CAFE: A NEW RELATIVISTIC MHD CODE

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ABSTRACT

We introduce CAFE, a new independent code designed to solve the equations of relativistic ideal magnetohydrodynamics (RMHD) in three dimensions. We present the standard tests for an RMHD code and for the relativistic hydrodynamics regime because we have not reported them before. The tests include the one-dimensional Riemann problems related to blast waves, head-on collisions of streams, and states with transverse velocities, with and without magnetic field, which is aligned or transverse, constant or discontinuous across the initial discontinuity. Among the two-dimensional (2D) and 3D tests without magnetic field, we include the 2D Riemann problem, a one-dimensional shock tube along a diagonal, the high-speed Emery wind tunnel, the Kelvin–Helmholtz (KH) instability, a set of jets, and a 3D spherical blast wave, whereas in the presence of a magnetic field we show the magnetic rotor, the cylindrical explosion, a case of Kelvin–Helmholtz instability, and a 3D magnetic field advection loop. The code uses high-resolution shock-capturing methods, and we present the error analysis for a combination that uses the Harten, Lax, van Leer, and Einfeldt (HLLE) flux formula combined with a linear, piecewise parabolic method and fifth-order weighted essentially nonoscillatory reconstructors. We use the flux-constrained transport and the divergence cleaning methods to control the divergence-free magnetic field constraint.

Key words: magnetohydrodynamics (MHD) – methods: numerical – relativistic processes

1. INTRODUCTION

Models of high-energy astrophysics are closely related to relativistic fluid dynamics because most of the sources are identified with the dynamics of a gas or plasma associated with a high-energy source. In the most complex cases, the models involve magnetic fields and cooling processes associated with various reactions taking place in the plasma. We know, moreover, that interesting sources of this sort also involve strong gravitational fields. In this sense, the most complete models that approach realistic scenarios within the field of high-energy astrophysics involve three main ingredients: relativistic hydrodynamics (RHD), magnetic fields (RMHD), and strong gravitational fields relativistic ideal magnetohydrodynamics (GRMHD), which are further complicated by the introduction of various cooling processes and realistic equations of state. In this way, the most modern relativistic models involve the solution of the coupled system of equations composed of the Einstein–Euler–Maxwell equations under very general conditions.

Given the complexity of this system of partial differential equations, these have been solved numerically for particular scenarios as a system of evolution equations, which requires the development of a code. Particular examples of high-energy phenomena with some of these ingredients are the propagation of jets on flat spacetimes, the accretion of tori around black holes, supernovae core-collapse processes, mergers of binary neutron stars, and so on (Font 2008; Abramowicz & Fragile 2013).

Astrophysical models of rapidly moving gas also involve a degree of idealization, in particular the gas equation of state, the conditions on magnetic fields, and sometimes the symmetries. The more powerful a code is, the fewer idealizations it assumes. Various codes have been presented that are distinguished in terms of what types of problems each one is able to solve. As examples, we mention some of the currently most used codes. The Cactus Einstein Toolkit, a multiusage package mounted on Cactus (Goode et al. 2003), is capable of solving the general relativistic MHD (Mösta et al. 2014). Whisky, a code that in its most sophisticated version can evolve general relativistic resistive magnetohydrodynamics (Dionysopoulou et al. 2013), also is mounted on Cactus. GENESIS is a code capable of solving the RMHD equations for relativistic and ultrarelativistic flows (Aloy et al. 1999; Leismann et al. 2005). HARM is a general relativistic code for a fixed spacetime (Gammie et al. 2012), and its latest version includes radiation terms (McKinney et al. 2014). HAD is capable of evolving binary compact stars in the presence of magnetic fields in general relativity (Palenzuela et al. 2013). There are also independent codes - dealing with general RHD, for instance the one in East et al. (2012). CoCoNuT evolves the general relativistic magnetohydrodynamics to simulate the core collapse of massive stars and the evolution of neutron stars (Cerdá-Durán et al. 2013). Specific-purpose codes also include Penner (2011), designed to evolve the accretion of magnetized winds onto black holes. The PLUTO code solves the RMHD equations (Mignone et al. 2012), as does that in Beckwith & Stone (2011). A certainly more complete list of codes designed for various purposes can be found for instance in Font (2008).

Even though the codes are extremely advanced, including coupling to general relativity and radiation cooling processes (e.g., Zanotti et al. 2011; Fragile et al. 2012), and knowing that new state-of-the-art numerical methods to handle the high-resolution shock-capturing (HRSC) schemes in relativistic and general RHD are now being studied and tested (e.g., Radice & Rezzolla 2012; Radice et al. 2014), it is our intention in this paper to present and certify our code CAFE, which in its first version focuses on the solution of RMHD equations, with the intent of extending it to general background spacetimes and applying it to the study of accretion processes on black holes.
Additionally, considering that we have not presented the tests of the RHD implementation before, we also include in this paper the tests in this regime.

The paper is organized as follows. In Section 2 we present the standard ideal RMHD equations that are solved. In Section 3 we show the tests for the zero magnetic field case, which reduces the system to the pure RHD regime and also shows how our implementation performs on the standard tests of the RMHD. In Section 4 we mention some final comments. Finally, in the Appendix we provide details of the implementation in cylindrical coordinates, as required in one of the tests.

2. SPECIAL RELATIVISTIC IDEAL MAGNETOHYDRODYNAMIC (SRMHD) EQUATIONS AND NUMERICAL METHODS

2.1. Ideal SRMHD Equations

SRMHD equations can be derived from the conservation of the rest mass, the local conservation of the stress-energy tensor $T^\mu{}^\nu$, and the Maxwell equations

$$\partial_\nu (\rho_0 u^\nu) = 0, \quad \partial_\nu (T^\mu{}^\nu) = 0, \quad \partial_\nu \left( \ast F^{\mu\nu} \right) = 0,$$  \hspace{1cm} (1)

where $\rho_0$ is the rest mass density of a fluid, $u^\mu$ is the four-velocity of the fluid elements, and $\ast F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\delta} F_{\lambda\delta}$ are the components of the components of the magnetic field measured by an Eulerian observer. Thus, the SRMHD Euler Equations (1) form a system of conservation laws, which can be written in cartesian coordinates as

$$\partial_0 f^0 + \partial_i f^i = 0, \quad \nabla \cdot B = \partial_i B^i = 0,$$  \hspace{1cm} (9)

where $f^0$ is the vector whose entries are the conservative variables, and the vector $f^i$ contains the fluxes along the spatial directions. All of these ingredients are explicitly

$$f^0 = [D, M_j, \tau, B^k],$$

$$f^i = [Dv^i, M_j v^i + p^* \delta_j^i - b_j B^k/W, \times \tau v^i + p^* v^i - b^0 B^k/W, v^k B^j - v^j B^k],$$

where $\delta_j^i$ is the Kronecker delta.

In these definitions, the components of the magnetic field measured by the comoving observer and an Eulerian observer are related as follows:

$$b^0 = WB^j v_j,$$

$$b^j = \frac{B^j}{W} + b^0 v^j,$$

where the magnitude of the magnetic field can be written as

$$b^2 = \frac{B^2 + (b^0)^2}{W^2},$$

with $B^2 = B^j B_j$. Finally, the RMHD system of Equations (8) and (9) is a set of eight equations for either the primitive variables $\rho_0, v_j, \epsilon, p, B^k$ or the conservative variables $D, S_j, \tau, B^k$. As usual, the system is closed with an equation of state relating $p = p(\rho_0, \epsilon)$. Specifically, we choose the fluid to obey an ideal gas equation of state $p = (\Gamma - 1)\rho_0 \epsilon$, where $\Gamma$ is the adiabatic index.

2.2. Numerical Methods

For a time-dependent PDE problem, a complete basic solver has several components: grid generation, initial conditions, boundary conditions, spatial discretization, and time integration. In our case, the relativistic magnetized Euler evolution equations are solved on a single uniform cell-centered grid. The integration in time uses the method of lines, with a third-order,
The total variation diminishing Runge–Kutta time integrator (RK3) (Shu & Osher 1988). The SRMHD equations are discretized using a finite volume approximation together with HRSC methods. Thus, the system of Equations (8) can be expressed in a semidiscrete form as follows:

$$\frac{d F_{i,j,k}}{dt} = -\frac{F_{i+1/2,j,k}^x - F_{i-1/2,j,k}^x}{\Delta x} - \frac{F_{i,j+1/2,k}^y - F_{i,j-1/2,k}^y}{\Delta y} - \frac{F_{i,j,k+1/2,z}^z - F_{i,j,k-1/2,z}^z}{\Delta z}, \quad (15)$$

where $F_{i,\pm1/2,j,k}^x$, $F_{i,j,\pm1/2,k}^y$, and $F_{i,j,k,\pm1/2,z}$ are the numerical fluxes at the respective cell interfaces. The right-hand side of this expression is what we will call the right-hand side of the conservative variables from now on.

CAFE provides different types of spatial reconstruction schemes, which are applied on the primitive variables $\{\rho, v^i, p, B^k\}$. Specifically, we use the MINMOD and MC linear piecewise reconstrucors, which are second-order methods. For higher-order reconstructions, we use the third-order piecewise parabolic method (PPM), which was developed by Colella & Woodward (1984) and adapted to the relativistic case by Martí and Müller (1996). We use this recipe with the parameters $K_0 = 1.0$, $\eta^1 = 5.0$, $\eta^2 = 0.5$, $\epsilon^1 = 0.1$, $\epsilon^2 = 1.0$, $\omega^{(1)} = 0.52$, and $\omega^{(1)} = 10.0$. In regions near contact discontinuities, the interpolation is modified in such a way that in the vicinity of local extrema, the scheme reduces to a piecewise constant approximation in order to avoid shock oscillations. We also use the fifth-order weighted essentially nonoscillatory reconstrucors (WENO5), which approaches the

### Table 1

| Test Type | $\Gamma$ | $\rho_0$ | $p$ | $v^1$ | $v^2$ |
|-----------|----------|---------|-----|-------|-------|
| Test 1    | 5/3      | 10.0    | 13.33 | 0.0   | 0.0   |
| Right state | 1.0 | $10^{-8}$ | 0.0 | 0.0 | 0.0 |
| Test 2    | 5/3      | 1.0     | 1000.0 | 0.0 | 0.0 |
| Right state | 1.0 | 0.01 | 0.0 | 0.0 | 0.0 |
| Test 3    | 4/3      | 1.0     | 0.001 | 0.99999995 | 0.0 | 0.0 |
| Right state | 1.0 | 0.001 | -0.99999995 | 0.0 | 0.0 |
| Test 4    | 4/3      | 1.0     | 1.0 | 0.9 | 0.0 |
| Right state | 1.0 | 10.0 | 0.0 | 0.0 | 0.0 |
| Test 5    | 5/3      | 1.0     | 1000.0 | 0.0 | 0.0 |
| Right state | 1.0 | 0.01 | 0.0 | 0.99 | 0.0 |
| Test 6    | 5/3      | 1.0     | 1000.0 | 0.0 | 0.9 |
| Right state | 1.0 | 0.01 | 0.0 | 0.9 | 0.0 |

Figure 1. Numerical cell centered at $(i, j, k)$ containing the detailed labels of the corners required in the calculations.

Figure 2. Numerical cell centered at $(i, j, k)$ and two adjacent faces. We show an example to illustrate the cell corner interpolation of the numerical fluxes using the four values computed using the HLLE flux formula. We can see that the fluxes $F_{i,j,k}^x$ and $F_{i,j,k}^z$ are computed along the $x$ and $z$ directions at points $(i + \frac{1}{2}, j, k)$, $(i + \frac{1}{2}, j, k)$, and $(i + \frac{1}{2}, j, k + 1)$. We use these values to compute $\Omega_{(i+1/2,j,k+1/2)}$ using the formula given by Equation (24).

Figure 3. Test 1: mildly relativistic blast wave explosion problem at $t = 0.4$. We show the proper rest mass density, pressure, and velocity.
variables with a high order of accuracy using polynomial interpolation (Harten et al. 1997; Titarev & Toro 2004) and is efficient at capturing the structure of turbulent fluxes (Radice & Rezzolla 2012). Our framework is such that other schemes can be incorporated easily.

2.2.1. Approximate Riemann Solver

In order to solve the system of Equation (15), we have implemented the Harten, Lax, van Leer, and Einfeldt (HLLE; Harten et al. 1983; Einfeldt 1988) approximate Riemann solver formula

\[
F^i = \frac{\xi^+ f_L^i - \xi^- f_R^i + \xi^+ \xi^- (f_R^0 - f_L^0)}{\xi^+ - \xi^-},
\]

where \(f_L^i\) and \(f_R^i\) are the fluxes, and \(f_L^0\) and \(f_R^0\) are the conservative variables at the left and right sides of the interface between cells; \(\xi^+ = \max(0, \xi_R^p, \xi_L^p)\), \(\xi^- = \min(0, \xi_R^p, \xi_L^p)\), and \(\xi_R^p, \xi_L^p\) denote the \(p\) eigenvalue of the Jacobian matrix for the left and right states, respectively.

Different approximate Riemann solvers require different characteristic information from the Jacobian matrix.
For instance, the Marquina (1994) and Roe (1981) approximate Riemann solvers require the eigenvalues and eigenvectors. One of the appealing properties of the HLLE approximate Riemann solver is that it requires only the eigenvalues of the Jacobian matrix. Specifically, the HLLE solver uses a two-wave approximation to compute the fluxes across the discontinuity at the cell interfaces. One disadvantage of this flux formula is that it does not properly resolve the contact discontinuity, and it is more dissipative than other methods like HLLC that actually do solve the contact discontinuity (Toro et al. 1994; Mignone & Bodo 2005, 2006). However, less dissipative formulas may produce undesirable effects, such as the carbuncle artifact along the axis of propagation of strong shocks (Wang et al. 2008).

Furthermore, in our HRSC scheme that solves the relativistic magnetized Euler Equation (8), in order to calculate the eigenvalue structure, we follow the formalism introduced in Anile (1989) and further developed in Antón et al. (2010, 2006). The expressions of the seven eigenvalues associated with entropic, Alfvén fast and slow magnetosonic waves are the following:

$$\xi = \nu^i,$$

$$\xi_{\pm} = \frac{b^i}{b^0} \pm \sqrt{E} W \nu^i,$$  

(17)  

(18)

where $E = \rho_0 h + b^2$ is the total energy density measured by an observer comoving with the fluid. The fast and slow magnetosonic waves, which are required in the computation of the numerical fluxes, can be obtained by solving the following quartic equation in each direction $i$ for the unknown $\xi$:

$$\rho_0 h \left( \frac{1}{c_1^2} - 1 \right) a^4 - \left( \rho_0 h + \frac{b^2}{c_1^2} \right) a^2 \mathcal{G} + \mathcal{H}^2 \mathcal{G} = 0,$$  

(19)

where

$$a = W \left( -\xi + \nu^i \right), \quad \mathcal{H} = b^i - b^0 \xi,$$

$$\mathcal{G} = -\xi^2 + 1,$$

and $c_1$ is the sound speed.

In order to solve this equation, we use an approximate method for the calculation of the eigenvalues related to the fast magnetosonic waves. The method was introduced by Leisman et al. (2005) and basically consists of reducing the original quartic Equation (19) to a quadratic equation, which can be solved analytically. Finally, because of the structure of the HLLE formula, which uses the upper and lower bounds of the eigenvalues, it is not necessary to incorporate the slow magnetosonic waves.

2.2.2. The Divergence-free Magnetic Field Constraint

Although the analytic solutions of Maxwell’s equations guarantee the constraint (9), experience shows that it is not the case when calculating numerical solutions of these equations. Instead, the numerical evolution of the initial data involving Maxwell equations may eventually lead to a violation of the divergence-free constraint (9), causing the development of unphysical results, like the presence of magnetic monopolar sources.

a. Flux Constraint Transport

There are various methods available to control the growth of the constraint violation (e.g., Tóth 2000); in our code we use a version of the constrained transport method “CT” originally proposed in Evans & Hawley (1988), which is based on the use of the fluxes computed with a conservative scheme. This algorithm is known as flux-CT and was proposed in Balsara (2001). Following Giacomazzo & Rezzolla (2007), the resulting discretized cell-faced evolution equations for the magnetic field components are given in terms of $\Omega \equiv v \times B$ as

$$\frac{dB^i_{(i+\hat{\xi}+\hat{\nu} \hat{k})}}{dt} = \frac{\Omega^\nu_{(i+\hat{\xi}+\hat{\nu} \hat{k})} - \Omega^\nu_{(i+\hat{\nu} \hat{k}-\hat{\xi})}}{\Delta \nu} \frac{\Delta \xi}{\Delta \nu}$$

$$- \frac{\Omega^\nu_{(i+\hat{\xi}+\hat{\nu} \hat{k})} - \Omega^\nu_{(i+\hat{\nu} \hat{k}-\hat{\xi})}}{\Delta \nu} \frac{\Delta \xi}{\Delta \nu}.$$

(20)
Table 2

$L_1$ Norm of the Error in Density for Four Schemes with Different Numerical Reconstructors

| Resolution | Test 1 | Test 2 | Test 3 | Test 4 | Test 5 | Test 6 |
|------------|--------|--------|--------|--------|--------|--------|
|            | MM     | MC     | WENO5  | PPM    | MM     | MC     | WENO5  | PPM    |
| Error      |        |        |        |        |        |        |        |        |
| $\Delta t_1$ | 1.33e-1 | 8.09e-2 | 8.05e-2 | 3.28e-1 | ...    | ...    | ...    | ...    |
| $\Delta t_1$ | 7.39e-2 | 4.22e-2 | 4.25e-2 | 1.98e-1 | 0.85    | 0.94    | 0.92    | 0.73    |
| $\Delta t_1$ | 3.71e-2 | 2.25e-2 | 2.28e-2 | 1.22e-1 | 0.99    | 0.90    | 0.90    | 0.70    |
| $\Delta t_1$ | 2.12e-2 | 1.19e-2 | 1.44e-2 | 7.49e-2 | 0.81    | 0.92    | 0.67    | 0.70    |
| $\Delta t_1$ | 1.17e-2 | 7.01e-3 | 9.02e-3 | 4.45e-2 | 0.86    | 0.76    | 0.68    | 0.75    |
| Order of Convergence |        |        |        |        |        |        |        |        |

Note. The $L_1$ norm is computed at $t = 0.4$, except for test 6, which is presented at $t = 0.6$. We also show the order of convergence between the different pairs of resolutions. The results achieve convergence as expected for problems with sharp discontinuities.

\[
\frac{dB_y}{dt} = \frac{\Omega_t^{(i,j,k+\frac{1}{2})} - \Omega_t^{(i,j,k-\frac{1}{2})}}{\Delta t} - \frac{\Omega_t^{(i+\frac{1}{2},j,k+\frac{1}{2})} - \Omega_t^{(i+\frac{1}{2},j,k-\frac{1}{2})}}{\Delta x}, \quad (21)
\]

\[
\frac{dB_z}{dt} = \frac{\Omega_t^{(i,j,k+\frac{1}{2})} - \Omega_t^{(i,j,k-\frac{1}{2})}}{\Delta t} - \frac{\Omega_t^{(i+\frac{1}{2},j,k+\frac{1}{2})} - \Omega_t^{(i+\frac{1}{2},j,k-\frac{1}{2})}}{\Delta y}, \quad (22)
\]
where \( B_{x(i+1/2,j,k)} \), \( B_{y(i+1/2,j,k)} \), and \( B_{z(i,j,k+1/2)} \) are the magnetic field components defined as the average on each surface \( A_1 \), \( A_2 \), and \( A_3 \), respectively; see Figure 1. Moreover, the \( \Omega \) terms in the right-hand side of Equations (20)–(22) are defined at the cell vertex. These terms are computed using the simple average of the neighboring values of the numerical fluxes at the intercells as follows:

\[
\Omega^x_{(i+1/2,j,k+1/2)} = \frac{1}{4} \left( F^{\Sigma}_{(i,j,k+1/2)} + F^{\Sigma}_{(i+1,j,k+1/2)} - F^{\Sigma}_{(i,j,k+1/2)} - F^{\Sigma}_{(i+1,j,k+1/2)} \right),
\]

\[ (23) \]

\[
\Omega^y_{(i+1/2,j,k+1/2)} = \frac{1}{4} \left( F^{\Sigma}_{(i,j,k+1/2)} + F^{\Sigma}_{(i+1,j,k+1/2)} - F^{\Sigma}_{(i,j,k+1/2)} - F^{\Sigma}_{(i+1,j,k+1/2)} \right),
\]

\[ (24) \]

\[
\Omega^z_{(i+1/2,j,k+1/2)} = \frac{1}{4} \left( F^{\Sigma}_{(i,j,k+1/2)} + F^{\Sigma}_{(i,j,k+1)} - F^{\Sigma}_{(i+1,j,k+1/2)} - F^{\Sigma}_{(i+1,j,k+1)} \right),
\]

\[ (25) \]

Table 3
Order of Convergence of the \( L_1 \) Norm of the Error in the Density for Four Schemes with Different Numerical Reconstructors

| Cells | MM | MC | WENO5 | PPM |
|-------|----|----|-------|-----|
| SMOOTH PROFILE TEST |
| 80    | ... | ... | ...   | ... |
| 160   | 2.11 | 2.57 | 2.27  | 1.90 |
| 320   | 2.09 | 2.37 | 2.30  | 1.86 |
| 640   | 2.04 | 2.18 | 2.48  | 1.91 |
| 1280  | 2.00 | 2.10 | 2.60  | 1.95 |

Note. The \( L_1 \) norm is computed at \( t = 0.8 \) for a test with smooth initial data. The left column indicates the number of cells used to cover the domain \([-0.35, 1]\) along the \( x \) direction.

Figure 9. Initial profile and a snapshot at \( t = 0.8 \) of the density for the smooth profile. The continuous line corresponds to the initial profile, the dashed line to the exact solution at \( t = 0.8 \), and the dots indicate the numerical solutions calculated using the MC reconstructor.

Figure 10. (Top) A 2D snapshot at \( t = 0.4 \) of the rest mass density for test 1 with the shock propagating along a diagonal direction. (Bottom) Comparison with the exact solution as seen along the diagonal perpendicular to that of the propagation.
Finally, the discretized magnetic field divergence, calculated at the corner of the grid, is given by

\[ \nabla \cdot \mathbf{B}(i + 1/2, j + 1/2, k + 1/2) \]
\[ = \frac{1}{4\Delta x} \sum_{j=k}^{j+1} \sum_{i=k}^{i+1} (B^z(i+1, jj, kk) - B^z(i, jj, kk)) \]
\[ + \frac{1}{4\Delta y} \sum_{k=i}^{k+1} \sum_{ii=k}^{ii+1} (B^y(ii, j+1, kk) - B^y(ii, j, kk)) \]
\[ + \frac{1}{4\Delta z} \sum_{jj=i}^{jj+1} \sum_{ii=j}^{ii+1} (B^x(ii, jj, k+1) - B^x(ii, jj, k)), \]

which is the discretized expression with which the constraint has to be monitored.

b. Hyperbolic Divergence Cleaning

We also implemented the divergence cleaning method, which preserves the magnetic field constraint by solving a modified version of the Maxwell equations

\[ \partial_t (\gamma \mathbf{F}^\mu + \eta^\mu \psi) = \kappa n^\mu \psi, \]

where a new field variable \( \psi \) and a diffusive term \( \kappa n^\mu \psi \) are added. The parameter \( \kappa \) may be adjusted in order to absorb errors in the constraint, and \( \psi \) vanishes when \( \nabla \cdot \mathbf{B} = 0 \) is exactly satisfied. Here, \( n^\mu \) is the normal vector to the hypersurface. The hyperbolic divergence cleaning method used was proposed by Dedner et al. (2002). To see how explicitly Maxwell equations are modified, we refer the reader to Liebling et al. (2010) and Penner (2011). The field variable \( \psi \) satisfies the damped wave equation

\[ \partial_t \partial_t \psi = -\kappa \partial_n (n^\mu \psi), \]

which means that the amplitude of \( \psi \) decreases in time during the evolution, recovering the unmodified Maxwell equations. Thus, the modified Maxwell equations in special relativity are written as

\[ \partial_t \psi + \partial_t \left( \nu^l \nu^l \psi - \nu^l B^l \right) = 0, \]  

and additionally, we have an evolution equation for the new variable \( \psi \) given by

\[ \partial_t \psi + \partial_l B^l = -\kappa \psi. \]

which is incorporated into the set of evolution equations.

2.2.3. Recovery of Primitive Variables

The code evolves the conservative \{ \( \mathbf{D} \), \( \mathbf{S}_\mu \), \( \tau \), \( \mathbf{B}_k \) \}, but not the primitive variables \{ \( \rho \), \( \nu^l \), \( p \), \( \mathbf{B}_k \) \}. However, the numerical fluxes depend on both sets of variables. Therefore, after each time step within the evolution scheme, one needs to recover the primitive variables out of the conservative ones. By definition, the conservative quantities can be written in terms of the primitives, but a solution to the inverse problem is not known, and a numerical algorithm is required.
The method we use is based on Mignone & Bodo (2006) and is as follows. As a starting point, the definitions of \( M', \tau \), and an auxiliary variable \( r = Z \frac{W^2}{2} \) are used. Then, it is necessary to compute \( M^2 = M_i \) and \( \tau \) in terms only of \( Z, W, \) and \( B \) as follows:

\[
M^2 = (Z + B^2)^2 \left(1 - W^{-2} \right) - \left( \frac{B \cdot M}{Z} \right)^2 \left(2Z + B^2 \right), \tag{28}
\]

\[
\tau = Z + B^2 - p - D - \frac{B^2}{2W^2} - \frac{1}{2} \left( \frac{B \cdot M}{Z} \right)^2, \tag{29}
\]

where \( B^2 = B_iB^i \) and \( p \) can be expressed in terms of \( Z \) and \( W \):

\[
p = \frac{\Gamma - 1}{\Gamma} \frac{Z - DW}{W^2}. \tag{30}
\]

From Equation (28) it is possible to express the Lorentz factor in terms of \( Z, B^2 \), and \( B \cdot M \):

\[
W = \frac{1}{\sqrt{1 - \frac{(B \cdot M)^2 \left(2Z + B^2 \right) + M^2Z^2}{\left(Z + B^2 \right)^2Z^2}}}, \tag{31}
\]

and then substituting this into Equation (29) we obtain

\[
f(Z) = Z + B^2 - (\tau + D) - p - \frac{B^2}{2W^2} - \frac{1}{2} \left( \frac{B \cdot M}{Z} \right)^2 = 0. \tag{32}
\]

In order to solve this algebraic equation, we use a numerical iterative algorithm, which is a combination of the Newton–Raphson and bisection methods (Press et al. 1992).
The Newton–Raphson method requires the derivative \( df(Z)/dZ \):

\[
\frac{df(Z)}{dZ} = 1 - \frac{dp}{dZ} + \frac{B^2}{W^3} \frac{dW}{dZ} + \frac{(B \cdot M)^2}{Z^3},
\]

(33)

where

\[
\frac{dp}{dZ} = \frac{\Gamma - 1}{\Gamma} W (1 + DdW/dZ) - 2ZdW/dZ,
\]

\[
\frac{dW}{dZ} = -W^3 \frac{M^2 Z^3 + (B^4 + 3B^2 Z + 3Z^2)(B \cdot M)^2}{Z^3 (B^2 + Z)^3}.
\]

Finally, after calculating \( Z \) it is possible to find the other primitive variables as follows. Once \( Z \) is known, \( W \) can be recovered from Equation (31) and then the pressure from Equation (30), \( \rho \) from the definition of \( D \), and the velocity components from the expression

\[
\nu' = \frac{S^2 + (B \cdot M)B'/Z}{Z + B^2},
\]

(34)

which is obtained from the definition of \( M_c \).

3. NUMERICAL TESTS

The first set of tests involve the evolution with the magnetic field switched off, which is the domain of the RHD, considered to be important because we have not shown prior evidence of the ability of our code to handle this system. A second set of tests involves nontrivial magnetic fields and includes a complete set of RMHD tests.

3.1. RHD Tests

In order to illustrate how our implementation handles the evolution of a relativistic gas, in this subsection we present the standard tests showing that our code works properly. The one-
dimensional (1D) tests are Riemann problems under various conditions, and we compare the numerical results with the exact solution we implemented based on Marti & Muller (1994), Marti & Muller (2003), and Lora-Clavijo et al. (2013).

We calculate the numerical solution of these tests using various limiters. However, unless otherwise specified, all of the results in the 1D test figures corresponding to Riemann problems use the HLLE formula and the MC limiter, and the problem is solved in the domain \([-0.5, 0.5]\) with \(N = 400\) identical cells, a Courant factor of \(CFL = 0.25\), and with the initial discontinuity located at \(x = 0\). The resolutions we have used for the error estimates are \(D_1 = x_1/2001\), \(D_2 = x_1/4002\), \(D_3 = x_1/8003\), \(D_4 = x_1/16004\), \(D_5 = x_1/32005\), and \(D_6 = x_1/64006\). For these tests we are using the 3D code with five cells along the transverse directions. The various parameters of the 1D tests are summarized in Table 1.

3.1.1. Test 1: Relativistic Blast Wave (a)

The first 1D Riemann problem test corresponds to a mildly relativistic blast wave explosion, characterized by an initial static state with higher pressure in the region on the left. The results can be seen in Figure 3, where we compare the numerical solution (points) with the exact solution (lines). The comparison between the exact and numerical solutions is as good as that obtained by other codes (e.g., Del Zanna & Bucciantini 2002; Martí & Müller 2003; Zhang & MacFadyen 2006). In this test the most important feature is that with a relatively small number of cells the shock speed is pretty much the exact one.

3.1.2. Test 2: Relativistic Blast Wave (b)

In this problem, unlike the previous one, the evolution of the initial discontinuity produces a sharper blast moving to the right. The standard initial data are those in Marti & Muller (2003). In Figure 4 we show our results and contrast them with the exact solution. Because of the important difference of pressure between the left and right states, behind the shock there is an extremely thin dense shell, which is a feature expected to be controlled by a code. The fact that the thin shell is not well resolved is a matter of resolution, and in this particular case \(\Delta x_3\) is not enough. However, with \(\Delta x_2\) the shell is well resolved and within the convergence regime.

3.1.3. Test 3: Head-on Stream Collision

In this case, the initial velocity in the two chambers is high and with opposite directions, and consequently two strong shocks form and propagate to the left and right, decelerating the gas to a very low speed. The evolution produces Lorentz factors of the order of 10,000, which tests the capability of an implementation to control extremely high fluid speeds. The numerical results compared with the exact solution are shown in Figure 5. Because of the strength of the shocks, unphysical oscillations may appear behind them, and in order to avoid these oscillations we use the MINMOD reconstructor, which is more dissipative than the MC.

3.1.4. Test 4: Strong Reverse Shock

In this problem, a strong reverse shock forms in which postshock oscillations are visible for the numerical methods used in our simulations. Specifically, these oscillations are more evident in the pressure and density. The numerical results compared with the exact solution are shown in Figure 6. Under these extreme conditions, none of the reconstructors...

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**Figure 15.** Morphology in logarithmic scale of rest mass density for the supersonic jet in Del Zanna & Bucciantini (2002) at time \(t = 40\) using MC. The injection is made with Lorentz factor 7 and relativistic Mach number \(\sim 18\).

**Figure 16.** Logarithm of the rest mass density for the hot jet model A1 in Martí et al. (1997) at time \(t = 48.82\). In this case we use the MINMOD (top), MC (middle), and WENO5 (bottom) reconstructors in the domain \([0, 15] \times [0, 60]\) covered with 576 \(\times\) 1920 cells.
used here is capable of diminishing the oscillations, but the amplitude of the oscillations converges to zero with resolution.

3.1.5. Test 5: Nonzero Transverse Velocity: Easy Test

Many problems of interest in hydrodynamics involve strong shear flows. For example, astrophysical jets include layers of ambient material shearing into the fast jet flow. It is therefore important to test the ability of numerical codes to handle Riemann problems with velocity components transverse to the direction of propagation of the main flow. In this first case, the problem is relatively easy because the transverse velocity is in the cold gas of the right state, not in the relativistically hot left state or in the rarefaction fan that subsequently propagates into it. The numerical results compared with the exact solution are shown in Figure 7, and there is no major difficulty in resolving the shock using resolution $\Delta x_1$.

3.1.6. Test 6: Nonzero Transverse Velocity: Hard Test

This is a very severe test requiring very high resolution to resolve the complicated structure of the transverse velocity. This test is particularly hard because the transverse velocity is high not only where the gas is cold but also in the hot region. The numerical results compared with the exact solution are shown in Figure 8, and the initial set of parameters can be seen in Table 1.

3.1.7. Error Estimates for the 1D RHD Tests

In order to systematically compare the numerical solutions under different combinations of limiters implemented in our code, we have calculated the error for each 1D Riemann test, and the results are summarized in Table 2. In all of the numerical solutions presented, we found consistency: the error decreases when resolution is increased. For this we calculate the $L_1$ norm of the error of these tests as compared with the exact solution. Because we always consider the resolution factors of two, the order of convergence is given by $\log(\text{error}_i/\text{error}_{i-1})/\log(2)$, where $i$ is the error calculated with the resolution $\Delta x_i$. We carried out the tests with five resolutions, and for the methods used in our code, involving the use of only the RK3 time integrator, a first-order convergence is expected because the problems start with discontinuous initial data. For tests 1, 2, 3, 4, and 5 we spotted the desirable nearly first-order convergence regime for at least a combination of the resolution and reconstructor used. The strong test 6, on the other hand, needs more resolution than the previous ones, but we nearly approach first-order convergence for the highest resolutions.

It is important to mention that with error estimates, we also locate the resolutions required to safely work on a convergence regime of resolution. In fact, these error estimates could be important in Adaptive Mesh Refinement (AMR) codes, in which the resolution in a certain region may be prescribed by the strength and type of local Riemann problems contained in the tests presented here.

3.1.8. Smooth Initial Profile

In order to know how the code performs in evolving initial smooth profiles, following Zhang & MacFadyen (2006) and Radice & Rezzolla (2012), we set the smooth initial density profile immersed on a reference ambient constant density.
Specifically, the density profile is

\[
\rho(x) = \begin{cases} 
1 + \exp\left[-1/(1 - x^2/L^2)\right], & \text{if } |x| < L \\
1, & \text{otherwise}
\end{cases}
\]

The fluid obeys the isentropic relation \( p = K \rho^{\frac{\Gamma}{\Gamma - 1}} \), whereas the initial velocity field is subject to the condition that the invariant

\[
J_\perp = \frac{1}{2} \ln \left( \frac{1 + v}{1 - v} \right) - \frac{1}{\sqrt{\Gamma - 1}} \ln \left( \frac{\sqrt{\Gamma - 1} - c_s}{\sqrt{\Gamma - 1} + c_s} \right)
\]

has to be constant in the whole domain. The construction of this invariant assumes that the velocity in the ambient region is set to zero. The parameters used for the test are \( L = 0.3 \), \( \Gamma = 5/3 \), and \( K = 100 \), and the domain along the x direction \([-0.35, 1]\) is covered with a number of cells. In Figure 9 we show the initial profile and a snapshot of the numerical solution using the MC reconstructor at \( t = 0.8 \), as shown in Zhang & MacFadyen (2006) and Radice & Rezzolla (2012). A convergence test was performed comparing the numerical with the exact solution for various resolutions and reconstructors. The order of convergence of the different combinations are collected in Table 3.

As expected, the various reconstructors approach the second-order convergences, unlike the first-order convergence achieved for the tests with initial shocks. The convergence found is comparable with that used in previous studies (e.g., Zhang & MacFadyen 2006) using RK3.
Multidimensional relativistic simulations are more difficult to carry out than the 1D ones because the components of the velocity, which are spatially interpolated separately, eventually may cause the velocity to be greater than \( v^2 > 1 \), especially in the ultrarelativistic regime, due to numerical errors in the reconstruction. For this reason, in some cases it is necessary to use more dissipative methods and in some regions low-order reconstructors. The first two-dimensional (2D) test consists of the evolution of a 1D shock-tube problem along a diagonal of a plane.

In order to check that the code is able to handle the fluxes along two different directions simultaneously, we implement the initial data of the 1D test 1, but with the initial shock propagating along the diagonal \( \hat{x} + \hat{y} \) direction. The initial data are set on a 2D domain \([-0.5, 0.5] \times [-0.5, 0.5]\) that is covered with 400 \( \times \) 400 cells. In Figure 10 we show the 2D profile as seen from the \( z \) axis and a snapshot of the density for the solution at \( t = 0.4 \) compared with the exact solution. In this example we use the MC reconstructor.

### 3.1.10. 2D Riemann Problem

The relativistic 2D Riemann problem basically involves the interaction of shock, rarefaction, and contact waves initially separated by four quadrants of constant values at initial time. In the context of classical hydrodynamics, this problem was formulated in Lax & Liu (1998) and its extension to the relativistic case in Del Zanna & Buciantini (2002), where the initial condition involves two shocks and two tangential discontinuities. In this simulation, we use the HLLE flux formula and the MINMOD limiter. The problem is defined in the domain \([-0.5, 0.5] \times [-0.5, 0.5]\), which is covered with 400 \( \times \) 400 cells. The integration was carried out with Courant factor \( CFL = 0.25 \), and we imposed outflow boundary conditions. Specifically, the initial state has the following parameters:

\[
\begin{pmatrix}
\rho, p, v_x, v_y
\end{pmatrix} =
\begin{cases}
(0.1, 1.0, 0.99, 0)^{\text{TL}} \\
(0.1, 0.01, 0, 0)^{\text{TR}} \\
(0.5, 1.0, 0, 0)^{\text{BL}} \\
(0.1, 1.0, 0, 0.99)^{\text{BR}}
\end{cases}
\]

where the labels correspond to the top left (TL), top right (TR), bottom left (BL), and bottom right (BR) quadrants of the \( xy \) plane. In Figure 11 we show the logarithm of the proper rest mass density and the pressure at \( t = 0.4 \).

The morphology shows various features, including a bow shock and a small jet moving diagonally in the initially high density region (BL quadrant), and in the opposite direction the fluid moves toward the region of lower density in a filamentary form.

### 3.1.11. Relativistic Emery’s Wind Tunnel

This is a test proposed for classical hydrodynamics in Emery (1968) and Woodward & Colella (1984) that has been
Figure 20. Test 1: Komissarov shock tube at $t = 1.0$. We use 3200 cells in the domain $[-2, 2]$ and CFL = 0.1.
Figure 21. Test 2: Komissarov collision test at $t = 1.22$. We cover the domain $[-2, 2]$ with 3200 cells and use CFL = 0.1.
Figure 22. Test 3: Balsara 1 test at time $t = 0.4$. We use $\Delta x = 1/1600$ in a domain $x \in [-0.5: 0.5]$ and CFL = 0.1.
Figure 23. Test 4: Balsara 2 blast wave at time $t = 0.4$. We use 1600 cells in the numerical domain $[-0.5, 0.5]$ and use CFL = 0.1.
Figure 24. Test 5: Balsara 3 blast wave along the x direction at time $t = 0.4$. We cover the numerical domain $[-0.5, 0.5]$ with 1600 cells and CFL = 0.1.
Table 5

$L_1$ Norm of the Error in Density for Each Reconstructor and the Flux-CT Method to Control the Magnetic Field Divergence-free Constraint

| Resolution | MM     | MC     | WENO5  | PPM | MM     | MC     | WENO5  | PPM |
|------------|--------|--------|--------|-----|--------|--------|--------|-----|
|            | Flux-CT Error | Order of Convergence |
| Test 1     |        |        |        |     |        |        |        |     |
| $\Delta t_1$ | 1.18e-1 | 1.00e-1 | 1.14e-1 | 1.18e-1 | ... | ... | ... | ... |
| $\Delta t_2$ | 8.09e-2 | 6.97e-2 | 7.14e-2 | 1.02e-1 | 0.54 | 0.65 | 0.67 | 0.21 |
| $\Delta t_3$ | 5.20e-2 | 3.96e-2 | 3.68e-2 | 8.21e-2 | 0.63 | 0.81 | 0.95 | 0.31 |
| $\Delta t_4$ | 3.05e-2 | 2.15e-2 | 1.95e-2 | 6.39e-2 | 0.77 | 0.88 | 0.91 | 0.36 |
| $\Delta t_5$ | 1.65e-2 | 1.12e-3 | 9.99e-3 | 4.95e-2 | 0.88 | 0.94 | 0.96 | 0.37 |
| Test 2     |        |        |        |     |        |        |        |     |
| $\Delta t_1$ | 3.39e-1 | 3.16e-1 | 3.15e-1 | 3.45e-1 | ... | ... | ... | ... |
| $\Delta t_2$ | 2.25e-1 | 2.50e-1 | 2.30e-1 | 2.35e-1 | 0.59 | 0.62 | 0.65 | 0.55 |
| $\Delta t_3$ | 1.47e-1 | 1.57e-1 | 1.42e-2 | 1.53e-1 | 0.63 | 0.67 | 0.69 | 0.62 |
| $\Delta t_4$ | 9.41e-2 | 9.80e-2 | 8.60e-2 | 9.87e-2 | 0.64 | 0.68 | 0.72 | 0.63 |
| $\Delta t_5$ | 5.91e-2 | 6.04e-2 | 4.97e-2 | 6.24e-2 | 0.67 | 0.69 | 0.79 | 0.66 |
| Test 3     |        |        |        |     |        |        |        |     |
| $\Delta t_1$ | 2.10e-2 | 1.56e-2 | 1.59e-2 | 2.09e-2 | ... | ... | ... | ... |
| $\Delta t_2$ | 1.33e-2 | 8.86e-3 | 9.36e-3 | 1.33e-2 | 0.65 | 0.81 | 0.76 | 0.65 |
| $\Delta t_3$ | 8.04e-3 | 4.76e-3 | 5.05e-3 | 8.00e-3 | 0.72 | 0.90 | 0.89 | 0.73 |
| $\Delta t_4$ | 4.83e-3 | 2.48e-3 | 2.68e-3 | 4.83e-3 | 0.73 | 0.89 | 0.91 | 0.73 |
| $\Delta t_5$ | 2.89e-3 | 1.39e-3 | 1.40e-3 | 2.89e-3 | 0.74 | 0.89 | 0.93 | 0.74 |
| Test 4     |        |        |        |     |        |        |        |     |
| $\Delta t_1$ | 1.65e-1 | 1.51e-1 | 1.49e-1 | 1.60e-1 | ... | ... | ... | ... |
| $\Delta t_2$ | 1.25e-1 | 1.14e-1 | 1.12e-1 | 1.22e-1 | 0.40 | 0.40 | 0.41 | 0.39 |
| $\Delta t_3$ | 7.98e-2 | 7.11e-2 | 6.76e-1 | 7.86e-1 | 0.64 | 0.68 | 0.72 | 0.63 |
| $\Delta t_4$ | 5.05e-2 | 4.37e-2 | 3.98e-2 | 5.00e-2 | 0.66 | 0.70 | 0.76 | 0.65 |
| $\Delta t_5$ | 3.15e-2 | 3.57e-2 | 2.32e-2 | 3.13e-2 | 0.68 | 0.76 | 0.83 | 0.67 |
| Test 5     |        |        |        |     |        |        |        |     |
| $\Delta t_1$ | 2.26e-1 | 2.24e-1 | 2.24e-1 | 2.94e-1 | ... | ... | ... | ... |
| $\Delta t_2$ | 1.84e-1 | 1.58e-1 | 1.48e-1 | 2.28e-1 | 0.47 | 0.50 | 0.59 | 0.37 |
| $\Delta t_3$ | 1.28e-1 | 1.04e-1 | 9.39e-2 | 1.63e-1 | 0.52 | 0.60 | 0.65 | 0.44 |
| $\Delta t_4$ | 8.70e-2 | 6.62e-2 | 5.74e-2 | 1.12e-1 | 0.55 | 0.65 | 0.71 | 0.54 |
| $\Delta t_5$ | 5.41e-2 | 3.96e-2 | 3.36e-2 | 6.80e-2 | 0.68 | 0.74 | 0.77 | 0.71 |
| Test 6     |        |        |        |     |        |        |        |     |
| $\Delta t_1$ | 2.31e0  | 2.15e0  | 2.24e0  | 2.31e0  | ... | ... | ... | ... |
| $\Delta t_2$ | 1.56e0  | 1.41e0  | 1.46e0  | 1.58e0  | 0.56 | 0.60 | 0.61 | 0.54 |
| $\Delta t_3$ | 1.05e0  | 8.89e-1 | 9.08e-1 | 1.07e0  | 0.57 | 0.66 | 0.68 | 0.56 |
| $\Delta t_4$ | 6.68e-1 | 5.33e-1 | 5.30e-1 | 6.89e-1 | 0.65 | 0.73 | 0.77 | 0.63 |
| $\Delta t_5$ | 4.13e-1 | 3.11e-1 | 2.99e-1 | 5.50e-1 | 0.71 | 0.77 | 0.82 | 0.69 |
| Test 7     |        |        |        |     |        |        |        |     |
| $\Delta t_1$ | 1.66e-1 | 1.64e-1 | 1.67e-1 | 1.75e-1 | ... | ... | ... | ... |
| $\Delta t_2$ | 1.14e-1 | 1.12e-1 | 1.13e-1 | 1.22e-1 | 0.54 | 0.55 | 0.56 | 0.52 |
| $\Delta t_3$ | 7.32e-2 | 6.96e-2 | 7.01e-2 | 7.94e-2 | 0.63 | 0.68 | 0.69 | 0.62 |
| $\Delta t_4$ | 4.47e-2 | 4.16e-2 | 4.01e-2 | 4.88e-2 | 0.71 | 0.74 | 0.80 | 0.70 |
| $\Delta t_5$ | 2.58e-2 | 2.36e-2 | 2.24e-2 | 2.89e-3 | 0.79 | 0.81 | 0.84 | 0.75 |
| Test 8     |        |        |        |     |        |        |        |     |
| $\Delta t_1$ | 1.85e-1 | 1.94e-2 | 1.96e-1 | 1.80e-1 | ... | ... | ... | ... |
| $\Delta t_2$ | 1.20e-1 | 1.24e-1 | 1.23e-1 | 1.16e-1 | 0.62 | 0.64 | 0.67 | 0.63 |
| $\Delta t_3$ | 7.21e-2 | 7.41e-2 | 7.22e-2 | 7.11e-2 | 0.73 | 0.74 | 0.76 | 0.70 |
Table 5
(Continued)

| Resolution | MM      | MC      | WENO5   | PPM      | MM      | MC      | WENO5   | PPM      |
|------------|---------|---------|---------|----------|---------|---------|---------|----------|
|            | Flux-CT Error | Order of Convergence |
| Δx1        | 4.11e-2 | 4.12e-2 | 4.00e-2 | 4.19e-2  | 0.82    | 0.84    | 0.85    | 0.76     |
| Δx4        | 2.23e-2 | 2.21e-2 | 2.11e-2 | 2.29e-2  | 0.88    | 0.90    | 0.92    | 0.87     |

Note. We use resolutions Δx₁ = 1/400, Δx₂ = 1/800, Δx₃ = 1/1600, Δx₄ = 1/3200, and Δx₄ = 1/6400. A dash indicates that the reconstructor was unable to carry out the simulation.

Table 6
L₃ Norm of the Error in Density for Each Reconstructor and the Divergence Cleaning Method to Control the Magnetic Field Divergence-free Constraint

| Resolution | MM      | MC      | WENO5   | PPM      | MM      | MC      | WENO5   | PPM      |
|------------|---------|---------|---------|----------|---------|---------|---------|----------|
|            | Divergence Cleaning Error | Order of Convergence |
| Test 1     |         |         |         |          |         |         |         |          |
| Δx₁        | 1.18e-1 | 1.10e-1 | 1.14e-1 | 1.38e-1  | ...     | ...     | ...     | ...      |
| Δx₄        | 8.10e-2 | 6.98e-2 | 7.15e-2 | 1.03e-1  | 0.54    | 0.65    | 0.67    | 0.42     |
| Δx₃        | 5.20e-2 | 3.96e-2 | 3.68e-2 | 5.50e-2  | 0.63    | 0.81    | 0.95    | 0.85     |
| Δx₄        | 3.05e-2 | 2.15e-2 | 1.95e-2 | 3.00e-2  | 0.76    | 0.88    | 0.92    | 0.87     |
| Δx₄        | 1.65e-2 | 1.16e-2 | 1.01e-2 | 1.63e-2  | 0.88    | 0.89    | 0.94    | 0.88     |

Test 2
| Δx₁        | 5.79e+1 | 3.18e+1 | 3.14e+1 | 1.48e+1  | ...     | ...     | ...     | ...      |
| Δx₄        | 3.98e+1 | 1.86e+1 | 1.76e+1 | 1.11e+1  | 0.54    | 0.77    | 0.83    | 0.41     |
| Δx₃        | 2.32e+1 | 1.03e+1 | 9.65e0  | 6.80e0   | 0.77    | 0.85    | 0.86    | 0.70     |
| Δx₄        | 1.35e+1 | 5.67e0  | 4.89e0  | 3.98e0   | 0.80    | 0.86    | 0.95    | 0.77     |
| Δx₄        | 7.51e0  | 3.03e0  | 2.51e0  | 2.24e0   | 0.84    | 0.90    | 0.96    | 0.82     |

Test 3
| Δx₁        | 2.16e-2 | 1.59e-3 | 1.61e-3 | 1.91e-2  | ...     | ...     | ...     | ...      |
| Δx₄        | 1.39e-2 | 9.25e-3 | 9.55e-3 | 1.38e-2  | 0.63    | 0.78    | 0.75    | 0.46     |
| Δx₃        | 8.45e-3 | 4.84e-3 | 5.02e-3 | 8.42e-3  | 0.71    | 0.93    | 0.92    | 0.71     |
| Δx₄        | 5.11e-3 | 2.57e-3 | 2.71e-3 | 5.10e-3  | 0.72    | 0.91    | 0.89    | 0.72     |
| Δx₄        | 3.06e-3 | 1.41e-3 | 1.40e-4 | 3.08e-3  | 0.73    | 0.86    | 0.95    | 0.72     |

Test 4
| Δx₁        | 1.65e-1 | 1.52e-1 | 3.02e-1 | 1.66e-1  | ...     | ...     | ...     | ...      |
| Δx₄        | 1.26e-1 | 1.15e-1 | 1.53e-1 | 1.27e-1  | 0.38    | 0.40    | 0.98    | 0.38     |
| Δx₃        | 8.29e-2 | 7.49e-2 | 9.31e-2 | 9.38e-2  | 0.61    | 0.61    | 0.71    | 0.43     |
| Δx₄        | 5.23e-2 | 4.62e-2 | 5.01e-2 | 6.42e-2  | 0.66    | 0.69    | 0.90    | 0.61     |
| Δx₄        | 3.03e-2 | 2.65e-2 | 2.55e-2 | 3.91e-2  | 0.78    | 0.80    | 0.97    | 0.71     |

Test 5
| Δx₁        | 2.56e-1 | 2.57e-1 | 2.55e-1 | 2.46e-1  | ...     | ...     | ...     | ...      |
| Δx₄        | 2.28e-1 | 2.04e-1 | 2.01e-1 | 2.00e-1  | 0.16    | 0.33    | 0.34    | 0.30     |
| Δx₃        | 1.74e-1 | 1.29e-1 | 1.19e-1 | 1.29e-1  | 0.38    | 0.66    | 0.75    | 0.63     |
| Δx₄        | 1.03e-1 | 7.42e-2 | 6.62e-2 | 8.04e-2  | 0.75    | 0.79    | 0.84    | 0.68     |
| Δx₄        | 6.13e-2 | 4.05e-2 | 3.58e-2 | 4.91e-2  | 0.75    | 0.87    | 0.88    | 0.71     |

Test 6
| Δx₁        | 2.32e-1 | 2.15e-1 | 2.23e-1 | 2.33e-1  | ...     | ...     | ...     | ...      |
| Δx₄        | 1.58e-1 | 1.42e-1 | 1.47e-1 | 1.72e-1  | 0.55    | 0.59    | 0.60    | 0.43     |
| Δx₃        | 1.08e-1 | 9.36e-2 | 9.39e-2 | 1.21e-1  | 0.54    | 0.60    | 0.64    | 0.50     |
| Δx₄        | 7.21e-2 | 6.00e-2 | 5.89e-2 | 8.08e-2  | 0.58    | 0.64    | 0.74    | 0.58     |
| Δx₄        | 4.42e-2 | 3.63e-2 | 3.39e-2 | 5.22e-2  | 0.70    | 0.72    | 0.79    | 0.63     |
extended to the relativistic case (Lucas-Serrano et al. 2004; Zhang & MacFadyen 2006). It consists of the flow entering from the left side of the domain and encountering a step. The standard initial conditions are $\rho = 1.4$, $p = 1.0$, $v^x = 0.999$, and $v^y = 0$ with $\Gamma = 1.4$, with the step starting at one-fifth from the horizontal and vertical domains (Zhang & MacFadyen 2006). The boundary conditions are inflow at the left boundary, outflow at the right, reflecting at the top and bottom and the step boundaries. The results using MC are shown in Figure 12. The global features of this test by $t = 1$ are that a reverse shock to the left is formed, which subsequently faces the constant entrance of the fluid from the left to form a bow shock. This shock then expands, and by $t = 2$ it reaches the upper boundary and gets reflected as shown in the snapshot at $t = 3$. Finally, it bounces back again from the step upward, as seen in $t = 4$. A Mach stem is also formed vertically at the top boundary. Unlike in the Newtonian case, the contact discontinuity caused by the corner of the step does not develop any Kelvin–Helmholtz (KH) instability near the Mach stem.  

### 3.1.12. Relativistic KH Instability in 2D

Another test in 2D is the response of the code to unstable initial conditions and the resolution of small structures with low resolution. The KH instability develops when the initial conditions of a gas in two different states separated by different membranes are perturbed. Specifically, a KH instability can occur when there is velocity shear in a continuous fluid or when there is a velocity difference across the interface between two states of the fluid. In this test, one assumes a chamber filled with gas in a given state and a strip in a different state. In our case we use the following setup for $\Gamma = 5/3$:

\[
(\rho, p, v^x, v^y) = \begin{cases} 
(2, 2.5, 0.5, 0), & \text{if } |y| < 0.25 \\
(1, 2.5, -0.5, 0), & \text{if } |y| \geq 0.25.
\end{cases}
\]

Additionally, the velocities are perturbed such that $v^x = v^y \times (1 + 0.01 \cos(10 \pi x) \cos(10 \pi y))$ and $v^y = v^x \times (0.01 \cos(10 \pi x) \cos(10 \pi y))$. In these numerical simulations, we cover the domain $x, y \in [-0.5, 0.5]$ with 400 $\times$ 400 cells and use a Courant factor of $CFL = 0.25$ and periodic boundary conditions in all faces.

In Figure 13, we show the KH instability test at $t = 1.5$. We present the proper rest mass density using different reconstructors. The figure shows the density computed with MINMOD (top left), MC (top right), PPM (bottom left), and WENO5 (bottom right) limiters in combination with the Hlle approximate Riemann solver. As we can see, MC and WENO5 present more substructure than do MINMOD and PPM because the latter introduce more dissipation. However, the less dissipative a limiter is, the more chances there are that unphysical oscillations appear, especially when the gas velocity approaches the speed of light. Thus, in order to avoid these oscillations, when the condition $v^2 < 1 \times 10^{-6}$ is violated, the code uses a constant piecewise reconstructor. In Figure 14, we show the morphology at various times for different stages of the instability using WENO5 at $t = 0.5, 1, 1.5, 2$.

Aside from the morphological tests, it would also be interesting to estimate the saturation time of the various initial perturbations and compare it with the linear perturbation theory as in Perucho et al. (2004). Nevertheless, this task would involve a more systematic analysis, and here we only point out the different features produced by the use of different reconstructors.

### 3.1.13. RHD Jets

The last of the 2D numerical RHD tests corresponds to an axisymmetric relativistic jet, in cylindrical coordinates, injected toward a homogeneous medium. The details of the SRMHD evolution equations in cylindrical coordinates can be found in the Appendix. The beam of the jet is injected with a velocity $v_{th}$.

#### Table 6 (Continued)

| Resolution | MM | MC | WENO5 | PPM | MM | MC | WENO5 | PPM |
|------------|----|----|-------|-----|----|----|-------|-----|
| $\Delta x_1$ | 1.10e-1 | 6.33e-2 | 6.61e-2 | 1.62e-1 | ... | ... | ... | ... |
| $\Delta x_2$ | 8.21e-2 | 4.39e-2 | 4.13e-2 | 1.14e-1 | 0.47 | 0.52 | 0.65 | 0.50 |
| $\Delta x_3$ | 5.21e-2 | 2.78e-2 | 2.50e-2 | 7.35e-2 | 0.65 | 0.66 | 0.72 | 0.63 |
| $\Delta x_4$ | 3.24e-2 | 1.72e-2 | 1.44e-2 | 4.78e-2 | 0.68 | 0.69 | 0.79 | 0.62 |
| $\Delta x_5$ | 1.98e-2 | 1.02e-2 | 8.16e-3 | 2.96e-2 | 0.71 | 0.75 | 0.82 | 0.69 |

Note. We use the resolutions $\Delta x_1 = 1/400$, $\Delta x_2 = 1/800$, $\Delta x_3 = 1/1600$, $\Delta x_4 = 1/3200$, and $\Delta x_5 = 1/6400$. A dash indicates that the reconstructor was unable to carry out the simulation.
Figure 25. Test 6: Balsara 4. We show the snapshot at $t = 0.4$ where the expected shocks have been formed. We use 1600 cells to cover the domain $[-0.5, 0, 5]$ and a Courant factor of $CFL = 0.1$. 
Figure 26. Test 7: Balsara 5. In this test we cover the numerical domain $[-0.5, 0.5]$ with 1600 cells and use CFL = 0.1. We show a snapshot at $t = 0.55$. 
Figure 27. Test 8: generic Alfvén test at $t = 1.5$. We use 3200 cells to cover the domain $[-2, 2]$. We use CFL = 0.1.
through a circular region of radius $r_b = 1$. The density of the beam $\rho_b$ and ambient density $\rho_m$ are related by $\eta = \rho_b/\rho_m$, where usually $\eta$ is less than 1. The relativistic Mach number in the beam is defined as $M_b = M_b W_b \sqrt{1 - c_s^2}$, where $M_b$ is the classical definition of the Mach number, $W_b$ is the Lorentz factor, and $c_s$ is the sound speed of the fluid. Finally, the pressure of the fluid is constant everywhere at the initial time. Outflow boundary conditions are used at the boundaries, except inside the beam radius, where the values of the variables are kept constant. In general, the resulting morphology of the relativistic jets shows a bow shock surrounding a central cocoon, which contains jet gas mixed with shocked ambient gas at the contact discontinuity between them while the mixing is enhanced by turbulent motions. Internal shocks are produced because of the lack of pressure equilibrium between the beam and the cocoon.

In Figure 15 we show the propagation of one of the hydrodynamical jets presented in Del Zanna & Bucciantini (2002). The domain is $[0, 8] \times [0, 20]$ in the $r$ and $z$ directions, respectively, where we use a uniform resolution of $\Delta r = \Delta z = 0.05$. The static medium density is $\rho_m = 10.0$, the pressure is $p_m = 0.01$, and the adiabatic index is $\Gamma = 5/3$. The jet is injected in a circular region with radius $r_b = 1.0$, $v_0 = 0.99$, and $\rho_b = 0.1$. In the figure we show a snapshot at time $t = 40$ using MC. The morphology is similar to that obtained in Del Zanna & Bucciantini (2002) when using CENO3.

In Figure 16 we present the hot model A1 in Martí et al. (1997) at time $t = 48.82$. The parameters of the injected jet are adiabatic index $\Gamma = 4/3$, $v_0 = 0.99$, $\rho_b = 0.01$, and $\eta = 0.01$, and the medium parameters are $\rho_m = 1.0$ and $M_b = 1.72$. The particular feature of this model is that the bow shock is extended and has a very thin cocoon, as can be seen in the figure. We compare the morphology of the rest mass density using (from top to bottom) MINMOD, MC, and WENO5 reconstructors, and we can see that the WENO5 method captures the turbulent shocks in the cocoon region much better than do the other reconstructors.
In Figure 17 we show the rest mass density and the Lorentz factor at time $t = 110.67$ of the model C2 in Martí et al. (1997). The jet parameters are $v_0 = 0.99$, adiabatic index $\Gamma = 5/3$, $\rho_b = 0.01$, and $\eta = 0.01$, and the ambient parameters are $\rho_\infty = 1.0$ and $M_b = 6.0$. We use the MC reconstructor in this case. This model has the extended bow shock surrounding the jet and also has a larger cocoon containing the spots with structure.

3.1.14. 3D Spherical Blast Wave

In order to observe the performance of the code in three dimensions, we consider the spherical blast wave test in RHD, which involves non-grid-aligned shocks. Specifically, the initial data involves the following parameters:

$$
(\rho, \rho, v^r) = \begin{cases} 
(1.0, 1.0, 0), & \text{if } r < 0.5 \\
(0.125, 0.1, 0), & \text{elsewhere},
\end{cases}
$$

with adiabatic index $\Gamma = 1.4$. Because the analytic solution for this problem is not known, we use as a reference solution the one calculated with our 1D spherically symmetric code. The simulation with the spherically symmetric code is done on the domain $r \subseteq [0, 1]$ covered with 2500 cells, whereas the 3D code uses the domain $x, y, z \subseteq [0, 1]$ covered with $100^3$ cells. Figure 18 shows the rest mass density of the fluid computed with two different reconstructors, MC and WENO5, at $t = 0.3$, compared with that obtained with the spherically symmetric code. Even though the resolution used by the 3D code is considerably lower than that of the spherically symmetric code, the numerical solution reproduces all of the features captured by the 1D code.

3.1.15. Relativistic $KH$ Instability in 3D

Tracking the development of turbulent zones depends highly on the dissipation of the numerical method used. This is especially important in 3D because the memory required may be restrictive. We illustrate the performance of our code with a variation of the initial conditions shown for the $K$H instabilities above. For this we follow Beckwith & Stone (2011) and Radice & Rezzolla (2012), where the authors propose a density profile and velocity field with the following components:

$$
\rho(y) = \begin{cases} 
\rho_0 + \rho_1 \tanh[(y - 0.5)/\alpha], & \text{if } y > 0 \\
\rho_0 - \rho_1 \tanh[(y + 0.5)/\alpha], & \text{if } y \leq 0.
\end{cases}
$$

$$
v^y(y) = \begin{cases} 
V_0 \tanh[(y - 0.5)/\alpha], & \text{if } y > 0 \\
V_0 \tanh[(y + 0.5)/\alpha], & \text{if } y \leq 0.
\end{cases}
$$

$$
v^z(x, y) = \begin{cases} 
A_0 V_0 \sin(2\pi x) \exp[-(y - 0.5)^2/\sigma], & \text{if } y > 0 \\
-A_0 V_0 \sin(2\pi x) \exp[-(y + 0.5)/\sigma], & \text{if } y \leq 0.
\end{cases}
$$

Figure 29. Variables of the 1D magnetic rotor test along the $x$ and $y$ axes using two resolutions. On the left we show the variables along $x$ and on the right the variables along $y$. The two resolutions we use are $200 \times 200$ and $400 \times 400$ cells that cover the numerical domain $[-0.5, 0.5] \times [-0.5, 0.5]$. 

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where $r = 0.5050$, $r = 0.4951$, $a = 0.1$, $V = 0.5s$, and $G = 4/3$.

The problem is solved in the domain $-0.5 \leq x \leq 0.5$, $-1 \leq y \leq 1$, $-0.5 \leq z \leq 0.5$ using $256 \times 512 \times 256$ cells.

The 3D nature of the test is given by the addition of a nontrivial $\nu^z$ component that has a random amplitude between 0 and 0.01 added at initial time. This random perturbation triggers a small asymmetry during the evolution. In Figure 19 we show the results at time $t = 3$ using three different reconstructors. In this situation, the WENO limiter is the one that best captures the development of small structures, whereas the linear reconstructors add enough dissipation to wash out the small structures.

3.2. Relativistic Magnetohydrodynamic Tests

Now we present the standard 1D and 2D tests for the RMHD according to the standard literature (Komissarov 1999; Balsara 2001; Del Zanna et al. 2003). The 1D tests, which are summarized in Table 4, consist of 1D Riemann problems with the special feature that now there is at least a nontrivial component of the magnetic field. Again, for 1D tests we are using the 3D code with five cells along the transverse directions. We compare the numerical solution with the exact solution, which was computed using Giacomazzo’s code (Giacomazzo & Rezzolla 2006, 2007). Unless otherwise stated, the results presented in the figures are obtained with the MINMOD limiter, HLLE Riemann solver, and flux-CT method to preserve the constraint (9). Furthermore, in order to compare the accuracy using different limiters, we have calculated an error for each 1D Riemann test using the limiters MINMOD, MC, PPM, and WENO5. In each figure, we show the proper rest mass density $\rho_0$, the total pressure $p + p_m$, the magnetic field components $B^y$, $B^z$, and the velocity components $\nu^x$, $\nu^y$.

We test our multidimensional scheme with the following 2D simulations in the presence of a magnetic field: a relativistic cylindrical explosion, a relativistic rigid rotor, a relativistic KH instability, and finally a 3D test regarding the magnetic field loop advection test.

3.2.1. Test 1: Komissarov Shock Tube

In the plots of 1D tests, we again use lines to represent the exact solution and points to represent the numerical solutions. The first test is a shock-tube test with pressure ratio of $p \sim 10^3$ and a constant magnetic field along the shock direction. In Figure 20, we show the numerical and exact solutions computed in the domain $x \in [-2, 2]$ at $t = 1.0$, using resolution $\Delta x = 1/800$. A fast rarefaction zone moves to the left and a fast shock to the right from the contact discontinuity.
The high difference of the pressure produces a thin shell in the density that moves with relativistic velocities in the shock direction. We can also verify in the figure that the transverse components of the magnetic and velocity fields are zero at all times during the evolution as expected. We obtain similar results using different reconstructors and divergence-control methods.

3.2.2. Test 2: Komissarov Collision Test

This is the collision of streams moving in opposite directions with initial head-on velocity $v^x = 0.98058$. In this test the dynamics of the fluid are immersed in a magnetic field with a constant $x$ component and a discontinuous $y$ component. We compare the results with the exact solution in Figure 21. As in the Komissarov shock-tube test, we use a numerical domain $x \in [-2, 2]$ at $t = 1.22$, using resolution $\Delta x = 1/800$. In this case, two slow and two fast shocks move to the left and to the right as expected. We experimented with different reconstructors and found that the MC limiter, unlike the other limiters, develops numerical oscillations.

3.2.3. Test 3: Balsara 1 Test

This RMHD test corresponds to the relativistic generalization of the classical Brio–Wu problem (Brio & Wu 1988). The results for the different variables and the various expected features are shown in Figure 22 at time $t = 0.4$: a left fast rarefaction wave in the region $x \in [-3.6, -2.5]$, at $x \sim 0.01$ a slow compound wave, a contact discontinuity at $x \sim 0.1$, a slow shock at $x \sim 0.15$, and a right fast rarefaction wave in the region $x = [0.3, 0.38]$. The slow shock appears in a strongly magnetically dominated region. In this zone, the magnetic energy is greater than the fluid rest mass energy or the fluid thermal energy. It is worth mentioning that the slow compound wave appears only in the numerical solution; the exact solution omits this by construction (Giacomazzo & Rezzolla 2006). However, the numerical solution is consistent with that of previous numerical RMHD codes.

3.2.4. Test 4: Balsara 2 Test

This corresponds to a weak blast wave; the initial configuration consists of a moderate initial discontinuity of the pressure and constant rest mass density. The ratio between pressures is $p_L/p_R = 30$. In Figure 23 we show a snapshot at $t = 0.4$. A slow shock wave is formed and propagates along the $x$ direction near the contact discontinuity with maximum Lorentz factor $W = 1.36$. In this case, all of the reconstructors and the constraint control methods produce well-behaved results. The numerical domain $x \in [-0.5, 0.5]$ is covered with resolution $\Delta x = 1/1600$.

3.2.5. Test 5: Balsara 3 Test

This is a strong blast wave with a high difference between the pressures in the initial discontinuity of nearly four orders of magnitude, constant density, and zero velocities at initial time. In Figure 24 we show the snapshot at $t = 0.4$. We see the typical peak of the density in the blast wave and the effects on the...
velocity and magnetic field. The Lorentz factor reaches values of $W \sim 3.5$. The numerical solution is consistent with the exact solution. In the rest mass density we can observe a fast and a slow rarefaction zone moving to the left, and a contact wave and two (fast and slow) shocks moving to the right. The presence of the magnetic field makes the slow and fast shocks propagate closely, giving as a result a thin density shell, which is difficult to capture with low resolution. However, in Tables 5 and 6 we show that a more accurate result is also obtained using MC or WENO5.

3.2.6. Test 6: Balsara 4 Test

This is again the case of a head-on collision of streams. However, unlike the Komissarov collision test, in this case the transversal components of the magnetic field are nonzero and the velocities are higher. The problem starts with two relativistic streams moving in opposite directions at nearly the speed of light, with initially constant pressure and rest mass density. In Figure 25, we show a snapshot at $t = 0.4$. In this particular case, in the plots some signals appear beyond $x > 0.4$, but these effects are due to numerical diffusion. The initial Lorentz factor is $W \sim 22.366$, and the initial pressure includes high values of $p \sim 1200$. We can also see that two slow waves are moving in opposite directions. On the other hand, in the strong shocks, spurious oscillations appear when the less dissipative reconstructors like MC and WENO5 are used.

3.2.7. Test 7: Balsara 5 Test

This test includes nonzero transversal and discontinuous components of the velocity and magnetic field. In Figure 26, we show the snapshot at $t = 0.55$. The Lorentz factor is rather small, of the order of $W \sim 1.86$. We can see also an Alfvén
wave moving to the left and another one moving to the right. In this test we obtain similar errors when using the MINMOD, MC, PPM, and WENO5 reconstructors.

3.2.8. Test 8: Alfvén Test

The last RMHD 1D test is the generic Alfvén wave. In Figure 27 we show the numerical results at $t = 1.5$. During the evolution, different regions are formed: a fast rarefaction region, an Alfvén wave, and a slow shock moving to the left, and a contact wave and two (slow and fast) shocks moving to the right. We reproduce similar results with different reconstructors, all of which are able to capture the thin shell formed in $B$ with a few cells.

3.2.9. Error Estimates for the 1D RMHD Tests

As in the RHD case, the numerical solution using the various limiters is consistent. We calculate the $L_1$ norm of the error of the 1D tests, and the results are shown in Tables 5 and 6. Convergence is considerably more difficult to achieve than in the RHD. For the tests 1–8 we obtain nearly first-order convergence for the constraint control methods and various reconstructors, as expected for initial data containing strong discontinuities and our RK3 integrator.

3.2.10. Magnetic Rotor Test

The first 2D test is a magnetic rotor defined on the $xy$ plane. The initial density within a cylinder of radius $r_m = 0.1$ is $\rho_{in} = 10$ and has angular velocity $\omega_z = 9.55$. The initial pressure is constant in the whole domain $\rho = 1$, and the magnetic field has components $B_x = 1$ and $B_y = 0$. The components of the initial velocity inside the cylinder are defined by $v^x_{in} = -\omega_z y$ and $v^y_{in} = \omega_z x$. In the exterior region ($r > r_m$), the fluid density is $\rho = 1$ and the velocity is zero. The results of the evolution for $\Gamma = 5/3$ are shown in Figure 28 at

Figure 33. Magnetized Kelvin–Helmholtz test. We show at $t = 2$ the proper rest mass density $\rho_0$ (top left), the magnetic pressure $p_{mag}$ (top right), the Lorentz factor $W$ (bottom left), and the divergence of the magnetic field $\nabla \cdot B$ (bottom right) calculated with WENO5. The simulations were calculated for $l = 3$ on the domain $[-0.5, 0.5] \times [-0.5, 0.5]$ covered with $600 \times 600$ cells, using a Courant factor of $\text{CFL} = 0.25$. 

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\( t = 0.4 \), where the different shocks and the rotational Alfvén waves can be observed and reproduce the morphology in Del Zanna et al. (2003). We show additionally the violation of the divergence of the magnetic field constraint calculated with the flux-CT method. In this case, the violation is bounded to be \( \nabla \cdot B \approx 10^{-11} \) for MINMOD. As expected, the magnetic field slows down the rotor velocity, reducing the Lorentz factor from \( \sim 10 \) at initial time to \( \sim 4 \) at time \( t = 0.4 \), and all of the variables are affected by the dragging of the initial angular velocity. In Figure 29, we show 1D slices for two different resolutions, showing the variable profiles along the \( x \) and \( y \) axes. The variable profiles reproduce those in Mösta et al. (2014).

### 3.2.11. Cylindrical Explosion Test

The second 2D test is the cylindrical explosion, starting the evolution with a cylindrical inner region with radius \( r_{\text{in}} = 0.8 \), where the rest mass density is \( \rho_{\text{in}} = 10^{-2} \) and the pressure is \( p_{\text{in}} = 1 \). In the outer region \( r > r_{\text{out}} = 1.0 \) the variables are \( \rho_{\text{out}} = 10^{-4} \) and \( p_{\text{out}} = 3 \times 10^{-5} \). The magnetic field is uniform in the whole domain, initially \( B_{r} = 0.1 \) and \( B_{\phi} = 0 \), whereas the adiabatic index is \( \Gamma = 4/3 \). In this configuration the fluid is initially at rest. We also use a smoothing function for the density and pressure for \( r_{\text{in}} < r < r_{\text{out}} \) as in Mösta et al. (2014),

\[
\rho_0 = \begin{cases} 
\rho_{\text{in}} & r \leq r_{\text{in}} \\
 e^{\frac{r_{\text{out}} - r_{\text{in}}}{r_{\text{out}} - r_{\text{in}}}} & r_{\text{in}} < r < r_{\text{out}} \\
\rho_{\text{out}} & r \geq r_{\text{out}},
\end{cases}
\]

(37)

where \( r = \sqrt{x^2 + y^2} \). A similar smoothing function is used for the pressure. We again practice the test in the \( xy \) plane. The results of the evolution are shown in Figure 30, and we reproduce those in Del Zanna et al. (2003). In this test, in order to compare the two different methods implemented to prevent the growth of the constraint violation of the magnetic field, we present the numerical calculations with the flux-CT and divergence-cleaning methods. In the first case, the divergence of the magnetic field...
remains on the order of $\nabla \cdot \mathbf{B} \sim 10^{-14}$, whereas in the second one the violation is of the order $\sim 10^{-2}$ in some regions, which is comparable with previous analyses (Neilsen et al. 2006). Additionally, like in the relativistic rigid rotor test, in Figure 31 we show 1D slices for two different resolutions, showing the variable profiles along the x and y axes. The variable profiles are better resolved with higher resolution, as in Mista et al. (2014). The evolution shows an exterior shock wave expanding radially at nearly the speed of light with a very small amplitude.

It is also possible to handle stronger magnetic fields. For instance, we evolved the cylindrical explosion test with $B_t = 1$ using the MC reconstructor, the flux-CT method to control the constraint, and a Courant factor of 0.1, but using a higher external pressure of $5 \times 10^{-3}$. In Figure 32 we show the results, which are consistent with those in Del Zanna et al. (2007) and Beckwith & Stone (2011) for such a strong field, and our code tolerates a magnetic field strength up to $B_t = 1.5$. Without increasing the external pressure, it has been possible to carry out this test, but using HLLC fluxes (Mignone & Bodo 2006).

3.2.12. MKH Test

Another standard 2D test is the MKH instability. The initial condition sets the fluid in three separate regions. In one of them it moves in one direction and in the other two in the opposite direction. Along the layers dividing the three regions the velocity is perturbed. We study this test in the $xy$ plane covering the domain $[-0.5, 0.5] \times [-0.5, 0.5]$, using 600 $\times$ 600 cells. The fluid in the strip $|y| \leq 0.25$ moves along the $x$ direction with velocity $v_x = 0.5$ and density $\rho = 1$. The fluid outside this area moves in the opposite direction with a speed $v_y = -0.5$ and density $\rho = 2$. The initial pressure is constant throughout the domain, the adiabatic index is $\Gamma = 1.4$, and the magnetic field is uniform along the $x$ direction, $B^z = 0.5$. The $y$ component of the velocity is also perturbed, and the velocity field including the perturbation at initial time is

$$v^i = v_0^i (1 + 0.01 \cos(2\pi y)\sin(2\pi x)), \quad (38)$$
$$v^f = 0.01 \cos(2\pi y)\sin(2\pi x), \quad (39)$$

where $l = 3$ is the number of nodes of the perturbation along the domain. The perturbation triggers the instabilities shown in Figure 33, where we show the proper rest mass density, magnetic pressure, Lorentz factor, and the constraint violation at $t = 2$ using WENO5. As we can see, the constraint violation is of the order of $10^{-12}$ when using the flux-CT method.

Like in the unmagnetized case, it would also be interesting to estimate the saturation time and to follow the evolution to the turbulent regime as in Bucciantini & Del Zanna (2006), but this would deserve a separate space elsewhere.

3.2.13. Relativistic Magnetic Field Loop Advection 3D Test

This is a test modeling a loop of magnetic field that is being advected similar to that in Beckwith & Stone (2011). The initial pressure gradients are zero, and the dynamics are ruled by the velocity field that carries the magnetic field with it. We set the initial constant density and pressure to $\rho = p = 1.0$ with adiabatic index $\Gamma = 4/3$.

The magnetic field is initialized using a vector potential defined by $A_3 = \text{MAX}([A(R_0 - r)], 0)$ with $r$ the 2D cylinder-type radius measured from an axis parallel to $A_3$. In order to have an oblique advection, we choose the vector potential such that its only component lies along the diagonal of the $xz$ plane. Because we choose the numerical domain to be $[-0.5, 0.5] \times [-0.5, 0.5] \times [-1, 1]$, the rotation $(x_1, x_2, x_3) = ((2x + y)/\sqrt{5}, y, -x + 2z)/\sqrt{5}$ makes the vector potential have the single component $A_3$ along this diagonal. The 2D radius perpendicular to this diagonal is thus $r = \sqrt{x_1^2 + x_2^2} = \sqrt{(2x + y)^2/5 + y^2}$.

We choose the amplitude $A$ to be small so that the field is weak compared to the gas pressure and thus maintains magnetostatic equilibrium. We use $A = 10^{-3}$ and the loop radius $R_0 = 0.3$. Face-centered magnetic fields are computed using finite differences to calculate $\mathbf{B} = \nabla \times \mathbf{A}$ to set $\nabla \cdot \mathbf{B} = 0$ initially up to the numerical error.

The velocity field is defined by $v_x = -0.3$, $v_y = 0.0$, and $v_z = 0.6$, such that the loop propagates along the diagonal of the $xz$ plane. In order to preserve the magnetic field constraint, we use the flux constraint transport method. The 3D simulation was calculated using $128 \times 128 \times 256$ cells, a Courant factor $\text{CFL} = 0.25$, and periodic boundary conditions. In Figure 34 we present snapshots of the squared magnitude of the magnetic field. This shows that the magnetic field is being advected across the domain. It is also shown that the magnetic field constraint is kept under control during the evolution.

4. DISCUSSION AND CONCLUSIONS

We have presented a new 3D code designed to solve the RMHD equations and shown that it passes the RHD and RMHD standard tests.

Among the various combinations of methods in an HRSC implementation, we have shown the tests only for a reduced set of linear reconstructors (MINMOD, MC), the parabolic PPM, and the fifth-order WENO5 limiter, all of them combined with the HLLE flux formula and the RK3 time integrator.

We have also presented an error analysis for the RHD and RMHD 1D tests. In all of the cases the numerical solutions are consistent. Furthermore, the convergence achieved is the expected nearly first order for initial data with shocks and second for smooth data, at least within a resolution regime and a particular linear reconstructor. In the 2D and 3D cases of RMHD, we have shown the ability of our code to keep the violation of the constraint under control, by using either the flux-CT or cleaning methods. In our case, in unigrid mode both methods are comparably easy to implement, but the flux-CT method may offer extra complication when implemented on adaptive mesh refinement, unlike the cleaning method, which is implemented as an extra evolution equation. Also, the flux-CT shows the advantage that the errors in the constraint are kept very low, of the order of the round-off error, whereas the divergence cleaning method is easy to implement but the errors are not as low, which on the other hand is consistent with previous experience.

Finally, we want to confirm some of the general limitations of the methods used here that have been described in the past. We confirm that MC captures the shocks better than MINMOD in most of the cases, but depending on the strength of the shocks, it introduces high-frequency noise on discontinuities. On the other hand, even though the PPM is a third-order method, the parameters we used for this reconstructor show
errors and convergence rates similar to those for the linear
reconstructors. Finally, the WENO5 is a fifth-order reconstruc-
tor, which is expensive, but the reconstruction is in most cases
free of unphysical oscillations.

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free of unphysical oscillations.

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APPENDIX

SRMHD IN CYLINDRICAL COORDINATES

In the particular test corresponding to the jets, our numerical
code requires the implementation of the SRMHD equations
in cylindrical coordinates $(r, \phi, z)$. In this case, the Min-
kowski metric becomes $\eta_{\alpha\beta} = \text{diag}(-1, 1, r, 1)$. Then
the SRMHD equations can be written in a conservative form as
follows:

$$\frac{\partial f^{(r)}}{\partial t} + \frac{1}{r} \frac{\partial (rf^\phi)}{\partial r} + \frac{\partial f^z}{\partial z} = s,$$

(40)

where the conservative variables, the fluxes in each direction,
and the source vector are

$$f^{(r)} = \begin{bmatrix}
D v^r \\
M_r v^r + p^r - b_r B^r/W \\
M_\phi v^\phi + b_\phi B^r/W \\
\tau v^r + p^r v^\phi - b_\phi B^r/W \\
0 \\
v^r B^\phi - v^\phi B^r \\
v^r B^r - v^r B^z \\
\end{bmatrix},$$

(42)

$$f^\phi = \begin{bmatrix}
D v^\phi \\
M_r v^\phi + b_r B^\phi/W \\
M_\phi v^\phi + p^\phi - b_\phi B^\phi/W \\
\tau v^\phi + p^\phi v^\phi - b_\phi B^\phi/W \\
0 \\
v^\phi B^r - v^r B^\phi \\
v^\phi B^\phi - v^\phi B^r \\
\end{bmatrix},$$

(43)

$$f^z = \begin{bmatrix}
D v^z \\
M_r v^z - b_r B^z/W \\
M_\phi v^z - b_\phi B^z/W \\
\tau v^z + p^z v^\phi - b_\phi B^z/W \\
0 \\
v^z B^\phi - v^\phi B^z \\
v^z B^r - v^r B^z \\
0 \\
\end{bmatrix}.$$
