Estimates of the Weak Annihilation Contributions to the Decays \( B \to \rho + \gamma \) and \( B \to \omega + \gamma \)

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Abstract

The dominant long-distance contributions to the exclusive radiative decays \( B \to \rho(\omega) + \gamma \) involve photon emission from the light quarks at large distances, which cannot be treated perturbatively. We point out that this emission can be described in a theoretically consistent way as the magnetic excitation of quarks in the QCD vacuum and estimate the corresponding parity-conserving and parity-violating amplitudes using the light-cone QCD sum rule approach. These are then combined with the corresponding short-distance contribution from the magnetic moment operator in the same approach, derived earlier, to estimate the decay rates \( \Gamma(B \to \rho(\omega) + \gamma) \). The implications of this result for the determination of the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements in radiative \( B \) decays are worked out.

Submitted to Physics Letters B
1. Exclusive radiative B decays provide a valuable source of information about the CKM matrix. Assuming that the short-distance (SD) contribution of the magnetic moment operator is dominating these transitions, one derives \[1\]

\[
\frac{\Gamma(B_{u,d} \to \rho \gamma)}{\Gamma(B_{u,d} \to K^* \gamma)} = \left( \frac{|V_{td}|}{|V_{ts}|} \right)^2 \xi_{u,d},
\]

where \(\xi_{u,d}\) takes into account SU(3) breaking in the decay form factors and the (nominal) phase space differences. With this assumption and isospin invariance one also expects:

\[
\Gamma(B_u \to \rho^+ \gamma) = 2 \Gamma(B_d \to \rho^0 \gamma) = 2 \Gamma(B_d \to \omega \gamma). \tag{2}
\]

These relations have been used to convert the experimental upper bound on the ratio of the exclusive radiative B decays \[2\]:

\[
\frac{B(B \to \rho(\omega)\gamma)}{B(B \to K^* \gamma)} < 0.34 (90\% C.L.), \tag{3}
\]

into a bound \(|V_{td}|/|V_{ts}| < 0.64 - 0.76\), depending on the estimate of the SU(3)-breaking parameters in the SD-piece \[1\ [3]\ [4]\. While this bound is at present not competitive with the corresponding bound from the CKM unitarity \[3\] and from the fits of the CKM matrix elements \[3\] which yield

\[
\frac{|V_{td}|}{|V_{ts}|} < 0.36, \tag{4}
\]

one anticipates that the increased sensitivity in the radiative B decay modes projected for the upgraded CLEO detector and the high luminosity B factories will allow to test the relationships \[1\ - \(2\) quantitatively.

The troublesome point, which has been raised in a number of papers \[7\ - \[13]\, is the possibility of significant long-distance (LD) contributions to radiative B decays from the light quark intermediate states. Their amplitudes necessarily involve other CKM matrix elements and hence the simple factorization of the decay rates in terms of the CKM factors involving \(|V_{td}|\) and \(|V_{ts}|\) no longer holds thereby invalidating the relationships \[1\] and \[2\]. A redeeming feature, however, is that the experiments by testing \[2\] can determine in a model-independent way the extent to which the radiative transitions in question are dominated by the SD contributions. In general, light quark \((u,c)\) intermediate states enter in the magnetic moment transitions through the corresponding Wilson coefficient \(C^\text{eff}_7(\mu)\) (see below), and through the transitions induced by the matrix elements of the four-Fermion operators. In fact, the effect of the light quarks to \(C^\text{eff}_7(\mu)\) is completely negligible as demonstrated through the explicit calculations \[14\ [15]\, being suppressed by powers \(O(m_i/m_W)^2\) and \(O(m_i/m_t)^2\) with \(m_i = m_u\) or \(m_c\) \[16\]. Estimates of the contribution of the four-Fermion operators require a certain non-perturbative technique and have been worked out using quark models and the vector meson dominance (VDM) approximation. The purpose of this letter is to suggest an alternative technique, which treats the photon emission from the light quarks in a theoretically consistent and model-independent way.

\[1\] See also \[13\] for a recent discussion of this point.
We then combine this treatment with the light-cone QCD sum rule approach to calculate both the SD and LD — parity conserving and parity violating — amplitudes in the decays $B_{u,d} \to \rho(\omega) + \gamma$. We find that the LD contributions are negligible in the neutral $B$-meson decays $B_d \to \rho(\omega) + \gamma$ but they may contribute up to $\pm 20\%$ corrections in the decay rate of the charged $B$ meson $B_u^\pm \to \rho^\pm + \gamma$. We work out the modified form of the relations ([1] and [2]), and work out the consequences of this result for the extraction of the CKM parameters from exclusive radiative $B$ decays.

2. For subsequent use, we collect the definitions used in this work. Radiative weak transitions at the $B$-meson scale are governed by the effective Hamiltonian
\[ \mathcal{H} = \frac{G_F}{\sqrt{2}} \left[ V_{ub} V^*_{ud} \left( C_1 O_1 + C_2 O_2 \right) - V_{tb} V^*_{td} C_7^{\text{eff}} O_7 + \ldots \right], \]
where we have shown only the contributions which will be important for what follows. The $O_1, O_2$ are the standard four-Fermion operators
\[ O_1 = (\bar{d}_\alpha \Gamma^\mu u_\beta)(\bar{u}_\beta \Gamma_\mu b_\alpha), \]
\[ O_2 = (\bar{d}_\alpha \Gamma^\mu u_\beta)(\bar{u}_\beta \Gamma_\mu b_\beta), \]
\[ \Gamma^\mu = \gamma^\mu(1 + \gamma_5) \] and $\alpha, \beta$ are color indices, the $C_1, C_2$ are the corresponding coefficients (depending on the scale $\mu$); $O_7$ is the magnetic moment operator
\[ O_7 = \frac{e m_b}{8\pi^2} \bar{d}\sigma_{\mu\nu}(1 - \gamma_5) F^{\mu\nu} b \]
and $F_{\mu\nu}$ is the electromagnetic field strength tensor, which we take, for the photon emission with momentum $q$ and polarization $\varepsilon^{(\gamma)}_\mu$, to be
\[ F_{\mu\nu}(x) = i(\varepsilon^{(\gamma)}_\nu q_\mu - \varepsilon^{(\gamma)}_\mu q_\nu)\varepsilon^{i\gamma x}. \]
The coefficient $C_7^{\text{eff}}$ includes also the effect of the four-Fermi operators $O_5$ and $O_6$. For details and numerical values of these coefficients, see [16, 17]. We concentrate on the $B_u^\pm$ decays, $B_u^\pm \to \rho^\pm + \gamma$ and take up the neutral $B$ decays $B_d \to \rho(\omega) + \gamma$ at the end. The SD contribution to the decay rate involves the matrix element
\[ \langle \rho|O_7|B \rangle = \frac{e m_b}{8\pi^2} (-2i)\varepsilon^{(\gamma)\mu} \langle \rho|\bar{d}\sigma_{\mu\nu}q^\nu(1 - \gamma_5)b|B(p)\rangle, \]
which is parametrized in terms of two invariant form factors
\[ \langle \rho(k)|\bar{d}\sigma_{\mu\nu}q^\nu(1 - \gamma_5)b|B(p)\rangle = \left[ \varepsilon^{(\rho)}_\mu (q \cdot p) - p_\mu (q \cdot \varepsilon^{(\rho)}) \right] \cdot 2F_1^S(q^2) + i\epsilon_{\mu\nu\alpha\beta}\varepsilon^{(\rho)\mu} p^\alpha q^\beta \cdot 2F_2^S(q^2). \]
Here $p$ and $k = p - q$ are, respectively, the $B$-meson and $\rho$-meson momentum and $\varepsilon^{(\rho)}$ is the polarization vector of the $\rho$ meson. For the real photon emission the two form factors coincide, $F_1^S = F_2^S \equiv F^S$. This form factor was calculated in [1] using the light-cone QCD sum rules, and we shall follow this approach in this paper as well.

\[ \text{Our conventions for the } \gamma_5\text{-matrix and } \epsilon_{\mu\nu\alpha\beta}\text{ tensor conform to those in [2].} \]
Combining the above expressions we get the SD decay amplitude

\[ A_{\text{short}} = - \frac{G_F}{\sqrt{2}} V_{tb} V_{td}^* C_7 \frac{e^2 m_b}{4 \pi^2} \varepsilon^{(\gamma)\mu} \varepsilon^{(\rho)\nu} \left\{ \epsilon_{\mu\alpha\beta} p^\alpha q^\beta - i \left[ g_{\mu\nu} (q \cdot p) - p_\mu q_\nu \right] \right\} \cdot 2 F^S(q^2 = 0). \tag{11} \]

The LD contributions of the four-Fermion operators \( O_1, O_2 \) involve two possibilities for the flavour flow, shown schematically in Fig. 1. In this paper we consider the contribution of the weak annihilation of valence quarks in the B meson, Fig. 1a,b. It is color-allowed for the decays of charged B mesons, and is expected to give the dominant LD contribution in this case. In the factorization approximation, we write

\[ \langle \rho \gamma | O_2 | B \rangle = \langle \rho | d \Gamma_{\mu} u | 0 \rangle \langle \gamma | \bar{u} \Gamma^\mu b | B \rangle + \langle \rho \gamma | d \Gamma_{\mu} u | 0 \rangle \langle 0 | \bar{u} \Gamma^\mu b | B \rangle, \tag{12} \]

and make use of the definitions of the decay constants

\[ \langle 0 | \bar{u} \Gamma^\mu b | B \rangle = i p_\mu f_B, \]
\[ \langle \rho | d \Gamma_{\mu} u | 0 \rangle = \varepsilon^{(\rho) \mu} m_\rho f_\rho, \tag{13} \]

to reduce the problem at hand to the calculation of simpler form factors induced by vector and axial-vector currents. The two terms in (12) correspond in an obvious way to the contributions of photon emission from the loop containing the b quark, Fig. 1a, and from the loop of the light quarks, Fig. 1b, respectively. The latter subprocess involves an (axial) vector transition with the the momentum transfer \( m_b^2 \). According to our estimates, this contribution is much smaller than the one coming from the first term, which is dominated by the photon emission from a soft u-quark and which we shall consider in detail. To that end, we write down

\[ f_\rho \langle \gamma | \bar{u} \Gamma^\mu b | B \rangle = - e \varepsilon^{(\gamma)\mu} \left\{ - i \left[ g_{\mu\nu} (q \cdot p) - p_\mu q_\nu \right] \cdot 2 F^L_1(q^2) + \epsilon_{\mu\alpha\beta} p^\alpha q^\beta \cdot 2 F^L_2(q^2) \right\} + \text{contact terms}, \tag{14} \]

where the two form factors describe parity-violating and parity-conserving amplitudes, respectively. Note that we have included the factor \( f_\rho \) on the l.h.s. of (14) to make the form factors dimensionless. The contact terms indicated on the r.h.s. of (14) depend on the gauge chosen for the electromagnetic field and should be omitted as they are identically cancelled by similar gauge-dependent contact terms contributing to the second term in (12). Physically, they correspond to contributions of photon emission from the \( W^\pm \) boson and arise in intermediate steps because the factorization approximation in (14) introduces charged weak currents. Adding the contribution of the operator \( O_1 \), we obtain the LD amplitude to the decay \( B_u \rightarrow \rho + \gamma \) in terms of the form factors \( F^L_1 \) and \( F^L_2 \),

\[ A_{\text{long}} = - \frac{e G_F}{\sqrt{2}} V_{ub} V_{ud}^* \left( C_2 + \frac{1}{N_c} C_1 \right) m_\rho \varepsilon^{(\gamma)\mu} \varepsilon^{(\rho)\nu} \left\{ - i \left[ g_{\mu\nu} (q \cdot p) - p_\mu q_\nu \right] \cdot 2 F^L_1(q^2) + \epsilon_{\mu\alpha\beta} p^\alpha q^\beta \cdot 2 F^L_2(q^2) \right\}. \tag{15} \]

We now proceed to estimate the form factors defined in (14).

3. The principal contribution to the form factors is due to the photon emission from the u-quark which involves both contributions of small (of order \( 1/m_b \)) and large (of order
\(1/\Lambda_{QCD}\) distances. A consistent tool to separate them is provided by the Operator Product Expansion, and is most easily implemented using the external field technique\(^\text{18}\). To this end we consider quark propagation in the background of the external electromagnetic field \(V_\mu\), adding to the QCD Lagrangian density an extra term \(\delta L = -e j_{\mu}^{em}(x)V_\mu(x)\) where \(j_{\mu}^{em}\) is the electromagnetic current. The photon emission is obtained as the linear term in the expansion of the relevant Green functions in the external field. The corresponding formalism has been worked out in detail in\(^\text{18, 19, 20, 22}\) and in this paper we only give a summary of the results.

To the leading twist accuracy, that is when the quark is propagating close to the light-cone, all the necessary information about the magnetic photon emission is contained in a single (nonperturbative) matrix element of the gauge-invariant nonlocal operator \(\bar{\psi}(0)\psi(x)\) at light-like separations \(x^2 = 0\)\(^\text{20, 22}\):

\[
\langle \bar{\psi}(0)\sigma_{\alpha\beta}\psi \rangle_F = e_\psi \chi \langle \bar{\psi}\psi \rangle_F \int_0^1 du \phi_\gamma(u) F_{\alpha\beta}(ux).
\]

(16)

The \(\langle \ldots \rangle_F\) denotes the vacuum expectation value in the external field \(F\). Note that the path-ordered exponential factor includes both the gluon field \(A_\mu\) and the photon field \(V_\mu\). In what follows we drop the gauge factors, assuming the Fock-Schwinger gauge \(x_\mu A_\mu = x_\mu V_\mu = 0\). The normalization is chosen in such a way that \(\int_0^1 du \phi_\gamma(u) = 1\), \(\langle \bar{\psi}\psi \rangle\) is the quark condensate and the dimensionful constant \(\chi\) has the physical meaning of being the magnetic susceptibility of the quark condensate. In the limit of a constant external magnetic field \(F_{\alpha\beta}(x) = \text{const.}\), the spins of quarks in the vacuum tend to get oriented along the field direction, with the average spin being proportional to the quark density \(\langle \bar{\psi}\psi \rangle\), the applied field \(F_{\alpha\beta}\), the quark charge \(e_\psi\) and the magnetic susceptibility of the medium\(^\text{18}\):

\[
\langle \bar{\psi}\sigma_{\alpha\beta}\psi \rangle_F = e_\psi \chi \langle \bar{\psi}\psi \rangle F_{\alpha\beta}.
\]

(17)

The value of the magnetic susceptibility is large\(^\text{24}\)

\[
\chi(\mu = 1 \text{ GeV}) = -4.4 \text{ GeV}^{-2},
\]

(18)

and because of this the nonperturbative photon emission is numerically very important.\(^\text{1}\)

The physical meaning of the response function \(\phi_\gamma(u)\) becomes transparent for the particular choice of the external field as a plain wave\(^\text{8}\). In this case\(^\text{10}\) can be rewritten as

\[
\int dy e^{iyq} \langle 0|T\{j_{\mu}^{em}(y)\bar{\psi}(0)\sigma_{\alpha\beta}\psi(x)\}|0\rangle = e_\psi \chi \langle \bar{\psi}\psi \rangle [q_\beta g_{\alpha\mu} - q_\alpha g_{\beta\mu}] \int_0^1 du e^{iuqx} \phi_\gamma(u).\]

(19)

 Comparing this expression with the usual definition of the light-cone hadron wave functions\(^\text{23}\) it is easy to see that \(\phi_\gamma(u)\) defines the distribution amplitude (photon wave function) describing the photon dissociation into a quark-antiquark pair with the variable \(u\) being just the momentum fraction carried by the quark. The shape of this distribution in general depends on the normalization scale for the quark fields. However, an accurate analysis

\(^\text{1}\) A larger value \(\chi = -5.7 \text{ GeV}^{-2}\) given in the first of refs.\(^\text{24}\) corresponds to a lower normalization point \(\mu = 500 \text{ MeV}\).
shows [20] that unlike meson wave functions [23], this distribution is close to its asymptotic form already at low virtualities \( \mu \sim 1 \text{ GeV} \):

\[
\phi_\gamma(u) = 6u(1 - u). \tag{20}
\]

Corrections to (16) come from higher-twist contributions which are formally suppressed by powers of the deviation from the light-cone \( x^2 \). For hadron wave functions these can be attributed to contributions of higher Fock-components in the wave function with a larger number of constituents. The case of the photon is still specific, since additional contributions arise related to operators involving the photon field instead of the gluon field\( ^4 \). These specific contributions are more important than those involving quark-antiquark-gluon operators (whose photon-to-vacuum matrix elements are numerically small, see [20, 22] for the list of all the contributions to twist-4 accuracy and numerical estimates) and can be calculated exactly in terms of the quark condensate.

The most general decomposition of the relevant matrix element to the twist-4 accuracy involves two new invariant functions (distribution amplitudes) \( g^{(1)}_\gamma(u) \) and \( g^{(2)}_\gamma(u) \):

\[
\langle \bar{\psi}(0) \sigma_{\alpha\beta} \psi(x) \rangle_F = e_\psi \langle \bar{\psi}\psi \rangle \int_0^1 du F_{\alpha\beta}(ux) \left[ \chi \phi_\gamma(u) + x^2 g^{(1)}_\gamma(u) \right] + e_\psi \langle \bar{\psi}\psi \rangle \int_0^1 du g^{(2)}_\gamma(u) \left[ x_\beta x_\eta F_{\alpha\eta} - x_\alpha x_\eta F_{\beta\eta} - x^2 F_{\alpha\beta} \right](ux). \tag{21}
\]

For one particular projection of Lorentz indices the answer is given in [20, 22]:

\[
\not{x} \sigma_{\alpha\beta} \not{x} \langle \bar{\psi}(0) \sigma_{\alpha\beta} \psi(x) \rangle_F = e_\psi \langle \bar{\psi}\psi \rangle \not{x} \sigma_{\alpha\beta} \not{x} \langle \bar{\psi}(0) \sigma_{\alpha\beta} \psi(x) \rangle_F = e_\psi \langle \bar{\psi}\psi \rangle \not{x} \sigma_{\alpha\beta} \not{x} \int_0^1 du F_{\alpha\beta}(ux) \left[ \chi \phi_\gamma(u) - \frac{x^2}{4}(1 - u) + \text{gluons + } O(x^4) \right], \tag{22}
\]

which implies

\[
g^{(1)}_\gamma(u) - \frac{1}{2} g^{(2)}_\gamma(u) = -\frac{1}{4}(1 - u). \tag{23}
\]

To determine the functions \( g^{(1)}_\gamma(u) \) and \( g^{(2)}_\gamma(u) \) separately, we use the operator identity

\[
\bar{\psi}(0) \sigma_{\alpha\beta} \psi(x) = \int_0^1 v dv \left[ \frac{\partial}{\partial x_\beta} \bar{\psi}(0) \sigma_{\alpha\eta} x_\eta \psi(vx) - (\alpha \leftrightarrow \beta) \right] - \epsilon_{\alpha\beta\eta\rho} \int_0^1 v dv x_\eta \frac{\partial}{\partial x_\rho} \bar{\psi}(0) \gamma_5 \psi(vx). \tag{24}
\]

To prove (24), consider a simpler identity

\[
\bar{\psi}(0) \sigma_{\alpha\beta} \psi(x) = \int_0^1 dv \left[ v^2 \frac{d}{dv} \bar{\psi}(0) \sigma_{\alpha\beta} \psi(vx) + 2v \bar{\psi}(0) \sigma_{\alpha\beta} \psi(vx) \right], \tag{25}
\]

which can easily be checked integrating by parts. Then substitute \( v(d/dv) = x_\xi (\partial/\partial x_\xi) \) and subtract from (25) the first term on the r.h.s. of (24). By simple algebra the difference can be written as

\[
\int_0^1 v^2 dv \bar{\psi}(0) \left[ x_\rho \partial_\rho \sigma_{\alpha\beta} - x_\eta \partial_\eta \sigma_{\alpha\eta} + x_\eta \partial_\alpha \sigma_{\beta\eta} \right] \psi(vx), \tag{26}
\]

\( ^4 \)These additional contributions can also be thought of as contact terms, produced by operators which vanish on using the equations of motion. The external field technique avoids all contact terms at the cost of introducing additional operators.
where \( \partial_\eta \equiv \partial/\partial x_\eta \). Since for arbitrary Lorentz vectors \( x_\alpha, y_\beta \), one has
\[
(x \cdot y)\sigma_{\alpha\beta} - y_\beta \sigma_{\alpha\eta} x_\eta + y_\alpha \sigma_{\beta\eta} x_\eta = -\epsilon_{\alpha\beta\eta\rho} x_\eta y_\rho \gamma_5 ,
\]
this expression coincides with the second term on the r.h.s. of (24).

To use (24), note that the vacuum-to-photon transition matrix element of the second term on the r.h.s. vanishes. Substituting the general expression (21) for the matrix elements in the first term, one obtains the (exact) relation
\[
2u^3 \int_0^1 \frac{dv}{v^4} g^{(1)}_\gamma(u) = g^{(2)}_\gamma(u) .
\]
(28)

Solving the system of equations (23), (28) we get:
\[
g^{(1)}_\gamma(u) = -\frac{1}{8}(1 - u)(3 - u) ,
\]
\[
g^{(2)}_\gamma(u) = -\frac{1}{4}(1 - u)^2 ,
\]
(29)
which is our final result for the photon emission to twist-4 accuracy. The contributions of the quark-antiquark-gluon operators can be added using (24) and the expressions given in [22].

They are numerically small and are omitted in the numerical analysis given below. Taking the external field in the form of the plain wave (8) we get an equivalent representation:
\[
\int dy e^{iqy} \langle 0 | T \{ j^e_m (y) \bar{\psi}(0) \sigma_{\alpha\beta} \psi(x) \} | 0 \rangle =
\]
\[
= e_\psi \langle \bar{\psi} \psi \rangle [q_3 g_{\alpha\mu} - q_\alpha g_{3\mu}] \int_0^1 du e^{iux} \left[ \chi \phi_\gamma(u) + x^2 g^{(1)}_\gamma(u) \right] + e_\psi \langle \bar{\psi} \psi \rangle \left\{ (qx) [x_\beta g_{\alpha\mu} - x_\alpha g_{3\mu}] \\
+ x_\mu [q_3 x_\alpha - q_\alpha x_3] - x^2 [q_3 g_{\alpha\mu} - q_\alpha g_{3\mu}] \right\} \int_0^1 du e^{iux} g^{(2)}_\gamma(u) .
\]
(30)

4. The expressions given above are sufficient for the description of real photon emission in the case that the kinematics of the particular process ensures that the quark propagates near the light-cone. To make a quantitative estimate of the form factors, we use the modification of the QCD sum rule technique, suggested in [20, 25, 22, 1].

The essence of this approach is to avoid introduction of model-dependent wave functions of the \( B \) meson, replacing it by a suitable interpolation operator, and using dispersion relations and duality to pick up the contribution of the \( B \) meson. Following this method, we consider the correlation function
\[
T_{\mu\nu}(p, q) = i \int dx e^{-ipx} \int dy e^{iqy} \langle 0 | T \{ j^e_m (y) \bar{u}(0) \Gamma_\nu b(0) \bar{b}(x) i\gamma_5 u(x) \} | 0 \rangle
\]
\[
= i[(p \cdot q) g_{\mu\nu} - p_\mu q_\nu] T_1(p^2) + \epsilon_{\mu\nu\alpha\beta} p_\alpha q_\beta T_2(p^2) + O(p_\nu) + \text{contact terms} ,
\]
(31)
with fixed \( q^2 = 0 \) and \( (p - q)^2 = m_B^2 \). The invariant functions \( T_1 \) and \( T_2 \) can be calculated in QCD at large negative \( p^2 \), in which case the above formalism to describe photon emission is
fully applicable: higher-twist components in the photon wave functions give rise to contributions suppressed by powers of $p^2$. On the other hand, the discontinuity of the functions $T_1(p^2)$, $T_2(p^2)$ at positive $p^2$ is saturated by contributions of meson states:

$$T_{1,2}(p^2) = \frac{f_B m_b^2}{m_b f_\rho} \frac{2F_{1,2}(0)}{m^2_B - p^2} + \ldots$$  \hspace{1cm} (32)

Making the usual assumption that the $B$-meson contribution corresponds to the spectral density integrated over the “region of duality”, one arrives at the sum rule for the form factors of interest expressed in terms of integrals of photon wave functions. In this letter we cannot give a detailed derivation and refer the reader to [1] for technical discussion.

Repeating the by now standard steps, we arrive at the sum rules:

$$\frac{f_B m_b^2}{m_b f_\rho} 2F_1^L(0)e^{-(m_\rho^2-m_b^2)/t} = \int_0^1 du \frac{e^{-(\bar{u}/u)m_\rho^2/t}\theta(s_0 - m_b^2/u)}{u} \left\{ e_u \langle \bar{\psi} \psi \rangle \left[ \chi \phi_\gamma(u) \right] ight. - \left. \frac{4(m_b^2 + ut)}{u^2 t^2} g_\gamma^{(1)}(u) \right\} + \frac{3m_b}{4\pi^2} \left( 2u - 1 \right) \left[ (e_u - e_b) \bar{u} - e_b \ln u \right],$$  \hspace{1cm} (33)

$$\frac{f_B m_B^2}{m_B f_\rho} 2F_2^L(0)e^{-(m_\rho^2-m_b^2)/t} = \int_0^1 du \frac{e^{-(\bar{u}/u)m_\rho^2/t}\theta(s_0 - m_b^2/u)}{u} \left\{ e_u \langle \bar{\psi} \psi \rangle \left[ \chi \phi_\gamma(u) \right] ight. - \left. \frac{4(m_b^2 + ut)}{u^2 t^2} \left[ g_\gamma^{(1)}(u) - g_\gamma^{(2)}(u) \right] \right\} + \frac{3m_b}{4\pi^2} \left( (e_u - e_b) \bar{u} - e_b \ln u \right),$$  \hspace{1cm} (34)

where we have introduced a shorthand notation $\bar{u} = 1 - u$ and neglected $m_\rho^2$ compared to $m_b^2$, $t$, and $s_0$. Here $t$ is the Borel parameter and $s_0$ is the continuum threshold which stands for the cutoff in the dispersion relation restricting the region of duality for the $B$ meson. In both the sum rules, the first term in the curly bracket corresponds to nonperturbative contributions of photon wave function, and the two last terms are perturbative contributions of the photon emission from $u$- and $b$-quark, respectively. Numerically, the nonperturbative contribution of leading twist dominates and is the same in both the form factors. Thus, in our approach the parity-violating and parity-conserving LD amplitudes are close to each other. For completeness, we quote the corresponding sum rule from [1] for the SD form factor:

$$\frac{f_B m_B^2}{m_B f_\rho} 2F^S(0)e^{-(m_\rho^2-m_b^2)/t} = \int_0^1 du \frac{e^{-(\bar{u}/u)m_\rho^2/t}\theta(s_0 - m_b^2/u)}{u} \left\{ m_b \phi_\rho^+(u) + um_\rho g_\perp^{(v)}(u) \right\} + \frac{m_b^2 + ut}{4ut} m_\rho g_\perp^{(a)}(u),$$  \hspace{1cm} (35)

where $\phi_\rho^+(u)$, $g_\perp^{(v)}(u)$ and $g_\perp^{(a)}(u)$ are the leading-twist $\rho$ meson wave functions, specified in Sec. 4 of [1]. It is seen that the structure of sum rules for the SD and LD form factors is
very similar, and many of the uncertainties (such as the dependence on $f_B$ and $f_\rho$) cancel in their ratio.

Before proceeding to present our numerical results, we would like to remark that the calculations presented here suggest that the VDM approach underestimates the LD amplitudes for the photon emission. The difference between the VDM approach and our method is two-fold. First, the value of the quark magnetic susceptibility in the VDM approximation $\chi_{VDM} = -2/m_\rho^2$ appears to be significantly below the result in (18). Second, the $\rho$-meson wave function is more “narrow” than the photon wave function, see [23, 20, 1] and since the decay kinematics picks up contributions with almost the entire momentum carried by one of the constituents, the $\rho$-meson contribution in VMD is additionally suppressed. Thus, we expect that the VDM-type relations between $B \to \rho\gamma$ and $B \to \rho\rho$ amplitudes receive significant ($\sim 50\%$) corrections from contributions of the excited states.

5. In performing the numerical analysis we conform to the values of the parameters given in [1]. We note that all the three sum rules are dominated by the first term in the curly brackets, and since also the wave functions $\phi_\gamma$ and $\phi_\rho^\perp$ are similar, the ratio of the SD- and LD-form factors takes a simple form:

$$F_L^1/F_S \approx F_L^2/F_S \approx \frac{e_u \chi\langle \bar{\psi}\psi \rangle}{m_b} \approx 0.01 .$$  \hspace{1cm} (36)

The only important correction to this estimate corresponds to the perturbative emission while nonperturbative twist-4 contributions are in fact negligible. Assuming the interval $t \sim 5 - 10 \text{ GeV}^2$ for the Borel parameter, we obtain:

$$F_L^1/F_S = 0.0125 \pm 0.0010 , \quad F_L^2/F_S = 0.0155 \pm 0.0010 ,$$  \hspace{1cm} (37)

where the errors correspond to the variation of the Borel parameter. Including other possible uncertainties, we expect an accuracy of the ratios in (37) to be of order 20%. This can be combined with the result of [1]

$$F_S^{B_{u\to \rho\gamma}} = \sqrt{2} F_S^{B_{d\to \rho\gamma}} = \sqrt{2} F_S^{B_{d\to \omega\gamma}} = 0.24 \pm 0.04 ,$$  \hspace{1cm} (38)

to extract the absolute values of the LD form factors. Since the parity-conserving and parity-violating amplitudes turn out to be close to each other, the ratio of the LD and the SD contributions reduces to a number

$$R_{L/S}^{B_{u\to \rho\gamma}} \equiv \frac{4\pi^2 m_\rho (C_2 + C_1/N_c)}{m_b C_7^{eff}} \cdot \frac{F_L}{F_S} \cdot \frac{V_{ub}V_{ud}^*}{V_{tb}V_{td}^*} .$$  \hspace{1cm} (39)

Using $C_2 = 1.10$, $C_1 = -0.235$, $C_7^{eff} = -0.306$ (at the scale $\mu = 5 \text{ GeV}$) [16, 17], we get

$$R_{L/S}^{B_{u\to \rho\gamma}} = -0.30 \pm 0.07 .$$  \hspace{1cm} (40)

Since the Wilson coefficients are scale-dependent, this ratio is in general also scale-dependent (unless cancelled by a compensating dependence in the form factors.) Varying the scale $\mu$ of the coefficients in the range $m_b/2 \leq \mu \leq 2m_b$, an additional dependence of $\pm 10\%$
is introduced in $R_{L/S}^{B_u \to \rho \gamma}$, which, however, is smaller than the error given above. To get a ball-park estimate of the ratio $A_{\text{long}}/A_{\text{short}}$, we take the central values of the CKM matrix elements, $V_{ud} = 0.9744 \pm 0.0010$ [3], $|V_{td}| = (1.0 \pm 0.045) \cdot 10^{-2}$, $|V_{cb}| = 0.041 \pm 0.004$ and $|V_{ub}/V_{cb}| = 0.08 \pm 0.02$ [4], yielding,

$$|A_{\text{long}}/A_{\text{short}}|_{B_u \to \rho \gamma} = |R_{L/S}^{B_u \to \rho \gamma}| \frac{|V_{ub}V_{ud}|}{|V_{tb}V_{td}|} \simeq 10\% .$$  \hspace{1cm} (41)

A quantitative analysis of the relative decay rates in terms of the CKM parameters is presented below.

The analogous LD-contributions to the neutral $B$ decays $B_d \to \rho \gamma$ and $B_d \to \omega \gamma$ are much smaller, a point that has also been noted in the context of the VMD and quark model based estimates. In our approach, the corresponding form factors for the decays $B_d \to \rho(\omega) \gamma$ are obtained from the ones for the decay $B_u \to \rho \gamma$ reported above by the replacement $e_u \to e_d$, which gives the factor $-1/2$; in addition, and more importantly, the LD-contribution to the neutral $B$ decays is colour-suppressed, which reflects itself through the replacement of the factor $a_1 \equiv C_2 + C_1/N_c$ in [2] by $a_2 \equiv C_1 + C_2/N_c$. This yields for the ratio

$$\frac{R_{L/S}^{B_d \to \rho \gamma}}{R_{L/S}^{B_u \to \rho \gamma}} = \frac{e_d a_2}{e_u a_1} \simeq -0.13 \pm 0.05,$$  \hspace{1cm} (42)

where the numbers are based on using $a_2/a_1 = 0.27 \pm 0.10$ [27]. This would then yield at most $R_{L/S}^{B_d \to \rho \gamma} \simeq R_{L/S}^{B_d \to \omega \gamma} = 0.05$, which in turn gives $A_{\text{long}}^{B_d \to \rho \gamma}/A_{\text{short}}^{B_d \to \rho \gamma} \leq 0.02$. We note that the LD-contribution just discussed is of the same order as the one expected from the diagrams in Fig. 1b,c which we do not consider in this paper. In what follows we shall therefore neglect the LD-contribution to the neutral $B$ decays. Restricting ourselves to the colour-allowed LD-contributions only, the relation (2) gets modified to:

$$\frac{\Gamma(B_u \to \rho \gamma)}{2\Gamma(B_d \to \rho \gamma)} = \frac{\Gamma(B_u \to \rho \gamma)}{2\Gamma(B_d \to \omega \gamma)} = \left(1 + R_{L/S}^{B_u \to \rho \gamma} \frac{V_{ub}V_{ud}^*}{V_{td}V_{tb}^*}\right)^2 = 1 + 2 \cdot R_{L/S} V_{ud} \left(\frac{\rho(1-\rho)}{(1-\rho)^2 + \eta^2} + (R_{L/S})^2 V_{ud}^2 \frac{\rho^2 + \eta^2}{(1-\rho)^2 + \eta^2}\right).$$  \hspace{1cm} (43)

where $R_{L/S} \equiv R_{L/S}^{B_u \to \rho \gamma}$ and $\rho$, $\eta$ are the Wolfenstein parameters [26]. The ratio $\Gamma(B_u \to \rho \gamma)/2\Gamma(B_d \to \rho \gamma)(= \Gamma(B_u \to \rho \gamma)/2\Gamma(B_d \to \omega \gamma))$ is shown in Fig. 2 as a function of the parameter $\rho$, with $\eta = 0.2$, 0.3 and 0.4. This suggests that a measurement of this ratio would constrain the Wolfenstein parameters ($\rho, \eta$), with the dependence on $\rho$ more marked. In particular, a negative value of $\rho$ leads to a constructive interference in $B_u \to \rho \gamma$ decays, while large positive values of $\rho$ give a destructive interference. This behaviour is in qualitative agreement with what has been also pointed out in [3].

The ratio of the CKM-suppressed and CKM-allowed decay rates in (1) for charged $B$ mesons likewise gets modified due to the LD contributions:

$$\frac{\Gamma(B_u \to \rho \gamma)}{\Gamma(B \to K^* \gamma)} = \xi_u \lambda^2 [(1-\rho)^2 + \eta^2] \times \left\{1 + 2 \cdot R_{L/S} V_{ud} \frac{\rho(1-\rho)}{(1-\rho)^2 + \eta^2} + (R_{L/S})^2 V_{ud}^2 \frac{\rho^2 + \eta^2}{(1-\rho)^2 + \eta^2}\right\},$$  \hspace{1cm} (44)
where $\lambda = 0.2205$. Using the central value from the estimate $\xi_u = 0.59 \pm 0.08$ [1] we show in Fig. 3a the ratio (14) as a function of $\rho$ for $\eta = 0.2$, 0.3 and 0.4. It is seen that the dependence of this ratio is rather weak on the parameter $\eta$ but it depends on $\rho$ rather sensitively. To show the effect of the LD contribution, we compare in Fig. 3b the predictions for the ratio (14) as a function of $\rho$ with and without the LD-contribution, fixing $\eta = 0.3$, the central value for this parameter obtained from the CKM fits [8]. It is seen that the effect of the LD contributions on this ratio is small, and is comparable to $\sim 15\%$ uncertainty in the normalization due to the $SU(3)$-breaking effects in the form factors. Neglecting the colour-suppressed LD contributions, as argued above, the ratio of the decay rates for neutral $B$ meson is not effected and to a good approximation the SD-dominated result

\[
\frac{\Gamma(B_d \to \rho\gamma, \omega\gamma)}{\Gamma(B \to K^{*}\gamma)} = \xi_d \lambda^2 [(1 - \rho)^2 + \eta^2]
\]  

still holds. Finally, combining the estimates presented here and in [1] for the form factors and restricting the Wolfenstein parameters in the range $-0.4 \leq \rho \leq 0.4$ and $0.2 \leq \eta \leq 0.4$, as suggested by the CKM-fits [3], we give the following range for the absolute branching ratios:

\[
\mathcal{B}(B_u \to \rho\gamma) = (1.9 \pm 1.6) \times 10^{-6}, \\
\mathcal{B}(B_d \to \rho\gamma) \simeq \mathcal{B}(B_d \to \omega\gamma) = (0.85 \pm 0.65) \times 10^{-6},
\]  

(46)

where we have used the experimental value for the branching ratio $\mathcal{B}(B \to K^{*} + \gamma) = (4.5 \pm 1.5 \pm 0.9) \times 10^{-5}$ [28], adding the errors in quadrature. The large error reflects the poor knowledge of the CKM matrix elements and hence experimental determination of these branching ratios will put rather stringent constraints on the parameters $\rho$ and $\eta$. Similar constraints would also follow from the measurement of the inclusive radiative decays $B \to X_d + \gamma$ [29], for which a branching ratio $\mathcal{B}(B \to X_d + \gamma) = (1.0 \pm 0.8) \times 10^{-5}$ is estimated using the measured inclusive rate $\mathcal{B}(B \to X_s + \gamma) = (2.32 \pm 0.57 \pm 0.35) \times 10^{-4}$ [30].

To summarize, we have presented a QCD sum rule-based calculation of the contribution of the weak annihilation to exclusive radiative rare $B$ decays, which is expected to dominate the LD contribution for the decays of charged $B$ mesons, $B^{\pm}_{d} \to \rho^{\pm} + \gamma$. The numerical effect of the weak annihilation amplitude depends on the value of the CKM-Wolfenstein parameter $\rho$, and is typically estimated in the 10% range for the presently allowed values of the parameters $\rho$ and $\eta$. The corresponding LD contributions in the neutral $B$-meson decays $B_d \to \rho(\omega)\gamma$ are found to be insignificant. The interference of LD and SD contributions provides a possibility to measure the sign of the Wolfenstein parameter $\rho$ from the ratio (13). The branching ratios $\mathcal{B}(B_u \to \rho\gamma)$ and $\mathcal{B}(B_d \to \rho(\omega)\gamma)$ likewise are sensitive to the sign and magnitude of $\rho$, increasing with large negative values of $\rho$. Our investigations strengthen the expectations that exclusive radiative rare $B$ decays, very much like their inclusive counterparts, are dominated by SD contributions, which in the context of the standard model implies that these decays are valuable in quantifying the CKM parameters.

Acknowledgements: One of us (A.A.) would like to thank Amarjit Soni and Giulia Ricciardi for vigorous discussions on radiative $B$ decays. The other (V.M.B.) would like to thank Alexander Khodzhamirian for a correspondence and helpful discussions.
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**Figure Captions**

Figure 1: Weak annihilation contributions in $B_u \rightarrow \rho \gamma$ involving the operators $O_1$ and $O_2$ denoted by $\otimes$ with the photon emission from a) the loop containing the $b$ quark, b) the loop containing the light quark, and c) the tadpole which contributes only with additional gluonic corrections.

Figure 2: Ratio of the neutral and charged $B$-decay rates $\Gamma(B_u \rightarrow \rho \gamma)/2\Gamma(B_d \rightarrow \rho \gamma)$ as a function of the Wolfenstein parameter $\rho$, with $\eta = 0.2$ (short-dashed curve), $\eta = 0.3$ (solid curve), and $\eta = 0.4$ (long-dashed curve).

Figure 3: Ratio of the CKM-suppressed and CKM-allowed radiative $B$-decay rates $\Gamma(B_u \rightarrow \rho \gamma)/\Gamma(B \rightarrow K^* \gamma)$ (with $B = B_u$ or $B_d$) as a function of the Wolfenstein parameter $\rho$, a) with $\eta = 0.2$ (short-dashed curve), $\eta = 0.3$ (solid curve), and $\eta = 0.4$ (long-dashed curve); b) comparison of the result obtained by neglecting the LD contribution (dashed curve) with the one including the LD contribution (solid curve), both evaluated with $\eta = 0.3$. 
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