Reverse isoperimetric inequalities for parallel sets

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Abstract

We consider the family of $r$-parallel sets in $\mathbb{R}^d$, that is sets of the form $A_r = A + rB_2^n$, where $B_2^n$ is the unit Euclidean ball and $A$ is arbitrary Borel set. We show that the ratio between the upper surface area measure of an $r$-parallel set and its volume is upper bounded by $d/r$. Equality is achieved for $A$ being a single point.

As a consequence of our main result we show that the Gaussian upper surface area measure of an $r$-parallel set is upper bounded by $18d\max(\sqrt{d}, r^{-1})$. Moreover, we observe that there exists a 1-parallel set with Gaussian surface area measure at least $0.28 \cdot d^{1/4}$.

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1 Introduction

For sets $A, B$ in $\mathbb{R}^d$ we define their Minkowski sum $A + B = \{a + b : a \in A, b \in B\}$. Suppose $K$ is some compact convex set and let $r > 0$. A set of the form $A_{r,K} = A + rK$ is called $(r, K)$-parallel. If $K = B_2^n$ is a unit Euclidean ball then $A_{r,K}$ will simply be called $r$-parallel and will be denoted by $A_r$. For a Borel set $A$ we shall write

$$|\partial A|_+ = \limsup_{\varepsilon \to 0^+} \frac{|A_\varepsilon| - |A|}{\varepsilon}, \quad |\partial A_r|_+ = \limsup_{\varepsilon \to 0^+} \frac{|A_{r,\varepsilon}| - |A_r|}{\varepsilon},$$

where $|\cdot|$ stands for the Lebesgue measure. The quantity $|\partial A|_+$ is called the upper surface area measure of $A$.

In [2] Jog considered reverse isoperimetric inequalities for parallel sets. He proved that for any compact set $A$ in $\mathbb{R}^d$ one has $|\partial A_r| \leq d 2^{2d-1} |A_r|$. In this note we prove the following sharp result.

Theorem 1. Let $A$ be a Borel set and let $K$ be compact and convex. Then

$$|\partial K A_{r,K}|_+ \leq \frac{d}{r} \cdot |A_{r,K}|,$$

which is tight for $A = \{0\}$. In particular $|\partial A_r|_+ \leq \frac{d}{r} \cdot |A_r|$, .

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In [2] also the Gaussian case was treated. The upper Gaussian surface area of a measurable set is defined as
\[
\gamma_d^+ (\partial A) = \limsup_{\varepsilon \to 0^+} \frac{\gamma_d (A_{\varepsilon}) - \gamma_d (A)}{\varepsilon},
\]
where \(\gamma_d\) stands for the standard Gaussian measure, that is measure with density \((2\pi)^{d/2} e^{-|x|^2/2}\). Jog proved the inequality \(\gamma_d^+ (\partial A_r) \leq 2^{2d-1} d^2 3^d \max (1, r^{-1})\). We shall prove this bound with a better dependence on the dimension.

**Theorem 2.** For any Borel set \(A\) we have
\[
\gamma_d^+ (\partial A_r) \leq 18d \max (\sqrt{d}, r^{-1}).
\]

We mention that it is not possible to remove the dimension dependence in the above estimate: there exists a 1-parallel set whose Gaussian surface area is of order \(d^{1/4}\). This follows from a simple observation: every set of the form \(K^c\) (complement of \(K\)), where \(K\) is open and convex, is \(r\)-parallel for every \(r > 0\). It is easy to verify that any closed halfspace \(H\) is \(r\)-parallel for every \(r > 0\) (in fact \(H = A_r\) for an appropriate halfspace \(A\)). Since every open convex set \(K\) is of the form \(K = \bigcap_{i \in I} H_i\) for some family of open halfspaces \((H_i)_{i \in I}\), we have
\[
K^c = \bigcup_{i \in I} H_i^c = \bigcup_{i \in I} (A_i)_r = \left( \bigcup_{i \in I} A_i \right)_r,
\]
where sets \(A_i\) satisfy \((A_i)_r = H_i^c\) (note that \(H_i^c\) are closed halfspaces). According to the result of Nazarov from [3], there exists a convex set \(K\) such that \(\gamma_d^+ (\partial K) \geq 0.28 \cdot d^{1/4}\).

2 Proofs

We first prove Theorem 1.

**Proof of Theorem 1** According to the result of Fradelizi and Marsiglietti from [1] (Proposition 2.1), for any compact set \(A\) in \(\mathbb{R}^d\) and any compact convex set \(K\) in \(\mathbb{R}^d\) the function
\[
(s, t) \mapsto sA + tK
\]
is non-decreasing on \(\mathbb{R}_+ \times \mathbb{R}_+\) in each coordinate. This is a consequence of the result of Stachó from [4]. Since for \(r > 0\) we have \(\frac{|A + rK|}{r^d} = |r^{-1} A + K|\), the left hand side is non-increasing. Thus, for any \(\varepsilon > 0\) we have
\[
0 \geq \frac{1}{\varepsilon} \left( \frac{|A + (r + \varepsilon)K| - |A + rK|}{(r + \varepsilon)^d} \right) = \frac{1}{(r + \varepsilon)^d} \cdot \frac{|A + (r + \varepsilon)K| - |A + rK|}{\varepsilon} + |A + rK| \cdot \frac{1}{(r + \varepsilon)^d} - \frac{1}{r^d}.
\]
Taking \(\varepsilon \to 0^+\) we arrive at
\[
0 \geq \frac{|\partial K A_r,K|}{r^d} - \frac{d}{r^{d+1}} |A_r,K|.
\]
The proof is completed. \(\square\)
Theorem 1 implies Theorem 2. For the proof we follow the strategy developed in [2].

Proof of Theorem 2. Through the proof c is a universal constant independent of the dimension, whose value may change from one line to the next. Note that for a measurable set \( A \) we have

\[
\gamma_d(A) = (2\pi)^{-\frac{d}{2}} \int_A e^{-|x|^2/2} dx = (2\pi)^{-\frac{d}{2}} \int \int_{|x|} \infty te^{-t^2/2} dt dx
\]

\[
= (2\pi)^{-\frac{d}{2}} \int_{|x|} \infty \int_{\mathbb{R}^d} te^{-t^2/2} 1_{|x| \leq t} 1_A dx dt = (2\pi)^{-\frac{d}{2}} \int_0^\infty te^{-t^2/2} |A \cap tB_2^d| dt.
\]

Let us fix \( \varepsilon_0 > 0 \) and take \( 0 < \varepsilon < \varepsilon_0 \). Let \( A^\varepsilon = A \cap (t + r + \varepsilon_0)B_2^d \). We have

\[
\gamma_d(A_{r+\varepsilon}) - \gamma_d(A_r) = (2\pi)^{-\frac{d}{2}} \int_0^\infty te^{-t^2/2} |(A_{r+\varepsilon} \setminus A_r) \cap tB_2^d| dt
\]

\[
= (2\pi)^{-\frac{d}{2}} \int_0^\infty te^{-t^2/2} |((A^\varepsilon)_{r+\varepsilon} \setminus (A^\varepsilon)_r) \cap tB_2^d| dt
\]

\[
\leq (2\pi)^{-\frac{d}{2}} \int_0^\infty te^{-t^2/2} |(A^\varepsilon)_{r+\varepsilon} \setminus (A^\varepsilon)_r| dt.
\]

Dividing by \( \varepsilon \), taking the limit \( \varepsilon \to 0^+ \) and applying Theorem 1 gives

\[
\gamma_d^+(\partial A_r) \leq (2\pi)^{-\frac{d}{2}} \int_0^\infty te^{-t^2/2} |\partial(A^\varepsilon)_r| + dt \leq (2\pi)^{-\frac{d}{2}} \int_0^\infty te^{-t^2/2} |(A^\varepsilon)_r| dt
\]

\[
\leq (2\pi)^{-\frac{d}{2}} \cdot \frac{d}{r} \int_0^\infty te^{-t^2/2} |(t + 2r + \varepsilon_0)B_2^d| dt
\]

\[
= \frac{|B_2^d|}{(2\pi)^{\frac{d}{2}}} \cdot \frac{d}{r} \int_0^\infty te^{-t^2/2} (t + 2r + \varepsilon_0)^d dt.
\]

Taking the limit \( \varepsilon_0 \to 0^+ \) yields

\[
\gamma_d^+(\partial A_r) \leq \frac{|B_2^d|}{(2\pi)^{\frac{d}{2}}} \cdot \frac{d}{r} \int_0^\infty te^{-t^2/2} (t + 2r)^d dt.
\]

For \( p > -1 \) we have

\[
m_p := \int_0^\infty e^{-t^2/2} t^p dt = 2^{\frac{p+1}{2}} \Gamma\left(\frac{p+1}{2}\right).
\]

We also have \( |B_2^d| = \frac{\pi^{d/2}}{\Gamma(d/2+1)} \). As a consequence \( m_{d+1} = 2^{\frac{d}{2}} \Gamma \left(\frac{d}{2} + 1\right) = (2\pi)^{\frac{d}{2}} |B_2^d| \) and thus

\[
\gamma_d^+(\partial A_r) \leq \frac{d}{r} \cdot \frac{1}{m_{d+1}} \cdot \int_0^\infty te^{-t^2/2} (t + 2r)^d dt = \frac{d}{r} \cdot \frac{1}{m_{d+1}} \cdot \sum_{i=0}^d \binom{d}{i} m_{i+1} (2r)^{d-i}
\]

\[
= \frac{d}{r} \cdot \sum_{i=0}^d \binom{d}{i} \frac{\Gamma\left(i+1\right)}{\Gamma\left(d/2 + 1\right)} 2^{i-d} (2r)^{d-i}.
\]
Using the standard bounds
\[ \sqrt{2\pi xx^x e^{-x}} \leq \Gamma(x + 1) \leq 2\sqrt{2\pi xx^x e^{-x}}, \quad x \in [1, \infty) \cup \{1/2\}, \]
and \( \frac{d^d}{(d-i)!} \) we get
\[
\sum_{i=0}^{d} \frac{d^d}{(d-i)!} \frac{\Gamma\left(\frac{i}{2} + 1\right)}{\Gamma\left(\frac{d}{2} + 1\right)} 2^{i/2} \leq 2 \sum_{i=0}^{d} \frac{d^d}{(d-i)!} \frac{(i/2)^{i/2} e^{-i/2}}{(d/2)^d e^{-d/2} (\sqrt{2r})^{d-i}} = 2 \sum_{i=0}^{d} \frac{i^{i/2}}{(d-i)!} \frac{\Gamma\left(\frac{i}{2}\right)}{d^{d/2}} (2\sqrt{er})^{d-i}.
\]
Let us now assume that \( r \leq r_* := \frac{d^{-1/2}}{2\sqrt{e}} \). Then
\[
2 \sum_{i=0}^{d} \frac{d^d}{(d-i)!} \frac{i^{i/2}}{d^{d/2}} (2\sqrt{er})^{d-i} \leq 2 \sum_{i=0}^{d} \frac{d^d}{(d-i)!} \frac{i^{i/2}}{d^{d/2}} \frac{1}{d^{d/2}} = 2 \sum_{i=0}^{d} \frac{1}{(d-i)!} \left( \frac{i}{d} \right)^{i/2} \leq 2 \sum_{i=0}^{d} \frac{1}{(d-i)!} \leq 2e.
\]
Therefore, for \( r \leq r_* \) we get \( \gamma^+_d(\partial A_r) \leq \frac{2de}{r} \).

If \( r > r_* \) one can use the bound for \( r = r_* \), since every \( r \)-parallel set is \( r' \)-parallel for every \( r' < r \). Thus in this case we get \( \gamma^+_d(\partial A_r) \leq 4e^{3/2} d^{3/2} < 18d^{3/2} \). We proved that always \( \gamma^+_d(\partial A_r) \leq \max(18d^{3/2}, \frac{2de}{r^2}) \leq 18d \max(\sqrt{d}, \frac{1}{r^2}) \).

\begin{thebibliography}{9}

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\end{thebibliography}