.

- **Periodic system**: One LE is equal to zero and all the other exponents are negative. A periodic system converges to a one-dimensional periodic orbit (also known as limit cycle) as \( t \to \infty \). Also, \( \psi_1^h(u_h) = f_h(u_h) \) is the CLV corresponding to \( \Lambda_1^h = 0 \) [20]. This is the case for periodic vortex shedding.

- **Aperiodic, non-chaotic system**: \( n_0 \geq 2 \) Lyapunov exponents vanish and all other exponents are negative. These systems converge to an \( n_0 \)-dimensional toroidal attractor as \( t \to \infty \). This is the case for non-chaotic, aperiodic vortex shedding.
• Chaotic system: \( n_+ \geq 1 \) exponents are positive, one exponent vanishes and all the other exponents are negative. The positive exponent(s) \( \Lambda^+_h = \{\Lambda^+_1, ..., \Lambda^+_n\} \) are responsible for the “butterfly effect”, a colloquial term to refer to the large sensitivity of chaotic systems to initial conditions. A chaotic system converges to a strange attractor as \( t \to \infty \). Examples of chaotic systems include turbulent flows and many separated flows.

Strictly speaking, the existence of positive exponents is only a necessary condition for chaos. Although there is no universally accepted definition of chaos, most definitions additionally require topological mixing and the existence of dense periodic orbits [26]. While non-chaotic systems with positive exponents are therefore possible, they are pathological in the case of fluid flows. For this reason, we shall adopt the existence of positive exponents as indicator of chaotic dynamics in the remainder of the paper.

The numerical simulation of unsteady flows requires further discretizing Eq. (1) in time. This yields an autonomous, discrete-time, first-order map

\[
u^{(i+1)} = F_{h,\Delta t}(u^{(i)}),
\]

where \( u^{(i)} \) denotes the solution at the end of the time step \( i \). The particular form of \( F_{h,\Delta t} \) depends on \( f_h \) –that is, on the spatial discretization–, as well as on the time-integration scheme and the time-step size \( \Delta t \). The discrete-time Lyapunov vectors \( \psi^i_{h,\Delta t} \) and exponents \( \Lambda^i_{h,\Delta t} \) of \( F_{h,\Delta t} \) are defined in an analogous way to their continuous counterparts.

A more formal discussion on ergodic theory, Lyapunov exponents and covariant Lyapunov vectors is presented in Appendix 7.3.

3. Methodology

3.1. Numerical discretization

High-order Hybridizable Discontinuous Galerkin (HDG) methods [15, 31] and diagonally implicit Runge-Kutta (DIRK) methods [1] are used for the spatial and temporal discretization of the compressible Navier-Stokes equations, respectively. The HDG method, as a discontinuous Galerkin method, allows for a systematic study of the effect of the accuracy order on the Lyapunov spectrum via \( p \)-refinement, where \( p \) denotes the polynomial order of the numerical approximation. The DIRK scheme, as an \( L \)-stable time-integration scheme, allows for a systematic study of the effect of the time-step size without running into numerical stability issues. A detailed description of the numerical discretization and the parallel iterative solver is presented in [13, 14, 15].

3.2. Lyapunov exponent algorithm

A non-intrusive version of the algorithm by Benettin et al. [4] is used to compute the \( l (\leq n) \) leading Lyapunov exponents. The original method is summarized in Algorithm 1. If the time integrals in Steps No. 5 and 6 of the algorithm are computed exactly, an estimator of the \( l \) leading continuous-time LEs \( \hat{\Lambda}^i_{h} \) of \( f_h \) are obtained. If the
time integrals are approximated using a numerical method, as it is always the case in practice, the algorithm computes an estimator of the $l$ leading discrete-time Lyapunov exponents $\hat{\Lambda}_h^{\text{disc}, \Delta t}$ of $F_h, \Delta t$.

The reader that is familiar with the QR algorithm for eigenvalue computation may find some similarities between both iterations. In particular, the periodic orthonormalization in Step 7 plays a similar role to that in the QR algorithm: Controlling the growth of round-off errors and thus making the iteration numerically stable. For this reason, the time segment $t_s$ needs to be such that $t_s \hat{\Lambda}_h^1 \ll \epsilon_{\text{tol}}^{-1}$, where $\epsilon_{\text{tol}}$ denotes machine epsilon.

Since the original algorithm requires the integration of the homogeneous tangent equation (4) in Step No. 6, it cannot be used with most existing computational fluid dynamics (CFD) solvers without modification of the source code. In the spirit of making the algorithm non-intrusive, we approximate the tangent map (4) by finite differences. In particular, let $u_h^{(i)} = \mathcal{F}_h, \Delta t(u_h^{(i-1)})$ denote the evolution operator of the Navier-Stokes system over a time segment of length $t_s$ computed by a CFD code starting from the initial condition $u_h^{(i-1)}$. We then replace Step No. 6 in Algorithm 1 by

$$v_j^{(i)} \approx \frac{1}{\epsilon} \left[ \mathcal{F}_h, \Delta t(u_h(t_{i-1}) + \epsilon q_j^{(i-1)}) - \mathcal{F}_h, \Delta t(u_h(t_{i-1})) \right],$$  \hspace{1cm} (6)

for $j = 1, \ldots, l$. The use of a forward finite difference approximation (FFDA) for the tangent field introduces some restrictions on $t_s$ and $\epsilon$. First, while the tangent field diverges exponentially and is unbounded for chaotic flows, $||\mathcal{F}_h, \Delta t||$ is bounded above for all $t_s$ since the attractor of the system itself is bounded. Hence, the length of the time segments must be small enough so that the perturbation lies within the linear region, that is,

$$||\mathcal{F}_h, \Delta t(u_h(t_{i-1}) + \epsilon q_j^{(i-1)}) - \mathcal{F}_h, \Delta t(u_h(t_{i-1}))|| \ll ||\mathcal{F}_h, \Delta t(u_h(t_{i-1}))||,$$  \hspace{1cm} (7)

and the FFDA provides an accurate representation of the tangent field (see footnote 3 for a clarification on this regard). Second, $\epsilon$ must satisfy $\epsilon_{\text{m}} \ll \epsilon \ll ||u_h||$. Third, the use of an implicit time integration scheme requires $\epsilon_{\text{tol}} \ll ||\partial F_h, \Delta t/\partial u_h|| \epsilon$, where $\epsilon_{\text{tol}}$ is the tolerance the nonlinear system arising from the time discretization is solved to.

### 3.3. Covariant Lyapunov vector algorithm

An analogous non-intrusive modification to the algorithm by Ginelli et al. [18] is used to compute the covariant Lyapunov vectors associated to the $l$ leading LEs. The online repository for the non-intrusive LE and CLV algorithms is freely available at https://github.com/qiqi/fds/

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Footnote 3: Equation (7) is only an approximate condition for an accurate FFDA. The actual conditions are given by the left-hand side in Eq. (7) being much larger than $\epsilon^2 ||\nabla^2 \mathcal{F}_h, \Delta t||$; which is certainly more difficult to check.
**Input:** Initial condition $u_{h,0}$, number of exponents to compute $l$, length of each time segment $t_s$, and number of time segments $K$.

**Output:** Estimators of the $l$ largest LEs $\hat{\Lambda}_h^j$, $j = 1, \ldots, l$.

1. Set $t_0 = 0$ and $u_h(t_0) = u_{h,0}$.
2. Compute an $n \times l$ random matrix $V^{(0)} \sim \mathcal{U}(0, 1)^{n \times l}$, where $\mathcal{U}(0, 1)$ denotes the continuous uniform distribution on the interval $(0, 1)$.
3. Compute the reduced QR decomposition $Q^{(0)} R^{(0)} = V^{(0)}$.

**Algorithm 1:** Original algorithm by Benettin et al. [4] to compute LEs. The superscript $(i)$ denotes the solution at the end of the time segment $i$.

4. Set $t_i = t_{i-1} + t_s$.
5. Time integrate the dynamical system (1) from $t_{i-1}$ to $t_i$ using the initial condition $u_{h}^{(i-1)} = u_h(t_{i-1})$.
6. Time integrate the homogeneous tangent equation (4) from $t_{i-1}$ to $t_i$ for each of the $l$ initial conditions given by the columns of $Q^{(i-1)}$ using the reference trajectory $u_h$, computed in Step No. 5,

\[
\frac{dv_j}{dt} = \frac{\partial f_h}{\partial u_h} |_{u_h(t)} v_j, \quad v_j(t_{i-1}) = q_j^{(i-1)},
\]

for $j = 1, \ldots, l$, and set $v_j^{(i)} = v_j(t_i)$. Here, $q_j^{(i)}$ and $v_j^{(i)}$ denote the $j$-th column of $Q^{(i)}$ and $V^{(i)}$, respectively.

7. Compute the reduced QR decomposition $Q^{(i)} R^{(i)} = V^{(i)}$.

8. Compute

\[
\hat{\Lambda}_h^i = \frac{1}{t_K - t_0} \sum_{i=1}^{K} \log |R_{jj}^{(i)}|. \quad (5)
\]
4. Case description and details of the numerical setup

We consider the two-dimensional, separated flow around the NACA 0012 airfoil at Reynolds number $\text{Re}_\infty = u_\infty c/\nu = 2400$, Mach number $M_\infty = u_\infty/a_\infty = 0.2$, and angle of attack $\alpha = 20$ deg. Here, $u_\infty$, $a_\infty$, $\nu$ and $c$ denote the freestream velocity, freestream speed of sound, kinematic viscosity and airfoil chord, respectively. The computational domain spans 10 chord lengths away from the airfoil and is partitioned using isoparametric triangular $c$-meshes. A non-slip, adiabatic wall boundary condition is imposed on the airfoil surface, and a characteristics-based, non-reflecting boundary condition is used on the outer boundary. A snapshot of the Mach number field for this flow is shown in Fig. 1.

A run-up time $t_0 = 2,000 c/a_\infty$ is used to ensure the system achieves its steady-state distribution on the attractor, and the LE and CLV algorithms are then applied for $K = 8,000$ time segments, each of them of length $t_s = c/a_\infty$. Also, the finite difference perturbation and the relative tolerance of the nonlinear solver are set to $\epsilon = 10^{-4}$ and $\epsilon_{\text{tol}} = 10^{-8}$, respectively. A parametric study with respect to $t_s$, $\epsilon$ and $\epsilon_{\text{tol}}$ showed that these values suffice for the numerical results not to be affected by finite precision arithmetic errors.

5. Lyapunov exponents and their dependence on numerical discretization

5.1. Effect of the temporal resolution: $t$-refinement study

First, we analyze the effect of the time-step size on the Lyapunov spectrum of the time-discrete system $F_{h,\Delta t}$. In particular, the continuous-time system $f_h$ associated to a fourth-order HDG discretization with $n = 115,200$ degrees of freedom (DOFs) is time-integrated using a third-order DIRK method with the following time-step sizes $\Delta t = (0.20, 0.10, 0.05, 0.025, 0.0125) c/a_\infty$. These correspond to global CFL numbers $\Delta t u_\infty/h_{\text{min}}$ of 59.94, 29.97, 14.98, 7.49 and 3.75, respectively, where $h_{\text{min}}$
denotes the smallest element size in the mesh. We emphasize that the time-step size affects the discrete-time map $F_{h,\Delta t}$ but it does not change $f_h$.

Figure 2 shows the six leading Lyapunov exponents $\Lambda_{j,\Delta t}^h$ for the time-step sizes considered, together with 90% confidence intervals (CIs). The confidence intervals are computed from the sample statistics of $\log |R_{ji}^{(i)}|$, $i = 1, \ldots, K$ in Eq. (5), as customary in statistical inference. While the samples are not independent here, the weak dependence version of the central limit theorem (CLT) and thus the usual procedure to compute CIs apply under the assumptions that the sequence \{log $|R_{ji}^{(i)}|\}_i$ is stationary, uniformly bounded, and $\alpha$-mixing with $\alpha_i = \mathcal{O}(i^{-5})$. For additional details on this generalization of the standard CLT, the interested reader is referred to Chapter 27 in [5].

From Figure 2, the time-step size in the range considered does not have a significant impact on the leading exponents of $F_{h,\Delta t}$. First, this gives us confidence that the discrete-time Lyapunov exponents approximate those of the continuous-time system, i.e. $\Lambda_{j,\Delta t}^h \approx \Lambda_j^h$. For this reason, we shall refer to $f_h$ and $\Lambda_j^h$ instead of $F_{h,\Delta t}$ and $\Lambda_{j,\Delta t}^h$ in the remainder of the paper. Second, the asymptotic spectrum of the discrete-time map as $\Delta t \downarrow 0$ is achieved with global CFL numbers $10^{-5}$ that are much larger than those used in engineering practice. The accurate prediction of the Lyapunov spectrum with such large CFL numbers is attributed to (1) the smallest element size (near the leading edge, where the flow is laminar) being much smaller than the element sizes in the separated region of the flow, and (2) these time steps sufficing to resolve all the vortical structures that are responsible for the chaotic dynamics of $f_h$. However, if the time step was much larger than $h_c/u_\infty$, where $h_c$ denotes the smallest length scale responsible for chaos in the continuous-time system, the discrete-time map might not accurately reproduce the continuous-time dynamics and thus $\Lambda_{j,\Delta t}^h \neq \Lambda_j^h$. A similar observation has been reported in [35] for the numerical integration of stiff ODEs with inadequate time steps.

![Figure 2: 90% confidence intervals of the six leading Lyapunov exponents for the discretizations in the $t$-refinement study.](image)

5.2. Effect of the spatial resolution: $h$-refinement study

We examine the effect of the spatial resolution on the number and magnitude of positive exponents. To this end, the Lyapunov spectrum of the flow is computed with
eleven \(c\)-meshes, each of them \(2^{1/3}\) times finer in each parametric coordinate direction than the previous one. The number of DOFs uniformly increases in logarithmic scale from \(n = 7,200\) (mesh No. 1) to \(726,240\) (mesh No. 11). Meshes No. 1 and 11 are shown in Fig. 3. We note that mesh No. 1 is intended to be pathologically coarse to analyze how very under-resolved systems behave.

The discretization scheme and time-step size are kept constant to analyze the effect of spatial resolution only. Again, fourth-order HDG and third-order DIRK methods are used for the spatial and temporal discretization, respectively. The time-step size is set to \(\Delta t = 0.05\, c/\alpha_{\infty}\). Figure 4 shows the fourteen leading LEs for the discretizations considered, whereas Table 1 collects the Strouhal number \(St\), lift coefficient \(c_l\), drag coefficient \(c_d\) and Kaplan-Yorke dimension \(D_{KY}\) for each discretization. The Strouhal number is defined as

\[
St = \frac{f v}{c/\alpha_{\infty}},
\]

where \(f\) denotes the frequency of the dominant vortex shedding, whereas the Kaplan-Yorke dimension [11] is an estimator of the strange attractor’s dimension and is defined as

\[
D_{KY} := j + \frac{\Lambda^1_h + \ldots + \Lambda^j_h}{|\Lambda^j_{h+1}|},
\]

where \(j\) is the largest integer such that \(\Lambda^1_h + \ldots + \Lambda^j_h \geq 0\). Due to chaotic dynamics, the Strouhal number can only be accurately computed to two significant digits in the chaotic discretizations. Also, we note that the missing vortex upwash due to the finite extent of the computational domain leads to a change in the effective angle of attack. This affects the pressure field around the airfoil and may have a small impact on the accuracy of the reported values for the lift coefficient. From these figures and tables, several remarks follow:

- The magnitude of the leading LE and the number of positive exponents increase above some spatial resolution threshold \(h^*\), corresponding to mesh No. 5. That is, the discrete system becomes more chaotic above this resolution as the mesh is refined. This is attributed to the fact that more vortical structures, which are responsible for the chaotic dynamics of the flow, are resolved as the numerical resolution increases.
Table 1: Strouhal number $St$, lift coefficient $c_l$, drag coefficient $c_d$ and Kaplan-Yorke dimension of the attractor $D_{K\&Y}$ for the discretizations in the $h$-refinement study

| Discretization No. | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  |
|--------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $St$               | 0.24| 0.23| 0.26| 0.24| 0.2642| 0.26| 0.26| 0.26| 0.26| 0.26| 0.26|
| $c_l$              | 1.0380| 1.0442| 1.1369| 1.1049| 1.1432| 1.1468| 1.1554| 1.1433| 1.1492| 1.1457| 1.1463|
| $c_d$              | 0.0934| 0.0990| 0.1034| 0.1029| 0.1000| 0.1031| 0.1023| 0.1043| 0.1038| 0.1031| 0.1027|
| $D_{K\&Y}$        | 4.73| 2.14| 2.68| 2.01| 1.00| 3.29| 5.37| 7.72| 8.27| 9.19| 10.90|

Figure 4: Fourteen leading Lyapunov exponents for the discretizations in the $h$-refinement study.

- Below the resolution threshold $h^*$, the discrete system poorly reproduces the dynamics of the continuous system, and this results in spurious dynamics. Here, spurious periodicity and chaoticity are observed. (No discretization results in stable dynamics.) Discretization No. 5, for example, has no positive LEs and is periodic. A time refinement study similar to that in Section 5.1 confirmed that the continuous-time system $f_{h_5}$ associated to this discretization—and not only the discrete-time map $f_{h_5, \Delta t}$—is indeed periodic. Hence, for the $h$-family of discrete dynamical systems considered here, a periodic orbit bifurcates into strange attractors above and below $h_5$. We hypothesize this is a numerical artifact and therefore discretization dependent.

- An approximately zero exponent $\hat{\lambda}_h \approx 0$ is present in all discretizations. Theoretical results show that at least one zero Lyapunov exponent $\Lambda_h = 0$ exists for non-stationary systems [20] and that $\psi_h = f_h(u_h)$ is a CLV for this exponent, that is, perturbations along the trajectory grow or decay at most sub-exponentially. This is expected to be such an LE.

- The positive Lyapunov exponents are created from bifurcations of the $\Lambda_h = 0$ exponent at discrete mesh resolutions. This explains the mechanism responsible for the increase of positive exponents above and below $h^*$.

The discrete Fourier transform (DFT) of the lift coefficient, drag coefficient and static pressure at the trailing edge for the discretizations No. 5 and 7 are displayed in Fig. 5. The trace of drag vs. lift coefficients over a time interval of length 20,000 $c/a_{\infty}$ is shown in Fig. 6, where the dots are colored by probability density function (PDF) in
$(c_d, c_l)$ space. Despite the chaotic dynamics of discretization No. 7, its PDF resembles the periodic trace of mesh No. 5. This indicates that, in some sense, the chaotic dynamics are similar to those of periodic vortex shedding and exemplifies the low dimensional nature of the chaotic attractor.

5.3. **Effect of the spatial accuracy order: $p$-refinement study**

Next, we investigate the effect of the accuracy order of the spatial discretization on the dynamics of $f_h$. To this end, third-, fourth-, and fifth-order (i.e. $p = \{2, 3, 4\}$)
HDG schemes are considered. The DIRK(3,3) method with $\Delta t = 0.05 \ c/a_\infty$ is used for the time integration, and the number of degrees of freedom is $n = 115,200$ in all cases. This corresponds to resolution No. 7 in the $h$-refinement study.

Figure 7 shows 90% confidence intervals of the six leading Lyapunov exponents for the accuracy orders considered. From this figure, the negative exponents get closer to zero, that is, perturbations along stable directions decay more slowly, as the accuracy order increases. Also, the fourth- and fifth-order methods lead to larger positive exponents than the third-order scheme, i.e. perturbations along unstable directions get more rapidly amplified. While the 90% confidence intervals of the two positive exponents in the fourth- and fifth-order discretizations overlap, increasing the accuracy order seems to lead to more chaotic dynamics. This is attributed to the lower numerical dissipation of high-order methods. For a discussion on the convergence rate and numerical dissipation of the HDG method for various polynomial orders, the interested reader is referred to [31, 32].

5.4. On the convergence of the Lyapunov spectrum

We consider three different notions of convergence of Lyapunov exponents:

1. Convergence of the exponents of the discrete-time map $F_{h,\Delta t}$ to the exponents of the continuous-time system $f_h$, i.e. $\Lambda^j_{h,\Delta t} \to \Lambda^j_h$, as the time-step size is refined $\Delta t \downarrow 0$.

2. Convergence of the exponents of the continuous-time system $f_h$ to the exponents of the actual flow $f$, i.e. $\Lambda^j_h \to \Lambda^j$, as the mesh is refined $h \downarrow 0$.

3. For a given discretization $F_{h,\Delta t}$, statistical convergence of the Lyapunov exponent estimators, i.e. $\hat{\Lambda}^j_{h,\Delta t} \to \Lambda^j_{h,\Delta t}$, as the sample size increases $t \to \infty$.

The convergence in the first sense was discussed and characterized in Section 5.1. Here, we investigate the second and third types of convergence, and compare them to that of other quantities of interest such as lift and drag.
Table 2: Relative error in the leading Lyapunov exponent $\Lambda^1 c/\infty$, Strouhal number $St$, lift coefficient $c_l$ and drag coefficient $c_d$ for the discretizations in the $h$-refinement study. The results from the fourth-order discretization with 2,880,000 DOFs are used as reference values.

| Discretization No. | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  |
|--------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\Lambda^1 c/\infty$ | 16.1| 40.8| 59.6| 66.7| 100.9| 49.9| 44.0| 38.8| 27.2| 16.7| 16.5|
| $St$               | $\sim 7\%$ | $\sim 12\%$ | $\sim 7\%$ | $\sim 5\%$ | $\sim 5\%$ | $\sim 5\%$ | $\sim 5\%$ | $\sim 5\%$ | $\sim 5\%$ | $\sim 5\%$ | $\sim 5\%$ |
| $c_l$              | 9.38% | 8.84% | 0.75% | 3.54% | 0.20% | 0.11% | 0.86% | 0.19% | 0.32% | 0.02% | 0.07% |
| $c_d$              | 9.35% | 3.92% | 0.39% | 0.10% | 2.91% | 0.10% | 0.68% | 1.26% | 0.78% | 0.10% | 0.29% |

5.4.1. Convergence $\Lambda^1_h \to \Lambda^1$ as $h \downarrow 0$

We investigate if an asymptotic spectrum is achieved with the numerical resolutions that can be afforded in engineering practice. To this end, we consider a fourth-order discretization with $n = 2,880,000$ degrees of freedom; which is vastly more than the best-practice meshes for this type of flows. The 90% confidence interval of the leading Lyapunov exponent for this discretization is $\Lambda^1_h c/\infty = 0.0473 \pm 0.0053$. Hence, the system seems to become more chaotic even for this discretization. While an asymptotic Lyapunov spectrum as $h \downarrow 0$ was obtained for simpler partial differential equations in other studies [42], this shows that such an asymptotic spectrum (if exists) is difficult to achieve in practice even for simple flows.

Table 2 shows the relative error in the leading exponent, Strouhal number, lift coefficient and drag coefficient for the discretizations in the $h$-refinement study. The results from the fourth-order discretization with 2,880,000 DOFs are used as reference values. As expected from the previous discussion, the Lyapunov exponents converge much slower as the mesh is refined than common quantities of interest in engineering applications such as the lift and drag coefficients.

Despite the previous results, and based on physical principles, we conjecture that an asymptotic Lyapunov spectrum does exist and that the leading exponent and covariant vector are associated to the smallest time and length scales responsible for chaos. This agrees with the leading Lyapunov exponent $\Lambda^1 \approx 0.01 - 0.05 c/\infty$ being of the same order as the frequency of the dominant vortex shedding $f_v \approx 0.05 c/\infty$. Hence, the (local) mesh size required to accurately compute the Lyapunov spectrum may need to be of the order of the smallest (local) chaotic length scale. In the case of turbulent flows, this corresponds to DNS resolutions and is consistent with the results in [30].

5.4.2. Convergence $\Lambda^1_{h,\Delta t} \to \Lambda^1_{h,\Delta t}$ as $t \to \infty$

Figure 8 shows the statistical convergence of the lift and drag coefficients (left) and the fourteen leading exponents (right) for discretization No. 7 in the $h$-refinement study. Unlike in the “$h \downarrow 0$” type of convergence, the Lyapunov exponents display similar statistical convergence as $t \to \infty$ as that of the lift and drag coefficients. Analogous results are observed for all the other discretizations considered in $t$-, $h$- and $p$-refinement studies.
6. Analysis of the nonlinear dynamics of the flow

6.1. Finite-time Lyapunov exponents

We define the $j$-th finite-time Lyapunov exponent (FTLE) of the system (1) over a time span of $0 < \tau < \infty$ as

$$\Lambda^j_{h,\tau}(u_h) := \frac{1}{\tau} \ln \left( \frac{||DF^\tau_h(u_h) \cdot \psi^j_h(u_h)||}{||\psi^j_h(u_h)||} \right),$$

where $DF^\tau_h(u_h)$ is the tangent propagator, that we shall approximate again by finite differences as described in Section 3.3. The finite-time Lyapunov exponents therefore describe the growth rate of the corresponding CLVs during a finite time span. Unlike the LEs, the FTLEs depend on the state of the system $u_h$ and can be interpreted as random variables with mean equal to the actual Lyapunov exponent $E[\Lambda^j_{h,\tau}(u_h)] = \Lambda^j_{h,\tau}$. Figure 9 shows a scatter plot of the mean $E[\Lambda^j_{h,\tau}(u_h)]$ and the skewness $\gamma_1$ of the $\tau = c/a_\infty$ FTLEs corresponding to the LEs of the discretizations in the $h$-refinement study. This figure shows that there is a positive correlation between the magnitude of the Lyapunov exponents and the skewness of their FTLE probability distributions. In particular, this indicates that the CLVs associated to the positive exponents become very unstable at some specific times. This in turn implies that it is some particular regions of the attractor (i.e. flow configurations) that are responsible for the existence of positive exponents and the chaotic dynamics of the flow.

6.2. Covariant Lyapunov vectors

Figure 10 shows a snapshot of the density, momentum and total specific energy magnitude fields of the leading covariant Lyapunov vector for the discretization No. 6 in the $h$-refinement study. From this figure, the most unstable perturbations to the system correspond to perturbations in the wake region. This is attributed to the vortical structures, responsible for the chaotic dynamics of the flow, being located in this region. This figure also points to the lack of resolution in the far-field wake as the main reason...
for the slow convergence of the Lyapunov exponents $\Lambda_j^h \to \Lambda_j$ as the mesh is refined. Also, Figure 11 shows the magnitude of the $x$-momentum field of the three leading CLVs for discretization No. 6. While these CLVs correspond to unstable, neutrally stable and stable modes, respectively, no significant differences are observed between them.

6.3. Homoclinic tangencies

We investigate the existence of homoclinic tangencies in the flow, that is, points in phase space where two or more covariant Lyapunov vectors become parallel. Figure 12 shows the time history of the angle between the second and third leading CLVs for the system associated to discretization No. 10 in the $h$-refinement study (top), as well as the corresponding $\tau = 1500 c/a_\infty$ FTLEs (bottom). The FTLEs on the bottom are plotted centered with respect to the averaging window, that is, the $y$-axis corresponds to $\Lambda_j^{h,\tau}(t-\tau/2)$, in order to facilitate the comparison with the image on the top. From this figure, the trajectory may be close to a homoclinic tangency from $4,500$ to $6,000$. Also, while the actual Lyapunov exponents differ significantly, the FTLEs are close to each other in this region of phase space. Quasi-homoclinic tangencies similar to that in Fig. 12 are also present in the other discretizations considered. From a flow physics perspective, this indicates that even this simple separated flow may not be a uniformly hyperbolic dynamical system. From an engineering perspective, this may pose challenges for sensitivity analysis as many sensitivity analysis methods become poorly conditioned near homoclinic tangencies [12].

7. Discussion

We conclude with a summary of the main findings in the paper and a discussion on their implications on flow physics and sensitivity analysis of chaotic flows.
Figure 10: Magnitude of the density (top left), total specific energy (top right), \( x \)-momentum (bottom left) and \( y \)-momentum (bottom right) fields of the leading CLV for discretization No. 6 in the \( h \)-refinement study.
Figure 11: Magnitude of the $x$-momentum field of the three leading CLVs for discretization No. 6 in the $h$-refinement study.

Figure 12: Top: Time history of the angle between the second and third CLVs. Bottom: Time-history of the corresponding $\tau = 1500$ $c/\alpha_\infty$ FTLEs. Results for discretization No. 10 in the $h$-refinement study.
7.1. Summary

We investigated the impact of the numerical discretization on the Lyapunov spectrum of the chaotic, separated flow around the NACA 0012 airfoil at a low Reynolds number and large angle of attack. Time, space and accuracy order refinement studies were performed to examine each of these effects separately. Numerical results showed that the time discretization has a small effect on the Lyapunov spectrum for the time-step sizes typically used in CFD practice. In particular, the asymptotic spectrum as the time-step size is refined was achieved for this wall-bounded flow with global CFL numbers of order $10^{-100}$. The spatial discretization, however, was shown to dramatically change the dynamics of the system. First, the discretized system poorly reproduced the dynamics of the flow, and spurious dynamics were observed, below some spatial resolution threshold $h^*$. Second, above this resolution threshold, the discrete system continued to become more and more chaotic as the mesh was refined, and an asymptotic spectrum was not achieved even with finer meshes than the best practice for this type of flows. In short, the asymptotic Lyapunov spectrum as the mesh is refined is difficult to achieve in practice even for this simple flow and, in particular, it requires significantly more resolution than that to accurately compute most quantities of interest in engineering applications, such as the lift and drag coefficients. Also, the additional positive exponents above and below $h^*$ were created from bifurcations of the $\Lambda_h = 0$ exponent at discrete mesh resolutions.

Other aspects of the nonlinear dynamics of this flow were investigated. An analysis of the short-time Lyapunov exponents showed that the CLVs associated to the positive LEs become very unstable at some particular times, i.e. for some particular flow configurations. Also, an analysis of the CLVs revealed the existence of quasi-homoclinic tangencies in the attractor of the system.

7.2. Implications on flow physics

From the results in this paper, we conjecture that the leading Lyapunov exponent and covariant vector are associated to the smallest time and length scales responsible for chaos. Hence, the accurate prediction of the Lyapunov spectrum may require the (local) mesh size to be of the same order as the smallest (local) chaotic length scale. In the case of turbulent flows, this would correspond to DNS resolutions. Also, the positive skewness of the FTLEs associated to unstable modes implies that it is some particular regions of the attractor, that is, some particular flow configurations, that are responsible for the existence of positive exponents and the chaotic dynamics of the flow. Finally, the existence of quasi-homoclinic tangencies indicates that chaotic separated flows may not be uniformly hyperbolic dynamical systems.

7.3. Implications on adjoint-based sensitivity analysis for chaotic flows

On the one hand, the discrete system being less chaotic than the actual flow makes sensitivity analysis more manageable. In particular, the cost of the two most popular adjoint-based sensitivity analysis methods, namely the Least Squares Shadowing (LSS) [46] and the Ensemble Adjoint (EA) [24] methods, increases with the number and/or magnitude of positive Lyapunov exponents in the discrete system. The computational cost of shadowing-based approaches, including the original LSS method, the Multiple
Shooting Shadowing (MSS) method [6] and the Non-Intrusive LSS (NILSS) method [7, 33, 34], is proportional to the number of positive Lyapunov exponents. Under optimistic assumptions, including uniform hyperbolicity and exponential decay of correlations, Chandramoorthy et al. [9, 10] showed that the mean squared sensitivity error $E$ in the EA method exhibits power-law convergence $E \sim C^{-\beta}$, where $C$ denotes computational cost and $\beta \equiv \beta(\lambda^1) \in (0, 1)$ is a decreasing function of the leading Lyapunov exponent.

On the other hand, the discrete system being less chaotic than the actual flow may raise questions on the accuracy of the computed sensitivities. In this regard, we note that most quantities of interest in engineering applications require only an accurate description of the large scales in the chaotic flow, that is, LES-type resolutions. Hence, even if reproducing the actual Lyapunov spectrum of the flow required DNS resolutions, accurate sensitivities for such quantities could be computed with discrete systems that are less chaotic than the underlying flow.

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Appendix A. Ergodic theorem, Lyapunov exponents and covariant Lyapunov vectors. An overview

In this appendix, we present an overview of ergodic theory, Lyapunov exponents and covariant Lyapunov vectors. While the discussion focuses on dynamical systems acting on a Riemannian manifold $\mathcal{M}$, the reader that is not familiar with Riemannian geometry may simply replace $\mathcal{M}$ and $\mathcal{T}_x \mathcal{M}$ below by (the less general case of) $\mathbb{R}^n$.

Preliminaries

Let $\mathcal{M}$ be a Riemannian manifold of finite dimension $n$ equipped on the tangent space $\mathcal{T}_x \mathcal{M}$ of each point $x \in \mathcal{M}$ with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{T}_x}$, and let $\mathcal{F}^t : \mathcal{M} \to \mathcal{M}$ with $t \in \mathcal{I}$ be the evolution operator from $t_0$ to $t_0 + t$ of an autonomous dynamical system in $\mathcal{M}$. Here, $\mathcal{I}$ denotes the index set and corresponds to $\mathbb{N}_0$ and $\mathbb{R}_{\geq 0}$ for discrete-time and continuous-time dynamics, respectively. Also, let $\mathcal{B}$ denote a $\sigma$-algebra on $\mathcal{M}$, and $\mu$ be an invariant probability measure under $\mathcal{F}^t$, that is,

\begin{align}
\text{Probability measure:} & \quad \mu(\mathcal{M}) = 1, \quad (9a) \\
\text{Invariant measure:} & \quad \mu((\mathcal{F}^t)^{-1}(A)) = \mu(A), \quad \forall A \in \mathcal{B} \quad \forall t \in \mathcal{I}, \quad (9b)
\end{align}

where $(\mathcal{F}^t)^{-1}(A) := \{ x \in \mathcal{M} : \mathcal{F}^t(x) \in A \}$. The triple $(\mathcal{M}, \mathcal{B}, \mu)$ is further assumed to be complete and isomorphic mod 0 to an interval with Lebesgue measure, that is, $(\mathcal{M}, \mathcal{B}, \mu)$ is a standard (Lebesgue-Rokhlin) probability space. We note that $(\mathcal{F}^t)^{-1} : \mathcal{B} \to \mathcal{M} \subset \mathcal{M}$ is different to the backward evolution operator $\mathcal{F}^{-t} : \mathcal{M} \to \mathcal{M}$; which is only well-defined for invertible dynamical systems. Also, Eq. (9a) implies $\mathcal{F}^t$ is a measurable function. The 4-tuple $(\mathcal{M}, \mathcal{B}, \mu, \mathcal{F}^t)$ completely defines a measure-preserving autonomous dynamical system.
Ergodic theorem

**Definition 1.** A measure-preserving autonomous dynamical system \((\mathcal{M}, \mathcal{B}, \mu, F_t)\) is ergodic if for every \(A \in \mathcal{B}\) such that \((F_t)^{-1}(A) = A\) (i.e. for every invariant set), either \(\mu(A) = 0\) or \(\mu(A) = 1\). Such \(\mu\) is usually referred to as an ergodic invariant probability measure.

**Theorem 1 (Birkhoff Ergodic Theorem).** Let \((\mathcal{M}, \mathcal{B}, \mu, F_t)\) be ergodic and let \(L^1(\mathcal{M}, \mu) \ni h : \mathcal{M} \to \mathbb{R}\) be a real-valued integrable function. Then, for \(\mu\)-almost every initial condition \(x \in \mathcal{M}\),

\[
\lim_{t \to \infty} \frac{1}{t} \int_{(0,t)} (h \circ F_t)(x) = \int_{\mathcal{M}} h \, d\mu,
\]

including the assertion that the integral and the limit on the left-hand side exist.

In other words, in an ergodic system, “time average” is equal to “phase space average” for \(\mu\)-almost every initial condition. In non-ergodic systems, a similar result holds for \(\mu_A\)-almost every initial condition \(x \in A\) if the right-hand side in Eq. (10) is replaced by \([\mu(A)]^{-1} \int_{\mathcal{M}} h \, d\mu_A\), where \(A \in \mathcal{B}\) is a positive \(\mu\)-measure, invariant set, and \(\mu_A\) is the probability measure restricted to \(A\). The ergodic theorem as presented above can be considered as a corollary of this more general result.

Lyapunov exponents

The Lyapunov stability theory studies how infinitesimal perturbations to a measure-preserving (ergodic or non-ergodic) autonomous dynamical system evolve over (continuous or discrete) time, that is, how infinitesimally close initial conditions evolve with respect to each other over time. The expansion or contraction rate of an infinitesimal perturbation in the direction \(0 \neq \xi \in T_x \mathcal{M}\) is given by

\[
\gamma(\xi, x, t) := \frac{||DF^t(x)\xi||_{T_{F^t(x)}\mathcal{M}}}{||\xi||_{T_x\mathcal{M}}},
\]

where \(DF^t(x) : T_x \mathcal{M} \to T_{F^t(x)} \mathcal{M}\) is the (linear) tangent propagator. Also, we say that \(DF^t\) is “\(\mu\)-log \(^+\)-bounded” if

\[
\int_{\mathcal{M}} \sup_{t \leq 1} \log^+ ||DF^t(x)||_B \, d\mu < \infty,
\]

where \(\log^+ z := \max(0, \log z)\) and

\[
||DF^t(x)||_B := \sup_{||\xi||_{T_x\mathcal{M}} = 1} ||DF^t(x)\xi||_{T_{F^t(x)}\mathcal{M}}
\]

denotes operator norm.

**Definition 2.** Provided the limit exists, the Lyapunov exponent associated to \(x \in \mathcal{M}\) and \(0 \neq \xi \in T_x \mathcal{M}\) is defined as

\[
\Lambda(\xi, x) := \lim_{t \to \infty} \frac{1}{t} \ln \left(\gamma(\xi, x, t)\right).
\]
Hence, the Lyapunov exponent $\Lambda(\xi, x)$ indicates the asymptotic rate of exponential growth (or decay) of infinitesimal perturbations $\delta x$ to the system at $x \in \mathcal{M}$ along the direction $\xi \in \mathcal{T}_x \mathcal{M}$, that is,

$$\|\delta x(t)\|_{\mathcal{T}_x \mathcal{F}^t(x)} \sim e^{\Lambda(\xi, x)t}.$$ 

**Theorem 2.** Let $(\mathcal{M}, \mathcal{B}, \mu, \mathcal{F}^t)$ be a measure-preserving autonomous dynamical system. If $D\mathcal{F}^t$ is “$\mu$-$\log^+$-bounded” in the sense in Eq. (11), then the limit in Eq. (12) exists, is finite and independent of the choice of norm $\| \cdot \|_{\mathcal{T}_x}$ for $\mu$-almost every $x \in \mathcal{M}$ and for every $0 \neq \xi \in \mathcal{T}_x \mathcal{M}$. For fixed $x$, $\Lambda(\xi, x)$ takes at most $m \leq n$ different values, $\Lambda^1(x) > \cdots > \Lambda^m(x)$, and there exists a filtration of the tangent space into subspaces $S_i(x)$, $\mathcal{T}_x \mathcal{M} = S_1(x) \supset \cdots \supset S_m(x) \supset S_{m+1}(x) := 0$ of dimension $g_i(x) = \dim S_i(x) - \dim S_{i+1}(x)$ such that if $\xi \in S_i(x) \setminus S_{i+1}(x)$ then $\Lambda(\xi, x) = \Lambda^i(x)$, $i = 1, \ldots, m$. Moreover, for all $t \in \mathcal{I}$, the Lyapunov exponents and their multiplicities are $\mathcal{F}^t$-invariant $\Lambda^i \circ \mathcal{F}^t = \Lambda^i$, $g_i \circ \mathcal{F}^t = g_i$, and the filtration is invariant under the dynamics in the sense that $[D\mathcal{F}^t(x)](S_i(x)) = S_i(\mathcal{F}^t(x))$.

**Corollary 1.** In an ergodic system, the Lyapunov exponents are $\mu$-almost everywhere constant, that is, there exists $A \in \mathcal{B}$ such that $\mu(A) = 1$ and $\Lambda^i(x) = \Lambda^i$, $i = 1, \ldots, m$, $\forall x \in A$.

**Definition 3.** Provided the limit exists, we define the forward Oseledets’ operator $\mathcal{D}^+(x) : \mathcal{T}_x \mathcal{M} \to \mathcal{T}_x \mathcal{M}$ as

$$\mathcal{D}^+(x) := \lim_{t \to \infty} \frac{1}{2t} \ln \left( [D\mathcal{F}^t(x)]^* [D\mathcal{F}^t(x)] \right),$$

where $*$ denotes the adjoint operator, $\ln(\cdot)$ is given by the functional calculus, and convergence is in operator norm.

We note that, by construction, this is a linear, positive-definite, self-adjoint operator on $\mathcal{T}_x \mathcal{M}$.

**Corollary 2.** For $\mu$-almost every $x \in \mathcal{M}$, the limit in Eq. (13) exists and the eigenvalues $\lambda^+_i(x)$ of $\mathcal{D}^+(x)$ coincide with the Lyapunov exponents, i.e. $\lambda^+_i(x) = \Lambda^i(x)$. Furthermore, the multiplicity $l^+_i(x)$ of $\lambda^+_i(x)$ is equal to the multiplicity of $\Lambda^i(x)$, i.e. $l^+_i(x) = g_i(x)$.

**Corollary 3.** If $\mathcal{F}^t$ is a diffeomorphism (see footnote 4) and $D\mathcal{F}^{-t}$ is “$\mu$-$\log^+$-bounded” in the sense in Eq. (11), then the backward Oseledets’ operator,

$$\mathcal{D}^-(x) := \lim_{t \to \infty} \frac{1}{2t} \ln \left( [D\mathcal{F}^{-t}(x)]^* [D\mathcal{F}^{-t}(x)] \right),$$

converges for $\mu$-almost every $x \in \mathcal{M}$, and its eigenvalues and multiplicities satisfy $\lambda^-_i(x) = -\Lambda^i(x)$ and $l^-_i(x) = g_i(x)$, $i = 1, \ldots, m$. Also, for all $t \in \mathcal{I}$, the Lyapunov exponents and their multiplicities are $\mathcal{F}^{-t}$-invariant $\Lambda^i \circ \mathcal{F}^{-t} = \Lambda^i$, $g_i \circ \mathcal{F}^{-t} = g_i$, and the filtration of the tangent space is invariant under the backward dynamics in the sense that $[D\mathcal{F}^{-t}(x)](S_i(x)) = S_i(\mathcal{F}^{-t}(x))$.

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4 Let $\mathcal{M}$ and $\mathcal{N}$ be smooth manifolds. A differentiable map $\mathcal{F}^t : \mathcal{M} \to \mathcal{N}$ is called a diffeomorphism if it is a bijection and its inverse, denoted by $\mathcal{F}^{-t}$, is differentiable as well.
Covariant Lyapunov Vectors

The covariant Lyapunov vectors (CLVs) are only defined for invertible dynamical systems. For this reason, we shall restrict to this case hereinafter. Also, we will denote the (orthogonal) subspaces spanned by the eigenvectors of $D(x)$ by $(U_x(i))^\pm$, $i = 1, ..., m$.

**Definition 4.** Provided they are well-defined, the Oseledets’ subspaces $\Omega_x(i)$ are given by

$$\Omega_x(i) = (\Gamma_x(i))^+ \cap (\Gamma_x(i))^-,$$

where

$$(\Gamma_x(i))^+ = (U_x(i))^+ \cup \cdots \cup (U_x(m))^+,$$

$$(\Gamma_x(i))^- = (U_x(1))^- \cup \cdots \cup (U_x(i))^-. $$

We note that the Oseledets’ subspaces are not orthogonal in general.

**Theorem 3.** Let $F^t$ be a diffeomorphism on $\mathcal{M}$ such that $DF^\pm$ are “$\mu$-log$^+$-bounded” in the sense in Eq. (11). Then, for $\mu$-almost every $x \in \mathcal{M}$, the Oseledets’ subspaces are well-defined, independent of the choice of norm $||\cdot||_{\mathcal{T}_x}$, have dimension $\dim \Omega_x(i) = g_i(x)$, and form a measurable splitting of the tangent space $\mathcal{T}_x\mathcal{M} = \Omega_x(1) \oplus \cdots \oplus \Omega_x(m)$.

Furthermore, they satisfy

$$\lim_{t \to \pm \infty} \frac{1}{t} \ln \left| |DF^t(x)\xi||_{\mathcal{T}_x(F^t(x))} \right| = \Lambda_x(i)(x), \quad \forall 0 \neq \xi \in \Omega_x(i),$$

$$\lim_{t \to \pm \infty} \frac{1}{t} \ln \left( \angle (\Omega_x(i), \Omega_x(j)) \right), \quad \forall i \neq j,$$

and are invariant under the forward and backward dynamics in the sense that $[DF^t(x)](\Omega_x(i)) = \Omega_x(i)$ for all $t \in I$.

All the pieces are finally in place to formally introduce the covariant Lyapunov vectors:

**Definition 5.** A set $\{\psi^j\}_{j=1,\ldots,n}$, $\psi^j : \mathcal{M} \to \mathcal{T}_x\mathcal{M}$, is a set of covariant Lyapunov vectors if it admits a partition consisting of bases for the $m$ Oseledets’ subspaces for every $x \in \mathcal{M}$ in which these subspaces are well-defined.

From this definition and the previous results, several remarks follow:

- The CLVs form a basis of the tangent space $\mathcal{T}_x\mathcal{M}$ $\mu$-almost everywhere.
- The CLVs are such that any infinitesimal perturbation $\delta x$ along the direction $\psi^j(x)$ will remain along $\psi^j(F^t(x))$ for all $t \in I$. Also, the magnitude of the perturbation increases or decreases exponentially at a rate $||\delta x(t)||_{\mathcal{T}_x(F^t(x))} \sim ||\delta x(0)||_{\mathcal{T}_x} \ e^{\Lambda_x(i)t}$ as $t \to \infty$, where $\Lambda_x(i) = \Lambda_x^i \mu$-almost everywhere in ergodic systems.
- Unlike the Lyapunov exponents, the CLVs are a function of $x$ even in ergodic systems.
If all the Lyapunov exponents have multiplicity one (i.e. $m = n$) and the system is ergodic, the set of CLVs is unique (up to normalization constant).

We note that the notion of CLVs can be generalized to non-invertible transformations by restricting $F^t$ to one “branch” of the inverse dynamics [2]. We shall omit the details here. Also, all the results in this appendix hold more generally if $D F^t(x)$ is replaced by an arbitrary cocycle $A(t, x)$ on $M$ (see footnote 5). Since it is the particular case $A(t, x) = D F^t(x)$ that leads to the definition of Lyapunov exponents and covariant vectors, all the theorems have been written in terms of $D F^t$ to simplify the discussion.

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5 Let $(M, B, μ, F^t)$ be a measure-preserving autonomous dynamical system. A measurable map $A : T \times M \rightarrow GL(s, \mathbb{R})$ is called a cocycle if it satisfies $A(0, x) = I_s$ and $A(t_1 + t_2, x) = A(t_1, F^{t_2}(x)) A(t_2, x)$, $\forall x \in M, \forall t_1, t_2 \in T$, where $GL(s, \mathbb{R})$ denotes the set of $s \times s$ invertible real matrices and $I_s$ is the $s \times s$ identity matrix.
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