A Generic Renormalization Method in Curved Spaces and at Finite Temperature

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Based only on simple principles of renormalization in coordinate space, we derive closed renormalized amplitudes and renormalization group constants at 1- and 2-loop orders for scalar field theories in general backgrounds. This is achieved through a generic renormalization procedure we develop exploiting the central idea behind differential renormalization, which needs as only inputs the propagator and the appropriate laplacian for the backgrounds in question. We work out this generic coordinate space renormalization in some detail, and subsequently back it up with specific calculations for scalar theories both on curved backgrounds, manifestly preserving diffeomorphism invariance, and at finite temperature.

UB-ECM-PF 94/10

I. GENERIC RENORMALIZATION OF SHORT-DISTANCE SINGULARITIES

Quantum field theories are often considered under classical external backgrounds. Two standard examples are thermal baths \footnote{But there will be, of course, leading singularities associated to the renormalization of new couplings like \(\xi R\phi^2\), etc.} and curved space-times \footnote{Research supported in part by CICYT grant #AEN93-0695, and NATO grant #910890. E-mail: comellas@ebubecm1, hagensen@ebubecm1, latorre@ebubecm1.} and, in general, the procedure of renormalization becomes much more involved due to the appearance of new dimensionful scales in the problem.

It is nevertheless known that some of the first coefficients of the renormalization group constants remain unaltered. For instance, finite temperature can be encoded by compactifying the Euclidean time, and this obviously affects the long-distance properties of the theory but not the leading short-distance ones. It is natural to expect no modification of the first coefficient of the \(\beta\)-function. Similarly, gravitational backgrounds do not affect the leading renormalization of flat field theories\footnote{But there will be, of course, leading singularities associated to the renormalization of new couplings like \(\xi R\phi^2\), etc.} due to the Equivalence Principle, which allows for a locally flat space-time and thus guarantees that short-distance singularities are those of Minkowski space-time. On the other hand, the complicated structure of thermal and gravitational backgrounds makes it often impossible in practice to get closed expressions for even the simplest Green functions. Infrared modifications of a theory do not interfere with the renormalization of leading singularities but make almost impractical any computation at and beyond one loop.

It is our aim to present a renormalization procedure adapted to quantum field theory defined at finite temperature or in curved space-times which produces closed explicit expressions for Green functions at low orders of perturbation theory. The method is based on the ideas behind differential renormalization (DR hereafter) \footnote{Research supported in part by CICYT grant #AEN93-0695, and NATO grant #910890. E-mail: comellas@ebubecm1, hagensen@ebubecm1, latorre@ebubecm1.} and we shall now expose it in a general setting. By way of example, we shall also check that the first two coefficients of the \(\beta\)-function and the first one of the anomalous dimension of the scalar field in \(\lambda\phi^4\) theory are independent of long-distance aspects of the theory.

Let us consider a massless Euclidean scalar field theory with a propagator given by

\[
\langle \phi(x)\phi(0) \rangle \equiv G(x, 0; \alpha)
\]  

where \(\alpha\) stands for dimensionful parameters such as temperature or curvature. The propagator obeys an equation of the type

\[
\mathcal{D} G(x, 0; \alpha) = \frac{\delta^{(4)}(x)}{\sqrt{g(x)}}.
\]
where \( \mathcal{D} \) is the proper scalar laplacian of the theory, for instance on curved space-time (with background metric \( g_{\mu\nu}(x) \)) or with nontrivial boundary conditions when necessary. For a preview of a particular form of this operator, one may consider for example

\[
\mathcal{D} = -g^{\mu\nu}(x)\nabla_\mu \partial_\nu - \alpha^2, \tag{1.3}
\]

where \( \nabla_\mu \) stands for a covariant derivative and \( \alpha^2 \) is related to the curvature of the background metric (see Sec. 2). For the sake of simplicity, we shall first consider massless \( \lambda \phi^4 \) theory. The massive case is discussed at the end of this section. We shall assume that our field theory is locally flat, that is,

\[
G(x, 0; \alpha) \xrightarrow{x \to 0} \frac{1}{4\pi^2 \, x^2}. \tag{1.4}
\]

This is the case of field theories both in curved spaces or in thermal baths. The long-distance behavior of the propagator will surely be complicated. This is at the heart of the problem of defining a Fourier transform \cite{6, 7} and computing it.

\[\text{Figure 1.}\]

As we turn to the simplest quantum correction of the amputated four-point vertex function, Fig. 1, we find the contribution

\[
\frac{\lambda^2}{2} \left( G(x, 0; \alpha) \right)^2, \tag{1.5}
\]

which displays a leading singularity at short distances of the type

\[
\left( G(x, 0; \alpha) \right)^2 \xrightarrow{x \to 0} \frac{1}{16\pi^4} \frac{1}{(x^2)^2} + \ldots. \tag{1.6}
\]

Due to this (logarithmic) singularity at \( x = 0 \), Eq. (1.5) does not accept a Fourier transform, it is not a good distribution upon integration against plane waves and needs to be renormalized. Typically, the short-distance expansion in Eq. (1.6) will contain subleading terms which diverge as the propagator itself and need not undergo renormalization. In the absence of nontrivial backgrounds, the method of differential renormalization gives a recipe to obtain right away the renormalized version of Eq. (1.4). The idea is to rewrite this equation by extracting a laplacian operator,

\[
\frac{1}{(x^2)^2} = \frac{1}{4} \, \square \ln \frac{x^2}{M^2}, \tag{1.7}
\]

where \( \square \equiv \partial_\mu \partial_\mu \). This expression is an identity at \( x \neq 0 \) and, furthermore, produces an extension of the too singular function of the l.h.s. into a proper distribution on the r.h.s., provided the operator \( \square \) is understood as acting onto the left. We should now generalize this simple identity to more complicated cases, when the infrared behavior of the propagator becomes very involved. We achieve this goal by noting that, in general, the logarithmic singularity in Eq. (1.7) is solved by writing it as

\[
G^2(x) = -\frac{1}{16\pi^2} \mathcal{D} \left( G(x) \ln G(x)/M^2 \right) + D(x). \tag{1.8}
\]

\( D(x) \equiv D(x, 0; \alpha) \) is then a bona fide distribution which will become explicit in each particular case, and \( G(x) \) is short-hand for \( G(x, 0; \alpha) \). Again, the function on the l.h.s. is turned into a distribution while nothing is changed away from the singularity. It is easy to see, by using the leading behavior of the propagator, that indeed the above formula encodes the renormalization of the singularity as shown in Eq. (1.3). The form of \( D(x) \) will depend on the particular problem we are addressing, and may look involved since it is encoding long-distance properties of the theory, but it is guaranteed to be at most as singular as the propagator itself.
There are basically three reasons why Eq. (1.8) produces the correct renormalization of the 1-loop diagram in thermal or gravitational deformations of massless $\lambda\phi^4$ theory:

1) Removal of singularities. As we just explained, our generalized renormalization of the bubble diagram is guaranteed by the short-distance limit of all the pieces of the r.h.s. of Eq. (1.8).

2) Renormalization group behavior. The renormalization scale only appears within the logarithm. The renormalization group equation at one loop

\[ \left( M \frac{\partial}{\partial M} + \beta(\lambda) \frac{\partial}{\partial \lambda} - 4\gamma(\lambda) \right) \Gamma^{(4)} = 0, \] (1.9)

where $\gamma(\lambda)$ stands for the anomalous dimension of the $\phi$ field, $\beta(\lambda)$ is the $\beta$-function and $\Gamma^{(4)}$ is the amputated four-point function, will thus contain the piece

\[ M \frac{\partial}{\partial M} \left( - \frac{1}{16\pi^2} D \left( G(x) \ln G(x)/M^2 \right) + D(x) \right) = \frac{1}{8\pi^2} DG(x) = \frac{1}{8\pi^2} \frac{\delta^{(4)}(x)}{\sqrt{g(x)}}. \] (1.10)

The full diagram carries an extra $\lambda^2/2$ factor plus the addition of the $s, t$ and $u$ channels. Therefore, the $M$ dependence can be reabsorbed by a change of the tree-level coupling dictated by the beta function

\[ \beta(\lambda) = M \frac{\partial \lambda}{\partial M} = \frac{3\lambda^2}{16\pi^2}, \] (1.11)

and the anomalous dimension receives no contribution from this diagram.

3) Unitarity. The imaginary part of the amplitude is related to the cross-section through the standard unitarity relations. In Eq. (1.8), the imaginary part is carried by the logarithm, which in turn gives rise to a delta function. As expected, the imaginary part of the 1-loop contribution is then proportional to the tree level structure.

It is also noteworthy that in the presence of a gravitational background, our procedure preserves diffeomorphism invariance since only covariant derivatives are manipulated. There is just the desired scale invariance breakdown. On the other hand, the above recipe has to be slightly modified in the case of massive theories as we discuss at the end of this section.

\[ \text{Figure 2.} \]

\[ \text{Figure 3.} \]

At two loops, the amputated 4-point function is given by the two diagrams in Figs. 2,3. The first diagram corresponds to a convolution and, thus, introduces no new kind of singularity. It just produces a promotion of leading logs to second order. To see this, note that the bare amplitude

\[ \text{\footnotesize Note: New $\beta$-functions associated to mass and background couplings will eventually appear, but at higher loop order.} \]

3
\[ \int d^4w \sqrt{g(w)} (G(x, w; \alpha))^2 (G(w, 0; \alpha))^2 \]  \tag{1.12}

simply gets renormalized using Eq. (1.8). The fact that this diagram brings no contribution to the 2-loop $\beta$-function stems from the fact that its renormalization scale dependence is non-local, which is readily checked

\[
M \frac{\partial}{\partial M} \left[ \int d^4w \sqrt{g(w)} \left( \frac{1}{16\pi^2} D(G(x, w; \alpha) \ln G(x, w; \alpha)/M^2) + D(x, w; \alpha) \right) \times \left( -\frac{1}{16\pi^2} D(G(w, 0; \alpha) \ln G(w, 0; \alpha)/M^2) + D(w, 0; \alpha) \right) \right] = \frac{1}{4\pi^2} \left( -\frac{1}{16\pi^2} D(G(x, 0; \alpha) \ln G(x, 0; \alpha)/M^2) + D(x, 0; \alpha) \right). \tag{1.13}
\]

On the other hand, the diagram in Fig. 3 displays a new kind of singularity, a generic 3-point one. The bare form of the amplitude reads

\[ G(x, 0; \alpha) G(y, 0; \alpha) (G(x, y; \alpha))^2. \tag{1.14} \]

Let us recall \[3\] that ordinary flat space renormalization goes in two steps, first renormalizing the inner 2-point singularity

\[
\frac{1}{x^2 y^2} = \frac{1}{x^2} \frac{1}{y^2} \ln(x - y)^2 M^2 \quad (x - y)^2. \tag{1.15}
\]

Then, pulling derivatives out with exact manipulations, one finds

\[
-\frac{1}{4} \frac{\partial}{\partial y_{\mu}} \left( \frac{1}{x^2 y^2} \frac{\partial}{\partial y_{\mu}} \ln(x - y)^2 M^2 \right) + x^2 \delta^{(4)}(y) \ln x^2 M^2 x^4. \tag{1.16}
\]

This final piece needs the following supplementary, genuine 2-loop renormalization

\[
\frac{\ln x^2 M^2}{x^4} = -\frac{1}{8} \ln^2 x^2 M^2 + 2 \ln x^2 \mu^2. \tag{1.17}
\]

The coefficient in front of this last expression is responsible for yielding the correct 2-loop $\beta$-function,

\[
\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} - \frac{17}{3} \frac{\lambda^3}{256\pi^4}, \tag{1.18}
\]

once the first term of the anomalous dimension of $\phi$ is introduced in the renormalization group equation. Coming back to our general discussion, it is easy to guess the correct generic renormalization of massless deformations of the flat theory. The first step is similar to the one in Eq. (1.17),

\[ G(x, 0; \alpha) G(y, 0; \alpha) \left( -\frac{1}{16\pi^2} D \left( G(x, y; \alpha) \ln G(x, y; \alpha)/M^2 \right) + D(x, y; \alpha) \right). \tag{1.19} \]

Through exact manipulations of covariant derivatives (recall that typically, $D = -g^{\mu\nu} \nabla_{\mu} \partial_{\nu} - \alpha^2$, where $\alpha^2$ is a constant), the expression will be transformed into a total derivative plus a generic 2-loop singularity which will be renormalized by

\[
G^2(x) \ln G(x)/M^2 = -\frac{1}{32\pi^2} D \left[ G(x) \left( \ln^2 \frac{G(x)}{M^2} - 2 \ln \frac{G(x)}{M^2} \right) \right] + D'(x; M). \tag{1.20}
\]

$D'(x; M)$ is again some left-over distribution. By taking the logarithmic $M$ derivative of the above, we find

\[
M \frac{\partial}{\partial M} D'(x; M) = -2D(x), \tag{1.21}
\]

where $D(x)$ is the distribution appearing in Eq. (1.8). This integrates to
\[ D'(x; M) = D(x) \ln \frac{G(x)}{M^2} + f(x), \quad (1.22) \]

where \( f(x) \) is some distribution independent of \( M \). It turns out that this dependence is precisely what is needed for consistency of the RG equations for \( \Gamma^{(4)} \) to order \( \lambda^3 \). This is another instance of the generality of our procedure. Both \( D(x) \) and \( f(x) \) give rise to benign singularities of the type

\[ \frac{1}{x^2} \frac{1}{y^2} \left( \frac{1}{x-y} \right)^2, \quad (1.23) \]

which are integrable. Another fine point in Eq. (1.20) is that the relative coefficient \(-2\) between the two powers of \( \log \) is fixed by the identity itself and eventually produces the same 2-loop \( \beta \)-function as in the flat case. The fact that we have renormalized logarithmic singularities has left no room for subleading divergences which might depend on \( \alpha \), leaving the result for the \( \beta \)-function unaltered. It is, indeed, remarkable how universal these manipulations are. They substantiate the common lore about independence of some short-distance singularities on long-distance physics, while keeping closed explicit expressions for the whole amplitudes.

\[ \text{Figure 4.} \]

Let us add a final and nontrivial example of this generic renormalization. The first non-local wave-function renormalization can again be treated in full generality. The relevant diagram is shown in Fig. 4 and we shall call it the “setting sun”. Its generic renormalization is

\[ G^3(x) = \frac{1}{512\pi^4} D(D+b) G(x) \ln \frac{G(x)}{M^2} + D''(x). \quad (1.24) \]

where \( b \) is a constant that depends on the details of the problem. It accompanies the term with a single \( D \) operator, which is needed to cure subleading divergences which, for the first time, may introduce temperature and curvature dependences. As in all previous cases, \( D''(x) \) is a left-over distribution which carries information on the long-distance properties of the theory but does not affect short-distance renormalizations. A small effort applying the renormalization group equation yields the result for the leading anomalous dimension of the field \( \phi \)

\[ \gamma(\lambda) = \frac{1}{12} \left( \frac{\lambda}{16\pi^2} \right)^2. \quad (1.25) \]

This generic renormalization again fulfills all three properties listed above: removal of singularities, correct renormalization group behavior and unitarity.

Let us briefly discuss our renormalization procedure in the presence of masses. First of all, the equation for the free propagator changes from Eq. (1.2) to

\[ (D + m^2)G(x,0; \alpha; m) = \frac{\delta^{(4)}(x)}{\sqrt{g(x)}}. \quad (1.26) \]

From this equation and the general form of the laplacian as previewed in Eq. (1.3), it becomes clear that the mass and the \( \alpha \) parameter play similar roles. Their presence does not interfere with the leading singularities of the theory. They first enter actively the game of renormalization to control the subleading renormalization of the 2-point function, which we called \( b \) in Eq. (1.24). Nonetheless, a blind application of massless recipes leads to unnecessarily complicated renormalized expressions due to the following observation. At long distances, the correct behavior of, say, the bubble diagram will be an exponential fall off as \( \exp(-2mr) \) since two particles circulate through the loop diagram. This is not the behavior of \( D(G \log G/M^2) \), which decays as \( \exp(-mr) \). Were we to use Eq. (1.8), the left-over distribution \( D \) would have to correct for this long-distance behavior. Although renormalization would be
carried out correctly, the answer achieved would not be minimally representing the long-distance fall-off. The solution to the problem is to substitute \[ \log G/M^2 \] by a function which shares the same short-distance behaviour but provides a better representation of long-distance physics. A first step in this direction is to notice through examples that the more appropriate differential operator to be used to renormalize the bubble diagram turns out to be \( \mathcal{D} + 4m^2 \). A production threshold naturally finds its place next to the laplacian (and eventually sneaks-in in momentum space as \( p^2 + 4m^2 \)), although it multiplies a distribution and thus there was no absolute need for it. The change in the distribution following the operator is of a more refined nature, and we postpone its discussion to the next section after an explicit example. Let us mention here that unitarity by itself demands further changes besides the one in the differential operator.

We have just sketched a program to get explicit Green functions at low orders of perturbation theory for \( \lambda \phi^4 \) on modified spaces which leave the leading behavior of the propagator unchanged. The universality of the first two coefficients of the \( \beta \)-function and the first one in the anomalous dimension of the field \( \phi \) also follows from the above. We shall now consider the explicit realization of these ideas. In Sec. 2, we treat constant curvature backgrounds, where we can in fact develop DR in an exact way and maintain manifest coordinate invariance, and in Sec. 3 we consider finite temperatures.

II. RENORMALIZATION IN MAXIMALLY SYMMETRIC SPACES

The renormalization effort in curved spaces, to date, has centered mainly on the techniques of point-splitting \([4]\), background field and heat kernel methods \([5,6]\). As opposed to flat space, in curved manifolds coordinate- rather than momentum-space is the natural setting for calculations, due to the lack of a generically-defined Fourier transform \([6,7]\). Because of that, it is reasonable to expect that differential renormalization will also be a natural method, beyond the abovementioned ones, since it is also based on coordinate space. The hope is that DR will present itself as an efficient renormalization procedure in curved space, and as we will see, this hope is brought to fruition. We also find that DR manifestly preserves the diffeomorphism invariance of amplitudes, given that all the typical manipulations with differential operators and their integration by parts are performed in this case in a covariant way. In this section, we will concentrate on the simplest cases, namely, maximally symmetric spaces, and we will give here the renormalized expressions for a number of loop diagrams. In these spaces, the \( \xi R \phi^2 \) coupling can be reabsorbed into the mass term and, thus, there is no need for a separate renormalization of \( \xi \).

a) The hyperboloid \( H_n \)

\( H_n \) is the maximally symmetric Euclidean space of constant negative curvature in \( n \) dimensions. The simplest construction of maximally symmetric spaces is made through the embedding into an \( (n + 1) \)-dimensional flat space, where the isometries of these spaces are very clearly put in evidence. Specifically for \( H_n \), this embedding is given by:

\[
\eta_{AB} y^A y^B = -(y^0)^2 + y^\mu y^\mu = -\frac{1}{a^2},
\]

(2.1)

where \( y^A, A = 0, \ldots, n \) are coordinates in \( R^{n+1} \) with metric \( \eta_{AB} = (\ + \ + \ +) \), \( \mu = 1, \ldots, n \), and \( R = -n(n-1)a^2 \) is the (constant) Ricci scalar of \( H_n \). With this embedding, the metric in these spaces is given by

\[
ds^2 = (\delta_{\mu\nu} - \frac{a^2 y^\mu y^\nu}{1 + a^2 \rho^2})dy^\mu dy^\nu,
\]

(2.2)

with \( y_\mu \equiv \delta_{\mu\nu} y^\nu \), and \( \rho^2 \equiv y^\mu y^\mu \). The action for a conformally coupled free scalar field is

\[
S = \frac{1}{2} \int d^n y \sqrt{g(y)} \left( g^{\mu\nu}(y) \partial_{\mu}\phi(y) \partial_{\nu}\phi(y) + m^2 \phi(y)^2 + \frac{(n-2)}{4(n-1)} R \phi(y)^2 \right).
\]

(2.3)

Denoting by \( \Box = g^{\mu\nu} \partial_{\mu} \partial_{\nu} \equiv -a^2 \Delta \) the scalar Laplace-Beltrami operator, the equation satisfied by the scalar propagator in this theory is:

\[
\left[ -\Box + \left( m^2 - \frac{n(n-2)a^2}{4} \right) \right] G_m(y) = \frac{\delta^{(n)}(y)}{\sqrt{g(y)}}.
\]

(2.4)
In maximally symmetric spaces it is also possible to arrive at these propagators \emph{via} a purely group-theoretical description based on highest weight representations of the isometry groups in these spaces [7]. For the hyperboloid $H_n$, this group is $SO(1,n)$, and if $L_{AB}$, $A, B = 0, 1, \ldots, n$, are the corresponding generators, then highest weight representations are labeled by the eigenvalue $\eta$ of $L_{00}$ (and possibly other eigenvalues related to the spin of the representation). Unitarity bounds $\eta$ to be 0 or $\geq (n-3)/2$ in a continuous range. Invariance under boosts fixes uniquely the propagator, without reference to the free action for the corresponding field. However, we do not need the details of such a construction; rather, for our purposes here we just use the fact that $\eta$ labels the scalar propagator just as the mass does, and we shall use this notation here for reasons which will become evident when we consider the renormalization of different diagrams. For now, the only difference that entails is that the equation satisfied by the scalar propagator, now labeled $G_n$, reads

$$[\Delta + \eta(\eta - n + 1)]G_\eta(y) = \frac{\delta^{(n)}(y)}{a^2 \sqrt{g(y)}}, \quad (2.5)$$

where $a^2\eta(\eta - n + 1) = m^2 - n(n-2)a^2/4$ is the quadratic Casimir eigenvalue for the representation $\eta$. In order to find the propagator explicitly, we first write down the laplacian:

$$-a^2\Delta = (1 + a^2\rho^2) \frac{\partial^2}{\partial \rho^2} + \left( \frac{n-1}{\rho} + na^2\rho \right) \frac{\partial}{\partial \rho}. \quad (2.6)$$

It is possible to put this in a more workable form by defining the reduced variable $z \equiv \sqrt{1 + a^2\rho^2}$, with range $1 \leq z < \infty$. With this definition, Eq. (2.5) becomes

$$\left[ -(z^2 - 1) \frac{\partial^2}{\partial z^2} - n z \frac{\partial}{\partial z} + \eta(\eta - n + 1) \right] G_\eta(z) = \frac{\delta^{(n)}(y)}{a^2}, \quad (2.7)$$

where we write $\delta^{(n)}(y)$ in the original variables $y$ for compactness, and where we have used $\sqrt{g(y)} |_{y=0} = 1$. It is then straightforward to find that

$$G_\eta(z) = (z^2 - 1)^{-\eta/2} h_\eta(z), \quad (2.8)$$

where $h_\eta$ satisfies an associated Legendre equation, with solutions given by $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$, the associated Legendre functions, with $\nu = \eta - n/2$ and $\mu = (n-2)/2$ (for this and all further references to these functions, cf. [8]). By examining the short distance limit $z \to 1$ of these functions it is easy to discard $P_\nu^\mu(z)$ as a solution since that leads to a propagator which is regular at the coincident point, and we finally find that the appropriate propagator is given by

$$G_\eta(z) = \frac{1}{2\pi^{n/2}} \left( \frac{a^2}{2e^{i\pi \sqrt{z^2 - 1}}} \right)^{(n-2)/2} Q_{\eta-n/2}^{(n-2)/2}(z). \quad (2.9)$$

From now on we consider $n = 4$ for simplicity, although it is clear that analogous procedures will be possible in any number of dimensions. The normalized propagator becomes

$$G_\eta(z) = \frac{a^2}{4\pi^2} \frac{1}{\sqrt{z^2 - 1}} Q_1^{\eta-2}(z), \quad (2.10)$$

and it satisfies the equation

$$[\Delta + \eta(\eta - 3)]G_\eta(z) = \frac{\delta^{(n)}(y)}{a^2}. \quad (2.11)$$

In particular, the massless propagator obtains for $\eta = 2$ (and for $\eta = 1$, but this cannot be reached continuously from a massive theory, and we discard it here), and the equation it satisfies is:

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3To be sure, this argument cannot discard a combination of both $Q_\nu^\mu$ and $P_\nu^\mu$, but the highest-weight construction indeed does.
\[ [\Delta - 2] G_{\eta=2}(z) = [\Delta - 2] \frac{a^2}{4\pi^2} \frac{a^2}{z^2 - 1} = \frac{\delta^{(4)}(y)}{a^2}. \]  
(2.12)

We now proceed to find the DR identity appropriate to the diagram of Fig. 1. As in the case of massive fields in flat space \[\Box\], we first study these identities away from contact, and then include the appropriate contact terms which lead, in this case, to a well-defined flat limit. The simplest case, the massless bubble diagram of Fig. 1, exemplifies well the procedure: we attempt to write that amplitude as (covariant) differential operators acting on less singular quantities, and find the identity

\[ 16\pi^4 G_{\eta=2}(z) = \left( \frac{a^2}{\sqrt{z^2 - 1}} Q_0^4(z) \right)^2 = \frac{a^2}{2} [\Delta - 2] \left[ \frac{a^2 z Q_0^4(z) Q_0^0(z)}{\sqrt{z^2 - 1}} \right], \quad z \neq 1. \]  
(2.13)

The differential operator on the r.h.s. is manifestly covariant, which implies we have preserved diffeomorphism invariance by this procedure. Also, for generic \( \eta \), \( Q_{n-2}^4(z) \) is expressed in terms of hypergeometric functions which we cannot write in terms of simple functions. Thus, although we could have derived Eq. (2.13) more easily with the known explicit forms for \( Q_0^0(z) \) and \( Q_0^4(z) \), in general we cannot do that, and have to rely solely on recursion relations among these functions in order to find the appropriate DR identities.

The contact terms for this DR identity will bring in a renormalization mass scale, and will determine the correct renormalization prescription for this diagram at all \( z \geq 1 \). They are gotten by examining the flat limit above \( z = \cosh ar, \ a \rho = \sinh ar \), where \( a \) is the geodesic distance to the origin in \( H_\eta \):

\[ 16\pi^4 G_{\eta=2}(z) = \frac{1}{\rho^4} \frac{a^2 \to 0}{1 \to r^4} \frac{1}{r^4}, \]

\[ \frac{a^2 Q_0^4(z)}{\sqrt{z^2 - 1}} = - \frac{1}{\rho^2} \frac{a^2 \to 0}{1 \to r^2}, \]

\[ z Q_0^0(z) = \frac{z}{2} \ln \frac{z + 1}{z - 1} - \frac{1}{2} \ln a^2 r^2, \]

\[ \Delta = \partial^2_\rho + \frac{1}{\rho} \partial \rho + a^2 (\rho^2 \partial^2_\rho + 4 \rho \partial \rho) \to \partial_{\text{flat}}^2. \]  
(2.14)

leading to:

\[ \frac{1}{r^4} = - \frac{1}{4} \frac{\ln a^2 r^2}{r^2} \quad \text{as} \quad a^2 \to 0. \]  
(2.15)

The r.h.s. of this is singular as \( a^2 \to 0 \), and we cure that by introducing a finite local quantity to the DR identity Eq. (2.13):

\[ 16\pi^4 G_{\eta=2}(z) = \frac{a^2}{2} (\Delta - 2) \left[ \frac{a^2 z Q_0^4(z) Q_0^0(z)}{\sqrt{z^2 - 1}} \right] + \pi^2 \ln \frac{M^2}{a^2} \delta^{(4)}(y). \]  
(2.16)

This then accomplishes the renormalization of the massless bubble diagram, with the usual DR prescription that the identity is now to be understood for all \( z \geq 1 \), and \( \Delta \) as acting through integration by parts. The full, renormalized 1-loop 4-point amplitude in \( \lambda \phi^4 \) theory is obtained by adding the tree-level amplitude to \( \lambda \phi^4 \) times the above one, for each of \( s, t, u \) channels. When this is required to satisfy the appropriate renormalization group equation, the correct value of the \( \beta \)-function, Eq. (1.11), is obtained. Again, the simplicity of \( H_n \) allowed us to circumvent an independent \( \xi \)-renormalization.

At this point, this renormalized amplitude should be compared with the general procedure outlined in the previous section. There we exploited the fact that, generically, the short-distance behavior is the same as in flat space, and thus ultraviolet renormalization is achieved by “imitating” as much as possible, given only \( G \) and \( D \) generically, the renormalization of the flat bubble diagram, Eq. (1.7). Thus, \( 1/x^2 \) is substituted by \( G \), \( \ln x^2 M^2 \) by \( -\ln G/M^2 \), and the result is Eq. (1.8). That does accomplish the renormalization of \( G^2 \) generically, but at the cost of introducing an extra distribution \( D \) which corrects for the long-distance discrepancies this “imitation” process entails. Indeed, we have independently checked that the same generic procedure works here, and does lead to the correct \( \beta \)-function. However, Eq. (2.16) shows clearly that in this specific case we can do better than that, i.e., we have a specific, explicit expression valid at all distances without complicated distributions left over. This will not happen in the finite temperature case, as we shall see in Sec. 3, but does also happen for massive fields in flat space \[\Box\]. The reason for this is that, loosely speaking, both in \( H_4 \) and for massive fields, just like \( Q_1^4 \) and \( K_1 \) (the modified Bessel function) are the respective equivalents of \( 1/x \) in flat space, so are \( Q_0^0 \) and \( K_0 \) the respective equivalents of the flat space quantity \( \ln x^2 M^2 \), and
thus in these settings renormalization logarithms enter naturally while still allowing us to exploit recursion relations among these special functions. On the other hand, in these cases the renormalization scale will always enter separately, determined through a limiting procedure as above. As a final comment, we note also that while $M^2$ enters Eq. 2.16 and the generic Eq. (1.8) in different ways, it does so in such a way that the logarithmic mass derivative of the renormalized amplitude gives a purely local term, which is uniquely responsible for the value of the 1-loop $\beta$-function.

We can now generalize the above to all $\ell$-loop 2-point functions of the form $(G_{\eta=2}(z))^{\ell+1}$. It is not difficult to verify the following identities for the corresponding 2-, 3-, and 4-loop diagrams, for $z \neq 1$:

$$
\left( \frac{a^2}{\sqrt{z^2 - 1}} Q_0^1(z) \right)^3 = -\frac{a^4}{16} (\Delta + 4)(\Delta - 2) \left[ a^2 z Q_0^1(z)^2 Q_0^0(z) \right]
$$

$$
\left( \frac{a^2}{\sqrt{z^2 - 1}} Q_0^1(z) \right)^4 = \frac{a^6}{384} (\Delta + 18)(\Delta + 4)(\Delta - 2) \left[ a^2 z\sqrt{z^2 - 1} Q_0^1(z)^3 Q_0^0(z) \right]
$$

$$
\left( \frac{a^2}{\sqrt{z^2 - 1}} Q_0^1(z) \right)^5 = -\frac{a^8}{18432} (\Delta + 40)(\Delta + 18)(\Delta + 4)(\Delta - 2) \left[ a^2 z(z^2 - 1)(Q_0^1(z))^4 Q_0^0(z) \right].
$$

(2.17)

The first identity above, when supplemented by the appropriate contact term, represents the renormalized setting sun amplitude in $\lambda \phi^4$ theory. Also in $\lambda \phi^4$ theory, the second identity is a vacuum energy diagram, while the third one, being made of 5-point vertices, would not occur. For the setting sun diagram, the same discussion as above for the bubble diagram and its generic renormalization applies here, with the generic renormalization given in Eq. (1.24), and the ensuing anomalous dimension for $\phi$ is given by Eq. (1.22).

A pattern is now seen to emerge here which is valid for all $\ell$; for the $\ell$-loop diagram, the differential operators which appear in the renormalization are given by:

$$
\prod_{k=1}^\ell \left[ \Delta + k\eta(k\eta - 3) \right],
$$

(2.18)

with $\eta = 2$, and the renormalizing quantity on the r.h.s. at each loop is written as

$$
a^2 z(z^2 - 1) \frac{\partial}{\partial z} (Q_0^1(z))^4 Q_0^0(z),
$$

(2.19)

which in fact is always the same as the 1-loop one, from $Q_0^1(z) = -1/\sqrt{z^2 - 1}$, but is written thus in order to preserve the same power of Legendre functions appearing on the l.h.s.. A comment is in order regarding the differential operators renormalizing these diagrams. In the flat massive case \[1\], it was noted that these operators indicate the particle production thresholds for the corresponding diagrams, thus $\mu - 4m^2$ for the bubble, $\mu - 9m^2$ for the setting sun diagram, etc.. Here, a new but related phenomenon is seen to occur: if we consider the propagators here to carry the representation $\eta = 2$ (of the isometry group $SO(1, 4)$ of $H_4$), then the differential operators in Eq. (2.18) will count the number of representations propagating along the diagram minus one. In curved spaces we cannot attach this the same meaning as in flat space, i.e., that this is an indication of the analyticity structure of the respective $p$-space amplitudes, but we nonetheless find this curious phenomenon worthy of mention. For conciseness, we do not present here the contact terms appropriate for the renormalization of these higher loop diagrams, but they are straightforward to find in the same fashion as for the bubble diagram.

We can now move on to the renormalization of massive diagrams. They are somewhat more difficult to work out, and we present here the results only for the bubble and setting sun diagrams. In what follows, we need to use two recursion relations for associated Legendre functions \[3\]:

$$
\frac{dQ_\nu^\mu(z)}{dz} = \mu^\frac{z}{\sqrt{z^2 - 1}} Q_\nu^\mu(z) + \frac{1}{\sqrt{z^2 - 1}} Q_\nu^{\mu+1}(z)
$$

$$
Q_\nu^{\mu+2}(z) = -2(\mu + 1)^\frac{z}{\sqrt{z^2 - 1}} Q_\nu^{\mu+1}(z) + (\nu - \mu)(\nu + \mu + 1)Q_\nu^\mu(z).
$$

(2.20)

For the bubble diagram, we find after some manipulations:

$$
\left( \frac{a^2}{\sqrt{z^2 - 1}} Q_1^1(z) \right)^2 = \frac{a^2}{2} \left( \Delta + 4 m^2 \frac{a^2}{z^2 - 1} \right) \left[ a^2 z Q_{\eta - 2}^1(z) Q_{\eta - 2}^0(z) \right] + a^2 m^2 (Q_{\eta - 2}^0(z))^2,
$$

(2.21)

where $m^2 = (\eta - 1)(\eta - 2)n^2$ (compare Eq. (2.4) with (2.5) for $n = 4$ to find this). We note that, again, in the differential operator appearing here, although we cannot speak of an analyticity structure for the corresponding $p$-space amplitude,
we also see a “production threshold” for two particles of mass $m$ appearing besides the curvature effect in the term $-2$ analyzed previously. Finally, to obtain the renormalized amplitude corresponding to this diagram, we need the appropriate contact terms. We find these in a two-step process: first, we take the limit $a^2 \to 0$ while holding $m^2$ fixed. This is done with the help of the following limit [10]:

$$
\lim_{\nu \to \infty} \nu^{-\mu} e^{-\nu \pi i} Q^\mu_n(\cos \frac{\xi}{\nu}) = K_\mu(\xi), \quad \xi > 0,
$$

(2.22)

where $K_\mu$ are modified Bessel functions. By using this in the DR identity above, with $z = \cosh ar$ and $\nu^2 \sim m^2/a^2$, we find precisely the DR identity for massive scalars on flat space [9]:

$$
\left( \frac{mK_1(mr)}{r} \right)^2 = \frac{1}{2} \left[ \left( \frac{mK_0(mr)}{r} \right)^2 - 4m^2 \right] K_1(mr).
$$

(2.23)

So far, no contact terms need be added. Now, we determine contact terms by the massless limit of the above equation. We simply borrow the result from [9], and finally write for the renormalized amplitude:

$$
\left( \frac{a^2}{\sqrt{z^2 - 1}} Q_{\eta-2}^1(z) \right)^2 = \frac{a^2}{2} \left( \Delta + 4 \frac{m^2}{a^2} - 2 \right) \left[ a^2 z Q_{\eta-2}^1(z) Q_{\eta-2}^0(z) + a^2 m^2 (Q_{\eta-2}^0(z))^2 + \pi^2 \ln(16M^2/\gamma^2 m^2) \delta^{(4)}(y) \right],
$$

(2.24)

where $\gamma$ is the Euler-Mascheroni constant. Naturally, we could also have taken the direct $a^2 \to 0, m^2 \to 0$ limit to determine these contact terms, and the result would be identical to what we found for the massless case, Eq. (2.16). These two procedures differ by a choice of scheme or, more specifically, by the finite contact term $\pi^2 \ln[(\eta - 1)(\eta - 2)\gamma^2/16]\delta^{(4)}(y)$.

We now present the DR identity that renormalizes the massive setting sun diagram. It represents somewhat more effort than previous cases, and we find:

$$
-16 \left( \frac{a^2}{\sqrt{z^2 - 1}} Q_{\eta-2}^1(z) \right)^3 = a^4 \left[ \Delta + 9 \frac{m^2}{a^2} + 4 \right] \left[ \Delta + \frac{m^2}{a^2} - 2 \right] \left[ a^2 z Q_{\eta-2}^0(z)(Q_{\eta-2}^1(z))^2 + m^2 z(Q_{\eta-2}^0(z))^3 \right] - 32a^4 m^2 \sqrt{z^2 - 1} \left( Q_{\eta-2}^1(z) \right)^3
$$

$$
-8m^2 a^2 (a^2 + 8m^2) \sqrt{z^2 - 1} \left( Q_{\eta-2}^0(z) \right)^2 Q_{\eta-2}^1(z) - 28m^2 a^2 \left( a^2 z Q_{\eta-2}^0(z)(Q_{\eta-2}^1(z))^2 + m^2 z(Q_{\eta-2}^0(z))^3 \right)
$$

(2.25)

By using the limit Eq. (2.22), the above identity again goes very smoothly in the $a^2 \to 0, m^2$ fixed limit into the corresponding amplitude in [9]. Borrowing the massless limit of that expression, like for the setting sun above, determines the contact terms appropriate for this DR identity. The result above stands as a success of DR, insofar as it produces the fully renormalized 2-loop 2-point function in a compact and explicit way, something which had not been achieved previously.

b) The sphere $S_n$

$S_n$ is the maximally symmetric space of constant positive curvature in $n$ dimensions. Because it is a more familiar space, and because most formulas here would be essentially redundant with formulas from the previous section on the hyperboloid, we simply give here a more condensed summary of our results. Essentially all the geometry formulas from last section can be adjusted to spheres by taking $a^2 \to -a^2$. The reduced variable to be used here, rather than $z$, is $x \equiv \pm \sqrt{1-a^2};$, with range $-1 \leq x \leq 1$, where the positive(negative) sign holds for the upper(lower) hemisphere. The massive propagator is

$$
G_m(x) = \frac{1}{2\pi^{n/2}} \left( \frac{a^2}{2e^{x^2/2} \sqrt{1-x^2}} \right)^{(n-2)/2} Q^{(n-2)/2}_\nu(x),
$$

(2.26)

where now $\nu(\nu + 1) = -m^2/a^2$, and it satisfies the equation

$$
\left[ -(1-x^2) \frac{\partial^2}{\partial x^2} + nx \frac{\partial}{\partial x} + \left( \frac{m^2}{a^2} + \frac{n(n-2)}{4} \right) \right] G_m(x) = \frac{\delta^{(n)}(y)}{a^2}.
$$

(2.27)
For the massless propagator in $n = 4$ in particular, we have:

$$G_{m=0}(x) = \frac{a^2}{4\pi^2} \frac{1}{\sqrt{1 - x^2}} Q_0^0(x),$$  \hspace{1cm} (2.28)

satisfying

$$[\Delta + 2] G_{m=0}(x) = \frac{\delta^{(4)}(y)}{a^2}. \hspace{1cm} (2.29)$$

The renormalization of $\ell$-loop 2-point functions, analogously to Eqs. (2.13) and (2.17), works as follows:

$$\left(\frac{a^2}{\sqrt{1 - x^2}} Q_0^0(x)\right)^2 = \frac{a^2}{2} (\Delta + 2) \left[ \frac{a^2 x Q_0^0(x)Q_0^0(x)}{\sqrt{1 - x^2}} \right]$$

and

$$\left(\frac{a^2}{\sqrt{1 - x^2}} Q_0^1(x)\right)^3 = -\frac{a^4}{16}(\Delta - 4)(\Delta + 2) \left[ a^2 x(Q_0^0(x))^2 Q_0^0(x) \right]$$

and

$$\left(\frac{a^2}{\sqrt{1 - x^2}} Q_0^1(x)\right)^4 = \frac{a^6}{384}(\Delta - 18)(\Delta - 4)(\Delta + 2) \left[ a^2 x\sqrt{1 - x^2}(Q_0^1(x))^3 Q_0^0(x) \right]$$

and

$$\left(\frac{a^2}{\sqrt{1 - x^2}} Q_0^1(x)\right)^5 = -\frac{a^8}{18432}(\Delta - 40)(\Delta - 18)(\Delta - 4)(\Delta + 2) \left[ a^2 x(1 - x^2)(Q_0^1(x))^4 Q_0^0(x) \right]. \hspace{1cm} (2.30)$$

The same pattern has emerged here as previously, with the only difference that all signs are reversed. Again, we do not work out the contact terms for these DR identities, as the procedure is identical to what was done previously. Also, one can verify that the first two DR identities above, which lead to the renormalization of the bubble and setting sun diagrams, respectively, again lead to the correct values for the $\beta$-function and anomalous dimension in $\lambda\phi^4$ theory at lowest nontrivial order.

With the hindsight of our work on the hyperboloid, it becomes straightforward to check the renormalization identities for the massive bubble and setting sun amplitudes on the sphere. They are given, respectively, by:

$$\left(\frac{a^2}{\sqrt{1 - x^2}} Q_1^0(x)\right)^2 = \frac{a^2}{2} \left( \Delta + 4 \frac{m^2}{a^2} + 2 \right) \left[ \frac{a^2 x Q_1^0(x)Q_0^0(x)}{\sqrt{1 - x^2}} \right] - a^2 m^2 (Q_1^0(x))^2, \hspace{1cm} (2.31)$$

and

$$-16 \left(\frac{a^2}{\sqrt{1 - x^2}} Q_1^0(x)\right)^3 =$$

$$= a^4 \Delta + 9 \frac{m^2}{a^2} - 4 \left[ \Delta + \frac{m^2}{a^2} + 2 \right] \left( a^2 x Q_1^0(x)Q_0^0(x))^2 + m^2 x(Q_0^0(x))^3 \right) + 32 a^4 m^2 \sqrt{1 - x^2} (Q_1^0(x))^3$$

and

$$-8a^2 m^2 (a^2 - 8m^2) \sqrt{1 - x^2} (Q_1^0(x))^2 Q_0^0(x) + 28m^2a^2 \left( a^2 x Q_1^0(x)(Q_1^0(x))^2 + m^2 x(Q_1^0(x))^3 \right), \hspace{1cm} (2.32)$$

and appropriate contact terms are again gotten by a straightforward massless and flat limit above.

### III. Renormalization at Finite Temperature

Although quantum field theory is usually defined at zero temperature, it is also possible to consider it at arbitrary nonzero temperatures [11,12], and finite temperature effects are relevant, for instance, in studying phase transitions in the early universe [13] or the thermal structure of QCD [14]. This implementation of temperature in quantum field theory is essentially made in one of two ways: either one proceeds in real Minkowskian time, or goes into Wick-rotated, compactified Euclidean time. The former suffers from some ambiguities [15], while the latter is a more appropriate setting for our approach, and is the one we shall follow here.

Again, to illustrate our general procedure, we work here with massless $\lambda\phi^4$ theory. It is known that finite temperatures induce masses through renormalization, and thus a consistent, renormalizable scalar model in this context must be massive. This phenomenon will be seen very clearly in our treatment from the non-closure of the renormalization group equations for the 2-point function. We choose, nonetheless, to start with a massless theory here in order to
illustrate our method in a simple way. Sure enough, the renormalization group equation will not be verified due to the lack of a bare mass to cancel a temperature-induced mass counterterm appearing at two loops. We then switch on a bare mass and treat it as a correction to the massless case in order to get its appropriate renormalization. The spirit of our approach remains, to show that in any theory one is able to isolate singular terms and correct them using differential identities.

In imaginary time formalism, time is rotated into Euclidean time $\tau$ and compactified to a cylinder of perimeter

$$\beta = \frac{1}{kT},$$  \hspace{1cm} (3.1)

where $k$ stands for the Boltzmann constant. The momentum space propagator is

$$\frac{1}{w_n - p^2}, \quad w_n = i \frac{2\pi n}{\beta},$$  \hspace{1cm} (3.2)

where the Euclidean zeroth component of the momentum vector takes only discrete values. In coordinate space this propagator can be written using the method of images as

$$G(x, \tau; \beta) = \frac{1}{4\pi^2} \sum_{n=\infty}^{\infty} \frac{1}{x^2 + (\tau + n\beta)^2},$$  \hspace{1cm} (3.3)

where $x = \sqrt{x^2}$. This expression can be resummed, or, alternatively, the Fourier transform of the momentum propagator can be computed to yield

$$G(x, \tau; \beta) = \frac{1}{4\pi\beta} \frac{\sinh \frac{2\pi}{\beta} x}{\cosh \nu x - \cos \frac{2\pi}{\beta} \tau} = \frac{\nu}{8\pi^2 x \cosh \nu x - \cos \nu \tau},$$  \hspace{1cm} (3.4)

where we have rewritten the last line using $\nu \equiv \frac{2\pi}{\beta} = 2\pi kT$. Note that this compact form of the propagator is manifestly periodic in Euclidean time and decreases at long distances as $1/x$, which is the characteristic decay in three dimensions. This is a hint at the effective reduction from four dimensions to three at high temperatures.

The simplest diagram is a pure tadpole. Its departure from $\beta = \infty$ can easily be computed in coordinate space using the first form of the propagator, Eq. (3.3). The computation is reduced to

$$\lim_{(x, \tau) \to (0,0)} (G(x, \tau; \beta) - G(x, \tau; \beta = \infty)) = \sum_{n \neq 0} \frac{1}{4\pi^2} \frac{1}{(n\beta)^2} = \frac{1}{12\beta^2}. $$  \hspace{1cm} (3.5)

This result agrees with the standard literature \[1,12\]. It can also be obtained through a simple expansion in Eq. \[3.4\]. Nontrivial renormalization first appears when trying to compute the vertex function at 1-loop order. We need to renormalize $G^2$, the square of the propagator, which presents a logarithmic singularity. We proceed as in Sec. 1, by noting that for $x \neq 0$

$$G^2(x) = \frac{1}{16\pi^2} \Box (G(x) \ln G(x)/M^2) + \frac{\nu^2}{16\pi^2} G(x) - \frac{\nu}{128\pi^4 x^3} (\nu x \cosh \nu x - \sinh \nu x)^2, $$  \hspace{1cm} (3.6)

where $G(x)$ is short-hand for the propagator and the laplacian operator is

$$\Box = \partial_x \partial_x + \frac{1}{x^2} \partial_x x^2 \partial_x . $$  \hspace{1cm} (3.7)

The renormalization of this diagram has been carried out according to the generic analysis presented in Sec. 1. The singularity has been corrected in the laplacian term, and the last two terms of the r.h.s. above correspond to the expected left-over distribution, $D$, which encodes extra long-distance information of the theory. One can easily check that this distribution $D$ has a leading singularity of the type $(x^2 + \tau^2)^{-1}$, which is Fourier-transformable.

Now it is easy to compute the first coefficient of the $\beta$ function, using the renormalization group equation for the 4-point amputated vertex function. At this order, there is no contribution coming from the anomalous dimension of the field $\phi$, thus Eq. \[1.11\] is readily obtained and we explicitly see that, to this order, the short-distance renormalization of the theory has not been affected by the compactification of Euclidean time. This corroborates the fact that we are
dealing with a scheme-independent term of the $\beta$-function. To compute this term of the $\beta$-function, only the coefficient of the logarithm matters, so we do not need the actual form of $D(x)$, provided it is a well-behaved distribution. Yet, the point is that we have explicitly obtained the finite parts of the diagram without much effort.

The 2-loop correction to the vertex function consists of two diagrams, Figs. 2, 3. Fig. 2, as explained in Sec. 1, is just a convolution of the 1-loop diagram and only promotes logarithms to their squares, as dictated by the first coefficient of the $\beta$-function, giving no contribution to the second coefficient of the $\beta$-function. The other one is computed exactly as in Sec. 1, that is, in a first step we renormalize the inner divergence through Eq. (1.19), with $D = -\Box$, and $D$ given in Eq. (3.4) above. Identical manipulations lead to Eqs. (1.20, 1.22), where now $f(x) = 0$. Again, the relative coefficient $-2$ between $\ln^2$ and $\ln$ above is dictated by short-distance behavior, and thus it is fixed and universal. Thus, collecting all the pieces, we finally obtain the renormalized expression

$$-\frac{\lambda^3}{3} \left\{ G(x-y) G(x) D(x) + \frac{1}{16\pi^2} \frac{\partial}{\partial y^\mu} \left[ G(x-y) G(x) \frac{\partial}{\partial y^\mu} \ln \frac{G(y)}{M^2} \right] \right\}$$

$$-\frac{1}{2} \frac{1}{(16\pi^2)^2} \delta^{(4)}(x-y) \left[ G(y) \left( \ln^2 \frac{G(y)}{M^2} - 2 \ln \frac{G(y)}{M^2} \right) - \frac{1}{16\pi^2} \delta^{(4)}(x-y) D(y) \right] \quad (3.8)$$

Where, again, we note that we have easily obtained explicitly the finite terms.

We now turn to the self-energy correction depicted in Fig. 4, and again use the general recipe given in Sec. 1. Without undue effort, one finds

$$G^3(x) = \frac{1}{512 \pi^4} \left( \Box + 2\nu^2 \right) G(x) \ln \frac{G(x)}{M^2} + R(x) \quad , \quad (3.9)$$

where

$$R(x) = -\frac{\nu}{4096\pi^6} \Box \left( \frac{1}{x^3} \cosh \nu x - \sinh \nu x \right) + \frac{\nu^4}{256\pi^4} G(x)$$

$$-\frac{\nu^2}{2048\pi^6} \frac{1}{x^4} \left( \cosh \nu x - \cos \nu \tau \right)^2 \sinh \nu x$$

$$\frac{2 \sinh \nu x + \nu x \cosh \nu x - \nu x \cos \nu \tau}{\sinh \nu x} \quad . \quad (3.10)$$

This has the same short-distance behavior as the propagator, and is Fourier-transformable.

As explained in Sec. 1, the appearance of a term linear in the laplacian (with coefficient $2\nu^2$) has to do with subleading singularities. In this case, it reflects the fact that the thermal bath screens the interaction. This will generate the inconsistency we are expecting: the renormalized 2-loop 2-point function will be

$$\Gamma^{(2)} = -\Box \delta^{(4)}(x) - \frac{\lambda^2}{6} \left( \frac{1}{512 \pi^4} \left( \Box + 2\nu^2 \right) G(x) \ln \frac{G(x)}{M^2} + R(x) \right) \quad , \quad (3.11)$$

and to check whether it satisfies the appropriate RG equation we compute

$$M \frac{\partial}{\partial M} \Gamma^{(2)} = \frac{\lambda^2}{6} \left( \frac{1}{16\pi^2} \right)^2 \left( \Box \delta^{(4)}(x) + 2\nu^2 \delta^{(4)}(x) \right) \quad . \quad (3.12)$$

The first term leads to the expected value of the anomalous dimension, Eq. (1.27), while the second term represents a mass renormalization. That means that, as mentioned previously, renormalizability requires a massive theory ab initio.

It is clear that masses modify the long-distance behavior of the propagator while preserving the $1/x^2$ singularity at the origin. Since renormalization consists in curing short-distance singularities, the possibility is open to treat masses as perturbations. More precisely, the exact massive propagator at finite temperature can be found to be

$$G(x, \tau; \beta; m) = \frac{\nu}{8\pi^2 x} \sum_{n=-\infty}^{\infty} e^{-\nu x \tau - x \sqrt{\nu^2 n^2 + m^2}} \quad . \quad (3.13)$$

Expanding around the massless case, we find

$$G(x, \tau; \beta; m) = \frac{\nu}{8\pi^2 x} \left( \frac{\sinh \nu x}{\cosh \nu x - \cos \nu \tau} - mx - \frac{m^2 x}{2\nu} \ln \left[ \nu^2 \left( x^2 + \tau^2 \right) \right] \right) + ... \quad (3.14)$$

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We are now ready to use this modified expression of the propagator to go through the renormalization procedure in all previous graphs. It is readily noticed that the 1- and 2-loop coefficient of the $\beta$-function and the 2-loop coefficient of the anomalous dimension of the scalar field are not modified by the presence of masses. The first and relevant appearance of a mass counterterm takes place in the setting sun diagram, as explained above. The expression for that diagram corresponds to Eq. (3.11) with the addition of the first massive correction

$$\left(\frac{\lambda}{16\pi^2}\right)^2 \left\{ \nu m G(x) \ln \frac{G(x)}{M^2} + \frac{m^2}{4} \left[ G(x) \left( \ln^2 \left[ \nu^2 (x^2 + \pi^2) \right] - 2 \ln \frac{G(x)}{M^2} \right) \right] \right\} + \ldots \quad (3.15)$$

The complete 2-point function then verifies the renormalization group equation

$$\left( M \frac{\partial}{\partial M} + \beta \frac{\partial}{\partial \lambda} + \gamma_m m^2 \frac{\partial}{\partial m^2} - 2 \gamma \right) \Gamma^{(2)}(x) = 0 \quad (3.16)$$

with

$$\gamma_m = \left(\frac{\lambda}{16\pi^2}\right)^2 \left( \frac{1}{3} \left( \frac{\nu}{m} \right)^2 - 2 \frac{\nu}{m} + \frac{7}{6} \right) + O(\lambda^3) \quad (3.17)$$

which is a scheme-dependent coefficient.

**IV. CONCLUSIONS**

The aim of the present paper has been to exploit the idea that the renormalization process amounts to replacing short-distance singularities by distributions and, more importantly, that this project can be carried out for any background a field theory might be defined on. Explicit examples on constant curvature and thermal backgrounds have been worked out.

We can draw several lessons from this investigation. In the finite temperature case, the absence of a compact form for the massive propagator in position space has forced us to treat masses in a perturbative way. This is a drawback, though renormalization group constants are obtained probably more easily than with other methods. On the other hand, constant curvature backgrounds are surprisingly suited for differential renormalization. Explicit closed expressions for 2-loop amplitudes are obtained with ease.

Further applications of this renormalization procedure are manifold, as for instance the extension to QED, along the lines of [16], or computations of the effective potential on curved backgrounds. Its very essence, based on the coordinate space propagator, makes it an extremely simple method to renormalize a field theory.

**ACKNOWLEDGMENTS**

We thank K. Kirsten and S. Odintsov for comments on the manuscript.

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