Exterior Differential Systems, Prolongations and the Integrability of Two Nonlinear Partial Differential Equations

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Abstract

A generalized KdV equation is formulated as an exterior differential system, which is used to determine the prolongation structure of the equation. The prolongation structure is obtained for several cases of the variable powers, and nontrivial algebras are determined. The analysis is extended to a differential system which gives the Camassa-Holm equation as a particular case. The subject of conservation laws is briefly discussed for each of the equations. A Bäcklund transformation is determined using one of the prolongations.

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1 Introduction.

Once nonlinear terms are included in linear dispersive equations, solitary waves can result which can be stable enough to persist indefinitely. It is well known that many important nonlinear evolution equations which have numerous applications in mathematical physics appear as sufficient conditions for the integrability of systems of linear partial differential equations of first order, and such systems are referred to as integrable [1-2]. This is not just an oddity, since algebraic structures such as those which appear in AKNS systems can arise very naturally from nonlinear evolution equations. This is very well exemplified by applying the so-called prolongation technique. Wahlquist and Estabrook [3-5] first constructed prolongations, or $sl(2)$ systems, both for the KdV and nonlinear Schrödinger equations, and Shadwick the former [6]. This procedure produces a nonclosed Lie algebra of vector fields which are defined on fibres above the base manifold that supports the exterior differential forms defining the nonlinear evolution equation. It has been shown that a simple linear prolongation of the KdV, sine-Gordon and nonlinear Schrödinger equation can be provided by $sl(2, \mathbb{C})$ [7]. What is more, it has been shown that the vanishing of the curvature form of a particular Cartan-Ehresmann connection is the necessary and sufficient condition for the existence of the prolongation.

These prolongations have a very useful application since Bäcklund transformations can be calculated based on them as well [8]. A Bäcklund transformation has important practical consequences, since such transformations can be used to calculate solutions to an associated equation, usually referred to as the potential equation, based on solutions of the initial equation. Sometimes these transformations can be used to obtain new solutions to the same initial equation, in which case they are referred to as auto-Bäcklund transformations.

Recently an exterior differential system which defines a generalized KdV equation on the transverse manifold was obtained [9]. A particular case of this equation has appeared in [10] recently. The symmetries of this equation were determined and some solutions were found as well [11]. This permitted the determination of a certain form of integrability. Also, a particular type of prolongation over a fibre bundle was found corresponding to this differential system, as well as a
specific form for a Bäcklund transformation with its associated potential equation. Here, the same differential system is studied, but a fully general calculation of the prolongation over the same bundle is carried out in detail for this generalized KdV equation. This allows the prolongation structure for any case of the given parameters in the equation. For completeness, the general theory for obtaining such prolongations based on the given exterior system of differential forms that defines the equation upon sectioning to a transversal integral manifold will be outlined first. Transversal integral manifolds give solutions of the equation. Finally this work is extended to a study of a differential system of one-forms which define an equation that includes the Camassa-Holm equation and Degasperis-Procesi equations as specific cases [12-14]. The Camassa-Holm equation has been of interest because it has been shown to have peaked soliton solutions. The Camassa-Holm equation has a lot in common with the KdV equation, but there are significant differences as well. The KdV equation is globally well-posed when considered on a suitable Sobolev space, while Camassa-Holm is in general not. The first derivative of a solution of the latter can become infinite in finite time. The associated prolongation equations are developed and found to be much more restrictive than the previous case. However, it is shown that at least one solution to the prolongation system can be found. Finally, for each system a brief discussion concerning how conservation laws arise and can be expressed in this context will be discussed based on the defining exterior differential system.

2 Introduction to Cartan Prolongations.

Cartan prolongations will be found for the equations mentioned in the Introduction, and to begin, a general outline of this subject is given. Consider the space $M = \mathbb{R}^n$ with the coordinates $(x, t, u, p, q, \cdots)$ and let there be given on $M$ a closed exterior differential system

$$\alpha_1 = 0, \quad \cdots, \quad \alpha_l = 0.$$  \hspace{1cm} (2.1)

Let $I$ be the ideal generated by the system (2.1), and so $I = \{ \chi = \sum_{i=1}^l \sigma_i \wedge \alpha_i : \sigma_i \in \Lambda_p(M), p = 0, 1, 2, \cdots \}$ with $\Lambda_p(M)$ the set of $p$-forms on $M$. Since (2.1) is closed, we have $dI \subset I$ and (2.1) is integrable.
The system (2.1) will be chosen such that solutions \( u = u(x,t) \) of an evolution equation
\[
 u_t = F(x,t,u,u_x,u_{xx},\cdots)
\]
correspond with two-dimensional transversal integral manifolds of (2.1). These integral manifolds can be written as sections \( S \) in \( M \) with \( S \) given by
\[
 (x,t) \rightarrow (x,t,u(x,t),p(x,t),q(x,t),\cdots),
\]
and due to transversality, \( dx\wedge dt|_S = \pi^*(dx\wedge dt) \neq 0 \) such that \( \pi : M \to \mathbb{R}^2 \) and \( \pi^* : \Lambda(\mathbb{R}^2) \to \Lambda(S) \).

Now introduce the fibre bundle \( (\tilde{M}, \tilde{p}, M) \) over \( M \), with \( M \subset \tilde{M} \) and \( \tilde{p} \) the projection of \( \tilde{M} \) onto \( M \), \( \tilde{p}(\tilde{M}) = M \). Points in \( \tilde{M} \) are written \( \tilde{m} \), those in \( M \) written \( m \) and \( \tilde{p}(\tilde{m}) = m \). The tangent and cotangent spaces of \( \tilde{M} \) and \( M \) are denoted by \( T(\tilde{M}) \) and \( T^*(M) \).

For reference, a Cartan-Ehresmann connection in the fibre bundle \( (\tilde{M}, \tilde{p}, M) \) is a system of 1-forms \( \tilde{\omega}_i, i = 1, 2, \cdots, k \) in \( T^*(\tilde{M}) \) with the property that the mapping \( \tilde{p}_* \) from the vector space \( H_{\tilde{m}} = \{ \tilde{X} \in T_{\tilde{m}}| \tilde{\omega}_i(\tilde{X}) = 0, i = 1, 2, \cdots, k \} \) onto the tangent space \( T_m \) is a bijection for all \( \tilde{m} \in \tilde{M} \).

At this point, consider the exterior differential system in \( \tilde{M} \)
\[
 \tilde{\alpha}_i = \tilde{p}^*\alpha_i = 0, \quad i = 1, \cdots, l, \quad \tilde{\omega}_i = 0, \quad i = 1, \cdots, k, \tag{2.2}
\]
with \( \{\tilde{\omega}_i\} \) a Cartan-Ehresmann connection in \( (\tilde{M}, \tilde{p}, M) \). The system (2.2) is called a Cartan prolongation if (2.2) is closed and whenever \( S \) is a transversal solution of (2.1), then there should also exist a transversal solution \( \tilde{S} \) of (2.2) with \( \tilde{p}(\tilde{S}) = S \). It follows from (2.2) closed that this prolongation condition may be written as
\[
 d\tilde{\omega}_i = \sum_{j=1}^{k} \tilde{\beta}^j_i \wedge \tilde{\omega}_j, \quad mod \quad \tilde{p}^*(I), \tag{2.3}
\]
where \( I \) is the ideal defined in (2.1).

For the considerations here, the fibre bundle will be the trivial fibre bundle given by \( \tilde{M} = M \times \mathbb{R}^k \) with \( y = (y_1, \cdots, y_k) \in \mathbb{R}^k \) and the connection used will have the form
\[
 \tilde{\omega}_i = dy_i - \eta_i, \quad \eta_i = A_i dx + B_i dt. \tag{2.4}
\]
In (2.4), \( A_i \) and \( B_i \) are defined as \( C^\infty \)-functions on \( \tilde{M} \), \( i = 1, \cdots, k \). The prolongation condition (2.3) applied to (2.4) is then
\[
 -d\eta_i = \tilde{\beta}^j_i \wedge (dy_j \wedge \eta_j), \quad mod \quad \tilde{p}^*(I), \quad i = 1, \cdots, k, \tag{2.5}
\]
with \( \eta_i \) given in (2.4) and \( A_i, B_i \) depend on \( x, t, u, p, q, \cdots, y_1, \cdots, y_k \). Comparing both sides of (2.5), \( \tilde{\beta}_i^j \) cannot contain differentials of the form \( dy_s - \eta_s \) for \( s \neq j \) and so

\[
\tilde{\beta}_i^j = a_i^j \, dx + \beta_i^j \, dt + c_i^j \, du + d_i^j \, dp + \cdots \quad \text{mod} \quad \gamma_i^j(dy_j - \eta_j).
\]

Comparing forms on both sides of (2.5) yields the results

\[
a_i^j = \frac{\partial A_i}{\partial y_j}, \quad b_i^j = \frac{\partial B_i}{\partial y_j},
\]

with \( c_i^j = d_i^j = \cdots = 0 \) because \( du \wedge dy_j, dp \wedge dy_j, \cdots \) do not occur on the left of (2.5). The prolongation condition reduces to

\[
-d\eta_i = \frac{\partial \eta_i}{\partial y_j} \wedge (dy_j - \eta_j), \quad \text{mod} \quad \tilde{p}^*(I).
\]  

Introducing the vertical valued one-form \( \eta = \eta_i \frac{\partial}{\partial y_i} \) as well as the definitions

\[
d\eta = (d_M \eta_i) \frac{\partial}{\partial y_i}, \quad [\eta, \omega] = (\eta_j \wedge \frac{\partial \omega_i}{\partial y_j} + \omega_j \wedge \frac{\partial \eta_i}{\partial y_j}) \frac{\partial}{\partial y_i},
\]

the prolongation condition reduces to the compact form,

\[
d\eta + \frac{1}{2}[\eta, \eta] = 0, \quad \text{mod} \quad \tilde{p}^*(I).
\]  

The form on the left of (2.8) is called the curvature form of the Cartan-Ehresmann connection \((dy_i - \eta_i)_{i=1}^k\). Thus, a sufficient condition for the existence of a Cartan prolongation of the set of exterior differential forms (2.1) is the vanishing of the curvature form of the Cartan-Ehresmann connection \((dy_i - \eta_i)_{i=1}^k\).

3 Cartan Prolongation of a Generalized KdV Equation.

3.1 Differential System and Associated Partial Differential Equation.

Let us introduce the exterior differential system defined over a base manifold \( M = \mathbb{R}^5 \) which supports the differential forms. Consider the system of two forms given by

\[
\alpha_1 = nu^{n-1} \, du \wedge dt - p \, dx \wedge dt = 0,
\]
\[ \alpha_2 = dp \wedge dt - q dx \wedge dt = 0, \]  
\[ \alpha_3 = du \wedge dt - dq \wedge dt - \gamma pu^s dx \wedge dt = 0, \]  
where \( \gamma \) is a nonzero, real constant. The exterior derivatives of the \( \alpha_j \) are given by

\[ d\alpha_1 = -dp \wedge dx \wedge dt = dx \wedge \alpha_2, \]
\[ d\alpha_2 = -dq \wedge dx \wedge dt = -dx \wedge \alpha_3, \]
\[ d\alpha_3 = -\gamma spu^{s-1} du \wedge dx \wedge dt - \gamma u^s dp \wedge dx \wedge dt = dx \wedge (\gamma \frac{s}{n} pu^{s-n} \alpha_1 + \gamma pu^s \alpha_2). \]

Therefore, the ideal \( I = \{ \omega | \omega = \sum_{i=1}^{3} \sigma_i \wedge \alpha_i : \sigma_i \in \Lambda(M) \} \) is closed, \( dI \subset I \) and the system \{\( \alpha_i \)\} given by (3.1) is integrable. On the transversal integral manifold, it follows that differential system (3.1) can be sectioned to give,

\[ 0 = \alpha_1|_S = S^* \alpha_1 = ((u^n)_x - p) dx \wedge dt, \]
\[ 0 = \alpha_2|_S = S^* \alpha_2 = (p_x - q) dx \wedge dt, \]
\[ 0 = \alpha_3|_S = S^* \alpha_3 = (u_t dt \wedge dx - q_x dx \wedge dt - \gamma pu^s dx \wedge dt). \]

The transversal integral manifolds correspond to the equations

\[ p = (u^n)_x, \quad q = p_x = (u^n)_{xx}, \quad u_t + q_x + \gamma pu^s = 0. \]  

(3.4)

Suppose that \( n + s \neq 0 \), then upon substituting \( p \) and \( q \) from the first two equations in (3.4) into the third, it can be seen that \( u \) must satisfy the following generalized KdV equation

\[ u_t + (u^n)_{xxx} + \gamma \frac{n}{n+s} (u^{n+s})_x = 0. \]  

(3.5)

A more compact form is obtained if we set \( m = n + s \) and define a new constant \( \beta = n\gamma/(n + s) \) so that (3.5) takes the form

\[ u_t + (u^n)_{xxx} + \beta (u^m)_x = 0. \]  

(3.6)

This is the partial differential equation defined by differential system (3.1) which was studied in [11].
3.2 Prolongations.

Based on the forms in system (3.1), the prolongation method outlined in Section 2 can be carried out, and the resulting system of equations can be solved quite generally. A very general prolongation corresponding to (3.6) can be calculated in terms of an algebra of vector fields which are defined on fibres above the base manifold that supports the forms (3.1). To do this, introduce the pseudopotentials and the Cartan-Ehresmann connection on the trivial fibre bundle \( \tilde{M} = M \times \mathbb{R}^k = \mathbb{R}^5 \times \mathbb{R}^k \) with coordinates \( y = (y_1, \ldots, y_k) \) on \( \mathbb{R}^k \). The connection forms are taken to be

\[
\tilde{\omega}_i = dy_i - \eta_i,
\eta_i = A_i dx + B_i dt,
A_i = A_i(x, t, u, p, q, y),
B_i = B_i(x, t, u, p, q, y).
\]

Substituting \( \eta = \eta_i \partial/\partial y_i \), \( A = A_i \partial/\partial y_i \), \( B = B_i \partial/\partial y_i \), the prolongation condition for the vectors \( A \) and \( B \) is

\[
(d_M A_i \wedge dx) \frac{\partial}{\partial y_i} + (d_M B_i \wedge dt) \frac{\partial}{\partial y_i} + \frac{1}{2} \{ A_j \frac{\partial B_i}{\partial y_j} dx \wedge dt + B_j \frac{\partial A_i}{\partial y_j} dt \wedge dx \} \frac{\partial}{\partial y_i} + \frac{1}{2} \{ B_j \frac{\partial A_i}{\partial y_j} dt \wedge dx + A_j \frac{\partial B_i}{\partial y_j} dx \wedge dt \} \frac{\partial}{\partial y_i} = dA \wedge dx + dB \wedge dt + [A, B] dx \wedge dt = 0, \quad \text{mod} \quad \tilde{p}^*(I),
\]

where \([A, B]\) denotes the ordinary Lie bracket of the vector fields \( A_i \partial/\partial y_i \) and \( B_i \partial/\partial y_i \) defined along fibres of the bundle. Using (3.1), it is found that the prolongation condition takes the form

\[
\frac{\partial A}{\partial t} dt \wedge dx + \frac{\partial A}{\partial u} du \wedge dx + \frac{\partial A}{\partial p} dp \wedge dx + \frac{\partial A}{\partial q} dq \wedge dx
\]

\[
+ \frac{\partial B}{\partial x} dx \wedge dt + \frac{\partial B}{\partial u} du \wedge dt + \frac{\partial B}{\partial p} dp \wedge dt + \frac{\partial B}{\partial q} dq \wedge dt + [A, B] dx \wedge dt
\]

\[= \lambda_1(nu^{n-1} du \wedge dt - pdx \wedge dt) + \lambda_2(dp \wedge dt - q dx \wedge dt) + \lambda_3(du \wedge dx - dq \wedge dt - \gamma pu^s dx \wedge dt).\]

Comparison of both sides of this equation yields the following set of conditions,

\[
A_u = \lambda_3, \quad A_p = 0, \quad A_q = 0,
B_u = n \lambda_1 u^{n-1}, \quad B_p = \lambda_2, \quad B_q = -\lambda_3,
\]

\[
-A_t + B_x + [A, B] = -p \lambda_1 - q \lambda_2 - \gamma pu^s \lambda_3.
\]
Subscripts indicate partial differentiation with respect to the variable indicated. Translations in $x$ and $t$ constitute symmetries of equation (3.6) \[11\], and so a simplifying assumption would be to suppose that $A$ and $B$ are independent of $x$ and $t$. Then it must be that $A$ and $B$ are also invariant under translations in these variables. The prolongation equations to be solved from (3.8) reduce to the following

\[ A_p = 0, \quad A_q = 0, \quad A_u = -B_q, \quad \frac{1}{n}u^{1-n}pB_u + qB_p - \gamma pu^sB_q = -[A, B]. \tag{3.9} \]

**Theorem 3.1** System (3.9) can be reduced to a single expression which specifies the algebra of brackets of a set of basis vector fields $X_i$. The structure of these algebras is dependent on the relative values of $m$ and $n$.

**Proof:** The first three differential equations in (3.9) imply the following results

\[ A = A(u, y), \quad B = B(u, p, q, y), \quad B = -q A_u(u, y) + \hat{B}(u, p, y). \tag{3.10} \]

Substituting $B$ from (3.10) into (3.9) and collecting terms in $q$ gives

\[ q(-\frac{1}{n}u^{-n+1}pA_{uu} + \hat{B}_p - [A, A_u]) + \frac{1}{n}pu^{-n+1}\hat{B}_u + \gamma pu^sA_u + [A, \hat{B}] = 0. \tag{3.11} \]

Since $A$ and $\hat{B}$ do not depend on $q$, it follows from (3.11) that

\[ \hat{B}_p = \frac{1}{n}u^{-n+1}pA_{uu} + [A, A_u]. \]

As $A$ does not depend on $p$, this can be integrated to give $\hat{B}$,

\[ \hat{B}(u, p, y) = \frac{1}{2n}u^{-n+1}p^2A_{uu} + [A, A_u]p + B''(u, y). \tag{3.12} \]

Substituting (3.12) into (3.11) as well as $\hat{B}_u$, there results

\[ \frac{1}{2n}u^{-2n+1}(-(n-1)A_{uu} + uA_{uuu})p^3 + u^{-n+1}[A, A_{uu}]p^2 + u^{-n+1}B''p + n\gamma pu^sA_u + \]

\[ + n[A, \frac{1}{2n}u^{-n+1}A_{uuu}p^2 + [A, A_u]p + B''] = 0. \tag{3.13} \]

Since $A$ and $B''$ do not depend on $p$, the coefficient of $p^3$ must vanish giving the equation

\[ uA_{uuu} - (n-1)A_{uu} = 0. \]
This can be solved for $A$ to give

$$A(u, y) = X_1(y) + X_2(y)u + X_3(y)u^{n+1}, \quad (3.14)$$

where the $X_i(y)$ are vertical vector fields. Consequently, (3.13) simplifies to

$$u^{-n+1}[A, A_{uu}] + \frac{1}{2}[A, A_{uu}]p^2 + (n\gamma u^s A_u + u^{-n+1}B''_u + n[A, A, A_u])p + n[A, B''] = 0. \quad (3.15)$$

The coefficient of $p^2$ implies that $[A, A_{uu}] = 0$, which using (3.14) immediately establishes two basic commutators of the vector fields $X_1, X_2,$ and $X_3$,

$$[X_1, X_3] = 0, \quad [X_2, X_3] = 0. \quad (3.16)$$

The coefficient of $p$ implies the condition,

$$n\gamma u^s A_u + u^{-n+1}B''_u + n[A, A, A_u] = 0.$$

Solving for $B''_u$ and putting $s = m - n$,

$$B''_u = n\gamma u^{m-1}A_u - nu^{n-1}[A, A, A_u].$$

Substituting $A$ from (3.14) and its derivative $A_u = X_2 + (n + 1)u^n X_3$ into $B''_u$ from above, we have

$$B''_u = n\gamma u^{m-1}X_2 + n(n + 1)\gamma u^{n+m-1}X_3 - nu^{n-1}[X_1 + uX_2 + u^{n+1}X_3, [X_1, X_2]]. \quad (3.17)$$

Suppose at this point that $X_1$ and $X_2$ do not commute with each other, then a new vector field can be defined as

$$X_7 = [X_1, X_2]. \quad (3.18)$$

Setting $X = X_3, Y = X_1$ and $Z = X_2$ in the Jacobi identity $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ gives

$$[X_3, [X_1, X_2]] + [X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] = 0. \quad (3.19)$$

However, using (3.16), the last two terms in (3.19) are zero, hence (3.19) implies that

$$[X_3, X_7] = 0. \quad (3.20)$$
Consequently, $B''_u$ reduces to the form

$$B''_u = n\gamma u^{m-1}X_2 + n(n + 1)\gamma u^{n+m-1}X_3 - nu^{n-1}X_5 - nu^nX_6,$$  \hspace{1cm} (3.21)

Two new commutators have been introduced to write (3.21) defined as

$$[X_1, X_7] = X_5, \quad [X_2, X_7] = X_6.$$  \hspace{1cm} (3.22)

Using (3.22) in the Jacobi identity, the following brackets result

$$[X_2, X_5] = [X_1, X_6], \quad [X_3, X_5] = 0.$$  \hspace{1cm} (3.23)

Finally, integrating $B''_u$ with respect to $u$ yields an expression for $B''$,

$$B'' = \frac{n}{m}\gamma u^mX_2 + \frac{n(n + 1)}{n + m}\gamma u^{m+n}X_3 - u^nX_5 - \frac{n}{n + 1}u^{n+1}X_6 + X_4.$$  \hspace{1cm} (3.24)

Only one term in (3.15) remains to be satisfied, namely $[A, B''] = 0$. Thus substituting $A$ and $B''$ into this bracket and using linearity to expand out, we have

$$[X_1 + uX_2 + u^{n+1}X_3, \frac{n}{m}\gamma u^mX_2 + \frac{n(n + 1)}{m + n}\gamma u^{m+n}X_3 - u^nX_5 - \frac{n}{n + 1}u^{n+1}X_6 + X_4]$$

$$= \frac{n}{m}\gamma u^m[X_1, X_2] - u^n[X_1, X_5] - \frac{n}{n + 1}u^{n+1}[X_2, X_5] + [X_1, X_4] - u^{n+1}[X_2, X_5]$$

$$- \frac{n}{n + 1}u^{n+2}[X_2, X_6] + u[X_2, X_4] - \frac{n}{n + 1}u^{2n+2}[X_3, X_6] + u^{n+1}[X_3, X_4].$$

Therefore, the vector fields must be interrelated in such a way that the following holds among the coefficients of each power of $u$,

$$[X_1, X_4] + u[X_2, X_4] + \frac{n}{m}\gamma u^m[X_1, X_2] - u^n[X_1, X_5] + u^{n+1}(-\frac{2n + 1}{n + 1}[X_2, X_5] + [X_3, X_4])$$

$$- \frac{n}{n + 1}u^{n+2}[X_2, X_6] - \frac{n}{n + 1}u^{2n+2}[X_3, X_6] = 0.$$  \hspace{1cm} (3.25)

This completes the proof.

**Theorem 3.2** There exist nontrivial algebras for the $X_i$ specified by (3.16), (3.18), (3.20), (3.23) and the coefficients of powers of $u$ in (3.25), which depend on the relative values of $m$ and $n.$
**Proof:** It is required to equate the independent powers of $u$ equal to zero. This has to be done on a case by case basis by putting individual restrictions on $m$ and $n$, and not all cases are given.

(i) Suppose none of the powers of $u$ in (3.25) are equal, hence $n \neq m \neq 1, 0$. Equating each power of $u$ to zero gives the following algebra

$$X_7 = [X_1, X_2] = 0, \quad [X_1, X_5] = 0, \quad [X_2, X_6] = 0, \quad \frac{2n+1}{n+1}[X_2, X_5] = [X_3, X_4],$$

$$[X_3, X_6] = 0, \quad [X_2, X_4] = 0, \quad [X_1, X_4] = 0.$$  

At this point, $X_1$ and $X_2$ have be required to commute, since $X_7 = 0$ must hold. However, from (3.22), it follows that $X_5 = X_6 = 0$. Moreover, $[X_1, X_3] = 0$ implies that $X_1$ and $X_3$ differ by a constant, hence $X_2$ and $X_3$ also differ by a constant. Finally, $[X_1, X_4] = 0$ implies that $X_1$ and $X_4$ differ by a constant. Therefore, we can put

$$X_1 = \kappa X, \quad X_2 = \sigma X, \quad X_3 = X, \quad X_4 = \alpha X. \quad (3.26)$$

Substituting these results into $A$ and $B$, they take the form

$$A = (\kappa + \sigma u + u^{n+1})X,$$

$$B = -(\sigma + (n+1)u^n)qX + \frac{1}{2}(n+1)p^2X + \frac{n}{m}\gamma u^mX + \frac{n(n+1)}{m+n}\gamma u^{m+n}X + \alpha X. \quad (3.27)$$

(ii) Suppose $n = 1$ and $m \neq 1, 2, 3, 4$. Then the same algebra as (3.26) results and $A$ and $B$ are given by (3.27) with $n$ set equal to one.

(iii) Suppose now that $n = m \neq 0, 1$, then prolongation equation (3.25) reduces to

$$[X_1, X_4] + u[X_2, X_4] + u^n(\gamma X_7 - [X_1, X_5]) + u^{n+1}(-[X_2, X_5] - \frac{n}{n+1}[X_1, X_6]$$

$$+[X_3, X_4]) - \frac{n}{n+1}u^{n+2}[X_2, X_6] - \frac{n}{n+1}u^{2n+2}[X_3, X_6] = 0.$$  

This equation is satisfied provided that the following brackets hold,

$$[X_3, X_6] = 0, \quad [X_2, X_6] = 0, \quad \frac{2n+1}{n+1}[X_2, X_5] = [X_3, X_4],$$

$$\gamma X_7 = [X_1, X_5], \quad [X_2, X_4] = 0, \quad [X_1, X_4] = 0, \quad (3.28)$$
in addition to the brackets given in (3.20), (3.22), and (3.23). This algebra has a simpler three-element realization which satisfies all the commutation relations provided that

\[ X_3 = 0, \quad X_4 = 0, \quad X_5 = \gamma X_2, \quad X_6 = X_2. \] (3.29)

The nonzero commutation relations are given by

\[
[X_1, X_2] = X_7, \quad [X_2, X_7] = X_2, \quad [X_1, X_7] = -\gamma X_2.
\] (3.30)

The algebra closes and a finite three-element algebra results.

(iv) Suppose that \( m = n + 1 \neq 0, 1 \), then prolongation equation (3.25) implies the algebra

\[
[X_1, X_4] = 0, \quad [X_2, X_4] = 0, \quad \gamma \frac{n}{n+1} [X_1, X_2] = \frac{2n+1}{n+1} [X_2, X_5] + [X_3, X_4] = 0,
\]

\[
[X_1, X_5] = 0, \quad [X_2, X_6] = 0, \quad [X_3, X_6] = 0.
\]

Recalling that (3.23) must be satisfied, a three element algebra results if we take

\[ X_2 = X_3, \quad X_4 = 0, \quad X_5 = -\frac{n\gamma}{2n+1} X_1, \quad X_6 = \frac{n\gamma}{2n+1} X_2. \] (3.31)

There is a closed algebra in this case with three nontrivial brackets,

\[
[X_1, X_2] = X_7, \quad [X_1, X_7] = -\frac{n\gamma}{2n+1} X_1, \quad [X_2, X_7] = \frac{n\gamma}{2n+1} X_2.
\] (3.32)

(v) The linear case \( m = n = 1 \) generates the following bracket relations

\[
[X_1, X_4] = [X_2, X_6] = [X_3, X_6] = 0, \quad \gamma X_7 + [X_2, X_4] - [X_1, X_5] = 0,
\]

\[
[X_3, X_4] - [X_2, X_5] - \frac{1}{2} [X_1, X_6] = 0.
\] (3.33)

(vi) The case \( m = 2, n = 1 \) corresponds to the classical KdV equation and the brackets must satisfy

\[
[X_3, X_6] = 0, \quad [X_2, X_6] = 0, \quad \frac{1}{2} \gamma X_7 = \frac{3}{2} [X_2, X_5] - [X_3, X_4],
\]

\[
[X_2, X_4] - [X_1, X_5] = 0, \quad [X_1, X_4] = 0.
\] (3.34)

Since (3.23) must be satisfied, this system is satisfied if we put

\[ X_3 = X_4 = 0, \quad X_5 = -\frac{\gamma}{3} X_1, \quad X_6 = \frac{\gamma}{3} X_2. \] (3.35)
There are three nontrivial commutators which take the form
\[ [X_1, X_2] = X_7, \quad [X_1, X_7] = -\frac{\gamma}{3} X_1, \quad [X_2, X_7] = \frac{\gamma}{3} X_2. \] (3.36)

This completes the proof.

### 3.3 Conservation Laws.

One way in which conservation laws can be associated with this equation is that they correspond to the existence of exact two-forms contained in the ring of the forms \( \alpha_i \) given by (3.1). Let us suppose we can find a set of functions \( g_i(x, t, u, p, q) \) such that the two-form
\[
\vartheta = g_1 \alpha_1 + g_2 \alpha_2 + g_3 \alpha_3
\] (3.37)
satisfies the condition for exactness, \( d\vartheta = 0 \). This is the integrability condition for the existence of a one-form, \( \omega \), such that
\[
\vartheta = d\omega.
\] (3.38)

Conversely, (3.38) implies that \( d\vartheta = 0 \) by the usual identity for double exterior derivatives. Differentiation of (3.37) and substituting (3.2) yields
\[
d\vartheta = (dg_1 + g_3 \gamma \frac{s}{n} pu^{s-n} dx) \wedge \alpha_1 + (dg_2 + (g_1 + g_3 \gamma pu^s) dx) \wedge \alpha_2 + (dg_3 - g_2 dx) \wedge \alpha_3.
\]

Therefore \( d\vartheta \in I \), and this clearly vanishes mod \( \tilde{\mathfrak{p}}^*(I) \).

As an example of a form \( \vartheta \) with the structure (3.37) corresponding to equation (3.6), consider the one-form \( \vartheta \) which is given in terms of the \( \alpha_i \) in (3.1) with \( g_1 = -\gamma u^s \), \( g_2 = 0 \) and \( g_3 = 1 \) as
\[
\vartheta = -\gamma u^s \alpha_1 + \alpha_3,
\] (3.39)
where \( s = m - n \). Exterior differentiation \( d\vartheta \) gives
\[
d\vartheta = \gamma su^{s-1}p du \wedge dx \wedge dt + \gamma u^s dp \wedge dx \wedge dt - \gamma spu^{s-1} du \wedge dx \wedge dt - \gamma u^s dp \wedge dx \wedge dt = 0.
\]

Thus the exterior derivative of (3.39) does vanish. Substituting \( \alpha_1 \) and \( \alpha_2 \) into (3.39), an explicit form for \( \vartheta \) is obtained
\[
\vartheta = -\gamma \frac{n}{m} d(u^m) \wedge dt + du \wedge dx - dq \wedge dt.
\]
If the one-form $\omega$ is defined to be

$$\omega = -\left(\gamma \frac{n}{m}u^m + q\right) dt + u \, dx,$$

(3.40)

Then it is easy to verify by differentiation that $\vartheta = d\omega$. The associated conservation law results from an application of Stokes theorem, which is written as

$$\oint_{M_1} \omega = \int_{M_2} d\omega.$$  

(3.41)

This has been written for any simply-connected, two-dimensional manifold $M_2$ with closed one-dimensional boundary $M_1$. The equation implies that $\omega$ and $d\omega$ are to be evaluated on their respective manifolds.

Returning to $\omega$ once more, we can of course add to $\omega$ any exact one-form $dv$, where $v$ is an arbitrary scalar function. Thus, $\omega$ can also be taken to be

$$\omega = dv - \left(\gamma \frac{n}{m}u^m + q\right) dt + u \, dx,$$

(3.42)

such that $\vartheta = d\omega$. Now $v$ may be regarded simply as a coordinate in an extended six-dimensional space of variables $\{x, t, u, p, q\}$, and the one-form $\omega$ may be included with the original set of forms. Since $d\omega$ is known to be in the ring of the original set, the new set of forms remains a closed ideal.

4 Prolongation of a Differential System Related to the Camassa-Holm Equation.

4.1 Exterior System and Associated Partial Differential Equation.

A differential system will be introduced which is related to several equations which are of interest in mathematical physics at the moment. In particular, the Camassa-Holm and Degasperis-Procesi equations are to be included in this group. Define the following system of two forms

$$\alpha_1 = du \wedge dt - p \, dx \wedge dt,$$

$$\alpha_2 = dp \wedge dt - q \, dx \wedge dt,$$

(4.1)

$$\alpha_3 = -du \wedge dx + dq \wedge dx + u \, du \wedge dt - u \, dq \wedge dt + \beta(u - q) \, du \wedge dt,$$
where $\beta$ in (4.1) is a real, non-zero constant. Exterior differentiation of the system of $\alpha_j$ in (4.1) yields
\[
\begin{align*}
d\alpha_1 &= -dp \wedge dx \wedge dt = dx \wedge \alpha_2, \\
d\alpha_2 &= \frac{1}{u} \ dx \wedge \left(-\alpha_3 + u((1+\beta)u - q)\alpha_1\right), \\
d\alpha_3 &= (1-\beta) \ dq \wedge du \wedge dt = (1-\beta)[dq \wedge \alpha_1 + pdt \wedge \alpha_3 - p\ dx \wedge \alpha_1].
\end{align*}
\]

Clearly all of the $d\alpha_i$ vanish modulo the set of $\alpha_j$ in (4.1). Therefore, the ideal $I = \{\omega | \omega = \sum_{i=1}^3 \sigma_i \wedge \alpha_i : \sigma_i \in \Lambda_p(M), p = 0,1,2,\ldots\}$ is closed, $dI \subset I$, and the system is integrable.

On the transversal manifold, it is determined that
\[
\begin{align*}
0 &= \alpha_1|_S = S^* \alpha_1 = (u_x - p) \ dx \wedge dt, \\
0 &= \alpha_2|_S = S^* \alpha_2 = (p_x - q) \ dx \wedge dt, \\
0 &= \alpha_3|_S = S^* \alpha_3 = (u_t - q_t + u(u_x - q_x) + \beta(u - q)u_x) \ dx \wedge dt.
\end{align*}
\]

Thus sectioning the differential system (4.1) generates the following set of equations
\[
p = u_x, \quad q = p_x, \quad (u - q)_t + u(u - q)_x + \beta(u - q)u_x = 0. \quad (4.4)
\]

The first two equations here imply that $q = u_{xx}$. Using this in the third equation of (4.4), the following partial differential equation results
\[
(u - u_{xx})_t + u(u - u_{xx})_x + \beta(u - u_{xx})u_x = 0.
\]

The reason for the interest in this equation is that it produces important, relevant equations which are of current interest when $\beta$ is picked appropriately. To write this in a concise form, it is usual to introduce the variable $\rho = u - u_{xx}$, which gives
\[
\rho_t + \rho_x u + \beta \rho u_x = 0. \quad (4.5)
\]

Setting $\beta = 2$ in (4.5), the Camassa-Holm equation results,
\[
\rho_t + \rho_x u + 2\rho u_x = 0. \quad (4.6)
\]
For the case in which $\beta = 3$, equation (4.5) takes the form of the Degasperis-Procesi equation,

$$\rho_t + \rho_x u + 3\rho u_x = 0. \quad (4.7)$$

The results which are obtained below will have implications for these two equations.

### 4.2 Prolongation Equations.

Upon substituting differential system (4.1) into (3.8), the prolongation relation takes the form,

$$\begin{align*}
\frac{\partial A}{\partial t} dt \wedge dx + \frac{\partial A}{\partial u} du \wedge dx + \frac{\partial A}{\partial p} dp \wedge dx + \frac{\partial A}{\partial q} dq \wedge dx \\
+ \frac{\partial B}{\partial x} dx \wedge dt + \frac{\partial B}{\partial u} du \wedge dt + \frac{\partial B}{\partial p} dp \wedge dt + \frac{\partial B}{\partial q} dq \wedge dt + [A, B] dx \wedge dt \\
= \lambda_1 (du \wedge dt - p dx \wedge dt) + \lambda_2 (dp \wedge dt - q dx \wedge dt) \\
+ \lambda_3 (-du \wedge dx + dq \wedge dx + u du \wedge dt - u dq \wedge dt + \beta(u - q) du \wedge dt).
\end{align*} \tag{4.8}$$

By comparing the coefficients of the forms on both sides of (4.8), the following system of prolongation equations is seen to hold,

$$\begin{align*}
\frac{\partial A}{\partial u} &= -\lambda_3, & \frac{\partial A}{\partial p} &= 0, & \frac{\partial A}{\partial q} &= \lambda_3, \\
\frac{\partial B}{\partial u} &= \lambda_1 + u\lambda_3 + \beta(u - q)\lambda_3, & \frac{\partial B}{\partial p} &= \lambda_2, & \frac{\partial B}{\partial q} &= -u\lambda_3, \\
-\frac{\partial A}{\partial t} + \frac{\partial B}{\partial x} + [A, B] &= -p\lambda_1 - q\lambda_2. \tag{4.9}
\end{align*}$$

It will be seen that prolongation system (4.9) is much more restrictive than that obtained for the previous case.

**Theorem 4.1** System (4.9) can be reduced to a single equation which relates the functions $A$ and $B$.

**Proof:** The results obtained in (4.9) imply the following system of equations,

$$\begin{align*}
\frac{\partial A}{\partial u} &= -\frac{\partial A}{\partial q}, & \frac{\partial B}{\partial q} &= -u\frac{\partial A}{\partial q} = u\frac{\partial A}{\partial u}, & \lambda_1 &= \frac{\partial B}{\partial u} - u\lambda_3 - \beta(u - q)\lambda_3. \tag{4.10}
\end{align*}$$
Making use of these in the last equation of (4.9), we obtain

\[-\frac{\partial A}{\partial t} + \frac{\partial B}{\partial x} + [A, B] = -p(\frac{\partial B}{\partial u} - u\lambda_3 - \beta(u - q)\lambda_3) - q\lambda_2.\] (4.11)

As with the previous equation studied in Section 3, A and B are taken to be independent of both x and t for the same reason. Moreover, the equation \(A_p = 0\) implies that A is independent of the variable \(p\), therefore,

\[A = A(u, q, y), \quad B = B(u, p, q, y).\] (4.12)

Integrating the equation \(B_q = -uA_q\) with respect to \(q\), it is found that B is related to A by

\[B = -uA(u, q, y) + B'(u, p, y).\] (4.13)

In (4.13), \(B'\) satisfies \(B'_p = \lambda_2\), but is arbitrary otherwise. Differentiating (4.13) with respect to \(u\), we obtain

\[B_u = -A(u, q, y) - uA_u(u, q, y) + B'_u(u, p, y).\] (4.14)

The first partial differential equation in (4.10)

\[\frac{\partial A}{\partial u} + \frac{\partial A}{\partial q} = 0,\] (4.15)

implies that A must be of the form

\[A(u, q, y) = A(u - q, y).\] (4.16)

Since \(\lambda_3 = -A_u\), we can write \(\lambda_1\) from (4.10) as

\[\lambda_1 = B_u + uA_u + \beta(u - q)A_u = -A - uA_u + B'_u + uA_u + \beta(u - q)A_u.\] (4.17)

The \(uA_u\) terms cancel in (4.17) and so substituting (4.17) and B from (4.13) into (4.11), we obtain that

\[[A, B'] = -p(-A + B'_u + \beta(u - q)A_u) - qB'_p.\] (4.18)

This is the required result, and finishes the proof.

It remains to find solutions to equation (4.18), which contains two unknown functions. One way to do this is to impose a condition on one of the unknown functions.
Theorem 4.2 There exists a nontrivial solution to system (4.15) and (4.18) under the condition \( B' = 0 \).

Proof: It follows from (4.15) that \( A \) must satisfy (4.16). Putting \( B' = 0 \) into (4.18), it simplifies to one equation in one unknown,
\[
\beta(u - q)A_u - A = 0.
\] (4.19)

On account of (4.16), we can introduce the variable \( \xi = u - q \), and (4.19) then becomes an ordinary differential equation for \( A \)
\[
\beta \xi A_\xi - A = 0.
\]
This equation has the nontrivial solution
\[
A(\xi, y) = (u - q)^{\frac{1}{2}}X(y),
\]
with the integration constant written as \( X(y) \). To summarize explicitly, by using (4.13), the solutions for \( A \) and \( B \) are given as
\[
A(u, q, y) = (u - q)^{\frac{1}{2}}X(y), \quad B(u, p, q, y) = -u(u - q)^{\frac{1}{2}}X(y).
\] (4.20)

Many prolongations can be specified by introducing different conditions on \( B' \). One more will be derived.

Theorem 4.3 There exists a nontrivial solution of (4.15) and (4.18) such that \( B' = \frac{1}{2}(p^2 - u^2)X_2(y) \) with \([A, X_2] = 0\).

Proof: Since \( B'_p = pX_2 \) and \( B'_u = -uX_2 \), and \([A, B'] = 0\), substituting these, equation (4.18) takes the form
\[
\beta(u - q)A_u - A = (u - q)X_2.
\]
Using (4.16), this reduces to an ordinary differential equation in the variable \( \xi = u - q \), namely,
\[
\beta \xi A_\xi - A = \xi X_2.
\]
This equation has the general solution which for $\beta \neq 1$ is given by

$$A(u, q, y) = (u - q)^{\frac{1}{\beta}} X_1(y) + \frac{u - q}{\beta - 1} X_2(y),$$

and from (4.13), $B$ is given by

$$B = -u(u - q)^{\frac{1}{\beta}} X_1(y) - \frac{u - q}{\beta - 1} X_2(y) + \frac{1}{2}(p^2 - u^2) X_2(y).$$

Here, $X_1$ and $X_2$ generate a commutative algebra such that $[X_1, X_2] = 0$. ♣

4.3 Conservation Laws.

With the $\alpha_i$ given by (4.1), we can define a form $\vartheta$ for this case as well. To do the calculations here, the exterior derivatives (4.2) of the forms (4.1) can be simplified to read

$$d\alpha_1 = -dp \wedge dx \wedge dt, \quad d\alpha_2 = -dq \wedge dx \wedge dt, \quad d\alpha_3 = (1 - \beta) dq \wedge du \wedge dt.$$

Consider the one-form $\vartheta$ which is defined to be

$$\vartheta = -(1 - \beta) q \alpha_1 + (1 - \beta) p \alpha_2 + \alpha_3. \quad (4.21)$$

Thus, the exterior derivative $d\vartheta$ is in ideal $I$ since

$$d\vartheta = -(1 - \beta) dq + (1 - \beta) p((1 + \beta) u - q) dx + (1 - \beta) dq - p dx) \wedge \alpha_1 + (1 - \beta)(-q dx + dp) \wedge \alpha_2$$

$$+(1 - \beta)(-\frac{p}{u} dx + p dt) \wedge \alpha_3.$$

Calculating the exterior derivative of $\vartheta$ in terms of the basic set of variables of $M$, it is found to vanish identically as well,

$$d\vartheta = (1 - \beta) dq \wedge du \wedge dt - (1 - \beta) dq \wedge du \wedge dt + (1 - \beta) p dq \wedge dx \wedge dt + (1 - \beta) q dp \wedge dx \wedge dt$$

$$-(1 - \beta) q dp \wedge dx \wedge dt - (1 - \beta) p dq \wedge dx \wedge dt = 0.$$

In fact, $\vartheta$ can be obtained directly from the one-form $\omega$ defined by

$$\omega = (q - u) dx + \frac{1}{2}(u^2 - 2uq + \beta u^2 + p^2) dt. \quad (4.22)$$
Upon differentiating \( \omega \) it is found that
\[
d\omega = -du \wedge dx + dq \wedge dx + u \, du \wedge dt - u \, dq \wedge dt + \beta u \, du \wedge dt - q \, du \wedge dt + (1 - \beta) p \, dp \wedge dt. \tag{4.23}
\]
This is precisely the two-form \( \vartheta \) given in (4.21). The associated conservation law results from an application of Stokes theorem, which is written in this case
\[
\oint_{M_1} \omega = \int_{M_2} d\omega. \tag{4.24}
\]
This has been written for any simply-connected, two-dimensional manifold \( M_2 \) with closed one-dimensional boundary \( M_1 \). The equation implies that \( \omega \) and \( d\omega \) are to be evaluated on their respective manifolds.

Returning to \( \omega \) again, we can again add to \( \omega \) any exact one-form \( dv \), where \( v \) is an arbitrary scalar function. Thus, \( \omega \) can also be taken to be
\[
\omega = dv + (q - u) \, dx + \frac{1}{2} (u^2 - 2uq + \beta u^2 + p^2) \, dt, \tag{4.25}
\]
such that \( \vartheta = d\omega \). As before, \( v \) may be regarded as a coordinate in an extended six-dimensional space of variables \( \{x, t, u, p, q, v\} \), and the one-form \( \omega \) may be included with the original set of forms. Since \( d\omega \) is known to be in the ring of the original set, the new set of forms remains a closed ideal.

5 Summary and Conclusions.

It has been seen that exterior differential systems have been constructed for some very important classes of partial differential equation. As well as giving some information about the associated integrability of these equations, it has been shown that the prolongation structure of these systems can be studied. This is more than just of theoretical interest, since Bäcklund transformations can be constructed based on these results. The relationship of differential systems to Bäcklund transformations has been discussed by Estabrook and Wahlquist [15], and the construction of such transformations has been done for some three element algebras in [16]. Let us show how to use the results of example (i) in Section 3 to obtain such a result.
Using (2.4), the connection \( \tilde{\omega} \) can always be chosen on \( \mathbb{R} \) with coordinate \( y \) and \( X = \partial/\partial y \)

\[
\tilde{\omega} = dy - \{ (\kappa + \sigma u + u^{n+1}) \, dx + (-\sigma + (n+1)u^n)q + \frac{1}{2}(n+1)p^2 + \frac{n}{m}\sigma u^m \\
+ \frac{n(n+1)}{m+n} \gamma^{m+n} + \alpha \} \, dt \}
\]

\( X(y), \kappa, \sigma, \alpha \in \mathbb{R}. \) \hfill (5.1)

Solutions of the system (3.6) determine transversal sections of the fibre bundle such that, upon substituting \( p \) and \( q \) from (3.4), we have

\[
y_x = \kappa + \sigma u + u^{n+1},
\]

\[
y_t = - (\sigma + (n+1)u^n) (u^n)_{xx} + \frac{1}{2}(n+1)((u^n)_x)^2 + \frac{n}{m}\gamma^{m+n} + \frac{n(n+1)}{m+n} \gamma^{m+n} + \alpha.
\] \hfill (5.2)

A similar result but for a different algebra was given in [9]. By solving the first of these for \( u \), it can be eliminated in the second equation of (5.2) to yield an equation for \( y = y(x,t) \). For \( \sigma = 0 \), this can be done in closed form, and to make the presentation more concise we put \( \kappa = 0 \) as well giving

\[
u = (y_x)^{\frac{1}{n+1}}. \] \hfill (5.3)

The positive root is taken if the exponent in (5.3) has an even denominator. Eliminating \( u \) from the second equation in (5.2), we have an equation for \( y \)

\[
y_t + (n+1)(y_x)^{\frac{n}{n+1}}((y_x)^{\frac{n}{n+1}})_{xx} - \frac{1}{2}(n+1)(((y_x)^{\frac{n}{n+1}})_x)^2 - \frac{n(n+1)}{m+n} \gamma^{(y_x)^{\frac{m+n}{m+1}}} - \alpha = 0. \] \hfill (5.4)

It follows that for \( \sigma = \kappa = 0 \), the potential equation in terms of \( y \) which results is given by (5.4).

In effect, a Bäcklund transformation has been determined and is expressed by (5.2). This set of equations transforms the original equation into the form of its potential equation (5.4).

References.

[1] A. C. Newell, Solitons in Mathematics and Physics, SIAM, Philadelphia, 1985.

[2] M. J. Ablowitz and P. A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, 1991.

[3] H. D. Wahlquist and F. B. Estabrook, Prolongation Structures of Nonlinear Evolution Equations, J. Math. Phys. 16, 1-7, (1975).
[4] H. D. Wahlquist and F. B. Estabrook, Prolongation Structures of Nonlinear Evolution Equations II, J. Math. Phys. 17, 1293-1297 (1976).
[5] F. B. Estabrook, Moving Frames and Prolongation Algebras, J. Math. Phys. 23, 2071-2076 (1982).
[6] W. F. Shadwick, The KdV Prolongation Algebra, J. Math. Phys., 21, 454-461 (1980).
[7] E. van Groesen and E. M. de Jager, Mathematical Structures in Continuous Dynamical Systems, Studies in Math. Phys., vol. 6, North Holland, 1994.
[8] E. M. de Jager and S. Spannenburg, Prolongation structures and Bäcklund transformations for the matrix Korteweg-de Vries and Boomeron equation, J. Phys. A: Math. Gen. 18, 2177-2189 (1985).
[9] P. Bracken, An Exterior Differential System for a Generalized Korteweg-de Vries Equation and its Associated Integrability, Acta Applicandae Mathematicae, 95, 223-231 (2007).
[10] T. Tao, Why are solitons stable?, Bulletin of the Amer. Math. Soc., 46, 1-33, (2009).
[11] P. Bracken, Symmetry Properties of a Generalized Korteweg-de Vries Equation and Some Explicit Solutions, Int. J. Math. and Math. Sciences, 2005, 13, 2159-2173 (2005).
[12] E. G. Reyes, Geometric Integrability of the Camassa-Holm Equation, Lett. Math. Phys., 59, 117-131 (2002).
[13] R. Camassa and D. Holm, An Integrable Shallow Water Equation with Peaked Solitons, Phys. Rev. Letts. 71, 1661-1664 (1993).
[14] J. Lenells, Conservation Laws of the Camassa-Holm Equation, J. Phys. A: Math. Gen. 38, 869-880 (2005).
[15] F. B. Estabrook and H. D. Wahlquist, ”Prolongation Structures, Connection Theory and Bäcklund Transformation”, in Nonlinear evolution equations solvable by the spectral transform, Ed. F. Calogero, Pitman, 1978.
[16] P. Bracken, The interrelationship of integrable equations, differential geometry and the geometry of their associated surfaces, to appear, 2009.