EMBEDDINGS OF HOMOGENEOUS SPACES INTO IRREDUCIBLE MODULES

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Abstract. Let $G$ be a connected reductive group. We find a necessary and sufficient condition for a quasiaffine homogeneous space of $G/H$ to be embeddable into an irreducible $G$-module. If $H$ is reductive we also find a necessary and sufficient condition for a closed embedding of $G/H$ into an irreducible module to exist. These conditions are stated in terms of the group of central automorphisms of $G/H$.

1. Introduction

The base field is the field $\mathbb{C}$ of complex numbers. Throughout the paper $G$ denotes a connected reductive algebraic group, $B$ its Borel subgroup and $T$ a maximal torus of $B$.

The celebrated theorem of Chevalley states that any homogeneous space can be embedded (as a locally-closed subvariety) into the projectivization of a $G$-module. If $H$ is an observable subgroup of $G$, that is, the homogeneous space $G/H$ is quasiaffine, then $G/H$ can be embedded even into a $G$-module itself, see, for example, [PV], Theorem 1.6. So it is natural to pose the following

Problem 1.1. Describe all observable subgroups $H$ such that $G/H$ can be embedded into an irreducible $G$-module.

To state the answer to that problem we need the definition of a central automorphism of a $G$-variety. Let $X$ be an irreducible $G$-variety. A subspace $\mathbb{C}(X)^{\lambda} \subset \mathbb{C}(X)$ consisting of all $B$-semiinvariant functions of weight $\lambda \in \mathfrak{X}(B)$ is stable under every $G$-equivariant automorphism of $X$. The following definition belongs to Knop, [K2].

Definition 1.2. A $G$-equivariant automorphism of $X$ is called central if it acts on $\mathbb{C}(X)^{\lambda}$ by the multiplication by a constant for any weight $\lambda$.

We denote the group of central automorphisms of $X$ by $\mathfrak{A}_G(X)$. We write $\mathfrak{A}_{G,H}$ instead of $\mathfrak{A}_G(G/H)$. It was shown by Knop, [K2], Section 5, that $\mathfrak{A}_{G,H}$ is an algebraic quasi-torus, that is a closed subgroup of an algebraic torus.

Theorem 1.3. Let $H$ be an observable subgroup of $G$. Then the following conditions are equivalent:

(a) $G/H$ can be embedded into an irreducible $G$-module.
(b) $\mathfrak{A}_{G,H}$ is a finite cyclic group or a one-dimensional torus.

For a given subgroup $H \subset G$ the group $\mathfrak{A}_{G,H}$ can be computed using techniques from [L2]. Namely, $\mathfrak{A}_{G,H}$ is the quotient of the weight lattice of $G/H$ by the root lattice of $G/H$. An algorithm for computing the weight lattice is the main result of [L2]. The computation of the root lattice can be reduced to that of the weight lattice by using [L2], Proposition 5.2.1.

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If $H$ is a reductive subgroup of $G$ or, equivalently, $G/H$ is affine, then one may ask whether there exists a closed embedding of $G/H$ into an irreducible $G$-module. Here is an answer.

**Theorem 1.4.** Let $H$ be a reductive subgroup of $G$. Then the following conditions are equivalent.

(a) There is a closed equivariant embedding of $G/H$ into a irreducible $G$-module.

(b) $\mathfrak{A}_{G,H}$ is a finite cyclic group.

We prove Theorems 1.3, 1.4 in Sections 3, 4. In Section 5 we present some examples of applications of our theorems.

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### 2. Notation and conventions

- $A_{\mu}^{(B)}$: the subspace of all $B$-semi-invariant functions of weight $\mu$ in a $G$-algebra $A$, where $G$ is a connected reductive group.
- $[g, g]$: the commutant of a Lie algebra $g$.
- $G^\circ$: the connected component of unit of an algebraic group $G$.
- $R_u(G)$: the unipotent radical of an algebraic group $G$.
- $G_x$: the stabilizer of a point $x \in X$ under an action $G : X$.
- $\text{Int}(g)$: the group of inner automorphisms of a Lie algebra $g$.
- $N_G(H)$: the normalizer of a subgroup $H$ in a group $G$.
- $V^g = \{v \in V | g^\circ v = 0\}$, where $g$ is a Lie algebra and $V$ is a $g$-module.
- $V(\mu)$: the irreducible module with highest weight $\mu$ over a reductive algebraic group or a reductive Lie algebra.
- $\mathfrak{X}(G)$: the character lattice of an algebraic group $G$.
- $X^G$: the fixed-point set for an action of $G$ on $X$.
- $\#X$: the cardinality of a set $X$.
- $Z(G)$ (resp., $Z_{\mathfrak{g}}$): the center of an algebraic group $G$ (resp., of a Lie algebra $\mathfrak{g}$).
- $Z_G(h)$ (resp., $Z_{\mathfrak{g}}(\mathfrak{h})$): the centralizer of a subalgebra $h \subset \mathfrak{g}$ in an algebraic group $G$ (resp., in its Lie algebra $\mathfrak{g}$).
- $\lambda^*$: the highest weight dual to $\lambda$.

If an algebraic group is denoted by a capital Latin letter, then we denote its Lie algebra by the corresponding small fraktur letter, for example, $\hat{h}$ denotes the Lie algebra of $\hat{H}$. All topological terms refer to the Zariski topology.

### 3. Proof of Theorem 1.3

First, we fix some notation and recall some definitions from the theory of algebraic transformation groups.

In this section $H$ denotes an observable subgroup of $G$. The group of $G$-equivariant automorphisms of $G/H$ is identified with $N_G(H)/H$. We consider $\mathfrak{A}_{G,H}$ as a subgroup in $N_G(H)/H$. Denote by $H^{\text{sat}}$ the inverse image of $\mathfrak{A}_{G,H}$ in $N_G(H)$. 

Let $X$ be an irreducible $G$-variety. An element $\lambda \in \mathfrak{X}(T)$ is said to be a *weight of $X$* if $\mathbb{C}(X)_{\lambda}^{(B)} \neq 0$. Clearly, all weights of $X$ form a subgroup of $\mathfrak{X}(T)$ called the *weight lattice* of $X$ and denoted by $\mathfrak{X}_{G,X}$. The rank of the weight lattice is called the *rank* of $X$ and is denoted by $\text{rk}_G(X)$. We put $\mathfrak{a}_{G,X} = \mathfrak{X}_{G,X} \otimes \mathbb{Z}$. If $X = G/G_0$, then we write $\mathfrak{X}_{G,G_0}$ instead of $\mathfrak{X}_{G,G/G_0}$. It is easy to see that the subspace $\mathfrak{a}_{G,G/G_0}$ depends only on the pair $(\mathfrak{g}, \mathfrak{g}_0)$. Thus we write $\mathfrak{a}_{\mathfrak{g}, \mathfrak{g}_0}$ instead of $\mathfrak{a}_{G,G/G_0}$. If $\hat{G}_0$ is a subgroup of $G$ containing $G_0$, then there exists a dominant $G$-equivariant morphism $G/G_0 \to G/\hat{G}_0$ and hence $\mathfrak{X}_{G,\hat{G}_0} \subset \mathfrak{X}_{G,G_0}$.

The codimension of a general $B$-orbit in $X$ is called the *complexity* of $X$ and is denoted by $c_G(X)$. Again, we write $c_{\mathfrak{g}, \mathfrak{g}_0}$ instead of $c_G(G/G_0)$. Let us note that $c_{\mathfrak{g}, \mathfrak{g}_0} \leq c_{\mathfrak{g}, \mathfrak{g}_0}$ whenever $G_0 \subset \hat{G}_0$.

Proceed to the proof of Theorem 1.3. The implication $(a) \Rightarrow (b)$ is easy.

**Proof of $(a) \Rightarrow (b)$.** By the Frobenius reciprocity, there is an $N_G(H)$-equivariant isomorphism $V(\lambda)^H \cong \mathbb{C}[G/H]_{\lambda}^{(B)}$. Clearly, $(a)$ implies that the action of $N_G(H)/H$ on $V(\lambda)^H$ is effective for some $\lambda$. Now $(b)$ follows easily from the definition of the subgroup $\mathfrak{A}_{G,H} \subset N_G(H)/H$.

The implication $(b) \Rightarrow (a)$ will follow from the following

**Proposition 3.1.** Suppose $\mathfrak{A}_{G,H}$ is a cyclic finite group or a one-dimensional torus. Then there is a highest weight $\lambda$ such that $V(\lambda)^H \neq \{0\}$ and the subset $\bigcap_{H \supseteq H'} V(\lambda)^{H'}$ is not dense in $V(\lambda)^H$.

The scheme of the proof of the proposition is, roughly speaking, as follows. On the first step we prove that for an appropriate highest weight $\lambda$ the complexity $c_{\mathfrak{g}, \mathfrak{g}_0}$ for a point $v \in V(\lambda)^H$ in general position coincides with $c_{\mathfrak{g}, \mathfrak{g}_0}$. On the second step we check that one may choose $\lambda$ such that $\mathfrak{g}_v = \mathfrak{h}$ for $v \in V(\lambda)^H$ in general position. At last, we show that $G_v = H$ for general $v \in V(\lambda)^H$.

We begin with some simple lemmas.

**Lemma 3.2.** $\dim V(\nu)^H \leq \dim V(\nu + \mu)^H$ for any highest weights $\mu, \nu$ such that $V(\mu)^H \neq 0$.

**Proof.** By the Frobenius reciprocity, $V(\nu)^H \cong \mathbb{C}[G/H]_{\nu}^{(B)}$, $V(\nu + \mu)^H \cong \mathbb{C}[G/H]_{\nu + \mu}^{(B)}$. The map $f_1 \mapsto ff_1 : \mathbb{C}[G/H]_{\nu}^{(B)} \hookrightarrow \mathbb{C}[G/H]_{\nu + \mu}^{(B)}$ is injective for any $f \in \mathbb{C}[G/H]_{\nu}^{(B)}$, $f \neq 0$.

In the sequel we will need some properties of central automorphisms.

**Lemma 3.3.**
1. An element $n \in N_G(H)/H$ is central iff it acts trivially on $K(G/H)^{(B)}$.
2. $\mathfrak{A}_{G,H} \subset Z(N_G(H)/H)$.

**Proof.** Let $X$ be an affine $G$-variety with open $G$-orbit $G/H$. Thanks to [PV], Theorem 3.3, to prove assertion 1 it is enough to check that $n$ acts on $\mathbb{C}[X]_{\lambda}^{(B)}$ by the multiplication by a constant for any highest weight $\lambda$ provided $n$ acts trivially on $\mathbb{C}(G/H)^B$. Since $X$ contains a dense $G$-orbit, we have $\mathbb{C}[X]^G = \mathbb{C}$. It follows from [PV], Theorem 3.24, that $\dim \mathbb{C}[X]_{\lambda}^{(B)} < \infty$. Now our claim is clear.

Assertion 2 follows from [K2], Corollary 5.6.

The following technical proposition is crucial in the proof of Proposition 3.1.

**Proposition 3.4.** Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ be proper subspaces of $\mathfrak{a}_{\mathfrak{g}, \mathfrak{g}_0}$ and $\mathfrak{X}_1, \ldots, \mathfrak{X}_l$ sublattices of $\mathfrak{X}_{G,H}$ such that $p_i := \#(\mathfrak{X}_{G,H}/\mathfrak{X}_i)$, $i = 1, \ldots, l$, are pairwise different primes. Put $c := c_{\mathfrak{g}, \mathfrak{g}_0}$. Then there exists a highest weight $\lambda$ satisfying condition (1), when $c$ is arbitrary, and conditions (2),(3), when $c > 0$. 
Proof. Let \( \lambda^* \not\subset \bigcup_{i=1}^k a_i \cup \bigcup_{i=1}^l \mathfrak{x}_i \).

(2) The codimension of the closure of the subset \( Z := (\bigcup V(\hat{\mathfrak{h}})) \cap V(\lambda)^H \subset V(\lambda)^H \),

where the union is taken over all algebraic subalgebras \( \mathfrak{h} \subset \mathfrak{g} \) such that \( \mathfrak{h} \supset \mathfrak{h}, c_{\mathfrak{g}, \mathfrak{h}} < c \),

is strictly bigger than \( 2 \dim G \).

(3) For any \( f \in \mathbb{C}(G/H)^B \) there exist \( f_1, f_2 \in \mathbb{C}[G/H]^B_{\lambda^*} \) such that \( f = \frac{f_1}{f_2} \).

Lemma 3.5. Let \( a_1, \ldots, a_k, \mathfrak{x}_1, \ldots, \mathfrak{x}_l \) be such as in Proposition 3.4. Let \( \mu' \in \Psi \) satisfy condition (1). Then there is \( n \in \mathbb{N} \) such that for any \( \lambda \in \Psi \) at least one of the weights \( \lambda + \mu', \lambda + 2\mu', \ldots, \lambda + n\mu' \) satisfies condition (1) of Proposition 3.4.

Proof. Set \( n := (k+1)p_1 \ldots p_l \). The proof is easy. \( \square \)

Proof of Proposition 3.4. Let us choose a norm \( | \cdot | \) on the space \( \mathfrak{a}_{\mathfrak{g}, \mathfrak{b}}(\mathbb{R}) := \mathfrak{X}_{G,H} \otimes \mathbb{R} \). It follows from Timashev’s theorem, \( [T] \), that the following assertions hold:

- There exists \( A_0 \in \mathbb{R} \) such that \( \dim V(\lambda)^\mathfrak{h} < A_0|\lambda|^{c-1} \) for any subalgebra \( \hat{\mathfrak{h}} \subset \mathfrak{g} \) with \( c_{\mathfrak{g}, \mathfrak{h}} < c \) and any highest weight \( \lambda \).
- For any \( A \in \mathbb{R} \) there exists a highest weight \( \lambda \) such that \( \dim V(\lambda)^H > A|\lambda|^{c-1} \).

Denote by \( Y \) the subvariety of \( \prod_{i=\dim \mathfrak{h}} \text{Gr}_i(\mathfrak{g}) \) consisting of all subalgebras \( \hat{\mathfrak{h}} \subset \mathfrak{g} \) containing \( \mathfrak{h} \). \( Y_0 := \{ \hat{\mathfrak{h}} \in Y | c_{\mathfrak{g}, \mathfrak{h}} < c \} \) is an open subvariety of \( Y \), because \( c_{\mathfrak{g}, \mathfrak{h}} = \min_{g \in G} \dim \mathfrak{g}/(\text{Ad}(g)\mathfrak{b} + \hat{\mathfrak{h}}) \). Put \( V := V(\lambda)^H, \tilde{Z} := \{ (\hat{\mathfrak{h}}, v) \in Y_0 \times V | v \in V(\lambda)^{\mathfrak{h}} \} \). The latter is a closed subvariety in \( Y_0 \times V \) of dimension at most \( \dim Y_0 + \max_{\hat{\mathfrak{h}} \in Y_0} \dim V(\lambda)^\mathfrak{h} \).

Note that \( \tilde{Z} \) is just the image of \( \tilde{Z} \) under the projection \( Y_0 \times V \rightarrow V \). Thus if \( c > 0 \), then the dimension of the closure of \( \tilde{Z} \) does not exceed \( A_0|\lambda|^{c-1} + \dim Y_0 \).

Note that there exists a highest weight \( \lambda_1 \) satisfying condition (3). Indeed, the field \( \mathbb{C}(G/H)^B \) is finitely generated and let \( f_1, \ldots, f_s \) be its generators. Analogously to \( [PV] \), Theorem 3.3, one proves that there are \( f_{i_1}, f_{i_2} \in \mathbb{C}(G/H)^{B}_{\nu_i}, i = 1, s \), such that \( f_i = \frac{f_{i_1}}{f_{i_2}} \). It is enough to take \( \sum_{i=1}^s \nu_i^* \) for \( \lambda_1 \). Note that for any highest weight \( \lambda_2 \) with \( \mathbb{C}(G/H)_{\lambda_2}^B \neq 0 \) the highest weight \( \lambda_2 + \lambda_1 \) also satisfies condition (3).

Note that there is a highest weight \( \lambda_2 \) satisfying condition (1) and such that \( V(\lambda_2)^H \neq \{0\} \). Indeed, otherwise \( \bigcup_{i=1}^k a_i \) contains an open cone in \( \mathfrak{a}_{\mathfrak{g}, \mathfrak{b}} \), which is absurd. So in case \( c = 0 \) we are done. Now suppose \( c > 0 \).

Let \( n \) be such as in Lemma 3.5. Choose \( A > 0 \) and a highest weight \( \nu \) such that \( \dim V(\nu)^H > A|\nu|^{c-1} \) and \( A|\nu|^{c-1} > A_0(|\nu| + |\lambda_1| + n|\lambda_2|)^{c-1} + \dim Y_0 + 2 \dim G \). Further, there is \( j \in \{1, \ldots, n\} \) such that \( \lambda := \nu + \lambda_1 + j\lambda_2 \) satisfies (1). For \( c > 0 \) it is easy to deduce from Lemma 3.2 that \( \lambda \) satisfies condition (2). Finally \( \lambda \) satisfies condition (3), for it is of the form \( \lambda_1 + \lambda_2 \) for some \( \lambda_2 \) with \( \mathbb{C}(G/H)_{\lambda_2}^B \neq 0 \). \( \square \)

The next proposition is used on the second step of the proof.

Proposition 3.6. The set \( \{ a_{\mathfrak{g}, \mathfrak{h}} | a_{\mathfrak{g}, \mathfrak{h}} = [\hat{\mathfrak{h}}, \mathfrak{h}] + R_u(\mathfrak{h}) + \hat{\mathfrak{h}}, \mathfrak{h} \text{ is algebraic} \} \) is finite.

Proof. Let \( \mathfrak{h} = \mathfrak{s} \oplus R_u(\mathfrak{h}), \hat{\mathfrak{h}} = \hat{\mathfrak{s}} \oplus R_u(\hat{\mathfrak{h}}) \) be Levi decompositions. We may assume that \( \mathfrak{s} \subset \hat{\mathfrak{s}} \). Denote by \( \hat{\mathfrak{H}}, \hat{\mathfrak{S}} \) the connected subgroups of \( G \) corresponding to \( \hat{\mathfrak{h}}, \hat{\mathfrak{s}} \). By the Weisfeller theorem, see \( [W] \), there is a parabolic subgroup \( P \subset G \) and a Levi subgroup \( L \subset P \) such that \( P \subset L, R_u(\hat{\mathfrak{H}}) \subset R_u(P) \). Conjugating \( \mathfrak{h}, \mathfrak{h} \) by an element of \( G \), we may assume that
There is an inclusion of $\mathcal{S}$-modules $R_u(p)/R_u(h) \hookrightarrow \mathfrak{g}/\mathfrak{s}$. So the set $\{(L, R_u(p)/R_u(h))\}$ is finite. It remains to check that $\mathfrak{s}$ belongs to $\mathfrak{g}$ following well-known lemma (which stems, for example, from [V], Proposition 3) allows us to replace Int(I)-conjugacy in the previous statement by Int($\mathfrak{g}$)-conjugacy.

**Lemma 3.7.** Let $\mathfrak{g}_0$ be a reductive subalgebra of $\mathfrak{g}$ and $\mathfrak{g}_1$ a reductive subalgebra of $\mathfrak{g}_0$. The set of subalgebras of $\mathfrak{g}_0$ that are Int($\mathfrak{g}$)-conjugate to $\mathfrak{g}_1$, decomposes into finitely many classes of Int($\mathfrak{g}_0$)-conjugacy.

The equality $\widehat{h} = [\hat{h}, \hat{h}] + R_u(\hat{h}) + h$ is equivalent to $\mathfrak{s} = [\widehat{\mathfrak{s}}, \widehat{\mathfrak{s}}] + \mathfrak{s}$. Therefore the statement on the finiteness of the set of Int($\mathfrak{g}$)-conjugacy classes stems from the following lemma

**Lemma 3.8.** Let $\mathfrak{s}$ be a reductive subalgebra of $\mathfrak{g}$. The set of Int($\mathfrak{g}$)-conjugacy of reductive subalgebras $\mathfrak{s} \subset \mathfrak{g}$ such that $\mathfrak{s} = [\widehat{\mathfrak{s}}, \widehat{\mathfrak{s}}] + \mathfrak{s}$ is finite.

**Proof of Lemma 3.8.** We may replace $\mathfrak{s}$ with its Cartan subalgebra and assume that $\mathfrak{s} \subset \mathfrak{t}$. In this case the proof is in three steps.

**Step 1.** Here we show that the set of subspaces of $\mathfrak{t}$, that are Cartan subalgebras of semisimple subalgebras of $\mathfrak{g}$, is finite. Note that there are finitely many conjugacy classes of semisimple subalgebras of $\mathfrak{g}$. Indeed, for $\mathfrak{g} = \mathfrak{gl}_n$ this is a consequence of the highest weight theory and in the general case one embeds $\mathfrak{g}$ into some $\mathfrak{gl}_n$ and uses Lemma 3.7. It follows that only finitely many subspaces of $\mathfrak{t}$ are G-conjugate to a Cartan subalgebra of a semisimple Lie algebra. Now it remains to note that G-conjugate subspaces of $\mathfrak{t}$ are W-conjugate. Here W denotes the Weyl group of $\mathfrak{g}$.

**Step 2.** Conjugating $\widehat{\mathfrak{s}}$ by an element of $Z_G(\mathfrak{s})$, one may assume that there is a Cartan subalgebra $\mathfrak{t}_0 \subset \widehat{\mathfrak{s}}$ contained in $\mathfrak{t}$. Since $\widehat{\mathfrak{s}} = [\widehat{\mathfrak{s}}, \widehat{\mathfrak{s}}] + \mathfrak{s}$, we see that $\mathfrak{t}_0$ is a sum of $\mathfrak{s}$ and a Cartan subalgebra of a semisimple subalgebra of $\mathfrak{g}$. By step 1, there are only finitely many possibilities for $\mathfrak{t}_0$.

**Step 3.** Clearly, $\mathfrak{z}(\mathfrak{s}) = \mathfrak{t}_0 \cap (\mathfrak{t}_0 \cap [\widehat{\mathfrak{s}}, \widehat{\mathfrak{s}}])^\perp$, where the orthogonal complement is taken with respect to some invariant non-degenerate symmetric form on $\mathfrak{g}$. Thus, by the previous steps, there are only finitely many possibilities for $\mathfrak{z}(\mathfrak{s})$. Obviously, $\widehat{\mathfrak{s}}$ is a direct sum of $\mathfrak{z}(\mathfrak{s})$ and a semisimple subalgebra of $\mathfrak{z}(\mathfrak{s})$. Thence, $\widehat{\mathfrak{s}}$ belongs to one of finitely many $Z_G(\mathfrak{s})$-conjugacy classes of subalgebras. To complete the proof of the lemma it remains to apply Lemma 3.7 to $\mathfrak{g}_0 = \mathfrak{z}(\mathfrak{s})$.

**Corollary 3.9.** There are proper subspaces $a_1, \ldots, a_m \subset a_{\mathfrak{g}, \mathfrak{h}}$ satisfying the following condition: if $\hat{\mathfrak{h}}$ is a subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ such that $c_{\mathfrak{g}, \mathfrak{h}} = c_{\mathfrak{g}, \mathfrak{h}}$ and $a_{\mathfrak{g}, \mathfrak{h}} \not\subset a_i$ for any $i$, then $\hat{\mathfrak{h}} \subset \mathfrak{h}_{\text{sat}}$.

**Proof.** For $a_i$ we take elements of the set $\{a_{\mathfrak{g}, \mathfrak{h}}| = [\hat{\mathfrak{h}}, \hat{\mathfrak{h}}] + R_u(\hat{\mathfrak{h}}) + \mathfrak{h}, a_{\mathfrak{g}, \mathfrak{h}} \not\subset a_{\mathfrak{g}, \mathfrak{h}}\}$. Put $\hat{\mathfrak{h}}_0 = [\hat{\mathfrak{h}}, \hat{\mathfrak{h}}] + R_u(\hat{\mathfrak{h}}) + \mathfrak{h}$. Clearly, $\hat{\mathfrak{h}}_0 = [\hat{\mathfrak{h}}, \hat{\mathfrak{h}}] + R_u(\hat{\mathfrak{h}}) + \mathfrak{h}$. If $a_{\mathfrak{g}, \mathfrak{h}}$ is not contained in any $a_i$, then $a_{\mathfrak{g}, \mathfrak{h}} = a_{\mathfrak{g}, \mathfrak{h}}$. Moreover, since $\mathfrak{h} \subset \hat{\mathfrak{h}}_0 \subset \hat{\mathfrak{h}}$, we get $c_{\mathfrak{g}, \mathfrak{h}} = c_{\mathfrak{g}, \mathfrak{h}} \leq c_{\mathfrak{g}, \mathfrak{h}} \leq c_{\mathfrak{g}, \mathfrak{h}}$. Applying the following lemma to $\mathfrak{g}_0 = \hat{\mathfrak{h}}_0, \mathfrak{h}$, we get $\hat{\mathfrak{h}}_0 = \mathfrak{h}$. 

**□**
Lemma 3.10. For any algebraic subgroup $G_0 \subset G$ we have
\[ 2(\dim \mathfrak{g} - \dim \mathfrak{g}_0) \geq 2c_{\mathfrak{g}, \mathfrak{g}_0} + 2 \dim \mathfrak{a}_{\mathfrak{g}, \mathfrak{g}_0} + \dim \mathfrak{g} - \dim \mathfrak{j}_\mathfrak{g}(\mathfrak{a}_{\mathfrak{g}, \mathfrak{g}_0}) \]
with the equality provided $G_0$ is observable.

Proof of Lemma 3.10. This follows from [K1], Sätze 7.1, 8.1, Korollar 8.2

It follows that $\mathfrak{h}$ is an ideal of $\hat{\mathfrak{h}}$ and that $\hat{\mathfrak{h}}/\mathfrak{h}$ is a commutative reductive algebraic Lie algebra. Let $\hat{H}$ denote the connected subgroup of $G$ corresponding to $\hat{\mathfrak{h}}$. By Proposition 4.7 from [L1], $\hat{H}/H^0$ acts on $G/H^0$ by central automorphisms, equivalently, $\hat{\mathfrak{h}} \subset \mathfrak{h}_{sat}$.

The following lemma is used on step 3 of the proof of Proposition 3.1.

Lemma 3.11. Let a highest weight $\lambda$ satisfy condition (3) of Proposition 3.4. Then

(3') Any subgroup $\hat{H} \subset G$ such that $H \subset \hat{H}$, $H^0 = \hat{H}^0$ and $V(\lambda)^H = V(\lambda)^{\hat{H}}$ is contained in $H_{sat}$.

Proof. By the Frobenius reciprocity, $\mathbb{C}[G/\hat{H}]^{(B)}_\lambda = \mathbb{C}[G/H]^{(B)}_\lambda$. By the choice of $\lambda$, $\mathbb{C}(G/H)^B = \mathbb{C}(G/\hat{H})^B$. Equivalently, $\mathbb{C}(G/B)^H = \mathbb{C}(G/B)^{\hat{H}}$. Applying the main theorem of the Galois theory to the field $\mathbb{C}(G/B)^H$, we see that the images of $H/H^0, \hat{H}/H^0$ in $\text{Aut}(\mathbb{C}(G/B)^H)$ (or, equivalently, $\text{Aut}(\mathbb{C}(G/H^0)^B)$) coincide. By assertion 1 of Lemma 3.3, $\hat{H}/H^0 = (H/H^0)\Gamma$, where $\Gamma \subset \mathfrak{a}_{G,H}$. Assertion 2 of Lemma 3.3 implies that $H$ is a normal subgroup in $\hat{H}$. In virtue of the natural inclusion $\mathbb{C}(G/H)^B \hookrightarrow \mathbb{C}(G/H^0)^B$, the group $\hat{H}/H$ acts trivially on $\mathbb{C}(G/H)^B$. It remains to apply assertion 1 of Lemma 3.3 one more time.

Now we define subspaces $a_1, \ldots, a_k \subset a_{\mathfrak{g}, \mathfrak{h}}$ and sublattices $\mathfrak{X}_1, \ldots, \mathfrak{X}_l \subset \mathfrak{X}_{G,H}$ satisfying the assumptions of Proposition 3.4.

Suppose that $\mathfrak{A}_{G,H}$ is a finite group. Take for $a_1, \ldots, a_k \subset a_{\mathfrak{g}, \mathfrak{h}}$ subspaces found in Corollary 3.9. Let $\mathfrak{A}_{G,H} \cong \bigoplus_{i=1}^l \mathbb{Z}/p_i^{n_i} \mathbb{Z}$, where $p_1, \ldots, p_l$ are distinct primes. Take for $\mathfrak{X}_i$ the lattice $\mathfrak{X}_{G,H}^{\tilde{H}_i}$, where $\tilde{H}_i$ denotes a unique subgroup of $\hat{H}$ such that $\#\tilde{H}_i/H = p_i$. Clearly, $\tilde{H}_i/H, i = 1, \ldots, l$ are all minimal proper subgroups of $\mathfrak{A}_{G,H}$.

Now suppose that $\mathfrak{A}_{G,H}$ is a one-dimensional torus. For $a_1, \ldots, a_{k-1}$ we take subspaces found in Corollary 3.9 and for $a_k$ we take the subspace $a_{\mathfrak{g}, \mathfrak{h}, sat}$.

Proposition 3.1 follows from Proposition 3.4, Lemma 3.11 and the following proposition.

Proposition 3.12. Let $\lambda$ be a highest weight satisfying conditions (1), (2) of Proposition 3.4 for $a_1, \ldots, a_k, \mathfrak{X}_1, \ldots, \mathfrak{X}_l$ defined above and condition (3') of Lemma 3.11 (or only condition (1) if $c_{\mathfrak{g}, \mathfrak{h}} = 0$). Then $\lambda$ has the properties indicated in Proposition 3.1.

Proof. Set $V := V(\lambda)^H$. By the choice of $\lambda$, $\mathfrak{g}_v = \mathfrak{h}$ and $G_v \cap H_{sat} = H$ for $v \in V$ in general position.

First of all, we consider the case $c_{\mathfrak{g}, \mathfrak{h}} = 0$. In this case $H_{sat} = N_G(H)$ (this stems directly from Definition 1.2 since $\dim \mathbb{C}(G/H)^B_\lambda = 1$ for any $\lambda \in \mathfrak{X}_{G,H}$). Further, $N_G(H^0)/H^0$ is commutative and thence $\hat{H} \subset N_G(H)$ for any $\hat{H}$ with $\hat{H}^0 = H^0$. Thus $G_v \subset H_{sat}$ for a non-zero vector $v \in V$. It follows from the choice of $\lambda$ that $G_v = \hat{H}$.

In the sequel we assume that $c_{\mathfrak{g}, \mathfrak{h}} > 0$. Let us prove that the set
\[ \bigcup_{\hat{H} \supseteq H, H^0 = H^0} V(\lambda)^{\hat{H}} \]
is not dense in $V$. Any subgroup $\tilde{H} \subset G$ with $\tilde{H}^\circ = H^\circ$ lies in $N_G(H^\circ)$. Denote by $Y_n$ the subset of $N_G(H^\circ)/H^\circ$ consisting of all elements $h$ such that $h$ and $H/H^\circ$ generate a finite subgroup in $N_G(H)$, whose order divide $n$. For $h \in Y_n$ we denote by $\tilde{H}(h)$ the inverse image in $N_G(H^\circ)$ of the subgroup of $N_G(H^\circ)/H^\circ$ generated by $h$ and $H/H^\circ$.

Note that for every $n$ the subset $Y_n \subset N_G(H^\circ)/H^\circ$ is closed. Put

$$Y_{n,i} = \{ h \in Y_n \mid \text{codim}_V V(\lambda)^{\tilde{H}(h)} = i \}.$$  

This is a locally closed subvariety of $Y_n$. Taking into account Lemma 3.11, we see that $Y_{n,0} = \{1\} \text{ or } \emptyset$.

It is enough to show that for all $n, i > 0$ the subset

$$(3.1) \quad \bigcup_{h \in Y_{n,i}} V(\lambda)^{\tilde{H}(h)}$$

is not dense in $V$.

Assume the converse: let $n, i \in \mathbb{N}$ be such that the subset $(3.1)$ is dense in $V$. Then (compare with the proof of Proposition 3.4) $\dim Y_{n,i} \geq i$. It follows that $i \leq \dim Y_{n,i} \leq \dim G$. For $h_1, h_2 \in Y_{n,i}$ the inequality

$$(3.2) \quad \dim V(\lambda)^{\tilde{H}(h_1)} \cap V(\lambda)^{\tilde{H}(h_2)} \geq \dim V - 2i \geq \dim V - 2 \dim G$$

holds. Let $\tilde{H}(h_1, h_2)$ denote the algebraic subgroup of $G$ generated by $\tilde{H}(h_1), \tilde{H}(h_2)$. Note that $\dim V(\lambda)^{\tilde{H}(h_1, h_2)} = V(\lambda)^{\tilde{H}(h_1)} \cap V(\lambda)^{\tilde{H}(h_2)}$. In virtue of (3.2) and condition (2) of Proposition 3.4, $V(\lambda)^{\tilde{H}(h_1, h_2)} \neq 0$. By the choice of $\lambda$, $a_{g,\tilde{h}(h_1, h_2)} = a_{g,\tilde{h}}$. Therefore, see Lemma 3.10, if $\tilde{h}(h_1, h_2) \neq \tilde{h}$, then $c_{g,\tilde{h}(h_1, h_2)} < c_{g,\tilde{h}}$. But in this case (3.2) contradicts condition (2) of Proposition 3.4. So $\tilde{h}(h_1, h_2) = \tilde{h}$ for any $h_1, h_2 \in Y_{n,i}$. In particular, any $h_1, h_2 \in Y_{n,i}$ generate a finite subgroup in $N_G(H^\circ)/H^\circ$. Choose an irreducible component $Y' \subset Y_{n,i}$ of positive dimension. Consider the map $\rho : Y' \times Y' \to N_G(H^\circ)/H^\circ, (h_1, h_2) \mapsto h_1h_2^{-1}$. Its image is a non-discrete constructible set, whose elements have finite order in $N_G(H^\circ)/H^\circ$. Note that 1 is a nonisolated point in $\text{Im } \rho$. Thus there is a locally closed subvariety $Z \subset \text{Im } \rho$ of positive dimension, whose closure contains 1. The subsets $Z_j := \{ z \in Z| z^j = 1 \}$ are closed in $Z$. Thus $1 \in \overline{Z}_j$ for some $j$. However, 1 is an isolated point in $\{ g \in N_G(H^\circ)/H^\circ| g^j = 1 \}$. Contradiction.

4. Proof of Theorem 1.4

Again, one implication in Theorem 1.4 is almost trivial.

Proof of $(a) \Rightarrow (b)$. Let $V(\lambda)$ be such a simple module. By Theorem 1.3, $\mathfrak{A}_{G,H}$ is either a finite cyclic group or a one-dimensional torus. Suppose that $\mathfrak{A}_{G,H} \simeq \mathbb{C}^\times$. As we noted in the proof of the implication $(a) \Rightarrow (b)$, $\mathfrak{A}_{G,H}$ acts on $V(\lambda)^H$ by constants. If $\mathfrak{A}_{G,H} \simeq \mathbb{C}^\times$, then $0 \in \mathfrak{A}_{G,H} v$ for any $v \in V(\lambda)^H$. Thus $0 \in N_G(H)v$ whence $0 \in \overline{Gv}$. Contradiction.

The proof of the other implication is much more complicated. Below we assume that $\mathfrak{A}_{G,H}$ is cyclic. At first, we prove $(b) \Rightarrow (a)$ for reductive subgroups $H \subset G$ satisfying the following condition.

(*) The group $T_0 := (N_G(H)/H)^\circ$ is a torus, equivalently, the Lie algebra $\mathfrak{g}^H$ is commutative.

The proof for $H$ satisfying (*) is based on the following technical proposition, which is analogous to Proposition 3.4.

**Proposition 4.1.** Let $H$ satisfy (*) and $a_1, \ldots, a_n, \bar{a}_1, \ldots, \bar{a}_l$ be such as in Proposition 3.4. Then there is a highest weight $\lambda$ satisfying conditions (1)-(3) of Proposition 3.4 (only (1) for $c_{a,b} = 0$) and the following condition:

(4) The cone spanned by the weights of $T_0$ in $V(\lambda)^H$ coincides with the whole space $X(T_0) \otimes_{\mathbb{Z}} \mathbb{Q}$.

We note that if $c_{a,b} = 0$, then (4) holds automatically.

**Proof of (b) ⇒ (a) for $H$ satisfying (*).** By Lemma 3.11 and Proposition 3.12, there is a dense subset $V^0 \subset V := V(\lambda)^H$ such that $G_v = H$ for any $v \in V^0$. By condition (4) of Proposition 4.1, a general orbit for the action $T_0 : V$ is closed. It follows that there is $v \in V$ such that $G_v = H$ and the orbit $N_G(H)v$ is closed. By the Luna theorem, see [PV], Theorem 6.17, $Gv$ is also closed.

In the proof of Proposition 4.1 we will need several lemmas. We may and will assume that $c_{a,b} > 0$.

Let us introduce some notation. Set $L := X(T_0) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $\Psi$ (resp., $\Psi^0$) denote the set of highest weights $\lambda$ with $V(\lambda)^H \neq 0$ (resp., satisfying condition (3)). By Lemma 3.2, $\Psi$ is a monoid. For $\lambda \in \Psi$ by $S(\lambda)$ we denote the set of weights of $T_0$ in $V(\lambda)^H$. Since $\mathbb{C}[G/H]_{\lambda}^B \subset \mathbb{C}[G/H]_{\mu}^{(B)} \subset \mathbb{C}[G/H]_{\lambda^* + \mu^*}^{(B)}$, we have $S(\lambda) + S(\mu) \subset S(\lambda + \mu)$. Finally, we denote by $\tilde{H}$ the inverse image of $T_0$ in $N_G(H)$ under the natural epimorphism $N_G(H) \twoheadrightarrow N_G(H)/H$.

**Lemma 4.2.** There is a highest weight $\nu$ satisfying conditions (1), (3), (4).

**Proof.** Step 1. Let us check that $a_{\bar{a},b} = a_{a,b}$. Since $\mathfrak{A}_{G,H}$ is finite, Lemma 3.3 implies that the action $T_0 : \mathbb{C}(G/H)^B$ is locally effective. It follows that $c_{a,b} = c_{a,b} - \dim T_0$. The required equality follows from the inclusion $a_{\bar{a},b} = a_{a,b}$ and Lemma 3.10.

Step 2. By step 1, elements $\lambda_0^\star$ with $\lambda_0 \in \Psi, 0 \in S(\lambda_0)$, span $a_{a,b}$. Clearly, $\Psi^0$ is an ideal in $\Psi$. Therefore even $\lambda_0^\star$ with $\lambda_0 \in \Psi_0 := \{\lambda_0 \in \Psi^0 | 0 \in S(\lambda_0)\}$ span $a_{a,b}$. Fix $\lambda_0 \in \Psi_0$. We claim that $S(\lambda_0)$ spans $L$. Indeed, otherwise there is a subgroup $\tilde{H}_0 \subset \tilde{H}$ such that $\dim \tilde{H}_0/H > 0$ and $\tilde{H}_0$ acts trivially on $V(\lambda)^H$. By (3), $\tilde{H}_0$ acts trivially on $C(G/H)^B$, which contradicts $\# \mathfrak{A}_{G,H} < \infty$.

Step 3. Set $\nu_0 := \lambda_0 + \lambda_0^\star$. Clearly, $V(\lambda_0)^H \cong (V(\lambda_0^\star))^*$. Thus $S(\lambda_0) = -S(\lambda_0^\star)$. It follows that $S(\nu_0) \supset S(\lambda_0), -S(\lambda_0)$ whence the cone spanned by $S(\nu_0)$ coincides with $L$.

Step 4. Let $\mu, n$ be such as in Lemma 3.5. For sufficiently large $m$ the cone spanned by $mS(\nu_0) + iS(\mu)$ coincides with $L$ for any $i = 1, n$. Thus for appropriate $\mu'$ the weight $\nu := m\nu_0 + i\mu'$ satisfies (1), (3), (4).

**Proof of Proposition 4.1.** Let $\nu$ be such as in Lemma 4.2, $n$ be such as in Lemma 3.5. We fix a norm $| \cdot |$ on $\mathfrak{a}_{a,b}(\mathbb{R})$ such that $|\lambda| = |\lambda^*|$ for any $\lambda \in \mathfrak{a}_{a,b}$. Let $A_0, Y_0$ be such as in the proof of Proposition 3.4.

We choose $\lambda \in \Psi$ and $A \in \mathbb{R}$ such that $\dim V(\lambda)^H > A|\lambda|^{-1}$, where $c := c_{a,b}$, and

$$A|\lambda|^{-1} > A_0(2|\lambda| + |\nu|n)^{c-1} + 2 \dim G + \dim Y_0.$$ 

By Lemma 3.5 there is $i = 1, n$ such that $\tilde{\lambda} := \lambda + \lambda^* + i\mu$ satisfies (1) and automatically (3). As in the proof of Proposition 3.4, $\tilde{\lambda}$ satisfies (2). Finally, note that $S(\lambda) = -S(\lambda^*)$.

It follows that $S(\nu) \subset S(\tilde{\lambda})$ whence $\tilde{\lambda}$ satisfies (4).
Proof of Theorem 1.4 in the general case. Now $H$ is a subgroup of $G$ such that $\mathfrak{A}_{G,H}$ is a finite cyclic group and the algebra $\mathfrak{g}^H$ is not commutative.

There is a finite cyclic subgroup $\Gamma \subset N_G(H)/H$ of such that $Z_{N_G(H)/H}(\Gamma)^{0}$ is a maximal torus of $N_G(H)/H$, $\#\Gamma$ is prime and does not divide $\#H/\Gamma^{0}$. Let $\overline{H}$ denote the inverse image of $\Gamma$ in $N_G(H)$. Clearly, $\overline{H} \cap H^{sat} = H$. Moreover, $(N_G(\overline{H})/\overline{H})^{0}$ is a torus. Choose a highest weight $\lambda$ satisfying conditions (1)-(4) of Propositions 3.4,4.1 (for $H$ instead of $H$). Let us check that $V(\lambda)$ has the required properties.

Choose $a_1, \ldots, a_k, x_1, \ldots, x_i$ as in Proposition 3.12 for $\overline{H}$ instead of $H$. Let us check that $\lambda$ satisfies conditions (1),(2) of Proposition 3.4 and condition (3') of Lemma 3.11 for $H$. Condition (1) is follows from the equality $\mathfrak{A}_{G,H} = \mathfrak{A}_{G,\overline{H}}$, which, in turn, stems from [K2], Theorem 6.3, and the choice of $\Gamma$. To check condition (2) it is enough to check that the subset $Z \subset V(\lambda)$ defined there is closed. This will follow if we check that $c_{\mathfrak{g},\mathfrak{h}} < c_{\mathfrak{g},\mathfrak{h}}$ for any algebraic subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{h}, V(\lambda)^{\mathfrak{h}} \neq \{0\}$. At first, suppose that $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] + R_{\mathfrak{h}}(\mathfrak{h}) + \mathfrak{h}$. Then, by the choice of $a_i$, we see that $a_{\mathfrak{g},\mathfrak{h}} = a_{\mathfrak{g},\mathfrak{h}}$. Contradiction with Lemma 3.10. Now let $\mathfrak{s}$ denote a maximal reductive subalgebra of $\mathfrak{h}$ containing $\mathfrak{h}$. Then $\mathfrak{s} \subset \mathfrak{s}_0 := \mathfrak{h} + 3(\mathfrak{s}) \nsubseteq \mathfrak{h}$. It follows that $c_{\mathfrak{s}_0,\mathfrak{h}} = c_{\mathfrak{g},\mathfrak{h}}$. Thanks to Lemma 3.3, the last equality contradicts $\#\mathfrak{A}_{G,H} < \infty$. So conditions (1),(2) for $\lambda$ and $H$ are checked.

Let us check condition (3'). Let $\tilde{H}$ be such a subgroup of $G$ strictly containing $H$ such that $H^{\circ} = \tilde{H}^{\circ}, V(\lambda)^{H} = V(\lambda)^{\tilde{H}}$. Let $\tilde{H}$ denote the algebraic subgroup of $G$ generated by $\overline{H}, \tilde{H}$. Then $V(\lambda)^{\tilde{H}} = V(\lambda)^{\overline{H}} \cap V(\lambda)^{\tilde{H}} = V(\lambda)^{\overline{H}}$. Thanks to Lemma 3.11, $\tilde{H} \subset \overline{H}^{sat}$. From the choice of $x_j$ it follows that $\tilde{H} \subset \overline{H} = \overline{H}$. By the choice of $\Gamma$, $\overline{H} = \tilde{H}$. So $V(\lambda)^{\overline{H}} = V(\lambda)^{\overline{H}}$. Choose a nilpotent element $\xi \in \mathfrak{g}^{\overline{H}}$. Then $exp(t\xi)\overline{H} exp(t\xi)^{-1} \neq \overline{H}$ but $exp(t\xi)V(\lambda)^{\overline{H}} = V(\lambda)^{\overline{H}}$. But, by the proof of Proposition 3.4, there is $v \in V(\lambda)^{\overline{H}}$ with $G_{\overline{H}} = G_{\overline{H}}$. However, $exp(t\xi)v \notin V(\lambda)^{\overline{H}}$. Contradiction. So condition (3') holds for $\lambda, H$. By Proposition 3.12, there is a dense open subset $V^{0} \subset V(\lambda)^{\overline{H}}$ such that $G_{\overline{H}} = H$ for any $v \in V^{0}$.

It remains to prove that there is $v \in V^{0}$ with closed $G$-orbit or, equivalently (by the Luna theorem, [PV], Theorem 6.17), $N_G(H)$-orbit. Let $u \in V(\lambda)^{\overline{H}}$ be such that $G_u = \overline{H}$ and $N_G(\overline{H})u$ is closed. Since $\#\Gamma$ does not divide $\#H/\Gamma^{0}$, we have $N_G(\overline{H}) \subset N_G(H)$. By the Luna theorem, $N_G(H)u$ is closed. Since there is a closed $N_G(H)$-orbit in $V(\lambda)^{H}$ of dimension $\dim N_G(H)/H$, a general orbit is also closed. \[\Box\]

5. Some examples

In Introduction we have noted that the group $\mathfrak{A}_{G,H}$ can be computed for any algebraic subgroup $H \subset G$. However, in general, the computation algorithm is rather involved. In this section we give examples when the application of our theorems is easy.

Example 5.1. Let $H$ be a spherical observable subgroup of $G$ (the former means that $G/H$ is spherical). In this case every automorphism of $G/H$ is central, so $\mathfrak{A}_{G,H} = N_G(H)/H$. The classification of reductive spherical subgroups is known and in this case groups $N_G(H)/H$ are easy to compute. Note also that $G/H$ can be embedded to any module $V(\lambda)$ provided $\lambda \notin \mathfrak{X}_{G,H}^{sat}$ for any subgroup $\tilde{H} \subset G$ containing $H$. For example, let $G = \text{SL}_{2n+1}, H = \text{Sp}_{2n}$. In this case $N_G(H)/H$ is a one-dimensional torus. In fact, $G/H$ can be embedded into $\wedge^3 \mathbb{C}^{2n+1}$ provided $n \geq 3$. 

Example 5.2. Let $H$ be a finite subgroup of $G$. It follows from results of [K2] that in this case $\mathfrak{z}_{G,H} \cong Z(G)/Z(G) \cap H$. So any homogeneous space $G/H$, where $Z(G)$ is a cyclic group or a one-dimensional torus, can be embedded into a simple module as a closed subvariety.

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