PERSISTENCE PROPERTIES AND UNIQUE CONTINUATION FOR A DISPERSIONLESS TWO-COMPONENT CAMASSA-HOLM SYSTEM WITH PEAKON AND WEAK KINK SOLUTIONS

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Abstract. In this paper, we study the persistence properties and unique continuation for a dispersionless two-component system with peakon and weak kink solutions. These properties guarantee strong solutions of the two-component system decay at infinity in the spatial variable provided that the initial data satisfies the condition of decaying at infinity. Furthermore, we give an optimal decaying index of the momentum for the system and show that the system exhibits unique continuation if the initial momentum $m_0$ and $n_0$ are non-negative.

1. Introduction. Recently, an integrable two-component Camassa-Holm system with both quadratic and cubic nonlinearity was proposed by Xia and Qiao [28]

\[
\begin{align*}
    m_t + \frac{1}{2} [m(uv - u_x v_x)]_x - \frac{1}{2} m(uv_x - u_x v) + bu_x &= 0 \\
    n_t + \frac{1}{2} [n(uv - u_x v_x)]_x + \frac{1}{2} n(uv_x - u_x v) + bv_x &= 0 \\
    m &= u - u_{xx}, \quad n = v - v_{xx}.
\end{align*}
\]

As shown in [28], this system has peakon and weak kink solutions as well as including some remarkable peakon equations such as the CH equation and the FORQ equation. For instance, letting $v = 2$ in Eq. (1.1) yields the Camassa-Holm (CH) equation, which models the unidirectional propagation of shallow water waves over a flat bottom while $u(t, x)$ stands for the fluid velocity at time $t$ in the spatial $x$ direction [2, 7, 20]. The CH equation has a bi-Hamiltonian structure [3, 16] and is completely integrable [2, 11, 4]. The Cauchy problem of the CH equation has been studied extensively. This equation is locally well-posed [6, 8, 21, 26] for initial data $u_0 \in H^s(\mathbb{S})$ with $s > \frac{3}{2}$. More interestingly, it has not only global strong solutions modelling permanent waves [8] and but also blow-up solutions modelling wave breaking [5, 9, 8, 10, 21, 26]. On the other hand, it has globally weak solutions with initial data $u_0 \in H^1$, cf. [1, 12, 30].

If choosing $v = 2u$ in Eq. (1.1), one may obtain the cubic CH equation which is also called the FORQ equation in the literature since it was developed independently.

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in [16, 22, 23, 24]. It might be derived from the two dimensional Euler equations, and its Lax pair, cuspon and other peaked solutions have been studied in [23, 24]. With \( v = k_1 u + k_2 \), Eq. (1.1) is able to be reduced to the generalized CH (gCH) equation. The gCH equation was first implied in the work of Fokas [17]. Its Lax pair, bi-Hamiltonian structure, peakons, weak kinks, kink-peakon interaction, and classical soliton solutions were investigated in [25].

Moreover, by imposing the constraint \( v = u^* \), equation (1.1) is reduced to a new integrable equation with cubic nonlinearity and linear dispersion

\[
m_t = bu_x + \frac{1}{2}[m(|u|^2 - |u_x|^2)]_x - \frac{1}{2} m(u u_x^* - u_x u^*), \quad m = u - u_{xx} \tag{1.2}
\]

where the symbol * denotes the complex conjugate. The above reduction of the two-component system (1.1) looks very like the reduction case of AKNS system, which embraces the KdV equation, the mKdV equation, the Gardner equation, and the nonlinear Schrödinger equation. Xia and Qiao [28, 29] proposed the complex-value N-peakon solution and weak kink wave solution to the cubic nonlinear equation (1.2).

Geometrically, system (1.1) describes pseudo-spherical surfaces. Integrability of the system, its bi-Hamiltonian structure, and infinitely many conservation laws were already presented by Xia and Qiao [28]. In the case \( b = 0 \) (dispersionless case), the authors showed that this system admits the single-peakon of travelling wave solution as well as multi-peakon solutions. The qualitative analysis for the integrable system (1.1) was investigated by Yan, Qiao and Yin [31].

In this paper, we consider the following Cauchy problem of system (1.1) with \( b = 0 \) on the line:

\[
\begin{align*}
    m_t + \frac{1}{2}[m(u v - u_x v_x)]_x - \frac{1}{2} m(u v_x - u_x v) &= 0, & t > 0, \quad x \in \mathbb{R}, \\
    n_t + \frac{1}{2}[n(u v - u_x v_x)]_x + \frac{1}{2} n(u v_x - u_x v) &= 0, & t > 0, \quad x \in \mathbb{R}, \\
    m(0, x) = m_0, \quad n(0, x) = n_0,
\end{align*}
\tag{1.3}
\]

where \( m = u - u_{xx} \) and \( n = v - v_{xx} \), and study the persistence properties and unique continuation of strong solutions for Eq. (1.3). There is a lot literatures concerning these problems. The persistence properties and unique continuation for the CH equation are proved in [19]. The unique continuation results about the Schrödinger and KdV equations were provided by Escauriaza, Kenig, Ponce and Vega in [14] and [15]. Persistence properties and infinite propagation for the modified 2-component Camassa-Holm equation and 3-component Camassa-Holm system were investigated in [18, 27].

As we mentioned at the very beginning of the paper, system (1.1) possesses peakons and weak kink solutions with both quadratic and cubic nonlinearity. It is quite interesting to study the persistence properties and unique continuation of strong solutions for system (1.1). Inspired by the method given by Himonas et al. in [19], we will show some persistence properties of the strong solutions, and furthermore present the optimal decay index of the momentum. Finally, by introducing a continuous family of diffeomorphisms of the line, we demonstrate that the system exhibits unique continuation if the initial momentum \( m_0 \) and \( n_0 \) are non-negative.

**Notation.** Throughout this paper, the convolution is denoted by \(*\). For \( 1 \leq p \leq \infty \), the norm in the Lebesgue space \( L^p(\mathbb{R}) \) is written by \( \| \cdot \|_{L^p} \), while \( \| \cdot \|_{H^s} \), \( s > 0 \), stands for the norm in the classical Sobolev spaces \( H^s(\mathbb{R}) \).
2. Persistence properties. For our convenience, let us first present the following well-posedness theorem given in \[31\].

**Theorem 2.1.** \[31\] Let \( s \geq 3 \). If \( z_0 = (u_0, v_0) \) belongs to the Sobolev space \( H^s \times H^s \) on the circle or the line, then there exists a maximal time \( T = T(z_0) > 0 \) and a unique solution \( z(t, x) \in C([0; T); H^s \times H^s) \cap C^1([0; T); H^{s-1} \times H^{s-1}) \) of the Cauchy problem for the equation (1.1). Furthermore, the data-to-solution map \( z(0) \to z(t) \) is continuous but not uniformly continuous.

From the above well-posedness result, we may now utilize it to the persistence properties and unique continuation to equation (1.3). Our basic assumption is that the initial data and its first spacial derivative decay exponentially. Then we have the following result based on the work \[19\] for the CH equation.

**Theorem 2.2.** Assume that \( s \geq 3 \), \( T > 0 \), and \( z \in C([0; T]; H^s \times H^s) \) is a solution of (1.3). If the initial data \( z_0(x) = z(0, x) \) decays at infinity, more precisely, if there is some \( \theta \in (0, 1) \) such that as \( |x| \to \infty \)
\[
|u_0(x)| \sim O(e^{-\theta |x|}), \quad |u_0'(x)| \sim O(e^{-\theta |x|}), \\
|v_0(x)| \sim O(e^{-\theta |x|}), \quad |v_0'(x)| \sim O(e^{-\theta |x|})
\]
then as \( |x| \to \infty \), we have
\[
|u(t, x)| \sim O(e^{-\theta |x|}), \quad |\partial_x u(t, x)| \sim O(e^{-\theta |x|}), \\
|v(t, x)| \sim O(e^{-\theta |x|}), \quad |\partial_x v(t, x)| \sim O(e^{-\theta |x|})
\]
uniformly with respect to \( t \in [0, T] \).

After establishing unique continuation for system (1.3) in the sense of Theorem 2.2, it is natural to ask the question of how the solution behaves at infinity when given compactly supported initial data. This qualitative behavior is examined by Theorem 4.1.

The paper is organized as follows. In Section 3, we prove the persistence properties of system (1.3) as listed in Theorem 2.2. Then we prove the optimal decay index of the momentum \( m \) and \( n \). In Section 4 we examine the behavior of strong solutions when the initial data have compact support.

3. Proof of Theorem 2.2. In the section, we prove the persistence properties of system (1.3). For our convenience, we rewrite Eq.(1.3) as the form of a quasi-linear evolution equation of hyperbolic type. Note that \( G(x) := \frac{1}{2} e^{-|x|} \) is the kernel of \((1 - \partial_x^2)^{-1}\). Then \((1 - \partial_x^2)^{-1} f = G * f \) for all \( f \in L^2(\mathbb{R}) \), \( G * m = u \) and \( G * n = v \). By these identities, Eq.(1.3) can be reformulated as follows:

\[
\begin{align*}
\begin{cases}
u_t + \frac{1}{2} (uv - u_x v_x) u_x = G * F_1 + \partial_x G * F_2, & t > 0, x \in \mathbb{R}, \\
v_t + \frac{1}{2} (uv - u_x v_x) v_x = G * H_1 + \partial_x G * H_2, & t > 0, x \in \mathbb{R},
\end{cases}
\end{align*}
\]

(3.1)

where
\[
M := (u_x n + v_x m) = (uv - u_x v_x)_x,
\]
\[
F_1 := -\frac{1}{2} (u M - (uv_x - u_x v)m),
F_2 := -\frac{1}{2} (u_x M),
\]
\[
H_1 := -\frac{1}{2} (v M + (uv_x - u_x v)n),
H_2 := -\frac{1}{2} (v_x M).
\]
Assume that $z \in C([0; T]; H^s \times H^s)$ is a strong solution to (1.3) with $s \geq 3$. Let 
\[ K = \sup_{t \in [0; T]} \| z(t) \|_{H^s} =: \sup_{t \in [0; T]} (\| u(t) \|_{H^s} + \| v(t) \|_{H^s}), \]

hence by Sobolev imbedding theorem, we have 
\[ \| u(t, \cdot) \|_{L^\infty} + \| u_x(t, \cdot) \|_{L^\infty} + \| u_{xx}(t, \cdot) \|_{L^\infty} \leq CK, \] (3.2) 
\[ \| v(t, \cdot) \|_{L^\infty} + \| v_x(t, \cdot) \|_{L^\infty} + \| v_{xx}(t, \cdot) \|_{L^\infty} \leq CK \] (3.3) 

Set 
\[ \varphi_N(x) = \begin{cases} e^\theta |x|, & |x| < N, \\ e^{\eta |x|}, & |x| \geq N, \end{cases} \] (4.4) 
where $N \in \mathbb{N}$ and $\theta \in (0, 1)$. Observe that for all $N$ we have 
\[ 0 \leq |\varphi_N| \leq \varphi(x), \ a.e. \ x \in \mathbb{R}. \] (5.5) 

Multiplying (3.1) by $(u \varphi_N)^{2q-1} \varphi_N$ for $q \in \mathbb{N}$ and integrating over the real line we obtain 
\[ \frac{1}{2q} \frac{1}{dt} \int (u \varphi_N)^{2q} dx = -\frac{1}{2} \int (uv - u_x v_x) u_x (u \varphi_N)^{2q-1} \varphi_N dx \] 
\[ + \int \partial_x (G * F_2)(u \varphi_N)^{2q-1} \varphi_N dx + \int (G * F_1)(u \varphi_N)^{2q-1} \varphi_N dx. \] (6.6) 

(3.2)-(3.3) and Hölder’s inequality lead us to achieve the following estimates 
\[ | -\frac{1}{2} \int (uv - u_x v_x) u_x (u \varphi_N)^{2q-1} \varphi_N dx | \leq CK^2 \| u \varphi_N \|_{2q}^{2q-1} \| u_x \varphi_N \|_{2q}, \] (7.7) 
\[ | \int (G * F_1)(u \varphi_N)^{2q-1} \varphi_N dx | \leq \| u \varphi_N \|_{2q}^{2q-1} \| (G * F_1) \varphi_N \|_{2q}. \] (8.8) 

and 
\[ | \int \partial_x (G * F_2)(u \varphi_N)^{2q-1} \varphi_N dx | \leq \| u \varphi_N \|_{2q}^{2q-1} \| (\partial_x G * F_2) \varphi_N \|_{2q}. \] (9.9) 

From (6.6) and the above estimates, this implies 
\[ \frac{d}{dt} \| u \varphi_N \|_{2q} \leq CK^2 \| u \varphi_N \|_{2q} + \| (G * F_1) \varphi_N \|_{2q} + \| (\partial_x G * F_2) \varphi_N \|_{2q}. \] (10.10) 

By Gronwall’s inequality, (10.10) implies the following estimate 
\[ \| u \varphi_N \|_{2q} \leq (\| u_0 \varphi_N \|_{2q} + \int_0^t \| (G * F_1) \varphi_N \|_{2q} + \| (\partial_x G * F_2) \varphi_N \|_{2q}|d\tau|) e^{CK^2 t}. \] (11.11) 

Now differentiating (3.1) with respect to the spacial variable $x$, multiplying by $(u_x \varphi_N)^{2q-1} \varphi_N$ and integrating over the real line yields 
\[ \frac{1}{2q} \frac{1}{dt} \int (u_x \varphi_N)^{2q} dx = -\frac{1}{2} \int (uv - u_x v_x) u_{xx} (u_x \varphi_N)^{2q-1} \varphi_N dx \] 
\[ + \int \partial_x^2 (G * F_2)(u_x \varphi_N)^{2q-1} \varphi_N dx + \int \partial_x (G * F_1)(u_x \varphi_N)^{2q-1} \varphi_N dx \] 
\[ - \frac{1}{2} \int Mu_x(u_x \varphi_N)^{2q-1} \varphi_N dx. \] (12.12) 

This leads us to obtain the following estimates 
\[ | \int \partial_x^2 (G * F_2)(u_x \varphi_N)^{2q-1} \varphi_N dx | \leq \| u_x \varphi_N \|_{2q}^{2q-1} \| (\partial_x^2 G * F_2) \varphi_N \|_{2q}, \]
For the first integral on the RHS of (3.12), we estimate as follows
\[ | \int (\partial_x G * F_1) (u_x \varphi_N) [2q] - 1 | \varphi_N dx | \leq \| u_x \varphi_N \|^{2q-1}_{2q} \| (\partial_x G * F_1) \varphi_N \|_{2q}, \]
\[ - \frac{1}{2} \int M u_x (u_x \varphi_N) [2q] - 1 | \varphi_N dx | \leq \| M \|_{L_{\infty}} \| u_x \varphi_N \|^{2q}_{2q} \leq CK^2 \| u_x \varphi_N \|^{2q}_{2q}. \]

(3.13)

For the first integral on the RHS of (3.12), we estimate as follows
\[ \int (uv - u_x v_x) u_{xx} (u_x \varphi_N)^{2q-1} \varphi_N dx \]
\[ = \int (uv - u_x v_x) (u_x \varphi_N)^2 - u_x \varphi_N dx \]
\[ = - \frac{1}{2q} \int M (u_x \varphi_N)^{2q} dx - \int (uv - u_x v_x) u_x \varphi_N dx \varphi_N^{2q-1} dx \]
\[ \leq CK^2 \| u_x \varphi_N \|^{2q}_{2q}. \]

(3.14)

From (3.12) - (3.14), we achieve the following differential inequality
\[ \frac{d}{dt} \| u_x \varphi_N \|_{2q} \leq CK^2 \| u_x \varphi_N \|_{2q} + \| (\partial^{2}_{x} G * F_2) \varphi_N \|_{2q} + \| (\partial_x G * F_1) \varphi_N \|_{2q}. \]

(3.15)

By Gronwall’s inequality, (3.15) implies the following estimate
\[ \| u_x \varphi_N \|_{2q} \leq (\| \partial_x u_0 \varphi_N \|_{2q} + \int_0^t \| (\partial^{2}_{x} G * F_2) \varphi_N \|_{2q} + \| (\partial_x G * F_1) \varphi_N \|_{2q}| d\tau | C K^2 t. \]

(3.16)

By adding (3.11) and (3.16), we have the following
\[ \| u_x \varphi_N \|_{2q} + \| u_x \varphi_N \|_{2q} \leq (\| u_0 \varphi_N \|_{2q} + \| \partial_x u_0 \varphi_N \|_{2q}) e^{CK^2 t} \]
\[ + (\int_0^t \| (\partial_x G * F_2) \varphi_N \|_{2q} + \| (G * F_1) \varphi_N \|_{2q} | d\tau | e^{CK^2 t} \]
\[ + (\int_0^t \| (\partial^{2}_{x} G * F_2) \varphi_N \|_{2q} + \| (\partial_x G * F_1) \varphi_N \|_{2q} | d\tau | e^{CK^2 t}. \]

(3.17)

Now, for any function \( f \in L^1 \cap L^\infty, \lim_{n \to \infty} \| f \|_{L^\infty} = \| f \|_{L^\infty}. \) Since we have that \( F_1, F_2 \in L^1 \cap L^\infty \) and \( G \in W^{1, \infty} \), we know that \( \partial_x^i G * F_1, \partial_x^j G * F_2 \in L^1 \cap L^\infty \) (for \( i = 0, 1 \) and \( j = 1, 2 \)). Thus, by the limit of (3.17) as \( q \to \infty \), we get
\[ \| u_x \varphi_N \|_{\infty} + \| u_x \varphi_N \|_{\infty} \leq (\| u_0 \varphi_N \|_{\infty} + \| \partial_x u_0 \varphi_N \|_{\infty}) e^{CK^2 t} \]
\[ + (\int_0^t \| (\partial_x G * F_2) \varphi_N \|_{\infty} + \| (G * F_1) \varphi_N \|_{\infty} | d\tau | e^{CK^2 t} \]
\[ + (\int_0^t \| (\partial^{2}_{x} G * F_2) \varphi_N \|_{\infty} + \| (\partial_x G * F_1) \varphi_N \|_{\infty} | d\tau | e^{CK^2 t}. \]

(3.18)

A simple calculation shows that for \( \theta \in (0, 1) \)
\[ \varphi_N(x) \int_R e^{-|x-y|} \frac{1}{\varphi_N(y)} dy \leq \frac{4}{1-\theta} = C_0. \]

(3.19)

Thus, for any function \( f, g, h \in L^\infty \), we have
\[ \| (G * f g h) \varphi_N \|_{\infty} = \frac{1}{2} \varphi_N \int_R e^{-|x-y|} (f g h)(y) dy \]
\[ \leq \frac{1}{2} (\varphi_N \int_R e^{-|x-y|} \frac{1}{\varphi_N(y)} dy) \| f \|_{\infty} \| g \|_{\infty} \| h \varphi_N \|_{\infty} \]
\[ \leq C_0 \| f \|_{\infty} \| g \|_{\infty} \| h \varphi_N \|_{\infty}. \]
Therefore, since \( u, v, u_x, v_x, m, n, M \in L^\infty \), we get
\[
\| (\partial_x^2 G * fgh) \varphi_N \|_\infty \leq C_0 \| M \|_\infty \| \varphi_N \|_\infty \leq C_0 K^2 \| u \varphi_N \|_\infty, \ j = 0, 1
\]
\[
\| (\partial_x^2 G * uM) \varphi_N \|_\infty \leq C_0 \| M \|_\infty \| u\varphi_N \|_\infty \leq C_0 K^2 \| u \varphi_N \|_\infty, \ j = 0, 1
\]
hence,
\[
\| (\partial_x^2 G * F_1) \varphi_N \|_\infty \leq C_0 K^2 (\| u \varphi_N \|_\infty + \| u_x \varphi_N \|_\infty) \ j = 0, 1. \tag{3.20}
\]
Similarly, we have
\[
\| (\partial_x^2 G * u_x M) \varphi_N \|_\infty \leq C_0 (\| M \|_\infty \| u_x \varphi_N \|_\infty \leq C_0 K^2 \| u_x \varphi_N \|_\infty, \ j = 0, 1
\]
For \( j = 2 \), noticing that \( \partial_x^2 G * f = G * f - f \), using the similar procedure, we have
\[
\| (\partial_x^2 G * u_x M) \varphi_N \|_\infty \leq C_0 K^2 \| u_x \varphi_N \|_\infty. \tag{3.21}
\]
Thus, we obtain
\[
\| (\partial_x^2 G * F_2) \varphi_N \|_\infty \leq C_0 K^2 \| u_x \varphi_N \|_\infty \ j = 1, 2. \tag{3.22}
\]
So, by estimates (3.18), (3.20) and (3.22) we achieve the following
\[
\| u \varphi_N \|_\infty + \| u_x \varphi_N \|_\infty \leq C (\| u_0 \varphi_N \|_\infty + \| u_{0,x} \varphi_N \|_\infty)
\]
\[
+ C \int_0^T (\| u_0 \varphi_N \|_\infty + \| u_{0,x} \varphi_N \|_\infty) d\tau \tag{3.23}
\]
where \( C \) is a constant depending on \( C_0, K \) and \( T \).

Multiplying (3.1) by \( (v \varphi_N)^{2q-1} \varphi_N \) for \( q \in \mathbb{N} \) and integrating over the real line, then differentiating (3.1) with respect to the spacial variable \( x \), multiplying by \( (v_x \varphi_N)^{2q-1} \varphi_N \) and integrating over the real line yields, using the similar steps above, we get
\[
\| v \varphi_N \|_\infty + \| v_x \varphi_N \|_\infty \leq C (\| v_0 \varphi_N \|_\infty + \| v_{0,x} \varphi_N \|_\infty)
\]
\[
+ C \int_0^T (\| v_0 \varphi_N \|_\infty + \| v_{0,x} \varphi_N \|_\infty) d\tau \tag{3.24}
\]
Adding (3.23) and (3.24), we have
\[
\| u \varphi_N \|_\infty + \| u_x \varphi_N \|_\infty + \| v \varphi_N \|_\infty \| v_x \varphi_N \|_\infty
\]
\[
\leq C (\| u_0 \varphi_N \|_\infty + \| v_0 \varphi_N \|_\infty + \| u_{0,x} \varphi_N \|_\infty + \| v_{0,x} \varphi_N \|_\infty)
\]
\[
+ C \int_0^T (\| u \varphi_N \|_\infty + \| v \varphi_N \|_\infty + \| u_x \varphi_N \|_\infty + \| v_x \varphi_N \|_\infty) d\tau.
\]
Hence, for any $N \in \mathbb{N}$ and any $t \in [0, T]$, we have by Gronwall’s inequality that
\[
\|u_N\|_\infty + \|u_N\|_\infty + \|v_N\|_\infty + \|v_N\|_\infty \leq C(\|u_0\|_\infty + \|v_0\|_\infty + \|u_0\|_\infty + \|v_0\|_\infty)
\]
\[
\leq C(\|u_0\|_\infty + \|v_0\|_\infty + \|u_0\|_\infty + \|v_0\|_\infty).
\]
(3.25)
with $f_\theta := \max(1, e^{\theta|x|})$. This concludes our proof of Theorem 2.2.

**Remark 3.1.** In fact, let $\theta \in (0, 1)$, and $j = 0, 1, 2, \ldots$, if the initial data $z_0$ satisfy
\[
\partial^j_t u_0, \partial^j_t v_0 \sim O(e^{-\theta|x|}), \quad \text{as } |x| \to \infty,
\]
then the solution $z$ also has the same exponential decay properties, i.e.
\[
\partial^j_t u, \partial^j_t v \sim O(e^{-\theta|x|}), \quad \text{as } |x| \to \infty.
\]

Theorem 2.2 tells us that the solution $z$ can decay as $e^{-\theta|x|}$, as $x \to \infty$ for $\theta \in (0, 1)$. Whether the decay is optimal? the next result tell us some information.

**Theorem 3.1.** Given $z_0 = (u_0, v_0) \in H^s \times H^s, s \geq 3$. Let $T = T(z_0)$ be the maximal existence time of the solutions $z(t, x) = (u(t, x), v(t, x))$ to system (1.3)/(or (3.1)) with the initial data $z_0$. If for some $\lambda \geq 0$ and $q \geq 1$,
\[
\|(m_0, n_0)e^{(1+\lambda)|x|}\|_{L^q} \leq C,
\]
then for all $t \in [0, T)$, we have
\[
\|(m, n)e^{(1+\lambda)|x|}\|_{L^q} \leq C,
\]
(3.26)
Moreover, if the initial data satisfy
\[
\partial^j_t u_0, \partial^j_t v_0 \sim O(e^{-\theta(1+\lambda)|x|}), \quad \text{as } |x| \to \infty, j = 0, 1, 2,
\]
(3.27)
then for all $t \in [0, T)$, we get
\[
m, n \sim O(e^{-\theta(1+\lambda)|x|}), \quad \text{as } |x| \to \infty,
\]
(3.28)
and there exists some $\theta \in (0, 1)$ such that
\[
\lim_{|x| \to \infty} |(\partial^j_t u, \partial^j_t v)e^{-\theta|x|}| \leq C, j = 0, 1, 2.
\]
(3.29)

**Proof.** Setting $\varphi_\lambda := e^{(1+\lambda)|x|}$, multiplying (3.1) by $(m\varphi_\lambda)^{2q-1}\varphi_\lambda$ for $q \in \mathbb{N}$ and integrating over the real line we obtain
\[
\int \frac{1}{2q} \frac{1}{dt} (m\varphi_\lambda)^{2q}dx = -\frac{1}{2}(uv - u_x v_x)m_x(m\varphi_\lambda)^{2q-1}\varphi_\lambda dx
\]
\[
-\frac{1}{2} \int Mm(m\varphi_\lambda)^{2q-1}\varphi_\lambda dx + \frac{1}{2} \int (uv - u_x v_x)(m\varphi_\lambda)^{2q-1}m\varphi_\lambda dx.
\]
(3.30)
For the first term on RHS of (3.31), we have
\[
\int (uv - u_x v_x)m_x(m\varphi_\lambda)^{2q-1}\varphi_\lambda dx
\]
\[
= \int (uv - u_x v_x)(m\varphi_\lambda)(m\varphi_\lambda)^{2q-1}dx
\]
\[
= -\frac{1}{2q} \int M(m\varphi_\lambda)^{2q}dx - \int (uv - u_x v_x)m\varphi_\lambda(m\varphi_\lambda)^{2q-1}dx
\]
\[
= -\frac{1}{2q} \int M(m\varphi_\lambda)^{2q}dx - (1+\lambda) \int sgn(x)(uv - u_x v_x)m\varphi_\lambda(m\varphi_\lambda)^{2q-1}dx,
\]
(3.31)
where we use the fact that
\[ \varphi_\lambda' = (1 + \lambda)\text{sgn}(x)\varphi_\lambda. \]

Hence we get
\[ |\int (uv - u_xv_x)m_x(m\varphi_\lambda)^2q^{-1}\varphi_\lambda dx| \leq CK^2\|m\varphi_\lambda\|_{2q}^{2q}, \tag{3.32} \]
Note that \(u, v, u_x, v_x, m, n, M \in L^\infty\). We achieve the following estimates
\[ \left|-\frac{1}{2}\int M(m\varphi_\lambda)^2q^{-1}m\varphi_\lambda dx\right| \leq CK^2\|m\varphi_\lambda\|_{2q}^{2q}, \tag{3.33} \]
and
\[ \left|-\frac{1}{2}\int (uv - u_xv_x)(m\varphi_\lambda)^2q^{-1}m\varphi_\lambda dx\right| \leq CK^2\|m\varphi_\lambda\|_{2q}^{2q}. \tag{3.34} \]

From (3.32)-(3.34), this implies
\[ \frac{d}{dt}\|m\varphi_\lambda\|_{2q} \leq CK^2\|m\varphi_\lambda\|_{2q}. \tag{3.35} \]

By Gronwall's inequality, (3.35) implies the following estimate
\[ \|m\varphi_\lambda\|_{2q} \leq \|m_0\varphi_\lambda\|_{2q}e^{CK^2t}. \tag{3.36} \]

As the process of the estimation to (3.36), we deal with system (3.1) is given by
\[ \|n\varphi_\lambda\|_{2q} \leq \|n_0\varphi_\lambda\|_{2q}e^{CK^2t}. \tag{3.37} \]

Add up (3.36) with (3.37), then by the Gronwall inequality yields that
\[ (\|m\varphi_\lambda\|_{2q} + \|n\varphi_\lambda\|_{\infty}) \leq (\|m_0\varphi_\lambda\|_{2q} + \|n_0\varphi_\lambda\|_{\infty})e^{CK^2t}. \tag{3.38} \]

By virtue of the assumption (3.26), it follows that (3.27).

In view of the assumption (3.28) to obtain
\[ (m_0, n_0) \sim O(e^{-(1+\lambda)|x|}), \text{ as } |x| \to \infty \]

Letting \(q \to \infty\) in (3.36) and (3.37) and combing the above relation, we get
\[ \|m\varphi_\lambda\|_{\infty} \leq \|m_0\varphi_\lambda\|_{\infty}e^{CK^2t}, \tag{3.39} \]

and
\[ \|n\varphi_\lambda\|_{\infty} \leq \|n_0\varphi_\lambda\|_{\infty}e^{CK^2t}. \tag{3.40} \]

Add up (3.39) with (3.40), then by the Gronwall inequality yields that
\[ (\|m\varphi_\lambda\|_{\infty} + \|n\varphi_\lambda\|_{\infty}) \leq (\|m_0\varphi_\lambda\|_{\infty} + \|n_0\varphi_\lambda\|_{\infty})e^{CK^2t}. \tag{3.41} \]

On the other hand, by virtue of (3.28) and Theorem 2.2, we deduce the last part of the theorem.

\[ \square \]

**Remark 3.2.** As long as the solution \(z(t, x)\) exists, the result of Theorem 3.1 tells us that the solutions \((z, z_x)\) decay as \(e^{-\theta|x|}\) when \(|x| \to \infty\) for \(\theta \in (0, 1)\). However, the momentum \((m, n)\) can decay as \(e^{-(1+\lambda)|x|}\) as \(|x| \to \infty\) for \(\lambda \in (0, \infty)\).
4. Compactly supported initial data. In this section, we reflect on the property of unique continuation which we have just shown the Cauchy problem for the system (3.1) to exhibit. In the case of compactly supported initial data unique continuation is essentially infinite speed of propagation of its support. Therefore, it is natural to ask the question: How will strong solutions behave at infinity when given compactly supported initial data? We will need two ingredients in order to provide a sufficient answer.

Given initial data \( z_0 \in H^s \times H^s, s \geq 3 \), Theorem 2.1 ensures the local well-posedness of strong solutions. Consider the following initial value problem

\[
\begin{cases}
q_t = (uv - u_xv_x)q(t, y), & t \in [0, T), y \in \mathbb{R}, \\
q(0, y) = x, & y \in \mathbb{R},
\end{cases}
\tag{4.1}
\]

where \( u, v \) denotes the two component of solution \( z \) to Eq.(3.1). Since \( z(t, \cdot) \in H^3 \times H^3 \subset C^m \times C^m \) with \( 0 \leq m \leq \frac{5}{2} \), thus \( z = (u, v) \in C^1([0, T) \times \mathbb{R}, \mathbb{R}) \), applying the classical results in the theory of ordinary differential equations, one can obtain the following results of \( q \) which is the key in the proof of unique continuation of strong solutions to Eq.(4.1).

We now present the following two lemmas for our goal.

**Lemma 4.1.** [31] Let \( z_0 \in H^s \times H^s, s \geq 2 \). Then Eq.(4.1) has a unique solution \( q \in C^1([0, T) \times \mathbb{R}, \mathbb{R}) \). Moreover, the map \( q(t, \cdot) \) is an increasing diffeomorphism of \( \mathbb{R} \) with

\[
q_x(t, x) = \exp \left( \int_0^t (u_xn + v_xm)(s, q(s, x))ds \right) > 0, \quad (t, x) \in [0, T) \times \mathbb{R}.
\]

**Lemma 4.2.** [31] Let \( z_0 \in H^s \times H^{s-1}, s \geq 2 \) and \( T > 0 \) be the maximal existence time of corresponding solution \( z \) to Eq.(3.1). Then for all \( (t, x) \in [0, T) \times \mathbb{R} \) we have

\[
m(t, q(t, x))q_x(t, x) = m_0(x) \exp \int_0^t (u_xn + v_xm)(\tau, q(\tau, x))d\tau,
\tag{4.2}
\]

\[
n(t, q(t, x))q_x(t, x) = n_0(x) \exp \int_0^t (u_xn + v_xm)(\tau, q(\tau, x))d\tau.
\tag{4.3}
\]

Now, utilizing the new form for the system (4.1) and our family of diffeomorphisms given by Lemma 4.1, we may now determine the behavior of our solutions at infinity when given compactly supported initial data. This is provided via the following theorem.

**Theorem 4.1.** Let \( z \in C([0, T) \times C([0, T), s \geq \frac{5}{2} \), be a nontrivial solution of (3.1), with maximal time of existence \( T > 0 \), which is initially compactly supported on an interval \([a, b] \). Then we have

\[
\begin{align*}
u(t, x) &= \begin{cases}
\frac{1}{2}E_+(t)e^{-\tau}, & x > q(t, b), \\
\frac{1}{2}E_-(t)e^{\tau}, & x < q(t, a), 
\end{cases} \\v(t, x) &= \begin{cases}
\frac{1}{2}F_+(t)e^{-\tau}, & x > q(t, b), \\
\frac{1}{2}F_-(t)e^{\tau}, & x < q(t, a), 
\end{cases}
\end{align*}
\tag{4.4}
\]

with \( E_+(t) := \int_{q(t, a)}^{q(t, b)} e^y m(t, y)dy \), \( E_-(t) := \int_{q(t, a)}^{q(t, b)} e^{-y} m(t, y)dy \), \( F_+(t) := \int_{q(t, a)}^{q(t, b)} e^y n(t, y)dy \), and \( F_-(t) := \int_{q(t, a)}^{q(t, b)} e^{-y} n(t, y)dy \). Moreover, \( E_+(t), E_-(t), F_+(t) \) and \( F_-(t) \) are continuous non-vanishing functions with \( E_+(0) = E_-(0) = F_+(0) = F_-(0) = 0 \).
Theorem 4.1 tells us that as long as the solution \( z(x, t) \) exists, then it is positive at infinity and negative at negative infinity. We now proceed to the proof of the above result.

Proof. If \( u_0 \) and \( v_0 \) are initially supported on the compact interval \([a, b]\) then so are \( m_0 \) and \( n_0 \). And from (4.2) and (4.3) it follows that \( m(t, \cdot), n(t, \cdot) \) is compactly supported with its support contained in the interval \([q(t, a), q(t, b)]\). We now use the relation \( u = \frac{1}{2} e^{-|x|} \ast m \) and \( v = \frac{1}{2} e^{-|x|} \ast n \) to write

\[
\begin{align*}
    u(t, x) &= \frac{e^x}{2} \int_{-\infty}^{x} e^y m(t, y) dy + \frac{e^x}{2} \int_{x}^{\infty} e^{-y} m(t, y) dy, \\
    u_x(t, x) &= -\frac{e^{-x}}{2} \int_{-\infty}^{x} e^y m(t, y) dy + \frac{e^x}{2} \int_{x}^{\infty} e^{-y} m(t, y) dy,
\end{align*}
\]

and

\[
\begin{align*}
    v(t, x) &= \frac{e^x}{2} \int_{-\infty}^{x} e^y n(t, y) dy + \frac{e^x}{2} \int_{x}^{\infty} e^{-y} n(t, y) dy, \\
    v_x(t, x) &= -\frac{e^{-x}}{2} \int_{-\infty}^{x} e^y n(t, y) dy + \frac{e^x}{2} \int_{x}^{\infty} e^{-y} n(t, y) dy.
\end{align*}
\]

Assume that \( m_0 \) and \( n_0 \) are non-negative, then we obtain

\[
\begin{align*}
    u(t, x) + u_x(t, x) &= \frac{e^x}{2} \int_{x}^{\infty} e^y m(t, y) dy \geq 0, \\
    u(t, x) - u_x(t, x) &= \frac{e^{-x}}{2} \int_{-\infty}^{x} e^y m(t, y) dy \geq 0, \\
    v(t, x) + v_x(t, x) &= \frac{e^x}{2} \int_{x}^{\infty} e^y n(t, y) dy \geq 0, \\
    v(t, x) - v_x(t, x) &= \frac{e^{-x}}{2} \int_{-\infty}^{x} e^y n(t, y) dy \geq 0.
\end{align*}
\]

i.e. \( |u_x| \leq u \) and \( |v_x| \leq v \). and then we define our functions

\[
\begin{align*}
    E_+(t) &= \int_{q(t, a)}^{q(t, b)} e^y m(t, y) dy, \\
    E_-(t) &= \int_{q(t, a)}^{q(t, b)} e^{-y} m(t, y) dy, \\
    F_+(t) &= \int_{q(t, a)}^{q(t, b)} e^y n(t, y) dy, \\
    F_-(t) &= \int_{q(t, a)}^{q(t, b)} e^{-y} n(t, y) dy.
\end{align*}
\]

we have that

\[
\begin{align*}
    u(t, x) &= \frac{e^{-x}}{2} E_+(t), \ x > q(t, b), \\
    u(t, x) &= \frac{e^x}{2} E_-(t), \ x < q(t, a), \\
    v(t, x) &= \frac{e^{-x}}{2} F_+(t), \ x > q(t, b), \\
    v(t, x) &= \frac{e^x}{2} F_-(t), \ x < q(t, a),
\end{align*}
\]
therefore from differentiating (4.10) directly we get

\[
\begin{align*}
\frac{e^{-x}}{2} E_+(t) &= u(t, x) = -u_x(t, x) = u_{xx}(t, x), \quad x > q(t, b), \\
\frac{e^x}{2} E_-(t) &= u(t, x) = u_x(t, x) = u_{xx}(t, x), \quad x < q(t, a), \\
\frac{e^{-x}}{2} F_+(t) &= v(t, x) = -v_x(t, x) = v_{xx}(t, x), \quad x > q(t, b), \\
\frac{e^x}{2} F_-(t) &= v(t, x) = v_x(t, x) = v_{xx}(t, x), \quad x < q(t, a).
\end{align*}
\]

(4.11)

Since \(u(0, \cdot)\) and \(v(0, \cdot)\) is supported in the interval \([a, b]\) this immediately gives us \(E_+(0) = E_-(0) = 0\) and \(F_+(0) = F_-(0) = 0\).

Since \(m(t, \cdot)\) is supported in the interval \([q(t, a), q(t, b)]\), for each fixed \(t\) we have

\[
\begin{align*}
\frac{dE_+(t)}{dt} &= \int_{q(t,a)}^{q(t,b)} e^y m(t, y) dy = \int_{-\infty}^{\infty} e^y m(t, y) dy.
\end{align*}
\]

Thus, we have

\[
\begin{align*}
\frac{dE_+(t)}{dt} &= \int_{q(t,a)}^{q(t,b)} e^y m(t, y) dy \\
&= \int_{-\infty}^{\infty} e^y m(t, y) dy \\
&= -\int_{-\infty}^{\infty} \frac{1}{2} [(uv - u_y v_y)m_y e^y dy + \int_{-\infty}^{\infty} \frac{1}{2} (uv_y - vu_y) m e^y dy \\
&= \int_{-\infty}^{\infty} \frac{1}{2} (u - u_y)(v + v_y) m e^y dy \geq 0.
\end{align*}
\]

Nevertheless,

\[
\begin{align*}
\frac{dE_-(t)}{dt} &= \int_{q(t,a)}^{q(t,b)} e^{-y} m(t, y) dy \\
&= \int_{-\infty}^{\infty} e^{-y} m(t, y) dy \\
&= -\int_{-\infty}^{\infty} \frac{1}{2} [(uv - u_y v_y)m_y e^{-y dy + \int_{-\infty}^{\infty} \frac{1}{2} (uv_y - vu_y) m e^{-y dy \\
&= \int_{-\infty}^{\infty} \frac{1}{2} (u + u_y)(v - v_y) m e^{-y dy \leq 0,
\end{align*}
\]

where the strict positivity of the relation above follows from our assumption that the solution is nontrivial. Using the similar process gives the properties of \(F_+\) and \(F_-\). This concludes the proof of Theorem 4.1.

\[\square\]

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