Laplace-Runge-Lenz vector in quantum mechanics in non commutative space

Samuel Kováčik

Commenius University

samuel.kovacik@gmail.com

21. August 2013
\[
\vec{A} \propto \vec{v} \times \vec{L} - k \frac{\vec{r}}{r}
\]
(is conserved in \(U(r) \propto 1/r\) potentials)

**Brief history of LRL vector**

- It was discovered by ???\(^a\) and ???\(^b\) (1710).
- After that Laplace in Celestial mechanics (1799 - 1825).
- After that Runge in (german) textbook (1919).
- After that Lenz in paper about (old) QM (1924).
- After that Pauli in paper about the hydrogen atom spectrum (1926).

\(^a\) Jakob Hermann
\(^b\) Johann Bernoulli
\[ \vec{A} \propto \vec{v} \times \vec{L} - k \frac{\vec{r}}{r} \]

**Interpretation (classical)**

The vector pointing towards the perihelion of the orbit is conserved

- **fixed direction** $\rightarrow$ the perihelion does not precess
- **constant magnitude** $\rightarrow$ the eccentricity does not change
Symmetry connected with this conserved quantity

- Kepler system $\leftrightarrow$ harmonic oscillator in $R^4$, $^a$
- There are conserved two scalar and two vector quantities (at least)
- If the scalar quantities satisfy certain restrictions, then the vector quantities are the components of $\vec{L}$ and the other three are the components of $\vec{A}$ $^b$
- Poisson brackets for $\vec{A}, \vec{L}$ forms the Lie algebra $so(4) = so(3) \oplus so(3)$ ($SO(4)$ is the group of $R^4$ isometrics) $^c$

---

$^a$see Kustaanheimo - Stiefel transformation

$^b$Bakaj, Stepanovskij

$^c$Bargmann, 1936
(from now on, $\hbar = m = 1$)

$$A_k^{QM} \propto \frac{1}{2} \varepsilon_{ijk} (L_i p_j + p_j L_i) + q \frac{x_k}{r}$$

(symmetrization is needed)

### Properties of LRL in QM

- Is conserved: $[H, A_k] = 0$ (for $U(r) = \frac{q}{r}$)
- $[L_i, A_j] = i \varepsilon_{ijk} A_k$
- $[A_i, A_j] \propto i \varepsilon_{ijk} L_k H$
- Together with $[H, L_k] = 0$, $[L_i, L_j] = i \varepsilon_{ijk} L_k$ this almost forms a so(4) representation.
- Is that bad? No, it’s great.
[L_i, A_j] = i\varepsilon_{ijk}A_k \ , \ [L_i, L_j] = i\varepsilon_{ijk}L_k \ , \ [A_i, A_j] \propto i\varepsilon_{ijk}L_kH

### Discrete energy spectrum (recipe)

1. To remove the Hamiltonian from these relations we restrict ourselves on subspace of energy eigenstates $H\psi_n = E_n\psi_n$ (let us assume it exists).

2. Rescale the $A_i$ to obtain $so(4)$ relations \(^a\). Note: Energy is now imprinted in the algebra constituents.

3. From group theories we know that the Casimir operators of $so(4)$ have integer eigenvalues.

4. Using this one can derive the hydrogen energy spectrum in a completely algebraical way.

\(^a\)Or $so(3,1)$ for positive energies
Brief introduction to NC QM

Main ideas

- The space is fuzzy on some small scale
- It is motivated by thoughts about quantum gravity
- It could remove divergences and eventually help to create quantum theory of gravity

\(^a\)Wheeler
NC relations

- NC coordinates satisfy \([x_i, x_j] = 2i\lambda\varepsilon_{ij} x_k\)
- \(x_i = \lambda\sigma^i_{\alpha\beta} a^+_{\alpha} a_{\beta}\), \(r = \lambda(a^+_{\alpha} a_{\alpha} + 1); \alpha, \beta = 1, 2\)

\[
[a_{\alpha}, a^+_{\beta}] = \delta_{\alpha\beta}, \quad [a_{\alpha}, a_{\beta}] = [a^+_{\alpha}, a^+_{\beta}] = 0, \quad |n_1, n_2\rangle = \frac{(a^+_{1})^{n_1} (a^+_{2})^{n_2}}{\sqrt{n_1! n_2!}} |0\rangle
\]

Hilbert space \(\mathcal{H}_\lambda\)

- Spanned on analytic functions of the form

\[
\psi = \sum C_{m_1m_2n_1n_2} (a^+_{1})^{m_1} (a^+_{2})^{m_2} (a_{1})^{n_1} (a_{2})^{n_2}
\]

- and is equipped with a norm

\[
\|\psi\|^2 = 4\pi \lambda^3 \text{Tr}[(N + 1) \psi^{\dagger} \psi] = 4\pi \lambda^2 \text{Tr}[r \psi^{\dagger} \psi]
\]
Operators in $\mathcal{H}_\lambda$

**Physical operators**

- $\hat{x}_i \psi = \frac{1}{2} (x_i \psi + \psi x_i)$
- $\hat{L}_i \psi = \frac{1}{2\lambda} [x_i, \psi]$
- $\hat{H}_0 \psi = \frac{1}{2\lambda r} [a^+_\alpha, [a_\alpha, \psi]]$
- $\hat{V}_i \psi = i[\hat{H}, \hat{X}_i] \psi = i[\hat{H}_0, \hat{X}_i] \psi$

**Auxiliary operators**

- $\hat{a}_\alpha \psi = a_\alpha \psi$, $\hat{b}_\alpha \psi = \psi a_\alpha$ (and similarly for creation operators)
- Simple combinations of the above-mentioned:
  - $\hat{w}_{\alpha\beta} = \hat{a}^+_\alpha \hat{b}_\beta - \hat{a}_\beta \hat{b}^+_\alpha$
  - $\hat{\zeta}_{\alpha\beta} = \hat{a}^+_\alpha \hat{b}_\beta + \hat{a}_\beta \hat{b}^+_\alpha$
**Definition of problem**

\[ \frac{\hbar^2}{2m\lambda r} [a_\alpha^+, [a_\alpha, \Psi]] - \frac{q}{r} \Psi = E \Psi \]

**Solution**

- Analytical solution found in terms of hypergeometric functions \(^a\)
- Energy spectrum derived

\[ E = -\frac{q^2 m}{2n^2\hbar^2} + \lambda^2 \frac{q^4 m^3}{24n^4\hbar^6} + \ldots \]

\(^a\)Gáliková, Prešnajder
The question is ...

Do the features of NC QM (fuzziness of space, energy cut-off,...) spoil the procedure of deriving the energy spectrum using the $SO(4)$ symmetry?
\[
\hat{A}_k = \frac{1}{2} \varepsilon_{ijk} (\hat{L}_i \hat{V}_j + \hat{V}_j \hat{L}_i) + q \frac{\hat{x}_k}{\hat{r}} \\
= -\frac{1}{2\lambda \hat{r}} (\hat{r} \hat{\zeta}_k - \hat{x}_k \hat{\zeta}) + q \frac{\hat{x}_k}{\hat{r}} \\
= \frac{1}{2\hat{r} \lambda} (\hat{r} \hat{W}'_k - \hat{x}_k (\hat{W}' - 2\lambda q))
\]

\(\hat{W}', \hat{W}'_k\)

- \(\hat{W}'_k\) and \(\hat{W}'\) are just simple linear combinations of \(\hat{r}, \hat{x}_k, \hat{\zeta}, \hat{\zeta}_k\).
- Schrödinger equation using those reads \((\hat{W}' - 2\lambda q)\Psi_E = 0\)
- So for energy eigenstates the LRL vector reduces to \(\hat{A}_k |_{\mathcal{H}_E^\lambda} = \frac{1}{2\lambda} \hat{W}'_k\)
LRL properties

\begin{itemize}
\item \( \hat{W}'_k = \left( \frac{2}{\lambda} - 2\lambda E \right) \hat{x}_k - \hat{\zeta}_k \)
\item \([\hat{A}_k, H] = \left[ \frac{1}{2\lambda} \hat{W}'_k, H \right] = \left[ \frac{1}{2\lambda} \hat{W}'_k, \hat{H}_0 - \frac{q}{r} \right] = 0 \)
\item \([\hat{A}_i, \hat{A}_j] = \frac{1}{4\lambda^2} [\hat{W}'_i, \hat{W}'_j] = -2iE\varepsilon_{ijk} \left( 1 - \frac{\lambda^2}{2} E \right) \hat{L}_k \)
\end{itemize}

To recover the \( \text{so}(4) \) relations we define

\[
\hat{K}_j = \frac{\hat{W}'_j}{\sqrt{\left( \frac{2}{\lambda} - 2\lambda E \right)^2 \lambda^2 - 4}}
\]
so(4) relations

\[
[\hat{L}_i, \hat{L}_j] = i\varepsilon_{ijk} \hat{L}_k \\
[\hat{L}_i, \hat{K}_j] = i\varepsilon_{ijk} \hat{K}_k \\
[\hat{K}_i, \hat{K}_j] = i\varepsilon_{ijk} \hat{L}_k \\
0 = [\hat{L}_i, H] = [\hat{K}_i, H]
\]

Casimir operators

- \( \hat{K}_j \hat{L}_j \psi_E = 0 \)
- \( (\hat{K}_i \hat{K}_i + \hat{L}_i \hat{L}_i + 1) \psi_E = \ldots = \frac{(\hat{W}')^2}{(\frac{2}{\lambda E} - 2\lambda E)^2\lambda^2 - 4} \psi_E \)
  \[ \text{SchEq} = \frac{q^2}{\lambda^2 E^2 - 2E} \]
- Eigenvalues of the second Casimir should be equal to \((2j + 1)^2\) for some integer or half-integer \(j\) (so let us denote \(2j + 1 = n\))
- Put those two things together and solve for \(E\)
Energy spectrum

\[
E = \frac{1}{\lambda^2} - \frac{1}{\lambda^2} \sqrt{1 + \left( \frac{q \lambda}{n} \right)^2} = \ldots \text{recover } \hbar \text{ and } m \ldots
\]

\[
= - \frac{q^2 m}{2 n^2 \hbar^2} + \lambda^2 \frac{q^4 m^3}{24 n^4 \hbar^6} + \ldots
\]
References

The End

S. Doplicher, K. Fredenhagen, J. F. Roberts, Comm. Math. Phys. 172 (1995) 187.

V. Gáliková, P. Prešnajder, Coulomb problem in NC quantum Mechanics., (2013) arXiv:1302.4623.

M. Bander, C. Itzykson, Group theory and the hydrogen atom, Reviews of modern physics (1966)