Sewing states of quantum field theory

Nima Lashkari

1Department of Physics and Astronomy, Purdue Quantum Science and Engineering Institute (PQSEI), Purdue University, West Lafayette, IN 47907, USA, School of Natural Sciences, Institute for Advanced Study, Princeton, New Jersey 08540, USA

Consider an $n$-partite system and denote by $\omega^{(i)}$ the local density matrix at site $A_i$. We say a pure $n$-partite state $|\Omega\rangle$ sews $\omega^{(i)}$ together if it reduces to $\omega^{(i)}$ on $A_i$ for all $i$. In finite quantum systems, density matrices can be sewn together only if their eigenvalues satisfy polygon inequalities.

We show that in quantum field theory there are no constraints on sewing local states. The reason is that rotating $\omega^{(i)}$ by a unitary $U_i$ in $A_i$, we come arbitrarily close to any other $\psi^{(i)}$. We construct explicit unitaries that sew local states.

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I. INTRODUCTION

Consider a pure state of an $n$-partite quantum system $|\Omega\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \mathcal{H}_n$ where $\mathcal{H}_i$ is the Hilbert space of site $A_i$. The reduced density matrix on $A_i$ is $\omega^{(i)} = \text{tr}_{j \neq i} |\Omega\rangle \langle \Omega|$. We refer to $\omega^{(i)}$ as the local state on $A_i$ and to $|\Omega\rangle$ as a global state. We keep all local density matrices fixed except for $\omega^{(i)}$ at site $A_i$ that we replace with $\psi^{(i)}$ such that the global state remains pure.

In other words, we sew $\psi^{(i)}$ to the $\omega^{(j)}$'s with $j \neq i$. The collection of density matrices $\omega^{(i)}$ are a mean field approximation to a pure global state. In finite quantum systems, there are constraints on the density matrices that can be sewn at site $A_i$. For instance, in a bipartite system, the density matrices $\omega^{(1)}$ and $\omega^{(2)}$ have the same eigenvalues that are the Schmidt coefficients of the global pure state $|\Omega\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$. The local state $\omega^{(1)}$ can be replaced with $\psi^{(1)}$ if and only if both density matrices have the same eigenvalues. The constraints become more complicated for $n > 2$, and imply that the eigenvalues of $\omega^{(i)}$ should satisfy polygon inequalities [1, 2]. The knowledge of $\omega^{(j)}$ for $j \neq i$ and the fact that the global state is pure implies some information about $\omega^{(i)}$, however this information is partial because it does not fix $\omega^{(i)}$. A local unitary rotation $U_i$ at site $A_i$ does not affect the state on the other sites, therefore $\omega^{(i)}$ can always be replaced with $U_i \omega^{(i)} U_i^\dagger$.

Consider a pure state of a Poincare invariant quantum field theory (QFT) in Minkowski spacetime of arbitrary dimensions on a constant time-slice. Split the time-slice into $n$ non-overlapping regions $A_1$ to $A_n$ such that $\cup_i A_i$ is the whole time-slice. Each region $A_i$ has a local algebra of observables $\mathcal{A}_i$. The analog of local density matrices are the restrictions of the the global state to the local algebras: $\omega_i(a_i) = \langle \Omega | a_i \Omega \rangle$ for all $a_i \in \mathcal{A}_i$. In this work, we argue that since the global algebra of QFT is not the tensor product $\mathcal{A}_1 \otimes \cdots \mathcal{A}_n$ we cannot sew all states perfectly. However, as opposed to finite quantum systems, if we allow for small errors it is possible to sew any collection of $\omega_i$'s in a pure state with arbitrary precision. The reason is that, in QFT, for any local state $\psi_i$ there exists a unitary $U_i$ in $\mathcal{A}_i$ such that $\psi_i(a_i) \approx \omega_i(U_i^\dagger a_i U_i)$ for all $a_i \in \mathcal{A}_i$. We use $\approx$ as opposed to equality to indicate that the unitarily rotated states approximate $\omega_i$ arbitrarily well in norm topology of operators. For instance, this procedure allows us to start with an arbitrary global state $|\Psi\rangle$ and act on $A$ with a unitary to prepare a state that is the same as vacuum on region $A$ and the same as $|\Psi\rangle$ outside of $A$. We use Modular theory, an alg-

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1 If the local Hilbert space $\mathcal{H}_i$ is infinite dimensional it suffices for $U_i$ to be an isometry, i.e. $U_i^\dagger U_i = 1$, and $U_i^\dagger a_i U_i$ a projection in $\mathcal{A}_i$. In finite dimensions, there are no distinctions between unitaries and isometries. Quantum field theory is a special infinite dimensional system in which every local isometry has a unitary arbitrarily close to it in norm topology. We prove this in appendix C.
braic framework to study general quantum systems, to construct the local unitaries that sew states.

II. SEWING QUDITS

It is worthwhile to analyze the problem in finite quantum systems first to identify the origin of the obstruction to sew arbitrary states. We focus on the bipartite case, and come back to the general n-partite case in section VII. Every density matrix of system $A$ written in its eigenbasis $\omega = \sum_k p_k |k\rangle \langle k|$ has a canonical purification in a pure global state $|\Omega\rangle = \sum_k \sqrt{p_k} |k,k\rangle \in \mathcal{H}_A \otimes \mathcal{H}_\bar{A}$, where $\mathcal{H}_\bar{A}$ is isomorphic to $\mathcal{H}_A$ and the reduced state on $\bar{A}$ is also $\omega$. The probability $p_k$ corresponds to the projections $\epsilon_k = |k\rangle \langle k|$ and if all $p_k > 0$ we say the state has the Reeh-Schlieder property. Reeh-Schlieder states have invertible density matrices and form a dense set, and we restrict ourselves to this set. The canonical purification of a density matrix $\omega$ sews it to a second identical copy on $A$. This symmetry of the canonical purification $|\Omega\rangle$ is captured by its invariance under the action of the anti-linear operator $J_\Omega$ that swaps the eigenbasis of $\omega_A$ and $\omega_{\bar{A}}$: $J_\Omega (c |k,k\rangle) = c^* |k',k\rangle$. This operator is called the modular conjugation and depends on $\omega$ of the anti-linear operator $J$. If both density matrices are the same $|\Omega\rangle \otimes |\Omega\rangle$ and if all $p_k > 0$ we say the state matches the Hilbert space distance between their canonical purifications:

$$d_F(\omega, \psi)^2 = \frac{1}{2} \| \sqrt{\omega} - \sqrt{\psi} \|^2 = 1 - \text{tr} \left( \sqrt{\omega} \sqrt{\psi} \right)$$

where $\|X\|^2_F = \text{tr} (X^\dagger X)$ is the Frobenius norm of a matrix. An important property of this distance measure is that it matches the Hilbert space distance between their canonical purifications:

$$d_F(\omega, \psi) = \| |\Psi_{\Omega}\rangle - |\Omega\rangle \|.$$ (4)

If we think of $|\chi\rangle = |\Psi_{\Omega}\rangle - |\Omega\rangle$ as an unnormalized vector in the Hilbert space we have

$$d_F(\omega, \psi) = \| |\chi\rangle \| = \sup_{|\Phi\rangle = 1} \| \langle \Phi | \chi \rangle \|.$$ (5)

The vector $|\Phi\rangle$ that achieves the supremum is parallel to $|\chi\rangle$ and saturates the Cauchy-Schwarz inequality. In finite dimensions, the infimum distance

$$\inf_U d_F(U |\omega U^\dagger, \psi)$$

can be calculated explicitly. The unitary $U_0 = \sum_k |\alpha_k\rangle \langle k|$ that rotates the eigenbasis of $\omega$ that of $\psi$ sets an upper bound on this infimum distance:

$$\inf_U d(U |\omega U^\dagger, \psi) \leq d(p, q)$$ (6)

where $d(p, q)$ is the classical analogs of our distance measure for probability distributions $\{p_k\}$ and $\{q_k\}$, i.e. $d(p, q)^2 = \frac{1}{2} \sum_k (\sqrt{p_k} - \sqrt{q_k})^2 = 1 - \sum_k \sqrt{p_k q_k}$. In general, this is not a tight bound because the unitaries that relabel the basis of $\omega$ can further reduce the distance. Each permutation in the symmetric group $\sigma \in S_d$, where $d$ is the dimension of $\mathcal{H}_A$, gives a relabelling unitary $U_\sigma = \sum_k |\alpha_{\sigma(k)}\rangle \langle \alpha_k|$. The action of these unitaries is equivalent to keeping the eigenvectors fixed and permuting the eigenvalues. We tighten our bound to

$$\inf_U d(\omega, \psi) \leq \inf_{\sigma \in S_d} d(p, \sigma(q)) \leq d(p, \sigma(q)).$$ (7)

The classical distance $d(p, \sigma(q))$ is minimized if the fidelity $\sum_k \sqrt{p_k q_k}$ is maximized. If we order $p_k$ in decreasing order $p_1 \geq p_2 \geq \cdots \geq p_d$ this maximum is achieved by the relabelling unitary that orders $q$ in decreasing order: $q_1 \geq q_2 \geq \cdots \geq q_d$, and then matches them by $\sigma(k) = k$:

$$\inf_{\sigma \in S_d} d_F^2(p, \sigma(q)) = 1 - \sum_k \sqrt{p_k q_k},$$ (8)

In fact, for finite $d$ the inequality in (7) is an equality [3], and the minimum of the classical distance we found is the same as the minimum of the quantum distance.

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2 The generalization of our discussion to non-Reeh-Schlieder states is straightforward.
The first indication that the bound above does not generalize to infinite dimensional Hilbert spaces comes from a Hilbert hotel type argument. There are one-to-one maps between any two countably infinite sets \( I_1 \) and \( I_2 \) of eigenvalues \( p_k \), even if \( I_1 \subset I_2 \). Therefore, there are labellings that map any countably infinite subset of eigenvalues to any larger countable subset. Furthermore, in an infinite dimensional Hilbert space, if we have a sequence of vectors \( |\chi_i\rangle \) the supremum in (5) and the limit of \( t \) might not commute. For example, consider \( |\chi_i\rangle = \frac{1}{\sqrt{2\pi}} e^{itx} \) with \( t \in \mathbb{N} \) in the Hilbert space of square-integrable functions on a circle. Taking the limit first gives

\[
\lim_{t \to \infty} \langle \Phi |\chi_i\rangle = \lim_{t \to \infty} \int_0^{2\pi} \frac{dx}{\sqrt{2\pi}} \phi(x)e^{itx} = \lim_{t \to \infty} \phi_t = 0 \tag{9}
\]

where \( \phi_t \) is the Fourier transform of \( \phi(x) \) and we have used the Riemann-Lebesgue lemma that says a square-integrable function cannot have an infinite Fourier mode. We say the sequence of vectors \( |\chi_i\rangle \) converge to zero in the weak topology. However, if we take the supremum first we obtain the norm of the vectors which is unity for all \( t \):

\[
\lim_{t \to \infty} \sup_{||\Phi||=1} \langle \Phi |\chi_i\rangle = \lim_{t \to \infty} |||\chi_i||| = 1 . \tag{10}
\]

This motivates introducing two different distance measures in infinite dimensions. Let us call the set of states \( U\omega U^\dagger \) the unitary orbit of \( \omega \), and denote by \( |\omega\rangle \) the equivalence class of \( \omega \) under unitary rotations. The global states \( |U\Omega\rangle = (U \otimes U_J) |\Omega\rangle \) and \( |\Psi\Omega\rangle \) are the canonical purifications of \( U\omega U^\dagger \) and \( \psi \), respectively. We define the strong unitary orbit distance

\[
d_s(\omega [\omega], |\psi\rangle) = \inf_{U \in A} \| |\Psi\Omega\rangle - |U\Omega\rangle \| = \inf_{U \in A} \sup_{|\Phi\rangle \in \mathcal{H}} \| \langle \Phi |\Psi\Omega\rangle - \langle \Phi |U\Omega\rangle \| . \tag{11}
\]

Since the supremum and the infimum do not commute, we also define a weak unitary orbit distance

\[
d_w(\omega [\omega], |\psi\rangle) = \sup_{|\Phi\rangle \in \mathcal{H}} \inf_{U \in A} \| \langle \Phi |\Psi\Omega\rangle - \langle \Phi |U\Omega\rangle \| . \tag{12}
\]

If the strong distance is small the weak distance is also small, whereas the opposite is incorrect. Both \( |U\Omega\rangle \) and \( |\Psi\Omega\rangle \) are invariant under the action of \( J \), therefore we can use (1) to rewrite the distance in terms of the relative modular operator. It is convenient to define the unnormalized vector \( |\chi_U\rangle = |\Psi\Omega\rangle - |U\Omega\rangle \) so that the weak and strong distances become

\[
d_s(\omega [\omega], |\psi\rangle) = \inf_{U \in A} \| |\chi_U\rangle \| = \inf_{U \in A} \| \left( \Delta_{\Psi\Omega}^{1/2} - 1 \right) U |\Omega\rangle \| = \inf_{|\Phi\rangle \in \mathcal{H}} \inf_{U \in A} \| \langle \Phi | \left( \Delta_{\Psi\Omega}^{1/2} - 1 \right) U |\Omega\rangle \| . \tag{13}
\]

Both unitary orbit distances are zero if and only if the state \( |U\Omega\rangle \) is invariant under the action of the relative modular operator. This ties the problem of sewing local states \( \psi \) of \( A \) to \( \omega \) of its complementary region \( A \) to the problem of locally preparing an invariant state of their relative modular operator.

III. INVARIANT STATES OF RELATIVE MODULAR OPERATOR

To find invariant states of the relative modular operator we look at its spectrum

\[
\Delta_{\Psi\Omega} = \sum_{\lambda} e^\lambda P_\lambda = \sum_{k,k'} \frac{q_{kk'}}{p_k} |\alpha_{k',k}\rangle \langle \alpha_{k,k'}| . \tag{14}
\]

where \( e^\lambda \) are the eigenvalues \( q_{kk'}/p_k \) for any pair \((k,k')\) and \( P_\lambda \) is the projection to the subspace of eigenvalue \( e^\lambda \). There can be degeneracies for each \( \lambda \):

\[
P_\lambda = \sum_{\gamma=1}^{d_\lambda} |\lambda;\gamma\rangle \langle \lambda;\gamma| . \tag{15}
\]

Invariant states of the relative modular operator are in the subspace \( P_0 \mathcal{H} \). If there are no \( q_{kk'} = p_k \) the projection \( P_0 = 0 \), and there are no invariant states. If \( S_0 \) is the subset of pairs \((k,k')\) with \( p_k = q_{kk'} \) then

\[
P_0 = \sum_{(k,k') \in S_0} |\alpha_{k',k}\rangle \langle \alpha_{k',k}| . \tag{16}
\]

The local operator \( v = \sum_{(k,k') \in S_0} |\alpha_{k',k}\rangle \langle k| \) is a partial isometry\(^4\) of system \( A \) that acts on \( |\Omega\rangle \) and prepares invariant states of the relative modular operator

\[
\Delta_{\Psi\Omega} (v \otimes 1) |\Omega\rangle = (v \otimes 1) |\Omega\rangle . \tag{17}
\]

The operator \( v \) rotates eigenvectors of \( \omega \) to those of \( \psi \) and further relabels them such that \( q_{kk'} = p_k \). To simplify the discussion, let us order \( p_k \) in descending order. The operator \( v \) is unitary if and only if for every eigenvalue \( p_k \) there exists a \( k' \) such that \( q_{kk'} = p_k \), in which case, \( v \) is

\[^4\text{A partial isometry } v \text{ is an operator that satisfies } v^\dagger v = p_1 \text{ and } vv^\dagger = p_2 \text{ where } p_1 \text{ and } p_2 \text{ are projections. An operator } \pi \text{ is a projection if } \pi = \pi^\dagger \text{ and } \pi^2 = \pi.\]

\(^3\text{In infinite dimensions, there are states for which the set of eigenvalues } p_k \text{ becomes uncountably infinite. Such states play an important role in section III.}\)
the unitary that sorts \( q_k \) in descending order. We recover the statement that one can sew two density matrices \( \omega \) and \( \psi \) if and only if they have the same eigenvalues. As we saw in the last section, when the eigenvalues do not match, the unitary \( v \) that sorts \( q_k \) minimizes the unitary orbit distance.

In infinite systems, the eigenvalues \( p_{k'}/p_k \) of the modular operator can become unbounded for small enough \( p_k \). Let us assume that our states of interest, \( |\Psi\rangle \) and \( |\Omega\rangle \), both have the property that the spectra of their modular operators are the entire continuum \([0, \infty)\) and the number of degeneracies at each \( \lambda \) goes to infinity:

\[
\Delta_{\Omega} = \sum_{kk'} p_k p_{k'}^{-1} \langle kk' | k k' \rangle = \int d\lambda_1 e^{\lambda_1} \sum_\gamma |\lambda_1; \gamma\rangle \langle \lambda_1, \gamma| \\
\Delta_{\Psi} = \sum_{kk'} q_k q_{k'}^{-1} |\alpha_k \alpha_{k'}\rangle \langle \alpha_k \alpha_{k'}| \\
= \int d\lambda_2 e^{\lambda_2} \sum_\gamma |\lambda_2; \gamma\rangle \langle \lambda_2, \gamma|, \tag{18}
\]

keeping in mind that the vectors \( |\lambda; \gamma\rangle \) are generalized eigenvectors that cannot be normalized: \( \langle \lambda; \gamma| \lambda'; \gamma'\rangle = \delta_{\lambda,\lambda'} \delta(\gamma - \gamma') \). In analogy with (17), we act on \( |\Omega\rangle \) with the partial isometry \( u_{kk'} = |\alpha_k\rangle \langle k| \) and prepare an eigenstate of the relative modular operator

\[
\Delta_{\Psi|\Omega}(v_{kk'} |\Omega\rangle) = q_k p_{k'}^{-1} (v_{kk'} |\Omega\rangle). \tag{19}
\]

The partial isometry \( f_{k'k} = |\alpha_{k'}\rangle \langle \alpha_k| \) that relabels \( q_k \) to \( q_{k'} \) acts on \( v_{kk'} |\Omega\rangle \) and creates a new eigenstate

\[
\Delta_{\Psi|\Omega} (f_{k'k}|v_{kk'} |\Omega\rangle) = q_k p_{k'}^{-1} (f_{k'k}|v_{kk'} |\Omega\rangle). \tag{20}
\]

This partial isometry sends an eigenspaces of relative modular operator to another one simply by changing the label:

\[
f_{k'k} P_\lambda \mathcal{H} \to P_{\lambda_{k'}} q_{k'} \mathcal{H}. \tag{21}
\]

Since we assumed that the spectrum of \( \Delta_{\Psi} \) is continuous and entire in \([0, \infty)\) we can tune the partial isometry \( f \) such that it brings any eigenspace \( P_k \) to an eigenstate \( P_\epsilon \) with \( \epsilon \) near zero. Take two partial isometries \( f_1 \) and \( f_2 \) that, respectively, map the eigenspace \( P_{\lambda_1} \) to \( P_{\epsilon_1} \) and \( P_{\lambda_2} \) to \( P_{\epsilon_2} \) such that \( \lambda_1 \neq \lambda_2 \) and \( \epsilon_1 \neq \epsilon_2 \) then the operator \( v_1 + v_2 \) is also partial isometry that maps \( P_{\lambda_1} + P_{\lambda_2} \) to \( P_{\epsilon_1} + P_{\epsilon_2} \) with \( \epsilon \) in a small neighborhood of zero. Adding such partial isometries and using a bijection from the eigenspace with \( \lambda \in (-\Lambda, \Lambda) \) to the one with \( \lambda \in (-\epsilon, \epsilon) \) we construct a partial isometry \( v_{\lambda} \) that compresses the spectrum of the relative modular operator from \((e^{-\Lambda}, e^\Lambda)\) to a narrow interval near one \((e^{-\epsilon}, e^\epsilon)\). Denote by \( \Pi_{\lambda} = \int_{\epsilon}^\Lambda d\lambda P_\lambda \) the projection in the spectrum to \((e^{-\epsilon}, e^{\epsilon})\). The state \( P_\lambda |\Omega\rangle \) is invariant under the action of the relative modular operator and the state \( \Pi_{\epsilon} |\Omega\rangle \) for small \( \epsilon \) is almost invariant in the sense that

\[
\left\| (\Delta_{\Psi|\Omega}^{1/2} - 1) \Pi_{\epsilon} |\Omega\rangle \right\| \leq (e^{\epsilon/2} - 1). \tag{22}
\]

At large \( \Lambda \), we can approximate \( |\Omega\rangle \approx \Pi_{\Lambda} |\Omega\rangle \), therefore the state \( v_{\Lambda} |\Omega\rangle \) is also almost invariant under the action of relative modular operator:

\[
|| (\Delta_{\Psi|\Omega}^{1/2} - 1) v_{\Lambda} |\Omega\rangle || \leq || (\Delta_{\Psi|\Omega}^{1/2} - 1) \Pi_{\Lambda} |\Omega\rangle || \\
+ || (1 - \Pi_{\Lambda})(\Delta_{\Psi|\Omega}^{1/2} - 1) |\Omega\rangle || \\
\leq \left( e^{\epsilon/2} - 1 \right) + 2 ||1 - \Pi_{\Lambda}|| \tag{23}
\]

Intuitively, one expects that in the limit \( \Lambda \to \infty \) the sequence of \( v_{\Lambda} \) tends to a unitary operator as the right-hand-side of the inequality above becomes vanishingly small. We postpone a discussion of this limit to section IV and conclude that in infinite dimensional systems, the states \( |\Omega\rangle \) and \( |\Psi\rangle \) for which the modular operator has an entire continuous spectrum there exists a local unitary such that \( || |\Psi\rangle - U \otimes |\Omega\rangle || \leq \epsilon \) for \( \epsilon \) arbitrarily small.\footnote{Such states can be constructed in infinite dimensional systems with density matrices (so-called type I systems). However, not all states have the property above. We thank Roberto Longo for pointing this out to us.}

Since any purification of local state \( \psi \) is related to the canonical one by a unitary in \( \mathcal{A} \) we have

\[
\inf_{U \in \mathcal{A}, U' \in \mathcal{A}} \| UU' |\Omega\rangle - |\Psi\rangle \| = 0 \tag{24}
\]

for \( |\Omega\rangle \) and \( |\Psi\rangle \) any two vectors in the Hilbert space.

The local algebra of QFT is the limit of a finite quantum system where for every state the individual probabilities \( p_k \) go to zero and only the ratio \( p_{k'}/p_k \) makes sense.\footnote{See [4] for the construction of the local algebra of quantum fields as the limit of finite quantum systems.}

Every eigenvalue \( p_{k'}/p_k \) repeats infinite times, therefore the sum over \( \gamma \) in (18) runs to infinity. In the continuum limit, it is more precise to replace the expression in these parenthesis in (14) with a projection operator valued measure \( dP_\lambda \) and write

\[
\Delta_{\Psi|\Omega} = \int e^{\lambda} dP_\lambda. \tag{25}
\]

In [5] Connes and Stormer used a generalization of the argument above to prove that for every pair of states \( |\Omega_{\Omega'}\rangle \) and \( |\Psi\rangle \) there exists a local unitary \( U \) that satisfies \( ||UU' |\Omega\rangle - |\Psi\rangle || \leq \epsilon \) for \( \epsilon \) arbitrarily small. In other words, they showed that the strong distance of unitary orbits of any two local states is zero, i.e. \( d_s(|\omega\rangle, |\psi\rangle) \equiv 0 \). We review the steps needed for the generalization in section IV. Unfortunately the argument in [5] just proves the existence of a unitary \( U \) that sews states. In the remainder of this section, we focus on the weak distance \( d_w(|\omega\rangle, |\psi\rangle) \) so that we can explicitly construct examples of unitaries that sew states of QFT.

Once again, we start with the example of density matrices to build intuition, and postpone the case of QFT to section ??.

Consider the sequence of operators

\[
\Delta_{\Psi|\Omega}^{it} = \psi^{it} \omega^{-it} \otimes 1_{\mathcal{A}} \equiv u_{\Psi|\Omega}(t) \otimes 1_{\mathcal{A}}, \tag{26}
\]
where $u_{\Psi t}(t)$ is called the Connes cocycle of $|\Psi\rangle$ and $|\Omega\rangle$. The cocycle is unitary if both density matrices $\omega$ and $\psi$ are full rank. The idea is to use the cocycle at large $t$ as a candidate unitary that sews states. To simplify notation define $u(t) \equiv u_{\Psi t}(t)$. One can check explicitly that $J\Delta_{\Psi t}|\Omega\rangle = \Delta_{\Psi t}^{-1}|\Omega\rangle$ and from the anti-linearity of $J$ follows that $J\Delta_{\Psi t}^{it}|\Omega\rangle = \Delta_{\Psi t}^{-it}|\Omega\rangle$ [6]. Therefore,

$$u(t) \otimes u_J(t) = \Delta_{\Psi t}^{it}|\Omega\rangle J\Delta_{\Psi t}^{it}J \Delta_{\Psi t}^{it}|\Omega\rangle \Delta_{\Psi t}^{it}J$$

$$= \Delta_{\Psi t}^{it}|\Omega\rangle \Delta_{\Psi t}^{it} \Delta_{\Psi t}^{it} \Delta_{\Psi t}^{it} = \Delta_{\Psi t}^{it} \Delta_{\Psi t}^{it}$$

where we have used the identity $\Delta_{\Psi t}^{it} \Delta_{\Psi t}^{-it} = \Delta_{\Psi t}^{it} \Delta_{\Psi t}^{-it}$ which follows from the definition in (26). The canonical purification of the state that the cocycle creates by acting on $|\Omega\rangle$ is

$$\vert u(t)\rangle = (u(t) \otimes u_J(t)) |\Omega\rangle = \Delta_{\Psi t}^{it} \Delta_{\Psi t}^{it} |\Omega\rangle = \Delta_{\Psi t}^{it} |\Omega\rangle (28)$$

where we have used $\Delta_{\Psi t} |\Omega\rangle = |\Omega\rangle$. In the distance measure in (13) we use the unnormalized vectors

$$|\chi_t\rangle = \left(\Delta_{\Psi t}^{1/2} - 1\right) |\psi_t\rangle = \left(\Delta_{\Psi t}^{1/2} - 1\right) \Delta_{\Psi t}^{it} |\Omega\rangle (29)$$

Denote by $f(t)$ the overlaps of $|\chi_t\rangle$ with an arbitrary state $|\Phi\rangle$:

$$f(t) = \langle \Phi | \chi_t \rangle = \sum_{kk'} \langle \Phi | (\Delta_{\Psi t}^{1/2} - 1) |\alpha_k\alpha_{k'}\rangle \langle \alpha_k\alpha_{k'} | \Delta_{\Psi t}^{it} |\Omega\rangle$$

$$= \sum_{kk'} \langle \Phi | (\Delta_{\Psi t}^{1/2} - 1) |\alpha_k\alpha_{k'}\rangle \langle \alpha_k\alpha_{k'} | q_{k}\langle\Omega| q_{k'}^{-it}$$

$$= \sum_{\lambda} \tilde{f}(\lambda) e^{it\lambda},$$

where we have inserted a resolution of identity in the eigenbasis of $\Delta_{\Psi t}$ and

$$\tilde{f}(\lambda) = \langle \Phi | (\Delta_{\Psi t}^{1/2} - 1) |\alpha_k\alpha_{k'}\rangle \langle \alpha_k\alpha_{k'} | \Omega\rangle (31)$$

is the Fourier transform of $f(t)$ with respect to $\lambda$. In a finite system, $f(t)$ oscillates erratically at large $t$. In an infinite systems, if the spectrum of the modular operator is continuous, we replace the sum over $\lambda$ in (30) with an integral. Then, $\tilde{f}(\lambda)$ is an integrable function of $\lambda$ because $\int d\lambda \tilde{f}(\lambda) = f(0)$ and

$$|f(0)| \leq \left| \langle \Phi | (\Delta_{\Psi t}^{1/2} - 1) |\Omega\rangle \right|$$

$$\leq \| \Delta_{\Psi t}^{1/2} - 1 \| \| |\Omega\rangle \| \leq 2,$$

where we have used the Cauchy-Schwarz inequality. The Riemann-Lebeque lemma says that $\tilde{f}(\lambda)$, as an integrable function, cannot have an infinite Fourier mode:

$$\lim_{t \to \infty} f(t) = \lim_{t \to \infty} (\Phi | \chi_t \rangle = 0 (32)$$

which implies that for all $|\Phi\rangle \in \mathcal{H}$

$$\lim_{t \to \infty} (\Phi | u(t)\rangle = |\Phi\rangle |\Omega\rangle \right).$$

We find that if the spectrum of the modular operator of $|\Psi\rangle$ is entirely continuous acting with the cocycle on $|\Omega\rangle$ for large $t$ prepares the local state of $|\Psi\rangle$ with arbitrary precision in the weak distance. Given any local unitary $V$ in $\mathcal{A}$ the operator $V_t(t) = e^{itV}$ is also a unitary, called the modular flow of $V$ with state $\omega$. At large $t$, the local unitary $u_{\Psi t}(t)$, $V_t(t)$ acting on $|\Omega\rangle$ also prepares the local state of $\psi$ in the weak distance. The argument is the same as above with $\langle \alpha_k\alpha_{k'} | V_t(t) = \langle \alpha_k\alpha_{k'} | VV_t(t) |\Omega\rangle.$

IV. CONNES-STORMER RESULT

In QFT, there are no local Hilbert spaces analogous to $\mathcal{H}_A$. Splitting a time-slice into two complementary regions $A$ and $\bar{A}$ there are local algebras on each region that we denote by $\mathcal{A}$ and $\bar{A}$. They are isomorphic and together they generate the global algebra. In the problem of sewing local states we are only interested in pure global states that we refer to as vectors in the Hilbert space. The local state on $\mathcal{A}$ is the restriction of the expectation values in the global vector $|\Omega\rangle$ to the local algebra $\omega(a) = \langle \Omega| a\Omega \rangle$ with $a \in \mathcal{A}$. We can sew local states $\omega$ on $A$ to $\psi$ on $\bar{A}$ if there exists a global vector $|\Phi\rangle$ such that

$$\forall a \in \mathcal{A} : \omega(a) = \langle \Phi | a\Phi \rangle$$

$$\forall a' \in \mathcal{A} : \psi(a') = \langle \Phi | a'\Phi \rangle .$$

We say an algebra $\mathcal{A} = A_1 \otimes A_2$ if for any local states $\omega$ on $A_1$ and $\psi$ on $A_2$ there exists a state $\phi$ on $A$ such that

$$\forall a \in A_1 : \omega(a) = \phi(a)$$

$$\forall b \in A_2 : \psi(b) = \phi(b) .$$

In QFT, as opposed to finite quantum systems, if $\bar{A}$ is the complement of $A$ the algebra does not split into a tensor product $\mathcal{A} \otimes \bar{A}$. The absence of tensor product between the algebra of region $A$ and that of $\bar{A}$ implies that it is impossible to sew all states perfectly. However, as we argue below it is possible to sew any two states with arbitrary precision.

Consider the set of vectors in the Hilbert space that reduce to $\omega$ on $\mathcal{A}$. Analogous to the case of qubits, there exists a unique canonical global vector $|\Omega\rangle$ that is invariant under the anti-linear $J_\Omega$ that swaps $\mathcal{A}$ and $\bar{A}$ [7]. Given a second local state $\psi$ on $\mathcal{A}$ there exists a unique global vector $|\Psi_{\Omega}\rangle$ that reduces to $\psi$ and is invariant under $J_\Omega$. We postpone a definition of $J_\Omega$ to the next section. In (4), we expressed the Frobenius distance of local states on $\mathcal{A}$ in terms of the Hilbert space distance of their corresponding global vectors. We take the Hilbert space distance as the definition of distance between local states

$$d_F(\omega, \psi) = \| |\Psi_{\Omega}\rangle - |\Omega\rangle \| \right.)$$

Rotating the local state by a local unitary corresponds to considering a new local state with $\omega_U(a) = \omega(UaU^\dagger)$. 

1
We are interested in the minimum distance in the unitary orbit distance of \( \omega \):

\[
\inf_{U \in \mathcal{A}} d(\omega_U, \psi) = \inf_{U \in \mathcal{A}} \| \Psi_{\Omega} - |U_{\Omega}\rangle \| .
\]

(37)

where \(|U_{\Omega}\rangle = U |U_{j}\rangle\) is the canonical purification of \(\omega_U\). In the last section, we argued that if the modular operators of both \(|\Psi_{\Omega}\rangle\) and \(|\Omega\rangle\) have an entirely continuous spectrum the infimum above is zero. We now generalize that argument to show that in QFT the infimum above is zero for any pair of states. The key observation is that QFT is a special type of infinite quantum system for which the spectrum of the modular operator of any state is the entire \([0, \infty)\). If \(U\) is the local unitary that makes the infimum in (37) vanishingly small, the state \(U |\Omega\rangle\) reduced to \(\mathcal{A}\) is arbitrarily close to \(\psi\), whereas outside of \(\mathcal{A}\) the state is the same as \(\omega\). The seeming contradiction with finite quantum systems has to do with the fact the there is no notion in QFT analogous to the eigenvalues of local states. There is only the modular operator whose spectrum is entirely continuous with no eigenvalues. That is the reason underlying the divergences that appear in a naive calculation of unitary invariant measures of local states such as entanglement or Renyi entropies.

An important step in the argument of the last section was the construction of the partial isometry \(v_{\Lambda}\). There is no local Hilbert space \(\mathcal{H}_A\) in QFT, therefore one needs to approximate the partial isometry \(f_{kk'} = |\alpha_k\rangle \langle \alpha_{k'}|\) that relates the projections \(f_{kk}\) and \(f_{kk'}\). The projection \(f_{kk}\) is special in that it commutes with the density matrix \(\psi\). In QFT, the projections in the local algebra that commute with the modular operator play the role of \(f_{kk}\), i.e. \(\Delta_{\psi}, f_{kk} = 0\). The other special feature of \(f_{kk}\) is that it is the smallest projection in the sense that there are no other non-zero projection \(f'\) such that \(f' \mathcal{H} \subset f_{kk} \mathcal{H}\). This aspect cannot be generalized to QFT. To run the argument in QFT we start with two projections \(f_k\) and \(f_{k'}\) that commute with the modular operator \(\Delta_{\psi}\) and take the partial isometry that relates them as a replacement of \(f_{kk}\). We repeat the argument of the last section by considering two partial isometries of this type \(v_1\) and \(v_2\) and their corresponding projections, respectively, \(f_1 = v_1^\dagger v_1\) and \(f'_1 = v_{1'}^\dagger v_{1'}\) for \(i = 1, 2\). If \(f_1 \perp f_2\) and \(f'_1 \perp f'_2\) the sum \(v_1 + v_2\) is also partial isometry with extended domain \((f_1 + f_2)\mathcal{H}\) and range \((f'_1 + f'_2)\mathcal{H}\). We continue adding such partial isometries until either the domain or the range of the partial isometry becomes the whole Hilbert space, at which point the resulting operator is either an isometry \(v^\dagger v = 1\) or a co-isometry \(vv^\dagger = 1\). The isometry \(\psi\) has the property that \(\| \Psi_{\Omega} - v |\Omega\rangle \| \approx 0\). The final step is to notice that in QFT any isometry is the limit of sequence of unitaries in the strong operator topology; see appendix C. This finishes the proof that in QFT any state \(\omega\) on \(\mathcal{A}\) can be sewn to any \(\psi\) on \(\mathcal{A}\).

V. MODULAR THEORY AND THE COCYCLE

In this section, our goal is to go beyond systems with density matrices and define the appropriate generalizations of the relative modular operator, modular conjugation and the cocycle in a general quantum system. The mathematical framework that accomplishes this is called the Tomita-Takesaki modular theory. See [4, 6] for a review of the modular theory. The remainder of this section applies to any general quantum system, from qubits to QFT.

Consider a local quantum system \(\mathcal{A}\) and the auxiliary quantum system \(\bar{A}\) that purifies its state. Their corresponding algebras \(\mathcal{A}\) and \(\bar{A}\) together generate a global algebra with a representation on a separable Hilbert space \(\mathcal{H}\). The pure global states are vectors \(|\Omega\rangle \in \mathcal{H}\). There are many vectors in the Hilbert space that reduce to \(\omega\) on \(\mathcal{A}\). Acting on a vector with any isometry \(W' \in \bar{A}\), i.e. \(W'W' = 1_\mathcal{A}\) and \(W'W' = \pi \in \mathcal{A}\) a projection, leaves the local state invariant

\[
|\Omega a\Omega\rangle = (W'\Omega|a\Omega)W' \Omega .
\]

(38)

A vector \(|\Omega\rangle\) has the Reeh-Schlieder property if the set of vector \(a|\Omega\rangle\) is dense in the Hilbert space. Every local operator \(a \in \mathcal{A}\) has a hermitian conjugate operator \(a^\dagger \in \mathcal{A}\). We define the anti-linear operator \(S_{\Omega}^A\) using its action on vectors

\[
\forall a \in \mathcal{A} : \quad S_{\Omega}^A a |\Omega\rangle = a^\dagger |\Omega\rangle .
\]

(39)

For vectors with the Reeh-Schlieder property \(S_{\Omega}^A\) in (39) is densely defined. Hereafter, we refer to \(S_{\Omega}^A\) as the closure of the operator above. The \(S_{\Omega}^A\) is called the Tomita operator and is the starting point of modular theory. The modular operator is the norm of the Tomita operator: \(\Delta_{\Omega} = S_{\Omega}^A S_{\Omega}\) where we have suppressed the region index \(\Lambda\). The modular conjugation is the anti-linear operator \(J_{\Omega} = \Delta_{\Omega}^{1/2} S_{\Omega}\). If \(|\Omega\rangle\) has the Reeh-Schlieder property

\[\text{Since the spectrum of } \Delta_{\psi} \text{ is the uncountable set } [0, \infty) \text{ the rigorous argument is as follows: If we have two sets of projections } \tilde{f} - f > 0 \text{ and } \tilde{f}' - f' > 0 \text{ and partial isometries } \tilde{v}^\dagger v = f \text{ and } \tilde{v}^\dagger v = f' \text{ and } \tilde{v}^\dagger v = f \text{ and } \tilde{v}^\dagger v = f' \text{, we say } \tilde{v} \succ v \text{ if } \tilde{v} f = v \text{ and } \tilde{v} f' = v' \text{. This define a partial order among partial isometries. Since every chain in this partially ordered set has a maximal element, by Zorn's lemma, there exists a maximal partial isometry that one proves to be either an isometry or a co-isometry [5].} \]
$J_\Omega$ is an anti-unitary. The results of the modular theory that we need here are the following two facts:\footnote{See [8] for proofs of these statements.}

1. The operator $\Delta_{\Omega}^it$ generates a unitary flow in the algebra

$$a \in A \to a_{\Omega}(t) \equiv \Delta_{\Omega}^it \cdot a \Delta_{\Omega}^{-it} \in A$$  \hspace{1cm} (40)

2. The modular conjugation is an anti-linear map that sends every operator in the $A$ to $A$: $a_J \equiv a_{\Omega}a_{\Omega}^* \in \bar{A}$. Every local state $\omega$ on $A$ has a unique canonical purificaiton $|\Psi_{\Omega}\rangle$ satisfying: $J_{\Omega} |\Psi_{\Omega}\rangle = |\Psi_{\Omega}\rangle$.

In the case of a density matrix $\omega$ and its canonical purification $|\Omega\rangle$, the modular conjugation is the anti-linear swap operator we discussed earlier and the modular operator is $\Delta_{\Omega} = \omega \otimes \omega^{-1}$. The modular flow of operators in the algebra is given by $\omega^{-it} \omega^{-it}$. The main difference in QFT is that, as opposed to the case of density matrices, the modular flow is not generated by unitaries in the algebra, but with the modular operator that is an operator in the global Hilbert space.

For two vectors $|\Omega\rangle$ and $|\Psi\rangle$ we define the relative Tomita operator and the relative modular operator as its norm

$$S_{\Psi|\Omega} a |\Omega\rangle = a^\dagger |\Psi\rangle, \quad \Delta_{\Psi|\Omega} = S_{\Psi|\Omega}^{1/2} S_{\Psi|\Omega}.$$  \hspace{1cm} (41)

If we choose the canonical purification $|\Psi_{\Omega}\rangle$ then $\Delta_{\Psi|\Omega}^{1/2} |\Omega\rangle = |\Psi_{\Omega}\rangle$. The definition of the weak and strong distance in (13) and definition of the cocycle $u_{\Psi|\Omega}(t) = \Delta_{\Psi|\Omega}^it \Delta_{\Psi|\Omega}^{-it}$ generalize trivially. However, we need to show that the cocycle is inside the algebra $A$. To prove this, we use a trick introduced by Connes in [9]. We add a qubit to the local algebra $A = A_A$ so that the new algebra $A_{AQ}$ is generated by $a \otimes |j\rangle$ (with $|j\rangle$ a vector in the Hilbert space of the qubit $Q$). We also add a new vector $Q$ to the algebra of the complementary region $\bar{A} = A_A$ so that we can purify local states of $A_{AQ}$ in $A_{AQ}$. Consider the global state

$$|\Theta\rangle_{AQ,AQ} = |\Omega\rangle_{A_A} \otimes |00\rangle_{Q} + |\Psi\rangle_{A_A} \otimes |11\rangle_{Q}$$  \hspace{1cm} (42)

where $|\Omega\rangle$ and $|\Psi\rangle$ are global field theory states which we assume to be Reeh-Schlieder.\footnote{It is straightforward to relax this assumption. In QFT, the set of states with the Reeh-Schlieder property is dense in the Hilbert space and includes the vacuum [4, 7].} The combined local state of $AQ$ is $\theta_{AQ} = \omega_{A} \otimes |0\rangle \langle 0|_{Q} + \psi_{A} \otimes |1\rangle \langle 1|_{A}$. The Tomita operator for this state acts as $S_{\Theta|x} |\Theta\rangle = x |\Theta\rangle$ with $x \in A_{AQ}$. The Tomita operator and the modular conjugation of this state decompose according to

$$S_{\Theta} = \begin{pmatrix} S_{\Omega} & 0 \\ 0 & S_{\Psi|\Omega} \\ S_{\Omega|\Psi} & 0 \\ S_{\Psi} & 0 \end{pmatrix}$$  \hspace{1cm} (43)

and

$$\Delta_{\Theta} = \begin{pmatrix} \Delta_{\Omega} & \Delta_{\Omega|\Psi} \\ \Delta_{\Psi|\Omega} & \Delta_{\Psi} \end{pmatrix}.$$  \hspace{1cm} (44)

The cocycle $u_{\Psi|\Omega}(t)$ is simply the modular flow of the operator $1_A \otimes |0\rangle \langle 1|_{Q}$ in state $|\Theta\rangle$:

$$\Delta_{\Theta}^it (1 \otimes |0\rangle \langle 1|_{Q} \otimes 1_{AQ}) \Delta_{\Theta}^{-it} = \Delta_{\Omega}^it \Delta_{\Psi|\Omega}^it \otimes |00\rangle + \Delta_{\Psi|\Omega}^it \Delta_{\Psi}^{-it} \otimes |01\rangle.$$  \hspace{1cm} (45)

Since the modular flow has to remain inside $A_{AQ}$ we find that the cocycle belongs to the local algebra of field theory in $A$ and satisfies $u_{\Psi|\Omega}(t) = \Delta_{\Psi|\Omega}^it \Delta_{\Psi|\Omega}^{-it} = \Delta_{\Psi|\Omega}^it \Delta_{\Psi|\Omega}^{-it}$\footnote{If we start with a three-level system for $Q$ instead of a qubit the same argument establishes the so-called cocycle chain rule: $u_{\Psi|\Omega}(t) u_{\Psi|\Omega}(t) = u_{\Psi|\Omega}(t)$ [9] which in the case $\Phi = \Psi$ gives $\Delta_{\Psi|\Omega}^it \Delta_{\Psi|\Omega}^{-it} = \Delta_{\Psi|\Omega}^it \Delta_{\Psi|\Omega}^{-it}$.}. More generally, the modular flow of the operator $V \otimes |0\rangle \langle 1|$ is

$$\Delta_{\Theta}^it (V \otimes |0\rangle \langle 1|_{Q}) \Delta_{\Theta}^{-it} = V_{\Omega}(t) u_{\Psi|\Omega}(t) V_{\Omega}^\dagger \otimes |0\rangle (1),$$

which is the conjugate of the operator we found in section III to sew states in the large $t$ limit. If both states are Reeh-Schlieder the cocycle generates a unitary flow in the algebra. For an operator in the complementary region $a' \in A_{\bar{A}}$ we have $u_{\Psi|\Omega}(t) u_{\Psi|\Omega} = a'$ which further implies

$$\Delta_{\Theta}^it \Delta_{\Theta}^{-it} a = \Delta_{\Theta}^it \Delta_{\Theta}^{-it} a,$$  \hspace{1cm} (45)

where in the second line we have used the relation $\Delta_{\Theta}^it = (\Delta_{\Theta})^{-1} \Delta_{\Theta}^it$ in [6].

As an example, consider state $u_{\Psi|\Omega}(t) |\Omega\rangle$ in the large $t$ limit. From (45) we have

$$\lim_{t \to -\infty} \langle \Omega | u(t) \rangle^t \langle u(t) | \Omega \rangle = \langle \Omega | u(t) \rangle \langle u(t) | \Omega \rangle$$

$$\lim_{t \to -\infty} \langle \Omega | u(t) \rangle \langle u(t) | \Omega \rangle = \lim_{t \to -\infty} \omega_{u(t)}(a).$$  \hspace{1cm} (46)

In section III we found that the weak distance $\lim_{t \to -\infty} d_w(\omega_{u(t)}(a)) = 0$. That is the sense in which the state is the same as $\psi$ on $A$.

Consider the algebra of half-space $x > |t|$ in the vacuum state of QFT in any dimension. The modular operator is $e^{-2\pi K_x}$ where $K_x$ is the boost operator in the $x$ direction [10]. If we pick $|\Psi\rangle$ to be the local state of the vacuum QFT on half space the modular flow $\Delta_{\Omega}^it$ is the boost operation. The authors of [11] argued that $\Delta_{\Omega}^it |\chi\rangle$ for some $|\chi\rangle$ is an infinitely boosted state that should look like vacuum. Here, we presented a generalization of that argument to an arbitrary state $|\Psi\rangle$. 


A benefit of enlarging the local algebra by a qubit is that it allows us to write the Frobenius distance measure in equation (3) in terms of the norm of a commutator between the local density matrix and an observable. In the next section, we use this rewriting to study invariant states of relative modular operator.

**VI. COMMUTATORS AND INVARIANT STATES**

In section IV, the projections that commute with the modular operator and the invariant states of relative modular operator played an important role in our construction of the unitary that sew states. Here, we discuss such projections in more detail. Let us go back to the Hilbert space of system AQ and consider the density matrix \( \theta_{AQ} = \omega_A \otimes |0\rangle\langle 0|_Q + \psi_A \otimes |1\rangle\langle 1|_Q \). For any \( a \in \mathcal{A} \) we can write

\[
(\sqrt{\omega} a - a \sqrt{\psi}) \otimes |0\rangle\langle 0|_Q = \left[ \sqrt{\theta_{AQ}}, a \otimes |0\rangle\langle 0|_Q \right].
\]

We rewrite the Frobenius distance of density matrices as

\[
d_F(\omega, \psi)^2 = \frac{1}{2} \left\| \left[ \sqrt{\theta_{AQ}}, a \otimes \sigma_X^j \right] \right\|_F^2
\]

\[
= -\frac{1}{2} \text{tr} \left( \left[ \sqrt{\theta_{AQ}}, a \otimes \sigma_X^j \right] \right) = \left\| \left( \Delta_{\Theta} \right)^{1/2} - 1 \right\| \omega a \text{, (49)}
\]

where \( \sigma_X^j = |0\rangle\langle 1| + |1\rangle\langle 0| \) is the X Pauli matrix in the algebra of the qubit Q. More generally, for an arbitrary operator \( a \in \mathcal{A} \) we have

\[
\frac{1}{2} \left\| \sqrt{\omega} a - a \sqrt{\psi} \right\|_F^2 = \left\| \omega a \left| \psi \right. \right\| - a \left. \left| \psi \right. \right\|, (50)
\]

where \( x_a \equiv a \otimes |0\rangle\langle 1| + a^\dagger \otimes |1\rangle\langle 0| \) is in the algebra of AQ. Our distance measure is the Frobenius norm of the commutator of density matrix \( \theta \) and an observable \( x_a \), otherwise known as the Wigner-Yanase skew information [12]. For unitary orbits we rewrite the infimum distance as

\[
d_F(\omega, \psi)^2 = \frac{1}{2} \inf \text{tr} \left( \left[ \sqrt{\theta}, x_U \right] \right)^2,
\]

where \( x_U = U \otimes |0\rangle\langle 1| + U^\dagger \otimes |1\rangle\langle 0| \) is a self-adjoint unitary operators. If there exists an \( x_U \) that commutes with \( \theta \) the density matrices \( \omega \) and \( \psi \) are simultaneously diagonalizable. If \( x_U \) almost commutes with \( \theta_{AQ} \) then \( U_\omega U^\dagger \) becomes close to \( \psi \). The almost invariant states of relative modular operator are prepared by \( x_U \) that almost commute with \( \theta \).

Consider the density matrix \( \omega = \sum_k e^{-\lambda_k} |k\rangle \langle k| \) and its canonical purification \( \Omega \). The orthonormal projection \( e_k = |k\rangle \langle k| \) commute with \( \omega \). If there are degenerate eigenvalues, i.e. \( \lambda_k = \lambda_{k'} \), then the partial isometry \( |k\rangle \langle k'| \) also commutes with \( \omega \). The sub-algebra of all operators that commute with \( \omega \) is called the centralizer of \( \omega \). If the algebra has a center \( Z \), the center is in the centralizer of all density matrices. Using the cyclicity of trace we find that any operator \( h \) that commutes with \( \omega \) satisfies: \( \text{tr}(\omega[h, a]) = \text{tr}(\omega[a, h]) = 0 \) for all \( a \in \mathcal{A} \). In QFT, we define the centralizer as the set of \( h \) such that \( \omega[h, a] = 0 \) for all \( a \in \mathcal{A} \). Any operator \( h \) in the centralizer of \( \omega \) commutes with the modular operator, and is invariant under modular flow [13]:

\[
(\Delta_{\Omega}^{1/2} - 1) h |\Omega\rangle = 0
\]

\[
\Delta_{\Omega}^{it} h \Delta_{\Omega}^{-it} = h. (51)
\]

Acting with \( h \) in the centralizer on \( |\Omega\rangle \) creates a new invariant state of \( \Delta_{\Omega} \). Moreover, every invariant vector \( |h\rangle \) is \( h |\Omega\rangle \) for some \( h \) affiliated with the centralizer [13]. Therefore, to understand invariant states of modular operator it suffices to study the centralizer of the state.

So far we have only discussed the invariant states of the relative modular operator \( \Delta_{\Omega|\Psi} \) that are locally prepared as \( u |\Omega\rangle \). It is natural to ask what does a general invariant state of the relative modular operator look like. Consider the state \( UU^\dagger |\Omega\rangle \) with \( U \in \mathcal{A} \) and \( U^\dagger \in \mathcal{A} \). Its relative modular operator is \( \Delta_{\Omega|UU^\dagger\Omega} = U^\dagger \Delta_{\Omega} U \) [14], therefore \( U^\dagger |\Omega\rangle \) is its invariant state. It is a state that looks like \( |\Omega\rangle \) on \( A \) and \( |UU^\dagger\Omega\rangle \) on \( \bar{A} \). As we showed in this work, every state \( |\Psi\rangle \) is well-approximated by some \( UU^\dagger |\Omega\rangle \). Hence, it is tempting to think that all states \( |\Omega\rangle \) that are invariant under \( \Delta_{\Omega|\Psi} \) look like \( |\Omega\rangle \) on \( A \) and \( |\Psi\rangle \) on \( \bar{A} \), and only differ by their long-range correlations across \( AA \). We show below that this is incorrect if the centralizers of the states are non-trivial.

Every operator \( h \) that commutes with \( \Delta_{\Omega} \) also commutes with \( \Delta_{\Omega|\Psi} \):

\[
\Delta_{\Omega|\Psi}^{it} h \Delta_{\Omega|\Psi}^{-it} = \Delta_{\Omega}^{it} h \Delta_{\Omega}^{-it} = h. (52)
\]

where we used (45). If \( |\Omega\rangle \) is an invariant state of relative modular flow so is \( h |\Omega\rangle \). This implies that if \( UU^\dagger |\Omega\rangle \simeq |\Psi_{\Omega}\rangle \) then \( (hU^\dagger)(hU) |\Omega\rangle \simeq hh^\dagger |\Psi_{\Omega}\rangle \). The physical interpretation of an arbitrary invariant state \( |\Omega_{\Psi}\rangle \) is that it is a state that is the same as \( h |\Omega\rangle \) with respect to \( A \) and \( \bar{h} |\Psi\rangle \) with respect to \( \bar{A} \) where \( h \) is in the centralizer of \( \Omega \) and \( \bar{h} \) is in the centralizer of \( |\Psi\rangle \). In the state \( |\Omega_{\Psi}\rangle \) the flow \( \Delta_{\Omega|\Psi} \) generates a symmetry of \( A \) and \( \Delta_{\bar{\Omega}|\Psi} \) generates a symmetry of \( \bar{A} \):

\[
\Delta_{\Omega|\Psi} |\Omega_{\Psi}\rangle = |\Omega_{\Psi}\rangle
\]

\[
\langle \Omega_{\Psi} | a \Omega_{\Psi} \rangle = \langle \Omega_{\Psi} | \Delta_{\Omega|\Psi}^{it} a \Delta_{\Omega|\Psi}^{-it} \Omega_{\Psi} \rangle = \langle \Omega_{\Psi} | \Delta_{\Omega}^{it} a \Omega_{\Psi} \rangle
\]

\[
\langle \Omega_{\Psi} | a' \Omega_{\Psi} \rangle = \langle \Omega_{\Psi} | \Delta_{\Omega|\Psi}^{it} a' \Delta_{\Omega|\Psi}^{-it} \Omega_{\Psi} \rangle = \langle \Omega_{\Psi} | \Delta_{\bar{\Omega}}^{it} a \Omega_{\Psi} \rangle
\]

\[
\Delta_{\Omega|\Psi}^{it} h \Delta_{\Omega|\Psi}^{-it} = h. (51)
\]

15 Affiliated with \( \mathcal{A} \) means that it commutes with \( \bar{A} \). If \( \bar{A} \) has trivial center any bounded operator that is affiliated with \( \mathcal{A} \) is in \( \mathcal{A} \).
for all \( a \in \mathcal{A} \) and \( a' \in \bar{\mathcal{A}} \). From the point of view of \( \mathcal{A} \) the state \( |\Omega\rangle \) is \( \hbar |\Omega\rangle \) and from the point of view of the algebra \( \bar{\mathcal{A}} \) the state is \( \hbar |\Psi\rangle \).

In the vacuum of QFT restricted to half-space the modular operator is the boost. Vacuum is the only boost invariant state, therefore its centralizer is trivial. The modular flow in such states is ergodic in the sense that in the large time limit \( \Delta^{\phi}_{\Omega} |\phi\rangle \) converges to vacuum in the weak norm for any state \( |\phi\rangle \). See [15] for a review. The algebra of QFT is not the tensor product of \( \mathcal{A} \otimes \bar{\mathcal{A}} \) which means that not all \( |\Omega\rangle \) and \( |\Psi\rangle \) have exact invariant states of their relative modular operator \( |\Omega\rangle \). However, there are always states that are almost invariant with arbitrary precision. We take this as evidence that, in general, the exact state \( |\Omega\rangle \) that sews \( \omega \) on the left with \( \psi \) on the right has a singularity (shockwave) at the boundary where the states are sewn together. As we saw this singular state \( |\Omega\rangle \) can be approximated arbitrarily well with normalizable states by acting with unitaries from the algebra. The approximation states are invariant under \( \Delta^{\phi}_{\Omega} \) in \( \mathcal{A} \) and almost invariant under \( \Delta^{\phi}_{\Omega} \) in \( \bar{\mathcal{A}} \).

The centralizer of the vacuum of QFT is trivial. To apply the argument of section III we need to find projections that are approximately boost-invariant projections in the algebra of the half-space. Once again, we start with finite quantum systems to obtain intuition. It is straightforward to construct operators that commute with density matrix \( \omega \) using the modular flow \( a_\omega(t) \) satisfies for all \( t \) to kill the off-diagonal terms:

\[
\sqrt{\omega} \int_{-\infty}^{\infty} dt \ e^{it\lambda} a_\omega(t) = \sum_k \int dt \ e^{it\lambda} a_{kk'} \ \sqrt{\omega} \ |k\rangle \langle k'\rangle = 0.
\]

The operator \( \sqrt{\omega} a_\omega(t) \) is the zero Fourier mode of \( a_\omega(t) \). A general Fourier mode \( \sqrt{\omega} \hat{a}_\omega(l) \) satisfies

\[
\sqrt{\omega} \hat{a}_\omega(l) = \sum_k \int dt \ e^{it\lambda} a_{k(k-l)} \ |k\rangle \langle k-l\rangle = \sum_k \int dt \ e^{-\lambda_\epsilon/2 \ (1-e^{-\epsilon/2}) \ |k\rangle \langle k|,}
\]

where \(|k\rangle \) is the eigenvector with \( \lambda_k = \lambda_k - l \). If there are no such eigenvector \( \hat{a}_\omega(l) = 0 \). In a finite quantum system the set of non-zero frequency modes is discrete, and the smallest frequency \( l \) corresponds to is the smallest eigenvalue gap. If the spectrum of the modular operator we have chosen is the full positive line all Fourier frequencies are non-zero and one can choose the Fourier mode \( \hat{a}_\omega(\epsilon) \) for some \( \epsilon \ll 1 \). Such a mode almost commutes with \( \omega \):

\[
\| \sqrt{\omega} \hat{a}_\omega(\epsilon) \| \leq \epsilon \| a \| \leq \| \sqrt{\omega} \ |k\rangle \langle k| \| \leq \epsilon \| a \| .
\]

If we take a set of Fourier modes \(|l\rangle < \epsilon \) and \( g(l) \) independent of \( \epsilon \) the operator \( \int_{|l| \leq \epsilon} dg(l) \hat{a}_\omega(l) \) almost commutes with \( \omega \). Fourier transforming back we find that for any square-integrable function \( g(t) \) whose Fourier transform \( g(l) \) is restricted to \(|l| \leq \epsilon \) the operator \( g_\omega(a) = \int dtg(t) a_\omega(t) \) almost commutes with \( \sqrt{\omega} \). It is self-adjoint if \( g(t) = g(-t) \). The Fourier modes of modular flow have been discussed in connection with the holographic duality in [16].

### VII. Sewing Multiple States

Finally, we come to the case of \( n \) subsystems. Consider a collection of \( n \) density matrices \( \{\omega^{(1)}, \cdots, \omega^{(n)}\} \) each corresponding to a qubit. The qubits can be sewn together if there exists a global pure state \(|\Omega\rangle \) such that \( \omega^{(i)} = tr_{î\not\in E} |\Omega\rangle \langle \Omega| \) for all \( i \). Local unitaries that rotate each qubit \( U_i \omega^{(i)} U_i^\dagger \) have no effect on whether or not the density matrices can be sewn. The constraints on sewing \( \omega^{(i)} \) only depend on their eigenvalues. Expand each density matrix in its diagonal eigenbasis:

\[
\omega^{(i)} = \sum_{\lambda} \lambda_i |0\rangle_i \langle 0|_i + (1 - \lambda_i) |1\rangle_i \langle 1|_i
\]

with \( 0 \leq \lambda_i \leq 1/2 \). The obstruction to sew density matrices is a constraint on the set of eigenvalues \( \lambda_i \). We briefly review the derivation of these constraints in [1].

In the case of two qubits, one can sew the qubits if and only if they have the same spectrum, i.e. \( \lambda_1 = \lambda_2 \). Next, consider the case of three qubits. The projection \( \epsilon_i = |0\rangle_i \langle 0|_i \) satisfy the identity:

\[
\epsilon_1 + \epsilon_2 = 1_{12} + \epsilon_1 \epsilon_2 - (1 - \epsilon_1)(1 - \epsilon_2).
\]

This identity implies that if \(|\Omega\rangle \) is a global vectors that sews them together

\[
\lambda_1 + \lambda_2 + \lambda_3 \geq 1 - \text{tr} \left( \omega^{(12)} (1 - \epsilon_1) (1 - \epsilon_2) \right),
\]

where we have discarded the positive term \(|\Omega\rangle \epsilon_1 \epsilon_2 \langle \Omega| \). Since \(|\Omega\rangle \) is pure the eigenvalues of \( \omega^{(12)} \) are the same as those of \( \omega^{(13)} \). Its largest eigenvalue is \( 1 - \lambda_3 \) which means \( \langle \Phi | \omega^{(12)} | \Phi \rangle \leq 1 - \lambda_3 \) for all \( |\Phi\rangle \) in the Hilbert space of three qubits. The projector \( (1 - \epsilon_1)(1 - \epsilon_2) \) is rank one, therefore \( \text{tr} (\omega^{(12)} (1 - \epsilon_1)(1 - \epsilon_2)) \leq 1 - \lambda_3 \). Plugging this back into (56) we find

\[
\lambda_1 + \lambda_2 \geq \lambda_3 .
\]

Permuting the qubits and repeating the same argument gives two more constraints:

\[
\sum_{i \neq j} \lambda_j \geq \lambda_i .
\]

Any three density matrices with \( \{\lambda_1, \lambda_2, \lambda_3\} \) that satisfy the above inequality can be sewn together in a global vector that is non-unique.

This argument generalizes to \( n \)-qubits in a straightforward manner, and the final constraint on the eigenvalues
\( \lambda_i \) are the same as (57) but with \( i = 1, \cdots n \). The eigenvalues of local density matrices of an \( n \)-partite system that can be sewn together form a convex polytope. See [17] for a review of the generalization to arbitrary finite dimensional systems.

We now turn to quantum field theory. Starting with any global state \( |\Psi\rangle \) acting with local unitaries in region \( A_1 \) we can prepare an arbitrary local state \( \omega_i \) on \( A_1 \). Then, we act locally on \( A_2 \) and prepare \( \omega_2 \), and repeat this for all subregions to \( A_n \) to obtain a global state that sews local states \( \omega_i \) for \( i = 1, \cdots, n \). There are no constraints on sewing \( \omega_i \). For instance, using local unitaries we can make every state look like vacuum on each \( A_i \). Every state \( |\Psi\rangle \) has the same entanglement structure as \( U_1U_2\cdots U_n |\Psi\rangle \). By classifying all states that sew vacuum reduced states \( \omega_i \) together we learn about various forms of multi-partite entanglement than can appear in QFT.

Consider the case \( n = 3 \) in QFT with the region \( A_2 \) separating \( A_1 \) and \( A_3 \). We take the mutual information between \( A_1 \) and \( A_3 \) in the vacuum as a measure of entanglement between \( A_1 \) and \( A_3 \). This mutual information is the same as the relative entropy of \( \omega_i \) with respect to \( \omega_1 \) and \( \omega_3 \):

\[
I(A_1 : A_3) = S(\omega_{13} \parallel \omega_1 \otimes \omega_3) .
\]

Since the relative entropy on \( A_1A_3 \) does not depend on the purification we write

\[
S_{\text{vac}}(\omega_{13} \parallel \omega_1 \otimes \omega_3) = - \langle \Omega | \log \Delta_{\omega_1 \otimes \omega_3 | \omega_{13} | \Omega} \rangle .
\]

Any other global state that sews \( \omega_i \) has the form \( |\tilde{\Omega}\rangle \simeq U_2U_{13} |\Omega\rangle \). The mutual information in this state is

\[
S(\tilde{\omega}_{13} \parallel \omega_1 \otimes \omega_3) = - \langle \Omega | U_{13}^\dagger \log \Delta_{\omega_1 \otimes \omega_3 | \omega_{13} U_{13} | \Omega} \rangle.
\]

The vacuum relative modular operator \( \Delta_{\omega_1 \otimes \omega_3 | \omega_{13}} \) teaches us about the mutual information in a dense set of states \( |\tilde{\Omega}\rangle \). The same principle extends to tri-partite entanglement and other measures of multi-partite entanglement for larger \( n \) that are invariant under local unitaries.

It is tempting to conclude from the discussion above that the vacuum relative modular operators \( \Delta_{\omega_1 \otimes \cdots \omega_n | \omega_{1 \cdots n}} \) contain the information about the multi-partite entanglement of all states. This is incorrect because the relative modular operator and its logarithm are unbounded and if we have a sequence of states \( \psi_n \rightarrow \psi \) the relative modular operators \( \lim_n \Delta_{\psi_n | \psi} \) need not converge. In fact, the relative entropy is not continuous, but just lower semi-continuous [18]:

\[
S(\phi \parallel \psi) \leq \lim n S(\phi \parallel \psi_n) .
\]

We postpone the study of the implications of sewing for the theory of multi-partite entanglement in QFT to future work.

VIII. DISCUSSION

In this work, we showed that in quantum field theory any collection of local reduced states in non-overlapping regions can be sewn together with arbitrary precision. We argued that the local unitary that acts on \( |\Psi\rangle \) and creates the vacuum state \( \omega \) is the cocycle \( u_{\Omega\Psi}(t) \) in the large \( t \) limit. Ideas similar to sewing states of quantum field theory have appeared in various context, recently [19, 20]. It would be interesting to explore the connection between our algebraic sewing prescription and sewing Euclidean path-integrals discussed in the literature. Here, we addressed the problem of sewing one-body local states of in a global pure state. The collection of one-body density matrices can be understood as a mean-field approximation to the state [1]. The problem can be generalized to sewing multi-body local states with overlapping regions, called the quantum marginal problem. See [17] for a discussion of the quantum marginal problem in the general setup. It is interesting to study the generalized marginal problem in QFT [21]. Finally, it is worthwhile to note that the sewing argument presented in this paper applied generically to any two states of a type III factor von Neumann algebra, independent of which quantum field theory they belong to. It would be interesting to study the physics of sewing states of different quantum field theories.

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The first inequality is the Fuchsvan de Graaf inequality [22]. The second inequality is due to the fact that \( \text{tr}(\sqrt{\rho \omega}) \leq \| \sqrt{\rho \omega} \| \). The last inequality uses \( S(\rho | \omega) \geq -2 \ln \text{tr}(\sqrt{\rho \omega}) \) [23].

**Appendix B: Skew information and relative entropy**

In defining the skew information in (49) we can use \( p \)-norms. This generalization of the Wigner-Yanase information is due to Dyson. For a self-adjoint operator \( x \) and \( \omega \) a density matrix purified canonically in \( |\Omega\rangle \) the Wigner-Yanase-Dyson \( p \)-skew information is

\[
I_p(x, \omega) = -\text{tr}(\sqrt{\sigma^p x [\sigma^{1-p} x, \omega]}). \tag{B1}
\]

In terms of the canonical purification \( |\Omega\rangle \) of the density matrix the skew information is

\[
I_p(x, \Omega) = \langle \Omega | x \left( \Delta_\Omega^p + \Delta_{\Omega^p} - 1 - \Delta_\Omega \right) x |\Omega\rangle. \tag{B2}
\]

The derivative at \( p = 0 \) is

\[
\lim_{p \to 0} I_p(x, \Omega) = 2 \langle \Omega \| x \log \Delta_\Omega x |\Omega\rangle. \tag{B3}
\]

If \( x \) is a unitary operator then this is the relative entropy

\[
S(u^\dagger \Omega | \Omega) = -\langle \Omega \| \log \Delta_{u^\dagger \Omega u | \Omega} \rangle. \tag{B4}
\]

The \( p \)-skew information is symmetric under \( p \to 1 - p \) with the symmetric point corresponding to the Wigner-Yanase measure of section VI.

**Appendix C: Isometries as a limit of unitaries**

In this section, we argue that in QFT the unitary orbit of any state passes through subspaces \( \pi \mathcal{H} \) for any projection \( \pi \in \mathcal{A} \). There are two parts to the argument. First, we observe that in QFT (in general, any type III von Neumann algebra) for every projection \( \pi \) there exists an isometry \( w \) such that \( ww^\dagger = \pi \) and \( w^\dagger w = 1 \) [7]. The state \( \langle \Phi_w | w | \Phi \rangle \) is an eigenstate of \( \pi \) for any \( |\Phi\rangle \). Therefore, by acting with an isometry we can bring any state to the subspace \( \pi \mathcal{H} \). Second, we notice that in QFT (any type III algebra) any isometry \( w \) is a limit of unitary operators \( u_n \) in strong operator norm, i.e. \( \lim_{n \to \infty} \| w - u_n \| = 0 \). To see this, consider a sequence of projections \( \pi_n \) in the algebra that converge to identity: \( \lim_{n \to \infty} |1 - \pi_n\rangle = 0 \). For any \( n \) the two projections \( 1 - \pi_n \) and \( 1 - w\pi_n w^\dagger \) belong to the algebra. A type III algebra is one in which for any two projections \( \pi \) and \( \tilde{\pi} \) there exists an operator \( v \) such that \( v^\dagger v = \pi \) and \( vv^\dagger = \tilde{\pi} \).

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operator $\pi_n w^\dagger v_n = 0$ is zero because its norm vanishes: $(\pi_n w^\dagger v_n)(v_n^\dagger w\pi_n) = 0$. Similarly, we have $v_n\pi_n = 0$. If we define the operator $u_n = w\pi_n + v_n$ we find that it is a unitary operator for all $n$:

$$
u_n^\dagger u_n = 1 + \pi_n w^\dagger v_n + v_n^\dagger w\pi_n = 1
$$

$$
u_n u_n^\dagger = 1 + w\pi_n v_n^\dagger + v_n\pi_n w^\dagger = 1.$$

(C1)

The sequence of unitaries $u_n$ tends to $w$ in strong operator topology, i.e. $\lim_n \|w - u_n\| = 0$. We have established that there exists a sequence of unitaries that bring any state to the eigenspace of any projector $\pi$ in the algebra: $\pi(\lim_n u_n |\Phi\rangle) = \lim_n u_n |\Phi\rangle$. Since this can be done for any two states, one might wonder if any two states can be brought arbitrarily close with unitaries. This happens only in a type III1 algebra.