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Quantized topological terms in weak-coupling gauge theories with a global symmetry and their connection to symmetry-enriched topological phases

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We study the quantized topological terms in a weak-coupling gauge theory with gauge group $G_s$ and a global symmetry $G_g$ in $d$ space-time dimensions. We show that the quantized topological terms are classified by a pair $(G,\nu_d)$, where $G$ is an extension of $G_s$ by $G_g$ and $\nu_d$ an element in group cohomology $H^d(G,\mathbb{Q}/\mathbb{Z})$. When $d = 3$ and/or when $G_s$ is finite, the weak-coupling gauge theories with quantized topological terms describe gapped symmetry enriched topological (SET) phases (i.e., gapped long-range-entangled phases with symmetry). Thus, those SET phases are classified by $H^d(G,\mathbb{Q}/\mathbb{Z})$, where $G/G_s = G_g$. We also apply our theory to a simple case $G_s = G_g = Z_2$, which leads to 12 different SET phases in $2 + 1$ dimensions $[(2 + 1)D]$, where quasiparticles have different patterns of fractional $G_s$ quantum numbers and fractional statistics. If the weak-coupling gauge theories are gapless, then the different quantized topological terms may describe different gapless phases of the gauge theories with a symmetry $G_s$, which may lead to different fractionalizations of $G_s$ quantum numbers and different fractional statistics [if in $(2 + 1)D$].

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I. INTRODUCTION

For a long time, we thought that Landau symmetry breaking theory1–3 described all phases and phase transitions. In 1989, through a theoretical study of high $T_c$ superconducting model, we realized that there exists a new kind of order—topological order—which cannot be described by Landau symmetry breaking theory.4–6 Recently, it was found that topological orders are related to long-range entanglements.7,8 In fact, we can regard topological order as a pattern of long-range entanglements9 defined through local unitary (LU) transformations.10–12

The notion of topological orders and long-range entanglements leads to a more general and also more detailed picture of phases and phase transitions (see Fig. 1).9 For gapped quantum systems without any symmetry, their quantum phases can be divided into two classes: short-range-entangled (SRE) states and long-range-entangled (LRE) states.

SRE states are states that can be transformed into direct product states via LU transformations. All SRE states can be transformed into each other via LU transformations. So all SRE states belong to the same phase [see Fig. 1(a)]; i.e., all SRE states can continuously deform into each other without closing the energy gap and without phase transition.

LRE states are states that cannot be transformed into direct product states via LU transformations. It turns out that, in general, different LRE states cannot be connected to each other through LU transformations. The LRE states that are not connected via LU transformations represent different quantum phases. Those different quantum phases are nothing but the topologically ordered phases.

Chiral spin liquids,13,14 fractional quantum Hall states,15,16 $Z_2$ spin liquids,17–19 non-Abelian fractional quantum Hall states,20–23 etc., are examples of topologically ordered phases. The mathematical foundation of topological orders is closely related to tensor category theory9,10,24,25 and simple current algebra.20,26 Using this point of view, we have developed a systematic and quantitative theory for nonchiral topological orders in two-dimensional (2D) interacting boson and fermion systems.9,10,25 Also for chiral 2D topological orders with only Abelian statistics, we find that we can use integer $K$ matrices to describe them.27–32

For gapped quantum systems with symmetry, the structure of phase diagram is even richer [see Fig. 1(b)]. Even SRE states now can belong to different phases. One class of nontrivial SRE phases for Hamiltonians with symmetry is the Landau symmetry breaking states. However, even SRE states that do not break the symmetry of the Hamiltonians can belong to different phases. The 1D Haldane phase for a spin-1 chain33–36 and topological insulators37–42 are nontrivial examples of phases with short-range entanglements that do not break any symmetry. We call these kinds of phases SPT phases. The term “SPT phase” may stand for symmetry protected topological phase.35,36 Since the known examples of those phases, the Haldane phase and the topological insulators, were already referred as topological phases. The term “SPT phase” may also stand for symmetry protected trivial phase, since those phases have no long-range entanglements and have trivial topological orders.

It turns out that there is no gapped bosonic LRE state in 1D.11 So all 1D gapped bosonic states are either symmetry breaking states or SPT states. This realization led to a complete classification of all 1D gapped bosonic quantum phases.43–45

In Refs. 46 and 47, the classification of 1D SPT phase is generalized to any dimensions: For gapped bosonic systems in $d$ space-time dimensions with an on-site symmetry $G_s$, we can construct distinct SPT phases that do not break the symmetry $G_s$ from the distinct elements in $H^d(G_s,U(1))$, the $d$-cohomology class of the symmetry group $G_s$ with $U(1)$ as coefficient. We see that we have a quite systematic understanding of SRE states with symmetry.48,49

For gapped LRE states with symmetry, the possible quantum phases should be much richer than SRE states. We may
FIG. 1. (Color online) (a) The possible gapped phases for a class of Hamiltonians $H(g_1, g_2)$ without any symmetry. (b) The possible gapped phases for the class of Hamiltonians $H_{\text{symm}}(g_1, g_2)$ with symmetry. The yellow regions in (a) and (b) represent the phases with long-range entanglement. Each phase is labeled by its entanglement properties and symmetry breaking properties. SRE stands for short-range entanglement, LRE for long-range entanglement, SB for symmetry breaking, SY for no symmetry breaking. SB-SRE phases are the Landau symmetry breaking phases. The SY-SRE phases are the SPT phases. The SY-LRE phases are the SET phases.

call those phases symmetry enriched topological (SET) phases. The projective symmetry group (PSG) was introduced to study the SET phases.\textsuperscript{50–52} The PSG describes how the quantum numbers of the symmetry group $G_s$ get fractionalized on the gauge excitations.\textsuperscript{51} When the gauge group $G_g$ is Abelian, the PSG description of the SET phases can be expressed in terms of group cohomology: The different SET states with symmetry $G_s$ and gauge group $G_g$ can be (partially) described by $H^2(G_s, G_g)$.\textsuperscript{53} Many examples of the SET states can be found in Refs. 48, 50, and 54–56.

Recently, Mesaros and Ran proposed a quite systematic understanding of a subclass of SET phases:\textsuperscript{55} One can use the elements of $H^d(G_s \times G_g, \mathbb{R}/\mathbb{Z})$ to describe the SET phases in $d$ space-time dimensions with a finite gauge group $G_g$ and a finite global symmetry group $G_s$. Here $H^d(G_s \times G_g, \mathbb{R}/\mathbb{Z})$ is the group cohomology class of group $G_s \times G_g$. This result is based on the group cohomology theory of the SPT phases\textsuperscript{47} and the Levin-Gu duality between the SPT phases and the “twisted” weak-coupling gauge theories.\textsuperscript{58–60} Also, Essin and Hermle generalized the results of Refs. 50, 51, 54, and 55 and studied quite systematically the SET phases described by a $G_g = \mathbb{Z}_2$ gauge group.\textsuperscript{53} They show that some of those SET phases can be classified by $H^2(G_s, G_g)$.

In this paper, we develop a somewhat systematic understanding of SET phases, following a path-integral approach developed for the group cohomology theory of the SPT phases\textsuperscript{47} and the topological gauge theory.\textsuperscript{60,61} The idea is to classify quantized topological terms in weak-coupling gauge theory with symmetry. If the weak-coupling gauge theory happens to have a gap, then the different quantized topological terms will describe different SET phases. This allows us to obtain and generalize the results in Refs. 53 and 57. Since weak-coupling gauge theories only describe some topological ordered states, our theory only describes some of the SET states.

We show that quantized topological terms in symmetric weak-coupling gauge theory in $d$ space-time dimensions with a gauge group $G_g$ and a global symmetry group $G_s$ can be described by a pair $(G_s, v_{ij})$, where $G_s$ is an extension of $G_s$ by $G_g$ and $v_{ij}$ is an element in $H^d(G_s, \mathbb{R}/\mathbb{Z})$. (An extension of $G_s$ by $G_g$ is group $G$ that contains $G_s$ as a normal subgroup and satisfy $G/G_s \simeq G_g$.) When $G_s$ is finite or when $d = 3$, the weak-coupling gauge theory is gapped. In this case, $(G_s, v_{ij})$ describe different SET phases. Note that the extension $G_s$ is nothing but the PSG introduced in Ref. 50. Also, when the symmetry group $G_s$ contains antiunitary transformations, those antiunitary transformations will act nontrivially on $\mathbb{R}/\mathbb{Z}$.

In Appendix B, we show that we can use $(y_0, y_1, \ldots, y_d)$ with

$$ y_k \in H^d_\text{SRE}(G_s, \mathbb{R}/\mathbb{Z}) $$

(1)

to label the elements in $H^d(G_s, \mathbb{R}/\mathbb{Z})$. However, such a labeling may not be one-to-one and it may happen that some of $(y_0, y_1, \ldots, y_d)$ correspond to the elements in $H^d(G_s, \mathbb{R}/\mathbb{Z})$. However, for every element in $H^d(G_s, \mathbb{R}/\mathbb{Z})$, we can find a $(y_0, y_1, \ldots, y_d)$ that corresponds to it. If we choose a special extension $G = G_g \rtimes G_s$, then we recover the result in Ref. 57 if $G$ is finite: A set of SET states can be can be described by $(y_0, y_1, \ldots, y_d)$ with an one-to-one correspondence [see Eq. (A10)]:

$$ H^d(G_s \rtimes G_g, \mathbb{R}/\mathbb{Z}) = \bigoplus_{p=0}^d H^{d-p}(G_s, H^p(G_g, \mathbb{R}/\mathbb{Z})) $$

(2)

The term $H^d(G_s, H^p(G_g, \mathbb{R}/\mathbb{Z})) = H^d(G_g, \mathbb{R}/\mathbb{Z})$ describes the quantized topological terms associated with only the symmetry $G_s$ which describes the SPT phases. The term $H^d(G_s, H^p(G_g, \mathbb{R}/\mathbb{Z})) = H^d(G_g, \mathbb{R}/\mathbb{Z})$ describes the quantized topological terms associated with pure gauge theory. Other terms $\bigoplus_{p=0}^d H^{d-p}(G_s, H^p(G_g, \mathbb{R}/\mathbb{Z}))$ describe the quantized topological terms that involve both gauge theory $G_g$ and symmetry $G_s$. Those terms describe how $G_s$, quantum numbers get fractionalized on gauge-flux excitations.\textsuperscript{57}

When $G_s$ is Abelian, the different extensions, $G_s$ of $G_g$ by $G_g$ is classified by $H^2(G_s, G_g)$. This reproduces a result in Ref. 53.

II. A SIMPLE FORMAL APPROACH

First let us describe a simple formal approach that allows us to quickly obtain the above results. We know that the SPT phases in $d$-dimensional discrete space-time are described by topological nonlinear $\sigma$ models with symmetry $G$:

$$ \mathcal{L} = \frac{1}{\lambda_s} [\partial g(x^\mu)]^2 + i W_{\text{top}}(g), \quad g \in G, $$

(3)

where $\lambda_s \rightarrow \infty$, and the $2\pi$-quantized topological term $\int W_{\text{top}}(g)$ is given by an element in $H^d(G, \mathbb{R}/\mathbb{Z})$. Different elements in $H^d(G, \mathbb{R}/\mathbb{Z})$ describe different SPT phases.\textsuperscript{47} If we “gauge” the symmetry $G$, the topological nonlinear $\sigma$ model will become a gauge theory,

$$ \mathcal{L} = \frac{1}{\lambda_s} [(\partial - i A)g(x^\mu)]^2 + i W_{\text{top}}(g, A) + \frac{(F_{\mu\nu})^2}{\lambda}, $$

(4)

where $W_{\text{top}}(g, A)$ is the gauged topological term. For those topological terms that can be expressed in continuous field theory, $W_{\text{top}}(g, A)$ can be obtained from $W_{\text{top}}(g)$ by replacing $\partial_{\mu}$ with $\partial_{\mu} - i A_{\mu}$. When $G_s$ and $G_g$ are finite, $W_{\text{top}}(g, A)$ can be constructed explicitly in discrete space-time.\textsuperscript{62}
If we further integrate out $g$, we get a pure gauge theory with a topological term,

$$
\mathcal{L} = \frac{(F_{\mu\nu})^2}{\lambda} + i W_{\text{top}}(A) .
$$

This line of thinking suggests that the quantized topological term $\int W_{\text{top}}(A)$ in symmetric gauge theory is classified by the same $\mathcal{H}^2(G,\mathbb{R}/\mathbb{Z})$ that classifies the $2\pi$-quantized topological term $\int W_{\text{top}}(g)$.

Now let us consider topological nonlinear $\sigma$ models with symmetry $G_s \times G_g$,

$$
\mathcal{L} = \frac{1}{\lambda_s} [\partial g(x^\mu)]^2 + i W_{\text{top}}(g), \quad g \in G = G_s \times G_g ,
$$

where the $2\pi$-quantized topological term $\int W_{\text{top}}(g)$ is classified by $\mathcal{H}^2(G_s \times G_g,\mathbb{R}/\mathbb{Z})$. If we gauge only a subgroup $G_g$ of the total symmetry group $G_s \times G_g$, we will get a gauge theory,

$$
\mathcal{L} = \frac{1}{\lambda_s} [\partial g(x^\mu)]^2 + i W_{\text{top}}(g,A) + \frac{(F_{\mu\nu})^2}{\lambda} ,
$$

with global symmetry $G_s$. This line of thinking suggests that the quantized topological term $\int W_{\text{top}}(g,A)$ is classified by the same $\mathcal{H}^2(G_s \times G_g,\mathbb{R}/\mathbb{Z})$.

We can generalize the above approach to obtain more general quantized topological terms in weak-coupling gauge theory with gauge group $G_g$ and symmetry $G_s$. We start with a group $G$ which is an extension of the symmetry group $G_s$ by the gauge group $G_g$:

$$
1 \rightarrow G_g \rightarrow G \rightarrow G_s \rightarrow 1 .
$$

In other words, $G$ contains a normal subgroup $G_g$ such that $G/G_g = G_s$. So we can start with a topological nonlinear $\sigma$ models with symmetry $G$,

$$
\mathcal{L} = \frac{1}{\lambda_s} [\partial g(x^\mu)]^2 + i W_{\text{top}}(g,A) + \frac{(F_{\mu\nu})^2}{\lambda} ,
$$

with global symmetry $G_s = G/G_g$. This line of thinking suggests that the quantized topological term $\int W_{\text{top}}(g,A)$ is classified by $\mathcal{H}^2(G_s,\mathbb{R}/\mathbb{Z})$.

So more generally, the SET states in $d$-dimensional space-time with gauge group $G_g$ and symmetry group $G_s$ are labeled by the elements in $\mathcal{H}^2(G_s,\mathbb{R}/\mathbb{Z})$, where $G$ is the extension of the symmetry group $G_s$ by the gauge group $G_g$, provided that the symmetric gauge theory (9) is gapped in a small $\lambda$ limit and $d \geq 3$. If the symmetric gauge Eq. (9) is gapless in the small $\lambda$ limit, then $\mathcal{H}^2(G_s,\mathbb{R}/\mathbb{Z})$ describes different gapless phases of the symmetric gauge theory.

The above approach is formal and hand-waving. When $G$ is finite, we can rigorously obtain the above results, which is described in Ref. 62. In the following, we discuss such an approach assuming $G_g$ is finite (but $G_s$ can be finite or continuous). Then we discuss another approach that allows us to obtain the above result more rigorously for the case $G = G_s \times G_g$, where $G_s, G_g$ can be finite or continuous.

### III. AN EXACT APPROACH FOR FINITE $G_g$

This approach is based on the formal approach (10) discussed above, where $G$ is an extension of the symmetry group $G_s$ by the gauge group $G_g$; $G/G_g = G_s$. We make the above approach exact by putting the theory on space-time lattice of $d$ dimensions.

#### A. Discretize space-time

We discretize the space-time $M$ by considering its triangulation $M_{\text{tri}}$ and define the $d$-dimensional gauge theory on such a triangulation. We call such a theory a lattice gauge theory. We call the triangulation $M_{\text{tri}}$ a space-time complex, and a cell in the complex a simplex.

In order to define a generic lattice theory on the space-time complex $M_{\text{tri}}$, it is important to give the vertices of each simplex a local order. A nice local scheme to order the vertices is given by a branching structure.47,63 A branching structure is a choice of orientation of each edge in the $d$-dimensional complex so that there is no oriented loop on any triangle (see Fig. 2).

The branching structure induces a local order of the vertices on each simplex. The first vertex of a simplex is the vertex with no incoming edges, and the second vertex is the vertex with only one incoming edge, etc. So the simplex in Fig. 2(a) has the following vertex ordering: 0,1,2,3.

The branching structure also gives the simplex (and its subcomplexes) an orientation denoted by $s_{ij \rightarrow k} = 1, \ast$. Figure 2 illustrates two 3-simplices with opposite orientations $s_{0123} = 1$ and $s_{0123} = \ast$. The red arrows indicate the orientations of the 2-simplices which are the subcomplexes of the 3-simplices. The black arrows on the edges indicate the orientations of the 1-simplices.

#### B. Gauged nonlinear $\sigma$ model on space-time lattice

To put Eq. (10) on space-time lattice, we put the $g(x^\mu) \in G$ field on the vertices of the space-time complex, which becomes $g_{ij}$, where $i$ labels the vertices. We also put the gauge field on the edges $ij$ which becomes $g_{ij} \in G_g$.

The action amplitude for a d cell $(ij \cdots k)$ is complex function of $g_{ij}$ and $g_{ij}$: $V_{ij \cdots k}(\{g_{ij}\},\{g_{ij}\})$ The partition function is given by

$$
Z = \sum_{\{g_{ij}\},\{g_{ij}\}} \prod_{(ij \rightarrow k)} [V_{ij \cdots k}(\{g_{ij}\},\{g_{ij}\})]^{\nu_{ij \cdots k}} ,
$$

![FIG. 2. (Color online) Two branched simplices with opposite orientations. (a) A branched simplex with positive orientation and (b) a branched simplex with negative orientation.](image-url)
where \( \prod_{i,j,k} \) is the product over all the \( d \) cells \((ij\cdots k)\). If the above action amplitude \( \prod_{i,j,k} [V_{ij-k}((g_{ij}),(g_{j}))]^{\nu_d} \) on closed space-time complex \((\partial M_{G\theta} = 0)\) is invariant under the gauge transformation

\[
g_{ij} \rightarrow g'_{ij} = h_i g_{ij} h_j^{-1}, \quad g_i \rightarrow g'_i = h_i g_i h_i^{-1}, \quad h_i \in G_g, \tag{12}
\]

then the action amplitude \( V_{ij-k}((g_{ij}),(g_{j})) \) defines a gauge theory of gauge group \( G_g \). If the action amplitude is invariant under the global transformation

\[
g_{ij} \rightarrow g'_{ij} = h g_{ij} h^{-1}, \quad g_i \rightarrow g'_i = h g_i h^{-1}, \quad h \in G, \tag{13}
\]

then the action amplitude \( V_{ij-k}((g_{ij}),(g_{j})) \) defines a gauge theory with a global symmetry \( G_g = G/G\theta \). (We need to mod out \( G_g \) since when \( h \in G_g \), it will generate a gauge transformation instead of a global symmetry transformation.)

Using a cocycle \( v_d(g_0,g_1,\ldots,g_d) \in \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z}) \), \( g_i \in G \), we can construct an action amplitude \( V_{ij-k}((g_{ij}),(g_{j})) \) that defines a gauge theory with gauge group \( G_g \) and global symmetry \( G_s \). First, we note that the cocycle satisfies the cocycle condition

\[
\tilde{\nu}_d = \nu_d(g_0,g_1,\ldots,g_d) = \nu_d(g_0 g_1,\tilde{g}_1 g_2,\ldots,\tilde{g}_d), \tag{16}
\]

where \( g_0,\ldots,\tilde{g}_i,\ldots,\tilde{g}_d+1 \) is the sequence \( g_0,\ldots,\tilde{g}_i,\ldots,\tilde{g}_d+1 \) with \( g_i \) removed. The gauge theory action amplitude is given by

\[
V_{01-d}((g_{ij}),(g_{j})) = 0, \quad \text{if } g_{ij} g_{jk} \neq g_{ik},
\]

\[
V_{01-d}((g_{ij}),(g_{j})) = v_d(g_0 g_1,\tilde{g}_1 g_2,\ldots,\tilde{g}_d), \quad \text{otherwise,} \tag{15}
\]

where \( \tilde{g}_i \) are given by

\[
\tilde{g}_0 = 1, \quad \tilde{g}_1 = \tilde{g}_0 g_0, \quad \tilde{g}_2 = \tilde{g}_1 g_1, \quad \tilde{g}_3 = \tilde{g}_2 g_2, \ldots \tag{16}
\]

One can check that the above action amplitude \( V_{01-d}((g_{ij}),(g_{j})) \) is invariant under the gauge transformation (12) and the global symmetry transformation (13). Thus, it defines a symmetric gauge theory.

We know that the action amplitude is nonzero only when \( g_{ij} g_{jk} = g_{ik} \). The condition \( g_{ij} g_{jk} = g_{ik} \) is the flat connection condition, and the corresponding gauge theory is in the weak-coupling limit (actually is at the zero-coupling). This condition can be implemented precisely only when \( G_g \) is finite. With the flat connection condition \( g_{ij} g_{jk} = g_{ik} \), \( \tilde{g}_i \)'s and the gauge equivalent sets of \( g_i \) have an one-to-one correspondence.

Since the total action amplitude \( \prod_{i,j,k} [V_{ij-k}((g_{ij}),(g_{j}))]^{\nu_d} \) on a sphere is always equal to 1 if the gauge flux vanishes, \( V_{ij-k}((g_{ij}),(g_{j})) \) describes a quantized topological term in weak-coupling gauge theory (or zero-coupling gauge theory). This way, we show that a quantized topological term in a weak-coupling gauge theory with gauge group \( G_g \) and symmetry group \( G_s \) can be constructed from each element of \( \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z}) \).

When \( G_g = \{1\} \) (or \( G = G_s \)),

\[
V_{01-d}((g_{ij}),(g_{j})) = v_d(g_0,g_1,\ldots,g_d) \tag{17}
\]

become the action amplitude for the topological nonlinear \( \sigma \) model, describing the SPT phase labeled by the cocycle \( v_d \in \mathcal{H}^d(G_g,\mathbb{R}/\mathbb{Z}) \). \(^{47}\)

When \( G_g = \{1\} \) (or \( G = G_s \)),

\[
V_{01-d}((g_{ij}),(g_{j})) = v_d(g_0,\tilde{g}_0 g_1,\ldots,\tilde{g}_d g_d). \tag{18}
\]

We can use the gauge transformation (12) to set \( g_i = 1 \) in the above and obtain

\[
V_{01-d}((g_{ij}),(g_{j})) = v_d(\tilde{g}_0,\tilde{g}_1,\ldots,\tilde{g}_d). \tag{19}
\]

This is the topological gauge theory studied in Refs. 60 and 61.

IV. AN APPROACH BASED ON CLASSIFYING SPACE

In this section, we consider the cases where \( G_s, G_g \) be finite or continuous. However, for the time being, we can only handle the situation where \( G = G_s \times G_g \). Our approach is based on the classifying space.

A. Motivations and results

Let us first review some known results. To gain a systematic understand of SRE states with on-site symmetry \( G_s \), we started with a nonlinear \( \sigma \) model,

\[
\mathcal{L} = \frac{1}{\lambda_s} [\partial g(x^\mu)]^2, \quad g \in G_s, \tag{20}
\]

with symmetry group \( G_s \) as the target space. The model can be in a disordered phase that does not break the symmetry \( G_s \), when \( \lambda \) is large. By adding different \( 2\pi \) quantized topological \( \theta \) terms to the Lagrangian \( \mathcal{L} \), we can get different Lagrangians that describe different disordered phases that do not break the symmetry \( G_s \). \(^{37}\) Those disordered phases are the symmetry protected topological (SPT) phases. \(^{35,36}\) So we can use the quantized topological terms to classify the SPT phases. (In general, topological terms, by definition, are the terms that do not depend on space-time metrics.)

We know that gauge theory

\[
\mathcal{L} = \frac{1}{\lambda} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) \tag{21}
\]

is one way to describe LRE states (i.e., topologically ordered states). In Refs. 60 and 61, different quantized topological terms in weak-coupling gauge theory with gauge group \( G_g \) and small \( \lambda \) in \( d \) space-time dimensions are constructed and classified, using the topological cohomology class \( H^{d+1}(BG_g,\mathbb{Z}) \) for the classifying space \( BG_g \) of the gauge group \( G_g \). By adding those quantized topological terms to the above Lagrangian for the weak-coupling gauge theory, we may obtain different phases of the weak-coupling gauge theory.

In this section, we plan to combine the above two approaches by studying the quantized topological terms in the combined theory,

\[
\mathcal{L} = \frac{1}{\lambda} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) + \frac{1}{\lambda_s} [\partial g(x^\mu)]^2, \quad g \in G_s, \tag{22}
\]

where \( F \) is the field strength with gauge group \( G_g \), and \( (\lambda,\lambda_s) \rightarrow (\text{small},\text{large}) \). Such a theory is a gauge theory with symmetry \( G_g \). We find that quantized topological terms in the combined theory can be constructed and classified by the topological cohomology class \( H^{d+1}(BG_g \times BG_g,\mathbb{Z}) \) for the
classifying space of the product $G_s \times G_t$. Those quantized topological terms give us a somewhat systematic understanding of different phases of weak coupling gauge theories with symmetry. If those symmetric weak coupling gauge theories are gapped (for example, for finite gauge groups), then the theories will describe topologically ordered states with symmetry. Those SET phases in $d$ space-time dimensions are described by elements in $H^{d+1}(BG_s \times BG_t, \mathbb{Z})$.

B. Gauge theory as a nonlinear $\sigma$ model with classifying space as the target space

To obtain the above result, we follow closely the approaches used in Refs. 60 and 47. We obtain our result in two steps.

1. Symmetric weak-coupling gauge theory as the nonlinear $\sigma$ model of $G_s \times BG_t$

As in Ref. 60, we may view a weak-coupling gauge theory with gauge group $G_s$ as a nonlinear $\sigma$ model with classifying space $BG_s$ as the target space. So the symmetric weak-coupling gauge theory in Eq. (22) can be viewed as a nonlinear $\sigma$ model with $G_s \times BG_t$ as the target space, where each path in the path integral is given by an embedding $\gamma: M_\text{top} \rightarrow G_s \times BG_t$ from the space-time complex $M_\text{top}$ to $G_s \times BG_t$. We can study topological terms in our symmetric weak-coupling gauge theory by studying the topological terms in the corresponding nonlinear $\sigma$ model.

Following Ref. 60, a total term $S_{\text{top}}$ corresponds to evaluating a cocycle $\alpha_d \in Z(G_s \times BG_t, \mathbb{R}/\mathbb{Z})$ on the complex $\gamma(M_\text{top}) \subset G_s \times BG_t$:

$$S_{\text{top}}[\gamma] = 2\pi(\alpha_d, \gamma(M_\text{top})) \mod 2\pi.$$  \hfill (23)

Such a topological term does not depend on any smooth deformation of $\gamma$ and is thus "topological." (Note that the evaluation of the $d$-cocycle on any $d$-cycles [i.e., $d$-dimensional closed complexes] are equal to $0 \mod 1$ if the $d$-cycles are boundaries of some $(d+1)$D complex.)

Here we would like to stress that the cocycle $\alpha_d$ on the group manifold is not the ordinary topological cocycle. It has a symmetry condition,

$$\langle \alpha_d, c \rangle = \langle \alpha_d, c_g \rangle,$$  \hfill (24)

where $c$ is a complex in $G_s$ and $c_g$ is the complex generated from $c$ by the symmetry transformation $G_s \rightarrow gG_s, g \in G_s$. Also, since $\lambda_s \rightarrow \infty$ and $g(x^n)$ have large fluctuations in Eq. (22), $\langle \alpha_d, c \rangle$ only depend on the vertices $g_0, g_1, \ldots$ of $c$:

$$\langle \alpha_d, c \rangle = \nu(g_0, g_1, \ldots), \quad \nu(g_0, g_0, g_1, \ldots) = \nu(g_0, g_1, \ldots);$$

$$g, g_1 \in G_s.$$  \hfill (25)

So, on $G_s$, $\alpha_d$ is actually a cocycle in the group cohomology $Z(G_s, \mathbb{R}/\mathbb{Z})$, \hfill (26)

while on $BG_s$, $\alpha_d$ is the usual cocycle in the topological cohomology $BG_s, \mathbb{R}/\mathbb{Z})$.

Since, on $G_s$, $\alpha_d$ is a cocycle in the group cohomology $Z(G_s, \mathbb{R}/\mathbb{Z})$, when $G_s$ contains antiunitary symmetry, such antiunitary symmetry transformation will have a nontrivial action on $\mathbb{R}/\mathbb{Z}$: $x \rightarrow -x, x \in \mathbb{R}/\mathbb{Z}$, \hfill (27)

If two $d$-cocycles, $\alpha_d, \alpha_d'$, differ by a coboundary: $\alpha_d - \alpha_d = d\mu_{d}$, $\mu_d \in C^{d+1}(BG_s, \mathbb{R}/\mathbb{Z})$, then, the corresponding action amplitudes, $e^{iS_{\text{top}}[\gamma]}$ and $e^{iS_{\text{top}}'[\gamma]}$, can smoothly deform into each other without phase transition. So $e^{iS_{\text{top}}[\gamma]}$ and $e^{iS_{\text{top}}'[\gamma]}$, or $\alpha_d$ and $\alpha_d'$, describe the same quantum phase. Therefore, we regard $\alpha_d$ and $\alpha_d'$ to be equivalent. The equivalent classes of the $d$-cocycles form the $d$ cohomology class $H^d(G_s \times BG_t, \mathbb{R}/\mathbb{Z})$. We conclude that the topological terms in symmetric weak-coupling gauge theories are described by $H^d(G_s \times BG_t, \mathbb{R}/\mathbb{Z})$ in $d$ space-time dimensions.

To calculate $H^d(G_s \times BG_t, \mathbb{R}/\mathbb{Z})$, let us first calculate $H^d(G_s \times BG_t, \mathbb{Z})$. Using the the K"unneth formula Eq. (A4) (with $M' = \mathbb{Z}$), we find

$$H^d(G_s \times BG_t, \mathbb{Z}) \cong \bigoplus_{p=0}^{d} H^p(G_s, \mathbb{Z}) \otimes H^{d-p}(BG_t, \mathbb{Z}) \oplus \bigoplus_{p=0}^{d} \text{Tor}_1^Z(H^p(G_s, \mathbb{Z}), H^{d-p+1}(BG_t, \mathbb{Z})).$$  \hfill (26)

In the above, we have used the fact that the cohomology on $G_s$ is the group cohomology $H$ and the cohomology on $BG_t$ is the usual topological cohomology $H$.

In Appendix A, we show that [see Eq. (A6)]

$$H^d(X, \mathbb{R}/\mathbb{Z}) \cong H^d(X, \mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z} \oplus \text{Tor}_1^Z(H^d(X, \mathbb{Z}), \mathbb{R}/\mathbb{Z}).$$  \hfill (27)

Using

$$Z \otimes \mathbb{R}/\mathbb{Z} = \mathbb{R}/\mathbb{Z}, \quad Z \otimes \mathbb{R}/\mathbb{Z} = 0,$$  \hfill (28)

we see that $H^d(X, \mathbb{R}/\mathbb{Z})$ has a form $H^d(X, \mathbb{R}/\mathbb{Z}) = \mathbb{R}/\mathbb{Z} \oplus \cdots \oplus \mathbb{R}/\mathbb{Z} \oplus Z \oplus \cdots$. So the discrete part of $H^d(X, \mathbb{R}/\mathbb{Z})$ is given by

$$\text{Dis}(H^d(X, \mathbb{R}/\mathbb{Z})) = Z \oplus Z \oplus \cdots = \text{Tor}(H^{d+1}(X, \mathbb{Z})).$$  \hfill (29)

where we have used

$$H^{d+1}(X, \mathbb{Z}) = \text{Free}(H^{d+1}(X, \mathbb{Z})) \oplus \text{Tor}(H^{d+1}(X, \mathbb{Z}))$$  \hfill (30)

with $\text{Tor}(H^{d+1}(X, \mathbb{Z}))$ the torsion part and $\text{Free}(H^{d+1}(X, \mathbb{Z}))$ the free part of $H^{d+1}(X, \mathbb{Z})$. Therefore, we have

$$\text{Dis}(H^d(G_s \times BG_t, \mathbb{R}/\mathbb{Z})) \cong \text{Tor}(H^d(G_s, \mathbb{Z}) \otimes H^{d-1}(BG_t, \mathbb{Z})) \oplus \bigoplus_{p=0}^{d+2} \text{Tor}_1^Z(H^p(G_s, \mathbb{Z}), H^{d-p+2}(BG_t, \mathbb{Z}))).$$  \hfill (31)

Since $H^d(G_s, \mathbb{Z}) = H^d(BG_t, \mathbb{Z})$, the above can be rewritten as

$$\text{Dis}(H^d(G_s \times BG_t, \mathbb{Z})) \cong \text{Tor}(H^d(BG_t, \mathbb{Z}) \otimes H^{d-1}(BG_t, \mathbb{Z})) \oplus \bigoplus_{p=0}^{d+2} \text{Tor}_1^Z(H^p(BG_t, \mathbb{Z}), H^{d-p+2}(BG_t, \mathbb{Z}))).$$  \hfill (32)
Each element in the above cohomology class describes a quantized topological term in the weakly coupled gauge theory with symmetry $G_s$.

2. Chern-Simons form

We note that

$$H^{d+1}(BG_s \times BG_g, \mathbb{Z})$$

$$= \left[ \mathfrak{d}_{p=0}^+ H^p(BG_s, \mathbb{Z}) \otimes \mathbb{Z} H^{d+1-p}(BG_g, \mathbb{Z}) \right]$$

$$\oplus \left[ \mathfrak{d}_{p=0}^+ \text{Tor}^2 \left( H^p(BG_s, \mathbb{Z}), H^{d-p+2}(BG_g, \mathbb{Z}) \right) \right].$$

(33)

So the result (32) is very close to our proposal that elements in $H^{d+1}(BG_s \times BG_g, \mathbb{Z})$ correspond to the quantized topological terms. The only thing missing is the free part of $H^d(BG_g, \mathbb{Z})$.

In fact, the free part of $H^{d+1}(BG_g, \mathbb{Z})$, denoted as $\text{Free}(H^{d+1}(BG_g, \mathbb{Z}))$, is nonzero only when $d = odd$. So in the following we consider only $d = odd$ cases. The free part $\text{Free}(H^{d+1}(BG_g, \mathbb{Z}))$ corresponds to the Chern-Simons forms in $d$ space-time dimensions.

To understand such a result, we first choose a $\omega \in \text{Free}(H^{d+1}(BG_g, \mathbb{Z}))$. We can find integers $K_i$ such that

$$-\omega + \frac{K_1}{2^{d+1}(2\pi)^{d+1}} \text{Tr} F^2 + \cdots$$

(34)

is an exact form $d\theta(A)$. Here $\theta(A)$ is called a Chern-Simons form in $d$ dimensions.

We can use a Chern-Simons form $\theta_{d-p}(A)$ and a cocycle $a_p \in H^p(G_s)$ to construct a quantized topological term,

$$S_{\text{top}}[\gamma] = 2\pi (\alpha_p \cup \theta_{d-p}(A), \gamma(M_{\text{top}})) \mod 2\pi.$$ (35)

Such kind of topological terms are labeled by the elements in

$$\oplus_{p=0}^d H^p(G_s, \mathbb{Z}) \otimes \mathbb{Z} \text{Free}(H^{d+1-p}(BG_g, \mathbb{Z}))$$

$$\oplus_{p=0}^d H^p(BG_g, \mathbb{Z}) \otimes \mathbb{Z} \text{Free}(H^{d+1-p}(BG_g, \mathbb{Z})).$$

(36)

Combining the above result with Eq. (32), we find that the elements in $H^{d+1}(BG_s \times BG_g, \mathbb{Z})$ correspond to the quantized topological terms.

V. AN EXAMPLE: $G_s = Z_2$ AND $G_g = Z_2$

In this section, we discuss a simple example with $G_s = Z_2$ and $G_g = Z_2$. There are two kinds of extensions $G$ of $G_s = Z_2$ by $G_g = Z_2$: $G = Z_2 \times Z_2$ and $G = Z_4$. So the quantized topological terms and the SET phases are described by $H^d(Z_2 \times Z_2, \mathbb{R}/\mathbb{Z})$ and $H^d(Z_4, \mathbb{R}/\mathbb{Z})$ in $d$ space-time dimensions.

In $d = 3$ space-time dimensions, we have

$$H^3(Z_2 \times Z_2, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2^3, \quad H^3(Z_4, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_4.$$ (37)

So there are $12$ SET phases for weak-coupling $Z_2$ gauge theory with $Z_4$ symmetry. However, at this stage, it is not clear if those $12$ SET phases are really distinct, since they could be smoothly connected via strong coupling gauge theory. Later, we will see that the $12$ SET phases are indeed distinct, since they have distinct physical properties.

A. A K-matrix approach

To understand the physical properties of those $12$ SET phases, we would like to use Levin-Gu duality to gauge the $G_s$ and turn the theory into gauge theory with gauge group $G$.

Let us first consider the $G = Z_2 \times Z_2$ case. A $G = Z_2 \times Z_2$ gauge theory can be described by $U^d(1)$ mutual Chern-Simons theory.$^{54,64}$

$$\mathcal{L} = \frac{1}{4\pi} K_{0,1} a_0^I a_0^I + \cdots,$$ (38)

with

$$K_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ (39)

The $G_s$ charge corresponds to the unit charge of $a_0^I$ gauge field and the $G_g$ gauge charge corresponds to the unit charge of $a_1^I$ gauge field. The $G_s$ flux excitation (in the $G = Z_2 \times Z_2$ gauge theory) corresponds to the end of the branch cut in the original theory along which we have a twist generated by a $G_s$ symmetry transformation (see Ref. 58 for a detailed discussion about the symmetry twist). Such $G_s$ flux corresponds to the flux of an $a_0^I$ gauge field.

The eight types of quantized topological terms are given by

$$W_{\text{top}} = \frac{n_1}{2\pi} a_0^I \partial_0 a_1^I + \frac{n_1 + n_2}{2\pi} a_0^I \partial_0 a_0^I + \frac{n_1}{2\pi} a_1^I \partial_0 a_1^I,$$ (40)

$n_1 = 0, 1$, $n_2 = 0, 1, n_2 = 0.1$. The total Lagrangian has a form

$$\mathcal{L} + W_{\text{top}} = \frac{1}{4\pi} K_{IJ} a_0^I a_1^J a_1^I + \cdots,$$ (41)

with

$$K_{IJ} = \begin{pmatrix} 2n_1 & 2 & n_1 & 0 \\ 2 & 0 & 0 & 0 \\ n_2 & 2n_2 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$ (42)

Two $K$ matrices are equivalent: $K_1 \sim K_2$ if $K_1 = U^T K_2 U$ for an integer matrix with det$(U) = \pm 1$. We find $K(n_1, n_1, n_2) \sim K(n_1 + 2, n_1, n_2) \sim K(n_1, n_2, n_2) \sim K(n_1, n_2, n_2 + 2)$. Thus, only $n_1, n_2, n_2 = 0.1$ give rise to inequivalent $K$ matrices.

A particle carrying $l_I a_1^I$ charge will have a statistics

$$\theta_I = \pi l_I (K^{-1})^{IJ} l_J.$$ (43)

A particle carrying $l_I a_1^I$ charge will have a mutual statistics with a particle carrying $l_I a_1^I$ charge:

$$\theta_{IJ} = 2\pi l_I (K^{-1})^{IJ} l_J.$$ (44)

We note that the $G_s$ charge is identified with the unit $a_0^I$ charge and the $G_g$ gauge charge is identified with the unit $a_0^I$ charge. Using

$$K^{-1} = \frac{1}{4} \begin{pmatrix} 0 & 2 & 2 & 0 \\ 2 & -2n_1 & 0 & -n_1 \\ 0 & -n_1 & -2 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}.$$ (45)
we find that the $G_0$ charge (the unit $a_\mu^0$ charge) and the $G_g$ gauge charge (the unit $a_\mu^1$ charge) remain bosonic after inclusion of the topological terms. This is actually a condition on the topological terms: The topological terms do not affect the statistics of the gauge charge.

The end of the branch cut in the original theory corresponds to $\pi$ flux in $a_\mu^0$. We note that a particle carrying $l_1 a_\mu^0$ charge created a $l_2\pi$ flux in $a_\mu^0$. So a unit $a_\mu^1$ charge always create a $G_0$ twist. However, what is the $G_0$ charge of the $l_1$ particle?

To measure the $G_0$ charge, we need to find the pure $G_0$ twist. Let us assume that the pure $G_0$ twist corresponds to $l^0 = (l_1^0, l_2^0, 0, 0) a_\mu^0$ charge. Then $l_2^0 = 1$ so that the $l^0$ particle produces $\pi a_\mu^1$ flux. For a pure $G_0$ twist, we also have

$$\pi (l^0)^T K^{-1} l^0 = 0.$$ (46)

This allows us to obtain

$$(l^0)^T = \left( \frac{n_1}{2}, 1, 0, 0 \right).$$ (47)

Note that some times, $l^0$ is not an allowed excitation. However, we can always use $l^0$ to probe the $G_0$ charge. Let

$$q = 2K^{-1}l^0 = \left( \begin{array}{c} 1 \\ -n_1/2 \\ 0 \\ -n_2/2 \end{array} \right).$$ (48)

Moving a pure $G_0$ twist around the $l_1$ particle will induce a phase

$$2\pi l^0 K^{-1} l^0 = \pi q^T l_1.$$ (49)

We find that the $G_0$ charge of the $l_1$ particle is

$$G_0 \text{ charge} = q^T l_1 \text{ mod } 2.$$ (50)

When $n_1 = 0$, those gauge excitations have a trivial mutual statistics with the unit $a_\mu^0$ charge (i.e., the end of the branch cut). This means that those gauge excitations carry a trivial $G_0$ quantum number. When $n_2 = 1$, the unit $a_\mu^0$ charge (the gauge-flux excitation) has a $\pi/2$ mutual statistics with the unit $a_\mu^1$ charge (i.e., the end of the branch cut). This means that the unit $a_\mu^1$ charge carries a fractional $G_0$ charge. Such a fractional-$G_0$-charge gauge excitation has a Bose/Fermi statistics if $n_2 = 0$ and a semion statistics if $n_2 = 1$. We see that both $n_1$ and $n_2$ are measurable. $n_1$ is also measurable, which describes the $G_0$ SPT phases.

To summarize, Tables I–VIII list the $G_i$ charges, the $G_i$ twists, the $G_g$ gauge sectors, and the statistics of the 16 kinds of quasiparticles/defects in the SET state where $(n_1,n_1,\pi) = (0,0)$.

Table I. The $G_i$ charges, the $G_g$ twists, the $G_g$ gauge sectors, and the statistics of the 16 kinds of quasiparticles/defects in the SET state where $(n_1,n_1,\pi) = (0,0)$.

| $(l_1,l_2,l_4)$ | $G_i$ charge | $G_i$ twist | $G_g$ gauge | Statistics |
|-----------------|--------------|-------------|-------------|------------|
| (0000)          | 0            | 0           | 0           | 0          |
| (1000)          | 1            | 0           | 0           | 0          |
| (0010)          | 0            | 0           | e           | 0          |
| (1010)          | 1            | 0           | e           | 0          |
| (0001)          | 0            | 0           | m           | 0          |
| (1001)          | 1            | 0           | m           | 0          |
| (0011)          | 0            | 0           | em          | 1          |
| (1011)          | 1            | 0           | em          | 1          |
| (0100)          | 0            | 1           | 0           | 0          |
| (1100)          | 1            | 1           | 0           | 1          |
| (0101)          | 0            | 1           | e           | 0          |
| (1101)          | 1            | 1           | e           | 1          |
| (0110)          | 0            | 1           | m           | 0          |
| (1110)          | 1            | 1           | m           | 1          |
| (0111)          | 0            | 1           | em          | 1          |
| (1111)          | 1            | 1           | em          | 0          |

excitations:

the trivial excitation $(l_1,l_4) = (0,0)$ → “0”;
the $G_g$ charge excitation $(l_1,l_4) = (1,0)$ → “e”;
the $G_g$ vortex excitation $(l_1,l_4) = (0,1)$ → “m”;
the $G_g$ charge vortex excitation $(l_1,l_4) = (1,1)$ → “em”.

We know that the above eight classes of SET states are classified by

$$\mathcal{H}^3(Z_2 \times Z_2, \mathbb{R}/\mathbb{Z})\oplus \mathcal{H}^3(G_g = Z_2, \mathbb{R}/\mathbb{Z})$$

$$= Z_2^3.$$ (51)

Table II. The $G_0$ charges, the $G_0$ twists, the $G_g$ gauge sectors, and the statistics of the 16 kinds of quasiparticles/defects in the SET state where $(n_1,n_1,\pi) = (1,0)$.

| $(l_1,l_2,l_4)$ | $G_i$ charge | $G_i$ twist | $G_g$ gauge | Statistics |
|-----------------|--------------|-------------|-------------|------------|
| (0000)          | 0            | 0           | 0           | 0          |
| (1000)          | 1            | 0           | 0           | 0          |
| (0010)          | 0            | 0           | e           | 0          |
| (1010)          | 1            | 0           | e           | 0          |
| (0001)          | 0            | 0           | m           | 0          |
| (1001)          | 1            | 0           | m           | 0          |
| (0011)          | 0            | 0           | em          | 1          |
| (1011)          | 1            | 0           | em          | 1          |
| (0100)          | 0            | 1           | 0           | 0          |
| (1100)          | 1            | 1           | 0           | 1          |
| (0101)          | 0            | 1           | e           | 0          |
| (1101)          | 1            | 1           | e           | 1          |
| (0110)          | 0            | 1           | m           | 0          |
| (1110)          | 1            | 1           | m           | 1          |
| (0111)          | 0            | 1           | em          | 1          |
| (1111)          | 1            | 1           | em          | 0          |
From the Tables I–VIII, we see that \( H^3(G_\xi) = Z_2, R/Z \) = \( Z_2 \) (labeled by \( n_2 \)) determine if the \( G_\xi \) gauge theory is a \( Z_2 \) gauge theory (for \( n_2 = 0 \)) or a double-semion theory (for \( n_2 = 1 \)). We also see that \( H^3(G_4) = Z_2, R/Z \) = \( Z_2 \) (labeled by \( n_1 \)) describes the \( G_4 \) SPT phases, and \( H^2(G_4, Z_2) = Z_2 \) (labeled by \( n_2 \)) determines if the \( G_\xi \) gauge-flux excitations can carry a \( 1/2 G_\xi \) charge.

From Tables I–VIII, we see that sometimes, a \( 1/2 G_\xi \) charge can and can only appear on a gauge-flux excitation with \( l_4 = 1 \). This implies that the symmetry of the gauge-flux excitations is described by a nontrivial PSG = \( Z_4 \). In all eight phases, the \( G_\xi \) gauge-charge excitations (the \( a^\mu_k \) charges) are always bosonic and always carry integer \( G_\xi \) charge. In other words,

**TABLE IV.** The \( G_\xi \) charges, the \( G_4 \) twists, the \( G_\xi \) gauge sectors, and the statistics of the 16 kinds of quasiparticles/defects in the SET state where \( (n_1, n_2) = (110) \).

| \((l_1 l_2 l_4)\) | \( G_\xi \) charge | \( G_4 \) twist | \( G_\xi \) gauge | Statistics |
|-----------------|-----------------|---------------|----------------|-----------|
| (0000)          | 0               | 0             | 0              | 0         |
| (1000)          | 1               | 0             | 0              | 0         |
| (0010)          | 0               | 0             | \( e \)        | 0         |
| (1010)          | 1               | 0             | \( e \)        | 0         |
| (0001)          | 0               | 0             | \( m \)        | 0         |
| (1001)          | 1               | 0             | \( m \)        | 0         |
| (0011)          | 0               | 0             | \( em \)       | 1         |
| (1011)          | 1               | 0             | \( em \)       | 1         |
| (0100)          | \(-1/2\)        | 1             | 0              | \(-1/2\) |
| (1100)          | \(1/2\)         | 1             | 0              | \(1/2\)  |
| (0110)          | \(-1/2\)        | 1             | \( e \)        | \(-1/2\) |
| (1110)          | \(1/2\)         | 1             | \( e \)        | \(1/2\)  |
| (0101)          | 1               | 1             | \( m \)        | 1         |
| (1101)          | 0               | 1             | \( m \)        | 0         |
| (0111)          | 1               | 1             | \( em \)       | 0         |
| (1111)          | 0               | 1             | \( em \)       | 1         |

and the statistics of the 16 kinds of quasiparticles/defects in the SET state where \( (n_1, n_2) = (101) \).

**TABLE V.** The \( G_\xi \) charges, the \( G_4 \) twists, the \( G_\xi \) gauge sectors, and the statistics of the 16 kinds of quasiparticles/defects in the SET state where \( (n_1, n_2) = (100) \).

| \((l_1 l_2 l_4)\) | \( G_\xi \) charge | \( G_4 \) twist | \( G_\xi \) gauge | Statistics |
|-----------------|-----------------|---------------|----------------|-----------|
| (0000)          | 0               | 0             | 0              | 0         |
| (1000)          | 1               | 0             | 0              | 0         |
| (0010)          | 0               | 0             | \( e \)        | 0         |
| (1010)          | 1               | 0             | \( e \)        | 0         |
| (0001)          | 0               | 0             | \( m \)        | 0         |
| (1001)          | 1               | 0             | \( m \)        | 0         |
| (0011)          | 0               | 0             | \( em \)       | 1         |
| (1011)          | 1               | 0             | \( em \)       | 1         |
| (0100)          | \(-1/2\)        | 1             | 0              | \(-1/2\) |
| (1100)          | \(1/2\)         | 1             | 0              | \(1/2\)  |
| (0110)          | \(-1/2\)        | 1             | \( e \)        | \(-1/2\) |
| (1110)          | \(1/2\)         | 1             | \( e \)        | \(1/2\)  |
| (0101)          | 1               | 1             | \( m \)        | 1         |
| (1101)          | 0               | 1             | \( m \)        | 0         |
| (0111)          | 1               | 1             | \( em \)       | 0         |
| (1111)          | 0               | 1             | \( em \)       | 1         |
TABLE VII. The $G_i$ charges, the $G_i$ twists, the $G_e$ gauge sectors, and the statistics of the 16 kinds of quasiparticles/defects in the SET state where $(n_1 n_2 n_2) = (101)$.

| $(l_1 l_2 l_4)$ | $G_i$ charge | $G_i$ twist | $G_e$ gauge | Statistics |
|-----------------|--------------|-------------|-------------|-----------|
| (0000)          | 0            | 0           | 0           | 0         |
| (1000)          | 1            | 0           | 0           | 0         |
| (0010)          | 0            | 0           | e           | 0         |
| (1010)          | 1            | 0           | e           | 0         |
| (0001)          | 0            | 0           | m           | $-1/2$    |
| (1001)          | 1            | 0           | m           | $-1/2$    |
| (0011)          | 0            | 0           | em          | 1/2       |
| (1011)          | 1            | 0           | em          | 1/2       |
| (0100)          | $-1/2$       | 1           | 0           | $-1/2$    |
| (1100)          | 1/2          | 1           | 0           | $1/2$     |
| (0110)          | $-1/2$       | 1           | e           | $-1/2$    |
| (1110)          | 1/2          | 1           | e           | $1/2$     |
| (0101)          | $-1/2$       | 1           | m           | $-1/2$    |
| (1101)          | 1/2          | 1           | m           | 0         |
| (0111)          | $-1/2$       | 1           | em          | 0         |
| (1111)          | 1/2          | 1           | em          | 1         |

A unit $G_e$ gauge charge corresponds to the unit charge of $a_i^1$ gauge field and a $G_e$ gauge-flux excitation corresponds to two-unit charge of $a_i^2$ gauge field. Note that a unit $G_e$ gauge charge carries $1/2 G_e$ charge. In other words, the symmetry of the gauge-charge excitations is described by a nontrivial PSG $Z_2$. Two-unit charge of $a_i^1$ gauge field carries no $G_e$ gauge charge, but a unit of $G_e$ charge.

The four types of quantized topological terms are given by

$$ W_{\text{top}} = \frac{m_1}{2\pi} a_i^1 \partial_0 a_i^1, $$

$m_1 = 0, 1, 2, 3$. The total Lagrangian has a form

$$ \mathcal{L} + W_{\text{top}} = \frac{1}{4\pi} K_{l_1 l_2 l_4} \partial_0 a_i^1 \partial_0 a_i^1 + \cdots, $$

TABLE VIII. The $G_i$ charges, the $G_i$ twists, the $G_e$ gauge sectors, and the statistics of the 16 kinds of quasiparticles/defects in the SET state where $(n_1 n_2 n_2) = (111)$.

| $(l_1 l_2 l_4)$ | $G_i$ charge | $G_i$ twist | $G_e$ gauge | Statistics |
|-----------------|--------------|-------------|-------------|-----------|
| (0000)          | 0            | 0           | 0           | 0         |
| (1000)          | 1            | 0           | 0           | 0         |
| (0010)          | 0            | 0           | e           | 0         |
| (1010)          | 1            | 0           | e           | 0         |
| (0001)          | 0            | 0           | m           | $-1/2$    |
| (1001)          | 1            | 0           | m           | $-1/2$    |
| (0011)          | 0            | 0           | em          | 1/2       |
| (1011)          | 1            | 0           | em          | 1/2       |
| (0100)          | 0            | 1           | 0           | $-1/2$    |
| (1100)          | 1/2          | 1           | 0           | $1/2$     |
| (0110)          | 0            | 1           | e           | $-1/2$    |
| (1110)          | 1/2          | 1           | e           | $1/2$     |
| (0101)          | 0            | 1           | m           | $-1/2$    |
| (1101)          | 1/2          | 1           | m           | 0         |
| (0111)          | 0            | 1           | em          | 0         |
| (1111)          | 1/2          | 1           | em          | 1         |

with

$$ K = \begin{pmatrix} 2m_1 & 4 \\ 4 & 0 \end{pmatrix}, \quad K^{-1} = \frac{1}{8} \begin{pmatrix} 0 & 2 \\ 2 & -m_1 \end{pmatrix}. $$

Since moving the $G_i$ charge (two units of $a_i^1$ charge) around a unit $a_i^2$ charge induced a phase $\pi$, a unit $a_i^2$ charge corresponds to the end of the branch cut in the original theory along which we have a $G_i$ symmetry twist. However, fusing two unit $a_i^2$ charges gives a nontrivial $G_e$ gauge excitation, a unit of $G_g$ gauge flux (described by two-unit charge of $a_i^2$ gauge field).

Therefore, a unit $a_i^2$ charge does not correspond to a pure $G_i$ twist. It is a bound state of $G_i$ twist, $G_g$ gauge excitation, and $G_e$ charge.

To calculate the $G_i$ charge for a generic quasiparticle with $l_ia_i^1$ charge, first we assume that the $G_i$ charge has the following form:

$$ G_i, \text{charge} = l^T \mathbf{q}. $$

The vector $\mathbf{q}$ must satisfy $(2,0)\mathbf{q} = \pm 1$ so that two units of $a_i^1$ charge carry a $G_i$ charge 1. To obtain another condition on $\mathbf{q}$, we note that the trivial quasiparticles are given by $l = (K_{11}, K_{12}) = (2m_1, 4)$ and $l = (K_{21}, K_{22}) = (4, 0)$. So we require that $(2m_1, 4)\mathbf{q} = 0$ or 2. We find that $\mathbf{q}$ has four choices:

$$ \mathbf{q}^T = (1/2, -m_1/4), \quad \mathbf{q}^T = (-1/2, m_1/4), $$

$$ \mathbf{q}^T = (1/2, (2 - m_1)/4), \quad \mathbf{q}^T = (-1/2, (2 + m_1)/4). $$

We may choose $\mathbf{q}^T = (1/2, -m_1/4)$ and obtain Tables IX-XII, which list the $G_i$ charges, the $G_i$ twists, the $G_g$ gauge sectors, and the statistics of the 16 kinds of quasiparticles/defects in the $Z_2$ gauge theory with $Z_2$ symmetry which contain a topological term labeled by $m_1$ and a mixing of the gauge $G_e$ and symmetry $G_i$ described by $G = Z_4$. Other choices of $\mathbf{q}$ sometimes regenerate the above four states and sometimes generate new states.

From Tables I-XII, we see the patterns of $G_i$ charges, $G_i$ twists, and statistics are all different, except the

TABLE IX. The $G_i$ charges, the $G_i$ twists, the $G_g$ gauge sectors, and the statistics of the 16 kinds of quasiparticles/defects in the SET state $m_1 = 0$ with $\mathbf{q}^T = (1/2, -m_1/4)$.
produces 11 different particles without fractional Gs.

\[
\begin{array}{cccc}
(l_1l_2) & G_1 \text{ charge} & G_1 \text{ twist} & G_2 \text{ gauge} & \text{statistics} \\
(00) & 0 & 0 & 0 & 0 \\
(20) & 1 & 0 & 0 & 0 \\
(10) & 0 & 1/2 & e & 0 \\
(30) & -1/2 & 0 & e & 0 \\
(02) & -1/2 & 0 & m & -1/2 \\
(22) & 1/2 & 0 & m & -1/2 \\
(12) & 0 & 0 & em & 1/2 \\
(32) & 1 & 0 & em & 1/2 \\
(01) & -1/4 & 1 & 0 & -1/8 \\
(21) & 3/4 & 1 & 0 & 7/8 \\
(11) & 1/4 & 1 & e & 3/8 \\
(31) & -3/4 & 1 & e & -5/8 \\
(03) & -3/4 & 1 & m & 7/8 \\
(23) & 1/4 & 1 & m & -1/8 \\
(13) & -1/4 & 1 & em & 3/8 \\
(33) & 3/4 & 1 & em & -5/8 \\
\end{array}
\]

(n_1n_2l_2) = (010) state and the m_1 = 0 state: The two states are related by an exchange e \leftrightarrow m. Thus, the construction produces 11 different Z_2 gauge theories with Z_2 symmetry.

Let us examine the quasiparticles without the Gs twist. We see that six states contain quasiparticles with bosonic and fermionic statistics. Those six states are described by standard \( G_2 = Z_2 \) gauge theory. However, the \( G_1 = Z_2 \) symmetry is realized differently. Some states contain quasiparticles with fractional \( G_1 = Z_2 \) charge, while others contain quasiparticles without fractional \( G_1 = Z_2 \) charge. In some states, the fermionic quasiparticles carry fractional \( G_1 = Z_2 \) charge, while in other states, the fermionic quasiparticles carry integer \( G_1 = Z_2 \) charge.

TABLE XI. The \( G_1 \) charges, the \( G_1 \) twists, the \( G_2 \) gauge sectors, and the statistics of the 16 kinds of quasiparticles/defects in the SET state \( m_1 = 2 \) with \( q^T = (1/2, -m_1/4) \).

\[
\begin{array}{cccc}
(l_1l_2) & G_1 \text{ charge} & G_1 \text{ twist} & G_2 \text{ gauge} & \text{Statistics} \\
(00) & 0 & 0 & 0 & 0 \\
(20) & 1 & 0 & 0 & 0 \\
(10) & 1/2 & 0 & e & 0 \\
(30) & -1/2 & 0 & e & 0 \\
(02) & -1/2 & 0 & m & -1/2 \\
(22) & 1/2 & 0 & m & -1/2 \\
(12) & 0 & 0 & em & 1/2 \\
(32) & 1 & 0 & em & 1/2 \\
(01) & -1/4 & 1 & 0 & -1/8 \\
(21) & 3/4 & 1 & 0 & 7/8 \\
(11) & 1/4 & 1 & e & 3/8 \\
(31) & -3/4 & 1 & e & -5/8 \\
(03) & -3/4 & 1 & m & 7/8 \\
(23) & 1/4 & 1 & m & -1/8 \\
(13) & -1/4 & 1 & em & 3/8 \\
(33) & 3/4 & 1 & em & -5/8 \\
\end{array}
\]

The other six states contain quasiparticles with semion statistics. Those states are twisted Z_2 gauge theory, which is also known as double-semion theory.\(^{10,24}\) Again some of those states have fractional \( G_1 = Z_2 \) charge while others are without fractional \( G_1 = Z_2 \) charge. Sometimes, the semions only carry integer \( G_2 = Z_2 \) charges, or only fractional \( G_1 = Z_2 \) charges, or both integer and fractional \( G_1 = Z_2 \) charges. Those results agree with those obtained in Refs.\(^65\) and \(^66\).

### B. Comparison with group cohomology construction

In Ref. \(^57\), SET phases are constructed using group cohomology, generalizing the toric code to include global symmetry. The physical excitations in phases with the group extension given by \( G = G_1 \times G_2 = Z_2 \times Z_2 \) were also explored there, and it is of interest to compare with the results above using a K matrix.

The group cohomology \( \mathcal{H}^1(Z_2 \times Z_2, \mathbb{R}/Z) = Z_2 \times Z_2 \times Z_2 \). The generators of each of the \( \mathbb{Z}_2 \) in the cohomology group are given by

\[
\omega_{11}(x,y,z) = \exp\left(\frac{\pi i}{2} x_1(y_1 + z_1 - \bar{y}_1 + \bar{z}_1)\right),
\]

\[
\omega_{22}(x,y,z) = \exp\left(\frac{\pi i}{2} x_2(y_2 + z_2 - \bar{y}_2 + \bar{z}_2)\right),
\]

\[
\omega_{22}(x,y,z) = \exp\left(\frac{\pi i}{2} x_1(y_2 + z_2 - \bar{y}_2 + \bar{z}_2)\right),
\]

where \( x, y, z \in Z_2 \times Z_2 \), and \( x = (x_1, x_2) \), where \( x_{1,2} = \{0, 1\} \), and similarly for \( y \) and \( z \). Also, \( a + b = a + b \mod 2 \). Note that

\[
\mathcal{H}^1(Z_2 \times Z_2, \mathbb{R}/Z) = \mathcal{H}^1(Z_2, \mathbb{R}/Z) \oplus \mathcal{H}^2(Z_2, \mathbb{R}/Z) \oplus \mathcal{H}^1(Z_2, \mathcal{H}^1(Z_2, \mathbb{R}/Z)) \oplus \mathcal{H}^1(Z_2, \mathcal{H}^1(Z_2, \mathbb{R}/Z)) \oplus \mathcal{H}^1(Z_2, \mathcal{H}^1(Z_2, \mathbb{R}/Z)) \oplus \mathcal{H}^1(Z_2, \mathcal{H}^1(Z_2, \mathbb{R}/Z)) \oplus \mathcal{H}^1(Z_2, \mathcal{H}^1(Z_2, \mathbb{R}/Z)).
\]
A phase is then characterized by three-cocycles of the form
\[ \Omega(x,y,z) = \omega_{l1}^{n1}(x,y,z) \omega_{l2}^{n2}(x,y,z) \omega_{l3}^{n3}(x,y,z), \]  
where \( n_{1,2,3} = [0,1] \), and they can be precisely identified with the \( n_1, n_2, n_3 \) in Eq. (42). This can be easily checked by computing the modular \( S \) matrix from the group cycles and comparing with the matrix of mutual statistics obtained from the \( K \) matrix. More explicitly, using the methods detailed in Refs. 61, 67, and 68, the modular \( S \) matrix evaluated on the cocycle \( \Omega(x,y,z) \) of the \( Z_2 \times Z_2 \) lattice gauge theory is given by

\[
S_{l}(n_1,n_2,n_3) = \frac{1}{4} \exp\left(-\pi i \left( \sum_{i=1}^{2} \alpha_i h_i + \beta_i g_i \right) + n_1 g_1 h_1 + n_2 g_2 h_2 + \frac{n_1 n_3}{2} (g_1 h_2 + h_1 g_2) \right),
\]  

where \( g,h,\alpha,\beta \) are all two-component vectors whose components each taking values \( \in [0,1] \). Here \( g,h \in Z_2 \times Z_2 \) are the flux excitations and \( \alpha,\beta \) denote irreducible representations of \( Z_2 \times Z_2 \), which correspond to charge excitations. The phase factor appearing in the modular matrix is related to the mutual statistics obtained in Eq. (44). It is clear that the phase factor indeed takes the form of Eq. (44) if we interpret \((\alpha_1,\alpha_2),(\beta_1,\beta_2)\) as our charge vectors \( l,l' \), respectively.\(^{6,69}\)

\[
S_{l,l'}(n_1,n_2,n_3) = \frac{1}{4} \exp\left(-2\pi i l l'^{-1}\right).
\]  

We can thus immediately read off the inverse of the \( K \) matrix from Eq. (64) to be

\[
K^{-1} = \frac{1}{4} \begin{pmatrix}
0 & 2 & 0 & 0 \\
2 & 0 & n_1 & 0 \\
0 & 0 & 0 & 2 \\
0 & n_1 & 2 & n_2
\end{pmatrix},
\]

which, up to a convention for the sign of \( n_1, n_1, n_3 \), is precisely Eq. (42).

In Ref. 57 the \( G_s \) charges of both flux and charge excitations of the gauge group \( G_s \) are computed, by explicitly constructing the \( G_s \) symmetry transformation operator and the (pair) creation operators (i.e., ribbon operators) of the excitations. In the language of the \( K \) matrix description, we again have to do the transformation (suppose we focus on the quasiparticle located at the end \( B \), and fixing \( u_A \) at the other end)

\[
| h, \alpha, \beta, u_A \rangle = \frac{1}{|G_s \times G_s|} \sum_{h,\tilde{g}} \rho_{\alpha}(h) \rho_{\beta}(\tilde{g}) | h, h, \tilde{g}, u_A \rangle,
\]

where \( \rho_{\alpha}(g) \) corresponds to characters of representations of \( G_s = Z_2 \) and \( \rho_{\beta}(\tilde{g}) \) corresponds to that of \( G_s = Z_2 \).\(^{70}\) One can check that in terms of the diagonalized basis vectors of the \( G_s \) transformation as specified in Table II in Ref. 57, the \( G_s \) charge matches up with the result obtained in the \( K \) matrix formulation given above.

The most important observation is that it is found in Ref. 57 (see Table II there) that only in the case where \( n_1 + l_4 \) (i.e., flux charge \( h = 1 \) there) are both nonvanishing that charge fractionalization occurs. In fact, the \( G_s \) transformation \( U \) for the flux charge squares to \(-1\), which is indeed the statement that the \( G_s \) charge is halved. This is in perfect agreement with the results in the previous section [see Eq. (50) or Tables I–VIII].

We note also that since the modular \( S \) matrix descending from the 3-cocycles agrees with that of the \( K \) matrix, the braiding statistics in Ref. 57 have to agree with that obtained using the \( K \) matrix when we turn off \( l_2 \) accordingly.

VI. SUMMARY

In this paper, we studied the quantized topological terms in a weak-coupling gauge theory with gauge group \( G_s \) and a global symmetry \( G_s \) in d-dimensional space-time. We showed that the quantized topological terms are classified by a pair \((G_s,G_d)\), where \( G_d \) is an extension of \( G_s \) by \( G_d \) and \( v_d \) is an element in group cohomology \( H^3(G_d;\mathbb{Z}) \). When \( d = 3 \) and/or when \( G_s \) is finite, the weak-coupling gauge theories with quantized topological terms describe gapped SET phases. Thus, those SET phases are classified by \( \tilde{H}^d(G_d;\mathbb{Z}) \), where \( G_s \) becomes the global symmetry. This result generalized the PSG description of the SET phases.\(^{50,51,54,55}\) It also generalized the recent results in Refs. 53 and 57. We also apply our theory to a simple case \( G_s = G_s = Z_2 \) to understand the physical meanings of the \( \tilde{H}^d(G_s;\mathbb{Z}) \) classification. Roughly, for the trivial extension \( G = G_s \times G_s \), \( \tilde{H}^d(G_s \times G_s;\mathbb{Z}) \) describes different ways in which the quantum number of \( G_s \) becomes fractionalized on gauge-flux excitations, while the nontrivial extensions \( G \) describe different ways in which the quantum number of \( G_s \) become fractionalized on gauge-charged excitations.

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APPENDIX A: CALCULATING $H^*(X,\mathbb{Z}/Z)$ FROM $H^*(X,Z)$

We can use the Künneth formula (see Ref. 71, page 247),

$$H^d(X \times X', M \otimes_R M') \simeq \left[ \bigoplus_{p=0}^{d-1} H^p(X,M) \otimes_R H^{d-2-p}(X',M') \right]$$

$$\oplus \left[ \bigoplus_{p=0}^{d-1} \text{Tor}^R_1 (H^p(X,M), H^{d-2-p}(X',M')) \right],$$

(A1)

to calculate $H^*(X,M)$ from $H^*(X,Z)$. Here $R$ is a principal ideal domain and $M,M'$ are $R$ modules such that $\text{Tor}^R_1(M,M') = 0$. Note that $Z$ and $R$ are principal ideal domains, while $R/Z$ is not. A $R$ module is like a vector space over $R$ (i.e., we can “multiply” a vector by an element of $R$). For more details on principal ideal domain and $R$ module, see the corresponding Wikipedia articles.

The tensor-product operation $\otimes_R$ and the torsion-product operation $\text{Tor}^R_1$ have the following properties:

$$A \otimes_R Z \simeq B \otimes_R A,$$

$$Z \otimes_R M \simeq M \otimes_R Z = M,$$

$$Z \otimes_R Z_n = Z \otimes Z_n = M/nM,$$

$$(A \otimes_R B) \otimes_R Z = (A \otimes_R Z) \otimes_R (B \otimes_R Z),$$

$$M \otimes_R (A \otimes_R B) = (M \otimes_R A) \otimes_R (M \otimes_R B),$$

and

$$\text{Tor}^R_1(A,B) \simeq \text{Tor}^R_1(B,A),$$

$$\text{Tor}^R_1(Z,M) = \text{Tor}^R_1(M,Z) = 0,$$

$$\text{Tor}^R_1(Z_n,M) = \{ m \in M | nm = 0 \},$$

$$\text{Tor}^R_1(Z_n,Z_n) = Z_{(m,n)},$$

$$\text{Tor}^R_1(A \oplus B, M) = \text{Tor}^R_1(A,M) \oplus \text{Tor}^R_1(B,M),$$

$$\text{Tor}^R_1(M,A \oplus B) = \text{Tor}^R_1(M,A) \oplus \text{Tor}^R_1(M,B),$$

where $(m,n)$ is the greatest common divisor of $m$ and $n$. These expressions allow us to compute the tensor-product $\otimes_R$ and the torsion-product $\text{Tor}^R_1$.

If we choose $R = M = Z$, then the condition $\text{Tor}^R_1(M,M') = \text{Tor}^R_1(Z,M') = 0$ is always satisfied. So we have

$$H^d(X \times X', M') \simeq \left[ \bigoplus_{p=0}^{d-1} H^p(X,Z) \otimes_R H^{d-2-p}(X',M') \right]$$

$$\oplus \left[ \bigoplus_{p=0}^{d-1} \text{Tor}^R_1 (H^p(X,Z), H^{d-2-p}(X',M')) \right].$$

(A4)

Now we can further choose $X'$ to be the space of one point and use

$$H^d(X',M') = \begin{cases} M', & \text{if } d = 0, \\ 0, & \text{if } d > 0, \end{cases}$$

(A5)

to reduce Eq. (A4) to

$$H^d(X,M) \simeq H^d(X,Z) \otimes Z M \oplus \text{Tor}^Z_1 (H^{d+1}(X,Z),M),$$

(A6)

where $M'$ is renamed as $M$. The above is a form of the universal coefficient theorem which can be used to calculate $H^*(BG,M)$ from $H^*(BG,Z)$ and the module $M$.

Now, let us choose $M = R/Z$ and compute $H^d(BG,R/Z)$ from $H^d(BG,Z)$. Note that $H^d(BG,Z)$ has a form $H^d(BG,Z) = Z \oplus \cdots \oplus Z \oplus Z_n \oplus Z_n \oplus \cdots$. A $Z$ in $H^d(BG,Z)$ will produce a $R/Z$ in $H^d(BG,R/Z)$ since $Z \otimes Z/R = R/Z$. A $Z_n$ in $H^{d+1}(BG,Z)$ will produce a $Z_n$ in $H^d(BG,R/Z)$ since $\text{Tor}^R_1(Z_n,R/Z) = Z_n$. So we see that $H^d(BG,R/Z)$ has a form $H^d(BG,R/Z) = R/Z \oplus \cdots \oplus R/Z \oplus Z_n \oplus Z_n \oplus \cdots$ and

$$\text{Dis}(H^d(X,R/Z)) \simeq \text{Tor}(H^{d+1}(X,Z)), \quad (A7)$$

where $\text{Dis}(H^d(X,R/Z))$ is the discrete part of $H^d(X,R/Z)$.

If we choose $M = R$, we find that

$$H^d(X,R) \simeq H^d(X,Z) \otimes Z R. \quad (A8)$$

So $H^d(X,R)$ has the form $R \oplus \cdots \oplus R$ and each $Z$ in $H^d(X,Z)$ gives rise to a $R$ in $H^d(X,R)$. Since $H^d(BG,Z) = 0$ for $d = odd$, we have

$$H^d(BG,Z) = \text{Tor}(H^d(BG,Z)), \quad \text{for } d = odd. \quad (A9)$$

Using the Künneth formula Eq. (A4) we can also rewrite $H^d(G,\times G_\mathbb{R}/Z)$ as

$$\mathcal{H}^d(G_\times G,\mathbb{R}/Z) = \mathcal{H}^d(BG_\mathbb{R}/Z) \oplus \mathcal{H}^d(BG_\mathbb{R}/Z)$$

$$\simeq \left[ \bigoplus_{p=0}^{d+1} \mathcal{H}^p(G_\times G,\mathbb{Z}/Z) \otimes \mathcal{H}^{d+1-p}(G_\mathbb{R}/Z) \right]$$

$$\oplus \left[ \bigoplus_{p=0}^{d+1} \text{Tor}^R_1(\mathcal{H}^p(G_\times G,\mathbb{Z}/Z), \mathcal{H}^{d+1-p}(G_\mathbb{R}/Z)) \right],$$

(A10)

where we have used $\mathcal{H}^n(G,\mathbb{R}/Z) = \mathcal{H}^{n+1}(G,\mathbb{R}/Z)$ for $n > 0$, and $\mathcal{H}^1(G,\mathbb{Z}/Z)$ is $0$ for compact or finite group $G$. We also used the universal coefficient theorem (A6)

$$\mathcal{H}^{d-p}(G,\mathcal{H}^p(G,\mathbb{R}/Z)) \simeq \mathcal{H}^{d-p}(G_\times G,\mathbb{R}/Z)$$

$$\oplus \text{Tor}^R_1(\mathcal{H}^{d-p+1}(G,\mathcal{H}^p(G,\mathbb{R}/Z))). \quad (A11)$$

APPENDIX B: A LABELING SCHEME OF SET STATES DESCRIEVED BY WEAK-COUPLING GAUGE THEORY

The Lyndon-Hochschild-Serre spectral sequence $\mathcal{H}^p(G,\mathcal{H}^q(G_\mathbb{R}/Z)) \Rightarrow \mathcal{H}^{p+q}(G,\mathbb{R}/Z)$ may help us to calculate the group cohomology $\mathcal{H}^d(G,\mathbb{R}/Z)$ in terms of $\mathcal{H}^d(G,\mathcal{H}^p(G,\mathbb{R}/Z))$. We find that $\mathcal{H}^d(G,\mathbb{R}/Z)$ contains a chain of subgroups,

$$[0] = H^d \subset H^{d-1} \subset \cdots \subset H^1 \subset H^0 = \mathcal{H}^d(G,\mathbb{R}/Z), \quad (B1)$$

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such that $\mathcal{H}^k / \mathcal{H}^{k+1}$ is a subgroup of a factor group of $\mathcal{H}^k[G_\mathcal{S}, \mathcal{H}^{d-k}[(G_\mathcal{S}, \mathbb{R}/\mathbb{Z})]]$, $\mathcal{H}^k / \mathcal{H}^{k+1} \subset \mathcal{H}^k[G_\mathcal{S}, \mathcal{H}^{d-k}((G_\mathcal{S}, \mathbb{R}/\mathbb{Z})]/\mathcal{H}^k$, $k = 0, \ldots, d$, \hfill (B2)

where $\mathcal{H}^k$ is a subgroup of $\mathcal{H}^k[G_\mathcal{S}, \mathcal{H}^{d-k}((G_\mathcal{S}, \mathbb{R}/\mathbb{Z})]$). Note that $G_\mathcal{S}$ has a nontrivial action on $\mathcal{H}^{d-k}((G_\mathcal{S}, \mathbb{R}/\mathbb{Z})$ as determined by the structure $G_\mathcal{S} = G/G_\mathcal{S}$. We also have

$$H^0 / H^1 \subset \mathcal{H}^0[G_\mathcal{S}, \mathcal{H}^{d}(G_\mathcal{S}, \mathbb{R}/\mathbb{Z})],$$

$$H^d / H^{d+1} = \mathcal{H}^d(G_\mathcal{S}, \mathbb{R}/\mathbb{Z})/\mathcal{H}^d. \hfill (B3)$$

In other words, the elements in $\mathcal{H}^d(G_\mathcal{S}, \mathbb{R}/\mathbb{Z})$ can be one-to-one labeled by $(x_0, x_1, \ldots, x_d)$ with $x_k \in \mathcal{H}^k / \mathcal{H}^{k+1} \subset \mathcal{H}^k[G_\mathcal{S}, \mathcal{H}^{d-k}((G_\mathcal{S}, \mathbb{R}/\mathbb{Z})]/\mathcal{H}^k$. \hfill (B4)

If we want to use $(y_0, y_1, \ldots, y_d)$ with $y_k \in \mathcal{H}^k[G_\mathcal{S}, \mathcal{H}^{d-k}((G_\mathcal{S}, \mathbb{R}/\mathbb{Z})]$, \hfill (B5)

to label the elements in $\mathcal{H}^d(G_\mathcal{S}, \mathbb{R}/\mathbb{Z})$, then such a labeling may not be one-to-one and it may happen that only some of $(y_0, y_1, \ldots, y_d)$ correspond to the elements in $\mathcal{H}^d(G_\mathcal{S}, \mathbb{R}/\mathbb{Z})$. However, for every element in $\mathcal{H}^d(G_\mathcal{S}, \mathbb{R}/\mathbb{Z})$, we can find a $(y_0, y_1, \ldots, y_d)$ that corresponds to it.