UNBOUNDED OPERATORS: (SQUARE) ROOTS, NILPOTENCE, CLOSABILITY AND SOME RELATED INVERTIBILITY RESULTS

MOHAMMED HICHEM MORTAD

Abstract. In this paper, we are mainly concerned with studying arbitrary unbounded square roots of linear operators as well as some of their basic properties. The paper contains many examples and counterexamples. As an illustration, we give explicit everywhere defined unbounded non-closable \( n \)th roots of the identity operator as well as the zero operator. We also show a non-closable unbounded operator without any non-closable square root. Among other consequences, we have a way of finding everywhere defined bijective operators, everywhere defined operators which are surjective without being injective and everywhere defined operators which are injective without being surjective. Some related results on nilpotence are also given.

Introduction

First, while we will recall most of the needed notions for readers’ convenience, we also assume readers have some familiarity with other very standard concepts and results of operator theory. Some useful references are [7], [23], [32], [38] and [40].

Let \( H \) be a Hilbert space and let \( B(H) \) be the algebra of all bounded linear operators defined from \( H \) into \( H \).

If \( S \) and \( T \) are two linear operators with domains \( D(S) \subset H \) and \( D(T) \subset H \) respectively, then \( T \) is said to be an extension of \( S \), written as \( S \subset T \), if \( D(S) \subset D(T) \) and \( S \) and \( T \) coincide on \( D(S) \). The restriction of some operator \( T \) to some subspace \( M \) is denoted by \( T_M \).

The product \( ST \) and the sum \( S + T \) of two operators \( S \) and \( T \) are defined in the usual fashion on the natural domains:

\[ 2020 \text{ Mathematics Subject Classification. Primary 47A05. Secondary 47A08, 47A99.} \]

\[ \text{Key words and phrases. Unbounded operators; Non-closable operators; (Square) roots of operators; Nilpotence; Spectrum; Bijective operators; Matrices of operators.} \]

Supported in part by PRFU project: C00L03UN310120200003.
\[
D(ST) = \{ x \in D(T) : Tx \in D(S) \}
\]
and
\[
D(S + T) = D(S) \cap D(T).
\]

When \( D(T) = H \), we say that \( T \) is densely defined. In this case, the adjoint \( T^* \) exists and is unique.

An operator \( T \) is called closed if its graph is closed in \( H \oplus H \). \( T \) is called closable if it has a closed extension. Equivalently, this signifies that for each sequence \( (x_n) \) in \( D(T) \) such that \( x_n \to 0 \) and \( Tx_n \to y \), then \( y = 0 \). If \( T \) is densely defined, then \( T \) is closable if and only if \( D(T^*) \) is dense. The smallest closed extension of \( T \) is called its closure, and it is denoted by \( \overline{T} \). When \( T \) is closable, then \( \overline{T} = (T^*)^* \). Recall also that if \( T \) is a bounded operator on some domain \( D(T) \), then \( T \) is closed if and only if \( D(T) \) is closed (see e.g. Theorem 5.2 in [10]).

If \( T \) is densely defined, we say that \( T \) is self-adjoint when \( T = T^* \); symmetric if \( T \subset T^* \); normal if \( T \) is closed and \( TT^* = T^*T \). A symmetric operator \( T \) is called positive if
\[
< Tx, x > \geq 0, \forall x \in D(T).
\]
Notice that unlike positive operators in \( B(H) \), unbounded positive operators need not be self-adjoint.

In the event of the density of all of \( D(S) \), \( D(T) \) and \( D(ST) \), then
\[
T^*S^* \subset (ST)^*,
\]
with the equality occurring when \( S \in B(H) \). Also, when \( S, T \) and \( S + T \) are densely defined, then
\[
S^* + T^* \subset (S + T)^*,
\]
and the equality holding again if \( S \in B(H) \).

The real and imaginary parts of a densely defined operator \( T \) are defined respectively by
\[
\text{Re} \ T = \frac{T + T^*}{2} \quad \text{and} \quad \text{Im} \ T = \frac{T - T^*}{2i}.
\]
Clearly, if \( T \) is closed, then \( \text{Re}T \) is symmetric but it is not always self-adjoint (it may even fail to be closed).

Let \( T \) be a densely defined operator with domain \( D(T) \subset H \). If there exist densely defined symmetric operators \( A \) and \( B \) with domains \( D(A) \) and \( D(B) \) respectively and such that
\[
T = A + iB \text{ with } D(A) = D(B),
\]
then \( T \) is said to have a Cartesian decomposition ([30]).
A densely defined operator $T$ admits a Cartesian decomposition if and only if $D(T) \subset D(T^*)$. In this case, $T = A + iB$ where $A = \text{Re} T$ and $B = \text{Im} T$.

Let $A$ be an injective operator (not necessarily bounded) from $D(A)$ into $H$. Then $A^{-1} : \text{ran}(A) \to H$ is called the inverse of $A$ with domain $D(A^{-1}) = \text{ran}(A)$.

If the inverse of an unbounded operator is bounded and everywhere defined (e.g. if $A : D(A) \to H$ is closed and bijective), then $A$ is said to be boundedly invertible. In other words, such is the case if there is a $B \in B(H)$ such that

$$AB = I \text{ and } BA \subset I.$$  

Clearly, if $A$ is boundedly invertible, then it is closed. Recall also that $T + S$ is closed if $S \in B(H)$ and $T$ is closed, and that $ST$ is closed if $S^{-1} \in B(H)$ and $T$ is closed or if $S$ is closed and $T \in B(H)$.

Based on the bounded case and the previous definition, we say that an unbounded $A$ with domain $D(A) \subset H$ is right invertible if there exists an everywhere defined $B \in B(H)$ such that $AB = I$; and we say that $A$ is left invertible if there is an everywhere defined $C \in B(H)$ such that $CA \subset I$. Clearly, if $A$ is left and right invertible simultaneously, then $A$ is boundedly invertible.

The spectrum of unbounded operators is defined as follows: Let $A$ be an operator on a complex Hilbert space $H$. The resolvent set of $A$, denoted by $\rho(A)$, is defined by

$$\rho(A) = \{ \lambda \in \mathbb{C} : \lambda I - A \text{ is bijective and } (\lambda I - A)^{-1} \in B(H) \}.$$  

The complement of $\rho(A)$, denoted by $\sigma(A)$, i.e.

$$\sigma(A) = \mathbb{C} \setminus \rho(A)$$

is called the spectrum of $A$.

Clearly, $\lambda \in \rho(A)$ iff there is a $B \in B(H)$ such that

$$(\lambda I - A)B = I \text{ and } B(\lambda I - A) \subset I.$$  

Also, recall that if $A$ is a linear operator which is not closed, then $\sigma(A) = \mathbb{C}$.

Kulkarni et al. showed using simple arguments in [18] that

$$\sigma(A^2) = [\sigma(A)]^2$$

when $A$ is closed.

Next, we recall the definition of unbounded nilpotent operators. We choose to use Ôta’s definition in [29] of nilpotence (S. Ôta gave the definition in the case $n = 2$).
Definition 1. Let $T$ be a non necessarily bounded operator with a dense domain $D(T)$. We say that $T$ is nilpotent if $T^n$ is well defined and

$$T^n = 0 \text{ on } D(T)$$

for some $n \in \mathbb{N}$ (hence necessarily $D(T^n) = D(T^{n-1}) = \cdots D(T)$).

Recall the following lemma:

Lemma 0.1. ([33] or [36]) If $H$ and $K$ are two Hilbert spaces and if $T : D(T) \subset H \to K$ is a densely defined closed operator, then

$$D(T) = D(T^*T) \iff T \in B(H, K).$$

Thanks to the previous lemma, if $T$ is some densely defined closed nilpotent operator with domain $D(T) \subset D(T^*) \subset H$, then $T \in B(H)$. In particular, if $T$ is a closed densely defined nilpotent symmetric or hyponormal operator, then $T = 0$ everywhere on $H$. See [12] for a proof and some closely related results. See also [36].

Now, we give a definition of square roots for general linear operators.

Definition 2. Let $A$ and $B$ be linear operators. Say that $B$ is a square root of $A$ if $B^2 = A$ (and so $D(B^2) = D(A)$). More generally, say that $B$ is an $n$th root of $A$ (where $n \in \mathbb{N}$) if $B^n = A$ (and so $D(B^n) = D(A)$).

Notice that for the case $A = 0$ on $D(A)$, this includes Definition 1 above. An objection, why not use a definition like $B^2 \subset A$? The issue then is that it is quite conceivable to have $D(B^2) = \{0\}$ (or higher powers as well). In such case, $B^2 \subset A$ holds trivially whilst $A$ can be anything in $B(H)$! Given the diversity of classes of examples such that $D(B^n) = \{0\}$ for some $n$ (as may be seen in [2], [5], [6], [11], [24] and [31]), a definition like $B^n \subset A$ would not therefore yield too informative conclusions in many situations.

It is well known that self-adjoint positive operators have a unique positive square root. This does not exclude the fact that a self-adjoint positive operator may well have other square roots. Self-adjoint positive square roots of self-adjoint positive operators play an important role. For instance, they intervene in the definition of the absolute value of unbounded operators (cf. [20]), and so in the polar decomposition. Recall here that the absolute value of a closed $A$ is given by $|A| = \sqrt{A^*A}$ where $\sqrt{\cdot}$ designates the unique positive square root of $A^*A$, which is positive by the closedness of $A$.

Positive self-adjoint square roots are also present in abstract wave and Schrödinger’s equations (see e.g. [32]). See also [20]. They, of course, have other utilizations. Here, we confine our attention to arbitrary square (or other) roots.
Finding counterexamples using matrices of non-necessarily bounded operators has been a success as demonstrates the recent papers: [11], [24], [25], [26] and [27]. Let us recall their definition briefly:

Let $H$ and $K$ be two Hilbert spaces and let $A : H \oplus K \to H \oplus K$ (we may equally use $H \times K$ instead of $H \oplus K$) be defined by

(1) \[ A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \]

where $A_{11} \in L(H)$, $A_{12} \in L(K, H)$, $A_{21} \in L(H, K)$ and $A_{22} \in L(K)$ are not necessarily bounded operators. If $A_{ij}$ has a domain $D(A_{ij})$ with $i, j = 1, 2$, then

$D(A) = (D(A_{11}) \cap D(A_{21})) \times (D(A_{12}) \cap D(A_{22}))$

is the natural domain of $A$. So if $(x_1, x_2) \in D(A)$, then

$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_{11}x_1 + A_{12}x_2 \\ A_{21}x_1 + A_{22}x_2 \end{pmatrix}.$

As is customary, we tolerate the abuse of notation $A(x_1, x_2)$ instead of $A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. The generalization to $n \times n$ matrices of operators is also clear.

Note that unlike matrices of everywhere defined bounded operators, not all unbounded operators admit such a decomposition (cf. [37]). Readers should also be wary when dealing with products of matrices of (unbounded) operators as they are not always well defined. However, when dealing with everywhere defined (unbounded) operators, all products are possible in this setting.

Recall that the adjoint of \( \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \) is not always \( \begin{pmatrix} A^*_{11} & A^*_{21} \\ A^*_{12} & A^*_{22} \end{pmatrix} \) (even when all domains are dense including the main domain $D(A)$) as many known counterexamples show. Nonetheless, e.g.

\[
\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}^* = \begin{pmatrix} A^* & 0 \\ 0 & B^* \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & D^* \\ C^* & 0 \end{pmatrix}
\]

if $A$, $B$, $C$ and $D$ are all densely defined. See e.g. [22] or [41] for more about the adjoint’s operation of general operator matrices.

The special case of the matrix \( \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \) may be denoted by $A \oplus B$.

Finally, since we will often be dealing with everywhere defined unbounded operators, we give an example (which is well known to most readers):
Example 0.2. Let $f : H \to \mathbb{C}$ be a discontinuous linear functional (this requires the axiom of choice as is known to readers). Let $x_0$ be any non-zero vector in $H$ and define $A : H \to H$ by

$$Ax = f(x)x_0.$$ 

Then, $A$ is clearly unbounded and everywhere defined. Obviously, $A$ is not closable for if it were, the Closed Graph Theorem would give $A \in B(H)$, which is impossible.

Remark. As observed above, an operator $A$ is closable iff $D(A^*)$ is dense. There are well known examples in the literature of densely defined unbounded operators $A$ such that even $D(A^*) = \{0\}$. See [24] for some "recent example". Even a non closable $A$ with $D(A) = H$ could be such that $D(A^*) = \{0\}$. This is a famous example by Berberian which may be consulted on Page 53 in [13].

1. Main Results

First, we give some examples of square roots as regards closedness. Let $\mathcal{F}_0$ be the restriction of the $L^2(\mathbb{R})$-Fourier transform to the dense subspace $C_0^\infty(\mathbb{R})$ which denotes here the space of infinitely differentiable functions with compact support. Then, it is well known that

$$D(\mathcal{F}_0^2) = \{0\}$$

because any function $f \in C_0^\infty(\mathbb{R})$ such that $\hat{f} \in C_0^\infty(\mathbb{R})$ is null. Hence $\mathcal{F}_0$ is an unclosed square root of the trivially closed operator $0$ on $\{0\}$.

On the other hand, there are unclosed operators having closed square roots. For example, on $\ell^2$ define the linear operator $A$ by

$$Ax = A(x_n) = (x_2, 0, 2x_4, 0, \ldots, nx_{2n}, 0, \ldots)$$

on the domain

$$D(A) = \{x = (x_n) \in \ell^2 : (nx_{2n}) \in \ell^2\}.$$ 

We may check that $D(A)$ is dense in $\ell^2$, that $A$ is unbounded and closed.

Then, it can readily be checked that

$$A^2 = 0 \text{ on } D(A^2) = D(A)$$

and so $A^2$ is bounded on $D(A)$. Finally, since we know that $A^2 = 0$ on $D(A)$ and that $D(A)$ is not closed, it follows that $A^2$ is a non closed operator. This example first appeared in [28].
As another example, take any unbounded closed operator $T$ on a dense domain $D(T) \subset H$. Then setting

$$A = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix},$$

we see that $D(A) = H \oplus D(T)$. Hence

$$A^2 = \begin{pmatrix} 0 & 0_{D(T)} \\ 0 & 0_{D(T)} \end{pmatrix} = \begin{pmatrix} 0 & 0_{D(T)} \\ 0 & 0 \end{pmatrix}$$

and so $A^2 = 0$ on $D(A^2) = D(A)$. Hence $A^2$ is unclosed.

Next, we provide an invertible bounded operator without any closable square root.

**Proposition 1.1.** There is a bounded subnormal invertible operator without any closable square root.

**Proof.** Let $D$ be the annulus $\{ z \in \mathbb{C} : r < |z| < R \}$ where $r, R > 0$. Let $\mu$ be a planar Lebesgue measure in $D$. Let $L^2(D)$ be the collection of all complex-valued functions which are analytic throughout $D$ and square-integrable w.r.t. $\mu$ (the Bergman space). That is, $f \in L^2(D)$ if $f$ is analytic in $D$ and $\int_D |f(z)|^2d\mu(z) < \infty$. Then $L^2(D)$ is a Hilbert space w.r.t. the inner product

$$< f, g > = \int_D f(z)\overline{g(z)}d\mu(z).$$

Define now an analytic position operator $A : L^2(D) \to L^2(D)$ by

$$Af(z) = zf(z).$$

Then $A$ is bounded, subnormal, invertible and without any (bounded) square root. In addition, $A$ does not have any bounded square root. This is utterly non trivial and was established by Halmos et al. in [14].

Assume now that there is a closable operator $B$ such that $B^2 = A$. Then

$$D(B^2) = D(A) = L^2(D) \subset D(B)$$

whereby $D(B) = L^2(D)$. The Closed Graph Theorem then tells us that $B$ is bounded and so $A$ would possess a square root, which contradicts the first part of the proof. Therefore, $A$ does not possess any closable square root. \qed

As is known to readers, there are finite square matrices which are rootless, i.e. not having any root of any order. Such is the case for instance with the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Next, we present two examples of
everywhere defined, unclosable and unbounded nilpotent operators. In other words, we supply two non-closable square roots of $0 \in B(H)$.

**Example 1.2.** Let $f : H \to \mathbb{C}$ be a *discontinuous* linear functional (where we allow $\dim H \geq \aleph_0$). Let $e$ be a normalized vector in $\ker f$. Now, define a linear operator $A$ on $H$ by $D(A) = H$ and $Ax = f(x)e$ for each $x \in H$. Then for $x \in H$

$$A^2x = A(f(x)e) = f(x)f(e)e = 0.$$  

Thus, $A^2 = 0$ everywhere on the whole of $H$. Accordingly, $A^2$ is self-adjoint! Now, $A$ cannot be closable. A way of seeing this, is that if $A$ were closable, the Closed Graph Theorem would make it bounded.

An alternative way of seeing that the operator $A$ is not closable is to invoke Proposition 4.5 of [35]. There, the writers showed that $D(A^*) = \{e\}^\perp$ (and $A^*$ is the zero operator on $\{e\}^\perp$), that is, $A^*$ is not densely defined and consequently, $A$ is not closable.

The second example is simple once readers are familiar with matrices of operators.

**Example 1.3.** Consider any unbounded non-closable operator $B$ with domain $D(B) = H$ (and so $D(B^*)$ is not dense). Then set

$$A = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$$

and so $D(A) = H \oplus H$. Clearly, $A$ is unbounded. Since

$$A^* = \begin{pmatrix} 0 & 0 \\ B^* & 0 \end{pmatrix},$$

we see that $D(A^*) = D(B^*) \oplus H$ is not dense in $H \oplus H$, making $A$ non-closable. Moreover,

$$D(A^2) = \{(x, y) \in H \times H : A(x, y) = (By, 0) \in H \times H\} = H \oplus H.$$  

Hence, we see that

$$A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(everywhere on $H \oplus H$).

**Remark.** As observed above, the foregoing two examples constitute non-closable unbounded square roots of $0 \in B(H)$. Incidentally, some compact operators may have unbounded square roots. Indeed, once we have seen one, non-zero compact operators may also have unbounded square roots. More precisely, we have:
Corollary 1.4. There are many compact operators having unbounded square roots.

Proof. Let $T$ be an unbounded square root of $0 \in B(H)$. Letting $B \in B(H)$ to be a square root of some compact operator $C$, we see that $T \oplus B$ is an unbounded square root of the non-zero compact operator $0 \oplus C$. □

Now, we treat the general case.

Theorem 1.5. Let $n \in \mathbb{N}$ be given. There are infinitely many everywhere defined non-closable unbounded operators $T$ such that $T^n = 0$ everywhere on $H$ while $T^{n-1} \neq 0$.

Proof. Let $B$ be an everywhere defined unbounded unclosable operator such that $B^2 \neq 0$. Perhaps some more details are desirable. Let $A$ be a non-closable operator such that $D(A) = H$ and $A^2 = 0$ as in Examples 1.2 & 1.3. Then set
\[ B = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \]
where $I$ is the identity operator on $H$ (hence $D(B) = H \oplus H$). It is seen that $B$ is unclosable, unbounded and
\[ B^2 = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} A^2 & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \]

Now, define
\[ T = \begin{pmatrix} 0 & B & B \\ 0 & 0 & B \\ 0 & 0 & 0 \end{pmatrix} \]
and so $D(T) = H \oplus H \oplus H$. Clearly, $T$ is unbounded and not closable. Then
\[ T^2 = \begin{pmatrix} 0 & 0 & B^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]
and
\[ T^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]
where all zeros are in $B(H)$. 
To deal with the general case, define on $H \oplus H \oplus \cdots \oplus H$ ($n$ copies of $H$) the unbounded non-closable

$$T = \begin{pmatrix}
0 & B & B & \cdots & B \\
0 & 0 & B & B & \vdots \\
0 & 0 & B & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & B \\
0 & 0 & \cdots & 0 & B \\
0 & 0 & \cdots & \cdots & 0
\end{pmatrix}.$$ 

Clearly, $D(T) = H \oplus H \oplus \cdots \oplus H$, and as above, it may be checked that

$$T^{n-1} \neq 0 \text{ whereas } T^n = 0$$

everywhere on $D(T^n) = H \oplus H \oplus \cdots \oplus H$. To obtain infinitely many of them, just change each $B$ by $\alpha B$ where $\alpha \in \mathbb{R}$, say. \qed

In an interesting preprint, I. D. Mercer [21] gave a way of constructing $n \times n$ matrices $B$ such that none of $B, B^2, \cdots, B^{n-1}$ has any zero entry yet $B^n = 0$. The general form (though not explicitly indicated in that preprint) is given by:

$$B = \begin{pmatrix}
2 & 2 & \cdots & \cdots & 2 & 1-n \\
\frac{n+2}{n} & 1 & \cdots & \cdots & 1 & -n \\
1 & n+2 & 1 & \cdots & 1 & \vdots \\
\vdots & 1 & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 1 & -n \\
1 & 1 & \cdots & 1 & n+2 & -n
\end{pmatrix}$$

By way of an example, consider the $6 \times 6$ matrix:

$$B = \begin{pmatrix}
2 & 2 & 2 & 2 & 2 & -5 \\
8 & 1 & 1 & 1 & 1 & -6 \\
1 & 8 & 1 & 1 & 1 & -6 \\
1 & 1 & 8 & 1 & 1 & -6 \\
1 & 1 & 1 & 8 & 1 & -6 \\
1 & 1 & 1 & 1 & 8 & -6
\end{pmatrix}.$$ 

Then it may be checked that $B^p \neq 0_{M_6}$ and none of their entries is null for all $1 \leq p \leq 5$, yet $B^n = 0_{M_6}$.

**Theorem 1.6.** Let $n \in \mathbb{N}$. There is a matrix of operators $A$ of size $n \times n$ whose entries are either all in $B(H)$ or all unbounded and unclosable and defined on all of $H$, such that all entries of all $A^p$ ($1 \leq p \leq n-1$) are non zero operators, yet $A^n = 0$ everywhere on $H \oplus H \oplus \cdots \oplus H$. 
Proof. The proof remains unchanged whether the entries are all in \( B(H) \) or are all unbounded, unclosable and everywhere defined on \( H \).

So let \( T \) be any linear operator defined on all of \( H \) such that \( T^p \neq 0 \) for \( 1 \leq p \leq n - 1 \) (as in Theorem 1.5). Set

\[
A = \begin{pmatrix}
2T & 2T & \cdots & \cdots & 2T & (1 - n)T \\
(n + 2)T & T & \cdots & \cdots & T & -nT \\
T & (n + 2)T & T & \cdots & T & \vdots \\
\vdots & T & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & T & -nT \\
T & T & \cdots & T & (n + 2)T & -nT
\end{pmatrix}
\]

which is defined on all \( H \oplus H \oplus \cdots \oplus H \) (\( n \) copies of \( H \)). Then none of the entries of \( A^p \) with \( 1 \leq p \leq n - 1 \) is the zero operator yet \( A^n = 0 \) everywhere on \( H \oplus H \oplus \cdots \oplus H \). \( \square \)

By borrowing an idea from \( \square \) used for bounded operators, we may give a way of finding non-closable roots of some particular non-closable operators:

**Proposition 1.7.** Let \( T \) be a non-closable operator such that \( D(T) = H \). Then \( T \oplus T \oplus \cdots \oplus T \), defined on \( H \oplus H \oplus \cdots \oplus H \) (\( n \) times), has always non-closable \( n \)th roots.

**Proof.** Let \( I \) be the identity operator on \( H \). An unclosable everywhere defined \( n \)th root of \( T \oplus T \oplus \cdots \oplus T \) is given by the \( n \times n \) matrix of operators:

\[
S := \begin{pmatrix}
0 & 0 & \cdots & 0 & T \\
I & 0 & \cdots & 0 & 0 \\
0 & I & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & I & 0
\end{pmatrix}.
\]

Indeed, \( S \) is clearly unclosable on \( D(S) = H \oplus H \oplus \cdots \oplus H \). Moreover,

\[
S^n = T \oplus T \oplus \cdots \oplus T,
\]

as wished. \( \square \)

Now, we give a non-closable operator without any closable square root.

**Proposition 1.8.** There exists a non-closable operator without any closable square root whatsoever.
Proof. Let \( A \) be a non-closable operator such that \( D(A) = H \) and \( A^2 = 0 \) everywhere on \( H \) as in Example 1.2 (or Example 1.3). Assume now that \( B \) is a closable square root of \( A \), that is, \( B^2 = A \). Hence \( B^4 = A^2 = 0 \) everywhere on \( H \). Therefore,
\[
H = D(A^2) = D(B^4) \subset D(B).
\]

This means that \( B \) would be everywhere defined on \( H \), and by remembering that \( B \) is closable, it would ensue that \( B \in B(H) \) and so \( B^2 \in B(H) \) too. Hence \( A^2 = 0 \) everywhere on \( H \). Therefore,
\[
H = D(A^2) = D(B^4) \subset D(B).
\]

This means that \( B \) would be everywhere defined on \( H \), and by remembering that \( B \) is closable, it would ensue that \( B \in B(H) \) and so \( B^2 \in B(H) \) too. Hence \( A^2 = 0 \) everywhere on \( H \). Therefore,
\[
H = D(A^2) = D(B^4) \subset D(B).
\]

This means that \( B \) would be everywhere defined on \( H \), and by remembering that \( B \) is closable, it would ensue that \( B \in B(H) \) and so \( B^2 \in B(H) \) too. Hence \( A^2 = 0 \) everywhere on \( H \). Therefore,
\[
H = D(A^2) = D(B^4) \subset D(B).
\]

This means that \( B \) would be everywhere defined on \( H \), and by remembering that \( B \) is closable, it would ensue that \( B \in B(H) \) and so \( B^2 \in B(H) \) too. Hence \( A^2 = 0 \) everywhere on \( H \). Therefore,
\[
H = D(A^2) = D(B^4) \subset D(B).
\]

This means that \( B \) would be everywhere defined on \( H \), and by remembering that \( B \) is closable, it would ensue that \( B \in B(H) \) and so \( B^2 \in B(H) \) too. Hence \( A^2 = 0 \) everywhere on \( H \). Therefore,
\[
H = D(A^2) = D(B^4) \subset D(B).
\]

This means that \( B \) would be everywhere defined on \( H \), and by remembering that \( B \) is closable, it would ensue that \( B \in B(H) \) and so \( B^2 \in B(H) \) too. Hence \( A^2 = 0 \) everywhere on \( H \). Therefore,
\[
H = D(A^2) = D(B^4) \subset D(B).
\]

This means that \( B \) would be everywhere defined on \( H \), and by remembering that \( B \) is closable, it would ensue that \( B \in B(H) \) and so \( B^2 \in B(H) \) too. Hence \( A^2 = 0 \) everywhere on \( H \). Therefore,
\[
H = D(A^2) = D(B^4) \subset D(B).
\]

This means that \( B \) would be everywhere defined on \( H \), and by remembering that \( B \) is closable, it would ensue that \( B \in B(H) \) and so \( B^2 \in B(H) \) too. Hence \( A^2 = 0 \) everywhere on \( H \). Therefore,
\[
H = D(A^2) = D(B^4) \subset D(B).
\]

This means that \( B \) would be everywhere defined on \( H \), and by remembering that \( B \) is closable, it would ensue that \( B \in B(H) \) and so \( B^2 \in B(H) \) too. Hence \( A^2 = 0 \) everywhere on \( H \). Therefore,
\[
H = D(A^2) = D(B^4) \subset D(B).
\]

This means that \( B \) would be everywhere defined on \( H \), and by remembering that \( B \) is closable, it would ensue that \( B \in B(H) \) and so \( B^2 \in B(H) \) too. Hence \( A^2 = 0 \) everywhere on \( H \). Therefore,
\[
H = D(A^2) = D(B^4) \subset D(B).
\]

This means that \( B \) would be everywhere defined on \( H \), and by remembering that \( B \) is closable, it would ensue that \( B \in B(H) \) and so \( B^2 \in B(H) \) too. Hence \( A^2 = 0 \) everywhere on \( H \). Therefore,
\[
H = D(A^2) = D(B^4) \subset D(B).
\]

This means that \( B \) would be everywhere defined on \( H \), and by remembering that \( B \) is closable, it would ensue that \( B \in B(H) \) and so \( B^2 \in B(H) \) too. Hence \( A^2 = 0 \) everywhere on \( H \). Therefore,
\[
H = D(A^2) = D(B^4) \subset D(B).
\]
Since
\[ T^* = \begin{pmatrix} I & 0 \\ A^* & -I \end{pmatrix}, \]
it ensues that
\[ T^{*2} = \begin{pmatrix} I_{D(A^*)} & 0 \\ 0 & I \end{pmatrix} \]
meaning that \( T^* \) is not a square root of \( S^* = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \). \( \square \)

What about closures? As is guessable, this is not the case either.

**Corollary 1.10.** There is a densely defined linear operator having a square root \( T \) but \( T \) is not a square root of its closure.

**Proof.** Consider the same example as before. Then, by considering \( (T^*)^\ast \) or else, it is seen that
\[ \overline{T} = \begin{pmatrix} I & \overline{A} \\ 0 & -I \end{pmatrix}. \]
Therefore,
\[ \overline{T}^2 = \begin{pmatrix} I & 0 \\ 0 & I_{D(\overline{A})} \end{pmatrix} \]
and so \( \overline{T} \) cannot be a square root of \( \overline{S} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \). \( \square \)

Non-closable operators may have closed square roots. This is maybe known to some readers, however, the approach here is different.

**Theorem 1.11.** There is a densely defined non-closable operator defined formally on \( L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \) possessing a densely defined closed square root.

**Proof.** First, consider two self-adjoint operators \( A \) and \( B \) defined on \( L^2(\mathbb{R}) \) and such that
\[ D(AB) = \{0\} \] and \( D(BA) = D(A) \)
where also \( B \in B[L^2(\mathbb{R})] \). This is highly non-trivial and most probably original. Let us proceed to obtain such a pair. Consider the operators \( C \) and \( A \):
\[ Cf(x) = e^{x^2} f(x) \]
defined on \( D(C) = \{f \in L^2(\mathbb{R}) : e^{x^2} f \in L^2(\mathbb{R})\} \) and \( A := \mathcal{F}^* C \mathcal{F} \), where \( \mathcal{F} \) designates the usual \( L^2(\mathbb{R}) \)-Fourier transform. Clearly \( C \) is boundedly invertible (hence so is \( A \)) and
\[ Bf(x) := C^{-1} f(x) = e^{-x^2} f(x) \]
is defined from $L^2(\mathbb{R})$ onto $D(C) \subset L^2(\mathbb{R})$.

We also know that $D(AB)$ is trivial if $D(A) \cap \text{ran}(B)$ is so and if $B$ is further assumed to be one-to-one (which is our case here). But,

$$D(A) \cap \text{ran}(B) = D(A) \cap D(C) = \{0\},$$

because this is already available to us from [17]. Accordingly

$$D(AB) = \{0\}.$$  

Since $B$ is everywhere defined and bounded, clearly

$$D(BA) = D(A)$$  

which is actually dense in $L^2(\mathbb{R})$.

Now, define

$$T = \begin{pmatrix} B^2 & BA \\ 0 & 0 \end{pmatrix}$$  

on $L^2(\mathbb{R}) \oplus D(A)$. Obviously, $T$ is densely defined. Since $B \in B[L^2(\mathbb{R})]$ and $A$ and $B$ are self-adjoint, it follows that

$$T^* = \left[ \begin{pmatrix} B^2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & BA \\ 0 & 0 \end{pmatrix} \right]^* = \begin{pmatrix} B^2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ AB & 0 \end{pmatrix},$$

that is,

$$T^* = \begin{pmatrix} B^2 & 0 \\ AB & 0 \end{pmatrix}.$$

Thus,

$$D(T^*) = \{0\} \oplus L^2(\mathbb{R})$$

which is not dense, i.e. $T$ is not closable.

Let us now exhibit a densely defined closed square root of $T$. Let

$$R = \begin{pmatrix} B & A \\ 0 & 0 \end{pmatrix}$$

be defined on $D(R) := L^2(\mathbb{R}) \oplus D(A)$. Then $R$ is closed on $D(R)$. In addition,

$$R^2 = \begin{pmatrix} B^2 & BA \\ 0 & 0 \end{pmatrix}$$

because

$$D(R^2) = \{ (f, g) \in L^2(\mathbb{R}) \times D(A) : (Bf + Ag, 0) \in L^2(\mathbb{R}) \times D(A) \}$$

$$= L^2(\mathbb{R}) \oplus D(A)$$

$$= D(T).$$

□
S. Ôta [29] introduced the concept of an unbounded projection or idempotent. Recall that if $T$ is a non necessarily bounded operator with a dense domain $D(T)$, then $T$ is said to be idempotent if $T^2$ is well defined and

$$T^2 = T \text{ on } D(T).$$

S. Ôta gave an example of an unclosable idempotent but did not show any closed idempotent operator. Here we give a different example of a non-closable idempotent as well as a closed idempotent. These two examples are in a close relationship to the main topic of the paper.

**Proposition 1.12.** There are non-closable unbounded idempotent operators as well as closed ones.

**Proof.** Let $A$ be an unbounded closed operator with domain $D(A) \subset H$ and let $I$ be the identity operator on all of $H$. Set

$$T = \begin{pmatrix} I & A \\ 0 & 0 \end{pmatrix}$$

and so $D(T) = H \times D(A)$. Then $T$ is densely defined, closed and unbounded. Since

$$D(T^2) = \{(x, y) \in H \times D(A) : (x + Ay, 0) \in H \times D(A)\} = D(T),$$

we see that

$$T^2 = \begin{pmatrix} I & A \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & A \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & A \\ 0 & 0 \end{pmatrix} = T.$$

In other words $T$ is idempotent. Once we have seen one example, others come to mind (e.g. $A$ may be replaced by $\alpha A$ say where $\alpha \neq 0$). For example, let $T$ be such that $T^2 = T$. If $U$ is unitary, then $U^*TU$ too is a densely defined closed idempotent. The density of $D(U^*TU)$ is easily seen. Since $TU$ is closed and $U^*$ is invertible, it follows that $U^*TU$ remains closed. Finally, observe that

$$(U^*TU)^2 = U^*TUU^*TU = U^*T^2U = U^*TU.$$

A similar idea is used to the non closable case. Indeed, define

$$T = \begin{pmatrix} I & A \\ 0 & 0 \end{pmatrix}$$

on $D(T) = H \times D(A)$ where this time $A$ is not closable. Then $T$ too is not closable. Indeed, there is a sequence $(x_n)$ in $D(A)$ such that $x_n \to 0$, $Ax_n \to y$ yet $y \neq 0$ (by the non closability of $A$). Next, $(0,x_n) \in D(T)$, $(0,x_n) \to (0,0)$ and $T(0,x_n) = (Ax_n,0) \to (y,0) \neq (0,0)$. This proves the non closability of $T$. That $T^2 = T$ may be checked as above. Therefore, $T$ is a densely defined non closable idempotent operator. $\square$
Now, we treat some related results to nilpotence and invertibility. Let $N \in B(H)$ be nilpotent and let $I \in B(H)$ be the identity operator. Then, it is known that $I \pm N$ are invertible. For example, the inverse of $I - N$ is given by $I + N + N + \cdots + N^p$ if $p + 1$ is the index of nilpotence of $N$.

What about unbounded nilpotent operators?

**Proposition 1.13.** There exist closed as well as non closable nilpotent unbounded operators $N$ such that $I + N$ is not boundedly invertible.

**Proof.** We start with the case of non closable nilpotent operators. Let $N$ be an unbounded non closable operator such that $D(N) = H$ and $N^2 = 0$ everywhere on $H$. Then, $I + N$ cannot be boundedly invertible for it were, it would ensue that $(I + N)^2$ too is boundedly invertible. However,

$$(I + N)^2 = I + 2N + N^2 = I + 2N$$

(all full equalities) is not even closable while boundedly invertible operators must be closed.

Consider now the case of a closed nilpotent operator. The simplest example to think of perhaps is:

$$N = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$$

defined on $D(N) = H \oplus D(A)$, where $A$ is an unbounded closed operator with domain $D(A)$. If $I_H \oplus H$ is the identity on $H \oplus H$, then

$$I_H \oplus H + N = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix}$$

is not boundedly invertible as it is not surjective (observe that it is injective). \hfill \square

**Remark.** In both cases above, $I + N$ are invertible.

Let us remain in the context of nilpotence a little longer. Recall that if $N \in B(H)$ is nilpotent, then

$$\sigma(N) = \{0\}.$$

Such is not always the case in case of unbounded closed operators.

**Proposition 1.14.** There are nilpotent closed operators $N$ such that $\sigma(N) \neq \{0\}$.

**Proof.** Let $T$ be any (unbounded) closed operator with a domain $D(T) \subset H$. Set $N = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}$ with $D(N) = H \oplus D(T)$. 

Since $N$ is closed, we know that $\sigma(N^2) = [\sigma(N)]^2$. But,

$$N^2 = \begin{pmatrix} 0 & 0_{D(T)} \\ 0 & 0 \end{pmatrix}$$

with $D(N^2) = D(N)$ and so $N$ is nilpotent. Since $N^2$ is clearly unclosed, it results that $\sigma(N^2) = \mathbb{C}$. If $\sigma(N) = \{0\}$, then we would have $[\sigma(N)]^2 = \{0\}$ as well, and this is absurd. Therefore, $\sigma(N) \neq \{0\}$, as needed. □

It is known that if $A, N \in B(H)$ are such that $AN = NA$ and $N$ is nilpotent, then $\sigma(A + N) = \sigma(A)$ (see e.g. Exercise 7.3.29 in [23] for an interesting proof).

Is the previous result valid in the context of one unbounded operator? Since commutativity in this case means $NA \subset AN$, we need to treat the case of the nilpotence of $N$ as well as that of $A$.

First, we have:

**Theorem 1.15.** Let $N \in B(H)$ be nilpotent and let $A$ be a densely defined closed operator such that $NA \subset AN$. Then

$$\sigma(A + N) = \sigma(A).$$

For the proof, we must fall back on the following auxiliary result:

**Theorem 1.16.** ([1]) If $B \in B(H)$ and commutes with an unbounded $A$, i.e. $BA \subset AB$, then

$$\sigma(A + B) \subset \sigma(A) + \sigma(B)$$

holds.

**Remarks.**

(1) In fact, the writers in [1] established the above result under the condition $\sigma(A) \neq \mathbb{C}$ which was also imposed for other subsequent results. However, the inclusion $\sigma(A + B) \subset \sigma(A) + \sigma(B)$ is trivial when $\sigma(A) = \mathbb{C}$.

(2) This result generalizes a well known result stating that if $A$ and $B$ are in $B(H)$ and $AB = BA$, then $\sigma(A + B) \subset \sigma(A) + \sigma(B)$ holds.

(3) How about the case of two unbounded operators? Without digging too much into the difficult notion of strong commutativity, we give a simple example by assuming readers have the necessary means to understand it. Let $A$ be an unbounded self-adjoint operator with domain $D(A)$ and such that $\sigma(A) = \mathbb{R}$, and let $B = -A$. Then $A$ commutes strongly with $B$ yet

$$\sigma(A + B) \not\subset \sigma(A) + \sigma(B)$$
because \(A + B\) is unclosed and hence \(\sigma(A + B) = \mathbb{C}\) whereas \(\sigma(A) + \sigma(B) = \mathbb{R}\).

So much for the digression, now we prove Theorem 1.15.

**Proof.** The proof is not difficult. By Theorem 1.16, we know that:

\[
\sigma(A + N) \subset \sigma(A) + \sigma(N) = \sigma(A).
\]

Conversely,

\[
\sigma(A) = \sigma(A + N - N) \subset \sigma(A + N) + \sigma(-N) = \sigma(A + N)
\]

for \(N\) is nilpotent and commutes with \(A + N\). Therefore,

\[
\sigma(A + N) = \sigma(A).
\]

\(\square\)

What about the case when the nilpotent operator is the unbounded one?

**Proposition 1.17.** There are linear operators \(A\) and \(N\), where \(A \in \mathcal{B}(H)\) and \(A\) is densely defined and closed, obeying \(NA \subset AN\) and yet

\[
\sigma(A + N) \neq \sigma(A).
\]

**Proof.** The simplest example is to take \(A = 0\) and \(N\) as in Proposition 1.14. Then trivially \(AN \subset NA\) holds. Besides, \(\sigma(N) \neq \{0\}\). Hence

\[
\sigma(A + N) = \sigma(N) \neq \{0\} = \sigma(A).
\]

Another "richer" example is based on one which appeared in [15] with a different aim. Let \(T\) be a closed, unbounded and boundedly invertible operator with domain \(D(T) \subset H\). Define on \(H \oplus H\)

\[
A = \begin{pmatrix} I & T \\ 0 & T \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} I & 0 \\ 0 & T^{-1} \end{pmatrix}
\]

with \(D(A) = H \oplus D(T)\) and \(D(B) = H \oplus H\). Then \(A\) is closed on \(D(A)\) and \(B\) is everywhere defined and bounded on \(H \oplus H\). Then

\[
BA = \begin{pmatrix} I & T \\ 0 & I \end{pmatrix}
\]

and \(\sigma(BA) = \mathbb{C}\) (see [15] for further details).

Now, write

\[
\begin{pmatrix} I & T \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} = \tilde{I} + N.
\]

Accordingly,

\[
\sigma(\tilde{I} + N) = \mathbb{C} \neq \{1\} = \sigma(\tilde{I}).
\]
yet $\tilde{I}$ is everywhere defined and bounded, and it plainly commutes with $N$. □

Now, we give some more results as regards invertibility.

**Theorem 1.18.** Let $T$ be a non necessarily bounded, closed and densely defined operator on a Hilbert space $H$. Assume that $T$ has a densely defined and closed square root $S$, that is, $S^2 = T$. Then $T$ is boundedly invertible if and only if $S$ is boundedly invertible. In such case, $S^{-1}$ is always a square root of $T^{-1}$.

Proof. If $S$ has an everywhere defined bounded inverse, then $S^2$ or $T$ too has an everywhere defined bounded inverse.

Conversely, assume that $T$ is boundedly invertible. Since $T$ is one-to-one and $S^2 = T$, it follows that $S$ is also one-to-one. By passing to adjoints and taking into account the "one-to-oness" of $T^*$, we easily see that $S^*$ is one-to-one. Now, clearly

$$S^2 = T \implies S(ST^{-1}) = S^2 T^{-1} = I$$

where $I$ the identity on $H$. The aim is to show that $ST^{-1} \in B(H)$. By the general theory, $ST^{-1}$ is closed for $T^{-1} \in B(H)$. Besides,

$$H = D(S^2 T^{-1}) \subset D(ST^{-1})$$

and so

$$D(ST^{-1}) = H.$$ 

By the Closed Graph Theorem, $ST^{-1}$ is in effect in $B(H)$. Therefore, $S$ is right invertible. As $\ker S = \ker(S^*)$, then Theorem 2.3 in [10] tells us that $S$ is (fully) invertible, marking the end of the proof. □

A related result is the following:

**Proposition 1.19.** Let $T$ and $S$ be two densely defined linear operators such that $S^2 = T$. If $T$ is right invertible, so is $\overline{S}$ whenever $S$ is closable.

Proof. Since $S^2 = T$ and $S \subset \overline{S}$, it follows that $T \subset \overline{S}^2$. By the right invertibility of $T$, we obtain $I \subset \overline{S}^2 B$ for some $B \in B(H)$. That is,

$$\overline{S} \overline{S} B = \overline{S}^2 B = I.$$ 

Since $\overline{S} B$ is closed and $H = D(\overline{S} B)$, clearly $\overline{S} B \in B(H)$. Therefore, $\overline{S}$ is right invertible. □

Remark. The converse being untrue as seen by taking $T = S = I_D$ (the identity restricted to some domain $D$). Then $S^2 = T$ yet $\overline{S} = I$ is right invertible whilst $T$ is not.

The next result is easily shown and so we omit its proof.
**Theorem 1.20.** Let $S$ and $T$ be two linear operators such that $S^2 = T$. If $S$ is right invertible, so is $T$. If $T$ is right invertible, so is $S$ if $S$ is closable. If $S$ is left invertible, then so is $T$.

Before stating and proving a result about normal and self-adjoint square roots, we give some auxiliary result whose proof is very simple and so it is omitted. It is worth noticing in passing that there are unbounded self-adjoint operators $A$ and $B$ such that $A + iB \subset 0$ (where 0 designates the zero operator on all of $H$), yet $A \not\subset 0$ and $B \not\subset 0$. For example, let $A$ and $B$ be unbounded self-adjoint operators such that $D(A) \cap D(B) = \{0_H\}$ (see e.g. [17]). Assuming $D(A) = D(B)$ makes the whole difference. Indeed:

**Proposition 1.21.** Let $A$ and $B$ be two densely defined symmetric operators with domains $D(A), D(B) \subset H$ respectively. Assume that $D(A) = D(B)$. If $A + iB \subset 0$, then $A \subset 0$ and $B \subset 0$. If $A$ (or $B$) is further taken to be closed, then $A = B = 0$ everywhere on $H$.

The following result generalizes one in [12].

**Theorem 1.22.** Let $T = A + iB$ where $A$ and $B$ are self-adjoint (one of them is also positive), $D(A) = D(B)$ and $D(AB) = D(BA)$. If $T^2 = S$, where $S$ is symmetric, then $T$ is normal. In particular, if $S$ is self-adjoint and positive, then $T$ is self-adjoint and positive, i.e. $T$ is the unique square root of $S$.

**Proof.** Assume that $A$ is positive (the proof in the case of the positivity of $B$ is similar). Let $T = A + iB$. Clearly,

$$A^2 - B^2 + i(AB + BA) \subset (A + iB)A + i(A + iB)B = T^2 \subset S$$

thereby

$$A^2 - B^2 - S + i(AB + BA) \subset 0.$$ 

Since $D(A) = D(B)$, it is seen that $D(A^2) = D(BA)$ and that $D(B^2) = D(AB)$. Thus,

$$D(A^2 - B^2) = D(AB + BA).$$

Since $D(AB) = D(BA)$, we have

$$D(A^2 - B^2 - S) = D(AB + BA) = D(A^2) = D(B^2).$$

Since $A$ is self-adjoint, so is $A^2$ and in particular $A^2$ is necessarily densely defined. Thus, $A^2 - B^2$ and $AB + BA$ are both densely defined. Now, by the symmetricity (only) of both $A$ and $B$ we have

$$AB + BA \subset A^*B^* + B^*A^* \subset (BA)^* + (AB)^* \subset (AB + BA)^*. $$
Similarly, \( A^2 - B^2 \subset (A^2 - B^2)^* \). Therefore, both \( AB + BA \) and \( A^2 - B^2 \) are symmetric. By Proposition \[22\], we get \( AB + BA \subset 0 \). Hence \( AB = -BA \) (for \( D(AB) = D(BA) \)) and so

\[
A^2 B = -ABA = BA^2.
\]

As \( A \) is positive, we obtain \( AB = BA \) by \[3\]. Hence \( AB + B = BA + B \). But \( AB + B = (A + I)B \) and \( BA + B \subset B(A + I) \). Hence \( (A + I)B \subset B(A + I) \). But

\[
D[B(A + I)] = \{ x \in D(A) : Ax + x \in D(B) \}.
\]

So, if \( x \in D[B(A + I)] \), it follows that \( x \in D(A) = D(B) \) and

\[
Ax = Ax + x - x \in D(B),
\]

i.e. \( Ax \in D(B) \), i.e. \( x \in D(BA) \). Since \( D(AB) = D(BA) \), we equally have \( x \in D(AB) = D[(A + I)B] \). This actually means that

\[
(A + I)B = B(A + I).
\]

Since \( A \) is self-adjoint and positive, it results that \( A + I \) is boundedly invertible. Right and left multiplying by \( (A + I)^{-1} \) yield \( (A + I)^{-1}B \subset B(A + I)^{-1} \). By Proposition 5.27 in \[32\], this means that \( A \) commutes strongly with \( B \). Accordingly \( T \) is normal.

Finally, we show the last statement. Assume that \( S \) is self-adjoint and positive (remember that \( T \) is still normal). Let \( \lambda \in \sigma(T) \). Then

\[
\lambda^2 \in [\sigma(T)]^2 = \sigma(T^2) = \sigma(S).
\]

That is, \( \lambda^2 \geq 0 \) and so the only possible outcome is \( \lambda \in \mathbb{R} \). Therefore, \( T \) is self-adjoint. Since in this case

\[
0 \leq A = \text{Re} T = \frac{T + T^*}{2} = T,
\]

it follows that \( T \) is also positive. This marks the end of the proof. \( \square \)

**Corollary 1.23.** Let \( T = A + iB \) where \( A \) and \( B \) are self-adjoint (one of them is also positive) where \( D(A) = D(B) \). If \( T^2 = 0 \) on \( D(T) \), then \( T \in B(H) \) is normal and so \( T = 0 \) everywhere on \( H \).

**Proof.** What prevents us a priori from using Theorem \[22\] is that the condition \( D(AB) = D(BA) \) is missing. But, writing \( A = (T + T^*)/2 \) and \( B = (T - T^*)/2i \) (and so \( D(T) \subset D(T^*) \)), we see that if \( x \in D(T) \), then

\[
Tx + T^*x \in D(T) \iff Tx - T^*x \in D(T)
\]

for \( Tx \in D(T) \) (because \( D(T^2) = D(T) \)). In other language, \( D(AB) = D(BA) \), as needed. \( \square \)
It is shown in ([10], Theorem 9.4) that if $A$ and $B$ are two self-adjoint positive operators with domains $D(A)$ and $D(B)$ respectively, then

$$D(A) = D(B) \implies D(\sqrt{A}) = D(\sqrt{B}).$$

It is therefore natural to wonder whether this property remains valid for arbitrary square roots? That is, if $A$ and $B$ are square roots of some $S$, i.e. $A^2 = B^2 = S$, is it true that $D(A) = D(B)$?

**Theorem 1.24.** There are square roots of self-adjoint operators $S$ having different domains. However, if the square roots are self-adjoint then they necessarily have equal domains.

**Proof.** Let $T$ be any unbounded self-adjoint operator with domain $D(T) \subsetneq H$ and set $S = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$, where $D(S) = D(T) \oplus D(T)$. Then both $A := \begin{pmatrix} 0 & T \\ I & 0 \end{pmatrix}$ and $B := \begin{pmatrix} 0 & I \\ T & 0 \end{pmatrix}$ are square roots of $S$ yet

$$D(A) = H \oplus D(T) \neq D(T) \oplus H = D(B).$$

By taking $T$ to be further positive, it is seen that $\begin{pmatrix} \sqrt{T} & 0 \\ 0 & \sqrt{T} \end{pmatrix}$ (where $\sqrt{T}$ represents here the unique positive square root of $T$) is yet another square root of $S$ whose domain is different from both $D(A)$ and $D(B)$.

To deal with the second statement, remember first that if $T$ is closed and densely defined, then $D(T) = D(|T|)$. Now, let $A$ and $B$ be two self-adjoint square roots of some (necessarily self-adjoint and positive) $S$, i.e. $A^2 = B^2 = S$. Then $D(A^2) = D(B^2)$ and so

$$D(A) = D(|A|) = D(\sqrt{A^2}) = D(\sqrt{B^2}) = D(|B|) = D(B),$$

as needed. \qed

**Remark.** It is well known that if $S$ is a positive self-adjoint which commutes with some $R \in B(H)$, i.e. $RS \subset SR$, then $R\sqrt{S} \subset \sqrt{SR}$ where $\sqrt{S}$ designates the unique positive self-adjoint square root of $S$. See e.g. [34] for a new proof.

What about arbitrary roots? The answer is again negative. For instance, take again $S = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$ as in the previous proof and set $U = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. Hence $U \in B(H \oplus H)$, in fact $U$ is a fundamental symmetry (it is both self-adjoint and unitary). Then $US \subset SU =$
\[
\begin{pmatrix}
0 & T \\
T & 0
\end{pmatrix}
\]. However, \(U\) does not commute with \(A\) for
\[
UA = \begin{pmatrix}
I & 0 \\
0 & T
\end{pmatrix}
\text{ while } AU = \begin{pmatrix}
T & 0 \\
0 & I
\end{pmatrix}.
\]

**Remark.** In fact, the previous question does not even hold on finite dimensional spaces. Just consider:
\[
S = \begin{pmatrix}
a & 0 \\
0 & a
\end{pmatrix}, \quad U = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix}
0 & a \\
1 & 0
\end{pmatrix}
\]
where \(a \in \mathbb{C}\).

**Proposition 1.25.** Let \(A\) and \(B\) be two (closed) quasinormal operators such that \(A^2 = B^2\). Then \(D(A) = D(B)\).

**Proof.** For the definition of quasinormality in the unbounded case, we refer readers to [16] or [39]. From either of the previous two references, we know that if \(T\) is a (closed) quasinormal operator, then \(|T|^n = |T^n|\) for any \(n \in \mathbb{N}\).

Since \(A\) and \(B\) are quasinormal, we have
\[
A^2 = B^2 \implies |A|^2 = |A^2| = |B^2| = |B|^2.
\]
Upon passing to the unique positive self-adjoint square root implies that \(|A| = |B|\). Hence \(D(A) = D(B)\) by the closedness of both \(A\) and \(B\). \(\square\)

It is unknown to me whether the previous result is valid for the weaker classes of subnormal (see e.g. [19] for its definition) or hyponormal closed operators. Recall that a densely defined operator \(A\) with domain \(D(A)\) is called hyponormal if
\[
D(A) \subset D(A^*) \text{ and } \|A^*x\| \leq \|Ax\|, \forall x \in D(A).
\]
However, we have:

**Proposition 1.26.** Let \(A\) and \(B\) be two (closed) hyponormal operators such that \(A^2 = B^2\). Assume further that \(A^2\) is self-adjoint. Then \(D(A) = D(B)\).

The proof relies on the following lemma:

**Lemma 1.27.** If \(A\) is a closed hyponormal operator such that \(A^2\) (resp. \(-A^2\)) is positive, then \(A\) is self-adjoint (resp. skew-adjoint, i.e. \(A^* = -A\)).
Proof. In view of the proof of Theorem 8 in [9], closed hyponormal operators having a real spectrum are self-adjoint. This result may be used to show that closed hyponormal operators with a purely imaginary spectrum are skew-adjoint. Indeed, let $A$ be a closed hyponormal operator such that $\sigma(A)$ is purely imaginary. Then set $B = iA$ and so $B$ remains hyponormal. Hence $\sigma(B) \subset \mathbb{R}$ since by hypothesis $\sigma(A) \subset i\mathbb{R}$. Thus $B$ is self-adjoint, i.e.
\[-iA^* = B^* = B = iA,
\]i.e. $A$ is clearly skew-adjoint.

Now, let $\lambda \in \sigma(A)$. Since $A$ is closed, we have that $\lambda^2 \in \sigma(A^2)$, i.e. $\lambda^2 \geq 0$ as $A^2$ is positive. But, this forces $\lambda$ to be real. Accordingly, $A$ is self-adjoint. When $-A^2$ is positive, it may be shown that $A$ is skew-adjoint, and the proof is over. \(\square\)

Now we prove Proposition 1.26:

**Proof.** Since $A^2 = B^2$ are self-adjoint and $A$ and $B$ are hyponormal, Lemma 1.27 says that $A$ and $B$ are self-adjoint or skew-adjoint. In all possible cases, we may obtain $D(A) = D(B)$. \(\square\)

What about $D(A) = D(B) \implies D(A^2) = D(B^2)$? This is not true even when $A$ and $B$ are self-adjoint. Let us give a counterexample.

**Proposition 1.28.** There exist unbounded self-adjoint positive operators $A$ and $B$ such that $D(A) = D(B)$ yet $D(A^2) \neq D(B^2)$.

**Proof.** There could be simpler counterexamples, but here we may construct lots of them. Indeed, let $T$ be a closed and densely defined operator such that $D(T) = D(T^*)$ but $D(TT^*) \neq D(T^*T)$.

Let $A$ be a densely defined closed and unbounded operator with domain $D(A)$ such that $D(A) = D(A^*) \subset H$. Define $T$ on $H \oplus H$ by
\[
T = \begin{pmatrix} A & I \\ 0 & 0 \end{pmatrix}
\]
with domain $D(T) = D(A) \oplus H$. It is plain that $T$ is closed.

\[
T^* = \left[ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \right]^* = \begin{pmatrix} A^* & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} = \begin{pmatrix} A^* & 0 \\ I & 0 \end{pmatrix}.
\]

Since $D(A) = D(A^*)$, it results that $D(T) = D(T^*)$. In addition
\[
D(TT^*) = \{(x, y) \in D(A) \times H : (A^* x, x) \in D(A) \times H \} = D(AA^*) \times H
\]
and also

\[ D(T^*T) = \{(x, y) \in D(A) \times H : (Ax + y, 0) \in D(A^*) \times H\}. \]

To see explicitly why \( D(TT^*) \neq D(T^*T) \), let \( \alpha \) be in \( H \) such that \( \alpha \notin D(A^*) \). If \( x_0 \in D(AA^*) \subset D(A^*) = D(A) \), then clearly \(-Ax_0 \in H\). Set \( y_0 = -Ax_0 + \alpha \). Then \((x_0, y_0) \in D(AA^*) \times H = D(TT^*)\). Nonetheless, \((x_0, y_0) \notin D(T^*T)\) for \( Ax_0 + y_0 = Ax_0 - Ax_0 + \alpha = \alpha \notin D(A^*) \).

To finish the proof, observe that \( TT^* \) and \( T^*T \) are both self-adjoint and positive. In particular, \(|T|\) and \(|T^*|\) are both self-adjoint. Moreover,

\[ D(|T|) = D(T) = D(T^*) = D(|T^*|). \]

However,

\[ D(|T|^2) = D(T^*T) \neq D(TT^*) = D(|T^*|^2), \]

as needed. \(\Box\)

**Remark.** We are aware now that \( D(A) = D(B) \) does not entail \( D(A^2) = D(B^2) \) even when \( A \) and \( B \) are self-adjoint. It is worth noting that the condition \( D(A) = D(B) \) does not even have to imply that \( D(A^2 - B^2) \) (or \( D(AB + BA) \)) is dense (cf. Theorem 1.22). Before giving a counterexample, we give a simple lemma:

**Lemma 1.29.** Let \( A \) and \( B \) two linear operators such that \( D(A) = D(B) \). Then

\[ D(AB + BA) = D(A^2 - B^2) \subset D[(A - B)^2] \text{ or } D[(A + B)^2]. \]

**Proof.** Write

\[ A^2 - B^2 + AB - BA \subset (A + B)(A - B). \]

Since \( D(A) = D(B) \), it follows that \( D(A^2) = D(BA) \) and that \( D(B^2) = D(BA) \). Hence

\[ D(AB + BA) = D(A^2 - B^2) \subset D[(A + B)(A - B)]. \]

But

\[ D[(A + B)(A - B)] = D[(A - B)^2] \]

for \( D(A + B) = D(A - B) \). The other inclusion can be shown analogously. \(\Box\)

Now, we give the promised counterexample.

**Corollary 1.30.** There are self-adjoint positive unbounded operators \( A \) and \( B \) such that \( D(A) = D(B) \) yet neither \( A^2 - B^2 \) nor \( AB + BA \) is densely defined.
Proof. First, observe that $D(A) = D(B)$ yields $D(AB + BA) = D(A^2 - B^2)$. So, it suffices to exhibit $A$ and $B$ with the claimed properties such that $D(A^2 - B^2)$ is not dense.

Consider a closed densely defined positive symmetric operator $T$ such that $D(T^2) = \{0\}$ (as in e.g. [6]), then set $A = T/2 + |T|$ and $B = |T|$. That $A$ and $B$ are positive is plain. Also, $D(A) = D(B)$ and $B$ is self-adjoint. As for the self-adjointness of $A$ one needs to call on the Kato-Rellich theorem (see e.g. [10]).

By Lemma 1.29 if $A^2 - B^2$ were densely defined, so would be $D[(A - B)^2]$. However,

$$D[(A - B)^2] = D(T^2) = \{0\},$$

and so $A^2 - B^2$ is not densely defined. \qed

Let us pass now to unclosable square (or other types of) roots of the identity operator $I : H \rightarrow H$.

**Proposition 1.31.** There exists an everywhere defined non-closable unbounded operator $T$ such that

$$T^2 = I.$$ 

**Proof.** Let $A$ be a non-closable unbounded operator defined on all of $H$ such that $A^2 = 0$ everywhere. Then, set

$$T = \begin{pmatrix} A & I \\ I & -A \end{pmatrix},$$

which is defined fully on $H \oplus H$. Then $T$ is unclosable and besides $D(T^2) = H \oplus H$. Since $A - A = 0$ and $A^2 = 0$ both everywhere on $H$, we may write

$$T^2 = \begin{pmatrix} A & I \\ I & -A \end{pmatrix} \begin{pmatrix} A & I \\ I & -A \end{pmatrix} = \begin{pmatrix} A^2 + I & A - A \\ A - A & A^2 + I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

i.e. $T^2 = I_{H \oplus H}$, as needed. \qed

**Remark.** The equation $T^2 = I$ says that $T : H \rightarrow H$ is a bijective or invertible (not boundedly though) non-closable operator which is everywhere defined.

**Corollary 1.32.** There are two everywhere defined unbounded non-closable operators $A$ and $B$ such that $AB = BA = I$ everywhere on some Hilbert space $K$, that is,

$$ABx = BAx = x, \ \forall x \in K.$$
Proof. From Proposition 1.31 we have a non-closable operator $T$ such that $T^2 = I$ everywhere on $H \oplus H$. Setting

$$A = \begin{pmatrix} 0 & T \\ I & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & I \\ T & 0 \end{pmatrix},$$

which are everywhere defined on $H \oplus H \oplus H \oplus H$, we see that

$$AB = \begin{pmatrix} T^2 & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & T^2 \end{pmatrix} = BA$$

everywhere.

Let us give a second example. Let $T$ be any unbounded non-closable everywhere defined operator on $H$ and let

$$A = \begin{pmatrix} I & T \\ 0 & I \end{pmatrix}.$$

It is seen that $A$, which is defined on all of $H \oplus H$, is bijective and so it is invertible (not boundedly) with an inverse given by $B = \begin{pmatrix} I & -T \\ 0 & I \end{pmatrix}$ for

$$AB = BA = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$ 

\[\square\]

Remark. Let $A, B, T$ be all everywhere defined and not closable. Note by $I$ the identity operator which need not act on the same space in each case. The existence of a $T$ such that $T^2 = I$ gave rise to two different operators $A$ and $B$ such that $AB = BA = I$.

Conversely the availability of a pair of two different operators $A$ and $B$ such that $AB = BA = I$ in turn leads to $T^2 = I$. This is easily seen by taking

$$T = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$$

which is defined on $D(T) := D(B) \oplus D(A) = H \oplus H$. Hence

$$T^2 = \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

as desired.

Remark. If we have a non-closable operator $T$ such that $D(T) = H$ and $T^2 = I$, then we can always manufacture a non-closable $S$ such that $S^2 = 0$. Just consider

$$S = \begin{pmatrix} I & T \\ -T & -I \end{pmatrix}.$$
on $D(S) = H \oplus H$. Then

$$S^2 = \begin{pmatrix} I - T^2 & T - T^2 \\ -T + T & -T^2 + I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

everywhere on $H \oplus H$ as all the resulting operations are carried out on all of $H$.

As alluded above, boundedly invertible operators are necessarily closed while invertible operators might even be unclosable in some cases (as when $T^2 = I$). What about the weaker notion of left or right invertibility?

**Theorem 1.33.** There is a left (resp. right) invertible operator which is not closed.

*Proof.* The simplest example in the left invertibility case is to restrict the identity operator on $H$ (noted $I_H$) to some non-closed domain $D \subset H$ and denote this restriction by $I_D$. Then $I_D$ is left invertible for $I_HI_D = I_D \subset I_H$.

Since $I_D$ is bounded on a non-closed domain, it follows that $I_D$ is unclosed.

As for the right invertibility case, there is an example of such an $A$ which is even everywhere defined in $H$ (there might not be any more explicit one). Start with $B$ in $B(H)$ such that its range $\text{ran}(B)$ is dense but is not all of $H$. Let $E$ be a linear subspace of $H$ which is complementary to $\text{ran}(B)$ in the algebraic sense (i.e. $\text{ran}(B) + E = H$, without taking closure, while the intersection is $\{0\}$). Then define $A$ on $\text{ran}(B)$ by

$$ABx = x,$$

and define $A$ on $E$ to be an arbitrary linear mapping of $E$ to $H$. $A$ then extends by linearity to all of $H$, and $AB = I$, but $A$ is not bounded (as it is not bounded on $\text{ran}(B)$ as if it were, then $\text{ran}(B)$ would be closed), so it cannot be closable.

We have given a way of finding everywhere defined bijective operators. A similar ideas applies to injectivity and surjectivity independently.

**Proposition 1.34.** There is an everywhere defined unbounded operator which is injective but not surjective, and there is an everywhere defined unbounded operator which is surjective but not injective.

*Proof.* Let $T$ be an everywhere defined operator such that $T^2 = I$. Hence $T$ is bijective.
(1) Let \( S \in B(H) \) be any injective operator which is not surjective. Set
\[
A := T \oplus S = \begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix},
\]
and so \( D(A) = H \oplus H \). Then \( A \) is unbounded and not closable. That \( A \) is injective is plain. As \( \text{ran} \, S \neq H \), it results that
\[
\text{ran} \, A = H \oplus \text{ran} \, S \neq H \oplus H,
\]
that is, \( A \) is not surjective.

(2) Consider a surjective \( R \in B(H) \) which is not injective. Then
\[
B := \begin{pmatrix} T & 0 \\ 0 & R \end{pmatrix}
\]
is unbounded, \( D(B) = H \oplus H \), \( \text{ran} \, B = H \oplus H \) and
\[
\text{ker} \, B \neq \{(0, 0)\},
\]
as needed.

Now, we deal with the general case. First, we provide a finite dimensional example:

**Example 1.35.** Let \( n \in \mathbb{N} \) be given. There is an \( n \times n \) matrix such that \( A^n = I \) with \( A^{n-1} \neq I \) (in fact, \( A^p \neq I \) for \( p = 1, 2, \cdots, n-1 \)).

There are many types of counterexamples. The simplest one is to take the following circulant permutation \( n \times n \) matrix
\[
A = \begin{pmatrix}
0 & 1 & 0 & \cdots & \cdots & 0 \\
0 & 0 & 1 & 0 & \ddots & \\
\vdots & 0 & 1 & \ddots & \ddots & \\
\vdots & & \ddots & \ddots & \ddots & 0 \\
0 & & & 0 & 1 & \\
1 & 0 & \cdots & \cdots & 0 & 0
\end{pmatrix}
\]
where the corresponding permutation being \( p(i) = i + 1 \). Then it is well known that \( A^n = I \) and \( A^p \neq I \) for \( p = 1, 2, \cdots, n-1 \).

In order to carry over this type of examples to matrices of unbounded operators, we need to place some parameter inside the previous matrix, and still obtain the same conclusions. So, a more general form of the
previous example reads:

\[
A = \begin{pmatrix}
0 & 1 & \alpha & \cdots & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & \\
\vdots & 0 & 1 & \ddots & \vdots & \\
\vdots & \ddots & \ddots & \ddots & 0 & \\
0 & \cdots & \cdots & \cdots & \cdots & 0 \\
1 & -\alpha & \cdots & \cdots & 0 & 0 \\
\end{pmatrix},
\]

where it can be again checked that \(A^n = I\) and that \(A^p \neq I\) for \(p = 1, 2, \ldots, n-1\) (all that holding for any \(\alpha\)).

**Theorem 1.36.** Let \(n \in \mathbb{N}\) be given. There are infinitely many everywhere defined non-closable unbounded operators \(T\) such that \(T^n = I\) everywhere on some Hilbert space while \(T^p \neq I\) for \(p = 1, 2, \ldots, n-1\).

**Proof.** Let \(A\) be a non-closable unbounded operator which is everywhere defined, i.e. \(D(A) = H\) and let \(I \in B(H)\) be the identity operator. Inspired by the example above, let

\[
T = \begin{pmatrix}
0 & I & A & 0 & \cdots & 0 \\
0 & 0 & I & 0 & \cdots & \\
\vdots & 0 & I & \ddots & \vdots & \\
\vdots & \ddots & \ddots & \ddots & 0 & \\
0 & \cdots & \cdots & \cdots & \cdots & 0 \\
I & -A & 0 & \cdots & 0 & 0 \\
\end{pmatrix}
\]

be defined on \(D(T) = H \oplus H \oplus \cdots \oplus H\) (\(n\) times). This means that \(T\) is everywhere defined. Notice also that \(T\) is clearly unbounded and not closable.

Readers may check that \(T^n = I\) on \(D(T^n) = H \oplus H \oplus \cdots \oplus H\) whereas \(T^p \neq I\) for \(p = 1, 2, \ldots, n-1\). As an illustration, we treat the special case \(n = 3\). In this case,

\[
T = \begin{pmatrix}
0 & I & A \\
0 & 0 & I \\
I & -A & 0 \\
\end{pmatrix}.
\]

Then

\[
T^2 = \begin{pmatrix}
A & -A^2 & I \\
I & -A & 0 \\
0 & I & 0 \\
\end{pmatrix} \neq I \oplus I \oplus I
\]
whereas
\[ T^3 = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} = I \oplus I \oplus I, \]
as wished.

To obtain an infinite family of such roots, just replace \( A \) by \( \alpha A \) where \( \alpha \) is real, say. \( \square \)

We finish with a digression which is in the spirit of the paper. In the case of matrices of operators, readers have already observed here an apparent resemblance to usual matrices with real or complex coefficients. In view of many examples treated here and elsewhere, it seems therefore reasonable to conjecture that:

If \( T \) is a matrix of operators defined formally on \( H \oplus H \oplus \cdots \oplus H \) (\( n \) times), that is, on \( H \times H \times \cdots \times H = H^n \) whether the entries are all in \( B(H) \) or not, and \( T^p = 0 \) for some integer \( p \geq n \), then necessarily \( T^n = 0 \).

The answer to this conjecture is negative. A counterexample is available on finite dimensional spaces!

**Example 1.37.** Let \( H = \mathbb{C}^2 \) and let
\[ A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \]
be both defined on \( H \). Then
\[ AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \]
and so \( ABA = BAB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \). Finally, set
\[ T = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \]
which is defined on \( H \times H \) (where \( 0 \in B(\mathbb{C}^2) \)). Thus,
\[ T^2 = \begin{pmatrix} 0 & 0 \\ 0 & BA \end{pmatrix} \quad \text{and} \quad T^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]
and so \( T^2 \neq 0 \) whilst \( T^3 = 0 \), marking the end of the proof.

2. **An open question**

A closable operator \( A \) such that \( \overline{A^2} \) is self-adjoint but \( A^2 \) is not self-adjoint exists. A simple example is to take \( A \) to be the restriction of the identity operator \( I \) (on \( H \)) to some dense (non closed) subspace \( D \).
of $H$. Then $\sqrt{A} = I$ fully on $H$ and so $\sqrt{A}^2$ is self-adjoint. However, $A^2$ is not self-adjoint for $A^2 = D$ and so $A^2$ is not even closed.

What about the converse, i.e. if $A$ is closable and $A^2$ is self-adjoint, then could it be true that $\sqrt{A}$ is self-adjoint? A positive answer can be obtained if one comes to show that if $A$ is a closable operator with a self-adjoint square $A^2$, then $A$ is closed.

Let us posit that we have in effect shown that a closable $A$ such that $A^2$ is self-adjoint is necessarily closed. Another natural question would then follow: What about when $A^2$ is normal? Another more general question is to see whether the self-adjointness or the normality of $A^n$ entails the closedness of $A$ whenever it is closable?

ACKNOWLEDGEMENTS

The second example in the proof of Theorem 1.33 was communicated to me by Professor A. M. Davie (The University of Edinburgh, UK) a while ago.

Thanks go to Dr S. Dehimi (University of Mohamed El Bachir El Ibrahimi, Algeria) with whom I discussed e.g. Proposition 1.13 and Theorem 1.15.

Thanks are also due to Professor Robert B. Israel (University of British Columbia, Canada). Indeed, for the purpose of Theorem 1.36 I asked him whether we can place a parameter inside the first matrix in Example 1.35 and still obtain the same conclusion? He kindly suggested to conjugate with some nonsingular matrix which does not commute with $A$, but contains a parameter $\alpha$, leading to the second example.

Finally, Example 1.2 is due to Professor Jan Stochel (Uniwersytet Jagielloński, Poland), who communicated it to me some time ago.

REFERENCES

[1] W. Arendt, F. Räbiger, A. Sourour. Spectral properties of the operator equation $AX + XB = Y$, *Quart. J. Math. Oxford Ser. (2)*, 45/178 (1994) 133-149.
[2] Y. Arlinski˘i, Y. Kovalev. Factorizations of nonnegative symmetric operators, *Methods Funct. Anal. Topology*, 19/3 (2013) 211-226.
[3] S. J. Bernau. The square root of a positive self-adjoint operator, *J. Austral. Math. Soc.*, 8 (1968) 17-36.
[4] I. Boucif, S. Dehimi and M. H. Mortad. On the absolute value of unbounded operators, *J. Operator Theory*, 82/2 (2019) 285-306.
[5] J. F. Brasche, H. Neidhardt. Has every symmetric operator a closed symmetric restriction whose square has a trivial domain?, *Acta Sci. Math. (Szeged)*, 58/1-4 (1993) 425-430.
[6] P. R. Chernoff. A semibounded closed symmetric operator whose square has trivial domain, *Proc. Amer. Math. Soc.*, 89/2 (1983) 289-290.
[7] J. B. Conway. *A course in operator theory*. Graduate Studies in Mathematics, 21, American Mathematical Society, Providence, RI, 2000.

[8] J. B. Conway, B. B. Morrel. Roots and logarithms of bounded operators on Hilbert space, *J. Funct. Anal.*, **70**/1 (1987) 171-193.

[9] S. Dehimi and M. H. Mortad, Bounded and Unbounded Operators Similar to Their Adjoints, *Bull. Korean Math. Soc.*, **54**/1 (2017) 215-223.

[10] S. Dehimi, M. H. Mortad. Right (or left) invertibility of bounded and unbounded operators and applications to the spectrum of products, *Complex Anal. Oper. Theory*, **12**/3 (2018) 589-597.

[11] S. Dehimi, M. H. Mortad. Chernoff like counterexamples related to unbounded operators, *Kyushu J. Math.*, **74**/1 (2020) 105-108.

[12] N. Frid, M. H. Mortad, S. Dehimi. On nilpotence of bounded and unbounded linear operators, (submitted).

[13] S. Goldberg. *Unbounded linear operators. Theory and applications*, Reprint of the 1985 corrected edition, Dover Publications, Inc., Mineola, NY, 2006.

[14] P. R. Halmos, G. Lumer, J. J. Schäffer. Square roots of operators, *Proc. Amer. Math. Soc.*, **4** (1953) 142-149.

[15] V. Hardt, A. Konstantinov, R. Mennicken. On the spectrum of the product of closed operators, *Math. Nachr.*, **215**, (2000) 91-102.

[16] Z. J. Jabłoński, Il B. Jung, J. Stochel. Unbounded quasi normal operators revisited, *Integral Equations Operator Theory*, **79**/1 (2014) 135-149.

[17] H. Kosaki. On intersections of domains of unbounded positive operators, *Kyushu J. Math.*, **60**/1 (2006) 3-25.

[18] S. H. Kulkarni, M. T. Nair, G. Ramesh. Some properties of unbounded operators with closed range, *Proc. Indian Acad. Sci. Math. Sci.*, **118**/4 (2008) 613-625.

[19] G. McDonald, C. Sundberg. On the spectra of unbounded subnormal operators, *Canad. J. Math.*, **38**/5 (1986) 1135-1148.

[20] A. McIntosh. Square roots of operators and applications to hyperbolic PDEs. *Miniconference on operator theory and partial differential equations (Canberra, 1983)*, 124-136, Proc. Centre Math. Anal. Austral. Nat. Univ., 5, Austral. Nat. Univ., Canberra, 1984.

[21] I. D. Mercer, Finding "nonobvious" nilpotent matrices (2005). [http://people.math.sfu.ca/~idmercer/nilpotent.pdf](http://people.math.sfu.ca/~idmercer/nilpotent.pdf)

[22] M. Möller, F. H. Szafraniec. Adjoints and formal adjoints of matrices of unbounded operators, *Proc. Amer. Math. Soc.*, **136**/6 (2008) 2165-2176.

[23] M. H. Mortad. *An operator theory problem book*, World Scientific Publishing Co., (2018). ISBN: 978-981-3236-25-7 (hardcover).

[24] M. H. Mortad. On the triviality of domains of powers and adjoints of closed operators, *Acta Sci. Math. (Szeged)*, **85** (2019) 651-658.

[25] M. H. Mortad. Counterexamples related to commutators of unbounded operators, *Results Math.*, **74** (2019), no. 4, Paper No. 174.

[26] M. H. Mortad. Simple examples of non closable paranormal operators. [arXiv:2002.06536](https://arxiv.org/abs/2002.06536)

[27] M. H. Mortad. Counterexamples related to unbounded paranormal operators, (submitted).

[28] S. Ôta. Closed linear operators with domain containing their range, *Proc. Edinburgh Math. Soc.*, (2) **27**/2 (1984) 229-233.
[29] S. Óta. Unbounded nilpotents and idempotents, *J. Math. Anal. Appl.*, 132/1 (1988) 300-308.
[30] S. Óta. On normal extensions of unbounded operators, *Bull. Polish Acad. Sci. Math.*, 46/3 (1998) 291-301.
[31] K. Schmüdgen. On domains of powers of closed symmetric operators, *J. Operator Theory*, 9/1 (1983) 53-75.
[32] K. Schmüdgen. *Unbounded self-adjoint operators on Hilbert space*, Springer. GTM 265 (2012).
[33] Z. Sebestyén, J. Stochel. On suboperators with codimension one domains, *J. Math. Anal. Appl.*, 360/2 (2009) 391-397.
[34] Z. Sebestyén, Zs. Tarcsey. On the square root of a positive selfadjoint operator, *Period. Math. Hungar.*, 75/2 (2017) 268-272.
[35] J. Stochel, J. B. Stochel. Composition operators on Hilbert spaces of entire functions with analytic symbols, *J. Math. Anal. Appl.*, 454/2 (2017) 1019-1066.
[36] Zs. Tarcsey. Operator extensions with closed range, *Acta Math. Hungar.*, 135 (2012) 325-341.
[37] A. E. Taylor, D. C. Lay. Introduction to functional analysis. Reprint of the second edition. *Robert E. Krieger Publishing Co.*, Inc., Melbourne, FL, 1986.
[38] Ch. Tretter. *Spectral Theory of Block Operator Matrices and Applications*. Imperial College Press, London, 2008.
[39] M. Uchiyama. Operators which have commutative polar decompositions. Contributions to operator theory and its applications, 197-208, *Oper. Theory Adv. Appl.*, 62, Birkhäuser, Basel, 1993.
[40] J. Weidmann. *Linear Operators in Hilbert Spaces*, Springer, 1980.
[41] D. Y. Wu, A. Chen. On the adjoint of operator matrices with unbounded entries II, *Acta Math. Sin. (Engl. Ser.)*, 31/6 (2015) 995-1002.

Department of Mathematics, Laboratoire d’analyse mathématique et applications, University of Oran 1, Ahmed Ben Bella, B.P. 1524, El Menouar, Oran 31000, Algeria.

E-mail address: mhmortad@gmail.com, mortad.hichem@univ-oran1.dz.