Zero forcing in iterated line digraphs

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Abstract

Zero forcing is a propagation process on a graph, or digraph, defined in linear algebra to provide a bound for the minimum rank problem. Independently, zero forcing was introduced in physics, computer science and network science, areas where line digraphs are frequently used as models. Zero forcing is also related to power domination, a propagation process that models the monitoring of electrical power networks.

In this paper we study zero forcing in iterated line digraphs and provide a relationship between zero forcing and power domination in line digraphs. In particular, for regular iterated line digraphs we determine the minimum rank/maximum nullity, zero forcing number and power domination number, and provide constructions to attain them. We conclude that regular iterated line digraphs present optimal minimum rank/maximum nullity, zero forcing number and power domination number, and apply our results to determine those parameters on some families of digraphs often used in applications.

Keywords: Minimum rank; Zero forcing; Power domination; Iterated line digraphs.
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1 Introduction

Zero forcing is a propagation process on a graph, independently introduced in linear algebra, physics, computer science and network science. In linear algebra, zero forcing was introduced in [2] to express a bound for the minimum rank problem, which consists of minimizing the rank of a matrix whose pattern of non-zero entries is determined by a given graph. The minimum rank problem appears frequently in engineering, where the order or the complexity of models can often be expressed as the rank of a matrix (see [16]). In physics, zero forcing was introduced to study controllability of quantum systems (see [9]); in computer science, it appears as the fast-mixed search model (see [32]) for some pursuit-evasion games (see [25]);
in network science, it models the spread of a disease over a population, or of an opinion in a social network (see [14]).

In addition to its intrinsic relation to minimum rank, zero forcing is closely related to power domination, a graph theory concept introduced in [20] to optimize the monitoring process of electrical power networks. From the definitions of power domination and zero forcing, it follows that the closed out neighborhood of a power dominating set is a zero forcing set, and a stronger relationship between zero forcing and power domination was established in [7].

Zero forcing, minimum rank and power domination are all $NP$-hard problems, as proven in [1], [10], and [20], respectively. Thus, it is important to obtain bounds for the minimum rank, the zero forcing and the power domination numbers, as well as closed formulas to calculate them for families of graphs. Although power domination and zero forcing where introduced on undirected graphs, they were extended to digraphs in [1] and [5], respectively. Further results on zero forcing on digraphs can be found in [5], [15], [21], [23] and [31]. Power domination in digraphs has not been so thoroughly explored, but recently zero forcing and power domination for de Bruijn and Kautz digraphs was studied in [19]. De Bruijn and Kautz digraphs are iterated line digraphs of the complete digraph, with and without loops, respectively. In this work, we extend the results in [19] to zero forcing and power domination of iterated line digraphs of any regular digraph.

The line digraph has been used in a broad range of disciplines, but its large number of applications precludes us from including an exhaustive summary here. In the context of this work, since the line digraph of a digraph described by a unitary matrix can also be described by a unitary matrix, iterated line digraphs are used to obtain arbitrarily large digraphs described by unitary matrices (see [24] and [28]). Such digraphs are frequently used to model quantum systems in physics, chemistry, and engineering (see [22]), and it was precisely to control quantum systems that zero forcing was introduced in physics. Indeed, line digraphs of digraphs described by unitary matrices are used in quantum computation and in the study of quantum walks (see [28]), as their statistical dynamics models that of random matrix theory (see [24] and [28]). In particular, the use of regular quantum graphs was studied in [29] and in [30], as the line digraph of a regular digraph is the digraph of the transition matrix of a coined quantum walk. In addition, line digraphs have been used in information theory as solutions to the *index coding with side information problem* (see [13]), where the minimum rank of a digraph represents the length of an optimal scalar linear solution of the corresponding instance of the problem (see [4]).

In this work, we extend to digraphs the relationship between zero forcing and power domination established for undirected graphs in [7]. We also present lower and upper bounds for the zero forcing and the power domination numbers of iterated line digraphs, and show that for regular digraphs, the corresponding lower and upper bounds coincide, providing expressions for the zero forcing and the power domination numbers. Combining our results with known properties of the minimum rank of line digraphs, we conclude that iterated line digraphs of regular digraphs present optimal properties in regards to the minimum rank, zero forcing and power domination problems. We apply our results to the Bruijn and Kautz digraphs, generalized de Bruijn and generalized Kautz digraphs, and wrapped butterflies. Through our work, we show that the relationship between minimum rank, zero forcing and
power domination is a powerful tool that permits us to combine results obtained separately for each problem to produce stronger results.

2 Definitions and notation

A digraph is a pair \( G = (V, A) \), where \( V = V(G) \) is a finite, non-empty set of vertices, and \( A = A(G) \) is a set of ordered pairs of vertices called arcs. The order of \( G \) is defined as \(|V(G)|\). An arc in the form \((u, u)\) is called a loop. The open out-neighborhood of a vertex \( v \) is \( N^+_G(v) = \{ u \in V(G) : (u, v) \in A(G) \} \) and the open in-neighborhood of \( v \) is \( N^-_G(v) = \{ w \in V(G) : (w, v) \in A(G) \} \). The closed out-neighborhood of \( v \) is \( N^+_G[v] = N^+_G(v) \cup \{ v \} \) and the closed in-neighborhood of \( v \) is \( N^-_G[v] = N^-_G(v) \cup \{ v \} \). The out-degree of \( v \) is \( d^+_G(v) = |N^+_G(v)| \) and the in-degree of \( v \) is \( d^-_G(v) = |N^-_G(v)| \). The digraph \( G \) is \( d \)-regular if \( d^+_G(v) = d^-_G(v) = d \), for every \( v \in V \). More generally, a digraph \( G \) is regular if there exists an integer \( d \) for which \( G \) is \( d \)-regular; otherwise \( G \) is said to be irregular. The maximum out-degree of \( G \) is \( \Delta^+(G) = \max\{ d^+_G(v) : v \in V(G) \} \) and the maximum in-degree of \( G \) is \( \Delta^-(G) = \max\{ d^-_G(v) : v \in V(G) \} \). The minimum out-degree of \( G \) is \( \delta^+(G) = \min\{ d^+_G(v) : v \in V(G) \} \) and the minimum in-degree of \( G \) is \( \delta^-(G) = \min\{ d^-_G(v) : v \in V(G) \} \). For a set of vertices \( T \subseteq V \), \( N^+_G[T] = \bigcup_{v \in T} N^+_G[v] \), and analogously for the other neighborhoods. We will omit the subindices when the digraph \( G \) is obvious from the context.

If \( u \) and \( v \) are two different vertices in \( G \), a path of length \( d \) from \( u \) to \( v \) is a sequence of distinct vertices \( u = x_0, \ldots, x_d = v \) such that \( (x_i, x_{i+1}) \in A(G) \) for every \( i = 0, \ldots, d - 1 \). A cycle of length \( \ell \) in \( G \), is a sequence of vertices \( x_0, \ldots, x_{\ell} \) such that \( x_0, \ldots, x_{\ell-1} \) are distinct, \( x_{\ell} = x_0 \) and \( (x_i, x_{i+1}) \in A(G) \) for every \( i = 0, \ldots, \ell - 1 \). A digraph \( G \) is strongly connected if for any two vertices \( u \) and \( v \) there is a path from \( u \) to \( v \) in \( G \). A digraph is weakly connected if its underlying graph (i.e. the graph obtained by replacing each arc \((u, v)\) or symmetric pair of arcs \((u, v), (v, u)\) by the edge \( uv \)) is connected. As suggested by their terms, strong connectivity implies weak connectivity, but they are not equivalent. If a digraph \( G \) is not weakly connected, each maximal weakly connected sub-digraph of \( G \) is a weak component of \( G \). Note that a digraph with exactly one vertex is weakly connected, so every vertex in \( G \) is in exactly one weak component. Thus, the vertex sets of all weak components of \( G \) form a partition of the vertex set of \( G \). In a weakly connected digraph that is not strongly connected, the notion of a strong component is analogous. For terminology about graphs or digraphs not defined above we refer the reader to [11].

Now we present the notion of zero forcing in digraphs, followed by its formal definition. Intuitively, zero forcing can be described through a coloring process on the vertices of a digraph. Initially, each vertex of a digraph \( G \) is arbitrarily colored in one of two colors, say blue and white. Then, apply a given color changing rule that establishes a condition for a white vertex to become blue. Iteratively apply the color changing rule until it fails to produce new blue vertices. At that moment, if all vertices in \( G \) are blue, then the initial set of blue vertices is a zero forcing set of \( G \). The zero forcing problem consists of finding a zero forcing set of minimum cardinality for a given digraph. The color changing rule to be applied in a digraph \( G \) depends on whether \( G \) has loops or not. For digraphs without loops, the color changing rule is: every white vertex that is the only white out-neighbor of a blue
vertex becomes blue. For digraphs that have at least one loop, the color changing rule is: if a white vertex is the only white out-neighbor of a vertex (blue or white), then the white vertex becomes blue. The difference in the color changing rule implies that in a digraph $G$, if there is a loop on a white vertex $v$ and all vertices in $N^+_G(v) \setminus \{v\}$ are blue, then $v$ becomes blue.

Next, we present the definition of zero forcing using a sequence of sets of vertices to describe the blue vertices after each application of the color changing rule. Note that in each application, the color changing rule is simultaneously applied to every white vertex.

Let $G = (V, A)$ be a digraph. For any non-empty set $S \subseteq V(G)$ and any non-negative integer $i$ we define $B^i(S)$ by the following rules.

1. $B^0(S) = S$.
2. If $G$ does not have any loops, then for every $i \geq 0$
   \[ B^{i+1}(S) = B^i(S) \cup \{v \in V(G) \setminus B^i(S) : \exists u \in B^i(S), N^+_G(u) \setminus B^i(S) = \{v\} \}. \]
3. If $G$ has at least one loop, then for every $i \geq 0$
   \[ B^{i+1}(S) = B^i(S) \cup \{v \in V(G) \setminus B^i(S) : \exists u \in V, N^+_G(u) \setminus B^i(S) = \{v\} \}. \]

Following [21], we say that $S \subseteq V(G)$ is a zero forcing set of $G$ if there exists a non-negative integer $m$ such that $B^m(S) = V(G)$. A minimum zero forcing set is a zero forcing set of minimum cardinality. The zero forcing number of $G$ is the cardinality of a minimum zero forcing set and is denoted by $Z(G)$. When rule 2 or rule 3 is applied, we say that $u$ forces $v$. Analogously, we say that a set $S \subseteq V$ forces a set of vertices $W$, when for every $w \in W$ there exists $u \in S$ such that $u$ forces $w$. Note that if a digraph $G$ is not weakly connected, then $G$ has $r \geq 2$ weak components $G_1, \ldots, G_r$. In this case, as observed in [3], $Z(G) = \sum_{i=1}^{r} Z(G_i)$. Since zero forcing must be studied independently in each weak component, in this paper we assume all digraphs are at least, weakly connected.

For a digraph $G = (V, A)$ of order $n$, the qualitative class of $G$, i.e. the matrix family of $G$, is the set of matrices $\mathcal{S}(G)$ defined as $\mathcal{S}(G) = \{ X \in \mathbb{R}^{n \times n} : \text{ for } i \neq j, X_{ij} \neq 0 \iff (i, j) \in A(G) \}$ if $G$ does not have loops, and as $\mathcal{S}(G) = \{ X \in \mathbb{R}^{n \times n} : X_{ij} \neq 0 \iff (i, j) \in A(G) \}$ if $G$ has at least one loop. The adjacency matrix of $G$ is the $n \times n$ matrix $A = A(G)$ where $A_{i,j} = 1$ if $(i, j) \in A(G)$ and $A_{i,j} = 0$ if $(i, j) \notin A(G)$. Then, $A(G) \in \mathcal{S}(G)$. The maximum nullity of $G$ is $\text{M}(G) = \max\{ \text{null} \ X : X \in \mathcal{S}(G) \}$, and the minimum rank of $G$ is $\text{mr}(G) = \min\{ \text{rank} X : X \in \mathcal{S}(G) \}$; clearly $\text{M}(G) + \text{mr}(G) = |V(G)|$. The concept of zero forcing models the process to force zeros in a null vector of a matrix $X \in \mathcal{S}(G)$, implying $\text{M}(G) \leq Z(G)$ [21]. As posed in [2], it is particularly interesting to identify classes of digraphs $G$ for which $\text{M}(G) = Z(G)$.

An important concept in this work is that of power domination. We first present the notion of power domination using a coloring process on the vertices of a digraph, and then give a formal definition. Initially, each vertex of a digraph $G$ is arbitrarily colored either blue or white. In power domination, there are two color changing rules. The first one is applied exactly once at the beginning of the process, and establishes that every white vertex in the out-neighborhood of a blue vertex becomes blue. Then, the coloring process continues with
the application of the second color changing rule of power domination, which coincides with the color changing rule used in zero forcing, and is also iteratively applied until it fails to produce new blue vertices. At that point, if all vertices in the digraph \( G \) are blue, then the original set of blue vertices is a \textit{power dominating set} of \( G \). The power domination problem consists of finding a power dominating set of minimum cardinality for a given digraph.

Let \( G = (V, A) \) be a digraph. For any non-empty set \( S \subseteq V(G) \) and any non-negative integer \( i \) we define \( P^i(S) \) by the following rules.

1. \( P^0(S) = S \).
2. \( P^1(S) = N^+[S] \).
3. If \( G \) does not have any loops, then for every \( i \geq 1 \)
   \[ P^{i+1}(S) = P^i(S) \cup \{v \in V(G) \setminus P^i(S) : \exists u \in P^i(S), N^+_G(u) \setminus P^i(S) = \{v\}\} . \]
4. If \( G \) has at least one loop, then for every \( i \geq 1 \)
   \[ P^{i+1}(S) = P^i(S) \cup \{v \in V(G) \setminus P^i(S) : \exists u \in V, N^+_G(u) \setminus P^i(S) = \{v\}\} . \]

We say that \( S \subseteq V(G) \) is a \textit{power dominating set} of \( G \) if there exists a non-negative integer \( t \) such that \( P^t(S) = V(G) \). A \textit{minimum power dominating set} is a power dominating set of minimum cardinality. The \textit{power domination number} of \( G \) is the cardinality of a minimum power dominating set and is denoted by \( \gamma_P(G) \). When rule 2, 3 or 4 is applied, we say that \( u \) \textit{propagates} to \( v \). Analogously, we say that a set \( S \subseteq V(G) \) \textit{propagates} to a set of vertices \( W \subseteq V(G) \), when for every \( w \in W \) there exists \( u \in S \) such that \( u \) propagates to \( w \). As in the case of zero forcing, it is sufficient to study power domination on weakly connected digraphs. If a digraph is not weakly connected, then power domination is studied independently in each weak component.

The power domination problem on digraphs was formally introduced in [1], and while not mentioned in the definition itself ([1, Definition 3.3.1]), it is clearly stated in [1, Pg.4] that only digraphs without loops were considered. Prior to our work, power domination in digraphs with loops had only been studied in [19]. It is important to remark that in [19], the authors studied power domination in digraphs with loops using the rules introduced in [1] for digraphs without loops, while we defined power domination using different rules for digraphs with at least one loop than for digraphs without loops. As a consequence, our definition of power domination differs from the one in [19] in the following situation. If there is a loop on a white vertex \( v \) and all vertices in \( N^+(v) \setminus \{v\} \) are blue, by our rules \( v \) becomes blue, while by the rules in [19] \( v \) remains white. By treating digraphs with loops in the same way as in zero forcing, our definition preserves an important relationship between zero forcing and power domination in digraphs without loops, also present in the case of undirected graphs (see [7]). Indeed, a careful observation of the definition of zero forcing and our definition of power domination in digraphs yields the conclusion that, a set \( S \subseteq V(G) \) is a power dominating set of digraph \( G \) if and only if \( N^+[S] \) is a zero forcing set of digraph \( G \).

For a digraph \( G = (V, A) \), the \textit{line digraph} of \( G \) is the digraph \( L(G) \) where \( V(L(G)) = \{uv : (u, v) \in A(G)\} \) and \( A(L(G)) = \{uxy : (u, x), (x, y) \in A(G)\} \). Iterated line digraphs are recursively defined by: \( L^0(G) = G \) and \( L^r(G) = L(L^{r-1}(G)) \) for every integer \( r \geq 1 \). Observe
that in particular, \( L^1(G) = L(G) \). Following [26], we say that a digraph \( G \) is \( L \)-convergent if the set \( \{ L^r : r \text{ is a non-negative integer} \} \) is finite; otherwise \( G \) is \( L \)-divergent. In this paper we are especially interested in digraphs that are \( L \)-divergent. In [6] it was proven that a digraph \( G \) is \( L \)-divergent if and only if at least one strong component of \( G \) is not a cycle or \( G \) has at least two cycles joined by a path. The class of \( L \)-divergent digraphs includes all strongly connected digraphs other than a cycle, which have been proven to be asymptotically dense in [17] for the regular case and in [12] for irregular digraphs.

In Section 3, we establish lower and upper bounds for the zero forcing number of iterated line digraphs, and determine the zero forcing number of iterated line digraphs of regular digraphs, for which we provide specific constructions of minimum zero forcing sets. We combine the results obtained on zero forcing with known results about minimum rank to prove that iterated line digraphs of regular digraphs are infinite families of digraphs with the property \( M(G) = Z(G) \). In Section 4, we establish a relationship between power domination and zero forcing in iterated line digraphs and determine the power domination number of iterated line digraphs of regular digraphs. In Section 5, we apply the results to special families of iterated line digraphs.

### 3 Zero forcing and minimum rank

We start this section by introducing the definitions of critical and strongly critical sets of vertices in a digraph. Intuitively, both concepts refer to the property of a set of vertices \( W \) in a digraph \( G \), that if all vertices in \( W \) are white and all vertices in \( V(G) \setminus W \) are blue, then the color changing rule fails to produce any additional blue vertices. If digraph \( G \) does not have loops, this means that no vertex in \( V(G) \setminus W \) forces a vertex in \( W \), and in this case we say that \( W \) is critical. However, in a digraph with at least one loop, since a vertex could force itself, it is necessary that no vertex, neither in \( V(G) \setminus W \) nor in \( W \), can force a vertex in \( W \), and in this case, the set \( W \) is strongly critical.

**Definition 3.1.** In a digraph \( G = (V, A) \), a non-empty set \( W \subseteq V(G) \) is called critical if every \( v \in V(G) \setminus W \) has either no out-neighbors in \( W \), or it has at least two out-neighbors in \( W \). That is, for every \( v \in V(G) \setminus W \), \( |N^+(v) \cap W| \neq 1 \). In addition, \( W \) is strongly critical if for every \( v \in V \), \( |N^+(v) \cap W| \neq 1 \).

**Remark 3.2.** Let \( G = (V, A) \) be a digraph and let \( S \) be a zero forcing set of \( G \). If \( G \) does not have any loops, then \( |S \cap W| \geq 1 \) for every critical set \( W \) in \( G \). Indeed, if \( W \) is critical in \( G \) and \( |S \cap W| = 0 \), then there is no \( v \in V(G) \setminus W \) such that \( |N^+(v) \cap W| = 1 \) so no \( v \in V(G) \setminus W \) can force a vertex in \( W \). Since \( W \) is non-empty and \( S \) is a zero forcing set, there must be at least one vertex in \( S \cap W \). Analogously, if \( G \) has at least one loop, then \( |S \cap W| \geq 1 \) for every strongly critical set \( W \) in \( G \).

**Observation 3.3.** Every strongly critical set is a critical set. As a consequence, in any digraph, the maximum number of pairwise disjoint strongly critical sets is less than or equal to the maximum number of pairwise disjoint critical sets.
Lemma 3.4. The zero forcing number of a digraph $G$ is at least the maximum number of pairwise disjoint strongly critical sets in $G$. Moreover, if $G$ does not have any loops, its zero forcing number is at least the maximum number of pairwise disjoint critical sets in $G$.

Proof. Let $S$ be a minimum zero forcing set of digraph $G$ and let $\{W_1, \ldots, W_r\}$ be a set of pairwise disjoint, strongly critical sets. By Remark 3.2, $|S \cap W_i| \geq 1$ for every $i = 1, \ldots, r$, and since the sets $W_1, \ldots, W_r$ are pairwise disjoint, hence $|S| \geq r$. If $G$ does not have any loops, then we apply the same argument with a collection of pairwise disjoint critical sets.

Next, we will show that the lower bound provided by Lemma 3.4 can be improved in the case of line digraphs.

Lemma 3.5. Let $G$ be a digraph and let $uv$ be a vertex of $L(G)$. If $d^+_{L(G)}(uv) \geq 2$, every subset $T \subseteq N^+_{L(G)}(uv)$ with $|T| \geq 2$ is a strongly critical set in $L(G)$.

Proof. Let $xy$ be any vertex of $L(G)$. If $y = v$, then $T \subseteq N^+_{L(G)}(xy)$, hence $|N^+_{L(G)}(xy) \cap T| = |T| \geq 2$. If $y \neq v$, then $|N^+_{L(G)}(xy) \cap T| = 0$. Therefore, $T$ is a strongly critical set.

Lemma 3.6. Let $G$ be a digraph and let $S$ be a zero forcing set of $L(G)$. If $uv$ is a vertex in $L(G)$ such that $d^+_{L(G)}(uv) \geq 2$, then $|N^+_{L(G)}(uv) \cap S| \geq d^+_{L(G)}(uv) - 1$.

Proof. Let $T = N^+_{L(G)}(uv) \setminus S$, then $|S \cap T| = 0$. If $|T| \geq 2$, then $T$ is strongly critical, by Lemma 3.5 which contradicts Remark 3.2. This implies $|T| \leq 1$, hence $|N^+_{L(G)}(uv) \cap S| \geq d^+_{L(G)}(uv) - 1$.

The following simple observation will be used in the proof of the next theorem.

Observation 3.7. In every digraph $G$, any vertex is in at most one cycle consisting only of vertices with in-degree 1 in $G$.

Theorem 3.8. Let $G = (V, A)$ be a digraph with $\delta^+(G) \geq 2$ and $\delta^-(G) \geq 1$. Then,

$$Z(L(G)) = |A(G)| - |V(G)|.$$

Proof. Let $V(G) = \{v_1, \ldots, v_n\}$ and let $S \subseteq V(L(G))$ be a zero forcing set of $L(G)$. Since $\delta^-(G) \geq 1$, there are vertices $u_1, \ldots, u_n \in V(G)$ such that $u_i v_i \in A(G)$ for every $i = 1, \ldots, n$. Then, by Lemma 3.6 we have $|S \cap N^+_{L(G)}(u_i v_i)| \geq d^+_{L(G)}(u_i v_i) - 1$ for every $i = 1, \ldots, n$. Further, if $x y \in N^+_{L(G)}(u_i v_i) \cap N^+_{L(G)}(u_j v_j)$ for some $i \neq j$ then $x = v_i$ and $x = v_j$, a contradiction. Thus, the sets $N^+_{L(G)}(u_i v_i)$ form a partition of $V(L(G))$ meaning $\sum_{i=1}^n d^+_{L(G)}(u_i v_i) = |V(L(G))| = |A(G)|$. Together this gives

$$|S| = \sum_{i=1}^n |S \cap N^+_{L(G)}(u_i v_i)| \geq \sum_{i=1}^n (d^+_{L(G)}(u_i v_i) - 1) = |A(G)| - |V(G)|.$$
This implies \( Z(L(G)) \geq |A(G)| - |V(G)| \). To find an upper bound on \( Z(L(G)) \) we will now provide a zero forcing set of \( L(G) \) of cardinality \( |A(G)| - |V(G)| \).

From Observation 3.7 for every \( i = 1, \ldots, n \) vertex \( v_i \) is in at most one cycle consisting only of vertices with in-degree 1, and as a consequence, at most one out-neighbor of \( v_i \) is in such cycle. Therefore, the condition \( \delta^+(G) \geq 2 \) guarantees that for every \( i = 1, \ldots, n \) there exists a vertex \( w_i \in N_G^+(v_i) \) such that vertex \( w_i \) is not in a cycle consisting only of vertices with in-degree 1 in \( G \). If this were not the case, then every \( w_i \) would be in such a cycle and arc \( v_iw_i \in A(G) \) must also be in that cycle. However, this means \( v_i \) is in at least two such cycles, a contradiction. Let

\[
S = \bigcup_{i=1}^n \{v_iw : w \in N_G^+(v_i) \setminus \{w_i\}\}.
\]

Then, \( |S| = |A(G)| - |V(G)| \) and \( S \) is a zero forcing set of \( G \). Indeed, for any vertex \( v_iv_j \) in \( V(L(G)) \setminus S \), since \( \delta^-(G) \geq 1 \) there exists \( v_t \in N_G^-(v_i) \) such that \( v_iv_j \in N_{L(G)}^+(v_i,v_t) \). If \( v_tv_i \in S \), then \( v_tv_j = v_iw_i \) so \( v_tv_j \) is the only out-neighbor of \( v_tv_i \) not forced, by construction of \( S \), and \( v_tv_i \) forces \( v_tv_j \). If \( v_tv_i \not\in S \), then proceed with \( v_tv_i \) as we did with \( v_tv_j \), and the selection of vertices \( w_1, \ldots, w_n \) guarantees that at some point, a vertex in \( S \) is reached.

**Corollary 3.9.** Let \( G = (V, A) \) be a digraph with \( \delta^+(G) \geq 2 \) and \( \delta^-(G) \geq 1 \). Then,

\[
(\delta - 1)|V(G)| \leq Z(L(G)) \leq (\Delta - 1)|V(G)|
\]

where \( \delta = \max \{\delta^+(G), \delta^-(G)\} \) and \( \Delta = \min \{\Delta^+(G), \Delta^-(G)\} \).

**Proof.** From Theorem 3.8 \( Z(L(G)) = |A(G)| - |V(G)| \). Since \( |A(G)| = \sum_{v \in V} d_G^+(v) \), then \( \delta^+(G)|V(G)| \leq |A(G)| \leq \Delta^+(G)|V(G)| \) and \( (\delta^+(G) - 1)|V(G)| \leq Z(L(G)) \leq (\Delta^+(G) - 1)|V(G)| \). Analogously, \( |A(G)| = \sum_{v \in V} d_G^-(v) \) implies \( (\delta^-(G) - 1)|V(G)| \leq Z(L(G)) \leq (\Delta^-(G) - 1)|V(G)| \), and the result is obtained by choosing the least upper bound and the greatest lower bound.

The next results extend Theorem 3.8 and Corollary 3.9 to iterated line digraphs.

**Theorem 3.10.** Let \( G = (V, A) \) be a digraph with \( \delta^+(G) \geq 2 \) and \( \delta^-(G) \geq 1 \). For every integer \( n \geq 1 \), \( Z(L^n(G)) = |V(L^n(G))| - |V(L^{n-1}(G))| \).

**Proof.** From Theorem 3.8 \( Z(L^n(G)) = |A(L^{n-1}(G))| - |V(L^{n-1}(G))| \). By definition of line digraph, \( |A(L^{n-1}(G))| = |V(L^n(G))| \) so \( Z(L^n(G)) = |V(L^n(G))| - |V(L^{n-1}(G))| \).

**Lemma 3.11.** Let \( G = (V, A) \) be a digraph with \( \delta^+(G) \geq 1 \) and \( \delta^-(G) \geq 1 \). For any integer \( n \geq 1 \),

\[
\text{a) } \delta^+(L^n(G)) = \delta^+(G), \quad \text{b) } \Delta^+(L^n(G)) = \Delta^+(G), \quad \text{c) } \delta^-(L^n(G)) = \delta^-(G), \quad \text{d) } \Delta^-(L^n(G)) = \Delta^-(G).
\]
Lemma 3.12. Let $G = (V, A)$ be a digraph with $\delta^+(G) \geq 1$ and $\delta^-(G) \geq 1$. For any integer $n \geq 0$,

a) $\delta^+(G)|V(L^n(G))| \leq |V(L^{n+1}(G))| \leq |V(L^n(G))|\Delta^+(G)$.

b) $\delta^-(G)|V(L^n(G))| \leq |V(L^{n+1}(G))| \leq |V(L^n(G))|\Delta^-(G)$.

Proof. Since $|V(L^{n+1}(G))| = |A(L^n(G))|$ and $|A(L^n(G))| = \sum_{v \in V(L^n(G))} d^+_L(v)$, from the observation

$$\delta^+(L^n(G))|V(L^n(G))| \leq \sum_{v \in V(L^n(G))} d^+_L(v) \leq |V(L^n(G))|\Delta^+(L^n(G)),$$

we conclude $\delta^+(L^n(G))|V(L^n(G))| \leq |V(L^{n+1}(G))| \leq |V(L^n(G))|\Delta^+(L^n(G))$.

By Lemma 3.11, $\delta^+(L^n(G)) = \delta^+(G)$ and $\delta^-(L^n(G)) = \delta^-(G)$, and as a consequence, the previous inequality can be written as $\delta^+(G)|V(L^n(G))| \leq |V(L^{n+1}(G))| \leq |V(L^n(G))|\Delta^+(L^n(G))$.

Analogously, from $|V(L^{n+1}(G))| = |A(L^n(G))|$ and $|A(L^n(G))| = \sum_{v \in V(L^n(G))} d^-_L(v)$, we conclude $\delta^-(L^n(G))|V(L^n(G))| \leq |V(L^{n+1}(G))| \leq |V(L^n(G))|\Delta^-(L^n(G))$. Since Lemma 3.11 also implies $\delta^-(L^n(G)) = \delta^-(G)$ and $\Delta^-(L^n(G)) = \Delta^-(G)$, the previous inequality is equivalent to $\delta^-(G)|V(L^n(G))| \leq |V(L^{n+1}(G))| \leq |V(L^n(G))|\Delta^-(L^n(G))$. \qed

Lemma 3.13. Let $G = (V, A)$ be a digraph with $\delta^+(G) \geq 1$ and $\delta^-(G) \geq 1$. For any integer $n \geq 1$,

a) $(\delta^+(G))^n|V(G)| \leq |V(L^n(G))| \leq |V(G)|\Delta^+(G)^n$.

b) $(\delta^-(G))^n|V(G)| \leq |V(L^n(G))| \leq |V(G)|\Delta^-(G)^n$.

Proof. For $n = 1$ the result holds, since $\delta^+(G)|V(G)| \leq |V(L(G))| \leq |V(G)|\Delta^+(G)$ and $\delta^-(G)|V(G)| \leq |V(L(G))| \leq |V(G)|\Delta^-(G)$ follow from replacing $n = 0$ in Lemma 3.12. We conclude the proof by induction on $n$.

Assume $(\delta^+(G))^n|V(G)| \leq |V(L^n(G))| \leq |V(G)|\Delta^+(G)^n$ for an integer $n \geq 1$. Then, $(\delta^+(G))^{n+1}|V(G)| \leq \delta^+(G)|V(L^n(G))| \leq |V(L^n(G))|\Delta^+(G) \leq |V(G)|\Delta^+(G)^{n+1}$. (1)
By Lemma 3.12, \( \delta^+(G)|V(L^n(G))| \leq |V(L^{n+1}(G))| \leq |V(L^n(G))|\Delta^+(G) \),
\[
(2)
\]
Combining (1) and (2), we obtain \((\delta^+)(n+1)|V(G)| \leq |V(L^{n+1}(G))| \leq |V(G)|(\Delta^+(G))^{n+1}\).

Now, assume \((\delta^-(G))|^n|V(G)| \leq |V(L^n(G))| \leq |V(G)|(\Delta^-(G))^n\) for an integer \(n \geq 1\). Observe that Lemma 3.12 implies \((\delta^-)(V(L^n(G)))| \leq |V(L^{n+1}(G))| \leq |V(L^n(G))|(\Delta^-)(G)\). Therefore, repeating the previous argument we conclude \((\delta^-)(n+1)|V(G)| \leq |V(L^{n+1}(G))| \leq |V(G)|(\Delta^-)(G)^{n+1}\).

**Corollary 3.14.** Let \(G = (V,A)\) be a digraph with \(\delta^+(G) \geq 2\) and \(\delta^-(G) \geq 1\). For every integer \(n \geq 1\),
\[
(\delta - 1)\delta^{n-1}|V(G)| \leq Z(L^n(G)) \leq (\Delta - 1)\delta^{n-1}|V(G)|,
\]
where \(\delta = \max\{\delta^+(G),\delta^-(G)\}\) and \(\Delta = \min\{\Delta^+(G),\Delta^-\}\).

**Proof.** Lemma 3.12 gives two lower bounds and two upper bounds for \(|V(L^n(G))|\). Choosing the greatest lower bound and the least upper bound we obtain \(\delta|V(L^{n-1}(G))| \leq |V(L^n(G))| \leq |V(L^n(G))|\Delta^n\). This chain of inequalities implies \((\delta - 1)|V(L^{n-1}(G))| \leq |V(L^n(G))| - |V(L^{n-1}(G))| \leq |V(L^n(G))|((\Delta - 1))\), and since by Theorem 3.10, \(Z(L^n(G)) = |V(L^n(G))| - |V(L^{n-1}(G))|\), we conclude
\[
(\delta - 1)|V(L^{n-1}(G))| \leq Z(L^n(G)) \leq |V(L^{n-1}(G))|((\Delta - 1)).
\]

By Lemma 3.13, again, selecting the greatest lower bound and the least upper bound, we have \(\delta^{n-1}|V(G)| \leq |V(L^{n-1}(G))| \leq |V(G)|\Delta^{n-1}\). From these inequalities we obtain \((\delta - 1)\delta^{n-1}|V(G)| \leq (\delta - 1)|V(L^{n-1}(G))|\) and \(|V(L^{n-1}(G))|((\Delta - 1)) \leq |V(G)|\Delta^{n-1}((\Delta - 1))\), which combined with (3) yield \((\delta - 1)\delta^{n-1}|V(G)| \leq Z(L^n(G)) \leq (\Delta - 1)\Delta^{n-1}|V(G)|\).  

In the remainder of this section we restrict ourselves to regular digraphs. The next result follows immediately from Theorem 3.10 or from Corollary 3.14.

**Corollary 3.15.** Let \(G = (V,A)\) be a \(d\)-regular digraph with \(d \geq 2\). For every integer \(n \geq 1\),
\begin{enumerate}
  \item \(Z(L^n(G)) = (d - 1)d^{n-1}|V(G)|\).
  \item \(Z(L^{n+1}(G)) = dZ(L^n(G))\).
\end{enumerate}

The following result, together with our results on zero forcing of line digraphs, allow us to show that for any \(d\)-regular digraph \(G\) and any positive integer \(n\), \(M(L^n(G)) = Z(L^n(G))\).

**Lemma 3.16.** [18] Lemma 1] A \(d\)-regular digraph of order \(p\) is a line digraph if and only if the rank of its adjacency matrix is equal to \(p/d\).

**Theorem 3.17.** Let \(G = (V,A)\) be a \(d\)-regular digraph with \(d \geq 2\). For every integer \(n \geq 1\),
\begin{enumerate}
  \item \(M(L^n(G)) = Z(L^n(G)) = (d - 1)d^{n-1}|V(G)|\).
  \item \(mr(L^n(G)) = d^{n-1}|V(G)|\).
\end{enumerate}
Proof. Let $A = A(L^2(G))$ be the adjacency matrix of $L^2(G)$. Since $G$ is $d$-regular, then $L^2(G)$ is also $d$-regular, and by Lemma 3.16, $A$ has rank $\frac{d^2 |V(G)|}{d} = d^{n-1}|V(G)|$. Then, the equality $\text{null}(A) + \text{rank}(A) = |V(L^2(G))| = d^n |V(G)|$ implies that $\text{null}(A) = d^n |V(G)| - d^{n-1}|V(G)| = (d-1)d^{n-1}|V(G)|$. Since $\text{null}(A) \leq M(L^2(G)) \leq Z(L^2(G))$ and, by Corollary 3.15, $Z(L^2(G)) = (d-1)d^{n-1}|V(G)|), we conclude $(d-1)d^{n-1}|V(G)| = \text{null}(A) \leq M(L^2(G)) \leq Z(L^2(G)) = (d-1)d^{n-1}|V(G)|$ and the result follows immediately. Replacing this value in $\text{mr}(L^2(G)) = d^n |V(G)| - M(L^2(G))$ completes the proof.

Remark 3.18. The only 1-regular, weakly connected, digraphs are cycles. If $G$ is a cycle, then $L^2(G) = G$ for any integer $n \geq 1$, so $Z(G) = 1, \text{mr}(G) = |V(G)| - 1, M(G) = 1$. As a consequence, for any regular digraph $G$ and for any integer $n \geq 1, M(L^2(G)) = Z(L^2(G))$. Thus, iterated line digraphs of regular digraphs have optimal values of zero forcing, minimum rank and maximum nullity. Furthermore, the minimum rank and maximum nullity are attained by the adjacency matrix of $L^2(G)$.

4 Power domination

Let $G$ be a digraph and $S$ a set of vertices of $G$. As observed in Section 2, $S$ is a power dominating set of $G$ if and only if $N^2[S]$ is a zero forcing set of $G$. Hence, $Z(G) \leq \gamma^P(G)(\Delta^+(G) + 1)$. The following result provides an improved upper bound in the case that $G$ is a line digraph.

Theorem 4.1. Let $G$ be a digraph with $\delta^+(G) \geq 1$. Then,

$$Z(L(G)) \leq \Delta^+(G)\gamma^P(L(G)).$$

Proof. Let $S = \{u_1v_1, \ldots, u_nv_n\}$ be a minimum power dominating set of $L(G)$. We prove $Z(L(G)) \leq \Delta^+(G)\gamma^P(L(G))$ by constructing a zero forcing set of $L(G)$ with cardinality $\Delta^+(G)|S|$.

Define $S_1 = \{u_iv_i \in S : v_i \neq v_j \text{ for all } j \text{ with } 1 \leq j < i\}$ and $S_2 = S \setminus S_1$. Since $S$ is a power dominating set of $L(G)$, $N^+_L(S) = \cup_{i=1}^nN^+_L[u_iv_i] = S_2 \cup \cup_{u_iw_i \in S_1}N^+_L[u_ivi \setminus \{v_iw_i\}]$ is a zero forcing set of $L(G)$. Since $\delta^+(G) \geq 2$, for every $u_iv_i \in S_1$ we can select an arbitrary vertex $v_iw_i \in N^+_L[u_ivi]$. Let $P = \cup_{u_iw_i \in S_1}(N^+_L[u_ivi \setminus \{v_iw_i\}] \cup S_2 \cup P$ forces $S_2 \cup \cup_{u_iw_i \in S_1}N^+_L(u_ivi \setminus \{v_iw_i\}] = N^+_L(S)$. Since $N^+_L(S)$ is a zero forcing set of $L(G)$, then $S_2 \cup P$ is also a zero forcing set of $L(G)$ and $Z(G) \leq |S_2 \cup P|$.

Now, $|S_2 \cup P| \leq |S_2| + \sum_{u_iw_i \in S_1} |N^+_L[u_ivi \setminus \{v_iw_i\}]| \leq |S_2| + |S_1|\Delta^+(G) \leq |S|\Delta^+(G)$. Therefore, $Z(L(G)) \leq \gamma^P(L(G))\Delta^+(G)$.

Corollary 4.2. Let $G$ be a digraph with $\delta^+(G) \geq 2$ and $\delta^-(G) \geq 1$. For every integer $n \geq 1$,

$$Z(L^n(G)) \geq \gamma^P(L^n(G)) \geq \left\lceil \frac{Z(L^n(G))}{\Delta^+(G)} \right\rceil.$$

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Corollary 4.3. For a regular digraph $G$, $Z(L(G)) \leq \gamma_P(L^2(G))$.

**Proof.** Assume $G$ is $d$-regular. Then, by definition of line digraph, $L(G)$ is also $d$-regular. If $d = 1$, then $G = L^0(G)$ are cycles, so $Z(L(G)) = 1$ and $\gamma_P(L^2(G)) = 1$. If $d \geq 2$, we apply Theorem [11] to $L(G)$ and obtain $Z(L^2(G)) \leq d\gamma_P(L^2(G))$. By Corollary 3.15, $Z(L^2(G)) = dZ(L(G))$ so $dZ(L(G)) \leq d\gamma_P(L^2(G))$, and this implies $Z(L(G)) \leq \gamma_P(L^2(G))$. \hfill $\Box$

Next, we will show that if $G$ is a $d$-regular digraph, then $Z(L(G)) = \gamma_P(L^2(G))$. First, we need to introduce additional terminology and obtain further results.

**Definition 4.4.** A digraph $H$ is a factor of a digraph $G$ if $V(H) = V(G)$ and $A(H) \subseteq A(G)$. In particular, if $H$ is a factor of $G$ and $H$ is 1-regular, then $H$ is a 1-factor of $G$.

**Observation 4.5.** A 1-factor in a digraph $G$ is either a cycle, or a set of pairwise vertex-disjoint cycles, containing every vertex in $G$.

**Theorem 4.6.** Let $G$ be a digraph with $\delta^+(G) \geq 2$ and $\delta^-(G) \geq 1$. If $G$ has a 1-factor in which every cycle contains at least one vertex $v$ with $d_G^-(v) > 1$, then $Z(L(G)) \geq \gamma_P(L^2(G))$.

**Proof.** Let $H$ be a 1-factor of $G$ with the condition in the hypothesis. Then, $H$ induces a permutation $f$ on the set of vertices of $G$ where $f(v) = u$ if and only if $(u, v)$ is an arc in $H$. Observe that the cycles in $H$ correspond to the orbits of the permutation $f$. Since each cycle in $H$ contains a vertex $v$ with $d_G^-(v) > 1$, then each orbit in the permutation $f$ contains a vertex $v$ with $d_G^-(v) > 1$. By definition, $V(L(G)) = \{uv : (u, v) \in A(G)\}$ and $V(L^2(G)) = \{uvw : (u, v) \in A(G)$ and $(v, w) \in A(G)\}$. Define $S = \{f(u)uv \in V(L^2(G)) : u \neq f(v)\}$. Observe $|S| = \sum_{v \in V(H)}(d_G^-(v) - 1)$ and $V(H) = V(G)$, therefore $|S| = \sum_{v \in V(G)}(d_G^-(v) - 1)$. Besides, $\sum_{v \in V(G)}(d_G^-(v) - 1) = (\sum_{v \in V(G)}d_G^-(v)) - |V(G)| = |A(G)| - |V(G)|$. By Theorem 3.8, $Z(L(G)) = |A(G)| - |V(G)|$, and as a consequence, it is sufficient to prove that $S$ is a power dominating set of $L^2(G)$ to conclude $\gamma_P(L^2(G)) \leq Z(L(G))$.

Given $xyz$ in $V(L^2(G))$, it is sufficient to show that either $xyz \in N^+_{L^2(G)}[S]$ or $xyz$ is obtained by a sequence of forces starting with a vertex in $N^+_{L^2(G)}[S]$.

**Case 1.** If $x \neq f(y)$ then $f(x)xy \in S$, hence $xyz \in N^+_{L^2(G)}(f(x)xy) \subseteq N^+_{L^2(G)}[S]$.

**Case 2.** If $x = f(y)$ and $y \neq f(z)$ then $xyz \in S \subseteq N^+_{L^2(G)}[S]$.

**Case 3.** If $x = f(y)$ and $y = f(z)$ then $xyz = f^2(z)f(z)z$.

In the last case, for notational convenience, denote the elements of the orbit of $z$ by $z_i = f^i(z)$, i.e., $z = z_0 = z_\ell$, $y = z_1 = z_{k+1}$, etc., where $\ell$ is the length of the orbit of $z$. By assumption, $d^-(z_k) \geq 2$ for some $k \geq 2$. This implies that there exists $a \in N^+_G(z_k)$ such that $a \neq z_{k+1}$. Consequently, $f(a)az_k \in S$ and $az_kz_{k-1} \in N^+_{L^2(G)}(f(a)az_k) \subseteq N^+_{L^2(G)}[S]$. If $d^+(z_{k-1}) = 1$ then $az_kz_{k-1}$ forces its only out-neighbor $z_kz_{k-1}z_{k-2}$. If $d^+_G(z_{k-1}) \geq 2$ every out-neighbor of $az_kz_{k-1}$ different from $z_kz_{k-1}z_{k-2}$ is in the form $z_kz_{k-1}b$ for some vertex $b \in N^+_G(z_{k-1}) \setminus \{z_{k-2}\}$. Since $f$ is bijective and $z_{k-2} \neq b$, we have $z_{k-1} \neq f(b)$, and therefore,
So every out-neighbor of \( az \) different from \( az_{k-1}z_{k-2} \) is in \( S \) and this means that \( az \) forces \( z_{k-1}z_{k-2} \). Repeating this argument, we obtain that all the vertices \( z_{i}z_{i-1}z_{i-2} \) for \( i = k, k-1, \ldots, 2 \) are forced, and in particular \( xyz = z_2z_1z_0 \).

\[ \begin{array}{c}
\text{Figure 1: Illustration of the argument for Case 3 in the proof of Theorem 4.7.}
\end{array} \]

In a digraph \( G \) with \( \delta^-(G) \geq 2 \), every 1-factor satisfies the condition in Theorem 4.6. As a consequence, we obtain the following corollary.

**Corollary 4.7.** Let \( G \) be a digraph with \( \delta^+(G) \geq 2 \) and \( \delta^-(G) \geq 2 \). If \( G \) has a 1-factor, then \( Z(L(G)) \geq \gamma_P(L^2(G)) \).

Next, we recall another definition and a property of regular digraphs from [27].

**Definition 4.8.** A cycle factorization in a digraph \( G \) is a set of 1-factors \( \{H_1, \ldots, H_k\} \) such that \( A(G) = \bigcup_{i=1}^k A(H_i) \) and \( A(H_i) \cap A(H_j) = \emptyset \) for every \( 1 \leq i < j \leq k \).

**Lemma 4.9.** [27] Let \( G \) be a d-regular digraph. Then, \( G \) has a cycle factorization consisting of exactly \( d \) 1-factors.

By Lemma 4.9, every regular digraph has a cycle factorization, and as a consequence, it has at a 1-factor. Therefore, when \( G \) is \( d \)-regular and \( d \geq 2 \) combining Corollary 4.3 and Corollary 4.7 we obtain the following result.

**Theorem 4.10.** If \( G \) is a regular digraph, then \( Z(L(G)) = \gamma_P(L^2(G)) \).

**Proof.** Assume \( G \) is \( d \)-regular. If \( d = 1 \), then as seen in the proof of Corollary 4.3, \( Z(L(G)) = 1 \) and \( \gamma_P(L^2(G)) = 1 \). If \( d \geq 2 \), then \( Z(L(G)) \leq \gamma_P(L^2(G)) \), by Corollary 4.3. By Lemma 4.9, \( G \) has a cycle factorization. As as a consequence, \( G \) has a 1-factor, and by Corollary 4.7, \( Z(L(G)) \geq \gamma_P(L^2(G)) \). Thus, \( Z(L(G)) = \gamma_P(L^2(G)) \).
We now extend Theorem 4.10 to a relationship between \( Z(L^n(G)) \) and \( \gamma_P(L^{n+1}(G)) \) for any regular digraph \( G \) and any positive integer \( n \).

**Corollary 4.11.** Let \( G \) be a \( d \)-regular digraph. For any integer \( n \geq 1 \),

\[
Z(L^n(G)) = \gamma_P(L^{n+1}(G)).
\]

Moreover, if \( d \geq 2 \), then \( Z(L^n(G)) = \gamma_P(L^{n+1}(G)) = (d - 1)d^{n-1}|V(G)| \).

**Proof.** If \( d = 1 \), then \( G \) is a cycle so \( G = L^n(G) \) for every \( n \geq 1 \) and it follows that \( Z(L^n(G)) = \gamma_P(L^{n+1}(G)) = 1 \). Assume \( d \geq 2 \). Observe that since \( G \) is \( d \)-regular, then \( L^n(G) \) is also \( d \)-regular, for every integer \( n \geq 1 \). By Corollary 4.10, \( Z(L^n(G)) = \gamma_P(L^{n+1}(G)) \) holds for \( n = 1 \). For \( n \geq 2 \), we apply Corollary 4.10 to \( L^{n-1}(G) \) and obtain \( Z(L(L^{n-1}(G))) = \gamma_P(L^2(L^{n-1}(G))) \). Observe that \( L(L^{n-1}(G)) = L^n(G) \) and \( L^2(L^{n-1}(G)) = L^n(G) = L^{n+1}(G) \). Then, \( Z(L(L^{n-1}(G))) = \gamma_P(L^2(L^{n-1}(G))) \) implies \( Z(L^n(G)) = \gamma_P(L^{n+1}(G)) \). By Corollary 3.15 if \( d \geq 2 \), then \( Z(L^n(G)) = (d - 1)d^{n-1}|V(L(G))| \). Since we have proven \( Z(L^n(G)) = \gamma_P(L^{n+1}(G)) \), we conclude that \( \gamma_P(L^{n+1}(G)) = (d - 1)d^{n-1}|V(G)| \).

Observe that Theorem 4.10 does not provide an expression to determine \( \gamma_P(L(G)) \). We show next that \( \gamma_P(L(G)) \) depends on properties of the digraph \( G \), and provide an expression to determine \( \gamma_P(L(G)) \) if \( G \) is a \( d \)-regular digraph. Since the only 1-regular digraphs are cycles, we assume \( d \geq 2 \), and when regularity is not necessary, \( \delta^+(G) \geq 2 \) and \( \delta^-(G) \geq 2 \).

**Lemma 4.12.** Let \( G \) be a digraph with \( \delta^+(G) \geq 2 \) and \( \delta^-(G) \geq 2 \). If there exists a set \( S \subseteq V(G) \) such that for any two different vertices \( x, y \in S \), \( N^+_G(x) \cap N^+_G(y) = \emptyset \), then, for each \( v \in N^+_G(S) \) there exists a unique vertex \( u \in S \cap N^-_G(v) \).

**Proof.** Suppose \( v \in N^+_G(S) \setminus S \) and there exist two different vertices \( u, w \in S \) such that \( u \in N^-_G(v) \) and \( v \in N^-_G(w) \). Then, \( v \in N^-_G(u) \cap N^-_G(w) \), a contradiction. Now suppose \( v \in N^+_G(S) \cap S \) and consider two cases depending on \( (v, v) \in A(G) \) or \( (v, v) \notin A(G) \). If \( (v, v) \in A(G) \), then \( v \in N^-_G(v) \). Suppose there exists \( u \neq v \) such that \( u \in N^-_G(v) \). Then, \( N^+_G(u) \cap N^+_G(v) = \{v\} \), a contradicting. If \( (v, v) \notin A(G) \) and there exist two different vertices \( u, w \in S \) such that \( u \in N^-_G(v) \) and \( v \in N^-_G(w) \). Then, \( v \in N^+_G(u) \cap N^+_G(w) \), another contradiction.

**Proposition 4.13.** Let \( G \) be a digraph with \( \delta^+(G) \geq 2 \) and \( \delta^-(G) \geq 2 \). If there exists a set \( S \subseteq V(G) \) such that for any two different vertices \( x, y \in S \), \( N^+_G(x) \cap N^+_G(y) = \emptyset \), and for every \( x \in S \), \( N^+_G(x) \cap S = \emptyset \) or \( N^+_G(x) \cap S = \{x\} \), then \( \gamma_P(L(G)) \leq |V(G)| - |S| \).

**Proof.** Assume \( V(G) = \{v_1, \ldots, v_n\} \) and \( S = \{v_r, \ldots, v_n\} \). For each integer \( i = 1, \ldots, n \) select one vertex \( u_i \in N^-_G(v_i) \) following this rule: if \( v_i \in N^-_G(S) \), then select as \( u_i \) the unique vertex, by Lemma 4.12 \( u_i \in (S \cap N^-_G(v_i)) \). Define \( P = \{u_iv_i : 1 \leq i \leq r - 1\} \). Observe that \( V(L(G)) = \bigcup_{i=1}^n N^+_L(u_iv_i) \) and \( N^+_L(P) = P \cup (\bigcup_{i=1}^n N^+_L(u_iv_i)) \).

If \( v_i \in S \) and there is no loop \((v_i, v_i) \in A(G)\), then the selection of vertices \( u_i \) implies that for each vertex \( v_j \in N^+_G(v_i) \), the vertex selected as \( u_j \) is \( v_i \). As a consequence, \( v_jv_i \in P \) and \( N^+_L(u_iv_i) \subseteq P \). If there is a loop \((v_i, v_i) \in A(G)\), then \( v_i,v_i \in V(L(G)) \) and \( v_i,v_i \notin P \).
However, by the rule for selecting \( u_i \), \( (N^+_L(G)(v_iv_i) \setminus \{v_iv_i\}) \subset P \). Then, \( v_iv_i \) forces itself, and we conclude that \( P \) forces all vertices in \( N^+_L(G)(v_iv_i) \). Then \( N^+_L(G)[P] \) is a zero forcing set of \( L(G) \), \( P \) is a power dominating set of \( L(G) \), and since \( |P| = |V(G)| - |S| \), we conclude \( \gamma_P(L(G)) \leq |V(G)| - |S| \).

\[ \square \]

In any digraph \( G \) with \( \delta^+(G) \geq 2 \) and \( \delta^-(G) \geq 2 \), for any \( v \in V(G) \) with \( d^+_G(v) \), the set \( S = \{v\} \) satisfies the conditions of Proposition 4.15 so \( \gamma_P(L(G)) = |V(G)| - 1 \).

**Corollary 4.14.** Let \( G \) be a \( d \)-regular digraph of order \( n \). If \( d \geq 2 \) and \( n < 2d \), then \( \gamma_P(L(G)) = |V(G)| - 1 \).

**Proof.** By Theorem 4.1 \( \left\lceil \frac{Z(L(G))}{d} \right\rceil \leq \gamma_P(L(G)) \). By Corollary 3.15 \( Z(L(G)) = n(d - 1) \). Then, \( n - \left\lfloor \frac{n}{d} \right\rfloor \leq \gamma_P(L(G)) \). Therefore, if \( \left\lfloor \frac{n}{d} \right\rfloor = 1 \), we conclude \( \gamma_P(L(G)) = |V(G)| - 1 \). \[ \square \]

**Corollary 4.15.** Let \( G \) be a \( d \)-regular digraph of order \( n \). If \( d \geq 2 \) and there exists \( S \subseteq V(G) \) such that 1) for any two different vertices \( x, y \in S \), \( N^-_G(x) \cap N^-_G(y) = \emptyset \), 2) for every \( x \in S \), \( N^+_G(x) \cap S = \emptyset \) or \( N^-_G(x) \cap S = \{x\} \), and 3) \( |S| = \left\lfloor \frac{n}{d} \right\rfloor \), then \( \gamma_P(L(G)) = \left\lceil \frac{n(d-1)}{d} \right\rceil \).

## 5 Applications

Next, we recall the definitions of some families of iterated line digraphs that have been extensively used in applications. In each family, we apply Corollary 3.15 to obtain their zero forcing number, Theorem 3.17 to obtain their maximum nullity and their minimum rank. The power domination number of \( L^n(G) \) follows from Corollary 4.11 when \( n \geq 2 \), and from Corollary 4.14 or Corollary 4.15 when \( n = 1 \). We refer the reader to [3] for additional details on the families of digraphs studied in this section.

### 5.1 de Bruijn digraphs

For any integers \( d \geq 2 \) and \( D \geq 1 \), the **de Bruijn digraph** \( B(d, D) \) has for vertices the set \( Z_{d^D} \), and each vertex \( x \) is adjacent to the vertices \( dx + t \) for any \( t \in Z_d \). Alternatively, \( B(d, D) \) can be iteratively defined by following rules: \( B(d, 1) = K_d \), the complete symmetric digraph of order \( d \) with a loop on each vertex, and \( B(d, D) = L(B(d, D - 1)) \), if \( D \geq 2 \). That is, \( B(d, D) = L^{D-1}(K_d) \), so \( B(d, D) \) has order \( d^D \).

**Corollary 5.1.** For any integers \( d \geq 2 \) and \( D \geq 2 \):

1. \( Z(B(d, D)) = (d - 1)d^{D-1} \).
2. \( M(B(d, D)) = (d - 1)d^{D-1} \).
3. \( \text{mr}(B(d, D)) = d^{D-1} \).
4. \( \gamma_P(B(d, D)) = (d - 1)d^{D-2} \).
For $D = 2$, $B(d, 2) = L(K_d)$ and $\gamma_P(B(d, 2)) = d - 1$ follows from Corollary 4.14.

Quantum systems based on de Bruijn digraphs were studied in [24] and [28]. Random walks on de Bruijn digraphs $B(d, n)$ have the fastest mixing rates among $d$-regular digraphs of order $d^n$ [28].

5.2 Kautz digraphs

For any integers $d \geq 3$ and $D \geq 1$, the vertices of the Kautz digraph $K(d, D)$ are the $D$-tuples in $Z_{d+1}$ in which any two consecutive elements are different; each vertex $x$ is adjacent to the vertices $dx + t$ for any $t \in Z_d$. The digraphs $K(d, D)$ can also be iteratively defined by following rules: $K(d, 1) = K_{d+1}^*$, the complete symmetric digraph of order $d+1$ without loops, and $K(d, D) = L(K(d, D - 1))$, if $D \geq 2$. Therefore, $K(d, D) = L^{D-1}(K_{d+1}^*)$, and $K(d, D)$ has order $d^{D-1}(d+1)$.

Corollary 5.2. For any integers $d \geq 3$ and $D \geq 3$:

1. $Z(K(d, D)) = (d-1)d^{D-2}(d+1)$.
2. $M(K(d, D)) = (d-1)d^{D-2}(d+1)$.
3. $\text{mr}(K(d, D)) = d^{D-2}(d+1)$.
4. $\gamma_P(K(d, D)) = (d-1)d^{D-3}(d+1)$.

5.3 Generalized de Bruijn digraphs

For any integers $d \geq 2$ and $n \geq 1$, the generalized de Bruijn digraph $GB(d, n)$, has $Z_n$ as its vertex set, and each vertex $x$ is adjacent to vertices $(dx + t)$ mod $n$ for any value of $t$ in $Z_d$. The name of this family of digraphs is due to the fact that it contains the de Bruijn digraphs as a sub-family. More precisely, $B(d, D) = GB(d, d^D)$. The generalized de Bruijn digraphs, also known as Reedy-Pradhan-Kuhl digraphs, have the property that $GB(d, dn) = L(GB(d, n))$, so for every integer $m \geq 1$, $GB(d, d^m n) = L^m(GB(d, n))$.

Corollary 5.3. For any integers $d \geq 2$, $n \geq 1$ and $m \geq 2$:

1. $Z(GB(d, d^m n)) = (d-1)d^{m-1}n$.
2. $M(GB(d, d^m n)) = (d-1)d^{m-1}n$.
3. $\text{mr}(GB(d, d^m n)) = d^{m-1}n$.
4. $\gamma_P(GB(d, d^m n)) = (d-1)d^{m-2}n$. 

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5.4 Generalized Kautz digraphs

For any integers \( d \geq 2 \) and \( n \geq 1 \), the \textit{generalized Kautz digraph} \( GK(d, n) \) has \( \mathbb{Z}_n \) as its vertex set, and each vertex \( x \) is adjacent to vertices \((-dx-t) \mod n\) for any value of \( t \) in \( \mathbb{Z}_d \). This family contains the Kautz digraphs as a sub-family. Indeed, \( K(d, D) = GK(d, d^D + d^{D-1}) \).

The generalized Kautz digraphs, also known as \textit{Imase-Itoh digraphs}, have the following property with respect to the line digraph: \( GK(d, dn) = L(GK(d, n)) \). Therefore, for every integer \( m \geq 1 \), \( GK(d, d^m n) = L^m(GK(d, n)) \).

**Corollary 5.4.** For any integers \( d \geq 2 \), \( n \geq 1 \) and \( m \geq 2 \):

1. \( Z(GK(d, d^m n)) = (d-1)d^{m-1}n \).
2. \( M(GK(d, d^m n)) = (d-1)d^{m-1}n \).
3. \( mr(GK(d, d^m n)) = d^{m-1}n \).
4. \( \gamma_P(Z(GK(d, d^m n))) = (d-1)d^{m-2}n \).

5.5 Directed wrapped butterfly

For any integers \( d, n \geq 2 \), the \textit{directed wrapped butterfly} \( WB(d, n) \) has for vertices the ordered pairs \((x, l)\) where \( x \) is an \( n \)-tuple of integers in \( \mathbb{Z}_d \) and \( l \) is an integer, \( 0 \leq l \leq n-1 \). A vertex \((x, l)\) is adjacent to \( d \) vertices in the form \((x_1 \ldots x_l x_{l+1} \ldots x_n, l + \alpha)\) where \( \alpha \in \mathbb{Z}_d \).

The wrapped butterflies \( WB(d, n) \) can be obtained as a line digraph, but we need some definitions prior to explain this in detail.

If \( G \) and \( H \) are two digraphs, the \textit{conjunction} of \( G \) and \( H \) is the digraph \( G \otimes H \), whose vertex set corresponds to \( V(G) \times V(H) \); a vertex \((g, h)\) is adjacent to a vertex \((g', h')\) if \((g, g')\) is an arc in \( G \) and \((h, h')\) is an arc in \( H \). Then, \( WB(d, n) = L^{n-1}(K_d \otimes C_n) \) where \( K_d \) denotes the complete symmetric digraph of order \( d \) with a loop on each vertex, and \( C_n \) denotes the cycle of order \( n \).

**Corollary 5.5.** For any integers \( d \geq 2 \) and \( n \geq 2 \):

1. \( Z(WB(d, n)) = (d-1)d^{n-1}n \).
2. \( M(WB(d, n)) = (d-1)d^{n-1}n \).
3. \( mr(WB(d, n)) = d^{n-1}n \).
4. \( \gamma_P(WB(d, n)) = (d-1)d^{n-2}n \).

For \( n = 2 \), \( WB(d, 2) = (K_d \otimes C_2) \) and \( \gamma_P(WB(d, 2)) = 2(d-1) \) follows from Corollary [1.15].
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