A Simple Algorithm for Minimum Cuts in Near-Linear Time

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Abstract

We consider the minimum cut problem in undirected, weighted graphs. We give a simple algorithm to find a minimum cut that 2-respects (cuts two edges of) a spanning tree $T$ of a graph $G$. This procedure can be used in place of the complicated subroutine given in Karger’s near-linear time minimum cut algorithm [21]. We give a self-contained version of Karger’s algorithm with the new procedure, which is easy to state and relatively simple to implement. It produces a minimum cut on an $m$-edge, $n$-vertex graph in $O(m \log^2 n)$ time with high probability. This performance matches that achieved by Karger, thereby matching the current state of the art.

2012 ACM Subject Classification Theory of computation → Graph algorithms analysis

Keywords and phrases minimum cut, near-linear time, sparsification, packing
1 Introduction

The minimum cut problem on an undirected (weighted) graph $G$ asks for a vertex subset $S$ such that the total number (weight) of edges from $S$ to $V \setminus S$ is minimized. The minimum cut problem is a fundamental problem in graph optimization and has received vast attention by the research community, across a number of different computation models [21, 8, 32, 13, 37, 17, 5, 15, 1, 20, 10, 24, 16, 6, 9, 30, 35, 11]. Its applications include network reliability [33, 19], cluster analysis [3], and a critical subroutine in cutting-plane algorithms for the traveling salesman problem [2].

The current fastest algorithm to compute a minimum cut in an undirected, weighted graph is an algorithm by Karger [21] which produces a minimum cut on an $m$-edge, $n$-vertex graph in $O(m \log^3 n)$ time with high probability. The main component of Karger’s algorithm is a subroutine that finds a minimum cut that $2$-respects (cuts two edges of) a spanning tree $T$ of a graph $G$. In other words, the cut found is minimal amongst all cuts of $G$ that cut exactly two edges of $T$. Despite the number of pairs of spanning tree edges totaling $\Omega(n^2)$, Karger shows this can be accomplished in $O(m \log^2 n)$ time. Unfortunately, the procedure developed is particularly complex, a detail Karger admits when comparing the algorithm to a simpler $O(n^2 \log n)$ algorithm he develops to find all minimum cuts [21]. Indeed, for this reason, implementation of the asymptotically fastest minimum cut algorithm has been avoided in practical performance analyses [5, 17].

In this paper, we give a simple algorithm to find a minimum cut that $2$-respects a spanning tree $T$ of a graph $G$. Our procedure runs in $O(m \log^2 n)$ time, matching the performance of Karger’s more-complicated subroutine. We achieve the simplification via a clever use of the heavy-light decomposition. Although our procedure requires the use of the top tree data structure [1] to achieve optimal performance, at the cost of an $O(\log n)$ factor, heavy-light decomposition can be used a second time so that only augmented binary search trees are a necessary data structure. We also give a self-contained version of Karger’s algorithm [21] with this new procedure, which avoids implementation issues associated with previous versions [21, 8].

Karger’s algorithm [21], as well as the edge-sampling technique it is based on [20], has been extended and adapted to results in a number of different settings [11, 6, 37, 10, 6, 50]. In particular, in the fully-dynamic setting, Thorup [37] uses the tree-packing technique developed by Karger [21], but maintains a larger set of trees so that the minimum cut $1$-respects at least one of them. In the parallel setting, Geissmann and Gianinazzi [10] are able to parallelize both the dynamic tree data structure and the necessary computation required by Karger’s algorithm [21]. This work is based off prior work in the cache-oblivious model [9], also based on Karger’s algorithm [21]. In the distributed setting, Ghaffari and Kuhn [11] achieve a $(2 + \epsilon)$-approximation probability to the minimum cut based off Karger’s sampling technique [20]. This is improved to a $(1 + \epsilon)$-approximation with similar runtime by Nanongkai and Su [30]. Nanongkai and Su develop their algorithm from Thorup’s fully-dynamic min-cut algorithm [37], Karger’s sampling technique [20], and Karger’s dynamic program to find the minimum cut that $1$-respects a tree [21]. Finally, Daga et al. [6] achieve a sublinear time distributed algorithm to compute the exact minimum cut in an unweighted undirected graph. This algorithm builds off a more recent development in minimum cut algorithms [24], combined again with the tree-packing technique introduced by Karger [21]. Specifically, a tree packing is found in an efficient number of distributed rounds, then Karger’s complicated algorithm to find a minimum $2$-respecting cut is applied in the distributed setting.

This vast amount of work based off Karger’s original near-linear time algorithm suggests
that simplifying it may yield additional techniques that can be applied in alternative settings. This has been shown to be the case in the dynamic setting, where techniques developed from our simplification yield new results in dynamic higher connectivity algorithms [27].

This paper is organized as follows. In Section 2, we discuss the history of the minimum cut problem, in particular discussing other algorithms with a claim to simplicity. In Section 3, we give a self-contained version of Karger’s algorithm to pack spanning trees, reducing the problem to finding minimum cuts that 1- and 2-respect a tree. In Section 4, we show how to find minimum cuts that 1-respect a tree using our new procedure. In Section 5, we extend the approach to find minimum cuts that 2-respect a tree. We give concluding remarks in Section 6. Appendix A describes a version of the Plotkin-Shmoys-Tardos algorithm for tree packing used in our procedure.

2 Related Work

Before we begin, we discuss a brief history of the minimum cut problem. The minimum cut problem was originally perceived as a harder variant of the maximum flow problem, and was solved by \( (n^2) \) flow computations. Gomory and Hu [12] showed how to compute all pairwise max flows in \( n - 1 \) flow computations, thus reducing the complexity of the minimum cut problem by a \( \Theta(n) \) factor. Hao and Orlin [14] further showed the minimum cut in a directed graph can be reduced to a single flow computation.

Nagamochi and Ibaraki [29, 28] developed a deterministic algorithm that is not based on computing maximum s-t flows. They achieve \( O(nm + n^2 \log n) \) time on a capacitated, undirected graph. This procedure was simplified by Stoer and Wagner [35], achieving the same runtime. The Stoer-Wagner algorithm gives a simple procedure to find an arbitrary minimum s-t cut. Vertices s and t are then merged, and the procedure repeats. Although the \( O(nm + n^2 \log n) \) time complexity requires an efficient priority queue such as a Fibonacci heap [7], a binary heap can be used to achieve runtime \( O(nm \log n) \).

Two algorithms based on edge contraction have been devised. The first is an algorithm of Karger [18] and is incredibly simple. The algorithm randomly contracts edges until only two vertices remain. Repeated \( O(n^2 \log n) \) times, the algorithm finds all minimum cuts on an undirected, weighted graph in \( O(n^2m \log n) \) time with high probability. This technique was improved by Karger and Stein [22] by observing an edge of the minimum cut is more likely to be contracted later in the contraction procedure. Their improvement thus branches the contraction procedure after a certain threshold has been reached, therefore spending more time to avoid contracting an edge of the minimum cut when fewer edges remain. The Karger-Stein algorithm achieves runtime \( O(n^2 \log n) \) Monte Carlo.

On an unweighted graph, Gabow [8] showed how to compute the minimum cut in \( O(cn \log(n^2/m)) \) time, where \( c \) is the capacity of the minimum cut. Karger [20] improved Gabow’s algorithm by applying random sampling, achieving runtime \( O(m \sqrt{c}) \) Las Vegas. The sampling technique developed by Karger [20], combined with the tree-packing technique devised by Gabow [8], form the basis of Karger’s near-linear time minimum cut algorithm [21]. As previously mentioned, this technique finds the minimum cut in an undirected, weighted graph in \( O(m \log^3 n) \) time with high probability.

A recent development uses low-conductance cuts to find the minimum cut in an undirected unweighted graph. This technique was introduced by Kawarabayashi and Thorup [24], who achieve near-linear deterministic time (estimated to be \( O(m \log^{1.2} n) \)). This was improved by

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1 The \( \tilde{O}(f) \) notation hides \( O(\log f) \) factors.
Henzinger, Rao, and Wang [16], who achieve deterministic runtime $O(m \log^2 n \log \log n)$.
Although the algorithm of Henzinger et al. is more efficient than Karger’s algorithm [21] on
unweighted graphs, the procedure, as well as the one it was based on [24] are quite involved,
thus making them largely impractical for implementation purposes.

3 Karger’s Algorithm for Packing Spanning Trees

We first formalize the definition mentioned earlier in this paper and first given by Karger.

\[ \textbf{Definition 1 (Karger [21])}. \text{ Let } T \text{ be a spanning tree of } G. \text{ We say that a cut in } G \text{ } k\text{-respects } T \text{ if it cuts at most } k \text{ edges of } T. \text{ We also say that } T \text{ } k\text{-constrains the cut in } G. \]

The basic idea of Karger’s near-linear time algorithm [21] is to exploit the following
combinatorial result. Recall that a tree packing of an undirected unweighted graph $G$ is a
set of spanning trees such that each edge of $G$ is contained in at most one spanning tree.

\[ \textbf{Theorem 2 (Nash-Williams [31])}. \text{ Any undirected unweighted graph with minimum cut } c \text{ contains a tree packing of value at least } c/2. \]

Now consider a minimum cut and a tree packing given by Theorem 2. Each edge of the
minimum cut can only be present in at most one spanning tree. As there are $c$ edges of the
minimum cut, this implies on average, each spanning tree contains at most $c/(c/2) = 2$ edges
of the minimum cut. In other words, a spanning tree chosen at random from a packing of
Theorem 2 is likely to $2$-constrain the minimum cut.

Suppose we are given a spanning tree $T$ of $G$ with each edge of $T$ marked if it crosses
the minimum cut. The endpoints of any marked edge must fall on opposite sides of the cut.
Conversely, the endpoints of any unmarked edge must be on the same side of the cut. It
follows that if we know the edges of $T$ in the minimum cut, we can determine the vertex
partition of the minimum cut and its total weight in $G$.

This gives the intuition behind Karger’s algorithm [21]. We sample spanning trees
from a tree packing of $G$ and for each tree $T$, we find the minimum cut that $2$-respects
$T$. Unfortunately, several obstacles need be overcome before this can be made into an
efficient algorithm. For one, all currently known approaches of determining a tree packing of
Theorem 2 have runtime $\Omega(cn)$, which for large values of $c$ is far more than the runtime we
seek. Further, Theorem 2 must be generalized to weighted graphs.

We first address the latter concern. We formally define a weighted tree packing as follows.

\[ \textbf{Definition 3 (Karger [21])}. \text{ A weighted tree packing is a set of spanning trees, each with an assigned overall weight, such that the total weight of trees containing a given edge of } G \text{ is no greater than the weight of that edge. The value of the packing is the total weight of the trees in it.} \]

As defined, Theorem 2 then applies unchanged to weighted graphs and weighted packings.
The proof is to simply consider weighted edges as multiple unit-cost edges and a particular
tree of the packing as a number of identical trees equal to its assigned weight.

\[ \textbf{Lemma 4 (Karger [21])}. \text{ Any undirected weighted graph with minimum cut } c \text{ contains a weighted tree packing of value at least } c/2. \]

To effectively use this lemma, we formally state the relationship between weighted packings
and trees that $2$-constrain small cuts.
Lemma 5 (Karger [21]). Consider a weighted graph $G$ and a weighted tree packing of value $\beta c$, where $c$ is the weight of the minimum cut in $G$. Then given a cut of weight $\alpha c$, a fraction at least $\frac{1}{2}(3 - \alpha / \beta)$ of the trees (by weight) 2-constrain the cut.

Proof. Apply the pigeonhole principle. Each spanning tree must cross any cut, and worst case is a large number of trees have three edges from our cut and the rest contain a single edge. Let $x$ denote the total weight of trees with 3 edges from the cut and $y$ the total weight of trees with a single edge. Then $x + y = \beta c$ and $3x + y = \alpha c$. Solving for $x$ and $y$ shows that $y/(\beta c) = 3/2 - \alpha/(2\beta)$.

3.1 Random Sampling

In order to avoid the $\Omega(cn)$ complexity of finding a packing of value $c/2$, we first apply random sampling to $G$. Specifically, the following is used from Karger’s earlier work.

Lemma 6 (Karger [20]). Let $p = 3(d + 2)(\ln n) / (c^2 \gamma c)$, where $c$ is the weight of the minimum cut of $G$ and $\gamma \leq 1$, $\gamma = \Theta(1)$. Then if we sample each edge of $G$ uniformly at random with probability $p$, the resulting graph $H$ has the following properties with probability $1 - 1/n^d$.

1. The minimum cut in $H$ is of size within a $(1 + \epsilon)$ factor of $c p = 3(d + 2)(\ln n) / (\gamma c^2)$, which is $O(\epsilon^{-2} \log n)$.

2. The minimum cut in $G$ corresponds (under the same vertex partition) to a $(1 + \epsilon)$-times minimum cut of $H$.

Let us consider the choice $\epsilon = 1/6$. Lemma 6 then allows us to reduce the weight of the minimum cut to $O(\log n)$. We can then apply existing algorithms [32, 8] to pack trees in $\tilde{O}(n)$ time. By Lemma 6, the minimum cut of $G$ is a cut of size $(1 + \epsilon)c = (7/6)c$ of $H$. Then by Lemma 5 a packing of size at least, say, $8c/18$ has the property that the minimum cut of $G$ will be 2-constrained by at least a 3/16ths fraction of the trees (by weight). We may then sample $\Theta(\log n)$ trees of the packing uniformly at random with probability proportional to their weight and find the minimum cut in $G$ that 2-respects each of them. If any sampled tree 2-constrains the minimum cut in $G$, the minimum cut will be found. Since each tree has constant probability of 2-constraining the minimum cut in $G$, the overall algorithm will return the minimum cut of $G$ with high probability.

There are still several issues to resolve. Lemma 6 requires knowing a constant-factor underestimate $c' = \gamma c$ for the minimum cut $c$. In particular, without $\gamma \leq 1$, property 2 of Lemma 6 is not guaranteed with high probability, and if $\gamma = o(1)$, the minimum cut of $H$ will be of size $\omega(\epsilon^{-2} \log n)$ with high probability. We may run a linear-time 3-approximation algorithm [26], with modifications to work on weighted graphs [28], to find this approximation. This is simple to state, but more difficult to implement.

A different approach is to start with a known upper bound $U$ for $c'$, say, the weight of edges attached to any single vertex. We can then halve this upper bound until “our algorithms succeed”, as stated by Karger [20]. This approach is taken by the implementation of Chekuri et al. [5]. Unfortunately, it is not rigorous. Lemma 6 indicates that with a constant-factor underestimate $c' = \gamma c$ for $c$, our algorithm can proceed. However, it does not give a process for rejecting a guess $c'$ that is not a constant-factor underestimate for $c$. We could try all powers of 2 for $c'$ within a known lower and upper bound of the value of the minimum cut, and run our algorithms for all possibilities. This is rigorous, but introduces an extra $O(\log n)$ factor in our runtimes, assuming the range of $c'$ we try is polynomial in $n$. We instead show the following:
Lemma 7. Let \( p = 3(d + 2)(\ln n)/(\epsilon^2 \gamma c) \) as in Lemma 6, but with \( \gamma \geq 6 \) and \( \epsilon \leq 1/3 \). Then if we sample each edge of \( G \) uniformly at random with probability \( p \), the resulting graph \( H \) has minimum cut of size less than \( (d + 2)(\ln n)/\epsilon^2 \) with probability at least \( 1 - 1/n^{d+2} \).

Proof. Consider the size of a minimum cut of \( G \) in \( H \). Let \( X \) be a random variable denoting this size. Then \( \mathbb{E}[X] = cp \). By a Chernoff bound, \( \Pr[X \geq (1 + \delta)cp] \leq e^{-\frac{1}{3}(cp\delta)} \) for \( \delta \geq 1 \).

Let \( \epsilon = \frac{\gamma c}{3} \). Then

\[
\Pr[X \geq (d + 2)(\ln n)/\epsilon^2] \leq e^{-\frac{1}{3}(cp(\frac{\gamma}{3} - 1))} \\
= e^{-(d+2)(\ln n)\gamma^{-1}c^{-1}(\frac{\gamma}{3} - 1)} \\
= n^{-\frac{1}{3}(d+2)c^{-2}+(d+2)\gamma^{-1}c^{-2}} \\
\leq n^{-\frac{1}{3}(d+2)c^{-2}} \\
< n^{-(d+2)}.
\]

Lemma 7 states that if our estimate \( c' = \gamma c \) satisfies \( \gamma \geq 6 \), the minimum cut will be less than a factor 3 from \( 3(d + 2)(\ln n)/\epsilon^2 \) with high probability. Recall that with \( \gamma = 1 \) and therefore \( c' = c \), we expect the minimum cut in \( H \) to be within a factor \((1 + \epsilon)\) from \( 3(d + 2)(\ln n)/\epsilon^2 \) with high probability. Lemma 7 gives us the necessary tool to reject \( c' \) that are not a constant factor underestimate of \( c \). We try a \( c' \), and if the size of the minimum cut in \( H \) is greater than \( (1 + \epsilon)^{-1}3(d + 2)(\ln n)/\epsilon^2 \), we know \( c' < 6c \). Therefore we can decrease \( c' \) by a factor of 6 and rerun the tree packing algorithm. The resulting graph \( H \) must satisfy the conditions of Lemma 6, therefore the algorithm may proceed. Since our tree packing algorithms determine the minimum cut up to constant factors, this procedure avoids the use of needing a different (or recursive!) minimum cut algorithm to run on \( H \).

The final choice is to pick a tree packing algorithm. Karger gives two options. The first is an algorithm by Gabow [8], which computes a \( c/2 \) packing. The second is a more general approach by Plotkin-Shmoys-Tardos [32], which can find a packing a factor \((1 + c')\) from the maximum packing, which has value in \([c/2, c]\). Karger describes the latter approach as simpler, using only minimum spanning tree computations. Unfortunately, the paper [32] does not explicitly give a routine for packing spanning trees and is relatively technical. This may be the reason that implementations have chosen the former option, despite requiring more code [3]. To remedy this, we have provided the specific pseudocode for packing spanning trees with Plotkin-Shmoys-Tardos (PST) in Appendix A.

We outline the required steps to obtain the spanning trees for the following sections in Algorithm 1.

Lemma 8. Algorithm 1 returns a collection of \( \Theta(\log n) \) spanning trees of \( G \) in time \( O(m \log^3 n) \) such that the minimum cut of \( G \) 2-respects at least one tree in the collection with high probability.

Proof. We first prove correctness. Let \( \epsilon = 1/6 \). Suppose for a particular \( c' \), \( c' \geq 6c \). Then by Lemma 7, \( H \) will have minimum cut of size less than \( 36(d + 2)(\ln n) \) with high probability. Since every spanning tree in the packing must contain at least one edge of the minimum cut, the maximum tree packing will have value at most \( 6c \), the weight of the minimum cut in \( H \), and thus the value of the tree packing found by PST will be at most \( 36(d + 2)(\ln n) < (6/7)^2 \cdot 54(d + 2)(\ln n) \). Therefore Algorithm 1 will proceed to the next iteration with \( c' = c'/2 \). Note that the overall probability of failure from any of the \( O(\log n) \) iterations of this step is at most \( O(\log n \cdot n^{-(d+2)}) \leq n^{-d} \) for sufficiently large \( n \).
Algorithm 1 Obtaining $\Theta(\log n)$ Spanning Trees for the 2-respect Algorithm

Let $U$ be an upper bound for the size of a minimum cut of $G$. Let $d$ denote the exponent in the probability of success $1 - 1/n^d$.

1. Let $G'$ be $G$ such that each edge weight is rounded to the nearest 100th and multiplied by 100.

2. Initialize $c' \leftarrow U$. Repeat the following:
   a. Construct $H$ in the following way: for each edge $e$ of $G'$, let $e$ have weight in $H$ drawn from the binomial distribution with $p = 3 \cdot 6^2(d+2)/(\ln n)/c'$ and number of trials the weight of $e$ in $G'$. Cap the weight of any edge in $H$ to at most $\lceil 24c'p \rceil$.
   b. Run PST on $H$ with $\epsilon' \leftarrow 1/6$. If the returned packing is of value $(6/7)^2 \cdot 54(d+2)/(\ln n)$ or greater, set $c' \leftarrow c'/6$, repeat steps 2a and 2b, and return $[7.43d\ln n]$ trees sampled uniformly at random proportional to their weights from the packing. If not, repeat steps 2a and 2b with $c' \leftarrow c'/2$.

Now suppose PST returns a tree packing of value $(6/7)^2 \cdot 54(d+2)/(\ln n)$ or greater. By the above, $c' < 6c$. If $c' \leq c$, Lemma 6 says that the weight of the minimum cut is at least $6/7 \cdot 3 \cdot 6^2(d+2)/(\ln n) = 6/7 \cdot 108(d+2)/(\ln n)$ with high probability. The tree packing is of value at least a factor $(1 + \epsilon') = 7/6$ from half the minimum cut. It follows the tree packing will be of value at least $(6/7)^2 \cdot 54(d+2)/(\ln n)$. The consequence of this is that if a tree packing of value $(6/7)^2 \cdot 54(d+2)/(\ln n)$ or greater is found in step 2b, in addition to the bound $c' < 6c$, we also know $c' > c/2$, since whenever $c' \leq c$, Lemma 6 says the packing will have value at least $(6/7)^2 \cdot 54(d+2)/(\ln n)$, and we decrease $c'$ by a factor of 2 in each iteration. Therefore if we set $c' = c'/6$, then in the next iteration we have $c/12 < c' < c$.

Now consider the next iteration when the tree packing is returned. In sampling $H$, we only preserve weights in $H$ up to $\lceil 24c'p \rceil$. It follows that this preserves all cuts of size at most a factor 2 larger than the minimum cut in $H$. The minimum cut in $G$ in $H$ will have size at most $101/100 \cdot 7/6$ times the minimum cut in $H$. Thus it is preserved, and by Lemma 5 with $\alpha \leq 101/100 \cdot 7/6$ and $\beta \geq 1/2 \cdot 6/7$, a fraction $43/500 = 8.6\%$ of the trees will 2-constrain the minimum cut of $G$. The probability that no tree in a sample of size $t$ 2-constrains the minimum cut is $(1 - 43/500)^t$. Solving for $t$ in $(457/500)^t = n^{-d}$ yields $t = -d\ln n/ \ln(457/500) < [7.43d\ln n]$. Therefore with probability at least $1 - 1/n^d$, at least one tree in the returned sample will 2-constrain the minimum cut.

Time complexity can be proven as follows. We bound the number of edges of $H$ in each iteration with the number of edges of $G$, $m$. Sampling $H$ can be done in about $O(m \log n)$ time, depending how the binomial distribution is sampled (more below). PST runs in $O(mc \log^2 n)$ time, where $c$ is the value of the minimum cut in $H$. In expectation, the value of the minimum cut doubles in each iteration. A similar high probability statement can be made via an argument similar to Lemma 7. Therefore the cost of running PST doubles in each iteration, with the final cost being $O(m \log^3 n)$, since $c = O(\log n)$ by Lemma 6. By a geometric series the entire cost is thus $O(m \log^3 n)$, therefore Algorithm 1 runs in $O(m \log^3 n)$ time with high probability.

The explicit constants chosen in Algorithm 1 were intended to make the presentation cleaner. Of course, a range of constants is possible.

Note that since $G'$ has integer weights and because we cap the weight of any edge in $H$ with value $O(\log n)$, this makes sampling from the binomial distribution easy with the inverse CDF method. This was described as a complication in [5], to which they substituted...
the Poisson distribution, which, as suggested, is not rigorous.

Finally, observe that this presentation is slightly different than that of Karger [21]. In particular, Karger sparsifies edges of $H$ to have $m = O(n \log n)$, and replaces an $O(m \log n)$ time minimum spanning tree computation in PST with an $O(m)$ one. This gives complexity $O(n \log^3 n)$ for finding the $\Theta(\log n)$ spanning trees. However, since the next section also takes $O(m \log n)$ time, we avoid these optimizations to simplify the algorithm.

4 Minimum Cuts that 1-Respect a Tree

We now show our algorithm for finding a minimum cut that 1-respects a spanning tree $T$ of a graph $G$. We first label the edges of $T$ in heavy-light decomposition order $e_1, \ldots, e_{n-1}$. We then iterate an index $i$ throughout the order and keep up-to-date the total weight of all edges of $G$ that cross the current cut. The minimum weight found is returned.

Call the edges of $G$ in $T$ tree edges and edges of $G$ not in $T$ non-tree edges. We have the following.

▶ Proposition 9. A non-tree edge $uv$ is cut if and only if exactly one tree edge is cut on the $uv$-path in $T$.

Proof. Recall that for any edge of $T$ in the cut, the components on each of its endpoints must fall on opposite sides of the cut. Therefore if the number of tree edges in the cut on the $uv$-path in $T$ is odd, the non-tree edge $uv$ is in the cut. Since we are only considering cuts that cut at most 2 edges of $T$, the proposition follows.

We use heavy-light decomposition.

▶ Lemma 10 (Sleator and Tarjan [34]). Given a tree $T$, there is an ordering of the edges of $T$ such that the edges of the path between any two vertices in $T$ consist of the union of up to $2 \log n$ contiguous subsequences of the order. The order can be found in $O(n)$ time.

Proof. We use heavy-light decomposition, credited to Sleator and Tarjan [34]. Note that the algorithm assumes $T$ is rooted. We can root $T$ arbitrarily.

We are now ready to describe the algorithm.

Algorithm 2 Minimum Cuts that 1-Respect $T$

1. Arrange the edges of $T$ in the order of Lemma 10 and label them $e_1, \ldots, e_{n-1}$.
2. For each non-tree edge $uv$, mark every $i$ such that $e_i$ is on the $uv$-path in $T$ and $e_{i+1}$ is not on the $uv$-path in $T$, or vice versa. Indicate whether edge $e_1$ is on the $uv$-path in $T$.
3. Iterate index $i$ from 1 to $n - 1$, in each iteration keeping track of the total weight of all non-tree edges $uv$ such that $e_i$ lies on the $uv$-path in $T$, totalled with the weight of edge $e_i$.
4. Return the minimum total weight found in step 3.

▶ Lemma 11. Algorithm 2 finds the value of the minimum cut that 1-respects a spanning tree $T$ of a graph $G$ in $O(m \log n)$ time.

Proof. Via Proposition 9, a non-tree edge $uv$ is cut if and only if the edge $e_i$ lies on the $uv$-path. Algorithm 2 keeps track of all such non-tree edges for each possible $e_i$ that is cut, therefore it finds the minimum cut of $G$ that cuts 1 edge of $T$. 
The time complexity can be determined as follows. Finding the heavy-light decomposition of step 1 takes $O(n)$ time. In doing so, we can label each edge and each chain so that every edge knows its index in the order as well as the chain to which it belongs. Each chain can store its starting and ending index in the order. With this information, step 2 can be completed by walking up from $u$ and $v$ in $T$ towards the root of $T$. We spend $O(1)$ work per chain from root to vertex, which is bounded by $O(\log n)$ via the heavy-light decomposition. In total this step takes $O(m \log n)$ time.

In step 3, we spend $O(n)$ total work plus $O(1)$ work for each transition of the current edge $e_i$ on or off the $uv$ path for all non-tree edges $uv$. Each non-tree edge transitions on or off $O(\log n)$ times as guaranteed by Lemma 10, therefore the time complexity of this step is $O(m \log n)$. Overall, Algorithm 2 takes $O(m \log n)$ time.

Note that if we wish to find the edges in the minimum cut, we can keep track of the minimum-achieving index $i$ so we know the vertex separation of the minimum cut. With the vertex separation, it is easy to find in $O(m \log n)$ time which non-tree edges cross the cut.

Further note that we need not know the identity of the non-tree edge $uv$ as $e_i$ falls on or off the $uv$-path. Thus the space required for step 2 need only be $O(m)$, since at each transition point we can just keep track of the total weight added or subtracted from the minimum cut.

5 Minimum Cuts that 2-Respect a Tree

We now discuss an extension of Algorithm 2 to find a minimum cut that 2-respects a tree. We still iterate $i$ through heavy-light decomposition order, but in addition to cutting $e_i$, we consider the best $j$ so that the cut resulting from cutting $e_i$ and $e_j$ is minimal. To find the best $j$, optimally, we need a clever data structure.

Lemma 12 (Alstrup et al. [1]). There is a data structure that supports the following operations on a weighted tree $T$ in $O(\log n)$ time:

- $\text{PathAdd}(u, v, x)$ := Add weight $x$ to all edges on the unique $uv$-path in $T$.
- $\text{NonPathAdd}(u, v, x)$ := Add weight $x$ to all edges not on the unique $uv$-path in $T$.
- $\text{QueryMinimum}()$ := Query for the minimum weight edge in $T$.

Proof. Operations $\text{PathAdd}()$ and $\text{QueryMinimum}()$ are just Theorems 3 and 4 of [1]. Operation $\text{NonPathAdd}(u, v, x)$ can be achieved by keeping a counter of global weight added to (subtracted from) $T$ and executing $\text{PathAdd}(u, v, -x)$ to undo this action on the $uv$-path. See also [36].

Note that the weight $x$ can be positive or negative.

If we seek to avoid implementing any sophisticated data structures, we can instead use heavy-light decomposition again and support the above two operations in $O(\log^2 n)$ time. To see how, by Lemma 10 each path of $T$ represents at most $O(\log n)$ contiguous segments of the total order of edges. Range add and a global minimum query can be supported in $O(\log n)$ time via an augmented binary search tree. Thus the total time complexity per operation is $O(\log^2 n)$.

We use the range operations as follows. As we iterate index $i$ through the order of Lemma 10, we keep up to date the cost of the cut resulting from cutting any other edge $e_j$ via the data structure of Lemma 12. Instead of querying each other edge directly, however, we just use a global minimum query. The algorithm is given below. The first two steps are the same as Algorithm 2.
Algorithm 3 Minimum Cuts that $2$-Respect $T$

1. Arrange the edges of $T$ in the order of Lemma 10 label them $e_1, \ldots, e_{n-1}$.
2. For each non-tree edge $uv$, mark every $i$ such that $e_i$ is on the $uv$-path in $T$ and $e_{i+1}$ is not on the $uv$-path in $T$, or vice versa. Indicate whether edge $e_i$ is on the $uv$-path in $T$.
3. Iterate index $i$ from 1 to $n-1$. Via the computation done in step 2, maintain the following invariants in the data structure of Lemma 12 as $i$ is iterated.
   a. When edge $e_i$ is on the $uv$-path in $T$, add the weight of non-tree edge $uv$ to all edges off the $uv$-path in $T$.
   b. When edge $e_i$ is off the $uv$-path in $T$, add the weight of non-tree edge $uv$ to all edges on the $uv$-path in $T$.
   Each time $i$ is incremented, after updating weights in Lemma 12 as per 3a and 3b, add $\infty$ to edge $e_i$, execute $\text{QueryMinimum}()$, then subtract $\infty$ from edge $e_i$. The value of the minimum cut found in each iteration is the result of $\text{QueryMinimum}()$ plus the weight of $e_i$.
4. Return the minimum cut found in step 3.

Lemma 13. Algorithm 3 finds the value of the minimum cut that $2$-respects a spanning tree $T$ of a graph $G$ in $O(m \log^2 n)$ time.

Proof. Via Proposition 9 a non-tree edge $uv$ is cut if and only if exactly one of $e_i$ or $e_j$ lies on the $uv$-path in $T$. Observe that the invariants enforced in step 3 guarantee that in each iteration the weight of the cut resulting from cutting any other edge $e_j$ along with $e_i$ is kept up-to-date in the data structure of Lemma 12. Since the minimum such $j$ is found for every $i$, it follows that the value returned in step 4 is the minimum cut of $G$ that $2$-respects $T$.

The time complexity follows similarly to Algorithm 2. Steps 1 and 2 take $O(m \log n)$ total time. However, step 3 now requires $O(\log n)$ time for non-tree edge $uv$ whenever edge $e_i$ falls on or off the $uv$-path in $T$, since the data structure of Lemma 12 takes $O(\log n)$ time per operation. For a given non-tree edge $uv$, edge $e_j$ falls on or off the $uv$-path in $T$ a total of $O(\log n)$ times by Lemma 10; thus step 3 takes $O(m \log^2 n)$ time. The total time taken is $O(m \log^2 n)$.

We make a few further remarks about Algorithm 3. To determine the edges of the minimum cut, the data structure of Lemma 12 can be augmented to return the index $j$ of the edge that achieves the minimum given in operation $\text{QueryMinimum}()$. With $e_i$ and $e_j$, we can determine the vertex partition in $G$ of the minimum cut and as stated in Section 4 from this we can find which non-tree edges cross the minimum cut easily in $O(m \log n)$ time.

The space complexity of Algorithm 3 was easily linear. In Algorithm 3 we must know the identity of each non-tree edge $uv$ in every transition point where edge $e_i$ falls on or off the $uv$-path. Naively this costs $O(m \log n)$ space. This can be improved to $O(m)$ space by performing steps 2 and 3 simultaneously. That is, we only need to know the next transition point where the non-tree edge $uv$ falls on or off the $uv$-path, and from the current transition point this can be determined in constant time.

From this we get our final theorem, equivalent to the result of Karger [21].

Theorem 14. The minimum cut in a weighted undirected graph can be found in $O(m \log^3 n)$ time with high probability.

Proof. We first find $\Theta(\log n)$ spanning trees by Algorithm 1. We then find the minimum cuts that $1$- and $2$-respect each of these trees by Algorithms 2 and 3. By Lemmas 8, 11 and
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13 this finds the minimum cut with high probability in \(O(m \log^3 n)\) time.

6 Conclusion

In this paper, we have discussed a simplification to Karger’s original near-linear time minimum cut algorithm [21]. In contrast to Karger’s original algorithm [21], finding spanning trees that have a constant probability of 2-respecting the minimum cut is now the complicated part of the algorithm, and finding minimum cuts that 2-respect a tree is now the simpler part. In actuality, both were complicated in Karger’s original algorithm, however the work to find the tree packing was largely abstracted to previous publications. Our version, on the other hand, is self-contained: the only procedure outside of Algorithms 1, 2, 3, and 4 required to implement the full algorithm is a minimum spanning tree subroutine and optionally a top tree. This shows a near-linear time implementation is practical.

The main contribution of our algorithm is a new, simple procedure to find a minimum cut that 2-respects a tree \(T\) in \(O(m \log^2 n)\) time. Karger advertises that the complexity of his near-linear time algorithm is \(O(m \log^3 n)\) and thus his routine to find a minimum cut that 2-respects a tree also takes \(O(m \log^2 n)\) time. However, he gives two small improvements to the algorithm to reduce the overall runtime to \(O(m \log^2 n \log(n^2/m) / \log \log n + n \log^6 n)\). The first uses the fact finding a 1-respecting cut can be done in linear time, and the other is an improvement which reduces an \(O(\log n)\) factor to an \(O(\log(n^2/m))\) factor in the 2-respect routine. For our algorithm, the first improvement can be applied by substituting our 1-respect algorithm with his. The second improvement can not be applied. Thus, when \(m = \Theta(n^2)\), his algorithm is faster by an \(O(\log n)\) factor. However, for this case, Karger gives a different, simpler algorithm [21] which finds the global minimum cut in \(O(n^2 \log n)\) time.

There are three algorithms that are referred to as simple min-cut algorithms; namely, the Stoer-Wagner algorithm [35] which runs in \(O(m \log n)\) time or \(O(mn + n^2 \log n)\) time with a Fibonacci heap [7], Karger’s randomized contraction algorithm [18] which runs in \(O(n^2 \log n)\) time, and the improvement to Karger’s algorithm by Karger and Stein [22], which runs in \(O(n^2 \log^3 n)\) time. In comparison to these, our approach is probably the least simple. However, our \(O(m \log^3 n)\) runtime is significantly better for sparse graphs. Further, it cannot be significantly improved as \(\Omega(m)\) time is necessary. Thus, with a fast implementation and large enough graph size, our algorithm should be not only faster than its competitors, but nearly as fast as possible. We suggest exploring a competitive implementation as important future work.

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A Simple Algorithm for Minimum Cuts in Near-Linear Time

A Plotkin-Shmoys-Tardos Algorithm for Tree Packing

Karger [21] cites an algorithm by Plotkin, Shmoys, and Tardos (PST) [32] to construct weighted tree packings. However, only a specific instance of the fairly general algorithm is needed in Karger’s approach. Further, Karger needs only a \((1 + \epsilon)\)-approximation for a fixed constant \(\epsilon\), which further simplifies the needed procedure.

The idea will be to maintain a convex combination of spanning trees and aim to minimize the maximum utilization of an edge. Our explanation assumes familiarity with the PST paper [32], but the algorithm itself should be understandable without.

We work in the vector space \(\mathbb{R}^m\), entries indexed by the edges of \(G\). We thus assume an ordering on the edges of \(G\) so that for vector \(x \in \mathbb{R}^m\), \(x_i\) corresponds to a value on the \(i\)th edge of \(G\).

We now use PST’s solution to the Packing problem. Let \(P \subseteq [0,1]^m \subseteq \mathbb{R}^m\) denote the convex closure of the encodings of spanning trees in \(G\) such that each vertex of \(P\) (a spanning tree) has every edge weight 1. Let \(A\) be the \(m \times m\) identity matrix and \(b \in \mathbb{R}^m\) be such that \(b_i = \text{the weight of the } i\text{th edge of } G\). PST solves the following minimization problem:

\[
\min(\lambda : Ax \leq \lambda b \text{ and } x \in P).
\]

Rather than computing an exact solution, however, PST terminates with a \(\lambda\) such that \(\lambda \leq (1 + \epsilon)\lambda^*,\) where \(\lambda^*\) denotes the optimal solution. By dividing \(x\) by \(\lambda\), we may find a packing within a factor \((1 + \epsilon)\) of the optimal packing.

As written, PST assumes the original goal is to find an \(x \in P\) such that \(Ax \leq b\), so that we are only interested in \(\lambda \geq 1\). If we knew the weight of the minimum cut \(c\) in \(G\), this could be used by setting the weight of every edge of a vertex of \(P\) to \(c/2\), so that the packing returned would be a \((1 + \epsilon)^{-1}c/2\) packing. However, their algorithm works just as well for the case \(\lambda < 1\), with a slight change in the runtime argument. This allows us to use the algorithm without knowing \(c\), and furthermore gives the advantage that the packing returned is within a factor \((1 + \epsilon)\) from the optimal packing, which has value within \([c/2, c]\).

We now give the simplified algorithm. Since \(\epsilon\) is a constant, we do not need to care about \(\epsilon\)-scaling. For a vector \(x \in \mathbb{R}^m\), \(x'\) denotes the transpose of \(x\).

Algorithm 4 (PST [32]). Algorithm for Finding a \((1 + \epsilon)\)-Optimal Tree Packing

1. Let \(\epsilon' = \epsilon/6\) and let \(x \in \mathbb{R}^m\) be an arbitrary spanning tree of \(G\) with each edge weight 1.
2. Repeat the following:
   a. Let \(\lambda \leftarrow \max_i x_i/b_i; \alpha \leftarrow 4\ln(2m)/\lambda; \sigma \leftarrow \epsilon'/4\alpha\).
   b. While \(\max_i x_i/b_i \geq \lambda\) do:
      i. For each \(i\), set \(y_i \leftarrow \frac{1}{b_i}e^{\alpha x_i/b_i}\).
      ii. Let \(\tilde{x} \leftarrow\) a minimum-cost spanning tree in \(G\) with edges weighted by \(y_i\).
      iii. If \(y'x - y'^{\prime}\tilde{x} \leq \epsilon' (y'x + \lambda y'^{\prime}b)\), then return the packing of weight \(1/\lambda\) where edge \(e_i\) has weight \(x_i/\lambda\); otherwise, set \(x \leftarrow (1 - \sigma)x + \sigma \tilde{x}\).

\[\textbf{Theorem 15} (\text{PST} [32]). \text{Given an undirected weighted graph } G \text{ with minimum cut } c \text{ and minimum edge weight at least 1, and any constant } 0 < \epsilon \leq 1, \text{ Algorithm 4 returns in } O(mc \log^2 n) \text{ time a weighted packing of value at most a } (1 + \epsilon) \text{ factor from the optimal packing.}\]

\[\textbf{Proof.}\] Let us first assume the weight of the minimum cut, \(c\), is known, and we weight each edge of each vertex of \(P\) with \(c\), similarly to as previously noted. As in PST [32], let
\[ \rho = \max_i \max_{x \in P} x_i/b_i = c. \]

Then Theorem 2.5 of PST [32] states that for constant \( \epsilon \), we can find a weighted packing within a factor of \( (1 + \epsilon) \) from optimal in \( O(\rho \log(m)) \) calls to the minimum spanning tree subroutine, thus, in \( O(cm \log^2 n) \) time. Now take the same algorithm but reduce the weight of edges of vertices of \( P \), and accordingly \( \lambda \) and \( \rho \) by a factor of \( c \) in the implementation. The condition in 2b, the costs \( y_i \), return condition 2biii and packing / weighting are identical, because the factors of \( c \) cancel (recall \( \sigma \leftarrow \frac{\epsilon}{4 \alpha \rho} \) in PST [32]). It follows that Algorithm 4 computes a packing within a factor \( (1 + \epsilon) \) of the optimal packing in time \( O(cm \log^2 n) \).

\[ \text{Algorithm 4} \]

A few points can be made about Algorithm 4. The values \( y_i \leftarrow \frac{1}{c} e^{\alpha a_i x/b_i} \) can be quite large. Since the minimum spanning tree is only determined by the relative ordering of edge weights [25], we can reduce via a logarithm so that \( y_i \leftarrow \ln\left(\frac{1}{b_i}\right) + \alpha a_i x/b_i \) and still determine the spanning tree \( \tilde{x} \). However, the stopping condition \( y'x - y'\tilde{x} \leq \epsilon (y'x + \lambda y'b) \) as written requires the original \( y \). This can be computed in low-precision, or alternatively, since the runtime of each iteration of 2b is proportional to twice that of the iteration before it unless the condition of 2biii is satisfied, we can simply stop improving the solution and return if the number of iterations of 2b is more than, say, thrice that of the previous iteration.

We finally note that while this returns an encoding of the convex sum of spanning trees (an element of \( P \)), we can also obtain the spanning trees used themselves by maintaining which spanning trees \( \tilde{x} \) get added to \( x \), and with what weights.