Contracted Bianchi Identity and Angle Relation on $n$-dimensional Simplicial Complex of Regge Calculus

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In this article, we prove the theorems concerning the trace relation of $\text{SO}(3)$, $\text{SU}(2)$, and $\text{SO}(n)$ which are representation of $\text{SO}(3)$ and $\text{SU}(2)$. An interesting fact we found is the trace relation of $\text{SU}(2)$ gives the spherical law of cosine which in turns is a dihedral angle relation, a constraint that must be satisfied by closed Euclidean simplices. Moreover, we applied our results on general group elements to holonomies on the simplicial complex of Regge Calculus, which is the main motivation of this article. Here, we found that: (1) in 4-dimensional Euclidean Regge Gravity, all the holonomy circling a single hinge are simple rotations, and (2) the dihedral angle relation represents the ‘contracted’ Bianchi identity for a simplicial complex.

I. INTRODUCTION

The discrete attempt on gravity was first introduced by Tullio Regge, written in the second order formulation where the generalized coordinate is the spatial metric [1, 2]. This discrete formulation, theoretically works for any dimension. In particular, for 3-dimensional gravity, the model is known as the Ponzano-Regge model [3, 4]. Another important development on general relativity was carried in [5, 6], where it is written in the form closer to gauge theory, usually known as the first order formulation. This is followed by the introduction of new variables in [7], which is important for the canonical quantization of gravity. The discrete version of the first order formulation of gravity immediately becomes an interest in the quantum gravity community, with the first development carried in [8]. This is followed by the Barret-Crane model for 4-dimensional Lorentzian gravity [9, 10]. Some corrections on the Barret-Crane model gives the EPRL-FK (or spinfoam) model [11–13], which could be derived from discrete BF theory. All these model use a simplicial complex to described spacetime, with the difference among them being the variations on action integral.

We are interested in discrete gauge theory of gravity, particularly, in the description of the possible simplicial complex of Regge gravity. It is well-known that discrete manifold in Regge Calculus is a special case of Riemannian manifold, in the sense that their Riemann tensor must be in the form of $R_{\mu\nu\beta} \sim \hat{\omega}_\beta \hat{\omega}_{\mu\nu}$ [14, 15]. With this restriction, it is natural to ask what would be the holonomy group for the discrete manifold in Regge Calculus. The importance of the holonomy in the theory is crucial if one consider the possibility of the discreteness of space in the Planck scale, as predicted by loop quantum gravity and other non-perturbative theories of gravity [16–18].

As a first step to solve the problem, we consider several important aspects of the theory: (1) The rotation group in 3-dimension, $\text{SO}(3)$ and its double-cover, $\text{SU}(2)$. These groups become our interest because it is the natural gauge group of the 3-dimensional spatial part of the spacetime bundle. In this article, we will show that the importance of these groups for a simplicial complex, particularly $\text{SU}(2)$, is not only restricted to dimension (3+1). The second is (2) the ‘dihedral’ angle relation of a simplex. One has the fact that the $d$ and $(d-1)$-dimensional angles in a closed $(n \geq d)$-simplex satisfy this relation as a constraint [19]. In three dimension, the relation is automatically satisfied by three bivectors meeting in a point, but this is not the case in four and larger dimension. In fact, it is shown in this article that one of the gauge groups in point (1) will give rise to the dihedral angle relation as its trace relation (or contracted Bianchi Identity for holonomy). Thus, one needs to take this relation into account. The last aspect related to the angle relation is (3) the simplicity of a bivector, and moreover the simplicity of the rotations. Similar with the angle relation, this is automatically satisfied in 3D but not in larger dimension. We will show that these three properties are related to one another and are important for the construction of a simplicial complex in $n$-dimension. Specifically, the gauge group in $n$-simplicial complex of Regge Calculus is $\text{SO}(n)$, but certain restrictions are needed, in order to described a theory of discrete gauge gravity.

The organization of the article is the following: Section II consists a discussion of the asymptotics of second Bianchi Identity, where the discreteness of space is described by holonomies o hinges: elements of Lie group attached as a variables on the loops. In the next two sections, we study a special case of gauge group $\text{SO}(n)$ in general, without referring to the loops. Specifically, Section III consists the proof of theorems concerning the trace relation of elements of $\text{SO}(3)$, $\text{SU}(2)$, and $\text{SO}(n)$ which are representation of $\text{SO}(3)$ and $\text{SU}(2)$. Here, we found that the trace relation of $\text{SU}(2)$ gives the spherical law of cosine which in turns, is equivalent with the angle relation. The application of some lemmas in Section III to the 4-dimensional case gives a classification of rotations in 4D Euclidean space, this is discussed in Section IV. In Section V, we applied our theorems on general group elements to holonomies on the simplicial complex of Regge Calculus, which is the main motivation of this article. Here, we found that: (1) in
Figure 1. (a). Given the loops on the theta graph configuration as follows: \( \gamma_1 = \gamma_a^{-1} \), \( \gamma_2 = \gamma_b^{-1} \), and \( \gamma_3 = \gamma_c^{-1} \), the holonomies on loops \( \gamma_i \), \( i = 1, 2, 3 \) satisfy the second Bianchi Identity (2). (b). Approximation of the theta configuration (a) by the square loops.

4-dimensional Euclidean Regge Gravity, all the holonomy circling a single hinge of a simplicial complex are simple rotations, and (2) the dihedral angle relation represents the 'contracted' Bianchi identity for a simplicial complex. Finally at the last section, we discuss the relevance of our findings to the established theory of discrete gauge gravity.

II. SECOND BIANCHI IDENTITY

A. Second Bianchi Identity in a point and finite loops

Given a fibre bundle \( E \) diffeomorphic to a standard fibre bundle \( M \times F \), with \( A \) is the connection on \( E \), the second Bianchi Identity is the condition that must be satisfied by the curvature 2-form \( F = dA \):

\[
d_D F = D_\mu F_{\nu\lambda} dx^\mu \wedge dx^\nu \wedge dx^\lambda = \frac{1}{3} (D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} + D_\lambda F_{\mu\nu}) dx^\mu \otimes dx^\nu \otimes dx^\lambda = 0. 
\]  

The geometrical interpretation of second Bianchi Identity is clear in the finite setting. Given three loops \( \gamma_i \) on \( M \) such that they meet together on two points as in FIG. 1a, the finite second Bianchi Identity is a product of three holonomies \( U_{\gamma_i} = U_{\gamma_i}(A, p) \):

\[
U_{\gamma_1} U_{\gamma_2} U_{\gamma_3} = 1, 
\]

with \( p \in M \) is the origin of the loop. Each holonomy is a solution to a parallel transport along the loop, \( \frac{dU_{\gamma_i}}{d\tau} = 0 \):

\[
U_{\gamma_i} = \hat{P} \exp \int_{\gamma_i = \partial S_i} A = \hat{P} \exp \int_{S_i} dA = \hat{P} \exp \int_{S_i} F,
\]

where the last equality is obtained by applying Stokes theorem on a closed loop \( \gamma_i \) as the boundary of a surface \( S_i \). A holonomy, by definition (3), is an element of a gauge group \( G \), which is attached as a variable on a loop \( \gamma \subset M \). In analog with the relation between a Lie group and its algebra, it comes from the ‘exponential map’ (3) of the connection \( A \), which is a Lie algebra valued 1-form on \( T^*_pM \).

For a case where loops \( \gamma_i \) are infinitesimally small, they can be approximated to a square loop with length \( \varepsilon \) as in FIG. 1b. Expanding (2) in terms of \( \varepsilon \) and taking only the first non-zeroth order, one obtains (1) such that it can be thought as a relation defined on an infinitesimal loop.

In the rest of the article, we will refer the second Bianchi Identity simply as the Bianchi Identity.

B. Asymptotics of Bianchi Identity

With the curvature and the scale \( \varepsilon \) as parameters, one could obtain asymptotics of the Bianchi Identity. Thus, one has four conditions, where the first is the case with no approximation, namely, the Bianchi Identity defined on a finite loop (finite scale \( \varepsilon \)) and finite (non-zero) curvature. The holonomies are exactly (3), and the Bianchi Identity is (2).
The second case is the nearly-flat case with finite loop and infinitesimally small curvature. For small $F$, the holonomy can be approximated as follows:

$$U_{S_{i}} \approx I + \int_{S_{i}} F,$$

where it is labeled by the surface $S_{i}$ enclosed by loop $\gamma_{i}$, and $I$ is the identity. The Bianchi Identity becomes a closure condition:

$$\int_{S_{i}} F + \int_{S_{j}} F + \int_{S_{k}} F \approx 0,$$

where the curvature $F$ is smeared on each finite surface.

The next case is the nearly-continuous case with infinitesimal scale and finite curvature. For the reason explained in Section II A, the holonomies could be approximated into:

$$U_{S_{\mu}} \approx I + \varepsilon^{2} \left( F_{\mu\nu} - F_{\lambda\mu} \right) + \varepsilon^{3} \frac{1}{3!} D_{\mu} F_{\nu\lambda},$$

and the Bianchi Identity (1) comes from the third order of $\varepsilon$.

The last case is the nearly-flat and nearly-continuous case. The holonomy is approximated into:

$$U_{S_{i}} \approx I + \varepsilon^{2} F(l_{j}, l_{k}),$$

with $S_{i} \cong l_{j} \wedge l_{k}$, and the corresponding Bianchi Identity comes from the second order of $\varepsilon$:

$$F(l_{j}, l_{k}) + F(l_{k}, l_{i}) + F(l_{i}, l_{j}) \approx 0.$$

A recent growing interest in discrete gauge theory of gravity and quantum gravity is the idea that space is fundamentally discrete in the Planck scale [16–18], and that curvature is quantized such that non-trivial loops could not be shrunk to a point [20]. Therefore the relevant case of holonomies on these non-contractible loops are the finite loop cases, namely (2)-(3) and (4)-(5). Up to the rest of the article, we will only consider these cases.

### III. CONTRACTED BIANCHI IDENTITY AND DIHEDRAL ANGLE FORMULA

Following the classification of manifold by their holonomy groups in [21], the holonomy group of a general, oriented, $n$-dimensional Riemannian manifold is (a subgroup of) the special orthogonal group SO(n). Some special condition on SO(n) will give rise to special properties on the Riemannian manifold. We will show that the holonomy group of an oriented, $n$-dimensional simplicial complex in Euclidean Regge Calculus needs to be SO(n) plus some certain condition. In order to show this, in the following section we study the holonomy as an element of a Lie Group, which could be defined independently from the loop on the base manifold. The theorem we obtain in this section will give important geometrical meaning if applied on loops, which are discussed in the last section.

#### A. The case of SO(3) as gauge group

In gauge theory, it is important to obtain the gauge-invariant quantities. One of them is the trace of holonomy, or in physics term, the Wilson loop:

$$\chi = \text{tr} U_{\gamma_{i}} = \text{tr} \hat{P} \exp \oint_{\gamma_{i}} A.$$

If $U_{i}$ is a matrix representation of an element of a group, the trace is the characteristics polynomials of the matrix. The physical importance of the Wilson loop is the Lagrangian (or action integral) in Chern-Simon theory.

A gauge-invariant quantity which become our interest is the trace relation of a gauge group as follows:

$$\text{tr} U_{3}^{-1} = \text{tr} U_{1} U_{2},$$

which is clearly equivalent with the contracted Bianchi Identity. Our pre-result in the previous paper [23] is:
Theorem 3.1 (Trace Relation of Gauge Group SO(3))

Given elements of group \( U_i \in SO(3) \), the trace relation or the contracted Bianchi Identity in the form of (6), gives the relation as follows:

\[
\cos \frac{\theta_3}{2} = \pm \left( \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} - \cos \phi_{12} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right),
\]

(7)

with \( \theta_i \) are the rotation angles of \( U_i \) and \( \phi_{12} \) is the angle between plane of rotation of \( U_1 \) and \( U_2 \).

**Proof.** The proof for this theorem is direct and straightforward, realizing that any element \( U_i \in SO(3) \) can always be written as:

\[
U_i = I_{3 \times 3} + \vec{J}_i \sin \theta_i + J_i^2 (1 - \cos \theta_i), \quad J_i \in \mathfrak{so}(3).
\]

(8)

Using the following relations for plane of rotations \( J_i = \vec{J}_i \cdot l \in \mathfrak{so}(3) \), with \( \vec{J}_i \in \mathbb{R}^3 \) and \( l \) are generator of \( \mathfrak{so}(3) \):

\[
\text{tr} (J_i J_j) = -2 \left\langle \vec{J}_i, \vec{J}_j \right\rangle
\]

\[
\text{tr} (J_i J_j J_k) = \vec{J}_i \cdot (\vec{J}_j \times \vec{J}_k)
\]

\[
\text{tr} (J_i J_j J_k J_l) = \left\langle \vec{J}_i, \vec{J}_j \right\rangle \left\langle \vec{J}_j, \vec{J}_k \right\rangle + \left\langle \vec{J}_i, \vec{J}_j \right\rangle \left\langle \vec{J}_j, \vec{J}_k \right\rangle + \left\langle \vec{J}_i, \vec{J}_k \right\rangle \left\langle \vec{J}_k, \vec{J}_l \right\rangle + \left\langle \vec{J}_i, \vec{J}_l \right\rangle \left\langle \vec{J}_l, \vec{J}_k \right\rangle,
\]

together with the half angle formula, one could obtain dihedral relation (8).

The aim of this article is to show that special restrictions on the gauge group \( SO(n) \) in any dimension will give rise to the same dihedral angle relation. This can be thought as an immersion of \( SO(3) \) on \( SO(n) \). It is clear that the image of the 'immersion' map \( \rho_n \):

\[
\rho_n : SO(3) \rightarrow SO(n),
\]

is a representation of \( SO(3) \) in \( n \)-dimension. The map induces a representation of the corresponding Lie algebra as follows:

\[
d\rho_n : \mathfrak{so}(3) \rightarrow \mathfrak{so}(n),
\]

The proof for this is simply to show that \( \rho_n \) is a group homomorphism.

Before arriving at the result, the followings are supporting definitions and lemmas used to obtain the theorems. Let \( \mathbb{R}^n \times \mathbb{R}^n \) be a space of matrices. Using terminologies in \( \mathbb{R}^n \), the coordinate basis in \( \mathbb{R}^n \times \mathbb{R}^n \) is \( \{ dx^i \otimes dx^j \} \), for \( i, j = 1, \ldots, n \), such that any element of the space can be written as their linear combination: \( u = u_{ij} dx^i \otimes dx^j \). The matrix space \( \mathbb{R}^n \times \mathbb{R}^n \) could be equipped with the Frobenius inner product defined as follows:

\[
\langle , \rangle : (\mathbb{R}^n \times \mathbb{R}^n) \times (\mathbb{R}^n \times \mathbb{R}^n) \rightarrow \mathbb{R}
\]

\[
(u, v) \mapsto \langle u, v \rangle,
\]

where the operation is given by:

\[
\langle u, v \rangle = u_{ij} v_{kl} \delta^{ik} \delta^{jl},
\]

(9)

with \( \delta^{ik} \delta^{jl} \) comes from the orthonormality condition of the coordinate basis:

\[
\delta^{ik} \delta^{jl} = \langle dx^i \otimes dx^k, dx^j \otimes dx^l \rangle.
\]

(10)

Other properties of matrices which are important particularly in this articles are the following: The trace of matrix \( \omega \) is defined as \( \text{tr} \, \omega = \omega_{ij} \delta^{ij} \), matrix multiplication of two matrices \( u \) and \( v \) is defined as \( uv = u_{ij} v_{kl} \delta^{ik} (dx^l \otimes dx^j) \), and transpose of a matrix \( \omega \) is defined as \( \omega^T = \omega_{ij} dx^i \otimes dx^j \). With these definitions, the Frobenius inner product (9) could be written as \( \langle u, v \rangle = \text{tr} (u^T v) \).

Let \( \{ e^{ij} \} \) be a non-coordinate basis (not necessarily orthonormal) of \( \mathbb{R}^n \times \mathbb{R}^n \) given by the Gramm-Schmidt procedure. \( \{ e^{ij} \} \) can always be written as:

\[
e^{ij} = e^{ij}_{kl} dx^k \otimes dx^l,
\]

(11)
where \( e_{ij} \) is a rank four tensor. Using the fact that \( \mathbb{R}^n \times \mathbb{R}^n \) is isomorphic with \( \mathbb{R}^{n^2} \), \( e_{ij} \) needs to be invertible on the pair of indices \((ij)\) and \((kl)\), namely the inverse \( (e^{-1})_{(ij)(kl)} = (e_{(ij)(kl)})^{-1} \) exists. Two sets of non-coordinate basis \( \{\hat{e}^{ij}\} \) and \( \{\hat{e}^{kl}\} \) are similar up to a general transformation \( \Omega \) as follows:

\[
\hat{e}^{ij} = \sum_{k,l} \Omega^{(ij)(kl)} e^{kl},
\]

(12)

Since \( \{\hat{e}^{ij}\} \) and \( \{\hat{e}^{kl}\} \) are basis of \( \mathbb{R}^n \times \mathbb{R}^n \), \( \Omega \) is invertible on the pair of indices \((ij)\) and \((kl)\).

A subset of matrix space \( \mathbb{R}^n \times \mathbb{R}^n \) which becomes an interest in this article is the space of bivectors in \( n \)-dimensional space, \( \Lambda^2(\mathbb{R}^n) \), namely a space of antisymmetric matrix in \( \mathbb{R}^n \times \mathbb{R}^n \). It is spanned by orthonormal coordinate basis: \( dx^i \otimes dx^j = dx^i \otimes dx^j - dx^j \otimes dx^i \), for \( i, j = 1, \ldots, n \). Any element of \( \Lambda^2(\mathbb{R}^n) \) could be written as a linear combination of the basis: \( \omega = \omega_{ij} dx^i \wedge dx^j \). A bivector \( \omega \in \Lambda^2(\mathbb{R}^n) \) is simple if it can be written as a wedge product of two vectors \( u, v \in \mathbb{R}^n \):

\[
\omega = u \wedge v.
\]

(13)

Equivalently, if (13) is satisfied, then \( \omega \wedge \omega = 0 \) [24]. From the definition, it is clear that the coordinate basis in bivector space is simple.

The following lemmas we proof in [25] are important to derive the main theorem in this article.

**Lemma 3.1 (Similarity of Simple Bivectors).**

Let \( \omega \in \Lambda^2(\mathbb{R}^n) \) be a simple bivector such that:

\[
\omega = u \wedge v = \frac{u \otimes v - (u \otimes v)^T}{2}, \quad u, v \in \mathbb{R}^n.
\]

(14)

The following transformation:

\[
\omega' = \Lambda \omega \Lambda^T, \quad \Lambda \in GL(n, \mathbb{R}),
\]

(15)

is the most general transformation preserving the simplicity of a bivector, namely:

\[
\omega' = u' \wedge v'.
\]

(16)

All simple bivectors are similar up to transformation (15).

The proof for Lemma 3.1 is sketched briefly as follows: The first step is to obtain the most general transformation which preserve the simplicity of matrix in \( \mathbb{R}^n \times \mathbb{R}^n \), namely \( \omega = u \otimes v \). The second step is to obtain the most general transformations in which preserve (anti)-symmetricity of a matrix, this transformation defines invariant subspaces in \( \mathbb{R}^n \times \mathbb{R}^n \), which are the space of symmetric and anti-symmetric matrices. Combining these two results proves Lemma 3.1. The detailed derivation is carried in [25].

The Frobenius inner product on \( \mathbb{R}^n \times \mathbb{R}^n \) induced an equivalent inner product in the bivector space. This cause the possibility to define matrix multiplication, and moreover, Lie derivative in \( \Lambda^2(\mathbb{R}^n) \).

**Definition 3.1 (Coordinate Generators of (Special) Orthogonal Group).**

The coordinate basis of bivector:

\[
\{dx^i \wedge dx^j\} \in \Lambda^2(\mathbb{R}^n) \quad i, j = 1, \ldots, n,
\]

is the coordinate generators of special orthogonal group \( SO(n) \), or the coordinate basis of Lie algebra \( so(n) \), satisfying the following closed algebra structure:

\[
[dx^i \wedge dx^j, dx^j \wedge dx^k] = \varepsilon^{ijk} dx^i \wedge dx^k, \quad i, j, k = 1, \ldots, n.
\]

(17)

The coordinate generator \( \{dx^i \wedge dx^j\} \) is simple and orthonormal, but a generator of \( SO(n) \) in general is not necessarily simple, as bivector in dimension \( n \geq 4 \) in general is not simple. Using Lemma 3.1, one could prove the following lemma.
Lemma 3.2 (Similarity of Simple Generators of (Special) Orthogonal Group).

Let a set of non-coordinate basis of bivectors \{e_{ij}\} for \(i, j = 1, \ldots, n\), satisfying the so(n) algebra structure relation (17), and:

\[
e_{ij} = \hat{\nu}^i \wedge \hat{\nu}^j,
\]

be a simple generators of SO(n), or equivalently, simple basis of Lie algebra so(n). The following transformation:

\[
e_{ij} = \Lambda e_{ij} \Lambda^{-1}, \quad \Lambda \in O(n),
\]

(18)
is the most general map preserving (simultaneously) anti-symmetricity, simplicity, and the closed algebra structure. All simple generators of so(n) are similar up to transformation (18).

The proof is rather direct. Using transformation (15) on (17) one could prove Lemma 3.2. The map (18), is known as the adjoint action in on the algebra:

\[
\text{Ad} : G \times g \to G
\]

\[
(g, \omega) \mapsto \omega' = \text{Ad}_g \omega,
\]

which is also called as the similarity transformation.

As consequences of Lemma 3.2, we could conclude the following two corollaries:

Corollary 3.2A (Simple Representation of Lie Algebra so(3) in n-Dimension).

The subsets

\[
\left\{dx^a \wedge dx^b, dx^b \wedge dx^c, dx^c \wedge dx^a\right\} \subset \left\{dx^i \wedge dx^j\right\}, \quad i, j = 1, \ldots, n,
\]

(19)

are coordinate generators of simple representation of so(3) in n-dimension. Any sets \{e^{ab}, \hat{\nu}^{bc}, \hat{\nu}^{ca}\} which are similar to (19) up to a similarity transformation (18) are non-coordinate generators of simple representation of so(3) in n-dimension.

Both (19) and \{e^{ab}, \hat{\nu}^{bc}, \hat{\nu}^{ca}\} spans \(d\rho_n[so(3)]_{\text{simp}}\): the space of simple representation of so(3) in n-dimension. All element of \(d\rho_n[so(3)]_{\text{simp}}\) are simple and similar up to a similarity transformation (18).

Corollary 3.2B (Representation of Lie Algebra so(3) in n-Dimension).

Let a set of general bivectors (not necessarily simple) labeled as follows:

\[
\left\{e^{ab}, \hat{\nu}^{bc}, \hat{\nu}^{ca}\right\} \in \bigwedge^2 R^n,
\]

satisfies the so(3) algebra

\[
[e^{ab}, \hat{\nu}^{bc}, \hat{\nu}^{ca}] = \epsilon^{abc} \hat{\nu}^{ca},
\]

(20)

then the set defines a generator of representation of so(3) in n-dimension. \{e^{ab}, \hat{\nu}^{bc}, \hat{\nu}^{ca}\} spans \(d\rho_n[so(3)]\): the space of representation of so(3) in n-dimension. If there exist a group element \(\Lambda \in O(n)\) such that \{e^{ab}, \hat{\nu}^{bc}, \hat{\nu}^{ca}\} is similar to \{e^{ab}, \hat{\nu}^{bc}, \hat{\nu}^{ca}\}, then the generator \{e^{ab}, \hat{\nu}^{bc}, \hat{\nu}^{ca}\} is simple, and so do their linear combinations.

As we have already mentioned in the previous section, the holonomy, which is an element of a Lie group, could be obtained from relation (3), which is an exponential of the Lie algebra-valued connection. But since holonomy, by its own right, is a group element (which could be defined independently from the loop), it can be written as an exponential of the Lie algebra element \(J \in so(n)\) (not to be confused with the Lie algebra-valued connection \(A\)):

\[
U_i = \exp J_i = \exp |J_i| \hat{J}_i \in SO(n).
\]

(21)

One could conclude easily that the adjoint action (18) commutes with the exponential map.

Lemma 3.3 (Simple Representation of Lie Group SO(3) in n-Dimension).

Let \(J \in d\rho_n[so(3)]\) be an element of representation of so(3) in n-dimension. The element of representation of the group SO(3) in n-dimension, \(\rho_n(SO(3))\), can be obtained from the exponential map:

\[
U = \exp J \in \rho_n(SO(3)).
\]

(22)
Figure 2. Types of angles inside a tetrahedron: the 2-dimensional angle $\phi_{ij}$ are the angle between segments, located on a point where these segments meet; and the 3-dimensional 'dihedral' angle $\theta_i$ are the angle between triangles, located on segments. These angles satisfy 'dihedral' angle relation in (2+1)-dimension.

If $J$ is simple, then any element of simple representation of $SO(3)$ in $n$-dimension, $\rho_n (SO(3))_{\text{sim}}$, can always be written as follows:

$$U = I_{n \times n} + J \sin \theta + J^2 (1 - \cos \theta) \in \rho_n (SO(3))_{\text{sim}}, \quad \theta = |J|$$  \hspace{1cm} (23)

up to the similarity transformation (18).

The proof is the following. Using Corollary 3.2A, all simple bivector in $n$-dimension are similar up to a similarity transformation (18), namely, there always exist $\Lambda \in SO(n)$ which bring $J$ to $J' \in \mathfrak{so}(n)$. Since the similarity transformation commutes with the exponential map, there always exists $\Lambda \in SO(n)$ which bring (22) to the form of (23), which proof the previous lemma. Lemma 3.3 is particularly important in deriving our main theorem of this article:

**Theorem 3.2 (Trace Relation of Gauge Group $SO(n)$: The $SO(3)$ Case)**

Let $U_1, U_2, U_3 \in SO(n)$. If $U_1, U_2, U_3$ satisfy the following condition:

1. $U_1, U_2, U_3 \in \rho_n [SO(3)] \subset SO(n)$, that is, they are elements of common $SO(3)$ subgroup representation in $n$-dimension as in Lemma 3.3,

2. $U_1, U_2, U_3$ are simple, that is, they are exponential of simple Lie Algebra, and

3. $U_1, U_2, U_3$ satisfy the Bianchi identity (2),

then the trace relation or the contracted Bianchi Identity in the form of (6) gives the angle relation (7).

**Proof.** From requirements (1) and (2), together with Corollary 3.2B, one can conclude that $U_1, U_2, U_3 \in \rho_n (SO(3))_{\text{sim}}$. From Lemma 3.3, every element of $\rho_n (SO(3))_{\text{sim}}$ could always be written as (23). Taking the trace of the second Bianchi Identity as in (6) and using the half angle formula gives the angle relation (7).

It must be kept in mind that in general, the trace relation does not gives the angle relation, only the one which satisfies point 1 and 2 of Theorem 3.2 does.

**B. The case of SU(2) as the gauge group**

The relation (7) is somewhat unsatisfying: in addition that it contains two solutions, the dihedral angle $\theta$’s are half of the angle of rotation, while the planar angle $\phi$ is fixed. As a comparison, the following is the original dihedral angle relation:

$$\cos \theta_3 = \cos \theta_1 \cos \theta_2 - \cos \phi_{12} \sin \theta_1 \sin \theta_2,$$  \hspace{1cm} (24)

with the geometrical interpretation in 3-dimension given in FIG. 2. Interestingly, if we choose the gauge group on Theorem 3.1 to be $SU(2)$ instead of $SO(3)$ so that $U_i \in SU(2)$, the trace relation gives exactly (24).

**Theorem 3.3 (Trace Relation of Gauge Group SU(2)).**

Given elements of group $U_i \in SU(2)$, the trace relation or the contracted Bianchi Identity in the form of (6) gives the dihedral angle relation (24), with $\theta_i$ are the rotation angles of $U_i$ and $\phi_{12}$ is the angle between plane of rotation of $U_1$ and $U_2$. 


Proof. The proof of this is direct and straightforward, realizing that any $U_i \in SU(2)$ can always be written as:

$$U_i = I_{2 \times 2} \cos \theta_i + \hat{J}_i \sin \theta_i, \quad \hat{J}_i \in \mathfrak{su}(2), \quad \theta_i = |J_i|.$$  \hspace{1cm} (25)

Using the following relations for $J_i = \hat{J}_i \cdot \hat{\sigma} \in \mathfrak{su}(2)$, with $\hat{J}_i \in \mathbb{R}^3$ and $\hat{\sigma}$ are generator of $\mathfrak{su}(2)$: $\text{tr}(J_iJ_j) = -2 \langle \hat{J}_i, \hat{J}_j \rangle$, one could obtain dihedral relation (24).

In fact, one could obtain (24) with a slight modification of the algebra of $\mathfrak{so}(3)$ as follows: Let $J_i = \hat{J}_i \cdot \hat{I} \in \mathfrak{so}(3)$, with $\hat{I}$ is the generator of $\mathfrak{so}(3)$. Let us define the transformation on the generator as follows: $I_a \mapsto \tau_a = 2I_a$, where the algebra of $\{ \hat{\tau} \}$ is:

$$[\tau_a, \tau_b] = -2\varepsilon_{abc}\tau_c,$$  \hspace{1cm} (26)

instead of (20). Using the new generator, an element of $\mathfrak{so}(3)$ could be written as $T = \hat{T}_i \cdot \tau_a \in \mathfrak{so}(3)$, and moreover its corresponding group element of $SO(3)$ could be written as:

$$U_i = I_{3 \times 3} + \hat{J}_i \sin 2\theta_i + \hat{J}_i^2 (1 - \cos 2\theta_i), \quad J_i \in \mathfrak{so}(3).$$  \hspace{1cm} (27)

(26) is exactly the algebra relation of $\mathfrak{su}(2)$. Specifically, $\tau_a$ are the irreducible representation of generator of $SU(2)$ in 3-dimension:

$$d\rho_3 : \mathfrak{su}(2) \to \mathfrak{so}(3) \subset \mathfrak{gl}(n, \mathbb{R}),$$  \hspace{1cm} (28)

since there exist a similarity transformation $d\rho_3(\hat{\sigma}) = \Lambda \hat{\sigma} \Lambda^{-1}$ by $\Lambda \in SU(2)$ which bring $\tau_a$ to the usual basis of irreducible representation of generator of $SU(2)$ in 3-dimension, namely, $d\rho_3(\hat{\tau})$. Using (27), the trace relation gives:

$$\cos \theta_3 = \pm (\cos \theta_1 \cos \theta_2 - \cos \phi_{12} \sin \theta_1 \sin \theta_2),$$  \hspace{1cm} (29)

where one of the solution is exactly the dihedral angle relation (24).

As a generalization to Theorem 3.3, we rederive Theorem 3.2, but now, the condition for gauge group $\rho_n[SO(3)]$ is replaced by $\rho_n[SU(2)]$:

**Theorem 3.4 (Trace Relation of Gauge Group SO(n): The SU(2) Case)**

Let $U_1, U_2, U_3 \in SO(n)$. If $U_1, U_2, U_3$ satisfy the following condition:

1. $U_1, U_2, U_3 \in \rho_n[SU(2)] \subset SO(n)$, that is, they are elements of common $SU(2)$ subgroup representation in $n$-dimension as in **Lemma 3.3**,

2. $U_1, U_2, U_3$ are simple, and

3. $U_1, U_2, U_3$ satisfy the Bianchi identity (2),

then the trace relation or the contracted Bianchi Identity in the form of (6) gives the angle relation (29).

**Proof.** The proof is started from writing the $\mathfrak{su}(2)$ real representation in 3-dimension as in (28). Since (28) sends $T \in \mathfrak{su}(2)$ to $\mathfrak{so}(3)$, the next step for proving Theorem 3.4 can be carried exactly as the proof for Theorem 3.2.

It is quite intriguing the fact that the one which gives the dihedral angle relation, which is an aspect of Euclidean 3-dimensional space, is the $SU(2)$ group instead of $SO(3)$. It is well-known that $SU(2)$ is the Spin(3) group which double covers the group $SO(3)$. As a manifold, $SO(3)$, topologically, is isomorphic to $\mathbb{RP}^3$: the real projective space in 3-dimension, which is not simply-connected. In the other hand, $SU(2)$ is a 3-sphere, which is topologically simpler than $\mathbb{RP}^3$. The fact that the dihedral angle relation (29) is exactly the spherical law of cosine if one takes $\phi = 2\pi - \phi$ may provide the reason why it is more natural to use $SU(2)$ instead of $SO(3)$ as the gauge-group for spatial geometries, in particular, Regge geometries. We will discuss this in the last section. Given the advantage of $SU(2)$ as a gauge group, it is relevant to continue to Section C where the geometrical meaning of the spherical law of cosine is discussed.

C. Geometrical Interpretation: Spherical Law of Cosine and Dihedral Angle Relation

Let us return to relation (21) in Section 2A. Rewriting $|J_i|$ as $\theta_i$, and approximating the dihedral angle relation (24) for small angle of rotation, one obtains:

$$\theta_3^2 \approx \theta_1^2 + \theta_2^2 + 2 \cos \phi_{12} \theta_1 \theta_2,$$  \hspace{1cm} (30)
A statistical generalization of a pure state is a mixed state: exactly, a simple and symmetric (possibly, infinite-dimensional) matrix. Its inner product in the complex Hilbert space is the statistical formulation of quantum mechanics [26], where a pure state |ψ⟩ is written by a density matrix |ψ⟩ ⟨ψ|, a simple and symmetric (possibly, infinite-dimensional) matrix. Its inner product in the complex Hilbert space is exactly:

\[ \langle \psi | \psi \rangle = \text{tr} |\psi⟩ ⟨\psi| .\]

A statistical generalization of a pure state is a mixed state:

\[ \rho_\psi = \sum_i \lambda_i |\psi_i⟩ ⟨\psi_i| = \int dx |\psi(x)⟩ ⟨\psi(x)| , \]

a linear combination of simple matrices, which in general is not simple. The (semi)-positive definiteness of the trace, and hence the possibility to use it as an inner product in complex Hilbert space, is a consequence of the symmetricity of the density matrices ρψ. In general, trace of an arbitrary matrix is not positive definite. We adopt the generalization of ‘inner product’ for the trace of SO(n) group, where the elements are not symmetric. The corresponding Lie algebra so(n), which are the infinitesimal rotations described by antisymmetric matrices, gives zero trace, thus the usual inner product is used to obtain the usual norm.

The condition that the elements of SO(n) needs to be simple representation of a common SU(2) in n-dimension can be thought of as the existence of an embedding of a 3-sphere on an n-dimensional manifold SO(n).

\[ \rho_n : SU(2) → \rho_n (SU(2))_{\text{sim}} ⊂ SO(n). \]

In the next paragraph, we will show that the trace relation of SO(n) satisfying Theorem 3.4, is a condition for the existence of a spherical triangle on the great 2-sphere of SU(2).

With the ‘inner product’ interpretation of the trace relation, the geometrical interpretation of the condition described by Theorem 3.4 is given as follows. The group Ui, on Theorem 3.4 could be written in the plane-angle representation:

\[ U_i = \exp \hat{J}_i \theta_i, \quad |J_i| = 1, \quad \theta_i \in \mathbb{R}^+. \]

Thus there exists a spherical triangle with length \( \theta_1, \theta_2, \theta_3 \) in a sphere \( S^2 \subset \wedge^2 \mathbb{R}^n \sim g \), where the center of the sphere is located in the origin \( O \) of \( \wedge^2 \mathbb{R}^n \). See FIG. 3a.

For a case where the angle \( \theta_1, \theta_2, \theta_3 \) is small, relation (30) becomes the closure condition (5), \( J_1 + J_2 + J_3 ≈ 0 \), which is the Bianchi Identity in the nearly-flat case with finite loop and infinitesimal curvature. It could be written as:

\[ \hat{J}_1 \theta_1 + \hat{J}_2 \theta_2 + \hat{J}_3 \theta_3 = 0, \quad |J_i| = 1, \quad \theta_i \in \mathbb{R}^+. \]

Then, using the theorem discovered by Minkowski [27], there exist a triangle with length \( \theta_1, \theta_2, \theta_3 \) in \( \wedge^2 \mathbb{R}^n \sim g \), where the center of the sphere is located in the origin \( O \) of \( \wedge^2 \mathbb{R}^n \). Thus the spherical triangle is approximated by a flat triangle.

As a conclusion for this section, we have already obtained Theorem 3.2 and particularly, Theorem 3.4. This theorem is important to defend the argument that the holonomy group of an oriented, n-dimensional simplicial complex in Euclidean Regge Calculus needs to be SO(n) plus some certain condition, which will be clear in the next section.

### IV. 4D CASE

For Euclidean gravity, one is interested, particularly, in the map:

\[ \rho_4 : SU(2) → \rho_4 (SU(2))_{\text{sim}} ⊂ SO(4). \]
The group SO(4) is topologically isomorphic to $\mathbb{RP}^3 \times S^3$, thus the representation map (33) is an immersion of a 3-sphere into $\mathbb{RP}^4 \times S^3$.

In 3-dimension and lower, all bivectors are simple, but this is not the case in four and larger dimension. Particularly in $\Lambda^2(\mathbb{R}^4) \sim \mathfrak{so}(4)$, one could define the notion of (anti) self-duality, since the Hodge star operator sends 2-forms to 2-forms. A bivector $\omega \in \mathfrak{so}(4)$ is a (anti) self-dual bivector if it satisfies $\omega_\pm = \pm \ast \omega_\pm$. Any bivector $\omega \in \mathfrak{so}(4)$ can be decomposed into its self-dual and anti self-dual parts, such that $\omega = \omega_+ + \omega_-$. Using the notion of self-duality, it is possible to write the components of any matrix $^4U \in SO(4)$ in terms of trigonometric functions as follows.

**Theorem 4.1 (Elements of SO(4))**

Let $J$ be an element of $\mathfrak{so}(4)$, written in its self and anti self-dual parts as follows:

\[
J = J_+ + J_- , \quad (J_\pm)_{\mu \nu} = \begin{cases} 
(J_\pm)_{00} = 0, & \text{if } \mu = \nu \\
(J_\pm)_{0i} = \pm (j_\pm)_i, & \text{if } \mu = i, \\
(J_\pm)_{i0} = \mp (j_\pm)_i, & \text{if } \nu = i, \\
(j_\pm)_{ij} = \varepsilon_{ijk} (j_\pm)^k & \text{else.}
\end{cases}
\]  

(34)

An element of SO(4) can be written as:

\[
^4U = U_+ U_- ,
\]  

(35)

with:

\[
U_\pm = \exp (J_\pm) = \exp (\hat{J}_\pm | J_\pm |) \in \rho_4 \left[ SO(3) \right] ,
\]

are the self and anti self-dual part of $^4U$. Moreover, it could be written as:

\[
^4U = I_{4 \times 4} \cos \varphi^+ \cos \varphi^- + \hat{J}^- \cos \varphi^+ \sin \varphi^- + \hat{J}^+ \sin \varphi^+ \cos \varphi^- + \hat{J}^+ \hat{J}^- \sin \varphi^+ \sin \varphi^- ,
\]  

(36)

with $\varphi^\pm = |J_\pm| = |j_\pm|$.

**Proof.** The proof for Theorem 4.1 is straightforward: Let $J \in \mathfrak{so}(4)$ be decomposed into its (anti) self-dual parts $J^\pm$. Since the self-dual and anti self-dual parts commute, the element of SO(4) can always be written as (35) using the exponential map. By Taylor expansion, (35) can be written as (36).

Now we are ready to classify all types of rotations in 4-dimensional Euclidean space. The classification is based on the corresponding Lie algebra of $^4U$, which can be obtained from the inverse of relation (21), namely $J = \ln U$.

It is a well-known fact that any $n \times n$ anti-symmetric matrix is similar with an anti-symmetric, block diagonal matrix by an orthogonal transformation $O(n)$. Using this result, a $4 \times 4$ anti-symmetric matrix $J \in \mathfrak{so}(4)$, can be written as an anti-symmetric $2 \times 2$ block diagonal matrix $J'$ as follows:

\[
\Lambda J \Lambda^{-1} = J' , \quad J' = \begin{pmatrix} \lambda_+ \sigma_z & 0_{2 \times 2} \\ 0_{2 \times 2} & \lambda_- \sigma_z \end{pmatrix} , \quad \lambda_\pm \in \mathbb{R} ,
\]  

(37)
by an orthogonal similarity transformation $\Lambda \in O(4)$. $\sigma_z$ is the $z$-components of Pauli matrix. Both $\lambda_+ \sigma_z$ and $\lambda_- \sigma_z$ describe geometrically the invariant planes of $4U$, where $\lambda_+ \sigma_z$ is fixed by a rotation of $\lambda_- \sigma_z$ and vice-versa. One could arrive to the conclusion that a bivector in 4-dimension, or an element of $\mathfrak{so}(4)$, can be written as a direct sum of two simple bivectors $\lambda_+ \sigma_z$, and $\lambda_- \sigma_z$, up to an orthogonal transformation. Using these fixed planes, one can defined the following classification for the rotations generated by $J \in \mathfrak{so}(4)$:

1. $4U$ describe a simple (or single) rotation, if one of the plane have zero norm: either $\lambda_+ = 0$, or $\lambda_- = 0$.
2. $4U$ describe a double (or Clifford) rotation, if $\lambda_+ \neq \lambda_-$.
3. $4U$ describe an isoclinic rotation, if $\lambda_+ = \lambda_-$.

Let us reviewed each case and see if it is possible to relate the classification with the self-duality of $J \in \mathfrak{so}(4)$. One could recall a remarkable relation between self-duality and simplicity of a bivector in 4-dimension.

Using this fact, we could classify the elements of $J \in \mathfrak{so}(4)$ as follows.

1. **The case** $j_+ = \pm j_-$. This automatically gives the simplicity condition $|j_+| = |j_-|$. Therefore, $J$ is simple and not self-dual. The constraint $(j_+, j_-) = (j_-, j_+)$ defines a three-dimensional subspace $\Omega \subset \mathfrak{so}(4)$. The subspace is spanned by generators $\{l\}$ satisfying the $\mathfrak{so}(3)$ (or $\mathfrak{su}(2)$) algebra relation. It is clear that $\Omega$ is isomorphic to $d \rho_n (\mathfrak{so}(3))_{\text{sim}}$, that is, the space of simple (irreducible) representation of $\mathfrak{so}(3)$ in 4-dimension. There exist only a single non-zero plane of rotation, which is either $\lambda_+ \sigma_z = 2 |j_+| \sigma_z$ or $\lambda_- \sigma_z = 2 |j_-| \sigma_z$, depending on the $\pm$ sign. The exponential map of such elements, say $U \in \rho_n (\mathfrak{so}(3))_{\text{sim}}$ can be obtained from Lemma 3.3. Any element $U \in \rho_n (\mathfrak{so}(3))_{\text{sim}}$ describe the simple rotation in 4-dimension.

2. **The case** $j_+ = 0$, or $j_- = 0$. Since the norms are not equal, $J$ is not simple, but is self or anti self-dual. The constraint defines a three-dimensional subspace $\Sigma_+ \subset \mathfrak{so}(4)$. Nevertheless, the degrees of freedom is three, spanned by generators $\{J^\pm\}$ satisfying the $\mathfrak{so}(3)$ (or $\mathfrak{su}(2)$) algebra relation. This is the space of semi-simple representation of $\mathfrak{so}(3)$ in 4-dimension. The planes of rotation have equal norm, which are either $\lambda_+ \sigma_z = |j_+| \sigma_z$ or $\lambda_- \sigma_z = |j_-| \sigma_z$, depending on which part is zero. Thus the exponential map of element of $\Sigma_\pm$ describe the left or right isoclinic rotation.

3. **The case** $|j_+| = |j_-|$. This caused $J$ to be simple, and the constraint $|j_+| = |j_-|$ defines a five-dimensional subspace $\mathfrak{so}(4)_{\text{sim}} \subset \mathfrak{so}(4)$, which is the space of simple bivectors in 4-dimension. In general, elements of $\mathfrak{so}(4)_{\text{sim}}$ is not a representation of $\mathfrak{so}(3)$ in 4-dimension, such that $\Omega \subset \mathfrak{so}(4)_{\text{sim}}$. Nevertheless, they have similar single plane of rotation as in the first case and therefore describe simple rotations.

4. **The case** $|j_+| \neq |j_-|$. This is the most general case where $J$ is semi-simple. The exponential map of such elements has two distinct planes $\lambda_+ \sigma_z$ satisfying (38), describing double (or Clifford) rotation in 4-dimension.

According to the (anti) self-dual pair $J^\pm$, we could conclude the following fact for rotations in 4-dimensional Euclidean space:

**Corollary 4.1** *(Classification of rotations in 4-dimensional Euclidean Space)*

*Given $U \in SO(4)$, satisfying $U = \exp J$ with $J \in \mathfrak{so}(4)$, the following statements are true:*

1. If $J$ is a simple bivector, then $U$ is a simple (or single) rotation.
2. If $J$ is either self-dual or anti self-dual, then $U$ is either a left or right isoclinic rotation.
3. If $J$ is semi-simple, then $U$ is a double (or Clifford) rotation.

Given three elements of group $U_1, U_2, U_3 \in \Omega$ of Case 1, they will automatically satisfy point 1 and 2 from Theorem 3.2 and 3.4. Moreover, if they satisfy the Bianchi Identity (2), their trace relation will gives angle relations. In the last chapter, we will show that the condition $|J^+| = |J^-|$, is crucial to obtain a (simplicial) complex in Regge gravity.
V. COMMENTS ON REGGE GEOMETRIES

At the end of this article, we apply the theorems concerning the elements of group to holonomies, that is, the group variables attached on the loops. The theorem gives condition on the loops, as well as on the holonomies, with the geometrical interpretations will also follows.

A. From Second Order to First Order Formulation

The spacetime in Regge gravity is modeled as a 4-dimensional manifold discretized by 4-dimensional simplicial complex, nevertheless the construction is valid for any dimension \( n \). Each simplex in the complex is uniquely defined by its edges length \( |l_m| \in \mathbb{R} \). Given these edges length, all higher geometrical variables such as the volume-form of the sub-simplex, as well as the angles between each two of them, are known. The curvature of the discretized manifold, defined as the deficit angles \( \delta \theta = 2\pi - \sum_m \theta_m \), are concentrated on the \((n - 2)\)-simplices, called as hinges.

Following the procedure described in [14], the Riemann tensor for each hinge \( h_i \) can be written as:

\[
R_{h_i} = \delta \theta_{\hat{\omega}_{h_i} \otimes \hat{\omega}_{h_i}},
\]

with \( \delta \theta_i \) is deficit angle on hinge \( h_i \) and \( \hat{\omega}_{h_i} \) is a generator of rotation, defined as a unit bivector, Hodge-dual to the direction of hinge \( h_i \) as follows:

\[
\hat{\omega}_{h_i} = \star \left( \frac{l_i \wedge \ldots \wedge l_{n-2}}{|l_i \wedge \ldots \wedge l_{n-2}|} \right).
\]

Using the fact that the deficit angle is proportional to the product of the modulus of rotation and area enclosed by the loop, \( \delta \theta_i \sim |\omega_{h_i}| \| \alpha_{h_i} \| \), (39) can be written as:

\[
R_{h_i} = \kappa \omega_{h_i} \otimes \alpha_{h_i},
\]

with \( \alpha_{h_i} \) is the (infinitesimal) loop orientation which is effectively a plane, \( \omega_{h_i} \) is the infinitesimal rotation parallel to \( \alpha_{h_i} \), and \( \kappa \) is a constant [28]. (40) guarantees the simplicity of \( \omega_{h_i} \) and \( \alpha_{h_i} \). Therefore, the Riemann tensor of a discretized Regge manifold where the curvature is concentrated on the hinge is:

\[
R(\mathbf{x}) = \rho(\mathbf{x}, x_{h_i}) R_{h_i},
\]

with \( \rho(\mathbf{x}, x_{h_i}) \) is the support for \( R_{h_i} \) which are constant along the hinges but vanish elsewhere.

The full contraction of Riemann tensor (39) is the Ricci scalar for each hinge, written as:

\[
R_{h_i} = \kappa \text{tr} (\omega_{h_i}^T \alpha_{h_i}) = 2\delta \theta_i.
\]

Moreover, the action of general relativity is \( S = \int \ast R(\mathbf{x}) = \int R(\mathbf{x}) \text{vol} \), such that inserting (42) gives:

\[
S = \int \rho(\mathbf{x}, x_{h_i}) R_{h_i} \text{vol} = \sum_i R_{h_i} \left( \int \rho(\mathbf{x}, x_{h_i}) \text{vol} = 2 \sum_i \delta \theta_i |V_{h_i}| \right),
\]

(43) is exactly the Regge action [1].

The condition (40) guarantees the simplicity of \( \alpha_{h_i} \) and \( \omega_{h_i} \), which in turns guarantees that the Riemann tensor (41) arise from a simplicial complex. In general, a non-simple bivectors do not have a well-defined and concrete geometrical interpretation, for example, it will be impossible to define a vector parallel (and perpendicular) to a non-simple bivector.

Let us proceed to the first order formulation, where the Riemann and the curvature 2-form is related by a local trivialization \( \mathbf{R} = e(\mathbf{F}) \) or:

\[
R_{a ij h} = e_a^l e_b^j F_{l ij h},
\]

where \( e_a^l \) is orthogonal. (44) could be written simply as \( \mathbf{R} = e^{\mathbf{F}} e^{\mathbf{T}} = e^{\mathbf{F} e^{-1}} \) by the orthogonality of \( e \). But from Lemma 3.2, (44) preserve the simplicity of a bivector, and this caused the curvature 2-form on hinge \( h_i \) to satisfy:

\[
F_{h_i} = \kappa e^{-1} \omega_{h_i} e \otimes \alpha_{h_i} = \kappa |\alpha_{h_i}| |\omega_{h_i}| \text{tr} (\omega_{h_i}^T \alpha_{h_i}),
\]

(45)
where \( \omega'_{h_i} \) is also simple. One could notice that \( \alpha_{h_i} \) and \( \omega'_{h_i} \) do not necessarily need to be parallel to each other. The curvature 2-form of discrete gauge gravity is:

\[
F(x) = \rho(x, x_{h_i}) F_{h_i},
\]

and the first order Regge action is unchanged since \( \epsilon \) is orthogonal.

For the reason concerning the fundamental discreetness in spacetime explained in the end of Chapter 2, one needs to apply a regularization procedure to Regge gravity, particularly, on the connection and curvature 2-form. The curvature 2-form is regularized into its corresponding holonomy on the hinge by relation (3). If the loop \( \gamma_i = \partial S_i \) only circles a single hinge \( h_i \), then:

\[
U_{S_i} = \hat{P} \exp_{\kappa |\alpha_{h_i}|} \omega'_{h_i} \int_{J_i} \rho(x, x_{h_i}) \hat{\omega}_{h_i}.
\]

(46)

The integrand on (46) gives a constant which is normalized to unity. From (46), it is clear that for a case where curvatures of the manifold are concentrated on hinges (conical singularity), the holonomy (or deficit angle) does not depend on the area of surface enclosed by the loop, \( |\alpha_{h_i}| \) (not to be confused with \( |\alpha_{h_i}| \), which is the weight of the infinitesimal 'loop' (plane orientation). An important fact one needs to notice is the simplicity of \( J_i \) as the algebra of \( U_{S_i} \). This, at least in 4D, cause the simplicity of \( U_{S_i} \). So we could conclude an important fact: Besides of the simplicity of all bivectors constructing the simplices, in 4-dimensional Euclidean Regge Gravity, all the holonomy circling a single hinge \( U_{h_i} \) are simple rotations.

### B. The Angle Relation as Contracted Bianchi Identity

Let us proceed further to an interesting geometrical fact on a simplicial complex. Without loosing of generality, let us consider a special \( n \)-dimensional simplicial complex known as the \((n + 1, 1)\)-Pachner move. Let us take a \( d \)-simplex inside this move, labeled as \( \Delta^{(d)} \), with \( d < n - 2 \). \( \Delta^{(d)} \) is shared by minimal three \((d + 1)\)-simplices. Let us consider three of them, say \( \Delta_i^{(d+1)} \) for \( i = 1, 2, 3 \). Each two of them, say \( \Delta_j^{(d+1)} \) and \( \Delta_j^{(d+1)} \), define a \((d + 2)\)-dimensional angle, which we label as \( \phi_{ij} \). Thus one has three \((d + 2)\)-angles \( \{\phi_{ij}\} \), \( j \neq i \) located on a \((d + 2)\)-hinge \( \Delta^{(d)} \). Moreover, each one of the three sets with elements \( \{\Delta_i^{(d+1)}, \Delta_j^{(d+1)}, \phi_{ij}\} \) belongs to a \((d + 2)\)-simplex, labeled as \( \Delta_i^{(d+2)} \). In a recursive manner, two of these \((d + 2)\)-simplices, say \( \Delta_i^{(d+2)} \) and \( \Delta_{ik}^{(d+2)} \), with \( i \neq j \neq k \), define a \((d + 3)\)-dimensional angle located on a \((d + 3)\)-hinge \( \Delta_i^{(d+1)} \), which we label as \( \theta_{ijk} \). Remarkably, these three sets of angle \( \{\phi_{ij}\} \) and \( \{\theta_{ijk}\} \) satisfy the dihedral angle relation (29), regardless of the dimension of the simplices [19].

We argue that the dihedral angle relation can be interpreted as the contracted Bianchi Identity in a simplicial complex. Let us consider three \( n \)-hinges on the \((n + 1, 1)\)-Pachner move. The three loops circling the hinges \( h_i \), say \( \gamma_{h_i} \), could be defined as the boundaries of faces \( S_i \) in Voronoï dual lattice [28]. As a consequence to this, three of these loops meets on a point \( \mathcal{O}_p \). See FIG 4(a).

One could attach elements of group to define holonomies on \( \gamma_{h_i} \):

\[
U_{\gamma_i} |\mathcal{O}_p = U_{S_i} |\mathcal{O}_p = \exp \hat{J}_i |\mathcal{O}_p \delta \theta_i \in SO(n).
\]

(47)

\( \hat{J}_i |\mathcal{O}_p \) is exactly \( \hat{\omega}_{h_i} = e^{-1} \hat{\omega}_{h_i} e \) as seen from a point \( \mathcal{O}_p \) inside simplex \( p \), where \( \hat{\omega}_{h_i} \) is defined as (40). On each hinge \( h_i \), the deficit angles are located, satisfying:

\[
\delta \theta_i = 2\pi - (\theta_{i,p} + \theta_{i,q} + \theta_{i,r}),
\]

(48)

where \( \theta_{i,p} \) is the \( n \)-dimensional angle of simplex-\( p \) located on \( n \)-hinge \( h_i \). One could define a special decomposition on the holonomy such that:

\[
U_{S_i} |\mathcal{O}_p = \exp - \hat{J}_i |\mathcal{O}_p \theta_{i,p} \exp - \hat{J}_i |\mathcal{O}_p \theta_{i,q} \exp - \hat{J}_i |\mathcal{O}_p \theta_{i,r}, \quad i = 1, 2, 3,
\]

(49)

this is illustrated in FIG. 4(a), with the following explanation. The holonomy on \( \gamma_i = \partial S_i \) is (47). Let us take \( U_{\gamma_1} |\mathcal{O}_p \) as an example. Moving the origin from \( p \) to \( a \), such that \( U_{\gamma_1} |\mathcal{O}_a = U_{\gamma_p} U_{\gamma_1} |\mathcal{O}_p U_{\gamma_p}^{-1} \), it is clear that \( U_{\gamma_1} |\mathcal{O}_a =
describe the decomposition defined in (49).

\( U \) special gauge fixing such that the holonomies on internal curves are identities, say

\[ U \] that it consists a product of holonomies on closed curve as follows:

\[ U_\theta \]

See FIG. 4(b). It is clear that the product of holonomies in (50) is equal to identity:

\[ \text{equivalent to the theta graph in FIG. 1(a), by an identification } a = b = c = p'. \]

\[ U_{\gamma_{apc}} U_{\gamma_{aqd}} U_{\gamma_{dra}} \]

which is a product of three holonomies on open curves. One could choose such that the three subholonomies have \( \theta_{1,p}, \theta_{1,q}, \theta_{1,c} \) from (48) as their modulus of rotation. Now we want to decompose \( U_{\gamma_1|O_a} \) such that it consists a product of holonomies on closed curve as follows: \( U_{\gamma_1|O_a} = U_{\gamma_{apc}} U_{\gamma_{aqd}} U_{\gamma_{dra}} \). Let us choose a special gauge fixing such that the holonomies on internal curves are identities, say \( U_{\gamma_{aqd}} = U_{\gamma_{dra}} = 1 \). Therefore, \( U_{\gamma_{apc}} = U_{\gamma_{aqd}} = U_{\gamma_{dra}} = U_{\gamma_{dra}} \). Sending back these holonomies from \( a \) to \( p \), they clearly describe the decomposition defined in (49).

Doing the same decomposition procedure to \( U_{\gamma_{1}|O_p} \) and \( U_{\gamma_{3}|O_p} \), one obtains three subholonomies meeting on \( p \), say \( U_{\gamma_{apc}} \), \( U_{\gamma_{aqd}} \), \( U_{\gamma_{dra}} \), or using more compact notations:

\[ \left\{ U_{1,p}|O_p = \exp - \hat{J}_i |O_p, U_{2,p}|O_p = \exp - \hat{J}_i |O_p, U_{3,p}|O_p = \exp - \hat{J}_i |O_p, \right\} \]

See FIG. 4(b). It is clear that the product of holonomies in (50) is equal to identity:

\[ U_{1,p}|O_p U_{2,p}|O_p U_{3,p}|O_p = 1, \quad U_{i,p}|O_p \in \rho_n [SU(2)]_\text{sim} \subset SO(n). \]

By an identification of point \( a = b = c \) in FIG. 4(c), the configuration of loops where \( U_{1,p}|O_p \), \( U_{2,p}|O_p \), and \( U_{3,p}|O_p \) are attached is topologically equivalent to a theta graph in FIG 1. Therefore relation (51) is indeed the Bianchi Identity.

One could realize the following facts that: (1) \( U_{i,p}|O_p \) are simple rotations since \( \hat{J}_i |O_p \) are simple, and (2) \( \left\{ \hat{\omega}_i |O_p \right\} \) and thus \( \left\{ \hat{J}_i |O_p \right\} \), \( i = 1, 2, 3 \) construct a trihedron, which cause \( \left\{ U_{i,p}|O_p \right\} i = 1, 2, 3 \) belongs to a common SU(2) (or SO(3)) subgroup of SO(n). With the Bianchi Identity (51), the set \( \left\{ U_{i,p}|O_p \right\} i = 1, 2, 3 \) satisfies either Theorem 3.4 or 3.2. As a consequence to this, the trace of (51) in the form of (25) gives angle relation (29). For consistency, one could check whether the angle \( \{ \theta_{i,p}, i = 1, 2, 3 \} \) really satisfy the angle relation from the geometries of the \((n+1,1)\)-Pachner move: In fact, the angle \( \{ \theta_{i,p}, i = 1, 2, 3 \} \) are the angles between \((n-1)\)-simplices located on hinges \( h_i \), say \( \Delta^{(n-2)} \), where these hinges meet on a common \((n-3)\) simplex \( \Delta^{(n-3)} \), and thus needs to satisfy the dihedral angle relation. With these arguments, the dihedral angle relation represents the 'contracted' Bianchi identity for a simplicial complex. Moreover, since the simplices satisfies dihedral angle relation, the gauge group of discrete gravity is a simple representation of SU(2), instead of SO(3).

VI. DISCUSSION AND CONCLUSION

As we had mentioned in the Introduction, there are three aspects which becomes our main interest in this article: (1) The gauge group SO(3) and SU(2), (2) the angle relation, or SU(2) trace relation, or spherical law of cosine, and...
(3) the simplicity of the bivectors, and more over the simplicity of the rotations. By the explanation from the previous sections, it had been clear that these three properties are related to one another, nevertheless, we will discuss these relations in more detailed manner.

Let us started from the simplicity of bivectors. As already been stated in [24], in order to construct a simplex from bivectors, each subsimplices need to be constructed from simple bivectors living in the same subspace. As we have mentioned earlier, the simplicity of a bivector guarantees the existence of a single plane defined by two non-parallel vectors. The existence or these vectors are crucial for the construction of a simplex, since it is uniquely determined by its edges. For 4D Regge simplicial complex, the simplicity condition is equivalent to demand that the norm of the self-dual and anti self-dual parts of the bivectors are equal. In this article, this is realized by elements of $\mathfrak{so}(4)$ satisfying Case 1 and Case 3 defined in Section IV. In fact, in spinfoam model of gravity, one has a more strict condition concerning the simplicity of the bivector, such that not only the norm of the self-dual and anti self-dual parts need to be equal, but also their directions. Precisely, they need to satisfy only Case 1 of Section IV. The condition is known as the linear simplicity constraint [15, 29], which originates from a specific gauge fixing, the time gauge [30]. Our work gives a similar condition for the holonomy representation of discrete gauge gravity. Through the derivation in Section V A, if we want our (lattice) gauge theory to describe Regge $n$-dimensional simplicial complex, each holonomy on the hinges needs to be simple.

The second important aspect is the angle relation. It should be kept in mind that besides being simple, the bivectors in an $n$-simplex needs to construct also the lower dimensional subsimplices recursively. In other words, the bivectors needs to be collected into sub-algebra space of $\mathfrak{so}(n)$, with $\mathfrak{so}(3)$ being the simplest, non-trivial case. This becomes the reason why SO(3) and SU(2) become an interest in this article, in the sense that SO(3) is the 'building blocks’ for higher dimensional orthogonal groups. We have found in Section V that all holonomies on the hinges are simple rotations, and moreover one could decompose in a proper way such that a holonomy on a single hinge is a product of (minimal) three simple subholonomies meeting on a point, satisfying the Bianchi Identity. Contracting the Bianchi Identity, one could obtain the trace relation of the holonomy. Since the subholonomies are simple and they belong to a common SO(3)/SU(2) group, the trace relation gives either (7) or the spherical law of cosine (24), by Theorem 3.2 or 3.4. But one has an interesting fact that the simplex in any dimension always satisfy dihedral angle relation (24). In this step we need to rule out the SO(3) group, since the one giving the angle relation, which is a constraint that must be satisfied by a closed Euclidean complex, is the spherical law of cosine originating from the trace relations of SU(2).

The last aspect is the SO(3)/SU(2) relation. As we had mentioned previously, SO(3) is the 'building blocks' for higher dimensional orthogonal groups. The importance of SU(2) is indirect: because it double covers the SO(3). But we have an interesting fact that the simplex satisfy SU(2) trace relation instead of SO(3) as its angle relation, and we found that this is not merely a coincidence. The reason for a simplex to satisfy SU(2) trace relation might be traced to the fact that an $n$-simplex, is a special case of convex polytopes, homeomorphic to an $n$-ball, with the boundary being homeomorphic to an $(n-1)$-sphere [31]. The trace of the Bianchi Identity carries information about the local curvature on the boundary. SU(2) is topologically isomorphic to a 3-sphere; a spherical triangle lies on its great 2-sphere. This is also the reason why SU(2) trace relation gives the spherical law of cosine. SO(3), although it describe a rotation in 3-dimensional Euclidean space, is not simply connected as an $\mathbb{R}^3$. These are also true for higher dimensional rotation: SO(n), in general is not simply connected. The Spin group, Spin(n), which double covers SO(n), is usually used as a substitute to SO(n) because of their simpler topological structure. Nevertheless, in a simplicial complex of Regge Calculus, the reason of using SU(2) instead of SO(3), is more than merely a factor of simplicity, but as a natural way which originates from a property of a simplex as a convex polytopes. In fact, the less natural feature of SO(3) in describing rotations in 3D, compared to SU(2), was already known, a common example are the problem of gimbal lock in navigation [32], and Dirac belt in a more abstract way [33], with a well-known example in physics includes the existence of spins in quantum mechanics. The fact that SU(2), defined in a complex and imaginary manner, provide a more compact and natural way to handle real-world problems is a fascinating fact of reality.

To conclude the article, we address the question concerning the holonomy group for discrete manifold in Regge Calculus. We found that the holonomy group is restricted such that the holonomies on the loop circling a single hinge are simple rotations, and that at each points where these loops meet, the angle relation as the contracted Bianchi Identity, is satisfied. One of the importance of this result, is that it provide a discrete and regularized version of the simplicity constraint for the Lie algebra-valued connection, while other importance are to be studied elsewhere.
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