Scattering of classical and quantum particles by impulsive fields

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Abstract
We investigate the scattering of classical and quantum particles in impulsive backgrounds fields. These fields model short outbursts of radiation propagating with the speed of light. The singular nature of the problem will be accounted for by the use of Colombeau’s generalized function which however give rise to ambiguities. It is the aim of the paper to show that these ambiguities can be overcome by implementing additional physical conditions, which in the non-singular case would be satisfied automatically. As example we discuss the scattering of classical, Klein–Gordon and Dirac particles in impulsive electromagnetic fields.

Keywords: relativity, scattering, distributions

(Some figures may appear in colour only in the online journal)

Introduction
In the following we consider the scattering of classical (point-like) as well as quantum (waves) particles by impulsive background fields. That is to say, the particles interact with a field that is solely concentrated on a null hyperplane described by a delta-like singularity. The physical motivation is to model the behavior of particles affected by extreme short outbursts of radiation such as observed in supernovae explosions, gamma-ray bursts or the fields of ultra-short laser pulses produced in the laboratory. Since, from the spacetime point of view, the particles move freely ‘above’ and ‘below’ the pulse-hyperplane the solution of the equations of motion is turned into a matching problem for free solutions.

The mathematical price for this simplified physical description comes in the form of non-linear operations performed on distributional objects. The adequate framework is provided by the algebra of new generalized functions $G$ of Colombeau [1]. It circumvents the Schwarz
impossibility result, that claims the non-existence of a (differential) algebra extending the continuous functions and containing the distributions, by requiring only the $C^\infty$-functions to be a sub-algebra.

Early work by DeVega and Sanchez [2] and Lousto and Sanchez [3] discuss the scattering of Klein–Gordon and Dirac fields in a special class of impulsive gravitational backgrounds. These geometries can be obtained from the ultra-relativistic limit of black hole space-times (AS-geometries [4] and generalizations thereof). The authors of notice a particular regularization dependence of their result. More recently scattering in shock-wave geometries has been discussed by Collas and Klein [10].

On the other hand Kunzinger and Steinbauer [6] have investigated the behavior of geodesics in general impulsive pp-wave backgrounds via rigorously embedding the equations into the Colombeau algebra. They show that a (unique) solutions to the geodesic as well as the geodesic deviation equation exist in $G$ which possess a reasonable macroscopic, that is distributional, ‘shadow’. These results are in accordance with earlier work by Balasin [7].

In the present paper we follow a similar strategy as in [7] whence extended to the field context. Distributional equality will be replaced by association which is the corresponding notion in $G$. We find that in spite of the singular character the nonlinear operations yield associated distributional objects containing, however, finite undetermined quantities, which is to be expected [1]. In order to define their values from a physical point of view we make use of conservation laws. These would follow in the smooth context via nonlinear operations which, in general however, break association. We believe that this method provides a systematic way to decide upon the ‘regularization dependence’ in [2] without relying on the heavy machinery used in [6] (in fact being closer to the approach used in [1]). In section 1 we briefly recall the definition of the Colombeau algebra and discuss the properties of association. As a warm up, we start in section 2 by considering a particle subject to a potential acting only at a single instant of time. Already in the Newtonian (classical) case, care must be taken in handling the non-linearities coming from the potential, but the solution is uniquely determined. In contrast, the corresponding Schrödinger problem, in section 3, leads to an undetermined quantity. The reason is the association of the product of the $\delta$ term in the potential with the $\theta$ functions of the matched free solutions. Rather than stipulating its value by hand, we require the conservation of probability through the pulse, which does not follow from the equation since it involves non-linear operations which in general break distributional equality, i.e. association. From this we find via a Cayley-transform the uniquely defined transition amplitude. In a next step, in section 4, we consider a classical particle in an electromagnetic pulse and the corresponding Klein–Gordon equation as its quantum version (section 5). Although the situation is more involved the strategy is precisely the same as in the Newton–Schrödinger case. Here already the classical problem gives rise to an ambiguity which can be fixed by requiring that the length of the tangent vector to be preserved.

The case of the Klein–Gordon field is still more involved because it leads to two undetermined constants, which at first sight seems hopeless for obtaining a unique solution. However, we show that by the physical requirement that the Klein–Gordon current is conserved across the pulse, the two constants are determined and thus the matching is unique. Finally we focus on relativistic particles with spin i.e. we consider the impact of an electromagnetic pulse on Dirac-particles. Again undefined quantities (constants) arise. It is natural to impose the conservation of the Dirac current across the pulse which then gives the unique transition amplitude.

3 For an initial value approach to impulsive gravitational waves see [5].
1. The method

The Colombeau algebra $\mathcal{G}$ consists of one-parameter families of $C^\infty$ functions, $(f_\epsilon(x))_{0<\epsilon<1}$ subject to certain growth-conditions in $\epsilon$. Its elements may be thought of as being regularizations of distributional (and even more singular) objects. Distributions form a linear subspace of $\mathcal{G}$. This subspace is not canonical in the same sense as $SU(2)$ is not a canonical subgroup of $SL(2, \mathbb{C})$. In particular this means that there are many different ‘$\delta$-functions’ in $\mathcal{G}$. This property is reflected by an equivalence relation on the algebra called association and denoted by $\approx$

\[
(f_\epsilon(x)) \approx (g_\epsilon(x)) \quad \text{iff} \quad \lim_{\epsilon \to 0} \int d^n x (f_\epsilon - g_\epsilon)(x) \varphi(x) = 0 \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n).
\]

(1)

Objects in the same equivalence class may differ in their micro-aspect. That is to say although they are in general different objects in $\mathcal{G}$, they all correspond (if it exists) to the same distribution. In this regard we may think of association as a kind of coarse-graining of $\mathcal{G}$. Distributionally well-defined, i.e. linear, operations have well-defined analogs in $\mathcal{G}$, which are compatible with their macro-aspect, meaning they do not break association

\[
(f_\epsilon) \approx (g_\epsilon) \quad \Rightarrow \quad f \cdot (f_\epsilon) \approx f \cdot (g_\epsilon) \quad \Rightarrow \quad (\partial_\alpha f_\epsilon) \approx (\partial_\alpha g_\epsilon).
\]

(2)

On the other hand, non-linear operations on different representatives of an association-class do in general break association. This means that upon non-linear operations different micro-aspects may get magnified to the macro-level. A simple, nevertheless important, example is given by

\[
\theta \cdot \delta \approx A\delta
\]

(3)

where $A$ denotes a constant. This simply states that $A$ is the result of the relative micro-aspects of the two elements of $\mathcal{G}$ that are associated to $\theta$ and $\delta$ respectively.

As a special case we have

\[
\theta \cdot \theta' = \frac{1}{2} (\theta^2)' \approx \frac{1}{2} \delta
\]

(4)

or more generally by

\[
\theta^n \theta' \approx \frac{1}{n+1} \delta.
\]

(5)

These results follow from $\theta^n \approx \theta$ and the compatibility of association with differentiation. In both the above cases the constant is determined regardless of the representative $\theta$. This reflects the fact that the relative micro-aspect between $\theta$ and $\theta'$, and $\theta^n$ and $\theta'$ respectively is independent of the representative. Notice that all $\theta^n$ are again $\theta$-functions, i.e. are associated to the $\theta$-distribution. However, their different micro-aspects relative to $\theta'$ a representative of $\delta$ gets magnified to the macro-level.

2. Newtonian particle

Having prepared the stage, let us apply the formalism to the simple-most classical system: a particle under the influence of a potential acting only at an instant of time, described by the Newtonian equation of motion.
\[ m\ddot{x}(t) + \delta(t)\partial_tV(x^m(t)) \approx 0. \]  
\quad (6)

We have chosen weak equality because the \( \delta \) term represents the idealized action of the force during the shortest possible period of time. Since the force acts only at \( t = 0 \) the trajectory is given by
\[ x^\prime(t) = \theta_+(t)x^\prime_+(t) + \theta_-(t)x^\prime_-(t), \quad x^\prime_+(t), x^\prime_-(t) \in C^\infty(\mathbb{R}) \]
\[ \theta_+(t) + \theta_-(t) = 1. \]  
\quad (7)

We require \( x^\prime_-(t) \) and \( x^\prime_+(t) \) to be solutions of the free equations of motions before and after the pulse respectively.

Upon insertion into (6) gives
\[ m(x^\prime_+(0) - x^\prime_-(0))\delta_\prime(t) + m(x^\prime_+(0) - x^\prime_-(0))\delta(t) \]
\[ + \theta_+(t)m\dddot{x}_+(t) + \theta_-(t)m\dddot{x}_-(t) + \delta(t)\partial_tV(x^m(t)) \approx 0. \]  
\quad (8)

The last term contains, via the association process, undetermined constants as pre-factors of \( \delta(t) \), i.e. \( \delta(t)\partial_tV(x^m(t)) \approx C_\delta(t) \).

In a first step, multiplication with the \( C^\infty \) function \( t \) ensures the vanishing of all the terms in (8) except the \( \delta_\prime \) term, since \( t\delta_\prime \) is not \( \approx 0 \). Since association becomes equality on the distributional level, this tells us that the coefficient has to vanish, i.e. \( x^\prime_+(0) = x^\prime_-(0) \). This then in turn determines the prefactor to be \( C_\delta = \partial_tV(x^m(0)) \).

So we are only left with the \( \delta \) function coefficient
\[ m\dddot{x}_+(0) - m\dddot{x}_-(0) = -\partial_tV(x^m(0)), \]  
\quad (9)

where \( x(0) \) has now a well-defined meaning. The junction condition has a simple physical interpretation as mapping the \( t = 0 \) conditions of for \( x_- \) onto the \( t = 0 \) condition for \( x_+ \), i.e.
\[ (x^\prime_-(0), \dddot{x}_-(0)) \mapsto (x^\prime_+(0), \dddot{x}_+(0)) = (x^\prime_-(-\delta_\prime(0)), \dddot{x}_-(0) - \frac{1}{m}\partial_tV(x^m(0)))). \]  
\quad (10)

As expected, the freely moving particle gets a kick at \( t = 0 \), thereby changing its velocity according to the applied force.

3. Schrödinger particle

The quantum description of the Newtonian particle is easily achieved, by using the standard Hamiltonian, which is the sum of kinetic and potential energy. In the position representation this gives rise to Schrödinger’s equation, which for the impulsive potential becomes
\[ i\dot{\psi}(x, t) + \left(-\frac{\partial^2}{2m} + \delta(t)V(x)\right)\psi(x, t) \approx 0. \]  
\quad (11)

Proceeding in the same way as with the classical system we combine two solutions before and after \( t = 0 \), i.e.
\[ \psi(x, t) = \theta_+(t)\psi_+(x, t) + \theta_-(t)\psi_-(x, t) \]  
\quad (12)

where we assume that \( \psi_-(x, t) \) and \( \psi_+(x, t) \) satisfy the free Schrödinger equation.

Inserting (12) into (11) we are left with
\[ i\delta(t)(\psi_+(x, 0) - \psi_-(x, 0)) \approx \delta(t)V(x)(A\psi_+(x, 0) + (1 - A)\psi_-(x, 0)). \]  
\quad (13)
The above relation made implicit use of \( \theta \cdot \delta \approx A \delta \), which expresses our ignorance about the microscopic relation between \( \theta \) and \( \delta \). As in the classical regime this relates the data \( \psi_-(x,0) \) before the shock to the data \( \psi_+(x,0) \) after the shock
\[
\psi_+(x,0) = \frac{1 - i(1 - A)V(x)}{1 + iAV(x)} \psi_-(x,0).
\] (14)

There is however an important difference, that manifests itself in the appearance of the arbitrary constant \( A \). It signals that we have oversimplified the physical description of the system by making it too singular. Following [1] the mathematical description would need further specification. It is precisely additional physical input that allows to determine \( A \). For general smooth solutions of the Schrödinger equation we have conservation of the probability current. However due to the weak nature of our equation current conservation is no longer one of its consequences, since it involves, as pointed out in the introduction, non-linear operations on (11). In order to preserve the physical interpretation of the Schrödinger equation we require
\[
\dot{\rho} + \partial_i f^i \approx 0, \quad \rho = \bar{\psi}\psi, \quad f^i = \frac{1}{2m} (\bar{\psi}\partial_i \psi - \psi\partial_i \bar{\psi}),
\] (15)
\[
\dot{\rho} = 0, \quad \rho = \theta_+ \bar{\psi}_+ \psi_+ + \theta_- \bar{\psi}_- \psi_-, \quad f^i = \frac{1}{2m} (\theta_+ (\bar{\psi}_+ \partial_i \psi_+ - \psi_+ \partial_i \bar{\psi}_+) + \theta_- (\bar{\psi}_- \partial_i \psi_- - \psi_- \partial_i \bar{\psi}_-)),
\]
\[
\dot{\rho} + \partial_i f^i \approx 0 = \delta (\bar{\psi}_+ \psi_+ - \bar{\psi}_- \psi_-).
\] (16)

Using (14) has the immediate consequence
\[
\bar{\psi}_+ \psi_+ = \left| \frac{1 - i(1 - A)V(x)}{1 + iAV(x)} \right|^2 \bar{\psi}_- \psi_-.
\] (17)

Therefore probability-current conservation is only achieved if the pre-factor in the last equation has unit length, thereby fixing \( A \) to be 1/2. With this, the data below the pulse are mapped uniquely via a Cayley-transform of the reduced potential into the data above the pulse.
\[
\psi_+(x,0) = \frac{1 - \frac{1}{2} V(x)}{1 + \frac{1}{2} V(x)} \psi_-(x,0).
\] (18)

We mention that our formalism also works in the more common situation of a spatially ‘impulsive’ potential \( V(x^m) = \delta(n^x)\tilde{V}(\tilde{x}^m) \) where similar ambiguities arise, but do not contribute to the result as for the Newtonian particle.

4. Lorentz particle

Turning to a relativistic setting it is natural to consider disturbances that travel with the fundamental velocity i.e. along null, rather than on \( t = \text{const} \) surfaces. In the following, we look at the scattering of charged particles by impulsive electromagnetic fields which are completely concentrated on a null hyperplane. The vector potential \( A_a \) and the field-strength \( F_{ab} \) take the form
\[
A_a = f(p\bar{x}^m)p_a, \quad F_{ab} = 2\delta_{[a|d|}p_{b]}(\bar{x}^m), \quad f(p\bar{x}^m) = \delta(p\bar{x})\tilde{f}(\tilde{x}^m)
\] (19)
where \( p^a \) denotes (a covariantly constant) null vector-field and \( \bar{x}^i \) denotes the spacelike coordinates of the two-dimensional subspace orthogonal to \( p^a \) and a conjugate null direction \( \bar{p}^a \).
We remark that the field strength in (19) can also be obtained by gluing two pure gauge potentials \( A_+ = 0 \) and \( A_- = \tilde{\mathcal{F}} \) along \( px = 0 \), which therefore is gauge-equivalent to (19). Here we follow the coordinate free notation of Penrose [8]. However, if one introduces the coordinates \((u, v, \tilde{x}^m)\) in Minkowski space
\[
\mathrm{d}s^2 = 2\mathrm{d}udv - \mathrm{d}\tilde{x}^m \mathrm{d}\tilde{x}^m
\]
and choosing \( p = \partial_v \) and \( \bar{p} = \partial_u \), then \( px = u \) and \( \bar{p}x = v \). \( F_{ab} \) satisfies the vacuum equations provided
\[
\Delta \tilde{f}(\tilde{x}^m) = 0
\]
(e.g. \( \tilde{f} = a, \tilde{x}^i \), which corresponds to a constant plane electromagnetic wave) The motion of test-particles with mass \( m \) and charge \( e, x^i(s) \), where \( s \) refers to the Eigenzeit, is described by the Lorentz-force law
\[
mx'' + e\bar{F}^m_{\;\;\bar{k}} \tilde{x}^\bar{k} = 0,
\]
\[
\bar{m}\tilde{x}'' + e((\bar{p}x)\tilde{\partial}f - (\tilde{x}\tilde{\partial})fp^a) = 0,
\]
which becomes upon decomposition with respect to \( p^i, \bar{p}^a \) and their orthogonal complement
\[
\bar{p}x = 0,
\]
\[
p\tilde{x} + \frac{e}{m} (\tilde{x}\tilde{\partial}) f = 0,
\]
\[
\tilde{x} + \frac{e}{m} (p\tilde{x})\tilde{\partial}f = 0,
\]
where we have suppressed the indices in the two-dimensional (tilde) part and made use of the normalization \( p \cdot \bar{p} = 1 \). The first equation of (23) tells us that \( px \) may be chosen as an ‘affine’ parameter for the trajectory \((\bar{p} x) = \alpha \) unless we consider motion within a hyper-plane orthogonal to \( p \). Taking into account the impulsive nature of the profile \( f \) the equation will be considered as weak equality within the Colombeau algebra\(^4\)
\[
(px)''(px) + \frac{e}{am} \delta(px)\tilde{x}''(px)\delta\tilde{f}(\tilde{x}^m(px)) \approx 0,
\]
\[
\tilde{x}''(px) + \frac{e}{am} \delta(px)\tilde{\partial}\tilde{f}(\tilde{x}^m(px)) \approx 0.
\]
(24)
Since the electromagnetic field is completely concentrated on the plane \( px = 0 \) the particle moves freely ‘above’ and ‘below’ the pulse, i.e.
\[
(px)''(px) = \theta_+(px)(px_+) + \theta_-(px)(px_-),
\]
\[
\tilde{x}''(px) = \theta_+(px)\tilde{x}_+(px) + \theta_-(px)\tilde{x}_-(px).
\]
(25)
The second equation of (24) is in its form identical to that for the Newtonian particle. Therefore the junction conditions become
\[
\tilde{x}''_+(0) = \tilde{x}''_-(0),
\]
\[
\tilde{x}'_+(0) = \tilde{x}'_-(0) - \frac{e}{am} \tilde{\partial}f(\tilde{x}^m(0)).
\]
(26)
Let us now take a closer look at the first equation of (24).

\(^4\) Here and in the following notation like \((px)(px) \equiv (px)(u)\) denotes the dependence of \( px \) and similar expressions on the affine parameter \( px = u \).
Due to the appearance of products like \( \theta'(px) \cdot \delta(px) \) the above expression makes only sense within the algebra. Multiplication with \( px \) and taking into account the junction conditions for \( \tilde{x}_i(0) \) along the lines of the Newtonian particle, shows that the coefficient of the \( \delta'(px) \) term has to vanish separately, i.e.

\[
(\tilde{p}x_+)(0) = (\tilde{p}x_-)(0).
\] (28)

From the remaining expression we obtain

\[
(\tilde{p}x_+)'(0) - (\tilde{p}x_-)'(0) + \frac{e}{\alpha m} (A\tilde{x}^i_+(0) + (1-A)\tilde{x}^i_-(0))\tilde{\delta}f(\tilde{x}^m(0)) = 0.
\] (29)

where, as before, the (remaining) constant \( A \) arises from \( \theta(px) \cdot \delta(px) \approx A\delta(px) \). Let us pause for a moment and compare our results with the Newtonian case. Although we have obtained the arbitrary constant \( A \) in very much the same way, this arbitrariness already appears at the classical level. In order to fix this constant we will invoke a consequence of the equation of motions for smooth solutions, namely the fact that the length of the tangent vector remains constant along the trajectory

\[
-2(\tilde{p}x)'(px) + (\tilde{x}'(px))^2 \approx \text{const},
\]

\[
-2(\theta_+(px)(\tilde{p}x_+)'(px) + \theta_-(px)(\tilde{p}x_-)'(px)) + (\theta_+(px)(\tilde{x}_+)(px) - \tilde{x}_-(px))
+ \theta_+(px)(\tilde{x}'_+)(px) + \theta_-(px)(\tilde{x}'_-(px))^2 \approx \text{const},
\]

\[
\theta_+(px)(-2\tilde{p}x_+(px) + (\tilde{x}'_+(px))^2) + \theta_-(px)(-2\tilde{p}x_-(px) + (\tilde{x}'_-(px))^2) \approx \text{const}.
\] (30)
Since differentiation does not break association, we have
\[
(-2px_e'(0) + (\dot{x}_+')(0))^2 + 2px'_p(0) - (\dot{x}_-')(0)^2)\delta(px) \approx 0,
\]
\[
2 \frac{e}{\alpha m}(\dot{x}_p(0) - A \frac{e}{\alpha m} \tilde{\delta f} \tilde{\delta f} + (\dot{x}_-')(0) - \frac{e}{\alpha m} \tilde{\delta f})^2 - (\dot{x}_-')(0)^2 = 0.
\]
This condition is only satisfied if \( A \) is taken to be 1/2. Thus, summing up and denoting the ‘jump’ at \( px = 0 \) by \([ \ ]\), we have:
\[
[\dot{x}] = 0,
\]
\[
[\ddot{x}] = -\frac{e}{\alpha m} \tilde{\delta f}(\dot{x}_m(0))
\]
\[
[px] = 0.
\]
\[
[p\dot{x}] = -\frac{e}{\alpha m}(\dot{x}_p(0) \tilde{\delta f}(\dot{x}_m(0))) + (\frac{e}{\alpha m})^2(\tilde{\delta f}(\dot{x}_m(0)))^2.
\]

Figure 1 displays the geometry of such a scattering process.

5. Klein–Gordon particle

In this and the following section we are interested in the quantum-mechanical analogue of the Lorentz-particle. For a spinless particle of mass \( m \) and charge \( e \) the wave-function has to satisfy the Klein–Gordon equation, which implements the relativistic energy-momentum condition
\[
(\gamma^{ab} \hat{p}_a \hat{p}_b - m^2)\phi = 0
\]
where \( \hat{p}_a \) is given by \( \hat{p}_a = (\hat{p}_a - eA_a) \), \( \hat{p}_a = \frac{1}{i} \partial_a \)
\[
(\gamma^{ab}(\partial_a - ieA_a)(\partial_b - ieA_b) + m^2)\phi = 0.
\]
Using the specific form of the potential (19) and taking into account the lightlike nature of \( \rho^a \), the above expression simplifies to
\[
(\partial^2 + m^2 - 2i e f(\rho \partial))\phi = 0.
\]
The standard decomposition \( \phi = \phi_+ + \phi_- \), resulting from the impulsive nature of \( f \), i.e. \( f = \delta(px) \tilde{f} \) yields upon insertion into (34)
\[
2((\rho \partial)\phi_+ - (\rho \partial)\phi_-)\delta + \phi_+((\partial^2 + m^2)\phi_+ + \phi_-((\partial^2 + m^2)\phi_-)
\]
\[
-2i e f(\partial f(\partial f\phi_+ + (1 - A)\phi_-)\phi_-) \approx 0
\]
where in order to have well-defined products of singular quantities (34) has been promoted to a weak statement within the Colombeau algebra. Since \( \phi_+, \phi_- \) satisfy the free Klein–Gordon equation ‘above’ and ‘below’ the pulse respectively, we find for the mapping from the final data of \( \phi_- \) to the initial data of \( \phi_+ \)
\[
(p \partial)\phi_+ - (p \partial)\phi_- = i e f(A(\rho \partial)\phi_+ + (1 - A)\phi_-).
\]
Once again we encounter the notorious parameter \( A \) resulting from \( \theta \cdot \phi \approx A \delta \). The (complex) Klein–Gordon equation gives rise to a conserved current \( j_\alpha = (1/2i)(\Sigma_\alpha \phi_\alpha \phi - \Sigma_\alpha \phi_\alpha) \), \( D_\alpha \phi = (\partial_\alpha - ieA_\alpha)\phi, \Sigma_\alpha \phi = (\partial_\alpha + ieA_\alpha)\phi \) for smooth initial data. Since current conservation may no longer be deduced from the singular equation of motion, which breaks association, we will require it to hold separately.
from the non-linear relation to the magnetic field. We therefore turn to Dirac’s equation, which is written in two-spinor form \( [9]\)

Consider a charged quantum particle with spin one-half subject to an impulsive electromagnetic field. We therefore turn to Dirac’s equation, which is written in two-spinor form \( [9]\)
\[ \hat{P}_{AA'}\psi^A = \frac{m}{\sqrt{2}} \chi_{A'} \quad \text{where} \quad \hat{P}_{AA'} = \frac{1}{i} \nabla_{AA'} \quad (42) \]

\[ \hat{P}^{AA'}\chi_{A'} = \frac{m}{\sqrt{2}} \psi^{A'} \quad \text{and} \quad \nabla_{AA'} = \partial_{AA'} - i e A_{AA'}. \quad (43) \]

For the specific form of the potential (19) this becomes

\[ \partial_{AA'} \psi^A - i \frac{m}{\sqrt{2}} \chi_{A'} = i e f o_A o_A \psi^A \]
\[ \partial_{AA'} \chi_{A'} - i \frac{m}{\sqrt{2}} \psi^{A'} = i e f o^A o^A \chi_{A'} \quad \text{where} \quad p^A = p^{AA'} = o^A o^{A'}. \quad (44) \]

For an impulsive profile \( f = \tilde{f} \delta (px) \) and the decomposition of \( \psi^A = \theta_+ \psi_+^A + \theta_- \psi_-^A \) and \( \chi^A = \theta_+ \chi_+^A + \theta_- \chi_-^A \) into solutions of the free equation above and below the pulse respectively, we find

\[ \delta (px) (o_A (o_A \psi_+^A - o_A \psi_-^A) - i \tilde{f} (\alpha_{o_A \psi_+^A} + (1 - A) o_A \psi_-^A)) \approx 0 \]
\[ \delta (px) (o^A (o^A \chi_+^A - o^A \chi_-^A) - i \tilde{f} (\alpha_{o^A \chi_+^A} + (1 - A) o^A \chi_-^A)) \approx 0 \quad (45) \]

which in turn yields

\[ o_A \psi_+^A - o_A \psi_-^A = -i \tilde{f} (\alpha_{o_A \psi_+^A} + (1 - A) o_A \psi_-^A) \]
\[ o^A \chi_+^A - o^A \chi_-^A = -i \tilde{f} (\alpha_{o^A \chi_+^A} + (1 - A) o^A \chi_-^A). \quad (46) \]

Once again we encounter the ‘ambiguity’ \( A \) arising from the product of \( \delta \) with \( \theta_+ \). As has been our strategy in the previous paragraphs we invoke the conservation law of the Dirac-current

\[ \partial_x J^x = 0 \quad J^x = \psi^A \tilde{\psi}^A + \chi^A \tilde{\chi}^A \quad (47) \]

which in the smooth context is a direct consequence of (42). The conservation requirement is equivalent to

\[ \delta (px) \left( (o_A \psi_+^A) (o_A \psi_+^A) - (o_A \psi_-^A) (o_A \psi_-^A) \right) + \left( o_A \chi_+^A) (o_A \tilde{\chi}_+^A) - (o_A \chi_-^A) (o_A \tilde{\chi}_-^A) \right) \approx 0. \quad (48) \]

Re-arranging terms and using (46) we find, not unexpectedly, \( A = 1/2 \), which in turn yields for the junction conditions at \( (px) = 0 \)

\[ o_A \psi_+^A = \frac{1 - i \tilde{f}}{1 + i \tilde{f}} o_A \psi_-^A. \]
\[ o_A \chi_+^A = \frac{1 - i \tilde{f}}{1 + i \tilde{f}} o_A \chi_-^A. \quad (49) \]

Note that in contrast to the Klein–Gordon particle no further constant appears. The current itself is simply associated to its classical parts above and below the pulse. In this regard the Dirac-particle is simpler than its Klein–Gordon analogue.
7. Concrete examples

Although implicitly constructed, some readers may want to see a direct application of the aforementioned formalism. Therefore we present two concrete examples for impulsive quantum scattering. We begin with the Schrödinger-particle subject to a spatially constant force-impulse and thereafter consider the analogous problem for the Klein–Gordon particle. In both cases we consider an momentum-eigenstate for the initial wave-function.

7.1. Schrödinger particle

Our initial wave-function is given by
\[ \psi_m(x^m, t) = e^{-i(k_0^2/2m)t} e^{i k_0 x_m}, \]  
which entails via the junction-conditions
\[ \psi_m(x^m, t = 0) = \frac{1 - (i/2)f_0z}{1 + (i/2)f_0z} e^{i k_0 x_m}, \]  
where we took \( V(x^m) = f_0z \) which corresponds to a constant force \( f^i = -\partial_i V = -f_0 e^z \) in the \( z \)-direction. The solution of the free Schrödinger-equation may be written via a Fourier integral as
\[ \psi_m(x^m, t) = \int \frac{d^3k}{(2\pi)^3} e^{-i(k^2/2m)t} e^{i k_0 x_m} \tilde{\psi}_+(k^m). \]  
Therefore we have
\[ \frac{1 - (i/2)f_0z}{1 + (i/2)f_0z} e^{i k_0 x_m} = \int \frac{d^3k}{(2\pi)^3} e^{i k_0 x_m} \tilde{\psi}_+(k^m), \]  
or equivalently
\[ \tilde{\psi}_+(k^m) = \int d^3x e^{-i(k^m - k_0^m)x} \frac{1 - (i/2)f_0z}{1 + (i/2)f_0z} \tilde{\psi}_+(k^m), \]  
\[ = (2\pi)^2 \delta^{(2)}(k^m - k_0^m) \theta(-k - k_0) \frac{8\pi}{f_0} e^{2i(k-k_0)/f_0}. \]  
Finally the solution after the pulse becomes
\[ \psi_m(x^m, t) = e^{i k_0 x_m} e^{-i(k^2/2m) t} \frac{8\pi}{f_0} \int_{-\infty}^{k_0} \frac{dk}{2\pi} e^{-i(k-k_0)^2/2m} e^{2i(k-k_0)/f_0}. \]  

7.2. Klein–Gordon particle

Let us now turn to corresponding relativistic case. For simplicity we will assume that the particle is massless. With respect to null-coordinates adapted to the geometry of the pulse, we may represent the solution of the free Klein–Gordon equation by a Fourier-integral
\[ \phi(x) = \phi(px, \bar{px}, \tilde{x}) = \int \frac{dpdk\epsilon^2\tilde{k}}{(2\pi)^3} e^{-i\epsilon(pk)(ps)} e^{-i\pi^2(x)(ps)} e^{i\tilde{k}\tilde{x}} \tilde{\phi}(pk, \tilde{k}). \quad (57) \]

For a plane incident wave \( \phi(px, \bar{px}, \tilde{x}) = e^{-i\epsilon(pk_0)(\bar{px})} e^{-\epsilon(\tilde{k}_0^2)(\tilde{x})} \) with \( k_0^2 = 0 \), we obtain for the Fourier-transform of the junction condition

\[ \tilde{\phi}((pk), \tilde{k}) = \int d^2\tilde{x} e^{i\tilde{k}x} \frac{1 - i\epsilon/2\tilde{f}}{1 + i\epsilon/2\tilde{f}} (2\pi)\delta((pk) - (pk_0)) \]

\[ = (2\pi)\delta((pk) - (pk_0))A(\tilde{k} - \tilde{k}_0) \quad (58) \]

where

\[ A(\tilde{k}) = \int d^2\tilde{x} e^{-i\tilde{k}\tilde{x}} \frac{1 - i\epsilon/2\tilde{f}}{1 + i\epsilon/2\tilde{f}}. \quad (59) \]

Therefore the solution ‘above’ the pulse becomes

\[ \phi_+(x) = e^{-i\epsilon(pk_0)(\bar{px})} \int d^2\tilde{k} A(\tilde{k} - \tilde{k}_0) e^{-i\pi^2(\bar{px})} e^{i\tilde{k}\tilde{x}} \]

which gives the scattered wave as a Fourier-integral over the transition amplitude.

8. Conclusion

We have considered the scattering of charged classical and quantum particles by impulsive electromagnetic waves. The problem is reduced to the matching of free solutions above and below the pulse. The theory of Colombeau generalized functions was applied to give a meaning to products of singular terms, however leading to undetermined constants. We have shown that by implementing conservation laws that follow automatically from the equation of motion for smooth solutions, allows one to determine these constants. As examples we discussed the scattering of relativistic charged point particles and their quantum analogues i.e. charged Klein–Gordon and Dirac fields. In all cases we obtained a unique scattering amplitude. A natural step further is to extend our approach to particles with vectorial charge structure as well as gravity. We think that this method can be applied to similar physical situations, whenever the scattering source can be modeled by impulsive waves.

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