Zeros of the electric field around a charged knot

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Consider a knotted loop of wire with charge distributed uniformly on it. The electric field around such a loop was recently proven to have at least $2t + 1$ equilibrium points, where $t$ is the tunnel number of the knot. Moreover, the symmetry of the situation suggests that the equipotential surfaces enclosing the circle are nested tori, at least sufficiently close to the circle. Far from the circle, they must approach spheres, because a circular loop of charge looks like a point source in the far-field.

Remarkably, a similar line of reasoning works even if the charged loop has knots tied in it. Although it becomes impossible to calculate the location of the equipotential points analytically in these cases, we can still prove that equilibria must exist in the surrounding electric field.

To do so, picture the equipotential surfaces again. Figure 1 shows what they look like for a trefoil knot, but the same idea works for any (sufficiently smooth) knot. One expects that close to the knot, the equipotential surfaces must be tubular versions of the knot itself—in other words, they must be knotted tori—whereas far from the knot, they must resemble spheres (because a charged knot, like any other compact charge distribution, appears point-like in the far-field). So if we imagine continuously varying the potential from high levels near the knot to low levels at infinity, the equipotential surfaces must continuously deform from knotted tori into spheres. To make this transition, the knotted tori swell up, collide with distant parts of themselves, and reconnect in ways that alter their topology. At such collisions, two patches of a single equipotential surface intersect tangentially. Because the electric field vector lies along the normal to each of the colliding patches, and because those normal vectors point in opposite directions at the point of collision, the electric field must vanish there.

This argument suggests that there must be one or more zeros in the electric field around any charged knot. Each zero represents an equilibrium point where a test charge could remain motionless. As is well known, there are no stable equilibria in an electrostatic field, so all these equilibrium points must be unstable. Indeed, they are all saddle points, with either one- or two-dimensional unstable manifolds.

These intuitive ideas have recently been sharpened and made rigorous with the help of Morse theory, algebraic topology, and geometric topology. In particular, one theorem provides a lower bound on the number of equilibrium points around any charged knot. It states that the electric field must have at least $2t + 1$ zeros,
To perform these computations, we replace the continuous knot by a thin tube around the knot. As we lower the potential (corresponding to moving from left to right in the figure), the surface inflates and swells up. At bifurcation values, it self-intersects and creates new zeros in the electric field, corresponding to new equilibrium points. The equipotential surfaces shown here have genus values of 1, 4, 3, and 0, moving from left to right.

where \( t \) is a topological invariant known as the tunnel number of the knot [40–42].

Unfortunately, this \( 2t+1 \) lower bound turns out to be somewhat loose. For example, the tunnel number for a trefoil knot is known to be 1 (indeed, the tunnel number is 1 for all prime knots with seven or fewer crossings [11, 12]), so the \( 2t+1 \) bound implies that a charged trefoil must always have at least 3 zeros in its surrounding electric field. Yet we have never seen fewer than 7 zeros in our computations, no matter how the trefoil is bent, twisted, or otherwise deformed. So is 7 the absolute minimum? Or could there be some needle-in-a-haystack conformation of a charged trefoil that has 3, 4, 5, or 6 zeros? The problem is currently unsolved, and is just one of many unsolved problems about charged knots.

To describe our computations, we introduce some notation and terminology. Let the knot \( K \) be parametrized by a vector-valued function \( \mathbf{r}(t) = (x(t), y(t), z(t)) \), where \( 0 \leq t \leq 2\pi \). Because the knot forms a closed loop, we also require that \( \mathbf{r}(0) = \mathbf{r}(2\pi) \). Then, from Coulomb’s law, the electric potential \( \phi \) at a point \( \mathbf{x} \in \mathbb{R}^3 \) away from the knot is given in dimensionless form by

\[
\phi(\mathbf{x}) = \int_{\mathbf{r} \in K} \frac{d\mathbf{r}}{|\mathbf{x} - \mathbf{r}|} = \int_0^{2\pi} \frac{|\mathbf{r}'(t)|dt}{|\mathbf{x} - \mathbf{r}(t)|}, \tag{1}
\]

where \( |\cdot| \) denotes the magnitude of a vector quantity. (We have written the potential in dimensionless form for convenience; one could include physical parameters like the vacuum permittivity \( \varepsilon_0 \) or the uniform charge density \( \rho \) along the knot, but we have chosen not to do so as they play no role in our analysis.) The electric field associated with the potential is given by \( \mathbf{E}(\mathbf{x}) = -\nabla \phi(\mathbf{x}) \). The zeros (i.e., the equilibrium points) of the electric field are equivalent to the critical points of the electric potential \( \phi \); as such, we will use the terms zeros, equilibrium points, and critical points interchangeably from now on.

For most knots, the potential \( \phi(\mathbf{x}) \) and its critical points cannot be calculated analytically. We must rely on numerical integration and rootfinding techniques. To perform these computations, we replace the continuous knot by \( N+1 \) unit point charges located at \( \mathbf{r}(t_0), \ldots, \mathbf{r}(t_N) \), where \( t_k = 2\pi k/(N+1) \), and use the following trapezoidal approximation of (1):

\[
\phi(\mathbf{x}) \approx \frac{2\pi}{N+1} \sum_{j=0}^N \frac{|\mathbf{r}'(t_j)|}{|\mathbf{x} - \mathbf{r}(t_j)|}. \tag{2}
\]

We use the same trapezoidal approximation for the electric field \( \mathbf{E}(\mathbf{x}) \). (There are more efficient approaches for evaluating \( \phi(\mathbf{x}) \) and \( \mathbf{E}(\mathbf{x}) \) when \( N \) is very large; these are based on multipole expansions such as those used in the fast multipole method [43].)

As mentioned above, we are interested in finding the zeros of the electric field. These are defined as points \( \mathbf{x}^* \in \mathbb{R}^3 \) such that \( \mathbf{E}(\mathbf{x}^*) = 0 \). To find them, we start with initial guesses and then refine the guesses iteratively, using a multivariable Newton method. To obtain reasonable initial guesses, we use an algorithm known as “3D Marching Cubes,” a computer graphics algorithm for finding level sets of a scalar function [44]. Here, we use Marching Cubes on each of the three components of \( \mathbf{E}(\mathbf{x}) \) to find their zero level sets. The algorithm partitions a large cube containing the knot into small cubes; then, on each cube, it uses a bilinear approximation of that component. If the bilinear approximations of all three components of \( \mathbf{E} \) pass through zero in the same cube, then we take the center of that cube as an initial guess for our multivariable Newton method. Some initial guesses to Newton end up diverging or converging to a far away critical point, and we throw these away. In contrast, the successful initial guesses quickly converge to the approximate locations of the zeros of \( \mathbf{E}(\mathbf{x}) \).

Along with the zeros, we are also interested in the equipotential surfaces. These are given by \( \phi^{-1}(v) \), where \( 0 < v < \infty \) is some given voltage level. The relevant values of \( v \) range from small positive values far from the knot, to large positive values close to the knot. Let \( v^* = \phi(\mathbf{x}^*) \) denote the potential at an equilibrium point. Recall that equipotential surfaces undergo self-collisions at \( \mathbf{x}^* \) and lose smoothness there. So to get a smooth surface, we perturb \( v^* \) to a nearby regular value \( v \) at which the Hessian matrix of second derivatives of \( \phi \) (equivalent to the Jacobian of \( \mathbf{E} \)) has full rank. By the implicit

\[\text{FIG. 1. Topologically different equipotential surfaces for a charged trefoil. Only “regular” values of the potential are shown, at which the equipotential surfaces are smooth manifolds. Bifurcations (not shown) lie in between these examples. In the leftmost panel, the potential is very large and positive close to the knot, and the corresponding equipotential surface resembles a thin tube around the knot. As we lower the potential (corresponding to moving from left to right in the figure), the surface inflates and swells up. At bifurcation values, it self-intersects and creates new zeros in the electric field, corresponding to new equilibrium points. The equipotential surfaces shown here have genus values of 1, 4, 3, and 0, moving from left to right.}\]
function theorem, this full rank condition ensures that \( \phi^{-1}(v) \) is a smooth, orientable, compact surface without boundary. We then use the Marching Cubes algorithm to render the surface. By repeating this process for a range of \( v \) values, we can explore how the equipotential surfaces change as we vary the level of the potential.

To illustrate the results obtained with this approach, consider the following parametrization of a trefoil knot:

\[
r(t) = (\sin t + 2 \sin 2t, \cos t - 2 \cos 2t, -\sin 3t).
\] (3)

In our numerical simulations, we sampled a cubic domain of \( 30 \times 30 \times 30 \) initial guesses in a mesh surrounding the knot and ran the multivariable Newton method to test for convergence to a zero. We rejected iterations that grew too large, or were within a small distance threshold from another computed zero, indicating a duplicate. Figure 2 shows that the electric field has 7 zeros for this particular embedding of a trefoil. By calculating the eigenvalues at these zeros, we can confirm that they are all saddle points and classify them by their indices (the dimensions of their stable manifolds).

Then, to obtain representatives of the equipotential surfaces around the trefoil, we compute the critical values \( v^* \) at the zeros, perturb them to nearby values \( v \), and take their inverse images \( \phi^{-1}(v) \). For the parametrization \( r(t) \), we find the outer triplet of zeros in Figure 2 has \( v^* \approx 12.79 \) and indices of 1; the inner triplet has \( v^* \approx 15.82 \) and indices of 2; and the origin has \( v^* \approx 15.42 \) and index 1. To see what the equipotential surfaces look like in between these critical cases, we compute the level sets \( \phi^{-1}(v) \) for the perturbed values \( v = 12.7, 15, 15.5, \) and \( 16 \). Figure 1 shows the resulting surfaces. Topologically, they are knotted tori with various numbers of holes.

In the example above, we assumed a highly symmetrical parametrization of a trefoil. What happens if we break the symmetry or, more generally, if we deform a knot continuously without allowing it to pass through itself? How does that affect the number of zeros in the electric field around the knot? Ideally, we would like to find deformations that cause as many zeros as possible to coalesce, thus bringing us closer to the absolute minimum number of zeros, whatever that may be, and perhaps allowing us to improve on the tunnel number bound.

One strategy is to deform the knot so as to reduce its complexity in some way, as quantified by an energy functional or a more general Lyapunov function. Two energy functionals in the literature on physical knot theory are the Möbius energy [45] and the Buck–Orloff energy [46], whose locally minimal configurations enjoy nice regularization properties. But we have found a simpler strategy to be useful: we slowly squash the knot from the top down and watch what happens to its zeros.

Figure 3 shows that by flattening a “figure-eight knot” we can reduce its number of zeros from 19 to 5; then, unexpectedly, it goes back up to 9. It is intriguing that the number of zeros can either increase or decrease as one flattens the knot. The staircase structure of the graph reveals that zeros can appear or disappear in pairs (which is what one expects generically [38]) or in two simultaneous pairs (due to non-generic symmetries in the particular parametrization of the knot). We suspect that 5 is the smallest number of zeros possible for any parametrization of a figure-eight knot, symmetrical or otherwise.

Let \( Z \) denote the minimum number of zeros around a charged knot \( K \), where the minimum is taken over all smooth embeddings of \( K \). Although we have not found a formula for \( Z(K) \), we conjecture that it can be bounded from above and below by two standard topological invariants of \( K \):

\[
2t + 1 \leq Z \leq 2c + 1.
\] (4)

We have already met the \( 2t + 1 \) lower bound, which uses the knot’s tunnel number \( t \). This lower bound has been proven [39]. By contrast, the \( 2c + 1 \) upper bound is conjectural. It involves the knot’s crossing number \( c \), defined as the minimal number of crossings possible in a planar projection of the knot. We have numerical evidence and a plausibility argument but not a proof, for reasons we will explain momentarily. Table I lists these bounds along with our conjectured values of \( Z \) for some simple knots.

Our plausibility argument for \( 2c + 1 \) as an upper bound on \( Z \) is based on the following observation: For any knot \( K \), we can always construct an embedding of \( K \) that has at least \( 2c + 1 \) zeros in its electric field. Unfortunately, that statement is weaker than what we need. To prove that \( 2c + 1 \) is an upper bound on \( Z \), we would need to
TABLE I. Conjectured minimum number of zeros $Z$ for knots with five or fewer crossings. In each case, our conjectured value of $Z$ lies between the proven lower bound $2t + 1$ and the conjectured upper bound $2c + 1$. Our conjectures are based on numerical experiments that used the following knot parametrizations: (1) unknot: $x(t) = \cos t, y(t) = \sin t, z(t) = 0$. Trefoil: $x(t) = \sin t + 2\sin 2t, y(t) = \cos t - 2\sin 2t, z(t) = -\gamma \sin 3t$. Figure-eight: $x(t) = (2 + \cos 2t)(\cos 3t), y(t) = (2 + \cos 2t)(\sin 3t), z(t) = \gamma \sin 4t$. Cinquefoil: $x(t) = (\cos 2t)(3 + \cos 5t)/2, y(t) = (\sin 2t)(2 + \cos 5t)/2, z(t) = (\gamma \sin 5t)/2$. Three-twist: $x(t) = 2\cos(2t + 0.2), y(t) = 2\cos(3t + 0.7), z(t) = \gamma \cos 7t$. The standard parametrization has height $\gamma = 1$.

To find a plausible conjecture for the minimum number of zeros, we flatten the knot by slowly decreasing $\gamma$ to 0, and watch the zeros coalesce.

The weakness of the “at least” part of the conclusion can be traced back to the Poincaré–Hopf index theorem: that is where the first “at least” qualifier popped up. If we could ensure exactly one zero in each hole, we would then be able to claim what we want: $Z \leq 2c + 1$. We suspect the uniqueness of the zero in each hole would follow if the holes were round enough (neither too elongated nor too non-convex), but this is what remains to be properly formulated and proven in future work.

As we have tried to show in this Letter, electrostatic knot theory opens up many new directions for exploration. The questions are mainly motivated by their simplicity and theoretical appeal, but they could have real-world implications. For instance, given that the zeros of electric and magnetic fields are relevant to problems of plasma confinement in nuclear fusion and to trapping of cold atoms, related questions may be of experimental interest in those settings [59–61]. The electric field of a charged knot may also have implications in biochemistry, where researchers study the knottedness of charged DNA molecules and the electric fields they generate [62].

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