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Abstract. In the paper a new approach to solving problem of optimal control of technological process for the optical fiber production is proposed. The process of optical fiber drawing from a preform are considered. The optimal control problem of this process is formulated as problem of controlling partial differential equation system. The problem of the geometric fiber shape optimization in controlling the winding speed of the resulted fiber is formulated and solved. The optimality system in the strong form is obtained with the help of an analog of the Lagrange method. The optimal control function is determined from solving of this system. To obtain optimality system the properties of convexity, lower semi-continuity and coercivity of objective functional are used. In conclusion, an algorithm for realizing the optimal control problem is proposed and the results are analyzed. The problem is solved for several types of initial conditions for the radius deviation function from its stationary (programmed) solution. In these cases, optimality systems are solved with the help of algorithms of multi-physical modeling, control functions are found. The obtained correction values for the winding speed of the resulted fiber correspond to the possibilities of real production.

1. Introduction

The theory of optimal control of distributed parameter systems is a much more complicated and challenge area of research than a similar problem for systems with lumped parameters. The main reason of the complexity of the distributed parameter system study is that the motion of such systems is described by complex functional equations and partial differential equations. These systems often have complex boundary and initial conditions, as well as control parameters. In production tasks real objects, that require the creation of optimal control algorithms, are complex processes from the point of view of mathematical modeling. These processes can not be described by ordinary differential equations. Thus, it is necessary to develop the theory of optimal control and generalize it to the case of distributed parameter systems. This direction of generalization of the basic theory finds practice in many technical applications.

The first results in the theory of optimal control of distributed systems, including the study of the existence and uniqueness of solutions, have been related to the work of J L Lyons [1, 2]. In the second half of the last century in the works of A G Butkovsky [3, 4] approximate methods for optimal control problem solution were showed in addition to exact mathematical formulations of these problems. Also, the optimization of systems, described by recurrence relations, and the moment method applied to control problems of distributed systems have been studied. The fundamentals of the theory of controllability, observability and identifiability of systems, including distributed ones, have been
presented in the works of A I Egorova, K A Lurie, T K Sirazetdinov [5-7]. It been shown that the control process of distributed systems can be described by boundary value problems for partial differential equations or infinite systems of ordinary differential equations. The monograph of A V Fursikov [8] has been devoted to obtaining the necessary and sufficient conditions for the solvability of optimal control problems for distributed systems in new forms, including in the forms of optimization systems. The first practical results of the presented theory have been related to the problems of hydrodynamics. These were problems of minimizing work during acceleration of a fluid at rest to a given speed, flow problems and inhomogeneous boundary value problems for the Navier-Stokes equations. A systematic presentation of the basic methods of analysis and synthesis of automatic control systems with distributed parameters and a review of the basic characteristics of controllability and observability of distributed control objects have been presented by E Y Rappoport [9]. Methods for studying control systems with moving sources of influence and control algorithms have been studied by V A Kubyshkin [10]. In modern studies of V V Provotorov [11], continuing the work of A V Fursikov [8], and O A Ladyzhenskaya [12] conditions for the existence and uniqueness of optimization problems in different formulations have been provided and the controllability of systems has been shown. The research of V V Provotorov has been based on an approach, which is built on a priori estimates of generalized solutions of the initial-boundary value problem for equations of parabolic type.

2. Materials and methods

The object of research in this paper is a distributed system that describes the production process of quartz optical fiber drawing. The production of quartz optical fibers is a complex high-precision technological process, which consists of several stages. At each of them, a specific aspect of quality control is the measurement of a number of the most important technological parameters, which include geometric and strength characteristics, as well as characteristics that determine the level of optical losses. At the final stage of production, the resulted fiber drawing, the fiber diameter is only measured continuously. As practice shows, there is a good correlation between the constancy of the resulted fiber diameter and the constancy of its other characteristics along the length. Therefore, the existing systems of monitoring and controlling the fiber drawing process are built on this basis. In the ideal case, under isothermal conditions, the constancy of the resulted fiber diameter can be obtained in the stationary mode, i.e. with constant feed speeds (preform) and fiber drawing. However, as practice shows in real production conditions, the constancy of the geometry of the resulted fiber can be influenced not only by micro changes of the preform and drawing feed rates, but also various defects of the preform. The general drawing scheme and its main geometrical characteristics are shown in Fig. 1.

![Figure 1. General scheme of optical fiber drawing](image-url)
Let us consider a one-dimensional coordinate system, the spatial axis $Oz$ of which is oriented downward, in the direction of the drawing movement (axis of symmetry). In the one-dimensional approximation, the technological drawing process is described by a system of partial differential equations (1) [13]:

$$
\frac{\partial R(t,z)}{\partial t} = V(t,z) \frac{\partial R(t,z)}{\partial z} + \frac{R(t,z)}{2} \frac{\partial V(t,z)}{\partial z},
$$

$$
R^2(t,z)\left(\frac{\partial V(t,z)}{\partial t} + V(t,z) \frac{\partial V(t,z)}{\partial z}\right) = \frac{\partial}{\partial z}\left(3\mu R^2(t,z) \frac{\partial V(t,z)}{\partial z}\right) + R^2(t,z)\rho g + \frac{\sigma}{2} \frac{\partial R(t,z)}{\partial z},
$$

with initial and boundary conditions that have the form:

$$
R|_{z=0} = R_0(z), \quad R|_{z=0} = R_0(t),
$$

$$
V|_{z=0} = V_s(z), \quad V|_{z=0} = V_0(t), \quad V|_{z=L} = V_L(t),
$$

here $t$ is time; $z$ is longitudinal coordinate; $R(t,z)$ and $V(t,z)$ are the radius and velocity of the quartz melt flow; $\rho$, $\mu$ are constant density and viscosity of the quartz melt; $\sigma$ is coefficient of surface tension; $g$ is gravitational acceleration; $L$ is the length of the studied drawing segment; $R_0(t)$ is the radius of the fiber preform; $R_s$, $V_s$ are given prescribed functions that depend on $z$; $V_0$ and $V_L$ are rates of the fiber preform feeding and drawing.

Let us suppose that $t \in [0, \tau]$, $z \in [0, L]$. Let us denote by $\Omega$ the domain $(0, L)$, by $\Omega_t$ the Cartesian product $[0, \tau] \times [0, L]$, by $\Gamma_t$ the piecewise smooth boundary of $\Omega_t$ ($\Gamma_0$ is the part of $\Gamma_t$ corresponding to $t = 0$, $\Gamma_1$ corresponds to the segment $z = 0$, $\Gamma_2$ is the segment $z = L$). Equations (1) are satisfied on the inside $\Omega_t$, conditions (2) and (3) are on the parts of the boundary $\Gamma_t$. Let us introduce the necessary spaces in the sequel: $L^2(\Omega_t)$ is the space of functions summable with a square on $\Omega_t$; $W^1_2(\Omega_t)$ is the space of functions from $L^2(\Omega_t)$ having the generalized first derivative also from $L^2(\Omega_t)$.

Let us formulate the control problem. It is required to keep the system near the state in which the deviations of the geometric characteristics of the quartz filament (fiber radius) from some optimal characteristics are minimal, provided that the winding speed of the fiber is given. In fact, this means that the control is the time function $V(t)$ described above. The function $V(t)$ is the boundary condition in (3). The observation is the radius of the resulted fiber $R(t, L)$.

Let us called the movement of the system at predetermined known values of velocities and radii as programmed movement, and the corresponding control as programmed control.

It is known, that in most cases the analysis of the state of a nonlinear system can be replaced by an analysis of the state of the system linearized in the locality of its stationary state. It is also convenient to consider the stationary state of the fiber radius as the mentioned above optimal geometric characteristic. Let us linearize equations (1), assuming that the real values of the fiber radius $R(t,z)$, the material velocity $V(t,z)$ and the control function in the general form $u(t,z)$ are described as follows dependencies:

$$
R(t,z) = R_0(z) + \tilde{R}(t,z),
$$

$$
V(t,z) = V_0(t) + \tilde{V}(t,z),
$$

$$
u(t,z) = u(t,z).
$$
where $R(t,z)$, $V(t,z)$ are programmed stationary movement; $u(t,z)$ is programmed control corresponding to the stationary mode; $\tilde{R}(t,z)$, $\tilde{V}(t,z)$ are deviations (disturbances) of the actual (real) movement from the programmed ones; $\tilde{u}(t,z)$ is deviation of the real control from the programmed one. The implementation of relations (4) - (6) in the problem (1) - (3), taking into account only the components that are in the first degrees, leads to a linear statement in the next form

$$
\begin{align*}
\frac{d\tilde{R}}{dt} &= V_\text{st} \frac{\partial V_\text{st}}{\partial z} + \frac{3\mu}{\rho} \frac{\partial^2 \tilde{V}}{\partial z^2}, \\
\frac{d\tilde{V}}{dt} &= \beta_1(z) \frac{\partial \tilde{V}}{\partial z} + \beta_2(z) \frac{\partial \tilde{V}}{\partial z} + \alpha_1(z) \frac{\partial \tilde{R}}{\partial z} + \alpha_2(z) \tilde{R}, \\
\tilde{R}(t=0) &= R_0(z), \quad \tilde{R}(z=0) = R_0(t), \\
\tilde{V}(t=0) &= V_0(z), \quad \tilde{V}(z=L) = \tilde{u}(t)
\end{align*}
$$

where the coefficients $\beta_1(z), \beta_2(z), \alpha_1(z), \alpha_2(z)$ have the following form

$$
\begin{align*}
\beta_1(z) &= -V_\text{st} + \frac{3\mu}{\rho} \frac{\partial V_\text{st}}{\partial z} + \frac{3\mu}{\rho R^2_\text{st} V_\text{st}} \frac{\partial}{\partial z} \left( R^2_\text{st} V_\text{st} \right), \\
\beta_2(z) &= -2 \frac{\partial V_\text{st}}{\partial z} + \frac{3\mu}{\rho R^2_\text{st} V_\text{st}} \frac{\partial}{\partial z} \left( R^2_\text{st} \frac{\partial V_\text{st}}{\partial z} \right), \\
\alpha_1(z) &= \frac{6\mu}{\rho V_\text{st}} \frac{\partial V_\text{st}}{\partial z} + \frac{\sigma}{2\rho R^2_\text{st} V_\text{st}}, \\
\alpha_2(z) &= -2 \frac{\partial V_\text{st}}{\partial z} + \frac{6\mu}{\rho R^2_\text{st} V_\text{st}} \frac{\partial}{\partial z} \left( R^2_\text{st} \frac{\partial V_\text{st}}{\partial z} \right) + \frac{2\sigma}{V_\text{st}} + \frac{\sigma}{2\rho R^2_\text{st} V_\text{st}} \frac{\partial R_\text{st}}{\partial z}.
\end{align*}
$$

Thus, the obtained initial-boundary value problem (7) has states whose values can be interpreted as deviations (in fractions) from the stationary programmed values of the states of the initial system (1) - (3).

Let us introduce into consideration the functional defined on $L_2^2(\Gamma_2) = \{ z(t) \in L_2(0,\tau); z(t) \geq 0 \}$ and given by

$$
F(\tilde{u}) = \int_0^\tau \tilde{R}^2(t,L) dt + \alpha \| \tilde{u}(t) \|^2_{L_2^2(\Omega)} = \int_0^\tau \tilde{R}^2(t,L) dt + \alpha \int_0^\tau \tilde{u}^2(t,L) dt \to \inf .
$$
The problem (7), (9) is a problem with boundary control and boundary observation, consisting in finding the exact lower bound of the functional (9) when the control parameter \( u \) runs through \( L^2_2(\Gamma_2) \), \( \alpha > 0 \).

Let us define a generalized solution of problem (1) - (3).

**Definition.** The generalized solution of problem (1) - (3) is the functions \( z_t, z_V \), which are bounded in \( t \) which satisfy the next conditions (let us omit the independent arguments \((t,z)\) for short):

1) for any function \( \varphi(t,z) \in W^1_2(\Omega_z) \) such that \( \varphi|_{\Gamma_z} = 0, \varphi|_{\Gamma_1} = 0 \) the relation holds
\[
\left( \tilde{R}, \varphi \right)_{L_2(\Omega_z)} + \frac{1}{2} \left( \tilde{V}, (\varphi \cdot V_{st})_z \right)_{L_2(\Omega_z)} + (R_z, \varphi)_{L_2(\Omega)} + \left( \varphi, V_0 \left( R_0 + \frac{1}{2} V_0 \right) \right)_{L_2(\Gamma_z)} = 0 ;
\]

2) for any function \( \psi(t,z) \in W^1_2(\Omega_z) \) such that \( \psi|_{\Gamma_z} = 0, \psi|_{\Gamma_1} = 0, \psi|_{\Gamma_2} = 0 \) the relation holds
\[
\left( \bar{V}, \psi \right)_{L_2(\Gamma_z)} - \frac{3\mu}{\rho} \left( \bar{V}, \alpha \cdot \psi \right)_{L_2(\Gamma_z)} + \left( \bar{R}, (\alpha \cdot \psi)_z - \alpha z \cdot \psi \right)_{L_2(\Omega)} =
\]
\[
= \left( V_z, \psi \right)_{L_2(\Gamma_z)} - \frac{3\mu}{\rho} \left( V_z, \bar{u} \right)_{L_2(\Gamma_z)} + \frac{3\mu}{\rho} \left( V_z, V_0 \right)_{L_2(\Gamma_z)}.
\]

Obviously that any classical solution of the initial-boundary value problem (1) - (3) is generalized and vice versa.

### 3. Results

The necessary conditions for the existence of a solution of both the initial optimization problem and the problem (7), (9) obtained as a result of linearization are called the optimality system or the optimization system. In this case, it is possible to obtain an optimization system in the form of a boundary-value problem written for the original equations and equations for the conjugate states [14].

Questions of the existence of optimization problem solutions for parabolic systems with objective functionals in special forms are discussed in [8, 14, 15, 16]. Note that the important properties of the functional (9) are the properties of convexity, lower semicontinuity and coercivity. Then there is at least one function \( \bar{u}_0(t) \), called the optimal element, on which the functional will reach its exact lower bound:

\[
F(\bar{u}_0) = \inf_{u \in L^2(\Gamma_z)} F(\bar{u}(t)).
\]

According to the optimality criterion [8, 14, 16] the value of Gateaux differential at the optimal element is zero:

\[
\frac{1}{2} \left\{ F'(\bar{u}_0), w - \bar{u}_0 \right\} + \int_{\Gamma_z} \left( \tilde{R}, \frac{d}{dt} \right)_{L^2} dt + \alpha \int_{\Gamma_z} \left( \tilde{u}_0 \delta \tilde{u}_0 \right)_{L^2} dt = 0 .
\]

where \( \tilde{R} = \left( \tilde{R}_0 \right)' \), \( <> \) is the operator of weak differentiation, \( \delta \tilde{u}_0 = w - \tilde{u}_0 \) is the variation of the optimal control.

Note that the differential problem in the formulation of (7) is linear with respect to the control function \( \tilde{u}(t) \), therefore one of its solutions, the function \( \tilde{R}(t,z) \), can be considered as a result of the action of some linear operator \( \Lambda \) on the function \( \tilde{u}(t) \):

\[
\tilde{R}(t,z) = \Lambda(\tilde{u}(t)) = \Lambda(\tilde{u}).
\]
The linearity conditions (distributivity and associativity) are fair for the operator $\Lambda$. Then the equality (11) can be rewritten in the following form:

$$\frac{1}{0} \Lambda \partial_w \Lambda (w-\partial_w) dt + a \frac{1}{0} \Lambda \partial_w (w-\partial_w) dt = 0. \quad (13)$$

Let us vary the original problem (7) [17,18]. As a result, we obtain:

$$\frac{\partial \hat{R}}{\partial t} = V_{\mu} \frac{\partial \hat{R}}{\partial x^\mu} + \frac{V_{\mu}}{2} \frac{\partial V}{\partial x^\mu}, \quad \frac{\partial \hat{V}}{\partial t} \bigg|_{t=0} = 0, \quad \frac{\partial^2 \hat{V}}{\partial t^2} \bigg|_{t=0} = 0, \quad \frac{\partial^2 \hat{V}}{\partial x^\mu \partial x^\nu} \bigg|_{t=0} = \delta_\mu^\nu, \quad \frac{\partial^2 \hat{V}}{\partial t \partial x^\mu} = 0. \quad (14)$$

Let us multiply the equation of continuity from (14) by a given function $q(t,z) \in L_2(\Omega)$, and the equation of movement by a given function $p(t,z) \in L_2(\Omega)$, and integrate both equations over the domain $\Omega$. The result of their addition is the following expression:

$$-\int_0^L \int_0^L \hat{R} q dx dt - \int_0^L \int_0^L \hat{V} p dx dt = \int_0^L \int_0^L \delta_\mu^\nu q dx dt + \int_0^L \int_0^L \delta_\mu^\nu p dx dt + \int_0^L \int_0^L \hat{R} V dx dt + \int_0^L \int_0^L \hat{V} q dx dt + \int_0^L \int_0^L \hat{V} p dx dt + \int_0^L \int_0^L \hat{R} p dx dt + \int_0^L \int_0^L \hat{V} V dx dt.$$
\[
\left\{ \begin{array}{l}
\frac{\partial q}{\partial t} + \frac{\partial (V_s q)}{\partial z} + \frac{\partial (\alpha_1(z)p)}{\partial z} - \alpha_2(z)p = 0, \\
\frac{\partial p}{\partial t} + \frac{\partial \left( \frac{V_s}{2} q \right)}{\partial z} + \frac{3\mu}{\rho} \frac{\partial^2 p}{\partial z^2} + \frac{\partial (\beta_1(z)p)}{\partial z} - \beta_2(z)p = 0,
\end{array} \right. \\
q|_{z=L} = R, \\
p|_{z=L} = p|_{z=0} = p|_{z=L} = 0.
\]

(15)

Then, taking into account (14) and (15), the previous integral relation has the form:

\[
\int_0^r \delta \left[ \frac{V_s}{2} R \right]_{z=L} dt - \frac{3\mu}{\rho} \int_0^r \delta \left[ \frac{\partial p}{\partial z} \right]_{z=L} dt + \int_0^r \left[ V_s R \right]_{z=L} dt = 0.
\]

Taking into account the relation (11), we obtain:

\[
\alpha \int_0^r \delta u_0 \delta u_0 dt = - \frac{3\mu}{\rho} \int_0^r \delta \left[ \frac{\partial p}{\partial z} \right]_{z=L} dt + \int_0^r \left[ \frac{V_s}{2} R \right]_{z=L} dt,
\]

from which it follows that

\[
u_0(t) = \frac{1}{\alpha} \left( \frac{V_s}{2} R - \frac{3\mu}{\rho} \delta \frac{\partial p}{\partial z} \right)_{z=L}.
\]

and the optimality system has the following form:

\[
\begin{align*}
\frac{\partial \bar{R}}{\partial t} &= V_s \frac{\partial \bar{R}}{\partial z} + \frac{V_s}{2} \frac{\partial \bar{V}}{\partial z}, \\
\bar{R}|_{z=0} &= R_s(z), \bar{R}|_{z=L} &= R_f(t), \\
\frac{\partial \bar{V}}{\partial t} &= \frac{3\mu}{\rho} \frac{\partial^2 \bar{V}}{\partial z^2} + \beta_1(z) \frac{\partial \bar{V}}{\partial z} + \beta_2(z) \bar{V} + \alpha_1(z) \frac{\partial \bar{R}}{\partial z} + \alpha_2(z) \bar{R}, \\
\bar{V}|_{z=0} &= V_s(z), \bar{V}|_{z=L} &= V_s(t), \bar{V}|_{z=L} = \frac{1}{\alpha} \left( \frac{V_s}{2} \bar{R} - \frac{3\mu}{\rho} \frac{\partial p}{\partial z} \right), \\
\frac{\partial q}{\partial t} + \frac{\partial (V_s q)}{\partial z} + \frac{\partial (\alpha_1(z)p)}{\partial z} - \alpha_2(z)p = 0, \\
q|_{z=L} = 0, q|_{z=L} = R, \\
- \frac{\partial p}{\partial t} + \frac{\partial \left( \frac{V_s}{2} q \right)}{\partial z} + \frac{3\mu}{\rho} \frac{\partial^2 p}{\partial z^2} + \frac{\partial (\beta_1(z)p)}{\partial z} - \beta_2(z)p = 0, \\
p|_{z=L} = 0, p|_{z=L} = p|_{z=0}.
\end{align*}
\]

(16)

The numerical implementation of the solution of the optimality system (16) was realized using the finite-element method in Comsol Multiphysics (multiphysics simulation software). The process of solving this system can be divided into several stages:

- searching for a stationary solution (functions \( R_s(z) \) and \( V_s(z) \)). These functions were found using the solution of system (1) with initial and boundary conditions (2), (3);
- finding functions $\alpha_1(z), \alpha_2(z), \beta_1(z), \beta_2(z)$, depending on stationary states;
- solving of the optimality system (16) and finding the optimal control function $\tilde{u}(t)$.
- analysis of obtained results.

Let us describe the implementation process in details.
First, the system (1) with the initial and boundary conditions (2), (3) was solved and stationary solutions $z_{Rst}$ and $z_{Vst}$ were found numerically (see Fig. 2).

![Figure 2. (a) Radius of the quartz melt flow, (b) Velocity of the quartz melt flow](image)

The optimality system was solved for the following input parameters: $t = 5s$, $\rho = 2200 \text{ kg/m}^3$, $\mu = 10000 \text{ Pa} \cdot \text{s}$, $\sigma = 0.3 \text{ N/m}$, $L = 0.3m$. The deviation of the fiber radius from its stationary solution was given by the convex up function shown in Fig. 3(a). The maximum deviation of the fiber radius from its stationary (programmed) solution was 10%.

![Figure 3. Radius drawing deviation (as a fraction of the stationary solution)](image)

Special attention should be paid to the choice of the value of the control price $\alpha$. The parameter $\alpha$ must be estimated in advance, or selected by solving test problems. In our case, its value is 0.5.
In Fig. 4(a) you can see the resulted control function $\tilde{F}(t, L)$. 
Figure 4. Optimal control function

Let us solve the same problem on the assumption that the function \( \Phi_R(z) \) defining the initial deviation of the fiber radius is now determined by the function convex downward and has the following form (Fig. 3(b)).

The maximum deviation of the fiber radius from its stationary (programmed) solution in this case is also 10\%. Then, having solved the optimality system (16) for the same values of the input parameters, we obtain the function \( \Phi(V, L) \) in the following form (Fig. 4(b)).

4. Discussion

Analyzing the results of solving the optimality system (16) with two different, but “mirror” relative to each other initial conditions for the equation of state, it can be noted that the obtained control functions (Fig. 4(a,b)) are also practically mirror reflections of each other. Note that the winding speed adjustment values (values of the control function) vary from -3\% to + 3\%, which corresponds to the capabilities of real production.

The objective functional (9) was calculated for various values of the control function. In particular, with a constant winding speed of the resulted fiber \( (\bar{u} = 0) \), the value of the functional exceeds the value obtained when implementing the optimal state \( \bar{u}(t) = \frac{\bar{V}}{2} \left( 3 \mu - \frac{\partial \rho}{\partial z} \right) \).

5. Conclusion

Thus, in the paper the problem of optimal stabilizing control of the process of quartz optical fiber production was formulated, justified and solved. The problem was solved in a one-dimensional formulation. The control function is the winding speed of the resulted fiber. For the control problem, linearization was performed and the optimality system was obtained in a strong form, i.e. in the form of a boundary value problem in partial derivatives for deviations of the radius and speed of drawing, as well as for their conjugate states. This problem is solved for two different initial conditions that specify the deviation of the radius with respect to its stationary (programmed) state. In both cases, the obtained solutions of the optimality systems were presented, as well as optimal control functions.

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