A note on scattering in deformed space with minimal length

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Abstract

We consider the elastic scattering in deformed space with minimal length. We give the basic relation for the elastic scattering in deformed space. We also investigate the partial wave method in deformed space. It is shown that the relations for the scattering amplitude and cross-section formally coincides with ordinary ones.

1 Introduction

In the recent years, a lot of works have been devoted to the quantum mechanical problems in a deformed space with minimal length. Such an interest was motivated by several independent lines of investigation as the string theory [1] and quantum gravity [2] where the existence of a finite lower bound for the possible resolution of length [3] was proposed. Kempf and collaborators showed that finite resolution of length can be obtained from the deformed commutation relations [4, 5, 6, 7]. One should note that deformed commutation relations were introduced earlier by Snyder [8].

The deformed algebra leading to the existence of a minimal length in a $D$-dimensional case takes form

$$
[X_i, P_j] = i\hbar(\delta_{ij}(1 + \beta P^2) + \beta' P_i P_j), \quad [P_i, P_j] = 0,
$$

$$
[X_i, X_j] = i\hbar\frac{(2\beta - \beta') + (2\beta + \beta')\beta P^2}{1 + \beta P^2}(P_i X_j - P_j X_i),
$$

(1)

where $\beta, \beta'$ are the deformation parameters. We assume that these quantities are nonnegative $\beta, \beta' \geq 0$. It is easy to show that minimal length equals $\hbar\sqrt{\beta + \beta'}$.

Deformed Heisenberg algebra (1) causes new complications in solving quantum mechanical problems. It is well known that algebra (1) does not have the position representation [5]. There are just a few known problems for which the energy spectra have been found exactly [9, 10, 11, 12, 13, 14].

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The several works were devoted to the hydrogen atom in the space with deformed Heisenberg algebra (1). In work [15] the partial case of deformation was considered when \(2\beta = \beta'\), the spectrum was found perturbatively. In paper [16] a general case \(2\beta \neq \beta'\) was considered, using perturbation theory the spectrum of the hydrogen atom was found but when \(l \neq 0\). For the \(s\)-states the corrections to the energy levels were found numerically. In work [17] a modified perturbation theory was proposed that allowed to obtain the analytical correction to the \(s\)-levels. In paper [18] the corrections to the energy for arbitrary \(s\)-levels of the hydrogen atom were calculated.

The scattering problem in deformed space was investigated in [19]. The elastic scattering was considered. Here we carry on the investigation of scattering problem. We develop the partial wave method and compare it with previous results.

\section{Scattering amplitude and differential cross-section}

To investigate the scattering problem it is essential to introduce the representation of operators satisfying the algebra (1). It is well known that such an algebra has the momentum representation, but we use the approximate representation fulfilling the algebra in the first order over the deformation parameters [17, 19]:

\[
\begin{align*}
X_i &= x_i + \frac{2\beta - \beta'}{4}(x_ip^2 + p^2x_i), \\
P_i &= p_i + \frac{\beta'}{2}pp^2,
\end{align*}
\] (2)

where \(p^2 = \sum_{j=1}^{3} p_j^2\) and the operators \(x_i, p_j\) satisfy the canonical commutation relation. The position representation \(x_i = x_i, \quad p_j = i\hbar \frac{\partial}{\partial x_j}\) can be taken for the ordinary Heisenberg algebra. We note that in the special case \(2\beta = \beta'\) the position operators commute in linear approximation over the deformation parameters, i.e. \([X_i, X_j] = 0\).

The Schrödinger equation with arbitrary potential \(U(R)\) in canonical variables takes the following form:

\[
\left( \frac{p^2}{2m} + \frac{\beta'p^4}{2m} + U(r, p) \right) \Psi = E\Psi.
\] (3)

We suppose that \(U(r, p) \to 0\) when \(r \to \infty\) and motion of a scattered particle at large distances form the scattering center is free. The kinetic energy of a free particle reads:

\[
E = \frac{\hbar^2 k^2}{2m}(1 + \beta^2\hbar^2k^2),
\] (4)

where \(k\) is the wave vector of an incident particle and \(P = \hbar k(1 + \beta^2\hbar^2k^2/2)\) is the momentum of the particle. We investigate the elastic scattering so we have that after the scattering \(k = k'\), where \(k'\) is the wave vector of scattered particle.

Then we can write a formal solution of equation (3) in the form:

\[
\Psi(r) = \psi_k(r) + \int G(r, r') \frac{2m}{\hbar^2} U(r', p') \Psi(r') dr',
\] (5)
where $G(r, r')$ is the Green’s function.

As was shown in paper [19] the asymptotic Green’s function takes form:

$$G(|r - r'|) = \frac{1}{4\pi|r - r'|(1 + 2\beta' \hbar^2 k^2)}e^{ik|r - r'|}. \quad (6)$$

Using this Green’s function (6) and after some simplifications we rewrite equation (5) in the form:

$$\Psi(r) = \psi_k(r) - \frac{m}{2\pi\hbar^2(1 + 2\beta' \hbar^2 k^2)} \frac{e^{ikr}}{r} \int e^{-ik'r'} U(r', p') \Psi(r') dr', \quad (7)$$

where $k' = kn$.

Equation (7) can be represented in the form:

$$\Psi(r) = e^{ikr} + \frac{e^{ikr}}{r} f, \quad (8)$$

where

$$f = -\frac{m}{2\pi\hbar^2(1 + 2\beta' \hbar^2 k^2)} \int e^{-ik'r'} U(r', p') \Psi(r') dr' \quad (9)$$

is the scattering amplitude.

As we see the first term in equation (8) corresponds to the wave function of an incident particle and the second term corresponds to the wave function of a scattered particle.

It is well known that central problem of the scattering theory is the calculation of the differential cross-section. It was shown in [19] that differential cross-section in deformed space takes the same form as in the ordinary quantum mechanics:

$$\frac{d\sigma}{d\Omega} = |f|^2. \quad (10)$$

The last equation allows to calculate the differential cross-sections for an arbitrary potential of scattering. But as we can see from expression (9) the scattering amplitude and as a consequence differential cross-section can not be calculated exactly. It necessary to use the approximate calculation. In the Born approximation we consider the scattering potential as a small perturbation and solve equation (7) by the method of successive approximation. In the first approximation we substitute the plane wave $\psi_k(r) = \exp(ikr)$ in relation (9).

As was shown in [19] the differential cross-section for the Coulomb potential $U = e^2/R$, where $R = \sqrt{\sum_i X_i^2}$ in deformed space takes the form:

$$\frac{d\sigma}{d\Omega} = \frac{me^4}{4\hbar^4 k^4 \sin^4 \frac{\theta}{2}} + \frac{me^2}{\hbar^2 k^2 \sin^2 \frac{\theta}{2}} \left( \frac{me^2}{2} (2\beta - \beta') \times \right.$$

$$\ln \left( \frac{h^2 (2\beta - \beta') k^2 \sin^2 \frac{\theta}{2}}{2} \right) + 2\gamma - 1 - \frac{1}{2 \sin^2 \frac{\theta}{4}} \right) - \beta' \frac{me^2}{\sin^2 \frac{\theta}{4}}. \quad (11)$$

The first term in (11) is the ordinary differential cross-section and other terms are caused by deformation of commutation relations. As we see the corrections to
the differential cross-section nonanalytically depend on the deformation parameters, but when the deformation parameters tend to zero the corrections also tend to zero and we return to the ordinary differential cross-section. We also note that as in ordinary quantum mechanics the scattering amplitude for the particle in the Coulomb potential is ill defined in the Born approximation, so it is necessary to calculate the scattering amplitude for the Yukawa potential $U(R) = -e^2 \exp(\lambda R)/R$ and then one should tend the parameter $\lambda$ to zero.

3 Partial wave method

When the potential of scattering is spherically symmetric then the angular momentum of scattering particle is the integral of motion, so it should to note that particles with different orbital quantum numbers are scattered independently. Then as in ordinary quantum mechanics we can represent the scattering cross-section as a sum of partial cross-sections for the fixed values of the orbital quantum numbers $l$. So we assume that we have a spherically symmetric scattering potential, and consider the scattering process on this potential. Arbitrary solution of the Shr¨odinger equation can be represented in the form:

$$\Psi(r) = \sum_{l=0}^{+\infty} A_l R_{kl}(r) P_l(\cos \vartheta), \quad (12)$$

where $R_{nl}(r)$ are the radial wave functions and $P_l(\cos \vartheta)$ are the Legendre polynomials.

Firstly consider free particles. The Shr¨odinger equation for a free particle:

$$-\frac{\hbar^2 \nabla^2}{2m} (1 - \beta' \hbar^2 \nabla^2) \psi = \frac{\hbar^2 k^2}{2m} (1 + \beta' \hbar^2 k^2) \psi, \quad (13)$$

or in equivalent form:

$$(\nabla^2 + k^2) (1 - \beta' \hbar^2 (\nabla^2 - k^2)) \psi = 0. \quad (14)$$

We represent the solution of last equation as sum of two functions $\psi_1$ and $\psi_2$ which are the solutions of the following equations:

$$(\nabla^2 + k^2) \psi_1 = 0, \quad (15)$$

$$(1 - \beta' \hbar^2 (\nabla^2 - k^2)) \psi_2 = 0. \quad (16)$$

Radial part of wave function $\psi_1$ takes the well known form:

$$R_{kl}(r) = \frac{1}{k^l} \sqrt{\frac{2}{\pi}} r^l \left( -\frac{1}{r} \frac{d}{dr} \right)^l \frac{\sin kr}{r}. \quad (17)$$

Equation (16) gives us unphysical solution and we reject it.

The asymptotic of solution (17) at large distances reads:

$$R_{kl}(r) = \sqrt{\frac{2}{\pi}} \frac{\sin (kr - l\frac{\pi}{2})}{r}. \quad (18)$$
We suppose similarly as in ordinary quantum mechanics that asymptotic of the wave function for the scattered particle is the same and differs from (18) by the phase shift only:

\[ R_{kl}(r) = \sqrt{\frac{2}{\pi}} \sin \left( \frac{kr - l\pi}{2} + \delta_l \right) r, \quad r \to \infty. \] (19)

We use decomposition (12) and instead of function \( R_{kl}(r) \) we take wave function (18) for the free particle and function (19) for the particle in the external field. Then we substitute aforementioned decompositions in (8), so we obtain:

\[ \psi_{\text{scatt}} = f \frac{e^{ikr}}{r} = \frac{1}{k r} \sum_{l=0}^{+\infty} i^l (2l + 1) P_l (\cos \theta) \times \left[ C_l \sin \left( kr - l\pi 2 + \delta_l \right) - \sin \left( kr - l\pi 2 \right) \right]. \] (20)

After simple transformations we obtain:

\[ f = i \frac{2}{2k} \sum_{l=0}^{+\infty} (2l + 1)(1 - e^{2i\delta_l}) P_l (\cos \theta). \] (21)

Last relation formally coincides with the expression for the scattering amplitude in the ordinary quantum mechanics, but we note that wave number \( k \) and phase shift \( \delta_l \) depend on the deformation parameters.

Using relation (10) and after integration over all spatial angles we have:

\[ \sigma = \frac{\pi}{k^2} \sum_{l=0}^{+\infty} (2l + 1) \sin^2 \delta_l. \] (22)

Similarly as in ordinary quantum mechanics the scattering cross-section relates to the imaginary part of the scattering amplitude in forward direction, so we have:

\[ \sigma = \frac{4\pi}{k} \text{Im} f(0). \] (23)

Partial wave method also allows to obtain the phase shifts. Using the radial functions \( \chi_{kl}(r) = r R_{kl}(r) \) and \( \overline{\chi}_{kl}(r) = r\overline{R}_{kl}(r) \) for the free and scattered particles respectively we find:

\[ \chi_{kl} \frac{\partial \overline{\chi}_{kl}}{\partial r} - \overline{\chi}_{kl} \frac{\partial \chi_{kl}}{\partial r} - \beta' \hbar^2 \left( \chi_{kl} \frac{\partial^2 \overline{\chi}_{kl}}{\partial r^2} - \overline{\chi}_{kl} \frac{\partial^2 \chi_{kl}}{\partial r^2} \right) - \frac{\partial \chi_{kl} \partial^2 \overline{\chi}_{kl}}{\partial r \partial r^2} + \frac{\partial \overline{\chi}_{kl} \partial^2 \chi_{kl}}{\partial r \partial r^2} \right) + 2\beta' \hbar^2 \frac{l(l+1)}{r^2} \times \left( \chi_{kl} \frac{\partial \overline{\chi}_{kl}}{\partial r} - \overline{\chi}_{kl} \frac{\partial \chi_{kl}}{\partial r} \right) = \int_0^r \chi_{kl}(r') U \chi_{kl}(r') \, dr'. \] (24)

It is a very complicated problem to solve the last equation, but most interesting for us is the phase shift at large distances from the scattering center, so we can use asymptotic wave functions (18) and (19). Taking into account that the last term in
the left hand side of relation (24) goes to zero at large distances from the scatterer we obtain:

$$\sin \delta_l = \frac{-1}{k(1 + 2\beta^2 k^2)} \int_0^r \sin \left(kr' - \frac{l\pi}{2}\right) U(r', p') \sin \left(kr' - \frac{l\pi}{2} + \delta_l\right) dr'. \quad (25)$$

In so far as we consider the phase shifts at large distances from the scattering center we can tend to infinity the upper bound in integral (25). In the first approximation we can put in right hand side of (25) asymptotic wave function of free particle $\chi_{kl}(r)$ instead of $\chi_{kl}(r)$, so we have:

$$\sin \delta_l = \frac{-1}{k(1 + 2\beta^2 k^2)} \int_0^{+ \infty} \sin \left(kr' - \frac{l\pi}{2}\right) U(r', p') \sin \left(kr' - \frac{l\pi}{2}\right) dr'. \quad (26)$$

Relation (26) corresponds the first Born approximation in relation (9). To obtain a more accurate estimation it is necessary to use expressions (25), but in practice enough good estimation we obtain in the Born approximation.

4 Conclusions

In this paper, the elastic scattering in deformed space with minimal length was considered. We reviewed main results obtained earlier. One should note that differential cross-section in deformed case equals to the square of module of the scattering amplitude, so we have the same expression as in the ordinary quantum mechanics. As was shown the correction to the differential cross-section for the Coulomb potential nonanalytically depends on deformation parameters but it tends to zero when the parameters go to zero. This dependence is caused by the shifted expansion for the distance operator. Then we also considered the partial wave method in deformed space. It was shown that in deformed case scattering on spherically symmetric potential is similar to the ordinary one. The particles that have different angular momentum (different orbital quantum numbers l) are scattered independently. So the expression for the scattering amplitude as sum of a partial amplitudes and the relation for cross-section formally coincide with the corresponding relations in ordinary quantum mechanics.

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References

[1] D. J. Gross and P. F. Mende, “String theory beyond the Planck scale”, Nucl. Phys. B 303, 407-454 (1988).

[2] M. Maggiore, “A generalized uncertainty principle in quantum gravity”, Phys. Lett. B 304, 65-69 (1993).
[3] E. Witten, “Reflections on the Fate of Spacetime”, Phys. Today 49, 24 (1996).
[4] A. Kempf, “Uncertainty relation in quantum mechanics with quantum group
symmetry”, J. Math. Phys. 35, 4483-4496 (1994).
[5] A. Kempf, G. Mangano and R. B. Mann, “Hilbert space representation of the
minimal length uncertainty relation”, Phys. Rev. D 52, 1108-1118 (1995).
[6] A. Kempf, “On quantum field theory with nonzero minimal uncertainties in
positions and momenta”, J. Math. Phys. 38, 1347-1372 (1997).
[7] A. Kempf, “Non-pointlike particles in harmonic oscillators”, J. Phys. A 30,
2093-2101 (1997).
[8] H. S. Snyder, “Quantized Space-Time”, Phys. Rev. 71, 38-41 (1947).
[9] C. Quesne and V. M. Tkachuk, “Harmonic oscillator with nonzero minimal
uncertainties in both position and momentum in a SUSYQM framework”, J.
Phys. A 36, 10373-10389 (2003).
[10] C. Quesne and V. M. Tkachuk, “More on a SUSYQM approach to the harmonic
oscillator with nonzero minimal uncertainties in position and/or momentum”,
J. Phys. A 37, 10095-10113 (2004).
[11] L. N. Chang, D. Minic, N. Okamura and T. Takeuchi, “Exact solution of the
harmonic oscillator in arbitrary dimensions with minimal length uncertainty
relations”, Phys. Rev. D 65, 125027 (8 pages) (2002).
[12] I. Dadić, L. Jonke and S. Meljanac, “Harmonic oscillator with minimal length
uncertainty relations and ladder operators”, Phys. Rev. D 67, 087701 (4 pages)
(2003).
[13] C. Quesne and V. M. Tkachuk, “Dirac oscillator with nonzero minimal uncer-
tainty in position”, J. Phys. A 38, 1747-1765 (2005).
[14] T. V. Fityo, I. O. Vakarchuk and V. M. Tkachuk, “One-dimensional Coulomb-
like problem in deformed space with minimal length”, J. Phys. A 39, 2143-2149
(2006).
[15] F. Brau, “Minimal length uncertainty relation and the hydrogen atom”, J.
Phys. A 32, 7691-7696 (1999).
[16] S. Benczik, L. N. Chang, D. Minic and T. Takeuchi, “Hydrogen-atom spectrum
under a minimal-length hypothesis”, Phys. Rev. A 72, 012104 (4 pages) (2005).
[17] M. M. Stetsko and V. M. Tkachuk, “Perturbation hydrogen-atom spectrum
in deformed space with minimal length”, Phys. Rev. A 74, 012101 (5 pages)
(2006).
[18] M. M. Stetsko, “ Corrections to the ns levels of the hydrogen atom in deformed
space with minimal length”, Phys. Rev. A 74, 062105 (5 pages) (2006); M.
M. Stetsko, “Erratum: Corrections to the ns levels of the hydrogen atom in
deformed space with minimal length [Phys. Rev. A 74, 062105 (2006)]”, Phys.
Rev. A 78, 029907 (1 page) (2008).
[19] M. M. Stetsko and V. M. Tkachuk, “Scattering problem in deformed space with
minimal length ”, Phys. Rev. A 76, 012707, (7 pages) (2007).