Semi-analytical postbuckling strength analysis of anisotropic shell structures

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Abstract. An investigation of the forms of shell buckling has been the subject of many experimental and theoretical studies. On the basis of analysing of the forms of equilibrium it is possible to determine the stability of a structure as a whole, especially if a statistical analysis is used. The numerical analysis of the shells considered is based on a semi-analytical treatment of displacement and stress field. This method is proven for static and dynamic nonlinear analysis of general shells of revolution and leads to important advantages in efficiency and accuracy compared with a common finite element analysis, especially in the case of geometrically imperfect shells. The method developed permits determination of stresses in a shell by means of an experimental deflection function. Failure criterion allows predicting the sites of fracture and maintenance of a shell upon loss of stability.

1. Introduction

Cylindrical shells find wider applications as primary structural members in various fields of engineering such as civil, mechanical, aerospace and nuclear engineering fields. The contribution of geometric imperfections due to manufacturing processes takes dominant role in decreasing the buckling load of cylindrical shells. Therefore investigation of initial imperfections and their influence on stress level in real shell structure is very important [1, 2].

Variable stiffness and thickness of shells are important factors for a numerical investigation into the buckling and post-buckling of cylinders under different cases of loading [3].

The study is intended to perform numerical analysis of the stress state of anisotropic shells of revolution in the case of short-time loading by means of experimental displacement function [4, 5] in postbuckling state. The radial displacement field of the shells was investigated and characterised by the Fourier series. The analysis has intended to be used in the development of the inner structure of the shell material. For strength related properties, a phenomenological failure criterion can supply the feedback in material permutations and provide guidance in the fabrication process of composite structures. Failure is interpreted here as the occurrence of discontinuity in the material.

2. Experimental investigations

In the test, fiberglass shells with a diameter of 300 mm and a nominal length of 300 mm were loaded. The thickness of the shells was 3 and 5 mm (figure 1a). The shells were loaded with a special hydraulic chamber (figure 1b). During the loading, shell displacements as well as initial imperfections were fixed throughout the perimeter and on 10 levels in the length.
Figure 1. Experimental equipment and loading scheme: a) geometrical characteristics and loading with hydrostatic pressure; b) hydraulic chamber.

To determine the stresses in the wall of the shell, the mechanical characteristics were also determined. The elastic moduli $E_1$ and $E_2$ were determined on shells but the shear moduli $G_{12}$ and $G_{23}$ on samples that were cut out of the shells. Strength characteristics were determined on small samples.

3. Determination of strains and stresses in anisotropic shell of revolution

In the analysis the length of cylindrical shell is $L$, the radius $R$, and the thickness $h$; axes $x_1$, $x_2$, $x_3$ are directed along the generatrix, the circumference, and the normal to the centre of curvature. Using the Kirchhoff-Love hypothesis, the total strains and stresses are determined as the sum of the strains and stresses in the middle surface and of the flexural strains and stresses:

\[
\varepsilon_{\alpha\beta}(x_1, x_2, x_3) = \varepsilon^0_{\alpha\beta}(x_1, x_2) + x_3 \left[ \kappa_{\alpha\beta}(x_1, x_2) - \kappa_{0\alpha\beta}(x_1, x_2) \right];
\]

\[
\sigma_{\alpha\beta}(x_1, x_2, x_3) = \sigma^0_{\alpha\beta}(x_1, x_2) + x_3 A_{\alpha\beta\gamma\delta} \left[ \kappa_{\gamma\delta}(x_1, x_2) - \kappa_{0\gamma\delta}(x_1, x_2) \right].
\]

Here $\varepsilon^0_{\alpha\beta}$, $\sigma^0_{\alpha\beta}$ are membrane strain and stress; $\kappa_{0\alpha\beta}$, $\kappa_{\alpha\beta}$ are the distortion tensor of initial geometrical imperfection and deformation of the shell; $A_{\alpha\beta\gamma\delta}$ is the stiffness tensor of the material. The Airy stress function $\Phi(x_1, x_2)$ is related uniquely with the displacement function $w(x_1, x_2)$ by the equation of strain compatibility and is determined on the basis of strain compatibility equation for shell by using compliance tensor components $S_{ijkl}$.
\[ S_{1111} \frac{\partial^4 \Phi (x_1, x_2)}{\partial y^4} + S_{2222} \frac{\partial^4 \Phi (x_1, x_2)}{\partial x^4} + (2S_{1122} + 4S_{1212}) \frac{\partial^4 \Phi (x_1, x_2)}{\partial x^2 \partial y^2} = -\frac{1}{R} \frac{\partial^2 w (x_1, x_2)}{\partial x^2} \]  

(3)

The solution of (3) is given by expressing the function of radial deflection \( w (x_1, x_2) \) and the stress function \( \Phi (x_1, x_2) \) by complex double Fourier series taking into account external load \( q \):

\[ w (x_1, x_2) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} (\alpha_{mn} - \alpha_{0mn}) \Psi_{mn} (x_1, x_2); \]

(4)

\[ \Phi (x_1, x_2) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} \varphi_{mn} \Psi_{mn} (x_1, x_2) - \frac{qRx_1^2}{2h} - \frac{qRx_2^2}{4h}, \]

(5)

where \( \Psi_{mn} (x_1, x_2) = e^{i(m\pi x_1/L + n\pi x_2/R)} \) (i – imaginary unit).

Here \( m \) and \( n \) are the numbers of half-waves along the generatrix and the number of waves along the circumference, respectively. According to the experimental data obtained the maximum number of harmonics of the series along the generatrix \( M = 4 \) and along the circumference \( N = 6 \). Here and bellow \( \alpha_{mn} \) and \( \alpha_{0mn} \) are the coefficients of displacement and imperfection function, respectively.

The membrane stresses in the middle surface of a shell were determined as

\[ \sigma_{11}^0 (x_1, x_2) = \frac{\partial \Phi (x_1, x_2)}{\partial x_2^2}; \quad \sigma_{22}^0 (x_1, x_2) = \frac{\partial \Phi (x_1, x_2)}{\partial x_1^2}; \quad \sigma_{12}^0 (x_1, x_2) = \frac{\partial \Phi (x_1, x_2)}{\partial x_1 \partial x_2}. \]

(6)

By using stiffness tensor \( A_{ijkl} \) the flexural stresses for \( x_3 = \pm h/2 \) were expressed in the form

\[ \sigma_{11}^f (x_1, x_2) = \pm A_{1111} \frac{\pi h}{2L^2} \times \sum_{m=-M}^{M} \sum_{n=-N}^{N} (\alpha_{mn} - \alpha_{0mn}) m^2 \Psi_{mn} (x_1, x_2) \pm \]

\[ \pm A_{1122} \frac{h}{2R^2} \times \sum_{m=-M}^{M} \sum_{n=-N}^{N} (\alpha_{mn} - \alpha_{0mn}) n^2 \Psi_{mn} (x_1, x_2); \]

(7)

\[ \sigma_{22}^f (x_1, x_2) = \pm A_{2222} \frac{\pi h}{2L^2} \times \sum_{m=-M}^{M} \sum_{n=-N}^{N} (\alpha_{mn} - \alpha_{0mn}) n^2 \Psi_{mn} (x_1, x_2) \pm \]

\[ \pm A_{1212} \frac{\pi h}{2R^2} \times \sum_{m=-M}^{M} \sum_{n=-N}^{N} (\alpha_{mn} - \alpha_{0mn}) m^2 \Psi_{mn} (x_1, x_2); \]

(8)

\[ \sigma_{12}^f (x_1, x_2) = \pm A_{1212} \frac{\pi h}{LR} \times \sum_{m=-M}^{M} \sum_{n=-N}^{N} (\alpha_{mn} - \alpha_{0mn}) mn \Psi_{mn} (x_1, x_2). \]

(9)

Figure 2 shows a scan of a part of the shell surface with lines of constant level of stresses \( \sigma_{22} \) and \( \sigma_{12} \). The dashed lines indicate the sites of the fracture. At the sites of maximum compressive stress \( \sigma_{22} \) the shear stress has minimum value, and vice versa but where the curvature \( \kappa_{\alpha\beta} \) of the deformed surface changes sign, the level of the shear stress is high.
Figure 2. Lines of constant level of stresses in MPa: a) $\sigma_{22}(x_1, x_2) = \text{const}$; b) $\sigma_{12}(x_1, x_2) = \text{const}$.

In order to take into account delaminating process in buckling process of a shell, the interlayer shear stresses $\sigma_{13}$ and $\sigma_{23}$ were determined as

$$
\sigma_{13}(x_1, x_2, x_3) = \frac{h^2 - 4x_3^2}{8} A_{1111} \frac{\pi^3}{L^3} \times \sum_{m=-M}^{M} \sum_{n=-N}^{N} (\alpha_{mn} - \alpha_{00mn}) m^3 i \Psi_{mn}(x_1, x_2) +
$$

$$
\frac{h^2 - 4x_3^2}{8} \left( A_{1111} \nu_{21} + 2A_{1212} \right) \frac{\pi}{LR^2} \times \sum_{m=-M}^{M} \sum_{n=-N}^{N} (\alpha_{mn} - \alpha_{00mn}) mn^2 i \Psi_{mn}(x_1, x_2); \quad (10)
$$

$$
\sigma_{23}(x_1, x_2, x_3) = \frac{h^2 - 4x_3^2}{8} A_{2222} \frac{1}{R^3} \times \sum_{m=-M}^{M} \sum_{n=-N}^{N} (\alpha_{mn} - \alpha_{00mn}) n^3 i \Psi_{mn}(x_1, x_2) +
$$

$$
\frac{h^2 - 4x_3^2}{8} \left( A_{1111} \nu_{21} + 2A_{1212} \right) \frac{\pi^2}{L^2 R} \times \sum_{m=-M}^{M} \sum_{n=-N}^{N} (\alpha_{mn} - \alpha_{00mn}) m^2 n i \Psi_{mn}(x_1, x_2). \quad (11)
$$

4. Strength analysis of shell in postbuckling state

For characterising of stress level in a shell depending on the co-ordinates the in-plane (membrane) stresses, couple stresses as well as transverse shear stresses were taken into account. The description of the surface of strength by quadratic polynomial criteria is proposed. In a general form the equation is given in [6–8].

In the case of orthotropic reinforced material the strength of the shell in the plane stress state has been considered with the retention of terms of the first and second order, the strength criterion depicts the surface (ellipsoid) in three-dimensional space of stresses and takes the following form:

$$
F_{11} \sigma_{11} + F_{22} \sigma_{22} + 2F_{12} \sigma_{12} + F_{1111} \sigma_{11}^2 + F_{2222} \sigma_{22}^2 + 4F_{1212} \sigma_{12}^2 + 2F_{1122} \sigma_{11} \sigma_{22} + 4F_{1112} \sigma_{11} \sigma_{22} + 4F_{2212} \sigma_{22} \sigma_{22} = 1. \quad (12)
$$

The coefficients of (12) are components of tensors $F_{ij}$ and $F_{ijkl}$. They are determined by the use of the strength tensor $R_{\alpha\beta\gamma\delta}$, where $\alpha = 0, 11, 1\overline{1}; \beta = 0, 22, 2\overline{2}; \gamma = 0, 12, 1\overline{2}$. The index 0
denotes that the given stress component is absent; the bar over the index shows the presence of a compressive component. Using the strength values obtained experimentally, the coefficients are represented in the following form:

\[ F_{11} = \frac{R_{1100} - R_{1100}}{R_{1100}R_{1100}}; \quad F_{22} = \frac{R_{2200} - R_{0220}}{R_{0220}R_{2200}}; \quad F_{1111} = \frac{1}{R_{1100}R_{1100}}; \quad F_{2222} = \frac{1}{R_{0220}R_{0220}}; \]
\[ 2F_{1122} = F_{11} - F_{22} \frac{1}{R_{11220}} + F_{1111} + F_{2222} - \frac{1}{R_{11220}^2}; \quad 4F_{1212} = \frac{1}{R_{0012}R_{0012}} \]
(13)

The axes 1 and 2 of the ellipsoid are in the plane \( \sigma_{11} - \sigma_{22} \) and axis 3 is parallel to axis \( \sigma_{12} \). The components of the tensor of the strength surface \( F_{11} \) and \( F_{22} \) express the displacement of the centre of ellipsoid in the direction of axes 1 and 2, respectively. The angle of rotation of the ellipsoid relative to axis 1 is a function of the component \( F_{1122} \).

Substituting into (12) the coefficients of (13) a quadratic equation was obtained. Solving the equation (12), the theoretical values of the ultimate strength (surface of failure) were determined. The analysis was performed with consideration of the following experimental strength (MPa):

\[ R_{1100} = 180; \quad R_{2200} = 300; \quad R_{0220} = 230; \quad R_{0012} = 95 \]

and characteristics of rigidity of the material (MPa):

\[ E_1 = 1.2 \times 10^4; \quad E_2 = 1.9 \times 10^4; \quad G_{12} = 0.25 \times 10^4 \]

Figure 3 shows the lines of the constant stress level over the shell according to the criterion (12).

Two types of shells were tested. The first type of shells was made by winding method using glass fibers and epoxy resin, the second one – by using glass fabric and epoxy as cohesive substance. It is fixed that after buckling of glass fibre/epoxy shell made by winding method, the delamination of material took place. The influence of transverse shear stresses on the stress level is shown in figure 4.

The boundary surface, determined by the criterion (12), depends considerably on the loading rate. The strength characteristics of material are established on loading at a certain constant rate. The loading of shell was stepwise, since at each step of loading the form of buckling was measured for 3 min. Consequently, an additional study of fracture toughness of the shell with consideration of time factors is necessary. The method developed permits determination of stresses in a shell by means of an experimental deflection function. It is established that the fracture of glass fabric plastic shells occurs at maximum couple-stresses.

For glass fibre/epoxy shells made by winding method the delaminating took place under action of interlayer shear stresses. Failure criterion permits predicting the sites of fracture and maintenance of a shell upon loss of stability.
Conclusions
The following main conclusions can be drawn:

1. The method developed permits determination of stresses in a shell by means of an experimental deflection function by using semi-analytical method.

2. It is established that the fracture of glass fabric plastic shells occurs at maximum couple-stresses. The fracture of glass fiber/epoxy shells made by winding method took place under action of transverse shear stress.

3. Failure criterion allows predicting the sites of fracture and maintenance of a shell upon loss of stability.

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