Wave Optics in Spacetimes with Compact Gravitating Object

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Abstract

We investigate the wave optics in spherically symmetric spacetimes: Schwarzschild black hole, spherical star with a perfect absorbing surface, and massless/massive Ellis wormholes. Assuming a point wave source, wave pattern and power spectrums for scattering waves are obtained by solving the scalar wave equation numerically. We found that the power spectrum at the observer in the forward direction shows oscillations with two characteristic periods determined by the interference effect associated with the photon sphere and the diffraction effect due to the absorbing boundary condition inside of the photon sphere.

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I. INTRODUCTION

The photon sphere is a set of circular unstable photon orbits around a gravitating object and it composes a two dimensional sphere with a constant radius for spherically symmetric static spacetimes. Recently, related to the existence of the photon sphere, bright ring and the ‘shadow’ of M87 have been observed [1]. The properties of shadows of strong gravitating object such as black holes have been studied in detail [2–10]. The shadow is defined as a region of absorption on the observer’s screen and its rim corresponds to the photon sphere projected onto the observer’s screen. By its definition, information inside of the photon sphere cannot be detectable by light rays unless an illuminating light source is placed inside of the photon sphere.

Although the photon sphere is introduced in terms of null geodesics, which are rays in the geometrical optics, the relation to the quasi-normal modes of the black hole has been also discussed so far [11]. The quasi-normal modes of black holes are obtained as poles of the scattering matrix in the complex frequency domain and its eikonal limit corresponds to light rays of the unstable photon orbits around black holes. Based on established treatment of wave scattering problems (partial wave decomposition, phase shift etc. see references [12,13]), the photon sphere is related to Regge poles which are poles of the scattering matrix in the complex angular momentum space. Thus it is possible to understand properties of spacetimes with strong gravitating objects using wave optics. As an application to this direction, imaging of black hole photon sphere with waves was investigated by Kanai and Nambu [14] and Nambu and Noda [15]. Reconstruction of black hole images from scattering waves was attempted by Fourier transform of scattered waves. Other approaches to the wave scattering by black holes such as evaluation of differential cross section have been investigated by many authors [16–29]. Recently, wave scattering by stars are also discussed [30,32].

Concerning the wave optical effect for the weak gravitational lensing, interference fringe patterns in the spatial domain (scattering amplitude) and the frequency domain (power spectrum) are expected. They are caused by interference between two coherent light rays (direct rays). For the gravitational lensing by a black hole, an additional interference effect associated with the photon sphere is expected. Light rays can go around the black hole an arbitrary number of times (orbiting) and the direct rays and these winding rays can interfere and additional component of fringe appears in the power spectrum. In the paper [15], the analytic expression for scattering waves by the Schwarzschild black hole was derived in the eikonal limit (the leading order of wave effect) and wave optical images of Einstein rings and the photon sphere were obtained. Moreover, modulation of power spectrums caused by the photon sphere was also clarified. As an astrophysical application of wave optics to gravitational lensing systems, Yoo et al. [33] investigated behavior of power spectrums from a point source and discussed possibility to distinguish Ellis wormhole spacetimes from spacetimes with a point mass. Their analysis is based on weak field approximation of gravitational field (weak lensing effect). They concluded that the Ellis wormhole spacetime shows different behavior of the power spectrum due to $r^{-2}$ law of the wormhole’s gravitational potential. However, their analysis lacks strong lensing effect associated with the photon sphere.

In this paper, we consider wave optical properties of spacetimes with the photon sphere. Let us consider a situation that a wave source is located outside of the photon sphere of the gravitating object and an observer detects scattered signal. In the geometrical optics,
as light rays captured by the photon sphere cannot escape from it, we cannot look inside of the photon sphere using a light source placed outside of the photon sphere if objects inside of the photon sphere do not emit and reflect light rays. However, in the wave optics, even if a part of wave propagates inside of the photon sphere, it can escape to outside due to the wave effect and extract information on inside of the photon sphere. This expectation is directly connected to the discrimination problem of gravitating objects called black hole mimickers such as ultra compact objects, which have photon spheres, using wave optical effects. As the black hole mimickers, we consider a spherical star with a perfect absorbing surface and massless/massive Ellis wormholes in this paper.

We mainly focus on behavior of power spectrums in the forward scattering case; the path difference between two direct rays is zero and if the gravitating object has no structure like photon sphere, we do not have any interference fringe in the power spectrum. However, if the gravitating object has the photon sphere or some structures, the modulation in the power spectrum caused by interference between direct rays and orbiting rays is expected.

The structure of the paper is as follows. In Sec. II, we introduce our setup of wave scattering problem in spherically symmetric spacetimes and explain our numerical methods. We present our numerical results in Sec. III for the Schwarzschild spacetime and the Ellis wormhole spacetime. In Sec. IV, we apply a formula in the wave optics to explain interference appeared in power spectrums. Sec. V is devoted to summary and conclusion.
II. WAVE OPTICS IN SPHERICALLY SYMMETRIC STATIC SPACETIMES

In this section, we introduce the setup of the problem and numerical method to obtain the scattered wave by spherical gravitating objects.

A. Wave equation with a point source

![Configuration of our wave scattering problem. An observer receives waves from a point wave source.](image)

**FIG. 1**: Configuration of our wave scattering problem. An observer receives waves from a point wave source.

Fig. 1 shows our setup of wave scattering problem. We consider a massless scalar field as the benchmark treatment for wave scattering problems and we do not consider polarization degrees of freedom which are necessary for the electromagnetic and gravitational waves. The background geometry is assumed to be static and spherically symmetric spacetimes with the metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2d\Omega^2.$$  

For a monochromatic stationary wave with time dependence $e^{-i\omega t}$, the wave equation for the massless scalar field reduces to the following Helmholtz type equation with a source term

$$-g^{00}\omega^2\Phi + \frac{1}{\sqrt{-g}}\partial_j\left(\sqrt{-g}g^{jk}\partial_k\Phi\right) = -\delta^3(\vec{r},\vec{r}_s), \quad i, j, k = r, \theta, \phi,$$

where $\delta^3(\vec{r},\vec{r}_s)$ denotes the invariant delta function $\frac{1}{\sqrt{-g}}\delta^3(\vec{r} - \vec{r}_s)$. We place a point source with a delta function profile at $r = r_s, \theta = \pi$. Owing to the symmetry of the spacetime, the wave function can be separated as

$$\Phi(r, \theta) = \frac{1}{r}\sum_{\ell=0}^{\infty} R_\ell(r)P_\ell(\cos \theta),$$

and the radial wave function $R_\ell$ obeys the following Schrödinger type equation

$$\frac{d^2R_\ell}{dx_{\text{tot}}^2} + \left(\omega^2 - V_{\text{eff}}\right)R_\ell = a_s(-)^\ell(\ell + 1/2)\delta(r - r_s),$$

where $a_s$ is a constant related to the mass of the source.
where a constant $a_s$ denotes the amplitude of the point source and the tortoise coordinate is introduced by

$$x_{\text{tot}} = \int \frac{dr}{\sqrt{fg}},$$

and the effective potential is defined by

$$V_{\text{eff}} = \ell (\ell + 1) \frac{r}{r^2} + \frac{fg}{2r}.$$  

In this paper, we consider the Schwarzschild spacetime and the Ellis wormhole spacetime. The metric of the Schwarzschild spacetime with the tortoise coordinate $x_{\text{tot}}$ is

$$ds_{\text{Schw}}^2 = \left(1 - \frac{2M}{r}\right)(-dt^2 + dx_{\text{tot}}^2) + r^2 d\Omega^2, \quad (7)$$

$$x_{\text{tot}} = r + 2M \ln \left(\frac{r}{2M} - 1\right), \quad -\infty < x_{\text{tot}} < +\infty. \quad (8)$$

The effective potential is

$$V_{\text{eff}} = \left(1 - \frac{2M}{r}\right)\left(\frac{\ell (\ell + 1)}{r^2} + \frac{2M}{r^3}\right). \quad (9)$$

The metric of the Ellis wormhole (massless case) is

$$ds_{\text{WH}}^2 = -dt^2 + dx^2 + r^2 d\Omega^2, \quad (10)$$

$$r = \sqrt{x^2 + a^2}, \quad -\infty < x < +\infty, \quad (11)$$

where the parameter $a$ represents size of the wormhole’s throat. The effective potential is

$$V_{\text{eff}} = \frac{\ell (\ell + 1)}{r^2} + \frac{a^2}{r^4}. \quad (12)$$

FIG. 2: Effective potentials for the Schwarzschild spacetime and the Ellis wormhole (massless) spacetime. The plot is with $\ell = 2, a = 3M$. The circumference radius of the photon sphere for both spacetimes is $3M$.

The metric for the massive Ellis wormhole is presented in Appendix (Eq. $[\text{A2}]$).
B. Our numerical methods

We present here numerical methods adopted in our analysis. We first obtain the solution of the radial equation (4) numerically. We impose two boundary conditions at \( r = r_{\text{in}} \) (an inner boundary corresponds to the black hole horizon, star’s surface, another asymptotic flat region of wormhole) and \( r = r_{\text{out}} \) (an outer boundary corresponds to the spatially far region). We prepare two solutions of the homogeneous radial equation without a source term:

\[
\begin{align*}
  &u_1(r), \quad r \in [r_{\text{in}}, r_s), \quad \text{BC is imposed at } r_{\text{in}}, \\
  &u_2(r), \quad r \in (r_s, r_{\text{out}}], \quad \text{BC is imposed at } r_{\text{out}}
\end{align*}
\]

where \( u_1 \) is obtained by integrating the radial equation from \( r_{\text{in}} \) to \( r_s \), and \( u_2 \) is obtained by integrating the radial equation from \( r_{\text{out}} \) to \( r_s \).

Radial functions \( u_1 \) and \( u_2 \) do not satisfy the boundary condition at the source \( r_s \) which is obtained by integrating (4) around \( r_s \):

\[
\left( \frac{dR_2}{dr} - \frac{dR_1}{dr} \right)_{r=r_s} = a_s(-)^\ell(\ell + 1/2) \equiv \Delta_\ell, \quad (R_2 - R_1)_{r=r_s} = 0,
\]

where \( R_1(r) = R(r \leq r_s) \) and \( R_2(r) = R(r \geq r_s) \). Using \( u_1 \) and \( u_2 \), we introduce new radial functions as

\[
R_1 = c_1 u_1, \quad R_2 = c_2 u_2,
\]

where \( c_1, c_2 \) are constants to be determined by the matching condition (13) at \( r_s \):

\[
c_2u_2'(r_s) - c_1u_1'(r_s) = \Delta_\ell, \quad c_1u_1(r_s) = c_2u_2(r_s).
\]

We obtain

\[
c_1 = \left( \frac{u_2}{W[u_1, u_2]} \right)_{r=r_s} \Delta_\ell, \quad c_2 = \left( \frac{u_1}{W[u_1, u_2]} \right)_{r=r_s} \Delta_\ell,
\]

where \( W = u_1u_2' - u_2u_1' \) is the Wronskian. Thus, \( R_1, R_2 \) are

\[
R_1 = \frac{\Delta_\ell}{W[u_1, u_2]_{r_s}} u_1(r_s)u_2(r_s), \quad R_2 = \frac{\Delta_\ell}{W[u_1, u_2]_{r_s}} u_1(r_s)u_2(r).
\]

To obtain numerical solutions \( u_1, u_2 \), we adopted the 4th-order Runge-Kutta method. We obtained the radial wave function within the relative errors \( 10^{-5} \). Concerning the value \( \ell_{\text{max}} \) for summation of the partial waves (3), we determined it by checking convergence of \( \Phi \) at \( r_{\text{obs}} = 20M, \theta = 0 \). For \( M\omega < 1 \), \( \Phi \) converges with \( \ell_{\text{max}} = 10 \sim 13 \). For \( 1 < M\omega \leq 10 \), we found that \( \ell_{\text{max}} = 9 \times (M\omega + 1) \) for the black hole cases. For calculations with stars and wormhole, we determined \( \ell_{\text{max}} \) using the same method.

III. RESULTS

We obtained the scattering wave from a monochromatic point source for the Schwarzschild spacetime (black hole, stars with a perfect absorbing surface) and the Ellis wormhole spacetime. A point source is placed at \( r_s = 6M, \theta = \pi \) with frequencies \( 0 < M\omega \leq 10 \). The observing point of power spectrum is \( r_{\text{obs}} = 20M \).
A. Black hole case

Numerical results for the Schwarzschild spacetime as follows.

Fig. 3 shows real part of scattering waves. For $M\omega = 10$, we can see the circle-like wave pattern corresponding to the photon sphere at $r \simeq 3M$ and the bright line (caustics) behind the black hole. While, for $M\omega = 2$ case, these features are blurred due to wave effect. Figs. 4, 5 and 6 are intensity of scattered waves at $r_{\text{obs}}$. They show interference fringes both in spatial domain ($\theta$) and frequency domain ($\omega$). Namely, a constant-$\theta$ slice is the power spectrum observed at the point and a constant-$M\omega$ slice represents the scattering amplitude for the fixed frequency.
FIG. 4: $|\Phi|^2$ at $r_{obs}$ as a function of $(\omega, \theta)$. Interference fringe appears in two dimensional parameter space.

FIG. 5: Sections of $M\omega = 2$ and $M\omega = 10$ of Fig. 4 (scattering amplitudes). These plots show interference fringes in the spatial domain.
FIG. 6: Section of $\theta = 0$ and $\theta = \pi/18$ of Fig. 4 (power spectrums). These plots show interference fringes in the frequency domain.

We explain basic features of power spectrums (Fig. 6). We observe interference fringes in the power spectrums. For $\theta = \pi/18$, we have two components of oscillations. The component with longer period is originated from interference between two light rays traveling far from the black hole (direct rays) and is associated with weak gravitational lensing effect. Its period depends on the scattering angle $\theta$. The other component of oscillation has a shorter period $M\Delta\omega \sim 0.2$ which is independent of the scattering angle $\theta$ and exists even for the forward direction $\theta = 0$; in which case the path difference between two direct rays becomes zero and we cannot expect interference fringe in the power spectrum. Thus we conclude that this oscillation of the power spectrum in the forward direction is caused by interference between winding rays and direct rays, and is peculiar to spacetimes with an unstable photon orbit (photon sphere).

To clarify the period $M\Delta\omega \sim 0.2$ corresponds to the scale of the unstable photon orbit, we consider a toy model of gravitational lensing by the black hole (Fig. 7).

FIG. 7: A model of gravitational lensing by a black hole. Ray 1 and ray 2 represent direct rays of which deflection angle is given by $\theta_{\text{defl}}$. Ray 3 and ray 4 represent winding rays.

We assume all rays follow straight lines as an approximation. Rays 1 and 2 are direct rays and their deflection angle is assumed to obey Einstein’s formula

$$\theta_{\text{defl}} = -\frac{4M}{b},$$

(18)
where \( b \) denotes the impact parameter of each rays. The position on the screen \( x \) and \( b \) are related by
\[
b_{1,2} = \frac{x \pm \sqrt{x^2 + 16LM}}{2}.
\] (19)

The rays 3 and 4 corresponds to winding rays with the impact parameter \( 3\sqrt{3}M \). The path length of each rays are
\[
r_1 = \sqrt{L^2 + (b_1 - x)^2}, \quad r_2 = \sqrt{L^2 + (b_2 - x)^2},
\]
\[
r_3 = \sqrt{L^2 + (b_3 - x)^2 + 2\pi \times |b_3|}, \quad r_4 = \sqrt{L^2 + (b_4 - x)^2 + 2\pi \times |b_4|},
\] (20)
where \( b_3 = 3\sqrt{3}M, b_4 = -3\sqrt{3}M \). We assume winding rays go around the black hole one round. Then ignoring difference of amplitudes for each rays, the wave on the screen is given by
\[
\Phi = e^{i\omega r_1} + e^{i\omega r_2} + c(e^{i\omega r_3} + e^{i\omega r_4}),
\] (22)
where \( c \approx 0.1 \) represents relative amplitude for winding rays but the value of this constant does not affect the period of interference in our estimation. For the forward direction \( \theta = 0 \), we obtain
\[
|\Phi|^2 \approx 1 + c^2 + 2c \cos \left[ \omega \left( 6\pi - 1/2 \right) \sqrt{3}M \right],
\] (23)
and the period of the power spectrum is given by
\[
M\Delta \omega = \frac{1}{\left( 3 - 1/(4\pi) \right) \sqrt{3}} \approx \frac{1}{3\sqrt{3}} \approx 0.2.
\] (24)
This value is consistent with the period of oscillation observed in our numerical calculation.
**B. Star case**

To clarify wave effects associated with the photon sphere, we investigate stars with a complete absorbing surface in the Schwarzschild spacetime. We consider the following form of effective potential

$$ V(r) = V_{BH}(r)\theta(r - r_{\text{star}}), $$

where $V_{BH}$ denotes the effective potential of the Schwarzschild spacetime and $r_{\text{star}}$ is the radius of the star. This form of the potential models complete absorption of incoming waves at the surface of the star:

$$ R_{\ell}(x_{\text{tot}}) \propto e^{-i\omega x_{\text{tot}}} \quad \text{for} \quad r \leq r_{\text{star}}. $$

We consider four different values of radius $r_{\text{star}} = 2.5M, 3M, 3.5M, 4M$. The obtained wave patterns in these models are shown in Fig. 8.

**FIG. 8:** Real part of $\Phi$ with $M\omega = 2$ for stars with a perfect absorbing surface. Four panels correspond to different values of radius of stars $2.5M, 3M, 3.5M, 4M$. The obtained wave patterns in these models are shown in Fig. 8.
The star with radius $2.5M$, which is smaller than the photon sphere of the Schwarzschild spacetime $3M$, is a model of black hole mimickers (gravastar, boson star etc.). For this case, the power spectrum shows oscillation with two different period (the upper left panel in Fig. 9). The shorter one is $M\Delta \omega_1 \sim 0.2$ exactly same value as the black hole case, and caused by interference between direct rays and winding rays associated with the photon sphere. In addition, oscillations with longer period $M\Delta \omega_2 \sim 2$ are superposed. We expect this component is due to diffraction effect caused by the surface of the star. To justify this interpretation, we also investigated power spectrums for stars with other radii.

![Fig. 9: The power spectrum in the forward direction for stars with different radius $2.5M, 3M, 3.5M, 4M$.](image)

For stars with radius larger than $3M$, the photon spheres are hidden by surface of stars and we do not have oscillation with $M\Delta \omega_1 \sim 0.2$. Power spectrums show oscillation with longer period $M\Delta \omega > 2$ depending on the radius of stars. It is possible to estimate this period based on diffraction effect of waves (Sec. IV).

Fig. 10 summarizes behavior of power spectrums for the black hole case and stars with different radii cases.
FIG. 10: Power spectrum at $r_{\text{obs}}$ for the black hole and stars with radius $2.5M, 3M, 3.5M, 4M$. The first panel is plotted in the range $0 \leq M\omega \leq 5$ to show the period of the oscillation clearly.
C. Wormhole case

We present numerical result for the Ellis wormhole spacetime.

a. Massless case  We choose parameters of the Ellis wormhole as \( m = 0, \ a = 3M \). In this case, the circumference radius of the wormhole throat and the photon sphere coincide.

![Graph](image)

**FIG. 11:** Real part of \( \Phi \) for the massless wormhole with \( M\omega = 2 \). The left and right panels correspond to wormhole regions \( x > 0 \) and \( x < 0 \), respectively. The source is placed in \( x > 0 \) region. The coordinates \( \bar{x}, \bar{y} \) are introduced by \( \bar{x} = \sqrt{x^2 + a^2 \cos \theta}, \bar{y} = \sqrt{x^2 + a^2 \sin \theta} \).

The power spectrum has the same behavior as that of the black hole: it shows an oscillation with a period \( M\Delta \omega \sim 0.4 \).

![Graph](image)

**FIG. 12:** Power spectrum at \( r_{\text{obs}} \) for the massless Ellis wormhole.

We can explain this value using the same toy model presented in Fig. 7. For the massless
Ellis wormhole, the deflection angle is given by

\[ \theta_{\text{defl}} = \frac{\pi}{4} \left( \frac{a}{b} \right)^2. \]  

(27)

Then the period of oscillation in the power spectrum becomes

\[ M \Delta \omega = M \left[ a - \frac{1}{4} \left( \frac{a}{2 \pi L} \right)^{2/3} \right]^{-1} \sim 0.36 \]

(28)

for \( a = 3M \) and \( L = 20M \) and well explains the value obtained by our numerical calculation. In this formula, the dependence of \( L \) (distance from the observer to the wormhole) appears due to \( b^{-2} \) behavior of the deflection angle, which is different from that for the black hole and stars.

**b. Massive case** We choose parameters of wormhole as \( m = M, a = 1.305716M \). For these parameters, the size of the throat is \( 3M \) and the photon sphere is \( 3.4823M \).

![FIG. 13: Real part of \( \Phi \) for the massive wormhole with \( M\omega = 2 \). The left and right panels correspond to worm hole regions \( x > 0 \) and \( x < 0 \), respectively. The coordinates \( \bar{x}, \bar{y} \) are introduced as \( \bar{x} = \sqrt{x^2 + a^2 - m^2 \cos \theta}, \bar{y} = \sqrt{x^2 + a^2 - m^2 \sin \theta} \).](image)

We notice that intervals of wave front is different for \( x > 0 \) region and \( x < 0 \) region. This is caused by different asymptotic behavior of metric (A2). For \( x \to \infty \), the metric is

\[ ds^2 \approx - \left( 1 - \frac{2m}{r} \right) dt^2 + \left( 1 + \frac{2m}{r} \right) dr^2 + r^2 d\Omega^2, \]

(29)

whereas for \( x \to -\infty \),

\[ ds^2 \approx -\alpha^{-2} \left( 1 + \frac{2\alpha m}{r} \right) dt^2 + \left( 1 - \frac{2\alpha m}{r} \right) dr^2 + r^2 d\Omega^2, \]

(30)
with $\alpha = \exp \left( \frac{\pi m}{\sqrt{a^2 - m^2}} \right)$ and this is a spacetime with negative gravitational mass $-\alpha m$. Let us consider the radial null vector $k_\mu = (\omega, k)$. Then $\omega$ and $k$ are connected by the relation

$$k = \omega \sqrt{\frac{|g^{tt}g^{rr}|}{|g^{rr}|}} = \begin{cases} \omega(1 + 2m/r) & \text{for } x \to \infty \\ \omega \alpha(1 - 2\alpha m/r) & \text{for } x \to -\infty \end{cases}$$

(31)

As the interval of wave front $\Delta r$ is determined by $k\Delta r = \text{const.}$, thus $\Delta r \propto 1/k$. For $\alpha \gg 1$ which holds for values of present parameters, the interval of wave front in the region $x < 0$ becomes much smaller compared to that in the region $x > 0$.

Fig. 14 shows power spectrum for the massive wormhole.

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Power spectrums has oscillations with two different period. The shorter one $M\Delta \omega_1 \sim 0.2$ is due to interference between direct rays and winding rays and the value coincides with that for the black hole because the radius of the photon sphere and the deflection angle for direct rays are same as the black hole. The longer one $M\Delta \omega_2 \sim 2$ comes from diffraction effect by the absorbing region: for the massive wormhole, this region corresponds to the wormhole throat $3M$ which is smaller than the radius of photon sphere $3.48M$. 

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Fig. 14: Left: Power spectrums at $r_{\text{obs}}$ for the massive Ellis wormhole. Right: power spectrums for the black hole and the massive Ellis wormhole.
IV. INTERPRETATION OF POWER SPECTRUM OSCILLATIONS

In this section, we will provide theoretical justification for behaviors of the power spectrums based on analytic formulas of wave optics. We have two key factors associated with wave effects in our scattering problem: interference and diffraction.

At a distant observing point \( r \), the scattering wave from a point source at \( r_s \) is represented as [15]

\[
\Phi(r, \theta) \approx \frac{e^{i \omega (x + x_s)}}{4\pi i\omega rr_s} \sum_{\ell=0}^{\infty} \lambda \, e^{i\lambda^2/2r} \, e^{2i\delta_{\ell}} \, P_{\ell}(\cos \theta), \quad \lambda := \ell + \frac{1}{2}, \tag{32}
\]

where \( x, x_s \) are tortoise coordinates corresponding to \( r, r_s \), and \( r = rr_s/(r+r_s) \). \( \delta_{\ell} \) represents the phase shift. Applying the Poisson’s sum formula, we can replace the sum with respect \( \ell \) to integral over continuous variable \( \lambda \). In the eikonal limit, it can be shown that

\[
\Phi(r, \theta) \approx \frac{e^{i \omega (x + x_s)}}{4\pi i\omega rr_s} \left[ \int_0^\infty d\lambda \, \lambda \, e^{i\lambda^2/2r} \, e^{2i\delta_{\lambda - 1/2}} \, J_0(\lambda \theta) + 2\pi i \sum_{n=0}^{\infty} \lambda_n \, \gamma_n \, e^{i\lambda_n^2/2r} \, J_0(\lambda_n \theta) \, f(\lambda_n) \right], \tag{33}
\]

where

\[
\lambda_n = \lambda_c + i \left( n + \frac{1}{2} \right), \quad \lambda_c = 3\sqrt{3}M\omega, \tag{34}
\]

\[
\gamma_n = -\frac{i}{\sqrt{2\pi}} \left( \frac{\lambda_n}{\lambda_c} \right) \left( n + \frac{1}{2} \right)^{n+1/2} \frac{e^{-(n+1/2)}}{n!}, \quad f(\lambda_n) = \frac{1}{1 - e^{-2i\pi(\lambda_n - 1/2)}}. \tag{35}
\]

The first term in (33) corresponds to the Fresnel-Kirchihoff diffraction formula in the wave optics [35] and the second term comes from contribution of poles in the S-matrix \( e^{2i\delta_{\lambda}} \) in the complex \( \ell \) plane (Regge poles). This term represents orbiting effect associated with the photon sphere. After taking \( n \) sum, we obtain [15]

\[
\Phi(r, \theta) \propto \frac{1}{\omega} \int_0^\infty d\lambda \, \lambda \, e^{i\lambda^2/2r} \, e^{2i\delta_{\lambda - 1/2}} \, J_0(\lambda \theta) + \frac{1}{\omega} \sqrt{\frac{\pi}{2}} \, e^{-\pi - i\pi/4 + i\pi\lambda_c} \, \lambda_c \, e^{i\lambda_c^2/2r} \, \sqrt{\omega r} \, J_0(\lambda_c \theta). \tag{36}
\]

For the forward direction \( \theta = 0 \),

\[
\Phi \propto \frac{1}{\omega} \int_0^\infty d\lambda \, \lambda \, e^{i\lambda^2/2r} \, e^{2i\delta_{\lambda - 1/2}} + \frac{1}{\omega} \sqrt{\frac{\pi}{2}} \, e^{-\pi - i\pi/4 + i\pi\lambda_c} \, \lambda_c \, e^{i\lambda_c^2/2r} \, \sqrt{\omega r} \\
= \omega \int_{b_0}^\infty db \, b \, e^{i\omega b^2/2r} \, e^{2i\delta_b} + \sqrt{\frac{\pi}{2}} \, e^{-\pi - i\pi/4 + i\pi\omega b_0} \, b_c \, e^{i\omega b_0^2/2r} \, \sqrt{\omega r}, \tag{37}
\]

where we introduced the impact parameter \( b = \lambda/\omega \) and a lower cutoff \( b_0 \) of the integral, which represents effect of a complete absorbing region (the black hole horizon, surface of the stars). Concerning the form of the phase shift for direct rays, we adopt \( \delta_b = -2M\omega \ln(b\omega) \) which results in the scattering angle in the eikonal limit as

\[
2 \frac{d}{d(b\omega)} \delta_b = -\frac{4M}{b}, \tag{38}
\]

and reproduces Einstein’s formula of deflection angle. Although this formula is correct only for rays with sufficiently large impact parameters compared to size of the photon sphere, it
is adequate for our purpose here to obtain qualitative understanding of oscillation of power spectrums. Then after performing the integral, the first term in Eq. (37) becomes
\[
\bar{r} (2i\bar{r} \omega)^{-2iM\omega} \left[ 2M\omega \Gamma(-2iM\omega) - i \left\{ \Gamma(1-2iM\omega) - \Gamma \left( 1 - 2iM\omega, -\frac{ib_0^2\omega}{2\bar{r}} \right) \right\} \right], \quad (39)
\]
where the 3rd term denotes the incomplete gamma function. We show behavior of the obtained analytic formula in Fig. 15.

![Diagram](image-url)

**FIG. 15:** Effect of interference and diffraction in power spectrums. The 1st panel is the power spectrum obtained by taking into account of interference between direct rays and winding rays. The 2nd panel is the power spectrum obtained by taking into account of the diffraction effect for direct rays. The 3rd panel is the power spectrum obtained by taking into account of all effects. We assume \(b_0 = 2.5M, b_c = 3\sqrt{3}M\) in this plot.

The analytic formula (37) well reproduces behaviors of the power spectrum obtained by our numerical calculation. We can estimate period of oscillations in the power spectrum. Using (37) and (39), the period of oscillation for diffraction effect is
\[
M\Delta \omega_2 \sim 2\pi \left( 2 - \frac{b_0^2}{2M\bar{r}} \right)^{-1} \sim 4.8 \quad \text{(for } b_0 = 2.5M, \bar{r} = 4.6M\text{).} \quad (40)
\]

Although this value is about two times larger than the value obtained by the numerical calculation, the formula shows increase of period for larger size of the diffraction region and
qualitatively explains behavior of the power spectrum obtained by numerical calculations. The period of the oscillation due to interference between direct rays and winding rays is

\[ M \Delta \omega_1 \sim \frac{M}{b_c - M/\pi} \sim 0.2 \quad \text{(for } b_c = 3\sqrt{3}M). \]  

(41)

These values are consistent with period of oscillations in the power spectrums obtained by the numerical calculation for the black hole and stars with smaller radius than the photon sphere.

V. SUMMARY AND CONCLUSION

By solving the scalar wave equation numerically, we obtained the scattering wave by rhw Schwarzschild black hole, spherical stara with a perfect absorbing surface, and the Ellis wormhole, then we investigated the wave pattern and the power spectrums. We focused on the case that an observer is located at the forward position where we cannot expect the interference between direct rays due to the difference of light paths length in the geometrical optics point of view. Even in this case, we found two kinds of oscillation in the power spectrums. When a gravitating object has the photon sphere (black hole, star with \( r_{\text{star}} \leq 3M \) and Ellis wormholes), we can see the oscillation of the power spectrum reflecting the interference between direct ray and winding ray. Moreover, diffraction effects were observed for stars and massive Ellis wormholes due to the absorption boundary condition. We have justified the periods of these oscillation by analytic evaluation of the wave scattering. As expected, it is possible to distinguish black holes from its mimickers by looking inside of the photon sphere by waves although we cannot tell the difference in the geometrical optics for the present source-object-observer configuration.

As other interesting models, we will consider stars with internal structure or reflecting surface and wormholes with double-peak effective potential \[37\], which may give echoes in the power spectrums of the scattered waves.

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Appendix A: Wormhole spacetimes

As an example of spacetimes with the photon unstable orbit without horizon, we consider the Ellis wormhole \[36\]. Although the wormhole spacetime is unstable and may not realized in our universe, we use them as benchmark models to detect wave optical effects for compact gravitating objects.

The Ellis wormhole spacetime is obtained as the solution of the Einstein-scalar system

\[ R_{\mu\nu} = 2\chi_{,\mu}\chi_{,\nu}, \quad \Box \chi = 0. \]  

(A1)
The metric is given by \[ ds^2 = -f dt^2 + \frac{1}{f} \left( dx^2 + (x^2 + a^2 - m^2) d\Omega^2 \right), \quad -\infty < x < +\infty, \tag{A2} \]
\[ f = \exp \left( -\frac{2m \chi(x)}{a} \right), \quad \chi(x) = \frac{a}{\sqrt{a^2 - m^2}} \left( \frac{\pi}{2} - \arctan \frac{x}{\sqrt{a^2 - m^2}} \right), \tag{A3} \]
where \( a \) and \( m \) are constants representing the throat size and the mass of the wormhole, respectively. The range of the scalar field is \( 0 (x = +\infty) \leq \chi \leq \frac{\pi a}{\sqrt{a^2 - m^2}} (x = -\infty) \). By introducing a new radial coordinate corresponding to the circumference radius \[ r = \frac{\sqrt{x^2 + a^2 - m^2}}{f^{1/2}}, \tag{A4} \]
the metric becomes
\[ ds^2 = -f dt^2 + \frac{dr^2}{g} + r^2 d\Omega^2, \quad g = f \left( \frac{dr}{dx} \right)^2. \tag{A5} \]
In terms of the scalar field \( \chi \),
\[ x = \sqrt{a^2 - m^2} \cot \left( \frac{\sqrt{a^2 - m^2}}{a} \chi \right), \tag{A6} \]
\[ r = \frac{\sqrt{a^2 - m^2} e^{m \chi/a}}{\sin \left( \frac{\sqrt{a^2 - m^2}}{a} \chi \right)}, \quad r_{\text{min}} = a \exp \left[ \frac{m}{\sqrt{a^2 - m^2}} \text{Arccot} \left( \frac{m}{\sqrt{a^2 - m^2}} \right) \right], \tag{A7} \]
\[ g = \left[ \cos \left( \frac{\sqrt{a^2 - m^2}}{a} \chi \right) - \frac{m}{\sqrt{a^2 - m^2}} \sin \left( \frac{\sqrt{a^2 - m^2}}{a} \chi \right) \right]^2. \tag{A8} \]
\( r_{\text{min}} \) is the circumference radius of the wormhole throat (at \( r = m \)). Asymptotic behavior of the metric \((A5)\) for \( x \to \infty (\chi \to 0) \) is
\[ x \sim \frac{a}{\chi}, \quad R \sim \frac{a}{\chi} \sim x, \]
\[ g \sim 1 - \frac{2m}{x} - \frac{a^2 - 2m^2}{x^2}, \quad f \sim 1 - \frac{2m \chi}{a} \sim 1 - \frac{2m}{x}. \]
Thus the metric for \( x \to \infty \) becomes
\[ ds^2 \approx - \left( 1 - \frac{2m}{r} \right) dt^2 + \left( 1 + \frac{2m}{r} + \frac{a^2 + 2m^2}{r^2} \right) dr^2 + r^2 d\Omega^2. \tag{A9} \]
On the other hand, asymptotic behavior of the metric for \( x \to -\infty (\chi \to \frac{\pi a}{\sqrt{a^2 - m^2}}) \) is
\[ x \sim \frac{-1}{\sqrt{a^2 - m^2} - \frac{\chi}{a}}, \quad r \sim -xe^{\frac{\pi m}{\sqrt{a^2 - m^2}}}, \quad g \sim 1 - \frac{2m}{x} \sim 1 + \frac{2m}{r} e^{\frac{\pi m}{\sqrt{a^2 - m^2}}}, \]
\[ f \sim e^{-\frac{2m}{\sqrt{a^2 - m^2}}} \left( 1 - \frac{2m}{x} \right) \sim e^{-\frac{2m}{\sqrt{a^2 - m^2}}} \left( 1 + \frac{2m}{r} e^{\frac{\pi m}{\sqrt{a^2 - m^2}}} \right). \]
Thus the metric for $x \to -\infty$ represents gravitating object with negative mass $-m e^{\frac{\pi m}{\sqrt{a^2 - m^2}}}$. For massless case $m = 0$, the metric reduces to

$$ds^2 = -dt^2 + dx^2 + (x^2 + a^2)d\Omega^2.$$ \hfill (A10)

In this case, the gravitational potential in the weak field region behaves as $\propto r^{-2}$ and the law of gravity is different from that for a point mass.

The shape of the effective potential for the wormhole is shown in Fig. 16.

![FIG. 16: Effective potentials for Ellis wormhole ($\ell = 10$) for massless case ($a = 3M, m = 0$) and massive case ($a = 1.305716M, m = M$). The throat of the wormhole is located at $x = m$ and the peak of the potential in the eikonal limit $\ell \gg 1$ is at $x \approx 2m$. This place corresponds to the photon sphere.

In our numerical calculations, we adopt the wormhole parameters as $a = 3M, m = 0$ (massless case) and $a = 1.305716M, m = M$ (massive case). The circumference radii of the throat for both wormholes are $3M$ and the circumference radii of the photo sphere are $3M$ (massless case) and $3.4823M$ (massive case).

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