Effects of turbulent environment on the surface roughening: The Kardar-Parisi-Zhang model coupled to the stochastic Navier–Stokes equation

N V Antonov¹, N M Gulitskiy¹,³, P I Kakin¹ and M M Kostenko¹,²

¹ Department of Physics, Saint Petersburg State University, 7/9 Universitetskaya nab., Saint Petersburg 199034, Russia
² L.D. Landau Institute for Theoretical Physics, 142432, Ak. Semenova 1-A, Chernogolovka, Moscow region, Russia

E-mail: n.antonov@spbu.ru, n.gulitskiy@spbu.ru, p.kakin@spbu.ru and m.m.kostenko@mail.ru

Received 26 February 2020, revised 10 June 2020
Accepted for publication 23 June 2020
Published 1 July 2020

Abstract

The Kardar-Parisi-Zhang model of non-equilibrium critical behaviour (kinetic surface roughening) with turbulent motion of the environment taken into account is studied by the field theoretic renormalization group approach. The turbulent motion is described by the stochastic Navier–Stokes equation with the random stirring force whose correlation function includes two terms that allow one to account both for a turbulent fluid and for a fluid in thermal equilibrium. The renormalization group analysis performed in the leading order of perturbation theory (one-loop approximation) reveals six possible types of scaling behaviour (universality classes). The most interesting values of the spatial dimension \( d = 2 \) and \( 3 \) correspond to the universality class of a pure turbulent advection where the nonlinearity of the Kardar-Parisi-Zhang model is irrelevant.

Keywords: surface roughening, non-equilibrium critical behaviour, turbulent advection, renormalization group

1. Introduction

The problem of random growth phenomena and fluctuating surfaces has been attracting constant attention over the past few decades [1–15]. Examples of such processes include dynamics of flame fronts, cancer tumours, bacterial colonies, spreading of cholera and other epidemics, earthquakes, social disturbances, etc. Various attempts have been undertaken to describe growth phenomena by microscopic models [12–15]. However, considerable success was achieved by focusing on one specific feature of those phenomena, i.e., on their universality. Indeed, systems with random surface growth display universal scaling behaviour; in this sense, they are similar to equilibrium nearly-critical systems. Such scaling behaviour is sometimes referred to as kinetic roughening [2] because the growing surfaces become increasingly rough over time.

The scaling is described by the power law for correlation functions asymptotic behaviour in the infrared (IR) range (large temporal \( t \) and spatial \( r \) differences when compared with characteristic microscopic scales) [2–4]:

\[
\langle [h(t, \mathbf{x}) - h(0, \mathbf{0})]^{n}\rangle \sim r^{\alpha n} F_n(r/t^{1/z}), \quad r = |\mathbf{x}|. \tag{1.1}
\]

The roughness exponent \( \chi \) and the dynamical exponent \( z \) define the universality class of the scaling behaviour. Here \( F_n(\cdot) \) are certain universal scaling functions, the averaging \( \langle \cdot \rangle \) is performed over the statistical ensemble, and \( h(t, \mathbf{x}) \) stands for the height of the surface profile (here and below, \( t \) and \( \mathbf{x} \) are the time and the space coordinates).

In the equilibrium critical state theory, most typical types of scaling behaviour belong to the universality class of the \( O(n) \)-symmetric \( \phi^4 \)-model [16]. As long as universal critical properties (like scaling laws and exponents) are concerned, lattice models (like the Ising or Heisenberg models) are equivalent to that continuous model. It is then natural to hope
that critical behaviour of various discrete models of surface roughening (Eden model, ballistic deposition, solid-on-solid models [12–15]) can be described by a smoothed height profile $h(t, x)$.

One of the widely accepted models of a smoothed profile is provided by the Kardar-Parisi-Zhang (KPZ) stochastic differential equation introduced in [1] $^4$. One of the possible ways to derive it is to use the idea that the growth velocity should be a smooth function of the height gradient. This assumption is rather general; at the same time, it determines the leading nonlinearity of the KPZ model. Therefore, it is not surprising that the KPZ model and its modifications appear in a variety of different physical circumstances.

There are many systems in nature besides growing surfaces in which nonlinear relationship between natural observations and underlying sources of random inputs and noise is manifested. As a consequence, they are not well described in terms of any of the classically developed statistical universality classes (e.g., Gaussian). The general features of these systems are smoothness (there is a force similar to surface tension that opposes roughening), the growth speed which is isotropic and lateral, and space-time uncorrelated noise.

These three properties are so basic that long-time behaviour should be the same for a variety of physical systems as well as mathematical models which share them. This is why KPZ universality class is sometimes celebrated as a fundamental class of stochastic processes [11]. As a result, the KPZ equation became a paradigmatic model of general non-equilibrium critical phenomena.

However, despite the relative simplicity of the formulation of the KPZ model, its paradigmatic reputation and numerous attempts made, a satisfactory theoretical understanding of the model has not been achieved yet. The value of the upper critical dimension for the KPZ model, and its very existence, is still disputed [18–26]. Numerous mappings onto various reaction-diffusion models raise questions about the meaning of imaginary random noise [27–34]. Rigorous mathematical studies, as a rule, deal with the case $d = 1$ and one-point statistics; see, e.g., [11, 35] and references therein; for an exception see, e.g., [36].

The perturbative renormalization group (RG) analysis within the $\varepsilon$ expansion (where $\varepsilon$ is a deviation from the logarithmic spatial dimension) proved to be extremely successful in the study of numerous equilibrium and non-equilibrium critical phenomena. In the case of the KPZ model, it shows that there is no IR attractive fixed point in the physical range of the parameters [37, 38]. The existence of the strong-coupling non-perturbative fixed point was established within the functional RG [39–42]. Although very convincing, it has not been confirmed by any other approach based on a systematic expansion in some (at least, formal) parameter.

Furthermore, one can expect that the KPZ model might be very sensitive to various extensions, disturbances and modifications. Indeed, the simple modification proposed by [43] immediately leads to a model with infinitely many coupling constants [44]. Another example was encountered in cosmological applications of the KPZ model, where it was applied to the description of a self-gravitating uniform medium [45–48]. Inclusion of non-potential degrees of freedom produces a typical IR attractive fixed point within the corresponding $\varepsilon$ expansion [48].

Experience with nearly-equilibrium nearly-critical systems suggests that they can be drastically affected by the motion of the constituting or surrounding medium. Indeed, the critical scaling behaviour can be destroyed in favour of the mean-field behaviour or completely new non-equilibrium universality classes [49–51]. Thus, it is highly desirable to study the effects of the medium motion (which can hardly be excluded in real experimental settings) on the critical behaviour of fluctuating surfaces. Two such attempts were undertaken recently. In [52], the KPZ model was coupled to the stochastic Navier–Stokes (NS) equation driven by a short-correlated random noise (which corresponds to the fluid in thermal equilibrium), an ensemble proposed in [17]. No physically acceptable nontrivial fixed points were found within the perturbative RG analysis.

In [53], the velocity was modelled by a ‘synthetic’ turbulent Gaussian ensemble with vanishing correlation time, known as the Kazantsev-Kraichnan ensemble; see [54] and references therein. It was shown that for incompressible case and for the most interesting values of the spatial dimension $d = 2$ and 3, the KPZ nonlinearity appears IR irrelevant in the sense of Wilson, that is, the IR scaling behaviour is completely determined by the turbulent advection.

Of course, it is desirable to consider more realistic velocity ensembles, in particular, to include finite correlation time. Unfortunately, synthetic Gaussian ensembles with finite correlation time suffer from the lack of Galilean symmetry, which plays an important role in the whole problem.

In this paper, we employ the stochastic NS equation for the incompressible viscous fluid, which implies finite correlation time, non-Gaussianity and, at the same time, is manifestly Galilean covariant. Moreover, now the velocity field has its own dynamics, which opens the possibility to study the feedback of the advected fields on the fluid dynamics itself.

The random stirring force has a power-like correlation function, namely, $\propto b(t) k^{4-d-\gamma}$, where $k$ is the wave number, $d$ is the spatial dimension, and $y$ is an exponent with the logarithmic value $y = 0$ and the physical value $y \to 4$. This choice is typical for the standard RG treatment of the problem [16, 55]. For renormalizability reasons, the correlation function should be modified by the inclusion of the local in-space term [56, 57]. This extended model allows one to consider both the turbulence and the fluid in thermal equilibrium.

We apply the field theoretic RG to the problem. In this approach, possible types of IR asymptotic behaviour are associated with the IR attractive fixed points of the corresponding RG equations. Practical calculations are performed in the leading one-loop approximation, but the critical exponents in the scaling relation (1.1) are found exactly.

Our main result is as follows: for the turbulent environment ($y \to 4$) and the most interesting physical values $d \geq 2$
the nonlinearity of the KPZ equation becomes irrelevant and the IR asymptotic behaviour is described by pure turbulent advection. Although rather disappointing, this result seems reliable and agrees with the one derived earlier in [53] for a simpler velocity ensemble.

The plan of the paper is as follows: detailed description of the model and its field theoretic formulation are given in sections 2 and 3, respectively. The RG analysis, one-loop RG functions and RG fixed points are discussed in section 4. Section 5 is reserved for the conclusion.

2. The KPZ model and the NS equation

The KPZ equation is a semi-phenomenological model described by the stochastic differential diffusion-type equation with the simplest nonlinear term that respects the symmetries \( h \to h+\text{const} \) and \( O(d) \):

\[
\partial_t h = \kappa_0 \partial^2 h + \lambda_0 (\partial h)^2 / 2 + f. \tag{2.1}
\]

Here \( \kappa_0 \) and \( \lambda_0 \) are the coefficients of surface tension and lateral growth respectively while \( f = f(t, x) \) is a random noise that represents small-scale perturbations. Here and below we denote \( \partial_t = \partial / \partial t, \partial_x = [\partial / \partial x], \partial^2 = (\partial \cdot \partial), (\partial h)^2 = (\partial h \cdot \partial h) \).

The statistics of \( f \) is implied to be Gaussian with a zero mean and the pair correlation function

\[
(f(t, x)f(t', x')) = C\delta(t - t')\delta^d(x - x'), \tag{2.2}
\]

where the positive amplitude can be set to \( C = 1 \) without loss of generality.

To include the advection by turbulent environment, the equation (2.1) should be modified by the ‘minimal’ replacement of the ordinary time derivative \( \partial_t \) with its Galilean covariant counterpart (Lagrangian derivative) \( \partial_t \to \nabla_t = \partial_t + (v \cdot \partial) \):

\[
\nabla_t h = \kappa_0 \partial^2 h + \lambda_0 (\partial h)^2 / 2 + f. \tag{2.3}
\]

The velocity field \( v \) is described by the stochastic NS equation for an incompressible viscous fluid:

\[
\nabla \cdot v = \nu_0 \partial^2 v - \partial \varphi + F. \tag{2.4}
\]

Here \( \varphi \) is the pressure, \( F \) is the transverse external random force, \( \nu_0 \) is the kinematic viscosity coefficient; \( v, \varphi, \) and \( F \) depend on \( \{t, x\} \). The field \( v \) is transverse due to the incompressibility condition \( (\partial \cdot v) = 0 \). The external stirring force \( F \) has a Gaussian statistics with a zero mean and the given correlation function:

\[
\langle F(t, x)F(t', x') \rangle = \delta(t - t') \int \frac{dk}{(2\pi)^d} P_0(k) D(k) e^{i\mathbf{k} \cdot (x - x')}, \tag{2.5}
\]

where \( P_0(k) = \delta_{0k} - k_jk_j/k^2 \) is the transverse projector, \( k \equiv |\mathbf{k}| \) is the wave number.

The function \( D(k) \) in (2.5) is usually chosen in the power-like form

\[
D(k) = D_{10} k^{2+\varepsilon-\gamma}, \quad D_{10} > 0, \tag{2.6}
\]

typical for the standard field theoretic approach to the fully developed turbulence; see, e.g., the monographs [16, 55] and references therein. The physical value of the exponent \( \gamma \) corresponds to the limit \( \varepsilon \to 0 \); however, the condition \( D_{10} > 0 \) is necessary to treat \( \varepsilon \) as a parameter of the same order \( \nu_0, \lambda_0 \).

The model (2.4)–(2.6) is logarithmic (the corresponding constant \( g_{10} = D_{10}\nu_0^{-3} \) is dimensionless) at \( \nu_0 \) and for arbitrary \( d \); the ultraviolet (UV) divergences in the perturbation theory have the forms of the poles in \( y \). However, the KPZ model (2.1)–(2.2) becomes logarithmic at \( d = 2 \), and its RG analysis should be performed within the expansion in \( \varepsilon = 2 - d \). In order to make the RG analysis of the full model internally consistent, it is necessary to treat \( \varepsilon \) and \( d \) as small parameters of the same order. Then the UV divergences take on the form of the poles in \( y, \varepsilon \), and their combinations, while the coordinates of the fixed points and various critical dimensions are calculated as double series in \( y \) and \( \varepsilon \).

In its turn, the RG analysis of the model (2.4)–(2.6) near \( d = 2 \) becomes rather delicate. It shows that, in order to ensure the multiplicative renormalizability, it is necessary to add a local term into the correlation function of the random force. That local term should be an even integer power of the wave number \( k \), namely

\[
D(k) = D_{10} k^{2+\varepsilon-\gamma} + D_{20} k^2, \tag{2.7}
\]

where both \( D_{10}, D_{20} \) are positive [56]; see also section 3.10 in the monograph [55]. Detailed discussion of this issue and the two-loop calculations in various renormalization schemes can be found in [57]. Similar situation, when a model is logarithmic for arbitrary \( d \) but additional UV divergences arise at some exceptional values of \( d \), and extension of the model (addition of local terms) is required, was encountered recently in [58–63].

Therefore, there are four coupling constants in the full model (2.2)–(2.7):

\[
g_{10} = D_{10}\nu_0^{-3}, \quad g_{20} = D_{20}\nu_0^{-3}, \quad g_{30} = \lambda_0 \nu_0^{-3/2}, \quad w_0 = \kappa_0\nu_0^{-1}. \tag{2.8}
\]

Although \( w_0 \) is not an expansion parameter, it is dimensionless and should be treated on equal footing with the other three couplings.

For the turbulent fluid, in which we are interested here, \( g_{20} = 0 \), but its renormalized counterpart does not vanish, and it is necessary to keep both terms in (2.7). It is also worth noting that the extended model includes the special case \( g_{10} = 0 \) which is closed with respect to renormalization. Physically, it corresponds to the model of a fluid in thermal equilibrium, studied earlier in [52].

6 Otherwise one of the interactions would be neglected from the very beginning as IR irrelevant (in the sense of Wilson) and some nontrivial asymptotic regimes could be lost.
3. Field theory

According to the general theorem by De Dominis-Janssen (see, e.g., chap. 5 in the monograph [16]) the original stochastic problem (2.2)—(2.7) can be reformulated as the field-theoretic model for the doubled set of fields $\Phi = \{v'_i, h', v, h\}$ with the action functional

$$S(\Phi) = v'_i D_F v'_i + v'_i \left\{ -\nabla_i + \nu_0 \alpha^2 h + \frac{1}{2} \lambda_0 (\partial h)^2 \right\}.$$  (3.1)

Here $D_F$ is the correlation function (2.7) and all the needed summations over repeated indices and integrations over $\{t, x\}$ are implied, for example,

$$v'_i \partial_v v_i = \frac{1}{d} \int dt d^d x v'_i(t, x) \partial_v v_i(t, x).$$  (3.2)

Analysis of UV divergences shows that the model (3.1) is multiplicatively renormalizable with the following renormalized action (in the symbolic notation):

$$S_R(\Phi) = v'_i \{ g_1 \mu^1 \nu \alpha^2 + y + Z_1 g_1 \nu \alpha^2 \} v'_i + v'_i \left\{ -\nabla_i + Z_2 \nu \alpha^2 \right\} v_i + \frac{1}{2} Z_3 h' h' + h' \left\{ -\nabla h + Z_4 \nu \alpha^2 h \right\} + \frac{1}{2} Z_5 g_3 \nu^3 (\partial h)^2.$$  (3.3)

Here the bare parameters are replaced by their renormalized counterparts (without the subscript '0'), while the momentum reference scale $\mu$ is an additional parameter of the renormalized theory. The first term is not renormalized being non-local, while the terms with $\nabla_i$ are not renormalized owing to the Galilean symmetry.

The renormalization constants $Z_1 - Z_6$ are chosen to absorb the UV divergences; they are related to the renormalization constants of the fields, and parameters as follows:

$$Z_1 = Z_2^{-3}, \quad Z_2 = Z_3^{-3}, \quad Z_3 = Z_4 Z_5^{1/2} Z_6^{-3/2}, \quad Z_4 = Z_1 Z_2^{-1}, \quad Z_5 = Z_1 Z_2^{-1}, \quad Z_6 = Z_1^{-1} Z_2, \quad Z_7 = Z_1^{-1} Z_2^{-1}, \quad Z_8 = Z_1^{-1} Z_2^{-1}.$$.  (3.4)

For brevity, we omit explicit one-loop expressions for the renormalization constants and the corresponding calculations; detailed presentation of similar calculations can be found, e.g., in [52].

4. RG analysis and the fixed points of the RG equations

The RG analysis allows one to establish possible types of IR asymptotic behaviour of the correlation (Green) functions; see, e.g., [16] for detailed discussion. The key role is played by the differential operator $D_i = \mu \partial_i$ taken at fixed bare parameters. For the model (3.1), it is expressed in the renormalized variables as follows:

$$\Delta \mu = \mu + \beta_{g_0} \partial_{g_0} + \beta_{g_1} \partial_{g_1} + \beta_{g_2} \partial_{g_2} + \beta_{g_3} \partial_{g_3} + \beta_w \partial_w - \gamma_e \gamma_0 \mu,$$  (4.1)

where we have written $D_i \equiv \partial_i$ for any variable $s$.

The anomalous dimension $\gamma_e$ of a certain parameter $e$ is defined as $\gamma_e = D_i \ln S_i$. The $\beta$ functions for all the coupling constants $g_i = \{g_1, g_2, g_3, w\}$ are defined as $\beta_{g_i} = D_i g_i$ and read (see (3.4))

$$\beta_{g_1} = g_1 (-\gamma - \gamma_1), \quad \beta_{g_2} = g_2 (-\gamma - \gamma_2), \quad \beta_{g_3} = g_3 (-\gamma - \gamma_3), \quad \beta_w = -w g_w.$$  (4.2)

The one-loop expressions for the anomalous dimensions have the forms:

$$\gamma_1 = \frac{(g_1 + g_2)^2}{32 \pi g_2}, \quad \gamma_2 = \frac{(g_1 + g_2)^2}{32 \pi}, \quad \gamma_3 = \frac{g_3^2}{16 \pi^2}, \quad \gamma_4 = \frac{(g_1 + g_2)^2}{8 \pi^2 (w + 1)}, \quad \gamma_5 = \frac{g_1 + g_2}{8 \pi^2 (w + 1)}.$$  (4.3)

It follows from (3.4) that the anomalous dimensions of the coupling constants, the fields, and the parameters are:

$$\gamma_1 = -\gamma_2; \quad \gamma_2 = -\gamma_3; \quad \gamma_3 = \gamma_5 = \frac{3}{2} \gamma_2 + \frac{1}{2} \gamma_3; \quad \gamma_4 = -\gamma_5 = -\gamma_3/2; \quad \gamma_5 = \gamma_0' = \gamma_w = \gamma_2.$$  (4.4)

The one-loop expressions for the $\beta$ functions are as follows:

$$\beta_{g_1} = g_1 \left\{ -\gamma + \frac{3 (g_1 + g_2)}{32 \pi} \right\},$$  (4.5)

$$\beta_{g_2} = g_2 \left\{ -\epsilon - \frac{(g_1 + g_2)^2}{32 \pi g_2} + \frac{3 (g_1 + g_2)}{32 \pi} \right\},$$  (4.6)

$$\beta_{g_3} = g_3 \left\{ -\epsilon - \frac{(g_1 + g_2)}{8 \pi^2 (w + 1)} + \frac{3 (g_1 + g_2)}{64 \pi} - \frac{g_5^2}{32 \pi w^3} \right\},$$  (4.7)

$$\beta_w = -\frac{w (g_1 + g_2)}{2 \pi} \left\{ -\frac{1}{4 w (w + 1)} - \frac{1}{16} \right\}.$$  (4.8)


\[7\] These expressions, and hence all the subsequent results, are in agreement with those derived earlier for various special cases in [17, 52, 56, 57].
Possible asymptotic scaling regimes of the model are determined by the fixed points of the RG equations. The coordinates of the fixed points are given by the zeroes of the $\beta$ functions. The type of a fixed point is determined by the matrix $\Omega_{ij} = \partial_{\beta_j} \beta_i$ where $\beta_i = \{g_1, g_2, g_3, w, \epsilon\}$ is the full set of couplings and $\beta_i$ is the full set of $\beta$ functions. For an IR attractive point the matrix $\Omega$ is non-negative, i.e., all its eigenvalues $\lambda_i$ have non-negative real parts.

Since the scalar field does not affect the velocity, the functions $\beta_{g_1}$ and $\beta_{g_2}$ do not depend on $g_3$ and $w$ and can be studied separately. They have three fixed points, corresponding to simple diffusion of the velocity field (regime number 1), fluid in thermal equilibrium (regime number 2), and turbulent fluid (regime number 3). In the full model, each of them splits into two points: with $g_3 = 0$ (the KPZ non-linearity is irrelevant) and with $g_3 \neq 0$; we label them by A and B, respectively. Thus, we have six fixed points.

The first pair of fixed points has the coordinates $g_{1A} = g_{2A} = 0$ and the corresponding eigenvalues $\lambda_1 = -y, \lambda_2 = -\epsilon$. It includes the points 1A and 1B:

1A. The fixed point with the coordinate $g_{1A} = 0$ and arbitrary $w_3$ (so it is rather a line of fixed points); two remaining eigenvalues are $\lambda_3 = -\epsilon/2$ and $\lambda_4 = 0$. The eigenvalues $\lambda_1$, $\lambda_2$, and $\lambda_3$ are positive when $y < 0$ and $\epsilon < 0$. This is a regime of ordinary diffusion both for the velocity field and for the scalar one.

1B. The fixed point with the coordinate $g_{1B} = -16\pi\epsilon w_3^3$ and arbitrary $w_3$; two remaining eigenvalues are $\lambda_1 = -y$ and $\lambda_2 = 0$. Due to the equality $\lambda_2 = -\lambda_3$ this fixed point is IR attractive only on the half-line $\epsilon < 0$, $y < 0$. Here $g_{1B} = 0$ which again leads to a simple diffusion.

The second pair of the fixed points has the coordinates $g_{1A} = 0$. $g_{2A} = 16\pi\epsilon w_3^3$ and corresponding eigenvalues $\lambda_1 = -\epsilon/2$ and $\lambda_2 = \epsilon$. This region corresponds to a fluid in thermal equilibrium. The pair includes the points 2A and 2B:

2A. The fixed point with the coordinates $g_{1A} = 0$ and $w_3 = (\sqrt{17} - 1)/2$; two remaining eigenvalues are $\lambda_3 = -\epsilon/4$ and $\lambda_4 = \epsilon/2 + 8\epsilon/(1 + \sqrt{17})$. This fixed point is also IR attractive only on the half-line $\epsilon < 0$, $y < 0$.

2B. The fixed point with the coordinates $g_{1B} = -16\pi\epsilon w_3^3$ and $w_3 = (\sqrt{17} - 1)/2$; two remaining eigenvalues are $\lambda_3 = \epsilon/2$ and $\lambda_4 = \epsilon/2 + 8\epsilon/(1 + \sqrt{17})$. When $\epsilon > 0$ and $y < 3\epsilon/2$ all the eigenvalues are positive and the fixed point is IR attractive. This is a regime where both the advection and the KPZ nonlinearity are irrelevant. However, $g_{1B}$ is negative, which is a typical feature of the perturbative RG approach to the KPZ model that requires a careful physical interpretation.

3. The third pair of the fixed points 3A and 3B has the coordinates

$g_{1A} = \frac{32\pi y(3\epsilon - 2y)}{9 \epsilon - y}, \quad g_{2A} = \frac{32\pi y^2}{9 \epsilon - y}$.

The corresponding eigenvalues are $\lambda_{1,2} = -\epsilon/2 + 2y/3 \pm \sqrt{9\epsilon^2 + 12\epsilon y - 8y^2}/6$. This is a regime with turbulent motion of the environment (the non-local term in (2.7) is relevant for $g_{1A} = 0$).

3A. The fixed point with the coordinates $g_{1A} = 0$ and $w_3 = (\sqrt{17} - 1)/2$; two remaining eigenvalues are $\lambda_3 = -\epsilon/2 + y/6$ and $\lambda_4 = (17 - \sqrt{17})y/24$. This is a regime of pure turbulent scalar field advection. This point is IR attractive for $3\epsilon/2 < y$. One can check that for $\lambda_2 > 0$ the expression $(9\epsilon^2 + 12\epsilon y - 8y^2)$ is negative, so that the eigenvalues $\lambda_1$ and $\lambda_3$ are complex. Therefore, this point is a focus in the $g_1 - g_2$ plane.

3B. The fixed point with the coordinates $g_{1B} = 2\pi(\epsilon - 3\epsilon)(\sqrt{17} - 1)/2$ and $w_3 = (\sqrt{17} - 1)/2$; two remaining eigenvalues are $\lambda_3 = \epsilon - y/3$ and $\lambda_4 = (17 - \sqrt{17})y/24$. For $3\epsilon/2 < y < 3\epsilon$ the fixed point is IR attractive. This a regime where both the turbulent advection and the KPZ nonlinearity are relevant. However, $g_{1B}$ is negative for these values of $\epsilon$ and $y$.

The general stability pattern of the fixed points in the plane $(\epsilon, y)$ is shown in figure 1. The existence of IR attractive solutions of the RG equations leads to the existence of the scaling behaviour of correlation functions. The critical exponents in the scaling relation (1.1) corresponding to these scaling regimes can be calculated in a standard fashion; see, e.g., [16, 55] for general scheme and [52, 53] for similar models. For the most interesting from physical point of view point 3A, corresponding to situation when $y \to 4$ (large-scale stirring) and $d = 2$ or 3, they read

$z = 2 - y/3, \quad \chi = \epsilon/2 - y/6$. \hfill (4.9)

These expressions are exact, that is, they have no higher-order corrections in $y$ and $\epsilon$. The absence of corrections follows
from direct calculations together with the fact that \( g_{1*} = 0 \) for this point.

5. Conclusion

We studied effects of turbulent environment on the scaling behaviour of a randomly growing surface. The latter was described by the KPZ model (2.3) which made our findings applicable to a wide class of non-equilibrium critical systems. The advecting field was modelled by the stochastic NS equation (2.4) with a power-like correlation function of the stirring force consisting of two terms (2.7): one term is non-local and, for \( y \rightarrow 4 \), represents the input of energy by the largest-scale motions, while the second term is local and was required for renormalizability near \( d = 2 \). As a byproduct, such choice of the correlation function allows one to consider both the case of a turbulent motion and the case of a fluid in thermal equilibrium.

The field theoretic RG analysis was applied in the leading order of perturbation theory (one-loop approximation). It was found that the model reveals six possible regimes of IR scaling behaviour associated with the six fixed points of the RG equations.

The diagram of the fixed points stability regions (see figure 1) shows that there are neither gaps nor overlaps between different regions. However, this can be an artefact of the one-loop approximation and the gaps or overlaps can appear in higher-order approximations [57].

It was found that for the case of a turbulent fluid \( (y \rightarrow 4) \) and the most interesting values of the spatial dimension \( (d = 2 \text{ or } 3), \text{i.e., } \varepsilon = 0 \text{ or } -1 \) the effects of the KPZ nonlinearity are ‘washed away’ by the flow and the IR behaviour is described by the regime of pure turbulent advection (point 3A) with exactly known scaling exponents (4.9) in the scaling representation (1.1). It is also worth mentioning that similar results were derived earlier for a simpler velocity statistics (the Kazantsev-Kraichnan ensemble), see [53]. In this sense, obtained result seems to be a feature of KPZ equation itself rather than an artefact of the model under consideration.

One can hope that these one-loop results will not change qualitatively when the higher-order corrections are taken into account. This statement is supported by the two-loop calculation [57] for the NS equation with the stirring force (2.7) employed in our paper. However, the two-loop analysis of the full model is welcome and it is an interesting problem for the future.

Another important question is the fate of the strong-coupling, essentially non-perturbative fixed point of the KPZ model (2.1)–(2.2) whose existence was hypothesized in the phenomenology and strongly supported by the functional RG [39–42]. If that point indeed exists, it should be necessarily present in our model (2.2)–(2.7). At the same time, it can become unstable with respect to the turbulent advection (i.e., it may be the saddle type point; as is shown to happen with the perturbative KPZ point). This means, that the resulting IR behaviour of the system is really governed either by pure turbulent advection (regime 3A) found by our analysis or by a new strong-coupling fixed point invisible in standard RG technique and corresponding to the regime where both the nonlinearity and advection are simultaneously important. In order to resolve this dilemma, one has to apply the functional RG to our model which clearly is a highly difficult task already for the pure NS equation itself [64].

Here it is instructive to discuss possible experimental realization of the studied model. The results reported for the characteristic scales of fully developed hydrodynamic turbulence [65–67] are rather controversial; see, e.g., [68–73] and references therein. However, in laboratory or atmospheric turbulence one can roughly estimate the inner (dissipation or Kolmogorov) length scale \( l \) as \( 1 \div 10 \text{ mm} \) and the outer (integral) scale \( L \) as \( 1 \div 100 \text{ m} \).

Therefore, the fully developed turbulence with the Reynolds number \( Re = (L/\nu)^{1/3} \gg 1 \) is ubiquitous in the ordinary ‘room’ or atmospheric motions of a fluid. Thus, a wide range of smoke front scales (from smoke of burning food or a cigarette to smoke of industrial plants or forest fires) lies well in between of the inertial-convective turbulence range. Thus, we hope that our predictions can be tested in real experiments. Another example can be provided by a liquid crystal growth in turbulent medium [74, 75].

The most prominent example of non-equilibrium critical behaviour is a flame propagation in forest-fires; see, e.g., [76–79]. In particular, effects of wind were discussed in [79]. Clearly, in a real forest fire the turbulent motion of the medium is unavoidable and very intense. However, in this case the turbulence itself results from the fire, so that the advection cannot be treated as passive.

To study such problems, it is necessary to take into account the feedback of the scalar field on the dynamics of the advecting velocity (‘active scalar’). The RG study of such problem for the linear advection-diffusion equation shows that, as far as the large-scale scaling behaviour is concerned, the active term in the NS equation appears IR irrelevant for both the turbulent case [80] and for the thermal equilibrium [81].

Thus, it might be possible that the results of the present paper are also applicable for that situation. On the other hand, the inclusion of the KPZ nonlinearity to the advection-diffusion equation can produce a fully nontrivial scaling behaviour where the velocity dynamics is affected by the scalar field.

This work is already in progress.

Acknowledgments

The reported study was funded by RFBR, project number 20-32-70139. The work by N V Antonov and P I Kakin was also supported by the Foundation for the Advancement of Theoretical Physics and Mathematics “BASIS.”
The authors thank the Organizers of the 8th International Conference on New Frontiers in Physics (Crete, ICNFP, 21–29 August 2019) for possibility to present results of the reported study.

ORCID iDs

N M Gulitskiy @ https://orcid.org/0000-0003-0684-328X

References

[1] Kardar M, Parisi G and Zhang Y-C 1986 Phys. Rev. Lett. 56 889
[2] Krug J and Spohn H 1990 Kinetic roughening of growing surfaces. Solids far from Equilibrium ed C Godreche (Cambridge: Cambridge University Press)
[3] Halpin-Healy T and Zhang Y-C 1995 Phys. Rep. 254 215
[4] Lässig M 1998 Journ. Phys.: Condens. Matter. 10 9905
[5] Takeuchi K A 2018 Physica A 504 77
[6] Drossel B and Kardar M 2003 Eur. Phys. Journ. B 36 401
[7] Strack P 2015 Phys. Rev. E 91 032131
[8] Niggemann O and Hinrichsen H 2018 Phys. Rev. E 97 062125
[9] Horowitz J M and Kardar M 2019 Phys. Rev. E 99 012134
[10] Xia H, Tang G and Lan Y 2020 J. Stat. Phys. J. Stat. Phys. 178 800
[11] Corwin I 2012 Random Matrices: Theory and Applications 1 1130001
[12] Eden M 1961 Berkeley Symp. on Math. Statist. and Prob. Proc. Fourth Berkeley Symp. on Math. Statist. and Prob. 4 (Cambridge: Cambridge University Press) 223
[13] Edwards S F and Wilkinson D R 1982 Proc. R. Soc. London Ser. A 381 17
[14] Kim J M, Kosterlitz J M and Ala-Nissila T 1991 J. Phys. A: Math. Gen. 24 5569
[15] Penrose D M 2008 J. Stat. Phys.J. Stat. Phys. 131 247
[16] Vasiliev A N 2004 The Field Theoretic Renormalization Group in Critical behaviour Theory and Stochastic Dynamics (Boca Raton: Chapman & Hall/CRC) [Translated from the Russian: 1998 (St Petersburg, Institute of Nuclear Physics, Gatchina, ISBN 5-86763-122-2)]
[17] Fortuin C M, Domb R D and Newell M J 1977 Phys. Rev. A 16 722
[18] Lässig M and Kizelier H 1997 Phys. Rev. Lett. 78 903
[19] Colaiori F and Moore M 2001 Phys. Rev. Lett. 86 3946
[20] Marinari E, Pagnani A, Parisi G and Rača Z 2002 Phys. Rev. E 65 026136
[21] Fogedby H C 2005 Phys. Rev. Lett. 94 195702
[22] Fogedby H C 2006 Phys. Rev. E 73 031104
[23] Fogedby H C 2008 J. Phys. (Pramana) 71 253
[24] Katzav E and Schwartz M 2002 Physica A 309 69
[25] Schwartz M and Perlman E 2012 Phys. Rev. E 85 050103(R)
[26] Alves S G, Oliveira T J and Ferreira S C 2014 Phys. Rev. E 90 020103(R)
[27] Täuber U C, Howard M and Vollmayr-Lee B P 2005 J. Phys. A: Math. Gen. 38 R79
[28] Zorzano M-P, Hochberg D and Morán F 2006 Phys. Rev. E 74 055102
[29] Andreanov A, Birolì G, Bouchaud J-P and Lefèvre A 2006 Phys. Rev. E 74 030101(R)
[30] Lefèvre A and Birolì G 2007 J. Stat. Mech.: Theory and Experiment 2007 P07024
[31] Le Doussal P and Wiese K J 2015 Phys. Rev. Lett. 114 110601
[32] Benitez F, Ducuit C, Chaté H, Delamotte B, Domínguez I and Muñoz M A 2016 Phys. Rev. Lett. 117 100601
[33] Wiese K J 2016 Phys. Rev. E 93 042117
[34] Cooper F and Dawson J F 2016 Ann. Phys. 365 118
[35] Hairer M 2013 Ann. Math. 178 559
[36] Kupiainen A and Marcozzi M 2017 J. Stat. Phys. 166 876
[37] Lässig M 1995 Nucl. Phys. B 448 559
[38] Wiese K J 1998 J. Stat. Phys. 93 143
[39] Canet L, Chaté H, Delamotte B and Wschebor N 2010 Phys. Rev. Lett. 104 150601
[40] Canet L, Chaté H, Delamotte B and Wschebor N 2011 Phys. Rev. E 84 061128
[41] Kloss T, Canet L and Wschebor N 2012 Phys. Rev. E 86 051124
[42] Mathey S, Agoritsas E, Kloss T, Lecomte V and Canet L 2017 Phys. Rev. E 95 032117
[43] Pavlik S I 1994 JETP 79 303 [Translated from the Russian: ZhETF 1994 94 533]
[44] Antonov N V and Vasil’ev A N 1995 JETP 81 485 [Translated from the Russian: ZhETF 108 885]
[45] Barbero J F, Dominguez A, Goldman T and Pérez-Mercader P 1997 Europhys. Lett. 38 637
[46] Dominguez A et al 1999 Astron. Astrophys. 349 343 https://ui.adsabs.harvard.edu/abs/1999A&A...349..343B/abstract
[47] Gaite J and Dominguez A 2007 J. Phys. A: Math. Theor. 40 6849
[48] Antonov N V 2004 Phys. Rev. Lett. 92 161101
[49] Imaeda T, Onuki A and Kawasaki K 1984 Progr. Theor. Phys. 71 16
[50] Sinai G and Ronis D 1986 Phys. Rev. A 33 2415
[51] Azenowitz A and Nelson D R 1984 Phys. Rev. A 29 1202
[52] Antonov N V, Kakin P I and Lebedev N M 2019 J. Phys. A: Math. Theor. 52 505002
[53] Antonov N V and Kakin P I 2015 Theor. Math. Phys 185 1391
[54] Falkovich G, Gawędzki K and Vergassola M 2001 Rev. Mod. Phys. 73 913
[55] Adzhemyan L T, Antonov N V and Vasiliev A N 1999 The Field Theoretic Renormalization Group in Fully Developed Turbulence (London: Gordon and Breach)
[56] Honkonen J and Nalimov M Y 1996 Z. Phys. B 99 297
[57] Adzhemyan L T, Honkonen J, Kompanietz M V and Vasiliev A N 2005 Phys. Rev. E 71 036305
[58] Antonov N V, Gulitskiy N M, Kostenko M M and Lucijansky V 2017 Phys. Rev. E 95 033120
[59] Antonov N V, Gulitskiy N M, Kostenko M M and Lucijansky V 2019 Universe 5 37
[60] Antonov N V, Gulitskiy N M, Kostenko M M and Lucijansky V 2019 Theor. Math. Phys. 200 1294
[61] Antonov N V, Gulitskiy N M, Kostenko M M and Lucijansky V 2016 EPJ Web. of Conf. 125 05006
[62] Antonov N V, Gulitskiy N M, Kostenko M M and Lucijansky V 2017 EPJ Web. of Conf. 137 10003
[63] Antonov N V, Gulitskiy N M, Kostenko M M and Lucijansky V 2017 EPJ Web. of Conf. 164 07044
[64] Canet L, Delamotte B and Wschebor N 2016 Phys. Rev. E 93 063101
[65] Monin A S and Yaglom A M 1975 Statistical Fluid Mechanics vol 2 (Cambridge, Mass: MIT Press)
[66] Frisch U 1995 Turbulence: The Legacy of A. N. Kolmogorov (Cambridge: Cambridge University Press)
[67] Davidson P A 2015 Turbulence: An Introduction for Scientists and Engineers (Oxford: Oxford University Press)
[68] Ziai A, Schöck M, Chanan G, Troy M, Dekany R G, Lane B F, Borgninio J and Martin F 2004 Appl. Opt. 43 2316
[69] Ochs G R and Hill R J 1985 Appl. Opt. 24 2430
[70] Azoulay E, Thiermann V, Jetter A, Kohnle A and Azar Z 1988 J. Phys. D: Appl. Phys. 21 S41
[71] Roddier F 1981 In Progress in Optics 19 281
[72] Kulikov V A, Andreeva M S, Koryabin A V and Shmalhausen V I 2012 Appl. Opt. 51 8505
[73] Osborn J 2010 Profiling the turbulent atmosphere and novel correction techniques for imaging and photometry in astronomy PhD Thesis Durham University, England section 2.1
[74] Takeuchi K and Sano M 2010 Phys. Rev. Lett. 104 230601
[75] Takeuchi K and Sano M 2012 J. Stat. Phys. 147 853
[76] Bak P, Chen K and Tang C 1990 Phys. Lett. A 147 297
[77] Grassberger P 2002 New J. Phys. 4 17
[78] Janssen H K and Tauber U C 2005 Ann. Phys. 315 147
[79] Song H-S and Lee S-H 2017 Forest Science and Technology 13 9
[80] Nandy M K and Bhattacharjee J K 1998 J. Phys. A: Math. Gen. 31 2621
[81] Antonov N V and Kostenko M M 2019 Zap. Nauch. Seminarov POMI 487 5 http://www.pdmi.ras.ru/znsl/2019/v487/abs005.html