Ashtekar-Barbero holonomy on the hyperboloid:
Immirzi parameter as a Cut-off for Quantum Gravity

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In the context of the geometrical interpretation of the spin network states of Loop Quantum Gravity, we look at the holonomies of the Ashtekar-Barbero connection on loops embedded in space-like hyperboloids. We use this simple setting to illustrate two points. First, the Ashtekar-Barbero connection is not a space-time connection, its holonomies depend on the spacetime embedding of the canonical hypersurface. This fact is usually interpreted as an inconvenience, but we use it to extract the extrinsic curvature from the holonomy and separate it from the 3d intrinsic curvature. Second, we show the limitations of this reconstruction procedure, due to a periodicity of the holonomy in the Immirzi parameter, which underlines the role of a real Immirzi parameter as a cut-off for general relativity at the quantum level in contrast with its role of a mere coupling constant at the classical level.

Introduction

Loop quantum gravity (LQG) proposes a framework for a canonical quantization of general relativity reformulated as a gauge field theory. It exchanges the usual metric canonical variables, the 3-metric and its conjugate extrinsic curvature tensor, by a new canonical pair defined from the first order formulation of general relativity: the Ashtekar-Barbero connection $A$ and its conjugate triad field $E$ (for an extensive review, see [1]). Then one proceeds to consider wave-functions of the Ashtekar-Barbero connection, $\psi(A)$, and to let geometric observables, constructed from the triad, act as differential operators. The loop quantization scheme actually relies on the choice of cylindrical functionals, which depend of the holonomies of the connection $A$ along the edges of an arbitrary chosen graph. The sum over all graphs embedded in the canonical spatial slice is then implemented by a projective limit, which yields the Hilbert space of spin network states [2].

In all this construction, the Immirzi parameter $\beta$ plays a crucial role. It can be seen as a new coupling constant entering the Palatini action for general relativity in front of an almost-topological term [3]. But, at a deeper level, it implements a canonical transformation from the original complex self-dual Ashtekar connection, which we will call $\mathcal{A}$, and the real Ashtekar-Barbero $su(2)$-connection $A$ [4]. This allowed both to work with a compact gauge group $SU(2)$ (instead of the non-compact Lorentz group $SL(2, \mathbb{C})$) and to avoid the issue of the reality conditions$^1$.

At a more effective level, it enters the loop quantum gravity dynamics in a non-trivial way and seems to be a crucial parameter in the description of quantum black holes (see the review [5]). It also appears to control the couplings to fermionic field and possible quantum gravity induced CP violation [7–10]. The main drawback of the Immirzi parameter is that the fact that Ashtekar-Barbero connection is not a space-time connection anymore and the resulting apparent loss of covariance [11] (see also the more recent [12, 13]). Nevertheless, this does not cause any problem in practice and one can perfectly define the kinematical Hilbert space of the theory and transition amplitudes between spin network states either by a canonical Hamiltonian [14, 15] or by a spinfoam path integral amplitude [16]. It can however be tempting to go back to the original complex formulation, given by the specific imaginary choice of Immirzi parameter $\beta = \pm i$, and attempt to define an analytic continuation of the real formulation of loop quantum gravity [17].

Here, we would like to underline the crucial difference between the role of the Immirzi parameter at the classical level and in the quantum theory. Classically, it appears as a coupling constant in the Holst-Palatini action for the first order formulation of general relativity [3]. In the effective field theory paradigm, one can then investigate its renormalisation flow, together with the Newton’s gravity constant and the cosmological constant, as proposed in [18, 19]. In the full quantum theory, it appears as a more essential parameter defining directly the fundamental quanta of geometry - scaling the discrete spectra of the area and volume operators- in Planck units. We would like to trace back, in the loop quantization procedure, where the Immirzi parameter acquires this deeper role.

We will use the very simple example of the holonomies of the Ashtekar-Barbero connection on a space-like 3-hyperboloid embedded in flat space-time and look at its dependence on both the hyperboloid curvature and the Immirzi parameter. As it was previously pointed out by Samuel in [20], this setting allows to illustrate that the

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$^1$As the Ashtekar connection $\mathcal{A}$ is complex and is thus not equal to its complex conjugate, it can not be simply quantized as a multiplicative operator if the scalar product is simply defined by the Gaussian measure. One needs to modify in a non-trivial way either the scalar product or the action of the connection operator (see e.g. [5]).
Ashtekar-Barbero connection is not a space-time connection and how the value of a Wilson loop depends on the space-time embedding of the canonical space-like hypersurface. We will push this analysis further and illustrate the effective compactification induced by the Immirzi parameter from the non-compact Lorentz group $\text{SL}(2,\mathbb{C})$ down to the compact loop gravity gauge group $\text{SU}(2)$. This compactification, which can be understood as the origin of the discrete spectra for areas and volumes, leads to a periodicity in the (extrinsic) curvature, that one can not fully reconstruct the metric from sampling the holonomy of the Ashtekar-Barbero connection, or in equivalent words, that the choice of observables in loop quantum gravity for a fixed Immirzi parameter does not allow to distinguish all points of the classical phase space. In that sense, the Immirzi parameter quits being a mere coupling constant but appears to play the new effective role of a cut-off, similarly to the energy scale cut-off in usual quantum field theory. This is consistent with the view that it determines the size of discrete quanta of geometry and points towards the perspective that the bare theory would conclude this short paper with the perspective of dealing with the Immirzi parameter as a cut-off for quantum gravity, combined with the Planck unit area, $\beta I_p^2$. 

\section{Ashtekar-Barbero Connection, Extrinsic Curvature and Holonomy}

Loop quantum gravity aims at a canonical quantization of general relativity. We start with the first order Palatini action, in terms of the vierbein 1-form $e^I_a$ and the Lorentz connection $\omega^J_{\mu}$, with the additional Holst term \cite{Holst}:

$$S[e, \omega] = \int_{\mathcal{M}} \star(e \wedge e)_{IJ} \wedge F[\omega]^{IJ} - \frac{1}{\beta} e \wedge e \wedge F[\omega]^{IJ}$$

(1)

with the curvature $F[\omega] = d\omega + \omega \wedge \omega$ or explicitly:

$$F[\omega]^{IJ}_{\mu\nu} = \partial_\mu \omega^J_{\nu I} - \partial_\nu \omega^J_{\mu I} + \delta_{KL} (\omega^I_{\mu} \omega^J_{\nu} - \omega^I_{\nu} \omega^J_{\mu})$$

(2)

The new term, whose coupling constant is the Immirzi parameter $\beta$, doesn’t affect the classical equations of motion as long as the tetrad $e$ is invertible and the resulting 4-metric non-degenerate. This term can be interpreted as a mass term for the torsion, through the Nie-Nyan topological invariant \cite{NieNyan}.

In the canonical quantization program, one considers a globally hyperbolic spacetime $\mathcal{M} \sim \mathbb{R} \times \mathbb{S}$ and proceeds to the 3+1 splitting and Hamiltonian analysis of the gravity action. As reviewed in \cite{Rovelli2}, we get a canonical pair of field variables, the densitized triad $E$, defining the intrinsic 3d geometry of the canonical hypersurface, and the extrinsic curvature $K$, which can be understood as the “speed” of the 3d geometry. Assuming the time-gauge $e^{0i} = \delta_{0i}$, these are defined as:

\begin{align*}
E^a_i &= \frac{1}{R} e^{abc} \epsilon_{ijk} e^j_i e^k_c \\
K^a_i &= \omega^a_i
\end{align*}

(3)

where $a$ are space coordinates indices and $i$ are $\text{SU}(2)$ labels.

They satisfy the following constraint by definition:

$$\forall i, \ G_i = \epsilon_{ijk} K^k_a E^a_j = 0$$

(4)

The Poisson bracket is:

$$\{K^a_i(x), E^b_j(y)\} = 8\pi G \delta^a_b \delta^i_j \delta(x-y)$$

(5)

The Ashtekar-Barbero variables for loop gravity are introduced by a simple canonical transformation on these variables, in order to recover the phase space of a gauge field theory. We define the Ashtekar-Barbero connection from the spin-connection compatible with the triad $E$:

$$A^i_a = \Gamma[E^a_i] + \beta K^i_a,$$

(6)

\footnote{The variables can be defined without the time-gauge but expressions are more involved. The densitized triad diagonalizes the 3d-metric and the extrinsic curvature is defined as $K_{ab} = g^c_{a} \overline{\nabla} n_c$ in terms of the normalized time normal $n$ and where $q$ is the projection operator to the tangent space of the 3d slice defined by $q_{ab} = g_{ab} - n_a n_b$. The triad in the 3d slice is then used to lift to tangent indices.}
with the spin connection $\Gamma$ satisfying the covariant derivative $\nabla_a E^i_b = \epsilon(\partial_a e^i_b - \Gamma^i_{ab} e^j_a + \epsilon_{ijk} \Gamma^j_{bc} e^k_b) = 0$, where $\epsilon$ is the determinant of cotriad $(e^i_a)$. $\Gamma^i_{ab}$ is the torsionless (symmetric in low indices) affine connection compatible with the 3-metric $h_{ab} = e^i_a e^j_b \delta_{ij}$. The compatibility condition is expressed $\nabla_a h_{bc} = 0$. Putting everything together, the spin connection reads explicitly:

$$\Gamma[E]_{ab} = \frac{1}{2} \epsilon^{ijk} E^i_j \left( \partial_a E^j_b - \partial_b E^j_a + E^j_c \partial_a E^k_c - E^k_c \partial_b E^j_a \right) + \frac{1}{4} \epsilon^{ijk} e^b_k \left( 2E^j_a \partial_b \det E - E^j_b \partial_a \det E \right).$$

(7)

The second term involving $\det E = \epsilon^{ijk} e^a_i e^b_j e^c_k = e^2$ comes from the fact that we are dealing with the densitized triad. The Poisson bracket still reads:

$$\{A^a_i(x), E^b_j(y)\} = 8\pi G \delta^a_b \delta^i_j \delta(x - y)$$

(8)

while the orthogonality constraint between $E$ and $K$ now becomes a Gauss law:

$$(D_A E)_i = \partial_a e^i_a + \epsilon_{ijk} A^j_a E^k_j = 0.$$  

(9)

This can be seen directly from the compatibility condition between the triad and the spin-connection and comes from the more general condition $\nabla_a e^b_i = 0$ (written earlier using the densitized triad). We underline that this equation, and thus the Gauss law, holds whatever the value of the Immirzi parameter $\beta$. The difference between the Ashtekar-Barbero connection and the spin-connection then resides in that fact that the connection $A$ carries a non-vanishing torsion $T[e, A]$, which is proportional to the Immirzi parameter $\beta$ and reflects the extrinsic curvature:

$$T[e, A]_{ab} = (d_A e^i)_{ab} = \partial_a e^i_b - \partial_b e^i_a + \epsilon_{ijk} (A^j_a e^k_b - A^j_b e^k_a) = \beta \epsilon_{ijk} (K^j_a e^k_b - K^j_b e^k_a) = \beta \epsilon_{ijk} (K^j \wedge e^k)_{ab}.$$  

(10)

In some sense, we can interpret the Gauss law $D_A e = 0$ as the longitudinal part of the parallel transport of the triad $e$ by the Ashtekar-Barbero connection while $T = d_A e = \beta K \wedge e$ is its transversal part.

After the choice of the Ashtekar-Barbero variables to describe the space-time geometry, the second prescription of loop quantum gravity is the choice of a specific set of observables, forming the holonomy-flux algebra \cite{22}. More precisely, we consider a class of cylindrical functionals of the connection, which generalizes the Wilson loop and depend on the holonomies of the connection $A$ along finite sets of edges within the canonical hypersurface $\Sigma$. Let us consider a horizontal $\gamma$ and consider the gauge-invariant observable defined by the trace of the holonomy around that loop:

$$W_{\gamma}[A] \equiv \text{Tr} U_{\gamma}[A] = \text{Tr} \mathcal{P} e^{i \int_{\Sigma} A^i_j (\frac{\pi}{2})^a},$$

(11)

where the $J_i$ are the generators of the $su(2)$ Lie algebra. In the fundamental representation, $J_a = \frac{a_i}{2}$ are the Pauli matrices. Of course, one must in principle choose a starting point, or root, for the loop, but the trace does not depend on that choice in the end. More generally, we consider a closed connected oriented finite graph $\mathcal{G}$ and construct a cylindrical functional of the connection $A$ which depends only on the holonomies of $A$ along the edges $e$ of the graph:

$$\Psi_{\mathcal{G}}[A] \equiv \psi \left( \{ U_e[A] \}_{e \in \mathcal{G}} \right)$$

(12)

such that the function $\psi$ is invariant under gauge transformations, which act by SU(2) transformations at the graph vertices:

$$\psi \left( \{ U_e \}_{e \in \mathcal{G}} \right) = \psi \left( \{ h_e U_e h_{-1}^{-1} \} \right), \forall h_e \in \text{SU(2)}.$$

(13)

These cylindrical wave-functions realize a sampling of the geometry at the classical level (see e.g. \cite{23}), but they will entirely define the state of geometry at the quantum level. Considering holonomies of the Ashtekar-Barbero connection for a fixed real Immirzi parameter irremediably introduces a periodicity in the extrinsic curvature $K$, which erases all the information about its high momentum fluctuations. Moreover the intrinsic curvature and the extrinsic curvature are also mixed in the Ashtekar-Barbero holonomies and it becomes a challenge to use the triad (or its discretized version as fluxes living of the graph edges) to distinguish the two contributions, intrinsic and extrinsic, to the connection in order to faithfully reconstruct the space-time geometry (at least around the canonical hypersurface). This is especially important when investigating the renormalization flow of loop quantum gravity as resulting from the coarse-graining of its geometry.

The two important points that we would like to underline in this short paper are:

- The connection $A$ is not a space-time connection, except in the special case of the (anti-)self-dual Ashtekar connection $\mathcal{A}$ for the purely imaginary choice $\beta = \pm i$. It depends on the space-time embedding of the canonical hypersurface $\Sigma$. Considering a Wilson loop $\gamma$, its value will change if we embed it in different canonical space-like hypersurfaces $\Sigma$. This apparently bad feature might be turned into an advantage: it could help us extract a simpler way the extrinsic data from the Ashtekar-Barbero holonomies and reconstruct the local embedding of the 3d space manifold.

- The Ashtekar-Barbero connection, for real $\beta$, is in some sense a projection of the non-compact Lorentz connection into the compact SU(2) group. We lose some information, due to the periodicity in the extrinsic curvature. At the classical level, different extrinsic curvatures will still lead to the same value of the Wilson loop. This appears to impose a cut-off on the possible excitations of the geometry, more precisely on the extrinsic curvature, i.e. on the speed/momentum of the 3d intrinsic geometry.
We will illustrate these two points in the following sections with the example of a closed loop embedded in space-like hyperboloids with variable curvature within the flat 4d space-time. We will discuss the dependence of the Wilson loop on the curvature of the hyperboloid, to show both how the Ashtekar-Barbero connection depends on the space-time embedding and how we can recover the extrinsic curvature from the value of the holonomy.

II. WILSON LOOPS ON THE HYPERBOLOID

Let us start with the flat 3+1d Minkowski space-time with signature \((-+++\)) and consider the upper sheet of the space-like hyperboloid,
\[-(t-t_0)^2 + (x^2 + y^2 + z^2) = -\kappa^2, \quad t \geq t_0,\]  
(14)
with an arbitrary curvature radius \(\kappa > 0\) and a possible time shift \(t_0 \in \mathbb{R}\). We would like to look at the Ashtekar-Barbero holonomy around a loop of radius \(R\).

\[\gamma \equiv \{ t = T, \ x^2 + y^2 + z^2 = R^2 \},\]  
(15)
where \(T\) and \(R\) are arbitrarily fixed. As illustrated on fig.1, we embed this loop in the whole family of hyperboloids of arbitrary curvature radius \(\kappa\) by adjusting their time shift in terms of \(\kappa\),
\[t_0 = T - \sqrt{R^2 + \kappa^2}.\]  
(16)
This setting is very similar to [20], but we extend that calculation explicitly to arbitrary curvature \(\kappa\).

Let us now compute the Ashtekar-Barbero connection of the hyperboloid and the value of the Wilson loop in terms of \(R, \kappa\) and the Immirzi parameter \(\beta\). Using the spherical coordinate on the hyperboloid, the canonical vector basis is:

\[
\begin{align*}
\partial_t &= \sin \theta \cos \varphi \partial_x + \sin \theta \sin \varphi \partial_y + \cos \theta \partial_z + \frac{r}{\sqrt{\kappa^2 + r^2}} \partial_t \\
\partial_\varphi &= -\cos \theta \cos \varphi \partial_x + \cos \theta \sin \varphi \partial_y - \sin \theta \partial_z \\
\partial_r &= -r \sin \varphi \sin \varphi \partial_x + r \sin \theta \cos \varphi \partial_y \\
\partial_\theta &= \frac{\sin \theta}{\kappa} \partial_r \\
\partial_\varphi &= \frac{1}{\sin \theta} \partial_\varphi 
\end{align*}
\]
(17)

The triad field is:

\[
\begin{align*}
\hat{e}_1 &= \sqrt{1 + \frac{\kappa^2}{\kappa^2}} \partial_t \\
\hat{e}_2 &= \frac{r}{\kappa} \partial_\varphi \\
\hat{e}_3 &= \frac{1}{\kappa \sin \theta} \partial_r \\
\end{align*}
\]
(18)
This allows to compute the affine connection:

\[
(\Gamma^a_{bc}) = \left( \begin{array}{ccc} 0 & 0 & 0 \\
0 & 0 & -\sqrt{1 + \frac{\kappa^2}{\kappa^2}} \\
-\cos \theta & -\sin \theta \sqrt{1 + \frac{\kappa^2}{\kappa^2}} & 0 \end{array} \right)
\]
(19)
where \(a\) is the metric. This gives the spin-connection:

\[
(K^a_i) = \left( \begin{array}{ccc} \frac{1}{\kappa} \sqrt{1 + \frac{\kappa^2}{\kappa^2}} & 0 & 0 \\
0 & \frac{\kappa}{\kappa} & 0 \\
0 & 0 & \frac{\kappa}{\kappa} \sin \theta \end{array} \right)
\]
(20)
These are combined into the Ashtekar-Barbero connection \(A^i_a = \Gamma^a_{bc} + \beta K^a_i\),

\[
(A^i_a) = \left( \begin{array}{ccc} \beta \frac{1}{\kappa} \sqrt{1 + \frac{\kappa^2}{\kappa^2}} & 0 & 0 \\
0 & \beta \frac{\kappa}{\kappa} & -\sqrt{1 + \frac{\kappa^2}{\kappa^2}} \\
-\cos \theta & -\sin \theta \sqrt{1 + \frac{\kappa^2}{\kappa^2}} & \beta \frac{\kappa}{\kappa} \sin \theta \end{array} \right)
\]
(21)
It is then straightforward to compute its holonomy around the loop \(\gamma\), the interested reader will find the detailed calculation in the appendix A. We obtain for the spin-1 Wilson loop (for the 3-dimensional representation, where the holonomy is represented as a SO(3) group element):

\[
W_{\kappa}(R) = 1 + 2 \cos \left( 2\pi \sqrt{1 + (1 + \beta^2) \frac{R^2}{\kappa^2}} \right)
\]
(23)
We see a clear dependence of the size of the loop in units of the curvature radius of the hyperboloid, as illustrated on the plots in fig.2.

This term further depends on the Immirzi parameter \(\beta\). For \(\beta^2 = -1\), this extra term vanishes and we recover FIG. 1: This shows several hyperboloids of different curvature all containing the same loop in flat spacetime. The curvature of the embedding hyperboloid affects however the curvature of the Ashtekar-Barbero connection.
The curvature is not uniquely fixed but determined up to a period $k \in \mathbb{Z}$.

The periodicity implies an ambiguity in the determination of the curvature from the Wilson loop. One could decide to take the lowest value of the curvature, i.e, the highest value of the curvature radius, typically given by the natural choice $k = 1$. But this would mean obviously neglecting the possibility of higher curvature fluctuations. In this sense, we see that fixing a real Immirzi parameter leads to a cut-off in curvature in the context of loop quantum gravity.

Here we have assumed a flat space-time and considered a homogeneous space slice. For later investigation, it would be interesting to extend this analysis to a homogeneous curved space-time with non-vanishing cosmological constant $\Lambda \neq 0$. Technically we would then have two length ratios to determine, the space curvature and the space-time curvature in terms of the Wilson loop size. Physically, this would allow to distinguish the intrinsic and extrinsic curvature of the canonical hypersurface and help clarifying the relationship between the torsion and the extrinsic curvature in loop quantum gravity. Finally, from the perspective of coarse-graining, it is necessary to see how the fluctuations of gravity renormalize the scalar curvature.

Moreover, we have focused on the Wilson loop observable, measuring the curvature of the Ashtekar-Barbero connection. In light of the interplay between curvature and torsion in loop quantum gravity, it might be interesting to also look at an observable discretizing the torsion, such as a surface integral $\int_{S^2} K \wedge e$ on the minimal surface supported by the loop. Having such an observable at our disposal would allow a more direct access to the extrinsic curvature. We think that this might be related to the holomorphic holonomy of the spinorial formalism for loop gravity $^{21,27}$. This link will be investigated in future work.

III. RECONSTRUCTING THE GEOMETRY FROM HOLONOMIES

Up to now, we have looked at a single Wilson loop. Of course, it would get much more information if we were to consider a mesh of Wilson loops and have several samples of the Ashtekar-Barbero connections through the holonomies around various loops. Having access to loops of various sizes $R_1, R_2, \ldots, R_n$, instead of a single loop unit $R$ would surely be very helpful. We could introduce a whole network graph, with loops of increasing sizes as illustrated on fig.3 on our canonical slice and consider all the data living on the corresponding spin network state.

We see three (inter-related) short-comings of this scenario:

- This very much resembles a discretization scheme and, if we don’t assume that the canonical slice
FIG. 3: This shows a network of loops on a 2d hyperboloid, with increasing loop size and transversal links between them creating a grid. When moving up to a 3d hyperboloid, the third dimension will make it harder to locate a loop “inside” or “outside” another one.

is homogeneous on the whole graph, we know that this will not pick up the curvature fluctuations with momentum higher than the network spacing.

- We can not assume to know all the loop sizes \( R_1, R_2, ..., R_n \). This would already mean assuming a lot of information on the geometry of the hypersurface! Working with a single loop, we don’t assume knowing its size \( R \) and we merely use it as a length unit for dimensionful observables.

- We can not locate a priori the loops with respect to each other, first, because we do not have an assumed background geometry to do so and evaluate the distances and so on, and, second, because we are actually considering equivalence of graph under diffeomorphisms. So there is no a priori notion of which loop is inside or smaller than another one and this has to be determined a posteriori from the geometry reconstruct from the spin network state itself.

The natural way to sidestep these obstacles is to introduce extra data, such as a sampling of the triad or torsion or extrinsic curvature and not only of the Ashtekar-Barbero connection. The natural arena for this is the twisted geometry framework and we will investigate in the future how to define further gauge-invariant observables probing the extrinsic curvature and space-time embedding of the canonical hypersurface in order to describe the space-time geometry around the spatial slice.

IV. THE IMMIRZI PARAMETER AS A CUT-OFF FOR QUANTUM GENERAL RELATIVITY

Focusing the value of the Wilson loop and the inversion formula to recover the curvature, let us push further the consequences of the boundedness of the Wilson loop for a real Immirzi parameter \( \beta \in \mathbb{R} \) and its associated periodicity in the curvature:

\[
R^2 \kappa^{-2} = \frac{1}{1 + \beta^2} \left[ \left( k + \frac{\cos^{-1} \left( \frac{W-1}{2} \right)}{2\pi} \right)^2 - 1 \right],
\]

with \( k \in \mathbb{Z} \). Different values of the periodicity parameter \( k \) define different slices for the curvature \( \kappa \), as we show with the plots on fig.4 For the value giving the minimal curvature sector, \( k = \pm 1 \), we have a simple bound of the curvature \( \kappa^{-1} \):

\[
R^2 \kappa^{-2} \leq \frac{1}{1 + \beta^2} \frac{5}{4}
\]

Therefore, if we naturally choose a single section for this inversion formula to uniquely determine the curvature from the Wilson loop, we do have a cut-off in the possible curvature.

This suggests a slight shift in perspective on the role of the Immirzi parameter in loop quantum gravity: it is not merely a flowing coupling constant to be renormalized, but it becomes the “energy” cut-off itself for the quantum theory. Indeed, it controls the maximal excitations of the extrinsic curvature, i.e. of the conjugate momentum to the intrinsic geometry.

Conclusion & Outlook

We have discussed the role of the Immirzi parameter \( \beta \) in loop quantum gravity. It plays at least two distinct roles. At the classical level, it controls the torsion of the Ashtekar-Barbero connection, which encodes the extrinsic curvature and thus the space-time embedding of the canonical hypersurface. However, through the choice of the holonomy observables, it further leads to a compactification of the gauge group (from the non-compact Lorentz group to the compact SU(2) group), at least for a real Immirzi parameter \( \beta \in \mathbb{R} \). This translates into a periodicity of the observables, on which the quantization is based, on the extrinsic curvature. This underlines the role of the Immirzi parameter as a cut-off for the
general relativity phase space and for the quantum fluctuations of the geometry: the limit $\beta \to \infty$ suppresses curvature fluctuations, while the limit $\beta \to 0$ allows to cover the whole phase space. This is very similar to the role of the energy cut-off in the renormalisation of quantum field theory. At the mathematical level, it implies looking at the renormalization flow of the other coupling constants (such Newton’s gravity constant and the cosmological constant) in terms of $\beta$. And at a more physical level, it means that one should adjust the Immirzi parameter according to the physical events for which one is trying to make predictions.

We illustrated this with the simple calculation of the Wilson loop of the Ashtekar-Barbero connection on a fixed loop living in various space-like hyperboloid embedded in the flat space-time and discussed how to invert the relation between the hyperboloid curvature and the value of the Wilson loop. Our computation is also relevant in the context of the coarse-graining of loop quantum gravity, when the goal is to evaluate the average curvature carried by the geometry fluctuations and attempting to reconstruct the mean geometry at large scale.

A direct possible technical improvement of our result would be to take into account the space-time curvature, as a non-vanishing cosmological constant, and not only the space curvature. This would allow to distinguish the intrinsic and extrinsic curvature of the spatial slice in the reconstruction process of the geometry from the algebraic data carried by the quantum states of geometry in loop quantum gravity. Another direction of investigation will be to look for discretized observables encoding in a more direct fashion the Ashtekar-Barbero torsion and one potential path we would like to explore in the future is the intrinsic and extrinsic curvature of the spatial slice in the context of the coarse-graining of loop quantum gravity. Another direction of investigation will be to test it in usual Quantum Field Theory. Indeed, loop-like quantization schemes for scalar fields have been developed [28, 29]. In this framework, the quantization is to test it in usual Quantum Field Theory. Indeed, loop-like quantization schemes for scalar fields have been developed [28,29]. In this framework, the quantization is developed [28,29]. In this framework, the quantization is.

Appendix A: Calculating the Wilson loop on the hyperboloid

We want to compute the Ashtekar-Barbero holonomy in space-time flat around a loop defined by:

$$\begin{align*}
  t &= T \\
  x &= R \cos \phi \\
  y &= R \sin \phi \\
  z &= 0 \\
  \phi \in [0, 2\pi] 
\end{align*}$$

(A1)

$\phi$ is the coordinate along the circle. Note here that $t$ is constant set to $T$ and is irrelevant in the following calculation. We embed this loop on the hyperboloid defined by $(t-t_0)^2 - x^2 - y^2 - z^2 = \kappa^2$ using $t_0 = T - \sqrt{\kappa^2 + R^2}$. We use the spherical coordinates on the hyperboloid and compute all the fields and the Ashtekar-Barbero connection as explained in the main text in section [1]. To compute the holonomy around the loop, we contract the connection with the tangent vector along the circle given by:

$$
(t^a) = \frac{dx^a}{d\phi} = (0 \ 0 \ 1)^T 
$$

(A2)

We can then compute the connection along the circle $A = t^a A^a_{\mu} 

\frac{\partial}{\partial x^\mu}$:

$$
A = \left( \begin{array}{ccc}
0 & \frac{1}{2} \left( \sqrt{\frac{\kappa^2 + R^2}{\kappa^2} - i \beta \frac{R}{\kappa}} \right) \\
\frac{1}{2} \left( \sqrt{\frac{\kappa^2 + R^2}{\kappa^2} + i \beta \frac{R}{\kappa}} \right) & 0 
\end{array} \right) 
$$

(A3)

where the $\sigma_i$ are the Pauli matrices. The holonomy $H(\phi)$ along the arc of the circle $(0, \phi)$ is then computed through:

$$
\frac{dH}{d\phi} = iAH(\phi) \quad H(0) = 1 
$$

(A4)

This differential equation is integrated into:

$$
H(\phi) = \cos(\alpha \phi) + i \sin(\alpha \phi) \frac{A}{\alpha} 
$$

(A5)

where $\alpha = \sqrt{-\det A} = \frac{1}{2} \sqrt{\frac{\kappa^2 + R^2}{\kappa^2} + \beta^2 \frac{R^2}{\kappa^2}}$.

The measurable quantity (that is the gauge invariant quantity) is the trace of the closed holonomy which is $2 \cos 2\pi \alpha$. However, our choice of coordinates is degenerate along the line $\theta \equiv 0(\pi)$. As a consequence, the loop is not contractible with this choice of coordinates and after one loop, in the flat case, we don’t get the identity but its opposite. We avoid this problem by using the vector representation of $SU(2)$ and take the trace in this representation which does not see this sign. We get:

$$
W_\kappa(R) = 4 \cos^2(2\pi \alpha) - 1 
$$

(A6)

which we finally write:

$$
W_\kappa(R) = 1 + 2 \cos \left( 2\pi \sqrt{1 + (1 + \beta^2) \left( \frac{R}{\kappa} \right)^2} \right). 
$$

(A7)
