Abstract. A convexification-based numerical method for a Coefficient Inverse Problem for a parabolic PDE is presented. The key element of this method is the presence of the so-called Carleman Weight Function in the numerical scheme. Convergence analysis ensures the global convergence of this method, as opposed to the local convergence of the conventional least squares minimization techniques. Numerical results demonstrate a good performance.

Key words. parabolic equation, coefficient inverse problem, globally convergent numerical method, convexification, Carleman estimate, numerical studies

AMS subject classifications. 35R30

1. Introduction. In this paper, we construct a globally convergent numerical method for a Coefficient Inverse Problem (CIP) for a parabolic PDE. This method is based on the so-called convexification concept. Both convergence analysis and numerical results are presented. The CIP, which is considered here, has applications in heat conduction [1], diffusion theory [33] and in medical optical imaging using the diffuse infrared light [10]. In addition, this CIP has applications in financial mathematics in the search of the volatility coefficient in the Black-Scholes equation using the market data [7, 25]. In the latter case, the volatility coefficient should be assumed to be dependent on the stock price.

The most challenging question one needs to address prior a numerical treatment of any CIP for a PDE is: How to choose such a starting point of iterations that the convergence of a corresponding iterative numerical method to the correct solution of that CIP would be rigorously guaranteed? The underlying reason of the importance of this question is that CIPs for PDEs are both nonlinear and ill-posed. These two factors cause the well known phenomenon of multiple local minima and ravines of conventional Tikhonov-like least squares cost functionals for CIPs, see, e.g. [34] for a convincing numerical example of this phenomenon. Therefore, the above question cannot be addressed within the framework of the conventional least squares minimization.

One option would be to choose that starting point in a small neighborhood of the solution. However, such a good first guess is rarely available in applications. In fact, in such a case, the rest of the numerical procedure would be a locally convergent numerical method. On the other hand, we call a numerical method for a CIP globally convergent method.
To address the above question, the first author with coauthors has been working since 1995 \[5, 19, 20, 21, 23\] on the concept of the so-called convexification method for CIPs. This concept leads to globally convergent numerical methods. Those initial works on the convexification were not concerned with numerical studies (although, see \[21\] for some numerical results in the 1D case). The main reason of this was the lack of some theorems at that time, which would ensure a proper behavior of iterates. These theorems were first proved in \[2\].

After \[2\], a number of works on the convexification was published by the first author with coauthors, in which the theory is combined with numerical results, see, e.g. \[16, 26, 27, 28, 29\]. We also refer to \[3\] where a different version of the convexification is developed for a CIP for the hyperbolic equation \(u_{tt} = \Delta u + q(x)u\) and numerical results are presented. Most recently the idea of \[3\] was explored in \[6, 32\] to develop globally convergent numerical methods for some inverse problems for quasilinear parabolic PDEs. We also refer to the most recent work \[11\] for another idea of a globally convergent numerical method for a discrete statement of a special version of the electrical impedance tomography problem.

The convexification is a concept rather than a ready-to-use algorithm. This means that each new CIP requires it own version of the convexification, and these versions differ from each other quite significantly. Currently the convexification is developed analytically and tested numerically for CIPs for the Helmholtz equation \[16, 26, 29\], two hyperbolic equations \[3, 5, 28\] and Electrical Impedance Tomography \[27\]. The goal of this paper is to develop analytically and implement computationally the convexification method for a CIP for a parabolic PDE. The first step towards this goal was made in \[23\]. However, there are some problems in \[23\], which prevent one from a numerical implementation of the idea of \[23\]. Indeed, although a weighted globally strictly convex Tikhonov-like functional is constructed in \[23\], the Carleman Weight Function (CWF) in it is too complicated since it depends on two large parameters rather than on a single one. This means that the CWF of \[23\] changes too rapidly. The latter does not allow a numerical implementation, see \[3\] for a similar conclusion regarding a different CIP. In addition, since \[23\] was published before \[2\], then uniqueness and existence of the minimizer as well as the global convergence of the gradient projection method are not proven in \[23\]. Besides, numerical studies were not conducted in \[23\].

Thus, in this paper we first prove a new Carleman estimate with a simpler CWF, which can be used for computations. Next, we prove the central result: the global strict convexity of our weighted Tikhonov-like functional. Next, we establish the existence and uniqueness of its minimizer, estimate the distance between that minimizer and the exact solution and prove the global convergence of the gradient projection method to the exact solution. Finally, we describe results of our numerical experiments.

In the convexification, one constructs a weighted Tikhonov-like functional \(J_\lambda\), where \(\lambda \geq 1\) is the parameter. The weight is the CWF, i.e. the function which is involved as the weight in the Carleman estimate for the underlying PDE operator. Given a convex bounded set \(B(d) \subset H^k\) of an arbitrary diameter \(d > 0\) in a certain Hilbert space \(H^k\), one can choose the parameter \(\lambda\) of the CWF such that the strict convexity of that functional on \(B(d)\) is ensured. Thus, the local minima do not exist. Furthermore, as stated above, starting from the publication \[2\], all works about the
convexification contain theorems, which claim the existence and uniqueness of the minimizer of $J_\lambda$ on the set $B(d)$ and convergence of the gradient projection method of the minimization of $J_\lambda$ to that minimizer, if starting from an arbitrary point of $B(d)$. Next, as long as the level of the noise in the data tends to zero, those minimizers converge to the correct solution of the corresponding CIP. In particular, the latter means the stability of minimizers with respect to a small noise in the data. Since the diameter $d$ of the convex set $B(d)$ is an arbitrary one, then the latter amounts to the global convergence. Even though the theory requires the parameter $\lambda$ to be sufficiently large, our rich computational experience with the convexification shows that in real computations $\lambda \in [1, 3]$ is sufficient [16, 26, 27, 28, 29], also, see (9.1). In other words, computations are far less pessimistic than the theory is.

In section 2, we formulate both forward and inverse problems. The first step of section 3 consists in obtaining a nonlinear integral differential equation in which the unknown coefficient is not present. In the second step of that section we construct the above mentioned weighted Tikhonov-like functional with a CWF in it. In section 4 we formulate our theorems related to this functional. These theorems are proved in sections 5-8. In section 9, we present results of our numerical studies.

2. Statement of the Coefficient Inverse Problem. Below $\mathbf{x} = (x, \mathbf{\pi}) \in \mathbb{R}^n$, where $\mathbf{\pi} = (x_2, ..., x_n)$ and $x = x_1$. Let the numbers $A, B > 0$ and $A < B$. We introduce the cube $\Omega \subset \mathbb{R}^n$ and a part $\Gamma$ of its boundary $\partial \Omega$ as

\begin{equation}
\Omega = \{ x : A < x, x_2, ..., x_n < B \}, \Gamma = \{ x = B, A < x_2, ..., x_n < B \}.
\end{equation}

Let the number $T > 0$. Denote

\begin{equation}
Q_T^+ = \Omega \times (-T, T), S_T^+ = \partial \Omega \times (-T, T), \Gamma_T^+ = \Gamma \times (-T, T).
\end{equation}

Below $\alpha \in (0, 1), m \geq 1$ is an integer and $C^{m+\alpha}(\Omega), C^{2m+\alpha, m+\alpha/2}(Q_T^\pm)$ are Hölder spaces [30]. Let $b_j(\mathbf{x}), c(\mathbf{x}) \in C^{2+\alpha}(\Omega) : j = 1, ..., n$.

We consider the elliptic operator $L$ in the following form:

\begin{equation}
Lu = \Delta u + \sum_{j=1}^n b_j(\mathbf{x}) u_{x_j} - c(\mathbf{x}) u, \mathbf{x} \in \Omega.
\end{equation}

We assume that

\begin{equation}
c(\mathbf{x}) \geq 0 \text{ in } \overline{\Omega}.
\end{equation}

The forward parabolic initial boundary value problem is stated as [30]:

**Forward Problem.** Let the initial condition $f(\mathbf{x}) \in C^{4+\alpha}(\overline{\Omega})$. Find a function $u(\mathbf{x}, t) \in C^{4+\alpha, 2+\alpha/2}(Q_T^\pm)$ satisfying the following conditions:

\begin{align}
(2.4) & \quad u_t = Lu \text{ in } Q_T^+, \\
(2.5) & \quad u(\mathbf{x}, -T) = f(\mathbf{x}), \\
(2.6) & \quad u|_{S_T^\pm} = g_0(\mathbf{x}, t).
\end{align}
If the domain $\Omega$ would have its boundary $\partial \Omega \in \mathcal{C}^{4+\alpha}$ and if the Dirichlet condition $g_0(x, t)$ would belong to $\mathcal{C}^{4+\alpha, 2+\alpha/2}(\overline{S_T^\pm})$ and also corresponding compatibility conditions would be satisfied [30], then the existence and uniqueness of the solution $u \in \mathcal{C}^{4+\alpha, 2+\alpha/2}(\overline{Q_T^\pm})$ of problem (2.2)-(2.6) would be ensured [30]. However, for the the convenience of our derivations for the inverse problem, we have chosen the case of a piecewise smooth boundary $\partial \Omega$. Hence, we can only assume the existence of the solution $u \in \mathcal{C}^{4+\alpha, 2+\alpha/2}(\overline{Q_T^\pm})$ of problem (2.4)-(2.6). As to its uniqueness, it follows immediately from (2.3) and the maximum principle for parabolic PDEs.

**Coefficient Inverse Problem (CIP).** Let the number $t_0 \in (-T, T)$. Suppose that the following two functions $g_1(x, t)$ and $f_0(x)$ are known:

(2.7) \[ u_x \big|_{\Gamma_T^\pm} = g_1(x, t), \]

(2.8) \[ u(x, t_0) = f_0(x). \]

Find the unknown coefficient $c(x)$.

If $n = 3$ and functions $b_j(x) \equiv 0$ for $j = 1, \ldots, n$, then $c(x)$ is the absorption coefficient in the case of medical optical imaging using the diffuse infrared light [10]. Uniqueness of this CIP for any value of $T$ was proven by the first author using the method of [8], see, e.g. theorem 1.10.7 in [4], theorem 2 in [17], theorem 3.10 in [18] and theorem 3.4 in [22]. We also refer to [13, 36] for the Lipschitz stability estimate for this CIP.

The data for our CIP are non redundant, so as for all CIPs for which the convexification method works. In other words, the number $m$ of free variables in the data equals the number $n$ of free variables in the unknown coefficient, $m = n$. As to the globally convergent numerical methods for CIPs with redundant data with $m > n$, see, e.g. [12, 14, 15].

3. **Weighted Globally Strictly Convex Tikhonov-like Functional.** We assume below that there exists a number $\mu > 0$ such that

(3.1) \[ f(x) \geq \mu, \ \forall x \in \overline{\Omega}, \]

(3.2) \[ g_0(x, t) \geq \mu, \ \forall (x, t) \in \overline{S_T^\pm}. \]

Then (2.3), (3.1), (3.2) and the maximum principle for parabolic PDEs [30] imply that

(3.3) \[ u(x, t) \geq \mu \text{ in } \overline{Q_T^\pm}. \]

3.1. **Nonlinear integral differential equation.** Using (3.3), we introduce a new function $v(x, t)$,

(3.4) \[ v(x, t) = \ln u(x, t) \rightarrow u = e^v. \]

Substituting (3.4) in (2.4)-(2.8), we obtain in $Q_T^\pm$:

(3.5) \[ v_t - \Delta v - (\nabla v)^2 - \sum_{k=1}^{n} b_j(x) v_{x_j} = c(x), \]
\begin{align}
\text{(3.6)} & \quad v \mid_{S_T^±} = \ln g_0 (x, t), \quad v_x \mid_{Γ_T^±} = (g_1 / g_0) (x, t), \\
\text{(3.7)} & \quad v (x, t_0) = \ln f_0 (x) := \tilde{f}_0 (x).
\end{align}

For brevity we set below \( t_0 := 0 \). The case \( t_0 \neq 0 \) can be considered along the same lines. Differentiate both sides of the nonlinear equation (3.5) with respect to \( t \) and denote \( w (x, t) = v_t (x, t) \). Since the function \( c(x) \) is independent on \( t \), then the right hand side of the resulting equation will be zero. By (3.7)

\begin{align}
\text{(3.8)} & \quad v (x, t) = \int_0^t w (x, \tau) \, d\tau + \tilde{f}_0 (x), \quad (x, t) \in Q_T^±.
\end{align}

Substituting (3.8) in (3.5) and (3.6), we obtain a nonlinear integral differential PDE with Volterra integrals, supplied by the lateral Cauchy data,

\begin{align}
\text{(3.9)} & \quad -2 \nabla w \int_0^t \nabla w (x, \tau) \, d\tau - 2 \nabla w \nabla \tilde{f}_0 = 0, \quad (x, t) \in Q_T^±,
\end{align}

\begin{align}
\text{(3.10)} & \quad w \mid_{S_T^±} = p_0 (x, t), \quad w_x \mid_{Γ_T^±} = p_1 (x, t),
\end{align}

where \( p_0 (x, t) = (g_0 / g_0) (x, t) \) and \( p_1 (x, t) = \partial_t (g_1 / g_0) (x, t) \).

3.2. The functional. There are many possible choices of the CWF for the parabolic operator, see, e.g. [4, 13, 22, 31, 36]. However, among all these choices, we should select such a CWF which would be simple and would work well computationally. Indeed, for example, the CWF of [4, 22, 31] depends on two large parameters, which means that it changes too rapidly. As it was stated in Introduction, that rapid change prevents one from a numerical implementation. Thus, we have chosen the CWF \( \varphi_λ (x, t) \) as:

\begin{align}
\text{(3.11)} & \quad \varphi_λ (x, t) = \exp \left( 2 \lambda \left( x^2 - t^2 \right) \right),
\end{align}

where \( \lambda \geq 1 \) is a parameter. This means that we need to prove the Carleman estimate with this CWF, see Theorem 1 in section 4. Let \( \lfloor (n + 1) / 2 \rfloor \) be the maximal integer which does not exceed \( (n + 1) / 2 \). Denote \( k_n = \lfloor (n + 1) / 2 \rfloor + 2 \). For example, we have for most popular cases of \( n = 1, 2, 3 \):

\[ k_n = \begin{cases} 
3 & \text{if } n = 1, 2, \\
4 & \text{if } n = 3.
\end{cases} \]

We have chosen the number \( k_n \) in such a way that

\begin{align}
\text{(3.12)} & \quad H^{k_n} (Q_T^±) \subseteq H^3 (Q_T^±), \\
\text{(3.13)} & \quad H^{k_n} (Q_T^±) \subseteq C^1 \left( \overline{Q_T^±} \right), \quad \| q \|_{C^1 \left( \overline{Q_T^±} \right)} \leq C_0 \| q \|_{H^{k_n} (Q_T^±)}, \quad \forall q \in H^{k_n} (Q_T^±),
\end{align}
where the number $C_0 = C_0 (Q_T^\pm) > 0$ depends only on the domain $Q_T^\pm$. Relations (3.13) follow from (3.12) and the embedding theorem.

Let $R > 0$ be an arbitrary number. We define the bounded set of functions $B (R, p_0, p_1)$ as follows:

$$
B (R, p_0, p_1) = \{ w \in H^{k_n} (Q_T^\pm) : \| w \|_{H^{k_n} (Q_T^\pm)} < R, w \mid_{S_T^\pm} = p_0, w_x \mid_{\Gamma_T^\pm} = p_1 \},
$$

where functions $p_0, p_1$ are taken from (3.10).

Let $\beta > 0$ be a small regularization parameter and $K (w)$ be the nonlinear integral differential operator defined in (3.9). We construct our weighted Tikhonov-like functional with the CWF (3.11) in it as:

$$
J_{\lambda, \beta} (w) = e^{-2\lambda B^2} \int_{Q_T^\pm} (K (w)) \varphi \lambda dx dt + \beta \| w \|_{H^{k_n} (Q_T^\pm)}^2.
$$

Since max $\varphi \lambda = e^{2\lambda B^2}$, then the multiplier $e^{-2\lambda B^2}$ is introduced in (3.15) to balance two terms in the right hand side of (3.15).

**Minimization Problem.** Minimize the functional $J_{\lambda, \beta} (w)$ on the set $B (R)$ defined in (3.14).

Assume for a moment that a minimizer $w_{\min, \lambda, \beta} (x, t)$ of functional (3.15) exists and is computed. Then we first calculate the corresponding function $v_{\text{comp}} (x, t)$ via (3.8). Next, substituting $v_{\min, \lambda, \beta} (x, t) = 0$ of functional (3.5), we calculate an approximation for the target unknown coefficient $c (x)$. However, due to the inevitable computational errors as well as the noise in the data, the resulting left hand side of (3.5) would depend on $t$. Hence, to calculate an approximation $c_{\text{comp}} (x)$ for $c (x)$, we set

$$
c_{\text{comp}} (x) = \frac{1}{2\gamma T} \int_{-\gamma T}^{\gamma T} \left( \partial_t v_{\text{comp}} - \Delta v_{\text{comp}} - (\nabla v_{\text{comp}})^2 - \sum_{k=1}^{n} b_j (x) \partial x_j v_{\text{comp}} \right) dt,
$$

where the number $\gamma \in (0, 1/\sqrt{3})$ is chosen in section 4. Thus, we focus below on the Minimization Problem.

4. **Theorems.** Introduce the subspaces $H^{2,1}_0 (Q_T^\pm) \subset H^{2,1} (Q_T^\pm)$ and $H^{k_n}_0 (Q_T^\pm) \subset H^{k_n} (Q_T^\pm)$ as

$$
H^{2,1}_0 (Q_T^\pm) = \{ u \in H^{2,1} (Q_T^\pm) : u \mid_{S_T^\pm} = 0, u_x \mid_{\Gamma_T^\pm} = 0 \},
$$

$$
H^{k_n}_0 (Q_T^\pm) = \{ u \in H^{k_n} (Q_T^\pm) : u \mid_{S_T^\pm} = 0, u_x \mid_{\Gamma_T^\pm} = 0 \}.
$$

Since it is well known that any Carleman estimate depends only on the principal part of the operator, see, e.g. [22, 31], then we consider in Theorem 1 only the principal part $\partial_t - \Delta$ of the parabolic operator $\partial_t - L$.

**Theorem 1** (Carleman estimate). Suppose that the domain $\Omega$ and the CWF $\varphi_\lambda (x, t)$ are the same as in (2.1) and (3.11) respectively. Then there exist numbers $\lambda_0, C$,

$$
\lambda_0 = \lambda_0 (\Omega) \geq 1, \quad C = C (\Omega, T) > 0
$$
depending only on listed parameters such that the following Carleman estimate holds

\[
\int_{Q_T^\pm} (u_t - \Delta u)^2 \varphi_\lambda \, dx dt \geq \frac{C}{\lambda} \int_{Q_T^\pm} \left( u_t^2 + \sum_{i,j=1}^n u_{x_i x_j}^2 \right) \varphi_\lambda \, dx dt
\]

\[
+ C\lambda \int_{Q_T^\pm} \left[ (\nabla u)^2 + \lambda^2 u^2 \right] \varphi_\lambda \, dx dt
\]

\[
- C \exp \left( 2\lambda \left( B^2 - T^2 \right) \right) \int_{\Omega} \left[ \left( u_t^2 + (\nabla u)^2 + \lambda^2 u^2 \right) (x, T) \right] \, dx
\]

\[
- C \exp \left( 2\lambda \left( B^2 - T^2 \right) \right) \int_{\Omega} \left[ \left( u_t^2 + (\nabla u)^2 + \lambda^2 u^2 \right) (x, -T) \right] \, dx,
\]

\[
\forall \lambda \geq \lambda_0, \forall u \in H^{2,1}_0 (Q_T^\pm).
\]

**Remarks 1:**

1. An analog of estimate (4.2) was proven in [24], although only for the 1D case, and terms with \( u_{xx}, u_t \) were not involved in the estimate of [24]. However, the presence in (4.2) of the terms with derivatives involved in the principal part of the parabolic operator is important for the proofs of Theorems 2-6. Thus, Carleman estimate (4.2) is new.

2. Since the normal derivative of the function \( u \in H^{2,1}_0 (Q_T^\pm) \) equals zero only on the part \( \Gamma_T^\pm \) of the lateral boundary \( S_T^\pm \) of the time cylinder \( Q_T^\pm \) rather than on the whole \( S_T^\pm \), then one should carefully analyze integrals over \( S_T^\pm \) which occur in the pointwise Carleman estimate: to make sure that these integrals equal zero.

**Theorem 2** (the central theorem of this paper). Assume that condition (3.3) holds. The functional \( J_{\lambda, \beta} (w) \) has the Fréchet derivative \( J'_{\lambda, \beta} (w) \in H^{1,0}_0 (Q_T^\pm) \) for all \( \lambda, \beta > 0, w \in B \left( 3R, p_0, p_1 \right) \). Let \( \lambda_0 \geq 1 \) be the constant of Theorem 1. There exist constants

\[
\lambda_1 = \lambda_1 \left( R, A, B, T, \max_j \| b_j \|_{C(\overline{\Omega})}, \| f_0 \|_{C^1(\overline{\Omega})}, \mu \right) \geq \lambda_0,
\]

\[
C_1 = C_1 \left( R, A, B, T, \max_j \| b_j \|_{C(\overline{\Omega})}, \| f_0 \|_{C^1(\overline{\Omega})}, \mu \right) > 0
\]

depending only on listed parameters such that if \( \lambda \geq \lambda_1 \) and the regularization parameter \( \beta \in \left[ 2e^{-\lambda T^2}, 1 \right) \), then the functional \( J_{\lambda, \beta} (w) \) is strictly convex on the set \( B \left( R, p_0, p_1 \right) \) for all \( \lambda \geq \lambda_1 \), i.e. for all \( w_1, w_2 \in B \left( R, p_0, p_1 \right) \) and for all \( \lambda \geq \lambda_1 \)

\[
J_{\lambda, \beta} (w_2) - J_{\lambda, \beta} (w_1) - J'_{\lambda, \beta} (w_1) (w_2 - w_1)
\]

\[
\geq \frac{C_1}{\lambda} \exp \left( -2\lambda \left( T^2 + B^2 - A^2 \right) \right) \| w_2 - w_1 \|^2_{H^{2,1} (Q_T^\pm)} + \frac{\beta}{2} \| w_2 - w_1 \|^2_{H^{1,0} (Q_T^\pm)}.
\]

Everywhere below \( C > 0 \) and \( C_1 > 0 \) denote different constants depending only on parameters listed in (4.1) and (4.4) respectively.
Theorem 3. Assume that condition (3.3) holds. Let parameters \( \lambda_1, \lambda \geq \lambda_1 \) and \( \beta \) be the same as the ones in Theorem 2. Then there exists unique minimizer \( w_{\min, \lambda, \beta} \in B(R) \) of the functional \( J_{\lambda, \beta} (w) \) on the set \( B(R) \). Furthermore, the following inequality holds:

\[
J'_{\lambda, \beta} (w_{\min, \lambda, \beta}) (w - w_{\min, \lambda, \beta}) \geq 0, \quad \forall w \in B(R).
\]

Following the regularization theory [35], we assume now that there exists an ideal, the so-called ‘exact’ solution \( c^* (x) \in C^{2+\alpha} (\Omega) \) of the CIP (2.3), (2.4)-(2.8), where the data (2.6)-(2.8) are noiseless. Also, let \( c_{\text{comp}} (x) \) be the coefficient \( c(x) \) reconstructed from the minimizer \( w_{\min, \lambda, \beta} (x, t) \) via backwards calculations, as outlined in the last paragraph of section 3 and, in particular, in (3.16). Having the function \( c^* (x) \), one can construct the noise free solution \( w^* \in H^{k_n} (Q_T^\pm) \) of equation (3.9) with the noiseless boundary data \( p_0^*, p_1^* \) in (3.10) and the noiseless function \( \tilde{f}_0^* (x) \) in (3.9).

We now want to estimate the distance between the minimizer \( w_{\min, \lambda, \beta} \) and the function \( w^* \) as well as between coefficients \( c^* (x) \) and \( c_{\text{comp}} (x) \). To do this, we first arrange zero boundary conditions in an analog of (3.10). More precisely, we assume that there exist functions \( G(x, t) \) and \( G^* (x, t) \) satisfying the same boundary conditions as those for \( w \) and \( w^* \) respectively and such that their norms in the space \( H^{k_n} (Q_T^\pm) \) are less than \( R \), i.e.

\[
G \in B(R, p_0, p_1), \quad G^* \in B(R, p_0^*, p_1^*)
\]

Let a small number \( \delta \in (0, 1) \) be the level of the noise in the functions \( G \) and \( f_0 \). More precisely, we assume that

\[
\|G - G^*\|_{H^{k_n} (Q_T^\pm)} < \delta, \quad \|f_0 - f_0^*\|_{C^1 (\Omega)} < \delta.
\]

Remark 2. By (4.8), we replace below \( \|f_0\|_{C^1 (\Omega)} \) with \( \|f_0^*\|_{C^1 (\Omega)} \) in (4.3) and (4.4).

We also assume that functions

\[
w^* \in B(R - \delta, p_0^*, p_1^*),
\]

\[
\|f_0^*\|_{C^1 (\Omega)} < R, \quad \min_{\Omega} f_0^* \geq \mu > 0,
\]

where the number \( \mu \) is the same as in (3.1), (3.3) and is independent on \( \delta \). Then (3.10), (4.7) and (4.9) imply that

\[
G \in B(R, p_0, p_1).
\]

Denote

\[
W = w - G, \quad W^* = w^* - G^*.
\]

Similarly with (3.14) denote

\[
B_0 (2R) = \left\{ W \in H^{k_n} (Q_T^\pm) : \|W\|_{H^{k_n} (Q_T^\pm)} < 2R, W|_{\partial T^\pm} = W_x |_{\Gamma_T^\pm} = 0 \right\}.
\]
Then (4.9)-(4.12) imply that
\begin{equation}
W \in B_0(2R), \forall w \in B(R,p_0,p_1) \text{ and also } W^* \in B_0(2R - 2\delta),
\end{equation}
\begin{equation}
W + G \in B(3R,p_0,p_1), \forall W \in B_0(2R).
\end{equation}

Due to (4.15), it is convenient to denote below \(\lambda_1(3R), \lambda(3R)\), which means that the values of the parameters \(\lambda_1 \) and \(\lambda \geq \lambda_1 \) correspond to \(B(3R,p_0,p_1)\) in Theorem 2 and, in particular, \(R\) is replaced with \(3R\) in (4.3) and (4.4). Consider the functional \(I_{\lambda,\beta}(W)\),
\begin{equation}
I_{\lambda,\beta} : B_0(2R) \to \mathbb{R}, I_{\lambda,\beta}(W) = J_{\lambda,\beta}(W + G).
\end{equation}

**Theorem 4.** Assume that condition (3.3) holds. Let parameters \(\lambda_1 \) and \(\beta\) be the same as in Theorem 2, except that \(R\) is replaced with \(3R\) in (4.3). Then the functional \(I_{\lambda,\beta}(W)\) is strictly convex on the ball \(B_0(2R)\) for all \(\lambda \geq \lambda_1(3R)\). Here, \(\lambda_1(3R)\) means (4.3), where \(R\) is replaced with \(3R\) and \(f_0\) is replaced with \(f_0^*\) (Remark 2). In other words, the following analog of (4.5) holds
\begin{equation}
I_{\lambda,\beta}(W_2) - I_{\lambda,\beta}(W_1) - I'_{\lambda,\beta}(W_1)(W_2 - W_1)
\geq \frac{C_1}{\lambda} \exp\left(-2\lambda(T^2 + B^2 - A^2)\right) \|W_2 - W_1\|_{H^{2,1}(Q_T)}^2 + \frac{\beta}{2} \|W_2 - W_1\|_{H^k(Q_T)}^2,
\end{equation}
for all \(\lambda \geq \lambda_1(3R)\) and for all \(W_1, W_2 \in B_0(2R)\), where \(I'_{\lambda,\beta}(W) \in H^k_0(Q_T)\) is the Frechét derivative of the functional \(I_{\lambda,\beta}(W)\) at the point \(W\), which exists due to Theorem 2 and (4.16). Furthermore, there exists unique minimizer \(W_{\min,\lambda(3R),\beta} \in B_0(2R)\) of the functional \(I_{\lambda,\beta}(W)\) and the following inequality holds:
\begin{equation}
I'_{\lambda(3R),\beta}(W_{\min,\lambda(3R),\beta})(W - W_{\min,\lambda(3R),\beta}) \geq 0, \forall W \in B_0(2R).
\end{equation}

**Theorem 5** (accuracy estimates). Assume that condition (3.3) holds. Suppose that conditions (4.6)-(4.12) hold and also let \(T > \sqrt{3(3^2 - A^2)}\). Choose a number \(\gamma \in (0,1/\sqrt{3})\) such that \(T^2(1 - 3\gamma^2) > 3(3^2 - A^2)\). Denote
\begin{equation}
\eta_1 = \gamma^2 T^2 + B^2 - A^2, \eta_2 = (1 - 3\gamma^2) T^2 - 3(3^2 - A^2), \rho = \frac{1}{2} \min\left(1, \frac{\eta_2}{\eta_1}\right).
\end{equation}
Let \(\lambda_1 = \lambda_1(3R)\) be the number of Theorem 4. Choose a sufficiently small number \(\delta_0 > 0\) such that \(\ln\left(\delta_0^{-1/\eta_1}\right) \geq \lambda_1\). For each \(\delta \in (0,\delta_0)\), let \(\lambda = \lambda(\delta,3R) = \ln\left(\delta^{-1/\eta_1}\right) > \lambda_1(3R)\). Let the regularization parameter \(\beta = \beta(\delta,3R) = 2e^{-\lambda(\delta,3R)T^2}\) (see Theorem 2). Let \(w_{\min,\lambda(\delta,3R),\beta(\delta,3R)} = W_{\min,\lambda(\delta,3R),\beta(\delta,3R)} + G\) (Theorem 4) and let \(c_{\text{comp}}(x)\) be the function \(c(x)\) computed from the function \(w_{\min,\lambda(\delta,3R),\beta(\delta,3R)}(x,t)\) by the procedure described in the last paragraph of section 3. Then the following accuracy estimates are valid
\begin{equation}
\|w^* - w_{\min,\lambda(\delta,3R),\beta(\delta,3R)}\|_{H^2(Q_T^c)} \leq C_2\delta^\rho,
\end{equation}
\begin{equation}
\|c^* - c_{\min,\lambda(\delta,3R),\beta(\delta,3R)}\|_{L^2(\Omega)} \leq C_2\delta^\rho.
\end{equation}
Here and below $C_2 > 0$ denotes different constants depending on the same parameters as ones in (4.4) as well as on the number $\gamma$.

We now construct the gradient projection method of the minimization of the functional $I_{\lambda, \beta}(W)$ defined in (4.16) on the set $\overline{B}_0(2R)$ defined in (4.13). Let $P_B : H^{k_o}(Q_T^+ \setminus \Omega) \rightarrow \overline{B}_0(2R)$ be the orthogonal projection operator. Let $W_0 \in B_0(2R)$ be an arbitrary point of the ball $B_0(2R)$. Let the number $\omega \in (0, 1)$. We arrange the gradient projection method of the minimization of the functional $I_{\lambda, \beta}(W)$ as:

\[(4.21) \quad W_n = P_B (W_{n-1} - \omega I'_{\lambda, \beta}(W_{n-1})), \quad n = 1, 2, ...\]

Note that since $W_{n-1}, I'_{\lambda, \beta}(W_{n-1}) \in H^{k_o}(Q_T^+ \setminus \Omega)$, then the function $W_{n-1} - \omega I'_{\lambda, \beta}(W_{n-1})$ has zero boundary conditions (3.10). The latter is important in the computational practice.

**Theorem 6** (global convergence of the gradient projection method). Assume that condition (3.3) holds. Let parameters $\lambda_1(3R)$ and $\beta$ be the same as in Theorem 2, except that $R$ is replaced with $3R$ in (4.3) and let $\lambda(3R)$. Then there exists a sufficiently small number $\omega_0 = \omega_0(\Omega, T, A, B, R, \lambda)$ such that for any $\omega [0, \omega_0)$ there exists a number $\theta = \theta(\omega) \in (0, 1)$ such that the sequence (4.21) converges to the unique minimizer $W_{\min, \lambda(3R), \beta} \in \overline{B}_0(2R)$ (Theorem 4) in the norm of the space $H^{k_o}(Q_T^+ \setminus \Omega)$.

More precisely,

\[(4.22) \quad \|W_{\min, \lambda(3R), \beta} - W_n\|_{H^{k_o}(Q_T^+ \setminus \Omega)} \leq \theta^n \|W_{\min, \lambda(3R), \beta} - W_0\|_{H^{k_o}(Q_T^+ \setminus \Omega)} \cdot\]

**Theorem 7** (global convergence to the exact solution of the gradient projection method). Suppose that assumptions of Theorem 5 hold and also that parameters $\lambda = \lambda(\delta, 3R)$ and $\beta = \beta(\delta, 3R)$ are the same as in that theorem. Let $w_n = W_n + G, n = 0, 1, ...$ and $w_{\min, \lambda(\delta, 3R), \beta(\delta, 3R)} = W_{\min, \lambda(\delta, 3R), \beta(\delta, 3R)} + G$ (Theorem 4). Let $c_{n, \text{comp}}(x)$ be the function $c(x)$ obtained from the function $w_n(x, t)$ by the procedure outlined in the end of section 3. Then there exists a sufficiently small number $\omega_1 = \omega_1(\Omega, T, A, B, R, \gamma, \lambda) \in (0, \omega_0)$ such that for any $\omega \in (0, \omega_1)$ there exists a number $\theta = \theta(\omega) \in (0, 1)$ such that the following convergence estimates are valid for $n = 1, 2, ...$

\[(4.23) \quad \|w^* - w_n\|_{H^{2,1}(Q_T^+ \setminus \Omega)} \leq C_2 \delta^n + \theta^n \|w_{\min, \lambda(\delta, 3R), \beta(\delta, 3R)} - w_0\|_{H^{k_o}(Q_T^+ \setminus \Omega)} \cdot\]

\[(4.24) \quad \|c^* - c_{\text{n,comp}}\|_{L^2(\Omega)} \leq C_2 \delta^n + \theta^n \|w_{\min, \lambda(\delta, 3R), \beta(\delta, 3R)} - w_0\|_{H^{k_o}(Q_T^+ \setminus \Omega)} \cdot\]

**Remarks:**

1. Since the starting point $W_0 \in B_0(2R)$ of the iterative process (4.21) is an arbitrary point of the ball $B_0(2R)$ and since $R > 0$ is an arbitrary number, then Theorem 7 ensures the **global convergence** of the gradient projection method (4.21) to the correct solution as long as the noise level $\delta$ tends to zero, see section 1 for our definition of the global convergence.

2. We omit the proof of Theorem 3 below since, by Lemma 2.1 of [2], Theorem 3 follows immediately from Theorem 2. In addition, we omit the proof of Theorem 6 since Theorem 2.1 of [2] implies that Theorem 6 follows immediately from Theorem 2.
5. Proof of Theorem 1. We prove this theorem only for functions $u(x, t)$ such that

\begin{equation}
\tag{5.1} u \in C^3 \left( \overline{Q_T^\pm} \right), u \big|_{\partial S_T^\pm} = u_x \big|_{\Gamma_T^\pm} = 0.
\end{equation}

The case $u \in H^{2,1}_0 \left( Q_T^\pm \right)$ follows immediately from this proof via density arguments. Below in this proof $O \left( 1/\lambda^k \right), k \geq 1$ denotes different smooth functions, which are independent on $u$ and for which the following estimate is valid $\| \cdot \|_{C^1 \left( \overline{Q_T^\pm} \right)} \leq C/\lambda^k, \forall \lambda \geq 1$.

Recall that by (3.11) $\varphi_\lambda(x, t) = \exp \left( 2\lambda \left( x^2 - t^2 \right) \right)$. Introduce a new function $v(x, t) = u(x, t) \exp \left( \lambda \left( x^2 - t^2 \right) \right)$. Then $u = v \exp \left( -\lambda \left( x^2 - t^2 \right) \right)$. Hence,

\begin{equation}
\tag{5.2} (u_t - \Delta u)^2 \varphi_\lambda \geq (2v_t + 8\lambda \lambda v_x) \left( -\Delta v - 4\lambda^2 x^2 \left( 1 + O \left( 1/\lambda \right) \right) v + 2\lambda tv \right).
\end{equation}

**Step 1.** Estimate from the below the following term in (5.2):

\[ 2v_t \left( -\Delta v - 4\lambda^2 x^2 \left( 1 + O \left( 1/\lambda \right) \right) v + 2\lambda tv \right), \]

\[ = -2 \sum_{i=1}^n v_{x_i} v_t + \left( -4\lambda^2 x^2 \left( 1 + O \left( 1/\lambda \right) \right) v^2 + 2\lambda tv^2 \right)_t - 2\lambda v^2 \]

\[ = \sum_{i=1}^n \left( -2v_{x_i} v_t \right)_{x_i} + \sum_{i=1}^n v_{x_i} v_{x_t} + \left( -4\lambda^2 x^2 \left( 1 + O \left( 1/\lambda \right) \right) v^2 + 2\lambda tv^2 \right)_t - 2\lambda v^2 \]

\[ = -2\lambda v^2 + \sum_{i=1}^n \left( -2v_{x_i} v_t \right)_{x_i} + \left( \nabla v \right)^2 - 4\lambda^2 x^2 \left( 1 + O \left( 1/\lambda \right) \right) v^2 + 2\lambda tv^2 \] \( \right)_t. \)

Thus, the desired estimate of Step 1 is:

\begin{equation}
\tag{5.3} 2v_t \left( -\Delta v - 4\lambda^2 x^2 \left( 1 + O \left( 1/\lambda \right) \right) v + 2\lambda tv \right) = -2\lambda v^2 + \text{div} \, U_1 + V_1, \]

\begin{equation}
\tag{5.4} \text{div} \, U_1 = \sum_{i=1}^n \left( -2v_{x_i} v_t \right)_{x_i}, \]

\begin{equation}
\tag{5.5} V_1 = \left( \nabla v \right)^2 - 4\lambda^2 x^2 \left( 1 + O \left( 1/\lambda \right) \right) v^2 + 2\lambda tv^2. \)

**Step 2.** Estimate from the below the following term in (5.2):

\[ 8\lambda v_x \left( -\Delta v - 4\lambda^2 x^2 \left( 1 + O \left( 1/\lambda \right) \right) v + 2\lambda tv \right), \]

\[ = -8\lambda v_x v_x + \sum_{i=2}^n \left( -8\lambda v_x v_{x_i} \right) \]

\[ = -8\lambda v_x v_x \sum_{i=2}^n \left( -8\lambda v_x v_{x_i} \right). \]
+ (−16λ³x³ (1 + O (1/λ)) v² + 8λ²xtu²) x + 48λ³x² (1 + O (1/λ)) v²
= (−4λxv₂) x + 4λv² + \sum_{i=2}^{n} (−8λxv_xv_{x_i}) x_i + \sum_{i=2}^{n} (8λxv_{xx_i}v_{x_i})
+ 48λ³x² (1 + O (1/λ)) v² + (−16λ³x³ (1 + O (1/λ)) v² + 8λ²xtu²) x
= 4λ \left( v_x² - \sum_{i=2}^{n} v_{x_i}² \right) + 48λ³x² (1 + O (1/λ)) v²
+ \left( −4λxv_x² + 4λx \sum_{i=2}^{n} v_{x_i}² - 16λ³x³ (1 + O (1/λ)) v² + 8λ²xtu² \right) x
+ \sum_{i=2}^{n} (−8λxv_xv_{x_i}) x_i.

Thus, we end up with the following estimate of Step 2:

\[ 8λxv_x (−Δv - 4λ²x² (1 + O (1/λ)) v + 2λtv) \]

\[ (5.6) = 4λ \left( v_x² - \sum_{i=2}^{n} v_{x_i}² \right) + 48λ³x² (1 + O (1/λ)) v² + \text{div} \, U_2, \]

\[ (5.7) \text{div} \, U_2 = \left( −4λxv_x² + 4λx \sum_{i=2}^{n} v_{x_i}² - 16λ³x³ (1 + O (1/λ)) v² + 8λ²xtu² \right) x
+ \sum_{i=2}^{n} (−8λxv_xv_{x_i}) x_i. \]

**Step 3.** Analysis of boundary integrals over \( S_T \).

Let \( ν = ν(x) \) be the unit outward looking normal vector to \( ∂Ω \) at the point \( x ∈ ∂Ω \). By Gauss’ formula, (5.4) and (5.7)

\[ T∫_{−T}^T ∫_{∂Ω} (\text{div} \, U_1 + \text{div} \, U_2) dx \, dt = \int_{−T}^T ∫_{∂Ω} ∑_{i=1}^{n} (U_{1,i} + U_{2,i}) cos (ν(x), x_i) dS dt, \]

where \( U_k = (U_{k,1}, ..., U_{k,n}), k = 1, 2 \). Obviously, \( U_{1i} = −2v_{x_i}v_t \). Since (5.1) holds and since

\[ v_t (x, t) = (u_t - 2λtu(x, t) \exp (λ (x² - t²))), \]

then \( v_t (x, t) = 0 \) for \( x ∈ ∂Ω \). Hence, in (5.8)

\[ (5.9) \int_{−T}^T ∫_{∂Ω} ∑_{i=1}^{n} U_{1,i} \cos (ν(x), x_i) dS dt = 0. \]

We now analyze the first term in the right hand side of (5.7). We have

\[ (5.10) \quad v_x (x, t) = (ux + 2λxu) (x, t) \exp (λ (x² - t²)), \]
By (5.7), (5.1), (5.10) and (5.11)

\[
\int_{\Gamma'} U^2_2 \, dS dt = 4\lambda A \int_{\Gamma'} u^2_2 \, dS \geq 0,
\]

where \( \Gamma' = \{x = A\} \cap \partial \Omega \). Similarly

\[
\int_{\Gamma'} \sum_{i=2}^{n} U^2_i \, dS dt = 0.
\]

Using (5.8), (5.9), (5.12) and (5.13), we obtain

\[
\int_{Q_T^\pm} (\text{div} U_1 + \text{div} U_2) \, dx dt \geq 0.
\]

**Step 4.** Integrate (5.2) over \( Q_T^\pm \). Then sum up (5.3) with (5.6), integrate the resulting inequality over \( Q_T^\pm \) and use that integral of (5.2), Gauss’ formula, (5.5) and (5.9)-(5.13). We obtain for all \( \lambda \geq \lambda_0 \) and all \( u \in C^2 (Q_T^\pm) \cap H^1_0 (Q_T^\pm) \)

\[
\int_{Q_T^\pm} (u_t - \Delta u)^2 \varphi_\lambda \, dx dt \geq -4\lambda \int_{Q_T^\pm} (\nabla u)^2 \varphi_\lambda \, dx dt + 47\lambda^3 \int_{Q_T^\pm} u^2 x^2 \varphi_\lambda \, dx dt
\]

\[
- C \exp \left( 2\lambda (B^2 - T^2) \right) \int_{\Omega} \left[ (\nabla u)^2 + \lambda^2 u^2 \right] (x, T) + \left[ (\nabla u)^2 + \lambda^2 u^2 \right] (x, -T) \, dx.
\]

The inconvenient point of (5.15) is the presence of the negative term in the first line of (5.15). Therefore, we continue.

**Step 5.** Estimate from the below \((u_t - \Delta u) u \varphi_\lambda\), and then estimate the corresponding integral over \( Q_T^\pm \).

\[
(u_t - \Delta u) u \varphi_\lambda = \left( \frac{u^2}{2} \varphi_\lambda \right)_t + 2\lambda u^2 \varphi_\lambda + (-u_x u \varphi_\lambda)_x + u_x^2 \varphi_\lambda + 4\lambda x u_x u \varphi_\lambda
\]

\[
+ \sum_{i=2}^{n} (-u_x u \varphi_\lambda)_{x_i} + \sum_{i=2}^{n} u_{x_i}^2 \varphi_\lambda
\]

\[
= (\nabla u)^2 \varphi_\lambda + \sum_{i=1}^{n} (-u_x u \varphi_\lambda)_{x_i} + \left( \frac{u^2}{2} \varphi_\lambda \right)_t + (2\lambda x u^2 \varphi_\lambda)_x
\]

\[
- 8\lambda^2 x^2 (1 + O(1/\lambda)) u^2 \varphi_\lambda.
\]

Hence,

\[
(u_t - \Delta u) u \varphi_\lambda \geq (\nabla u)^2 \varphi_\lambda - 9\lambda^2 x^2 u^2 \varphi_\lambda + \text{div} U_3 + V_{2t},
\]

\[
\text{div} U_3 = \sum_{i=1}^{n} (-u_x u \varphi_\lambda)_{x_i} + (2\lambda x u^2 \varphi_\lambda)_x,
\]
\begin{equation}
(5.18) \quad V_2 = \frac{u^2}{2} \varphi_\lambda.
\end{equation}

Hence, by (5.1), (5.17) and Gauss formula

\begin{equation}
(5.19) \quad \int_{Q_T^+} \text{div} U_3 dx dt = 0.
\end{equation}

Integrate (5.16) over $Q_T^+$ using (5.18) and (5.19). Then multiply the resulting inequality by $5\lambda$ and sum up with (5.15). We obtain

\begin{equation}
(5.20) \quad 5\lambda \int_{Q_T^+} (u_t - \Delta u) u \varphi_\lambda dx dt + \int_{Q_T^+} (u_t - \Delta u)^2 \varphi_\lambda dx dt + 5\lambda^3 \int_{Q_T^+} u^2 x^2 \varphi_\lambda dx dt \geq \lambda^3 \int_{Q_T^+} u^2 x^2 \varphi_\lambda dx dt + \frac{5}{2} \lambda^2 \int_{Q_T^+} u^2 \varphi_\lambda dx dt.
\end{equation}

Next, by the Cauchy-Schwarz inequality

\begin{equation}
(5.21) \quad 5\lambda \int_{Q_T^+} (u_t - \Delta u) u \varphi_\lambda dx dt + \int_{Q_T^+} (u_t - \Delta u)^2 \varphi_\lambda dx dt \leq \frac{7}{2} \int_{Q_T^+} (u_t - \Delta u)^2 \varphi_\lambda dx dt + \frac{5}{2} \lambda^2 \int_{Q_T^+} u^2 \varphi_\lambda dx dt.
\end{equation}

Since for sufficiently large $\lambda_0 > 1$ and for $\lambda \geq \lambda_0$

\begin{equation}
(5.22) \quad 2\lambda^3 \int_{Q_T^+} u^2 x^2 \varphi_\lambda dx dt - \frac{5}{2} \lambda^2 \int_{Q_T^+} u^2 \varphi_\lambda dx dt \geq \lambda^3 \int_{Q_T^+} u^2 x^2 \varphi_\lambda dx dt,
\end{equation}

then (5.20)-(5.22) imply that for all $u \in H_0^{2,1} (Q_T^+)$ and for all $\lambda \geq \lambda_0$

\begin{equation}
(5.23) \quad \int_{Q_T^+} (u_t - \Delta u)^2 \varphi_\lambda dx dt \geq C\lambda \int_{Q_T^+} \left[ (\nabla u)^2 + \lambda^2 u^2 \right] \varphi_\lambda dx dt + C \exp \left( 2\lambda \left( B^2 - T^2 \right) \right) \int_{Q_T^+} \left[ \left( \nabla u \right)^2 + \lambda^2 u^2 \right] (x, T) + \left( \nabla u \right)^2 + \lambda^2 u^2 \right] (x, -T) dx,
\end{equation}

which is a part of estimate (4.2). We now need to incorporate in our estimate terms with $u_t^2, u_{x,x}^2$.

**Step 6.** Incorporating terms with $u_t^2, u_{x,x}^2$.

We have

\begin{equation}
(5.24) \quad (u_t - \Delta u)^2 \varphi_\lambda = u_t^2 \varphi_\lambda - 2u_t u_{xx} \varphi_\lambda - \sum_{i=2}^{n} 2u_t u_{x,i} \varphi_\lambda + (\Delta u)^2 \varphi_\lambda.
\end{equation}
Denote
\[
(z_1 = -2u_t u_{xx} \varphi_\lambda, \quad z_2 = -\sum_{i=2}^n 2u_t u_{x_i} \varphi_\lambda, \quad z_3 = (\Delta u)^2 \varphi_\lambda)
\]
and estimate each of terms in (5.25). First, we have
\[
z_1 = -2u_t u_{xx} \varphi_\lambda = (-2u_t u_x \varphi_\lambda)_x + 2u_{tx} u_x \varphi_\lambda + 8\lambda x u_t u_x \varphi_\lambda
\]
\[
= (-2u_t u_x \varphi_\lambda)_x + (u^2_x \varphi_\lambda)_t + 2\lambda u^2_x \varphi_\lambda + 8\lambda x u_t u_x \varphi_\lambda
\]
\[
\geq -\frac{1}{2} u^2_t \varphi_\lambda - C\lambda^2 u^2_x \varphi_\lambda + (-2u_t u_x \varphi_\lambda)_x + (u^2_x \varphi_\lambda)_t.
\]
Thus,
\[
z_1 \geq -\frac{1}{2} u^2_t \varphi_\lambda - C\lambda^2 u^2_x \varphi_\lambda + (-2u_t u_x \varphi_\lambda)_x + (u^2_x \varphi_\lambda)_t.
\]
We now estimate \(z_2\),
\[
z_2 = \sum_{i=2}^n (-2u_t u_{x_i} \varphi_\lambda)_x + \sum_{i=2}^n 2u_{tx_i} u_{x_i} \varphi_\lambda
\]
\[
= \sum_{i=2}^n (u^2_{x_i} \varphi_\lambda)_t + 4\lambda t \sum_{i=2}^n u^2_{x_i} \varphi_\lambda + \sum_{i=2}^n (-2u_t u_{x_i} \varphi_\lambda)_x
\]
\[
\geq -C\lambda \sum_{i=2}^n u^2_{x_i} \varphi_\lambda + \sum_{i=2}^n (u^2_{x_i} \varphi_\lambda)_t + \sum_{i=2}^n (-2u_t u_{x_i} \varphi_\lambda)_x.
\]
Thus,
\[
z_2 \geq -C\lambda \sum_{i=2}^n u^2_{x_i} \varphi_\lambda + \sum_{i=2}^n (u^2_{x_i} \varphi_\lambda)_t + \sum_{i=2}^n (-2u_t u_{x_i} \varphi_\lambda)_x.
\]
Now we estimate \(z_3\),
\[
z_3 = (\Delta u)^2 \varphi_\lambda = \left(u_{xx} + \sum_{i=2}^n u_{x_i x_i}\right)^2 \varphi_\lambda
\]
\[
= \sum_{i=1}^n u^2_{x_i x_i} \varphi_\lambda + 2 \sum_{i=2}^n u_{xx} u_{x_i x_i} \varphi_\lambda + 2 \sum_{i=2}^n u_{x_i x_i} u_{x_j x_j} \varphi_\lambda
\]
\[
= \sum_{i=1}^n u^2_{x_i x_i} \varphi_\lambda + \left(2 \sum_{i=2}^n u_{x_i x_i} \varphi_\lambda\right)_x - 8\lambda x \sum_{i=2}^n u_{x_i x_i} \varphi_\lambda
\]
\[
(5.28) \quad - 2 \sum_{i=2}^n u_{x_i x_i} \varphi_\lambda + \left(2 \sum_{i=2}^n u_{x_i x_i} \varphi_\lambda\right)_x - 2 \sum_{i,j=2, i \neq j}^n u_{x_i x_j x_j} \varphi_\lambda
\]
\[
= \sum_{i=1}^n u^2_{x_i x_i} \varphi_\lambda + \left(2 \sum_{i=2}^n u_{x_i x_i} \varphi_\lambda\right)_x - 8\lambda x \sum_{i=2}^n u_{x_i x_i} \varphi_\lambda
\]
\[ + \left( -2 \sum_{i=2}^{n} u_{x} u_{xx, i} \varphi_{\lambda} \right)_{x_i} + \sum_{i=2}^{n} u_{x}^{2} \varphi_{\lambda} + \left( 2 \sum_{i,j=2, i \neq j}^{n} u_{x, x_j} \varphi_{\lambda} \right)_{x_i} \]
\[ + \left( -2 \sum_{i,j=2, i \neq j}^{n} u_{x, x_i} \varphi_{\lambda} \right)_{x_j} + 2 \sum_{i,j=2, i \neq j}^{n} u_{x, x_j}^{2} \varphi_{\lambda} \]

Since by the Cauchy-Schwarz inequality
\[ -8 \lambda \sum_{i=2}^{n} u_{x} u_{x, x_i} \varphi_{\lambda} \geq -C \lambda^{2} (\nabla u)^{2} \varphi_{\lambda} - \frac{1}{2} \sum_{i=2}^{n} u_{x}^{2} \varphi_{\lambda}, \]
then (5.28) implies that
\[ z_{3} \geq \frac{1}{2} \sum_{i,j=1}^{n} u_{x, x_j}^{2} \varphi_{\lambda} - C \lambda^{2} (\nabla u)^{2} \varphi_{\lambda} \]

Combining (5.24)-(5.29), we obtain
\[ \frac{1}{4 \lambda} (u_{t} - \Delta u)^{2} \varphi_{\lambda} \geq \frac{1}{8 \lambda} \left( u_{t}^{2} + \sum_{i,j=1}^{n} u_{x, x_j}^{2} \right) \varphi_{\lambda} - \frac{C \lambda}{2} (\nabla u)^{2} \varphi_{\lambda} \]
\[ + \text{div } U_{4} + \left( u_{x}^{2} \varphi_{\lambda} \right)_{t}, \]

Using (5.31), integrate (5.30) over \( Q_{T}^{\pm} \). Then sum up the resulting inequality with (5.23). Then we obtain the target estimate (4.2) of this theorem. \( \square \)

6. Proofs of Theorems 2 and 4. Lemma 1 follows immediately either from Lemma 1.10.3 of [4] or from Lemma 3.1 of [22].

Lemma 1. The following estimate holds for every function \( q \in L_{2} \left( Q_{T}^{\pm} \right) \) and for every \( \lambda \geq 1 \):
\[ \int_{Q_{T}^{\pm}} \left( \int_{0}^{t} q(x, \tau) \, d\tau \right)^{2} \varphi_{\lambda}(x, t) \, dx \, dt \leq \frac{1}{4 \lambda} \int_{Q_{T}^{\pm}} q^{2}(x, t) \varphi_{\lambda}(x, t) \, dx \, dt. \]
6.1. Proof of Theorem 2. Let \( w_1, w_2 \in B(R, p_0, p_1) \) be two arbitrary functions. Denote \( h = w_2 - w_1 \). Then \( w_2 = w_1 + h \) and also

\[ h \in B_0(2R). \]

First, we evaluate the expression \((K(w_1 + h))^2 - (K(w_1))^2\), where the nonlinear operator \( K \) is given in (3.9). We have

\[
(K(w_1 + h))^2 = \left[ h_t - Lh - 2\nabla h \int_0^t \nabla w_1(x, \tau) d\tau - 2\nabla w_1 \int_0^t \nabla h_1(x, \tau) d\tau - 2\nabla h \int_0^t \nabla h(x, \tau) d\tau + K(w_1) \right]^2
\]

\[
= \left( h_t - Lh - 2\nabla h \int_0^t \nabla w_1(x, \tau) d\tau - 2\nabla w_1 \int_0^t \nabla h_1(x, \tau) d\tau - 2\nabla h \int_0^t \nabla h(x, \tau) d\tau \right)^2
\]

\[+ 4K(w_1) \nabla h \int_0^t \nabla h(x, \tau) d\tau + (K(w_1))^2 \]

Let \( \text{Lin}(h) \) be the linear, with respect to \( h \), part of the above expression,

\[ \text{Lin}(h) = 2K(w_1) \left( h_t - Lh - 2\nabla h \int_0^t \nabla w_1(x, \tau) d\tau - 2\nabla w_1 \int_0^t \nabla h_1(x, \tau) d\tau \right). \]

Then

\[ (K(w_1 + h))^2 - (K(w_1))^2 = \text{Lin}(h) \]

\[+ \left( h_t - Lh - 2\nabla h \int_0^t \nabla w_1(x, \tau) d\tau - 2\nabla w_1 \int_0^t \nabla h_1(x, \tau) d\tau - 2\nabla h \int_0^t \nabla h(x, \tau) d\tau \right)^2 \]

\[ - 4K(w_1) \nabla h \int_0^t \nabla h(x, \tau) d\tau. \]

Using (3.13), (3.14), (6.1) and the Cauchy-Schwarz inequality, we obtain

\[ \left( h_t - Lh - 2\nabla h \int_0^t \nabla w_1(x, \tau) d\tau - 2\nabla w_1 \int_0^t \nabla h_1(x, \tau) d\tau - 2\nabla h \int_0^t \nabla h(x, \tau) d\tau \right)^2 \]

\[ - 4K(w_1) \nabla h \int_0^t \nabla h(x, \tau) d\tau \]
\[ \geq \frac{1}{2} (h_t - Lh)^2 - C_1 (\nabla h)^2 - C_1 \left( \int_0^t \nabla h(x, \tau) \, d\tau \right)^2. \]

Hence, (3.15) and (6.2)-(6.4) lead to

\[ J_{\lambda, \beta} (w_1 + h) - J_{\lambda, \beta} (w_1) - e^{-2\lambda B^2} \int_{Q_T^+} \text{Lin} (h) \varphi_\lambda d\mathbf{x} d\mathbf{t} + 2\beta \{ w, h \} \]

(6.5)

\[ \geq \frac{1}{2} e^{-2\lambda B^2} \int_{Q_T^+} (h_t - Lh)^2 \varphi_\lambda d\mathbf{x} d\mathbf{t} \]

\[ -C_1 e^{-2\lambda B^2} \int_{Q_T^+} \left[ \frac{1}{2} (\nabla h)^2 + \left( \int_0^t \nabla h(x, \tau) \, d\tau \right)^2 \right] \varphi_\lambda d\mathbf{x} d\mathbf{t} + \beta \| h \|^2_{H^{kn} (Q_T^+)}. \]

Here and below \( \{,\} \) is the scalar product in \( H^{kn} (Q_T^+) \).

Consider now the functional \( S (h) : H_0^kn (Q_T^+) \rightarrow \mathbb{R} \) defined as

(6.6)

\[ S (h) = e^{-2\lambda B^2} \int_{Q_T^+} \text{Lin} (h) \varphi_\lambda d\mathbf{x} d\mathbf{t} + 2\beta \{ w, h \}. \]

It is clear from (6.2) that \( S (h) \) is a bounded linear functional. Hence, by Riesz theorem there exists a function \( Z \in H_0^kn (Q_T^+) \) such that \( S (h) = \{ Z, h \}, \forall h \in H_0^kn (Q_T^+) \).

Furthermore, it follows from (6.3) that

\[ \lim_{\| h \|_{H^{kn} (Q_T^+)} \rightarrow 0} | J_{\lambda, \beta} (w_1 + h) - J_{\lambda, \beta} (w_1) - S (h) | = 0. \]

Hence, \( S (h) \) is the Frechét derivative of the functional \( J_{\lambda, \beta} (w) \) at the point \( w_1 \),

(6.7)

\[ S (h) = J'_{\lambda, \beta} (w_1) (h) = \{ Z, h \} = \{ J'_{\lambda, \beta} (w_1), h \}, \forall h \in H_0^kn (Q_T^+) \],

i.e. we can set \( Z = J'_{\lambda, \beta} (w_1) \). Note that the proof of the existence of the Frechét derivative on the set \( B (3R, p_0, p_1) \), as claimed in this theorem, is basically the same as the one above. Thus, (6.5)-(6.7) imply that

\[ J_{\lambda, \beta} (w_1 + h) - J_{\lambda, \beta} (w_1) - J'_{\lambda, \beta} (w_1) (h) \]

(6.8)

\[ \geq \frac{1}{2} e^{-2\lambda B^2} \int_{Q_T^+} (h_t - Lh)^2 \varphi_\lambda d\mathbf{x} d\mathbf{t} \]

\[ -C_1 e^{-2\lambda B^2} \int_{Q_T^+} \left[ \frac{1}{2} (\nabla h)^2 + \left( \int_0^t \nabla h(x, \tau) \, d\tau \right)^2 \right] \varphi_\lambda d\mathbf{x} d\mathbf{t} + \beta \| h \|^2_{H^{kn} (Q_T^+)}. \]
Applying Theorem 1 and Lemma 1, we obtain for all \( \lambda \geq \lambda_0 \geq 1 \)
\[
\frac{1}{2} e^{-2\lambda B^2} \int_{Q^+_T} (h_t - Lh)^2 \varphi_\lambda dx dt
\]

(6.9) \[-C_1 e^{-2\lambda B^2} \int_{Q^+_T} \left[ (\nabla h)^2 + \left( \int_0^t \nabla h (x, \tau) d\tau \right)^2 \right] \varphi_\lambda dx dt + \beta \|h\|_{H^k(\Omega)}^2 \]

\[\geq \frac{C}{\lambda} e^{-2\lambda B^2} \int_{Q^+_T} \left( h_t^2 + \sum_{i,j=1}^n h_{x_i x_j}^2 \right) \varphi_\lambda dx dt
\]

\[+ C\lambda e^{-2\lambda B^2} \int_{Q^+_T} \left[ (\nabla h)^2 + \lambda^2 h^2 \right] \varphi_\lambda dx dt - C_1 e^{-2\lambda B^2} \int_{Q^+_T} (\nabla h)^2 \varphi_\lambda dx dt
\]

\[-C e^{-2\lambda T^2} \int_{\Omega} \left[ (h_t^2 + (\nabla h)^2 + \lambda^2 h^2) (x, T) + (h_t^2 + (\nabla h)^2 + \lambda^2 h^2) (x, -T) \right] dx
\]

\[+ \beta \|h\|_{H^k(\Omega)}^2 (Q^+_T) . \]

Choose \( \lambda_1 \geq \lambda_0 \) so large that \( C\lambda \geq 2C_1 \) and also \( 2e^{-\lambda T^2} \geq C\lambda^2 e^{-2\lambda T^2} \), for all \( \lambda \geq \lambda_1 \).

Also, we keep in mind that by trace theorem
\[
\|u(x, \pm T)\|_{H^1(\Omega)}, \|u_t(x, \pm T)\|_{L^2(\Omega)} \leq C \|u\|_{H^k(\Omega)} , \forall u \in H^k (Q^+_T) . \]

Then, taking \( \beta \in [2e^{-\lambda T^2}, 1] \) and using (6.8) and (6.9), we obtain

(6.10) \[
J_{\lambda, \beta} (w_1 + h) - J_{\lambda, \beta} (w_1) - J'_{\lambda, \beta} (w_1) (h)
\]

\[\geq \frac{C_1}{\lambda} e^{-2\lambda B^2} \int_{Q^+_T} \left( h_t^2 + \sum_{i,j=1}^n h_{x_i x_j}^2 \right) \varphi_\lambda dx dt + C\lambda e^{-2\lambda B^2} \int_{Q^+_T} \left[ (\nabla h)^2 + \lambda^2 h^2 \right] \varphi_\lambda dx dt
\]

\[+ \beta \|h\|_{H^k(\Omega)}^2 (Q^+_T) , \forall h \in B_0 (2R), \forall \lambda \geq \lambda_1 , \]

also, see (6.1). Finally, since \( \varphi_\lambda (x, t) \geq \exp (-2\lambda (T^2 - A^2)) \) for \( x \in [A, B], t \in [-T, T] \), then the target estimate (4.5) follows immediately from (6.10). \( \square \)

6.2. Proof of Theorem 4. Since by (4.16) \( I_{\lambda, \beta} (W) = J_{\lambda, \beta} (W + G), W \in B_0 (2R) \) and also since \( W + G \in B(3R, p_0, p_1), \forall W \in B_0 (2R) \), then we take in Theorems 2 and 3 \( \lambda_1 = \lambda_1 (3R), \lambda (3R) \geq \lambda_1 (3R) \) meaning that we replace in (4.3) \( R \) with \( 3R \). Denote \( w_1 = W_1 + G, w_2 = W_2 + G \). Then \( w_1, w_2 \in B(3R, p_0, p_1) \). The rest of the proof follows immediately from Theorems 2 and 3. \( \square \)
7. Proof of Theorem 5. Recall that by (4.14) $w^* - G^* = W^* \in B_0 (2R - \delta)$. Hence, by (3.14), (4.6), (4.7) and (4.11) $W^* + G \in B (3R, p_0, p_1)$. We temporarily denote $I_{\lambda, \beta} (W, f_0)$ the functional $I_{\lambda, \beta} (W) = J_{\lambda, \beta} (W + G)$, in which we emphasize the presence of the vector function $f_0 = \nabla \ln f_0$ in the operator $K (w)$. We also temporarily denote this operator as $K (w, f_0) = K (W + G, f_0)$ in (3.9) and (3.15). Similarly, we also temporally denote $J_{\lambda, \beta} (W + G, f_0) := I_{\lambda, \beta} (W, f_0)$. Let

\[
I_{\lambda, \beta}^0 (W, f_0) = J_{\lambda, \beta}^0 (W + G, f_0) = e^{-2AB^2} \int_{Q_T^+} (K (W + G, f_0))^2 \varphi_A dx dt.
\]

By (3.9) $K (W^* + G^*, f_0^*) = 0$. Hence,

\[
I_{\lambda, \beta}^0 (W^*, f_0^*) = 0.
\]

Hence, it follows from (3.9), (4.7), (4.8), (4.10), (4.12), (7.1) and (7.2) that

\[
I_{\lambda, \beta} (W^*, f_0) = J_{\lambda, \beta} (W^* + G, f_0) = J_{\lambda, \beta} (W^* + G^* + (G - G^*), f_0 + (f_0 - f_0^*))
= J_{\lambda, \beta}^0 (W^* + G^*, f_0^*) + \rho \| W^* + G \|_{H^{k_n} (Q_T^+)} = \rho \| W^* + G \|_{H^{k_n} (Q_T^+)}^2
\]

where $| \rho | \leq C_1 \delta^2$. Thus,

\[
I_{\lambda, \beta} (W^*, f_0) \leq C_1 \delta^2 + \beta \| W^* + G \|_{H^{k_n} (Q_T^+)}^2.
\]

By (4.7) and (4.9)

\[
\| W^* + G \|_{H^{k_n} (Q_T^+)} \leq \| (W^* + G^*) + (G - G^*) \|_{H^{k_n} (Q_T^+)} \leq \| w^* \|_{H^{k_n} (Q_T^+)} + \delta < R.
\]

Hence,

\[
W^* + G \in B (R, p_0, p_1).
\]

Let $W_{\min, \lambda (3R), \beta (3R)} \in \overline{B_0 (2R)}$ be the minimizer of the functional $I_{\lambda, \beta} (W, f_0)$, the existence and uniqueness of which on the set $\overline{B_0 (2R)}$ is guaranteed by Theorem 3. We will choose the dependencies on $\delta$ of parameters $\lambda$ and $\beta$ later in this proof. Thus, we can apply (4.17) now as

\[
I_{\lambda (3R), \beta (3R)} (W^*, f_0) - I_{\lambda (3R), \beta (3R)} (W_{\min, \lambda (3R), \beta (3R)}, f_0)

= -I_{\lambda (3R), \beta (3R)} (W_{\min, \lambda (3R), \beta (3R)}, f_0) (W^* - W_{\min, \lambda (3R), \beta (3R)})
\]

\[
\geq C_1 \exp (-3\lambda (3R) (\gamma^2 T^2 + B^2 - A^2)) \| W^* - W_{\min, \lambda (3R), \beta (3R)} \|_{H^{k_n} (Q_T^+)}^2

+ \frac{\beta (3R)}{2} \| W^* - W_{\min, \lambda (3R), \beta (3R)} \|_{H^{k_n} (Q_T^+)}^2.
\]

By (4.18) $-I_{\lambda (3R), \beta (3R)} (W_{\min, \lambda (3R), \beta (3R)}, f_0) (W^* - W_{\min, \lambda (3R), \beta (3R)}) \leq 0$. Recall that

\[
\eta_1 = \gamma^2 T^2 + B^2 - A^2.\]

Hence, (7.3)-(7.5) imply that

\[
\| W^* - W_{\min, \lambda (3R), \beta (3R)} \|_{H^{k_n} (Q_T^+)}^2 \leq C_1 (\delta^2 + \beta) \exp (3\lambda (3R) \eta_1).
\]
We now specify dependencies of $\lambda$ and $\beta$ on the noise level $\delta$. Choose $\lambda = \lambda (\delta, 3 R)$ such that
\[
\exp (3 \lambda (\delta, 3 R) \eta_1) = \exp (3 \lambda (\delta, 3 R) (\gamma^2 T^2 + B^2 - A^2)) = \frac{1}{\delta}.
\]
Then $\lambda (\delta, 3 R) = \ln (\delta^{-1/(3 \eta_1)})$ and
\[(7.7) \quad \delta^2 \exp (3 \lambda (\delta, 3 R) \eta_1) = \delta.
\]
Since $\delta \in (0, \delta_0)$ and since $\ln (\delta^{-1/(3 \eta_1)}) \geq \lambda_1 (3 R)$, then $\lambda (\delta, 3 R) \geq \lambda_1 (3 R)$. Next, by Theorem 2, we can take $\beta = 2 e^{-\lambda (3 R) T^2}$. Hence, in (7.6)
\[(7.8) \quad \beta \exp (3 \lambda (3 R) \eta_1) = \delta^{\eta_2/\eta_1}, \eta_2 = (1 - 3 \gamma^2) T^2 - 3 (B^2 - A^2) > 0.
\]
Recalling that $2 \rho = \min (1, \eta_2/\eta_1)$ and using (7.6)-(7.8), we obtain
\[
\| W^* - W_{\min, \lambda(3 R), \beta(3 R)} \|_{H^{2,1}(Q^+_{\gamma/T})} \leq C_1 \delta^\rho.
\]
Hence,
\[
\| w^* - w_{\min, \lambda(3 R), \beta(3 R)} \|_{H^{2,1}(Q^+_{\gamma/T})} \leq \| W^* - W_{\min, \lambda(3 R), \beta(3 R)} \|_{H^{2,1}(Q^+_{\gamma/T})}
\]
\[(7.9) \quad + \| G^* - G \|_{H^{2,1}(Q^+_{\gamma/T})} \leq C_2 \delta^\rho + \delta \leq (C_2 + 1) \delta^\rho,
\]
which proves (4.19). Finally, since (4.19) holds, then (4.20) follows immediately from (3.16) and the rest of the discussion in the last paragraph of section 3. \(\square\)

8. Proof of Theorem 7. Recall that Theorem 6 is valid: see item 2 in Remarks 3 (section 4). By the triangle inequality, (4.7), (4.19) and (4.22)
\[
\| w^* - w_n \|_{H^{2,1}(Q^+_{\gamma/T})} = \| w^* - w_{\min, \lambda(3 R), \beta(3 R)} + (w_{\min, \lambda(3 R), \beta(3 R)} - w_n) \|_{H^{2,1}(Q^+_{\gamma/T})}
\leq C_2 \delta^\rho + \| w_{\min, \lambda(3 R), \beta(3 R)} - w_n \|_{H^{2,1}(Q^+_{\gamma/T})} \leq C_2 \delta^\rho + \| w_{\min, \lambda(3 R), \beta(3 R)} - w_n \|_{H^{2,1}(Q^+_{\gamma/T})}
\]
\[= C_2 \delta^\rho + \| W_{\min, \lambda(3 R), \beta} - W_n \|_{H^{k_n}(Q^+_{\gamma/T})}
\leq C_2 \delta^\rho + \theta^n \| W_{\min, \lambda(3 R), \beta} - W_0 \|_{H^{k_n}(Q^+_{\gamma/T})} = C_2 \delta^\rho + \theta^n \| w_{\min, \lambda(3 R), \beta} - w_0 \|_{H^{k_n}(Q^+_{\gamma/T})},
\]
which proves (4.23). Estimate (4.24) follows immediately from (4.23) and the discussion in the last paragraph of section 3. \(\square\)

9. Numerical Testing. In the following tests, we set the domain $\Omega = (1, 2) \times (1, 2)$ and also
\[Lu = \Delta u - c(x) u.
\]
To solve the inverse problem, we should first computationally simulate the data (2.7), (2.8) via the numerical solution of the forward problem (2.4). To solve problem (2.4), computationally, we have used the standard finite difference method. The spatial mesh size is $1/640 \times 1/640$ while the temporal one $T/512$. For the forward problem, we use the implicit scheme to compute the data needed for the inverse problem.
In computations of the inverse problem, the spatial mesh size is 1/16 × 1/16 and the temporal one T/16. When minimizing the functional \( J_{\lambda,\beta}(w) \) in the discrete sense, we formulate the right hand side of (3.15) via finite differences and minimize with respect to the values of the function \( w \) at grid points. To minimize the discretized functional, we use Matlab’s built-in function \texttt{fminunc} with its option of \texttt{quasi-newton algorithm}. This procedure calculates the gradient \( \nabla J_{\lambda,\beta}(w) \) automatically and iterations stop when the condition \( |\nabla J_{\lambda,\beta}(w)| < 1 \times 10^{-2} \) holds. Note that even though our theory requires the application of the gradient projection method, we have established numerically that we can avoid the use of the projection operator \( P_B \) and to use just the conjugate gradient method. In fact, the use of the operator \( P_B \) would complicate the matter. The same observation took place in all our works on the convexification, which contain numerical studies [16, 26, 27, 28, 29]. Also, we have minimized the functional \( J_{\lambda,\beta}(w) \) rather than \( I_{\lambda,\beta}(W) \) and it worked quite well.

As to (3.16), we have numerically discovered that rather than taking an average over \( t \in [-\gamma T, \gamma T] \), better to use (3.5) at \( \{t = t_0\} \). In numerical tests below, we took

\[
\lambda = 1, \quad k_n = 3, \quad \beta = 0.01.
\]

In the process of the minimization of the functional \( J_{\lambda,\beta}(w) \), the starting point of iterations is always chosen to be the null function of value zero everywhere.

In the following three tests, we show the results of the recovery of the coefficients \( c(x) \) with sophisticated structures. We choose the tested coefficients \( c(x) \) having the shapes of the letters ‘A’ and ‘Ω’. We measure \( g_1(x_1, x_2, t) \) on 16 × 32 detectors uniformly distributed on the rectangle \( T/2 \) and ‘measure’ the function \( f_0(x_1, x_2, t_0) \) on 16 × 16 detectors uniformly distributed on the square \( (1, 2) \times (1, 2) \times \{t = t_0\} \). As initial and Dirichlet boundary conditions for the data simulations in (2.5), (2.6), we took

\[
\begin{align*}
\lambda &= 1, \quad k_n = 3, \quad \beta = 0.01. \\
\text{In the process of the minimization of the functional } J_{\lambda,\beta}(w), \text{ the starting point of iterations is always chosen to be the null function of value zero everywhere.} \\
\text{In the following three tests, we show the results of the recovery of the coefficients } c(x) \text{ with sophisticated structures. We choose the tested coefficients } c(x) \text{ having the shapes of the letters ‘A’ and ‘Ω’. We measure } g_1(x_1, x_2, t) \text{ on } 16 \times 32 \text{ detectors uniformly distributed on the rectangle } T/2 \text{ and ‘measure’ the function } f_0(x_1, x_2, t_0) \text{ on } 16 \times 16 \text{ detectors uniformly distributed on the square } (1, 2) \times (1, 2) \times \{t = t_0\}. \text{ As initial and Dirichlet boundary conditions for the data simulations in (2.5), (2.6), we took}
\end{align*}
\]

\[
u(x, -T) = 1 + \sin(\pi(x_1 - 1)) \sin(\pi(x_2 - 1)) \quad \text{and} \quad u |_{S^+_{T/2}} = 1.
\]

We allow in our tests the function \( c(x) \) to be both positive and negative. Indeed, we have imposed the positivity condition (2.3) only to ensure that the function \( u(x, t) \neq 0 \) in \( \Omega_T \). However, we have not observed any zeros of this function in our numerical studies.

**Test 1.** First, we test the reconstruction by our method of the coefficients \( c(x) \) with the shapes of letters ‘A’ and ‘Ω’. In this test, we measure the data at time \( \{t_0 = 0\} \) for the cases \( T = 1 \) and \( T = 0.1 \). The numerical results are shown in Figure 9.1.

**Test 2.** In this test, we set \( T = 0.1 \). We show the results in the case when the data are measured at a time \( \{t_0\} \) which is close to the initial time \( \{t = -T = -0.1\} \). We take \( t_0 = -T + \epsilon \) with \( \epsilon = 0.02 \) and \( \epsilon = 0.01 \). We test the reconstruction by our method of the coefficients \( c(x) \) with the shapes of the letters ‘A’ and ‘Ω’. The numerical results are shown in Figure 9.2. In this test, we demonstrate the results when one measures the data at some time close to the initial time. It is numerically shown that the closer \( t_0 \) is to the initial time \( t = -T \), the worse the result is.

**Test 3.** We now want to see how the random noise in the data influences our reconstruction. We add 5% relative random noise to each detector on \( \Gamma^+_T \) as well as on \( (1, 2) \times (1, 2) \times \{t = 0\} \), i.e. we work now with the noisy data,

\[
u^{\text{noise}}_x |_{\Gamma^+_T} = g_1(x, t) + \sigma \xi_x(t) g_1(x, t),
\]

where \( \sigma \) is the magnitude of the noise.
CONVEXIFICATION OF A PARABOLIC INVERSE PROBLEM

(a) $c(x)$ with the shape of the letter ‘A’
(b) Recovered $c(x)$ for $T = 1$
(c) Recovered $c(x)$ for $T = 0.1$

(d) $c(x)$ with the shape of the letter ‘Ω’
(e) Recovered $c(x)$ for $T = 1$
(f) Recovered $c(x)$ for $T = 0.1$

Fig. 9.1. Results of Test 1. Here $t_0 = 0$ in (2.8). (a) The coefficient $c(x)$ with the shape of the letter ‘A’. (d) The coefficient $c(x)$ with the shape of the letter ‘Ω’. (b) and (c) are the recovered $c(x)$ for $T = 1$ and $T = 0.1$ respectively for coefficient with the shape of the letter ‘A’. (e) and (f) are the recovered $c(x)$ for $T = 1$ and $T = 0.1$ respectively for coefficient with the shape of the letter ‘Ω’.

(9.3) $u^{\text{noise}}(x, t_0) = f_0(x) + \sigma \xi_x f_0(x)$.

Here $\sigma = 5\%$ is the noise level, $\xi_x$ and $\xi_t$ are independent normally distributed random variables. To preprocess the noisy data, we use the thin plate spline smoother developed in [9]. The algorithm proposed in [9] provides a good approximation to the true function without knowing neither the noise level nor any other a priori information of the true function to be approximated. Then the cubic B-splines are employed to approximate the first and second order derivatives of the noisy data. In this test, we ‘measure’ $g_1(x_1, x_2, t)$ on $16 \times 32$ detectors uniformly distributed on the plane $\Gamma_T$ and also ‘measure’ $f_0(x_1, x_2, t_0)$ on $160 \times 160$ detectors uniformly distributed on the plane $(1, 2) \times (1, 2) \times \{t = 0\}$. We now set $T = 1$, in (2.8) $t_0 = 0$, and the noise is added to the data as in (9.2), (9.3). We test the reconstruction by our method of the coefficients with the shape of the letters ‘A’ and ‘Ω’. The numerical results are shown in Figure 9.3. We see that our method works still very well in the mild noisy case.

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Fig. 9.2. Results of Test 2. Here $T = 0.1$ and in (2.8) $t_0 = 0$. We show the results in the case when the data are measured at a time $t_0$, which is close to the initial time moment $\{t = -T = -0.1\}$. (a) and (b) are the recovered $c(x)$ for $\epsilon = 0.02$ and $\epsilon = 0.01$ respectively for coefficient with the shape of the letter 'A'. (c) and (d) are the recovered $c(x)$ for $\epsilon = 0.02$ and $\epsilon = 0.01$ respectively for coefficient with the shape of the letter 'Ω'. Comparison with Figure 1 shows that the quality of images is better if $t_0$ is not too close to the initial time moment $\{t = -T\}$.

Fig. 9.3. Results of Test 3. Here $T = 1$ and $t_0 = 0$ in (2.8). (a) and (b) are the recovered coefficients $c(x)$ with the shapes of the letters 'A' and 'Ω' respectively. The measured data contain 5% relative random noise.
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