Primordial Power Spectra of Cosmological Fluctuations with Generalized Uncertainty Principle and Maximum Length Quantum Mechanics

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The existence of the cosmological particle horizon as the maximum measurable length \( l_{\text{max}} \) in the universe leads to a generalization of the quantum uncertainty principle (GUP) to the form \( \Delta x \Delta p \geq \frac{\hbar}{2(1+\alpha \Delta p^2)} \), where \( \alpha \equiv l_{\text{max}}^{-2} \). The implication of this GUP and the corresponding generalized commutation relation \( [x,p] = i\hbar \frac{1}{1-\alpha x^2} \) on simple quantum mechanical systems has been discussed recently [1] by one of the authors and shown to have extremely small (beyond current measurements) effects of the energy spectra of these systems due to the extremely large scale of the current particle horizon. This is not the case in the Early Universe during the quantum generation of the inflationary primordial fluctuation spectrum. Here we estimate the effects of such GUP on the primordial fluctuation spectrum and on the corresponding spectral index. In particular motivated by the above GUP we generalize the field commutation (GFC) relation to \( [\varphi(k),\pi_\varphi(k')] = i\hbar \frac{1}{1-\alpha x^2} \), where \( \mu \simeq \alpha^2 \equiv l_{\text{max}}^{-4} \) is a GFC parameter, \( \varphi \) denotes a scalar field and \( \pi_\varphi \) denotes its canonical conjugate momentum. In the context of this GFC we use standard methods to obtain the primordial scalar perturbations spectrum and show that it is of the form \( P_S(k) = P_{S}^{(0)}(k)(1+\frac{\mu}{k^4}) \) where \( \mu \equiv \mu V_s \simeq \sqrt{\alpha} \approx l_{\text{max}} \) (here \( V_s \approx l_{\text{max}}^3 \)) is the volume corresponding to the maximum measurable scale \( l_{\text{max}} \) and \( P_{S}^{(0)}(k) \) is the standard primordial spectrum obtained in the context of the Heisenberg uncertainty principle (HUP \( \mu = 0 \)). We show that the scalar spectral index predicted by the model, defined from \( P_S(k) = A_s k^{n_s-1} \) is running and may be written as \( n_s = 1 - \lambda - \frac{\mu}{k^2} \) with \( \lambda = 6 \epsilon - 2 \eta \) (where \( \epsilon \) and \( \eta \) are the slow-roll parameters). Using observational constraints on the scale dependence of the spectral index \( n_s \) a cosmological constraint may be imposed on \( \mu \) as \( \mu \approx (0.9 \pm 7.6) \cdot 10^{-56} \text{GeV} \). Using this result we estimate the GUP parameter \( \alpha \lesssim 10^{-56} \text{m}^{-2} \) and the maximum measurable scale \( l_{\text{max}} \gtrsim 10^{22} \text{m} \) which is two orders of magnitude larger than the current particle horizon.

I. INTRODUCTION

A central issue of fundamental research is the unification of quantum theory (QT) and general relativity (GR) in the framework of quantum gravity (QG). A critical scale in the context of this unification is the Planck scale defined as \( l_{\text{pl}} = \sqrt{\frac{\hbar c}{G}} = 10^{-35} \text{m} \) (see Ref.[2] for a review) which has been shown to be the minimum measurable scale if both QT and GR are applicable. Indeed it may be shown [3] that the high energies required to probe scales smaller than the Planck scale would lead to the formation of a black hole through the gravitational disturbances of spacetime structure which would prohibit any measurement on smaller scales. The existence of such a minimum measurable length would lead to a modification of the Heisenberg Uncertainty Principle [4, 5] to the so-called Generalized (Gravitational) Uncertainty Principle (GUP)(see Ref.[6] for a review)

\[
\Delta x \Delta p \geq \frac{\hbar}{2(1+\beta \Delta p^2)}
\]

where \( \beta \) is the GUP parameter defined as \( \beta = \beta_0/M_{\text{pl}} c^2 = \beta_0 l_{\text{pl}}^2 / \hbar^2 \), \( M_{\text{pl}} c^2 = 10^{19} \text{GeV} \), \( l_{\text{pl}} \) is the 4-dimensional fundamental Planck scale and \( \beta_0 \) is a dimensionless parameter expected to be of order unity. Such a GUP is closely related to the concept of noncommutative geometry [7] and has been extensively investigated in Refs. [8–18]. In particular interest in a minimum measurable length or equivalently in a ultraviolet cutoff has been motivated by studies of string theory [19–25], loop quantum gravity [26–32], quantum geometry [33], doubly special relativity (DSR)[34–39] and by black hole physics [10, 40–42] or even Gedanken experiments [43] and thermodynamic properties of gravity [44]. Several phenomenological implications of minimal length theories and quantum gravity phenomenology were investigated and a number of researchers have studied phenomenological aspects of GUP effects in several contexts (e.g. in Refs. [45, 46] atomic physics experiments such as Lamb’s shift and Landau levels have been considered and constraints on the minimum length scale parameter \( \beta \) have been estimated ). In Refs. [47–51] a model that is consistent with string theory, black hole physics and DSR is presented and discussed. This model of GUP predicts both a minimal observable length and a maximal momentum simultaneously [48, 52].

The existence of a minimum measurable length is closely related to the existence of the black hole horizon which tends to form if length scales below the Planck scale are probed. Correspondingly, there is a maximum measurable length associated with the cosmological particle horizon [53, 54] which provides due to causality a maximum measurable length scale in the Universe. The particle horizon corresponds to the length scale of the
boundary between the observable and the unobservable regions of the universe. This scale at any time defines the size of the observable universe. The physical distance to this maximum observable scale at the cosmic time \( t \) is given by (see e.g. [55, 56])

\[
l_{\text{max}}(t) = a(t) \int_0^t \frac{c \, dt}{a(t)}
\]

(1.2)

where \( a(t) \) is the cosmic scale factor. For the best fit ΛCDM cosmic background at the present time \( t_0 \) we have

\[
l_{\text{max}}(t_0) \simeq 14Gpc \simeq 10^{26} m
\]

(1.3)

This existence of such a maximum measurable length would lead to modified version of the GUP of the form [1]

\[
\Delta x \Delta p \geq \frac{\hbar}{2} \frac{1}{1 - \alpha \Delta x^2}
\]

(1.4)

where \( \alpha = l_{\text{max}}^{-2} \). Such a GUP originates from a commutation relation of the form

\[
[x, p] = i\hbar \frac{1}{1 - \alpha x^2}
\]

(1.5)

which may be represented by position and momentum operators of the form

\[
p = \frac{1}{1 - \alpha x_0^2} p_0 = (1 + \alpha x_0^2 + \alpha^2 x_0^4 + ...) p_0
\]

(1.6)

\[
x = x_0
\]

(1.7)

where \( x_0 \) and \( p_0 \) are the usual position and momentum operators satisfying the Heisenberg commutation relation \([x_0, p_0] = i\hbar \). The representation (1.6), (1.7) may be used to solve the Schrödinger equation for simple quantum systems to find the dependence of the energy spectrum on the maximum measurable scale \( l_{\text{max}} \). Such an analysis has indicated [1] that the current cosmic particle horizon is too large to lead to any observable effects in present day quantum systems. This however is not necessarily the case in the Early Universe when the particle horizon scale is much smaller and could leave an observable signature in the quantum generation of the primordial fluctuations during inflation. Thus, in the present analysis we wish to address the following questions:

- What is the deformation of the scale invariant spectrum of perturbations produced during inflation due to the Heisenberg algebra deformation (1.5) corresponding to the existence of a maximum measurable scale?
- What constraints can be imposed on the fundamental parameter \( \alpha = l_{\text{max}}^{-2} \) from the observed power spectrum of primordial fluctuations?

The structure of this paper is the following: In the next section II we consider a simple harmonic oscillator in the presence of a large maximum measurable scale and find the variance of the position as a function of the parameter \( \alpha \) and the corresponding variance in the context of the HUP (\( \alpha = 0 \)). In section III we generalize this analysis to the case of systems with infinite degrees of freedom (fields) and derive the spectrum and the spectral index of tensor and scalar perturbations generated during inflation as a function of the parameter \( \alpha \) and of the corresponding spectrum obtained in the context of the HUP. In section IV we use the derived theoretical expression for the (running) spectral index along with the corresponding observationally allowed range of the index as a function of the scale \( k \) to derive constraints on the fundamental parameter \( \alpha \) of the GUP. Finally in section V we conclude, summarize and discuss the implications and possible extensions of our analysis.

II. TOY MODEL: THE POSITION VARIANCE OF THE HARMONIC OSCILLATOR UNDER GUP

In order to quantize the simple harmonic oscillator under the assumption of the GUP (1.4) we need to generalize the expressions of the creation and annihilation operators \( \hat{a} \) and \( \hat{a}^\dagger \) in terms of \( x, p \) so that the commutation relation [57]

\[
[\hat{a}, \hat{a}^\dagger] = 1
\]

(2.1)

is retained while at the same time the GUP commutation relation (1.5) is also respected. Thus, in order to satisfy these conditions, we generalize the analysis of Refs. [58, 59] which applies to the GUP (1.1) and define

\[
\hat{a} = \frac{1}{\sqrt{2\hbar\omega}} (\omega [x + f(\alpha, x)] + ip)
\]

(2.2)

\[
\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar\omega}} (\omega [x + f(\alpha, x)] - ip)
\]

(2.3)

where \( f(\alpha, x) \) is a function chosen so that the commutation relations (2.1) and (1.5) are respected.

It is straightforward to show that the following function satisfies the aforementioned conditions simultaneously

\[
f(\alpha, x) = \sum_{n=1}^{\infty} \left( \frac{-\alpha}{\sqrt{2n+1}} \right)^n x^{2n+1}
\]

(2.4)

while it reduces to 0 in the limit \( \alpha \to 0 \) as it should. Thus, we can rewrite eqs. (2.2) and (2.3) as

\[
\hat{a} = \frac{1}{\sqrt{2\hbar\omega}} \left( \omega \frac{1}{\sqrt{\alpha}} \arctan(\sqrt{\alpha}x) + ip \right)
\]

(2.5)

\[
\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar\omega}} \left( \omega \frac{1}{\sqrt{\alpha}} \arctan(\sqrt{\alpha}x) - ip \right)
\]

(2.6)

and the \( p \) and \( x \) operators are

\[
p = -i \sqrt{\frac{\hbar\omega}{2}} (\hat{a} - \hat{a}^\dagger)
\]

(2.7)
where
\[ x = x_0 + \alpha x_0^3 \right. \frac{2}{3} + \left. \frac{2\alpha^2 x_0^5}{15} + ... \right. \]
\[ (\langle x|^2) \equiv \langle 0|x^\dagger x|0 \rangle \Rightarrow \langle |x|^2 \rangle = |v(\omega, t)|^2 \left[ 1 + 2\alpha |v(\omega, t)|^2 \right] \]

which reduces to the familiar result for $\alpha = 0$ (see e.g. [60, 61]).

In the next section we generalize the above analysis to the case of quantum field fluctuations involving infinite degrees of freedom aiming to derive the perturbation power spectrum generated during inflation in the context of the GUP.

III. PRIMORDIAL SPECTRA OF COSMOLOGICAL FLUCTUATIONS WITH GUP

According to the decomposition theorem [62] the perturbations of each type evolve independently (at the linear level) and we can treat tensor (T) and scalar (S) perturbations of the metric separately. Therefore for spatially flat Friedmann-Robertson-Walker (FRW) background plus the perturbations we can write
\[ ds^2 = ds_T^2 + ds_S^2 \]
with
\[ ds_T^2 = a^2 [-d\tau^2 + (\delta_{ij} + H_{ij})dx^i dx^j] \]
and in conformal Newtonian gauge [63]
\[ ds_S^2 = a^2 [-(1 + 2\Psi)d\tau^2 + \delta_{ij}(1 + 2\Phi)dx^i dx^j] \]
where $a$ is the scale factor, $\tau$ is the conformal time, $\Psi$ corresponds to the gravitational potential of the perturbations, $\Phi$ is the perturbation of the spatial curvature\(^1\) and $H_{ij}$ is the tensor perturbation which has the form \(^2\)
\[ [H_{ij}] = \begin{bmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

The classical evolution equations for the tensor mode perturbations $h_T$ (where $T = +, \times$ for two polarization states [65]) of the FRW metric during inflation in conformal time are obtained from the linearized Einstein equations and may be written as [66]
\[ h''_T + \frac{2a'}{a}h'_T + k^2 h_T = 0 \]
where primes denote derivatives with respect to $\tau$. This becomes a simple harmonic oscillator equation by defining
\[ \tilde{h}_T \equiv \frac{ah_T}{\sqrt{16\pi G}} \]
and eq. (3.5) takes the form
\[ \tilde{h}_T'' + \omega^2 \tilde{h}_T = 0 \]

\(^1\) In the absence of anisotropic stress ($\Pi = 0$) we have $\Psi = -\Phi$
\(^2\) It has this form in a coordinate system where wavevector $k$ points along the $z$-axis.
where
\[ \omega^2 = k^2 - \frac{a''}{a} \]  
(3.8)

During slow roll inflation when the Hubble rate \( H \) is nearly constant \[67\], the conformal time is \[61, 68\]
\[ \tau \simeq -\frac{1}{aH} \]  
(3.9)

Thus we obtain
\[ \omega^2 = k^2 - \frac{2}{\tau^2} \]  
(3.10)

We now quantize the tensor field fluctuations by promoting them to operators and imposing a generalized field commutation (GFC) relation \[69, 70\] corresponding to (1.5). This GFC takes the form \( \hat{\mu} \equiv \mu \).

\[ \left[ \hat{h}_T(k), \pi_{h_T}(k') \right] = i\delta(k - k') \left( \frac{1}{1 - \mu \hat{h}_T^2(k)} \right) \]  
(3.11)

where \( \pi_{h_T} \) is the conjugate momentum to \( \hat{h}_T \) which is given by
\[ \pi_{h_T} = \hat{h}_T' - \frac{a'}{a} \hat{h}_T \]  
(3.12)

Thus we have an infinite number of decoupled harmonic oscillators corresponding to eq. (3.7) which may be quantized in accordance with the GFC (3.11).

Using the results of the previous section we connect the field normal modes with the creation and annihilation operators which satisfy the commutation relation
\[ [\hat{a}_k, \hat{a}^+_k] = \delta^3(k - k') \], as
\[ \begin{align*}
\hat{h}_T(k) &= \frac{1}{\sqrt{\mu}} \tan \left( \sqrt{\frac{\mu}{2\omega}} (\hat{a}_k + \hat{a}^+_k) \right) \\
\pi_{h_T}(k) &= -i \sqrt{\frac{\omega}{2}} (\hat{a}_k - \hat{a}^+_k)
\end{align*} \]  
(3.14)

and obtain the variance of the perturbations as
\[ \langle h_T^4(k, \tau) h_T(k', \tau) \rangle = \frac{16\pi G}{a^2} |v(k, \tau)|^2 \left[ 1 + 2\hat{\mu}|v(k, \tau)|^2 \right] (2\pi)^3 \delta^3(k - k') \equiv (2\pi)^3 P_h(k) \delta^3(k - k') \]  
(3.16)

where
\[ P_h^{(o)}(k) = \frac{16\pi G}{a^2} |v(k, \tau)|^2 \]  
(3.21)

Once \( k|\tau| < 1 \), the mode leaves the horizon, after which \( h \) remains constant. Thus, using eqs. (3.19) and (3.21) we obtain
\[ P_h^{(o)}(k) = \frac{16\pi G}{a^2} \frac{1}{2k^3 \tau^2} = \frac{8\pi G H^2}{k^3} \]  
(3.22)

where the equality on the second line holds because we have assumed that \( H \) is constant and \( \tau = -\frac{1}{aH} \).

In a similar manner we may investigate scalar perturbations induced by quantum fluctuations of the inflaton scalar field \( \phi \) \[61, 78, 79\] of the form
\[ \phi(x, t) = \phi^{(o)}(t) + \delta \phi(x, t) \]  
(3.23)

where \( \phi^{(o)} \) is the zero-order part and \( \delta \phi \) is the first-order perturbation.

\[ \mu \equiv \alpha^2 = l_{max}^{-4} \]  
(3.13)

where \( \alpha \) is the parameter of the GUP (1.4).
The fluctuations $\delta \phi$ of the scalar field driving inflation evolve in conformal time $\tau$ according to the equation (see e.g. [55])

$$\delta \phi'' + 2\frac{a'}{a} \delta \phi' + k^2 \delta \phi = 0$$  \hspace{1cm} (3.24)

Using the definition

$$\varphi = a \delta \phi$$  \hspace{1cm} (3.25)

eq (3.24) becomes

$$\varphi'' + \omega^2 \varphi = 0$$  \hspace{1cm} (3.26)

with $\omega^2 = k^2 - \frac{a''}{a}$.

In the context of the maximal measurable length GUP as applied to the case of the inflaton fluctuations, the field commutation relation gets generalized as

$$[\varphi(k), \pi_{\varphi}(k')] = i\delta(k-k') \frac{1}{1-\mu k^2(k)}$$  \hspace{1cm} (3.27)

where $\pi_{\varphi}$ is the conjugate momentum to $\varphi$ which is given by

$$\pi_{\varphi} = \varphi' - \frac{a'}{a} \varphi$$  \hspace{1cm} (3.28)

Since eq. (3.24) is identical to eq. (3.5) we can use the result of eq. (3.20) without the factor $16\pi G$ in order to turn the dimensionless $h$ into a field $\delta \phi$ with dimensions of mass

$$P_{\delta \phi}(k) = P_{\delta \phi}^{(0)}(k) \left[ 1 + 2\mu a^2 P_{\delta \phi}^{(0)}(k) \right]$$  \hspace{1cm} (3.29)

where

$$P_{\delta \phi}^{(0)}(k) = \frac{H^2}{2k^3}$$  \hspace{1cm} (3.30)

In the case $\mu = 0$ eqs. (3.20) and (3.29) reduce to the familiar results of HUP [63].

The perturbation from the scalar field driving inflation $\delta \phi$ gets transferred to the gravitational potential $\Phi$. The post inflation power spectrum of $\Phi$ is related to the horizon-crossing power spectrum of $\delta \phi$ via [60]

$$P_\Phi = \frac{16\pi G}{9\epsilon} P_{\delta \phi}$$  \hspace{1cm} (3.31)

where $\epsilon$ is the Hubble slow-roll parameter, defined as

$$\epsilon \equiv \frac{d}{dt} \left( \frac{1}{H} \right)$$  \hspace{1cm} (3.32)

We note that the Hubble slow-roll parameter $\epsilon$ is equal to the first potential slow-roll parameter $\epsilon_V$, to leading order in the slow-roll approximation [61, 68, 80–82]

$$\epsilon \simeq \epsilon_V \equiv \frac{1}{16\pi G} \left( \frac{V'}{V} \right)^2$$  \hspace{1cm} (3.33)

where $V'$ is defined as the derivative of the potential $V$ with respect to the field $\phi^{(0)}$.

In the case of single-field slow-roll models of inflation for modes which are outside the horizon ($k|\tau| \ll 1$) at the end of inflation, the primordial spectra of scalar and tensor perturbations do not depend on time$^4$ and it is conventional to write [68]

$$P_S(k) \equiv k^3 P_\Phi(k) \equiv A_S k^{n_s-1}$$  \hspace{1cm} (3.34)

$$P_T(k) \equiv k^3 P_h(k) \equiv A_T k^{n_T}$$  \hspace{1cm} (3.35)

where $A_S(A_T)$ is the scalar (tensor) amplitude and $n_s(n_T)$ is the scalar (tensor) spectral index. The special case with $n_s = 1$ ($n_T = 0$) results in the scale-invariant spectrum.

From eqs. (3.20) and (3.35) we obtain

$$P_T(k) = P_T^{(0)}(k) \left[ 1 + \frac{\bar{\mu} a^3 P_{\delta \phi}^{(0)}(k)}{8\pi G k^3} \right]$$  \hspace{1cm} (3.36)

where (for $k|\tau| \ll 1$)

$$P_T^{(0)}(k) = \frac{8\pi G}{a^3 k^2} = 8\pi G H^2$$  \hspace{1cm} (3.37)

It is straightforward to show at the horizon crossing time ($k = aH$)

$$P_T(k) = P_T^{(0)}(k) \left( 1 + \frac{\bar{\mu}}{k} \right)$$  \hspace{1cm} (3.38)

In eq. (3.35) the tensor spectral index is defined as

$$n_T \equiv \frac{d\ln P_T}{d\ln k}$$  \hspace{1cm} (3.39)

Also by virtue of eq. (3.32) we have that the logarithmic derivative of Hubble rate at horizon crossing is

$$\frac{d\ln H}{d\ln k} = -\epsilon$$  \hspace{1cm} (3.40)

Therefore using eqs. (3.37), (3.38) and (3.39) we obtain that the tensor spectral index runs as

$$n_T = -2\epsilon - \frac{\bar{\mu}}{k}$$  \hspace{1cm} (3.41)

Similarly, from eq. (3.29) and using $P_S = k^3 \frac{16\pi G}{9\epsilon} P_{\delta \phi}$ we obtain at horizon crossing time ($k = aH$)

$$P_S(k) = P_S^{(0)}(k) \left[ 1 + \frac{9\bar{\mu} a^3 k^3 P_{\delta \phi}^{(0)}(k)}{8\pi G H^2 k} \right]$$  \hspace{1cm} (3.42)

where

$$P_S^{(0)}(k) = \frac{8\pi G H^2}{9\epsilon}$$  \hspace{1cm} (3.43)

$^4$ We assume that non-adiabatic pressure terms are negligible.
HUP vs GUP best fit on the observed data

\[ \mu = 0 \]

\[ \mu = 0.9 \times 10^{-6} \]

\[ -\sigma: \mu = -6.7 \times 10^{-6} \]

\[ +\sigma: \mu = 8.5 \times 10^{-6} \]

\[ ns = 1 - \lambda - \bar{\mu} \]

\[ k \left[ \frac{h}{\text{Mpc}} \right] \]

\[ \text{FIG. 1. The best fit forms of the scalar spectral index eq. (4.2) (blue dashed curve for HUP and red dashed curve for GFC eq. (3.27)) on the observed data (thick dots). The green and brown continuous curves correspond to } -\sigma \text{ and } +\sigma \text{ deviation of the parameter } \bar{\mu} \text{ respectively.} \]

It is straightforward to show that the

\[ P_S(k) = P_S^{(0)}(k) \left( 1 + \frac{\bar{\mu}}{\bar{k}} \right) \]  

(3.44)

In eq. (3.34) the scalar spectral index is defined as

\[ n_s - 1 = \frac{d \ln P_{\Phi}}{d \ln k} \]  

(3.45)

Now using the eq. (3.33) and the Hubble slow-roll parameter [82]

\[ \delta = \frac{1}{H} \frac{d^2 \phi^{(0)}}{dt^2} \]  

(3.46)

we have that the logarithmic derivative of the slow-roll parameter \( \epsilon \) is

\[ \frac{d \ln \epsilon}{d \ln k} = 2(\epsilon + \bar{\delta}) \]  

(3.47)

Therefore using eqs. (3.43), (3.44) and (3.45) we obtain that the scalar spectral index runs as

\[ n_s = 1 - 4\epsilon - 2\delta - \bar{\mu} \]  

(3.48)

Alternatively using the second potential slow-roll parameter \( \eta \equiv \frac{1}{8\pi G} \frac{V''}{V} \) and the relation \( \delta = \epsilon - \eta \) [68], we obtain

\[ n_s = 1 - 6\epsilon + 2\eta - \bar{\mu} \]  

(3.49)

In the next subsection we use observational scalar spectral index data to obtain bounds on \( \bar{\mu} \).

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\[ ^5 \] The second slow-roll parameter \( \delta \) and the second potential slow-roll parameter \( \eta \) are sometimes defined as \( \eta \) and \( \eta V \) respectively, so that the relation has the form \( \eta = \epsilon V - \eta V \).

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IV. OBSERVATIONAL CONSTRAINTS

The predicted form of the running spectral index eq. (3.49) reduces to the standard form [67, 68] for the HUP (\( \bar{\mu} = 0 \)) and may be used along with observational constraints of the spectral index to impose constraints on the GFC parameter \( \bar{\mu} \).

The parameters that can lead to deviations from scale
invariance of the spectral index are the GFC parameter \(\mu\) and the slow-roll parameter \(\lambda\) defined as

\[\lambda = 6\epsilon - 2\eta\] (4.1)

Thus using eq. (3.49), the scalar spectral index takes the form

\[n_s = 1 - \lambda - \frac{\bar{\mu}}{k}\] (4.2)

In order to impose constraints on the parameters \(\lambda, \bar{\mu}\) we use constraints on the scalar spectral index of Ref. [83] which are based on the angular power spectrum data of the 5 year Wilkinson Microwave Anisotropy Probe (WMAP5) Cosmic Microwave Background (CMB) temperature and polarization, the Large Scale Structure (LSS) data of the Sloan Digital Sky Survey (SDSS) data release 7 (DR7) Luminous Red Galaxy (LRG) power spectrum, and the Lyman-alpha forest (Lya) power spectrum constraints. The allowed range on \(n_s\) is shown in Fig. 1.

Expressing this range as a set of \(N = 60\) datapoints leads to constraints on the parameters \(\lambda, \bar{\mu}\) through the maximum likelihood method [84]. As a first step for the construction of \(\chi^2\), we consider the vector [85]

\[V^i(k_i, \lambda, \bar{\mu}) = n_{s,i}^{\text{obs}}(k_i) - n_{s,i}^{\text{th}}(k_i, \lambda, \bar{\mu})\] (4.3)

where \(n_{s,i}^{\text{obs}}(k_i)\) and \(n_{s,i}^{\text{th}}(k_i, \lambda, \bar{\mu})\) are the observational and the theoretical spectral index at wavenumber \(k_i\) respectively (\(i = 1, \ldots, N\) with \(N\) corresponds to the number of datapoints). Then we obtain \(\chi^2\) as

\[\chi^2 = V^i F_{ij} V^j\] (4.4)

where \(F_{ij}\) is the Fisher matrix [86] (the inverse of the covariance matrix \(C_{ij}\) of the data).

The \(N \times N\) covariance matrix is assumed to be of the form

\[C_{ij} = \begin{bmatrix} \sigma_1^2 & 0 & \cdots \\ 0 & \sigma_2^2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}\] (4.5)

where \(\sigma_i\) denotes the 1\(\sigma\) error of data point \(i\).

The 68.3% (1\(\sigma\)), 95.4% (2\(\sigma\)) and 99.7% (3\(\sigma\)) confidence contours in the \(\lambda\) and \(\bar{\mu}\) parametric space are shown in Fig. 2. The contours correspond to confidence regions obtained from the full data set (left panel), the large scales \((k < 0.015\ h/\text{Mpc})\) data (middle panel), and the small scales \((k > 0.015\ h/\text{Mpc})\) data (right panel). The 1\(\sigma\)-3\(\sigma\) contours for parameters \(\lambda\) and \(\bar{\mu}\) correspond to the curves \(\chi^2(\lambda, \bar{\mu}) = \chi^2_{\text{min}} + 2.3\), \(\chi^2(\lambda, \bar{\mu}) = \chi^2_{\text{min}} + 6.17\) and \(\chi^2(\lambda, \bar{\mu}) = \chi^2_{\text{min}} + 9.21\) respectively. Notice that the large scales are most efficient in constraining the GFC parameter \(\bar{\mu}\).

In Table I we show the best fit values of parameters \(\lambda\) and \(\bar{\mu}\) with the corresponding 1\(\sigma\) standard deviations. In the case of HUP \((\bar{\mu} = 0)\) the result agrees with the current best fit values of the scalar spectral index from Planck which indicate that \(\lambda \approx 0.04\) [87].
Using eq. (3.13) and the value of GFC parameter $\bar{\mu} = 0.9 \times 10^{-6} h/\text{Mpc}$ we can obtain the single GUP free parameter as

$$\alpha = \bar{\mu}^2 \lesssim 10^{-56} m^{-2} \quad (4.6)$$

and the corresponding maximum measurable scale as

$$l_{\text{max}} = \bar{\mu}^{-1} \gtrsim 10^{28} m \quad (4.7)$$

This result is two orders of magnitude larger than the present day particle horizon ($l_{\text{max}}(l_0) \simeq 10^{26} m$) given in eq. (1.3).

V. CONCLUSIONS-DISCUSSION

We have derived the generalized form of the primordial power spectrum of cosmological perturbations generated during inflation due to the quantum fluctuations of scalar and tensor degrees of freedom in the context of a generalization of quantum mechanics involving a maximum measurable length scale. The existence of such a scale is motivated by the existence of the particle horizon in cosmology and would lead to a generalization of the uncertainty principle (GUP) to the form $\Delta x \Delta p \gtrsim \frac{\hbar}{2(1-\alpha \Delta x)^2}$, which implies the existence of a maximum position and a minimum momentum uncertainty (infrared cutoff) [1]. The GUP implies a generalization of the commutation relation between conjugate operators including fields and their conjugate momenta. For example we showed that the generalized field commutation (GFC) relation between a scalar field and its conjugate momentum $[\varphi(k), \pi_\varphi(k')] = i\delta(k - k') \frac{1}{1 - \mu \varphi^* (k)}$ which is implied by the GUP leads to a modified primordial spectrum of scalar perturbation $P_S(k) = P_{S}^{(0)}(k) \left(1 + \frac{\bar{\mu}}{k}\right)$ with a running spectral index of the form $n_s = 1 - \lambda - \frac{4}{\pi}$ with $\lambda = 6 \epsilon - 2 \eta$.

Using cosmological constraints of the scalar perturbations spectral index as a function of the scale $k$ [85] we imposed constraints on the parameter of the GFC $\bar{\mu} \simeq l^{-1}_{\text{max}}$. We found that $\bar{\mu} = (0.9 \pm 7.6) \times 10^{-6} h/\text{Mpc}$ which corresponds to an upper bound scale much larger than the present horizon scale. Thus, the derived constraints are consistent with the current maximum measurable scale which is the current cosmological particle horizon and are much more powerful than the corresponding constraints obtained using laboratory data measuring the energy spectrum of simple quantum systems obtained in Ref. [1].

An interesting extension of our analysis would be the consideration of other types of GUP (e.g. the UV cut-off GUP of eq. (1.1)) and the derivation of constraints on the corresponding fundamental parameters using cosmological data and constraints on the power spectrum index.

An alternative approach in deriving the effects of a GUP on the primordial perturbation spectrum involves the generalization of the position and momentum operators as described in the Introduction, but with an ultraviolet rather than infrared cutoff, while keeping the field theoretical commutation relations unchanged [88, 89]. According to [88, 89], this approach would also lead to a modification of the evolution of the field perturbation modes eq. (3.24) even though this equation is derived before quantization at the classical level. This approach is questionable as it is implemented at the classical level. Nevertheless, it would be of interest to extend our analysis to include such effects of modification of the classical evolution of field perturbations due to a generalization of position and momentum operators.

Supplemental Material: The Mathematica file used for the numerical analysis and for construction of the figures can be found in [90].

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