ON $K$-THEORY AUTOMORPHISMS RELATED TO BUNDLES OF FINITE ORDER

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Abstract. In the present paper we describe the action of (not necessarily line) bundles of finite order on the $K$-functor in terms of classifying spaces. This description might provide with an approach for more general twistings in $K$-theory than ones related to the action of the Picard group.

Introduction

The complex $K$-theory is a generalized cohomology theory represented by the $\Omega$-spectrum $\{K_n\}_{n \geq 0}$, where $K_n = \mathbb{Z} \times BU$, if $n$ is even and $K_n = U$ if $n$ is odd. $K_0 = \mathbb{Z} \times BU$ is an $E_\infty$-ring space, and the corresponding space of units $K_\otimes$ (which is an infinite loop space) is $\mathbb{Z}/2\mathbb{Z} \times BU_\otimes$, where $BU_\otimes$ denotes the space $BU$ with the $H$-space structure induced by the tensor product of virtual bundles of virtual dimension 1. Twistings of the $K$-theory over a compact space $X$ are classified by homotopy classes of maps $X \to B(\mathbb{Z}/2\mathbb{Z} \times BU_\otimes) \simeq K(\mathbb{Z}/2\mathbb{Z}, 1) \times BB\otimes$ (where $B$ denotes the functor of classifying space). The theorem that $BU_\otimes$ is an infinite loop space was proved by G. Segal. Moreover, the spectrum $BU_\otimes$ can be decomposed as follows: $BU_\otimes = K(\mathbb{Z}, 2) \times BSU_\otimes$. This implies that the twistings in the $K$-theory can be classified by homotopy classes of maps $X \to K(\mathbb{Z}/2\mathbb{Z}, 1) \times KSU_\otimes$, $[X, KSU_\otimes] = bsu_1(X)$, where $\{bsu_3\}_n$ is the generalized cohomology theory corresponding to the infinite loop space $BSU_\otimes$.

The twisted $K$-theory corresponding to the twistings coming from $H^1(X, \mathbb{Z}/2\mathbb{Z}) \times H^3(X, \mathbb{Z}) \times [X, BSU_\otimes]$, $[X, BSU_\otimes] = bsu_1(X)$, where $\{bsu_3\}_n$ is the generalized cohomology theory corresponding to the infinite loop space $BSU_\otimes$ given by the infinite matrix grassmannian $Gr^\infty$ (see also subsection 4.5 below).

A brief outline of this paper is as follows. In section 1 we recall the well-known result that the action of the projective unitary group of the separable Hilbert space $PU(\mathcal{H})$ on the space of Fredholm operators $Fred(\mathcal{H})$ (which is the representing space of $K$-theory) by conjugation corresponds to the action of the Picard group $Pic(X)$ on $K(X)$ by group automorphisms (Theorem 1). The key result of section 2 is Theorem 7 which is in some sense a counterpart of Theorem 1. Roughly speaking, it asserts that in terms of representing space $Fred(\mathcal{H})$ the tensor multiplication of $K$-functor by (not necessarily line) bundles of finite order $k$ can be described by some maps $\gamma_{klm, l\rightarrow m}^{k}: Fr_{klm, l\rightarrow m} \times Fred(\mathcal{H}) \rightarrow Fred(\mathcal{H})$, where $Fr_{klm, l\rightarrow m}$ are the spaces parametrizing unital $\ast$-homomorphisms of matrix algebras $M_{klm}(\mathbb{C}) \rightarrow M_{klm}(\mathbb{C})$. Then by arranging these maps...
we should construct an action of the $H$-space $\lim_{n} Fr_{k,l,m}$ on $\text{Fred}(\mathcal{H})$. We also have shown that $\lim_{n} Fr_{k,l,m}$ is a well-pointed grouplike topological monoid and therefore there exists the classifying space $B \lim_{n} Fr_{k,l,m}$ (see subsection 3.2). More precisely, we consider the direct limit of matrix algebras $M_{k,l,m}(\mathbb{C}) = \lim_{m} M_{k,l,m}(\mathbb{C})$ and the monoid of its unital endomorphisms. We fix an increasing filtration $A_{k,l,m} \subset A_{k,l,m+1} \subset \ldots$, $A_{k,l,m} = M_{k,l,m}(\mathbb{C})$ such that $A_{k,l,m+1} = M_{l}(A_{k,l,m})$. Then we consider endomorphisms of $M_{k,l,m}(\mathbb{C})$ that are induced by unital homomorphisms of the form $h: A_{k,l,m} \rightarrow A_{k,l,n}$ (for some $m, n$), i.e. have the form $M_{l}(h)$. Such endomorphisms form the above mentioned topological monoid which is homotopy equivalent to the direct limit $Fr_{k,l,m} := \lim_{m,n} Fr_{k,l,m,n}$, which is not contractible provided $(k, l) = 1$. Moreover, the automorphism subgroup in the monoid (corresponding to $n = 0$) is $\lim_{m} PU(k,l,m)$. Furthermore, this monoid naturally acts on the space of Fredholm operators and this action corresponds to the tensor multiplication of the $K$-functor by bundles of order $k$. In subsection 3.3 we also sketch the idea of the definition of the corresponding version of the twisted $K$-theory.

Roughly speaking, the “usual” (Abelian) twistings of order $k$ correspond to the group of automorphisms while the nonabelian ones correspond to the monoid of endomorphisms of $M_{k,l}(\mathbb{C})$. Note that these endomorphisms act on the localization of the space of Fredholm operators over $l$ by homotopy auto-equivalences, i.e. they are invertible in the sense of homotopy.

Although some technical difficulties remain we hope that this approach will be useful in order to define a general version of the twisted $K$-theory.

1. $K$-THEORY AUTOMORPHISMS RELATED TO LINE BUNDLES

In this section we describe well-known results about the action of $Pic(X)$ on the group $K(X)$. We also consider the special case of the subgroup of line bundles of finite order.

Let $X$ be a compact space, $Pic(X)$ its Picard group consisting of isomorphism classes of line bundles with respect to the tensor product. The Picard group is represented by the $H$-space $BU(1) \cong \mathbb{C}P^{\infty} \cong K(\mathbb{Z}, 2)$ whose multiplication is given by the tensor product of line bundles or (in the appearance of the Eilenberg-MacLane space) by addition of two-dimensional integer cohomology classes. In particular, the first Chern class $c_1$ defines the isomorphism $c_1: Pic(X) \rightarrow H^{2}(X, \mathbb{Z})$. The group $Pic(X)$ is a subgroup of the multiplicative group of the ring $K(X)$ and therefore it acts on $K(X)$ by group automorphisms. This action is functorial on $X$ and therefore it can be described in terms of classifying spaces (see Theorem 1).

As a representing space for the $K$-theory we take $\text{Fred}(\mathcal{H})$, the space of Fredholm operators in the separable Hilbert space $\mathcal{H}$. It is known [2] that for a compact space $X$ the action of $Pic(X)$ on $K(X)$ is induced by the conjugate action

$$\gamma: PU(\mathcal{H}) \times \text{Fred}(\mathcal{H}) \rightarrow \text{Fred}(\mathcal{H}), \gamma(g, T) = gTg^{-1}$$

of $PU(\mathcal{H})$ on $\text{Fred}(\mathcal{H})$. More precisely, there is the following theorem (recall that $PU(\mathcal{H}) \cong \mathbb{C}P^{\infty} \cong K(\mathbb{Z}, 2)$).
Theorem 1. If \( f_\xi : X \to \text{Fred}(\mathcal{H}) \) and \( \varphi_\zeta : X \to \text{PU}(\mathcal{H}) \) represent \( \xi \in K(X) \) and \( \zeta \in \text{Pic}(X) \) respectively, then the composite map

\[
\begin{align*}
&X \xrightarrow{\text{diag}} X \times X \xrightarrow{\varphi_\zeta \times f_\xi} \text{PU}(\mathcal{H}) \times \text{Fred}(\mathcal{H}) \xrightarrow{\gamma} \text{Fred}(\mathcal{H})
\end{align*}
\]

represents \( \zeta \otimes \xi \in K(X) \).

Proof see [2]. \( \square \)

It is essential for the theorem that the group \( \text{PU}(\mathcal{H}) \), on the one hand having the homotopy type of \( \mathbb{C}P^\infty \) is the base of the universal \( U(1) \)-bundle (which is related to the exact sequence of groups \( U(1) \to U(\mathcal{H}) \to \text{PU}(\mathcal{H}) \), because \( U(\mathcal{H}) \) is contractible in the considered norm topology), on the other hand being a group acts in the appropriate way on the representing space of \( K \)-theory (the space of Fredholm operators).

Then in order to define the corresponding version of the twisted \( K \)-theory one considers the \( \text{Fred}(\mathcal{H}) \)-bundle \( \tilde{\text{Fred}}(\mathcal{H}) \to \text{BPU}(\mathcal{H}) \) associated (by means of the action \( \gamma \)) with the universal \( \text{PU}(\mathcal{H}) \)-bundle over the classifying space \( \text{BPU}(\mathcal{H}) \simeq K(\mathbb{Z}, 3) \), i.e. the bundle

\[
\begin{align*}
\text{Fred}(\mathcal{H}) \xrightarrow{\text{EPU}(\mathcal{H})} \text{BPU}(\mathcal{H}).
\end{align*}
\]

Then for any map \( f : X \to \text{BPU}(\mathcal{H}) \) the corresponding twisted \( K \)-theory \( K_f(X) \) is the set (in fact the group) of homotopy classes of sections \( [X, f^*\text{Fred}(\mathcal{H})]' \) of the pullback bundle (here \([..., ...]'\) denotes the set of fiberwise homotopy classes of sections). The group \( K_f(X) \) depends up to isomorphism only on the homotopy class \([f]\) of the map \( f \), i.e. in fact on the corresponding third integer cohomology class.

In this paper we are interested in the case of bundles (more precisely, of elements in \( bsu_0^0 \)) of finite order, therefore let us consider separately the specialization of the mentioned result to the case of line bundles of order \( k \) in \( \text{Pic}(X) \). For this we should consider subgroups \( \text{PU}(k) \subset \text{PU}(\mathcal{H}) \). Let us describe the corresponding embedding.

Let \( \mathcal{B}(\mathcal{H}) \) be the algebra of bounded operators on the separable Hilbert space \( \mathcal{H} \), \( M_k(\mathcal{B}(\mathcal{H})) := M_k(\mathbb{C}) \otimes \mathcal{B}(\mathcal{H}) \) the matrix algebra over \( \mathcal{B}(\mathcal{H}) \) (of course, it is isomorphic to \( \mathcal{B}(\mathcal{H}) \)). Let \( U_k(\mathcal{H}) \subset M_k(\mathcal{B}(\mathcal{H})) \) be the corresponding unitary group (which is isomorphic to \( U(\mathcal{H}) \)). It acts on \( M_k(\mathcal{B}(\mathcal{H})) \) by conjugations (which are \(*\)-algebra isomorphisms), moreover, the kernel of the action is the center, i.e. the subgroup of scalar matrices \( \cong U(1) \). The corresponding quotient group we denote by \( \text{PU}_k(\mathcal{H}) \) (of course, it is isomorphic to \( \text{PU}(\mathcal{H}) \)).

\( M_k(\mathbb{C}) \otimes \text{Id}_{\mathcal{B}(\mathcal{H})} \) is a \( k \)-subalgebra (i.e. a unital \(*\)-subalgebra isomorphic to \( M_k(\mathbb{C}) \)) in \( M_k(\mathcal{B}(\mathcal{H})) \). Then \( \text{PU}(k) \subset \text{PU}_k(\mathcal{H}) \) is the subgroup of automorphisms of this \( k \)-subalgebra. Thereby we have defined the injective group homomorphism

\[
i_k : \text{PU}(k) \hookrightarrow \text{PU}_k(\mathcal{H})
\]

induced by the group homomorphism \( U(k) \hookrightarrow U_k(\mathcal{H}), g \mapsto g \otimes \text{Id}_{\mathcal{B}(\mathcal{H})} \).
Let \([k]\) be the trivial \(C^k\)-bundle over \(X\).

**Proposition 2.** For a line bundle \(\zeta \to X\) satisfying the condition
\[(3)\]
the classifying map \(\varphi_\zeta: X \to PU_k(\mathcal{H}) \cong PU(\mathcal{H})\) can be lifted to a map \(\tilde{\varphi}_\zeta: X \to PU(k)\) such that
\[i_k \circ \tilde{\varphi}_\zeta \cong \varphi_\zeta.\]

**Proof.** Consider the exact sequence of groups
\[(4)\]
and the fibration
\[(5)\]
obtained by its extension to the right. In particular, \(\psi_k: PU(k) \to BU(1) \cong \mathbb{C}P^\infty\) is the classifying map for the \(U(1)\)-bundle \(\chi_k\) (4). It is easy to see that the diagram
\[
\begin{array}{ccc}
PU(k) & \xrightarrow{\psi_k} & BU(1) \\
\downarrow{i_k} & & \downarrow{\cong} \\
PU_k(\mathcal{H}) & & \\
\end{array}
\]
commutes.

Let \(\zeta \to X\) be a line bundle satisfying the condition (3), \(\varphi_\zeta: X \to BU(1)\) its classifying map. Since \(\omega_k\) (see (5)) is induced by taking the direct sum of a line bundle with itself \(k\) times (and the extension of the structural group to \(U(k)\)), we see that \(\omega_k \circ \varphi_\zeta \cong \ast\). Now it is easy to see from exactness of (5) that \(\varphi_\zeta: X \to BU(1)\) can be lifted to \(\tilde{\varphi}_\zeta: X \to PU(k)\). \(\square\)

Note that the choice of a lift \(\tilde{\varphi}_\zeta\) corresponds to the choice of a trivialization (3): two lifts differ up to a map \(X \to U(k)\). Thus, a lift is defined up to the action of \([X, U(k)]\) on \([X, PU(k)]\). The subgroup in \(Pic(X)\) consisting of line bundles satisfying the condition (3) is \(\text{im}\{\psi_{k*}: [X, PU(k)] \to [X, \mathbb{C}P^\infty]\}\) or the quotient \([X, PU(k)]/[X, U(k)]\).

Let \(\text{Fred}_k(\mathcal{H})\) be the subspace of Fredholm operators in \(M_k(B(\mathcal{H}))\). Clearly, \(\text{Fred}_k(\mathcal{H}) \cong \text{Fred}(\mathcal{H})\). Acting on \(M_k(\mathcal{C})\) by \(*\)-automorphisms, the group \(PU(k)\) acts on \(M_k(\mathcal{C}) \otimes B(\mathcal{H}) = M_k(B(\mathcal{H}))\) through the first tensor factor. Let \(\gamma_k^*: PU(k) \times \text{Fred}_k(\mathcal{H}) \to \text{Fred}_k(\mathcal{H})\) be the restriction of this action on \(\text{Fred}_k(\mathcal{H})\). Then the diagram
\[
\begin{array}{ccc}
PU(\mathcal{H}) \times \text{Fred}_k(\mathcal{H}) & \xrightarrow{\gamma_k} & \text{Fred}_k(\mathcal{H}) \\
\downarrow{i_k \times \text{id}} & & \downarrow{\gamma_k^*} \\
PU(k) \times \text{Fred}_k(\mathcal{H}) & & \\
\end{array}
\]
commutes. Now one can consider the $\text{Fred}_k(\mathcal{H})$-bundle

$$
\begin{array}{ccc}
\text{Fred}_k(\mathcal{H}) & \longrightarrow & \text{EPU}(k) \times_{\text{PU}(k)} \text{Fred}_k(\mathcal{H}) \\
\downarrow & & \downarrow \\
\text{BPU}(k) & & \text{BPU}(k)
\end{array}
$$

associated by means of the action $\gamma'_k$. This bundle is the pullback of (2) by $B_i k$.

It is easy to see from the definition of the embedding $i_k$ that the action $\gamma'_k$ is trivial on elements of the form $k\xi$. Indeed, a classifying map for $k\xi$ can be decomposed into the composite $X \overset{f}{\longrightarrow} \text{Fred}(\mathcal{H}) \overset{\text{diag}}{\longrightarrow} \text{Fred}_k(\mathcal{H})$.

From the other hand, $(1 + (\zeta - 1)) \cdot k\xi = k\xi + 0 = k\xi$ or $\zeta \otimes ([k] \otimes \xi) = (\zeta \otimes [k]) \otimes \xi = [k] \otimes \xi$.

**Remark 3.** Note that if we choose an isomorphism $B(\mathcal{H}) \cong M_{k\infty}(B(\mathcal{H}))$ and hence the isomorphism $\text{Fred}(\mathcal{H}) \cong \text{Fred}_{k\infty}(\mathcal{H})$, we can define the limit action $\gamma'_{k\infty}: \text{PU}(k\infty) \times \text{Fred}_{k\infty}(\mathcal{H}) \to \text{Fred}_{k\infty}(\mathcal{H})$, etc.

### 2. The case of bundles of dimension $\geq 1$

As was pointed out in the previous section, the group $\text{PU}(\mathcal{H})$, from one hand acts on the representing space of $K$-theory $\text{Fred}(\mathcal{H})$, from the other hand it is the base of the universal line bundle. This two facts lead to the result that the action of $\text{PU}(\mathcal{H})$ on $K(X)$ corresponds to the tensor product by elements of the Picard group $\text{Pic}(X)$ (i.e. classes of line bundles). This action can be restricted to subgroups $\text{PU}(k) \subset \text{PU}(\mathcal{H})$ which classify elements of finite order $k$, $k \in \mathbb{N}$.

In what follows the role of groups $\text{PU}(k)$ will play some spaces $\text{Fr}_{k,l}$ (defined below). From one hand, they “act” on $K$-theory (more precisely, their direct limit (which has the natural structure of an $H$-space) acts), from the other hand, they are bases of some nontrivial $l$-dimensional bundles of order $k$. We will show that their “action” on $K(X)$ corresponds to the tensor product by those $l$-dimensional bundles (see Theorem 7). The key result of this section is Theorem 7 which can be regarded as a counterpart of Theorem 1.

**Fix a pair of positive integers $k, l > 1$.** Let $\text{Hom}_{\text{alg}}(M_k(\mathbb{C}), M_{kl}(\mathbb{C}))$ be the space of all unital $\ast$-homomorphisms $M_k(\mathbb{C}) \to M_{kl}(\mathbb{C})$. It follows from Noether-Skolem’s theorem that it can be represented in the form of a homogeneous space of the group $\text{PU}(kl)$ as follows:

$$\text{Hom}_{\text{alg}}(M_k(\mathbb{C}), M_{kl}(\mathbb{C})) \cong \text{PU}(kl)/(E_k \otimes \text{PU}(l))$$

(here $\otimes$ denotes the Kronecker product of matrices). This space we shall denote by $\text{Fr}_{k,l}$. We will be interested in the case $(k, l) = 1$ (we have to impose a condition of such a kind to make the direct limit of the above spaces noncontractible and the construction below nontrivial).

**Proposition 4.** A map $X \to \text{Fr}_{k,l}$ is the same thing as an embedding

$$\mu: X \times M_k(\mathbb{C}) \hookrightarrow X \times M_{kl}(\mathbb{C}),$$

whose restriction to a fiber is a unital $\ast$-homomorphism of matrix algebras.
Proposition 5. (Cf. Proposition 2) For an $M_k(\mathbb{C})$-bundle $B_l \to X$ such that

\[ [M_k] \otimes B_l \cong X \times M_{kl}(\mathbb{C}) \]

(cf. [4]) a classifying map $\varphi_{B_l} : X \to \text{BPU}(l)$ can be lifted to $\tilde{\varphi}_{B_l} : X \to \text{Fr}_{k,l}$ (i.e. $\psi'_k \circ \tilde{\varphi}_{B_l} = \varphi_{B_l}$ or $B_l = \tilde{\varphi}_{B_l}(\tilde{B}_{k,l})$).

Proof follows from the analysis of fibration (11). □

Moreover, the choice of such a lift corresponds to the choice of trivialization (12) and we return to the interpretation of the map $X \to \text{Fr}_{k,l}$ given in Proposition 3. We stress that a map $X \to \text{Fr}_{k,l}$
is not just an $M_l(\mathbb{C})$-bundle, but an $M_l(\mathbb{C})$-bundle together with a particular choice of trivialization \cite{12}.

It is not difficult to show \cite{8} that the bundle $B_l \to X$ as in the statement of Proposition 5 has the form $\operatorname{End}(\eta_l)$ for some (unique up to isomorphism) $\mathbb{C}^l$-bundle $\eta_l \to X$ with the structural group $\operatorname{SU}(l)$ (here the condition $(k, l) = 1$ is essential).

Let $\tilde{\zeta} \to \operatorname{Fr}_{k,l}$ be the line bundle associated with the universal covering $\rho_k \to \tilde{\operatorname{Fr}}_{k,l} \to \operatorname{Fr}_{k,l}$, where $\rho_k$ is the group of $k$th roots of unity. Note that $\tilde{\operatorname{Fr}}_{k,l} = \operatorname{SU}(kl)/(E_k \otimes \operatorname{SU}(l))$. Put $\zeta' := \tilde{\varphi}_{B_l}^*(\tilde{\zeta}) \to X$ and $\eta_l' := \eta_l \otimes \zeta'$.

Recall that $\operatorname{Fred}_n(\mathcal{H})$ is the subspace of Fredholm operators in $M_n(\mathcal{B}(\mathcal{H}))$. The evaluation map
\begin{equation}
\operatorname{ev}_{k,l} : \operatorname{Fr}_{k,l} \times M_k(\mathbb{C}) \to M_{kl}(\mathbb{C}), \quad \operatorname{ev}_{k,l}(h, T) = h(T)
\end{equation}
(recall that $\operatorname{Fr}_{k,l} := \operatorname{Hom}_{\text{alg}}(M_k(\mathbb{C}), M_{kl}(\mathbb{C}))$) induces the map
\begin{equation}
\gamma_{k,l} : \operatorname{Fr}_{k,l} \times \operatorname{Fred}_k(\mathcal{H}) \to \operatorname{Fred}_{kl}(\mathcal{H}).
\end{equation}

**Remark 6.** Note that map \eqref{13} can be decomposed into the composition
\[\operatorname{Fr}_{k,l} \times M_k(\mathbb{C}) \to \operatorname{Fr}_{k,l} \times M_{kl}(\mathbb{C}) = \mathcal{A}_{k,l} \to M_{kl}(\mathbb{C}),\]
where the last map is the tautological embedding $\mu : \mathcal{A}_{k,l} \to \operatorname{Gr}_{k,l} \times \operatorname{M}_{kl}(\mathbb{C})$ followed by the projection onto the second factor.

Let $f_\xi : X \to \operatorname{Fred}_n(\mathcal{H})$ represent some element $\xi \in K(X)$.

**Theorem 7.** (Cf. Theorem \cite{11}.) With respect to the above notation the composite map (cf. \cite{7})
\[X \xrightarrow{\operatorname{diag}} X \times X \xrightarrow{\tilde{\varphi}_{B_l} \times f_\xi} \operatorname{Fr}_{k,l} \times \operatorname{Fred}_k(\mathcal{H}) \xrightarrow{\gamma_{k,l}'} \operatorname{Fred}_{kl}(\mathcal{H})\]
represents the element $\eta_l' \otimes \xi \in K(X)$.

**Proof** (cf. \cite{2}, Proposition 2.1). By assumption the element $\xi \in K(X)$ is represented by a family of Fredholm operators $F = \{F_x\}$ in a Hilbert space $\mathcal{H}^k$. Then the element $\eta_l' \otimes \xi \in K(X)$ is represented by the family of Fredholm operators $\{\operatorname{Id}_{B_l}, 0 \otimes F_x\}$ in the Hilbert bundle $\eta_l' \otimes (\mathcal{H}^k)$ (recall that $\operatorname{End}(\eta_l') = B_l \Rightarrow \operatorname{End}(\eta_l' \otimes \xi) = B_l$). A trivialization $\eta_l' \otimes (\mathcal{H}^k) \cong \mathcal{H}^{kl}$ is the same thing as a map $\tilde{\varphi}_{B_l} : X \to \operatorname{Fr}_{k,l}$, i.e. a lift of the classifying map $\varphi_{B_l} : X \to \operatorname{BPU}(l)$ for $B_l$ (see \cite{11}). \hfill \Box

**Remark 8.** In order to separate the “SU”-part of the “action” $\gamma_{k,l}'$ from its “line” part, one can use the space $\tilde{\operatorname{Fr}}_{k,l} = \operatorname{SU}(kl)/(E_k \otimes \operatorname{SU}(l))$ \cite{8} in place of $\operatorname{Fr}_{k,l}$. Then one would have the representing map for $\eta_l' \otimes \xi \in K(X)$ instead of $\eta_l' \otimes \xi$ in the statement of Theorem \cite{7}.

**Remark 9.** Note that $\operatorname{Fr}_{k,l} = \operatorname{PU}(k)$ and the action $\gamma_{k,l}'$ coincides with the action $\gamma_k'$ from the previous section.

Now using the composition of algebra homomorphisms we are going to define maps $\phi_{k,l} : \operatorname{Fr}_{k,l} \times \operatorname{Fr}_{k,l} \to \operatorname{Fr}_{k,l}$, i.e.
\[\phi_{k,l} : \operatorname{Hom}_{\text{alg}}(M_k(\mathbb{C}), M_{kl}(\mathbb{C})) \times \operatorname{Hom}_{\text{alg}}(M_k(\mathbb{C}), M_{kl}(\mathbb{C})) \to \operatorname{Hom}_{\text{alg}}(M_k(\mathbb{C}), M_{kl}(\mathbb{C})).\]

First let us define a map
\[\iota_{k,l} : \operatorname{Hom}_{\text{alg}}(M_k(\mathbb{C}), M_{kl}(\mathbb{C})) \to \operatorname{Hom}_{\text{alg}}(M_k(\mathbb{C}), M_{kl}(\mathbb{C})), \quad \iota_{k,l}(h) = h \otimes \operatorname{id}_{M_{kl}(\mathbb{C})}.\]
Then $\phi_{k,l}$ is defined as the composition of homomorphisms: $\phi_{k,l}(h_2, h_1) = \iota_{k,l}(h_2) \circ h_1$, where $h_1 \in \text{Hom}_{\text{alg}}(M_k(\mathbb{C}), M_{kl}(\mathbb{C}))$. Then we have $ev_{k,l,2}(\phi_{k,l}(h_2, h_1), T) = ev_{k,l,1}(\iota_{k,l}(h_2), ev_{k,l,1}(h_1, T))$, i.e. $\phi_{k,l}(h_2, h_1)(T) = \iota_{k,l}(h_2)(h_1(T))$, where $T \in M_k(\mathbb{C})$.

Now suppose there is an $M_i(\mathbb{C})$-bundle $C_l \to X$ with the corresponding vector bundle $\theta_l'$ such that $C_l \cong \text{End}(\theta_l')$ (cf. a few paragraphs after Proposition 5). Suppose that $\tilde{\varphi}_{C_l} : X \to Fr_{k,l}$ is its classifying map.

**Proposition 10.** (cf. Theorem \[\square\]) The composition

$$X \xrightarrow{\text{diag}} X \times X \xrightarrow{\bar{\varphi}_{C_l} \times \bar{\varphi}_{B_{kl}} \times \phi_{k,l}} Fr_{k,l} \times Fr_{k,l} \times \text{Fred}_k(\mathcal{H}) \to \text{Fred}_{kl}(\mathcal{H})$$

where the last map is the composition $\gamma'_{k,l} \circ (\phi_{k,l} \times \text{id}_{\text{Fred}_k(\mathcal{H})}) = \gamma'_{k,l} \circ (\iota_{k,l} \times \gamma_{k,l})$ represents the element $\eta_1' \otimes \theta_l' \otimes \xi \in K(X)$.

**Proof** is evident. $\square$

Clearly, the results of this section can be generalized to the case of spaces $Fr_{kl^m,l^n}$, $m, n \in \mathbb{N}$. In the next section we will construct a genuine action of their direct limit $Fr_{k\infty,l\infty}$ on the space of Fredholm operators.

3. A CONSTRUCTION OF THE CLASSIFYING SPACE

A simple calculation with homotopy groups shows that the direct limit $Fr_{k\infty,l\infty} := \varinjlim Fr_{kl^m,l^n}$ for $(k, l) = 1$ is not contractible because its homotopy groups are $\mathbb{Z}/k\mathbb{Z}$ in odd dimensions (and 0 in even ones). In this section we show that it is a topological monoid and construct its classifying space.

3.1. The category $C_{k,l}$. First, we define some auxiliary category $C_{k,l}$. Fix a pair of positive integers $\{k, l\}$, $k, l > 1, (k, l) = 1$. By $C_{k,l}$ denote the category with the countable number of objects which are matrix algebras of the form $M_{kl^m}(\mathbb{C})$ ($m = 0, 1, \ldots$) and morphisms in $C_{k,l}$ from $M_{kl^m}(\mathbb{C})$ to $M_{kl^{m+1}}(\mathbb{C})$ are all unital $\ast$-homomorphisms $M_{kl^m}(\mathbb{C}) \to M_{kl^{m+1}}(\mathbb{C})$ (this set is nonempty iff $m \leq n$). (Since $k > 1$ we see that this category does not contain the initial object, therefore there is no reason to expect that its classifying space (i.e. the “geometric realization”) is contractible). Thus, morphisms Mor($M_{kl^m}(\mathbb{C}), M_{kl^{m+1}}(\mathbb{C})$) form the space $Fr_{kl^m,l^n-m} = \text{Hom}_{\text{alg}}(M_{kl^m}(\mathbb{C}), M_{kl^{m+1}}(\mathbb{C}))$, hence $C_{k,l}$ is a topological category. (Note that for $n = m$ the space $Fr_{kl^m,l^n-m} = Fr_{kl^m,1}$ is the group $\text{PU}(k^{l^m})$.) In particular, there is the collection of continuous maps $Fr_{kl^m+n,l^{n-r}} \times Fr_{kl^{m+1},l^n} \to Fr_{kl^m,l^{n+r}}$ for all $m, n, r \geq 0$ given by the composition of morphisms.

Recall (\[\square\], Chapter IV) that there is an appropriate modification of the construction of the geometric realization $B\mathcal{C}$ for topological categories $\mathcal{C}$. More precisely, the nerve in this case is a simplicial **topological space** and $B\mathcal{C}$ is its appropriate geometric realization. Now we are going to describe the classifying space of the category $B\mathcal{C}_{k,l}$.

So let $B\mathcal{C}_{k,l}$ be the classifying space of the topological category $C_{k,l}$. Its 0-cells (vertices) are objects of $C_{k,l}$, i.e. actually positive integers. Its 1-cells (edges) are morphisms in $C_{k,l}$ (excluding identity morphisms) attached to their source and target, i.e. $\bigsqcup_{m, n \geq 0} Fr_{kl^m,l^n}$ (recall that for
For each pair of composable morphisms \( h_0, h_1 \) in \( C_{k,l} \) there is a 2-simplex:

\[
\begin{array}{ccc}
0 & \overset{h_0}{\rightarrow} & 1 \\
& \swarrow & \nearrow \\
& h_1 \circ h_0 & \rightarrow 2
\end{array}
\]

attached to the 1-skeleton, etc. Thus, at the third step we obtain the space
\[
\coprod_{m,n,r,0} (Fr_{k+m,n,r} \times Fr_{k+m,n})
\]
consisting of all pairs of composable morphisms. The face maps \( \partial_0, \partial_2 \) are defined by the deletion of the corresponding morphism \((h_0\) and \( h_1 \) respectively in the above simplex) and \( \partial_1 \) is defined by the composition map
\[
\coprod_{m,n,r,0} (Fr_{k+m,n,r} \times Fr_{k+m,n}) \rightarrow \coprod_{m,n+r,0} Fr_{k+m,n+r} = \coprod_{m,n,0} Fr_{k+m,n}.
\]
The nerve \( NC_{k,l} \) of the category \( C_{k,l} \) is

\[
(\mathbb{N}, \coprod_{m,n,0} Fr_{k+m,n}, \coprod_{m,n,r,0} (Fr_{k+m,n+r} \times Fr_{k+m,n}), \ldots).
\]

Recall that there is a construction of the classifying space of a (topological) group \( G \) as the geometric realization of the simplicial topological space \((pt, G, G \times G, \ldots)\). Then the total space \( EG \) of the universal principal \( G \)-bundle is the geometric realization of the simplicial space \((G, G \times G, \ldots)\). Consider the simplicial topological space
\[
\mathcal{E}_{k,l} := (\coprod_{m,n,0} Fr_{k+m,n}, \coprod_{m,n,r,0} (Fr_{k+m,n+r} \times Fr_{k+m,n}), \ldots),
\]
whose faces and degeneracies are defined by analogy with the construction of \( EG \). There is the map of simplicial spaces \( p: \mathcal{E}_{k,l} \rightarrow NC_{k,l} \).

Now we are going to give a construction of the corresponding “universal bundle”. The idea is to construct a “simplicial bundle” associated with the “universal principal bundle” \( p \).

More precisely, again consider the space \( \mathcal{E}_{k,l} \):
\[
(\coprod_{m,n,0} Fr_{k+m,n}, \coprod_{m,n,r,0} (Fr_{k+m,n+r} \times Fr_{k+m,n}), \coprod_{m,n,r,0} (Fr_{k+m,n+r} \times Fr_{k+m,n+r} \times Fr_{k+m,n}), \ldots).
\]

Applying the natural (“evaluation”) maps \( Fr_{k+m,n} \times M_{k+m}(\mathbb{C}) \rightarrow M_{k+m+n}(\mathbb{C}) \), we obtain

\[
(\coprod_{m,0} M_{k+m}(\mathbb{C}), \coprod_{m,n,0} (Fr_{k+m,n} \times M_{k+m}(\mathbb{C})), \coprod_{m,n,r,0} (Fr_{k+m,n+r} \times Fr_{k+m,n} \times M_{k+m}(\mathbb{C})), \ldots).
\]

The obtained object can be regarded as a “simplicial bundle” over \( \mathcal{E}_{k,l} \). Now performing the appropriate factorizations we should define its geometric realization. Namely, the matrix algebras from the first disjoint union in \( \mathcal{E}_{k,l} \) are fibers of our bundle over 0-cells of the space \( BC_{k,l} \), from the second over 1-cells, etc. The attachment of 1-cells to their source corresponds to the projection \( Fr_{k+m,n} \times M_{k+m}(\mathbb{C}) \rightarrow M_{k+m}(\mathbb{C}) \) onto the second factor, and the one to their target corresponds to the natural map \( Fr_{k+m,n} \times M_{k+m}(\mathbb{C}) \rightarrow M_{k+m+n}(\mathbb{C}) \). \( M_{k+m}(\mathbb{C}) \)-bundles over 2-simplices correspond to the products \( Fr_{k+m,n} \times Fr_{k+m,n} \times M_{k+m}(\mathbb{C}) \) and they are identified over the boundary as follows:

\[
\partial_0: Fr_{k+m+n} \times Fr_{k+m,n} \times M_{k+m}(\mathbb{C}) \rightarrow Fr_{k+m+n} \times M_{k+m+n}(\mathbb{C}) \quad \text{induced by the natural map} \quad Fr_{k+m,n} \times M_{k+m}(\mathbb{C}) \rightarrow Fr_{k+m,n} \times Fr_{k+m,n} \times M_{k+m}(\mathbb{C}) \quad \text{by the composition of morphisms} \quad Fr_{k+m,n} \times Fr_{k+m,n} \rightarrow Fr_{k+m,n+r},
\]
The definition of $\mathcal{C}_{k,l}$ one can define categories $\mathcal{C}_{km,l}$, $m \geq 1$. The objects of $\mathcal{C}_{km,l}$ are matrix algebras $M_{km,l}(\mathbb{C})$, $n \geq 0$. Note that the tensor product of matrix algebras gives rise to the bifunctor $T_{m,n}: \mathcal{C}_{km,l} \times \mathcal{C}_{kn,l} \rightarrow \mathcal{C}_{km+n,l}$. Thus, on objects we have: $T_{m,n}(M_{km,l}(\mathbb{C}), M_{kn,l}(\mathbb{C})) = M_{km+n,l}(\mathbb{C})$, and on morphisms $h_1: M_{km,l}(\mathbb{C}) \rightarrow M_{km+l,++}(\mathbb{C})$, $h_2: M_{kn,l}(\mathbb{C}) \rightarrow M_{k+n,++}(\mathbb{C})$.

Moreover, the bifunctor $T_{m,n}$ determines the continuous map of the topological spaces $\text{Fr}_{km+l,++} \times \text{Fr}_{kn+l,++} \rightarrow \text{Fr}_{km+n,l,++}$, $(h_1, h_2) \mapsto h_1 \otimes h_2$.

For the topological bicategory $\mathcal{C}_{km,l} \times \mathcal{C}_{kn,l}$ one can define the bisimplicial topological space $X = \{X_{p,q}\}$, where $X_{p,q}$ consists of all pairs of functors $p+1 \rightarrow \mathcal{C}_{km,l}$, $q+1 \rightarrow \mathcal{C}_{kn,l}$. There are two types of face and degeneracy maps: “horizontal” and “vertical” which correspond to the category $\mathcal{C}_{km,l}$ and $\mathcal{C}_{kn,l}$ respectively. Clearly that its geometric realization $B X$ (to any point of $X_{p,q}$ we attach $\Delta^p \times \Delta^q$) is $B \mathcal{C}_{km,l} \times B \mathcal{C}_{kn,l}$. The bifunctor $T_{m,n}$ determines the continuous map $T_{m,n}: B \mathcal{C}_{km,l} \times B \mathcal{C}_{kn,l} \rightarrow B \mathcal{C}_{km+n,l}$.

In conclusion of this subsection we describe yet another two properties of the category $\mathcal{C}_{k,l}$.

Let $\mathcal{C}_{k,l}$ be the category with the same objects as $\mathcal{C}_{k,l}$ but with morphisms that are automorphisms in $\mathcal{C}_{k,l}$. Clearly, $B \mathcal{C}_{k,l} \simeq \coprod_{n \geq 0} BPU(k^n)$ and the embedding $\coprod_{n \geq 0} BPU(k^n) \rightarrow B \mathcal{C}_{k,l}$ corresponds to the inclusion of the subcategory $\mathcal{C}_{k,l} \rightarrow \mathcal{C}_{k,l}$.

Let $\mathbb{N}$ be the category with countable set of objects $\{0, 1, 2, \ldots\}$ and there is a morphism from $i$ to $j$ iff $i \leq j$, and such morphism is unique. It is easy to see that its classifying space $B \mathbb{N}$ is the infinite simplex $\Delta$.

There is the obvious functor $F: \mathcal{C}_{k,l} \rightarrow \mathbb{N}$, $F(M_{kl}(\mathbb{C})) = m$ and for a morphism $h: M_{kl}(\mathbb{C}) \rightarrow M_{kl+m}(\mathbb{C})$ the morphism $F(h)$ is the unique morphism $m \rightarrow n$ in $\mathbb{N}$. Therefore there is the corresponding map of classifying spaces $B \mathcal{C}_{k,l} \rightarrow B \mathbb{N}$. The subspace $\coprod_{n \geq 0} BPU(k^n) \subset B \mathcal{C}_{k,l}$ corresponds to the vertices of the simplex $B \mathbb{N}$ (more precisely, to the corresponding discrete category).

In general, simplices degenerate under this map of classifying spaces, moreover, their degenerations correspond to automorphisms of objects of $\mathcal{C}_{k,l}$ and nondegenerate simplices correspond to chains $\{h_0, h_1, \ldots, h_n\}$ of composable morphisms such that for all $h_r$ the source and the target are different objects (in other words, $h_i \in \text{Fr}_{kl+m}^r$, $n \neq 0$).

3.2. The definition of $B \text{Fr}_{k\infty, l\infty}$. Now consider new topological category $\overline{\mathcal{C}}_{k,l}$. It has a unique object $M_{kl}(\mathbb{C}) = \lim_m M_{kl}(\mathbb{C})$ (i.e. $\overline{\mathcal{C}}_{k,l}$ is actually a monoid), where the direct limit is taken over unital *-homomorphisms $M_{kl}(\mathbb{C}) \rightarrow M_{kl+1}(\mathbb{C})$, $X \mapsto X \otimes E_l$. More precisely, we assume
that the matrix algebra $M_{kl}(\mathbb{C})$ is given together with the infinite family of $*$-subalgebras $A_k \subset A_{kl} \subset A_{kl^2} \subset \ldots$, where $A_{kl} = M_{kl}(\mathbb{C})$ and $A_{kl+1} = M_I(A_{kl})$ for every $m \geq 0$, which form its filtration.

By definition, for each morphism $h: M_{kl}(\mathbb{C}) \to M_{kl}(\mathbb{C})$, $h \in \text{Mor}(\mathcal{C}_{k,l})$ there exists a pair $m, n$, $n \geq m$ such that 1) $h|_{A_{kl}}$ is a unital $*$-homomorphism $h|_{A_{kl}}: A_{kl} \to A_{kl^n}$ and 2) $h = M_I(h|_{A_{kl}})$, i.e. $h|_{A_{kl+m+1}} = M_I(h|_{A_{kl}}): A_{kl+m+1} \to A_{kl+m+1} \to M_I(A_{kl})$, etc. In other words, $h$ is induced by some $h' \in \text{Fr}_{kl,m,n}$, $h_2: A_k \to A_{kl}$ is defined by the following diagram

as the class of the homomorphism $M_{kl}(h_2) \circ h_1$. Clearly, the composition of morphisms is well-defined and associative and the identity morphism is $M_I\text{id}_{A_k}$, i.e. the family \{id$_A$, id$_{A_{kl}}$, id$_{A_{kl^2}}$, $\ldots$\}.

Now we define the functor $\Phi$: $\mathcal{C}_{k,l} \to \mathcal{C}_{k,l}$ which sends every object $M_{kl}(\mathbb{C})$ in $\mathcal{C}_{k,l}$ to the unique object $M_{kl}(\mathbb{C})$ in $\mathcal{C}_{k,l}$. For a morphism $h \in \text{Fr}_{kl,m,n}$ we put $\Phi(h) = M_I(h)$. Thus, we see that $\text{Mor}(\mathcal{C}_{k,l})$ is the well-pointed grouplike (because $\pi_0(\text{Fr}_{kl,m,n}) = 0$) topological monoid $\text{Fr}_{kl,m,n}$. Recall [4] that for such a monoid $M$ there exists the classifying space $BM$. Thus we have the classifying space $B\text{Fr}_{kl,m,n}$ which is defined uniquely up to CW-equivalence and there is the Whitehead equivalence $\text{Fr}_{kl,m,n} \to \Omega B\text{Fr}_{kl,m,n}$; in particular, $\pi_i(\text{Fr}_{kl,m,n}) = \pi_{i+1}(\Omega B\text{Fr}_{kl,m,n})$. Note that $\Phi$ defines a continuous map $\bar{\Phi}: B\mathcal{C}_{k,l} \to B\text{Fr}_{kl,m,n}$. Moreover, the maps $\bar{T}_{m,n}$ correspond to the maps $B\text{Fr}_{km,n+1} \to B\text{Fr}_{km+n+1}$ which are given by maps $\text{Fr}_{km,n} \times \text{Fr}_{km+n} \to \text{Fr}_{km+n+1}$ induced by the tensor product of matrix algebras.

Note that there is the subgroup $\text{PU}(kl^n) = \lim_m \text{PU}(kl^m)$ in the monoid $\text{Fr}_{kl,m,n}$ which corresponds to automorphisms of $M_{kl}(\mathbb{C})$. The corresponding map $B\text{PU}(kl^n) \to B\text{Fr}_{kl,m,n}$ has the homotopy fiber $\text{Gr}_{kl,m,n}$ (this follows from the fibration

\[ \text{PU}(kl^n) \to \text{Fr}_{kl,m,n} = \text{PU}(kl^m+n)/E_{kl^m} \to \text{Gr}_{kl,m,n}. \]
Remark 11. Retrospectively, we note that we have used the maps \( \text{Fr}_{kl^{m+n}, l^r} \times \text{Fr}_{kl^m, l^n} \to \text{Fr}_{kl^{m+n+r}, l'^t} \) (given by the composition of morphisms) in order to define the monoidal structure on \( \text{Fr}_{kl^{≈}, l^{≈}} \) and thereby the classifying space \( B \text{Fr}_{kl^{≈}, l^{≈}} \), and the maps \( \text{Fr}_{k^{m+r}, l^t} \times \text{Fr}_{k^{n+t}, l^u} \to \text{Fr}_{k^{m+n+r+t}, l^{u+t}} \) given by the tensor product in order to define the additional structure on the spaces \( \text{Fr}_{k^{m+l}, l^{≈}} \) (which gives rise to the \( H \)-space structure on \( \lim_m \text{Fr}_{k^{m+l}, l^{≈}} \)). From the category-theoretic point of view the first corresponds to the composition of morphisms in the category and the second to the monoidal structure on it.

3.3. The action of \( \text{Fr}_{kl^{≈}, l^{≈}} \) on the space of Fredholm operators. Recall (13) that there are evaluation maps

\[
e_{vl^{km}, l^{m-n}} : \text{Fr}_{kl^{km}, l^{m-n}} \times M_{kl^m} (\mathbb{C}) \to M_{kl^n} (\mathbb{C})
\]

and the corresponding maps (recall that \( \text{Fred}_{kl^m}(\mathcal{H}) \) is the subspace of Fredholm operators in \( M_{kl^m}(B(\mathcal{H})) \))

\[
\gamma'_{kl^m, l^{m-n}} : \text{Fr}_{kl^{km}, l^{m-n}} \times \text{Fred}_{kl^m}(\mathcal{H}) \to \text{Fred}_{kl^{n}}(\mathcal{H})
\]

(see (14)). Using the filtration in \( M_{kl^{≈}}(B(\mathcal{H})) \) (and hence in \( \lim_m \text{Fred}_{kl^{m}}(\mathcal{H}) \)) corresponding to the above filtration \( A_k \subset A_{kl} \subset A_{kl^2} \subset \ldots \) in the matrix algebra \( M_{kl^{≈}}(\mathbb{C}) \) one can define the action of the monoid \( \text{Fr}_{kl^{≈}, l^{≈}} \) on \( \lim_m \text{Fred}_{kl^m}(\mathcal{H}) \). Note that since the direct limit is taken over maps induced by the tensor product, we see that \( \text{Fred}_{kl^{≈}}(\mathcal{H}) := \lim_m \text{Fred}_{kl^m}(\mathcal{H}) \) is the localization in which \( l \) becomes invertible (in particular, the index takes values in \( \mathbb{Z} [\frac{1}{l}] \), not in \( \mathbb{Z} \)). Thereby we have defined the required action

\[
\gamma'_{kl^{≈}, l^{≈}} : \text{Fr}_{kl^{≈}, l^{≈}} \times \text{Fred}_{kl^{≈}}(\mathcal{H}) \to \text{Fred}_{kl^{≈}}(\mathcal{H})
\]

of the monoid \( \text{Fr}_{kl^{≈}, l^{≈}} \).

Note that the action (17) gives rise to the action on \( K \)-theory (which is recall represented by the space of Fredholm operators) which corresponds to the action of the \( k \)-torsion subgroup in \( BU \) by tensor products (cf. Proposition (14) and Theorem (7)). In fact, this action is defined on \( K \)-theory \( K[\frac{1}{l}] \) localized over \( l \) (in the sense that \( l \) becomes invertible). This is not surprising because in (14) we take the tensor product of \( K(X) \) by some \( l \)-dimensional bundle, \( l > 1 \). It is not difficult to show that in fact our construction does not depend on the choice of \( l \), \( (k, l) = 1 \).

Note that the restriction of the action \( \gamma'_{kl^{≈}, l^{≈}} \) on \( \text{Fr}_{kl^m, 1} \cong PU(kl^m) \) coincides with the composition of the action \( \gamma'_{kl^m} \) (see Remark 3) and the localization map \( \text{Fred}_{kl^m}(\mathcal{H}) \to \text{Fred}_{kl^{≈}}(\mathcal{H}) \) on \( l \).

Using this action (17) we can define the \( \text{Fred}_{kl^{≈}}(\mathcal{H}) \)-bundle

\[
\begin{CD}
\text{Fred}_{kl^{≈}}(\mathcal{H}) @> \text{EFr}_{kl^{≈}, l^{≈}} \times \text{Fred}_{kl^{≈}}(\mathcal{H}) >> \text{Fr}_{kl^{≈}, l^{≈}} \times \text{Fred}_{kl^{≈}}(\mathcal{H}) \\
\downarrow @. \\
B \text{Fr}_{kl^{≈}, l^{≈}}
\end{CD}
\]

"associated" with the universal principal \( \text{Fr}_{kl^{≈}, l^{≈}} \)-bundle (more precisely, with the universal principal quasi-fibration, see (4)) over \( B \text{Fr}_{kl^{≈}, l^{≈}} \). This allows us to define a more general version of
the twisted $K$-theory than the one given by the action of $Pic(X)$ on $K(X)$. The above defined maps $B Fr_{k^n,l^∞} × B Fr_{k^n,l^∞} → B Fr_{k^n+l^n,l^∞}$ give rise to the operation on twistings which is an analog of the one induced by maps $BPU(k^m) × BPU(k^n) → BPU(k^{m+n})$ in the Abelian case (i.e. the Brauer group), etc.

4. APPENDIX: $H$-space $Fr_{k^∞,l^∞}$

In this section we give a category-theoretic description of the structure of $H$-space on $Fr_{k^∞,l^∞}$ and $Gr_{k^∞,l^∞}$.

4.1. $k^m$-frames.

**Definition 12.** A $k^m$-frame $α$ in the algebra $M_{k^{m,n}}(ℂ)$ is an ordered collection of $k^{2m}$ linearly independent matrices $\{α_{i,j}\}_{1 ≤ i, j ≤ k^m}$ such that

(i) $α_{i,j}α_{r,s} = δ_{j,r}α_{i,s}$ for all $1 ≤ i, j, r, s ≤ k^m$;

(ii) $\sum_{i=1}^{k^m} α_{i,i} = E$, where $E = E_{k^{m,n}}$ is the unit $k^m l^n × k^m l^n$-matrix which is the unit of the algebra $M_{k^{m,n}}(ℂ)$;

(iii) matrices $\{α_{i,j}\}$ form an orthonormal basis with respect to the hermitian inner product $(x, y) := \text{tr}(x\overline{y})$ on $M_{k^{m,n}}(ℂ)$.

For instance, the collection of “matrix units” $\{e_{i,j}\}_{1 ≤ i, j ≤ k^m}$ (where $e_{i,j}$ is the $k^m × k^m$-matrix whose only nonzero element is 1 on the intersection of $i$th row with $j$th column) is a $k^m$-frame in $M_{k^m}(ℂ)$, and the collection $\{e_{i,j} ⊗ E_{l^n}\}_{1 ≤ i, j ≤ k^m}$ is a $k^m$-frame in $M_{k^{m,n}}(ℂ)$. Clearly, every $k^m$-frame in $M_{k^{m,n}}(ℂ)$ is a linear basis in some $k^m$-subalgebra.

**Proposition 13.** The set of all $k^m$-frames in $M_{k^{m,n}}(ℂ)$ is the homogeneous space $PU(k^m l^n)/(E_{k^m} ⊗ PU(l^n))$.

**Proof** follows from two facts: 1) the group $PU(k^m l^n)$ of $*$-automorphisms of the algebra $M_{k^{m,n}}(ℂ)$ acts transitively on the set of $k^m$-frames, and 2) the stabilizer of the $k^m$-frame $\{e_{i,j} ⊗ E_{l^n}\}_{1 ≤ i, j ≤ k^m}$ is the subgroup $E_{k^m} ⊗ PU(l^n) ⊂ PU(k^m l^n)$. $\square$

In fact, the space of $k^m$-frames $Fr_{k^m,l^n}$ in $M_{k^{m,n}}(ℂ)$ is isomorphic to the space of unital $*$-homomorphisms $\text{Hom}_{alg}(M_{k^m}(ℂ), M_{k^{m,n}}(ℂ))$. More precisely, let $\{e_{i,j}\}_{1 ≤ i, j ≤ k^m}$ be the frame in $M_{k^m}(ℂ)$ consisting of matrix units. Then the isomorphism $Fr_{k^m,l^n} \cong \text{Hom}_{alg}(M_{k^m}(ℂ), M_{k^{m,n}}(ℂ))$ is given by the assignment

$$α \mapsto h_α: M_{k^m}(ℂ) → M_{k^{m,n}}(ℂ), \quad (h_α)_∗(\{e_{i,j}\}) = α \quad ∀α ∈ Fr_{k^m,l^n}.$$

Let $β$ be a $k^r$-frame in $M_{k^r l^n}(ℂ)$ and $m ≤ r$. Then one can associate with $β$ some new $k^m$-frame $α := π^m_1(β)$ as follows:

$$α_{i,j} = \beta_{(i-1)k^r-m+1,(j-1)k^r-m+1} + \beta_{(i-1)k^r-m+2,(j-1)k^r-m+2} + \ldots + \beta_{i(k^r-m),j(k^r-m)}, \quad 1 ≤ i, j ≤ k^m.$$

Also one can associate with $β$ some $k^{r-m}$-frame $γ := π^{r-m}_2(β)$ by the following rule:

$$γ_{i,j} = \beta_{i,j} + \beta_{i+k^r-m,j+k^r-m} + \ldots + \beta_{i+(k^r-1)k^r-m,j+(k^r-1)k^r-m}, \quad 1 ≤ i, j ≤ k^{r-m}.$$

The idea of the definition of $π^m_1(β)$ and $π^{r-m}_2(β)$ is the following. If one takes the $k^r$-frame $ε$ in $M_{k^r}(ℂ) = M_{k^m}(ℂ) ⊗ M_{k^{r-m}}(ℂ)$ consisting of the matrix units, then the $k^m$ and $k^{r-m}$-frames in
subalgebras $M_{km}(\mathbb{C}) \otimes CE_{kr-m} \subset M_{kr}(\mathbb{C})$ and $CE_{km} \otimes M_{kr-m}(\mathbb{C}) \subset M_{kr}(\mathbb{C})$ consisting of the matrix units tensored by the corresponding unit matrices are $\pi_1^m(\epsilon)$ and $\pi_2^{r-m}(\epsilon)$ respectively. From the other hand it is easy to see that the frame $\epsilon$ (under the appropriate ordering) is the tensor product of the frames of matrix units in the tensor factors $M_{km}(\mathbb{C})$ and $M_{kr-m}(\mathbb{C})$. The matrices from $\pi_1^m(\epsilon)$ commute with the matrices from $\pi_2^{r-m}(\epsilon)$, moreover, all possible pairwise products of the matrices from $\pi_1^m(\epsilon)$ by the matrices from $\pi_2^{r-m}(\epsilon)$ (we have exactly $k^{2m} \cdot k^{2(r-m)} = k^{2r}$ such products) give all matrices from the frame $\epsilon$. If we order the collection of products in the appropriate way, we get the frame $\epsilon$. The operation which to a pair consisting of commuting $k^m$ and $k^{r-m}$-frames assigns (according to this rule) the $k^r$-frame we will denote by dot $\cdot$. In particular, $\beta = \pi_1^m(\beta) \cdot \pi_2^{r-m}(\beta)$ for any $k^r$-frame $\beta$.

Thereby we have defined the continuous maps $\pi_1^m: Fr_{kr, l^*} \to Fr_{km, kr-m^*}$ and $\pi_2^{-m}: Fr_{kr, l^*} \to Fr_{kr-m, km^*}$. In terms of algebra homomorphisms they correspond to the assignment to a homomorphism $h: M_{kr}(\mathbb{C}) \to M_{kr'}(\mathbb{C})$ its compositions with homomorphisms $M_{km}(\mathbb{C}) \to M_{kr}(\mathbb{C})$, $X \mapsto X \otimes E_{kr-m}$ and $M_{kr-m}(\mathbb{C}) \to M_{kr}(\mathbb{C})$, $X \mapsto E_{km} \otimes X$ respectively.

4.2. Functor Fr. In this subsection we define a functor Fr from some monoidal category $C_{k,l}$ to the category of topological spaces with a chosen basepoint.

Let us fix an ordered pair of positive integers $k, l$, $(k, l) = 1$, $k, l > 1$. Define the category $C_{k,l}$ whose objects are pairs of the form $(M_{km^*}(\mathbb{C}), \alpha)$, consisting of a matrix algebra $M_{km^*}(\mathbb{C})$, $m, n \geq 0$ and a $k^m$-frame $\alpha$ in it. A morphism $f: (M_{km^*}(\mathbb{C}), \alpha) \to (M_{kr^*}(\mathbb{C}), \beta)$ is a unital $*$-homomorphism of matrix algebras $f: M_{km^*}(\mathbb{C}) \to M_{kr^*}(\mathbb{C})$ such that $f_s(\alpha) = \pi_1^m(\beta)$, where by $f_s$ we denote the map induced on frames by $f$. Equivalently, we have the commutative diagram

$$
\begin{array}{ccc}
M_{km^*}(\mathbb{C}) & \xrightarrow{f} & M_{kr^*}(\mathbb{C}) \\
\uparrow h_\alpha & & \uparrow h_\beta \\
M_{km}(\mathbb{C}) & \xrightarrow{i_{r,m}} & M_{kr}(\mathbb{C}),
\end{array}
$$

where $i_{r,m}: M_{km}(\mathbb{C}) \to M_{kr}(\mathbb{C})$ is the homomorphism $X \mapsto X \otimes E_{kr-m}$.

Note that $C_{k,l}$ is a symmetric monoidal category with respect to the bifunctor $\otimes$:

$$
((M_{km^*}(\mathbb{C}), \alpha), (M_{kr^*}(\mathbb{C}), \beta)) \mapsto (M_{km^*}(\mathbb{C}) \otimes M_{kr^*}(\mathbb{C}), \alpha \otimes \beta)
$$

and the unit object $\epsilon := (M_1(\mathbb{C}) = \mathbb{C}, \epsilon)$, where $\epsilon = 1$ is the unique $k^0 = 1$-frame.

Now let us define a functor Fr from $C_{k,l}$ to the category of topological spaces with a chosen basepoint. On objects Fr($M_{km^*}(\mathbb{C}), \alpha$) is the space of $k^m$-frames in $M_{km^*}(\mathbb{C})$, where $\alpha$ gives the basepoint. For a morphism $f: (M_{km^*}(\mathbb{C}), \alpha) \to (M_{kr^*}(\mathbb{C}), \beta)$ put

$$
\text{Fr}(f): \text{Fr}(M_{km^*}(\mathbb{C}), \alpha) \to \text{Fr}(M_{kr^*}(\mathbb{C}), \beta), \quad \text{Fr}(f)(\alpha') = f_s(\alpha') \cdot \pi_2^{r-m}(\beta).
$$

Then Fr$(f)$ is a well-defined continuous map preserving basepoints.

Consider a few particular cases.
1) Suppose \( m = 0 \), then \( \mathrm{Fr}(M_{m}(\mathbb{C}) , \varepsilon) = \{ \varepsilon \} \) is the space consisting of one point, and for a morphism \( f: (M_{m}(\mathbb{C}) , \varepsilon) \rightarrow (M_{k'p}(\mathbb{C}) , \beta) \) the induced map

\[
\mathrm{Fr}(f): \mathrm{Fr}(M_{m}(\mathbb{C}), \varepsilon) \rightarrow \mathrm{Fr}(M_{k'p}(\mathbb{C}), \beta), \quad \varepsilon \mapsto \varepsilon \cdot \beta = \beta
\]

is the inclusion of the basepoint (note that \( \pi_0^k(\beta) = \varepsilon \), cf. Definition \([12]\) (ii)).

2) Suppose \( n = 0 \), then \( \mathrm{Fr}(M_{k^{m}}(\mathbb{C}), \alpha) = \mathrm{PU}(k^{m}) \) and \( \alpha \) corresponds to the unit in the group \( \mathrm{PU}(k^{m}) \). For a morphism \( f: (M_{k^{m}}(\mathbb{C}), \alpha) \rightarrow (M_{k^{r}}(\mathbb{C}), \beta) \) the diagram

\[
\begin{array}{ccc}
\mathrm{Fr}(M_{k^{m}}(\mathbb{C}), \alpha) & \xrightarrow{\mathrm{Fr}(f)} & \mathrm{Fr}(M_{k^{r}}(\mathbb{C}), \beta) \\
\downarrow & & \downarrow \\
\mathrm{PU}(k^{m}) & \xrightarrow{\ldots \otimes E_{k^{r}-m}} & \mathrm{PU}(k^{r})
\end{array}
\]

is commutative (the lower row corresponds to the homomorphism \( X \mapsto X \otimes E_{k^{r}-m} \)).

3) \( r = m \), \( f: (M_{k^{m}n}(\mathbb{C}), \alpha) \rightarrow (M_{k^{m}l}(\mathbb{C}), \beta) \). Then \( \beta = f_{*}(\alpha) \), \( \pi_0^k(\beta) = \varepsilon \) (cf. Definition \([12]\) (ii)) \( \mathrm{Fr}(f)(\alpha') = f_{*}(\alpha') \).

4.3. **Natural transformation** \( \mu: \mathrm{Fr}(\ldots) \times \mathrm{Fr}(\ldots) \rightarrow \mathrm{Fr}(\ldots \otimes \ldots) \). Using the bifunctor \( \otimes \) on the category \( C_{k,i} \) we define a natural transformation of functors \( \mu: \mathrm{Fr}(\ldots) \times \mathrm{Fr}(\ldots) \rightarrow \mathrm{Fr}(\ldots \otimes \ldots) \) from the category \( C_{k,i} \times C_{k,i} \) to the category of topological spaces with a chosen basepoint. More precisely,

\[
\mu: \mathrm{Fr}(M_{k^{m}n}(\mathbb{C}), \alpha) \times \mathrm{Fr}(M_{k^{p}l}(\mathbb{C}), \varphi) \rightarrow \mathrm{Fr}(M_{k^{m}n}(\mathbb{C}) \otimes M_{k^{p}l}(\mathbb{C})), \alpha \otimes \varphi),
\]

(recall that \( M_{k^{m}n}(\mathbb{C}) \otimes M_{k^{p}l}(\mathbb{C}) \cong M_{k^{m+p}n+q}(\mathbb{C}) \)), where \( \alpha' \otimes \varphi' \) is the \( k^{m+p} \)-frame which is the tensor product of the \( k^{m} \)-frame \( \alpha' \) and the \( k^{p} \)-frame \( \beta' \).

In fact, \( \mu \) is a natural transformation, because for any two morphisms in \( C_{k,i} \)

\[
f: (M_{k^{m}n}(\mathbb{C}), \alpha) \rightarrow (M_{k^{r}l}(\mathbb{C}), \beta), \quad g: (M_{k^{p}l}(\mathbb{C}), \varphi) \rightarrow (M_{k^{t}u}(\mathbb{C}), \psi)
\]

the diagram

\[
\begin{array}{ccc}
\mathrm{Fr}(M_{k^{m}n}(\mathbb{C}), \alpha) \times \mathrm{Fr}(M_{k^{p}l}(\mathbb{C}), \varphi) & \xrightarrow{\mu} & \mathrm{Fr}(M_{k^{m}n}(\mathbb{C}) \otimes M_{k^{p}l}(\mathbb{C})), \alpha \otimes \varphi) \\
\mathrm{Fr}(f) \times \mathrm{Fr}(g) & & \mathrm{Fr}(f) \times \mathrm{Fr}(g)
\end{array}
\]

\((18)\)

is commutative. Indeed, \( \mu \circ (\mathrm{Fr}(f) \times \mathrm{Fr}(g))(\alpha', \varphi') = \mu(f_{*}(\alpha') \cdot \gamma, g_{*}(\varphi') \cdot \chi) = (f_{*}(\alpha') \cdot \gamma) \otimes (g_{*}(\varphi') \cdot \chi) \), where \( \gamma, \chi \) are \( \pi_{-}^{k-r}(\beta) \) and \( \pi_{-}^{k-p}(\psi) \) respectively. On the other hand, \( \mathrm{Fr}(f \otimes g) \circ \mu(\alpha', \varphi') = \mathrm{Fr}(f \otimes g)(\alpha' \otimes \varphi') = (f_{*}(\alpha') \otimes g_{*}(\varphi')) \cdot \Xi \), where \( \Xi \) is the unique \( k^{r+l-m-p} \)-frame such that \( (f_{*}(\alpha) \otimes g_{*}(\varphi)) \cdot \Xi = \beta \otimes \psi \). But \( (f_{*}(\alpha') \cdot \gamma) \otimes (g_{*}(\varphi') \cdot \chi) = (f_{*}(\alpha') \otimes g_{*}(\varphi')) \cdot (\gamma \otimes \chi) \) by virtue of the commutativity of frames, and moreover \( (f_{*}(\alpha) \cdot \gamma) \otimes (g_{*}(\varphi) \cdot \chi) = \beta \otimes \psi \). Hence \( \Xi = \gamma \otimes \chi \) and the diagram commutes, as claimed.
4.4. Properties of the natural transformation $\mu$. First, the natural transformation $\mu$ is associative in the sense that the functor diagram

$$
\begin{align*}
\operatorname{Fr}(\ldots) \times (\operatorname{Fr}(\ldots) \times \operatorname{Fr}(\ldots)) & \cong (\operatorname{Fr}(\ldots) \times \operatorname{Fr}(\ldots)) \times \operatorname{Fr}(\ldots) \\
\mu \times \operatorname{id}_\mu & \\
\operatorname{Fr}(\ldots) \times \operatorname{Fr}(\ldots) & \rightarrow \operatorname{Fr}(\ldots) \times \operatorname{Fr}(\ldots)
\end{align*}
$$

commutes (we have used the natural isomorphism $\operatorname{Fr}((\ldots) \times ((\ldots) \times (\ldots))) \cong \operatorname{Fr}(((\ldots) \times (\ldots)) \times (\ldots))$ in the lower right corner).

Secondly, we need the diagram for identity. Recall that in the monoidal category $\mathcal{C}_{k,l}$ $e = (\mathbb{C}, \varepsilon)$ is the unit object, and it is also the initial object. In particular, for any object $A = (M_{km}m(C), \alpha)$ there is a unique morphism $\iota_A: e \to A$, i.e. $\iota_A: (\mathbb{C}, \varepsilon) \to (M_{km}m(C), \alpha)$. The identity diagram has the following form:

$$
\begin{align*}
\operatorname{Fr}(e) \times \operatorname{Fr}(\ldots) & \rightarrow \operatorname{Fr}(\ldots) \times \operatorname{Fr}(\ldots) \\
\mu & \\
\operatorname{Fr}(\ldots) \times \operatorname{Fr}(\ldots) & \rightarrow \operatorname{Fr}(\ldots)
\end{align*}
$$

(note that $\operatorname{Fr}(\iota): \operatorname{Fr}(e) \to \operatorname{Fr}(\ldots)$ is the inclusion of the basepoint). It is easy to see that (for any pair of objects $A, B$ of $\mathcal{C}_{k,l}$) the slanted arrows are homeomorphisms on their images.

There is also the commutativity diagram

$$
\begin{align*}
\operatorname{Fr}(\ldots) \times \operatorname{Fr}(\ldots) & \rightarrow \operatorname{Fr}(\ldots) \times \operatorname{Fr}(\ldots) \\
\mu & \\
\operatorname{Fr}(\ldots) \times \operatorname{Fr}(\ldots) & \rightarrow \operatorname{Fr}(\ldots)
\end{align*}
$$

(where $\tau$ is the map which switches the factors) which is commutative up to isomorphism. This gives us a homotopy $\mu \circ \tau \simeq \mu$.

For any pair $A, B$ of objects of $\mathcal{C}_{k,l}$ the natural transformation $\mu$ determines a continuous map $\overline{\mu}_{A,B}: \operatorname{Fr}(A) \times \operatorname{Fr}(B) \to \operatorname{Fr}(A \otimes B)$ of topological spaces.

Put $\operatorname{Fr}_{k,l,\infty} := \lim_{i} \operatorname{Fr}(M_{k}m_{l}m(C), \alpha)$. Then the above diagrams show that $\operatorname{Fr}_{k,l,\infty}$ is a homotopy associative and commutative $H$-space with multiplication given by $\overline{\mu} := \lim_{i} \overline{\mu}_{A,B}$ and with the homotopy unit $\overline{\eta} := \lim_{i} \operatorname{Fr}(\iota_A): * = \operatorname{Fr}(e) \to \operatorname{Fr}_{k,l,\infty}$.

4.5. $H$-space structure on the matrix grassmannian. Note that the analogous construction can be applied to matrix grassmannians (in place of frame spaces). Namely, consider the category $\mathcal{D}_{k,l}$ whose objects are pairs of the form $(M_{km}m(C), A)$, consisting of a matrix algebra $M_{km}m(C)$, $m, n \geq 0$ and a $k^m$-subalgebra $A \subset M_{km}m(C)$ in it (recall that a $k$-subalgebra is a unital $*$-subalgebra isomorphic $M_k(C)$). A morphism $f: (M_{km}m(C), A) \to (M_{k}m_{l}m(C), B)$ in $\mathcal{D}_{k,l}$
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is a unital $*$-homomorphism of matrix algebras $f: M_{k^m,l^m}(C) \to M_{k^r,l^r}(C)$ such that $f(A) \subset B$. We define the $k^r-m$-subalgebra $C \subset M_{k^r,l^r}(C)$ as the centralizer of the subalgebra $f(A)$ in $B$.

Define the functor $\text{Gr}$ from the category $\mathcal{D}_{k,l}$ to the category of topological spaces with a chosen basepoint as follows. On objects the space $\text{Gr}(M_{k^m,l^m}(C), A)$ is the space of all $k^m$-subalgebras in $M_{k^m,l^m}(C)$ and $A$ corresponds to its basepoint. For a morphism $f: (M_{k^m,l^m}(C), A) \to (M_{k^r,l^r}(C), B)$ as above we put

$$\text{Gr}(f): \text{Gr}(M_{k^m,l^m}(C), A) \to \text{Gr}(M_{k^r,l^r}(C), B), \quad \text{Gr}(f)(A') = f(A') \cdot C,$$

where $C$ is the centralizer of the subalgebra $f(A)$ in $B$ and $f(A') \cdot C$ denotes the subalgebra in $M_{k^r,l^r}(C)$ generated by subalgebras $f(A')$ and $C$ (clearly, $B = f(A) \cdot C$). Then one can define the analog $\mu': \text{Gr}(\ldots) \times \text{Gr}(\ldots) \to \text{Gr}(\ldots \otimes \ldots)$ of the natural transformation $\mu$, etc. (For example, the commutativity of the analog of diagram [IS] follows from the coincidence $Z_B \otimes \Psi(f(A) \otimes g(\Phi)) = Z_B(f(A)) \otimes Z_\Psi(g(\Phi))$ for any two morphisms $f: (M_{k^m,l^m}(C), A) \to (M_{k^r,l^r}(C), B)$ and $g: (M_{k^r,l^r}(C), \Phi) \to (M_{k^r,l^r}(C), \Psi)$ (which is an analog of the above formula $\Xi = \gamma \otimes \chi$ for frames), where $Z_B(A)$ denotes the centralizer of a subalgebra $A$ in an algebra $B$.)

This allows us to equip the direct limit $\text{Gr}_{k^\infty, l^\infty} := \lim_{(f)} \text{Gr}(M_{k^m,l^m}(C), A)$ with the structure of a homotopy associative and commutative $H$-space with a homotopy unit.

Note that there is the functor

$$\lambda: \mathcal{C}_{k,l} \to \mathcal{D}_{k,l}, \quad (M_{k^m,l^m}(C), \alpha) \mapsto (M_{k^m,l^m}(C), M(\alpha)),$$

where $M(\alpha)$ is the $k^m$-subalgebra spanned on the $k^m$-frame $\alpha$. There is the obvious natural transformation of functors $\theta: \text{Fr} \to \text{Gr} \circ \lambda$ from the category $\mathcal{C}_{k,l}$ to the category of topological spaces with a chosen basepoint which gives rise to the $H$-space homomorphism $\text{Fr}_{k^\infty, l^\infty} \to \text{Gr}_{k^\infty, l^\infty}$.

Recall that $\text{Gr}_{k^\infty, l^\infty} = \text{Gr} \cong \text{BSU}_\otimes$, and the image of the just constructed homomorphism is the $k$-torsion subgroup in it, as the next proposition claims.

**Proposition 14.** Let $X$ be a compact space. Then the image of the homomorphism $[X, \text{Fr}_{k^\infty, l^\infty}] \to [X, \text{Gr}_{k^\infty, l^\infty}]$ is the $k$-torsion subgroup in the group $\text{BSU}_\otimes^0(X)$.

**Proof.** This proposition follows from Proposition [5]. Another way is to pass to the direct limit in fibration

$$\text{Fr}_{k^m, l^m} \to \text{Gr}_{k^m, l^m} \to \text{BPU}(k^n),$$

and to notice that the limit map $\text{Gr}_{k^\infty, l^\infty} \to \text{BPU}(k^\infty) := \lim_n \text{BPU}(k^n)$ actually is a localization on $k$. □

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