Topological barriers for locally homeomorphic quasiregular mappings in 3-space

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Abstract

We construct a new type of locally homeomorphic quasiregular mappings in the 3-sphere and discuss their relation to the M.A.Lavrentiev problem, the Zorich map with an essential singularity at infinity and a quasiregular analogue of domains of holomorphy in complex analysis. The construction of such mappings comes from our construction of non-trivial compact 4-dimensional cobordisms $M$ with symmetric boundary components and whose interiors have complete 4-dimensional real hyperbolic structures. Such locally homeomorphic quasiregular mappings are defined in the 3-sphere $S^3$ as mappings equivariant with the standard conformal action of uniform hyperbolic lattices $\Gamma \subset \text{Isom} H^3$ in the unit 3-ball and its complement in $S^3$ and with its discrete representation $G = \rho(\Gamma) \subset \text{Isom} H^4$. Here $G$ is the fundamental group of our non-trivial hyperbolic 4-cobordism $M = H^4/G$ and the kernel of the homomorphism $\rho : \Gamma \to G$ is a free group $F_3$ on three generators.

1 Introduction

Liouville’s rigidity of spatial conformal geometry shows that conformal mappings in domains in $S^n = R^n \cup \{\infty\}, n \geq 3$ are restrictions of Möbius transformations. However this rigidity no longer persists in quasiconformal geometry intensively studied since 1930s after its introduction by H.Grötzsch [15] and M.A.Lavrentiev [16]. First assertions reflecting spatial specifics in this quasiconformal geometry were made by M.A.Lavrentiev [17], on removability of some singularities of quasiconformal mappings and on locally homeomorphic mappings in $R^3$. V.A.Zorich’s 1967 solution [26] of the last Lavrentiev’s problem shows that locally homeomorphic quasiregular mappings of $R^n, n \neq 3$ into itself are homeomorphisms of $R^n$, and thus quasiconformal mappings.

In addition to his proof of Lavrentiev’s problem Zorich gave an example of a quasiregular mapping $R^3 \to R^3$ omitting the origin and having an essential singularity at infinity. This so-called Zorich map is a spatial analogue of the exponential function in $C$ and is based on P.P.Belinskii’s construction of a quasiconformal mapping of a half space $R^3_+$ onto a round solid cylinder. Due to the previous Zorich theorem, the branching of the map (along parallel lines orthogonally intersecting the boundary plane at integer points) cannot be avoided. In a general sense all quasiregular mappings topologically have a
branched covering type. Namely by Reshetnyak’s theorem, quasiregular mappings are (generalized) branched covers, that is, discrete and open mappings and hence local homeomorphisms modulo an exceptional set of (topological) codimension at least two. Their intensive study, especially after the mentioned Zorich results and conjectures in [26], resulted in a rich theory of quasiregular mappings which is a natural and beautiful generalization of the geometric aspects of the theory of holomorphic functions in the plane to higher dimensions. It is covered by several papers and a number of monographs - see [9]-[11], [18], [24], [14], [19], [25], and most recently by a fundamental paper on sharpness of Rickman’s Picard theorem in all dimensions [13].

Here we address two sides of the mentioned Lavrentiev-Zorich assertions: on locally homeomorphic spatial quasiregular mappings defined in the (almost) whole sphere $S^3$ and their essential singularities. Despite a relative rigidity of quasiregular mappings without branching, in Theorem 4.1 we present a new (flexible) way for constructions of such locally homeomorphic quasiregular mappings $S^3 \to S^3$ defined in the whole sphere $S^3$ except a dense Cantor subset in $S^2 \subset S^3$ (or in a quasi-sphere $S^2_q \subset S^3$). Moreover this exceptional dense Cantor subset in a quasi-sphere $S^2_q \subset S^3$ creates a barrier for our quasiregular mapping (of a topological nature) since it consists of essential singularities of our mapping allowing no continuous extension of it to any neighbourhood of an arbitrary point $x \in S^2_q$.

The construction of such quasiregular mappings in $S^3$ having $S^2 \subset S^3$ as their barrier is based on our construction of non-trivial compact 4-dimensional cobordisms $M^4$ with symmetric boundary components. The interiors of these 4-cobordisms have complete 4-dimensional real hyperbolic structures and universally covered by the real hyperbolic space $H^4$, while the boundary components of $M^4$ have (symmetric) 3-dimensional conformally flat structures obtained by deformations of the same hyperbolic 3-manifold whose fundamental group $\Gamma$ is a uniform lattice in $\text{Isom} H^3$. Such conformal deformations of hyperbolic manifolds are well understood after their discovery in [3], see [4]. Nevertheless till recently such "symmetric" hyperbolic 4-cobordisms with described properties were unknown despite our well known constructions of non-trivial hyperbolic homology 4-cobordisms with very assymmetric boundary components - see [8] and [4]-[6]. In [7] we presented a method of constructing such non-trivial "symmetric" hyperbolic 4-cobordisms $M^4 = H^4/G$ whose fundamental groups $\pi_1(M^4)$ act discretely in the hyperbolic 4-space $H^4$ by isometries, $\pi_1(M^4) \cong G \subset \text{Isom} H^4$, and can be obtained from the hyperbolic 3-lattice $\Gamma \subset \text{Isom} H^3$ by a homomorphism $\rho : \Gamma \to G \subset \text{Isom} H^4$ with non-trivial kernel (in our construction such kernel of $\rho$ is a free subgroup $F_3 \subset \Gamma$ on three generators). In Section 2 we present all necessary details of our construction of such "symmetric" hyperbolic 4-cobordisms and used discrete groups (Theorem 2.2 and Proposition 2.4). By using such "symmetric" hyperbolic 4-cobordisms our locally homeomorphic quasiregular mappings $F$ are defined in the 3-sphere $S^3 = \mathbb{R}^3$ as mappings equivariant with the standard conformal action of uniform hyperbolic lattices $\Gamma \subset \text{Isom} H^3$ in the unit 3-ball $B^3(0, 1) = \{ x \in \mathbb{R}^3 : |x| < 1 \}$ and in its complement in $S^3$ and with the discrete representation $G = \rho(\Gamma) \subset \text{Isom} H^4$. In other words such $\Gamma$-equivariance of our quasiregular mappings $F$ can be described as $F(\Gamma(x)) = \rho(\Gamma)(F(x)) = G(F(x)), x \in S^3$. Another essential element of our construction is a direct building in Section 3 of the so called bending quasiconformal homeomorphisms between polyhedra which preserve combinatorial structure of polyhedra and their dihedral angles.
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2 Non-trivial ”symmetric” hyperbolic 4-cobordisms

Since the construction of the fundamental group \( \pi_1(M^4) \cong G \subset \text{Isom}H^4 \) of a non-trivial ”symmetric” hyperbolic 4-cobordisms \( M^4 = H^4/G \) acting discretely in the hyperbolic 4-space \( H^4 \) is very essential for our construction of a locally homeomorphic quasiregular mapping \( F : S^3 \to S^3 \) having a quasi-sphere \( S^2_\rho \subset S^3 \) as a barrier, we start with a detailed construction of such discrete group \( G \subset \text{Isom}H^4 \) and the corresponding discrete representation \( \rho : \Gamma \to G \) of a uniform hyperbolic lattice \( \Gamma \subset \text{Isom}H^3 \).

These discrete groups \( G \) and \( \Gamma \) negatively answer a conjecture: If one had a hyperbolic 4-cobordism \( M^4 \) whose boundary components \( N_1 \) and \( N_2 \) are highly (topologically and geometrically) symmetric to each other it would be in fact an h-cobordism, possibly not trivial, i.e. not homeomorphic to the product of \( N_1 \) and the segment \([0, 1] \).

Namely the boundary components \( N_1 \) and \( N_2 \) of \( M^4 = M(G) = \{ H^4 \cup \Omega(G) \} \) are covered by the discontinuity set \( \Omega(G) \subset S^3 \) of \( G \) with two connected components \( \Omega_1 \) and \( \Omega_2 \), where the conformal action of \( G = \rho(\Gamma) \) is symmetric and has contractible fundamental polyhedra \( P_1 \) and \( P_2 \) of the same combinatorial type allowing to realize them as a compact polyhedron \( P_0 \) in the hyperbolic 3-space, i.e. the dihedral angle data of these polyhedra satisfy the Andreev’s conditions \([1]\). Nevertheless this geometric symmetry of boundary components of our hyperbolic 4-cobordism \( M(G) \) is not enough to ensure that the group \( G = \pi_1(M^4) \) is quasi-Fuchsian and our 4-cobordism \( M \) is trivial.

Here a Fuchsian group \( \Gamma \subset \text{Isom}H^3 \subset \text{Isom}H^4 \) conformally acts in the 3-sphere \( S^3 = \partial H^4 \) and preserves a round ball \( B^3 \subset S^3 \) where it acts as a cocompact discrete group of isometries of \( H^3 \). Due to the Sullivan structural stability (see Sullivan \([22]\) for \( n = 2 \) and Apanasov \([4]\), Theorem 7.2)), the space of quasi-Fuchsian representations of a hyperbolic lattice \( \Gamma \subset \text{Isom}H^4 \) is an open connected component of the Teichmüller space of \( \Gamma \) or the variety of conjugacy classes of discrete representations \( \rho : \Gamma \to \text{Isom}H^4 \).

Points in this (quasi-Fuchsian) component correspond to trivial hyperbolic 4-cobordisms \( M(G) \) where the discontinuity set \( \Omega(G) = \Omega_1 \cup \Omega_2 \subset S^3 = \partial H^4 \) is the union of two topological 3-balls \( \Omega_i \), \( i = 1, 2 \), and \( M(G) \) is homeomorphic to the product of \( N_1 \) and the closed interval \([0, 1] \).

To simplify the situation we may consider the hyperbolic 4-cobordisms \( M(\rho(\Gamma)) \) corresponding to uniform hyperbolic lattices \( \Gamma \subset \text{Isom}H^3 \) generated by reflections (or cobordisms related to their finite index subgroups). Natural inclusions of these lattices into \( \text{Isom}H^4 \) act at infinity \( \partial H^4 = S^3 \) as Fuchsian groups \( \Gamma \subset \text{Möb}(3) \) preserving a round ball in the 3-sphere \( S^3 \). In this case the above conjecture can be reformulated as the following question on the Möbius action of corresponding reflection groups \( G = \rho(\Gamma) \subset \text{Isom}H^4 \) on the 3-sphere \( S^3 = \partial H^4 \):

**Question 2.1.** Is any discrete Möbius group \( G \) generated by finitely many reflections with respect to spheres \( S^2 \subset S^3 \) and whose fundamental polyhedron \( P(G) \subset S^3 \) is the union of two contractible polyhedra \( P_1, P_2 \subset S^3 \) of the same combinatorial type (with equal corresponding dihedral angles) quasiconformally conjugate in the sphere \( S^3 \) to some Fuchsian group preserving a round ball \( B^3 \subset S^3 \)?
Our construction of the mentioned discrete groups $\Gamma$ and $G = \rho(\Gamma)$ gives a negative answer to this question and proves the following (see Apanasov [7]):

**Theorem 2.2.** There exists a discrete Möbius group $G \subset \text{Möb}(3)$ on the 3-sphere $S^3$ generated by finitely many reflections such that:

1. Its discontinuity set $\Omega(G)$ is the union of two invariant components $\Omega_1, \Omega_2$;

2. Its fundamental polyhedron $P \subset S^3$ has two contractible components $P_i \subset \Omega_i$, $i = 1, 2$, having the same combinatorial type (of a compact hyperbolic polyhedron $P_0 \subset H^3$);

3. For the uniform hyperbolic lattice $\Gamma \subset \text{Isom} H^3$ generated by reflections in sides of the hyperbolic polyhedron $P_0 \subset H^3$ and acting on the sphere $S^3 = \partial H^4$ as a discrete Fuchsian group $\rho(\Gamma) \subset \text{Isom} H^4 = \text{Möb}(3)$ preserving a round ball $B^3$ (where $\rho : \text{Isom} H^3 \subset \text{Isom} H^4$ is the natural inclusion), the group $G$ is its image under a homomorphism $\rho : \Gamma \to G$ but it is not quasiconformally (topologically) conjugate in $S^3$ to $\rho(\Gamma)$.

**Proof:** For our construction of the desired Möbius group $G \subset \text{Möb}(3)$ generated by reflections it is enough to define its finite collection $\Sigma$ of reflecting 2-spheres $S_i \subset S^3$, $1 \leq i \leq N$. As the first four spheres we consider mutually orthogonal spheres centered at the vertices of a regular tetrahedron in $\mathbb{R}^3$. Let $B = \bigcup_{1 \leq i \leq 4} B_i$ be the union of the closed balls bounded by these four spheres, and let $\partial B$ be its boundary (a topological 2-sphere) having four vertices which are the intersection points of four triples of our spheres. Applying a Möbius transformation in $S^3 \cong \mathbb{R}^3 \cup \{\infty\}$, we may assume that the first three spheres $S_1, S_2$ and $S_3$ correspond to the coordinate planes $\{x \in \mathbb{R}^3 : x_i = 0\}$, and $S_4 = S^3(0, R)$ is the round sphere of some radius $R > 0$ centered at the origin. The value of the radius $R$ will be determined later.

On the topological 2-sphere $\partial B$ with four vertices we consider a simple closed loop $\alpha \subset \partial B$ which does not contain any of our vertices and which symmetrically separates two pairs of these vertices from each other as the white loop does on the tennis ball shown in Figure 1. This loop $\alpha$ can be considered as the boundary of a topological 2-disc $\sigma$ embedded in the complement $D = S^3 \setminus B$ of our four balls. Our geometric construction needs a detailed description of such a 2-disc $\sigma$ and its boundary loop $\alpha = \partial \sigma$ obtained as it is shown in Figure 3.

The desired disc $\sigma \subset D = S^3 \setminus B$ can be described as the boundary in the domain $D$ of the union of a finite chain of adjacent blocks $Q_i$ (regular cubes) with disjoint interiors whose centers lie on the coordinate planes $S_1$ and $S_2$ and whose sides are parallel to the coordinate planes. This chain starts from the unit cube whose center lies in the second coordinate axis, in $e_2 \cdot \mathbb{R}_+ \subset S_1 \cap S_3$. Then our chain goes up through small adjacent cubes centered in the coordinate plane $S_1$, at some point changes its direction to the horizontal one toward the third coordinate axis, where it turns its horizontal direction by a right angle again (along the coordinate plane $S_2$), goes toward the vertical line passing through the second unit cube centered in $e_1 \cdot \mathbb{R}_+ \subset S_2 \cap S_3$, then goes down along that vertical line and finally ends at that second unit cube, see Figure 3. We will define the size of small cubes $Q_i$ in our block chain and the distance of the centers of two unit cubes to the origin in the next step of our construction.
Figure 1: White loop separating two pairs of vertices on a tennis ball.

Figure 2: Big and small cube sizes and ball covering
Figure 3: Configuration of blocks and the loop $\alpha \subset \partial B$. 
Let us consider one of our cubes $Q_i$, i.e. a block of our chain, and let $f$ be its square side having a nontrivial intersection with our 2-disc $\sigma \subset D$. For that side $f$ we consider spheres $S_j$ centered at its vertices and having a radius such that each two spheres centered at the ends of an edge of $f$ intersect each other with angle $\pi/3$. In particular, for the unit cubes such spheres have radius $\sqrt{3}/3$. From such defined spheres we select those spheres that have centers in our domain $D$ and then include them in the collection $\Sigma$ of reflecting spheres. Now we define the distance of the centers of our big (unit) cubes to the origin. It is determined by the condition that the sphere $S_4 = S^2(0, R)$ is orthogonal to the sphere $S_j \in \Sigma$ centered at the vertex of such a cube closest to the origin.

As in Figure 2 let $f$ be a square side of one of our cubic blocks $Q_i$ having a nontrivial intersection $f_\sigma = f \cap \sigma$ with our 2-disc $\sigma \subset D$. We consider a ring of four spheres $S_i$ whose centers are interior points of $f$ which lie outside of the four previously defined spheres $S_j$ centered at vertices of $f$ and such that each sphere $S_i$ intersects two adjacent spheres $S_{i-1}$ and $S_{i+1}$ (we numerate spheres $S_i \mod 4$) with angle $\pi/3$. In addition these spheres $S_i$ are orthogonal to the previously defined ring of bigger spheres $S_j$, see Figure 2. From such defined spheres $S_i$ we select those spheres that have nontrivial intersections with our domain $D$ outside the previously defined spheres $S_j$, and then include them in the collection $\Sigma$ of reflecting spheres. If our side $f$ is not the top side of one of the two unit cubes we add another sphere $S_k \in \Sigma$. It is centered at the center of this side $f$ and is orthogonal to the four previously defined spheres $S_i$ with centers in $f$, see Figure 2.

Now let $f$ be the top side of one of the two unit cubes of our chain. Then, as before, we consider another ring of four spheres $S_k$. Their centers are interior points of $f$, lie outside of the four previously defined spheres $S_i$, closer to the center of $f$ and such that each sphere $S_k$ intersects two adjacent spheres $S_{k-1}$ and $S_{k+1}$ (we numerate spheres $S_k \mod 4$) with angle $\pi/3$. In addition these new four spheres $S_k$ are orthogonal to the previously defined ring of bigger spheres $S_i$, see Figure 2. We note that the centers of these new spheres $S_k$ are vertices of a small square $f_s \subset f$ whose edges are parallel to the edges of $f$, see Figure 2. We set this square $f_s$ as the bottom side of the small cubic box adjacent to the unit one. This finishes our definition of the family of twelve round spheres whose interiors cover the square ring $f_\sigma \setminus f_s$ on the top side of one of the two unit cubes in our cube chain and tells us which two spheres among the four new defined spheres $S_k$ were already included in the collection $\Sigma$ of reflecting spheres (as the spheres $S_j \in \Sigma$ associated to small cubes in the first step).

This also defines the size of small cubes in our block chain. Now we can vary the remaining free parameter $R$ (which is the radius of the sphere $S_4 \in \Sigma$) in order to make two horizontal rows of small blocks with centers in $S_1$ and $S_2$, correspondingly, to share a common cubic box centered at a point in $e_3 : \mathbb{R}_+ \subset S_1 \cap S_2$, see Figure 3.

The constructed collection $\Sigma$ of reflecting spheres $S_j$ bounding round balls $B_j$, $1 \leq j \leq N$, has the following properties:

1. The closure of our 2-disc $\sigma \subset D$ is covered by balls $B_j$: $\bar{\sigma} \subset \operatorname{int} \bigcup_{j \geq 5} B_j$;
2. Any two spheres $S_j, S_j' \in \Sigma$ either are disjoint or intersect with angle $\pi/2$ or $\pi/3$;
3. The complement of all balls, $S^3 \setminus \bigcup_{j=1}^N B_j$ is the union of two contractible polyhedra $P_1$ and $P_2$ of the same combinatorial type.
Therefore we can use the constructed collection $\Sigma$ of reflecting spheres $S_i$ to define a discrete group $G = G_\Sigma \subset \text{M"ob}(3)$ generated by $N$ reflections in spheres $S_j \in \Sigma$. The fundamental polyhedron $P = P_1 \cup P_2 \subset S^3$ for the action of this discrete reflection group $G$ on the sphere $S^3$ is the union of two connected polyhedra $P_1$ and $P_2$ which are disjoint topological balls. So the discontinuity set $\Omega(G) \subset S^3$ of $G$ consists of two invariant connected components $\Omega_1$ and $\Omega_2$:

$$\Omega(G) = \bigcup_{g \in G} g(\bar{P}) = \Omega_1 \cup \Omega_2, \quad \Omega_i = \bigcup_{g \in G} g(\bar{P}_i), \quad i = 1, 2.$$  \hspace{1cm} (2.1)

**Lemma 2.3.** The splitting of the discontinuity set $\Omega \subset S^3$ of our discrete reflection group $G = G_\Sigma \subset \text{M"ob}(3)$ into $G$-invariant components $\Omega_1$ and $\Omega_2$ in \cite{APA2} defines a Heegaard splitting of the 3-sphere $S^3$ of infinite genus with ergodic word hyperbolic group $G$ action on the separating boundary $\Lambda(G)$ which is quasi-self-similar in the sense of Sullivan.

**Proof:** In fact, despite the contractibility of polyhedra $P_1$ and $P_2$ both components $\Omega_1$ and $\Omega_2$ are not simply connected and even are mutually linked. To show this it is enough to see that the union of the bounded polyhedron $P_1$ (inside of our block chain) and its image $g_3(P_1)$ under the reflection $g_3$ with respect to the plane $S_3$ has a non-contractible loop $\beta_1$ which represents a non-trivial element of the fundamental group $\pi_1(\Omega_1)$. This loop is linked with the loop $\beta_2$ in the unbounded component $\Omega_2$ which goes around $P_1 \cup g_3(P_1)$ and represents a non-trivial element of the fundamental group $\pi_1(\Omega_2)$.

This fact is illustrated by Figure 4 where one can see a handlebody obtained from our initial chain of building blocks in Figure 3 by the union of the images of this block chain by first generating reflections in the group $G$ (in $S_1, S_2$ and $S_3$). Then our non-contractible loop $\beta_1 \subset \Omega_1$ lies inside of this handlebody in Figure 4 and is linked with the second loop $\beta_2 \subset \Omega_2$ which goes around one of the handles of the handlebody in Figure 4. The resulting handlebodies $\Omega_1$ and $\Omega_2$ are the unions of the corresponding images $g(\bar{P}_i)$ of the polyhedra $P_1$ and $P_2$, so they have infinitely many mutually linked handles. Their fundamental groups $\pi_1(\Omega_1)$ and $\pi_1(\Omega_2)$ have infinitely many generators, and some of those generators correspond to the group $G$-images of the linked loops $\beta_1 \subset \Omega_1$ and $\beta_2 \subset \Omega_2$. The limit set $\Lambda(G)$ is the common boundary of $\Omega_1$ and $\Omega_2$. Since the group $G \subset \text{M"ob}(3)$ acts on the hyperbolic 4-space $H^4$, $\partial H^4 = S^3$, as a convex cocompact isometry group, its action on the limit set $\Lambda(G)$ is ergodic. Moreover, the common boundary $\Lambda(G)$ of the handlebodies $\Omega_1$ and $\Omega_2$ is quasi-self-similar in the sense of D.Sullivan, that is each arbitrary small piece of $\Lambda(G)$ can be expanded to a standard size and then mapped into $\Lambda(G)$ by a $K$-quasi-isometry. More precisely, there are uniform constants $K$ and $r_0$ such that, for any $x \in \Lambda(G)$ and for any ball $B(x,r)$ centered at $x$ with radius $r$, $0 < r < r_0$, there exists a $K$-quasi-isometric bijection $f$,

$$f : \Lambda(G) \cap B(x,r) \overset{r}{\rightarrow} \Lambda(G)$$  \hspace{1cm} (2.2)

which distorts distances in the interval between $1/K$ and $K$. In other words, the distortion of an unlimited “microscoping”\cite{APA2} of the limit set $\Lambda(G)$ can be uniformly bounded, see Corollary 2.66 in Apanasov \cite{APA2}.

To finish the proof of Theorem 2.2 we notice that the combinatorial type (with magnitudes of dihedral angles) of the bounded component $P_1$ of the fundamental polyhedron
Figure 4: Handlebody obtained by the first 3 reflections of the cub chain.
Let \( P \subset S^3 \) coincide with the combinatorial type of its unbounded component \( P_2 \). Applying Andreev’s theorem on 3-dimensional hyperbolic polyhedra \( \square \), one can see that there exists a compact hyperbolic polyhedron \( P_0 \subset H^3 \) of the same combinatorial type with the same dihedral angles (\( \pi/2 \) or \( \pi/2 \)). So one can consider a uniform hyperbolic lattice \( \Gamma \subset \text{Isom} \, H^3 \) generated by reflections in sides of the hyperbolic polyhedron \( P_0 \). This hyperbolic lattice \( \Gamma \) acts in the sphere \( S^3 \) as a discrete co-compact Fuchsian group \( i(\Gamma) \subset \text{Isom} \, H^3 = \text{Möb}(3) \) (i.e. as the group \( i(\Gamma) \subset \text{Isom} \, H^4 \) where \( i: \text{Isom} \, H^3 \subset \text{Isom} \, H^4 \) is the natural inclusion) preserving a round ball \( B^3 \) and having its boundary sphere \( S^2 = \partial B^3 \) as the limit set. Obviously there is no self-homeomorphism of the sphere \( S^3 \) conjugating the action of the groups \( G \) and \( i(\Gamma) \) because the limit set \( \Lambda(G) \) is not a topological 2-sphere. So the constructed group \( G \) is not a quasi-Fuchsian group.

One can construct a natural homomorphism \( \rho: \Gamma \rightarrow G, \rho \in R_3(\gamma) \), between these two Gromov hyperbolic groups \( G \subset \text{Isom} \, H^4 \) and \( \Gamma \subset \text{Isom} \, H^3 \) defined by the correspondence between sides of the hyperbolic polyhedron \( P_0 \subset H^3 \) and reflecting spheres \( S_i \) in the collection \( \Sigma \) bounding the fundamental polyhedra \( P_i \) and \( P_2 \).

**Proposition 2.4.** The homomorphism \( \rho \in R_3(\gamma), \rho: \Gamma \rightarrow G \), in Theorem \( \square \) is not an isomorphism. Its kernel \( \ker(\rho) = \rho^{-1}(e_G) \) is a free rank 3 subgroup \( F_3 \subset \Gamma \).

**Proof:** The homomorphism \( \rho \) cannot be an isomorphism since its kernel \( \rho^{-1}(e_G) \) is not trivial, \( \rho^{-1}(e_G) \neq \{e_G\} \). In fact this kernel is a free rank 3 group \( F_3 = \langle x, y, z \rangle \) generated by three hyperbolic translations \( x, y, z \in \Gamma \). The first hyperbolic translation \( x = a_1 b_1 \) in \( H^3 \) is the composition of reflections \( a_1 \) and \( b_1 \) in two disjoint hyperbolic planes \( H_1, H_1' \subset H^3 \) containing those two 2-dimensional faces of the hyperbolic polyhedron \( P_0 \) that correspond to two sides of the polyhedron \( P_2 \) which are disjoint parts of the sphere \( S_4 \). The second hyperbolic translation \( y = a_2 b_2 \) in \( H^3 \) is the composition of reflections \( a_2 \) and \( b_2 \) in two disjoint hyperbolic planes \( H_2, H_2' \subset H^3 \) containing those two 2-dimensional faces of the hyperbolic polyhedron \( P_0 \) that correspond to two sides of the polyhedron \( P_3 \) which are disjoint parts of the sphere \( S_3 \). And the third generator \( z \) is a hyperbolic translation in \( H^3 \) which is \( a_1 \)-conjugate of \( y, z = a_1 y a_1 \). The fact that these hyperbolic 2-planes \( H_1 \) and \( H_1' \) (correspondingly, the 2-planes \( H_2 \) and \( H_2' \)) are disjoint follows from Andreev’s result \( \square \) on sharp angled hyperbolic polyhedra. Restricting our homomorphism \( \rho \) to the subgroup of \( \Gamma \) generated by reflections \( a_1, a_2, b_1, b_2 \in \Gamma \), we can formulate its properties as the following statement in combinatorial group theory:

**Lemma 2.5.** Let \( A = \langle a_1, a_2 \mid a_1^2, a_2^2, (a_1 a_2)^2 \rangle \cong B = \langle b_1, b_2 \mid b_1^2, b_2^2, (b_1 b_2)^2 \rangle \cong C = \langle c_1, c_2 \mid c_1^2, c_2^2, (c_1 c_2)^2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \), and let \( \varphi: A \ast B \rightarrow C \) be a homomorphism of the free product \( A \ast B \) into \( C \) such that \( \varphi(a_1) = \varphi(b_1) = c_1 \) and \( \varphi(a_2) = \varphi(b_2) = c_2 \). Then the kernel \( \ker(\varphi) = \varphi^{-1}(e_C) \) of \( \varphi \) is a free rank 3 subgroup \( F_3 \subset A \ast B \) generated by elements \( x = a_1 b_1, y = a_2 b_2 \) and \( z = a_1 a_2 b_2 a_1 = a_1 y a_1 \).

**Proof:** It is obvious that \( K_0 = \langle x, y, z \rangle \) is a subgroup in \( \ker(\varphi) \). From the definition of \( \varphi \) on the generators of \( A \ast B \) it is also clear that \( \ker(\varphi) \) is \( \langle (x, y) \rangle \), the normal closure of elements \( x = a_1 b_1 \) and \( y = a_2 b_2 \). Therefore in order to prove that \( K_0 = \ker(\varphi) \) it is enough to show that \( K_0 \) contains all elements which are conjugate in \( A \ast B \) to \( x \) and \( y \).

As any element \( w \in A \ast B \) is a product of generators of \( A \ast B \), the conjugation by any such \( w \) may be regarded as a consequent conjugation by the generators of \( A \ast B \). So, it is enough to prove that \( K_0 \) contains any element conjugate to \( x, y, z \) by \( a_1, a_2, b_1, b_2 \).
In fact it is easy to verify that
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\begin{align*}
 a_1^{-1} x a_1 &= x^{-1}, & a_1^{-1} y a_1 &= z, & a_1^{-1} z a_1 &= y, \\
 a_2^{-1} x a_2 &= z y^{-1}, & a_2^{-1} y a_2 &= y^{-1}, & a_2^{-1} z a_2 &= z^{-1}, \\
 b_1^{-1} x b_1 &= x^{-1}, & b_1^{-1} y b_1 &= x^{-1} z x, & b_1^{-1} z b_1 &= x^{-1} y x, \\
 b_2^{-1} x b_2 &= y^{-1} z x, & b_2^{-1} y b_2 &= y^{-1}, & b_2^{-1} z b_2 &= y^{-1} z^{-1} y.
\end{align*}
\]

Now we should show that the elements \(x, y, z\) form a free basis for \(K_0\). Let us check that any reduced word \(w(x, y, z)\) represents a nontrivial element of \(A \ast B\). By a “letter” we mean any of symbols \(x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\). We claim that for the element \(g\) represented by a reduced word \(w(x, y, z)\) the following holds: the last syllable of \(g\) written in the normal form is always equal to the last syllable of the last letter of the word \(w\) (and so is nontrivial) except for the case when the last letter is \(x\) and the preceding letter is \(z\) (cf. [20], §4.1). In this case the last syllable equals \(b_1 b_2\) (and so it is also nontrivial). Besides that, if the last letter of \(w\) is \(z\) then the two last syllables of \(w\) are the ones of \(z\).

This statement can be easily verified by induction on the length of \(w(x, y, z)\). So it obviously implies nontriviality of the element \(g\) represented by \(w(x, y, z)\).

Now the claim that \(\ker(\rho) \subset \Gamma\) is a free rank 3 subgroup \(F_3 = \langle x, y, z \rangle\) generated by our hyperbolic translations \(x, y, z \in \text{Isom} \mathbb{H}^3\) follows directly from Lemma 2.5 which completes the proof of Proposition 2.4.

Therefore the configuration of reflecting spheres \(S_j \subset \Sigma\) shows that one can deform our discrete co-compact Fuchsian group \(i(\Gamma) \subset \text{Isom} \mathbb{H}^4 = \text{Möb}(3)\) preserving a round
In this section we should construct quasiconformal homeomorphisms $\phi_3$ bending homeomorphisms between polyhedra dihedral angles unchanged. $\phi$ and $P \subset 2$-spheres of the Fuchsian group $i(\Gamma)$ corresponding to the pairs of hyperbolic planes $H_1, H'_1 \subset H^3$ and $H_2, H'_2 \subset H^3$ into the reflecting spheres $S_4$ and $S_3$ while keeping all dihedral angles unchanged.

3 Bending homeomorphisms between polyhedra

In this section we should construct quasiconformal homeomorphisms $\phi_1 : P_1 \to P_0$ and $\phi_2 : P_2 \to \widehat{P}_0$ between components $P_i, i = 1, 2$, of the fundamental polyhedron $P \subset \Omega(G) \subset S^3$ for the group $G$ and the corresponding components $P_i$ and $\widehat{P}_0$ of the fundamental polyhedron for conformal action in $S^3$ of our hyperbolic lattice $\Gamma \subset Isom H^3$ from Theorem 2.2. These mappings $\phi_i$ are compositions of finitely many elementary "bending homeomorphisms" and should map faces to faces and preserve the combinatorial structure of the polyhedra and their corresponding dihedral angles.

First we observe that to each cube $Q_j, 1 \leq j \leq m$, used in the previous section for our construction of the group $G$ (see Figure 3 and Figure 4), we may associate a round ball $B_j$ centered at the center of the cube $Q_j$ and such that its boundary sphere is orthogonal to the reflection spheres $S_i$ from our generating family $\Sigma$ whose centers are at vertices of the cube $Q_j$. In particular for the unit cubes $Q_1$ and $Q_m$, the reflection spheres $S_i$ centered at their vertices have radius $\sqrt{3}/3$, so the balls $B_1$ and $B_m$ (whose boundary spheres are orthogonal to those corresponding reflection spheres $S_i$) should have radius $\sqrt{5}/12$. Also we add another extra ball $B^3(0, R)$ (which we consider as two balls $B_0$ and $B_{m+1}$) whose boundary is the reflection sphere $S^2(0, R) = S_4 \in \Sigma$ centered at the origin and orthogonal to the closest reflection spheres $S_i$ centered at vertices of two unit cubes $Q_1$ and $Q_m$. Our different enumeration of this ball will be used when we consider different faces of our fundamental polyhedron $P_1$ lying on that reflection sphere $S_4$.

Now for each cube $Q_j, 1 \leq j \leq m$, we may associate a discrete subgroup $G_j \subset G \subset \text{Möb}(3) \cong Isom H^4$ generated by reflections in the spheres $S_i \in \Sigma$ associated to that cube $Q_j$ - see our construction in Theorems 2.2. One may think about such a group $G_j$ as a result of quasiconformal bending deformations (see [4], Chapter 5) of a discrete Möbius group preserving the round ball $B_j$ associated to the cube $Q_j$ (whose center coincides with the center of the cube $Q_j$). As the first step in such deformations, let us define two quasiconformal "bending" self-homeomorphisms of $S^3$, $f_1$ and $f_{m+1}$, preserving the balls $B_1, \ldots, B_m$ and the set of their reflection spheres $S_i, i \neq 4$, and transferring $\partial B_0$ and $\partial B_{m+1}$ into 2-spheres orthogonally intersecting $\partial B_1$ and $\partial B_m$ along round circles $b_1$ and $b_{m+1}$, respectively.

To construct the bending $f_1$ ($f_{m+1}$ is similar), we may assume that the balls $B_0$ and $B_1$ are half-spaces with boundary planes $\partial B_0$ and $\partial B_1$ and such that $b_1 = \{ x \in \mathbb{R}^3 : x_1 = x_3 = 0 \}$ is their intersection line. From our construction of the group $G$, we have that the dihedral angle of the intersection $B_0 \cap B_1$ has a magnitude $\alpha$, $0 < \alpha < \pi/2$, and there exists a dihedral angle $V_1 \subset \mathbb{R}^3$ with the edge $b_1$ and magnitude $2\zeta$, where $0 < \zeta < \pi/4$ and $\alpha < \pi - 2\zeta$, such that $V_1$ contains all the reflection spheres in $\Sigma$ disjoint from $b_1$. Let us assume the natural complex structure in the orthogonal to $b_1$ plane $\mathbb{R}^2 = \{ x \in \mathbb{R}^3 : x_3 = 0 \}$. Then the quasiconformal homeomorphism $f_1 : S^3 \to S^3$ is described by its restriction to this plane $\mathbb{C} = \mathbb{R}^2$ (where $-\pi < \arg z \leq \pi$ is the principal value of the argument of $z \in \mathbb{C}$) as follows, see Figure 6.
We remark that $f_1 = \text{id}$ in $V_1$ and hence it is the identity on all reflection spheres $S_i \in \Sigma$ disjoint from $b_1 = \partial B_0 \cap \partial B_1$. Also all spheres $S_k \in \Sigma$ intersecting $b_1$ and the exterior dihedral angles of their intersections with other spheres $S_i$ are still invariant with respect to $f_1$.

In the next steps in our bending deformations, for two adjacent cubes $Q_{j-1}$ and $Q_j$, let us denote $G_{j-1,j} \subset G$ the subgroup generated by reflections with respect to the spheres $S_i \subset \Sigma$ centered at common vertices of these cubes. This subgroup preserves the round circle $b_j = b_{j-1,j} = \partial B_{j-1} \cap \partial B_j$. This shows that our group $G$ is a result of the so called "block-building construction" (see [4], Section 5.4) from the block groups $G_j$ by sequential amalgamated products:

$$G = G_1 \ast G_{2,1} \ast \cdots \ast G_{j-1,j-1} \ast G_{j-1} \ast G_{j,j+1} \ast \cdots \ast G_{m-1,m} \ast G_m$$  \hspace{1cm} (3.2)

Then the chain of these building balls $\{B_j\}, 1 \leq j \leq m$, contains the bounded polyhedron $P_1 \subset \Omega_1$, and the unbounded polyhedron $P_2 \subset \Omega_2$ is inside of the chain of the balls $\{\tilde{B}_j, 1 \leq j \leq m\}$, which are the complements in $S^3$ to the balls $B_j$.

Now for each pair of balls $B_{i-1}$ and $B_i$ with the common boundary circle $b_i = \partial B_{i-1} \cap \partial B_i, 1 \leq i \leq m$, we construct a quasi-conformal bending homeomorphism $f_i$ that transfers $B_i \cup B_{i-1}$ onto the ball $B_i$ and which is conformal in dihedral neighborhood of the spherical disks $\partial B_i \setminus B_{i-1}$ and $\partial B_{i-1} \setminus B_i$. Namely, let $B_i$ and $B_{i-1}$ be half-spaces whose boundary planes $\partial B_i$ and $\partial B_{i-1}$ contain the origin and intersect along the third coordinate axis $b_i = \{x \in \mathbb{R}^3 : x_1 = x_2 = 0\}$ at an angle $\alpha$, $0 < \alpha < \pi$, and let $\zeta$ be a fixed number such that $0 < \zeta < \pi/2$ and $0 < \alpha < \pi - 2\zeta$. Assuming the natural complex

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In the next steps in our bending deformations, for two adjacent cubes $Q_{j-1}$ and $Q_j$, let us denote $G_{j-1,j} \subset G$ the subgroup generated by reflections with respect to the spheres $S_i \subset \Sigma$ centered at common vertices of these cubes. This subgroup preserves the round circle $b_j = b_{j-1,j} = \partial B_{j-1} \cap \partial B_j$. This shows that our group $G$ is a result of the so called "block-building construction" (see [4], Section 5.4) from the block groups $G_j$ by sequential amalgamated products:

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Then the chain of these building balls $\{B_j\}, 1 \leq j \leq m$, contains the bounded polyhedron $P_1 \subset \Omega_1$, and the unbounded polyhedron $P_2 \subset \Omega_2$ is inside of the chain of the balls $\{\tilde{B}_j, 1 \leq j \leq m\}$, which are the complements in $S^3$ to the balls $B_j$.

Now for each pair of balls $B_{i-1}$ and $B_i$ with the common boundary circle $b_i = \partial B_{i-1} \cap \partial B_i, 1 \leq i \leq m$, we construct a quasi-conformal bending homeomorphism $f_i$ that transfers $B_i \cup B_{i-1}$ onto the ball $B_i$ and which is conformal in dihedral neighborhood of the spherical disks $\partial B_i \setminus B_{i-1}$ and $\partial B_{i-1} \setminus B_i$. Namely, let $B_i$ and $B_{i-1}$ be half-spaces whose boundary planes $\partial B_i$ and $\partial B_{i-1}$ contain the origin and intersect along the third coordinate axis $b_i = \{x \in \mathbb{R}^3 : x_1 = x_2 = 0\}$ at an angle $\alpha$, $0 < \alpha < \pi$, and let $\zeta$ be a fixed number such that $0 < \zeta < \pi/2$ and $0 < \alpha < \pi - 2\zeta$. Assuming the natural complex
structure in the plane $\mathbb{R}^2 = \{ x \in \mathbb{R}^3 : x_3 = 0 \}$, we define the quasi-conformal elementary bending homeomorphism $f_i$ by its restriction to the plane $C = \mathbb{R}^2$ (see Figure 7), where

$$f_i(z) = \begin{cases} 
  z & \text{if } |\arg z| \geq \pi - \zeta \\
  z \exp \left( i(\pi - \alpha) \right) & \text{if } |\pi - \alpha - \arg z| \leq \zeta \\
  z \exp \left( i(\pi - \alpha)(1 - \frac{\arg z - \zeta}{\pi - 2\zeta}) \right) & \text{if } \alpha - \pi + \zeta < \arg z < \pi - \zeta \\
  z \exp \left( i(\pi - \alpha)(1 + \frac{\zeta + \arg z}{\pi - 2\zeta}) \right) & \text{if } \zeta - \pi < \arg z < \alpha - \pi - \zeta 
\end{cases} \quad (3.3)$$

Note that in each $i$-th step, $2 \leq i \leq m$, we reduce the number of balls $B_j$ in our chain by one. The constructed quasiconformal homeomorphisms $f_i$ have the properties:

1. $f_i = \text{id}$ in a neighborhood of reflection spheres from our collection $\Sigma$ that are disjoint from the circle $b_i$ and intersect some balls $B_j$, $j \geq i$.

2. The composition $f_{m+1}f_i f_{i-1} \cdots f_2 f_1$ transfers all spheres from $\Sigma$ to spheres orthogonal to the boundary spheres of some balls $B_j$, $i \leq j \leq m$, where all intersection angles between these spheres do not change.

Finally, renormalizing our last ball $B_m$ as the unit ball $B(0,1)$, we define our desired quasiconformal homeomorphism $\phi_1 : P_1 \to P_0$ as the restriction of the composition $f_{m+1}f_m f_{m-1} \cdots f_2 f_1$ of our bending homeomorphisms $f_j$ on the fundamental polyhedron $P_1 \subset \Omega_1$. Similarly (working with the balls $\hat{B}_j$) we define the second quasiconformal homeomorphism $\phi_2 : P_2 \to \hat{P}_0$. Both mappings preserve the combinatorial structure of the polyhedra and their dihedral angles.

## 4 Locally homeomorphic quasiregular mappings

Now we can apply results of the previous Section 3 to define our quasiregular mapping $F$ from $S^3$ to $S^3$.

**Theorem 4.1.** Let the hyperbolic lattice $\Gamma \subset \text{Isom} H^3$ and its discrete representation $\rho : \Gamma \to G \subset \text{Isom} H^4$ with the kernel as a free subgroup $F_3 \subset \Gamma$ be as in Theorem
2.2 Then there is a locally homeomorphic quasiregular mapping \( F : S^3 \to S^3 \) whose all singularities lie in the unit sphere \( S^2 = \{ x \in \mathbb{R}^3 \} \), form a dense Cantor subset in \( S^2 \), and these (essential) singularities create a barrier for \( F \) in the sense that \( F \) cannot be continuously extended from either side of \( S^2 \subset S^3 \) in any neighbourhood of an arbitrary point \( x \in S^2 \).

**Proof:** First we define our quasiregular mapping \( F \) in the complement \( S^3 \setminus S^2 \) of the unit sphere, \( F : S^3 \setminus S^2 \to \Omega(G) \subset S^3 \). In fact in the previous Section 3 we have constructed quasiconformal homeomorphisms \( \phi_1^{-1} : P_0 \to P_1 \) and \( \phi_2^{-1} : \widetilde{P}_0 \to P_2 \). Here \( \widetilde{P}_0 \) is the symmetric image of \( P_0 \subset B^3(0,1) \) with respect to the reflection in the unit sphere \( S^2 \), the polyhedron \( P_0 \cup \widetilde{P}_0 \) is a fundamental polyhedron (having two connected components which are convex in the induced there hyperbolic metrics) for conformal and discontinuous action of our hyperbolic lattice \( \Gamma \subset \text{Isom} \, H^3 \) in its discontinuity set \( \Omega(\Gamma) = S^3 \setminus S^2 \). Also \( P = P_1 \cup P_2 \) is the fundamental polyhedron for conformal and discontinuous action of the discrete group \( G = \rho(\Gamma) \subset \text{Isom} \, H^4 \cong \text{Möb}(3) \) in \( \Omega(G) \subset S^3 \). These two homeomorphisms map polyhedral sides to polyhedral sides and preserve combinatorial structures of polyhedra and their dihedral angles. Equivariantly extending these homeomorphisms, we define a quasiregular mapping \( F : S^3 \setminus S^2 \to \Omega(G) \):

\[
F(x) = \begin{cases} 
\rho(\gamma) \circ \phi_1^{-1} \circ \gamma^{-1}(x) & \text{if } |x| < 1, x \in \gamma(P_0), \gamma \in \Gamma \\
\rho(\gamma) \circ \phi_2^{-1} \circ \gamma^{-1}(x) & \text{if } |x| > 1, x \in \gamma(\widetilde{P}_0), \gamma \in \Gamma
\end{cases}
\]

(4.1)

Since the initial quasiconformal homeomorphisms \( \phi_1^{-1} \) and \( \phi_2^{-1} \) preserve combinatorial structures of polyhedra and their dihedral angles, the tessellations of \( \Omega(\Gamma) = S^3 \setminus S^2 \) and \( \Omega(G) \) by corresponding \( \Gamma \)- and \( G \)-images of fundamental polyhedra of the reflection groups \( \Gamma \) and \( G \) around all sides of polyhedra including their edges and vertices are perfectly similar. This implies that our quasiregular mapping \( F \) defined by (4.1) is locally homeomorphic.

It follows from Lemma 2.3 that the limit set \( \Lambda(G) \subset S^3 \) of the group \( G \subset \text{Möb}(3) \) defines a Heegaard splitting of infinite genus of the 3-sphere \( S^3 \) into two connected components \( \Omega_1 \) and \( \Omega_2 \) of the discontinuity set \( \Omega(G) \). Moreover the action of \( G \) on the limit set \( \Lambda(G) \) is an ergodic word hyperbolic action (quasi-self-similar in the sense of Sullivan). For this ergodic action the set of fixed points of loxodromic elements \( g \in G \) (conjugate to similarities in \( \mathbb{R}^3 \)) is dense in \( \Lambda(G) \). Preimages \( \gamma \in \Gamma \) of such loxodromic elements \( g \in G \) for our homomorphism \( \rho : \Gamma \to G \) are loxodromic elements in \( \Gamma \) with two fixed points \( p, q \in \Lambda(G) = S^2, p \neq q \). This and arguments of the group completion (see [23] and [4], Section 4.6) show that our mapping \( F \) can be continuously extended to the set of fixed points of such elements \( \gamma \in \Gamma \), \( F(Fix(\gamma)) = Fix(\rho(\gamma)) \). The sense of this continuous extension is that if \( \gamma \in \Gamma \) is a loxodromic preimage of a loxodromic element \( g \in G \), \( \rho(\gamma) = g \), and if \( x \in S^3 \setminus S^2 \) tends to its fixed points \( p \) or \( q \) along the hyperbolic axis of \( \gamma \) (in \( B(0,1) \) or in its complement \( \overline{B(0,1)} \)) then \( \lim_{|x| \to 1} F(x) \) exists and equals to the corresponding fixed point of the loxodromic element \( g = \rho(\gamma) \in G \). In that sense one can say that the limit set \( \Lambda(G) \) (the common boundary of the connected components \( \Omega_1, \Omega_2 \subset \Omega(G) \)) is the \( F \)-image of the unite sphere \( S^2 \subset S^3 \).

Nevertheless not all loxodromic elements \( \gamma \in \Gamma \) in the hyperbolic lattice \( \Gamma \subset \text{Isom} \, H^3 \) have their images \( \rho(\gamma) \in G \) as loxodromic elements. Proposition 2.4 shows that \( \ker \rho \cong F_3 \).
is a free subgroup on three generators in the lattice $\Gamma$, and all elements $\gamma \in F_3$ are loxodromic. We should look at $\lim_{x \to p} F(x)$ when $x$ tends to a fixed point $p \in S^2$ of this loxodromic element $\gamma \in F_3 \subset \Gamma$.

For a group $\Gamma$ with a finite set $\Sigma = \{\gamma_1, \ldots, \gamma_k\}$ of generators we consider its Cayley graph $K(\Gamma, \Sigma)$, i.e. a 1-complex whose set of vertices is $\Gamma$ and such that $a, b \in \Gamma$ are joined by an edge if and only if $a = bg^{\pm 1}$ for some $g \in \Sigma$. Since our $\Gamma$ is a co-compact lattice acting in the hyperbolic space, we may define an embedding $\varphi$ of its Cayley graph $K(\Gamma, \Sigma)$ in the hyperbolic space $H^3$ (model in the unit ball $B(0,1)$ or in its complement $\overline{B(0,1)}$).

For a point $0 \in H^3$ not fixed by any $\gamma \in \Gamma \setminus \{1\}$, vertices $\gamma \in K(\Gamma, \Sigma)$ are mapped to $\gamma(0)$, and edges joining vertices $a, b \in K(\Gamma, \Sigma)$ are mapped to the hyperbolic geodesic segments $[a(0), b(0)]$. In other words, $\varphi(\Gamma(\Gamma, \Sigma))$ is the graph that is dual to the tessellation of $H^3$ by polyhedra $\gamma(P_0)$ (or $\overline{P_0}$), $\gamma \in \Gamma$. Obviously, the map $\varphi$ is a $\Gamma$-equivariant proper embedding: for any compact $C \subset H^3$, its pre-image $\varphi^{-1}(\varphi(\Gamma(\Gamma, \Sigma)) \cap C)$ is compact. Moreover this embedding is a pseudo-isometry (see [12] and [3], Theorem 4.35):

**Theorem 4.2.** Let $\Gamma \subset \operatorname{Isom} H^n$ be a convex co-compact group. Then the map $\varphi : K(\Gamma, \Sigma) \rightarrow H^n$ is a pseudo-isometry of the word metric $(\ast, \ast)$ on $K(\Gamma, \Sigma)$ and the hyperbolic metric $d$, that is, there are positive constants $K$ and $K'$ such that

$$(a, b)/K \leq d(\varphi(a), \varphi(b)) \leq K \cdot (a, b) \quad (4.2)$$

for all $a, b \in K(\Gamma, \Sigma)$ satisfying one of the following two conditions: either $(a, b) \geq K'$ or $d(\varphi(a), \varphi(b)) \geq K'$. 

This Theorem 4.2 implies (see [3], Theorem 4.38), that the limit set of any convex-cocompact group $\Gamma \subset \operatorname{Mob}(n)$ can be identified with its group completion $\Gamma = K(\Gamma, \Sigma) \setminus K(\Gamma, \Sigma)$. Namely there exists a continuous and $\Gamma$-equivariant bijection $\varphi_\Gamma : \Gamma \rightarrow \Lambda(\Gamma)$.

Now for the kernel subgroup $F_3 = \ker \rho \subset \Gamma \subset \operatorname{Isom} H^3$ and for the pseudo-isometric embedding $\varphi$ from Theorem 4.2 we consider its Cayley subgraph in $\varphi(K(\Gamma, \Sigma)) \subset H^3$ which is a tree - see Figure 5. Since the limit set of $\ker \rho = F_3 \subset \Gamma$ corresponds to the group completion $F_3$, it is a Cantor subset of the unit sphere $S^2$. Moreover it is a dense subset of $S^2 = \Lambda(\Gamma)$ because of density in the limit set $\Lambda(\Gamma)$ of the $\Gamma$-orbit $\Gamma\{p, q\}$ of fixed points $p$ and $q$ of a loxodromic element $\gamma \in \Gamma$ (images of $p$ and $q$ are fixed points of $\Gamma$-conjugates of such loxodromic element $\gamma \in F_3 \subset \Gamma \subset \operatorname{Isom} H^3$).

On the other hand let $x \in l_\gamma$ where $l_\gamma$ is the hyperbolic axis of an element $\gamma \in F_3 \subset \Gamma$ (either in $B(0,1)$ or in its complement $\overline{B(0,1)}$). Denoting $d_\gamma$ the translation distance of $\gamma$, we have that any segment $[x, \gamma(x)] \subset l_\gamma$ is mapped by our quasiregular mapping $F$ to a non-trivial closed loop $F([x, \gamma(x)]) \subset \Omega(G) = \Omega_1 \cup \Omega_2$, inside of a handle of the mutually linked handlebodies $\Omega_1$ or $\Omega_2$ (similar to the loops $\beta_1 \subset \Omega_1$ and $\beta_2 \subset \Omega_2$ constructed in the proof of Lemma 2.3). Therefore when $x \in l_\gamma$ tends to a fixed point $p$ (in $\operatorname{fix}(\gamma) \in S^2$) of such $\gamma \in F_3$, its image $F(x)$ goes around that closed loop $F([x, \gamma(x)]) \subset \Omega(G)$ because $F(\gamma(x)) = \rho(\gamma)(F(x)) = F(x)$. Immediately it implies that $\lim_{x \rightarrow p} F(x)$ does not exist. This shows that fixed points of any element $\gamma \in F_3 \subset \Gamma$ are essential singularities of our quasiregular mapping $F$. Since the set of such fixed points is a dense subset of the unit sphere $S^2 \subset S^3$, our quasiregular mapping $F$ has no continuous extension into any neighbourhood of an arbitrary point in $S^2$ (from both its sides in $S^3$).
Figure 8: Locally inextensible wild embedding of a closed ball into $\mathbb{R}^3$. 
Remark 4.3. In terms of the holomorphic function theory of several complex variables, both components of the complement $S^3 \setminus S^2$ play the role of the so called domain of holomorphy for the constructed in Theorem 4.1 locally homeomorphic quasiregular mapping $F$. Obviously instead of the sphere $S^2 \subset S^3$ one may consider any quasi-sphere $S_q^2 \subset S^3$ which the image of $S^2$ under a quasiconformal homeomorphism of $S^3$. In other words, The complement $S^3 \setminus S_q^2$ consisting of two quasi-balls has the same property of domain of holomorphy for a locally homeomorphic quasiregular mapping $F$ similar to our mapping $F$ the constructed in Theorem 4.1, and the quasi-sphere $S_q^2 \subset S^3$ is a barrier for it with a dense subset of essential singularities.

Remark 4.4. The constructed in Theorem 4.1 barrier $S^2$ for our locally homeomorphic quasiregular mapping $F : S^3 \to S^3$ has completely different nature from the topological barrier $S^2$ for the quasisymmetric embedding $f : B(0,1) \hookrightarrow \mathbb{R}^3$ of the closed unit ball $B(0,1)$ into $\mathbb{R}^3$ constructed in [5]. That topological barrier for the embedding $f$ was due to wild knotting of the boundary topological sphere $f(S^2) \subset \mathbb{R}^3$ on its dense subset (the fundamental group of the complement $\mathbb{R}^3 \setminus f(B(0,1))$ is infinitely generated near wild knotting points) - see Figure 8.

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