On the general structure of gauged Wess–Zumino–Witten terms

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Abstract

The problem of gauging a closed form is considered. When the target manifold is a simple Lie group $G$, it is seen that there is no obstruction to the gauging of a subgroup $H \subset G$ if we may construct from the form a cocycle for the relative Lie algebra cohomology (or for the equivariant cohomology), and an explicit general expression for these cocycles is given. The common geometrical structure of the gauged closed forms and the D’Hoker and Weinberg effective actions of WZW type, as well as the obstructions for their existence, is also exhibited and explained.

1 Introduction

Wess–Zumino–Witten (WZW) terms [1, 2, 3] may be described by closed forms on an $n$–dimensional manifold $D$, the boundary $\partial D$ of which is spacetime $M$. More generally (as will be the case throughout this paper) they are given by closed $n$–forms $\Omega$ on a certain manifold $P$, the target manifold, and it is their pull–back to $D$ by $\phi : D \rightarrow P$ which defines the integrand of the WZW action, $I_{WZW} = \int_D \phi^* \Omega$. Quite often, the target manifold is a Lie group $G$\(^2\). WZW terms may be called topological in the sense that they depend on the properties of the manifold on which they are defined (and not e.g. on the metric). Since the variation of a closed form is in turn closed, the classical Euler–Lagrange equations on the $(n – 1)$–dimensional spacetime are unambiguous when certain topological conditions (which will not be of our concern here) are met (the quantum theory requires [2, 3, 6] the quantisation of the coefficient with which $\Omega$ appears in the action).

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\(^2\)It is also possible to define WZ terms $h$ on supergroups as, e.g. on superspace. The Grassmann sector is topologically trivial, $h = db$ ($h$ is exact), and the quasi–invariance of $b$ ($L_X h = 0 \Rightarrow \exists \gamma / L_X b = d\gamma$) becomes their characteristic property (see [4, 5]).
The gauging of WZW terms, i.e. the introduction of the Yang–Mills fields via minimal coupling, does not preserve the closedness condition and, as a result, it is not always possible. The case of the two-dimensional sigma model was solved in [7], and the possible obstructions to the process of gauging rigid symmetries were discussed in [8, 9, 10, 11, 12, 13] and in [14], which also included in its analysis the new topological terms in [15]. It was realised that the obstructions to the gauging process found in [10] have an elegant geometric interpretation [13, 14] in terms of equivariant cohomology [16, 17, 18, 19, 20], a fact that had also been noticed by Witten [12]. Equivariant cohomology has also appeared [21, 22, 23] in other physical theories of recent interest, and particularly in topological/cohomological field theories [24, 25] (for reviews and references see, e.g., [26, 27]).

Recently, D’Hoker and Weinberg [6, 28] have studied the structure of the most general effective actions with symmetry group $G$ broken down to a subgroup $H$. These include those based on $G$–invariant Lagrangian densities [29, 30, 31, 32] as well as those given by quasi–invariant Lagrangians (which change by a total derivative) as is the case of the WZW terms. It has been pointed out [33] how, for $G$ simple, it is possible to derive a general expression for the effective actions of WZW type by looking for cocycles of the relative Lie algebra cohomology $H^\ast(G, H; \mathbb{R})$.

We show explicitly in this paper that the existence of both types of action terms, gauged WZW terms and the effective actions of WZW type of D’Hoker and Weinberg, have the same cohomological origin. As a result, the obstructions encountered in their construction are given in terms of the same type of ‘anomaly’. This common structure is the result of the mathematical equivalence of the classes of projectable closed invariant forms, the relative Lie algebra cohomology $H^\ast(G, H; \mathbb{R})$ (see [34, 35]) and the equivariant cohomology $H^\ast_H(G)$. Thus, this paper may also be seen as a proof of these mathematical equivalences in terms of physical theories. To make this explicit, we shall first recover in Secs. 3 and 4 the obstructions to the gauging process [10] in terms of equivariant cohomology, but using the Kalkman BSRT operator [36] rather than the Mathai-Quillen [20] isomorphism used in [13, 14]. We shall then find in Sec. 5 the general form of an $H$–gauged WZW term with target manifold $G$, and discuss in Sec. 6 why this problem and that of finding effective actions on a coset manifold are similar, making the comparison explicit. Sec. 7 is devoted to exploit the larger (left×right) $G \times G$ symmetry of the cocycles on a compact group $G$, and to give a general expression of the gauged WZW term which is then applied to an example. Some calculations are indicated in the Appendix; the mathematical background for the paper is summarised in the next section.

2 Mathematical preliminaries. The Weil algebra

Let $P(H, K)$ be a principal bundle with structure group $H$ acting on $P$ from the right and let $\{X_\alpha\}$ ($\{\omega^\alpha\}$, $\omega^\alpha(X_\beta) = \delta^\alpha_\beta$) ($\alpha, \beta = 1, \ldots, \dim H$) be a basis for its Lie algebra $\mathcal{H}$ (dual $\mathcal{H}^\ast$). Let $\mathcal{W}(\mathcal{H})$ be the Weil algebra [37, 38, 19, 20], $\mathcal{W}(\mathcal{H}) = \wedge(\mathcal{H}^\ast) \otimes S(\mathcal{H}^\ast)$, where $\wedge(\mathcal{H}^\ast)$ is the algebra of multilinear antisymmetric mappings on $\mathcal{H}$ and $S(\mathcal{H}^\ast)$ the symmetric algebra on $\mathcal{H}^\ast$ (or symmetric polynomials on $\mathcal{H}$). Endowed with the Weil
differential $d_W$, $\mathcal{W}(\mathcal{H})$ becomes a differential graded commutative algebra freely generated by the elements $\theta^\alpha$ (of degree 1) and $u^\alpha$ (of degree 2) in $\wedge(\mathcal{H}^*)$ and $\mathcal{S}(\mathcal{H}^*)$ respectively, satisfying the relations

$$d_W \theta^\alpha + \frac{1}{2} C^\alpha_{\beta \gamma} \theta^\beta \wedge \theta^\gamma = u^\alpha, \quad d_W u^\alpha = -C^\alpha_{\beta \gamma} \theta^\beta \wedge u^\gamma; \quad i_{W X_\alpha} \theta^\beta = \delta^\beta_\alpha, \quad i_{W X_\alpha} u^\beta = 0; \quad (2.1)$$

$d_W^2 = 0$ follows from the Jacobi identity. An element in $\wedge^q(\mathcal{H}^*) \otimes \mathcal{S}^s(\mathcal{H}^*)$ has degree $(q + 2s)$.

Let $A$ be a connection on $P$ and $F$ its curvature; $A$ and $F$ are $\mathcal{H}$-valued forms $A = A^\alpha X_\alpha$, $F = F^\alpha X_\alpha$. Eqs. (2.1) are the same as those satisfied by $A^\alpha$ and $F^\alpha$. Thus, the mapping $\phi_W : (\theta^\alpha, u^\alpha) \mapsto (A^\alpha, F^\alpha)$ induces a homomorphism $\phi_W : \mathcal{W}(\mathcal{H}) \to \wedge(P)$ of differential algebras which is called the Weil homomorphism determined by the connection $A$ on $P$. As a result, the Weil algebra provides a universal model for the relations satisfied by any connection $A$ and curvature $F$ on $P(H, K)$,

$$d_W A^\alpha + \frac{1}{2} C^\alpha_{\beta \gamma} A^\beta \wedge A^\gamma = F^\alpha \quad (F = d_W A + \frac{1}{2} [A, A]),$$

$$d_W F^\alpha = -C^\alpha_{\beta \gamma} A^\beta \wedge F^\gamma \quad (d_W F + [A, F] = 0), \quad (2.2)$$

$$i_{W X_\alpha} A^\beta = \delta^\beta_\alpha, \quad i_{W X_\alpha} F^\beta = 0,$$

where we keep $i_W$ and $d_W$ to denote the corresponding differential operators acting on $A$, $F$, cf. (2.1). Forms which are $H$-invariant and horizontal ($i_{X_\alpha}(\_ \_ ) = 0$) are called basic. Thus, the forms on the base manifold $K$ may be identified with the basic forms on the total manifold $P$, i.e. with those that are ‘projectable’ to $K$ (a term which, more precisely, indicates that the bundle projection $\pi$ induces an embedding $\pi^* : \wedge(K) \to \wedge(P)$ which determines the basic forms on $P$).

The Weil homomorphism is compatible with the differentials, the contraction and the action of $H$ ([$\phi_W, d_W$] = 0, [$\phi_W, i_W$] = 0, [$\phi_W, L_W X$] = 0 where $L_W X$ is the Lie derivative with respect the vector field $X$) [37, 38, 19, 20]. Using (2.2), $L_W X = i_{W X} d_W + d_W i_{W X}$ gives

$$L_W X_\alpha A^\beta = -C^\beta_{\alpha \gamma} A^\gamma \quad (L_W X A = -[X, A]), \quad L_W X_\alpha F^\beta = -C^\beta_{\alpha \gamma} F^\gamma \quad (L_W X F = -[X, F]); \quad (2.3)$$

$$[L_W X_\alpha, i_W X_\beta] = i_W [X_\alpha, X_\beta].$$

Since

$$L_{\zeta^\alpha} X_\alpha = \zeta^\alpha L_X + d\zeta^\alpha \wedge i_{X_\alpha}, \quad (2.4)$$

we see that

$$L_W \zeta^\alpha X_\alpha A^\beta = d\zeta^\beta - \zeta^\alpha C^\beta_{\alpha \gamma} A^\gamma, \quad L_W \zeta^\alpha X_\alpha F^\beta = -\zeta^\alpha C^\beta_{\alpha \gamma} F^\gamma \quad (2.5)$$

i.e., the action of $L_W \zeta^\alpha X_\alpha$ on $A$ and $F$ generates the gauge transformation $\delta_\zeta$ associated with the group parameters $\zeta^\alpha$ of $H$. If we add the zero and one forms $\zeta^\alpha$ and $d\zeta^\alpha$ to the generators $A$, $F$ of the Weil algebra, the resulting one (as $\mathcal{W}(\mathcal{H})$ itself) is a contractible
[39] free differential algebra, and hence it has trivial de Rham cohomology (see also [37]).

If we have (matter) fields \( \phi^i \) defined on \( P \) or through some associated bundle on which \( H \) acts, \( \delta \zeta \phi^i = L_{\zeta^a X_a} \phi^i = \zeta^a L_{X_a} \phi^i \). For a linear action \( T_\alpha \), we have

\[
L_{\zeta^a X_a} \phi^i = -\zeta^a (T_\alpha)^i_j \phi^j \quad \text{with} \quad X_\alpha = X^i_\alpha(\varphi) \frac{\partial}{\partial \varphi^i}, \quad X^i_\alpha(\varphi) = (T_\alpha)^i_j \phi^j,
\]

where \( T \) is in the representation of \( H \) provided by the fields \( \phi^i \). Strictly speaking, the gauge transformations are not (2.5), (2.6), but their pull backs to a suitable spacetime. Nevertheless we can use these expressions to discuss the universal obstructions to the gauging process (i.e., there will have a solution if these obstructions are absent).

The Lie derivative property (2.4) shows immediately why horizontality plays an essential role in the discussion: if the form is horizontal, the term containing \( d \) will not contribute, and \( L_{\zeta^a X_a} \phi^i = \zeta^a L_{X_a} \phi^i \) if the parameters \( \zeta^a \) are not constant. For instance, \( L_{\zeta^a X_a} F^i = \zeta^a L_{X_a} F^i \) but this is of course not the case for \( d \phi^i \) (see (3.2) below).

A comment on notation. Being \( A \) a connection on the principal bundle \( P(H, K) \), \( A \) and \( F \) are forms on the manifold \( P \), and hence in \( \wedge(P) \). However, for the purposes of this paper and to make the gauge mechanism clearer, it is practical to consider \( A \) and \( F \) as the generators of a separate algebra. Since \( A, F \) are a copy of the generators of the universal Weil algebra, we shall treat them as generators of \( \mathcal{W}(H) \). In this way, a form with components \( \in \mathcal{W}(H) \)' will indicate that it includes terms in the connection \( A \) and/or curvature \( F \), and a form \( \in \wedge(P) \) will refer to an ungauged form, with no components in the Yang–Mills fields or strengths (alternatively, we could keep the generators \( \theta^a \) and \( u^a \) of \( \mathcal{W}(H) \) in (2.1) throughout and replace them by \( A, F \) using \( \phi_W \) at the end). In the above framework, it will be convenient to distinguish between the operators on \( \mathcal{W}(H) \) and those acting on \( \wedge(P) \). To this aim, we shall keep the subindex \( w \) for the operators in (2.2), and reserve the notation \( d, i_{X_a} \) (or \( i_{a} \)) for their counterparts acting on \( \wedge(P) \) or on the exterior algebra on an associated bundle. The total \( d \) and \( i \) will denote the sums \( d = d_w \otimes 1 + 1 \otimes d \) and \( i_a = i_{w_a} \otimes 1 + 1 \otimes i_a \) (\( d^2 = 0 = i^2 \)) acting on \( \mathcal{W}(H) \otimes \wedge(P) \). Similarly, \( L_a \equiv i_a d + di_a = L_{w_a} \otimes 1 + 1 \otimes L_a \); on horizontal forms \( L_{\zeta^a X_a} = \zeta^a L_a \). The \( \otimes \) symbol will be often omitted if no confusion arises.

3 Gauging closed forms

Let \( \Omega \) be an \( n \)-form on \( P \) (or on an associated bundle) and let \( \phi^i \) be the coordinates of \( P \). Let \( H \) be the compact and simply connected group to be gauged. The minimal coupling substitution has the form

\[
d \mapsto D := d - A^a L_a,
\]

(\( i.e. \), \( 1 \otimes d \rightarrow 1 \otimes d - A^a \otimes L_a \)), where \( L_a \) is the Lie derivative with respect to the vector field associated with the right action of \( H \) on \( P \) \((X_\alpha = X^i_\alpha(\varphi) \partial/\partial \varphi^i, \ L_\alpha \phi^i = X^i_\alpha(\varphi))\).

Indeed, we may check that in the present language
\[ L_{\zeta^\alpha X_\alpha} D\varphi = (d\zeta^\alpha \wedge i_\alpha + \zeta^\alpha L_\alpha) (d - A^\beta L_\beta) \varphi \]
\[ = (d\zeta^\alpha \wedge i_\alpha d - d\zeta^\alpha \wedge (i_W A^\beta) L_\beta + d\zeta^\alpha \wedge A^\beta i_\alpha L_\beta) \varphi + \zeta^\alpha L_\alpha (D\varphi) = \zeta^\alpha L_\alpha (D\varphi) . \] (3.2)

Under (3.1), \( \Omega \rightarrow \tilde{\Omega} \) i.e.,
\[ \Omega = \frac{1}{n!} \Omega_{i_1...i_n} d\varphi^{i_1} \wedge \ldots \wedge d\varphi^{i_n} \rightarrow \tilde{\Omega} = \frac{1}{n!} \Omega_{i_1...i_n} D\varphi^{i_1} \wedge \ldots \wedge D\varphi^{i_n} . \] (3.3)

To see how the minimal coupling affects the closedness of \( \Omega \) we have to compute \( d\tilde{\Omega} \).

Using
\[ dD\varphi^i = -F^\alpha L_\alpha \varphi^i + A^\alpha DL_\alpha \varphi^i = -F^\alpha i_\alpha d\varphi^i + A^\alpha DL_\alpha \varphi^i , \] (3.4)
we find
\[ d\tilde{\Omega} = \tilde{d}\tilde{\Omega} + \frac{1}{n!} \left\{ \partial_j \Omega_{i_1...i_n} A^\alpha L_\alpha \varphi^j \wedge D\varphi^{i_1} \wedge \ldots \wedge D\varphi^{i_n} \right. \]
\[ + \sum_{s=1}^n (-1)^{s+1} \Omega_{i_1...i_n} [A^\alpha DL_\alpha \varphi^i_s - F^\alpha i_\alpha d\varphi^{i_s}] D\varphi^{i_1} \wedge \ldots \wedge \tilde{D}\varphi^{i_s} \wedge \ldots \wedge D\varphi^{i_n} \right\} \] (3.5)
where, in general, the tilde indicates the result of performing the minimal substitution (3.1) in the expression underneath. The last term in (3.5) is clearly \( -F^\alpha i_\alpha \tilde{\Omega} \) \( (i_\alpha d\varphi = i_\alpha \tilde{d}\varphi) \) and the second and the third are easily identified with \( A^\alpha L_\alpha \tilde{\Omega} \). Hence,
\[ d\tilde{\Omega} = \tilde{d}\tilde{\Omega} + A^\alpha L_\alpha \tilde{\Omega} - F^\alpha i_\alpha \tilde{\Omega} = \tilde{d}\tilde{\Omega} + A^\alpha L_\alpha \tilde{\Omega} - F^\alpha i_\alpha \tilde{\Omega} \] (3.6)
which is [10, eq. (4.2)]. Since the different terms in (3.6) are independent, it follows that a closed form \( \tilde{\Omega} \) will remain closed after gauging the group \( H \) iff

a) it is horizontal \((i_\alpha \Omega = 0)\)

b) \( \Omega \) is invariant under the right translations of \( H \) generated by the vector fields \( X_\alpha \in \mathcal{H} \).

If \( \Omega \) satisfies a) and b), \( d\Omega \) also satisfies them. These are also the conditions that guarantee the existence of WZW–type effective actions on coset spaces [6, 28, 33] and will explain the formal similarity of their general expressions in [33] with those which will be found later for the present case.

However if a form \( \Omega \) is \((1 \otimes i_\alpha)\)-horizontal the minimal coupling (3.1) does not act since \( D\varphi^i = d\varphi^i - A^\alpha L_\alpha d\varphi^i = (1 - A^\alpha i_\alpha) d\varphi^i \) and \( \tilde{\Omega} = \Omega \). In fact, a horizontal and \( H \)–invariant form is automatically gauge invariant. Thus, to obtain a non–trivial result and incorporate the Yang–Mills fields we need ‘extending’ \( \Omega \) to a form \( \tilde{\beta} \in \mathcal{W}(\mathcal{H}) \otimes \wedge (P) \) such that \( \tilde{\beta}(A = 0, F = 0) = \Omega \). In this case, (3.6) is trivially modified to read
\[ d\tilde{\beta} = \tilde{d}\beta + \tilde{A}^\alpha L_\alpha \beta - F^\alpha i_\alpha \beta \equiv \delta \tilde{\beta} , \] (3.7)
where
\[ \delta := d_W \otimes 1 + 1 \otimes d + A^\alpha \otimes L_\alpha - F^\alpha \otimes i_\alpha \quad (\delta := d + A^\alpha L_\alpha - F^\alpha i_\alpha) \] (3.8)
it may be easily checked that \( \delta^2 = 0 \). This is the BRST operator of [36] which we now discuss in the present context.
4 The Mathai–Quillen and Kalkman isomorphisms and the gauging of forms

The minimal coupling (3.1) defines a one–to–one correspondence, the gauging map \( \psi : \beta \mapsto \tilde{\beta} \), between ungauged (\( \beta \)) and gauged (\( \tilde{\beta} \)) forms. Since \( d\tilde{\beta} = \psi(\delta\beta) \) (eq. (3.7)), \( \psi^{-1}d\psi\beta = \delta\beta \). Hence

\[
\psi^{-1}d\psi = \delta = d_W \otimes 1 + 1 \otimes d + A^\alpha \otimes L_\alpha - F^\alpha \otimes i_\alpha .
\]  

(4.1)

The map \( \psi \) is given by \([36]\) (cf. \([20]\))

\[
\psi = \exp \left( - \sum_{\alpha=1}^{\dim \mathcal{H}} A^\alpha \otimes i_\alpha \right) = \prod_\alpha (1 - A^\alpha \otimes i_\alpha) ,
\]  

(4.2)

where in the last term there is no sum in \( \alpha \); \( \psi^{-1} = \exp \left( \sum_\alpha A^\alpha \otimes i_\alpha \right) \). Eq. (4.1) may be checked using \( \psi^{-1}d\psi = (ad(A^\alpha \otimes i_\alpha))d \) and the relations

\[
[A^\alpha \otimes i_\alpha, d] = -dA^\alpha \otimes i_\alpha + A^\alpha \otimes L_\alpha , \quad [A^\alpha \otimes i_\alpha, dA^\alpha \otimes i_\alpha] = 0 ,
\]

\[
[A^\alpha \otimes i_\alpha, A^\beta \otimes L_\beta] = -C^\gamma_{\beta\alpha} A^\beta \otimes i_\gamma ,
\]

\[
[A^\alpha \otimes i_\alpha, [A^\beta \otimes i_\beta, d]] = -C^\gamma_{\beta\alpha} A^\beta \otimes i_\gamma
\]

(higher order terms are zero), and taking into account that, in \( \mathcal{W}(\mathcal{H}) \otimes \wedge(P), (u_1 \otimes v_1)(u_2 \otimes v_2) = (-1)^{v_1u_2}(u_1u_2 \otimes v_1v_2) \). As an example of the action of \( \psi \) we may check easily that, on \( d\varphi \) [i.e., on \((1 \otimes d)(1 \otimes \varphi)\)], \( \psi : d\varphi \mapsto D\varphi \) since \( \prod_\alpha (1 - A^\alpha \otimes i_\alpha)d\varphi \) (no sum in \( \alpha \)) is given (restoring the summation convention) by \( d\varphi - A^\alpha \otimes i_\alpha d\varphi = (d - A^\alpha \otimes L_\alpha)\varphi \) i.e., by \( (d - A^\alpha L_\alpha)\varphi = D\varphi \). Hence (4.2) implements the minimal coupling.

Let us now take two copies \( \mathcal{A}, \mathcal{B} \) of the algebra \( \mathcal{W}(\mathcal{H}) \otimes \wedge(P) \) endowed with the differential operators \( \delta \) and \( d \) respectively. (\( \mathcal{A}, \delta \)) and (\( \mathcal{B}, d \)) are not equal as differential algebras, but \( \psi : \mathcal{A} \to \mathcal{B} \) makes them isomorphic \([36]\). Thus, their cohomology rings coincide, \( H^*_\mathcal{A}(\mathcal{A}) = H^*_d(\mathcal{B}) \): if \( \beta \in \mathcal{A} \) and \( \delta\beta = 0 \), then \( d\tilde{\beta} = 0 \) for \( \psi\beta = \tilde{\beta} \in \mathcal{B} \). Moreover, these rings are both equal to \( H_{DR}(P) \) because, being contractible, the \( \mathcal{W}(\mathcal{H}) \) part in \( (\mathcal{B}, d) \) has trivial de Rham cohomology.

Let us go back to (4.1) and eqs. (2.5), (2.6) and restrict \( \mathcal{B} \) to the subalgebra of the horizontal and invariant (hence gauge invariant) forms. These forms \( \tilde{\alpha} \) fulfill the conditions

\[
i_\alpha \tilde{\alpha} \equiv (i_{W\alpha} \otimes 1 + 1 \otimes i_\alpha)\tilde{\alpha} = 0 , \quad L_\alpha \tilde{\alpha} \equiv (L_{W\alpha} \otimes 1 + 1 \otimes L_\alpha)\tilde{\alpha} = 0
\]

(4.4)

which are also satisfied by \( d\tilde{\alpha} \), since \( L_\alpha = d i_\alpha + i_\alpha d \). Thus, these forms constitute a subalgebra of \( (\mathcal{W}(\mathcal{H}) \otimes \wedge(P), d) \), the subalgebra of basic forms \( ([\mathcal{W}(\mathcal{H}) \otimes \wedge(P)]_{\text{basic}}, d) \) of the Weil model for the equivariant cohomology \( H^*_\mathcal{H}(P) \). It is easy to check that \(^3\)

\[\text{These expressions follow by noticing that } [A^\alpha \otimes i_\alpha, i_{W\beta} \otimes 1 + 1 \otimes i_\beta] = -1 \otimes i_\beta, \quad [A^\alpha \otimes i_\alpha, 1 \otimes i_\beta] = 0 \quad \text{and } [A^\alpha \otimes i_\alpha, L_{W\beta} \otimes 1 + 1 \otimes L_\beta] = 0.\]
ψ^{-1}(i_{Wα}⊗1 + 1 ⊗ i_α)ψ = i_{Wα}⊗1 , \quad [ψ, L_{Wα}⊗1 + 1 ⊗ L_α] = 0 \quad . (4.5)

Hence, the algebra ([W(H) ⊗ ∧(P)]_{basic}, d) is isomorphic to the H–invariant subalgebra ([S(H^*) ⊗ ∧(P)]^H, d_C) of the Cartan model, where

\[ d_C = 1 ⊗ d - F^α ⊗ i_α \quad , \quad (4.6) \]

(or, simply, \(d - F^α_i_α\); on \([S(H^*) ⊗ ∧(P)]^H\), \(d_C^2 = 0\). This is the Mathai–Quillen isomorphism \([20]\) and ([S(H^*) ⊗ ∧(P)]^H, d_C) is the complex for the Cartan model of equivariant cohomology \([37]\). The expression of \(d_C\) follows from (4.1) restricting it to horizontal forms: since \(i_{Wα} ⊗ 1\) \(α = 0\), \(d_W = A^αL_{Wα}(A^αL_{Wα}F_β = -A^αC_β^γF_γ = d_WF_β)\). Then, since \(L_αA = 0\), \(δ\) reduces to \(d_C\). The above results may be summarised in the diagram

\[
\begin{array}{ccc}
\mathfrak{A} = \mathcal{W}(H) \otimes ∧(P), δ & \xrightarrow{ψ} & \mathfrak{B} = \mathcal{W}(H) \otimes ∧(P), d
\\
\uparrow & & \uparrow
\\
(i_{Wα}⊗1) & (L_{Wα}⊗1 ⊗ L_α) & (i_{Wα}⊗1 ⊗ i_α)
\\
([S(H^*) ⊗ ∧(P)]^H, d_C) & \xrightarrow{ψ} & ([\mathcal{W}(H) \otimes ∧(P)]_{basic}, d)
\\
\text{Intermediate scheme} & & \text{Weil model}
\end{array}
\]

(4.7)

It is easy to see that the image of \(\bar{α} ∈ [\mathcal{W}(H) \otimes ∧(P)]_{basic}\) in \([S(H^*) ⊗ ∧(P)]^H\) by \(ψ^{-1}\) is obtained by setting \(A^α = 0\) \([36]\) in \(\bar{α}\). Let \(\bar{α}A = 0\) \(≡ α\). Then the previous analysis shows that the \(d\)-closed elements which determine the \(n\)–cocycles of \(H^H_P(P)\) for the Weil model are represented in the Cartan model by \(d_C\)–cocycles in \(\sum_{s=1}^\infty [S^s(H^*) \otimes ∧_{n-2s}(P)]^H\). A \(d\)–closed form \(Ω ∈ ∧(P)\) will be gaugeable \([12, 13, 14]\) iff it admits an equivariant extension \(α ∈ [S(H^*) ⊗ ∧(P)]^H\) in the Cartan model \((d_Cα = 0)\). The gauged, closed and gauge invariant form is then the associated \(d\)–cocycle \(\bar{α} = ψ(α)\) in the Weil model.

Let \(Ω\) be a closed \(n\)–form. As we have seen, performing in it the minimal substitution (3.1) does not solve the problem of gauging \(Ω\) as it stands. However, let \(α ∈ S(H^*) ⊗ ∧(P)\) be the \((i_{Wα} ⊗ 1\)–horizontal) form

\[ α = Ω + \sum_{s=1}^p F^{α_1} ∧ \ldots ∧ F^{α_s}v_{α_1…α_s} ≡ Ω + \sum_{s=1}^p v^{(s,n-2s)} , \quad α(F = 0) = Ω \quad , \quad (4.8) \]

where \(p\) is the integer part of \(n/2\) and \(v_{α_1…α_s}\) is a \((n - 2s)\)–form on \(P\),

\[ v_{α_1…α_s} = \frac{1}{(n - 2s)!}v_{α_1…α_s, j_1…j_{n-2s}}dφ_{j_1} ∧ \ldots ∧ dφ_{j_{n-2s}} . \quad (4.9) \]

If \(α ∈ [S(H^*) ⊗ ∧(P)]^H\), the \(∧_{n-2s}(P)\)–valued symmetric polynomials \(v_{α_1…α_s}\) must be \(H\)–invariant, \(L_βv^{(s,n-2s)} = 0\), \(i.e.,\)

\[ L_βv^{(s,n-2s)} = (L_βv_{α_1…α_s} - C_β^γv_{γα_2…α_s} - \ldots - C_β^γv_{α_1…α_{s-1}γ})F^{α_1} ∧ \ldots ∧ F^{α_s} = 0 \quad . \quad (4.10) \]
Since the second part in (4.10) is simply \((\text{coad} X_\beta)^{\boxtimes s}\), we see that our \(L_\beta\) may be identified with the `covariant derivative’ in [10], \(L_\beta^{\text{cov}} := L_\beta + (\text{coad} X_\beta)^{\boxtimes s}\). The cocycle condition \(d_C \alpha = (d - F^\beta i_\beta) \alpha = 0\) now gives
\[
d_C \alpha = \sum_{s=1}^{p} F^{\alpha_1} \wedge \ldots \wedge F^{\alpha_s} dv_{\alpha_1 \ldots \alpha_s} - F^\beta i_\beta \Omega - \sum_{s=1}^{p} F^\beta \wedge F^{\alpha_1} \wedge \ldots \wedge F^{\alpha_s} i_\beta v_{\alpha_1 \ldots \alpha_s} = 0 \quad , \tag{4.11}
\]
and equating equal powers in \(F\) the descent equations of Hull and Spence [10] are recovered [13, 14]
\[
dv_{\alpha_1} = i_{\alpha_1} \Omega \quad , \quad dv_{\alpha_1 \alpha_2} = i_{\{\alpha_2 \alpha_1\}} \quad , \quad \ldots \quad dv_{\alpha_1 \ldots \alpha_s} = i_{\{\alpha_s \ldots \alpha_1\}} \quad , \quad \tag{4.12}
\]
where the symmetrisation, represented by the curly brackets \(\{ \}\), is imposed by the commuting \(F\)’s and includes a factor \(1/s!\). These equations contain the possible obstructions to the problem of gauging the form \(\Omega\), i.e., to finding an equivariant extension \(\tilde{\alpha}\) such that \(d \tilde{\alpha} = 0\), \(\tilde{\alpha}\rvert_{A=0=F} = \Omega\) (or an \(\alpha\) such that \(d_C \alpha = 0\), \(\alpha\rvert_{F=0} = \Omega\))

5 Gauging cocycles on simple groups: general solution

The descent equations (4.11) from the \(n\)-form \(\Omega\) correspond to the pattern

\[
\begin{array}{c}
\Omega^{(0,n)} \\ \downarrow \quad \downarrow d \\
(i_{\alpha_1} \Omega)^{(1,n-1)} \\ \ldots \\
(i_{\alpha_2} v_{\alpha_1})^{(2,n-3)} \\ \ldots \\
(i_{\alpha_m} v_{\alpha_1 \ldots \alpha_{m-1}})^{(m,n-2m+1)} \\
\end{array}
\quad \longrightarrow
\begin{array}{c}
(v_{\alpha_1})^{(1,n-2)} \\ \downarrow \\
(v_{\alpha_2} v_{\alpha_1})^{(2,n-3)} \\ \downarrow d \\
(v_{\alpha_1 \alpha_2})^{(2,n-4)} \\
\ldots
\end{array}
\tag{5.1}
\]

Each step \(v_{\alpha_1 \ldots \alpha_s} F^{\alpha_1} \wedge \ldots \wedge F^{\alpha_s} \rightarrow v_{\alpha_1 \ldots \alpha_{s+1}} F^{\alpha_1} \wedge \ldots \wedge F^{\alpha_{s+1}}\) takes a \((s, n-2s)\)-type \(n\)-form in \(\mathcal{W}(\mathcal{H}) \otimes \wedge (P)\) to a \((s+1, n-2s-2)\) one. We may distinguish two cases:

a) \(n\ odd, \ n=2m-1\). Then, \((m-1)\)-steps will lead \(\Omega^{(0,2m-1)}\) to \(v^{(m-1,1)}\ i.e., to \(v_{\alpha_1 \ldots \alpha_{m-1}} d\phi^j\). The \(i_{\alpha_m}\) contraction in the \(m\)-th step will then produce \(i_{\{\alpha_m \ldots \alpha_1\}} = c_{\alpha_1 \ldots \alpha_m}\) which is a symmetric zero–form. Then, the last term of eq. (4.11) is
\[
-c_{\alpha_1 \ldots \alpha_m} F^{\alpha_1} \wedge \ldots \wedge F^{\alpha_m} \quad . \tag{5.2}
\]

Thus, the form \(\alpha\) will be a Cartan cocycle if (4.10) holds and
\[
dv_{\alpha_1 \ldots \alpha_s} = i_{\{\alpha_s \ldots \alpha_1\}} \quad (s = 1, \ldots, m - 1) \quad , \quad i_{\{\alpha_m \ldots \alpha_1\}} = c_{\alpha_1 \ldots \alpha_m} = 0 \quad . \tag{5.3}
\]
b) \( n \) even, \( n = 2m \). In this case, a succession of \( m \) steps brings \( \Omega^{(0,2m)} \) to \( v^{(m,0)} \) i.e., to the \( 2m \)-form \( v_{a_1...a_m} F^{a_1} \cdots \wedge F^{a_m} \). Since \( i_\alpha v^{(m,0)} = 0 \) necessarily, \( d_C \alpha \) will be zero if (4.10) is satisfied and

\[
dv_{a_1...a_s} = i_{\{a_s v_{a_1...a_{s-1}}\}} \quad (s = 1, \ldots, m) .
\]

Finding an \( \alpha \in [S(H^*) \otimes \wedge(P)]^G \) such that eqs. (5.3), (5.4) are fulfilled is tantamount to saying that \( \Omega \) may be gauged. This means that we can obtain from \( \alpha \) a (\( d \)-closed, gauge invariant) form \( \tilde{\alpha} \) [10] given by \( \psi(\alpha) \), i.e. by (4.8) with the replacements \( \Omega \to \tilde{\Omega} \) and \( v_{a_1...a_s} \to \tilde{v}_{a_1...a_s} \):

\[
\tilde{\alpha} = \psi(\alpha) = \tilde{\Omega} + \frac{1}{(n-2s)!} \sum_{s=1}^{p} (s, n-2s) v_{a_1...a_s j_1...j_{n-2s}} (5.6)
\]

where \( p = (n-1)/2 \) (\( n \) odd) or \( p = n/2 \) (\( n \) even). For reasons which will be apparent in a moment, we shall be concerned here with the odd \( n = 2m - 1 \) case only.

Let now \( P = G \) where \( G \) is a simple, simply connected compact Lie group of algebra \( \mathcal{G} \) with basis \( \{X_i\} \). We may construct on it WZW terms on spacetimes of suitable dimension by means of Witten’s procedure [3] and using the forms on \( \mathcal{G} \) closed \( \alpha \)–invariant) form \( \tilde{\omega} \) [10] given by \( \psi(\alpha) \), i.e. by (4.8) with the replacements \( \Omega \to \tilde{\Omega} \) and \( v_{a_1...a_s} \to \tilde{v}_{a_1...a_s} \):

\[
\tilde{\omega} = \psi(\omega) = \tilde{\Omega} + \frac{1}{(n-2s)!} \sum_{s=1}^{p} (s, n-2s) v_{a_1...a_s j_1...j_{n-2s}} F^{a_1} \cdots \wedge F^{a_s} D\varphi^j_1 \cdots \wedge D\varphi^j_{n-2s} \quad (5.5)
\]

The primitive cocycles are given by the closed\(^5\) odd \( (2m-1) \)-forms on \( \mathcal{G} \):

\[
\Omega = k_{i_1...i_{m-1} i_m} d\omega^{i_1} \wedge \cdots \wedge d\omega^{i_{m-1}} \wedge \omega^{i_m} \quad (i = 1, \ldots, \dim \mathcal{G}) ,
\]

where \( \omega = g^{-1}dg = \omega^i T_i \) is the left–invariant (LI) canonical form on \( \mathcal{G} \) (\( \omega^i(X_j) = \delta^i_j \) and \( T_i \in \mathcal{G} \) is the generator in the representation of \( g \)) and \( k_{i_1...i_m} \) is one of the \( l \) primitive symmetric invariant polynomials. We may restrict ourselves to primitive cocycles since they generate the cohomology ring on \( \mathcal{G} \). We may also express (5.6) in the form

\[
\Omega \propto C_{i_1 i_2}^{j_1} \cdots C_{i_{2m-3} i_{2m-2}}^{j_{2m-1}} k_{j_1...j_{m-1} k_{m-1}} \omega^{i_1} \wedge \cdots \wedge \omega^{i_{2m-1}} ,
\]

omitting a factor \((-1/2)^{m-1}\) coming from \( d\omega^i = (-1/2)C_{jk}^i \omega^j \wedge \omega^k \). The form \( \Omega \) (after a suitable pull–back) may be used to define a \((2m-1)\)-dimensional WZW term on a manifold \( D \) with a \((2m-2)\)-spacetime \( M \) as its boundary, \( M = \partial D \) (provided certain topological conditions are met; we shall not discuss these nor the quantisation conditions for the WZW term coefficient [3, 43, 6]). We shall now prove the following

**Proposition 5.1**

Let \( \Omega \) be the closed odd form on a simple, compact and simply connected Lie group \( G \) associated with a primitive cocycle in \( H^{2m-1}(\mathcal{G}, \mathbb{R}) \). Let \( H \) be a non–trivial Lie subgroup of \( G \). Then the symmetry group \( H \) may be gauged if the polynomial \( k \) defining \( \Omega \) (eq. (5.6)) is zero on its Lie algebra \( \mathcal{H} \).

\(^4\)For details and background references on these topics see, e.g., [40, 41, 42].

\(^5\)For an explicit check see [40, Lemma 3.1].
Proof. Since this corresponds to the odd case, we have to show that all conditions (5.3) are verified by virtue of $\Omega$ being a cocycle. Introduce now the $[2(m - p) + 1]$-forms on $G$ given by

$$\Omega'_p(\alpha_1...\alpha_p - 1):= k_{\alpha_1...\alpha_p - 1 i p_1+1...i m_1} d\omega^{i p} \wedge \ldots \wedge d\omega^{i m-1} \wedge \omega^{i m}$$

$$(\alpha = 1, \ldots, \dim H \quad ; \quad i = 1, \ldots, \dim G) \quad ;$$

(5.8)

$\Omega'_p(\alpha_1...\alpha_p - 1)$ is symmetric by construction and $\Omega'_p(1) = \Omega$. Clearly if we insert $F^{\alpha_1} \wedge \ldots \wedge F^{\alpha_p - 1}$ in (5.8), the result is a $(2m - 1)$-form in $\mathcal{S}(\mathcal{H}^*) \otimes \wedge (G)$. Using that $i_{X_{\alpha}} d\omega^i = -C_{\alpha i}^j \omega^j$, where $X_{\alpha} \in \mathcal{H}$ is now given by a LI vector field on $G$, we find

$$i_{(\alpha p} \Omega'_p(\alpha_1...\alpha_p - 1)} = - (m - p) C_{i_1}^{i p} k_{\alpha_1...\alpha_p - 1 i_1} \omega^j \wedge \omega^{i m} \wedge d\omega^{i p+1} \wedge \ldots \wedge d\omega^{i m-1} + \Pi'_p(\alpha_1...\alpha_p) ,$$

where we have introduced the $2(m - p)$-form

$$\Pi'_p(\alpha_1...\alpha_p) \equiv k_{\alpha_1...\alpha_p i p_1+1...i m_1} d\omega^{i p+1} \wedge \ldots \wedge d\omega^{i m} \quad .$$

(5.9)

$\Pi'_p(\alpha_1...\alpha_p)$ is obviously exact,

$$d\Omega'_p(\alpha_1...\alpha_p - 1) = \Pi'_p(\alpha_1...\alpha_p - 1) ; \quad \Pi'_p(\alpha_1...\alpha_p) = \Pi'_p(\alpha_1...\alpha_p) F^{\alpha_1} \wedge \ldots \wedge F^{\alpha_p} \quad .$$

(5.11)

We now use the $G$-invariance of $k_{i_1...i_m}$ to write

$$-C_{i_1}^{j} k_{\alpha_1...\alpha_p - 1 j i p_1+1...i m_1} = \sum_{s=1}^{p-1} C_{i_1}^{j} k_{\alpha_1...\alpha_s j...\alpha_{p-1} \alpha_p i p_1+1...i m_1}$$

$$+ \sum_{s=p+1} C_{i_1}^{j} k_{\alpha_1...\alpha_{p-1} \alpha_p i p_1+1...i s j...i m_1} + C_{s}^{j i_1} k_{\alpha_1...\alpha_p i p_1+1...i m-1 j} \quad .$$

(5.12)

The second term in the r.h.s. does not contribute to (5.9) by the Jacobi identity. Symmetrising the $\alpha$’s and using that $k$ is symmetric, eq. (5.12) gives

$$p C_{i_1}^{j} k_{\alpha_1...\alpha_p - 1 j i p_1+1...i m} \omega^j \wedge \omega^{i m} \wedge d\omega^{i p+1} \wedge \ldots \wedge d\omega^{i m-1} =$$

$$2 k_{\alpha_1...\alpha_p i p_1+1...i m} d\omega^{i p+1} \wedge \ldots \wedge d\omega^{i m} = 2\Pi'_p(\alpha_1...\alpha_p) \quad .$$

(5.13)

Thus, eq. (5.9) becomes

$$i_{(\alpha p} \Omega'_p(\alpha_1...\alpha_p - 1)} = \frac{2m - p}{p} \Pi'_p(\alpha_1...\alpha_p) = \frac{2m - p}{p} d\Omega'_p(\alpha_1...\alpha_p)$$

(5.14)

(a rapid way of seeing that exactness holds in each step is to notice that $i_{\alpha} [\text{Tr}(T_{\alpha_1} \ldots T_{\alpha_{p-1}} d\omega \wedge \ldots \wedge d\omega \wedge \omega)]$ is exact on account of the Maurer–Cartan equations). The proof is now

\textsuperscript{6}The primes are unnecessary at this stage, but will facilitate in Sec. 6 the comparison with the results in [33].
almost complete: the first steps of the descent are

\[ i_{\alpha_1} \Omega'_{(1)} = \frac{2m - 1}{1} d\Omega'_{(2)\alpha_1} \equiv dv_{\alpha_1}, \]

\[ i_{(\alpha_2 v_{\alpha_1})} = \frac{2m - 1}{1} i_{(\alpha_2 v_{\alpha_1})} = \frac{2m - 1}{2} 2m - 2 \frac{1}{1} d\Omega'_{(3)\alpha_1\alpha_2} \equiv dv_{\alpha_1}\alpha_2, \]

etc. Hence, the first set of equations in (5.3) is fulfilled with

\[ v_{\alpha_1...\alpha_p} := (\prod_{r=1}^{p} \frac{2m - r}{r}) \Omega'_{(p+1)\alpha_1...\alpha_p}. \] (5.16)

Thus, there is only one possible obstruction to the gauging of \( H \), which will be overcome iff

\[ i_{\{\alpha_m} \Omega'_{(m)\alpha_1...\alpha_{m-1}\}} = \Pi'_{(m)\alpha_1...\alpha_{m}} = k_{\alpha_1...\alpha_{m}} = 0 \] (5.17)

i.e., if the polynomial \( k_{i_1...i_m} \) on \( G \) is zero on \( H \), q.e.d. Clearly, the group \( G \) itself may never be gauged since by hypothesis \( k \) is non-zero on the whole \( G \) (but see below and Sec. 7).

The above procedure is a constructive one and, under the sole assumption that \( \Omega \) is a primitive cocycle for \( G \), provides through (5.16) and (5.8) the explicit solution for the form \( \alpha \) in (4.8) which is a \( d_C \)-cocycle when Proposition 5.1 holds. To find a closed expression for it, let us write

\[ \alpha = \sum_{p=1}^{m} \alpha_m(p)\Omega'_{(p)} \quad , \quad \alpha_m(p) = \prod_{r=1}^{p-1} \frac{2m - r}{r} \quad , \quad \Omega'_{(p)} = \Omega'_{(p)\alpha_1...\alpha_{p-1}} F^{\alpha_1} \wedge ... \wedge F^{\alpha_{p-1}}. \] (5.18)

Hence, and since \( d\omega^i = -(\omega \wedge \omega)^i \equiv -(\omega^2)^i \), the gauged form is given by

\[ \tilde{\alpha} = \sum_{p=1}^{m} \alpha_m(p) \tilde{\Omega}'_{(p)} \]

where

\[ \tilde{\Omega}'_{(p)} = \frac{(-1)^{m-p}}{2^{m-p}} k_{\alpha_1...\alpha_{p-1}i_1...i_{m-1}i_m} C^{i_1}_{j_2} C^{j_2}_{j_2} \cdots C^{i_m}_{j_2} F^{\alpha_1} \wedge ... \wedge F^{\alpha_{p-1}} \wedge \tilde{\omega}^{j_2} \wedge ... \wedge \tilde{\omega}^{j_2} \wedge \tilde{\omega} \] (5.19)

and \( \tilde{\omega} = g^{-1}(d - A^a L_a)g \). If we look at the coordinates of the symmetric polynomial in (5.19) as the symmetric trace \( \frac{1}{m!} \text{Tr}(T_{\alpha_1} \cdots T_{\alpha_{p-1}} T_{\alpha_{p}} \cdots T_{\alpha_{m}}) \), we may rewrite \( \tilde{\Omega}'_{(p)} \) as

\[ \tilde{\Omega}'_{(p)} = s \text{Tr}(F^{p-1} d\omega^{m-p} \tilde{\omega}) = (-1)^{m-p} s \text{Tr}(F^{p-1}(\tilde{\omega}^2)^{m-p} \tilde{\omega}) \] (5.20)

The symmetric trace over the \( m \) factors in (5.20) may be replaced by the trace of the sum \( S \) over all different ‘words’ which can be made from \( (p - 1) \) \( F \)'s and \( (m - p) \) \( \omega^2 \)'s by adding a weight \( (p - 1)!/(m - p)!/(m - 1)! \)

\[ \tilde{\Omega}'_{(p)} = (-1)^{m-p}(p-1)!/(m-p)!/(m-1)! \text{Tr}(S[F^{p-1}(\tilde{\omega}^2)^{m-p}]) \] (5.21)
expressions (5.20) and (5.21) may be compared with [33, eqs. (5.3) and (5.4)]. Thus, the gauged form \( \tilde{\alpha} \) is given by

\[
\tilde{\alpha} = \sum_{p=1}^{m} (-1)^{m-p} \alpha_m(p) \frac{(p-1)!(m-p)!}{(m-1)!} \Tr \{ S[F^{p-1}(\tilde{\omega}_2)^{m-p}] \}
\]

or, ignoring global factors,

\[
\tilde{\alpha} \propto \int_0^1 dt \, \Tr \left( \tilde{\omega} (tF + t(t-1)(\tilde{\omega}_2)^{m-1}) \right) .
\]

Eq. (5.22) provides explicitly the general form of the Weil cocycle \( \tilde{\alpha} \). It is formally equivalent to the expression [33, eq. (5.8)] of the relative Lie algebra cohomology cocycle \( \Omega^{(2m-1)} \in H^{2m-1}(G, \mathcal{H}; \mathbb{R}) \), which may be given as a form on the coset \( K = G/H \). This is now not surprising: it realises the isomorphism \( H^*(G, \mathcal{H}; \mathbb{R}) = H^*_{\text{DR}}(G/H) = H^*_H(G) \).

If (5.17) is not satisfied, \( \tilde{\alpha} \) will not be closed; instead (eq. (5.2)),

\[
d\tilde{\alpha} = -i_{(\alpha_m v_{a_2...a_m})} F^{a_1} \wedge \ldots \wedge F^{a_m} \equiv -\alpha_m(m) k_{a_1...a_m} F^{a_1} \wedge \ldots \wedge F^{a_m} .
\]

Let us denote by \( Q(A, F) \) the Chern–Simons (2m – 1)–form which is the local potential of (5.24) on \( K \), \( dQ(A, F) \propto \Tr(F \wedge \cdots \wedge F) \propto ch_m(F) \) (Chern character). \( Q(A, F) \) has formally the same structure as \( \tilde{\alpha} \) in (5.22); in fact, eq. (5.22) provides the expression of \( Q(A, F) \) if we replace \( \tilde{\omega} \) by the connection \( A \). Then, the form [10] \( \tilde{\alpha}' = \tilde{\alpha} - Q(A, F) \) will be closed and hence acceptable for an action leading to (2m – 2)–dimensional equations of motion. However, \( \tilde{\alpha}' \) will no longer be gauge–invariant due to \( Q(A, F) \); in fact, \( \delta_G \tilde{\alpha}' \) is proportional to the non–abelian anomaly which is tied to the existence of the polynomial \( k_{i_1...i_m} \) which is non–zero on \( \mathcal{H} \).

6 Gauging of forms and effective actions

Eq. (5.22) has the same structure as the general expression [33] which gives the WZW–type effective actions \( \text{`a la} \) D’Hoker and Weinberg [6, 28] on the coset \( K = G/H \) which for \( G \) simple are obtained from certain cocycles \( \Omega \in \wedge_{2m-1}(G) \). The key notion in all these
constructions is the projectability of forms (Secs. 2, 3) i.e., their horizontality and their $H$–invariance. Both in the case of gauging WZW terms which have a Lie group $G$ as the target manifold or in the expressions for the effective actions in [6, 28, 33], what matters at the end is the cohomology of $G/H$. The Chern–Simons–like appearance of the terms of all these formulae is due to eq. (5.24); note, however, that the transgression expression which gives the Chern–Simons form $Q(A, F)$ of $ch_m(F)$, $dQ(A, F) \propto \text{Tr}(F \wedge \cdots \wedge F)$, is not projectable.

The WZW effective actions or, equivalently, the relative cohomology cocycles on the coset $G/H$ are given by [33, eq. (3.7)] (cf. (5.18))

$$\Omega = \sum_{p=1}^{m} \alpha_m(p) \Omega_{(p)}$$

(6.1)

in terms of the forms [33, eq. (3.2)]

$$\Omega_{(p)} = (-1)^{p-1}2^{p-1}k^{a_1 \ldots a_p}_{a_{p+1} \ldots a_{2p}}C_{a_{2p-1} a_{2p}}^{i_1 \ldots i_{2p}} \cdots C_{a_{2m-3} a_{2m-2}}^{i_{m-1}} \wedge \mathcal{W}^{a_1} \wedge \cdots \wedge \mathcal{W}^{a_{2p-1}} \wedge \omega^{a_{2p}} \wedge \cdots \wedge \omega^{a_{2m-2}} \wedge \omega^b,$$

(6.2)

where $\mathcal{W}^a = -\frac{1}{2} C_{ab}^{\gamma} \omega^a \wedge \omega^b$ is the curvature of the LI–connection on $G(H, G/H) \omega^a = \mathcal{V}^a$ ($\mathcal{W}^a = (d\mathcal{V} + \mathcal{V} \wedge \mathcal{V})^a$) and $a, b$ are coset indices in $G \setminus H$. They are analogous to the present $\Omega'_{(p)}$'s here, eq. (5.8). The $\Omega_{(p)}$'s in (6.2) satisfy the (Weil model) condition [33, eq. (3.5)]

$$d \Omega_{(p)} = -\frac{1}{2} \Pi_{(p-1)} + \frac{2m-p}{2p} \Pi_{(p)} ,$$

(6.3)

where ([33, eq. (3.6)]; cf. (5.10))

$$\Pi_{(p)} = (-1)^{p-1}2^{p-1}k^{a_1 \ldots a_p}_{a_{p+1} \ldots a_{2p}} C_{a_{2p-1} a_{2p} \ldots}^{i_1 \ldots i_{2p}} \mathcal{W}^{a_1} \wedge \cdots \wedge \mathcal{W}^{a_{2p-1}} \wedge \omega^{a_{2p}} \wedge \cdots \wedge \omega^{a_{2m}} .$$

(6.4)

Notice that, since the exterior derivative in (6.3) acts on all forms on $G$ and hence on $\mathcal{W}^a$'s and $\omega^a$'s in (6.2), it corresponds to $d$ in the notation of this paper.

We see that we can relate $\Omega_{(p)}$, $\Pi_{(p)}$ (relevant in the analysis of the effective actions in [33]) with their $\Omega'_{(p)}$, $\Pi'_{(p)}$ counterparts (used here in the analysis of the gauged WZW terms) by means of the replacements $\mathcal{W}^a \rightarrow F^a$, $\omega^a \rightarrow \tilde{\omega}^i$. Hence, we move from the expressions of the effective actions in [33] to those of the present WZW gauged terms by the replacements $\mathcal{W}^a \rightarrow F^a$, $\frac{1}{2} C_{ab}^{\gamma} \omega^a \wedge \omega^b \rightarrow \frac{1}{2} C_{ik}^{\gamma} \tilde{\omega}^i \wedge \tilde{\omega}^k$. The reason for this common structure may be also understood in terms of the equivariant cohomology (in the Cartan model, l.h.s. of diagram (4.7)). In the problem of gauging WZW forms discussed in the

7In fact, if the action of a group on a manifold is locally free, the equivariant and relative (coset) cohomology rings coincide.

8The $H$–horizontal forms $\omega^a \ (i_{\psi, \omega^a} = 0, \ a \ in \ H, \ a \ in \ the \ coset \ G \setminus H)$, appearing in the relative cohomology cocycles which define effective actions, find their counterparts here in $\psi(\omega^i) = \tilde{\omega}^i = [g^{-1}(d - A^j L_j)]g$. The $\tilde{\omega}^i$ (in contrast with $\omega^i$) are horizontal by construction; explicitly, $i_a(g^{-1}(d - A^j L_j)g) = (g^{-1}(L_\alpha - d i_\alpha - L_\alpha))g = 0.$
previous sections, the Weil algebra $\mathcal{W}(\mathcal{H})$ is generated by the gauge fields $A^\alpha$ and the curvature $F^\alpha$. Therefore, the forms $\omega$ spanning the exterior algebra $\wedge(G)$ are (minimally) coupled by $\psi(\omega) = \tilde{\omega}$ to $A$. However, the structure of the Cartan model expressions does not depend on the specific connection and curvature, and the above effective actions also correspond to equivariant cocycles. The only difference is that, when we are interested in effective actions and relative cohomology, we take as generators of the Weil algebra $\mathcal{W}(\mathcal{H})$ the $H$–connection $V^\alpha$ and its curvature $W^\alpha$. The ‘minimal coupling’ analogue to (4.2) is then given by

$$\psi(\omega) = \prod_\alpha (1 - V^\alpha i_\alpha) \omega = \omega - V \equiv U$$

(6.5)

where $U$ is, clearly, the coset or $(\mathcal{G} \setminus \mathcal{H})$–component of the LI canonical form $\omega = \omega|_H + \omega|_K \equiv \mathcal{V} + \mathcal{U}$.

This explains the similarity of the final expressions. For instance, eq. (6.3) may be computed in the Cartan model. Using that $\psi^{-1}(\Omega(p)) = \Omega(p)(V = 0)$ we see that

$$\psi^{-1}(\Omega(p)) = (-1)^{m-1}2^{m-1}\Omega'(p)_{\alpha_1...\alpha_{p-1}} W^{\alpha_1} \ldots W^{\alpha_{p-1}}$$

(6.6)

Thus, with (5.8) and (5.14) we find

$$d_C(\psi^{-1}(\Omega(p))) = d_C((-1)^{m-1}2^{m-1}\Omega'(p)_{\alpha_1...\alpha_{p-1}} W^{\alpha_1} \ldots W^{\alpha_{p-1}}) =$$

$$(-1)^{m-1}2^{m-1}\left(\prod_{p-1}^{\prime} W^{\alpha_1} \ldots W^{\alpha_{p-1}} - \frac{2m - p}{p} \prod_{p-1}^{\prime} W^{\alpha_1} \ldots W^{\alpha_{p-1}} \right)$$

(6.7)

which corresponds to (6.3) once $\psi$ has been applied to it (note that, for (6.5), $\psi\Pi^{\prime}(p) = \frac{1}{(-1)^{m-1}2^{m-1}}\Pi(p)$). This shows that the Cartan derivative $d_C$ is equivalent to $d$ once the minimal coupling has been performed.

Summarising, we may express these results in the following general form

**Theorem 6.1**

Let $\Omega$ be a closed $(2m - 1)$–Lie algebra cocycle given by a $(2m - 1)$–form on the manifold $G$, and let $H$ be the structure group of the principal bundle $G(H, K)$, $K = G/H$. Let $F$ (resp. $W$) be the curvature associated with $A$ (resp. with the LI $H$–connection $V$). Then it will be possible to construct from $\Omega$ a) an effective action $\bar{\Omega}$ on the coset manifold $K$ and b) an $H$–gauged, closed and gauge invariant form $\tilde{\alpha}$ if $\Omega$ is a Lie algebra cocycle in $H^{2m-1}(\mathcal{G}, \mathbb{R})$ defined by a symmetric invariant polynomial which vanishes on $H$. In this case, $\bar{\Omega}$ (eq. (6.1)) and $\alpha$ (eq. (5.18)) will be respectively, cocycles in the relative Lie algebra $H^{2m-1}(\mathcal{G}, \mathcal{H}; \mathbb{R})$ and equivariant $H_{H}^{2m-1}(G)$ cohomologies.

The above constructions constitute, in fact, a physics inspired proof of the isomorphism between the relative and equivariant (for the action of $H$ on $G$) cohomologies. The obstruction to constructing the effective action and to gauging the WZW term has the same geometrical origin; it is given in terms of an anomaly, which appears when $k_{\alpha_1...\alpha_m}$ is non–zero.
7 The gauging of left and right symmetries

It was stated in Sec. 5 that $G$ itself may never be gauged. But being $G$ a compact group the cocycles $\Omega$ on $G$ are both LI and right invariant (RI): there is a $G^L \times G^R$ symmetry. Thus, although a simple factor $G$ may not be gauged, we may expect to have the unwanted contributions to $d\tilde{\alpha}$ from each factor to cancel each other. Moreover, even if $k_{a_1\ldots a_m}$ is non-zero on $\mathcal{H}$, we may use this fact to overcome the obstruction which would be present for $H \times 1$ or $1 \times H$ separately. We shall now do this and provide general expressions for the gauged WZW terms following the above pattern.

The $R$ and $L$ actions are generated, respectively, by the LI and RI vector fields $X^L_\alpha$ and $-X^R_\alpha$ (the sign is introduced to compensate for the $-$ sign in $[X^L_\alpha, X^R_\beta] = -C^\gamma_{\alpha\beta} X^R_\gamma$ which is required if we conventionally adopt $[X^L_\alpha, X^L_\beta] = C^\gamma_{\alpha\beta} X^L_\gamma$). Let us denote by $i_{X_L^\alpha}$ the inner product $i_{X_L^\alpha}$ and by $i_{R^\alpha}$ the inner product $i_{(-X^R_\alpha)}$. Let $(A^L_\alpha, F^L_\alpha)$ and $(A^R_\alpha, F^R_\alpha)$ be two different copies of the Weil algebra $\mathcal{W}(\mathcal{H})$ (the indices $L$ and $R$ in $(A, F)$ correspond to the accompanying LI or RI vector fields and hence to the $R$ and $L$ actions). Then,

\[ F^\alpha_L i_{L^\alpha} \omega = F_L, \quad F^\alpha_R i_{R^\alpha} \omega = -g^{-1} F_R g \]  

In the sequel, the following relations will be useful

\[ F^\alpha_L i_{L^\alpha} \omega^2 = [F, \omega], \quad F^\alpha_R i_{R^\alpha} \omega^2 = [-g^{-1} F_R g, \omega], \]  

\[ (1 \otimes d)(g^{-1} F_R g) \equiv d(g^{-1} F_R g) = [g^{-1} F_R g, \omega]. \]  

Let us now introduce the $(2m-1)$-forms $\Upsilon_{[p,q]}$ and the $(2m)$-forms $\Pi_{[p,q]}$ and $\Gamma_{[p,q+1]}$ by\(^9\)

\[ \Upsilon_{[p,q]} = s\text{Tr}(\omega F^p_L (g^{-1} F_R g)^q (\omega^2)^{m-p-q-1}) \]  

\[ \Pi_{[p,q]} = s\text{Tr}(F^p_L (g^{-1} F_R g)^q (\omega^2)^{m-p-q}) \]  

\[ \Gamma_{[p,q+1]} = s\text{Tr}(\omega F^p_L (g^{-1} F_R g)^q [g^{-1} F_R g, \omega] (\omega^2)^{m-p-q-2}) \]  

We show in the Appendix that

\[ \Gamma_{[0,q+1]} = -\frac{2}{q+1} \Pi_{[0,q+1]} \]  

\[ s\text{Tr}(\omega F^p_L [F_L, \omega] (g^{-1} F_R g)^q (\omega^2)^{m-p-q-2}) = -\frac{1}{p+1} (2 \Pi_{[p+1,q]} + q \Gamma_{[p+1,q]}) \]  

To find now the equivariant extension of $\Omega$ we need to compute $d_C \Upsilon_{[p,q]}$, and hence the action of $d$, $F^\alpha_L i_{L^\alpha}$ and $F^\alpha_R i_{R^\alpha}$ on $\Upsilon_{[p,q]}$. It is not difficult to check, using (7.1), (7.2), (7.3) and (7.6), that

\[ d\Upsilon_{[p,q]} = -\Pi_{[p,q]} - q \Gamma_{[p,q]} \]  

\(^9\) The previous forms $\Omega'_{(p)}$, $\Pi'_{(p)}$, are particular cases of $\Upsilon_{[p,q]}$, $\Pi_{[p,q]}$: $\Upsilon_{[0,0]} = (-1)^m \Omega'$ (eq. (5.6)) and, for $F_L \equiv F$, $\Upsilon_{[p,0]} = (-1)^{m-p-1} \Omega'_{(p+1)}$ (cf. (5.20)) and $\Pi_{[p,0]} = (-1)^{m-p} \Pi'_{(p)}$ (cf. (5.10)).
Let $\Omega$ be a cocycle on $\text{Cartan model}$. We may then state the following

\begin{equation}
F^\alpha_\omega i_{L_\omega} Y_{[p,q]} = \Pi_{[p+1,q]} + \frac{m-p-q-1}{p+1}(2\Pi_{[p+1,q]} + q\Gamma_{[p+1,q]}) \quad ,
\end{equation}

\begin{equation}
F^\alpha_\omega i_{R_\omega} Y_{[p,q]} = -\Pi_{[p,q+1]} + (m-p-q-1)\Gamma_{[p,q+1]} \quad .
\end{equation}

Now, let us introduce $(s \geq 1)$

$$
\Omega_{[s]} := \sum_{p+q=s-1} \frac{(m-p-1)!(m-q-1)!}{p!q!} Y_{[p,q]} \quad , \quad \Omega_{[1]} = (-1)^{m-1}(m-1)!\Omega \quad .
$$

Using (7.7), (7.8) and (7.9) we find (see (A.7))

\begin{equation}
(F^\alpha_\omega i_{L_\omega} + F^\alpha_\omega i_{R_\omega})\Omega_{[s]} = -(2m-s)(m-s)d\Omega_{[s+1]} \quad .
\end{equation}

We now observe that eq. (7.11) has the same structure as (5.14). Hence, the cohomological descent shows that the form

$$
\alpha = \sum_{s=1}^{m} \frac{(-1)^{m-s}}{(2m-s)!(m-s)!}\Omega_{[s]} \quad (7.12)
$$

verifies $(d\Omega_{[1]} = 0)$

\begin{equation}
d_C\alpha \equiv (d - F^\alpha_\omega i_{L_\omega} - F^\alpha_\omega i_{R_\omega})\alpha = \sum_{s=2}^{m} \frac{(-1)^{m-s}}{(2m-s)!(m-s)!}d\Omega_{[s]}
\end{equation}

$$
+ \sum_{s=1}^{m-1} \frac{(-1)^{m-s}}{(2m-s-1)!(m-s-1)!}\Omega_{[s+1]} - (F^\alpha_\omega i_{L_\omega} + F^\alpha_\omega i_{R_\omega}) \frac{1}{m!}\Omega_{[m]}
$$

\begin{equation}
= -(F^\alpha_\omega i_{L_\omega} + F^\alpha_\omega i_{R_\omega}) \frac{1}{m!}\Omega_{[m]} \quad .
\end{equation}

Now, using (A.8)

\begin{equation}
(F^\alpha_\omega i_{L_\omega} + F^\alpha_\omega i_{R_\omega})\Omega_{[m]} = \Pi_{[m,0]} - \Pi_{[0,m]} \equiv s\text{Tr}(F^m_L) - s\text{Tr}((g^{-1}F_Rg)^m) \quad ,
\end{equation}

so that

\begin{equation}
d_C\alpha = -\frac{1}{m!} \left(s\text{Tr}(F^m_L) - s\text{Tr}((g^{-1}F_Rg)^m)\right) = -\frac{1}{m!} \left(s\text{Tr}(F^m_L) - s\text{Tr}(F^m_R)\right) \quad .
\end{equation}

In particular, if $F_L = F_R \equiv F$ eq. (7.15) is zero and $\alpha$ is an equivariant cocycle in the Cartan model. We may then state the following

**Proposition 7.1**

Let $\Omega$ be a cocycle on $G$, $H \times H$ the symmetry to be gauged. Then, the extension $\alpha$ of $\Omega$ given by

\begin{equation}
\alpha = \sum_{s=1}^{m} \sum_{p+q=s-1} \frac{(-1)^{m-s}(m-p-1)!(m-q-1)!}{(2m-s)!(m-s)!p!q!} s\text{Tr}(\omega F^p_L (g^{-1}F_Rg)^q (\omega^2)^{m-p-q-1})
\end{equation}

is an equivariant extension of $\Omega$ for $F_L = F_R = F$. 

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To make contact with the work in [10] let us note that the symmetric trace and the double sum may be replaced by a ‘trinomial’ using that
\[
\text{Tr}(F_L + g^{-1} F_R g + (\omega^2))^{m-1} = \sum_{p,q} \frac{1}{p!q!(m-p-q-1)!} s \text{Tr}(F_L^p(g^{-1} F_R g)^q(\omega^2)^{m-p-q-1}).
\]
(7.17)
Thus, splitting the sum over \(p\) and \(q\) in a sum over \(s \equiv p + q + 1\) and a sum over \(s\) and recalling that the Beta function
\[
B(v + 1, w + 1) = \int_0^1 dt \, t^v (1 - t)^w = \frac{v!w!}{(v+w+1)!},
\]
(7.18)
we obtain the expression
\[
\sum_{s=1}^m \sum_{p+q=s-1} \frac{(-1)^{m-s}}{(2m-s)!(m-s)!} \frac{(m-p-1)!(m-q-1)!}{p!q!} s \text{Tr}(\omega F_L^p(g^{-1} F_R g)^q(\omega^2)^{m-p-q-1})
\]
\[
= \int_0^1 dt \text{Tr}\left(\omega(tF_L + (1-t)g^{-1} F_R g + t(t-1)\omega^2)^{m-1}\right),
\]
(7.19)
which recovers [10, eq. (7.41) (ignoring the Chern–Simons terms there)] once the minimal coupling has been performed. For \(1 \times H \equiv H, F = F_L, F_R = 0\,\text{we obtain eq. (5.23) (before minimal coupling).}

In the present framework, the minimal coupling is implemented by means of the gauging map (4.2) which here (for \(A_L \neq A_R\)) takes the form
\[
\psi = \prod_\alpha (1 - A_L^\alpha i_{L\alpha} - A_R^\alpha i_{R\alpha})
\]
(7.20)
so that
\[
\tilde{\omega} = \psi(\omega) = \omega - A_L + g^{-1} A_R g = g^{-1}(d - gA_L g^{-1} + A_R) g \equiv g^{-1} D g.
\]
(7.21)

For \(A_L = A, A_R = 0\,\text{eq. (7.21) reduces to } D = d - A^\alpha L_\alpha \) (eq. (3.1)).

Example 7.1
Let us illustrate the above with the lowest example for \(A_L = A_R = A, F_L = F_R = F\) (the three-cocycle, \(m = 2\) [10, 12]. In this case, our expression above has three terms, one corresponding to \(s = 1\) \((p = q = 0)\) and two for \(s = 2\) \((p = 1, q = 0\,\text{and } p = 0, q = 1)\). Explicitly (cf. [10, eq. (7.18)]),
\[
\alpha = -\frac{1}{3!} s \text{Tr}(\omega(\omega^2)) + \frac{1}{2!} s \text{Tr}(\omega F) + \frac{1}{2!} s \text{Tr}(\omega g^{-1} Fg)
\]
(7.22)
where the first term corresponds to the original three-cocycle \(\Omega \propto \text{Tr}(\omega^3)\). Substituting \(\omega\) by its gauged version \(\tilde{\omega} = \omega - A + g^{-1} Ag\) (cf. (7.21)) we obtain the gauged WZW action in a two–dimensional spacetime.
8 Concluding remarks

WZW terms on a simple group $G$ as the target manifold are obtained from odd forms $\Omega \in H^{2m-1}(G, \mathbb{R})$, $m \geq 2$, which define non-trivial primitive cocycles in the Lie algebra cohomology. These are in turn characterised by primitive symmetric invariant polynomials of order $m$. In this paper we have given a closed and general expression for the forms $\alpha$ which provide the $H$–gauged version or ‘extension’ ($H \subset G$) of such Lie algebra cocycles $\Omega$. This expression is explicitly constructed from the invariant polynomials on $G$ (which are all known for $G$ simple). We have also explained the similarity between the gauged extensions of the various $\Omega$’s and the expression of the WZW–type $G$–invariant effective actions of D’Hoker and Weinberg relative to the coset $G/H$. The correspondence among these two types of action terms constitutes a physical realisation of the mathematical isomorphism $H^*_H(G) \sim H^*(G, H; \mathbb{R})$ between the equivariant and relative cohomologies.

Since we have been concerned here with simple algebras, only odd forms $\Omega$ on $G$ have entered into our discussion since the primitive $2m$–cocycles on $G$ are coboundaries (exact forms on $G$). We might, of course, remove the semisimplicity condition. The cohomology theory in the non–semisimple case, however, is not complete, so that a general constructive process (similar to the one presented here) does not exist (nevertheless, we wish to mention here that the contraction of Lie algebras may provide a systematic procedure to discuss the cohomology of non–semisimple algebras, a first step to extend the physical considerations of the present paper). Also, in the non–semisimple case it is possible to introduce non–trivial cocycles which take values in a representation space $V$ of $G$, i.e., elements in $H^k_\rho(G, V)$ where $\rho$ is a representation of $G$ (by the Whitehead Lemma, $H^k_\rho(G, V) = 0 \forall k \geq 0$ if $\rho$ is non–trivial and $G$ is semisimple). In particular, this approach might lead to a different class of topological terms [15] (which are not obtained by gauging a WZW term $\Omega$ and hence are zero for $A = 0 = F$) in even dimensional spacetime and which may be added to the kinetic term for gauge theories with noncompact groups. We may come back to these problems and to their extension to the supersymmetric case (see e.g., [44, 45]) in the future.

Note added. On the subject of the topological terms in [15] we have just become aware of [46].

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We prove here some expressions used in Sec. 7. Eq. (7.5) may be derived using that the forms $\Upsilon_{[p,q]}$ are defined by a symmetrised trace. Thus,

$$0 = s\text{Tr}(\omega (g^{-1}F_R g)^{q+1}\omega^2)^{m-q-2}) = s\text{Tr}([\omega,\omega](g^{-1}F_R g)^{q+1}\omega^2)^{m-q-2}) = 0$$

(A.1)

which, since $[\omega^2,\omega] = 0$ (Jacobi), reduces to eq. (7.5)

$$2\Pi_{[0,q+1]} + (q+1)\Gamma_{[0,q+1]} = 0 .$$

(A.2)

Eq. (7.6) follows similarly from $s\text{Tr}(\omega (g^{-1}F_R g)^{q}(\omega^2)^{m-p-q-2}) = 0$.

The relations (5.14) (used to find $\alpha$ when gauging the (left) symmetry group $H$) may be recovered from (7.7) and (7.8). For $q = 0$, these equations give

$$d\Upsilon_{[p,0]} = -\Pi_{[p,0]} , \quad F^\alpha_i L_\alpha \Upsilon_{[p,0]} = \Pi_{[p+1,0]} + \frac{2(m-p-1)}{p+1} \Pi_{[p+1,0]} .$$

(A.3)

For $F_L \equiv F$, and recalling (see footnote 9) that $\Upsilon_{[p,0]} = (-1)^{m-p-1}\Omega'_{(p+1)}, \Pi_{[p,0]} = (-1)^{m-p}\Pi'_{(p)}$ we find

$$d\Omega'_{(p+1)} = \Pi'_{(p)} , \quad F^\alpha_i \Omega'_{(p+1)} = \frac{2m-p-1}{p+1} \Pi'_{(p+1)}$$

(A.4)

which reproduce (5.14). An equivalent result is obtained for the right symmetry group by simply setting $p = 0$ in (7.7) and (7.9)

$$d\Upsilon_{[0,q]} = -\Pi_{[0,q]} - q\Gamma_{[0,q]} , \quad F^\alpha_R i_R \alpha \Upsilon_{[0,q]} = -\Pi_{[0,q+1]} + (m-q-1)\Gamma_{[0,q+1]} .$$

(A.5)

Then, using (A.2) (cf. (A.3))

$$d\Upsilon_{[0,q]} = \Pi_{[0,q]} , \quad F^\alpha_R i_R \alpha \Upsilon_{[0,q]} = -\Pi_{[0,q+1]} - \frac{2(m-q-1)}{q+1} \Pi_{[0,q+1]} .$$

(A.6)
Eq. (7.11) may be computed as follows

\[
(F_L^\alpha i_{L\alpha} + F_R^\alpha i_{R\alpha})\Omega_{[s]} = \sum_{p+q=s-1}^{p+q=s-1} \frac{(m-p-1)!(m-q-1)!}{p!q!} \left[ \left( 1 + \frac{2(m-s)}{p+1} \right) \Pi_{[p+1,q]} - \Pi_{[p,q+1]} \right] + \frac{q(m-s)}{p+1} \Gamma_{[p+1,q]} - \Pi_{[p,q+1]} + (m-s)\Gamma_{[p,q+1]}
\]

\[
= \frac{(m-s-1)!(m-1)!}{s!} \left\{ (2m-s)(m-s)\Pi_{[s,0]} - s(m-s)(\Pi_{[0,s]} - (m-s)\Gamma_{[0,s]}) \right\} + \sum_{p+q=s-1}^{p+q=s-1} \frac{(m-p-1)!(m-q-2)!}{p!(q+1)!} \left[ (m-p)(2m-2s+p) \right. \\
\left. - (m-q-1)(q+1) \right] \Pi_{[p,q+1]} + (q+1) \left[ (m-p)(m-s) + (m-q-1)(m-s) \right] \Gamma_{[p,q+1]}
\]

\[
= (2m-s)(m-s) \left\{ \frac{(m-s-1)!(m-1)!}{s!} \left( \Pi_{[s,0]} - \Pi_{[0,s]} \right) \right\} + \sum_{p+q=s-1}^{p+q=s-1} \frac{(m-p-1)!(m-q-2)!}{p!(q+1)!} \left[ \Pi_{[p,q+1]} + (q+1)\Gamma_{[p,q+1]} \right]
\]

\[
= -(2m-s)(m-s) \left\{ \frac{(m-s-1)!(m-1)!}{s!} \left( \gamma_{[s,0]} + \gamma_{[0,s]} \right) \right\} + \sum_{p+q=s-1}^{p+q=s-1} \frac{(m-p-1)!(m-q-2)!}{p!(q+1)!} \gamma_{[p,q+1]}
\]

\[
= -(2m-s)(m-s)d\Omega_{[s+1]},
\]

(A.7)

where we have used (7.8), (7.9) in the first equality, then we have changed the summation indices \( p+1 \rightarrow p \) and \( q \rightarrow q+1 \) in the first and second terms to obtain the second equality, and eq. (A.2) has been used in the third one. Finally, eq. (7.7) and the first identity in (A.6) lead to the fourth equality which trivially rearranges into \(-2m-s)(m-s)d\Omega_{[s+1]}\).

We also note that the first equality of this calculation also shows that

\[
(F_L^\alpha i_{L\alpha} + F_R^\alpha i_{R\alpha})\Omega_{[m]} = \sum_{p+q=s-1}^{p+q=s-1} \frac{(m-p-1)!(m-q-1)!}{p!q!} \left[ \Pi_{[p+1,q]} - \Pi_{[p,q+1]} \right] = \Pi_{[m,0]} - \Pi_{[0,m]}
\]

(A.8)

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