A classical calculation of the leptonic magnetic moment

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In this paper we will show that purely classical concepts based on a few heuristic considerations about extended field configurations are enough to compute the leptonic magnetic moment with corrections up to the order $\alpha$ perturbatively.

I. HISTORICAL INTRODUCTION

1947's Shelter Island's conference is known to have raised awareness around two effects, the anomalous magnetic moment of the electron [1] and the Lamb shift in hydrogen atoms [2,3], which were shortly later explained in terms of $n$-loop contributions; the following year another effect, the Casimir effect [4], was explained in terms of zero-point energies: as the former two can be interpreted by means of creation and re-absorption of virtual particles, while the latter one can be interpreted by means of vacuum fluctuations, these three effects are altogether taken as proof to justify the interpretation for which quantum fluctuations are real [5] — nevertheless, if we wish to avoid meaningless interpretations, it is necessary to take into account these figures rather gingerly.

A first instance indicating that quantum fluctuations, and more precisely vacuum fluctuations, are not necessarily real should have been seen already in the seminal paper about the Casimir effect: Casimir calculated the pull of two plates using zero-point energies only after Bohr suggested to follow a method simpler than the original one, in terms of which Casimir and Polder calculated the attraction between paired conductors employing retarded van der Waals forces; the Casimir force has also been computed in terms of radiative processes connected to external legs by Jaffe [6], or fields in interaction with external sources by Schwinger [7]. In none of these calculations is the zero-point energy found: if Jaffe replaced the ground-state with higher-order corrections but always dealing with operators while Schwinger replaced operators with sources in a formalism that was essentially a path-integral formulation, Casimir and Polder considered no quantum concept. So we may ask if also for the other two effects field quantization may be avoided.

The reality of quantum fields could also be questioned in view of what might give rise to the Lamb shift for the hydrogen atom, with the hyper-fine splitting described by Bethe in terms of the quantum setting: nevertheless, it has also been shown in a semi-classical treatment by Welton that it is possible to describe these splittings as differences of the potential due to oscillations in the position of the electron [8]: moreover, we can make entirely classical this description if the displacement in the location of the electron is due its Zitterbewegung [9].

That the hyper-fine splittings could be re-interpreted by assuming that electrons have a trembling motion is important for the fact that in their decay rates, para-positronium and ortho-positronium display a discrepancy in the fine-structure constant [10]: the reason may be that in the case of positronium, electrons and positrons have the elementary dynamics, but singlet and triplet states of positronium may receive different contributions if electrons and positrons were to have non-trivial dynamics.

Finally, also the electron magnetic moment correction has been calculated in terms of Zitterbewegung [10].

This is important because the calculated and measured values of the anomalous magnetic moment, if in the case of the electron they agree, in the case of the muon they disagree for 3.4 standard deviations [11]: this discrepancy might be quenched for leptons of finite extension.

That such corrections have something to do with a finite extension is clear since the most precise tests of QED strongly depend on the precision about the measurement of the Compton wave-length of particles [12].

Hence, precision tests of QED do show discrepancies between experiments and theory. But still worse is the fact that QED is known to have problems internally in its theoretical structure: the most well-known and important is that (for the energy shifts and the anomalous magnetic moment of leptons) the calculations are done by using a cut-off that is not intrinsic to QED, therefore suggesting that the theory would have to fail beyond a certain energy scale; also (in the case of the anomalies for the magnetic moment of leptons) calculations are based on perturbative expansions which, despite being finite term-by-term, do not converge. And an additional requirement in terms of which all calculations are done is by assuming the existence of expressions of the type

$$\hat{A}_I = \hat{O}_I \hat{A}_0 \hat{O}_I^{-1}$$  \hspace{1cm} (1)

$$|I\rangle = \hat{O}_I |0\rangle$$  \hspace{1cm} (2)

which gives the possibility to write operators and states in interaction in terms of an operator $\hat{O}_I$ containing the information about the interaction acting upon operators and states in the free case: this is called interaction picture, but it does not exist in a Lorentz-covariant form of quantum field theory [12]. Such result is known as Haag theorem and, establishing the theoretical proof that expressions such as (12) cannot hold, it implies that quantum field theory may make no sense whatsoever [13].

However, this does not imply that there is no physical meaning we can extract from the quantization prescriptions anyway: for instance, when quantization prescriptions are considered in path-integral formalism, the most
common and natural interpretation is by thinking that the quantum particle follows all of the possible weighted trajectories; nevertheless, this interpretation may be seen as the fact that the quantum particle has an oscillatory motion about a single trajectory, and again it is reasonable to view the quantum particle as an extended field.

Assuming classical fields with a finite size and calculating what are the consequences for the leptonic magnetic moment correction is what we will do in this paper.

II. CLASSICAL EXTENDED FIELDS

In the introduction, we have re-called and high-lighted that quantum electrodynamics in its fundamental structure may be not well defined at all [13], and consequently it is wise to do calculations avoiding any form of field quantization prescription; also we have remarked that the presence of the Compton wave-length of the particle is ubiquitous [5], which suggests that considering point-like particles is restrictive: a leptonic magnetic moment correction up to the lowest-order was calculated for a classical particle looking like an extended field because of its Zitterbewegung in [10], although in this paper there are additional arbitrary assumptions. As a consequence, it is preferable to have a classical particle as a real extended field distribution and with no additional assumption.

An investigation of the Dirac field that was purely classical and in which the extension of the field was considered has been done previously in [14], where we showed that the Dirac field equation had a leptonic magnetic moment correction, although the lack of explicit solutions forbade to compute its magnitude; a different strategy would be to compute the magnetic moment correction directly, but still we need heuristic assumptions about the structure of the matter dispersion. But while in the previous work exact solutions were needed, here a few simple features of the material distribution would be enough.

To see what these properties are, we start from the fundamental observation that, despite the field is fundamental, nevertheless it is not irreducible: 1/2-spin spinors are given by two complementary parts, the left-handed and the right-handed semi-spinor projections, and although they cannot be disentangled in the massive case, nonetheless they are independent components; if we think at these two components as represented by wave-packets, then we can imagine they are localized in two small regions, and that the two peaks are separated by a distance equaling the size of the particle, that is the Compton wave-length associated to the mass of the particle itself.

The mathematical model we employ is for fermions described by 1/2-spin spinors of charge and m mass, Clifford matrices \( \{ \gamma^\mu, \gamma^5 \} = 2\gamma^\mu \gamma^5 \) are introduced and the Dirac spinors \( \psi \equiv \psi^\dagger \gamma^0 \) are defined, with gauge-covariant derivative \( D_\mu \psi = \nabla_\mu \psi + iq A_\mu \psi \) and where the electrodynamics strength is \( F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \) as usual; action

\[
\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{i}{2} \left( \bar{\psi} \gamma^\mu D_\mu \psi - D_\mu \bar{\psi} \gamma^\mu \psi \right) - m \bar{\psi} \psi
\]  

(3)
determines the dynamics of the model: variation with respect to the gauge potential and the spinor field yields

\[
\nabla_\alpha F^{\alpha\nu} = \bar{\psi} \gamma^\nu \gamma^\alpha \psi
\]  

(4)

\[
i \gamma^\mu D_\mu \psi - m \psi = 0
\]  

(5)
or explicitly in Lorentz-gauge \( \nabla A = 0 \) according to

\[
\nabla^2 A^\nu = q \bar{\psi} \gamma^\nu \psi
\]  

(6)

\[
i \gamma^\mu \nabla_\mu \psi - q A_\mu \gamma^\mu \psi - m \psi = 0
\]  

(7)

and these are the field equations we will have to solve.

The Maxwell field equations (8) have solution

\[
A^\nu = \frac{q}{4 \pi} \int \frac{\bar{\psi} \gamma^\nu \psi'}{|r - r'|} d^3 r'
\]  

(8)

where \( \psi' = \psi'(t - |r - r'|, \vec{r}') \) and \( \psi = \psi(t, \vec{r}) \) were used and called retarded potentials; notice also that the Dirac field equation (7) can be written in the Gordon form as

\[
\nabla_\mu \left( \frac{i}{2} \bar{\psi} \gamma^\alpha \gamma^\mu \psi \right) + \frac{1}{2} \left( \bar{\psi} \gamma^\alpha \psi - \nabla^\alpha \psi \right) - q A^\mu \bar{\psi} \psi - m \bar{\psi} \gamma^\alpha \psi \equiv 0
\]  

(9)
decomposed into the divergence of the leptonic spin plus the current the lepton would have had if it were a scalar plus the retarded potentials plus the leptonic current.

The retarder potentials (8) have the purely spatial part

\[
\tilde{A} = \frac{q}{4 \pi} \int \frac{\bar{\psi} \gamma^\nu \psi'}{|r - r'|} d^3 r'
\]  

(10)

while the decomposition (9) has the purely spatial part that can be written in the standard representation as

\[
\tilde{\nabla} \times (\phi \frac{\vec{r}}{2} \psi) - \frac{i}{2} (\phi \vec{\nabla} \phi - \vec{\nabla} \phi \cdot \phi - \phi \vec{\nabla} \phi) - \phi' \phi q \tilde{A} \equiv m \bar{\psi} \gamma \psi
\]  

(11)
as the spinor is such that \( \tilde{\psi} = (\phi^\dagger, 0) \) with a dependence of the type \( \phi' = \phi'(\vec{r}') \) and \( \phi = \phi(\vec{r}) \) in the non-relativistic low-speed regime we consider, with no retardation all for the potentials (8), being instantaneous potentials, and where in decomposition (11), we may drop the current the lepton would have had if it were a scalar because such a term would account for the linear momentum solely.

When (10) is plugged in (11) we get the relationship

\[
\tilde{\nabla} \times (\psi \frac{\vec{r}}{2} \phi) \equiv m \bar{\psi} \gamma \psi + \frac{q^2}{4 \pi m} \int \frac{\bar{\psi} \gamma \psi'}{|r - r'|} d^3 r'
\]  

(12)
in which the electrodynamic potentials disappeared from an expression yielding the curl of the spin density as the momentum density plus the momentum density due to the electrodynamic interaction of the field distribution.

Up to now all is general, but from this moment on we may take advantage of the heuristic interpretation of extended fields given above: in it, the lepton is described in terms of two localized wave-packets, separated by the Compton wave-length; the two wave-packets are to be identified with the left-handed and right-handed semi-spinor components localized in \( \vec{r} \) and \( \vec{r}' \) so that they can
be written as in $\psi^\dagger = (L^\dagger, 0)$ and $\psi^\dagger = (0, R^\dagger)$ respectively, 
with the condition $|\vec{r} - \vec{r}'| = \lambda$ and $\lambda$ being the Compton wave-length associated to the mass of the particle.

Because in the non-relativistic limit we are taking into account it is known that $L \approx R$ then it is possible to prove that $\mathcal{M} = m \mathcal{\overrightarrow{\gamma}} \psi^\dagger \approx -m \mathcal{\overrightarrow{\gamma}} \psi$ and $\phi^\dagger \phi = \phi^\dagger \phi$ as it is clear since the two opposite helicity states must have opposite spatial momentum densities; additionally, for wave-packets we have $m \mathcal{\overrightarrow{\gamma}} \psi^\dagger = \phi^\dagger \phi \mathcal{\overrightarrow{r}}$ and $m \mathcal{\overrightarrow{\gamma}} \psi = \phi^\dagger \phi \mathcal{\overrightarrow{r}}$ as it is easy to see by expanding in plane-waves: then (12) becomes
\[ \vec{\nabla} \times (\phi^\dagger \frac{\vec{\sigma}}{2} \phi) \approx m \mathcal{\overrightarrow{\gamma}} \psi \left(1 - \frac{\alpha}{2\pi}\right) \] \[ (13) \]

having set $\alpha = \frac{q^2}{2m}$ and as $m\lambda = 2\pi$ by definition of Compton wave-length associated to the mass of the particle and yielding the curl of the spin density in terms of the momentum density in a very simple relationship indeed.

III. LEPTONIC MAGNETIC MOMENT CORRECTION

From general electrodynamic considerations about the multi-pole expansion, the magnetic moment is defined as
\[ \vec{\mu} = \frac{1}{2} \int \mathcal{\overrightarrow{r}} \times (q\mathcal{\overrightarrow{\gamma}} \psi) \, d^3r \] \[ (14) \]

for the leptonic current; plugging (13) into (14) gives
\[ \vec{\mu} = \frac{q}{2m} \left(1 - \frac{\alpha}{2\pi}\right)^{-1} \int \phi^\dagger \frac{\vec{\sigma}}{2} \phi \, d^3r \] \[ (15) \]

having used $\mathcal{\overrightarrow{r}} \times [\mathcal{\overrightarrow{\nabla}} \times (\phi^\dagger \frac{\vec{\sigma}}{2} \phi)] \equiv \phi^\dagger \frac{\vec{\sigma}}{2} \phi$ and where the result is given up to surface terms that can be neglected at infinity inside the integral: because the integral of the spin density is the spin we may then write
\[ \vec{\mu} = \frac{2}{\pi} \frac{q}{2m} \left(1 - \frac{\alpha}{2\pi}\right)^{-1} \] \[ (16) \]

where $\frac{2}{\pi}$ is known to be much smaller than unity.

Therefore, it is possible to make the approximation
\[ \vec{\mu} \approx \frac{2}{\pi} \frac{q}{2m} \left(1 + \frac{\alpha}{2\pi}\right) \] \[ (17) \]

to the lowest-order of $\frac{\alpha}{2\pi}$ in perturbation: we notice that the leptonic magnetic moment is given by the spin, times the factor $\frac{q}{2m}$ as it should be expected, times the gyro-magnetic factor $2(1 + \frac{\alpha}{2\pi})$ itself being the product of the factor 2 that recovers the prediction of the Dirac theory and the fine structure factor $1 + \frac{\alpha}{2\pi}$ in which the unity is corrected by the factor $\frac{2}{\pi}$ in perturbation that recovers the prediction coming from Schwinger’s calculations.

In our heuristic interpretation the gyro-magnetic factor has this meaning: factor 2 comes from the two-fold multiplicity of spin spinors; in the fine-structure factor, the unity term is determined by the mechanical moment and it is corrected by powers of $\alpha$ as the result of the mutual electrodynamic interaction between left-handed and right-handed semi-spinorial components: more precisely, the lowest-order power is given by $\frac{2}{\pi}$ as the result of the fact that each semi-spinorial component is a charged field moving in the electrodynamic potential induced by the other semi-spinorial component at the distance of the Compton wave-length; higher-order powers would be due to the fact that each semi-spinorial component moves in the electrodynamic potential induced by the other semi-spinorial component which itself moves in the electrodynamic potential induced by the initial semi-spinorial component, so that each would respond through the other to the action produced by itself; nevertheless, we have not accounted for these corrections because in the case of electrodynamic self-coupling the retarded potentials could no longer be approximated as instantaneous, and the non-relativistic approximation would no longer be a valid approximation. Of course, this situation is already subject of investigations, for a forthcoming paper.

IV. THE INTERPRETATION

To summarize what we have been doing, we may say that we have considered the lepton no longer as a point-like particle but as an extended field with an internal structure constituted by two chiral projections themselves taken as point-like particles, ans separated by the associated Compton wave-length; this picture may look naïve but it is merely the application for leptons of a picture that for hadrons is successful: like hadrons are composite of quarks and their chromodynamic interactions, similarly leptons can be thought as composite of two chiral projections and the reciprocal electrodynamic interactions, and as for hadrons the size of the particle is that of its Compton wave-length, so for leptons the separation of the two components can be taken to be given by its Compton wave-length; for both the magnetic moment correction is due to the same process, although the simpler structure of leptons compared to hadrons and the weaker coupling of photons compared to gluons are the reason why the correction of leptons compared to hadrons is much less dramatic. And intriguingly, in both cases the particle is composite of parts that have no individual existence, although for two entirely different reasons.

V. CONCLUSION

In this paper, we have considered the heuristic interpretation of leptons as extended objects with an internal structure given by the two chiral projections localized in two small regions, separated by the Compton wave-length of the mass of the particle, and we have considered non-relativistic regime; we have calculated the leptonic magnetic moment correction to the lowest-order, that is the only order that could be meaningfully computed in
that approximation: we have remarked that the leptonic magnetic moment correction and the hadronic magnetic moment anomalies have an analogous interpretation and therefore they might have the same physical meaning.

In this picture, we have considered no tool whose existence might be questioned in the same way in which one may question the existence of the interaction picture in view of the Haag theorem: the only truly different assumption we made is that leptons despite being fundamental nevertheless are reducible, whose two irreducible components constitute an internal structure, but this too is not in debate; nor is it in debate the fact that the extensiveness of the field is the Compton wave-length associated to the mass of the particle. Then, the correction to the gyro-magnetic factor of leptons is described in terms of the electrodynamic interaction between the two chiral projections of the field; this result has to be taken together with the fact that the hyper-fine splitting can be described in terms of a displacement in the location of the electron due to its Zitterbewegung, and with the original description of the Casimir effect as due to the retardation in van der Waals forces. These threes results describe in terms of extended fields and their retarded interactions but with no field quantization the three effects that are commonly described by field quantization.

In this perspective, the concept of extended fields replaces field quantization; in QED, the common procedure is that of considering the particle to be point-like although quantization would give rise to a surrounding cloud of virtual photons that would make point-like particles look like an extended field: but of course it may be that quantum particles actually are extended fields.

That implementing field quantization for point-like particles may merely mean considering extended fields is an idea underlying the entire framework of QED in the accepted interpretation: we suggest that this is no coincidence and that it should be taken as real.

Retaining the description of extended fields is theoretically simpler, although it will take time for this idea to get the precision that is obtained in QED.

It may be curious to think at what might have happened if this new idea came in 1947.

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