 SEMIDEFINITE PROGRAMMING FOR CHANCE OPTIMIZATION OVER SEMIALGEBRAIC SETS *

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Abstract. In this paper, a novel method for solving “chance optimization” problems based on the theory of measures and moments is proposed. In this type of problems, one aims at maximizing the probability of a set defined by polynomial inequalities. These problems are, in general, nonconvex and computationally hard. With the objective of developing systematic numerical procedures to solve such problems, a sequence of convex relaxations is provided, whose optimal value is shown to converge to the optimal value of the original problem. Indeed, we provide a sequence of semidefinite programs of increasing dimension and complexity which can arbitrarily approximate the solution of the original problem. To be able to solve large scale chance constrained problems, a first-order augmented Lagrangian algorithm is implemented to solve the resulting semidefinite relaxations. Numerical examples are presented to illustrate the computational performance of the proposed approach.

Key words. Semialgebraic set, Chance Optimization, SDP relaxation, augmented Lagrangian, First-order Methods.

1. Introduction. In this paper, we aim at solving chance optimization problems; i.e. problems which involve maximizing of the probability of a semialgebraic set defined by polynomial inequalities. More precisely, given a probability space \((\mathbb{R}^m, \Sigma_q, \mu_q)\) with \(\Sigma_q\) denoting the Borel \(\sigma\)-algebra of \(\mathbb{R}^m\) and \(\mu_q\) denoting a finite Borel measure on \(\Sigma_q\), we focus on the problem given in (1.1) over decision variable \(x \in \mathbb{R}^n\).

\[
P^* := \sup_{x \in \mathbb{R}^n} \mu_q \left( \bigcup_{j=1}^N \bigcap_{k=1}^N \left\{ q \in \mathbb{R}^m : P_j^{(k)}(x, q) \geq 0 \right\} \right), \tag{1.1}
\]

where \(P_j^{(k)} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \ j = 1, 2, \ldots, \ell_k\) and \(k = 1, \ldots, N\) are given polynomials. Let \(K_k := \{(x, q) : P_j^{(k)}(x, q) \geq 0, \ j = 1, \ldots, \ell_k\}\) and \(K := \bigcup_{k=1}^N K_k\). Under the assumption that \(K\) is bounded, we show that by solving a sequence of semidefinite programming (SDP) problems of growing dimension, we can construct a sequence \(\{y^d_n\}_{d \in \mathbb{Z}_+} \subset \mathbb{R}^N\) that has an accumulation point in the weak \(*\) topology of \(\sigma(\ell_\infty, \ell_1)\) and for every accumulation point \(y^*_n \in \mathbb{R}^N\), there is a representing finite Borel measure \(\mu^*_n\) such that any \(x^* \in \text{supp}(\mu^*_n)\) is an optimal solution to (1.1), i.e. the supremum \(P^*\) is attained at \(x^*\), where \(\mathbb{R}^N\) denotes the vector space of real sequences.

First, the emphasis will be placed on the following special case of (1.1), where \(N = 1\),

\[
P^* := \sup_{x \in \mathbb{R}^n} \mu_q \left( \left\{ q \in \mathbb{R}^m : P_j(x, q) \geq 0, \ j = 1, \ldots, \ell \right\} \right), \tag{1.2}
\]

and then the results for the special will be extended to the case \(N > 1\).

This class of problems is quite large and encompasses many problems in different areas. For example, designing probabilistic robust controllers \([23]\) and model predictive controllers in presence of random disturbances \([11, 40, 45]\), and optimal path planning and obstacle avoidance problems in robotics \([12, 13, 17]\) can be cast as special cases of this framework. Moreover, problems in the area of economy, finance, and trust design \([32, 47, 50]\) can also be formulated as (1.2) and (1.1). Although, in some particular cases, the problem in (1.1) is convex (e.g., see \([26, 42]\)), in general, chance constrained problems are not convex; e.g, see \([26]\) for examples of non-convex chance constrained linear programs. In this paper, we use previous results on moments of measures (e.g., see \([29, 30]\)) to develop a sequence of SDP problems whose solutions converge to the solution of (1.1).

1.1. Previous Work. Several approaches have been proposed to solve chance constrained problems. The main idea behind most of the proposed methods is to find a tractable approximation for chance constraints. One particular method is the so-called scenario approach; see \([14, 15, 34, 38, 48]\) and the references

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In this approach, the probabilistic constraint is replaced by a (large) number of deterministic constraints obtained by drawing iid samples of random parameters. Being a randomized approach, there is always a (perhaps small) probability of failure.

In [34, 6, 8, 9], robust optimization is used to deal with uncertain linear programs (LP). In this method, the uncertain LP is replaced by its robust counterpart, where the worst case realization of uncertain data is considered. The proposed method is not computationally tractable for all types of uncertainty sets. A specific case that is tractable is LP with ellipsoidal uncertainty set [4].

In [10, 33, 37], an alternative approach is proposed where one analytically determines an upper bound on the probability of constraint violation. Although this method does provide a convex approximation, it can only be applied to specific uncertainty structures. In [35, 41], the authors propose a so-called Bernstein approximation where a convex conservative approximation of chance constraints is constructed using generating functions. Although approximation is efficiently computable, it is only applicable to problems with convex constraints that are affine in random vector \( q \in \mathbb{R}^m \). Moreover, components of \( q \) need to be independent and have computable finite generating functions.

In [16, 19, 20], convex relaxations of chance constrained problems are presented. The concept of polynomial kinship function is used to estimate an upper bound for the probability of constraint violation. Solutions to a sequence of relaxed problems converge to a solution of the original problem as the degree of the polynomial kinship function increases along the sequence. In [20, 25], an equivalent convex formulation is provided based on the theory of moments. In this method the probability of a polynomial being negative is approximated by computing polynomial approximations for univariate indicator functions [25].

In this paper, we take a different approach to deal with chance constrained problems. The proposed method is based on volume approximation results in [22] and the theory of moments [29, 30]. In [22], a hierarchy of SDP problems are proposed to compute the volume of a given compact semialgebraic set. It is shown in [22] that the volume of a semialgebraic can be computed solving a maximization problem over Borel measures supported on the given set and restricted by the Lebesgue measure on a simple set containing the semialgebraic set of interest. Building on this result, we propose the chance optimization problem over semialgebraic sets—see our preliminary results in [24]. In particular, we address the problem of probability maximization over the union of semialgebraic sets defined by intersections of finite number of polynomial inequalities as in (1.1). In this case, one needs to search for the Borel measure with maximum possible mass over given semialgebraic set and, at the same time, for an upper bound probability measure over a set containing the semialgebraic set of interest and restricting the Borel measure.

1.2. The Sequel. The outline of the paper is as follows. In Section 2, the notation used in this paper and preliminary results on measure theory are presented. In sections 3 and 4, we propose equivalent problems and sequences of SDP relaxations to (1.2) and (1.1) and show that the optimal solution sequences converge to the solutions of the original problems. In Section 5, an efficient optimization algorithm to solve regularized SDP relaxations of the chance constrained problems is proposed and numerical examples are presented. Moreover, in our numerical experiments, we show that regularizing the SDP problems with nuclear norm to induce rank-one solutions, improves the quality of solutions to relaxed problems in practice. Finally, concluding remarks are given in Section 6.

2. Notation and Preliminary Results.

2.1. Notations and Definitions. Let \( \mathbb{R}[x] \) be the ring of real polynomials in the variables \( x \in \mathbb{R}^n \). Given \( \mathcal{P} \in \mathbb{R}[x] \), we will represent \( \mathcal{P} \) as \( \sum_{\alpha \in \mathbb{N}^n} p_\alpha x^\alpha \) using the standard basis \( \{x^\alpha\}_{\alpha \in \mathbb{N}^n} \) of \( \mathbb{R}[x] \), and \( \mathbf{p} = \{p_\alpha\}_{\alpha \in \mathbb{N}^n} \) denotes the sequence of polynomial coefficients. Throughout the paper, we assume that for \( \mathcal{P} \in \mathbb{R}[x] \), the elements of its coefficient sequence \( \mathbf{p} = \{p_\alpha\}_{\alpha \in \mathbb{N}^n} \) are sorted according to a graded reverse lexicographic order (grevlex) of the corresponding monomials. Given \( \mathbf{y} = \{y_\alpha\}_{\alpha \in \mathbb{N}^n} \) in \( \mathbb{R} \), of which elements are also sorted according to grevlex order, let \( L_y : \mathbb{R}[x] \to \mathbb{R} \) be a linear map defined as

\[
\mathcal{P} \left( = \sum_{\alpha \in \mathbb{N}^n} p_\alpha x^\alpha \right) \mapsto L_y(\mathcal{P}) = \sum_{\alpha \in \mathbb{N}^n} p_\alpha y_\alpha.
\]

Moreover, let \( \Sigma^2[x] \subset \mathbb{R}[x] \) be the set of sum of squares (SOS) polynomials. \( \sigma : \mathbb{R}^n \to \mathbb{R} \) is a SOS polynomial if it can be written as a sum of finitely many squared polynomials, i.e. \( \sigma(x) = \sum_{j=1}^\ell h_j(x)^2 \) for some \( \ell < \infty \).
and \( h_j \in \mathbb{R}[x] \) for \( 1 \leq j \leq \ell \). Given \( n \) and \( d \) in \( \mathbb{N} \), we define \( S_{n,d} := \binom{d+n}{n} \) and \( \mathbb{N}_d = \{ \alpha \in \mathbb{N}^n : \|\alpha\|_1 \leq d \} \). Let \( \mathbb{R}_d[x] \subset \mathbb{R}[x] \) denote the set of polynomials of degree at most \( d \in \mathbb{N} \), which is indeed a vector space of dimension \( S_{n,d} \). Given \( \mathcal{P} \subset \mathbb{R}_d[x] \), \( \mathbf{p} = \{ p_\gamma \}_{\gamma \in \mathbb{N}_d} \) is sorted according to grevlex order so that we have \( \mathbb{R}^n \ni \alpha = \alpha(1) < \ldots < \alpha(S_{n,d}) \), where \( S_{n,d} \) is the number of components in \( \mathbf{p} \).

Let \( \mathbb{R}^n \) denote the space of real sequences, and let \( \mathcal{M}(K) \) be the set of finite Borel measures \( \mu \) such that \( \text{supp}(\mu) \subset K \), where \( \text{supp}(\mu) \) denotes the support of the measure \( \mu \); i.e., the smallest set that contains all measurable sets with strictly positive measure. A sequence \( \mathbf{y} = \{ y_\alpha \}_{\alpha \in \mathbb{N}^n} \in \mathbb{R}^\mathbb{N} \) is said to have a representing measure, if there exists a finite Borel measure \( \mu \) on \( \mathbb{R}^n \) such that \( y_\alpha = \int x^n d\mu \) for every \( \alpha \in \mathbb{N}^n \) - see \([29, 30]\). In this case, \( \mathbf{y} \) is called the moment sequence of the measure \( \mu \). Given two measures \( \mu_1 \) and \( \mu_2 \) on a Borel \( \sigma \)-algebra \( \Sigma \), the notation \( \mu_1 \ll \mu_2 \) means \( \mu_1(S) \leq \mu_2(S) \) for any set \( S \subset \Sigma \). Moreover, if \( \mu_1 \) and \( \mu_2 \) are both measures on Borel \( \sigma \)-algebras \( \Sigma_1 \) and \( \Sigma_2 \), respectively, then \( \mu = \mu_1 \times \mu_2 \) denotes the product measure satisfying \( \mu(S_1 \times S_2) = \mu_1(S_1)\mu_2(S_2) \) for any measurable sets \( S_1 \subset \Sigma_1, S_2 \subset \Sigma_2 \). Let \( C \subset \mathbb{R}^n \), \( \sigma(C) \) denotes the Borel \( \sigma \)-algebra over \( C \). Given two square symmetric matrices \( A \) and \( B \), the notation \( A \succeq B \) denotes positive semidefiniteness of \( A \) and \( A \succeq B \) stands for \( A - B \) being positive semidefinite.

**Putinar’s property:** A closed semialgebraic set \( K = \{ x \in \mathbb{R}^n : \mathcal{P}_j(x) \geq 0, j = 1, 2, \ldots, \ell \} \) defined by polynomials \( \mathcal{P}_j \in \mathbb{R}[x] \) satisfies Putinar’s property if there exists \( \mathcal{U} \subset \mathbb{R}[x] \) such that \( \{ x : \mathcal{U}(x) \geq 0 \} \) is compact and \( \mathcal{U} = \mathcal{U}_0 + \sum_{j=1}^\ell \sigma_j \mathcal{P}_j \) for some SOS polynomials \( \sigma_j \subset \Sigma^2[x] \) see \([27, 30, 42]\). Putinar’s property holds if the level set \( \{ x : \mathcal{P}_j(x) \geq 0 \} \) is compact for some \( j \), or if all \( \mathcal{P}_j \) are affine and \( K \) is compact - see \([27]\). Clearly these results imply that if there exists \( M > 0 \) such that the polynomial \( \mathcal{P}_{\ell+1}(x) := M - \| x \|^2 \geq 0 \) for all \( x \in K \), then \( K \cap \{ x : \mathcal{P}_{\ell+1} \geq 0 \} \) satisfies Putinar’s property.

**Moment matrix:** Given \( d \geq 1 \) and the sequence \( \{ y_\alpha \}_{\alpha \in \mathbb{N}^d} \), the moment matrix \( M_d(\mathbf{y}) \in \mathbb{R}^{S_{n,d} \times S_{n,d}} \), containing all the moments up to order \( 2d \), is a symmetric matrix defined as follows \([29, 30]\):

\[
M_d(\mathbf{y})(i,j) := L_\mathbf{y} \left( x^{\alpha(i)+\alpha(j)} \right) = y_{\alpha(i)+\alpha(j)}, \quad 1 \leq i,j \leq S_{n,d},
\]

where the elements of the moment sequence \( \mathbf{y} = \{ y_\alpha \}_{\alpha \in \mathbb{N}^n} \) are sorted according to a graded reverse lexicographic order of the corresponding monomials so that we have \( \mathbb{R}^n \ni \alpha = \alpha(1) < \ldots < \alpha(S_{n,2d}) \) and \( S_{n,2d} \) is the number of moments in \( \mathbb{R}^n \) up to order \( 2d \).

For \( d = 2 \) and \( n = 2 \), the moment matrix containing moments up to order \( 2d \) is

\[
M_2(\mathbf{y}) = \begin{bmatrix}
 y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\
 y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\
 y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\
 y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\
 y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\
 y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04}
\end{bmatrix}.
\]

**Localizing matrix:** Given a polynomial \( \mathcal{P} \in \mathbb{R}[x] \) with coefficient vector \( \mathbf{p} = \{ p_\gamma \}_{\gamma \in \mathbb{N}^n} \) and degree \( \delta \), localizing matrix \( M_{\delta}(\mathbf{p} \mathbf{y}) \) with respect to \( \mathbf{y} \) and \( \mathbf{p} \) is as follows \([29, 30]\):

\[
M_{\delta}(\mathbf{p} \mathbf{y})(i,j) := L_\mathbf{y} \left( \mathcal{P} x^{\alpha(i)+\alpha(j)} \right) = \sum_{\gamma \in \mathbb{N}^n} p_\gamma y_{\gamma+\alpha(i)+\alpha(j)}, \quad 1 \leq i,j \leq S_{n,d}.
\]

For example, given \( \mathbf{y} = \{ y_\alpha \}_{\alpha \in \mathbb{N}^2} \) and the coefficient sequence \( \mathbf{p} = \{ p_\alpha \}_{\alpha \in \mathbb{N}^2} \) corresponding to polynomial \( \mathcal{P} \),

\[
\mathcal{P}(x_1, x_2) = a - b x_1 - c x_2^2,
\]

the localizing matrix for \( d = 1 \) is formed as follows

\[
M_1(\mathbf{p} \mathbf{y}) = \begin{bmatrix}
 a y_{00} - b y_{10} & a y_{10} - b y_{20} - c y_{02} & a y_{01} - b y_{11} - c y_{03} \\
 a y_{10} - b y_{02} & a y_{02} - b y_{20} - c y_{01} & a y_{11} - b y_{21} - c y_{12} \\
 a y_{01} - b y_{11} - c y_{03} & a y_{11} - b y_{21} - c y_{13} & a y_{02} - b y_{22} - c y_{04}
\end{bmatrix}.
\]
2.2. Preliminary Results. In this section, we state some standard results found in the literature that will be referred to later in Sections 3 and 4.

Lemma 2.1. Let \( \mu \) be a Borel probability measure supported on hyper-cube \([-1, 1]^n\). Its moment sequence \( y \in \mathbb{R}^N \) satisfies \( \|y\|_\infty \leq 1 \).

Proof. Since \( \text{supp}(\mu) \subseteq [-1, 1]^n \) and \( \mu \) is a probability measure, we have \( |y_\alpha| \leq \int |x^\alpha|d\mu \leq \int |x|d\mu \leq 1 \) for each \( \alpha \in \mathbb{N}^n \). Hence, \( \|y\|_\infty \leq 1 \).

The following lemmas give necessary, sufficient conditions for \( y \) to have a representing measure \( \mu \) – for details see [22, 28, 30].

Lemma 2.2. Let \( \mu \) be a finite Borel measure on \( \mathbb{R}^n \), and \( y = \{y_\alpha\}_{\alpha \in \mathbb{N}^n} \) such that \( y_\alpha = \int x^\alpha d\mu \) for all \( \alpha \in \mathbb{N}^n \). Then \( M_d(y) \geq 0 \) for all \( d \in \mathbb{N} \).

Lemma 2.3. [30] Let \( y = \{y_\alpha\}_{\alpha \in \mathbb{N}^n} \) be a real sequence. If \( M_d(y) \geq 0 \) for some \( d \geq 1 \), then

\[
|y_\alpha| \leq \max \left\{ y_0, \max_{i=1,...,n} L_x(x_i^{2d}) \right\} \quad \forall \alpha \in \mathbb{N}^n_{2d}.
\]

Lemma 2.4. If there exist a constant \( c > 0 \) such that \( M_d(y) \geq 0 \) and \( |y_\alpha| \leq c \) for all \( d \in \mathbb{N} \) and \( \alpha \in \mathbb{N}^n \), then there exists a representing measure \( \mu \) supported on \([-1, 1]^n\).

Given polynomials \( P_j \in \mathbb{R}[x] \) for \( j = 1, 2, \ldots, \ell \), consider the semialgebraic set \( K \) defined as

\[
K = \{ x \in \mathbb{R}^n : P_j(x) \geq 0, j = 1, 2, \ldots, \ell \}.
\]

The following lemma gives a necessary and sufficient condition for \( y \) to have a representing measure \( \mu \) supported on \( K \) – see [22, 28, 29, 30].

Lemma 2.5. If \( K \) defined in (2.7) satisfies Putinar’s property, then the sequence \( y = \{y_\alpha\} \) has a representing finite Borel measure \( \mu \) on the set \( K \), if and only if

\[
M_d(y) = 0, \quad M_d(P_jy) = 0, \quad j = 1, \ldots, \ell, \quad \text{for all } d \in \mathbb{N}.
\]

Finally, the following lemma shows that the Borel measure of a compact set can be represented as an optimization problem.

Lemma 2.6. [22] Let \( \Sigma \) be the Borel \( \sigma \)-algebra on \( \mathbb{R}^n \), and \( \mu_1 \) be a measure on a compact set \( B \subseteq \Sigma \). Then for any given \( \mathcal{K} \in \Sigma \) such that \( \mathcal{K} \subseteq B \), one has

\[
\mu_1(\mathcal{K}) = \int_{\mathcal{K}} d\mu_1 = \sup_{\mu_2 \in \mathcal{M}(\mathcal{K})} \left\{ \int d\mu_2 : \mu_2 \ll \mu_1 \right\},
\]

where \( \mathcal{M}(\mathcal{K}) \) is the set of finite Borel measures on \( \mathcal{K} \).

3. Chance Optimization over a Semialgebraic Set. In this section we focus on the chance optimization problem [12]. We first provide an equivalent problem over the variables of finite Borel measures and then we will consider its relaxations in the moment space. Given polynomials \( P_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) with degree \( \delta_j \) for \( j = 1, \ldots, \ell \), we define

\[
\mathcal{K} = \{(x, q) \in \mathbb{R}^n \times \mathbb{R}^m : P_j(x, q) \geq 0, j = 1, 2, \ldots, \ell \}.
\]

Assumption 1. \( \mathcal{K} \) satisfies Putinar’s property.

Remark 3.1. Assumption [7] implies that \( \mathcal{K} \) is a compact set; hence the projections of \( \mathcal{K} \) onto \( x \)-coordinates and onto \( q \)-coordinates, i.e. \( \Pi_1 := \{x \in \mathbb{R}^n : \exists q \in \mathbb{R}^m \text{ s.t. } (x, q) \in \mathcal{K} \} \) and \( \Pi_2 := \{q \in \mathbb{R}^m : \exists x \in \mathbb{R}^n \text{ s.t. } (x, q) \in \mathcal{K} \} \), are also compact. Therefore, after rescaling of polynomials, we assume without loss of generality that \( \Pi_1 \subseteq \chi := [-1, 1]^n \) and \( \Pi_2 \subseteq Q := [-1, 1]^m \). Furthermore, instead of working on the original probability space \( (\mathbb{R}^m, \Sigma_q, \mu_q) \), we can adopt a smaller probability space \( (Q, \Sigma_q, \mu_q) \), where \( \Sigma_q := \{ S \cap Q : S \in \Sigma_q \} \) and \( \mu_q(S) := \frac{\mu_q(S)}{\mu_q(Q)} \) for all \( S \in \Sigma_q \). Therefore, we can take for granted that \( \mu_q \in \mathcal{M}(Q) \) without loss of generality, i.e. \( \mu_q \) is a finite Borel measure with \( \text{supp}(\mu_q) \subseteq Q \). We assume that moments of any order of \( \mu_q \) can be computed.
3.1. An Equivalent Problem. As an intermediate step in the development of convex relaxations of the original problem, a related infinite dimensional problem in the measure space is provided below. Consider the following problem

\[
P^*_\mu \triangleq \sup_{\mu \in \mathcal{M}(\chi)} \int d\mu,
\]

s.t. \[ \mu \leq \mu_x \times \mu_q, \] \[ \mu_x \text{ is a probability measure}, \] \[ \mu_x \in \mathcal{M}(\chi), \quad \mu \in \mathcal{M}(\mathcal{K}). \]

\[ (3.2) \]

\[ (3.2a) \]

\[ (3.2b) \]

\[ (3.2c) \]

\[ \]

**Theorem 3.1.** The optimization problems in \((1.2)\) and \((3.2)\) are equivalent in the following sense:

i) The optimal values are the same, i.e., \(P^* = P^*_\mu\).

ii) If an optimal solution to \((3.2)\) exists, call it \(\mu_x^*\), then any \(x^* \in \text{supp}(\mu_x^*)\) is an optimal solution to \((1.2)\).

iii) If an optimal solution to \((1.2)\) exists, call it \(x^*\), then Dirac measure at \(x^*\) \(\mu_x = \delta_{x^*}\) and \(\mu = \delta_{x^*} \times \mu_q\) is an optimal solution to \((3.2)\).

**Proof.** Let \((Q, \Sigma, \mu_q)\) be the probability space defined in Remark 3.1. Note that since \(\mathcal{P}_j(x, q)\) is a polynomial in random vector \(q \in \mathbb{R}^m\) for all \(x \in \mathbb{R}^n\), it is continuous in \(q\); hence \(\mathcal{P}_j(x, \cdot)\) is Borel measurable for all \(x \in \mathbb{R}^n\) and \(j = 1, \ldots, \ell\). As discussed in Remark 3.1, it can be assumed that \(\mathcal{K} \subset \chi \times Q = [-1,1]^n \times [-1,1]^m\). Define \(\mathcal{F} : \mathbb{R}^n \to \Sigma\) as follows

\[ \mathcal{F}(x) = \{ q \in \mathbb{R}^m : \mathcal{P}_j(x, q) \geq 0, \quad j = 1, 2, \ldots, \ell \}, \]

and consider the following problem over the probability measures in \(\mathcal{M}(\chi)\):

\[ P := \sup_{\mu_x \in \mathcal{M}(\chi)} \left\{ \int \mu_q(\mathcal{F}(x)) \, d\mu_x : \mu_x(\chi) = 1 \right\} \]

\[ \]

(3.4)

Note that the optimal value of \((1.2)\) can be written as \(P^* = \sup_{x \in \chi} \mu_q(\mathcal{F}(x))\).

Let \(\mu_x\) be a feasible solution to \((3.4)\). Since \(\mu_q(\mathcal{F}(x)) \leq P^*\) for all \(x \in \chi\), we have \(\int \mu_q(\mathcal{F}(x)) \, d\mu_x \leq P^*\). Thus, \(P \leq P^*\). Conversely, let \(x \in \mathbb{R}^n\) be a feasible solution to the problem in \((1.2)\) and \(\delta_x\) denote the Dirac measure on \(x\). The objective value of \(x\) in \((1.2)\) and \(\mu_x = \delta_x\) is an optimal solution to the problem in \((3.2)\) with objective value equal to \(\mu_q(\mathcal{F}(x))\). This implies that \(P^* \leq P\). Hence, \(P^* = P\), and \((3.4)\) can be rewritten as

\[ P^* = \sup_{\mu_x \in \mathcal{M}(\chi)} \left\{ \int \int \mu_q d\mu_x : \mu_x(\chi) = 1 \right\} = \sup_{\mu_x \in \mathcal{M}(\chi)} \left\{ \int \mu_x \mu_q : \mu_x(\chi) = 1 \right\}. \]

and using the epigraph formulation shown in Lemma 2.6 we finally obtain

\[ P^* = \sup_{\mu_x \in \mathcal{M}(\chi), \mu \in \mathcal{M}(\mathcal{K})} \int d\mu \quad \text{s.t.} \quad \mu \leq \mu_x \times \mu_q, \quad \mu_x(\chi) = 1. \]

Therefore, \(P^* = P^*_\mu\). \(\square\)

As an example consider the following chance constrained problem. A semialgebraic set \(\mathcal{K}\) is given as Fig.1.a in the space of \((x, q) \in \mathbb{R} \times \mathbb{R}\). Our objective is to find the decision variable \(x^*\) that attains \(P^* = \sup_{x \in \chi} \mu_q(\mathcal{F}(x))\), in presence of random variable \(q\) with known probability measure \(\mu_q\). In other words, \(x^*\) should be chosen such that the probability that the point \((x^*, q)\) belongs to \(\mathcal{K}\) becomes maximum. Fig.1.b shows the problem in the measure space, where a probability measure \(\mu_x\) is assigned to decision variable \(x\). To calculate the desired probability an integral with respect to measure \(\mu_x \times \mu_q\) over the set \(\mathcal{K}\) should be taken as \((3.1)\) (Fig.1.c). This integral is equal to the volume of a measure whose support is defined on the \(\mathcal{K}\) and has the same distribution as the measure \(\mu_x \times \mu_q\) (Fig.1.d). Therefore, for the fixed \(\mu_x\), one needs to look for the measure \(\mu\) defined on \(\mathcal{K}\) with maximum volume and bounded with measure \(\mu_x \times \mu_q\), which leads to the optimization problem \((3.2)\) in the measure space.
Fig. 3.1: a) Chance constrained example with random parameter $q$, decision variable $x$, and semi algebraic set $K$, b) Equivalent problem in the measure space with given probability measure $\mu_q$, and unknown probability measure $\mu_x$, c) Probability of given semi algebraic set $K$ for a fixed $\mu_x$ is equal to integral with respect to the measure $\mu_x \times \mu_q$ over set $K$, d) The probability is equal to the volume of the measure $\mu$ who has defined on the set $K$ and has the same distribution as the measure $\mu_x \times \mu_q$

### 3.2. Semidefinite Relaxations

In this section, we provide an infinite dimensional SDP of which feasible region is defined over real sequences in $\mathbb{R}^N$. Unlike the problem (3.2) in which we are looking for a measure, in the SDP formulation given in (3.5), we aim at finding a sequence of moments corresponding to a measure that is optimal to (3.2). After proving the equivalence of (3.2) and (3.5), we next provide a sequence of finite dimensional SDPs and show that the corresponding sequence of optimal solutions can arbitrarily approximate the optimal solution of (3.5), which characterizes the optimal solution of (3.2).

Consider the following infinite dimensional SDP:

$$
P_{y_q}^* := \sup_{y, y_x \in \mathbb{R}^N} y_0, \tag{3.5}
$$

subject to:

$$M_\infty(y) \succeq 0, \quad M_\infty(p_jy) \succ 0, \quad j = 1, \ldots, \ell, \tag{3.5a}
$$

$$M_\infty(y_x) \succeq 0, \quad \|y_x\|_\infty \leq 1, \quad y_{x_0} = 1, \tag{3.5b}
$$

$$M_\infty(Ay_x - y) \succ 0, \tag{3.5c}
$$

where $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a linear map depending only on $\mu_q$. Indeed, let $y_q := \{y_{q_\alpha}\}_{\alpha \in \mathbb{N}^m}$ be the moment sequence of $\mu_q$. Then for any given $y_x = \{y_{x_\alpha}\}_{\alpha \in \mathbb{N}^n}$, $y = Ay_x$ such that $y_\theta = y_{q_{\beta}}x_\alpha$ for all $\theta = (\beta, \alpha) \in \mathbb{N}^m \times \mathbb{N}^n$. Given $y \in \mathbb{R}^N$, $M_\infty(y) \succeq 0$ means that $M_d(y) \succeq 0$ for all $d \in \mathbb{Z}_+$. 

The following lemma establishes the equivalence of (3.2) and (3.5).

**Lemma 3.2.** Suppose that $K$ satisfies Assumption 1. If an optimal solution to (3.2) exists, call it $(\mu^*, \mu_x^*)$, then their moment sequences $(y^*, y_x^*)$ is an optimal solution to (3.5). Conversely, if an optimal
solution to (3.5) exists, call it \((y^*, y_2^*)\), then there exists representing measures \(\mu^*\) and \(\mu_2^*\) such that \((\mu^*, \mu_2^*)\) is an optimal solution to (3.2). Moreover, the optimal values of (3.2) and (3.5) are the same, i.e. \(P^*_{\mu_q} = P^* y_q\).

**Proof.** Suppose that \((\mu, \mu_2)\) is feasible to (3.2). Let \(y\) and \(y_x\) be the moment sequences corresponding to \(\mu\) and \(\mu_2\), respectively. Lemma 2.7 implies (3.5a); Lemma 2.1 and Lemma 2.2 imply (3.5b). Moreover, let \(\bar{y} = \{y_i\}_{\alpha \in N^{m+n}}\) be the moment sequence corresponding to the product measure \(\mu := \mu_2 \times \mu_q\). (3.2a) implies that \(\mu - \mu\) is a measure; hence, Lemma 2.2 implies \(M_\infty (\bar{y} - y) \geq 0\). Moreover, the definition of \(A\) implies that \(\bar{y} = Ay_x\), which gives (3.5a). Since \(y\) is chosen to be the moment sequence of \(\mu\), we have \(\int dy = y_0\). This shows that for each \((\mu, \mu_2)\) feasible to (3.2), one can construct a feasible solution to (3.5) with the same objective value. Therefore, \(P^*_{\mu_q} \geq P^*_{\mu_2}\). Note that Assumption I is not used for this argument.

Next, suppose that \((y, y_x)\) is a feasible solution to (3.5). Since \(S\) satisfies Assumption II (3.5a) and Lemma 2.5 together imply that \(y\) has a representing finite Borel measure \(\mu\) supported on \(\mathcal{M}\). The sequence \(\bar{y}\) has a representing probability measure \(\mu_2\) supported on \(\chi\), i.e., \(\mu_2 \in \mathcal{M}(\chi)\) such that \(\mu_2(\chi) = 1\). Hence, the sequence \(Ay_x\) has a representing measure \(\mu\) which is the product measure of \(\mu_x\) and \(\mu_q\), i.e., \(\mu_x = \mu_2 \times \mu_q\). Moreover, since \(\mathbb{K} \subset \chi \times \mathcal{Q} = [-1,1]^{n+m}\), (3.5a) implies that \(\mu \leq \mu_2\), which is (3.2a). Finally, the fact that \(\mu\) is a representing measure of \(y\) implies that \(\int dy = y_0\). Therefore, \(P^*_{\mu_q} \leq P^*_{\mu_2}\). Combining this with the above result gives us \(P^*_{\mu_q} = P^*_{\mu_2}\).

In order to have tractable approximations to the infinite dimensional SDP in (3.5), we consider the following sequence of SDPs defined in (3.6).

\[
P_d := \sup_{y \in \mathbb{R}^{S_{n+m-2d}, y_x \in \mathbb{R}^{S_{n-2d}}} \mid \begin{align}
M_d(y) &\geq 0, \quad M_{d-j}(y) \geq 0, \quad j = 1, \ldots, \ell, \\
M_d(y_x) &\geq 0, \quad \|y_x\|_\infty \leq 1, \quad y_{x_0} = 1, \\
M_d(A_d y_x - y) &\geq 0
\end{align} \right. \tag{3.6}
\]

where \(\delta_j\) is the degree of \(\mathcal{P}_j\), \(r_j := \left\lceil \frac{n}{2} \right\rceil\) for all \(1 \leq j \leq \ell\), and \(A_d : \mathbb{R}^{S_{n-2d}} \rightarrow \mathbb{R}^{S_{n+m-2d}}\) is defined similarly to \(A\) in (3.5). Indeed, let \(y_q := \{y_{q_\alpha}\}_{\alpha \in N_{2d}^{m+n}}\) be the truncated moment sequence of \(\mu_q\). Then for any given \(y_x = \{y_{x, \alpha}\}_{\alpha \in N_{2d}^m}\), \(y = A_d y_x\) such that \(y_0 = y_q y_{x, \alpha}\) for all \(\theta = (\beta, \alpha) \in N_{2d}^{m+n}\).

In the following theorem, it is shown that the sequence of optimal solutions to the SDPs in (3.6) converges to the solution of the infinite dimensional SDP in (3.5). In essence, the following theorem is very similar to Theorem 3.2 in [23]; however, for the sake of completeness we give its proof below.

**Theorem 3.3.** For all \(d \geq 1\), there exists an optimal solution \((y^d, y_x^d) \in \mathbb{R}^{S_{n+m-2d} \times R^{S_{n-2d}}}\) to (3.6) with the optimal value \(P_d\). Moreover,

i) \(\lim_{d \rightarrow 2} P_d = P^*\), the optimal value of (1.2).

ii) Let \(S := \{y^d, y_x^d\}_{d \in \mathbb{N}} \subset \mathbb{R}^{S_{n+m} \times R^n}\) such that each element is obtained by zero-padding both \(y^d\) and \(y_x^d\). Then \(S\) is an accumulation point of \(\mathcal{S}\) in the weak * topology of \(\sigma(\ell, \ell_1)\) and every accumulation point of \(S\) is an optimal solution to (3.5). Hence, there exists corresponding representing measures \((\mu^*, \mu_2^*)\) that is optimal to (3.2) and any \(x^* \in \text{supp}(\mu_2^*)\) is optimal to (1.2).

**Proof.** First, we will show that for all \(d \geq 1\), the corresponding feasible region of (3.6) is bounded. Fix \(d \geq 1\). Let \((y, y_x)\) be a feasible solution to (3.6). Then from (3.6d), we have \(\|y_x\|_\infty \leq 1\). Since \(\mu_q\) is a probability measure supported on \(Q = [-1,1]^m\), Lemma 2.1 implies that \(\|y_q\|_\infty \leq 1\) as well. Moreover, the definition of \(A_d\) further implies that \(\|A_d y_x\|_\infty \leq 1\). Let \(\bar{y} := A_d y_x\). It follows from (3.6b) that the diagonal elements of \(M_d(\bar{y} - y)\) are nonnegative, i.e. \(\bar{y}_{2i} - y_{2i} \geq 0\) for all \(\alpha \in N_{2d}^{n+m}\). This implies that

\[
\max \left\{y_0, \max_{i=1, \ldots, n+m} L_y (x^{2d}_i) \right\} \leq \max_{\alpha \in N_{2d}^{n+m}} y_2\alpha \leq \max_{\alpha \in N_{2d}^{n+m}} \bar{y}_{2\alpha} \leq \|\bar{y}\|_\infty \leq 1, \tag{3.7}
\]

where the first inequality follows from the fact that

\[
\{y_0\} \cup \left\{L_y (x^{2d}_i) : i = 1, \ldots, n+m\right\} \subset \{y_{2\alpha} : \alpha \in N_{2d}^{n+m}\}.
\]

From (3.6a), we have \(M_2(y) \geq 0\). Hence, using Lemma 2.3 (3.7) implies that \(\|y_0\| \leq \|\bar{y}\|_\infty \leq 1\) for all \(\alpha \in N_{2d}^{n+m}\). Therefore, the feasible region is bounded. Since the cone of positive semidefinite matrices is a
closed set and all the mappings in \( \mathbb{X} \) is linear, we also conclude that the feasible region is compact. Hence, there exists an optimal solution \((\mathbf{y}^d, \mathbf{y}^\star_d)\) to the problem \((3.6)\) for all \(d \geq 1\).

Fix \(d \geq 1\). Clearly, for any given feasible solution \((\mathbf{y}, \mathbf{y}_x)\) to \((3.5)\), by truncating the both sequences to vectors \(\mathbf{y} \in \mathbb{R}^{S_{n+m,2d}}\) and \(\mathbf{y}_x \in \mathbb{R}^{S_{n,2d}}\), we can construct a feasible solution to \((3.6)\) with the same objective value. Hence, it can be concluded that \(P_d \geq P_{d'}\) for all \(d \geq 1\). Moreover, the same argument also shows that \(P_d \geq P_{d'}\) for all \(d' \geq d\). Hence, \(\{P_d\}_{d \in \mathbb{Z}_+}\) is a decreasing sequence bounded below by \(P_{y^\star}\). Therefore, it is a convergent sequence and has a limit such that \(\lim_{k \to \infty} P_k \geq P_{y^\star}\).

In order to collect all the optimal solutions corresponding to different \(d\) in one space, we extend \((\mathbf{y}^d, \mathbf{y}^\star_d)\) to vectors in \(\ell_\infty\) (The Banach space of bounded sequences, equipped with the sup-norm) by zero-padding, i.e. we set \(\mathbf{y}^d = 0\) for all \(\alpha \in \mathbb{N}^{n+m}\) such that \(\|\alpha\|_1 > 2d\), and \(\mathbf{y}_x^d = 0\) for all \(\alpha \in \mathbb{N}^\ell\) such that \(\|\alpha\|_1 > 2d\). Let \(\mathbb{B}_\infty\) be the unit ball of \(\ell_\infty\), which is weak * sequentially compact for weak * topology \(\sigma(\ell_\infty, \ell_1)\). Since \(\{\mathbf{y}^d\}_{d \in \mathbb{Z}_+} \subset \mathbb{B}_\infty\) and \(\{\mathbf{y}^d_x\}_{d \in \mathbb{Z}_+} \subset \mathbb{B}_\infty\), there exists a subsequence \(\{d_k\} \subset \mathbb{Z}_+\) such that \(\{\mathbf{y}^d_{d_k}\}_{k \in \mathbb{Z}_+}\) and \(\{\mathbf{y}^d_{x,d_k}\}_{k \in \mathbb{Z}_+}\) weakly converge to \(\mathbf{y}^* \in \mathbb{B}_\infty\) and \(\mathbf{y}_x^* \in \mathbb{B}_\infty\), respectively. Hence,

\[
\lim_{k \to \infty} \mathbf{y}^d_{\alpha,d_k} = \mathbf{y}^\star_{\alpha}, \quad \forall \alpha \in \mathbb{N}^{n+m}, \quad \lim_{k \to \infty} \mathbf{y}^d_{x,d_k} = \mathbf{y}_x^\star, \quad \forall \alpha \in \mathbb{N}^\ell.
\]  

Fix \(d \geq 1\), then for all \(k \in \mathbb{Z}_+\) such that \(d_k \geq d\), we have

\[
M_d(\mathbf{y}^{d_k}) \geq 0, \quad M_{d-r_j}(\mathbf{P}_j \mathbf{y}^{d_k}) \geq 0, \quad j = 1, \ldots, \ell, \\
M_d(\mathbf{y}_{x,d_k}^d) \geq 0, \quad \|\mathbf{y}_{x,d_k}^d\|_\infty \leq 1, \quad y_{x,d_k}^{d_0} = 1, \\
M_d(\mathbf{A} \mathbf{y}_{x,d_k}^d - \mathbf{y}_{x,d_k}^d) \geq 0.
\]

Since \(d \in \mathbb{Z}_+\) is arbitrary, by taking the limit as \(k \to \infty\), we see that \((\mathbf{y}^*, \mathbf{y}_x^*)\) satisfies all the constraints in \((3.6)\). Therefore, \(\mathbf{y}_0^* \leq \mathbf{P}_{y^\star}\). On the other hand, \(\mathbf{y}_0^* = \lim_{k \to \infty} \mathbf{y}^{d_k}_{\alpha} = \lim_{k \to \infty} \mathbf{P}_{d_k}\). Moreover, since every subsequence of a convergent sequence converges to the same point, we have \(\lim_{k \to \infty} \mathbf{P}_k = \lim_{k \to \infty} \mathbf{P}_{d_k} = \mathbf{P}_{y^\star}\). This shows that the subsequential limit \((\mathbf{y}^*, \mathbf{y}_x^*)\) is an optimal solution to \((3.5)\). The rest of the claims follow from our previous results: Theorem 3.1 and Lemma 3.2.

\[\square\]

### 3.3. Example

We now present an example that illustrates the effectiveness of the proposed methodology in finding good approximations for the optimal solution to the chance optimization problem in \((1.2)\) even with lower order relaxations. The example is low dimensional for illustrative purposes. Consider the problem

\[
\sup_{x \in \mathbb{R}} \mu_q(q \{q \in \mathbb{R} : \mathcal{P}(x, q) \geq 0 \}), \tag{3.9}
\]

where

\[
\mathcal{P}(x, q) = 0.5 q \left( q^2 + (q - 0.5)^2 \right) - \left( q^4 + q^2(q - 0.5)^2 + (q - 0.5)^4 \right). \tag{3.10}
\]

The uncertain parameter \(q \in \mathbb{R}\) has a uniform distribution on \([-1, 1]\). To obtain an approximate solution, we solve the SDP in \((3.6)\) with the minimum relaxation order \(d = 2\) since the degree of the polynomial in \((3.10)\) is 4. The moment vectors \(\mathbf{y}_q, \mathbf{y}_x\), and \(\mathbf{y}\) for the measures \(\mu_q\) and \(\mu_x\), and \(\mu\) up to order four are

\[
\mathbf{y}_q^T = [1, 0, 1/3, 0, 1/5], \quad \mathbf{y}_x^T = [1, y_{x1}, y_{x2}, y_{x3}, y_{x4}] \\
\mathbf{y}^T = [y_{00} \mid y_{10}, y_{01} \mid y_{20}, y_{11}, y_{02} \mid y_{30}, y_{21}, y_{12}, y_{03} \mid y_{40}, y_{31}, y_{22}, y_{13}, y_{04}]
\]

Given moment vectors \(\mathbf{y}_q\) and \(\mathbf{y}_x\), the moment vector \(\bar{\mathbf{y}}\) for the measure \(\mathbf{P} = \mu_x \times \lambda_q\) has the form

\[
\bar{\mathbf{y}}^T = [1 \mid y_{x1}, y_{q1} \mid y_{x2}, y_{x1}, y_{q1}, y_{q2} \mid y_{x3}, y_{x2}, y_{x1}, y_{q1}, y_{q2}, y_{q3} \mid y_{x4}, y_{x3}, y_{x2}, y_{x1}, y_{q1}, y_{q2}, y_{q3}, y_{q4}, y_{q5}] \\
= [1 \mid y_{x1}, 0 \mid y_{x2}, 0, 1/3 \mid y_{x3}, 0, 1/3 y_{x1}, 0 \mid y_{x4}, 0, 1/3 y_{x2}, 0, 1/5]
\]

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To solve the SDP in (3.6) with \( d = 2 \), GloptiPoly was used which is a Matlab-based toolbox aimed at optimizing moments of measures [21]. Using the Matlab code provided in Fig 3.2, the following solution was obtained

\[
(y^*)^T = [0.66, 0.3, 0.14, 0.16, 0.07, 0.1, 0.08, 0.03, 0.05, 0.04, 0.02, 0.02, 0.02, 0.02]
\]

\[
(y_k^*)^T = [1, 0.50, 0.25, 0.13, 9.61].
\]

Since the solution obtained satisfies \( M_d(y_x) \geq 0 \) and \( \text{rank}(M_d(y_x)) = 1 \), we approximate the corresponding measure \( \mu_x^* \) with a Dirac measure on \( y_x^* \). Therefore, the approximate solution solution to (1.2) is \( x^* = 0.5 \). Also, the obtained estimate of the optimal probability is \( y_{60}^0 = 0.66 \).

To test the accuracy of the results obtained, a Monte Carlo simulation was used to estimate the true optimum. This computationally intensive method estimated that \( x^* = 0.5 \) with optimal probability of 0.25. Therefore, even with this very low relaxation order, one is able to accurately estimate the optimum value of the decision variable. To obtain better estimates of the optimum probability, one needs to increase the relaxation order \( d \). Figure 3.3 displays the optimum probability obtained by solving SDP relaxations \( 3.6 \) of increasing degrees \( d = 2, \ldots, 25 \) with two different objective functions. One is obtained by maximizing the \( \int d\mu \) and the other is obtained by maximizing the \( \int pd\mu \). When \( \int pd\mu \) instead of \( \int d\mu \) is maximized a much faster convergence is observed [22].

### 4. Chance Optimization over a Union of Sets

We now focus on the more general setting of the chance optimization problem in (1.1). Given polynomials \( P^k_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) with degree \( \delta^{(k)}_j \) for \( j = 1, \ldots, \ell_k \) and \( k = 1, \ldots, N \), the semi-algebraic set of interest is \( \mathcal{K} = \bigcup_{k=1}^N \mathcal{K}_k \), where

\[
\mathcal{K}_k = \left\{(x, q) \in \mathbb{R}^n \times \mathbb{R}^m : P^{(k)}_j(x, q) \geq 0, \ j = 1, \ldots, \ell_k \right\}, \ k = 1, \ldots, N. \tag{4.1}
\]

Similar to the previous section, we need Putinar’s property to hold for \( \mathcal{K}_k \) for all \( k = 1, \ldots, N \). With the following assumption, we can ensure this.

**Assumption 2.** \( \mathcal{K} = \bigcup_{k=1}^N \mathcal{K}_k \) is bounded, where \( \mathcal{K}_k \) is defined in (4.1).

Hence, as discussed in Remark 3.1, we can assume without loss of generality that \( \mathcal{K} \subseteq \chi \times \mathcal{Q} \) and the probability measure \( \mu_q \in \mathcal{M}(\mathcal{Q}) \), where \( \chi = [-1,1]^n \) and \( \mathcal{Q} = [-1,1]^m \). Therefore, for all \( (x, q) \in \mathcal{K} \), we
have \(|x|^2 + |q|^2 \leq m + n\). Define \(P_0^{(k)}(x, q) := m + n - \sum_{i=1}^m x_i^2 - \sum_{i=1}^m q_i^2\) for all \(k = 1, \ldots, N\), and redefine \(K_k\) to be \(K_k = \{(x, q) : P_j^{(k)}(x, q) \geq 0, j = 0, \ldots, \ell_k\}\) — note that index \(j\) starts from 0. Since polynomials are continuous in \((x, q)\), by redefining \(K_k\) for all \(k\), we make sure that \(K_k\) satisfies Putinar’s property for each \(k\) and we still have \(K = \cup_{k=1}^N K_k\).

The objective of this section is to provide a sequence of SDP relaxations to the chance optimization problem in (1.1) with \(N > 1\), and show that the results presented in the previous sections can be easily extended for this case. More precisely, we start by providing an equivalent problem in the measure space and then develop relaxations based on moments of measures.

### 4.1. An Equivalent Problem

As an intermediate step in the development of convex relaxations of (1.1), an equivalent problem in the measure space is provided below.

\[
P_{\mu_q}^* := \sup_{\mu_x, \mu_q} \sum_{k=1}^N \int d\mu_k, \tag{4.2}
\]

subject to

\[
\sum_{k=1}^N \mu_k \preceq \mu_x \times \mu_q, \tag{4.2a}
\]

\[
\mu_x \text{ is a probability measure}, \tag{4.2b}
\]

\[
\mu_x \in \mathcal{M}(\chi), \quad \mu_k \in \mathcal{M}(K_k) \quad k = 1, \ldots, N. \tag{4.2c}
\]

This problem is equivalent to the problem addressed in this paper in the following sense.

**Theorem 4.1.** The optimization problems in (1.1) and (4.2) are equivalent in the following sense:

i) The optimal values are the same, i.e. \(P^* = P_{\mu_q}^*\).

ii) If an optimal solution to (4.2) exists, call it \(\mu_x^*\), then any \(x^* \in \text{supp}(\mu_x^*)\) is an optimal solution to (1.1).

iii) If an optimal solution to (1.1) exists, call it \(x^*\), then Dirac measure at \(x^*\), \(\mu_x = \delta_{x^*}\) and \(\mu = \delta_{x^*} \times \mu_q\) is an optimal solution to (4.2).

**Proof.** Let \(P^*\) denote the optimal value of (1.1), and \(K = \cup_{k=1}^N K_k\), where \(K_k\) is defined in (4.1). It can be proven as in Theorem 3.1 that

\[
P^* = \sup_{\mu_x \in \mathcal{M}(\chi)} \sup_{\mu_q \in \mathcal{M}(K)} \int d\mu \quad \text{s.t.} \quad \mu \preceq \mu_x \times \mu_q, \quad \mu_x(\chi) = 1. \tag{4.3}
\]
Lemma 2.3 implies that the infinite dimensional SDP in (4.2). More precisely, we have the following result. For all \( P \) and any \( \mu \), let \( \delta \) be a feasible solution to (4.2) with objective value \( \delta \). Since \( \delta \in M(\mathcal{K}) \) for all \( k = 1, \ldots, N \), we have \( \sum_{k=1}^{N} \mu_{k} \in M(\mathcal{K}) \). Hence, \( \left( \sum_{k=1}^{N} \mu_{k} \right) \) is a feasible solution to (4.3) with objective value \( \delta \), as well. Clearly, this shows that \( \mathbf{P}_{\mu_{n}} \leq \mathbf{P}^{\ast} \), where \( \mathbf{P}_{\mu_{n}} \) denotes the optimal value of (4.2).

Suppose that \( (\mu, \mu_{x}) \) is a feasible solution to (4.3) with objective value \( \delta \). Define \( \{ \mu_{k} \}_{k=1}^{N} \) as follows

\[
\mu_{k}(S) := \mu \left( S \cap \left( \mathbb{K}_{k} \setminus \bigcup_{j=0}^{k-1} \mathbb{K}_{j} \right) \right), \quad \forall S \in \sigma(\mathcal{K}),
\]

(4.4)

for all \( k = 1, \ldots, N \), where \( \mathbb{K}_{0} := \emptyset \) and \( \sigma(\mathcal{K}) \) denotes the Borel-algebra over \( \mathcal{K} \). Definition in (4.4) implies that \( \mu_{k} \in \mathcal{M}(\mathcal{K}) \) for all \( k = 1, \ldots, N \), and \( \sum_{k=1}^{N} \mu_{k}(S) = \mu(S) \) for all \( S \in \sigma(\mathcal{K}) \). Hence, \( \{ \mu_{k} \}_{k=1}^{N} \) and \( \mu_{x} \) form a feasible solution to (4.2) with objective value equal to \( \delta \). Therefore, \( \mathbf{P}_{\mu_{n}} = \mathbf{P}^{\ast} \).

4.2. Semidefinite Relaxations. In this section, a sequence of semidefinite programs is provided which can arbitrarily approximate the optimal solution of (4.2). As before, this is done by considering moments of measures instead of the measures themselves. Define the following optimization problem indexed by the relaxation order \( d \).

\[
\mathbf{P}_{d} := \sup_{y_{k} \in \mathbb{R}^{s_{n+m,2d}}, \ y_{x} \in \mathbb{R}^{s_{n,2d}}} \sum_{k=1}^{N} y_{k_{0}}, \quad \text{s.t.} \quad M_{d}(y_{k}) \succeq 0, \quad M_{d-x}(y_{k}) \succeq 0, \quad j = 1, \ldots, l_{k}, \quad k = 1, \ldots, N
\]

(4.5a)

\[
M_{d}(y_{x}) \succeq 0, \quad \|y_{x}\|_{\infty} \leq 1, \quad y_{x_{0}} = 1,
\]

(4.5b)

\[
M_{d} \left( A_{d}y_{X} - \sum_{k=1}^{N} y_{k} \right) \succeq 0,
\]

(4.5c)

where \( s_{j}^{(k)} \) is the degree of \( \mathcal{P}_{j}^{(k)} \), \( r_{j}^{(k)} := \left[ \frac{s_{j}^{(k)}}{2} \right] \) for all \( 1 \leq j \leq \ell_{k} \) and \( 1 \leq k \leq N \); and \( A_{d} : \mathbb{R}^{s_{n,2d}} \rightarrow \mathbb{R}^{s_{n+m,2d}} \) is defined similarly to \( A \) in (3.5). Indeed, let \( y_{q} := \{ y_{q_{\beta}} \}_{\beta \in \mathcal{N}_{n+m,2d}} \) be the truncated moment sequence of \( \mu_{q} \). Then for any given \( y_{x} = \{ y_{x_{\alpha}} \}_{\alpha \in \mathcal{N}_{n,2d}} \), \( y = A_{d}y_{X} \) such that \( y_{\theta} = y_{q_{\beta}} y_{x_{\alpha}} \) for all \( \theta = (\beta, \alpha) \in \mathcal{N}_{n+m,2d} \).

Next, we show that the sequence of optimal solutions to the SDPs in (4.5) converges to the solution of the infinite dimensional SDP in (4.2). More precisely, we have the following result.

THEOREM 4.2. For all \( d \geq 1 \), there exists an optimal solution \( (\{ y_{k}^{d} \}_{k=1}^{N}, y_{X}^{d}) \) to (4.5) with the optimal value \( \mathbf{P}_{d} \). Moreover,

i) \( \lim_{d \rightarrow \infty} \mathbf{P}_{d} = \mathbf{P}^{\ast} \), the optimal value of (1.1).

ii) Let \( S \) be such that each element is obtained by zero-padding \( y^{d} \) and \( y_{k}^{d} \) for \( 1 \leq k \leq N \). There exists an accumulation point of \( S \) in the weak \( * \) topology of \( \sigma(\ell_{\infty}, \ell_{1}) \) and for every accumulation point of \( S \), there exists corresponding representing measures \( \{ \mu_{k}^{*} \}_{k=1}^{N} \) that is optimal to (4.2) and any \( x^{*} \in \text{supp}(\mu_{k}^{*}) \) is optimal to (1.1).

Proof. Let \( \{ y_{k} \}_{k=1}^{N} \subset \mathbb{R}^{s_{n+m,2d}} \) and \( y_{x} \in \mathbb{R}^{s_{n,2d}} \) be a feasible solution to (4.5). As in Theorem 3.3, it can be shown that

\[
\max \left\{ y_{0}, \max_{i=1,\ldots,n+m} L_{X_{i}}(x_{i}^{2d}) \right\} \leq 1, \quad \text{(4.6)}
\]

where \( y := \sum_{k=1}^{N} y_{k} \). Note that \( L_{X}(x_{i}) = \sum_{k=1}^{N} L_{y_{k}}(x_{i}^{2d}) \), and \( \{ L_{y_{k}}(x_{i}^{2d}) \}_{i=1}^{n+m} \) is a subset of diagonal elements of \( M_{d}(y_{k}) \succeq 0 \) for each \( k \in \{1, \ldots, N\} \). Hence, \( L_{y_{k}}(x_{i}^{2d}) \succeq 0 \) for all \( i \in \{1, \ldots, n+m\} \) and \( k \in \{1, \ldots, N\} \). Therefore, (4.6) implies that \( \max\{ y_{k}, \max_{i=1,\ldots,n+m} L_{y_{k}}(x_{i}^{2d}) \} \leq 1 \) for all \( k \in \{1, \ldots, N\} \). Lemma 2.3 implies that \( \|y_{k}\|_{1} \leq 1 \) for all \( \alpha \in \mathcal{N}_{n+m,2d} \). Therefore, the feasible region is bounded. The rest of the proof is exactly the same as in Theorem 3.3. \( \square \)
5. Implementation and Numerical Results. In previous sections, we showed that chance optimization problem in (1.1) can be relaxed to a sequence of SDPs. In this section, we go one step further to improve approximation quality of the relaxed problems in practice and implement an efficient first-order algorithm to solve SDPs.

5.1. Regularized Chance Optimization Using Trace Norm. As shown in Theorem 3.1 and Theorem 4.1 if the chance optimization problems in (1.2) and (1.1) have unique optimal solution \( x^* \), then the optimal distribution \( \mu^*_x \) is a Dirac measure whose mass is concentrated on the single point \( x^* \), i.e. its support is the singleton \{x^*\}. Such distributions, have moment matrices with rank one. To improve the solution quality of the algorithm, one can incorporate this observation in the formulation of the relaxed problem. For the sake of notational simplicity, in this section we will consider the regularized version of chance optimization problem (3.1) for presenting the algorithm:

\[
\min_{y \in \mathbb{R}^{s \times n \times m \times d}, \ y \in \mathbb{R}^{s \times n \times m \times d}} \omega_r \, \text{Tr}(M_d(y_x)) - y_t \quad \text{subject to} \quad (3.6a), (3.6b), (3.6c)
\]  

(5.1)

for some \( \omega_r > 0 \), where \( \text{Tr}(\cdot) \) denotes the trace function. Our objective is to achieve the maximum probability with a low-rank moment matrix \( M_d(y^*_x) \), hopefully with rank 1. To this end, we regularize the objective with trace norm. Since, \( M_d(y^*_x) \geq 0, \text{Tr}(M_d(y^*_x)) = \text{trace of singular values of } M_d(y^*_x) \), which is called the nuclear norm of \( M_d(y^*_x) \). This is a well known approach for obtaining low-rank solutions. Indeed, the nuclear norm is the convex envelope of the rank function and, in practice, produces good results; see [18] and [44] for details.

To be able to solve the SDP in (5.1) involving large scale matrices, one need to implement an efficient and fast convex optimization algorithm. Recently, a first-order augmented Lagrangian algorithm ALCC has been proposed in [2] to deal with regularized conic convex problems. We will adapt this algorithm to solve SDPs of the form in (5.1). In the following section, we briefly discuss the algorithm ALCC.

5.2. First-Order Augmented Lagrangian Algorithm. Consider the conic convex problems of the form

\[
\min \{ \gamma(x) : Ax - b \in \mathcal{C}, \ x \in \chi \},
\]  

(5.2)

where \( \gamma : \mathbb{R}^n \rightarrow \mathbb{R} \) is a convex function with a Lipschitz continuous gradient \( \nabla \gamma \) of which Lipschitz constant is \( L_\gamma \), \( A \in \mathbb{R}^{m \times n} \), \( \mathcal{C} \subset \mathbb{R}^n \) is a nonempty, closed, convex cone, and \( \chi \subset \mathbb{R}^n \) is a compact, convex set. Let \( B > 0 \) be the diameter of \( \chi \), i.e. \( \|x - y\|_2 \leq \beta \) for all \( x, y \in \chi \); and we assume that \( B \) is given. Indeed, the conic convex problems considered in [2, 3] are more general allowing non-smooth regularizers in the objective function, \( \rho(x) + \gamma(x) \), where \( \rho : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\} \) a proper, closed, convex function. ALCC iterate sequence \( \{x_k\} \) for (5.2) is constructed by inexactly solving a sequence of subproblems of the form:

\[
\min_{x \in \chi} f_k(x) := \frac{1}{\nu_k} \gamma(x) + \frac{1}{2} d_C^2(Ax - \theta_k),
\]  

(5.3)

where \( d_C : \mathbb{R}^n \rightarrow \mathbb{R} \) is the distance function to the cone \( \mathcal{C} \), i.e. \( d_C(x) := \|x - \Pi_C(x)\|_2 \) and \( \Pi_C(z) = \text{argmin}\{\|z - \tilde{z}\|_2 : \ z \in \mathcal{C}\} \). Note that \( f_k \) is a convex function with Lipschitz continuous gradient \( \nabla f_k(x) = -A^*\Pi_C(\theta_k - Ax + b) \) of which Lipschitz constant is \( L_k := \frac{1}{\nu_k} L_\gamma + \sigma_{\max}^2(A) \), where \( \mathcal{C}^* = \{z \in \mathbb{R}^n : \langle z, y \rangle \geq 0, \ \forall y \in \mathcal{C}\} \) is the dual cone of \( \mathcal{C} \) and \( A^*(\cdot) : \mathbb{R}^m \rightarrow \mathbb{R}^n \) is the adjoint operator of \( A : \mathbb{R}^n \rightarrow \mathbb{R}^m \). These subproblems are inexactly solved using an Accelerated Proximal Gradient (APG) algorithm [4, 38, 39, 49]. ALCC algorithm for (5.2) is presented in Figure 5.1. Step 11 and Step 12 in Figure 5.1 are the bottleneck steps (one \( \nabla \gamma \) evaluation and two projections: one onto \( \mathcal{C}^* \), and one onto \( \chi \)) -- in Step 11 \( \nabla f_k \) is evaluated at \( x_\ell \), and then in Step 12 \( x_\ell^{(1)} \) is computed via a projected gradient step of length \( 1/L_k \). It has been shown in [2] that \( \lim_{\ell \rightarrow \infty} \theta_k \) is an optimal dual solution to (5.2), and every limit point of ALCC iterate sequence \( \{x_k\} \) is an optimal solution of the conic convex program (5.2) and for all \( \varepsilon > 0 \), \( x_k \) is \( \varepsilon \)-feasible and \( \varepsilon \)-optimal within \( O\left(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon}\right) \) projections and gradient computations. Moreover, if \( A \) is surjective, then \( O\left(\frac{1}{\varepsilon} \right) \) projections are sufficient [3]. In our numerical experiments in Section 5.3 we used \( \|x_k - x_{k-1}\|_2/(1 + \|x_{k-1}\|_2) \leq \varepsilon \) as the stopping condition for ALCC.
Semidefinite program of (5.1) is a special case of the conic convex problem in (5.2), where \( \gamma(y_x, y) = c^T y_x + c_p^T y \) for some \( c_e \in \mathbb{R}^{S_d \times 2d} \) and \( c_p \in \mathbb{R}^{S_e \times m \times 2d} \) since the objective of (5.1) is linear in \( (y, y_x) \), hence \( L_{\gamma} = 0 \), the conic constraint \( A(\cdot) - b \in C \) in (5.2) is replaced by linear matrix inequalities (LMI), with \( C = C^* \) being the cone of positive semidefinite matrices \( S_+ \), and the compact set \( \chi = \{ (y, y_x) : \|y\|_\infty \leq 1, \|y_x\|_\infty \leq 1, (y_x)_{0} = 0 \} \). Hence, \( \Pi_{C}(\cdot) = \Pi_{C}(.\cdot) \) can be computed using one eigenvalue decomposition, and \( \Pi_{C}(\cdot) \) is very efficient and can be done in linear time.

5.3. Numerical Examples. We now present four numerical examples that illustrate the effectiveness of the proposed semidefinite relaxation in (5.1) and the augmented Lagrangian algorithm presented in Section 5.2 in finding good approximate solutions to the chance constrained problems in (1.2) and (1.1), even with lower order relaxations. In all the tables, \( d \) denotes the relaxation order, \( \mathbf{P}_d \) denotes the objective value of the solution computed corresponding to relaxation order \( d \), \( \text{iter} \) denotes the total number of algorithm iterations, \( n_{\text{var}} \) denotes the number of variables, i.e. total number of moments used, and \( \text{cpu} \) denotes the computing time required in seconds. For ALCC \( \text{iter} \) is the total number of APG iterations, and for GloptiPoly it denotes the total number of SeDuMi \( \text{itr} \) iterations.

5.3.1. Example 1: A Simple Semialgebraic Set. Consider the chance optimization problem

\[
\sup_{x \in \mathbb{R}^5} \mu_q \left( \{ q \in \mathbb{R}^5 : \mathcal{P}(x, q) \geq 0 \} \right) \tag{5.4}
\]

where

\[
\mathcal{P}(x, q) = 0.185 + 0.5x_1 - 0.5x_2 + x_3 - x_4 + 0.5q_1 - 0.5q_2 + q_3 - q_4 - x_1^2 - 2x_1q_1 - x_2^2 - 2x_2q_2 - x_3^2 - 2x_4q_4 - x_5^2 + 2x_5q_5 - q_1^2 - q_2^2 - q_3^2 - q_4^2 - q_5^2,
\]

and the uncertain parameters \( q_1, q_2, q_3, q_4, q_5 \) have a uniform distribution: \( q_1 \sim U[-1, 0], q_2 \sim U[0, 1], q_3 \sim U[-0.5, 1], q_4 \sim U[-1, 0.5], q_5 \sim U[0, 1], \) where \( U[a, b] \) denotes the uniform distribution between \( a \) and
Consider the chance optimization problem \( q \), where \( \nu \). The optimum solution and probability obtained by Monte Carlo method is \( x_1^* = 0.75, x_2^* = -0.75, x_3^* = 0.25, x_4^* = -0.25, x_5^* = 0.5 \), and \( P^* = 0.75 \). To obtain an approximate solution, we solve the SDP in (5.6) using GloptiPoly and ALCC. For ALCC, we set \( \nu_0 \) to 1, 5 \( \times \) \( 10^{-2} \) and 5 \( \times \) \( 10^{-3} \) when \( d \) is equal to 1, 2, and 3, respectively, and \( \text{tol} = 1 \times \) \( 10^{-2} \). The results for relaxation order \( d = 1, 2, 3 \) are shown in Table 5.1. As in Figure 4.8, \( P_d \) approximates \( P^* \) better when the objective function is max \( \int P d\mu \) instead of max \( \int d\mu \). In Table 5.2, we report the results for max \( \int P d\mu \). For both objective functions, we reported results up to order \( d = 3 \), because GloptiPoly did not terminate in 24 hours.

### 5.3.2. Example 2: Union of Simple Sets

Given the following polynomials

\[
P^{(1)}(x, q) = -0.263 + 0.4x_1 - 0.4x_2 + 0.8x_3 - 0.8x_4 + 1.2x_5 + 0.1q_1 + 0.08q_2 + 0.04q_3
\]

\[
+ 0.4q_1 + 0.6q_5 - x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_5^2 - 0.5q_1^2 - 0.4q_2^2 - 0.1q_3^2 - q_4^2 - q_5^2,
\]

\[
P^{(2)}(x, q) = -2.06 + 0.4x_1 - 0.8x_2 + 3.2x_3 - 1.6x_4 + 3.6x_5 - 0.4q_1 - 0.4q_2 - 0.2q_3
\]

\[
- 0.2q_4 - 0.8q_5 - x_1^2 - 2x_2^2 - 4x_3^2 - 2x_4^2 - 3x_5^2 - q_1^2 - q_2^2 - q_3^2 - q_4^2 - q_5^2,
\]

consider the chance optimization problem

\[
\sup_{x \in \mathbb{R}^5} \mu_q \left( \bigcup_{j=1,2} \left\{ q \in \mathbb{R}^5 : P^{(j)}(x, q) \geq 0 \right\} \right), \tag{5.5}
\]

where \( q_i \sim U[-0.5, 0.5] \) for all \( i = 1, \ldots, 5 \), i.e. the uncertain parameters \( q_i \) are uniformly distributed on \([-0.5, 0.5]\). The optimum solution and probability obtained by Monte Carlo method is \( x_1^* = 0.2, x_2^* = -0.2, x_3^* = 0.4, x_4^* = -0.4, x_5^* = 0.6 \), and \( P^* = 0.80 \). To obtain an approximate solution, we solve the SDP in (5.5) using ALCC, where we set \( \nu_0 \) to 1, \( 1 \times 10^{-1} \) and \( 1 \times 10^{-3} \) when \( d \) is equal to 1, 2, and 3, respectively, and \( \text{tol} = 1 \times \) \( 10^{-2} \). The results for relaxation order \( d = 1, 2, 3 \) are shown in Table 5.3. When the objective function is chosen as \( \int P^{(1)}d\mu_1 + \int P^{(2)}d\mu_2 \), the optimal probability \( P_d \) is equal to 0.992, 0.887 and 0.852 corresponding to \( d \) equal to 1, 2 and 3, respectively. For this example, GloptiPoly fails to extract the optimum solution.

### Table 5.1: ALCC and GloptiPoly results for Example 1 with objective function \( \int d\mu \)

| \( d \) | ALCC | GloptiPoly |
|-------|------|------------|
| \( d \) | 1    | 2          | 3          |
| \( d \) | 1    | 2          | 3          |
| \( n_{\text{var}} \) | 87   | 1127       | 8463       |
| \( x_1 \) | 0.742 | 0.745       | 0.757       |
| \( x_2 \) | -0.777 | -0.701     | -0.721     |
| \( x_3 \) | 0.213  | 0.226       | 0.216       |
| \( x_4 \) | -0.239 | -0.250     | -0.236     |
| \( x_5 \) | 0.500  | 0.551       | 0.557       |
| \( P_d \) | 0.991 | 0.971       | 0.961       |
| \( \text{iter} \) | 169  | 624        | 1207       |
| \( \text{cpu} \) | 0.9  | 28.1       | 785.9       |

### Table 5.2: ALCC and GloptiPoly results for Example 1 using objective function \( \int P d\mu \)

| \( d \) | ALCC | GloptiPoly |
|-------|------|------------|
| \( d \) | 1    | 2          | 3          |
| \( d \) | 1    | 2          | 3          |
| \( P_d \) | 0.990 | 0.804       | 0.734       |
| \( \text{iter} \) | 156  | 91         | 1209       |
| \( \text{cpu} \) | 0.8  | 24.5        | 760.2       |

### Table 5.3: ALCC and GloptiPoly results for Example 2 with objective function \( \int P d\mu \)

| \( d \) | ALCC | GloptiPoly |
|-------|------|------------|
| \( d \) | 1    | 2          | 3          |
| \( d \) | 1    | 2          | 3          |
| \( P_d \) | 0.992 | 0.887       | 0.852       |
| \( \text{iter} \) | 11   | 25         | 41         |
| \( \text{cpu} \) | 0.8  | 21.0        | 4938.9       |
Table 5.3: ALCC results for Example 2 using objective function \( f_d \mu_1 + f_d \mu_2 \)

| d   | 1   | 2   | 3   |
|-----|-----|-----|-----|
| nvar| 151 | 2128| 16475|
| x1  | 0.209| 0.328| 0.201|
| x2  | -0.202| -0.174| -0.201|
| x3  | 0.397| 0.466| 0.430|
| x4  | -0.400| -0.405| -0.401|
| x5  | 0.667| 0.638| 0.591|
| \( P_d \) | 1 | 0.997 | 0.981 |
| iter | 979 | 1467 | 1875 |
| cpu  | 6.5 | 102.2 | 434.7 |

Table 5.3: Example 3: Portfolio Selection Problem. We aim at selecting a portfolio of financial assets to maximize the probability of achieving a return higher than a specified amount \( r^* \). Let \((Q, \Sigma, \mu_q)\) be a probability space, and for each asset \( i = 1, \ldots, N \), let \( \xi_i(q) \) be a random variable denoting uncertain rate of return and \( x_i \) denote the percentage of money invested in. More precisely, we solve the following problem:

\[
\sup_{x \in \mathbb{R}^N} \mu_q \left( \{ q \in \mathbb{R}^N : \sum_{i=1}^{N} \xi_i(q)x_i \geq r^* \} \right) \quad \text{s.t.} \quad \sum_{i=1}^{N} x_i \leq 1, \ x_i \geq 0 \ \forall \ i. \tag{5.6}
\]

For both objective functions, we reported results up to order \( d = 3 \), because GloptiPoly did not terminate in 24 hours.

\[
\begin{array}{|c|c|c|c|}
\hline
x_1 & 0.004 & 0.009 & 0.002 \\
\hline
x_2 & 0.012 & 0.009 & 0.006 \\
\hline
x_3 & 0.438 & 0.449 & 0.299 \\
\hline
x_4 & 0.5007 & 0.522 & 0.677 \\
\hline
\end{array}
\]

Table 5.4: ALCC results for Example 3 with objective function \( \sum_{j=1}^{N} \xi_j(q)x_j \)

| d   | 1   | 2   | 3   |
|-----|-----|-----|-----|
| nvar| 60  | 565 | 3213|
| x1  | 0.004| 0.009| 0.002|
| x2  | 0.012| 0.009| 0.006|
| x3  | 0.438| 0.449| 0.299|
| x4  | 0.5007| 0.522| 0.677|
| \( P_d \) | 0.996| 0.994| 0.980|
| iter | 573 | 388 | 2227 |
| cpu  | 3.625| 16.426| 756.798 |

Table 5.4: ALCC and GloptiPoly results for Example 3 with objective function \( f_d \mu_1 + f_d \mu_2 \)

| d   | 1   | 2   | 3   |
|-----|-----|-----|-----|
| nvar| 60  | 565 | 3213|
| x1  | 0.143| 0.0462| 0.003|
| x2  | 0.192| 0.154| 0.075|
| x3  | 0.295| 0.297| 0.210|
| x4  | 0.325| 0.493| 0.710|
| \( P_d \) | 1 | 1 | 0.999 |
| iter | 15 | 20 | 48 |
| cpu  | 0.509| 2.617| 1025.045 |

Table 5.4: GloptiPoly results for Example 3 with objective function \( f_d \mu_1 + f_d \mu_2 \)

Table 5.4: GloptiPoly results for Example 3 with objective function \( f_d \mu_1 + f_d \mu_2 \)

\[
\begin{array}{|c|c|c|c|}
\hline
x_1 & 0.143 & 0.0462 & 0.003 \\
\hline
x_2 & 0.192 & 0.154 & 0.075 \\
\hline
x_3 & 0.295 & 0.297 & 0.210 \\
\hline
x_4 & 0.325 & 0.493 & 0.710 \\
\hline
\end{array}
\]

Table 5.4: GloptiPoly results for Example 3 with objective function \( f_d \mu_1 + f_d \mu_2 \)

5.3.4. Example 4: Nonlinear Control Problem. In this example, we consider the controller design problem for the following uncertain nonlinear dynamical system. For a given control parameter vector...
The following optimal solution and the corresponding maximum probability is computed by Monte Carlo method:

\[
\begin{array}{ccc}
  d & 2 & 3 \\
  \text{P}_d & 0.992 & 0.782 \\
  \text{iter} & 1714 & 2395 \\
  \text{cpu} & 12.113 & 864.411
\end{array}
\]

Table 5.5: ALCC and GloptiPoly results for Example 3 with objective function \( \int P^{(7)}d\mu \)

\[
\begin{array}{ccc}
  d & 2 & 3 \\
  \text{P}_d & 0.999 & 0.876 \\
  \text{iter} & 30 & 51 \\
  \text{cpu} & 0.7 & 5.7
\end{array}
\]

Table 5.6: ALCC and GloptiPoly results for Example 4 with objective function \( \int d\mu \)

\[ K \in \mathbb{R}^3, \] let the system \( x(k)^T = [x_1(k), x_2(k), x_3(k)] \in \mathbb{R}^3 \) satisfy

\[
\begin{align*}
  u(k) &= K_1 x_1(k) + K_2 x_2(k) + K_3 x_3(k), \\
  x_1(k+1) &= \Delta x_2(k), \\
  x_2(k+1) &= x_1(k) x_3(k), \\
  x_3(k+1) &= 1.2 x_1(k) - 0.5 x_2(k) + x_3(k) + u(k),
\end{align*}
\]

for \( k = 0, 1, \) where \( x_1(0) \sim U[-1,1], x_2(0) \sim U[-1,1], x_3(0) \sim U[-1,1], \Delta \sim U[-0.4,0.4], \) i.e. initial state vector \( x(0), \) and model parameter \( \Delta \) are uncertain and uniformly distributed. The objective is to lead the system using state feedback control \( u(k) \) to the cube centered at the origin with the edge length of 0.2 in at most 2 steps by properly choosing the control decision variables \( \{K_i\}_{i=1}^3 \) such that \(-1 \leq K_i \leq 1. \) The equivalent chance problem is stated in (5.8),

\[
\begin{align*}
  \sup_{K \in \mathbb{R}^3} & \mu_\rho \left( \left\{ (x(0), \Delta) : -0.1e \leq x(2) \leq 0.1e \right\} \right), \\
  \text{s.t.} & \ (x(k), u(k))_{k=0}^2 \text{ satisfy (5.7),} \\
  & \ -e \leq K \leq e.
\end{align*}
\]

The following optimal solution and the corresponding maximum probability is computed by Monte Carlo method: \( K_1^* = -1, K_2^* = 0.5, K_3^* = -0.9, \) and \( P^* = 0.84. \) To obtain an equivalent SDP formulation for the chance constrained problem in (5.8), \( x(2) \) is explicitly written in terms of control vector \( K \in \mathbb{R}^3 \) and uncertain parameters \( x(0) \) and \( \Delta \) using the dynamic of the system in (5.7):

\[
\begin{align*}
  x_1(2) &= x_1(0) x_3(0) \Delta, \\
  x_2(2) &= (1.2 + K_1) \Delta x_1(0) x_2(0) + (K_2 - 0.5) \Delta x_2(0)^2 + (1 + K_3) \Delta x_2(0) x_3(0), \\
  x_3(2) &= (1 + 2K_3 + K_3^2) x_3(0) + (K_2 - 0.5K_3 - 0.5 + 1.2 \Delta + K_1 \Delta + K_2 K_3) x_2(0) \\
  &\quad + (1.2 + K_1 + 1.2 K_3 + K_1 K_3) x_1(0) + (K_2 - 0.5) x_3(0) x_3(0).
\end{align*}
\]

Based on the obtained polynomials, the minimum relaxation order for this problem is 2. To obtain an approximate solution, we solve the SDP in (5.6) using GloptiPoly and ALCC. For ALCC, we set \( \nu_0 \) to \( 5 \times 10^{-3}, 5 \times 10^{-3} \) and \( 1 \times 10^{-3} \) when \( d = 2 \) and \( d = 3 \) and \( d = 4, \) respectively, and \( \text{tol} = 1 \times 10^{-3}. \) The results for relaxation order \( d = 2, 3, 4 \) are shown in Table 5.6.
6. Conclusion. In this paper, we present a novel approach for solving a large class of chance optimization problems defined on semialgebraic sets. A sequence of semidefinite relaxations is provided whose optimal value is shown to converge to the optimal value of the original problem. To solve the semidefinite programs obtained by relaxing the original chance optimization problem, a first-order augmented Lagrangian algorithm is implemented which enables us to solve much larger size semidefinite programs that interior point methods can deal with. Numerical examples are provided that show that, even for low order relaxations, one obtains good approximations for the optimal solution and the optimum probability.

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