Dynamics of Non-Gaussian Entanglement of Two Magnetically Coupled Modes

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Abstract

This paper surveys the quantum entanglement of two coupled harmonic oscillators via angular momentum generating a magnetic coupling $\omega_c$. The corresponding Hamiltonian is diagonalized by using three canonical transformations and then the stationary wave function is obtained. Based on the Schmidt decomposition, we explicitly determine the Schmidt modes $\lambda_k$ with $k \in \{0,1,\ldots,n+m\}$, $n$ and $m$ being two quantum numbers associated to the two oscillators. By studying the effect of the anisotropy $R = \omega_1^2/\omega_2^2$, $\omega_c$, asymmetry $|n-m|$ and dynamics on the entanglement, we summarize our results as follows. (i)– The entanglement becomes very large with the increase of $(n,m)$. (ii)– The sensistivity to $\omega_c$ depends on $(n,m)$ and $R$. (iii)– The periodic revival of entanglement strongly depends on the physical parameters and quantum numbers.

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1 Introduction

The entanglement of quantum particles is a physical phenomenon and remains among the most amazing features of the quantum world [1]. It accounts for the inseparability of the wavefunction describing a system of $n$ particles to the tensor product of the state of each particle, i.e. $\psi(x_1, \cdots, x_n) \neq \psi(x_1) \otimes \cdots \otimes \psi(x_n)$. Moreover, the entanglement is a fundamental resource of the quantum information processing and by which the quantum technologies can go beyond the classical protocols [2–4]. In entanglement theory, the most important questions can be summarized as follows. 

(i)– How to generate a large entanglement between particles in a given physical set up. (ii)– How to mathematically compute the corresponding entanglement content [5, 6]. As for (i), the generation of entangled particles can be done for example with spontaneous parametric down-conversion, nuclear or atomic sources [7–9]. Regarding (ii), it depends on the type of the quantum states under consideration, that is continuous or discrete, bipartite or multipartite, Gaussian or non-Gaussian, pure or mixed [10, 11]. In the case of pure continuous bipartite states, the quantification of entanglement can be faithfully done via the von Neumann entropy $S_v = -\sum_k \lambda_k \ln(\lambda_k)$ with $\lambda_k$ being the Schmidt modes of the state at hand. Thereby, computing the modes $\lambda_k$ is not an easy task because it involves complex integrals [5,6].

The entanglement of systems made of coupled harmonic oscillators becomes a very active domain of research. In this respect, the pairwise entanglement of the ground state is largely studied, for instance we refer to [12–15]. However, the entanglement of excited states of coupled harmonic oscillators is less studied, which is due to the mathematical complexity of computing the Schmidt modes $\lambda_k$ [10, 16]. If the Schmidt modes are computed, then the quantification of entanglement can easily be obtained through Schmidt parameter $K = \left(\sum_k \lambda_k^2\right)^{-1}$ or von Neumann entropy. Recently, in a seminal works, Makarov has exactly obtained the analytical expressions of the Schmidt modes of two harmonic oscillators connected via position-position coupling type, i.e. $\hat{J}\hat{x}_1\hat{x}_2$ [6], as well as with a coupling velocity-position interaction type, i.e. $\alpha x_1p_2$ [5]. As a result, Makarov showed that in both cases the entanglement becomes very important for large quantum numbers and under some specific choices of physical parameters.

Motivated by the Makarov works [5, 6], we address the problem of two harmonic oscillators connected via an angular momentum coupling type. As a matter of clarity, this kind of coupling term is extensively used in several studies, for instance trapping ions [17–19] and quantum invariant theory [20–22]. We diagonalize the Hamiltonian by using three canonical transformations and give the Schmidt decomposition of the obtained stationary and non-stationary wavefunctions. In addition, we harness the derived modes to compute the entanglement content. As a result, we numerically investigate the effect of the anisotropy, asymmetry, magnetic coupling and dynamics on the entanglement. Our obtained results are important not only from a theoretical point of view but also they lead to study a generation of entangled photons via magnetic coupled waveguide beam splitters [23–25].

The layout of the paper is given as follows. In Sec 2, we define our model and diagonalize it by involving three canonical transformations. The normal modes together with the stationary wave function of Schrödinger equation will be obtained. The Schmidt decomposition will be analyzed in Sec 3 where the Schmidt modes will be explicitly computed. In Sec 4, we quantify the entanglement via two pure state quantifiers that are von Neumann entropy $S_v$ and Schmidt parameter $K$. We
analyze the effect of the asymmetry $|n - m|$, anisotropy $R$ and the mixing angle $\theta$ on the entanglement content. In Sec 5, we discuss the dynamics of entanglement by using the dynamical Schrödinger equation. Finally, we conclude our work.

## 2 Hamiltonian of two modes

We consider two coupled harmonic oscillators connected via a transversal magnetic field $\mathbf{B} = B_z \mathbf{u}_z$ generating a rotation described by the angular momentum $L_z$. Our system can be described by the Hamiltonian

$$
\hat{H} = \frac{\hat{p}_1^2}{2} + \frac{\hat{p}_2^2}{2} + \frac{1}{2} \omega_1^2 \hat{x}_1^2 + \frac{1}{2} \omega_2^2 \hat{x}_2^2 + \omega_c L_z
$$

(1)

where $\omega_c$ stands for the coupling frequency and $L_z = \hat{x}_1 \hat{p}_2 - \hat{x}_2 \hat{p}_1$. To diagonalize (1), we proceed by introducing the transformation

$$
\hat{p}_1 = \frac{\hat{p}_1 + \hat{p}_2}{\sqrt{2} \epsilon}, \quad x_1 = \frac{\epsilon}{\sqrt{2}} (\hat{q}_1 + \hat{q}_2)
$$

(2)

$$
\hat{p}_2 = \frac{\hat{q}_2 - \hat{q}_1}{\sqrt{2} \gamma}, \quad x_2 = \frac{\gamma}{\sqrt{2}} (\hat{p}_1 - \hat{p}_2)
$$

(3)

with the commutation relations $[\hat{q}_j, \hat{p}_k] = i \delta_{jk}$ and $[\hat{q}_j, \hat{q}_k] = [\hat{p}_j, \hat{p}_k] = 0$. By choosing the involved parameters as $\epsilon = 1$ and $\gamma = \omega_c^{-1}$, then after substituting into (1) we get the Hamiltonian

$$
\hat{H}_1 = \frac{\hat{p}_1^2}{2m_+} + \frac{\hat{p}_2^2}{2m_-} + \frac{1}{2} \Omega_1^2 \hat{q}_1^2 + \frac{1}{2} \Omega_2^2 \hat{q}_2^2 + J \hat{q}_1 \hat{q}_2
$$

(4)

where we have set $\Omega_1^2 = \frac{1}{2} (\omega_1^2 + \omega_2^2) \mp \omega_c \omega_2$, $J = \frac{\omega_1^2 - \omega_2^2}{2}$ and $m_\pm = \frac{1}{1 \pm \frac{\omega_c}{\omega_1 \omega_2}}$. Note that $H_1$ is decoupled when both modes are in resonance and remain coupled out of it. To simplify the Hamiltonian diagonalization, we perform a second transformation by reducing the kinetic matrix to unity

$$
(\hat{y}_1, \hat{\Pi}_2) = \sqrt{\frac{m_+}{m_-}} (\hat{q}_1, \hat{p}_2), \quad (y_2, \Pi_1) = \sqrt{\frac{m_-}{m_+}} (\hat{q}_2, \hat{p}_1)
$$

(5)

and then obtain the Hamiltonian

$$
\hat{H}_2 = \frac{\hat{\Pi}_1^2}{2m_+} + \frac{\hat{\Pi}_2^2}{2m_-} + \frac{1}{2} m \varpi_+^2 \hat{y}_1^2 + \frac{1}{2} m \varpi_-^2 \hat{y}_2^2 + J \hat{y}_1 \hat{y}_2
$$

(6)

where $m = \sqrt{m_+ m_-}$ and $\varpi_\pm = \Omega_\pm / \sqrt{m_\pm}$. Because of the last term in (6), $H_2$ is not decoupled and then a third transformation is required. It can be realized by rotating our system

$$
\hat{Q}_1 = \cos \left( \frac{\theta}{2} \right) \hat{y}_1 + \sin \left( \frac{\theta}{2} \right) \hat{y}_2, \quad \hat{Q}_2 = -\sin \left( \frac{\theta}{2} \right) \hat{y}_1 + \cos \left( \frac{\theta}{2} \right) \hat{y}_2
$$

(7)

$$
\hat{P}_1 = \cos \left( \frac{\theta}{2} \right) \hat{\Pi}_1 + \sin \left( \frac{\theta}{2} \right) \hat{\Pi}_2, \quad \hat{P}_2 = -\sin \left( \frac{\theta}{2} \right) \hat{\Pi}_1 + \cos \left( \frac{\theta}{2} \right) \hat{\Pi}_2
$$

(8)

with $[\hat{Q}_j, \hat{P}_k] = i \delta_{jk}$ and $[\hat{Q}_j, \hat{Q}_k] = [\hat{P}_j, \hat{P}_k] = 0$. We conclude that for a mixing angle fulfilling the condition

$$
\tan \theta = \frac{(1 - R) \sqrt{1 - r^2}}{(1 + 3R)r}
$$

(9)
with \( r = \omega_1/\omega_2 \) and \( R = \omega_1^2/\omega_2^2 \), a diagonalized Hamiltonian can be obtained

\[
\hat{H}_3 = \frac{\hat{P}_1^2}{2m} + \frac{\hat{P}_2^2}{2m} + \frac{1}{2}m\sigma_1^2 \hat{Q}_1^2 + \frac{1}{2}m\sigma_2^2 \hat{Q}_2^2
\]

where the involved frequencies are given by

\[
\sigma_{1,2}^2 = \frac{1}{2}(\omega_1^2 + \omega_2^2 + 2\omega_r^2) \pm \frac{1}{2} \sqrt{r^2(3\omega_2^2 + \omega_1^2)^2 + (\omega_1^2 - \omega_2^2)^2}.
\]

Now, it is easy to solve the eigenvalue equation to derive the energies \([26]\)

\[
E_{n,m} = \sigma_1(n + 1/2) + \sigma_2(m + 1/2)
\]

and the corresponding wave functions read as

\[
\psi_{n,m}(Q_1, Q_2) = \left(\pi^\frac{\sigma_1 \sigma_2}{\omega} \right)^{\frac{1}{4}} \frac{1}{\sqrt{2^{n+m}n!m!}} \exp \left(\frac{-r^2}{2} e^{-\eta}Q_1^2 e^{-\eta}Q_2^2\right) \hat{P}_n \left(\sqrt{\omega e^{-\eta}}Q_1\right) \hat{P}_m \left(\sqrt{\omega e^{-\eta}}Q_2\right)
\]

where \( \omega = \sqrt{\sigma_1 \sigma_2} \) and \( e^{-\eta} = \sigma_1/\omega \). In terms of the old coordinates, \(13\) can be mapped as

\[
\psi_{n,m}(x_1, p_2) = \left(\pi^\frac{\sigma_1 \sigma_2}{\omega} \right)^{\frac{1}{4}} \frac{1}{\sqrt{2^{n+m}n!m!}} \exp \left(\frac{-r^2}{2} e^{-\eta}(S_{11} x_1 + S_{14} p_2)^2 + e^{-\eta}(S_{21} x_1 + S_{24} p_2)^2\right) \times \hat{H}_n \left(\sqrt{\omega e^{-\eta}}(S_{11} x_1 + S_{14} p_2)\right) \hat{H}_m \left(\sqrt{\omega e^{-\eta}}(S_{21} x_1 + S_{24} p_2)\right)
\]

and \( S_{ij} \) are matrix elements of \( S \), see the Appendix A. To write the wave functions in term of \( x_1 \) and \( x_2 \), we use the following Fourier transform

\[
\psi_{n,m}(x_1, x_2) = \int \frac{dp_2}{\sqrt{2\pi}} e^{ip_2 x_2} \psi(x_1, p_2).
\]

We point out here that the calculation of \(15\) is not an obvious task. Fortunately, its analytical expression is not needed to achieve our object as we will see in the forthcoming analysis.

### 3 Schmidt decomposition

To study the entanglement of our system, let us proceed by applying the Schmidt decomposition techniques. Indeed, we decompose the stationary wave function as

\[
\psi_{n,m}(x_1, p_2) = \sum_{l,k=0}^{\infty} A_{n,m}^{k,l} \hat{\varphi}_k(x_1) \hat{\phi}_l(p_2)
\]

such that \( \varphi_k(x_1) \) and \( \phi_l(p_2) \) are the vectors of the Schmidt basis

\[
\hat{\phi}_l(x_1) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\omega_1}{\pi}} e^{-\frac{\omega_1}{2} x_1^2} \hat{H}_l(\sqrt{\omega_1} x_1)
\]

\[
\hat{\varphi}_k(p_2) = \frac{1}{\sqrt{2\pi^3}} \sqrt{\frac{1}{\omega_2}} e^{-\frac{1}{2\omega_2} p_2^2} \hat{H}_k \left(\frac{p_2}{\sqrt{\omega_2}}\right)
\]

which correspond to the unbound oscillators and satisfy \( \langle \hat{\phi}_k | \hat{\phi}_l \rangle = \langle \hat{\varphi}_k | \hat{\varphi}_l \rangle = \delta_{kl} \). Consequently, the coefficients \( A_{n,m}^{k,l} \) will be obtained by using the orthogonality properties

\[
A_{n,m}^{k,l} = \int dx_1 dp_2 \psi_{n,m}(x_1, p_2) \varphi_k(x_1) \phi_l(p_2).
\]
This can be computed by employing the Rodrigues formula

\[ H_n(\omega x) = \frac{d^n}{du^n} e^{-u^2 + 2\omega xu} \big|_{u=0} \]  

(20)

to get the result

\[ A_{n,m}^{k,l} = \frac{1}{\pi} \sqrt{\frac{\omega}{\omega_1}} \frac{\sqrt{\omega_2}}{2^{n+m+k+l} n! m! k! l!} \int \int \frac{d^k}{dx^k} \frac{d^l}{dx^l} \frac{d^m}{du^m} \frac{d^n}{du^n} \left[ e^{-\omega(x_1^2 + \frac{1}{\omega} p_2^2)} - e^{-u^2 + 2\sqrt{\omega} e^{-\eta} (S_{11} x_1 + S_{14} p_2) u} - e^{-u^2 + 2\sqrt{\omega} e^{-\eta} (S_{21} x_1 + S_{24} p_2) v} - e^{-u^2 + 2(\sqrt{\omega} x_1 w) - e^{-s^2 + \frac{2}{\omega} p_2 s}} \right]_{u,v,w,s=0} dx_1 dp_2. \]  

(21)

Performing the integration over \( x_1 \) and \( p_2 \), we end up with

\[ A_{n,m}^{k,l} = \frac{1}{\sqrt{2^{n+m+k+l} n! m! k! l!}} \frac{d^k}{dx^k} \frac{d^l}{dx^l} \frac{d^m}{du^m} \frac{d^n}{du^n} \left[ 2u (S_{11} w + \pi S_{14} s) + 2v (S_{21} w + \pi S_{24} s) \right]_{u,v,w,s=0}. \]  

(23)

Using the fact that \( \frac{d^n}{du^n} u^k |_{u=0} = k! \delta_{k,n} \) and the following identity [27]

\[ \frac{d^n}{du^n} [u^\alpha (a-u)^\beta] = n! a^n u^{\alpha-n} (a-u)^{\beta-n} P_n^{(\alpha-n,\beta-n)} (1 - \frac{2u}{a}) \]  

(24)

we obtain

\[ A_{n,m}^{k,l} = (-1)^m \sqrt{\frac{2^{n+m}}{2^{n+m} n! m! k! l!}} S_{11}^{m} S_{21}^{l} \left[ (w + \frac{S_{11}}{S_{11}} s)^n \left( \frac{2}{\cos \theta} s - w - \frac{S_{11}}{S_{11}} s \right)^m \right]_{w,s=0}. \]  

(25)

Let us introduce the change \( W = w + \frac{S_{11}}{S_{11}} s \) and then we get

\[ A_{n,m}^{k,l} = (-1)^m \sqrt{\frac{2^{n+m}}{2^{n+m} n! m! k! l!}} P_l^{(n-l,m-l)} (\sin \theta) (1 - \sin \theta)^{\frac{n-l}{2}} (1 + \sin \theta)^{\frac{m-l}{2}} \delta_{n+m-l,k}. \]  

(26)

where \( P_n^{(\alpha,\beta)} (x) \) stands for Jacobi polynomials. Using the conservation formula of indexes \( n+m = k+l \) one can rewrite \( A_{n,m}^{k,l} \) as

\[ A_{n,m}^{k,l} = (-1)^m \sqrt{\frac{l! (n+m-l)!}{2^{n+m-2l} n! m!}} P_l^{(n-l,m-l)} (\sin \theta) (1 - \sin \theta)^{\frac{n-l}{2}} (1 + \sin \theta)^{\frac{m-l}{2}}. \]  

(27)
Now, it is time to discuss the physical meaning of the coefficients $A_{n,m}^{k,l}$. Indeed, if the system is initially in the state $|k,l\rangle$ it will be detected in the state $|p_1,p_2\rangle$ with the probability

$$a_{p_1,p_2}(t) = \sum_{n=0}^{k+l} A_{k,l}^{n,k+l-n} A_{p_1,p_2}^{*n,k+l-n} e^{-i\Delta E_{n,k+l-n}}$$

(28)

such that the energy variation reads as

$$\Delta E_{n,k+l-n} = E_{n,k+l-n} - \omega_1 (k + \frac{1}{2}) - \omega_2 (l + \frac{1}{2})$$

(29)

and $E_{n,k+l-n}$ is given in (12). In the next, we will see how the above results will be employed to discuss the entanglement of our system.

4 Quantum entanglement

4.1 Entanglement quantifiers

Two quantifiers can be used to study the entanglement and to do one needs to involves the reduced matrices. Then according to our Schmidt decomposition (16), we straightforwardly derive the required densities as

$$\rho_1(x_1,x'_1,t) = \sum_{l=0}^{m+n} |A_{n,m}^{l,n+m-l}|^2 \varphi_l(x_1)\varphi_l^*(x'_1)$$

(30)

$$\rho_2(p_2,p'_2,t) = \sum_{l=0}^{m+n} |A_{n,m}^{l,n+m-l}|^2 \phi_{m+n-l}(p_2)\phi_{m+n-l}^*(p'_2)$$

(31)

where $\varphi_l$ and $\phi_{m+n-l}$ are vectors of the Schmidt basis, with $n+m$ is the Schmidt number. From the above considerations, one can derive the Schmidt modes as

$$\lambda_k = |A_{n,m}^{l,n+m-l}|^2.$$  

(32)

Consequently, quantifying entanglement becomes a simple task. Indeed, this can be done using the von Neumann entropy $S_v$ or the Schmidt parameter $K$ defined as [16,25]

$$S_v = -\sum_{k=0}^{m+n} \lambda_k \ln(\lambda_k), \quad K = \left(\sum_{k=0}^{m+n} \lambda_k^2\right)^{-1}.$$  

(33)

These results will be used to show the asymmetry, anisotropy and magnetic coupling effects on the entanglement.

4.2 Asymmetry and anisotropy effect

It should be noted here that the study of entanglement can be performed only by using the angle $\theta$ given in (9). In Figure 1, we plot both quantifiers ($S_v$ and $K$) versus $\sin \theta$ for various quantum numbers $(n,m)$. We observe that the entanglement strongly depends on the quantum pairs $(n,m)$ because its hierarchy is clear since it increases by increasing $(n,m)$. Subsequently as expected, the entanglement disappears for $\theta \to \pm \pi/2$, which corresponds to uncoupled of the two modes, i.e. $\omega_c = 0$, 


with ± reflects the sign of frequency \((\omega_1 - \omega_2)\). Additionally for different quantum numbers \(n \neq m\), the entanglement reaches maximal values for \(\theta = 0\), which tells us that our system is isotropic in this case, i.e. \(\omega_1 = \omega_2\).

\[
\text{Figure 1} \quad \text{(color online)} \quad \text{The quantum entanglement via von Neumann entropy } S_v \text{ and Schmidt parameter } K \text{ versus the mixing angle } \sin \theta. \text{ The pairs stand for the quantum numbers } (n, m).
\]

To show the entanglement when both oscillators have the same state \(n = m\), we plot in (left panel) of Figure 2 the entanglement versus \(\sin \theta\) for various pairs with \((n, n)\). The optimal values of the entanglement is not obtained for the isotropic case, but it depends strongly on the quantum number \(n\) and approaches \(\theta_{op} \sim 0\) as \(n\) increases. As example, for \(n = 2\), the optimal value of entanglement is obtained for \(\sin \theta_{op} \sim 0.4\), while for \(n = 3\) we have \(\sin \theta_{op} \sim 0.3\). In the (right panel) of Figure 2, we show the effect of the state asymmetry defined as \(|n - m|\). Assuming for instance that \(n + m = 10\) and varying \(n\) and \(m\), we observe that the entanglement has not a monotonically behavior. However, it becomes more important for the symmetry case \(n = m\) in the full range of \(\theta\) except at vicinity of the isotropic regime. Now, by increasing the asymmetry \(|n - m|\), the entanglement in the vicinity of isotropic regime for non-symmetric state becomes more important than that of symmetric ones.

\[
\text{Figure 2} \quad \text{(color online)} \quad \text{The quantum entanglement versus } \sin \theta \text{ with the same states in left panel and } n + m = 10 \text{ in right panel.}
\]
4.3 Magnetic coupling effect

In Figure 3, we present the effect of the magnetic coupling \( r = \frac{\omega_c}{\omega_2} \) on the entanglement. In panels (a,b) we choose \( R = \frac{\omega_1^2}{\omega_2^2} = 0.97 \), as observed after its revival, the entanglement is frozen after a fast oscillatory behavior. This indicates that the entanglement of large quantum numbers is more sensitive to the coupling \( r \). For the maximal asymmetric states \( |n - m| = \max(m, n) \), the entanglement revives exponentially and will be immediately frozen, which indicates that for these states the entanglement is indifferent to \( r \). Additionally, to show the magnetic coupling effect in the vicinity of resonance, i.e. \( R \to 1 \), we plot in the panels (c,d) the case for \( R = 0.999 \). Consequently, we notice that the sensitivity to \( r \) decreases and after its revival, the entanglement will be rapidly frozen.

5 Dynamics of entanglement

Now, we consider the dynamical Schrödinger equation associated with the Hamiltonian (1). Initially, we assume both oscillators are separable (\( \omega_c = 0 \)) and having the quantum states \( |m_1, m_2 \rangle \), i.e. the wave functions of the uncoupled oscillators. In order to quantify the entanglement encoded in our state, we reapply the Schmidt decomposition of the non-stationary Schrödinger equation. The expansion of the wave function is

\[
\psi(x_1, x_2, t) = \sum_{k,m} d_{k,p}(t) \phi_k(x_1, t) \varphi_p(p_2, t)
\]
where $\phi_k(x_1,t)$ and $\varphi_p(p_2,t)$ are the wave functions related to the unbound oscillators. The time-dependent expansion coefficients $d_{k,p}(t)$ can be computed by using the integral form (19) to get

$$d_{k,p}(t) = \sum_{m=0}^{m_1+m_2} A_{m_1,m_2}^{m_1+m_2-m} A_{k,p}^{m_1+m_2-m} e^{-i\Delta E_{m_1+m_2-m} t}$$

with energy difference

$$\Delta E_{m_1+m_2-m} = E_{m_1+m_2-m} - \omega_1 (m_1 + \frac{1}{2}) - \omega_2 (m_2 + \frac{1}{2}).$$

(36)

By virtue of (26), we find new conservation formula $k + p = m_1 + m_2$. By making use of the reduced density matrices, we straightforward end up with the time-dependent Schmidt modes

$$\lambda_k(t) = |d_{k,m_1+m_2-k}(t)|^2$$

(37)

Under the assumption of the weak coupling, i.e. $r \ll 1$, and $\omega_1 \sim \omega_2$, one can show that the energy difference depends only on the quantum number $m$

$$\Delta E_{m_1+m_2-m} \sim \epsilon m t$$

(38)

with $\epsilon = \sigma_1 - \sigma_2$. In what follows, we will use the dimensionless time $\tilde{t} = \epsilon t$ to study the dynamics of entanglement.

![Figure 4](image-url)

**Figure 4** — (color online) The dynamics of the quantum entanglement for different values of $\sin \theta = (0.01, 0.1, 0.25, 0.5, 0.75)$ and quantum states $(n,m)$.

In Figure 4, we plot the dynamics of entanglement for the quantum states $(0,3), (3,3), (5,5), (10,5)$ and for several choices of $\sin \theta$. As expected, the entanglement exhibits an $2\pi$–periodic behavior and
becomes $\pi-$periodic as $\theta \to 0$. We mention also, that the fast generation of the maximal entanglement requires the resonance $R \to 1$ and a time $t \sim \epsilon^{-1}$. In contrast, the death of entanglement will be rapid for the resonance case, which is consistent with the results of position-velocity coupling, i.e. $x_1p_2$ obtained by Makarov [5]. As a result, rapidly generating entanglement and maintaining it during the period requires small but not too small values of $\theta$ (e.g. $\sin \theta \sim 0.1$). An interesting remark, is that the maximum of the entanglement entropy $S_{v, \text{max}}$ depends only on the quantum states $n + m$, and this value is reached during the dynamics. We observe that the physical parameters accelerate or decelerate the generation of entanglement.

6 Conclusion

We have dealt with a specific quadratic Hamiltonian based on two magnetically coupled harmonic oscillators. First, we have diagonalized the Hamiltonian by using three canonical transformations and obtained the exact stationary wave function as well as the associated energies. Secondly, we have derived the analytical expression of the Schmidt modes by using Schmidt decomposition. The non-Gaussian entanglement has been studied by using two quantifiers such that von Neumann entropy and Schmidt parameter $K$.

We have analyzed the obtained results by investigating the effect of state asymmetry, anisotropy and the dynamics. It is found that the generation of entanglement is possible, by choosing the physical parameters and the quantum states $(n, m)$. Thereby, accelerate and decelerate the generation of the maximal entanglement for a given quantum state is possible by adjusting the physical parameters. In addition, we have shown that the asymmetry of the quantum state affects the entanglement. The magnetic coupling and the anisotropy effects are discussed, and it is found that the most excited states are more sensible to the magnetic coupling and the sensitivity increases as we go away from resonance.

The obtained results show the possibility to obtain a very large quantum entanglement probability, which can be used for example to study the magnetically coupled wave guides beam splitters.

A Appendix: Transformation matrix $S$

To express the new coordinates $\hat{Q}_j, \hat{P}_j$ in terms of the old ones $\hat{x}_j, \hat{p}_j$, we use three canonical transformations in their matrix forms

\[
S_1 = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\
0 & \frac{\omega}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & -\frac{\omega}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0
\end{pmatrix}, \quad S_2 = \begin{pmatrix}
\frac{\sqrt{m_-}}{m_+} & 0 & 0 & 0 \\
0 & \frac{\sqrt{m_+}}{m_-} & 0 & 0 \\
0 & 0 & \sqrt{\frac{m_+}{m_-}} & 0 \\
0 & 0 & 0 & \sqrt{\frac{m_-}{m_+}}
\end{pmatrix}
\]

\[
S_3 = \begin{pmatrix}
\cos \frac{\theta}{2} & \sin \frac{\theta}{2} & 0 & 0 \\
-\sin \frac{\theta}{2} & \cos \frac{\theta}{2} & 0 & 0 \\
0 & 0 & \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\
0 & 0 & -\sin \frac{\theta}{2} & \cos \frac{\theta}{2}
\end{pmatrix}
\]
to obtain the mapping

\[
\begin{pmatrix}
\hat{Q}_1 \\
\hat{Q}_2 \\
\hat{P}_1 \\
\hat{P}_2
\end{pmatrix} = S \begin{pmatrix}
\hat{x}_1 \\
\hat{x}_2 \\
\hat{p}_1 \\
\hat{p}_2
\end{pmatrix}
\] (A.3)

where we have set \( S = S_3 S_2 S_1 \). More explicitly, one can write

\[
\hat{Q}_1 = \frac{\sqrt{2}}{2} \left( \sqrt{\frac{m}{m_+}} \cos \frac{\theta}{2} + \sqrt{\frac{m}{m_-}} \sin \frac{\theta}{2} \right) \hat{x}_1 + \frac{\sqrt{2}}{2\omega} \left( \sqrt{\frac{m}{m_+}} \sin \frac{\theta}{2} - \sqrt{\frac{m}{m_-}} \cos \frac{\theta}{2} \right) \hat{p}_2
\] (A.4)

\[
\hat{Q}_2 = \frac{\sqrt{2}}{2} \left( -\sqrt{\frac{m_+}{m}} \sin \frac{\theta}{2} + \sqrt{\frac{m_-}{m}} \cos \frac{\theta}{2} \right) \hat{x}_1 - \frac{\sqrt{2}}{2\omega} \left( \sqrt{\frac{m_+}{m}} \sin \frac{\theta}{2} + \sqrt{\frac{m_-}{m}} \cos \frac{\theta}{2} \right) \hat{p}_2
\] (A.5)

\[
\hat{P}_1 = \omega \sqrt{\frac{2}{m}} \left( -\sqrt{\frac{m_+}{m}} \sin \frac{\theta}{2} + \sqrt{\frac{m_-}{m}} \cos \frac{\theta}{2} \right) \hat{x}_2 + \frac{\sqrt{2}}{2\omega} \left( \sqrt{\frac{m_+}{m}} \sin \frac{\theta}{2} + \sqrt{\frac{m_-}{m}} \cos \frac{\theta}{2} \right) \hat{p}_1
\] (A.6)

\[
\hat{P}_2 = -\omega \sqrt{\frac{2}{m}} \left( \sqrt{\frac{m_+}{m}} \cos \frac{\theta}{2} + \sqrt{\frac{m_-}{m}} \sin \frac{\theta}{2} \right) \hat{x}_2 + \frac{\sqrt{2}}{2\omega} \left( \sqrt{\frac{m_+}{m}} \cos \frac{\theta}{2} - \sqrt{\frac{m_-}{m}} \sin \frac{\theta}{2} \right) \hat{p}_1
\] (A.7)

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