ON A DETERMINANTAL INEQUALITY ARISING FROM DIFFUSION TENSOR IMAGING

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Abstract. In comparing geodesics induced by different metrics, Audenaert formulated the following determinantal inequality
\[ \det(A^2 + |BA|) \leq \det(A^2 + AB), \]
where \( A, B \) are \( n \times n \) positive semidefinite matrices. We complement his result by proving
\[ \det(A^2 + |AB|) \geq \det(A^2 + AB). \]
Our proofs feature the fruitful interplay between determinantal inequalities and majorization relations. Some related questions are mentioned.

1. Introduction

In the mathematical framework of interpolation methods for image processing in diffusion tensor imaging, one needs to compare geodesics induced by different metrics. The following determinantal inequality, formulated by Audenaert [1], arises in this setting
\[ \det(A + U^*B) \leq \det(A + B), \]
where \( A, B \) are positive semidefinite matrices and \( U \) is a unitary matrix that appears in the polar decomposition of \( BA \). For readers interested in how the determinantal inequality (1.1) comes into being, we refer to [1] and references therein. In this article, we are mainly interested in determinantal inequalities that are inspired by (1.1).

If one does not want the “specified” unitary matrix to come into the play, an equivalent formulation of (1.1) is the following

Theorem 1.1. Let \( A, B \) be \( n \times n \) positive semidefinite matrices. Then
\[ \det(A^2 + |BA|) \leq \det(A^2 + AB). \]

Here, for a complex matrix \( X \), the absolute value of \( X \) is defined as \( |X| = (X^*X)^{1/2} \), the unique positive semidefinite square root of \( X^*X \).

We complement Theorem 1.1 by proving

\[ \text{2010 Mathematics Subject Classification.} \quad 15A45, 15A60. \]
\[ \text{Key words and phrases.} \quad \text{determinantal inequality, positive semidefinite matrix, log-majorization.} \]
Theorem 1.2. Let $A, B$ be $n \times n$ positive semidefinite matrices. Then
\[
\det(A^2 + |AB|) \geq \det(A^2 + AB).
\]

The paper is organized as follows. In the next section, we review Audenaert’s proof of Theorem 1.1 with a special attention paid to the fruitful interplay between determinantal inequalities and majorization relations (to be introduced). In Section 3 we present a proof of Theorem 1.2. We conclude with some remarks and open problems in Section 4.

2. Preliminaries

For a vector $x \in \mathbb{R}^n$, we denote by $x^\downarrow = (x_1^\downarrow, \ldots, x_n^\downarrow) \in \mathbb{R}^n$ the vector with the same components as $x$, but sorted in descending order. Given $x, y \in \mathbb{R}^n$, we say that $x$ weakly majorizes $y$, written as $x \succ_w y$, if
\[
\sum_{i=1}^k x_i^\downarrow \geq \sum_{i=1}^k y_i^\downarrow \quad \text{for} \quad k = 1, \ldots, n.
\]
We say $x$ majorizes $y$, denoted by $x \succ y$, if $x \succ_w y$ and the sum of all elements of $x$ equal to the sum of all elements of $y$. Now given $x, y \in \mathbb{R}_+^n$, we say that $x$ weakly log-majorizes $y$, written as $x \succ_{w \log} y$, if
\[
\prod_{i=1}^k x_i^\downarrow \geq \prod_{i=1}^k y_i^\downarrow \quad \text{for} \quad k = 1, \ldots, n.
\]
We say $x$ log-majorizes $y$, denoted by $x \succ_{\log} y$, if $x \succ_{w \log} y$ and the product of all elements of $x$ equal to the product of all elements of $y$. For an $n \times n$ matrix $A$ with all eigenvalues real, we denote the vector of eigenvalues by $\lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A))$, and we assume that the components of $\lambda(A)$ are in descending order. The eigenvalues of $|A|$ are called the singular values of $A$. A concise treatment of majorization relations for eigenvalues or singular values can be found in [3, Chapter II], [10, Chapter 3] and [11, Chapter 10].

It has been long known that majorization is a powerful tool in establishing determinantal inequalities; see [9], [3, p. 183] and for recent examples see [4, 5, 7]. Moreover, many classical determinantal inequalities (for example, the Hadamard-Fischer inequality, the Oppenheim inequality) can find their majorization counterparts [2]. The following are two prototypes that we apply majorization relations to derive determinantal inequalities. Assume that $X$ and $Y$ are $n \times n$ matrices:

(P1) If $\lambda(X), \lambda(Y) \in \mathbb{R}_+^n$ such that $\lambda(X) \succ \lambda(Y)$, then $\det X \leq \det Y$.

(P2) If $\lambda(X), \lambda(Y) \in \mathbb{R}_+^n$ such that $\lambda(X) \succ_{w \log} \lambda(Y)$, then
\[
\det(I + X) \geq \det(I + Y).
\]
The proof of (P1) makes use of the convexity of the function $f(x) = -\log x$ and [11] Theorem 10.13, while the proof of (P2) makes use of the convexity and monotonicity of the function $f(x) = \log(1 + e^x)$ and [11] Theorem 10.14.

To review Audenaert’s proof of Theorem 1.1, we present a slightly more general result. Such a generalization also motivates our thinking in Section 4.

**Theorem 2.1.** Let $A$ and $B$ be $n \times n$ positive semidefinite matrices. Then
\[
\det(A^2 + |BA|^p) \leq \det(A^2 + A^pB^p), \quad 0 \leq p \leq 2.
\]

**Remark 2.2.** As $\text{trace}(A^2 + |BA|^p) \neq \text{trace}(A^2 + A^pB^p)$ in general, it is less likely that (P1) would apply. We therefore try to apply (P2).

**Proof of Theorem 2.1.** We may assume without loss of generality that $A$ is positive definite by a standard perturbation argument. The following majorization relation is known (e.g., [4, Eq. (17)])
\[
\lambda(A^t B) \prec_{\log} \lambda(A^{1-t}B^t), \quad 0 \leq t \leq 1,
\]
where $A^t B := A^{1/2}(A^{-1/2}BA^{-1/2})^tA^{1/2}$. Thus by (P2) we get
\[
\det(I + A^{1/2}(A^{-1/2}BA^{-1/2})^tA^{1/2}) \leq \det(I + A^{1-t}B^t).
\]
Repeating (2.3) with $A$ replaced by $A^{-2}$, $B$ with $B^2$ and $t$ with $p/2$, respectively, in (2.3) yields
\[
\det(I + A^{-1}(AB^2A)^{p/2}A^{-1}) \leq \det(I + A^{p-2}B^p)
= \det(I + A^{p-1}B^pA^{-1}).
\]
Pre-post multiplying both sides by $\det A$ yields the desired result. □

The quantity $A^t B$ is sometimes called the weighted geometric mean of $A$ and $B$. In the sequel, if $t = 1/2$, we simply put $A^t B$ for $A^t_{1/2} B$.

The next example shows that (2.1) may not be valid for $p > 2$.

**Example 2.3.** Take two positive definite matrices
\[
A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix}.
\]
A simple calculation gives $\det(A^2 + |AB|^3) = 100 > \det(A^2 + A^3B^3) = 71$.

**3. Proof of Theorem 1.2**

Based on a reason similar to Remark 2.2, we would apply (P2) for our purpose. Define
\[
A^t B := A^{1/2}(B^{1/2}A^{-1/2}B^{1/2})^{1/2}A^{1/2}
\]
for positive definite matrices $A$ and $B$. Clearly, $A^t B$ is positive definite. We will establish the following majorization relation.
Lemma 3.1. Let $A, B$ be $n \times n$ positive definite matrices. Then

$$\lambda(A \# B) \succeq_{\log} \lambda(A^{1/2}B^{1/2}).$$

Proof. First of all, using a standard argument via the anti-symmetric product (see, e.g., [3, p.18]), it suffices to show

$$\lambda_1(A \# B) \geq \lambda_1(A^{1/2}B^{1/2}),$$

which would follow from the spectral norm inequality

(3.1) \[ \|A \# B\| \geq \|A^{1/2}B^{1/2}\|. \]

Using the Schur complement [11, p. 227], it is easy to see that if an $n \times n$ matrix $H$ is positive definite, then $\begin{bmatrix} H & X^* \\ X & XH^{-1}X^* \end{bmatrix}$ is positive semidefinite for any $m \times n$ matrix $X$.

This implies (see [11, p. 352])

(3.2) \[ \|X\| \leq \|H\|\|XH^{-1}X^*\|. \]

Observe that

$$A \# B = A^{1/2}B^{1/2}(A \# B)^{-1}B^{1/2}A^{1/2}.$$ 

Now letting $X = A^{1/2}B^{1/2}$, $H = A \# B$ in (3.2) yields

$$\|A^{1/2}B^{1/2}\|^2 \leq \|A \# B\|\|A \# B\|.$$

The required inequality (3.1) follows by noting

$$\|A \# B\| \leq \lambda_1(A^{1/2}B^{1/2}) \leq \|A^{1/2}B^{1/2}\|$$

from (2.2).

Now we are ready to present

Proof of Theorem 1.2. We may assume without loss of generality that $A, B$ are positive definite by a standard perturbation argument. Thus by Lemma 3.1 and (P2) we get

(3.3) \[ \det(I + A^{1/2}(B^{1/2}A^{-1}B^{1/2})^{1/2}A^{1/2}) \geq \det(I + A^{1/2}B^{1/2}). \]

Replacing $A$ with $A^{-2}$, $B$ with $B^2$, respectively, in (3.3) yields

$$\det(I + A^{-1}(BA^2B)^{1/2}A^{-1}) \geq \det(I + A^{-1}B).$$

Pre-post multiplying both sides by $\det A$ leads to the desired result. \qed
4. Remarks and Open Problems

We can get a complement of Theorem 2.1 when \( p = 2 \).

**Theorem 4.1.** Let \( A \) and \( B \) be \( n \times n \) positive semidefinite matrices. Then
\[
\det(A^2 + |AB|^2) \geq \det(A^2 + A^2B^2).
\]

**Proof.** Again, we assume \( A \) is positive definite. After dividing both sides by \( \det A^2 \), the claimed inequality reduces to
\[
\det(I + |ABA^{-1}|^2) \geq \det(I + B^2).
\]
By (P2), it suffices to observe the following majorization relation
\[
\lambda(|ABA^{-1}|) \succ_{\log} \lambda(ABA^{-1}) = \lambda(B),
\]
which is ensured by a result of Weyl (see [11, p. 353]). □

Clearly, (4.1) could be written as
\[
\det(A^2 + |AB|^2) \geq \det(A^2 + |BA|^2).
\]

On the other hand, in [8, Corollary 2.3] the authors observed that for any positive integer \( k \), it holds
\[
\lambda(A^2 + |BA|^{2k}) \succ \lambda(A^2 + |AB|^{2k}).
\]
Thus by (P1), it follows
\[
\det(A^2 + |AB|^{2k}) \geq \det(A^2 + |BA|^{2k}).
\]
This says (4.1) could be proved using either (P1) or (P2).

As an analogue of Theorem 2.1 and with the evidence of Theorem 1.2 as well as Theorem 4.1, we make the following conjecture.

**Conjecture 4.2.** Let \( A \) and \( B \) be \( n \times n \) positive semidefinite matrices. Then
\[
\det(A^2 + |AB|^p) \geq \det(A^2 + A^pB^p), \quad 0 \leq p \leq 2.
\]

For any \( n \times n \) positive definite matrices \( A \) and \( B \), in Section 3 we defined the quantity \( A^\#_t B \). A weighted version seems to be
\[
A^\#_t B := A^{1/2}(B^{1/2}A^{-1}B^{1/2})^t A^{1/2}, \quad 0 \leq t \leq 1.
\]
This quantity should be closely related to the previous conjecture and may deserve further investigation.

Theorem 1.1 and Theorem 1.2 immediately lead to
\[
\det(A^2 + |AB|) \geq \det(A^2 + |BA|),
\]
a situation not covered by (4.2). We may ask whether it is true
\[ \det(A^2 + |AB|^p) \geq \det(A^2 + |BA|^p) \text{ for all } p > 0? \]
Indeed, some simulations suggest a corresponding majorization relation is true. That is,

**Conjecture 4.3.** Let \( A \) and \( B \) be \( n \times n \) positive semidefinite matrices. Then
\[ \lambda(A^2 + |BA|^p) \succ \lambda(A^2 + |AB|^p) \text{ for all } p > 0. \]

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