Weyl-type bounds for twisted GL(2) short character sums

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Abstract
Let $f$ be a Hecke–Maass or holomorphic primitive cusp form of full level for $SL(2, \mathbb{Z})$ with normalized Fourier coefficients $\lambda_f(n)$. Let $\chi$ be a primitive Dirichlet character of modulus $p$, a prime. In this article, we shorten the range of cancellation for $N$ in the twisted $GL(2)$ short character sum. Here, we consider the problem of cancellation in short character sum of the form
\[ S_{f,\chi}(N) := \sum_{n \in \mathbb{Z}} \lambda_f(n) \chi(n) W\left( \frac{n}{N} \right). \]
We show that, for $0 < \theta < \frac{1}{10}$,
\[ S_{f,\chi}(N) \ll f, \varepsilon N^{3/4+\theta} p^{1/6} (pN)^{\varepsilon} + N^{1-\theta} (pN)^{\varepsilon}, \]
which is non-trivial if $N \geq p^{2/3+\alpha+\varepsilon}$ where $\alpha = \frac{4\theta}{1-6\theta}$. Previously, such a bound was known for $N \geq p^{3/4+\varepsilon}$.

Keywords Automorphic forms · Short character sums · Weyl bound · Maass forms · Holomorphic forms · Fourier coefficients of $GL(2)$ cusp forms

Mathematics Subject Classification Primary 11F66 · 11F30 · Secondary 11F55 · 11M41

1 Introduction

A problem which arises in a variety of contexts is the cancellation in sums of the form
\[ S_{\chi}(N) = \sum_{n \in \mathbb{Z}} \chi(n) W\left( \frac{n}{N} \right). \]
and

\[ S_{f, \chi}(N) = \sum_{n \in \mathbb{Z}} \lambda_f(n) \chi(n) W\left( \frac{n}{N} \right), \quad (2) \]

where \( \chi \) is a character of conductor \( p \), \( \lambda_f(n) \)’s are normalized Fourier coefficients of a Hecke–Maass or holomorphic primitive cusp form \( f \) for \( SL(2, \mathbb{Z}) \) and \( W \) is a smooth bump function supported on \([1, 2]\).

By applying the Mellin inversion formula

\[
W(x) = \frac{1}{2\pi i} \int_{\sigma} \hat{W}(s)x^{-s} \, ds, \quad \sigma > 1,
\]

we see that Eq. (1) becomes

\[
S_{\chi}(N) = \frac{1}{2\pi i} \int_{\sigma} N^s \hat{W}(s)L(s, \chi) \, ds, \quad (3)
\]

where \( L(s, \chi) \) is the Dirichlet \( L \)-function. We can shift the contour to the central line \( \sigma = \frac{1}{2} \). As the Mellin transform \( \hat{W}(s) \) decays rapidly on the vertical line, the main contribution to the integral comes from the points near the centre \( \sigma = \frac{1}{2} \).

For example, plugging in a bound

\[
|L\left( \frac{1}{2} + it, \chi \right)| \ll p^{\varepsilon}(2 + |t|)^A,
\]

we get

\[
S_{\chi}(N) \ll \sqrt{N}p^{\varepsilon}.
\]

In particular, if we take the convexity bound

\[
L(1/2 + it, \chi) \ll \varepsilon p^{1/4+\varepsilon}(2 + |t|)^{1/4+\varepsilon},
\]

then we conclude that \( S_{\chi}(N) \ll N^{1/2}p^{1/4}p^{\varepsilon} \) which is non-trivial if and only if \( N > p^{1/2+\varepsilon} \). Here, one can note that the convexity bound recovers the conclusion of the Polya–Vinogradov inequality. Hence, subconvexity corresponds to cancellation in shorter sums. Burgess (see [6]) proved that \( L(1/2, \chi) \ll \varepsilon p^{3/16+\varepsilon} \) which yields a non-trivial bound if and only if \( N > p^{3/8+\varepsilon} \). Burgess’s method required new ideas, in particular it uses the Riemann Hypothesis for curves over finite fields. Note that the Burgess exponent of \( 3/16 \) falls short of the exponent \( 1/6 \) found by Weyl. However, Burgess’s method yields a non-trivial bound for \( S_{\chi}(N) \) for any \( N \gg p^{1/4+\varepsilon} \) if \( p \) is cube free (especially for primes). This does not come through the passage to \( L \)-functions as we have sketched above. But this basic idea is applicable to the scenarios as well, invoking higher rank harmonics. Recently, Petrow and Young (see [24]) proved a Weyl-exponent subconvex bound for any Dirichlet \( L \)-function of cube-free conductor.
They also got a bound of the same strength for certain $L$-functions of self-dual $GL(2)$ automorphic forms that arise as twists of forms of smaller conductor. One can also see the recent work of Nelson (see [22]). Curiously, the exponent $3/16$ often re-occurs in the modern incarnations of these problems, see [1, 3, 5, 7, 11, 25, 26] as examples. Also related work on the Burgess-type bounds can be found in the paper of Munshi (see [20]). For the case of Dirichlet $L$-functions, the Burgess bound has only been improved in some limited special cases. In a breakthrough, B. Conrey and H. Iwaniec (see [8]) obtained a Weyl quality bound for quadratic characters of odd conductor using techniques from automorphic forms and P. Deligne’s solution of the Weil conjectures for varieties over finite fields. Another class of results, such as [2, 12], consider situations where the conductor $q$ of $\chi$ runs over prime powers or otherwise has some special factorizations. Notably, D. Milicic (see [18]) recently obtained a sub-Weyl subconvex bound when $q = p^n$ with $n$ large. Also for the Weyl bound for short twisted sums in the prime power case, one can look the papers by Blomer and Milicevic (see [4]) and a related paper of Munshi and Singh (see [21]). Here a subconvex bound of the form $L(1/2, \pi) \ll_{\epsilon} Q(\pi)^{1/6+\epsilon}$ is the Weyl bound, where $Q(\pi)$ is the analytic conductor of the automorphic $L$-function $L(1/2, \pi)$. The Weyl bound is only known in a few cases, notably for quadratic twists of certain self-dual $GL(2)$ automorphic forms; see [8, 13, 23, 27].

For the $GL(2)$ case we have the following:

$$S_{f,\chi}(N) = \frac{1}{2\pi i} \int_{(\sigma)} N^3 \tilde{W}(s)L(s, f \otimes \chi)ds. \tag{4}$$

Again by shifting the contour to the central line $\sigma = 1/2$, as $\tilde{W}(s)$ decays rapidly on the vertical line, the main contribution to the integral comes from the points near the contour $\sigma = 1/2$. Similarly plugging in a bound

$$|L\left(\frac{1}{2} + it, f \otimes \chi\right)| \ll p^\epsilon (3 + |t|)^A,$$

we have

$$S_{f,\chi}(N) \ll_{f,\epsilon} \sqrt{N} p^\epsilon. \tag{5}$$

In this context, the convexity bound is $L\left(\frac{1}{2} + it, f \otimes \chi\right) \ll_{f,\epsilon} p^{\frac{1}{2}+\epsilon} (3 + |t|)^{\frac{1}{2}+\epsilon}$. So we have $S_{f,\chi}(N) \ll_{f,\epsilon} \sqrt{N} p^{\frac{1}{2}+\epsilon}$ which becomes non-trivial if and only if $N > p^{1+\epsilon}$. Further improvement can be done. By the Burgess exponent (see [7]), we have

$$L\left(\frac{1}{2}, f \otimes \chi\right) \ll_{f,\epsilon} p^{3/8+\epsilon}.$$ 

Hence, we have

$$S_{f,\chi}(N) \ll_{f,\epsilon} \sqrt{N} p^{\frac{3}{8}+\epsilon} < N \iff N > p^{3/4+\epsilon}.$$
which may be called the Burgess range.

In this paper, we will analyise the sum \( S_{f,\chi}(N) \) using a version of \( \delta \)-method, without going into \( L \)-functions. Our method improves the range of cancellation from \( N > p^{3/4+\varepsilon} \) (Burgess range) to \( N > p^{2/3+\varepsilon} \) (Weyl range). Here, we get the following result:

**Theorem 1** Let \( f \) be a Hecke–Maass or holomorphic primitive cusp of form full level for \( SL(2, \mathbb{Z}) \). Let \( \chi \) be a primitive Dirichlet character of modulus \( p \), a prime. Then for any \( \varepsilon > 0 \) and \( 0 < \theta < \frac{1}{10} \), we have

\[
S_{f,\chi}(N) \ll f, \varepsilon N^{3/4+\theta/2} p^{1/6}(pN)^\varepsilon + N^{1-\theta}(pN)^\varepsilon,
\]

which becomes non-trivial if \( p^{2/3+\varepsilon} \leq N \leq p \), where \( \alpha = \frac{4\theta}{1-6\theta} \).

Though this result is implicit in the paper of Munshi (see [19]), we are doing here explicitly. Actually in that paper (see [19]), his aim is to get a subconvexity bound for \( L(1/2 + it, f \otimes \chi) \) but here our aim is to get a range for \( N \) to have a non-trivial bound or more precisely getting cancellation in our twisted \( GL(2) \) short character sum. Here, we are using the same strategy and ideas developed in the paper of Munshi (see [19]). We will only present the case of holomorphic cusp forms for \( SL(2, \mathbb{Z}) \) as for the Maass forms one can see Munshi’s paper (see [19]) which carried out the Maass form case in details. The case for Maass forms is just similar as we only need the Ramanujan bound in the \( L^2 \)-sense.

**Remark 1** Here we are considering \( p \) to be a prime number for simplicity but also one can do for \( p \) when \( p \) is not a prime (one has to handle coprimality issue carefully) using the same method.

### 1.1 Sketch of the proof

We shall describe our method briefly by taking \( f \) to be a holomorphic primitive cusp form for \( SL(2, \mathbb{Z}) \). Here, we are using the method of Munshi (see [19]). At first we consider the sum

\[
S := \sum_{n \sim N} \lambda_f(n) \chi(n),
\]

for \( N > p^{2/3+\varepsilon} \) for some \( \varepsilon > 0 \), where \( \lambda_f(n) \)'s are the normalized Fourier coefficients of \( f \) and \( p \) is the conductor of \( \chi \). Here in the sketch, we will suppress the weight function for notational simplicity. Then, we write this sum as

\[
S = \sum_{n,m \sim N} \lambda_f(n) \chi(m) \delta_{n,m},
\]

where \( \delta_{n,m} \) is the Kronecker \( \delta \)-symbol. Here to get an inbuilt bilinear structure in the circle method itself, we need to use a more flexible version of the circle method—the
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one investigated by Jutila (see [14, 15]). This version comes with an error term which is satisfactory, as we shall find out, as long as we allow the moduli to be slightly larger than $\sqrt{N}$ (see Section 3). Up to an admissible error we see that $S$ is given by

$$ S = \sum_{n,m \sim N} \lambda_f(n) \chi(m) \int_{\mathbb{R}} \tilde{I}(\alpha) e((n-m)\alpha) d\alpha, $$

where $\tilde{I}(\alpha) := \frac{1}{2\delta L} \sum_{q \in \Phi} \sum_{(\mod{q})} I_{d/q}(\alpha)$ and $I_{d/q}$ is the indicator function of the interval $[\frac{d}{q} - \delta, \frac{d}{q} + \delta]$, $Q := N^{1/2+\varepsilon}$ and $L \asymp Q^{2-\varepsilon}$ (see Sect. 2.3).

Trivial bound at this stage yields $N^{1/2+\varepsilon}$ and we need to establish the bound $N^{1-\theta}$ for some $\theta > 0$, i.e. roughly speaking we need to save $N$. Observe that by our choice of $Q$, there is no analytic oscillation in the weight function $e((n-m)\alpha)$. Hence their weights can be dropped in our sketch. At first using the $GL(2)$ Voronoi summation formula on the $n$ sum, we get that

$$ \sum_{n \sim N} \lambda_f(n) e\left(\frac{na}{q}\right) \approx N q \sum_{n \sim \frac{Q^2}{N}} \lambda_f(n) e\left(\frac{-n\bar{a}}{q}\right), $$

where $q$ is of size $Q = N^{1/2+\theta}$. The left hand side is trivially bounded by $N$, whereas the right hand side is trivially bounded by $Q$. Hence, we have “saved” $N Q = \sqrt{N}$. Now applying the Poisson summation formula to the $m$ sum, we arrive at

$$ \sum_{m \sim N} \chi(m) e\left(-\frac{ma}{q}\right) \approx \frac{N \tau \chi}{p} \sum_{|m| \leq \frac{pQ}{N}} \chi(m) \chi(q) \mathbb{1}_{a \equiv \bar{m} \bar{p} (\mod{q})}, $$

where $\mathbb{1}_{a \equiv \bar{m} \bar{p} (\mod{q})}$ is the indicator function for $a \equiv \bar{m} \bar{p} (\mod{q})$ on $\mathbb{Z}$. Comparing the trivial contribution of the two sides, we observe that we have “saved”

$$ \frac{N}{\sqrt{pQ}} \times \sqrt{Q} = \frac{N}{\sqrt{p}}. $$

With this the above sum roughly reduce to

$$ S \approx \frac{N^2}{Q^3 p^{1/2}} \sum_{q \in \Phi} \sum_{n \sim N^{1/2}} \sum_{m \sim \frac{pN}{\sqrt{N}}} \lambda_f(n) \tilde{\chi}(m) \chi(q) e\left(-\frac{\bar{m}np}{q}\right). $$

So far we have “saved” $N^{1/2-\theta} \times N = \frac{N^{3/2-\theta}}{\sqrt{p}}$. Hence our job is to “save” $\frac{N}{\sqrt{N^{3/2-\theta}}} = \frac{N}{\sqrt{p}}$ in the above sum.

Next we choose $Q = Q_1 Q_2$ and take the set of moduli $\Phi$ to be a product of two sets of primes so that (as was done in the Sects. 2.3, 5) $q = q_1 q_2$ in a certain unique
way with $q_1 \leq Q_1$ and $q_2 \leq Q_2$ (see Sect. 5). Then applying the Cauchy–Schwarz inequality, we arrive at

$$\sum_{q_1 \sim Q_1} \sum_{m \sim pN^\theta} \left| \sum_{n \sim N^{2\theta}} \sum_{q_2 \sim Q_2} \lambda_f(n) \chi(q_2) e\left(-\frac{\overline{mnp}}{q_1q_2}\right) \right|^2.$$ 

Next we open the absolute value square and apply the Poisson summation formula to the $m$-sum (after appropriate smoothing). Here the diagonal is of length $Q_2 N^{2\theta}$ and so the contribution of the zero frequency is given by $\ll pN^{4\theta}$. Hence, the diagonal contribution is satisfactory if

$$Q_2 N^{2\theta} > \frac{p}{N^{1-2\theta}}, \quad \text{i.e.} \quad Q_2 > \frac{p}{N}.$$ 

Also the contribution of the off-diagonal is given by $\ll pN^{6\theta}$. Note that this is satisfactory if

$$\frac{pN^{\theta/2}}{N^{3/4} \sqrt{Q_2}} > \frac{p}{N^{1-2\theta}} \iff Q_2 < N^{1/2-3\theta}.$$ 

So we have a choice for $Q_2$ if

$$\frac{p}{N} < N^{1/2-3\theta} \Rightarrow p < N^{3/2-3\theta}.$$ 

Hence as long as $N > p^{2/3+\epsilon}$ for some $\epsilon > 0$, then the above method yields a non-trivial bound for $S$.

**Notation** In this article, ‘$\ll$’ will mean that whenever it occurs, the implied constants will depend on $f$, $\epsilon$ only and the notation ‘$X \asymp Y$’ will mean that $Y p^{-\epsilon} \leq X \leq Y p^\epsilon$.

### 2 Preliminaries

#### 2.1 Preliminaries on holomorphic cusp forms

Let $f : \mathbb{H} \rightarrow \mathbb{C}$, be a holomorphic cusp form with normalized Fourier coefficients $\lambda_f(n)$. Also we take $\chi$, a primitive Dirichlet character of modulus $p$ where $p$ is a prime.

#### 2.2 Voronoi summation formula

We will use the following Voronoi summation formula. This was first established by Meurman (see [17]) in the case of full level.
Lemma 2.1 Let $f$ be as above, and $v$ be a compactly supported smooth function on $(0, \infty)$. Also consider $(a, q) = 1$. Then, we have

$$
\sum_{n=1}^{\infty} \lambda_f(n)e_q(an)v(n) = \frac{1}{q} \sum_{n=1}^{\infty} \lambda_f(n)e_q(-\bar{a}n)V(n),
$$

where $\bar{a}$ is the multiplicative inverse of $a$ mod $q$, and $V(n)$ is a certain integral Hankel transform of $v$.

Here note that, if we take $v$ to be supported in $[Y, 2Y]$ and satisfying $y^j v(j)(y) \ll j^1$, then one can see that the sum on the right hand side of (6) becomes being supported essentially on $n \ll q^2(qY)^\epsilon / Y$ (the implied constant depends only on $f$ and $\epsilon$). Also note that the terms with $n \gg q^2(qY)^\epsilon / Y$ contribute an amount which is negligibly small. For smaller values of $n$, one can consider the trivial bound $V(n/q^2) \ll Y$. For more details, one can see the paper of Munshi (see [19]).

2.3 Circle method

Here, in this paper, we shall use Jutila’s circle method (see [14–16]). For any set $S \subset \mathbb{R}$, let $I_S$ denote the associated characteristic function, i.e. $I_S(x) = 1$ for $x \in S$ and 0 otherwise. For any collection of positive integers $\Phi \subset [Q, 2Q]$ (which we call the set of moduli), where $Q \geq 1$ and a positive real number $\delta$ in the range $Q^{-2} \ll \delta \ll Q^{-1}$, we define the function

$$
\tilde{I}_{\Phi, \delta}(x) := \frac{1}{2\delta L} \sum_{q \in \Phi} \sum_{d \mod q} I_{[\frac{d}{q} - \delta, \frac{d}{q} + \delta]}(x),
$$

where $I_{[\frac{d}{q} - \delta, \frac{d}{q} + \delta]}$ is the indicator function of the interval $[\frac{d}{q} - \delta, \frac{d}{q} + \delta]$. Here $L := \sum_{q \in \Phi} \phi(q)$ (then roughly we have $L \asymp Q^2$) and we will choose $\Phi$ in such a way that $L \asymp Q^{2-\epsilon}$.

Then this becomes an approximation of $I_{[0, 1]}$ in the following sense:

Lemma 2.2 We have

$$
\int_{\mathbb{R}} \left| I_{[0, 1]}(x) - \tilde{I}_{\Phi, \delta}(x) \right|^2 dx \ll \frac{Q^{2+\epsilon}}{\delta L^2},
$$

where $I$ is the indicator function of $[0, 1]$.

This is a consequence of the Parseval theorem from Fourier analysis (see [14]).

3 Setting-up the circle method

Let us apply the circle method directly to the smooth sum
\[ S(N) = \sum_{n \in \mathbb{Z}} \lambda_f(n) \chi(n) h_1\left(\frac{n}{N}\right), \]

where the function \( h_1 \) is smooth, supported in \([1, 2]\) with \( h_1^{(j)}(x) \ll j \). Now we will approximate the above sum \( S(N) \) using Jutila’s circle method (see \([14, 15]\)) by the following sum:

\[ \tilde{S}(N) = \frac{1}{L} \sum_{q \in \Phi} \sum_{a \mod q} \lambda_f(m) e\left(\frac{a(n-m)}{q}\right) F(n, m), \]

where \( e_q(x) = e^{2\pi i x/q} \), and

\[ F(x, y) = h_1\left(\frac{x}{N}\right) h_2\left(\frac{y}{N}\right) \frac{1}{2\delta} \int_{-\delta}^{\delta} e(\alpha(n-m)) d\alpha. \]

Here, \( h_2 \) is another smooth function having compact support in \((0, \infty)\), with \( h_2(x) = 1 \) for \( x \) in the support of \( h_1 \). Also we choose \( \delta = N^{-1} \) so that we have

\[ \frac{\partial^{i+j}}{\partial x^i \partial y^j} F(x, y) \ll i, j \frac{1}{N^{i+j}}. \]

Then, we have the following lemma:

**Lemma 3.1** Let \( \Phi \subset [Q, 2Q] \), with

\[ L = \sum_{q \in \Phi} \phi(q) \ll Q^{-\epsilon}, \]

and \( \delta = \frac{1}{N} \gg \frac{N^{2\theta}}{Q^2} \). Then we must have

\[ S(N) = \tilde{S}(N) + O_{f, \epsilon} \left(\sqrt{N} N(QN)^\epsilon Q^{-\theta - \epsilon}\right). \]

**Proof** For the proof of this lemma one can see Lemma 3 of \([19]\). \(\square\)

We will choose the set of moduli in Sect. 5. We shall pick the set of the moduli to be \( Q = N^{1/2 + \theta} \). Hence the error term getting from the previous lemma is \( O(N^{1-\theta+\epsilon}) \) for some \( \theta > 0 \). Now we shall proceed towards the estimation of \( \tilde{S}(N) \).

### 4 Estimation of \( \tilde{S}(N) \)

#### 4.1 Applying summation formulae

Let us now assume that each member of \( \Phi \) is coprime to \( p \), the modulus of the character \( \chi \). Let us define

\[ \tilde{S}(N) = \frac{1}{L} \sum_{q \in \Phi} \sum_{a \mod q} \lambda_f(m) e\left(\frac{a(n-m)}{q}\right) F(n, m), \]

where \( e_q(x) = e^{2\pi i x/q} \), and

\[ F(x, y) = h_1\left(\frac{x}{N}\right) h_2\left(\frac{y}{N}\right) \frac{1}{2\delta} \int_{-\delta}^{\delta} e(\alpha(n-m)) d\alpha. \]

Here, \( h_2 \) is another smooth function having compact support in \((0, \infty)\), with \( h_2(x) = 1 \) for \( x \) in the support of \( h_1 \). Also we choose \( \delta = N^{-1} \) so that we have

\[ \frac{\partial^{i+j}}{\partial x^i \partial y^j} F(x, y) \ll i, j \frac{1}{N^{i+j}}. \]

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**Lemma 3.1** Let \( \Phi \subset [Q, 2Q] \), with

\[ L = \sum_{q \in \Phi} \phi(q) \ll Q^{-\epsilon}, \]

and \( \delta = \frac{1}{N} \gg \frac{N^{2\theta}}{Q^2} \). Then we must have

\[ S(N) = \tilde{S}(N) + O_{f, \epsilon} \left(\sqrt{N} N(QN)^\epsilon Q^{-\theta - \epsilon}\right). \]

**Proof** For the proof of this lemma one can see Lemma 3 of \([19]\). \(\square\)

We will choose the set of moduli in Sect. 5. We shall pick the set of the moduli to be \( Q = N^{1/2 + \theta} \). Hence the error term getting from the previous lemma is \( O(N^{1-\theta+\epsilon}) \) for some \( \theta > 0 \). Now we shall proceed towards the estimation of \( \tilde{S}(N) \).
\[
\tilde{S}_x(N) = \frac{1}{L} \sum_{q \in \Phi} \sum_{a \pmod{q}}^* S(a, q, x, f) T(a, q, x, \chi), \quad (7)
\]

where

\[
S(a, q, x, f) := \sum_{n \in \mathbb{Z}} \lambda_f(n) h_1\left(\frac{n}{N}\right)e\left(\frac{an}{q}\right)e(nx),
\]

and

\[
T(a, q, x, \chi) := \sum_{m \in \mathbb{Z}} \chi(m)e\left(-\frac{am}{q}\right) h_2\left(\frac{m}{N}\right)e(-xm),
\]

with \(|x| < \delta\). Then, we have

\[
\tilde{S}(N) = (2\delta)^{-1} \int_{-\delta}^{\delta} \tilde{S}_x(N) dx.
\]

Let us first study the \(n\)-sum using the Voronoi summation formula.

\[
S(a, q, x, f) = \sum_{n=1}^{\infty} \lambda_f(n) h_1\left(\frac{n}{N}\right)e\left(\frac{an}{q}\right)e(nx). \quad (8)
\]

Then, we have the following lemma:

**Lemma 4.1** We have

\[
S(a, q, x, f) = \frac{N^{3/4}}{q^{1/2}} \sum_{|n| \leq \frac{Q^2}{\sqrt{N}}} \lambda_f(n) n^{1/4} e\left(-\frac{\tilde{a}n}{q}\right) \mathcal{I}_1(n, x, q) + O(N^{-2021}), \quad (9)
\]

where \(q \in [Q, 2Q]\), coprime with \(p\) and \(\mathcal{I}_1(n, x, q)\) is given by

\[
\mathcal{I}_1(n, x, q) := \int_{\mathbb{R}} h_1(y)e\left(Nxy \pm \frac{4\pi}{q} \sqrt{Nny}\right) W\left(\frac{4\pi \sqrt{Nny}}{q}\right) dy,
\]

where \(W\) is a smooth nice function.

**Proof** Applying the Voronoi summation formula (6) to the \(n\)-sum of Eq. (8), then we have

\[
\sum_{n \in \mathbb{Z}} \lambda_f(n) e\left(\frac{an}{q}\right) e(nx) h_1\left(\frac{n}{N}\right)
= \frac{1}{q} \sum_{n \in \mathbb{Z}} \lambda_f(n) e\left(-\frac{\tilde{a}n}{q}\right) \int_{\mathbb{R}} h_1\left(\frac{y}{N}\right) e(xy) J_{k-1}\left(\frac{4\pi \sqrt{Nny}}{q}\right) dy,
\]

where \(J_k\) is a Bessel function of the first kind.
where $J_{k-1}$ is the Bessel function. By changing $y \mapsto Ny$ and using the decomposition,

$$J_{k-1}(x) = \frac{W(x)}{\sqrt{x}} e(x) + \frac{\bar{W}(x)}{\sqrt{x}} e(-x),$$

where $W(x)$ is a nice function, the right-hand side integral becomes

$$N^{3/4} q^{1/2} \int_{\mathbb{R}} h_1(y) e\left(Nxy \pm \frac{4\pi}{q} \sqrt{N}y\right) W\left(\frac{4\pi \sqrt{N}y}{q}\right) dy.$$ 

By repeated integral by parts we see that, this integral is negligibly small if $|n| \gg \frac{Q^2}{N}N^\epsilon$. Hence the lemma follows.

**Remark 2** Note that $x \asymp \frac{\sqrt{n}}{\sqrt{N}q}$, otherwise $I_1(n, x, q)$ is negligibly small.

Now let us consider the $m$-sum of (7) given by

$$T(a, q, x, \chi) = \sum_{m \in \mathbb{Z}} \chi(m) e\left(-\frac{am}{q}\right) h_2\left(\frac{m}{N}\right) e(-xm),$$

for which we have the following lemma:

**Lemma 4.2** We have

$$T(a, q, x, \chi) = \frac{N \tau_x}{\sqrt{p}} \sum_{\substack{|m| \ll \frac{Q}{N} \chi(\bar{m} \mod q) \equiv a \mod q}} \chi(m) \chi(q) I_2(m, x, q) + O(N^{-2021}),$$

where

$$I_2(m, x, q) := \int_{\mathbb{R}} h_2(y) e(-Nxy) e\left(-\frac{mNy}{pq}\right) dy.$$ 

**Proof** To the $m$-sum in Eq. (10), we apply the Poisson summation formula to get that

$$T(a, q, x, \chi) = \frac{N}{pq} \sum_{m \in \mathbb{Z}} I_2(m, x, q) \sum_{\beta \equiv a \mod q} \chi(\beta) e\left(-\frac{a\beta}{q}\right) e\left(\frac{m\beta}{pq}\right),$$

where

$$I_2(m, x, q) := \int_{\mathbb{R}} h_2(y) e(-Nxy) e\left(-\frac{mNy}{pq}\right) dy.$$ 

Here note that, this integral is negligibly small if $|m| \gg \frac{Q}{N}N^\epsilon$. 

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So we have
\[
T(a, q, x, \chi) = \frac{N}{pq} \sum_{|m| \ll \frac{pq}{N}} \mathcal{I}_2(m, x, q) \sum_{\beta \pmod{pq}} \chi(\beta) e\left(-\frac{a\beta}{q}\right) e\left(\frac{m\beta}{pq}\right)
+ O(N^{-2021}).
\]

As we know \((p, q) = 1\), so that we can write \(\beta = \beta_1 q\bar{q} + \beta_2 p\bar{p}\), where \(\beta_1, \beta_2\) run through a complete set of residue classes congruent to \(p, q\) respectively. Then substituting these in the place of \(\beta\), we have
\[
T(a, q, x, \chi) = \frac{N\tau_x}{p} \sum_{|m| \ll \frac{pq}{N}} \overline{\chi}(m) \chi(q) \mathcal{I}_2(m, x, q) + O(N^{-2021}).
\]

This completes the proof. \(\square\)

From (9) and (11), we get

**Proposition 1** We have
\[
\tilde{S}_x(N) = \frac{N^{7/4}}{\sqrt{pL}} \sum_{q \in \Phi} \frac{\chi(q)}{q^{1/2}} \sum_{|n| \ll \frac{q^{1/4}}{N}} \sum_{\lambda_f(n) \ll \frac{q^{1/4}}{N}} \frac{\lambda_f(n)}{n^{1/4}} \overline{\chi}(m) e\left(-\frac{p\bar{m}n}{q}\right)
\times \mathcal{I}_1(n, x, q) \mathcal{I}_2(m, x, q) (N^{-2021}),
\]

(12)

where \(\mathcal{I}_1(n, x, q), \mathcal{I}_2(m, x, q)\) are given by (9), (11) respectively.

**5 Further estimation**

**5.1 Applying the Cauchy and Poisson summation formulae**

We choose the set of moduli \(\Phi\) to be the product set \(\Phi_1 \Phi_2\), where \(\Phi_i\) consists of primes in the dyadic segment \([Q_i, 2Q_i]\) (and not dividing \(p\)) for \(i = 1, 2\) and \(Q_1 Q_2 = Q = N^{1/2+\theta}\). Also we pick \(Q_1\) and \(Q_2\) (whose optimal sizes will be determined later) so that the collections \(\Phi_1\) and \(\Phi_2\) are disjoint. Now consider \(M_0 := \frac{pQ}{N}, N_0 := \frac{Q^2}{N}\). Here we note that, as \(0 < \theta < 1/10\) so that we have \(Q_2 > N_0\) and also we have \(Q_1 > N_0\).

Now applying the Cauchy–Schwarz inequality to Eq. (12), we arrive at
\[
\tilde{S}_x(N) \ll \frac{N^{7/4} \sqrt{M_0}}{\sqrt{pL} \sqrt{Q_1}}
\times \sum_{q_1 \in \Phi_1} \left( \sum_{|m| \ll M_0} \sum_{q_2 \in \Phi_2} \frac{\chi(q_2)}{q_2^{1/2}} \sum_{|n| \ll N_0} \frac{\lambda_f(n)}{n^{1/4}} \mathcal{I}_1(n, x, q_1 q_2) \mathcal{I}_2(m, x, q_1 q_2) e\left(-\frac{p\bar{m}n}{q_1 q_2}\right) \right)^{1/2}
\]
where \(\Omega(N_0, q_1, Q_2, x)\) is defined as

\[
\Omega(N_0, q_1, Q_2, x) = \sum_{q_2, q_2' \in \Phi_2} \frac{\chi(q_2 q_2')}{(q_2 q_2')^{1/2}} \times \sum_{|n|, |n'| \ll N_0} \frac{\lambda_f(n) \lambda_f(n')}{(nn')^{1/4}} \mathcal{I}_1(n, x, q_1 q_2) \mathcal{I}_2(m, x, q_1 q_2) e \left( -\frac{p_m n}{q_1 q_2} \right),
\]

where

\[
\Delta = \sum_{M_1 \ll M_0} \sum_{m \in \mathbb{Z}} W'(\frac{m}{M_1}) e \left( \frac{m p(q_2' n - n' q_2)}{q_1 q_2 q_2'} \right) \mathcal{I}_2(m, x, q_1 q_2) \mathcal{I}_2(m, x, q_1 q_2'),
\]

\(W'(x)\) is a non-negative smooth function supported on \([2/3, 3]\) with \(W'(x) = 1\) for \(x \in [1, 2]\) and \(W^{(j)}(x) \ll j\).

Now applying the Poisson summation formula to the \(m\)-sum it transforms into

\[
\frac{M_1}{q_1 q_2 q_2'} \sum_{m \in \mathbb{Z}} S(p(q_2' n - n' q_2), m; q_1 q_2 q_2') \mathcal{I}(m, x, q_1, q_2, q_2'),
\]

where

\[
\mathcal{I}(m, x, q_1, q_2, q_2') := \int_{\mathbb{R}} W'(y) \mathcal{I}_2(M_1 y, x, q_1 q_2) \mathcal{I}_2(M_1 y, x, q_1 q_2') e \left( -\frac{m M_1 y}{q_1 q_2 q_2'} \right) dy.
\]

Here note that the integral \(\mathcal{I}\) is negligilbly small if \(|m| \gg \frac{Q_1 Q_1^2}{M_1} N^\epsilon = \frac{O_2 Q_1}{M_1} N^\epsilon\).

Let \(R_1 = \frac{O_2 Q_1}{M_1}\). So we get

\[
\hat{S}_x(N) = \frac{N^{7/4} \sqrt{M_0}}{\sqrt{p} L \sqrt{Q_1}} \sum_{q_1 \in \Phi_1} \Omega(N_0, q_1, Q_2, x)^{1/2} + O(N^{-2021}),
\]

\(\hat{S}_x(N)\) is negligilbly small if \(|m| \gg \frac{Q_1 Q_1^2}{M_1} N^\epsilon = \frac{O_2 Q_1}{M_1} N^\epsilon\).
where

\[
\Omega(N_0, q_1, Q_2, x) = \sum_{M_1 \ll M_0} \frac{M_1}{q_1} \sum_{q_2, q_2' \in \Phi_2} \frac{\chi(q_2 q_2')}{(q_2 q_2')^{3/2}} \sum_{|n|, |n'| \ll N_0} \frac{\lambda_f(n) \lambda_f(n')}{(nn')^{1/4}} \\
\times \mathcal{I}(n, x, q_1 q_2) \mathcal{I}(n', x, q_1 q_2') \\
\times \sum_{|m| \ll R_1} S(p(q_2' n - n' q_2), m; q_1 q_2 q_2').
\]

**Remark 3** Let us recall the Rankin–Selberg bound for Fourier coefficients. If \(\lambda_f(n)\) be the normalised Fourier coefficients of a holomorphic cusp form, or of a Maass form \(f\). Then for any real number \(x \geq 1\), we have

\[
\sum_{1 \leq n \leq x} |\lambda_f(n)|^2 \ll_f x.
\]

Moreover, by the work of Deligne (see [9]) and Deligne and Serre (see [10]) (the latter is for \(k = 1\)), the Ramanujan conjecture for holomorphic cusp forms is now well known:

\[
\lambda_f(n) \ll n^\epsilon.
\]

**Lemma 5.1** We have

\[
\Omega(N_0, q_1, Q_2, x) \ll M_0 N_0^{1/2} + N_0^{3/2} Q_2^2 \sqrt{Q_1}.
\]  \quad (16)

The proof of this lemma is given below. The first term of right hand side of (16) is coming from \(m = 0\) and the second term is coming from other \(m\)’s, i.e. for the terms with \(m \neq 0\). For the proof, at first we consider the zero frequency case, i.e. when \(m = 0\).

**The Zero Frequency** The zero frequency \(m = 0\) has to be treated differently. Let \(\Sigma_0\) denote the contribution of the zero frequency to \(\tilde{S}_x(N)\), i.e.

\[
\Sigma_0 = \sum_{M_1 \ll M_0} \frac{M_1}{q_1} \sum_{q_2, q_2' \in \Phi_2} \frac{\chi(q_2 q_2')}{(q_2 q_2')^{3/2}} \sum_{|n|, |n'| \ll N_0} \frac{\lambda_f(n) \lambda_f(n')}{(nn')^{1/4}} S(p(q_2' n - n' q_2), 0; q_1 q_2 q_2') \\
\times \mathcal{I}(n, x, q_1 q_2) \mathcal{I}(n', x, q_1 q_2') \mathcal{I}(0, x, q_1, q_2, q_2').
\]

\[
= \sum_{M_1 \ll M_0} \frac{M_1}{q_1} \sum_{q_2, q_2' \in \Phi_2} \frac{\chi(q_2 q_2')}{(q_2 q_2')^{3/2}} \sum_{|n|, |n'| \ll N_0} \frac{\lambda_f(n) \lambda_f(n')}{(nn')^{1/4}} \\
\times \int_{\{(q_1 q_2 q_2', p(q_2' n - n' q_2))\}} d\mu \left(\frac{p(q_2' n - n' q_2)}{d}\right)
\]

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\[ \times I_1(n, x, q_1q_2) \overline{I_1}(n', x, q_1q_2') I(0, x, q_1, q_2, q_2') \]

\[ = \sum_{M_1 \ll M_0} \sum_{q_1}^{M_1} \sum_{q_2, q_2' \in \Phi_2} \frac{\chi(q_2q_2')}{(q_2q_2')^{3/2}} \sum_{d|q_1q_2q_2'} \frac{\lambda_f(n)\lambda_f(n')}{(nn')^{1/4}} \sum_{|n|, |n'| \ll N_0} \chi(n') \times I_1(n, x, q_1q_2) \overline{I_1}(n', x, q_1q_2') I(0, x, q_1, q_2, q_2'). \tag{17} \]

**Lemma 5.2** We have

\[ \Sigma_0 \ll M_0 N_0^{1/2}. \tag{18} \]

**Proof** For \( m = 0 \), we have six cases according to the divisors of \( q_1q_2q_2' \) and note that \( Q = Q_1Q_2 \).

**Case 1** Let \( d = q_1q_2q_2' \). Then note that size of \( d \) for this case is \( Q_1Q_2^3 \). But size of \( q_2'n - n'q_2 \) is \( Q_2N_0 \). So for this case

\[ q_2'n - n'q_2 \equiv 0 \pmod{d} \iff q_2 = q_2' \text{ and } n = n', \]

as size of \( n \) is smaller than size of \( Q_2 \). Hence we have, using the well-known pointwise Ramanujan bound, given in remark 3,

\[ \Sigma_0 \ll \sup_{M_1 \ll M_0} M_1 N_0^{1/2}, \]

as there are atmost \( \log M_0 (\ll p^6) \) many \( M_1 \)'s. Hence, we have

\[ \Sigma_0 \ll M_0 N_0^{1/2}. \]

**Case 2** Let \( d = 1 \). Then we get that, as done in the previous case,

\[ \Sigma_0 \ll \sup_{M_1 \ll M_0} \frac{M_1 N_0^{3/2}}{Q} \ll \frac{M_0 N_0^{3/2}}{Q}. \]

But as \( N_0 \ll Q \) so for this case, we must have

\[ \Sigma_0 \ll M_0 N_0^{1/2}. \]

**Case 3** Let \( d = q_1 \). For this case, we have

\[ \Sigma_0 \ll \sup_{M_1 \ll M_0} \frac{M_1 N_0^{3/2}}{Q_2} \ll \frac{M_0 N_0^{3/2}}{Q_2}. \]
But as \( N_0 < Q_2 \) so that for this case again, we have
\[
\Sigma_0 \ll M_0 N_0^{1/2}.
\]

**Case 4** Now consider \( d = q_2 \). For this case, we have
\[
\Sigma_0 \ll \sup_{M_1 \ll M_0} \frac{M_1 N_0^{3/2}}{Q_1} \ll \frac{M_0 N_0^{3/2}}{Q_1}.
\]
But as \( N_0 < Q_1 \) so for this case, we must have
\[
\Sigma_0 \ll M_0 N_0^{1/2}.
\]

For the case \( d = q'_2 \), we have to process similarly and we will get the same bound.

**Case 5** Now take \( d = q_1 q_2 \). But as size of \( q'_2 n - n' q_2 \) is \( Q_2 N_0 \) which is less than the size of \( d \), i.e. \( Q_1 Q_2 \) so for this case, we have
\[
q'_2 n - n' q_2 \equiv 0 \pmod{d} \iff q_2 = q'_2 \text{ and } n = n'.
\]
Hence, we have
\[
\Sigma_0 \ll \sup_{M_1 \ll M_0} \frac{M_1 N_0^{1/2}}{Q_2} \ll \frac{M_0 N_0^{1/2}}{Q_2}.
\]
But then again we have, for this case,
\[
\Sigma_0 \ll M_0 N_0^{1/2}.
\]

For \( d = q_1 q'_2 \) if we process similarly then we shall get the same bound.

**Case 6** For the last case, we have \( d = q_2^2 \). This case will be similar as case 5. By considering the size of \( d \) for this case again we can say that
\[
q'_2 n - n' q_2 \equiv 0 \pmod{d} \iff q_2 = q'_2 \text{ and } n = n'.
\]
So for this case we have
\[
\Sigma_0 \ll \sup_{M_1 \ll M_0} \frac{M_1 N_0^{1/2}}{Q_1} \ll \frac{M_0 N_0^{1/2}}{Q_1}.
\]
Hence we have, for this case,
\[
\Sigma_0 \ll M_0 N_0^{1/2}.
\]
This completes the proof of the Lemma 5.2.
Non-zero frequency Now we will consider the non-zero frequency case, i.e. when \( m \neq 0 \). In this case, we will need the following basic lemma:

**Lemma 5.3** For any \( x, y \in \mathbb{R} \) with \( x, y \geq 1 \) and \( c \in \mathbb{N} \), we have

\[
\sum_{1 \leq a \leq x} \sum_{1 \leq b \leq y} (a, b, c) \leq (xy)^{1+\epsilon}.
\]

**Proof** Let \((a, b) = d\) so that \( d \geq 1\). Then, we have

\[
\sum_{1 \leq a \leq x} \sum_{1 \leq b \leq y} (a, b, c) \leq \sum_{1 \leq a \leq x} \sum_{1 \leq b \leq y} (a, b) \leq \left( \sum_{1 \leq d \leq x} \sum_{1 \leq b \leq \frac{y}{d}} 1 \right) (xy)^{\epsilon} = (xy)^{1+\epsilon}.
\]

This completes the proof of this lemma. \( \square \)

Here, the contribution of the non-zero frequency to \( \tilde{S}_x(N) \) is given by the following:

\[
\text{Sigma}_{\neq 0} = \sum_{M_1 < M_0} \frac{M_1}{q_1} \sum_{q_2, q_2' \in \Phi_2} \frac{\chi(q_2 q_2')}{(q_2 q_2')^{3/2}} \\
\times \sum_{|n|, |n'| \ll N_0} \frac{\lambda_f(n) \lambda_f(n')}{(nn')^{1/4}} I_1(n, x, q_1 q_2) I_1(n', x, q_1 q_2') \\
\times \sum_{1 \leq |m| \ll R_1} S(p(q_2' n - n' q_2), m; q_1 q_2 q_2') I(m, x, q_1, q_2, q_2'). \tag{19}
\]

By the Weil’s bound for Kloosterman sums, we arrive at

\[
\sum_{1 \leq |m| \ll R_1} S(p(q_2' n - n' q_2), m; q_1 q_2 q_2') I(m, x, q_1, q_2, q_2') \\
\ll Q_2 \sqrt{Q_1} \sum_{1 \leq |m| \ll R_1} (p(q_2' n - n' q_2), m; q_1 q_2 q_2')^{1/2}.
\]

Then by the previous lemma, we have

\[
\sum_{1 \leq |m| \ll R_1} (p(q_2' n - n' q_2), m; q_1 q_2 q_2')^{1/2} \ll R_1^{1+\epsilon}.
\]

Hence, we get that

\[
\sum_{|m| \ll R_1} S(p(q_2' n - n' q_2), m; q_1 q_2 q_2') I(m, x, q_1, q_2, q_2') \ll R_1^{1+\epsilon} Q_2 \sqrt{Q_1}. \tag{20}
\]
Now putting values of \( R_1 \) we get that, using the well-known pointwise Ramanujan bound, given in Remark 3,

\[
\Sigma_{\neq 0} \ll \sup_{M_1 \ll M_0} \frac{M_1}{Q_1} \times \frac{1}{Q_2} \times N_0^{3/2} \times \frac{Q_2 Q_1}{M_1} \times Q_2 \sqrt{Q_1}
\]
as there are atmost \( \log M_0 (\ll p^\epsilon) \) many \( M_1 \)'s. So we have

\[
\Sigma_{\neq 0} \ll N_0^{3/2} Q_2^2 \sqrt{Q_1}.
\]

This completes the proof of the Lemma 5.1.

### 6 Final estimation

From Eqs. (15) and (16), we get that

\[
\tilde{S}_x(N) \ll \frac{N^{7/4} \sqrt{M_0}}{\sqrt{pL} \sqrt{Q_1}} \sum_{q_1 \sim Q_1} \left( M_0 N_0^{1/2} + N_0^{3/2} Q_2^2 \sqrt{Q_1} \right)^{1/2}
\]

\[
\ll \frac{N^{7/4} \sqrt{Q_1 M_0}}{\sqrt{pL}} \left( M_0^{1/2} N_0^{1/4} + N_0^{3/4} Q_2 Q_1^{1/4} \right).
\]

This completes the proof of the Theorem 1.
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