Abstract

For discretely observed functional data, estimating eigenfunctions with diverging index is essential in nearly all methods based on functional principal components analysis. In this paper, we propose a new approach to handle each term appeared in the perturbation series and overcome the summability issue caused by the estimation bias. We obtain the moment bounds for eigenfunctions and eigenvalues for a wide range of the sampling rate. We show that under some mild assumptions, the moment bound for the eigenfunctions with diverging indices is optimal in the minimax sense. This is the first attempt at obtaining an optimal rate for eigenfunctions with diverging index for discretely observed functional data. Our results fill the gap in theory between the ideal estimation from fully observed functional data and the reality that observations are taken at discrete time points with noise, which has its own merits in models involving inverse problem and deserves further investigation.

Keywords: Kernel smoothing; perturbation series; phase transition; optimal convergence.
1 Introduction

With the rapid evolution of modern data collection technologies, functional data emerge ubiquitously and have been extensively developed over the past decades. In general, functional data are typically regarded as stochastic processes with certain smoothness conditions or realizations of Hilbert space valued random elements. These perspectives convey two essential natures of functional data, smoothness and infinite dimensionality, which distinguish functional data from high-dimensional and Euclidean data. For a comprehensive treatment on functional data, we recommend monographs by Ramsay and Silverman (2006), Ferraty and Vieu (2006), Horváth and Kokoszka (2012), Hsing and Eubank (2015) and Kokoszka and Reimherr (2017).

Although functional data provide information over a continuum, which is often time, or spatial location, real data are mostly collected in a discrete form with measurement errors. For instance, we usually use \( n \) to denote the sample size and \( N_i \) the number of observations for the \( i \)th subject. Thanks to the smoothness nature of functional data, having the number of observations per subject large is a blessing rather than a curse in contrast to the high-dimensional data (Hall et al., 2006). There is an extensive literature on the nonparametric methods in accessing the smoothness nature of functional data, including kernel method (Yao et al., 2005a; Hall et al., 2006) and various kinds of spline methods (Rice and Wu, 2001; Yao and Lee, 2006; Paul and Peng, 2009; Cai and Yuan, 2011).

For a given smoothing method, there are two typical strategies. When the observed points per subject are relatively dense, pre-smoothing each curve before further analysis is suggested by Ramsay and Silverman (2006) and Zhang and Chen (2007). Otherwise, pooling observations together from all subjects is more recommended when the sampling scheme is rather sparse (Yao et al., 2005a). Whether using the individual or pooled information affects the convergence rate and phase transition in estimating population quantities such as mean and covariance functions. When \( N \geq O(n^{5/4}) \), the reconstructed curves by pre-smoothing are \( \sqrt{n} \)-consistency and thus, the estimated mean and covariance functions based on the pre-smoothed curves have the optimal parametric rate. In contrast, by borrowing information from all subjects, the pooling method only requires \( N \geq O(n^{1/4}) \) for mean and covariance estimation reaching optimal (Cai and Yuan, 2010, 2011; Zhang and Wang, 2016), which provides theoretical insight for the supremacy of pooling strategy over the pre-smoothing strategy.

However, estimating mean and covariance only reflects the smoothness nature of functional data but leaves the infinite dimensionality out of consideration. Due to the decaying eigenvalues, covariance operators for functional random objects are non-invertible and consequently, regularization is needed in models involving inverse problem with functional covariates, including functional linear regression (FLR) (Yao et al., 2005b; Hall and Horowitz, 2007; Yuan and Cai, 2010), functional generalized linear model (fGLM) (Müller and Stadtmüller, 2005; Dou et al., 2012) and functional Cox model (Qu et al., 2016). Truncation on the functional principal components (FPC) is a well developed approach to do regularization (Hall and Horowitz, 2007; Dou et al., 2012), which requires the asymptotic behavior of the estimated eigenfunctions in theoretical analysis. In order to suppress the estimation bias,
the number of principal components used in estimating the regression function should grow with sample size. Therefore, convergence rate for a diverging number of the estimated eigenfunctions is a fundamental issue in functional data analysis (FDA). It is not only interested in its own theoretical merits but also embraced in most inverse models involving functional covariates.

For fully observed functional data, the seminal work Hall and Horowitz (2007) obtained the convergence rate $j^2/n$ for the $j$th eigenfunction, which has been proofed optimal in the minimax sense by Wahl (2020). Subsequently, this result becomes the keystone in establishing the optimal convergence in FLR (Hall and Horowitz, 2007) and fGLM (Dou et al., 2012). For discretely observed functional data, stochastic bounds derived by the existing literature are only considered for a fixed number of eigenfunctions. Specifically, applying the local linear smoother, Hall et al. (2006) show that the $L^2$ convergence rate for a fixed eigenfunction with finite $N_i$ is $O_p(n^{-4/5})$. For another line of research, under the reproducing kernel Hilbert space (RKHS) framework, Cai and Yuan (2010) claimed that eigenfunctions with fixed indices admit the same convergence rate as the covariance function, which is $O_p((nN_i \log n)^{-4/5} + n^{-1})$. We point out here that even though both of their results are one-dimensional non-parametric rates (at most differ by a factor of $(\log n)$), the methodologies and techniques they used are completely disparate. On one hand, Hall et al. (2006) utilized the fact that the additional integration enables the eigenfunctions to be estimated at a faster rate than the covariance function itself and adopted a detailed perturbation expansion in Bosq (2000). On the other hand, the one-dimensional rate for the covariance function in Cai and Yuan (2010) is facilitated by the tensor product space, which is much smaller than the $L^2$ space on $[0,1]^2$, and their result for eigenfunctions is a direct application of the perturbation bound in Bhatia et al. (1983). A detailed discussion on the perturbation bounds can be found in Section 2. Moreover, Paul and Peng (2009) proposed a reduced rank model and studied its asymptotic properties under a particular setting. We emphasize that the theoretical results in literature mentioned above are only for a fixed number of eigenfunctions, and to our best knowledge, there is no progress in obtaining the convergence rate for a diverging number of eigenfunctions when the data are discretely observed with noise.

An intrinsic difference between estimating the diverging and fixed number of eigenfunctions lies in the infinite-dimensional nature of functional data. Specifically, the decaying eigenvalues make it challenging in analyzing the eigenfunctions with diverging index, which yields all the existing techniques failed. Moreover, the perturbation results used in Hall and Horowitz (2007) and Dou et al. (2012) are facilitated by the cross-sectional sample covariance, which reduces each term in the perturbation series to the FPC scores. This virtue no longer exists when the trajectories are observed at discrete time points, and a summability issue occurs due to the estimation bias, see Section 2 for further elaboration. Combination of smoothness and infinite dimensionality reveals the elevated difficulties in estimating a diverging number of eigenfunctions, which remains an open problem in FDA.

In view of the aforementioned significance and difficulties, we present in this paper a unified theory in estimating a diverging number of eigenfunctions from discretely observed functional data. The main contributions of this paper are summarized as follows. First, we bring up a new approach to handle each term appeared in the perturbation series, which
overcomes the summability issue caused by the estimation bias. This methodology can be also applied in other perturbation involved problems, such as estimation and prediction in FLR. Second, under some mild assumptions, we obtain the moment bound for the eigenfunctions with diverging indices. Our unified theory allows \(N_i\) varying in a wide range, and we show that when \(N_i\) reaches the magnitude \(n^{1/4+\delta}\), where \(\delta\) is determined by the smoothness of the underlying trajectory and the frequency of the estimated eigenfunction, this rate reaches optimal in the minimax sense as if the curves are fully observed. This result is the first attempt at obtaining an optimal convergence rate for eigenfunctions with diverging index of discretely observed functional data. Third, we obtain the asymptotic normality for eigenvalues with diverging index. In summary, the results obtained in this paper is not merely a theoretical progress, it builds the bridge between “ideal” and “reality” in models involving inverse problem, which has its own merits deserving further investigation.

In what follows, we use \(A_n = O_p(B_n)\) stands for \(\mathbb{P}(A_n < M B_n) \geq 1 - \epsilon\) whereas \(A_n = o_p(B_n)\) stands for \(\mathbb{P}(A_n < \epsilon B_n) \to 0\) as \(n \to \infty\) for each \(\epsilon > 0\) and a positive constant \(M\). A non-random sequence \(a_n\) is said to be \(O(1)\) if it is bounded and for each non-random sequence \(b_n\), \(b_n = O(a_n)\) stands for \(b_n/a_n = O(1)\) and \(b_n = o(a_n)\) stands for \(b_n/a_n\) converges to zero. Throughout this paper, we use \(\text{Const}\). stands for a positive constant which may vary from place to place. The relation \(a \lesssim b\) if \(a \leq \text{Const.}\) and the relation \(\gtrsim\) can be defined analogously. We write \(a \asymp b\) if \(a \lesssim b\) and \(b \lesssim a\). For \(a \in \mathbb{R}\), we use \([a]\) to denote the largest integer smaller or equal to \(a\). For a function \(p(s) \in \mathcal{L}^2[0, 1]\), we use \(\|p\|^2\) denotes \(\int_{[0,1]} p(s)^2 ds\) and \(\|p\|\) stands for \(\sup_{s \in [0,1]} |p(s)|\). For a function \(A(s, t) \in \mathcal{L}^2[0, 1]^2\), define \(\|A\|^2_{\text{HS}} = \iint_{[0,1]^2} A(s, t)^2 ds dt\) and \(\|A\|^2_{(ij)} = \iint_{[0,1]} \{\int_{[0,1]} A(s, t) \phi_j(s) ds\}^2 ds dt\). Write \(\int pq\) and \(\int Apq\) for \(\int p(u)q(u) du\) and \(\int A(u, v)p(u)q(v) dv\).

The reminder of the paper is organized as follows. Section 2 introduces the model and theoretical results are presented in Section 3. The proofs are collected together in Section 4.

2 Eigenfunctions estimation for discretely observed data

Denote \(X(t)\) a square integrable stochastic process on \([0, 1]\) and \(X_i(t)\) independent and identically distributed (i.i.d.) copies of \(X(t)\). The mean and covariance functions of \(X(t)\) are \(\mu(t) = \mathbb{E}\{X(t)\}\) and \(C(s, t) = \mathbb{E}\{\{X(s) - \mu(s)\}(X(t) - \mu(t))\}\), respectively. By the Mercer’s Theorem (Indritz, 1963, Chapter 4), \(C(s, t)\) admits the spectral decomposition

\[
C(s, t) = \sum_{k=1}^{\infty} \lambda_k \phi_k(s) \phi_k(t),
\]

where \(\lambda_1 > \lambda_2 > \ldots > 0\) are eigenvalues and \(\phi_j\)’s are the corresponding eigenfunctions. The eigenfunctions \(\{\phi_j\}_{j=1}^{\infty}\) form a complete orthonormal system (CONS) on \(\mathcal{L}^2[0, 1]\), which denotes the space of square-integrable functions on \([0, 1]\). For each \(i\), the process \(X_i\) admits the so-called Karhunen-Loève expansion

\[
X_i(t) = \mu(t) + \sum_{j=1}^{\infty} \xi_{ik} \phi_k(t),
\]
where $\xi_{ik} = \int_0^1 \{X_i(t) - \mu(t)\} \phi_k(t)dt$ are uncorrelated zero mean random variables with variance $\lambda_k$. In this paper we focus on the eigenfunction estimation thus we assume $\mu(t) = 0$ without loss of generality (W.L.O.G.).

In practice, we cannot observe the full trajectory $X_i(t)$. Instead, measurements are taken at $N_i$ discrete time points with noise contamination. The actual observations for each $X_i$ are

$$\{(T_{ij}, X_{ij})|X_{ij} = X_i(T_{ij}) + \varepsilon_{ij}, j = 1, \ldots, N_i\}, \quad (3)$$

where $\varepsilon_{ij}$’s are i.i.d. copies of $\varepsilon$ with $E(\varepsilon) = 0$ and $\text{Var}(\varepsilon) = \sigma^2$. We further assume that $\{T_{ij}\}$ are i.i.d. copies of $T$, which follows the uniform distribution on $[0, 1]$ and $N_i = N$.

Kernel method has been widely developed as a smoothing approach in FDA due to its attractive theoretical features (Yao et al., 2005a; Hall et al., 2006; Li and Hsing, 2010; Zhang and Wang, 2016). The main motivation of this paper is to derive a unified theory in estimating a diverging number of eigenfunctions for discretely observed functional data and thus, we adopt the local constant smoother here to avoid distractions from complicated calculations. Denote $\delta_{ijt} = X_{ij}X_{it} = \{X_i(T_{ij}) + \varepsilon_{ij}\}\{X_i(T_{il}) + \varepsilon_{il}\}$ and the covariance estimator is obtained through

$$\hat{C}(s, t) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{N(N-1)} \frac{1}{h^2} \sum_{1 \leq j, l \leq N} K\left(\frac{T_{ij} - s}{h}\right) K\left(\frac{T_{il} - t}{h}\right) \delta_{ijt}. \quad (4)$$

where $K$ is a symmetric density kernel. The function $\hat{C}(s, t)$ admits an empirical version of decomposition (1),

$$\hat{C}(s, t) = \sum_{k=1}^{\infty} \hat{\lambda}_k \hat{\phi}_k(s) \hat{\phi}_k(t), \quad (5)$$

where $\hat{\lambda}_k$ and $\hat{\phi}_k$ are estimators for $\lambda_k$ and $\phi_k$. We further assume $(\hat{\phi}_k, \phi_k) \geq 0$.

Before introducing our theoretical, we shall give a comprehensive synopsis for eigenfunction estimation problem in FDA. We start with the following resolvent series (6) and illustrate how it is applied in statistical analysis.

$$E(\|\hat{\phi}_j - \phi_j\|^2) \asymp \sum_{k \neq j} E\{(\hat{C} - C)\hat{\phi}_j, \phi_k\}^2 \left(\frac{\hat{\lambda}_k}{\lambda_k - \lambda_j}\right)^2. \quad (6)$$

Such kind of expansions can be found in Bosq (2000), Dou et al. (2012) and Li and Hsing (2010), see chapter 5 in Hsing and Eubank (2015) for details. Denote $\eta_j := \min_{k \neq j} |\lambda_k - \lambda_j|$ the eigengap of $\lambda_j$, then a rough bound for $E\|\hat{\phi}_j - \phi_j\|^2$ is derived by (6) and the Bessel’s inequality directly

$$E\|\hat{\phi}_j - \phi_j\|^2 \lesssim \eta_j^2 E\|\hat{C} - C\|^2_{\text{HS}}, \quad (7)$$

which is exactly the bound in Bhatia et al. (1983). However, this bound is clearly suboptimal for eigenfunction estimation in two aspects. First, $\eta_j$ is away from 0 when $j$ is regarded as a fixed number, and (7) indicates that eigenfunctions admit the same convergence rates as the covariance function. This is counter-intuitive since $\phi_j = \lambda_j^{-1} \int C(s, t) \phi_j(t)dt$ and the integration usually brings extra smoothness (Cai and Hall, 2006). Thus, the eigenfunction
should admit a rate faster than the two-dimensional rate of \(\|\hat{C} - C\|_{\text{HS}}^2\). Second, when \(j\) is diverging to infinity, there is \(\eta_j \to 0\) and \(\eta_j^2\) appears in (7) could not be ignored. Assume \(\lambda_j \asymp j^{-a}\), then the rate for \(\mathbb{E}(\|\hat{\phi}_j - \phi_j\|^2)\) obtained by (7) is slower than \(j^{2a+2}/n\), which is clearly suboptimal (Wahl, 2020). Therefore, in order to obtain a sharp rate for \(\mathbb{E}(\|\hat{\phi}_j - \phi_j\|^2)\), we should resort to the original perturbation series (6) rather than its approximation (7).

When each trajectory \(X_i(t)\) is observed for all \(t \in [0, 1]\), which refers to the fully observed case, the cross-sectional sample covariance \(\hat{C}(s, t) = n^{-1}\sum_{i=1}^n X_i(s)X_i(t)\) is a canonical estimator of \(C(s, t)\). Then the numerators in each term of equation (6) can be reduced to the FPC scores under some mild assumptions, e.g. \(\mathbb{E}((\hat{C} - C)\phi_j, \phi_k)^2 \leq n^{-1}\lambda_j\lambda_k\) (Hall and Horowitz, 2007; Dou et al., 2012). Then \(\mathbb{E}((\hat{\phi}_j - \phi_j)^2)\) is bounded by \((\lambda_j/n)\sum_{k \neq j} \lambda_k/(\lambda_k - \lambda_j)^2\). Assume the polynomial decay of eigenvalues, the aforementioned summation is dominated by \(\lambda_j/n\sum_{j/2 < k < 2j} \lambda_k/(\lambda_k - \lambda_j)^2\), which is \(O(j^2/n)\) and optimal in the minimax sense (Wahl, 2020), see Lemma 7 in Dou et al. (2012) for detailed elaboration.

However, we emphasize that when tackling with discretely observed data, all the existing literature use a bound analogous to (7), which excludes the diverging indices as a consequence. Specifically, the result in Cai and Yuan (2010) is obtained by applying the bound (7) directly and their one-dimensional rate is facilitated by the tensor product space. In contrast, the one-dimensional rate derived by Hall et al. (2006); Li and Hsing (2010) is profited by a detailed calculation and the perturbation series (6). However, their theoretical analysis are based on the assumption that \(\eta_j\) is away from zero, which indicates that \(j\) must be a fixed constant. Moreover, the restricted maximum likelihood estimator proposed by Paul and Peng (2009) also has a similar boundedness assumption on \(\lambda_j^{-1}\). When the data are discretely observed, how to utilize (6) effectively is the key in obtaining a sharp bound for a diverging number of eigenfunctions.

The main challenges comes from quantifying the summation (6) without the fully observed sample covariance. For pre-smoothing method, the reconstructed \(\hat{X}_i\) achieves a \(\sqrt{n}\) convergence in \(L^2\) sense when each \(N_i\) reaches the magnitude \(n^{5/4}\) and thus, the estimated covariance function \(\hat{C}(s, t) = n^{-1}\sum_{i=1}^n \hat{X}_i(s)\hat{X}_i(t)\) admits \(\|\hat{C} - C\|_{\text{HS}} = O_p(n^{-1/2})\). However, this does not guarantee the optimal convergence of a diverging number of eigenfunctions by the existing techniques. Specifically, the numerator in each term of (6) is no longer the component scores and such a complicated form makes it infeasible to quantify this infinite summation when \(|\lambda_k - \lambda_j| \to 0\). Besides, for pooling method, it is also highly nontrivial to sum up all \(\mathbb{E}((\hat{C} - C)\phi_j, \phi_k)^2\) with respect to \(j, k\), see comments below Theorem 1 for more detail.

### 3 \(L^2\) convergence for eigenfunctions and eigenvalues

Based on the aforementioned issues, we propose a new technique to handle the perturbation series (6) for the discretely observed functional data. We shall make the following regularity assumptions.

**A.1** \(X(t)\) has finite fourth moment \(\int \mathbb{E}X^4(t)dt < \infty\); \(\mathbb{E}X_4^\alpha \leq \lambda_j^2\) for all \(j\) and \(\sigma_X^4 < \infty\).
The covariance function admits an expansion
\[ C(s, t) = \sum_{k=1}^{\infty} \lambda_k \phi_k(s) \phi_k(t), \]
where the eigenvalues \( \lambda_j \) are decreasing with \( \text{Const.} j^{-a} \geq \lambda_j \geq \lambda_{j+1} + (a/\text{Const.}) j^{-a-1} \) for \( a > 1 \) and each \( j \geq 1 \).

For each \( j \in \mathbb{N}^* \), the eigenfunctions \( \phi_j \) satisfies
\[
\sup_{t \in [0,1]} |\phi_j(t)| = O(1) \quad \text{and} \quad \sup_{t \in [0,1]} |\phi_j^{(k)}(t)| \leq j^{c/2} \sup_{t \in [0,1]} |\phi_j^{(k-1)}(t)|, \quad \text{for } k = 1, 2,
\]
where \( c \) is a positive constant and assume \( \phi_j(0) = \phi_j(1), \phi_j^{(1)}(0) = \phi_j^{(1)}(1) \) W.L.O.G..

Assumptions (A.1) and (A.2) are commonly adopted in the FDA literature (Yao et al., 2005a; Hall and Horowitz, 2007; Cai and Yuan, 2010; Dou et al., 2012). The number of eigenfunctions that can be well estimated for exponential decaying eigenvalues is at the order of \( \log n \) and thus, a polynomial decaying rate is majorly considered in FDA. In mean and covariance estimation, it is typical to assume that the covariance function has bounded second order derivatives (Zhang and Wang, 2016). However, this assumption might be inappropriate in estimating a specific eigenfunction with diverging index. In general, the frequency of \( \phi_j \) is higher for larger \( j \), which requires a smaller bandwidth to capture its local variation. The boundary assumption on \( \phi_j \) and its first order derivative eliminates the edge effect caused by the local constant smoother for technical convenience, and may be relaxed with more technicality. Assumption (A.3) characterizes the frequency increment of a specific eigenfunction via the smoothness of its derivatives, and for some common used basis, Fourier, Legendre and wavelet basis for instance, \( c = 2 \). In Hall et al. (2006), the authors assume that \( \max_{1 \leq j \leq r} \max_{s=0,1,2} \sup_{t \in [0,1]} |\phi_j^{(s)}(t)| \leq \text{Const.} \) which is achievable only for fixed \( r \) and assumption (A.3) can be regarded as its generalized version. For the kernel function \( K \) and sampling points \( T_{ij} \), we shall make the following assumptions (Zhang and Wang, 2016).

**K.1** \( K(\cdot) \) is a bounded symmetric probability density function on \([-1,1]\).

**K.2** \( X, T \) and \( \varepsilon \) are mutually independent.

The following theorem quantifies \( \mathbb{E}\{(\hat{C}-C)\phi_j, \phi_k)^2\} \), which is fundamental in most problems in FDA that involving inverse issues, such as eigenfunction estimation with diverging index, estimation and prediction in FLR.

**THEOREM 1.** Under assumptions (A.1) to (A.3), (K.1), (K.2), and \( h^4 j^{2a+2c} = O(1) \), denote \( \Delta = \hat{C} - C \). For all \( 1 \leq k \leq 2j \)
\[
\mathbb{E}(\Delta \phi_j, \phi_k)^2 \leq \frac{1}{n} \left( j^{-a} k^{-a} + \frac{j^{-a} + k^{-a}}{N} + \frac{1}{N^2} \right) + h^4 k^{2c-2a}
\]
and
\[
\sum_{k=j+1}^{\infty} \mathbb{E}(\Delta \phi_j, \phi_k)^2 \leq \frac{1}{n} \left( j^{1-2a} + \frac{h^{-1} j^{-a} + j^{1-a}}{N} + \frac{1}{h N^2} \right) + h^4 j^{1+2c-2a}.
\]
When \( \hat{C} \) is obtained from a kernel type smoother similar to (4), the convergence rate for \( \|\Delta\|_{HS} \) should be a two-dimensional kernel smoothing rate with variance \( n^{-1}(1 + (Nh)^{-2}) \) (Zhang and Wang, 2016). Similarly, one can show that \( \mathbb{E}(\|\Delta\|_{(j)}^2) \) admits a one-dimensional rate with variance \( n^{-1}(1 + (Nh)^{-1}) \). The first assertion of Theorem 1 reveals that the convergence rate for \( \mathbb{E}(\langle \Delta \phi_j, \phi_k \rangle^2) \) is a degenerated kernel smoothing rate and the variance terms \((Nh)^{-1}, (Nh)^{-2}\) vanish after twice integrations. However, by Bessel’s equality, there is \( \sum_{k=1}^{\infty} \mathbb{E}(\langle \Delta \phi_j, \phi_k \rangle^2) = \mathbb{E}(\|\Delta\|_{(j)}^2) \) and \( \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \mathbb{E}(\langle \Delta \phi_j, \phi_k \rangle^2) = \mathbb{E}(\|\Delta\|_{HS}^2) \), which indicates that one cannot sum up all \( \mathbb{E}(\langle \Delta \phi_j, \phi_k \rangle^2) \) with respect to \( j, k \) due to the estimation bias. In the fully observed case, the summation \( \sum_{k=j}^{\infty} \mathbb{E}(\langle \Delta \phi_j, \phi_k \rangle^2) \) dominates by \( \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \mathbb{E}(\langle \Delta \phi_j, \phi_k \rangle^2) = (\text{Hall and Horowitz, 2007; Dou et al., 2012}) \). This inspires us that the convergence rate caused by the inverse issue can be captured by the summation on the set \( \{k \leq 2j\} \), and the tail sum on \( \{k > 2j\} \) can be treated as a unity. The second assertion in Theorem 1 reveals that \( \sum_{k=j+1}^{\infty} \mathbb{E}(\langle \Delta \phi_j, \phi_k \rangle^2) \) admits a one-dimensional rate caused by kernel smoothing. The terms appear in the variance term with \( j^{-a} \) are facilitated by the additional integration and the term involving \( j^c \) is caused by the increasing frequency for larger \( j \).

Based on Theorem 1, the following theorem gives a unified theory for estimating eigenfunctions with diverging indices, which is the main result of this paper.

**Theorem 2.** Under assumptions (A.1) to (A.3), (K.1) and (K.2), for \( m \in \mathbb{N}_+ \) admits

\[ m \frac{2a+2}{n} \to 0, m \frac{2a}{n} \to 0, h^4 m^{2a+2} \to 0 \text{ and } h^4 m^{2a+2c} = O(1). \]

Define \( \Omega_m(n, N, h) := \{ \|\Delta\| \leq \eta_m/2 \} \), then \( P(\Omega_m(n, N, h)) \to 1 \) and on \( \Omega_m(n, N, h) \)

\[ \mathbb{E}\|\hat{\phi}_j - \phi_j\|^2 \leq \frac{j^2}{n} \left\{ 1 + \left( \frac{j^a}{N} \right)^2 \right\} + \frac{j^a}{nNh} \left( 1 + \frac{j^a}{N} \right) + h^4 j^{2c+2} \quad j = 1, \ldots, m. \] (8)

The event \( \Omega_m(n, N, h) \) denotes the set of all realizations such that \( \|\Delta\|_{HS} < \eta_m/2 \) for sample size \( n \), sampling rate \( N \) and bandwidth \( h \) (Hall and Horowitz, 2007). It is worth mentioning that “on \( \Omega_m(n, N, h) \)” is not a statement that relates to a conditioning argument in the sense of probability theory. It should be interpreted as stating that all realizations for which \( \|\Delta\|_{HS} < \eta_m/2 \). Alternatively, \( m \) can be regarded as the maximum number of eigenfunctions that can be well estimated based on the observed data. Note that \( m \) is not fixed but can diverge to infinity and jointly determined by the observed data \( (n, N) \), smoothing strategy \( (h) \) and the decaying eigengap \( \langle \eta_j \rangle \) or \( a \). Specifically, the growing \( n \) and \( N \) make the covariance function well estimated and thus the eigenfunctions. The frequency of \( \phi_j \) is higher for larger \( j \), which requires smaller \( h \) to capture its local variations. For larger \( a \), the eigengap \( \eta_j \) shrinks to 0 rapidly, which makes two adjacent eigenfunctions difficult to distinguish. The \( m \) in Assumptions (M) could cover the most cases in the subsequent analysis, such as regression problems.

Theorem 2 is a good illustration for both infinite dimensionality and smoothness nature of functional data. To understand this result, note that \( j^2/n \) is the convergence rate in the fully observed case while remaining terms appear in the right-hand of (8) can be viewed as the contamination caused by the discrete observations and measurement errors. Specifically, the
term involving $h^4$ represents the smoothing bias, and $(nNh)^{-1}$ is a typical one-dimensional variance by kernel smoothing. The terms containing $N^{-1}$ are owing to the discrete approximation and those involving $j$ with its positive powers arise from the decaying eigengaps for increasing number of eigen components. Rather than stochastic bounds (in the form of $O_p$) obtained by Hall et al. (2006), Paul and Peng (2009) and Cai and Yuan (2010), (8) is a moment bound and more feasible for subsequent analysis in inverse-involved problems, where $\sum_{j=1}^{m} \| \hat{\phi}_j - \phi_j \|^2$ is commonly appeared with $m \rightarrow \infty$.

Theorem 2 is a unified result for eigenfunction estimation and has no restriction on the sampling rate $N$. As the phase transition in mean and covariance functions estimation (Cai and Yuan, 2011; Zhang and Wang, 2016), we hope to derive a systematic partition in estimating a diverging number of eigenfunctions. The following corollary discusses the convergence rate of $\mathbb{E}(\| \hat{\phi}_j - \phi_j \|^2)$ under different $N$ after selecting the optimal bandwidth $h$.

**Corollary 1.** Under assumptions (A.1) to (A.3), (K.1) and (K.2), given $h_{opt}(m) = (nN)^{-1/5}m^{(a-2c-2)/5}(1+m^a/N)^{1/5}$, assume $m \in \mathbb{N}_+$ satisfies (M). For each $j \leq m$,

(a) If $N \geq \text{Const.} j^a$,

$$\mathbb{E}(\| \hat{\phi}_j - \phi_j \|^2) \leq \frac{j^2}{n} + \frac{j^{(4a+2c+2)/5}}{(nN)^{4/5}}.$$  \hspace{1cm} (9)

In addition, if $N \geq n^{1/4}j^{a+c/2-2}$, this rate becomes $\mathbb{E}(\| \hat{\phi}_j - \phi_j \|^2) \leq j^2/n$.

(b) If $N = o(j^a)$,

$$\mathbb{E}(\| \hat{\phi}_j - \phi_j \|^2) \leq \frac{j^{2a+2}}{nN^2} + \frac{j^{(8a+2c+2)/5}}{(nN^2)^{4/5}}.$$  \hspace{1cm} (10)

**Remark 1.** Different from the mean and covariance function estimation, the optimal bandwidth in estimating the $j$th eigenfunction is varying with $j$. When $a > 2c+2$, the optimal bandwidth is larger for growing $j$. This refers to the case that the eigenvalues are decaying rapidly with respect to the frequency increment, and over-smoothing is needed to balance the increasing estimation variance caused by the decaying eigengaps. On the contrary, when the frequency of the target eigenfunction is relatively large compare to its smoothness, that is $a \leq 2c+2$, the optimal bandwidth becomes smaller for growing $j$ to capture the local variations, which stands for the under-smoothing case.

**Remark 2.** When $j$ is fixed and $N$ is finite, the convergence rate derived by Corollary 1 becomes $(nh)^{-1} + h^4$, which is a typical one-dimensional kernel smoothing rate and reaches optimal at $h \approx n^{-1/5}$. This result is consistent with those in Hall and Hosseini-Nasab (2006) and optimal in the minimax sense. When $j$ is not fixed but diverging to infinity, the knowing lower bound $j^2/n$ for fully observed data is derived by applying the van Tree’s inequality on the special orthogonal group (Wahl, 2020). For discretely observed functional data, there are no lower bounds for the eigenfunctions with diverging indices and it is not straightforward to extend the results in Wahl (2020) to the discrete case. Lower bounds of eigenfunctions with diverging indices for discretely observed functional data is beyond the scope of this paper and remains an open problem deserving investigation.
Remark 3. In case of the common used basis where $c = 2$, the convergence rate for the $j$th eigenfunction reaches optimal as if the curves are fully observed when $N > \max\{j^a, n^{1/4}j^{a-1}\}$. When $j$ is fixed, the phase transition occurs at $n^{1/4}$, which is same as the mean and covariance function and consistent with those in Hall et al. (2006) and Cai and Yuan (2010). For $n$ subjects, the largest index of eigenfunction that can be well estimated is smaller than $j_{\max} := n^{1/(2a+2)}$, which is a direct consequence from Assumption (M). It can be check that $j_{\max}' = n^{1/4}j_{\max}^{-1}$ and in this case, the phase transition occurs at $n^{1/4+(a-1)/(2a+2)}$, which can be interpreted from two aspects. On one hand, more observations per subject are needed in order to obtain a optimal rate for eigenfunctions with diverging indices compared to the mean and covariance estimation, which reveals the elevated difficulties caused by the infinite dimensionality. On the other hand, $n^{1/4+(a-1)/(2a+2)}$ is slightly larger than $1/4$, which provides the merits of the pooling method as well as our theoretical analysis.

Next, we focus on the asymptotic normality for the estimated eigenvalues. Before stating our results, we shall make the following assumption.

N.1 $\mathbb{E}(\|X\|^6) < \infty$ and $\mathbb{E}(\varepsilon^6) < \infty$.

N.2 For any sequence $j_1, \ldots, j_4$, $\mathbb{E}(\xi_{j_1}\xi_{j_2}\xi_{j_3}\xi_{j_4}) = 0$ unless each index $j_k$ is repeated.

Assumption (N.1) gives the additional moment condition needed in deriving the asymptotic normality, which is typical in the FDA literature (Zhang and Wang, 2016). Assumption (N.2) is a standard assumption in functional principal components analysis (Cai and Hall, 2006; Hall and Hosseini-Nasab, 2009), which simplifies the moment calculation.

Theorem 3. Under assumptions (A.1) to (A.3), (K.1), (K.2), (N.1) and (N.2), for $m \in \mathbb{N}_+$ admits

\[ \sqrt{n}(m^a + N)[m^2n^{-1}\{1 + (j^a/N)^2\} + m^a(nNh)^{-1}(1 + m^a/N) + h^4m^{2c+2}] = o(1) \text{ and } h(m^{2c} + m^a) = o(1). \]

For all $j \leq m$

\[ \Sigma_n^{-1/2} \left( \frac{\hat{\lambda}_j - \lambda_j}{\lambda_j} - 2\lambda_j\sigma_K^2h^2 \int_h \phi_j^{(2)}(u)\phi_j(u)du + o(j^{-a}h^2) \right) \xrightarrow{d} \mathcal{N}(0,1), \]

where

\[ \Sigma_n = \frac{1}{n} \left[ \frac{(N-2)(N-3)}{N(N-1)} \mathbb{E}(\xi_j^4) + 4(N-2) \mathbb{E}\{\xi_j^2(\|X\phi_j\|^2 + \sigma_X^2)\} \right] \]

\[ + \frac{2}{N(N-1)} \mathbb{E}\{\|X\phi_j\|^2 + \sigma_X^2\}^2 \frac{1}{\lambda_j^2} - 1 \right]. \]

Note that $\lambda_j \to 0$ as $j \to \infty$, we need to do regularization such that the eigenvalues serve on a common scale of variabilities. After twice integration, $(\hat{\lambda}_j - \lambda_j)/\lambda_j$ admit a degenerated kernel smoothing rate and for fixed $j$, $(\hat{\lambda}_j - \lambda_j)/\lambda_j$ is $\sqrt{n}$ consistency for small enough $h$. For slowly diverging $j$, three types of asymptotic normality emerge from Theorem 3 depending on the order of $N$. 

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Corollary 2. Under the assumption of Theorem 3, for all \( j \leq m \)

(a) If \( N\lambda_j \to \infty, \sqrt{n}\lambda_j h^2 \int_0^{1-h} \phi_j^{(2)}(u)\phi_j(u)du \to 0 \),

\[
\sqrt{n} \left( \frac{\hat{\lambda}_j - \lambda_j}{\lambda_j} \right) \xrightarrow{d} \mathcal{N} \left( 0, \frac{\mathbb{E}(\xi_j^4) - \lambda_j^2}{\lambda_j^2} \right).
\]

(b) If \( N\lambda_j \to C_1 \),

\[
\sqrt{n} \left( \frac{\hat{\lambda}_j - \lambda_j}{\lambda_j} - 2\lambda_j \sigma_k^2 h^2 \int_0^{1-h} \phi_j^{(2)}(u)\phi_j(u)du \right) \xrightarrow{d} \mathcal{N} \left( 0, \frac{\mathbb{E}(\xi_j^4) - \lambda_j^2}{\lambda_j^2} + \frac{4\mathbb{E}(\xi_j^2(\|X\phi_j\|^2 + \sigma_X^2))}{C_1\lambda_j} + \frac{2\mathbb{E}(\|X\phi_j\|^2 + \sigma_X^2)^2}{C_1^2} \right).
\]

(c) If \( N\lambda_j \to 0 \),

\[
\sqrt{n}N\lambda_j \left( \frac{\hat{\lambda}_j - \lambda_j}{\lambda_j} - 2\lambda_j \sigma_k^2 h^2 \int_0^{1-h} \phi_j^{(2)}(u)\phi_j(u)du \right) \xrightarrow{d} \mathcal{N} \left( 0, \frac{2N}{N-1} \mathbb{E}(\|X\phi_j\|^2 + \sigma_X^2)^2 \right).
\]

4 Proofs of theorems in Section 3

In this section, we will give the proofs of theorems in Section 3. In order to avoid the summability issues mentioned in Section 2, we propose a new approach to treat the dominating terms in the summation.

To begin with, denote \( T_h f(x) = h^{-1} \int K((x - y)/h)f(y)dy \) and it is clear that \( T_h \) is a self-adjoint and bounded operator. We first give the bound for \( \|T_h\phi_k - \phi_k\|^2 \),

\[
|T_h\phi_k(v) - \phi_k(v)| = \left| \frac{1}{h} \int K \left( \frac{v - t}{h} \right) \phi_k(t)dt - \phi_k(v) \right|
\]

\[
= \left| \int_{-1}^{1} K(u) \left\{ \phi_k(v) - hu\phi_k^{(1)}(v) + \frac{h^2u^2}{2}\phi_k^{(2)}(v^*) \right\} du - \phi_k(v) \right|
\]

\[
\leq h^2|\phi_k^{(2)}(t)|_{\infty} \leq h^2k^c,
\]

where \( v^* \in [v - hu, v] \) and the last inequality comes form assumption (A.3). Then there is

\[
\|T_h\phi_k - \phi_k\|^2 \lesssim h^4k^{2c} \text{ for all } k.
\]

4.1 Proof of Theorem 1

Proof. Recall \( \delta_{ij} = X_{ij}X_{il} = (X_i(T_{ij}) + \varepsilon_{ij})(X_i(T_{il}) + \varepsilon_{il}) \) and note that

\[
(\hat{C}\phi_j, \phi_k) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{N(N-1)} \sum_{t_i \neq t_2} \delta_{ilt}T_h\phi_j(T_{it_1})T_h\phi_k(T_{it_2}).
\]
Denote $A_i(\phi_j, \phi_k) = \sum_{l_1,l_2} \delta_{l_1l_2} T_h \phi_j(T_{il_1}) T_h \phi_k(T_{il_2})$, then
\[
\text{Var}(\langle \Delta \phi_j, \phi_k \rangle) = \text{Var}(\langle \hat{\phi}_j, \phi_k \rangle) \leq \frac{1}{n N^2 (N - 1)^2} \mathbb{E} A_i^2(\phi_j, \phi_k).
\]

The second moment of each $A_i$ can be decomposed as
\[
\mathbb{E} A_i^2(\phi_j, \phi_k) = 4! \binom{N}{4} A_{i1}(\phi_j, \phi_k) + 3! \binom{N}{3} A_{i2}(\phi_j, \phi_k) + 2! \binom{N}{2} A_{i3}(\phi_j, \phi_k)
\]
with
\[
A_{i1}(\phi_j, \phi_k) = \mathbb{E}(\langle X_i, T_h \phi_j \rangle^2(X_i, T_h \phi_k)^2),
\]
\[
A_{i2}(\phi_j, \phi_k) = \mathbb{E}(\langle X_i, T_h \phi_j \rangle(X_i, T_h \phi_k)(X_i, T_h \phi_j, X_i, T_h \phi_k))
\]
\[
+ \mathbb{E}\left\{\|X_i, T_h \phi_j\|^2 \|X_i, T_h \phi_k\|^2 + \sigma_X^2 \|T_h \phi_k\|^2\right\}
\]
\[
+ \mathbb{E}\left\{\|X_i, T_h \phi_k\|^2 \|X_i, T_h \phi_j\|^2 + \sigma_X^2 \|T_h \phi_j\|^2\right\},
\]
\[
:= A_{i21}(\phi_j, \phi_k) + A_{i22}(\phi_j, \phi_k) + A_{i23}(\phi_j, \phi_k)
\]
\[
A_{i3}(\phi_j, \phi_k) = \mathbb{E}(\|X_i, T_h \phi_j\|^2 + \sigma_X^2 \|T_h \phi_j\|^2)(\|X_i, T_h \phi_k\|^2 + \sigma_X^2 \|T_h \phi_k\|^2)
\]
\[
+ \mathbb{E}(\langle X_i, T_h \phi_j, X_i, T_h \phi_k \rangle + \sigma_X^2 \langle T_h \phi_j, T_h \phi_k \rangle)^2
\]
\[
:= A_{i31}(\phi_j, \phi_k) + A_{i32}(\phi_j, \phi_k).
\]

By AM-GM inequality,
\[
A_{i21}(\phi_j, \phi_k) \leq 2\mathbb{E}(\langle X_i, T_h \phi_j \rangle \langle X_i, T_h \phi_k \rangle \|X, T_h \phi_j\| \|X, T_h \phi_k\|)
\]
\[
\leq \mathbb{E}(\langle X_i, T_h \phi_j \rangle^2 \|X, T_h \phi_j\|^2 + \langle X, T_h \phi_k \rangle^2 \|X, T_h \phi_j\|^2)
\]
\[
:= A_{i22}(\phi_j, \phi_k) + A_{i23}(\phi_j, \phi_k).
\]

Similarly $A_{i32}(\phi_j, \phi_k) \leq A_{i31}(\phi_j, \phi_k)$ and thus $A_{i2}(\phi_j, \phi_k) \leq 2\{A_{i22}(\phi_j, \phi_k) + A_{i23}(\phi_j, \phi_k)\}$, $A_{i3}(\phi_j, \phi_k) \leq 2A_{i31}(\phi_j, \phi_k)$. In summary,
\[
\mathbb{E} \langle \Delta \phi_j, \phi_k \rangle^2 \leq \left(\mathbb{E} \langle \Delta \phi_j, \phi_k \rangle\right)^2
\]
\[
+ \frac{1}{n} \left\{ A_{i1}(\phi_j, \phi_k) + \frac{A_{i22}(\phi_j, \phi_k) + A_{i23}(\phi_j, \phi_k)}{N} + \frac{A_{i31}(\phi_j, \phi_k)}{N^2}\right\}.
\] (12)

The following lemmas give the bounds for $\|\mathbb{E} \Delta\|_{\text{HS}}^2$ and the fourth moment of $(X, T_h \phi_k)$, which are useful in qualifying the bias and variance terms in equation (12) and their proofs can be found in the supplementary material.

**Lemma 4.** Under assumptions (A.1) to (A.3) and (K.1), (K.2),
\[
\|\mathbb{E} \Delta\|_{\text{HS}}^2 \leq \text{Const.} \begin{cases} h^4, & 2a - 2c > 1, \\ h^4 \ln h, & 2a - 2c = 1, \\ h^2(2a-1)\epsilon, & 2a - 2c < 1. \end{cases}
\]

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**Lemma 5.** Under assumptions (A.1) to (A.3), (K.1), (K.2) and $h^4 j^{2c+2a} = O(1)$, there are $\mathbb{E}(\|X\|^4) < \infty$, $\mathbb{E}((X, T_h \phi_k)^4) \leq k^{-2a}$ for $1 \leq k \leq 2j$ and

$$\mathbb{E} \left| \sum_{k>j} |(X, T_h \phi_k)|^2 \right|^2 \leq \text{Const.} j^{2-2a}.$$  

**Step 1: bound the bias term.**

In the following, we first give the bounds for the bias term $(\mathbb{E}(\Delta \phi_j, \phi_k))^2$ for each $1 \leq k \leq 2j$ and the summation $\sum_{k=j+1}^\infty (\mathbb{E}(\Delta \phi_j, \phi_k))^2$. Note that

$$\mathbb{E}(\Delta \phi_j, \phi_k) = \langle C T_h \phi_j, T_h \phi_k \rangle - \langle C \phi_j, \phi_k \rangle$$

$$= \langle C(T_h \phi_j - \phi_j), T_h \phi_k - \phi_k \rangle + \langle C(T_h \phi_j - \phi_j), \phi_k \rangle + \langle C \phi_j, T_h \phi_k - \phi_k \rangle.$$  

(13)

For the first part in equation (13),

$$|\langle C(T_h \phi_j - \phi_j), T_h \phi_k - \phi_k \rangle| \leq \lambda_1 \|T_h \phi_j - \phi_j\| \|T_h \phi_k - \phi_k\| \leq \text{Const.} h^2 j^-h^2 k^c.$$  

As $\langle T_h \phi_j, \phi_k \rangle = \langle T_h \phi_k + \phi_k \rangle$, there is

$$|\langle T_h \phi_j - \phi_j, \phi_k \rangle| = |\langle \phi_j, T_h \phi_k - \phi_k \rangle| \leq \|\phi_j\| \|T_h \phi_k - \phi_k\| \leq \text{Const.} h^2 k^c.$$  

Thus the last two terms in equation (13) are bounded by

$$|\langle C \phi_j, T_h \phi_k - \phi_k \rangle| \leq \lambda_j |\langle \phi_j, T_h \phi_k - \phi_k \rangle| \leq \text{Const.} j^{-a} h^2 k^c,$$

$$|\langle C(T_h \phi_j - \phi_j), \phi_k \rangle| \leq \lambda_k |\langle T_h \phi_j - \phi_j, \phi_k \rangle| \leq \text{Const.} k^{-a} h^2 k^c.$$  

(14)

Then we conclude that if $1 \leq k \leq 2j$ and under the assumption $h^4 j^{2c+2a} = O(1)$,

$$\mathbb{E}(\Delta \phi_j, \phi_k) \leq C(\|h^2 c h^2 k^c + j^{-a} h^2 k^c + k^{-a} h^2 k^c) \leq \text{Const.} h^2 k^{c-a}.$$  

Furthermore, by Lemma 4, $h^4 j^{2c+2a} = O(1)$ and the fact $\min(4, 2(2a - 1)/c) > 4(2a - 1)/(2a + 2c)$, we deduce that $\|T_h C - C\|_{\text{HS}}^2 \leq \text{Const.} j^{1-2a}$. Then the summation of the first term in equation (13) over $k > j$ can be bounded by

$$\sum_{k>j} \langle C(T_h \phi_j - \phi_j), T_h \phi_k - \phi_k \rangle^2 \leq \|T_h C - C\|_{\text{HS}}^2 \|T_h \phi_j - \phi_j\|^2 \leq \text{Const.} j^{1-2a} h^4 j^{2c}.$$  

Furthermore, note that

$$\langle C(T_h \phi_j - \phi_j), \phi_k \rangle + \langle C \phi_j, T_h \phi_k - \phi_k \rangle = \lambda_k \langle T_h \phi_j - \phi_j, \phi_k \rangle + \lambda_j \langle \phi_j, T_h \phi_k - \phi_k \rangle$$

$$= (\lambda_k + \lambda_j) \langle T_h \phi_j - \phi_j, \phi_k \rangle.$$  

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There is
\[
\sum_{k>j} |(C(T_h\phi_j - \phi_j), \phi_k) + (C\phi_j, T_h\phi_k - \phi_k)|^2 = \sum_{k>j} (\lambda_k + \lambda_j)(T_h\phi_j - \phi_j, \phi_k)^2
\]
\[
\leq (2\lambda_j)^2 \sum_{k>j} (T_h\phi_j - \phi_j, \phi_k)^2 \leq (2\lambda_j)^2 \|T_h\phi_j - \phi_j\|^2 \leq \text{Const.} j^{-2a} h^4 j^{2c}.
\]

Then under assumptions (A.1) to (A.3), (K.1), (K.2) and \(h^4 j^{2c+2a} = O(1)\), there is
\[
\sum_{k=j+1}^{\infty} \langle \Delta \phi_j, \phi_k \rangle^2 \leq \text{Const.} h^4 j^{2c+1-2a}.
\]

**Step 2: bound the variance term.**

Next, we shift to variance terms \(A_{i1}, A_{i22}, A_{i23}, A_{i31}\) for each \(1 \leq k \leq 2j\) and their corresponding summations on \(k \geq j\). We start with the bounds for \(A_{i1}\). By Lemma 5 and Cauchy-Schwarz inequality
\[
A_{i1}(\phi_j, \phi_k) = \mathbb{E}((X, T_h\phi_j)^2)(X, T_h\phi_k)^2)
\]
\[
\leq \mathbb{E}(X, T_h\phi_j)^4 \mathbb{E}(X, T_h\phi_k)^4)^{1/2} \leq \text{Const.} j^{-a} k^{-a} \text{ and}
\]
\[
\sum_{k>j} A_{i1}(\phi_j, \phi_k) = \mathbb{E} \left( (X, T_h\phi_j)^2 \sum_{k>j} (X, T_h\phi_k)^2 \right)
\]
\[
\leq \left( \mathbb{E}(X, T_h\phi_j)^4 \mathbb{E} \left( \sum_{k>j} (X, T_h\phi_k)^2 \right)^2 \right)^{1/2} \leq \text{Const.} j^{1-2a}.
\]

For \(A_{i22}\). By Lemma 5, (A.3) and Cauchy-Schwarz inequality,
\[
A_{i22}(\phi_j, \phi_k) = \mathbb{E} \left\{ (X, T_h\phi_j)^2 \|XT_h\phi_k\|^2 + \sigma_X^2 \|T_h\phi_k\|^2 \right\}
\]
\[
\leq \|T_h\phi_k\|^2_{\infty} \mathbb{E} \left\{ (X, T_h\phi_j)^2 (\|X\|^2 + \sigma_X^2) \right\}
\]
\[
\leq \|\phi_k\|^2_{\infty} \left\{ \mathbb{E}(X, T_h\phi_j)^4 \mathbb{E}(\|X\|^2 + \sigma_X^2)^2 \right\}^{1/2}
\]
\[
\leq \text{Const.} \left( \mathbb{E}(X, T_h\phi_j)^4 \right)^{1/2} \leq \text{Const.} j^{-a}.
\]

For the summation \(\sum_{k>j} A_{i22}(\phi_j, \phi_k)\), note that
\[
\sum_{k=1}^{\infty} |T_h\phi_k(x)|^2 = \frac{1}{h^2} \int \left| K \left( \frac{x-y}{h} \right) \right|^2 \text{ dy} \leq \text{Const.} h^{-1} \text{ for all } x \in [0, 1].
\]

Thus,
\[
\sum_{k=1}^{\infty} A_{i22}(\phi_j, \phi_k) = \mathbb{E} \left\{ (X, T_h\phi_j)^2 \sum_{k=j}^{\infty} (\|XT_h\phi_k\|^2 + \sigma_X^2 \|T_h\phi_k\|^2) \right\}
\]
\[
\leq \sum_{k=1}^{\infty} \|T_h\phi_k(x)\|^2 \mathbb{E} \left\{ (X, T_h\phi_j)^2 (\|X\|^2 + \sigma_X^2) \right\}
\]
\[
\leq \text{Const.} h^{-1} \left( \mathbb{E}(X, T_h\phi_j)^4 \right)^{1/2} \leq \text{Const.} h^{-1} j^{-a}.
\]
For $A_{i23}$, by symmetry and Lemma 5, there is

$$A_{i23}(\phi_j, \phi_k) = A_{i22}(\phi_k, \phi_j) \leq \|T_h \phi_j\|_2^2 E \left\{ (X, T_h \phi_k)^2 (\|X\|^2 + \sigma_X^2) \right\}$$

$$\leq \text{Const.} \left( E(X, T_h \phi_k)^4 \right)^{1/2} \leq \text{Const.} k^{-a}, \quad \forall \ 1 \leq k \leq 2j, \quad (19)$$

and

$$\sum_{k>j} A_{i23}(\phi_j, \phi_k) \leq \|T_h \phi_j\|_2^2 E \left[ \sum_{k>j} (X, T_h \phi_k)^2 (\|X\|^2 + \sigma_X^2) \right]$$

$$\leq \text{Const.} \left\{ E \left( \sum_{k>j} (X, T_h \phi_k)^2 \right)^2 E(\|X\|^2 + \sigma_X^2)^2 \right\} \leq \text{Const.} j^{1-a}. \quad (20)$$

For the last term $A_{i31}$,

$$A_{i31}(\phi_j, \phi_k) = E \left\{ (\|XT_h \phi_j\|^2 + \sigma_X^2 \|T_h \phi_j\|^2) (\|XT_h \phi_k\|^2 + \sigma_X^2 \|T_h \phi_k\|^2) \right\}$$

$$\leq \text{Const.} E(\|X\|^2 + \sigma_X^2)^2 \leq \text{Const.}, \quad (21)$$

and

$$\sum_{k=j}^\infty A_{i31}(\phi_j, \phi_k)$$

$$\leq E \left\{ (\|XT_h \phi_j\|^2 + \sigma_X^2 \|T_h \phi_j\|^2) \sum_{k=1}^\infty (\|XT_h \phi_k\|^2 + \sigma_X^2 \|T_h \phi_k\|^2) \right\}$$

$$\leq E \left\{ (\|XT_h \phi_j\|^2 + \sigma_X^2 \|T_h \phi_j\|^2) \int \{X^2(u) + \sigma_X^2 \} \sum_{k=1}^\infty |T_h \phi_k(u)|^2 du \right\}$$

$$\leq E \left\{ (\|XT_h \phi_j\|^2 + \sigma_X^2 \|T_h \phi_j\|^2) (\|X\|^2 + \sigma_X^2) \frac{\|K\|_h^2}{h} \right\}$$

$$\leq \text{Const.} h^{-1} E(\|X\|^2 + \sigma_X^2)^2 \leq \text{Const.} h^{-1}. \quad (21)$$

Combine equation (16) to (21), under assumptions (A.1) to (A.3), (K.1), (K.2) and $h^{4j^{2c+2a}} = O(1)$, there are

$$A_{i11}(\phi_j, \phi_k) \leq \text{Const.} j^{-a} h^{-a}, \quad A_{i22}(\phi_j, \phi_k) \leq \text{Const.} j^{-a}, \quad A_{i23}(\phi_j, \phi_k) \leq \text{Const.} k^{-a}, \quad A_{i31}(\phi_j, \phi_k) \leq \text{Const.} k^{-a}, \quad \forall \ 1 \leq k \leq 2j;$$

$$\sum_{k=j+1}^\infty A_{i11}(\phi_j, \phi_k) \leq \text{Const.} j^{1-a}, \quad \sum_{k=j+1}^\infty A_{i22}(\phi_j, \phi_k) \leq \text{Const.} j^{-a} h^{-1}, \quad (22)$$

$$\sum_{k=j+1}^\infty A_{i23}(\phi_j, \phi_k) \leq \text{Const.} j^{-a}, \quad \sum_{k=j+1}^\infty A_{i31}(\phi_j, \phi_k) \leq \text{Const.} h^{-1}.$$

The proof of Theorem 1 is complete by combing equations (15), (22) and (12).
4.2 Proof of Theorem 2

Proof. By the proof of Theorem 5.1.8 in Hsing and Eubank (2015), for $j \in \Omega(n, N, h)$, we have the following expansion,

$$
\hat{\phi}_j - \phi_j = \sum_{k \neq j} \frac{\int (\hat{C} - C) \phi_j \phi_k}{(\lambda_j - \lambda_k)} \phi_k + \sum_{k \neq j} \frac{\int (\hat{C} - C) (\hat{\phi}_j - \phi_j) \phi_k}{(\lambda_j - \lambda_k)} \phi_k + \sum_{k \neq j} \frac{(\lambda_j - \lambda_j)^s}{(\lambda_j - \lambda_k)^{s+1}} \left\{ \int (\hat{C} - C) \hat{\phi}_j \phi_k \right\} \phi_k + \left\{ \int (\hat{\phi}_j - \phi_j) \phi_j \right\} \phi_j,
$$

(23)

such kind of expansion can also be found in Hall and Hosseini-Nasab (2006) and Li and Hsing (2010). We first show that $E \| \hat{\phi}_j - \phi_j \|^2$ is dominated by the $L^2$ norm of the first term in the right hand side of equation (23). By Bessel’s inequality, we see that

$$
E \left\| \sum_{k \neq j} \frac{\int (\hat{C} - C) (\hat{\phi}_j - \phi_j) \phi_k}{(\lambda_j - \lambda_k)} \phi_k \right\|^2 \leq E \frac{\| \hat{C} - C \|^2_{HS} \| \hat{\phi}_j - \phi_j \|^2}{(2\eta_j)^2} < \frac{1}{16} E \| \hat{\phi}_j - \phi_j \|^2,
$$

(24)

where the last equality comes from the fact $\eta_j^{-1} \| \hat{C} - C \| < 1/2$ on $\Omega_n(n, N, h)$. Similarly,

$$
E \left\| \sum_{k \neq j} \frac{(\lambda_j - \lambda_j)^s}{(\lambda_j - \lambda_k)^{s+1}} \left\{ \int (\hat{C} - C) \hat{\phi}_j \phi_k \right\} \phi_k \right\|^2 
= E \sum_{k \neq j} \frac{(\lambda_j - \lambda_j)^2}{(\lambda_j - \lambda_k)^2} \left\{ \int (\hat{C} - C) \hat{\phi}_j \phi_k \right\}^2 
\leq E \frac{\| \hat{C} - C \|^2_{HS}}{(2\eta_j - \| \hat{C} - C \|_{HS})^2} \left[ \sum_{k \neq j} \frac{\| \int (\hat{C} - C) \hat{\phi}_j \phi_k \|^2}{(\lambda_j - \lambda_k)^2} \right] + \sum_{k \neq j} \frac{\{ \int (\hat{C} - C) (\hat{\phi}_j - \phi_j) \phi_k \}^2}{(\lambda_j - \lambda_k)^2} 

\leq \frac{8}{9} E \frac{\| \hat{C} - C \|^2_{HS}}{\eta_j^2} \sum_{k \neq j} \frac{\| \int (\hat{C} - C) \hat{\phi}_j \phi_k \|^2}{(\lambda_j - \lambda_k)^2} + \frac{1}{18} E \| \hat{\phi}_j - \phi_j \|^2,
$$

(25)

Combing (23) to (25) and the fact $\| \{ \int (\hat{\phi}_j - \phi_j) \phi_j \} \| = 1/2 \| \hat{\phi}_j - \phi_j \|$, $E \| \hat{\phi}_j - \phi_j \|^2$ is dominated by the first term in the right hand side of equation (23). The proof is complete by

$$
E \| \hat{\phi}_j - \phi_j \|^2 \leq \sum_{k \neq j} \frac{E \| \Delta \phi_j, \phi_k \|^2}{(\lambda_j - \lambda_k)^2} = \sum_{k \neq j \leq 2j} \frac{E \| \Delta \phi_j, \phi_k \|^2}{(\lambda_j - \lambda_k)^2} + \sum_{k > 2j} \frac{E \| \Delta \phi_j, \phi_k \|^2}{(\lambda_j - \lambda_k)^2} 
\leq \text{Const.} \left\{ h^4 j^{2c+2} + \frac{j^2}{N} \left\{ 1 + \frac{j^a}{N} + \left( \frac{j^a}{N} \right)^2 \right\} \right\} 
\leq \text{Const.} \left\{ h^4 j^{2c+1} + \frac{1}{N} \left\{ j + \frac{j^a}{Nh} + \left( \frac{j^a}{N} \right)^2 \right\} \right\} 
\leq \text{Const.} \left\{ \frac{j^2}{N} \left\{ 1 + \left( \frac{j^a}{N} \right)^2 \right\} + \frac{j^a}{nNh} \left( 1 + \frac{j^a}{N} \right) + h^4 j^{2c+2} \right\},
$$

(26)

where the first inequality comes from Theorem 1 and Lemma 7 in Dou et al. (2012).
4.3 Proof of Theorem 3

Proof Proof of Theorem 3. By equation (5.22) in Hsing and Eubank (2015), \( \hat{\lambda}_j - \lambda_j \) admits the following expansion,

\[
\hat{\lambda}_j - \lambda_j = \langle \Delta \phi_j, \phi_j \rangle + \langle (\hat{P}_j - P_j)(\hat{C} - \hat{\lambda}_j)(\hat{P}_j - P_j)\phi_j, \phi_j \rangle, \tag{27}
\]

where \( \hat{P}_j = \hat{\phi}_j \otimes \hat{\phi}_j \), \( P_j = \phi_j \otimes \phi_j \) and \( I \) is the identity transformation on \( L^2[0,1] \). By Lemma 5.1.7 and Taylor expansion of \( \sqrt{1-x} \) at 0,

\[
\|\hat{\phi}_j - \phi_j\|^2 = 2\{1 - (1 - \|\hat{P}_j - P_j\|^2)^{1/2}\}
\.
= 2\left\{1 - 1 + \frac{\|P_j - P_j\|^2}{2} + o(\|P_j - P_j\|^2)\right\}
\.
= 2\left\{1 - 1 + \|P_j - P_j\|^2 + o(\|P_j - P_j\|^2)\right\} \tag{28}
\.

Combine (27), (28) and Cauchy-Schwarz inequality,

\[
\hat{\lambda}_j - \lambda_j = \langle \Delta \phi_j, \phi_j \rangle + O(\|\hat{\phi}_j - \phi_j\|^2) + o(\|\hat{\phi}_j - \phi_j\|^2). \tag{29}
\]

We first focus on the asymptotic behavior of \( \langle \Delta \phi_j, \phi_j \rangle \). For the bias term \( \mathbb{E}(\langle \Delta \phi_j, \phi_j \rangle) \),

\[
\mathbb{E}(\langle \Delta \phi_j, \phi_j \rangle)
= \mathbb{E}\left\{\int X_i(u) \frac{1}{h} \int K\left(\frac{u-s}{h}\right)\phi_j(s)ds du \int X_i(v) \frac{1}{h} \int K\left(\frac{v-t}{h}\right)\phi_j(t)dt dv\right\}
= \int C(u, v) \left\{\frac{1}{h} \int K\left(\frac{u-s}{h}\right)\phi_j(s)ds - \phi_j(u)\right\}
\times \left\{\frac{1}{h} \int K\left(\frac{v-t}{h}\right)\phi_j(t)dt - \phi_j(v)\right\} dudv
+ 2 \int C(u, v) \left\{\frac{1}{h} \int K\left(\frac{u-s}{h}\right)\phi_j(s)ds - \phi_j(u)\right\} \phi_j(v) dv \tag{30}
\]

For each \( u \in [h, 1-h] \),

\[
\frac{1}{h} \int K\left(\frac{u-s}{h}\right)\phi_j(s)ds - \phi_j(u)
= \int_{-1}^{1} K(v) \left\{\phi_j(u-hv) - \phi_j(v)\right\} dv - \phi_j(u)
= \int_{-1}^{1} K(v) \left\{-h v \phi_j^{(1)}(u) + \frac{h^2 v^2}{2} \phi_j^{(2)}(u) - \frac{h^3 v^3}{3!} \phi_j^{(3)}(u) + \frac{h^4 v^4}{4!} \phi_j^{(4)}(u)\right\} dv
= \frac{\sigma^2 h^2}{2} \phi_j^{(2)}(u) + o(h^2), \tag{31}
\]
where $\sigma^2_K = \int v^2 K(v) dv$ and the last equality holds under condition $hj^c = o(1)$. For each $u \in [0, h]$,

$$\frac{1}{h} \int K\left(\frac{u-s}{h}\right) \phi_j(s) ds - \phi_j(u) = \int_{-1}^{h} K(v)\phi_j(u-hv) dv - \phi_j(v)$$

$$= -h\phi^{(1)}_j(u) \int_{-1}^{h} vK(v) dv + \frac{h^2}{2}\phi^{(2)}_j(u) \int_{-1}^{h} v^2 K(v) dv$$

$$- \frac{h^3}{3!}\phi^{(3)}_j(u) \int_{-1}^{h} v^3 K(v) dv + o(h^2).$$

Similarly, for each $u \in [1-h, 1]$,

$$\frac{1}{h} \int K\left(\frac{u-s}{h}\right) \phi_j(s) ds - \phi_j(u)$$

$$= -h\phi^{(1)}_j(u) \int_{\frac{u}{h}}^{1} vK(v) dv + \frac{h^2}{2}\phi^{(2)}_j(u) \int_{\frac{u}{h}}^{1} v^2 K(v) dv$$

$$- \frac{h^3}{3!}\phi^{(3)}_j(u) \int_{\frac{u}{h}}^{1} v^3 K(v) dv + o(h^2).$$

Combine equation (31)–(33), for each $k \in \mathbb{N}_+$

$$\int_{0}^{1} \left\{ \frac{1}{h} \int K\left(\frac{u-s}{h}\right) \phi_j(s) ds - \phi_j(u) \right\} \phi_k(u) du$$

$$= \int_{h}^{1-h} \frac{\sigma^2_K h^2}{2}\phi^{(2)}_j(u) \phi_k(u) du - h \int_{0}^{1} \left\{ \int_{-1}^{h} vK(v) dv \right\} \phi^{(1)}_j(u) \phi_k(u) du$$

$$- h \int_{1-h}^{1} \left\{ \int_{\frac{u}{h}}^{1} vK(v) dv \right\} \phi^{(1)}_j(u) \phi_k(u) du + o(h^2),$$

where the last equality is due to

$$\int_{0}^{h} \frac{h^2}{2}\phi^{(2)}_j(u) \int_{-1}^{h} v^2 K(v) dv \phi_k(u) du \lesssim h^3 j^c = o(h^2),$$

$$\int_{1-h}^{1} \frac{h^2}{2}\phi^{(2)}_j(u) \int_{\frac{u}{h}}^{1} v^2 K(v) dv \phi_k(u) du \lesssim h^3 j^c = o(h^2)$$

under $hj^c = o(1)$. Combine equation (30) and (34), under $h(j^{2c} + j^a) = o(1)$, the bias term is derived by

$$\mathbb{E}(\langle \Delta \phi_j, \phi_j \rangle)$$

$$\leq \sum_{k=1}^{\infty} \lambda_k \left[ \int_{0}^{1} \left\{ \frac{1}{h} \int K\left(\frac{u-s}{h}\right) \phi_j(s) ds - \phi_j(u) \right\} \phi_k(u) du \right]^2$$

$$+ 2\lambda_j \int_{0}^{1} \left\{ \frac{1}{h} \int K^*\left(\frac{u-s}{h}\right) \phi_j(s) ds - \phi_j(u) \right\} \phi_j(u) du$$

$$= 2\lambda_j \sigma^2_K h^2 \int_{h}^{1-h} \phi^{(2)}_j(u) \phi_j(u) du + o(j^{-a}h^2).$$
Next we focus on $\text{Var}((\Delta \phi_j, \phi_j))$,

$$\text{Var}((\Delta \phi_j, \phi_j)) = \frac{1}{n} \text{Var} \left[ \frac{1}{N(N-1)} \sum_{i_1 \neq i_2} \{ \delta_{i_1i_2} T_h \phi_j(T_{i_1}) T_h \phi_j(T_{i_2}) - \lambda_j \} \right]$$

$$= \frac{1}{n} \mathbb{E} \left( \left[ \frac{1}{N(N-1)} \sum_{i_1 \neq i_2} \{ \delta_{i_1i_2} T_h \phi_j(T_{i_1}) T_h \phi_j(T_{i_2}) - \lambda_j \} \right]^2 \right) - \frac{1}{n} \mathbb{E}((\Delta \phi_j, \phi_j))^2. \tag{36}$$

For the first term in the right hand side of equation (36),

$$\mathbb{E} \left( \left[ \frac{1}{N(N-1)} \sum_{i_1 \neq i_2} \{ \delta_{i_1i_2} T_h \phi_j(T_{i_1}) T_h \phi_j(T_{i_2}) - \lambda_j \} \right]^2 \right)$$

$$= \mathbb{E} \left( \left[ \frac{1}{N(N-1)} \sum_{i_1 \neq i_2} \{ \delta_{i_1i_2} T_h \phi_j(T_{i_1}) T_h \phi_j(T_{i_2}) \} \right]^2 \right) - 2\lambda_j \mathbb{E} \left( \left[ \frac{1}{N(N-1)} \sum_{i_1 \neq i_2} \{ \delta_{i_1i_2} T_h \phi_j(T_{i_1}) T_h \phi_j(T_{i_2}) \} \right] \right) + \lambda_j^2. \tag{37}$$

By similar calculation of equation (12),

$$\mathbb{E} \left[ \left[ \frac{1}{N(N-1)} \sum_{i_1 \neq i_2} \{ \delta_{i_1i_2} T_h \phi_j(T_{i_1}) T_h \phi_j(T_{i_2}) \} \right]^2 \right]$$

$$= \frac{4! \binom{N}{4} B_1 + 3! \binom{N}{3} B_2 + 2! \binom{N}{2} B_3}{N^2(N-1)^2}. \tag{38}$$

with

$$B_1 = \mathbb{E} \left[ \left\{ \int X(u) T_h \phi_j(u) du \right\}^4 \right]$$

$$B_2 = 4 \mathbb{E} \left[ \left\{ \int X(u) T_h \phi_j(u) du \right\}^2 \left\{ \int \{ X^2(u) + \sigma_X^2 \} T_h \phi_j^2(u) du \right\} \right]$$

$$B_3 = 2 \mathbb{E} \left[ \left\{ \int \{ X^2(u) + \sigma_X^2 \} T_h \phi_j^2(u) du \right\}^2 \right].$$

For $B_1$,

$$\left\{ \int X(u) T_h \phi_j(u) du \right\}^4 = \xi_j^4 + \langle X, T_h \phi_j - \phi_j \rangle^4 + 6 \xi_j^2 \{ X, T_h \phi_j - \phi_j \}^2$$

$$+ 4 \xi_j \{ X, T_h \phi_j - \phi_j \}^3 + 4 \xi_j^3 \{ X, T_h \phi_j - \phi_j \}. \tag{39}$$
Note that \( \mathbb{E}(\xi_j^4) \geq \{ \mathbb{E}(\xi_j^2) \}^2 = \lambda_j^2 \) and
\[
\mathbb{E}(X, T_h \phi_j - \phi_j)^4 \leq \mathbb{E}(\| X \|^4) \| T_h \phi_j - \phi_j \|^4 \leq h^8 j^{4c} = o(j^{-2a});
\]
\[
\mathbb{E}(\xi_j^2(X, T_h \phi_j - \phi_j)^2) \leq \sqrt{\mathbb{E}(\xi_j^4)\mathbb{E}((X, T_h \phi_j - \phi_j)^4)} = o(j^{-2a});
\]
\[
\mathbb{E}(\xi_j(X, T_h \phi_j - \phi_j)^3) \leq \sqrt{\mathbb{E}(\xi_j^2)\mathbb{E}(\| X \|^6)\mathbb{E}((X, T_h \phi_j - \phi_j)^6)} = o(j^{-2a});
\]
\[
\mathbb{E}(\xi_j^3(X, T_h \phi_j - \phi_j)) \leq \sqrt{\mathbb{E}(\xi_j^4)\mathbb{E}(\xi_j^2(X, T_h \phi_j - \phi_j)^2)} = o(j^{-2a}).
\]
Thus,
\[
B_1 = \mathbb{E}(\xi_j^4) + o(\mathbb{E}(\xi_j^4)). \tag{40}
\]

For \( B_2 \), we start with \( \mathbb{E}((X, T_h \phi_j)^2 \| XT_h \phi_j \|^2), \)
\[
\mathbb{E}((X, T_h \phi_j)^2 \| XT_h \phi_j \|^2) = \mathbb{E}((\| X \|^4) \| T_h \phi_j - \phi_j \|^2)\| T_h \phi_j \|^2
\]=\(\mathbb{E}(\xi_j^2 \| XT_h \phi_j \|^2) + \mathbb{E}((X, T_h \phi_j - \phi_j)^2 \| XT_h \phi_j \|^2) + 2\mathbb{E}\{\xi_j(X, T_h \phi_j - \phi_j) \| XT_h \phi_j \|^2\}. \tag{41}\)

For the first term in the right hand side of equation (41),
\[
\mathbb{E}(\xi_j^2 \| XT_h \phi_j \|^2)
=\mathbb{E}(\xi_j^2 \| X \|^2) + \mathbb{E}(\xi_j^2 \| X(T_h \phi_j - \phi_j) \|^2) + 2\mathbb{E}(\xi_j^2 \| X \phi_j, X(T_h \phi_j - \phi_j) \|).
\]
Under Assumption \( (N.2), \)
\[
\mathbb{E}(\xi_j^2 \| X \|^2) = \mathbb{E}\left\{ \xi_j \int \sum_{k=1}^{\infty} \xi_k^2 \phi_k^2(u) \phi_j^2(u) du \right\} \geq \text{Const.} \lambda_j
\]
and the remaining terms in the right hand side of equation (41) admit
\[
\mathbb{E}(\xi_j^2 \| X(T_h \phi_j - \phi_j) \|^2) \leq \sqrt{\mathbb{E}(\xi_j^4)\mathbb{E}(\| X \|^4)\| T_h \phi_j - \phi_j \|^2} = o(\lambda_j);
\]
\[
\mathbb{E}(\xi_j^2 \| X \phi_j, X(T_h \phi_j - \phi_j) \|) \leq \mathbb{E}(\xi_j^2 \| X \| \| T_h \phi_j - \phi_j \|) = o(\lambda_j).
\]
Thus,
\[
\mathbb{E}(\xi_j^2 \| XT_h \phi_j \|^2) = \mathbb{E}(\xi_j^2 \| X \|^2) + o(\mathbb{E}(\xi_j^2 \| X \|^2)). \tag{42}\)

For the last two terms in (41),
\[
\mathbb{E}((X, T_h \phi_j - \phi_j)^2 \| XT_h \phi_j \|^2) \leq \mathbb{E}(\| X \|^4)\| T_h \phi_j - \phi_j \|^2\| T_h \phi_j \|^2
\leq h^4 j^{2c} = o(j^{-a});
\]
\[
\mathbb{E}\{\xi_j(X, T_h \phi_j - \phi_j) \| XT_h \phi_j \|^2\} \leq \sqrt{\mathbb{E}(\xi_j^2)\mathbb{E}(\| X \|^6)\| T_h \phi_j - \phi_j \|}
\leq j^{-\frac{a}{2}} h^{2c} = o(j^{-a}). \tag{43}\)

Combine (41) and (43),
\[
\mathbb{E}((X, T_h \phi_j)^2 \| XT_h \phi_j \|^2) = \mathbb{E}(\xi_j^2 \| X \|^2) + o(\mathbb{E}(\xi_j^2 \| X \|^2)). \tag{44}\)

By similar arguments there is

$$\mathbb{E}(\langle X, T_h \phi_j \rangle^2 \| T_h \phi_j \|^2) = \mathbb{E}(\xi_j^2) + o(\mathbb{E}(\xi_j^2)).$$

Thus,

$$B_2 = 4\mathbb{E}\{\xi_j^2(\| X \phi_j \|^2 + \sigma_X^2)\} + o(\mathbb{E}\{\xi_j^2(\| X \phi_j \|^2 + \sigma_X^2)\}). \quad (45)$$

For $B_3$, note that

$$B_3 = 2\mathbb{E}\left[ \left\{ \int (X^2(u) + \sigma_X^2)T_h \phi_j^2(u)du \right\}^2 \right]$$

$$= 2\mathbb{E}\left[ \left\{ \| XT_h \phi_j \|^2 + \sigma_X^2 \| T_h \phi_j \|^2 \right\}^2 \right]$$

$$= 2\mathbb{E}\left\{ (\| X \phi_j \|^2 + \sigma_X^2)^2 \right\} + o(1). \quad (46)$$

Combine equation (38), (40), (45) and (46),

$$\mathbb{E}\left( \left\{ \frac{1}{N(N-1)} \sum_{t_1 \neq t_2} \delta_{it_1t_2} T_h \phi_j(T_{it_1}) T_h \phi_j(T_{it_2}) \right\}^2 \right)$$

$$= \{1 + o(1)\} \left[ \frac{(N-2)(N-3)}{N(N-1)} \mathbb{E}\xi_j^4 + \frac{4(N-2)}{N(N-1)} \mathbb{E}\{\xi_j^2(\| X \phi_j \|^2 + \sigma_X^2)\} \right]$$

$$+ \frac{2}{N(N-1)} \mathbb{E}\{ (\| X \phi_j \|^2 + \sigma_X^2)^2 \}. \quad (47)$$

For the last two terms in equation (37), by equation (35)

$$\mathbb{E}\left( \left\{ \frac{1}{N(N-1)} \sum_{t_1 \neq t_2} \delta_{it_1t_2} T_h \phi_j(T_{it_1}) T_h \phi_j(T_{it_2}) \right\} \right) = \lambda_j + o(j^{-a}).$$

Thus

$$\frac{1}{n} \mathbb{E}\left\{ \left\{ \frac{1}{N(N-1)} \sum_{t_1 \neq t_2} \delta_{it_1t_2} T_h \phi_j(T_{it_1}) T_h \phi_j(T_{it_2}) - \lambda_j \right\}^2 \right\} = \Sigma'_n + o(\Sigma'_n) \quad (48)$$

with

$$\Sigma'_n = \frac{(N-2)(N-3)}{N(N-1)} \mathbb{E}\xi_j^4 - \lambda_j^2 + \frac{4(N-2)}{N(N-1)} \mathbb{E}\{\xi_j^2(\| X \phi_j \|^2 + \sigma_X^2)\}$$

$$+ \frac{2}{N(N-1)} \mathbb{E}\{ (\| X \phi_j \|^2 + \sigma_X^2)^2 \}. \quad (49)$$

The proof is complete by combing equation (28), (35), (48) and $\Sigma_n^{-1/2} \| \hat{\phi}_j - \phi_j \| = o_p(1)$ under Assumption (M.2).
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