Abstract. Recently, Giles et al. [14] proved that the efficiency of the Multilevel Monte Carlo (MLMC) method for evaluating Down-and-Out barrier options for a diffusion process \( (X_t)_{t \in [0,T]} \) with globally Lipschitz coefficients, can be improved by combining a Brownian bridge technique and a conditional Monte Carlo method provided that the running minimum \( \inf_{t \in [0,T]} X_t \) has a bounded density in the vicinity of the barrier. In the present work, thanks to the Lamperti transformation technique and using a Brownian interpolation of the drift implicit Euler scheme of Alfonsi [2], we show that the efficiency of the MLMC can be also improved for the evaluation of barrier options for models with non-Lipschitz diffusion coefficients under certain moment constraints. We study two example models: the Cox-Ingersoll-Ross (CIR) and the Constant of Elasticity of Variance (CEV) processes for which we show that the conditions of our theoretical framework are satisfied under certain restrictions on the models parameters. In particular, we develop semi-explicit formulas for the densities of the running minimum and running maximum of both CIR and CEV processes which are of independent interest. Finally, numerical tests are processed to illustrate our results.

1. Introduction

Barrier options are one of the most widely traded exotic options in the financial markets. Pricing and hedging such path-dependent option can quickly become very challenging especially when we need to achieve a good precision for the approximation. Evaluating barrier options by a classic Monte Carlo method introduces a systematic bias when approximating the continuous running maximum (resp. minimum) in the crossing-barrier indicator by a discrete running maximum (resp. minimum). To overcome this difficulty, several numerical strategies exist in the literature among them the popular Brownian bridge technique introduced in [3] well known for its efficiency and ease of use (see also [16] for related refinements). The Brownian bridge technic uses an analytic expression for the probability of hitting the barrier between two known values in a simulated path of the underlying asset. More recently, a combination of the Multilevel Monte Carlo (MLMC) method with the Brownian bridge technique has been developed in [14] for pricing barrier options. The Multilevel Monte Carlo method introduced in Giles [12] as an extension of the two-level Monte Carlo method of [20], significantly reduces the time complexity of the classical Monte Carlo method. More precisely, for a given precision \( \varepsilon > 0 \) and a Lipschitz payoff function, if the underlying asset \( (X_t)_{t \in [0,T]} \) is approximated using a discretization scheme \( (\bar{X}_t)_{t \in [0,T]} \) with time step \( h > 0 \) satisfying \( \mathbb{E}|X_t - \bar{X}_t|^\beta = O(h^\beta) \) and \( \mathbb{E}|X_t - \bar{X}_t| = O(h^\alpha) \) with \( \alpha \geq \frac{1}{2} \), then the time complexity of the MLMC methods is: \( O(\varepsilon^{-2}) \) when \( \beta > 1 \), \( O(\varepsilon^{-2}(\log \varepsilon)^2) \) when \( \beta = 1 \) and \( O(\varepsilon^{-2}\frac{1-\beta}{\alpha}) \) when \( \beta \in (0,1) \). However, for the same precision \( \varepsilon > 0 \) the optimal time complexity of a classic Monte Carlo method is \( O(\varepsilon^{-3}) \). As the payoff function of a barrier option is not Lipschitz, Giles et al. [14] take advantage of the Brownian bridge to run the MLMC method for pricing such options, since this technique substitutes...
the barrier-crossing indicators by the probabilities that the approximation scheme \((\bar{X}_t)_{t \in [0,T]}\) hits the barrier between each two consecutive discretization times \(t_i\) and \(t_{i+1}\) and which are represented as smooth functions of the realized points \(\bar{X}_{t_i}\) and \(\bar{X}_{t_{i+1}}\). More precisely, Giles et al. [14] consider an underlying asset solution to a one-dimensional stochastic differential equation (SDE) with globally Lipschitz smooth coefficients that is approximated by a high order strong approximation scheme namely the Milstein scheme \((\bar{X}^{\text{Milstein}}_t)_{t \in [0,T]}\) that satisfies \(\mathbb{E}|X_t - \bar{X}^{\text{Milstein}}_t|^2 = O(h^2)\). For this case, they prove that the MLMC method reaches its optimal time complexity \(O(\varepsilon^{-2})\) for pricing a Down-and-Out barrier option \(^1\) provided that \(\inf_{t \in [0,T]} X_t\) has a bounded density in the neighborhood of the barrier. This latter condition cannot be easily checked even when the SDE coefficients are Lipschitz except for very specific cases.

In the current paper, we are interested in studying the MLMC method for pricing barrier options when the underlying asset is solution to a SDE with a non-Lipschitz diffusion coefficient such as the popular Cox-Ingerson-Ross (CIR) and the Constant of Elasticity of Variance (CEV) processes. Only few works exist in the literature that studied the problem of pricing path-dependent options under such singular models (see e.g. [8]). To analyze the performance of the MLMC method, we consider in Section 2 a general framework of models with non-Lipschitz diffusion coefficients and use a Lamperti transformation to focus our study on a new process \((\bar{Y}_t)_{t \in [0,T]}\) with an additive noise diffusion but in counterpart with a possibly singular drift coefficient \(L\). Then, we introduce a Brownian interpolation scheme \((\bar{Y}_t)_{t \in [0,T]}\) associated to the drift implicit Euler scheme of Alfonsi [2] for which we prove a strong convergence result with order one (see Theorem 2.1). In Section 3, we use the Brownian bridge technic that substitutes the crossing-indicators with smooth functions of realized points in the path of the scheme \((\bar{Y}_t)_{t \in [0,T]}\) to build the corresponding MLMC estimator. Next, under suitable assumptions on the drift \(L\), we prove that the obtained MLMC method for pricing Down-and-Out (resp. Up-and-Out) reaches its optimal time complexity \(O(\varepsilon^{-2})\) provided that \(\inf_{t \in [0,T]} Y_t\) (resp. \(\sup_{t \in [0,T]} Y_t\)) has a bounded density in the neighborhood of the barrier (see Theorem 3.3 and Remark 3.4). In Sections 4 and 5, we provide two examples of processes satisfying Theorem 3.3 conditions, namely the CIR and the CEV models. It turns out that under additional constraints on the parameters of these two models ensuring the existence of finite negative moments up to a certain order, the MLMC method behaves exactly like a classical unbiased Monte Carlo estimator despite the use of approximation schemes. To show that the conditions of our theoretical framework are satisfied for these two models, we develop using fine asymptotic properties of confluent hypergeometric type functions, semi-explicit formulas for the densities of the running minimum and running maximum of both CIR and CEV processes which are of independent interest (see Theorems 4.1, 4.2, 5.2 and 5.3). Finally, we proceed to several numerical tests illustrating our results.

2. General framework

Let us consider a process \((X_t)_{t \in [0,T]}\) solution to

\[
\begin{align*}
    dX_t &= b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x, \\
\end{align*}
\]

where \((W_t)_{t \geq 0}\) is a standard Brownian motion, \(b : \mathbb{R} \to \mathbb{R}\) and \(\sigma : \mathbb{R} \to \mathbb{R}_+^*\) are locally Lipschitz-functions such that \(\frac{1}{\sigma}\) is locally integrable. For \(\phi(y) = \int_{y_0}^y \frac{1}{\sigma(x)} \, dx\), if \(\sigma \in \mathcal{C}^1\) then by the Lamperti transform \(Y_t = \phi(X_t)\) solves the stochastic differential equation

\[
    dY_t = L(Y_t)dt + dW_t, \quad Y_0 = \phi(x),
\]

\(^1\)A Down-and-Out barrier Call (resp. Put) is the option to buy (resp. sell), at maturity \(T\), the underlying with a fixed strike if the underlying value never falls below the barrier before time \(T\).
with \( L(x) = \left( \frac{b}{\sigma} - \frac{\sigma'}{T} \right) (\phi^{-1}(x)) \). In this work, we are interested in approximating barrier option prices such as the Down-and-Out (D-O) and the Up-and-Out (U-O) barrier options

\[
\pi_{B_D} = \mathbb{E}\left[ f(X_T) \mathbb{1}_{\{ \inf_{t \in [0,T]} X_t > B_D \}} \right] \quad \text{and} \quad \pi_{B_U} = \mathbb{E}\left[ f(X_T) \mathbb{1}_{\{ \sup_{t \in [0,T]} X_t < B_U \}} \right].
\]

The other types of barrier options such as the Down-and-In and the Up-and-In can be easily deduced from the price of the vanilla option \( \mathbb{E}[f(X_T)] \). As the function \( \phi \) is monotonic, by the Lamperti transformation we reduce ourselves to a pricing problem with the process \( (Y_t)_{t \in [0,T]} \). More precisely, we first introduce the discrete version of the drift implicit scheme given by

\[
\text{scheme.}
\]

and introduce the following interpolated drift implicit scheme

\[
\forall t \in [t_i, t_{i+1}], \quad \text{for} \quad 0 \leq i \leq n-1.
\]

The main advantages of this Brownian interpolation is that it preserves the rate of strong convergence of the original drift implicit scheme (4) and allows at the same time the use of the Brownian
bridge technique for pricing Barrier options (see Section 3.1 below). In what follows, we strengthen
our assumption on the drift coefficient $L$ as follows:

$$L : I \to \mathbb{R}$$

is $C^2$ such that: $L$ is decreasing on $(0, A)$ for $A > 0$,
and $L'$ the first derivative of $L$ satisfies $\exists L_A' > 0$ s.t. $\forall y \in (A, \infty), |L'(y)| \leq L_A'$. (H3)

**Theorem 2.1.** Assume that conditions (H2) and (H3) hold true for a given $p > 1$ and with
$L_A' < \frac{n}{2}$. Then, there exists a constant $K_p > 0$ such that

$$\mathbb{E}^F \left[ \sup_{t \in [0, T]} |Y^n_t - Y_t|^p \right] \leq K_p \frac{T}{n}.$$

**Proof.** At first, for $p \geq 1$ and $t \in [0, T]$, we denote $e_t = Y^n_t - Y_t$. By (2), we have for $0 \leq i \leq n - 1$,

$$e_{t_{i+1}} = e_i + L(Y^n_{t_{i+1}})(t_{i+1} - t_i) - \int_{t_i}^{t_{i+1}} L(Y_s)ds$$

since for all $0 \leq i \leq n - 1$, $Y^n_{t_i} \in I$. As $L$ is of class $C^2$ there exists a point $\xi_{t_{i+1}}$ lying between $Y_{t_{i+1}}$ and $Y^n_{t_{i+1}}$ such that $L(Y^n_{t_{i+1}}) - L(Y_{t_{i+1}}) = \beta_{t_{i+1}}(Y^n_{t_{i+1}} - Y_{t_{i+1}})$ with $\beta_{t_{i+1}} = L'(\xi_{t_{i+1}})$. Besides,
according to the proof [2, Proposition 3], we know that

$$\mathbb{E} \left[ \sup_{1 \leq i \leq n} |e_t|^p \right] \leq K \left( \frac{T}{n} \right)^p \mathbb{E} \left[ \left( \int_0^T |L'(Y_s)L(Y_s) + \frac{\gamma^2}{2} L''(Y_s)|du \right)^p \right]$$

$$+ |\gamma|^p \mathbb{E} \left[ \left( \int_0^T (L'(Y_s))^2du \right)^\frac{p}{2} \right],$$

where $K$ is a positive constant that depends on $T$ and $p$. On the one hand, we first use (6) to write

$$e_{t_{i+1}} = e_i + L(Y^n_{t_{i+1}})(t_{i+1} - t_i) - \int_{t_i}^{t_{i+1}} L(Y_s)ds$$

$$= e_i + \left[ L(Y^n_{t_{i+1}}) - L(Y_{t_{i+1}}) \right](t_{i+1} - t_i) + \int_{t_i}^{t_{i+1}} \left[ L(Y_{t_{i+1}}) - L(Y_s) \right]ds$$

$$= e_i + \beta_{t_{i+1}}e_{t_{i+1}}(t_{i+1} - t_i) + \int_{t_i}^{t_{i+1}} \left[ L(Y_{t_{i+1}}) - L(Y_s) \right]ds.$$

It follows that

$$(1 - \beta_{t_{i+1}}(t_{i+1} - t_i))e_{t_{i+1}} = e_i + \int_{t_i}^{t_{i+1}} \left[ L(Y_{t_{i+1}}) - L(Y_s) \right]ds.$$

(8)

On the other hand, we have for all $t \in [t_i, t_{i+1})$

$$\overline{Y}^n_t = \overline{Y}^n_{t_i} + L(Y^n_{t_{i+1}})(t - t_i) + \gamma(W_t - W_{t_i})$$

$$= \overline{Y}^n_{t_i} + L(Y^n_{t_{i+1}})(t_{i+1} - t_i) + \gamma(W_{t_{i+1}} - W_{t_i}) - L(Y^n_{t_{i+1}})(t_{i+1} - t) - \gamma(W_{t_{i+1}} - W_t)$$

$$= \overline{Y}^n_{t_{i+1}} - L(Y^n_{t_{i+1}})(t_{i+1} - t) - \gamma(W_{t_{i+1}} - W_t).$$

Then, it follows that for all $t \in [t_i, t_{i+1})$

$$\overline{Y}^n_t - Y_t = \overline{Y}^n_{t_{i+1}} - Y_{t_{i+1}} + Y_{t_{i+1}} - Y_t - L(Y^n_{t_{i+1}})(t_{i+1} - t) - \gamma(W_{t_{i+1}} - W_t)$$

$$e_t = e_{t_{i+1}} + \int_t^{t_{i+1}} L(Y_s)ds + \gamma(W_{t_{i+1}} - W_t) - L(Y^n_{t_{i+1}})(t_{i+1} - t) - \gamma(W_{t_{i+1}} - W_t).$$
By assumption (H3), on $p \in \sigma(t_{i+1}) + 1$,
$$
\left| e^{t_{i+1}} - \beta_{t_{i+1}}(Y^{\sigma}_{i+1} - Y_{i+1})(t_{i+1} - t) + \int_{t}^{t_{i+1}} L(Y_s) - L(Y_{i+1}) ds \right|
= e^{t_{i+1}} - \beta_{t_{i+1}}(Y^{\sigma}_{i+1} - Y_{i+1})(t_{i+1} - t) + \int_{t}^{t_{i+1}} L(Y_s) - L(Y_{i+1}) ds.
$$

So, we deduce that for all $t \in [t_i, t_{i+1})$
$$
e_t = (1 - \beta_{t_{i+1}}(t_{i+1} - t))e^{t_{i+1}} + \int_{t}^{t_{i+1}} L(Y_s) - L(Y_{i+1}) ds. \tag{9}
$$

By assumption (H3), on $(0, A)$ $L$ is decreasing, so it is easy to see that $1 < 1 - \beta_{t_{i+1}}(t_{i+1} - t) < 1 - \beta_{t_{i+1}}(t_{i+1} - t_i)$. On $(A, \infty)$, as $L'$ is bounded and since $n > 2L_A T$, we have $|1 - \beta_{t_{i+1}}(t_{i+1} - t)| \leq \frac{3}{2}$ and $1 - \beta_{t_{i+1}}(t_{i+1} - t_i) > \frac{1}{2}$. Then, it follows that $\frac{1 - \beta_{t_{i+1}}(t_{i+1} - t)}{1 - \beta_{t_{i+1}}(t_{i+1} - t_i)} \leq 3$. Now, combining (8) and (9) we easily get
$$
e_t = \frac{1 - \beta_{t_{i+1}}(t_{i+1} - t)}{1 - \beta_{t_{i+1}}(t_{i+1} - t_i)} \left( e_t + \int_{t_i}^{t_{i+1}} (L(Y_{i+1}) - L(Y_s)) ds \right) + \int_{t}^{t_{i+1}} L(Y_s) - L(Y_{i+1}) ds. \tag{10}
$$

Then, by Itô’s formula and Fubini theorem we get
$$
|e_t| \leq 3 \left( |e_{t_i}| + \int_{t_i}^{t_{i+1}} (L(Y_{i+1}) - L(Y_s)) ds \right) + \int_{t}^{t_{i+1}} L(Y_s) - L(Y_{i+1}) ds
\leq 3 \left( |e_{t_i}| + \frac{T}{n} \int_{t_i}^{t_{i+1}} \left| L'(Y_u)L(Y_u) + \frac{\gamma^2}{2} L''(Y_u) \right| du + |\gamma| \int_{t_i}^{t_{i+1}} (u - t_i)L'(Y_u) dW_u \right)
+ \frac{T}{n} \int_{t_i}^{t_{i+1}} L'(Y_u)L(Y_u) + \frac{\gamma^2}{2} L''(Y_u) du + |\gamma| \int_{t}^{t_{i+1}} (u - t)L'(Y_u) dW_u |
.$$

Therefore, there exists a positive constant $C_p$ such that
$$
|e_t|^p \leq C_p \left[ |e_{t_i}|^p + 2 \left( \frac{T}{n} \right)^p \left( \int_{t_i}^{t_{i+1}} \left| L'(Y_u)L(Y_u) + \frac{\gamma^2}{2} L''(Y_u) \right|^p du \right)^2 + |\gamma|^p \left( \int_{t_i}^{t_{i+1}} (u - t_i)L'(Y_u) dW_u \right)^p 
+ |\gamma|^p \left( \int_{t}^{t_{i+1}} uL'(Y_u) dW_u \right)^p \right]
$$

and thus,
$$
\sup_{t \in [0, T]} |e_t|^p \leq C_p \left[ \sup_{0 \leq s \leq n} |e_{t_i}|^p + 2 \left( \frac{T}{n} \right)^p \left( \int_{0}^{T} \left| L'(Y_u)L(Y_u) + \frac{\gamma^2}{2} L''(Y_u) \right|^p du \right)^2
+ |\gamma|^p \sup_{0 \leq s \leq T} \left( \int_{s}^{t} (u - t_{\bar{u}(u)})L'(Y_u) dW_u \right)^p
+ |\gamma|^p T \sup_{0 \leq s \leq T} \left( \int_{s}^{t} L'(Y_u) dW_u \right)^p \right]
\leq C_p \left[ \sup_{0 \leq s \leq n} |e_{t_i}|^p + 2 \left( \frac{T}{n} \right)^p \left( \int_{0}^{T} \left| L'(Y_u)L(Y_u) + \frac{\gamma^2}{2} L''(Y_u) \right|^p du \right)^2
+ 2^{p-1} |\gamma|^p \sup_{0 \leq t \leq T} \left( \int_{0}^{t} (u - t_{\bar{u}(u)})L'(Y_u) dW_u \right)^p
+ 2^{p-1} |\gamma|^p T \sup_{0 \leq t \leq T} \left( \int_{0}^{t} uL'(Y_u) dW_u \right)^p \right].
$$
The result follows using (7) and the Burkholder-Davis-Gundy inequality.

\[ \exists \alpha > 0 \text{ such that } \forall y \in I, \ yL(y) \leq \alpha(1 + |y|^2) \quad \text{(H4)} \]

Corollary 2.2. Assume that conditions of Theorem 2.1 hold true for a given \( p > 1 \) and \( 0 < L' < \frac{p}{2p} \). If in addition the drift coefficient \( L \) satisfies the following one-sided linear growth assumption:

\[ \exists \alpha > 0 \text{ such that } \forall y \in I, \ yL(y) \leq \alpha(1 + |y|^2) \quad \text{(H4)} \]

then \( \mathbb{E}[\sup_{0 \leq t \leq T} |Y^n_t|^p] < \infty \).

Proof. Under assumption (H4), [17, Lemma 3.2] ensures that for all \( q > 0 \) \( \mathbb{E}[\sup_{0 \leq t \leq T} |Y_t|^q] < \infty \). Thus, by Theorem 2.1 we get

\[ \mathbb{E}[\sup_{0 \leq t \leq T} |Y^n_t|^p] \leq 2^{p-1} \left( \mathbb{E}[\sup_{0 \leq t \leq T} |Y_t|^p + \mathbb{E}[\sup_{0 \leq t \leq T} |Y^n_t - Y_t|^p] \right) < \infty. \]

\[ \square \]

3. THE MULTILEVEL MONTE CARLO METHOD FOR PRICING BARRIER OPTIONS WITH THE INTERPOLATED DRIFT IMPLICIT SCHEME.

We are interested to approximate the following quantities of interest of the form

\[ \pi_D = \mathbb{E}\left[g(Y_T)1_{\{\tau_D > T\}}\right] \quad \text{and} \quad \pi_U = \mathbb{E}\left[g(Y_T)1_{\{\tau_U > T\}}\right], \]

where \( \tau \) denotes the first passage time given by:

- \( \tau_D = \inf\{t \in [0, T], Y_t \leq D\} \) with \( y > D > 0 \), for a Down-Out (D-O) option, and
- \( \tau_U = \inf\{t \in [0, T], Y_t \geq U\} \) with \( 0 < y < U \), for an Up-Out (U-O) option.

3.1. Brownian bridge and drift implicit scheme for pricing Barrier options. For a given time grid \( t_i = \frac{t}{n}, i \in \{1, \ldots, n\} \), we consider the Brownian interpolation of the drift implicit scheme \((Y^n_t)_{t \in [0, T]}\) defined in (6). Then, the above option prices can be approximated respectively by

\[ \pi_D := \mathbb{E}\left[g(\bar{Y}_T^n)\prod_{i=0}^{n-1}1_{\{\inf_{t \in [t_i, t_{i+1}]} Y^n_t > D\}}\right] \quad \text{and} \quad \pi_U := \mathbb{E}\left[g(\bar{Y}_T^n)\prod_{i=0}^{n-1}1_{\{\sup_{t \in [t_i, t_{i+1}]} Y^n_t < U\}}\right]. \]

To get more accurate approximations, we use the Brownian bridge technique to substitute the barrier-crossing indicators by the probabilities that the approximation scheme \((\bar{Y}_t^n)_{t \in [0, T]}\) do not cross the barrier in each time interval \([t_i, t_{i+1}], i \in \{1, \ldots, n\}\). In what follows, for \( x \in \mathbb{R}, (x)_{+} \) stands for \( \sup(x, 0) \).

Proposition 3.1. Under the above notation, for \( h = \frac{T}{n} \), we have

\[ \pi_D = \mathbb{E}\left[g(\bar{Y}_T^n)\prod_{i=0}^{n-1}(1 - \bar{q}_i)\right], \text{ where } \bar{q}_i := \exp\left(-\frac{2(\bar{Y}_{t_i} - D)(\bar{Y}_{t_{i+1}} - D)}{\gamma^2 h}\right) \]

and

\[ \pi_U = \mathbb{E}\left[g(\bar{Y}_T^n)\prod_{i=0}^{n-1}(1 - \bar{p}_i)\right], \text{ where } \bar{p}_i := \exp\left(-\frac{2(U - \bar{Y}_{t_i})+ (U - \bar{Y}_{t_{i+1}})_{+}}{\gamma^2 h}\right). \]
Proof. For the U-O barrier option, we first notice that conditionally on \((Y^n_0, Y^n_{t_1}, \ldots, Y^n_T)\), the barrier-crossing indicators \(\{1_{\{\sup_{t\in[t_i,t_{i+1}]} Y^n_t < U\}}; i \in \{1, \ldots, n\}\}\) are independent, we write

\[
\pi_U = E\left[ g(Y^n_T) \prod_{i=0}^{n-1} 1_{\{\sup_{t\in[t_i,t_{i+1}]} Y^n_t < U\}} | Y^n_0, Y^n_{t_1}, \ldots, Y^n_T \right]
\]

\[
= E\left[ g(Y^n_T) \prod_{i=0}^{n-1} E\left[ 1_{\{\sup_{t\in[t_i,t_{i+1}]} Y^n_t < U\}} | Y^n_{t_i}, Y^n_{t_{i+1}} \right] \right]
\]

\[
= E\left[ g(Y^n_T) \prod_{i=0}^{n-1} (1 - \varphi(Y^n_{t_i}, Y^n_{t_{i+1}})) \right],
\]

where, for \(y_i, y_{i+1} \in I\), \(\varphi(y_i, y_{i+1}) = P\left( \sup_{t\in[t_i,t_{i+1}]} Y^n_t \geq U | Y^n_{t_i} = y_i, Y^n_{t_{i+1}} = y_{i+1} \right).\) Without loss of generality, we may assume \(\gamma > 0\), the same arguments below work for \(\gamma < 0\) using that \((W_t)_{t\geq 0}\) and \((-W_t)_{t\geq 0}\) have the same law. By (6), we write

\[
\sup_{t\in[t_i,t_{i+1}]} Y^n_t = Y^n_{t_i} + \gamma \sup_{t\in[t_i,t_{i+1}]} \left[ W_t - W_{t_i} + \frac{1}{\gamma} L(Y^n_{t_{i+1}})(t - t_i) \right],
\]

\[
W_{t_{i+1}} - W_{t_i} + \frac{1}{\gamma} L(Y^n_{t_{i+1}})(t_{i+1} - t_i) = \frac{1}{\gamma}(Y^n_{t_{i+1}} - Y^n_{t_i}).
\]

By the stationarity property of the brownian increments and using a change of probability measure, we easily get that the law of \(\sup_{t\in[t_i,t_{i+1}]} \left[ W_t - W_{t_i} + \frac{1}{\gamma} L(y_{i+1})(t - t_i) \right]\) given \(W_{t_{i+1}} - W_{t_i} + \frac{1}{\gamma} L(y_{i+1})(t_{i+1} - t_i) = \frac{1}{\gamma}(y_{i+1} - y_i)\) is equal to the law of \(\sup_{t\in[0,t_1]} W_t\) given \(W_{t_1} = \frac{1}{\gamma}(y_{i+1} - y_i)\) which is given by

\[
P\left( \sup_{t\in[0,t_1]} W_t \geq y \big| W_{t_1} = x \right) = e^{\frac{-2(y-x)(y-x)}{\gamma}} (\text{see e.g. [19, p. 265]}).\]

Thus, we get

\[
\varphi(y_i, y_{i+1}) = P\left( \sup_{t\in[0,t_1]} W_t \geq \frac{1}{\gamma}(U - y_i) | W_{t_1} = \frac{1}{\gamma}(y_{i+1} - y_i) \right)
\]

\[
= \exp\left( -\frac{2(U - y_i)(U - y_{i+1})}{\gamma^2 h} \right).
\]

The same arguments applied to \((-W_t)_{t\geq 0}\) work for the Down-Out barrier option.

\[\square\]

3.2. The interpolated drift implicit Euler scheme MLMC method analysis. We consider the drift implicit scheme \((Y^{2^\ell}_{t^{\ell}_i})_{0\leq i \leq 2^\ell}\) given in (5) that approximates \((Y_t)_{0\leq t \leq T}\) solution to (2) using a time step \(h_\ell = 2^{-\ell}T\) for \(\ell \in \{0, \ldots, L\}\), with \(L = \log n/\log 2\), where \(n\) denotes the finest time step number. Let \((Y^{2^\ell}_{t^{\ell}_i})_{0\leq i \leq 2^\ell}\) denote the Brownian interpolation of the drift implicit scheme defined in (6) with time step \(h_\ell\). As the same arguments work for both Down-Out and Up-Out barrier options, we give details only for the latter one. To do so, we introduce

\[
P_\ell := g(Y^{2^\ell}_{t^{\ell}_i}) \prod_{i=0}^{2^\ell - 1} 1_{\{\sup_{t\in[t^{\ell}_i,t^{\ell}_{i+1}]} Y^{2^\ell}_t < U\}}, \quad \text{where } t^{\ell}_i = \frac{iT}{2^\ell} \quad \text{for } \ell \in \{0, \ldots, L\},
\]

and write

\[
\pi_U = E[\pi_L] = E[\pi_0] + \sum_{\ell=1}^L E[\pi_\ell - \pi_{\ell-1}],
\]
where $\pi_u$ is introduced in subsection 3.1. On the one hand, applying Proposition 3.1 yields

$$\mathbb{E}[\mathcal{P}_\ell] = \mathbb{E}[\bar{\mathcal{P}}^f_{\ell}],$$

where $\bar{\mathcal{P}}^f_{\ell} := g(\mathcal{Y}^f_T) \prod_{i=0}^{2^{\ell}-1} (1 - \bar{p}_{i}^{2^{\ell}})$ with

$$\bar{p}_{i}^{2^{\ell}} = \exp \left( \frac{-2(U - \mathcal{Y}^{f}_{2^{\ell}i}) + (U - \mathcal{Y}^{f}_{2^{\ell}i+1})}{\gamma^2 h_\ell} \right).$$

On the other hand, we write

$$\mathbb{E}[\mathcal{P}_{\ell-1}] = \mathbb{E}[g(\mathcal{Y}^{2^{\ell-1}}_T) \prod_{i=0}^{2^{\ell-1}-1} \mathbb{I}_{\{\sup_{t \in [t_i^{\ell-1}, t_{i+1}^{\ell-1}]} \mathcal{Y}^{2^{\ell-1}}_t < U\}} \mathbb{I}_{\{\sup_{t \in [t_{i+1}^{\ell-1}, t_{i+2}^{\ell-1}]} \mathcal{Y}^{2^{\ell-1}}_t < U\}}],$$

which can be rewritten as

$$\mathbb{E}[\mathcal{P}_{\ell-1}] = \mathbb{E}[g(\mathcal{Y}^{2^{\ell-1}}_T) \prod_{i=0}^{2^{\ell-1}-1} \mathbb{I}_{\{\sup_{t \in [t_i^{\ell-1}, t_{i+1}^{\ell-1}]} \mathcal{Y}^{2^{\ell-1}}_t < U\}}],$$

where the coarse scheme $\mathcal{Y}^{2^{\ell-1}}_{t_{i+2}}$ is computed using our Brownian interpolation scheme (6) that is

$$\mathcal{Y}^{2^{\ell-1}}_{t_{i+2}} = \mathcal{Y}^{2^{\ell-1}}_{t_i} + L(\mathcal{Y}^{2^{\ell-1}}_{t_i+1})(t_{i+2} - t_i) + \gamma (W_{t_{i+2}} - W_{t_i}).$$

Thus, we rewrite $\sup_{t \in [t_i^{\ell-1}, t_{i+1}^{\ell-1}]} \mathcal{Y}^{2^{\ell-1}}_t$ and $\sup_{t \in [t_{i+1}^{\ell-1}, t_{i+2}^{\ell-1}]} \mathcal{Y}^{2^{\ell-1}}_t$ as follows

$$\sup_{t \in [t_i^{\ell-1}, t_{i+1}^{\ell-1}]} \mathcal{Y}^{2^{\ell-1}}_t = \mathcal{Y}^{2^{\ell-1}}_{t_i} + \gamma \sup_{t \in [t_i^{\ell-1}, t_{i+1}^{\ell-1}]} \left( W_t - W_{t_i^{\ell-1}} + \frac{1}{\gamma} L(\mathcal{Y}^{2^{\ell-1}}_{t_{i+1}^{\ell-1}})(t - t_{i+1}^{\ell-1}) \right),$$

and

$$\sup_{t \in [t_{i+1}^{\ell-1}, t_{i+2}^{\ell-1}]} \mathcal{Y}^{2^{\ell-1}}_t = \mathcal{Y}^{2^{\ell-1}}_{t_{i+2}} + \gamma \sup_{t \in [t_{i+1}^{\ell-1}, t_{i+2}^{\ell-1}]} \left( W_t - W_{t_{i+2}} + \frac{1}{\gamma} L(\mathcal{Y}^{2^{\ell-1}}_{t_{i+1}^{\ell-1}})(t - t_{i+2}^{\ell-1}) \right).$$

Then, using the independence of the Brownian increments and the same arguments as in the proof of Proposition 3.1, we get

$$\mathbb{E}[\mathcal{P}_{\ell-1}] = \mathbb{E}[\bar{\mathcal{P}}^{f}_{\ell-1}],$$

where $\bar{\mathcal{P}}^{f}_{\ell-1} := g(\mathcal{Y}^{2^{\ell-1}}_T) \prod_{i=0}^{2^{\ell-1}-1} (1 - \bar{p}_{i}^{2^{\ell-1}})$ with

$$\bar{p}_{i}^{2^{\ell-1}} = \exp \left( \frac{-2(U - \mathcal{Y}^{2^{\ell-1}}_{i}) + (U - \mathcal{Y}^{2^{\ell-1}}_{i+1})}{\gamma^2 h_\ell} \right),$$

which can be rewritten as

$$\bar{\mathcal{P}}^{f}_{\ell-1} := g(\mathcal{Y}^{2^{\ell-1}}_T) \prod_{i=0}^{2^{\ell-1}-1} (1 - \bar{p}_{i}^{2^{\ell-1}}) \quad \text{with} \quad \bar{p}_{i}^{2^{\ell-1}} = \exp \left( \frac{-2(U - \mathcal{Y}^{2^{\ell-1}}_{i}) + (U - \mathcal{Y}^{2^{\ell-1}}_{i+1})}{\gamma^2 h_\ell} \right),$$

(15)
Lemma 3.2. Assume that conditions (H2), (H3) and (H4) are satisfied for a given $p > 1$ and $0 < L_l' < \frac{1}{2h_l}$, with $h_l = 2^{\ell} T$ sufficiently small. Let $\eta \in (0, 1)$, the following extreme path events satisfy

$$\mathbb{P}\left( \max_{0 \leq i \leq 2^\ell} \left( \sup_{0 \leq t \leq 2^\ell} \left( |Y_{t_i}^\ell|, |Y_{t_i}^{2^\ell-1}| \right) > h_l^{1-\eta} \right) \right) = o(h_l^{q})$$

(18)

$$\mathbb{P}\left( \max_{0 \leq i \leq 2^\ell} \left( \sup_{0 \leq t \leq 2^\ell} \left( |Y_{t_i}^\ell - Y_{t_i}^{2^\ell-1}|, |Y_{t_i}^{2^\ell-1} - Y_{t_i}^{2^\ell-2}| \right) > h_l^{1-\eta} \right) \right) = o(h_l^{q})$$

(19)

$$\sup_{0 \leq i \leq 2^\ell} \mathbb{P}\left( \int_{t_i}^{t_{i+1}} |L(Y_s)| ds > h_l^{1-\eta} \right) = o(h_l^{q})$$

(20)

$$\mathbb{P}\left( \sup_{t \in [0, T]} |Y_{t_i}^{2^\ell} - Y_{t_i}| > h_l^{1-\eta} \right) = o(h_l^{q})$$

(21)

for all $0 < q < pn$, and

$$\sup_{0 \leq i \leq 2^\ell} \mathbb{P}\left( \sup_{t \in [t_i, t_{i+1}]} |W_t - W_{t_i}^\ell| > h_l^{1-\eta} \right) = o(h_l^{q}), \text{ for all } q > 0.$$ 

(22)

Proof. For the first extreme path property, we have
\begin{align*}
\mathbb{P}(\max_{0 \leq i \leq 2^t} (|Y_{t_i}^\ell|, |\bar{Y}_{t_i}^{2\ell}|, |\bar{Y}_{t_i}^{2\ell-1}|)) > h_\ell^{-\eta})
\leq \mathbb{P}(\sup_{0 \leq i \leq 2^t} |Y_{t_i}| > h_\ell^{-\eta}) + \mathbb{P}(\sup_{0 \leq i \leq 2^t} |\bar{Y}_{t_i}^{2\ell}| > h_\ell^{-\eta}) + \mathbb{P}(\sup_{0 \leq i \leq 2^t} |\bar{Y}_{t_i}^{2\ell-1}| > h_\ell^{-\eta}).
\end{align*}

Then, by Markov’s inequality we get for $m \geq 1$

\begin{align*}
\mathbb{P}(\max_{0 \leq i \leq 2^t} (|Y_{t_i}^\ell|, |\bar{Y}_{t_i}^{2\ell}|, |\bar{Y}_{t_i}^{2\ell-1}|)) > h_\ell^{-\eta}) & \leq h_\ell^{m\eta}\left(\mathbb{E}[\sup_{0 \leq t \leq T} |Y_t|^m] + \mathbb{E}[\sup_{0 \leq t \leq T} |\bar{Y}_t^{2\ell}|^m] + \mathbb{E}[\sup_{0 \leq t \leq T} |\bar{Y}_t^{2\ell-1}|^m]\right).
\end{align*}

The result follows by Corollary 2.2 for $h_\ell$ sufficiently small with choosing $m$ such that $0 < \frac{q}{\eta} < m \leq p$. For the second extreme property, we proceed in the same way to get for $m \geq 1$

\begin{align*}
\mathbb{P}\left(\max_{0 \leq i \leq 2^t} (|Y_{t_i}^\ell - \bar{Y}_{t_i}^{2\ell}|, |Y_{t_i}^\ell - \bar{Y}_{t_i}^{2\ell-1}|, |\bar{Y}_{t_i}^{2\ell} - \bar{Y}_{t_i}^{2\ell-1}|)) > h_\ell^{-\eta}\right)
& \leq h_\ell^{m(1-\eta)}\left(\mathbb{E}[\sup_{0 \leq t \leq T} |Y_t - \bar{Y}_t^{2\ell}|^m] + \mathbb{E}[\sup_{0 \leq t \leq T} |Y_t - \bar{Y}_t^{2\ell-1}|^m] + \mathbb{E}[\sup_{0 \leq t \leq T} |\bar{Y}_t^{2\ell} - \bar{Y}_t^{2\ell-1}|^m]\right).
\end{align*}

Thus, we deduce the result using Theorem 2.1 with choosing $m$ such that $0 < \frac{q}{\eta} < m \leq p$, for $h_\ell$ sufficiently small. For the third extreme property, we proceed in the same way to get that for all $0 \leq i \leq 2^\ell$, using Jensen’s inequality

\begin{align*}
\mathbb{P}\left(\int_{t_i}^{t_{i+1}} |L(Y_s)| ds > h_\ell^{1-\eta}\right) & \leq h_\ell^{m(\eta-1)}\mathbb{E}\left(\left(\int_{t_i}^{t_{i+1}} |L(Y_s)| ds\right)^m\right)
& \leq h_\ell^{m\eta-1}\mathbb{E}\left(\int_{t_i}^{t_{i+1}} |L(Y_s)|^m ds\right)
& \leq h_\ell^{m\eta}\sup_{t \in [0, T]} \mathbb{E}\left(|L(Y_s)|^m\right).
\end{align*}

Then we conclude using (H2) by choosing $m$ such that $0 < \frac{q}{\eta} < m \leq p$, for $h_\ell$ sufficiently small. The fourth extreme property follows in the same way as the second one. Finally, the last property follows using that $\mathbb{E}\left[\sup_{t \in [0, T]} |W_t|^m\right]$ is finite for any positive power $m$.

Now, we are able to state our main theorem for the MLMC method to price barrier options when the underlying asset has possibly non-Lipschitz coefficients.

**Theorem 3.3.** Let $g$ denote a payoff function satisfying: \(\exists C > 0\ s.t. \forall x, y > 0,\)

\[|g(x) - g(y)| \leq C|x - y|(1 + |x|^\nu + |y|^\nu) \text{ and } |g(x)| \leq C(1 + |x|^\nu+1), \text{ with } \nu \in \mathbb{R}_+.\] \hspace{1cm} (23)

Moreover, assume that conditions (H2), (H3) and (H4) are satisfied for $p > \frac{(1+\delta)(1+\gamma)}{2-\delta}$, with $\varepsilon, \gamma > 0$, $\delta \in (0, 1/2)$ and $0 < L'_A < \frac{1}{2h_\ell}$, with $h_\ell = 2^{-\ell} T$ sufficiently small. If in addition $\inf_{t \in [0, T]} Y_t$ (resp. $\sup_{t \in [0, T]} Y_t$) has a bounded density in the neighborhood of the barrier $\mathcal{D}$ (resp. $\mathcal{U}$), then the multilevel estimator $\hat{Q}_D$ given by (17) (resp. $\hat{P}_U$ given by (16) ) for the D-O (resp. U-O) barrier option satisfies $\text{Var}(\hat{Q}_\ell - Q_\ell) = O(h_\ell^{1+\delta})$ (resp. $\text{Var}(\hat{P}_\ell - P_\ell) = O(h_\ell^{1+\delta})$).
Remark 3.4. \( \bullet \) Combining the complexity theorem in [12, Theorem 3.1] with the above result, we deduce that for any \( \delta \in (0, \frac{1}{2}) \) the MLMC estimators \( Q_D \) and \( P_t \) reach the optimal time complexity \( O(\varepsilon^{-2}) \), for a given precision \( \varepsilon > 0 \), and behave like an unbiased Monte Carlo estimator.

\( \bullet \) Taking \( \delta \) close to \( \frac{1}{2} \) achieves a smaller variance of the difference between the finer and coarse approximations which is of order \( O(h_\ell^2) \) with \( \beta \) close to \( \frac{3}{2} \) similar to the case of diffusion with Lipschitz coefficients studied in [14, Theorem 3.15], but clearly leads to very restrictive conditions on the finiteness of the moments of \((Y_t)_{t \in [0,T]} \) and \((\bar{Y}_t^n)_{t \in [0,T]} \).

Proof. We only give a proof for the D-O barrier option since the proof for the U-O barrier option is quite similar. At first, following the extreme path approach given in [15, 14], we write

\[
\text{Var}[Q^{\ell}_{\ell} - \bar{Q}_{\ell}] \leq E[(Q^{\ell}_{\ell} - \bar{Q}_{\ell})^2] = E[(Q^{\ell}_{\ell} - \bar{Q}_{\ell})^2 1_{A_1}] + E[(Q^{\ell}_{\ell} - \bar{Q}_{\ell})^2 1_{A_2}] + E[(Q^{\ell}_{\ell} - \bar{Q}_{\ell})^2 1_{A_3}]
\]

where we split the paths into the following three events.

**First event** \( A_1 \). We consider any of the extreme path events given in Lemma 3.2 that satisfy (18)-(22), with some \( \eta > 0 \) to be fixed later on. For \( \gamma > 0 \), we use Hölder’s inequality to get

\[
E[(Q^{\ell}_{\ell} - \bar{Q}_{\ell})^2 1_{A_1}] \leq E^{\frac{1}{1+\gamma}}[(Q^{\ell}_{\ell} - \bar{Q}_{\ell})^2 1_{A_1}] \leq 2^{\frac{2(1+\gamma)}{1+\gamma}} (E^{\frac{1}{1+\gamma}}[(Q^{\ell}_{\ell})^2 1_{A_1}] + E^{\frac{1}{1+\gamma}}[\bar{Q}_{\ell}^{2(1+\gamma)}]) (E[|A_1|])^{\frac{1}{1+\gamma}}.
\]

As the payoff function \( g \) satisfies assumption (23), we deduce using Corollary 2.2, that for \( h_\ell \) sufficiently small \( \mathbb{E}[(Q^{\ell}_{\ell})^2 1_{A_1}] \) and \( \mathbb{E}[(\bar{Q}_{\ell}^c)^{2(1+\gamma)}] \) are finite. By Lemma 3.2, we get that

\[
\mathbb{E}[(Q^{\ell}_{\ell} - \bar{Q}_{\ell})^2 1_{A_1}] = o(h_\ell^{\frac{q}{1+\gamma}}) \quad \text{for all} \quad q \quad \text{such that} \quad 0 < \frac{q}{\eta} \leq p.
\]

**Second event** \( A_2 \). This event corresponds to the non-extreme paths satisfying

\[
\inf_{t \in [0,T]} |Y_t - D| > h_\ell^{\frac{1}{2} - \eta(1+\varepsilon)} \quad \text{for} \quad \eta \in (0, 1/2(1 + \varepsilon)) \quad \text{with} \quad \varepsilon > 0.
\]

Let us assume that \( \inf_{t \in [0,T]} Y_t = Y_\tau \) with \( \tau \in [t^{\ell}_i, t^{\ell}_{i+1}] \) for a given \( i \in \{0, \ldots, 2^\ell\} \).

- First case: for \( Y_\tau < D - h_\ell^{\frac{1}{2} - \eta(1+\varepsilon)} \), we write

\[
|Y^{2^\ell}_{t^{\ell}_i} - Y_\tau| \leq |Y^{2^\ell}_{t^{\ell}_i} - Y^{t^{\ell}_i}_{t^{\ell}_i}| + |Y^{t^{\ell}_i}_{t^{\ell}_i} - Y_\tau| \leq |Y^{2^\ell}_{t^{\ell}_i} - Y^{t^{\ell}_i}_{t^{\ell}_i}| + \int_{t^{\ell}_i}^{t^{\ell}_{i+1}} |L(Y_s)|ds + \gamma \sup_{t \in [t^{\ell}_i, t^{\ell}_{i+1}]} |W_t - W_{t^{\ell}_i}|.
\]

Then, as we work on the non-extreme paths events (see Lemma 3.2 for the extreme paths events), for \( h_\ell \) sufficiently small we have that \( |Y^{2^\ell}_{t^{\ell}_i} - Y_\tau| = O(h_\ell^{\frac{1}{2} - \eta}) \) and then \( |Y^{2^\ell}_{t^{\ell}_i} - Y_\tau| < h_\ell^{\frac{1}{2} - \eta(1+\varepsilon)} \), which yields that \( Y^{2^\ell}_{t^{\ell}_i} < D \). Once again, as we are in the case of non extreme paths we have \( |Y^{2^\ell}_{t^{\ell}_i} - Y^{2^\ell_{i-1}}| \leq h_\ell^{1 - \eta} \) and so \( |Y^{2^\ell_{i-1}} - Y_\tau| = O(h_\ell^{\frac{1}{2} - \eta}). \) Hence, we also have \( Y^{2^\ell_{i-1}} < D \) for sufficiently small \( h_\ell \) which yields that \( \bar{Q}_{\ell}^{2^\ell_{i-1}} = 0 \).

- Second case: for \( Y_\tau > D + h_\ell^{\frac{1}{2} - \eta(1+\varepsilon)} \), we proceed in the same way and we easily check that for \( h_\ell \) sufficiently small \( \prod_{i=0}^{2^\ell-1} (1 - \bar{Q}_{t^{\ell}_i}) \) and \( \prod_{i=0}^{2^\ell-1} (1 - Q^{2^\ell-1}_{t^{\ell}_i}) \) are both equal to \( 1 + o(h_\ell^a) \) for all \( a > 0 \).
Consequently, as the payoff function $g$ satisfies condition (23) and as we work with the non-extreme paths events, we deduce that $E[(\bar{Q}_t - \bar{Q}_t)^2 1_{A_2}] = O(h_t^{2(1-\eta)-2\nu}).$

**Third event $A_3.$** This last event corresponds to the rest of the non extreme paths. At first, let us note that

$$|\bar{Y}_{t+1}^{2_d} - \bar{Y}_t^{2_d}| \leq |\bar{Y}_{t+1}^{2_d} - Y_{t+1}^{2_d}| + |Y_{t+1}^{2_d} - Y_t^{2_d}| + |\bar{Y}_t^{2_d} - Y_t^{2_d}|$$

$$\leq |\bar{Y}_{t+1}^{2_d} - Y_{t+1}^{2_d}| + |\bar{Y}_t^{2_d} - Y_t^{2_d}| + \int_{t}^{t+1} |L(Y_s)|ds + |\gamma| \sup_{t \leq t' \leq t+1} |W_t - W_{t'}|.$$  

Then, similarly as above we deduce that $|\bar{Y}_{t+1}^{2_d} - \bar{Y}_t^{2_d}| = O(h_t^{1/2-\eta})$ since we work on the non-extreme paths events. Thus, it is clear that if any one of $\bar{Y}_t^{2_d}, \bar{Y}_{t+1}^{2_d}, \bar{Y}_{t+1}^{2_d}, \bar{Y}_{t+1}^{2_d}$ is greater than $D + h_t^{1/2-\eta(1+\varepsilon)}$, then the others will be greater than $D + h_t^{1/2-\eta(1+\varepsilon)}$, for $h_t$ sufficiently small. In this case, if $R$ denotes the set of indices for which none of $\bar{Y}_t^{2_d}, \bar{Y}_{t+1}^{2_d}, \bar{Y}_{t+1}^{2_d}, \bar{Y}_{t+1}^{2_d}$ is greater than $D + h_t^{1/2-\eta(1+\varepsilon)}$ and $R^c$ its complementary set, then similarly as above we get $\prod_{i \in R^c} (1 - \bar{q}_t^{2_d}) = 1 + o(h_t^a)$ and $\prod_{i \in R} (1 - \bar{q}_t^{2_d}) = 1 + o(h_t^a)$, for all $a > 0$. Thus, we have

$$\prod_{i=0}^{2d-1} (1 - \bar{q}_t^{2_d}) = \prod_{i \in R} (1 - \bar{q}_t^{2_d}) + o(h_t^a)$$

and

$$\prod_{i=0}^{2d-1} (1 - \bar{q}_t^{2_d}) = \prod_{i \in R} (1 - \bar{q}_t^{2_d}) + o(h_t^a),$$

for all $a > 0$. (25)

Now, for $i \in R$ we have

$$| \log \bar{q}_t^{2_d} - \log \bar{q}_t^{2_d-1} | = \frac{2}{\gamma^2 h_t} \left| (\bar{Y}_{t+1}^{2_d} - D) + (\bar{Y}_{t+1}^{2_d} - D) - (\bar{Y}_t^{2_d} - D) - (\bar{Y}_t^{2_d-1} - D) \right|$$

$$\leq \frac{1}{\gamma^2 h_t} \left| (\bar{Y}_{t+1}^{2_d} - D) + (\bar{Y}_{t+1}^{2_d} - D) - (\bar{Y}_t^{2_d} - D) - (\bar{Y}_t^{2_d-1} - D) \right|$$

$$+ \frac{1}{\gamma^2 h_t} \left| (\bar{Y}_{t+1}^{2_d} - D) + (\bar{Y}_{t+1}^{2_d} - D) - (\bar{Y}_t^{2_d} - D) - (\bar{Y}_t^{2_d-1} - D) \right|,$$

where we used the relation $f_1 g_1 - f_2 g_2 = \frac{1}{2} (f_1 - f_2) (g_1 + g_2) + \frac{1}{2} (f_1 + f_2) (g_1 - g_2)$. By the Lipschitz property of the map $x \mapsto (x - D)_+$ and as $i \in R$, there exists a positive constant $C$ that may vary from line to line such that

$$| \log \bar{q}_t^{2_d} - \log \bar{q}_t^{2_d-1} | \leq C h_t^{1/2-\eta(1+\varepsilon)} \left| (\bar{Y}_{t+1}^{2_d} - \bar{Y}_t^{2_d}) + (\bar{Y}_{t+1}^{2_d} - \bar{Y}_t^{2_d-1}) \right|$$

$$\leq C h_t^{1/2-\eta(2+\varepsilon)}.$$  

So, for $h_t$ sufficiently small we have $| \log \bar{q}_t^{2_d} - \log \bar{q}_t^{2_d-1} | < h_t^{1/2-2\eta(1+\varepsilon)}$. Thus, we have

$$1 - \bar{q}_t^{2_d-1} = (1 - \bar{q}_t^{2_d}) + \bar{q}_t^{2_d} \left( 1 - \exp(\log \bar{q}_t^{2_d-1} - \log \bar{q}_t^{2_d}) \right)$$

$$\leq (1 - \bar{q}_t^{2_d}) + \bar{q}_t^{2_d} \left( 1 - \exp(-h_t^{1/2-2\eta(1+\varepsilon)}) \right).$$

Consequently, we get

$$\prod_{i \in R} (1 - \bar{q}_t^{2_d-1}) \leq \prod_{i \in R} \left( (1 - \bar{q}_t^{2_d}) + \bar{q}_t^{2_d} \left( 1 - \exp(-h_t^{1/2-2\eta(1+\varepsilon)}) \right) \right)$$

$$\leq \prod_{i \in R} \left( (1 - \bar{q}_t^{2_d}) + \left( 1 - \exp(-h_t^{1/2-2\eta(1+\varepsilon)}) \right) \right),$$
Moreover, assume that conditions (H2), (H3) and (H4) are satisfied for 
\[ p > \frac{(1 + \delta)(1 + \gamma)(1 + \varepsilon)}{\frac{1 - \delta}{2 + 2\nu}} \] 
with \( \varepsilon, \gamma > 0, \delta \in (0, 1/2) \) and \( 0 < L_A' < \frac{1}{2\nu} \), with \( h_{\ell} = 2^{-\ell}T \) sufficiently small. If in addition \( \inf_{t \in [0,T]} Y_t \)
has a bounded density in the neighborhood of the barrier, then the multilevel estimator \( \hat{Q}_\ell \) given by (17) for the D-O barrier option satisfies \( \text{Var}(\hat{Q}_\ell - \hat{Q}_\ell) = O(h_\ell^{1+\delta}) \).

**Proof.** We use a similar decomposition as in the proof of Theorem 3.3

\[
\text{Var}(\hat{Q}_\ell - \hat{Q}_\ell) \leq E[(\hat{Q}_\ell - \hat{Q}_\ell)^2] = E[(\hat{Q}_\ell - \hat{Q}_\ell)^2 1_{A_1}] + E[(\hat{Q}_\ell - \hat{Q}_\ell)^2 1_{A_2}] + E[(\hat{Q}_\ell - \hat{Q}_\ell)^2 1_{A_3}]
\]

where we split the paths into the following three events.

**First event** \( A_1 \). We consider any of the extreme path events given in Lemma 3.2 that satisfy (18)-(22), with some \( \eta > 0 \) to be fixed later on. For \( \eta > 0 \), we use Hölder’s inequality to get

\[
E[(\hat{Q}_\ell - \hat{Q}_\ell)^2 1_{A_1}] \leq E^{\frac{\gamma}{1+\gamma}}[(\hat{Q}_\ell - \hat{Q}_\ell)^{2(1+\gamma)}] \leq 2^{\frac{2+\gamma}{1+\gamma}}(E^{\frac{1}{1+\gamma}}[(\hat{Q}_\ell - \hat{Q}_\ell)^{2(1+\gamma)}])^{\frac{1}{1+\gamma}}
\]

As the payoff function \( f \) is bounded, we deduce by Lemma 3.2 that for \( h_\ell \) sufficiently small

\[
E[(\hat{Q}_\ell - \hat{Q}_\ell)^2 1_{A_1}] = o(h_\ell^{1+\gamma}) \quad \text{for all } q \text{ such that } 0 < \frac{q}{\eta} \leq p.
\]

**Second event** \( A_2 \). This event corresponds to the non-extreme paths satisfying

\[
| \inf_{t \in [0,T]} Y_t - D | > h_\ell^{\frac{1}{2}-\eta(1+\epsilon)} \quad \text{for } \eta \in (0, 1/2(1 + \epsilon)) \quad \text{with } \epsilon > 0.
\]

Let us assume that \( \inf_{t \in [0,T]} Y_t = Y_\tau \) with \( \tau \in [t_i^\ell, t_{i+1}^\ell] \) for a given \( i \in \{0, \ldots, 2^\ell \} \). Now, we write

\[
|\overline{Y}_{t_i^\ell}^{2i^\ell} - Y_\tau| \leq |\overline{Y}_{t_i^\ell}^{2i^\ell} - Y_{t_i^\ell}| + |Y_{t_i^\ell} - Y_\tau|
\]

\[
\leq |\overline{Y}_{t_i^\ell}^{2i^\ell} - Y_{t_i^\ell}| + \int_{t_i^\ell}^{t_{i+1}^\ell} |L(Y_s)| ds + \gamma \sup_{t \in [t_i^\ell, t_{i+1}^\ell]} |W_t - W_{t_i^\ell}|.
\]

Then, as we work on the non-extreme paths events then for \( h_\ell \) sufficiently small we have that

\[
|\overline{Y}_{t_i^\ell}^{2i^\ell} - Y_\tau| = O(h_\ell^{\frac{1}{2}-\eta}) \quad \text{and therefore} \quad |\overline{Y}_{t_i^\ell}^{2i^\ell} - Y_\tau| < h_\ell^{\frac{1}{2}-\eta(1+\epsilon)}. \]

Similarly, we deduce that \( |\overline{Y}_{t_i^\ell}^{2i^\ell-1} - Y_\tau| < h_\ell^{\frac{1}{2}-\eta(1+\epsilon)} \) since on the non-extreme paths events we have that \( |\overline{Y}_{t_i^\ell}^{2i^\ell} - \overline{Y}_{t_i^\ell}^{2i^\ell-1}| < h_\ell^{1-\eta} \).

- First case: for \( Y_\tau < D - h_\ell^{\frac{1}{2}-\eta(1+\epsilon)} \), the above estimate yields that \( \overline{Y}_{t_i^\ell}^{2i^\ell} < D \). and so

\[
|\overline{Y}_{t_i^\ell}^{2i^\ell-1} - Y_\tau| = O(h_\ell^{\frac{1}{2}-\eta}). \]

Hence, we also have \( \overline{Y}_{t_i^\ell}^{2i^\ell-1} < D \), for sufficiently small \( h_\ell \) which yields that \( \overline{Q}_{t_i^\ell}^{2i^\ell} - \overline{Q}_{t_i^\ell} = 0 \).

- Second case: for \( Y_\tau > D + h_\ell^{\frac{1}{2}-\eta(1+\epsilon)} \), we have

\[
\overline{Y}_{t_i^\ell}^{2i^\ell} > D + h_\ell^{\frac{1}{2}-\eta(1+\epsilon)}.
\]

So as \( |\overline{Y}_{t_i^\ell}^{2i^\ell} - Y_{t_i^\ell}| < h_\ell^{1-\eta} \) then \( \overline{Y}_{t_i^\ell}^{2i^\ell} > D + h_\ell^{\frac{1}{2}-\eta(1+\epsilon)}. \) Using similar arguments we also get \( \overline{Y}_{t_i^\ell}^{2i^\ell-1} > D + h_\ell^{\frac{1}{2}-\eta(1+\epsilon)} \). Then, we easily check that for \( h_\ell \) sufficiently small \( \prod_{i=0}^{2^\ell} (1 - q_{t_i^\ell}) \) and \( \prod_{i=0}^{2^\ell-1} (1 - q_{t_i^\ell}) \) are both equal to \( 1 + o(h_\ell^a) \) for all \( a > 0 \). Besides, by Taylor formula we have \( |g(x) - g(y)| \leq \frac{|x-y|}{\nu} |x-y|^{-\nu-1} + |y|^{-\nu-1} \) for all \( x, y \in I = (0, +\infty) \). Therefore, using that \( \overline{Y}_{t_i^\ell}^{2i^\ell} > D \) and \( \overline{Y}_{t_i^\ell}^{2i^\ell-1} > D \) we deduce that \( E[(\overline{Q}_{t_i^\ell}^{2i^\ell} - \overline{Q}_{t_i^\ell}^{2i^\ell-1})^2 1_{A_2}] = O(h_\ell^{2(1-\eta)}). \)
**Third event** \( A_3. \) For this last event, we proceed exactly as in Theorem 3.3 to get

\[
\left| \prod_{i=0}^{2^\ell-1} \left( 1 - \frac{\ell}{\delta} \right) - \prod_{i=0}^{2^\ell-1} \left( 1 - \frac{\ell}{\delta} \right) \right| = O(h_\ell^{\frac{1}{2} - 2\eta(1+\varepsilon)}).
\]

Therefore, as we work on the non-extreme paths events, we deduce using condition (23) on the payoff function \( g \) that

\[
|\mathcal{Q}_\ell^f - \mathcal{Q}_\ell^c|^2 = \left| g(Y_T^2f) \prod_{i=0}^{2^\ell-1} \left( 1 - \frac{\ell}{\delta} \right) - g(Y_T^2e^{-1}) \prod_{i=0}^{2^\ell-1} \left( 1 - \frac{\ell}{\delta} \right) \right|^2 \\
\leq C|\mathcal{Y}_T^{2\ell} - \mathcal{Y}_T^{2\ell-1}|^2 \left( 1 + |\mathcal{Y}_T^{2\ell}|^{-2\nu} + |\mathcal{Y}_T^{2\ell-1}|^{-2\nu} \right) + h_\ell^{1-4\eta(1+\varepsilon)}|g(\mathcal{Y}_T^{2\ell})|^2.
\]

where \( C \) is a positive constant that may vary from line to line. The second term of the above upper bound is clearly \( O(h_\ell^{1-4\eta(1+\varepsilon)}) \) since \( g \) is bounded. For the first term, let us recall that we are in the case where \( Y_T \geq Y_T \geq D - h_\ell^{1-\eta(1+\varepsilon)} \) and since on the non-extreme paths events we have that

\[
\mathcal{Y}_T^{2\ell} - Y_T > -h_\ell^{1-\eta(1+\varepsilon)} \quad \text{and} \quad \mathcal{Y}_T^{2\ell-1} - Y_T > -h_\ell^{1-\eta(1+\varepsilon)}
\]

then

\[
\mathcal{Y}_T^{2\ell} > D - h_\ell^{1-\eta(1+\varepsilon)} - h_\ell^{1-\eta(1+\varepsilon)} > D \quad \text{for } h_\ell \text{ small enough.}
\]

By similar arguments we get that \( \mathcal{Y}_T^{2\ell-1} > D \) for \( h_\ell \) small enough. Then, since \( \nu > 0 \), we get

\[
|\mathcal{Y}_T^{2\ell} - \mathcal{Y}_T^{2\ell-1}|^2 \left( 1 + |\mathcal{Y}_T^{2\ell}|^{-2(\nu+1)} + |\mathcal{Y}_T^{2\ell-1}|^{-2(\nu+1)} \right) = O(h_\ell^{2-2\eta})
\]

Therefore, we have

\[
\mathbb{E}[(\mathcal{Q}_\ell^f - \mathcal{Q}_\ell^c)^2 1_{A_3}] = O(h_\ell^{1-4\eta(1+\varepsilon)}) \times \mathbb{P}(\inf_{t \in [0,T]} Y_t - D \leq h_\ell^{1-\eta(1+\varepsilon)}) \\
= O(h_\ell^{3-5\eta(1+\varepsilon)})
\]

since the random variable \( \inf_{t \in [0,T]} Y_t \) has a bounded density on the neighborhood of \( D \). To complete the proof, we choose

\[
\eta = \frac{1}{2} - \frac{\delta}{5(1+\varepsilon)},
\]

which yields \( \mathbb{E}[(\mathcal{Q}_\ell^f - \mathcal{Q}_\ell^c)^2 1_{A_3}] = O(h_\ell^{1+\delta}) \). Now concerning the second event noticing that \( \eta \leq \frac{1-\delta}{5} \) we easily see that \( 2(1-\eta) > 1+\delta \), for \( \delta \in (0, 1/2) \), which yields \( \mathbb{E}[(\mathcal{Q}_\ell^f - \mathcal{Q}_\ell^c)^2 1_{A_2}] = o(h_\ell^{1+\delta}) \). Finally, for the first event, we choose \( q = (1+\gamma)(1+\delta) \) to guarantee that \( \mathbb{E}[(\mathcal{Q}_\ell^f - \mathcal{Q}_\ell^c)^2 1_{A_1}] = O(h_\ell^{1+\delta}) \) which is satisfied as soon as \( p > \frac{(1+\gamma)(1+\delta)\varepsilon(1+\varepsilon)}{2-\delta} \). \( \square \)

4. **APPLICATION TO THE CIR PROCESS**

In this section, we consider the problem of pricing D-O and U-O barrier options

\[
\pi_D = \mathbb{E}\left[f(X_T)1_{\{\inf_{t \in [0,T]} X_t > D\}}\right] \quad \text{and} \quad \pi_U = \mathbb{E}\left[f(X_T)1_{\{\sup_{t \in [0,T]} X_t < U\}}\right],
\]

where \( f \) is a Lipschitz payoff function with Lipschitz constant \( [f]_{\text{Lip}} \) and \( (X_t)_{0 \leq t \leq T} \) denotes the Cox-Ingersoll-Ross (CIR) process solution to

\[
\begin{aligned}
\frac{dX_t}{X_t} &= (a - \kappa X_t)dt + \sigma \sqrt{X_t}dW_t \\
X_0 &= x > 0,
\end{aligned}
\]

(28)
with \(a \geq \sigma^2/2\), \(\kappa \in \mathbb{R}\), \(\sigma > 0\), \(X_0 = x > 0\). It is well known that this SDE admits a unique strong positive solution. Applying the Lamperti transformation, the process \((Y_t)_{0 \leq t \leq T}\) given by \(Y_t = \sqrt{X_t}\) satisfies
\[
\begin{cases}
  dY_t = L(Y_t)dt + \gamma dW_t \\
  Y_0 = \sqrt{x},
\end{cases}
\]
where \(L(y) = \frac{a - \sigma^2/4}{2y} - \frac{\kappa}{2} y\) and \(\gamma = \frac{\sigma}{2}\). Thus, for \(g : x \in \mathbb{R} \mapsto g(x) = f(x^2)\) we get
\[
\pi_D = \mathbb{E}\left[g(Y_T)\mathbf{1}_{\{\inf_{t \in [0,T]} Y_t > \sqrt{T}\}}\right] \quad \text{and} \quad \pi_U = \mathbb{E}\left[g(Y_T)\mathbf{1}_{\{\sup_{t \in [0,T]} Y_t < \sqrt{T}\}}\right].
\]
As \(a - \sigma^2/4 > 0\), we easily check assumptions (H1) and (H4). Besides, noticing that \(\lim_{y \to 0^+} L'(y) = -\frac{(a - \sigma^2/4)}{2y^2} - \frac{\kappa}{2} = -\infty\), we deduce that \(L\) is decreasing on \((0, \tilde{\varepsilon})\) for \(\tilde{\varepsilon} > 0\) small enough. It is also globally Lipschitz on \([\tilde{\varepsilon}, +\infty)\) so that assumption (H3) is satisfied with \(A = \tilde{\varepsilon}\) and \(L'_A = \frac{|a - \sigma^2/4|}{2 \tilde{\varepsilon}^2} + \frac{\kappa}{2}\).

Now, to check (H2) it is enough to show that
\[
\sup_{t \in [0,T]} \mathbb{E}\left[|L'(Y_t)L(Y_t)|^p + |L''(Y_t)|^p + |L'(Y_t)|^{(2\nu)p} + |L(Y_t)|^p\right] < \infty
\]
which is clearly satisfied as soon as
\[
\sup_{t \in [0,T]} \mathbb{E}\left[Y_t^{-(4\nu^3p)}\right] = \sup_{t \in [0,T]} \mathbb{E}\left[X_t^{-(2\nu^3p)}\right] < \infty \quad \text{and} \quad \sup_{t \in [0,T]} \mathbb{E}\left[Y_t^2\right] = \sup_{t \in [0,T]} \mathbb{E}\left[X_t^2\right] < \infty.
\]
Recalling that \(\sup_{t \in [0,T]} \mathbb{E}\left[X_t^q\right] < \infty\) for all \(q > -\frac{2a}{\sigma^2}\) (see e.g. [9, 6]), we easily conclude that condition (30) is satisfied when \(\sigma^2 < a\) and \(p < \frac{4}{\delta^2}\). Now, as \(|g(x) - g(y)| \leq |f|_{\text{Lip}} |x-y|(|x| + |y|)\) for all \(x, y \in I = (0, +\infty)\), then \(g\) satisfies condition (23) with \(\nu = 1\). Consequently, for \(\delta \in (0, 1/2)\), if we choose \(\frac{4}{3} > P > \frac{(1+\delta)(1+\gamma)[(1+\varepsilon)+2\delta]}{\varepsilon - \delta} > 18\) for \(\varepsilon, \delta\) close to zero and \(h_t\) sufficiently small s.t.
\[
L'_A = \frac{|a - \sigma^2/4|}{2 \tilde{\varepsilon}^2} + \frac{\kappa}{2} < \frac{1}{2h_t},
\]
then Theorem 3.3 is valid provided that \(\inf_{t \in [0,T]} Y_t\) (resp. \(\sup_{t \in [0,T]} Y_t\)) has a bounded density in the neighborhood of the barrier \(\sqrt{D}\) (resp. \(\sqrt{H}\)). More precisely, by the monotone property of \(x \in \mathbb{R}^*_+ \mapsto \sqrt{x}\) we have the relationship \(\inf_{t \in [0,T]} Y_t = \sqrt{\inf_{t \in [0,T]} X_t}\) (resp. \(\sup_{t \in [0,T]} Y_t = \sqrt{\sup_{t \in [0,T]} X_t}\)), then it is sufficient to prove that \(\inf_{t \in [0,T]} X_t\) (resp. \(\sup_{t \in [0,T]} X_t\)) has a continuous density in the neighborhood of the barrier which is the aim of the following subsection.

4.1. Running maximum of the CIR process. The aim of this subsection is to prove that the maximum of the CIR process (28) admits a continuous density. To do so, let us introduce firstly the confluent hypergeometric function \(1F1(x, b, y)\) defined for all \(y, x \in \mathbb{C}\) and \(b \in \mathbb{C} \setminus \{0, -1, -2, \cdots\}\) by
\[
1F1(x, b, y) = \sum_{n=0}^{\infty} \frac{(x)_n}{(b)_n n!} y^n,
\]
where \((x)_n = x(x+1)...(x+n-1)\) stands for the Pochhammer symbol.

**Theorem 4.1.** Let \((X_t)_{0 \leq t \leq T}\) denote the CIR process solution to (28) with \(\kappa > 0\). Then \(\sup_{t \in [0,T]} X_t\) has a continuous density on any compact set \(K \subset (X_0, +\infty)\), given by
\[
z \in K \mapsto P_{\text{CIR,Max}}(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{(1+iu)T} \hat{\phi}(u, z) du
\]
\[
\hat{\phi}(u, z) = -\frac{1}{2\pi i} \int_{1-\infty}^{1+\infty} e^{is} \frac{F_1(s/\kappa, 2a/\sigma^2, 2\kappa X_0/\sigma^2)}{F_1(s/\kappa, 2a/\sigma^2, 2\kappa z/\sigma^2)} ds,
\]

with

\[
\phi(u, z) := \frac{F_1((1 + iu)/\kappa, 2a/\sigma^2, 2\kappa X_0/\sigma^2)}{F_1((1 + iu)/\kappa, 2a/\sigma^2, 2\kappa z/\sigma^2)},
\]

where we recall that for the CIR process, we have

\[
\text{Proof. At first, let us recall that for the CIR process, we have}
\]

\[
\mathbb{P}[\sup_{0 \leq s \leq t} X_s > z] = \frac{1}{2\pi} \int_{1-\infty}^{1+\infty} e^{is} \frac{F_1(s/\kappa, 2a/\sigma^2, 2\kappa X_0/\sigma^2)}{F_1(s/\kappa, 2a/\sigma^2, 2\kappa z/\sigma^2)} ds,
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{(1+iu)t} \phi(u, z) du,
\]

which gives that

\[
\frac{\partial \phi}{\partial z}(u, z) = -\frac{F_1((1 + iu)/\kappa, 2a/\sigma^2, 2\kappa X_0/\sigma^2)}{a_1 F_1((1 + iu)/\kappa, 2a/\sigma^2, 2\kappa z/\sigma^2)^2}.
\]

On the one hand, by formula (13.4.8) in [10] the derivative of the Kummer confluent hypergeometric function is given by

\[
\frac{\partial F_1(a, b, z)}{\partial z} = \frac{a}{b} F_1(a + 1, b + 1, z),
\]

which gives that

\[
\frac{\partial \phi}{\partial z}(u, z) = -\frac{F_1((1 + iu)/\kappa, 2a/\sigma^2, 2\kappa X_0/\sigma^2)}{a_1 F_1((1 + iu)/\kappa, 2a/\sigma^2, 2\kappa z/\sigma^2)^2}.
\]

which is valid inside the sector \( |\arg(x)| < \pi \) and uniformly in bounded \( b \) and \( y \)-domains, where \( I_\nu \) stands for the modified Bessel functions of the first kind. By formula (9.3.14), we have \( I_\nu(x) \sim \frac{\exp(x)}{\sqrt{2\pi x}} \) as \( x \to \infty \) uniformly in the sector \( |\arg(x)| < \frac{\pi}{2} \), we then deduce that

\[
F_1((1 + iu)/\kappa + j, 2a/\sigma^2 + j, 2\kappa v/\sigma^2) \sim \frac{\exp(\frac{\kappa v}{\sigma})}{\sqrt{4\pi}} \left( \frac{2\kappa v}{\sigma^2} \right)^{\frac{j}{2} - \frac{1}{2} \frac{2a}{\sigma^2}} \Gamma\left( \frac{2a}{\sigma^2} + j \right) \Gamma\left( \frac{iu + 1}{\kappa} + j \right) \exp\left( 2\sqrt{\frac{2\kappa v}{\sigma^2}} \frac{1 + iu}{\kappa} + j \right)
\]

as \( u \to \infty \) uniformly on \( v \)-bounded domain. Using that \( \Gamma(x + a)/\Gamma(x + b) \sim x^{a - b} \) as \( x \to \infty \) uniformly inside the sector \( |\arg(x)| < \pi \), (see e.g. (6.5.72) of [24]), and that

\[
\exp\left( 2\sqrt{\frac{2\kappa v}{\sigma^2}} \frac{1 + iu}{\kappa} + j \right) \sim \exp\left( \frac{2\sqrt{\kappa v}}{\sigma} (1 + i)\sqrt{u} \right)
\]

as \( u \to +\infty \) uniformly in bounded \( v \)-domain, we get

\[
F_1((1 + iu)/\kappa + j, 2a/\sigma^2 + j, 2\kappa v/\sigma^2) \sim \frac{\exp(\frac{\kappa v}{\sigma})}{2\sqrt{\pi}} \left( \frac{2\kappa v}{\sigma^2} \right)^{\frac{j}{2} - \frac{1}{2} \frac{2a}{\sigma^2}} \Gamma\left( \frac{2a}{\sigma^2} + j \right) \Gamma\left( \frac{iu + 1}{\kappa} + j \right) \exp\left( 2\sqrt{\frac{2\kappa v}{\sigma^2}} \frac{1 + iu}{\kappa} + j \right)
\]

\[
\times e^{2(1+i)\sqrt{\sigma^2 u}}
\]

\[(34)\]
as \( u \to \infty \) uniformly in bounded \( v \)-domain. Thus, we deduce that
\[
\frac{\partial \phi}{\partial z}(u, z) \sim - \frac{\sqrt{2}}{\sigma} \left( X_0 \right)^{-\frac{\alpha}{\sigma^2} + \frac{1}{2}} (z)^{\frac{\alpha}{\sigma^2} - \frac{3}{4}} \left( 1 + iu \right)^{\frac{\alpha}{\sigma^2} - \frac{1}{4}} ((1 + \kappa) + iu)^{-\frac{\alpha}{\sigma^2} - \frac{1}{4}} \exp \left( (\kappa/\sigma^2)(X_0 - z) \right) \exp \left[ \frac{2}{\sigma} \sqrt{u}(1 + i)(\sqrt{X_0 - \sqrt{z}}) \right],
\]
and therefore
\[
\left| \frac{\partial \phi}{\partial z}(u, z) \right| \sim \frac{\sqrt{2}}{\sigma} \left( X_0 \right)^{-\frac{\alpha}{\sigma^2} + \frac{1}{2}} (z)^{\frac{\alpha}{\sigma^2} - \frac{3}{4}} \exp \left[ (\kappa/\sigma^2)(X_0 - z) \right] u^{-\frac{1}{4}} \exp \left[ \frac{2}{\sigma} \sqrt{u}(\sqrt{X_0 - \sqrt{z}}) \right]
\]
as \( u \to \infty \), uniformly in bounded \( z \)-domain. Hence, \( \int_1^{+\infty} e^{(1+iu)t} \frac{\partial \phi}{\partial z}(u, z) \, du < \infty \) is uniformly convergent in bounded \( z \)-domain too. On the other hand, for the integral from \(-\infty \) to \(-1 \), by (10.3.58) of [24] we have for \( x, y \in \mathbb{C} \)
\[
1F_1(-x, b, y) \sim \left( \frac{y}{x} \right)^{\frac{1-b}{2}} \frac{\Gamma(b) \Gamma(x + 1)}{\Gamma(x + b)} e^{\frac{\pi}{2}y} J_{b-1}(2\sqrt{xy}), \text{ as } x \to +\infty
\]
which is valid inside the sector \(|\arg(x)| < \pi \) and uniformly in bounded \( b \) and \( y \)-domains, where \( J_{\nu} \) stands for the Bessel functions of the first kind. By (9.2.1) of [1], as \(|x| \to \infty \) we have that
\[
J_{\nu}(x) = \sqrt{\frac{2}{\pi x}} \left( \cos \left( x - \nu \frac{\pi}{2} - \frac{\pi}{4} \right) + e^{|\nu(x)|} O \left( \frac{1}{|x|} \right) \right), \ |\arg(x)| < \pi.
\]
Combining this with the following standard asymptotic expansions valid for any \( \alpha \in \mathbb{R}, \beta > 0 \) and \( u \to +\infty \cos(\alpha + i\beta u) = \frac{1}{2} e^{-\alpha u + o(\beta |u|)} \), we get
\[
1F_1((1-iu)/\kappa + j, 2a/\sigma^2 + j, 2\kappa v/\sigma^2) \sim \frac{e^{\kappa v}}{\sqrt{\pi}} \left( \frac{2\kappa v}{\sigma^2} \right)^{\frac{1}{2} - \nu - \frac{\beta}{2}} \frac{\Gamma((\nu+1)/\kappa) - j + 1)}{\Gamma((\nu+1)/\kappa) + \frac{2a}{\sigma^2}} \left( \frac{\kappa v}{\sigma^2} \left( \frac{iu - 1}{\kappa} - j \right) \right)^{-\frac{1}{2}} \cos \left( 2 \sqrt{\frac{2\kappa v}{\sigma^2} \left( \frac{iu - 1}{\kappa} - j \right) + \frac{\pi}{2} \frac{\sqrt{2a}}{\sigma^2} + j + 1} + \frac{\pi}{4} \right).
\]
Using that
\[
\cos \left( 2 \sqrt{\frac{2\kappa v}{\sigma^2} \left( \frac{iu - 1}{\kappa} - j \right) + \frac{\pi}{2} \frac{\sqrt{2a}}{\sigma^2} + j + 1} + \frac{\pi}{4} \right) \sim \frac{1}{2} e^{-i\pi \left( \frac{\beta}{\sigma^2} + \frac{1}{2} \right)} e^{2(1-i)\sqrt{\frac{\kappa v}{\sigma^2} u} \sqrt{a}} ,
\]
as \( u \to \infty \) uniformly on \( v \)-bounded domain and that \( \Gamma(z + a)/\Gamma(z + b) \sim z^{a-b} \) as \( z \to \infty \) uniformly inside the sector \(|\arg(z)| < \pi \), (see e.g. (6.5.72) of [24]), we get
\[
1F_1((1-iu)/\kappa + j, 2a/\sigma^2 + j, 2\kappa v/\sigma^2) \sim \frac{e^{\kappa v}}{2\sqrt{\pi}} \left( \frac{2\kappa v}{\sigma^2} \right)^{\frac{1}{2} - \nu - \frac{\beta}{2}} \frac{\Gamma((\nu+1)/\kappa) - j + 1}{\Gamma((\nu+1)/\kappa) + \frac{2a}{\sigma^2}} \left( \frac{\kappa v}{\sigma^2} \left( \frac{iu - 1}{\kappa} - j \right) \right)^{-\frac{1}{2}} \left( \frac{2\kappa v}{\sigma^2} \left( \frac{iu - 1}{\kappa} - j \right) + \frac{\pi}{2} \frac{\sqrt{2a}}{\sigma^2} + j + 1} + \frac{\pi}{4} \right) e^{-i\pi \left( \frac{\beta}{\sigma^2} + \frac{1}{2} \right)} e^{2(1-i)\sqrt{\frac{\kappa v}{\sigma^2} u} \sqrt{a}}
\]
\[
\times e^{-i\pi \left( \frac{\beta}{\sigma^2} + \frac{1}{2} \right)} e^{2(1-i)\sqrt{\frac{\kappa v}{\sigma^2} u} \sqrt{a}}
\]
(35)
and therefore
\[
\left| \frac{\partial \phi}{\partial z}(-u, z) \right| \sim \frac{\sqrt{2}}{\sigma} (X_0)^{\frac{1}{2} - \frac{a}{\sigma}^2} (z)^{\frac{a}{\sigma} - \frac{1}{2}} \exp \left( \frac{\kappa}{\sigma^2} (X_0 - z) \right) u^{-\frac{1}{2}} \exp \left( \frac{2}{\sigma} \sqrt{u} (\sqrt{X_0} - \sqrt{z}) \right)
\]
as \(u \to \infty\), uniformly in bounded \(z\)-domain. Hence, we deduce that \(\int_{-\infty}^{-1} \left| e^{(1+iu)t} \frac{\partial \phi}{\partial z}(u, z) \right| du < \infty\) uniformly in bounded \(z\)-domain. Finally, we complete the proof by noticing that \((u, z) \in \mathbb{R} \times K \mapsto e^{(1+iu)t} \frac{\partial \phi}{\partial z}(u, z)\) is a continuous function for any compact set \(K \subset (X_0, +\infty)\). (see e.g. [23, Theorem B.3]) □

4.2. Running minimum of the CIR process. In the current subsection, we focus on studying the density of the running minimum of the CIR process (28). For this aim, we introduce the Tricomi confluent hypergeometric function \(U(a, b, z)\) defined for all \(a, z \in \mathbb{C}\) and \(b \in \mathbb{C} \setminus \{\pm 0, \pm 1, \pm 2, \ldots\}\) by
\[
U(a, b, z) = \frac{\Gamma(1 - b)}{\Gamma(1 + a - b)} F_1(a, b, z) + \frac{\Gamma(b - 1)}{\Gamma(a)} z^{1-b} F_1(1 + a - b, 2 - b, z). \tag{36}
\]
Let us denote by \(\tau_{X_0,z} := \inf\{t \geq 0 : X_t = z\}\) the first time that the CIR process \((X_t)_{t \geq 0}\) starting at \(X_0\) hits the level \(z\) satisfying \(0 < z < X_0\). By [7, Theorem 3], the Laplace Transform of \(\tau_{X_0,z}\) is explicit and given by
\[
\mathbb{E}[e^{-s\tau_{X_0,z}}] = \frac{U(s/\kappa, 2a/\sigma^2, 2\kappa X_0/\sigma^2)}{U(s/\kappa, 2a/\sigma^2, 2\kappa z/\sigma^2)}, \quad \text{for } s > 0. \tag{37}
\]

Theorem 4.2. Let \((X_t)_{0 \leq t \leq T}\) denote the CIR process solution to (28) with \(\kappa > 0\). Then \(\inf_{t \in [0, T]} X_t\) has a continuous density on any compact set \(K \subset (0, X_0)\), given by
\[
z \in K \mapsto \text{P}_{\text{CIR, Min}}(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{(1+iu)t} \hat{\psi}(u, z) du \tag{38}
\]
with
\[
\hat{\psi}(u, z) = \frac{2U((1 + (i+u)/\kappa, 2a/\sigma^2, 2\kappa X_0/\sigma^2)U((1 + iu)/\kappa + 1, 2a/\sigma^2 + 1, 2\kappa z/\sigma^2))}{\sigma^2 U((1 + iu)/\kappa, 2a/\sigma^2, 2\kappa z/\sigma^2)^2}.
\]

Proof. Making use of an inverse Laplace transform, the cumulative distribution function of the running minimum CIR process can be expressed as
\[
\mathbb{P}\left[ \inf_{0 \leq s \leq t} X_s \leq z \right] = \mathbb{P}[\tau_{X_0,z} \leq t] = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} e^{ts} U(s/\kappa, 2a/\sigma^2, 2\kappa X_0/\sigma^2) ds \tag{39}
\]
with \(\psi(u, z) := \frac{U((1 + iu)/\kappa, 2a/\sigma^2, 2\kappa X_0/\sigma^2)}{(1 + iu)U((1 + iu)/\kappa, 2a/\sigma^2, 2\kappa z/\sigma^2)}\)

and where we recall that for \(b, y > 0\), the \(x\)-zeros of \(U(x, b, y)\) are negative real and simple (see e.g. [22]). By formula (13.4.21) in [10] the derivative of the Tricomi confluent hypergeometric function is given by
\[
\frac{\partial U(a, b, z)}{\partial z} = -a U(a + 1, b + 1, z) \tag{39}
\]
which gives that
\[
\frac{\partial \psi(u, z)}{\partial z} = 2U((1 + iu)/\kappa, 2a/\sigma^2, 2\kappa X_0/\sigma^2)U((1 + iu)/\kappa + 1, 2a/\sigma^2 + 1, 2\kappa z/\sigma^2).
\]

Now using formulas (10.3.37), (10.3.31) and (9.1.3) in [24], we get for \( x, b \) and \( y \in \mathbb{C} \)
\[
U(x, b, y) \sim \frac{\sqrt{\pi}}{\Gamma(x)} x^{-\frac{1}{2} + \frac{i}{2} y + \frac{1}{2} b} e^y 2^{\sqrt{y}}, \text{ as } x \to +\infty,
\]
inside the sector \( |\arg(x)| < \pi \) and uniformly in bounded \( b \) and \( y \)-domains. For \( j \in \{0, 1\} \) and \( \nu \in \{X_0, z\} \), we get
\[
U((1 + iu)/\kappa + j, 2a/\sigma^2 + j, 2\kappa v/\sigma^2) \sim
\]
\[
\frac{\sqrt{\pi}}{\Gamma(\frac{1+iu}{\kappa} + j)} \exp \left( \frac{\kappa v}{\sigma^2} - 2\sqrt{\frac{2\kappa v}{\sigma^2}(\frac{1+iu}{\kappa} + j)} \right) \left( \frac{2\kappa v}{\sigma^2} \right)^{\frac{1}{2} \frac{\kappa}{\sigma^2} - \frac{1}{2} \frac{\sigma^2}{\kappa(v + j)}} \left( \frac{1 + iu}{\kappa} + j \right)^{-\frac{3}{2} \frac{\sigma^2}{\kappa} + \frac{1}{2}} \tag{41}
\]
as \( u \to \infty \) uniformly in bounded \( v \)-domain. Besides, we easily check that
\[
\exp \left( \frac{\kappa v}{\sigma^2} - 2\sqrt{\frac{2\kappa v}{\sigma^2}(\frac{1+iu}{\kappa} + j)} \right) \sim \exp \left( \frac{\kappa v}{\sigma^2} - \frac{2\sqrt{u}}{\sigma}(1 + i)\sqrt{u} \right), \text{ as } u \to +\infty, \tag{42}
\]
also uniformly in bounded \( v \)-domain. We deduce that
\[
\frac{\partial \psi(u, z)}{\partial z} \sim \frac{2}{\sigma^2} \left( \frac{1+iu}{\kappa} \right)^{-\frac{1}{2} \frac{\sigma^2}{\kappa}} \left( \frac{2\kappa X_0}{\sigma^2} \right)^{\frac{1}{2} \frac{\kappa}{\sigma^2}} \left( \frac{1 + iu}{\kappa} \right)^{-\frac{3}{2} \frac{\sigma^2}{\kappa} + \frac{1}{2}} \exp \left( \left( \frac{\kappa v}{\sigma^2} \right)(X_0 - z) \right) \exp \left( \frac{2}{\sigma} (1 + i)(\sqrt{z} - \sqrt{X_0})\sqrt{u} \right)
\]
and then
\[
\left| \frac{\partial \psi(u, z)}{\partial z} \right| \sim \frac{\sqrt{2}}{\sigma} \left( \frac{\kappa v}{\sigma^2} \right)^{-\frac{1}{2} \frac{\sigma^2}{\kappa}} \exp \left( \left( \frac{\kappa v}{\sigma^2} \right)(X_0 - z) \right) u^{-\frac{1}{2}} \exp \left( \frac{2}{\sigma} (\sqrt{z} - \sqrt{X_0})\sqrt{u} \right)
\]
as \( u \to \infty \) uniformly in bounded \( z \)-domain in \((0, X_0)\). Hence, \( \int_{1}^{+\infty} \left| e^{(1+iu)t} \frac{\partial \psi(u, z)}{\partial z} \right| \,du < \infty \) is uniformly convergent in any bounded \( z \)-domain in \((0, X_0)\).

On the other hand, for the integral from \(-\infty \) to \(-1 \), we have by formula (10.3.68) of [24] that for \( x, b \) and \( y \in \mathbb{C} \)
\[
U(-x, b, y) \sim \left( \frac{y}{x} \right)^{-\frac{1}{2} \frac{\sigma^2}{\kappa}} \Gamma(x + 1) e^y (\cos(\pi x)J_{\nu-1}(2\sqrt{xy}) + \sin(\pi x)Y_{\nu-1}(2\sqrt{xy})), \text{ as } x \to +\infty, \tag{43}
\]
which is valid inside the sector \( |\arg(x)| < \pi \) and uniformly in bounded \( b \) and \( y \)-domains, where \( J_{\nu} \) and \( Y_{\nu} \) stand for the Bessel functions of respectively the first and the second kind. By (9.2.1) and (9.2.2) of [1] we have as \( |x| \to +\infty \)
\[
J_{\nu}(x) = \sqrt{\frac{2}{\pi x}} \left( \cos \left( x - \frac{\nu\pi}{2} \right) - \frac{\pi}{4} \right) + e^{[\text{O}(|x|^{-1})]} \text{, } |\arg(x)| < \pi \tag{44}
\]
\[
Y_{\nu}(x) = \sqrt{\frac{2}{\pi x}} \left( \sin \left( x - \frac{\nu\pi}{2} \right) + \frac{\pi}{4} \right) + e^{[\text{O}(|x|^{-1})]} \text{, } |\arg(x)| < \pi. \tag{45}
\]
Combining all this with the following standard asymptotic expansions valid for any \( \alpha \in \mathbb{R}, \beta > 0 \) and \( u \to +\infty \) \( \cos(\alpha + i\beta u) = O(e^{\beta u}) \), \( \sin(\alpha + i\beta u) = O(e^{\beta u}) \) and with the relation \( \cos(z_1) \cos(z_2) +
\[
\sin(z_1) \sin(z_2) = \cos(z_1 - z_2), \quad z_1, z_2 \in \mathbb{C}, \quad \text{we get}
\]
\[
U \left( -\left( \frac{iu - 1}{\kappa} - j \right), \frac{2a}{\sigma^2} + j, \frac{2 \kappa v}{\sigma^2} \right) \sim \frac{e^{\frac{2i \pi}{\kappa}}}{\sqrt{\pi}} \left( \frac{2 \kappa v}{\sigma^2} \right)^{\frac{1 - j - a}{\kappa^2}} \left( \frac{2 \kappa v}{\sigma^2} \left( \frac{iu - 1}{\kappa} - j \right) \right)^{-\frac{1}{4}} \times \Gamma \left( \frac{iu - 1}{\kappa} - j + 1 \right) \cos \left( \pi \left( \frac{iu - 1}{\kappa} - j \right) - 2 \sqrt{\frac{2 \kappa v}{\sigma^2} \left( \frac{iu - 1}{\kappa} - j \right)} + \frac{\pi}{2} \left( \frac{2a}{\sigma^2} + j + 1 \right) + \frac{\pi}{4} \right)
\]
as \( u \to +\infty \) uniformly in bounded \( v \)-domain. Using that
\[
\cos \left( \pi \left( \frac{iu - 1}{\kappa} - j \right) - 2 \sqrt{\frac{2 \kappa v}{\sigma^2} \left( \frac{iu - 1}{\kappa} - j \right)} + \frac{\pi}{2} \left( \frac{2a}{\sigma^2} + j + 1 \right) + \frac{\pi}{4} \right) \sim \frac{1}{2} e^{-i \pi \left( \frac{2a}{\sigma^2} \cdot \frac{1}{\kappa} - \frac{1}{\kappa^2} + \frac{1}{4} \right) + \frac{\pi}{2} u - 2(1-i) \sqrt{\frac{2 \kappa v}{\sigma^2} u}}
\]
we get
\[
U \left( -\left( \frac{iu - 1}{\kappa} - j \right), \frac{2a}{\sigma^2} + j, \frac{2 \kappa v}{\sigma^2} \right) \sim \frac{e^{\frac{2i \pi}{\kappa}}}{2 \sqrt{\pi}} e^{-i \pi \left( \frac{2a}{\sigma^2} \cdot \frac{1}{\kappa} - \frac{1}{\kappa^2} + \frac{1}{4} \right) + \frac{\pi}{2} u - 2(1-i) \sqrt{\frac{2 \kappa v}{\sigma^2} u}} \left( \frac{2 \kappa v}{\sigma^2} \right)^{\frac{1 - j - a}{\kappa^2}} \times \Gamma \left( \frac{iu - 1}{\kappa} - j + 1 \right) \left( \frac{iu - 1}{\kappa} - j \right)^{-\frac{3}{4} + \frac{1}{2} + \frac{a}{\kappa^2}},
\]
as \( u \to +\infty \) uniformly in bounded \( v \)-domain. Thus,
\[
\left| \frac{\partial \psi(u, z)}{\partial z} \right| \sim \frac{\sqrt{2}}{\sigma} (X_0)^{\frac{1}{4} + \frac{a}{\kappa^2}} (z)^{-\frac{3}{4} + \frac{a}{\kappa^2}} \exp \left( (\kappa/\sigma^2)(X_0 - z) \right) u^{-\frac{1}{2}} \exp \left( \frac{2}{\sigma}(\sqrt{\sigma} - \sqrt{X_0}) \sqrt{u} \right)
\]
as \( u \to \infty \) uniformly in bounded \( z \)-domain in \((0, X_0)\). Hence, we deduce that
\[
\int_{-\infty}^{-1} e^{(1+iu)t} \left| \frac{\partial \psi(u, z)}{\partial z} \right| du < \infty
\]
uniformly in any bounded \( z \)-domain subset of \((0, X_0)\). Finally, we complete the proof by noticing that \((u, z) \in \mathbb{R} \times K \mapsto e^{(1+iu)t} \frac{\partial \psi(u, z)}{\partial z}\) is a continuous function for any compact set \( K \subset (0, X_0) \) (see e.g. [23, Theorem B.3]).

4.3. Numerical tests. For these numerical tests, we consider the problem of pricing D-O and U-O barrier options \( \pi_D = \mathbb{E} \left[ f(X_T) 1_{\{\inf_{t \in [0,T]} X_t > D\}} \right] \) and \( \pi_U = \mathbb{E} \left[ f(X_T) 1_{\{\sup_{t \in [0,T]} X_t < U\}} \right] \), where the payoff function \( f(x) = e^{-rT}(x - K)_+ \) and \((X_t)_{0 \leq t \leq T}\) is the CIR process solution to (28). By the Lamperti transform we get
\[
\pi_D = \mathbb{E} \left[ g (Y_T) 1_{\{\inf_{t \in [0,T]} Y_t > \sqrt{D}\}} \right] \quad \text{and} \quad \pi_U = \mathbb{E} \left[ g (Y_T) 1_{\{\sup_{t \in [0,T]} Y_t < \sqrt{U}\}} \right],
\]
where \( g(x) = e^{-rT}(x^2 - K)_+ \) and \((Y_t)_{t \in [0,T]}\) is solution to (29). We approximate \( \pi_D \) (resp. \( \pi_U \)) by the improved MLMC algorithm \( \hat{Q}_D \) given in (17) (resp. \( \hat{P}_U \) given in (16)), where we used our interpolated drift implicit scheme
\[
Y^n_t = Y^n_{t_i} + \left( \frac{a - \gamma^2}{2Y^n_{t_{i+1}}} - \frac{\kappa Y^n_{t_{i+1}}}{2} \right) (t - t_i) + \gamma (W_t - W_{t_i}), \quad \text{for} \ t \in [t_i, t_{i+1}]
\]
\[
Y_0 = \sqrt{X_0},
\]
with $\gamma = \frac{\sigma^2}{2}$. For $n$ large enough, the positive solution to the above implicit scheme is explicit and given by

$$
Y^n_{t_{i+1}} = \frac{\sqrt{(2 + \kappa \frac{T}{n})(a - \gamma^2)\frac{T}{n} + (\gamma(W_{t_{i+1}} - W_{t_i}) + \bar{Y}^n_{t_i})^2 + \gamma(W_{t_{i+1}} - W_{t_i}) + \bar{Y}^n_{t_i}}}{2 + \kappa \frac{T}{n}}.
$$

To illustrate the performance of our MLMC algorithms we consider the same comparison procedure as in [13]. We considered the same model and option parameters proposed by [8]. We take $r = 0.1$, $X_0 = 100$, $a = 0$, $\kappa = -0.1$, $\sigma = 2.5$ and $T = 0.5$. For the D-O option the strike is $K = 95$, and for the U-O option the strike is $K = 105$ and the barrier $U = 120$. The benchmark prices given in [8] for the D-O (resp. U-O) option is 10.6013 (resp. 0.7734). The performance of the improved MLMC is given in the tables and figure below.

| Accuracy  | Price   | MLMC cost | MC cost   | Saving  |
|-----------|---------|-----------|-----------|---------|
| $10^{-3}$ | 10.669  | 2.588 $\times 10^5$ | 6.752 $\times 10^8$ | 260.91  |
| $5 \times 10^{-3}$ | 10.668 | 1.051 $\times 10^7$ | 3.376 $\times 10^8$ | 32.13   |
| $10^{-2}$ | 10.668  | 2.510 $\times 10^6$ | 4.220 $\times 10^7$ | 16.81   |
| $2 \times 10^{-2}$ | 10.677 | 6.187 $\times 10^5$ | 5.275 $\times 10^6$ | 8.52    |

**Table 1.** MLMC complexity tests for D-O barrier option pricing of $\pi_D$

| Accuracy  | Price   | MLMC cost | MC cost   | Saving  |
|-----------|---------|-----------|-----------|---------|
| $10^{-3}$ | 0.77200 | 4.674 $\times 10^6$ | 4.221 $\times 10^8$ | 90.32   |
| $5 \times 10^{-3}$ | 0.76926 | 1.571 $\times 10^5$ | 2.11 $\times 10^6$ | 13.44   |
| $10^{-2}$ | 0.77015 | 3.809 $\times 10^4$ | 2.638 $\times 10^5$ | 6.93    |
| $2 \times 10^{-2}$ | 0.78168 | 1.463 $\times 10^4$ | 6.596 $\times 10^4$ | 4.51    |

**Table 2.** MLMC complexity tests for U-O barrier option pricing $\pi_U$

![Figure 1](image1.png)

**Figure 1.** Comparison for the performances of MLMC vs classical MC algorithm under the CIR model.

The numerical results illustrates Theorem 3.3, that is the improved MLMC algorithm reaches for a given precision $\varepsilon$ the optimal time complexity $O(\varepsilon^{-2})$ for D-O and U-O barrier options under the CIR model.
5. Application to CEV Process

In this section, we consider the CEV process solution to

$$dX_t = \mu X_t \, dt + \sigma X_t^\alpha \, dW_t, \quad t \geq 0, \ X_0 > 0, \ \mu \in \mathbb{R} \quad \text{and} \ \alpha > 1. \quad (46)$$

We consider the problem of pricing an U-O barrier option $\Pi_{D}^{U-O,X} := \mathbb{E} \left[ f(X_T) \mathbb{I}_{\{\sup_{t \in [0,T]} X_t < D\}} \right]$ with barrier $D$ where $f$ is a bounded Lipschitz function with Lipschitz constant $|f|_{\text{Lip}}$. For $\alpha > 1$, by Feller’s test the solution of $(46)$ is known to be positive (see e.g. [18, Lemma 6.4.4.1]). Applying the Lamperti transformation, the process $(Y_t)_{0 \leq t \leq T}$ given by $Y_t = X_t^{1-\alpha}$ is well defined on $I = (0, +\infty)$ and satisfies

$$\begin{cases}
    dY_t &= L(Y_t) \, dt + \gamma dW_t \\
    Y_0 &= X_0^{1-\alpha},
\end{cases} \quad (47)$$

where $L(y) = (1 - \alpha) \left( \mu y - \alpha \frac{\sigma^2}{2} y^{-1} \right)$ and $\gamma = \sigma (1 - \alpha)$. Thus, as the map $x \mapsto x^{1-\alpha}$ is decreasing, our initial pricing problem is transformed as follows on the Lamperti transform space

$$\Pi_{D}^{U-O,X} = \mathbb{E} \left[ g(Y_T) \mathbb{I}_{\{\inf_{t \in [0,T]} Y_t > D^{1-\alpha}\}} \right], \quad (48)$$

with $g : x \in \mathbb{R} \mapsto f(x^{1-\alpha})$. As $\lim_{y \to 0^+} L'(y) = \lim_{y \to 0^+} (1 - \alpha)(\mu + \alpha \frac{\sigma^2}{2} y^{-2}) = -\infty$, we deduce that $L$ is decreasing on $(0, \bar{\varepsilon})$ for $\bar{\varepsilon} > 0$ small enough and it is clearly globally Lipschitz on $[\bar{\varepsilon}, +\infty)$ so that assumption (H3) is satisfied with $A = \bar{\varepsilon}$ and $L_A = (\alpha - 1) \left( |\mu| + \alpha \frac{\sigma^2}{2} \varepsilon^{-2} \right)$. Also, we easily check assumptions (H1) and (H4). On the one hand, by Itô’s formula the process $(Z_t)_{0 \leq t \leq T}$ given by $Z_t = X_t^{2(\alpha - 1)} / 4(\alpha - 1)^2$ is a CIR process solution to

$$\begin{cases}
    dZ_t &= (a - \kappa Z_t) \, dt - \sigma \sqrt{Z_t} \, dW_t \\
    Z_0 &= X_0^{2(\alpha - 1)} / 4(\alpha - 1)^2,
\end{cases} \quad (49)$$

with $a = \frac{\sigma^2(2\alpha - 1)}{4(\alpha - 1)}$ and $\kappa = 2\mu(\alpha - 1)$. Thanks to this second transformation we deduce that $\sup_{t \in [0,T]} \mathbb{E}[Y_t^q] < \infty$ for $q > -\frac{2\alpha - 1}{2(\alpha - 1)}$. On the other hand to check assumption (H2) it is enough to show that

$$\sup_{t \in [0,T]} \mathbb{E} \left[ |L'(Y_t) L(Y_t)|^p + |L''(Y_t)|^p + |L'(Y_t)|^{(2\nu p)} + |L(Y_t)|^p \right] < \infty \quad (50)$$

which is satisfied if $\sup_{t \in [0,T]} \mathbb{E}[Y_t^{-(4\nu p)}] < \infty$. This condition is satisfied when $4 < \frac{2\alpha - 1}{2(\alpha - 1)}$ (which corresponds to have $\alpha \in (1, \frac{6}{5})$) and $p < \frac{2\alpha - 1}{6(\alpha - 1)}$.

Besides, since by Taylor formula we have $|g(x) - g(y)| \leq \frac{|f|_{\text{Lip}}}{\alpha - 1} |x - y| (|x|^{-\frac{\alpha}{\alpha - 1}} + |y|^{-\frac{\alpha}{\alpha - 1}})$ for all $x, y \in I = (0, +\infty)$, then $g$ satisfies condition (26) with $\nu = -\frac{\alpha}{\alpha - 1}$. Hence, for $\delta \in (0, 1/2)$, if we choose $\alpha$ such that $1 < \alpha < \frac{50}{53} < \frac{7}{6}$ then we can find $p$ such that $\frac{2\alpha - 1}{6(\alpha - 1)} > p > \frac{(1+\delta)(1+\gamma)5(1+\varepsilon)}{2 - \delta} > 10$. Finally, if we choose $h_\delta$ sufficiently small such that $L_A' = (\alpha - 1) \left( |\mu| + \alpha \frac{\sigma^2}{2} e^{-2} \right) < \frac{1}{2\delta}$, then Theorem 3.5 is valid provided that $\inf_{t \in [0,T]} Y_t$ has a bounded density in the neighbourhood of the barrier $D^{1-\alpha}$. By the monotone property of $x \in \mathbb{R}^+ \mapsto x^{1-\alpha}$ we have the relationship $\inf_{t \in [0,T]} Y_t = (\sup_{t \in [0,T]} X_t)^{1-\alpha}$, then it is sufficient to prove that $\sup_{t \in [0,T]} X_t$ has a continuous density in the neighborhood of the barrier which is the aim of the following subsection.
Remark 5.1. One can also consider the CEV process for \( \alpha \in (\frac{1}{2}, 1) \) solution to
\[
d X_t = (a - \kappa X_t)dt + \sigma Y_t^\alpha dW_t, \text{ with } X_0 > 0, a > 0. \tag{51}
\]
It can be easily checked that for \( a > 0 \) this SDE is well defined on \( I = (0, +\infty) \) (see [2, Section 3]). The associated drift implicit Euler scheme is well defined on \( I \) too and satisfy the conditions of our theoretical setting. However, the condition that \( \inf_{t \in [0,T]} X_t \) or \( \sup_{t \in [0,T]} X_t \) admits a continuous density in the neighborhood of the barrier seems to be a challenging problem, since we dont have explicit Laplace transform of the corresponding hitting times as it is the case for the previous CEV process solution to (46). In counterpart, the efficiency of the MLMC method is still confirmed by our numerical tests for the model (51).

5.1. Running maximum of the CEV process. Let us denote by \( \tau_{X_0^+} := \inf\{t \geq 0 : X_t = z\} \) the first time that the CEV process \( (X_t)_{t \geq 0} \) starting at \( X_0 \) hits the level \( z > X_0 \). From [18, subsections 5.3.6 and 6.4.5], the Laplace transform of the hitting time \( \tau_{X_0^+} \) is given by
\[
E[e^{-s\tau_{X_0^+}}] = \left( \frac{X_0}{z} \right)^{\beta + \frac{1}{2}} \exp\left( \frac{c}{2} \left( X_0^{-2\beta} - z^{-2\beta} \right) \right) \frac{W_{k,n}(cX_0^{-2\beta})}{W_{k,n}(cz^{-2\beta})}, \tag{52}
\]
with \( c = \text{sign}(\mu \beta), n = \frac{1}{4\beta}, k = c \left( \frac{1}{2} + \frac{1}{4\beta} \right) - \frac{s}{2c\mu \beta} \) and \( W_{k,n} \) the Whittaker’s function \( W_{k,n}(y) = y^{n+\frac{1}{2}} e^{-y/2} U(n - k + \frac{1}{2}, 2n + 1, y) \), where \( U \) denotes the confluent hypergeometric function of second kind defined in (36) and with \( \beta = \alpha - 1 \) and \( c = \frac{\lvert \mu \rvert}{\beta \sigma^2} \).

Theorem 5.2. Let \((X_t)_{0 \leq t \leq T}\) denotes the CEV process solution to (46). Then \( \sup_{t \in [0,T]} X_t \) has a continuous density on any compact set \( K \subset (X_0, +\infty) \), given by
\[
z \in K \mapsto P_{CEV, \text{Max}}(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{(1+iu)T} \hat{\Phi}(z, u) du, \tag{53}
\]
with
\[
\hat{\Phi}(z, u) = -c z^{-2\beta - 1} \frac{U\left( \frac{1+iu}{2\mu \beta}, 1 + \frac{1}{2\beta}, cX_0^{-2\beta} \right) U\left( \frac{1+iu}{2\mu \beta} + 1, 2 + \frac{1}{2\beta}, cz^{-2\beta} \right)}{U\left( \frac{1+iu}{2\mu \beta}, 1 + \frac{1}{2\beta}, cz^{-2\beta} \right)^2}, \text{ for } \mu > 0
\]
and
\[
\hat{\Phi}(z, u) = -c z^{-2\beta - 1} \left( \frac{2\beta + 1}{1 + iu} - \frac{1}{\mu} \right) \frac{U\left( 1 + \frac{1}{2\beta} - \frac{1+iu}{2\mu \beta}, 1 + \frac{1}{2\beta}, cX_0^{-2\beta} \right) U\left( 2 + \frac{1}{2\beta} - \frac{1+iu}{2\mu \beta}, 2 + \frac{1}{2\beta}, cz^{-2\beta} \right)}{U\left( 1 + \frac{1}{2\beta} - \frac{1+iu}{2\mu \beta}, 1 + \frac{1}{2\beta}, cz^{-2\beta} \right)^2}, \text{ for } \mu < 0.
\]

Proof. • Case \( \mu > 0 \). Let us recall that the process given by \( Z_t = \frac{X_t^{-2(a-1)} - 4(a-1)^{-2}}{4} \) is solution to (49). Then, as \( \alpha > 1 \) we get
\[
P\left( \sup_{t \in [0,T]} X_t \geq z \right) = P\left( \left( \sup_{t \in [0,T]} X_t \right)^{-2(a-1)} \leq z^{-2(a-1)} \right) = P\left( \inf_{t \in [0,T]} X_t^{-2(a-1)} \leq z^{-2(a-1)} \right) = P\left( \inf_{t \in [0,T]} Z_t \leq \frac{z^{-2(a-1)}}{4(a-1)^2} \right).
\]
Now, using the same arguments in the proof of Theorem 4.2 with $Z_0 = \frac{X_0^{2(\alpha-1)}}{4(\alpha-1)^2}$, $a = \frac{\sigma^2(2\alpha-1)}{4(\alpha-1)}$, $\kappa = 2\mu(\alpha - 1)$, we get for $\beta = \alpha - 1$ and $c = \frac{\mu}{\beta \sigma^2}$

$$
P\left( \sup_{t \in [0,T]} X_t \geq z \right) = \mathbb{P}\left[ \inf_{0 \leq s \leq t} Z_s \leq \frac{z^{-2(\alpha-1)}}{4(\alpha-1)^2} \right]
= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{(1+iu)t} \Phi(u,z) du,
$$

with $\Phi(u,z) := \frac{1}{1+iu} \mathbb{P}\left[ \sup_{0 \leq s \leq t} X_s > z \right]$. Using the same arguments in the proof of Theorem 4.2 with $\beta = \alpha - 1$ and $c = \frac{\mu}{\beta \sigma^2}$,

$$
\partial \Phi\left( z, u \right) = -cz^{-2\beta-1} \left( \frac{2\beta + 1}{1+iu} - \frac{1}{\mu} \right) \frac{U\left( 1 + \frac{1}{2\beta} - \frac{1+iu}{2\mu}, 1 + \frac{1}{2\beta}, cX_0^{-2\beta} \right) U\left( 2 + \frac{1}{2\beta} - \frac{1+iu}{2\mu}, 1 + \frac{1}{2\beta}, cz^{-2\beta} \right)}{U\left( 1 + \frac{1}{2\beta} - \frac{1+iu}{2\mu}, 1 + \frac{1}{2\beta}, cX_0^{-2\beta} \right)^2}.
$$

Proceeding in the same way as for the previous case, we first apply (40) to get for $j \in \{0,1\}$ and $v \in \{X_0, z\}$

$$
U(1 + j + \frac{1}{2\beta} - \frac{1+iu}{2\mu}, 1 + j + \frac{1}{2\beta}, cv^{-2\beta}) \sim \sqrt{\pi} \left( \frac{1 + j + \frac{1}{2\beta} - \frac{1+iu}{2\mu}}{1 + \frac{1}{2\beta} - \frac{1+iu}{2\mu}} \right)^{-\frac{1}{2} + \frac{1}{4\beta}} \left( \frac{1 + j + \frac{1}{2\beta} - \frac{1+iu}{2\mu}}{1 + \frac{1}{2\beta} - \frac{1+iu}{2\mu}} \right)^{-\frac{3}{4} + \frac{1}{4\beta}} \Gamma\left( 1 + j + \frac{1}{2\beta} - \frac{1+iu}{2\mu} \right) \left( cv^{-2\beta} \right)^{-\frac{1}{2} + \frac{1}{4\beta}} \exp\left( \frac{cv^{-2\beta} - 2}{2} \right) \exp\left( \sqrt{cv^{-2\beta}} \left( 1 + j + \frac{1}{2\beta} - \frac{1+iu}{2\mu} \right) \right)
$$

and then we use that $\exp\left( -2\sqrt{cv^{-2\beta}} \left( 1 + j + \frac{1}{2\beta} - \frac{1+iu}{2\mu} \right) \right) \sim \exp\left( -\sqrt{\frac{cv^{-2\beta}}{\mu^2}} (1+i)\sqrt{u} \right)$ as $u \to +\infty$, uniformly in bounded $\nu$-domain. We then deduce that

$$
\partial \Phi\left( z, u \right) \sim -cz^{-2\beta-1} \left( \frac{2\beta + 1}{1+iu} - \frac{1}{\mu} \right) \left( 1 + \frac{1}{2\beta} - \frac{1+iu}{2\mu} \right)^{-\frac{3}{4} + \frac{1}{4\beta}} \left( 2 + \frac{1}{2\beta} - \frac{1+iu}{2\mu} \right)^{\frac{1}{4} + \frac{1}{4\beta}} \left( cX_0^{-2\beta} - z^{-2\beta} \right)^{-\frac{1}{2} + \frac{1}{4\beta}} \exp\left( \frac{c}{25} \left( X_0^{-2\beta} - z^{-2\beta} \right) \right) \exp\left( \sqrt{-\frac{c}{\mu^2}} (1+i)\sqrt{u} (z^{-\beta} - X_0^{-\beta}) \right).
$$
and
\[
\frac{\partial \Phi}{\partial z}(z, u) \sim \sqrt{-\frac{2\beta c}{\mu}} (X_0)^{\frac{1}{2}(\beta+1)} z^{-\frac{1}{2}(\beta+1)} u^{-\frac{1}{2}} \exp \left( -\frac{c}{\mu \beta} \sqrt{u(z^{-\beta} - X_0^{-\beta})} \right) \exp \left( \frac{c}{2}(X_0^{-2\beta} - z^{-2\beta}) \right)
\]
as \( u \to +\infty \) uniformly in any bounded \( z \)-domain subset of \((X_0, +\infty)\). As \( c, \beta \) and \(-\mu\) are positive constants it follows that \( \int_{-1}^{+\infty} |e^{(1+iu)t} \frac{\partial \Phi}{\partial z}(z, u)| \, du < \infty \) is uniformly convergent in any bounded \( z \)-domain in \((X_0, +\infty)\). Concerning the integral from \(-\infty \) to \(-1\), we proceed as above and use (43), (44) and (45) to get for \( j \in \{0, 1\} \) and \( v \in \{X_0, z\} \)
\[
U(-(-1 - j - \frac{1}{2\beta} + \frac{1}{2\mu \beta}), 1 + j + \frac{1}{2\beta}, cv^{-2\beta}) \sim \frac{e^{cv^{-2\beta}}}{\sqrt{\pi}} \left( -\frac{cv^{-2\beta}}{-1 - j - \frac{1}{2\beta} + \frac{1}{2\mu \beta}} \right)^{-\frac{1}{2} - \frac{1}{4\beta}}
\]
\times \cos \left( \pi \left( -1 - j - \frac{1}{2\beta} + \frac{1}{2\mu \beta} \right) \right) - 2 \sqrt{cv^{-2\beta}} \left( -1 - j - \frac{1}{2\beta} + \frac{1}{2\mu \beta} \right) + \frac{\pi}{2} (j + \frac{1}{2\beta}) + \frac{\pi}{4} \right)
\]
as \( u \to +\infty \) uniformly in bounded \( v \)-domain. Using that
\[
\cos \left( \pi \left( -1 - j - \frac{1}{2\beta} + \frac{1}{2\mu \beta} \right) \right) - 2 \sqrt{cv^{-2\beta}} \left( -1 - j - \frac{1}{2\beta} + \frac{1}{2\mu \beta} \right) + \frac{\pi}{2} (j + \frac{1}{2\beta}) + \frac{\pi}{4} \right)
\]
as \( u \to +\infty \) uniformly in bounded \( v \)-domain. Thus, we get
\[
\frac{\partial \Phi}{\partial z}(z, u) \sim \sqrt{-\frac{2\beta c}{\mu}} (X_0)^{\frac{1}{2}(\beta+1)} z^{-\frac{1}{2}(\beta+1)} u^{-\frac{1}{2}} \exp \left( -\frac{c}{\mu \beta} \sqrt{u(z^{-\beta} - X_0^{-\beta})} \right) \exp \left( \frac{c}{2}(X_0^{-2\beta} - z^{-2\beta}) \right)
\]
as \( u \to +\infty \) uniformly in any bounded \( z \)-domain subset of \((X_0, +\infty)\). As \( c, \beta \) and \(-\mu\) are positive constants it follows that \( \int_{-1}^{-\infty} |e^{(1+iu)t} \frac{\partial \Phi}{\partial z}(z, u)| \, du < \infty \) is uniformly convergent in any bounded \( z \)-domain in \((X_0, +\infty)\). We complete the proof by noticing that \((u, z) \in \mathbb{R} \times K \mapsto e^{(1+iu)t} \frac{\partial \Phi}{\partial z}(u, z) \) is a continuous function for any compact set \( K \subset (X_0, +\infty) \) (see e.g. [23, Theorem B.3]).

5.2. Minimum of CEV process. Let us denote by \( \tau_{X_0 z} := \inf\{t \geq 0 : X_t = z\} \) the first time that the CEV process \((X_t)_{t \geq 0}\) starting at \( X_0 \) hits the level \( 0 < z < X_0 \). By [18, subsections 5.3.6 and 6.4.5], the Laplace transform of the hitting time \( \tau_{X_0 z} := \inf\{t \geq 0 : X_t = z\} \) is given by
\[
\mathbb{E}[e^{-\epsilon \tau_{X_0 z}}] = \left( \frac{X_0}{z} \right)^{\beta + \frac{1}{2}} \exp \left( \frac{\epsilon}{2} c (X_0^{-2\beta} - z^{-2\beta}) \right) \frac{M_{k,n}(cX_0^{2\beta})}{M_{k,n}(cz^{-2\beta})}
\]
with \( \epsilon = \text{sign}(\mu \beta) \), \( n = \frac{1}{4\beta} \), \( k = \epsilon \left( \frac{1}{2} + \frac{1}{4\beta} \right) - \frac{\mu}{2\beta |\mu|} \) and the Whittaker function \( M_{k,n}(y) = y^{n+\frac{1}{2}} e^{-\frac{y}{2}} 1_F(n-k + \frac{1}{2}, 2n + 1, y) \), where \( 1_F \) denotes the confluent hypergeometric function of the first kind defined in (31) and with \( \beta = \alpha - 1 \) and \( c = \frac{|\mu|}{\beta \sigma^2} \).
Theorem 5.3. Let \((X_t)_{0 \leq t \leq T}\) denotes the CEV process solution to (46). Then \(\inf_{t \in [0,T]} X_t\) has a continuous density on any compact set \(K \subset (0, X_0),\) given by

\[
z \in K \mapsto P_{CEV, \text{Mfn}}(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{(1+iu)T} \hat{\Psi}(z, u) du,
\]

with

\[
\hat{\Psi}(z, u) = -cz^{-2\beta - 1} \frac{1}{\mu(1 + \frac{1}{2\beta})} \frac{1}{1 + \frac{1}{2\beta}} \frac{F_1(1 + \frac{1}{2\beta}, cX_0^{-2\beta})}{1 F_1(\frac{1}{2\beta}, 1 + \frac{1}{2\beta}, cz^{-2\beta})}, \text{ for } \mu > 0
\]

and

\[
\hat{\Psi}(z, u) = -cz^{-2\beta - 1} \left( \frac{2\beta}{1 + iu} - \frac{1}{\mu(1 + \frac{1}{2\beta})} \right) \frac{1}{1 + \frac{1}{2\beta}} \frac{F_1(1 + \frac{1}{2\beta} - \frac{1}{2\beta}, 1 + \frac{1}{2\beta}, cX_0^{-2\beta})}{1 F_1(\frac{1}{2\beta}, 1 + \frac{1}{2\beta}, cz^{-2\beta})}, \text{ for } \mu < 0.
\]

Proof. • Case \(\mu > 0\). Recalling that the process given by \(Z_t = \frac{X_t^{-(\alpha - 1)}}{4(\alpha - 1)^2}\) is solution to (49), then, as \(\alpha > 1\) we get

\[
\mathbb{P}\left( \inf_{t \in [0,T]} X_t \leq z \right) = \mathbb{P}\left( \inf_{t \in [0,T]} X_t^{-2(\alpha - 1)} \geq z^{-2(\alpha - 1)} \right)
= \mathbb{P}\left( \sup_{t \in [0,T]} X_t^{-2(\alpha - 1)} \geq z^{-2(\alpha - 1)} \right)
= \mathbb{P}\left( \sup_{t \in [0,T]} Z_t \geq \frac{z^{-2(\alpha - 1)}}{4(\alpha - 1)^2} \right).
\]

Now, using the same arguments in the proof of Theorem 4.1 with \(Z_0 = \frac{X_0^{-(\alpha - 1)}}{4(\alpha - 1)^2}, a = \frac{\sigma^2(2\alpha - 1)}{4(\alpha - 1)}, \kappa = 2\mu(\alpha - 1),\) we get for \(\beta = \alpha - 1\) and \(c = \frac{\mu}{\beta\sigma^2}\)

\[
\mathbb{P}\left( \inf_{t \in [0,T]} X_t \leq z \right) = \mathbb{P}\left[ \sup_{0 \leq s \leq t} Z_s \geq \frac{z^{-2(\alpha - 1)}}{4(\alpha - 1)^2} \right]
= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{(1+iu)t} \hat{\Psi}(u, z) du,
\]

with \(\hat{\Psi}(u, z) := \frac{1}{1 + iu} \frac{1}{1 F_1(\frac{1}{2\beta}, 1 + \frac{1}{2\beta}, cX_0^{-2\beta})} \frac{1}{1 F_1(\frac{1}{2\beta}, 1 + \frac{1}{2\beta}, cX_0^{-2\beta})},\)

and thus we get (55).

• Case \(\mu < 0\). In this case by (54) we have,

\[
\mathbb{E}[e^{-\gamma X_{t_0}}] = \exp\left( -c(X_0^{-2\beta} - z^{-2\beta}) \right) \frac{1}{1 F_1(\frac{1}{2\beta} + 1 - \frac{\sigma^2}{2\beta}, \frac{1}{2\beta} + 1, cX_0^{-2\beta})}.
\]

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and
\[ P \left[ \inf_{0 \leq s \leq t} X_s \geq z \right] = \frac{\exp \left( -c(X_0 - 2\beta - z^{-2\beta}) \right)}{2\pi} \int_{-\infty}^{\infty} e^{(1+iu)t} \Psi(z,u) du \]

with \( \Psi(z,u) = \frac{1}{1+iu} F_1 \left( \frac{1}{2\beta} + 1 - \frac{1+iu}{2\mu\beta}, 1, \frac{1}{2\beta} + 1, cX_0^{-2\beta} \right) \). By (33), we have
\[
\frac{\partial \Psi}{\partial z}(z,u) = -cz^{-2\beta-1} \left( \frac{2\beta}{1+iu} - \frac{1}{\mu(1+2\beta)} \right) \times \frac{F_1(1 + \frac{1}{2\beta} - \frac{1+iu}{2\mu\beta}, 1 + \frac{1}{2\beta}, cX_0^{-2\beta}) F_1(2 + \frac{1}{2\beta} - \frac{1+iu}{2\mu\beta}, 2 + \frac{1}{2\beta}, cz^{-2\beta})}{F_1(1 + \frac{1}{2\beta} - \frac{1+iu}{2\mu\beta}, 1 + \frac{1}{2\beta}, cz^{-2\beta})^2}.
\] (56)

For \( j \in \{0,1\} \) and \( v \in \{X_0,z\} \), we follow the similar steps that led us to get (34) and we obtain
\[
F_1(1 + j + \frac{1}{2\beta} - \frac{1+iu}{2\mu\beta}, 1 + j + \frac{1}{2\beta}, cv^{-2\beta}) \sim \exp \left( 2 \sqrt{cv^{-2\beta}} \left( 1 + j + \frac{1}{2\beta} - \frac{1+iu}{2\mu\beta} \right) \right)
\]
\[
\times \exp \left( \sqrt{-\frac{cv^{-2\beta}}{\mu\beta}} (1+i)\sqrt{u} \right), \text{ as } u \to +\infty
\]
uniformly in bounded \( v \)-domain. Now, we use that
\[
\exp \left( 2 \sqrt{cv^{-2\beta}} \left( 1 + j + \frac{1}{2\beta} - \frac{1+iu}{2\mu\beta} \right) \right) \sim \exp \left( \sqrt{-\frac{cv^{-2\beta}}{\mu\beta}} (1+i)\sqrt{u} \right), \text{ as } u \to +\infty
\]
uniformly in bounded \( v \)-domain, to get
\[
F_1(1 + j + \frac{1}{2\beta} - \frac{1+iu}{2\mu\beta}, 1 + j + \frac{1}{2\beta}, cv^{-2\beta}) \sim \exp \left( 2 \sqrt{cv^{-2\beta}} \left( 1 + j + \frac{1}{2\beta} - \frac{1+iu}{2\mu\beta} \right) \right)
\]
\[
\times \exp \left( \sqrt{-\frac{cv^{-2\beta}}{\mu\beta}} (1+i)\sqrt{u} \right), \text{ as } u \to +\infty
\]
uniformly in bounded \( v \)-domain. Hence, by (56) we deduce that
\[
\left| \frac{\partial \Psi}{\partial z}(z,u) \right| \sim \sqrt{-\frac{2c}{\mu\beta}} (X_0)^{\frac{1}{2}(\beta+1)} z^{-\frac{3}{2}(\beta+1)} u^{-\frac{1}{2}} \exp \left( -\frac{c}{\mu\beta}\sqrt{u}(X_0 - z) \right) \exp \left( \frac{c}{2} (X_0 - 2\beta - z^{-2\beta}) \right), \text{ as } u \to +\infty
\]
in any bounded \( z \)-domain subset of \((0, X_0)\). As \( c, \beta \) and \( -\mu \) are positive constants it follows that \( \int^{+\infty}_1 |e^{(1+iu)t} \frac{\partial \Psi}{\partial z}(z,u)| du < \infty \) is uniformly convergent in any bounded \( z \)-domain in \((0, X_0)\). Besides, for the integral from \(-\infty\) to \(-1\), we use similar steps as the ones that led us to get (35), and we have
\[
F_1(1 + j + \frac{1}{2\beta} - \frac{1-iu}{2\mu\beta}, 1 + j + \frac{1}{2\beta}, cv^{-2\beta}) \sim \frac{e^{\frac{1}{2} cv^{-2\beta}}}{\sqrt{\pi}} \left( \frac{cv^{-2\beta}}{-1 - j - \frac{1}{2\beta} - \frac{iu-1}{2\mu\beta}} \right)^{\frac{1}{2} - \frac{1}{4\beta}}
\]
For the CEV model, we don't have any benchmark price. To illustrate the MLMC complexity as a continuous function for any compact set and given by

\[\frac{\Gamma(1+j + \frac{1}{2\beta})\Gamma(-j - \frac{1}{2\beta} - \frac{iu - 1}{2\mu\beta})}{\Gamma(-\frac{iu - 1}{2\mu\beta})} \times \left( cv^{-2\beta} \left( -1 - j - \frac{1}{2\beta} - \frac{iu - 1}{2\mu\beta} \right) \right) \frac{-\frac{1}{4}}{4} \cos \left( 2\sqrt{cv^{-2\beta} \left( -1 - j - \frac{1}{2\beta} - \frac{iu - 1}{2\mu\beta} \right)} - \frac{\pi}{2} \left( j + \frac{1}{2\beta} - \frac{1}{4} \right) \right) \]

as \( u \to \infty \) uniformly on \( v \)-bounded domain. Now, using that

\[\cos \left( 2\sqrt{cv^{-2\beta} \left( -1 - j - \frac{1}{2\beta} - \frac{iu - 1}{2\mu\beta} \right)} - \frac{\pi}{2} \left( j + \frac{1}{2\beta} - \frac{1}{4} \right) \right) \sim \frac{1}{2} e^{i\pi \left( \frac{3}{4} + \frac{1}{4\beta} + \frac{1}{4} \right)} e^{(1-i)\sqrt{\frac{-cv^{-2\beta}}{4\beta}}u},\]

as \( u \to \infty \) uniformly on \( v \)-bounded domain, we get

\[1F_{1} \left( 1 + j + \frac{1}{2\beta} - \frac{1}{2\beta} - \frac{iu - 1}{2\mu\beta}, 1, j + \frac{1}{2\beta}, cv^{-2\beta} \right) \sim \frac{e^{\frac{1}{2}cv^{-2\beta}}}{2\sqrt{\pi}} \left( cv^{-2\beta} \right)^{-\frac{1}{2} - \frac{1}{4\beta} - \frac{1}{4}} \Gamma(1+j+\frac{1}{2\beta}) \left( -1 - j - \frac{1}{2\beta} - \frac{iu - 1}{2\mu\beta} \right)^{-\frac{1}{4} - \frac{1}{4\beta} - \frac{1}{4}} e^{i\pi \left( \frac{3}{4} + \frac{1}{4\beta} + \frac{1}{4} \right)} \times \exp \left( (1-i)\sqrt{\frac{-cv^{-2\beta}}{4\beta}}u \right)\]

and that

\[|\frac{\partial \Psi}{\partial z}(z,-u)| \sim \sqrt{-\frac{2\beta c}{\mu} (X_{0})^{\frac{1}{2}(\beta+1)} z^{-\frac{1}{2}(\beta+1)} u^{-\frac{1}{2}} \exp \left( \sqrt{-\frac{c}{\mu\beta} \sqrt{u}(X_{0}^{-\beta} - z^{-\beta})} \right) \exp \left( \frac{c}{2} (X_{0}^{-\beta} - z^{-\beta}) \right)},\]

as \( u \to +\infty \) uniformly in any bounded \( z \)-domain subset of \((0,X_{0})\). Thus, as \( c, \beta \) and \( -\mu \) are positive constants we deduce that \( \int_{-\infty}^{1} e^{(1+iu)t} \frac{\partial \Psi}{\partial z}(z,u) \, du < \infty \) is uniformly convergent in any bounded \( z \)-domain in \((0,X_{0})\). We complete the proof by noticing that \((u,z) \in \mathbb{R} \times K \mapsto e^{(1+iu)t} \frac{\partial \Psi}{\partial z}(u,z)\) is a continuous function for any compact set \( K \subset (0,X_{0}) \) (see e.g. [23, Theorem B.3]). \( \square \)

### 5.3. Numerical tests.

For the CEV case, we consider the pricing problem of the quantities introduced in (48). More precisely, we approximate \( \Pi_{D}^{1,0,X} \) (resp. \( \Pi_{t}^{1,0,X} \)) by the improved MLMC algorithm \( \tilde{Q}_{D}^{1-\alpha} \) given in (17) (resp. \( \tilde{P}_{t}^{1-\alpha} \) given in (16)), where we used our interpolated drift implicit scheme

\[Y_{t}^{n} = Y_{t_{i}}^{n} + (1 - \alpha) \left( \mu Y_{t_{i+1}}^{n} - \alpha \frac{\sigma^{2}}{2Y_{t_{i+1}}^{n}} \right) (t - t_{i}) + \gamma(W_{t} - W_{t_{i}}), \text{ for } t \in [t_{i}, t_{i+1}], 0 \leq i \leq n - 1,\]

\[Y_{0} = X_{0}^{1-\alpha},\]

with \( \gamma = \sigma(1 - \alpha) \). For \( n \) large enough, the positive solution to the above implicit scheme is explicit and given by

\[\frac{\sqrt{2\sigma^{2}\alpha(\alpha - 1)(1 + \mu(\alpha - 1)\frac{T}{n}) + \gamma(W_{t_{i+1}} - W_{t_{i}}) + Y_{t_{i}}^{n})^{2} + \gamma(W_{t_{i+1}} - W_{t_{i}}) + Y_{t_{i}}^{n}}}{2 + 2\mu(\alpha - 1)\frac{T}{n}}.\]

For the CEV model, we don’t have any benchmark price. To illustrate the MLMC complexity performance we choose a set of parameter (50), namely \( \alpha = 1.2, X_{0} = 100, \mu = 0.1, \sigma = 0.2, T = 1 \). The payoff function \( g(x) = e^{-rT} (x^{1/\alpha} - K)_{+} \) is a discounted call function with \( r = 0.1 \).
For the U-O option the strike is $K = 90$, and the barrier $D = 150$. For the D-O option the strike is $K = 100$ and the barrier $U = 90$. The tables and the figures below confirm the high performance of the improved MLMC.

| Accuracy | Price  | MLMC cost | MC cost    | Saving   |
|----------|--------|-----------|------------|----------|
| $10^{-4}$| 3.0390 | $8.226 \times 10^9$ | $7.34 \times 10^{13}$ | 8922.33  |
| $5 \times 10^{-4}$ | 3.0391 | $3.17 \times 10^8$ | $3.67 \times 10^{11}$ | 1155.67  |
| $10^{-3}$ | 3.041  | $7.436 \times 10^7$ | $4.587 \times 10^{10}$ | 616.91   |
| $10^{-2}$ | 3.0452 | $6.539 \times 10^5$ | $5.734 \times 10^7$   | 87.69    |

Table 3. MLMC complexity tests for the U-O barrier option pricing of $\Pi_{\mathcal{D},X}^{\text{U-O}}$

| Accuracy | Price  | MLMC cost | MC cost    | Saving   |
|----------|--------|-----------|------------|----------|
| $5 \times 10^{-4}$ | 11.102 | $6.483 \times 10^9$ | $1.642 \times 10^{13}$ | 2532.83  |
| $10^{-3}$  | 11.103 | $1.608 \times 10^9$ | $2.053 \times 10^{12}$ | 1276.66  |
| $5 \times 10^{-3}$ | 11.106 | $6.379 \times 10^7$ | $2.053 \times 10^{10}$ | 321.77   |
| $10^{-2}$  | 11.094 | $1.587 \times 10^7$ | $2.566 \times 10^9$   | 161.69   |

Table 4. MLMC complexity tests for the D-O barrier option pricing of $\Pi_{\mathcal{U},X}^{\text{D-O}}$

For pricing barrier options under the popular CEV model, the numerical results confirm the supremacy of the improved MLMC algorithm that reaches the optimal time complexity $O(\varepsilon^{-2})$ for a given precision $\varepsilon > 0$.

6. Conclusion

In this paper, we proved that the MLMC method for pricing barrier options reaches its optimal time-complexity regime, when the underlying asset has non-Lipschitz coefficients. To apply our theoretical results for the popular CIR and CEV processes, we developed semi-explicit formulas for the densities of the running minimum and running maximum of these processes that are of independent interest. It turns out that under some constraints on the parameters of these models guaranteeing the existence of finite negative moments up to some order, the MLMC method behaves like an unbiased classic Monte Carlo estimator despite the use of approximation schemes. It may
be interesting to extend this study by combining this improved version of the MLMC method with importance sampling technics for variance reduction as proposed in [4, 21, 5], we leave this for a possible future work.

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