The axial N to Δ transition form factors from Lattice QCD

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We evaluate the N to Δ axial transition form factors in lattice QCD with no dynamical sea quarks, with two degenerate flavors of dynamical Wilson quarks, and using domain wall valence fermions with three flavors of staggered sea quarks. We predict the ratio \( C_6^A(q^2)/C_3^V(q^2) \) relevant for parity violating asymmetry experiments and verify the off-diagonal Goldberger-Treiman relation.

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The electromagnetic structure of the nucleon including the transition form factors for electroproduction of the Δ has been the subject of recent experimental \cite{1} and theoretical studies \cite{2,3}. The N to Δ transition has the advantages that the \( \Delta(1232) \) is the dominant, clearly accessible nucleon resonance, and the isovector spin-flip transition provides selective information on hadron structure. The weak structure functions provide valuable input often complimentary to that obtained from electromagnetic probes. In the nucleon, for example, measurements of the elastic parity violating asymmetry yield information on strange quark contributions. The N to Δ transition filters out the isoscalar \( s\bar{s} \) contributions, so the parity violating asymmetry in weak neutral and charge changing N to Δ transitions probes isovector structure not accessible in the study of strange isoscalar quark currents. Furthermore its isovector nature could be used to detect physics beyond the standard model since it is sensitive to additional heavy particles not appearing in the standard model. However, in order to compete with other low-energy semi-leptonic measurements, the N to Δ transition requires a determination of the parity violating asymmetry to significantly better than 1% precision \cite{4}. Hence, a lattice prediction for the axial form factors provides valuable input for ongoing experiments \cite{5}. In this work, we evaluate the dominant contribution to the parity violating asymmetry, determined by the ratio \( C_6^A/C_3^V \). This is the off-diagonal analogue of the \( g_A/g_V \) ratio extracted from neutron \( \beta \)-decay and therefore tests low-energy consequences of chiral symmetry, such as the off-diagonal Goldberger-Treiman relation. In addition, the ratio of axial form factors \( C_6^A/C_3^A \) provides a measure of axial current conservation.

Since this is the first lattice computation of the axial N to Δ transition form factors, the starting point is an evaluation in the quenched theory using the standard Wilson action. A quenched calculation allows us to use a large lattice in order to minimize finite volume effects and obtain accurate results at small momentum transfers reaching pion mass, \( m_\pi \), down to about 410 MeV. In order to study the role of the pion cloud, which is expected to provide an important ingredient in the description of the properties of the nucleon system, one requires dynamical configurations with light quarks on large volumes. In this work, the light quark regime is studied in two ways. First, we use configurations with the lightest available dynamical two flavor Wilson fermions spanning approximately the same pion mass range as in the quenched calculation \cite{6,7}. Second, we use a hybrid combination of domain wall valence quarks, which have chiral symmetry on the lattice, and MILC configurations generated with three flavors of staggered sea quarks using the Asqtad improved action \cite{8}. The effectiveness of this hybrid combination has recently been demonstrated in the successful precision calculation of the axial charge, \( g_A \) \cite{9}. Since Wilson fermions have discretization errors in the lattice spacing, \( a \), of \( O(a^4) \) and break chiral symmetry whereas the hybrid action has discretization errors of \( O(a^2) \) and chirally symmetric valence fermions, agreement between calculations using these two lattice actions provides a non-trivial check of consistency of the lattice results. The hybrid calculation is the most computationally demanding, since it requires propagators on a five-dimensional lattice. The bare quark mass for the domain wall fermions, the size of the fifth dimension and the renormalization factors, \( Z_A \), for the four-dimensional axial current are taken from Ref. \cite{9}. As in the case of Wilson fermions, we consider three values of light quark mass with the strange sea quark mass fixed to approximately its physical value \cite{8}. In all cases we use Wuppertal smeared \cite{10} interpolating fields at the source and sink. In the unquenched Wilson case, to minimize fluctuations \cite{11} we use hypercubic (HYP) smearing \cite{12} on the spatial links entering in the Wuppertal smearing function at the source and sink whereas for the hybrid case all gauge links in the fermion action are HYP smeared. We list the parameters used in our computations in Table \ref{tab:parameters}. The value of the lattice spacing is determined from the nucleon mass at the physical limit for the case of Wilson fermions and for the staggered sea quark configurations, we take the value determined from heavy quark spectroscopy \cite{13}.

The invariant N to Δ weak matrix element, expressed in terms of four transition form factors \cite{12,13} can be
written as

\[<\Delta(p', s')|A^\mu_3|N(p, s)> = \frac{1}{\sqrt{3}} \left( \frac{M_{\Delta} M_N}{E_{\Delta}(p') E_N(p)} \right)^{1/2} \]

\[i \bar{\psi}(p', s') \left[ \left( \frac{C_A^\Lambda(q^2)}{M_N^2} \gamma^\nu + \frac{C_4^\Lambda(q^2)}{M_N^2} q^\nu \right) (g_{\lambda\mu} q^\nu - g_{\lambda\nu} q^\mu) q^\tau + C_5^\Lambda(q^2) g_{\lambda\mu} + \frac{C_4^\Lambda(q^2)}{M_N^2} q_{\lambda\mu} \right] u(p, s) \]

(1)

where \(q_\mu = p'_\mu - p_\mu\) is the momentum transfer, \(A^\mu_3(x) = \bar{\psi}(x)\gamma_\mu\gamma_5\gamma^3\psi(x)\) is the isovector part of the axial current, and \(\tau^3\) is the third Pauli matrix. We evaluate this matrix element on the lattice by computing the nucleon two-point function \(G_N^t(t; p; \Gamma_4)\), the \(\Delta\) two-point function \(G_\Delta^t(t; p; \Gamma_4)\), and the three point function \((G_\Lambda^\tau N(t_2, t_1; p', p; \Gamma))\) and forming the ratio [2]

\[R_n(t_2, t_1; p', p; \Gamma; \mu) = \frac{\langle G_\Lambda^\tau N(t_2, t_1; p', p; \Gamma) \rangle}{\langle G_N^t(t_2, t_1; p; \Gamma_4) \rangle} \frac{\langle G_\Delta^t(t_2, t_1; p; \Gamma_4) \rangle}{\langle G_\Lambda^\tau N(t_2, t_1; p; \Gamma_4) \rangle} \frac{1}{1/2} \Pi_n(p', p; \Gamma; \mu) \]

(2)

where the indices \(i\) are summed, \(\Gamma_4 = \frac{1}{2} \left[ \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right] \), and \(\Gamma_j = \frac{1}{2} \left[ \begin{array}{cc} \sigma_j & 0 \\ 0 & \sigma_j \end{array} \right] \). For large time separations between \(t_2, t_1\), the time when a photon interacts with a quark, and \(t_2, t_1\), the time when the \(\Delta\) is annihilated, the ratio of Eq. (2) becomes time independent and yields the transition matrix element of Eq. (1). The source-sink time separation is optimized as in Ref. [11] so that a plateau is clearly identified when varying \(t_1\). We use kinematics where the \(\Delta\) is produced at rest and \(Q^2 = -q^2\) is the Euclidean momentum transfer squared. There are various choices for the Rarita-Schwinger spinor index, \(\sigma\), and projection matrices, \(\Gamma\), that can be used in the computation of the three point function, each requiring a sequential inversion. We use this freedom to construct optimized \(\Delta\) sources that maximize the number of lattice momentum vectors contributing to a given value of \(Q^2\) in analogy with the evaluation of the electromagnetic \(N\) to \(\Delta\) transition form factors [2].

In Fig. 1 we show quenched and unquenched results obtained with Wilson fermions. All errors shown are obtained using a jackknife analysis. We observe that \(C_3^A\) is consistent with zero and that unquenching effects are small for the dominant form factors, \(C_5^A\) and \(C_6^A\). In contrast, for the form factor \(C_{4\tau}^A\), dynamical fermions produce a dramatic increase at low momentum transfer relative to the quenched results. Such large deviations between quenched and full QCD results for these relatively heavy quark masses are unusual, making this an interesting quantity with which to study effects of unquenching. In Fig. 2 we compare the Wilson and hybrid results for the two dominant form factors for \(m_\pi \sim 500\) MeV. The hybrid results thus corroborate the small unquenching effects, and similar behavior is observed at the heavier and lighter quark masses. For \(C_4^A\), also shown in Fig. 2 we observe the same large unquenching effects observed for dynamical Wilsons. In addition, although they are not shown, hybrid calculations yield \(C_5^A \sim 0\). A dipole Ansatz \(C_4^A(0)/(1 + Q^2/M_\Delta^2)\) describes the \(Q^2\)-dependence of \(C_4^A\) well, as shown by the curves in Fig. 2 yielding, at this pion mass, an axial mass \(M_A \sim 1.8(1)\) GeV. In this range of pion masses, we observed a weak quark mass dependence for \(M_A\) that however needs to be checked for lighter quarks before comparing to the experimental result of \(1.28 \pm 0.10\) GeV [17].

In the chiral limit, axial current conservation leads to the relation \(C_4^A(Q^2) = M_N^2 C_4^N(Q^2)/Q^2\). In Fig. 3 we show the ratio \((Q^2/M_\Delta^2) C_4^N(Q^2)/C_4^A(Q^2)\) for Wilson and domain wall fermions at the lightest quark mass available in each case. The expected value in the chiral limit for this ratio is one. For finite quark mass, the axial current is not conserved and for Wilson fermions chiral symmetry is broken, so that deviations from one are expected. We observe that this ratio differs from unity at low \(Q^2\) but approaches unity at higher values of \(Q^2\). For the quenched case, where we have accurate results, a linear extrapolation in \(m_\pi^2\) yields values that are consistent with unity for \(Q^2 \sim 0.5\) GeV\(^2\). It is reassuring that this chiral restoration is seen on the lattice even for Wilson fermions, confirming that the discrete theory is correctly representing continuum physics. For finite pion mass, deviations from unity are expected to be proportional to \(m_\pi^2/(Q^2 + m_\pi^2)\) for chiral lattice fermions. As can be seen by the dashed line in the figure, this form describes well the \(Q^2\)-dependence of the results obtained with domain

| TABLE I: The number of configurations, the hopping parameter, \(\kappa\), for Wilson fermions or the light-quark mass, \(m_l\), for staggered quarks, and the pion, nucleon and \(\Delta\) masses in lattice units. |
|---|---|---|---|---|
| no. confs | \(\kappa\) or \(a_{\kappa}\) | \(a_{\kappa}\) | \(a_{M_N}\) | \(a_{M_{\Delta}}\) |
| Quenched Wilson 24\(^3\) \times 40 | \(a^{-1} = 2.14(6)\) GeV |
| 200 | 0.1554 | 0.263(2) | 0.592(5) | 0.687(7) |
| 200 | 0.1558 | 0.229(2) | 0.556(6) | 0.668(6) |
| 200 | 0.1562 | 0.192(2) | 0.518(6) | 0.646(9) |
| Unquenched Wilson 24\(^3\) \times 32 \([2]\) | \(a^{-1} = 2.56(10)\) GeV |
| 185 | 0.1575 | 0.270(3) | 0.580(7) | 0.645(5) |
| 157 | 0.1580 | 0.199(3) | 0.500(10) | 0.581(14) |
| Unquenched Wilson 24\(^3\) \times 32 \([2]\) | \(a^{-1} = 2.56(10)\) GeV |
| 200 | 0.15825 | 0.150(3) | 0.423(7) | 0.533(8) |
| \(\kappa_c = 0.1585\) | 0.366(13) | 0.486(14) |
| MILC 20\(^4\) \times 64 | \(a^{-1} = 1.58\) GeV |
| 150 | 0.03 | 0.373(3) | 0.886(7) | 1.057(14) |
| 150 | 0.02 | 0.306(3) | 0.800(10) | 0.992(16) |
| MILC 28\(^3\) \times 64 | \(a^{-1} = 1.58\) GeV |
| 118 | 0.01 | 0.230(3) | 0.751(7) | 0.988(26) |
For finite mass pions, partial conservation of the axial current \( \left( \partial_\mu A_\mu^A(x) = f_\pi m_\pi \bar{\psi}(x) \gamma_\mu \psi(x) \right) \) and pole dominance lead to the off-diagonal Goldberger-Treiman relation \( C_3^A(Q^2) = f_\pi g_{\pi N A}(Q^2)/2M_N \) where \( g_{\pi N A}(Q^2) \) is determined from the matrix element of the pseudoscalar density \( <\Delta^+|\bar{\psi}(x)\gamma_\mu \psi(x)|\pi> \) and the pion decay constant, \( f_\pi \), is calculated from the two-point function \( <0|A_\mu(x)|\pi> \). To relate the lattice pion matrix element to its physical value, we need the pseudoscalar renormalization constant, \( Z_P \), and we use for quenched \([19]\) and dynamical Wilson fermions \([20]\) the value \( Z_P(p^2,a^2 \sim 1) = 0.5 \). In Fig. 4 we show the ratio \( f_\pi g_{\pi N A}(Q^2)/(2M_N C_3^A) \) for Wilson fermions, which is almost independent of \( Q^2 \), indicating similar \( Q^2 \) dependence for \( g_{\pi N A} \) and \( C_3^A \). The ratio becomes consistent with unity for \( Q^2 \lesssim 1 \text{ GeV}^2 \), in agreement with the off-diagonal Goldberger-Treiman relation. The large deviations seen for the lightest quark mass for dynamical fermions point to lattice artifacts that become more dominant for small quark masses.

The presently unmeasured ratio \( C_5^A/C_3^V \) is an interesting prediction of our lattice calculation. The form factor \( C_3^V \) can be obtained from the electromagnetic N to \( \Delta \) transition. Using our lattice results for the dipole and electric quadrupole Sachs factors, \( G_{M1} \) and \( G_{E2} \), we extract \( C_3^V \) using the relation

\[
C_3^V = \frac{3}{2} \frac{M_\Delta (M_N + M_\Delta)}{(M_N + M_\Delta)^2 + Q^2} (G_{M1} - G_{E2}).
\]

From the ratio \( C_3^A/C_3^V \) shown in Fig. 5, we observe that quenched and unquenched results at \( m_\pi \sim 500 \text{ MeV} \) agree within errors, and that the quenched data have a sufficiently weak mass dependence that when extrapolated to the physical pion mass, the results nearly coincide with the 500 MeV data within errors. Hence, it is
low energy consequences of chiral symmetry \cite{4}, this ratio provides the basis for a physical estimate of the parity violating asymmetry. Under the assumptions that $C_3^A \sim 0$ and $C_4^A$ is suppressed as compared to $C_5^A$, both of which are supported by our lattice results, the parity violating asymmetry can be shown to be proportional to this ratio \cite{4}. Thus, our lattice results show that this ratio and, to a first approximation, the parity violating asymmetry is non-zero at $Q^2 = 0$ and increases for $Q^2 \gtrsim 1.5 \text{ GeV}^2$.

In summary, we have provided a first lattice calculation of the axial transition form factors, which are to be measured at Jefferson Lab \cite{5}. The first conclusion is that $C_3^A$ is consistent with zero whereas $C_4^A$ is small but shows unusually high sensitivity to unquenching effects. The two dominant form factors are $C_5^A$ and $C_6^A$, which are related in the chiral limit by axial current conservation. The ratio $(Q^2/M_N^2) C_6^A/C_5^A$ is shown to approach unity as the quark mass decreases, as expected from chiral symmetry. For any quark mass, the strong coupling, $g_{A\pi N\Delta}$, and the axial form factor, $C_5^A$, show similar $Q^2$ dependence, and the off diagonal Goldberger-Treiman relation is reproduced as the quark mass decreases. The ratio of $C_5^A/C_3^A$, which provides a first approximation to the the parity violating asymmetry, is predicted to be non-zero at $Q^2 = 0$ with a two-fold increase when $Q^2 \sim 1.5 \text{ GeV}^2$.

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