Classification of degenerate 4-dimensional matrices with semi-group structure and polarization optics

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Abstract

In polarization optics, an important role play Mueller matrices – real four-dimensional matrices which describe the effect of action of optical elements on the polarization state of the light, described by 4-dimensional Stokes vectors. An important issue is to classify possible classes of the Mueller matrices. In particular, of special interest are degenerate Mueller matrices with vanishing determinants.

Earlier, it was developed a special technique of parameterizing arbitrary 4-dimensional matrices with the use of four 4-dimensional vectors \((k, m, l, n)\). In the paper, a classification of degenerate 4-dimensional real matrices of rank 1, 2, 3. is elaborated. To separate possible classes of degenerate matrices of ranks 1 and 2, we impose linear restrictions on \((k, m, l, n)\), which are compatible with the group multiplication law. All the subsets of matrices obtained by this method, are either subgroups or semigroups. To obtain singular matrices of rank 3, we specify 16 independent possibilities to get the 4-dimensional matrices with zero determinant.

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(matrices with vanishing determinant, for which the law of multiplication holds, but there is no inverse elements). In [2], [3], it was developed a special technique of parameterizing arbitrary 4-dimensional matrices on the base of the Dirac matrices.

In particular, in the spinor basis, an arbitrary $4 \times 4$ matrix can be parameterized by four 4-dimensional vectors $(k, m, l, n)$

$$G = \begin{bmatrix} k_0 + k \sigma & n_0 + n \sigma \\ l_0 + l \sigma & m_0 + m \sigma \end{bmatrix} = \begin{bmatrix} K & N \\ L & M \end{bmatrix}.$$  

(1)

The matrices $G$ will be real, if the second components of parameters are imaginary

$$k^*_2 = -k_2, \quad m^*_2 = -m_2, \quad l^*_2 = -l_2, \quad n^*_2 = -n_2,$$

and all remaining components are real.

The law of multiplication in explicit form is

$$k''_0 = k''_0 k_0 + k' k + n'_0 l_0 + n' 1, \quad m''_0 = m''_0 m_0 + m' m + l'_0 n_0 + l' n, \quad n''_0 = n''_0 k_0 + k' n + n'_0 m_0 + n' m, \quad l''_0 = l''_0 k_0 + l' k + m'_0 l_0 + m' l,$$

$$k'' = k''_0 k + k' k + i k' \times k + n'_0 l + n' l + i n' \times l,$$

$$m'' = m''_0 m + m' m + i m' \times m + l'_0 n + l' n + i l' \times n,$$

$$n'' = n''_0 n + k' n + i k' \times n + n'_0 m + n' m + i n' \times m,$$

$$l'' = l''_0 k + l' k + i l' \times k + m'_0 l + m' l + i m' \times l.$$  

(2)

In this paper, we will study degenerate 4-dimensional matrices of rank 1, 2, 3. To separate possible classes of degenerate matrices of ranks 1 and 2, we impose linear restrictions on $(k, m, l, n)$, which are compatible with the multiplication law (2). All the subsets of matrices obtained by this method, are either sub-groups or semigroups. To obtain singular matrices of rank 3, we just specify 16 independent possibilities to get the 4-dimensional matrices with zero determinant.

First, consider the variants with one independent vector.
Variant I(k):

\[ n = A k, \quad n_0 = \alpha k_0, \]
\[ m = B k, \quad m_0 = \beta k_0, \]
\[ l = D k, \quad l_0 = t k_0; \quad (3a) \]

there exist only 7 solutions:

\[ (K - 1) \quad G = \begin{vmatrix} K & 0 \\ 0 & 0 \end{vmatrix}, \]
\[ (K - 2) \quad G = \begin{vmatrix} k_0 + k\bar{\sigma} & 0 \\ 0 & k_0 + k\bar{\sigma} \end{vmatrix}, \]
\[ (K - 3) \quad G = \begin{vmatrix} K & 0 \\ DK & 0 \end{vmatrix}, \]
\[ (K - 4) \quad G = \begin{vmatrix} K & AK \\ 0 & 0 \end{vmatrix}, \]
\[ (K - 5) \quad G = \begin{vmatrix} K & AK \\ DK & ADK \end{vmatrix}, \]
\[ (K - 6) \quad G = \begin{vmatrix} k_0 + \bar{k}\bar{\sigma} & Ak_0 + A\bar{k}\bar{\sigma} \\ t\bar{k} - A^{-1}\bar{k}\bar{\sigma} & A\bar{k}\bar{\sigma} \end{vmatrix}, \]
\[ (K - 7) \quad G = \begin{vmatrix} k_0 + \bar{k}\bar{\sigma} & \alpha k_0 + A\bar{k}\bar{\sigma} \\ -A^{-1}(k_0 + \bar{k}\bar{\sigma}) & -A^{-1}\alpha k_0 - \bar{k}\bar{\sigma} \end{vmatrix}. \quad (3b) \]

Here there are only 7 types of solutions, among them there is only one subgroup \((K - 2)\), the remaining 6 cases lead to the structure of the semigroup (matrices with rank 2).

Variant I(m):

\[ n = A m, \quad n_0 = \alpha m_0, \]
\[ m = B m, \quad m_0 = \beta m_0, \]
\[ l = D m, \quad l_0 = t m_0; \quad (4a) \]

there exist only 7 solutions:

\[ (M - 1) \quad G = \begin{vmatrix} 0 & 0 \\ 0 & M \end{vmatrix}. \]

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\[(M-2) \quad G = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix},\]

\[(M-3) \quad G = \begin{bmatrix} 0 & 0 \\ DM & M \end{bmatrix},\]

\[(M-4) \quad G = \begin{bmatrix} 0 & AM \\ 0 & M \end{bmatrix},\]

\[(M-5) \quad G = \begin{bmatrix} (Atm_0 - \vec{m}\vec{\sigma}) & (Am_0 + A\vec{m}\vec{\sigma}) \\ (tm_0 - A^{-1}\vec{m}\vec{\sigma}) & (m_0 + \vec{m}\vec{\sigma}) \end{bmatrix},\]

\[(M-6) \quad G = \begin{bmatrix} (-\alpha A^{-1}tm_0 - \vec{m}\vec{\sigma}) & (\alpha m_0 + A\vec{m}\vec{\sigma}) \\ (A^{-1}m_0 + A^{-1}\vec{m}\vec{\sigma}) & (m_0 + \vec{m}\vec{\sigma}) \end{bmatrix},\]

\[(M-7) \quad G = \begin{bmatrix} ADM & AM \\ DM & M \end{bmatrix}. \quad (4b)\]

All cases, except for \((M-2)\), describe the semigroups of the rank 2.

**Variant I(n)**

\[k = A \, n, \quad k_0 = \alpha \, n_0,\]

\[m = B \, n, \quad m_0 = \beta \, n_0,\]

\[l = D \, n, \quad l_0 = t \, n_0; \quad (5a)\]

there exist only 4 solutions:

\[(N-1) \quad G = \begin{bmatrix} AN & N \\ 0 & 0 \end{bmatrix},\]

\[(N-2) \quad G = \begin{bmatrix} AN & N \\ A^2N & AN \end{bmatrix},\]

\[(N-3) \quad G = \begin{bmatrix} \alpha n_0 + A\vec{n}\vec{\sigma} & n_0 + \vec{n}\vec{\sigma} \\ -\alpha An_0 - A^2\vec{n}\vec{\sigma} & -An_0 - A\vec{n}\vec{\sigma} \end{bmatrix},\]

\[(N-4) \quad G = \begin{bmatrix} A\alpha_0 + A\vec{n}\vec{\sigma} & n_0 + \vec{n}\vec{\sigma} \\ \beta A\alpha_0 - A^2\vec{n}\vec{\sigma} & \beta n_0 - A\vec{n}\vec{\sigma} \end{bmatrix}. \quad (5b)\]

All four solutions describe the semigroup of the rank 2.

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Variant I(l):
\[
\begin{align*}
\textbf{k} &= A \textbf{l}, \quad k_0 = \alpha l_0, \\
\textbf{m} &= B \textbf{l}, \quad m_0 = \beta l_0, \\
\textbf{n} &= D \textbf{l}, \quad n_0 = t l_0; \\
\end{align*}
\]

there exist only 4 solutions:
\[
(L - 1) \quad G = \begin{bmatrix} AL & 0 \\ L & 0 \end{bmatrix},
\]
\[
(L - 2) \quad G = \begin{bmatrix} AL & A^2L \\ L & AL \end{bmatrix},
\]
\[
(L - 3) \quad G = \begin{bmatrix} \alpha l_0 + A\bar{\sigma} & -\alpha Al_0 - A^2 l \\ l_0 + l\bar{\sigma} & -Al_0 - A\bar{\sigma} \end{bmatrix},
\]
\[
(L - 4) \quad G = \begin{bmatrix} Al_0 + A\bar{\sigma} & \beta Al_0 - A^2 l\bar{\sigma} \\ l_0 + l\bar{\sigma} & \beta l_0 - A\bar{\sigma} \end{bmatrix}.
\]

All four solutions describe the semigroup of rank 2.

We now consider the cases of two independent vectors.

Variant II(k, m)
\[
\begin{align*}
\textbf{n} &= Ak + B\textbf{m}, \quad n_0 = \alpha k_0 + \beta m_0, \\
\textbf{l} &= C\textbf{k} + D\textbf{m}, \quad l_0 = sk_0 + tm_0; \\
\end{align*}
\]

there exist only 5 solutions:
\[
(KM - 1) \quad G = \begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix},
\]
\[
(KM - 2) \quad G = \begin{bmatrix} K \\ D(M - K) \end{bmatrix} 
\begin{bmatrix} 0 \\ M \end{bmatrix},
\]
\[
(KM - 3) \quad G = \begin{bmatrix} K \\ B^{-1}K \end{bmatrix} 
\begin{bmatrix} BM \\ M \end{bmatrix},
\]
\[
(KM - 4) \quad G = \begin{bmatrix} K \\ 0 \end{bmatrix} 
\begin{bmatrix} A(K - M) \\ M \end{bmatrix}.
\]
\[(KM - 5) \quad G = \begin{vmatrix} K & A(K - M) \\ C(K - M) & M \end{vmatrix}. \quad (7b)\]

All solutions except for the \((KM - 1)\) describe the semigroup of rank 2.

**Variant II(l, n)**

\[k = (Al + Bn) , \quad k_0 = (\alpha l_0 + \beta n_0) ,\]

\[m = (Dl + Cn) , \quad m_0 = (tl_0 + sn_0) ; \quad (8a)\]

there exist only 2 solutions:

\[(LN - 1) \quad G = \begin{vmatrix} AL & N \\ L & A^{-1}N \end{vmatrix}, \]

\[(LN - 2) \quad G = \begin{vmatrix} BN & N \\ L & B^{-1}L \end{vmatrix}. \quad (8b)\]

The two solutions describe the semigroup of rank 2.

**Variant II(k, n)**

\[l = (Ak + Bn) , \quad l_0 = (\alpha k_0 + \beta n_0) ,\]

\[m = (Dn + Ck) , \quad m_0 = (tn_0 + sk_0) ; \quad (9a)\]

solutions:

\[(KN - 1) \quad G = \begin{vmatrix} K & N \\ AK & AN \end{vmatrix}, \]

\[(KN - 2) \quad G = \begin{vmatrix} K & N \\ 0 & K \end{vmatrix}. \quad (9b)\]

The first solution describes the group, the second – the semigroup of rank 2.

**Variant II(m, l)**

\[n = Am + Bl , \quad l_0 = \alpha m_0 + \beta l_0 ,\]

\[k = Dl + Cm , \quad k_0 = tl_0 + sm_0 ; \quad (10a)\]
there exist only 2 solutions:

\[ (ML - 1) \quad G = \begin{vmatrix} AL & AM \\ L & M \end{vmatrix}, \]

\[ (ML - 2) \quad G = \begin{vmatrix} M & 0 \\ L & M \end{vmatrix}. \quad (10b) \]

The first solution describes the group, the second – the semigroup of rank 2.

Now consider the case 3 independent vectors.

**Variant I(k, m, n):**

\[ l = Ak + Bm + Cn, \quad l_0 = ak_0 + \beta m_0 + s n_0; \quad (11a) \]

there exist only 2 solutions:

\[ (KMN - 1) \quad G = \begin{vmatrix} K & N \\ 0 & M \end{vmatrix}, \]

\[ (KMN - 2) \quad G = \begin{vmatrix} K & N \\ -K + M + N & M \end{vmatrix}. \quad (11b) \]

The first solution describes the group, the second - the semigroup of rank 2.

There are similar solutions for **variant I(k, m, l):**

\[ (KML - 1), \quad G = \begin{vmatrix} K & 0 \\ L & M \end{vmatrix}, \]

\[ (KML - 2), \quad G = \begin{vmatrix} K & -M + K + L \\ L & -M + K + L \end{vmatrix}. \quad (11c) \]

**Variant I(n, l, k)**

\[ m = An + Bl + Ck, \quad m_0 = an_0 + \beta l_0 + sk_0; \quad (12a) \]

there exists only 1 solution:

\[ (NLK - 1) \quad G = \begin{vmatrix} K & N \\ L & (K + AN - A^{-1}L) \end{vmatrix} \quad (12b) \]
it describes a semigroup of rank 2.

There is a similar solution for \textbf{variant I}(n, l, m)

\[
(NLM - 1) G = \begin{vmatrix}
(M + AL - A^{-1}N) & N \\
L & M
\end{vmatrix} ;
\tag{12c}
\]

it describes a semigroup of rank 2.

In all cases above, from the semigroups of the rank 2 one can easily to obtain semi-groups of the rank 1, for this it suffices to add a requirement that the determinant of the basic $2 \times 2$-matrices be equal to zero.

Let us consider singular Mueller matrices of the rank 3. Given the explicit form of the matrix $G$, it is easy to understand that there are 16 simple ways to get the semigroups of rank 3. It is enough to have vanishing any $i$-line and any $j$-column in the original 4-dimensional matrix. Compatibility of the law of multiplication with this constrain is obvious.

All 16 possibilities are listed below.

\textbf{Variant (00)}

\[
G = \begin{vmatrix}
0 & 0 & 0 & 0 \\
0 & 2k_0 & 2n_1 & 2n_0 \\
0 & 2l_1 & m_0 + m_3 & m_1 - im_2 \\
0 & 2l_0 & m_1 + im_2 & m_0 - m_3
\end{vmatrix} ,
\]

$k_1 = 0, \quad k_2 = 0, \quad k_0 = -k_3 ,

n_0 = -n_3, \quad l_0 = -l_3, \quad +in_2 = n_1, \quad -il_2 = l_1 .

\textbf{Variant (01)}

\[
G = \begin{vmatrix}
0 & 0 & 0 & 0 \\
2k_1 & 0 & 2n_1 & 2n_0 \\
2l_0 & 0 & m_0 + m_3 & m_1 - im_2 \\
2l_1 & 0 & m_1 + im_2 & m_0 - m_3
\end{vmatrix} ,
\]

$k_0 = 0, \quad k_3 = 0, \quad k_1 = ik_2 ,

l_1 = il_2, \quad l_0 = l_3, \quad n_0 = -n_3, \quad n_1 = in_2 .

\textbf{Variant (02)}

\[
G = \begin{vmatrix}
0 & 0 & 0 & 0 \\
2k_1 & 2k_0 & 0 & 2n_0 \\
l_0 + l_3 & l_1 - il_2 & 0 & 2m_1 \\
l_1 + il_2 & l_0 - l_3 & 0 & 2m_0
\end{vmatrix} .
\]
\[
\begin{align*}
n_1 &= 0, \quad n_2 = 0, \quad n_0 = -n_3, \\
m_0 &= -m_3, \quad m_1 = -im_2, \quad k_0 = -k_3, \quad k_1 = ik_2.
\end{align*}
\]

**Variant (03)**

\[
G = \begin{vmatrix}
0 & 0 & 0 & 0 \\
2k_1 & 2k_0 & 2n_1 & 0 \\
l_0 + l_3 & l_1 - il_2 & 2m_0 & 0 \\
l_1 + il_2 & l_0 - l_3 & 2m_1 & 0
\end{vmatrix},
\]

\[
n_0 = 0, \quad n_3 = 0, \quad n_1 = in_2, \\
m_0 = m_3, \quad m_1 = im_2, \quad k_0 = -k_3, \quad k_1 = ik_2.
\]

**Variant (10)**

\[
G = \begin{vmatrix}
0 & 2k_1 & 2n_0 & 2n_1 \\
0 & 0 & 0 & 0 \\
0 & 2l_1 & m_0 + m_3 & m_1 - im_2 \\
0 & 2l_0 & m_1 + im_2 & m_0 - m_3
\end{vmatrix},
\]

\[
k_0 = 0, \quad k_3 = 0, \quad k_1 = -ik_2, \\
l_0 = -l_3, \quad l_1 = -il_2, \quad n_1 = -in_2, \quad n_0 = n_3.
\]

**Variant (11)**

\[
G = \begin{vmatrix}
2k_0 & 0 & 2n_0 & 2n_1 \\
0 & 0 & 0 & 0 \\
2l_0 & 0 & m_0 + m_3 & m_1 - im_2 \\
2l_1 & 0 & m_1 + im_2 & m_0 - m_3
\end{vmatrix},
\]

\[
k_1 = 0, \quad k_2 = 0, \quad k_0 = k_3, \\
l_0 = l_3, \quad l_1 = il_2, \quad n_1 = -in_2, \quad n_0 = n_3.
\]

**Variant (12)**

\[
G = \begin{vmatrix}
2k_0 & 2k_1 & 0 & 2n_1 \\
0 & 0 & 0 & 0 \\
l_0 + l_3 & l_1 - il_2 & 0 & 2m_1 \\
l_1 + il_2 & l_0 - l_3 & 0 & 2m_0
\end{vmatrix},
\]

\[
n_0 = 0, \quad n_3 = 0, \quad n_1 = -in_2, \\
m_0 = -m_3, \quad m_1 = -im_2, \quad k_1 = -ik_2, \quad k_0 = k_3.
\]
Variant (13)

$$G = \begin{vmatrix} 2k_0 & 2k_1 & 2n_0 & 0 \\ 0 & 0 & 0 & 0 \\ l_0 + l_3 & l_1 - il_2 & 2m_0 & 0 \\ l_1 + il_2 & l_0 - l_3 & 2m_1 & 0 \end{vmatrix} ,$$

$$n_1 = 0, \quad n_2 = 0, \quad n_0 = n_3 ,$$

$$m_0 = m_3, \quad m_1 = im_2, \quad k_1 = -ik_2, \quad k_0 = k_3 .$$

Variant (20)

$$G = \begin{vmatrix} 0 & 2k_1 & n_0 + n_3 & n_1 - in_2 \\ 0 & 2k_0 & n_1 + in_2 & n_0 - n_3 \\ 0 & 0 & 0 & 0 \\ 2l_0 & 2m_1 & 2m_0 \end{vmatrix} ,$$

$$l_1 = 0, \quad l_2 = 0, \quad l_0 = -l_3 ,$$

$$m_0 = -m_3, \quad m_1 = im_2, \quad k_1 = -ik_2, \quad k_0 = -k_3 .$$

Variant (21)

$$G = \begin{vmatrix} 2k_0 & 0 & n_0 + n_3 & n_1 - in_2 \\ 2k_1 & 0 & n_1 + in_2 & n_0 - n_3 \\ 0 & 0 & 0 & 0 \\ 2l_1 & 0 & 2m_1 & 2m_0 \end{vmatrix} ,$$

$$l_0 = 0, \quad l_3 = 0, \quad l_1 = il_2 ,$$

$$m_0 = -m_3, \quad m_1 = im_2, \quad k_1 = ik_2, \quad k_0 = k_3 .$$

Variant (22)

$$G = \begin{vmatrix} k_0 + k_3 & k_1 - ik_2 & 0 & 2n_1 \\ k_1 + ik_2 & k_0 - k_3 & 0 & 2n_0 \\ 0 & 0 & 0 & 0 \\ 2l_1 & 2l_0 & 0 & 2m_0 \end{vmatrix} ,$$

$$m_1 = 0, \quad m_2 = 0, \quad m_0 = -m_3 ,$$

$$n_0 = -n_3, \quad n_1 = -in_2, \quad l_1 = il_2, \quad l_0 = -l_3 .$$
### Variant (23)

\[
G = \begin{vmatrix}
  k_0 + k_3 & k_1 - ik_2 & 2n_0 & 0 \\
  k_1 + ik_2 & k_0 - k_3 & 2n_1 & 0 \\
  0 & 0 & 0 & 0 \\
  2l_1 & 2l_0 & 2m_1 & 0 \\
\end{vmatrix}
\]

\[
m_0 = 0, \quad m_3 = 0, \quad m_1 = im_2, \\
n_0 = n_3, \quad n_1 = in_2, \quad l_1 = il_2, \quad l_0 = -l_3.
\]

### Variant (30)

\[
G = \begin{vmatrix}
  0 & 2k_1 & n_0 + n_3 & n_1 - in_2 \\
  0 & 2k_0 & n_1 + in_2 & n_0 - n_3 \\
  0 & 2l_1 & 2m_0 & 2m_1 \\
  0 & 0 & 0 & 0 \\
\end{vmatrix},
\]

\[
l_0 = 0, \quad l_3 = 0, \quad l_1 = -il_2, \\
k_0 = -k_3, \quad k_1 = -ik_2, \quad m_1 = -im_2, \quad m_0 = m_3.
\]

### Variant (31)

\[
G = \begin{vmatrix}
  2k_0 & 0 & n_0 + n_3 & n_1 - in_2 \\
  2k_1 & 0 & n_1 + in_2 & n_0 - n_3 \\
  2l_0 & 0 & 2m_0 & 2m_1 \\
  0 & 0 & 0 & 0 \\
\end{vmatrix},
\]

\[
l_1 = 0, \quad l_2 = 0, \quad l_0 = l_3, \\
k_0 = k_3, \quad k_1 = ik_2, \quad m_1 = -im_2, \quad m_0 = m_3.
\]

### Variant (32)

\[
G = \begin{vmatrix}
  k_0 + k_3 & k_1 - ik_2 & 2n_1 & 0 \\
  k_1 + ik_2 & k_0 - k_3 & 2n_0 & 0 \\
  2l_0 & 2l_1 & 0 & 2m_1 \\
  0 & 0 & 0 & 0 \\
\end{vmatrix},
\]

\[
m_0 = 0, \quad m_3 = 0, \quad m_1 = -im_2, \\
l_0 = l_3, \quad l_1 = -il_2, \quad n_1 = -in_2, \quad n_0 = -n_3.
\]

(14c)
Variant (33)

\[ G = \begin{vmatrix}
  k_0 + k_3 & k_1 - ik_2 & 2n_0 & 0 \\
  k_1 + ik_2 & k_0 - k_3 & 2n_1 & 0 \\
  2l_0 & 2l_1 & 2m_0 & 0 \\
  0 & 0 & 0 & 0 \\
\end{vmatrix}, \]

\[ m_1 = 0, \quad m_2 = 0, \quad m_0 = m_3, \]

\[ l_0 = l_3, \quad l_1 = -il_2, \quad n_1 = in_2, \quad n_0 = n_3. \]

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