NAVARRO’S GALOIS–MCKAY CONJECTURE FOR THE PRIME 2

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Abstract. We complete the proof of the McKay–Navarro conjecture (also known as the Galois–McKay conjecture) for the prime 2, by completing the proof of the inductive McKay–Navarro conditions introduced by Navarro–Späth–Vallejo in this situation.

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In the last several decades, much of the progress in the character theory of finite groups has been inspired by the so-called “local-global” conjectures, which posit various relationships between the character theory of a finite group $G$ and the character theory of (or properties of) certain “local” subgroups, such as the normalizer $N_G(Q)$ of a Sylow $\ell$-subgroup $Q$ for a prime $\ell$. One of the most famous of these conjectures is the McKay conjecture, which suggests that there should exist a bijection $\text{Irr}_\ell(G) \leftrightarrow \text{Irr}_\ell(N_G(Q))$ between the set of irreducible complex characters of $G$ whose degrees are not divisible by the prime $\ell$ and the corresponding set for $N_G(Q)$. Despite its seemingly innocent formulation, this conjecture has proven to be extremely elusive, with experts still seeking a fundamental reason for its validity. In a landmark paper by Isaacs–Malle–Navarro [IMN07], the McKay conjecture was reduced to a set of stronger conditions on simple groups, now often known as the “inductive McKay” conditions. With this in place, Malle–Späth and Späth [MS16, Spä22] were able to complete the proof of the McKay conjecture for the primes 2 and 3, respectively, by proving the inductive conditions for simple groups.

In 2004, Navarro noted that a bijection in the McKay conjecture should exist that holds a much stronger property, relating not just the character degrees but also the other character values. To be more precise, let $\mathcal{G}$ denote the Galois group $\text{Gal}(\overline{\mathbb{Q}}(e^{2\pi i/|G|})/\mathbb{Q})$, which acts naturally on the set $\text{Irr}(G)$ via $\chi^\sigma(g) := \sigma(\chi(g))$ for each $g \in G$ and $\sigma \in \mathcal{G}$, and let $\mathcal{H}_\ell \leq \mathcal{G}$ denote the subgroup consisting of those $\sigma \in \mathcal{G}$ for which there is some nonnegative integer $e$ such that $\sigma(\zeta) = \zeta^{\ell^e}$ for every root of unity $\zeta$ of order not divisible by $\ell$. Then Navarro’s refinement of the McKay conjecture [Nav04], which is often known as the McKay–Navarro conjecture or the Galois–McKay conjecture, is as follows:

McKay–Navarro Conjecture (Navarro 2004). Let $G$ be a finite group, let $\ell$ be a prime, and let $Q \in \text{Syl}_\ell(G)$. Then there exists an $\mathcal{H}_\ell$-equivariant bijection $\text{Irr}_\ell(G) \leftrightarrow \text{Irr}_\ell(N_G(Q))$.

This conjecture has recently inspired a large amount of study on the action of Galois automorphisms on $\text{Irr}(G)$ for various groups and the fields of values of characters, see e.g. [SV20, SFT22, SFV19]. It has also led to the proof of many interesting statements about properties of $G$ that one can determine from the character table, which would have followed from the McKay–Navarro conjecture or its blockwise version, see e.g. [NTT07, SF19, NV17, NTV19, SFT18, NTT19, Mal19, RSF20, NRSF21].

In [NSV20], Navarro–Späth–Vallejo reduced the McKay–Navarro conjecture to proving the so-called *inductive Galois–McKay or inductive McKay–Navarro* conditions for the Schur covering...
The Inductive McKay–Navarro Conditions

1. Let $q$ be a power of an odd prime. Then the inductive McKay–Navarro conditions hold for the prime $\ell = 2$ for the following simple groups:

1. sporadic groups and groups of Lie type defined over $\mathbb{F}_q$ with exceptional Schur multipliers;
2. the alternating groups $\mathfrak{A}_n$ with $n \geq 5$;
3. the groups of Lie type $D_n(q)$ and $3D_n(q)$ for $n \geq 4$; $A_n(q)$ and $2A_n(q)$ for $n \geq 2$; and $E_6(q)$ and $F_4(q)$.

Corollary B. The McKay–Navarro conjecture holds for the prime $\ell = 2$.

We make a few remarks. We first note that, thanks to the results mentioned above, Corollary B follows from Theorem A. We also remark that we include $\mathfrak{E}_6(q)$ and $\mathfrak{F}_4(q)$, since it can be addressed naturally alongside the others listed. The inductive McKay–Navarro conditions for $\ell = 2$ for sporadic and alternating groups were originally checked by C. Vallejo (private communication), and we discuss them briefly in Section 2.

The remainder of the paper is structured as follows. In Section 1, we recall the inductive McKay–Navarro conditions. In Section 2, we prove Theorem A (1) and (2). In Section 3, we introduce some notation to be used in the remaining sections, which are devoted to the groups in Theorem A. We discuss odd-degree characters of these groups and their rationality in Section 4, which is used in Section 5 to prove the “equivariant” part of the inductive McKay–Navarro conditions. Finally, we prove the “extension” part of the conditions in Section 6, completing the proof of Theorem A.

1. The Inductive McKay–Navarro Conditions

Fix a prime $\ell$, and write $\mathcal{H} := \mathcal{H}_\ell$. For a character $\chi$, let $\chi^\mathcal{H}$ denote the orbit under $\mathcal{H}$ of $\chi$. When a group $A$ acts on a set $\mathcal{X}$, we will write $A_\omega$ for the stabilizer of the element $\omega \in \mathcal{X}$ under $A$. So, for example, for a finite group $G$, $Q \subset \mathcal{X}$, and $\chi \in \text{Irr}(G)$, the set $\text{Aut}(G)_Q$ is the set of automorphisms normalizing $Q$, and $\text{Aut}(G)_{Q,\chi^\mathcal{H}}$ is the stabilizer of the orbit $\chi^\mathcal{H}$. That is, $\text{Aut}(G)_{Q,\chi^\mathcal{H}}$ is the set of $\alpha \in \text{Aut}(G)_Q$ such that $\chi^\alpha = \chi^\sigma$ for some $\sigma \in \mathcal{H}$.

The inductive McKay–Navarro conditions [NSV20, Definition 3.1] require two main parts. The “equivariant” condition can be phrased as follows

**Condition ‡.** Let $G$ be a finite quasisimple group and let $Q \subset \text{Syl}_\ell(G)$. Then there is a proper $\text{Aut}(G)_Q$-stable subgroup $M$ of $G$ with $N_G(Q) \leq M$ and an $\text{Aut}(G)_Q \times \mathcal{H}$-equivariant bijection $\Omega : \text{Irr}_{\ell}(G) \rightarrow \text{Irr}_{\ell}(M)$ such that corresponding characters lie over the same character of $\mathbb{Z}(G)$.

For the inductive McKay–Navarro conditions to hold for a finite nonabelian simple group $S$, we require that its universal covering group $\tilde{G}$ satisfies Condition ‡ and that

$$(G \rtimes \text{Aut}(G)_{Q,\chi^\mathcal{H}}, G, \chi)_\mathcal{H} \succeq (M \rtimes \text{Aut}(G)_{Q,\chi^\mathcal{H}}, M, \Omega(\chi))_\mathcal{H}$$
for all \( \chi \in \text{Irr}_P(G) \) in the notation of [NSV20, Definition 1.5]. The latter is equivalent (see [Joh22a, Lemma 1.32]) to the following “extension” condition, which corresponds to [NSV20, Definition 1.5 (iii) and (iv)].

**Condition \( \mathcal{S} \).** For all \( \chi \in \text{Irr}_P(G) \), there exist projective representations \( \mathcal{P} \) and \( \mathcal{P}' \) of \( G \rtimes \text{Aut}(G) Q, \chi \) and \( M \rtimes \text{Aut}(G) Q, \chi \) associated to \( \chi \) and \( \Omega(\chi) \), respectively, with entries in \( Q^{ab} \) satisfying:

1. The two corresponding factor sets take root of unities and coincide on \( (M \rtimes \text{Aut}(G) Q, \chi) \times (M \rtimes \text{Aut}(G) Q, \chi) \) and the scalar matrices \( \mathcal{P}(c) \) and \( \mathcal{P}'(c) \) correspond to the same scalar for each \( c \in C_{G\rtimes\text{Aut}(G)} Q, \chi \) \( (G) \) (see [NSV20, Lemma 1.4]) agree on \( M \rtimes \text{Aut}(G) Q, \chi \).

In practice, we are often able to replace the groups \( G \rtimes \text{Aut}(G) Q, \chi \) and \( M \rtimes \text{Aut}(G) Q, \chi \) with more convenient groups inducing the same set of automorphisms, using [NSV20, Theorem 2.9]. This will be especially useful in our proof of Theorem A.3.

2. **Alternating Groups, Sporadic Groups, and Groups with Exceptional Schur Multipliers**

We collect here some results that can be completed more computationally and/or follow directly from previous results.

**Proposition 2.1.** Theorem A holds for the groups listed in Theorem A(1) and (2).

*Proof.* Let \( S \) be one of the simple groups listed. We first note that it suffices to consider the 2'-covering group of \( S \), using e.g. [Joh22a, Lemma 4.15].

First let \( S = A_n \) be an alternating group with \( n \geq 8 \), so that \( \text{Aut}(S) = \mathfrak{S}_n \), both \( S \) and \( \text{Aut}(S) \) have self-normalizing Sylow 2-subgroups, and the abelianization of these Sylow 2-subgroups are elementary abelian (see e.g. [NRSFV21, Lemma 4.1]). The group \( S \) is known to satisfy the inductive McKay conditions for the prime 2 by [Mal08, Theorem 1.1] and to satisfy the McKay–Navarro conjecture by [Nat09]. In fact, every member of \( \text{Irr}_2(S) \) is fixed by \( H_2 \) by [Nat09, Theorem 2.3]. Since it is well known that the characters of \( \mathfrak{S}_n \) are rational-valued and since the same is true for the abelianization of a Sylow 2-subgroup by the first sentence, it follows that the inductive McKay–Navarro conditions are satisfied.

Johannson [Joh22a] has completed the cases of the groups \( A_6 \cong B_2(2)' \), \( G_2(3) \), and the groups that can be identified with groups of Lie type with exceptional Schur multiplier in defining characteristic. This leaves the sporadic groups, \( A_7 \), \( PSU_4(3) \), and \( B_2(3) \). (We remark that the sporadic groups were also checked by C. Vallejo - private communication.) These groups can be dealt with using the character tables in GAP, building on the fact that they satisfy the inductive McKay conditions by [Mal08]; the knowledge of the action of \( \text{Aut}(S) \) on the Schur multiplier (see [GLS98, Table 6.3.1]); the fact that in many cases, the Sylow 2-subgroup is self-normalizing; and arguing as in [Joh22a, Proposition 5.13].

We discuss more on the case of \( PSU_4(3) \), as the Schur multiplier and outer automorphism group in this case are more complicated. For \( S = PSU_4(3) \), we consider the odd covering group \( G := 3^2.S \), and note \( \text{Out}(S) = \text{Dih}_8 \) is the dihedral group of size 8. In this case, \( S \) has a self-normalizing Sylow 2-subgroup, so for \( Q \in Syl_2(G) \), we have \( N := N_G(Q) = Q \rtimes Z(G) \). We also remark that the members of \( \text{Irr}_2(S) \) and \( \text{Irr}_2(Q) \) are \( H \)-invariant. Hence we can see the action of \( H \) directly from the character table for \( G \), and from the action on \( C_3 \times C_3 \) for \( N \). We can partition \( \text{Irr}_2(G) \) and \( \text{Irr}_2(N) \) into 5 sets each, based on the kernels of the characters when restricted to \( Z(G) \), which are either \( Z(G) \) or a copy of \( C_3 \), and we can see that such a partitioning can be chosen equivariant with respect to the automorphisms that preserve that kernel, using that \( S \) satisfies the inductive McKay conditions [Mal08] and we can further see directly from the action of \( H \) that this is \( H \)-equivariant.

Note that \( \text{Aut}(S) \) acts faithfully on \( Z(G) \) (see [GLS98, Table 6.3.1]), and this action is described, for example, in [LI71]. Namely, if we let \( \delta \) denote a diagonal automorphism inducing
the action of PGU\(_4(3)/\) PSU\(_4(3)\) and \(\varphi\) a field automorphism of order 2, so that Dih\(_8\) = \(\langle \delta, \varphi \rangle\), we have \(\delta\) acts on one copy of \(C_3\) by inversion and trivially on the other, and \(\varphi\) interchanges the copies of \(C_3\). Further, the members of Irr\(_\tau\)(\(S\)) are stable under \(\varphi\) and there are two pairs (of degrees 35 and 315) that are interchanged by \(\delta\). Observing the corresponding values for Irr\(_\tau\)(\(G\)), we are able to see that it is possible to define our map in such a way that it is Aut\(\langle S\rangle \times H\)-equivariant. From here, since |Out\(\langle S\rangle\)| is a 2-group and Irr\(_\tau\)(\(N\)) is comprised of linear characters, we may argue similarly to the case of \(B_2(2)'\) in [Joh22a] Proposition 5.13 to obtain the desired extensions, completing the proof. \(\square\)

3. Notation and Preliminaries

From now on, we set our sights on the groups of Lie type defined in odd characteristic. We begin by setting some notation that we will use throughout.

3.1. Characters. Given finite groups \(X \leq Y\), we write Irr\((X)\) for the irreducible ordinary characters of \(X\), Irr\((X|\chi)\) for the irreducible constituents of the restriction Res\(\chi^X\) for \(\chi \in\) Irr\((Y)\), and Irr\((Y|\psi)\) for the set of irreducible constituents of the induced character Ind\(\chi^X\) for \(\psi \in\) Irr\((X)\). If \(X \not< Y\) and \(\mathcal{X}\) is a subset of Irr\((X)\), an extension map for \(\mathcal{X}\) with respect to \(X \not< Y\) is a map \(\Lambda: \mathcal{X} \to \bigcup_{\psi \in \mathcal{X}}\text{Irr}(Y|\psi)\) such that for each \(\psi \in \mathcal{X}\), the character \(\Lambda(\psi)\) is an extension of \(\psi\) to \(Y|\psi\).

3.2. Groups of Lie Type. Throughout, we let \(G\) be a connected, reductive algebraic group over \(\overline{\mathbb{F}}_p\) for some prime \(p\), and let \(q\) be a power of \(p\). Let \(F: G \to \mathbb{G}\) be a Frobenius endomorphism giving \(G\) an \(\mathbb{F}_q\)-rational structure, and write \(G := G^F\) for the set of fixed points, giving a finite group of Lie type defined over \(\mathbb{F}_q\). (Note that the case of Suzuki and Ree groups has been completed in [Joh22a], so we do not need to consider the case of more general Steinberg endomorphisms here.)

We let \(T \leq B\) be an \(F\)-stable maximal torus inside an \(F\)-stable Borel subgroup, and further write \(T := T^F\). We will call such a \(T\) a maximally split torus of \(G\). We will write \(W := \text{NG}(T)/T\) for the Weyl group of \(G\), let \(W := W^F\), and let \(\Phi\) denote the root system for \(G\) with respect to \(T\). We use the notation \(n\alpha(t')\), \(h\alpha(t')\), and \(x\alpha(t)\) to denote the Chevalley generators as in the Chevalley relations [GLS98] Theorem 1.12.1] for \(t \in \mathbb{F}_p\), \(t' \in \mathbb{F}_p^\times\), and \(\alpha \in \Phi\).

Letting \(N :=\text{NG}(T)\), we have \(N = TV\) and \(\text{NG}(T) = \langle T, V \rangle\) for the so-called extended Weyl group \(V := \{n\alpha(1) \mid \alpha \in \Phi\} \leq \text{NG}(T)\) defined as in [MS13] Section 3.A]. We then have \(H := V \cap T := \langle h\alpha(-1) \mid \alpha \in \Phi\rangle\) is an elementary abelian 2-group.

3.3. Harish-Chandra Theory. More generally, we can consider an \(F\)-stable Levi subgroup \(L \leq P\) inside an \(F\)-stable parabolic subgroup of \(G\), and write \(L := L^F\). If \(\lambda \in \text{Irr}(L)\), we write \(R^\lambda_L(\lambda)\) for the Harish-Chandra induced character - that is, the character obtained by inflating \(\lambda\) to \(P^F\) and then inducing to \(G\). If \(\lambda\) is moreover cuspidal, we call the pair \((L, \lambda)\) a cuspidal pair for \(G\).

Given a cuspidal pair \((L, \lambda)\), we have a bijection between the set \(E(G, L, \lambda)\) of irreducible constituents of \(R^\lambda_L(\lambda)\) and the irreducible characters of the relative Weyl group \(W(\lambda) := \text{NG}_\mathbb{F}_2(L)/L\), and we fix such a bijection throughout. (See [Car93] Section 10.6). With this established, we denote by \(R^\lambda_L(\eta)\) the constituent of \(R^\lambda_L(\lambda)\) corresponding to the character \(\eta \in \text{Irr}(W(\lambda))\). Now, the group \(W(\lambda)\), which is not necessarily a reflection group, can be decomposed as a semidirect product \(W(\lambda) = R(\lambda) \rtimes C(\lambda)\), where \(R(\lambda)\) is a Weyl group with some root system \(\Phi_\lambda\) and \(C(\lambda)\) is the stabilizer of a simple system in \(\Phi_\lambda\). (See [Car93] Section 10.6.)

3.4. Duality. Now, associated to \((G, F)\) is a dual pair \((G^*, F^*)\) with maximally split torus \(T^*\) dual to \(T\), and we write \(G^* := (G^*)^F\). To ease notation, we write \(F\) also for the Frobenius \(F^*\). Given a semisimple element \(s \in G^*\), we may define \(W^s(\lambda)\) to be the Weyl group of \(C^s_G(\lambda) := C^s_G(\lambda)^{s}\) with respect to \(T^*\) and \(W(\lambda)\) to be the Weyl group of \(C^s_G(\lambda)\) with respect to \(T^*\). Then we have \(W(s) = W^s(\lambda) \rtimes A(s)\), where \(A(s) \cong C^s_G(\lambda)/C^s_G(\lambda)\). (See e.g. [Bon05] Proposition 1.3.) If \(s \in T^* := T^*\), there is an associated \(\lambda \in \text{Irr}(T)\), and the decompositions of
$W(\lambda)$ and $W(s)$ are closely related (this will be made more precise when needed - see e.g. [SFT22, Lemma 3.4] and [SF19, Lemma 4.5]).

By results Lusztig [Lus88], the set $\text{Irr}(G)$ can be partitioned into subsets $\mathcal{E}(G, s)$ called rational series, indexed by $G^*$-conjugacy class representatives of semisimple elements $s \in G^*$. For $\chi \in \mathcal{E}(G, s)$, we have $\chi(1)$ is divisible by $[G^* : C_{G^*}(s)]_{q'}$, implying that if $\chi \in \text{Irr}_{2'}(G)$, it must be that $s \in G^*$ is 2-central.

3.5. Automorphisms. Now, when $G$ is simple of simply connected type and $G$ is perfect (so that $G/Z(G)$ is a simple group of Lie type), we may obtain the group $\text{Aut}(G)$ in a relatively natural way. Namely, we may take a regular embedding $\iota : G \rightarrow \tilde{G}$ as in [GM20, Section 1.7] so that, among other properties, $[\tilde{G} : \tilde{G}^*]$ restrictions of the characters in $\mathcal{E}(G, s)$ are induced by $\iota^*$. Then the automorphisms $\text{Aut}(G)$ are induced by $\tilde{G} \rtimes D$, where $D$ is an appropriate set of graph-field automorphisms of $G$, see [GLS98, Theorem 2.5.1].

Further, $\iota$ induces a dual map $\iota^* : G^* \rightarrow \tilde{G}^*$ such that the characters in $\mathcal{E}(G, s)$ are isomorphic to $\text{Irr}(\tilde{G}/G)$, and if $\tilde{z} \in \text{Irr}(\tilde{G}/G)$ corresponds to $z \in \text{Irr}(\tilde{G}^*)$, then $\mathcal{E}(\tilde{G}, z) \cong \mathcal{E}(\tilde{G}, \tilde{z})$ (see e.g. [DM20, Prop. 11.4.12, Rem. 11.4.14]).

4. Odd-degree characters and rationality

In this section, we discuss the action of $\mathcal{H} := \mathcal{H}_2$ on the sets $\text{Irr}_{2'}(G)$ for quasisimple groups $G = G^F$ of Lie type where $G$ is simple of simply connected type $A$, $D$, or $E_6$.

4.1. Recalling Previous Results. We begin with some previous results on the sets $\text{Irr}_{2'}(G)$ for the groups $G = G^F$ under consideration.

A key result from [MS16] (completed later for groups of type $A$ in [Mal19]) is that if $G$ is of simply connected type, with some exceptions, any $\chi \in \text{Irr}_{2'}(G)$ lies in the principal series. Here the groups $\text{Sp}_{2n}(q)$ are the main exception.

Proposition 4.1 ([MS16, Mal19]). Let $G = G^F$ with $G$ of simply connected type defined over $\overline{F}_q$ with $q$ odd and $F$ a Steinberg endomorphism. Suppose $G \neq \text{Sp}_{2n}(q)$ for $n \geq 1$, and let $\chi \in \text{Irr}_{2'}(G)$. Then $\chi$ lies in the principal series. In particular, $\chi = R^G_T(\lambda, \eta)$ with $T$ a maximally split torus of $G$, $\lambda \in \text{Irr}(T)$, $\eta \in \text{Irr}_{2'}(W(\lambda))$, and $2 | [W : W(\lambda)]$.

Proof. That $\chi$ lies in the principal series is [MS16, Theorem 7.7] and [Mal19, Theorem 3.3]. The second statement is [MS16, Lemma 7.9], which now follows for $G$ of type $A$ as well by the same proof and [Mal19, Theorem 3.3].

Let $\mathcal{N} := N_G(T)$ and let $\Lambda : \text{Irr}(T) \rightarrow \bigcup_{\psi \in \text{Irr}(T) \text{ Irr}(N_\psi)}$ be an extension map for $\text{Irr}(T)$ with respect to $T \lhd \mathcal{N}$, which exists by [Gec93] and [Lus84, Theorem 8.6]. That is, for each $\psi \in \text{Irr}(T)$, we have $L(\psi)$ is an extension of $\psi$ to $N_\psi$. For $\sigma \in G$, let $\delta_{\lambda, \sigma}$ be the linear character of $W(\lambda)$ such that $L(\lambda) = \delta_{\lambda, \sigma} \Lambda(\lambda^\sigma)$, guaranteed by Gallagher’s theorem [Isa06, Corollary 6.17], and let $\delta_{\lambda, \sigma}^{-1} \in \text{Irr}(W(\lambda))$ be the truncated character such that $\delta_{\lambda, \sigma}^{-1}(w) = \delta_{\lambda, \sigma}(w)$ for $w \in C(\lambda)$ and $\delta_{\lambda, \sigma}^{-1}(w) = 1$ for $w \in R(\lambda)$.

Let $\gamma_{\lambda, \sigma}$ be the character of $W(\lambda)$ such that $\gamma_{\lambda, \sigma}$ is trivial if $\sigma$ fixes $\sqrt{q}$ and otherwise satisfies $\gamma_{\lambda, \sigma}(w) = (-1)^{l(w)}$ for $w = w_rw_c$ with $w_r \in R(\lambda)$ and $w_c \in C(\lambda)$, where $l(w_c)$ is the length $l(w_c)$ in $W$ of $w_c$. (Note that $\gamma_{\lambda, \sigma}$ is defined more generally in [SF19], but can be viewed as stated here in the case of finite groups of Lie type, see [SFT22, Lemma 3.5].)
For $\eta \in \text{Irr}(W(\lambda))$, we denote by $\eta^{(\sigma)}$ the character $j(\tilde{\eta}^{\sigma})$, where $j$ is an isomorphism between $\text{Irr}(\text{End}_{\mathbb{C}}(R_T^G(\lambda)))$ and $\text{Irr}(W(\lambda))$ induced by the standard specializations and $\tilde{\eta} \in \text{Irr}(\text{End}_{\mathbb{C}}(R_T^G(\lambda)))$ is such that $j(\tilde{\eta}) = \eta$. (See [SF19, Section 3.5].) Note that this is not necessarily the same as $\eta^{\sigma}$, although we see in [SF22] Section 5, combined with Lemma 4.3 below, that this is the case in the situations in which we are interested. The following is [SF19] Theorem 3.8 in our situation.

**Theorem 4.2** ([SF19]). *Keep the notation above and let $\sigma \in \mathcal{G}$ and $\chi = R_T^G(\lambda)_\eta$. Then $\chi^\sigma = R_T^G(\lambda)'_\eta$, where $\eta' \in \text{Irr}(W(\lambda)) = \text{Irr}(W(\lambda'))$ is defined by $\eta'(w) = \gamma_{\lambda,\sigma}(w)\delta_{\lambda,\sigma}(w^{-1})\eta^{(\sigma)}(w)$ for each $w \in W(\lambda)$.*

From this, we see that the main obstruction to understanding the action of $\mathcal{G}$ on $\text{Irr}_2(G)$ is in understanding the characters $\eta^{(\sigma)}$, $\gamma_{\lambda,\sigma}$, and $\delta_{\lambda,\sigma}$. Significant progress was made on this in [SF22],[SFT22]. In the next subsections, we adapt these results to the cases that $\mathcal{G}$ is of type $A$ or $D$ omitted there.

The following will also be useful for the case that $\mathcal{G}$ is of type $A$.

**Lemma 4.3.** *Let $q$ be odd and let $G = SL_n^*(q)$ and $\tilde{G} = GL_n^*(q)$ with $n \geq 3$. Let $\chi \in \text{Irr}_2(G)$ and $\tilde{\chi} \in \text{Irr}(\tilde{G}|\chi)$. Then either $\tilde{\chi}|G = \chi$ or we have $\tilde{\chi}|G = \chi + \chi'$ is the sum of two characters and $n = 2^r$ is some power of 2. In the first case, $\chi \in \mathcal{E}(G,s)$ with $C_{G^r}(s)$ connected, and in the second case, one can choose $\tilde{\chi} \in \mathcal{E}(\tilde{G},\tilde{s})$ with $\tilde{s} = \text{diag}(1,\ldots,1,\ldots,1)$ with $2^r-1$ copies of each eigenvalue.*

**Proof.** The first statement is [SFT18a] Lemma 10.2, which is shown as a consequence of [NT16, Lemmas 4.5 and 4.6], and the second statement follows from [NT16, Lemmas 4.5 and 4.6]. □

4.2. **The Characters $\eta^{(\sigma)}$ and $\gamma_{\lambda,\sigma}$.** We now let $\mathcal{G}$ be simple of simply connected type and continue to write $G := G^F$, where now $F \colon \mathcal{G} \to \mathcal{G}$ is a Frobenius endomorphism defining $G$ over $\mathbb{F}_q$ with $q$ odd.

Thanks to the statements in Section 4.1 and the work of [SF22] Section 5, we can say that in our situation, the character $\eta^{(\sigma)}$ is just $\eta$.

**Lemma 4.4.** *Let $\mathcal{G}$ be simply connected of type $A_{n-1}$ with $n \geq 3$ or $D_n$ with $n \geq 4$ and write $G = G^F$. Let $\chi \in \text{Irr}_2(G)$, so that $\chi = R_T^G(\lambda)_\eta$ as in Proposition 4.1 Then $|C(\lambda)| \leq 2$ or $\mathcal{G}$ is type $D_n$ and $C(\lambda)$ is elementary abelian of size 4.*

**Proof.** Let $s \in G^*$ be semisimple such that $\chi \in \mathcal{E}(G,s)$. Let $\iota \colon \mathcal{G} \to \tilde{G}$ be a regular embedding as in [GM20, Section 1.7], let $\tilde{G} := \tilde{G}^F$, and let $\tilde{s} \in \tilde{G}^*$ be such that $\iota^*(\tilde{s}) = s$. Further, let $\tilde{\lambda} \in \text{Irr}(\tilde{T}|\lambda)$, where $\tilde{T}$ is a maximally split torus of $\tilde{G}$ containing $T$. Then by [SFT22] Lemma 3.4 and [SF19, Lemma 4.5], we have $W(\lambda) \cong W(s)^F$ and $R(\lambda) = W(\lambda) = R(\lambda) \cong W(\tilde{s})^F = \tilde{W}(s)^F$, where the last equality follows by [Bon05,(2.2)]. In particular, we have $C(\lambda)$ is isomorphic to $A(s)^F$, where $A(s) = W(s)/W(s) \cong C_{G^r}(s)/C_{G^r}^*(s)$. For $G$ of type $D$, the result about $C(\lambda)$ now follows from the description of disconnected $C_{G^r}(s)$ for 2-central semisimple elements $s \in G^*$ in [MS16, Table 1] and [Mal19, Table 2], which follows from [Bon05, Table 2]. For $G$ of type $A$ the statement about $C(\lambda)$ follows since $|A(s)^F| \leq 2$ using Lemma 4.3. □

**Corollary 4.5.** *Let $G = G^F$ where $\mathcal{G}$ is simple of simply connected type. Let $\chi \in \text{Irr}_2(G)$ be of the form $\chi = R_T^G(\lambda)_\eta$ as before. Then $\eta^{(\sigma)} = \eta^{(\eta)} = \eta$ for each $\sigma \in \mathcal{G}$.***

**Proof.** For the groups not of type $A$ and $D$, this was established in [SF22] Sections 5 and 8 using [SF22 Corollary 5.6]. The full statement can now be seen using Lemma 4.3 and [SF22 Corollary 5.6]. We remark that the result still holds for $A_1$ in the case of principal series characters. □

We next consider the character $\gamma_{\lambda,\sigma}$.

**Proposition 4.6.** *Let $G = G^F$ where $\mathcal{G}$ is simple of simply connected type, not of type $A_1,C_n$. Let $\chi = R_T^G(\lambda)_\eta \in \text{Irr}_2(G)$ as before. Then $\gamma_{\lambda,\sigma} = 1$ for every $\sigma \in \mathcal{G}$.***
Proof. For $G$ not of type $A_{n-1}$, this is [SF22 Proposition 3.4]. So assume that $G$ is type $A_{n-1}$ with $n \geq 3$. If $C(\lambda) = 1$, then certainly the statement holds. Otherwise, we have $n = 2^r \geq 4$ and $C(\lambda)$ has size 2 by Lemmas 3.3 and 4.3. From the description there of $\bar{s}$ and $R(\lambda)$, we see $R(\lambda)$ is a subgroup of $\mathfrak{S}_{2r-1} \times \mathfrak{S}_{2r-1}$, with $C(\lambda)$ inducing the element $\prod_{i=1}^{r-1} (i, 2^{r-1} + i)$, which has even length in $W$ since $r \geq 2$. Hence the statement follows from [SF19 Lemma 4.10].

4.3. Extension Maps and Galois Automorphisms. In this section, we assume that $G$ is simple of simply connected type $(G, F)$. We consider the stabilizers of 2-central elements in $G^*$. Throughout this section, we will work with a semisimple $s \in T^*$ and the associated linear character $\lambda \in \text{Irr}(T)$, where $\lambda$ corresponds to $s$ under the isomorphism $\text{Irr}(T) \rightarrow T^*$ induced by duality. Now, for $\sigma \in \mathcal{G}$ and $s \in T^*$, we may define $s^\sigma$ to be $s^k$, where $\sigma$ maps $s$'th roots of unity to their $k$th power. (For $\sigma \in \mathcal{H}$, we may be more specific, see [SFT18a, Lemma 3.4] and [SFT18b, Lemma 3.4].) Further, given $\alpha \in D$, we may find a dual automorphism $\alpha^* \in D^*$, and following the proofs of [SFT18a, Lemma 3.4] and [Lay18 Proposition 7.2], we see that the isomorphism $\text{Irr}(T) \rightarrow T^*$ is compatible with the action of $D \times \mathcal{G}$.

We will write $\text{Irr}_{2\text{cen}}(T)$ for the set of $\lambda \in \text{Irr}(T)$ such that the corresponding $s \in G^*$ is 2-central. The structure of 2-central elements is given in the proof of [Mal19 Theorem 3.4] (see also Lemma 4.3). We further let $\mathcal{H} := \mathcal{H}_2$.

Lemma 4.7. Keep the notation and assumptions above. Then every $\lambda \in \text{Irr}_{2\text{cen}}(T)$ is $N_G(T)$-conjugate to some $\lambda_0 \in \text{Irr}_{2\text{cen}}(T)$ satisfying

$$N_{G \times \mathcal{H}}(T)\lambda_0 = N_G(T)\lambda_0(D \times \mathcal{H})\lambda_0.$$

Proof. We assume first that $G$ is of type $A$. If $A(s) \neq 1$ then by [Mal19 Theorem 3.4], up to conjugacy, $s = \text{diag}(-1, \ldots, -1, 1, \ldots, 1)$. In particular, $s$ is stable under all automorphisms. If $A(s) = 1$ we have that $s = \text{diag}(\mu_1, \ldots, \mu_k, \ldots)$ where we assume that each distinct eigenvalue $\mu_i$ appears $n_i$ times with $n_1 \geq n_2 \geq \cdots \geq n_k$. Since $s$ is 2-central it follows that all $n_i$ must necessarily be distinct. Any automorphism stabilizing the orbit of $s$ permutes the eigenvalues. Therefore, it must stabilize $s$.

For type $D$, we observe by [Mal19 Theorem 3.4] that $s$ is quasi-isolated with disconnected centralizer and $s^2 = 1$. Moreover, in all cases the associated character $\lambda$ is conjugate to a $(D \times \mathcal{H})$-stable character.

For $G$ of type $E_6(q)$ we observe that the 2-central elements are (up to conjugation) contained in the center $Z(L^*)$ of a Levi subgroup $L^*$ of $G^*$ of type $D_5(q) = q^2 - q$, see [Mal19 Theorem 3.4]. The relative Weyl group $W_{G^*}(L^*)/L^*$ is trivial and so any automorphism stabilizing the orbit of $s$ in $Z(L^*)$ must centralize $s$. This implies again the stabilizer condition.

Lemma 4.8. Keep the notation and assumptions above, and further assume $G$ is not of type $D$. Then there exists an $N_{G \times \mathcal{H}}(T)$-equivariant extension map for $\text{Irr}_{2\text{cen}}(T)$ with respect to $T \triangleleft N$.

Proof. Let $s$ be the semisimple element associated to $\lambda \in \text{Irr}_{2\text{cen}}(T)$. According to [Mal19 Theorem 3.4], $C_{G^*}^0(s)$ is an $F$-stable Levi subgroup of $G^*$ and we let $T \subset L$ be the $F$-stable Levi subgroup of $G$ corresponding to it under duality. The character $s \in \text{Irr}(L^F)$ is an extension of the character $\lambda$ and $W(\lambda)$ corresponds to $W(s)$ under the isomorphism $W \cong W^*$ induced by duality. We can therefore assume that $C_{G^*}(s)$ is disconnected, since otherwise $s \in \text{Irr}(L^F)$ is a linear character with the desired properties.

If $C_{G^*}(s)$ is disconnected, then $G$ is of type $A_{n-1}$ with $n$ a power of 2 and $\lambda \in \text{Irr}(T^F)$ (an $N_G(T)$-conjugate of) the linear character associated to $s = \text{diag}(-1, -1, 1, \ldots)$. Then $N_G(L)\lambda = L(\varphi)$, where $\varphi \in V$ is the preimage of the longest element of the Weyl group. We deduce that $\varphi^2 = -1$ and since $\lambda$ is trivial on $Z(G)$, the character $\lambda$ extends to a character $\hat{\lambda} \in \text{Irr}(L(\varphi))$ of order 2. In particular, $\hat{\lambda}$ is rational.

We remark that the omitted case of type $D$ is addressed below in Proposition 4.11. The following lemma slightly generalizes the idea from Lemma 4.8 to $\text{Irr}(T)$, in the case of $\text{SL}_n(q)$.
However, we remark that the case of $\text{SU}_n(q)$ is more complicated and is being considered elsewhere.

**Lemma 4.9.** Assume that $G$ is of type $A_{n-1}(q)$. Let $\lambda \in \text{Irr}(T)$, with associated semisimple element $s \in T^*$. Then $\lambda$ has no extension $\tilde{\lambda}$ to $\text{N}_G(T)_{\lambda}$ such that $\text{Res}_{\tilde{\lambda}}^G(\tilde{\lambda})$ is rational if and only if $(n) = (q - 1)/2 = |\text{A}(s)^F|/2 = 2$.

**Proof.** We can extend $\lambda$ to the preimage of $\text{W}(s)$ in $V$ by setting $\lambda(n_{\lambda}(1)) = 1$ for all $\alpha \in \Phi^0(s)$. Let $w$ be a generator of $\text{A}(s)^F$ and $m \in V_{\tilde{\lambda}}$ with image $w$ in $W$. According to [Spa12, Lemma 2.11] the character $\lambda$ has an extension as in the statement of the lemma precisely when $\lambda$ has a rational extension to $T(m)$.

Let $r$ be the order of $w$ and let $s \in T^*$ corresponding to $\lambda \in \text{Irr}(T)$ by duality. We have $\langle m \rangle \cap T = \langle m^r \rangle$. Hence, $\lambda$ has an extension $\lambda' \in \text{Irr}(T(m))$ with $\lambda'(m^r) \in \mathbb{Q}$ if and only if $r$ is odd or $m^r \in \ker(\lambda)$. Moreover, $m^r = 1$ if $q$ is even since $H \cap V = 1$. Hence, we can assume now that $r$ is even and $q$ is odd. One can find an involution $i \in T$ and a permutation matrix $\bar{m}$ such that $m = \bar{m}i$ has order $r$. Since $\det(m) = 1$, we have $\det(i) = \text{sgn}(w) = (-1)^{\ell(w)}$. We have $\bar{m} = \tilde{\lambda} \tilde{\lambda}$. A computation now shows that

$$\lambda(m^r) = \tilde{\lambda}(\bar{m}i)^r = \tilde{\lambda}'(i)^{2(r + 1)/2} = \tilde{\lambda}'(i)^{r + 1},$$

where we use that $i^2 = 1$. Since $\tilde{\lambda}$ has order $r$, we deduce that $\tilde{\lambda}'(i)^{r + 1} = -1$ precisely when $r \equiv 2 \mod 4$ and $i$ generates the Sylow 2-subgroup of $G/GZ(G)$. The latter is equivalent to $(n) = (q - 1)/2 = 2$ and $\det(i) = -1$. If the first two conditions are met, then $w$ has order 2 and is conjugate to the longest element of the Weyl group, so $\det(i) = -1$ is always satisfied. \hfill $\square$

We now let $\Lambda$ be an ND-equivariant extension map with respect to $T \leq N$, which exists by [CS17, Prop. 5.9], [MS16, Cor. 3.13], [CS13 Thm. 3.6], and [Spa09], and we consider the character $\chi$. Recall that this is the linear character of $\text{ND}(q)$ and it is conjugate to the longest element of the Weyl group, so $\det(i) = -1$ is always satisfied. \hfill $\square$

**Lemma 4.10.** Assume that $G$ is simple of simply connected type, but not of type $A_1$. Let $\lambda \in \text{Irr}_{2\text{even}}(T)$ and $\sigma \in G$. Then $\Lambda_\sigma^\lambda \downarrow \Lambda_\sigma^\lambda = \Lambda_\sigma^\lambda \downarrow \Lambda_\sigma^\lambda$. In particular, $R(\lambda) \leq \ker \delta_{\lambda,\sigma}$.

**Proof.** For $G$ not of type $A$, this is [SF22, Lemma 4.3]. So let $G$ be type $A_{n-1}$ with $n \geq 3$. Then in this case, the statement follows from Lemma 4.8. \hfill $\square$

The next proposition complements Lemma 4.8.

**Proposition 4.11.** Let $G = G^F$ with $G$ simple of simply connected type $D_n$ for $n \geq 4$, and let $\chi = R_{G^F}(\lambda)_0 \in \text{Irr}_2(G)$. Then the extension $\Lambda_\chi$ of $\lambda$ to $N_\chi := \text{N}_G(T)_{\chi}$ satisfies $\Lambda_\chi^\alpha = \Lambda_\chi$, and hence $\delta_{\lambda,\sigma} = 1$, for any $\sigma \in G$.

**Proof.** We may assume $G \neq D_4(q)$, as in this case the result follows from the fact that $C(\lambda) = 1$. Let $\chi \in \mathcal{E}(G, s)$. Then by [MS16, Lemma 7.6], [Mal19, Table 2], we see that $s^2 = 1$ (and hence $\lambda^2 = 1$) and $A(s)$ is either trivial; or it induces the graph automorphisms simultaneously on the two components of $C_2^G(s) \cong D_k \times D_{n-k}$ for some $1 \leq k \leq n/2$; or $n = 4m$, $C_2^G(s) \cong D_{2m} \times D_{2m}$, and $A(s) \cong C_2^G(s)$ is generated by an element inducing the graph automorphisms simultaneously on the two factors of $C_2^G(s)$ by an element that permutes the two factors.

Let $\sigma \in G$. Since $\lambda^2 = 1$ and hence $\lambda^2 = \lambda$, it suffices by Lemma 4.10 to show that $\Lambda_\lambda(c)$ is fixed by $\sigma$ for an inverse image $\tilde{c}$ of any $1 \neq c \in C(\lambda)$ in $N$. Since $\chi$ has odd degree and $Z(G)$ is a 2-group, we see $\chi$, and hence $\lambda$ (see e.g. [SF22, Proposition 2.7(i)], is trivial on $Z(G)$. Then it suffices to know that such a $\tilde{c}$ satisfies $\tilde{c}^2 \in Z(G)$. Since such a $c$ can be represented as the image in $W$ of the element

$$n_{\alpha_1}(1)n_{\alpha_2}(1) \cdots n_{\alpha_{n-2}}(1)n_{\alpha\alpha}(1)n_{\alpha_{n-1}}(1)n_{\alpha_{n-2}} \cdots n_{\alpha_1}(1)$$
(this also corresponds to the analogous element \( u_1 \) in the notation of [SFT22, 7.2] for the case of \( \text{SO}_{2n}^+(q) \)), the element \( \prod_{i=1}^{n/2} e_{n_i} - e_{n/2+i}(1) \), or a product of the two, we see by computation with the Chevalley relations [GLS98, Theorem 1.12.1] that this is indeed the case. \( \Box \)

**Corollary 4.12.** Let \( G = G^F \) with \( G \) simple of simply connected type \( D_n \) with \( n \geq 4 \) and let \( \chi \in \text{Irr}_{2}(G) \). Or, let \( G \) be simple of simply connected type \( A_{n-1} \) with \( n = 2r \geq 4 \) and let \( \chi \in \text{Irr}_{2}(G) \) be such that \( \chi \) does not extend to \( \tilde{G} \). Then \( \chi \) is rational-valued.

**Proof.** Recall that \( \chi^2 = 1 \) in the case \( G \) is type \( D_n \), and similarly the same is true by Lemma 4.3 for the characters in type \( A_{n-1} \) with \( n = 2r \) that do not extend to \( \tilde{G} \). Hence the statement follows from Theorem 4.2, by combining Corollary 4.5 with Propositions 4.6 and 4.11 and Lemma 4.8. \( \Box \)

We remark that Corollary 4.12 already shows that the McKay–Navarro conjecture holds for the groups of type \( D_n \), complementing [SF22, Theorem B], since a Sylow 2-subgroup \( P \) of \( G \) is self-normalizing in this case and satisfies \( P/[P,P] \) is elementary abelian.

### 4.4. More on Stabilizers and Extensions.

**Proposition 4.13.** Let \( G \) be simple of simply connected type \( E_6, A_{n-1} \) with \( n \geq 3 \), or type \( D_n \) with \( n \geq 4 \). Let \( \chi \in \text{Irr}_2(G) \) and let \( \mathcal{H} := \mathcal{H}_2 \). Then \((\tilde{G} \times \mathcal{H})_{\chi} = \tilde{G}_{\chi} \times \mathcal{H}_{\chi} \).

**Proof.** Let \( \chi \in \text{Irr}_2(G) \cap \mathcal{E}(G,s) \), with \( \lambda \in \text{Irr}(T) \) corresponding to \( s \), so that \( \chi = R^G_s(\lambda) \eta \) for some \( \eta \in \text{Irr}_2(W(\lambda)) \) as before. Let \( \alpha \in \tilde{G} \) and \( \sigma \in \tilde{H} \) such that \( \alpha \sigma \in (\tilde{G} \times \tilde{H})_{\chi} \). If \( \chi \) is as in one of the situations of Corollary 4.12, then \( \mathcal{H}_{\chi^\alpha} = \mathcal{H} \) for any \( \alpha \in \tilde{G} \), and the result follows.

If \( G \) is type \( E_6 \), then \( C_{G^s}(s) \) is a Levi subgroup from before, and hence connected, so \( \chi \) extends to \( \tilde{G} \) and the statement again follows, as \( \chi = \chi^{\alpha \sigma} = \chi^\sigma \). The same holds if \( G \) is of type \( A_{n-1} \) and \( \chi \) extends to \( \tilde{G} \), completing the proof by Lemma 4.3. \( \Box \)

**Remark 4.14.** Let \( G \) be as above. Recall that for \( G \) of type \( D_n \) or when \( \chi \in \text{Irr}_2(G) \) with \( G \) of type \( A_n \) and \( \chi \) does not extend to \( \tilde{G} \), we have that \( \chi \) is \( \mathcal{H} \)-invariant, by Corollary 4.12. In particular, we have

\[
(\tilde{G}D \times \mathcal{H})_{\chi} = (\tilde{G}D)_{\chi} \times \mathcal{H}_{\chi}
\]

in those cases.

In the remaining cases being considered, \( \chi \) extends to \( \tilde{G} \), so

\[
(\tilde{G}D \times \mathcal{H})_{\chi} = \tilde{G}(DH)_{\chi}.
\]

### 5. Equivariant Bijections for the Prime 2

In previous work, the second author [SFT22] showed that the bijections used by Malle and Spåth in the proof of Isaacs–Malle–Navarro’s inductive McKay conditions for \( \ell = 2 \) are also equivariant with respect to \( \mathcal{H}_2 \), thus proving Condition 3 with a few exceptions. The aim of this section is to complete the case of the main exceptions in loc. cit.: groups with root systems of type \( A \) and \( D \).

**Theorem 5.1.** Let \( q \) be a power of an odd prime and let \( G \) be a group of Lie type of simply connected type defined over \( \mathbb{F}_q \) of one of the following types: \( A_{n-1}(q), D_{n-1}(q) \) with \( n \geq 3 \) or \( D_n(q), D_{2n}(q) \) with \( n \geq 4 \). Then Condition 7 holds for \( G \) and the prime \( \ell = 2 \).

**Proof.** Throughout, let \( \mathcal{H} := \mathcal{H}_2 \). First, suppose that \( q \equiv 1 \mod 4 \). By [SFT22, Theorem 6.2], we may assume that \( G \) is type \( A_{n-1} \). With the work in the previous sections, the proof follows exactly as in [SFT22, Prop. 6.1 and Thm. 6.2], but we provide it for completeness.

For \( \lambda \in \text{Irr}(T) \), recall that the Harish-Chandra series \( \mathcal{E}(G,T,\lambda) \) is the set of irreducible constituents of \( R^G_s(\lambda) \) and that \( \Lambda \) is an \( ND \)-equivariant extension map with respect to \( T < N \), which is further \( \mathcal{H} \)-equivariant on \( \text{Irr}_{2cen}(T) \) by Lemma 4.8. From [MS16, Theorem 5.2] (using the corresponding results for type \( A \) in [CS14]), the map

\[
\Omega : \bigcup_{\lambda \in \text{Irr}(T)} \mathcal{E}(G,T,\lambda) \to \text{Irr}(N)
\]
given by

\[(1) \quad R^G_T(\lambda)_{\eta} \mapsto \text{Ind}_{N_{\lambda}}^N(\Lambda(\lambda)_{\eta})\]

defines an \(\tilde{ND}\)-equivariant bijection, which induces a bijection (which we will also call \(\Omega\)) between \(\text{Irr}_2(G)\) and \(\text{Irr}_2(N)\), using Proposition 4.1. Further, the group \(M := N\) and map \(\Omega\) satisfy the stated properties (other than \(H\)-equivariance) by [Mal07, Theorem 7.8] and [CST13, Proposition 2.5].

Now, we have

\[\Omega(\chi^\sigma) = \Omega(R^G_T(\lambda^\sigma)_{\gamma_{\lambda,\sigma}(\delta_{\lambda,\sigma}^{-1}\eta(\sigma))}) = \text{Ind}_{N_{\lambda}}^N(\Lambda(\lambda^\sigma)_{\gamma_{\lambda,\sigma}(\delta_{\lambda,\sigma}^{-1}\eta(\sigma))}),\]

by Theorem 4.2 and the definition (1) of \(\Omega\). But \(\eta(\sigma) = \eta = \eta_{\gamma_{\lambda,\sigma}} = 1\), and \(\delta_{\lambda,\sigma} = \delta_{\lambda,\sigma} = 1\), by Corollary 4.3, Proposition 4.6 and Lemma 4.8 respectively. Hence, we have

\[\Omega(\chi^\sigma) = \text{Ind}_{N_{\lambda}}^N(\Lambda(\lambda^\sigma)_{\delta_{\lambda,\sigma}\eta^\sigma}) = \text{Ind}_{N_{\lambda}}^N(\Lambda(\lambda)_{\eta^\sigma}) = \Omega(\chi)^\sigma,\]

completing the proof when \(q \equiv 1 \pmod{4}\).

We are now left to consider the case that \(q \equiv 3 \pmod{4}\). Recall that \(G\) is type \(A_{n-1}\) with \(n \geq 3\) or \(D_n\) with \(n \geq 4\). We use the same notation and considerations made in [SF22, Section 8], which we now summarize.

We may write \(T = C_G(S)\) for some Sylow 2-torus \(S\) of \((G,vF)\), where \(v \in V\) is a representative for the longest element of \(W\) as in [MS16, 3.A]. Writing \(T_1 := T^{vF}\) and \(N_1 := N_G(S)^{vF} = N^{vF}\), [Mal07, Theorem 7.8] yields that \(N_1\) contains \(N_G(Q)\) for a Sylow 2-subgroup \(Q\) of \(G\) (which we now identify with the isomorphic group \(G^{vF}\)) and there is a bijection

\[\Omega_1: \text{Irr}_2(G) \to \text{Irr}_2(N_1),\]

where corresponding characters lie over the same character of \(Z(G)\). Then \(N_1\) is also \(\text{Aut}(G)_Q\)-stable by [CST13, Proposition 2.5]), and \(\Omega_1\) is moreover \(\text{Aut}(G)_Q\)-equivariant by [MS16, Theorem 6.3] and [CST13, Theorem 6.1], combined with [Spr12, Theorem 2.12]. Hence, as in [SF22, Section 8] it suffices to show that this bijection can be chosen to further be \(H\)-equivariant.

For \(\lambda_1 \in \text{Irr}(T_1)\), we write \(W_1(\lambda_1)\) for the group \((N_1)_{\lambda_1}/T_1\) and \(R_1(\lambda_1)\) for the reflection group generated by the simple reflections \(s_\alpha\) for \(\alpha \in \Phi\) such that \(s_\alpha \in W_1(\lambda_1)\) and \(T_1 \cap (X_\alpha, X_{-\alpha})\) is in the kernel of \(\lambda_1\). Here \(X_\alpha\) denotes the root subgroup of \(G\) associated to \(\alpha\). Then by the proof of [Mal07, Theorem 7.8], both \(\text{Irr}_2(G^{vF})\) and \(\text{Irr}_2(N_1)\) are parametrized by pairs \((\lambda_1, \eta_1)\) or \((s, \eta_1)\), where \(s \in T_1^*\) is a semisimple element (up to \(N_1^* := (N^*)^{vF}\)-conjugation) centralizing a Sylow 2-subgroup of \(G^{vF}\); \(\lambda_1 \in \mathcal{E}(T_1, s)\) with \(2 \nmid \frac{[N_1 : (N_1)_{\lambda_1}]}{\text{and}} \eta_1 \in \text{Irr}_2(W_1(\lambda_1))\).

Letting \(W_1(s)\) and \(W_1(s)^{\eta_1}\) denote the fixed points under \(vF\) of the Weyl groups of \(C_G(s)\) and \(C_G(s)^{\eta_1}\), respectively, we have

\[(2) \quad W_1(\lambda_1) \cong W_1(s); \quad W_1(s)^{\eta_1} \cong R_1(\lambda_1); \quad \text{and} \quad W_1(s)/W_1(s)^{\eta_1} \cong W_1(\lambda_1)/R_1(\lambda_1).\]

The member of \(\text{Irr}_2(N_1)\) corresponding to \((\lambda_1, \eta_1)\) is of the form \(\text{Ind}_{N_1}^{N_1}(\lambda_1_{\eta_1})\), where \(\lambda_1\) is an extension map with respect to \(T_1 < N_1\).

First suppose that \(\text{Irr}_2(N_1)\) is of type \(D_n\) with \(n \geq 4\). Then by Corollary 4.12 and the discussion above, it suffices to show that every member of \(\text{Irr}_2(N_1)\) is rational. But in this case, recall that \(s^2 = 1\), so \(\lambda_1\) is fixed by \(G\). Further, the same arguments as in Proposition 4.11 yield that \(\lambda_1(\lambda_1)\) will be rational-valued, as \(\lambda_1(\lambda_1)\) is also rational-valued on the elements \(n_\alpha(-1)\) for \(s_\alpha \in R_1(\lambda_1)\) since \(\lambda_1\) is trivial on \(h_\alpha(-1)\) for such elements. Further, using (2) and the proof of Lemma 4.4 we have \(W_1(\lambda_1)\) is the semidirect product of the reflection group \(R(\lambda_1)\) with an elementary abelian 2-group. As a Weyl group, every character of \(R(\lambda_1)\) must be rational-valued, and therefore we may apply [SF22, Lemma 5.5] to see that any \(\eta_1 \in \text{Irr}_2(W_1(\lambda_1))\) is necessarily fixed by \(G\). This yields that \(\psi := \text{Ind}_{N_1}^{N_1}(\lambda_1(\lambda_1)_{\eta_1})\) is fixed by \(G\) for each \(\psi \in \text{Irr}_2(N_1)\), and we are done with this case.

Finally, let \(G\) be of type \(A_{n-1}\) with \(n \geq 3\). Let \(\chi = R^G_T(\lambda)_{\eta} \in \text{Irr}_2(G) \cap \mathcal{E}(G, s)\) and \(\sigma \in \mathcal{H}\). As before, we have \(\eta(\sigma) = \eta\) and \(\gamma_{\lambda,\sigma} = 1 = \delta_{\lambda,\sigma}\). This tells us that \(\chi^\sigma\) is
determined entirely by $\lambda^p$, or equivalently, by $s^p$. But the same argument as in Lemma 5.3 shows that the extension map for $T_1 < N_1$ is $H$-equivariant when considering those $\lambda_1$ corresponding to $s$ with $E(G^F, s) \cap \text{Irr}_2(G^F) \neq \emptyset$, and $\eta^s = \eta_1$ for the same reasons as above. Hence $\Omega(\chi) := \text{Ind}_{(N_1)_{\lambda_1}}^{N_1}(\Lambda(\lambda_1)\eta_1)$ is also determined entirely by $s^p$, completing the proof. \[
\]

Theorem 5.1 completes Condition 3 for groups of Lie type for the prime $\ell = 2$. In the next sections, we will deal with Condition 3 (i.e., parts (iii) and (iv) of the relation on $H$-triples [NSV20, Definition 1.5]), which was addressed for nontwisted groups without nontrivial graph automorphisms in [RSF22, Theorem A].

5.1. An $H$-equivariant map for $\tilde{G}$. We end this section with the corresponding statement at the level of $\tilde{G}$. Let $G = G^F$ be such that $G$ is simply of connected type $A_{n-1}$ with $n \geq 3$, $D_n$ with $n \geq 4$, or $E_6$ and let $Q \in SL_2(G)$. Write $\mathcal{H} := H_2$ and let $d$ be the order of $q$ modulo 4.

From Theorem 5.1 and [SF22 Theorem A], we have an $\text{Aut}(G)_Q \times H$-equivariant bijection $\Omega: \text{Irr}_2(G) \rightarrow \text{Irr}_2(M)$, where $M = N_G(S)^F$ for a Sylow $d$-torus $S$ of $(G, F)$, which is the group $N$ in the case $q \equiv 1 \mod 4$ and can be identified in $G^F \cong G$ with the group $N_1$ in the case $q \equiv 3 \mod 4$. From [MS16 Theorem 6.3] and [CS17 Theorem 6.1], we have a $(GD)_S$-equivariant bijection

(3) $\tilde{\Omega}: \text{Irr}_2(\tilde{G}|\text{Irr}_2(G)) \rightarrow \text{Irr}_2(M|\text{Irr}_2(M))$,

where $\tilde{M} := MN_G(S)$, such that $\tilde{\chi}$ and $\tilde{\Omega}(\chi)$ always lie over the same character of $Z(\tilde{G})$; $\tilde{\Omega}(\tilde{\chi})\beta = \tilde{\Omega}(\chi)\beta$ for each $\beta \in \text{Irr}(\tilde{G}|\tilde{G}G) \cong \text{Irr}(M/M)$; and such that $\tilde{\chi}$ lies over $\chi \in \text{Irr}_2(G)$ if and only if $\tilde{\Omega}(\tilde{\chi})$ lies over $\Omega(\chi) \in \text{Irr}_2(M)$ (see [CS17 Remark 6.5]). We next prove that $\tilde{\Omega}$ is further $H$-equivariant.

Proposition 5.2. Let $G$ be one of the quasisimple groups of Lie type coming from a simply connected group of the types in Theorem 3. Then the map $\tilde{\Omega}$ from (3) is $H_2$-equivariant.

Proof. Let $\chi = R^G_F(\lambda)\eta \in \text{Irr}_2(G) \cap E(G, s)$. Let $\bar{T} := G_T$ for $T := TZ(\tilde{G})$ and $\tilde{\eta} := N_G(T)$. Then we may find $\tilde{\lambda} \in \text{Irr}(\tilde{T}|\bar{T})$ and $\tilde{\eta} \in \text{Irr}(\tilde{N}_\lambda/\bar{T})$ such that $\tilde{\chi} := R^G_{\tilde{T}}(\tilde{\lambda})\tilde{\eta} \in E(\tilde{G}, \tilde{s})$ lies above $\chi$. The other members of $\text{Irr}(\tilde{G}|\chi)$ are then of the form $\tilde{\chi}\beta$ for $\beta \in \text{Irr}(\tilde{G}|\tilde{G}G)$, since $\tilde{G}/G$ is abelian. (Recall then that $\beta = \hat{z}$ for some $z \in Z(\tilde{G}^s)$, and $\tilde{\chi}\beta = R^G_{\tilde{T}}(\tilde{\lambda}\hat{z})\tilde{\eta} \in E(\tilde{G}, \tilde{s}z)$.) Let $\sigma \in H := H_2$. Since $\tilde{N}_\lambda/\bar{T}$ is a Weyl group containing no component of type $G_2$, $E_7$, or $E_8$, we have $\tilde{\eta}^s = \tilde{\eta} = \tilde{\eta}^s$ (see e.g. [SF22 Proposition 5.4]). Since the other characters appearing in the description of the action of $\sigma$ in Theorem 122 are necessarily trivial since $C_{\tilde{G}}(\tilde{s})$ is connected, we see that $\tilde{\chi}^s = R^G_{\tilde{T}}(\tilde{\lambda}^s\hat{z})$ is determined only by $\tilde{\chi}^s$, or $\tilde{s}^s$. (Alternatively, one sees this using [SV20] and the fact that unipotent characters of odd degree are rational, see [SF19 Proposition 4.4].) We aim to show that the same is true on the local side.

Recall from the proof of Theorem 5.1 that the $MD$-equivariant extension map $\Lambda$ or $\Lambda_1$ with respect to $T < N$ or $T_1 < N_1$ is $H$-equivariant on the characters $\text{Irr}_{2cen}(T)$ or $\text{Irr}_{2cen}(T_1)$ under consideration. To ease notation, let $C := C_{G}(S)$, which we identify with $T$, respectively $T_1$, and let $\Lambda$ denote the corresponding extension map with respect to $C < M$ on $\text{Irr}_{2cen}(C)$. Let $\check{C} := C_{G}(S) \subset \check{M}$. Then using [CS17 Lemma 5.8(a)], we can define an extension map $\check{\Lambda}$ with respect to $\check{C} < \check{M}$ such that for $\lambda \in \text{Irr}(C)$ and $\check{\lambda} \in \text{Irr}(\check{C}|\lambda)$, $\check{\Lambda}(\check{\lambda})$ is the unique extension of $\check{\lambda}$ to $\check{M}$ such that $\text{Res}^{\check{M}}_{\check{\Lambda}}(\check{\lambda}) = \text{Res}^{\check{M}}_{\check{\Lambda}}(\Lambda(\lambda))$. This is the extension map used to construct the map $\check{\Omega}$ from (3) (see [CS17 Proposition 6.3]). Further, from the uniqueness and the $H$-equivariance of $\Lambda$, we see that $\check{\Lambda}$ is also $H$-equivariant on characters of $\check{C}$ above $\text{Irr}_{2cen}(C)$.

Let $\tilde{\psi} := \tilde{\Omega}(\tilde{\chi}) \in \text{Irr}(\tilde{M}|\tilde{\Omega}(\chi))$ and recall that $\psi := \Omega(\chi)$ is of the form $\text{Ind}^{M}_{\Lambda_1}(\Lambda(\lambda_1)\eta_1)$ with $\lambda_1 \in \text{Irr}(C)$ corresponding to $s$ and $\eta_1 \in \text{Irr}(M_{\Lambda_1}/C)$. Following the construction in [CS17]...
Section 6 [which is then used in [MS16 Theorem 6.3]], we may write \( \tilde{\psi} := \text{Ind} \tilde{M}_{\tilde{\lambda}_1} (\tilde{\Lambda} (\tilde{\lambda}_1) \tilde{\eta}_1) \) for \( \tilde{\lambda}_1 \in \text{Irr}(\tilde{C}|\lambda_1) \) corresponding to \( \tilde{s} \) and \( \tilde{\eta}_1 \in \text{Irr}(\tilde{M}_{\tilde{\lambda}_1}/\tilde{C}) \). For the same reason as before, we have \( \tilde{\eta}_1^a = \tilde{\eta}_1 \), so since \( \tilde{\Lambda} (\tilde{\lambda}_1)^a = \tilde{\Lambda} (\tilde{\lambda}_1^a) \), the image of \( \psi \) under \( \sigma \) is again determined by the image of \( \tilde{\lambda}_1 \), or equivalently, of \( \tilde{s} \), completing the proof. \( \square \)

6. Extensions and the Proof of Theorem \([A]\)

Keep the notation from Section 5.1. We now wish to complete the proof of Theorem \([A]\).

Thanks to Section 2, we may assume that \( G = G^F \) is a group of Lie type such that \( G \) is simple of simply connected type, \( G \) is the full Schur covering group of \( S = G/Z(G) \), and that \( S \) is one of the simple groups listed in Theorem \([A]\). We keep the notation from the previous section.

Given the \( \text{Aut}(G)_Q \times H \)-equivariant bijection \( \Omega: \text{Irr}_2(G) \to \text{Irr}_2(M) \) for \( G \) from Theorem 5.1 and [SF22 Theorem A], it suffices to show that

\[
(G \times \text{Aut}(G)_Q, \chi)_H \geq_c (M \times \text{Aut}(G)_Q, \chi, M, \Omega(\chi))_H
\]

for all \( \chi \in \text{Irr}_2(G) \), in the notation of [NSV20 Definition 1.5] (see also Condition \([S]\)). Let \( \tilde{M} := MN_G(S) \). Since \( \tilde{G}D \) induces all automorphisms of \( G \), and \( M \) can be chosen to be \( D \)-stable, [NSV20 Theorem 2.9] implies that it suffices to show

\[
((\tilde{G}D)_\chi, M, \Omega(\chi))_H \geq_c ((\tilde{M}D) \chi, M, \Omega(\chi))_H
\]

for all \( \chi \in \text{Irr}_2(G) \). Further, [NSV20 Lemma 2.2] yields that it suffices to show

\[
((\tilde{G}D)_\chi, \Omega(\chi))_H \geq_c ((\tilde{M}D) \chi, \Omega(\chi))_H
\]

for some \( \tilde{G} \)-conjugate \( \chi_0 \) of each \( \chi \in \text{Irr}_2(G) \). In particular, we will work with such a \( \chi_0 \) satisfying that \( (\tilde{G}D)_{\chi_0} = \tilde{G}_{\chi_0}D_{\chi_0} \) and that \( \chi_0 \) extends to \( GD_{\chi_0} \), which exists by [CS17 Theorem 4.1] and [MS16 Theorem 6.4 and Proof of Theorem 1]. We also have \( (\tilde{M}D)_{\Omega(\chi_0)} = \tilde{M}_{\Omega(\chi_0)}D_{\Omega(\chi_0)} \) and \( \Omega(\chi_0) \) extends to \( MD_{\Omega(\chi)} \) by using [MS16] Theorems 3.1 and 3.18 (see also [CS17 Section 5] for the case \( G \) is type A) and by considering the proof of [MS16 Corollary 5.3].

Note that parts (i) and (ii) of [NSV20 Definition 1.5] hold for these groups by construction and by the equivariance of \( \Omega \). Further, part (iii), which corresponds to the first bullet in Condition \([S]\), holds by [Spa12 Lemmas 2.11 and 2.13] and their proofs, once we prove (iv), corresponding to the second bullet in Condition \([S]\) using the constructions in loc. cit. and using [RSF22 Lemma 4.2].

Then it suffices to prove part (iv) of [NSV20 Definition 1.5] for \( \chi_0 \). Note that by [NSV20 Lemma 1.8], it suffices to prove part (iv) for a complete set of coset representatives for \( (MD)_{\chi_0} \) in \( (MD \times H)_{\chi_0} \). In particular, from the considerations in Remark 4.14, we see that it suffices to show (iv) for elements of \( (DH)_{\chi_0} \).

From now on, we will write \( \psi_0 := \Omega(\chi_0) \). Let \( D_1 := O_{2^2}(D_{\chi_0}) \) and \( D_2 \in \text{Syl}_2(D_{\chi_0}) \). Note that if \( G \neq D_1(g) \), we have \( D_{\chi_0} = D_1 \times D_2 \) is abelian.

**Proposition 6.1.** Let \( G \) be one of the groups in Theorem \([A]\), let \( \chi_0 \in \text{Irr}_2(G) \) be chosen as above and write \( \psi_0 := \Omega(\chi_0) \). Then there exist \( (N_D(D_2) \times H)_{\chi_0} \)-invariant extensions \( \chi_2 \) and \( \psi_2 \) of \( \chi_0 \) and \( \psi_0 \) to \( GD_2 \) and \( MD_2 \), respectively.

**Proof.** Let \( \mu \) be the linear character extending \( \psi_0 \) to \( MD_2 \) such that \( \mu \) is trivial on \( D_2 \). Then \( \mu^a = \mu \) for \( a \in (N_D(D_2) \times H)_{\chi_0} \), and by [Isa06 Lemma 6.24], we have a unique extension \( \psi_2 \) of \( \psi_0 \) to \( MD_2 \) such that \( \text{det} \psi_2 = \mu \). Then \( \psi_2^2 \) is another extension, with \( \text{det} \psi_2^2 = \mu^a = \mu \), so \( \psi_2^2 = \psi_2 \). Arguing similarly, we obtain an extension \( \chi_2 \) of \( \chi_0 \) to \( GD_2 \) with \( \chi_2^2 = \chi_2 \). \( \square \)

The next proposition, found in [Joh22a Proposition 8.7], is a generalized version of Proposition 2.6 and Corollary 2.7 from [RSF22] (minor corrections at arXiv:2106.14745v2).

**Proposition 6.2.** Suppose that \( G \) is a connected reductive group and \( F \) is a Frobenius endomorphism such that \( F = F_0^k \rho \) for some Frobenius endomorphism \( F_0 : G \to G \), graph automorphism
Proposition 6.4. Let \( \rho : G \to \mathbb{G} \) commuting with \( F_0 \), and positive integer \( k \). Let \( L \preceq P \) be \( \langle F_0, F \rangle \)-stable Levi and parabolic subgroups of \( G \) with \( (\mathbb{N}_G(L)/L)^{F_0} = N_G(L)/L \), where \( L := L^F \). Suppose that \( \lambda \in \text{Irr}(L) \) is an \( F_0 \)-invariant cuspidal character that extends to \( \text{Irr}(N_G(L \lambda(F_0))) \), and let \( \chi \in \text{Irr}(G) \) be a character in the Harish–Chandra series of \( \lambda \). Then:

1. For \( \chi \in \text{Irr}(G(F_0)) \) \( | \chi \rangle \), there exists a unique \( \hat{\lambda} \in \text{Irr}(L(F_0) | \lambda) \) such that \( \langle R^{G(F_0)}_{L(F_0)} \hat{\lambda}, \chi \rangle \neq 0 \). (Here we define \( R^{G(F_0)}_{L(F_0)} := \text{Ind}^{G(F_0)}_{L(F_0)} \circ \text{Infl}_{L(F_0)} \).

2. If \( \chi \) is \( F_0 \)-invariant, then \( \chi \) is \( \hat{\chi} \beta \) for some \( \beta \in \text{Irr}(G(F_0)/G) \). In particular, \( \beta \) is such that \( \lambda^{\alpha \beta} = \lambda \) and \( \hat{\lambda}^{\alpha \beta} = \lambda \beta \) for some \( x \in N_G(L) \).

For extensions to \( GD_1 \) in the case of \( D_4 \), the following will be useful, from [NT08 Corollary 2.2].

Lemma 6.3. Let \( X \prec Y \) be finite groups with \( |Y/X| \) odd, and let \( \theta \in \text{Irr}(X) \) be rational-valued. Then there exists a unique rational-valued character \( \hat{\theta} \in \text{Irr}(Y/\theta) \).

Proposition 6.4. Let \( G \) be one of the groups in Theorem 4.13, let \( \chi_0 \in \text{Irr}(G) \) be chosen as above and write \( \psi_0 := \Omega(\chi_0) \). Then there exist \( (D\mathbb{H})_{\chi_0} \)-invariant extensions \( \chi_1 \) and \( \psi_1 \) of \( \chi_0 \) and \( \psi_0 \) to \( GD_1 \) and \( MD_\psi \), respectively.

Proof. First, suppose that \( G \) is of type \( D_n \) with \( n \geq 4 \). Then by Corollary 4.12 and the proof of Theorem 5.1, \( \chi_0 \) and \( \psi_0 \) are rational-valued. Then by Lemma 6.3, there exists a unique rational character \( \chi_1 \), resp. \( \psi_1 \), in \( \text{Irr}(GD_1 | \chi_0) \), resp. \( \text{Irr}(MD_\psi | \psi_0) \). Since \( D_1 \) is abelian and \( \chi_0 \) and \( \psi_0 \) extend to the respective groups, these must be extensions. Then it now suffices to note that \( \chi_0 ^\alpha = \chi_1 \psi_0 \), and hence \( \chi_1 = \chi_1 \), and similarly \( \psi_1 = \psi_1 \).

Now suppose that \( G \) is not of type \( D_n \) (in fact, the following argument also works for \( D_n \) with \( n \neq 4 \)) and let \( \chi_0 \) lie in the series \( R^G(\lambda) \). Then there exists a Frobenius map \( F_0 \) such that \( GD_1 = G(F_0) \), \( F_0 = F \) for some positive integer \( r \), and \( W^{F_0} = W^F \), where \( W \) is the Weyl group of \( G \). Further, \( \chi \) (and, in the case \( d = 2 \), \( \lambda_1 \) ), is invariant under \( F \) and \( a \) by the proof of Lemma 4.7. In particular, the hypotheses of Proposition 6.2 hold in this case, as do the hypotheses in [RSF22] Lemmas 3.1(c), 3.3 with \( r \in \mathbb{G} \) replaced with \( a \). Then the exact same arguments as in loc. cit. apply here and give the desired extensions.

Corollary 6.5. Let \( G \) be one of the groups in Theorem 4.13, let \( \chi_0 \in \text{Irr}(G) \) be chosen as above and write \( \psi_0 := \Omega(\chi_0) \). Then there are \( (D\mathbb{H})_{\chi_0} \)-invariant extensions \( \hat{\chi}, \hat{\psi} \) of \( \chi_0 \) and \( \psi_0 \) to \( GD_\chi \) and \( MD_\psi \), respectively.

Proof. If \( G \) is not \( D_4(q) \), this follows from Propositions 6.1 and 6.4 by considering the unique common extensions of \( \chi_1 \) and \( \chi_2 \), respectively \( \psi_1 \) and \( \psi_2 \), to \( GD_\chi \), respectively \( MD_\psi \).

So now assume that \( G = D_4(q) \). As before, by Lemma 6.3 we can extend \( \chi_0 \) to a unique rational valued character \( \chi_1 \in \text{Irr}(GD_1) \). Moreover, we let \( \chi_2 \in \text{Irr}(GD_2) \) be the \( H \)-stable extension as in Proposition 6.1. The properties of \( \chi_1 \) imply that \( \chi_1 \) is \( D_2 \)-stable. As in the proof of [Ruh21] Corollary 6.10], we conclude that there exists an extension \( \hat{\chi} \in \text{Irr}(GD_\chi) \) which extends both \( \chi_1 \) and \( \chi_2 \). Now by Remark 4.13, we have \( (H\mathbb{H})r = H\mathbb{H}_r \). For \( a \in H \) there exists a linear character \( \beta \in \text{Irr}(D_{\chi_0}) \) such that \( \chi_0 ^\alpha = \beta \). Arguing as in [Ruh21] Corollary 6.10, we observe that \( D_\chi \) is \( H \)-stable. Replacing \( G \) by \( M \), we may argue the same to obtain the extension \( \hat{\psi} \).

Corollary 6.6. Let \( G \) be one of the groups in Theorem 4.13, let \( \chi_0 \in \text{Irr}(G) \) be chosen as above and write \( \psi_0 := \Omega(\chi_0) \). Let \( a \in (D\mathbb{H})_{\chi_0} \). Then there are extensions \( \hat{\chi}, \hat{\psi} \) of \( \chi_0 \) and \( \psi_0 \) to \( GD_\chi \) and \( MD_\psi \), respectively, satisfying that for \( a \in (D\mathbb{H})_{\chi_0} \), if \( \hat{\chi}^\alpha = \tilde{\chi} \mu_a \) and \( \hat{\psi}^\alpha = \tilde{\psi} \mu_a \) for \( \mu_a, \hat{\mu}_a \in \hat{\text{Irr}}(GD_{\chi_0}/G) \cong \hat{\text{Irr}}(M_{\chi_0}/M) \), then \( \mu_a = \hat{\mu}_a \).

In particular, Condition 3 holds for \( \chi_0 \) and for any \( a \in (D\mathbb{H})_{\chi_0} \).

Proof. First, note that the second statement follows from the first and from Corollary 5.5 by using [RSF22] Lemma 4.2 to obtain the desired extensions in [NSV20 Definition 1.5(iv)].
Recall that \( \chi_0 \) either extends to \( \tilde{G} \) or we have \( |Z(G)| \) and \( |\tilde{G}/GZ(\tilde{G})| \) are each a power of 2 and \( \chi_0 \) is stabilized by \( \mathcal{H} \). In the latter case, note that \( \chi_0 \) and \( \psi_0 \) may be extended trivially to \( GZ(\tilde{G}) \) and \( MZ(\tilde{G}) \), respectively, since they are trivial on \( Z(G) \) (as they have odd degree). So, we may identify \( \chi_0 \) and \( \psi_0 \) with these extensions. Then since \( \chi_0 \) has odd degree, \( o(\chi_0) = 1 \) (since \( G \) is perfect), and \( |\tilde{G}/GZ(\tilde{G})| \) is a power of 2, there is a unique extension \( \tilde{\chi}_0 \) of \( \chi_0 \) to \( \tilde{G} \), with \( o(\tilde{\chi}_0) = 1 \), by [Isa06, Corollary 8.16]. Then since \( \tilde{\chi}_0 \) is another extension with \( o(\tilde{\chi}_0) = 1 \), we have \( \tilde{\chi}_0 = \tilde{\chi} \). Further, in these cases, recall that \( \chi_0 \) is rational-valued, so that \( a \in D_{X_0} \mathcal{H} \). Then \( \chi_0 := \text{Ind}_{\tilde{G}}^{\tilde{\tilde{G}}}(\tilde{\chi}) \) is also \( a \)-invariant. Then \( \psi_0 := \tilde{\Omega}(\tilde{\chi}_0) \) is \( a \)-invariant since \( \tilde{\Omega} \) is \((\tilde{G}D \times \mathcal{H})\)-equivariant by Proposition 5.2 and the discussion before it. Then since \( \tilde{\psi}_0 \) must lie above \( \psi_0 \) (by the properties of \( \tilde{\Omega} \) discussed above), \( \tilde{M}/M \) is abelian, we have \( \tilde{\psi}_0 = \text{Ind}_{\tilde{M}}^{\tilde{\tilde{M}}}(\tilde{\psi}) \) for some extension \( \tilde{\psi} \) of \( \psi_0 \) to \( \tilde{M} \). Now, since \( \psi_0^\alpha = \psi_0 \), we have \( \tilde{\psi}^\alpha = \tilde{\psi} \beta \) for some linear \( \beta \in \text{Irr}(\tilde{M}/\psi_0 \tilde{M}) \). But then \( \tilde{\psi}_0 = \psi_0^\alpha = \text{Ind}_{\tilde{M}}^{\tilde{\tilde{M}}}(\tilde{\psi}^\alpha) = \text{Ind}_{\psi_0 \tilde{M}}^{\tilde{M}}(\tilde{\psi} \beta) = \tilde{\psi}_0 \beta \), where the second-to-last equality follows from [Isa06, Problem (5.3)]. Then \( \beta = 1 \) and \( \tilde{\psi}^\alpha = \tilde{\psi} \).

So, we are left with the case that \( \chi_0 \) and \( \psi_0 \) extend to \( \tilde{G} \) and \( \tilde{M} \), respectively. In this case, let \( \tilde{\chi}_0 \) be an extension of \( \chi_0 \), so that \( \tilde{\chi}_0 = \tilde{\chi}_0 \beta \) for some \( \beta \in \text{Irr}(\tilde{G}/G) \) and let \( \psi_0 := \tilde{\Omega}(\tilde{\chi}_0) \). Then \( \tilde{\psi}_0 \) is an extension of \( \psi_0 \) as well, since it lies above \( \psi_0 \) and \( \tilde{M}/\tilde{M} \) is abelian. Then, recalling that \( \tilde{\Omega} \) is \((\tilde{G}D \times \mathcal{H})\)-equivariant by Proposition 5.2 and the properties discussed after [GU], we have \( \psi_0^\alpha = \tilde{\Omega}(\tilde{\chi}_0 \alpha) = \tilde{\Omega}(\tilde{\chi}_0) = \tilde{\Omega}(\tilde{\chi}_0 \beta) = \tilde{\psi}_0 \beta \). The result now follows, taking \( \tilde{\chi}_0 := \tilde{\chi}_0, \tilde{\psi}_0 := \psi_0 \), and \( \mu_a = \beta = \mu_a' \). \( \square \)

From here, we see that we have proved the inductive McKay–Navarro conditions for the groups in Theorem A.3, which combined with Proposition 2.1 completes the proof of Theorem A. In conjunction with those results discussed in the introduction, this completes the proof of the McKay–Navarro conjecture for \( \ell = 2 \).

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