Multifractal cross wavelet analysis

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Complex systems are composed of mutually interacting components and the output values of these components are usually long-range cross-correlated. We propose a method to characterize the joint multifractal nature of such long-range cross correlations based on wavelet analysis, termed multifractal cross wavelet analysis (MFXWT). We assess the performance of the MFXWT method by performing extensive numerical experiments on the dual binomial measures with multifractal cross correlations and the bivariate fractional Brownian motions (bFBMs) with monofractal cross correlations. For binomial multifractal measures, the empirical joint multifractality of MFXWT is found to be in approximate agreement with the theoretical formula. For bFBMs, MFXWT may provide spurious multifractality because of the wide spanning range of the multifractal spectrum. We also apply the MFXWT method to stock market indexes and uncover intriguing joint multifractal nature in pairs of index returns and volatilities.

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I. INTRODUCTION

A series of multifractal cross-correlation analysis methods have been developed in the past few years, which have been applied to diverse fields to unveiling possible multifractal long-range cross correlations between two time series. An early method called joint multifractal analysis was invented in 1990 to study the relationship between the dissipation rates of kinetic energy and passive scalar fluctuations in fully developed turbulence, which handles the joint partition function of two multifractal measures [1]. This method can also be called the multifractal cross-correlation analysis based on partition function approach (MFXPF) [2]. A special case of MFXPF was independently reinvented in 2012 to study financial volatility time series, which was termed as the multifractal cross-correlation analysis based on statistical moments (MFS-XA) [3]. In 2015, the main properties of the joint multifractal nature of binomial measures were derived and numerically validated [2].

Another multifractal cross-correlation analysis is the multifractal height cross-correlation analysis (MF-HXA) [4], which is a bivariate generalization of the height-height correlation analysis [5]. The MF-HXA method also has its origin in turbulence and can be regarded as an extension of the cross-correlation analysis of the structure functions of temperature and velocity dissipation fields in a heated turbulent jet [6]. Hence, we can also name it the multifractal cross-correlation analysis based on structure function approach (MFXSF). Other multifractal cross-correlation analysis methods include the multifractal detrended cross-correlation analysis based on the detrended fluctuation analysis (MFXDFA) [7], which is a multifractal version of the detrended cross-correlation analysis (DCCA) [8], the multifractal detrended cross-correlation analysis based on the detrending moving-average analysis (MFXDMA) [9] based on the multifractal detrending moving-average analysis (MF-DMA) [10] and the detrending moving-average analysis (DMA) [11,12], the multifractal cross-correlation analysis (MFCCA) [13,20], and the multifractal detrended partial correlation analysis (MFDPXA) [21].

Wavelet transform has long been applied to study fractals and multifractals [22,23] and a partition function approach based on wavelet transform has been proposed [24]. In this work, we generalize the multifractal wavelet analysis to the bivariate case and propose a new joint multifractal analysis based on the wavelet transform of two time series, which is a multifractal generalization of the cross wavelet transform [25,26]. We can thus also call it the multifractal cross wavelet analysis (MFXWT). Similar to the MFXPF method, we introduce two orders in the MFXWT method. We test the validity of the method by conducting numerical experiments of two mathematical models with explicit analytical results. At last, we apply the method to handle empirical time series.

II. METHODS

Following Refs. [28,29], the wavelet transform of a given time series \( x(t) \) is defined as

\[
w(s, i) = \frac{1}{s} \sum_{i=1}^{n} x(t) \psi[(t - i)/s], \quad i = 1, \ldots, n. \quad (1)
\]
where the function $\psi(x)$ is the wavelet kernel shifted by $i$, $s$ is the scale, and $n$ is the length of $x(t)$. The wavelet transform can be used to decompose the signals in the time-scale plane. The resulting wavelet coefficients can serve as an indicator for detecting the singular behavior of signals, if the wavelet kernel is required to satisfy $\int x^m \psi(s) dx = 0$, which allows us to approximate the signal trends by polynomials up to the $m$-th order.

Usually, a good choice of $\psi(x)$ is the $m$-th derivative of a Gaussian, $\psi^m(x) = d^m e^{-x^2/2}/dx^m$. In this paper, we use $m = 2$, known as the Mexican hat.

Inspired by the approaches of wavelet-based scaling (or multimulti-scaling) estimator [31,32] and the methods of cross correlation (or multifractal) analysis [2,4,7,8], we propose a new method for detecting the multifractal cross correlations in a pair of series $x(t)$ and $y(t)$ based on wavelet analysis, namely multifractal cross wavelet analysis with two moment orders (MFXWT $(p,q)$).

Firstly, we perform wavelet transform of the two time series and obtain the wavelet coefficients $w_x(s, i)$ and $w_y(s, i)$. Then we define the joint partition function with moment $p$ and $q$ based on the obtained wavelet coefficients,

$$\chi_{xy}(p, q, s) = \sum_{i=1}^{n} |w_x(s, i)|^{p/2} |w_y(s, i)|^{q/2}. \quad (2)$$

There will be a part of wavelet coefficients very close to 0, which presents a problem that the partition function diverges for $p < 0$ or $q < 0$. When $w_x = w_y$ and $p = q$, we recover the traditional partition function based on wavelet analysis. One can also expect the following scaling behavior if the underlying processes are joint multifractal,

$$\chi_{xy}(p, q, s) \sim s^{T_{xy}(p,q)}. \quad (3)$$

where $T_{xy}(p,q)$ is the joint mass exponent function. Obviously, we can estimate $T_{xy}(p,q)$ through regressing $ln \chi_{xy}(p,q,s)$ against $ln s$ in the scaling range for a given pair $(p,q)$.

In analogy to the double Legendre transforms in the joint multifractal analysis based on the partition function approach MFXPF$(p,q)$ [4], we define the joint singularity strength function $h_x$ and $h_y$,

$$h_x(p,q) = 2\partial T_{xy}(p,q)/\partial p, \quad (4)$$

$$h_y(p,q) = 2\partial T_{xy}(p,q)/\partial q, \quad (5)$$

and the multifractal spectrum $D_{xy}(h_x, h_y) \equiv ph_x/2 + qh_y/2 - T_{xy}. \quad (6)$

The quantities $h_x(p,q)$, $h_y(p,q)$, and $D_{xy}(h_x, h_y)$ from the MFXPF$(p,q)$ method differ from the joint singularity strengths $\alpha_x(p,q)$, $\alpha_y(p,q)$ and the joint multifractal spectrum $f_{xy}(\alpha_x, \alpha_y)$ obtained from the MFXPF$(p,q)$ method in their values. Taking $p = 0$ and $q = 0$ for example, the joint partition function in Eq. (2) always equals to the number of wavelet coefficients, corresponding to the total number of data points in the original series, which means that $T_{xy}(0,0) = 0$, while in the MFXPF$(p,q)$ method $T_{xy}(0,0) = -1$. We also find that all the estimated $h_x$ and $h_y$ are less than 0. This violates our intuition that the singularity strength should be positive. However, these differences in values do not mean that our method is useless, because the joint multifractal quantities obtained from both methods still share the same physical meanings and geometric features, which allows us to uncover the cross correlations in time series pairs. Following most of the numerical experiments and empirical analysis, in which the multifractal analysis based wavelet estimators are performed on integral series, we have also tested our MFXWT$(p,q)$ method on integral series. However, the obtained results are not easy to explain and very hard to link to the theoretical values for the $p$-model [33]. Hence, our investigations focus only on non-cumulative series, for example, stock returns rather than stock prices.

As pointed out in Ref. [24], the estimation of joint singularity strength and joint multifractal spectrum based on the Legendre transform may have various errors because of its innate disadvantages. They also propose an alternative method to compute $h$ and $D(h)$ from the perspective of canonical method, which is known as a direct estimation method. This inspires us to directly estimate the joint singularity strength $h_x$ and $h_y$ and the joint multifractal spectrum $D_{xy}(p,q)$ through the following equations,

$$h_x(p,q) = \lim_{s \to 0} \frac{1}{\ln s} \sum_i \mu_{xy}(p,q,s,i) \ln |w_x(s,i)|, \quad (7)$$

$$h_y(p,q) = \lim_{s \to 0} \frac{1}{\ln s} \sum_i \mu_{xy}(p,q,s,i) \ln |w_y(s,i)|, \quad (8)$$

$$D_{xy}(p,q) = \lim_{s \to 0} \frac{1}{\ln s} \sum_i \mu_{xy}(p,q,s,i) \ln \mu_{xy}(p,q,s,i), \quad (9)$$

where

$$\mu_{xy}(p,q,s,i) = |w_x(s,t)|^{p/2} |w_y(s,i)|^{q/2} \chi_{xy}(p,q,s).$$

Thus we can directly determine the joint singularity strength functions $h_x(p,q)$ and $h_y(p,q)$ and the joint multifractal function $D_{xy}(p,q)$ from Eqs. (7-9).

### III. NUMERICAL EXPERIMENTS

We conduct two numerical experiments including binomial measures generated from the multiplicative $p$-model [33] and bivariate fractional Brownian motions (bFBMs) [34,36] to test the validity and performance of the proposed MFXWT$(p,q)$ method.
A. Binomial measures

As a first example, we conduct a numerical experiment of testing the validity of the MFXWT($p,q$) method using two binomial measures $\{x(i) : i = 1, 2, \cdots, 2^k\}$ and $\{y(i) : i = 1, 2, \cdots, 2^k\}$ from the $p$-model with known analytic multifractal properties $^{32,33}$. Each binomial measure is generated in an iterative manner. We start with the zeroth iteration $k = 0$, where the data set $z(i)$ consists of one value, $z(0)(1) = 1$. In the $k$-th iteration, the data set $\{z(k)(i) : i = 1, 2, \cdots, 2^k\}$ is obtained from

\[
\begin{align*}
  z^{(k)}(2i-1) &= p_x z^{(k-1)}(i) \\
  z^{(k)}(2i) &= (1 - p_x) z^{(k-1)}(i)
\end{align*}
\]  

(10)

for $i = 1, 2, \cdots, 2^{k-1}$. When $k \to \infty$, $z^{(k)}(i)$ approaches to a binomial measure, whose scaling exponent function $H_{zz}(q)$ and mass exponent function $\tau_{zz}(q)$ have an analytic form $^{32,33}$

\[
\begin{align*}
  H_{zz}(q) &= 1/q - \log_2[p_x^q + (1 - p_x)^q]/q, \quad (11) \\
  \tau_{zz}(q) &= -\log_2[p_x^q + (1 - p_x)^q]. \quad (12)
\end{align*}
\]

In our numerical experiment, the parameters of the two binomial measures from $p$-model are set as $p_x = 0.3$ for $x(i)$ and $p_y = 0.4$ for $y(i)$ with an iteration step $k = 16$. The analytic scaling exponent functions $H_{xx}(q)$ and $H_{yy}(q)$ of $x$ and $y$ are expressed in Eq. (11). Because the two series are generated in terms of the same rule, the two series $x$ and $y$ exhibit a strong correlation with a coefficient of 0.82.

In Ref. $^2$, Xie et al have analytically derived the joint multifractal properties for two binomial measures constructed from the $p$-model. The joint mass exponent function $\tau_{xy}(p,q)$,

\[
\tau_{xy}(p,q) = \frac{p\gamma}{2 \ln 2} - \frac{\ln \left[p_x^q + (1 - p_y)^q\right]}{\ln 2}, \quad (13)
\]

the two joint singularity strength functions $\alpha_x(p,q)$ and $\alpha_y(p,q)$,

\[
\begin{align*}
  \alpha_x(p,q) &= \frac{\gamma}{\ln 2} - \frac{\beta p_x^q \ln p_y + (1 - p_y)^q \ln(1 - p_y)}{p_x^q + (1 - p_y)^q}, \\
  \alpha_y(p,q) &= -\frac{1}{\ln 2} \frac{p_y^q \ln p_y + (1 - p_y)^q \ln(1 - p_y)}{p_y^q + (1 - p_y)^q}
\end{align*}
\]  

(14)

and the joint multifractal spectrum $f_{xy}(p,q)$ are expressed as follows,

\[
 f_{xy}(\alpha_x, \alpha_y) = \frac{QZ^Q \ln Z + (1 + Z^Q) \ln(1 + Z^Q)}{\ln 2(1 + Z^Q)}, \quad (16)
\]

where $\beta = \frac{\ln p_x - \ln(1 - p_x)}{-\ln p_x - \ln(1 - p_x)}$, $\gamma = \beta \ln(1 - p_y) - \ln(1 - p_x)$, $Q = \beta p/2 + q/2$ and $Z = \frac{1 - p_x}{p_y}$. These theoretical formulas have been numerically confirmed to agree nicely with the empirical results from the MFXPF($p,q$) method $^2$, which allows us to check whether such theoretical formulas can be employed as a benchmark to test the performance of the MFXWT($p,q$) algorithm when it is applied to binomial measures. By comparing with the scaling behaviors of the joint partition functions from both meth-
methods, we can simply give the theoretical formulas of the joint mass exponent function $\tau_{xy}(p, q)$, the joint singularity strength functions $h_x(p, q)$ and $h_y(p, q)$, and the joint multifractal spectrum $D_{xy}(h_x, h_y)$ for MFXWT($p, q$).

Fig. 2 illustrates the scaling behaviors of the joint partition functions obtained from the MFXWT($p, q$) and MFXWT($p, q$) methods with different values of $p$ and $q$. The joint partition functions of MFXWT($p, q$) are scaled by a factor of $s^{p/2+q/2-1}$ and one can find that the scaled joint partition functions share the same scaling pattern as the joint partition functions of MFXWT($p, q$), which allows us to obtain the following equations to connect the theoretical joint multifractal formulas of binomial measures to the empirical joint multifractal features of MFXWT($p, q$):

$$\tau_{xy}(p, q) = p/2 + q/2 - 1 = \tau_{xy}(p, q),$$

$$h_x(p, q) + 1 = \alpha_x(p, q),$$

$$h_y(p, q) + 1 = \alpha_y(p, q),$$

$$D_{xy}(h_x, h_y) + 1 = f_{xy}(\alpha_x, \alpha_y),$$

where $\tau_{xy}(p, q)$, $\alpha_x(p, q)$, $\alpha_y(p, q)$, and $f_{xy}(\alpha_x, \alpha_y)$ are given by Eqs. [13–16].

These formulas provide us an efficient way to test the estimation accuracy of the MFXWT($p, q$) method in the joint multifractal analysis of two binomial measures. For the partition function approach and the wavelet analysis for detecting multifractal nature of a single time series, it has been reported that $\tau_{xx}(q) = \tau_{xx}(q) + q$ and $\alpha_x(q) = h_x(q) + 1$ [33–40].

We first take into account the case of $p = q$. Fig. 2(a) illustrates the scaling behavior between the joint partition functions $\chi_{xy}(q, s)$ and the scale $s$. The most intriguing observation is the significant power-law dependence over more than three orders of magnitude. By estimating the power-law exponents between $\chi_{xy}(q, s)$ and $s$ for different $q$, we obtain the joint mass exponent function $T(q)$, which is plotted in Fig. 2(d). In Fig. 2(d), we also sketch the theoretical values of $T(q)$ obtained from Eq. (17). The two curves match with each other very well, suggesting the high accuracy of our MFXWT($p, q$) algorithm on the analysis of joint multifractal nature in two binomial measures. A nonlinear behavior of $T(q)$ against $q$ also demonstrates the joint multifractality in binomial measures, which is in accordance with our expectation.

Fig. 2(b) and Fig. 2(c) present the power-law scaling behaviors of two quantities ($\sum \mu_{xy} \ln |w_x w_y|^{1/2}$ and $\sum \mu_{xy} \ln \mu_{xy}$) against the scale $s$, whose power-law exponents are the estimates of the joint singularity strength $h_{xy}$ and the joint multifractal function $D(h_{xy})$.

In Fig. 2(e), we show the comparison of the joint singularity strength $h_{xy}(q)$ obtained from different methods. The solid line corresponds to the theoretical values. The squares and circles are obtained from the first derivation of the joint mass exponent $\tau_{xy}(q)$ and the direct estimating method, respectively. One can see that the empirical $h_{xy}(q)$ of both estimating methods perfectly coincide with each other. However, both empirical curves only agree well with the theoretical values when $q \geq 1$. Devi-
analyzing the joint multifractality in binomial measures. Fig. 3(a) presents the dependence of the joint partition spectra of binomial measures, in which both theoretical quantities are observed when \( p < 1 \) and the reason generating such phenomenon is not clear.

Fig. 2(f) illustrates the corresponding joint multifractal spectra of binomial measures, in which both theoretical values and estimated values are plotted. The \( D_{xy}(h_{xy}) \) values obtained from Eq. (8) and Eq. (10) are in good agreement with each other and also collapse approximately on the theoretical curves. Our results suggest that the accuracy of the MFXWT \((p,q)\) is acceptable for analyzing the joint multifractality in binomial measures.

We then release the restriction of \( p = q \) posed in Fig. 2. Fig. 3(a) presents the dependence of the joint partition function \( \chi_{xy}(2,q,s) \) on the scale \( s \) for different \( q \) with fixed \( p = 2 \). Very nice power-law behaviors are observed. For each pair of \((p,q)\), the slope of the straight line gives the estimate of the corresponding joint mass exponent \( \mathcal{T}_{xy}(p,q) \). Fig. 3(e) plots the joint mass exponent function \( \mathcal{T}_{xy}(p,q) \) with respect to \( p \) and \( q \). Again, one can see that there are nonlinear features between \( \mathcal{T}_{xy}(p,q) \) and \((p,q)\), which verifies the presence of joint multifractality nature in the two binomial measures. Following Eqs. (14) and (15), the joint singularity strength functions \( h_x(p,q) \) and \( h_y(p,q) \) and the joint multifractal function \( D_{xy}(p,q) \) can be numerically computed once we have the mass exponent \( \mathcal{T}_{xy}(p,q) \). Fig. 3(f), Fig. 3(g) and Fig. 3(h) illustrate the corresponding \( h_x(p,q) \), \( h_y(p,q) \), and \( D_{xy}(p,q) \), respectively. Wide spanning range of \( h_x, h_y \) and \( D_{xy} \) further corroborate the presence of joint multifractal nature in binomial measures.

The direct estimation method presented in Eqs. (14) provides an alternative way to estimate the joint singularity strength \( h_x(p,q) \) and \( h_y(p,q) \) and the joint multifractal function \( D_{xy}(p,q) \). By estimating the three quantities \( \sum \mu_{xy} \ln |w_x|, \sum \mu_{xy} \ln |w_y|, \) and \( \sum \mu_{xy} \ln \mu_{xy} \), we find very good power-law scaling behaviors between these quantities and the scale \( s \), as shown in Fig. 3(b-d). Their power-law exponents correspond to the joint singularity strength function \( h_x(p,q) \) in Fig. 3(i) and \( h_y(p,q) \) in Fig. 3(j) and the joint multifractal function \( D_{xy}(p,q) \) in Fig. 3(k). As we can see, in Fig. 3(f) and Fig. 3(i), \( h_y(p,q) \) in Fig. 3(g) and Fig. 3(j), and \( D_{xy}(p,q) \) in Fig. 3(h) and Fig. 3(k) obtained from both methods agree well with each other.

The nice agreement of both methods is also observed in Fig. 3(l), which illustrates the joint multifractal spectra \( D_{xy}(h_x,h_y) \) of both methods, as well as the theoretical values (magenta curve) expressed in Eq. (14). In Ref. [2], Xie et al. showed that the joint multifractal spectrum \( f_{xy}(\alpha_x, \alpha_y) \) of binomial measures is a univariate function of \( Q \), and thus of \( \alpha_x \) or of \( \alpha_y \) due to Eqs. (14) and
suggesting the efficiency of the MFXWT over 

Fig. 3(l), the plots of joint multifractal spectrum 

As described in Ref. [2], if the two time series is 

A bivariate fractional Brownian motion \([x(t), y(t)]\) with parameters \([H_{xx}, H_{yy}] \in (0, 1)^2\) is a self-similar Gaussian process with stationary increments, where \(x(t)\) and \(y(t)\) are two univariate fractional Brownian motions with Hurst indices \(H_{xx}\) and \(H_{yy}\) and are the two components of the bFBM [34–36]. The basic properties of multivariate fractional Brownian motions have been extensively studied [34–36]. Extensive numerical experiments of multifractal cross-correlation analysis algorithms have been performed on bFBMs [2, 9, 21]. The two Hurst indexes \(H_{xx}\) and \(H_{yy}\) of the two univariate FBMs and their cross-correlation coefficient \(\rho\) are input arguments of the simulation algorithm. By using the simulation procedure describe in Ref. [34–36], we have generated a realization of bFBM with \(H_{xx} = 0.1, H_{yy} = 0.5\), and \(\rho = 0.5\). The length of the bFBM is 2^{16}.

As described in Ref. [2], if the two time series is monofractal, their joint singularity strength \(h_x(p, q)\) and \(h_y(p, q)\) will become constants, and their joint multifractal spectrum \(D_{xy}(h_x, h_y) = 0\). We must note that Eqs. (17-20) obtained from the \(p\)-model are no longer valid here. Because these equations are derived based on conservative measures, while the increments of both components \(x(t)\) and \(y(t)\) in bFBM are not conservative.

Fig. 4 illustrates the results of the joint multifractal analysis of the bFBM using the MFXWT algorithm. In Fig. 4(a), the joint partition functions \(\chi_{xy}(2, q, s)\) of the wavelet coefficients are plotted with respect to the scale \(s\) for fixed \(p = 2\) and different \(q\). Again, excellent power-law scaling behaviors are observed, which allow us to estimate the joint mass exponents \(T_{xy}\) by the least square estimation. Fig. 4(b) illustrates the joint mass exponent function against different \(p\) and \(q\). In accordance with the monofractality of bFBMs, a plane is observed for \(T(p, q)\). The bivariate regression gives that

\[
T_{xy}(p, q) = -0.485p - 0.268q + 0.135. \tag{21}
\]

According to Eq. (21), we can infer that \(T_x = -0.970\), \(T_y = -0.536\), and \(D_{xy} = -0.135\) deviating from the theoretical value \(D_{xy}(0, 0) = 0\). When \(p = q = 0\), Eq. (21) gives that \(T_{xy}(0, 0) = 0.135\), also deviating from the theoretical value \(T_{xy}(0, 0) = 0\).

Alternatively, Eq. (18) and (19) provides an approximate way to estimate \(h_x(p, q)\) and \(h_y(p, q)\) by means of numerical differentiation of \(T_{xy}\). Fig. 4(c) and Fig. 4(d) plot the estimated joint singularity strength functions \(h_x(p, q)\) and \(h_y(p, q)\) obtained from taking the forward difference...
of $\mathcal{T}_{xy}(p, q)$. One can see that the singularity strength function $h_x(p, q)$ and $h_y(p, q)$ obtained from the numerical methods vary in a relatively small range. The corresponding average value is $-0.968$ for $h_x$ and $-0.534$ for $h_y$, which nicely agree with $\mathcal{T}_x$ and $\mathcal{T}_y$ obtained from the plane equation of $\mathcal{T}_{xy}(p, q)$ in Eq. (21). Based on the double Legendre transform in Eq. (6), we can further obtain the joint multifractal function $D_{xy}(p, q)$, which is plotted with respect to $p$ and $q$ in Fig. 5(c) and with respect to $h_x$ and $h_y$ in Fig. 5(f). The average value of $D_{xy}$ is $-0.178$, also close to $\overline{D}_{xy} = -0.135$. However, unlike $h_x$ and $h_y$ locating in narrow range, $D_{xy}$ spans in a relatively large range, from 0 to 0.5. This indicating that the MFXWT method may give spurious multifractality for bFBM if we determine the joint multifractality only according to the spanning range of $D_{xy}$. Such spurious multifractality is often observed in the partition function approach, which usually stems from the finite size effect [41]. It suggests the necessity of performing statistical tests on checking multifractality based on bootstrapping [42, 43].

IV. APPLICATION TO STOCK MARKET INDEXES

We now apply the MFXPF$(p, q)$ method to unveil the joint multifractality of the daily return series of the dow Jones industrial average (DJIA) and the National Association of Securities Dealers Automated Quotations (NASDAQ) index. In order to compare with the results in Ref. [2], we also conduct similar analysis on the volatility time series of the two indexes. The daily return is defined as the logarithmic difference of daily closing price:

$$R(t) = \ln I(t) - \ln I(t - 1),$$

where $I(t)$ is the closing price of the DJIA index or the NASDAQ index on day $t$. Both indexes are retrieved from “Yahoo! Finance”. The spanning period of both indexes is from 5 February 1971 to 17 June 1 2016, containing 11430 data points in total. The volatilities are determined by the absolute values of the daily returns.

A. Daily return time series

Firstly, we analyze the joint multifractality of the daily returns of both indexes using the MFXWT$(p, q)$ method. The results are illustrated in Fig. 5.

Fig. 5(a) plots the joint partition function $\chi_{xy}(2, q, s)$ as a function of the scale $s$ for fixed $p = 2$ and different $q$. Very good power-law behaviors are observed with the scaling range spanning over three orders of magnitude. The results for other $(p, q)$ pairs are similar. By regressing $\ln \chi_{xy}(p, q, s)$ with respect to $\ln s$ for a given pair of $(p, q)$, we obtain the joint mass exponents $\mathcal{T}_{xy}(p, q)$, which are plotted in Fig. 5(b). One can see that the joint mass exponents are a nonlinear function of $p$ and $q$, indicating the presence of joint multifractal features in the daily returns of two indexes.

Fig. 5(c) and Fig. 5(d) illustrate the joint singularity strength functions $h_x(p, q)$ and $h_y(p, q)$, which are numerically estimated from $\mathcal{T}(p, q)$. We find that both singularity strength function are well dispersed with spanning
FIG. 6. (Color online) Multifractal cross wavelet analysis of the joint multifractality between the daily volatility series of DJIA index and NASDAQ index using the MFXWT\((p, q)\) method. (a) Power-law dependence of \(\chi_{xy}(p, q, s)\) on scale \(s\) for fixed \(p = 2\) and different \(q\). (b) Joint mass exponent function \(T(p, q)\). (c) Joint singularity strength function \(h_x(p, q)\). (d) Joint singularity strength function \(h_y(p, q)\). (e) Joint multifractal function \(D_{xy}(p, q)\). (f) Joint multifractal singularity spectrum \(D_{xy}(h_x, h_y)\).

ranges greater than 0.3. In addition, the joint singularity strength functions are monotonic with respect to \(p\) and \(q\). Fig. 6(c) plots the joint multifractal function \(D_{xy}(p, q)\) obtained from the double Legendre transform. It is observed that the joint multifractal function locates in the range of \((-1, 0)\). The maximum point of \(D_{xy}(p, q)\) is reached at the point of \((p, q) = (0, 0)\). In Fig. 6(f), we show the joint multifractal spectrum \(D_{xy}(h_x, h_y)\), which does not collapse into the neighborhood of a fixed point. Our empirical findings favor the existence of joint multifractality in daily returns of DJIA and NASDAQ.

B. Daily volatility time series

We next apply the MFXWT\((p, q)\) method to perform multifractal cross wavelet analysis on the daily volatilities of the two indexes. The results are illustrated in Fig. 6

Fig. 6(a) plots in log-log scale the dependence of the joint partition function \(\chi_{xy}(2, q, s)\) with respect to the scale \(s\) for fixed \(p = 2\) and different \(q\). Excellent power-law behaviors are observed with more than two orders of magnitude. The resulting joint mass exponents \(T_{xy}(p, q)\) are shown in Fig. 6(b). The mass exponents are monotonically and nonlinearly increasing, implying that the cross correlations between the two index volatilities exhibit joint multifractal nature.

We further numerically calculate the joint singularity strength functions \(h_x(p, q)\) and \(h_y(p, q)\) and plot them in Fig. 6(c) and Fig. 6(d), respectively. We observe that the widths of both singularity strength functions are significantly greater than 0, further confirming the existence of joint multifractality in the cross correlations of the two volatility time series. Fig. 6(e) and Fig. 6(f) show the joint multifractal function \(D_{xy}(p, q)\) and the joint multifractal spectrum \(D_{xy}(h_x, h_y)\), respectively, which again consolidates the joint multifractal characteristics in the cross correlations between the two index volatilities. Our results also reveal that the joint multifractal nature in volatilities is stronger than that in returns, because their widths of joint singularity strength functions and joint multifractal functions are larger.

V. CONCLUSION AND DISCUSSION

In this work, we have developed a new method of joint multifractal analysis with two moment orders based on wavelet transform, termed MFXWT\((p, q)\). Due to the fact that there is a part of wavelet coefficients very close to 0, the values of \(p\) and \(q\) are restricted to be greater than 0. We have checked the performances of the MFXWT\((p, q)\) method by extensive numerical experiments on pairs of time series generated from binomial measures and bivariate fractional Brownian motions. We also test the capability of this method in uncovering the potential joint multifractal nature in return pairs and volatility pairs in the US stock markets.

For binomial measures from \(p\)-model, the theoretical expressions of the joint multifractality is derived by comparing with the scaling behaviors of the joint partition functions between the MFXWT\((p, q)\) and MFXPF\((p, q)\) methods. It is found that the joint multifractal quantities \((T_{xy}, h_x, h_y,\) and \(D_{xy})\) extracted from the MFXWT
method approximately agree with the theoretical values. Our finding indicates that the accuracy of MFXWT\((p, q)\) is acceptable in detecting the joint multifractal nature in binomial measures.

For bivariate fractional Brownian motions, the joint mass exponent function \(T_{xy}\) of the cross correlations is found to be linearly dependent on the orders \(p\) and \(q\), which is the hallmark of monofractal. The result nicely reveals the inherent monofractal characteristics in bFBMs. The singularity strength functions \(h_x\) and \(h_y\) are found to locate in a very narrow range, which again support the fact that bFBMs are monofractal. However, we should be cautious of the multifractal function \(D_{xy}\) given by the MFXWT method, because this method gives biased outcomes for the bFBMs. We should also be cautious of the multifractal nature determined only based on the multifractal function \(D_{xy}\) given by the MFXWT algorithm, because it may provide the results of spurious multifractality, especially for the cases that we do not know \textit{a priori} the underlying fractal properties. Such shortcomings can be compensated by performing statistical tests based on the bootstrap method.

Comparing with the MFXPF\((p, q)\) method which can be applied only to conservative measures (volatility), the MFXWT\((p, q)\) method has the advantage of analyzing both conservative measures and non-conservative measures. We thus applied the MFXWT\((p, q)\) method to analyze the joint multifractal nature in the returns and volatilities of two US stock market indexes. The joint multifractality is detected both in the returns and in the volatilities, and the joint multifractal nature in the volatilities is stronger than that in the returns.

The shortcoming of wavelet analysis of multifractals is well known that the moment order should be positive due to the presence of small wavelet coefficients and one should use the wavelet transform modulus maxima (WTMM) method since all the modulus maxima are significantly greater than 0 [24, 30, 32, 44]. Unfortunately, the WTMM method cannot be generalized to bivariate cases, since at each scale \(s\) the numbers of modulus maxima of two time series are usually different.

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