I. INTRODUCTION

The one-gluon exchange interaction between quarks is attractive in the color-antitriplet channel and leads to color superconductivity in cold and dense quark matter [1]. Due to asymptotic freedom of QCD, the strength of the gluon-exchange interaction around the Fermi surface changes with the quark number density. At asymptotically high densities, the interaction is sufficiently weak to apply perturbation theory and the mean-field approximation, and color superconductivity is very similar to standard BCS superconductivity. The gap parameter is \( \Delta \sim \mu \exp(-1/g) \), where \( g \) is the QCD coupling constant. Quark Cooper pairs are large, with a correlation length \( \xi \sim 1/\Delta \) which is parametrically larger than the interparticle distance \( d \sim 1/\mu \). On the other hand, at intermediate densities which may be realized in the cores of compact stars and/or in the intermediate stages of heavy-ion collisions, the quark-quark interaction is relatively strong and the properties of the quark Cooper pairs will be modified. In particular, we expect their size to become of the same order as the interparticle distance, \( \xi \sim 1/\Delta \sim d \). Moreover, it was argued that, due to the strong coupling, the fluctuations of the diquark-pair field become large around the critical temperature \( T_c \) and that they give rise to a pseudogap region in the normal phase above \( T_c \).

If the quark-quark interaction becomes strong enough before the quarks are confined at lower density, quark Cooper pairs become small enough to be considered as diquark molecules which are tightly bound states of two quarks. The color-superconducting ground state of quark matter at low temperature turns into a Bose-Einstein condensed phase of diquark molecules and their Bose-Einstein condensation (BEC) in strongly coupled quark matter

We explore the formation of diquark molecules and their Bose-Einstein condensation (BEC) in the phase diagram of three-flavor quark matter at nonzero temperature, \( T \), and quark chemical potential, \( \mu \). Using a quark model with a four-fermion interaction, we identify possible diquark excitations as poles of the microscopically computed diquark propagator. The quark masses are obtained by solving a dynamical equation for the chiral condensate and are found to determine the stability of the diquark excitations. The stability of diquark excitations is investigated in the \( T-\mu \) plane for different values of the diquark coupling strength. We find that bound diquark molecules appear at small quark chemical potentials at intermediate coupling and that BEC of non-strange diquark molecules occurs if the attractive interaction between quarks is sufficiently strong.

In this work, we explore the appearance of diquark molecules and their BEC in the phase diagram of quark matter using a low-energy effective model. This model features an attractive quark-quark interaction with a constant coupling strength \( G_D \) that is regarded as a free parameter of the model. We show that diquark molecules appear at low density at intermediate values of \( G_D \). It is also shown that BEC of diquark molecules can occur for large values of \( G_D \).

In the normal phase above \( T_c \), the strongest decay mode of diquarks is that into two quarks. Since the excitation energy of a quark at rest is \( M - \mu \), with \( M \) being the mass of the quark, the threshold energy for this decay process is \( \omega_{\text{thr}} = 2(M - \mu) \), where \( M \) and \( \mu \) are the average mass and chemical potential of the quarks in the diquark. Recalling that the energy of the diquark excitations should be positive to ensure the stability of the system, one finds that the necessary condition for the existence of stable diquarks is \( \omega_{\text{thr}} > 0 \), or

\[
M > \mu. \tag{1}
\]

As we will see later, at \( T = T_c \), Eq. (1) is also a sufficient condition [5, 4]. From Eq. (1), we conclude that the stability of diquark excitations is determined by the quark masses. Quark masses are dynamically generated by chiral symmetry breaking and change as functions of \( T \) and \( \mu \). In this work, we solve the gap equations for the chiral condensates and incorporate this effect in our
calculation. Equation (11) also indicates that diquarks composed of heavier quarks tend to be more stable than those composed of light quarks, provided they exist at all. One thus expects that diquarks including a strange quark are more stable than those composed of up and down quarks.

Usually, BEC is discussed using the canonical ensemble, i.e., the particle number density is fixed as an external parameter. In this work, however, we employ the grand canonical ensemble and draw the phase diagram in the \( T - \mu \) plane, as is usually done in the literature when exploring the QCD phase diagram \([1]\). In order to decide whether BEC of diquarks occurs in this ensemble, we regard the region of the superconducting phase satisfying Eq. (11) as Bose-Einstein condensed phase \([2]\).

In this exploratory study, we employ a common chemical potential \( \mu \) for all flavors and colors. For quark matter in compact stars, this is probably not a very good assumption, as the chemical potentials should be determined to satisfy the neutrality and beta-equilibrium conditions. It is known that a rich phase structure can appear under these conditions \([10]\). However, as we shall see in the following, diquark excitations play an important role even at high temperatures and small chemical potentials in the range relevant for heavy-ion collisions. In this case, our assumption of equal chemical potentials for all quark flavors and colors is applicable to very good approximation.

This statement warrants a few remarks. Thermal model fits of hadron yields at chemical freeze-out show that, because of strangeness and isospin conservation, neither the strangeness chemical potential \( \mu_S \) nor the isospin chemical potential \( \mu_I \) are zero. The strangeness chemical potential is nonzero because of associated production channels in hadronic matter at nonzero baryon chemical potential. For quark matter, however, \( \mu_S \) should be strictly zero if the system has zero strangeness, and we may neglect \( \mu_S \). The isospin chemical potential is nonzero because of the initial isospin asymmetry of the colliding nuclei. However, at all collision energies it has been demonstrated \([17]\) that \( \mu_B \equiv 3 \mu \gg |\mu_I| \). Thus, leading order we may set \( \mu_I = 0 \).

## II. FORMALISM

In order to study the phase diagram of quark matter, we employ a three-flavor quark model with four-fermion interactions. The Lagrangian is given by

\[
\mathcal{L} = \bar{\psi} (i \partial - \hat{m}) \psi + G_S \sum_{a=0}^{8} \left[ (\bar{\psi} \lambda_a \psi)^2 + (\bar{\psi} i \gamma_5 \lambda_a \psi)^2 \right] + G_D \sum_{\gamma, \epsilon} \left[ \bar{\psi}_u \gamma_\alpha \epsilon^{\alpha \beta} \epsilon_{abc} (\bar{\psi} C)^c_{\beta} \right] \left[ (\bar{\psi} C)^c_{\beta} \gamma_5 \epsilon^{\sigma \gamma} \epsilon_{rsc} \psi^r_s \right],
\]

where the quark field \( \psi^a_\alpha \) has color \( (a = r, g, b) \) and flavor \( (\alpha = u, d, s) \) indices. The matrix of current quark masses is given by \( \hat{m} = \text{diag}(m_u, m_d, m_s); \lambda_0 = \sqrt{2/3} \) and \( \lambda_a, a = 1, \ldots, 8, \) are the Gell-Mann matrices in flavor space. The charge conjugate spinors are \( \psi_C = C \bar{\psi}^T \) and \( \bar{\psi}_C = \bar{\psi}^T C \), where \( C = i \gamma^2 \gamma^0 \) is the charge conjugation matrix. In the following, we only consider diquark condensates and diquark excitations in the color anti-triplet channel. For the numerical calculations, we employ a three-dimensional momentum cutoff \( \Lambda \). We treat the diquark coupling constant \( G_D \) as a free parameter. For the other parameters, we use the values of Ref. \([19]\), \( m_u = m_d = 5 \text{ MeV}, m_s = 120 \text{ MeV}, G_S = 6.41 \text{ GeV}^{-2} \) and \( \Lambda = 600 \text{ MeV} \).

We evaluate the thermodynamic potential in the mean-field approximation:

\[
\Omega = \frac{1}{4 G_D} \sum_{c=1}^{3} |\Delta_c|^2 + \frac{1}{8 G_S} \sum_{a=1}^{3} (M_a - m_a)^2 - \frac{T}{2} \sum_n \int \frac{d^3 p}{(2\pi)^3} \text{Tr}_{D,f,c} \ln \left[ S^{-1}(i\omega_n, \mathbf{p}) \right],
\]

where the trace is taken over Dirac, flavor, and color indices, \( \omega_n = (2n + 1)\pi/T \) is the Matsubara frequency for fermions, and

\[
M_a = m_a - 4 G_S \langle \bar{\psi}_a \psi_a \rangle, \quad \Delta_c = 2 G_D \langle \bar{\psi}_C P_c \psi_C \rangle,
\]

are the constituent quark masses and the gap parameters for color superconductivity, respectively, with \( (P_c)_{\alpha \beta}^{ab} = i \gamma^\alpha \epsilon^{\alpha \beta \epsilon} \epsilon_{abc} \). The 72 \( \times \) 72 Nambu-Gor’kov propagator is defined by

\[
S^{-1}(i\omega_n, \mathbf{p}) = \begin{pmatrix}
\hat{p} + \mu \gamma_0 - \hat{M} & \sum_{\eta} P_{\eta} \Delta_{\eta} \\
\sum_{\eta} \gamma^0 P_{\eta} \gamma_0 \Delta_{\eta} & \hat{p} + \mu \gamma_0 + \hat{M}
\end{pmatrix},
\]

with \( \hat{p} = i\omega_n \gamma_0 - \mathbf{p} \cdot \gamma \) and \( \hat{M} = \text{diag}(M_u, M_d, M_s) \). The quark chemical potential \( \mu \) has a common value for all flavors and colors, as mentioned above.

The physical values of the variational parameters \( \Delta_c \) and \( M_a \) satisfy the stationary conditions (the gap equations)

\[
\frac{\partial \Omega}{\partial \Delta_c} = 0 \quad \text{and} \quad \frac{\partial \Omega}{\partial M_a} = 0.
\]

As we will see later, the color-superconducting phase transitions at nonzero temperature are of second order. Thus, the critical temperatures are determined by solving the following equation:

\[
1 - \left| \frac{\partial \Omega}{\partial \Delta_c} \right| \bigg|_{\Delta_c = 0} = 0.
\]

We shall see later that this equation determines the poles of the diquark propagator at vanishing energy and momentum.

Since up and down flavors are degenerate in our model, we always have \( M_u = M_d \) and \( \Delta_1 = \Delta_2 \). In the following,
we refer to the phase with \( \Delta_3 \neq 0 \) and \( \Delta_{1,2} = 0 \) as the 2SC phase; the phase with \( \Delta_3 \neq 0 \) and \( \Delta_{1,2} \neq 0 \) is the CFL phase [1]. Unpaired quark matter has \( \Delta_1 = \Delta_2 = \Delta_3 = 0 \). Because of the explicit chiral symmetry breaking by the nonzero current quark masses, the chiral condensates \( \langle \bar{\psi}\psi \rangle \) never vanish.

At nonzero temperature, the order parameter fields, \( \Delta_c(x, t) \) and \( M_c(x, t) \), have fluctuations around the values determined by the mean-field approximation. In the following, we consider the amplitude fluctuations of \( \Delta_c(x, t) \) in unpaired quark matter. The propagation of the fluctuations is characterized by the retarded propagator

\[
D_c^R(p, \omega) = \frac{1}{2} G_D Q_c^R(p, \omega),
\]

where \( Q_c^R(p, \omega) \) is the one-loop quark-quark polarization function. In the imaginary-time formalism, it is given by

\[
Q_c(p, i\nu_n) = -2T \sum_m \int \frac{d^3q}{(2\pi)^3} \text{Tr}_{D.c}[P_cS_0(p-q, i\nu_n - i\omega_m)P_cCS_p^T(q, i\omega_m)C],
\]

where \( \nu_n = 2\pi n/T \) denotes the Matsubara frequency for bosons, and \( S_0(p, i\omega_n) = [\hat{p} + \mu \gamma_0 - \hat{M}]^{-1} \). Taking the analytic continuation, \( Q_c^R(p, \omega) = Q_c(p, i\nu_n) \big|_{i\nu_n \to -i\eta} \), we obtain

\[
Q_c^R(p, \omega) = -2 \sum_{\beta, \gamma = 1}^3 \left| e^{i\beta\gamma} \right| \int \frac{d^3q}{(2\pi)^3} \sum_{s, t = \pm} \frac{(sE_\beta + tE_\gamma)^2 - |p|^2 - (\delta M_c)^2}{stE_\beta E_\gamma} \left[ f(tE_\omega - \mu) - f(-sE_\beta + \mu) \right] \frac{\omega + 2\mu - sE_\beta - tE_\gamma + i\eta}{\omega + 2\mu - sE_\beta - tE_\gamma + i\eta},
\]

where \( E_\beta = \sqrt{|q - p|^2 + M_\beta^2}, E_\gamma = \sqrt{|q|^2 + M_\gamma^2}, \delta M_c = |M_\beta - M_\gamma| \), and \( f(E) = [\exp(E/T) + 1]^{-1} \) is the Fermi distribution function. The imaginary part of \( Q_c^R(p, \omega) \) denotes the difference of decay and production rates of the diquark field. At \( p = 0 \), it is given by

\[
\text{Im} Q_c^R(0, \omega) = 2\pi \sum_{\beta, \gamma = 1}^3 \left| e^{i\beta\gamma} \right| \int \frac{d^3q}{(2\pi)^3} \left( \omega + 2\mu \right)^2 - (\delta M_c)^2
\]

\[
\times \left\{ \left[ (1 - f_\beta^+)(1 - f_\gamma^+ - f_\beta^+ f_\gamma^+) \right] \delta(\omega + 2\mu - E_\beta - E_\gamma) + \left[ (1 - f_\beta^-)(1 - f_\gamma^- - f_\beta^- f_\gamma^-) \right] \delta(\omega + 2\mu + E_\beta + E_\gamma) - \left[ f_\beta^-(1 - f_\gamma^-) - (1 - f_\beta^-) f_\gamma^- \right] \delta(\omega + 2\mu + E_\beta - E_\gamma) - \left[ f_\beta^+(1 - f_\gamma^+) - (1 - f_\beta^+) f_\gamma^+ \right] \delta(\omega + 2\mu - E_\beta + E_\gamma) \right\},
\]

where \( f_\beta^\pm = [\exp((E_\omega + \mu)/T) + 1]^{-1} \). The first (second) term in the bracket in Eq. (13) includes the decay processes of the diquark into two quarks (anti-quarks) and takes nonzero values at \( \omega > 2\tilde{M}_c - 2\mu \) (\( \omega < -2\tilde{M}_c - 2\mu \)), with \( \tilde{M}_c = (M_\beta + M_\gamma)/2 \). The third and fourth terms represent Landau damping of the diquark. These terms become nonzero at \( -\delta M_c - 2\mu < \omega < \delta M_c - 2\mu \).

The poles of the diquark propagator \( D_c^R(p, \omega) \) are determined by solving \( D_c^R(p, \omega) = 0 \), or equivalently

\[
1 + G_D Q_c^R(p, \omega) = 0,
\]

in the complex-energy plane. Applying \( \omega = |p| = 0 \) to this equation, one can easily show that Eq. (13) is equivalent to the critical condition Eq. (8). Therefore, \( D_c^R(p, \omega) \) has a pole at the origin at \( T = T_c \), of the second order transition. This property is known as the Thouless criterion in condensed matter physics [18]. Above \( T_c \), the pole moves continuously from the origin to the fourth quadrant. This mode is called the soft mode. If \( M_c < \mu \) at \( T = T_c \), \( \omega = 0 \) is in the continuum of the decay process into two quarks and the imaginary part at the pole starts growing just above \( T_c \) [2]. When \( M_c > \mu \), on the
other hand, the soft mode is stable against spontaneous breaking into a pair of quarks and the pole moves on the real axis in the vicinity of $T_c$. This mode is nothing but a bound state of two quarks: the diquark molecule \cite{8}. As $T$ increases, the pole eventually arrives at the threshold of the decay process into two quarks $\omega_{thr} = 2(M_c - \mu)$ at the dissociation temperature $T_{diss}$, and the soft mode is no longer a bound state at $T > T_{c diss}$. Since the pole is at $\omega_{thr}$ at $T_{diss}$, the dissociation temperature is determined by solving

$$1 + G_D Q_c(p = 0, \omega_{thr})|_{T = T_{c diss}} = 0. \quad (15)$$

Although the diquark modes can acquire decay rates due to Landau damping, i.e., the third and fourth term in Eq. (13), our numerical results show that the soft modes never appear in the range of energies where Landau damping is nonzero. Therefore, these processes do not contribute to the decay rate of the soft modes in the parameter range employed in the present study. There can appear another pole of $D^0(S, p, \omega)$ instead of the soft mode at the energy $-2 M_c - 2 \mu < \omega < -\delta M_c - 2 \mu$. This mode does not have a decay rate and should be identified as a bound anti-diquark \cite{8}. For lower $\mu$, thermal excitations of bound anti-diquarks play an important role.

If bound diquarks are formed at $T = T_c$, it is natural to identify the color-superconducting phase below $T_c$ as a Bose-Einstein condensed phase of diquark molecules \cite{8}. In the following, therefore, we regard the color-superconducting phase satisfying $M_c < \mu$ as a Bose-Einstein condensate. Notice, however, that this is just a rough guide to separate the BEC and BCS regions; these two limits are connected continuously and there is no sharp phase boundary between them \cite{8}.

**III. NUMERICAL RESULTS**

In this section, we show the phase diagram in the $T-\mu$ plane for several values of the diquark coupling $G_D$. In Fig. 1 we first discuss the case $G_D/G_S = 0.75$, which is a value commonly used in the literature \cite{1, 20}. The bold and thin solid lines represent first- and second-order phase transitions, respectively. One sees that there appear two types of color-superconducting phases, the 2SC and CFL phase, at high $\mu$ and low $T$. At $T = 0$, these phases are separated by a first-order phase transition; the first-order transition terminates at a critical point for some nonzero value of temperature. The dissociation temperatures of diquark molecules $T_{diss}^c$ are shown by the dashed lines. We see that there exists a region at small chemical potential where stable diquark molecules are formed.

In order to see whether diquark molecules undergo BEC, we plot the conditions $\mu = M_c$ by the dash-dotted lines in Fig. 1. The regions to the left of these lines satisfy $\mu < M_c$. We see that these lines terminate at the first-order transition and we do not have a color-superconducting phase satisfying $\mu < M_c$. In other words, BEC does not appear for this value of $G_D/G_S$. The behavior of $M_c$ and $\Delta_c$ as functions of $\mu$ at $T = 0$ are shown in the upper panel of Fig. 2. One observes that $M_3 = M_{c, d}$ has a discontinuity at $\mu \simeq 343$ MeV corresponding to a first-order transition, and $M_3$ is larger than $\mu$ to the left of the discontinuity. The diquark condensate $\Delta_3$ assumes nonzero values only for $\mu > M_3$. This is a typical property at weak coupling: when $\mu > M_3$, the Fermi surfaces of up and down quarks exist and the Cooper instability leads to a diquark condensate, while if not, the ground state is nothing other than the vacuum.

It is worth mentioning that bound diquark molecules appear in the phase diagram even though BEC does not exist in the phase diagram. The diquark coupling used in Fig. 1 is strong enough to form bound diquarks, but it is still too weak to lead to their BEC.

Next, we show the phase diagrams with much stronger diquark couplings. In Fig. 3 the phase diagrams with $G_D/G_S = 1.1$ and 1.2 are shown. We see that, as $G_D$ becomes larger, the regions of the 2SC and CFL phases expand toward lower $\mu$ and higher $T$. For $G_D/G_S = 1.1$, there appears BEC of up-down diquarks in the region of the 2SC phase satisfying $\mu < M_3$, shown by the shaded area in Fig. 3. The BEC region becomes wider for $G_D/G_S = 1.2$. One also observes that the dissociation temperatures $T_{diss}^c$ become higher as $G_D$ increases. In the phase diagrams in Fig. 3 $T_{diss}^c$ at $\mu = 0$ are comparable or much higher than the critical temperature of the QCD phase transition, which is predicted to be in the range $T_c \sim 150 - 190$ MeV in lattice QCD simulations \cite{20}. This result shows that diquark molecules can exist even in the quark-gluon plasma phase if the diquark coupling is strong enough.
FIG. 2: Order parameters $M_\alpha$ and $\Delta_c$ at $T = 0$ as functions of $\mu$ for various values of the diquark coupling $G_D/G_S = 0.75, 1.1$ and $1.2$. The chemical potential is shown by the dashed line.

In order to see the diquark coupling dependence of the dissociation temperatures, we show $T_{\text{diss}}^{\mu}$ at $\mu = 0$ as functions of $G_D/G_S$ in Fig. 4. At weak coupling, $T_{\text{diss}}^{\mu} = 0$ and bound diquarks do not exist. As $G_D/G_S$ becomes larger, $T_{\text{diss}}^{\mu}$ eventually become nonzero and increase rapidly. The dissociation temperatures for diquarks including the strange quark, $T_{\text{diss}}^{1,2}$, are always higher than that for the up-down diquark. This feature comes from the difference of the threshold energy $2(\bar{M}_c - \mu)$.

The other interesting feature shown in Figs. 1 and 3 is the behavior of the line of first-order phase transitions. The first-order phase transition at lower density is shorter for $G_D/G_S = 1.1$ than that for $G_D/G_S = 0.75$, and disappears at $G_D/G_S = 1.2$. To understand this behavior, we display the order parameters for $G_D/G_S = 1.1$ and $1.2$ in the middle and lower panels of Fig. 2 respectively. The figure shows that, as $G_D$ becomes larger, $\Delta_c$ increase while the quark masses $\bar{M}_c$ become smaller, and the discontinuity of $\bar{M}_c$ disappears at $G_D/G_S = 1.2$. The decrease of $\bar{M}_c$ can be understood as the interplay between the chiral and diquark condensates $[21]$; the energy gain due to diquark condensation is proportional to the surface area of the Fermi sphere. Since the radius of the
the vacuum, i.e., $T = \mu = 0$, eventually becomes a Bose-Einstein condensate of diquark molecules, which is clearly unphysical.

IV. SUMMARY AND DISCUSSIONS

In this paper, we explored the phase diagram of three-flavor quark matter focusing on the appearance of diquark molecules and their Bose-Einstein condensation under variation of the diquark coupling constant $G_D$. We found that diquark molecules can appear at small $\mu$ and (probably realistically large) intermediate values of the diquark coupling, while BEC of up-down diquarks is realized for (probably unphysically) large values of the diquark coupling. The dissociation temperatures of diquarks become higher as $G_D$ increases. At strong coupling, the dissociation temperatures could be higher than the critical temperature of the deconfinement transition.

In this work, we employed the random-phase approximation for the calculation of the diquark propagator. In this approximation, the propagators in the polarization function Eq. (11) are those for non-interacting quarks, i.e., the effect of diquark excitations is not self-consistently incorporated. An extension of the present work to include this effect is an interesting subject for further study, because we expect that the formation of diquark molecules would modify the result especially at high temperatures. Similarly, the mean-field approximation used to draw the phase diagram is no longer applicable due to the existence of well-developed soft modes in a strongly coupled system. The incorporation of these effects has already been partially made in Ref. [15].

M.K. thanks H. Abuki and T. Kunihiro for discussions.
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