Metabelian Wreath Products are LERF

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1. Introduction

The subgroup $S$ of $\Gamma$ is separable means that $S$ is closed in the profinite topology of $\Gamma$. A finitely generated (f.g.) group is LERF if all its f.g. subgroups are separable.

Gruenberg’s theorem [G] asserts that the wreath product $A \wr Q$ is residually finite (r.f.) iff $A$ and $Q$ are r.f. and either $A$ is abelian or $Q$ is finite. We seek general conditions which will characterize the wreath products which are LERF. Since LERF entails r.f. Gruenberg’s theorem gives a first restriction. Since subgroups of LERF groups are LERF we shall assume $A$ and $Q$ are LERF.

In the case where $Q$ is finite and $A$ is a finitely generated LERF group then up to finite index $\Gamma = A \wr Q$ is direct product of finitely many LERF groups.

QUESTION 1. For which f.g. groups $X$ is $X^n$ LERF for all $n \geq 1$.

Polycyclic groups are LERF by a theorem of Malcev and thus $X$ polycyclic gives a positive answer to the question. By a theorem of M. Hall the free group $F_n$ of rank $n$ is LERF while Mihailova has shown that $F_2 \times F_2$ is not L-S; so any group $X$ which contains $F_2$ cannot give a positive answer to this question.

Our main result concerns the other case when $A$ is f.g. abelian; it generalizes the recent result of [de C]: $A \wr Z$ is LERF for any f.g. abelian group. We show that when $A, Q$ are f.g. abelian then $\Gamma = A \wr Q$ is LERF. Using the Magnus embedding it follows that free metabelian groups are LERF.

2. Lemma

Here is small modification of an important lemma from [de C]. We say a subgroup $S$ of $\Gamma$ is ‘strongly separable’ if all the finite index subgroups of $S$ are separable in $\Gamma$.

LEMMA 1. Let $\Gamma$ be f.g and r.f. Suppose $f : \Gamma \rightarrow Q$ is a surjective homomorphism with abelian kernel $K$. If $S$ is a f.g. subgroup such that $f(S)$ is of finite index in $Q$ then $S$ is strongly separable in $\Gamma$. 

PROOF. If $S'$ is a subgroup of finite index in $S$ then its image $Q'$ is finite index in $Q$. We can show $S'$ is separable in $\Gamma$ using $f': f'^{-1}(Q') \to Q'$ with kernel $K$. If $S'$ is closed in a subgroup of finite index in $\Gamma$ then it is closed in $\Gamma$ so $S$ is strongly closed. Also, if $f(S) = Q'$ of finite index in $Q$ then $f^{-1}(Q')$ is a subgroup of finite index in $\Gamma$; since separable in a subgroup, say $f^{-1}(Q')$, of finite index in $\Gamma$, it suffices to prove the case when $f(S) = Q$. We may suppose then that $S' = S$ and $f(S) = Q$. It now follows that $\Gamma = KS$; consequently $S \cap K$ is normal in $\Gamma$. If we show $S/S \cap K$ is separable in $\Gamma/S \cap K$ then since $S$ is f.g we can also separate $S$ in $\Gamma$. Thus we need only consider the group $\bar{\Gamma} = \Gamma/S \cap K$ which is a split extension; we assume then $S \cap K = \{1\}$ and consequently $\Gamma = K \times S$.

It now suffices to show that the profinite closure $\bar{S}$ of $S$ in $\bar{\Gamma}$ meets $K$ trivially, $\bar{S} \cap K = \{1\}$. For then, if $x \in \bar{S}$ then $f(x)x^{-1} \in \bar{S} \cap K = \{1\}$; hence $x \in S$ so $S$ is closed. Consider then $x \in \bar{S} \cap K$; if $x \neq 1$ then since $\Gamma$ is r.f. we can choose a normal subgroup $N$ of finite index in $\Gamma$ so that $x \notin N$; consider $L = N \cap K$ and $T = N \cap S$; then $x \notin L$. Since $T$ normalizes $L$ then $M = \bigcap_{t \in S \text{ mod } T} L^t$ is finite index in $L$, and $x \notin M$; since $M \rtimes T$ is finite index in $\Gamma$ and does not contain $x$, then $x \notin \bar{S}$; thus we must have $x = 1$. ■

3. Main Results

We now assume $A, Q$ are f.g. abelian. Any subgroup $R$ of $Q$ is contained as a subgroup of finite index in $R_1$ and there is a (retract) homomorphism $\pi : Q \to R_1$ so that $\pi(r) = r$, $r \in R$. We can in fact achieve the case that $\pi$ is split and that $Q = R_1 \rtimes R_2$ with $R_1, R_2$ infinite, when $R$ is non-trivial, not of finite index and $Q$ has rank greater than 1. Then

$$\Gamma = A \wr Q = A^{R_1 \rtimes R_2} \rtimes (R_1 \times R_2)$$

$$= (A^{R_2 \rtimes R_1} \rtimes R_1) \times (A^{R_1 \rtimes R_2} \rtimes R_2)$$

$$= (A^{R_2 \wr R_1}) \rtimes (A^{R_1 \wr R_2})$$

$$= \Gamma^{(2)}_{R_1} \rtimes \Gamma^{(1)}_{R_2}.$$

PROPOSITION 2. Suppose that $A, Q$ are f.g. abelian then $\Gamma^{(2)}_{R_1}, \Gamma^{(1)}_{R_2}$ are f.g and r.f.

PROOF. Since $\Gamma$ is f.g. these quotient groups are also f.g. By Grunen-berg’s theorem these are also r.f. since any direct sum of cyclics is r.f. ■

THEOREM 3. Suppose $A$ and $Q$ are f.g. abelian groups then $A \wr Q$ is LERF.

PROOF. Let $K = A^Q$, $\Gamma = A \wr Q$; we have the natural homomorphism $f : \Gamma \to Q$. Let $x \notin S$, a f.g. subgroup of $\Gamma$; $R = f(S)$. We may assume that $R$ is not of finite index in $Q$ for then the Theorem follows from the Lemma. Also the case of $Q$ having rank one is proven in [de C]. As in the remarks above we consider after passing to a subgroup of finite index
in $S$ if necessary, a retract $R_1$ with $R_1 \times R_2$ a subgroup of finite index in $Q$, each $R_i$ infinite. Since a subgroup $S$ is separable if a subgroup of finite index is separable in a subgroup of finite index in $\Gamma$, we may now assume that $R_1 \times R_2 = Q$.

If $x$ has a non-trivial image in $R_2$ we project $\Gamma$ to $\Gamma^{(1)}_{R_2}$. In this group the image of $S$ is trivial and the image of $x$ is not so we can map to a finite quotient to separate because $\Gamma^{(1)}_{R_2}$ is r.f. by the Proposition.

Suppose then $x$ has trivial image in $R_2$, then we can replace $\Gamma$ by $\Gamma^{(2)}_{R_1}$ and consider there $x \notin S$; we can separate them since $S$ is closed in $\Gamma^{(2)}_{R_1}$ by the Lemma since $f(S)$ is of finite index in $R_1$ and since $\Gamma^{(2)}_{R_1}$ is f.g and r.f. by the Proposition.

**Corollary 4.** Free metabelian groups are LERF.

**Proof.** Via the Magnus embedding, [M], the free metabelian group can be embedded in a group $A \wr Q$ where $A$ and $Q$ are f.g. abelian groups. The corollary follows now since subgroups of LERF groups are LERF.

**References**

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