Vanishing diffusion in a dynamic boundary condition for the Cahn–Hilliard equation

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Abstract. The initial boundary value problem for a Cahn–Hilliard system subject to a dynamic boundary condition of Allen–Cahn type is treated. The vanishing of the surface diffusion on the dynamic boundary condition is the point of emphasis. By the asymptotic analysis as the diffusion coefficient tends to 0, one can expect that the solutions of the surface diffusion problem converge to the solution of the problem without the surface diffusion. This is actually the case, but the solution of the limiting problem naturally loses some regularity. Indeed, the system we investigate is rather complicated due to the presence of nonlinear terms including general maximal monotone graphs both in the bulk and on the boundary. The two graphs are related each to the other by a growth condition, with the boundary graph that dominates the other one. In general, at the asymptotic limit a weaker form of the boundary condition is obtained, but in the case when the two graphs exhibit the same growth the boundary condition still holds almost everywhere.

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1. Introduction

In this paper, we deal with the initial boundary value problem for a Cahn–Hilliard system subject to a dynamic boundary condition of Allen–Cahn type. We study the limiting behavior of the system as the coefficient of the diffusive term in the boundary condition goes to 0 and we investigate the limit problem.

The Cahn–Hilliard system [5] is a celebrated model describing the spinodal decomposition by the simple framework of partial differential equations. It is a phenomenological model that finds its root in the work of Cahn [4], who studied the effects of interfacial energy on the stability of spinodal states in
solid binary solutions, and this took origin from the previous collaboration with Hilliard [5]. In the recent decades, a lot of research contributions has been devoted to Cahn–Hilliard and viscous Cahn–Hilliard [25,26] systems. An impressive number of related references can be found in the literature and it would be worthless to report a list here. Let us simply refer to the recent review paper [23].

Let $0 < T < \infty$, and $\Omega \subset \mathbb{R}^d$ ($d = 2$ or $3$) be a bounded domain with smooth boundary $\Gamma := \partial \Omega$. Let $u, \mu : Q := (0,T) \times \Omega \rightarrow \mathbb{R}$ be two unknowns, representing the order parameter and the chemical potential, respectively. They have to satisfy

\begin{align}
\partial_t u - \Delta \mu &= 0 \quad \text{in } Q, \\
\mu &= \tau \partial_t u - \Delta u + F'(u) \quad \text{in } Q,
\end{align}

where $\tau \geq 0$ is a constant coefficient (the viscous Cahn–Hilliard system corresponds to the case $\tau > 0$); $\partial_t$ denotes the time derivative; $\Delta$ denotes the Laplacian. The information on the potential $F$ and its derivative $F'$ is given later. From the viewpoint of partial differential equations, in order to solve the system (1.1)–(1.2) we need some auxiliary conditions, namely the boundary and initial conditions. Here we set up the following boundary conditions and initial condition:

\begin{align}
\partial_\nu \mu &= 0 \quad \text{on } \Sigma, \\
u_\Gamma &= u_\Gamma \quad \text{on } \Sigma, \\
\mu(0) &= \mu_0 \quad \text{in } \Omega,
\end{align}

where $\partial_\nu$ denotes the outward normal derivative on $\Gamma$; $u_\Gamma$ stands for the trace of $u$ on the boundary $\Gamma$. The point of emphasis is now the boundary condition for $u$. It seems that (1.4) is setting a non-homogeneous Dirichlet boundary condition for $u$, instead $u_\Gamma : \Sigma := (0,T) \times \Gamma \rightarrow \mathbb{R}$ is also unknown and is required to satisfy the dynamic boundary condition

\begin{equation}
\partial_t u_\Gamma + \partial_\nu u - \kappa \Delta_\Gamma u_\Gamma + F'_\Gamma(u_\Gamma) = 0 \quad \text{on } \Sigma,
\end{equation}

where $\kappa > 0$ is a coefficient, intended to tend to 0, and it is the most important parameter in this paper. Besides, $\Delta_\Gamma$ denotes the Laplace–Beltrami operator on $\Gamma$ (see, e.g., [19, Chapter 3]). We observe that the coupling of (1.1)–(1.2) and (1.6) gives rise to a sort of transmission problem. We can also set the initial condition for $u_\Gamma$, i.e.,

\begin{equation}
u_\Gamma(0) = u_{0\Gamma} \quad \text{on } \Gamma
\end{equation}

to complete the problem.

The nonlinear terms $F'$ and $F'_\Gamma$ are usually referred as the derivatives of the double-well potentials $F$ and $F_\Gamma$. Therefore, the problem (1.1)–(1.7) yields the Cahn–Hilliard system when $\tau = 0$ (resp. the viscous Cahn–Hilliard system if $\tau > 0$) with the Neumann homogeneous boundary condition (1.3) for the chemical potential $\mu$ and the dynamic boundary condition of Allen–Cahn type (1.6) for the trace $u_\Gamma$ of the order parameter $u$. Typical and physically significant examples for potentials like $F$ and $F_\Gamma$ are the so-called classical
regular potential, the logarithmic potential, and the double obstacle potential, which are defined by
\[
F_{\text{reg}}(r) := \frac{1}{4} (r^2 - 1)^2, \quad r \in \mathbb{R},
\]
\[
F_{\text{log}}(r) := (1 + r) \ln(1 + r) + (1 - r) \ln(1 - r) - c_1 r^2, \quad r \in (-1, 1),
\]
\[
F_{\text{obs}}(r) := c_2 (1 - r^2) \quad \text{if } |r| \leq 1 \quad \text{and } F_{\text{obs}}(r) := +\infty \quad \text{if } |r| > 1,
\]
where \(c_1 > 1\) and \(c_2 > 0\) are positive constants, fixed so that \(F_{\text{log}}\) and \(F_{\text{obs}}\) are nonconvex. In this paper, we treat the nonlinear terms \(F'\) in (1.2) and \(F'_{\Gamma}\) in (1.6) by separating them into two parts, i.e., by assuming that \(F' = \beta + \pi\) and \(F'_{\Gamma} = \beta_{\Gamma} + \pi_{\Gamma}\), where \(\beta, \beta_{\Gamma}\) are the monotone parts (derivatives or subdifferentials of the convex parts of \(F\) and \(F_{\Gamma}\)), while \(\pi, \pi_{\Gamma}\) play as the (smooth) anti-monotone parts. For example, in the case of the classical regular potential, \(F'_{\text{reg}} = \beta_{\text{reg}} + \pi_{\text{reg}}\) is specified as the derivative of \(F_{\text{reg}}\), that is
\[
F'_{\text{reg}}(r) = r^3 - r, \quad \text{with } \beta_{\text{reg}}(r) := r^3, \quad \pi_{\text{reg}}(r) := -r.
\]
On the other hand, in the case of the non-smooth double obstacle potential \(F_{\text{obs}}, \beta_{\text{obs}}\) is defined by the subdifferential of the indicator function of \([-1, 1]\), so to have
\[
F'_{\text{obs}}(r) = \partial I_{[-1,1]}(r) - 2 c_2 r, \quad \text{with } \beta_{\text{obs}}(r) := \partial I_{[-1,1]}(r), \quad \pi_{\text{obs}}(r) := -2 c_2 r. \tag{1.8}
\]
We point out that, as a general rule, we can always use the subdifferentials for \(\beta, \beta_{\Gamma}\), and these subdifferentials reduce to the derivatives whenever a derivative exists. Please note that the subdifferentials may also be multivalued graphs, as it happens in (1.8). Thus, we generally interpret the equation in (1.2) by
\[
\mu = \tau \partial_t u - \Delta u + \xi + \pi(u), \quad \xi \in \beta(u) \quad \text{in } Q \tag{1.9}
\]
and rewrite the boundary condition (1.6) as
\[
\partial_t u_{\Gamma} + \partial_{\nu} u - \kappa \Delta_{\Gamma} u_{\Gamma} + \xi_{\Gamma} + \pi_{\Gamma}(u_{\Gamma}) = 0, \quad \xi_{\Gamma} \in \beta_{\Gamma}(u_{\Gamma}) \quad \text{on } \Sigma. \tag{1.10}
\]
Of course, we can chose different potentials for \(F\) and \(F_{\Gamma}\), which may lead in particular to different graphs \(\beta\) and \(\beta_{\Gamma}\); about possible relations among them, according to a rather usual setting (cf., e.g., [6,8–14,22,28]) here it is supposed that \(\beta_{\Gamma}\) dominates \(\beta\) in the sense of assumption (A2) in Sect. 2. We note that in our framework it is possible to choose similar or even equal graphs \(\beta\) and \(\beta_{\Gamma}\).

Next, let us emphasize a property of the PDE system under consideration. Indeed, we see from (1.1), (1.3), (1.5) that the following mass conservation holds:
\[
\int_{\Omega} u(t) dx = \int_{\Omega} u_0 dx \quad \text{for all } t \in [0, T].
\]
Of course, in the analysis of the problem expressed by (1.1), (1.3)–(1.7), (1.9)–(1.10) this property of mass conservation plays a role. Let us review some contribution to this class of problems. As some pioneering result for the Cahn–Hilliard system with a dynamic boundary condition of heat equation type,
the global existence and uniqueness of solutions was treated in [27] and the convergence to equilibrium was shown in [30]. Moreover, the Cahn–Hilliard system with a semi-linear equation as dynamic boundary condition (including the Allen–Cahn equation), was addressed from different viewpoints: the long time behaviour was studied in [17, 24], while other investigations treated the coupling with the heat equation in [16], the problem with memory in [7], some regularity results for the problem with a singular potential in [12, 18], the boundary mass constraint in [8], and so on.

In this paper, we focus on the asymptotic analysis of the surface diffusion term on the dynamic boundary condition (1.10). By the asymptotic limit as $\kappa \downarrow 0$, one can wonder whether the solution of the problem with surface diffusion converges to the one of the same problem without surface diffusion, i.e., with (1.10) replaced by

$$\partial_t u + \partial_\nu u + \xi u + \pi(u) = 0, \quad \xi \in \beta(u) \quad \text{on } \Sigma.$$  

(1.11)

The answer is in the affirmative, but it turns out that the solution of the limiting problem apparently looses regularity, due to the absence of the diffusive term on the boundary. As the reader will see, in general the terms $\partial_\nu u$ and $\xi u$ in (1.11) are not functions but elements of a dual space, and the inclusion $\xi \in \beta(u)$ has to be suitably reinterpreted and generalized. However, in the case when the graphs $\beta$ and $\beta(u)$ have the same growth, it is proven that the boundary condition (1.11) holds almost everywhere on $\Sigma$.

In the light of our approach, we aim to quote the recent paper [28], which deals with a highly nonlinear extension of the Cahn–Hilliard system with dynamic boundary condition, including nonlinear viscosity terms in the equation corresponding to (1.9) and in the boundary condition. In fact, our asymptotic results can be compared with the ones contained in [28, Theorem 2.11], where a convergence statement similar to our Theorem 2.2 below is given in terms of a subsequence $\kappa_n$ going to 0, both for a viscous Cahn–Hilliard type system and, under suitable regularity properties on data, for a nonlinear variation of the Cahn–Hilliard system. However, due to the presence of additional nonlinearities, nothing is investigated in [28] about convergence to solutions satisfying the boundary condition almost everywhere (as instead we do here).

A brief outline of the present paper along with a short description of the various items is as follows. In Sect. 2, after setting up the notation and technical tools, as well as the known results for the case $\kappa > 0$, we will state the main theorems. These theorems are the convergence–existence result and the continuous dependence with respect to data when $\kappa = 0$, the uniqueness of the solution being included in the second theorem. In Sect. 3, we work with the approximate solutions and collect the uniform estimates, being able to check the existence of solutions for the problem with $\kappa > 0$. In Sect. 4, we consider the limiting procedure as $\kappa \downarrow 0$ and give the proof of the continuous dependence result. In Sect. 5, we deal with an improvement of the convergence–existence theorem, including some regularity properties for the solution, under the stronger assumption (A2)′.
2. Notation and main results

In this paper, we deal with the following spaces

\[ H := L^2(\Omega), \quad H_\Gamma := L^2(\Gamma), \quad V := H^1(\Omega), \quad V_\Gamma := H^1(\Gamma), \quad W_\Gamma := H^{1/2}(\Gamma), \]

that are all Hilbert spaces with respect to the usual norms and inner products, denoted by \(|\cdot|_H\) and \(\langle\cdot,\cdot\rangle_H\), and so on (see, e.g., [21] for precise definitions). Moreover, \(V^*\) and \(V_\Gamma^*\) stand for the dual spaces of \(V\) and \(V_\Gamma\), respectively. The notation \(\langle\cdot,\cdot\rangle_{V^*,V}\) is used for the duality pairing between \(V^*\) and \(V\). It is understood that \(H\) (resp. \(H_\Gamma\)) is embedded in \(V^*\) (resp. \(V_\Gamma^*\)) in the usual way, i.e., \(\langle w, z \rangle_{V^*,V} = \langle w, z \rangle_H\) for all \(w \in H\) and \(z \in V\) (resp. \(\langle w_\Gamma, z_\Gamma \rangle_{V_\Gamma^*,V_\Gamma} = \langle w_\Gamma, z_\Gamma \rangle_{H_\Gamma}\) for all \(w_\Gamma \in H_\Gamma\) and \(z_\Gamma \in V_\Gamma\)). We also use \(W_\Gamma^*\) for the dual space of \(W_\Gamma\) and, in view of the actual identification between \(H\) and \(H_\Gamma\), it turns out that \(V_\Gamma^*\) and \(W_\Gamma^*\) are completely isomorphism to \(H^{-1}(\Gamma)\) and \(H^{-1/2}(\Gamma)\), respectively (see, [21, Theorem 7.6, p. 36]).

We start from the following Cahn–Hilliard system with the dynamic boundary condition of Allen–Cahn type: for all \(\tau \in [0,1]\) and \(\kappa \in (0,1]\) (the upper bounds are taken as 1 for simplicity), one has to find a quintuplet \((u, \mu, \xi, u_\Gamma, \xi_\Gamma)\) such that

\[ \partial_\tau u - \Delta \mu = 0 \quad \text{a.e. in } Q, \]
\[ \mu = \tau \partial_\tau u - \Delta u + \xi + \pi(u) - g, \quad \xi \in \beta(u) \quad \text{a.e. in } Q, \]
\[ \partial_\nu \mu = 0 \quad \text{a.e. on } \Sigma, \]
\[ u|_{\Sigma} = u_\Gamma \quad \text{a.e. on } \Sigma, \]
\[ \partial_\nu u_\Gamma + \partial_\nu u - \kappa \Delta u_\Gamma + \xi_\Gamma + \pi(u_\Gamma) = g_\Gamma, \quad \xi_\Gamma \in \beta_\Gamma(u_\Gamma) \quad \text{a.e. on } \Sigma, \]
\[ u(0) = u_0 \quad \text{a.e. in } \Omega, \quad u_\Gamma(0) = u_{0\Gamma} \quad \text{a.e. on } \Gamma, \]

where \(g : Q \to \mathbb{R}, \; g_\Gamma : \Sigma \to \mathbb{R}, \; u_0 : \Omega \to \mathbb{R}, \; u_{0\Gamma} : \Gamma \to \mathbb{R}\) are given functions. Hereafter we assume that:

(A1) \(\beta, \beta_\Gamma\) are maximal monotone graphs in \(\mathbb{R} \times \mathbb{R}\), which coincide with the subdifferentials \(\beta = \partial \hat{\beta}, \beta_\Gamma = \partial \hat{\beta}_\Gamma\) of some proper, lower semicontinuous, and convex functions \(\hat{\beta}, \hat{\beta}_\Gamma : \mathbb{R} \to [0, +\infty]\) such that \(\hat{\beta}(0) = \hat{\beta}_\Gamma(0) = 0\), with the corresponding effective domains denoted by \(D(\beta)\) and \(D(\beta_\Gamma)\), respectively;

(A2) \(D(\beta_\Gamma) \subseteq D(\beta)\) and there exist two constants \(p \geq 1\) and \(c_0 > 0\) such that

\[ |\beta^o(r)| \leq p|\beta^o_\Gamma(r)| + c_0 \quad \text{for all } r \in D(\beta_\Gamma), \]

where \(\beta^o\) and \(\beta^o_\Gamma\) denote the minimal sections of \(\beta\) and \(\beta_\Gamma\), specified by \(\beta^o(r) := \{r^* \in \beta(r) : |r^*| = \min_{s \in \beta(r)} |s|\}, r \in D(\beta), \) for \(\beta\);

(A3) \(\pi, \pi_\Gamma : \mathbb{R} \to \mathbb{R}\) are Lipschitz continuous functions with their Lipschitz constants \(L\) and \(L_\Gamma\), respectively;

(A4) \(u_0 \in V, \; u_{0\Gamma} \in V_\Gamma\) satisfy \(\hat{\beta}(u_0) \in L^1(\Omega), \; \hat{\beta}_\Gamma(u_{0\Gamma}) \in L^1(\Gamma)\), and \((u_0)|_{\Sigma} = u_{0\Gamma}\) a.e. on \(\Gamma\). Moreover, let

\[ m_0 := \frac{1}{|\Omega|} \int_{\Omega} u_0 dx \in \text{int } D(\beta_\Gamma); \]
g ∈ L^2(0, T; H), g_Γ ∈ L^2(0, T; H_Γ); in the case τ = 0 let g ∈ H^1(0, T; H) or g ∈ L^2(0, T; V).

As a remark, we deduce that 0 ∈ \( \beta(0) \) and 0 ∈ \( \beta_Γ(0) \) as consequences from the assumption (A1). Next, we set |Ω| := \( \int_Ω 1dx \) and point out that the condition (2.7) in (A2) is very useful in the treatment of two different potentials, \( \beta \) in the bulk Ω and \( \beta_Γ \) on the boundary Γ (to our knowledge, this condition has been used for the first time in [6]). Moreover, let us note that in the case when \( \beta \) and \( \beta_Γ \) satisfy (2.7) for some \( \varrho \in (0, 1) \), then (A2) also works for \( \varrho = 1 \).

2.1. Well-posedness for \( \kappa \in (0, 1] \)

The system (2.1)–(2.6) and some extensions of it have been shown to be well posed [8,12] provided that \( \kappa > 0 \). More precisely, in view of the results proved in [8,12] we can state the following proposition.

**Proposition 2.1.** Under the assumptions (A1)–(A5), there exists a unique (weak) solution \((u_κ, μ_κ, ξ_κ, u_Γ, κ, ξ_Γ, κ)\) such that

- \( u_κ ∈ H^1(0, T; V^*) ∩ L^∞(0, T; V) ∩ L^2(0, T; H^2(Ω)) \),
- \( τu_κ ∈ H^1(0, T; H) ∩ C([0, T]; V) \),
- \( μ_κ ∈ L^2(0, T; V) \), \( ξ ∈ L^2(0, T; H) \),
- \( u_Γ, κ ∈ H^1(0, T; H_Γ) ∩ C([0, T]; V_Γ) ∩ L^2(0, T; H^2(Γ)) \),
- \( ξ_Γ, κ ∈ L^2(0, T; H_Γ) \)

and satisfying (2.1)–(2.6); if \( τ = 0 \), then Eq. (2.1) and boundary condition (2.3) make sense in terms of the variational formulation

\[
\langle \partial_t u(t), z \rangle_{V^*, V} + \int_Ω \nabla μ(t) \cdot \nabla z dx = 0
\]

for all \( z ∈ V \), for a.a. \( t ∈ (0, T) \).  \( \text{ (2.8) } \)

2.2. Asymptotic analysis as \( κ \searrow 0 \)

We aim to discuss the asymptotic analysis of the system (2.1)–(2.6) as the coefficient \( κ \) of the term with the Laplace–Beltrami operator in (2.5) tends to 0. Then, we will obtain the singular limit problem, with the equality in (2.5) replaced by

\[
\partial_t u_Γ + \partial_ν u + ξ_Γ + π_Γ(u_Γ) = g_Γ,
\]

meant in some dual space, and the characterization of the inclusion in (2.5), i.e.,

\[
ξ_Γ ∈ β_Γ(u_Γ) \quad \text{a.e. on } Σ,
\]

in a weaker form as well. For this purpose, we introduce the maximal monotone operator \( β_Γ,W_Γ : W_Γ → 2^{W_Γ^*} \), which is the subdifferential of the functional \( \hat{β}_Γ,H_Γ : W_Γ → [0, +∞] \) defined below (cf., e.g., [6, Section 5]). In fact, we set

\[
\hat{β}_Γ,H_Γ(z_Γ) := \begin{cases} 
\int_Γ \hat{β}_Γ(z_Γ) & \text{if } z ∈ H_Γ \text{ and } \hat{β}_Γ(z_Γ) ∈ L^1(Γ), \\
+∞ & \text{if } z ∈ H_Γ \text{ and } \hat{β}_Γ(z_Γ) \notin L^1(Γ),
\end{cases}
\]
and, subsequently,
\[ \tilde{\beta}_{\Gamma,W_T}(z_T) := \tilde{\beta}_{\Gamma,H_T}(z_T) \quad \text{if } z_T \in W_T. \]

We notice that both functionals \( \tilde{\beta}_{\Gamma,H_T} \) and \( \tilde{\beta}_{\Gamma,W_T} \) are proper (equal to 0 for \( z_T = 0 \)), lower semicontinuous, and convex on \( H_T \) and \( W_T \), respectively. Now, if we set
\[ \beta_{\Gamma,H_T} := \partial \tilde{\beta}_{\Gamma,H_T} : H_T \to 2^{H_T}, \]
then we have that \( z_T^* \in \beta_{\Gamma,H_T}(z_T) \) in \( H_T \) if and only if \( z_T^* \in H_T \), \( z_T \in D(\tilde{\beta}_{\Gamma,H_T}) \), and
\[ (z_T^*, \tilde{z}_T - z_T)_{H_T} \leq \tilde{\beta}_{\Gamma,H_T}(\tilde{z}_T) - \tilde{\beta}_{\Gamma,H_T}(z_T) \quad \text{for all } \tilde{z}_T \in H_T. \]

We emphasize that \( \beta_{\Gamma,H_T} \) is nothing but the operator induced by \( \beta_{\Gamma} \) on \( H_T \), so that (see, e.g., [2]) the inclusion (2.10) can be equivalently rewritten as
\[ \xi_T(t) \in \beta_{\Gamma,H_T}(u_T(t)) \quad \text{in } H_T, \quad \text{for a.a. } t \in (0,T). \]

On the other hand, for
\[ \beta_{\Gamma,W_T} := \partial \tilde{\beta}_{\Gamma,W_T} : W_T \to 2^{W_T}, \]
we remark that \( z_T^* \in \beta_{\Gamma,W_T}(z_T) \) in \( W_T^* \) if and only if \( z_T^* \in W_T^*, z_T \in D(\tilde{\beta}_{\Gamma,W_T}) \), and
\[ (z_T^*, \tilde{z}_T - z_T)_{W_T} \leq \tilde{\beta}_{\Gamma,W_T}(\tilde{z}_T) - \tilde{\beta}_{\Gamma,W_T}(z_T) \quad \text{for all } \tilde{z}_T \in W_T. \]

Hence, it is clear that \( z_T^* \in \beta_{\Gamma,H_T}(z_T) \) in \( H_T \) entails \( z_T^* \in \beta_{\Gamma,W_T}(z_T) \) in \( W_T^* \) whenever \( z_T^* \in H_T \subset W_T^* \) and \( z_T \in D(\tilde{\beta}_{\Gamma,W_T}) \subset W_T \subset H_T \). Then a possible extension of (2.10) is
\[ \xi_T(t) \in \beta_{\Gamma,W_T}(u_T(t)) \quad \text{in } W_T^*, \quad \text{for a.a. } t \in (0,T), \]
in the case when \( \xi_T(t) \notin H_T \) for all \( t \) of a set which is not negligible.

2.3. Convergence–existence theorem

Our main result is stated here.

**Theorem 2.2.** Let \( \tau \geq 0 \) and assume that (A1)–(A5) hold. For all \( \kappa \in (0,1] \) let \((u_\kappa,\mu_\kappa,\xi_\kappa,u_{\Gamma,\kappa},\xi_{\Gamma,\kappa})\) denote the solution to (2.1)–(2.6) defined by Proposition 2.1. Then exists a quintuplet \((u,\mu,\xi,u_\Gamma,\xi_\Gamma)\) such that
\[ u_\kappa \to u \quad \text{weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;V), \]
and strongly in \( C([0,T];H) \), \hspace{1cm} (2.11)
\[ \tau u_\kappa \to \tau u \quad \text{weakly in } H^1(0,T;H), \]
\[ \mu_\kappa \to \mu \quad \text{weakly in } L^2(0,T;V), \]
\[ \xi_\kappa \to \xi \quad \text{weakly in } L^2(0,T;H), \]
\[ \Delta u_\kappa \to \Delta u \quad \text{weakly in } L^2(0,T;H), \]
\[ u_{\Gamma,\kappa} \to u_\Gamma \quad \text{weakly star in } H^1(0,T;H_\Gamma) \cap L^\infty(0,T;W_T), \]
and strongly in \( C([0,T];H_\Gamma) \), \hspace{1cm} (2.16)
\[
\frac{\partial}{\partial \nu} u_\kappa \to \frac{\partial}{\partial \nu} u \quad \text{weakly in } L^2(0, T; W^*_{\Gamma}), \quad (2.17)
\]

\[
\xi_{\Gamma, \kappa} \to \xi_{\Gamma} \quad \text{weakly in } L^2(0, T; W^*_{\Gamma}) \quad (2.18)
\]
as \kappa \searrow 0. Moreover, the limit functions \( u, \mu, \xi, u_{\Gamma}, \xi_{\Gamma} \) satisfy

\[
\langle \partial_t u(t), z \rangle_{V^*, V} + \int_{\Omega} \nabla \mu(t) \cdot \nabla z \, dx = 0
\]

for all \( z \in V \), for a.a. \( t \in (0, T) \),

\[
\mu = \tau \partial_t u - \Delta u + \xi + \pi(u) - g, \quad \xi \in \beta(u) \quad \text{a.e. in } Q, \quad (2.19)
\]

\[
u(t) = u_{\Gamma} \quad \text{a.e. on } \Gamma, \quad (2.20)
\]

\[
\int_{\Gamma} \partial_t u_{\Gamma}(t) z_{\Gamma} d\Gamma + \langle \partial_{\nu} u(t), z_{\Gamma} \rangle_{W^*_{\Gamma}, W_{\Gamma}} + \langle \xi_{\Gamma}(t), z_{\Gamma} \rangle_{W^*_{\Gamma}, W_{\Gamma}} + \int_{\Gamma} \pi_{\Gamma}(u_{\Gamma}(t)) z_{\Gamma} d\Gamma = \int_{\Gamma} g_{\Gamma}(t) z_{\Gamma} d\Gamma \quad \text{for all } z_{\Gamma} \in W_{\Gamma}, \quad \text{for a.a. } t \in (0, T), \quad (2.21)
\]

\[
\xi_{\Gamma}(t) \in \beta_{\Gamma, W_{\Gamma}}(u_{\Gamma}(t)) \quad \text{in } W^*_{\Gamma}, \quad \text{for a.a. } t \in (0, T), \quad (2.22)
\]

\[
u(0) = u_0 \quad \text{a.e. in } \Omega, \quad u_{\Gamma}(0) = u_{\Gamma 0} \quad \text{a.e. on } \Gamma. \quad (2.23)
\]

The above result is simultaneously a convergent theorem of the solutions to (2.1)–(2.6) toward the solution of the limiting problem, which then exists and turns out to be unique, as the next result will confirm.

Note that the Eq. (2.9) on the boundary is expressed in the weak form (2.22). We remark that the regularity of the solution to (2.19)–(2.24) is not enough to conclude that \( \xi_{\Gamma} \) belong to \( L^2(0, T; H_{\Gamma}) \) (and in this case (2.9) may hold a.e. on \( \Sigma \)), although \( \Delta u \in L^2(0, T; H) \). Indeed, if we recall the elliptic regularity theorem [3, Theorem 3.2, p. 1.79] for \( u(t) \in V \)

\[
\left\{ \begin{array}{l}
-\Delta u(t) = \tilde{g}(t) \quad \text{a.e. in } \Omega, \\
\nu_{\Gamma}(t) = u_{\Gamma}(t) \quad \text{a.e. on } \Gamma,
\end{array} \right.
\]

for a.a. \( t \in (0, T) \), where \( \tilde{g} = \mu - \tau \partial_t u - \xi - \pi(u) + g \in L^2(0, T; H) \) and \( u_{\Gamma} \in L^2(0, T; W_{\Gamma}) \), we can just infer that \( \partial_{\nu} u \in L^2(0, T; W^*_{\Gamma}) \). In this regard, we announce that an improvement of this theorem is given in Sect. 5, under a special assumption on the graphs \( \beta \) and \( \beta_{\Gamma} \).

### 2.4. Continuous dependence on data

For the solutions to the problem (2.19)–(2.24) we can prove a continuous dependence result, which in particular implies uniqueness.

**Theorem 2.3.** Take two sets of data \( u_{0,1}, u_{0\Gamma,1}, g_{1}, g_{\Gamma,1} \) for \( i = 1, 2 \) satisfying (A4)–(A5) and

\[
\frac{1}{|\Omega|} \int_{\Omega} u_{0,1} \, dx = \frac{1}{|\Omega|} \int_{\Omega} u_{0,2} \, dx = m_0. \quad (2.25)
\]

Let \( (u_i, \mu_i, \xi_i, u_{\Gamma,i}, \xi_{\Gamma,i}), i = 1, 2, \) denote the corresponding solutions to (2.19)–(2.24). Then there exists a positive constant \( C \), depending only on \( L, L_{\Gamma}, \) and \( T \), such that
\begin{equation}
|u_1 - u_2|^2_{C([0,T];V^*)} + \tau |u_1 - u_2|^2_{C([0,T];H)} + |u_1 - u_2|^2_{L^2(0,T;V)} \\
+ |u\Gamma,1 - u\Gamma,2|^2_{C([0,T];H_T)} \\
\leq C\left(|u_{0,1} - u_{0,2}|^2_{V^*} + \tau |u_{0,1} - u_{0,2}|^2_H + |u_{0\Gamma,1} - u_{0\Gamma,2}|^2_{H_T} \\
+ |g_1 - g_2|^2_{L^2(0,T;H)} + |g\Gamma,1 - g\Gamma,2|^2_{L^2(0,T;H_T)} \right). 
\tag{2.26}
\end{equation}

Of course, this theorem implies the uniqueness of the solution $(u, \mu, \xi, u\Gamma, \xi\Gamma)$ obtained by the limit procedure in Theorem 2.2. The theorem will be proved in Sect. 4.

### 3. Approximate solutions

In this section we sketch the main steps for proving the existence of a solution stated in Proposition 2.1 and, at the same time, we will derive uniform estimates that will be useful in the proof of Theorem 2.2.

Thus, we approximate the problem (2.1)–(2.6) by introducing the Yosida regularizations $\beta_\varepsilon$ for $\beta$ and $\beta\Gamma,\varepsilon$ for $\beta\Gamma$ (see, e.g., [1,2]): for each $\varepsilon \in (0,1]$ and for all $r \in \mathbb{R}$ we set

\begin{align}
\beta_\varepsilon(r) &:= \frac{1}{\varepsilon} \left( r - J_\varepsilon(r) \right), \quad J_\varepsilon(r) := (I + \varepsilon\beta)^{-1}(r), 
\tag{3.1}
\\
\beta\Gamma,\varepsilon(r) &:= \frac{1}{\varepsilon} \left( r - J\Gamma,\varepsilon(r) \right), \quad J\Gamma,\varepsilon(r) := (I + \varepsilon\beta\Gamma)^{-1}(r). 
\tag{3.2}
\end{align}

Then, we point out that the condition (2.7) in (A2) implies that

\begin{equation}
|\beta_\varepsilon(r)| \leq \varrho |\beta\Gamma,\varepsilon(r)| + c_0 \quad \text{for all } r \in \mathbb{R}, 
\tag{3.3}
\end{equation}

for all $\varepsilon \in (0,1]$, with the same constants $\varrho$ and $c_0$ (see “Appendix”). We also have $\beta_\varepsilon(0) = \beta\Gamma,\varepsilon(0) = 0$.

Then, the problem in terms of the $\varepsilon$-approximation reads as follows: find a triplet $(u_\varepsilon, \mu_\varepsilon, u\Gamma,\varepsilon)$, with at least the same regularity as $(u_\kappa, \mu_\kappa, u\Gamma,\kappa)$ in Proposition 2.1, satisfying

\begin{align}
\partial_t u_\varepsilon - \Delta \mu_\varepsilon & = 0 \quad \text{a.e. in } Q, 
\tag{3.4}
\\
\mu_\varepsilon & = \tau \partial_t u_\varepsilon - \Delta u_\varepsilon + \beta_\varepsilon(u_\varepsilon) + \pi(u_\varepsilon) - g \quad \text{a.e. in } Q, 
\tag{3.5}
\\
\partial_\nu \mu_\varepsilon & = 0 \quad \text{a.e. on } \Sigma, 
\tag{3.6}
\\
(u_\varepsilon)|_\Gamma & = u\Gamma,\varepsilon \quad \text{a.e. on } \Sigma, 
\tag{3.7}
\\
\partial_t u\Gamma,\varepsilon + \partial_\nu u\Gamma,\varepsilon - \kappa \Delta u\Gamma,\varepsilon + \beta\Gamma,\varepsilon(u\Gamma,\varepsilon) + \pi\Gamma(u\Gamma,\varepsilon) & = g\Gamma \quad \text{a.e. on } \Sigma, 
\tag{3.8}
\\
u_\varepsilon(0) & = u_0 \quad \text{a.e. in } \Omega, \quad u\Gamma,\varepsilon(0) = u_{0\Gamma} \quad \text{a.e. on } \Gamma, 
\tag{3.9}
\end{align}

where (3.4) and (3.6) have to be collected into the proper variational formulation if $\tau = 0$ (cf. (2.8)). From the results shown in [8,12] it follows that there exists such a triplet $(u_\varepsilon, \mu_\varepsilon, u\Gamma,\varepsilon)$ and, in addition, it is unique. We observe that for the proof one can use the abstract theory of doubly nonlinear evolution equations presented in [15] and argue in the function spaces

$$H_0 := \{ z \in H : m(z) = 0 \},$$
Then we can introduce the norm $N$:

$$m(z) := \frac{1}{|\Omega|} \int_{\Omega} z dx.$$  

As a remark, we can identify $V_{0\ast}$ by $V_0^*$ (see, [9, Remark 2]). Then, from the Poincaré inequality, we have that there exists a positive constant $C_P$ such that

$$|z|_V \leq C_P |\nabla z|_H \quad \text{for all } z \in V_0,$$  

(3.10) that is, we see that $|\cdot|_V := |\nabla \cdot|_H$ and the standard $|\cdot|_V$ are equivalent norms on $V_0$. Moreover, we introduce the linear, bijective, and symmetric operator $N : V_{0\ast} \to V_0$ by $v = Nv^*$ if and only if $m(v) = 0$ and

$$\int_{\Omega} \nabla N v^* \cdot \nabla z dx = \int_{\Omega} \nabla v \cdot \nabla z dx = \langle v^*, z \rangle_{V^*, V} \quad \text{for all } z \in V.$$  

Then we can introduce the norm

$$|z^*|_{V_{0\ast}} := \left( \int_{\Omega} |\nabla N z^*|^2 dx \right)^{1/2} = |Nz^*|_{V_0} \quad \text{for all } z^* \in V_{0\ast}.$$  

Next, in the light of (3.4)–(3.9) we prove and collect some estimates for $(u_\varepsilon, \mu_\varepsilon, u_\Gamma, \varepsilon)$ that are independent of $\varepsilon$ and $\kappa \in (0, 1]$. The dependence on $\tau$ will be explicitly mentioned when needed.

3.1. Uniform estimates

**Lemma 3.1.** Let $\tau \in (0, 1]$; then there exists a positive constant $M_1 := M_1(\tau)$ independent of $\varepsilon, \kappa \in (0, 1]$ such that

$$|u_\varepsilon|_{H^1(0,T;V^*)} + |u_\varepsilon|_{L^\infty(0,T;V)} + \sqrt{\tau}|\partial_t u_\varepsilon|_{L^2(0,T;H)}$$

$$+ |u_{\Gamma,\varepsilon}|_{H^1(0,T;H_T)} + |u_{\Gamma,\varepsilon}|_{L^\infty(0,T;W_T)} + \sqrt{\kappa}|u_{\Gamma,\varepsilon}|_{L^\infty(0,T;V_T)}$$

$$+ |\beta_\varepsilon(u_\varepsilon)|_{L^\infty(0,T;L^1(\Omega))} + |\beta_{\Gamma,\varepsilon}(u_\varepsilon)|_{L^\infty(0,T;L^1(\Gamma))} \leq M_1. \quad (3.11)$$

Otherwise, if $\tau = 0$, then there exists a positive constant $M_2$ independent of $\varepsilon, \kappa \in (0, 1]$ such that

$$|u_\varepsilon|_{H^1(0,T;V^*)} + |u_\varepsilon|_{L^\infty(0,T;V)} + |u_{\Gamma,\varepsilon}|_{H^1(0,T;H_T)}$$

$$+ |u_{\Gamma,\varepsilon}|_{L^\infty(0,T;W_T)} + \sqrt{\kappa}|u_{\Gamma,\varepsilon}|_{L^\infty(0,T;V_T)}$$

$$+ |\beta_\varepsilon(u_\varepsilon)|_{L^\infty(0,T;L^1(\Omega))} + |\beta_{\Gamma,\varepsilon}(u_\varepsilon)|_{L^\infty(0,T;L^1(\Gamma))} \leq M_2. \quad (3.12)$$

**Proof.** By integrating (3.4) over $\Omega$ and using (3.6), we obtain $\int_{\Omega} \partial_t u_\varepsilon = 0$ a.e. in $(0,T)$. Hence, by integrating with respect to time and taking (3.9) and (A2) into account, we easily have

$$\frac{1}{|\Omega|} \int_{\Omega} u_\varepsilon(t) dx = \frac{1}{|\Omega|} \int_{\Omega} u_0 dx = m_0 \quad (3.13)$$

for all $t \in [0,T]$. Thus, we see that $m(u_\varepsilon(t)) = m_0$ and $\partial_t m(u_\varepsilon(s)) = m(\partial_t u_\varepsilon(s)) = 0$, that is, $\partial_t u_\varepsilon(s) \in V_0$, for all $t \in [0,T]$ and a.a. $s \in (0,T)$.  

Now, multiplying (3.4) by $\mathcal{N}(\partial_t u_\varepsilon)$, integrating the resultant over $\Omega$, and using (3.6) we obtain

$$
(\partial_t u_\varepsilon(s), \mathcal{N}(\partial_t u_\varepsilon)(s))_{H_0} + \int_\Omega \nabla u_\varepsilon(s) \cdot \nabla \mathcal{N}(\partial_t u_\varepsilon)(s) \, dx = 0 \quad (3.14)
$$

for a.a. $s \in (0, T)$. The continuation of the estimate is formal, at least in the case $\tau = 0$; however the reader may refer to [8,12] for the details of a rigorous proof. Hence, adding $u_\varepsilon$ to both sides of (3.5), testing it by $\partial_t u_\varepsilon$ and using (3.8) we obtain

$$
\int_\Omega \mu_\varepsilon(s) \partial_t u_\varepsilon(s) \, dx = \tau |\partial_t u_\varepsilon(s)|^2_H + \frac{1}{2} \frac{d}{dt} |u_\varepsilon(s)|^2_V + \frac{d}{dt} \int_\Omega \beta_\varepsilon(u_\varepsilon(s)) \, dx + |\partial_t u_{\Gamma, \varepsilon}(s)|^2_{H_\Gamma}
+ \frac{\kappa}{2} \frac{d}{dt} \int_\Gamma |\nabla u_{\Gamma, \varepsilon}(s)|^2 \, d\Gamma + \frac{d}{dt} \int_\Gamma \beta_{\Gamma, \varepsilon}(u_{\Gamma, \varepsilon}(s)) \, d\Gamma
- \int_\Omega \{ g(s) + u_\varepsilon(s) - \pi(u_\varepsilon(s)) \} \partial_t u_\varepsilon(s) \, dx
- \int_\Gamma \{ g_{\Gamma}(s) - \pi_{\Gamma}(u_{\Gamma, \varepsilon}(s)) \} \partial_t u_{\Gamma, \varepsilon}(s) \, d\Gamma \quad (3.15)
$$

for a.a. $s \in (0, T)$. Here, let us remark that

$$
(z, \mathcal{N}z)_{H_0} = (z, \mathcal{N}z)_{V^*, V} = \int_\Omega \nabla N z \cdot \nabla N z \, dx = |N z|^2_{V_0} = |z|^2_{V^*} \quad \text{for all} \quad z \in H_0,
$$

and

$$
\int_\Omega \nabla z_1 \cdot \nabla (\mathcal{N}z_2) \, dx = (z_2, z_1)_{V^*, V} = (z_2, z_1)_{H} \quad \text{for all} \quad z_1 \in V, \ z_2 \in H_0.
$$

Therefore, by subtracting (3.15) from (3.14) we cancel two terms and obtain

$$
\left| \partial_t u_\varepsilon(s) \right|^2_{V^*} + \tau |\partial_t u_\varepsilon(s)|^2_H + \frac{1}{2} \frac{d}{dt} |u_\varepsilon(s)|^2_V + \frac{d}{dt} \int_\Omega \beta_\varepsilon(u_\varepsilon(s)) \, dx
+ \left| \partial_t u_{\Gamma, \varepsilon}(s) \right|^2_{H_\Gamma} + \frac{\kappa}{2} \frac{d}{dt} \int_\Gamma |\nabla u_{\Gamma, \varepsilon}(s)|^2 \, d\Gamma + \frac{d}{dt} \int_\Gamma \beta_{\Gamma, \varepsilon}(u_{\Gamma, \varepsilon}(s)) \, d\Gamma
\leq \int_\Omega \{ g(s) + u_\varepsilon(s) - \pi(u_\varepsilon(s)) \} \partial_t u_\varepsilon(s) \, dx
+ \int_\Gamma \{ g_{\Gamma}(s) - \pi_{\Gamma}(u_{\Gamma, \varepsilon}(s)) \} \partial_t u_{\Gamma, \varepsilon}(s) \, d\Gamma \quad (3.16)
$$

for a.a. $s \in (0, T)$. Then, we remark that there exists some positive constant $C_1$, depending on $L_{\Gamma}$, such that

$$
\left| \int_\Gamma \{ g_{\Gamma}(s) - \pi_{\Gamma}(u_{\Gamma, \varepsilon}(s)) \} \partial_t u_{\Gamma, \varepsilon}(s) \, d\Gamma \right|
\leq \frac{1}{2} \left| \partial_t u_{\Gamma, \varepsilon}(s) \right|^2_{H_\Gamma} + \left| g_{\Gamma}(s) \right|^2_{H_\Gamma} + C_1 \left( 1 + \left| u_{\Gamma, \varepsilon}(s) \right|^2_{H_\Gamma} \right) \quad (3.17)
$$
for a.a. \( s \in (0, T) \). In the case when \( \tau > 0 \), the first term on the right-hand side of (3.16) can be handled with the help of the Young inequality, as

\[
\left| \int_{\Omega} \left\{ g(s) + u_\varepsilon(s) - \pi(u_\varepsilon(s)) \right\} \partial_t u_\varepsilon(s) dx \right| \\
\leq \frac{\tau}{2} |\partial_t u_\varepsilon(s)|^2_{H_\Gamma} + \frac{3}{2\tau} |g(s)|^2_H + \frac{C_2}{\tau} \left( 1 + |u_\varepsilon(s)|^2_H \right),
\]

(3.18)

while if \( \tau = 0 \) and \( g \in L^2(0, T; V) \), then we have that

\[
\left| \int_{\Omega} \left\{ g(s) + u_\varepsilon(s) - \pi(u_\varepsilon(s)) \right\} \partial_t u_\varepsilon(s) dx \right| \\
\leq \frac{1}{2} |\partial_t u_\varepsilon(s)|^2_{V^*} + C_2 \left( 1 + |g(s)|^2_V + |u_\varepsilon(s)|^2_V \right)
\]

(3.19)

for a.a. \( s \in (0, T) \), where \( C_2 \) is a positive constant, depending on \( L \). On the other hand, if \( \tau = 0 \) and \( g \in H^1(0, T; H) \), then we treat separately the related term and integrate by parts in time, so to obtain

\[
\int_0^t \int_{\Omega} g(s) \partial_t u_\varepsilon(s) dx ds \\
= \int_{\Omega} g(t) u_\varepsilon(t) dx - \int_{\Omega} g(0) u_0 dx - \int_0^t \int_{\Omega} \partial_t g(s) u_\varepsilon(s) dx ds
\]

(3.20)

on the right-hand side, when integrating (3.16) from 0 to \( t \in [0, T] \). Then, one has to estimate the three terms above.

Hence, let us integrate (3.16) with respect to time. In the case \( \tau > 0 \), we use (3.17) and (3.18) to find that there exists a positive constant \( C_3 \), depending on \( C_1 \), such that

\[
\frac{1}{2} |\partial_t u_\varepsilon|^2_{L^2(0, T; V^*)} + \frac{\tau}{2} |\partial_t u_\varepsilon|^2_{L^2(0, t; H)} + \frac{1}{2} |\partial_t u_\Gamma, \varepsilon|^2_{L^2(0, t; H_\Gamma)} + \frac{1}{2} |u_\varepsilon(t)|^2_V \\
+ \frac{\kappa}{2} \int_{\Gamma} |\nabla_\Gamma u_\Gamma, \varepsilon(t)|^2 d\Gamma \int \beta_\varepsilon(u_\varepsilon(t)) dx + \int_{\Gamma} \beta_\varepsilon(u_\Gamma, \varepsilon(t)) d\Gamma \\
\leq \frac{1}{2} |u_0|^2_V + \frac{\kappa}{2} \int_{\Gamma} |\nabla_\Gamma u_0|^2 d\Gamma + \int \beta(u_0) dx + \int_{\Gamma} \beta_\Gamma(u_0 \varepsilon(t)) d\Gamma \\
+ |g_\Gamma|^2_{L^2(0, T; H_\Gamma)} + C_3 \int_0^t \left( 1 + |u_\varepsilon(s)|^2_V \right) ds \\
+ \frac{3}{2\tau} |g|^2_{L^2(0, T; H)} + \frac{C_2}{\tau} \int_0^t \left( 1 + |u_\varepsilon(s)|^2_H \right) ds,
\]

(3.21)

where we used the linear continuity of the trace operator from \( V \) to \( H_\Gamma \) and the fundamental property of the Moreau–Yosida regularizations

\[ 0 \leq \beta_\varepsilon(r) \leq \beta(r), \quad 0 \leq \beta_\Gamma, \varepsilon(r) \leq \beta_\Gamma(r) \quad \text{for all} \quad \varepsilon \in (0, 1] \quad \text{and} \quad r \in \mathbb{R}. \]

Thus, we can apply the Gronwall lemma and infer the estimate (3.11). As a remark, we used the continuity of the trace operator from \( V \) to \( W_\Gamma \) to obtain the estimate of \( u_\Gamma, \varepsilon \) in \( L^\infty(0, T; W_\Gamma) \).
In the case when \( \tau = 0 \), if \( g \in L^2(0, T; V) \), then we use (3.19) and the last two terms of the estimate (3.21) modify into
\[
C_2 |g|^2_{L^2(0, T; V)} + C_2 \int_0^T (1 + |u_\varepsilon(s)|^2_V) ds.
\]
Thus, the estimate (3.12) can be obtained still by applying the Gronwall lemma, observing that \( C_2 \) is independent of \( \tau \). Instead, if \( g \in H^1(0, T; H) \), then in the right-hand side in the estimate (3.21) we can find the terms
\[
\frac{1}{4} |u_\varepsilon(t)|^2_V + |g|^2_{C([0, T]; H)} + |g|_{C([0, T]; H)} |u_0|_H
+ \frac{1}{2} |\partial_t g|^2_{L^2(0, T; H)} + \frac{1}{2} \int_0^T |u_\varepsilon(s)|^2_V ds,
\]
derived from (3.20) by applying the Young inequality and the embedding inequality \( |z|_H \leq |z|_V \), \( z \in V \). Then, the estimate (3.12) follows easily also in this case. \( \square \)

In the next lemmas we will not deal with constants as upperbounds, but with time functions whose boundedness in \( L^2(0, T) \) (or \( L^\infty(0, T) \)) is understood to be uniform with respect to \( \varepsilon \) and \( \kappa \in (0, 1] \).

**Lemma 3.2.** There exists a function \( \Lambda_0 \), bounded in \( L^2(0, T) \), such that
\[
|\mu_\varepsilon(t) - m(\mu_\varepsilon(t))|_V \leq \Lambda_0(t)
\]
for a.a. \( t \in (0, T) \).

**Proof.** We recall the Poincaré inequality (3.10) and test (3.4) by \( \mu_\varepsilon(t) - m(\mu_\varepsilon(t)) \in V_0 \). In view of (3.11) or (3.12), we can deduce that
\[
|\mu_\varepsilon(t) - m(\mu_\varepsilon(t))|^2_V \leq \left( C_P |\nabla(\mu_\varepsilon(t) - m(\mu_\varepsilon(t)))|_H \right)^2
= C_P^2 \langle \partial_t u_\varepsilon(t), \mu_\varepsilon(t) - m(\mu_\varepsilon(t)) \rangle_{V^*, V}
\leq C_P^2 |\partial_t u_\varepsilon(t)|_{V^*} |\mu_\varepsilon(t) - m(\mu_\varepsilon(t))|_V
\]
for a.a. \( t \in (0, T) \), whence (3.22) follows with \( \Lambda_0 = C_P^2 |\partial_t u_\varepsilon|_{V^*} \). \( \square \)

**Lemma 3.3.** There exists a function \( \Lambda_1 \), bounded in \( L^2(0, T) \), such that
\[
|\beta_\varepsilon(u_\varepsilon(t))|_{L^1(\Omega)} + |\beta_{\Gamma, \varepsilon}(u_{\Gamma, \varepsilon}(t))|_{L^1(\Gamma)} \leq \Lambda_1(t)
\]
for a.a. \( t \in (0, T) \).

**Proof.** First, we recall the useful inequality proved in [18, Section 5] and holding for both graphs under the assumptions (A2) and (A4): there exist two positive constants \( \delta_0 \) and \( c_1 \) such that
\[
\beta_\varepsilon(r - m_0) \geq \delta_0 |\beta_\varepsilon(r)| - c_1, \quad \beta_{\Gamma, \varepsilon}(r - m_0) \geq \delta_0 |\beta_{\Gamma, \varepsilon}(r)| - c_1
\]
for all \( r \in \mathbb{R} \) and \( \varepsilon \in (0, 1] \). Then, we test (3.5) by \( u_\varepsilon - m_0 \) and take advantage of (3.8) in order to deduce that
\[ \delta_0 \int_\Omega |\beta_\varepsilon(u_\varepsilon)| \, dx + \delta_0 \int_\Gamma |\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon})| \, d\Gamma - c_1 (|\Omega| + |\Gamma|) \]

\[ \leq \int_\Omega \beta_\varepsilon(u_\varepsilon)(u_\varepsilon - m_0) \, dx + \int_\Gamma \beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon})(u_{\Gamma,\varepsilon} - m_0) \, d\Gamma \]

\[ \leq \int_\Omega (\mu_\varepsilon - m(\mu_\varepsilon))(u_\varepsilon - m_0) \, dx + \int_\Gamma (g - \tau \partial_t u_\varepsilon - \pi(u_\varepsilon))(u_\varepsilon - m_0) \, dx \]

\[ - \int_\Omega |\nabla u_\varepsilon|^2 \, dx + \int_\Gamma (g_\Gamma - \partial_t u_{\Gamma,\varepsilon} - \pi_\Gamma(u_{\Gamma,\varepsilon}))(u_{\Gamma,\varepsilon} - m_0) \, d\Gamma \]

\[ - \kappa \int_\Gamma |\nabla u_{\Gamma,\varepsilon}|^2 \, d\Gamma \]

ea.e. on \((0,T)\), where based on (3.13) we used the following equality:

\[ \int_\Omega m(\mu_\varepsilon)(t)(u_\varepsilon(t) - m_0) \, dx = m(\mu_\varepsilon)(t) \int_\Omega (u_\varepsilon(t) - m_0) \, dx = 0 \]

for a.a. \(t \in (0,T)\). Hence, by squaring we arrive at

\[ \delta_0^2 \left( \int_\Omega |\beta_\varepsilon(u_\varepsilon)| \, dx + \int_\Gamma |\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon})| \, d\Gamma \right)^2 \leq 4\varepsilon_1^2 (|\Omega| + |\Gamma|)^2 + 4|\mu_\varepsilon - m(\mu_\varepsilon)|^2_{L^2} |u_\varepsilon - m_0|^2_{H} \]

\[ + 4 \left( |g|_H + \tau |\partial_t u_\varepsilon|_H + \sqrt{2} (L |u_\varepsilon|_H + |\pi(0)||\Omega|^{1/2}) \right)^2 |u_\varepsilon - m_0|^2_{H} \]

\[ + 4 \left( |g_{\Gamma}|_{H_{\Gamma}} + |\partial_t u_{\Gamma,\varepsilon}|_{H_{\Gamma}} + \sqrt{2} (L_\Gamma |u_{\Gamma,\varepsilon}|_{H_{\Gamma}} + |\pi_\Gamma(0)||\Gamma|^{1/2}) \right)^2 |u_{\Gamma,\varepsilon} - m_0|^2_{H_{\Gamma}} \]
a.e. on \((0,T)\). Now, note that (3.11) or (3.12), (3.22) and assumption (A5) enable us to infer that the right-hand side of the last inequality is a summable function in \((0,T)\). Hence, it follows that there is a function \(\Lambda_1\), bounded in \(L^2(0,T)\), such that (3.23) holds. \(\square\)

**Lemma 3.4.** There exist functions \(\Lambda_2\) and \(\Lambda_3\), bounded in \(L^2(0,T)\), such that

\[ |m(\mu_\varepsilon(t))| \leq \Lambda_2(t), \quad (3.25) \]

\[ \sqrt{\kappa} ||\partial_\nu u_\varepsilon(t)||_{H_{\Gamma}} + ||\partial_\nu u_\varepsilon(t)||_{W^1_{\Gamma}} \leq \Lambda_3(t) \quad (3.26) \]

for a.a. \(t \in (0,T)\).

**Proof.** Integrating (3.5) over \(\Omega\) directly and using (3.8) and (3.13) lead to

\[ |\Omega|m(\mu_\varepsilon) = \int_\Omega (\beta_\varepsilon(u_\varepsilon) + \pi(u_\varepsilon) - g) \, dx \]

\[ + \int_\Gamma (\partial_t u_{\Gamma,\varepsilon} + \beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}) + \pi_{\Gamma}(u_{\Gamma,\varepsilon}) - g_\Gamma) \, d\Gamma \]

\[ \leq |\beta_\varepsilon(u_\varepsilon)|_{L^1(\Omega)} + L |u_\varepsilon|_H |\Omega|^{1/2} + |\pi(0)||\Omega| + |g|_H |\Omega|^{1/2} + |\partial_t u_{\Gamma,\varepsilon}|_{H_{\Gamma}} |\Gamma|^{1/2} \]

\[ + |\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon})|_{L^1(\Gamma)} + L_{\pi} |u_{\Gamma,\varepsilon}|_{H_{\Gamma}} |\Gamma|^{1/2} + |\pi_{\Gamma}(0)||\Gamma| + |g_{\Gamma}|_{H_{\Gamma}} |\Gamma|^{1/2} \]
a.e. on \((0, T)\), whence Lemmas 3.1 and 3.3 allow us to conclude that a function \(\Lambda_2\), bounded in \(L^2(0, T)\), exists such that (3.25) holds. Next, we can combine (3.22) and (3.25) to deduce that

\[
\left| \mu_\varepsilon(t) \right|_V \leq \left| \mu_\varepsilon(t) - m(\mu_\varepsilon(t)) \right|_V + \left| m(\mu_\varepsilon(t)) \right|_V \\
\quad \leq \Lambda_0(t) + |\Omega|^{1/2} \Lambda_2(t) =: \Lambda_3(t)
\]

for a.a. \(t \in (0, T)\), where the function \(\Lambda_3\) is bounded in \(L^2(0, T)\) as well. \(\square\)

**Lemma 3.5.** There exists a function \(\Lambda_4\), bounded in \(L^2(0, T)\), such that

\[
\left| \beta_\varepsilon(\mathbf{u}_\varepsilon(t)) \right|_H + \left| \beta_\varepsilon(\mathbf{u}_{\Gamma, \varepsilon}(t)) \right|_{H_T} \leq \Lambda_4(t)
\]

(3.27) for a.a. \(t \in (0, T)\).

**Proof.** We test (3.5) by \(\beta_\varepsilon(\mathbf{u}_\varepsilon)\) and exploit (3.8) to obtain

\[
\int_\Omega \left| \beta_\varepsilon(\mathbf{u}_\varepsilon) \right|^2 dx + \int_\Omega \beta_\varepsilon(\mathbf{u}_\varepsilon) \|
abla \mathbf{u}_\varepsilon \|^2 dx \\
+ \int_\Gamma \beta_{\Gamma, \varepsilon}(\mathbf{u}_{\Gamma, \varepsilon}) \beta_\varepsilon(\mathbf{u}_{\Gamma, \varepsilon}) d\Gamma + \kappa \int_\Gamma \beta'_{\Gamma, \varepsilon}(\mathbf{u}_{\Gamma, \varepsilon}) \|
\nabla_{\Gamma} \mathbf{u}_{\Gamma, \varepsilon} \|^2 d\Gamma \\
\leq \int_\Omega (\mu_\varepsilon + g - \tau \partial_t \mathbf{u}_\varepsilon - \pi(\mathbf{u}_\varepsilon)) \beta_\varepsilon(\mathbf{u}_\varepsilon) dx \\
+ \int_\Gamma (g_{\Gamma} - \partial_t \mathbf{u}_{\Gamma, \varepsilon} - \pi_{\Gamma}(\mathbf{u}_{\Gamma, \varepsilon})) \beta_\varepsilon(\mathbf{u}_{\Gamma, \varepsilon}) d\Gamma
\]

(3.28)

a.e. in \((0, T)\), where we used that the trace of \(\beta_\varepsilon(\mathbf{u}_\varepsilon)\) is equal to \(\beta_\varepsilon(\mathbf{u}_{\Gamma, \varepsilon})\). In order to treat the gap between \(\beta\) and \(\beta_{\Gamma}\), we recall (3.3) and observe that

\[
\int_\Gamma \beta_{\Gamma, \varepsilon}(\mathbf{u}_{\Gamma, \varepsilon}) \beta_\varepsilon(\mathbf{u}_{\Gamma, \varepsilon}) d\Gamma = \int_\Gamma \left| \beta_{\Gamma, \varepsilon}(\mathbf{u}_{\Gamma, \varepsilon}) \right| \left| \beta_\varepsilon(\mathbf{u}_{\Gamma, \varepsilon}) \right| d\Gamma \\
\geq \frac{1}{\varrho} \int_\Gamma \left| \beta_\varepsilon(\mathbf{u}_{\Gamma, \varepsilon}) \right|^2 d\Gamma - \frac{c_0}{\varrho} \int_\Gamma \left| \beta_\varepsilon(\mathbf{u}_{\Gamma, \varepsilon}) \right| d\Gamma \\
\geq \frac{1}{2\varrho} \int_\Gamma \left| \beta_\varepsilon(\mathbf{u}_{\Gamma, \varepsilon}) \right|^2 d\Gamma - \frac{c_0}{2\varrho} |\Gamma|
\]

a.e. in \((0, T)\), because \(\beta_\varepsilon\) and \(\beta_{\Gamma, \varepsilon}\) have the same sign. Therefore, applying the Young inequality in (3.28) we deduce that

\[
\left| \beta_\varepsilon(\mathbf{u}_\varepsilon) \right|^2_H + \frac{1}{\varrho} \left| \beta_\varepsilon(\mathbf{u}_{\Gamma, \varepsilon}) \right|^2_{H_T} \\
\leq \frac{5}{2} \left( |\mu_\varepsilon|^2_H + g|^2_H + \tau^2 \|\partial_t \mathbf{u}_\varepsilon\|^2_H + L^2 |\mathbf{u}_\varepsilon|^2_H + |\pi(0)|^2 |\Omega| \right) + \frac{1}{2} \left| \beta_\varepsilon(\mathbf{u}_\varepsilon) \right|^2_H \\
+ 4\varrho \left( |g_{\Gamma}|^2_{H_T} + |\partial_t \mathbf{u}_{\Gamma, \varepsilon}|^2_{H_T} + L^2 |\mathbf{u}_{\Gamma, \varepsilon}|^2_{H_T} + |\pi_{\Gamma}(0)|^2 |\Gamma| \right) + \frac{1}{4\varrho} \left| \beta_\varepsilon(\mathbf{u}_{\Gamma, \varepsilon}) \right|^2_{H_T}
\]

a.e. in \((0, T)\), that is, by virtue of Lemmas 3.1 and 3.4, there is a function \(\Lambda_4\), which is bounded in \(L^2(0, T)\), such that (3.27) holds. \(\square\)

**Lemma 3.6.** There exist functions \(\Lambda_5\) and \(\Lambda_6\), bounded in \(L^2(0, T)\), such that

\[
\left| \Delta \mathbf{u}_\varepsilon(t) \right|_H \leq \Lambda_5(t),
\]

(3.29)
Lemma 3.7. There exist functions $\Lambda_7$, $\Lambda_8$, $\Lambda_{10}$, $\Lambda_{11}$, which are bounded in $L^2(0,T)$, and $\Lambda_9$, which is bounded in $L^\infty(0,T)$, such that

\[
\sqrt{\kappa} |\partial_\nu u_\varepsilon(t)|_{H^1_\Gamma} + |\partial_\nu u_\varepsilon(t)|_{W^{1,2}_\Gamma} \leq \Lambda_6(t)
\]  

(3.30)

for a.a. $t \in (0,T)$.

Proof. We write the Eq. (3.5) as

\[- \Delta u_\varepsilon = \mu_\varepsilon + g - \tau \partial_t u_\varepsilon - \beta_\varepsilon(u_\varepsilon) - \pi(u_\varepsilon) \quad \text{a.e. in } Q,
\]

(3.31)

and observe that, by a comparison in (3.31) and recalling (3.26), (A5), (3.11) (or (3.12) if $\tau = 0$), and (3.27), we deduce that there is a function $\Lambda_5$, which is bounded in $L^2(0,T)$, such that (3.29) holds. Next, we recall [3, Theorem 2.27, p. 1.64] to derive the estimate (3.30). Concerning (3.30), we remark that the coefficient $\sqrt{\kappa}$ is only in the first term since the control of $|\partial_\nu u_\varepsilon(t)|_{W^{1,2}_\Gamma}$ just needs the bound of $|u_\varepsilon(t)|_{V'}$ and (3.29). $\square$

Lemma 3.7. There exist functions $\Lambda_7$, $\Lambda_8$, $\Lambda_{10}$, $\Lambda_{11}$, which are bounded in $L^2(0,T)$, and $\Lambda_9$, which is bounded in $L^\infty(0,T)$, such that

\[
\sqrt{\kappa} |\beta_\Gamma u_\varepsilon(t)|_{H^1_\Gamma} \leq \Lambda_7(t),
\]

(3.32)

\[
\kappa^{3/2} |\Delta u_\varepsilon(t)|_{H^1_\Gamma} \leq \Lambda_8(t),
\]

(3.33)

\[
\sqrt{\kappa} |\Delta u_\varepsilon(t)|_{V^*_\Gamma} \leq \Lambda_9(t),
\]

(3.34)

\[
\kappa |\beta_\Gamma u_\varepsilon(t)|_{W^{1,2}_\Gamma} \leq \Lambda_{10}(t),
\]

(3.35)

\[
|\beta_\Gamma u_\varepsilon(t)|_{W^{1,2}_\Gamma} \leq \Lambda_{11}(t)
\]

(3.36)

for a.a. $t \in (0,T)$.

Proof. We write (3.8) as

\[-\kappa \Delta_\Gamma u_{\Gamma,\varepsilon} + \beta_\Gamma u_{\Gamma,\varepsilon} = g_\Gamma - \partial_t u_{\Gamma,\varepsilon} - \partial_\nu u_\varepsilon - \pi_\Gamma(u_{\Gamma,\varepsilon}) \quad \text{a.e. on } \Sigma.
\]

Multiplying it by $\beta_\Gamma u_{\Gamma,\varepsilon}$ and integrating over $\Gamma$, we obtain

\[
\kappa \int_\Gamma \beta_\Gamma^2 u_{\Gamma,\varepsilon} |\nabla_\Gamma u_{\Gamma,\varepsilon}|^2 d\Gamma + |\beta_\Gamma u_{\Gamma,\varepsilon}|_{H^1_\Gamma}^2
\]

\[
\leq \left| g_\Gamma - \partial_t u_{\Gamma,\varepsilon} - \pi_\Gamma(u_{\Gamma,\varepsilon}) \right|_{H^1_\Gamma} |\beta_\Gamma u_{\Gamma,\varepsilon}|_{H^1_\Gamma} + |\partial_\nu u_\varepsilon|_{H^1_\Gamma} |\beta_\Gamma u_{\Gamma,\varepsilon}|_{H^1_\Gamma}
\]

(3.37)

a.e. in $(0,T)$. Next, using the Young inequality in both terms of the right hand side, we find that

\[
\frac{1}{2} |\beta_\Gamma u_{\Gamma,\varepsilon}|_{H^1_\Gamma}^2 \leq \left| g_\Gamma - \partial_t u_{\Gamma,\varepsilon} - \pi_\Gamma(u_{\Gamma,\varepsilon}) \right|_{H^1_\Gamma}^2 + |\partial_\nu u_\varepsilon|_{H^1_\Gamma}^2
\]

whence (A5), (3.11) (or (3.12) if $\tau = 0$), (A3), and (3.30) enable us to deduce that
\[
\sqrt{\kappa} |\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}(t))|_{H_\Gamma} \\
\leq \sqrt{2} \left( |g_{\Gamma}(t)|_{H_\Gamma} + |\partial_t u_{\Gamma,\varepsilon}(t)|_{H_\Gamma} + |\pi_{\Gamma}(u_{\Gamma,\varepsilon}(t))|_{H_\Gamma} + \sqrt{\kappa} |\partial_\nu u_\varepsilon(t)|_{H_\Gamma} \right)
\]
for a.a. \( t \in (0, T) \), as \( 0 < \kappa \leq 1 \) in our setting. The above right-hand side is uniformly bounded in \( L^2(0, T) \), then there is a function \( \Lambda_7 \) such that (3.32) holds.

Next, multiplying (3.8) by \( \sqrt{\kappa} \) and comparing the terms, we obtain
\[
\kappa^{3/2} |\Delta_{\Gamma} u_{\Gamma,\varepsilon}(t)|_{H_\Gamma} \leq \sqrt{\kappa} |\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}(t))|_{H_\Gamma} + |g_{\Gamma}(t)|_{H_\Gamma} + |\partial_t u_{\Gamma,\varepsilon}(t)|_{H_\Gamma} + \sqrt{\kappa} |\partial_\nu u_\varepsilon(t)|_{H_\Gamma} + |\pi_{\Gamma}(u_{\Gamma,\varepsilon}(t))|_{H_\Gamma} \leq \Lambda_8(t)
\]
for a.a. \( t \in (0, T) \), where \( \Lambda_8 \) is bounded in \( L^2(0, T) \). Moreover, since \( \Delta_{\Gamma} \) is a linear and bounded operator from \( V_\Gamma \) to \( V_\Gamma^* \), then from (3.11) (or (3.12) if \( \tau = 0 \)) it follows that
\[
\sqrt{\kappa} |\Delta_{\Gamma} u_{\Gamma,\varepsilon}(t)|_{V_\Gamma^*} \leq \sqrt{\kappa} C_6 |u_{\Gamma,\varepsilon}(t)|_{V_\Gamma} \leq \Lambda_9(t)
\]
for a.a. \( t \in (0, T) \), where \( C_6 \) is a positive constant and \( \Lambda_9 \) is bounded in \( L^\infty(0, T) \). Then, by interpolation \( H_\Gamma \hookrightarrow W_\Gamma^* \hookrightarrow V_\Gamma \), there exists a positive constant \( C_7 \) such that
\[
|\kappa \Delta_{\Gamma} u_{\Gamma,\varepsilon}|_{W_\Gamma^*} \leq C_7 |\kappa^{3/2} \Delta_{\Gamma} u_{\Gamma,\varepsilon}|_{H_\Gamma} |\sqrt{\kappa} \Delta_{\Gamma} u_{\Gamma,\varepsilon}|_{V_\Gamma^*}^{1/2}
\]
a.e. on \( (0, T) \). Therefore, we can find a function \( \Lambda_{10} \), depending on \( \Lambda_8 \) and \( \Lambda_9 \), which is bounded in \( L^2(0, T) \) (actually, up to \( L^4(0, T) \)) such that (3.35) holds. Consequently, we recall (3.35), (A5), (3.11) (or (3.12) if \( \tau = 0 \), (3.30) and make a comparison of terms in (3.8) to deduce that
\[
|\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}(t))|_{W_\Gamma^*} \leq |\kappa \Delta_{\Gamma} u_{\Gamma,\varepsilon}(t)|_{W_\Gamma^*} + |g_{\Gamma}(t)|_{W_\Gamma^*} + |\partial_t u_{\Gamma,\varepsilon}(t)|_{W_\Gamma^*} + |\pi_{\Gamma}(u_{\Gamma,\varepsilon}(t))|_{W_\Gamma^*} + |\partial_\nu u_\varepsilon(t)|_{W_\Gamma^*} \leq \Lambda_{11}(t)
\]
for a.a. \( t \in (0, T) \), with \( \Lambda_{11} \) bounded in \( L^2(0, T) \) so that (3.36) holds. \( \square \)

3.2. Proof of Proposition 2.1

For the proof of the complete statement we refer to [8,12]. Here we just detail the limit procedure as \( \varepsilon \searrow 0 \), so as to show existence of the solution. Of course, we keep \( \kappa \in (0, 1) \) fixed and let the triplet \( (u_\varepsilon, \mu_\varepsilon, u_{\Gamma,\varepsilon}) \) solve (3.4)-(3.9) for \( \varepsilon > 0 \). By virtue of Lemmas 3.1 and 3.4-3.7, there exist some limit functions \( u_{\kappa}, \mu_\kappa, u_{\Gamma,\kappa}, \xi_\kappa, \xi_{\Gamma,\kappa} \) such that
\[
\begin{align*}
&u_\varepsilon \rightharpoonup u_\kappa \quad \text{weakly star in } H^1(0, T; V^*) \cap L^\infty(0, T; V), \\
&\sqrt{\tau} u_\varepsilon \rightharpoonup \sqrt{\tau} u_\kappa \quad \text{weakly in } H^1(0, T; H), \\
&\mu_\varepsilon \rightharpoonup \mu_\kappa \quad \text{weakly in } L^2(0, T; V), \\
&u_{\Gamma,\varepsilon} \rightharpoonup u_{\Gamma,\kappa} \quad \text{weakly star in } H^1(0, T; H_\Gamma) \cap L^\infty(0, T; V_\Gamma), \\
&\Delta u_\varepsilon \rightharpoonup \Delta u_\kappa \quad \text{weakly in } L^2(0, T; H),
\end{align*}
\]
\[ \Delta \Gamma \mu_{\Gamma, \varepsilon} \to \Delta \mu_{\Gamma, \kappa} \text{ weakly in } L^2(0, T; H_G), \]
\[ \partial_{\nu} u_{\varepsilon} \to \partial_{\nu} u_{\kappa} \text{ weakly in } L^2(0, T; H_G), \]
\[ \beta_{\varepsilon}(u_{\varepsilon}) \to \xi_{\kappa} \text{ weakly in } L^2(0, T; H), \]
\[ \beta_{\Gamma, \varepsilon}(u_{\Gamma, \varepsilon}) \to \xi_{\Gamma, \kappa} \text{ weakly in } L^2(0, T; H_G) \]
as \( \varepsilon \searrow 0 \), in principle for a subsequence, then for the entire family due to the uniqueness of the limits. Moreover, from a well-known compactness theorem (see, e.g. [29, Section 8, Corollary 4]), we obtain
\[ u_{\varepsilon} \to u_{\kappa} \text{ strongly in } C([0, T]; H) \cap L^2(0, T; V), \]
\[ u_{\Gamma, \varepsilon} \to u_{\Gamma, \kappa} \text{ strongly in } C([0, T]; H_G) \cap L^2(0, T; V_G) \]
as \( \varepsilon \searrow 0 \). This and the Lipschitz continuities of \( \pi \) and \( \pi_G \) give us
\[ \pi(u_{\varepsilon}) \to \pi(u_{\kappa}) \text{ strongly in } C([0, T]; H), \]
\[ \pi_G(u_{\Gamma, \varepsilon}) \to \pi_G(u_{\Gamma, \kappa}) \text{ strongly in } C([0, T]; H_G) \]
as \( \varepsilon \searrow 0 \). Now, applying [1, Proposition 2.2, p. 38] it is straightforward deduce that
\[ \xi_{\kappa} \in \beta(u_{\kappa}) \text{ a.e. in } Q, \quad \xi_{\Gamma} \in \beta_G(u_{\Gamma, \kappa}) \text{ a.e. on } \Sigma. \]
Therefore, passing to the limit as \( \varepsilon \searrow 0 \) in (3.4)–(3.9), we infer that the limit quintuplet \( (u_{\kappa}, \mu_{\kappa}, \xi_{\kappa}, u_{\Gamma, \kappa}, \xi_{\Gamma, \kappa}) \) solves (2.1)–(2.6) and thus conclude the existence proof. \( \square \)

4. Proof of main theorems

In view of Proposition 2.1 and with the aim of summarizing the previous section, we see that for \( \kappa \in (0, 1] \) the above limit \( (u_{\kappa}, \mu_{\kappa}, \xi_{\kappa}, u_{\Gamma, \kappa}, \xi_{\Gamma, \kappa}) \) satisfies
\[ u_{\kappa} \in H^1(0, T; V^*) \cap L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega)), \]
\[ \tau u_{\kappa} \in H^1(0, T; H) \cap C([0, T]; V), \]
\[ \mu_{\kappa} \in L^2(0, T; H), \quad \xi_{\kappa} \in L^2(0, T; H), \]
\[ u_{\Gamma, \kappa} \in H^1(0, T; H_G) \cap C([0, T]; V_G) \cap L^2(0, T; H^2(\Gamma)), \]
\[ \xi_{\Gamma, \kappa} \in L^2(0, T; H_G) \]
and solves
\[ \partial_t u_{\kappa} - \Delta \mu_{\kappa} = 0 \text{ a.e. in } Q, \quad (4.1) \]
\[ \mu_{\kappa} = \tau \partial_t u_{\kappa} - \Delta u_{\kappa} + \xi_{\kappa} + \pi(u_{\kappa}) - g, \quad \xi_{\kappa} \in \beta(u_{\kappa}) \text{ a.e. in } Q, \quad (4.2) \]
\[ \partial_{\nu} \mu_{\kappa} = 0 \text{ a.e. on } \Sigma, \quad (4.3) \]
\[ (u_{\kappa})_{|\Sigma} = u_{\Gamma, \kappa} \text{ a.e. on } \Sigma, \quad (4.4) \]
\[ \partial_t u_{\Gamma, \kappa} + \partial_{\nu} u_{\kappa} - \kappa \Delta_{\Gamma} u_{\Gamma, \kappa} + \xi_{\Gamma, \kappa} + \pi_G(u_{\Gamma, \kappa}) = g_{\Gamma}, \quad \xi_{\Gamma, \kappa} \in \beta_{\Gamma, \kappa}(u_{\Gamma, \kappa}) \text{ a.e. on } \Sigma, \quad (4.5) \]
\[ u_{\kappa}(0) = u_0 \text{ a.e. in } \Omega, \quad u_{\Gamma, \kappa}(0) = u_{0\Gamma} \text{ a.e. on } \Gamma. \quad (4.6) \]
If \( \tau = 0 \), then (4.1) and (4.3) are replaced by the variational equality
\[
\langle \partial_t u_\kappa(t), z \rangle_{V^*, V} + \int_{\Omega} \nabla u_\kappa(t) \cdot \nabla z \, dx = 0
\]
for all \( z \in V \), for a.a. \( t \in (0, T) \). Moreover, owing to Lemma 3.1 and the weak and weak star lower semicontinuity of norms we have the following uniform estimates: if \( \tau > 0 \), then from (3.11) it follows that
\[
|u_\kappa|_{H^1(0,T;V^*)} + |u_\kappa|_{L^\infty(0,T;V)} + \sqrt{\tau} |\partial_t u_\kappa|_{L^2(0,T;H)} \\
+ |u_{\Gamma,\kappa}|_{H^1(0,T;H)} + |u_{\Gamma,\kappa}|_{L^\infty(0,T;W_T)} + \sqrt{\kappa} |u_{\Gamma,\kappa}|_{L^\infty(0,T;V_T)} \leq M_1(\tau),
\]
(4.8)
while in the case \( \tau = 0 \), then by (3.12) we have that
\[
|u_\kappa|_{H^1(0,T;V^*)} + |u_\kappa|_{L^\infty(0,T;V)} + |u_{\Gamma,\kappa}|_{H^1(0,T;H)} \\
+ |u_{\Gamma,\kappa}|_{L^\infty(0,T;W_T)} + \sqrt{\kappa} |u_{\Gamma,\kappa}|_{L^\infty(0,T;V_T)} \leq M_2.
\]
(4.9)
Furthermore, in both cases from (3.26), (3.27), (3.29), (3.30), and (3.36) we obtain
\[
|\mu_\kappa|_{L^2(0,T;V)} \leq \liminf_{\varepsilon \to 0} |\mu_\varepsilon|_{L^2(0,T;V)} \leq |\Lambda_3|_{L^2(0,T)},
\]
(4.10)
\[
|\xi_\kappa|_{L^2(0,T;H)} \leq \liminf_{\varepsilon \to 0} |\beta_\varepsilon(u_\varepsilon)|_{L^2(0,T;H)} \leq |\Lambda_4|_{L^2(0,T)},
\]
(4.11)
\[
|\Delta u_\kappa|_{L^2(0,T;H)} \leq \liminf_{\varepsilon \to 0} |\Delta u_\varepsilon|_{L^2(0,T;H)} \leq |\Lambda_5|_{L^2(0,T)},
\]
(4.12)
\[
|\partial_\nu u_\kappa|_{L^2(0,T;W^*_T)} \leq \liminf_{\varepsilon \to 0} |\partial_\nu u_\varepsilon|_{L^2(0,T;W^*_T)} \leq |\Lambda_6|_{L^2(0,T)},
\]
(4.13)
\[
|\xi_{\Gamma,\kappa}|_{L^2(0,T;W_T)} \leq \liminf_{\varepsilon \to 0} |\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon})|_{L^2(0,T;W_T)} \leq |\Lambda_{11}|_{L^2(0,T)}.
\]
(4.14)
Based on these uniform estimates, we can prove Theorem 2.2.

4.1. Proof of Theorem 2.2

Thanks to the uniform estimates (4.8)–(4.14) and by compactness we deduce the following convergence properties: there exist a subsequence of \( \kappa \) (not relabeled) and some limit functions \( u, u_{\Gamma,\kappa}, \mu, \xi, \xi_{\Gamma} \) such that the convergences (2.11)–(2.18) and
\[
\kappa u_{\Gamma,\kappa} \to 0 \quad \text{strongly in } L^\infty(0,T;V_T)
\]
(4.15)
hold as \( \kappa \searrow 0 \). Indeed, we used the fact that the embedding \( W_T \subset H_T \) is compact, to obtain the strong convergence in (2.16). Now, by using (2.11) and (2.13), taking the limit in the variational formulation (4.7) yields (2.19). About the equation in (4.2), we can pass to the limit directly with respect to \( \kappa \) and obtain
\[
\mu = \tau \partial_t u - \Delta u + \xi + \pi(u) - g \quad \text{a.e. in } Q,
\]
(4.16)
whereas for the inclusion in (4.2) it is easy to infer that
\[
\xi \in \beta(u) \quad \text{a.e. in } Q,
\]
by (2.11), (2.14) and the demi-closedness of \( \beta \). Hence, also (2.20) is proved. The trace condition (2.21), i.e., \( u_{\mid \Sigma} = u_{\Gamma} \) a.e. on \( \Sigma \), is a consequence of (2.11)
and (2.16). The initial conditions (2.24) follow from (2.11) and (2.16) as well. It remains to pass to the limit in (4.5) in order to show (2.22) and (2.23). Then, testing the equality in (4.5) by an arbitrary $z_{\Gamma} \in V_{\Gamma}$, we have that

$$
\int_{\Gamma} \partial_t u_{\Gamma, \kappa} z_{\Gamma} d\Gamma + \langle \partial_{\nu} u_{\kappa}, z_{\Gamma} \rangle_{W_{\Gamma}^*, W_{\Gamma}} + \kappa \int_{\Gamma} \nabla u_{\Gamma, \kappa} \cdot \nabla z_{\Gamma} d\Gamma
+ \langle \xi_{\Gamma, \kappa}, z_{\Gamma} \rangle_{W_{\Gamma}^*, W_{\Gamma}} + \int_{\Gamma} \pi_{\Gamma}(u_{\Gamma, \kappa}) z_{\Gamma} d\Gamma = \int_{\Gamma} g_{\Gamma} z_{\Gamma} d\Gamma
$$

a.e. in $(0, T)$. Taking the limit as $\kappa \searrow 0$, with the help of (2.11), (2.16)–(2.18) and (4.15) we obtain (2.22) for all $z_{\Gamma} \in V_{\Gamma}$. As a further step, the density of $V_{\Gamma}$ in $W_{\Gamma}$ helps us to definitively show (2.22). Let us point out that this implies that the equation

$$
\partial_t u_{\Gamma} + \partial_{\nu} u + \xi_{\Gamma} + \pi_{\Gamma}(u_{\Gamma}) = g_{\Gamma} \quad \text{holds in } L^2(0, T; W_{\Gamma}^*). \quad (4.18)
$$

Next, we have to prove (2.23). Note here that from the inclusion in (4.5), we have that

$$
\xi_{\Gamma, \kappa}(t) \in \beta_{\Gamma, H_{\Gamma}}(u_{\Gamma, \kappa}(t)) \quad \text{in } H_{\Gamma}, \text{ for a.a. } t \in (0, T).
$$

Thus, due to the convergences (2.16) and (2.18), it is enough to prove that

$$
\limsup_{\kappa \searrow 0} \int_{0}^{T} \langle \xi_{\Gamma, \kappa}(t), u_{\Gamma, \kappa}(t) \rangle_{W_{\Gamma}^*, W_{\Gamma}} dt \leq \int_{0}^{T} \langle \xi_{\Gamma}(t), u_{\Gamma}(t) \rangle_{W_{\Gamma}^*, W_{\Gamma}} dt. \quad (4.19)
$$

Indeed, taking $z_{\Gamma} := u_{\Gamma, \kappa}$ in (4.17) and integrating it with respect to time, we obtain

$$
\int_{0}^{T} \langle \xi_{\Gamma, \kappa}(t), u_{\Gamma, \kappa}(t) \rangle_{W_{\Gamma}^*, W_{\Gamma}} dt = \int_{0}^{T} (g_{\Gamma}(t) - \partial_t u_{\Gamma, \kappa}(t) - \pi_{\Gamma}(u_{\Gamma, \kappa}(t)), u_{\Gamma, \kappa}(t))_{H_{\Gamma}} dt
- \int_{0}^{T} \langle \partial_{\nu} u_{\kappa}(t), u_{\Gamma, \kappa}(t) \rangle_{W_{\Gamma}^*, W_{\Gamma}} dt - \kappa \int_{0}^{T} \int_{\Gamma} |\nabla u_{\Gamma, \kappa}(t)|^2 d\Gamma dt. \quad (4.20)
$$

Next, we observe that an integration by part formula gives

$$
- \int_{0}^{T} \langle \partial_{\nu} u_{\kappa}(t), u_{\Gamma, \kappa}(t) \rangle_{W_{\Gamma}^*, W_{\Gamma}} dt
= - \int_{0}^{T} \int_{\Omega} \Delta u_{\kappa}(t) u_{\kappa}(t) dx dt - \int_{0}^{T} \int_{\Omega} |\nabla u_{\kappa}(t)|^2 dx dt
$$

and, in view of (2.11) and (2.15), taking the limit superior yields

$$
\limsup_{\kappa \searrow 0} \left\{ - \int_{0}^{T} \langle \partial_{\nu} u_{\kappa}(t), u_{\Gamma, \kappa}(t) \rangle_{W_{\Gamma}^*, W_{\Gamma}} dt \right\}
= - \int_{0}^{T} \int_{\Omega} \Delta u(t) u(t) dx dt - \liminf_{\kappa \searrow 0} \int_{0}^{T} \int_{\Omega} |\nabla u_{\kappa}(t)|^2 dx dt
\leq - \int_{0}^{T} \int_{\Omega} \Delta u(t) u(t) dx dt - \int_{0}^{T} \int_{\Omega} |\nabla u(t)|^2 dx dt
$$
$$\frac{1}{2} \frac{d}{dt} |u_1 - u_2|^2_H + (\mu_1 - \mu_2, u_1 - u_2)_H = 0$$

(4.21)
a.e. in $(0, T)$. Now, with the help of (2.22), we have that
\[
- \left\langle \partial_{\nu}(u_1 - u_2), u_{\Gamma,1} - u_{\Gamma,2} \right\rangle_{W_{\Gamma}^r, W_{\Gamma}^s}
= \frac{1}{2} \frac{d}{dt} |u_{\Gamma,1} - u_{\Gamma,2}|_{H_r}^2 + \langle \xi_{\Gamma,1} - \xi_{\Gamma,2}, u_{\Gamma,1} - u_{\Gamma,2} \rangle_{H_r}
+ (\pi_{\Gamma}(u_{\Gamma,1}) - \pi_{\Gamma}(u_{\Gamma,2}), u_{\Gamma,1} - u_{\Gamma,2})_{H_r}
- (g_{\Gamma,1} - g_{\Gamma,2}, u_{\Gamma,1} - u_{\Gamma,2})_{H_r}
\]
a.e. in $(0, T)$. Combining (4.21)–(4.23), adding $|u_1 - u_2|_{L^2}^2$, applying the Young inequality, and using the monotonicity of $\beta$ and $\beta_{\Gamma}$, we obtain
\[
\frac{1}{2} \frac{d}{dt} |u_1 - u_2|_{V^*}^2 + \frac{\tau}{2} \frac{d}{dt} |u_1 - u_2|_{H}^2 + |u_{\Gamma,1} - u_{\Gamma,2}|_{H_r}^2
\leq |u_1 - u_2|_{L^2}^2 + L|u_1 - u_2|_{H}^2 + |u_{\Gamma,1} - u_{\Gamma,2}|_{H_r}^2 + \frac{1}{4} |g_1 - g_2|_{H_r}^2
+ L_{\Gamma}|u_{\Gamma,1} - u_{\Gamma,2}|_{H_r}^2 + |u_{\Gamma,1} - u_{\Gamma,2}|_{H_r}^2 + \frac{1}{4} |g_{r,1} - g_{r,2}|_{H_r}^2
\leq (2 + L)(\delta |u_1 - u_2|_{V^*} + C(\delta)|u_1 - u_2|_{V^*}^2) + (1 + L_{\Gamma})|u_{\Gamma,1} - u_{\Gamma,2}|_{H_r}^2
+ \frac{1}{4} |g_1 - g_2|_{H_r}^2 + \frac{1}{4} |g_{r,1} - g_{r,2}|_{H_r}^2
\]
a.e. in $(0, T)$, where to obtain this we used the compactness inequality (see, e.g., [20, Lemme 5.1, p. 58] or [21, Theorem 16.4, p. 102]) with the parameter $\delta > 0$ and some constant $C(\delta)$. Now we take $\delta = 1/2(2 + L)$, then integrate from 0 to $t \in [0, T]$ using the initial conditions and finally apply the Gronwall lemma. We deduce that
\[
\left| u_1(t) - u_2(t) \right|_{V^*}^2 + \tau \left| u_1(t) - u_2(t) \right|_{H_r}^2
+ \int_0^t \left| (u_1 - u_2)(s) \right|_{V^*}^2 ds + |u_{\Gamma,1}(t) - u_{\Gamma,2}(t)|_{H_r}^2
\leq C_8 \left\{ |u_{0,1} - u_{0,2}|_{V^*}^2 + \tau |u_{0,1} - u_{0,2}|_{H_r}^2 + |u_{0,\Gamma,1} - u_{0,\Gamma,2}|_{H_r}^2
+ |g_1 - g_2|_{L^2(0, T; H_r)} + |g_{r,1} - g_{r,2}|_{L^2(0, T; H_r)} \right\}
\]
for all $t \in [0, T]$, where $C_8$ is a positive constant depending on $L, L_{\Gamma}$, and $T$. Finally, by the last inequality we arrive at the desired estimate (2.26). \(\square\)

5. Improvement of the convergence–existence theorem

The aim of this section is an improvement of Theorem 2.2 in the case when the two graphs $\beta$ and $\beta_{\Gamma}$ grow at the same kind of rate, more precisely:

(A2)’ $D(\beta_{\Gamma}) = D(\beta)$ and there exist two constants $\varrho \geq 1$ and $c_0 > 0$ such that
\[
\frac{1}{\varrho} |\beta^r_{\Gamma}(r)| - c_0 \leq |\beta^r(\varrho)| \leq \varrho |\beta^r_{\Gamma}(r)| + c_0 \quad \text{for all } r \in D(\beta) \equiv D(\beta_{\Gamma}). \quad (5.1)
\]
Under this assumption, the same inequalities as (5.1) hold for the Yosida regularizations $\beta_\varepsilon$ and $\beta_{\Gamma,\varepsilon}$, that is,
\[
\frac{1}{\varrho} |\beta_{\Gamma,\varepsilon}(r)| - c_0 \leq |\beta_\varepsilon(r)| \leq \varrho |\beta_{\Gamma,\varepsilon}(r)| + c_0 \quad \text{for all } r \in \mathbb{R}, \tag{5.2}
\]
for all $\varepsilon \in (0, 1]$, with the same constants $\varrho$ and $c_0$ (see “Appendix”).

Under the assumption $(A2)'$, the conclusion of Theorem 2.2 can be improved. In particular, the solution $(u, \mu, \xi, u_\Gamma, \xi_\Gamma)$ to (2.19)–(2.24) is more regular.

**Theorem 5.1.** Let $\tau \geq 0$ and assume that $(A1), (A2)'$, $(A3)$–$(A5)$ hold. For all $\kappa \in (0, 1]$ let $(u_\kappa, \mu_\kappa, \xi_\kappa, u_{\Gamma,\kappa}, \xi_{\Gamma,\kappa})$ denote the solution to (2.1)–(2.6) defined by Proposition 2.1. Then the convergences (2.11)–(2.17) and
\[
\xi_{\Gamma,\kappa} \to \xi_\Gamma \quad \text{weakly in } L^2(0, T; H_\Gamma) \tag{5.3}
\]
hold as $\kappa \searrow 0$, where $(u, \mu, \xi, u_\Gamma, \xi_\Gamma)$ is the solution to (2.19)–(2.24) and satisfies
\[
\begin{align*}
&u \in H^1(0, T; V^*) \cap C([0, T]; H) \cap L^\infty(0, T; V) \cap L^2(0, T; H^3/2(\Omega)), \tag{5.4} \\
&\tau u \in H^1(0, T; H), \quad \mu \in L^2(0, T; V), \quad \xi \in L^2(0, T; H), \tag{5.5} \\
&u_\Gamma \in H^1(0, T; H_\Gamma) \cap C([0, T]; W_\Gamma) \cap L^2(0, T; V_\Gamma), \tag{5.6} \\
&\partial_\nu u \in L^2(0, T; H_\Gamma), \quad \xi_\Gamma \in L^2(0, T; H_\Gamma). \tag{5.7}
\end{align*}
\]
Moreover, we have that
\[
\partial_t u_\Gamma + \partial_\nu u + \xi_\Gamma + \pi_\Gamma(u_\Gamma) = g_\Gamma, \quad \xi_\Gamma \in \beta_\Gamma(u_\Gamma) \quad \text{a.e. on } \Sigma \tag{5.8}
\]
as improvement of (2.22) and (2.23).

**Proof.** We refer to the estimate (3.36) in Lemma 3.7 and show that $(A2)'$ and consequently (5.2) allow us to produce a better estimate. Indeed, thanks to (5.2) and (3.27) we can obtain
\[
|\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}(t))|^2_{H_\Gamma} \leq \int_{\Gamma} \left( \varrho |\beta_\varepsilon(u_{\Gamma,\varepsilon}(t))| + c_0 \right)^2 d\Gamma \\
\leq 2\varrho^2 |\beta_\varepsilon(u_{\Gamma,\varepsilon}(t))|^2_{H_\Gamma} + 2c_0^2 |\Gamma| \\
\leq C_0 (1 + \Lambda_4(t)^2)
\]
for a.a. $t \in (0, T)$, where the right-hand side is bounded in $L^1(0, T)$. Thus, the estimate (3.36) in Lemma 3.7 is replaced by the key estimate
\[
|\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}(t))|^2_{H_\Gamma} \leq \Lambda_{12}(t)
\]
for a.a. $t \in (0, T)$, where $\Lambda_{12}$ is bounded in $L^2(0, T)$. Hence, the convergence in (2.18) can be improved to (5.3). Using the demi-closedness of the maximal monotone operator induced by $\beta_\Gamma$ on $L^2(0, T; H_\Gamma)$, in the light of (2.16) and (5.3) we can show that
\[
\xi_\Gamma \in \beta_\Gamma(u_\Gamma) \quad \text{a.e. on } \Sigma,
\]
which improves (2.23). Moreover, from a comparison of the terms in (4.18) it turns out that
\[ \partial_{\nu} u \in L^2(0, T; H_{\Gamma}), \] (5.9)
thus the condition (4.18) on the boundary actually holds in \( L^2(0, T; H_{\Gamma}) \). The additional information (5.9), along with
\[ u \in L^2(0, T; V), \quad \Delta u \in L^2(0, T; H), \]
implies that \( u \in L^2(0, T; H^{3/2}(\Omega)) \). Indeed, one can apply the elliptic regularity theorem [3, Theorem 3.2, p. 1.79] to \( u(t) \in V \) solving
\[
\begin{align*}
-\Delta u(t) &= \tilde{g}(t) \quad \text{a.e. in } \Omega, \\
\partial_{\nu} u(t) &= \tilde{g}_{\Gamma}(t) \quad \text{a.e. on } \Gamma
\end{align*}
\]
for a.a. \( t \in (0, T) \), where
\[
\tilde{g} = \mu - \tau \partial_t u - \xi - \pi(u) + g \in L^2(0, T; H), \quad \tilde{g}_{\Gamma} = g_{\Gamma} - \partial_t u_{\Gamma} - \xi_{\Gamma} - \pi_{\Gamma}(u_{\Gamma}) \in L^2(0, T; H_{\Gamma}).
\]
As a consequence of \( u \in L^2(0, T; H^{3/2}(\Omega)) \), the trace theory enables us deduce that \( u_{\Gamma} = u|_{\Gamma} \in L^2(0, T; V_{\Gamma}) \), whence (5.6) follows by interpolation. \( \square \)

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## Appendix A

We use the same setting as in the previous sections.

**Lemma A.1.** Assume (A2). Then (3.3) holds, that is,
\[
|\beta_\varepsilon(r)| \leq \varrho |\beta_{\Gamma, \varepsilon}(r)| + c_0 \quad \text{for all } r \in \mathbb{R},
\]
for all \( \varepsilon \in (0, 1] \) with the same constants \( \varrho \geq 1 \) and \( c_0 > 0 \).
Proof. Thanks to [6, Lemma 4.4], it is known that
\[ |\beta_{\varepsilon}(r)| \leq \varrho |\beta_{\Gamma,\varepsilon \varrho}(r)| + c_0 \quad \text{for all } r \in \mathbb{R}, \]
where \( \beta_{\Gamma,\varepsilon \varrho} \) denotes the Yosida approximation of \( \beta_{\Gamma} \) with parameter \( \varepsilon \varrho \), i.e.,
\[ \beta_{\Gamma,\varepsilon \varrho}(r) := \frac{1}{\varepsilon \varrho} (r - (I + \varepsilon \varrho \beta_{\Gamma})^{-1}(r)). \]  
(5.10)
Then, recalling that \( \varrho \geq 1 \), we may invoke the fundamental property [2, Proposition 2.6, p. 28] of Yosida approximations, which implies that
\[ |\beta_{\Gamma,\varepsilon \varrho}(r)| \leq |\beta_{\epsilon \varrho}(r)| \quad \text{for all } r \in \mathbb{R}, \]
because \( \varepsilon \leq \varepsilon \varrho \). Thus we get the conclusion. \( \square \)

Lemma A.2. Assume (A2)'. Then (5.2) holds, that is,
\[ \frac{1}{\varrho} |\beta_{\Gamma,\varepsilon}(r)| - c_0 \leq |\beta_{\varepsilon}(r)| \leq \varrho |\beta_{\Gamma,\varepsilon}(r)| + c_0 \quad \text{for all } r \in \mathbb{R}, \]
for all \( \varepsilon \in (0,1] \) with the same constants \( \varrho \geq 1 \) and \( c_0 > 0 \).

Proof. In view of Lemma A.1, is enough to prove that
\[ \frac{1}{\varrho} |\beta_{\Gamma,\varepsilon}(r)| - c_0 \leq |\beta_{\varepsilon}(r)| \quad \text{for all } r \in \mathbb{R}, \]
which is the same as
\[ |\beta_{\Gamma,\varepsilon}(r)| \leq \varrho |\beta_{\varepsilon}(r)| + \varrho c_0 \quad \text{for all } r \in \mathbb{R}. \]
But this follows immediately from Lemma A.1 again. \( \square \)

Remark A.3. Comparing to previous works (see, e.g., [8,9,12]) in which the same kind of property (2.7) was assumed for the two maximal monotone graphs, the parameter of the Yosida regularizations is here the same for both graphs (see also [11, Section 2]) . Instead, in the approach devised in [6, Lemma 4.4] exactly the approximation \( \beta_{\Gamma,\varepsilon \varrho} \) defined by (5.10) was introduced and used for \( \beta_{\Gamma} \).

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