Quantum models related to fouled Hamiltonians of the harmonic oscillator

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Abstract We study a pair of canonoid (fouled) Hamiltonians of the harmonic oscillator which provide, at the classical level, the same equation of motion as the conventional Hamiltonian. These Hamiltonians, say $K_1$ and $K_2$, result to be explicitly time-dependent and can be expressed as a formal rotation of two cubic polynomial functions, $H_1$ and $H_2$, of the canonical variables $(q, p)$.

We investigate the role of these fouled Hamiltonians at the quantum level. Adopting a canonical quantization procedure, we construct some quantum models and analyze the related eigenvalue equations. One of these models is described by a Hamiltonian admitting infinite self-adjoint extensions, each of them has a discrete spectrum on the real line. A self-adjoint extension is fixed by choosing the spectral parameter $\varepsilon$ of the associated eigenvalue equation equal to zero. The spectral problem is discussed in the context of three different representations. For $\varepsilon = 0$, the eigenvalue equation is exactly solved in all these representations, in which square-integrable solutions are explicitly found. A set of constants of motion corresponding to these quantum models is also obtained. Furthermore, the algebraic structure underlying the quantum models is explored. This turns out to be a nonlinear (quadratic) algebra, which could be applied for the determination of approximate solutions to the eigenvalue equations.
I Introduction

A few years ago, in Ref. [1] a method was devised to find alternative Lagrangians for the time dependent oscillator

$$\ddot{q} + \omega(t)^2 q = 0, \quad (1)$$

where \( q = q(t), \omega(t) \) is a given differentiable function, and the dot stands for time-derivative. The method, which is based on the concept of folding transformation \[2\], is reviewed in Section II.

In this paper we study certain nonconventional quantum Hamiltonians corresponding to the classical fouled Hamiltonians associated with the Lagrangians derived in Ref. [1]. This investigation is motivated by the fact that these nonconventional quantum Hamiltonians, having a polynomial structure in the operators \( a \) and \( a^\dagger \), may play an important role in the context of quantum optics especially in handling coherence and squeezing of multiphoton systems.

All the fouled Lagrangians of the hierarchy found in Ref. [1] lead to the same equation of motion (1) as it occurs for the conventional Lagrangian \( L_1 = \frac{1}{2}(\dot{q}^2 - \omega(t)^2 q^2) \) (see (5)). In Section III the fouled Hamiltonians \( K_\pm \), related to the simplest fouled Lagrangians, \( L_2^{(1)} \) and \( L_2^{(2)} \), given by (23) and (24) are written down. By way of example, we have limited ourselves to consider the standard (harmonic) oscillator where \( \omega(t) \equiv \lambda \) is a constant. Furthermore, it is shown that (at the classical level), as one expects, \( K_\pm \) reproduce the same equation of motion coming from the conventional Hamiltonian \( H_0 = \frac{1}{2}(p^2 + \lambda^2 q^2) \). For brevity, we shall deal with \( K_+ \) only. It turns out that \( K_+ \) (see Section IV) can assume two possible forms, denoted by \( K_+^{(1)} \equiv K_1 \) and \( K_+^{(2)} \equiv K_2 \) (see (39) and (40)), which are independent but depend explicitly on time via the coefficients \( \cos \lambda t \) and \( \sin \lambda t \). Consequently, \( K_1 \) and \( K_2 \) can be formally interpreted as the result of a rotation (of an angle \( \lambda t \)) of the quantities \( H_1 = \sqrt{\lambda}(p^2 + \lambda^2 q^2)q \) and \( H_2 = \frac{2}{3\sqrt{\lambda}}p^3 \), which are not explicitly dependent on time. In other words, we have \( K_1^2 + K_2^2 = H_1^2 + H_2^2 \), and

$$\{K_1, K_2\}_{q,p} = \{H_1, H_2\}_{q,p} \quad (2)$$

(see Section IV), where the symbol \{ , \} denotes the Poisson bracket with respect to the canonical
variables \((q, p)\), namely

\[ \{A, B\}_{q,p} = \frac{\partial A}{\partial q} \frac{\partial B}{\partial p} - \frac{\partial A}{\partial p} \frac{\partial B}{\partial q}. \]  

(3)

In Section V the canonical quantization procedure based on a bosonic couple of annihilation and creation operators, \(a\) and \(a^\dagger\) (with \([a, a^\dagger] = 1\)), is applied to the fouled Hamiltonians \(K_1\) and \(K_2\). In such a way \(K_1\) and \(K_2\) are converted into a pair of Hamiltonian operators, say \(\mathcal{K}_1\) and \(\mathcal{K}_2\). (Classically, as we know, \(K_1\) and \(K_2\) provide the same equation of motion \((1)\)). Furthermore, since the canonical quantization procedure does affect only the cubic polynomial functions of \((q, p)\), namely \(H_1\) and \(H_2\), here we limit ourselves to study the operators \(\mathcal{H}_1\) and \(\mathcal{H}_2\), which are the quantized versions of \(H_1\) and \(H_2\), respectively. The arising quantum models are investigated in Sections VI and VII.

Precisely, in Section VI we show that the quantum model coming from \(\mathcal{H}_1\) gives rise to a linear second order eigenvalue equation of the Sturm-Liouville type (in the harmonic oscillator excitation number representation, or \(n\)-rep).

An interesting feature of the operator \(\mathcal{H}_1\) is that it is connected with an undetermined Hamburger moment problem \([3, 4, 5]\). We show that this operator has deficiency indices \((1,1)\) and allows a one-parameter family of self-adjoint extensions whose spectra are discrete and with no point in common. This goal has been achieved essentially by associating the operator \(\mathcal{H}_1\) with a Jacobi matrix \([3, 5]\). A possible physical interpretation of the operator \(\mathcal{H}_1\) is provided by fixing, among the infinite self-adjoint extensions, the extension corresponding to \(\epsilon = 0\), where \(\epsilon\) denotes the spectral parameter. In this case the eigenvalue equations for \(\mathcal{H}_1\) can be solved exactly in all the following representations: the harmonic oscillator excitation number representation \((n\)-rep), the coordinate space representation \((q\)-rep), and the Fock-Bargmann holomorphic function representation \((z\)-rep) \([3, 5, 6]\). In all these representations, for \(\epsilon = 0\) square-integrable solutions of the eigenvalue equations are explicitly obtained.

In Section VII we deal with the eigenvalue equation for \(\mathcal{H}_2\). The analysis of this Hamiltonian is trivial. It is exactly solvable \([3, 4]\), its spectrum is on the whole line and the related eigenfunctions can be easily found. Nevertheless, \(\mathcal{H}_2\) has been involved in the construction of two quantum models
described by the operators $\mathcal{H}_3$ and $\mathcal{H}_4$ (see (57) and (58)), where $\mathcal{H}_2 = \mathcal{H}_3 + \mathcal{H}_4$. One can attribute to $\mathcal{H}_3$ a physical meaning. Specifically, $\mathcal{H}_3$ can be interpreted as a special case of a class of Hamiltonians appearing in the higher order nonlinear optical processes [3, 11, 12]. This Hamiltonian is involved in the construction of third power squeezed states [3, 11, 13].

On the other hand, as we can see in Section VII, the operator $\mathcal{H}_4$ is closely related to $\mathcal{H}_1$ so that the solutions of the corresponding eigenvalue equation can be derived from the solutions of the eigenvalue equation for $\mathcal{H}_1$.

In Section VIII some constants of motion involving the operators $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4$ and $\mathcal{H}_5 \sim a^3 + a^{13}$ are derived. Section IX contains a discussion on a possible algebraic framework which could be employed to analyze Eq. (69). This approach resorts to a quadratic algebra in terms of which the operator $\mathcal{H}_1$ can be naturally expressed. Finally, in Section X a few concluding remarks are presented, while in the Appendix a detail of a calculation is reported.

II Fouled Lagrangians

We recall that a fouling transformation is a transformation under which the coordinates in configuration space are preserved:

$$Q = q,$$  \hspace{1cm} (4)

while

$$P_n = P_n(q, p, t),$$  \hspace{1cm} (5)

$P_n$ being a polynomial of degree $n$ in the variables $q$ and $p = \frac{\partial L_1}{\partial \dot{q}} = \dot{q}$, i.e.

$$P_n = \sum_{j=0}^{n} a_j p^{n-j} q^j,$$  \hspace{1cm} (6)

where

$$L_1 = \frac{1}{2}(q^2 - \omega(t)^2 \dot{q}^2)$$  \hspace{1cm} (7)
is the conventional Lagrangian, and \( a_j = a_j(t) \) are (real) time-dependent coefficients.

In [1] it was proven that the function \( L_n = L_n(q, \dot{q}, t) \) expressed by

\[
L_n = \sum_{j=0}^{n} \frac{1}{n-j+1} a_j \dot{q}^{n-j+1} q^j + (\dot{a}_n - \omega^2 a_{n-1}) \frac{q^{n+1}}{n+1}
\]

(8)
satisfies the equations

\[
\frac{\partial L_n}{\partial \dot{q}} = P_n, \quad \frac{\partial L_n}{\partial q} = \dot{P}_n.
\]

(9)

The compatibility condition for these equations provides

\[
\dot{a}_o = -\frac{n-1}{n} a_1, \quad \dot{a}_j = (n-j+1)\omega^2 a_{j-1} - (j+1)\frac{n-j-1}{n-j} a_{j+1},
\]

(10)

with \( j = 1, 2, ..., n-1 \). Furthermore, we have

\[
\frac{d}{dt} \frac{\partial L_n}{\partial \dot{q}} = \frac{\partial L_n}{\partial q} = I_n(t)(\dot{q} + \omega(t)^2 q),
\]

(11)

where \( I_n(t) \) is a time-dependent constant of motion, viz. \( \frac{d}{dt} I_n(t) = 0 \), given by

\[
I_n(t) = \sum_{j=0}^{n} (n-j) a_j p^{n-j-1} q^j = \frac{\partial^2 L_n}{\partial \dot{q}^2},
\]

(12)

for any \( n \geq 1 \).

From Eq. (9) we deduce that corresponding to the solution \( q \) of the generalized oscillator (1), \( L_n \) satisfies the Euler-Lagrange equation. We notice that for \( n = 1 \), Eq. (6) gives the conventional Lagrangian (5) (with \( a_0 = 1 \) and \( a_1 = 0 \)). The related invariant is \( I_1 = \frac{\partial^2 I_1}{\partial q^2} = 1 \).

Hereafter, by way of example, we are interested in the case \( n = 2 \). Then, from (6) and (10) we obtain

\[
L_2 = \frac{1}{3} a_0 \dot{q}^3 + \frac{1}{2} a_1 \dot{q}^2 q + a_2 \dot{q} q^2 + \frac{1}{3}(\dot{a}_2 - \omega^2 a_1) q^3,
\]

(13)

\[
I_2 = 2a_0 p + a_1 q,
\]

(14)

where (see (8))

\[
\dot{a}_0 = -\frac{1}{2} a_1, \quad \dot{a}_1 = 2\omega^2(t) a_0,
\]

(15)

namely

\[
\dot{a}_0 + \omega^2(t) a_0 = 0.
\]

(16)
The general solution of Eq. (14) can be written in the form

\[ a_0 = \sqrt{2} \sigma (c_1 \cos \frac{\theta}{2} + c_2 \sin \frac{\theta}{2}), \]  
\[ (17) \]

where \( c_1, c_2 \) are constants, and \( \sigma, \theta \) are defined by

\[ \dot{\sigma} + \omega^2(t) \sigma = \frac{1}{4 \sigma^3}, \]  
\[ (18) \]

and

\[ \dot{\theta} = \frac{1}{\sigma^2}. \]  
\[ (19) \]

In the following we shall limit ourselves to the choice \( \omega(t) = \lambda = \text{const} \). Consequently, Eqs. (16) and (17) admit the solution

\[ \sigma = \frac{1}{\sqrt{2\lambda}} \]  
\[ (20) \]

and

\[ \theta = 2\lambda t. \]  
\[ (21) \]

Equation (15) has the two independent solutions

\[ a_0^{(1)} = \frac{1}{\sqrt{\lambda}} \cos \lambda t, \quad a_0^{(2)} = \frac{1}{\sqrt{\lambda}} \sin \lambda t, \]  
\[ (22) \]

while the corresponding values \( a_1^{(1)}, a_1^{(2)} \) are (see (13))

\[ a_1^{(1)} = 2\sqrt{\lambda} \sin \lambda t, \quad a_1^{(2)} = -2\sqrt{\lambda} \cos \lambda t. \]  
\[ (23) \]

On the other hand, the expressions of \( a_2^{(1)}, a_2^{(2)} \) turn out to be (see [1])

\[ a_2^{(1)} = \lambda^\frac{3}{2} \cos \lambda t, \quad a_2^{(2)} = \lambda^\frac{3}{2} \sin \lambda t. \]  
\[ (24) \]

Now, using (20), (21) and (22), from (11) we obtain the two alternative fouled Lagrangians

\[ L_2^{(1)} = \frac{1}{3\sqrt{\lambda}} \cos \lambda t \dot{q}^3 + \sqrt{\lambda} \sin \lambda t \dot{\theta}^2 q + \lambda^\frac{3}{2} \cos \lambda t \dot{q}^2 q^2 - \lambda^\frac{3}{2} \sin \lambda \dot{q}^3, \]  
\[ (25) \]

\[ L_2^{(2)} = \frac{1}{3\sqrt{\lambda}} \sin \lambda t \dot{q}^3 - \sqrt{\lambda} \cos \lambda t \dot{\theta}^2 \dot{q} + \lambda^\frac{3}{2} \sin \lambda t \dot{q}^2 q^2 + \lambda^\frac{3}{2} \cos \lambda \dot{q}^3, \]  
\[ (26) \]
which furnish the two independent invariants

\[ I_2^{(1)} = \frac{\partial^2 L^{(1)}}{\partial \dot{q}^2} = \frac{2}{\sqrt{\lambda}} \cos \lambda t \dot{q} + 2\sqrt{\lambda} \sin \lambda t \ q, \]  

\[ I_2^{(2)} = \frac{\partial^2 L^{(2)}}{\partial \dot{q}^2} = \frac{2}{\sqrt{\lambda}} \sin \lambda t \dot{q} - 2\sqrt{\lambda} \cos \lambda t \ q, \]  

respectively (see (10)).

Equations (25) and (26) provide the general solution \( q \) and the momentum \( p = \dot{q} \) of the harmonic oscillator \( (\omega \equiv \lambda) \), i.e.

\[ q = \frac{1}{2\sqrt{\lambda}} (I_2^{(1)} \sin \lambda t - I_2^{(2)} \cos \lambda t), \]  

\[ p = \dot{q} = \frac{\sqrt{\lambda}}{2} (I_2^{(1)} \cos \lambda t + I_2^{(2)} \sin \lambda t), \]  

from which, as one expects,

\[ H_0 = \frac{1}{2} (p^2 + \lambda^2 q^2) = \frac{\lambda}{8} [(I_2^{(1)})^2 + (I_2^{(2)})^2] = \text{const}, \]  

where \( H_0 \) is the conventional Hamiltonian.

### III Fouled Hamiltonians

At this point, we build up the Hamiltonian corresponding to the Lagrangian \( L_2 \) given by (11). Following the procedure of Ref. [1], we get

\[ K_\pm = -\frac{a_1}{2a_0} qP \pm \frac{2}{3} a_0 \left( \frac{a_1^2}{4a_0^2} - \frac{a_2}{a_0} \right) q^2 + \left( \frac{a_1}{6a_0} (3a_2 - \frac{a_1^2}{2a_0}) - \frac{1}{3} (a_2 - \lambda^2 a_1) \right) q^3, \]  

where

\[ P = a_0 p^2 + a_1 q p + a_2 q^2. \]

For the sake of definiteness, later we shall study the Hamiltonian \( K_+ \) only.

The Hamilton equations for \( K_+ \) read

\[ \dot{q} = -\frac{a_1}{2a_0} q + \frac{I_2}{2a_0}, \]  

7
\[ P = \frac{a_1}{2a_0} P - \frac{a_1^2}{4a_0} q^2 - \frac{4a_2^2}{4a_0} I_2 q - 3 a_1 (\lambda^2 - \frac{a_1^2}{12a_0^2}) q^2, \]  

where the relation

\[ \left( \frac{a_2^2}{4a_0^2} - \frac{a_2}{a_0} \right) q^2 + \frac{P}{a_0} = \left( \frac{I_2}{2a_0} \right)^2 \]  

with \( a_2 = \lambda^2 a_0 \) has been used (see [1]).

By expliciting Eqs. (31) and (32) with the help of (12) and (30), we arrive at the equation

\[ I_2 (\dot{q} + \lambda^2 q) = 0, \]  

which tells us that the fouled Hamiltonian \( K_+ \), as it occurs for the related fouled Lagrangian \( L_2 \) (see (9)), gives rise to the same equation of motion emerging from the conventional Hamiltonian (29).

By virtue of (33), the Hamiltonians (30) take the form

\[ K_+ = \frac{1}{2} a_1 (p^2 + \lambda^2 q^2) q + \frac{2}{3} a_0 p^3, \]  

\[ K_- = -\frac{3}{2} a_1 p^2 q - \frac{a_2^2}{a_0} q^2 p + \left( \frac{a_1^2}{2} \lambda^2 - \frac{a_1^3}{6a_0^2} \right) q^3 - \frac{2}{3} a_0 p^3. \]  

One can see that

\[ K_\pm = Pp - L_2, \]  

where \( P \) and \( L_2 \) are given by (31) and (11).

## IV The fouled Hamiltonians \( K^{(1)}_+ \) and \( K^{(2)}_- \)

By using (20), (21) and (22), from (36) we derive the pair of (explicitly time-dependent) Hamiltonians

\[ K^{(1)}_+ \equiv K_1 = H_1 \sin \lambda t + H_2 \cos \lambda t, \]  

\[ K^{(2)}_+ \equiv K_2 = -H_1 \cos \lambda t + H_2 \sin \lambda t, \]  

where \( H_1 \) and \( H_2 \) are defined by

\[ H_1 = \sqrt{\lambda} (p^2 + \lambda^2 q^2) = 2 \sqrt{\lambda} H_0 q, \]  

\[ H_2 = \frac{2}{3 \sqrt{\lambda}} p^3. \]
One can check straightforwardly that the following evolution equations

\[ \dot{K}_1 = \{K_1, H_0\}_{q,p} + \frac{\partial K_1}{\partial t}, \quad (45) \]

\[ \dot{K}_2 = \{K_2, H_0\}_{q,p} + \frac{\partial K_2}{\partial t}, \quad (46) \]

\[ \dot{H}_1 = \{H_1, H_0\}_{q,p}, \quad (47) \]

\[ \dot{H}_2 = \{H_2, H_0\}_{q,p}, \quad (48) \]

hold, where \( H_0 \) is given by (29) and

\[ \{H_1, H_0\}_{q,p} = 2\sqrt{\lambda}pH_0, \quad (49) \]

\[ \{H_2, H_0\}_{q,p} = -2\frac{\chi^2}{\sqrt{\lambda}}p^2q. \quad (50) \]

We also have that the Poisson brackets between \( K_1, K_2 \) and \( H_1, H_2 \) coincide, i.e.

\[ \{K_1, K_2\}_{q,p} = \{H_1, H_2\}_{q,p}. \quad (51) \]

This is a direct consequence of the rotation form of the transformations (39) and (40).

**V  Quantization**

Hereafter, we shall put for simplicity \( \lambda = \hbar = 1 \).

In order to quantize the fouled Hamiltonians (39) and (40), let us introduce the operators

\[ \hat{q} = \frac{1}{\sqrt{2}}(a + a^\dagger), \quad \hat{p} = -\frac{i}{\sqrt{2}}(a - a^\dagger), \quad (52) \]

where \( a \) and \( a^\dagger \) denote a (boson) annihilation and a creation operator, respectively.

By means of (50), we can write the operators \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) corresponding to the classical functions (41) and (42). We obtain

\[ \mathcal{H}_1 = \sqrt{2}(a^{12}a + a^1a^2 + a^1 + a), \quad (53) \]

\[ \mathcal{H}_2 = \frac{i}{\sqrt{2}}\left[ \frac{1}{3}(a^3 - a^1a^2a^1) + (a^{12}a - a^1a^2 + a^1 - a) \right]. \quad (54) \]
with the help of the commutation rule

$$[a, a^\dagger] = 1$$  \hspace{1cm} (55)

or, in terms of the Heisenberg commutation relation: \(\hat{q}, \hat{p} = i\).

Now let us use the representation \(\hat{q} = x, \hat{p} = -i\frac{d}{dx}\), so that the operators \(a\) and \(a^\dagger\) can be written as (see (50))

\[
a = \frac{1}{\sqrt{2}}(x + \frac{d}{dx}), \quad a^\dagger = \frac{1}{\sqrt{2}}(x - \frac{d}{dx}).
\]  \hspace{1cm} (56)

In terms of these quantities, \(\mathcal{H}_1\) and \(\mathcal{H}_2\) take the forms

\[
\mathcal{H}_1 = -(x\frac{d^2}{dx^2} + \frac{d}{dx}) + x^3,
\]  \hspace{1cm} (57)

and

\[
\mathcal{H}_2 = \frac{2i}{3}\frac{d^3}{dx^3}.
\]  \hspace{1cm} (58)

For later convenience, we shall report also in the representation (54) the following operators appearing in (52):

\[
\mathcal{H}_3 \equiv \frac{i}{\sqrt{2}}\frac{1}{3}(a^3 - a^{13}) = \frac{i}{2}\frac{1}{3}\frac{d^3}{dx^3} + (x^2 \frac{d}{dx} + x)],
\]  \hspace{1cm} (59)

\[
\mathcal{H}_4 \equiv \frac{i}{\sqrt{2}}(a^{12}a - a^1a^2 + a^\dagger - a) = \frac{i}{2}\frac{d^3}{dx^3} - (x^2 \frac{d}{dx} + x)].
\]  \hspace{1cm} (60)

In the next Section, we shall study the operators \(\mathcal{H}_1\) and \(\mathcal{H}_2\) by dealing with the corresponding eigenvalue problems.

**VI  Self-adjoint extensions of the operator \(\mathcal{H}_1\)**

In order to clarify the quantum-mechanical meaning of the Hamiltonian \(\mathcal{H}_1\), it is crucial to establish whether \(\mathcal{H}_1\) enjoys self-adjoint type properties. In doing so, first let us recall that one can provide different representations for the operator (51), which correspond to various forms of the related eigenvalue equations. We shall consider the following representations: the harmonic oscillator excitation
number representation \((n-\text{rep})\), the coordinate representation \((q-\text{rep})\), and the Fock-Bargmann holomorphic function representation \((z-\text{rep})\).

For reader’s convenience, we shall summarize below the main properties of these representations.

Let us \(H\) denote the Hilbert space where the operators \(\hat{q}, \hat{p}\) and \(a^\dagger, a\) act. In the \(n-\text{rep}\), the vectors \(\{|n>\}\) form a basis in \(H\). The following relations
\[
a = \sqrt{n} \mid n - 1 \rangle, \quad a^\dagger \mid n \rangle = \sqrt{n+1} \mid n + 1 \rangle, \quad a^\dagger a \mid n \rangle = n \mid n \rangle
\]
hold.

On the other hand, in the \(q-\text{rep}\) a vector \(\mid \psi \rangle\) belonging to the space \(H\) is represented by a coordinate function \(<q \mid \psi >= \psi(q)\) which is square-integrable:
\[
\int_{-\infty}^{+\infty} |\psi(q)|^2\, dq < \infty.
\]
The basic vector \(\mid n \rangle\) is described by the function
\[
<q \mid n > = \varphi_n(q) = N_n H_n(q) \exp(-\frac{q^2}{2}),
\]
where \(N_n = (\sqrt{\pi} 2^n n!)^{-\frac{1}{2}}\) and \(H_n(q)\) is the Hermite polynomial of degree \(n\). In the \(q-\text{rep}\), formulas (59) become the standard recursion relations for the Hermite polynomials.

In order to introduce the Fock-Bargmann representation (or \(z-\text{rep}\), let \(\mid \psi \rangle\) be an arbitrary normalized vector in \(H\), namely
\[
\mid \psi \rangle = \sum_{n=0}^{\infty} c_n \mid n \rangle
\]
with \(<\psi \mid \psi >= \sum_{n=0}^{\infty} |c_n|^2 = 1\). Furthermore, taking into account the Glauber form
\[
\mid z \rangle = \exp\left(-\frac{|z|^2}{2}\right) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \mid n \rangle
\]
the state \(\mid \psi \rangle\) is completely determined by
\[
<z \mid \psi >= \exp\left(-\frac{|z|^2}{2}\right) \psi(z),
\]
where
\[ \psi(z) = \sum_{n=0}^{\infty} c_n u_n(z), \] (67)
with \( u_n(z) = \frac{z^n}{\sqrt{n!}} \). Owing to the condition \( \sum_{n=0}^{\infty} |c_n|^2 = 1 \), the series in (65) converges uniformly in any compact domain of the complex \( z \) plane. Consequently, \( \psi(z) \) turns out to be an entire holomorphic function in the \( z \) plane, and
\[ \| \psi \|^2 = \int \exp(-|z|^2) |\psi(z)|^2 d\mu(z) < \infty, \] (68)
where \( d\mu(z) = \pi^{-1} dx dy \), \( z = x + iy \). [8]

The scalar product of two entire functions \( \psi_1(z) \) and \( \psi_2(z) \) obeying the condition (66) is given by
\[ \langle \psi_1 | \psi_2 \rangle = \int \exp(-|z|^2) \bar{\psi}_1(z) \psi_2(z) d\mu(z). \] (69)
As it was proven by Bargmann [7], the Fock-Bargmann representation space with a scalar product provided by (67), is really a Hilbert space. We observe also that in the \( z \)-rep, the operator solution for the commutation relation \([a, a^\dagger] = 1\) [6] is
\[ a \rightarrow \frac{d}{dz}, \quad a^\dagger \rightarrow z. \] (70)

Now let us consider the eigenvalue equation
\[ (a^\dagger a^2 + a^\dagger a + a + a^\dagger) |\chi\rangle = \varepsilon |\chi\rangle. \] (71)
Starting from the \( n \)-rep and following the lines of Ref. [8], let us put
\[ |\chi\rangle = \sum_{n=0}^{\infty} f_n(\varepsilon) |n\rangle \] (72)
into Eq. (69). By using Eqs. (59), after simple calculations we obtain the recursion formula
\[ (n + 1)^2 f_{n+1}(\varepsilon) - \varepsilon f_n(\varepsilon) + n^2 f_{n-1}(\varepsilon) = 0 \] (73)
for \( n \geq 1 \), and
\[ f_1(\varepsilon) = \varepsilon f_0(\varepsilon) \] (74)
(the boundary condition).
We remark that the sequence \( \{ f_n \} \) is such that the series \( \sum_{n=0} |f_n|^2 \) converges, i.e. the sequence \( \{ f_n \} \equiv (f_0, f_1, f_2, \ldots) \) belongs to the Hilbert space \( l^2 \).

To show this, first let us consider the case \( \varepsilon = 0 \). Then, Eqs. (71)-(72) provide

\[
f_{2n} = (-1)^n \left( \frac{(2n-1)!!}{(2n)!!} \right)^{3/2} f_0,
\]

the odd terms being zero. Thus, from the asymptotic formula of Gamma function ([14], p. 257), we deduce that for \( n \to \infty \), \( |f_{2n}|^2 \) behaves as \( n^{-3/2} \), so that \( \sum_{n=0} |f_{2n}|^2 < \infty \). In general, i.e. for \( \varepsilon \neq 0 \), Eq. (71) tells us that for great values of \( n \), the sum of the first and the last term is leading with respect to the second term. This allows us to see easily that both even and odd terms, \( |f_{2n}|^2 \) and \( |f_{2n+1}|^2 \), behave asymptotically as \( n^{-3/2} \). To conclude, the series \( \sum_{n=0} |f_n|^2 \) is convergent, namely the sequence \( \{ f_n \} \) belongs to the Hilbert space \( l^2 \) for any value of the spectral parameter \( \varepsilon \). Then, the equation \( H\chi = \varepsilon \chi \), for \( \Im \lambda \neq 0 \), has nontrivial solutions ([3], p. 140).

Now, by introducing the notation \( b_n = (n+1)^{3/2} \), we see that it is possible to associate with the difference equation (71) the (infinite) Jacobi matrix

\[
A = \begin{pmatrix}
0 & b_0 & 0 & 0 & \cdots \\
b_0 & 0 & b_1 & 0 & \cdots \\
0 & b_1 & 0 & b_2 & \cdots \\
0 & 0 & b_2 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

so that Eq. (69) is equivalent to the eigenvalue equation \( Af = \varepsilon f \), with \( f = (f_0, f_1, \ldots)^T \).

The Jacobi matrix (74) plays a crucial role in the study of the Hamburger moment problem (see, for example, [8, 9, 13]). Precisely, let us consider the moments

\[
s_n = \int_{-\infty}^{+\infty} x^n d\sigma(x), \quad n = 0, 1, 2, \ldots,
\]

where \( \sigma \) denotes a (positive) measure on \( \mathbb{R} \) ([15], p. 145). The Hamburger moment problem is to determine conditions on a sequence of real numbers \( \{ s_n \}_{n=0}^\infty \), so that there exists a measure satisfying...
One can show that a sequence of real numbers \( \{s_n\} \) are the moments of a positive measure on \( \mathcal{R} \) if and only if for all \( N \) and all \( \alpha_0, \alpha_1, ..., \alpha_N \in \mathbb{C} \), one has
\[
\sum_{n,m=0}^{N} \alpha_n \alpha_m s_{n+m} \geq 0.
\] (78)

From the Jacobi matrix (74) we get the limitations
\[
\sum_{n=0}^{\infty} \frac{1}{b_n} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} < \infty, \quad b_{n-1}b_{n+1} < b_n^2.
\] (79)

Consequently, the Jacobi matrix (74) belongs to the type \( C \) (limit circle case), and corresponds to an undetermined Hamburger moment problem \([\text{3, 5}]\). So, the properties of the operator \( \mathcal{H}_1 \sim a^2a + a^\dagger a^2 + a + a^\dagger \) are similar to the properties of the operator \( a^k + a^\dagger k \) \((k = 3)\) discussed by Nagel \([\text{5}]\). In other words, the operator \( \mathcal{H}_1 \) has deficiency indices \((1,1)\) and allows a one-parameter family of self-adjoint extensions, each having a purely discrete spectrum on the real line \([\text{4, 16}]\). The spectra of two different extensions turn out to have no point in common \((\text{3, p. 152})\). We have that different self-adjoint extensions correspond to different dynamics \([\text{13, 17}]\).

Since every self-adjoint extension of \( \mathcal{H}_1 \) has a discrete spectrum on the real line, and taking one eigenvalue determines the corresponding extension uniquely, let us choose \( \varepsilon = 0 \).

In this case the eigenvalue equation underlying the operator \( \mathcal{H}_1 \) can be solved exactly in all the representations mentioned at the beginning of this Section.

To show this, let us deal with the \( q- \) rep. With the help of (54), the eigenvalue equation (69) reads
\[
x\phi'' + \phi' + (\sqrt{2}\varepsilon - x^3)\phi = 0,
\] (80)

where \( \phi' \equiv \frac{d}{dx} \phi \).

Equation (78) can be written as the Sturm-Liouville equation \((\text{18, p. 59})\)
\[
\mathcal{L}[\phi(x)] = -\sqrt{2}\varepsilon \phi(x),
\] (81)

where \( \mathcal{L} \) denotes the Sturm-Liouville operator
\[
\mathcal{L} = \frac{d}{dx} \left( x \frac{d}{dx} \right) - x^3.
\] (82)
By using the transformation

\[ \phi(x) = \exp(-x^2/2)\psi(x), \tag{83} \]

Equation (78) becomes

\[ x\psi'' + (1 - 2x^2)\psi' + (\sqrt{2}\xi - 2x)\psi = 0. \tag{84} \]

This equation is satisfied by

\[ \psi(x) = \sum_{n=0}^{\infty} f_n N_n H_n(x), \tag{85} \]

where the coefficients \( f_n \) fulfill the recursion relations (71) and (72). We have already shown that \( \{f_n\} \in l^2 \). Then, the function \( \psi(x) \) belongs to the Hilbert space \( L^2_{e^{-x^2}}(-\infty, \infty) \) (\( \exp(-x^2) \) is the weight function). We point out that generally the relation (83) is not valid in the pointwise sense, but it holds in accordance with the metric of \( L^2_{e^{-x^2}}(-\infty, \infty) \), namely

\[ \lim_{n \to \infty} \int_{-\infty}^{+\infty} |\psi(x) - \sum_{k=0}^{n} f_k N_k H_k(x)|^2 \exp(-x^2)dx = 0. \tag{86} \]

For \( \varepsilon = 0 \), via the change of variable \( \xi = x^2 \) Eq. (82) can be written as a special case of a Kummer equation, whose independent solutions are \( M(\frac{1}{2}, 1; x^2) \) and \( U(\frac{1}{2}, 1; x^2) \) (\[14\], p. 504). In the case the solution \( \psi(x) \) of Eq. (82) with the property \( \psi(x) \in L^2_{e^{-x^2}}(-\infty, \infty) \) is given by

\[ \psi(x) = c U(\frac{1}{2}, 1; x^2) = \sum_{n=0}^{\infty} f_{2n} N_{2n} H_{2n}(x), \tag{87} \]

where the constant \( c \) is such that \( c\pi^{\frac{1}{4}} = f_0 \), and

\[ f_{2n} = f_0 \pi^{-\frac{1}{4}} N_{2n} \int_{-\infty}^{+\infty} U(\frac{1}{2}, 1; x^2)H_{2n}(x)\exp(-x^2)dx. \tag{88} \]

One can easily check that (86) is satisfied for any \( n \in \mathbb{N} \).

Finally, in the \( z- \) rep, i.e. for \( |\chi > \to \chi, a^\dagger \to z, a \to \frac{d}{dz}, \) Eq. (69) gives

\[ z\chi_{zz} + (1 + z^2)\chi_z + (z - \varepsilon)\chi = 0, \tag{89} \]

where

\[ \chi = \sum_{n=0}^{\infty} f_n \frac{z^n}{\sqrt{n!}} \tag{90} \]
and $f_n$ satisfies the recursion relations (71) and (72). For $\varepsilon = 0$, the eigenvalue equation (87) as well can be exactly solved. In fact, by setting $z^2 = y$, $\zeta = -\frac{y}{2}$, this equation becomes a special case of the Kummer equation.

Therefore, in the $z-$ rep, where the eigenfunction should be a holomorphic (and normalizable) function in the whole $z-$ plane, one has the solution $M\left(\frac{1}{2}, 1; -\frac{z^2}{2}\right)$. The other solution, namely the Kummer function $U\left(\frac{1}{2}, 1; -\frac{z^2}{2}\right)$, is not holomorphic at $z = 0$.

To conclude this Section, we observe that the solutions of the eigenvalue equation for $H_1$ in the case $\varepsilon = 0$ can be also found from Eqs. (71) and (72), obtained within the $n-$ rep, by means of the standard integral representations of the confluent hypergeometric functions. An example of this procedure is displayed in [19].

VII The $H_2$, $H_3$, $H_4$ models

The eigenvalue problem for the operator $H_2$ can be written as

$$\mathcal{L}\psi = \varepsilon\psi, \quad (91)$$

where $\mathcal{L} = \frac{2i}{\sqrt{3}} D_x^3 \ (D_x = \frac{d}{dx})$. The study of Eq. (89) is trivial. It is exactly solvable [3], its spectrum is on the whole line, and the related eigenfunctions are of the exponential type.

The operator $H_3$ (see (57)) belongs to the class of Hamiltonians

$$\mathcal{H} = i\kappa_n (a^n - a^\dagger n) \quad (92)$$

appearing in the higher order nonlinear optical processes. In particular, (57) describes a subharmonic generation process, in which a photon from a strong pump beam produces $n$ photons of the signal beam in a nonlinear medium [1]. The constant $\kappa_n$ is related to the $n$th nonlinear susceptibility coefficient and to the amplitude of the pump field, while $a$ and $a^\dagger$ are the annihilation and the creation operators for the signal field. In this context, the evolution of an arbitrary initial state $|\Psi(0)>$ of the signal field to the state $|\Psi(t)>$ is governed by

$$|\Psi(t)> = \exp[i\kappa_n(a^n - a^\dagger n)] \ |\Psi(0)> \quad (93)$$
The squeezing of this state was examined by Hillery, Zubairy and Wódkiewicz \cite{11}. They showed that to any order in the coupling constant $\kappa_n$, the vacuum state is not squeezed in the higher order nonlinear optical processes ($n \geq 3$).

This important result stimulated the analysis of $n$th power squeezed states. Interesting (and, generally, not yet completely explored) questions arise in connection with this argument. Some of them are discussed in \cite{5, 13, 19} and references therein.

Now let us make some comments about the Hamiltonian $H_4$. This operator is closely related to $H_1$. This can be seen by means of the phase transformation $a' = ia$, $a'^\dagger = -ia'^\dagger$, so that $H_4$ takes the form

$$H_4 = -\frac{1}{\sqrt{2}}(a'^2 a' + a'^\dagger a'^2 + a'^\dagger + a').$$

(94)

In other words, one has $H_4 = \frac{1}{2}H_1$ (in terms of the primed operators). This corresponds to pick up $q' = -p$ and $p' = q$. In such a way $H_4$ turns out to be the inverse Fourier transform of (55). Therefore, the solutions of the eigenvalue equation for the Hamiltonian operator $H_4$ can be derived from the solutions of the eigenvalue equation for the Hamiltonian operator $H_1$ (see (69)).

The eigenvalue equation for $H_1$ ($H_4$) can be investigated by means of the algebraic approach outlined in Section IX.

\section*{VIII Equations and constants of motion related to the operators $H_j$}

The equations of motion for the Hamiltonians $H_j$ ($j = 1, 2, 3, 4$) arise immediately by using the Heisenberg representation. In other words, by putting $a(t) = a(0)\exp(-it)$ and $a'^\dagger(t) = a'^\dagger(0)\exp(it)$ in the expressions (51) and (58), we easily find (as one expects) that $H_1$ and $H_4$ satisfy the same equation of motion (i.e., the equation for the harmonic oscillator of frequency $\lambda = 1$):

$$\frac{d^2}{dt^2}H_1 + H_1 = 0, \quad \frac{d^2}{dt^2}H_4 + H_4 = 0.$$ 

(95)
On the other hand, for the operators $H_3 = \frac{i}{\sqrt{2}}(a^3 - a^\dagger)$ (see (57)) and $H_5 = \frac{1}{\sqrt{2}}(a^3 + a^\dagger)$, the same considerations made for $H_1$ and $H_4$ in Section VII hold. One has that $H_3$ and $H_5$ obey the same equation of motion (i.e., the equation for the harmonic oscillator of frequency 3):

$$\frac{d^2}{dt^2} H_3 + 9H_3 = 0, \quad \frac{d^2}{dt^2} H_5 + 9H_5 = 0. \quad (96)$$

Some comments on Eqs. (93) and (94) are presented in Section X.

At this point, we observe that in addition to the constants of motion

$$\tilde{q} = e^{-i\hat{H}_0} \hat{q} e^{i\hat{H}_0} = \hat{q} \cos t - \hat{p} \sin t, \quad (97)$$

$$\tilde{p} = e^{-i\hat{H}_0} \hat{p} e^{i\hat{H}_0} = \hat{q} \sin t + \hat{p} \cos t, \quad (98)$$

where $[\hat{q}, \hat{p}] = i$, $\hat{H}_0 = \frac{1}{2}(\hat{p}^2 + \hat{q}^2)$ and $\tilde{q} = -\frac{1}{2}L_2^{(2)}$, $\tilde{p} = \frac{1}{2}L_2^{(1)}$, we obtain also the following set of constants:

$$\tilde{H}_1 = H_1 \cos t - 2H_4 \sin t, \quad (99)$$

$$\tilde{H}_4 = \frac{1}{2}H_1 \sin t + H_4 \cos t, \quad (100)$$

$$\tilde{H}_3 = H_3 \cos 3t - H_5 \sin 3t, \quad (101)$$

$$\tilde{H}_5 = H_3 \sin 3t + H_5 \cos 3t. \quad (102)$$

In terms of the arbitrary constants $\tilde{H}_j$ ($j = 1, 4, 3, 5$), one can express the general solutions of Eqs. (93) and (94), which can be easily found by inverting the transformations (97), (98) and (99), (100), respectively.

**IX A possible algebraic framework for the study of Eq. (69)**

For $\varepsilon \neq 0$, Eq. (69) could be investigated following different analytical techniques. Among these, an important role is played by the algebraic approach, which allows one to express Eq. (69) in terms of the generators of a \textit{quadratic} algebra [20, 21].

As an example of a quadratic algebra, we can consider a nonlinear deformation of the (linear) $su(1,1)$ algebra, defined by the relations
\[ [J_0, J_\pm] = \pm J_\pm, \]  
\[ [J_+, J_-] = P(J_0), \]  
where the Jacobi identity holds. \( J_\pm \) are the ladder operators, and \( P(J_0) \) is a second-degree polynomial function of the diagonal operator \( J_0 \). \( P(J_0) \) can be written in the form

\[ P(J_0) = \alpha_1 + \alpha_2 J_0 + \alpha_3 J_0^2, \]  
where \( \alpha_i \) (\( i = 1, 2, 3 \)) are arbitrary coefficients (\( \alpha_3 \neq 0 \)).

Now let us introduce the (bosonic) realization

\[ J_0 = a^\dagger a + \frac{1}{2}, \]  
\[ J_- = -(a^\dagger a^2 + a), \]  
\[ J_+ = a^{12} a + a^\dagger, \]  
with \( J_+ = -(J_-)^\dagger \). Then, the operators (104)–(106) turn out to obey the commutation relations

\[ [J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = \frac{1}{4} + 3 J_0^2. \]

We remark that the quadratic algebra (107) is a finite \( W^{(2)}_3 \)-algebra [20, 21], which corresponds to choose \( \alpha_1 = \frac{1}{4}, \alpha_2 = 0, \) and \( \alpha_3 = 3 \) in (103).

By virtue of (104)–(106), Eq. (69) can be re-expressed as

\[ \Omega(\varepsilon) \mid \chi > = 0, \]  
with

\[ \Omega(\varepsilon) = J_+ - J_- - \varepsilon. \]

Therefore, the eigenvalue problem (69) can be formulated in terms of the generators \( J_\pm \) of a quadratic algebra of the type \( W^{(2)}_3 \). The Casimir operator of the quadratic algebra (107) is considered in the Appendix.
X Conclusions

In the context of the existence of an infinite set of fouled Lagrangians and Hamiltonians for the generalized (time-dependent) oscillator, we have studied a special case where the frequency of the oscillator is assumed to be a constant $\lambda$ (harmonic oscillator). By way of example, we have considered a pair of independent fouled Lagrangians ($L_2^{(1)}$, $L_2^{(2)}$) (see (23)-(24)) and, in correspondence, a pair of fouled independent Hamiltonians ($K_1$, $K_2$) (see (39)-(40)) which lead, at the classical level, to the same equation of motion provided by the conventional Lagrangian and Hamiltonian. Both ($L_2^{(1)}$, $L_2^{(2)}$) and ($K_1$, $K_2$) are explicitly time-dependent. The method followed to find these alternative Lagrangians and Hamiltonians implies the construction of two independent invariants (constants of motion), $I_2^{(1)}$ and $I_2^{(2)}$; in terms of them the canonical variables $q$ and $p$ can be expressed. These invariants, which are of the Nöther type [22], are connected with the "quadrature-phase amplitudes" appearing in the problem of generation of squeezed states in certain optical devices [23].

In this paper we have focused our attention mainly on the quantized version of the fouled Hamiltonians ($K_1$, $K_2$). Since $K_1$ and $K_2$ are related to the cubic polynomials $H_1 = \sqrt{\lambda}(p^2 + \lambda^2 q)q$ and $H_2 = \frac{2}{3\sqrt{\lambda}}p^3$ by a formal rotation, the canonical quantization prescription affects essentially $H_1$ and $H_2$. Our purpose has been to study, at the quantum level, the Hamiltonians $\mathcal{H}_1$ and $\mathcal{H}_2$ (see (51), (52)) corresponding to $H_1$ and $H_2$. The model described by $\mathcal{H}_2$ is associated with an eigenvalue problem given by a (linear) third order differential equation, with constant coefficients, which is exactly solvable [3]. Furthermore, we have put $\mathcal{H}_2 = \mathcal{H}_3 + \mathcal{H}_4$, where $\mathcal{H}_4$ (see (58)) turns out to be closely related to the quantum model $\mathcal{H}_1$ in the sense discussed in Section VII. On the other hand, the quantum model $\mathcal{H}_3$ has a well-defined physical interpretation. It belongs to a class of Hamiltonians which finds applications in the field of $n$th power squeezed states.

Finally, the operator $\mathcal{H}_1$, considered in Section VI, has deficiency indices (1,1) and allows a one-parameter family of self-adjoint extensions, each having a purely discrete spectrum on the real line. The spectra of two different extensions have no point in common. Since different self-adjoint extensions correspond to different dynamics, we needed to fix a given dynamics. This has been carried
out by choosing the value $\varepsilon = 0$ for the eigenvalue parameter. In this case all the differential equations coming from (69) in all the representations: $n-$ rep, $q-$ rep and $z-$ rep, can be solved exactly. In this case, square-integrable solutions of the eigenvalue equations are explicitly determined.

In order to investigate some properties of the Hamiltonians $H_j$ ($j = 1, 2, 3, 4$), in Section VIII we have introduced the operator $H_5 \sim (a^3 + a^3)\dagger$, which is connected with $H_3$, as one can see by using the transformation $a' = ia, a'^\dagger = -ia\dagger$. We have built up a set of constants of motion involving $(H_1, H_4)$ and $(H_3, H_5)$. These constants play the role of the arbitrary constants present in the general solution of two equations of the harmonic oscillator-type of frequencies 1 and 3, respectively, which are satisfied by $(H_1, H_4)$ and $(H_3, H_5)$ (see (93) and (94)).

To conclude, we point out that quadratic (and, more in general, polynomially deformed algebras) take place in quantum optics in relation to the construction of coherent states and in the description of multiphoton processes (see, for examples, [24, 25] and references therein). Keeping in mind these problems, it should be of interest to deal with the quantization of fouled Hamiltonians (of the generalized oscillator) expressed by polynomials in the canonical variables $(q, p)$ of degree higher than three.

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XI Appendix: the Casimir invariant for the quadratic algebra

\begin{equation}
(107)
\end{equation}

We remind the reader that a standard form of a quadratic algebra, which is important in the treatment of coherent states of trilinear boson Hamiltonians [26], is [24, 25, 27]

\begin{equation}
[N_0, N_\pm] = \pm N_\pm, \quad [N_+, N_-] = \pm 2N_0 + \delta N_0^2,
\end{equation}
where the positive (negative) sign of $2N_0$ indicates a polynomially deformed $su(2)$ ($su(1, 1)$), and $\delta$ is a parameter.

The Casimir operator is given by \cite{24, 25, 27}

$$C = N_+ N_+ + N_0(N_0 + 1)\left[1 + \frac{\delta}{6}(2N_0 + 1)\right].$$  \hfill (113)

By setting

$$J_0 = a_0 N_0 + b_0,$$  \hfill (114)
$$J_- = k_1 N_-,$$  \hfill (115)
$$J_+ = k_2 N_+,$$  \hfill (116)

and choosing, for example,

$$a_0 = 1, \quad b_0 = \frac{i}{2 \sqrt{3}}, \quad k_1 k_2 = \frac{i \sqrt{3}}{2},$$  \hfill (117)

the commutator relations (107) are converted into

$$[N_0, N_{\pm}] = \pm N_{\pm}, \quad [N_+, N_-] = 2N_0 + \delta N_0^2,$$  \hfill (118)

with $\delta = -2i \sqrt{3}$. Hence, the Casimir operator \cite{113} reads

$$C = N_+ N_+ + N_0(N_0 + 1)\left[1 - \frac{i}{\sqrt{3}}(2N_0 + 1)\right].$$  \hfill (119)

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