EXPLICIT NON-ALGEBRAIC LIMIT CYCLES FOR
POLYNOMIAL SYSTEMS

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ABSTRACT. We consider a system of the form 
\[ \dot{x} = P_n(x, y) + xR_m(x, y), \quad \dot{y} = \]
\[ Q_n(x, y) + yR_m(x, y), \] where \( P_n(x, y), Q_n(x, y) \) and \( R_m(x, y) \) are homogeneous
polyomials of degrees \( n, n \) and \( m, \) respectively, with \( n \leq m. \) We prove that this
system has at most one limit cycle and that when it exists it can be explicitly
found. Then we study a particular case, with \( n = 3 \) and \( m = 4. \) We prove that
this quintic polynomial system has an explicit limit cycle which is not algebraic.
To our knowledge, there are no such type of examples in the literature.
The method that we introduce to prove that this limit cycle is not algebraic can
be also used to detect algebraic solutions for other families of polynomial vector
fields or for probing the absence of such type of solutions.

1. Introduction and Main Results

Examples of planar polynomial vector fields having explicit algebraic limit cycles
appear in most textbooks of ordinary differential equations. One of the simplest
examples is the one of a cubic system that in polar coordinates writes as \( \dot{r} = r(1 - r^2), \) \( \dot{\theta} = 1, \) see for instance [8]. On the other hand, it seems intuitively
clear that “most” limit cycles of planar polynomial vector fields have to be non-
algebraic. Nevertheless, until 1995 it was not proved that the limit cycle of the
van der Pol equation is not algebraic, see [7].

The goal of this paper is to give a planar polynomial vector field for which we
can get an explicit limit cycle which is not algebraic. As far as we know there are
no examples of this situation in the literature.

Recall that a real or complex polynomial \( F(x, y) \) is an algebraic solution of a
real polynomial system \((\dot{x}, \dot{y}) = (X(x, y), Y(x, y))\) if
\[ \frac{\partial F(x, y)}{\partial x} X(x, y) + \frac{\partial F(x, y)}{\partial y} Y(x, y) = K(x, y)F(x, y), \]
for some polynomial \( K(x, y), \) called the cofactor of \( F. \) Notice that when \( F(x, y) \) is
real, the curve \( F(x, y) = 0 \) is invariant under the flow of the differential equation.
Observe also that the degree of the cofactor is one less than the degree of the
vector field. A limit cycle is called algebraic if it is an oval of a real algebraic
solution.

Our main result is:

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Theorem 1.1. The planar differential system
\[
\begin{align*}
\dot{x} &= -(x-y)(x^2 - xy + y^2) + x(2x^4 + 2x^2y^2 + y^4), \\
\dot{y} &= -(x+y)(2x^2 - xy + 2y^2) + y(2x^4 + 2x^2y^2 + y^4),
\end{align*}
\]
has exactly one limit cycle which is hyperbolic and non-algebraic. In polar coordinates, this limit cycle is
\[
r = e^{\frac{3}{2}(\psi(\theta) - \theta)} \left( a + 2 \int_0^\theta \frac{\cos^4(s) + 1}{\cos^2(s) + 1} e^{3\psi(s) - 2s} ds \right)^{-1/2},
\]
where \( \psi(\theta) = 2 \int_0^\theta \frac{1 + \tan^2 s}{2 + \tan^2 s} ds \) and
\[
a = \frac{2e^{4\pi - 6\sqrt{2}/\pi}}{1 - e^{4\pi - 6\sqrt{2}/\pi}} \int_0^{2\pi} \frac{\cos^4(s) + 1}{\cos^2(s) + 1} e^{3\psi(s) - 2s} ds \approx 1.19903.
\]

The main steps of the proof are:
(i) We consider the family of systems,
\[
\begin{align*}
\dot{x} &= P_n(x, y) + xR_m(x, y), \\
\dot{y} &= Q_n(x, y) + yR_m(x, y),
\end{align*}
\]
where \( P_n(x, y), Q_n(x, y) \) and \( R_m(x, y) \) are homogeneous polynomials of degrees \( n, n \) and \( m, \) respectively, with \( n \leq m. \) In Section 2 we prove that all of them have at most one limit cycle and that when it exists it can be explicitly found.

(ii) We then consider a concrete system of the form (3) with \( n = 3 \) and \( m = 4, \) having an explicit, unique and hyperbolic limit cycle, see Section 4.

(iii) Finally, we use the method developed in Section 5 for studying the algebraic solutions of the fixed system. Once all of them are found, none of which has ovals, we can prove that the limit cycle is non-algebraic, as shown in Section 6.

In Section 3 we study system (3) with \( n = 1. \) In this case we prove that when the limit cycle exists it is always algebraic. This is a short preliminary that we include for the sake of completeness and that helps to understand how we have arrived to the final example studied in our main theorem.

We end this introduction by showing a system of the form (3), almost equal to the one studied in Theorem 1.1, having also a unique hyperbolic limit cycle, but algebraic. Concretely, it is easy to check that the system
\[
\begin{align*}
\dot{x} &= -(x-y)(x^2 - xy + y^2) + x(x^4 + 3x^2y^2 + 2y^4), \\
\dot{y} &= -(x+y)(2x^2 - xy + 2y^2) + y(x^4 + 3x^2y^2 + 2y^4)
\end{align*}
\]
have the algebraic limit cycle \( 1 - x^2 - y^2 = 0. \) The results when \( n = 1 \) and systems (2) and (4) illustrate that the problem of distinguishing whether the limit cycle for system (3) is algebraic or not is quite hidden in the coefficients of the system.

2. Systems with explicit limit cycles

This section is devoted to study system (3). In polar coordinates it writes as
\[
\begin{align*}
\dot{r} &= f(\theta)r^n + h(\theta)r^{m+1}, \\
\dot{\theta} &= g(\theta)r^{n-1},
\end{align*}
\]
where
\[ f(\theta) = \cos \theta P_n(\cos \theta, \sin \theta) + \sin \theta Q_n(\cos \theta, \sin \theta), \]
\[ g(\theta) = \cos \theta Q_n(\cos \theta, \sin \theta) - \sin \theta P_n(\cos \theta, \sin \theta), \]
\[ h(\theta) = R_m(\cos \theta, \sin \theta). \]

**Theorem 2.1.** System (3) has at most one limit cycle. When it exists it is hyperbolic and in polar coordinates it writes as
\[ r = \left( \exp \left[ \int_0^\theta \frac{f(s)}{g(s)} ds \right] a + \int_0^\theta \frac{h(s)}{g(s)} \exp \left( - \int_0^s \frac{f(w)}{g(w)} dw \right) ds \right)^{\frac{1}{n-m-1}}, \]
where \( a = AB/(1 - A) \), being
\[ A = \exp \left( \int_0^{2\pi} \frac{f(s)}{g(s)} ds \right) \quad \text{and} \quad B = \int_0^{2\pi} \frac{h(s)}{g(s)} \exp \left( - \int_0^s \frac{f(w)}{g(w)} dw \right) ds. \]

**Proof.** Consider the expression of system (3) in polar coordinates, i.e. system (5). If \( g(\theta) \) vanishes for some \( \theta = \theta^\ast \) then it has \( \{ \theta = \theta^\ast \} \) as an invariant straight line. From the uniqueness of solutions we get that system (3) has no limit cycles. If \( g(\theta) \neq 0 \) then we can write the system as
\[ \frac{dr}{d\theta} = \frac{f(\theta)}{g(\theta)} r + \frac{h(\theta)}{g(\theta)} r^{m-n+2}, \]
which is a Bernoulli equation. By introducing the standard change of variables \( \rho = r^{n-m-1} \) we obtain the linear equation
\[ \frac{d\rho}{d\theta} = (n - m - 1) \frac{f(\theta)}{g(\theta)} \rho + (n - m - 1) \frac{h(\theta)}{g(\theta)}. \tag{6} \]

Notice that system (3) has a periodic orbit if and only if equation (6) has a strictly positive 2\pi periodic solution.

The general solution of equation (6), with initial condition \( \rho(0) = \rho_0 \), is
\[ \rho(\theta; \rho_0) = \exp \left( \int_0^\theta \frac{f(s)}{g(s)} ds \right) \left( \rho_0 + \int_0^\theta \frac{h(s)}{g(s)} \exp \left( - \int_0^s \frac{f(w)}{g(w)} dw \right) ds \right) := G(\theta, \rho_0). \tag{7} \]

The condition that the solution starting at \( \rho = \rho_0 \) is periodic reads as \( A(\rho_0 + B) = \rho_0 \). Hence, if \( A = 1 \) and \( B = 0 \), system (3) has a continuum of periodic orbits, otherwise it has at most the solution starting at \( \rho_0 = AB/(1 - A) := a \).

In order to prove the hyperbolicity of the limit cycle notice that the Poincaré return map is \( \Pi(\rho_0) = \rho(2\pi; \rho_0) \). Thus \( \Pi'(\rho_0) = \exp \left( \int_0^{2\pi} \frac{f(s)}{g(s)} ds \right) = A \neq 1 \) for all \( \rho_0 \), and in particular we get that the limit cycle is hyperbolic, whenever it exists. \( \Box \)

> From the proof of the above theorem we also get the following remark.

**Remark 2.2.** When system (3) has a limit cycle it can be written in the form \( F(r, \theta) := r^{m-n+1}G(\theta, a) - 1 = 0 \). As we will see, the function \( F(r, \theta) \) can be algebraic or not in cartesian coordinates, depending on the concrete system considered. In any case the expression given in (7) can be also extended to non-algebraic functions \( F \), and in this case the cofactor \( K \) is not necessarily a polynomial. Curiously
enough, independently of the algebraicity of $F$ its corresponding cofactor satisfying \[ K(x,y) = (m-n+1)R_m(x,y). \] Examples of non-algebraic solutions having a polynomial cofactor have been already given in the literature, see for instance \[ 2, 4. \] The example presented in Theorem 1.1 provides a non-algebraic limit cycle having a polynomial cofactor.

The next remark explains why our proof that the limit cycle of system (2) is not algebraic does not use the explicit expression of the limit cycle. On the contrary, in Section 5, we develop a method for studying all the algebraic solutions of a system having at least a solution of the form $y - \alpha(x) = 0$, where $\alpha(x)$ is a rational function, and we apply this method to determine all the algebraic solutions of system (2).

**Remark 2.3.** Although for system (3) we know that the expression of the limit cycle is $r^{m-n+1}G(\theta,a) - 1 = 0$, it is not an easy task to elucidate whether this curve is algebraic or not in cartesian coordinates. As an example of this difficulty we recall the Filippsov’s example, see \[ 1: \]

\[
\begin{align*}
\dot{x} &= 6(1+a)x + 2y - 6(2+a)x^2 + 12xy, \\
\dot{y} &= 15(1+a)y + 3a(1+a)x^2 - 2(9+5a)xy + 16y^2,
\end{align*}
\]

which has the algebraic solution $3(1+a)(ax^2 + y^2) + 2y^2(2y - 3(1+a)x)$. This algebraic solution contains a limit cycle for $0 < a < 3/13$. For the sake of simplicity we fix $a = 1/6$. For this value the limit cycle is $rG(\theta) - 1 = 0$, where

$$G(\theta) = \frac{7(\sin^4 \theta - 2\sin^2 \theta + 1)}{6\sin(-17\sin^2 \theta + 42\sin \theta \cos \theta - 7 \pm 2\sqrt{\varphi(\theta)})}$$

and

$$\varphi(\theta) = 60\sin^4 \theta - 357\sin^3 \theta \cos \theta + 84\sin^2 \theta + 441\sin^2 \theta \cos^2 \theta - 147\sin \theta \cos \theta.$$

Note that is not easy at all to realize that the expression $rG(\theta) - 1 = 0$ corresponds to a polynomial in cartesian coordinates.

It is not difficult to see that system (3) has always algebraic solutions.

**Lemma 2.4.** System (3) has $F(x,y) = yP_n(x,y) - xQ_n(x,y)$ as an algebraic solution with cofactor $(n+1)R_m + \text{div}(P_n, Q_n)$. Notice that it is formed by a product of (complex or real) invariant straight lines through the origin.

**Proof.** By using the homogeneity of $P_n$ and $Q_n$, we know from the Euler’s formula that $nP_n(x,y) = x\frac{\partial P_n}{\partial x} + y\frac{\partial P_n}{\partial y}$ and $nQ_n(x,y) = x\frac{\partial Q_n}{\partial x} + y\frac{\partial Q_n}{\partial y}$. Thus

$$\left( y\frac{\partial P_n}{\partial x} - Q_n - x\frac{\partial Q_n}{\partial x} \right) (P_n + xR_m) + \left( P_n + y\frac{\partial P_n}{\partial y} - x\frac{\partial Q_n}{\partial y} \right) (Q_n + yR_m)$$

$$= \left( (n+1)R_m + \frac{\partial P_n}{\partial x} + \frac{\partial Q_n}{\partial y} \right) F.$$

\[ \square \]

**Lemma 2.5.** Let $F(x,y)$ be an algebraic solution of degree $\ell$ of the system

\[
\begin{align*}
\dot{x} &= P(x,y) + xR_m(x,y), \\
\dot{y} &= Q(x,y) + yR_m(x,y),
\end{align*}
\]
where \(P(x, y)\) and \(Q(x, y)\) are polynomials of degree less or equal than \(n\) and \(R_m(x, y)\) is a homogeneous polynomial of degree \(m\), with \(n \leq m\). Thus the homogeneous part of maximum degree of its cofactor is \(\ell R_m(x, y)\).

**Proof.** Since \(F\) is an algebraic solution of system (3) we know that
\[
\frac{\partial F}{\partial x} (P + xR_m) + \frac{\partial F}{\partial y} (Q + yR_m) = K F,
\]
where \(K\) is the cofactor of \(F\).

Denote by \(F_\ell(x, y)\) and by \(K_m(x, y)\) the homogeneous parts of maximum degree of \(F(x, y)\) and \(K(x, y)\), respectively. By using the homogeneity of \(F\) we know, from the Euler’s formula, that 
\[
\ell F_\ell = x \frac{\partial F_\ell}{\partial x} + y \frac{\partial F_\ell}{\partial y}.
\]
By equating the higher degree terms in the above equation we obtain
\[
\ell F_\ell R_m = x \frac{\partial F_\ell}{\partial x} x R_m + y \frac{\partial F_\ell}{\partial y} y R_m = K_m F_\ell.
\]
Thus \(K_m(x, y) = \ell R_m(x, y)\) as we wanted to prove.

Next lemma collects some easy remarks on the structure of the cofactors.

**Lemma 2.6.** Let
\[
\begin{align*}
\dot{x} &= X(x, y), \\
\dot{y} &= Y(x, y),
\end{align*}
\]
be a real planar polynomial system. The following holds:

(i) If it has a complex algebraic solution, then it also has a real algebraic solution.

(ii) Assume the vector field satisfies
\[
(X(-x, -y), Y(-x, -y)) = (-1)^s(X(x, y), Y(x, y)), \quad \text{being } s \text{ either 0 or 1}.
\]
Then if it has a real algebraic solution then it has another real algebraic solution with cofactor \(K\) satisfying \(K(-x, -y) = (-1)^{s+1} K(x, y)\).

**Proof.** The proof is elementary.

Finally we give an integrating factor for system (3).

**Lemma 2.7.** Consider system (3) and define
\[
V(x, y) = (r^{m-n+1} G(\theta, a) - 1)(y P_n(x, y) - x Q_n(x, y)),
\]
where \(G(\theta, \rho_0)\) is the function given in (7) and \(\rho_0 = a\) is the value for which this function is \(2\pi\)-periodic. Then, whenever it is defined, \(1/V(x, y)\) is an integrating factor of the system and we call \(V(x, y)\) an inverse integrating factor.

**Proof.** We use the following formula: let \(F_1\) and \(F_2\) be two solutions of \((\dot{x}, \dot{y}) = (X(x, y), Y(x, y))\) with cofactors \(K_1\) and \(K_2\), respectively. Thus
\[
\text{div} \left( \frac{(X,Y)}{F_1 F_2} \right) = \frac{1}{F_1 F_2} \text{div}(X, Y) - (K_1 + K_2).
\]
We remark that the above formula, taking a denominator of the form \(\prod F_i^{\alpha_i}\), for some real or complex constants \(\alpha_i\), is indeed the key point of the Darboux theory of integrability, see [6]. Take \(F_1(x, y) = y P_n(x, y) - x Q_n(x, y)\) and \(F_2(x, y) = r^{m-n+1} G(\theta, a) - 1\). By using Lemma 2.4 and Remark 2.2 we know that their
associated cofactors are being non-Liouvillean $K_1(x, y) = (n + 1)R_m(x, y) + \text{div}(P_n(x, y), Q_n(x, y))$ and $K_2(x, y) = (m - n + 1)R_m(x, y)$, respectively. On the other hand, taking the vector field associated to system (3) we get

$$\text{div}(X, Y) = \text{div}(P_n, Q_n) + 2R_m + x\frac{\partial R_m}{\partial x} + y\frac{\partial R_m}{\partial y} = \text{div}(P_n, Q_n) + (2 + m)R_m,$$

where we have used again Euler’s formula. Collecting all the above results we get $\text{div}((X, Y)/(F_1F_2)) \equiv 0$ as we wanted to prove. \hfill \qed

**Remark 2.8.** (i) When we apply the above Lemma to systems (2) and (4) we get both non-algebraic and algebraic inverse integrating factors.

(ii) In [5] it is proved that when $1/V(x, y)$ is an integrating factor of $(\dot{x}, \dot{y}) = (X(x, y), Y(x, y))$ and $V(x, y)$ is defined in the whole plane, all the limit cycles of the system are included in the curve $V(x, y) = 0$. This is the case of system (3): the limit cycle, whenever it exists, is given by the expression $F_2(x, y) = r^{m-n+1}G(\theta, a) - 1 = 0$.

(iii) The equality $\text{div}((X, Y)/(F_1F_2)) \equiv 0$ also holds when instead of $F_2(x, y) = r^{m-n+1}G(\theta, \rho_0)|_{\rho_0=0} - 1$ we take a different value of $\rho_0$, but in this case $F_2$ is indeed a multivaluated function and the result of [5] can not be applied.

3. A FAMILY OF SYSTEMS WITH EXPLICIT ALGEBRAIC LIMIT CYCLES

The existence of limit cycles for a subfamily of system (3) has been studied in [4]. Here we prove that the limit cycle found there is algebraic.

**Proposition 3.1.** Consider the system

$$\begin{cases}
\dot{x} &= -y + x(a + R_m(x, y)), \\
\dot{y} &= x + y(a + R_m(x, y)),
\end{cases}$$

where $a$ is a real parameter and $R_m(x, y)$ is a homogeneous polynomial of degree $m$. Then it has only two algebraic invariant curves $x^2 + y^2$ and $H(x, y) = G_m(x, y) - 1$, where $G(\theta) = G_m(\cos \theta, \sin \theta)$ satisfies $G' + maG + mR_m(\cos \theta, \sin \theta) = 0$. Furthermore, when the limit cycle exists, $m$ is even and $H(x, y)$ contains a real oval which is the limit cycle of the system.

**Proof.** From Lemma 2.4 we can see that $x^2 + y^2$ is an algebraic solution with cofactor $K(x, y) = 2a + 2R_m(x, y)$. Now, we study other possible algebraic solutions. Write the Fourier expansion of $R_m$:

$$R_m(\cos \theta, \sin \theta) = \sum_{k=-m}^{k=m} c_k e^{k\theta i},$$

where $\overline{c_k} = c_{-k} \in \mathbb{C}, c_k = 0$ when $k \not\equiv m(\text{mod } 2)$.

Following the steps of the proof of Theorem 2.1 we obtain that, in polar coordinates, the solution of (3) starting at $r = r_0$ when $\theta = 0$ can be written as

$$r^{-m} = \left( r_0^{-m} + m \sum_{k=-m}^{k=m} \frac{c_k}{ki + ma} \right) e^{-ma\theta} + G_m(\cos \theta, \sin \theta),$$

or

$$1 = \left( r_0^{-m} + m \sum_{k=-m}^{k=m} \frac{c_k}{ki + ma} \right) r^m e^{-ma\theta} + G_m(r \cos \theta, r \sin \theta),$$
where $G_m(x, y)$ is the homogeneous polynomial of degree $m$ defined by its Fourier expansion as

$$G_m(\cos \theta, \sin \theta) := -m \sum_{k=-m}^{k=m} \frac{c_k}{k^2 + ma} e^{k \theta i},$$

and $G(\theta) = G_m(\cos \theta, \sin \theta)$ satisfies $G' + maG + mR_m(\cos \theta, \sin \theta) = 0$.

By using the above expression we get that the only algebraic solution of system (8) is the one that satisfies

$$r_0^{-m} + m \sum_{k=-m}^{k=m} \frac{c_k}{k^2 + ma} = 0.$$

Moreover, it is easy to check that the cofactor of this algebraic solution, $H(x, y) = G_m(x, y) - 1$, is $K(x, y) = mR_m(x, y)$, see also the proof of Lemma 2.7

Notice that a necessary and sufficient condition for the existence of a real algebraic solution is that

$$\sum_{k=-m}^{k=m} \frac{c_k}{k^2 + ma} < 0.$$

Finally, the limit cycle exists when $G_m(\cos \theta, \sin \theta) > 0$ for $\theta \in [0, 2\pi]$. This only can happen when $m$ is even, see also [3]. $\square$

4. The examples

In this section we will prove that the system given in Theorem 1.1 has an explicit limit cycle. That the limit cycle is not algebraic will be proved in Section 6.

The system (2) is

$$\begin{align*}
\dot{x} &= -(x-y)(x^2 - xy + y^2) + x(2x^4 + 2x^2y^2 + y^4), \\
\dot{y} &= -(x+y)(2x^2 - xy + 2y^2) + y(2x^4 + 2x^2y^2 + y^4),
\end{align*}$$

and in polar coordinates it can be written as

$$\begin{align*}
\dot{r} &= (\cos^4 \theta + 1)r^5 + (\cos^2 \theta - 2)r^3, \\
\dot{\theta} &= -(\cos^2 \theta + 1)r^2.
\end{align*}$$

By following the same steps than in the proof of Theorem 2.1 we introduce the change of variables $r = 1/\sqrt{\rho}$, obtaining:

$$\rho' = 2\frac{\cos^2 \theta - 2}{\cos^2 \theta + 1} \rho + 2\frac{\cos^4 \theta + 1}{\cos^2 \theta + 1}.$$

The solution satisfying that $\rho = \rho_0 > 0$ when $\theta = 0$ is

$$\rho(\theta; \rho_0) = e^{-3\psi(\theta) + 2\theta} \left( \rho_0 + 2 \int_0^\theta \frac{\cos^4(s) + 1}{\cos^2(s) + 1} e^{3\psi(s) - 2s} ds \right) > 0,$$

where $\psi(\theta) = 2 \int_0^\theta \frac{1 + \tan^2 s}{2 + \tan^2 s} ds$.

The initial condition of the limit cycle is given by the equation $\rho(2\pi) = \rho(0) = \rho_0^*$. Hence

$$\rho_0^* = 2e^{4\pi - 6\sqrt{2} \pi} \int_0^{2\pi} \frac{\cos^4(s) + 1}{\cos^2(s) + 1} e^{3\psi(s) - 2s} ds > 0.$$
This value can be computed numerically, giving \( \rho_0^* \approx 1.1990 \). The intersection of the limit cycle with the \( \text{OX}^+ \) axis is the point having \( r_0^* = 1/\sqrt{\rho_0} \approx 0.9132 \).

Since the Poincaré return map is \( \Pi(\rho_0) = \rho(2\pi; \rho_0) \) we have \( \Pi'(\rho_0) = e^{(4-6\sqrt{2})\pi} < 1 \) for all \( \rho_0 \) and \( \dot{\theta} < 0 \), we get that the limit cycle of system (2) is hyperbolic and unstable.

5. A METHOD FOR STUDYING THE EXISTENCE OF ALGEBRAIC SOLUTIONS

Let \( F(x, y), K(x, y), X(x, y) \) and \( Y(x, y) \) be real analytic functions such that
\[
\frac{\partial F}{\partial x} X(x, y) + \frac{\partial F}{\partial y} Y(x, y) = K(x, y) F(x, y).
\]
(9)

Thus it is clear that the set \( \{(x, y) \in \mathbb{R}^2 : F(x, y) = 0\} \) is formed by solutions of the system
\[
\begin{cases}
  \dot{x} = X(x, y), \\
  \dot{y} = Y(x, y).
\end{cases}
\]
(10)

Fixed an analytic solution of (10) of the form \( y = \alpha(x) \), we can consider the following Taylor expansions in \( z \),
\[
F(x, z + \alpha(x)) = F_0(x) + zF_1(x) + z^2F_2(x) + \ldots,
\]
\[
K(x, z + \alpha(x)) = K_0(x) + zK_1(x) + z^2K_2(x) + \ldots,
\]
\[
X(x, z + \alpha(x)) = X_0(x) + zX_1(x) + z^2X_2(x) + \ldots,
\]
\[
Y(x, z + \alpha(x)) = Y_0(x) + zY_1(x) + z^2Y_2(x) + \ldots.
\]

Notice that \( \alpha'(x) = Y_0(x)/X_0(x) \). Then equation (9) can be written as
\[
\sum_{k=0}^{\infty} \left( \sum_{i=0}^{k} \left( X_{k-i}(x)F_i'(x) + (iY_{k-i+1} - i\alpha'(x)X_{k-i+1} - K_{k-i})F_i(x) \right) \right) z^k = 0.
\]

The functions \( F_k(x) \) can be obtained recurrently from the above relation by solving the linear differential equations in \( F_k(x) \), obtained vanishing each coefficient in \( z^k \).

In particular, for \( k = 0 \) and \( k = 1 \), we get
\[
X_0(x)F_0'(x) - K_0(x)F_0(x) = 0,
\]
\[
X_0(x)F_1'(x) + (Y_1(x) - \alpha'(x)X_1(x) - K_0(x))F_1(x) + X_1(x)F_0'(x) - K_1(x)F_0(x) = 0.
\]
We obtain \( F_0(x) = C_0 \exp(\int_0^x \frac{K_0(s)}{X_0(s)} ds) \), where \( C_0 \) is an arbitrary constant, and similarly we could get \( F_1(x) \).

When \( \alpha(x) \) is a rational function and \( F(x, y), K(x, y), X(x, y) \) and \( Y(x, y) \) are polynomials, with real or complex coefficients, the linear differential equations for each \( F_k(x) \) described in the above algorithm give us a collection of necessary conditions for the existence of an algebraic solution \( F(x, y) \). The conditions are that, for each \( k \), the functions \( F_k(x) \) must be polynomials. For instance, for \( k = 0 \), the first necessary condition is that the primitive of the rational function
\[
\frac{K_0(x)}{X_0(x)} = \frac{K(x, \alpha(x))}{X(x, \alpha(x))}
\]
must be a linear combination of logarithms of polynomials. Furthermore the coefficients of the logarithms have to be natural numbers.
The necessary conditions obtained for the existence of algebraic solutions restrict the possible cofactors of \( F \). These restrictions give the key for searching the possible algebraic solutions of system (10), see Remark 5.2 and Section 6. As we will see, in our case we only need to apply the described method for \( k = 0 \) but we remark that in other situations, by using it for bigger \( k \), it can give more information about the existence or non-existence of algebraic solutions.

**Remark 5.1.** Notice that the above method can only be applied when the candidate \( F(x, y) \) to be an algebraic solution of system (10) does not contain the factor \( y - \alpha(x) \).

**Remark 5.2.** Assume that system (10) is fixed and it is polynomial. Notice that the equation (9) that gives the possible set of algebraic solutions of system (10) is equivalent to a set of quadratic equations where the unknowns are the coefficients of \( F \) and the coefficients of \( K \). In general, it is very hard to solve this system of equations, even by using algebraic manipulators. On the other hand, the method developed in this section imposes restrictions on the cofactor \( K \) for the existence of \( F \). Ideally, if \( K \) is totally known, the system to be solved is be linear and so the problem of knowing the existence or not of algebraic solutions of a given degree would be a much easier task. In any case, any information on \( K \) makes the problem simpler.

### 6. A non-algebraic limit cycle

This section is devoted to prove that the limit cycle of system (2) is not algebraic. Indeed, by using the method introduced in Section 5 we will prove that the only algebraic solutions of the system are the ones given in Lemma 2.4. The algebraic solution given by this lemma is \( y^P_n - x^Q_n = (2x^2 + y^2)(x^2 + y^2) \). Concretely, the curves \( x^2 + y^2 = 0 \) and \( 2x^2 + y^2 = 0 \) have the cofactors \( 2(2x^4 + 2x^2y^2 + y^4 - x^2 - 2y^2) \) and \( 2(2x^4 + 2x^2y^2 + y^4 - x^2 + xy - 2y^2) \), respectively. These two curves coincide with the four complex lines \( y = \pm ix \) and \( y = \pm \sqrt{2}ix \).

As we will see, the first step (\( k = 0 \)) of the method developed in Section 5 applied to each one of the four complex lines, \( y = \pm ix \) and \( y = \pm \sqrt{2}ix \), will give enough restrictions to prove that the only algebraic solutions of system (2) are the ones described above.

Assume that the differential system has a real or complex algebraic solution \( F \) and that it does not contain none of the given four lines as a factor. By using Lemma 2.6 it is not restrictive to assume that \( F \) is real and that its cofactor is an even function, i.e. \( K(-x, -y) = K(x, y) \).

Since the degree of the vector field (2) is 5 we know that the degree of \( K(x, y) \) is at most 4. By the above restrictions on \( K(x, y) \) and by using also Lemma 2.5 we can write it as the real polynomial

\[
K(x, y) = a_{00} + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \ell(2x^4 + 2x^2y^2 + y^4),
\]

where \( \ell \) is the degree of the corresponding algebraic curve \( F(x, y) = 0 \).
We apply the first step of our method, i.e. we take \( k = 0 \). By considering the cases \( \alpha(x) = \pm ix \) we obtain

\[
\int \frac{K(x, \alpha(x))}{X(x, \alpha(x))} \, dx = \int \frac{K(x, \pm ix)}{X(x, \pm ix)} \, dx = \int \frac{K_0(x)}{X_0(x)} \, dx =
\]

\[
\frac{a_{00}(-1 \pm i)}{4x^2} + \frac{1}{2}(a_{20} + a_{11} - a_{02} \pm i(-a_{20} + a_{11} + a_{02} + a_{00})) \log(x) +
\]

\[
\frac{1}{8}(-a_{20} - a_{11} + a_{02} + 2\ell \pm i(a_{20} - a_{11} - a_{02} - a_{00})) \log(2 + 2x^2 + x^4) +
\]

\[
\frac{1}{4}(a_{20} - a_{11} - a_{02} - a_{00} \pm i(a_{20} + a_{11} - a_{02} - 2\ell)) \arctan(x^2 + 1).
\]

By forcing \( F(x, \alpha(x)) = F(x, \pm ix) = F_0(x) = C_0 \exp\left(\int K_0(x, \pm ix)/X_0(x, \pm ix) \, dx\right) \) to be a polynomial, with \( C_0 \) an arbitrary constant, we obtain a first set of necessary conditions:

\[
\begin{cases}
  a_{20} - a_{11} - a_{02} - a_{00} &= 0, \\
  a_{20} + a_{11} - a_{02} - 2\ell &= 0, \\
  a_{00} &= 0.
\end{cases}
\]

The same computations can be done for the other pair of algebraic solutions, \( y = \pm \sqrt{2}ix \), that is

\[
\int \frac{K(x, \alpha(x))}{X(x, \alpha(x))} \, dx = \int \frac{K(x, \pm i\sqrt{2}x)}{X(x, \pm i\sqrt{2}x)} \, dx = \int \frac{K_0(x)}{X_0(x)} \, dx =
\]

\[
-\frac{a_{00}}{6x^2} + \frac{1}{9}(3a_{20} - 6a_{02} - 2a_{00} \pm 3\sqrt{2}ia_{11}) \log(x) +
\]

\[
\frac{1}{18}(-3a_{20} + 6a_{02} + 9\ell + 2a_{00} \pm 3\sqrt{2}ia_{11}) \log(3 + 2x^2).
\]

As in the previous case, we obtain a second set of necessary conditions:

\[
\begin{cases}
  a_{11} &= 0, \\
  a_{00} &= 0.
\end{cases}
\]

Collecting all the obtained equations we get that the degree of the invariant algebraic curve is \( \ell = 0 \), or in other words that such a curve does not exist.

**Proof of Theorem 1.1** The proof of the theorem follows from the results of Sections 2, 4 and 6.

**Remark 6.1.** From [1] Thm 1, if a planar system has an explicit non-algebraic solution which is in the zero level set of a Liouvillian function then it has a Darboux integrating factor and therefore the whole system is integrable by quadratures. Notice that this is the case for system (2): it has a non-algebraic Liouvillian limit cycle and it can be transformed into a Bernoulli equation. Consequently, if we would like to have an explicit non-algebraic limit cycle for a planar system, which is not integrable by quadratures, we should look for a limit cycle given by a non-Liouville function.
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