THE DEGREE OF THE THIRD SECANT VARIETY OF A SMOOTH CURVE OF GENUS 2

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Abstract. We give a new method of computation of the degree of the third secant variety \( \text{Sec}_3(C) \) of a smooth curve \( C \subseteq \mathbb{P}^{d-2} \) of genus 2 and degree \( d \geq 8 \), using the presentation of \( \text{Sec}_3(C) \) as the union of all scrolls that are defined via a \( g_1^3 \) on \( C \).

1. Introduction

Berzolari’s formula from 1895 (cf. [2], Section 4) computes the number of trisecant lines to a smooth curve of genus \( g \) and degree \( d \) in \( \mathbb{P}^4 \), in the case this number is finite. This number is equal to \( \binom{d-2}{3} - g(d-4) \).

In this paper \( C \) denotes a smooth and irreducible curve of genus 2 and degree \( d \geq 8 \) embedded in \( \mathbb{P}^{d-2} \).

The third secant variety of \( C \), \( \text{Sec}_3(C) \), is defined as the closure of the union of all trisecant planes to \( C \):

\[
\text{Sec}_3(C) = \bigcup_{D \in C_3} \text{span}(D),
\]

where \( C_3 := (C \times C \times C)/S_3 \) parametrizes effective divisors of degree 3 on \( C \), and \( \text{span}(D) \) denotes the plane spanned by the three points in \( D \).

The dimension of \( \text{Sec}_3(C) \) is equal to \( \dim(C_3) + \dim(\text{span}(D)) = 5 \), i.e. in order to find the degree of \( \text{Sec}_3(C) \) we have to intersect with five general hyperplanes.

Let now \( V \) denote the intersection of five general hyperplanes in \( \mathbb{P}^{d-2} \), i.e. \( V \) is a general space of codimension 5 in \( \mathbb{P}^{d-2} \). Since \( \dim(\text{Sec}_2(C)) = 3 \) and \( \text{codim}(V) = 5 \), \( V \) and \( \text{Sec}_2(C) \) do not intersect. This implies that \( V \) cannot intersect any trisecant plane to \( C \) in a line, since every trisecant plane to \( C \) contains three lines in \( \text{Sec}_2(C) \), and so if \( V \) intersects a trisecant plane to \( C \) in a line \( L \), then \( L \) intersects at least one of those lines in \( \text{Sec}_2(C) \) in a point which obviously lies in \( \text{Sec}_2(C) \).

Projecting from \( V \) to \( \mathbb{P}^4 \) gives us the equality of the degree of \( \text{Sec}_3(C) \) and the number of trisecant lines to a curve \( C \subseteq \mathbb{P}^4 \) of genus 2 and the same degree \( d \) in the following way: Since \( V \) was chosen to be a general space of codimension 5, \( V \) does not intersect the curve \( C \), and thus the image of \( C \) under the projection from \( V \) is a curve of degree \( d \) and genus 2 in \( \mathbb{P}^4 \). Moreover, the fact that \( V \) does not intersect \( \text{Sec}_2(C) \) implies that the image curve is smooth as well.
A trisecant plane to $C \subseteq \mathbb{P}^{d-2}$ which intersects $V$ in one point projects down to a trisecant line to the image curve in $\mathbb{P}^4$.

Summarizing, the number of trisecant planes to $C \subseteq \mathbb{P}^{d-2}$ that intersect $V$ in one point is equal to the number of trisecant lines to the image curve in $\mathbb{P}^4$, and thus it follows that the degree of $\text{Sec}_3(C)$ is equal to the number of trisecant lines to the image curve in $\mathbb{P}^4$.

Consequently, the degree of $\text{Sec}_3(C)$ is equal to $(d-2) - 2(d-4)$, and our motivation is now to compute the degree of $\text{Sec}_3(C)$ in a different way, identifying $\text{Sec}_3(C)$ as the union of all scrolls defined via a $g^1_3$ on $C$.

Any abstract curve $C$ of genus 2 can be embedded as a smooth curve of degree $d \geq 5$ into $\mathbb{P}^{d-2}$.

In this paper we restrict ourselves to the case $d \geq 8$, since for a curve $C$ of genus 2 and degree $d = 6$ or $d = 7$, although Berzolari’s formula of course being valid, taking Berzolari’s formula to compute the degree of $\text{Sec}_3(C)$ does not make sense, since for these values of $d$ the third secant variety $\text{Sec}_3(C)$ is equal to the ambient space $\mathbb{P}^{d-2}$. For $d = 5$ the following holds: There are infinitely many trisecant lines to a curve $C \subseteq \mathbb{P}^3$ of genus 2 and degree 5, since $C$ lies on a quadric on which there exists a one-dimensional family of lines that each intersects $C$ in three points.

2. Preliminaries

Let $C$ be a smooth curve of genus 2 and degree $d \geq 8$ embedded in projective space $\mathbb{P}^{d-2}$. For each $g^1_3$ on $C$, which we denote by $|D|$, we set

$$V_{|D|} := \bigcup_{D' \in |D|} \text{span}(D'),$$

where $\text{span} D'$ denotes the plane spanned by the three points in $|D|$. Each $V_{|D|}$ is a threedimensional rational normal scroll. (For general theory about rational normal scrolls we refer to [3].)

We set $G^1_3(C) := \{g^1_3 \text{‘s on } C\}$. Since our aim is to identify $\text{Sec}_3(C) = \bigcup_{|D| \in G^1_3(C)} V_{|D|}$, we want to find the dimension of $\bigcup_{|D| \in G^1_3(C)} V_{|D|}$, and for this purpose we need the dimension of the family $G^1_3(C)$, which we will now compute:

**Proposition 2.1.** Let $C$ be a curve of genus 2. The family $G^1_3(C) = \{g^1_3 \text{‘s on } C\}$ is two-dimensional.

**Proof.** If $D$ is a divisor of degree 3 on $C$, then $h^0(\mathcal{O}_C(D)) = 2$ by the Riemann-Roch theorem for curves (see e.g. [5], Thm. 1.3 in Chapter IV.1), i.e. each linear system $|D|$ of degree 3 is a $g^1_3$ on $C$. The set of all effective divisors of degree 3 on $C$ is given by $C_3 := (C \times C \times C)/S_3$, where $S_3$ denotes the symmetric group on 3 letters. The dimension of this family is equal to 3, and since each linear system $|D|$ of degree 3 on $C$ has dimension 1, as shown above, the family of $g^1_3$’s on $C$ has to be two-dimensional. □
We obtain that the dimension of $\bigcup_{|D| \in G^2_3(C)} V_{|D|}$ is equal to 5, which is also the dimension of $\text{Sec}_3(C)$, as we have seen in the introduction, and since each scroll $V_{|D|}$ obviously is contained in $\text{Sec}_3(C)$, we obtain equality:

$$\text{Sec}_3(C) = \bigcup_{|D| \in G^2_3(C)} V_{|D|}.$$  

For an integer $k \geq 0$ we denote by $\text{Pic}^k(C)$ the set of all line bundles of degree $k$ on $C$ modulo isomorphism. In this paper we will consider $k = 0$ and $k = 3$.

We use the definition of the Jacobian variety of $C$, $\text{Jac}(C)$, as in [6], namely that $\text{Jac}(C)$ is defined as the abelian variety that represents the functor $T \to \text{Pic}^0(C/T)$ from schemes over the base field $k$ to abelian groups (cf. [6], Theorem 1.1).

By fixing a divisor $D_0$ of degree 3 we obtain an isomorphism

$$\mu : \text{Pic}^0(C) \to \text{Pic}^3(C),$$

$$[\mathcal{O}_C(D)] \mapsto [\mathcal{O}_C(D + D_0)].$$

Hence $\text{Pic}^3(C)$ is isomorphic to the Jacobian variety $\text{Jac}(C)$. Fixing a point $P_0$ on $C$ gives an embedding

$$\nu : C \to \text{Jac}(C),$$

$$R \mapsto [\mathcal{O}_C(R - P_0)].$$

The dimension of $\text{Jac}(C)$ is equal to the genus of $C$, which is equal to 2. Hence $\text{Jac}(C)$ is an abelian surface. The theta divisor $\Theta$ on $\text{Jac}(C)$ is the image of $C$ under the above map $\nu$.

For fixed points $P$ and $Q$ on $C$ we define

$$\Theta_{P,Q} := \{[\mathcal{O}_C(P + Q + R)] | R \in C\}.$$  

$\Theta_{P,Q}$ is a divisor on $\text{Pic}^3(C)$, and using the above isomorphism $\mu$ with $D_0 = P + Q + P_0$ we see that the divisor $\Theta_{P,Q}$ is isomorphic to $\Theta$. It is this $\Theta_{P,Q}$ we will use in Sections 4 and 5 when we consider $\Theta$ on $\text{Pic}^3(C)$.

**Proposition 2.2.** The divisor $\Theta$ has self-intersection $\Theta^2 = 2$.

**Proof.** Choose points $P$, $P'$, $Q_1$ and $Q_2$, $Q_1 \neq Q_2$, on $C$ such that $P + P'$ is a divisor in the canonical system on $C$, $|K_C|$, and such that $Q_1 + Q_2$ is not a divisor in $|K_C|$. There exist points $Q'_1$ and $Q'_2$ on $C$ such that $Q_1 + Q'_1 \in |K_C|$ and $Q_2 + Q'_2 \in |K_C|$. We obtain the following:

$$\Theta^2 = \Theta_{P,Q_1} \cdot \Theta_{P',Q_2} = \#\{[\mathcal{O}_C(Q_1 + Q_2 + R)] | R \in \{Q'_1, Q'_2\}\} = 2.$$  

$\square$
Consider now the following projections:

\[
\begin{array}{c}
C 	imes \text{Pic}^3(C) \\
\downarrow p \\
C
\end{array}
\quad \quad \quad
\begin{array}{c}
C 	imes C_3 \\
\downarrow \pi \\
C_3
\end{array}
\quad \quad \quad
\begin{array}{c}
\text{Pic}^3(C) \\
\downarrow q \\
C
\end{array}
\]

Let \( P \) be a point on \( C \) such that \( 2P \) is a divisor in the canonical system \( |K_C| \), and set \( f := p^*(P) \).

In the rest of this paper we will use the notation \( P \) and \( f \) both as varieties and as classes.

As before, we define \( C_3 \) to be the threedimensional family of all effective divisors of degree 3 on \( C \).

Let \( \Delta \) be the universal divisor on \( C \times C_3 \), i.e. \( \Delta|_{C \times \{D\}} \cong D \) for all \( D \in C_3 \).

For any point \( Q \) on \( C \) set \( X_Q := \{ D \in C_3 | Q \in D \} \), which is a divisor on \( C_3 \).

Finally, let \( u : C_3 \to \text{Pic}^3(C) \) be the map given by \( u(D) := [O_C(D)] \).

Now we are able to define a line bundle \( L \) on \( C \times \text{Pic}^3(C) \) which turns out to be a Poincaré line bundle. In Section 5 we will compute the degree of \( \text{Sec}^3(C) \) by identifying \( \text{Sec}^3(C) \) as a degeneracy locus of a map of vector bundles involving this Poincaré line bundle \( L \).

3. The Poincaré line bundle \( \mathcal{L} \)

We will first give the definition of a Poincaré line bundle:

**Definition 3.1.** A Poincaré line bundle of degree \( k \) is a line bundle \( \mathcal{L} \) on \( C \times \text{Pic}^3(C) \) such that \( \mathcal{L}|_{C \times [O_C(D)]} \cong O_C(D) \) for all points \( [O_C(D)] \) in \( \text{Pic}^3(C) \).

Set \( \mathcal{L} := (1 \times u)_* (O_{C \times C_3} (\Delta - \pi^*(X_Q))) \). \( \mathcal{L} \) is a Poincaré line bundle of degree 3 (cf. [1], Chapter IV, §2, p. 167).

Let \( |H| \) be the linear system of degree \( d \) that embeds \( C \) into projective space \( \mathbb{P}^{d-2} \). Set \( \mathcal{H} := q_* (\mathcal{L}) \) and \( \mathcal{G} := q_* (p^* O_C(H) \otimes \mathcal{L}^{-1}) \).

Since the fiber of \( \mathcal{H} \) over a point \( [O_C(D)] \in \text{Pic}^3(C) \) is equal to \( H^0(O_C(D)) \), the rank of \( \mathcal{H} \) is equal to \( h^0(O_C(D)) = 2 \), and since the fiber of \( \mathcal{G} \) over a point \( [O_C(D)] \in \text{Pic}^3(C) \) is equal to \( H^0(O_C(H - D)) \), the rank of \( \mathcal{G} \) is equal to \( h^0(O_C(H - D)) = d - 4 \).

We will use these vector bundles \( \mathcal{H} \) and \( \mathcal{G} \) in Section 3 to define a map of vector bundles which degeneracy locus is equal to the third secant variety of \( C \), \( \text{Sec}^3(C) \). We will need the Chern classes of \( \mathcal{H} \) and \( \mathcal{G} \), and for this purpose we need the Chern classes of \( \mathcal{L} \). We will find all of these Chern classes in the next section.

4. The Chern classes of \( \mathcal{L}, \mathcal{H} \) and \( \mathcal{G} \)

In this section we will find the Chern classes of \( \mathcal{L}, \mathcal{H} \) and \( \mathcal{G} \) as defined in Section 3.
4.1. The Chern class of $L$. By [1], Chapter VIII, §2 (pp. 333-336) we obtain that the first Chern class of $L$ is equal to $c_1(L) = 3f + \gamma$, where $\gamma$ is the diagonal component of $c_1(L)$ in the term $H^1(C) \otimes H^1(\text{Pic}^3(C))$ of the Künneth decomposition

$$H^2(C \times \text{Pic}^3(C)) = (H^2(C) \otimes H^0(\text{Pic}^3(C)))$$

$$\oplus (H^1(C) \otimes H^1(\text{Pic}^3(C)))$$

$$\oplus (H^0(C) \otimes H^2(\text{Pic}^3(C))).$$

The following is satisfied: $\gamma^2 = -2f.q^*(\Theta)$, $\gamma^3 = f.\gamma = 0$, where now $\Theta$ on $\text{Pic}^3(C)$ is equal to $\Theta_{P,Q}$ as defined in Section 2.

Thus for the Chern character of $L$ we obtain:

$$\text{ch}(L) = e^{c_1(L)} = 1 + 3f + \gamma - f.q^*(\Theta).$$

4.2. The Chern classes of $H$. Recall that we had defined $H := q_*L$. The Chern character of $H$ we obtain by the Grothendieck-Riemann-Roch Theorem (cf. [4], Thm. 15.2):

$$\text{ch}(q_*(L)) \cdot \text{td}(\text{Pic}^3(C)) = q_*(\text{ch}(L) \cdot \text{td}(C \times \text{Pic}^3(C))).$$

Before we can continue our computation of $\text{ch}(H)$ we need some Todd classes and pushforwards.

**Definition 4.1.** (cf. [4], Example 3.2.4) The Todd class of a vector bundle $E$ of rank $r$ on a variety $X$ is defined as

$$\text{td}(E) = \prod_{i=1}^{r} \frac{\alpha_i}{1 - e^{\alpha_i}},$$

where $\alpha_1, \ldots, \alpha_r$ are the Chern roots of $E$.

If $Y$ is a variety, then by $\text{td}(Y)$ we denote $\text{td}(T_Y)$, the Todd class of the tangent bundle of $Y$.

We will need Todd classes only in the cases when the dimension of $X$ is equal to 1 or 2. In these cases $c_i(E) = 0$ for $i \geq 3$, and expanding the above product yields:

$$\text{td}(E) = 1 + \frac{1}{2}c_1(E) + \frac{1}{12}(c_1^2(E) + c_2(E)).$$

**Lemma 4.2.** We have the following Todd classes:

1. $\text{td}(\text{Pic}^3(C)) = 1$.
2. $\text{td}(C) = 1 - P$.
3. $\text{td}(C \times \text{Pic}^3(C)) = 1 - f$.

**Proof.**

1. Since $\text{Pic}^3(C) \cong \text{Jac}(C)$ is an abelian variety, we have $K_{\text{Pic}^3(C)} = 0$ and thus also $c_1(T_{\text{Pic}^3(C)}) = 0$.
2. $\text{td}(C) = 1 + \frac{1}{2}c_1(T_C) = 1 - \frac{1}{2}[K_C] = 1 - P.$
\( (3) \, \text{td}(C \times \text{Pic}^3(C)) = \text{td}(p^*(C)). \text{td}(q^* \text{Pic}^3(C)) = 1 - f. \) \\

**Lemma 4.3.** We have the following pushforwards:

(1) \( q_*(1) = 0. \)

(2) \( q_*(f) = 1. \)

(3) \( q_*(\gamma) = 0. \)

**Proof.**

(1) \( q_*(1) = q_*([C \times \text{Pic}^3(C)]) = 0, \) since \( \dim(q(C \times \text{Pic}^3(C))) = \dim(\text{Pic}^3(C)) = 2 < 3 = \dim(C \times \text{Pic}^3(C)). \)

(2) Since \( q(f) = \text{Pic}^3(C) \) has the same dimension as \( f, \) we have \( q_*(f) = a[\text{Pic}^3(C)] \) for a positive integer \( a. \) By the projection formula (cf. \[4\], Prop. 2.5(c)) we obtain for every point \( [O_C(D_0)] \in \text{Pic}^3(C): \)

\[
\begin{align*}
a &= q_*(f).[O_C(D_0)] = q_*(f.q^*[O_C(D_0)]) = f.q^*[O_C(D_0)] \\
&= [P \times \text{Pic}^3(C)].[C \times O_C(D_0)] = 1,
\end{align*}
\]

where we could use the equality \( q_*(f.q^*[O_C(D_0)]) = f.q^*[O_C(D_0)], \) since \( f.q^*[O_C(D_0)] \) is 0-dimensional.

(3) Since \( \gamma \) is of codimension 1 on \( C \times \text{Pic}^3(C), q_*(\gamma) = a[\text{Pic}^3(C)] \) for some non-negative integer \( a. \) By the projection formula we have for every point \( [O_C(D_0)] \in \text{Pic}^3(C): \)

\[
\begin{align*}
a &= q_*(\gamma).[O_C(D_0)] = q_*(\gamma.q^*[O_C(D_0)]) = \gamma.q^*[O_C(D_0)] \\
&= c_1(\mathcal{L}).q^*[O_C(D_0)] - q_*(3f) = 3 - 3 = 0,
\end{align*}
\]

where, analogously to (2), we could use the equality \( q_*(\gamma.q^*[O_C(D_0)]) = \gamma.q^*[O_C(D_0)] \) since \( \gamma.q^*[O_C(D_0)] \) is 0-dimensional.

Now, by Lemma 4.2 we obtain:

\[
\begin{align*}
\text{ch}(\mathcal{H}) &= \text{ch}(q_*(\mathcal{L})) \\
&= q_*(\text{ch}(\mathcal{L}).(1 - f)) = q_*((1 + 3f + \gamma - f.q^*(\Theta)).(1 - f)) \\
&= q_*(1 + 2f + \gamma - f.q^*(\Theta)).
\end{align*}
\]

By Lemma 4.3 and the projection formula we can conclude:

\[
\text{ch}(\mathcal{H}) = 2 - q_*(f).\Theta = 2 - \Theta.
\]

Consequently we obtain for the Chern polynomial of \( \mathcal{H}: \)

\[
(1) \quad c_t(\mathcal{H}) = e^{-\Theta t}.
\]
4.3. The Chern classes of $\mathcal{G}$. Now we want to find the Chern classes of the vector bundle $\mathcal{G} := q_*(p^*(\mathcal{O}_C(H) \otimes \mathcal{L}^{-1}))$, where $|H|$ denotes the linear system of degree $d$ that embeds $C$ into projective space. In order to do so we use again the Grothendieck-Riemann-Roch formula:

$$\text{ch}(q_*(p^*(\mathcal{O}_C(H) \otimes \mathcal{L}^{-1}))). \text{td}(\text{Pic}^3(C)) = q_*(\text{ch}(p^*(\mathcal{O}_C(H) \otimes \mathcal{L}^{-1}))). \text{td}(C \times \text{Pic}^3(C)).$$

By Lemma 4.2, Lemma 4.3 and the projection formula we obtain

$$\text{ch}(\mathcal{G}) = q_*(\text{ch}(p^*(\mathcal{O}_C(H) \otimes \mathcal{L}^{-1})).(1 - f))$$

$$= q_*(p^*(\text{ch}(\mathcal{O}_C(H))). \text{ch}(\mathcal{L}^{-1}).(1 - f))$$

$$= q_*(1 + p^*(H)).(1 - 3f - \gamma - f.q^*(\Theta)).(1 - f))$$

$$= q_*(1 + df).(1 - 4f - \gamma - f.q^*(\Theta)))$$

$$= q_*(1 + (d - 4)f - \gamma - f.q^*(\Theta))$$

$$= d - 4 - \Theta.$$ 

This yields for the Chern polynomial of $\mathcal{G}$:

$$c_t(\mathcal{G}) = e^{-\Theta t}. \quad (2)$$

5. The degree of Sec$_3(C)$

Set $E := \mathcal{G} \boxtimes \mathcal{O}_{\mathbb{P}^{d-2}}(-1)$ and $F := \mathcal{H}^* \boxtimes \mathcal{O}_{\mathbb{P}^{d-2}}$. These are two vector bundles on $\text{Pic}^3(C) \times \mathbb{P}^{d-2}$. The rank of $E$ is equal to $d - 4$, and the rank of $F$ is equal to 2.

The multiplication of fibers

$$H^0(\mathcal{O}_C(H - D)) \otimes H^0(\mathcal{O}_C(D)) \to H^0(\mathcal{O}_C(H))$$

induces a map of vector bundles $\Phi : E \to F$. Set

$$X_1 := X_1(\Phi) := \{x \in \text{Pic}^3(C) \times \mathbb{P}^{d-2} | \text{rk}(\Phi_x) \leq 1\}$$

Consider the two projections

$$\begin{array}{ccc}
\text{Pic}^3(C) \times \mathbb{P}^{d-2} & \xrightarrow{p_1} & \text{Pic}^3(C) \\
& & \text{Pic}^3(C)
\end{array}$$

$$\begin{array}{ccc}
\text{Pic}^3(C) \times \mathbb{P}^{d-2} & \xrightarrow{p_2} & \mathbb{P}^{d-2} \\
& & \mathbb{P}^{d-2}
\end{array}$$

We have the following:

(i) Over every point $[\mathcal{O}_C(D)] \in \text{Pic}^3(C)$ the fiber of $p_1|X_1$ is a 3-dimensional rational normal scroll $V_{[D]} \subseteq [\mathcal{O}_C(D)] \times \mathbb{P}^{d-2} \cong \mathbb{P}^{d-2}$.

(ii) The image of such a fiber under the projection $p_2$ is thus the rational normal scroll $V_{[D]}$ in $\mathbb{P}^{d-2}$. 

(iii) Consequently, $p_2(X_1)$ is the union of all $g_3^1(C)$-scrolls $V_D$ in $\mathbb{P}^{d-2}$ which again is equal to Sec$_3(C)$.

Set $x_1$ to be the class of $X_1$. From the above we have $(p_2)_*(x_1) = [\text{Sec}_3(C)]$. Let $h' \subseteq \mathbb{P}^{d-2}$ be a hyperplane class and set $h := (p_2)^*(h') \subseteq \text{Pic}^3(C) \times \mathbb{P}^{d-2}$. Since Sec$_3(C) \subseteq \mathbb{P}^{d-2}$ has dimension 5, we obtain the degree of Sec$_3(C)$ by intersecting with $(h')^5$. Now we have the following:

\[
\text{deg}(\text{Sec}_3(C)) = [\text{Sec}_3(C)](h')^5 = (p_2)_*(x_1)^5 = (p_2)_*(x_1.h^5) = x_1.h^5.
\]

That is, now we have to find the class $x_1$ of $X_1(\Phi)$. Since $X_1(\Phi)$ has expected dimension $5 = \dim(\text{Pic}^3(C) \times \mathbb{P}^{d-2}) - (d - 4 - 1)(2 - 1)$, by Porteous’ formula ([1], Chapter II, (4.2)) we obtain the following:

\[
x_1 = \Delta_{1,d-5}(c_t(F - E))
\]

where $c_i := c_i(F - E)$, and $c_i(F - E)$ is defined via $c_t(F - E) := \frac{c_t(F)}{c_t(E)}$.

Again we use $\Theta_{P,Q}$ as defined in Section 2 when we talk about $\Theta$ on $\text{Pic}^3(C) \cong \text{Jac}(C)$. Equations (1) and (2) gave us the following Chern polynomials:

\[
c_t(H) = c_t(G) = e^{-\Theta t}.
\]

We thus obtain

\[
c_t(F) = c_t(p_1^*H) = c_{-t}(p_1^*H) = e^{p_1^*\Theta t}.
\]

We compute $c_t(E)$:

Let $\alpha_i$ be the Chern roots of $G$, i.e. $c_t(G) = \prod_{i=1}^{d-4} (1 + \alpha_i t)$, and set $\beta_i := p_1^*(\alpha_i)$. Then we obtain the following:
\[
c_i(E) = \prod_{i=1}^{d-4} (1 + (\beta_i - h)t) = \prod_{i=1}^{d-4} (1 - ht) \left( 1 + \beta_i \frac{t}{1 - ht} \right)
\]
\[
= (1 - ht)^{d-4} \prod_{i=1}^{d-4} \left( 1 + \beta_i \frac{t}{1 - ht} \right) = (1 - ht)^{d-4} c_{\frac{t}{1-h}} \left( p^* \mathcal{G} \right)
\]
\[
= (1 - ht)^{d-4} e^{-\frac{p^* \Theta t}{1-h t}}.
\]

In the following we will identify \( \Theta \) with \( p^* (\Theta) \), it will be clear from the context if we mean \( \Theta \) on \( \text{Pic}^3(C) \) or \( \Theta \) on \( \text{Pic}^3(C) \times \mathbb{P}^{d-2} \).

We conclude now:

\[
c_i(F - E) = e^{\Theta t} (1 - ht)^{4-d} e^{\frac{\Theta t}{1-h t}} = (1 - ht)^{4-d} e^{\frac{2\Theta t - \Theta h t^2}{1-h t^2}}
\]
\[
= (1 - ht)^{4-d} \sum_{j=0}^{\infty} \frac{1}{j!} (2\Theta t - \Theta h t^2)^j (1 - ht)^{-j}
\]
\[
= \sum_{j=0}^{\infty} (1 - ht)^{4-d-j} \frac{1}{j!} (2\Theta t - \Theta h t^2)^j.
\]

Since \( \Theta^3 = 0 \), we only get some contribution from \( j = 0, 1, 2 \) and thus obtain the following:

\[
c_i(F - E) = (1 - ht)^{4-d} + (1 - ht)^{3-d}(2\Theta t - \Theta h t^2)
\]
\[
+ \frac{1}{2} (1 - ht)^{2-d}(4\Theta^2 t^2 - 4\Theta^2 h t^3 + \Theta^2 h^2 t^4)
\]
\[
= \sum_{k=0}^{\infty} \begin{pmatrix} d + k - 3 \\ k \end{pmatrix} h^k t^k
\]
\[
+ \sum_{k=0}^{\infty} \begin{pmatrix} d + k - 3 \\ k \end{pmatrix} (2\Theta h^k - 2h^{k+1}) t^{k+1}
\]
\[
+ \sum_{k=0}^{\infty} \begin{pmatrix} d + k - 3 \\ k \end{pmatrix} (2\Theta^2 h^k - 3\Theta h^{k+1} + h^{k+2}) t^{k+2}
\]
\[
+ \sum_{k=0}^{\infty} \begin{pmatrix} d + k - 3 \\ k \end{pmatrix} (\Theta h^{k+2} - 2\Theta^2 h^{k+1}) t^{k+3}
\]
\[
+ \sum_{k=0}^{\infty} \frac{1}{2} \begin{pmatrix} d + k - 3 \\ k \end{pmatrix} \Theta^2 h^{k+2} t^{k+4}.
\]
This implies that
\[
c_i(F - E) = \left( \binom{d - 3 + i}{i} - 2 \binom{d - 4 + i}{i - 1} + \binom{d - 5 + i}{i - 2} \right) h^i
+ \left( 2 \binom{d - 4 + i}{i - 1} - 3 \binom{d - 5 + i}{i - 2} + \binom{d - 6 + i}{i - 3} \right) \Theta h^{i-1}
+ \left( 2 \binom{d - 5 + i}{i - 2} - 2 \binom{d - 6 + i}{i - 3} + \frac{1}{2} \binom{d - 7 + i}{i - 4} \right) \Theta^2 h^{i-2}
= \binom{d - 5 + i}{i} h^i
+ \left( \binom{d - 5 + i}{i - 1} + \binom{d - 6 + i}{i - 1} \right) \Theta h^{i-1}
+ \left( 2 \binom{d - 6 + i}{i - 2} + \frac{1}{2} \binom{d - 7 + i}{i - 4} \right) \Theta^2 h^{i-2}.
\]

Now the last step in the computation of \( x_1 \) is to find the determinant of the matrix \( A_{d-5} \):

**Proposition 5.1.** Set \( c_i := c_i(F - E) \), where \( c_i(F - E) \) is defined via \( c_t(F - E) := \frac{c_t(F)}{c_t(E)} \).

For \( d \geq 8 \) the determinant of the matrix

\[
A_{d-5} = \begin{pmatrix}
c_1 & c_2 & c_3 & \cdots & c_{d-6} & c_{d-5} \\
1 & c_1 & c_2 & \cdots & c_{d-7} & c_{d-6} \\
0 & 1 & c_1 & \cdots & c_{d-8} & c_{d-7} \\
& & & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & c_1 & c_2 \\
0 & 0 & 0 & \cdots & 1 & c_1 \\
\end{pmatrix}
\]
is equal to

\[
D_{d-5} = \left( \frac{1}{2} \binom{d - 2}{3} - (d - 4) \right) \Theta^2 h^{d-7}
+ \left( \binom{d - 3}{2} - 1 \right) \Theta h^{d-6}
+ (d - 4) h^{d-5}.
\]

**Proof.** Let \( d \) be fixed. Set \( d_0 := 1 \) and for \( n = 1, \ldots, d - 5 \), \( k = 2, \ldots, d - 6 \), set
THE DEGREE OF THE THIRD SECANT VARIETY OF A SMOOTH CURVE OF GENUS 2

\[ d_n := \det \begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_n \\ 1 & c_1 & c_2 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & c_1 & c_2 \\ 0 & 0 & \cdots & 1 & c_1 \end{pmatrix} \]

and

\[ b_{n,k} := \det \begin{pmatrix} c_k & c_{k+1} & c_{k+2} & \cdots & c_{n-1} & c_n \\ 1 & c_1 & c_2 & \cdots & c_{n-(k+1)} & c_{n-k} \\ 0 & 1 & c_1 & \cdots & c_{n-(k+2)} & c_{n-(k+1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & c_1 & c_2 \\ 0 & 0 & 0 & \cdots & 1 & c_1 \end{pmatrix}. \]

By expansion with respect to the first column we have for each \( n \) and \( k \):

\[ d_n = c_1d_{n-1} - b_{n,2} \]

and

\[ b_{n,k} = c_kd_{n-k} - b_{n,k+1}. \]

This gives us by induction:

\[ d_n = \sum_{i=1}^{n} (-1)^{i-1}c_id_{n-i}. \]

Computing \( d_n \) for low \( n \) leads us to the following statement:

**Lemma 5.2.** For \( n \geq 3 \) we have

\[
\begin{align*}
d_n &= \binom{d-4}{n} h^n + \left( \binom{d-3}{n} - \binom{d-5}{n} \right) \Theta h^{n-1} \\
&\quad + \left( \frac{1}{2} \binom{d-2}{n} - \binom{d-4}{n} + \frac{1}{2} \binom{d-6}{n} \right) \Theta^2 h^{n-2}.
\end{align*}
\]
Proof. By induction over $n$:

\[
d_n = \sum_{i=1}^{n} (-1)^{i-1} c_i d_{n-i}
\]

\[
= \sum_{i=1}^{n} (-1)^{i-1} \binom{d-5+i}{i} \binom{d-4}{n-i} h^n
\]

\[
+ \sum_{i=1}^{n} (-1)^{i-1} \left( \binom{d-5+i}{i} \binom{d-3}{n-i} - \binom{d-5+i}{i} \binom{d-5}{n-i} \right)
\]

\[
+ \left( \binom{d-4}{n-i} \binom{d-6+i}{i-1} + \binom{d-4}{n-i} \binom{d-5+i}{i-1} \right) \Theta h^{n-1}
\]

\[
+ \sum_{i=1}^{n} (-1)^{i-1} \left( \frac{1}{2} \left( \binom{d-5+i}{i} \binom{d-2}{n-i} - \binom{d-5+i}{i} \binom{d-4}{n-i} \right) \right.
\]

\[
+ \frac{1}{2} \left( \binom{d-5+i}{i} \binom{d-6}{n-i} + \binom{d-6+i}{i-1} \binom{d-3}{n-i} \right)
\]

\[
- \left( \binom{d-6+i}{i-1} \binom{d-5}{n-i} + \binom{d-5+i}{i-1} \binom{d-3}{n-i} \right)
\]

\[
- \left( \binom{d-5+i}{i-1} \binom{d-5}{n-i} + 2 \binom{d-6+i}{i-2} \binom{d-4}{n-i} \right)
\]

\[
+ \frac{1}{2} \left( \binom{d-7+i}{i-4} \binom{d-4}{n-i} \right) \Theta^2 h^{n-2}.
\]

Using the binomial identities

(a) Upper negation: $\binom{-r}{m} = (-1)^m \binom{r+m-1}{m}$ for $r, m \in \mathbb{N}$,

(b) Vandermonde’s identity: $\sum_{k=0}^{r} \binom{m}{k} \binom{s}{r-k} = \binom{m+s}{r}$ for $m, r, s \in \mathbb{N}$

we obtain the formula for $d_n$ as given in Lemma 5.2. \qed

To finish the proof of Proposition 5.1 we use Lemma 5.2 taking $n = d - 5 \geq 3$:
\[ D_{d-5} = d_{d-5} = \binom{d-4}{d-5} h^{d-5} + \left( \binom{d-3}{d-5} - \binom{d-5}{d-5} \right) \Theta \cdot h^{d-6} + \left( \frac{1}{2} \binom{d-2}{d-5} - \binom{d-4}{d-5} + \frac{1}{2} \binom{d-6}{d-5} \right) \Theta^2 \cdot h^{d-7} \]

\[ = (d-4) h^{d-5} + \left( \binom{d-3}{2} - 1 \right) \Theta \cdot h^{d-6} + \left( \frac{1}{2} \binom{d-2}{3} - (d-4) \right) \Theta^2 \cdot h^{d-7}. \]

Now we are able to deduce the formula for the degree of \( \text{Sec}_3(C) \) where \( C \) is a curve of genus 2 and degree \( d \geq 8 \) in \( \mathbb{P}^{d-2} \).

**Proposition 5.3.** The degree of the third secant variety \( \text{Sec}_3(C) \) of a smooth curve of genus 2 and degree \( d \geq 8 \) in \( \mathbb{P}^{d-2} \) is equal to

\[ \binom{d-2}{3} - 2(d-4). \]

**Proof.** Since \( \text{Sec}_3(C) \) has dimension 5, we have to intersect with \( (h')^5 \) where \( h' \) is a hyperplane class in \( \mathbb{P}^{d-2} \) in order to obtain the degree of \( \text{Sec}_3(C) \). From the above remarks we now have to find \( \deg x_1.h^5 \), where \( h = (p_2)^*(h') \).

We have

\[ \deg x_1.h^5 = \deg D_{d-5}.h^5 = \deg \left( \frac{1}{2} \binom{d-2}{3} - (d-4) \right) \Theta^2 \cdot h^{d-2}. \]

Since \( \deg \Theta^2 \cdot h^{d-2} = 2 \), (cf. Proposition 2.2) we finally obtain

\[ \deg D_{d-5}.h^5 = 2 \left( \frac{1}{2} \binom{d-2}{3} - (d-4) \right) = \binom{d-2}{3} - 2(d-4). \]

\[ \square \]
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