STATISTICAL MOMENTS AND INTEGRABILITY PROPERTIES OF MONATOMIC GAS MIXTURES WITH LONG RANGE INTERACTIONS

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ABSTRACT. This document presents a priori estimates related to statistical moments and integrability properties for solutions of systems of monatomic gas mixtures modelled with the homogeneous Boltzmann equation with long range interactions for hard potentials. We detail the conditions for the generation and propagation of polynomial and exponential moments, and the integrability in Lebesgue spaces.

1. Introduction

In this paper we study a priori estimates for the homogeneous Boltzmann system describing multi-component monatomic gas mixtures for binary interactions in three dimension. We focus in long range and hard interactions using techniques that have been developed recently for the scalar equation combining recent result and notation presented in the works [18, 11] for the case of cutoff angular kernels. The Cauchy problem will be addressed elsewhere.

The method of proof follows a classical approach where the generation and propagation of polynomial and exponential moments use, at its core, a Povzner inequality associated to the binary interaction of different monatomic species. The Povzner inequality for monatomic species is similar to the scalar version, yet less symmetric. This loss of symmetry poses important difficulties that modify the classical mathematical treatment. The mathematical analysis at the level of moments, polynomial and exponential, follows ideas from refined versions for the scalar case given in [18, 22, 15], however, it is worthwhile to mention that the moment analysis requires considering the system as a whole and not component by component independently. In other words, a global moment inequality has to be made for the system involving all species at once. This is particularly challenging for the case of exponential moment analysis which turns out to be subtle.

In a second stage, once the statistical moments are understood, a theory of generation and propagation of higher integrability norms is performed. We implement an approach given in [3] for the treatment of $L^p$-norm generation in the range $p \in (1, \infty]$. Interestingly, for higher integrability generation and propagation, each species of the gas mixture can be considered independently by interpreting the rest of the components as given. This is related to the fact that the type of scattering considered in this document is only forward; in such a case only statistical moments are needed to prove higher Lebesgue integrability generation and propagation.

Let us mention here that regarding the homogeneous scalar equation, the study of moments have been addressed for the long range regime with hard and Maxwell potentials interactions in references such as [23, 19, 22, 15, 20]. The generation and propagation of higher integrability for the scalar problem have been addressed in [2, 3, 4, 14] using methods initially presented in [1, 2]. Boundedness of solutions for the homogeneous equation has been studied in [3, 21, 17] using different approaches and including polynomial and exponential weights, a probabilistic numerical method can be found in [16]. In addition, the gas mixture for monatomic components has been addressed in the so-called cutoff with hard interactions regime in references such as [9, 18, 11], the spatially inhomogeneous setting near thermal equilibrium is addressed in [10], the BGK approximation for multi-species can be found in [8], hydrodynamics expansions can be referred to [7].
chemical reactions of multi-components is addressed in [13], fine mathematical properties of the linear multi-component model are addressed in [12].

The paper is organised in the following way: In section 2, we give some notations and preliminaries and state the main results of this manuscript. Then, in section 3, we prove Theorem 2.2 and give some preliminary Lemma including Povzner inequality that we need to prove our first result. Section 4, is devoted to prove the uniform coercive estimate and our two last Theorem 2.5 and 2.6. Finally, in section 5, we give some general Lemma which help us to prove ours results.

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2. Preliminaries and Main Results

2.1. Preliminaries. Let us consider a gas mixture of $I$ species where each component of the mixture, namely $A_i$, with $1 \leq i \leq I$, is described by its distribution density function denoted with $f_i := f_i(t, v) \geq 0$ depending on time $t > 0$ and particle velocity $v \in \mathbb{R}^3$. The evolution of the gas mixture $\mathbb{F} = [f_i]_{1 \leq i \leq I}$ satisfies the Boltzmann system

\begin{equation}
\partial_t F(t, v) = Q(F, F),
\end{equation}

where $Q(F, F)$ is the vector of collision operator defined for each component as

\begin{equation}
[Q(F, F)]_i := \sum_{j=1}^{I} Q_{ij}(f_i, f_j).
\end{equation}

Here the collision Boltzmann operators $Q_{ij}(f_i, f_j)$ measure the influence that the species $f_j$ exerts on the species $f_i$. Consequently, for any $i$ fixed the distribution function $f_i$ solves a Boltzmann equation where the collision operator takes into account the influence of all the species $A_i$ as

\begin{equation}
\partial_t f_i(t, v) = [Q(F, F)]_i = \sum_{j=1}^{I} Q_{ij}(f_i, f_j), \quad 1 \leq i \leq I.
\end{equation}

2.1.1. Collision process. Consider two interacting particles from the species $A_i$ with associated particle mass $m_i > 0$ and $A_j$ with associated particle mass $m_j > 0$. The shorthand ($v_{ij}, v_{*ij}$) denotes the particles velocities before collision. The dependence of such pre-collisional velocities with respect to the post-collisional velocities $(v, v_*)$ is given by

\begin{equation}
\begin{aligned}
v_{ij} &= \frac{m_i v + m_j v_*}{m_i + m_j} + \frac{m_j}{m_i + m_j} |v - v_*| \sigma = \frac{2 m_i u^-}{m_i + m_j} \\
v_{*,ij} &= \frac{m_i v + m_j v_*}{m_i + m_j} - \frac{m_j}{m_i + m_j} |v - v_*| \sigma = \frac{2 m_j u^-}{m_i + m_j}.
\end{aligned}
\end{equation}

Here $\sigma \in S^2$ is the scattering direction and $u^\pm = \frac{u \pm |u| \sigma}{2}$. After collision, the species remain in the same species with the same associated mass. The collision laws (2.4) conserve momentum and kinetic energy, that is, the interactions are elastic. In this way one has that pre and post collisional velocities are interchangeable ($v_{ij}, v_{*,ij}$) = ($v_{ij}', v_{*,ij}'$) and the conservation laws

\begin{equation}
\begin{aligned}
m_i v_{ij}' + m_j v_{*,ij}' &= m_i v + m_j v_* \quad \text{conservation of momentum,} \\
m_i |v_{ij}'|^2 + m_j |v_{*,ij}'|^2 &= m_i |v|^2 + m_j |v_*|^2 \quad \text{conservation of energy},
\end{aligned}
\end{equation}

are valid. Now, note that in the collision laws (2.4) (or in the conservation laws (2.5)) one can divide the laws by $\sum_{j=1}^{I} m_j$ giving normalised component masses $\tilde{m}_i = \frac{m_i}{\sum_{j=1}^{I} m_j} \in (0, 1)$, for each species $1 \leq i \leq I$. For such reason, one can assume without loss of generality that the component masses add up to unity $\sum_{j=1}^{I} m_j = 1$. 
Remark 2.1. To simplify notation we eliminate the subindices from the shorthand \((v'_{ij}, v''_{s,ij})\) and simply write \((v', v_s')\). Just keep in mind that the post and pre collisional relations will differ for each collision operator \(Q_{ij}\).

2.1.2. Collision operators. The collision operator \(Q_{ij}\), describing the binary interaction between the particles of species \(A_i\), denoted by \(f\), and the particles of species \(A_j\), denoted by \(g\), is explicitly defined by

\[
Q_{ij}(f, g)(v) = \int \int_{\mathbb{R}^2 \times S^2} \left( f(v')g(v_s') - f(v)g(v_s) \right) B_{ij}(\|v\|, \hat{u} \cdot \sigma) d\sigma dv,
\]

where \(u = v - v_s\) is the relative velocity between interacting particles, \(\hat{u} = \frac{u}{\|u\|}\) its direction, \(\sigma = \frac{\hat{v}'}{\|\hat{v}'\|}\) the scattering direction, and \(B_{ij}\) the collisional cross section. The \(B_{ij}\) are positive functions satisfying the symmetric condition \(B_{ij} = B_{ji}\) and the following micro-reversibility assumption

\[
B_{ij}(v, v_s, \sigma) = B_{ij}(v', v'_s, \sigma') = B_{ij}(v_s, v, -\sigma),
\]

with \(\sigma' = \frac{\hat{v}}{\|\hat{v}\|}\).

In this document we concentrate our efforts in hard potential interactions without angular cutoff. More precisely, the collision cross sections \(B_{ij}\), for \(1 \leq i, j \leq I\), are assumed to satisfy the explicit expression

\[
B_{ij}(\|v\|, \hat{u} \cdot \sigma) = |\|v\|^{\lambda_{ij}} b_{ij}(\hat{u} \cdot \sigma), \quad \lambda_{ij} \in (0, 2],
\]

where \(b_{ij}\) is the angular scattering kernel. Writing \(\hat{u} \cdot \sigma = \cos \theta\) with \(\theta \in (0, \frac{\pi}{2}]\), the angular scattering satisfies the condition

\[
\exists \kappa_{ij}, \kappa_{ij}' \in (0, \infty) \quad \kappa_{ij}' \theta^{-s_{ij}'} - 1 \leq \beta_{ij}(\theta) = \sin \theta b_{ij}(\cos \theta) \leq \kappa_{ij} \theta^{-s_{ij}} - 1, \quad s_{ij} \in (0, 2).
\]

The condition on the support of \(b\) implies that the regime we treat in this document is the so-called peaked forward scattering. Note that the symmetric condition implies that

\[
\lambda_{ij} = \lambda_{ji} \quad \text{and} \quad b_{ij} = b_{ji} \quad \text{(that is} \quad s_{ij} = s_{ji}).
\]

The following parameters, related to the minimum and the maximum value of the potential rates \(\lambda_{ij}\), will be important when considering the different properties of solutions,

\[
\underline{\lambda}_i := \min_{1 \leq j \leq I} \lambda_{ij}, \quad \overline{\lambda}_i := \max_{1 \leq j \leq I} \lambda_{ij},
\]

\[
\underline{\lambda} := \min_{1 \leq i, j \leq I} \lambda_{ij}, \quad \overline{\lambda} := \max_{1 \leq i, j \leq I} \lambda_{ij},
\]

and

\[
\lambda^\pm := \min_{1 \leq i, j \leq I} \max_{1 \leq i, j \leq I} \lambda_{ij}.
\]

Additionally, we will consider the following parameters related to the angular singularities,

\[
\underline{s}_i := \min_{1 \leq j \leq I} s_{ij}, \quad \overline{s}_i := \max_{1 \leq j \leq I} s_{ij},
\]

\[
\underline{s} := \min_{1 \leq i, j \leq I} s_{ij}, \quad \overline{s} := \max_{1 \leq i, j \leq I} s_{ij},
\]

and

\[
s^\pm := \min_{1 \leq i, j \leq I} \max_{1 \leq i, j \leq I} s_{ij}.
\]

Before continuing, let us mention that condition (2.10) is only needed in a vicinity of \(\theta = 0\). Away from this point \(b\) can be assumed just integrable. Additionally, let us remark that for the scalar Boltzmann equation the support of the angular scattering \(b(\cos(\theta))\) can be assumed in the upper scattering sphere \(\theta \in (0, \pi/2]\) with no loss of generality since \(v' \leftrightarrow v'_s\) under the change \(\sigma \leftrightarrow -\sigma\). This is not the case for monatomic gas mixtures due to lack of symmetry in the collision laws. We point out, however, that the arguments and results of Section 3 are all valid, without modification, with a kernel \(b(\cos(\theta))\) defined in \((0, \pi]\). Moreover, from a physical perspective, the most generic assumption for the backscattering is \(b(\hat{e} \cdot \sigma)1_{\{\hat{e} \cdot \sigma \leq 0\}} \in L^2(S^2)\). In other words, terms associated
to the backscattering are of lower order in comparison to the forward and can be included in the analysis of Section 4 at a technical price, which may slightly impact some of the arguments in this section. For the interested reader, we refer to [11] for the nuances of the cutoff scattering analysis.

2.1.3. Functional spaces. First, we recall some notations and definitions that will be important along the manuscript.

- The mixture’s bracket form is defined as
  \[
  (v)_i := \sqrt{1 + \frac{m_i}{\sum_{j=1}^{n} m_j}} |v|^2 = \sqrt{1 + \frac{m_i}{|v|^2}}, \quad v \in \mathbb{R}^3, \quad 1 \leq i \leq I.
  \]
- The scalar $k^{th}$-polynomial moment, with $k \geq 0$, is defined for any $0 \leq t \in L_k^1$ as
  \[
  m_{k,i}[f](t) := \int_{\mathbb{R}^3} f(t,v) (v)_i^k dv, \quad 1 \leq i \leq I.
  \]

For the gas mixture $F = [f_i]_{1 \leq i \leq I}$ we use the shorthand
\[
m_{k,i}(t) = m_{k,i}[f_i](t) := \int_{\mathbb{R}^3} f_i(t,v) (v)_i^k dv,
\]
that is, the subindex $i$ in the shorthand $m_{k,i}(t)$ prescribes the bracket and the associate component in $F$. The cumulative $k^{th}$ moment of the gas mixture is defined as
\[
m_k[F](t) := \sum_{i=1}^{I} m_{k,i}[f_i](t) = \sum_{i=1}^{I} \int_{\mathbb{R}^3} f_i(t,v) (v)_i^k dv.
\]

The Banach $L^n_p$ spaces associated to the mixture are defined as follow, refer to [11],
\[
L^n_p = \left\{ F = [f_i]_{1 \leq i \leq I} \left| \sum_{i=1}^{I} \int_{\mathbb{R}^3} (v)_i^n |f_i(v)|^p dv < \infty, \quad n \geq 0, \quad 1 \leq p < \infty \right. \right\}
\]
with associated norm
\[
\|F\|_{L^n_p} := \left( \sum_{i=1}^{I} \int_{\mathbb{R}^3} (v)_i^n |f_i(v)|^p dv \right)^{\frac{1}{p}}.
\]

For the special case $p = +\infty$, the space is defined as
\[
L^n_{\infty} = \left\{ F = [f_i]_{1 \leq i \leq I} \left| \sum_{i=1}^{I} \text{ess sup} |f_i(v)| \leq \infty, \quad n \geq 0 \right. \right\}
\]
with associated norm
\[
\|F\|_{L^n_{\infty}} := \sum_{i=1}^{I} \text{ess sup} |f_i(v)|.
\]

The $L \log L$ space is defined as
\[
L \log L = \left\{ F = [f_i]_{1 \leq i \leq I} \left| \sum_{i=1}^{I} \int_{\mathbb{R}^3} |f_i(v)| \log (1 + |f_i(v)|) dv < \infty \right. \right\}
\]
with associated norm
\[
\|F\|_{L \log L} := \sum_{i=1}^{I} \int_{\mathbb{R}^3} |f_i(v)| \log (1 + |f_i(v)|) dv.
\]

We also work with the Sobolev spaces
\[
H^k := \left\{ F = [f_i]_{1 \leq i \leq I} \left| \sum_{i=1}^{I} \int_{\mathbb{R}^3} \left| (1 + (-\Delta))^{\frac{k}{2}} f_i(v) \right|^2 dv < \infty, \quad k \geq 0 \right. \right\},
\]
and endowed with the norm
\[ \|F\|_{H^k} = \left( \sum_{i=1}^{l} \int_{\mathbb{R}^3} \left| (1 + \Delta)^k f_i(v) \right|^2 dv \right)^{\frac{1}{2}} = \left( \sum_{i=1}^{l} \int_{\mathbb{R}^3} (1 + |\xi|^2)^k |\mathcal{F}(f_i)(\xi)|^2 d\xi \right)^{\frac{1}{2}}. \]

2.1.4. Weak form of the Boltzmann collision operator. Testing the collision operator against a suitable test vector function \([\psi_i]_{1 \leq i \leq I}\) it holds that
\[
\int_{\mathbb{R}^3} Q_{ij}(f_i, f_j)(v) \psi_i(v) dv = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} f_1(v) f_j(v_*) (\psi_i(v') - \psi_i(v)) B_{ij}(|u|, \hat{u} \cdot \sigma) d\sigma dv_* dv,
\]
and
\[
\int_{\mathbb{R}^3} Q_{ji}(f_j, f_i)(v) \psi_j(v) dv = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} f_j(v) f_i(v_*) (\psi_j(v') - \psi_j(v)) B_{ji}(|u|, \hat{u} \cdot \sigma) d\sigma dv_* dv.
\]
Thus, splitting the sum in the sets \(\{ i \geq j \}\) and \(\{ i < j \}\) we can interchange indexes \(i \leftrightarrow j\) to obtain that
\[
\sum_{i=1}^{l} \int_{\mathbb{R}^3} [Q(F, F)]_i \psi_i(v) dv = \sum_{i=1}^{l} \sum_{j=1}^{l} \int_{\mathbb{R}^3} Q_{ij}(f_i, f_j)(v) \psi_i(v) dv
\]
\[
= \int_{\mathbb{R}^3} \left( \sum_{i=1}^{l} \sum_{j=1}^{l} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} f_i(v) f_j(v_*) B_{ij}(|u|, \hat{u} \cdot \sigma) d\sigma dv_* dv \right) \psi_i(v).
\]
Recalling that \(B_{ii} = B_{ij}\) and the micro reversibility condition \((2.7)\), we can interchange variables \((v, v_*, \sigma) \rightarrow (v_*, v, -\sigma)\) in the term associated to \(Q_{ij}(f_j, f_i)\) (it holds that \(v_{ji} = v'_{ji}\)) to obtain that
\[
2 \sum_{i=1}^{l} \int_{\mathbb{R}^3} [Q(F, F)]_i \psi_i(v) dv = \sum_{i=1}^{l} \sum_{j=1}^{l} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} f_i(v) f_j(v_*)
\]
\[
\times (\psi_i(v') + \psi_j(v') - \psi_i(v) - \psi_j(v_*)) B_{ij}(|u|, \hat{u} \cdot \sigma) d\sigma dv_* dv.
\]
The weak form of Boltzmann collision operator imply the conservation properties of the system. In particular,
\[
\int_{\mathbb{R}^3} Q_{ij}(f_i, f_j)(v) dv = 0, \quad 1 \leq i, j \leq I,
\]
which leads to conservation of mass for each single component and for the system as a whole
\[
\int_{\mathbb{R}^3} [Q(F, F)]_i dv = 0 \quad \text{and} \quad \int_{\mathbb{R}^3} [Q(F, F)] dv = 0.
\]
Moreover, when we choose the test functions \(\psi_i = m_i v\) and \(\psi_i = m_i |v|^2\) in the weak form \((2.10)\) we obtain the conservation of momentum and energy for the system as a whole (but not for each single species)
\[
\sum_{i=1}^{l} \int_{\mathbb{R}^3} [Q(F, F)]_i m_i v dv = 0 \quad \text{and} \quad \sum_{i=1}^{l} \int_{\mathbb{R}^3} [Q(F, F)]_i m_i |v|^2 dv = 0.
\]
More precisely, if \(F\) is a solution of Boltzmann gas mixture system then
\[
\int_{\mathbb{R}^3} f_i(v) dv = \int_{\mathbb{R}^3} f_{0,i}(v) dv, \quad 1 \leq i \leq I, \quad \text{conservation of mass per species},
\]
and
\[
\sum_{i=1}^{l} \int_{\mathbb{R}^3} m_i v f_i(v) dv = \sum_{i=1}^{l} \int_{\mathbb{R}^3} m_i v f_{0,i}(v) dv, \quad \text{conservation of total momentum},
\]
and
\[ \sum_{i=1}^{I} \int_{\mathbb{R}^3} m_i |v|^2 f_i(v) dv = \sum_{i=1}^{I} \int_{\mathbb{R}^3} m_i |v|^2 f_{0,i}(v) dv, \quad \text{conservation of total energy}. \]

This latter yields
\[ \int_{\mathbb{R}^3} m_i |v|^2 f_i(v) dv \leq \sum_{i=1}^{I} \int_{\mathbb{R}^3} m_i |v|^2 f_{0,i}(v) dv, \quad 1 \leq i \leq I. \]

Also, we recall the gas mixture entropy
\[ \mathcal{H}[F] := \sum_{i=1}^{I} \int_{\mathbb{R}^3} f_i(v) \log(f_i)(v) dv, \]
and the gas mixture entropy production, given by
\[ D[F] := \sum_{i=1}^{I} \int_{\mathbb{R}^3} [Q(F,F)]_i(v) \log(f_i)(v) dv. \]

It is possible to deduce from the weak form (2.10) that \( D[F] \leq 0 \), refer to [13], which yields that
\[ \mathcal{H}[F](t) \leq \mathcal{H}[F](0) = H_0, \quad \forall t > 0. \]

Consequently,
\[ \|F\|_{L \log L} = \sum_{i=1}^{I} \int_{\mathbb{R}^3} f_i(v) \log(1 + f_i(v)) dv = \sum_{i=1}^{I} \int_{\mathbb{R}^3} f_i(v) \log(1 + f_i(v)) dv + \sum_{i=1}^{I} \int_{\mathbb{R}^3} f_i(v) \log(1 + f_i(v)) dv \]
\[ \leq \log(2) \sum_{i=1}^{I} \int_{\mathbb{R}^3} f_i(v) dv + \sum_{i=1}^{I} \int_{\mathbb{R}^3} f_i(v) \log(f_i(v)) dv - \sum_{i=1}^{I} \int_{\mathbb{R}^3} f_i(v) \log(f_i(v)) dv. \]

Note that
\[ \frac{1}{\rho} \int f_i(v) \log(f_i(v)) dv \leq \frac{3}{4} \sum_{i=1}^{I} \int_{\mathbb{R}^3} |f_i(v)|^2 dv \leq C \left( \sum_{i=1}^{I} \int_{\mathbb{R}^3} f_i(v)^2 dv \right)^{\frac{1}{2}} \]
for \( C = \frac{3}{2} \left( \sum_{i=1}^{I} \langle v \rangle_i^{-4} \right)^{\frac{1}{2}} \). Therefore, by conservation of total mass, energy, and dissipation of entropy, it holds that
\[ \|F\|_{L \log L} \leq \log(2) \sum_{i=1}^{I} \int_{\mathbb{R}^3} f_{0,i}(v) dv \]
\[ \quad + \sum_{i=1}^{I} \int_{\mathbb{R}^3} f_{0,i}(v) \log(f_{0,i}(v)) dv + C \left( \sum_{i=1}^{I} \int_{\mathbb{R}^3} f_{0,i}(v)^2 dv \right)^{\frac{1}{2}}. \]

Thus, we consider initial data satisfying that
\[ m_0 = \sum_{i=1}^{I} \int_{\mathbb{R}^3} f_{0,i}(v) dv < \infty, \quad m_1 = \sum_{i=1}^{I} \int_{\mathbb{R}^3} m_i v f_{0,i}(v) dv = 0, \]
\[ m_2 = \sum_{i=1}^{I} \int_{\mathbb{R}^3} f_{0,i}(v)^2 dv < \infty, \quad H_0 = \sum_{i=1}^{I} \int_{\mathbb{R}^3} f_{0,i}(v) \log(f_{0,i})(v) dv < \infty, \]
so that $\|F\|_{L^\infty L^1} \leq \log(2)\mathbf{m}_0 + \mathcal{H}_0 + C m^2 \lambda < \infty$ as argued in (2.12). In this way, solutions to the gas mixture will formally lie in the space

$$U(D_0, E_0) = \{ G \in (L_2^2 \cap L^\infty L^1), \ G \geq 0, \| g_i \|_{L^1} \geq D_0, \| G \|_{L^1} + \| G \|_{L^\infty L^1} \leq E_0 \},$$

where $D_0$, $E_0 > 0$. In particular, we take initial data $F_0 \in U(D_0, E_0)$.

2.2. Main Results. Recall that we address a priori estimates related to statistical moments and higher integrability for a system of spatially homogeneous Boltzmann equations describing a gas mixture of monatomic components $A_1, ..., A_I$, where the component $A_i$ has associated particle mass $m_i > 0$.

$$(2.14)\begin{cases}
\partial_t F(t,v) = Q(F,F)(t,v), & t > 0, \ v \in \mathbb{R}^3, \\
F(0,v) = F_0(v).
\end{cases}$$

The $i^{th}$ entry of the vector $F := F(t,v) = [f_i(t,v)]_{1 \leq i \leq I}$ is the velocity distribution function $f_i := f_i(t,v) \geq 0$ associated to each component $A_i$ of the mixture. Recall that

$$Q(F,F) = \left[ \sum_{j=1}^I Q_{ij}(f_i,f_j) \right]_{1 \leq i \leq I}$$

was defined in (2.2). Also recall in the following that $\lambda_{ij} \in (0, 2]$ and $s_{ij} \in (0, 2)$ with $1 \leq i, j \leq I$. Keep in mind that the following are a priori estimates, so we consider solutions $F(t)$ to (4.12) with sufficient regularity so that all computations performed are allowed. Finally, in the results it is explicit that constants depend on the parameters of the model, that is, on the $\lambda_{ij}$, $s_{ij}$, $\kappa^1_{ij}$, $\kappa^2_{ij}$, $m_i$, and $\|\theta^2 h_i\|_{L^1(\mathbb{R}^3)}$. Such constant will depend on the mass, energy and entropy $D_0$ and $E_0$ of the gas mixture initial configuration as well.

**Proposition 2.2.** Take $\lambda^3 > 0$ and $F_0 \in U(D_0, E_0)$. For any integer $n \geq 2$, the statistical moments satisfy the equation

$$\mathbf{m}'_n[F(t)] = \sum_{i=1}^I \sum_{j=1}^I \partial_t \mathbf{m}_n^{ij}(t), \quad t > 0,$$

where, for each $1 \leq i, j \leq I$,

$$\partial_t \mathbf{m}_n^{ij}(t) := \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^\pi \left( \langle v' \rangle^2_i + \langle v' \rangle^2_j - \langle v \rangle^2_i - \langle v \rangle^2_j \right) d\varphi \beta_{ij}(\theta) d\theta \times |v - v^*|^{\lambda_{ij}} f_i(v) f_j(v^*) dv_* dv.$$

Furthermore, the following estimates hold for all $t > 0$.

(i) For any fixed $1 \leq i, j \leq I$ there exist positive constants $C_{ij}^1$ and $C_{ij}^2$ depending on the parameters of the model, $D_0$ and $E_0$, such that the pair $(f_i(t), f_j(t))$ satisfies

$$\partial_t \mathbf{m}_n^{ij}(t) \leq -C_{ij}^1 n^{s_{ij}/2} (m_{2n+\lambda_{ij},i}(t) + m_{2n+\lambda_{ij},j}(t)) + C_{ij}^2 S_n^{ij}(t),$$

where

$$S_n^{ij}(t) := \sum_{a=1}^{[n/2]} \int_{\mathbb{R}^3} \left[ n^{s_{ij}/2} \right] (m_{2a,i}(t)m_{2(n-a)+\lambda_{ij},i}(t) + m_{2a,i}(t)m_{2(n-a)+\lambda_{ij},j}).$$

(ii) In addition, the system $F(t)$ satisfies the inequality

$$\mathbf{m}'_n[F(t)] \leq -C_1 n^{2} m_{2n+\lambda^3}[F(t)] + \left( C_2 2^+ n^{\frac{\theta}{\theta - \frac{n^\theta}{n^\theta - 1}}} \right)^{\frac{1}{\theta}}, \quad \theta = \frac{2n + \lambda - 4}{2n + \lambda^3 - 2} < 1.$$

The positive constants $C_1$ and $C_2$ have the aforementioned dependence.

Based on Proposition 2.2 item (ii) one can prove the following results about higher algebraic/exponential tail integrability.
Theorem 2.3 (Polynomial Moments). Take $\lambda^3 > 0$ and $F_0 \in U(D_0, E_0)$. Then, for any $r > 2$ there exist positive constants $C_1$ and $C_2$ depending on $r$, the parameters of the model, $D_0$ and $E_0$, such that
\[
m_t[F](t) \leq C_1 \left(1 + t^{-\frac{-r}{8r}}\right), \quad t > 0.
\]
Furthermore, in the particular case of even integers $r = 2n \geq 4$ if $m_{2n}(0) < \infty$ then
\[
\sup_{t \geq 0} m_{2n}[F](t) \leq \max \{m_{2n}(0), C_2\}.
\]

Using the two aforementioned results, we deduce the following theorem.

Theorem 2.4 (Exponential Moments). Let $\lambda^3 > 0$, $s^3 > 0$, and $F_0 \in U(D_0, E_0)$.
(i) Let $\rho = \min \left\{\frac{\lambda^3}{s^3}, 2\right\}$. There exist constants $T > 0$ and $\sigma > 0$ depending only on the parameters of the model, $D_0$ and $E_0$, such that
\[
\sup_{[0,T]} \int_{\mathbb{R}^3} \exp \left[\frac{\sigma \rho^i \lambda^i(v)^2}{2}\right] f_i(v) dv \leq 4 m_0 \quad \text{(Generation of Moments)}.
\]
(ii) For any $A > 0$, $\sigma_0 > 0$ and $\rho \in (0, 2]$, there exists a (computable) constant $\sigma > 0$ depending on the parameters of the model, $D_0$, $E_0$, $\rho$, $\sigma_0$, and $A$, such that if
\[
\sum_{i=1}^{l} \int_{\mathbb{R}^3} \exp \left[\sigma_0 (v)^p\right] f_i(0, v) dv \leq A,
\]
then
\[
\sup_{t \geq 0} \sum_{i=1}^{l} \int_{\mathbb{R}^3} \exp[\sigma (v)^p] f_i(t, v) dv \leq 6 I e (m_0 + 1) \quad \text{(Propagation of Moments)}.
\]

Once statistical moments are understood for the system, it is possible to obtain higher integrability generation and propagation as stated in the following results. We remark that the results now refer to individualised components.

Theorem 2.5 (Higher integrability). Fix $1 \leq i \leq l$, and let $\lambda^3 > 0$ and $\overline{s}^3 > 0$. Assume also that $F_0 \in U(D_0, E_0)$. Then, it holds for any $p \in (1, \infty)$ and $t_0 > 0$ that
\[
\|f_i(t)\|_{L^\infty} \leq C_{t_0}, \quad t \geq t_0.
\]
The constant $C_{t_0} \lesssim t_0^{-\beta} + 1$ (for some $\beta > 0$) depends only on $t_0$, the parameters of the model, $D_0$ and $E_0$. Furthermore, if $f_{i,0} \in L^1_{2n} \cap L^p$ with sufficiently large number of moments $n \geq 2$, then
\[
\sup_{t \geq 0} \|f_i(t)\|_{L^p} \leq \max \left\\{\|f_{i,0}\|_{L^p}, C\right\}.
\]
The positive constant $C$ depends, in addition, on the $L^1_{2n}$ norm of $f_{i,0}$. The number of moments can be taken as $2n \geq r_s := \frac{\overline{s}^3}{\lambda^3}$.

In light of Theorem 2.5, one has the following result as a consequence.

Corollary 2.6 (Baseline regularity). Fix $1 \leq i \leq l$. Assume $\overline{s}_i > 0$ and $F_0 \in U(D_0, E_0)$. Then,
\[
\int_{t_0}^{t} \|v_i\|_{H^\infty} f_i(\tau) d\tau \leq C_{t_0} (1 + t), \quad t \geq t_0 > 0,
\]
where the constant $C_{t_0} \lesssim t_0^{-\beta} + 1$ (for some $\beta > 0$) depends only on $t_0$, the parameters of the model, $D_0$ and $E_0$.

Furthermore, if $f_{i,0} \in L^1_{2n} \cap L^2$ with sufficiently large number of moments $2n \geq \frac{s_i}{\lambda^3}$, then
\[
\int_{t_0}^{t} \|v_i\|_{H^\infty} f_i(\tau) d\tau \leq C (\|f_{i,0}\|_{L^2} + t), \quad t \geq 0,
\]
where \( C > 0 \) depends, in addition, on the \( L_{2n}^1 \) norm of \( f_{i,0} \).

The case \( p = +\infty \) is treated in our last result.

**Theorem 2.7** (Boundedness). Fix \( 1 \leq i \leq I \), and let \( \lambda^2 > 0 \) and \( c_i > 0 \). Assume also that \( F_0 \in U(D_0, E_0) \). Then, given \( t_0 > 0 \) there exists a constant \( C_{i,0} > 0 \) depending on the parameters of the model, \( D_0, E_0 \), and \( t_0 \), such that

\[
\| f_i(t) \|_{L^\infty} \leq C_{i,0}, \quad t \geq t_0.
\]

The constant is controlled as \( C_{i,0} \lesssim (t_0^{-\beta} + 1) \) (for some \( \beta > 0 \)). Furthermore, if additionally \( f_{i,0} \in L_{2n}^1 \cap L^\infty \) with sufficiently large number of moments \( 2n \geq \frac{18}{\beta^4} \), then

\[
\sup_{t \geq 0} \| f_i(t) \|_{L^\infty} \leq \max \left\{ 2 \| f_{i,0} \|_{L^\infty}, C \right\},
\]

for a positive constant \( C \) depending additionally on the \( L_{2n}^1 \) norm of \( f_{i,0} \).

Let us remark that the fact that we can infer properties of higher integrability/regularity emergence and propagation for individualised components in Theorem 2.7 Corollary 2.6 and Theorem 2.7 is a consequence of the forward scattering hypothesis in the \( b^β \). As previously mentioned, including the backscattering in the interactions adds lower order terms that can be managed at a technical price. We refer to [11] for a detailed analysis on the handling of “cutoff” terms.

3. Generation and propagation of statistical moments

In this section, the proof are inspired from classical approaches, in particular we extend the Povzner lemma given in [15] to this setting of system of equations.

3.1. Povzner Lemma for long range interactions.

**Lemma 3.1.** Let the cross section \( B_{ij} \) satisfied (2.8) and (2.9). Let \( v' \) and \( v_\ast' \) the collision laws defined in (2.4) with \( m_i, m_j > 0 \). Then, for all integers \( n \geq 2 \) the following estimate holds

\[
\int_0^\pi \int_0^\pi \left( \langle v' \rangle_i^{2n} + \langle v_\ast' \rangle_i^{2n} - \langle v \rangle_i^{2n} - \langle v_\ast \rangle_i^{2n} \right) d\varphi_\ast \beta_\ast(\theta) d\theta \leq -\lambda_{ij}^1 n^{s_{ij}/2} \left( \langle v \rangle_i^{2n} + \langle v_\ast \rangle_i^{2n} \right)
\]

\[
+ \lambda_{ij}^2 \sum_{a=1}^{n-1} \binom{n}{a} \left( \frac{n^{s_{ij}/2}}{(n-a)^{s_{ij}/2+1}} + \frac{1}{a} \right) \times \left( \langle v \rangle_i^{2n} \langle v_\ast \rangle_j^{2(n-a)} + \langle v \rangle_i^{2n} \langle v_\ast \rangle_j^{2(n-a)} \right),
\]

where

\[
\lambda_{ij}^1 = \frac{\kappa_{ij}}{2 - s_{ij}} \frac{m_i m_j (m_i^2 + m_j^2 + m_i m_j)}{(m_i + m_j)^2}, \quad \lambda_{ij}^2 = \zeta \frac{(m_i + m_j)^2}{m_i m_j}.
\]

**Proof.** In this proof we need to parametrize the collision laws (2.4), see for example [15] or [16]. For every \( X \in \mathbb{R}^3 \setminus \{0\} \), we introduce \( I(X), J(X) \in \mathbb{R}^3 \) such that \( \left( \frac{X}{|X|}, \frac{I(X)}{|X|}, \frac{J(X)}{|X|} \right) \) is an orthonormal basis of \( \mathbb{R}^3 \). We also put \( I(0) = J(0) = 0 \). For \( X, v, v_\ast \in \mathbb{R}^3 \), \( \theta \in [0, 2\pi) \) and \( \varphi \in [0, 2\pi) \), we set

\[
\Gamma(X, \varphi) = \cos \varphi I(X) + \sin \varphi J(X),
\]

\[
\sigma = \cos \theta \frac{v - v_\ast}{|v - v_\ast|} + \sin \theta \frac{(v - v_\ast) \times v}{|v - v_\ast|},
\]

\[
v' = v - \frac{m_j}{m_i + m_j} (1 - \cos \theta) (v - v_\ast) + \frac{m_i}{m_i + m_j} \sin \theta \Gamma(v - v_\ast, \varphi),
\]

\[
v_\ast' = v_\ast + \frac{m_i}{m_i + m_j} (1 - \cos \theta) (v - v_\ast) + \frac{m_j}{m_i + m_j} \sin \theta \Gamma(v - v_\ast, \varphi),
\]

\[
\Gamma(v, \varphi) = \cos \varphi I(v) + \sin \varphi J(v),
\]

\[
\sigma = \cos \theta \frac{v - v_\ast}{|v - v_\ast|} + \sin \theta \frac{(v - v_\ast) \times v}{|v - v_\ast|},
\]

\[
v' = v - \frac{m_j}{m_i + m_j} (1 - \cos \theta) (v - v_\ast) + \frac{m_i}{m_i + m_j} \sin \theta \Gamma(v - v_\ast, \varphi),
\]

\[
v_\ast' = v_\ast + \frac{m_i}{m_i + m_j} (1 - \cos \theta) (v - v_\ast) + \frac{m_j}{m_i + m_j} \sin \theta \Gamma(v - v_\ast, \varphi),
\]
where \(|\Gamma(v - v_*, \varphi)| = |v - v_*|\) and \((v - v_*) \cdot \Gamma(v - v_*, \varphi) = 0\). Then, we find that

\[
|v'|^2 = |v|^2 + \left(\frac{m_j}{m_i + m_j}\right)^2 (1 - \cos \theta)^2 |v - v_*|^2 + \left(\frac{m_j}{m_i + m_j}\right)^2 \sin^2 \theta |v - v_*|^2
- 2 \frac{m_j}{m_i + m_j} (1 - \cos \theta) v \cdot (v - v_*) + 2 \frac{m_j}{m_i + m_j} \sin \theta \cdot \Gamma(v - v_*, \varphi)
\]

\[
= |v|^2 \left[1 + 2 \left(\frac{m_j}{m_i + m_j}\right)^2 (1 - \cos \theta)^2 - 2 \frac{m_j}{m_i + m_j} (1 - \cos \theta)\right] + |v_*|^2 \left[2 \left(\frac{m_j}{m_i + m_j}\right)^2 (1 - \cos \theta)\right]
+ v \cdot v_* \left[- 4 \left(\frac{m_j}{m_i + m_j}\right)^2 (1 - \cos \theta) + 2 \frac{m_j}{m_i + m_j} (1 - \cos \theta)\right] + 2 \frac{m_j}{m_i + m_j} \sin \theta v \cdot \Gamma(v - v_*, \varphi)
\]

\[
= \left[\frac{m_i^2 + m_j^2 + 2m_jm_i \cos \theta}{(m_i + m_j)^2}\right] |v|^2 + \left[2 \left(\frac{m_j}{m_i + m_j}\right)^2 (1 - \cos \theta)\right] |v_*|^2
+ \left[\frac{2m_i^2m_j - 2m_jm_i}{(m_i + m_j)^2 (1 - \cos \theta)}\right] v \cdot v_* + 2 \frac{m_j}{m_i + m_j} \sin \theta v \cdot \Gamma(v - v_*, \varphi).
\]

Multiplying the above equality by \(\sum_{m_i}^m\) we get that

\[
\sum_{m_i}^m |v'|^2 = \left[\frac{m_i^2 + m_j^2 + 2m_jm_i \cos \theta}{(m_i + m_j)^2}\right] \sum_{m_i}^m |v|^2 + \sum_{m_i}^m \left[2 \left(\frac{m_j}{m_i + m_j}\right)^2 (1 - \cos \theta)\right] |v_*|^2
+ \sum_{m_i}^m \left[\frac{2m_i^2m_j - 2m_jm_i}{(m_i + m_j)^2 (1 - \cos \theta)}\right] v \cdot v_* + 2 \sum_{m_i}^m \frac{m_j}{m_i + m_j} \sin \theta v \cdot \Gamma(v - v_*, \varphi),
\]

which yields,

\[
\langle v' \rangle_i^2 = \left[\frac{m_i^2 + m_j^2 + 2m_jm_i \cos \theta}{(m_i + m_j)^2}\right] \langle v \rangle_i^2 + \left[2 \left(\frac{m_j}{m_i + m_j}\right)^2 (1 - \cos \theta)\right] \langle v_* \rangle_i^2
+ \sum_{m_i}^m \left[\frac{2m_i^2m_j - 2m_jm_i}{(m_i + m_j)^2 (1 - \cos \theta)}\right] v \cdot v_* + 2 \sum_{m_i}^m \frac{m_j}{m_i + m_j} \sin \theta v \cdot \Gamma(v - v_*, \varphi)
+ 1 \frac{m_i^2 + m_j^2 + 2m_jm_i \cos \theta}{(m_i + m_j)^2}
= 0
\]

\[
= \left[\frac{m_i^2 + m_j^2 + 2m_jm_i \cos \theta}{(m_i + m_j)^2}\right] \langle v \rangle_i^2 + \left[2 \left(\frac{m_j}{m_i + m_j}\right)^2 (1 - \cos \theta)\right] \langle v_* \rangle_i^2
+ \sum_{m_i}^m \left[\frac{2m_i^2m_j - 2m_jm_i}{(m_i + m_j)^2 (1 - \cos \theta)}\right] v \cdot v_* + 2 \sum_{m_i}^m \frac{m_j}{m_i + m_j} \sin \theta v \cdot \Gamma(v - v_*, \varphi).
\]

Fixing \(n \geq 2\) and letting \(B_n = \{(p, q, k, l) \in \mathbb{N}^4 : p + q + k + l = n\}\), we obtain using Newton’s expansion that

\[
\langle v' \rangle_i^n = \sum_{p,q,k,l} \frac{n!}{p!q!l!k!} \left[\frac{m_i^2 + m_j^2 + 2m_jm_i \cos \theta}{(m_i + m_j)^2}\right]^p
\times \left[2 \left(\frac{m_j}{m_i + m_j}\right)^2 (1 - \cos \theta)\right]^q \left[\frac{2m_i^2m_j - 2m_jm_i}{(m_i + m_j)^2 (1 - \cos \theta)}\right]^k
\times \sum_{m_i}^m \frac{m_j}{m_i + m_j} \sin \theta v \cdot v_*^2 \langle v_* \rangle_i^2 v \cdot v_*^k \langle v \cdot v_* \rangle_i^l \langle v' \rangle_i^m \langle v_* \rangle_i^n \langle v \rangle_i^p \langle v \rangle_i^q \langle v \rangle_i^k \langle v \rangle_i^l.
\]
Also, we have that
\[
\langle v'_{ij} \rangle^{2n} = \left[ \frac{m_i^2 + m_j^2 + 2m_im_j \cos \theta}{(m_i + m_j)^2} \right] \langle v_{ij} \rangle^2 + \left[ \frac{2m_im_j}{(m_i + m_j)^2} (1 - \cos \theta) \right] \langle v \rangle^2
\]

which implies that
\[
\langle v'_{ij} \rangle^{2n} = \sum_{p,q,k,l} \frac{n!}{p!q!k!l!} \left[ \frac{m_i^2 + m_j^2 + 2m_im_j \cos \theta}{(m_i + m_j)^2} \right]^p \left[ \frac{2m_im_j}{(m_i + m_j)^2} (1 - \cos \theta) \right]^q \left[ \frac{m_i^2 - 2m(im_j - m_j^2)}{(m_i + m_j)^2} \right] \sin \theta \cdot \Gamma(\theta)
\]

Then, combining (3.1) and (3.2) we get the following equality
\[
\frac{1}{2\pi} \int_0^{2\pi} \left( \langle v' \rangle_{ij}^{2n} + \langle v' \rangle_j^{2n} - \langle v \rangle_{ij}^{2n} \right) d\varphi \beta_{ij}(\theta) d\theta = -a_n + b_n
\]
where
\[
a_n := \int_0^{\pi} \left( 1 - \left[ \frac{m_i^2 + m_j^2 + 2m_im_j \cos \theta}{(m_i + m_j)^2} \right] \right) \left( \langle v \rangle_{ij}^{2n} + \langle v \rangle_j^{2n} \right) \beta_{ij}(\theta) d\theta,
\]
and
\[
b_n := \int_0^{\pi} \left( \Theta(v, v, \theta) + \Theta(v, v, -\theta) \right) d\varphi \beta_{ij}(\theta) d\theta,
\]
with
\[
\int_0^{2\pi} \Theta(v, v, \theta) d\varphi = \frac{1}{2\pi} \sum_{(p,q,k,l) \in A_n} \frac{n!}{p!q!k!l!} \left[ \frac{m_i^2 + m_j^2 + 2m_im_j \cos \theta}{(m_i + m_j)^2} \right]^p \left[ \frac{2m_im_j}{(m_i + m_j)^2} (1 - \cos \theta) \right]^q \left[ \frac{m_i^2 - 2m(im_j - m_j^2)}{(m_i + m_j)^2} \right] \sin \theta \cdot \Gamma(\theta)
\]

such that \( A_n = \{(p, q, k, l) \in \mathbb{N}^4; p + q + k + l = n/\{(0, 0, 0, 0), (0, n, 0, 0)\} \} \).

We proceed in two steps.

**Step 1.** We show that there exists a constant \( \lambda_{ij}^1 > 0 \) such that
\[
a_n \geq \lambda_{ij}^1 n^{2n} \left( \langle v \rangle_{ij}^{2n} + \langle v \rangle_j^{2n} \right).
\]

We remark that the integral in \( a_n \) is nonnegative because for \( X = \frac{m_i^2 + m_j^2 + 2m_im_j \cos \theta}{(m_i + m_j)^2} \in [0, 1] \) we have that
\[
X^n + (1 - X)^n \leq 1.
\]

Then, using (2.9) we get that
\[
a_n \geq \kappa_{ij}^1 \int_0^{\pi} \left( 1 - \left[ \frac{m_i^2 + m_j^2 + 2m_im_j \cos \theta}{(m_i + m_j)^2} \right] \right) \left[ \frac{2m_im_j}{(m_i + m_j)^2} (1 - \cos \theta) \right]^n \theta^{-s_{ij} - 1} d\theta \left( \langle v \rangle_{ij}^{2n} + \langle v \rangle_j^{2n} \right).
\]
Furthermore, for all \( \theta \in [0, n^{-\frac{1}{2}}] \) we have that
\[
\cos \theta \leq 1 - \frac{\theta^2}{2},
\]
which yields
\[
\frac{m_i^2 + m_j^2 + 2m_im_j \cos \theta}{(m_i + m_j)^2} \leq 1 - \frac{m_im_j}{(m_i + m_j)^2} \theta^2,
\]
whence
\[
\left[ \frac{m_i^2 + m_j^2 + 2m_im_j \cos \theta}{(m_i + m_j)^2} \right]^n \leq \left[ 1 - \frac{m_im_j}{(m_i + m_j)^2} \theta^2 \right]^n \leq e^{-n \frac{m_im_j}{(m_i + m_j)^2} \theta^2}.
\]

Then
\[
1 - \left[ \frac{m_i^2 + m_j^2 + 2m_im_j \cos \theta}{(m_i + m_j)^2} \right]^n \geq n \frac{m_im_j}{(m_i + m_j)^2} \theta^2.
\]
Next, still for \( \theta \in [0, n^{-\frac{1}{2}}] \) we have that
\[
1 - \cos \theta \leq \frac{\theta^2}{2},
\]
which implies that
\[
\left[ \frac{2m_im_j}{(m_i + m_j)^2} (1 - \cos \theta) \right]^n \leq \left[ \frac{m_im_j}{(m_i + m_j)^2} \theta^2 \right]^n,
\]
whence, recalling that \( n \geq 2, \)
\[
1 - \left[ \frac{m_i^2 + m_j^2 + 2m_im_j \cos \theta}{(m_i + m_j)^2} \right]^n - \left[ \frac{2m_im_j}{(m_i + m_j)^2} (1 - \cos \theta) \right]^n
\geq n \frac{m_im_j}{(m_i + m_j)^2} \theta^2 - \left[ \frac{m_im_j}{(m_i + m_j)^2} \theta^2 \right]^n
\geq \frac{m_im_j}{(m_i + m_j)^2} \left[ n - \frac{nm_im_j}{2(m_i + m_j)^2} \right] \theta^2 \geq \frac{m_im_j(m_i^2 + m_j^2 + m_im_j)}{2(m_i + m_j)^4} n \theta^2 \geq c n \theta^2.
\]
Consequently,
\[
a_n \geq \kappa_{ij}^1 c n \int_0^{n^{-\frac{1}{2}}} \theta^{1-s_{ij}} d\theta \left( (v_j^i)_{2n} + (v_j^i)_{2n} \right) \geq \lambda_{ij}^1 n \frac{n}{\sin^{2\theta/2}} \left( (v_j^i)_{2n} + (v_j^i)_{2n} \right)
\]
with \( \lambda_{ij}^1 = \frac{\kappa_{ij}^1 m_im_j(m_i^2 + m_j^2 + m_im_j)}{(m_i + m_j)^4}. \)

**Step 2.** We estimate \( b_n. \) We start by recalling that
\[
\int_0^{2\pi} \Theta(v, v_*, \theta) d\varphi = \frac{1}{2\pi} \sum_{(p, q, k, l) \in A_n} \frac{n!}{p!q!k!l!} \left[ \frac{m_i^2 + m_j^2 + 2m_im_j \cos \theta}{(m_i + m_j)^2} \right]^p \left[ \frac{2m_im_j}{(m_i + m_j)^2} (1 - \cos \theta) \right]^q
\times \left[ \frac{2m^2_j + 2m_i^2 + 2m_im_j \cos \theta}{(m_i + m_j)^2} (1 - \cos \theta) \right]^k \left[ \frac{2m_im_j}{(m_i + m_j)^2} \sin \theta \right]^l
\times (v_j^i)_{2p}(v_j^i)_{2q}(v_j^i)_{2k} \int_0^{2\pi} [v \cdot \Gamma(v - v_*, \varphi)]^l d\varphi
\]
such that \( A_n = \{(p, q, k, l) \in \mathbb{N}^4; p + q + k + l = n\} \setminus \{(n, 0, 0, 0), (0, n, 0, 0)\}. \) For the last term we have that
\[
\frac{1}{2\pi} \int_0^{2\pi} [v \cdot \Gamma(v - v_*, \varphi)]^l d\varphi = 1_{[l \in 2\mathbb{N}]} \frac{l!}{2![(\frac{1}{2})]^l} \left( |v|^2 |v_*|^2 - (v \cdot v_*) \right)^\frac{l}{2}.
\]
Let \( C_n = \{(p, q, k) \in A_n; l \in 2\mathbb{N}\} \), we compute that

\[
\int_0^{2\pi} \Theta_1(v, u, \theta) d\varphi = \sum_{(p, q, k, l) \in C_n} \frac{n!}{p!q!k!l!} \left[ \frac{m_j^2 + m_j^2 + 2m_ij_j \cos \theta}{(m_i + m_j)^2} \right]^p \left[ \frac{m_i m_j}{(m_i + m_j)^2} (1 - \cos \theta) \right]^q \\
\times \left[ 2m_i m_j - 2m_i m_j \right] \left( \sum m_i (m_i + m_j)^2 \right) \left[ \frac{m_i m_j}{(m_i + m_j)^2} (1 - \cos \theta) \right]^k \left( \frac{2}{(m_i + m_j)^2} (v \cdot v_s)^2 \right)^{\frac{q}{2}} \\
\times \left[ \sum m_i (m_i + m_j) \sin \theta \right] \left[ \frac{1}{2} |v|^2 |v| - (v \cdot v_s)^2 \right]^{\frac{q}{2}} \\
\leq \sum_{(p, q, k, l) \in C_n} \frac{n!}{p!q!k!l!} \left[ \frac{m_i^2 + m_j^2 + 2m_i m_j \cos \theta}{(m_i + m_j)^2} \right]^p \left[ \frac{m_i m_j}{(m_i + m_j)^2} (1 - \cos \theta) \right]^q \\
\times \left( \frac{2}{(m_i + m_j)^2} (v \cdot v_s)^2 \right)^{\frac{q}{2}} \left[ \frac{m_i^2 + m_j^2}{(m_i + m_j)^2} (1 - \cos \theta) \right]^k \left( \frac{2}{(m_i + m_j)^2} (v \cdot v_s)^2 \right)^{\frac{q}{2}} \\
\times \left[ \frac{m_i^2 + m_j^2}{(m_i + m_j)^2} (1 - \cos \theta) \right]^k \left[ \frac{m_i^2 + m_j^2}{(m_i + m_j)^2} \sin \theta \right] \left( \frac{1}{2} |v|^2 |v| - (v \cdot v_s)^2 \right) \beta_{ij}(\theta) d\theta
\]

Then,

\[
\int_0^{2\pi} \int_0^{2\pi} \left( \Theta_1(v, u, \theta) + \Theta_2(v, u, -\theta) \right) d\varphi \beta_{ij}(\theta) d\theta \\
\leq \sum_{(p, q, k, l) \in C_n} \frac{n!}{p!q!k!l!} \left[ \frac{m_i^2 + m_j^2 + 2m_i m_j \cos \theta}{(m_i + m_j)^2} \right]^p \left[ \frac{m_i m_j}{(m_i + m_j)^2} (1 - \cos \theta) \right]^q \\
\times \left[ \frac{m_i^2 + m_j^2}{(m_i + m_j)^2} (1 - \cos \theta) \right]^k \left[ \frac{m_i^2 + m_j^2}{(m_i + m_j)^2} \sin \theta \right] \beta_{ij}(\theta) d\theta \\
\leq \sum_{(p, q, k, l) \in C_n} \frac{n!}{p!q!k!l!} I_{p, q, k, l} \left( \frac{2}{(m_i + m_j)^2} (v \cdot v_s)^2 \right)^{\frac{q}{2}} \left( \frac{2}{(m_i + m_j)^2} (v \cdot v_s)^2 \right)^{\frac{q}{2}}
\]

where

\[
I_{p, q, k, l} = \int_0^{2\pi} \left( \frac{m_i^2 + m_j^2 + 2m_i m_j \cos \theta}{(m_i + m_j)^2} \right)^p \left[ \frac{m_i m_j}{(m_i + m_j)^2} (1 - \cos \theta) \right]^q \\
\times \left[ \frac{m_i^2 + m_j^2}{(m_i + m_j)^2} (1 - \cos \theta) \right]^k \left[ \frac{m_i^2 + m_j^2}{(m_i + m_j)^2} \sin \theta \right] \beta_{ij}(\theta) d\theta
\]

This can be rewritten as

\[
\int_0^{2\pi} \int_0^{2\pi} \left( \Theta_1(v, u, \theta) + \Theta_2(v, u, -\theta) \right) d\varphi \beta_{ij}(\cos \theta) d\theta \leq \sum_{a=0}^{n} K_{n,a} \left( \frac{1}{2} \left( \frac{2}{(m_i + m_j)^2} (v \cdot v_s)^2 \right)^{\frac{q}{2}} \right)^{2a}
\]

where, setting \( A_{n,a} = \{(p, q, k, l) \in C_n; p + \frac{q}{2} + \frac{l}{2} = a \} \), whence \( q + \frac{q}{2} + \frac{l}{2} = n - a \), one has that

\[
K_{n,a} = \sum_{(p, q, k, l) \in A_{n,a}} \frac{n!}{p!q!k!l!} I_{p, q, k, l} \left( \frac{2}{(m_i + m_j)^2} (v \cdot v_s)^2 \right)^{\frac{q}{2}}
\]
Let us show that consequently, we deduce that

where

From the definition of $I_{a-r-s,n-a-r-s,2r,2s}$, we deduce that

we deduce that

we deduce that

From the definition of $J_{n,a}$ one has that

where

and

Let us show that $L_{s_{ij},n,a}$ is estimated in terms of $K_{0,n,n-a}$. Indeed, using the substitution $\theta \to \pi - \theta$ we see that

and

Consequently, we deduce that

Fix $a \in [1, n-1]$, then $(p,q,k,l) \in A_{n,a}$ if and only if there exist $r, s \in \mathbb{N}$ such that $k = 2r$, $l = 2s$, $p = a - r - s$, and $q = n - a - r - s$. Consequently,

Using the fact that

we deduce that

Thus,

From the definition of $J_{n,a}$ one has that

where

and

Finally, we deduce that
Now, we estimate $K_{s_{ij},n,a}$. For $\theta \in (0, \pi/2]$ we have that
\[
\theta \leq 2 \sin \theta \quad \text{and} \quad \theta^{-1} \leq \left[ \frac{2}{(m_i + m_j)^2} (1 - \cos \theta) \right]^{-1/2},
\]
so that
\[
\theta^{-s_{ij} - 1} \leq 2^{s_{ij} - 2} \sin \theta \leq 2 \left[ \frac{2}{(m_i + m_j)^2} (1 - \cos \theta) \right]^{-s_{ij}/2 - 1},
\]
and thus
\[
K_{s_{ij},n,a} \leq 2 \left[ \frac{2}{(m_i + m_j)^2} (1 - \cos \theta) \right]^{-s_{ij}/2 - 1} \sin \theta d\theta
\]
where we used the change of variables $x = \frac{m_i^2 + m_j^2 + 2m_i m_j \cos \theta}{(m_i + m_j)^2}$ in the latter. Hence
\[
K_{s_{ij},n,a} \leq \left[ \frac{2}{m_i m_j} \right] \frac{(m_i + m_j)^2}{(n-a-s_{ij}/2)n} u_{s_{ij},n,a}
\]
with $u_{s_{ij},k} = \frac{\Gamma(k+1)}{\Gamma(k+1-s_{ij}/2)}$. By Sirling’s Formula $\Gamma(x + 1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x$ as $x \to \infty$; one can verify that $u_{s_{ij},k} \sim \frac{k}{s_{ij}/2}$ as $k \to \infty$ so that there exists $A_{s_{ij}} \in (0, \infty)$ such that
\[
\left[ \frac{2}{m_i m_j} \right] \frac{(m_i + m_j)^2}{(n-a-s_{ij}/2)n} u_{s_{ij},n,a} \leq (2/\pi)^{s_{ij}/2} A_0 \left[ \frac{2}{m_i m_j} \right] \frac{(m_i + m_j)^2}{(n-a-s_{ij}/2)n},
\]
and
\[
\left[ \frac{2}{m_i m_j} \right] \frac{(m_i + m_j)^2}{(n-a-s_{ij}/2)n} u_{s_{ij},n,a} \leq (2/\pi)^{s_{ij}/2} A_0 \left[ \frac{2}{m_i m_j} \right] \frac{(m_i + m_j)^2}{(n-a-s_{ij}/2)n}.
\]
Then
\[
K_{n,a} \leq \zeta \left[ \frac{2}{m_i m_j} \right] \frac{(n-a)!}{(n-a-s_{ij}/2)!} \left[ \frac{2}{m_i m_j} \right] \left[ \frac{2}{m_i m_j} \right] A_0 \left[ \frac{2}{m_i m_j} \right] \frac{(m_i + m_j)^2}{(n-a-s_{ij}/2)n},
\]
where $\zeta \equiv \frac{(n-a)!}{(n-a-s_{ij}/2)!} \left[ \frac{2}{m_i m_j} \right] \left[ \frac{2}{m_i m_j} \right] A_0 \left[ \frac{2}{m_i m_j} \right] \frac{(m_i + m_j)^2}{(n-a-s_{ij}/2)n}$.

This concludes the proof. \hfill \Box

3.2. Generation and Propagation of polynomial moments. The purpose of Proposition 2.2 is to derive from the Boltzmann system (2.14) an ordinary differential inequality for polynomial moment of order $2n$, with $n \geq 2$, based on the Povzner estimate from Lemma 3.1 to implement statistical moment analysis. In particular, the key idea in regard of exponential tails generation and propagation is to realise that the mixture system’s control inequality is obtained by individualising the $ij$-scalar components satisfying the Povzner lemma, Lemma 3.1, specifically described in Proposition 2.2 item (i).

Proof of Proposition 2.2 Let $n \geq 2$ and $1 \leq i, j \leq I$, and recall that
\[
\partial_t \mathbf{m}_{2n}^{ij} = \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^{2\pi} \left( \langle v \rangle_i^{2n} + \langle v \rangle_i^{2n} - \langle v \rangle_i^{2n} - \langle v \rangle_i^{2n} \right) d\varphi \beta_{ij}(\theta) d\theta \times |v - v_\ast|^{\lambda_{ij}} f_i(v_\ast) f_j(v) dv_\ast dv.
\]
Consequently, a direct consequence of the weak formulation of the Boltzmann mixture equation (2.10) with $\psi_i = \langle v_i \rangle^{2n}$ is that

\[(3.4) \quad m_{2n}^i(t) = \sum_{i=1}^{I} \sum_{j=1}^{l} \partial_t m_{2n}^{ij}(t), \quad t > 0.\]

Using Povzner Lemma 3.1, we get that $\partial_t m_{2n}^{ij}(t) \leq -A_n^{ij}(t) + B_n^{ij}(t)$ where

\[A_n^{ij}(t) = \frac{1}{2} \lambda_n^{ij} n^{2n} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( \langle v_i \rangle^{2n} + \langle v_j \rangle^{2n} \right) |v - v_*|^{\lambda_n^{ij}} f_i(v) f_j(v_*) dv_* dv,
\]

and

\[B_n^{ij}(t) = \frac{1}{2} \lambda_n^{ij} n^{2n} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( \langle v_i \rangle^{2n} + \langle v_j \rangle^{2n} \right) |v - v_*|^{\lambda_n^{ij}} f_i(v) f_j(v_*) dv_* dv.
\]

Let us begin proving the first result by proceeding in two steps.

**Step 1.** Finding a lower bound for $A_n$. Since $F \in U(D_0, E_0)$, one can use a slight modification of the proof of [6, Lemma 9] to obtain an explicit constant $c := c(D_0, E_0) > 0$ such that

\[\int_\mathbb{R}^3 |v - z|^{\lambda} f_i(z) dz \geq c(v)^\lambda, \quad \lambda \in [0, 2], \quad 1 \leq i \leq I.
\]

Consequently,

\[A_n^{ij}(t) = 2\lambda_n^{ij} n^{2n} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( \langle v_i \rangle^{2n} + \langle v_j \rangle^{2n} \right) |v - v_*|^{\lambda_n^{ij}} f_i(v) f_j(v_*) dv_* dv 
\]

\[\geq 2\lambda_n^{ij} n^{2n} c \left( m_{2n+\lambda_n^{ij},i}(t) + m_{2n+\lambda_n^{ij},j}(t) \right).
\]

**Step 2.** Finding a convenient upper bound for $B_n^{ij}$. Using that, see for example [18],

\[|v - v_*|^{\lambda_n^{ij}} \leq (m_{m})^{\lambda_n^{ij}} \left( \langle v_i \rangle^{\lambda_n^{ij}} + \langle v_j \rangle^{\lambda_n^{ij}} \right),
\]

we get that

\[B_n^{ij}(t) \leq 2(m_{m})^{\lambda_n^{ij}} \lambda_n^{ij} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( \langle v_i \rangle^{2n} + \langle v_j \rangle^{2n} \right) |v - v_*|^{\lambda_n^{ij}} f_i(v) f_j(v_*) dv_* dv
\]

\[+ m_{2a,i}(t)m_{2(n-a)+\lambda_n^{ij},i}(t) + m_{2a,i}(t)m_{2(n-a)+\lambda_n^{ij},j}(t) + m_{2a,i}(t)m_{2(n-a)+\lambda_n^{ij},j}(t) \leq 4(m_{m})^{\lambda_n^{ij}} \lambda_n^{ij} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( m_{2a,i}(t)m_{2(n-a)+\lambda_n^{ij},j}(t) + m_{2a,i}(t)m_{2(n-a)+\lambda_n^{ij},j}(t) \right),
\]

where we performed the change of index $a \to n - a$ in the intermediate step, and the fact that $\frac{1}{a} \leq \frac{n^{2n}}{a^{2n+2}}$ for $a \leq n$. Furthermore, the terms in parenthesis are symmetric with respect to $a$, that is

\[m_{2a+\lambda_n^{ij},i}(t)m_{2(n-a),j}(t) \quad \text{for} \quad a = 1, 2, 3, \cdots \quad \text{equals}
\]

\[m_{2a,i}(t)m_{2(n-a)+\lambda_n^{ij},i}(t) \quad \text{for} \quad a = n - 1, n - 2, n - 3, \cdots.
\]
And similarly for the second group of moments. Note, in addition, that the first half of weights controls the second half because \( a \to \frac{a}{2} \) is decreasing. Hence,

\[
B_i^n(t) \leq 8(m_i m_j)^{-\frac{a}{2}} \lambda_i^2 \sum_{a=1}^{[n/2]} \binom{n}{a} \frac{n^{s/2}}{a^{s/2+1}} \left( m_{2a+\lambda_i,i}(t) m_{2(n-a),i}(t) + m_{2a,i}(t) m_{2(n-a)+\lambda_i,i}(t) \right).
\]

Now, since \( n - a \geq a \) for \( 1 \leq a \leq [n/2] \) we can invoke the Lemma 5.1 with \( b = 2(n-a) \) and \( \beta = \lambda_i \), which is a generalisation of the classical lemma [15, Lemma 7], to conclude that

\[
m_{2a+\lambda_i,i}(t) m_{2(n-a),i}(t) \leq \theta_i m_{2(n-a)+\lambda_i,i}(t) m_{2a,i}(t) + (1 - \theta_i) m_{2(n-a)+\lambda_i,i}(t) m_{2a,i}(t),
\]

resulting in the estimation

\[
B_i^n(t) \leq 16(m_i m_j)^{-\frac{a}{2}} \lambda_i^2 \sum_{a=1}^{[n/2]} \binom{n}{a} \frac{n^{s/2}}{a^{s/2+1}} \left( m_{2a,i}(t) m_{2(n-a)+\lambda_i,i}(t) + m_{2a,i}(t) m_{2(n-a)+\lambda_i,j}(t) \right).
\]

Consequently,

\[
\theta_i m_{2a,i}(t) m_{2(n-a)+\lambda_i,i}(t) + m_{2(n-a)+\lambda_i,j}(t)) + C_{ij}^2 S_i^n(t).
\]

where the constants can be taken as

\[
C_{ij}^1 = 2 \lambda_i^2 \quad \text{and} \quad C_{ij}^2 = 16(m_i m_j)^{-\frac{a}{2}} \lambda_i^2.
\]

This is precisely statement (i).

Now, statement (ii) follow from (i) after some observations. First, for the coercive term it readily follows that

\[
\sum_{i=1}^{l} \sum_{j=1}^{l} C_{ij}^1 n^{s/2} (m_{2n+\lambda_i,i}(t) + m_{2n+\lambda_i,j}(t)) \geq 2 \min_{i,j} \{ C_{ij}^1 n^{s/2} \} \sum_{i=1}^{l} \sum_{j=1}^{l} m_{2n+\lambda_i,i}(t) \geq 2 C_1 n^{\frac{s}{2}} \sum_{i=1}^{l} m_{2n+\lambda_i,i}(t).
\]

The upper estimate related to \( S_i^n \) follows after some moment interpolations. Note that (since \( n/a \geq 1 \))

\[
\sum_{i=1}^{l} \sum_{j=1}^{l} \frac{C_{ij}^2 S_i^n(t)}{2} \leq \max_{1 \leq i,j \leq l} C_{ij}^2 \sum_{a=1}^{[n/2]} \binom{n}{a} \frac{n^{s/2}}{a^{s/2+1}} m_{2a}(t) m_{2(n-a)+\lambda}(t).
\]

Furthermore, \( 2 < 2(n-a) + \overline{\lambda} < 2n \) (recalling that \( n \geq 2 \)), therefore, the following interpolations hold

\[
m_{2(n-a)+\overline{\lambda}}(t) \leq m_{2n+\lambda_{1}}(t), \quad \theta_1 = \frac{2(n-a) + \overline{\lambda} - 2}{2n + \lambda^2 - 2},
\]

\[
m_{2\lambda}(t) \leq m_{2n+\lambda_{2}}(t), \quad \theta_2 = \frac{2n + \lambda - 2a - 2}{2n + \lambda^2 - 2},
\]

which yields that (since \( \overline{\lambda} - \lambda^2 < 2 \))

\[
m_{2(n-a)+\overline{\lambda}}(t) m_{2(n-a)}(t) \leq m_{2n+\lambda_{3}}(t), \quad \theta_1 + \theta_2 = \frac{2n + \overline{\lambda} - 4}{2n + \lambda^2 - 2} < 1.
\]
Consequently,
\[
\sum_{i=1}^{I} \sum_{j=1}^{L} C_{ij}^2 \sum_{n} S_n^i(t) \leq C(m_2) 2^{\frac{\theta}{2}} n^\frac{n}{2} m_\theta^{2n+\lambda\theta}(t), \quad \theta = \frac{2n + \lambda - 4}{2n + \lambda^2 - 2} < 1,
\]
where in the latter we used Young’s inequality. Consequently,
\[
\sum_{i=1}^{I} \sum_{j=1}^{L} \partial_t m_{ij}^{2n}(t) \leq -C_1 n^\frac{n}{2} m_{2n+\lambda\theta}(t) + \left( C 2^{\frac{\theta}{2}} n^\frac{n}{2} \right)^{\frac{1}{1+\theta}},
\]
which proves item (ii) of the proposition.

\[\square\]

Proof of Theorem 2.3. Use the interpolation
\[
m_{2n}(t) \leq m_{2n}^{1-\theta}(t), \quad \theta = \frac{2n - 2}{2n + \lambda^2 - 2} < 1,
\]
in Proposition 2.2 item (ii) to conclude that
\[
m_{2n}'(t) \leq -C_1 n^\frac{n}{2} m_{2n}^{1-\theta} + \left( C 2^{\frac{\theta}{2}} n^\frac{n}{2} \right)^{\frac{1}{1+\theta}}.
\]
Thus, a comparison principle for order, see [6, Lemma 18], gives that the super solution \( m_{2n}'(t) \) controls the \( 2n \)-th-moment
\[
m_{2n}(t) \leq m_{2n}(t) := \left( \frac{B_n}{A_n} \right)^{\frac{1}{1+\theta}} + \left( \frac{1}{c_n A_n} \right)^{\frac{1}{1+\theta}} t^{-\frac{2n-2}{\lambda^2}}.
\]
Similarly, if \( m_{2n}(0) < \infty \) one deduces the propagation bound
\[
m_{2n}(t) \leq \max \left\{ m_{2n}(0), \left( \frac{B_n}{A_n} \right)^{\frac{1}{1+\theta}} \right\}.
\]
For a general \( r > 2 \), consider the minimal integer \( n \geq 2 \) such that \( 2 < r \leq 2n \), then
\[
m_{r}(t) \leq m_{2n}^{1-\theta}(t), \quad \theta = \frac{r - 2}{2n - 2} \leq 1.
\]
Thus, we conclude that
\[
m_{r}(t) \leq \left( \frac{B_n}{A_n} \right)^{\frac{\theta}{1+\theta}} + \left( \frac{1}{c_n A_n} \right)^{\frac{\theta}{1+\theta}} t^{-\frac{r-2}{\lambda^2}}.
\]
This concludes the proof of the result. \[\square\]

3.3. Exponential moments. Let us now focus in the generation and propagation of exponential moments result of Theorem 2.4. Our goal is to show that the partial sums
\[
\sum_{n=0}^{P} \frac{(\sigma t)^m m_{2n}[F](t)}{(n!)^\alpha)
\]
are bounded uniformly in time \( t \) and summation truncation \( P \) for some explicit \( \rho > 0 \) and \( \alpha > 0 \). To this end we use a methodology reminiscent of [15, 5]. The proof relies on generation and propagation of polynomial moments stated in Theorem 2.3 and the following lemma.

Lemma 3.2 (Equivalence property). Let \( G = [g_i]_{1 \leq i \leq I} \in L^1(\mathbb{R}^3) \), then we have the following equivalence:
(i) (System) For $\sigma_0 \in (0, \infty)$, $\alpha \in [1, \infty)$ and $K \in [1, \infty)$, we have

$$\sup_{n \geq 0} \frac{\sigma_0^n}{(n!)^\alpha} m_{2n}[G] \leq K \Rightarrow \sum_{i=1}^I \int_{\mathbb{R}^3} \exp \left( \frac{\left( 2^{1/\alpha} (v)^{2/\alpha} \right)}{2} \right) g_i(v)dv \leq 2 m_0^{1-\frac{1}{K}} K^{\frac{n}{\alpha}}.$$

(ii) (Component) Let $\rho \in (0, 2]$, $\sigma_0 \in (0, 1]$ and $K \geq 1$, there exists $\sigma_1 := \sigma_1(\rho, \sigma_0, K)$ such that

$$\int_{\mathbb{R}^3} \exp \left( \sigma_0(v)^\rho \right) g_i(v)dv \leq \frac{K}{T} \Rightarrow \sup_{n \geq 0} \frac{\sigma_1^n}{(n!)^{2/\rho}} \leq \frac{1}{T}.$$

Proof. We start with (i) noticing that by Holder's inequality (recall that $\alpha \geq 1$),

$$m_{2n}[G] \leq m_{2n}[G]^{1/2} m_0^{1/2} \leq m_0^{1-\frac{1}{K}} K^{1/2} \sigma_0^{-\frac{n}{\alpha}} n!.$$

Then,

$$\sum_{i=1}^I \int_{\mathbb{R}^3} \exp \left( \frac{\left( 2^{1/\alpha} (v)^{2/\alpha} \right)}{2} \right) g_i(v)dv = \sum_{n \geq 0} \frac{\sigma_0^n}{2^n n!} m_{2n}[G] \leq m_0^{1-\frac{1}{K}} K^{\frac{n}{\alpha}} \sum_{n \geq 0} 2^{-n} = 2 m_0^{1-\frac{1}{K}} K^{\frac{n}{\alpha}}.$$

Regarding to (ii), note that

$$\sup_{n \geq 0} \frac{\sigma_0^n m_{m, i}[g_i]}{n!} \leq \sum_{n \geq 0} \frac{\sigma_0^n}{n!} m_{m, i}[g_i] = \int_{\mathbb{R}^3} \exp \left( \sigma_0(v)^\rho \right) g_i(v)dv \leq \frac{K}{T}.$$

For $n \geq 1$ we set $k_n = \lceil \frac{2n}{\rho} \rceil \in \{ \frac{2n}{\rho}, \frac{2n}{\rho} + 1 \}$. Using that $\rho k_n \geq 2n$ we obtain that

$$m_{2n, i}[g_i] \leq m_{k_n, i}[g_i] \leq \frac{K k_n!}{I \sigma_0^k}.$$

and consequently,

$$\frac{\sigma_1^n}{(n!)^{2/\rho}} \leq \frac{\sigma_1^n}{(n!)^{2/\rho}} \frac{K k_n!}{I \sigma_0^k}.$$

Invoking Stirling’s formula $n! \sim \sqrt{2\pi n}(n/e)^n$ and since $k_n = \lceil \frac{2n}{\rho} \rceil$ we conclude that for some constant $A \in (0, \infty)$ depending only on $\rho$ it holds that

$$\frac{k_n!}{(n!)^{2/\rho}} \leq A \left( \frac{4}{\rho} \right)^{\frac{2n}{\rho}}, \quad n \geq 1.$$

We refer to [13] for additional details. Moreover, since $\sigma_0 \in (0, 1]$ then $\sigma_0^k \geq \sigma_0^{\frac{2n+1}{\rho}}$, and consequently

$$\frac{\sigma_1^n}{(n!)^{2/\rho}} \leq \frac{K A \sigma_0^k}{I \sigma_0^{\frac{2n+1}{\rho}}} \frac{4^n}{\rho} < 1, \quad n \geq 1, \quad \frac{1}{7},$$

after choosing, for example,

$$\sigma_1 \leq \frac{1}{2} \inf_{n \geq 1} \frac{I \sigma_0^{\frac{2n+1}{\rho}}}{(K A)^{\frac{1}{\rho}} \left( \frac{4}{\rho} \right)^{\frac{2n}{\rho}}}.$$

This concludes the proof. □
Lemma 3.3

The following lemma is crucial as it expresses the convolution structure of the moments in the We start by time differentiation of \( E_p[F](t) \) and then using Proposition 2.2 item (i). Since \( m_0 = m_2 = 0 \), we deduce that

\[
\frac{d}{dt} E_p[F](t) = \sum_{n=0}^{p} \left[ \frac{(\sigma t)^{2n/\lambda^3} m_{2n,i}(F_i)(t)}{(n!)^\alpha} \right] + \sum_{n=2}^{p} \left[ \frac{n^{s_{ij}/2}(\sigma t)^{2n/\lambda^3} m_{2n+l_{ij},i}(F_i)(t)}{(n!)^\alpha} \right],
\]

Additionally, we define the objects, for \( 1 < i, j \leq I \),

\[
E_p^i[f_i](t) := \sum_{n=0}^{p} \left[ \frac{(\sigma t)^{2n/\lambda^3} m_{2n,i}(f_i)(t)}{(n!)^\alpha} \right], \quad F_p^i[f_i](t) := \sum_{n=2}^{p} \left[ \frac{n^{s_{ij}/2}(\sigma t)^{2n/\lambda^3} m_{2n+l_{ij},i}(f_i)(t)}{(n!)^\alpha} \right],
\]

\[
G_p^i[f_i, f_j](t) := \sum_{n=2}^{p} \left[ \frac{(\sigma t)^{2n/\lambda^3} S_{ij}(n)}{(n!)^\alpha} \right], \quad H_p^i[f_i](t) := \sum_{n=0}^{p} \left[ \frac{n^{s_{ij}/2}(\sigma t)^{2n/\lambda^3} m_{2n,i}(f_i)(t)}{(n!)^\alpha} \right].
\]

We start by time differentiation of \( E_p[F](t) \) and then using Proposition 2.2 item (i). Since \( m_0 = m_2 = 0 \), we deduce that

\[
\frac{d}{dt} E_p[F](t) = \sum_{n=0}^{p} \left[ \frac{(\sigma t)^{2n/\lambda^3} m_{2n,i}(F_i)(t)}{(n!)^\alpha} \right] + \frac{2 \sigma n(\sigma t)^{2n/\lambda^3-1} m_{2n,i}(F_i)(t)}{(n!)^\alpha}
\]

\[
\leq \sum_{i,j=1}^{I} \left[ -C^i_{ij} \left( F_p^i[f_i](t) + F_p^j[f_j](t) \right) + C^i_{ij} \left( G_p^i[f_i, f_j](t) \right) + \frac{\sigma^2}{\lambda^3} \sum_{i=1}^{I} \left( H_p^i[f_i](t) \right) 
\]

The following lemma is crucial as it expresses the convolution structure of the moments in the summation. This is where the nonlinear part is estimated.

Lemma 3.3 (Moment Convolution). Let \( 1 \leq i, j \leq I, \lambda^3 > 0, s_{ij} \in (0, 2) \), and \( F_0 \in U(D_0, E_0) \). Given \( \epsilon > 0 \) there are explicit constants \( B_\epsilon \) and \( D_\epsilon \), depending on the parameters of the model, \( E_0 \) and \( D_0 \), such that for \( \sigma \in (0, 1], \alpha \geq 1 \) and \( p \geq 2 \), the following estimate holds

\[
G_p^i[f_i, f_j](t) \leq B_\epsilon \sigma^{2/\lambda^3} \left( F_p^i[f_i](t) + F_p^j[f_j](t) \right) + \epsilon \left( F_p^i[f_i](t) E_p^i[f_i](t) + F_p^j[f_j](t) E_p^j[f_j](t) \right) + \sigma^{2/\lambda^3} D_\epsilon, \quad t \in (0, 1].
\]

Proof. Using the definition of \( S_{ij}(n) \) it follows that

\[
G_p^i[f_i, f_j](t) = \sum_{n=2}^{p} \left[ (\sigma t)^{2n/\lambda^3} \sum_{a=0}^{n/2} \binom{n}{a} \frac{n^{s_{ij}/2}}{(a!)^\alpha \alpha^{s_{ij}/2+1} (n!)^\alpha} \left( m_{2a,i}(t) m_{2(n-a)+l_{ij},j}(t) \right) + m_{2a,j}(t) m_{2(n-a)+l_{ij},i}(t) \right],
\]

For \( 1 \leq a \leq \lfloor n/2 \rfloor \) one has that \( n^{s_{ij}/2} \leq 2(n-a)^{s_{ij}/2} \). Therefore,

\[
G_p^i[f_i, f_j](t) \leq 2 \sum_{n=2}^{p} \left[ (\sigma t)^{2n/\lambda^3} \sum_{a=1}^{n-1} \binom{n}{a} \frac{(n-a)^{s_{ij}/2}}{(a!)^\alpha \alpha^{s_{ij}/2+1} (n!)^\alpha} \left( m_{2a,i}(t) m_{2(n-a)+l_{ij},j}(t) \right) + m_{2a,j}(t) m_{2(n-a)+l_{ij},i}(t) \right] \]

\[
= 2 \sum_{n=2}^{p} \sum_{a=1}^{n-1} \frac{(\sigma t)^{2n/\lambda^3} m_{2a,i}(t) (n-a)^{s_{ij}/2} (\sigma t)^{2(n-a)/\lambda^3} m_{2(n-a)+l_{ij},j}(t)}{(a!)^\alpha \alpha^{s_{ij}/2+1} ((n-a)!)^\alpha} + m_{2a,j}(t) m_{2(n-a)+l_{ij},i}(t) \]

\[
= G_p^i[f_i, f_j](t) + G_p^j[f_j, f_i](t),
\]

The following lemma is crucial as it expresses the convolution structure of the moments in the summation. This is where the nonlinear part is estimated.
with obvious definitions for each term. Now, note that interchanging the summation it holds that
\[ G_{p,1}^{ij}[f_t, f_j](t) = \frac{2 \sum_{a=1}^{\infty} \sum_{\lambda>0} \lambda^{\frac{1}{2}} \left( \sigma(t) \right)^{2/\lambda^3} \left( \frac{m_{2+\lambda,ij}}{t} \right) \left( \frac{m_{2a,ij}}{t} \right)}{a^{\lambda^3/2+1}} \left( \sum_{a=1}^{\infty} \frac{\lambda^{\frac{1}{2}} \left( \sigma(t) \right)^{2/\lambda^3} \left( \frac{m_{2a,ij}}{t} \right)}{a^{\lambda^3/2+1}} \right) \]

Thus,

\[ I_{p,1}^{ij}(t) = \sum_{a=1}^{\infty} \frac{\lambda^{\frac{1}{2}} \left( \sigma(t) \right)^{2/\lambda^3} \left( \frac{m_{2a,ij}}{t} \right)}{a^{\lambda^3/2+1}} \]


Setting \( N_{\epsilon} = [2 \epsilon^{-1}] \), it holds that for \( a \geq N_{\epsilon} \) implies \( 2 a^{-1} \leq \epsilon \). Hence,

\[ I_{p,1}^{ij}(t) \leq \epsilon E_{p[i]}^{ij}(t) + J_{p[i]}^{ij}(t) \quad \text{where} \quad J_{p[i]}^{ij}(t) = \sum_{a=1}^{N_{\epsilon}} \frac{\lambda^{\frac{1}{2}} \left( \sigma(t) \right)^{2/\lambda^3} \left( \frac{m_{2a,ij}}{t} \right)}{a^{\lambda^3/2+1}} \]

Using the first result of Theorem 2.3, we deduce that there are explicit constants \( C \) and \( C_{\epsilon} \) such that

\[ \left( \sigma(t) \right)^{2/\lambda^3} m_{2+\lambda,ij}(t) \leq C(\sigma(t))^{2/\lambda^3} \left( 1 + t^{-\frac{\lambda^3}{\lambda^3}} \right) \leq 2 C \sigma^{2/\lambda^3}, \quad \sigma \in (0, 1], \quad t \in (0, 1], \]

and

\[ J_{p[i]}^{ij}(t) \leq C_{\epsilon} \sum_{a=1}^{N_{\epsilon}} \frac{\lambda^{\frac{1}{2}} \left( \sigma(t) \right)^{2/\lambda^3} \left( \frac{m_{2a,ij}}{t} \right)}{a^{\lambda^3/2+1}} \leq C_{\epsilon} \sum_{a=1}^{\infty} \frac{2}{a^{\lambda^3/2}} = : A_{\epsilon} \sigma^{2/\lambda^3}. \]

Thus,

\[ G_{p,1}^{ij}[f_t, f_j](t) \leq \left( F_{p,1}^{ij}(f_j)(t) + 2 C \sigma^{2/\lambda^3} \right) \left( E_{p,1}^{ij}(t) + A_{\epsilon} \sigma^{2/\lambda^3} \right). \]

Similarly, we also have that

\[ G_{p,2}^{ij}[f_t, f_j](t) \leq \left( F_{p,2}^{ij}[f_j](t) + 2 C \sigma^{2/\lambda^3} \right) \left( E_{p,2}^{ij}(t) + A_{\epsilon} \sigma^{2/\lambda^3} \right) \]

In this way, adding (3.7) and (3.8), it holds that

\[ G_{p}^{ij}[f_t, f_j](t) \leq G_{p,1}^{ij}[f_t, f_j](t) + G_{p,2}^{ij}[f_t, f_j](t) \]

\[ \leq \sigma^{2/\lambda^3} A_{\epsilon} \left( F_{p}^{ij}[f_t](t) + F_{p}^{ij}[f_j](t) \right) + 2 \sigma^{2/\lambda^3} \epsilon C \left( E_{p}^{ij}[f_t](t) + E_{p}^{ij}[f_j](t) \right) \]

\[ + \epsilon \left( F_{p}^{ij}[f_t](t) E_{p}^{ij}[f_j](t) + F_{p}^{ij}[f_j](t) E_{p}^{ij}[f_t](t) \right) + 4 \sigma^{4/\lambda^3} A_{\epsilon} C. \]

Furthermore, for \( t \in (0, 1] \) it holds that

\[ E_{p}^{ij}[f_t](t) + E_{p}^{ij}[f_j](t) \leq m_{0}(t) + \sigma m_{2}(t) + F_{p}^{ij}[f_t](t) + F_{p}^{ij}[f_j](t). \]

Consequently, we are led to

\[ G_{p}^{ij}[f_t, f_j](t) \leq B_{\epsilon} \sigma^{2/\lambda^3} \left( F_{p}^{ij}[f_t](t) + F_{p}^{ij}[f_j](t) \right) \]

\[ + \epsilon \left( F_{p}^{ij}[f_t](t) E_{p}^{ij}[f_j](t) + F_{p}^{ij}[f_j](t) E_{p}^{ij}[f_t](t) \right) + \sigma^{2/\lambda^3} D_{\epsilon}, \quad t \in (0, 1), \]

where

\[ B_{\epsilon} := 2 A_{\epsilon} + 2 \epsilon C, \quad \text{and} \quad D_{\epsilon} := 4 \sigma^{2/\lambda^3} A_{\epsilon} C + 2 \epsilon C \left( m_{0} + \sigma m_{2} \right). \]

This concludes the proof.
Lemma 3.4. Let \( 1 \leq i, j \leq I, \lambda^2 > 0, \nu^2 > 0, \alpha = \max \left\{ 1, \frac{2 - \lambda^2}{\lambda^4} \right\} \), and \( F_0 \in \mathcal{U}(D_0, E_0) \). There are explicit constants \( K \) and \( L \), depending on the parameters of the model, \( E_0 \) and \( D_0 \), such that for any \( \sigma \in (0, 1) \) and \( p \geq 2 \), it holds that

\[
H_p^i[f_i](t) \leq K F_p^i[f_i](t) + L, \quad t \in (0, 1),
\]

where \( F_p^i[f_i](t) := \sum_{n=2}^{p} \frac{n^{1/2} (\sigma t)^{2n/\lambda^3}}{(n!)^\alpha} m_{2n+\lambda_i, i}(t) \).

Proof. First observe that for \( \kappa \geq 1 \), to be chosen later sufficiently large,

- If \( m_{2n,i}(t) \leq \left( \frac{n^{1/2}}{\kappa \sigma t} \right) \), then \( \frac{n m_{2n,1}(t)}{\sigma t} \leq \frac{n}{\kappa \sigma t} \).
- If \( m_{2n,i}(t) \geq \left( \frac{n^{1/2}}{\kappa \sigma t} \right) \), then \( \frac{n m_{2n,1}(t)}{\sigma t} \leq \kappa n^{1/2} m_{2n,1}(t)^{1+\frac{\nu^2-2}{\nu}} \).

Thus, overall we get that

\[
\frac{n m_{2n,1}(t)}{\sigma t} \leq \kappa n^{1/2} m_{2n,1}(t)^{1+\frac{\nu^2-2}{\nu}} + \frac{n}{\kappa \sigma t} \left( \frac{n^{1-\frac{\nu}{2}}}{\kappa \sigma t} \right)^{2n-2}.
\]

Interpolation gives that

\[
m_{2n,1}(t)^{1+\frac{\nu^2-2}{\nu}} \leq m_{2n+\nu, i}(t) m_{2,1}^{\frac{\nu^2}{\nu}}(t) \leq m_{2n+\nu, i}(t) \max \{1, m_2\},
\]

and consequently,

\[
H_p^i[f_i](t) \leq \kappa m_{2,1}(t) \sigma t^{2/\lambda^3} + \kappa \max \{1, \kappa, m_2\} F_p^i[f_i](t) + \sigma t^{2/\lambda^3} - \frac{p}{\kappa} \sum_{n=1}^{p} \frac{n^{2-\frac{\nu}{2}}}{(n!)^{\alpha} K(2n-2)/\lambda_i}.
\]

Conservation of mass and energy implies that

\[
m_{2,1}(t) \sigma t^{2/\lambda^3} \leq m_2 \sigma t^{2/\lambda^3}, \quad t \in (0, 1).
\]

Moreover, recall that \( \alpha = \max \left\{ 1, \frac{2 - \lambda^2}{\lambda^4} \right\} \geq \frac{2 - \frac{\nu}{2}}{\lambda_i} \) and \( \kappa \geq 1 \), then

\[
\left( \sigma t \right)^{2/\lambda_i} - \frac{p}{\kappa} \sum_{n=1}^{p} \frac{n^{2-\frac{\nu}{2}}}{(n!)^{\alpha} K(2n-2)/\lambda_i} \leq \kappa^{3/\lambda^3} \sum_{n=1}^{\infty} \frac{n^{\alpha n+1}}{(n!)^{\alpha} K n} =: S,
\]

where the series \( S \) is convergent provided \( \kappa > 2e^\alpha \) (invoking the Stirling’s formula \( n! \sim \sqrt{2\pi n}(n/e)^n \)).

Therefore, for all \( 1 \leq i \leq I \),

\[
H_p^i[f_i](t) \leq 3e^\alpha \max \{1, \kappa, m_2\} F_p^i[f_i](t) + 3e^\alpha m_2 + S, \quad \sigma \in (0, 1), \quad t \in (0, 1),
\]

which is the statement of the lemma for \( K = 3e^\alpha \max \{1, m_2\} \) and \( L = 3e^\alpha m_2 + S \). \( \square \)

We can finally proceed with the proof of exponential moment generation.

Proof of the generation result of Theorem 2.4. We devise the proof in two steps.

**Step 1.** We show first that for \( \lambda^2 > 0, \nu^2 > 0 \) and \( \alpha = \max \left\{ 1, \frac{2 - \lambda^2}{\lambda^4} \right\} \), there exist \( \sigma \in (0, 1) \) and \( T \in (0, 1] \), depending only on the parameters of the model such that

\[
\sup_{t \in [0, T]} \sum_{n=0}^{\infty} \frac{(\sigma t)^{2n/\lambda^3}}{(n!)^\alpha} m_{2n}(t) \leq 2 m_0.
\]

Invoking Theorem 2.3 it holds that for some constant \( C_p \in (0, \infty) \),

\[
m_0 \leq E_p[F](t) \leq m_0 + C_p t^{2/\lambda^3},
\]

then \( \lim_{t \to 0} E_p[F](t) = m_0 \).
Now, using identity \( \text{[3.6]} \) and lemmas \( \text{[3.3]} \) and \( \text{[3.4]} \), we get that

\[
\frac{d}{dt} E_p[\mathcal{F}](t) \leq \sum_{i,j=1}^{l} \left[ - C_{ij}^l \left( F_p^{ij}[f_i](t) + F_p^{ji}[f_j](t) \right) + C_{ij}^2 G_p^{ij}[f_i](t) \right] + \frac{2\sigma}{\lambda^2} \sum_{i=1}^{l} H_p^i(t)
\]

\[
\leq \sum_{i,j=1}^{l} \left[ - C_{ij} \left( F_p^{ij}[f_i](t) + F_p^{ji}[f_j](t) \right) + C_{ij}^2 \left( B_i \sigma^{2/\lambda^2} [F_p^{ij}[f_i](t) + F_p^{ji}[f_j](t)] \right) \right. \\
\left. + \epsilon \left( F_p^{ij}[f_i](t) + F_p^{ji}[f_j](t) \right) \left( E_p^{ij}[f_i](t) + E_p^{ji}[f_j](t) \right) + \frac{2\sigma}{\lambda^2} \sum_{i=1}^{l} \left( K F_p^i[f_i](t) + L \right) \right]
\]

\[
\leq \sum_{i,j=1}^{l} \left[ - C_{ij} \left( F_p^{ij}[f_i](t) + F_p^{ji}[f_j](t) \right) + R_{ij}^l \sigma^{2/\lambda^2} [F_p^{ij}[f_i](t) + F_p^{ji}[f_j](t)] \right.
\left. + \epsilon C_{ij}^2 \left( F_p^{ij}[f_i](t) + F_p^{ji}[f_j](t) \right) \left( E_p^{ij}[f_i](t) + E_p^{ji}[f_j](t) \right) + \frac{2\sigma}{\lambda^2} \sum_{i=1}^{l} F_p^i[f_i](t) + Q \right]
\]

\[
= I^1 + I^2 + Q \epsilon,
\]

where \( R_{ij}^l = C_{ij}^2 B_i \), \( Q \epsilon = D_i \sum_{i,j=1}^{l} C_{ij}^2 + \frac{2\sigma}{\lambda^2} IL \), and

\[
I^1 := \sum_{i,j=1}^{l} \left[ - C_{ij} \left( F_p^{ij}[f_i](t) + F_p^{ji}[f_j](t) \right) + R_{ij}^l \sigma^{2/\lambda^2} [F_p^{ij}[f_i](t) + F_p^{ji}[f_j](t)] \right.
\left. + \epsilon C_{ij}^2 \left( F_p^{ij}[f_i](t) + F_p^{ji}[f_j](t) \right) \left( E_p^{ij}[f_i](t) + E_p^{ji}[f_j](t) \right) \right].
\]

\[
I^2 := \frac{2\sigma}{\lambda^2} K \sum_{i=1}^{l} F_p^i[f_i](t).
\]

Choosing

\[
\epsilon := \min_{1 \leq i,j \leq l} \frac{C_{ij}^l}{8I C_{ij}^2 m_0}, \quad \text{and} \quad \sigma_0 := \min_{1 \leq i,j \leq l} \left( \frac{C_{ij}^l}{4 R_{ij}^l} \right)^{\lambda^2/2},
\]

we get for any \( \sigma \leq \min\{1, \sigma_0\}\)

\[
I^1 \leq -\frac{3 \min_{1 \leq i,j \leq l} C_{ij}^l}{4} \sum_{i,j=1}^{l} \left( F_p^{ij}[f_i](t) + F_p^{ji}[f_j](t) \right) \\
+ \frac{\min_{1 \leq i,j \leq l} C_{ij}^l}{8I m_0} \sum_{i,j=1}^{l} \left( F_p^{ij}[f_i](t) + F_p^{ji}[f_j](t) \right) \left( E_p^{ij}[f_i](t) + E_p^{ji}[f_j](t) \right).
\]

Similar, for \( \sigma \leq \sigma_1 := \min_{1 \leq i,j \leq l} \frac{C_{ij}^l \lambda^2}{8K} \), we obtain that

\[
I^2 \leq \frac{\min_{1 \leq i,j \leq l} C_{ij}^l}{4} \sum_{i=1}^{l} F_p^i[f_i](t) \leq \frac{\min_{1 \leq i,j \leq l} C_{ij}^l}{4} \sum_{i,j=1}^{l} F_p^{ij}[f_i](t),
\]

where the latter inequality is a direct consequence of the definitions of \( F_p^{ij}[f_i] \) and \( F_p^{ij}[f_j] \). Thus, overall we deduce that

\[
\frac{d}{dt} E_p[\mathcal{F}](t) \leq I^1 + I^2 + Q \leq -\frac{\min_{1 \leq i,j \leq l} C_{ij}^l}{2} \sum_{i,j=1}^{l} \left( F_p^{ij}[f_i](t) + F_p^{ji}[f_j](t) \right) \\
+ \frac{\min_{1 \leq i,j \leq l} C_{ij}^l}{8I m_0} \sum_{i,j=1}^{l} \left( F_p^{ij}[f_i](t) + F_p^{ji}[f_j](t) \right) \left( E_p^{ij}[f_i](t) + E_p^{ji}[f_j](t) \right) + Q.
\]
Recalling that $E$ and observe that $\alpha$. Then, we conclude that for all $$ which is precisely the statement of the exponential moment generation.

Thus, for $$ we have that

\[
\frac{d}{dt} E_p[F](t) \leq -\frac{\min_{1 \leq i,j \leq t} C_{ij}^l}{2} \sum_{i,j=1}^l \left( F^{ij}_p[f_i](t) + F^{ji}_p[f_j](t) \right) + \frac{\min_{1 \leq i,j \leq t} C_{ij}^l}{8/m_0} \sum_{i,j=1}^l \left( F^{ij}_p[f_i](t) + F^{ji}_p[f_j](t) \right) + Q
\]

Then, we conclude that for all $t \in (0, T_p]$ it holds that

\[
\frac{d}{dt} E_p[F](t) \leq Q , \quad \text{and thus } E_p[F](t) \leq E_p[F](0) + Q \ t .
\]

Recalling that $E_p[F](0) = m_0$, we deduce directly from the definition of $T_p$ that

\[
T_p = \min\{1, \frac{m_0}{q} \} =: T, \quad \text{independent of } p .
\]

Thus, for $\sigma \leq \min\{1, \sigma_0, \sigma_1\}$,

\[
E[F](t) = \lim_{p \to \infty} E_p[F](t) \leq 2 \ m_0 , \quad t \in (0, T] .
\]

**Step 2.** Let us proceed with the concluding argument. To this end, fix the rate

\[
\rho = \min\left\{ \frac{2\lambda^3}{2 - 8^\alpha} , 2 \right\} ,
\]

and observe that $\alpha = \max\{1, \frac{2 - 8^\alpha}{\lambda^3}\} = \frac{2}{\lambda^3}$. Using the result of Step 1, we obtain that

\[
\frac{(\sigma t)^{2n/\lambda^3} m_{2n}(F)(t)}{(n!)^\alpha} \leq E[F](t) = \sum_{n=0}^{\infty} \frac{(\sigma t)^{2n/\lambda^3} m_{2n}(F)(t)}{(n!)^\alpha} \leq 2m_0 , \quad t \in (0, T] .
\]

Then, Lemma 3.2 item (i) with $\sigma_0 = (\sigma t)^{2/\lambda^3}$ and $K = 2m_0$ implies that

\[
\sum_{i=1}^l \int_{\mathbb{R}^3} \exp \left[ \frac{(\sigma t)^{2/\lambda^3} (v)_{i}^{2/\alpha}}{2} \right] f_i(v) dv = \sum_{i=1}^l \int_{\mathbb{R}^3} \exp \left[ \frac{(\sigma t)^{2/\lambda^3} (v)_{i}^{2/\alpha}}{2} \right] f_i(v) dv \leq 2^{1+1/\alpha} m_0 \leq 4m_0 , \quad t \in (0, T] ,
\]

which is precisely the statement of the exponential moment generation. \qed
3.3.2. Propagation of exponential moment. The argument to propagate exponential moments follows a similar path to that of propagation; we present the main steps for completeness. Let us introduce for \( \sigma \in (0,1] \), to be chosen sufficiently small, and \( p \geq 2 \),

\[
\mathcal{E}_p[F](t) := \sum_{n=0}^{p} \frac{\sigma^{2n/\lambda^3} m_{2n}[F](t)}{(n!)^\alpha}, \quad t \geq 0.
\]

Analogously, for all \( 1 \leq i, j \leq I \), we denote

\[
\mathcal{E}_p^{ij}[f_i](t) := \sum_{n=0}^{p} \sigma^{2n/\lambda^3} m_{2n,[f_i]}(t), \quad \mathcal{P}_p^{ij}[f_i](t) := \sum_{n=2}^{p} \frac{n^{2}\sigma^{2n/\lambda^3} m_{2n+\lambda_{ij},[f_i]}}{(n!)^\alpha},
\]

and

\[
\mathcal{G}_p^{ij}[f_i, f_j](t) := \sum_{n=2}^{p} \sigma^{2n/\lambda^3} S_{ij}^{(t)}.
\]

Note that the analogous to (3.6) in this context is given by

\[
\frac{d}{dt} \mathcal{E}_p[F](t) \leq \sum_{i,j=1}^{I} \left[ -C
\frac{1}{p} \left( \mathcal{P}_p^{ij}[f_i](t) + \mathcal{P}_p^{ji}[f_j](t) \right) + C_2 \mathcal{G}_p^{ij}[f_i, f_j](t) \right].
\]

As before, we have the key estimate on the convolution structure of moments.

**Lemma 3.5** (Moment Convolution). Let \( 1 \leq i, j \leq I \), \( \lambda^I > 0 \), \( s_{ij} \in (0,2) \), \( F_0 \in U(D_0, E_0) \), and \( \tilde{\sigma}_0 = \sup_{n \geq 0} \frac{\sigma_0^a m_{2n}(0)}{(n!)^{\alpha}} \leq 1 \), for some \( \sigma_0 > 0 \).

Given \( \epsilon > 0 \) there are explicit constants \( B_\epsilon \) and \( D_\epsilon \), depending on the parameters of the model, \( E_0 \), \( D_0 \), and \( \tilde{\sigma}_0 \), such that for any \( \sigma \in (0,1] \), \( p \geq 2 \), and \( \alpha \geq 1 \), the following estimate holds

\[
\mathcal{G}_p^{ij}[f_i, f_j](t) \leq B_\epsilon \sigma^{2/\lambda^3} \left[ \mathcal{P}_p^{ij}[f_i](t) + \mathcal{P}_p^{ji}[f_j](t) \right] + \epsilon \left( \mathcal{P}_p^{ij}[f_i](t) \mathcal{E}_p^{ij}[f_j](t) + \mathcal{P}_p^{ij}[f_j](t) \mathcal{E}_p^{ij}[f_i](t) \right) + \sigma^{2/\lambda^3} D_\epsilon, \quad t \geq 0.
\]

**Proof.** Using a copycat argument in the proof of Lemma 3.3 we have that

\[
\mathcal{G}_p^{ij}[f_i, f_j](t) \leq \left( \mathcal{P}_p^{ij}[f_i](t) + \sigma^{2/\lambda^3} m_{2+\lambda_{ij},(t)} \right) \left( \epsilon \mathcal{E}_p^{ij}[f_i](t) + \mathcal{J}^{ij}_{\epsilon}[f_i](t) \right) + \left( \mathcal{P}_p^{ij}[f_j](t) + \sigma^{2/\lambda^3} m_{2+\lambda_{ij},(t)} \right) \left( \epsilon \mathcal{E}_p^{ij}[f_j](t) + \mathcal{J}^{ij}_{\epsilon}[f_j](t) \right),
\]

where, setting \( N_\epsilon = [2 \epsilon^{-1}] \),

\[
\mathcal{J}^{ij}_{\epsilon}[f_i](t) := 2 \sum_{a=1}^{N_\epsilon} \frac{\sigma^{2a/\lambda^3} m_{2a,[f_i]}(t)}{(a!)^{\alpha}} \leq A_\epsilon \sigma^{2/\lambda^3}.
\]

For the later inequality in the right we have used the propagation of polynomial moments given in Theorem 2.23 which applies since the initial datum satisfies \( m_{2n}(0) \leq \frac{(n!)^{\alpha}}{\bar{\sigma}_0} \), for \( n \geq 1 \). Then, we can simply take the constant

\[
A_\epsilon := A_{\epsilon}(\tilde{\sigma}_0) = 2 \sum_{a=1}^{\infty} \frac{\sigma^{2a/\lambda^3} m_{2a}(\mathbb{F})(t)}{(a!)^{\alpha}} < \infty.
\]

Thus,

\[
\mathcal{G}_p^{ij}[f_i, f_j](t) \leq \left( \mathcal{P}_p^{ij}[f_i](t) + 2 C \sigma^{2/\lambda^3} \right) \left( \epsilon \mathcal{E}_p^{ij}[f_i](t) + A_\epsilon \sigma^{2/\lambda^3} \right) + \left( \mathcal{P}_p^{ij}[f_j](t) + 2 C \sigma^{2/\lambda^3} \right) \left( \epsilon \mathcal{E}_p^{ij}[f_j](t) + A_\epsilon \sigma^{2/\lambda^3} \right).
\]

And, since

\[
\mathcal{E}_p^{ij}[f_i](t) + \mathcal{E}_p^{ij}[f_j](t) \leq \mathcal{E}_0 + \sigma \mathcal{E}_2 + \mathcal{P}_p^{ij}[f_i](t) + \mathcal{P}_p^{ij}[f_j](t),
\]
we conclude that
\[
\mathcal{G}^i_j(f_i, f_j)(t) \leq \mathcal{B}_e \sigma^{2/3} \left[ \mathcal{P}^i_j(f_i)(t) + \mathcal{P}_p^j(f_j)(t) \right] + \epsilon \left[ \mathcal{P}^i_j(f_i)(t) \mathcal{E}^i_j(f_j)(t) + \mathcal{P}_p^j(f_j)(t) \mathcal{E}^i_p(f_i)(t) \right] + \sigma^{2/3} \mathcal{D}_e,
\]
where
\[
\mathcal{B}_e = 2A_e + 2\epsilon C \quad \text{and} \quad \mathcal{D}_e = 4\sigma^{2/3} A_e C + 2\epsilon C \left( m_0 + \sigma m_2 \right),
\]
which is the statement of the result. \qed

**Lemma 3.6.** Let \(1 \leq i, j \leq I\), \(\lambda^2 > 0\), \(s_{ij} \in (0, 2)\) and \(p > 2\). For any \(\sigma \in (0, 1]\), \(\alpha \geq 1\) it holds that
\[
\mathcal{P}^i_j(f_i)(t) + \mathcal{P}_p^j(f_j)(t) \geq \frac{1}{\sigma^{\lambda/2}} \left[ \mathcal{E}^i_p(f_i)(t) + \mathcal{E}^i_p(f_j)(t) - 2\epsilon \max \left\{ m_0[f_i], m_0[f_j] \right\} \right], \quad t \geq 0.
\]

**Proof.** Using that \(x^b \geq x^a - 1\) for any \(b \geq a \geq 0\), we can write
\[
\mathcal{P}^i_j(f_i)(t) + \mathcal{P}_p^j(f_j)(t) = \sum_{n=2}^{\infty} \frac{n^{s_{ij}}}{(n!)^\alpha} \left( m_{2n+i} f_i(t) + m_{2n+j} f_j(t) \right)
\]
\[
\geq \frac{1}{\sigma^{\lambda/2}} \sum_{n=2}^{\infty} \left[ \int_{\mathbb{R}^3} (\sigma v_j^2)^{n+\lambda/2} f_i(t, v) dv \right] + \frac{1}{\sigma^{\lambda/2}} \sum_{n=2}^{\infty} \left[ \int_{\mathbb{R}^3} (\sigma v_i^2)^{n+\lambda/2} f_j(t, v) dv \right]
\]
\[
\geq \frac{1}{\sigma^{\lambda/2}} \sum_{n=2}^{\infty} \left( m_{2n+i} f_i(t) + m_{2n+j} f_j(t) \right) - \frac{1}{\sigma^{\lambda/2}} \sum_{n=2}^{\infty} \left( m_0[f_i] + m_0[f_j] \right)
\]
Then, since \(\sigma \in (0, 1]\) and \(\alpha \geq 1\), it holds that
\[
\mathcal{P}^i_j(f_i)(t) + \mathcal{P}_p^j(f_j)(t) \geq \frac{1}{\sigma^{\lambda/2}} \left[ \mathcal{E}^i_p(f_i)(t) + \mathcal{E}^i_p(f_j)(t) - \sigma m_2[f_i](t) - m_0[f_i](t) \right] \quad \text{and} \quad \alpha \geq 1,
\]
\[
\mathcal{P}^i_j(f_i)(t) + \mathcal{P}_p^j(f_j)(t) \geq \frac{1}{\sigma^{\lambda/2}} \left[ \mathcal{E}^i_p(f_i)(t) + \mathcal{E}^i_p(f_j)(t) - 2\epsilon \max \left\{ m_0[f_i], m_0[f_j] \right\} \right].
\]
\qed

Let us proceed with the proof the propagation of exponential moments.

**Proof of the propagation result of Theorem 2.2.** As before, we present two steps.

**Step 1.** First we prove that for any \(\sigma_0 > 0\) and \(\alpha \geq 1\) there exists \(\sigma > 0\) (sufficiently small) depending only on the parameters of the model, \(D_0, E_0, \sigma_0\) and \(\alpha\) such that
\[
\text{if} \quad \sup_{n \geq 0} \frac{\sigma_0^2 m_{2n}(0)}{(n!)^\alpha} \leq 1 \quad \Rightarrow \quad \sup_{t \geq 0} \sum_{n=0}^{\infty} \frac{\sigma^{2n/3} m_{2n}(t)}{(n!)^\alpha} \leq 3\epsilon (m_0 + 1).
\]

We fix \(\alpha \geq 1\), \(\sigma_0 > 0\), and assume that \(\sup_{n \geq 0} \frac{\sigma_0^2 m_{2n}(0)}{(n!)^\alpha} \leq 1\). Note that if \(\sigma \in (0, \frac{\sigma_0}{2a})\) then \(\mathcal{E}_p^i(t)[0] \leq 1 + m_0\), for \(p \geq 2\). Indeed,
\[
\mathcal{E}_p^i(t)[0] \leq m_0 + \sum_{n=1}^{p} \frac{(\sigma_0/2)^n m_{2n}(0)}{(n!)^\alpha} \leq m_0 + \sum_{n=1}^{2^n} = m_0 + 1.
\]
Using identity \(3.39\) and lemmas \(3.38\) and \(3.39\) for any \(\sigma \in (0, 1]\) and \(p \geq 2\), it follows that
\[
\frac{d}{dt} \mathcal{E}_p[F](t) \leq \sum_{i,j=1}^{I} \left( -C_{ij}^1 \left[ \mathcal{P}_p^{ij}[f_i](t) + \mathcal{P}_p^{ji}[f_j](t) \right] + C_{ij}^2 B_{ij} \sigma^{2/\lambda} \left[ \mathcal{P}_p^{ij}[f_i](t) + \mathcal{P}_p^{ji}[f_j](t) \right] \right)
+ \epsilon C_{ij}^2 \left[ \mathcal{E}_p^{ij}[f_i](t) + \mathcal{E}_p^{ji}[f_j](t) \right] + \mathcal{D}_i, \quad t > 0.
\]
Choosing
\[
\epsilon = \min_{1 \leq i,j \leq I} \frac{C_{ij}^1}{12 I e C_{ij}^2 (m_0 + 1)} \quad \text{and} \quad \sigma_1 = \min_{1 \leq i,j \leq I} \frac{C_{ij}^1}{2 C_{ij}^2 B_{ij}} \lambda_i^{1/2},
\]
we conclude that for any choice of \(\sigma \in (0, \sigma_1]\)
\[
\frac{d}{dt} \mathcal{E}_p[F](t) \leq -\min_{1 \leq i,j \leq I} \frac{C_{ij}^1}{4} \sum_{i,j=1}^{I} \left( \mathcal{P}_p^{ij}[f_i](t) + \mathcal{P}_p^{ji}[f_j](t) \right) + I^2 \mathcal{D}
\]
\[
\leq -\min_{1 \leq i,j \leq I} \frac{C_{ij}^1}{4 \sigma^{1/2}} \sum_{i,j=1}^{I} \left( \mathcal{E}_p^{ij}[f_i](t) + \mathcal{E}_p^{ji}[f_j](t) - 2 \epsilon \max \{m_0[f_i], m_0[f_j]\} \right) + I^2 \mathcal{D}, \quad t \in (0, T_p],
\]
and consequently,
\[
\frac{d}{dt} \mathcal{E}_p[F](t) \leq -\min_{1 \leq i,j \leq I} \frac{C_{ij}^1}{4} \left[ \mathcal{E}_p[F](t) - 2 I e m_0 \right] + I^2 \mathcal{D}, \quad t \in (0, T_p].
\]
Then,
\[
\mathcal{E}_p[F](t) \leq 2 I e m_0 + \frac{2 \sigma^{1/2} I^2 \mathcal{D}}{\min_{1 \leq i,j \leq I} C_{ij}^1}, \quad t \in [0, T_p].
\]
Choosing \(\sigma \leq \min\{\frac{2}{\lambda}, \sigma_1\}\) sufficiently small so that
\[
\frac{2 \sigma^{1/2} I^2 \mathcal{D}}{\min_{1 \leq i,j \leq I} C_{ij}^1} \leq I e m_0, \quad \text{that is} \quad \sigma \leq \left( \frac{e m_0 \min_{1 \leq i,j \leq I} C_{ij}^1}{2 I \mathcal{D}} \right)^{2/\lambda} =: \sigma_2,
\]
we conclude that, for all \(p \geq 2\), it holds that
\[
\mathcal{E}_p[F](t) \leq 3 I e m_0 \quad t \in [0, T_p].
\]
Then, by continuity of the polynomial moments and the definition of \(T_p\), we must have \(T_p = \infty\) for all \(p \geq 2\). We conclude sending \(p \to \infty\) that, for \(\sigma \leq \min\{1, \sigma_0/2, \sigma_1, \sigma_2\}\),
\[
\sum_{n=0}^{\infty} \sigma^{2n/\lambda} m_2^n[F](t) \leq 3 I e (m_0 + 1), \quad t \geq 0.
\]
**Step 2.** Let us conclude. Fix \(\rho \in (0, 2], \sigma_0 \in (0, 1], \) and \(A > 1\). Assume that
\[
\sum_{i=1}^{I} \int_{\mathbb{R}^3} \exp (\sigma_0 \langle v \rangle^p) f_i(0, v) dv \leq A,
\]
and set $\alpha = \frac{2}{\rho} \geq 1$. Using Lemma 3.2 item (ii) it follows that for some $\tilde{\sigma} := \tilde{\sigma}(\rho, \sigma_0, A)$
\[
\sup_{n \geq 0} \frac{\tilde{\sigma}^n m_{2n}(0)}{(n!)^{2/\rho}} \leq \frac{1}{I}, \quad \text{and then,} \quad \sup_{n \geq 0} \frac{\tilde{\sigma}^n m_{2n}(0)}{(n!)^{2/\rho}} \leq 1.
\]
Thus, we apply Step 1 to conclude that there exists $\sigma \in (0, \tilde{\sigma}/2]$ (computed is Step 1) such that
\[
\sup_{n \geq 0} \frac{\sigma^{2n/\lambda^3} m_{2n}(t)}{(n!)^{2/\rho}} \leq \sum_{n=0}^{\infty} \frac{\sigma^{2n/\lambda^3} m_{2n}(t)}{(n!)^{2/\rho}} \leq 3 I e (\sigma_0 + 1), \quad t \geq 0.
\]
We deduce from Lemma 3.2 item (i) (applied with $\sigma_0 = \sigma^{2/\lambda^3}$) that, for $t \geq 0$,
\[
\sum_{i=1}^{I} \int_{\mathbb{R}^3} \exp\left(\frac{\sigma^{2n/\lambda^3} (v_i)^{\rho}}{2}\right) f_i(t, v)dv \leq 2 m_0^{1-\rho/2} \left(3 I e (\sigma_0 + 1)\right)^{\rho/2} \leq 6 I e (\sigma_0 + 1),
\]
which is the statement of the theorem.

\[\square\]

4. Lebesgue Integrability Generation and Propagation

After some technical generalization of a coercivity estimate for the collision operator originated in [1] for Maxwell Molecules and refined in [2] for other potentials, it is possible to implement an energy estimate method, see [13] [3], to prove generation and propagation of higher Lebesgue integrability.

4.1. A Uniform Coercive Estimate. Recall that the dynamic of the gas mixture satisfies $F(t) \in U(D_0, E_0)$. Let us make an important observation related to such space $U(D_0, E_0)$. Set $B(R) = \{v \in \mathbb{R}^3 \mid |v| \leq R\}$ and $B_{\delta}(R, r) = \{v \in B(R) \mid |v - v_0| \geq r\}$ for $R > 0$, $r > 0$, and $v_0 \in \mathbb{R}^3$. It follows from the compactness of $U(D_0, E_0)$ in $L^1$ that there exist positive constants $\tilde{R}$ and $\tilde{r}$ depending only on $D_0$ and $E_0$ (independent of $v_0$) such that

\[
G \in U(D_0, E_0) \Rightarrow \chi_{B_{\delta}(\hat{R}, \hat{r})} G \in U(D_0/2, E_0), \quad \forall v_0 \in \mathbb{R}^3,
\]

where $\chi_A$ denotes a characteristic function of the set $A \subset \mathbb{R}^3$. Of course one can take $\tilde{R} > 1 > \tilde{r}$ with no loss of generality.

Proposition 4.1. Fix $1 \leq i \leq I$ and assume that $s_{ij} \in (0, 2)$ and $\lambda_{ij} \in (0, 2]$ for all $1 \leq j \leq I$. Assume also that $D_0$ and $E_0 > 0$, and $G = \{g_j\}_{1 \leq j \leq I} \in U(D_0, E_0)$. Then there exist positive constants $c_i, C_i$ depending only on the parameters of the model, $D_0$ and $E_0$, such that

\[
- \sum_{j=1}^{I} \left( Q_{ij}(f, g_j), f \right)_{L^2} \geq c_i \|\langle v \rangle_i f \|_{H^{3\sigma}}^2 - C_i \|\langle v \rangle_i f \|_{L^2}^2.
\]

Proof. Compute with a suitable $f$
\[
\sum_{j=1}^{I} \left( Q_{ij}(f, g_j), f \right)_{L^2} = \sum_{j=1}^{I} \int_{\mathbb{R}^3} Q_{ij}(f, g_j) f dv
\]
\[
= \sum_{j=1}^{I} \int_{\mathbb{R}^3 \times \mathbb{R}^3} g_j(v_*) f(v) (f(v') - f(v)) B_{ij}(v, v_*; \sigma) d\sigma dv_* dv
\]
\[
= \sum_{j=1}^{I} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} g_j(v_*) f(v) (f(v') - f(v)) |v - v_*|^\lambda_{ij} b_{ij}(\hat{u}, \sigma) d\sigma dv_* dv.
\]

Using the identity
\[
f(v)(f(v') - f(v)) = -\frac{1}{2} (f(v') - f(v))^2 + \frac{1}{2} (f(v')^2 - f(v)^2)
\]
we get that
\[
\sum_{j=1}^{I} (Q_{ij}(f,g), f)_{L^2} = -\frac{1}{2} \sum_{j=1}^{I} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v - v^*|^\lambda g_j(v^*)(f(v') - f(v))^2 b_{ij}(\tilde{u}, \sigma) \sigma dv dv
\]
\[
+ \frac{1}{2} \sum_{j=1}^{I} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v - v^*|^\lambda g_j(v^*)(f(v')^2 - f(v)^2) b_{ij}(\tilde{u}, \sigma) \sigma dv dv
\]
\[
= \frac{1}{2} I_{1,i}(g_j,f) + \frac{1}{2} I_{1,2}(g_j,f)
\]
where the terms are given by
\[
I_{1,i}(g_j,f) := \sum_{j=1}^{I} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v - v^*|^\lambda g_j(v^*)(f(v') - f(v))^2 b_{ij}(\tilde{u}, \sigma) \sigma dv dv,
\]
and
\[
I_{1,2}(g_j,f) := \sum_{j=1}^{I} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v - v^*|^\lambda g_j(v^*)(f(v')^2 - f(v)^2) b_{ij}(\tilde{u}, \sigma) \sigma dv dv.
\]

**Step 1.** This step is based on the scalar case given in [2]. Let us estimate $I_1(g_j,f)$ in a region of high velocities $|v^*| \leq R \ll |v|$. Choose $R$ and $\tilde{r}$ such that (4.1) holds. Let $\phi_{\tilde{R}}$ be a non-negative smooth function not greater than one, which is 1 for $\{|v| \geq 4R\}$ and 0 for $\{|v| \leq 2\tilde{R}\}$. In view of
\[
\frac{\langle v \rangle_i}{4\sqrt{m_i}} \leq |v - v^*| \leq \frac{2\langle v \rangle_i}{\sqrt{m_i}} \quad \text{in the support of} \quad \chi_{B_{\tilde{R}}(v)}(v) \phi_{\tilde{R}}(v),
\]
we have that
\[
4^\lambda |v - v^*|^\lambda g_j(v^*)(f(v') - f(v))^2
\]
\[
\geq \min_{1 \leq j \leq I} m_i \frac{\lambda^2}{4} (g_j \chi_{B_{\tilde{R}}(v)})(v^*)(\langle v \rangle_i \phi_{\tilde{R}}(v))^2 (f(v') - f(v))^2
\]
\[
\geq \min_{1 \leq j \leq I} m_i \frac{\lambda^2}{4} (g_j \chi_{B_{\tilde{R}}(v)})(v^*) \left[ \frac{1}{2} \left( \langle \langle i \rangle \phi_{\tilde{R}}(v') \rangle \right)^2 - \left( \langle \langle i \rangle \phi_{\tilde{R}}(v) \rangle \right)^2 f(v') \right]^2.
\]
Observe that
\[
|v' - v| \leq 2|v - v^*| \sin \frac{\theta}{2}, \quad \text{and}
\]
\[
\frac{|v - v^*|}{\sqrt{2}} \leq |v' - v^*| \leq |v - v^* + \tau(v' - v)| \leq 6|v - v^*| \quad \text{for} \quad \cos \theta \geq 0,
\]
where the latter follow after using that $u^+ u^- = 0$ and $|u|^2 = |u^+|^2 + |u^-|^2$ in the definition of $v'$. Using the mean value theorem there exists $\tau \in (0,1)$ such that the inequalities hold
\[
\left| \langle \langle i \rangle \phi_{\tilde{R}}(v') \rangle - \langle \langle i \rangle \phi_{\tilde{R}}(v) \rangle \right| \lesssim \langle v + \tau(v' - v) \rangle_i \frac{\lambda^2}{4} |v - v^*| \sin \frac{\theta}{2}
\]
\[
\lesssim \frac{1}{\sqrt{m_i}} (v^*_i)^{1-\frac{\lambda^2}{4}} \langle v^* - v^*_i \rangle_i^{\frac{\lambda^2}{4} - 1} |v - v^*| \sin \frac{\theta}{2} \lesssim \frac{1}{\sqrt{m_i}} (v^*_i)^{1-\frac{\lambda^2}{4}} \langle v^* - v^*_i \rangle_i^{\frac{\lambda^2}{4} - 1} \sin \frac{\theta}{2}.
\]
In the second inequality we used the uniform equivalences in (4.2). Since
\[
\langle v^*_i \rangle_i \leq \left( \max_{1 \leq j \leq I} \frac{m_i}{m_j} \right) \langle v^*_j \rangle_j
\]
it implies that
\[
\left| \langle \langle i \rangle \phi_{\tilde{R}}(v') \rangle - \langle \langle i \rangle \phi_{\tilde{R}}(v) \rangle \right| \lesssim \left( \max_{1 \leq j \leq I} m_i \right) \langle v^*_j \rangle_i^{1-\frac{\lambda^2}{4}} \langle v^* - v^*_i \rangle_i^{\frac{\lambda^2}{4} - 1} \sin \frac{\theta}{2}.\]
One concludes that
\[ I_{i,1}(g_j, f) \]
\[ \geq \min_{1 \leq j \leq l} m_i^{-\frac{\lambda_j}{2}} \max_{1 \leq j \leq l} \int_{\mathbb{R}^2 \times \mathbb{R}^2 \times [0, \frac{\pi}{2}]} \left( g_j(x_B(\tilde{r})) \right)(v_*) \left[ \frac{1}{2} \left( (\langle \sigma \rangle_{j}^2 \phi_{\tilde{R}} f)(v') - (\langle \sigma \rangle_{j}^2 \phi_{\tilde{R}} f)(v) \right)^2 \right] \]
\[ - \left( \max_{1 \leq j \leq l} m_j^{-\frac{\lambda_j}{2}} \right) \left( v_* \right)_{j} \left( v' \right)_{j}^2 \sin \frac{\theta}{2} f(v') \right]^2 \right] b_{ij}(\cos \theta) d\theta dv_* \text{d}v. \]
The second term in the right side, due to the second order cancellation, is controlled by the expression
\[ C \min_{1 \leq j \leq l} m_j^{-\frac{\lambda_j}{2}} \max_{1 \leq j \leq l} \max_{1 \leq j \leq l} \left\{ \| \theta^2 b_{ij} \|_{L^1} \right\} E_0 \| \langle \sigma \rangle_{j} f \|_{L^2}. \]
As for the first term, it holds, thanks to Lemma 5.7
\[ \frac{1}{2} \min_{1 \leq j \leq l} m_j^{-\frac{\lambda_j}{2}} \sum_{j=1}^{l} \int_{\mathbb{R}^2 \times \mathbb{R}^2 \times [0, \frac{\pi}{2}]} \left( g_j(x_B(\tilde{r})) \right)(v_*) \times \left( \left( \langle \sigma \rangle_{j}^2 \phi_{\tilde{R}} f)(v') - (\langle \sigma \rangle_{j}^2 \phi_{\tilde{R}} f)(v) \right)^2 \right) b_{ij}(\cos \theta) d\theta dv_* \text{d}v \]
\[ \geq c_0 \min_{1 \leq j \leq l} m_j^{-\frac{\lambda_j}{2}} \min_{1 \leq j \leq l} K^{ij}(s, D_0, E_0) \int_{|\xi| \geq 1} \left\{ |\xi|^{s_0} F((\langle \sigma \rangle_{j} f)(\xi)) \right\}^2 d\xi \]
\[ \geq c_0 \min_{1 \leq j \leq l} m_j^{-\frac{\lambda_j}{2}} \min_{1 \leq j \leq l} K^{ij}(s, D_0, E_0) \| \langle \sigma \rangle_{j} f \|_{H^{s_0}}, \]
where \( c_0 = \frac{1}{4\pi}. \) Thus, overall
\[ (4.3) \quad I_{i,1}(g_j, f) \geq c_0 \min_{1 \leq j \leq l} m_j^{-\frac{\lambda_j}{2}} \min_{1 \leq j \leq l} K^{ij}(s, D_0, E_0) \| \langle \sigma \rangle_{j} f \|_{H^{s_0}}^2 \]
\[ - C \min_{1 \leq j \leq l} m_j^{-\frac{\lambda_j}{2}} \max_{1 \leq j \leq l} \max_{1 \leq j \leq l} \left\{ \| \theta^2 b_{ij} \|_{L^1} \right\} E_0 \| \langle \sigma \rangle_{j} f \|_{L^2}. \]
Now we estimate \( I_1 \) in a region where \( v_* \) and \( v \) have comparable sizes, yet, separated. For the set \( B(4\tilde{R}) \) we take the finite covering
\[ B(4\tilde{R}) \subset \bigcup_{v_* \in B(4\tilde{R})} A_k, \quad A_k = \left\{ v \in \mathbb{R}^3; |v - v_*| \leq \frac{\tilde{R}}{4} \right\}. \]
For each \( A_k \) choose a non-negative smooth function \( \phi_{A_k} \) which is 1 on \( A_k \) and 0 on \( \{|v - v_*| \geq \frac{\tilde{R}}{2}\} \).
Note that
\[ \frac{3}{2} \leq |v - v_*| \leq 6\tilde{R} \quad \text{on the support of } \chi_{B_{v_*}(\tilde{R}, \tilde{r})}(v_*) \phi_{A_k}(v). \]
Then, recalling that \( m_i < 1 \) and \( \tilde{R} \geq 1, \)
\[ |v - v_*|^{\lambda_i} g_j(v_*)(f(v') - f(v))^2 \geq \tilde{r}^{\lambda_i} \left( g_j \chi_{B_{v_*}(\tilde{R}, \tilde{r})}(v_*) \right)_{\phi_{A_k}}(f(v'))^2 \]
\[ \geq \left( \min_{1 \leq j \leq l} \tilde{r}^{\lambda_i} \tilde{R}^{\lambda_i} \right) \left( g_j \chi_{B_{v_*}(\tilde{R}, \tilde{r})}(v_*) \right)_{\phi_{A_k}} \left[ \frac{1}{2} \left( (\langle \sigma \rangle_{i}^2 \phi_{\tilde{R}} f)(v') - (\langle \sigma \rangle_{i}^2 \phi_{\tilde{R}} f)(v) \right)^2 \right] \]
\[ - \left( (\langle \sigma \rangle_{i}^2 \phi_{\tilde{R}} f)(v') - (\langle \sigma \rangle_{i}^2 \phi_{\tilde{R}} f)(v) \right)^2 f(v')^2 \right]. \]
Again, using the mean value theorem it holds for \( |v_*| \leq \tilde{R} \) that
\[ \left| (\langle \sigma \rangle_{i}^2 \phi_{\tilde{R}} f)(v') - (\langle \sigma \rangle_{i}^2 \phi_{\tilde{R}} f)(v) \right| \leq \frac{\tilde{R}}{\sqrt{m_i}} |(v')_i \sin \theta|^2 \]
\[ \leq \frac{\tilde{R}}{\sqrt{m_i}} |v'| \sin \theta \]
\[ \leq \frac{\tilde{R}}{\sqrt{m_i}} |v| \sin \theta \]
Consequently, we obtain in a similar fashion as before that
\[
I_{1}(g_{j}, f) \geq \min_{1 \leq j \leq l} \frac{F_{j}^{\lambda}}{R^{\lambda}} \left[ \min_{1 \leq j \leq l} K^{ij}(s_{ij}, D_{0}, E_{0}) \| \langle v \rangle_{v}^{i} \phi_{A_{k}} f \|_{H^{rac{2}{p}}}^{2} \right]
\]
\[
- C \left( \max_{1 \leq j \leq l} R \delta_{ij} \| \theta_{i} b_{j} \|_{L^{2}} \right) E_{0} \| \langle v \rangle_{v}^{rac{2}{p}} f \|_{L^{2}}^{2} \right].
\]

Now observe that \( \phi_{R}^{2} + \sum_{k=1}^{N} \phi_{A_{k}}^{2} \geq 1 \) where \( N \sim (R/\theta)^{3} \) is the number of sets in the covering. Consequently,
\[
(1 + N)I_{1}(g_{j}, f) \geq I(g_{j}, f) + NI_{1}(g_{j}, f) \geq \tilde{c}_{i} \| \langle v \rangle^{rac{2}{p}}_{v} \phi_{R} f \|_{H^{rac{2}{p}}}^{2} - \tilde{C}_{i} \| \langle v \rangle^{rac{2}{p}}_{v} f \|_{L^{2}}^{2}
\]
\[
+ c_{i} \sum_{k=1}^{N} \| \langle v \rangle^{rac{2}{p}}_{v} \phi_{A_{k}} f \|_{H^{rac{2}{p}}}^{2} - NC_{i} \| \langle v \rangle^{rac{2}{p}}_{v} f \|_{L^{2}}^{2}
\]
\[
\geq \min \{ \tilde{c}_{i}, c_{i} \} \| \langle v \rangle^{rac{2}{p}}_{v} f \|_{H^{rac{2}{p}}}^{2} - (\tilde{C}_{i} + NC_{i}) \| \langle v \rangle^{rac{2}{p}}_{v} f \|_{L^{2}}^{2},
\]
where we used (4.3) for the first term and (4.4) for the \( N \) remaining.

**Step 2.** Let us estimate \( I_{2}(g_{j}, f) \). To this purpose use the Cancellation Lemma 5.1 specifically Remark 5.3 that gives
\[
I_{2}(g_{j}, f) = \sum_{j=1}^{l} \int_{\mathbb{R}^{3} \times \mathbb{R}^{3} \times S^{2}} |v - v_{i}|^{\lambda_{ij}} g_{j}((v_{j})^{2} - f(v_{j})^{2})b_{ij}(\tilde{u}, \sigma) d\sigma dv_{i} dv
\]
\[
\leq |S| \sum_{j=1}^{l} \int_{\mathbb{R}^{3} \times \mathbb{R}^{3}} \int_{0}^{\frac{\pi}{2}} |v - v_{i}|^{\lambda_{ij}} g_{j}(v_{i})^{2} \left[ \frac{1}{\beta(\cos \theta)^{3 + \lambda_{ij}}} - 1 \right] b_{ij}(\cos \theta) \sin \theta d\theta dv_{i} dv
\]
\[
= \sum_{j=1}^{l} \int_{\mathbb{R}^{3} \times \mathbb{R}^{3}} \int_{0}^{\frac{\pi}{2}} |v - v_{i}|^{\lambda_{ij}} g_{j}(v_{i})^{2} f_{i}(v_{i}) b_{i}(\cos \theta) d\theta dv_{i} dv
\]
where we defined
\[
b_{i}(\cos \theta) := |S| \left[ \frac{1}{\beta(\cos \theta)^{3 + \lambda_{ij}}} - 1 \right] \sin \theta b_{ij}(\cos \theta) \in L^{1}(0, \frac{\pi}{2}].
\]
In addition, recall that
\[
|v - v_{i}|^{\lambda_{ij}} \leq (m_{i}m_{j})^{-\lambda_{ij}} \langle v \rangle^{\lambda_{ij}}_{v} \langle v \rangle^{\lambda_{ij}}_{v},
\]
therefore, since \( \lambda_{ij} \leq 2 \),
\[
I_{2}(g_{j}, f) \leq \max_{1 \leq j \leq l} (m_{i}m_{j})^{-\lambda_{ij}} \| b_{i} \|_{L^{1}} \| \langle \cdot \rangle^{\lambda_{ij}}_{v} g_{j} \|_{L^{1}} \| \langle \cdot \rangle^{\lambda_{ij}}_{v} f_{i} \|_{L^{2}}^{2}
\]
\[
\leq \max_{1 \leq j \leq l} (m_{i}m_{j})^{-\lambda_{ij}} \| b_{i} \|_{L^{1}} E_{0} \| \langle \cdot \rangle^{\lambda_{ij}}_{v} f_{i} \|_{L^{2}}^{2} =: \tilde{C}_{i} \| \langle \cdot \rangle^{\lambda_{ij}}_{v} f_{i} \|_{L^{2}}^{2}.
\]
This proves the proposition with \( c_{i} := \min \{ \tilde{c}_{i}, c_{i} \} \) and \( C_{i} := \tilde{C}_{i} + NC_{i} \).

The following corollary of Proposition 4.1 extends the \( L^{2} \) frame to \( L^{p} \).
where

Consequently,

and

Given that $\lambda_{ij}$ depending on the parameters of the model, $D_0$, $E_0$, and $p$.

**Proof.** This argument is based on [3] Lemma 1. Using the weak formulation, it holds for any fixed $1 \leq i \leq I$ that

$$
\sum_{j=1}^{I} \int_{\mathbb{R}^3} Q_{ij}(f, g_j)(v)f(v)^{p-1} dv
$$

$$
= \sum_{j=1}^{I} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} g_j(v_i)f(v)(f(v'))^{p-1} - f(v)^{p-1})|v - v_*|^{\lambda_{ij}}b_{ij}(\vec{u} \cdot \sigma)d\sigma dv_* dv.
$$

Using that, with notation $p' = \frac{p}{p-1}$,

$$
f(v)[f(v')^{p-1} - f(v)^{p-1}] = f(v)\left[\left(\frac{f(v')^{\frac{p}{p'}}}{f(v)^{\frac{p}{p'}}}\right)^{\frac{p}{p'}} - 1\right]
$$

$$
\leq p'f(v)^{p'}\left[\frac{f(v')^{\frac{p}{p'}}}{f(v)^{\frac{p}{p'}}} - 1\right] - \frac{1}{\max\{p', p\}} f(v)^p\left[\frac{f(v')^{\frac{p}{p'}}}{f(v)^{\frac{p}{p'}}} - 1\right]^2
$$

$$
\leq \left[\frac{1}{\max\{p', p\}}\right] f(v)^p - f(v)^{p'} - \frac{1}{\max\{p', p\}} \left[ f(v')^{\frac{p}{p'}} - f(v)^{\frac{p}{p'}} \right]^2,
$$

we conclude that

$$
\sum_{j=1}^{I} \int_{\mathbb{R}^3} Q_{ij}(f, g_j)(v)f(v)^{p-1} dv \leq J_{1,i}(g_j, f) - J_{2,i}(g_j, f),
$$

where

$$
J_{1,i}(g_j, f) := \frac{1}{p'}\sum_{j=1}^{I} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} g_j(v_i)\left[ f(v')^{p} - f(v)^{p} \right]|v - v_*|^{\lambda_{ij}}b_{ij}(\vec{u} \cdot \sigma)d\sigma dv_* dv,
$$

and

$$
J_{2,i}(g_j, f) := \frac{1}{\max\{p, p'\}}\sum_{j=1}^{I} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} g_j(v_i)\left[ f(v')^{\frac{p}{p'}} - f(v)^{\frac{p}{p'}} \right]^2|v - v_*|^{\lambda_{ij}}b_{ij}(\vec{u} \cdot \sigma)d\sigma dv_* dv.
$$

For the term $J_1(g_j, f)$ we use the Cancellation Lemma [5,1] to obtain that

$$
p'J_{1,i}(g_j, f) = |S| \sum_{j=1}^{I} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{0}^{\pi} |v - v_*|^{\lambda_{ij}}g_j(v_i)f(v)^p
$$

$$
\times \left[ \frac{1}{\beta(\cos \theta)^{1+\lambda_{ij}}} - 1 \right] \sin \theta b_{ij}(\cos \theta)d\theta dv_* dv
$$

$$
= \sum_{j=1}^{I} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{0}^{\pi} |v - v_*|^{\lambda_{ij}}g_j(v_i)f(v)^p b_i(\cos \theta)d\theta dv_* dv,
$$

where recall that the scattering $b_i$ was defined as

$$
b_i(\cos \theta) = |S| \sin \theta \left[ \frac{1}{\beta(\cos \theta)^{1+\lambda_{ij}}} - 1 \right]b_{ij}(\cos \theta) \in L^1([0, \pi]).
$$

Consequently,

$$
p'J_{1,i}(g_j, f) \leq \max_{1 \leq j \leq I} (m_i, m_j) - \frac{\lambda_{ij}}{p'} \left\| b_i \right\|_{L^1} \sum_{j=1}^{I} \left\| \frac{1}{\beta} g_j \right\|_{L^1} \left\| \frac{\lambda_{ij}}{p'} \right\|_{L^p} \leq C' \left\| \frac{\lambda_{ij}}{p'} \right\|_{L^p}.
$$

Given that $\lambda_{ij} \leq 2$, we can take

$$
C' = \max_{1 \leq j \leq I} (m_i, m_j) - \frac{\lambda_{ij}}{p'} \left\| b_i \right\|_{L^1} E_0.
$$
Focusing in the term \( J_{2,i}(g_j, f) \) now, we use the coercive estimate (4.5)

\[
\max \{ p, p' \} J_{2,i}(g_j, f) \geq \tilde{c}_i \| \langle v \rangle^2 f \|_{H^{-\frac{2}{p}}}^2 - \tilde{C}_i \| \langle v \rangle^\frac{p}{p'} f \|_{L^p}^p.
\]

The result follows from (4.6), (4.7), (4.8) with \( c_i = \frac{\tilde{c}_i}{\max \{ p, p' \} \tilde{C}_i} \) and \( C_i = \frac{\tilde{C}_i}{\max \{ p, p' \}} \).

4.2. Generation and propagation of \( L^p \) integrability. In this section, we study the \( L^p \) integrability generation and propagation property for solutions of the monatomic Boltzmann system in the case \( 1 < p < \infty \).

**Proof of Theorem 2.5.** Take \( 1 \leq i \leq I \) and multiplying the \( i \)th row of the Boltzmann equation (2.13) by \( p f_i \). Integrating on \( v \in \mathbb{R}^3 \) and using Corollary 4.2 we obtain that

\[
\frac{d}{dt} \| f_i \|_{L^p} \leq \sum_{j=1}^I \int_{\mathbb{R}^3} Q_{ij}(f_j, f_i)(v) f_i(v) p^{-1} dv \leq -c_i \| \langle v \rangle^\frac{p}{p'} f \|_{H^{-\frac{2}{p}}}^2 + C_i \| \langle v \rangle^\frac{p}{p'} f \|_{L^p}^p
\]

with the corresponding constants \( c_i \) and \( C_i \) given in the statement of the corollary. We use below the weighted interpolation a couple of times

\[
\| \langle v \rangle^\theta g \|_{L^p} \leq \| \langle v \rangle^{\theta_1} g \|_{L^p}^{\theta_1} \| \langle v \rangle^{\theta_2} g \|_{L^p}^{1-\theta_1}
\]

with

\[
\frac{1}{\theta} = \frac{\theta_1}{r_1} + \frac{\theta_2}{r_2}, \quad a = \theta a_1 + (1-\theta) a_2, \quad \theta \in (0, 1).
\]

Now, by Sobolev inequality it follows that

\[
\| \langle v \rangle^\theta f \|_{L^p} \leq \| \langle v \rangle^{\theta_1} f \|_{L^{p_1}}^{\theta_1} \| \langle v \rangle^{\theta_2} f \|_{L^{p_2}}^{1-\theta_1},
\]

where \( \frac{1}{p_1} = (1-\theta_2) + \frac{\theta_2 \theta_3 - \theta_2}{\theta_3} \), namely, \( \theta_3 := \frac{\theta_3 - \theta_2}{\theta_3} \in (0, 1) \), and \( \alpha_1 \leq \frac{\theta_1 \theta_3 - \theta_2}{\theta_3} \). Then, using Young’s inequality it holds that

\[
\frac{d}{dt} \| f_i \|_{L^p}^p + \tilde{c}_i \| \langle v \rangle^\frac{p}{p'} f_i \|_{L^\frac{p_1}{p_2}}^\theta \| f_i \|_{L^\frac{p_2}{p_3}}^{1-\theta} \leq \tilde{C}_i \| \langle v \rangle^\frac{p}{p'} f_i \|_{L^\frac{p_1}{p_2}}^{\theta_1} \| f_i \|_{L^\frac{p_2}{p_3}}^{\theta_2}.
\]

Using the same interpolation, with no weights, it holds then for the same \( \theta_3 \) that

\[
\frac{d}{dt} \| f_i \|_{L^p}^p + \tilde{c}_i \| \langle v \rangle^\frac{p}{p'} f_i \|_{L^\frac{p_1}{p_2}}^\theta \| f_i \|_{L^\frac{p_2}{p_3}}^{1-\theta} \leq \tilde{C}_i \| \langle v \rangle^\frac{p}{p'} f_i \|_{L^\frac{p_1}{p_2}}^{\theta_1} \| f_i \|_{L^\frac{p_2}{p_3}}^{\theta_2} \left( 1 + t^{-\frac{\alpha_3}{\theta_3 - 1}} \right),
\]

for \( t > 0 \).

For the latter inequality in the right side we used Theorem 2.3. Thus, introducing \( X := X(t) = \| f_i(t) \|_{L^p}^p \) one has, after invoking conservation of mass, that

\[
\frac{dX}{dt} + a X^{\frac{1}{p}} \leq B_{t_*} := C_i \| f_i \|_{L^p}^p \left( 1 + t^{-\frac{\alpha_3}{\theta_3 - 1}} \right), \quad t > t_*
\]

where \( a := \frac{\tilde{c}_i}{\max \{ p, p' \} \tilde{C}_i} \). Therefore, using [6, Lemma 18] in the interval \( t > t_* \) it follows that

\[
\| f_i(t) \|_{L^p}^p = X(t) \leq \left( \frac{B_{t_*}}{a} \right)^{\theta_1} + \left( \frac{\theta_2}{(1-\theta_3) a} \right)^{\theta_2} (t - t_*)^{-\frac{\alpha_3}{\theta_3 - 1}} := K_{t_*}, \quad t > t_*
\]

This proves the estimate (2.13) when particularised to \( t_* = \frac{1}{2} \) with \( C_{t_0} = K_{t_0/2}^{\frac{1}{2}} \). Next, take an
integer $n \geq 2$ such that $2n \geq \frac{q}{p}$ and assume $\|\langle \cdot \rangle^2 f_i(t)\|_{L^2} < \infty$. Then, Theorem 2.3 implies that

$$\sup_{t \geq 0} \|\langle \cdot \rangle^{2n} f_i(t)\|_{L^2} \leq \max \left\{ m_{2n}(0), C_2 \right\}.$$ 

Consequently, estimate (1.11) changes to

$$\frac{dX}{dt} + aX \leq B := C_1 \max \left\{ m_{2n}(0), C_2 \right\} \quad t > 0,$$

which leads to

$$X(t) \leq \max \left\{ \|f_{i,0}\|_{L^{\infty}}, \left( \frac{B}{a} \right)^{\frac{1}{\theta_i}} \right\}.$$ 

This proves estimate (2.16) since $\frac{a}{p} \leq \frac{a}{q_i} - 2 = \frac{3}{\theta_i}$. \hfill \Box

**Proof of Corollary 2.6.** In one hand using estimate (1.9) with $p = 2$, we get that

$$\frac{d}{dt} \|f_i\|_{L^2}^2 + c_i \|\langle v \rangle f_i\|_{H^1}^2 \leq C_i \|\langle v \rangle f_i\|_{L^2}^2.$$ 

In the other hand, the interpolation (1.10) with $p = 2$ give us

$$\|\langle \cdot \rangle^2 f_i\|_{L^2} \leq \|\langle \cdot \rangle^{\frac{q}{p}} f_i\|_{L^1}^{1-\frac{q}{p}} \|\langle \cdot \rangle^{\frac{q}{2p}} f_i\|_{L^{2q}}^{\frac{q}{p}} \leq \|\langle \cdot \rangle^{\frac{q}{2p}} f_i\|_{L^1}^{1-\frac{q}{p}} \|\langle \cdot \rangle^{\frac{q}{2p}} f_i\|_{L^{2q}}^{\frac{q}{p}},$$

with $q_s, \theta_s, \alpha_i$ previously defined. Consequently,

$$\frac{d}{dt} \|f_i\|_{L^2}^2 + c_i \|\langle v \rangle f_i\|_{H^1}^2 \leq \tilde{C}_i \|\langle \cdot \rangle^{\frac{q}{2p}} f_i\|_{L^2}^2,$$

which yields after time integration in the interval $(t_0, t)$

$$\tilde{c}_i \int_{t_0}^{t} \|\langle v \rangle f_i(\tau)\|_{H^1}^2 d\tau \leq \|f_i(t_0)\|_{L^2}^2 + \tilde{C}_i \int_{t_0}^{t} \|\langle \cdot \rangle^{\frac{q}{2p}} f_i(\tau)\|_{L^2}^2 d\tau \leq \tilde{C}_{t_0}(1 + t).$$

In the last inequality we used theorems 2.3 and 2.5 so that $\tilde{C}_{t_0} \lesssim t_0^{-\beta} + 1$ for some $\beta > 0$. This is the first statement of the corollary with $C_{t_0} = \frac{\tilde{C}_{t_0}}{c_i}$. The second statement is also clear from this last estimate using the propagation part of theorems 2.3 and 2.5. \hfill \Box

4.3. $L^\infty$ Theory. Let us study in this section the particular case of $p = \infty$. We start with a coercive estimate for the levels of levels of each component. We set in the sequel

$$f_K^+(v) := (f(v) - K)1_{\{f_K \geq 0\}}.$$

**Lemma 4.3.** Take $G \in U(D_0, E_0)$, $f$ sufficiently smooth, $\lambda_i \in (0, 2]$ and $s_{ij} \in (0, 2)$. Then,

$$\sum_{j=1}^{r} \int_{\mathbb{R}^3} Q_{ij}(f, g_j)(v) f_K^+(v) dv \leq -c_i \|\langle v \rangle f_K^+ \|_{H^{-\infty}}^2 + \tilde{C}_i \left( \|\langle v \rangle f_K\|_{L^1}^2 + K \|\langle \cdot \rangle f_K^+\|_{L^1} \right), \quad K \geq 0,$$

for positive constant

$$\tilde{C}_i \lesssim C_i + \max_{1 \leq j \leq 1} \left[ \left( m_i m_j \right)^{-\frac{\lambda_{ij}}{2}} \right] \|b_i\|_{L^{1}, E_0},$$

where the constants $c_i$ and $C_i$ are defined in Proposition 4.7.

**Proof.** Set $K \geq 0$, define $f_K(v) := f(v) - K$, and recall that $f_K^+(v) = f_K(v)1_{\{f_K \geq 0\}}$. Note that

$$f(v)[f_K^+(v') - f_K^+(v)] = f_K(v)[f_K^+(v') - f_K^+(v)] + K[f_K^+(v') - f_K^+(v)]$$

$$= f_K(v)(1_{\{f_K \geq 0\}} + 1_{\{f_K < 0\}}) \left[ f_K^+(v') - f_K^+(v) \right] + K[f_K^+(v') - f_K^+(v)]$$

$$\leq f_K^+(v) \left[ f_K^+(v') - f_K^+(v) \right] + K[f_K^+(v') - f_K^+(v)],$$

where we used in the last step that

$$f_K^+(v)1_{\{f_K < 0\}} (f_K^+(v') - f_K^+(v)) = f_K(v)1_{\{f_K < 0\}} f_K^+(v') \leq 0.$$
Therefore, it holds that
\[
\begin{align*}
\sum_{j=1}^{J} J_{i} &= -\frac{1}{2} \sum_{j=1}^{J} \int_{R^{3} \times R^{3} \times S^{2}} g_{j}(v_{*}) f(v) \left[ f_{K}^{+}(v') - f_{K}^{+}(v) \right] \left[ v - v_{*} \right]^{\lambda_{ij}} b_{ij} \hat{\nu} \cdot \hat{\sigma} d\sigma dv_{*} dv \\
&= J_{1} + J_{2} + J_{3},
\end{align*}
\]
where
\[
J_{1} = -\frac{1}{2} \sum_{j=1}^{J} \int_{R^{3} \times R^{3} \times S^{2}} g_{j}(v_{*}) \left[ f_{K}^{+}(v') - f_{K}^{+}(v) \right]^{2} \left[ v - v_{*} \right]^{\lambda_{ij}} b_{ij} \hat{\nu} \cdot \hat{\sigma} d\sigma dv_{*} dv,
\]
\[
J_{2} = \frac{1}{2} \sum_{j=1}^{J} \int_{R^{3} \times R^{3} \times S^{2}} g_{j}(v_{*}) \left[ \left( f_{K}^{+}(v') \right)^{2} - \left( f_{K}^{+}(v) \right)^{2} \right] \left[ v - v_{*} \right]^{\lambda_{ij}} b_{ij} \hat{\nu} \cdot \hat{\sigma} d\sigma dv_{*} dv,
\]
\[
J_{3} = K \sum_{j=1}^{J} \int_{R^{3} \times R^{3} \times S^{2}} g_{j}(v_{*}) \left[ f_{K}^{+}(v') - f_{K}^{+}(v) \right] \left[ v - v_{*} \right]^{\lambda_{ij}} b_{ij} \hat{\nu} \cdot \hat{\sigma} d\sigma dv_{*} dv.
\]
Using the coercive estimate given in the proof of Proposition 4.1 one deduces that
\[
J_{1} \leq -c_{i} \| \langle \cdot \rangle^{\frac{s+1}{d}} \left[ f_{K}^{+} \right]^{2} \|_{L^{\infty}} + C_{i} \| \langle \cdot \rangle^{\frac{s}{d}} f_{K}^{+} \|_{L^{2}}^{2}.
\]
Furthermore, using the Cancellation Lemma [5.1] for the terms $J_1$ and $J_2$, it follows that
\[
J_{2} \leq \max_{1 \leq j \leq l} \left( \left( m_{i} m_{j} \right)^{-\frac{s}{d}} \| b_{i} \|_{L^{1}} \| \langle \cdot \rangle^{\frac{s}{d}} f_{K}^{+} \|_{L^{2}}^{2} \right) \quad \text{and}
\]
\[
J_{3} \leq K \max_{1 \leq j \leq l} \left( \left( m_{i} m_{j} \right)^{-\frac{s}{d}} \| b_{i} \|_{L^{1}} \| \langle \cdot \rangle^{\frac{s}{d}} f_{K}^{+} \|_{L^{1}} \right).
\]
The result follows adding the estimates. \(\square\)

**Lemma 4.4.** Fix dimension $d \geq 2$, $s \in (0,2)$ and $1 - \frac{d}{s} \leq \alpha \leq 1$. Then,
\[
\| \varphi \|_{L^{2}}^{2} \leq \| \varphi \|_{L^{2}}^{2(1-\frac{d}{s})} \| \varphi \|_{H^{\frac{d}{s}}}^{\frac{d}{s}},
\]
where
\[
q := q(s, \alpha) = 2 \left( 1 + \frac{s}{d} \right) > 2, \quad \theta := \theta(s, \alpha) = \frac{d}{s + d\alpha} \in (0,1].
\]

**Proof.** Note that for $2 \leq q \leq p_{s} = \frac{2}{1 - s/d}$ Lebesgue’s interpolation and Sobolev inequality gives that
\[
\| \varphi \|_{L^{q}}^{\theta q} \leq \| \varphi \|_{L^{2}}^{(1-\theta)q} \| \varphi \|_{L^{p_{s}}}^{\theta p_{s}} \leq \| \varphi \|_{L^{2}}^{(1-\theta)q} \| \varphi \|_{H^{\frac{d}{s}}}^{\theta p_{s}}, \quad \theta = \frac{d}{s} \left( 1 - \frac{2}{q} \right).
\]
Choosing $\theta q \alpha = 2$ we arrive to
\[
q = 2 \left( 1 + \frac{s}{d\alpha} \right) \in (2,p_{s}], \quad \text{and consequently} \quad \theta = \frac{d}{s + d\alpha} > 0.
\]
The proof follows by writing $(1 - \theta)q \alpha = 2\left(\frac{1}{q} - 1\right)$. \(\square\)
Proof of Theorem 2.7. We follow the spirit of the argument used in the proof of [3] Theorem 2 after fixing 1 ≤ i ≤ T. For solution of the Boltzmann system we apply Lemma 4.3 with \( \mathbb{G} = \mathbb{F} \) and \( f = f_i \) to conclude that

\[
\frac{1}{2} \frac{d}{dt} \| f K \|_{L^2}^2 + c_i \| \langle v \rangle_i^2 f K \|_{H^\alpha f}^2 \leq \tilde{C} \left( \| \langle \cdot \rangle_i \| f \|_{L^2}^2 + K \| \langle \cdot \rangle_i \| f \|_{L^1} \right).
\]

Let us introduce the levels and times (\( K > 0 \) and \( t_\ast > 0 \))

\[
K_k := K \left( 1 - \frac{1}{2^k} \right), \quad t_k = t_\ast \left( 1 - \frac{1}{2^{k+1}} \right), \quad k = 0, 1, 2, \cdots.
\]

Here \( K > 0 \) will be chosen later sufficiently large. Define also the energy functional (with notation \( f_k = f_{K_k} \))

\[
W_k := \frac{1}{2} \sup_{\tau \in [t_k, T]} \| f_k(\tau) \|_{L^2}^2 + c_i \int_{t_k}^T \| \langle \cdot \rangle_i \| f_k(\tau) \|_{H^\alpha f}^2 \, d\tau, \quad T > t_\ast > 0.
\]

Integrating estimate (4.12) we deduce that for \( t_k-1 \leq \tau \leq t_k \)

\[
W_k \leq \frac{1}{2} \| f_k(\tau) \|_{L^2}^2 + \tilde{C} \int_{t_k-1}^T \left( \| \langle \cdot \rangle_i \| f_k \|_{L^2}^2 + K \| \langle \cdot \rangle_i \| f_k \|_{L^1} \right),
\]

and taking the mean over \( \tau \in [t_k-1, t_k] \) (noticing that \( t_k - t_{k-1} = \frac{1}{2^k} \)), it follows that

\[
W_k \leq \left( \frac{2^k}{L^k} + \tilde{C} \right) \int_{t_{k-1}}^T \left( \| \langle \cdot \rangle_i \| f_k \|_{L^2}^2 + K \| \langle \cdot \rangle_i \| f_k \|_{L^1} \right)
\]

\[
\leq 2^k \left( \frac{2^k}{L^k} + \tilde{C} \right) \int_{t_{k-1}}^T \| \langle \cdot \rangle_i \| f_{k-1} \|_{L^2} \right).
\]

where in the last step we used the key observation that in the set \( \{ f_k \geq 0 \} \) one has that \( f_{k-1} \geq 2^{-k} K \), thus

\[
K \| \langle \cdot \rangle_i \| f_k \|_{L^1} \leq 2^k \| \langle \cdot \rangle_i \| f_{k-1} \|_{L^2} \right).
\]

In fact, keep in mind that for any \( \beta > 0 \)

\[
1_{\{ f_{k} \geq 0 \}} \leq \left( \frac{2^k}{K} f_{k-1} \right)^{\beta}.
\]

Now, interpolation give us that

\[
\| \langle \cdot \rangle_i \| f_{k-1} \|_{L^2} \leq \| \langle \cdot \rangle \| f_{k-1} \|_{L^2}^{\xi} \| f_{k-1} \|_{L^\xi}^{\theta}, \quad \xi > 2, \quad \theta = \frac{\xi}{2(\xi - 1)}.
\]

Moments are controlled by Theorem 2.3 so that

\[
\| \langle \cdot \rangle_i \| f \|_{L^1} \leq C \left( 1 + t_\ast \right)^{\frac{\xi}{\xi - 1}}
\]

for a constant \( C := C(E_0, \xi) > 0 \). Also, using (4.14) with \( \beta = q - \xi > 0 \) with \( q > 2 \) given in Lemma 4.3 that is \( q = 2 \left( 1 + \frac{\xi}{\xi - 1} \right) \), it holds that

\[
\| f_{k-1} \|_{L^\xi} \leq \frac{2^k (q - 1)}{K^{q - 1}} \| f_{k-1} \|_{L^2}^{\frac{q}{q - 1}}.
\]

In this way, we are led from (4.13) and some algebra to the estimate

\[
W_k \leq \frac{2^k (q - 1)}{K^{q - 1}} \tilde{C} \left( 1 + t_\ast \right)^{\frac{\xi}{\xi - 1}} \int_{t_{k-1}}^T \| f_{k-1} \|_{L^\xi}^{\frac{q}{q - 1}}, \quad 2 < \xi < q.
\]
for a constant \( \hat{C} \) depending on \((E_0, \xi)\) (so the limit \( \xi \to 2 \) is not allowed) and \( \alpha_0 := \frac{\sqrt{2} E_0}{\bar{\lambda}^3} \).

Invoking Lemma 4.4 with \( s = \bar{\lambda} \) and \( \alpha = \frac{1}{\bar{\lambda}^3} < 1 \) we conclude that for \( k = 1, 2, \ldots \)

\[
W_k \leq \frac{2t^{\frac{2}{1-s} - 1}}{K_{t, \bar{\lambda}}^{(L^2 + H^2)^{\frac{1}{k-1}}}} \sup_{t \in [t_{k-1}, T]} \| f_{k-1} \|_{L^2} \frac{2t^{\frac{2}{1-s} - 1}}{K_{t, \bar{\lambda}}^{(L^2 + H^2)^{\frac{1}{k-1}}}} \int_{t_{k-1}}^{T} \| f_{k-1} \|_{H^2} \, d\tau,
\]

\[
K = \left[ 4t^{\frac{2}{1-s} - 1} C_{c_k^{-1}} W_0^{(a-1)} (1 + t_0^{-\alpha_0}) \right]^{\frac{1}{k-1}}.
\]

Then, \( W_k = W_0 \) is a super solution of the aforementioned recursion inequality. Thus,

\[
W_k \leq W_0^* \to 0 \quad \text{as} \quad k \to \infty,
\]

provided \( W_0 \) is finite. The fact that \( W_0 \) is finite is clear invoking Theorem 2.5 and Corollary 2.6

\[
W_0 = \frac{1}{2} \sup_{t \in [\bar{\lambda}^2 T]} \| f(t) \|_{L^2}^2 + c_i \int_{\bar{\lambda}^2 T}^{T} \langle \bar{\lambda}^2 f(\tau) \rangle \| H^2 \, d\tau \leq C_{t_*} (1 + T), \quad T > t_*, > 0,
\]

where \( C(t_*) \leq (t_*^{-\beta} + 1) \) (for some \( \beta > 0 \)) is defined in such theorem and corollary.

As a consequence, since \( K_k \to K \) and \( t_k \to t_* \) as \( k \to \infty \),

\[
\sup_{t \in [T_*]} \| f_{K}^+(t) \|_{L^2} = 0,
\]

and thus, using the choice for \( K \),

\[
f(t) \leq K \sim C_{t_*} (1 + T^{(a-1)\frac{1}{2}}), \quad T \geq t \geq t_* \to 0,
\]

with a constant \( C_{t_*} \leq (t_*^{-\beta} + 1) \) (for some \( \beta > 0 \)) depending only on \((D_0, E_0, \xi)\). In order to make estimate \((1.15)\) independent of \( T \) we recall that the Boltzmann system is time-invariant, so we can perform previous analysis in any time interval \([t_0, t_0 + T]\). In this case, \( T > 0 \) plays the role of the time interval length which can be set, for example, equal 1. It is, of course, essential that the constants involved only depend on the conserved quantities \((D_0, E_0)\).

Regarding the propagation part of the result one uses a slight modification of the previous argument, we refer to [3] for the details. We only mention that the condition \( f_{t,0} \in L_{2n}^2 \) for \( 2n \geq \frac{1}{\bar{\lambda}^3} \) is necessary to guarantee the finiteness of the constants involved near time zero. \( \square \)

5. Appendix

5.1. Interpolation Lemma.

**Lemma 5.1** (Mixed interpolation). For any \( b \geq a \geq 0 \) and \( \beta > 0 \)

\[
\mathbf{m}_{a, \beta}(f) \mathbf{m}_{b, \xi}'[f] \mathbf{m}_{a, \beta}(g) \leq \theta \mathbf{m}_{b, \beta}(f) \mathbf{m}_{a, \beta}(g) + (1 - \theta) \mathbf{m}_{b, \beta}(g) \mathbf{m}_{a, \beta}(f),
\]

with \( \theta = \frac{\beta}{b + \beta - a} \in (0, 1) \).

\[1\]This condition follows after specifically choosing \( \xi = \frac{2 - \sqrt{\beta}}{1 - \sqrt{\beta}} \) in the argument.
Proof. Standard interpolation gives that
\[
\begin{align*}
\mathbf{m}_{a+b, i}[f] & \leq \mathbf{m}_{a+b, i}^\theta [f] \mathbf{m}_{a, i}^{1-\theta} [f], \\
\mathbf{m}_{b, j}[g] & \leq \mathbf{m}_{b, j}^{1-\theta} [g] \mathbf{m}_{b, j}^\theta [g],
\end{align*}
\]
and
\[
\theta = \frac{\beta}{b + \beta - a}.
\]
The result follows from here associating the common powers and applying Young’s inequality. □

5.2. Cancellation Lemma. We show in this section the cancellation Lemma of Boltzmann equation for monatomic gas mixtures. All results are proven assuming a integrable approximation of the scattering kernel, so that the collision operator can be separated, and then arguing by density. For the classical Boltzmann equation the reader can consult [1] where technical details are filled.

In all the results below we assume the scattering is forward, that is, \(B_{ij}\) has support in the set \(\{\hat{u} \cdot \sigma \geq 0\}\).

**Lemma 5.2** (Cancellation lemma). For a.e. \(v_\ast \in \mathbb{R}^3\) and \(1 \leq i, j \leq I\) we have that
\[
\begin{align*}
\int_{\mathbb{R}^3 \times S^2} B_{ij}(u, \hat{u} \cdot \sigma)(f(v') - f(v))dv\sigma &= (f * S_{ij})(v_\ast), \quad u = v - v_\ast,
\end{align*}
\]
where
\[
S_{ij}(u) = |S| \int_0^{\frac{\pi}{2}} \frac{1}{\beta(\cos \theta)} B_{ij} \left( \frac{|u|}{\beta(\cos \theta)}, \cos(\theta) \right) - B_{ij}(\|u\|, \cos(\theta)) \sin \theta d\theta.
\]
Here \(\beta(x) = \sqrt{\alpha^2 + \beta^2} + 2\alpha(1 - \alpha)x \in (0, 1]\).

Proof. For each \(\sigma\) and \(v_\ast\) fixed we perform the change of variables \(v \rightarrow v'\). Recall that
\[
v' = v_\ast + \frac{m_i}{m_i + m_j} (v - v_\ast) + \frac{m_j}{m_i + m_j} |v - v_\ast| \sigma.
\]
This change of variables is well defined on the set \(\{\cos \theta \geq 0\}\). Indeed, it follows by a direct calculation that the Jacobian is given by
\[
\left| \frac{dv'}{dv} \right| = \alpha^2 (\alpha + (1 - \alpha) \hat{u} \cdot \sigma), \quad \alpha = \frac{m_i}{m_i + m_j},
\]
where \(\hat{u} = \frac{v - v_\ast}{|v - v_\ast|}\). One can also find the relations between magnitude and scattering angles
\[
\begin{align*}
\frac{|v' - v_\ast|}{|v - v_\ast|} &= \sqrt{\alpha^2 + \beta^3} + 2\alpha(1 - \alpha)\hat{u} \cdot \sigma =: \beta(\hat{u} \cdot \sigma) \in (0, 1],
\end{align*}
\]
and
\[
\hat{u} \cdot \sigma = \frac{\alpha \hat{u} \cdot \sigma + 1 - \alpha}{\sqrt{\alpha^2 + \beta^3} + 2\alpha(1 - \alpha)\hat{u} \cdot \sigma} =: \phi(\hat{u} \cdot \sigma),
\]
where \(\hat{u}' = \frac{v' - v_\ast}{|v' - v_\ast|}\). Since \(\left| \frac{dv'}{dv} \right| \geq \alpha^3 > 0\) in the set \(\{\hat{u} \cdot \sigma = \cos \theta \geq 0\}\), the inverse transformation \(v' \rightarrow \psi_\sigma(v') = v\) is, then, well defined. Similarly, note that \(\phi\) is also invertible and its derivative is given by
\[
\phi'(x) = \frac{\alpha^2 (\alpha + (1 - \alpha)x)}{(\alpha^2 + \beta^3 + 2\alpha(1 - \alpha)x)^2} \geq \alpha^3 > 0, \quad x \in [0, 1].
\]

Applying this change of variable to the term \(\int B_{ij} f'\) in the left-hand of (5.1) we find that
\[
\begin{align*}
\int_{\mathbb{R}^3 \times S^2} B_{ij}(v - v_\ast, \hat{u} \cdot \sigma)f(v')dv\sigma &= \int_{\mathbb{R}^3 \times S^2} B_{ij}(\psi_\sigma(v') - v_\ast, \phi^{-1}(\hat{u} \cdot \sigma))f(v')\left| \frac{dv}{dv'} \right| d\nu'd\sigma
\end{align*}
\]
\[
\begin{align*}
&= \int_{\mathbb{R}^3} f(v) \int_{\phi^{-1}(\hat{u} \cdot \sigma) \geq 0} \frac{1}{\alpha^2 (\alpha + (1 - \alpha)\phi^{-1}(\hat{u} \cdot \sigma))} B_{ij}(\frac{|u|}{\beta\phi^{-1}(\hat{u} \cdot \sigma)}\phi^{-1}(\hat{u} \cdot \sigma))d\sigma dv.
\end{align*}
\]
We renamed \( v' \) to \( v \) before interchanging integrals in the last step. The inner integral is further expanded with polar coordinates performing the change of variables \( x = \phi^{-1}(\tilde{u} \cdot \sigma) \). In this way,

\[
\int_{\phi^{-1}(\tilde{u} \sigma) \geq 0} \frac{1}{\alpha^2(\alpha + (1 - \alpha)x)} B_{ij}\left(\frac{|u|}{\beta(\phi^{-1}(\tilde{u} \cdot \sigma))}, \phi^{-1}(\tilde{u} \cdot \sigma)\right) d\sigma
\]

\[
= |S| \int_{x \geq 0} \frac{\phi'(x)}{\alpha^2(\alpha + (1 - \alpha)x)} B_{ij}\left(\frac{|u|}{\beta(x)}, x\right) dx
\]

\[
= |S| \int_0^{\pi} \frac{\sin \theta}{\beta(\cos \theta)^3} B_{ij}\left(\frac{|u|}{\beta(\cos \theta)} \cos(\theta), \cos(\theta)\right) d\theta.
\]

In the last step we used that

\[
\frac{\phi'(x)}{\alpha^2(\alpha + (1 - \alpha)x)} = \frac{1}{\beta(x)^3}.
\]

Therefore, estimate \( 34 \) follows with

\[
S_{ij}(u) = |S| |u|^{\lambda_j} \int_0^{\pi/2} \left[ \frac{1}{\beta(\cos \theta)^{3+\lambda_j}} - 1 \right] \sin \theta b_{ij}(\cos \theta) d\theta \geq 0.
\]

### Remark 5.3.

Note that for cross sections of the form \( B_{ij}(|u|, \cos \theta) = |u|^{\lambda_j} \cos \theta \), one has the explicit form

\[
S_{ij}(u) = |S| |u|^{\lambda_j} \int_0^{\pi/2} \left[ \frac{1}{\beta(\cos \theta)^{3+\lambda_j}} - 1 \right] \sin \theta b_{ij}(\cos \theta) d\theta.
\]

### 5.3. Coercive estimate of the Dirichlet form.

In this subsection we explore a coercive estimate for Dirichlet form of the collision operator for monatomic gas mixtures. The argument is inspired in the classical case developed in \([1]\) and is based on a series of lemmata.

Recall the weak formulation for a suitable function \( f, g \), and \( \varphi(v) \),

\[
(5.3) \quad \int_{\mathbb{R}^3} Q_{ij}^+(f, g)\varphi(v) dv = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} B_{ij}(v - v_*, \sigma) g(v_*) f(v) \varphi'(v') dv_0 dv dv_0 d\sigma.
\]

#### Lemma 5.4 (Bobylev identity).

Let \( f \in L^2(\mathbb{R}^3) \) and \( g \in L^1(\mathbb{R}^3) \). For

\[
B_{ij}(v - v_*, \sigma) = b_{ij}\left(\frac{v - v_*}{|v - v_*|}, \sigma\right),
\]

it holds that

\[
\mathcal{F}(Q_{ij}^+(f, g)) (\xi) = \int_{\mathbb{S}^2} \mathcal{F}(g)(\xi_+^-) \mathcal{F}(f)(\xi_+^-) b_{ij}\left(\frac{\xi}{|\xi|}, \sigma\right) d\sigma, \quad 1 \leq i, j \leq I,
\]

where \( \xi = \xi_+^+ + \xi_+^- \). More explicitly,

\[
\xi_+^+ := \frac{m_j \xi}{m_i + m_j} + \frac{m_i \xi}{m_i + m_j} \quad \text{and} \quad \xi_+^- := \frac{m_i \xi}{m_i + m_j} - \frac{m_j \xi}{m_i + m_j}.
\]

**Proof.** Plugging \( \varphi(v) = e^{-iv \cdot \xi} \) in the weak formulation \((5.3)\), we get that

\[
\mathcal{F}(Q_{ij}^+(f, g)) (\xi) = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} g(v_*) f(v) B_{ij}(v - v_*, \sigma) e^{-iv \cdot \xi} dv_0 dv dv_0 d\sigma
\]

\[
= \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} g(v_*) f(v) B_{ij}(v - v_*, \sigma) e^{-\frac{m_j \xi}{m_i + m_j} |v - v_*| |\xi|} e^{-\frac{m_i \xi}{m_i + m_j} |v - v_*| \sigma} dv_0 dv dv_0 d\sigma.
\]

Note that, a key remark by Bobylev,

\[
\int_{\mathbb{S}^2} b_{ij}\left(\frac{v - v_*}{|v - v_*|}, \sigma\right) e^{-\frac{m_j \xi}{m_i + m_j} |v - v_*| |\xi|} d\sigma = \int_{\mathbb{S}^2} b_{ij}\left(\frac{\xi}{|\xi|}, \sigma\right) e^{-\frac{m_j \xi}{m_i + m_j} |\sigma| |v - v_*|} d\sigma.
\]
Thus,
\[
\mathcal{F}(Q_{ij}^+(f,g))(\xi) = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} g(v_*) f(v) b_{ij} \left( \frac{\xi \cdot \cdot \cdot}{|\xi|} \sigma \right) e^{-i\xi \cdot \sigma} \frac{me^{i\pi} + m_e^{i\pi}}{m_e^{i\pi}} e^{-i\eta \cdot \sigma} dv_* d\sigma \\
= \int_{S^2} \left( \int_{\mathbb{R}^3} f(v) e^{-i\xi \cdot \sigma} dv \right) \left( \int_{\mathbb{R}^3} g(v_*) e^{-i\eta \cdot \sigma} dv_* \right) b_{ij} \left( \frac{\xi \cdot \cdot \cdot}{|\xi|} \sigma \right) d\sigma.
\]

The result follows from here. \[\square\]

**Lemma 5.5.** Let \( g \in L^1(\mathbb{R}^3) \) and \( f \in L^2(\mathbb{R}^3) \). Then,
\[
\int_{\mathbb{R}^3} \int_{S^2} b_{ij}(\hat{u} \cdot \sigma) g(v_*) (f(v'_j) - f(v))^2 dv_* d\sigma = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \int_{S^2} b_{ij} \left( \frac{\xi \cdot \cdot \cdot}{|\xi|} \sigma \right) \left[ \mathcal{F}(g)(0) |\mathcal{F}(f)(\xi)|^2 + \mathcal{F}(g)(0) |\mathcal{F}(f)(\xi^+_i)|^2 \\
- \mathcal{F}(g)(\xi^-_j) \mathcal{F}(f)(\xi^+_j) \mathcal{F}(f)(\xi^-_j) \mathcal{F}(f)(\xi) \right] d\xi d\sigma,
\]
with the same definitions of \( \xi^\pm \) of Lemma 5.4.

**Proof.** Expanding the quadratic term gives three terms, namely,
\[
(f(v'_j) - f(v))^2 = f(v'_j)^2 + f(v)^2 - 2f(v'_j)f(v).
\]

We begin with the middle term. By the pre-post collisional change of variables and Parseval’s identity,
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} b_{ij}(\hat{u} \cdot \sigma) g(v_*) f(v) f(v'_j) dv_* d\sigma = \int_{\mathbb{R}^3} Q_{ij}^+(f,g) f dv = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \mathcal{F}[Q_{ij}^+(f,g)] \mathcal{F}(f) d\xi.
\]

Furthermore, using Bobylev’s identity Lemma 5.4 in the right-side it holds that
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} b_{ij}(\hat{u} \cdot \sigma) g(v_*) f(v) f(v'_j) dv_* d\sigma = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3 \times S^2} b_{ij} \left( \frac{\xi \cdot \cdot \cdot}{|\xi|} \sigma \right) \mathcal{F}(g)(\xi^-_j) \mathcal{F}(f)(\xi^+_j) \mathcal{F}(f)(\xi^-_j) \mathcal{F}(f)(\xi) d\xi d\sigma.
\]

The right-side is equal to its complex conjugate since the left-side is real valued. Now, we note that
\[
\int_{S^2} b_{ij}(\hat{u} \cdot \sigma) d\sigma \text{ does not depend on the unit vector } \hat{u} \in S^2, \text{ consequently it follows that}
\]
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} b_{ij}(\hat{u} \cdot \sigma) g(v_*) f(v) dv_* d\sigma = \int_{S^2} b_{ij}(\hat{u} \cdot \sigma) d\sigma \int_{\mathbb{R}^3} g(v_*) dv_* \int_{\mathbb{R}^3} f(v)^2 dv = \frac{1}{(2\pi)^3} \int_{S^2} b_{ij} \left( \frac{\xi \cdot \cdot \cdot}{|\xi|} \sigma \right) d\sigma \mathcal{F}(g)(0) \int_{\mathbb{R}^3} |\mathcal{F}(f)(\xi)|^2 d\xi,
\]
where we have applied the usual Plancherel identity. Finally, for the term involving \( f(v'_j)^2 \) we first make the change of variables \((v, v_*) \rightarrow (v - v_*, v_*)\), and then \( v \rightarrow v'_j \) (and rename as \( v \)) as in
Cancellation lemma to obtain that
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} b_{ij}(\tilde{v} \cdot \sigma) g(v_*) f(v'_{ij})^2 d\sigma dv_* = \int_{\mathbb{R}^3 \times S^2 \times \mathbb{R}^3} b_{ij}(\tilde{v} \cdot \sigma) g(v_*) |f(v'_{ij}) + v_*|^2 d\sigma dv_*
\]
\[
= \int_{\mathbb{R}^3} g(v_*) \int_{S^2 \times \mathbb{R}^3} b_{ij}(\phi^{-1}(\tilde{v} \cdot \sigma)) |f(v)|^2 d\sigma dv_*
\]
\[
= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} g(v_*) dv_* \int_{\mathbb{R}^3 \times S^2} b_{ij}(\phi^{-1}(\tilde{\xi} \cdot \sigma)) |F(f)(\xi)|^2 d\xi d\sigma
\]
which achieve the proof of Lemma 5.3 after adding the three respective terms. \qed

**Corollary 5.6.** Let \( f \in L^2(\mathbb{R}^3) \) and \( g \in L^1(\mathbb{R}^3) \) with \( g \geq 0 \). Then,
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{S^2} b_{ij}(\tilde{v} \cdot \sigma) g(v_*) (f(v'_{ij}) - f(v))^2 d\sigma dv_*
\]
\[
\geq \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} |F(f)(\xi)|^2 \int_{S^2} b_{ij} \left( \frac{\xi}{|\xi|} \cdot \sigma \right) \left| F(g)(0) - |F(g)(\xi'_{ij})| \right| d\sigma d\xi
\]

**Proof.** In Lemma 5.5 use Cauchy-Schwarz inequality to control the latter two terms as
\[
F(g)(\xi'_{ij}) F(f)(\xi'_{ij}) F(f)(\xi) + F(g)(\xi'_{ij}) F(f)(\xi'_{ij}) F(f)(\xi)
\]
\[
\leq |F(f)(\xi)| \left( |F(f)(\xi)|^2 + |F(f)(\xi')|^2 \right)
\]
The result follows after observing that for \( g \geq 0 \) one has that \( F(g)(0) - |F(g)(\xi'_{ij})| \geq 0 \). \qed

**Lemma 5.7.** Suppose that \( b_{ij} \) satisfies assumption (2.9) and (2.10). Then, there exists a constant \( K^{ij} > 0 \) such that
\[
\int_{S^2} b_{ij} \left( \frac{\xi}{|\xi|} \cdot \sigma \right) \left| F(g)(0) - |F(g)(\xi'_{ij})| \right| d\sigma \geq K^{ij} (|\xi|^2 \land |\xi'|^2) .
\]
The constant can be taken as \( K^{ij} := K^{ij}(g) = \left( \frac{m_j}{m_i + m_j} \right)^2 \frac{1}{2-s_j} \kappa^{ij}_j C_g |S| \) with the constant \( C_g > 0 \) depending only on \( \|g\|_{L^1} \) and \( \|g\|_{L^\log L} \).

**Proof.** Using (1.2) Lemma 3], there exists a constant \( C_g \) depending on \( \|g\|_{L^1} \) and \( \|g\|_{L^\log L} \) such that
\[
F(g)(0) - |F(g)(\xi'_{ij})| \geq C_g (|\xi'_{ij}|^2 \land 1) \geq 0
\]
Then,
\[
\int_{S^2} b_{ij} \left( \frac{\xi}{|\xi|} \cdot \sigma \right) \left( F(g)(0) - |F(g)(\xi'_{ij})| \right) d\sigma \geq C_g \int_{S^2} b_{ij} \left( \frac{\xi}{|\xi|} \cdot \sigma \right) (|\xi'_{ij}|^2 \land 1) d\sigma
\]
We note that
\[
|\xi'_{ij}|^2 = 2 \left( \frac{m_j}{m_i + m_j} \right)^2 |\xi|^2 \left( 1 - \frac{\xi}{|\xi|} \cdot \sigma \right)
\]
Thus, using spherical coordinates and recalling assumption (2.9) it follows that
\[
\int_{S^2} b_{ij} \left( \frac{\xi}{|\xi|} \cdot \sigma \right) (|\xi'_{ij}|^2 \land 1) d\sigma \geq \left( \frac{m_j}{m_i + m_j} \right)^2 |S| \int_0^{\pi} \sin \theta b_{ij}(\cos \theta) \left[ |\xi|^2 (1 - \cos \theta) \land 1 \right] d\theta
\]
\[
\geq \left( \frac{m_j}{m_i + m_j} \right)^2 \kappa^{ij}_j |S| \int_0^{\pi} \left[ (|\xi|^2 \theta^2 \land 1) \right] d\theta
\]
Using the change of variables \( \tilde{\theta} = |\xi| \theta \), this latter integral can be estimated as
\[
|\xi|^{s_j} \int_0^{\pi} \frac{\tilde{\theta}^2 \land 1}{\tilde{\theta}^{1+s_j}} \frac{d\tilde{\theta}}{\tilde{\theta}^{1+s_j}} \geq |\xi|^{s_j} \int_0^{\pi} \left( \tilde{\theta}^2 \land 1 \right) \frac{d\tilde{\theta}}{\tilde{\theta}^{1+s_j}} \geq \frac{|\xi|^{s_j}}{2 - s_j}, \quad \text{for } |\xi| \geq 1.
\]
Meanwhile, when $|\xi| \leq 1$ the integral is estimated as
\[
|\xi|^{s_{ij}} \int_{0}^{2|\xi|} (\tilde{\theta}^2 + 1) \frac{d\tilde{\theta}}{\tilde{\theta}^{1+s_{ij}}} \geq |\xi|^{s_{ij}} \int_{0}^{2|\xi|} (\tilde{\theta}^2 + 1) \frac{d\tilde{\theta}}{\tilde{\theta}^{1+s_{ij}}} = \frac{|\xi|^2}{2 - s_{ij}}.
\]
This concludes the proof with the constant
\[
K_{ij} = \left( \frac{m_i}{m_i + m_j} \right)^2 \frac{\kappa_{ij}}{2 - s_{ij}} C_g |S|.
\]

\[\square\]

Corollary 5.6 and Lemma 5.7 readily prove the main coercive estimate.

**Proposition 5.8** (Coercivity estimate). Let $f \in L^2(\mathbb{R}^3)$ and $g \in (L^1 \cap L \log L)(\mathbb{R}^3)$ with $g \geq 0$. Then,
\[
\int_{\mathbb{R}^3} \int_{S^2} b_{ij}(\tilde{\theta} \cdot \sigma) g(v_s) (f(v_{ij}) - f(v))^2 dv dv_s d\sigma \geq K_{ij} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} |\mathcal{F}(f)(\xi)|^2 \left( |\xi|^2 + |\xi|^{s_{ij}} \right) d\sigma d\xi,
\]
with constant $K_{ij}$ given in Lemma 5.7. And, as a consequence, it holds that
\[
\int_{\mathbb{R}^3} \int_{S^2} b_{ij}(\tilde{\theta} \cdot \sigma) g(v_s) (f(v_{ij}) - f(v))^2 dv dv_s d\sigma \geq K_{ij} \frac{1}{(2\pi)^3} \left( 2 \cdot \frac{\|f\|_{H^{s_{ij}}}^2}{\|f\|_{L^2}^2} \right).
\]

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