Accelerator modes and their effect in the diffusion properties in the kicked rotator
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Abstract: We highlight a few recent results on the effect of the diffusion process in deterministic area preserving maps with noncompact phase space, namely the standard map. In more detail, we focus on the anomalous diffusion arising due to the accelerator modes, i.e. resonant-like features of the phase space which are transported in a super-diffusive (ballistic) manner. Their presence affects also the trajectories lying in the immediate neighborhood resulting in anomalous (non-Gaussian) diffusion. We aim to shed some light on these special properties of the phase space by utilizing the diffusion exponent (rate of diffusion) and the momentum distribution in terms of the Lévy stable distributions, with the goal to reach an understanding of the global behaviour. To this end we consider a rather large interval for the kick parameter ($0 \leq K \leq 70$) of the standard map where accelerator modes of different periodicity exist.

1. INTRODUCTION

In this paper we present a small review of our recent findings on the diffusion properties in area preserving maps, exemplified by the standard map [1]. This work has been linked to the extensive studies in [2,3] of the quantum kicked rotator (see e.g. [4]), where the classical diffusion process also plays a significant role in the quantum semiclassical level, where the (exponential) quantum (or dynamical) localization is observed and studied. We here present a few results on how the presence of accelerator modes affects the dynamics of the classical phase space, and in more detail its diffusion rates and transport. We pursue this goal by studying the diffusion process of ensembles of initial conditions for a rather large value-interval of the control (kick) parameter ($0 \leq K \leq 70$). Within this interval the phase space of the model shows a very rich variety of dynamics, namely from the almost entirely regular phase space to gradually more and more chaotic dynamics until there are practically no stable regions anymore. In this picture there are some resonance features, the so-called accelerator modes, that may affect the diffusion of trajectories rather drastically.

Here, we are trying to shed some light on these effects by calculating the diffusion exponent $\mu$, the corresponding diffusion constant $D_\mu$ and the parameters of the relevant Lévy stable distribution in the case of non-Gaussian (anomalous) diffusion for different $K$ parameter values. Comparing to previous studies, here and mainly in [5] we do not restrict ourselves to local effects and isolated accelerator modes but we are interested in a rather more global point of view. Namely, on how these features may influence the transport in their neighborhood. That is the case when one, for example, considers mixed ensembles of initial conditions, e.g. ensembles that contain accelerator modes together with chaotic trajectories around them.

Let us also stress here that the time scales we are interested in in this particular study are relatively short and we are not interested at this point on the asymptotic diffusion rates. This has to do with our initial motivation in understanding the quantum effects associated with such features, namely the quantum localization properties (see [2,3] for more details).

The paper is structured as follows: In Sec. 2 we give brief description of the model (standard map), while in Sec. 3 we present our main results on the dynamical effect of accelerator modes on the diffusion exponent and the stable Lévy distribution parameters. Finally, in Sec. 4 we discuss the main findings of this work.

2. THE MODEL

For the purposes of our study we use a rather well-known time-dependent system, the paradigm of classical chaos in Hamiltonian (nondissipative) systems, namely the kicked rotator [1]. Its Hamiltonian function is given by

$$H = \frac{p^2}{2I} + V_0 \delta_T(t) \cos \theta. \quad (1)$$

Here $p$ is the (angular) momentum, $I$ the moment of inertia, $V_0$ is the strength of the periodic kicking, $\theta$ is the (canonically conjugate, rotation) angle, and $\delta_T(t)$ is the periodic Dirac delta function with period $T$. Taking into account that between the kicks the rotation is free, the Hamiltonian equations of motion can be easily integrated, and thus the dynamics can be reduced to the standard mapping, or so-called Chirikov-Taylor mapping, given by

$$\begin{align*}
p_{n+1} &= p_n + V_0 \sin \theta_{n+1}, \\
\theta_{n+1} &= \theta_n + \frac{T}{I} p_n.
\end{align*} \quad (2)$$

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Note that the quantities \((\theta_n, P_n)\) refer to their values just immediately after the \(n\)-th kick. Then, by introducing new dimensionless momentum \(P_n = p_n T/I\), we get

\[
\begin{align*}
P_{n+1} &= P_n + K \sin \theta_{n+1}, \\
\theta_{n+1} &= \theta_n + P_n,
\end{align*}
\]

where the system is now governed by a single classical dimensionless kick parameter \(K = V_0 T/I\), and the mapping is area preserving.

The generalized diffusion process of the standard map [Eq. (3)] is then defined as

\[
\langle (\Delta P)^2 \rangle = D_\mu(K)n^\mu,
\]

where \(n\) is the number of iterations (kicks), and the exponent \(\mu\) is in the interval \([0, 2]\), and all variables \(P, \theta\) and \(K\) are dimensionless. By \(D_\mu(K)\) we define the generalized classical diffusion constant and then according to the \(\mu\)-value one may distinguish further the different types (rates) of diffusion. In more detail: in the case \(\mu = 1\) we have the normal diffusion, and \(D_1(K)\) is then the normal diffusion constant, while in the case of anomalous diffusion we observe subdiffusion when \(0 < \mu < 1\) or superdiffusion if \(1 < \mu \leq 2\). In the case \(\mu = 2\) we have the ballistic transport which is associated strictly - as we found - with the presence of accelerator modes.

Whenever the diffusion is normal (\(\mu = 1\)) the theoretical value of \(D_1(K)\) is found to be [4]

\[
D_1(K) = \begin{cases} 
\frac{2}{3} K^2 [1 - 2J_2(K)(1 - J_2(K))], & \text{if } K \geq 4.5 \\
0.15(K - K_{cr})^2, & \text{if } K_{cr} < K \leq 4.5
\end{cases}
\]

where \(K_{cr} \approx 0.9716\) and \(J_2(K)\) is the Bessel function (the higher terms of order \(K^{-2}\) are here neglected).

The dependence of the diffusion constant for the growth of the variance of the momentum on \(K\) is not trivial and fails to be in a good agreement around the period 1 accelerator mode intervals

\[
(2\pi n) \leq K \leq \sqrt{(2\pi n)^2 + 16},
\]

where \(n\) is any positive integer (see [4, 5]). In more detail, in these intervals for the accelerator modes \(n = 1\) there are two stable fixed points located at \(p = 0, \theta = \pi - \theta_0\) and \(p = 0, \theta = \pi + \theta_0\), where \(\theta_0 = \arcsin(2\pi/K)\). There are also two unstable fixed points at \(p = 0, \theta = \theta_0\) and \(p = 0, \theta = 2\pi - \theta_0\). Furthermore, as the diffusion might even be anomalous, we have recalculated the effective diffusion constant \(D_{eff} = \langle (\Delta P)^2 \rangle / n\) numerically, which in general is not equal to the \(D_\mu\) defined in Eq. (4) (more detail about this process can be found in [5]).

3. RESULTS

We have calculated the diffusion exponent \(\mu\) as a function of \(K\). The final number of iterations is here \(n = 5000\) and the total number of initial conditions is \((\approx 100000)\) spread uniformly on a \(314 \times 314\) grid on the plane \((\theta, P) = (0, 2\pi)\) (Fig. 1). The diffusion exponent \(\mu\) was calculated as described in Sec. II [Eq. (4)] in [5], i.e.

![Fig. 1 (Color online) The diffusion exponent \(\mu\) vs. \(K\) for an ensemble of \((\approx 100000)\) \((314 \times 314)\) initial conditions uniformly spread on the plane \((\theta, P) = (0, 2\pi)\). The intervals of stable accelerator modes of period 1 are depicted on the black horizontal line \(\mu = 0.9\) while all the intervals of \(K\) where the diffusion is normal are found to have a diffusion exponent \(\mu \approx 1\). Note that the larger relatively peaks (for \(K > 2\pi\) - black full circles) correspond accelerator modes (mainly of period 1). Those peaks of smaller amplitude and for \(K < 2\pi\) (enlarged in the inset) as well as those for \(2\pi < K < 4\pi\) marked with empty circle are related to accelerator modes of higher period. There are few \(K\) values marked with the symbol (\(\times\)) associated with normal diffusion. All these cases where thoroughly studied in [5] where the interested reader may find more details. The power law describing the decay of the exponent \(\mu\) of the main peaks’ amplitude due to accelerator modes of period 1 (see text for more details) are shown by the blue dotted line.](image-url)

by the slopes of the lines of the variance of the momentum \(P\) as a function of iterations. The black horizontal line \(\mu = 0.9\) indicates the intervals of stable accelerator modes of period 1 [Eq. (6)]. All intervals of \(K\) associated with normal diffusion processes have \(\mu \approx 1\). The large peaks, marked with full black circles, that appear for \(K > 2\pi\) are due to the anomalous diffusion raised by the accelerator modes of period 1. On the other hand, there are a few relatively smaller peaks for \(K < 2\pi\) (see the inset panel in Fig. 1), whose origin is accelerator modes of higher period, and also for \(2\pi < K < 4\pi\), both these sets are marked with empty circles. With the symbol (\(\times\)) we mark few typical examples, close to those peaks, for which the diffusion is normal. All these cases where thoroughly studied in [5] where the interested reader may find more details.

In [5] it was found that the size of the large peaks for \(K > 2\pi\) (full black circles in Fig. 1 associated with regimes with accelerator modes of period 1) decays monotonically with a power law

\[
f(x) = c x^d,
\]

where \(c = 2.41645\) and \(d = -0.195896\) [blue dotted
line in Fig. 1, with asymptotic standard error ±0.04294 (1.7777%) and ±0.00537 (2.741%) respectively. Hence, this shows that their effect decreases significantly for large $K$-values, becoming probably entirely insignificant for $K > 70$. Moreover, the size of the successive accelerator modes of period 1 intervals decays with a power law defined simply and analytically by the Eq. (6).

3.1. The Lévy stable distribution

Let us here give a brief description of the Lévy stable distribution. For a random variable $X$ with distribution function $F(x)$, the characteristic function is defined by $\phi(u) = E \exp(iux) = \int_{-\infty}^{\infty} \exp(iux) dF(x)$. Then, a random variable $X$ is stable if and only if $X = \delta + aZ + b$, with $a > 0$, $b \in \mathbb{R}$ and $Z$ is a random variable with characteristic function

$$E \exp(iuZ) = \begin{cases} e^{-|u|^\alpha |1-i\beta \tan(\pi\alpha/2)|}, & \alpha \neq 1 \\ e^{-|u|^\alpha |1+i\beta \tan(\pi\alpha/2)| \log |u|}, & \alpha = 1 \end{cases},$$

(8)

where $0 < \alpha \leq 2$ and $-1 \leq \beta \leq 1$ (the symbol $\delta$ indicates that both expressions have the same probability law). We then adopt the parametrization $k = 0, S(\alpha, \beta, \gamma, \delta; 0)$ for which the random variable $X$ given by

$$X = \begin{cases} \gamma(Z - \beta \tan(\pi\alpha/2)) + \delta, & \alpha \neq 1 \\ \gamma Z + \delta, & \alpha = 1 \end{cases},$$

(9)

has characteristic function

$$S(\alpha, \beta, \gamma, \delta; 0) \equiv E \exp(iuX) = \begin{cases} e^{i\alpha\delta - \gamma u \beta |1+i\beta(1+|\beta|^2)^{\gamma}| \log(u) \tan(\pi\alpha/2)}, & \alpha \neq 1 \\ e^{i\alpha\delta - \gamma u \beta |1+i\beta(1+|\beta|^2)^{\gamma}| \log(u) \tan(\pi\alpha/2)}, & \alpha = 1 \end{cases},$$

(10)

where $Z = Z(\alpha, \beta)$ is defined as described in Eq. (8), $\alpha \in (0, 2)$ is the index of stability or characteristic exponent, $\beta \in [-1, 1]$ the skewness parameter, $\gamma > 0$ the scale parameter and $\delta \in \mathbb{R}$ location parameter. Two important special cases are the Gaussian distribution with $\alpha = 2$ and the Cauchy-Lorentz with $\alpha = 1$ which are the only ones with an explicit closed formula. More details on the Lévy stable distribution can be found in [6].

Let us now see how one may utilize the Lévy stable distribution in order to capture the effect of the anomalous diffusion due to the accelerator modes. In Fig. 2 we depict the stable Lévy distribution with parameters $(\alpha, \beta, \gamma, \delta) \approx (1.59, 0.164, 3.7, 83.63)$ for $K = 6.5$. In the inset color-map we show the exact grid of mixed initial conditions (i.e. covering regular and chaotic regions) used $(314 \times 314 \approx 100000)$. This particular cell contains: (i) trajectories that are transported ballistically by the unstable accelerator mode around $(\theta_0, P) = (1.31179, 0)$ $(\times)$ and (ii) more evidently around the stable one at $(\theta_1, P) = (\pi - \theta_0, 0)$ $(\ast)$ together with (iii) chaotic ones in their vicinity after $n = 5000$. The fit for the Lévy distribution was performed by excluding the last peak in the positive large momenta due to the ballistic transport by the accelerator mode. Generally speaking, one should expect two peaks in the distribution of the diffusive variable $P$ (in the positive and negative direction) depending on the choice of the ensemble of initial conditions. The color-plot inset panel shows the diffusion exponent $\mu$ for a grid of cells on the subspace $(\theta, P) \in [\pi/3, 2\pi/3] \times [0, \pi/3]$ of the phase space. The yellow (light gray in b/w) color areas are related to ballistic motion due to the two accelerator modes pointed by the symbols $(\times)$ and $(\ast)$ respectively. Those initial conditions being inside “pure” chaotic regime diffuse normally with $\mu \approx 1$. In the edges of the accelerator mode areas where $\mu \approx 0.8$ one may find sticky subdiffusive transport (“belt”-like darker zone). The ballistic transport on the cylindrical phase space of the two above mentioned accelerator modes positioned initially at $(\theta_{0,1}, 0)$ and boosted by $\delta P = 2\pi$ at every successive kick is shown in the small inset panel.
the ballistic transport takes place on the cylindrical phase space of the two above-mentioned accelerator modes positioned initially at \((\theta_0, 0)\) and boosted by \(\delta P = 2\pi\) at every (only four here) successive kick. Furthermore, let us point out that the three yellow (or light gray in black and white) areas, surrounding the stable accelerator mode (+), are ballistically transported as well due to accelerator mode of higher period.

Color maps like the one presented in the inset of Fig. 2 can be used for the (numerical) detection of accelerator modes in the phase space. In Fig. 3 we show how one can distinguish efficiently the regions with accelerator modes from the islands of stability with the use of diffusion exponent \(\mu\) color-plots. Let us first consider a grid of \(500 \times 500\) cells (on the entire phase space \((\theta, P) \in [0, 2\pi] \times [0, 2\pi]\) calculated after \(n = 5000\) iterations. Then, for each ensemble of each cell separately, we calculate numerically the diffusion exponent \(\mu\) and append a color to this cell (color bar of Fig. 3(a)) that characterizes the different kind of diffusion of this small area. Depending on the location of each ensemble and the color assigned to the cell, one may expect to find: (i) normal diffusion \((\mu = 1)\) inside chaotic regimes without the presence of accelerator modes, (ii) subdiffusion \((0 < \mu < 1)\) inside islands of stability, (iii) superdiffusion \((1 < \mu < 2)\) inside chaotic regimes with the presence of accelerator modes in the phase space and (iv) ballistic transport \((\mu \approx 2)\) inside and in the very close vicinity of accelerator modes.

Note that the stable regions around \((\pi/2, 0)\), \((3\pi/2, 0)\), \((\pi/2, \pi)\), \((3\pi/2, \pi)\) and \((\pi/2, 2\pi)\), \((3\pi/2, 2\pi)\) are correctly identified as islands of stability since their diffusion exponent \(\mu\) is smaller than 1. The remaining tiny yellow areas correspond to stable higher period accelerator modes with \(\mu \approx 2\) (see [5] for more on the relation between the diffusion process and the stability of the different regimes of the phase space). Let us stress that following the above procedure, one manages to locate the accelerator modes not only of period 1 but also of higher periods 2, 3, 4,... The latter ones are in general hard to be calculated analytically. Finally, in Fig. 3(b) we show two examples (marked with a square and a circle in panel a) which present a rather different diffusion processes, i.e., for a trajectory oscillating between islands of stability (squares) and for one transported ballistically (with \(P < 0\)) by the effect of an accelerator mode (circles). Let us notice here that the period of both is 4 (when projected on the \((\theta, P) \in [0, 2\pi]\), as it can be seen when we iterate them for a few steps.

4. SUMMARY

In this short paper we tried to give a small flavor of the accelerator modes’ impact in the diffusion process in the phase space of a system. These effects are not limited only to the standard map but also to other models that can have such dynamical features. One of the main results is the link between the superdiffusive transport and the existence of the accelerator modes for certain control \(K\) parameter interval values. The interval size as well as the intensity (i.e. the deviation from normal diffusion expressed by the deviation of the diffusion exponent \(\mu\) from \(\mu = 1\)) is found to decrease as a power law, as \(K\) increases (see Fig. 1). The effect of these modes seems to be strong even when their relative size in the phase space is small and even when one considers averages of initial conditions taken over the whole phase space. In the case where \(K > 2\pi\) the accelerator modes are in general of period one, while for \(K < 2\pi\) they may be of higher period, and their influence on \(\mu\) decays with \(K\) much faster than for period one accelerator modes.

Looking into these resonances of the phase space even further, we tried to use an adequate probability distribution function of the momenta \(P\) that would allow us to distinguish between normal and anomalous diffusion for different ensembles of initial conditions. It turns out that the stable Lévy distribution and its \(\alpha\) parameter can serve for this goal quite well, despite the fact that this distribution is mainly introduced and broadly used to capture random (Lévy) walks in the literature. Here, we showed...
only one exemplary case for $K = 6.5$ (see Fig. 2) where the $\alpha$-Lévy parameter was found to be $\approx 1.59$ while the $\mu$ diffusion exponent lies on a range of values depending on the location of the sub-cell of initial conditions (inset of Fig. 2).

Finally, by considering a fine grid in the phase space and then taking ensembles of many initial conditions for the averages inside the small cells, we managed to produce diffusion “chart-map” encoding their different (numerically estimated) $\mu$ value as shown for example in Fig. 3. With this kind of plots one may detect and separate accelerator modes ($\mu \approx 2$) of different periods from even small islands of stability ($0 < \mu < 1$), or chaotic regions ($\mu \approx 1$), or “sticky” intermediate regimes.

We should mention that another important method to study the structure of the phase portrait rests upon calculating the Lyapunov exponents in order to distinguish the regular from the chaotic regions in the phase space, and also allows us to quantify the degree of chaos by the value of the Lyapunov exponents [7] at the sufficiently densely and uniformly chosen initial conditions. However, the calculation of the Lyapunov exponents can be quite laborious task, while the so-called GALI method, a generalization of the original SALI method, introduced and developed by Skokos et al [8-11], and recently reviewed by Skokos and Manos in reference [12], is an extremely efficient method, which distinguishes quickly the regular from chaotic orbits, although it does not quantify the degree of chaos to such an extent as the Lyapunov exponents. This method has been used in our main work [5] and it was found that there is excellent agreement between the $\mu$-landscape and the GALI-landscape in the phase space of the standard map. We believe that our approach and the method can be applied in other smooth Hamiltonian dynamical systems and mappings.

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