Partial quasi likelihood analysis

Nakahiro Yoshida

Graduate School of Mathematical Sciences, University of Tokyo †
CREST, Japan Science and Technology Agency

December 29, 2017

Summary The quasi likelihood analysis is generalized to the partial quasi likelihood analysis. Limit theorems for the quasi likelihood estimators, especially the quasi Bayesian estimator, are derived in the situation where existence of a slow mixing component prohibits the Rosenthal type inequality from applying to the derivation of the polynomial type large deviation inequality for the statistical random field. We give two illustrative examples.

Keywords and phrases Partial quasi likelihood analysis, large deviation, quasi maximum likelihood estimator, quasi Bayesian estimator, mixing, partial mixing.

1 Introduction

The Ibragimov-Has’minskii theory enhanced the asymptotic decision theory by Le Cam and Hájek by convergence of the likelihood ratio random field, and was programed by Kutoyants to statistical inference for semimartingales. The core of the theory is the large deviation inequality for the associated likelihood ratio random field. Asymptotic properties of the likelihood estimators are deduced from those of the likelihood ratio random field. Precise estimates of the tail probability and hence convergence of moments of the estimators follow in a unified manner once such a strong mode of convergence of the likelihood ratio random field is established. For details, see Ibragimov and Has’minskii [3, 4, 5] and Kutoyants [9, 8, 10, 11].

The quasi likelihood analysis (QLA) descended from the Ibragimov-Has’minskii-Kutoyants
In Yoshida [30], it was showed that a polynomial type large deviation (PLD) inequality universally follows from certain separation of the random field, such as the local asymptotic quadraticity of the random field, and $L_p$ estimates of easily tractable random variables. Since the PLD inequality is no longer a bottleneck of the program, the QLA applies to various complex random fields.

The QLA is a framework of statistical inference for stochastic processes. It features the polynomial type large deviation of the quasi likelihood random field. Through QLA, one can systematically derive limit theorems and precise tail probability estimates of the associated QLA estimators such as quasi maximum likelihood estimator (QMLE), quasi Bayesian estimator (QBE) and various adaptive estimators. The importance of such precise estimates of tail probability is well recognized in asymptotic decision theory, prediction, theory of information criteria for model selection, asymptotic expansion, etc. The QLA is rapidly expanding the range of its applications: for example, sampled ergodic diffusion processes (Yoshida [30]), contrast-based information criterion for diffusion processes (Uchida [24]), approximate self-weighted LAD estimation of discretely observed ergodic Ornstein-Uhlenbeck processes (Masuda [12]), jump diffusion processes Ogihara and Yoshida ([17]), adaptive estimation for diffusion processes (Uchida and Yoshida [24]), adaptive Bayes type estimators for ergodic diffusion processes (Uchida and Yoshida [27]), asymptotic properties of the QLA estimators for volatility in regular sampling of finite time horizon (Uchida and Yoshida [25]) and in non-synchronous sampling (Ogihara and Yoshida [18]), Gaussian quasi-likelihood random fields for ergodic Lévy driven SDE (Masuda [15]), hybrid multi-step estimators (Kamatani and Uchida [6]), parametric estimation of Lévy processes (Masuda [13]), ergodic point processes for limit order book (Clinet and Yoshida [1]), a non-ergodic point process regression model (Ogihara and Yoshida [19]), threshold estimation for stochastic processes with small noise (Shimizu [21]), AIC for non-concave penalized likelihood method (Umezu et al. [28]), Schwarz type model comparison for LAQ models (Eguchi and Masuda [2]), adaptive Bayes estimators and hybrid estimators for small diffusion processes based on sampled data (Nomura and Uchida [16]), moment convergence of regularized least-squares estimator for linear regression model (Shimizu [22]), moment convergence in regularized estimation under multiple and mixed-rates asymptotics (Masuda and Shimizu [14]), asymptotic expansion in quasi likelihood analysis for volatility (Yoshida [31]) among others.

As already mentioned, the PLD inequality is the key to the QLA. Once a PLD inequality is established, we can obtain a very strong mode of convergence of the random field and the associated estimators. However, in the present theory, boundedness of high order of moments of functionals is assumed. On the other hand, for example, if the statistical model has a component with a slow mixing rate, the Rosenthal inequality does not serve to validate the boundedness of moments of very high order. How do QMLE and QBE behave in such a situation? This question motivates us to introduce the partial quasi likelihood analysis (PQLA).

The aim of this short note is to formulate the PQLA and to exemplify it. The basic idea is conditioning by partial information. Easy to understand is a situation where there are two components $(L, U)$ of stochastic processes and $U$ has a fast mixing rate but $L$ has a slow mixing rate. The QLA is not in the sense of Robert Wedderburn. Since exact likelihood function can rarely be assumed in inference for discretely sampled continuous time processes, quasi likelihood functions are quite often used there. Further, the word “QLA” also implies a new framework of inferential theory for stochastic processes within which the polynomial type large deviation is easily available today and plays an essential role in the theory.
rate. Suppose that the Rosenthal inequality may control the moments of a functional of $U$ but cannot control the moments of a functional of $L$. In this situation, we cannot apply the present QLA theory or the way of derivation of the PLD inequality to the random fields expressed by $U$ and $L$. However, if there is a partial mixing structure in that $U$ possesses a very good mixing rate conditionally on $L$, then we can apply a conditional version of the QLA theory for given $L$. Even if $L$ has a bad mixing rate and its temporal impact on the system is unbounded, there is a possibility that we can recover limit theorems for the QLA estimators. Technically, a method of truncation is essential to detach the slow mixing component’s effects from the main body of the randomness.

Partial QLA naturally emerges in the structure of the partial mixing. The notion of partial mixing was used in Yoshida [29] to derive asymptotic expansion of the distribution of an additive functional of the conditional $\epsilon$-Markov process admitting a component with long-range dependency.

The organization of this note is as follows. Section 2 presents a frame of the partial quasi likelihood analysis. The asymptotic properties of the QMLE and QBE are provided there. The conditional polynomial type large deviation inequality is the key to the partial QLA. Section 3 gives a set of sufficient conditions for it. A conditional version of a Rosenthal type inequality is stated in Section 4. Section 5 illustrates a diffusion process having slow and fast mixing components. Statistics is ergodic in Section 5, while a non-ergodic statistical problem will be discussed in Section 6.

2 Partial quasi likelihood analysis

2.1 Quasi likelihood analysis

Given a probability space $(\Omega, \mathcal{F}, P)$, we consider a sequence of random fields $H_T : \Omega \times \Theta \rightarrow \mathbb{R}$, $T \in \mathbb{T}$, where $\mathbb{T}$ is a subset of $\mathbb{R}_+$ with $\sup \mathbb{T} = \infty$, $\Theta$ is a bounded domain in $\mathbb{R}^p$ and $\overline{\Theta}$ is its closure. We assume that $H_T$ is $\mathcal{F} \otimes \mathcal{B}[\mathbb{R}^p]$-measurable and that the mapping $\Theta \ni \theta \mapsto H_T(\omega, \theta)$ is continuous for every $\omega \in \Omega$. By convention, $H_T(\omega, \theta)$ is simply denoted by $H_T(\theta)$.

The random field $H_T$ serves like the log likelihood function in the likelihood analysis, but does more. A measurable mapping $\hat{\theta}_T : \Omega \rightarrow \overline{\Theta}$ is called a quasi maximum likelihood estimator (QMLE) if

$$H_T(\hat{\theta}_T) = \max_{\theta \in \overline{\Theta}} H_T(\theta)$$

for all $\omega \in \Omega$. The mapping $\tilde{\theta}_T : \Omega \rightarrow \mathcal{C}[\Theta]$, the convex hull of $\Theta$, is defined by

$$\tilde{\theta}_T = \left[ \int_{\Theta} \exp \left( H_T(\theta) \varpi(\theta) d\theta \right) \right]^{-1} \int_{\Theta} \theta \exp \left( H_T(\theta) \varpi(\theta) d\theta \right)$$

and called the quasi Bayesian estimator (QBE) with respect to the prior density $\varpi$. We assume $\varpi$ is continuous and satisfies $0 < \inf_{\theta \in \Theta} \varpi(\theta) \leq \sup_{\theta \in \Theta} \varpi(\theta) < \infty$. We call these estimators together quasi likelihood estimators.

The quasi likelihood analysis (QLA) is formulated with the random field

$$Z_T(u) = \exp \left( H_T(\theta^* + a_T u) - H_T(\theta^*) \right) \quad (u \in \mathbb{R})$$
Here \( \theta^* \in \Theta \) is the target value of \( \theta \) in estimation and \( \cup_T = \{ u \in \mathbb{R}^p; \theta^*_T(u) \in \Theta \} \), where \( \theta^*_T(u) = \theta^* + a_T u \). The matrix \( a_T \in \text{GL}(\mathbb{R}^p) \) satisfies \( a_T \to 0 \) as \( T \to \infty \). It is possible to extend \( Z_T \) to \( \mathbb{R}^p \) so that the extension has a compact support and \( \sup_{u \in \mathbb{R}^p \cup_T} Z_T(u) \leq \max_{u \in \partial u_T} Z_T(u) \).

We denote this extended random field by the same \( Z_T \). Let \( \hat{\mathcal{C}} = \{ f \in \mathcal{C}(\mathbb{R}^p); \lim_{|u| \to \infty} f(u) = 0 \} \). Then \( Z_T \in \hat{\mathcal{C}} \).

Consider \( \sigma \)-fields \( \mathcal{C} \) and \( \mathcal{G} \) such that \( \mathcal{C} \subset \mathcal{G} \subset \mathcal{F} \). We introduce \( \mathcal{C} \)-measurable variables \( \Psi_T : \Omega \to \{0, 1\} \). These functionals are helpful to localize QLA.

### 2.2 Quasi maximum likelihood estimator

Let \( L \) be a positive constant. We start with the so-called polynomial type large deviation inequality, which plays an essential role in the theory of QLA as in [30]. Let \( \forall_T(r) = \{ u \in \cup_T; |r| \geq r \} \). Let \( B_{c,T} = \{ u \in \mathbb{R}^p; |u| < c, \theta^*_T(u) \in \Theta \} \) for \( c > 0 \). The modulus of continuity of \( \log Z_T \) is

\[
w_T(\delta, c) = \sup \left\{ \left| \log Z_T(u_2) - \log Z_T(u_1) \right|; u_1, u_2 \in B_{c,T}, |u_2 - u_1| \leq \delta \right\}.
\]

Let \( W_T(\delta, c, \epsilon) = \{ w_T(\delta, c) > \epsilon \} \) and let

\[
S_T(r, \epsilon) = \left\{ \sup_{u \in \forall_T(r)} Z_T(u) \geq \epsilon \right\}.
\]

Let \( T_0 > 0 \). Let \( \mathcal{I} \) be the set of sequences \( (T_n)_{n \in \mathbb{N}} \) of numbers in \( \mathbb{T} \) such that \( T_n \geq T_0 \) for all \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} T_n = \infty \). Let \( (\Psi_T)_{T \in \mathcal{I}} \) be a sequence of \( [0, 1] \)-valued \( \mathcal{C} \)-measurable random variables.

- **[A1]** There exists a sequence of positive \( \mathcal{C} \)-measurable random variables \( (\epsilon(r))_{r \in \mathbb{N}} \) such that \( \lim_{r \to \infty} \epsilon(r) = 0 \) a.s. and that

  \[
  \sup_T \sup_{r \in \mathbb{N}} r^L P_C[S_T(r, \epsilon(r))] \Psi_T < \infty \quad a.s.
  \]

  for every \( T \in \mathcal{T} \).

- **[A1]** For a sequence of positive numbers \( (\epsilon(r))_{r \in \mathbb{N}} \) with \( \lim_{r \to \infty} \epsilon(r) = 0 \) and a sequence of positive random variables \( (\eta(r))_{r > 0} \) with \( \lim_{r \to \infty} \eta(r) = 0 \) a.s., it holds that

  \[
  P_C[S_T(r, \epsilon(r))] \Psi_T \leq \eta(r) \quad a.s. \quad (T \in \mathcal{T}, r \in \mathbb{N})
  \]

  for every \( T \in \mathcal{T} \).

- **[A2]** \( \limsup_{T \to \infty, T \in \mathcal{T}} P_C[W_T(\delta, c, \epsilon)] \Psi_T \to^P 0 \) as \( \delta \downarrow 0 \) for every \( \epsilon > 0, c > 0 \) and \( T \in \mathcal{T} \).

**Remark 2.1.** The estimate of modulus of continuity is used only countable times to prove tightness.
We consider and its extension \((\Omega, \mathcal{F}, \mathcal{P})\), that is, \(\Omega \subset \overline{\Omega}, \mathcal{F} \subset \overline{\mathcal{F}}\) and \(\mathcal{P} = \mathcal{P}|_{\mathcal{F}}\). Let \(Z(u)\) be a \(\hat{C}\)-valued random variable defined on \((\Omega, \overline{\mathcal{F}}, \overline{\mathcal{P}})\).

[A3] (i) For any \(k \in \mathbb{N}\), \(u_i \in \mathbb{R}^p\) \((i = 1, \ldots, k)\), \(f \in C_b(\mathbb{R}^k)\) and any bounded \(\mathcal{G}\)-measurable random variable \(Y\),

\[
E_{\hat{C}}[f((Z(u_i))_{i=1,\ldots,k})\Psi_T Y] \to^P \overline{E}_{\hat{C}}[f((Z(u_i))_{i=1,\ldots,k})Y]
\]

(ii) \(\Psi_T \to^P 1\).

[A4] With probability one, there exists a unique element \(\hat{u} \in \mathbb{R}^p\) that maximizes \(Z\).

Remark: From [A3] (i), we can remove \(\Psi_T\) but keeping it explicitly is helpful in applications. We may assume \(\hat{u}\) is \(\overline{\mathcal{F}}\)-measurable; the given mapping \(\hat{u}\) has a measurable version. The following theorems claim \(C\)-conditional \(\mathcal{G}\)-stable convergence of \(Z_T\) and \(\hat{u}_T = a^{-1}_T(\hat{\theta}_T - \theta^*)\).

**Theorem 2.1.** Suppose that \([A1\hat{\flat}], [A2]\) and \([A3]\) are satisfied. Then

\[
E_{\hat{C}}[F(Z_T)Y] \to^P \overline{E}_{\hat{C}}[F(Z)Y]
\]

(2.1)

for any \(F \in C_b(\hat{\mathcal{C}})\) and any bounded \(\mathcal{G}\)-measurable random variable \(Y\). In particular,

\(Z_T \to^{d(\mathcal{G})} Z\)

as \(T \to \infty\).

**Theorem 2.2.** Suppose that \([A1\hat{\flat}], [A2], [A3]\) and \([A4]\) are satisfied. Then

\[
E_{\hat{C}}[f(\hat{u}_T)Y] \to^P \overline{E}_{\hat{C}}[f(\hat{u})Y]
\]

(2.2)

for any \(f \in C_b(\mathbb{R}^p)\) and any bounded \(\mathcal{G}\)-measurable random variable \(Y\). In particular,

\(\hat{u}_T \to^{d(\mathcal{G})} \hat{u}\)

as \(T \to \infty\).

**Proof of Theorems 2.1 and 2.2** (a) We may assume that \(\|Y\|_\infty \leq 1\). Let

\[
\tilde{Z}_T(u) = \begin{cases} 
Z_T(u) & (\Psi_T = 1) \\
e^{-|u|^2} & (\Psi_T = 0)
\end{cases}
\]

In view of [A3] (ii), we may show

\[
E_{\hat{C}}[F(\tilde{Z}_T)Y] \to^P \overline{E}_{\hat{C}}[F(Z)Y]
\]

(2.3)

for \(F \in C_b(\hat{\mathcal{C}})\) in order to show (2.1). Then, by subsequence argument, it suffices to show that for any sequence \((T_n)\) with \(0 \leq T_1 < T_2 < \cdots \to \infty\), there exists a subsequence \((T_{n'})\) of \((T_n)\)

\(^2\)Continuity is not necessary to assume as a matter of fact. Without it, the convergence of \(Z_T\) ensures the limit distribution is supported by \(\hat{C}\); we may assume \(Z\) is a continuous process after modification.
such that (2.31) holds along \((T_n')\). For \(k \in \mathbb{N}\), let \(\mathfrak{G}_k\) be a countable subset of \(C_b(\mathbb{R}^{k+1})\) that determines probability measures on \(\mathbb{R}^{k+1}\).

Let \(P^T_n(dx, dy)\) be a regular conditional distribution of \((\mathcal{Z}_{T_n}, Y)\) on \(\hat{C} \times [-1, 1]\) given \(C\). Let \(P_\omega\) be a regular conditional distribution of \((Z, Y)\) given \(C\). Moreover let

\[
w(\delta, c, x) = \sup \{ |x(u_2) - x(u_1)|; u_1, u_2 \in B_{C,T}, |u_2 - u_1| \leq \delta\}
\]

for \(x \in \hat{C}\), and let

\[
\bar{w}(c, x) = \sup \{ |x(u)|; u \in \mathbb{R}^c, |u| \geq c\}.
\]

According to [A3] (ii) and [A2], there exists a subsequence \((T_n^{(i)})\) of \((T_n)\) such that \(\lim_{n \to \infty} \Psi_{T_n^{(i)}} = 1\) a.s. and that

\[
\lim_{m \in \mathbb{N}, m \to \infty} \limsup_{n \to \infty} P_C[W_{T_n^{(i)}}(m^{-1}, j, k^{-1})] \Psi_{T_n^{(i)}} = 0 \quad \text{a.s.}
\]

for all \(j, k \in \mathbb{N}\). Moreover, from [A1\textsuperscript{p}], for \(k \in \mathbb{N}\), there exists an \(r_0 > 0\) such that \(\epsilon(r) \leq k^{-1}\) for all \(r \geq r_0\). Then

\[
\lim\limsup_{r \to \infty} P_C \left[ S_{T_n^{(i)}}(r, k^{-1}) \right] \leq \lim\limsup_{r \to \infty} P_C \left[ S_{T_n^{(i)}}(r, \epsilon(r)) \right] \Psi_{T_n^{(i)}} \leq \lim_{r \to \infty} \eta(r) = 0 \quad \text{a.s.}
\]

Thus, thanks to [A1\textsuperscript{p}], [A2] and [A3], there exist an event \(\Omega_0 \in \mathcal{F}\) with \(P[\Omega_0] = 1\) and a subsequence \((T_n')\) of \((T_n^{(i)})\) such that for any \(\omega \in \Omega_0\), the following conditions hold:

(i) \(\lim\limsup_{n' \to \infty} P_{T_n'}^{T_n'} \left[ \left\{ (x, y); w(m^{-1}, j, \log x) > k^{-1}\right\} \right] = 0 \quad (\forall j, k \in \mathbb{N})\)

(ii) \(\lim\limsup_{n' \to \infty} P_{T_n'}^{T_n'} \left[ \left\{ (x, y); \bar{w}(j, x) > k^{-1}\right\} \right] = 0 \quad (\forall k \in \mathbb{N})\)

(iii) For every \(k \in \mathbb{N}\),

\[
\int_{\hat{C} \times [-1, 1]} g \left( (x(u_i))_{i=1,..,k}, y \right) P_{T_n'}^{T_n'}(dx, dy) \to \int_{\hat{C} \times [-1, 1]} g \left( (x(u_i))_{i=1,..,k}, y \right) P_\omega(dx, dy)
\]

as \(n' \to \infty\) for all \(g \in \mathfrak{G}_k\).

(iv) \(\Psi_{T_n'} \to 1\) as \(n' \to \infty\).

(v) \(\int_{\hat{C} \times [-1, 1]} y P_{T_n'}^{T_n'}(dx, dy) = \int_{\hat{C} \times [-1, 1]} y P_\omega(dx, dy)\) for all \(n'\).

For \(\omega \in \Omega_0\), \(\epsilon > 0\) and \(j, k \in \mathbb{N}\), there exist \(m(k), j(k) \in \mathbb{N}\) such that

\[
\sup_{n'} P_{T_n'}^{T_n'}[A_{\omega, j, k}] < 2^{-j-k-1}\epsilon
\]
and

$$\sup_{n'} P^{T_{n'}}_\omega[B_{\omega,k}], \quad < 2^{-k-1}\epsilon$$

where

$$A_{\omega,j,k} = \left\{ (x,y) ; w(m(k)^{-1}, j, \log z) > k^{-1} \right\}$$

and

$$B_{\omega,k} = \left\{ (x,y) ; \tilde{w}(j(k), x) > k^{-1} \right\},$$

respectively. Let

$$A_\omega = \cap_{j,k \in \mathbb{N}} A_{\omega,j,k} \cap \cap_{k \in \mathbb{N}} B_{\omega,k}.$$ Then

$$A_\omega \text{ is a compact set in } \hat{C} \times [-1,1] \text{ and }$$

$$\sup_n P^{T_{n'}}_\omega[A_\omega] \geq 1 - \epsilon.$$ Therefore the family of probability measures \(\{P^{T_{n'}}_\omega\}_{n'}\) is tight since \(\bar{Z}_{T_{n'}}(0) = 1\). Let \(n^1\) be any subsequence of \((n')\). Then there exist a subsequence \((n'')\) of \((n^1)\), \((n'')\) depending on \(\omega\), and a probability measure \(P^*_\omega\) on \(\hat{C} \times [0,1]\) such that \(P^{T_{n''}}_\omega \to P^*_\omega\) as \(n'' \to \infty\). In particular,

$$\int_{\hat{C} \times [-1,1]} g((x(u_i))_{i=1,\ldots,k}, y) P^{T_{n''}}_\omega(dx, dy) \to \int_{\hat{C} \times [-1,1]} g((x(u_i))_{i=1,\ldots,k}, y) P^*_\omega(dx, dy)$$

as \(n'' \to \infty\) for every \(\omega \in \Omega_0\) and every \(g \in \mathfrak{G}_k, k \in \mathbb{N}\). Therefore

$$\int_{\hat{C} \times [-1,1]} g((x(u_i))_{i=1,\ldots,k}, y) P^*_\omega(dx, dy) = \int_{\hat{C} \times [-1,1]} g((x(u_i))_{i=1,\ldots,k}, y) P_\omega(dx, dy)$$

for all \(g \in \mathfrak{G}_k, k \in \mathbb{N}\). Since all finite-dimensional marginal distributions coincide, \(P^*_\omega = P_\omega\). This implies \(P^{T_{n''}}_\omega \to P^*_\omega\) as \(n'' \to \infty\), and hence

$$P^{T_{n''}}_\omega \to P_\omega$$ \hspace{1cm} (2.4)

for every \(\omega \in \Omega_0\). In particular, we obtain (2.3) along \((T_{n'})\), which gives Theorem 2.1.

(b) We may assume \(0 \leq Y \leq 1\). Consider sequences \((T_n)\) and \((T_{n'})\) in Step (a). Let \(\mathfrak{F} = \{\Delta_q(x); q \in \mathbb{Q}^p\}\) with

$$\Delta_q(x) = (-1) \vee \left( \sup_{u \in R_q} x(u) - \sup_{u \in R_q} x(u) \right) \land 1$$

where \(R_q = \{u = (u_i)_{i=1,\ldots,q}; u_i \leq q_i (i = 1,\ldots,p)\}\) for \(q = (q_i)_{i=1,\ldots,p} \in \mathbb{Q}^p\). If \(\omega \in \Omega_0\) satisfies \(J_\omega := \int_{\hat{C} \times [0,1]} yP_\omega(dx, dy) > 0\), then the already obtained (2.3) yields

$$\tilde{P}^{T_{n'}}_\omega \to \tilde{P}_\omega$$ \hspace{1cm} (2.5)

as \(n' \to \infty\), where

$$\tilde{P}^{T_{n'}}_\omega(B) = \int_{\hat{C} \times [0,1]} 1_B(x) y dP^{T_{n'}}_\omega / J_\omega \quad \text{and} \quad \tilde{P}_\omega(B) = \int_{\hat{C} \times [0,1]} 1_B(x) y dP_\omega / J_\omega$$

7
for \( B \in \mathcal{B}[\hat{C}] \). We notice that \( \tilde{P}_{\omega}^{T_{n'}} \) as well as \( \tilde{P}_\omega \) is a probability measure by (v) of Part (a). The convergence (2.5) gives

\[
\tilde{P}_\omega[\Delta_q(x) > 0] \leq \liminf_{n' \to \infty} \tilde{P}_{\omega}^{T_{n'}}[\Delta_q(x) > 0] \\
\leq \limsup_{n' \to \infty} \tilde{P}_{\omega}^{T_{n'}}[\Delta_q(x) \geq 0] \\
\leq \tilde{P}_\omega[\Delta_q(x) \geq 0],
\]

for all \( q \in Q^p \), and hence

\[
E_C[1_{\{\Delta_q(Z) > 0\}} Y] \leq \liminf_{n' \to \infty} E_C[1_{\{\Delta_q(\tilde{Z}_{T_{n'}}) > 0\}} Y] \\
\leq \limsup_{n' \to \infty} E_C[1_{\{\Delta_q(\tilde{Z}_{T_{n'}}) \geq 0\}} Y] \\
\leq E_C[1_{\{\Delta_q(Z) \geq 0\}} Y] \quad \text{a.s.}
\]

(2.6)

for all \( q \in Q^p \), since this is obvious when \( J_\omega = 0 \). By definition,

\[
\{ \hat{u}_{T} \leq q \} \cap \{ \Psi_T = 1 \} \subset \{ \Delta_q(\tilde{Z}_{T}) \geq 0 \}
\]

and

\[
\{ \hat{u}_{T} \leq q \} \cup \{ \Psi_T = 0 \} \supset \{ \Delta_q(\tilde{Z}_{T}) > 0 \}
\]

Therefore (2.6) implies

\[
\limsup_{n' \to \infty} E_C[1_{\{\hat{u}_{T_{n'}} \leq q\}} Y] \leq \limsup_{n' \to \infty} E_C[1_{\{\Delta_q(\tilde{Z}_{T_{n'}}) \geq 0\}} Y] \\
\leq E_C[1_{\{\Delta_q(Z) \geq 0\}} Y] \\
\leq E_C[1_{\{\hat{u} \leq q\}} Y] \quad \text{a.s.}
\]

(2.7)

for all \( q \in Q^p \), and

\[
\liminf_{n' \to \infty} E_C[1_{\{\hat{u}_{T_{n'}} \leq q\}} Y] \geq \liminf_{n' \to \infty} E_C[1_{\{\Delta_q(\tilde{Z}_{T_{n'}}) > 0\}} Y] \\
\geq E_C[1_{\{\Delta_q(Z) > 0\}} Y] \\
\geq E_C[1_{\{\hat{u} < q\}} Y] \quad \text{a.s.}
\]

(2.8)

for all \( q \in Q^p \), where \((a_i) < (b_i)\) means \( a_i < b_i \) for all \( i = 1, \ldots, p \), and we used uniqueness of \( \hat{u} \) in the last part of each.

Denote by \( Q_{\omega}^{T_{n'}} \) [resp. \( Q_{\omega} \)] a regular conditional probability of \((\hat{u}_{T_{n'}}, Y) \) [resp. \((\hat{u}, Y) \)] given \( C \). From (2.7) and (2.8), there exists \( \Omega_1 \in \mathcal{F} \) with \( P[\Omega_1] = 1 \) such that

\[
I_\omega := \int_{\mathbb{R}^p \times [0,1]} y Q_{\omega}^{T_{n'}}(du, dy) = \int_{\mathbb{R}^p \times [0,1]} y Q_{\omega}(du, dy)
\]
for all $n'$ and all $\omega \in \Omega_1$, and that
\[
\int_{\mathbb{R}^p \times [0,1]} 1_{\{u < q\}} y \omega (du, dy) \leq \liminf_{n' \to \infty} \int_{\mathbb{R}^p \times [0,1]} 1_{\{u \leq q\}} \omega Q^{T_{n'}} (du, dy)
\]
\[
\leq \limsup_{n' \to \infty} \int_{\mathbb{R}^p \times [0,1]} 1_{\{u \leq q\}} \omega Q^{T_{n'}} (du, dy)
\]
\[
\leq \int_{\mathbb{R}^p \times [0,1]} 1_{\{u \leq q\}} \omega (du, dy)
\]
for all $\omega \in \Omega_1$ and all $q \in \mathbb{Q}_p$. If $I_\omega > 0$, then
\[
\int_{\mathbb{R}^p} 1_{\{u < q\}} \omega (du) \leq \liminf_{n' \to \infty} \int_{\mathbb{R}^p} 1_{\{u \leq q\}} \omega \tilde{Q}^{T_{n'}} (du)
\]
\[
\leq \limsup_{n' \to \infty} \int_{\mathbb{R}^p} 1_{\{u \leq q\}} \omega \tilde{Q}^{T_{n'}} (du)
\]
\[
\leq \int_{\mathbb{R}^p} 1_{\{u \leq q\}} \omega (du)
\]
(2.9)
for all $q \in \mathbb{Q}_p$ and all $\omega \in \Omega_1$, where the probability measures $\omega \tilde{Q}^{T_{n'}}$ and $\omega \tilde{Q}$ on $\mathbb{R}^p$ are given by
\[
\tilde{Q}(B) = \int_{\mathbb{R}^p \times [0,1]} 1_B(u) y \omega (du, dy) / \int_{\mathbb{R}^p \times [0,1]} y \omega (du, dy) \quad (B \in \mathcal{B}[\mathbb{R}^p])
\]
for $Q = Q^{T_{n'}}$ and $Q$.$\omega$. For any continuity point $r \in \mathbb{R}^p$ of $\omega \tilde{Q}$, we take $q_1, q_2 \in \mathbb{R}^p$ with $q_1 < r \leq q_2$ so that both are sufficiently close to $r$, and apply (2.9) to conclude $Q^{T_{n'}} \omega \to \omega \tilde{Q}$ for such $r$. Thus
\[
\int_{\mathbb{R}^p \times [0,1]} f(u) y Q^{T_{n'}} (du, y) \to \int_{\mathbb{R}^p \times [0,1]} f(u) y \omega (du, y)
\]
(2.10)
for $f \in C_b(\mathbb{R}^p)$, $\omega \in \Omega_1$ with $I_\omega > 0$. In the case $I_\omega = 0$, it is obvious, so (2.10) holds for all $\omega \in \Omega_1$. This concludes the proof of Theorem 2.2.

Conditional type PLD provides convergence of the conditional moments of $\omega T$ under truncation.

**Theorem 2.3.** Suppose that $q > 0$ and $L > p \lor q \lor 1$. Suppose that $[A1]$, $[A2]$, $[A3]$ and $[A4]$ are satisfied. Then (2.7) of Theorem 2.1 holds. Moreover
\[
E_C[f(\hat{u}_T)Y] \to^P E_C[f(\hat{u})Y]
\]
(2.11)
for any bounded $\mathcal{G}$-measurable random variable $Y$ and any $f \in C(\mathbb{R}^p)$ such that $\limsup_{u \to \infty} |f(u)|u^{-q} < \infty$. In particular, $\omega T \to^{d,(\Theta)} \hat{u}$ as $T \to \infty$.

**Remark 2.2.** The conditional expectation $E_C[W]$ of a random variable $W$ is defined as the integral $\int_{\mathbb{R}^p} wT^W (dw)$ with respect to a regular conditional probability $\tilde{P}_\omega$ of $W$ given $C$. If $W \in L^1(\tilde{P})$, then it coincides with the ordinary conditional expectation $E[W|C]$ almost surely. However in general we do not assume $W \in L^1(\tilde{P})$ nor $E_C[W] \in L^1(\tilde{P})$ in this article. We should be careful when applying the formula $E_C[W\Psi_T] = E_C[W]\Psi_T$; it is possible only when $E_C[W]$ is well defined. The same remark applies to $E_C[W]$. On the other hand, each $\hat{u}_T$ is bounded because $\Theta$ is bounded, so $E_C[f(\hat{u}_T)Y]$ in (2.11) is well defined in any sense.
Proof of Theorem 2.3. Let $p \in (q \lor 1, L)$. We may assume $T \to \infty$ along $T \in \mathcal{T}$. Almost surely

\[
E_p[|\hat{u}_T|^p \Psi_T] = \int_0^\infty p^{p-1} E_p[1_{|\hat{u}_T|>t}] \Psi_T dt
\]

\[
\leq 1 + p \sum_{r=1}^\infty (r+1)^{p-1} E_p[1_{|\hat{u}_T|>r}] \Psi_T
\]

\[
\leq 1 + p \sum_{r=1}^\infty (r+1)^{p-1} P^c \left[ \sup_{u \in V_T(r)} Z_T(u) \geq 1 \right] \Psi_T
\]

\[
\leq 1 + p \sum_{r=1}^\infty (r+1)^{p-1} \left( 1_{\{r \geq 1\}} + \frac{V_L}{r^{\ell}} \right) =: V
\]

where $V_L$ is a random variable bounding the right-hand side of the inequality of [A1]. The variable $V < \infty$ a.s. because $L > p$. By the convergence (2.2) of Theorem 2.2, we have

\[
\overline{E}_p[\hat{|u}|^p \land A] = \lim_{n \to \infty} E_p[|\hat{u}|_n|^p \land A] \Psi_{T_n} \leq V \quad \text{a.s.}
\]

for $A \in \mathcal{B}$ and some sequence $(T_n)_{n \in \mathbb{N}} \uparrow \infty$, and then the conditional monotone convergence theorem gives

\[
\overline{E}_p[\hat{|u}|^p] \leq V \quad \text{a.s.}
\]

by letting $A \uparrow \infty$. For some constant $C > 0$, $|f(u)| \leq C(1 + |u|^q)$ for all $u \in \mathbb{R}^p$. Let $f_A(u) = (-A) \lor f(u) \land A$ for $u \in \mathbb{R}^p$ and $A > 0$. Then for $\epsilon > 0$,

\[
P \left[ E_p[f(\hat{u}_T)\Psi_T Y] - \overline{E}_p[f(\hat{u})\Psi_T Y] > \epsilon \right]
\]

\[
\leq P \left[ E_p[f_A(\hat{u}_T) Y] - \overline{E}_p[(f_A)(\hat{u}) Y] \right]
\]

\[
+ P \left[ \frac{E_p[|f - f_A|(\hat{u}_T)\Psi_T]}{3(||Y||_\infty + 1)} > \frac{\epsilon}{3(||Y||_\infty + 1)} \right]
\]

\[
+ P \left[ \frac{E_p[|f - f_A|(\hat{u})]}{3(||Y||_\infty + 1)} > \frac{\epsilon}{3(||Y||_\infty + 1)} \right].
\]

We have

\[
|f(u) - f_A(u)| \leq |f(u)|_1 \{ |f(u)| \geq A \} \leq C(1 + |u|^q) 1_{\{ C(1 + |u|^q) \geq A \}}
\]

\[
\leq C(1 + |u|^p) \delta(A)
\]

for $A > C$ and

\[
\delta(A) = \sup_{u:|u|^q \geq (A/C-1)^{1/q}} \frac{1 + |u|^q}{1 + |u|^p}.
\]

Then

\[
\lim_{T \to \infty} \sup_{T \to \infty} P \left[ E_p[f(\hat{u}_T)\Psi_T Y] - \overline{E}_p[f(\hat{u})\Psi_T Y] > \epsilon \right] \leq 2P \left[ C(1 + V) \delta(A) > \frac{\epsilon}{3(||Y||_\infty + 1)} \right].
\]

Since $\lim_{A \to \infty} \delta(A) = 0$ and $\Psi_T \to P 1$, we obtain the convergence (2.11). \qed
2.3 Quasi Bayesian estimator

There exists $C$-measurable random variables $U$ and $V$ such that

$$P_C \left[ S_T \left( r, \frac{U}{r} \right) \right] \Psi_T \leq \frac{V}{r^L} \quad \text{a.s.} \quad (T \in \mathcal{T}, \ r \in \mathbb{N})$$

for every $T \in \mathcal{T}$.

**Proposition 2.3.** Suppose that $q \geq 0$, $D > p + q$ and $L > 1$. Suppose that $[A1^\dagger], [A2]$ and $[A3]$ are fulfilled. Then

$$E_C \left[ f \left( \int_{U_T} h(u)Z_T(u)du \right) Y \right] \rightarrow P E_C \left[ f \left( \int_{\mathbb{R}^p} h(u)Z(u)du \right) Y \right]$$

(2.12)

as $T \to \infty$ for any $f \in C_b(\mathbb{R}^k)$, any bounded $\mathcal{G}$-measurable variable $Y$, and any $\mathbb{R}^k$-valued measurable mapping $h$ satisfying $|h(u)| \leq C(1 + |u|^q)$ for some constant $C$.

**Proof.** We may show the convergence along every sequence $T = (T_n)_{n \in \mathbb{N}}$ in $\mathcal{T}$. Choosing a sufficiently small positive constant $c_1$ depending on $(p, D - q - p)$, we obtain

$$P_C \left[ \int_{\{u \geq N\} \cap U_T} u^q Z_T(u)du > c_1 UN^{-(D-q-p)} \right] \Psi_T$$

$$\leq \sum_{r=N}^{\infty} P_C \left[ \sup_{u \in V_T(r)} Z_T(u) > Ur^{-(D-q-p+1)} \right] \Psi_T$$

$$\leq \sum_{r=N}^{\infty} \frac{V}{r^T} \leq (N - 1)^{-(L-1)} V \quad \text{a.s.}$$

(2.13)

for $T > 1$ and $N \in \mathbb{N}$.

For any $[0,1]$-valued $C$-measurable random variable $\Phi$ and sufficiently large $T$,

$$E \left[ \frac{1}{\{\int_{\{u \geq N\} \cap B[0,N]} u^q Z_T(u)du > c_1 UN^{-(D-q-p)}\} \Psi_T \Phi} \right] \leq (N - 1)^{-(L-1)} E[V \Phi].$$

(2.14)

Letting $T \to \infty$ with Theorem 2.1, next letting $R \uparrow \infty$, we obtain

$$P_C \left[ \int_{\{u \geq N\}} u^q Z(u)du > c_1 UN^{-(D-q-p)} \right] \leq (N - 1)^{-(L-1)} V \quad \text{a.s.}$$

(2.14)

We have

$$\int_{\{u \geq N\} \cap U_T} u^q Z_T(u)du \leq N^q |B(0, N)| \exp (Nw_T(1, N)).$$

(2.15)

In particular, this property is transferred to the limit as

$$\int_{\{u \leq N\}} u^q Z(u)du \leq N^q |B(0, N)| \exp (Nw_T(1, N)) \quad \text{a.s.}$$

(2.16)
Let $\epsilon > 0$. Then by (2.13), (2.15), (2.14) and (2.16), there exists a number $K_0$ such that
\[
P\left[\sup_n P_C \left[ \int_{U_{T_n}} |h(u)| Z_{T_n}(u) du > K_0 \right] \right] < \frac{\epsilon}{4(1 + \| f \|_{\infty})} \geq 1 - \epsilon
\]
and
\[
P\left[ P_C \left[ \int_{U_{\eta \rho}} |h(u)| Z(u) du > K_0 \right] \right] \leq \frac{\epsilon}{4(1 + \| f \|_{\infty})} > 1 - \epsilon
\]

Take $\eta \in (0, 1)$ such that $|f(x_2) - f(x_1)| \leq \epsilon/2$ for all $x_1, x_2 \in \mathbb{R}^k$ such that $|x_1|, |x_2| \leq K_0 + 1$ and $|x_2 - x_1| \leq \eta$. From (2.13) and from (2.14), there exists $N_0 \in \mathbb{N}$ such that
\[
P\left[ \sup_n P_C \left[ \int_{\{ |u| \geq N_0 \} \cap U_{T_n}} |h(u)| Z_{T_n}(u) du > \eta \right] \right] \leq \frac{\epsilon}{4(1 + \| f \|_{\infty})} > 1 - \epsilon
\]
and
\[
P\left[ P_C \left[ \int_{\{ |u| \geq N_0 \} \cap \mathbb{R}^p} |h(u)| Z(u) du > \eta \right] \right] \leq \frac{\epsilon}{4(1 + \| f \|_{\infty})} > 1 - \epsilon
\]

Write
\[
J_T(S) = \int_{S \cap U_T} h(u) Z(u) du
\]
and
\[
J(S) = \int_S h(u) Z(u) du
\]
for $S \subset \mathbb{R}^p$. Let $B_0 = B(0, N_0)$. We may assume $\| Y \|_{\infty} \leq 1$. We will consider $n$ such that $B_0 \subset U_{T_n}$. Then
\[
P\left[ E_C \left[ f(J_{T_n}(U_{T_n})) Y \right] - E_C \left[ f(J_{T_n}(B_0)) Y \right] > \frac{\epsilon}{2} \right]
\leq P\left[ 2\| f \|_{\infty} P_C \left[ |J_{T_n}(U_{T_n} - B_0)| > \eta \right] \right] > \frac{\epsilon}{2}
\]
\[
+ P\left[ 2\| f \|_{\infty} P_C \left[ \int_{U_{T_n}} |h(u)| Z_{T_n}(u) du > K_0 \right] \right] > \frac{\epsilon}{2}
\]
\[
< 2\epsilon
\]

Similarly,
\[
P\left[ E_C \left[ f(J(\mathbb{R}^p)) Y \right] - E_C \left[ f(J(B_0)) Y \right] > \frac{\epsilon}{2} \right] < 2\epsilon
\]

Now we apply Theorem 2.1 to the functional
\[
F(x) = f \left( \int_{B_0} h(u)x(u) du \right)
\]
to obtain (2.12).

Let $\tilde{u}_T = a_T^{-1}(\hat{\theta}_T - \theta^*)$. \hfill \Box

Let $\tilde{u}_T = a_T^{-1}(\hat{\theta}_T - \theta^*)$.\hfill 12
Theorem 2.4. Let $D > p + 1$ and $L > 1$. Suppose that $[A1^\sharp], [A2]$ and $[A3]$ are fulfilled. Then

$$E_C [f(\tilde{u}_T)Y] \to^P E_C [f(\tilde{u})Y]$$

for any $f \in C_b(\mathbb{R}^p)$ and any bounded $\mathcal{G}$-measurable variable $Y$. In particular, $\tilde{u}_T \to^{d,(\mathcal{G})} \tilde{u}$ as $T \to \infty$.

Proof. Let $J_T = \int_{\cup_r} u Z_T(u) \varpi(\theta^1_T(u)) du$, $I_T = \int_{\cup_r} Z_T(u) \varpi(\theta^1_T(u)) du$, $J_\infty = \int_{\mathbb{R}^p} u Z(u) \varpi(\theta^*_u) du$ and $I_\infty = \int_{\mathbb{R}^p} Z(u) \varpi(\theta^*_u) du$. Proposition 2.3 provides convergence $(I_T, J_T) \to^{d*} (I_\infty, J_\infty)$. Therefore, for $\epsilon \in (0, 1]$, we can find $a > 0$ such that

$$\limsup_{T \to \infty} P \left[ I_T \leq a \right] > 3^{-1} (1 + 2 \|f\|_\infty \|Y\|_\infty)^{-1} \epsilon \leq 3 \limsup_{T \to \infty} \epsilon^{-1} P \left[ I_T \leq a \right] \leq 3 \epsilon^{-1} P \left[ J_T \leq a \right] < \epsilon / 2.$$

We have

$$\left| E_C [f(\tilde{u}_T)Y] - E_C \left[ f \left( \frac{J_T}{I_T \vee a} \right) Y \right] \right| \leq 2 \|f\|_\infty \|Y\|_\infty P_C [I_T \leq a] \quad a.s.$$

for $T \in (0, \infty]$, writing $\tilde{u}_\infty = \tilde{u}$.

Apply Proposition 2.3 to $q = 1$ and the function $(x, y) \mapsto f((x/y \vee a)) (x, y \in \mathbb{R})$ and $h(u) = (u, 1)$, we obtain

$$\limsup_{T \to \infty} P \left[ \left| E_C [f(\tilde{u}_T)Y] - E_C [f(\tilde{u})Y] \right| > \epsilon \right] < \epsilon. \quad \square$$

[A5] $E_C \left[ \frac{1}{\int_{\cup_r} Z_T(u) du} \right] = O_p(1)$ as $T \to \infty$.

Theorem 2.5. Suppose that $D > p + q$, $q \geq 1$ and $L > q + 1$ and that $[A1^\sharp], [A2]$ and $[A3]$ are fulfilled. Then

$$E_C [f(\tilde{u}_T)Y] \to^P E_C [f(\tilde{u})Y]$$

as $T \to \infty$ for any $\mathcal{G}$-measurable bounded random variable $Y$ and any $f \in C(\mathbb{R}^p; \mathbb{R}^k)$ satisfying

$$\sup_{u \in \mathbb{R}^p} (1 + |u|^q)^{-1} |f(u)| < \infty.$$

Proof. We may assume $\|Y\|_\infty \leq 1$. Let $b \in (q, (D - p) \wedge (L - 1))$. On $\{\Psi_T = 1\}$,

$$E_C \left[ \left| \tilde{u}_T \right|^b Y \right] \leq \sum_{r=0}^{\infty} E_C \left[ \int_{\{r < |u| \leq r+1\} \cap \cup_r} \left| u \right|^b Z_T(u) \varpi(\theta^1_T(u)) du \right]$$

$$\leq 1 + \sum_{r=1}^{\infty} (r + 1)^b \left\{ P_C \left[ \sup_{u \in \Psi_T(r)} Z_T(u) \geq \frac{U}{r^D} \right] + C(\varpi) U r^{p-1-D} E_C \left[ \frac{1}{\int_{\cup_r} Z_T(u) du} \right] \right\}$$

$$\leq 1 + C \left\{ \sum_{r=1}^{\infty} r^{b-L} + U \sum_{r=1}^{\infty} r^{p+b-D-1} E_C \left[ \frac{1}{\int_{\cup_r} Z_T(u) du} \right] \right\}$$

$$= O_p(1) \quad (2.17)$$
as $T \to \infty$ under [A5]. Let $f_A = (-A) \vee f \wedge A$ for $A > 0$. From (2.17), for any $T = (T_n) \in \mathcal{T}$,

$$
\limsup_{n \to \infty} E \left[ |E_C[f(\tilde{u}_T^n)Y] - E_C[f_A(\tilde{u}_T^n)Y]| \wedge 1 \right] \leq C \limsup_{n \to \infty} E \left[ \left\{ A^{-\alpha/(\beta - 1)} E_C \left[ 1 + |\tilde{u}_T^n|^b \right] \right\} \wedge 1 \right] \to 0 \quad (2.18)
$$

as $A \to \infty$. Moreover, by Theorem 2.4, we have

$$
E_C[f_A(\tilde{u}_T)Y] \to P E_C[f_A(\bar{u})Y] \quad (2.19)
$$

as $T \to \infty$. Then (2.18) and (2.19) give the desired convergence.

\[\square\]

**Remark 2.4.** Localization is essential. If the effect of a slow component to the fast component is unbounded, sophisticated construction of $\Psi_T$ is required and any way using $E_C[f(\tilde{u}_T)]$ without localization for unbounded $f$ is banned in general.

**Remark 2.5.** Generalization to the multi-scaling case is straightforward though we only treated a single scaling $a_T$.

### 3 Conditional polynomial type large deviation

As seen in Section 2, the polynomial type large deviation inequality under conditional probability plays an essential role in the partial quasi likelihood analysis. We present a set of conditions that induces a conditional polynomial type large deviation (CPLD) inequality though there are various versions of sufficient conditions as [30] in unconditional cases.

Suppose that $H_T$ is of class $C^3$. $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimum and maximum eigenvalues of a symmetric matrix $A$, respectively. Let $L > 0$ and let $b_T = (\lambda_{\min}(a_T^*a_T))^{-1}$. We assume that $b_T^{-1} \leq \lambda_{\max}(a_T^*a_T) \leq C_1 b_T^{-1}$ for some constant $C_1 \in [1, \infty)$. Let $\alpha \in (0, 1)$ and $\beta = \alpha/(1 - \alpha)$. Let $\rho$ be a positive constant; practically $\rho = 2$ in most cases.

[B1] Parameters $\beta_1$, $\rho_1$, $\rho_2$ and $\beta_2$ satisfy the following inequalities

$$
0 < \beta_1 < \frac{1}{2}, \quad 0 < \rho_1 < \min \left\{ 1, \beta, \frac{2\beta_1}{1 - \alpha} \right\},
\alpha \rho < \rho_2, \quad \beta_2 \geq 0, \quad 1 - 2\beta_2 - \rho_2 > 0.
$$

For example, the following two sets of conditions respectively satisfy [B1].

(i) $\beta_1 = \alpha/2$, $\rho_1 = \alpha$, $\rho_2 = 3\alpha$ and $\beta_2 = \alpha$ for $\rho = 2$ and $\alpha \in (0, 1/5)$.

(ii) $\beta_1 = \alpha/2$, $\rho_1 = \alpha$, $\rho_2 = 3\alpha$ and $\beta_2 = 0$ for $\rho = 2$ and $\alpha \in (0, 1/3)$.

Let $\Psi : \Omega \times \Theta \to \mathbb{R}$ be a random field, i.e., a measurable mapping.

[B2] There exists a positive random variable $\chi_0$ satisfying the following conditions.
(i) With probability one,
\[ \Psi(\theta) = \Psi(\theta) - \Psi(\theta^*) \leq -\chi_0 |\theta - \theta^*|^\rho \]
for all \( \theta \in \Theta \).

(ii) \( \sup_{r \in \mathbb{N}} r^L P_C \left[ \chi_0 \leq r^{-(\rho_2-\alpha \rho)} \right] < \infty \) a.s.

Let \( \Gamma \) be a \( p \times p \) positive definite random matrix.

[B3] \( \sup_{r \in \mathbb{N}} r^L P_C \left[ \lambda_{\min}(\Gamma) < 4r^{-\rho} \right] < \infty \) a.s.

Let
\[ \Psi_T(\theta) = \frac{1}{b_T} (\mathbb{H}_T(\theta) - \mathbb{H}_T(\theta^*)) \]
Define a \( p \)-dimensional random variable \( \Delta_T \) and a \( p \times p \) random matrix \( \Gamma_T \) by
\[ \Delta_T[u] = \partial_\theta \mathbb{H}_T(\theta^*)[a_T u] \quad (u \in \mathbb{R}^p) \]
and
\[ \Gamma_T[u^{\otimes 2}] = -\partial_\theta^2 \mathbb{H}_T(\theta^*)[(a_T u)^{\otimes 2}] \quad (u \in \mathbb{R}^p) \]
respectively.

[B4] For \( M_1 = L(1-\rho_1)^{-1} \),
\[ \sup_{T \in \mathcal{T}} E_C \left[ \left| \Delta_T \right|^{M_1} \right] \Psi_T < \infty \) a.s.

for every \( \mathcal{T} \in \mathcal{F} \). Moreover, for \( M_2 = L(1-2\beta_2-\rho_2)^{-1} \),
\[ \sup_{T \in \mathcal{T}} E_C \left[ \left( \sup_{\theta \in \Theta} b_T^{1-\beta_2} \left| \Psi_T(\theta) - \Psi(\theta) \right| \right)^{M_2} \right] \Psi_T < \infty \) a.s.

for every \( \mathcal{T} \in \mathcal{F} \).

[B5] For \( M_3 = L(\beta - \rho_1)^{-1} \),
\[ \sup_{T \in \mathcal{T}} E_C \left[ \left( b_T^{-1} \sup_{\theta \in \Theta} |\partial_\theta^3 \mathbb{H}_T(\theta)| \right)^{M_3} \right] \Psi_T < \infty \) a.s.

for every \( \mathcal{T} \in \mathcal{F} \). Moreover, for \( M_4 = L(2\beta_1(1-\alpha)^{-1}-\rho_1)^{-1} \),
\[ \sup_{T \in \mathcal{T}} E_C \left[ \left( b_T^{\beta_1} |\Gamma_T - \Gamma| \right)^{M_4} \right] \Psi_T < \infty \) a.s.

for every \( \mathcal{T} \in \mathcal{F} \).
Theorem 3.1. Suppose that Conditions [B1]-[B5] are satisfied. Then

\[
\sup_{T \in \mathcal{T}} \sup_{r \in \mathbb{N}} \mathbb{P}[\mathbb{C}] \left[ \sup_{u \in \mathcal{U}(r)} \mathbb{Z}_T(u) \geq \exp \left( -2^{-1} r^{2-(\rho_1 + \rho_2)} \right) \right] \Psi_T < \infty \quad \text{a.s.} \quad (3.1)
\]

for every \( \mathcal{T} \in \mathbb{T} \). Moreover, \( \mathbb{Z}_T \) has a LAQ representation

\[
\mathbb{Z}_T(u) = \exp \left( \Delta_T[u] - \frac{1}{2} \Gamma[u^\otimes 2] + r_T(u) \right)
\]

with \( r_T(u) \to P^0 \) as \( T \to \infty \) for every \( u \in \mathbb{R}^p \).

Proof. Arbitrarily given \( \mathcal{T} \in \mathbb{T} \), we will follow the proof of Theorems 1 and 2 of Yoshida [30] under \( P_C[\cdot, \Psi_T] \). This is valid because positivity of the expectation used in the proof in [30] is obviously valid for \( P_C[\cdot, \Psi_T] \) a.s. in the present case. Condition [A4’] of [30] is the present Condition [B1]. Condition [A1’] of [30] is satisfied by [B5] in conditional version. Then as Lemma 1 of [30] deduced [A1’] therein, we obtain

\[
\sup_{T \in \mathcal{T}} \sup_{r \in \mathbb{N}} \mathbb{P}[\mathbb{C}] \left[ \mathbb{S}_T(r)^c \right] \Psi_T < \infty \quad \text{a.s.} \quad (3.2)
\]

where

\[
\mathbb{S}'_T(r) = \begin{cases} 
\omega; & \sup_{\theta \in \Theta} \left| \Gamma_T(\theta) - \Gamma \right| < \epsilon_1(r) \\
\end{cases}
\]

with \( \delta_T = b_T^{-\alpha/2} \) and \( \epsilon_1(r) = r^{-\rho_1} \). \( \Gamma_T(\theta) \) is defined by

\[
\Gamma_T(\theta)[u^\otimes 2] = -\partial^2_{\theta \theta^\dagger T}(\theta)[(a_T u)^\otimes 2] \quad (u \in \mathbb{R}^p).
\]

The variable \( r_T(u) \) is defined by the LAQ representation of \( \mathbb{Z}_T(u) \):

\[
\mathbb{Z}_T(u) = \exp \left( \Delta_T[u] - \frac{1}{2} \Gamma[u^\otimes 2] + r_T(u) \right) \quad (3.3)
\]

where \( r_T(u) \) admits the expression

\[
r_T(u) = \int_0^1 (1 - s) \left\{ \Gamma[u^\otimes 2] - \Gamma_T(\theta^\dagger_T(su))[u^\otimes 2] \right\} ds
\]

for every \( u \in \mathbb{R}^p \) and sufficiently large \( T \) depending on \( u \). Then from (3.2), we obtain, as a counterpart of [A1] of [30],

\[
\sup_{T \in \mathcal{T}} \sup_{r \in \mathbb{N}} \mathbb{P}[\mathbb{C}] \left[ \mathbb{S}_T(r)^c \right] \Psi_T < \infty \quad \text{a.s.} \quad (3.4)
\]

where

\[
\mathbb{S}_T(r) = \begin{cases} 
\omega; & \sup_{u \in \mathcal{U}(r)} (1 + |u|^2)^{-1} \left| r_T(u) \right| < \epsilon_1(r) \\
\end{cases}
\]
with $\cup_T(r) = \{ u \in \cup_T; r \leq |u| \leq \delta_T b_T^{1/2} \}$.

Condition [B3] serves as [A2] in [30]. Condition [B2] (i) as [A3], Condition [B2] (ii) as [A5], and Condition [B4] as [A6], respectively. As already mentioned, [B1] is $[A4']$ that is stronger than [A4]. Now by using (3.4) and [B1]-[B4], we follow the line of the proof of Theorem 1 of [30] given under [A1]-[A6], to obtain (3.1).

Condition [B5] implies $r_T(u) \rightarrow^P 0$ as $T \rightarrow \infty$.

Obviously the inequality (3.1) ensures [A1’], [A1] and [A1’’].

4 Partial mixing

Partial mixing is a structure we often meet in applications of the partial quasi likelihood analysis, though it is not the all. We state a Rosenthal type inequality under conditional expectation.

**Lemma 4.1.** Let $2 \leq p < r$. Given a probability space $(\Omega, \mathcal{F}, P)$ and a sub $\sigma$-fields $\mathcal{C}$ of $\mathcal{F}$, let $\mathcal{G}_j$ and $\mathcal{H}_j$ ($j = 1, 2, ..., n$) be sub $\sigma$-fields of $\mathcal{F}$ such that $\mathcal{G}_j \cap \mathcal{H}_j \supset \mathcal{C}$ for all $j = 1, ..., n$. Let $X = (X_j)_{j=1,...,n}$ be a sequence of random variables such that $X_j \in L_r(\Omega, \mathcal{G}_j \cap \mathcal{H}_j, P)$ and $E_C[X_j] = 0$ a.s. Suppose that $[0, 1/2]$-valued $\mathcal{C}$-measurable random variables $\alpha_C(h)$ satisfy

$$\alpha_C(h) \geq \sup_{k=1,...,n-h} \sup \{ |P_C[A \cap B] - P_C[A]P_C[B]|; A \in \mathcal{G}_k, B \in \mathcal{H}_{k+h} \}$$

for $h = 1, ..., n - 1$ on some $\Omega_0 \in \mathcal{F}$ such that $P[\Omega_0] = 1$. Then

$$E_C \left[ \max_{k=1,...,n} \left| \sum_{j=1}^k X_j \right|^p \right] \leq C(p, r) \max_{j=1,...,n} E_C \left[ \left| X_j \right|^r \right]^{p/r} \times n^{p/2} \left( 1 + \sum_{h=1}^{n-1} \alpha_C(h)^{1-2/r} \right)^{p/2} + n \sum_{h=1}^{n-1} (h + 1)^{p-2} \alpha_C(h)^{1-p/r} \text{ a.s.}$$

where $C(p, r)$ is a constant depending only on $p$ and $r$.

**Proof.** Denoted by $Q_j$ the random upper quantile function of a regular conditional distribution $P^{[X_j]}_\omega$ of $|X_j|$ given $\mathcal{C}$, i.e., an inverse of the function $t \mapsto P^{[X_j]}([t, \infty])$. Let $\alpha_C(0) = 1/2$. The random function $\alpha_C^{-1} : (0, 1) \rightarrow \mathbb{Z}_+$ is defined by $\alpha_C^{-1}(u) = \sum_{h \in \mathbb{Z}_+} 1_{\{u < \alpha_C(h)\}}$. We apply Theorem 6.3 of Rio [20] under the conditional probability $P_C$ to obtain

$$E_C \left[ \max_{k=1,...,n} \left| \sum_{j=1}^k X_j \right|^p \right] \leq C(p) \left\{ s_C(n)^p + n \int_0^1 [\alpha_C^{-1}(u) \wedge n]^{p-1} Q_C(u)^p du \right\} \text{ a.s.} \ (4.1)$$

where $C(p)$ is a constant depending only on $p$, $Q_C = \max_{j=1,...,n} Q_j$ is a $\mathcal{C}$-measurable random function of $u \in (0, 1)$ and

$$s_C(n)^2 = \sum_{i,j=1}^n \left| \text{Cov}_C[X_i, X_j] \right|,$$

$\text{Cov}_C$ denoting $\mathcal{C}$-conditional covariance. We shall estimate the right-hand side of (4.1).
Since \( Q_C(u) \leq \rho u^{-1/r} \) with \( \rho = \max_{j=1,...,n} E_C[|X_j|^r]^{1/r} \), we have

\[
\int_0^1 [\alpha_C^{-1}(u) \wedge n]^{p-1} Q_C(u)^p du \leq \int_0^{1/2} \left[ \left( \sum_{h \in \mathbb{Z}_+} \mathbb{1}_{\{u < \alpha_C(h)\}} \right) \wedge n \right]^{p-1} \rho^p u^{-p/r} du \\
\leq \int_0^{\alpha_C(n)} n^{p-1} \rho^p u^{-p/r} du + \sum_{\ell=1}^n \int_{\alpha_C(\ell)}^{\alpha_C(\ell-1)} \ell^{p-1} \rho^p u^{-p/r} du \\
\leq \rho^p \left(1 - \frac{p}{r}\right)^{-1} \sum_{\ell=1}^n \ell^{p-2} \alpha_C(\ell - 1)^{1-p/r}
\]

and move the term for \( \ell = 1 \) into the error bound we will consider.

By the covariance inequality applied to the conditional situation,

\[
s_C(n)^2 \leq \sum_{i,j=1}^n 2\alpha_C(|i-j|)^{1-2/r} \rho^2 \leq 4\rho^2 n \sum_{h=0}^{n-1} \alpha_C(h)^{1-2/r}.
\]

Bring the above two estimates into (4.1), we complete the proof. \( \square \)

In the following two sections, we will present applications of the partial quasi likelihood analysis.

5 Diffusion process having a component with a slow mixing rate

5.1 Partial QLA for a stochastic regression model

Given a stochastic basis \((\Omega, \mathcal{F}, \mathbf{F}, P)\), \(\mathbf{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}\), we consider a stochastic regression model

\[
Y_t = Y_0 + \int_0^t b(X_s, \theta^*) ds + \int_0^t \sigma(X_s) dw_s \quad (t \in \mathbb{R}_+).
\]

Here \(X = (X_t)_{t \in \mathbb{R}_+}\) is a stochastic process taking values in a measurable space \((X, \mathcal{B}_X)\). \(\Theta\) is a bounded domain in \(\mathbb{R}^p\). We assume that the boundary of \(\Theta\) is as good as it admits the ordinary Sobolev’s inequality for the embedding \(W^{1,p}(\Theta) \hookrightarrow C_b(\Theta)\) for \(p > p\). Moreover, \(b : X \times \Theta \to \mathbb{R}^m\) and \(\sigma : X \to \mathbb{R}^m \otimes \mathbb{R}^r\) are given functions. \(w = (w_t)_{t \in \mathbb{R}_+}\) is an \(r\)-dimensional standard \(\mathbf{F}\)-Wiener process. We assume that \(b(X_t, \theta)\) and \(\sigma(X_t)\) are almost surely locally integrable \(\mathbf{F}\)-progressively measurable processes.

The model (5.1) can express a fairly general class of models. For example, consider a system

\[
Y_t = Y_0 + \int_0^t b(\gamma, s, L, \xi_s, \theta^*) ds + \int_0^t \sigma(\gamma, s, L, \xi_s) dw_s \\
\xi_t = \xi_0 + \int_0^t \tilde{b}(\xi_s) ds + \int_0^t \tilde{\sigma}(\xi_s) d\tilde{w}_s
\]

18
where $\gamma$ expresses a measurable random scenery taking values in a measurable space $(G, \mathcal{B}_G)$, and $(b(\gamma, \cdot), \sigma(\gamma, \cdot))$ are regarded as a random environment in space-time. $\xi_t$ is a latent diffusion process having a good mixing property. The process $L_t$ is a process with long memory. The process $(Y_t, \xi_t)$ is like a diffusion process but it does not enjoy a fast decay of mixing coefficient due to the component $L_t$. In this example, we may set $X_s = (\gamma, s, L_s, \xi_s)$. It is also possible to incorporate feedback of $Y_t$ as $X_s = (\gamma, s, L_s, \xi_s, Y_s)$. If the whole path $(L_t)$ is included in $\gamma$, then a simplified expression $b(\gamma, s, \xi_s, \theta^*)$ is possible for $b(\gamma, s, L_s, \xi_s, \theta^*)$.

We estimate the true value $\theta^*$ of the parameter $\theta \in \Theta$ based on observations $((Y_t, X_t)_{t \in [0,T]})$. Let $S = \sigma \sigma^*$ and assume that $S(X_t)$ is invertible a.s. Define a random function $\mathbb{H}_T$ by

$$\mathbb{H}_T(\theta) = \int_0^T S(X_t)^{-1}[b(X_t, \theta), dY_t] - \frac{1}{2} \int_0^T S(X_t)^{-1}[b(X_t, \theta)^{\otimes 2}] dt.$$

By (5.1), $\mathbb{H}_T$ has the following representation:

$$\mathbb{H}_T(\theta) = M_T(\theta) + N_T(\theta),$$

where

$$M_T(\theta) = \int_0^T S(X_t)^{-1}[b(X_t, \theta), \sigma(X_t)dw_t]$$

and

$$N_T(\theta) = \int_0^T S(X_t)^{-1}\left[b(X_t, \theta) \otimes b(X_t, \theta^*) - \frac{1}{2} b(X_t, \theta)^{\otimes 2}\right] dt.$$

Define a $(r + 1)$-dimensional function $H$ by

$$H(x, \theta) = \left(S(x)^{-1}[b(x, \theta), \sigma(x) \cdot ], S(x)^{-1}\left[b(x, \theta) \otimes b(x, \theta^*) - \frac{1}{2} b(x, \theta)^{\otimes 2}\right]\right)$$

for $x \in X$ and $\theta \in \Theta$.

[C1] The mapping $\Theta \ni \theta \mapsto H(x, \theta)$ is four times continuously differentiable and

$$\sup_{\theta \in \Theta} \sum_{i=0}^4 \left| \partial_i^p H(x, \theta) \right| \leq H_1(x) \quad (x \in X)$$

for some measurable function $H_1 : X \to \mathbb{R}_+$ such that

$$\sup_{t \in \mathbb{R}_+} E[H_1(X_t)^p] < \infty \quad a.s.$$

for all $p > 1$.

Let $C$ be a sub $\sigma$-field of $\mathcal{F}_0$, and let $B_t = C \vee \sigma[X_t, w_t - w_{\inf t}; t \in I]$ for $I \subset \mathbb{R}_+$. A partial mixing coefficient $\alpha_C(h)$ is a $C$-measurable $[0,1/2]$-valued random variable satisfying the inequality

$$\alpha_C(h) \geq \sup_{t \in \mathbb{R}_+} \left\{ |P_C[A \cap B] - P_C[A] P_C[B]| \ ; A \in B_{[0,t]}, B \in B_{(t+h, \infty)} \right\}$$
Let $X = \mathbb{R}^d$. Suppose that a regular conditional probability $\mu_t = P^{X_t}[\cdot]$ of $X_t$ given $C$ exists. The measure-valued process $\mu_t$ is a “basso continuo”, which may only admit a very weak ergodic property. We will consider the following two situations.

[C2] (i) There exists a positive constant $L_0$ such that for every $L > 0$,

$$\limsup_{h \to \infty} h^L \| \alpha_C(h)^{L_0} \|_1 < \infty.$$ 

(ii) There exist a probability measure $\nu$ on $\mathbb{R}^d$ and a positive constant $\epsilon_1$ such that

$$T \epsilon_1 \left| \frac{1}{T} \int_0^T \mu_t(f) dt - \nu(f) \right| \to 0 \quad \text{a.s.}$$

as $T \to \infty$ for any measurable function $f : \mathbb{R}^d \to \mathbb{R}$ satisfying $|f(x)| \leq C(1 + H_1(x)^C)$ ($x \in \mathbb{R}^d$) for some positive constant $C$.

Here we wrote $\mu_t(f) = \int f(x) \mu_t(dx)$ for a measurable function $f : \mathbb{R}^d \to \mathbb{R}$. The strong mixing coefficient of the measure valued process $\mu = (\mu_t)_{t \in \mathbb{R}^+}$ is defined by

$$\alpha^\mu(h) = \sup \left\{ \left| P[A \cap B] - P[A]P[B] \right| ; A \in \mathcal{C}_{[0,t]}, B \in \mathcal{C}_{[t,\infty)} \right\}$$

where $\mathcal{C}_I = \sigma[\mu_t(f); t \in I, f \in C_b(\mathbb{R}^d)]$ for $I \subset \mathbb{R}^+$. 

[C2*] (i) There exists a positive constant $L_0$ such that for every $L > 0$,

$$\limsup_{h \to \infty} h^L \| \alpha_C(h)^{L_0} \|_1 < \infty.$$ 

(ii) For some $\epsilon_0 > 0$, $\alpha^\mu(h) = O(h^{-\epsilon_0})$ as $h \to \infty$.

(iii) There exist a probability measure $\nu$ on $\mathbb{R}^d$ and a positive constant $\epsilon_1$ such that

$$T \epsilon_1 \left| \frac{1}{T} \int_0^T \mathbb{E} \left[ \int_{\mathbb{R}^d} f(x) \mu_t(dx) \right] dt - \nu(f) \right| \to 0$$

as $T \to \infty$ for any measurable function $f : \mathbb{R}^d \to \mathbb{R}$ satisfying $|f(x)| \leq C(1 + H_1(x)^C)$ ($x \in \mathbb{R}^d$) for some constant $C$.

Define $\mathcal{Y} : \Theta \to \mathbb{R}$ by

$$\mathcal{Y}(\theta) = -\frac{1}{2} \int_{\mathbb{R}^d} S(x)^{-1} \left[ (b(x, \theta) - b(x, \theta^*)) \otimes 2 \right] \nu(dx).$$

[C3] There exists a positive constant $\chi_0$ such that $\mathcal{Y}(\theta) \leq -\chi_0|\theta - \theta^*|^2$ for all $\theta \in \Theta$.  

20
Under [C3], the matrix
\[ \Gamma := -\partial_\theta^2 Y(\theta^*) = \int_{\mathbb{R}^d} S(x)^{-1} \left[ \left( \partial_x b(x, \theta^*) \right) \otimes^2 \right] \nu(dx) \]
is a positive-definite $p \times p$ symmetric matrix. Let \( \hat{\theta}_M^T = \hat{\theta}_T \) and \( \hat{\theta}_B^T = \tilde{\theta}_T \), and let \( \hat{u}_M^T = \hat{u}_T \) and \( \hat{u}_B^T = \tilde{u}_T \).

**Theorem 5.1.** (i) Suppose that Conditions [C1], [C2] and [C3] are satisfied. Then
\[ \hat{u}_T^A \rightarrow^d \Gamma^{-1/2} \zeta \quad (5.2) \]
as \( T \to \infty \) for \( A = M \) and \( B \), where \( \zeta \) is a \( p \)-dimensional standard Gaussian random vector.

(ii) The convergence (5.2) holds under Conditions [C1], [C2] and [C3].

### 5.2 Proof of Theorem 5.1

Let \( \epsilon_* > 0 \). Define \( \Psi_T \) by
\[ \Psi_T = 1_{A_T} \]
where
\[ A_T = \left\{ \max_{j=1,\ldots,\lfloor T \rfloor} E_C \left[ \int_{j-1}^j H_1(X_t)^* \right. dt \left. \leq T^{\epsilon_*} \right] \right\}, \]
where we fix a sufficiently large but finite constant \( r_* \) in what follows, since we only aim at asymptotic normality of the QLA estimators.

Let \( \mathcal{Y}_T(\theta) = T^{-1}(H_T(\theta) - H_T(\theta^*)) \). Then
\[ \mathcal{Y}_T(\theta) = \mathcal{Y}_T^{(0)}(\theta) + \mathcal{Y}_T^{(1)}(\theta), \]
where
\[ \mathcal{Y}_T^{(0)}(\theta) = \frac{1}{T} \left\{ M_T(\theta) - M_T(\theta^*) \right\} \]
and
\[ \mathcal{Y}_T^{(1)}(\theta) = -\frac{1}{2T} \int_0^T S(X_t)^{-1} \left[ (b(X_t, \theta) - b(X_t, \theta^*)) \otimes^2 \right] dt. \]

Let
\[ \Delta_T = T^{-1/2} \partial_\theta H_T(\theta^*) = T^{-1/2} \partial_\theta M_T(\theta^*) \]
\[ = T^{-1/2} \int_0^T S(X_t)^{-1} \left[ \partial_\theta b(X_t, \theta^*), \sigma(X_t)dw_t \right] \]
\[ = T^{-1/2} \int_0^T S(X_t)^{-1} \left[ \partial_\theta b(X_t, \theta^*) \right] \sigma(X_t)dw_t, \]

\[ + T^{-1/2} \int_0^T S(X_t)^{-1} \left[ \partial_\sigma b(X_t, \theta^*) \right] \partial_\theta \sigma(X_t)dw_t \]
and let
\[ \Gamma_T = -T^{-1} \partial^2_T b_T(\theta^*) - T^{-1} \partial^2_T N_T(\theta^*) \]
\[ = -T^{-1} \int_0^T S(X_t)^{-1} [\partial^2_T b(X_t, \theta^*), \sigma(X_t) dw_t] \]
\[ + T^{-1} \int_0^T S(X_t)^{-1} [\partial^2 b(X_t, \theta^*)]^2 dt. \]

Lemma 5.1. (i) \( \sup_{T \in \mathcal{T}} E_C[|\Delta_T|^M] < \infty \) a.s. for every \( \mathcal{T} \in \mathfrak{S} \) and every \( M > 0 \).

(ii) Let \( \eta \in (0, 1/2) \) and \( M > 0 \). Then for sufficiently large \( r_* \), one has
\[ \sup_{T \in \mathcal{T}} E_C \left[ \left( \sup_{\theta \in \Theta} T^{\frac{1}{2}-\eta} |\mathcal{Y}_T(\theta) - \mathcal{Y}(\theta)| \right)^M \right] \Psi_T < \infty \] a.s.
for every \( \mathcal{T} \in \mathfrak{S} \).

Proof. Suppose that \( |g(x)| \leq C(1 + H_1(x)^C) \) for some constant \( C > 0 \). Then
\[ \sup_{t \in \mathbb{R}_+} E[|\mu_t(g)|^p] \leq \sup_{t \in \mathbb{R}_+} E[|g(X_t)|^p] < \infty \] for any \( p \geq 1 \).

Suppose that \( [C2^p] \) holds, for a while. Let \( r \in (1, \min\{1 + \epsilon_0, 2\}) \). In the notation of Rio [20],
for the tail-quantile function \( Q_j(u) \) of \( \int_{j-1}^j \mu_t(g) dt \), we have \( Q_j(u) \lesssim u^{-1/L} \) for arbitrarily large \( L \) due to \( L_{\infty} \)-boundedness of \( H_1(X_t) \) uniform in \( t \). Moreover, \( (\alpha^U)^{-1}(u) \lesssim u^{-1/\epsilon_0} \). Therefore,
\[ M_{r, \alpha^U}(Q) = \int_0^1 [((\alpha^U)^{-1}(u))]^{r-1} Q(u)^r du \lesssim \int_0^1 u^{-\epsilon_0^{-1}(r-1)-L^{-1}} du < \infty \]
if we take a sufficiently large \( L \). We apply Corollary 3.2 (i) of Rio [20] to conclude
\[ \frac{1}{n^{1/r}} \left( \int_0^n \mu_t(g) dt - \int_0^n E[\mu_t(g)] dt \right) \to 0 \] a.s.
as \( T \to \infty \). Further, since
\[ \sum_{n \in \mathbb{N}} P \left[ \sup_{T : n \leq T < n+1} \int_n^T |\mu_t(g)| dt > n^{1/(2r)} \right] < \infty, \]
we obtain
\[ \frac{1}{T^{1/r}} \left( \int_0^T \mu_t(g) dt - \int_0^T E[\mu_t(g)] dt \right) \to 0 \] a.s.
(5.4)
as \( T \to \infty \). We choose a sufficiently large constant \( \eta \in (0, 1/2) \). Under \( [C2^p] \) (iii) with (5.3),
we have
\[ T^{\frac{1}{2}-\eta} \left| \frac{1}{T} \int_0^T E[\mu_t(g)] dt - \nu(g) \right| \to 0 \]
as $T \to \infty$. Then (5.4) gives
\[
T^{\frac{1}{2} - \eta} \left| \frac{1}{T} \int_0^T \mu_t(g) dt - \nu(g) \right| \to 0 \text{ a.s.} \tag{5.5}
\]
as $T \to \infty$. Under $[C2]$, the convergence (5.5) is obvious for a suitable $\eta$.

By the Burkholder-Davis-Gundy inequality,
\[
E_{\mathcal{C}} \left[ \left| \Delta_T \right| M \right] \lesssim E_{\mathcal{C}} \left[ \left| \frac{1}{T} \int_0^T H_1(X_t)^2 dt \right|^{M/2} \right] \\
\lesssim \frac{1}{T} \int_0^T \mu_t(H_1^M) dt \to \nu(H_1^M) \text{ a.s.} \tag{5.6}
\]
as $T \to \infty$ for $M \geq 2$, which proved (i).

In this situation, we can exchange the differentiation in $\theta$ and the stochastic integral, and
\[
\partial_\theta \Psi_T^{(0)}(\theta) = \frac{1}{T} \int_0^T S(X_t)^{-1} \left[ \partial_\theta b(X_t, \theta), \sigma(X_t)dw_t \right] .
\]

Then with Sobolev's inequality and following the way in (5.6), we obtain
\[
E_{\mathcal{C}} \left[ \left\| T^{2 - \eta} \Psi_T^{(0)}(\cdot) \right\|_{\mathcal{C}(\Theta)}^M \right] \lesssim \int_{\Theta} \left\{ E_{\mathcal{C}} \left[ \left| T^{2 - \eta} \Psi_T^{(0)}(\cdot) \right| ^M \right] + E_{\mathcal{C}} \left[ \left| T^{2 - \eta} \partial_\theta \Psi_T^{(0)}(\cdot) \right| ^M \right] \right\} d\theta \rightarrow 0 \text{ a.s.}
\]
as $T \to \infty$.

Next, we apply Lemma 4.1 to $\Psi_T^{(1)}(\theta) - \Psi(\theta)$ and $\partial_\theta (\Psi_T^{(1)}(\theta) - \Psi(\theta))$ with the help of $\Psi_T$, as well as Sobolev's inequality, to show (ii). More precisely, let
\[
g(x, \theta) = \frac{1}{2} S(x)^{-1} \left[ (b(x, \theta) - b(x, \theta^*)) \otimes 2 \right] .
\]

Then for $M > p$,
\[
E_{\mathcal{C}} \left[ \left| T^{2 - \eta} \left( \Psi_T^{(1)} - \Psi \right) \right| \Psi_T \right|_{\mathcal{C}(\Theta)}^M \right] \\
\lesssim \sum_{i=0,1} \int_{\Theta} E_{\mathcal{C}} \left[ \left| T^{2 - \eta} \partial_\theta \left( \Psi_T^{(1)}(\cdot) - \Psi(\cdot) \right) \Psi_T \right| ^M \right] d\theta \\
\lesssim I_T + J_T
\]
where
\[
I_T = \sum_{i=0,1} \int_{\Theta} E_{\mathcal{C}} \left[ \left| T^{2 - \eta} \left( \partial_\theta \Psi_T^{(1)}(\cdot) - E_{\mathcal{C}} \left[ \partial_\theta \Psi_T^{(1)}(\cdot) \right] \right) \Psi_T \right| ^M \right] d\theta
\]
and
\[
J_T = \sum_{i=0,1} \int_{\Theta} \left| T^{2 - \eta} \left( E_{\mathcal{C}} \left[ \partial_\theta \Psi_T^{(1)}(\cdot) \right] - \partial_\theta \Psi(\cdot) \right) \Psi_T \right| ^M \right] d\theta.
\]
We notice that \( \nu(H_p) < \infty \) for every \( p > 1 \) by (5.3) under both \([C2]\) and \([C2]^f\); in particular, 
\( \partial_\theta \Upsilon(\theta) = \nu(\partial_\theta g(\cdot, \theta)) \). By (5.5), we have \( J_T \to 0 \) as \( T \to \infty \) a.s. Applying Lemma 4.1, we see

\[
I_T \leq T^{M(\eta + \frac{\rho}{r_*})} V_*
\]
for suitably set \( (M, r_*) \) so that \( r_* > M \geq 2 \) and \( -\eta + \epsilon_* / r_* < 0 \), where

\[
V_* \leq C(M, r_*) \left[ \left( 1 + \sum_{h=1}^{\infty} \alpha_C(h)^{1-\frac{\rho}{r_*}} \right)^{M/2} + \sum_{h=1}^{\infty} (h + 1)^{M-2} \alpha_C(h)^{1-\frac{\rho}{r_*}} \right].
\]

Let \( K = L_0(1 - M / r_*)^{-1} \). Since \( \alpha_C(h) \leq 1/2 \leq 1 \), we have

\[
\left\| \left( \sum_{h=1}^{\infty} \alpha_C(h)^{1-\frac{\rho}{r_*}} \right)^K \right\|_1 \leq \left\| \left( \sum_{h=1}^{\infty} (h + 1)^{M-2} \alpha_C(h)^{1-\frac{\rho}{r_*}} \right)^K \right\|_1 
\leq \sum_{h=1}^{\infty} (h + 1)^{KM-2} \| \alpha_C(h)^{L_0} \|_1 < \infty.
\]

Therefore \( V_* < \infty \) a.s. and hence \( I_T \to 0 \) as \( T \to \infty \) a.s. \( \square \)

In a similar fashion to Lemma 5.1 we can show the following lemma.

**Lemma 5.2.** (i) Let \( M > 0 \). Then for a sufficiently large \( r_* \), for any \( \mathcal{T} \in \mathfrak{T} \),

\[
\sup_{T \in \mathcal{T}} E^C \left[ \left( T^{-1} \sup_{\theta \in \Theta} |\partial_\theta^2 \Upsilon_T(\theta)| \right)^M \right] \Psi_T < \infty \quad \text{a.s.}
\]

(ii) Let \( M > 0 \) and \( \eta \in (0, 1/2) \). Then for a sufficiently large \( r_* \),

\[
\sup_{T \in \mathcal{T}} E^C \left[ \left( T^\eta |\Gamma_T - \Gamma| \right)^M \right] \Psi_T < \infty \quad \text{a.s.}
\]

for any \( \mathcal{T} \in \mathfrak{T} \).

As before, the random field \( Z_T \) is defined by

\[
Z_T(u) = \exp \left( \Upsilon_T(\theta^*_T(u)) - \Upsilon_T(\theta^*) \right) \quad (u \in \mathbb{R}^p)
\]

**Lemma 5.3.** Let \( L > 0 \). Then there exist \( r_* > 0 \) (in \( \Psi_T \)) and \( \varphi \in (1, 2) \) such that

\[
\sup_{T \in \mathcal{T}} \sup_{r \in \mathbb{N}} r^L P_C \left[ \sup_{u \in \mathcal{V}_T(r)} Z_T(u) \geq \exp \left( -2^{-1} r^\varphi \right) \right] \Psi_T < \infty \quad \text{a.s.}
\]

for every \( \mathcal{T} \in \mathfrak{T} \).

**Proof.** Apply Theorem 5.1 with the help of Lemmas 5.1 and 5.2. \( \square \)
Lemma 5.4. For any $\epsilon > 0$ and $c > 0$, \( \limsup_{T \to \infty} P_c[W_T(\delta, c, \epsilon)] \Psi_T \to^P 0 \) as $\delta \downarrow 0$.

Proof. We apply Lemma 5.2 (i) to estimate $r_T(u)$ in the LAQ representation of $Z_T$. \qed

Define a random field $Z : \Omega \times \mathbb{R}^p \to \mathbb{R}$ on an extension $(\Omega, \mathcal{F}, P)$ of $(\Omega, \mathcal{F}, P)$ by

$$Z(u) = \exp \left( \Delta[u] - \frac{1}{2} \Gamma[u \otimes 2] \right),$$

where $\Delta = \Gamma^{1/2} \zeta$ and $\zeta$ is a $p$-dimensional standard Gaussian random variable defined on $\Omega$ and independent of $\mathcal{F}$.

Lemma 5.5. (i) For any $k \in \mathbb{N}$, $u_i \in \mathbb{R}^p$ ($i = 1, ..., k$) and $f \in C_b(\mathbb{R}^k)$,

$$E_C[f((Z_T(u_i))_{i=1,...,k}) \Psi_T] \to^P E_C[f((Z(u_i))_{i=1,...,k})]$$

as $T \to \infty$.

(ii) $\Psi_T \to^P 1$ as $T \to \infty$.

Proof. The conditional version of martingale central limit theorem gives

$$E_C[g(\Delta_T)] \to^P \mathbb{P}[g(\Delta)]$$

for $g \in C_b(\mathbb{R}; \mathbb{R}^k)$. Indeed, the quadratic variation of the martingale associated with $\Delta_T$ is $\frac{1}{T} \int_0^T g(X_t, \theta^* \otimes 2) dt$ if evaluated at $T$, and it converges to $\Gamma$ in probability. Then we have the convergence $E_C[\Psi_T \exp (\Delta_T[iu] + 2^{-1}\Gamma[u \otimes 2])] \to^P 1$ as $T \to \infty$ for every $u \in \mathbb{R}^p$. We obtain (5.7) with uniform approximation of $g$ on a compact set by trigonometric functions.

In the representation (5.5) of $Z_T$, the convergence $E_C[|r_T(u)| \wedge 1] \to^P 0$ follows from e.g. Lemma 5.2 (i). Thus we obtain (i). The property (ii) is easy to show by definition of $\Psi_T$. \qed

Condition [A5] is verified e.g. with Lemma 2 of [30]. Now Theorem 5.1 follows from Theorems 2.2 and 2.4 together with Theorem 3.1 as well as Lemmas 5.3, 5.4 and 5.5.

5.3 An example

On a stochastic basis $(\Omega, \mathcal{F}, F, P)$, $F = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$, let us consider stochastic processes $Y = (Y_t)_{t \in \mathbb{R}_+}$, $L = (L_t)_{t \in \mathbb{R}_+}$ and $U = (U_t)_{t \in \mathbb{R}_+}$ satisfying

$$Y_t = Y_0 + \int_0^t b_0(L_s)b_1(U_s, \theta^*) ds + \int_0^t \sigma_0(L_s)\sigma_1(U_s) dw_s$$

where $w = (w_t)_{t \in \mathbb{R}_+}$ is a one-dimensional standard $F$-Wiener process. We assume

(i) $L$ is càdlàg $\mathcal{F}_0$-measurable, stationary and independent of $(U, w, Y_0)$. The $\alpha$-mixing coefficient $\alpha^L$ of $L$ satisfies $\alpha^L(h) \lesssim h^{-a}$ as $h \to \infty$ for some positive constant $a$. For every $p > 1$, $\|L_0\|_p < \infty$.  

25
(ii) $U$ is a càdlàg $\mathbf{F}$-progressively measurable stationary process satisfying $\|U_0\|_p < \infty$ for every $p > 1$. The $\alpha$-mixing coefficient $\alpha_U$ of $U$ satisfies $\alpha_U(h) \leq b^{-1}e^{-bh}$ for some positive constant $b$.

(iii) $b_0$, $\sigma_0$ and $\sigma_1$ are measurable functions of at most polynomial growth. The function $b_1$ is measurable, four times continuously differentiable in $\theta$ and $\partial_\theta^2 b(\cdot, \theta)$ is of at most polynomial growth uniformly in $\theta \in \Theta$ for $i = 0, ..., 4$. Moreover $\inf_{z,u} \sigma_0(z)\sigma_1(u) > 0$.

The variable $X_t = (L_t, U_t)$ for this model. The random field $\mathbb{H}_T$ is given by

$$
\mathbb{H}_T(\theta) = \int_0^T \frac{b_0(L_t)b_1(U_t, \theta)}{\sigma_0(L_t)^2\sigma_1(U_t)^2}dY_t - \frac{1}{2} \int_0^T \frac{b_0(L_t)b_1(U_t, \theta)^2}{\sigma_0(L_t)^2\sigma_1(U_t)^2}dt.
$$

It has a representation

$$
\mathbb{H}_T(\theta) = \int_0^T \frac{b_0(L_t)b_1(U_t, \theta)}{\sigma_0(L_t)\sigma_1(U_t)}dw_t + \int_0^T \left\{ \frac{b_0(L_t)b_1(U_t, \theta)b_1(U_t, \theta^*)}{\sigma_0(L_t)^2\sigma_1(U_t)^2} - \frac{1}{2} \frac{b_0(L_t)b_1(U_t, \theta)^2}{\sigma_0(L_t)^2\sigma_1(U_t)^2} \right\} dt.
$$

Let $C = \sigma[L_t; t \in \mathbb{R}_+]$. Since $\mathcal{B}_t = C \cap \sigma[L_t, U_t, w_t - w_{int}; t \in I] = C \cap \sigma[U_t, w_t - w_{int}; t \in I]$ for $I \subset \mathbb{R}_+$ and $C$ is independent of $\sigma[U_t, w_t; t \in \mathbb{R}_+]$, we can take $\alpha_C(h) = \alpha^U_{dw}(h)$, which is the $\alpha$-mixing coefficient associated with $\mathcal{B}_t = \sigma[U_t, w_t - w_{int}; t \in I]$ for $I \subset \mathbb{R}_+$. The coefficient $\alpha^U_{dw}$ enjoys an exponential decay; see Kusuoka and Yoshida \cite{Kusuoka2002}.

For any bounded measurable function $f$ on $\mathbb{R}^2$,

$$
\mu_t(f) = \mathbb{E}_C[f(L_t, U_t)] = \mathbb{E}[f(\ell, U_t)]_{\ell=L_t} = \mathbb{E}[f(\ell, U_0)]_{\ell=L_t} \text{ a.s.}
$$

In particular, \([C2\sharp]\) (iii) holds for $\nu(f) = \mathbb{E}[f(L_0, U_0)]$. Moreover,

$$
\Upsilon(\theta) = -\frac{1}{2} \mathbb{E} \left[ \frac{b_0(L_0)^2(b_1(U_0, \theta) - b_1(U_0, \theta^*))^2}{\sigma_0(L_0)^2\sigma_1(U_0)^2} \right]
$$

Thus, if \([C3]\) is satisfied, then $\hat{a}_T^A$ ($A = M, B$) are asymptotically normal with variance

$$
\Gamma = \mathbb{E} \left[ \frac{b_0(L_0)^2(\partial_\theta b_1(U_0, \theta^*))^2}{\sigma_0(L_0)^2\sigma_1(U_0)^2} \right].
$$

6 Stochastic regression model for volatility in random environment

Let $(\Omega', \mathcal{F}', P')$ be a probability space and let $(\Omega'', \mathcal{F}'', P)$ be a measurable space having a right-continuous filtration $\mathbf{F} = (\mathcal{F}_t)_{t \in [0,T]}$. We consider a transition kernel $Q_{\omega'}(d\omega'')$ from $\Omega'$ to $(\Omega'', \mathcal{F}'')$ The extension $(\Omega, \mathcal{F}, P)$ of $(\Omega', \mathcal{F}', P')$ is defined by $\Omega = \Omega' \times \Omega''$, $\mathcal{F} = \mathcal{F}' \times \mathcal{F}''$ and $P(d\omega', d\omega'') = P'(d\omega')Q_{\omega'}(d\omega'')$. Let $\mathbb{T} = [0, T]$. We consider measurable processes $b : \Omega \times \mathbb{T} \to$
$\mathbb{R}^m$, $X : \Omega \times \mathbb{T} \rightarrow \mathbb{R}^d$, $Y : \Omega \times \mathbb{T} \rightarrow \mathbb{R}^m$ and $w : \Omega \times \mathbb{T} \rightarrow \mathbb{R}$. A random variable $\gamma$ takes values in a measurable space $(G, \mathcal{B}_G)$ defined on a probability space $(\Omega', \mathcal{F}', P')$.

For each $\omega' \in \Omega'$, on the stochastic basis $\mathcal{B}_{\omega'} = (\Omega'', \mathcal{F}'', \mathbf{F}, Q_{\omega'})$, we suppose that $b(\omega', \cdot) = (b_l(\omega', \cdot))_{t \in [0,T]}$ and $X(\omega', \cdot) = (X_l(\omega', \cdot))_{t \in [0,T]}$ become progressively measurable processes, $w(\omega', \cdot) = (w_l(\omega', \cdot))_{t \in [0,T]}$ is an $r$-dimensional $\mathbf{F}$-Wiener process, and the process $Y(\omega', \cdot) = (Y_t(\omega', \cdot))_{t \in [0,T]}$ satisfies

$$Y_t = Y_0 + \int_0^t b_s ds + \int_0^t \sigma(\gamma, X_s, \theta^*) dw_s, \quad t \in [0, T].$$

Here the function $\sigma$ is an $\mathbb{R}^m \otimes \mathbb{R}^r$-valued measurable function defined on $G \times \mathbb{R}^d \times \Theta$ and $\Theta$ is a bounded domain in $\mathbb{R}^d$, $\mathbf{F}$ is well defined. Detailed conditions for it will be specified below.

We observe $(\gamma, (X_{t_j}, Y_{t_j}))_{j=0, \ldots, n}$, where $t_j = t^n_j = jT/n$, and want to estimate $\theta^*$ from the data. It is regarded that $\omega'$ denotes the state of a random environment. The variable $\gamma$ describes the partially observed state of the random environment. The process $b$ is assumed to be completely unobservable.

The random field $\mathbb{H}_n$ is defined by

$$\mathbb{H}_n(\theta) = -\frac{1}{2} \sum_{j=1}^n \left\{ \log \det S(\gamma, X_{t_{j-1}}, \theta) + h^{-1} S(\gamma, X_{t_{j-1}}, \theta)^{-1} \left[ (\Delta_j Y)^{\otimes 2} \right] \right\},$$

where $h = T/n$ and $\Delta_j Y = Y_{t_j} - Y_{t_{j-1}}$. The QMLE and QBE are defined by $\mathbb{H}_n$ as in Section 2.1. The existence of continuous extension of $\mathbb{H}_n$ to $\tilde{\Theta}$ for the QMLE, and the conditions for the prior density $\varpi$ of the QBE are assumed, as before. Moreover, we assume that the boundary of $\Theta$ is good as in Section 5.

Let $\mathcal{C} = \mathcal{F}'$ and denote by $E_C[V](\omega')$ the integral $\int_{\Omega'} V(\omega', \omega'') Q_{\omega'}(d\omega'')$ of a measurable function $V$ on $(\Omega, \mathcal{F})$. Let $S = \sigma \sigma^*$. We will work with the following conditions. Suppose that a $\mathcal{C}$-measurable random variable $K_p : \Omega' \rightarrow \mathbb{R}_+$ is given for every $p \geq 1$.

[D1] (i) $\sup_{t \in [0,T]} E_C[|b_t|^p] \leq K_p$.

(ii) The mapping $(x, \theta) \mapsto \sigma(\gamma, x, \theta)$ is continuously differentiable twice in $x$ and four times in $\theta$ and

$$\sum_{i=0,1,2} \sum_{j=0,1,2,3,4} |\partial^i_x \partial^j_\theta \sigma(\gamma, x, \theta)| \leq K_1(1 + |x|)^{K_1}$$

Furthermore, $\left( \inf_{x, \theta} \det S(\gamma, x, \theta) \right)^{-1} \leq K_1$.

(iii) On each $\mathcal{B}_{\omega'}$, the process $X(\omega', \cdot) = (X_l(\omega', \cdot))_{t \in [0,T]}$ admits a representation

$$X_t = X_0 + \int_0^t \tilde{b}_s ds + \int_0^t \sigma_s dw_s + \int_0^t \tilde{a}_s d\tilde{w}_s,$$

where $\tilde{w} = (\tilde{w}_t)_{t \in [0,T]}$ is an $r_1$-dimensional $\mathbf{F}$-Wiener process independent of $w$, and $\tilde{b} = (\tilde{b}_t)_{t \in [0,T]}$, $a = (a_t)_{t \in [0,T]}$ and $\tilde{a} = (\tilde{a}_t)_{t \in [0,T]}$ are progressively measurable processes taking values in $\mathbb{R}^d$, $\mathbb{R}^d \otimes \mathbb{R}^r$ and $\mathbb{R}^d \otimes \mathbb{R}^{r_1}$, respectively, satisfying

$$E_C[|X_0|^p] + \sup_{t \in [0,T]} (E_C[|\tilde{b}_t|^p] + E_C[|a_t|^p] + E_C[|\tilde{a}_t|^p]) \leq K_p$$

27
for every $p > 1$.

Let
\[
\Psi(\theta) = \frac{1}{2T} \int_0^T \left\{ \log \frac{\det S(\gamma, X_t, \theta)}{\det S(\gamma, X_t, \theta^*)} + \text{Tr} \left( S(\gamma, X_t, \theta)^{-1} S(\gamma, X_t, \theta^*) - I_m \right) \right\} dt.
\]

Define $\chi_0$ by
\[
\chi_0 = \inf_{\theta \neq \theta^*} \frac{-\Psi(\theta)}{\|\theta - \theta^*\|^2}.
\]

[D2] For every $L > 0$,
\[
\sup_{r \in \mathbb{N}} r^LP_c[\chi_0 \leq r^{-1}] < \infty \quad a.s.
\]

**Remark 6.1.** If $\sigma$ does not depend on $\gamma$, then estimation with $\mathbb{H}_n$ needs no information about $\gamma$.

**Remark 6.2.** We do not assume unconditional $L^p$ integrability of the functionals. For example, consider $X_t = \int_0^t e^{B_s^t}X_s ds + \tilde{w}$ and the diffusion coefficient $\sigma(\gamma, X_t, \theta) = \theta \sqrt{1 + X_t^2}$ for a Wiener process $B = (\tilde{B}_t)_{t \in [0, T]}$ living in $\mathcal{C}$. Then $\sigma(\gamma, X_t, \theta)$ is not integrable. This situation is not formally treated in Uchida and Yoshida [26]. We can obtain a limit theorem for the QBE even in such a case.

**Remark 6.3.** An analytic criterion and a geometric criterion for Condition [D2] are provided by [26].

The random matrix $\Gamma$ is defined by
\[
\Gamma = \frac{1}{2T} \int_0^T \text{Tr}((\partial_\theta S)S^{-1}(\partial_\theta S)S^{-1}(\gamma, X_t, \theta^*)) dt.
\]

We are writing $\hat{\theta}_n^M = \hat{\theta}_T$ and $\hat{\theta}_n^B = \tilde{\theta}_T$, and also $\hat{u}_n^M = \hat{u}_T$ and $\hat{u}_n^B = \tilde{u}_T$. We consider an extension $(\Omega, \mathcal{F}, P)$ of $(\Omega, \mathcal{F}, P)$. $\zeta$ denotes a random vector defined on this extension, having the $p$-dimensional standard normal distribution $N_p(0, I_p)$ independent of $\mathcal{F}$.

**Theorem 6.1.** Suppose that [D1] and [D2] are fulfilled. Then, for every $A \in \{M, B\}$, it holds that
\[
E_{\mathcal{C}}[f(\hat{u}_T^A)Y] \rightarrow^P E_{\mathcal{C}}[f(\Gamma^{-1/2}\zeta)Y]
\]
as $T \rightarrow \infty$ for any $\mathcal{F}$-measurable bounded random variable $Y$ and any $f \in C(\mathbb{R}^p)$ of at most polynomial growth. In particular, $\hat{u}_T^A \rightarrow^{d_{\mathcal{F}}} \Gamma^{-1/2}\zeta$ as $T \rightarrow \infty$ for $A = M, B$.

\[\text{More precisely, the processes } \hat{b} = (b_t)_{t \in [0, T]}, a = (a_t)_{t \in [0, T]}, \tilde{a} = (\tilde{a}_t)_{t \in [0, T]} \text{ and } \tilde{w} = (\tilde{w}_t)_{t \in [0, T]} \text{ are measurable mappings defined on } (\Omega, \mathcal{F}), \text{ and for each } \omega' \in \Omega, \text{ the processes } b(\omega', \cdot) = (b_t(\omega', \cdot))_{t \in [0, T]}, a(\omega', \cdot) = (a_t(\omega', \cdot))_{t \in [0, T]}, \tilde{a}(\omega', \cdot) = (\tilde{a}_t(\omega', \cdot))_{t \in [0, T]} \text{ and } \tilde{w}(\omega', \cdot) = (\tilde{w}_t(\omega', \cdot))_{t \in [0, T]} \text{ satisfy the required conditions. } \tilde{w}(\omega', \cdot) \text{ is independent of } w(\omega', \cdot), \text{i.e., } w \text{ and } w \text{ are } \mathcal{C}\text{-conditionally independent, though this independency is not indispensable.} \]
Proof. This result can be proved if we follow the proof of Theorems 4 and 5 of Uchida and Yoshida [26] in their Section 8, with the expectation $E$ replaced by the conditional expectation $E_C$. We omit details. $L_p$-boundedness of functionals are necessary, but it is possible under $E_C$ since the semimartingale structure is assumed under each $\mathcal{B}_t$.

Remark 6.4. Seemingly, we only considered time-independent scenario of the random field $\sigma$ represented by $\gamma$. However, it is possible to consider a time-dependent coefficient $\sigma(t, \gamma, X_t, \theta)$ if we take $(t, X_t)$ for $X_t$. Then, this model includes also the model $\sigma(t, \gamma_t, X_t, \theta)$ having a time-varying component $\gamma = (\gamma_t)$. If we only assume discrete time observations $(\gamma_t)$ of $\gamma$, then some condition for continuity of $\gamma$ would give similar results for the estimators.

References

[1] Clinet, S., Yoshida, N.: Statistical inference for ergodic point processes and application to limit order book. arXiv preprint arXiv:1512.01899 (2015)

[2] Eguchi, S., Masuda, H.: Schwarz type model comparison for laq models. arXiv preprint arXiv:1606.01627 (2016)

[3] Ibragimov, I.A., Has’minskiı̆, R.Z.: The asymptotic behavior of certain statistical estimates in the smooth case. I. Investigation of the likelihood ratio. Teor. Verojatnost. i Primenen. 17, 469–486 (1972)

[4] Ibragimov, I.A., Has’minskiı̆, R.Z.: Asymptotic behavior of certain statistical estimates. II. Limit theorems for a posteriori density and for Bayesian estimates. Teor. Verojatnost. i Primenen. 18, 78–93 (1973)

[5] Ibragimov, I.A., Has’minskiı̆, R.Z.: Statistical estimation, Applications of Mathematics, vol. 16. Springer-Verlag, New York (1981). Asymptotic theory, Translated from the Russian by Samuel Kotz

[6] Kamatani, K., Uchida, M.: Hybrid multi-step estimators for stochastic differential equations based on sampled data. Statistical Inference for Stochastic Processes 18(2), 177–204 (2014)

[7] Kusuoka, S., Yoshida, N.: Malliavin calculus, geometric mixing, and expansion of diffusion functionals. Probab. Theory Related Fields 116(4), 457–484 (2000)

[8] Kutoyants, Y.: Identification of dynamical systems with small noise, Mathematics and its Applications, vol. 300. Kluwer Academic Publishers Group, Dordrecht (1994)

[9] Kutoyants, Y.A.: Parameter estimation for stochastic processes, Research and Exposition in Mathematics, vol. 6. Heldermann Verlag, Berlin (1984). Translated from the Russian and edited by B. L. S. Prakasa Rao

[10] Kutoyants, Y.A.: Statistical inference for spatial Poisson processes, Lecture Notes in Statistics, vol. 134. Springer-Verlag, New York (1998)
[11] Kutoyants, Y.A.: Statistical inference for ergodic diffusion processes. Springer Series in Statistics. Springer-Verlag London Ltd., London (2004)

[12] Masuda, H.: Approximate self-weighted LAD estimation of discretely observed ergodic Ornstein-Uhlenbeck processes. Electronic Journal of Statistics 4, 525–565 (2010)

[13] Masuda, H.: Parametric estimation of lévy processes. In: Lévy Matters IV, pp. 179–286. Springer (2015)

[14] Masuda, H., Shimizu, Y.: Moment convergence in regularized estimation under multiple and mixed-rates asymptotics. Mathematical Methods of Statistics 26(2), 81–110 (2017)

[15] Masuda, H., et al.: Convergence of gaussian quasi-likelihood random fields for ergodic lévy driven sde observed at high frequency. The Annals of Statistics 41(3), 1593–1641 (2013)

[16] Nomura, R., Uchida, M.: Adaptive bayes estimators and hybrid estimators for small diffusion processes based on sampled data. Journal of the Japan Statistical Society 46(2), 129–154 (2016)

[17] Ogihara, T., Yoshida, N.: Quasi-likelihood analysis for the stochastic differential equation with jumps. Stat. Inference Stoch. Process. 14(3), 189–229 (2011). DOI 10.1007/s11203-011-9057-z. URL http://dx.doi.org/10.1007/s11203-011-9057-z

[18] Ogihara, T., Yoshida, N.: Quasi-likelihood analysis for nonsynchronously observed diffusion processes. Stochastic Processes and their Applications 124(9), 2954–3008 (2014)

[19] Ogihara, T., Yoshida, N.: Quasi likelihood analysis of point processes for ultra high frequency data. arXiv preprint arXiv:1512.01619 (2015)

[20] Rio, E.: Asymptotic Theory of Weakly Dependent Random Processes. Springer (2017)

[21] Shimizu, Y.: Threshold estimation for stochastic processes with small noise. arXiv preprint arXiv:1502.07409 (2015)

[22] Shimizu, Y.: Moment convergence of regularized least-squares estimator for linear regression model. Annals of the Institute of Statistical Mathematics 69(5), 1141–1154 (2017)

[23] Uchida, M.: Contrast-based information criterion for ergodic diffusion processes from discrete observations. Ann. Inst. Statist. Math. 62(1), 161–187 (2010). DOI 10.1007/s10463-009-0245-1. URL http://dx.doi.org/10.1007/s10463-009-0245-1

[24] Uchida, M., Yoshida, N.: Adaptive estimation of an ergodic diffusion process based on sampled data. Stochastic Process. Appl. 122(8), 2885–2924 (2012). DOI 10.1016/j.spa.2012.04.001. URL http://dx.doi.org/10.1016/j.spa.2012.04.001

[25] Uchida, M., Yoshida, N.: Quasi likelihood analysis of volatility and nondegeneracy of statistical random field. Stochastic Process. Appl. 123(7), 2851–2876 (2013). DOI 10.1016/j.spa.2013.04.008. URL http://dx.doi.org/10.1016/j.spa.2013.04.008
[26] Uchida, M., Yoshida, N.: Quasi likelihood analysis of volatility and nondegeneracy of statistical random field. Stochastic Processes and their Applications 123(7), 2851–2876 (2013)

[27] Uchida, M., Yoshida, N.: Adaptive bayes type estimators of ergodic diffusion processes from discrete observations. Statistical Inference for Stochastic Processes 17(2), 181–219 (2014)

[28] Umezu, Y., Shimizu, Y., Masuda, H., Ninomiya, Y.: Aic for non-concave penalized likelihood method. arXiv preprint arXiv:1509.01688 (2015)

[29] Yoshida, N.: Partial mixing and Edgeworth expansion. Probability Theory and Related Fields 129(4), 559–624 (2004)

[30] Yoshida, N.: Polynomial type large deviation inequalities and quasi-likelihood analysis for stochastic differential equations. Annals of the Institute of Statistical Mathematics 63(3), 431–479 (2011)

[31] Yoshida, N.: Asymptotic expansion in quasi likelihood analysis for volatility. Preprint (2017)