DIRAC EIGENVALUES FOR GENERIC METRICS ON THREE-MANIFOLDS

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ABSTRACT. We show that for generic Riemannian metrics on a closed spin manifold of dimension three the Dirac operator has only simple eigenvalues.

1. INTRODUCTION

The spectrum of the Dirac operator has been explicitly computed for quite a few compact spin manifolds, for example spherical space forms, flat manifolds, spheres with Berger metrics, see [8, 7, 2, 3, 12]. These examples exhibit high multiplicities of the eigenvalues, coming from the high degree of symmetry of the spaces, a phenomenon one would not expect for general Riemannian manifolds. The purpose of this note is to prove the following theorem.

Theorem 1.1. For a generic metric on a compact spin three-manifold the Dirac operator has only simple eigenvalues.

The Dirac operator $D$ (or $D_g$ to show its dependence on the Riemannian metric) is a first order elliptic operator acting on sections of the spinor bundle $\Sigma M = \text{Spin}(M, g) \times_\rho \Sigma$. It is formally self-adjoint and on a compact manifold it has discrete real spectrum. Here $\text{Spin}(M, g) \to \text{SO}(M, g)$ is a spin structure on $M$ and $\rho : \text{Spin}(n) \to \text{End}(\Sigma)$, $\Sigma = \mathbb{C}^{2^{[n/2]}}$, is the spinor representation. In dimensions $n \equiv 3, 4, 5 \mod 8$ the spinor representation is quaternionic; there is an $\mathbb{R}$-linear endomorphism $J$ of $\Sigma$ with $J^2 = -1$, $Ji = -iJ$, which commutes with the action of $\text{Spin}(n)$. In particular in three dimensions the spinor bundle is a quaternionic line bundle and the eigenspaces $E_\lambda$ of $D$ are quaternionic vectorspaces. So in three dimensions an eigenvalue $\lambda$ is simple if the quaternionic dimension $\dim_{\mathbb{H}} E_\lambda = 1$.

For the details of the constructions of spin geometry here omitted see [10].

For a compact spin manifold $M$ denote by $R(M)$ the space of Riemannian metrics on $M$ equipped with the $C^1$-topology. Denote by $S(M)$ the subset consisting of Riemannian metrics for which all eigenvalues of the Dirac operator are simple.

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Recall that a subset of a topological space is called residual if it contains a countable intersection of open and dense sets. The precise theorem we will prove is the following.

**Theorem 1.2.** Let $M$ be a compact spin manifold of dimension three. Then $S(M)$ is residual.

It is reasonable to conjecture that the conclusion of Theorem 1.2 holds in any dimension. The restriction to dimension three comes from using the simple approach of making conformal variations of the metric. Perhaps the techniques of [4, 11] can be used to extend the result to higher dimensions.

**Remark 1.3.**

1. The proof will show that in each conformal class the set of metrics for which all non-zero eigenvalues are simple is residual.

2. The proof will also show that in any dimension the set of metrics for which $\dim E_{\lambda} \leq \text{rank } \Sigma M$ for all non-zero $\lambda$ is residual. Only in dimension three is $\text{rank } \Sigma M = 1$.

## 2. Conformal Deformations and Dirac Eigenvalues

Let $M$ be an $n$-dimensional manifold with a Riemannian metric $g$ and a spin structure $\text{Spin}(M, g) \to SO(M, g)$. We are going to study the deformation of $g$ given by $g^t = e^{2tf} g$, where $t$ is a real parameter and $f$ is a function on $M$.

There is an isomorphism $\gamma_{tf} : SO(M, g) \to SO(M, g^t)$ given by

$$\gamma_{tf}(\{e_1, \ldots, e_n\}) = \{e^{-tf} e_1, \ldots, e^{-tf} e_n\}.$$  

Composition with $\gamma_{tf}$ turns the spin structure $\text{Spin}(M, g) \to SO(M, g)$ for $g$ into a spin structure for $g^t$:

$$\text{Spin}(M, g) \to SO(M, g) \xrightarrow{\gamma_{tf}} SO(M, g^t),$$

and the associated spinor bundle $\Sigma M = \text{Spin}(M, g) \times_{\rho} \Sigma$ is independent of $t$ and $f$. Let $D^t$ denote the Dirac operator acting on sections of $\Sigma M$ defined using the metric $g^t$. Then $D^t$ is related to $D = D^0$ by

$$D^t \varphi = e^{-tf} \left( D \varphi + \frac{n-1}{2} \text{grad } f \cdot \varphi \right),$$

where the Clifford multiplication by the gradient of $f$ is defined using the metric $g$, see for instance section 3.2.4. in [5]. Define the Hilbert space $L^2(\Sigma M, g^t)$ as the completion of the smooth sections of $\Sigma M$ with respect to the inner product

$$(\varphi, \psi)_{g^t} = \int_M \langle \varphi, \psi \rangle d\mu_{g^t} = \int_M \langle \varphi, \psi \rangle e^{ntf} d\mu.$$  

The proof of the following theorem is analogous to that of Theorem A.3 of [11], see also [9] and [6].
Theorem 2.1. Suppose \( \lambda \) is an eigenvalue of \( D \) with \( \dim_C E_\lambda = p \). Then there exist real analytic functions \( \lambda^t_1, \ldots, \lambda^t_p \) and curves of smooth spinor fields \( \varphi^t_1, \ldots, \varphi^t_p \) (defined for small \( t \)) such that

\[
\begin{align*}
  \bullet & \quad D^t \varphi^t_i = \lambda^t_i \varphi^t_i, \quad i = 1, \ldots, p, \\
  \bullet & \quad \lambda^0_i = \lambda, \quad i = 1, \ldots, p, \\
  \bullet & \quad \varphi^t_1, \ldots, \varphi^t_p \text{ are real analytic as maps } t \mapsto \varphi^t_i \in L^2(\Sigma M, g), \\
  \bullet & \quad \varphi^t_1, \ldots, \varphi^t_p \text{ are orthonormal in } L^2(\Sigma M, g^t).
\end{align*}
\]

Remark 2.2. If the spinor representation is quaternionic \( \dim_C \) can be replaced by \( \dim_H \), in which case “orthonormal” should be interpreted as

\[
(\varphi_i, \varphi_i)_g = 1, \quad (\varphi_i, J \varphi_i)_g = (\varphi_i, \varphi_j)_g = (\varphi_i, J \varphi_j)_g = 0, \quad i \neq j.
\]

Suppose that \( t \mapsto (\lambda^t, \varphi^t) \) is a one-parameter family of eigenspinors as given by Theorem 2.1; \( D^t \varphi^t = \lambda^t \varphi^t \) and \( \|\varphi^t\|_{g^t} = 1 \) for small \( t \). Then

\[
\begin{align*}
\lambda^t & = (D^t \varphi^t, \varphi^t)_g^t, \\
& = \int_M \langle e^{-tf} \left( D \varphi^t + \frac{n-1}{2} t \text{ grad } f \cdot \varphi^t \right), \varphi^t \rangle e^{ntf} d\mu \\
& = \int_M \langle D \varphi^t, \varphi^t \rangle e^{(n-1)tf} + \frac{n-1}{2} t \langle \text{ grad } f \cdot \varphi^t, \varphi^t \rangle e^{(n-1)tf} \rangle d\mu \\
& = \text{Re} \int_M \langle D \varphi^t, \varphi^t \rangle e^{(n-1)tf} d\mu,
\end{align*}
\]

where the last equality follows since the second term is purely imaginary. Set \( \lambda = \lambda^0, \varphi = \varphi^0 \) and denote by prime the derivative with respect to \( t \) at \( t = 0 \). We have

\[
\begin{align*}
\lambda' & = \text{Re} \int_M \langle D \varphi', \varphi \rangle + \langle D \varphi, \varphi' \rangle + \langle D \varphi, \varphi \rangle (n-1) f \rangle d\mu \\
& = \lambda \int_M 2 \text{Re} \langle \varphi', \varphi \rangle + (n-1) f |\varphi|^2 d\mu
\end{align*}
\]

and differentiating \( \|\varphi^t\|_{g^t}^2 = 1 \) at \( t = 0 \) we get

\[
\int_M 2 \text{Re} \langle \varphi', \varphi \rangle + n f |\varphi|^2 d\mu = 0
\]

which together give

\[
(1) \quad \lambda' = -\lambda \int_M f |\varphi|^2 d\mu.
\]

3. PROOF OF THEOREM 1.2

Let \( (M, g) \) be a three dimensional compact Riemannian spin manifold. The main technical point in the proof of Theorem 1.2 is the following lemma.
Lemma 3.1. Let \( \lambda \) be a non-zero eigenvalue of \( D_g \) with \( \dim \mathcal{H} E_\lambda = p > 1 \). Then there is a conformal deformation of \( g \) for which \( E_\lambda \) splits into lower-dimensional eigenspaces.

Proof. For each conformal deformation \( g^t = e^{2tf} g \) Theorem 2.1 provides real analytic parametrizations \( \lambda_1^t, \ldots, \lambda_p^t \) of eigenvalues such that \( \lambda_i^t = \lambda, i = 1, \ldots, p \). If for some \( f, i \) and \( j \) we have \( \lambda_i^t \neq \lambda_j^t \) for all \( t \neq 0 \) we are done. Assume for a contradiction that this is not the case, but instead that \( \lambda_i^t \) is independent of \( i, t \) and \( f \). Then for a given \( f \) we can replace the eigenspinors \( \varphi_i^t \) from Theorem 2.1 by eigenspinors \( \overline{\varphi}_i = \sum_{j=1}^p U_{ij} \varphi_j^t \) where \( U_{ij} \) is a (constant) unitary matrix, which have the same properties. We may thus assume that \( \varphi_1^0 \) and \( \varphi_2^0 \) are the same for all conformal deformations.

For fixed \( f \) and \( p, q = 0, 1 \) let
\[
\alpha_{p,q}^t = 2^{-1/2}(\varphi_1^t + i^p J^q \varphi_2^t), \quad \beta_{p,q}^t = 2^{-1/2}(\varphi_1^t - i^p J^q \varphi_2^t).
\]
Then
\[
D\alpha_{p,q}^t = \lambda \alpha_{p,q}^t, \quad D\beta_{p,q}^t = \lambda \beta_{p,q}^t,
\]
where \( \lambda = \lambda_1^t = \lambda_2^t \). By (1) we have
\[
\int_M f|\alpha_{p,q}^t|^2 \, d\mu = \lambda' = \int_M f|\beta_{p,q}^0|^2 \, d\mu.
\]
Since by assumption this holds for all \( f \) \( \alpha_{p,q}^0 \) and \( \beta_{p,q}^0 \) are indepedent of \( f \). We conclude that \( |\alpha_{p,q}^0|^2 = |\beta_{p,q}^0|^2 \) at each point. It follows that
\[
|\varphi_1^0|^2 + |i^p J^q \varphi_2^0|^2 - 2 \Re \langle \varphi_1^0, i^p J^q \varphi_2^0 \rangle = |\varphi_1^0|^2 + |i^p J^q \varphi_2^0|^2 - 2 \Re \langle \varphi_1^0, i^p J^q \varphi_2^0 \rangle
\]
so \( \Re \langle \varphi_1^0, i^p J^q \varphi_2^0 \rangle = 0 \) for \( p, q = 0, 1 \), and thus \( \langle \varphi_1^0, \varphi_2^0 \rangle = \langle \varphi_1^0, J \varphi_2^0 \rangle = 0 \) at each point. Since \( \text{rank}_{\mathbb{H}} \Sigma M = 1 \) one of \( \varphi_1^0 \) and \( \varphi_2^0 \) has to vanish on an open set, and by unique continuation vanish identically. This is a contradiction which proves the lemma.

Enumerate the non-zero eigenvalues of \( D_g \) as
\[
\ldots \leq \lambda_{-3} \leq \lambda_{-2} \leq \lambda_{-1} < 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots
\]
with repetitions according to quaternionic multiplicity. Let \( S^k(M) \), \( k \geq 1 \), be the subset of \( R(M) \) consisting of Riemannian metrics for which \( \lambda_{\pm 1}, \lambda_{\pm 2}, \ldots, \lambda_{\pm k} \) are all different.\(^1\)

Proposition 3.2. The sets \( S^k(M) \), \( k \geq 1 \), are open and dense in \( R(M) \).

\(^1\)Comment added July 2013: For Proposition 3.2 and its proof to hold one must add the condition that the metrics in \( S^k(M) \) have no harmonic spinors. This assumption is formulated slightly later here, in the proof of Theorem 1.2 on the next page. The fact that there is a problem in the published version of this paper was pointed out by Y. Canzani and R. Ponge, see the preprint http://arxiv.org/abs/1207.0548. The author wishes to express his thanks to them for their interest and kind correspondence concerning this paper. Note that the argument works exactly as written to prove Remark 1.3 for perturbations within a conformal class the dimension of the kernel of \( D \) is constant.
Proof: The eigenvalues of $D_g$ depend continuously on $g$, see Proposition 7.1 in [3]. It follows that the $S^k(M)$ are open.

We prove that the $S^k(M)$ are dense by induction, for $k = 1$ there is nothing to prove. Assume that $S^k(M)$ is dense in $R(M)$ and let $U$ be an open set in $R(M)$. We need to show that $U \cap S^{k+1}(M)$ is non-empty. By assumption we can find a metric $g \in U \cap S^k(M)$. Consider first the positive eigenvalues of $D_g$. The first $k$ are distinct, denote the multiplicity of $\lambda_k$ by $p$ so that

$$0 < \lambda_1 < \ldots < \lambda_{k-1} < \lambda_k = \lambda_{k+1} = \ldots = \lambda_{k+p-1} < \lambda_{k+p} \leq \ldots .$$

If $p = 1$ we are done with the first step, so assume $p > 1$. Then by Lemma 3.1 there is a conformal deformation which decreases the multiplicity of $\lambda_k$. By choosing the deformation parameter $t$ small enough we can guarantee to get a metric in $U \cap S^k(M)$ for which $\dim H^0_{\lambda_k} < p$. Repeating this process we end up with a metric $\tilde{g} \in U \cap S^k(M)$ for which $\lambda_k$ has multiplicity 1.

The second step is to go through the same procedure with the negative eigenvalues of $D_{\tilde{g}}$. When we do this we get a metric $\check{g}$ in $U \cap S^k(M)$ for which both $\lambda_k$ and $\lambda_{-k}$ have multiplicity 1. The metric $\check{g}$ is thus in $U \cap S^{k+1}(M)$ and we have proved that $S^{k+1}(M)$ is dense in $R(M)$. □

Proof of Theorem 1.2 Let $S^0(M)$ denote the set of metrics on $M$ for which $\ker D = 0$. Then $S(M)$ contains the intersection $\bigcap_{i=0}^{\infty} S^i(M)$. By Theorem 1.2 in [11] we know that $S^0(M)$ is open and dense in $R(M)$ and together with Proposition 3.2 we conclude that $\bigcap_{i=0}^{\infty} S^i(M)$ is a residual set. □

Proof of Remark 1.3 We know from Lemma 3.1 that we can split multiple eigenvalues within a conformal class. It follows that the intersection of $S^k(M)$ with a given conformal class is open and dense in the conformal class. This proves the first statement. For the second statement note that with the assumption $\dim E_\lambda > \rank \Sigma M$ the conclusion of Lemma 3.1 holds in any dimension. □

REFERENCES

1. S. Bando and H. Urakawa, Generic properties of the eigenvalue of the Laplacian for compact riemannian manifolds, Tohoku Math. J. 35 (1983), 155–172.
2. C. Bär, The Dirac operator on space forms of positive curvature, J. Math. Soc. Japan 48 (1996), 69–83.
3. ________, Metrics with harmonic spinors, Geom. Func. Anal. 6 (1996), 899–942.
4. C. Bär and M. Dahl, Surgery and the spectrum of the Dirac operator, Preprint, Universität Hamburg, 2001.
5. H. Baum, Spin-Strukturen und Dirac-Operatoren über pseudoriemannischen Mannigfaltigkeiten, BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1981.
6. J.-P. Bourguignon and P. Gauduchon, Spineurs, opérateurs de Dirac et variations de métriques, Comm. Math. Phys. 144 (1992), 581–599.
7. T. Friedrich, Zur Abhängigkeit des Dirac-Operators von der Spin-Struktur, Coll. Math. 48 (1984), 57–62.
8. N. Hitchin, Harmonic spinors, Adv. Math. 14 (1974), 1–55.
9. T. Kato, *Perturbation theory for linear operators*, second ed., Springer-Verlag, Berlin, 1976, Grundlehren der Mathematischen Wissenschaften, Band 132.
10. H. B. Lawson and M.-L. Michelsohn, *Spin geometry*, Princeton University Press, Princeton, 1989.
11. S. Maier, *Generic metrics and connections on spin- and spin*-manifolds*, Commun. Math. Phys. 188 (1997), 407–437.
12. F. Pfäffle, *The Dirac spectrum of Bieberbach manifolds*, J. Geom. Phys. 35 (2000), 367–385.

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