Optimum Allocation Rule for Accelerated Degradation Tests With a Class of Exponential-Dispersion Degradation Models

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Optimum allocation problem in accelerated degradation tests (ADTs) is an important task for reliability analysts. Several researchers have attempted to address this decision problem, but their results have been based only on specific degradation models. Therefore, they lack a unified approach toward general degradation models. This study proposes a class of exponential dispersion (ED) degradation models to overcome this difficulty. Assuming that the underlying degradation path comes from the ED class, we analytically derive the optimum allocation rules (by minimizing the asymptotic variance of the estimated \( q \) quantile of product’s lifetime) for two-level and three-level ADT allocation problems whether the testing stress levels are prefixed or not. For a three-level allocation problem, we show that all test units should be allocated into two out of three stresses, depending on certain specific conditions. Two examples are used to illustrate the proposed procedure. Furthermore, the penalties of using nonoptimum allocation rules are also addressed. This study demonstrates that a three-level compromise plan with small proportion allocation in the middle stress, in general, is a good strategy for ADT allocation. Supplementary materials for this article are available online.

KEY WORDS: Accelerated degradation tests (ADTs); Optimum allocation; V-optimality; Exponential dispersion model.

1. INTRODUCTION

Assessing the reliability information (e.g., the mean time to failure or the \( q \) quantile of lifetime distribution) of products is an essential task in the continual enhancement of a product’s quality and reliability. For highly reliable products, it is difficult to assess the lifetime distribution using traditional life tests (which may have very few or even no failures to be observed). In this case, an accelerated life test (ALT) that uses higher levels of stress (including elevated temperatures or voltages) to extrapolate the lifetime information at normal use conditions can provide timely lifetime information for highly reliable products (Meeker and Escobar 1998; Nelson 2004). When designing an efficient ALT plan, several authors such as Chernoff (1962) and Meeker and Hahn (1977, 1985) have studied optimum and good-compromise ALT plans for the simple linear model and have outlined practical guidelines for planning an ALT.

For very highly reliable products, even using ALT, it may happen that few or even no failures are observed at higher testing stresses. Under these conditions, if the products have quality characteristics (QCs) whose degradation over time is related to their reliability, then collecting the data from an accelerated degradation test (ADT) can also provide useful lifetime information for highly reliable products. Two real examples (stress relaxation data, Yang 2007; and Device B data, Meeker and Escobar 1998) in Section 5 will illustrate this purpose. Further applications can also be found in Tang and Chang (1995), Liu and Tang (2010), Meeker, Escobar, and Lu (1998), and Shi and Meeker (2012).

Similar to designing an efficient ALT plan, the optimum ADT allocation problem (including the determinations of stress levels, total testing times, and sample size allocations) needs to be addressed very carefully. Prefixing the stress levels, Tseng, Tsai, and Balakrishnan (2011) proposed the optimum sample size allocation in two-level ADT design by using three well-known criteria, \( D \)-optimality, \( A \)-optimality, and \( V \)-optimality (Myers, Montgomery, and Anderson-Cook 2009). Furthermore, Lim and Yum (2011) used the \( V \)-optimality criterion to address the problem of how to simultaneously determine the optimal settings of two-level and three-level stresses and the allocation of test units. These results, however, are restricted to the assumption that the underlying degradation path follows a Wiener degradation model.

Generally speaking, the Wiener process may not be appropriate for modeling the fatigue data because the degradation behavior tends to be monotone. In this situation, Gamma and inverse Gaussian (IG) degradation models (Singpurwalla 1995; Lawless and Crowder 2004; Park and Padgett 2005; Wang and Xu 2010; Ye and Chen 2014) are more appropriate. Except for these well-known continuous-type degradation processes, a discrete-type compound Poisson process may be appropriate to model a leakage current of thin gate oxides in nano-technology.
(Hsieh and Jeng 2007). In this study, we propose a class of exponential dispersion (ED; Jørgensen 1997) degradation model in such a way that the above-mentioned degradation models turn out to be the special cases of this class. Two examples in Section 5 also demonstrate that some specific ED degradation models even outperform the existing and well-known degradation models. Therefore, the class of ED models can provide us more suitable degradation models to describe the product’s degradation path.

Assuming that the underlying degradation path comes from the ED class, we propose a unified approach to address the optimum ADT allocation problem. When the testing stress levels are prefixed, the optimum sample size allocations for two-level and three-level ADT problems are analytically derived by minimizing the asymptotic variance of the estimated $q$ quantile ($\hat{E}_q$) of the product’s lifetime. For a three-level allocation problem, we define a “generalized distance” to measure the distance of any two standardized stresses and this study further demonstrates that all test units should be assigned into two out of three stresses, depending on some specific conditions (which can be expressed in terms of the generalized distances of three standardized stresses). Furthermore, the parameters $b$ and $d$ in an ED accelerated degradation model (which will be explained later in Section 3) play important roles for determining the optimum allocation strategy when the standardized stresses and their corresponding termination times are prefixed. We will discuss this result later in Section 5. In addition, when the testing stress levels are not prefixed, we provide a proactive way to determine the testing stress levels and their corresponding sample size allocations, simultaneously. Finally, some examples are used to illustrate the proposed procedure.

The rest of this study is organized as follows. Section 2 presents a class of ED degradation models. Section 3 introduces the assumptions and problem formulation of this study. Section 4 presents the optimum allocation rules for two-level and three-level ADT whether the testing stress levels are prefixed or not. Section 5 uses two examples to illustrate the proposed procedure. Section 6 addresses the penalties of using nonoptimum allocation rules. Some concluding remarks are addressed at the end of this article and all the appendices are given in an online supplementary material.

2. A CLASS OF ED DEGRADATION MODELS

A stochastic process \(\{Y(t)\mid t > 0\}\) is called an exponential dispersion (ED) degradation model and is denoted by \(Y(t) \sim \text{ED}(\mu, \lambda)\) if (i) \(Y(0) = 0\); (ii) \(\{Y(t)\mid t > 0\}\) is stationary and independent increments with mean drift rate $\mu$ and dispersion parameter $\lambda$; (iii) each increment $\Delta Y_j = Y(t_j) - Y(t_{j-1})$ has the following probability density function (pdf):

\[
f(\Delta y_j \mid \mu, \lambda) = c(\Delta y_j \mid \lambda, \Delta t_j) \exp \left\{ \lambda \left[ \Delta y_j \sigma(\mu) - \Delta t_j \kappa(\sigma(\mu)) \right] \right\},
\]

where $c$ and $\kappa$ are suitable functions and $\Delta t_j = (t_j - t_{j-1}) > 0$, for all $j \geq 1$. The motivation of Equation (1) is modified slightly from the ED distribution in eq. (3.7) of Jørgensen (1997) by setting $\theta = \sigma(\mu)$. It can be easily shown that $E(Y(t)) = \mu t$ and $\text{var}(Y(t)) = \sigma^2(\mu) t$, where $\mu = \kappa'(\sigma(\mu))$ and $\sigma(\mu) = \kappa''(\sigma(\mu))$. Note that $\kappa'(\theta)$ and $\kappa''(\theta)$ are the first and second derivatives of $\kappa(\theta)$ with respect to $\theta$, and $\sigma(\theta)$ is essentially an inverse function of $\kappa'$. Furthermore, $V(\mu)$ is known as the variance function. An important class of ED models can be proposed by the following power function (Jørnsten 1997):

\[
V(\mu) = \mu^d, \quad d \in (-\infty, 0] \cup [1, \infty).
\]

This class of ED models is also known as Tweedie model (Tweedie 1984). Table 1 summarizes that the Wiener, Gamma, and IG processes are the special cases of ED class with $d = 0, 2$, and $3$. In addition, the Poisson and compound Poisson processes can also be found in this class, with $d = 1$ and $1 < d < 2$, respectively. For example, the independent increment of the Wiener process follows a normal distribution with $N(\eta \Delta t, \sigma^2 \Delta t)$. That is,

\[
f_W(\Delta y) = \frac{1}{\sqrt{2\pi \sigma^2 \Delta t}} \exp \left\{ -\frac{(\Delta y - \eta \Delta t)^2}{2 \sigma^2 \Delta t} \right\}.
\]

Set $\lambda = \sigma^{-2}$, and then we have $c(\Delta y \mid \lambda, \Delta t) = \frac{\sqrt{\lambda}}{\sqrt{2\pi \Delta t}} \exp\left\{ -\frac{1}{2} \left( \frac{\Delta y - \eta \Delta t}{\lambda \Delta t} \right)^2 \right\}$. Furthermore, let $\eta = \sigma(\mu) = \mu$ and $\kappa(z) = \frac{z^2}{2}$. Then, $f_W(\Delta y)$ can be expressed as Equation (1).

In the following sections, we mainly focus on the optimum sample-size allocation of an ADT when the degradation path comes from this class of ED models.

3. ASSUMPTIONS AND PROBLEM FORMULATION

Suppose that $N$ test units (prefixed) are available for conducting a single accelerating variable with $k$-level ADT. Then, the test conditions are described as follows:

1. Let $S_0$ denote the use-stress level, and ADT uses $k$ stress levels \(\{S_l\}_{l=1}^k\) with \(S_0 < S_1 < S_2 < \cdots < S_k\). For $l = 1, \ldots, k$, let $X(S_l)$ be a suitable function of $S_l$ and $x_l$ be the standardized stress. That is,

\[
x_l = \frac{X(S_l) - X(S_0)}{X(S_k) - X(S_0)}.
\]

Under the standardization, $x_0 = 0 < x_1 < \cdots < x_k = 1$.

2. For $1 \leq l \leq k$, assign $n_l$ units to the standardized stress level $x_l$, and $p_l = (n_l/N)$ denotes the proportion of test units that is allocated to $x_l$, where $\sum_{l=1}^k n_l = N$. Assume that each test unit in $x_l$ has $m$ measurements and the total time on test (TTT) is $t^*_l$, where \(\{t^*_1, \ldots, t^*_m\} = t^*_l\) are the corresponding measurement times.

| Table 1. Three well-known degradation models |
|---------------------------------------------|
| Unknown parameters | Wiener process \(N(\eta, \sigma^2 t)\) | Gamma process \(\Gamma(\alpha, \beta)\) | IG process \(\mathcal{IG}(\delta_1, \delta_2, t^2)\) |
| \(\mu\) | \(\eta\) | \(\alpha\beta\) | \(\delta_1\) |
| \(\lambda\) | \(1/\sigma^2\) | \(\alpha\) | \(\delta_2\) |
| \(\sigma(\zeta)\) | \(z\) | \(-1/z\) | \(-1/\delta_2^2\) |
| \(\kappa(\zeta)\) | \(z^2/2\) | \(-\ln(-z)\) | \(-(\delta_2^2)^{1/2}\) |
| \(d\) | 0 | 2 | 3 |

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3. For $i = 1, \ldots, n_l$, $j = 1, \ldots, m$, and $l = 1, \ldots, k$, let $Y_i(t_{jl}|x_i)$ denote the degradation path of the $i$th test unit under standardized stress $x_i$ at time $t_{jl}$ and assume that $Y_i(t_{jl}|x_i) \sim ED(\mu(x_i), t_{jl}, \lambda)$, where

$$\ln(\mu(x_i)) = a + bx_i.$$  

The product’s lifetime $T$ (under $x_0$) can be defined as the first passage time when $Y(t|x_0)$ crosses a critical value $\omega_0$. There is an inexact assumption that $Y(t|x_0)$ has an increasing trend. Hence, $T$ can be expressed as follows:

$$T = \inf \{t | Y(t|x_0) \geq \omega_0 \}.$$

Assume that $F(t; \lambda) = \Pr(T \leq t; \lambda)$, and $\hat{\xi}_q(= F^{-1}(q; \lambda))$ is the $q$ quantile of products, where $\lambda = (a, b, \lambda)$. Let $\hat{\lambda} = (\hat{a}, \hat{b}, \hat{\lambda})$ be the maximum likelihood estimator (MLE) of $\lambda$ and $\hat{\xi}_q(= F^{-1}(q; \hat{\lambda}))$ denote the corresponding estimator of $\xi_q$. In this study, we adopt the $V$-optimality to find the optimum ADT test plan. The criterion considers the minimization of asymptotic variance of $\hat{\xi}_q$ over a set of available allocation plans. Hence, given $N$ and $\{t_{jl}\}_{l=1}^k$, the goal of this study is to simultaneously determine the optimum allocations of $x^* = (x_1^*, \ldots, x_k^*)$ and $p^* = (p_1^*, \ldots, p_k^*)$ such that

$$(x^*, p^*) = \arg \min_{x \in \mathcal{X}, p \in \mathcal{P}} \text{Avar}(\hat{\xi}_q|x, p),$$

where

$$\mathcal{X} = \{x = (x_1, \ldots, x_k)|0 \leq x_1 < x_2 < \cdots < x_k = 1\},$$

and

$$\mathcal{P} = \{p = (p_1, \ldots, p_k)\} \sum_{l=1}^k p_l = 1, p_l \geq 0, l = 1, \ldots, k.$$ 

In the following, $(x^*, p^*)$ is called the optimum ED-ADT allocation rule.

4. Optimum ED-ADT Allocation Rule

4.1 Expression of Asymptotic Variance of $\hat{\xi}_q$

For $i = 1, \ldots, n_l$, $j = 1, \ldots, m$, and $l = 1, \ldots, k$, let $\Delta Y_{jl} = Y_i(t_{jl}|x_i) - Y_i(t_{(j-1)l}|x_i)$ and set $\Delta t_{jl} = t_{jl} - t_{(j-1)l}$. Then, the pdf of $\Delta Y_{jl}$ is

$$f(\Delta Y_{jl}|\mu(x_i), \lambda) = c(\Delta Y_{jl}|\lambda, \Delta t_{jl}) \exp \left[ \lambda \left( \sigma(\mu(x_i)) \Delta Y_{jl} - \Delta t_{jl} \mu(\sigma(\mu(x_i))) \right) \right],$$

and the log-likelihood function of $\lambda$ can be expressed as follows:

$$\ln L(\lambda) = C_0 + \sum_{l=1}^k \sum_{j=1}^{n_l} \sum_{i=1}^m \lambda \sigma(\mu(x_i)) \Delta Y_{ijl} - \lambda \sum_{l=1}^k \sum_{j=1}^{n_l} \sum_{i=1}^m \lambda \sigma(\mu(x_i)) \Delta t_{ijl},$$

where $C_0 = \sum_{l=1}^k \sum_{j=1}^{n_l} \sum_{i=1}^m \ln c(\Delta Y_{ijl}|\lambda, \Delta t_{ijl})$. Then, $\hat{\lambda}$ can be obtained directly by maximizing Equation (5). As $m$ and $n_l$ are sufficiently large for all $l = 1, \ldots, k$, by the delta method and the invariance property of MLE, $\hat{\xi}_q$ is asymptotically normal with mean $\xi_q$ and the approximate variance (the proof is given in Appendix S1)

$$\text{Avar}(\hat{\xi}_q|x, p) = \frac{1}{(f(\hat{\xi}_q))^2 N} \left( \frac{h_{1,q}^2 e^{b(\hat{\xi}_q) - 2} \sum_{l=1}^k e^{-b(d-2)x_l^2} \sigma_l^2 p_l}{\sum_{l=1}^k e^{-b(d-2)x_l^2} (x_l - x_u)^2 \sigma_l^2 \sigma_u^2 p_u p_e} + 2\lambda h_{2,q}^2 \right),$$

where $f(\hat{\xi}_q)$ is the pdf of the product’s lifetime distribution at $\hat{\xi}_q$ under normal use condition, $h_{1,q} = \frac{\partial f(\hat{\xi}_q)}{\partial a}$, and $h_{2,q} = \frac{\partial f(\hat{\xi}_q)}{\partial b}$. Note that $h_{1,q}$ and $h_{2,q}$ are independent of $\{p_l\}_{l=1}^k$. Therefore, the optimum allocation for the ED-ADT problem can be further expressed as the following constrained optimization problem.

Minimize

$$G(x, p) = \sum_{l=1}^k e^{-b(\hat{\xi}_q) - 2} x_l^2 \sigma_l^2 p_l,$$

subject to

$$\sum_{l=1}^k p_l = 1, p_l \geq 0, 1 \leq l \leq k,$$

and

$$0 \leq x_1 < x_2 < \cdots < x_k = 1.$$ 

Note that the objective function is independent of the distribution of $\hat{\xi}_q$. In practice, $k = 2$ (two-level) and $k = 3$ (three-level) are commonly used for designing an ADT plan. In the following two subsections, we will address the optimum ED-ADT allocation problems whether the testing stress levels are prefixed or not.

4.2 Optimum ED-ADT Allocation Rule When All Stresses Are Prefixed

Case a: $k = 2$

Prefixing $x_1$, and substituting $x_2 = 1 - p_2$ into Equation (7), it is easy to have the following result by taking derivative with respect to $p_1$.

Result 1.

For a two-level ED-ADT allocation problem, the optimum setting for $p_1$ is

$$p_1^* = 1 + x_1(t_1^*/t_2^*)^{1/2} e^{b(\hat{\xi}_q)(1-x_1)/2}.$$ 

Setting $p_2^* = 1 - p_1^*$, the optimum allocations for the Wiener ($d = 0$), Gamma ($d = 2$), and IG processes ($d = 3$) are given in Table 2. It shows an interesting result that the optimum allocation for Gamma process is independent of $b$ (which means that no matter how large the value of $b$ is, the allocation always remains the same). Furthermore, when $t_1^* = t_2^*$, the result for $d = 0$ is essentially the same as Equation (9) of Lim and Yum (2011).

Case b: $k = 3$
Applying the method of Karush–Kuhn–Tucker (KKT) conditions (Bigi and Castellani 2004) to minimize the constrained optimization problem in Equations (7) and (8), Appendix S2 shows that no nontrivial optimum allocation rule \((p^0_1, p^0_2, p^0_3)\) exists such that \(p^0_i > 0, 1 \leq i \leq 3\), and \(\sum_{i=1}^{3} p^0_i = 1\). Therefore, for a three-level allocation problem, we only need to allocate all test units into two out of three stresses and we have the following result:

Result 2.

For a three-level ED-ADT allocation problem, the optimum allocation rule is

\[
(p^*_1, p^*_2, p^*_3) = \begin{cases} 
(0, (1 + r_{23})^{-1}, r_{23}(1 + r_{23})^{-1}), & \text{if } g_{21} + g_{13} < g_{23} \\
((1 + r_{13})^{-1}, 0, r_{13}(1 + r_{13})^{-1}), & \text{if } g_{21} + g_{13} > g_{23} \\
((1 + r_{12})^{-1}, r_{12}(1 + r_{12})^{-1}, 0), & \text{if } g_{13} + g_{32} < g_{12}
\end{cases}
\]

where

\[
r_{uv} = (x_u/x_v)(\sqrt{\tau_u^*/\tau_v^*}) \exp\{b(d - 2)(|x_u - x_v|)/2\},
\]

and

\[
g_{uv} = |x_u - x_v|\sqrt{\tau_u^*/\tau_v^*} \exp\{-b(d - 2)(x_u + x_v)/2\},
\]

for \(u, v \in \{1, 2, 3\}\).

Appendix S2 provides the proof.

Note that \(g_{uv}\) stated in Equation (11) can be viewed as a generalized distance between \(x_u\) and \(x_v\). For all \(i \neq j \neq k \in \{1, 2, 3\}\), \(x_j\) is not adopted if \(g_{ij} < g_{ik}\). That is, the generalized distances violate the rule of triangle inequality. Result 2 demonstrates that \(g_{21} + g_{13} < g_{23}\) and \(g_{13} + g_{32} < g_{12}\) are the conditions for the cases of \(p^*_1 = 0\) and \(p^*_3 = 0\), respectively. For the case of \(p^*_2 = 0\) in Result 2, we can explain in the following way. Note that \(p^*_2 = 0\) implies that \(p^*_1 \neq 0\) and \(p^*_3 \neq 0\). Therefore, \(g_{21} + g_{13} > g_{23}\) and \(g_{13} + g_{32} > g_{12}\) are two conditions for \(p^*_2 = 0\). The results are similar to Peng (2012).

Taking the role of \(d\) into consideration, some interesting results can be further obtained from Result 2.

Result 3.

For a three-level ED-ADT allocation problem,

1. if \(d \geq 2\) and \(t^*_1 \geq t^*_2 \geq t^*_3\), then the optimum allocation rule is the following

\[
(p^*_1, p^*_2, p^*_3) = \begin{cases} 
(0, (1 + r_{23})^{-1}, r_{23}(1 + r_{23})^{-1}), & \text{if } g_{21} + g_{13} < g_{23} \\
((1 + r_{13})^{-1}, 0, r_{13}(1 + r_{13})^{-1}), & \text{if } g_{21} + g_{13} > g_{23} \\
((1 + r_{12})^{-1}, r_{12}(1 + r_{12})^{-1}, 0), & \text{if } g_{13} + g_{32} < g_{12}
\end{cases}
\]

2. if \(d \geq 2\) and \(t^*_1 \geq t^*_3 \geq t^*_2\), then the optimum allocation rule is the following

\[
(p^*_1, p^*_2, p^*_3) = \begin{cases} 
((1 + r_{13})^{-1}, 0, r_{13}(1 + r_{13})^{-1}), & \text{if } g_{13} + g_{32} > g_{12} \\
((1 + r_{12})^{-1}, r_{12}(1 + r_{12})^{-1}, 0), & \text{if } g_{13} + g_{32} < g_{12}
\end{cases}
\]

Appendix S3 provides the proof.

Result 3 can be applied in the following way. When the underlying degradation process comes from (a) the compound Poisson or Wiener processes (with \(t^*_1 \leq t^*_2 \leq t^*_3\)), then the highest stress should always be used; (b) the inverse Gaussian process (with \(t^*_1 \geq t^*_2 \geq t^*_3\)), then the lowest stress should always be used; and (c) the Gamma process (\(d = 2\)), the optimum allocation rule flexibly depends on the ordering sequences of \(\{t^*_i\}_{i=1}^{3}\), as shown in Result 3. Note that, if \(t^*_1 = t^*_2 = t^*_3\), the condition \(g_{21} + g_{13} > g_{23}\) and \(g_{13} + g_{32} > g_{12}\) will be automatically satisfied, and we have \((p^*_1, p^*_2, p^*_3) = ((1 + x_1)^{-1}, 0, x_1(1 + x_1)^{-1})\). That is, the middle stress should not be used in planning an ADT for the Gamma process.

Note that all \(g_{uv}\) in Equation (11) are the functions of \(b\) and \(d\) when \([x_i]_{i=1}^{3}\) and \([t^*_i]_{i=1}^{3}\) are prefixed. Therefore, the values of \(b\) and \(d\) play important roles for determining the optimum allocation rule and we will discuss it later in Section 5.

4.3 Optimum ED-ADT Allocation Rule for \(k = 2\) When One Stress Level Is Prefixed

In case (i) of Result 3, when the testing stresses are prefixed, the experimenters face the problem that the highest stress level should be used, whereas the lowest stress level may be either \(x_L = x_1\) or \(x_L = x_2\). Conversely, in case (ii) of Result 3, we may face a dual problem in which the lowest stress level is fixed, even as the highest stress level remains to be determined. Furthermore, in practical applications, the experimenters may also face the problem that some testing stresses are not fixed in advance. That is, the combination of stress levels and their corresponding sample-size allocation should be considered simultaneously. In the following, we propose a proactive way to determine the optimum ED-ADT allocation rule.

When \([x_i]_{i=1}^{3}\) are prefixed, Appendix S2 shows that there is no nontrivial optimum allocation rule \((p^0_1, p^0_2, p^0_3)\) such that \(p^0_i > 0, 1 \leq i \leq 3\), and \(\sum_{i=1}^{3} p^0_i = 1\). Now, if \([x_i]_{i=1}^{3}\) are unknown decision variables, we need to impose another constraint on the Lagrange function stated in Appendix S2. Therefore, the above result still holds and we only need to consider the case of two-level ED-ADT allocation problem.

For \(\omega > 0\), let the Lambert W function (Corless et al. 1996) denote the inverse function of \(z = \omega e^{\omega}\). That is, \(\omega = W(z)\).
Set $\Omega_1 = 1 + W(e^{-1}\sqrt{t_L^*/t_H^*})$, and $\Omega_2 = 1 + W(e^{-1}\sqrt{t_H^*/t_L^*})$. Then, we have the following result.

**Result 4.**

1. Prefixed the highest stress ($x_H = 1$), the optimum ED-ADT allocation rule is $(x_H^*, x_L^*, p_L^*, p_H^*)$, where

$$x_L^* = \begin{cases} 1 - \frac{\Omega_1}{(b^2 - d)} & \text{if } d < 2 \left(1 - \frac{\Omega_1}{\Omega_2}\right) \\ 0 & \text{if } d \geq 2 \left(1 - \frac{\Omega_1}{\Omega_2}\right) \end{cases},$$

(12)

$p_L^* = (1 + r_{LH})^{-1}$ and $p_H^* = 1 - p_L^*$.

Note that $r_{LH}$ (which stated in Equation (10)) is evaluated at $x_L = x_H^*$ and $x_H = 1$.

2. Prefixed the lowest stress ($x_L = x_a$), the optimum ED-ADT allocation rule is $(x_a, x_H^*, p_L^*, p_H^*)$, where

$$x_H^* = \begin{cases} 1 & \text{if } d \leq 2 \left(1 + \frac{\Omega_2}{b(1-x_a)}\right) \\ x_a + \frac{\Omega_1}{b(d-2)} & \text{if } d \geq 2 \left(1 + \frac{\Omega_2}{b(1-x_a)}\right) \end{cases}.$$  

(13)

$p_L^* = r_{LH}/(1 + r_{LH})$ and $p_H^* = 1 - p_L^*$.

Appendix 1 provides the proof.

**Remark 1:** In Equation (12), if $x_L^* = 0$, then $p_L^* = 1$ (which follows directly from Result 1). This result means that there is no need to conduct an ADT when the underlying degradation model follows an ED process with $d \geq 2 \left(1 - \frac{\Omega_1}{\Omega_2}\right)$. However, it will suffer from a longer testing time under the normal use condition. To handle this controversy, based on the experimenter’s experience, we may set an appropriate lower bound for the lower stress. We will adopt this approach in Section 5.3.

**Remark 2:** For the Wiener process (i.e., $d = 0$), Lim and Yum (2011) used a numerical computation to obtain the optimum settings of $x_L^*$ and $p_L^*$, which turned out to be a special case of Result 4.

5. TWO ILLUSTRATIVE EXAMPLES

In the following, we use two examples to illustrate the proposed procedure.

5.1 Example 1: Stress Relaxation Data (Yang 2007)

Stress relaxation is the loss (in percent) of stress in a component subjected to a constant strain over time (hours). Due to excessive stress relaxation, the contacts of electrical connectors often fail. A connector is considered to have failed when its stress relaxation exceeds a predefined failure level $\omega_0$ (say, $\omega_0 = 30\%$). To assess the reliability of electrical connectors at the normal-use condition ($S_0 = 40^\circ$C), a sample of 18 test units was randomly selected and divided into three equal groups, which were tested at 65$^\circ$C, 85$^\circ$C, and 100$^\circ$C. The dataset is given in Ye et al. (2014) and the degradation paths of the losses of stress relaxation at three temperatures were shown in Figure 1(a). For illustrative purposes, the original time scale $(t)$ was taken as an appropriate power transformation with $t = s^{0.45}$ (Ye et al. 2014). Figure 1(b) shows the transformed degradation paths and their corresponding TTTs for the following three temperatures: $t_0^* = 34.68$, $t_1^* = 33.41$, and $t_2^* = 23.87$. The plots demonstrate that the degradation paths have linear patterns for all three stress levels.

The Arrhenius model is adopted in this study. That is, $X(S_i) = 1/(273.15 + S_i)$. Then, the standardized stresses are the following:

$$x_1 = \frac{X(65) - X(40)}{X(100) - X(40)} = \frac{1/338.15 - 1/313.15}{1/373.15 - 1/313.15} = 0.46,$$

$$x_2 = \frac{X(85) - X(40)}{X(100) - X(40)} = \frac{1/358.15 - 1/313.15}{1/373.15 - 1/313.15} = 0.78,$$

and $x_3 = 1$. By using the package “nweedle” in R software (R Development Core Team 2013), the optimum maximum likelihood estimates (MLEs) of $a$, $b$, $\lambda$, and $d$ can be obtained via the maximization of Equation (5). The second column of Table 3 shows that the compound Poisson process with $d^* = 1.4$ is the best ED model for the stress relaxation data. In addition, three sub-models with $d = 0$, 2, and 3 are considered. The MLEs for unknown parameters $\Lambda = (a, b, \lambda)$ and the values of likelihood ratio (LR) test $(-2(\ln L(A_0) - \ln L(\hat{A})))$ for $H_0 : d = d_0$, with $d_0 = 0, 2$, and 3, are also given in Table 3. The results show that all of the above null hypotheses are rejected at the 0.05 level.
Table 3. MLEs and optimum allocation rules for some well-known ED processes

| Compound | $d^*$ | $d_0 = 0$ | $d_0 = 2$ | $d_0 = 3$ |
|----------|-------|-----------|-----------|-----------|
| Poisson  | 1.4985| -2.0709   | -1.9135   | -1.8704   |
| Wiener   | 1.8266| 1.9745    | 1.7795    | 1.7159    |
| Gamma    | 2.2019| 3.7481    | 1.4434    | 0.5459    |
| IG       |       |           |           |           |
| LR test  | 40.1732 | 8.5496*   | 57.4188*  |
| $H_0 : d = d_0$ | 0.71, 0, 0.29 | 0.84, 0.16 | 0.64, 0, 0.36 | 0.53, 0, 0.47 |

Furthermore, the Q-Q plot in Figure 2 also demonstrates that the ED model with $d^* = 1.4$ is better than the other three sub-models.

Now, if we treat the MLEs as the true process parameters, the optimum ADT allocation rule for the ED process with $d^* = 1.4$ is

$$p_1^* = \frac{1}{1 + 0.46e^{1.8266(1.4 - 2)(1 - 0.46)/2} \sqrt{34.687/23.87}} = 0.71,$$

and $p_3^* = 0.29$, since $g_{21} + g_{13}(56.07) > g_{23}(16.48)$ and $g_{13} + g_{32}(51.06) > g_{12}(21.49)$.

If 50 test units are randomly selected for an ADT, then it is necessary to allocate 36 test units to the stress level $65^\circ C$ and 14 test units to the stress level $100^\circ C$, respectively. Moreover, for illustrative purposes, we treat the MLEs as the true process parameters for the other three sub-models and their corresponding optimum ADT allocation rules are shown in the last row of Table 3. In addition, based on Result 2, Figure 3 also demonstrates the optimal allocation strategy under various combinations of $(b, d)$. It shows that the settings of $(b, d)$ in Table 3 are all inside the region of $p_2^* = 0$. Therefore, the optimum allocation rules of all four models illustrated in Table 3 are always arranged in the smallest and largest stresses.
5.2 Example 2: Device B Data (Meeker and Escobar 1998, p. 564)

Power drop of Device B is the loss (in dB) of power in a sample of integrated circuit devices over time. Failure is defined as power output more than 0.5 dB below the initial output (say, ω = 0.5 dB) and the operating temperature $S_0 = 80\degree C$. A sample of 33 test units was randomly selected and divided into three groups, which were tested at 150\degree C, 195\degree C, and 237\degree C. The dataset can be found in the software of JMP 10 (SAS Institute Inc. 2013). Figure 4(a) shows the original degradation paths. According to Meeker and Escobar (1998), the following model is appropriate to describe the degradation paths of Device-B power drop.

$$Y(t|x_l) = D_\infty \times (1 - e^{-\mu(x_l)t}), \quad l = 1, 2, 3.$$  

Figure 4(b) shows the transformed degradation paths $(-\ln (1 - Y(t|x_l)/D_\infty))$ over time (hours), where $D_\infty = -1.42$ (which was adopted directly from eq. (21.4) in the book of Meeker and Escobar 1998). Note that two measurement points (with power drop in db, which are less than $-1.42$) are deleted to ensure the linearity of the transformed data. Their corresponding TTTs for these three temperatures are 4000, 2000, and 1000 hr, respectively. The Arrhenius model is used and the standardized stresses are $x_1 = 0.54$, $x_2 = 0.80$, and $x_3 = 1$.

Similar to the analysis in Table 3, it demonstrates that the ED model with $d^* = 2.22$ is the best model for Device B data. In addition, the Q-Q plot in Figure 5 also demonstrates this result. The MLEs of two sub-models, the Gamma and IG processes, and their corresponding optimum allocation rules are also given in Table 4.

Again, Figure 6 also demonstrates the optimal allocation strategy under various combinations of $(b, d)$. It shows that all settings of $(b, d)$ in Table 4 fall inside either in the region of $p^*_2 = 0$ or $p^*_3 = 0$. Typically, the largest stress of the IG process in this example is not adopted for an ADT experiment.

$$d^* = 2.22$$

$$d = 2$$

$$d = 3$$

Figure 5. The Q-Q plot for stress relaxation data.
5.3 Optimum Two-Level Allocation Rule When One Stress Level Is Not Prefixed

Adopting the estimates of two examples in Tables 3 and 4 as the true process parameters, we illustrate the proposed method stated in Section 4.3.

Example 1 (continued):
Prefixing the highest stress $S_H = 100^\circ C$, $\tau^*_L = 34.68$, and $\tau^*_H = 23.87$, we have

$$\Omega_1 = 1 + W(e^{-1} \sqrt{34.68/23.87}) = 1.32.$$  

1. Under the case of Wiener process, the optimum setting for the lowest stress level (from Equation (12)) is

$$x^*_L = 1 - \frac{2\Omega_1}{b(2 - d)} = 1 - \frac{2 \times 1.32}{1.9745 \times 2} = 0.33,$$

since $2 \left(1 - \frac{\Omega_1}{b}\right) = 2 \left(1 - \frac{1.32}{1.9745}\right) = 0.66 > 0.$

2. Under the case of ED degradation model (with $d = 1.4$), we have $x^*_L = 0$, since $2(1 - \frac{\Omega_1}{b}) = 2(1 - \frac{1.32}{1.9745}) = 0.55 < 1.4$. As mentioned in Remark 1 of Result 4 and setting a lower bound, say $x^*_L = 0.46$, we use $65^\circ C$ and $100^\circ C$ for an ADT and the optimum allocations are $p^*_L = 0.71$ and $p^*_H = 0.29$, respectively.

3. Under the cases of Gamma and IG processes, we have $x^*_L = 0$. Similarly to the case (b), we use $65^\circ C$ and $100^\circ C$ for an ADT and the optimum allocations for the two processes are $(p^*_L, p^*_H) = (0.64, 0.36)$ and $(p^*_L, p^*_H) = (0.53, 0.47)$, respectively.

Example 2 (continued):
Prefixing the lowest stress $x_L = 0.54$, $\tau^*_L = 4000$, and $\tau^*_H = 1000$, we have

$$\Omega_2 = 1 + W \left(e^{-1} \sqrt{1000/4000}\right) = 1.16.$$  

1. Under the case of IG process, the optimum setting for the highest stress level (from Equation (13)) is

$$x^*_H = x_L + \frac{2 \times \Omega_2}{b(4 - 2)} = 0.54 + \frac{2.32}{7.2482 \times 1} = 0.86,$$

since

$$2 \left(1 + \frac{1.16}{b(1 - x_L)}\right) = 2 \left(1 + \frac{1.16}{7.2482(1 - 0.54)}\right) = 2.70 < 3.$$

That is, $150^\circ C$ and $207^\circ C$ are used for an ADT and the corresponding optimum allocations are $p^*_L = 0.15$ and $p^*_H = 0.85$, respectively.

2. Under the cases of ED degradation model with $d = 2.22$, the optimum setting for the highest stress level (from Equation (13)) is $x^*_H = 1$, since

$$2 \left(1 + \frac{\Omega_2}{b(1 - x_L)}\right) = 2 \left(1 + \frac{1.16}{7.1126(1 - 0.54)}\right) = 2.71 > 2.21.$$  

That is, we should use $150^\circ C$ and $237^\circ C$ for an ADT and the corresponding optimum allocations are $p^*_L = 0.40$ and $p^*_H = 0.60$, respectively.

Table 4. MLEs and optimum allocation rules for some well-known ED processes

| Process | $d^* = 2.22$ | $d = 2$ | $d = 3$ |
|---------|-------------|---------|---------|
| $\hat{a}$ | $-13.1868$ | $-13.1509$ | $-13.2714$ |
| $\hat{b}$ | $7.1126$ | $7.0618$ | $7.2482$ |
| $\hat{\lambda}$ | $0.0065$ | $0.0374$ | $7.2329 \times 10^{-6}$ |

LR test for $H_0: d = d_0$ $(\hat{p}^*_1, \hat{p}^*_2, \hat{p}^*_3)$ $(0.40, 0, 0.60)$ $(0.48, 0, 0.52)$ $(0.29, 0.71, 0)$

Table 5. REs of the allocation rules $\hat{p}_1$ and $\hat{p}_2$

| Process | Compound Poisson | $d = 0$ | $d = 2$ | $d = 3$ |
|---------|----------------|---------|---------|---------|
| $\Gamma = 1.4$ | $(0.71, 0.29)$ | $(0.84, 0.16)$ | $(0.64, 0.36)$ | $(0.53, 0.47)$ |
| RE of rule | $0.852$ | $0.613$ | $0.924$ | $0.996$ |

That is, $58^\circ C$ and $100^\circ C$ are used for an ADT and the corresponding optimum allocations are $p^*_L = 0.90$ and $p^*_H = 0.10$, respectively.
Under the cases of Gamma process, the optimum for the high stress level is $x^*_H = 1$. That is, we should use $150^\circ C$ and $237^\circ C$ for an ADT and the corresponding optimum allocations are $p^*_L = 0.48$ and $p^*_H = 0.52$, respectively.

6. THE EFFECTS OF USING NONOPTIMUM ALLOCATION RULES

In the following, we will further investigate the effects of using nonoptimum allocation rule $(\hat{x}, \hat{p})$ on the performance of $G(\hat{x}, \hat{p})$ in Equation (7). Define an index to measure the relative efficiency (RE) of using the nonoptimum allocation rule $\hat{p}$ as

$$RE = \frac{G(x^*, p^*)}{G(\hat{x}, \hat{p})},$$

where $(x^*, p^*)$ is the optimum ED-ADT allocation.

6.1 Comparison With Two Well-Known Plans

Under $k = 2$

When the testing stresses are prefixed, the equal-sample-size allocation rule (i.e., $\hat{p}_1: (p_1, p_2) = (0.5, 0.5)$) and the 2:1 allocation rule (i.e., $\hat{p}_2: (p_1, p_2) = (0.67, 0.33)$) are considered here. Adopting the settings in Example 1 (i.e., $S_L = 65^\circ C$, $S_H = 100^\circ C$, $\tau^*_L = 34.68$, and $\tau^*_H = 23.87$), Table 5 shows the REs of these rules under well-known ED processes. It demonstrates that $p^*_1$ is a decreasing function of $d$. Typically, when $d = 0$, we have $p^*_1 = 0.84$, which is far away from $\hat{p}_1 = 0.5$ or $\hat{p}_1 = 0.67$. Therefore, the heuristic rule $\hat{p}_1$ (or $\hat{p}_2$) for this example is not a good allocation rule if the underlying degradation model follows a Wiener process.

6.2 Comparison With Some Well-Known Compromise Plans for $k = 3$

Section 4 demonstrates that a two-level experiment is the best strategy for planning an ED-ADT. However, two-level designs do not allow us to check the validity of the relationship between $\ln \mu(x_1)$ and the stress level $x_1$. To remedy this weakness, several three-level compromise plans have been widely proposed in the literature (Meeker 1984; Meeker and Hahn 1985). Let $x_1$, $x_2$, and $x_3^* = 1$ denote the standardized stress levels. In the following, we always adopt that $x_2 = (x_1 + x_3^*)/2$. Three-candidate compromise plans are considered as follows:

$R_1$: The proportion of allocation $4:2:1$ is used, while $\tilde{x}_1$ (the optimum stress level for $x_1$) should be determined.

$R_2$: Prefixed $p_2 = 0.1$, determine the optimum settings $\tilde{x}_1$ and $\tilde{p}_1$, simultaneously.

$R_3$: Prefixed $p_2 = 0.2$, determine the optimum settings $\tilde{x}_1$ and $\tilde{p}_1$, simultaneously.

For $i = 1, 2, 3$, substitute the optimum settings $\tilde{x}_1$ and $\tilde{p}_1$ of the rule $R_i$ into the denominator of Equation (14), then it allows us to measure the RE of the rule $R_i$. For illustrative purposes, we only address the REs of these rules when the underlying degradation process follows a Wiener process. Under various combinations of $1.5 \leq b \leq 3$, Table 6 shows the optimum settings of these compromise plans and their corresponding REs with respect to the optimum allocation rule $R^*$ (which is obtained from Equation (12) directly).

From Table 6, it demonstrates that the REs for all available plans are greater than 82%. In addition, the RE performance of $R_1$ is worse than $R_2$ and $R_3$, while the performance of $R_2$ is slightly better than that of $R_3$, and its REs are greater than 94%, for all $1.5 \leq b \leq 3$. Note that if we choose a compromise rule with $p_2 = 0.05$, it will have a better RE performance, comparing with the compromise rule with $p_2 = 0.1$. However, it is more difficult to validate the relationship between $\ln \mu(x)$ and the standardized stress $x$ due to only the half of the sample size is used. Rule $R_{22}$ serves as a good compromise strategy based on this viewpoint and this result is very similar to the case of optimum ALT allocation plans, as reported in Meeker and Hahn (1985).

7. CONCLUSION

By minimizing the asymptotic variance of the estimated $q$ quantile of the product’s lifetime, this article presents analytical results for the optimum allocation rules of two-level and three-level ED-ADT allocation problems when the underlying degradation path comes from the ED class. Results 2 and 3 provide us the optimum allocation rules of planning an ADT when the testing stress levels are prefixed. The results demonstrate that a two-level experiment is the best strategy for planning ED-ADT. Result 4 provides us a more proactive way to simultaneously determine both the testing stress levels and their corresponding sample size allocations. A two-level design does not allow us to check the validity of the relationship between $\ln \mu(x)$ and the stress level $x$. To overcome this difficulty, several well-known three-level compromise plans are considered and we also address their REs with respect to the optimum allocation rules.

Table 6. REs of three candidate compromise plans under the case of Wiener process

| $b$  | $R^*$ | $R_1$ | $R_2$ | $R_3$ |
|------|-------|-------|-------|-------|
|      | $x^*_1$ | $p^*_1$ | $\tilde{x}_1$ | $RE$ | $\tilde{x}_1$ | $p^*_1$ | $RE$ | $\tilde{x}_1$ | $p^*_1$ | $RE$ |
| 1.5  | 0.12  | 0.96  | 0.06  | 0.82 | 0.09  | 0.86  | 0.94 | 0.04  | 0.75  | 0.89 |
| 1.75 | 0.24  | 0.93  | 0.17  | 0.84 | 0.22  | 0.82  | 0.94 | 0.18  | 0.72  | 0.89 |
| 2    | 0.34  | 0.90  | 0.25  | 0.85 | 0.31  | 0.80  | 0.94 | 0.28  | 0.69  | 0.89 |
| 2.25 | 0.41  | 0.88  | 0.32  | 0.86 | 0.39  | 0.78  | 0.94 | 0.36  | 0.68  | 0.89 |
| 2.5  | 0.47  | 0.87  | 0.38  | 0.86 | 0.45  | 0.77  | 0.94 | 0.42  | 0.66  | 0.90 |
| 2.75 | 0.52  | 0.86  | 0.43  | 0.86 | 0.50  | 0.76  | 0.95 | 0.47  | 0.65  | 0.90 |
| 3    | 0.56  | 0.85  | 0.48  | 0.86 | 0.54  | 0.75  | 0.95 | 0.51  | 0.64  | 0.90 |
two-level allocation rule. The results demonstrate that the REs of the rule $R_2$ (with $p_2 = 0.1$) are greater than 94% under various settings of parameter $b$, and this rule is a good three-level compromise allocation rule.

At the end of this article, the following concluding remarks are offered.

1. Figures 2 and 5 demonstrate that the Wiener, Gamma, and inverse Gaussian processes are the potential degradation models of these two examples. Therefore, an interesting scenario arises: “if the correct process follows an ED model with $d^* = 1.4$ in Example 1 (or $d^* = 2.22$ in Example 2), what is the RE of using an incorrect degradation model (such as Wiener, Gamma, or IG model)?” The following table summarizes the RE of using incorrect degradation models. The RE performances of these three wrong models are acceptable in Example 1. However, Example 2 demonstrates the model misspecification cannot be negligible, especially when a wrong model (IG process) is adopted.

| Incorrect | Example 1 ($d^* = 1.4$) | Example 2 ($d^* = 2.22$) |
|-----------|--------------------------|--------------------------|
| Wiener    | 0.89                     | —                        |
| Gamma     | 0.98                     | 0.97                     |
| IG        | 0.89                     | 0.60                     |

2. For $l = 1, \ldots, k$, when the test units in $x_l$ have unequal numbers of measurements $m_l$, then Equation (6) shall be slightly modified as

\[
\text{AVar}(\hat{\xi}_q | x, p) = \frac{1}{(f(\hat{\xi}_q))^2} \left\{ \frac{h^2_{1,q} e^{q(d-2)}}{h \sum_{k=1}^{v} e^{-q(k+1)}} + \frac{\lambda^2 h^2_{2,q}}{\sum_{l=1}^{v} m_l p_l} \right\} .
\]

This equation means that the “quantile” will affect optimum allocation rules only when numbers of measurements are unequal. Therefore, “how to analytically determine the optimum allocation rule with unequal numbers of measurements” shall be a challenging issue for further research.

3. Except for simultaneously determining the testing stress levels and their corresponding sample size allocations, we may also interest in finding the optimum settings of the total sample size ($N$) and all TTTTs of testing stresses. By substituting the optimum ED-ADT allocation rule (stated in Result 4) into Equation (6), then AVar$(\hat{\xi}_q)$ turns out to be a function of $N$ and TTTTs. Similar to Yu and Tseng (1999), Tsai, Tseng, and Balakrishnan (2012), and Boulanger and Escobar (1994), under the budget constraint of experimental cost, it is not difficult to obtain the optimum settings for $N$ and TTTTs numerically.

4. Applying the KKT conditions to a similar version of Result 2 for the case of $k = 4$, it is easy to show that nontrivial optimum allocation does not exist. More specifically, under $t^*_1 = t^*_2 = \ldots = t^*_k$, we have the following result:

\[
\begin{cases}
(p^*_1, p^*_2, 0, 0), \text{ if } g_{13} + g_{32} < g_{12} \text{ and } g_{14} + g_{42} < g_{12} \\
(0, p_2, p_3, 0), \text{ if } g_{12} + g_{13} < g_{23} \text{ and } g_{24} + g_{43} < g_{23} \\
(0, 0, p_3, p_4), \text{ if } g_{31} + g_{14} < g_{34} \text{ and } g_{32} + g_{24} < g_{34} \\
(p^*_1, 0, p^*_3, 0), \text{ if } g_{21} + g_{13} > g_{23}, g_{13} + g_{32} > g_{12}, \text{ and } g_{14} + g_{43} < g_{13} \\
(0, p_2, 0, p_4), \text{ if } g_{32} + g_{24} > g_{34}, g_{24} + g_{43} > g_{23}, \text{ and } g_{31} + g_{14} < g_{34} \\
(p^*_1, 0, 0, p^*_4), \text{ if } g_{21} + g_{14} > g_{24}, g_{14} + g_{43} > g_{13}, \text{ and } g_{41} + g_{14} > g_{43}, \text{ and } g_{14} + g_{43} > g_{13},
\end{cases}
\]

where

\[
(p^*_u, p^*_v) = \left( \frac{1}{1 + r_{uv}}, \frac{r_{uv}}{1 + r_{uv}} \right), \text{ for all } u, v \in \{ 1, 2, 3, 4 \}.
\]

This result is more generalized than Result 2 and it can be reduced to Result 2 if the above expressions with sub-index 4 are ignored completely. The proof for the case of $k \geq 5$ is a mathematically interesting issue, although conducting an ADT with $k \geq 5$ is not very practical.

5. In this study, we did not consider the issue of censored observations (at the end of test) and we did not use the combined degradation-failure model (such as Pettit and Young 1999; Padgett and Tomlinson 2004; Park and Padgett 2005; Yum, Lim, and Seo 2007) to address our ADT planning problem. Of course, it shall be an interesting research topic for further research.

6. Accelerated destructive degradation tests have also been widely used (Shi, Escobar, and Meeker 2009). In the future research, we can deal with allocation problems for an accelerated destructive degradation test based on the ED degradation model.

SUPPLEMENTARY MATERIALS

The online supplementary materials include the derivation of the asymptotic variance of the estimated $q$ quantile of lifetime in Equation (6) and the proofs of Results 2–4 in Section 4.

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