LOWER BOUNDS ON THE ARITHMETIC SELF-INTERSECTION NUMBER OF THE RELATIVE DUALIZING SHEAF ON ARITHMETIC SURFACES

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Abstract. We give an explicitly computable lower bound for the arithmetic self-intersection number $\omega^2$ of the dualizing sheaf on a large class of arithmetic surfaces. If some technical conditions are satisfied, then this lower bound is positive. In particular, these technical conditions are always satisfied for minimal arithmetic surfaces with simple multiplicities and at least one reducible fiber, but we also use our techniques to obtain lower bounds for some arithmetic surfaces with non-reduced fibers.

1. Introduction

Let $K$ be a number field, let $O_K$ denote the ring of integers of $K$. Let $X/K$ denote a smooth, projective, geometrically irreducible curve of genus $g > 1$, let $\pi : X \to \text{Spec}(O_K)$ be a proper regular model of $X$ and let $\omega = \omega_X$ denote the relative dualizing sheaf on $X$ over $\text{Spec}(O_K)$, equipped with the Arakelov metric. The arithmetic self-intersection $\omega^2$ is one of the most fundamental objects in arithmetic intersection theory; see for instance [22] for a discussion. In this note we show how to effectively compute lower bounds on $\omega^2$ in many situations including, but not limited to, semistable $X$. To each $\mathbb{Q}$-divisor $D \in \text{Div}_\mathbb{Q}(X) = \text{Div}(X) \otimes \mathbb{Q}$ of degree one we attach in Definition 4.2 a hermitian line bundle $L_D$. We show that the height $h_{L_D}(\cdot)$ with respect to $L_D$ is closely related to the Néron-Tate height induced by the embedding $j_D$ of $X$ into its Jacobian via $D$. More precisely, we define a certain finite set $T(X)$ of closed points on $X$ and prove the following result in Section 4, where we write $D_X$ for the Zariski closure in $X$ of an irreducible divisor $D \in \text{Div}(X)$ and extend this to $\text{Div}_\mathbb{Q}(X)$ by linearity.

Theorem 1.1. Suppose that $D \in \text{Div}_\mathbb{Q}(X)$ has degree one and $\text{supp}(D_X) \cap T(X) = \emptyset$. Then, if $E = \sum_{j=1}^e (P_j)$ is an irreducible divisor on $X$, where $P_j \in X(\bar{K})$, and $\text{supp}(E_X) \cap T(X) = \emptyset$, we have

$$h_{L_D}(E) = \frac{1}{e} \sum_{j=1}^e h_{\text{NT}}(j_D(P_j)) \geq 0.$$ 

In particular, we have $h_{L_D}(P) = h_{\text{NT}}(j_D(P))$ for all $P \in X(K)$.

If $X$ is semistable, then $T(X)$ is simply the set of singular points on the special fibers of $X$. Note that in the proof of [23, Theorem 5.6], Zhang proves an analogue (with $T(X) = \emptyset$) of Theorem [1.1] in the language of his admissible intersection theory. Since we want to be able to compute lower bounds on $\omega^2$ for non-semistable $X$, we cannot use the admissible theory and have to work with hermitian line bundles throughout. In order to use Theorem [1.1] to...
derive a lower bound on $\omega^2$, we follow Zhang’s approach from [25]. The idea is to show that under certain conditions, we have $L_D^2 \geq 0$. If these conditions are satisfied and $(2g - 2)D$ is a canonical $\mathbb{Q}$-divisor on $X$, then we can relate $L_D^2$ to $\omega^2$ and use this inequality to obtain lower bounds on $\omega^2$. As in Zhang’s theory (cf. [24, Theorem 6.5]), the crucial condition on $L_D$ is relative semipositivity, where we call a hermitian line bundle relatively semipositive if its restriction to every irreducible vertical divisor has nonnegative arithmetic degree (see Definition 3.2). The proof of the following result is similar to the proof of [24, Theorem 6.5], but rather more complicated.

Proposition 1.2. If $L_D$ is relatively semipositive and $D_X \cap T(X) = \emptyset$, then we have $L_D^2 \geq 0$.

In Section 2 we locally define certain vertical $\mathbb{Q}$-divisors $V_D$ and $U_D$ attached to $D$; they are the main ingredients in the construction of $L_D$, see Definition 4.2. Moreover, we set $\beta_D = 1 - g \mathcal{O}(2V_D + U_D)^2 + 2(\omega \cdot \mathcal{O}(U_D))$.

Theorem 1.3. Let $D \in \text{Div}_\mathbb{Q}(X)$ be a $\mathbb{Q}$-divisor such that $(2g - 2)D$ is a canonical $\mathbb{Q}$-divisor on $X$ and such that $D_X$ satisfies $D_X \cap T(X) = \emptyset$. If the hermitian line bundle $L_D$ is relatively semipositive, then we have $\omega^2 \geq \beta_D$.

Note that by a result of Chinburg and Rumely [5, Lemma A.1], a $\mathbb{Q}$-divisor $D$ as in Theorem 1.3 always exists. In order to derive a nontrivial lower bound on $\omega^2$ from Theorem 1.3 for a given $X$, we need to show that $L_D$ is relatively semipositive and that for some choice of $D$ as in the statement of the theorem, we have $\beta_D \geq 0$.

Theorem 1.4. If $X$ is minimal and all special fibers of $X$ are reduced, then the following are satisfied:

(i) $L_D$ is relatively semipositive for every $\mathbb{Q}$-divisor $D \in \text{Div}_\mathbb{Q}(X)$ of degree one;
(ii) $\beta := \beta_D$ does not depend on the choice of $D$;
(iii) $\beta \geq 0$, with equality if and only if all special fibers of $X$ are irreducible.

The proof of Theorem 1.4 essentially follows from a sequence of local lemmas proved in Section 2. We provide an explicit formula for $\beta$ in Lemma 2.19, making it very simple to compute $\beta$ for a given minimal model $X$ with reduced fibers. As an immediate corollary we recover the following result from [24, 19] and [4].

Corollary 1.5. If $X$ is semistable and minimal and has at least one reducible fiber, then there is an effectively computable positive lower bound on $\omega^2$.

In the semistable case lower bounds on $\omega^2$ can be derived by means of the admissible intersection theory due to Zhang, cf. [25, Theorem 5.5]. This method requires the computation of admissible Green’s functions on the reduction graphs of the special fibers of $X$ and has been employed by Abbes-Ullmo [1] to find lower bounds for certain modular curves (see also Subsection 6.1), but such an approach does not work for non-semistable arithmetic surfaces. The positivity of $\omega^2$ for non-semistable $X$ with at least one reducible has been proven by Sun in [21]. However, his result is often not suitable for explicit computations of such bounds in practice, since it requires computing a global semistable model over an extension of $K$. We believe that for $D$ as in Theorem 1.3, $\beta_D$ is a lower bound on $\omega^2$ for all minimal arithmetic surfaces, even those with non-reduced fibers. Indeed, if we are given a minimal $X$ having
components of multiplicity $> 1$, we can still check whether Theorem 1.3 is applicable. As an example, we prove that the conditions of Theorem 1.3 are satisfied for the minimal regular model $\mathcal{F}_p^{\text{min}}$ of the Fermat curve of prime exponent $p > 3$ over the field of $p$-th cyclotomic numbers and that the resulting lower bound is positive. This does not follow from Theorem 1.4 since the irreducible components of $\mathcal{F}_p^{\text{min}}$ need not have multiplicity one.

**Theorem 1.6.** The arithmetic self-intersection $\varpi^2$ of the relative dualizing sheaf on $\mathcal{F}_p^{\text{min}}$ satisfies

$$\varpi^2 \geq p \log p + O(\log p).$$

A more precise statement is provided in Theorem 6.6. The paper is organized as follows: In Section 2 we define the $\mathbb{Q}$-divisors $V_D$ and $U_D$ locally and prove that they have certain properties with respect to the intersection multiplicity. We then switch to a global perspective in Section 3 where we prove some general results on hermitian line bundles. Section 4 contains the definition of $\mathcal{L}_D$ and the proof of Theorem 1.1. The results of Sections 2, 3 and 4 are then used in Section 5 to prove Proposition 1.2 and Theorems 1.3 and 1.4. At the end of that section, we also discuss a possible application of our results to the effective Bogomolov conjecture. In Section 6 we first use Theorem 1.4 to prove an asymptotic lower bound for $\varpi^2$ on minimal regular models of modular curves $X_1(N)$ for certain $N$. Finally, we use Theorem 1.3 to prove Theorem 1.6 here we also compare the resulting lower bound to the upper bound computed by Curilla and the first author in [8].

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2. **Intersection properties of certain vertical $\mathbb{Q}$-divisors**

Let $\mathcal{O}$ be a strictly Henselian discrete valuation ring with field of fractions $K$. Let $\mathcal{X}_s$ be the special fiber of a proper regular model $\mathcal{X}/\mathcal{O}$ of a smooth projective geometrically irreducible curve $X/K$ of genus $g > 1$. In this section we define certain vertical $\mathbb{Q}$-divisors $V_D$ and $U_D$ with support in the $\mathcal{X}_s$ which are attached to $\mathbb{Q}$-divisors $D \in \text{Div}_\mathbb{Q}(X)$ and study their properties.

Suppose that, as a divisor on $\mathcal{X}$, the special fiber $\mathcal{X}_s$ is given by

$$\mathcal{X}_s = \sum_{i=1}^{r} b_i \Gamma_i,$$

where $\{\Gamma_1, \ldots, \Gamma_r\}$ is the set of irreducible components of $\mathcal{X}_s$ and the multiplicities $b_i$ are positive integers. We fix a canonical $\mathbb{Q}$-divisor $\mathcal{K}$ on $\mathcal{X}$, and set

$$a_i = (\Gamma_i \cdot \mathcal{K})$$

for $i \in \{1, \ldots, r\}$, where $(\cdot)$ is the rational-valued intersection multiplicity on $\mathcal{X}$. Note that by the adjunction formula [15, Theorem 9.1.37], we have

$$a_i = -\Gamma_i^2 + 2p_a(\Gamma_i) - 2,$$

where $p_a(\Gamma_i)$ is the arithmetic genus of $\Gamma_i$. Given a prime divisor $D \in \text{Div}(X)$, we denote the Zariski closure of $D$ in $\mathcal{X}$ by $D_\mathcal{X}$ and we extend this to $\text{Div}_\mathbb{Q}(X)$ by linearity.
Proposition 2.1. For every $D \in \text{Div}_\mathbb{Q}(X)$ there exists a vertical $\mathbb{Q}$-divisor $V_D \in \text{Div}_\mathbb{Q}(X)$, which is unique up to addition of rational multiples of $\mathcal{X}_s$, such that

$$((D_X + V_D) \cdot \Gamma_i) = \frac{\deg(D)}{2g - 2} a_i$$

for all $i \in \{1, \ldots, r\}$. Moreover, the assignment

$$D \mapsto V_D \mod \mathcal{X}_s$$

is linear in $D$.

Proof. The assignment

$$E \mapsto \left(\left(\frac{\deg(D)}{2g - 2} \mathcal{K} - D_X\right) \cdot E\right)$$

defines a linear map on $Z^1(\mathcal{X}_s)_\mathbb{Q} = Z^1(\mathcal{X}_s) \otimes \mathbb{Z}$. By the non-degeneracy of the intersection pairing on $Z^1(\mathcal{X}_s)_\mathbb{Q}$ modulo the entire fiber, this map is representable by a cycle $V_D \in Z^1(\mathcal{X}_s)_\mathbb{Q}$. As this assignment is also a linear map in $D$, the two claims follow immediately. $\square$

Corollary 2.2. The pairing $[E_1, E_2]$ extends the local Néron pairing (see [14, §III.5]) to $\mathbb{Q}$-divisors of arbitrary degree.

To our knowledge, the pairing $[\cdot, \cdot]$ is the first extension of the local Néron pairing to $\mathbb{Q}$-divisors of arbitrary degree that is not based on the reduction graph as in [5] or [25].

Corollary 2.3. Let $l \in \{1, \ldots, r\}$.

(i) There exists a $\mathbb{Q}$-divisor $D_l \in \text{Div}_\mathbb{Q}(X)$ of degree 1 which satisfies

$$(D_l \cdot \mathcal{X}) \cdot \Gamma_i = \frac{\delta_{il}}{b_i}$$

for all $i \in \{1, \ldots, r\}$, where $\delta_{il}$ is the Kronecker delta and the multiplicity $b_i$ is as in (2.1).

(ii) There exists a vertical $\mathbb{Q}$-divisor $V_l := V_{D_l}$ on $X$ such that

$$V_l \cdot \Gamma_i = a_i' - \frac{\delta_{il}}{b_i},$$

where $a_i' = \frac{1}{2g - 2} a_i$.

Proof. The existence of $D_l$ follows from [2, Corollary 9.1.10]. Proposition 2.1 proves (ii). $\square$

Our applications require explicit formulas for the $V_l$. We define a matrix $M = (m_{ij})_{i,j}$ as minus the intersection matrix of $\mathcal{X}_s$:

$$m_{i,j} = -(b_i \Gamma_i, b_j \Gamma_j)$$

Let $M^+ = (n_{ij})_{i,j}$ denote the Moore-Penrose pseudoinverse of $M$, cf. [7, §3]. For fixed $l \in \{1, \ldots, r\}$ we define a vector

$$c_l = (c_{l1}, \ldots, c_{lr})^t = -M^+ w_l,$$
where
\[ w_l = (w_{l1}, \ldots, w_{lr})^t, \quad w_{lj} = b_j a'_j - \delta_{lj}. \]

**Proposition 2.4.** If \( l \in \{1, \ldots, r\} \), then we have
\[ V_l = \sum_{i=1}^{r} b_i c_i \Gamma_i. \]

**Proof.** This is an easy consequence of the relations
\[ MM^+ M = M \quad \text{and} \quad M^+ MM^+ = M^+. \]

**Definition 2.5.** If \( D \in \text{Div}_Q(X) \) has degree \( d > 0 \), then we define a vertical \( Q \)-divisor \( U_D \) on \( X \) associated to \( D \) as follows: For all \( i \in \{1, \ldots, r\} \) we set
\[ \gamma_{D,i} = \frac{b_i}{d} \left(V_i^2 - (V_l - dV_i)^2\right) \]
and define
\[ U_D = \sum_{i=1}^{r} \gamma_{D,i} \Gamma_i. \]

It is easy to see that the association \( D \mapsto U_D \) is linear in \( D \). Our main motivation for the definition of \( U_D \) is Proposition 2.11 below, which plays a crucial part in the proof of Theorem 1.1. If \( D \) has degree 0, then we write
\begin{equation}
\Phi_X(D) := V_D
\end{equation}
in accordance with the classical literature (see for instance [14, Theorem III.3.6]).

**Lemma 2.6.** Let \( E \in \text{Div}_Q(X) \) with \( \deg(E) = 1 \) such that
\[ (E \cdot \Gamma_i) = \frac{\delta_{il}}{b_i} \]
for all \( i \in \{1, \ldots, r\} \), where \( l \in \{1, \ldots, r\} \). Then we have
\[ \Phi_X(dE - D)^2 = V_D^2 - d(E \cdot U_D). \]

for any \( D \in \text{Div}_Q(X) \) with degree \( d > 0 \).

**Proof.** The condition on \( E \) implies
\begin{equation}
\Phi_X(dE - D)^2 = d^2 V_i^2 - 2d(V_i \cdot V_D) + V_D^2 = -\frac{d}{b_i} \gamma_{D,l} + V_D^2.
\end{equation}
Hence the lemma follows from
\[ (E \cdot U_D) = \frac{\gamma_{D,l}}{b_i}. \]
surfaces we refer to [16]; in particular, we do not require a semistable arithmetic surface to be minimal.

**Lemma 2.7.** ([Liu, [16]]) There exists a finite extension $F_0/K$ such that $\mathcal{X}^F$ is semistable for every finite extension $F/F_0$.

**Definition 2.8.** Let $F_0/K$ be as in Lemma [2.7] We denote the smooth locus of $\mathcal{X}^{F_0}$ by $\mathcal{X}^{F_0}_{\text{sm}}$ and we denote the exceptional locus of $\varphi_{F_0}$ by $\text{Exc}(\varphi_{F_0})$. With this notation we define

$$T(\mathcal{X}) = \varphi_{F_0}((\mathcal{X}^{F_0} \setminus \mathcal{X}^{F_0}_{\text{sm}}) \cup \text{Exc}(\varphi_{F_0})).$$

**Remark 2.9.** If $\mathcal{X}$ is semistable, then we have $T(\mathcal{X}) = \mathcal{X} \setminus \mathcal{X}_{\text{sm}}$.

**Lemma 2.10.** Every irreducible divisor $E$ on $X$ such that $\text{supp}(E_X) \cap T(\mathcal{X}) = \emptyset$ satisfies

$$\varphi^*_F E_X = E_{\mathcal{X}^F} := (E \otimes F)_{\mathcal{X}^F}$$

for every finite extension $F/F_0$.

**Proof.** Let $E$ be an irreducible divisor on $X$ whose closure $E_X$ does not contain an element of $T(\mathcal{X})$ in its support and let $F/K$ be a finite extension containing $F_0$. Note that if $\varphi^*_F E_X \neq E_{\mathcal{X}^F}$, then there is an irreducible component $\Gamma \subset \text{Exc}(\varphi_F)$ such that $\varphi_F(\Gamma) \in \text{supp}(E_X)$. But this means that either $\Gamma \subset \text{Exc}(\varphi_{F_0})$, implying that $\varphi_F(\Gamma) \in T(\mathcal{X})$, or $\Gamma$ is contracted to a point by the desingularization morphism $\mathcal{X}_F \to \mathcal{X}_{F_0} \otimes \mathcal{O}_F$. In this case $\Gamma$ maps to a singular point of $\mathcal{X}_{F_0} \otimes \mathcal{O}_F$, whence $\varphi_F(\Gamma) \in T(\mathcal{X})$. $\square$

**Proposition 2.11.** Let $D \in \text{Div}_Q(X)$ be a $Q$-divisor of degree $d > 0$ and let $E = \sum_{j=1}^e (P_j) \neq 0$ be an effective divisor on $X$, where $P_j \in X(F)$ and $F/K$ is a finite extension such that $F_0 \subset F$. Furthermore, suppose that

$$\text{supp}(D_X) \cap T(\mathcal{X}) = \emptyset = \text{supp}(E_X) \cap T(\mathcal{X}).$$

Then we have

$$\frac{1}{[F : K]} \sum_{j=1}^e \Phi_{\mathcal{X}^F} (dP_j - D)^2 = eV_D^2 - d(E_X \cdot U_D)$$

**Proof.** As $\text{supp}(E_X) \cap T(\mathcal{X}) = \emptyset$, we have

$$\varphi^*_F (E_X) = E_{\mathcal{X}^F} = \sum_{j=1}^e P_j \cdot \mathcal{X}^F$$

and hence

$$\frac{1}{[F : K]} (E_X \cdot U_D) = (E_{\mathcal{X}^F} \cdot \varphi^*_F U_D) = \sum_{j=1}^e (P_j \cdot \mathcal{X}^F \cdot \varphi^*_F U_D).$$

Therefore it suffices to show that

$$(2.6) \quad \Phi_{\mathcal{X}^F} (dP - D)^2 = [F : K] \cdot V_D^2 - d(P_{\mathcal{X}^F} \cdot \varphi^*_F U_D).$$

holds for a point $P \in X(F)$ such that $P_{\mathcal{X}^F}$ intersects $\varphi^*_F(\Gamma_i)$ for precisely one irreducible component $\Gamma_i$ of $\mathcal{X}_s$. Let $P$ be such a point.

From Corollary [2.3] we get the existence of a $Q$-divisor $D_i \in \text{Div}_Q(X)$ of degree 1 such that $D_i$ intersects precisely the component $\Gamma_i$ on $\mathcal{X}_s$. For the purpose of the present proof, we refine Corollary [2.3] so that, moreover, $\text{supp}(D_i \cdot \mathcal{X})$ is disjoint from $T(\mathcal{X})$. Such a $Q$-divisor
$D_i$ exists because, by the proof of [2 Corollary 9.1.9], we can choose $D_i, X$ so that it intersects $X_s$ in $U \cap X_s$, where $U$ is any open subscheme of $X$ satisfying

$$U \cap X_s \subset \Gamma_i.$$ 

Because both $P$ and $D_i$ have degree one, their respective closures $P_X$ and $D_i, X$ have the same intersection multiplicity with every irreducible component of $X \otimes \mathcal{O}_F$ by [2 Lemma 9.1.5].

As $\text{supp}(D_i, X) \cap T(X) = \emptyset$ implies (2.7) $\varphi^*_F D_i, X = D_i, X F$,

we have (2.8) $\Phi_{X_F} (dP - D)^2 = \Phi_{X_F} (dD_i - D)^2$

by the projection formula and furthermore (2.9) $(P_{X_F} \cdot \varphi^*_F U_D) = (D_i, X F \cdot \varphi^*_F U_D) = [F : K] \cdot (D_i, X . U_D)$,

Now we can complete the proof. Using [14 Theorem III.4.5] and (2.7), we get

$$\Phi_{X_F} (dD_i - D)^2 = (\varphi^*_F (\Phi_X (dD_i - D)))^2.$$ 

Applying Lemma 2.6 to $D_i$ we therefore see that

$$\Phi_{X_F} (dD_i - D)^2 = [F : K] \cdot \Phi_X (dD_i - D)^2$$

$$= [F : K] \cdot (V_D^2 - d(D_i, X . U_D))$$

Combining this observation with (2.8) and (2.9), we get (2.6) and the claim follows.

We are now ready to define a local version of what will be our lower bound on $\overline{\omega}^2$.

**Definition 2.12.** If $D \in \text{Div}_Q(X)$ is a $Q$-divisor of degree 1, we define

$$\beta_D = \frac{1 - g}{g} (2V_D + U_D)^2 + 2(\mathcal{K} . U_D).$$

**Example 2.13.** Suppose that the special fiber of $X$ consists of two irreducible components $\Gamma_1$ and $\Gamma_2$ of multiplicity 1 and identical arithmetic genus $p_a$ which intersect transversally in $s \geq 1$ points. Let $D = \frac{1}{2} D_1 + \frac{1}{2} D_2$. Then it is easy to see that we can take $V_D = 0$ and

$$V_1 = \frac{1}{4s} \Gamma_1 - \frac{1}{4s} \Gamma_2 = -V_2.$$ 

This leads to

$$U_{D_1} = \frac{3}{4s} \Gamma_2 - \frac{1}{4s} \Gamma_1,$$

$$U_{D_2} = \frac{3}{4s} \Gamma_1 - \frac{1}{4s} \Gamma_2$$

and

$$U_D = \frac{1}{4s} \Gamma_1 + \frac{1}{4s} \Gamma_2.$$ 

A simple computation reveals

$$\beta_D = \frac{1}{2s} (s + 2p_a - 2).$$
In order to show that (a global version of) $\beta_D$ indeed provides a non-trivial lower bound for $\omega^2$ in many situations, we first need to prove some further intersection-theoretic properties of $U_D$. To this end, we define a metrized graph $\mathcal{G}_X$ as follows: The vertex set of $\mathcal{G}_X$ is given by $\{\Gamma_1, \ldots, \Gamma_r\}$. There are no self-loops or multiple edges; two vertices $\Gamma_i$ and $\Gamma_j$ are connected by an edge if and only if $m_{ij} \neq 0$, in which case the length of the edge is $-1/m_{ij}$. We also need some facts about the matrix $M$ and its pseudoinverse $M^+$.

**Lemma 2.14.** The following properties are satisfied:

(i) Both $M$ and $M^+$ are symmetric and positive semidefinite.
(ii) We have $\sum_{j=1}^r m_{ij} = \sum_{j=1}^r n_{ij} = 0$ for all $i \in \{1, \ldots, r\}$.
(iii) We have $\sum_{j=1}^r n_{ij}m_{jk} = -\frac{1}{r} + \delta_{ik}$ for all $i, k \in \{1, \ldots, r\}$.
(iv) We have $n_{ii} - \sum_{j,k} n_{ij}n_{kk}m_{jk} = \frac{\operatorname{Tr}(M^+)}{r}$ for all $i \in \{1, \ldots, r\}$.
(v) $M$ is the discrete Laplacian matrix associated to $\mathcal{G}_X$.

**Proof.** These properties are proved in [7].

**Remark 2.15.** Note that when $\mathcal{X}$ is minimal and semistable, $\mathcal{G}_X$ need not coincide with the reduction graph $R(X)$ associated to $X$ in [5] and [25]. For instance, suppose that $\mathcal{X}_s$ is given by two curves $\Gamma_1$ and $\Gamma_2$ intersecting transversally in $n$ points. In this case, $R(X)$ is the banana graph with $n$ edges of length 1, whereas $\mathcal{G}_X$ is the complete graph with two vertices which are connected by a single edge of length $1/n$.

From now on, we suppose that $D \in \operatorname{Div}_\mathbb{Q}(X)$ has degree one. For $i \in \{1, \ldots, r\}$ we set

$$v_i(D) = (b_i \Gamma_i \cdot D_X) \quad \text{and} \quad w_i(D) = b_i a'_i - v_i(D) = \frac{b_i a_i}{2g - 2} - v_i(D).$$

**Lemma 2.16.** Let $i \in \{1, \ldots, r\}$.

(a) We have

$$b_i(U_D \cdot \Gamma_i) = -\sum_{j=1}^r n_{jj}m_{ij} + 2v_i(D) - \frac{2}{r}.$$ 

(b) If the special fiber of $\mathcal{X}$ is reduced, then

$$(K \cdot \Gamma_i) + 2(D_X \cdot \Gamma_i) - (U_D \cdot \Gamma_i)$$

is independent of $D$ and nonnegative.
Proof. Assertion (a) is an easy computation using Lemma 2.14:

\[ b_i(U_D . \Gamma_i) = - \sum_{l=1}^{r} \frac{\gamma_{D,l}}{b_j} m_{li} \]

\[ = - \sum_{l=1}^{r} (2(V_D \cdot V_i) - V_i^2) m_{li} \]

\[ = \sum_{l=1}^{r} \sum_{j=1}^{r} c_{lj}(w_{lj} - 2w_j(D)) m_{li} \]

\[ = \sum_{l=1}^{r} \left( \sum_{j=1}^{r} c_{lj}(2v_j(D) - a_j' - \delta_{lj}) \right) m_{li} \]

\[ = - \sum_{l,j,k} (2b_k a_k' v_j(D) - b_k b_j a_k' a_j' - b_k a_k' \delta_{lj} - 2\delta_{kl} v_j(D) + b_j \delta_{kl} a_j' + \delta_{kl} \delta_{lj}) n_{kj} m_{li} \]

\[ = - \sum_{j=1}^{r} n_{jj} m_{ij} + 2 \sum_{l=1}^{r} \sum_{j=1}^{r} n_{lj} v_j(D) m_{li} \]

\[ = - \sum_{j=1}^{r} n_{jj} m_{ij} + 2 \sum_{j=1}^{r} v_j(D) \left( \delta_{ij} - \frac{1}{r} \right) \]

\[ = - \sum_{j=1}^{r} n_{jj} m_{ij} + 2v_i(D) - \frac{2}{r}. \]

Now we turn to assertion (b) of the lemma and compute, using (a) and the adjunction formula:

\[ (K . \Gamma_i) + 2(D_X . \Gamma_i) - (U_D . \Gamma_i) = m_{ii} + 2p_a(\Gamma_i) - 2 + \sum_{j=1}^{r} n_{jj} m_{ij} + \frac{2}{r}. \]

We deduce the first part of (b) and furthermore:

\[ \sum_{j} n_{jj} m_{ij} = \sum_{j} (n_{jj} + n_{ii} - 2n_{ij}) m_{ij} - \frac{2}{r} + 2 \]

Note that by [7, Lemma 4.1] we have

\[ n_{jj} + n_{ii} - 2n_{ij} = r(\Gamma_i, \Gamma_j), \]

where \( r(\Gamma_i, \Gamma_j) \) is the effective resistance between the nodes \( \Gamma_i \) and \( \Gamma_j \) if we consider the metrized graph \( G_X \) as a resistive electric circuit, where the resistance along an edge \( e \) is given by the length \( \ell(e) \). Hence, using (2.10) and (2.11), it suffices to show

\[ m_{ii} + \sum_{j} r(\Gamma_i, \Gamma_j) m_{ij} \geq 0 \]

in order to prove assertion (b). But we can rewrite the left hand side of (2.12) as

\[ \sum_{j \neq i} m_{ij}(r(\Gamma_i, \Gamma_j) - 1) \]
because of Lemma 2.14 (ii). A component $\Gamma_j$ can only contribute a negative summand to this sum if $m_{ij} \neq 0$, which means that the nodes on $G_X$ corresponding to $\Gamma_i$ and $\Gamma_j$ are connected by an edge $e$ of length $\ell(e) = -\frac{1}{m_{ij}} \leq 1$. But in this case the effective resistance $r(\Gamma_i, \Gamma_j)$ is bounded from above by $\ell(e)$. Hence all terms in the sum are nonnegative, proving (2.12) and thus the lemma.

If $\mathcal{X}_{s}$ is reduced, then we can give a formula for the intersection of $U_D$ with a canonical $\mathbb{Q}$-divisor. This result will be important in order to show that for reduced $\mathcal{X}$, our lower bound $\beta_D$ does not depend on $D$ and that $\beta_D$ is nonnegative if $\mathcal{X}$ is also minimal.

**Lemma 2.17.** Suppose that the special fiber of $\mathcal{X}$ is reduced. Then we have

$$(U_D \cdot K) = -\sum_{i=1}^{r} V_i^2 a_i.$$ 

If, furthermore, $\mathcal{X}$ is minimal, then $(U_D \cdot K)$ is nonnegative.

**Proof.** Since $D \mapsto U_D$ is linear in $D$, we may assume that we have $D = D_l$ for some $l \in \{1, \ldots, r\}$, that is, $(D \cdot \Gamma_j) = \delta_{lj}$ for all $j \in \{1, \ldots, r\}$. By definition of $U_D$, we have

$$(U_D \cdot K) = -\sum_{i=1}^{r} V_i^2 a_i + 2\sum_{i=1}^{r} (V_i \cdot V_l) a_i,$$

so we have to show that

(2.13) \[ \sum_{i=1}^{r} (V_i \cdot V_l) a_i = 0. \]

Note that

$$(V_i \cdot V_l) = \sum_{j=1}^{r} c_{lj} (V_i \cdot \Gamma_j) = \sum_{j=1}^{r} c_{lj} w_{ij},$$

so that, using $c_i^t = M^+ w_i^t$, we find

$$\sum_{i=1}^{r} (V_i \cdot V_l) a_i = -(2g - 2) w_l^t M^+ \sum_{i=1}^{r} a_i^t w_i.$$ 

Therefore the proof of (2.13) follows from the fact that for each $j \in \{1, \ldots, r\}$ we have

$$\sum_{i=1}^{r} w_{ij} a_i^t = a_j^t \sum_{i=1}^{r} a_i^t - a_j^t = 0,$$

since $\sum_{i=1}^{r} a_i^t = 1$. If $\mathcal{X}$ is minimal, then the adjunction formula implies that

$$a_i = (K \cdot \Gamma_i) = -\Gamma_i^2 + 2p_a(\Gamma_i) - 2 \geq 0.$$

for all $i$, which completes the proof of the lemma. \qed

The following result can be deduced immediately from Lemma 2.17.

**Corollary 2.18.** Suppose that $\mathcal{X}$ is minimal with reduced special fiber. Then $\beta_D$ is nonnegative.

It is natural to ask whether $\beta_D$ depends on $D$. We now show that this is not the case when $\mathcal{X}$ has reduced special fiber; furthermore, we provide a rather explicit formula for $\beta_D$. 

Lemma 2.19. If $\mathcal{X}$ has reduced special fiber, then $\beta = \beta_D$ is independent of $D$. More precisely, we have the following formula for $\beta$:

$$\beta = \frac{4(g-1)}{gr} \text{tr}(M^+) + \frac{g-1}{g} \sum_{i=1}^{r} \sum_{j=1}^{r} n_{ii} n_{jj} m_{ij} + \frac{2(g-1)}{g} \sum_{i=1}^{r} a_{i} n_{ii}$$

$$- \frac{1}{g} \sum_{i=1}^{r} \sum_{j=1}^{r} a_{i} a_{j} n_{ij}.$$

Proof. The proof is essentially a straightforward, but tedious computation using the properties of $M$ and $M^+$ listed in Lemma 2.14, so we do not present all details. Suppose that $\mathcal{X}$ has reduced special fiber. We first give an expression for $(K.U_D)$. By Lemma 2.17, we have

$$(K.U_D) = - \sum_{i=1}^{r} V_{i}^{2} a_{i}.$$ 

A simple computation shows that for a fixed $i \in \{1, \ldots, r\}$ we have

$$V_{i}^{2} = 2 \sum_{j} a'_{j} n_{ij} - n_{ii} - \sum_{j,k} a'_{j} a'_{k} n_{jk};$$

using Lemma 2.14 and $\sum_{i} a'_{i} = 1$, this implies

$$(2.14) \quad (K.U_D) = (2g-2) \left( \sum_{i} a'_{i} n_{ii} - \sum_{i,j} a'_{i} a'_{j} n_{ij} \right).$$

Now we rewrite $(U_D + 2V_D)^2$. From the definition of $V_D$ we get

$$(2.15) \quad V_{D}^{2} = 2 \sum_{i,j} a'_{i} v_{j}(D)n_{ij} - \sum_{i,j} a'_{i} a'_{j} n_{ij} - \sum_{i,j} v_{i}(D)v_{j}(D)n_{ij}.$$ 

Next we compute, using Lemma 2.14 and omitting details:

$$(V_{D}.U_{D}) = \sum_{i} (2(V_{D}.V_{i}) - V_{i}^{2}) w_{i}(D)$$

$$= \sum_{i,j,k} a'_{j} a'_{k} v_{i}(D)n_{jk} - 2 \sum_{i,j,k} a'_{k} v_{i}(D)v_{j}(D)n_{jk} + 2 \sum_{i,j} v_{i}(D)v_{j}(D)n_{ij}$$

$$+ \sum_{i,j,k} a'_{j} a'_{k} a'_{i} n_{jk} - 2 \sum_{i,j} a'_{j} a'_{i} n_{ij} + \sum_{i} a'_{i} n_{ii} - \sum_{i} v_{i}(D)n_{ii}$$

$$(2.16) \quad = 2 \sum_{i,j} v_{i}(D)v_{j}(D) + \sum_{i} a'_{i} n_{ii} - 2 \sum_{i,j} a'_{i} v_{j}(D)n_{ij} - \sum_{i} v_{i}(D)n_{ii}.$$
The computation of $U_D^2$ is more complicated than the previous one, so we only provide a rough sketch:

$$U_D^2 = -4 \sum_{i,j} (V_D \cdot V_i)(V_D \cdot V_j)m_{ij} + 4 \sum_{i,j} (V_D \cdot V_i)V_j^2m_{ij} - \sum_{i,j} V_i^2V_j^2m_{ij}$$

$$= 4 \sum_{i,j} v_i(D)n_{ij}n_{kk}m_{jk} - \sum_{i,j} n_{ii}n_{jj}m_{ij} - 4 \sum_{i,j} v_i(D)v_l(D)n_{ij}n_{kl}m_{jk}$$

$$(2.17) = 4 \sum_{i,j} v_i(D)n_{ij}n_{kk}m_{jk} - \sum_{i,j} n_{ii}n_{jj}m_{ij} - 4 \sum_{i,j} v_i(D)v_j(D)n_{ij}.$$  

Combining $(2.15)$, $(2.16)$ and $(2.17)$, we find

$$(U_D + 2V_D)^2 = 4 \sum_i a'_i n_{ii} - \sum_{i,j} n_{ii}n_{jj}m_{ij} - 4 \sum_{i,j} a'_i a'_j n_{ij} + 4 \sum_{i,j,k} v_i(D)n_{ij}n_{kk}m_{jk}$$

$$- 4 \sum_i v_i(D)n_{ii}.$$  

Part $(iv)$ of Lemma 2.14 implies

$$(2.18) (U_D + 2V_D)^2 = 4 \sum_i a'_i n_{ii} - \sum_{i,j} n_{ii}n_{jj}m_{ij} - 4 \sum_{i,j} a'_i a'_j n_{ij} - \frac{4\text{Tr}(M^+)}{r},$$

which does not depend on $D$. The result now follows from $(2.14)$ and $(2.18)$.  

**Example 2.20.** Keeping the notation of Example 2.13, Lemma 2.19 immediately implies that

$$\beta = \frac{1}{2s}(s + 2p_a - 2).$$

**Remark 2.21.** Note that the first two terms in the formula for $\beta$ given in Lemma 2.19 only depend on $M$, so they only depend on the combinatorial configuration of $X_s$. The last two terms, however, depend on the arithmetic genera of the irreducible components; more precisely, we have

$$\frac{2(g-1)}{g} \sum_{i=1}^r a_i n_{ii} - \frac{1}{g} \sum_{i=1}^r \sum_{j=1}^r a_i a_j n_{ij} = \frac{2g-2}{g} \sum_{i=1}^r a_i \left(n_{ii} - \sum_{j=1}^r a'_j n_{ij}\right).$$

Therefore, if $X$ is semistable and minimal, then $\beta$ can be viewed as an invariant of the polarized metrized graph $(R(X), q)$ associated to $X_s$, where the polarization $q$ assigns to each component its arithmetic genus, see [6, §4].

**Remark 2.22.** It seems worthwhile to relate $\beta$ to other invariants of $(R(X), q)$, such as Zhang’s invariants $\varepsilon$ (called $r$ in [25]), $\varphi$ and $\lambda$. See [6] for definitions of and some relations between these invariants. If $X$ is hyperelliptic, then it would also be interesting to compare $\beta$ to the invariant $\chi$ studied, for instance, in [11]. Because of its potential relevance for an effective version of the Bogomolov conjecture for curves over number fields (see Remark 5.2), it is especially interesting to compare $\beta$ to $\varepsilon$. We have computed $\beta$ for all semistable reduction types of genus 2 curves. Table 1 contains the values of $\beta$ and the values of $\varepsilon$, computed by de Jong, cf. [10], §2. We find that in genus 2, we always have $\beta \leq \varepsilon$. 

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LOWER BOUNDS ON $\omega^2$

| Type       | $\varepsilon$ | $\beta$ |
|------------|---------------|---------|
| I          | 0             | 0       |
| II(a)      | $a$           | $a - 1$ |
| III(a)     | $\frac{1}{6}a$ | $\frac{1}{6}a - \frac{1}{6a}$ |
| IV(a, b)   | $a + \frac{1}{6}b$ | $a + \frac{1}{6}b - \frac{1}{6b}$ |
| V(a, b)    | $\frac{1}{6}(a + b)$ | $\frac{1}{6}(a + b) - \frac{1}{6a} - \frac{1}{6b}$ |
| VI(a, b, c)| $\frac{1}{6}(a + b + c)$ | $\frac{1}{6}(a + b + c) + \frac{1}{6(ab+ac+bc)}$ |

Table 1. The invariants $\varepsilon$ and $\beta$ in genus 2

3. Semipositive hermitian line bundles

Let $K$ be a number field with ring of integers $\mathcal{O}_K$ and let $\mathcal{X}$ be a proper regular arithmetic surface over $\mathcal{O}_K$ whose generic fiber $X = \mathcal{X}_K$ is a smooth projective geometrically irreducible curve of genus $g > 1$. In this section we prove several general lemmas about certain hermitian line bundles on $\mathcal{X}$. All of these will be used in the proof of Proposition 1.2. Several results of this section are quite similar to results from [24]. We start with a number of definitions.

**Definition 3.1.** Let $\mathcal{L}$ be a hermitian line bundle on $\mathcal{X}$. If $E$ is an irreducible effective divisor on $X$ with Zariski closure $E$, then the height of $E$ with respect to $\mathcal{L}$ is defined by

$$h_{\mathcal{L}}(E) = \frac{(\mathcal{L} \cdot \mathcal{O}(E))}{[K: \mathbb{Q}] \deg(E_K)},$$

where the metric on $\mathcal{O}(E)$ is admissible in the Arakelov-theoretic sense and $(\cdot, \cdot)$ is the arithmetic intersection pairing on $\mathcal{X}$, see for instance [20]. We extend this to arbitrary effective divisors on $\mathcal{X}$ by linearity.

**Definition 3.2.** We say that a hermitian line bundle $\mathcal{L}$ on $\mathcal{X}$ is **relatively semipositive** if it has nonnegative intersection with all irreducible vertical components of $\mathcal{X}$. If $\mathcal{L}$ has nonnegative (resp. positive) intersection with all irreducible horizontal divisors on $\mathcal{X}$, then we call $\mathcal{L}$ **horizontally semipositive** (resp. **horizontally positive**).

**Definition 3.3.** Let $\mathcal{L}$ be a hermitian line bundle on $\mathcal{X}$. We call a nonzero section $s$ of $\mathcal{L}$ **effective** (resp. **strictly effective**) if $\|s\|_{\sup} \leq 1$ (resp. $\|s\|_{\sup} < 1$). We say that $\mathcal{L}$ is **ample** if $\mathcal{L}$ is ample and $H^0(\mathcal{X}, \mathcal{L}^n)$ has a basis consisting of strictly effective sections for $n \gg 0$. If this holds for $n = 1$, then we call $\mathcal{L}$ **very ample**.

**Lemma 3.4.** Let $\mathcal{L}$ be a hermitian line bundle on $\mathcal{X}$. For any hermitian line bundle $\mathcal{M}$ on $\mathcal{X}$ and $a, b \in \mathbb{N}$ we set

$$\mathcal{M}_{a,b} = \mathcal{M}^a \otimes \mathcal{L}^b.$$

If $\mathcal{L}^2 < 0$, then there exists no ample hermitian line bundle $\mathcal{M}$ on $\mathcal{X}$ such that the following holds for all $a, b \in \mathbb{N}$:

$$\text{If } \mathcal{M}_{a, b}^2 > 0, \text{ then } (\mathcal{L}, \mathcal{M}_{a, b}) \geq 0.$$ 

**Proof.** This proof is somewhat similar to the first part of the proof of [24, Theorem 6.3]. Suppose that $\mathcal{L}^2 < 0$ and that $\mathcal{M}$ is an ample hermitian line bundle on $\mathcal{X}$ satisfying (3.1).
Since $\overline{M}$ is ample, it has positive arithmetic self-intersection by [24, Theorem 1.3]. Therefore, if $p(t)$ denotes the polynomial

$$p(t) = \overline{L}^2 + 2(\overline{L} \cdot \overline{M})t + \overline{M}^2 t^2,$$

then there is a positive real number $t_0$ satisfying $p(t_0) = 0$ and $p(t) > 0$ for every $t > t_0$. Let $a, b \in \mathbb{N}$ such that $a/b > t_0$. Then we find

$$\overline{M}^2_{a,b} = b^2 p(a/b) > 0.$$

By (3.1), we know that

$$\frac{1}{b} (\overline{L} \cdot \overline{M}_{a,b}) = \overline{L}^2 + \frac{a}{b} (\overline{L} \cdot \overline{M}) \geq 0.$$

In particular, our assumption that $\overline{L}^2 < 0$ implies (3.2)

$$(\overline{L} \cdot \overline{M}) > 0,$$

and also, since $a/b$ can be arbitrary close to $t_0$,

(3.3) $\overline{L}^2 + (\overline{L} \cdot \overline{M})t_0 \geq 0$.

Now we can derive a contradiction as in the proof of [24, Theorem 6.3]. Namely, combining $p(t_0) = 0$ and (3.3) implies

$$(\overline{L} \cdot \overline{M})t_0 + \overline{M}^2 t_0^2 \leq 0.$$

But using $\overline{M}^2 > 0$ and (3.2), we see that this is impossible. $\square$

The following result provides us with a method to show that two hermitian line bundles on $\mathcal{X}$ have nonnegative intersection.

**Lemma 3.5.** Let $\overline{L}$ and $\overline{M}$ be hermitian line bundles on $\mathcal{X}$. If $\overline{M}$ has an effective global section $s$ such that $h^{\overline{L}}(\text{div}(s)^{\text{hor}}) \geq 0$ and $\overline{L}$ is relatively semipositive, then $(\overline{L} \cdot \overline{M}) \geq 0$.

**Proof.** According to [3, §3.2.2], we have

$$(\overline{L} \cdot \overline{M}) = \frac{(\mathcal{O}(\text{div}(s)^{\text{ver}}) \cdot \overline{L})}{[K : \mathbb{Q}]} + h^{\overline{L}}(\text{div}(s)^{\text{hor}}) - \int \log \|s\| \sup c_1(\overline{L}).$$

Since $\text{div}(s)^{\text{ver}}$ is effective, the claim follows. $\square$

Suppose we want to show that a hermitian line bundle $\overline{L}$ on $\mathcal{X}$ satisfies $\overline{L}^2 \geq 0$. By Lemma 3.4, it suffices to find some ample hermitian line bundle $\overline{M}$ on $\mathcal{X}$ such that under the assumption $\overline{L}^2 < 0$ we have $(\overline{L} \cdot \overline{M}_{a,b}) \geq 0$ whenever $\overline{M}^2_{a,b} > 0$. If $\overline{L}$ is horizontally semipositive, then we can take any ample $\overline{M}$ and any effective section $s$ of $\overline{M}^{\otimes n}$ (which exists for $n \gg 0$ by [24, Theorem 2.1], see [24, §8]) and apply Lemma 3.5. However, in the proof of Proposition 1.2 we will apply Lemma 3.5 to a hermitian line bundle $\overline{L} = \overline{L}_D$ which is not in general horizontally semipositive, but only satisfies $h^{\overline{L}_D}(E) \geq 0$ if the Zariski closure $E_{\mathcal{X}}$ avoids a certain finite set of points, so we have to be more careful with our choice of $s$. Lemma 3.6 (i) below tells us that, under the hypothesis $\overline{L}^2 < 0$, we can find $\overline{M}$ such that for $n \gg 0$ there are many effective sections of $\overline{M}^{\otimes n}_{a,b}$ whenever $\left(\overline{M}^{\otimes n}_{a,b}\right)^2 > 0$. Intuitively, it should be possible to find an effective section $s$ avoiding a finite set of points if $\overline{M}_{a,b}$ has enough effective sections. Lemma 3.6 (ii) makes this intuition precise.
Lemma 3.6. Suppose $\mathcal{L}$ is a relatively semipositive hermitian line bundle on $X$ such that $\deg(\mathcal{L}) > 0$ and $\mathcal{L}^2 < 0$. Then we have:

(i) There exists an ample hermitian line bundle $\mathcal{M}$ on $X$ such that every $\mathcal{M}_{a,b}$ with positive arithmetic self-intersection is already ample.

(ii) In particular, for such a $\mathcal{M}_{a,b}$ and a finite set $P_1, \ldots, P_r$ of closed points on $X$, there exists a strictly effective global section $s$ of $\mathcal{M}_{a,b}^{\otimes n}$, for some $n \gg 0$, such that $s(P_i) \neq 0$ for $i = 1, \ldots, r$.

Proof. Let $\mathcal{M}$ be an ample hermitian line bundle on $X$. We scale $\mathcal{M}$ by $\alpha \in \mathbb{Q}_{\geq 0}$ such that (i) holds for $\mathcal{M}(\alpha)$. By [24, Theorem 1.5], it suffices to show that $\mathcal{M}_{a,b}(\alpha)^2 > 0$ implies that $\mathcal{M}_{a,b}(\alpha)$ is horizontally positive, since relative semipositivity is automatic. Let $p_\alpha(t)$ denote the polynomial

$$p_\alpha(t) = \mathcal{L}^2 + 2(\mathcal{L} \cdot \mathcal{M}(\alpha))t + \mathcal{M}(\alpha)^2t^2.$$  

As in the proof of Lemma 3.4, there is some positive real number $t_0 = t_0(\alpha)$ such that $p_\alpha(t_0) = 0$ and $p_\alpha(t) > 0$ for all $t > t_0$. Now let $m_\mathcal{L} = -\inf_D h_{\mathcal{L}}(D)$ and $m_\mathcal{M} = \inf_D h_{\mathcal{M}}(D)$, where we take the infima over all irreducible divisors $D$ on $X$. If $m_\mathcal{L} \geq 0$, then we can take $\alpha = 0$, so we may assume that $m_\mathcal{L} < 0$. Let $D$ be some irreducible divisor on $X$. We construct $\alpha$ such that

$$h_{\mathcal{M}_{a,b}(\alpha)}(D) \geq 0$$

whenever $a/b > t_0(\alpha)$, which will prove (i). Because of

$$h_{\mathcal{M}_{a,b}(\alpha)}(D) = ah_{\mathcal{M}(\alpha)}(D) + bh_{\mathcal{L}}(D) \geq am_\mathcal{M} + a\alpha - bm_\mathcal{L},$$

we need a nonnegative $\alpha$ such that $\frac{a}{b} \geq t_0(\alpha) \Rightarrow \frac{a}{b} \geq \frac{m_\mathcal{L}}{\alpha + m_\mathcal{M}}$, so $\alpha$ must satisfy

$$t_0(\alpha) \geq \frac{m_\mathcal{L}}{\alpha + m_\mathcal{M}}.$$ 

Hence (3.4) is easily seen to follow from

$$\alpha + m_\mathcal{M}r(\alpha) \geq m_\mathcal{L} \left( \mathcal{L}^2 + 2\alpha \deg(\mathcal{M}) \right) + (\alpha + m_\mathcal{M}) \left( (\mathcal{M} \cdot \mathcal{L}) + \alpha \deg(\mathcal{L}) \right),$$

where

$$r(\alpha) = \sqrt{\alpha^2 \deg(\mathcal{L})^2 + 2\alpha \deg(\mathcal{M}) \left( (\mathcal{M} \cdot \mathcal{L}) - \mathcal{L}^2 \right) + (\mathcal{M} \cdot \mathcal{L})^2 - \mathcal{L}^2 \mathcal{M}^2}.$$ 

Note that $r(\alpha)$ is real since $p_\alpha$ always has real roots. Hence the left hand side of (3.5) is always nonnegative. If the right hand side is negative for some $\alpha$, then (3.5) holds and we are done, so we may assume that the right hand side is also nonnegative. We find that (3.5) holds if and only if $w(\alpha) \geq 0$, where $w$ is a cubic polynomial in $\alpha$, obtained by subtracting the square of the right hand side of (3.5) from the square of the left hand side. The leading coefficient of $w$ is

$$-2 \deg(\mathcal{M})(\mathcal{L}^2 + 2m_\mathcal{L} \deg(\mathcal{L}))$$

which is positive by our assumptions on $\mathcal{L}$. Hence (3.5) holds for $\alpha \gg 0$, proving (i).

To prove (ii), we may assume, without loss of generality, that $\mathcal{M}_{a,b}$ is very ample. In particular, the lattice $H^0(X, \mathcal{M}_{a,b})$ is spanned by strictly effective sections $s_1, \ldots, s_m$. Moreover, since $\mathcal{M}_{a,b}$ is also very ample in the geometric setting, it is globally generated, i.e., for each point $P_i$ there exists at least one section $s_j$ such that $s_j(P_i) \neq 0$. Let $\{s_1, \ldots, s_k\}$ be a minimal
set of such sections. Then for every even \( n \) the global section \( s' = \sum_{j=1}^{k} s_j^n \) of \( \mathcal{M}_{a,b}^{\otimes n} \) satisfies \( s'(P_i) \neq 0 \) for all \( i = 1, \ldots, r \). But now for some even \( n \gg 0 \) the global section \( s = \sum_{j=1}^{k} s_j^n \) also satisfies the condition
\[
\|s\|_{\text{sup}} \leq k \cdot \max_j \|s_j\|^n < 1.
\]

4. Heights and intersections

We keep the notation of the previous section. If \( F \) is a finite extension of \( K \), then we let \( \varphi_F = \text{pr}_1 \circ \pi_F : \mathcal{X}^F \to \mathcal{X} \), where \( \pi_F : \mathcal{X}^F \to \mathcal{X} \otimes \mathcal{O}_F \) denotes the minimal desingularization.

We need a global version of Lemma 2.7 and Lemma 2.10.

**Lemma 4.1.** (Liu, [16])

(i) There exists a finite extension \( F_0/K \) such that \( \mathcal{X}^F \) is semistable for every finite extension \( F/F_0 \).

(ii) Let
\[
T(\mathcal{X}) = \varphi_{F_0} \left( \mathcal{X}^{F_0} \setminus \mathcal{X}_{\text{sm}}^{F_0} \cup \text{Exc}(\varphi_{F_0}) \right).
\]
Then every irreducible divisor \( E \) on \( X \) such that \( \text{supp}(E_X) \cap T(\mathcal{X}) = \emptyset \) satisfies \( \varphi_F^* E_X = E_{\mathcal{X}^F} \) for every finite extension \( F/F_0 \).

**Proof.** The proof follows at once from the local results, namely Lemma 2.7 and Lemma 2.10. □

Let \( h_{\text{NT}} \) denote the Néron-Tate height on the Jacobian \( J \) of \( X \) with respect to the symmetrized theta divisor \( \Theta + [-1]^* \Theta \). For every \( \mathbb{Q} \)-divisor \( D \in \text{Div}_\mathbb{Q}(X) \) of degree one, let \( j_D : X \hookrightarrow J \) be the embedding which maps a point \( Q \in X \) to the class of \( Q - D \).

**Definition 4.2.** Let \( D \in \text{Div}_\mathbb{Q}(X) \) have degree one. For every non-archimedean place \( v \) of \( K \) we define \( D_v = D \otimes K_v^{nr} \), where \( K_v^{nr} \) is the maximal unramified extension of the completion of \( K \) at \( v \) and we fix the proper regular model \( \mathcal{X} \otimes \mathcal{O}_v^{nr} \) of \( X \) over the ring \( \mathcal{O}_v^{nr} \) of integers of \( K_v^{nr} \). With these choices, we define vertical \( \mathbb{Q} \)-divisors
\[
V_D = \sum_v V_{D_v}
\]
and
\[
U_D = \sum_v U_{D_v}
\]
on \( \mathcal{X} \), where both sums are over all non-archimedean places of \( K \). Moreover, we define a hermitian line bundle \( \underline{L_D} \) on \( \mathcal{X} \) by
\[
\underline{L_D} = \underline{w} \otimes \underline{O}(2D_X) \otimes O(U_D)^{-1} \otimes \underline{O}(-a\mathcal{X}_\infty),
\]
where \( a = \underline{O}(D_X)^2 - O(V_D)^2 \in \mathbb{R} \). Here \( \underline{O}(D_X)^2 \) is the self-intersection of \( \mathcal{O}(D_X) \), equipped with the Arakelov metric, and \( \underline{O}(-a\mathcal{X}_\infty) = (O_X, | \cdot |^a) \). Finally, we set
\[
\beta_D = \sum_v \beta_{D_v}.
\]
Unless otherwise stated, all metrics are Arakelov metrics (so that the Arakelov adjunction formula holds, cf. [14, §IV.5]), except for vertical line bundles, which are equipped with the trivial metric. Now we prove Theorem 1.1 stating that the height with respect to $L_D$ is closely related to the Néron-Tate height on the Jacobian of $X$. Recall that we can express $h_{NT}$ in terms of arithmetic intersection theory on $X$ using the Hodge Index Theorem on arithmetic surfaces due to Faltings [12] and Hriljac [13] as follows: If $Z$ is a divisor of degree zero on $X$ and $P$ is the corresponding point on the Jacobian, then we have

$$h_{NT}(P) \cdot [K : \mathbb{Q}] = -(\mathcal{O}(Z_X) \otimes \mathcal{O}(\Phi_X(Z)))^2,$$

where $\Phi_X(Z) \in \text{Div}_\mathbb{Q}(X)$ is a vertical $\mathbb{Q}$-divisor such that $Z_X + \Phi_X(Z)$ has trivial intersection multiplicity with all vertical $\mathbb{Q}$-divisors on $X$. But because $\Phi_X(Z)$ is a vertical $\mathbb{Q}$-divisor itself, we have

$$(\mathcal{O}(\Phi_X(Z)) \cdot \mathcal{O}(Z_X)) = -\mathcal{O}(\Phi_X(Z))^2,$$

implying

$$(4.1) \quad h_{NT}(P) \cdot [K : \mathbb{Q}] = -\mathcal{O}(Z_X)^2 + \mathcal{O}(\Phi_X(Z))^2.$$

**Proof of Theorem 1.1.** Let $E$ be an irreducible divisor on $X$ such that $\text{supp}(E_X) \cap T(X) = \emptyset$. Let $F_0$ be as in Lemma 4.1 and let $F/K$ be a finite extension containing $F_0$ such that $E$ has pointwise $F$-rational support. Let $e = \deg(E)$ and $E = \sum_{j=1}^{e} (P_j)$, where $P_j \in X(F)$. By Lemma 4.1 (ii) we have

$$(4.2) \quad \varphi_F^* E_X = E_X^F \quad \text{and} \quad \varphi_F^* D_X = D_X^F.$$

Moreover, Proposition 2.11 implies that

$$(4.3) \quad \sum_{j=1}^{e} \mathcal{O}(\Phi_X^F(P_j - D))^2 = [F : K] (e\mathcal{O}(V_D)^2 - (\mathcal{O}(E_X) \cdot \mathcal{O}(U_D))).$$
We now compare \( h_{\overline{\varphi_D}} \) to \( h_{NT} \) and find
\[
\sum_{j=1}^{e} h_{NT}(jD(P_j)) \cdot [F : \mathbb{Q}]
= \sum_{j=1}^{e} \left( -\mathcal{O}(P_{j,X}^F - D_{X}^F)^2 + \mathcal{O}(\Phi_{X}^F(P_j - D))^2 \right)
= \sum_{j=1}^{e} \left( -\mathcal{O}(P_{j,X}^F)^2 + 2(\mathcal{O}(P_{j,X}^F) \cdot \mathcal{O}(D_{X}^F)) + \mathcal{O}(\Phi_{X}^F(P_j - D))^2 \right)
= \left( \mathcal{O}(E_{X}^F) \cdot \mathcal{O}(2D_{X}^F) \right) + \sum_{j=1}^{e} \mathcal{O}(\Phi_{X}^F(P_j - D))^2 - e\mathcal{O}(D_{X}^F)^2
= (\mathcal{O}(E_{X}^F) \cdot \mathcal{O}(2D_{X}^F) + \sum_{j=1}^{e} \mathcal{O}(\Phi_{X}^F(P_j - D))^2 - e\mathcal{O}(D_{X}^F)^2)
= (\mathcal{O}(E_{X}) \cdot \mathcal{O}(2D_{X}) - (\mathcal{O}(E_{X}) \cdot \mathcal{O}(U_D)) - e\mathcal{O}(D_{X})^2 + e\mathcal{O}(V_D)^2) \cdot [F : K]
= e \cdot [F : \mathbb{Q}] \cdot h_{\overline{\varphi_D}}(E).
\]

Here the first equality holds by \( (4.1) \), the third equality holds because of the Arakelov adjunction formula (see\( \cite{14,IV.5} \)) and the fifth equality holds because of \( (4.2), (4.3) \) and because, by assumption, \( E_{X} \) does not intersect any vertical divisors contracted by \( \varphi_F \).

The first assertion of the proposition is now immediate since the Néron-Tate height only takes nonnegative values. The second assertion follows if we put \( E = (P) \), where \( P \in X(K) \). \( \square \)

Remark 4.3. If \( X \) is a smooth projective geometrically irreducible curve defined over an archimedean local field and \( E_1, E_2 \in \text{Div}(X) \) have disjoint support, then we set \([E_1, E_2] = (E_1, E_2)_{\text{ad}}\), where the latter denotes the admissible pairing on \( X \), see \( \cite{25,IV.5} \). Now suppose that \( X \) is defined over a number field \( K \). We can use \( (2.2) \) and Corollary \( 2.2 \) to define a pairing on divisors \( E_1, E_2 \) on \( X \) with disjoint support as
\[
[E_1, E_2] = \sum_v [E_{1,v}, E_{2,v}]_v,
\]
where \( E_{i,v} = E_i \otimes_K K_v \) for archimedean \( v \) and the sum is over all places of \( K \). This global pairing has the following properties, which may be of independent interest:

(i) \([\cdot, \cdot]\) is bilinear and symmetric.
(ii) \([E_1, \text{div}(f)] = 0\) for any \( f \in K(X)^* \). Hence \([\cdot, \cdot]\) induces a well-defined pairing on divisor classes.
(iii) If \( \deg(E_1) = \deg(E_2) = 0 \), then we have \([E_1, E_2] = -(E_1, E_2)_{NT}\), where the latter is the Néron-Tate height pairing.
(iv) If \( E_1 \) and \( E_2 \) are canonical divisors on \( X \), then we have \([E_1, E_2] = \varepsilon^2\).
5. Proofs of Proposition 1.2, Theorem 1.3 and Theorem 1.4

We finally get to our original problem, namely the derivation of lower bounds on \( \omega^2 \).

Proof of Proposition 1.2. Let \( D \) be such that \( \underline{\mathcal{L}}_D \) is relatively semipositive and such that \( D_X \cap T(\mathcal{X}) = \emptyset \). Assume that \( \underline{\mathcal{L}}_D^2 < 0 \). By Lemma 3.6 (i), there exists an ample hermitian line bundle \( \mathcal{M} \) on \( \mathcal{X} \) such that every \( \mathcal{M}_{a,b} \) with positive arithmetic self-intersection is already ample, where
\[
\mathcal{M}_{a,b} = \mathcal{M}^{\otimes a} \otimes \underline{\mathcal{L}}_D^{\otimes b}
\]
for positive integers \( a, b \). If we can show that for all these \( \mathcal{M}_{a,b} \), we have
\[
(\underline{\mathcal{L}}_D \cdot \mathcal{M}_{a,b}) \geq 0,
\]
then we get a contradiction from Lemma 3.4.

So let \( a, b \) be positive integers such that \( \mathcal{M}_{2a,b} > 0 \). By Lemma 3.6 (ii), there exists a strictly effective global section \( s \) of \( \mathcal{M}^{\otimes n}_{a,b} \), for some \( n \gg 0 \), such that \( \text{div}(s) \text{hor} \) does not intersect \( T(\mathcal{X}) \). Therefore Theorem 1.1 implies
\[
h_{\underline{\mathcal{L}}_D}(\text{div}(s) \text{hor}) \geq 0.
\]
It follows from Lemma 3.5 that we have
\[
(\underline{\mathcal{L}}_D \cdot \mathcal{M}^{\otimes n}_{a,b}) \geq 0
\]
and thus
\[
(\underline{\mathcal{L}}_D \cdot \mathcal{M}_{a,b}) \geq 0.
\]
But by Lemma 3.4 this leads to the desired contradiction and proves the proposition. \( \square \)

Next we prove Theorem 1.3. It follows from [5, Lemma A.1] that there exists a \( \mathbb{Q} \)-divisor \( D \in \text{Div}_\mathbb{Q}(X) \) such that \( D_X \cap T(\mathcal{X}) = \emptyset \) and such that \( (2g - 2)D \) is a canonical \( \mathbb{Q} \)-divisor on \( X \). Moreover, it is shown in [8] that
\[
(5.1) \quad K = (2g - 2)(D_X + V_D) \in \text{Div}_\mathbb{Q}(\mathcal{X})
\]
is a canonical \( \mathbb{Q} \)-divisor on \( \mathcal{X} \). By the latter we mean a \( \mathbb{Q} \)-divisor such that \( \mathcal{O}(K) = \omega \).

Proof of Theorem 1.3. From (5.1) we get \( D_X = \frac{1}{2g-2}K - V_D \). We can use this to rewrite \( \underline{\mathcal{L}}_D \) (cf. Definition 4.2) as
\[
\underline{\mathcal{L}}_D = \omega \otimes \frac{1}{2g-2} \otimes \mathcal{O}(-2V_D - U_D) - 4a,
\]
where
\[
a = \overline{1}(D_X)^2 - \mathcal{O}(V_D)^2 = \frac{1}{4(g-1)^2} \omega^2 - \frac{1}{g-1}(\omega \cdot \mathcal{O}(U_D)).
\]
Hence we have
\[
\underline{\mathcal{L}}_D^2 = \frac{1}{g-1} (g \omega^2 + (g-1)\mathcal{O}(2V_D + U_D)^2 - 2g(\omega \cdot \mathcal{O}(U_D)))
\]
Since \( \underline{\mathcal{L}}_D^2 \geq 0 \) by Proposition 1.2, Theorem 1.3 follows. \( \square \)

Remark 5.1. We have developed the theory of \( \underline{\mathcal{L}}_D \) for rather general degree one divisors \( D \in \text{Div}_\mathbb{Q}(X) \). The main reason why we choose to work with \( D \) as in Theorem 1.3 is that \( D_X \) has an obvious relation with \( \mathbb{Z} \). But there are other promising choices for \( D \); for instance, we could take \( D = \frac{1}{g^2-g}W \), where \( W \) is the divisor of Weierstrass points on \( X \). This was
suggested by Ariyan Javanpeykar. In fact, it is easy to see that the divisor \( V \) used in [9, Lemma 5.1] to extend \( W \) to a divisor on \( X \) with good properties is a valid choice for \( V_W \).

**Proof of Theorem 1.4.** Let \( D \in \text{Div}_\mathbb{Q}(X) \) have degree one and let \( \Gamma \) be an irreducible component of a special fiber of \( X \). By definition of \( \mathcal{L}_D \), we have

\[
(\mathcal{L}_D \cdot \mathcal{O}(\Gamma)) = (K \cdot \Gamma) + 2(D_X \cdot \Gamma) - (U_D \cdot \Gamma)
\]

up to a positive constant, where \( K \) is a canonical \( \mathbb{Q} \)-divisor on \( X \). Hence relative semipositivity of \( \mathcal{L}_D \) follows from part (b) of Lemma 2.16, proving (i).

Lemma 2.19 implies (ii); assertion (iii) is an immediate consequence of Corollary 2.18. \( \square \)

**Remark 5.2.** One of the main original motivations to look at lower bounds for \( \omega^2 \) was a conjecture of Bogomolov. Building on earlier work of Zhang [25], the conjecture was finally proved by Ullmo [23], who proved the positivity of the admissible self-intersection \( \omega^2_a \) of \( \omega \) for curves over number fields. For curves over function fields of characteristic 0, the conjecture was reduced by Zhang [26] to a conjecture about invariants of polarized metrized graphs and the latter was proved by Cinkir [6]. However, Cinkir actually proved an effective version of the Bogomolov conjecture. Such an effective version can also be conjectured over number fields, but in this situation it has not been proved yet (it would follow from a proof of the arithmetic standard conjectures of Gillet-Soulé, see [26, §1.4]). Using [25], it suffices to find an effectively computable nontrivial lower bound for \( \omega^2_a \). If \( X \) is semistable and minimal, then we have

\[
\omega^2 = \omega^2_a + \sum_v \varepsilon_v(X),
\]

where \( \varepsilon_v(X) \geq 0 \) is Zhang’s admissible constant associated to \( X \otimes K_v \) (see Remark 2.21) and the sum is over all non-archimedean places of \( K \). Hence it suffices to find an effectively computable lower bound \( b \) for \( \omega^2 \) such that \( \sum_v \varepsilon_v(X) < b \). Therefore our work provides a possible approach to the effective Bogomolov Conjecture; however, we have \( \beta \leq \sum_v \varepsilon_v \) for \( g = 2 \) by Remark 2.22.

### 6. Applications

Now we apply our results to compute lower bounds on the self-intersection of the relative dualizing sheaf for certain families of curves.

#### 6.1. Modular curves.

Let \( N = N'QR \) be a squarefree integer such that \( N' \in \mathbb{N}, Q, R \geq 4 \) and \( \gcd(Q, R) = 1 \). Consider the modular curve \( X_1(N) \) over the cyclotomic field \( \mathbb{Q}[\zeta_N] \) and its minimal regular model \( X = X_1(N)/\mathbb{Z}[\zeta_N] \). Then \( X \) is semistable. More precisely, the special fibers \( X_p \) are smooth if \( p \nmid N \). If \( p \mid N \) with residue characteristic \( p \), then the special fiber \( X_p \) consists of two isomorphic curves intersecting in

\[
s_p = \frac{p - 1}{24} \varphi(N/p)N \prod_{q \mid N/p} (1 + \frac{1}{q})
\]

points, all of which are rational over the residue field at \( p \), see [17, Proposition 7.3]. The arithmetic genera of these components are given by

\[
q_p = \frac{1}{2} (g_N - s_p + 1),
\]
where
\[ g_N = 1 + \frac{1}{24} \varphi(N) \prod_{p \mid N} (1 + \frac{1}{p}) - \frac{1}{4} \sum_{d \mid N} \varphi(d) \varphi(N/d) \]
is the genus of \( X_1(N) \). We can use Example 2.20 to compute an asymptotic lower bound for \( \omega^2 \) quite easily:

**Proposition 6.1.** The arithmetic self-intersection \( \omega^2 \) of the relative dualizing sheaf on \( X_1(N) \) satisfies
\[ \omega^2 \geq \frac{1}{2} \varphi(N) \log N + o(1), \]

*Proof.* Let \( n_p = \log \# k(p) \). Then we have \( \sum_{p \mid N} n_p = \varphi(N/p) \log(p) \) and hence, by Example 2.20,
\[
\beta = \sum_{p \mid N} \frac{n_p}{2s_p}(s_p + 2q_p - 2)
= \sum_{p \mid N} \frac{n_p}{2s_p}(g_N - 1)
= \frac{g_N - 1}{2} \sum_{p \mid N} \sum_{p \mid p} n_p
= 12(g_N - 1) \frac{\varphi(N)}{\prod_{p \mid N} p^2 - 1} \sum_{p \mid N} \frac{p + 1}{p - 1} \log p
= \frac{1}{2} \varphi(N) \log N + o(1),
\]
since \( \frac{24(gN - 1)}{\prod_{p \mid N} p^2 - 1} = 1 + o(1). \)

**Remark 6.2.** In [17, Theorem 7.7], Mayer obtains the asymptotic formula
\[ \omega^2 = 3g_N \log(N) + o(g_N \log(N)). \]

Our lower bound is much smaller than this asymptotic value.

6.2. **Fermat curves of prime exponent.** In Section 5, we derived a nontrivial lower bound \( \beta_D \) on \( \omega^2 \) for minimal arithmetic surfaces with simple multiplicities. In the present subsection we compute lower bounds on \( \omega^2 \) in a situation where Theorem 1.4 is not applicable, namely for minimal regular models of Fermat curves of prime exponent. Along the way, we construct \( U_D \) and show that \( \Sigma_D \) is relatively semipositive, where \( D = (S_x) \) for a certain rational point \( S_x \) such that \( (2g - 2)D \) is a canonical divisor on \( X \). We start with a brief review of the notation from [8]. Let \( p > 3 \) be a prime number, let
\[ F_p : X^p + Y^p = Z^p \]
denote the Fermat curve with exponent \( p \) over \( K = \mathbb{Q}(\zeta_p) \), where \( \zeta_p \) is a primitive \( p \)-th root of unity. Let \( F_p^{\min} \) denote the minimal regular model of \( F_p \) over \( \mathbb{Z}[\zeta_p] \) as computed by McCallum [18]. We denote the components of the only non-reduced special fiber of \( F_p^{\min} \) by \( L_i, i \in I \), where
\[ I = \{ x, y, z, \alpha_1, \alpha_1, \ldots, \alpha_{1,p}, \alpha_2, \alpha_2, \ldots, \alpha_{2,p}, \ldots, \alpha_r, \alpha_r, \ldots, \alpha_{r,p}, \beta_1, \ldots, \beta_s \} \]
and \(2r + s = p - 3\), see [8]. In Figure 1 the configuration of the only reducible special fiber (occurring at the unique prime above \(p\)) of a certain non-minimal model \(F_p\) of \(F\) is shown. It has the property that contracting the unique exceptional component \(L\) on \(F_p\) yields \(F^{\text{min}}_p\). For every component \(L_i\), we also list the pair \((m_i, L_i^2)\), where \(m_i\) is the multiplicity of \(L_i\) in \(F_p\). All components have genus 0 and the only component with self-intersection number -1 is \(L\). See [IS, S] for further details.

**Figure 1.** The configuration of the only reducible special fiber of \(F_p\).

Note that \(L_{\alpha_i}\) has multiplicity two for all \(i\). Therefore Theorem 1.4 does not apply in general and we have to show relative semipositivity of \(\overline{L_D}\) and nontriviality of the bound from Theorem 1.3 directly for a suitable \(\mathbb{Q}\)-divisor \(D\). Since there is only one place of bad reduction, we omit it from the notation for the sake of simplicity. It is shown by Curilla and the first author in [S] that a canonical divisor on \(F\) yields \((2g - 2)D\), where \(D = S_x\) for a certain \(K\)-rational point \(S_x\) on \(F\) whose Zariski closure \(S_x := S_x,F^{\text{min}}_p\) in \(F^{\text{min}}_p\) intersects only \(L_\alpha\). In order to compute a lower bound on \(\overline{L}_D\) using the results of the previous sections, we will find \(U_D\). This means that we first have to compute the \(\mathbb{Q}\)-divisor \(V_i\) for each \(i \in I\).

**Lemma 6.3.** We can take

- \(V_i = \frac{1}{p} L_i, \text{ if } i \in \{x, y, z, \beta_j\}\),
- \(V_{\alpha_i} = \frac{1}{p} L_{\alpha_i} + \frac{1}{2p} \sum_{j=1}^p L_{\alpha_{ij}}\),
- \(V_{\alpha_{ij}} = -\frac{1}{p} L_{\alpha_i} + \left(\frac{1}{2} - \frac{1}{2p}\right) L_{\alpha_{ij}} - \frac{1}{2p} \sum_{k \neq j} L_{\alpha_{ik}}\).

**Proof.** Recall that the \(\mathbb{Q}\)-divisor \(V_i\) must satisfy

(6.1) \[ (V_i, L) = a'_j - \frac{\delta_{ij}}{b_j} \]

for all \(j \in I\), where \(b_j\) denotes the multiplicity of \(L_j\), we have \(a'_j = \frac{1}{2g - 2}(-L_j^2 + 2p_a(L_j) - 2)\) and \(\delta_{ij}\) is the Kronecker delta function on \(I\). Since all components have genus zero and self-intersection \(-p\) except for the components \(L_{\alpha_{ij}}\), which have genus zero and self-intersection \(-2\), we get \(a'_j = 0\) for the components \(L_{\alpha_{ij}}\) and \(a'_j = \frac{1}{p}\) for all other components. Checking the validity of the Lemma reduces to checking (6.1) for each \(i \in I\) which is a simple computation that we leave to the reader. Note that for \(V_x\) this was essentially shown in [S] Proposition 8.3. \(\square\)
Corollary 6.4. A canonical $\mathbb{Q}$-divisor on $F^\text{min}_p$ is given by $(2g - 2)S_x + \frac{p}{p^2}L_x$ and we have

\[ U_D = \frac{1 - p}{p^2}L_x + \frac{1 + p}{p^2} \left( L_y + L_z + \sum_{j=1}^{s} L_{\beta_j} \right) + \frac{2 + p}{p^2} \sum_{i=1}^{r} L_{\alpha_i} \]

\[ + \frac{p^2 + p - 6}{2p^2} \sum_{i=1}^{r} \sum_{j=1}^{p} L_{\alpha_{ij}}. \]

**Proof.** The first assertion follows immediately from (5.1), the fact that $(2g - 2)S_x$ is a canonical $\mathbb{Q}$-divisor on $F_p$, and Lemma 6.3. For the second assertion, recall the definition of $U_D$:

\[ U_D = \sum_{i \in I} b_i (2(V_i \cdot V_x) - V_i^2) L_i \]

Next we compute

\[ V_i^2 = \left( \frac{1}{p} L_i \right)^2 = \frac{1}{p^2} (1 - p) \text{ for } i \in \{x, y, z, \beta_j\} \]

\[ (V_i \cdot V_x) = \left( \frac{1}{p} L_i \cdot L_x \right) = \frac{1}{p^2} \text{ for } i \in \{y, z, \beta_j\} \]

\[ (V_{\alpha_i} \cdot V_x) = \left( \left( \frac{1}{p} L_{\alpha_i} + \frac{1}{2p} L_{\alpha_{ij}} \right) \cdot \frac{1}{p} L_x \right) = \frac{1}{p^2} \]

\[ (V_{\alpha_{ij}} \cdot V_x) = \left( \left( -\frac{1}{p} L_{\alpha_i} + \left( \frac{1}{2} - \frac{1}{2p} \right) L_{\alpha_{ij}} - \frac{1}{2p} \sum_{k \neq j} L_{\alpha_{ik}} \right) \cdot \frac{1}{p} L_x \right) = -\frac{1}{p^2} \]

\[ (V_{\alpha_i})^2 = \left( \frac{1}{p} L_{\alpha_i} + \frac{1}{2p} L_{\alpha_{ij}} \right)^2 = \frac{1}{p^2} (1 - p) + \frac{p}{p^2} - \frac{2p}{4p^2} = \frac{1}{p^2} - \frac{1}{2p} \]

\[ (V_{\alpha_{ij}})^2 = \left( -\frac{1}{p} L_{\alpha_i} + \left( \frac{1}{2} - \frac{1}{2p} \right) L_{\alpha_{ij}} - \frac{1}{2p} \sum_{k \neq j} L_{\alpha_{ik}} \right)^2 \]

\[ = \frac{1}{2p^2} (1 - p) - \frac{1}{2} \left( 1 - \frac{1}{p} \right)^2 - \frac{1}{2p^2} (p - 1) - \frac{1}{p} \left( 1 - \frac{1}{p} \right) + \frac{p - 1}{p^2} \]

\[ = \frac{1}{2p^2} (2 - p - p^2) \]

A simple computation proves the corollary. \(\square\)

**Lemma 6.5.** The hermitian line bundle $\overline{L_D}$ is relatively semipositive.

**Proof.** Recalling Definition 4.2 of $\overline{L_D}$ we see that we need to show

\[ a_i + 2(S_x \cdot L_i) - (U_D \cdot L_i) \geq 0 \]

for all $i \in I$, where $a_i = 0$ for $i = \alpha_{ij}$ and $a_i = p - 3$ for all other components. As usual, we distinguish between the different components $L_i$ to prove (6.2). Throughout, we use
s = p − 3 − 2r to eliminate s. We start with L_x and find, using Corollary 6.4 that

\[(U_D \cdot L_x) = \frac{(1 - p)^2}{p^2} + \frac{(1 + p)(s + 2)}{p^2} + \frac{r(2 + p)}{p^2} = 2 - \frac{2}{p} - \frac{r}{p}.\]

Let \(i \in \{y, z, \beta_j\}\). Then we get

\[(U_D \cdot L_i) = 1 - \frac{p}{p^2} + \frac{1 - p^2}{p^2} + \frac{(1 + p)(s + 1)}{p^2} + \frac{r(2 + p)}{p^2} = -\frac{2}{p} - \frac{r}{p}.\]

Moreover, we compute

\[(U_D \cdot L_{\alpha_i}) = -\frac{3}{p} - \frac{r}{p}\]

and

\[(U_D \cdot L_{\alpha_{ij}}) = \frac{8}{p^2} - 1.\]

This suffices to show that (6.2) is satisfied for all components \(L_i\), which proves the lemma. \(\square\)

Theorem 1.6 is an immediate consequence of the following result:

**Theorem 6.6.** Let \(p > 3\) be a prime number and let \(\overline{\omega}^2\) denote the arithmetic self-intersection of the relative dualizing sheaf on \(\mathcal{F}_p^{\text{min}}\). Then we have

\[\overline{\omega}^2 \geq (p - 2 - \frac{3}{p}) \log p > 0.\]

**Proof.** By Theorem 1.3 and Lemma 6.5 we know that

\[\overline{\omega}^2 \geq \beta_D = \frac{1 - g}{g} \mathcal{O}(2V_D + U_D)^2 + 2(\overline{\omega} \cdot \mathcal{O}(U_D)).\]

We compute the terms on the right hand side.

First note that

\[2V_D + U_D = \frac{1 + p}{p^2} \left( L_x + L_y + L_z + \sum_{j=1}^{s} L_{\beta_j} \right) + \frac{2 + p}{p^2} \sum_{i=1}^{r} L_{\alpha_i} + \frac{p^2 + p - 6}{2p^2} \sum_{i=1}^{r} \sum_{j=1}^{p} L_{\alpha_{ij}}.\]

In particular, we have

\[2V_D + U_D = \left( \frac{1 + p}{p^2} \right) \mathcal{F}_p^{\text{min}}\]

when \(r = 0\), so in general we only get the trivial bound

\[\frac{1 - g}{g} (2V_D + U_D)^2 \geq 0.\]

For the computation of \((\mathcal{K} \cdot U_D)\), where \(\mathcal{K}\) is a canonical \(\mathbb{Q}\)-divisor on \(\mathcal{F}_p^{\text{min}}\), we use the adjunction formula. Namely, if \(t_i\) denotes the multiplicity of \(L_i\) in \(U_D\), then we have

\[(\mathcal{K} \cdot U_D) = \sum_{i \in I} t_i a_i\]
and hence
\[(K \cdot U_D) = (p - 3) \left( \frac{1 - p}{p^2} + \frac{(1 + p)(s + 2)}{p^2} + \frac{r(2 + p)}{p^2} \right)\]
\[= p - 4 + \frac{3}{p} + r \left( \frac{3}{p} - 1 \right).\]

As we have
\[0 \leq r \leq \frac{1}{2} p - \frac{3}{2},\]
this proves the claim. □

Remark 6.7. The upper bound computed by Curilla and the first author in [8] is of order \(O(gp \log p)\), i.e. of order \(O(p^3 \log p)\). We can slightly improve the bound in Theorem 6.6 by bounding \((2V_D + U_D)^2\) in terms of \(r\), but this does not change the asymptotic behavior.

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