Massless spinning particle and null-string on $AdS_d$: projective-space approach

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Abstract

Massless spinning particle and tensionless string models on $AdS_d$ background in the projective-space realization are proposed as constrained Hamiltonian systems. Various forms of particle and string Lagrangians are derived and classical mechanics is studied including the Lax-type representation of the equations of motion. After that transition to the quantum theory is discussed. Analysis of potential anomalies in the tensionless string model necessitates introduction of ghosts and BRST charge. It is shown that quantum BRST charge is nilpotent for any $d$ if coordinate-momentum ordering for the phase-space bosonic variables, Weyl ordering for the fermions and $cb$ $(\gamma\beta)$ ordering for ghosts is chosen, while conformal reparametrizations and space-time dilatations turn out to be anomalous for the ordering in terms of positive and negative Fourier modes of the phase-space variables and ghosts.

1 Introduction

The study of field dynamics in (anti-)de Sitter and Minkowski spaces viewed as the manifolds embedded into flat space-times with extra dimension(s) was initiated in the seminal works of Dirac [1]. In the context of point-particle mechanics the embedding of $d$–dimensional anti-de Sitter space into $(d + 1)$–dimensional flat space-time with metric signature $(2, d - 1)$ parametrized by the homogeneous coordinates on which the $SO(2, d - 1)$ isometry group transformations act linearly was considered in Ref. [2]. It is also possible to consider particle and string models [3], [4] in $(d + 2)$–dimensional space with linearly realized $SO(2, d)$ transformations, which describe propagation on either $d$–dimensional Minkowski or anti-de Sitter space depending on the conditions imposed to partially fix extended gauge symmetries. To examine conformal field theories widely used is a realization of Minkowski (Euclidean) space as a projective light-cone in the space-time with two extra dimensions [5], [6]. Massless (spinning) particle and null string models on the projective light-cone (named conformal space there) were considered in [7], [8], [9]. More recently projective light-cone (embedding space) approach has been applied to obtain correlation functions in $d$-dimensional conformal field theories taking advantage of the AdS/CFT inspired techniques [10], [11], [12], as well as to study higher-spin field dynamics on $AdS_d$ and its conformal boundary [13], [14]. In [15] there was considered the possibility of applying the twistor methods to the $AdS/CFT$ duality relying on the projective-space description of $AdS_d$ that naturally combines linear realization of $SO(2, d - 1)$ isometry with the projective light-cone description of the $(d - 1)$-dimensional conformal boundary.

Utility of the projective-space realization of anti-de Sitter space from the viewpoint of the canonical description of massless particle (tensionless string) mechanics can be illustrated as follows. Description of $AdS_d$ as an embedded hyperboloid assumes imposition of the constraint $x^2 + 1 \approx 0$ on the ambient-space coordinates $x^\mu$, where the $SO(2, d - 1)$-invariant scalar product $x^2 = (x \cdot x) = x^\mu\eta_{\mu\nu}x^\nu$ is taken using Minkowski

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metric \( \eta_{mn} = \text{diag}(-,+,\ldots,+,+) \) and \( AdS_d \) radius is set to unity. In the canonical approach the mass-shell constraint for the massless particle (tensionless string zero modes) in its simplest form is \( p^2 \approx 0 \) and its Poisson bracket (P.B.) relations with the above constraint imply that \( (x \cdot p) \approx 0 \) is also a constraint forming with \( x^2 + 1 \approx 0 \) the pair of the second-class constraints. Quantization in the presence of the second-class constraints necessitates introduction of the Dirac brackets (D.B.) that in general essentially complicates further analysis so it is convenient to consider \( x^2 + 1 \approx 0 \) as a gauge-fixing condition for the constraint \( (x \cdot p) \approx 0 \) that generates dilatations of the embedding-space coordinates \([16]\). But gauging dilatations precisely implements the projective-space realization of \( AdS_d \) so the set of two first-class constraints \( (x \cdot p) \approx 0 \) and \( p^2 \approx 0 \) can be taken as the starting point for describing massless particle (tensionless string zero modes) Hamiltonian on \( AdS_d \) in such an approach.

In the Lagrangian formalism important feature of the projective-space description of \( AdS_d \) is that the object that can be identified with the metric, taking into account the form of the line element,

\[
ds^2 = \frac{1}{|x|^2} dx^m \theta_{mn} dx^n, \quad \theta_{mn} = \eta_{mn} + \frac{1}{|x|^2} x_m x_n, \tag{1.1}
\]
is degenerate \( \text{det} \theta = 0 \). It can be easily shown to have the eigenvector \( x^m \) with zero eigenvalue \( \theta_{mn} x^m = 0 \). So in this approach one is led to consider particle (string, brane) mechanics in the space with degenerate metric. Hamiltonian mechanics of particle and string models in curved spaces with degenerate metrics has been previously treated in \([17]\), though the results appear to strongly depend on the particular form of the metric indicating favorability of the case by case study. In the higher-spin theory on \( AdS_d \) \( \theta_{mn} \) is known as a projection operator that enters field equations in the embedding-space formulation \([18]\), \([19]\), \([20]\), \([21]\).

Here we adhere to the bottom-up approach (see, e.g., Ref. \([22]\))): starting with the world-line (world-sheet) supersymmetry generator that is the classical analogue of the Dirac-type equation on \( AdS_d \) in the projective-space formulation we seek to close the P.B. (D.B.) algebra of the constraints that includes the generators of the world-line (world-sheet) reparametrizations and space-time dilatations and then proceed to write down corresponding phase-space Lagrangian based on the (weakly vanishing) Hamiltonian. Upon integrating out the space-time momentum and the Lagrange multipliers for the first-class constraints this Lagrangian can further be expressed in terms of the configuration-space variables in various forms.

In the next section the approach outlined above is applied to the spinning particle on the \( AdS_d \) space-time in the projective-space formulation. After suggesting closed algebra of the first-class constraints including the generators of \( 1d \) supersymmetry, world-line reparametrizations and space-time dilatations we discuss various representations of the particle Lagrangian. Further it is considered quantum realization of the classical constraint algebra and obtained are the first-order (Dirac-type) and second-order (Klein-Gordon-type) equations for the wave function of the spinning particle. Section 3 is devoted to the study of the null string on \( AdS_d \). Compatibility with the world-sheet conformal reparametrizations essentially restricts the form of admissible constraints. We have found closed classical algebra for the quadratic first-class constraints which include world-sheet supersymmetry, space-time dilatations and Virasoro generators and examine various representations of the associated null string action. Then we discuss two quantum realizations of the constraint algebra: one corresponding to the generalization of the coordinate-momentum ordering and another in terms of positive and negative frequency modes of the operators associated with the classical phase-space variables. The former realization is anomaly free for any dimension of the anti-de Sitter space-time, while for the latter dilatations and conformal reparametriza-
tions are anomalous. Complete values of these anomalies are computed in the framework of the BRST approach.

2 Spinning particle

In what follows we consider $AdS_d$ space-time to be parametrized by $d + 1$ homogeneous coordinates $x^m$ ($m = 0, \ldots, d$), for which the canonical momenta are $p_m$. Spinning degrees of freedom of the particle are described by the $SO(2, d-1)$ vector $\xi^m$ with the Grassmann-odd components.

2.1 Classical mechanics

We start with the classical analogue of the Dirac equation

$$\Phi = |x|(\xi \cdot p) \approx 0, \quad (2.1)$$

where $|x| = \sqrt{-x^2}$ and $(\xi \cdot p) = \xi^m p_m$ are the $SO(2, d-1)$-invariant norm and scalar product in the embedding space. With the standard definition of the P.B. and D.B. relations

$$\{p_m, x^n\}_{P.B.} = \delta^n_m, \quad \{\xi^m, \xi^n\}_{D.B.} = i\eta^{mn} \quad (2.2)$$

it is possible to obtain the closed algebra of three constraints: odd $\Phi$ and bosonic $T$ and $D$

$$T = |x|^2 p^2 + 2i(\xi \cdot x)(\xi \cdot p) \approx 0, \quad D = (x \cdot p) \approx 0. \quad (2.3)$$

The only non-zero D.B. relation of this algebra is

$$\{\Phi, \Phi\}_{D.B.} = iT \quad (2.4)$$

so that it can be recognized as a 1-dimensional supersymmetry algebra. In the bosonic limit our constraint algebra reduces to that spanned by $D \approx 0$ and $T \approx 0$ [16].

This allows to write the spinning particle action in terms of the phase-space variables

$$S = \int d\tau \mathcal{L}_{ph}, \quad \mathcal{L}_{ph} = (p \cdot \dot{x}) + \frac{i}{2}(\xi \cdot \dot{\xi}) - \frac{e}{2}T + aD + i\chi \Phi, \quad (2.5)$$

where the constraints (2.1) and (2.3) are introduced with the Lagrange multipliers $e$ and $a$ which are even and $\chi$ which is odd. Integrating out the momentum $p_m$ yields configuration-space form of the particle’s Lagrangian

$$\mathcal{L}_c = \frac{1}{2e|x|^2}(\dot{x} + ax)^2 + \frac{i}{2}(\xi \cdot \dot{\xi}) - \frac{i}{|x|^2}(\xi \cdot x)(\xi \cdot \dot{x}) + \frac{i\chi}{e|x|^2}\xi \cdot (\dot{x} + ax). \quad (2.6)$$

The form of the Lagrangian $\mathcal{L}_c$ suggests that the Lagrange multiplier $a$ plays the role of the 1d gauge field for the scale transformations of $x$ and $p$. The non-supersymmetric Lagrangian coincides with that of the conformal particle model of Ref. [2]. Further integrating out $a$ one arrives at the Lagrangian

$$\mathcal{L}_{RP^d} = \frac{1}{2e|x|^2}(\dot{x} \theta \dot{\theta} + \frac{i}{2}(\xi \cdot \dot{\xi}) - \frac{i}{|x|^2}(\xi \cdot x)(\xi \cdot \dot{x}) + \frac{i\chi}{e|x|^2}(\xi \theta \dot{x}), \quad (2.7)$$

where $\theta^{\underline{m}} = \eta^{\underline{m}} + \frac{1}{|x|^2}x^m x^m$ is the degenerate metric tensor corresponding to the realization of $AdS_d$ as the projective space $RP^d$ parametrized by the homogeneous coordinates.
In the gauge $e = 1$, $\chi = 0$ spinning particle equations

$$
\frac{d}{dr} \left( \frac{1}{|x|}(\theta \dot{x})^m - \frac{i}{|x|}(\xi \cdot x)\xi^m \right) - \frac{1}{|x|^2}(\dot{x}\theta \dot{x}) x^m - \frac{i}{|x|^2}(\xi \theta \dot{x})\xi^m
- \frac{2i}{|x|^2}(\xi \cdot \dot{x})(\xi \cdot x)x^m = 0,
$$

(2.8)

$$
\dot{\xi}^m + \frac{1}{|x|^2}(\xi \cdot x)\dot{x}^m - \frac{i}{|x|^2}(\xi \cdot \dot{x})x^m = 0
$$

(2.9)

admit Lax representation

$$
\dot{L}_r - M_r L_r = \dot{\xi} - M_r \xi = 0
$$

(2.10)

with

$$
M^m_{\tau n} = \frac{1}{|x|^2}(x^m \dot{\xi}^n - \dot{x}^m x^n) - \frac{i}{|x|^2}(\xi \cdot x)(x^m \xi^n - \xi^m x^n) - i\xi^m \xi^n.
$$

(2.11)

Observe that $M^m_{\tau m}$ coincide up to the sign with the $SO(2, d - 1)$ generators. The first two summands in the gauge $e = 1$, $\chi = 0$ equal $x^m p^m - x^m p^m$ and give the orbital part of the $SO(2, d - 1)$ generators. The last summand represents the spin part.

### 2.2 Quantization

Quantization consists in replacing classical observables with the Hermitian operators. Operators associated with the phase-space variables satisfy the (anti)commutation relations\(^2\)

$$
[p^m_{\tau}, x^n] = -i\delta^m_{\tau n}, \quad \{\xi^m, \xi^n\} = \gamma^{mn}.
$$

(2.12)

It is clear that $\xi^m = 2^{-1/2}\gamma^m$ and in what follows we use $\gamma$—matrices in $(d + 1)$ dimensions. Hermiticity of $\xi^m$ is understood in the same sense as that of $\gamma^m$, i.e. $(\gamma^m)^d = (-)^t A \gamma^m A^{-1}$, where $A = \gamma^{01} \gamma^{02} \ldots \gamma^{0t}$ and $t$ is the number of time-like dimensions ($t = 2$ for the ambient space of $AdS_d$).

Hermitian operator corresponding to the classical constraint $\Phi$ (2.1)

$$
\Phi \rightarrow \frac{1}{\sqrt{2}} \Phi_H,
$$

(2.13)

is defined by the expression

$$
\Phi_H = |x|(\gamma \cdot p) + \frac{i}{2|x|}(\gamma \cdot x) \approx 0,
$$

(2.14)

where the last summand comes from moving the momentum operator to the right in the manifestly Hermitian expression $\frac{1}{2}(|x|p^m + p^m |x|)$ and the overall factor of $2^{-1/2}$ has been extracted from $\Phi_H$ to simplify the form of the quantum counterpart of the classical world-line supersymmetry algebra (2.1)

$$
\Phi_H^2 = T_H.
$$

(2.15)

Fulfilment of (2.13) allows to fix unambiguously the form of the Hermitian operator for the bosonic constraint (2.3)\(^3\)

$$
T_H = |x|^2 p^2 + i(\gamma \cdot x)(\gamma \cdot p) + i(\gamma \cdot p) + \frac{2d + 1}{4} \approx 0.
$$

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\(^2\)Hats are not placed over the operator quantities to simplify the notation.

\(^3\)Further discussion of ambiguities in the definition of Hermitian operators in supersymmetric models can be found, e.g., in [23], [24].
For another bosonic constraint Hermitian operator realization is chosen to be
\[
D_H = (x \cdot p) - \frac{i(d + 1)}{2} \approx 0.
\] (2.16)

To apply quantum first-class constraints (2.13), (2.15) and (2.16) to the configuration-space wave function \(\Psi(x)\) the momentum operator has to be realized as the differential operator. For the case of curved configuration space appropriate expression for the Hermitian momentum operator is known to be
\[
p_m = -i(-g)^{-1/4}\partial_m(-g)^{1/4},
\] (2.17)

where \(g\) is the determinant of the configuration space metric tensor. In the projective description of anti-de Sitter space-time scale-invariant measure can be defined as
\[
|x|^{-d-1}\varepsilon_{m_1m_2\cdots m_{d+1}}dx^{m_1}dxd^{m_2} \wedge \cdots \wedge dx^{m_{d+1}}
\]
so as the definition of the Hermitian momentum operator we take
\[
p_m = -i|x|^{d+1/2}\partial_m|x|^{-d+1/2} = -i\left(\partial_m + \frac{(d + 1)}{2|x|^2}x_m\right).
\] (2.18)

As a result the operator (2.13) gives the Dirac-type equation
\[
D_H\Psi(x) = \left(|x| (\gamma \cdot \partial) + \frac{d}{2|x|} (\gamma \cdot x)\right)\Psi(x) = 0,
\] (2.19)

while the quantum bosonic constraints (2.15) and (2.16) in the configuration space acquire the form
\[
T_H = -|x|\partial_m(|x|\partial_m) + (\gamma \cdot x)(\gamma \cdot \partial) - (d + 1)(\gamma \cdot \partial) - \frac{\partial^2}{4} \approx 0, \\
D_H = -i(x \cdot \partial) \approx 0.
\] (2.20)

So that the wave function \(\Psi(x)\) has homogeneity degree zero in \(x^m\) and satisfies the second-order equation
\[
D_H^2\Psi(x) = \left(|x|\partial_m(|x|\partial_m) - (\gamma \cdot x)(\gamma \cdot \partial) + \frac{\partial^2}{4}\right)\Psi(x) = 0.
\] (2.21)

In terms of non-homogeneous embedding coordinates \(y^m = x^m/|x|\) equations (2.19) and (2.21) become
\[
\left((\gamma \cdot \nabla) + \frac{d}{2}(\gamma \cdot y)\right)\Psi(y) = 0, \\
\left((\gamma \cdot \nabla)^2 + \frac{d^2}{4}\right)\Psi(y) = 0,
\] (2.22)

where \(\nabla = \partial_y + y(y \cdot \partial_y), \partial_y = \partial/\partial y\) and \(D_H\) trivializes.

### 3 Null spinning string

#### 3.1 Classical theory

In case of the null string P.B. relations (2.2) generalize to
\[
\{P_m(\sigma), X^n(\sigma')\}_{P.B.} = \delta_m^n \delta(\sigma - \sigma'), \quad \{\Xi^m(\sigma), \Xi^n(\sigma')\}_{D.B.} = i\eta^{mn}\delta(\sigma - \sigma').
\] (3.1)

We concentrate on the closed null string model with the space-like world-sheet coordinate \(\sigma\) ranging from 0 to \(2\pi\). Since extended nature of the null string puts severe restrictions on
and the Lagrange particle model. We can integrate it out to obtain the Lagrangian

$$\Phi(\sigma) = (\Xi \cdot P) \approx 0$$

and

$$T(\sigma) = P^2 \approx 0, \quad -L(\sigma) = (P \cdot \partial_\sigma X) + \frac{i}{2}(\Xi \cdot \partial_\sigma \Xi) \approx 0, \quad D(\sigma) = (X \cdot P) \approx 0$$

that form closed P.B. algebra

$$\{\Phi(\sigma), \Phi(\sigma')\}_{P.B.} = iT(\sigma)\delta(\sigma - \sigma');$$

$$\{\Phi(\sigma), D(\sigma')\}_{P.B.} = \Phi(\sigma)\delta(\sigma - \sigma'),$$

$$\{\Phi(\sigma), L(\sigma')\}_{P.B.} = \frac{2}{3}\Phi(\sigma)\partial_\sigma\delta(\sigma - \sigma') + \partial_\sigma\Phi(\sigma)\delta(\sigma - \sigma');$$

$$\{T(\sigma), D(\sigma')\}_{P.B.} = 2T(\sigma)\delta(\sigma - \sigma'),$$

$$\{T(\sigma), L(\sigma')\}_{P.B.} = 2T(\sigma)\partial_\sigma\delta(\sigma - \sigma') + \partial_\sigma T(\sigma)\delta(\sigma - \sigma'),$$

$$\{D(\sigma), L(\sigma')\}_{P.B.} = D(\sigma)\partial_\sigma\delta(\sigma - \sigma') + \partial_\sigma D(\sigma)\delta(\sigma - \sigma'),$$

$$\{L(\sigma), L(\sigma')\}_{P.B.} = 2L(\sigma)\partial_\sigma\delta(\sigma - \sigma') + \partial_\sigma L(\sigma)\delta(\sigma - \sigma').$$

Note that $L(\sigma)$ can be identified with the Virasoro generator density, $D(\sigma)$ – with the dilatation generator density and the algebra (3.4) – with the semidirect sum of generators of dilatations, 2$d$ supersymmetry and Virasoro transformations.

### 3.1.1 Various representations of null string Lagrangian

Now we can write down the null-string Lagrangian and the action in terms of the phase-space variables

$$S = \int d\tau d\sigma \mathcal{L}_{ph}(\tau, \sigma), \quad \mathcal{L}_{ph}(\tau, \sigma) = (P \cdot \partial_\tau X) + \frac{i}{2}(\Xi \cdot \partial_\tau \Xi) - \frac{e}{2}T - vL + aD + i\chi \Phi, \quad (3.5)$$

where $a$, $e$ and $v$ are even Lagrange multipliers for the bosonic constraints and $\chi$ is an odd Lagrange multiplier for the fermionic constraint $\Phi$. Integrating out the momentum $P^m(\tau, \sigma)$ gives configuration-space Lagrangian

$$\mathcal{L}_c = \frac{1}{2e}((\partial_\tau X^m + v\partial_\sigma X^m + aX^m)^2 + \frac{i}{2}(\Xi \cdot \partial_\tau \Xi) + \frac{iv}{2}(\Xi \cdot \partial_\sigma \Xi))$$

$$+ \frac{iv}{\xi} \Xi^m(\partial_\tau X^m + v\partial_\sigma X^m + aX^m).$$

(3.6)

This Lagrangian can be written in the manifestly 2$d$–covariant form

$$\mathcal{L}_{2dcov} = \frac{1}{2}(\rho^m\partial_\mu X^m + aX^m)^2 + \frac{1}{2}(\Xi \cdot \rho^m\partial_\mu \Xi) + i\chi \Xi^m(\rho^m\partial_\mu X^m + aX^m)$$

(3.7)

by introducing $\rho^m = e^{-1/2}(1, v)$, $\mu = (\tau, \sigma)$ and redefining vector $\Xi^m$ and the Lagrange multipliers $a$ and $\chi$

$$\Xi^m \rightarrow e^{-1/4}\Xi^m, \quad a \rightarrow e^{1/2}a, \quad \chi \rightarrow e^{3/4}\chi.$$  

(3.8)

Configuration-space form of the null string Lagrangian makes obvious the role of the Lagrange multiplier $a$ as a gauge field for the scaling symmetry similarly to the spinning particle model. We can integrate it out to obtain the Lagrangian

$$\mathcal{L}_{RPd} = \frac{1}{2}(\rho^m\partial_\mu X^m \theta \partial_\mu X + \frac{i}{2}(\Xi \cdot \rho^m\partial_\mu \Xi) + i\chi(\Xi \theta \rho^m\partial_\mu X)$$

(3.9)
that manifests projective space realization of the $AdS_d$ space-time and is the generalization of the conformal null spinning string Lagrangian of Ref. 9 (see also 8). This Lagrangian can be rewritten in yet other forms that exhibit dependence on the world-sheet vector density $\rho^\mu$
\[
\mathcal{L}_{RP^d} = \frac{1}{2}(\theta^{mn}\rho^\mu\partial_\mu X_n + i\chi\Xi^m)^2 + \frac{1}{2}(\Xi \cdot \rho^\mu \partial_\mu \Xi)
\]

(3.10)

where $g_{\mu\nu} = (\partial_\mu X \partial_\nu X)$ is the induced world-sheet metric and $\zeta_\mu = \frac{i}{2}(\Xi \cdot \partial_\mu \Xi) + i\chi(\Xi \partial_\mu X)$. It follows that the $SO(2, d - 1)$ vector $\theta^{mn}\rho^\mu\partial_\mu X_n + i\chi\Xi^m$ is null, whereas equations for $\rho^\mu$
\[
g_{\mu\nu}\rho^\nu + \zeta_\mu = 0
\]

(3.11)

imply that induced world-sheet metric is non-degenerate (see [25]). Thus Eq. (3.11) has the unique solution $\rho^\mu = -g^{-1}\mu\nu\zeta_\nu$ and its substitution back into (3.10) yields non-polynomial form of the null spinning string Lagrangian
\[
\mathcal{L}'_{RP^d} = \frac{1}{2}g_{\mu\nu}\epsilon^{\mu\nu}g_{\mu\nu}\epsilon^\rho \zeta_\rho, \quad g = \det g_{\mu\nu}.
\]

(3.12)

This is in contrast with the bosonic null string for which the equation
\[
g_{\mu\nu}\rho^\nu = 0
\]

(3.13)

implies that $g = 0$. Solving (3.13) under assumption that $g_{\sigma\sigma} \neq 0$ and $\rho^\sigma \neq 0$ and substituting the solution into the bosonic null string Lagrangian
\[
\mathcal{L}_{bosnull} = \rho^\mu \rho^\nu g_{\mu\nu}
\]

(3.14)

brings it to the form
\[
\mathcal{L}_{bosnull} = \frac{g}{E}, \quad E = \frac{g_{\sigma\sigma}}{(\rho^\sigma)^2}
\]

(3.15)

which directly generalizes massless particle Lagrangian and for the case of the null string in flat space-time was examined in [26], [27].

3.1.2 Equations of motion and gauge conditions

Linear equations for the null-string phase-space variables and the constraints are obtained from the variation of the first-order action (3.5)
\[
\dot{X}^m + v\dot{X}^m + aX^m + i\chi\Xi^m = eP^m,
\]
\[
\dot{P}^m + (vP^m)' = aP^m,
\]
\[
\dot{\Xi}^m + v\dot{\Xi}^m + \frac{1}{2}i\Xi^m = \chi P^m
\]

(3.16)

Above equations simplify if one takes into account gauge symmetries of the action (3.5) generated by the first-class constraints (3.2) and (3.3). Fermionic constraint (3.2) is responsible for local world-sheet supersymmetry
\[
\delta_\varepsilon X^m = -\varepsilon\Xi^m, \quad \delta_\varepsilon\Xi^m = -i\varepsilon P^m, \quad \delta_\varepsilon e = -2\varepsilon\chi, \quad \delta_\varepsilon\chi = -i\left(\dot{\varepsilon} + v\varepsilon + a\varepsilon - \frac{1}{2}\dot{\varepsilon}\varepsilon\right),
\]

(3.17)
where $\varepsilon(\tau,\sigma)$ is the odd parameter. Bosonic constraint $D(\sigma)$ generates local dilations with the parameter $\Delta(\tau,\sigma)$

$$\delta_\Delta X^m = -\Delta X^m, \quad \delta_\Delta P^m = \Delta P^m, \quad \delta_\Delta a = \dot{\Delta} + v \dot{\Delta}, \quad \delta_\Delta e = -2\Delta e, \quad \delta_\Delta \chi = -\Delta \chi. \quad (3.18)$$

$T(\sigma)$ and $L(\sigma)$ generate time-like and space-like world-sheet reparametrizations with parameters $\mu(\tau,\sigma)$ and $\lambda(\tau,\sigma)$

$$\delta_\mu X^m = -2\mu P^m, \quad \delta_\mu e = -2(\mu + v\dot{\mu} + 2a\mu - \dot{e}\mu) \quad (3.19)$$

and

$$\delta_\lambda X^m = \lambda \dot{X}^m, \quad \delta_\lambda P^m = (\lambda P^m)', \quad \delta_\lambda \Xi^m = \lambda \Xi^m + \frac{1}{2} \dot{\lambda} \Xi^m, \quad \delta_\lambda a = \lambda \dot{a}, \quad \delta_\lambda e = \lambda e - \dot{\lambda} e, \quad \delta_\lambda v = -\dot{\lambda} - \dot{\lambda} v + \lambda \dot{v}, \quad \delta_\lambda \chi = \lambda \dot{\chi} - \frac{1}{2} \dot{\lambda} \chi. \quad (3.20)$$

Below we consider equations (3.16) in a number of widely used gauges. In all these gauges Lagrange multipliers $a$ and $\chi$ are set to zero.

Gauge 1 (particle-like) : $v = 0, \quad e = 1. \quad (3.21)$

In this gauge e.o.m. reduce to

$$\dot{X}^m = P^m, \quad \dot{P}^m = \dot{\Xi}^m = 0. \quad (3.22)$$

Their solutions are

$$X^m(\tau,\sigma) = X^m_0(\sigma) + \tau P^m_0(\sigma), \quad P^m(\tau,\sigma) = P^m_0(\sigma), \quad \Xi^m(\tau,\sigma) = \Xi^m_0(\sigma), \quad (3.23)$$

where $X^m_0(\sigma), P^m_0(\sigma)$ and $\Xi^m_0(\sigma)$ represent initial data. For the closed null string $X^m_0(\sigma)$ and $P^m_0(\sigma)$ are sigma-periodic $\sigma \simeq \sigma + 2\pi$, while $\Xi^m_0(\sigma)$ can also be antiperiodic to account for the Neveu-Schwarz (NS) sector states. In this gauge constraints (3.2) and (3.3) acquire the on-shell form

$$T(\tau,\sigma) = T_0(\sigma) \approx 0, \quad D(\tau,\sigma) = D_0(\sigma) + \tau T_0(\sigma) \approx 0, \quad L(\tau,\sigma) = L_0(\sigma) - \frac{1}{2} \tau \dot{T}_0(\sigma) \approx 0; \quad \Phi(\tau,\sigma) = \Phi_0(\sigma) \approx 0, \quad (3.24)$$

where

$$T_0(\sigma) = P_0^2 \approx 0, \quad D_0(\sigma) = (X_0 \cdot P_0) \approx 0, \quad -L_0(\sigma) = (P_0 \cdot \dot{X}_0) + \frac{1}{2} (\Xi_0 \cdot \dot{\Xi}_0) \approx 0; \quad \Phi_0(\sigma) = (P_0 \cdot \Xi_0) \approx 0 \quad (3.25)$$

represent initial data on the constraint surface. Eq. (3.24) illustrates the fact that time evolution does not bring the constraints out of the constraint surface defined by the above initial data.

Gauge 2 (static) : $v = 0, \quad e = 0. \quad (3.26)$

In this gauge e.o.m. trivialize $\dot{X}^m = \dot{P}^m = \dot{\Xi}^m = 0$ so that the solution is defined by the null string initial profile

$$X^m(\tau,\sigma) = X^m_0(\sigma), \quad P^m(\tau,\sigma) = P^m_0(\sigma), \quad \Xi^m(\tau,\sigma) = \Xi^m_0(\sigma) \quad (3.27)$$
and the constraints are 'frozen'

\[ T(\tau, \sigma) = T_0(\sigma) \approx 0, \quad D(\tau, \sigma) = D_0(\sigma) \approx 0, \quad L(\tau, \sigma) = L_0(\sigma) \approx 0; \quad \Phi(\tau, \sigma) = \Phi_0(\sigma) \approx 0 \]  \hspace{1cm} (3.28)

with the initial data on the constraint surface defined in (3.25).

Next consider two gauges for which Lagrange multiplier \( v \) is a non-zero constant.

Gauge 3 (string-like) : \( v = -1, \quad e = 2 \). \hspace{1cm} (3.29)

E.o.m. acquire the string-like form

\[ \partial_- X^m = P^m, \quad \partial_- P^m = 0, \quad \partial_- \Xi^m = 0, \]  \hspace{1cm} (3.30)

with \( \partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma) \), and their solution has the form similar to that in the particle-like gauge

\[ X^m(\tau, \sigma) = X^m_0(\sigma^+) + \sigma^- P^m_0(\sigma^+), \quad P^m(\tau, \sigma) = P^m_0(\sigma^+), \quad \Xi^m(\tau, \sigma) = \Xi^m_0(\sigma^+), \]  \hspace{1cm} (3.31)

where \( \sigma^- = \tau - \sigma \) plays the role of the world-sheet time-like variable and \( \sigma^+ = \tau + \sigma \) of the space-like variable. Superficially in this gauge \( \sigma \)-periodicity is lost but looking at the constraints, whose modes in quantum theory are used to identify physical subspace of the state space, it is clear that they are weakly periodic

\[ T(\tau, \sigma) = T_0(\sigma^+) \approx 0, \quad D(\tau, \sigma) = D_0(\sigma^+) + \sigma^- T_0(\sigma^+) \approx 0, \]
\[ L(\tau, \sigma) = L_0(\sigma^+) + T_0(\sigma^+) - \frac{1}{2} \sigma^-(T_0) \approx 0; \]
\[ \Phi(\tau, \sigma) = \Phi_0(\sigma^+) \approx 0. \]  \hspace{1cm} (3.32)

In (3.32)

\[ T_0(\sigma^+) = P^2_0 \approx 0, \quad D_0(\sigma^+) = (X_0 \cdot P_0) \approx 0, \quad -L_0(\sigma^+) = (P_0 \cdot \dot{X}_0) + \frac{i}{2}(\Xi_0 \cdot \dot{\Xi}_0) \approx 0; \]
\[ \Phi_0(\sigma^+) = (P_0 \cdot \dot{\Xi}_0) \approx 0 \]  \hspace{1cm} (3.33)

define initial data on the constraint surface similarly to (3.25) and \( ' \) stands for the differentiation w.r.t. the function’s argument.

Yet another gauge is

Gauge 4 (CFT-like) : \( v = -1, \quad e = 0 \). \hspace{1cm} (3.34)

In this gauge e.o.m. yield that

\[ X^m(\tau, \sigma) = X^m_0(\sigma^+), \quad P^m(\tau, \sigma) = P^m_0(\sigma^+), \quad \Xi^m(\tau, \sigma) = \Xi^m_0(\sigma^+) \]  \hspace{1cm} (3.35)

in analogy with the family of the world-sheet \( \beta \gamma \) CFT’s with the action (see, e.g., [28])

\[ S_{\beta \gamma} = \int d\tau d\sigma \beta \partial_\gamma \]  \hspace{1cm} (3.36)

and \( \psi \) CFT defined by

\[ S_\psi = i \int d\tau d\sigma \psi \partial_\psi. \]  \hspace{1cm} (3.37)
Constraints are defined by their initial values (3.33), in particular $L_0$ has the form of the sum of the stress-tensor of $\beta\gamma$ CFT with conformal weights $w_\beta = 1$, $w_\gamma = 0$ and $\psi$ CFT with $w_\psi = 1/2$.

As a concluding remark of this subsection note that the null-string action corresponding to the second-order Lagrangian (3.9) is a convenient framework for discussing integrability of the e.o.m. In the gauge $\rho^x = 1$, $\rho^\sigma = \chi = 0$ equations for the space-time coordinates admit Lax representation similar to that of the spinning particle (2.9), (2.10)

$$\frac{d}{d\tau} L^m_T - M^m_T L^m_T = 0; \quad L^m_T = \frac{1}{|X|} (\theta X)^m_T, \quad M^m_{mn} = \frac{1}{|X|^2} (X^m X^n - \dot{X}^m X^n).$$

(3.38)

### 3.1.3 Fourier expansions of the phase-space variables and constraints

In all of the above gauges initial data for the phase-space variables are taken $\sigma -$ periodic and can be Fourier expanded \(^4\)

$$P^{0m}_0(\sigma) = \frac{i}{\sqrt{2\pi}} \sum_m e^{im\sigma} P^m_0, \quad X^{0m}_0(\sigma) = \frac{1}{\sqrt{2\pi}} \sum_m e^{im\sigma} x^m_0, \quad \Xi^{0m}_0(\sigma) = \frac{1}{\sqrt{2\pi}} \sum_{m \in \mathbb{Z}+\nu} e^{im\sigma} \xi^m_0.$$ 

(3.39)

Fermions are allowed to be either periodic ($\nu = 0$, Ramond (R) sector) or antiperiodic ($\nu = 1/2$, NS sector). Then one can define Fourier modes of the initial data for the constraints. For the Virasoro generator density

$$L_0(\sigma) = -(P_0 \cdot \dot{X}_0) - \frac{i}{2} (\Xi_0 \cdot \dot{\Xi}_0) \approx 0$$

(3.40)

we define

$$L_0(\sigma) = \frac{1}{2\pi} \sum m e^{im\sigma} I^m_0, \quad I^m_0 = I^\text{bos}_m + I^\text{ferm}_m,$$

(3.41)

where the superscript $m$ labels modes of the matter part of the Virasoro generator that in the next section will be augmented by the contributions of ghosts. $L^\text{bos}_m$ and $L^\text{ferm}_m$ read

$$L^\text{bos}_m = \sum_n n (p_{m-n} \cdot x_n), \quad I^\text{ferm}_m = \frac{1}{4} \sum_{n \in \mathbb{Z}+\nu} (2n - m) (\xi_{m-n} \cdot \xi_n).$$

(3.42)

Similarly for the dilatation generator density

$$D_0(\sigma) = (X_0 \cdot P_0) \approx 0$$

(3.43)

we find

$$D_0(\sigma) = \frac{1}{2\pi} \sum m e^{im\sigma} D^m_0, \quad D^m_0 = i \sum_n (p_{m-n} \cdot x_n).$$

(3.44)

For the bosonic constraint

$$T_0(\sigma) = P^2_0 \approx 0$$

(3.45)

Fourier-mode expansion reads

$$T_0(\sigma) = \frac{1}{2\pi} \sum m e^{im\sigma} T^m_0, \quad T^m_0 = - \sum_n (p_{m-n} \cdot p_n).$$

(3.46)

\(^4\)If it is not specified explicitly all sums are over integers ranging from minus to plus infinity.
Finally for the world-sheet supersymmetry generator density
\[ \Phi_0(\sigma) = (P_0 \cdot \Xi_0) \approx 0 \] (3.47)
we have
\[ \Phi_0(\sigma) = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}+\nu} e^{im\sigma} \Phi_m \cdot \Phi_m^\dagger = i \sum_{n \in \mathbb{Z}+\nu} (p_{m-n} \cdot \xi_n). \] (3.48)

### 3.2 Quantum theory

Upon quantization P.B. relations (3.1) become (anti)commutators
\[ [P_m(\sigma), X_n(\sigma')] = -i \delta_{m}^{\dagger} \delta_n \delta(\sigma - \sigma'), \quad \{ \Xi_m(\sigma), \Xi_n(\sigma') \} = \eta^{mn} \delta(\sigma - \sigma') \] (3.49)
which for the modes translate into the relations
\[ [p_m, x_n] = -\delta_m^{\dagger} \delta_{m-n}, \quad \{ \xi_m, \xi_n \} = \eta^{mn} \delta_{m-n}. \] (3.50)

Since operators \( P_m(\sigma), X_m(\sigma) \) and \( \Xi_m(\sigma) \) are assumed to be Hermitian their modes have the following conjugation properties
\[ (p_m)^\dagger = -p_m^{\dagger}, \quad (x_m)^\dagger = x_m^{\dagger}, \quad (\xi_m)^\dagger = \xi_m^{\dagger}. \] (3.51)

In quantum theory P.B. relations of the classical null string constraint algebra (3.4) may acquire anomalous contributions depending on the choice of the vacuum and ordering of the constituent Fourier modes (3.39). Below we discuss two vacua commonly used in the quantization of null strings: one that is annihilated by the momentum conjugate to the space-time coordinates \([29], [30]\) and corresponds to the coordinate-momentum or \(xp\)-ordering, and another – annihilated by the positive-frequency modes of coordinates and momenta \([31]\) that corresponds to placing positive-frequency modes to the right of the negative-frequency ones that is to the \((-+)\)-ordering\(^5\). For each choice of the vacuum for the bosonic variables there is a natural associated vacuum for the fermions and ghosts. Potentially anomalous are (anti)commutation relations between operators for which, using the parlance of the Wick theorem, multiple cross-contractions of the non-commuting constituent operators are possible, in our case of the coordinates and momenta and/or fermionic operators between themselves. Since all the constraints of the null spinning string are quadratic possible double contractions may lead to anomalous contributions. Convenient way of identifying such terms is to consider commutators of the corresponding Fourier modes of the constraints (3.42), (3.44), (3.46) and (3.48). By inspection potentially anomalous appear to be the commutators
\[ [L_m^m, L_m^{\dagger}], \quad [D_m^m, D_m^{\dagger}], \quad [D_{\pm}^m, L_{\pm}^m]. \] (3.52)
To calculate them it is convenient to present Fourier modes of involved generators in the form, in which contributions of the positive and negative modes are written down explicitly.

---

\(^5\)For the recent comparative analysis of those vacua in the context of ambitwistor strings see \([32]\). The relationship between (ambi)twistor strings and the tensionless (limit of) string was also discussed in \([33]\) (see also \([34]\)).
3.2.1 Quantum algebra of the constraints

Consider in detail the contribution of the bosonic modes in (3.42) to the first commutator in (3.52). For positive and negative modes of $L^\text{bos}_{m}$ we get

$$
L^\text{bos}_{m>0} = \sum_{n=1}^{m} n(x_n \cdot p_{m-n}) + \sum_{n=1}^{\infty} (m + n)(x_{m+n} \cdot p_n) - \sum_{n=1}^{\infty} n(x_n \cdot p_{m+n}),
$$

$$
L^\text{bos}_{-m} = - \sum_{n=1}^{m} n(x_n \cdot p_{-(m-n)}) - \sum_{n=1}^{\infty} (m + n)(x_{-(m+n)} \cdot p_n) + \sum_{n=1}^{\infty} n(x_n \cdot p_{-(m+n)}).
$$

(3.53)

These are Hermitian conjugate $(L^\text{bos}_{m})^\dagger = L^\text{bos}_{-m}$ and are free from the ordering ambiguities unlike $L^\text{bos}_{0}$. Hermitian expression for which will emerge as an outcome of the commutator calculation. Ordering violation arises only from commuting finite sums in (3.53) for the $(\pm)$-ordering leading to the anomalous contribution

$$
[L^\text{bos}_{m}, L^\text{bos}_{-m}](\pm)-\text{ordering} = \frac{(d + 1)}{6} m^3 + 2mL^\text{bos}_{0(\pm)}
$$

(3.54)

and $L^\text{bos}_{0(\pm)}$ is defined by the following expression

$$
L^\text{bos}_{0(\pm)} = \sum_{n=1}^{\infty} n((p_n \cdot x_n) - (x_n \cdot p_n)) - \frac{(d + 1)}{12}
$$

(3.55)

which is manifestly Hermitian with the freedom of adding a real constant $-\frac{(d + 1)}{12}$ that has been chosen in accordance with the conventional string-theoretic value for $2(d + 1)$ periodic bosons (see, e.g., [28]). In contrast, for the $xp$-ordering above commutator takes the same form as in the classical theory

$$
[L^\text{bos}_{m}, L^\text{bos}_{-m}](xp)-\text{ordering} = 2mL^\text{bos}_{0xp},
$$

(3.56)

where

$$
L^\text{bos}_{0xp} = \sum_{n=1}^{\infty} n((x_n \cdot p_n) - (x_n \cdot p_n)).
$$

(3.57)

Checking its Hermiticity requires Fourier modes to be commuted producing terms proportional to the infinite sum $\sum_{n=1}^{\infty} n$ that can be given finite value $-1/12$ via the $\zeta$-function regularization conventionally used in string theory. In the final expression these terms cancel out.

Similarly can be calculated other two commutators in (3.52). Making explicit contributions of positive and negative modes in $D^m_{m}$

$$
D^m_{m>0} = i \left( \sum_{n=0}^{m} (x_n \cdot p_{m-n}) + \sum_{n=1}^{\infty} (x_{m+n} \cdot p_n) + \sum_{n=1}^{\infty} (x_n \cdot p_{m+n}) \right),
$$

$$
D^m_{-m} = i \left( \sum_{n=0}^{m} (x_n \cdot p_{-(m-n)}) + \sum_{n=1}^{\infty} (x_{-(m+n)} \cdot p_n) + \sum_{n=1}^{\infty} (x_n \cdot p_{-(m+n)}) \right)
$$

(3.58)

we find $^6$

$$
[D^m_{m}, D^m_{-m}](\pm)-\text{ordering} = (d + 1)m
$$

(3.59)

and

$$
[D^m_{m}, D^m_{-m}](xp)-\text{ordering} = 0.
$$

(3.60)

$^6$Observe that $(D^m_{m})^\dagger = D^m_{-m}$. 

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Commuting further, e.g., $D^m_{m>0}$ with $L^\text{bos}_{-m}$ gives

$$[D^m_{m}, L^\text{bos}_{-m}](---)\text{-ordering} = \frac{i(d + 1)}{2}m^2 + mD^m_{0(---)},$$

where

$$D^m_{0(---)} = i\left((x_0 \cdot p_0) + \sum_{n=1}^{\infty}((x_n \cdot p_n) + (p_n \cdot x_n)) - \frac{(d + 1)}{2}\right)$$

is the Hermitian zero-mode operator. For the $xp$-ordering no anomalous terms arise

$$[D^m_{m}, L^\text{bos}_{-m}](xp)\text{-ordering} = mD^m_{0xp}$$

and $D^m_{0xp}$ equals

$$D^m_{0xp} = i\left((x_0 \cdot p_0) + \sum_{n=1}^{\infty}((x_n \cdot p_n) + (p_n \cdot x_n))\right).$$

Now let us find the contribution to the first commutator in (3.52) of the part of the Virasoro generators (3.41) determined by the fermionic modes (3.42). Precise form of the commutator depends on the periodicity of the fermions. So consider first the Ramond sector. Explicit form of $L^\text{form}_{m}$ in terms of the positive and negative frequency modes is

$$L^\text{form}_{m>0} = \frac{1}{4} \sum_{n=0}^{m} (2n - m)(\xi_{m-n} \cdot \xi_n) - \frac{1}{2} \sum_{n=1}^{\infty} (m + 2n)(\xi_{m+n} \cdot \xi_{-n}),$$

$$L^\text{form}_{-m} = \frac{1}{4} \sum_{n=0}^{m} (m - 2n)(\xi_{-(m-n)} \cdot \xi_{-n}) + \frac{1}{2} \sum_{n=1}^{\infty} (m + 2n)(\xi_{-(m+n)} \cdot \xi_{-n}).$$

Like for bosons their commutator is anomalous for the $(+-)$-ordering

$$[L^\text{form}_{m}, L^\text{form}_{-m}](+-)\text{-ordering} = \frac{(d + 1)}{24}m^3 + 2mL^\text{form}_{0(+-)}$$

with the Hermitian zero-mode operator

$$L^\text{form}_{0(+-)} = \sum_{n=1}^{\infty} n(\xi_{-n} \cdot \xi_n) + \frac{(d + 1)}{24},$$

where the numerical constant $\frac{(d+1)}{24}$ is that prescribed for $(d+1)$ periodic fermions [28]. In contrast there is no anomaly if the Weyl ordering is chosen for $\xi$-modes in $L^\text{form}_{0}$

$$[L^\text{form}_{m}, L^\text{form}_{-m}](\text{Weyl-ordering}) = 2mL^\text{form}_{0W},$$

$$L^\text{form}_{0W} = \frac{1}{2} \sum_{n=1}^{\infty} n ((\xi_{-n} \cdot \xi_n) - (\xi_n \cdot \xi_{-n})).$$

Analogous conclusions hold also in the NS sector. $\xi$-modes here are half-integer-valued so that Fourier modes of the Virasoro generator acquire the form

$$L^\text{form}_{m>0} = \frac{1}{4} \sum_{r=1/2}^{m-1/2} (2r - m)(\xi_{m-r} \cdot \xi_r) - \frac{1}{2} \sum_{r=1/2}^{\infty} (m + 2r)(\xi_{m+r} \cdot \xi_{-r}),$$

$$L^\text{form}_{-m} = \frac{1}{4} \sum_{r=1/2}^{m-1/2} (m - 2r)(\xi_{-(m-r)} \cdot \xi_{-r}) + \frac{1}{2} \sum_{r=1/2}^{\infty} (m + 2r)(\xi_{-(m+r)} \cdot \xi_{r}).$$
The value of the commutator of $L_{m>0}^{\text{ferm NS}}$ and $L_{-m}^{\text{ferm NS}}$ for the (−+)−ordering is the same as in (3.66)

$$\left[ L_{m}^{\text{ferm NS}}, L_{-m}^{\text{ferm NS}} \right]_{(−+)−ordering} = \frac{(d + 1)}{24} m^3 + 2mL_{0}^{\text{ferm NS}}$$

with $L_{0}^{\text{ferm NS}}$ defined by

$$L_{0}^{\text{ferm NS}} = \sum_{r=1/2}^{\infty} r(\xi_{-} \cdot \xi_{r}) - \frac{(d + 1)}{48},$$

where each antiperiodic fermion contributes $1/48$ to the $c$–number term. For the Weyl-ordering similarly to (3.68) we find

$$\left[ L_{m}^{\text{ferm NS}}, L_{-m}^{\text{ferm NS}} \right]_{\text{Weyl-ordering}} = 2mL_{0}^{\text{ferm NS}}$$

and

$$L_{0}^{\text{ferm NS}} = \frac{1}{2} \sum_{r=1/2}^{\infty} r ((\xi_{-} \cdot \xi_{r}) - (\xi_{r} \cdot \xi_{-})).$$

So we come to the preliminary conclusion that no anomalies are observed in the matter sector of the quantum algebra of the constraints for $xp$–ordering (Weyl-ordering for fermions), while for the (−+)−ordering values of the anomalous contributions are the same as for the corresponding world-sheet CFTs [28]. Complete calculation of the anomalies and study of the possibility of their cancelation requires also the contributions of ghosts to be taken into account and should be carried out in the framework of the BRST quantization to which we now turn.

### 3.2.2 BRST quantization

With each of the constraints (3.40), (3.43), (3.45) and (3.47) a canonical pair of ghost and antighost fields is associated that form the triads

$$(L_{0}(\sigma), c^{L}(\sigma), b^{L}(\sigma)), \quad (D_{0}(\sigma), c^{D}(\sigma), b^{D}(\sigma)), \quad (T_{0}(\sigma), c^{T}(\sigma), b^{T}(\sigma)), \quad (\Phi_{0}(\sigma), \gamma(\sigma), \beta(\sigma)).$$

Classically ghosts satisfy the P.B. relations

\[
\{c^{L}(\sigma), b^{L}(\sigma')\}_{P.B.} = \{c^{D}(\sigma), b^{D}(\sigma')\}_{P.B.} = \{c^{T}(\sigma), b^{T}(\sigma')\}_{P.B.} = i\delta(\sigma - \sigma');
\]

\[
\{\gamma(\sigma), \beta(\sigma')\}_{P.B.} = -\delta(\sigma - \sigma').
\]

Ghosts $\gamma(\sigma)$ and $\beta(\sigma)$ associated with the world-sheet supersymmetry generator $\Phi_{0}(\sigma)$ are even, while other ghosts associated with the bosonic constraints are odd, as is the BRST charge

$$\Omega = \int d\sigma \Omega(\sigma), \quad \Omega(\sigma) = c^{L}L^{\text{ext}} + c^{D}D^{\text{ext}} + c^{T}T_{0} + \gamma\Phi^{\text{ext}}$$

such that $\{\Omega, \Omega\}_{P.B.} = 0$. The density of the BRST charge in (3.77) has been presented in convenient form as the sum of the constraints (3.40), (3.43), (3.45), (3.47) extended by the ghost contributions and multiplied by the associated ghost fields

$$L^{\text{ext}}(\sigma) = L_{0} + \frac{1}{2}L^{ghL} + L^{ghT} + L^{ghD} + L^{gh\Phi};$$

$$L^{ghL}(\sigma) = 2i(c^{L}(T)b^{L}(T))' - i\xi^{L}(T)\gamma^{L}(T), \quad L^{ghD}(\sigma) = i\xi^{D}b^{D}, \quad L^{gh\Phi}(\sigma) = \gamma\beta - \frac{3}{2}(\gamma\beta)'$$

(3.78)
and
\[ D^\text{ext}(\sigma) = D_0 - 2i\epsilon^T b^T + \gamma/\beta, \]
(3.79)
as well as
\[ \Phi^\text{ext}(\sigma) = \Phi_0 - \frac{1}{2} \gamma b^T. \]
(3.80)
Observe certain arbitrariness in the definition of extended gauge generators.

By appropriate choice of the gauge fermion, for which P.B. relations with the BRST charge give the BRST Hamiltonian, it is possible to make ghost fields satisfy the same equations as the matter fields do for each of the considered gauge conditions (3.21), (3.26), (3.29) and (3.34). Besides that they have the same periodicity as the associated constraints and admit Fourier expansions similar to those of the matter fields (3.39)

\[ c(\sigma) = \frac{1}{\sqrt{2\pi}} \sum_m e^{i\sigma} c_m, \quad b(\sigma) = \frac{1}{\sqrt{2\pi}} \sum_m e^{i\sigma} b_m, \]
\[ \gamma(\sigma) = \frac{1}{\sqrt{2\pi}} \sum_m e^{i\sigma} \gamma_m, \quad \beta(\sigma) = \frac{1}{\sqrt{2\pi}} \sum_m e^{i\sigma} \beta_m, \]
(3.81)
where \( c \) and \( b \) stand for any of the odd ghosts and both R and NS periodicity conditions for even ghost fields \( \gamma \) and \( \beta \) are taken into account. This enables one to write the BRST charge in terms of modes

\[ \Omega = c_0 L_0^\text{ext} + \sum_{m=1}^\infty (c_m L_m^\text{ext} + c_{-m} L_m^\text{ext}) + c_0^D D_0^\text{ext} + \sum_{m=1}^\infty (c_m^D D_{-m}^\text{ext} + c_{-m}^D D_m^\text{ext}) + c_0^T T_0^\text{ext} + \sum_{m=1}^\infty (c_m^T T_m^\text{ext} + c_{-m}^T T_m^\text{ext}) + \delta_{e,0} \alpha_0 \Phi^\text{ext} + \sum_{m=1}^\infty (\gamma_m \Phi_m^\text{ext} + \gamma_{-m} \Phi_m^\text{ext}). \]
(3.82)

Fourier modes of the extended constraints are given by the sums of contributions of the matter modes and ghosts. For instance, modes of the extended Virasoro generator (3.78) can be presented in the form

\[ L_m^\text{ext} = L_m + \frac{1}{2} L_m^\text{gh L} + L_m^\text{gh T} + L_m^\text{gh D} + L_m^\text{gh} \Phi. \]
(3.83)
For the matter part (3.41) expressions in terms of positive and negative frequency modes have already been found in (3.33), (3.65) and (3.70). Contributions of \((c^T, b^T)\) and \((c^D, b^D)\) ghost pairs have the form

\[ L_{m}^{\text{gh T}} = -\sum_{n=0}^{m} (m+n)c_{-n}^{T} b_{m-n}^{T} - \sum_{n=1}^{\infty} (2m+n)c_{-n}^{T} b_{m-n}^{T} - \sum_{n=0}^{m} (m-n)c_{n}^{T} b_{m+n}^{T}, \]
\[ L_{-m}^{\text{gh T}} = \sum_{n=0}^{m} (m+n)c_{n}^{T} b_{m-n}^{T} + \sum_{n=1}^{\infty} (2m+n)c_{n}^{T} b_{m-n}^{T} + \sum_{n=0}^{m} (m-n)c_{-n}^{T} b_{m+n}^{T}, \]
(3.84)
and

\[ L_{m}^{\text{gh D}} = -\sum_{n=1}^{m} nc_{n}^{D} b_{m-n}^{D} - \sum_{n=1}^{\infty} (m+n)c_{n}^{D} b_{m-n}^{D} + \sum_{n=1}^{\infty} nc_{-n}^{D} b_{m+n}^{D}, \]
\[ L_{-m}^{\text{gh D}} = \sum_{n=1}^{m} nc_{-n}^{D} b_{m-n}^{D} + \sum_{n=1}^{\infty} (m+n)c_{-n}^{D} b_{m-n}^{D} - \sum_{n=1}^{\infty} nc_{-n}^{D} b_{m+n}^{D}. \]
(3.85)
For the Virasoro ghosts \((c^L, b^L)\) the contribution of \(\frac{1}{2} L_m^{\text{gh L}}\) is the same as that of \((c^D, b^D)\) ghosts above since \(L_m^\text{ext}\) is multiplied by \(c_m^L\) in the BRST charge. For the bosonic ghosts modes of the Virasoro generator in each sector are

\[ L_{n}^{\text{gh } \Phi \text{ R}} = \sum_{m=0}^{n} (\frac{1}{2} m + n) \gamma_{m} \beta_{-m-n} + \sum_{n=1}^{\infty} (\frac{3}{2} m + n) \gamma_{m+n} \beta_{-n} + \sum_{n=1}^{\infty} (\frac{1}{2} m - n) \gamma_{m} \beta_{m+n}, \]
\[ L_{-n}^{\text{gh } \Phi \text{ R}} = -\sum_{m=0}^{n} (\frac{1}{2} m + n) \gamma_{-m} \beta_{n-m-n} - \sum_{n=1}^{\infty} (\frac{3}{2} m + n) \gamma_{-(m+n)} \beta_{n} - \sum_{n=1}^{\infty} (\frac{1}{2} m - n) \gamma_{n} \beta_{-(m+n)}, \]
(3.86)
and

\[ L_{m}^{gh\Phi_{NS}} = \sum_{r=1/2}^{m-1/2} (\frac{1}{2} m + r) \gamma_{r}, \beta_{m-r} + \sum_{r=1/2}^{\infty} (\frac{3}{2} m + r) \gamma_{m+r}, \beta_{r} + \sum_{r=1/2}^{m-1/2} (\frac{1}{2} m - r) \gamma_{-r}, \beta_{m+r}, \]

\[ L_{-m}^{gh\Phi_{NS}} = \sum_{r=1/2}^{m-1/2} (\frac{1}{2} m + r) \gamma_{-r}, \beta_{(m-r)} - \sum_{r=1/2}^{\infty} (\frac{3}{2} m + r) \gamma_{-(m+r)}, \beta_{r} - \sum_{r=1/2}^{m-1/2} (\frac{1}{2} m - r) \gamma_{r}, \beta_{-(m+r)}. \]

Similarly Fourier modes of the extended dilatation generator

\[ D_{m}^{ext} = D_{m}^{m} + D_{m}^{ghT} + D_{m}^{gh\Phi} \]

include contributions of matter variables (3.58) and ghosts. For the fermionic (c^T, b^T) ghosts we obtain

\[ D_{m}^{ghT} = -2i \left( \sum_{n=0}^{m} c_{n}^T b_{m-n}^{T} + \sum_{n=1}^{\infty} c_{m+n}^T b_{n}^{T} + \sum_{n=1}^{\infty} c_{n}^T b_{m+n}^{T} \right), \]

\[ D_{-m}^{ghT} = -2i \left( \sum_{n=0}^{m} c_{n}^T b_{-(m-n)}^{T} + \sum_{n=1}^{\infty} c_{-(m+n)}^T b_{n}^{T} + \sum_{n=1}^{\infty} c_{n}^T b_{-(m+n)}^{T} \right). \]

For the (γ, β) ghosts in R and NS sectors one finds

\[ D_{m}^{gh\Phi_{R}} = i \left( \sum_{n=0}^{m} \gamma_{n} \beta_{m-n} + \sum_{n=1}^{\infty} \gamma_{m+n} \beta_{n} + \sum_{n=1}^{\infty} \gamma_{n} \beta_{m+n} \right), \]

\[ D_{-m}^{gh\Phi_{R}} = i \left( \sum_{n=0}^{m} \gamma_{-n} \beta_{-(m-n)} + \sum_{n=1}^{\infty} \gamma_{-(m+n)} \beta_{n} + \sum_{n=1}^{\infty} \gamma_{n} \beta_{-(m+n)} \right) \]

and

\[ D_{m}^{gh\Phi_{NS}} = i \left( \sum_{r=1/2}^{m-1/2} \gamma_{r} \beta_{m-r} + \sum_{r=1/2}^{\infty} \gamma_{m+r} \beta_{r} + \sum_{r=1/2}^{\infty} \gamma_{-r} \beta_{m+r} \right), \]

\[ D_{-m}^{gh\Phi_{NS}} = i \left( \sum_{r=1/2}^{m-1/2} \gamma_{-r} \beta_{-(m-r)} + \sum_{r=1/2}^{\infty} \gamma_{-(m+r)} \beta_{r} + \sum_{r=1/2}^{\infty} \gamma_{r} \beta_{-(m+r)} \right). \]

Taking into account above mode expansions of the extended constraints it is possible to check the nilpotency of the quantum BRST charge. In the case of xp-ordering for the space-time coordinates and momenta, Weyl-ordering for odd coordinates, cb- and γ/β-ordering for ghosts BRST charge indeed appears to be nilpotent

\[ \Omega^2{|_{xp, Weyl, cb−ordering}} = 0 \]

extending the result of the previous section on the anomaly absence in the matter sector of the quantum constraint algebra. The choice of the ordering and nilpotency condition of the BRST charge fix the form of the zero-mode parts of the extended Virasoro

\[ L_{0}^{ext} = L_{0}^{m} + \frac{1}{2} L_{0}^{ghL} + L_{0}^{ghT} + L_{0}^{ghD} + L_{0}^{gh\Phi} \]

and dilation generators

\[ D_{0}^{ext} = D_{0}^{m} + D_{0}^{ghT} + D_{0}^{gh\Phi}. \]

For the xp-ordering (Weyl-ordering for fermions) zero modes of the Virasoro generator for the matter variables were given in (3.57), (3.69) and (3.74). For contributions of the fermionic ghosts to the extended Virasoro generator we get

\[ L_{0_{cb}}^{ghT(D)} = \sum_{n=1}^{\infty} n(c_{-n}^T b_{n}^{T(D)} - c_{n}^T b_{-n}^{T(D)}), \]
and the same expression holds for the contribution of $\frac{1}{2} L_0^{ghL}$. For the bosonic ghosts depending on their periodicity one finds

$$L_0^{\Phi R}_{\gamma \beta} = \sum_{n=1}^{\infty} n(\gamma_n \beta_{-n} - \gamma_{-n} \beta_n) \quad \text{or} \quad L_0^{\Phi NS}_{\gamma \beta} = \sum_{r=1/2}^{\infty} r(\gamma_r \beta_{-r} - \gamma_{-r} \beta_r). \quad (3.96)$$

Similarly the contribution of the $xp$-ordered matter variables to the zero mode of the dilatation generator were obtained in (3.64). Contribution of the $(c^T, b^T)$ ghosts is

$$D_0^{ghT}_{\gamma \beta} = -2i \left( c_0^T b_0^T + \sum_{n=1}^{\infty} (c_n^T b_n^T + c_{-n}^T b_{-n}^T) \right) \quad (3.97)$$

and that of $(\gamma, \beta)$ ghosts is

$$D_0^{ghR}_{\gamma \beta} = i \left( \gamma_0 \beta_0 + \sum_{n=1}^{\infty} (\gamma_n \beta_{-n} + \gamma_{-n} \beta_n) \right) \quad \text{or} \quad D_0^{ghNS}_{\gamma \beta} = i \sum_{r=1/2}^{\infty} (\gamma_r \beta_{-r} + \gamma_{-r} \beta_r). \quad (3.98)$$

In case of the $(-+)$-ordering quantum BRST charge is not nilpotent

$$\Omega^2|_{(-+)-ordering} = \left( \frac{5(d+1)}{24} - \frac{43}{12} \right) \sum_{n=1}^{\infty} n^3 c_n^L c_n^L + (d - 2) \sum_{n=1}^{\infty} n c_n^D c_n^D$$

$$+ \frac{i}{2} (d - 3) \sum_{n=1}^{\infty} n^2 (c_n^D c_n^L - c_n^L c_n^D). \quad (3.99)$$

The obstacle on the r.h.s. is given by three anomalous contributions. The first infinite sum is proportional to the Virasoro anomaly to which each $(X, P)$-pair contributes $+2$ and each odd $\Xi$-variable contributes $+1/2$, $(c^L, b^L)$ and $(c^T, b^T)$ ghosts each contribute $-26$, $(c^D, b^D)$ ghosts $-2$ and $(\gamma, \beta)$ ghosts $+11$ in accordance with the standard Virasoro anomaly calculus (see, e.g., [23]). The second sum is proportional to the dilatation anomaly. $(X, P)$-variables contribute to its value $+(d + 1)$, $(\gamma, \beta)$ ghosts $+1$ and $(c^T, b^T)$ ghosts $-4$. The last term is proportional to the mixed dilatation-Virasoro anomaly. If contribution of the $(X, P)$-variables is normalized to $+(d + 1)$, then that of $\gamma \beta$ ghosts is $+2$ and that of $(c^T, b^T)$ ghosts is $-6$. It is clear that for such ordering not all anomalies can be canceled suggesting a modification of the model by adding extra variables and/or constraints.

Let us finally remark that the above expression for the square of the BRST charge corresponds to the following form of the zero mode of the extended Virasoro generator

$$L_{0(-+)}^\text{ext} = L_{0(-+)}^m + \frac{1}{2} L_{0(-+)}^{ghL} + L_{0(-+)}^{ghT} + L_{0(-+)}^{ghD} + L_{0(-+)}^{gh\Phi} \quad (3.100)$$

where the matter contributions have been defined in (3.55), (3.67) and (3.72). Contributions of the fermionic ghosts read

$$L_{0(-+)}^{ghT(D)} = \sum_{n=1}^{\infty} n(c^{T(D)}_n b^{T(D)}_{-n} + b^{T(D)}_n c^{T(D)}_{-n}) + \frac{1}{12}. \quad (3.101)$$

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7 In the recently proposed chiral world-sheet CFT models [45] (that we believe are also null by their nature) with Minkowski target-space realized as a projective light-cone embedded into flat space-time with extra time-like and space-like dimensions the set of the constraints was adjusted in such a way that dilatation and mixed anomalies have the same value and can be simultaneously canceled fixing the value of the space-time dimension.
with the same expression for $1/2T_{0(+-)}^{ghL}$ and that of bosonic ghosts is

$$L_{0(+-)}^{gh\Phi R} = \sum_{n=1}^{\infty} n(\beta_n \gamma_n - \gamma_n \beta_n) - \frac{1}{12} \quad \text{or} \quad L_{0(+-)}^{gh\Phi NS} = \sum_{r=1/2}^{\infty} r(\beta_r \gamma_r - \gamma_r \beta_r) + \frac{1}{24}. \quad (3.102)$$

The values of numeric constants equal those prescribed for the (anti)periodic bosons/fermions by the $\zeta$-function regularization of the relevant infinite sums [28]. Zero mode of the extended dilatation generator for the $(+-)$-ordering

$$D_{0(+-)}^{ext} = D_{0(+-)}^{m} + D_{0(+-)}^{ghT} + D_{0(+-)}^{gh\Phi} \quad (3.103)$$

is defined by the matter term (3.62) and ghosts

$$D_{0(+-)}^{ghT} = 2i \left( c_0 b_0^T + \sum_{n=1}^{\infty} (c_n^T b_n^T - b_n^T c_n^T) - \frac{1}{2} \right), \quad (3.104)$$

$$D_{0(+-)}^{gh\Phi R} = i \left( \gamma_0 \beta_0 + \sum_{n=1}^{\infty} (\gamma_n \beta_n + \beta_n \gamma_n) - \frac{1}{2} \right) \quad \text{or} \quad D_{0(+-)}^{gh\Phi NS} = i \sum_{r=1/2}^{\infty} (\gamma_r \beta_r + \beta_r \gamma_r). \quad (3.105)$$

4 Conclusion and outlook

Viewing anti-de Sitter space-time as a projective space combines linearization of the $SO(2, d - 1)$ isometry group action with the absence of any restrictions on the homogeneous coordinates parametrizing it which suggests an interesting perspective for studying models of point-like and extended objects. Here our attention has been restricted to the consideration of the simplest models of the massless point particle and the null string with minimal world-line (world-sheet) supersymmetry. At the classical level starting with the phase-space representation of the action as the integral of the pullback of the symplectic 1-form supplemented by the sum of the first-class constraints with the Lagrange multipliers various forms of the massless point particle and null string Lagrangians have been obtained by integrating out the momenta and the part of Lagrange multipliers and compared with other models in which conformal symmetry is linearly realized.

Quantization of the spinning particle model has been shown to produce the Dirac-type equation in homogeneous embedding-space coordinates of the form similar to that of the Fang-Fronsdal equations for the fermionic fields on $AdS_d$.

The form of the possible constraints for the null string is severely restricted by the compatibility with the world-sheet conformal reparametrizations. So we have considered the realization of the 2d minimal supersymmetry algebra extended by the space-time dilatation and Virasoro generators by the quadratic constraints. At the quantum level we examined two realizations of this classical algebra: one corresponding to the generalization of the coordinate-momentum ordering and another based on the ordering in terms of the positive/negative Fourier modes of the constituent operators. These orderings are associated with two vacua commonly used when quantizing null strings: one annihilated by the momentum conjugate to the space-time coordinates and the other annihilated by the positive modes of both coordinate and momentum operators. In the case of the generalized coordinate-momentum ordering BRST charge remains nilpotent, while for the ordering in terms of the positive/negative Fourier modes the square of the BRST charge is proportional
to the sum of the Virasoro, dilatation and mixed anomalies. Their cancelation necessitates a modification of the model adding extra variables and/or constraints.

The fact that the quantum BRST charge is nilpotent in any space-time dimension for the generalized coordinate-momentum ordering invites further exploration of this case. Experience in quantizing tensionless string models on the flat and $AdS$ backgrounds [36], [37], [38], [16], [39], [40] hints at the presence in the spectrum of the supermultiplets of massless higher-spin fields on $AdS_d$ in the projective-space formulation. For the ordering in terms of positive/negative Fourier modes of the operators further examination of the associated theory (upon certain amendments to cancel Virasoro and dilatation anomalies) could consist of considering correlation functions of the vertices. In the ambitwistor string model [32] and its predecessors [41], [42] which have a similar structure such correlators are known to reproduce tree-level scattering amplitudes for a variety of field theories on the flat background [43], [44]. In our case it is tempting to suggest that they could give correlators for field theories on $AdS_d$ that in the boundary limit coincide with the dual free $CFT^8$ correlators, whose stringy interpretation has been recently probed in [45].

Other direction of extension of our results is to consider particle and string models for which the projective description of anti-de Sitter space-time combines with the space-time supersymmetry. It is known that in the case of 4−dimensional anti-de Sitter space its $SO(2,3)$ isometry is the subgroup of $OSp(4|N)$ supergroup and $SO(2,4) \sim SU(2,2)$ isometry group of the 5−dimensional anti-de Sitter space is the subgroup of $SU(2,2|N)$. For flat 4-dimensional space-time it is well known that linear realization of $SU(2,2)$ and $SU(2,2|N)$ is achieved in the framework of the (super)twistor theory [46], [47] and for the higher-dimensional spaces linear realization of respective (generalized) superconformal symmetries using higher-dimensional generalizations of supertwistors [50], [51], [52], [53], [54], [55]. Generalization of $SU(2,2)$ twistors (actually ambitwistors) to the case of $AdS_5$ space was recently considered in [15] and it is quite plausible that the above mentioned higher-dimensional (super)twistors can be applied to the linearization of isometries of the higher-dimensional $AdS$ spaces compatible the space-time supersymmetry [10].

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9For the embedding space description of $D = 4$ superconformal theories that uses supertwistors see, e.g., [48], [49].

10For various previous approaches to the construction of (super)twistors for $AdS$ spaces see, e.g., [50], [51], [52], [53], [54], [55] and also [60], [61].
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