Large $N$, $\mathbb{Z}_N$ Strings and Bag Models

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Abstract

We study $\mathbb{Z}_N$ strings in nonabelian gauge theories, when they can be considered as domain walls compactified on a cylinder and stabilized by the flux inside. To make the wall vortex approximation reliable, we must take the 't Hooft large $N$ limit. Our construction has many points in common with the phenomenological bag models of hadrons.

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1 Introduction

In a previous paper [1], we explored the idea that a flux tube can be thought, under particular circumstances, as a domain wall compactified on a cylinder. To make it possible, the theory must admit two degenerate vacua: one in the Coulomb phase, that is kept inside the cylinder, and the other one in the Higgs (or confining) phase, that is kept outside the cylinder. For the simplest example, the abelian Higgs-Coulomb model, we were able to determine a condition under which the wall vortex approximation is quantitative reliable. When we increase the number \( n \) of quanta of magnetic flux, the radius of the vortex grows like \( n^{\frac{1}{2}} \), while the thickness of the wall remains fixed. When \( n \) is enough large, the thickness of the wall becomes negligible and the tension of the vortex is given by the simple minimization of the energy density.

The present paper started with the following question: can we find a nonabelian version of the wall vortex? In [1] we found examples where \( \mathbb{Z}_N \) strings in a confining \( SU(N) \) gauge theory can be qualitative thought as wall vortices. What we are locking for, is some realization in which the wall vortex in quantitative reliable, so that we...
can apply the simple argument of the energy minimization used for the abelian Higgs-Coulomb model.

If we want to mimic what we obtained for the abelian model, we must increase the number $n$ of flux quanta, until the radius of the $n$-string becomes much larger than the thickness of the wall. A problem immediately arise: in a $SU(N)$ gauge theory, $n$ is limited to be smaller than $N$. This means that, to have a chance of obtaining some result, we must explore the large $N$ limit of the nonabelian gauge theory [2].

In the first part of the paper we will explore the case of degenerate vacua, one in the Coulomb phase and the other one in the confining phase, and we will consider two examples of supersymmetric gauge theories. In the second part of the paper we face the more general situation in which the Coulomb vacuum is metastable (or in the extreme case unstable). Our hope is to apply these ideas to non supersymmetric gauge theories and we will try to do that for large $N$ QCD.

The ideas we will expose are not new to particle physics. Almost thirty years ago a lot of works have been made on the bag models of hadrons. The first one was the so called MIT bag model [3], where the quarks where modelled as free fermions bounded in region of finite volume and non zero energy density. This model had a great success in explaining static properties of hadrons [4]. A lot of different version of the bag model have been proposed after that. For example, in the so called SLAC bag model [5], the bag had finite tension and the interior of the bag had zero energy density. Another interesting approach was the Friedberg-Lee model of hadrons [6]. In this works, by means of an auxiliary scalar field, they derived the bag as a domain wall interpolating between a metastable vacuum and a true vacuum. In the metastable vacuum the quarks have a small mass, while in the true vacuum they have a great (or infinite) mass. In the Friedberg-Lee model, hadrons where realized as nontopological solitons. Despite their success in explaining static properties of hadrons, bag model had some difficulties in describing interactions and so they where though only as phenomenological modelling of the real theory, QCD. Attempts to derive the bag model from QCD first principles have been made in [7] and [8].

In this paper we will argue that some confining gauge theory can share, in the large $N$ limit, some properties in common with bag models. To be more clear, we give now a definition of what we mean by a bag model. The definition is not precise but sufficiently large to include all the possible realizations. In a bag model we have two phases: one in the interior of the bag that can contain quarks and gluons, the other
in the exterior of the bag where only gauge singlets can live. The interior Coulomb phase can have some energy density $\varepsilon_0$ and the bag can have a tension $T_W$. For example, in the MIT bag model $\varepsilon_0 \neq 0$ and $T_W = 0$, while in the SLAC bag model $\varepsilon_0 = 0$ and $T_W \neq 0$. In general we can define a bag model action:

$$L = \int_V d^3 x \mathcal{L}_{int} - \varepsilon_0 V - T_W \int_S d^2 \xi \sqrt{\text{det} h},$$

(1.1)

where $\mathcal{L}_{int}$ describes the dynamics in the interior Coulomb phase. Another important ingredient, to complete the definition of the theory, consist in specifying the boundary conditions of the fields at the bag surface. This condition is that color must be trapped inside the bag. In Figure 1 there is a meson in the bag model. When the meson is rotating very fast, it becomes approximatively the wall vortex with a quark and an antiquark at the ends. The connection between the bag and the string was first recognized in [9].

Confinig Phase

![Figure 1](image)

**Figure 1:** A meson in the bag model. Quarks and gauge fields are in a Coulomb phase trapped inside the bag. The Coulomb phase has energy density $\varepsilon_0$ and the bag has surface tension $T_W$. The confining vacuum is outside the bag.

In the large $N$ limit of a $SU(N)$ gauge theory, Feymann graphs organize themselves into a genus expansion in powers if $\frac{1}{N}$ [2]. For this reason it is believed that some dual string theory should describe nonabelian gauge theories, and this string
theory should become weakly coupled as $N$ goes to infinity. The AdS/CFT correspondence is a concrete realization of this duality [10] for the $\mathcal{N}=4$ superconformal gauge theory. After that, some examples have been found where the gauge theory is not superconformal but confining [11, 12]. For the above mentioned reasons it may be sound a bit strange that a confining gauge theory becomes well approximated by a bag model in the large $N$ limit. An important point, is that our claims regard $n$-strings with $n$ of order $N$, and not the fundamental string over which is supposed to be built the dual string theory. So the hadrons that becomes a bag are the exotic mesons $q^n-\bar{q}^n$ with $n \sim N$.

The string tensions $T_V(n)$ has been much investigated, both theoretically and numerically (see [14] for a review), in particular the ratio of string tensions defined by $R(n, N) = \frac{T_V(n)}{T_V(1)}$. From the theoretical side there are two important predictions: the Casimir scaling and the sine formula:

**Casimir scaling.** In an intermediate range of distance, the force between two static sources is proportional to the quadratic Casimir of the representation [15, 16]. As $N$ becomes large, this range goes to infinity and the extrapolation suggest the ratio for string tensions

$$R(n, N) = \frac{n(N-n)}{N-1}.$$ (1.2)

This procedure is not rigorous, since we should first make the distance to go to infinity, and then make the large $N$ limit. It has be noted [17] that the Casimir scaling, when $n$ is kept fixed and $N$ goes to infinity, has corrections $\frac{1}{N}$ instead of the expected $\frac{1}{N^2}$. This should rule out the Casimir scaling but there are also different opinions on that [18].

**Sine formula.** This formula first appeared first in [20]

$$R(n, N) = \frac{\sin \left(\frac{\pi n}{N}\right)}{\sin \left(\frac{\pi}{N}\right)}.$$ (1.3)

studying $\mathcal{N}=2$ $SU(N)$ gauge theory, softly broken by the adjoint mass term $\mu \text{Tr} \Phi^2$. In [21], in the MQCD contest, has been shown that this formula is true also in the opposite limit $\mu \to \infty$. In [22] it has been shown that the sine formula

1This should also overcome previous problems founded in trying to relate large $N$ QCD and bag models [13].
doesn’t hold in the intermediate regime but has non universal corrections. The formula reappeared in [23] in the contest of cascading gauge theories. Another meaning of the sine formula has appeared recently in [24]. The conclusion is that the sine formula, even if not directly derived in the QCD contest, present some universal characteristics that makes it a good candidate for QCD, or for any confining gauge theory.

On the lattice side, a lot of works have been done to compute the tensions in pure $SU(N)$ Yang-Mills [25, 26]. The most recent work on the subject [27] gives results up to $N = 8$. In the end of the paper we will confront our theory with these results.

The paper is organized as follows. In Section 2 we consider the wall vortex in the abelian Higgs model. First we review the results of [1] in the case where the Coulomb phase has the same energy of the Higgs phase. Then we extend the results to the most general case where the Coulomb phase is a stationary point of the potential (this must always be due to the $U(1)$ symmetry). In Section 3 we study a model in which there are $Z_N$ solitonic magnetic strings that in the large $N$ limit become wall vortices. In Section 4 we try to apply our ideas to confining $Z_N$ string in ordinary QCD. Even if our reasoning doesn’t follows directly from QCD first principles, it gives a sharp prediction for the ratio of string tensions in the large $N$ limit. The future lattice computations should easily prove or disprove our hypothesis.

2 The Abelian Theory and the Wall Vortex

We consider the abelian gauge theory coupled to a charged scalar field

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu - i A_\mu) q |^2 - V(|q|). \quad (2.1)$$

In [1] we considered the case in which the potential has two degenerate vacua, one in the Coulomb phase and the other in the Higgs phase. In 2.1 we briefly review the results that we have obtained. In 2.2 we make a step forward trying to generalize the wall vortex idea to the most general case, where the Coulomb phase is not a true vacuum but only a stationary point.
2.1 Coulomb-Higgs model and the surface bag

Now we consider the potential of Figure 2. There are two degenerate vacua: one in the Higgs phase where $|q| = q_0$, and the other in the Coulomb phase. For this reason we call it abelian Coulomb-Higgs model. The magnetic vortex [29, 30] in the Higgs vacuum is nothing but the wall interpolating between the two vacua, compactified on a cylinder and with the Coulomb vacuum left inside. The energy density as function of the radius is

$$T(R) = \frac{\Phi_B^2}{2\pi e^2 R^2} + T_W 2\pi R ,$$

where $\Phi_B$ is the magnetic flux. The magnetic flux is quantized in integer values $\Phi_B = 2\pi n$. The stable configuration is the one that minimizes the tension:

$$T_V = 3\sqrt{2}\pi \left( \frac{n T_W}{e} \right)^\frac{3}{2}, \quad R_V = \sqrt{2} \sqrt[3]{\frac{n^2}{e^2 T_W}} .$$

In this simple calculation we have neglected the thickness of the wall $\Delta_W$ and this is the crucial point. We can trust (2.3) only if the radius if the vortex $R_V$ is much greater than $\Delta_W$. In [1] we argued that (2.3) can be used in a self-consistent way to determine its region of validity. Taking $R_V = \sqrt{2} \frac{n^\frac{3}{2}}{e^2 T_W^\frac{3}{4}}$ for true, we increasing $n$ keeping fixed all the other parameters of the theory. Note that $\Delta_W$ does not depend on $n$. There will be some value $n^*$ above which $R_V \gg \Delta_W$ and so (2.3) can be trusted.
2.2 Higgs model and the volume bag

Now we discuss an abelian gauge theory with a potential like Figure 3. The Coulomb vacuum is metastable and has energy density $\varepsilon_0$.

Figure 3: The vacuum $q = 0$ is in the Coulomb phase, it is metastable and its energy density is $\varepsilon_0$. The other vacuum at $|q| = q_0$ is in the Higgs phase and has zero energy density. When $\varepsilon_0 = 0$ the potential becomes the one of Figure 2 and when $\varepsilon_0 = v_0$ it becomes that of Figure 4.

Also in this case is convenient to think of the flux tube as a domain wall compactified in a cylinder, with the metastable Coulomb vacuum inside and the true Higgs vacuum outside. Neglecting the thickness of the wall, we can write the tension as function of the radius:  

$$T(R) = \frac{2\pi n^2}{e^2 R^2} + T_W 2\pi R + \varepsilon_0 \pi R^2. \quad (2.4)$$

There are two regimes in which (2.4) can be easily solved.

**Surface (or SLAC) bag.** This region is when $n$ satisfies the conditions

$$\frac{q_0^2 e}{\sqrt{v_0}} \ll n \ll \frac{q_0^2 e}{\sqrt{\varepsilon_0}}. \quad (2.5)$$

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To derive this formula we could first think of the vortex tension as function of two parameters, the radius $R$ and the thickness $\Delta$ of the shell where the $q$ field goes from 0 to $q_0$. The tension is roughly $T(R, \Delta) = \frac{n^2}{e^2 R^2} + \frac{q_0^2 R}{\Delta} + R\Delta v_0 + \varepsilon_0 R^2$. Minimizing with respect to $\Delta$ we obtain $\Delta_W \sim \frac{q_0}{\sqrt{v_0}}$. So we can interpret it as a domain wall since is independent on $n$ and $R$. 

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In this limit the surface term in (2.4) dominates over the volume term and the minimization gives:

\[ T_V = 3\sqrt[3]{2\pi} \left( \frac{nT_W}{e} \right)^{\frac{2}{3}}, \quad R_V = \sqrt[3]{2} \sqrt[3]{\frac{n^2}{e^2T_W}}. \] (2.6)

Note that the surface region (2.5) exists only if \( \varepsilon_0 \ll v_0 \). As \( \varepsilon_0 \) is increased until it reaches \( v_0 \), the SLAC region is eaten by the MIT region.

**Volume (or MIT) bag.** This region is when \( n \) satisfies the condition

\[ \frac{q_0^2e}{\sqrt{\varepsilon_0}} \ll n. \] (2.7)

In this limit the volume term in (2.4) dominates over the surface term and the minimization gives:

\[ T_V = 2\sqrt{2\pi} \frac{n\sqrt{\varepsilon_0}}{e}, \quad R_V = \sqrt{2} \sqrt[3]{\frac{n}{e^2\sqrt{\varepsilon_0}}}. \] (2.8)

Note that the tension is proportional to \( n \), as happens in the BPS case.

**General conjecture**

The general conjecture is that the result (2.8) works for every potential, even if the Coulomb phase is not metastable but instable like in Figure 4. Let’s write the conjecture for clarity. Consider the abelian Higgs model (2.1) with a general potential that has a true vacuum at \( |q| = q_0 \neq 0 \) and a Coulomb phase with energy density \( V(0) = \varepsilon_0 \neq 0 \). Call \( T_V(n) \) the tension of the vortex with \( n \) units of magnetic flux. The claim is that

\[ \lim_{n \to \infty} T_V(n) = 2\sqrt{2\pi} \frac{n\sqrt{\varepsilon_0}}{e}. \] (2.9)

We will give a substantial check of this statement in [28] using numerical computations.

**A check**

Now we make a non trivial check of the conjecture using the famous example solved by Bogomoln’y [41]. When the potential is

\[ V(|q|) = \frac{e^2}{2}(|\phi|^2 - \xi)^2, \] (2.10)
the tension is

$$T_V = 2\pi n \xi$$

(2.11)

for every value of \(n\). Solving the model with our trick, the result must coincide with (2.11). For the BPS potential (2.10), the energy density of the Coulomb vacuum is \(\varepsilon_0 = \frac{e^2 \xi^2}{2}\) and, using (2.8), we find exactly (2.11). This can hardly be considered only a coincidence.

3 Solitonic \(\mathbb{Z}_N\) Strings

The principal aim of this paper is to see if it possible to have a nonabelian generalization of the wall vortex. In this section we consider a toy model of solitonic and magnetic \(\mathbb{Z}_N\) strings. To have this we need an \(SU(N)\) gauge theory with scalar fields that are not charged under the center of the group. If in some vacuum the scalar vevs brake completely \(SU(N)\), then the solitonic strings are stabilized by the homotopic group

$$\pi_1 \left( \frac{SU(N)}{\mathbb{Z}_N} \right) = \mathbb{Z}_N.$$  

(3.1)

In the following we will consider the a broken version of the \(\mathcal{N} = 4\) supersymmetric gauge theory. There are three adjoint scalar fields that break completely the gauge group and are responsible for the formation of the \(\mathbb{Z}_N\) strings.
3.1 $\mathcal{N} = 1^*$ and the surface bag

The first case that we consider is the $\mathcal{N} = 1^*$ $SU(N)$ gauge theory, that is $\mathcal{N} = 4$ broken to $\mathcal{N} = 1$ by mass terms for the chiral superfields $\Phi_1$, $\Phi_2$ and $\Phi_3$. The superpotential is the $\mathcal{N} = 4$ interaction plus the mass terms:

$$ W_{\mathcal{N}=1^*} = \frac{N}{g^2} \text{Tr} \left( \Phi_1 [\Phi_2, \Phi_3] + \frac{m_1}{2} \Phi_1^2 + \frac{m_2}{2} \Phi_2^2 + \frac{m_3}{2} \Phi_3^2 \right) . $$

(3.2)

The stationary equations are, a part from numerical factors, the commutation relations of the $SU(2)$ algebra [32, 33]. For example, deriving (3.2) with respect to $\Phi_3$, we obtain $[\Phi_1, \Phi_2] = -m_3 \Phi_3$. Making the rescalings

$$ \Phi_1 = i \sqrt{m_2 m_3} \tilde{\Phi}_1 , \quad \Phi_2 = i \sqrt{m_3 m_1} \tilde{\Phi}_2 , \quad \Phi_3 = i \sqrt{m_1 m_2} \tilde{\Phi}_3 , $$

(3.3)

we obtain exactly the $SU(2)$ algebra as equations of motion:

$$ [\tilde{\Phi}_i, \tilde{\Phi}_j] = i \epsilon_{ijk} \tilde{\Phi}_k . $$

(3.4)

The vacua of the theory are obtained choosing a partition of $N$

$$ \sum_{d=1}^{N} d k_d = N , $$

(3.5)

so that the $N \times N$ matrices $\Phi_i$ are covered with spin $\frac{d-1}{2}$ representations of the $SU(2)$ algebra. The gauge group is classically broken

$$ SU(N) \rightarrow \bigotimes_{d=1}^{N} U(k_d) \bigotimes U(1) , $$

(3.6)

and the vacua of the theory are divided into two classes:

**Massive Vacua.** These vacua are called massive because there is a mass gap. For every divisor of $N$ we must cover the matrices $\tilde{\Phi}_i$ only with representation of the same dimensionality. In this case the gauge group is classically broken to $SU \left( \frac{N}{d} \right)$ and there are no $U(1)$ factors. The $SU \left( \frac{N}{d} \right)$ group confines and there is a mass gap. This vacua are in one to one correspondence with the possible phases of a general confining gauge theory [34]. The works [32, 33] showed that the $SL(2, \mathbb{Z})$ duality of the original $\mathcal{N} = 4$ theory, exchange the massive vacua among them.
**Coulomb Vacua.** When there are at least two representations with different dimensionality, the unbroken gauge group has at least one $U(1)$ factor. Since there is no strong dynamics for the $U(1)$'s, they survive in the infrared giving some massless particles.

Before proceeding to the central point, we need to show some qualitative properties of the domain walls in the $\mathcal{N} = 1$ theory. What is really important for us in the $N$ dependence of the tensions and of the thicknesses of the walls. If we assume that the walls are BPS saturated, their tension is governed by the difference of the superpotential in the two vacua [31]. For a general vacuum (3.5) the superpotential (3.2) is proportional to the sum of the Casimirs of the spin representations

\[ W \sim \tilde{m}^3 N^2 \left( \sum_{d=1}^{N} k_d (d-1)(d+1) \right), \quad (3.7) \]

where for simplicity we have used $\tilde{m} = \sqrt{m_1 m_2 m_3}$. And so the superpotential, goes like $\sim N^4$. Choosing two generic vacua, the interpolating wall scales like [36, 37]:

\[ T_W \sim N^4 \tilde{m}^3, \quad \Delta W \sim \frac{1}{\tilde{m}}. \quad (3.8) \]

There are some exceptions to this scalings. Take for example the domain wall between the Higgs vacuum and the Coulomb vacuum where the $N$ is partitioned into a $N-1$ and a 1 representation (call it $(N-1,1)$ for simplicity). The leading terms in the superpotential cancel each other and the wall goes like:\(^3\)

\[ T_W \sim N^3 \tilde{m}^3, \quad \Delta W \sim \frac{1}{N \tilde{m}}. \quad (3.9) \]

Now we make our claim. The solitonic $\mathbb{Z}_N$ strings in the Higgs vacuum are made by a domain wall compactified on a cylinder with a Coulomb vacuum inside. When $N$ becomes large the energetically favorite Coulomb vacuum is the one where $N$ is partitioned into a $N-1$ and a 1 spin representation. For sufficiently large $N$ the radius of the $n$-strings becomes much larger than the thickness of the wall, and so that the wall vortex condition (namely $\Delta W \ll R_V$) is satisfied.

First of all we check that the wall vortex condition is satisfied. The Coulomb vacuum of interest, has only one $U(1)$ factor, the one generated by the matrix

\(^3\)This mechanism is very similar to what operates in $\mathcal{N} = 1$ SYM where the tension of the domain wall between adjacent vacua goes like $\frac{1}{N}$ instead of the expected $\frac{1}{N^2}$ [38]. This point has been clarified in [39] and [40].
diag \((1, \ldots, 1, -(N - 1))\). This is the only generator in the Cartan subalgebra that, when exponentiated, passes through all the elements of the center of \(SU(N)\). In particular the \(n\)-string has charge

\[
\frac{n}{N} \begin{pmatrix}
1 & & \\
& \ddots & \\
& & 1 \\
\end{pmatrix} - (N - 1).
\]

The tension as function of the radius \(R\) is

\[
T(R) \sim \frac{n^2 N}{R^2} + N^3 \tilde{m}^3 R
\]

(3.11)

and, minimizing with respect to \(R\), we obtain:

\[
T_V \sim \sqrt[3]{n^2 N^7 \tilde{m}^2}, \quad R_V \sim \left(\frac{n}{N}\right)^{\frac{2}{3}} \frac{1}{\tilde{m}}.
\]

(3.12)

Since \(\Delta_W \sim \frac{1}{N}\), for \(N\) sufficiently large \(R_V\) is much greater than \(\Delta_W\) and the wall vortex approximation works. The ratio of string tensions is thus:

\[
R(n, N) = \min \left(\frac{2}{3}, \frac{N-n}{\tilde{m}} \right).
\]

(3.13)

Now we consider the other point: why the Coulomb vacuum \((N - 1, 1)\) should be preferred with respect to the others? Or again: why the \(\mathbb{Z}_N\) strings should all choose the same Coulomb vacuum? We don’t have a rigorous proof for these questions but only an argument in favor of it. For example consider the \(n\)-string. Its flux could also be generated by the \(U(1)\) of the Coulomb vacuum \((N - n, n)\):

\[
\frac{1}{N} \begin{pmatrix}
n1_{N-n} & \\
& -(N-n)1_n \\
\end{pmatrix}.
\]

(3.14)

Note that this \(U(1)\) cannot reach all the elements of the center of \(SU(N)\) so, if this generator comes out to be energetically favorite, we would have a different \(U(1)\) inside every \(n\)-string. What we are going to do is to evaluate the tension and compare it with the one obtained with the Coulomb vacuum \((N - 1, 1)\). The tension as function of the radius is

\[
T(R) \sim \frac{n(N - n)}{R^2} + \tilde{m}^3 N^4 R.
\]

(3.15)
The flux term has an $\frac{n(N-n)}{N}$ from the trace of the square of (3.14), and a power $N$ from the 't Hooft scaling. The domain wall is an ordinary soliton that scales like (3.8). The minimization of (3.15) with respect to $R$ gives:

$$T_V \sim \sqrt{n(N-n)N^8 \tilde{m}^2}, \quad R_V \sim \sqrt{\frac{n(N-n)}{N^4}} \frac{1}{\tilde{m}}. \quad (3.16)$$

Note that in this case the radius scales like $N^{-\frac{2}{3}}$ if we send $n$ to infinity like $N$. So we can not apply our approximation since the thickness of the wall (3.8) remains finite. Note that (3.16) would give a tension that scales like $N^{\frac{10}{3}}$ would be greater than the other one (3.12). But, as we said, the wall vortex approximation doesn’t work and we have to find another way to estimate the tension. The minimization of (3.16) would give $R_V \sim N^{-\frac{2}{3}}$ that is much lesser than the thickness of the wall $\Delta W \sim 1$. This means that the scalar fields are spread all over the radius of the vortex and so their kinetic energy goes like $\frac{N^4}{R_V}$ and dominates over the magnetic energy term that goes like $\frac{N^2}{R_V}$. This imply that the vortex tension is essentially given by the minimization of the scalar field action $N^4 (\partial \phi \partial \phi + V(\phi))$ and so it scales like $N^4$. This shows that the $(N-1, 1)$ vacuum gives a lighter tension than the $(N-n, n)$ vacuum in the large $N$ limit.

4 Large $N$ QCD

Now we try to see if there is any chance that the wall vortex scenario is realized in large $N$ QCD.

First we recall some well established results regarding the QCD string tension at large $N$. Consider the interaction between two fundamental strings. It has been shown in [17] that interaction vanishes like $\frac{1}{N}$. Thus the tension of the $n$-string, when $n$ is fixed and $N$ becomes large, is equal to $n$ times the tension of the fundamental string

$$T_V(n) = nT_V(1) + O\left(\frac{1}{N^2}\right). \quad (4.1)$$

If instead we keep $n$ of order $N$, the interactions are of order 1. In fact, there are $\binom{n}{2}$ ways of making a fundamental interaction between any couple of strings, and so in total we have an interaction of order $\binom{n}{2} \frac{1}{N}$. Another information we will use is that the tension of the fundamental string remains of order 1 as $N$ is increased. In fact its tension determines the mass of the
meson and, for large $N$, these masses remains finite $^{42,43}$.

Now we explore the possibility that the $\mathbb{Z}_N$ strings of QCD becomes wall vortices in the large $N$ limit with $\frac{n}{N}$ kept fixed. A sort of magnetic dual of what happened in the $\mathcal{N} = 1^*$ theory, but with a Coulomb energy density different from zero. We write the tension as function of the radius of the vortex considering only the dependence on $n$ and $N$:

$$T(R) \sim \frac{n^2}{NR^2} + N^\alpha \Lambda^4 R^2. \tag{4.2}$$

Note that the energy of the electric flux term is $\frac{n^2}{NR^2}$ instead of the $\frac{n^2N}{R^2}$ obtained for the solitonic magnetic vortex in (3.11). The reason is this: normalizing the gauge kinetic term as we have done, $\frac{N}{g^2} FF$, the magnetic monopoles have charge of order 1 while the electric particles have charge of order $\frac{g^2}{N}$ and this brings the $N$ factor in the denominator of the electric field energy density. The Coulomb energy density is instead $N^\alpha \Lambda^4 R^2$ where $\Lambda^4$ is just for dimensional reasons and $\alpha$ is a parameter that we are going to fix.

Minimizing (4.2) with respect to $R$ we obtain

$$T_V \sim \frac{n}{\sqrt{N^{1-\alpha}}} \Lambda^2, \quad R_V \sim \sqrt{\frac{n^2}{N^{1+\alpha}}} \frac{1}{\Lambda}. \tag{4.3}$$

We could have a spectrum like (4.1), only if the $\alpha$ parameter in (4.2) would be equal to 1. This in fact is the only way to match (4.3) with (4.1) since $T_V(1)$ is of order 1. In this way the tension is of order $nN^0$ and so, keeping fixed $n$ and sending $N$ to infinity, we obtain a finite limit.

Now there are two left points to explain. Let us rewrite the energy density, the tension and the radius now that we have fixed $\alpha = 1$:

$$\epsilon_0 \sim N^4 \Lambda, \quad T_V \sim n \Lambda^2, \quad R_V \sim \sqrt{\frac{n}{N}} \frac{1}{\Lambda}. \tag{4.4}$$

First we have to explain how is possible to have a Coulomb energy density that goes like $N \Lambda^4$. Second we have to explain how the wall vortex approximation can work, that is we have to find a domain wall between the confining vacuum and the Coulomb vacuum with a thickness that goes like $\Delta W \sim \frac{1}{N}$. Both of these points can be explained by an effective Lagrangian that scales with an overall factor of $N^2$

$$\mathcal{L}_{eff} = N^2 F[B, \partial B, \ldots], \tag{4.5}$$

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and a distance between the confining vacuum and the Coulomb phase that is $\delta B \sim \frac{1}{N}$.

Essentially is the same mechanism that worked in Section 3.4

4.1 Lattice data and saturation

Now we confront our theory with the lattice data. First of all we make a brief discussion of the saturation limit. This is the best way to confront the experimental results for different gauge groups. Given any ratio of string tensions $R(n, N) = \frac{T_V(n)}{T_V(1)}$, we rewrite it as a function of the ratio $x = \frac{n}{N}$, and then rescale with a $\frac{1}{N}$ factor. For example the Sine formula and the Casimir formula become respectively:

\[ \frac{1}{N} \sin \left( \frac{\pi n}{N} \right) \rightarrow \frac{1}{\pi} \sin (\pi x) + O \left( \frac{1}{N^2} \right) \] (4.6)

\[ \frac{1}{N} \frac{n(N-n)}{N-1} \rightarrow x(1-x) + O \left( \frac{1}{N} \right) \] . (4.7)

Note that in this way we can plot $\frac{1}{N}R(n, N)$ with respect to $x$ in the same graph for all the values of $N$. The saturation limit is when the $\frac{1}{N}$ corrections can be neglected. This also the limit in which the various data becomes dense and describe a continuous curve. The Sine formula saturates to $\frac{1}{\pi} \sin (\pi x)$ while the Casimir formula saturate to $x(1-x)$. An important thing to note is that both the Sine and the Casimir formula saturate from above, that is the $o(\frac{1}{N})$ corrections are positive. Our formula is instead

\[ \min (x, 1-x) + O \left( \frac{1}{N} \right) \]. (4.8)

Since the deviation from an exact wall vortex is given by the wall thickness of order $\frac{1}{N}$, is natural to have $\frac{1}{N}$ correction. It’s also natural to expect a negative $\frac{1}{N}$ correction and so a saturation from below.

Finally we confront with the lattice experiments that are plotted Figure 5. Up to now the largest $N$ for which computations have been done is $N = 8$. In principle these data could be consistent with the formula $\min (x, 1-x)$. To explain the observed deviation we should have $\frac{1}{N}$ corrections with coefficients of order 1. When the saturation will be reached it will be easy to prove or disprove our formula.

\[ \text{It’s also the same mechanism that works for the pure } \mathcal{N} = 1 \text{ theory for domain walls between two adjacent vacua [38, 39, 40]. The effective Lagrangian can be written as } \mathcal{L}_{\text{eff}} = N^2 \left[ \frac{\delta B}{\Delta} + \Delta \right]. \text{ If } \delta B \sim \frac{1}{N}, \text{ minimizing with respect to } \Delta \text{ we obtain } \Delta_W \sim \frac{1}{N} \text{ and } T_W \sim N. \]
Figure 5: In the three graphs are reported the lattice data taken from [26]. They refer respectively to $SU(4)$, $SU(6)$ and $SU(8)$. The lines plotted are respectively the Casimir formula (green line), the Sine formula (red line) and the $\min(x, 1-x)$ formula (blue line).
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