Flip Distance Between Triangulations of a Planar Point Set is APX-Hard

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May 1, 2014

Abstract

In this work we consider triangulations of point sets in the Euclidean plane, i.e., maximal straight-line crossing-free graphs on a finite set of points. Given a triangulation of a point set, an edge flip is the operation of removing one edge and adding another one, such that the resulting graph is again a triangulation. Flips are a major way of locally transforming triangular meshes. We show that, given a point set $S$ in the Euclidean plane and two triangulations $T_1$ and $T_2$ of $S$, it is an APX-hard problem to minimize the number of edge flips to transform $T_1$ to $T_2$.

1 Introduction

Given a finite set $S$ of $n$ points in the Euclidean plane, a triangulation $T$ of $S$ is a maximal straight-line crossing-free graph on $S$. An edge flip is the operation of removing an edge $e$ of $T$ and adding a different edge $f$ such that the resulting graph $\tilde{T}$ is again a triangulation of $S$. This requires the two empty triangles incident to $e$ to form a convex quadrilateral, which is the same as the one formed by the triangles at $f$ in $\tilde{T}$. The flip operation defines the graph of triangulations $G$ of $S$, also called the flip graph of $S$. For a given set $S$, the vertex set of $G$ is the set of all triangulations of $S$. Two vertices in $G$ are adjacent if the corresponding triangulations can be transformed into each other by a single edge flip. Lawson [17] showed that $G$ is connected for any $S$ with diameter $O(n^2)$. Hurtado, Noy, and Urrutia [15] proved that this bound is tight.

Bose and Hurtado [8] give an extensive survey on the flip operation. Flips in triangulations are used for enumeration and as a local operation to generate meshes of good quality according to a predefined criterion. For example, Lawson [18] showed that one can always obtain the Delaunay triangulation after $O(n^2)$ locally improving flips. The Delaunay triangulation optimizes several criteria. Also, heuristic methods for improving other properties of triangular meshes may apply local optimization using flips in combination with techniques like simulated annealing. See [7] [14] for information on the topic of mesh optimization. Another reason for

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the continuing interest in flips in triangulations is the bijection between binary trees and triangulations of convex point sets. There, a flip corresponds to a rotation in the binary tree. Properties of the flip graph for convex point sets were studied in the landmark paper of Sleator, Tarjan, and Thurston [22].

For general point sets, Hanke, Ottmann, and Schuierer [13] show that the length of a shortest path between two triangulations in $G$ (i.e., the flip distance) can be bounded from above by the number of crossings between the edges of the two triangulations. Eppstein [12] gives a polynomial-time algorithm for computing a lower bound; note that the point sets for which Eppstein’s result is tight may not contain empty convex 5-gons, which requires more than two points being placed on a common line for sets of 10 or more points (see, e.g., [1]). Throughout this paper, we make the common assumption that $S$ is in general position, i.e., that no three points are collinear.

Despite these results, the complexity of determining the flip distance between two triangulations has been open. Our main result is that the problem is APX-hard, which sheds light on a “fundamental open issue” [8] in the study of flip graphs. It has been addressed as an open problem in [13] already in 1996, and, most recently, in a monograph by Devadoss and O’Rourke [11, p. 71]. APX-hardness of the problem implies that no polynomial time approximation scheme (PTAS) exists (i.e., there is no polynomial time algorithm that approximates the flip distance by a ratio of at most $1 + \epsilon$ for every constant $\epsilon > 0$), unless $P = NP$. Most recently, NP-completeness of the problem has simultaneously and independently been shown by Lubiw and Pathak [19]. However, their reduction is from the PLANAR CUBIC VERTEX COVER problem, for which a PTAS exists [3, 6] (see also [4, p. 369]), and the reduction can therefore not be adapted directly to show APX-hardness. For triangulations of simple polygons, Aichholzer, Mulzer, and Pilz [2] recently showed that the corresponding problem is NP-complete. Interestingly, the question is still open for point sets in convex position (or equivalently, convex polygons), regardless of the intensive investigation of that structure within the last 25 years (see, e.g., [20, 21] for two exemplary results, one long-standing, the other recent). A previous preprint version of this paper (arXiv:1206.3179v1) only showed NP-completeness of the corresponding decision problem.

Clearly, the problem is an NP optimization problem. Our reduction is from the well-known MINIMUM VERTEX COVER problem on cubic (3-regular) graphs. In the next section, we show certain properties of triangulations of a class of point sets called double chains, which will be subsets in our construction. In Section 3, we present the gadgets used in our reduction and analyze the construction. In that section, we only present a rough overview on how the points of the set are placed, a more detailed description of how to calculate their coordinates is given in Appendix [A].

2 Double-Chain Constructions

A main ingredient of our reduction will be gadgets consisting of subsets that, being considered on their own, would require a number of flips that is quadratic in their size.
Figure 1: A double chain. The points are divided in an upper and lower chain, each chain being in convex position in a way that every point of the lower chain “sees” every vertex of the convex hull of the upper chain, and vice-versa.

2.1 A Single Double Chain

The construction shown in [15] to have quadratic flip graph diameter is the so-called double chain. In a double chain $D$, there are $2n$ points, $n$ on the upper chain and $n$ on the lower chain. Let these points be $\langle u_1, \ldots, u_n \rangle$ and $\langle l_1, \ldots, l_n \rangle$, respectively, ordered from left to right. The upper chain is reflex w.r.t. any point of the lower chain, and vice-versa. See Figure 1. Let $P_D$ be the polygon $\langle l_1, \ldots, l_n, u_n, \ldots, u_1 \rangle$. The edges $u_iu_{i+1}$ and $l_il_{i+1}$ for $1 \leq i < n$ have to be part of every triangulation of $D$, since there does not exist a straight-line segment between two points of $D$ that crosses any of them (such edges are called unavoidable). Therefore, we only need to consider the triangulation inside $P_D$ for the following result.

Theorem 1 ([15]). Consider any triangulation $T_1$ of $D$ where $u_1$ is adjacent to each of $l_1, \ldots, l_n$, and any other one, called $T_2$, where $l_1$ is adjacent to $u_1, \ldots, u_n$. The flip distance between $T_1$ and $T_2$ is at least $(n-1)^2$.

See Figure 2 for the relevant parts of the two triangulations. In their proof, Hurtado et al. [15] label the triangles that have two points on the upper chain with $1$ and the ones with two points on the lower chain with $0$. Consider a horizontal line $\ell$ that separates the two chains. The triangles crossed by $\ell$ from left to right define a sequence $\sigma$ of $(n-1)$ elements labeled $0$ and $(n-1)$ elements labeled $1$, see Figure 3. Note that there are no triangles of a third type stabbed by $\ell$. Further note that we do not care about the triangulation of the convex hull of either chain; the lower bound on the flip distance stems from the part stabbed by $\ell$. It is easy to see that only an edge with two differently labeled triangles can be flipped in the stabbed part. This corresponds to exchanging an adjacent pair of $0$ and $1$. Flipping the first triangulation to the second one corresponds to transforming the sequence $\sigma_1 = ((0)^{n-1}(1)^{n-1})$ to $\sigma_2 = ((1)^{n-1}(0)^{n-1})$, which leads to the desired bound.

Our next step will be to gain more insight into the way the flip graph is altered by the addition of points. For the following definition refer to Figure 4.

Definition 1. Let $D$ be a double chain of $n$ points, and let $W_1$ be the double wedge defined by the supporting lines of $u_1u_2$ and $l_1l_2$ whose interior does not contain a point of $D$. $W_n$ is defined analogously by the supporting lines of $u_nu_{n-1}$ and $l_nl_{n-1}$. Let $W = W_1 \cup W_n$ be called the wedge of the double chain. The unbounded set $W \cup P_D$ therefore is defined by four rays and the two chains. A point is outside of $D$ if it is not contained in $W \cup P_D$. Let $\ell$ be a horizontal line that separates the two chains. All points in the same half-plane of $\ell$ as $u_1$
Figure 2: Two (partial) triangulations of the double chain with a flip distance of at least \((n - 1)^2\).

Figure 3: An illustration of the labeling argument for the lower bound. By the flip, the sequence changes from \langle 11001010 \rangle to \langle 11001001 \rangle.

are called the upper points. The lower points are defined analogously. The kernel of a double chain \(D\) is the intersection of the closed half-planes below \(u_1u_2\) and \(u_{n-1}u_n\), as well as above \(l_1l_2\) and \(l_{n-1}l_n\).

Figure 4: The polygon and the wedge (gray) of a double chain. The diamond-shaped kernel can be stretched arbitrarily by flattening the bend of the chains.

Let us add a point \(s\) inside the kernel of \(D\). From any triangulation of the resulting set \(D \cup \{s\}\), we can flip the edges between the chains such that they are incident to \(s\). Reaching this canonical triangulation only requires a linear number of flips. This behavior is well-known, see, e.g., [23]. Consider the case where \(s\) is placed outside of \(P_D\) but inside the kernel of \(D\). Add edges from \(s\) to \(u_1\) and \(l_1\) to again have a triangulation, as shown in Figure 5. Then, for flipping all possible edges to be incident to \(s\), we need at most \(2n - 2\) flips.

Proposition 2. Let \(S\) be any set of points outside of \(D\). Then, the diameter of the flip graph of \(S \cup D\) remains in \(\Omega(n^2)\), where \(|D| = 2n\).

1Note that the kernel of \(D\) might not be completely inside the polygon \(P_D\) (but no point in the kernel is outside \(D\)). This is in contrast to the common use of the term “kernel” in visibility problems for polygons.
An extra point in the kernel of $D$ allows flipping one triangulation of $P_D$ to the other in $4n - 4$ flips. Note that an edge common to source and target triangulation is temporarily flipped.

Figure 6: Mapping a triangulation to a local triangulation of a double chain. To the left, all triangles intersecting with $W$ are shown. Visually, one can think of “cutting” the edges at the boundary of $P_D \cup W$ (middle) and moving (and merging) the endpoints to the next point (right).

In order to prove the proposition, we consider a mapping $L$ from the set of triangulations of $S \cup D$ to the set of triangulations of the polygon $P_D$. When flipping an edge in a triangulation $T$ of $D \cup S$, at most one edge is flipped in the corresponding triangulation $L(T)$ of $P_D$.

Consider any triangulation $T$ of $S \cup D$. If all edges of $P_D$ are present, $L(T)$ equals the triangulation of $P_D$ in $T$ (note that this is also the case for the triangulations $T_1$ and $T_2$ of Theorem 1). Otherwise, consider the following construction (see Figure 6 for an example). For any edge $e$ of $T$ that intersects $W$, we draw a new edge $e'$ of $L(T)$ in the following way. If one of the endpoints of $e$ is on a vertex of $P_D$, $e'$ has the same endpoint. If $e$ passes through an edge $u_iu_{i+1}$ or $l_jl_{j+1}$, then the corresponding upper or lower endpoint of $e'$ is set to $u_{i+1}$ or $l_{j+1}$, respectively. If $e$ passes through one of the rays defining $W \cup P_D$, e.g., the one through $u_n$, then the upper endpoint of $e'$ is placed at $u_n$, such that $e'$ is contained in $P_D$.

Let $T' = L(T)$ be the graph induced by the new edges, and let the edges of $T$ that pass through $W$ but do not have an endpoint in $D$ be called wide edges. We call the construction $T'$ the local triangulation of $D$ when $T$ is clear from the context. The following lemmata show that $T'$ actually is a triangulation. Observe that, informally, the construction corresponds to continuously introducing the edges of the chains along the arrows drawn in Figure 6, while continuously sliding the edges of $T$ accordingly.
Lemma 3. For every wide edge \( e \in T \) that is mapped to \( e' \in T' \), there is an edge \( \tilde{e} \in T \) that has an endpoint \( p \in D \) that is also mapped to \( e' \).

Proof. Let \( e' \) be \( u_i l_j \). Consider first the case where both endpoints of \( e' \) are on the same side of (the directed line supporting) \( e \). Consider the empty triangle \( t \) of \( T \) incident to \( e \) that has its apex \( a \) on the same side of \( e \) as \( e' \). If \( a \) is outside of \( D \), then another wide edge \( f \) of \( t \) is also mapped to \( e' \). In that case we continue the argumentation with \( f \), as \( e \) and \( f \) are both mapped to the same edge. If \( a \) is not outside, then \( a \) equals either \( u_i \) or \( l_j \), as otherwise \( t \) would contain one of them. Hence, one of the edges of \( t \) incident to \( a \) is also mapped to \( e' \).

For the case where the two endpoints of \( e' \) are on different sides of \( e \) (i.e., one of the endpoints of \( e' \) is \( u_n \) or \( l_n \)), the argumentation is almost the same. W.l.o.g. let \( i = n \) and \( l_j \) be to the right of \( e \) (note that \( j \) might be \( n \)). Therefore, \( u_n \) is to the left of \( e \). Again, consider the empty triangle \( t \) of \( T \) incident to \( e \) with apex \( a \) to the right of \( e \). Again, if \( a \) is outside of \( D \), there is another wide edge \( f \) of \( t \) that is also mapped to \( e' \). If \( a \) is not outside of \( D \), then \( a = l_j \); this follows from the construction of \( D \) and the fact that the lower endpoint of \( e \) is outside of \( D \). Hence, an edge of \( t \) incident to \( a \) is also mapped to \( e' \).

Lemma 4. Every point \( p \) of \( D \) is incident to at least one edge \( e \) of \( T \) such that \( e \) disconnects \( W \).

Proof. This follows directly from the construction of \( D \): If there were no such edge, then there would be an angle larger than \( \pi \) incident to \( p \), and the wedge defined by this angle would contain points. This would contradict the fact that \( T \) is a triangulation.

Lemma 5. \( T' \) is a triangulation of \( P_D \).

Proof. We have to prove that \( T' \) is crossing-free and maximal in \( P_D \).

Lemma 3 allows us to only consider non-wide edges. With all relevant remaining edges of \( T \) being incident to a point in \( D \), the fact that \( T' \) is crossing-free follows from \( T \) being crossing-free, as the mapping only "moves" the endpoints of the edges of \( T \) to the next point of \( D \).

If \( T' \) were not maximal, there would exist a quadrilateral \( q \) inside \( P_D \) that is spanned by points of \( D \) and whose interior does not intersect any edge. If \( q \) is not convex, this would mean that no edge of \( T \) is incident to the reflex vertex of the quadrilateral. But this can not happen due to Lemma 4. If \( q \) is convex, it is of the form \( l_i l_{i+1} u_{j+1} u_j \). If an edge of \( T \) would have passed through the side \( l_i u_j \), the quadrilateral would not be empty of edges. Hence, \( l_i u_j \) must have been a part of \( T \). Also, neither \( l_i u_{j+1} \) nor \( u_{j+1} l_{i+1} \) was present in \( T \), and therefore, there was at least one edge crossing the segment \( l_i u_{j+1} \), as well as one crossing \( u_{j+1} l_{i+1} \). None of these edges was incident to \( l_i \) or \( u_j \), since otherwise \( q \) would not be empty. If there have been edges that have crossed both these diagonals of \( q \), choose the leftmost one. As in the proof of Lemma 3, this is a wide edge that is adjacent to a triangle with either \( l_i \) or \( u_j \). One of the other triangle edges would have been mapped to \( l_i u_{j+1} \) or \( u_{j+1} l_{i+1} \), respectively. But if there were two different "leftmost" edges that prevented visibility along the diagonals, these edges would have crossed in \( T \) (since none of them has crossed \( l_i u_j \) or was incident to one of \( l_i u_j \)), which contradicts the fact that \( T \) was a triangulation. Hence, there is no empty quadrilateral in \( P_D \), which completes the proof.
Since any flip in $T$ results in at most one edge being flipped in $T'$, the lower bound construction holds: a shorter flip sequence with outside points would immediately imply a shorter flip sequence between $T_1$ and $T_2$ in the proof of Theorem 1. This completes the proof of Proposition 2.

2.2 Multiple Double Chains

Proposition 2 is, however, of little use when we try to construct a point set that contains many double chains and try to argue that the flip distance between two triangulations of the set is bounded by the sum of the distances between the local triangulations of these double chains. One could imagine that a flip in the overall triangulation leads to changes in the local triangulations of several double chains. In this section, we prove that this is not possible. Keep in mind that it is a necessary condition that, for any double chain, all other double chains are outside and their polygons do not intersect.

**Lemma 6.** Let $e$ be a flippable edge of any triangulation $T$ of $D \cup S$ that is mapped to the edge $e'$ in the corresponding local triangulation $T'$ of a double chain $D$. Then flipping $e$ changes the local triangulation only if there is no other edge mapped to $e'$.

**Proof.** Suppose $e$ would not be the only edge mapped to $e'$. If we remove $e$ from $T$, the graph on $D$ defined by the mapping is still the local triangulation $T'$. If we add the new edge $f$ after the removal of $e$, $f$ must also be mapped to some existing edge $f'$ in $T'$, as otherwise $T'$ would not be a triangulation. Note that because of Lemma 6, flipping an edge that is wide for a double chain does not change the local triangulation of that double chain. Therefore, a flip can only represent a flip in at most four local triangulations (which would already be enough for our reduction, as will become clear). However, we can actually prove the following, more accurate result.

**Lemma 7.** Let $D_1$ and $D_2$ be two double chains in a point set $S$. If each of $D_1$ and $D_2$ is outside of the other and $P_{D_1} \cap P_{D_2} = \emptyset$, each flip in a triangulation of $S$ affects at most one of the two local triangulations.

**Proof.** If the flipped edge $e$ or its replacement $f$ would not both have an endpoint in the same double chain $D$, at least one of $e$ or $f$ would either not dissect the corresponding wedge $W$ or be a wide edge of $D$. It follows from Lemma 6 that such a flip does not influence the local triangulation of $D$. Hence, in the only remaining case there is a quadrilateral that has two adjacent points in $D_1$ and two adjacent points in $D_2$ and contains a flippable edge. Let the quadrilateral be $abcd$. W.l.o.g., let $a$ and $b$ be part of $D_1$ and $e = ac$. Suppose, for the sake of contradiction, that we flip the edge $ac$ and the flip changes the local triangulation of $D_1$. Then $ac$ has to dissect the wedge of $D_1$. However, then also $ad$ dissects the wedge and crosses the same edge of $P_{D_1}$ as $ac$. Hence, $ac$ and $ad$ are mapped to the same edge in the local triangulation of $D_1$, a contradiction due to Lemma 6.

**Corollary 8.** If a point set consists of $m$ double chains, each of size $2n$, and for every double chain all other points are outside, then the flip graph diameter of the whole set is in $\Omega(mn^2)$.  


3 The Reduction

Up to now we have gathered enough knowledge about double chains as sub-configurations in order to use them as the main building blocks in a reduction. We reduce from CUBIC MINIMUM VERTEX COVER, which is known to be APX-complete [3].

Problem 1 (CUBIC MINIMUM VERTEX COVER). Given a simple cubic (3-regular) graph $G = (V, E)$ with $n = |V|$, choose a set $C \subseteq V$ such that every edge in $E$ has at least one vertex in $C$ and such that $|C|$ is minimized.

We follow the common approach of embedding the graph $G$ and transforming its elements to geometric gadgets. The gadgets consist of points together with the corresponding edges in the source triangulation $T_1$ and in the target triangulation $T_2$. We give the overall idea of how to embed the gadgets, for a detailed description on how to exactly place the points with rational coordinates having a representation bounded by a polynomial in the input size using polynomial time can be found in Appendix A.

3.1 Gadgets

Given an instance $(G = (V, E), k)$ of the CUBIC MINIMUM VERTEX COVER problem, with $n = |V|$ and $m = |E|$ (note that $m = \frac{3n^2}{2}$), we place the elements of $V$ as the vertices of a convex $n$-gon and draw the straight-line edges between them (as a mean of orientation for our construction of the point set). Let $x'$ be the maximum number of crossings a single edge has, and let $x = \lceil \frac{x'}{2} \rceil$. It is easy to see that $x' \leq m - 5$. For each edge $e$ mark a point $c_e \in e$ that has no more than $x$ crossings on either side along $e$. Let $\vec{t}$ be a vector perpendicular to $e$ of sufficiently small length (which will be specified in Appendix A). Make two copies of $e$ and translate them by $\vec{t}$ and $-\vec{t}$, respectively, to obtain the tunnel of the edge, i.e., the quadrilateral defined by the two copies of $e$. Then slightly “bend” the copies towards the midpoint of $e$ to obtain two circular arcs $A_e$ and $A'_e$. The endpoints of $e$ have to see any point on $A_e$ and $A'_e$. See Figure 7.

3.1.1 Edge Centers

Cf. Figure 8. Instances of the double chain are the main ingredient in our reduction. Let $e$ be a straight-line edge of $G$, drawn between the points $v$ and $v'$. In a close neighborhood of $c_e$,
place a double chain $D_e$, the edge center, of $2d$ points (we will fix the value of $d$ later) along $A_e$ and $A'_e$, such that the two chains are separated by the supporting line of $e$. Note that the endpoints $v$ and $v'$ of $e$ are the only ones that are not outside $D_e$, and they are also in the kernel of $D_e$ (remember that $A_e$ and $A'_e$ can be chosen sufficiently flat). The edge centers are the only gadgets that have different edges in the source and in the target triangulation. Draw the edges that define the polygon $P_{D_e}$ in both $T_1$ and $T_2$. Let the first point of the upper chain be adjacent to all points of the lower chain in the source triangulation $T_1$, and do the analogous for the last one in the target triangulation $T_2$. We call the execution of flips that remove and add edges of an edge center transforming an edge center.

### 3.1.2 Crossings

If two straight-line edges $e$ and $f$ of $G$ cross, also their corresponding circular arcs cross. Place one point at each of their four crossings (we will actually place them close to these crossings, see Appendix A). In both source and target triangulation draw the edges connecting two points that are consecutive on any circular arc, which results in a crossing being represented by a convex quadrilateral, to which we add an arbitrary diagonal.

### 3.1.3 Wirings

Cf. Figure 9. Wirings are gadgets that represent the elements of $V$. Consider any vertex $v$ of $G$ and a small circle $C$ with $v$ in the embedding as its center. This point $v$ is part of the triangulated point set. Place points on the crossings of $C$ with the arcs of the edges incident to $v$ in $G$. Since the graph is embedded on a convex $n$-gon and due to the small length of the vector $\ell$, these points occupy strictly less than half of $C$. This allows us to place two chains $L$ and $R$, each of $w$ points (the value of $w$ to be defined later) on $C$ in a way that any line between one point of $R$ and one point of $L$ separates $v$ from the remaining construction. In both the source and target triangulation draw the edges between consecutive points on $C$. Draw a zig-zag path through the $2w$ points of $L$ and $R$. Connect $v$ to the two first points of the chains. The remaining part may be triangulated arbitrarily.

The remaining faces in the two plane graphs we obtained so far are triangulated arbitrarily, however in a way that the resulting triangulations $T_1$ and $T_2$ have the same edges except at the edge centers.

### 3.2 Analysis

The basic idea of the construction is that an effective flipping algorithm has to choose which wirings to flip (requiring $4w - 2$ flips each for flipping the edges of a wiring away and back
Figure 9: A wiring with its initial (and final) triangulation and a triangulation that allows to quickly transform the edge centers.

again) in order that the triangulation of an edge center can be transformed using the point in its kernel at the chosen wiring. Also, the edges between and at the crossings need to be flipped away. We will fix the values of \( w \) and \( d \) to force this behavior of any flipping algorithm that requires less flips than a trivial upper bound. Every edge of \( G \) will be covered; using a vertex of \( G \) for covering corresponds to flipping all edges in the corresponding wiring.

Let \( v \) and \( v' \) be any two adjacent vertices in \( G \). The exact number of edges in \( T_1 \) or \( T_2 \) intersected by the segment \( vv' \) in the drawing might differ according to the choice of \( v \) and \( v' \) due to the triangulation of the wiring gadget at the region where the edge gadgets enter it. For every wiring there is at most a constant number \( c \) of such edges, giving in total at most \( \tau = cn \) edges.

The following lemma shows how to deduce a flip sequence in our construction from a vertex cover of size \( k \). Note that we do not claim that this is the optimum if \( k \) is optimal.

**Lemma 9.** If there exists a vertex cover of size \( k \) in \( G \), then there exists a flip sequence between \( T_1 \) and \( T_2 \) of length at most

\[
\delta = 2(k(2w - 1) + m(4x + 2d) + \tau) .
\]

**Proof.** Let \( C \) be a vertex cover of \( G \) with \( k = |C| \). Let \( v \) be the vertex used to cover an edge. We use \( v \) to transform the edge centers of the adjacent edges in \( G \) (if they have not already been transformed). We need to flip all edges in the wiring to \( v \), which takes \( 2w - 1 \) flips. Then we need at most \( c \) flips for the remaining wiring edges, as well as 2 further flips for the edges before the first crossing and 2 flips for the first crossing itself (all in all, with this method we need up to \( 4x + 2 \) flips for the crossing gadgets to “reach” the edge center). There, we need \( 2d - 2 \) flips to make the edges incident to \( v \) (as in Figure 5). Flipping in the desired way we would need at most \( \delta \) flips.

On the other hand, a flip sequence should define a vertex cover. For the following lemma, we fix \( w > m(4x + 2d) + \tau + 1/2 \); further, we choose \( d \) such that \((d - 1)^2 > 2(n(2w - 1) + m(4x + 2d) + \tau) \) (note that since the term to the right is linear in \( d \), such a value of \( d \) clearly exists and is polynomial in the problem size).

**Lemma 10.** If there exists a flip sequence between \( T_1 \) and \( T_2 \) of length \( \delta \), then there exists a vertex cover of size \( k = \left\lfloor \frac{\delta}{4w - 2} \right\rfloor \).
Proof. We argue that the choice of \( d \) forces an effective algorithm to flip the wirings (which corresponds to covering vertices), and that the choice of \( w \) allows to transform the number of flips to the size of the corresponding vertex cover.

If \( \delta \geq (d - 1)^2 \), then the choice of \( d \) implies a vertex cover of size \( n \), which is trivially true. We therefore assume that \( \delta < (d - 1)^2 \). For every edge center, if we do not use the corresponding central points \( v \) or \( v' \) of a wiring, we would need at least \( (d - 1)^2 \) flips due to Lemma \( 7 \). Suppose we want to transform an edge center \( D \) using a point \( v \). As in the proof of Lemma \( 9 \), we need to flip all edges in the wiring to \( v \), taking \( 2w - 1 \) flips. Note that this is optimal since only one of the edges can be removed with each flip. The values of \( d \) and \( w \) have been chosen in a way that flipping the edges of all wirings, crossings, and edge centers to the corresponding central point and back, as described, uses less flips than transforming one edge center, due to the bound of Lemma \( 9 \). For any algorithm, this means that flipping all edges at wirings and crossings twice and transforming the edge centers with a point at the wiring is cheaper than transforming one edge center without a point at a wiring. Due to Lemma \( 7 \) we know that we need a point at a wiring for each edge center to be transformed in fewer than \( (d - 1)^2 \) flips. Therefore, we know that the (optimal) flip distance \( \delta_{\text{opt}} \) is given by

\[
\delta_{\text{opt}} = k_{\text{opt}} (4w - 2) + R \quad \text{for some } R > 0.
\]  

Equation (1) shows how to deduce \( k_{\text{opt}} \) from \( \delta_{\text{opt}} \). Note that if \( 4w - 2 > R \), the size of the minimum vertex cover can be calculated from the flip distance as stated. For the reasoning about approximation ratios later in this section, we actually want to have \( 4w - 2 > 2R \). Lemma \( 9 \) gives us an upper bound on the flip distance, and hence \( R \leq 2(m(4x + 2d) + \tau) \). Thus, we have chosen \( w \) such that flipping the edges of one wiring needs more flips than twice the number of all other flips in total in an optimal flip sequence. Therefore the flip distance \( \delta_{\text{opt}} \) gives

\[
k_{\text{opt}} = \left\lfloor \frac{\delta_{\text{opt}}}{4w - 2} \right\rfloor.
\]

No matter how well an algorithm performs, it has to flip the wirings at least \( k_{\text{opt}} \) times when using less than \( (d - 1)^2 \) flips, and Lemma \( 9 \) tells us that the flips of the edges not in a wiring are in total fewer than those used for one wiring when the algorithm is optimal.

The choice of \( w \) with respect to \( R \) and the proof of the previous lemma gives us the following fact.

**Corollary 11.** For the flip distance \( \delta_{\text{opt}} \) between \( T_1 \) and \( T_2 \) and the minimum vertex cover of size \( k_{\text{opt}} \) we have \( k_{\text{opt}} = \frac{\delta_{\text{opt}} - R}{4w - 2} \) for some positive \( R < 2w - 1 \).

To show APX-hardness of the flip distance problem, we show that we have an AP-reduction \([4, pp. 256–261]\) from CUBIC MINIMUM VERTEX COVER using the previous lemmata. Let \( k_{\text{opt}} \) be the size of a minimum vertex cover for \( G \) and \( \delta_{\text{opt}} \) be the flip distance between \( T_1 \) and \( T_2 \). The performance ratio of a minimization problem is the measure value of the approximation divided by the optimal value, e.g., \( k/k_{\text{opt}} \) for an approximate vertex cover of size \( k \).

**Definition 2** \([4, pp. 257–258]\). Let \( P_1 \) and \( P_2 \) be two NP optimization problems. \( P_1 \) is AP-reducible to \( P_2 \) if two functions \( f \) and \( g \) and a constant \( \alpha \geq 1 \) exist such that:
1. For any instance \( x \) of \( P_1 \) and any rational \( r > 1 \), \( f(x, r) \) is an instance of \( P_2 \).

2. For any instance \( x \) of \( P_1 \) and any rational \( r > 1 \), if there is a feasible solution of \( x \), then there is a feasible solution of \( f(x, r) \).

3. For any instance \( x \) of \( P_1 \) and any rational \( r > 1 \), and for any \( y \) that is a feasible solution of \( f(x, r) \), \( g(x, y, r) \) is a feasible solution of \( x \).

4. \( f \) and \( g \) are computable by two algorithms whose running time is polynomial for any fixed rational \( r \).

5. For any instance \( x \) of \( P_1 \) and any rational \( r > 1 \), and any feasible solution \( y \) for \( f(x, r) \), a performance ratio of at most \( r \) for \( y \) implies a performance ratio of at most \( 1 + \alpha(r-1) \) for \( g(x, y, r) \).

In our case, \( f \) corresponds to the construction of the point set and the two triangulations. Requirements 1 and 2 follow from our construction. A vertex cover can be extracted from a flip sequence \( y \) from the edges flipped at the wirings; this analysis corresponds to \( g \), and requirement 3 is therefore fulfilled. Both are polynomial-time algorithms (the parameter \( r \) is actually not used), as demanded by requirement 4.

Intuitively, Lemmata 9 and 10 give evidence that the reduction described so far fulfills also requirement 5 of Definition 2. However, there is a technical difficulty with the remainder term \( R \) that we have to take a closer look at. Let \( \delta \) be an approximate solution for the flip distance such that \( \delta \leq \delta_{\text{opt}} r \). Further, let \( R' \) be the remainder produced by the floor function in Lemma 10, i.e., \( \delta \equiv R'(\text{mod } 4w - 2) \). By Lemma 10, we get

\[
k \leq \frac{\delta - R'}{4w - 2} \leq \frac{\delta_{\text{opt}} r - R'}{4w - 2} .
\]

Let \( R \) be the remainder term for the optimal solution \( \delta_{\text{opt}} \) as in Corollary 11. Then we have

\[
k \leq r \frac{\delta_{\text{opt}} - R}{4w - 2} + \frac{rR - R'}{4w - 2} = rk_{\text{opt}} + \frac{rR - R'}{4w - 2} \leq rk_{\text{opt}} + \frac{rR}{4w - 2} < rk_{\text{opt}} + \frac{r}{2} ,
\]

(2)

where the equality and the latter inequality are due to Corollary 11. Let \( \alpha = 4 \) and suppose first that \( r - 1 = \epsilon \geq \frac{1}{2k_{\text{opt}}+1} \). Then

\[
rk_{\text{opt}} + \frac{r}{2} = k_{\text{opt}} + \frac{\alpha \epsilon k_{\text{opt}}}{2} = k_{\text{opt}} + \alpha \epsilon k_{\text{opt}} + \frac{1}{2} - \epsilon \left(3k_{\text{opt}} - \frac{1}{2}\right) .
\]

(3)

To get rid of the last part we use

\[
\epsilon \left(3k_{\text{opt}} - \frac{1}{2}\right) \geq \frac{3k_{\text{opt}} - 1/2}{2k_{\text{opt}} + 1} > \frac{1}{2} ,
\]

which, by (2) and (3), implies

\[
k \leq k_{\text{opt}} + \alpha \epsilon k_{\text{opt}} .
\]
On the other hand, suppose that $r - 1 = \epsilon < \frac{1}{2k_{\text{opt}}+1}$. Then from (2), we get

$$k < rk_{\text{opt}} + \frac{r}{2} = k_{\text{opt}} + \epsilon k_{\text{opt}} + \frac{\epsilon}{2} = k_{\text{opt}} + \epsilon\left(k_{\text{opt}} + \frac{1}{2}\right) + \frac{1}{2}$$

$$< k_{\text{opt}} + \frac{k_{\text{opt}} + 1/2}{2k_{\text{opt}} + 1} + \frac{1}{2} = k_{\text{opt}} + 1.$$  

Since the solutions to vertex cover are integers, this implies that $k = k_{\text{opt}} \leq k_{\text{opt}} + \alpha \epsilon k_{\text{opt}}$. Hence, in both cases $k \leq k_{\text{opt}}[1 + \alpha(r - 1)]$ and our reduction fulfills all properties of an AP-reduction from Cubic Minimum Vertex Cover.

**Theorem 12.** The problem of determining a shortest flip sequence between two triangulations of a point set is APX-hard.

### 4 Conclusion

In this paper, we showed that it is APX-hard to minimize the number of flips to transform two triangulations $T_1$ and $T_2$ of a point set $S$ into each other. As a by-product, Corollary 8 revealed an interesting aspect on distances in the flip graph.

In presence of the recent NP-completeness result for simple polygons [2], the main remaining open problem is the one for triangulations of convex point sets and its dual problem, the computation of the binary tree rotation distance [22].

**Acknowledgements.** The author wants to express his gratitude to Oswin Aichholzer, Thomas Hackl, and Pedro Ramos, as well as anonymous referees for valuable suggestions on improving the presentation of the result.

### A A Tedious Calculation of the Coordinates

Section 3 already contained a description of the gadgets we used in our reduction. However, the validity of gadget-based reductions when proving NP- or APX-hardness for problems on point sets requires that the coordinates of the points used can be calculated in polynomial time. While this is often, implicitly or explicitly, left to the reader, we perform this step in high detail in this appendix.

The reader might have noticed that our high-level construction involves points placed at the crossing of circular arcs, which, in general, leads to irrational coordinates, even if the circular arcs are defined by rational points. We will give a construction that slightly varies from the one described that uses only rational coordinates, with both the numerator and denominator bounded by a polynomial in the input size.

The gadgets in our reduction do not make use of collinear points. However, we did not explicitly mention how to avoid three points on a line when the construction is assembled. In

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2This was explicitly demanded in a review of a conference submission of this paper.
this appendix we give an explicit construction of the point set in a way that it is in general position, i.e., that no three points are collinear.

Note that the construction might not be “economic” in the sense that the construction might be possible with smaller values. We will always prefer constructions that are easy to prove. We will place the points on and close to the unit disc (meaning that a coordinate will never exceed 2), and therefore can give the size of a coordinate in terms of the size of its denominator.

A.1 Placing the Points of the Convex Polygon

As a first step, we give a simple construction of a convex $n$-gon for placing the central points of the wiring gadgets with all vertex coordinates being rational and the denominators being in $O(n^{10})$. Further, we want to assure that no three diagonals cross in the same point. For doing so, we will first choose $n^5$ candidate points on the unit circle and then select $n$ points out of them.

Rational points on the unit circle are known to be given by $(\frac{1-t^2}{t^2+1}, \frac{2t}{t^2+1})$ with $t \in \mathbb{Q}$, see, e.g., [10]. We define a sequence $K$ of candidate points with $t = i/n^5$ for the integers $1 \leq i \leq n^5$. (For consistency with later parts and ease of presentation therein we choose the candidate points from the upper-right quadrant in counterclockwise ascending order; hence, the value of $t$ is between 0 and 1.) Now we select $n$ points out of $K$ such that there are no three diagonals that cross at a single point. We choose the first five points of our final set from the candidate points. Suppose we have chosen $j \geq 5$ points such that no three diagonals cross at a single point. We have $n^5 - j$ points in $K$ to choose the next point from. Look at all possible combinations of five points among the already chosen ones. Each combination gives exactly five points on the unit circle that can not be chosen, and this point can be one of the $j$ already chosen candidate points. Hence, we have $5 \binom{j}{5} + j$ “forbidden” points (which might not be among the candidate points). See Figure 10. However, we have $n^5 \geq j^5 > 5 \binom{j}{5} + j$ candidate points to choose from, and therefore we for sure can choose point number $(j + 1)$.

We denote this set of points by $P_V$; the elements of $P_V$ are the points representing the vertices of the input graph.
Proposition 13. A point set of $n$ points in convex position with all coordinates rational having a denominator in $O(n^{10})$ and no three diagonals crossing in the same point can be found in polynomial time.

Note that the facts that no three diagonals of the resulting $n$-gon intersect and that the coordinates are bounded also give us a lower bound on the distance between intersection points and other diagonals, which we will use in the next part.

A.2 A Sufficiently Small Value

Find the minimum squared distance from every of the $\binom{n}{4}$ crossings of the diagonals of the $n$-gon to the diagonals not involved in the corresponding crossing. Since this squared distance $\delta'^2_e$ is given by $(x_a - x_b)^2 + (y_a - y_b)^2$ between two points $a$ and $b$, we can set $\delta'_e = |x_a - x_b|$ to obtain a “small”, rational and positive distance $\delta'_e \leq \delta_e$ (at least one of the horizontal or vertical distance is non-zero, in particular, up to here no two points can have the same $x$- or $y$-coordinate). When we construct the tunnels that are formed around an edge of the drawing of the input graph, we can choose, say, $\delta'_e/3$ as an upper bound for the distance between the edge and the edges defining the tunnel. Then no three tunnels have a non-empty intersection. (Our actual tunnels will be even narrower.)

The vertex gadgets used “small” circles around each point in $P_V$. Let $u, v, w$ be a consecutive triple of vertices on the $n$-gon defined by $P_V$. Let $\delta'^2_v$ denote the smallest squared distance between $v$ and the line through $u$ and $w$ for every choice of the triple. As with the tunnels, we can choose a rational $\delta'_v \leq \delta_v$ by choosing only the horizontal or vertical distance between $v$ and the closest point on the supporting line of $u$ and $w$. Let $n_{ov}$ be the line through $v$ that is perpendicular to the supporting line of the origin $o$ and $v$. Consider the distances from $u$ and $w$ to $n_{ov}$. Let $\delta'^2_n$ be the smallest squared distance for all choices of $v$ (and corresponding $u$ and $w$), and choose a rational $\delta'_n \leq \delta_n$ as before. Further, let $\delta_r$ be the smallest horizontal or vertical distance between two points in $P_V$ (which is non-zero by construction). See Figure 11.

We define $\delta = \min\{\delta'_e/3, \delta'_v, \delta'_n, \delta_r\}$. If we now choose the radius of the cycle centered at each vertex by $r_V = \delta/6$, then no two circles intersect (there is actually a distance of at least $4r_V$ between two circles), and each circle only intersects the edges of the input graph that are incident to the vertex it is centered at. Further, no circle intersects the convex hull of two other circles.
A.3 Tunnel Construction

For each edge \( e \) of the input graph connecting two vertices \( v \) and \( w \), we now give the construction of the tunnels. Let \( C_v \) and \( C_w \) be the circles around \( v \) and \( w \), respectively. The tunnel for the edge between \( v \) and \( w \) is given by two segments, each having one endpoint on \( C_v \) and one endpoint on \( C_w \). We want to get rational points on \( C_v \) and \( C_w \). Since these cycles are not only defined by a rational center point, but also have a rational radius, the problem boils down to find a rational point on the unit circle, or, equivalently, a (possibly irrational) angle \( \alpha \) such that \( \sin(\alpha) \) and \( \cos(\alpha) \) are rational, within some interval given by quadratic irrationals. Sines with this property are called rational sines, and correspond with the parametrization of the unit circle that we already used before. Canny, Donald and Ressler [10] give an algorithm for finding a rational sine for a parameter \( t = p/q \), such that \( |p/q - x| < \epsilon \), for given \( x \) and \( \epsilon \) (we will use an extended method for non-rational radii later). Their algorithm gives a denominator \( q \) in \( O(1/\epsilon) \), and the running time is polynomial in \( q \). However, the input \( x \) is an approximation as well, and their goal is to get rational sines with small binary representation. Our angle intervals, however, are given by rational points and their relative position to the circle center. For finding a point within this interval, the Farey approximation as used in [10] for \( t = p/q \) is sufficient and easy to apply for our setting, as we do not need an explicit approximation of the angle and the interval as input (this algorithm searches a point inside the interval in the fashion of binary search, calculating the mediant of two rational values in each step). We, however, need an upper bound on the denominator \( q \) derived from the points defining the angle.

Now we show how to use the results of [10] for our needs. Consider the unit circle and two points \( a \) and \( b \). Let \( \angle a \) and \( \angle b \) be the polar angles of these points, and w.l.o.g. let \( \angle a < \angle b \). We describe only the case where both angles are within \([0, \ldots, \pi/2]\), the other cases are similar (and can easily be distinguished). To approximate an angle between \( \angle a \) and \( \angle b \) using a rational number \( t \), we reason about the (possibly irrational) values \( t_a \) and \( t_b \). For \( t_a \) and \( \angle a \) we define

\[
\sin(\angle a) = \frac{2t_a}{t_a^2 + 1},
\]

which, when choosing the appropriate root, gives

\[
t_a = \frac{1}{\sin(\angle a)} - \sqrt{\frac{1}{\sin^2(\angle a)} - 1}.
\]

The sine of \( \angle a \) is given by \( a_y/\sqrt{a_x^2 + a_y^2} \). We define the analogous for \( t_b \). The values of \( \angle b \) and \( t_b \) are defined analogously. We therefore need to find a rational number \( t \) with \( t_a \leq t \leq t_b \). The Cauchy bound (see [24]) for an algebraic number \( g \) being the root of a polynomial \( \sum_{i=0}^{m} c_i x^i \) with rational coefficients \( c_i \) is given by

\[
|g| \geq \frac{|c_0|}{|c_0| + \max\{|c_1|, \ldots , |c_m|\}}.
\]

The difference \( |t_a - t_b| \) is therefore bounded from below by a rational that has a denominator polynomial in the problem size. Using Farey approximation, we can find a rational \( t \) whose denominator exceeds the denominator of the bound only by a polynomial factor.
Since we can choose rational points on the unit circle inside an interval (and therefore on instances of the unit circle that are translated and scaled by rational values), we now have the tools to choose the endpoints of the tunnels. For two vertices \( v \) and \( w \), let these be called \( p_v \) and \( q_v \) (placed on \( C_v \)), as well as \( p_w \) and \( q_w \) (placed on \( C_w \)). Hence, a tunnel between \( v \) and \( w \) consists of the quadrilateral \( p_v q_v q_w p_w \). In order to prevent collinear triples of points, we again select a set of candidate points on \( C_v \) and \( C_w \) and choose the four points among them. Note that this results in tunnels that might not be exactly rectangular, but this is irrelevant for our final construction. See Figure 12 for an accompanying illustration.

W.l.o.g. suppose that \( x_v < x_w \). To obtain the set of candidate points for \( p_v \), consider the segment between \( w \) and the point \((x_w, y_w + r_v)\), where \( r_v \) is the radius of the circles, which we call the upper spoke of \( w \). Let the lower spoke of \( w \) be defined analogously between \( v \) and the point \((x_w, y_w - r_v)\). Find the two parameters \( t_1 \) and \( t_2 \) for rational points \( p_{t_1} \) and \( p_{t_2} \) on \( C_v \) such that the line through \( v \) and \( p_{t_1} \) intersects the upper spoke of \( w \) at a point in the topmost eighth of the upper spoke and the line through \( v \) and \( p_{t_2} \) intersects the upper spoke in its fifth eighth from below. We can now select our set \( K_v \) of candidate points from the interval \([t_1, t_2]\). The same can be done for two parameters \( t_3 \) and \( t_4 \), with the roles of \( v \) and \( w \) changed. We select a point \( p_v \in K_v \) and a point \( p_w \in K_w \) as the endpoints of one side of the tunnel gadget between \( v \) and \( w \). Note that we do not need to require the sides to be parallel to the supporting line of \( v \) and \( w \) (we could do so by increasing the number of candidate points). The crucial property of the points we need is that \( p_v v w p_w \) forms a convex quadrilateral and we therefore have to prove that \( p_v \) is always left of the directed line through \( v \) and \( p_w \) (and, analogously, that \( p_w \) is right of the directed line through \( w \) and \( p_v \)). Let \( l_w = (x_w, y_w + r_v/2) \) and \( u_w = (x_v, y_v + r_v) \). The diagonals \( v l_w \) and \( u_w w \) of the trapezoid \( u_v v w l_w \) intersect each other at a ratio of \((r_v/2)/r_v\), i.e., at two thirds of the interval \([x_v, x_w]\). Recall that the radius \( r_v \) was chosen in a way that the disc centers have a horizontal distance of at least \( 6r_v \). Hence, the segments intersect outside \( C_w \); the topmost candidate point on \( C_w \) is below the line through \( v \) and the lowest candidate point on \( C_v \), and vice versa.
Note that since the candidate points on \( C_v \) are chosen inside the convex hull of \( C_w \) and \( v \), no two tunnels from \( v \) can intersect.

It remains to find the correct number of candidate points. Suppose we already constructed all but one tunnel point. Since at every circle there are at most six tunnel points there are less than \( \binom{7}{2} \) lines on which we are not allowed to place a point. Every line intersects the circle we place the last point on at most twice. Hence, if we choose more than twice the number of points as we have lines, we can always choose a point to remain in general position.

### A.4 Points in the Tunnels

For each tunnel, we construct two circular arcs, one for each segment defining the tunnel, on which we place the points of the edge center. The crucial property of such an arc is that for two wire centers \( v \) and \( w \), these points are the only ones in the kernel. Let \( q_v \) and \( q_w \) be the two endpoints of a tunnel edge, s.t. \( q_w \) is to the right of \( q_v \) and the interior of the tunnel is above the line \( v \). See Figure 13. The constructed arc will start at \( q_w \) and end at \( q_v \). W.l.o.g. let the angle between \( q_v \) and \( q_w \) be smaller than the one between \( q_v \) and \( q_w \). Then we consider four rays, namely the ones that leave \( q_w \) to the right in an angle of \( 0, -\pi/4 \), \( \pi/4 \) with the \( x \)-axis and the upward vertical ray at \( q_v \). Let \( r \) be the one that opens the smallest positive angle \( \alpha \) with \( q_v q_w \). If the angle between \( q_v \) and \( q_w \) and \( q_v w \) is smaller than \( \alpha \), then let \( s \) be the ray through \( w \) starting at \( q_v \); otherwise, let \( s = \pi \). Construct the circle \( A \) that has its center on the line through the midpoint of \( q_v q_w \) perpendicular to \( q_v q_w \) and that has \( s \) as a tangent. The coordinates of the center of \( A \) are still rational. It is well-known that, when given any rational point \( p \) on \( A \) and a line \( \ell \) with rational slope that intersects \( A \) at \( p \) and a second point \( p' \), the point \( p' \) is rational as well, see, e.g., [16, p. 5]. Hence, we need to appropriately choose lines through a point \( p \). The crossings of the segments that define all the tunnels identify the region where to place the edge center. Let \( R \) be the region we have to place the points in (marked gray in Figure 13). By the choice of \( r \), we constructed \( A \) in a way that we can mirror and rotate the plane orthogonally such that the intersection of \( A \) and \( R \) is within an angle of \( 0 \) and \( \pi/4 \) from \( q_v \). This means that any line \( \ell \) through \( q_v \) and this intersection will have a slope \( t \) between 0 and 1. This reasoning is similar to the one of Burnikel [9] to adapt the techniques of [10] for such rational circles (i.e., circles given by three rational points). As before, we can use, e.g., Farey approximation for the slope \( t \) of \( \ell \). At each iteration, we check whether the second intersection of \( \ell \) with \( A \) is inside the quadrilateral \( R \), and, if not, on which side it is. Since the denominators of the coordinates of the points defining \( A \) and \( R \) are polynomial, there is a polynomial lower bound on the difference between the (possibly non-rational) parameters for the two points where \( A \) enters and leaves \( R \) (as for the construction of the tunnel endpoints). Hence, after a polynomial number of steps, we have a rational slope for \( \ell \) such that \( \ell \) passes through \( A \) inside \( R \); therefore, also this intersection point has rational coordinates and its denominator is polynomial in the problem size. To obtain a second such point, the process can be continued. Now we have two points in the intersection of \( A \) and \( R \) which define two slopes of lines through \( q_v \). Any line through \( q_v \) with a slope in the interval between these two slopes gives a rational point on \( A \cap R \). Hence, we can choose our candidate points by dividing that interval.

The remaining problem is the one of choosing the points for the crossing gadgets. Two arcs in crossing tunnels will, in general, cross at a point that does not have rational coordinates.
Figure 13: We want to choose rational points on the (blue) arc inside the gray region by Farey approximation on the slope $t$. Note that the gray region actually is, for presentational reasons, drawn too close to $w$.

Figure 14: Construction of tunnel crossings. The drawing shows the lower part of a tunnel between $v$ and $w$ and a part of another tunnel (indicated by the near-horizontal strokes).

The crucial property of the points of the crossing gadgets, however, is that they are outside the edge centers (recall Definition 1) and that the edges between them can “quickly” be flipped to the center of the corresponding wiring gadget. Placing the points for the crossing gadgets at the crossings of the segments that define the tunnels would satisfy these constraints, but would lead to collinear triples. So we have to slightly perturb each point $p$ to obtain a point $p'$ without losing these properties. See Figure 14. Between every consecutive pair of crossing points on a tunnel segment $q_vq_w$ we can choose the rational midpoint. If the perturbed point $p'$ remains on the same side of the line through the wire center and the midpoint as $p$, the order around the wire center is maintained. Further, the perturbed points have to remain on the same sides of the lines that define the wedges of the edge centers involved. Together with the tunnel edges, these constraints give a convex region from which we can choose our perturbed point. We may again place a circular arc inside this region (marked gray in Figure 14) on which we select a sufficiently large number of candidate points, analogously to the construction of the other gadgets.

A.5 Points for the Wiring

Finally, we place the points at the wiring gadgets that allows to draw the wiring edges, see Figure 15. The circles for the wiring gadgets are scaled versions of the unit circle. For a vertex $v$, let $n_{ov}$ be the line through $v$ that is perpendicular to the supporting line of the origin $o$ and $v$. Since the coordinates of $v$ are rational sines, the intersection points of $n_{ov}$ with the circle $C_v$ are rational as well. Due to the choice of $\delta'_v$, all points on $C_v$ that define tunnels are on the same side of $n_{ov}$ as $o$. We are given two intervals, each between two rational sines, i.e., between the “extremal” tunnel endpoints on $C_v$ and the intersection points of $n_{ov}$
Figure 15: Construction of the wiring points. The small gap (indicated by the arrow) on the circle $C_v$ of $v$ between the intersection point with $n_{ov}$ and the neighboring tunnel endpoint can be used for the candidate points.

with $C_v$. Therefore, we can choose a sufficient number of rational candidate points on $C_v$ to choose the points for the wiring from.

A.6 Concluding Remarks on the Embedding

The crucial part throughout the whole embedding procedure is that each (intermediate) point that is not a candidate point is constructed using only a constant number of other points. The candidate points were constructed with polynomial parameters. Hence, all denominators are polynomial in the input. In particular, note that even though some intervals were defined by points with algebraic coordinates, a lower bound on the interval can be given in terms of the other, rational coordinates that were used in the construction. This allowed us to find rational points with polynomial denominators within these intervals.

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