Complex conjugate for second-order derivative of 2D analytic signal of magnetic field anomaly

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Abstract. The complex conjugate approach for the second-order derivative of a function was obtained by truncating the Taylor series whose function has been expanded around \( x \) in the \( h \)-term in the complex conjugate argument. This approach was composed to resemble the Central-Difference form of the ordinary Finite-Difference Method. After being tested by comparing it with manual solution, this method was very accurate with relative errors obtained in the range of \( 0.088271927039904 \leq \text{RE} \leq 6.67999344897438 \times 10^{-9} \). The combination of step \( h \) and interval \( \delta x \) in the formula of this approach that was appropriate for obtaining optimal computational results were \( h = 1.0 \times 10^{-13} \) and \( \delta x = 1.0 \times 10^{-10} \). Finding the appropriate combination of \( h \) and \( \delta x \) could be done easily and quickly, requiring no special treatment. The computational results of the derivatives were then used to compute second-order 2D Analytic Signals of magnetic field anomalies. Illustratively, Analytic Signal could transform bipolarity of magnetic data into positive polarity so that it would be very useful in processing and interpreting actual data.

1. Introduction

To solve derivatives in geophysics, a variety of methods have been developed for both first and second-order methods. Among the methods that are commonly used is Finite-Difference Method. This method is widely used because it is easily implemented in computing by just writing simple codes.

In this study, a new approach was developed by utilizing complex variable to calculate second-order derivative of magnetic field anomaly function in the form of complex conjugate. This approach built on the philosophy of the Finite-Difference Method. While the function in question was assumed to be caused by a 2D finite prism. The approach was conducted by truncating the Taylor series expansion from the function of the magnetic field to a certain term. For derivative estimate, the complex variable approach was first introduced by [4,5] and further simplified by [6]. Whereas in the form of complex conjugate it has basically been investigated by [19] for first order, and the computational results obtained show that this approach was very accurate and highly precision when compared with manual or analytical solution with relative errors ranging from \( 3.159607284251312 \times 10^{-17} \) to \( 7.079603533496899 \times 10^{-12} \). Besides that, it was also known that series truncation did not have a significant effect on computational results. This could be seen when \( h \) got smaller, then the truncation error was stable at zero. In other words, the computational accuracy for first-order derivatives did not depend on the choice of step \( h \).

Furthermore, the complex conjugate approach was further developed for the second-order by truncating the Taylor series from the function of the magnetic field anomaly that expanded around \( x \).
in the $h$-term, and for the formula to followed the Central-Difference approach of the ordinary Finite-Difference method. The second-order derivative which has been mathematically derived was implemented using simple codes in the Matlab® programming language. The computational results were then used to compute 2D analytic signals. The amplitude of the 2D analytic signal is the square root of the second-order derivative of a function.

Analytic signals are widely used in geophysical data processing especially to identify the edges of anomalous source objects. Analytic signal is complex variable with imaginary component being Hilbert transform from the real one [2,3], and it is able to transform bipolarity of magnetic data into monopolarity so that it will ultimately facilitate the processing of geophysical data. In processing the data, analytic signal can be developed into a transformation of Reducion to Pole (RTP). For interpretation of magnetic data, analytic signals are first utilized by [2]. Several other researchers who have used analytic signals in the field of geophysics include [15,16,17,18].

2. 2D Analytic Signal

In general, the analytic signal $a(x)$ of the function $f(x)$ based on complex variable is given by [3]

$$a(x) = f(x) + iH_t[f(x)]$$

where $H_t[f(x)]$ represents Hilbert transform of $f(x)$ and lies in the imaginary component of equation (1). [2] has applied and used this concept to analyze 2D magnetic bodies. For a magnetic field $T(x)$ caused by a 2D body parallel to the $y$-axis and measured on surface along the $x$-axis, then the analytic signal is rewritten as

$$a(x,z) = \frac{\partial T}{\partial x} + i \frac{\partial T}{\partial z}$$

which satisfies the Cauchy-Riemann conditions [3], so the two equations above are interrelated

$$f(x) = \frac{\partial T}{\partial x} \quad \text{and} \quad H_t[f(x)] = -\frac{\partial T}{\partial z}$$

The absolute value of the analytic signal in equation (2) yields an amplitude

$$|a(x,z)| = \left[ \left( \frac{\partial T}{\partial x} \right)^2 + \left( \frac{\partial T}{\partial z} \right)^2 \right]^{1/2}$$

For second-order analytic signal, the amplitude is defined here by the following expression [15]

$$|a_2(x,z)| = \left[ \left( \frac{\partial^2 T}{\partial x^2} \right)^2 + \left( \frac{\partial^2 T}{\partial z^2} \right)^2 \right]^{1/2}$$

In the equation it appears that the analytic signal using second-order derivative. The amplitude depends only on the magnetic field $T$ measured at each measurement point $x$ on the surface.

3. Complex CONJUGATE Approach

The complex conjugate is expressed as $z^*=x-i\cdot h$. If $T$ is a real and analytical function, then $T$ in complex conjugate argument can be expanded around $x$ using the Taylor series. The result of expansion is

$$T(z^*) = T(x-i\cdot h) = T(x) - i\cdot h \cdot \frac{T'(x)}{!} - h^2 \cdot \frac{T''(x)}{2!} + i\cdot h^2 \cdot \frac{T''(x)}{3!} + h^2 \cdot \frac{T^4(x)}{4!} - i\cdot h^2 \cdot \frac{T^4(x)}{5!} - ...$$

Equation (6) can be found in a paper published by [19]. In this paper, this equation is used to find the first-order derivative formula and to estimate its original function. For the second-order can be developed from this equation with several changes and treatments, especially using an approach that resembles the ordinary Finite-Difference Method.
4. Magnetic Field Anomaly

The total magnetic field measured on the surface is assumed to be caused by a 2D finite prism. This model can be approximated from the \(2^3_2D\) prism model by following the definition given by [13]. The \(2^3_2D\) model of vertical infinite-prism can be seen in Figure 1. Basically, the requirements for the \(2^3_2D\) model to be considered as a 2D model are when the strike length is greater than or equal to 10 times its width \((2L \geq 10b)\) [12] and perpendicular to the profile [13]. The magnitude of the \(2^3_2D\) prism magnetic field is expressed as follows [12]

\[
T = kF_i \sin 2I \sin \beta \left[ \ln \left( r_1^2 + L^2 \right)^{1/2} + L \right] - \ln \left( r_2^2 + L^2 \right)^{1/2} - L - \ln \left( r_3^2 + L^2 \right)^{1/2} + L \right] - \ln \left( r_3^2 + L^2 \right)^{1/2} + L}
\]

\[
- \left( \cos^2 \beta \sin^2 I \right) \left[ \tan^{-1} \left( \frac{L}{x-b} \right) - \tan^{-1} \left( \frac{Ld}{x(z^2 + L^2)^{1/2}} \right) + \tan^{-1} \left( \frac{Ld}{(x-b)(z^2 + L^2)^{1/2}} \right) \right]
\]  

\[
(7)
\]

and for prism like Figure 2

\[
T = 2kF_i \left[ \ln \left( r_1^2 + r_2^2 \right) \sin \left( 2I \right) \sin \beta + \left( \cos^2 I \right) \left( \sin \beta \right) \sin^2 \left( \phi - \phi_2 + \phi_1 \right) \right]
\]

where \(T\) denotes the measured total field on the surface, \(k\) for magnetic susceptibility, \(F_i\) intensity of the earth's magnetic field, \(I\) the inclination of the earth's magnetic field, \(\beta\) angle of the prism strike relative to magnetic north, \(b\) width of the prism, \(d\) depth of the prism, and \(L\) for the half-strike length of the prism. The distances and angles that establish the prism geometry are determined by \(r\), and \(\phi\) with \(i=1,2,3\) and 4.

![Figure 1. \(2^3_2D\) infinite-prism cross section](image1)

![Figure 2. 2D finite-prism cross section](image2)

To determine the magnetic field anomaly generated by a 2D prism, it can be conducted by first calculating each total magnetic field caused by the Upper and Lower of the Prism using equation (7), then subtracting the results of both. Equation (8) must be considered when choosing \(r\) and \(\phi\) values in Figure 2. This technique was proposed and first applied by [14], and [13] applied it to the modeling and inversion of magnetic data.

5. Research Framework

In this study, the first thing to do is to construct a second-order derivative equation based on a complex conjugate approach. This approach was developed from equation (6), and the results can be found in Section 6.1. Furthermore, it is used to estimate the function of the total field anomaly which is assumed to be caused by a 2D prism. The total field anomaly is obtained as a result of subtracting the total magnetic field caused by the Upper and Lower of the prism using equation (7) and following [12,13,14]. The choice of this technique is the first step to create an inversion of the magnetic field anomaly in determining the position \((x)\) and depth \((d)\) of the source of the anomaly \((d)\).
after being derived. Then the derivative results are used to compute the second-order analytic signal using equation (5). Analytic signal is one of the most reliable tools in processing and interpreting actual magnetic data. All this process is carried out by making codes using the Matlab® programming language.

Parameters for establishing prism geometry using parametric equations follow [19] based on Figures 1 and 2. These parametric equations can generally be written

\[ r_i^2 = d_i^2 + (x + d_i \cot \xi - b_i)^2 \]  \tag{9}

\[ \tan \phi_i = \left[ \frac{d_i}{x + d_i \cot \xi - b_i} \right] \]  \tag{10}

On the surface, total fields are assumed to be measured at measurement points ranging from -100 to 100 km (-100≤x≤100 km) with spaces of 0.5 km. While the other parameter values used are \( F_e = 45.000 \) nT, \( k = 7 \times 10^3 \) SI (chromite), \( I = 180^\circ \), \( \beta = 65^\circ \), \( b = 150 \) km, \( L = 8b \) km, and \( d = 2 \) km for Upper Prism and 12 km for Lower Prism. In this study, the prism dip angle was chosen \( \xi = 90^\circ \).

To test the accuracy of this new approach, the relative errors (RE) of this approach toward analytical or manual solution are calculated using equation

\[ RE = \frac{\Delta^2 T_{complex \ conjugate} / \Delta x^2 - \Delta^2 T_{manual} / \Delta x^2}{\Delta^2 T_{manual} / \Delta x^2} \]  \tag{11}

The research flowchart can be seen in Figure 3.

![Research flowchart](image)

**Figure 3.** Research flowchart

6. Results and Discussion

6.1. Constructing Complex CONJUGATE Approach for Second-Order Derivative

Derivative approach using complex conjugate has been published by [19] for the first-order. For the second-order, this approach was developed following the ordinary Finite-Difference Method for function that is expanded around \( x \). Let \( T \) be shifted backward from +\( \delta x \) to \( x \), then the complex conjugate changes to

\[ z_i^* = (x + \delta x) - ih \quad \text{or} \quad z_i^* = \Delta x_i^* - ih \]  \tag{12}
Similarly if shifted forward from \(-\delta x\) to \(x\)

\[ z'_i = (x - \delta x) - ih \quad \text{or} \quad z''_h = \Delta x_h - ih \]  

(13)

Taylor series for function \(T\) with arguments in the form of equations (12) and (13) are respectively expressed by

\[ T(z''_h) = T(\Delta x_h - ih) = T(\Delta x_h) - \frac{ih}{!} T'(\Delta x_h) \frac{h^2}{2!} T''(\Delta x_h) + \frac{ih^3}{3!} T'''(\Delta x_h) + \frac{h^4}{4!} T^4(\Delta x_h) - \ldots \]  

(14)

and

\[ T(z'_i) = T(\Delta x_i - ih) = T(\Delta x_i) - \frac{ih}{!} T'(\Delta x_i) \frac{h^2}{2!} T''(\Delta x_i) + \frac{ih^3}{3!} T'''(\Delta x_i) + \frac{h^4}{4!} T^4(\Delta x_i) - \ldots \]  

(15)

Subtract equations (1) and (2)

\[ T'(\Delta x_f) \text{ and } T'(\Delta x_b) \text{ which are in the } [T'(\Delta x_f) - T'(\Delta x_b)] \text{ term of equation (16) above can be decomposed again respectively, then subtract the results, it will be obtained} \]

\[ [T'(\Delta x_f) - T'(\Delta x_b)] = 2\delta x T'(x) \]  

(17)

Substitute equation (17) into (16), and truncate the result to term containing order \(h\) only

\[ \delta T(z''_h) = \delta T(\Delta x_h) - ih2\delta x T'(x) \]  

(18)

If what is taken in equation (18) is an imaginary part of both sides, rearrange it again, then the second-order derivative equation will be obtained as follows

\[ T''(x) = \frac{d^2T(x)}{dx^2} = -\frac{\text{Im}[T(z''_h) - T(z'_i)]}{2h\delta x} + E_i \]  

(19)

which \(E_i\) denotes the truncation error of the Taylor series. Equation (19) is different from the first-order which does not have a multiplication of \(1/\delta x\) as can be found in [19], and not the same as the simple form of the complex variable approach that has been formulated by [6]. However, this is analogous to the Central-Difference Approach of the ordinary Difference-Finite Method.

Figure 4. (a) Total magnetic field, and (b) Total field anomaly

6.2. Profile of the total magnetic field anomaly

Analytically calculates the total magnetic fields generated by each of the Upper and Lower parts of the Prism where the results can be seen in Figure 4a. Next subtract the total magnetic field generated by the Upper of the prism with the one generated by the Lower, and the results are shown in Figure 4b. The magnetic profile of the figure appears in the N-S direction with the strike made in the E25°W
direction. If noted, the figure shows the presence of magnetic bipolarity namely positive and negative polarity, and this is the nature of the magnetic field which will certainly complicate the interpretation of the actual data.

Figure 5. (a) Second-order derivative, and (b) Relative Error

Figure 6. Some combination of step $h$ and interval $\delta x$ (a) $h=1.0 \times 10^{-13}$, $\delta x = 1.0 \times 10^{-15}$ (b) $h=1.0 \times 10^{-25}$, $\delta x = 1.0 \times 10^{-15}$, (c) $h=1.0 \times 10^{-25}$, $\delta x = 1.0 \times 10^{-15}$, and (d) $h=1.0 \times 10^{-25}$, $\delta x = 1.0 \times 10^{-20}$
6.3. Compare Complex CONJUGATE Approach with Manual Solution

Figure 5a displays a comparison of the completion of a second-order derivative of the total field anomaly between using a complex conjugate approach and manual completion. Manual completion is a second-order derivative of Figure 4b. Whereas derivative completion based on complex conjugate approach uses equation (19). To solve the equation, a combination of parameter values $h=1.0\times10^{-13}$ and $\delta x = 1.0\times10^{-10}$ are used when computing.

The relative error of this approach can be seen in Figure 5b. Based on this figure it is known that the maximum $RE$ is 0.088271927039904 and the minimum is 6.679993448987438x10^-07. The difficulty in using this approach is to find the right combination between $h$ and $\delta x$. This is done because in the approach there is a multiplication with $1/h\delta x$ as can be seen in equation (19), different from the first-order which does not suffer from this multiplication so it is insensitive to the choice of steps $h$ or $\delta x$ [19]. Figure 6 shows some combination of step $h$ and interval $\delta x$ in solving equation (19).

As in Figure 6, although in its use the complex conjugate approach has a little complexity because it has to find the right combination between $h$ and $\delta x$ due to the appearance of $1/h\delta x$ in the equation, but after obtaining the right combination, everything becomes easy. In practice, finding the right combination is not difficult and can be done relatively quickly. Besides that, making a program using the Matlab\textsuperscript{\textregistered} Programming Language for computing this approach is very easy and simple to do just by creating simple codes, it does not require a long and complicated declaration.

6.4. Second-Order 2D Analytic Signal Amplitude

The 2D analytic signal for Figure 5a is shown in Figure 7. This analytic signal is only an illustration based on a synthetic model with a 2D finite prism-shaped anomalous source measured on the surface (-100≤$x$≤100 km). In actual data processing, this kind of transformation is very important especially for big data because it can eliminate the bipolarity of the data into positive polarity only and the anomaly source is repositioned below the peak of the anomaly graph, so that finally the position and depth of the source can be known. The advantage of this method is that the required parameter are only second-order derivatives of potential magnetic fields, do not require other parameters as mentioned by [2] and [7], namely invariant both in the direction of the magnetization vector and in the induction field vector.

![Figure 7. Second-order 2D analytic signal amplitude](image)

7. Conclusion

The complex conjugate approach for the derivative is obtained by truncating the Taylor series of function of the total magnetic field which is expanded around $x$ in the complex conjugate argument
\( T(x-i\eta) \). Second-order derivative is derived following the type of the Central-Difference approach of the ordinary Finite-Difference method. In this type, the function of the magnetic field anomaly located at \( x \) is approached from the front forward as far as +\( \delta x \) and from the back backward as far as -\( \delta x \). To test the accuracy of this approach, it is compared with analytical or manual solution. The results obtained, the accuracy of this approach is very high which can be seen from the relative errors in the range of \( 0.088271927039904 \leq \frac{RE}{6.679993448987438 \times 10^{-07}} \). Implementing this approach in the Matlab\(^{b}\) programming language is relatively easy and fast because it only makes simple codes. A little difficulty that arises is when looking for the right combination between the values of \( h \) and \( \delta x \). This is because in the equation of this approach it has multiplication with \( 1/h\delta x \). But finding the right combination between \( h \) and \( \delta x \) is not something difficult because it can be done quickly and easily. Once the right combination is obtained, everything becomes easy. The optimal combination obtained in this study is \( h = 1.0 \times 10^{-13} \) and \( \delta x = 1.0 \times 10^{-10} \). Furthermore, the computational results of second-order derivative are used to estimate second-order 2D Analytic Signal. As a result, the magnetic profile becomes monopolarity or positive polarity only, negative polarity is reduced from the anomaly graph. This situation can facilitate interpretation of actual data, especially if the data is very large without requiring further treatment.

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