Jump-preserving polynomial interpolation in non-manifold polyhedra

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Abstract

We construct a piecewise-polynomial interpolant \( u \mapsto \Pi u \) for functions \( u: \Omega \setminus \Gamma \to \mathbb{R} \), where \( \Omega \subset \mathbb{R}^d \) is a Lipschitz polyhedron and \( \Gamma \subset \Omega \) is a possibly non-manifold \((d-1)\)-dimensional hypersurface. This interpolant enjoys approximation properties in relevant Sobolev norms, as well as a set of additional algebraic properties, namely, \( \Pi^2 = \Pi \), and \( \Pi \) preserves homogeneous boundary values and jumps of its argument on \( \Gamma \). As an application, we obtain a bounded discrete right-inverse of the “jump” operator across \( \Gamma \), and an error estimate for a Galerkin scheme to solve a second-order elliptic PDE in \( \Omega \) with a prescribed jump across \( \Gamma \).

1 Main results

For any open set \( U \subset \mathbb{R}^d, \ d \geq 2 \), we denote by \( H^1(U) \) the completion of \( C^\infty(U) \) with respect to the norm

\[
\|u\|_{H^1(U)}^2 := \sum_{|\alpha| \leq l} \|D^\alpha u\|_{L^2(U)}^2 .
\]

When \( U \) is a Lipschitz domain, we define \( H^s(U) \) for all real \( s \) as in [28]. For a closed set \( F \subset U \), let \( H^1_0,F(U) \) be the closure of \( C^\infty_c(U \setminus F) \) in \( H^1(U) \). Let \( \Omega \subset \mathbb{R}^d \) be a Lipschitz polyhedron and let \( \Omega_h \) be a conforming simplicial mesh of \( \Omega \). Let \( \Gamma \subset \mathbb{R}^d \) be a \((d-1)\)-dimensional hypersurface resolved by \( \Omega_h \), in the sense that there is a conforming mesh \( \Gamma_h \) of \( \Gamma \) whose elements are faces of elements of \( \Omega_h \). We assume that the elements of \( \Omega_h \) intersecting \( \Gamma \) do not intersect \( \partial \Omega \). We denote by \( V^p(\Omega_h; \Gamma) \) the finite-dimensional subspace of \( H^1(\Omega \setminus \Gamma) \) consisting of functions which are polynomial of degree \( p \) on each mesh element of \( \Omega_h \). Note that we do not assume any regularity for the set \( \Gamma \). It can be a non-manifold \((d-1)\)-dimensional polyhedral surface, such as the one represented in Figure 1 for \( d = 3 \). We also make no assumption on the uniformity of the mesh \( \Omega_h \).

Theorem 1.1. There exists a linear operator

\[
\Pi_h : H^1(\Omega \setminus \Gamma) \to V^p(\Omega_h; \Gamma)
\]

satisfying the following properties.

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1Precise definitions of all technical terms appearing in Sections 1 to 4 are given in Section 5.
(i) \( \Pi_h u_h = u_h \) for all \( u_h \in V^p(\Omega_h; \Gamma) \)
(ii) \( \Pi_h u \in H^1_{0,1}(\Omega) \) for all \( u \in H^1_{0,1}(\Omega) \),
(iii) \( \Pi_h u \in H^1(\Omega) \) for all \( u \in H^1(\Omega) \).

We call such an operator a *jump-aware interpolant on* \( V^p(\Omega_h; \Gamma) \). We say that \( \Omega_h \) is \( \gamma \)-shape regular if for every mesh element \( K \) of \( \Omega_h \), \( \frac{h_K}{\rho_K} \leq \gamma \), where \( h_K \) is the diameter of \( K \) and \( \rho_K \) is the radius of its inscribed sphere.

![Figure 1: A non-manifold surface \( \Gamma \subset \mathbb{R}^3 \) and its conforming triangular mesh.](image)

**Theorem 1.2.** Given an integer \( p \geq 1 \), and real numbers \( \frac{1}{2} < t \leq p + 1 \) and \( \gamma_0 > 0 \), there exists a real number \( C(\gamma_0, p, t) > 0 \) such if \( \Omega_h \) is \( \gamma_0 \)-shape regular mesh, there is a jump-aware interpolant \( \Pi_h \) with the following continuity properties

\[
\| \Pi_h u \|_{H^t(K)}^2 \leq C(\gamma_0, p, t) \sum_{K' \in \omega_K} \| u \|_{H^t(K')}^2,
\]

\[
\| u - \Pi_h u \|_{H^t(K)}^2 \leq C(\gamma_0, p, t) h_K^{2(t-s)} \sum_{K' \in \omega_K} \| u \|_{H^t(K')}^2,
\]

for any \( s \in [0, t] \) and for all \( u \in H^t_{pw}(\omega_K) \cap H^1(\Omega \setminus \Gamma) \), where for every mesh element \( K \) of \( \Omega_h \), \( \omega_K \subset \Omega_h \) is the set of mesh elements sharing at least one vertex with \( K \).

Here, \( H^t_{pw}(\omega_K) \) is the set of functions in \( L^2(\Omega) \) whose restriction to any \( K' \in \omega_K \) belongs to \( H^t(K') \). When \( 0 \leq s < \frac{1}{2} \), it is possible to obtain global bounds on the \( H^s \) norm of the interpolation error:

**Corollary 1.3 (Estimate with global norms).** Under the same assumptions as in Theorem 1.2, \( \Pi_h \) satisfies

\[
\| u - \Pi_h u \|_{H^s(\Omega)}^2 \leq \frac{5}{2} C(\gamma_0, p, 1) h^{2(2s)} \| u \|_{H^1(\Omega \setminus \Gamma)}^2
\]

for all \( 0 \leq s < \frac{1}{2} \) and \( u \in H^1(\Omega \setminus \Gamma) \), where \( h = \max_K h_K \), and

\[
\| u \|_{H^s(\Omega)}^2 = \| \nabla u \|_{L^2(\Omega \setminus \Gamma)}^2
\]

where \( \nabla u \) is the weak gradient of \( u \) in \( \Omega \setminus \Gamma \) defined in Remark 1.4; Eq. (1.3).

**Proof.** Use [32, Lemma 4.1.49] and [28, Lemma 3.33] and the fact that \( H^s_0(\Omega) = H^s(\Omega) = \tilde{H}^s(\Omega) \) for these values of \( s \).
Remark 1.4 (The space $H^1(\Omega \setminus \Gamma)$). Contrarily to other common notations, the definition taken here, implies that

$$H^1(\Omega \setminus \Gamma) \neq H^1(\Omega)$$

in general. For instance, taking $\Gamma = \partial \Omega_0$ where $\Omega_0 \subset \Omega$ are both bounded Lipschitz polyhedra, the indicator function $1_{\Omega_0}$ belongs to $H^1(\Omega \setminus \partial \Omega_0)$ but not $H^1(\Omega)$. In this regard, our notation differs e.g. from [38] (compare Theorem 6.1 below with Proposition 1 of this reference, keeping in mind that $\Omega$ is a not assumed to be a regular polyhedron therein). By the “$H = W$” theorem of Meyers and Serrin [29], $H^1(\Omega \setminus \Gamma)$ as defined here is equal to the set of functions in $L^2(\Omega \setminus \Gamma)$ which possess a weak gradient $p \in L^2(\Omega \setminus \Gamma)$, that is

$$\exists p \in L^2(\Omega \setminus \Gamma)^3 : \forall \varphi \in D(\Omega \setminus \Gamma)^3, \quad \int_{\Omega \setminus \Gamma} u(x) \text{div} \varphi(x) \, dx = - \int_{\Omega \setminus \Gamma} p(x) \cdot \varphi(x) \, dx. \quad (1.3)$$

Note that in general, the weak gradient of $u$ is not equal to the gradient of $u$ in the sense of distributions on $\Omega$. Another common, different definition for $H^1(U)$ is

$$H^1_{\text{restrict}}(U) := \{ f \mid f \in H^1(\mathbb{R}^d) \}.$$ 

If $U$ is an extension domain, then $H^1_{\text{restrict}}(U) = H^1(U)$ with equivalent norms, see [28] Thm 3.30, (iii)], but here, $\Omega \setminus \Gamma$ is not an extension domain in general. Finally, care must be taken when defining a fractional Sobolev space $H^s(\Omega \setminus \Gamma)$. If one uses, for instance, a definition via the Gagliardo semi-norm as in [17], then, perhaps surprisingly, the inclusion $H^1(\Omega \setminus \Gamma) \subset H^s(\Omega \setminus \Gamma)$ for $\frac{1}{2} < s < 1$ does not hold in general; this is because the Gagliardo semi-norm does not “see” the set of zero $d$-dimensional measure $\Gamma$, so that $H^s(\Omega \setminus \Gamma) \neq H^s(\Omega \setminus \Gamma)$. For the purposes of this work, we will not consider $H^s(U)$ if $U$ is not a Lipschitz domain.

1.1 Relation to other results in the literature

If $\Omega_0$ is a Lipschitz polyhedron, then one can choose a polyhedron $\Omega_0 \Subset \Omega$ and apply Theorems 1.1 and 1.2 with $\Gamma = \partial \Omega_0$ (i.e. our result applies also if $\Gamma$ is in fact a manifold boundary). It can be seen from the proof below that $\Pi_h u = 0$ on $\Omega \setminus \Omega_0$ if $u = 0$ on this set. This way, our results extend several results of the literature, including [33] Theorems 2.1, 4.1 and Corollary 4.1 (the original construction by Scott and Zhang), by giving the full range of continuity properties in fractional Sobolev norms as in [12], and giving the estimate in terms of broken norms of $u$ as in [9], (instead of norms over patches). The latter feature, first discovered by Veeser in [38], has recently come to attention in several works, see e.g. [23, 8, 10] and references therein.

1.2 Background and motivation

The motivation for this work is the discretization of partial differential equations (PDEs) in complex geometries, in particular, the complement in $\mathbb{R}^d$ of a thin “screen” or “crack” $\Gamma$ represented by a $(d-1)$-dimensional surface. These types of geometries occur in a broad range of applications, including antennas [25], microwaves [11], satellites [2], elasticity [35], geosciences [27, 21], water waves [40] or computer graphics [34].

2 Note, however, that we have restricted our analysis to the $L^2$ setting for simplicity, while these references cover the general $L^p$ setting.
This work is especially connected to the development, for such geometries, of boundary element methods (BEM) [28, 32], also widely known as Method of Moments (MoM) in computational electromagnetics [24]. The BEM is usually well-suited for “obstacle problems”, that is, PDEs set in $\mathbb{R}^d \setminus \Omega$, mainly because it allows to recast a $d$-dimensional, unbounded problem, into an integral equation on the boundary $\partial \Omega$ of the obstacle.

In real-life applications with thin obstacles (e.g., metallic plates, screens, fractures), one is led to apply the BEM in the degenerate case where the obstacle becomes an open surface in $\mathbb{R}^3$. This is by now well-understood if the surface is an orientable manifold with boundary, see e.g. [36, 39, 7]. The manifoldness makes it possible to regard the fundamental quantities of the BEM – namely, the “jumps” $[u]_\Gamma$ of functions $u$ defined in $\mathbb{R}^3 \setminus \Gamma$ – as genuine functions on $\Gamma$. One puts $[u]_\Gamma(x) := \lim_{\varepsilon \to 0^+} u(x + \varepsilon n(x)) - u(x - \varepsilon n(x))$, where $n(x)$ is the normal vector at $x$ specified by the orientation.

When $\Gamma$ is not an orientable manifold, $\Omega$ may be on more than 2 sides of $\Gamma$ at a given point, so jumps cannot be interpreted as single-valued functions on $\Gamma$. There is nevertheless a more abstract point of view, proposed by Claeyts and Hiptmair in [14, 15], which stems from the observation that, when $\Gamma$ is a manifold, the jump $[u]_\Gamma$ of a function $u$ in $H^1(\Omega \setminus \Gamma)$ vanishes if, and only if, $u \in H^1(\Omega)$. Hence, $[u]_\Gamma$ can be equivalently defined as the equivalence class of $u$ for the relation “differing by a $H^1(\Omega)$ function”, thus extending the definition of $[u]_\Gamma$ to the case where $\Gamma$ is no longer a manifold. It is proved in [14, 15] that the PDE can be recast into a uniquely solvable variational problem set in the quotient space associated to this equivalence relation. This has paved the way for the rigorous analysis of non-manifold BEM methods in recent works [13, 4, 16, 3].

The latest of these works [16, 3] have highlighted the need, as a key ingredient for the stability analysis, for a boundary-element analog to the Scott and Zhang interpolant [33]. The latter is a fundamental tool in the analysis of finite-element methods (see e.g. [1, Thm 1.1]) and many works are concerned with its study and generalizations, see e.g. [12, 19, 22] and references therein. For the aforementioned BEM applications, the needed generalization of the Scott-Zhang operator is the one given by Theorem 2.2 below, see [3, Proposition 6.9] and [16, Theorem 1].

The remainder of this paper is organized as follows. In the next section, we prove a consequence of Theorems 1.1 and 1.2 regarding polynomial interpolation on non-manifold polyhedral hypersurfaces. We then give two other applications of Theorems 1.1 and 1.2 in Section 3. An outline of the construction of $\Pi_h$ is given in Section 4. We then collect definitions and notations in Section 5 and prove Theorems 1.1 and 1.2 in Section 6.

2 Polynomial interpolation on polyhedral hypersurfaces

From Theorems 1.1 and 1.2 one can derive a result concerning piecewise polynomial interpolation of functions defined on the non-manifold hypersurface $\Gamma$. We allow the interpolated functions to be multi-valued on $\Gamma$, as is typically the case when considering restrictions to $\Gamma$ of elements of $H^1(\Omega \setminus \Gamma)$.

To set a rigorous framework for this problem, we follow [14] by defining the multi-trace space $H^{1/2}(\Gamma) := H^1(\Omega \setminus \Gamma)/H^1_0(\Omega)$ (equipped with the quotient norm) and let $\gamma_\Gamma : H^1(\Omega \setminus \Gamma) \to H^{1/2}(\Gamma)$ be the canonical surjection corresponding to this quotient. The polyhedral setup that we have adopted here also allows us to view $H^{1/2}(\Gamma)$ as a subspace of $L^2(\Gamma) \times L^2(\Gamma)$ in the following way:

**Lemma 2.1.** For each $F \in \Gamma_h$, let $K^+(F)$ and $K^-(F)$ be the two elements of $\Omega_h$ that are incident to $F$, (where the $+/-$ labels are chosen arbitrarily). Given $u \in H^1(\Omega \setminus \Gamma)$, define the function
\( \text{Tr}^\pm(u) \) on \( \Gamma \) as
\[
(\text{Tr}^\pm u)(x) := \sum_{F \in \mathcal{M}_G} 1_{\{x \in F\}}(\gamma^\pm_F u|_{K^\pm(F)})(x)
\]
where \( \gamma^\pm_F : H^1(K^\pm(F)) \to L^2(F) \) is the trace operator, and \( L^2(F) \) is identified to a subspace of \( L^2(\Gamma) \) in the obvious way. Let
\[
\text{Tr} : H^1(\Omega \setminus \Gamma) \to L^2(\Gamma) \times L^2(\Gamma)
\]
be defined by \( \text{Tr}(u) := (\text{Tr}^+(u), \text{Tr}^-(u)) \). Then \( \text{Tr} \) is linear and continuous, does not depend on the choice of mesh \( \Omega_h \), and satisfies \( \operatorname{Ker}(\text{Tr}) = H^1_0(\Gamma) \).

**Proof.** The independence with respect to the choice of the mesh \( \Omega_h \) follows from the fact that \( \text{Tr} \) is invariant under mesh subdivision of \( \Omega_h \) and the fact that two meshes of the same polyhedron admit a common subdivision, see [26, Corollary 1.6]. The continuity of \( \text{Tr} \) follows from the trace Theorem, see [26 Thm 3.37]. The equality \( H^1_0(\Gamma) = \operatorname{Ker}(\text{Tr}) \) can be shown as a particular case of the (much more general) result [37 Theorem 2.2] applied to the open set \( \Omega := \mathbb{R}^d \setminus \Gamma \). □

Let \( \iota : \mathbb{H}^{1/2}(\Gamma) \to L^2(\Gamma) \times L^2(\Gamma) \) be defined by \( \iota(u) := \text{Tr}(f) \) for any \( f \in H^1(\Omega \setminus \Gamma) \) such that \( \gamma_\Gamma(f) = u \). By Theorem 2.1, this is a well-defined injective linear map. We can thus identify \( \mathbb{H}^{1/2}(\Gamma) \) to \( \iota(\mathbb{H}^{1/2}(\Gamma)) \). Under this identification, one has \( \gamma_\Gamma = \iota \).

Let \( H^{1/2}(\Gamma) := \gamma_\Gamma(\mathbb{H}^{1}(\Omega)) \). This space is called the single-trace space, and is the set of elements of \( \mathbb{H}^{1/2}(\Gamma) \) of the form \( (u, u) \). We can thus identify it to a subspace of \( L^2(\Gamma) \), which coincides (with equivalent norms) with the usual \( H^{1/2}(\Gamma) \) when \( \Gamma \) is the boundary of a Lipschitz domain (because the trace operator \( \gamma_{\partial \Omega} : H^1(\Omega) \to H^{1/2}(\partial \Omega) \) has a continuous right inverse in this case, see [26 Thm. 3.37]). Finally, let \( \mathbb{H}^{1/2}(\Gamma) := \mathbb{H}^{1/2}(\Gamma)/H^{1/2}(\Gamma) \) (the jump space) and let \( [\cdot]_\Gamma : \mathbb{H}^{1/2}(\Gamma) \to \mathbb{H}^{1/2}(\Gamma) \) be the corresponding canonical surjection. Define the space of piecewise-polynomial multi-traces by
\[
\mathbb{V}^p(\Gamma_h) := \text{Tr}(\mathbb{V}^p(\Omega_h; \Gamma)).
\]
At least if \( d \in \{2, 3\} \), \( p = 1 \), and when \( d = 3 \), if all the vertices of \( \Gamma_h \) have “edge-connected stars” (but this is probably a more general fact), this space only depends on \( \Gamma_h \) and not on the tetrahedral mesh \( \Omega_h \), because it is equal to the space of “continuous piecewise linear functions on the inflated mesh”, see [4 Theorem 5.5]. Following this reference, we define
\[
\Gamma_h^* := \{(K, F) \in \Omega_h \times \Gamma_h \mid K \text{ is incident to } F\},
\]
and for \( F^* = (K, F) \in \Gamma_h^* \) and \( u = (u^+, u^-) \in L^2(\Gamma) \times L^2(\Gamma) \), we write
\[
\|u\|_{H^1(F^*)} := \|u^\pm\|_{H^1(F)}
\]
if \( K = K^\pm(F) \).

**Theorem 2.2** (Multi-trace interpolant). For all \( \gamma_0 > 0 \) and integer \( p \geq 1 \), there exist a constant \( C'(\gamma_0, p) > 0 \) such that for all real \( s \in [0, \frac{1}{2}] \), the following holds. If \( \Omega_h \) is \( \gamma_0 \)-shape-regular, there exists a linear operator \( \Phi_h : \mathbb{H}^{1/2}(\Gamma) \to \mathbb{V}^p(\Gamma_h) \) satisfying

(i) \( \Phi_h u_h = u_h \) for all \( u_h \in \mathbb{V}^p(\Gamma_h) \).

(ii) \( [u]_\Gamma = 0 \Longrightarrow [\Phi_h u]_\Gamma = 0 \) for all \( u \in \mathbb{H}^{1/2}(\Gamma) \).
(iii) For all \( u \in \mathbb{H}^{1/2}(\Gamma) \),
\[
\|\Phi_h u\|_{\mathbb{H}^{1/2}(\Gamma)} \leq C'(\gamma_0, p) \|u\|_{\mathbb{H}^{1/2}(\Gamma)},
\]

(iv) For all \( u \in \mathbb{H}^{1/2}(\Gamma) \)
\[
\sum_{F^* \in \Gamma^*_h} h_{F^*}^{2s-1} \|u - \Phi_h u\|_{H^s(F^*)}^2 \leq C'(\gamma_0, p) \|u\|_{\mathbb{H}^{1/2}(\Gamma)}^2
\]

where \( h_{F^*} = \max(h_{K^+}, h_{K^-(F^*)}) \) with \( F^* = (K, F) \).

Remark 2.3. The operator \( \Phi_h \) is analogous the Scott-Zhang interpolant \([33]\), with the multi-trace space \( \mathbb{H}^{1/2}(\Gamma) \) playing the role of the volume Sobolev spaces, and with the jump operator \([\cdot]\) \( \Gamma \) playing the role of the trace operator.

Proof of Theorem 2.2. Let \( \Pi_h : H^1(\Omega \setminus \Gamma) \to V^p(\Omega_h; \Gamma) \) be given by Theorem 1.2. Let \( L : \mathbb{H}^{1/2}(\Gamma) \to H^1(\Omega \setminus \Gamma) \) be the linear isometry satisfying
\[
\|L u\|_{H^1(\Omega \setminus \Gamma)} = \|u\|_{\mathbb{H}^{1/2}(\Gamma)} \quad \text{and} \quad \gamma T L u = \gamma T u
\]
for all \( u \in \mathbb{H}^{1/2}(\Gamma) \). We define \( \Phi_h := \gamma T (\Pi_h \circ L) \), and show that properties (i)-(iv) hold. The property (ii) is an immediate consequence of the property (iii) of Theorem 1.1. Given \( u_h \in V^p(\Gamma_h) \), let \( U_h \in V^p(\Omega_h; \Gamma) \) be such that \( u_h = \gamma T (U_h) \). On the other hand, let \( U = L u_h \). Noting that \( U - U_h \in H^1_{0,1}(\Omega) \), we deduce that \( \Pi_h(U - U_h) \in H^1_{0,1}(\Omega) \), hence
\[
\Phi_h u_h - u_h = \gamma T (\Pi_h(U - U_h)) = 0.
\]
This proves (i). Next, notice that the first inequality of Theorem 1.2 with \( t = 1 \) implies that
\[
\|\Pi_h u\|_{H^1(\Omega \setminus \Gamma)} \leq C(\gamma_0, p, 1) \|u\|_{H^1(\Omega \setminus \Gamma)}.
\]
Thus \( \|\Phi_h\|_{\mathbb{H}^{1/2} \rightarrow \mathbb{H}^{1/2}} \leq \|\gamma T\|_{H^1(\Omega \setminus \Gamma) \rightarrow H^{1/2}} C(\gamma_0, p, 1) \|L\|_{\mathbb{H}^{1/2} \rightarrow H^1(\Omega \setminus \Gamma)} = C(\gamma_0, p, 1) \), which proves (iii). Finally, given \( s \in (0, \frac{1}{2}] \), we have, for all \( F \in \mathcal{M}_h^* \), by the scaled trace theorem (see Theorem 6.13 below)
\[
\|\gamma F v\|_{H^s(F)}^2 \leq C_{\text{tr}} \left( h_K^{-1} \|v\|_{L^2(K)}^2 + h_K^{1-2s} |v|_{H^1(K)}^2 \right);
\]
\[
\|\gamma F v\|_{L^2(F)}^2 \leq C_{\text{tr}} \left( h_K^{-1} \|v\|_{L^2(K)}^2 + h_K |v|_{H^1(K)}^2 \right),
\]
for all \( v \in H^1(K) \), for some constant \( C_{\text{tr}} \) depending only on \( \gamma_0 \). Applying these inequalities to \( v = Lu - \Pi_h Lu \), we obtain
\[
\sum_{F^* \in \Gamma^*_h} h_{F^*}^{2s-1} \|u - \Phi_h u\|_{H^s(F^*)}^2 \leq C_{\text{tr}} \sum_{K \in \Omega_h} h_K^{2} \|Lu - \Pi_h u\|_{L^2(K)}^2 + \|Lu - \Pi_h Lu\|_{H^1(K)}^2
\]
\[
\leq C_{\text{tr}} C(\gamma_0, p, 1) \sum_{K \in \Omega_h} \|Lu\|_{H^1(K)}^2
\]
\[
\leq C_{\text{tr}} C(\gamma_0, p, 1) \|Lu\|_{H^1(\Omega \setminus \Gamma)}^2
\]
\[
= C_{\text{tr}} C(\gamma_0, p, 1) \|u\|_{\mathbb{H}^{1/2}(\Gamma)}^2
\]
and,
\[
\sum_{F \in \mathcal{T}_h} h_F^{2s-1} \left\| u - \Phi_h u \right\|_{L^2(F)}^2 \\
\leq C \max_{K \in \mathcal{T}_h} h_K^{2s} \left\| \mathcal{L} u - \Pi_h u \right\|_{L^2(K)}^2 + h_K^{2s} \left\| \mathcal{L} u - \Pi_h \mathcal{L} u \right\|_{H^1}^2 \\
\leq C \max_{K \in \mathcal{T}_h} C(\gamma_0, p, 1) \max_{K \in \mathcal{T}_h} h_K^{2s} \left\| \mathcal{L} u \right\|_{H^1(K)}^2 \\
\leq C \max_{K \in \mathcal{T}_h} C(\gamma_0, p, 1) \text{diam}(\Omega)^{2s} \left\| \mathcal{L} u \right\|_{H^1(\Omega \setminus \Gamma)}^2 \\
= C \max_{K \in \mathcal{T}_h} C(\gamma_0, p, 1) \text{diam}(\Omega)^{2s} \left\| \mathcal{L} u \right\|_{H^{1/2}(\Gamma)}^2
\]
where diam(\Omega) is the diameter of \Omega. Summing these two estimates gives (iv). \qed

3 Applications

We present two applications of Theorems 1.1 and 1.2 which parallel the ones given in Section 5 by treating jumps instead of traces.

3.1 Bounded discrete right inverse for the jump operator

With the notation of Section 2, define the discrete jump space
\[
\tilde{V}^p(\Gamma_h) := [V^p(\Omega_h, \Gamma)]_r = \{ [u_h]_r : u_h \in V^p(\Omega_h; \Gamma) \};
\]
in other words, \( \tilde{V}^p(\Gamma_h) \) is the set of equivalence classes in \( H^1(\Omega \setminus \Gamma) \) of the elements of \( V^p(\Omega_h; \Gamma) \) for the relation “differing by a function in \( H^1(\Omega) \).”

Corollary 3.1. With the same assumptions as in Theorem 1.2, there exists a linear operator \( E_h : \tilde{H}^{1/2}(\Gamma) \rightarrow V^p(\Omega_h; \Gamma) \) such that

(i) \( [E_h \bar{\varphi}_h]_r = \bar{\varphi}_h \) for all \( \bar{\varphi}_h \in \tilde{V}^p(\Gamma_h) \),

(ii) for all \( \bar{\varphi} \in \tilde{H}^{1/2}(\Gamma) \),

\[
\left\| E_h \bar{\varphi} \right\|_{H^1(\Omega \setminus \Gamma)} \leq C(\gamma_0, p, 1) \left\| \bar{\varphi} \right\|_{\tilde{H}^{1/2}(\Gamma)}.
\]

Proof. Let \( \mathcal{E} : \tilde{H}^{1/2}(\Gamma) \rightarrow H^1(\Omega \setminus \Gamma) \) be the linear isometry such that

\[
[\mathcal{E} \bar{\varphi}]_r = \bar{\varphi} \quad \text{and} \quad \left\| \mathcal{E} \bar{\varphi} \right\|_{H^1(\Omega \setminus \Gamma)} = \left\| \bar{\varphi} \right\|_{\tilde{H}^{1/2}(\Gamma)} \quad \forall \bar{\varphi} \in \tilde{H}^{1/2}(\Gamma).
\]

We put \( E_h := \Pi_h \circ \mathcal{E} \). Given \( \bar{\varphi}_h \in \tilde{V}^p(\Gamma_h) \), let \( w_h \in V^p(\Omega_h; \Gamma) \) be such that \([w_h]_r = \bar{\varphi}_h\). Then, one has \([\mathcal{E} \bar{\varphi}_h - w_h]_r = 0\), which exactly means that \( \mathcal{E} \bar{\varphi}_h - w_h \in H^1(\Omega) \). Therefore, the same is true of \( \Pi_h(\mathcal{E} \bar{\varphi}_h - w_h) \), hence

\[
0 = [\Pi_h(\mathcal{E} \bar{\varphi}_h - w_h)]_r = [\Pi_h(\mathcal{E} \bar{\varphi}_h) - \Pi_h w_h]_r = [E_h \bar{\varphi}_h]_r - [w_h]_r = [E_h \bar{\varphi}_h]_r - \bar{\varphi}_h.
\]

This proves (i). The property (ii) is immediate. \qed
3.2 Boundary value problems with prescribed jumps

In [33, Section 5], Scott and Zhang show that their interpolant provides a “systematical way for averaging [a prescribed] boundary data” in a boundary value problem. Here we translate this idea to solve an elliptic boundary value problem with a prescribed jump

\[- \sum_{i,j=1}^{3} \frac{\partial}{\partial x_j} \left( \alpha_{ij} \frac{\partial u}{\partial x_i} \right) = 0 \quad \text{in} \quad \Omega \setminus \Gamma, \]

\[ [u]_\Gamma = [g]_\Gamma, \]

\[ u = 0 \quad \text{on} \quad \partial \Omega, \]

for some given \( g \in H^1(\Omega \setminus \Gamma) \) satisfying \( g = 0 \) on \( \partial \Omega \). As in [33, Section 5], we assume that \( \alpha \) is bounded and symmetric positive definite a.e. on \( \Omega \). The weak solution \( u \) of (3.1) is defined as the unique element of \( H^1(\Omega \setminus \Gamma) \) satisfying \( u - g \in H^1_0(\Omega) \) and \( a(u - g, v) = 0 \) for all \( v \in H^1_0(\Omega) \), where

\[ a(u, v) := \int_{\Omega} \sum_{i,j=1}^{3} \alpha_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx. \]

Define \( V^p_0(\Omega_h) := V^p(\Omega_h; \Gamma) \cap H^1_0(\Omega) \), and let

\[ V^{p,g}(\Omega_h; \Gamma) := \{ v_h \in V^p(\Omega_h; \Gamma) \mid v_h - \Pi_h g \in V^p_0(\Omega_h) \} \]

(3.2)

We can then define an approximation \( u_h \in V^{p,g}(\Omega_h; \Gamma) \) by

\[ a(u_h, v_h) = 0 \quad \forall v_h \in V^p_0(\Omega_h) \]

and using the same method of proof as in [33], we obtain

\[ \| u - u_h \|_{H^1(\Omega \setminus \Gamma)} \leq \max \left\{ \| \alpha_{i,j} \|_{L^\infty(\Omega)} \mid 1 \leq i, j \leq 3 \right\} h^{l-1} \| u \|_{H^l(\Omega \setminus \Gamma)}, \quad 1 \leq l \leq p + 1, \]

where \( h = \max_{K \in \Omega_h} \bar{h}_K \).

4 Outline of the construction of \( \Pi_h \)

Replacing \( \Gamma \) by \( \partial \Omega \) and ignoring property (iii) of Theorem 1.1, an operator meeting the remaining requirements is given by the Scott and Zhang interpolant \( \mathcal{Z}_h \) [33]. This operator acts on a function \( u \) as

\[ \mathcal{Z}_h u := \sum_{i=1}^{N} \left( \int_{\sigma_i} \psi_i(x) u(x) \, dx \right) \phi_i =: \sum_{i=1}^{N} N_i(u) \phi_i. \]

(4.1)

Here, \( \{ \phi_i \}_{i=1}^{N} \) is the Lagrange nodal basis of \( V^p(\Omega_h) \), the space of continuous piecewise polynomial functions of degree \( p \) on \( \Omega_h \). For Lagrange nodes \( x_i \) lying in the interior of some mesh element \( K_i \in \Omega_h \), we take \( \sigma_i := K_i \). For \( x_i \in \partial K_i \), \( \sigma_i \) is instead a \((d-1)\)-simplex, freely chosen among the faces of \( \Omega_h \) incident to \( x_i \), with the sole restriction that \( \sigma_i \subset \partial \Omega \) if \( x_i \in \partial \Omega \). Finally, \( \psi_i \) is a “dual Lagrange polynomial” with the property that

\[ \int_{\sigma_i} \psi_i(x) P(x) \, dx = P(x_i) \]
for any polynomial $P$ of degree at most $p$. It is immediate that $Z_h u_h = u_h$ for all $u_h \in V^p(\Omega_h)$, and the restriction on $\sigma_i$ also ensure $(\Pi_h u)|_{\partial \Omega} = 0$ when $u|_{\partial \Omega} = 0$.

One may try to obtain a jump-aware interpolant by adapting the definition in (4.1). The first important change is that for $x_i \in \Gamma$, there generally needs to be more than just one basis function associated to $x_i$, to account for the fact that elements of $V^p(\Omega_h; \Gamma)$ can have several distinct limits at $x_i$ depending on the considered “side” of $\Gamma$, see Figure 2.

Labeling the sides of $\Gamma$ around $x_i$ from 1 to $q_i$, one can construct a nodal basis $\{\phi_{i,j}\}_{1 \leq i \leq N, 1 \leq j \leq q_i}$ with the following properties. Any element $u \in V^p(\Omega_h; \Gamma)$ can be written as

$$u = \sum_{i,j} u(x_{i,j}) \phi_{i,j},$$

where the coefficient $u(x_{i,j})$ is equal to the limit of $u(x)$ as $x$ approaches $x_i$ from side $j$. Furthermore,

$$\phi_i = \sum_{j=1}^{q_i} \phi_{i,j} \quad \text{and} \quad \sum_{j=1}^{q_i} \lambda_j \phi_{i,j} \in H^1(\Omega) \iff \lambda_j = \lambda_{j'} \quad \forall 1 \leq j, j' \leq q_i.$$

With these definitions, we set

$$\Pi_h u := \sum_{1 \leq i \leq N} \sum_{1 \leq j \leq q_i} N_{i,j}(u) \phi_{i,j},$$

where the $N_{i,j}$ are suitably chosen linear form. For $x_i$ away from $\Gamma$, $q_i = 1$ and $N_{i,1}(u)$ will be defined exactly as $N_i$ in eq. (4.1); all the difficulty resides in the case where $x_i \in \Gamma$ and $q_i > 1$. In this case, one can attempt to define

$$N_{i,j}(u) := \int_{\sigma_{i,j}} \psi_{i,j}(x) u_{i,j}(x) \, dx.$$

Here $\sigma_{i,j} \subset \Gamma$ is any face $F \in \mathcal{F}(\Omega_h)$ which contains $x_i$ and lies in the boundary of the $j$-th side of $\Gamma$ around $x_i$. Like above, $\psi_{i,j}$ is the dual Lagrange polynomial associated to $x_i$ on $\sigma_{i,j}$, and $u_{i,j}$ is the restriction of (the polynomial function) $u|_{K_{i,j}}$ to $\sigma_{i,j}$.

The formula (4.2) comes close to satisfying the requirements. It guarantees the conditions $\Pi_h u_h = u_h$ and $(\Pi_h u)|_{\Gamma} = 0$ when $u|_{\Gamma} = 0$. To investigate the condition about jumps, suppose that $u \in H^1(\Omega)$. To fulfill requirement (iii) of Theorem 1.2, we need to ensure that $N_{i,j}(u) = N_{i,j'}(u)$ for
all \(1 \leq i \leq N\), \(1 \leq j, j' \leq q_i\). If \(q_i = 2\), i.e., if there are just two sides of \(\Gamma\) around \(x_i\) (as for example when \(\Gamma\) is a manifold with boundary), this can be arranged by choosing \(\sigma_{i,1} = \sigma_{i,2}\) in eq. (4.2). However, when there are more than two sides, there is always a pair \(j, j'\) such that \(\sigma_{i,j} \neq \sigma_{i,j'}\). As soon as this is the case, one can easily find a smooth function \(u\) such that that \(N_{i,j}(u) = N_{i,j'}(u)\), showing that eq. (4.2) cannot meet the requirements.

A remedy is to define

\[
N_{i,j}(u) = N_i(u) + \sum_{i,k,\ell} \mu_{i,k,\ell} N_{i,k,\ell}
\]

where \(N_i\) is as in (4.1), and the \(N_{i,k,\ell}\), that we call bridge functions, are of the form

\[
N_{i,k,\ell}(u) = \int_{\sigma_{i,k,\ell}} \psi_{i,k,\ell}(x)(u_{i,k}(x) - u_{i,\ell}(x)) \, dx,
\]

where \(\sigma_{i,k,\ell} \subset \Gamma\) is a bridge that is, a mesh face \(F \subset \Gamma\) incident to \(x_i\), and located at the interface between the sides \(k\) and \(\ell\) of \(\Gamma\) around \(x_i\) (see Figure 2). The key property is that \(N_{i,k,\ell}(u) = 0\) when \(u \in H^1(\Omega)\), so that bridge functions do not contribute to the value of \(N_{i,j}(u)\) in this case. This ensures that \(\Pi h u \in H^1(\Omega)\) whenever \(u \in H^1(\Omega)\).

It remains to make sure that \(\Pi h u_h = u_h\) for all \(u_h \in V^p(\Omega_h; \Gamma)\), and it turns out that one can always find a set of coefficients \(\mu_{i,k,\ell}\) such that this property holds; as we shall see, this is related to the fact that the graph with nodes the sides \(1, \ldots, q_i\) and with edges the bridges \(\{k, \ell\}\), is connected.

5 Definitions and notation

We now gather definitions and notations in preparation for the proof of Theorem 1.1 and Theorem 1.2.

Simplices

An \(n\)-simplex \(S \subset \mathbb{R}^d\), \(n \leq d\), is the closed convex hull of \(n + 1\) affinely independent points in \(\mathbb{R}^d\) called its vertices. A \(k\)-subsimplex \(\sigma\) of \(S\) is a \(k\)-simplex, \(k \leq n\), whose vertices are also vertices of \(S\). If \(k = (n - 1)\), \(\sigma\) is called a face of \(S\). We denote by \(\mathcal{F}(S)\) the set of faces of \(S\). We say that \(\sigma\) is incident to \(S\) if it is a subsimplex of \(\sigma\), and that two \(n\)-simplices are adjacent if they share a face.

Simplicial meshes

An \(n\)-dimensional simplicial mesh \(\mathcal{M}\) is a finite set of \(n\)-simplices, called (mesh) elements, such that if \(K, K' \in \mathcal{M}\), then \(K \cap K'\) is either empty, or equal to a subsimplex of both \(K\) and \(K'\). The set of faces of \(\mathcal{M}\) is \(\mathcal{F}(\mathcal{M}) := \bigcup_{K \in \mathcal{M}} \mathcal{F}(K)\). We say that \(\mathcal{M}\) is face-connected if, for any elements \(K, K' \in \mathcal{M}\), there exists a sequence \(K_1, \ldots, K_N\) of elements of \(\mathcal{M}\) such that \(K_1 = K\), \(K_N = K'\) and for all \(i \in \{1, \ldots, N - 1\}\), the elements \(K_i\) and \(K_{i+1}\) are adjacent. The boundary \(\partial \mathcal{M}\) is the \((n - 1)\)-dimensional mesh whose elements are the faces of \(\mathcal{M}\) incident to only one element of \(\mathcal{M}\).

Conforming meshes

For a \(n\)-dimensional simplicial mesh \(\mathcal{M}\) in \(\mathbb{R}^d\), we denote by \(|\mathcal{M}|\) the closed subset of \(\mathbb{R}^d\) defined by

\[
|\mathcal{M}| := \bigcup_{K \in \mathcal{M}} K.
\]
Given a (closed) set $M \subset \mathbb{R}^d$, we say that $\mathcal{M}$ is a conforming mesh of $M$ if $|\mathcal{M}| = M$.

**Regular meshes**

An $n$-dimensional simplicial mesh $\mathcal{M}$ is regular if it is a conforming mesh of some $n$-dimensional submanifold of $\mathbb{R}^d$ (possibly with boundary). In other words, $\mathcal{M}$ is regular if, for every point of $|\mathcal{M}|$, there is an open set $U_x \subset \mathbb{R}^d$ containing $x$ such that $U_x$ is homeomorphic to either $\mathbb{R}^n$ or $\mathbb{R}^{n-1} \times \mathbb{R}_+$. Recall that for a positive real number $\gamma > 0$, we say that $\mathcal{M}$ is $\gamma$-shape-regular if

$$\max_{K \in \mathcal{M}} \frac{h_K}{\rho_K} \leq \gamma,$$

where $h_K$ is the diameter of $K$ and $\rho_K$ is the radius of the largest $n$-dimensional closed ball contained in $K$.

We will use the following well-known facts about meshes. We haven’t found a complete proof of the first one in the literature, so we give one for completeness. The second one is proven in [5, Lemma 11.1.3] for connected “d-triangulations”, and by [26, Corollary 1.16] and the remark following it, since $\mathcal{M}$ is a regular mesh, $\text{st}(v)$ (or more precisely, the underlying simplicial complex) is indeed a $d$-triangulation.

**Proposition 5.1** (Solid angles in shape-regular meshes). Let $K \subset \mathbb{R}^d$ be a $d$-simplex and $v$ a vertex of $K$. The normalized solid angle $\Omega_v(K)$ of the simplex $K$ from $v$ satisfies

$$\Omega_v(K) \geq \frac{1}{d} \frac{\rho_K^d}{h_K^d} \frac{\omega_{d-1}}{\omega_d},$$

where $\omega_n$ is the volume of the Euclidean unit ball in $\mathbb{R}^n$. Therefore, if $\mathcal{M}$ is a $\gamma$-shape regular $d$-dimensional mesh in $\mathbb{R}^d$, the number $N$ of mesh elements sharing a given vertex is bounded by

$$N \leq d\gamma^d \frac{\omega_d}{\omega_{d-1}}.$$

For the definition of the normalized solid angle, refer to [30].

**Proof.** To show the first inequality, we consider the solid $d$-dimensional cone $C$ with apex $v$ and base $S$, where $S$ is a $(d-1)$-dimensional ball centered at the the circumcenter $c$ of $K$, with radius $\rho_K$, and lying in the plane orthogonal to the line $(vc)$. This is illustrated in Figure 3. The solid angle spanned by $K$ from $v$ is bounded from below by the solid angle spanned by $C$, for which we can readily prove the claimed estimate. If $K_1, \ldots, K_n$ are mesh elements incident to $v$, the normalized solid angle of $\text{int}(K_1) \cup \ldots \cup \text{int}(K_n)$ (where $\text{int}(E)$ stands for the interior of the set $E$) from $v$ is at most 1, by definition of the normalized solid angle, and equal to $\sum_{i=1}^n \Omega_v(K_i)$ due to the convexity of the simplices $K_i$ and the fact that their interior are pairwise disjoint. This immediately leads to the second inequality. $\square$

**Proposition 5.2** (Local face-connectedness in regular meshes). Let $\mathcal{M}$ be a regular $n$-dimensional simplicial mesh with $n \geq 1$, let $v$ be a vertex of some mesh element $K$ and let $\text{st}(v)$ be the set of mesh elements incident to $v$. Then $\text{st}(v)$ is face-connected.
Figure 3: Figure for the proof of Theorem 5.1. The cone \( C \) is shaded in gray.

**The sets \( \Omega \) and \( \Gamma \)**

For the remainder of this paper, we fix \( \Omega \subset \mathbb{R}^d \) a Lipshitz polyhedron, that is, a bounded open set with Lipshitz regular boundary (see [20, Definition 4.4]) such that \( \overline{\Omega} \) admits a conforming, regular \( d \)-dimensional mesh \( \mathcal{M}_\Omega \). We also fix a (possibly non-regular) mesh \( \mathcal{M}_\Gamma \) such that

\[
\mathcal{M}_\Gamma \subset \mathcal{F}(\mathcal{M}_\Omega) \setminus \partial \mathcal{M}_\Omega ,
\]

and let \( \Gamma = |\mathcal{M}_\Gamma| \).

**Continuous function spaces**

For an open set \( U \subset \mathbb{R}^d \), \( C^\infty(U) \) (resp. \( C^\infty_c(U) \)) is the set of infinitely differentiable functions on \( U \) (resp. which are furthermore compactly supported on \( U \)), and \( C^\infty(U) \) is the set of restrictions to \( U \) of elements of \( C^\infty(\mathbb{R}^d) \). For \( l \in \mathbb{N} \), let \( H^l(U) \) be the completion of \( C^\infty(U) \) for the norm

\[
\|f\|^2_{H^l(U)} := \sum_{|\alpha| \leq l} \|D^\alpha f\|_{L^2(U)}^2 .
\]

Here, we use the multi-index notation \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \), where \( \alpha_1, \alpha_2, \alpha_3 \in \mathbb{N} = \{0, 1, 2, \ldots\} \), \( |\alpha| = \sum_{i=1}^d \alpha_i \) and \( D^\alpha = (\partial/\partial x_1)^{\alpha_1} (\partial/\partial x_2)^{\alpha_2} \ldots (\partial/\partial x_d)^{\alpha_d} \). A semi-norm is defined by

\[
|f|_{H^s(U)} := \sum_{|\alpha| = l} |D^\alpha f|_{L^2(U)}^2 .
\]

Finally, let \( H^s_{\partial, \Gamma}(\Omega) \) be the closure of \( C^\infty_c(\Omega \setminus \Gamma) \) in \( H^s(\Omega \setminus \Gamma) \). If \( K \subset \mathbb{R}^d \) is a \( d \)-simplex, we write \( H^s(K) \) in place of \( H^s(\text{int}(K)) \) for all \( s \geq 0 \).
Traces

If $K \subset \mathbb{R}^d$ is a $d$-simplex, the restriction operator $C^\infty(K) \to L^2(\partial K)$, $u \mapsto u|_{\partial K}$ has a unique continuous linear extension $\gamma_K : H^1(K) \to L^2(\partial K)$, and for all $u \in H^1(K)$ and $\varphi \in C^\infty(\mathbb{R}^d)^3$,

$$\int_{\Omega} u(x) \operatorname{div} \varphi(x) + \nabla u(x) \cdot \varphi(x) \, dx = \int_{\partial K} (\gamma_K u)(x) \varphi(x) \cdot n_K(x) \, d\mathcal{H}^{d-1}(x), \quad (5.3)$$

where $n_K$ is the unit outer normal vector and $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure, see [20, Theorem 4.6].

We shall also require fractional Sobolev spaces defined over faces of $d$-simplices. Let $F$ be a $n$-simplex in $\mathbb{R}^d$, with $n \geq 2$ (but possibly $n < d$) we define $H^s(F)$ for $s \in (0,1)$ as the subset of $L^2(F)$ of functions satisfying

$$|u|^2_{H^s(F)} := \int_F \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, d\mathcal{H}^n(x) \, d\mathcal{H}^n(y),$$

and let

$$||u||^2_{H^s(F)} := ||u||^2_{L^2(F)} + |u|^2_{H^s(F)} .$$

If $K$ is a $d$-simplex such that and $F \in \mathcal{F}(K)$ is a face of $K$, $s \in (\frac{1}{2},1]$, then there exists a constant $C$ (depending on $K$ and $s$) such that the restriction operator $\gamma_{[K,F]} : C^\infty(K) \to L^2(F)$ defined by $\gamma_{[K,F]} u := u|_{F}$ satisfies

$$||\gamma_{[K,F]} u||_{H^{s-\frac{1}{2}}(F)} \leq C \||u||_{H^s(K)} \quad (5.4)$$

for all $u \in H^s(K)$, see [25, Theorem 3.37]. The operator $\gamma_{[K,F]}$ admits a unique linear continuous extension from $H^s(K) \to H^{s-\frac{1}{2}}(F)$, which we denote again by $\gamma_{[K,F]}$. Furthermore, if $U \subset \mathbb{R}^d$ is an open set such that $\text{int}(K) \subset U$, then for any $u \in H^1(U)$, one has $u|_K \in H^1(K)$ and we still denote

$$\gamma_{[K,F]} u := \gamma_{[K,F]} u|_K .$$

Discrete spaces

Given a conforming mesh $\Omega_h$ of $\Omega$ (possibly other than $\mathcal{M}_\Omega$ itself), we define

$$V^p(\Omega_h; \Gamma) := \{ u \in H^1(\Omega \setminus \Gamma) \mid u|_K \in \mathbb{P}_p \text{ for all } K \in \Omega_h \} \quad (5.5)$$

where $\mathbb{P}_p$ is the set of polynomials of degree $p$. We also define

$$V^p(\Omega_h) := \{ u \in H^1(\Omega) \mid u|_K \in \mathbb{P}_p \text{ for all } K \in \Omega_h \} = V^p(\Omega_h; \Gamma) \cap H^1(\Omega) .$$

Mesh conditions

Given a conforming mesh $\Omega_h$ of $\Omega$, we say that $\Omega_h$ resolves $\Gamma$ if there is a conforming mesh of $\Gamma$ made of faces of $\Omega_h$, that is

$$\Gamma = |\Gamma_h| \quad \text{with} \quad \Gamma_h \subset \mathcal{F}(\Omega_h) .$$

Furthermore, $\Omega_h$ strictly encloses $\Gamma$ if

$$(K \cap \partial \Omega \neq \emptyset) \Rightarrow (K \cap \Gamma = \emptyset) \quad \forall K \in \Omega_h .$$

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Lagrange nodes

Let $S \subset \mathbb{R}^d$ be a $n$-simplex, and label its vertices $v_1, \ldots, v_{n+1}$. The Lagrange nodes of $S$ are the points of $S$ defined by

$$\mathcal{L}_p(S) := \left\{ \frac{1}{p} \sum_{i=1}^{n+1} \alpha_i v_i \bigg| \alpha \in \mathbb{N}^{n+1} \text{ s.t. } |\alpha| = p \right\}.$$ 

Note that if $S'$ is a subsimplex of $S$, then the Lagrange nodes on $S'$ are Lagrange nodes on $S$, i.e. $\mathcal{L}_p(S') \subset \mathcal{L}_p(S)$. The Lagrange nodes of a mesh $M$ are the Lagrange nodes of its elements, i.e.

$$\mathcal{L}_p(M) := \bigcup_{K \in M} \mathcal{L}_p(K).$$

6 Construction of $\Pi_h$

In this section we define $\Pi_h$ and prove Theorems 1.1 and 1.2. The construction involves a primal and a dual basis, which are defined in Section 6.1 and Section 6.2, respectively. The definition of $\Pi_h$ and the proof are given in Section 6.3.

From now on, we fix integers $d \geq 2$ and $p \geq 1$ and real numbers $\gamma_0 > 0$ and $t > \frac{1}{2}$. $\Omega$ and $\Gamma$ are fixed as in the previous section. We write $a \lesssim b$ when there exists $C > 0$ whose value depends only on $\Omega, \Gamma, d, p, \gamma_0$ and $t$, such that $a \leq Cb$. We fix a conforming $\gamma_0$-shape-regular mesh $\Omega_h$ of $\Omega$ which resolves and strictly encloses $\Gamma$, and let $\Gamma_h$ be the mesh of $\Gamma$ constituted of faces of $\Omega_h$.

6.1 Primal basis of $V^p(\Omega_h; \Gamma)$

It is well-known that a piecewise-polynomial function $u_h$ on $\Omega_h$ belongs to $H^1(\Omega)$ if, and only if, its polynomial values in any two adjacent mesh elements “match” at the common face. The condition $u_h \in H^1(\Omega \setminus \Gamma)$ similarly involves the continuity of traces at faces of adjacent elements, except when this face is in $\Gamma$.

**Lemma 6.1.** Let $f : \Omega \setminus \Gamma \to \mathbb{R}$ be such that for all $K \in \Omega_h$, there exists $f_K \in H^1(\Omega)$ such that $f|_K = f_K$. Then the following properties are equivalent

(i) $f \in H^1(\Omega \setminus \Gamma),$

(ii) For any pair of adjacent mesh elements $K, K' \in \Omega_h$ sharing a face $F \notin \Gamma_h$,

$$\gamma_{[K,F]} f_K = \gamma_{[K',F]} f_{K'}.$$ 

**Proof.** Recall that by the $H = W$ theorem [29], $f \in H^1(\Omega \setminus \Gamma)$ if and only if there exists a square-integrable vector field $g \in L^2(\Omega \setminus \Gamma)^d$ such that

$$\int_{\Omega \setminus \Gamma} f(x) \div(\varphi(x)) \, dx = -\int_{\Omega \setminus \Gamma} g(x) \cdot \varphi(x) \, dx$$

for all $\varphi(x) \in C_c^\infty(\Omega \setminus \Gamma)^d$. The proof that this happens if and only if (ii) holds is classical. \qed
One sees from this result that elements of $V^p(\Omega_h; \Gamma)$ may be discontinuous, and especially have non-zero jumps through faces in $\Gamma$. On the other hand, not all polynomial (of degree $p$) jumps are possible through a given face $F$. Indeed, if for some face $F \in \Gamma_h$ and some Lagrange node $x$ on $F$, there is a chain of mesh elements $K_1, \ldots, K_r \in \Omega_h$ with $K_1 = K_-(F)$, $K_r = K_+(F)$, such that $K_i$ and $K_{i+1}$ share a face $F_i$ not in $\Gamma$ containing $x$ (see Figure 4), then the polynomial functions $v_{h|K_i}$ and $v_{h|K_{i+1}}$ agree at $x$. Thus, $v_{h|K_+(F)}(x) = v_{h|K_-(F)}(x)$, hence the jump $[v_h]_F$ vanishes at $x$, for all $v_h \in V^p(\Omega_h; \Gamma)$.

To construct a basis of $V^p(\Omega_h; \Gamma)$, it is thus necessary to capture the couplings produced by this kind of chain of elements. In order to do this, denote by $\{x_1, \ldots, x_M\}$ the set of Lagrange nodes on $\Omega_h$, with

$$\{x_1, \ldots, x_M\} = \mathcal{L}_p(\Gamma_h) \quad \text{and} \quad \{x_{M+1}, \ldots, x_N\} = \mathcal{L}_p(\Omega_h) \setminus \mathcal{L}_p(\Gamma_h).$$

Moreover, let $\{\phi_i\}_{1 \leq i \leq N}$ be the nodal basis of $V^p(\Omega_h)$, that is, the set of elements of $V^p(\Omega_h)$ defined by

$$\phi_i(x_{i'}) = \delta_{i,i'}, \quad 1 \leq i, i' \leq N.$$

Define the star of a subsimplex $\sigma$ of $K \in M$, as the mesh

$$\text{st}(\sigma, M) := \{K \in M \mid K \cap \sigma \neq \emptyset\},$$

and let $\text{st}(x_i) := \text{st}(x_i, \Omega_h)$ be the set of mesh elements $K \in \Omega_h$ incident to $x_i$. We define the following (unoriented) graph $G(x_i)$ (see also [4]):

- **Nodes:** The mesh elements $K \in \text{st}(x_i)$,
- **Edges:** The pairs $\{K, K'\} \subset \text{st}(x_i)$ such that $K$ and $K'$ share a face $F \notin \Gamma_h$.

Let $q_i$ be the number of connected components of $G(x_i)$. We can then partition the mesh elements $K \in \Omega_h$ incident to $x_i$ as

$$\text{st}(x_i) = \text{st}(x_i; 1) \cup \ldots \cup \text{st}(x_i; q_i).$$

This is illustrated in Figure 5. Each of the disjoint sets $\text{st}(x_i; j)$ corresponds to one connected component of $G(x_i)$, and is thus a face-connected mesh, with face-connectedness holding through faces not in $\Gamma$. The quantity $\max_{1 \leq i \leq N} q_i$ is invariant by mesh subdivision of $\Omega_h$ resolving $\Gamma$, and thus

$$\max_{1 \leq i \leq N} q_i \lesssim 1. \quad (6.1)$$
Figure 5: A Lagrange node $\mathbf{x}_i$ on the hypersurface $\Gamma$ (solid black line), its star $\text{st}(\mathbf{x}_i)$ (filled in gray), and the components $\text{st}(\mathbf{x}_i; j)$ for $j = 1, 2, 3$. The gray solid lines represent the boundary of some mesh elements of $\Omega_h$.

Let $I := \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i \leq N, 1 \leq j \leq q_i\}$. For $(i, j) \in I$, we define the split basis function $\phi_{i,j} : \Omega \setminus \Gamma \to \mathbb{R}$ by

$$
\phi_{i,j}(x) := \begin{cases} 
\phi_i(x) & \text{for } x \in \text{int}(K) \text{ such that } K \in \text{st}(\mathbf{x}_i; j), \\
0 & \text{otherwise}.
\end{cases}
$$

Note that $\sum_{j=1}^{q_i} \phi_{i,j} = \phi_i$ and by Lemma 6.1, $\phi_{i,j} \in V^p(\Omega_h; \Gamma)$ for all $(i, j) \in I$.

**Lemma 6.2.** The family $\{\phi_{i,j}\}_{(i,j) \in I}$ is a basis of $V^p(\Omega_h; \Gamma)$.

**Proof.** Suppose that there exist coefficients $\lambda_{i,j}$ such that

$$
\forall x \in \Omega \setminus \Gamma, \quad \sum_{(i,j) \in \mathcal{H}(\Omega)} \lambda_{i,j} \phi_{i,j}(x) = 0.
$$

(6.2)

Fix $(i_0, j_0) \in \mathcal{H}(\Omega)$ and let $K \in \text{st}(\mathbf{x}_{i_0}; j_0)$. Let $(y_n)_{n \in \mathbb{N}}$ be a sequence of points in the interior of $K$, such that

$$
\lim_{n \to \infty} y_n = \mathbf{x}_{i_0}.
$$

It is easy to show that $\lim_{n \to \infty} \phi_{i,j}(y_n) \to \delta_{i,i_0} \delta_{j,j_0}$. Therefore, applying eq. (6.2) to $y_n$ and passing to the limit, we conclude that $\lambda_{i_0,j_0} = 0$. This proves that the functions $\phi_{i,j}$ are linearly independent.

Next, we consider $u_h \in V^p(\Omega_h; \Gamma)$. For each $(i, j) \in \mathcal{H}(\Omega)$, we fix a mesh element $K_{i,j} \in \text{st}(\mathbf{x}_i; j)$ and a sequence $(y_n^{i,j})_{n \in \mathbb{N}}$ converging to $\mathbf{x}_i$ from $K_{i,j}$, as above. Let $\lambda_{i,j} := \lim_{n \to \infty} u_h(y_n^{i,j})$. We prove that

$$
u_h = \sum_{(i,j) \in \mathcal{H}(\Omega)} \lambda_{i,j} \phi_{i,j}.
$$

(6.3)

by showing that this equality holds on the interior of each mesh element $K \in \Omega_h$. Thus let us fix an element $K$ and write its Lagrange nodes as $\mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_R}$. For each $r \in \{1, \ldots, R\}$, let $j_r$ be such that $K \in \text{st}(\mathbf{x}_{i_r}; j_r)$. Let $y_n^r$ be a sequence of points in the interior of $K$ converging to $\mathbf{x}_{i_r}$. We claim that

$$
\lim_{n \to \infty} \phi_{i,j}(y_n^r) = \delta_{i,i_r} \delta_{j,j_r}, \quad \lim_{n \to \infty} u_h(y_n^r) = \lambda_{i_r,j_r}.
$$

(6.4)
Using that there holds Corollary 6.3, and Theorem 6.2, one deduces that we identify $V$ and for $(i,j) \in I$ we define the discrete scalar product

$$\forall (u_h, v_h) \in V^p(\Omega_h; \Gamma), \quad [u_h, v_h]_{i^2} := \sum_{(i,j) \in I} u_h(x_{i,j}) v_h(x_{i,j}).$$

We identify $V^p(\Omega_h)$ with the subspace of $V^p(\Omega_h; \Gamma)$ defined by $u_h(x_{i,j}) = u_h(x_{i,j'})$ for all $1 \leq j, j' \leq q_i$ and let $\Psi^p_k(\Omega; \Gamma)$ be the $[,]_{i^2}$ orthogonal complement of $V^p(\Omega_h)$ in $V^p(\Omega_h; \Gamma)$. Let

$$\Tilde{I} := \{(i,j) \in I | q_i > 1 \text{ and } 1 \leq j \leq q_i - 1 \},$$

and for $(i,j) \in \Tilde{I}$, define

$$\psi_{i,j} := \phi_{i,j} - \phi_{i,q_i}.$$ 

Using that $\{\phi_i\}_{1 \leq i \leq N}$ is a basis of $V^p(\Omega_h)$, together with the property

$$\phi_i = \sum_{j=1}^{q_i} \phi_{i,j}$$

and Theorem 6.2 one deduces that $\{\psi_{i,j}\}_{(i,j) \in \Tilde{I}}$ is a basis of $\Psi^p(\Omega_h; \Gamma)$.

**Corollary 6.3.** There holds

$$V^p(\Omega_h; \Gamma) = V^p(\Omega_h) \oplus \Psi^p(\Omega_h; \Gamma),$$

and $\{\phi_i\}_{1 \leq i \leq N}$, $\{\psi_{i,j}\}_{(i,j) \in \Tilde{I}}$ are bases of $V^p(\Omega_h)$ and $\Psi^p(\Omega_h; \Gamma)$, respectively.

### 6.2 Dual basis of $V^p(\Omega_h; \Gamma)$

With the primal basis $\{\phi_i\}_{1 \leq i \leq N} \cup \{\psi_{i,j}\}_{(i,j) \in \Tilde{I}}$ at hand, we now set out to define a “dual basis” $\{N_i\}_{1 \leq i \leq N} \cup \{N_{i,j}\}_{(i,j) \in \Tilde{I}}$, consisting of linear forms on $N_i, N_{k,l} : H^1(\Omega \setminus \Gamma) \rightarrow \mathbb{R}$ such that, among other properties,

$$N_i(\phi_i') = \delta_{i,i'}, \quad N_i(\psi_{k,j'}) = 0, \quad N_{k,l}(\phi_i') = 0, \quad N_{k,l}(\psi_{k',j'}) = \delta_{k,k',l,l'},$$

3If $x_{i,r}$ is an interior Lagrange node, that is, $x_{i,r} \in \text{int}(K)$, then one always has $K = K_{i_r,j_r}$. However, if $x_{i_r,j_r}$ lies on $\partial K$, then $K_{i_r,j_r}$ can be any other mesh element in $st(x_{i_r,j_r})$. 

Only the second limit deserves attention. It is immediate if $K = K_{i_r,j_r}$ If $K$ shares a face $F \notin \mathcal{M}_r$ with $K_{i_r,j_r}$, then it is a consequence of Lemma 6.1. Otherwise, by definition of $G(x_i)$, one can find a face-connected (through faces not in $\Gamma$) path of mesh elements from $K$ to $K_{i_r,j_r}$, and the desired limit is established by repeating the previous argument for each pair of consecutive elements in this path.

Having established the equalities in (6.4), we conclude that the polynomials defined on $K$ by each side of eq. (6.3) coincide at all the Lagrange points in $K$, and are both of degree $p$, thus they are equal on $K$ by unisolvency of $L_p(K)$.

For $u_h \in V^p(\Omega_h; \Gamma)$, we will denote by $u_h(x_{i,j})$ the coefficient of $u_h$ on the split basis function $\phi_{i,j}$, so that

$$u_h = \sum_{(i,j) \in I} u_h(x_{i,j}) \phi_{i,j}.$$
for all $1 \leq i, i' \leq N$ and $(k, l), (k', l') \in \tilde{I}$. As in Scott and Zhang's original construction [33], we will use Lagrange dual polynomials:

**Definition 6.4 (Lagrange dual polynomials).** For any $\gamma_0$ shape-regular $n$-simplex $S \subset \mathbb{R}^d$ and any Lagrange node $x \in \mathcal{L}_p(S)$, there exists a polynomial $\psi_{[S,x]} \in \mathbb{P}_p$ such that

$$
\int_S \psi_{[S,x]}(y) P(y) \, dH^n(y) = P(x)
$$

for all $P \in \mathbb{P}_r$, and furthermore,

$$
\|\psi_{[S,x]}\|_{L^2(F)} \lesssim \text{diam}(S)^{-n/2}.
$$

This is shown by considering the dual basis of the vector space $\mathbb{P}_r$ on the reference element, representing it under the form (6.6) (with $S$ replaced by the reference element) via the Riesz representation theorem, and mapping back to $S$, tracking the scaling of the $L^2$ norm accordingly, see [33, Lemma 3.1] and [9, eq. (3.3)].

**Lemma 6.5.** Let $u_h \in V^p(\Omega_h; \Gamma)$ and let $K \in \Omega_h$, $F \in \mathcal{F}(K)$ and $x_i \in \mathcal{L}_p(F)$ a Lagrange node on $F$. Let $1 \leq j \leq q_i$ be such that $K \in \text{st}(x_i; j)$. Then

$$
\int_F \psi_{[F,x_i]} \gamma_{[K,F]} u_h(x) \, dH^{d-1}(x) = u_h(x_{i,j}).
$$

**Proof.** This is immediate from the definition of $\phi_{i,j}$ and $\psi_{[F,x_i]}$. $\square$

The next lemma deals with the construction of $\{N_i\}_{1 \leq i \leq N}$.

**Lemma 6.6.** For every $i \in \{1, \ldots, N\}$, there exists a linear form $N_i : H^1(\Omega \setminus \Gamma) \to \mathbb{R}$, such that the following properties hold:

(i) $N_i(u_h) = 0$ for all $u_h \in \Psi^p(\Omega_h; \Gamma)$

(ii) $N_i(\phi_{i'}) = \delta_{i,i'}$ for all $i, i' \in \{1, \ldots, N\}$.

(iii) If $u \in H_0^1(\Omega)$, then $N_i(u) = 0$ for all $i$ such that $x_i \in \Gamma \cup \partial\Omega$.

(iv) $N_i$ is a linear combination of terms of the form

$$
w \mapsto \int_K \psi_{[K,x_i]} w(x) \, dH^d(x)
$$

and

$$
w \mapsto \int_F \psi_{[F,x_i]}(x) \gamma_{[K,F]} w(x) \, dH^{d-1}(x),
$$

where $K$ (resp. $F$) is a mesh element of $\Omega_h$ (resp. of $\mathcal{F}(\Omega_h)$) incident to $x_i$ and $K_F \in \text{st}(x_i)$ is such that $F \in \mathcal{F}(K_F)$. The number $N$ of terms in this linear combination satisfies $N \lesssim 1$. 

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Proof. Let \( i \in \{1, \ldots, N\} \) be such that \( x_i \notin \Gamma \). We then choose a “control simplex” \( \sigma_i = \sigma_{i,1} \) such that \( x_i \in \sigma_i \) as in \([33]\). That is, if \( x_i \) is an interior Lagrange node, then we take for \( \sigma_i \) the unique mesh element \( K_i \in \Omega_h \) such that \( x_i \in K_i \), and otherwise, we choose for \( \sigma_i \) any mesh face in \( F(\Omega_h) \) containing \( x_i \), with the restriction that \( F \subset \partial \Omega \) if \( x_i \in \partial \Omega \). We then put

\[
N_i(u) := \int_{\sigma_i} \psi_{[\sigma_i,x_i]}(x) u(x) \, d\mathcal{H}^{n_i}(x),
\]

where \( n_i \in \{d-1,d\} \) is the dimension of \( \sigma_i \), and where we understand \( u(x) \) as \( \gamma_{[K_i,\sigma_i]} u(x) \) if \( \sigma_i \) is a face of \( K_i \). Note that \( (i) \) holds since, when \( q_i = 1 \), there is no \( j \) such that \( (i,j) \in \bar{I} \).

On the other hand if \( x_i \in \Gamma \), then according to Theorem 6.7 we can choose \( K_{i,1} \in \text{st}(x_i;1), \ldots, K_{i,q_i} \in \text{st}(x_i;q_i) \) such that for all \( j = 1, \ldots, q_i \), \( K_{i,j} \) has a face \( F_{i,j} \in \Gamma \) with \( x_i \in F_{i,j} \). We then set

\[
N_i(u) := \frac{1}{q_i} \sum_{j=1}^{q_i} \int_{F_{i,j}} \psi_{[F_{i,j},x_i]}(x) \gamma_{[K_{i,j},F_{i,j}]} u(x) \, d\mathcal{H}^{d-1}(x).
\]

One has \( N_i(u_h) = \frac{1}{q_i} \sum_{j=1}^{q_i} u_h(x_{i,j}) \) for all \( u_h \in V^p(\Omega_h;\Gamma) \) by Theorem 6.5 which implies \( (i) \) and \( (ii) \). The property \( (iii) \) is also immediate due to the choice of the simplices \( \sigma_i \) and \( F_{i,j} \). Finally, Theorem 6.4 implies that the number of elements in \( \text{st}(x_i) \) is \( \lesssim 1 \), which, in combination with the upper bound \( 6.1 \) on \( q_i \), establishes the claim about \( N \) in \( (iv) \).

We have used the following lemma.

Lemma 6.7. For each \( (i,j) \in I \) such that \( q_i > 1 \), there exists an element \( K^* \in \text{st}(x_i;j) \) which possesses a face \( F \subset \Gamma \) and such that \( x_i \in F \).

Proof. Let \( K \in \text{st}(x_i;j) \) and let \( K' \in \text{st}(x_i) \) such that \( K' \notin \text{st}(x_i;j) \). Since \( \Omega_h \) is a regular mesh, by Theorem 5.2 there is a face-connected path \( K = K_0, K_1, \ldots, K_Q = K' \in \text{st}(x_i) \). Let \( q \in \{1, \ldots, Q\} \) be the first index such that \( K_q \notin \text{st}(x_i;j) \). Then \( K^* := K_q \) satisfies the requirements.

We now deal with the construction of the second family of linear forms. The goal is to prove the following Lemma:

Lemma 6.8. There exists a family of linear forms \( \{N_{i,j}\}_{(i,j) \in \bar{I}} \) such that the following properties hold:

(i) \( N_{i,j}(u) = 0 \) for all \( u \in H^1(\Omega) \),

(ii) \( N_{i,j}(\psi_{i',j'}) = \delta_{i,i'} \delta_{j,j'} \) for all \( (i,j), (i',j') \in \bar{I} \).

(iii) \( N_{i,j} \) is a linear combination of terms of the form

\[
w \mapsto \int_F \psi_{[F,x_i]}(x) \gamma_{[K,F]} w(x) \, d\mathcal{H}^{d-1}(x),
\]

where \( F \) is a face incident to \( x_i \) and \( K \in \text{st}(x_i) \) is such that \( F \in F(K) \). The number \( N \) of terms in this linear combination satisfies \( N \lesssim 1 \).

Each \( N_{i,j} \) will be constructed as a linear combination of “bridge functions” that we define now.
Definition 6.9 (Bridge functions). Given \( k, \ell \in \{1, \ldots, q_i\} \), we say that \( \{k, \ell\} \) is a bridge around \( x_i \) if there exist two mesh elements \( K_k \in \text{st}(x_i; k), K_\ell \in \text{st}(x_i; \ell) \) sharing a face \( F_{k\ell} := K_k \cap K_\ell \subset \Gamma \). The set of bridges around \( x_i \) is denoted by \( \mathcal{B}(i) \). Given a bridge \( \{k, \ell\} \in \mathcal{B}(i) \) with \( k < \ell \), we define the bridge function \( N_{i,\{k,\ell\}} \) by

\[
N_{i,\{k,\ell\}}(u) := \int_{F_{k\ell}} \psi[F_{k\ell},x_i](x)(\gamma|_{K_k,F_{k\ell}} - \gamma|_{K_\ell,F_{k\ell}}) u(x) \, d\mathcal{H}^{d-1}(x).
\]

Note that, by Theorem 6.5, for all \( u_h \in V^p(\Omega_i;\Gamma) \),

\[
N_{i,\{k,\ell\}}(u_h) = u_h(x_{i,k}) - u_h(x_{i,\ell}).
\]

We now fix \( i \) such that \( q_i > 1 \) and consider the vector space

\[
F_i := \text{Span}(\{\psi_{i,j}\}_{1 \leq j \leq q_i-1}).
\]

Let \( F^*_i \) be the set of linear forms \( L_i : F_i \to \mathbb{R} \), and let \( n_{i,\{k,\ell\}} \in F^*_i \) be the restriction of \( N_{i,\{k,\ell\}} \) to \( F_i \).

Lemma 6.10. The bridge functions around \( x_i \) span \( F^*_i \), i.e.,

\[
F^*_i = \text{Span}(\{n_{i,\{k,\ell\}}\}_{\{k,\ell\} \in \mathcal{B}(i)})\).
\]

Proof. By a classical result on linear forms (see e.g. [31, Lemma 3.9]), it suffices to show that if \( u_h \in F_i \) satisfies

\[
n_{i,\{k,\ell\}}(u_h) = 0 \quad \forall \{k, \ell\} \in \mathcal{B}(i),
\]

then \( u_h = 0 \). We first note that \( u_h \in F_i \) implies that \( \sum_{j=1}^{q_i} u_h(x_{i,j}) = 0 \) since this property holds for every \( \psi_{i,j} \). Hence to conclude it remains to show that \( u_h(x_{i,j}) = u_h(x_{i,j'}) \) for \( 1 \leq j, j' \leq q_i \).

For this, introduce the graph \( G^*(x_i) \) defined by

- **Nodes**: the numbers \( 1, \ldots, q_i \),
- **Edges**: the bridges \( \{k, k'\} \).

The key point is that \( G^*(x_i) \) is connected, because \( \text{st}(x_i) \) is face-connected by Theorem 5.2. Eq. (6.7) immediately implies \( u_h(x_{i,j}) = u_h(x_{i,j'}) \) if \( \{j, j'\} \in \mathcal{B}(i) \), and the same follows for a general pair \( \{j, j'\} \in \{1, \ldots, q_i\} \) by using a path from \( j \) to \( j' \) in \( G^*(x_i) \). This concludes the proof of the Lemma.

We denote by \( \{n_{i,j}\}_{1 \leq j \leq q_i-1} \) the basis of \( F^*_i \) such that \( n_{i,j}(\psi_{i,j'}) = \delta_{j,j'} \).

Lemma 6.11. For every \( (i, j) \in \overline{\mathcal{I}} \), there exists

\[
\{k_1, \ell_1\}, \ldots, \{k_{q_i-1}, \ell_{q_i-1}\} \in \mathcal{B}(i), \quad \lambda_{i,1}, \ldots, \lambda_{i,q_i-1} \leq 1
\]

such that

\[
n_{i,j} = \sum_{q=1}^{q_i-1} \lambda_{i,q} n_{i,\{k_q,\ell_q\}}.
\]
Proof. Let \( Q = \max_{1 \leq i \leq N} (q_i - 1) \). Since \( \dim(F^*_i) = \dim(F_i) = q_i - 1 \), one can find a set of pairs \( \{k_q, \ell_q\}_{1 \leq q \leq q_i - 1} \) such that \( \{n_{i,\{k_q,\ell_q\}}\}_{1 \leq q \leq q_i - 1} \) is a basis of \( F^*_i \). One can write \( n_{i,j} \) in a unique way as

\[
n_{i,j} = \sum_{q=1}^{q_i-1} \lambda_{i,q} n_{i,\{k_q,\ell_q\}}.
\]

The coefficients \( \lambda_{i,q} \) can be found by solving the linear system

\[
A \begin{bmatrix} \lambda_{i,1} & \cdots & \lambda_{i,q_i-1} \end{bmatrix}^T = e_j,
\]

where \( e_j \) is the \( j \)th vector of the canonical basis of \( \mathbb{R}^{q_i-1} \) and the matrix coefficients

\[
A_{p,p'} := \psi_{i,p'}(x_{i,k_q}) - \psi_{i,q'}(x_{i,\ell_q})
\]

are integer between \(-2\) and \( 2 \). The set \( S_Q \) of invertible square matrices \( A \) of size at most \( Q \) with coefficients in \( \{-2,-1,0,1,2\} \) is finite. Therefore, we can write

\[
|\lambda_{i,q}| \leq \sup_{A \in S_Q} \|A^{-1}\|_\infty, \quad \forall q \in \{1, \ldots, q_i - 1\},
\]

where, for an element \( v \) of \( \mathbb{R}^r \) and a square \( r \times r \) matrix \( M \), we have denoted

\[
\|v\|_\infty := \max_{1 \leq q \leq r} |v_q|, \quad \|M\|_\infty := \sup_{v \in \mathbb{R}^r} \|Mv\|_\infty / \|v\|_\infty.
\]

Since \( Q \lesssim 1 \), the right-hand side of the inequality (6.8) is \( \lesssim 1 \). \( \square \)

**Proof of Theorem 6.8** We define

\[
N_{i,j} := \sum_{q=1}^{q_i-1} \lambda_{i,q} n_{i,\{k_q,\ell_q\}}.
\]

The properties (i)-(iii) follow immediately by construction and the uniform bound on the number of elements in \( \text{st}(x_i) \). \( \square \)

### 6.3 Definition of \( \Pi_h \) and proof of Theorems 1.1 and 1.2

We define

\[
\Pi_h u := \sum_{i=1}^N N_i(u) \phi_i + \sum_{(i,j) \in \mathbb{I}} N_{i,j}(u) \psi_{i,j}.
\]

Theorem 1.1 is a consequence of the properties of \( N_i \) and \( N_{i,j} \). The rest of this section is devoted to the proof of Theorem 1.2

**Lemma 6.12** (Bramble-Hilbert lemma). Let \( p, d \geq 1 \) be integers, \( \gamma_0 > 0 \) a real number. Then for any \( \gamma_0 \)-shape-regular \( d \)-simplex \( K \) and \( 0 < t \leq p + 1 \), and \( u \in H^t(K) \), there exists \( p_K \in P_p \) such that

\[
\|u - p_K\|_{H^t(K)} \leq C(\gamma_0) h^{t-s} |u|_{H^s(K)}
\]

(6.10)
Lemma 6.14. For any $K$ (6.10) on $K$ every pulling back, using the scaling properties of the (semi-) norms $\|\cdot\|$.

This is shown by mapping to the reference element, using the trace inequality (5.4) and pulling back, using the scaling properties of the (semi-) norms $\|\cdot\|_{L^2(F)}$, $\|\cdot\|_{H^s(K)}$, and $\|\cdot\|_{L^2(F)}$. □

Lemma 6.13 (Scaled trace theorem). For every $\gamma_0 > 0$, $t \in (\frac{1}{2}, 1]$, there exists $C(\gamma_0, t) > 0$ such that, if $K \subset \mathbb{R}^d$ is a $\gamma_0$-shape-regular d-simplex ($d \geq 2$) and $F \in \mathcal{F}(K)$, then for all $u \in H^t(K)$

$$\|u\|_{L^2(F)} \leq C(\gamma_0, t) \left( \left( \text{diam}(K) \right)^{\frac{t}{2}} \|u\|_{H^1(K)} + \text{diam}(K)^{-\frac{1}{2}} \|u\|_{L^2(K)} \right)$$

This is well-known, see the seminal work [6] (the proof for fractional Sobolev exponents is obtained by interpolation), and [18]. □

Proof. Let $\tilde{N}_{i,j}$ be the linear forms such that

$$\Pi_h z = \sum_{(i,j) \in I} \tilde{N}_{i,j}(z) \phi_{i,j}.$$  

We can write

$$\|u - \Pi_h u\|_{H^s(K)} \leq \|u - v_h\|_{H^s(K)} + \|\Pi_h (u - v_h)\|,$$

$$\leq h^{\frac{t}{2} - s} \|u\|_{H^t} + \sum_{(i,j)} |\tilde{N}_{i,j}(u - v_h)| \|\phi_i\|_{H^s(K)}$$

$$\leq h^{\frac{t}{2} - s} \|u\|_{H^t} + h^{3/2 - s} \sum_{(i,j)} |\tilde{N}_{i,j}(u - v_h)|$$

where the sum runs over the pairs $(i,j) \in I$ such that $K \in \text{st}(x_i; j)$. For all $(i,j) \in I$ and $w \in H^1(\Omega \setminus \Gamma)$, $\tilde{N}_{i,j}(w)$ is a linear combination of $N_i(w)$ and $N_{i,j'}(w)$ for $1 \leq j' \leq q_i - 1$, which in turn are both defined via linear combinations of terms of the form

$$\int_{F_{i,j'}} \psi_{[i,j'; F_{i,j'}]}(x) \gamma_{[K_{i,j'}; F_{i,j'}]} w(x) \, d\mathcal{H}^{d-1}(x)$$

with $x_{i,j}$, $F_{i,j}$ and $K_{i,j}$ as in the statement of the Lemma, or terms of the form

$$\int_K \psi_{[K; x_{i,j}]} w(x) \, dx$$

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when \( x_i \) is an interior Lagrange node. In the latter case, we can write

\[
\int_K \psi_{[K,x_i]} (u(x) - v_h(x)) \, dx = \int_K \psi_{[K,x_i]} (u(x) - p_K(x)) \, dx \\
\leq \| \psi_{[K,x_i]} \|_{L^2(K)} \| u - p_K \|_{L^2(K)} \\
\leq h^{-3/2} |u|_{H^1}.
\]

We conclude the proof using again that \( \text{card}(\text{st}(x_i)) \lesssim 1 \).

Define \( v_h \in V^p(\Omega_h; \Gamma) \) by

\[
v_h := \sum_{(i,j)} p_{K_{i,j}}(x_i) \phi_{i,j}
\]

where the sum ranges over the pairs \((i, j)\) such that \( x_i \in \mathcal{L}_p(K) \), and where the element \( K_{i,j} \in \Omega_h \) is chosen freely in \( \text{st}(x_i; j) \), with the only constraint that \( K_{i,j} = K \) whenever possible, i.e. whenever \( \text{st}(x_i; j) \ni K \). This constraint then ensures \( (v_h)|_K = p_K \) by unisolvence, so that Theorem 6.14 applies to \( v_h \).

We now estimate a term of the form

\[
\int_{F_{i'}} \psi_{[F_{i'},y_{i'}]}(x) \gamma_{[K_{i'},F_{i'}]}(u(x) - v_h(x)) \, d\mathcal{H}^{d-1}(x)
\]

as defined in that Lemma. Let \((i_0, j_0)\) be the unique element of \( \mathcal{I} \) determined by

\[
y_{i'} = x_{i_0} \quad \text{and} \quad K_{i'} \in \text{st}(x_{i_0}; j_0)
\]

Then for any \((i, j)\) in \( \mathcal{I} \), one has

\[
\int_{F_{i'}} \psi_{[F_{i'},y_{i'}]}(x) \gamma_{[K_{i'},F_{i'}]}(u(x) - p_{K_{i_0,j_0}}(x)) \, d\mathcal{H}^{d-1}(x) = \phi_{i}(x_{i_0}) \quad \text{if} \ K_{i'} \in \text{st}(x_{i}; j),
\]

\[
= 0 \quad \text{otherwise},
\]

\[
= \delta_{i,i_0} \delta_{j,j_0}.
\]

Therefore, for any \( u_h \in V^p(\Omega_h; \Gamma) \),

\[
\int_{F_{i'}} \psi_{[F_{i'},y_{i'}]}(x) \gamma_{[K_{i'},F_{i'}]} u_h(x) \, d\mathcal{H}^{d-1}(x) = u_h(x_{i_0,j_0}).
\]

Thus,

\[
\int_{F_{i'}} \psi_{[F_{i'},y_{i'}]}(x) \gamma_{[K_{i'},F_{i'}]}(u(x) - v_h(x)) \, d\mathcal{H}^{d-1}(x)
\]

\[
= \int_{F_{i'}} \psi_{[F_{i'},y_{i'}]}(x) \gamma_{[K_{i'},F_{i'}]}(u(x) - p_{K_{i_0,j_0}}(x)) \, d\mathcal{H}^{d-1}(x).
\]
This quantity can be estimated using Veeser’s trick from [38, Theorem 1] (see Eq. (23)), stated as an independent result in [9, Lemma 3.1]. In our notation, this reads

\[
\int_{F_\nu} \psi_{[F_{\nu};y_{\nu}]}(x) \gamma_{[K_{\nu},F_{\nu}]}(u(x) - p_{K_{\nu},j_0}(x)) \, d\mathcal{H}^{d-1}(x)
\]

\[
= \int_{F_\nu} \psi_{[F_{\nu};y_{\nu}]}(x) \gamma_{[K_{\nu},F_{\nu}]}(u(x) - p_{K_{\nu}}(x)) \, d\mathcal{H}^{d-1}(x) + p_{K_{\nu}}(x_{i_0}) - p_{K_{\nu},j_0}(x_{i_0})
\]

\[
= \int_{F_\nu} \psi_{[F_{\nu};y_{\nu}]}(x) \gamma_{[K_{\nu},F_{\nu}]}(u(x) - p_{K_{\nu}}(x)) \, d\mathcal{H}^{d-1}(x) + \sum_{l=1}^{M-1} p_{K_l}(x_{i_0}) - p_{K_{l+1}}(x_{i_0})
\]

using a telescoping sum, where \(K_1, \ldots, K_M\) is a sequence of mesh elements in \(\text{st}(x_{i_0};j_0)\) such that \(K_{\nu} = K_1, K_{i,j} = K_M,\) and \(K_l\) and \(K_{l+1}\) share a face \(F_l\) not in \(\Gamma\) containing \(x_i\) for every \(l = 0, \ldots, M - 1.\) The first term is estimated via the scaled trace theorem (Theorem 6.13)

\[
\left| \int_{F_\nu} \psi_{[F_{\nu};y_{\nu}]}(x) \gamma_{[K_{\nu},F_{\nu}]}(u(x) - p_{K_{i,j}}(x)) \, d\mathcal{H}^{d-1}(x) \right|
\]

\[
\leq \left\| \psi_{[F_{\nu};x_i]} \right\| \left\| \gamma_{[K_{\nu},F_{\nu}]}(u - p_{K_{i,j}}) \right\|_{L^2(F_\nu)}
\]

\[
\lesssim h^{-3/2} \left\| u - p_{K_{i,j}} \right\|_{L^2(K_{i,j})} + h^{-3/2 + t} \left| u - p_{K_{i,j}} \right|_{H^t}
\]

\[
\lesssim h^{-3/2 + t} |u|_{H^t(K_{i,j})}.
\]

Similarly, for all \(l \in \{1, \ldots, M - 1\},\) we write

\[
p_{K_l}(x_i) - p_{K_{l+1}}(x_i) = \int_{F_l} \psi_{[F_l;x_i]}(p_{K_l}(x) - p_{K_{l+1}}(x)) \, d\mathcal{H}^{d-1}(x)
\]

\[
= \int_{F_l} \psi_{[F_l;x_i]} \gamma_{[K_{l+1},F_l]}(u - p_{K_{l+1}})(x) - \gamma_{[K_l,F_l]}(u - p_{K_l})(x) \, d\mathcal{H}^{d-1}(x)
\]

where we have used Theorem 6.1. Therefore,

\[
\left| p_{K_l}(x_i) - p_{K_{l+1}}(x_i) \right| \lesssim h^{-3/2 + t} |u|_{H^t(K_{l+1})}
\]

In view of Theorem 6.14, this establishes Theorem 1.2

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