Conformal invariance and the expressions for $C_F^4\alpha_s^4$ contributions to the Bjorken polarized and the Gross-Llewellyn Smith sum rules

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ABSTRACT

Considering massless VVA triangle diagram in the conformal invariant limit and the the results of recent distinguished analytical calculations of the 5-loop single-fermion loop corrections to the QED $\beta$-function, we derive the analytical expressions for the $C_F^4\alpha_s^4$-contributions to the Bjorken polarized and Gross-Llewellyn Smith sum rule. The asymptotic structure of the series obtained is briefly discussed.

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1 Introduction

Quite recently the complicated analytical expression of the non-singlet order $\alpha_s^4$ contribution to the $e^+e^-$ annihilation Adler function function

$$D^{NS}(Q^2) = Q^2 \int_0^\infty \frac{R(s)}{(s+Q^2)^2} ds = 3 \sum_F Q_F^2 C_D^{NS}(a_s(Q^2)) = 3 \sum_F Q_F^2 \left[ 1 + \sum_{n=1}^{n=4} d_n^{NS} a_s^n \right]$$

became available in the literature [1]. $R(s)$ here is the well-known $e^+e^-$ ratio, $Q_F$ are the quarks charges, $a_s = \alpha_s(Q^2)/\pi$ and $\alpha_s(Q^2)$ is the $\overline{MS}$-scheme QCD coupling constant, which obeys the property of asymptotic freedom at large $Q^2 > 0$. The calculation of $d_4^{NS}$ [1] is the third step after complete analytical calculations of the $\alpha_s^2$ [2] and $\alpha_s^3$ corrections [3], [4] to the Adler function of vector currents. The analytical result for the $\alpha_s^3$ coefficient, obtained in Refs.[3], [4] was confirmed later on in Ref. [5]. Its $SU(N)$-group structure was analyzed in detail in Ref. [6]. Unfortunately, on the contrary to the the results for the $\alpha_s^2$ and $\alpha_s^3$ coefficients to the Adler function of vector currents (see [2] and Refs.[3], [4] respectively), the expression for the $\alpha_s^4$ contribution to Eq.(1) was presented in the case of $SU(3)$ group only, without singling out the corresponding Casimir operators $C_F$ and $C_A$ [1]. This does not allow one to study special theoretical features of both $\alpha_s^4$-coefficient to $D^{NS}(Q^2)$ and to the photon vacuum polarization constant $Z_{ph}$ in particular. The latter consideration was done at the $\alpha_s^3$-level in Ref. [3] and the special emphasis was made to the cancellation of $\zeta_3$-term in the case of $SU(N)$ gauge group with $C_A = C_F = T f/2 = N$, which corresponds to the case of $SU(4)$ supersymmetric Yang-Mills theory [7]. However, rather interesting part of analytical result of Ref.[1], namely the the 5-loop single-fermion contribution to the QED $\beta$-function, which is proportional to the single-fermion QED contribution to Eq.(1), was already presented in the literature some time ago [8]. The main result of this work is [8]:

$$\beta_{QED}^{[1]} = \frac{4}{3} A + 4A^2 - 2A^3 - 46A^4 + \left( \frac{4157}{6} + 128 \zeta_3 \right) A^5$$

$$= \frac{4}{3} A \times C_D^{NS}(A)$$

(2)

(3)

where $A = \alpha/(4\pi)$ and $\alpha$ is the QED coupling constant.

It should be stressed, that the coefficients of Eq.(2) are scheme-independent, at least in the schemes, not related to the lattice regularization. The performed by different methods calculations of the order $O(A^4)$-approximation to Eq.(2) [9], [10], are clarifying this feature. The analytical structure of the 5-loop result of Ref. [1] differs from the previously known terms: it contains $\zeta_3$-term in the 5-loop coefficient.

We will start this Letter not from discussions of this interesting to our point of view feature, but from more theoretically-motivated considerations, which are allowing to extract the part of the $\alpha_s^4$-correction to the Bjorken polarized and Gross-Llewellyn Smith sum rules from the result of Ref. [8] with the help of the relation, derived by Crewther in Ref. [11] using the concept of conformal symmetry, typical to quark-parton model of strong interactions. The consequences of this relation were studied in Ref. [12] and Ref. [6] in the conformal-invariant limit of QCD and were generalized by different ways to the case
of full QCD with non-zero QCD β-function in the works of Ref.[6], Ref. [13] - Ref. [16] (for the reviews see [17], [18]).

2 Crewther relation in the momentum space

Let us translate the investigations of Refs. [11], [12], performed in the $x$-space, to the language of the momentum space following the presentations, given in Ref.[13], and in Ref. [17] in particular.

Consider first the 3-point function

$$T_{\mu\alpha\beta}^{abc}(p,q) = i \int <0|TA_{\mu}^{a}(y)V_{\alpha}^{b}(x)V_{\beta}^{c}(0)|0> e^{ipx+iqy}dxdy = d^{abc}T_{\mu\alpha\beta}(p,q)$$

where $A_{\mu}^{a}(x) = \bar{\psi}\gamma_{\mu}5(\lambda^{a}/2)\psi$, $V_{\mu}^{\alpha}(x) = \bar{\psi}\gamma_{\mu}(\lambda^{a}/2)\psi$ are the axial and vector non-singlet quark currents. The r.h.s. of Eq.(4) can be expanded in a basis of 3 independent tensor structures under the condition $(pq) = 0$ as

$$T_{\mu\alpha\beta}(p,q) = \xi_{1}(q^{2},p^{2})\epsilon_{\mu\alpha\beta\gamma}p^{\gamma}$$
$$+ \xi_{2}(p^{2},q^{2})(q_{a}\epsilon_{\mu\beta\rho\tau}p^{\rho}q^{\tau} - q_{\beta}\epsilon_{\mu\alpha\rho\tau}p^{\rho}q^{\tau})$$
$$+ \xi_{3}(p^{2},q^{2})(p_{a}\epsilon_{\mu\beta\rho\tau}p^{\rho}q^{\tau} + p_{\beta}\epsilon_{\mu\alpha\rho\tau}p^{\rho}q^{\tau})$$

Taking now the divergency of axial current one can get the following relation for the invariant amplitude $\xi_{1}(q^{2},p^{2})$:

$$q_{\beta}T_{\mu\alpha\beta}(p,q) = \epsilon_{\mu\alpha\rho\tau}q^{\rho}p^{\tau}\xi_{1}(q^{2},p^{2})$$

while the property of the conservation of the vector currents implies that

$$\lim_{p^{2}\rightarrow\infty}p^{2}\xi_{3}(q^{2},p^{2}) = -\xi_{1}(q^{2},p^{2})$$

(see Ref.[19] for the discussions of the details of the derivation of Eqs.(5)-(7)).

In order to clarify the meaning of the second invariant amplitude $\xi_{2}(q^{2},p^{2})$, let us first define the characteristics of the deep-inelastic processes, namely the polarized Bjorken sum rule

$$B_{jp}(Q^{2}) = \int_{0}^{1} [g_{1}^{ep}(x,Q^{2}) - g_{1}^{en}(x,Q^{2})]dx = \frac{1}{6}g_{A}\frac{g_{V}}{C_{Bjp}(a)}$$

and the Gross-Llewellyn Smith sum rule

$$GLS(Q^{2}) = \frac{1}{2} \int_{0}^{1} \left[F_{3}^{ep}(x,Q^{2}) + F_{3}^{en}(x,Q^{2})\right]dx = 3C_{GLS}(a)$$

where $a_{s} = \alpha_{s}/\pi$. The coefficient function $C_{Bjp}(a)$ can be found from the operator-product expansion of two non-singlet vector currents [20]

$$i \int TV_{a}^{\alpha}(x)V_{b}^{\beta}(0)e^{ipx}dx|_{p^{2}\rightarrow\infty} \approx C_{\alpha\beta\rho}^{P,abc} A_{\rho}^{c}(0) + other\ structures$$

\[ 2 \]
with
\[ C_{\alpha\beta\rho}^{\mu} \sim i\epsilon_{\alpha\beta\rho\sigma} \frac{p^\sigma}{p^2} C_{B\mu}(a_s) \]  
and \( P^2 = -p^2 \). In the case of the definition of the coefficient function of the Gross-Llewellyn Smith sum rule one should consider the operator-product expansion of the axial and vector non-singlet currents
\[ i \int T A_\mu^a(x) V_\nu^b(0) e^{iqx} dx |_{q^2 \to \infty} \approx C_{\mu\nu}^{V,ab} V_\nu(0) + \text{other structures} \]
where
\[ C_{\mu\nu}^{V,ab} \sim i\delta^{ab} \epsilon_{\mu\nu\alpha\beta} \frac{q^\beta}{Q^2} C_{GLS}(a_s) \]
and \( Q^2 = -q^2 \). The third important quantity, which will enter into our analysis, is the QCD coefficient function \( C_{NS}(a_s) \) of the Adler function of the non-singlet axial currents
\[ D_{NS}(a_s) = -12\pi^2 q^2 \frac{d}{dq^2} \Pi_{NS}(q^2) = 3 \sum_F Q_F^2 C_{NS}(a_s) \]
with \( \Pi_{NS}(q^2) \) defined as
\[ i \int < 0 | TA_\mu^a(x) A_\nu^b(0) |0 > e^{iqx} dx = q^{\mu\nu} q^2 - q_\mu q_\nu \Pi_{NS}(q^2) \]  
At this point we will stop with definitions of the basic quantities and return to the consideration of the 3-point function of Eq.(4). Following original work [11] one can apply to this correlation function an operator-product expansion in the limit \( |p^2| >> |q^2| \), \( p^2 \to \infty \), namely expand first the \( T \)-product of two non-singlet vector currents via Eq.(10) and then take the vacuum expectation value of the \( T \)-product of two remaining non-singlet axial currents defined through Eq.(15). These studies imply that [13]
\[ \xi_2(q^2,p^2)|_{|p^2| \to \infty} \to \frac{1}{p^2} C_{B\mu}(a_s) \Pi_{NS}(a_s) \]  
and thus
\[ q^2 \frac{d}{dq^2} \xi_2(q^2,p^2)|_{|p^2| \to \infty} \to \frac{1}{p^2} C_{B\mu}(a_s) C_{NS}(a_s) \]
Equations (16),(17) reflect the physical meaning of the invariant amplitude \( \xi_2(q^2,p^2) \) and should be considered together with the relations for the invariant amplitudes \( \xi_1(q^2,p^2) \) of Eq.(10) and \( \xi_3(q^2,p^2) \) from Eq.(17).
On the other hand, it was shown in Ref.[21] that in a conformal invariant (c-i) limit the three-index tensor of Eq.(11) is proportional to the fermion triangle one-loop graph, constructed from the massless fermions:
\[ T^{\mu\nu\rho}_{a\alpha\beta}(p,q)|_{c-i} = d^{abc} K(a_s) \Delta_{\mu\nu\rho}^{1}\text{-loop}(p,q) \]
In other words, in a conformal invariant limit one has
\[ \xi_1^{c-i}(q^2,p^2) = K(a_s) \xi_1^{1}\text{-loop}(q^2,p^2) \]  
\[ \xi_2^{c-i}(q^2,p^2) = K(a_s) \xi_2^{1}\text{-loop}(q^2,p^2) \]  
\[ \xi_3^{c-i}(q^2,p^2) = K(a_s) \xi_3^{1}\text{-loop}(q^2,p^2) \]
Moreover, in view of the Adler-Bardeen theorem \cite{22} the invariant amplitude $\xi_1(q^2, p^2)$, related to the divergency of axial current (see Eq.\(6\)), has no radiative corrections. Therefore one has $K(a_s) = 1$. The 3-loop light-by-light-type scattering graphs, which were calculated in Ref. \cite{23} and analyzed in Ref. \cite{24}, do not affect this conclusion. Indeed, in the case of the 3-point function of the non-singlet axial-vector-vector currents they are contributing to the higher order QED corrections, while the QCD corrections of the similar origin are appearing only in the 3-point function with the singlet axial current in one of the vertexes, which will be not discussed here.

Taking into account the property $K(a_s) = 1$ for the 3-point function of the non-singlet axial-vector-vector currents allows us to derive the fundamental Crewther relation

$$C_{Bjp}(a_s(Q^2))C_{NS}^{V}(a_s(Q^2))|_{c-i} = 1 ,$$

which is be valid in the conformal invariant limit in all orders of perturbation theory. The similar relation is also true for the coefficient function $C_{GLS}(a_s)$, defined by Eqs.\(9\),\(12\),\(13\) \cite{12}. Indeed, considering first the operator-product expansion of the axial and vector non-singlet currents in the 3-point function of Eq.\(4\) (see Eq.\(12\),\(13\)) taking the $T$-product of the remaining vector currents and repeating the above discussed analysis, one can find that in the conformal invariant limit the second identity takes place:

$$C_{GLS}(a_s(Q^2))C_{D}^{V}(a_s(Q^2))|_{c-i} = 1$$

where $C_{D}^{V}(a_s)$ is the coefficient function of the Adler $D$-function of two vector currents, which coincide with $C_{NS}^{V}(a_s(Q^2))$.

### 3 Application of the Crewther relation

The results of calculations of Ref.\cite{8} are equivalent to the following expression for the $C_{NS}^{V}(a_s)$ function

$$C_{NS}^{V}(a_s) = \left[ 1 + \frac{3}{4} C_F a_s - \frac{3}{32} C_F^2 a_s^2 - \frac{69}{128} C_F^3 a_s^3 + \left( \frac{4157}{2048} + \frac{96}{256} \zeta_3 \right) C_F^4 a_s^4 \right]$$

where $C_F = (N^2 - 1)/(2N)$. Using now Eq.\(22\) and Eq.\(23\) we get the new analogous scheme-independent contributions to the Bjorken polarized sum rule and Gross-Llewellyn Smith sum rule

$$C_{Bjp}(a_s) = C_{GLS}(a_s) = 1 - \frac{3}{4} C_F a_s + \frac{21}{32} C_F^2 a_s^2 - \frac{3}{128} C_F^3 a_s^3 - \left( \frac{8873}{2048} + \frac{3}{8} \zeta_3 \right) C_F^4 a_s^4$$

Note, that the order $a_s$, $a_s^2$ and $a_s^3$ coefficients are in agreement with the result of explicit calculations, performed in Refs.\cite{25}, \cite{20} and \cite{26} respectively. Thus, the direct calculation of the predicted $a_s^4$ coefficient may be rather useful for the independent cross-checks of the results of Ref.\cite{8} and the verification of the appearance of $\zeta_3$-term at the $a_s^4$-level. Indeed, at the previous stage of understanding definite claims on the pure rationality of the related to renormalization group functions QED series were made (see e.g. \cite{27}). However, this
statement may be not valid in higher orders of perturbation theory. In view of this it is highly desirable to get additional theoretical (or calculational) support in favour of the appearance of $\zeta_3$-term in the result of Eq.(24).

It is interesting to note, that after application of the the Crewther relation the perturbative series with unclear sign structure (see Eq.(2)) is transformed to the the sign-alternating series of Eq.(25). This feature may clarify, why the Lipatov-type estimates originally proposed in Ref. [28] for the sign-alternating series in the $g^{4}\Phi$-theory, are not working properly in QED in the case of Eq.(2 (see Ref. [29]). One may hope, that this disadvantage may be solved in the case of the sign-alternating series of Eq.(25), as obtained above and the QED considerations, analogous to ones of Ref. [29], will give better estimates for the coefficients of the series of Eq.(25).

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