HARNACK TYPE INEQUALITIES FOR OPERATORS IN
LOGARITHMIC SUBMAJORISATION

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Abstract. The aim of this paper is to study the Harnack type logarithmic
submajorisation and Fuglede-Kadison determinant inequalities for operators
in a finite von Neumann algebra. In particular, the Harnack type determinant
inequalities due to Lin-Zhang[17] and Yang-Zhang[28] are extended to the
case of operators in a finite von Neumann algebra.

1. Introduction

The classical Harnack inequality, named after Carl Gustav Axel von Harnack,
gives an estimate from above and an estimate from below for a positive har-
monic function in a domain. Even though the classical Harnack inequality is
almost trivially derived from the Poisson formula, the consequences that may be
deduced from Harnack inequality are particularly of great importance. Later,
these inequalities became an important tool in the general theory of harmonic
functions and partial differential equations. There exist as yet extensive works
on generalized Harnack inequalities in various forms, see [19, 27, 28] for a nice
introduction about the inequality. The purpose of this paper is to investigate
the Harnack type determinant inequality for operators and matrices.

With the help of Lagrange multiplier method, the following Harnack type
determinant inequality was established by Tung[23], as a tool to study Har-
nack inequality: If $Z \in M_n$ is a complex matrix with singular values $r_k$ with
$0 \leq r_k < 1, k = 1, 2, ..., n$, then

$$\prod_{k=1}^{n} \frac{1 - r_k}{1 + r_k} \leq \frac{\det(I - Z^*Z)}{\det(I - UZ)^2} \leq \prod_{k=1}^{n} \frac{1 + r_k}{1 - r_k}, U \in U_n,$$

(1.1)

where $U_n$ denotes the set of all $n \times n$ unitary matrices $U$. From these bounds
Tung obtained upper and lower bounds of a Poisson kernel on $U_n$(see [23]), hence
that the so-called Harnack’s first and second theorems are established. Tung’s
work drew immediate attention of Hua and Marcus. Using majorisation theory
and singular value (eigenvalue) inequalities of Weyl, Marcus [18] gave another
proof of (1.1) and gave an equivalent form of (1.1). Almost at the same time, a
proof of (1.1) was also given by Hua[11] based on the determinantal inequality he
had previously obtained in[10]. In the past decades, Tung’s work has attracted
attentions of mathematicians and been extended to various setting (see [15, 17,
19, 27, 28] and the references therein for more details). Among these outstanding

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works we will be interested in Lin- -Zhang’s and Yang- -Zhang’s work. Specifically, with \( A = U Z \), (1.1) is equivalently rewritten in terms of eigenvalues ([15, 28]) as

\[
\prod_{k=1}^{n} \frac{1 - r_k}{1 + r_k} \leq \prod_{k=1}^{n} \lambda_k((I - A^*)^{-1}(I - A^*A)(I - A)^{-1}) \leq \prod_{k=1}^{n} \frac{1 + r_k}{1 - r_k}, \tag{1.2}
\]

where \( Z \in M_n \) is a complex matrix with singular values \( r_k \) with \( 0 \leq r_k < 1, k = 1, 2, \ldots, n \) and \( U \in U_n \). (1.2) leads to the study of inequalities of logarithmic submajorisation of eigenvalues and singular values. Following this line, an interesting generalization of (1.2) is presented by Yang- -Zhang[28] and Jiang--Lin[15] as follows:

\[
\prod_{k \in K} \lambda_k((I - A^*)^{-1}(I - A^*A)(I - A)^{-1}) \leq \prod_{k \in K} \frac{1 + r_k}{1 - r_k}, \tag{1.3}
\]

\[
\prod_{i \in K} \lambda_{n-k+1}((I - A^*)^{-1}(I - A^*A)(I - A)^{-1}) \geq \prod_{k \in K} (1 - r_k^2) \prod_{i=1}^{|K|} (1 + r_i)^2, \tag{1.4}
\]

where \( K \) is a subset of \( \{r_1, r_2, \ldots , r_n\} \) and \( |K| \) denotes the number of terms in \( K \). The main theme of the paper is to continue with Jiang- -Lin and Yang- -Zhang’s work and to show their results hold in the case of operators in finite von Neumann algebras.

We are concerned with the Harnack type logarithmic submajorisation inequality and Fuglede-Kadison determinant inequality for operators in a finite von Neumann algebra. The properties of the logarithmic submajorisation and Fuglede--Kadison determinant for operators in a finite von Neumann algebra was investigated by many authors, see for example [4, 2, 14]. Those properties are important, for example, in investigation of noncommutative Hardy spaces and invariant subspaces for operators in von Neumann algebras. By adapting the techniques in [28, 9, 21], we obtain some inequalities which is related to the Harnack type logarithmic submajorisation inequality and Fuglede--Kadison determinant inequality. In particular, we show that the inequalities (1.3) and (1.4) hold for operators in a finite von Neumann algebra. We will conclude this paper with a series of logarithmic submajorisation inequalities which is related to Cayley transform.

2. Preliminaries

2.1. von Neumann algebras. Suppose that \( \mathcal{H} \) is a separable Hilbert space over the field \( \mathbb{C} \) and \( I \) is the identity operator in \( \mathcal{H} \). We will denote by \( B(\mathcal{H}) \) the \( * \)-algebra of all linear bounded operators in \( \mathcal{H} \). Let \( \mathcal{M} \) be a \( * \)-subalgebra of \( B(\mathcal{H}) \) containing the identity operator \( I \). Then \( \mathcal{M} \) is called a von Neumann algebra if \( \mathcal{M} \) is weak operator closed. Let \( \mathcal{M}^+ \) denote the positive part of \( \mathcal{M} \). We recall that a weight on \( \mathcal{M} \) is a map \( \tau : \mathcal{M}^+ \to [0, \infty] \) satisfying

1. \( \tau(x + y) = \tau(x) + \tau(y) \), for all \( x, y \in \mathcal{M}^+ \);
2. \( \tau(\alpha x) = \alpha \tau(x) \) for all \( x \in \mathcal{M}^+ \) and \( \alpha \in [0, \infty) \), with the convention \( 0 \cdot \infty = 0 \).

The weight \( \tau \) is called faithful if \( \tau(x^*x) = 0 \) implies \( x = 0 \), normal if \( x_i \uparrow_i x \) in \( \mathcal{M}^+ \) implies that \( 0 \leq \tau(x_i) \uparrow \tau(x) \), tracial if \( \tau(x^*x) = \tau(xx^*) \) for all \( x \in \mathcal{M} \).
Note that since \((x_i)\) is bounded there is \(x \in \mathcal{M}^+\) such that, for any \(h \in \mathcal{H}\), \(\langle x_i h, h \rangle \to \langle xh, h \rangle\), which implies that \(x_i\) tends to \(x\) weak* and hence \(x \in \mathcal{M}^+\). The operator \(x\) is obviously the least upper bound of \((x_i)\), it is natural to denote it by \(\sup_i x_i\). The self-adjoint part of \(\mathcal{M}, \mathcal{M}^{sa}\), is a partially ordered vector space under the ordering \(x \geq 0\) defined by \(\langle x \xi, \xi \rangle \geq 0, \xi \in \mathcal{H}\). Recall that \(x \in \mathcal{M}\) is contractive if \(\|x\| \leq 1\) and strictly contractive if \(\|x\| < 1\). Moreover, if \(x\) is strict contractive, then \(\mathcal{I} - x^*x\) is invertible and \(\mathcal{I} - x^*x \geq 0\).

It is also customary to say trace instead of tracial weight. A trace \(\tau\) is called finite if \(\tau(\mathcal{I}) < \infty\). A finite trace \(\tau\) is extended uniquely to a positive linear functional on \(\mathcal{M}\) which will also be denoted by \(\tau\). A positive linear functional \(\tau\) on a von Neumann algebra is said to be a state if \(\tau(\mathcal{I}) = 1\).

A von Neumann algebra \(\mathcal{M}\) is called finite if the family formed of the finite normal traces separates the points of \(\mathcal{M}\). Clearly this happens if \(\mathcal{M}\) admits a single faithful normal finite trace. But a finite \(\mathcal{M}\) may fail to have any faithful finite trace, for instance \(\mathcal{M} = \ell^\infty(\mathbb{R})\) where \(\mathbb{R}\) is equipped with counting measure. However, on a separable Hilbert space (i.e. if \(\mathcal{M}\) is weak*-separable) the converse is also true i.e., \(\mathcal{M}\) is finite if and only if it admits a faithful normal finite trace.

In what follows, we will keep all previous notations throughout the paper, and \(\mathcal{M}\) will always denote a finite von Neumann algebra acting on a separable Hilbert space \(\mathcal{H}\), with a normal faithful finite tracial state \(\tau\), i.e., a normal faithful finite trace \(\tau\) satisfies that \(\tau(\mathcal{I}) = 1\). We refer to [24] for von Neumann algebras.

### 2.2. The eigenvalue function and generalized singular value function.

**Definition 2.1.** Let \(x \in \mathcal{M}\) and \(t > 0\). The “\(t\)-th singular number (or generalized singular number) of \(x\)” \(\mu_t(x)\) is defined by

\[
\mu_t(x) = \inf\{|\|xe\| : e \text{ is a projection in } \mathcal{M} \text{ with } \tau(e^\perp) \leq t\}.
\]

We denote simply by \(\mu(x)\) the function \(t \to \mu_t(x)\). The generalized singular number function \(t \to \mu_t(x)\) is decreasing right-continuous. For convenience to discuss the properties of \(\mu_t(x)\) we define \(\mu^t_t(x)\) by

\[
\mu^t_t(x) = \inf\{|\|xe\| : e \text{ is a projection in } \mathcal{M} \text{ with } \tau(e^\perp) < t\}.
\]

Then \(t \to \mu^t_t(x)\) is non-increasing and right-continuous and \(\mu^t_t(x) = \mu_t(x)\) holds for almost every \(t \in [0, 1]\). See [6, 9, 25, 26] for basic properties and detailed information on \(\mu_t(x)\) and \(\mu^t_t(x)\).

If \(x\) is self-adjoint and \(x = \int_{-\infty}^\infty tde_t(x) \in \mathcal{M}\) is the spectral resolution of \(x\) then for any Borel subset \(B \subseteq (-\infty, \infty)\) we denote by \(e_B(x)\) the corresponding spectral projection. However, we write \(e_{s}(x) = e_{(-\infty,s]}(x)\). Given \(x \in \mathcal{M}^{sa}\), the spectral scale \(\lambda_t(x)\) on \((0, \tau(\mathcal{I}))\) is defined by

\[
\lambda_t(x) = \inf\{s \in \mathbb{R} : \tau(\mathcal{I} - e_s(x)) \leq t\}.
\]

Obviously, if \(0 \leq x \in \mathcal{M}\) then \(\lambda_t(x) = \mu_t(x)\) for \(0 < t < 1\). The spectral scale \(\lambda_t(x)\) is non-increasing and right-continuous. For the properties of \(\lambda_t(\cdot)\), it is important to note that \(\lambda_t(x + a\mathcal{I}) = \lambda_t(x) + a\) for every \(x \in \mathcal{M}^{sa}\) and \(a \in \mathbb{R}\). This property enables us to deduce estimations for \(\lambda_t(x)\) from formulas on \(\mu_t(x)\).
To achieve our main results, we state some properties of $\lambda(\cdot)$ and $\mu(\cdot)$ without proof (see [12, 9]).

**Proposition 2.2.** (see [12, 9]) Let $x, y \in \mathcal{M}$ and $v \in \mathcal{M}$. Then

1. $\mu(|x|) = \mu(x) = \mu(x^*)$ and $\mu(\alpha x) = |\alpha|\mu_t(x)$, for $t > 0$ and $\alpha \in \mathbb{C}$.
2. Let $f$ be a bounded continuous increasing function on $[0, \infty)$ with $f(0) = 0$. Then $\mu(f(x)) = f(\mu(x))$ and $\tau(f(x)) = \int_0^{\tau(1)} f(\mu_t(x)) dt$.
3. $\mu_{s+t}(x + y) \leq \mu_t(x) + \mu_{s_t}(y)$, $s, t > 0$.
4. If $0 \leq x \leq y$, then $\mu_t(x) \leq \mu_t(y)$.
5. $\mu_{s+t}(xy) \leq \mu_t(x)\mu_{s_t}(y)$, $s, t > 0$.
6. If $x, y$ are self-adjoint, then $\lambda_{s+t}(x + y) \leq \lambda_t(x) + \lambda_{s_t}(y), t, s \geq 0, t + s \leq 1$.
7. If $0 \leq t \leq 1$ and $x, y$ are self-adjoint, then $\lambda_t(x) \geq 0$ implies that $\lambda_t(v^*av) \leq \|v\|^2\lambda_t(x)$.
8. If $x, y$ are self-adjoint and $x \leq y$, then $\lambda_t(x) \leq \lambda_t(y)$.
9. If $x$ is self-adjoint, then $\lambda_t(f(x)) = f(\lambda_t(x)), t \in (0, \tau(\mathbb{I}))$, for every increasing continuous function $f$ on $\mathbb{R}$.

**Example 2.3.** Let $\mathcal{H} = \mathbb{C}^n$ and let $\mathcal{M} = \mathcal{B}(\mathcal{H}) \cong \mathbb{M}_n(\mathbb{C})$ equipped with the normalized trace $\tau_n : \triangleq \frac{1}{n}tr_n$ where $tr_n$ is the standard trace on $\mathbb{M}_n(\mathbb{C})$. If $x \in \mathcal{B}(\mathcal{H}) = \mathbb{M}_n(\mathbb{C})$ is self-adjoint, then $x$ can be written as $x = \sum_{j=1}^{n} \alpha_j p_j$, where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ is the sequence of eigenvalues of $x$ in which each is repeated according to its multiplicity and $\sum_{i=1}^{n} p_j = \mathbb{I}$. Therefore,

$$
\lambda_t(x) = \sum_{j=1}^{n} \alpha_j \chi_{[\frac{j-1}{n}, \frac{j}{n})}, t \in [0, 1) .
$$

If $x \geq 0$, then $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0$, $\lambda_t(x) = \mu_t(x)$ and $\mu^\ell_t(x) = \sum_{j=1}^{n} \alpha_j \chi_{[\frac{j-1}{n}, \frac{j}{n})}$.

If $x \in \mathbb{M}_n(\mathbb{C})$ is arbitrary, then $\mu_t(x) = \mu_t(|x|)$ and the eigenvalues of $|x|$ are usual called the singular values of $x$. It follows that

$$
\mu_t(x) = \sum_{j=1}^{n} s_j \chi_{[\frac{j-1}{n}, \frac{j}{n})},
$$

and

$$
\mu^\ell_t(x) = \sum_{j=1}^{n} s_j \chi_{[\frac{j-1}{n}, \frac{j}{n})},
$$

where $s_1 \geq s_2 \geq \cdots \geq s_n \geq 0$ is the sequence of singular values of $x$, repeated according to multiplicity. It is clear that $\mu_{\frac{t}{n}}(x) = \mu_{\frac{t}{n}}^\ell(x)$.

Note that if $x \in \mathbb{M}_n(\mathbb{C})$ is self-adjoint, then $x$ can also be written as $x = \sum_{j=1}^{m} \beta_j p_j$, where $\beta_1 > \beta_2 > \cdots > \beta_m (m \leq n)$. Then

$$
\lambda_t(x) = \sum_{j=1}^{m} \beta_j \chi_{[d_{j-1}, d_j)},
$$

where $d_j = \sum_{i=1}^{j} \tau(p_i)$ for $j = 1, 2, \cdots, m$ and $d_0 = 0$. For each $j$, the length of the interval $[nd_{j-1}, nd_j)$ is $n\tau_n(p_j)$, which is the dimension of the eigenspace corresponding to $\beta_j$. See [13, 6] for more details of $\mu_t(\cdot)$ and $\lambda_t(\cdot)$ of operators and
matrices (Note: the generalized singular values \( \mu_k \), as defined in [13], is denoted by \( \mu_n^k \), in this paper; the generalized singular values \( \mu_n^k \) and \( \mu_n^k \) is noting but \( \mu_{k+1} \) and \( \mu_k \), respectively, in [13])

**Example 2.4.** Consider the algebra \( \mathcal{M} = L^\infty([0,1]) \) of all Lebesgue measurable essentially bounded functions on \([0,1]\). Algebra \( \mathcal{M} \) can be seen as an abelian von Neumann algebra acting via multiplication on the Hilbert space \( \mathcal{H} = L^2([0,1]) \), with the trace given by integration with respect to Lebesgue measure \( m \). For a real measurable function \( f \in L^\infty([0,1]) \), the decreasing rearrangement \( f^* \) of the function \( f \) is given by

\[
f^*(t) = \inf \{ s \in \mathbb{R} : m(\{ h \in [0,1] : f(h) > s \}) \leq t \}, \quad 0 < t < 1.
\]

Then \( \mu_t(f) = |f|^*(t) \) and \( \lambda_t(f) = f^*(t) \). Suppose that \( f = \sum_{i}^{n} \alpha_i \chi_{B_i} \), where \( B_i \subseteq [0,1] \) with \( B_i \cap B_j = \emptyset \) whenever \( i \neq j \), and \( 0 < \alpha_j \in \mathbb{R} \) \( (j = 1, 2, \ldots, n) \) are such that \( \alpha_i \neq \alpha_j \) whenever \( i \neq j \). For the computation of \( \mu_t(f) \), it may be assumed that \( \alpha_1 > \alpha_2 > \cdots > \alpha_n \). Then

\[
\lambda_t(f) = \sum_{j=1}^{n} \alpha_j \chi_{[d_{j-1}, d_j]},
\]

where \( d_j = \sum_{i=1}^{j} m(B_i) \) for \( j = 1, 2, \ldots, n \) and \( d_0 = 0 \). If \( f \geq 0 \), then \( \alpha_1 > \alpha_2 > \cdots > \alpha_n \geq 0 \), \( \lambda_t(f) = \mu_t(f) \) and \( \mu_t(f) = \sum_{j=1}^{n} \alpha_j \chi_{[d_{j-1}, d_j]} \). See [6, 21] for more details.

### 2.3. Fuglede-Kadison determinant

Let \( \mathcal{M} \) be a finite von Neumann algebra acting on a separable Hilbert space \( \mathcal{H} \), with a normal faithful finite tracial state \( \tau \). Recall that the Fuglede-Kadison determinant \( \Delta = \Delta_\tau : \mathcal{M} \to \mathbb{R}^+ \) is defined by \( \Delta_\tau(x) = \tau(\log |x|) \) if \( |x| \) is invertible; and otherwise, we define \( \Delta_\tau(x) = \inf \Delta_\tau(\|x\| + \varepsilon \|I\|) \), the infimum takes over all scalars \( \varepsilon > 0 \). We define Fuglede-Kadison determinant-like function of \( x \) by

\[
\Lambda_t(x) = \exp \{ \int_0^t \log \mu_s(x) ds \}, \quad t > 0.
\]

Since \( \tau(\|I\|) = 1 \), if \( |x| \) is invertible, then

\[
\Delta_\tau(x) = \Lambda_1(x) = \exp \{ \int_0^1 \log \mu_s(x) ds \}.
\]

We understanding that \( \Delta(x) = 0 \) if

\[
\int_0^{\tau(\|I\|)} \log \mu_s(x) ds = -\infty.
\]

Recall that \( x \) is said to be logarithmically submajorised by \( y \)(see [7, 14]), denoted by \( x \prec \log y \) (or \( \mu(x) \prec \log \mu(y) \)), if \( \Lambda_t(x) \leq \Lambda_t(y) \) for all \( t > 0 \).

We state for easy reference the following fact, obtained from [1, 4] for Fuglede-Kadison determinant which will be applied below.

**Proposition 2.5.** Let \( x, y \in \mathcal{M} \). Then

(1) \( \Delta_\tau(\|I\|) = 1, \Delta_\tau(xy) = \Delta_\tau(x) \Delta_\tau(y) \).
In this section, we will consider some Logarithmic submajorisation inequalities. Indeed, let \( \psi \) which implies that (1) \( \Rightarrow \) which yields (2)

\[
(4) \Delta_r(x) \leq \Delta_r(y), \text{ if } 0 \leq x \leq y
\]

(5) \( \lim_{\varepsilon \to 0^+} \Delta_r(x + \varepsilon 1) = \Delta_r(x), \text{ if } 0 \leq x. \)

(6) \( \Delta_r(x) \leq \Delta_r(y), \text{ if } x \prec \log y. \)

See [1, 4, 2] for basic properties and detailed information on Fuglede-Kadison determinant of \( x \in \mathcal{M}. \)

**Example 2.6.** Let \( \mathcal{H} = \mathbb{C}^n \) and let \( \mathcal{M} = \mathcal{B}(\mathcal{H}) \cong M_n(\mathbb{C}) \) equipped with the normalized trace \( \tau_n : \triangleq \frac{1}{n} tr_n \) where \( tr_n \) is the standard trace on \( M_n(\mathbb{C}) \). If \( x \in \mathcal{B}(\mathcal{H}) \), then \( \Delta_{\tau_n}(x) = (\det(|x|))^\frac{1}{n} \). See [13] for more information on determinant of matrices.

If \( x, y \in \mathcal{M} \) and \( 0 < p < \infty \), then \( x \) is said to be \( p \)-submajorised by \( y \), denoted by \( x \prec \prec_p y \), if \( \int_0^t \mu_s(x)^p ds \leq \int_0^t \mu_s(y)^p ds \) for all \( t > 0 \).

**Remark 2.7.** Let \( x, y \in \mathcal{M}^+ \) be invertible. Then the following conditions are equivalent:

\[
(1) \mathbb{I} + rx \prec \prec \log \mathbb{I} + ry, \text{ for all } r \in \mathbb{R}^+; \\
(2) x \prec \prec_p y, \text{ if } 0 < p < 1; \\
(3) x \prec \prec_p y; \\
(4) \int_0^t \phi(\mu_s(x)) ds \leq \int_0^t \phi(\mu_s(y)) ds \text{ for all } t > 0 \text{ and all nondecreasing functions } \phi \text{ on } [0, \infty) \text{ such that } \phi(0) = 0 \text{ and } t \to \phi(e^t) \text{ is convex.}
\]

Indeed, let \( \psi \) is a bounded positive measurable function on \( [0, \infty) \) and \( \pi_t(r) = \exp\{\int_0^t \log(1 + r\psi(s)) ds\} \). By [8, Lemma 3.2], we have

\[
\int_0^t \psi(s)^p ds = \frac{p \sin(\pi p)}{\pi} \int_0^\infty \log \pi_t(r) \frac{dr}{r^{p+1}},
\]

which implies that (1) \( \Rightarrow \) (2) holds.

Note that if \( \left( \int_0^t \varphi(s)^p ds \right)^\frac{1}{p} < \infty, t > 0 \) for some \( p > 0 \), then from [22, p.71] we obtain

\[
\exp\left\{ \int_0^t \log |\varphi(s)| \frac{ds}{t} \right\} = \lim_{p \to 0} \left( \int_0^t |\varphi(s)|^p \frac{ds}{t} \right)^\frac{1}{p}, t > 0,
\]

which yields (2) \( \Rightarrow \) (3). (3) \( \Rightarrow \) (4) follows from the fact that \( t \to \varphi(e^t) \) is convex and \( \varphi(e^{\log \mu(x)}) = \varphi(\mu(x)) \) (see [20, p.22, Theorem D.2]). It is easy to check that (4) \( \Rightarrow \) (1).

3. **Unitary approximation and Logarithmic submajorisation**

Our starting point is the following inequality for complex numbers:

\[
||z| - 1| \leq ||z| - v| \leq ||z| + 1|, z, v \in \mathbb{C} \text{ with } |v| = 1.
\]

(3.1)

In this section, we will consider some Logarithmic submajorisation inequalities for operator version of (3.1). We start with a lemma which will be used in our proof.
Lemma 3.1. Let $x \in \mathcal{M}^+$. Then

$$
\lambda_s(-x) = -\mu_{1-s}^\ell(x), 0 < s < 1.
$$

Proof. Let $x = \sum_{i=1}^n \alpha_i p_i$ with $\alpha_1 > \alpha_2 > \cdots > \alpha_n \geq 0$ and $p_ip_j = 0, i \neq j$. Without loss of generality we can assume $\sum_{i=1}^n p_i = \mathbb{I}$. Indeed, if $\sum_{i=1}^n p_i \neq \mathbb{I}$ we write $p_n = \mathbb{I} - \sum_{i=1}^{n-1} p_i$. Replacing $p_n$ by $p_n + p_{n+1}$ in the equation $x = \sum_{i=1}^n \alpha_i p_i$ if $\alpha_n = 0$, and replacing $x = \sum_{i=1}^n \alpha_i p_i$ by $x = \sum_{i=1}^{n+1} \alpha_i p_i$ if $\alpha_n \neq 0$. Set $d_i = \sum_{j=1}^n \tau(p_j), 1 \leq i \leq n$ and $d_0 = 0$. Then $d_{n+1} = \sum_{j=1}^n \tau(p_j) = \tau(\mathbb{I}) = 1$.

$$
\mu_{1-s}^\ell(x) = \sum_{i=1}^n \alpha_i \chi(d_{i-1}, d_i)(s), 0 < s < 1,
$$

and

$$
\lambda_{1-s}(x) = \sum_{j=1}^n -\alpha_{n-j+1} \chi(1-d_j, 1-d_{j-1}), 0 < s < 1.
$$

Thus,

$$
\mu_{1-s}^\ell(x) = -\lambda_{1-s}(x), 0 < s < 1,
$$

For the general case, let $0 \leq x \in \mathcal{M}$ and let $x = \int_0^{|x|} \lambda d\ell(x)$ be the spectral decomposition of $x$. Put

$$
f_k(t) = \sum_{j=1}^{2^n} \frac{j}{2^n} \lambda_k^{\ell}(t, \frac{|x|}{2^n}).
$$

Write $x_n = f_n(|x|)$. It follows that $||x - x_n|| \leq \frac{|x|}{2^n}$ and $x_n \geq x_{n+1} \geq x$. The proof is completed by showing that

$$
\lim_{n \to \infty} \mu_{1-s}^\ell(x_n) = \mu_{1-s}^\ell(x) \quad \text{and} \quad \lim_{n \to \infty} \lambda_{1-s}(x_n) = \lambda_{1-s}(x).
$$

For $\epsilon > 0$, we obtain

$$
\mu_{1-s}^\ell(x_n) \leq \mu_{1-s}^\ell(x) + \mu_{1-s}^\ell(x_n - x) \leq \mu_{1-s}^\ell(x) + ||x_n - x||, 0 < s < 1
$$

and

$$
\lambda_{s+\epsilon}(-x) - ||x_n - x|| \leq \lambda_{s+\epsilon}(-x) - \lambda_{\epsilon}(x_n - x) \leq \lambda_{s}(-x_n), 0 < s < 1.
$$

Taking the $n \to \infty$ of the both side, we get

$$
\limsup_{n \to \infty} \mu_{1-s}^\ell(x_n) \leq \mu_{1-s}^\ell(x), 0 < s < 1
$$

and

$$
\liminf_{n \to \infty} \lambda_{1-s}(x_n) \geq \lambda_{1-s}(x), 0 < s < 1.
$$

Since $\lambda(-x)$ is right-continuous and $\mu^\ell(x)$ is left-continuous on $(0, 1)$, letting $\epsilon \downarrow 0$, we obtain

$$
\limsup_{n \to \infty} \mu_{1-s}^\ell(x_n) \leq \mu_{1-s}^\ell(x), \quad \liminf_{n \to \infty} \lambda_{1-s}(x_n) \geq \lambda_{1-s}(x), 0 < s < 1.
$$

On the other hand, since $x \leq x_n$, $-x \geq -x_n$, moreover,

$$
\liminf_{n \to \infty} \mu_{1-s}^\ell(x_n) \geq \mu_{1-s}^\ell(x), \quad \limsup_{n \to \infty} \lambda_{1-s}(x_{-n}) \leq \lambda_{1-s}(x), 0 < s < 1.
$$
Hence \( \lim_{n \to \infty} \mu^\ell_s(x_n) = \mu^\ell_s(x) \), \( \limsup_{n \to \infty} \lambda_s(-x_n) = \lambda_s(-x) \), 0 < s < 1. This completes the proof.

\[ \square \]

**Proposition 3.2.** Let \( x \in \mathcal{M} \). Then
\[ -\mu^\ell_{1-s}(x) \leq \lambda_s(\text{Rex}) \leq \mu_s(x), \quad -\mu^\ell_{1-s}(x) \leq \lambda_s(\text{Imx}) \leq \mu_s(x), 0 < s < 1. \]

**Proof.** The proof is adapted from [15, Lemma 2.1]. For any \( t > 0 \), we have
\[
t^2 x^* x + \frac{1}{t^2} \| - (x^* + x) = (tx - \frac{1}{t})^*(tx - \frac{1}{t}) \geq 0.
\]
which tell us that
\[
t^2 x^* x + \frac{1}{t^2} \| \geq 2 \text{Rex} \geq -(t^2 x^* x + \frac{1}{t^2} \|). \tag{3.2}
\]
Proposition 2.2(8) now yields
\[
\lambda_s(t^2 x^* x + \frac{1}{t^2} \|) \geq \lambda_s(2 \text{Rex}) \geq \lambda_s(-(t^2 x^* x + \frac{1}{t^2} \|)), 0 < s < 1 \tag{3.3}
\]
for all \( s > 0 \). An easy calculation shows that \( \mu_s(y + \|) = \mu_s(y) + 1 \) and \( \mu^\ell_s(y + \|) = \mu^\ell_s(y) + 1 \) for \( y \geq 0 \) and \( 0 < s < 1 \). Combining this with (3.3) and Proposition 2.2 we can assert that
\[
t^2 \mu_s(x)^2 + \frac{1}{t^2} 1 = \mu_s(t^2 x^* x + \frac{1}{t^2} \|)
\]
\[
= \lambda_s(t^2 x^* x + \frac{1}{t^2} \|)
\]
\[
\geq \lambda_s(2 \text{Rex}), 0 < s < 1.
\]
If it was true that \( \mu_s(x) = 0 \) for some \( s > 0 \), there would be \( 2 \lambda_s(\text{Rex}) = \lambda_s(2 \text{Rex}) \leq 0 \) by take \( t \to \infty \). Otherwise, we take \( t = \frac{1}{\mu_s(x)^2} \), it follows that \( \lambda_s(\text{Rex}) \leq \mu_s(x), 0 < t < 1 \).

On the other hand, combining (3.3) with Proposition 2.2 and Lemma 3.1 yields
\[
\lambda_s(2 \text{Rex}) \geq \lambda_s(-(t^2 x^* x + \frac{1}{t^2} \|))
\]
\[
= -\mu^\ell_{1-s}(t^2 x^* x + \frac{1}{t^2} \|)
\]
\[
= -t^2 \mu^\ell_{1-s}(x^* x) - \frac{1}{t^2}, 0 < s < 1.
\]
We now apply the above argument again, with \( \mu(x) \) replaced by \( \mu^\ell(x) \), to obtain
\[
\lambda_s(\text{Rex}) \geq -\mu^\ell_{1-s}(x), 0 < s < 1. \]
Finally, since \( \text{Re}(-ix) = \text{Imx} \), from what has already been proved we see that \( -\mu^\ell_{1-s}(x) = -\mu^\ell_{1-s}(-ix) \leq \lambda_s(\text{Imx}) \leq \mu_s(-ix) = \mu_s(x), 0 < s < 1. \]

\[ \square \]

**Remark 3.3.** (1) Let \( x \in \mathcal{M} \). From the proof of Lemma 3.1 we have
\[
\lambda_s(\text{Rex}) \leq \mu_s(x), \quad \lambda_s(\text{Imx}) \leq \mu_s(x), 0 < s < 1.
\]
(2) Let \( x \in \mathcal{M} \). It follows from inequality (3.2) that
\[
t^2 x^* x + \frac{1}{t^2} \| \geq 2 \text{Rex} \geq -(t^2 x^* x + \frac{1}{t^2} \|).\]
Moreover, [14, Lemma 4.2] means that $\mu(2Rex) \prec \log \mu(t^2x^*x + \frac{1}{t^2}I)$ for all $t > 0$, with $-\infty$ allowed for values. Moreover, we have
\[
\Delta_r(2Rex) \leq \Delta_r(t^2x^*x + \frac{1}{t^2}I), t > 0.
\]

**Corollary 3.4.** Let $x, y \in \mathcal{M}$ and let $\alpha \in \mathbb{R}$. If $x^* = x$, then
\[
\lambda_s(iRey - iy) \leq \mu_s(y - \alpha x), 0 < s < 1
\]
and
\[
\lambda_s(y - iImy) \leq \mu_s(y - i\alpha x), 0 < s < 1.
\]

**Proof.** The results follow from Remark 3.3(1) along with the fact that $-i(y - Rey) = Imy = Im(y - \alpha x)$ and $y - iImy = Rey = Re(y - i\alpha x)$.

**Proposition 3.5.** Let $0 \leq x \in \mathcal{M}$ such that $\|x\| > 1$.

1. If $u \in \mathcal{M}$ is an unitary operator, then
\[
\mu(x - Reu) \prec \log \mu(x + I),
\]
which implies that
\[
\Delta_r(x - Reu) \leq \Delta_r(x + I).
\]

2. If $u \in \mathcal{M}$ is an unitary operator and $\tau(\|x - I\|) = \tau(\|x - u\|)$, then
\[
\Delta_r(x - u) \leq \Delta_r(x - I).
\]

**Proof.** (1). From $-I \leq -Reu \leq I$, we deduce that $-(x + I) \leq x - I \leq x - Reu \leq x + I$. Then we conclude from [14, Lemma 4.2] that
\[
\mu(x - Reu) \prec \log \mu(x + I).
\]

Hence we see that
\[
\Delta_r(x - Reu) \leq \Delta_r(x + I).
\]

(2). Note that [5, Corollary 2.6] leads to
\[
\mu(x - I) \prec \mu(x - u).
\]

Since $\tau(\|x - I\|) = \tau(\|x - u\|), [3, Theorem 3.3]$ shows that $\tau(\|x - u\|^p) \leq \tau(\|x - I\|^p), 0 < p < 1$, i.e.
\[
\int_0^1 \mu_t(x - u)^p \leq \int_0^1 \mu_t(x - I)^p dt, 0 < p < 1.
\]

Hence, from $\int_0^1 |f(s)|^p ds < \infty$ and [22, p.74] we obtain
\[
\exp\{\int_0^1 \log |f(s)||ds\} = \lim_{p \to 0}(\int_0^1 |f(s)|^p ds)^{\frac{1}{p}},
\]
which force $\Delta_r(x - u) \leq \Delta_r(x - I)$.

**Lemma 3.6.** Let $0 \leq x \in \mathcal{M}$ be invertible. Then
\[
\mu_t(x^{-1}) = \mu_{1-t}(x)^{-1}, 0 < t < 1.
\]
Proof. Without loss of generality, we may assume that $\mathcal{M}$ has no minimal projections (otherwise we consider the von Neumann algebra $\mathcal{M} \otimes L^\infty([0,1])$). First we assume that $x = \sum_{i=1}^{n} \alpha_i p_i$ with $\alpha_1 > \alpha_2 > \cdots > \alpha_n > 0$ and $\sum_{i=1}^{n} p_i = 1, p_i p_j = 0, i \neq j$. Thus $x^{-1} = \sum_{i=1}^{n} \frac{1}{\alpha_i} p_i$. Let $d_i = \sum_{j=1}^{i} \tau(p_j), 1 \leq i \leq n$. Then $d_n = \tau(1) = 1$, 

$$\mu_t(x) = \alpha_1 \chi(0,d_1) + \sum_{i=2}^{n} \alpha_i \chi(d_{i-1},d_i), 0 < t < 1,$$

and 

$$\mu^\ell_t(x^{-1}) = \sum_{i=2}^{n} \frac{1}{\alpha_i} \chi(1-d_{i-1},1-d_i) + \frac{1}{\alpha_1} \chi(1-d_1,1), 0 < t < 1.$$ 

Therefore, 

$$\mu^\ell_t(x^{-1}) = \frac{1}{\mu_{1-t}(x)}, 0 < t < 1.$$

For the general case, let $0 \leq x \in \mathcal{M}$. Since $x^{-1} \in \mathcal{M}$, there exists $\delta > 0$ such that $x = \int_{|x|}^{\infty} \lambda d\epsilon_\lambda(x)$ is the spectral decomposition of $x$. Put 

$$f_k(t) = \sum_{j=1}^{2^n} \left( \delta + \frac{(j-1)a}{2^n} \right) \chi_{[\delta + (j-1)a, \delta + \frac{ja}{2^n}]},$$

where $a = ||x|| - \delta > 0$. Obviously, $0 \leq f_k(t) \leq f_{k+1}(t) \leq t$. Set 

$$x_n = f_n(|x|) = \sum_{j=1}^{2^n} \left( \delta + \frac{(j-1)a}{2^n} \right) e_{[\delta + (j-1)a, \delta + \frac{ja}{2^n}]}(x).$$

Then 

$$x_n^{-1} = f_n(|x|) = \sum_{j=1}^{2^n} \left( \delta + \frac{(j-1)a}{2^n} \right)^{-1} e_{[\delta + (j-1)a, \delta + \frac{ja}{2^n}]}(x).$$

It follows that $||x - x_n|| \leq \frac{\delta}{2^n}$ and 

$$||x^{-1} - x_n^{-1}|| \leq (\delta + \frac{(j-1)a}{2^n})^{-1} - (\delta + \frac{ja}{2^n})^{-1} \leq \frac{1}{\delta^2} \frac{a}{2^n}.$$

Hence we infer from [9, Lemma 3.4] that $\mu_t(x) = \lim_{t \to \infty} \mu_t(x_n)$. On the other hand, picking up a small $\epsilon > 0$, we obtain 

$$\mu^\ell_t(x_n^{-1}) \leq \mu^\ell_{t-\epsilon}(x^{-1}) + \mu^\ell_t(x^{-1} - x_n^{-1}) \leq \mu^\ell_{t-\epsilon}(x^{-1}) + ||x^{-1} - x_n^{-1}||.$$ 

Letting $\epsilon \downarrow 0$ we get 

$$\mu^\ell_t(x_n^{-1}) \leq \mu^\ell_t(x^{-1}) + ||x^{-1} - x_n^{-1}||.$$ 

In consequence, $\limsup_{n \to \infty} \mu^\ell_t(x_n^{-1}) \leq \mu^\ell_t(x^{-1})$. Therefore, $x^{-1} \leq x_n^{-1}$ tells us that 

$$\liminf_{n \to \infty} \mu^\ell_t(x_n^{-1}) \geq \mu^\ell_t(x^{-1}).$$ 

Hence $\mu^\ell_t(x) = \lim_{n \to \infty} \mu^\ell_t(x_n)$. This completes the proof.  \[\square\]
Example 3.7. Let $\mathcal{H} = \mathbb{C}^n$ and let $\mathcal{M} = \mathcal{B}(\mathcal{H}) \cong M_n(\mathbb{C})$ equipped with the normalized trace $\tau_n : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ where $\tau_n$ is the standard trace on $M_n(\mathbb{C})$. If $x \in M_n(\mathbb{C})$ is positive and invertible, then $x$ can be written as $x = \sum_{i=1}^{n} \alpha_j p_j$, where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n > 0$ is the sequence of eigenvalues of $x$ in which each is repeated according to its multiplicity and $\sum_{i=1}^{n} p_j = \mathbb{I}$. The proof of Lemma 3.6 tells us that

$$\mu^{s}_{\tau}(x) = (\alpha_{n+1-k})^{-1} = \mu_{\tau}^{s}(\frac{1}{n}(x^{-1})).$$

We conclude this section with a series of inequalities of generalized singular value function.

Lemma 3.8. Let $x,y \in \mathcal{M}$.

(1) If $x^* = x$, then $\lambda_t(x) \leq \mu_t(x)$.

(2) If $s,t > 0$ such that $s + t < 1$, then $1 \leq \mu_t(x) + \mu_s(\mathbb{I} - x)$ and $1 \leq \mu^s_t(x) + \mu^s_s(\mathbb{I} - x)$.

(3) For any $t > 0$ we have $1 \leq \mu_t(x) + \mu^t_{1-t}(\mathbb{I} - x)$, $1 \leq \mu^t_t(x) + \mu^t_{1-t}(\mathbb{I} - x)$ and $1 \leq \mu^t_t(x) + \mu^t_{1-t}(x \pm i\mathbb{I})$.

(4) If $0 \leq x \in \mathcal{M}$ and $\|x\| \leq 1$, then

$$\mu_t(1-x) = 1 - \mu^t_{1-t}(x), \quad \mu^t_t(1-x) = 1 - \mu^t_{1-t}(x).$$

Proof. (1). Since $-|x| \leq x \leq |x|$, $\lambda_t(x) \leq \lambda_t(|x|) = \mu_x(x)$. (2)-(4) follow from the fact $\mu_{s+t}(x + y) \leq \mu_t(x) + \mu_s(y)$ and $\mu^t_{s+t}(x + y) \leq \mu^t_t(x) + \mu^t_s(y)$. (5). This follows by the same method as in Lemma 3.6. $\square$

4. Harnack type inequality for operator

In this section Harnack type inequalities for operators in Logarithmic submajorisation are stated and proved. We will extend the results of Yang--Zhang[28] and Lin--Zhang[17] to the case of finite von Neumann algebra. We start with a lemma which follows by the same method as in [28, Proposition 2].

Lemma 4.1. Let $x \in \mathcal{M}$. If $\mathbb{I} - x$ is invertible, then

$$(\mathbb{I} - x^*)^{-1}(\mathbb{I} - x^*x)(\mathbb{I} - x)^{-1} = 2Re((\mathbb{I} - x)^{-1} - \mathbb{I})$$

$$= 2Re((\mathbb{I} - x)^{-1} - \frac{1}{2}\mathbb{I})$$

$$= Re((\mathbb{I} + x)(\mathbb{I} - x)^{-1}) = S^*S,$$

where $S = (\mathbb{I} - x^*x)^{\frac{1}{2}}(\mathbb{I} - x)^{-1}$. Moreover, if $x \in \mathcal{M}$ with $\|x\| < 1$, then $\mathbb{I} - x$ is invertible, which implies that the equalities above are true.

Theorem 4.2. Let $x \in \mathcal{M}$ with $\|x\| < 1$. Then

$$\mu_t((\mathbb{I} - x^*)^{-1}(\mathbb{I} - x^*x)(\mathbb{I} - x)^{-1}) \leq \frac{1 + \mu_t(x)}{1 - \mu_t(x)}, 0 < t < 1.$$ (4.1)
Moreover, for any subset $K \subseteq [0, 1]$ we have
\[
\int_K \log \mu_t((\mathbb{I} - x^*)^{-1}(\mathbb{I} - x^*x)(\mathbb{I} - x)^{-1}) dt \leq \int_K \log \frac{1 + \mu_t(x)}{1 - \mu_t(x)} dt \\
\leq \int_0^1 \log \frac{1 + \mu_t(x)}{1 - \mu_t(x)} dt.
\]
In particular,
\[
\frac{\Delta_r(\mathbb{I} - x^*x)}{\Delta_r(\mathbb{I} - x)^2} \leq \exp \int_0^1 \log \frac{1 + \mu_t(x)}{1 - \mu_t(x)} dt.
\]

Proof. We conclude from the definition of $\mu_t(\cdot)$ and $\lambda_t(\cdot)$ that
\[
\mu_t((\mathbb{I} - x^*)^{-1}(\mathbb{I} - x^*x)(\mathbb{I} - x)^{-1}) = \lambda_t((\mathbb{I} - x^*)^{-1}(\mathbb{I} - x^*x)(\mathbb{I} - x)^{-1}) \\
= \lambda_t(2Re((\mathbb{I} - x)^{-1}) - \mathbb{I}) \quad (\text{Lemma } 4.1) \\
= \lambda_t(2Re((\mathbb{I} - x)^{-1})) - 1 \quad (\text{Proposition } 2.2(9)) \\
\leq \mu_t(2(\mathbb{I} - x)^{-1}) - 1 \quad (\text{Remark } 3.3) \\
= \frac{2}{\mu_t^{1-s}(\mathbb{I} - x)} - 1 \quad (\text{Lemma } 3.6) \\
\leq \frac{2}{1 - \mu_t(x)} - 1 \quad (\text{Lemma } 3.8) \\
= \frac{1 + \mu_t(x)}{1 - \mu_t(x)}, \quad 0 < t < 1.
\]

Furthermore, since $\frac{1 + \mu_t(x)}{1 - \mu_t(x)} \geq 1$, (4.1) means that
\[
\int_K \log \mu_t((\mathbb{I} - x^*)^{-1}(\mathbb{I} - x^*x)(\mathbb{I} - x)^{-1}) dt \leq \int_K \log \frac{1 + \mu_t(x)}{1 - \mu_t(x)} dt \\
\leq \int_0^1 \log \frac{1 + \mu_t(x)}{1 - \mu_t(x)} dt.
\]

Finally, by (4.1) and Proposition 2.5(1)-(3), we have
\[
\frac{\Delta_r(\mathbb{I} - x^*x)}{\Delta_r(\mathbb{I} - x)^2} = \Delta_r((\mathbb{I} - x^*)^{-1}(\mathbb{I} - x^*x)(\mathbb{I} - x)^{-1}) \\
= \exp \int_0^1 \log \mu_t((\mathbb{I} - x^*)^{-1}(\mathbb{I} - x^*x)(\mathbb{I} - x)^{-1}) dt \\
\leq \exp \int_0^1 \log \frac{1 + \mu_t(x)}{1 - \mu_t(x)} dt.
\]

To achieve one of our main results, we state for easy reference the following fact, which will be applied below.
**Lemma 4.3.** Let \( x, y \in \mathcal{M} \) be invertible. If \( K \) is a Borel subset of \([0, 1]\) with \( m(K) = t \) (\( m(K) \) denotes the Lebesgue measure of \( K \)), then
\[
\int_K \log \mu_s(x) \, ds + \int_0^t \log \mu_{1-s}(y) \, ds \leq \int_K \log \mu_s(xy) \, ds.
\]

**Proof.** Let \( K^c \) denote the set \( \{ t \in [0, 1] : t \notin E \} \). Then \( m(K^c) = 1 - t \). We conclude from [21, Theorem 2] that
\[
\int_{K^c} \log \mu_s(xy) \, ds \leq \int_{K^c} \log \mu_s(x) + \int_0^{1-t} \log \mu_s(y) \, ds. \tag{4.2}
\]

Note that \( x, y \in \mathcal{M} \) are invertible. By Proposition 2.5(1) and (3) we have \( \Delta(x) \neq 0, \Delta(y) \neq 0 \) and
\[
-\infty < \int_0^1 \log(\mu_s(x)) \, ds + \int_0^1 \log \mu_s(y) \, ds = \int_0^1 \log \mu_s(xy) \, ds < \infty \tag{4.3}
\]
Subtracting (4.2) from (4.3) yields
\[
\int_E \log \mu_s(x) \, ds + \int_{1-t}^1 \log \mu_s(y) \, ds \leq \int_E \log \mu_s(xy) \, ds,
\]
i.e.,
\[
\int_E \log \mu_s(x) \, ds + \int_0^t \log \mu_{1-s}(y) \, ds \leq \int_E \log \mu_s(xy) \, ds. \tag{4.4}
\]

**Remark 4.4.** (1) Let \( x, y \in \mathcal{M} \) and let \( K \) be a Borel subset of \([0, 1]\) with \( m(K) = t \) (here \( m(K) \) denotes the Lebesgue measure of \( K \)). Then
\[
\int_K \log \mu_s(x) \, ds + \int_0^t \log \mu_{1-s}(y) \, ds \leq \int_K \log \mu_s(xy) \, ds.
\]
Indeed, if \( x, y \) are invertible, then it follows from Lemma 4.3. We write \( x = u|x| \) and \( y = v|y| \) for unitary operators \( u, v \in \mathcal{M} \). Then \( z = u|x||y^*|v^* \) and \( \mu_s(x) = \mu_s(|x|) \), \( \mu_s(y) = \mu_s(|y^*|) \), \( \mu_s(z) = \mu_s(|x||y^*|) \). Thus, we may without loss of generality assume \( x \geq 0, y \geq 0 \) and let
\[
z(\epsilon_1, \epsilon_2) = (x + \epsilon_1 \mathbb{I})(y + \epsilon_2 \mathbb{I}).
\]
Note that \( \mu_s(x + \epsilon_1 \mathbb{I}) = \mu_s(x) + \epsilon_1 \) and \( \mu_s(y + \epsilon_2 \mathbb{I}) = \mu_s(y) + \epsilon_2 \). From Lemma 4.3 we see that
\[
\int_K \log(\mu_s(x) + \epsilon_1) \, ds + \int_0^t \log(\mu_{1-s}(y) + \epsilon_2) \, ds \leq \int_K \log \mu_s(z(\epsilon_1, \epsilon_2)) \, ds. \tag{4.4}
\]
Moreover, for any projection operators \( e \in \mathcal{M} \), we have
\[
\|z(\epsilon_1, \epsilon_2)e\|^2 = \|e(y + \epsilon_2 \mathbb{I})(x^2 + 2\epsilon_1 x + \epsilon_1^2 \mathbb{I})(y + \epsilon_2 \mathbb{I})e\|
\]
which implies that \( \mu_s(z(\epsilon_1, \epsilon_2)) \) is decreasing in \( \epsilon_1 \). Similarly, \( \mu_s(z(\epsilon_1, \epsilon_2)) \) is decreasing in \( \epsilon_2 \). Letting \( \epsilon_i \to 0 \) and using the monotone convergence theorem in (4.4), we obtain the desired inequality.
(2) Let $x, y \in \mathcal{M}$ and let $K$ be a Borel subset of $[0, 1]$ with $m(K) = t$ ($m(K)$ denotes the Lebesgue measure of $K$). Combining [21, Theorem 2] with Lemma 4.3 we can assert that

$$
\int_K \log \mu_s(x)ds + \int_0^t \log \mu_{1-s}(y)ds \leq \int_K \log \mu_s(xy)ds
$$

$$
\leq \int_K \log \mu_s(x) + \int_0^t \log \mu_s(y)ds.
$$

In particular, if $K = [0, t]$, then

$$
\int_0^t \log \mu_s(x)ds + \int_0^t \log \mu_{1-s}(y)ds \leq \int_0^t \log \mu_s(xy)ds
$$

$$
\leq \int_0^t \log \mu_s(x) + \int_0^t \log \mu_s(y)ds.
$$

**Theorem 4.5.** Let $x \in \mathcal{M}$ with $\|x\| < 1$. If $K$ is a Borel subset of $[0, 1]$ with $m(K) = t$ ($m(K)$ denotes the Lebesgue measure of $K$), then

$$
\int_K \log \mu_s((I - x^*)^{-1}(I - x^*x)(I - x)^{-1})ds
$$

$$
\geq \int_0^t 2 \log \frac{1}{1 + \mu_s(x)}ds + \int_K \log(1 - \mu_{1-s}(x)^2)ds, \ t > 0.
$$

**Proof.** For convenience, we write $A := (I - x^*)^{-1}(I - x^*x)(I - x)^{-1}$. Since $\|x\| < 1$, $A$ is invertible, hence that $\Delta(A) > 0$. Therefore, $\int_0^1 \log \mu_s(A)ds > -\infty$. Using Lemma 4.3 twice, we have

$$
\int_K \log \mu_s(A)ds \geq \int_0^t 2 \log \mu_{1-s}((I - x)^{-1})ds + \int_K \log \mu_s(I - x^*x)ds.
$$

It follows from Lemma 3.8(3)-(5) and Lemma 3.6 that

$$
\int_K \log \mu_s(A)ds \geq \int_0^t 2 \log \mu_{1-s}((I - x)\) ^{-1})ds + \int_K \log \mu_s(I - x^*x)ds
$$

$$
= \int_0^t 2 \log \frac{1}{\mu_{s}(I - x)} ds + \int_K \log \mu_s(I - x^*x)ds
$$

$$
\geq \int_0^t 2 \log \frac{1}{1 + \mu_{s}(x)} ds + \int_K \log(1 - \mu_{1-s}(x)^2)ds
$$

$$
= \int_0^t 2 \log \frac{1}{1 + \mu_{s}(x)} ds + \int_K \log(1 - \mu_{1-s}(x)^2)ds,
$$

because $\mu_{s}(x) = \mu_{s}(x)$ holds for almost every $t \in [0, 1]$. \[\Box\]

**Corollary 4.6.** Let $x \in \mathcal{M}$ with $\|x\| < 1$. Then

$$
\int_0^t \log \mu_{1-s}((I - x^*)^{-1}(I - x^*x)(I - x)^{-1})ds \geq \int_0^t \log \frac{1 - \mu_{s}(x)}{1 + \mu_{s}(x)} ds, \ t > 0.
$$
In particular,
\[
\frac{\Delta_t(\|I - x^* x\|)}{\Delta_t(\|I - x\|)^2} \geq \exp \int_0^1 \log \frac{1 - \mu_s(x)}{1 + \mu_s(x)} ds.
\]

**Proof.** Replacing \(K\) by \([1 - t, 1]\), in Theorem 4.5 we have
\[
\int_0^t \log \mu_{1-s}(A) ds = \int_0^1 \log \mu_{1-s}(A) ds
\]
\[
\geq \int_0^t 2 \log \frac{1}{1 + \mu_s(x)} ds + \int_{1-t}^1 \log(1 - \mu_{1-s}(x)^2) ds
\]
\[
= \int_0^t 2 \log \frac{1}{1 + \mu_s(x)} ds + \int_0^t \log(1 - \mu_s(x)^2) ds
\]
\[
= \int_0^t \log \frac{1 - \mu_s(x)}{1 + \mu_s(x)} ds.
\]
Therefore, letting \(t \to 1\) yields
\[
\int_0^1 \log \mu_s(A) ds = \int_0^1 \log \mu_{1-s}(A) ds \geq \int_0^1 \log \frac{1 + \mu_s(x)}{1 - \mu_s(x)} ds.
\]
This completes the proof. \(\square\)

**Theorem 4.7.** Let \(0 \leq x_i \in \mathcal{M}\) with \(\|x_i\| < 1\), \(i = 1, 2, \ldots, n\). Then for any unitary operator \(u \in \mathcal{M}\) and positive scalars \(\omega_i, i = 1, 2, \ldots, n\), \(\sum^n_i \omega_i = 1\), we have
\[
\prod^n_{i=1} [\exp \int_0^1 \log \frac{1 - \mu_t(x_i)}{1 + \mu_t(x_i)} dt]^{\omega_i} \leq \frac{\Delta_t(\|I - W\|^2)}{\Delta_t(\|I - uW\|^2)} \leq \prod^n_{i=1} [\exp \int_0^1 \log \frac{1 + \mu_t(x_i)}{1 - \mu_t(x_i)} dt]^{\omega_i},
\]
where \(W = \sum^n_{i=1} \omega_i x_i\).

**Proof.** An easy calculation shows that \(1 - W^2\) and \(1 - uW\) are invertible and \(W \geq 0\) with \(\|W\| < 1\). Theorem 4.2 and Corollary 4.6 tell us that
\[
\exp \int_0^1 \log \frac{1 - \mu_t(x)}{1 + \mu_t(x)} dt \leq \frac{\Delta_t(\|I - x^* x\|^2)}{\Delta_t(\|I - x\|^2)} \leq \exp \int_0^1 \log \frac{1 + \mu_t(x)}{1 - \mu_t(x)} dt. \quad (4.5)
\]
Note that [9, Theorem 4.4] tells us that
\[
\int_0^t \mu_s(W) ds \leq \int_0^t \sum^n_{i=1} \omega_i \mu_s(x_i) ds.
\]
The rest of the proof run as [17, Theorem 5]. For the convenience of the reader, we add a proof. Indeed, the convexity and the monotonicity of the function \(f(t) = \log \frac{1 + t}{1 - t}, 0 \leq t < 1\) mean that
\[
\int_0^t f(\mu_s(W)) ds \leq \int_0^t f(\sum^n_{i=1} \omega_i \mu_s(x_i)) ds.
\]
On the other hand, by Lewent’s inequality ([17, 16]), we obtain
\[
\frac{1 + \sum_{i=1}^{n} \omega_i \mu_s(x_i)}{1 - \sum_{i=1}^{n} \omega_i \mu_s(x_i)} \leq \prod_{i=1}^{n} \left( \frac{1 + \mu_s(x_i)}{1 - \mu_s(x_i)} \right)^{\omega_i}.
\]
Thus
\[
\int_0^t f(\mu_s(W)) ds \leq \int_0^t \log \prod_{i=1}^{n} \left( \frac{1 + \mu_s(x_i)}{1 - \mu_s(x_i)} \right)^{\omega_i} ds = \sum_{i=1}^{n} \omega_i \int_0^t \log \left( \frac{1 + \mu_s(x_i)}{1 - \mu_s(x_i)} \right) ds.
\]
It follows that
\[
\exp \{ \int_0^t \log \left( \frac{1 + \mu_s(W)}{1 - \mu_s(W)} \right) ds \} \leq \prod_{i=1}^{n} \left[ \exp \int_0^1 \log \left( \frac{1 + \mu_s(x_i)}{1 - \mu_s(x_i)} \right) dt \right]^{\omega_i} \quad (4.6)
\]
Moreover, the inequalities in (4.6) reverse by taking reciprocals, which implies
\[
\exp \{ \int_0^t \log \left( \frac{1 - \mu_s(W)}{1 + \mu_s(W)} \right) ds \} \geq \prod_{i=1}^{n} \left[ \exp \int_0^1 \log \left( \frac{1 - \mu_s(x_i)}{1 + \mu_s(x_i)} \right) dt \right]^{\omega_i} \quad (4.7)
\]
Combining (4.5) with (4.6) and (4.7) yields
\[
\prod_{i=1}^{n} \left[ \exp \int_0^1 \frac{1 - \mu_t(x_i)}{1 + \mu_t(x_i)} dt \right]^{\omega_i} \leq \frac{\Delta_t(I - W^2)}{\Delta_t(I - uW)^2} \leq \prod_{i=1}^{n} \left[ \exp \int_0^1 \frac{1 + \mu_t(x_i)}{1 - \mu_t(x_i)} dt \right]^{\omega_i}.
\]
\[
\square
\]
5. Cayley transform with logarithmic submajorisation

In this section, we will consider some logarithmic submajorisation inequalities related to Cayley transform. We will extend some results of Yang- Zhang [28] to the case of finite von Neumann algebra.

Let \( x \in \mathcal{M} \). If \( x + iI \) is invertible, we call \( C(x) = (x - iI)(x + iI)^{-1} \) the Cayley transform of \( x \).

**Theorem 5.1.** Let \( x, y \in \mathcal{M} \) with \( \|x\| < 1, \|y\| < 1 \) and let \( C(x) \) and \( C(y) \) be the Cayley transforms of \( x \) and \( y \), respectively. If \( K \) is a Borel subset of \([0, 1]\) with \( m(K) = t \) (\( m(K) \) denotes the Lebesgue measure of \( K \)), then
\[
\int_K \log(1 - \mu_{1-s}(x)) ds - \int_0^t \log(1 + \mu_s(x)) ds
\]
\[
\leq \int_K \log \mu_s(C(x)) ds
\]
\[
\leq \int_K \log(1 + \mu_s(x)) - \int_0^t \log(1 - \mu_s(x)) ds
\]
and
\[
\int_E \log \mu_s(C(x) - C(y)) ds
\]
\[
\leq \int_E \log 2\mu_s(x - y) ds - \int_0^t \log[(1 - \mu_s(x))(1 - \mu_s(y))] ds.
\]
Proof. Let us first compute the upper bounds. Remark 4.4 shows that

\[ \int_K \log \mu_s(C(x))ds = \int_K \log \mu_s((x - i\mathbb{I})(x + i\mathbb{I})^{-1})ds \leq \int_K \log \mu_s(x - i\mathbb{I})ds + \int_0^t \log \mu_s((x + i\mathbb{I})^{-1})ds. \]

Together with Lemma 3.8 this gives

\[ \int_0^t \log \mu_s((x + i\mathbb{I})^{-1})ds \leq \int_0^t \log[\mu_{1-s}^\ell(x + i\mathbb{I})]^{-1}ds \leq \int_0^t \log(1 - \mu_s(x))^{-1}ds = -\int_0^t \log(1 - \mu_s(x))ds. \]

Thus

\[ \int_K \log \mu_s(C(x))ds \leq \int_K \log(1 + \mu_s(x)) - \int_0^t \log(1 - \mu_s(x))ds. \]

The lower bound follows easily by using Remark 4.4. Indeed, from Remark 4.4 we obtain

\[ \int_E \log \mu_s(C(x))ds \geq \int_0^t \log \mu_1^{-s}(x - i\mathbb{I})ds + \int_E \log \mu_s((x + i\mathbb{I})^{-1})ds \geq \int_0^t \log(1 - \mu_s^\ell(x))ds - \int_E \log \mu_{1-s}^\ell(x + i\mathbb{I})ds \geq \int_0^t \log(1 - \mu_s^\ell(x))ds - \int_E \log(1 + \mu_s^\ell(x))ds \geq \int_0^t \log(1 - \mu_s(x))ds - \int_E \log(1 + \mu_s(x))ds. \]

For the second part, an easy calculation shows that $C(x) = 1 - 2i(x + i\mathbb{I})^{-1}$ and

\[ C(x) - C(y) = 2i(y + i\mathbb{I})^{-1}(x - y)(x + i\mathbb{I})^{-1}. \]
Hence, Remark 4.4 implies that
\[ \int_E \log \mu_s(C(x) - C(y)) ds = \int_E \log 2 \mu_s((y + i\mathbb{I})^{-1}(x - y)(x + i\mathbb{I})^{-1}) ds \]
\[ \leq \int_0^t \log \mu_s((y + i\mathbb{I})^{-1}) ds + \int_E \log 2 \mu_s(x - y) ds \]
\[ + \int_0^t \log \mu_s((x + i\mathbb{I})^{-1}) ds \]
\[ \leq \int_0^t \log [\mu_{1-s}^t (y + i\mathbb{I})^{-1}] ds + \int_E \log 2 \mu_s(x - y) ds \]
\[ + \int_0^t \log [\mu_{1-s}^t (x + i\mathbb{I})^{-1}] ds \]
\[ \leq \int_0^t \log [1 - \mu_s(y)]^{-1} ds + \int_E \log 2 \mu_s(x - y) ds \]
\[ + \int_0^t \log [1 - \mu_s(x)]^{-1} ds \]
\[ = \int_E \log 2 \mu_s(x - y) ds \]
\[ - \int_0^t \log [(1 - \mu_s(x))(1 - \mu_s(y))] ds. \]
\[ \square \]

If we replace \( K \) by \([0, 1]\), in Theorem 5.1 we have the following corollary.

**Corollary 5.2.** Let \( x, y \in \mathcal{M} \) with \( \|x\| < 1, \|y\| < 1 \) and let \( C(x) \) and \( C(y) \) be
the Cayley transforms of \( x \) and \( y \), respectively. Then
\[ \int_0^1 \log \frac{1 - \mu_{1-s}^t (x)}{1 + \mu_s(x)} ds \leq \int_0^1 \log \mu_s(C(x)) ds \leq \int_0^1 \log \frac{1 + \mu_s(x)}{1 - \mu_s(x)} ds \]
and
\[ \int_0^1 \log \mu_s(C(x) - C(y)) ds \leq \int_0^1 \log \frac{2 \mu_s(x - y)}{(1 - \mu_s(x))(1 - \mu_s(y))} ds. \]

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REFERENCES

1. Arveson, W. B.: Analyticity in operator algebras. Amer. J. Math. 89, 578-642 (1967)
2. Blecher, D. P., Labuschagne, L. E.: Applications of the Fuglede-Kadison determinant: Szego’s theorem and outer for noncommutative $H^p$. Trans. Amer. Math. Soc. 360, 6131-6147 (2008)
3. Bekjan, T. N., Raikhan, M.: An Hadamard-type inequality. Linear Algebra and its Application, 443, 228-234 (2014)
4. Brown, L. G.: Lidskii theorem in the type II case, Geometric methods in operator algebras. (Kyoto, 1983), 1-35, Pitman Res. Notes Math. Ser., 123, Longman Sci. Tech., Harlow, (1986)
5. Dodds, P. G., Dodds, T. K.-Y.: Unitary approximation and submajorization. Proc. Centre Math. Appl. Austral. Nat. Univ., Austral. Nat. Univ., Canberra, 29, 42-57, (1992)
6. Dodds, P. G., de Pagter, B., Sukochev, F.: Theory of noncommutative integration. unpublished manuscript.
7. Dodds, P. G., Dodds, T. K., Sukochev, F. A., Zanin, D.: Logarithmic submajorization, uniform majorization and H"older type inequalities for $\tau$-measurable operators, Indag. Math. http://doi.org/10.1016/j.indag.2020.02.004, (2020)(in press)
8. Fack, T.: Proof of the conjecture of A. Grothendieck on the Fuglede-Kadison determinant. J. Funct. Anal. 50, 215-228 (1983)
9. Fack, T., Kosaki, H.: Generalized s-numbers of $\tau$-measurable operators. Pac. J. Math. 123, 269-300 (1986)
10. Hua, L. -K.: Inequalities involving determinants. Acta Math. Sinica 5 (4), 463-470 (1955) (in Chinese). Translated into English: Transl. Amer. Math. Soc. Ser. II 32 265-272 (1963)
11. Hua, L. -K.: On an inequality of Harnack’s type. Sci. Sin. 14, 791 (1965).
12. Hiai, F.: Majorization and stochastic maps in von Neumann algebras. J. Math. Anal. Appl., 127, 18-48 (1987)
13. Hiai, F.: Log-majorizations and norm inequalities for exponential operators. Banach Center Publications, 38, 119-181 (1997)
14. Huang, J., Sukochev, F., Zanin, D.: Logarithmic submajorisation and order-preserving linear isometries. Journal of Functional Analysis 278, 108352(2020)
15. Jiang, Z., Lin, M.: A Harnack type eigenvalue inequality. Linear Algebra Appl. 585, 45-49 (2020)
16. Lin, M.: A Lewent type determinantal inequality. Taiwanes J. Math., 17, 1303-1309 (2013)
17. Lin, M., Zhang, F.: An extension of Harnack type determinantal inequality. Linear Multilinear Algebra 65, 2024-2030 (2017)
18. Marcus, M.: Harnack’s and Weyl’s inequalities. Proc. Amer. Math. Soc. 16, 864-866 (1965)
19. Kassmann, M.: Harnack inequalities: an introduction. Boundary Value Problems, 2007, 21 (2007)
20. Marshall, A. W., Olkin, I.: Inequalities: theory of majorization and its applications, Academic Press, New York, (1979)
21. Nakamura, Y.: An inequality for generalized s-numbers. Integral Equations and Operator Theory, 10, 140-145 (1987)
22. Rudin, W.: Real and complex analysis. McGraw-Hill, (1974)
23. Tung, S. H.: *Harnack’s inequality and theorems on matrix spaces.* Proc. Amer. Math. Soc. 15, 375-381 (1964)
24. Takesaki, M.: *Theory of Operator Algebras I.* Springer-Verlag, New York, (1979)
25. Ovchinnikov, V. I.: *s-numbers of measurable operators.* Functional Analysis and Its Applications, 4, 236-242 (1970)
26. V. I. Ovchinnikov, *Symmetric spaces of measurable operators.* Dokl. Akad. Nauk SSSR, 191, 769-771 (1970)
27. Wang, F.-Y.: *Harnack inequalities for stochastic partial differential equations.* Springer (2013)
28. Yang, C., Zhang, F.: *Harnack type inequalities for matrices in majorization.* Linear Algebra and its Applications, 588, 196-209 (2020)

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