COMPLEX DIMENSION OF ADDITIVE SUBGROUPS OF $\mathbb{R}^n$

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Abstract. In this paper, we define the complex dimension of any additive subgroup of $\mathbb{R}^n$ which generalize the euclidean dimension given for the vector space. We give an explicit method to calculate this dimension.

1. Introduction

The additive groups are seen as weaker than vector space structures. The stability by scalar multiplication in an additive group is not totally or non-existent in general. The linear independence defined the dimension of vector spaces. By analogy, we define the dimension of a discrete additive group as a dimension of the vector space that can generates. As if we assume that it is stable by virtual scalar multiplication. For this reason, this dimension is given purely complex. The difference between these two structures is given by the complex dimension which will be defined in the following.

In \cite{2} M.Waldschmidt gave the form of any closed subgroup $F$ of $\mathbb{R}^n$ as $F = E + D$ with $E$ is a vector space and $D$ is a discrete additive group. By Theorem 2.1, in \cite{2}, this means that there is a basis $(u_1, \ldots, u_n)$ of $\mathbb{R}^n$ and $0 \leq r, p \leq n$ with $p + r \leq n$ such that

$$F := \begin{cases} 
\sum_{k=1}^{p} \mathbb{R}u_k + \sum_{k=p+1}^{p+r} \mathbb{Z}u_k, & \text{if } p > 0 \text{ and } r > 0 \\
\sum_{k=1}^{p} \mathbb{R}u_k, & \text{if } r = 0 \\
\sum_{k=1}^{r} \mathbb{Z}u_k, & \text{if } p = 0
\end{cases}$$

Let $W$ be the vector space generated by $D$. We define the complex dimension of $F$ as $\tilde{\dim}(F) = \dim(E) + i\dim(W)$ (i.e. $\dim(F) = p + ir$). We want to define a new dimension of the discrete groups of $\mathbb{R}^n$ which generalizes that given as a manifold. This dimension can not be rational for the discrete group, for example if $u, v$ are free in $\mathbb{R}^n$ then $\dim(\mathbb{Z}u) = \dim(\mathbb{Z}u + \mathbb{Z}v) = 0$ as manifolds, We can not distinguish between them, but with the complex dimension we have $\tilde{\dim}(\mathbb{Z}u) = i$ and $\tilde{\dim}(\mathbb{Z}u + \mathbb{Z}v) = 2i$. For any subset $A$ of $\mathbb{R}^n$, denote by $\text{vect}(A)$ the vector subspace of $\mathbb{R}^n$ generated by $A$. Therefore the dimension of any closed subgroup of $\mathbb{R}^n$ (as a manifold) is equal the real part of the complex dimension. As a manifold

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we can verify that
\[ \dim(V) = \max\{\dim(V); \text{ } V \text{ vector space } V \subset F\}. \]

For any additive subgroup \( H \) of \( \mathbb{R}^n \), we call the complex dimension of \( H \) the number
\[ \dim(H) := p + i(q - p) \]
where
\[ p = \max\{\dim(V); \text{ } V \text{ vector space } V \subset H\} \quad \text{and} \quad q = \min\{\dim(V); \text{ } V \text{ vector space } H \subset V\}. \]

This means that \( q = \dim(\text{vect}(H)) \).

For a number \( z \in \mathbb{C} \), we write \( z = \Re(z) + i\Im(z) \), where \( \Re(z) \) and \( \Im(z) \in \mathbb{R} \).

The \( \Re(\dim(H)) \) represents the dimension of the vector space generated by all points of \( H \) which are stable by scalar multiplication and \( \Im(\dim(H)) \) represents the dimension of the vector space generated by all points of \( H \) which are stable only by addition.

Our principal results are the following:

**Theorem 1.1.** Let \( H \) be an additive subgroup of \( \mathbb{R}^n \). If \( \overline{H} = E + D \) with \( E \) is a vector space and \( D \) is a discrete additive group such that \( \text{vect}(D) \oplus E = \text{vect}(H) \), then \( H = H_1 + D \) with \( H_1 \) is an additive subgroup dense in \( E \).

**Corollary 1.2.** If \( \Re(\dim(H)) \neq 0 \) then \( H \) can not be dense in \( \mathbb{R}^n \).

**Corollary 1.3.** Let \( H \) and \( K \) be two additive subgroups of \( \mathbb{R}^n \). One has:
(i) If \( \dim(H) = p + ir \) then \( \dim(\overline{H}) = p' + ir \) with \( p' \geq p \).
(ii) If \( K \subset H \) then \( \overline{\dim(K)} \leq \overline{\dim(H)} \).

**Remark 1.4.** (i) \( H \) is closed and discrete if and only if \( \Re(\dim(H)) = 0 \).
(ii) If \( E \) is a vector space then \( \dim(E) = \overline{\dim(E)} \).
(iii) If \( E \) is a connected component of \( \overline{H} \) containing 0 then \( \overline{\dim(E)} \in \mathbb{N} \).

We will use the following notations and definitions to characterize the density of any subgroup \( H \) of \( \mathbb{R}^n \).

We say that a matrix \( M = (a_{i,j})_{1 \leq i \leq p, 1 \leq j \leq q} \) satisfies the rational property at level \( r \), \( 0 \leq r \leq n \), if and only if there exists \( k_1, \ldots, k_s \in \{1, \ldots, q\} \) such that each following list is formed by rationally independent numbers:

\[
\begin{align*}
1, & a_{k_1, i_{k_1}}, \ldots, a_{k_1, i_{k_{s_1}}} \\
1, & a_{k_2, i_{k_2}}, \ldots, a_{k_2, i_{k_{s_2}}} \\
\vdots & \vdots \vdots \vdots \\
1, & a_{k_s, i_{k_s}}, \ldots, a_{k_s, i_{k_{s_s}}} 
\end{align*}
\]

and \( i_{kj} \in \{1, \ldots, 2n\} \) such that the cardinal
\[ \text{Card}\{i_{kj}, \text{ } j = 1, \ldots, s_k, \text{ } k = 1, \ldots, s\} = r. \]

Denote \( L(M) = r \) the level of the rational property of \( M \). For example:

- The matrix \( A = \begin{bmatrix} 2 & 5 & \sqrt{2} & 3 \\ \sqrt{2} & 0 & 2 & \sqrt{3} \\ 1 & \sqrt{2} & 2 & 1 \end{bmatrix} \) satisfies the rational properties at level
4 \ (L(A) = 4), since there are three family of rational independent numbers with 1 in each row:

\[
\begin{bmatrix}
0 & 0 & \sqrt{2} & 0 \\
\sqrt{2} & 0 & 0 & \sqrt{3} \\
0 & \sqrt{2} & 0 & 0
\end{bmatrix}
\]

We can symbolize this table by:

\[
\begin{array}{cccc}
0 & 0 & \times & 0 \\
\times & 0 & 0 & \times \\
0 & \times & 0 & 0 \\
\times & \times & \times & \times
\end{array}
\]

The last row contains the union of all up lines, so if it is complete then the indices chosen of each family contains the set \( \{1, \ldots, q\} \), where \( q = 4 \) is the number of \( A \) colons.

- The matrix \( B = \begin{bmatrix} 2 & 5 & \sqrt{2} & 3 \\ \sqrt{2} & 0 & \sqrt{3} & 0 \\ 1 & \sqrt{2} & 2 & 1 \end{bmatrix} \) has \( L(B) = 3 \), since the three family of rational independent numbers with 1 in each row are:

\[
\begin{bmatrix}
0 & 0 & \sqrt{2} & 0 \\
\sqrt{2} & 0 & \sqrt{3} & 0 \\
0 & \sqrt{2} & 0 & 0
\end{bmatrix}
\]

We can symbolize this table by:

\[
\begin{array}{cccc}
0 & 0 & \times & 0 \\
\times & 0 & \times & 0 \\
0 & \times & 0 & 0 \\
\times & \times & \times & 0
\end{array}
\]

Suppose that \( H \) is finitely generated. We can write \( H = \sum_{k=1}^{p} Z u_k \), where \( u_k \in \mathbb{R}^n \). Suppose that \( u_1, \ldots, u_q \) generate \( \text{vect}(H) \) and write \( u_k = \sum_{i=1}^{q} \alpha_{k,i} u_i \) for every \( q + 1 \leq k \leq p \). Denote by

\[
M_H = \begin{bmatrix}
\alpha_{q+1,1} & \ldots & \alpha_{q+1,q} \\
\vdots & \ddots & \vdots \\
\alpha_{p,1} & \ldots & \alpha_{p,q}
\end{bmatrix}
\]

**Theorem 1.5.** Let \( H \) be an additive subgroup of \( \mathbb{R}^n \) generated by \( u_1, \ldots, u_m \). Then \( \overline{\dim(H)} = L(M_H) + i(q - L(M_H)) \), where \( q = \dim(\text{vect}(H)) \).

**Theorem 1.6.** Let \( H \) be an additive subgroup of \( \mathbb{R}^n \) with complex dimension \( p + ir \). Then there exists \( u \in \mathbb{R}^n \) such that \( H + Zu \) is dense in \( \text{vect}(H) \). (i.e. \( \overline{\dim(H + Zu)} = p + r \)).
Corollary 1.7. Let $H$ be an additive subgroup of $\mathbb{R}^n$ with complex dimension $p + ir$. If $p + r < n$ then for every $u \in \mathbb{R}^n$, $H + Zu$ cannot be dense in $\mathbb{R}^n$.

Let $H = E + D$ and $K = E' + D'$ be two additive subgroup of $\mathbb{R}^n$, where $E$ and $E'$ are two vector spaces, $D$ and $D'$ are two discrete groups. A map $f : H \rightarrow K$ called homomorphism of closed additive group if $f = f_1 \oplus f_2$ with $f_1 : E \rightarrow E'$ is a linear map and $f_2 : D \rightarrow D'$ is a homomorphism of group, (i.e. $f(\lambda x + py) = \lambda f_1(x) + pf_2(y)$ for every $\lambda \in \mathbb{R}$, $p \in \mathbb{Z}$, $x \in E$ and $y \in D$). An homomorphism of closed additive group is called isomorphism of closed additive group if it is invertible.

Theorem 1.8. Let $H$ and $K$ be two closed additive subgroups of $\mathbb{R}^n$ and $f : H \rightarrow K$ be an homomorphism of closed additive group. One has:

(i) If $f$ is injective then $|\dim(H)| \leq |\dim(K)|$.
(ii) If $f$ is surjective then $|\dim(H)| \geq |\dim(K)|$.
(iii) If $f$ is invertible then $\dim(H) = \dim(K)$.
(iv) $f(H)$ is a closed additive subgroup of $K$.
(v) $f^{-1}(L)$ is a closed additive subgroup of $H$, for every closed subgroup of $K$.

2. Proof of Theorem 1.1 and Corollaries 1.2, 1.3

Lemma 2.1. ([2], Theorem 2.1) Let $H$ be an additive subgroup of $\mathbb{R}^n$. Then there exist a vector space $E$ and a discrete additive group $D$ such that $\overline{H} = E + D$ with $E \oplus \text{vect}(D) = \text{vect}(H)$.

Proof of Theorem 1.1. By Lemma 2.1, we can write $\overline{H} = E + D$ with $E$ is a vector space and $D$ is a discrete additive subgroup of $\overline{H}$. Let $W$ be the vector space generated by $D$, then $E \cap W = \{0\}$. Therefore we define $p_1 : E \oplus W \rightarrow E$ the first projection and $p_2 : E \oplus W \rightarrow W$ the second projection. Now, $H \subset \overline{H}$, then $H = H_1 + H_2$ where $H_1 = p_1(H)$ and $H_2 = p_2(H)$. Since $E \cap W = \{0\}$, then $\overline{H} = \overline{H_1} + \overline{H_2} = E + D$, which yields that $\overline{H_1} = E$ and $H_2 = D$ because $D$ is closed and discrete so is $H_2$. \[\Box\]

Proof of Corollary 1.2. The proof follows directly from Theorem 1.1. \[\Box\]

Proof of Corollary 1.3. By Lemma 2.1 we can write $\overline{H} = E + D$ with $E$ is a vector space and $D$ is a discrete additive group such that $E \oplus W = \text{vect}(H)$, where $W := \text{vect}(D)$. Then by Theorem 1.1 $H = H_1 + D$ with $H_1$ is an additive subgroup dense in $E$.

(i) If $\overline{\dim}(H) = p' + ir$ with $p' \leq \dim(E)$ and $r = \dim(W)$. As $\overline{\dim}(\overline{H}) = \dim(E) + \dim(W)$, we have the results.
(ii) Write $\dim(H) = p + ir$ and $\dim(H) = p' + ir'$. Then if $E$ (resp. $E'$) is the smaller vector space contained in $H$ (resp. $K$) then $p' = \dim(E') \leq p = \dim(E)$. Now, if $W$ (resp. $W'$) is the smaller vector space containing $H$ (resp. $K$) then $p' + r' = \dim(W') \leq p + r = \dim(W)$. It follows that $r' \leq r$ and so $p'^2 + r'^2 \leq p^2 + r^2$. \[\Box\]
3. Proof of Theorems 1.5 and Corollary 1.7

Proposition 3.1. ([1], Proposition 4.1) Let $u_1, \ldots, u_p \in \mathbb{R}^n$, $(p \geq n + 1)$. Suppose that $(u_1, \ldots, u_n)$ be a basis of $\mathbb{R}^n$ and $u_k = \sum_{i=1}^{n} \alpha_{k,i} u_i$ for every $n + 1 \leq k \leq p$. Then the additive group $H = \bigoplus_{k=1}^{p} \mathbb{Z} u_k$ is dense in $\mathbb{R}^n$ if and only if the matrix $L(M_H) = n$.

Lemma 3.2. Let $H = \sum_{k=1}^{m} \mathbb{Z} u_k$, $u_k \in \mathbb{R}^n$. Suppose that vect$(H)$ is generated by $u_1, \ldots, u_p$, $p \leq m < n$. Let $v_{p+1}, \ldots, v_n \in \mathbb{R}^n$ such that $(u_1, \ldots, u_p, v_{p+1}, \ldots, v_n)$ forms a basis of $\mathbb{R}^n$. Then for every $1 \leq r \leq n - m$ we have $L(M_H) = L(M_{H'})$ where $H' = H + \sum_{k=1}^{r} \mathbb{Z} v_{p+k}$. In particular, $\dim \overline{\dim(H)} = \dim \overline{\dim(H')}$. 

Proof. Write $u_k = \sum_{j=1}^{p} \alpha_{k,j} u_j$ for every $p + 1 \leq k \leq m$. Then $(\alpha_{k,1}, \ldots, \alpha_{k,p})$ are the coordinate of $u_k$ in the basis $(u_1, \ldots, u_p)$ of vect$(H)$ and $(\alpha_{k,1}, \ldots, \alpha_{k,p}, 0, \ldots, 0)$ are the coordinate of $u_k$ in the basis $(u_1, \ldots, u_p, v_{p+1}, \ldots, v_{p+r})$ of vect$(H')$. Therefore $M_{H'} = [ M_H, \ 0 ] \in M_{m-p,p+r}(\mathbb{R})$. It follows that $L(M_H) = L(M_{H'})$. Since $\dim \overline{\dim(H)} = \dim \overline{\dim(H')}$ and by using the definition of $\dim \overline{\dim(H)}$ as the greatest dimension of all vector subspaces contained in $\overline{H}$, we obtain $\dim \overline{\dim(H)} = \dim \overline{\dim(H')}$. □

Lemma 3.3. Let $H = \sum_{k=1}^{m} \mathbb{Z} u_k$, $u_k \in \mathbb{R}^n$. Then for every $P \in GL(n, \mathbb{R})$ we have $L(M_H) = L(M_{P(H)})$. In particular $\overline{\dim(H)} = \overline{\dim(P(H))}$. 

Proof. Suppose that $(u_1, \ldots, u_p)$ is a basis of vect$(H)$ with $1 \leq p \leq m$ and write $u_k = \sum_{j=1}^{p} \alpha_{k,j} u_j$ for every $p + 1 \leq k \leq m$. Then $Pu_k = \sum_{i=1}^{n} \alpha_{k,i} Pu_i$. It follows that $u_k$ and $Pu_k$ have the same coordinate respectively in the basis $(u_1, \ldots, u_p)$ and $(Pu_1, \ldots, Pu_p)$. We conclude that $M_H = M_{P(H)}$. Moreover, $P$ is viewed as an isomorphism, so $\dim \overline{\dim(H)} = \overline{\dim(P(H))}$, because $\dim \overline{\dim(H)} = \dim \overline{\dim(P(H))}$ and $\dim \overline{\dim(E)} = \dim \overline{\dim(P(E))}$, where $E$ is the greater vector space contained in $\overline{H}$. □

Proof of Theorem 1.5. By Lemma 2.4 we can write $\overline{H} = E + D$ with $E$ is a vector space and $D$ a discrete additive subgroup of $\overline{H}$. Let $W$ be the vector space generated by $D$, then $E \cap W = \{0\}$. Denote by $\overline{\dim(H)} = p + ir$. Since $H = \sum_{k=1}^{n} \mathbb{Z} u_k$ then there exists a basis $B := (v_1, \ldots, v_n)$ of $\mathbb{R}^n$ such that $(v_1, \ldots, v_p)$ forms a basis of $E$ and $D = \sum_{k=1}^{r} \mathbb{Z} v_{p+k}$. Denote by $P \in GL(n, \mathbb{R})$ the matrix of basis change from
Let that $f: H_1 + D$ be a homomorphism of closed additive group. Then:

Proof. By Lemma 2.1 we can write $L(M_{p(H_1)}) = p$ and by Lemma 3.3 $L(M_{H_1}) = p$, hence by Lemma 3.4 $L(M_H) = p$. Since $r = q - p$ with $q = \text{dim}(\text{vect}(H))$ then $\text{dim}(H) = L(M_H) + i(q - L(M_H))$. □

Proposition 3.4. Let $(u_1, \ldots, u_n)$ be a basis of $\mathbb{R}^n$ then there exists $u \in \mathbb{R}^n$ such that $H := \sum_{k=1}^n Zu_k + Zu$ is dense in $\mathbb{R}^n$.

Proof. let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that $1, \alpha_1, \ldots, \alpha_n$ are rationally independent and $u = \sum_{k=1}^n \alpha_k u_k$. By Lemma 3.3 $L(M_H)$ is invariant by basis change, then in the basis $(u_1, \ldots, u_n)$ we have $M_H = [\alpha_1, \ldots, \alpha_n]$, so $L(M_H) = n$. By applying Proposition 3.1 $H$ is a vector space, with $p, p$ and $L(M_H) = p$, hence by Proposition 3.4 $L(M_H) = p$. Since $r = q - p$ with $q = \text{dim}(\text{vect}(H))$ then $\text{dim}(H) = L(M_H) + i(q - L(M_H))$. □

Proof of Theorem 1.6. Suppose that $(u_1, \ldots, u_q)$ is a basis of $\text{vect}(H)$ and let $\alpha_1, \ldots, \alpha_q \in \mathbb{R}$ such that $1, \alpha_1, \ldots, \alpha_q$ are rationally independent and $u = \sum_{k=1}^q \alpha_k u_k$. Denote by $H' = \sum_{k=1}^q Zu_k + Zu$. By Proposition 3.4 $H'$ is dense in $\mathbb{R}^n$.

Proof of Corollary 1.7. Let $u \in \mathbb{R}^n$ and $H' = H + Zu$. Since $\text{dim}(\text{vect}(H)) = p + r < n$ then $\text{dim}(\text{vect}(H')) \leq p + r + 1 \leq n$. Denote by $\text{dim}(H') = p' + ir'$ with $p' \geq p$ and $r' \leq p + r + 1 - p'$. Then there are two cases:
- if $r' \neq 0$ then by Corollary 1.2 $H'$ can not be dense in $\mathbb{R}^n$.
- if $r' = 0$, so $p' = p + r + 1 = n$ then $\mathbb{R}u \oplus \text{vect}(H) = \mathbb{R}^n$. Denote by $p_1 : \mathbb{R}u \oplus \text{vect}(H) \rightarrow \mathbb{R}u$ the first projection. Then $p_1(H') = Zu$, so $\mathbb{R}u = p_1(H') \subset p_1(H) = Zu$, a contradiction. □

4. Proof of Theorem 1.8

Lemma 4.1. Let $H$ and $K$ be two closed additive subgroup of $\mathbb{R}^n$ and $f : H \rightarrow K$ be a homomorphism of closed additive group. Then:

(i) $f(H)$ is a closed additive group of $\mathbb{R}^n$.

(ii) $f^{-1}(K)$ is a closed additive group of $\mathbb{R}^n$.

Proof. By Lemma 2.1 we can write $H = E + D$ and $K = E' + D'$ with $E, E'$ are two vector spaces with dimension respectively $p, p'$ and $D, D'$ are two discrete additive groups with dimension respectively $ir$, $ir'$ such that $E \oplus \text{vect}(D) = \text{vect}(H)$ and $E' \oplus \text{vect}(D') = \text{vect}(K)$. Write $f = f_1 \oplus f_2$ given by $f(x + y) = f_1(x) + f_2(y)$ for every $x \in E$ and $y \in D$ with $f_1$ is linear and $f_2$ is an homomorphism of additive group.

(i) The proof follows directly from the fact $f_1(E) \subset E'$ is a vector space, $f_2^{-1}(D') \subset D$ is an additive group and $f(E + D) = f_1(E) + f_2(D)$ is a closed additive group.
Let $K$ be a homomorphism of closed additive group. Then:

(i) $\ker f$ is a closed additive subgroup of $\mathbb{R}^n$.

(ii) $\text{im}(f)$ is a closed additive subgroup of $\mathbb{R}^n$.

Proof. The proof results directly from Lemma 4.1, since $\ker f = f^{-1}(0)$ and $\text{im}(f) = f(H)$. \hfill \square

Corollary 4.2. Let $H$ and $K$ be two closed additive subgroup of $\mathbb{R}^n$ and $f : H \rightarrow K$ be a homomorphism of closed additive group. Then:

(i) $\ker f$ is a closed additive subgroup of $\mathbb{R}^n$.

(ii) $\text{im}(f)$ is a closed additive subgroup of $\mathbb{R}^n$.

Proposition 4.3. Let $H$ and $K$ be two closed additive subgroup of $\mathbb{R}^n$ and $f : H \rightarrow K$ be a homomorphism of closed additive group. Then:

(i) If $f$ is injective then $f : H \rightarrow f(H)$ is an isomorphism of closed additive group of $\mathbb{R}^n$.

(ii) If $f$ is surjective and $F$ is a closed additive group supplement of $\ker f$ in $H$ with defect 0, then the restriction $f_f : F \rightarrow K$ of $f$ to $F$ is an isomorphism of closed additive group of $\mathbb{R}^n$. Moreover, $\Re(\dim(K)) = \Re(\dim(H)) - \Re(\dim(Ker(f)))$.

Proof. (i) It is clear that $f : H \rightarrow f(H)$ is invertible. Write $H = E + D$ as above and $f = f_1 + f_2$ with $f_1 : E \rightarrow f_1(E)$ is linear and $f_2 : D \rightarrow f_2(D)$ is an homomorphism of group. Then $f_1$ and $f_2$ are also invertible and $f^{-1} = f_1^{-1} \oplus f_2^{-1}$. It follows that $f^{-1}$ is an homomorphism of closed additive group.

(ii) Let $F$ be a closed additive group supplement of $\ker f$ in $H$ with defect 0. Then

\[ \Re(\dim(\ker f \cap F)) = \Im(\dim(\ker f \cap F)) = 0, \]

so $\ker f \cap F = \{0\}$. It follows that $f_f$ is injective, so it is invertible because it is surjective. On the other hand, we have $\Re(\dim(K)) = \Re(\dim(F)) = \Re(\dim(H)) - \Re(\dim(Ker(f)))$. \hfill \square
Proof of Theorem 1.8. The proof of (iv) and (v) results from Lemma 4.1.

Proof of (iii) The proof follows directly from the fact that $f$ is an isomorphism of closed additive group.

Proof of (i): Since $f$ is injective then by Proposition 4.3(i) we have $f : H \rightarrow f(H)$ is an isomorphism of closed additive group. Then by (iii), $\dim(H) = \dim(f(H))$.

By Lemma 4.1(i), $f(H)$ is a closed additive subgroup of $K$, so by Corollary 1.3 $|\dim(f(H))| \leq |\dim(K)|$.

Proof of (i): Since $f$ is injective then by Proposition 4.3(i) we have $f : H \rightarrow f(H)$ is an isomorphism of closed additive group. Then by (iii), $\dim(H) = \dim(f(H))$.

By Lemma 4.1(i), $f(H)$ is a closed additive subgroup of $K$, so by Corollary 1.3 $|\dim(f(H))| \leq |\dim(K)|$.

Proof of (ii): Since $f$ is surjective then by Proposition 4.3(ii), for every supplement $F$ of Ker($f$) in $H$ with 0 defect, we have $f_F : F \rightarrow K$ is an isomorphism of closed additive group. Then by (iii), $\dim(F) = \dim(K)$. By Corollary 1.3 $|\dim(F)| \leq |\dim(K)|$, So $|\dim(K)| \leq |\dim(H)|$.

□

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