On the L-Grundy Domination Number of a Graph

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Abstract. In this paper, we continue the study of the L-Grundy domination number of a graph introduced and first studied in [Grundy dominating sequences and zero forcing sets, Discrete Optim. 26 (2017) 66–77]. A vertex in a graph dominates itself and all vertices adjacent to it, while a vertex totally dominates another vertex if they are adjacent. A sequence of distinct vertices in a graph \(G\) is called an L-sequence if every vertex \(v\) in the sequence is such that \(v\) dominates at least one vertex that is not totally dominated by any vertex that precedes \(v\) in the sequence. The maximum length of such a sequence is called the L-Grundy domination number, \(\gamma_{gr}^L(G)\), of \(G\). We show that the L-Grundy domination number of every forest \(G\) on \(n\) vertices equals \(n\), and we provide a linear-time algorithm to find an L-sequence of length \(n\) in \(G\). We prove that the decision problem to determine if the L-Grundy domination number of a split graph \(G\) is at least \(k\) for a given integer \(k\) is NP-complete. We establish a lower bound on \(\gamma_{gr}^L(G)\) when \(G\) is a regular graph, and investigate graphs \(G\) on \(n\) vertices for which \(\gamma_{gr}^L(G) = n\).

1. Introduction

In the last decade, four Grundy domination invariants were introduced [2, 6, 7], which arise from two standard domination invariants, the domination number and the total domination number. The idea behind these invariants is to determine the worst case that can appear during online creation of (total) dominating sets. When building a dominating set, we may form a sequence \(D\) by adding vertices one by one, and require that when a vertex \(x\) is added to \(D\) it dominates at least one vertex that was not dominated before \(x\) was added. The length of a longest such sequence in a graph \(G\) is the Grundy domination number, \(\gamma_{gr}(G)\). The Grundy domination number of a graph was introduced in [6] and has subsequently attracted much attention [2, 3, 5, 9]. In [6] it was shown that the decision version of the problem is NP-complete, even when restricted to chordal graphs, and the Grundy domination number was established for some well-known classes of graphs. The concept was further studied in [5], where exact formulas for Grundy domination numbers of Sierpinski graphs were proven, and a linear algorithm for determining these numbers in arbitrary interval graphs was presented. The Grundy domination number was also studied in Kneser graphs [9] and graph products [3, 15].
Total domination, introduced for the first time in [10], is an intensively studied area of graph theory; see the monograph surveying the topic [13]. A vertex in a graph totally dominates another vertex if they are adjacent. Let $D$ be a sequence of vertices such that every vertex in the sequence totally dominates at least one vertex that was not totally dominated by preceding vertices in the sequence. Such sequence is an open neighborhood sequence, and the maximum length of an open neighborhood sequence in a graph $G$ is the Grundy total domination number, $\gamma_{gr}^T(G)$. The Grundy total domination number of a graph $G$ was introduced in [7], where it was also proved that the decision version of the Grundy total domination number is NP-complete, even when restricted to bipartite graphs. (One of the motivations for studying Grundy total domination is the total domination game introduced in [12].) In addition, several lower and upper bounds for the Grundy total domination number were presented and the concept was also studied in regular graphs. Grundy total domination number was investigated in some well-known graph classes, such as trees [8], Kneser graphs [9] and graph products [4]. For split graphs it was shown that the problem is NP-complete [8]. In [11] the graphs with equal total and Grundy total domination numbers were studied.

A connection of Grundy domination number to the zero forcing number of a graph was presented in [2], and led to the introduction of additional two Grundy domination invariants, namely the Z-Grundy domination and L-Grundy domination number of a graph. The Z-Grundy domination number is dual to the concept of the zero forcing number of a graph, as introduced in [1]. The latter presents a useful lower bound for the minimum rank of a graph. Recently, all four types of Grundy domination invariants were studied by Lin [14], and for each of them a minimum rank type parameter was considered, which is an upper bound for the Grundy type parameter.

In the next section we establish the necessary notation and graph theory terminology. In Section 3, we provide a lower bound for the L-Grundy domination number in regular graphs, in a similar way as it
was done for Grundy total domination in [7]. In Section 4 we consider upper bounds for the L-Grundy domination number in arbitrary graphs. It is clear that if \( G \) is a graph, then the bound \( \gamma^L_{gr}(G) \leq n(G) \) is sharp, but the question remains if it can be improved by involving other parameters. We pose the following conjecture, where \( n(G) \) and \( \delta(G) \) denote the order and minimum degree, respectively, of the graph \( G \).

**Conjecture 1.1.** If \( G \) is a graph, then \( \gamma^L_{gr}(G) \leq n(G) - \delta(G) + 1 \) always holds.

In Section 5, we prove that somewhat surprisingly for every forest \( T \), we have \( \gamma^L_{gr}(T) = n(T) \). Further, we prove that every graph \( G \) with \( \gamma^L_{gr}(G) = n(G) \) satisfies \( \delta(G) = 1 \), implying that Conjecture 1.1 is true for graphs with \( \gamma^L_{gr}(G) = n(G) \). It was shown in [2] that the decision version of L-Grundy domination number is NP-complete even when restricted to bipartite graphs. In Section 6, we prove that this holds also when restricted to split graphs.

2. Basic Concepts and Notation

This section is devoted to notational and preliminary issues. For notation and terminology, we will typically follow [13]. Specifically, let \( G \) be a (a directed or undirected) graph with vertex set \( V(G) \) and edge set \( E(G) \). The order and size of \( G \) will be denoted by \( n(G) \) and \( m(G) \), respectively. A loop is an edge in \( G \) that joins a vertex to itself. A multiple edge in \( G \) presents two or more edges that are incident to the same two vertices of \( G \); in particular, a double edge means two edges that join the same pair of vertices. Unless explicitly stated otherwise, graphs in this article will be without loops and multiple edges.

The open neighborhood of a set \( S \subseteq V(G) \) is the set of all neighbors of vertices in \( S \), denoted \( N_G(S) \), whereas the closed neighborhood of \( S \) is \( N_G[S] = N_G(S) \cup S \). If \( A \) is a set of vertices in \( G \), then the set of neighbors of a vertex \( v \) that belong to the set \( A \) in \( G \) is denoted by \( N_A(v;G) \); that is, \( N_A(v;G) = N_G(v) \cap A \). For a set \( S \subseteq V(G) \), we let \( N_A(S;G) = \bigcup_{v \in S} N_A(v;G) \). If the graph \( G \) is clear from the context, we omit the label \( G \) and write \( N(S) \), \( N[S] \), \( N_A(v) \) and \( N_A(S) \) rather than \( N_G(S) \), \( N_G[S] \), \( N_G(v) \) and \( N_G(S) \), respectively.

The degree of a vertex \( v \) in \( G \) is denoted by \( d_G(v) = |N_G(v)| \). A \( k \)-regular graph is a graph in which every vertex has degree \( k \). A graph \( G \) is regular if it is \( k \)-regular for some integer \( k \geq 0 \). The maximum degree among all the vertices of \( G \) is denoted by \( \Delta(G) \). If the graph \( G \) is clear from the context, we omit the subscript \( G \) and write \( d(v) \) rather than \( d_G(v) \).

A directed graph (or digraph) \( D = (V(D), A(D)) \) is a graph where the edge set \( A(D) \) is a set of ordered pairs of vertices, called arcs or directed edges. If \( (x, y) \) is an arc of \( D \), then we say that the arc \( (x, y) \) is from \( x \) to \( y \) in \( D \), and that \( x \) is adjacent to \( y \), and \( y \) is adjacent from \( x \). The indegree of a vertex \( v \) is the number of arcs incoming to a vertex; that is, the number of vertices adjacent to \( v \). The outdegree of \( v \) is the number of arcs outgoing from a vertex; that is, the number of vertices adjacent from \( v \). A directed cycle \( C \) is a connected digraph where all vertices have indegree and outdegree equal to 1.

A forest is a graph with no cycles. A tree is a connected forest. An independent set in a graph \( G \) is a set of pairwise non-adjacent vertices. The independence number of a graph \( G \), denoted by \( \alpha(G) \), is the cardinality of the largest independent set in \( G \). An independent set of size \( \alpha(G) \) will be called an \( \alpha \)-set of \( G \). A complete graph is a graph whose vertices are pairwise adjacent. The complete graph with \( n \) vertices is denoted by \( K_n \). A set \( C \subseteq V(G) \) is a clique in a graph \( G \) if its induced subgraph is complete. A graph \( G \) is bipartite if \( V(G) \) is the union of two disjoint independent sets. A complete bipartite graph is a bipartite graph, such that every pair of vertices in the two independent sets are adjacent. If the two maximal independent sets are of size \( n \) and \( m \) (where \( n, m \in \mathbb{N} \)), then the complete bipartite graph is denoted by \( K_{n,m} \).

3. Lower Bounds in Regular Graphs

The Grundy total domination number of regular graphs was studied in [7], where the following lower bound for regular graph was proven.

**Theorem 3.1.** ([7]) For \( k \geq 3 \), if \( G \) is a connected, \( k \)-regular graph different from \( K_{k,k} \), then \( \gamma^L_{gr}(G) \geq \frac{n(G)}{k-1} \) with strict inequality if \( k \geq 5 \).
In [11] it was proved that every Grundy total dominating sequence of a bipartite graph has the same number of vertices in each partite set of the graph.

**Theorem 3.2.** ([11]) If $G$ is a bipartite graph with bipartition $A \cup B$, then the Grundy total domination number of $G$ is even and for any Grundy total dominating sequence $S = (v_1, \ldots, v_{2k})$ we have $|A \cap S| = |B \cap S| = k$.

Since $\gamma^{L}_{gr}(G) \leq \gamma^{T}_{gr}(G)$, a lower bound for the Grundy total domination number is also a lower bound for the L-Grundy domination number. For a regular graph, the following result improves this lower bound on the L-Grundy domination number, which follows from Theorem 3.1.

**Theorem 3.3.** For $k \geq 3$ if $G$ is a $k$-regular connected bipartite graph different from $K_{k,k}$, then
\[
\gamma^{L}_{gr}(G) \geq \frac{k n(G)}{2(k-1)}.
\]

*Proof.* Let $n = n(G)$. We construct an $L$-sequence of length $\frac{nk}{2(k-1)}$ in the $k$-regular connected bipartite graph $G$ of order $n$ as follows. Let $A$ and $B$ be the partite sets of the bipartite graph $G$. Since $G$ is $k$-regular, we note that $|A| = |B| = \frac{n}{2}$. First, we put in the sequence $S$ all vertices from the set $A$ in an arbitrary order. We note that each such vertex footprints itself and possibly some vertices in $B$. Secondly, we continue the sequence by adding to it vertices from $B$ such that each added vertex from $B$ footprints a vertex of $A$. Thus, the vertices in $B$ that belong to the sequence $S$ totally dominate the set $A$. Theorem 3.2 implies that the number of vertices from $B$ in $S$ is $\frac{1}{2} \gamma^{T}_{gr}(G)$. By Theorem 3.1, we have $\gamma^{T}_{gr}(G) \geq \frac{n}{k-1}$ for every $k$-regular bipartite graph $G$ different from $K_{k,k}$, implying that
\[
\gamma^{L}_{gr}(G) \geq |A| + \frac{1}{2} \left( \frac{n}{k-1} \right) = \frac{n}{2} + \frac{1}{2} \left( \frac{n}{k-1} \right) = \frac{nk}{2(k-1)},
\]
as desired. □

We remark that the bound in Theorem 3.3 is best possible in the case when $k = 3$, as may be seen by considering the graph $G = K_{4,4} - M$, where $M$ is a perfect matching of $K_{4,4}$. In this case, $G$ is a 3-regular graph of order $n = 8$ and $\gamma^{L}_{gr}(G) = 6 = \frac{nk}{2(k-1)}$.

**Remark 3.4.** For $k \geq 1$, the following holds.

(a) If $G = K_{k+1}$, then $\gamma^{L}_{gr}(G) = 2$.

(b) If $G = K_{k,k}$, then $\gamma^{L}_{gr}(G) = k + 1$.

We establish next a lower bound on the L-Grundy domination number of a $k$-regular connected graph that is different from $K_{k,k}$ and $K_{k+1}$.

**Theorem 3.5.** For $k \geq 3$, if $G$ is a $k$-regular connected graph different from $K_{k,k}$ and $K_{k+1}$, then
\[
\gamma^{L}_{gr}(G) \geq \frac{2(k-2)n(G) + (4-k)\alpha(G)}{(k-1)^2}.
\]

*Proof.* Let $G$ be a $k$-regular connected graph of order $n$ different from $K_{k,k}$ and $K_{k+1}$. Let $A$ be an $\alpha$-set of $G$ and $B = V(G) \setminus A$. Let $B_1 \cup B_2 \cup \cdots \cup B_k$ be a partition of $B$, such that each vertex of $B_i$ has exactly $i$ neighbors in $A$ for $i \in [k]$. Let $n_i = |B_i|$ for $i \in [k]$, and let $E(A,B)$ be the set of edges from $A$ to $B$. Since $|E(A,B)| = k\alpha(G) = n_1 + 2n_2 + \cdots + kn_k$ and $n_1 + n_2 + \cdots + n_k = n - \alpha(G)$ we get
\[
n_1 = 2n - (2 + k)\alpha(G) + \sum_{i=3}^{k} (i-2)n_i.
\]

Let $A_1 = N(B_1) \cap A$. Since each vertex in $B_1$ has exactly one neighbor in $A_1$, we note that no two vertices in $A_1$ have a common neighbor in $B_1$. We state this formally as follows.
Observation 3.6. If \( a, a' \in A_1 \), then the set of neighbors of \( a \) in \( B_1 \) is disjoint with the set of neighbors of \( a' \) in \( B_1 \).

We proceed further with the following claim.

Claim 3.7. \( |A_1| \geq \frac{n_1}{k-1} \).

Proof. Suppose that there exists \( a \in A_1 \) that has \( k \) neighbors \( x_1, \ldots, x_k \) in \( B_1 \). Since \( G \) does not contain a subgraph isomorphic to \( K_{k+1} \), there exists \( x_i, x_j \in \{ x_1, \ldots, x_k \} \) such that \( x_i x_j \notin E(G) \), implying that \( (A \setminus \{a\}) \cup \{x_i, x_j\} \) is an independent set larger than \( A_1 \), a contradiction. Hence every \( a \in A_1 \) has at most \( k-1 \) neighbors in \( B_1 \) and \( N(a) \cap B_1 \) is a clique. Therefore, \( (k-1)|A_1| \geq |B_1| = n_1 \) and, consequently, \( |A_1| \geq \frac{n_1}{k-1} \). \( \square \)

We construct an L-sequence \( S \) as follows. Initially, we let the sequence consist of all vertices from \( A \) in any order. Since when \( a \in A \) is added to \( S \) it footprints at least itself, the resulting sequence \( S \) is an L-sequence of length \( a(G) \). Let \( A_1 = \{a_1, \ldots, a_{|A_1|}\} \). For each \( a_i \in A_1 \), let \( b_i \) be an arbitrary neighbor of \( a_i \) in \( B_1 \) and let \( B' = \{b_i : i \in [|A_1]|\} \). From Observation 3.6 we note that \( B' \) contains \( |A_1| \) pairwise different vertices, and so \( \{a_1b_1, \ldots, a_{|A_1|}b_{|A_1|}\} \) is a matching in \( G \). We now add to \( S \) all vertices from \( B' \) in any order. Since when \( b_i \in B' \) is added to \( S \) it footprints the vertex \( a_i \in A_1 \), the sequence \( S \) is still an L-sequence. The length of the resulting sequence \( S \) is \( a(G) + |A_1| \).

Claim 3.8. There exists a sequence \( S' = (s_1, \ldots, s_{\ell}) \) of vertices in \( B \setminus B_1 \), such that \( N_{A \setminus A_1}(S') = A \setminus A_1 \) and each \( s_i \in S' \) footprints at least one and at most \( k - 1 \) vertices in \( A \setminus A_1 \) for \( i \in [\ell] \); that is, each \( s_i \in S' \) is adjacent to at least one and at most \( k - 1 \) vertices from \( A \setminus (A_1 \cup (N_{A \setminus A_1}(s_1, \ldots, s_{i-1})) \).

Proof. For the purpose of contradiction, suppose to the contrary that we already chose a subset \( X \subseteq S' \) of vertices in \( B \setminus B_1 \) that satisfies the statement of the claim, but in \( B_X = B \setminus (B_1 \cup X) \) every vertex has either no neighbor in \( A_X = A \setminus N_A(B_1 \cup X) \) (which is not yet empty set) or has all its \( k \) neighbors in \( A_X \). Let \( B'_X \) be the set of vertices from \( B_X \) with no neighbor in \( A_X \) and let \( B''_X \) be the set of vertices from \( B_X \) with all \( k \) neighbors in \( A_X \). We note that \( B_X = B'_X \cup B''_X \). In this case, the vertices in the set \( A_X \cup B''_X \) induce a bipartite \( k \)-regular subgraph of \( G \), which implies that \( G \) is disconnected, a contradiction. \( \square \)

By Claim 3.8, there exists a sequence \( S' = (s_1, \ldots, s_{\ell}) \) of vertices in \( B \setminus B_1 \), such that \( N_{A \setminus A_1}(S') = A \setminus A_1 \) and each \( s_i \in S' \) footprints at least one and at most \( k - 1 \) vertices in \( A \setminus A_1 \) for \( i \in [\ell] \). We now consider the expanded sequence \( S = S \oplus S' \). The resulting sequence \( S \) is an L-sequence. Thus, letting \( \alpha = a(G) \) we have

\[
\gamma_{Gr}^L(G) \geq |A_1| + |B'| + \ell
\]

\[
= |A_1| + |A_1| + \ell \quad \text{(Claim 3.8)}
\]

\[
\geq \left( \frac{k}{k-1} \right)|A_1| + \left( \frac{k}{k-2} \right)|A_1| \quad \text{(Claim 3.7)}
\]

\[
\geq \left( \frac{k}{k-1} \right) \alpha + \left( \frac{k}{k-2} \right) n_1 \quad \text{ (Equation 1)}
\]

\[
= \frac{1}{(k-1)^2} \left( 2(k-2)n + (4-k)\alpha + (k-2) \sum_{i=3}^{k} (i-2)n_i \right).
\]

Since \( n_i \geq 0 \) for \( i \in [k] \setminus [1, 2] \), this implies that

\[
\gamma_{Gr}^L(G) \geq \frac{1}{(k-1)^2} (2(k-2)n + (4-k)\alpha),
\]

which is the desired lower bound on the L-Grundy domination number. \( \square \)
If \( G \) is a \( k \)-regular graph on \( n \) vertices, then by double counting the edges joining a maximum independent set \( A \) in \( G \) and its complement \( V(G) \setminus A \), we have \( \alpha(G) = |A| \leq \frac{n}{2} \). Further if \( G \) is a bipartite graph, then both partite sets of \( G \) have cardinality \( \frac{1}{2}n \), implying that \( \alpha(G) = \frac{1}{2}n \). Thus in the proof of Theorem 3.5, we can choose in this case the \( \alpha \)-set \( A \) to be either of the two partite sets of \( G \), implying that \( n_1 = n_2 = \cdots = n_{k-1} = 0 \) and \( n_k = n - \alpha(G) = \frac{1}{2}n \). Hence, as an immediate consequence of the proof of Theorem 3.5 we have

\[
\gamma_{gr}^L(G) \geq \frac{1}{(k-1)^2} \left( 2(k-2)n + (4-k)\alpha + (k-2)n_k \right) = \left( \frac{k}{2(k-1)} \right)n.
\]

Theorem 3.3 is therefore a corollary of Theorem 3.5. In the special case of Theorem 3.5 when \( k = 3 \), we have the following lower bound on the L-Grundy domination number of a cubic graph.

**Corollary 3.9.** If \( G \) is a cubic graph different from \( K_{3,3} \) and \( K_4 \), then

\[
\gamma_{gr}^L(G) \geq \frac{1}{2}n(G) + \frac{1}{4}\alpha(G).
\]

### 4. General Upper Bound

From the definition of the Grundy domination number, Grundy total domination number and L-Grundy domination number it follows that L-Grundy domination number of a graph \( G \) is bounded below by \( \gamma_{gr}^L(G) \) and \( \gamma_{gr}(G) + 1 \), as first observed in [2]. The Grundy domination number and Grundy total domination number have trivial upper bounds in terms of the minimum degree of a graph \( G \), namely \( \gamma_{gr}(G) \leq n(G) - \delta(G) \) and \( \gamma_{gr}^L(G) \leq n(G) - \delta(G) + 1 \). In the case of the L-Grundy domination number, it is not necessary true that \( n(G) - \delta(G) \) is an upper bound. Although we strongly suspect that \( \gamma_{gr}^L(G) \leq n(G) - \delta(G) + 1 \), we have yet to prove that this upper bound always holds since there exist vertices that are footprinted twice, first by itself and then by one of its neighbors. However, we were able to establish the following result.

**Theorem 4.1.** If \( G \) is a graph with \( \gamma_{gr}^L(G) = n(G) \), then \( \delta(G) \leq 1 \).

**Proof.** For the purpose of contradiction, let \( G \) be a graph of order \( n \) satisfying \( \gamma_{gr}^L(G) = n \) and suppose, to the contrary, that \( \delta(G) \geq 2 \). In particular, \( G \) has no isolated vertex. Let \( S \) be a Grundy \( \gamma_{gr}^L \)-sequence of a graph \( G = (V, E) \) and for \( x \in S \) let \( (x) \) denote the position of \( x \) in \( S \). Since \( \gamma_{gr}^L(G) = n \), we note that \( |S| = n \).

We construct next a directed graph \( D \) as follows. Let \( V(D) = V \), where \( xy \in A(D) \) if and only if \( x \) footprints \( y \) with respect to the sequence \( S \) in \( G \). Since \( |S| = n \), we note that each vertex of \( D \) has outdegree at least 1 (noting that loops are also possible, which contribute 1 to the outdegree of each vertex). Once the sequence \( S \) is constructed, all vertices are footprinted, implying that each vertex of \( D \) has indegree at least 1. Since the graph \( G \) has no isolated vertex, we note that if a vertex footprints itself, then it will also be subsequently footprinted by one of its neighbors. Let \( G' \) be the (undirected) graph obtained from \( D \) by removing the orientation of each arc in \( D \) to form an edge. Next, we list some properties of directed graph \( D \) and the graph \( G' \).

**Claim 4.2.** If we delete from \( G' \) all loops and multiple edges, then the resulting graph is a forest.

**Proof.** Let \( F \) be the graph obtained from \( G' \) by deleting all loops and multiple edges. For the purpose of contradiction, suppose to the contrary that \( F \) has a cycle \( C \). Let \( C' \) be a subgraph of the directed graph \( D \) obtained from \( C \) by restoring the orientation of the edges \( xy \in E(C) \). Since each vertex is footprinted by exactly one vertex different from itself, the oriented cycle \( C' \) is a directed cycle in \( D \) (in which every vertex has indegree 1 and outdegree 1). Let \( V(C') = \{x_1, x_2, \ldots, x_\ell\} \) where \( \ell \geq 3 \) and \( (x_i, x_{i+1}) \) is an arc from \( x_i \) to \( x_{i+1} \) for \( i \in [\ell] \) and where addition is taken modulo \( k \). Renaming vertices of \( C' \) if necessary, we may assume that \( x_1 \) is the vertex with the smallest index in \( S \); that is, among all vertices in \( V(C') \), the vertex \( x_1 \) appears first in \( S \). According to our orientation of the directed cycle \( C' \), the vertex \( x_i \) is footprinted by the vertex \( x_{i-1} \) for \( i \in [\ell] \setminus \{1\} \). In particular, the vertex \( x_1 \) is footprinted by the vertex \( x_{\ell-1} \). However since \( x_1 \) is a neighbor of
It follows from Claim 4.2 that the directed graph $D$ has a forest structure (with addition of loops and multiple edges). Therefore we will call $D$ a pseudo forest. We note that $D$ has no vertices with outdegree 0 and all vertices of $D$ have indegree 1 or 2. Further we note that if we delete all loops in $D$, then all vertices of the resulting directed graph have indegree 1.

We call a vertex $x$ in $D$ that has outdegree 1 and such that $(x, x) \in A(D)$ a leaf of type 1 in $D$, and we call a vertex $x$ that has outdegree 1 and such that $(x, y) \in A(D)$ where $x \neq y$ and $(y, x) \in A(D)$ a leaf of type 2 in $D$. We call a leaf of type 1 in $D$ or a leaf of type 2 in $D$ a leaf of $D$.

Recall that a component of a graph is a maximal connected subgraph of the graph. In particular, we note that a component of a graph is by definition connected. A component $T$ of a pseudo forest $D$ is called a pseudo tree.

Let $T$ be a pseudo tree in $D$, and let $x$ be a vertex of $T$ with the smallest index in $S$. We call the vertex $x$ the root of the component $T$. Let $T'$ be the (undirected) multigraph obtained from $T$ by removing the orientation of each arc in $T$ and removing any resulting loops.

Claim 4.3. The following holds for the pseudo tree $T$ with the root $x$:

(a) $(x, x) \in A(D)$,

(b) the multigraph $T'$ has exactly one multiple edge, namely a double edge incident with the vertex $x$.

Proof. Since $G$ has no isolated vertex and each vertex that footprints itself is also footprinted by one of its neighbors, we note that $n(T) \geq 2$. Since $T$ is a component of $D$, we note that the vertex $x$ footprints itself and all its neighbors $N = \{x_1, \ldots, x_\ell\}$ in $G$ that are not adjacent to any vertex that precedes $x$ in $S$. Hence, $(x, x) \in A(T)$, which proves Part (a).

Further, $(x, x_i) \in A(T)$ for all $i \in [k]$. Since $\gamma_{gr}^T(G) = n$, the vertex from $N$ with the smallest index in $S$ footprints $x$. Renaming indices if necessary, we may assume that $x_1$ is the vertex in $N$ with the smallest index in $S$ that footprints $x$, and so $(x_1, x) \in A(T)$ but no other vertex in $N$ footprints $x$. Thus, $T'$ has at least one multiple edge, namely the edge $xx_1$ (corresponding to the arcs $(x, x_1)$ and $(x_1, x)$ in $T$).

Suppose now that there is a multiple edge $yz$ in $T'$ different from the edge $xx_1$. As observed earlier, no other vertex in $N \setminus \{x_1\}$ footprints $x$, implying that $x \neq y$ and $x \neq z$. Since $yz$ is a multiple edge in $T'$, we note that $(y, z) \in A(T)$ and $(z, y) \in A(T)$; that is, $y$ footprints $z$ and $z$ footprints $y$. Renaming $y$ and $z$ if necessary, we may assume that the distance from $x$ to $y$ in $T'$ is less than or equal to the distance from $x$ to $z$ in $T'$. Let $P$: $x, y_1, \ldots, y_\ell = y$ be a shortest path from $x$ to $y$ in $T'$. In the component $T$, we note that $(x, y_1) \in A(T)$ and $(y_\ell, y_{\ell+1}) \in A(T)$ for $i \in \{\ell - 1\}$. This implies that $x$ footprints $y_1$, and if $\ell \geq 2$, then the vertex $y_\ell$ footprints $y_{\ell+1}$ for all $i \in \{\ell - 1\}$. Since $d_{T'}(x, y) \leq d_{T'}(x, z)$, we note that $z \notin V(P)$. This implies that the vertex $y$ is footprinted by at least two vertices different from itself, namely the vertices $y_{\ell+1}$ and $z$, a contradiction. Hence, the multiple edge $xx_1$ is the only multiple edge in $T'$. This proves Part (b).

Claim 4.4. If $u$ and $v$ are vertices in $D$ with $(u, u) \in A(D)$ and $(v, v) \in A(D)$, then $u$ and $v$ are not adjacent in $G$.

Proof. Suppose, to the contrary, that $uv \in E(G)$. Renaming vertices if necessary, we may assume that $i(u) < i(v)$. Hence when $v$ is added to $S$, the vertex $v$ is already footprinted, implying that $(v, v) \notin A(D)$, a contradiction.

As a consequence of Claim 4.4, we have the following results.

Claim 4.5. The following holds.

(a) If $u$ and $v$ are leaves of type 1 in $D$, then $u$ and $v$ are not adjacent in $G$.

(b) If $y$ is a leaf of type 1 in $D$ and $x$ a root of a component $T$ of $D$, then $xy \notin E(G)$.

(c) If $T_1$ and $T_2$ are components of $D$ with roots $x_1$ and $x_2$, respectively, then $x_1x_2 \notin E(G)$.

Claim 4.6. If $u$ is a leaf of type 2 in $D$ which is adjacent to a vertex $v$ in $D$, then $i(v) < i(u)$ and $v$ is a root of the component in $D$ that contains $u$. 

Proof. Let \( u \) be a leaf of type 2 in \( D \). Thus, \( u \) has outdegree 1 and there exists exactly one vertex \( v \in V(D) \) with \((u, v) \in A(D)\) and \((v, u) \notin A(D)\). It follows from Claim 4.3 that \( v \) is a root of a component of \( D \) that contains the vertex \( u \). By Claim 4.3(a), the vertex \( v \) footprints itself, implying that \( v \) is in \( S \) before \( u \).

**Claim 4.7.** Let \( T_1 \) and \( T_2 \) be two components of order 2 in \( D \) with roots \( x_1 \in V(T_1), x_2 \in V(T_2) \) and leaves \( y_1 \in V(T_1), y_2 \in V(T_2) \). If \( i(y_1) < i(y_2) \), then \( x_2y_1 \notin E(G) \).

Proof. Let \( x_1 \) be the root and \( y_1 \) the leaf of \( T_1 \), and let \( x_2 \) be the root and \( y_2 \) the leaf of \( T_2 \). Thus, \( x_1 \) footprints \( y_1 \), and \( x_2 \) footprints \( y_2 \). We note that \( y_i \) is a leaf of \( T_i \) of type 2 for \( i \in [2] \) and hence \( y_1 \) footprints \( x_1 \) and \( y_2 \) footprints \( x_2 \). Suppose that \( x_2y_1 \notin E(G) \). Since \( i(y_1) < i(y_2) \), \( x_2 \) is footprinted by \( y_1 \), a contradiction. (\( \Box \))

**Claim 4.8.** Let \( P: \ldots x_t \ldots \) be a directed path in \( T \) where \( t \geq 2 \), and so \((x, x_i) \in A(D)\) and \((x_i, x_{i+1}) \in A(D)\) for \( i \in [t - 1] \). If \( x_t \) is a leaf of type 1, then

\[
\max\{i(x_1), i(x_2), \ldots, i(x_t)\} = i(x_{t-1}).
\]

Proof. Since \( P: \ldots x_t \ldots \) is a directed path in \( T \), we note that the vertex \( x \) footprints the vertex \( x_1 \), and the vertex \( x_t \) footprints the vertex \( x_{t-1} \) for \( i \in [t - 1] \). Since \( x_t \) is a leaf of type 1 in \( D \), the vertex \( x_t \) footprints only itself when it is added to \( S \). This implies that \( i(x_t) < i(x_{t-1}) \). However, \( x_t \) does not footprint \( x_{t-1} \), since \( x_t \) footprints only itself. Thus, \( x_{t-1} \) is already footprinted when \( x_t \) is added to \( S \). Since \( x_{t-2} \) footprints the vertex \( x_{t-1} \), this implies that \( i(x_{t-2}) < i(x_{t-3}) \). Since \( x_{t-3} \) footprints the vertex \( x_{t-4} \) (and \( x_{t-2} \) is not footprinted by the vertex \( x_{t-2} \)), we have that \( i(x_{t-4}) < i(x_{t-2}) \). Continuing in this way, we deduce that the vertex \( x_{t-2j} \) where \( j \in [0, 1, \ldots, \lfloor \frac{t}{2} \rfloor] \) is in \( S \) before \( x_{t-1} \).

Since \( x_{t-1} \) footprints \( x_t \) and does not footprint \( x_{t-2} \), as \( x_{t-2} \) is footprinted by \( x_{t-3} \), we note that \( i(x_{t-3}) < i(x_{t-1}) \). Since \( x_{t-3} \) footprints the vertex \( x_{t-4} \) (and \( x_{t-3} \) is not footprinted by the vertex \( x_{t-3} \)), we have that \( i(x_{t-3}) < i(x_{t-3}) \). Analogously as before we deduce that the vertex \( x_{t-(j+1)} \) where \( j \in [1, \ldots, \lceil \frac{t}{2} \rceil] \) is in \( S \) before \( x_{t-(j+1)} \) (as \( x_{t-2j} \) is footprinted by \( x_{t-(j+1)} \) and not by \( x_{t-(j+1)} \)) and hence before \( x_{t-1} \). Thus, every vertex in \( V(P) \setminus \{x_{t-1}\} \) is in \( S \) before \( x_{t-1} \). (\( \Box \))

Let \( L \) be the set of vertices in \( D \) that are adjacent to at least one leaf in \( D \). Among all vertices in \( L \), let \( y \) have the largest index in \( S \) and let \( y_r \) be a leaf of \( D \) that is adjacent to \( y \), i.e., \( y \) footprints \( y_r \). Since \( \delta(G) \geq 2 \), there exists a vertex \( z \in V(G) \) different from \( y \) that is adjacent to \( y \). The leaf \( y_r \) can of be two types, which we deal with separately.

**Case 1.** The leaf \( y_r \) is of type 1.

By Claim 4.5, the vertex \( z \) is not a leaf of type 1 in \( D \) and is not the root of a component of \( D \).

Suppose that \( z \) is not a leaf of type 2 in \( D \). Let \( T_z \) be a component of \( D \) that contains \( z \) and let \( z_1, z_2, \ldots, z_k \) be a directed path from \( z \) to a leaf \( z_k \) of type 1 in \( T_z \) (note that this path does not contain the root of \( T_z \)). Since \( y \) and not \( z \), is a footprint of \( y_r \), we note that \( i(y) < i(z) \). By Claim 4.8, we have that \( i(z) < i(z_{k-1}) \), implying that \( i(y) < i(z_{k-1}) \), contradicting our choice of the vertex \( y \). Hence, \( z \) is a leaf of type 2 in \( D \).

Since \( z \) is a leaf of type 2 in \( D \), there exists a vertex \( z' \in V(D) \) distinct from \( z \) such that \((z, z') \in A(D)\) and \((z', z) \in A(D)\), and the outdegree of \( z \) in \( D \) is 1. By Claim 4.6, the vertex \( z' \) is a root of the component \( T_z \) in \( D \) that contains \( z \) and \((z') \notin i(z) \). Since \( y_r \) is footprinted by \( y \) and not by \( z \), we have \( i(y) < i(z) \).

Suppose that \( n(T_z) > 2 \). In this case, we let \( z, z', z_2, \ldots, z_k \) be a directed path in \( T_z \) from \( z \) to a leaf \( z_2 \) of type 1 in \( T_z \). Since \( z' \) is footprinted by \( y \) and not by \( z \), the vertex \( z_2 \) is in \( S \) after \( z \) and thus after \( y \). Thus \( n(T_z) > 2 \) implies that \( z_{k-1} \) is in \( S \) after \( z_2 \) and thus after \( y \), a contradiction with the choice of \( y \). Hence, \( n(T_z) = 2 \).

Recall that \( i(y) < i(z) \). Since \( z' \) has only one neighbor in \( D \), there exists a vertex \( z_2 \in V(G) \) different from \( z \) that is adjacent to \( z' \) in \( G \). Since \( z' \) is footprinted by \( y \) (and not \( z_2 \)), the vertex \( z_2 \) is in \( S \) after \( z \) and thus after \( y \). If the component \( T_2 \) of \( D \) that contains \( z_2 \) has order more than 2, then Claim 4.8 implies that there is a vertex from \( L \) which is in \( S \) after \( y \), a contradiction. Hence, \( n(T_2) = 2 \).

By Claim 4.5, \( z_2 \) is a leaf of type 2 and the neighbor \( z'_2 \) of \( z_2 \) in \( T_2 \) is a root of \( T_2 \). Since the degree of \( z'_2 \) in \( G \) is at least 2, the vertex \( z'_2 \) has a neighbor \( z_3 \neq z_2 \) in \( G \), implying as before that \( z_3 \) is in \( S \) after \( z_2 \), noting
that \( z_1 \) is footprinted by \( z_2 \). Claim 4.7 implies that this procedure will eventually produce a component \( T_i \) of order more than 2, i.e., \( z_{i-1}, z_{i-1}' \in V(T_{i-1}) \), \( n(T_{i-1}) = 2 \) and \( z_{i-1} \) is a leaf of type 2 in \( T_{i-1} \) and \( z_{i-1}' \) a root of \( T_{i-1} \). Since the degree of \( z_{i-1}' \) in \( G \) is more than 1, there exists \( z_i \in V(T_i) \) that is adjacent to \( z_{i-1}' \) in \( G \). We therefore have \( i(y) < i(z_2) < \cdots < i(z_{i-1}) \) and as \( z_{i-1}' \) is footprinted by \( z_{i-1} \) and not by \( z_i \), this implies that \( i(z_{i-1}) < i(z_i) \). Since \( T_i \) has order more than 2, Claim 4.8 guaranties the existence of a vertex from \( L \) that is in \( S \) after \( z_i \) and hence after \( y \), a contradiction.

**Case 2.** The leaf \( y_\ell \) is of type 2.

By Claim 4.3, the vertex \( y \) is the root of the component \( T_y \) that contains \( y_\ell \) and \( i(y) < i(y_\ell) \).

If \( n(T_y) > 2 \), then let \( z_2 \) be a neighbor of \( y \) in \( D \) different from \( y_\ell \) and let \( y_\ell, z_2, \ldots, z_k \) be a directed path from \( y \) to a leaf \( z_k \) of type 1 in \( T_y \). By Claim 4.3, we note that the vertex \( y_\ell \) is the only leaf of type 2 in \( T_y \). Since \( y \) is footprinted by \( y_\ell \) and not by \( z_2 \), we have \( i(y_\ell) < i(z_2) \leq i(z_{i-1}), \) where the last inequality follows from Claim 4.8. Hence, \( i(y) < i(z_k-1) \), contradicting the choice of the vertex \( y \). Thus, \( n(T_y) = 2 \). Since \( y_\ell \) is footprinted by \( y \), we have \( i(y) < i(z) \). If \( z \) is not a leaf of type 2 in a component of order 2, then we infer, using also Claim 4.8, that there exists a vertex from \( L \) that is in \( S \) after \( y_\ell \), a contradiction. Therefore, \( z \) is a leaf of type 2 in \( T_1 \), with \( n(T_1) = 2 \). Hence the neighbor \( z' \) of \( z \) in \( T_1 \) has another neighbor in \( G \). We continue in the same way as in Case 1 and obtain a contradiction. This completes the proof of Theorem 4.1.

For every graph \( G \) we have \( \gamma_{gr}^L(G) \leq n(G) \). In particular, if \( \delta(G) \leq 1 \), then \( \gamma_{gr}^L(G) \leq n(G) - \delta(G) + 1 \) holds.

If \( \delta(G) = 2 \), then Theorem 4.1 implies that \( \gamma_{gr}^L(G) \leq n(G) - 1 = n(G) - \delta(G) + 1 \). Thus, we have the following result.

**Corollary 4.9.** If \( G \) is a graph with \( \delta(G) \leq 2 \), then \( \gamma_{gr}^L(G) \leq n(G) - \delta(G) + 1 \).

We believe that Corollary 4.9 holds for all graphs, and pose the following conjecture.

**Conjecture 4.10.** If \( G \) is a graph, then \( \gamma_{gr}^L(G) \leq n(G) - \delta(G) + 1 \).

### 5. Forests

In this section we show that the L-Grundy domination number of any forest equals the number of vertices in the forest.

A leaf of a graph \( G \) is a vertex of degree 1 in \( G \), while its only neighbor is a support vertex. We denote by \( L(G) \) the set of leaves of \( G \). For each vertex \( u \in V(G) \), we let \( L(u) \) be the set of leaf neighbors of \( u \) in \( G \); that is, \( L(u) = L(G) \cap N(u) \). We note that if \( L(u) \neq \emptyset \), then \( u \) is a support vertex of \( G \).

**Theorem 5.1.** For any forest \( T \),
\[
\gamma_{gr}^L(T) = n(T).
\]

Algorithm 1 returns a \( \gamma_{gr}^L \)-sequence of an arbitrary forest \( T \). The complexity of Algorithm 1 is \( O(n(T)) \).

**Proof.** Let us start the proof by showing that the sequence \( S \) produced by Algorithm 1 is an L-sequence; i.e., every vertex in the sequence footprints some vertex in \( T \). When \( w \in V(T') \) such that \( L(w) \neq \emptyset \) is chosen in some step of the algorithm (line 5 of the algorithm), then the vertex \( u \in L(w) \) footprints itself and maybe also its unique neighbor \( w \) (for later purposes denoted also by \( w_i \)), and so the first part of the sequence \( S \) is an L-sequence. When \( T' \) contains just isolated vertices, each vertex of \( T' \) footprints itself. We note that all vertices that are added to \( S \) in the first and the second part (until line 13 of the algorithm) form an independent set. Hence, all vertices are footprinted only by themselves and therefore, can be footprinted again by some other vertex. The last part is constructed from vertices \( w_i \) that are footprinted in the first part, only that they are listed in the reverse order. At the time vertex \( w_i \) appears in \( S \), vertices in \( L(w_i) \) (of a forest \( T' \) before \( w_i \) was removed), are footprinted only by themselves. Hence, \( w_i \) footprints vertices in \( L(w_i) \). This yields that \( S \) is an L-sequence.

Clearly, the time complexity of the algorithm is \( O(n(T)) \) and all vertices from \( T \) are added to the sequence \( S \). Hence, \( |S| = n(T) \), \( \gamma_{gr}^L(T) = n(T) \) and \( S \) is a \( \gamma_{gr}^L \)-sequence.

\[ \square \]
6. Split graphs

A split graph is a graph in which the vertices can be partitioned into a clique and an independent set. The decision version of L-Grundy domination is the following problem:

**L-Grundy Domination Number Problem**

**Input:** $G = (V, E), k \in \mathbb{N}$.

**Question:** Is there a $\gamma^l_{Gr}$-sequence of $G$ of length at least $k$?

We prove that this problem remains NP-complete when restricted to split graphs.

In the proof, we will rely on the NP-completeness result [6, Corollary 8.5] concerning the Grundy Total Domination Number Problem. Given a graph $G$ with no isolated vertices and a positive integer $k$ the problem is to determine whether $\gamma^t_G(G) \geq k$. We also use the concept of the $t$-footprinter of a vertex, which is defined with respect to an open neighborhood sequence in $G$. If $S = (v_1, \ldots, v_k)$ is such a sequence, then $v_i$ t-footprints the vertices in $N(v_i) \setminus \bigcup_{j=1}^{i-1} N(v_j)$, and so $v_i$ is the t-footprinter of $v_i$ vertices.

**Theorem 6.1.** L-Grundy Domination Number Problem is NP-complete, even when restricted to split graphs.

**Proof.** It is clear that the problem is in NP.

Given a graph $G = (V, E)$ with no isolated vertices, we construct the split graph $G' = (V_1 \cup V_2, E')$ as follows: $V_1 = \{v^1 : v \in V\}$ is an independent set, $V_2 = \{v^2 : v \in V\}$ induces a clique and $N_G(v^1) = \{v^2 \in V_2 : v \in N_G(v^1)\}$.

We will prove that $\gamma^l_{Gr}(G') = \gamma^t_G(G) + \gamma^l_{Gr}(G)$, which by the NP-completeness result on Grundy Total Domination Number Problem from [6] readily implies that the problem is NP-complete even when restricted to split graphs.

Let $S^1$ be any sequence of all vertices in $V_1$. If $(v_1, \ldots, v_t)$ is a $\gamma^l_{Gr}$-sequence of $G$, then $S^1 \oplus (v^2_1, \ldots, v^2_t)$ is an $L$-sequence of $G'$. Hence $\gamma^l_{Gr}(G') \geq n(G) + \gamma^l_{Gr}(G)$.

Now, let $S = (w_1, \ldots, w_k)$ be a $\gamma^l_{Gr}$-sequence of $G'$. Let $t = \min\{j : w_j \in V_2\}$ and $r = \min\{j : j > t, w_j \in V_2\}$. The vertex $w_t$ L-footprints some vertex from $V_1$, while vertices from $(w_{t+1}, \ldots, w_k)$ L-footprint only vertices from $V_1$. 

---

**Algorithm 1:** A $\gamma^l_{Gr}$-sequence of a forest.

**Input:** A forest $T$.

**Output:** A $\gamma^l_{Gr}$-sequence $S$ of $T$.

1. $S = \emptyset$;
2. $T' = T$;
3. $i = 0$;
4. **while** $T'$ has non-isolated vertices **do**
   5. Choose $w \in V(T')$, such that $L(w) \neq \emptyset$;
   6. $S = S \oplus (L(w))$;
   7. $i = i + 1$;
   8. $w_i = w$;
   9. $T' = T' \setminus (L(w) \cup \{w\})$;
5. **while** $V(T') \neq \emptyset$ **do**
6. Choose a vertex $v \in V(T')$;
7. $S = S \oplus \{v\}$;
8. $T' = T' \setminus \{v\}$
9. **while** $i > 0$ **do**
10. $S = S \oplus (w_i)$;
11. $i = i - 1$;
Now we distinguish two cases. Suppose the vertex $w_r$ L-footprints some vertex in $V_1$. Hence, the subsequence of the vertices in $S$ that are taken from $\tilde{S} \cap V_2$ forms an open neighborhood sequence in $G'$ (all vertices $t$-footprint some vertex in $V_1$). Hence, the corresponding sequence forms an open neighborhood sequence in $G$ and $|\tilde{S} \cap V_2| \leq \gamma_{fr}'(G)$. Thus, $\gamma_{fr}'(G') = |\tilde{S}| = |\tilde{S} \cap V_1| + |\tilde{S} \cap V_2| \leq n(G) + \gamma_{fr}'(G).

Suppose next that $w_r$ does not L-footprint any vertex in $V_1$. In this case, the vertex $w_r$ L-footprints only the vertex $w_t$. Hence, the subsequence of the vertices in $S$ that are taken from $(\tilde{S} - w_r) \cap V_2$ form an open neighborhood sequence in $G'$. The corresponding sequence therefore forms an open neighborhood sequence in $G$ and $|\tilde{S} \cap V_2| - 1 \leq \gamma_{fr}'(G)$. Let $u \in V_1 \cap N(w_t)$. We claim that $u \notin S$. Indeed, if $u$ is in $S$ before $w_r$, then $w_r$ does not L-footprint $w_r$. If $u$ is in $S$ after $w_r$, then $u$ does not L-footprint any vertex. This is a contradiction with $S$ being a $\gamma_{fr}'$-sequence of $G$. Hence, $|\tilde{S} \cap V_1| \leq |V_1| - 1 = n(G) - 1$. We again infer that $\gamma_{fr}'(G') = |\tilde{S}| = |\tilde{S} \cap V_1| + |\tilde{S} \cap V_2| \leq n(G) + \gamma_{fr}'(G)$. □

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