Generalised co-spherical classes

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Abstract

Suppose $0 \leq k < l < +\infty$ and $m, n \geq 0$. Does there exist $f : \Omega^l S^{n+l} \to \Omega^k S^{m+k}$ with $f_* \neq 0$? We call such maps generalised co-spherical classes. Working at the prime $p = 2$, in the case of $k = 0$, our results completely determine the image of the Hurewicz map

$$h : [\Omega^l S^{n+l}, \Omega^k S^{m+k}] \to \text{Hom}_{\mathbb{Z}/2}(H_4 \Omega^l S^{n+l}, H_4 \Omega^k S^{m+k})$$

defined by $h(f) = f_*$ for (i) $m = n \geq 0$ and $l > 0$; (ii) $m > n$ with $m = 2^k n$, $k > 0$, and $l = 1$.

The problem in its general form (arbitrary $m, n, k, l$) could be interesting, challenging, and worth studying for which our computation provide three motivations: (1) It is a kind of dual to the problem of computing spherical classes in $H_4 \Omega^l S^{n+l}$; (2) In many cases, the image is trivial; (3) In cases such as $m = n$ the existence or non-existence of a map $\Omega^l S^{n+l} \to S^n$ (for various $l > 0$) is related/determined by existence of Hopf invariant one elements as well as existence of $H_n$-structures on spheres. This note aims to collect some observation to begin the investigation on this problem.

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1 Introduction and statement of results

Our motivation in this work comes from studying spherical classes in $\mathbb{Z}/2$-homology loop spaces on spheres $\Omega^l S^{n+l}$ where $0 < l \leq +\infty$ and $n \geq 0$. We often state results integrally and if we work...
locally it would be at the prime $p = 2$.

Let’s recall that a spherical class $x \in H_nX$ is any homology class which is in the image of the Hurewicz map $h : \pi_nX \to H_nX$, that is if there exists $f : S^n \to X$ so that $f_*(x_n) = x$ where $x_n \in \tilde{H}_nS^n$ is a generator. The problem of determining spherical classes in $H_*X$ is not always an easy problem, e.g. in the case of $X$ the Hurewicz map $h$ is shown. Let’s recall that a spherical class. The problem of determining spherical classes in $H_*X$ is not always an easy problem, e.g. in the case of $X = QS^0 = \operatorname{colim} \Omega^lS^l$ it is an open problem (see for example [1], [4], [6]). The problem of determining spherical classes in finite loop spaces $\Omega^lS^{n+l}$ also is an open, although some progress for small values of $l$ has been made where we have achieved complete classification of these classes (see [7] and [5]).

**Motivation and programme.** A complete classification of spherical classes in $H_*\Omega^lS^{n+l}$ with $l < l_0$ and $n \geq 0$ for some fixed value $l_0 > 0$ (see [7] and [5] for a complete classification when $l < 9$ and $n > 0$) tempts one to reduce studying spherical classes in $H_*\Omega^lS^{n+l}$ to the study of spherical classes in $H_*\Omega^kS^{m+k}$ where $0 \leq k < l$. This might be very optimistic, but motivates one to look for maps $f : \Omega^lS^{n+l} \to \Omega^kS^{m+k}$ with $f_* \neq 0$ where $0 \leq k < l$ and $m, n \geq 0$. This motivates the following definition.

**Definition 1.1.** A generalised co-spherical class in $H_*X$ is determined by a map $f : X \to \Omega^kS^{m+k}$, $k \geq 0$, so that $f_* \neq 0$.

Here, we are speaking loosely as preimage of $f_*$ could contain more than one element. If one is hesitated by our definition, then the may call any element in the preimage of $f_*$ a cospherical class. The problem of determining co-spherical classes in $H_*X$ is also a kind of dual problem (in the case $k = 0$) to determining spherical classes in $H_*X$. Let’s note that if we have a duality, say if we work over field so that $H_*X$ and $H^*X$ are dual, then one may decide to define $z \in H^*X$ to be a cospherical class if for some $f : X \to S^m$ we have $f^*(x_m) = z$ which is the point of view considered in [3] where the authors consider maps $X \to S^m$ which induce nontrivial maps in $KO$-theory; hence they have studied co-spherical classes in $KO(X)$ for specific choices of $X$. However, we consider the classes of maps themselves as cospherical classes.

Let’s note that the problem that we consider is really about computing the image of a ‘Hurewicz map’ (homomorphism if $k > 0$)

$$h : [\Omega^lS^{n+l}, \Omega^kS^{m+k}] \to \operatorname{Hom}_{\mathbb{Z}/2}(H_*\Omega^lS^{n+l}, H_*\Omega^kS^{m+k})$$

defined by $h(f) = f_*$. Now, the source of this homomorphism is the ‘unstable’ group $[\Omega^lS^{n+l}, \Omega^kS^{m+k}]$ whose complete computation needs a suitable unstable Adams spectral sequence (ASS). However, we do not attempt working with any unstable ASS and instead we try to use available geometric techniques to study the image of this homomorphism.

The following is our main result.

**Theorem 1.2.** Suppose $k = 0$ the following statement hold.

(i) For $m = n = 0$, for any $l > 0$, the image of $h$ is isomorphic to $\mathbb{Z}/2\{\iota\}$ where $\iota \in H^1QS^1$ is the fundamental class.

(ii) For $m = n > 0$ and $l = 1$, the image of $h$ is nontrivial if and only if $n \in \{1, 3, 7\}$ and in this case the image is isomorphic to $\mathbb{Z}/2\{\theta\}$ where $\theta : \Omega S^{n+1} \to S^n$ is the boundary map in the Barratt-Puppe sequence for one of the Hopf maps $\eta, \nu, \sigma$.

(iii) For $m = n > 1$ and $l > 1$ the image of $h$ is trivial.

(iv) For $m = n = 1$ and any $l > 1$ the image of $h$ is isomorphic to $\mathbb{Z}/2\{\theta_{S^1}\}$ where $\theta_{S^1} : QS^1 \to S^1$ corresponds to the structure map of $S^1$ as an infinite loop space.

(v) For $m > n$ with $m = 2^kn$, $k > 0$, and $l = 1$, the image of $h$ is trivial.
2 Preparatory observations

From now on we only consider the case of $\Omega^l S^{n+l} \to S^m$. These computations serve in two directions. First, they are about the simplest case of the problem with $k = 0$. Second, although we do expect all maps $\Omega^l S^{n+l} \to \Omega^k S^{m+k}$ to be loop maps, but given a map $f : \Omega^{l-k} S^{n+l} \to S^{m+k}$ with $f_* \neq 0$ then $\Omega^k f : \Omega^l S^{n+l} \to \Omega^k S^{m+k}$ is a natural candidate to look at and check whether $(\Omega^k f)_* \neq 0$ is satisfied or not, hence a natural source to produce some examples. Although, we shall observe that in the case of $k = 0$ they are not many such $f$ to begin with.

We begin with a well known argument on decompositions arising from homology. For the purpose of future reference, we record the following.

Lemma 2.1. Suppose $f : X \to S^m$ is given with $f_* \neq 0$. Let $X$ be a CW-complex whose $k$-skeleton we denote by $X^{[k]}$ and write $X_{k+1} = X/X^{[k]}$. Then, $f$ extends to a map $\tilde{f} : X_m \to S^m$ and we have a decomposition

$$X_m \simeq S^m \times \text{Fib}(\tilde{f}).$$

Proof. For dimensional reasons $[X^{[m-1]}, S^m]$ is a singleton, so the composition $X^{[m-1]} \to X \to S^m$ is null. Therefore, $f$ admits an extension $\tilde{f} : X^m \to S^m$. Writing $x_n : S^n \to X_m$ for the inclusion of the bottom cell, the composition $f \circ x_m$ is nontrivial. This gives the claimed decomposition (at least after localisation at a suitable prime).

Suppose there exists $f : \Omega^l S^{n+l} \to S^m$ with $f_* \neq 0$. For dimensional reasons, if $n > m$ then $f_* = 0$. We consider the remaining cases separately.

3 Case of $m = n > 0$

If $l = 1$ then there are examples at hand which are provided by Hopf fibrations, namely maps $\Omega S^{n+1} \to S^n$ for $n = 1, 3, 7$. The existence of these maps also provides a decomposition

$$\Omega S^{n+1} \simeq S^n \times \Omega S^{2n+1}.$$ 

We show these are the only possible cases (at least modulo 2). We have the following formulation of Adams' Hopf invariant one element result.

Lemma 3.1. The followings are equivalent.

(i) There is a map $f : \Omega S^{n+1} \to S^n$ with $f_* \neq 0$ (modulo 2).

(ii) There is a map $g : S^{2n} \to \Omega S^{n+1}$ with $g_* \neq 0$ (modulo 2).

(iii) There is a map $h : S^{2n+1} \to \Omega S^{n+1}$ of unstable Hopf invariant one, and $n \in \{1, 3, 7\}$.

Proof. (i) $\Rightarrow$ (ii): Let $i : S^n \to \Omega S^{n+1}$ be the inclusion adjoint the to identity $S^{n+1} \to S^{n+1}$. Then $(f \circ i)_* \neq 0$, hence (at $p = 2$) $f \circ i$ is homotopic to the identity. Together with James fibration $S^n \to \Omega S^{n+1} \xrightarrow{H} \Omega S^{2n+1}$ it follows that

$$(f, H) : \Omega S^{n+1} \to S^n \times \Omega S^{2n+1}$$

is a homotopy equivalence. The inclusion $S^{2n} \to \Omega S^{2n+1}$ gives rise to a spherical class. Consequently, the composition

$$g : S^{2n} \to \Omega S^{2n+1} \to \Omega S^{n+1}$$
gives rise to a spherical class in \( H_*(\Omega S^{2n+1}; \mathbb{Z}/2) \), so \( g_* \neq 0 \).

(ii) \( \Rightarrow \) (iii): If \( g_* \neq 0 \) then from James’ description of \( H_*\Omega S^{2n+1} \) we see that \( g_*(x_{2n}) = x_n^2 \). It is well known that the adjoint of \( g \), say \( h : S^{2n+1} \to S^{n+1} \) has unstable Hopf invariant one.

(iii) \( \Rightarrow \) (i): As noted above, (ii) and (iii) are equivalent. It follows that the adjoint of \( h \), say \( g : S^{2n} \to \Omega S^{2n+1} \) maps nontrivially under \( H_* : \pi_{2n}\Omega S^{2n+1} \to \pi_{2n}\Omega S^{2n+1} \). This implies that \( H \circ g \) is homotopy the identity. The claimed decomposition follows immediately.

This settles down the case with \( l = 1 \). In this case, the case of \( l > 1 \) reduces to the case of \( l = 1 \) in the following sense.

**Lemma 3.2.** Suppose \( f : \Omega^l S^{n+l} \to S^n \) is given with \( f_* \neq 0 \). Then, \( n \in \{1, 3, 7\} \). Moreover, writing \( \text{Fib}(f) \) for the homotopy fibre of \( f \), we have

\[ \Omega^l S^{n+l} \simeq S^n \times \text{Fib}(f). \]

\[ \pi_i \text{Fib}(f) \simeq \pi_{i+1} \text{Fib}(f). \]

**Proof.** Suppose there is an unstable map \( f : \Omega^l S^{n+l} \to S^n \) which is nontrivial in homology. It is immediate that the composition

\[ \Omega S^{n+1} \xrightarrow{\iota} \Omega^l S^{n+l} \xrightarrow{f} S^n \]

is nontrivial in homology. By Lemma we have \( n = 1, 3, 7 \). Similar to above, it is immediate that \( \Omega^l S^{n+l} \) decomposes as a product of \( S^n \) and the homotopy fibre of \( f \), inclusion \( \iota : S^n \to \Omega^l S^{l+1} \). Extending the commutative square fibrations

Note that in the case of \( l = 1 \), whenever \( n \in \{1, 3, 7\} \), there exists a map \( f : \Omega S^{n+1} \to S^n \) with \( f_* \neq 0 \). The existence of such maps when \( l > 0 \) is not so immediate, however. For \( n = 1 \) we may choose \( f : \Omega^l S^{n+l} \to S^1 \) to be any representative of the identity element of \( H^1(\Omega^l S^{n+l}; \mathbb{Z}) \simeq \mathbb{Z} \). It is immediate that \( f \) is nonzero in \( H_*(-; k) \) for \( k = \mathbb{Z}, \mathbb{Z}/2 \). There is another way to see existence of such maps. Since \( S^1 \) is an infinite loop space, let \( \theta : QS^1 \to S^1 \) be the structure map which has the property that the composition \( S^1 \to QS^1 \to S^1 \) is identity. In particular, \( \theta_* \neq 0 \). Now, the composition

\[ f : \Omega^l S^{l+1} \to QS^1 \to S^1 \]

satisfies \( f_* \neq 0 \).

For the remaining cases, we have the following nonexistence result.

**Lemma 3.3.** (i) Suppose \( n = 3 \). Then, there exists no map \( f : \Omega^2 S^5 \to S^3 \) with \( f_* \neq 0 \). Consequently, for \( 2 \leq l \leq +\infty \) there exists no map \( f : \Omega^l S^{l+3} \to S^3 \) with \( f_* \neq 0 \).

(ii) Suppose \( n = 7 \). Then, there exists no map \( f : \Omega^2 S^9 \to S^7 \) with \( f_* \neq 0 \). Consequently, for \( 2 \leq l \leq +\infty \) there exists no map \( f : \Omega^l S^{l+7} \to S^3 \) with \( f_* \neq 0 \).

**Proof.** Proof of (i) and (ii) are similar. First note that the general case follows from our claim for double loop spaces as follows. For instance, note that for \( l \geq 2 \), the composition \( \Omega^2 S^5 \to \Omega^l S^{l+3} \) is nonzero in \( H_3(-; k) \) with \( k = \mathbb{Z}, \mathbb{Z}/2 \). Hence, existence of any map \( f : \Omega^l S^{l+3} \to S^3 \) with \( f_* \neq 0 \) would imply that the composition \( \Omega^2 S^5 \to \Omega^l S^{l+3} \to S^3 \) is nonzero in homology, giving the desired contradiction.

Now we show there is no map \( f : \Omega^2 S^{n+2} \to S^n \) (with \( n = 3, 7 \)) so that \( f_* \neq 0 \). We work at
the prime 2. Given a map $f : \Omega^2 \Sigma^2 X \to X$ we may define $\mu : X \times X \to X$ by the following composition
\[
\mu : X \times X \to * \times_{\Sigma^2} (X \times X) \to F(\mathbb{R}^2, 2) \times_{\Sigma^2} (X \times X) \to \Omega^2 \Sigma^2 X \to X
\]
where the first map on the left is projection, second and third maps are inclusion, and the last map is $f$. This map is a commutative multiplication on $X$. Since the composition $S^n \to \Omega^2 \Sigma^2 S^n \to S^n$ is nonzero in homology (we may assume it is multiplication of degree 1), hence it is homotopic to the identity. On the other hand, by construction, for a based path connected space $X$, the inclusion $X \to \Omega^2 \Sigma^2 X$ can be decomposed as a composition $X \xrightarrow{\alpha} X \times X \to \Omega^2 \Sigma^2 X$ where $\alpha$ can be taken as either $(1, *)$ or $(*, 1)$ with $*$ being the base point of $X$. This implies that, for $X = S^3, S^7$, $(X, \mu, *)$ is a commutative $H$-space in the sense of [2]. But this is a contradiction as it is known that $S^3$ and $S^7$ do not admit any commutative $H$-space structure. \[\square\]

4 Case of $m = n = 0$

Lemma 4.1. For any $l > 0$ there exists a map $f : \Omega^l S^l \to S^0$ with $f_* \neq 0$. Moreover, any such $f$ we have $f = \Omega \iota$ where $\iota : \Omega^{l-1} S^l \to P = K(\mathbb{Z}/2, 1)$ represents the fundamental class.

Proof. Let $l > 0$, and take the fundamental class $\iota \in H^1 QS^1 \simeq \mathbb{Z}/2$ which can be realised as a map $\iota : QS^1 \to P$. Define $f$ to be the composition $\Omega^l S^l \to QS^0 \xrightarrow{\Omega \iota} \mathbb{Z}/2 = S^0$.

Clearly, $f_* \neq 0$. On the other hand, if $f : \Omega^l S^l \to S^0 = \Omega P$ is given with $f_* \neq 0$, then the adjoint of $f$, say $\tilde{f} : \Sigma \Omega^l S^l \to P$ is nontrivial in homology. Also, note we may consider the composition $\Sigma \Omega^l S^l \xrightarrow{e} \Omega^{l-1} S^l \xrightarrow{\iota} P$ which is nontrivial in homology where $e$ is the evaluation map (adjoint to the identity). Since $f, \iota \circ e \in [\Sigma \Omega^l S^l, P] \simeq \mathbb{Z}/2$ and both elements are nontrivial, therefore $\tilde{f} = \iota \circ e \Rightarrow f = \Omega \iota \circ e = \Omega \iota \circ 1 = \Omega \iota$.

This completes the proof. \[\square\]

5 Case of $m > n$

Similar to the case of $m = n$, we first look at the case $l = 1$.

5.1 Case of $l = 1$

Our main result in this section is the following.

Theorem 5.1. For $m = 2^k n$ with $k > 0$ there is no map $f : \Omega S^{n+1} \to S^m$ with $f_* \neq 0$. 

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If \( f : \Omega S^{n+1} \to S^m \) is given with \( f_* \neq 0 \) then \((\Sigma f)_* \neq 0\). In the light of James splitting, \( \Sigma \Omega S^{n+1} \simeq \bigvee_{t=1}^m S^{tn+1}, \) \( m = tn \) for some \( t > 1 \) are the only possible choices for which \((\Sigma f)_* \neq 0\) and consequently \( f_* \neq 0 \) can happen. This early use of stable results may give us hope that one may use (semi) stable maps, i.e. maps of the sort \( \Sigma^i \Omega S^{n+1} \to \Sigma^i S^n \) with \( 0 < i \leq +\infty \) which are known to be nontrivial, and then try to destabilise them. For instance, after James splitting, we have stable projection maps \( \Sigma \Omega S^{n+1} \to S^{tn+1} \) and one may ask whether or not any of these could be destabilised? The following provides a partial (negative) answer to this question.

**Lemma 5.2.** Suppose \( t > 1 \) is even with \( t = 2^k \) and \( f : \Omega S^{n+1} \to \Omega S^{tn+1} \) is given so that \( H_{tn}(f) \neq 0 \); equivalently assume \( f : \Sigma \Omega S^{n+1} \to S^{tn+1} \) is given with \( f_* \neq 0 \). Then, \( f \) does not pull back to a map \( \Omega S^{n+1} \to S^m \).

**Proof.** The proof is by contradiction. We work at the prime \( p = 2 \). For \( r > 0 \), we write \( x_r \) for a generator of \( H_r S^r \) so that \( H_* \Omega S^{r+1} = T_{\mathbb{Z}/2}(x_r) \) is the tensor algebra over \( \mathbb{Z}/2 \) generated by \( x_r \). Suppose \( f : \Omega S^{tn+1} \to \Omega S^{tn+1} \) so that \( H_{tn}(f) \neq 0 \) then \( f_*(x^n_r) = x_m \). Consider the James fibration sequence \( S^m \xrightarrow{E} \Omega S^{tn+1} \xrightarrow{H} \Omega S^{2tn+1} \). We show that the composition \( H \circ f \) is essential, so \( f \) does not pull back to \( S^m \). First note that the map \( H \) is adjoint to the projection \( \Sigma \Omega S^{tn+1} \to \Sigma S^{2tn+1} \) to the second factor in James splitting, hence in homology it satisfies \( H_*(x^n_{2tn}) = x_{2tn} \). We loop the composition \( H \circ f \) and consider

\[
\Omega^2 S^{n+1} \xrightarrow{\Omega H} \Omega^2 S^{tn+1} \xrightarrow{\Omega H} \Omega^2 S^{2tn+1}.
\]

We use lower indexed homology operations, and working with double loop spaces only \( Q_0 \) and \( Q_1 \) exist in the homology. We write \( Q^k \) for \( k \)-iteration of \( Q_1 \). If we write \( \sigma_* \) for the homology suspension then we have \( \sigma_* Q_1 x_{nt-1} = x_{n}^2 \). We have \( H_* Q_0 x_{nt} = x_{2nt} \) which by dimensional arguments implies that \( (\Omega H)_* Q_1 x_{nt-1} = x_{2nt-1} \). Similarly, since \( x^n = Q^k_0 x_n = \sigma_* Q^k_1 x_{n-1} \) then from the equation

\[
\sigma_*(\Omega f)_* Q^k_1 x_{n-1} = f_* \sigma_* Q^k_1 x_n = x_{nt}
\]

together with the fact that \( \Omega^2 S^{nt+1} \) has it bottom cell in dimension \( nt - 1 \) we conclude that

\[
(\Omega f)_* Q^k_1 x_{n-1} = x_{nt-1}.
\]

We claim that \( (\Omega f)_* (Q^k_1 x_{n-1}) = Q_1 x_{nt-1} \) which then would imply that \( (\Omega H \circ \Omega f)_* Q^k_1 x_{n-1} = x_{2nt-1} \) showing that \( H \circ f \) is essential. To prove our claim note that by Nishida relations we have

\[
S_{Q_1} (Q^k_1 x_{n-1}) = (Q^k_1 x_{n-1})^2.
\]

Since \( \Omega f \) is a loop map, together with naturality of the Steenrod operations we have

\[
S_{Q_1} (\Omega f)_* Q^k_1 x_{n-1} = (\Omega f)_* S_{Q_1} Q^k_1 x_{n-1} = (\Omega f)_* (Q^k_1 x_{n-1})^2 = x_{nt-1}^2 = S_{Q_1} Q_1 x_{nt-1}
\]

Hence, we deduce that modulo \( \ker S_{Q_1} \) we have

\[
(\Omega f)_* Q^k_1 x_{n-1} = Q_1 x_{nt-1}.
\]

However, it is easy to see that \( H_{2nt-1} \Omega^2 S^{nt+1} \simeq \mathbb{Z}/2 \{Q_1 x_{nt-1}\} \) which implies that \( (\Omega f)_* Q^k_1 x_{n-1} = Q_1 x_{nt-1} \). This completes the proof. \(\square\)

The above lemma allows us to solve the unstable problem with the aid of homology of the stabilisation map \( S^m \to \Omega S^{tn+1} \). We have the following.
Proof of Theorem 5.1. By comments after Theorem 5.1, it is enough to eliminate the cases \( f : \Omega S^{n+1} \to S^{tn} \) with \( t > 1 \) even. If \( f_* \neq 0 \) then \( f_*(x^t_n) = x_{tn} \). The inclusion \( E : S^{tn} \to \Omega S^{tn+1} \) is a monomorphism in homology, so \( E \circ f : \Omega S^{n+1} \to \Omega S^{tn+1} \) is nonzero in homology with \( (E \circ f)_*(x^t_n) = x_{tn} \). On the other hand, by construction \( E \circ f \) pulls back to \( f \) through \( E \) which contradicts Lemma 5.2. This completes the proof. 

Note that the existence of \( f : \Omega S^{n+1} \to S^{tn} \) for some \( t > 1 \) would have implied, by Lemma 2.1, that the map \( f \) extends to a map \( \tilde{f} : (\Omega S^{n+1})_tn \to S^{tn} \) and there is a decomposition \( (\Omega S^{n+1})_tn \simeq S^{tn} \times \text{Fib}(\tilde{f}) \).

We therefore obtain the following non-decomposition result.

**Corollary 5.3.** There is no power of 2, \( t = 2^k \), so that we have a decomposition \( (\Omega S^{n+1})_tn \simeq S^{tn} \times \text{Fib}(\tilde{f}) \).

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