The associated random walk and martingales in random walks with stationary increments

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Abstract

We extend the notion of the associated random walk and the Wald martingale in random walks where the increments are independent and identically distributed to the more general case of stationary ergodic increments. Examples are given where the increments are Markovian or Gaussian, and an application in queueing is considered.

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1 Introduction and definition

Let $X_1, X_2, \ldots$ be independent identically distributed (i.i.d.) random variables with distribution function (d.f.) $F$ and positive mean. If $S_n = X_1 + X_2 + \cdots + X_n$ for each $n = 0, 1, 2, \ldots$ then the process $\{S_n\}$ is called the random walk with increments $X_1, X_2, \ldots$. Because the increments have positive mean, by the strong law of large numbers the random walk will in the long run drift upwards to infinity.

There may exist $\theta \neq 0$ such that

$$\hat{F}(\theta) := E(e^{-\theta X_1}) = \int_{-\infty}^{\infty} e^{-\theta x} dF(x) = 1,$$

in which case $\theta$ is unique, and necessarily positive because of the upward drift. In this case, if we define a new increment distribution by

$$dF^\theta(x) := e^{-\theta x} dF(x),$$

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then we obtain the associated random walk, which has downward drift. Because of the definition, probabilities in one random walk may easily be expressed in terms of those in the other. For instance, renewal theory in the associated random walk yields the Cramér estimate of the probability of ruin in the original random walk, the parameter $\theta$ determining the rate of exponential decay (Feller (1971), XI.7, XII.6; see also Asmussen (2000)). Note also that since $\hat{F}^*(-\theta) = \int_{-\infty}^{\infty} e^{\theta x} dF^*(x) = 1$ the association is a duality relationship in the sense that if we perform the analogous transformation on the associated random walk with $\theta$ replaced by $-\theta$ then we obtain the original one.

For the same value of $\theta$, it may easily be shown that $V_n := e^{-\theta S_n}$ defines a martingale, known as the Wald martingale. This is also useful in the investigation of hitting probabilities.

The question arises to what extent the concepts of the associated random walk and the Wald martingale may be generalized to the case where the increments are no longer necessarily i.i.d. but merely stationary and ergodic. In such cases, because of the ergodic theorem, the random walk still drifts upwards to infinity and so we might still be interested, for example, in the probability of ruin (hitting a low level). The following is a suggested way forward. It should be noted that, once one goes beyond the ergodic theorem, the general stationary ergodic process behaves quite differently from the independent-increments case. For example, the convergence rate in the ergodic theorem may be arbitrarily slow, however many moments may be finite. For background, see for example Eberlein & Taqqu (1986).

The work below is motivated by that of Lu (1991) on branching processes in random environments (Smith & Wilkinson (1969), Athreya & Karlin (1971)) and by a convergence result given a straightforward proof by the author (Grey (2001)). We obtain generalizations of the associated random walk and the Wald martingale, under certain assumptions. Three applications are considered in Section 2, to the Markov and Gaussian cases, and to a queueing problem. Our work is also relevant to random walks in random environments; see for example Révész (1990), Part III.

To proceed, we need to make the following assumptions. In Section 2 we shall show that these assumptions are satisfied in some important cases of interest.
Assumption 1  There exists \( \theta > 0 \) such that
\[
q := \lim_{n \to \infty} E(e^{-\theta S_n})
\]
exists and is positive and finite.

This assumption is trivially satisfied in the i.i.d. case, with \( \theta \) as identified earlier and \( q = 1 \). It is important to note that since \( S_n \to \infty \) with probability one, there can be at most one value of \( \theta \) satisfying the assumption. This is because, if such \( \theta \) exists, for any positive constant \( K \)
\[
E(e^{-\theta S_n}; S_n \leq -K) \to q
\]
as the remaining contribution to the expectation tends to zero, by the bounded convergence theorem. From here it is easy to see that if \( 0 < \phi < \theta \),
\[
\limsup_{n \to \infty} E(e^{-\phi S_n}; S_n \leq -K) \leq e^{-(\theta-\phi)K}q
\]
whence by similar reasoning
\[
\limsup_{n \to \infty} E(e^{-\phi S_n}) \leq e^{-(\theta-\phi)K}q
\]
and so, since \( K \) is arbitrary, \( E(e^{-\phi S_n}) \to 0 \). Similarly if \( \phi > \theta \) then \( E(e^{-\phi S_n}) \to \infty \).

For our next assumption we extend our sequence of increments to a doubly infinite one \( \ldots, X_{-1}, X_0, X_1, \ldots \), as is always possible with a stationary sequence (Breiman (1968), Proposition 6.5). We also define the more general partial sum
\[
S_{m,n} := \sum_{r=m}^{n} X_r.
\]
Let \( \mathcal{F}_{m,n} \) denote the \( \sigma \)-field generated by \( \{X_r; r = m, \ldots, n\} \).

Assumption 2  For all \( k = 1, 2, \ldots \) and for all \( B \in \mathcal{F}_{-k,k} \),
\[
q(B) := \lim_{m,n \to \infty} E(e^{-\theta S_{m,n}}; B)
\]
events, where \( \theta \) is as defined in Assumption 1.

Again we refer immediately to the i.i.d. case, where the assumption is easily seen to be satisfied, the limiting operation being essentially trivial, and we may write explicitly
\[
q(B) = E(e^{-\theta S_k}; B).
\]
If we now fix $k$ and define for $m, n \geq k$
\[ P_{m,n}^*(B) := \frac{E(e^{-\theta S_{m,n}}; B)}{E(e^{-\theta S_{m,n}})} \]
for $B \in \mathcal{F}_{-k,k}$, then clearly $P_{m,n}^*$ is a probability measure on $\mathcal{F}_{-k,k}$. Moreover if Assumptions 1 and 2 hold then
\[ P_{m,n}^*(B) \to \frac{q(B)}{q} \text{ as } m, n \to \infty \]
for all $B \in \mathcal{F}_{-k,k}$. We make use of the result, given a straightforward proof in Grey (2001), rather simpler than for the more general case of signed measures considered by Halmos (1950, p. 170), and a special case of the Vitali–Hahn–Saks theorem (Dunford & Schwartz (1958), III.7.2–4), that if $\{P_n\}$ is a sequence of probability measures on a space $(\Omega, \mathcal{F})$ such that the limit $P(A) := \lim_{n \to \infty} P_n(A)$ exists for all $A \in \mathcal{F}$, then $P$ is a probability measure on $(\Omega, \mathcal{F})$. It follows that
\[ P^*(B) := \frac{q(B)}{q} \]
defines a probability measure on $\mathcal{F}_{-k,k}$. Since this definition obviously does not depend upon $k$, we have consistency between different values of $k$ and therefore a probability measure defined on $\bigcup_{k=1}^{\infty} \mathcal{F}_{-k,k}$. The Carathéodory extension theorem (Durrett (1996), Appendix A.2) now ensures that $P^*$ can be extended to a probability measure defined on the whole $\sigma$-field $\mathcal{F} := \mathcal{F}_{-\infty,\infty}$.

It is the probability measure $P^*$ which we use to define the distribution of the increments of the associated random walk. Note that, because the original process is stationary and because of the double-ended limiting process involved in the definition of $P^*$, the associated process of increments is also stationary; it would not have been possible to achieve this with a single-ended sequence. Whether the associated process is necessarily ergodic and whether duality occurs is left open here; some further remarks are given in Section 3. Note that ergodicity and duality do occur in the two special cases studied in detail in Section 2. Note also that in the i.i.d. case the associated random walk as defined here coincides with the one which we have already met, since, for example, in the discrete case
\[ P^*(X_1 = x_1, \ldots, X_k = x_k) = e^{-\theta \sum_{i=1}^k x_i} P(X_1 = x_1, \ldots, X_k = x_k) \]
\[ = \prod_{i=1}^k (e^{-\theta x_i} P(X_i = x_i)) \]
As expected.

To construct our martingale, we need to replace Assumption 2 by the following one-sided equivalent, which again is to be read in conjunction with Assumption 1. Write \( \mathcal{F}_k := \mathcal{F}_{1,k} \).

**Assumption 2** For all \( k = 1, 2, \ldots \) and for all \( B \in \mathcal{F}_k \),

\[
\mathcal{r}(B) := \lim_{n \to \infty} \mathbb{E}(e^{-\theta S_n}; B)
\]

exists, where \( \theta \) is as defined in Assumption 1.

If this assumption holds, then by the aforementioned convergence theorem, for each \( k \), \( \mathcal{r} \) is a measure on \( \mathcal{F}_k \) with total mass \( q \). Also it is absolutely continuous with respect to \( P \), since

\[
P(B) = 0 \implies \int_B e^{-\theta S_n} \, dP = 0 \implies \mathcal{r}(B) = 0.
\]

So \( \mathcal{r} \) restricted to \( \mathcal{F}_k \) has a Radon–Nikodym derivative \( V_k \) with respect to \( P \):

\[
\int_B V_k \, dP = \mathcal{r}(B) \quad \text{for all} \quad B \in \mathcal{F}_k
\]

where \( V_k \) is \( \mathcal{F}_k \)-measurable.

But if \( B \in \mathcal{F}_k \), then \( B \in \mathcal{F}_{k+1} \), and therefore

\[
\int_B V_k \, dP = \mathcal{r}(B) = \int_B V_{k+1} \, dP \quad \text{for all} \quad B \in \mathcal{F}_k.
\]

So, by definition of conditional expectation, \( V_k = \mathbb{E}(V_{k+1}|\mathcal{F}_k) \) almost surely, which shows that \( \{V_k\} \) is a martingale with respect to \( \{\mathcal{F}_k\} \).

**Note** In many cases it will be true that \( V_k = \lim_{n \to \infty} \mathbb{E}(e^{-\theta S_n}|\mathcal{F}_k) \) almost surely, but it seems difficult to try to use this equation as a definition of \( V_k \) in general.

In the case of i.i.d. increments it is easy to see that \( V_k = e^{-\theta S_k} \) almost surely, and so our definition generalizes that of the Wald martingale.

### 2 Three examples

In this section we demonstrate the existence of the associated random walk in two important cases of interest: stationary Markov chain increments and stationary Gaussian increments. In both cases, as indeed in the simpler i.i.d. case, certain regularity conditions will be required. The
corresponding martingale is mentioned more briefly in each case. We also consider an application in queueing theory.

2.1 Stationary Markov chain increments

Here we suppose that the increments \( \{X_n\} \) perform a stationary irreducible (and therefore ergodic) aperiodic Markov chain with countable state space \( S \). We shall use labels such as \( i \) and \( j \) to represent the actual sizes of the increments, so that they need not be integer-valued or non-negative; however this will not prevent us from also using them to denote positions in vectors and matrices, since this non-standard notation will not lead to confusion. The associated Markov chain constructed here has been considered in a rather more general context by, for example, Arjas & Speed (1973).

Let the transition matrix of the Markov chain be \( P = (p_{ij}) \) and let its equilibrium distribution be given by the column vector \( \pi = (\pi_i) \).

Note that if \( 1 \) denotes the vector consisting entirely of ones, then \( P \) has Perron–Frobenius eigenvalue 1 with \( \pi^T \) and \( 1 \) as corresponding left and right eigenvectors respectively; also \( \pi^T 1 = 1 \) and \( P^n \to 1 \pi^T \) as \( n \to \infty \).

The regularity conditions which we impose are as follows. For some \( \theta > 0 \) the matrix \( Q \) with elements \( (p_{ij}e^{-\theta j}) \) has Perron–Frobenius eigenvalue 1 and corresponding left and right eigenvectors \( v^T \) (with components \( (v_i) \)) and \( c \) (with components \( (c_i) \)) respectively; also \( v^T c = 1 \) and \( Q^n \to cv^T \) as \( n \to \infty \). This requirement is not especially stringent when the state space \( S \) is finite; the Perron–Frobenius eigenvalue \( \lambda(\theta) \) of \( Q \) for general \( \theta \) behaves rather like the Laplace transform \( \hat{F}(\theta) \) in the i.i.d. case (Lu (1991)).

We firstly show that Assumption 1 holds, with \( \theta \) as defined above.

\[
E(e^{-\theta S_n}) = \sum_{i_0 \in S} \sum_{i_1 \in S} \cdots \sum_{i_n \in S} \pi_{i_0} p_{i_0 i_1} \cdots p_{i_{n-1} i_n} e^{-\theta (i_1 + \cdots + i_n)}
\]

\[
= \pi^T Q^n 1 \to \pi^T cv^T 1 \quad \text{as} \quad n \to \infty.
\]

To check Assumption 2, we shall evaluate

\[
E(e^{-\theta S_{m-n}}; X_{-k} = i_{-k}, \ldots, X_k = i_k)
\]

for given \( k < m, n \) and \( i_{-k}, \ldots, i_k \in S \). By the Markov property, this may be written as the product of the three factors \( e^{-\theta (i_{-k} + \cdots + i_k)} \)

\[
P(X_{-k} = i_{-k}, \ldots, X_k = i_k) \text{ together with } E(e^{-\theta S_{k+1,n}} | X_k = i_k) \text{ and } E(e^{-\theta S_{m-k-1}} | X_{-k} = i_{-k}).
\]

The first factor may be written

\[
e^{-\theta (i_{-k} + \cdots + i_k)} \pi_{i_{-k}} p_{i_{-k}i_{-k+1}} \cdots p_{i_k-1 i_k}.
\]
The second factor may be written
\[
\sum_{i_{k+1}\in S} \cdots \sum_{i_n\in S} e^{-\theta(i_{k+1}+\cdots+i_n)} p_{i_{k+1}} \cdots p_{i_{n-1}i_n}.
\]
Because the reverse Markov chain has transition probabilities \( \pi_j p_{ji}/\pi_i \), the third factor may be written
\[
\sum_{i_{-k-1}\in S} \cdots \sum_{i_{-m}\in S} e^{-\theta(i_{-k-1}+\cdots+i_{-m})} \frac{\pi_{i_{-k-1}}}{\pi_{i_{-k}}} p_{i_{-k-1}i_{-k}} \cdots \frac{\pi_{i_{-m}}}{\pi_{i_{-m+1}}} p_{i_{-m}i_{-m+1}}.
\]
The second factor is seen to be the \( i_k \) component of the vector \( Q^{n-k}1 \) and so converges to \( c_i v^T 1 \) as \( n \to \infty \).

Writing \( \mu_i := \pi_i e^{-\theta i} \) for each \( i \in S \) and letting \( \mu \) be the corresponding vector, after cancellation and rearrangement the third factor is seen to be \( \mu_{i_{-k}}^{-1} \) times the \( i_{-k} \) component of the vector \( \mu^T Q^{n-k} \) and so converges to \( \mu_{i_{-k}}^{-1} v_{i_{-k}} \mu^T c \) as \( m \to \infty \). Note that
\[
\mu^T c = \sum_{j\in S} \pi_j e^{-\theta j} c_j
\]
\[
= \sum_{j\in S} \sum_{i\in S} \pi_i p_{ij} e^{-\theta j} c_j = \sum_{i\in S} \pi_i \sum_{j\in S} p_{ij} e^{-\theta j} c_j = \sum_{i\in S} \pi_i c_i = \pi^T c.
\]

Using this fact and putting all the preceding results together, after cancellation we see that \( P^*(X_{-k} = i_{-k}, \ldots, X_k = i_k) \) exists and is equal to
\[
e^{-\theta(i_{-k+1}+\cdots+i_k)} p_{i_{-k+1}i_{-k}} \cdots p_{i_{-1}i_{-1}} c_i v_{i_{-k}}.
\]
Writing \( p^*_{ij} := p_{ij} e^{-\theta j} c_j / \pi_i \) and \( \pi^*_i := c_i v_i \) for each \( i, j \in S \), the above may also be written
\[
P^*(X_{-k} = i_{-k}, \ldots, X_k = i_k) = \pi^*_i p^*_{i_{-k+1}i_{-k}} \cdots p^*_{i_{-1}i_k}.
\]

It is a routine matter to check that the numbers \( \{p^*_{ij}\} \) form the transition probabilities of a Markov chain and that \( \{\pi^*_i\} \) is an equilibrium distribution for it. We have thus established that the associated random walk exists and its increments perform a stationary Markov chain, which is also obviously irreducible and aperiodic like the original. Duality is left as an exercise.

The martingale may be constructed similarly. Letting \( B = \{X_1 = i_1, \ldots, X_k = i_k\} \) it may be calculated that
\[
E(e^{-\theta S_B}; B) \to P(B) e^{-\theta(i_1+\cdots+i_k)} c_i v^T 1 \quad \text{as} \quad n \to \infty
\]
and so Assumption 2* holds; moreover, since \( B \) is an atom of the \( \sigma \)-field
it follows that \( V_k = e^{-\theta S_k}c_X^T v^T 1 \). We may take \( v^T 1 = 1 \) since \( v \) and \( c \) have so far only been scaled relative to each other. This gives the martingale \( V_k = c_X e^{-\theta S_k} \) which has also been used by Lu (1991).

### 2.2 Stationary Gaussian increments

Now let \( \{X_n\} \) be a stationary Gaussian process in which each \( X_n \) has normal distribution with \( \mu > 0 \) and variance \( \sigma^2 > 0 \). For each \( r = 1, 2, \ldots \) let \( \rho_r \) be the correlation coefficient between \( X_n \) and \( X_{n+r} \) for any \( n \). These parameters completely determine the behaviour of the process, since the joint distribution of any finite collection of the \( X_n \) is multivariate normal.

The regularity condition we need here is that

\[
\sum_{r=1}^{\infty} r |\rho_r| < \infty.
\]

This is an asymptotic independence property more than sufficient for ergodicity, and one which is easily satisfied by commonly studied processes such as autoregressive and moving average processes.

Under this condition, let \( R := \sum_{r=1}^{\infty} \rho_r \) and let \( S := \sum_{r=1}^{\infty} r \rho_r \); these will both be finite. Below we shall see that \( R \geq -\frac{1}{2} \) necessarily, and that we need to exclude the extreme case \( R = -\frac{1}{2} \).

We firstly find \( \theta \) such that Assumption 1 is satisfied. Since \( S_n = X_1 + \cdots + X_n \) has a normal distribution with mean \( n \mu \) and variance \( \sigma^2 [n + 2 \sum_{r=1}^{n-1} (n-r) \rho_r] \), by the standard formula for the Laplace transform of the normal distribution we have that

\[
E(e^{-\theta S_n}) = \exp \left\{ -n \mu \theta + \frac{1}{2} \sigma^2 [n + 2 \sum_{r=1}^{n-1} (n-r) \rho_r] \theta^2 \right\}.
\]

Under our regularity condition

\[
\sum_{r=1}^{n-1} (n-r) \rho_r = nR - S + o(1) \quad \text{as} \quad n \to \infty
\]

and so in particular

\[
\text{var } S_n = \sigma^2 (n[1 + 2R] - 2S) + o(1) \quad \text{as} \quad n \to \infty,
\]

whence \( 1 + 2R \geq 0 \). It is possible to construct examples with \( 1 + 2R = 0 \) (such as \( X_n := \mu + Z_n - Z_{n-1} \) where \( \{Z_n\} \) are i.i.d. \( N(0, \frac{1}{2} \sigma^2) \) random
variables) but if we exclude this rather extreme case then we see that convergence of \( E(e^{-\theta S_n}) \) to a positive limit will occur if and only if

\[
-\mu \theta + \frac{1}{2} \sigma^2 [1 + 2R] \theta^2 = 0.
\]

This yields

\[
\theta = \frac{2\mu}{\sigma^2 [1 + 2R]}
\]

and then it is easy to compute that for this value of \( \theta \)

\[
E(e^{-\theta S_n}) \to \exp \left\{ -\frac{4\mu^2 S}{\sigma^2 [1 + 2R]^2} \right\} \quad \text{as} \quad n \to \infty.
\]

We turn to Assumption 2. Fix \( k \), and take \( \theta \) as just identified. For \( m, n > k \) it is evidently relevant to look at the distribution of \( Y := S_{-m,-k+1} + S_{k+1,n} \) conditional on \( X_{-k} = x_{-k}, \ldots, X_k = x_k \), or \( X = x \) say. If the unconditional distribution of \( Y \) is denoted by \( \mathcal{N}(\nu, \tau^2) \), the vector of covariances of \( Y \) with the components of \( X \) is denoted by \( v \), and the mean and covariance matrix of \( X \) are denoted by \( \mu \) and \( \Sigma \) respectively, then by multivariate normal theory (Mardia, Kent & Bibby (1979), Theorem 3.2.4), the conditional distribution is \( \mathcal{N}(\nu + v^T \Sigma^{-1} (x - \mu), \tau^2 - v^T \Sigma^{-1} v) \). (For ease of notation, we suppress the dependence of \( Y, \nu, \tau^2 \) and \( v \) on \( m \) and \( n \).)

A typical component of \( v \) is of the form

\[
\sigma^2 \left( \sum_{r=i+k+1}^{i+m} \rho_r + \sum_{r=k+1-i}^{n-i} \rho_r \right)
\]

for some \( i \), and so \( v \) converges to a finite limit as \( m, n \to \infty \). Also \( \nu = (m + n - 2k) \mu \). Then, since

\[
\text{var } S_{-m,n} = \text{var } (1^T X + Y) = 1^T \Sigma 1 + 21^T v + \tau^2,
\]

we can use the estimate of \( \text{var } S_n \) obtained in checking Assumption 1 to deduce that

\[
\tau^2 - \sigma^2 [1 + 2R] (m + n + 1)
\]

converges to a finite limit as \( m, n \to \infty \). Putting these results together we see that

\[
E(\exp(-\theta Y) | X = x) = \exp \left\{ -[\nu + v^T \Sigma^{-1} (x - \mu)] \theta + \frac{1}{2} [\tau^2 - v^T \Sigma^{-1} v] \theta^2 \right\}
\]

converges to a positive limit as \( m, n \to \infty \), since because of the value of \( \theta \) the difference between the large terms in \( \nu \) and \( \tau^2 \) converges to a finite limit, and the other terms also converge to finite limits. We may denote
the limit in the above by \( \exp(\alpha^T x + \beta) \) for some constants \( \alpha \) and \( \beta \). It is then easy to see that for any \( B \in \mathcal{F}_{-k,k} \),

\[
E(e^{-\theta S_{m,n}}; B) \to E(\exp\{-\theta^T X + \alpha^T X + \beta\}; B) \quad \text{as} \quad m, n \to \infty,
\]

and we have established Assumption 2. It is not hard in this case to see that the associated random walk has the same covariance structure as the original, but downward drift \(-\mu\).

Calculations similar to the above may be used to check that Assumption 2\* holds in this case also, and that the associated martingale is of the form \( V_k = \exp(\gamma^T X + \delta) \) for some \( \gamma, \delta \), where now \( X = (X_1, \ldots, X_k)^T \).

### 2.3 A queueing application

Suppose we have a G/GI/1 queue in which the inter-arrival times \( \{T_n\} \) form a stationary ergodic sequence and the independent service times \( \{U_n\} \) are i.i.d. with \( E T_n > EU_n > 0 \). Then the waiting times \( \{W_n\} \) satisfy

\[
W_{n+1} = (W_n + U_n - T_n)^+
\]

and it follows by a standard argument, dating back to Lindley (1952) in the case of i.i.d. inter-arrival times and exploited, among others, by Kingman (1964), that \( W_n \) has an equilibrium distribution which is the same as the distribution of minus the all-time minimum of an unrestricted random walk started at zero in state zero, with increments \( X_n := T_n - U_n \).

The tail of this distribution is therefore intimately related to the probability of ruin in this random walk, and, in particular, the parameter \( \theta \), if it exists, has an important part to play.

The simplest example is the M/M/1 queue where the \( T_n \) are independent exponential with parameter \( \lambda \) and the \( U_n \) are exponential with parameter \( \mu \), where \( \mu > \lambda > 0 \). In this case the \( X_n \) are i.i.d. and

\[
E(\exp(-\theta X_n)) = \lambda \mu / (\lambda + \theta)(\mu - \theta),
\]

which is easily seen to be equal to one when \( \theta = \mu - \lambda \). So the key parameter \( \theta \) depends upon both the arrival rate and the service rate in a simple and obvious way.

As another example, suppose that the \( U_n \) have some arbitrary distribution with Laplace transform \( \phi(\theta) = E(\exp(-\theta U_n)) \), and that there is a regular appointments system such that customer \( n \) arrives at clock time \( \lambda^{-1} n + \epsilon_n \), where \( \{\epsilon_n\} \) is a sequence of i.i.d. errors with Laplace transform \( \psi(\theta) = E(\exp(-\theta \epsilon_n)) \). In this case it is possible to compute

\[
E(\exp(-\theta S_n)) = [\phi(-\theta)]^n \exp(-\lambda^{-1} n \theta) \psi(\theta) \psi(-\theta)
\]
and so the key parameter $\theta$ satisfies the equation

$$\phi(-\theta) \exp(-\lambda^{-1}\theta) = 1$$

which does not involve the distribution of the $\epsilon_n$. By considering the special case $\phi(\theta) = \mu/(\mu + \theta)$, so that the $U_n$ are exponentially distributed, and the mean inter-arrival time and mean service time are $\lambda^{-1}$ and $\mu^{-1}$ respectively as in the previous $M/M/1$ example, it is possible to compare an appointments system with random (Poisson) arrivals. For the value $\theta = \mu - \lambda$ found in the case of random arrivals, we may compute $\phi(-\theta) \exp(-\lambda^{-1}\theta) = (\mu/\lambda) \exp(1 - (\mu/\lambda)) < 1$, and so the actual value of the key parameter $\theta$ for the appointments system must be larger than for random arrivals. This suggests a thinner tail for the equilibrium waiting time distribution, and a more efficient system.

3 Some remarks on duality and asymptotic independence

If we wish for duality to occur, then, replacing $\theta$ by $-\theta$ and denoting expectation with respect to $P^*$ by $E^*$, we require for $B \in \mathcal{F}_{-k,k}$ that

$$\frac{E^*(e^{\theta S_{-m,n}}; B)}{E^*(e^{\theta S_{-m,n}})} \to P(B) \quad \text{as} \quad m, n \to \infty.$$ 

Now, denoting $E(e^{-\theta S_{-r,s}})$ by $q_{r,s}$ and noting that $P^*_{r,s}$ has Radon–Nikodým derivative $q_{r,s}^{-1}e^{-\theta S_{-r,s}}$ with respect to $P$, we have that

$$E^*(e^{\theta S_{-m,n}}; B) = \int_B e^{\theta S_{-m,n}} dP^*$$

$$= \lim_{r,s \to \infty} \int_B e^{\theta S_{-m,n}} dP^*_{r,s}$$

$$= \lim_{r,s \to \infty} \int_B e^{\theta S_{-m,n}} q_{r,s}^{-1}e^{-\theta S_{-r,s}} dP$$

$$= q_{r,s}^{-1} \lim_{r,s \to \infty} \int_B e^{-\theta S_{-r,s}} e^{-\theta S_{n+1,s}} dP.$$ 

For the required convergence to occur, it seems therefore that for large $m$ and $n$ there must be approximate independence between $S_{-r,-m-1}$, $S_{n+1,s}$ and the event $B$, so that we can say that the above integral is approximately $P(B)E(e^{-\theta S_{-r,-m-1}})E(e^{-\theta S_{n+1,s}})$ which converges to
$P(B)q^2$ as $r, s \to \infty$. Hence under these circumstances

$$\frac{E^*(e^{g_{S-m,n}; B})}{q^{-1}P(B)q^2} \sim \frac{q^{-1}P(B)q^2}{q^{-1}q^2} \to P(B) \quad \text{as} \quad m, n \to \infty.$$ 

The asymptotic independence is a kind of mixing condition which is already stronger than ergodicity, suggesting that the latter is not the most appropriate property to be considering in this context. See Bradley (2005) on mixing conditions.

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