Reconstructing quantum theory from diagrammatic postulates

John H. Selby*1,2,3, Carlo Maria Scandolo†1, and Bob Coecke‡1

1Department of Computer Science, University of Oxford, UK
2Department of Physics, Imperial College London, UK
3Perimeter Institute for Theoretical Physics, Waterloo, Ontario, Canada

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Abstract

A reconstruction of quantum theory refers to both a mathematical and a conceptual paradigm that allows one to derive the usual formulation of quantum theory from a set of primitive assumptions. The motivation for doing so is a discomfort with the usual formulation of quantum theory, a discomfort that started with its originator John von Neumann.

We present a reconstruction of finite-dimensional quantum theory where all of the postulates are stated entirely in diagrammatic terms, making them intuitive. Equivalently, they are stated in category-theoretic terms, making them mathematically appealing. Again equivalently, they are stated in process-theoretic terms, establishing the fact that the conceptual bare-bones of quantum theory concerns the manner in which systems and processes compose.

Aside from the diagrammatic form, the key novel aspect of this reconstruction is the introduction of a new postulate, symmetric purification. Unlike the ordinary purification postulate, symmetric purification applies equally well to classical theory as well as quantum theory. We therefore first reconstruct the full process theoretic description of quantum theory, consisting of composite classical-quantum systems and their interactions, before restricting ourselves to just the ‘fully quantum’ systems in a final step.

We propose two novel alternative manners of doing so, ‘no-leaking’ (roughly that information gain causes disturbance) and ‘purity of cups’ (roughly the existence of entangled states). Interestingly, these turn out to be equivalent in any process theory with cups and caps. Additionally, we show how the standard purification postulate can then be seen as an immediate consequence of the symmetric purification postulate and purity of cups.

Other tangential results concern the specific frameworks of generalised probabilistic theories (GPTs) and process theories (a.k.a. CQM). Firstly, we provide a diagrammatic presentation of GPTs, which, henceforth, can now be subsumed under process theories. Secondly, we have now characterised necessary additional axioms for a process theory to correspond to the Hilbert space model, and in particular, that a ‘sharp dagger’ is indeed the right choice of a dagger structure.
1 Introduction

Reconstructions of quantum theory aim to reproduce the standard quantum formalism from assumptions of some desired flavour, which in our case means diagrammatic. The idea of reconstructing quantum theory is not at all new, indeed, the first to contribute to the endeavour was John von Neumann. Merely three years after the publication of his book [91] that cemented the mathematical formalism of quantum theory, he made it clear in a letter to the mathematician Garrett Birkhoff that he was no longer satisfied with the Hilbert space formalism [78]. However, rather than just aiming to reconstruct this formalism, his hope was in fact to find a different formalism that may also produce new physics. The actual reconstruction program, building further on von Neumann’s work, was outlined by George Mackey [72], and mostly completed by Constantin Piron [77] (an academic ancestor of two of the authors) within the arena of so-called property lattices, and by Günther Ludwig [71] within the arena of generalised probabilistic theories (GPTs). The main motivation for this was a dissatisfaction with accepting the abstract Hilbert space as a given. Instead, the aim was to start off with conceptually justified structures and axioms from which the Hilbert space could then be derived. A more modern perspective is that reconstructions help to see the physical principles that underlie the Hilbert space structure. This in turn allows one both to derive more applications of quantum theory, and to understand how it can, and should, be modified in order to reconcile it with general relativity.

The flavour of the assumptions that go into a reconstruction has varied substantially, ranging from counterfactual ontology to instrumentalism. However, broadly, there are two kinds of approaches.

The first wave of reconstructions [91, 72, 77, 71], surveyed in [37], took place in the previous century, and assumptions were taken to be axioms in the mathematical sense. Nothing was left implicit, and there were no givens. As a result, the mathematical sophistication of these reconstructions was substantial. For example, Piron’s Theorem [77] was actually only finalised in 1995 by Solér [84], while arguably it already took off from von Neumann’s book [91] in 1932. In this first wave different reconstructions correspond to different conceptually motivated mathematical structures, e.g. lattices [77] or generalised probability spaces [71].

This millennium, a new wave of quantum reconstructions emerged under Lucien Hardy’s impetus [52, 42, 73, 23, 55]. In these, there was a broad shift away from the mathematical axiomatics of the first wave. Instead, they took inspiration from Einstein’s derivation of relativity theory from two ‘principles’, i.e. the limited speed of light and the invariance of laws for different observers. This shift means that many pieces of structure are often sneaking into the reconstructions —like in Einstein’s derivation of relativity theory— for example, the structure of the real numbers. To distinguish the assumptions used in these approaches from the mathematical axioms of the first wave, the term ‘postulates’ was adopted. Now, of key importance is the conceptual grounding of the postulates, which are typically cast within some ‘interpretative school’. For example, principles should only make reference to measurement [52], should be information-theoretic [29, 23], or, should make reference to only properties of single systems [11]. To speed up the job only finite dimensional Hilbert spaces were considered. This stance was not only for mathematical convenience, but also justified by the many new results —within the context of quantum foundations, information and computation— demonstrating that the essential quantum phenomena were all apparent within the finite-dimensional setting.

One clear outcome of this new wave of reconstructions was the role of composite systems. For example, principles such as tomographic locality [13] concerned pairs of systems, rather than just a single one. This was in sharp contrast with the first wave of reconstructions where the focus was on single systems under observation. Indeed, for this first wave, the Achilles heel was the description of composite systems at a general axiomatic level: while these approaches were able to recover Hilbert space, they all failed to reproduce the tensor product as part of the formalism. This is probably why —with the emergence of quantum computation and information, where composition is of course vital— these approaches have more or less vanished. But, despite this significant advantage over the first wave, the postulate-based approaches lacked the mathematical clarity of the earlier attempts, and in many cases the proofs were not so insightful.

Also learning from the failures of the earlier axiomatic reconstructions, a rigorous focus on composition of quantum systems and processes has been the subject of categorical quantum mechanics (CQM) [11, 36] for some 15 years now. Its initial goal was to recast quantum theory in high-level terms, making reasoning and computing more intuitive. While the first reconstructions of the second
wave focused on states of physical systems and their geometry, the categorical approach changed the paradigm completely, from states to processes \[32\]. Consequently, one now refers to the theories that CQM is concerned with as process theories. The particular nature of the categorical structures that make up process theories have the great upshot that they admit a full and faithful diagrammatic representation \[30, 31, 31\] (a.k.a. quantum picturalism). This has provided an intuitive picture of many aspects of quantum information and computation \[43, 60, 43, 46\] and, moreover, lends itself well to computational automation \[65, 47, 19, 45\].

Meanwhile, borrowing from CQM, the reconstructionists of the second wave adopted ‘diagrams of processes’ as their starting point \[56, 22, 23, 55\], hence embracing composition of processes as one of the core ingredients of quantum theory. More recently, a third wave of reconstruction attempts took off \[8, 11, 94\], which can be seen as resurrecting the mathematical spirit of the first wave while still embracing the principled underpinning that guides the second wave. Recently, there have been various other reconstructions of quantum theory from other perspectives, for example \[51, 59, 58, 58, 48, 87, 89\].

1.1 Main result

This paper provides a reconstruction that is conceptually grounded whilst still being based on crisp mathematical axioms. This is achieved by exploiting the correspondence:

\[
\text{diagrams} \simeq \text{category theory} \simeq \text{process theory}
\]

Our postulates are now entirely diagrammatic, i.e. entirely category-theoretic, i.e. entirely process-theoretic. This provides them with an intuitive, elegant as well as principled underpinning. In short, what we prove is:

\[
\text{classical interface} + \text{postulates about how processes compose} \implies \text{quantum processes}
\]

More explicitly, the complete list of postulates (introduced formally in section \[3\]) that we use to reconstruct quantum theory are as follows:

1. the theory is a process theory (Def. \[1\]),
2. with a finite local classical interface (Defs. \[2.21\] & \[2.25\]),
3. cups and caps (Def. \[3\]),
4. a sharp-dagger (Def. \[4\]),
5. and in which all processes admit symmetric purifications (Def. \[5\]).

To understand these postulates and the properties derived from them, one must be familiar with the process theory framework and associated diagrammatic language. In section \[2\] we therefore provide an introduction to this framework and the concepts necessary to read the rest of the paper.

Informally, that the theory is a process theories states that nature is fundamentally about processes and how they compose. The classical interface describes how we interact with, control, and learn about the theory. Cups and caps represent a fundamental symmetry which ensures that the theory has maximal correlations. The sharp dagger represent another fundamental symmetry, namely time reversal. Informally, the time reverse of a process can be thought of as “playing the video of that process in reverse”, in particular, for a state this corresponds to a ‘test’ for that state. Finally, the existence of symmetric purifications ensures that any lack of purity can always be traced to lack of access to the past or future of certain systems.

The proof of this result is remarkably simple in contrast to many other reconstructions, largely owing to the use of the diagrammatic proofs. Indeed, the entire reconstruction can be presented in a simple flowchart (see Fig. \[1\]) showing that the structure of the reconstruction consists of a few lemmas and a standard result, the Koecher-Vinberg theorem \[66, 66\]. The latter was borrowed from the works of Barnum, Wilce et al., for example \[9, 12, 61, 7, 8\].
1.2 Purification

The standard purification postulate was first used in [22] as an operational generalisation of the Stinespring dilation theorem [85]. It roughly states that any mixed process can be represented as a pure process with an extra output that is discarded. Importantly, this postulate is satisfied by quantum theory but not by classical probability theory, so it can be used to single out quantum theory [22].

Our symmetric purification postulate holds for both quantum and classical theory. We therefore reconstruct quantum, classical and hybrid systems all together. One can then ask how to single out the fully quantum systems. There are many ways to do so, but two of them are of particular interest: the existence of a pure ‘cup’ (roughly speaking the existence of entanglement), or, the lack of non-trivial ‘leaks’ (essentially that information gain causes disturbance). Whilst at first glance these appear to be unrelated postulates involving different diagrammatic concepts, they can actually be shown to be equivalent in any process theory with cups and caps.

Moreover, it can then be shown that the standard purification postulate is implied by the conjunction of the symmetric purification postulate and the existence of a pure cup. Our work can therefore be seen as deconstructing the standard purification postulate into two parts— one which applies to both quantum and classical theory, and the other only to quantum— therefore refining exactly what it is that is uniquely quantum about the postulate.

1.3 Connection to GPTs, OPTs and CQM

While initially GPTs only appealed to single systems, under the impetus of CQM a new hybrid form was proposed (a.k.a. OPTs), where multiple systems are supported by means of a diagrammatic backbone. We demonstrate that, within the context of process theories, the essential structure of OPTs can be derived from the classical interface postulate. Hence, OPTs are subsumed within the process theory framework. Independently, a similar results was also obtained in [50].

Cups and caps have been part of the structure of CQM from the start [1] and provide a cup- and a cap-shaped wire for each system: allowing for inputs to be connected to inputs and outputs to outputs. Daggers have also been part of CQM since its very beginning [2, 81]. Unlike the transpose, which is constructed using cups and caps, a dagger did not have any other structural requirements besides being compositional. In order to fully characterise the Hermitian adjoint of quantum theory, a sharpness constraint was added in [79]. We demonstrate that once we have a classical interface, cups and caps, and a sharp dagger, that the only additional postulate to be imposed, for a process theory to correspond to the Hilbert space model, is the aforementioned symmetric purification postulate.

2 Process theoretic concepts

Process theories [35, 36] are theories that have a particular diagrammatic representation. A comprehensive introduction is in [36].

Definition 2.1 (Process theory). Process theories consist of a collection of systems, denoted by labelled wires, and a collection of processes, denoted by labelled boxes with input wires (at the bottom) and output wires (at the top). These processes can be wired together, for example:

where the resulting diagram must also be a process in the theory; the relevant data for a diagram are:

i. the processes that appear in the diagram, and

ii. how the diagram is connected including the overall ordering of free inputs and outputs,

and formation of diagrams is constrained by:
1. connected systems must be the same (i.e. the same label),
2. outputs are wired up to inputs, and
3. no loops are created.

Remark 2.2. For those who favour an information-based characterisation of quantum theory, one can think of the wires in the diagrams as information flows. For those who favour a more operational interpretation, one can think of the processes as corresponding to a single use of a piece of lab equipment in a single run of an experiment.

There are three particular types of processes within a theory that are often distinguished: those with no inputs, which are state preparation procedures or states for short; those with no outputs, corresponding to the outcome of some destructive measurement, or effects for short; and those with neither, known as scalars, which typically correspond to probabilities. These are respectively denoted as:

\[
\begin{align*}
&\begin{array}{c}
\text{\quad } \\
\text{\quad } \\
\end{array}
&
\begin{array}{c}
\text{\quad } \\
\text{\quad } \\
\end{array}
\quad \text{ and } \\
\begin{array}{c}
\text{\quad } \\
\text{\quad } \\
\end{array}
\end{align*}
\]

Note that we often drop various labels and the box around scalars when they are clear from context.

It is often useful — particularly in connecting these theories to standard mathematical models — to introduce two primitive forms of composition: sequential composition, symbolically denoted as \( g \circ f \), and parallel composition, denoted as \( f \otimes g \), which diagrammatically correspond to:

\[
\begin{align*}
&\begin{array}{c}
\text{\quad } \\
\text{\quad } \\
\end{array}
&
\begin{array}{c}
\text{\quad } \\
\text{\quad } \\
\end{array}
\quad \text{ and } \\
\begin{array}{c}
\text{\quad } \\
\text{\quad } \\
\end{array}
\end{align*}
\]

We also represent the parallel composition of systems as \( A \otimes B \). Moreover, to translate to standard symbolic notation, we denote a process \( f \) with input \( A \) and output \( B \) as \( f : A \rightarrow B \). To describe processes lacking inputs and/or outputs we must therefore introduce a fictitious ‘trivial’ system denoted \( I \), so that we can, for example, denote the state \( s \) of a system \( C \) as \( s : I \rightarrow C \). To be consistent with our diagrammatic notation, that is, \( I \) representing the lack of an input, it must satisfy the equation of \( I \otimes A = A = A \otimes I \), i.e. appending a trivial system does nothing.

A simple example of a process theory is classical probability theory:

Example 2.3 (Classical). Here we restrict ourselves to finite probabilistic models. We can characterise every classical system with a natural number \( n \) which we call its dimension. The composite of systems is given by the product of their dimensions, i.e. \( n \otimes m = nm \), and so the trivial system corresponds to \( n = 1 \). Processes \( f : n \rightarrow m \) then correspond to \( m \times n \) matrices with non-negative matrix elements. Sequential composition is given by matrix multiplication and parallel composition by the standard tensor product. States therefore correspond to \( 1 \times n \) matrices, i.e. vectors, and effects \( m \times 1 \) matrices, i.e. covectors, the sequential composition of a state and effect therefore gives a scalar valued in \( \mathbb{R}^+ \).

A more involved example is the process-theoretic description of quantum theory, which includes quantum, classical, composite, and hybrid systems. The not fully quantum systems in this process theory can be interpreted in two ways: either as the systems that arise from the branching structure of measurements, or as the systems that are obtained from a generalised decoherence mechanism. In particular, the classical systems are necessary for a process-theoretic description of quantum measurements.

Example 2.4 (Quantum). Here we restrict ourselves to finite-dimensional quantum theory. Systems correspond to finite dimensional C*-algebras, which can be represented as direct sums of complex matrix algebras \( \bigoplus_k M(C^{n_k}) \). The standard tensor product providing composition, hence, the trivial system is given by \( M(C) \). Processes are completely positive of C*-algebras, \( f : \bigoplus_k M(C^{n_k}) \rightarrow \bigoplus_k M(C^{n_k}) \). States therefore correspond to block-diagonal density matrices and effects to block-diagonal POVM elements. Again, scalars correspond to non-negative real numbers.

We can retrieve the fully quantum or fully classical systems by restricting to those of the form \( M(C^n) \) or \( \bigoplus_{k=1}^m M(C) \) respectively, the former system being equivalent to an \( n \)-level quantum system.
and the latter to an \( m \)-dimensional classical system. The processes between systems of these respective types are precisely those that we would expect. For example, states of \( M(C^n) \) are just \( n \)-level density matrices, and states of \( \bigoplus_{k=1}^n M(C) \) are probability distributions over an \( m \)-element set.

2.1 Discarding and causal processes

A process theory often comes with a discarding effect for each system which provides a way to ‘throw away’ or even simply ‘ignore’ systems. We denote this by:

\[
\overline{A}
\]

where it should be the case that discarding two systems independently is the same as discarding the composite system:

\[
\overline{A \otimes B} = \overline{A} \overline{B}.
\]

In theories with discarding we can elegantly define causality of processes.

**Definition 2.5 (Causal processes).** A process \( f \) is causal \([22, 36]\) if it satisfies:

\[
\overline{f} = \overline{A} \overline{B}.
\]

By the causal subtheory we then mean the theory of all causal processes.

The connection between this definition and standard notions of causality may not be immediately apparent. However, it can be shown that this is equivalent to the notion that future measurement choices do not effect current experiments \([22]\). More general, it implies that there is no superluminal signalling in the theory \([33]\), and even full compatibility with relativity \([63]\).

Note that the definition of a causal subtheory automatically implies that the only effects in the subtheory are the discarding effects themselves:

\[
\overline{f} = \overline{B}.
\]

It therefore turns out to be beneficial not to work with the causal subtheory, but with the full theory including non-causal processes. This is standard practice within quantum theory which besides Dirac’s kets also has non-causal Dirac’s bras. That is, allowing for non-causal processes gives access to the specific outcomes in a measurement (cf. “in a measurement we obtain the outcome corresponding to bra \( \langle x \rangle \)”), or more general, processes that can only occur probabilistically. Moreover, just like in the case of kets and bras, the larger theory admits additional symmetry which allows one to more easily characterise it. We discuss these symmetries below in Sec. 2.3.

We can however ‘bunch’ together a collection of non-causal processes to represent all ‘branches’ of a measurement, or more general non-deterministic process, as a single causal process. In order to do so we introduce classical systems —represented diagrammatically by thin gray lines\(^1\) — to hold the resultant probability distribution over the possible outcomes. Then, destructive and non-destructive measurements are just processes of the form

\[
\overline{m} \quad \text{and} \quad \overline{M}
\]

respectively.

Such classical systems also allow us to represent controlled state preparation procedures as

\[
\overline{S}
\]

\(^1\)Note that in this paper, whilst a thin gray wire always indicates a classical system, a thicker solid wire indicates any system allowed by the theory, which could be in fact classical.
where the choice of classical state input into $S$ will determine the quantum state that is prepared. Such interactions between classical systems and the general systems of our theory form the basis of the classical interface for a theory that we formally introduce in section 2.4.

**Example 2.6** (Classical). We define the discarding maps to be the covector with every matrix element $1$. It is therefore clear that the causal subtheory theory is the restriction to stochastic matrices as they will satisfy Eq. (1). In particular, this condition implies that states correspond to probability distributions over an $n$ element set, and that processes are maps from probability distributions over $n$ elements to those over $m$ elements. In particular we denote the states and effects that are everywhere $0$ except for a $1$ at position $i$ as:

$\downarrow$ and $\boxed{\Lambda}$ respectively.

**Example 2.7** (Quantum). We define the discarding maps in this theory to be the (partial) trace, hence causal states correspond to block-diagonal trace-$1$, density matrices with blocks of dimension $\{n_k\}$. General processes are causal iff they are trace-preserving. This theory contains both quantum and classical theory as subtheories. Moreover it contains processes mapping between these sectors, such as $f : M(C^n) \rightarrow \bigoplus_{k=1}^{m} M(C)$ mapping from a quantum to a classical system. Such processes can be shown to correspond to POVMs in the standard presentation of quantum theory, where the causality constraint for these measurements is equivalent to the constraint that POVM elements sum to the identity. General processes with both quantum and classical inputs and outputs have similarly clear interpretations: the classical input can be seen as a control system determining which process to implement, the implemented processes can involve some (potentially non-destructive) measurement, the result of which is then encoded in the classical output.

### 2.2 Leaks and purity

We will need the process-theoretic definition of purity first presented in [80], and for this purpose we introduce the notion of leaks.

**Definition 2.8.** A leak is a process

$$
\begin{align*}
\begin{array}{c}
\text{A} \\
\big\downarrow \\
\text{L} \\
\big\uparrow \\
\text{A}
\end{array}
\end{align*}
$$

such that

$$
\begin{align*}
\begin{array}{c}
\text{A} \\
\big\downarrow \\
\text{L} \\
\big\uparrow \\
\text{A}
\end{array}
\end{align*} = \begin{array}{c}
\text{A} \\
\big\downarrow \\
\text{A}
\end{array}
$$

Note that when we have multiple leaks we typically will label them with a different colour.

**Example 2.9** (Trivial). A leak is said to be trivial if there exists a state $s$ such that

$$
\begin{align*}
\begin{array}{c}
\big\downarrow \\
\big\uparrow \\
\text{A}
\end{array}
\end{align*} = \begin{array}{c}
\big\downarrow \\
\big\uparrow \\
\text{A}
\end{array}
$$

Note that any $s$ which is causal will define a trivial leak in this way.

**Example 2.10** (Broadcasting). A broadcasting map

$$
\begin{align*}
\begin{array}{c}
\big\downarrow \\
\big\uparrow \\
\text{A}
\end{array}
\end{align*}
$$

is one that ‘leaks both ways’, that is:

$$
\begin{align*}
\begin{array}{c}
\big\downarrow \\
\big\uparrow \\
\text{A}
\end{array}
\end{align*} = \begin{array}{c}
\big\downarrow \\
\big\uparrow \\
\text{A}
\end{array} = \begin{array}{c}
\big\downarrow \\
\big\uparrow \\
\text{A}
\end{array}
$$

8
We provide a formal classification of the leaks for quantum theory in Prop. B.1. However, informally, the leaks for a quantum system \( A = \bigoplus_i A_i \) can be seen as leaking the ‘which branch’ information. Therefore, for fully quantum systems all leaks are trivial, whilst for classical systems all leaks are based on the broadcasting map, which simply is the ‘copier’:

\[
\begin{align*}
\vdots & \quad \mapsto \quad \quad & \vdots & \quad \quad & \vdots
\end{align*}
\]

Informally, a process is pure if it plays nicely with leaks. More precisely:

**Definition 2.11.** \( f \) is pure if and only if

\[
f = \quad g \quad \quad \quad \Rightarrow \quad \exists \quad \quad g = f
\]

This condition is best thought of as two independent conditions: the first being that any dilation of a pure process must be due to a leak, and the second, that pure processes do not interact with ‘leakable’ information, so leaking before or after is equivalent. This definition is motivated by the fact that it gives the right notion of purity of processes both in quantum and classical theories [80]. We characterize the pure processes for quantum theory in Prop. B.2 in the case of fully quantum systems we obtain the standard notion of purity, i.e. that the process is Kraus rank 1.

**Example 2.12.** For states this definition of purity reduces to

\[
\vdots = \quad g \quad \quad \quad \Rightarrow \quad \exists \quad \quad g = \quad \vdots
\]

We now formalise the idea of a set of states prepared by a causal state preparation \( S \) being pure and jointly distinguishable. This gives an important information theoretic property of each system, namely, the amount of classical information which they can reliably store. Recall that causality for a state preparation means:

\[
\overline{A} \quad \vdots \quad S \quad \vdots \quad n
\]

**Definition 2.13 (Testability).** A causal state preparation \( S : n \rightarrow A \) is said to be testable if

\[
\vdots \quad S \quad \vdots \quad n
\]

is pure and if there is a measurement \( M \) such that

\[
\vdots \quad M \quad \vdots \quad S \quad \vdots \quad n
\]

We say that the state preparation \( S \) is maximal testable if \( n \) is moreover maximal.

**Example 2.14.** In quantum theory the maximal testable state preparations correspond to orthonormal bases of the Hilbert space.
2.3 Cups, caps and sharp daggers

We now formalise what it means for a process theory to be ‘blind w.r.t. inputs and outputs’ i.e. that inputs can be ‘bent’ into outputs and vice versa, essentially relaxing the basic constraints on forming diagrams for a process theory (cf. point 2 of Def. 2.1). In terms of diagrams this is usually referred to as string diagrams [6, 36] or compact structure [62, 1]. This ability to bend wires can also be represented within a process theory as a particular bipartite state, the cap, and effect, the cap, for each system:

\[
\begin{align*}
\cup & := \cup \\
\cap & := \cap 
\end{align*}
\]

A detailed discussion can be found in [38, 36].

**Definition 2.15 (Cups and caps).** A theory has cups and caps if for each system it has processes:

\[
\begin{align*}
\cup & := \cup \\
\cap & := \cap 
\end{align*}
\]

which satisfy:

\[
\begin{align*}
\cup & = \cap \\
\cap & = \cup \\
\cup & = \cap
\end{align*}
\]

Equivalently, this means that in diagrams inputs can be connected to inputs, outputs to outputs, and also that loops are allowed.

Cups and caps have a very clear conceptual meaning: they assert that the theory has ‘maximal’ correlations. Firstly, that the theory must have correlations is easily shown by a simple diagrammatic argument. Assume, for the sake of contradiction, that there are no correlations, i.e. that all bipartite states separate:

\[
\begin{align*}
\cup & = \cup \\
\cap & = \cap \\
\cup & = \cap
\end{align*}
\]

then, in particular, the cup separates:

\[
\begin{align*}
\cup & = \cup \\
\cap & = \cap \\
\cup & = \cap
\end{align*}
\]

so we have:

\[
\begin{align*}
\cup & = \cup \\
\cap & = \cap \\
\cup & = \cap
\end{align*}
\]

That is, all wires separate and the theory is trivial. Any non-trivial theory with cups must therefore have correlations. Intuitively, maximality can then be deduced from the first of Eqs. (4) in that ‘via a cup and cap an identity can be realised’, so that the cup and cap must allow for a flow of all state-data at the input. Formally this intuition is substantiated in [34].

Similarly to the above proof, it can now easily be seen that the cap cannot be causal, confirming that this simple diagrammatic structure is lost when passing to the causal subtheory. Indeed, if the cap were causal then the Eq. (2) becomes:

\[
\begin{align*}
\cup & = \cup \\
\cap & = \cap \\
\cup & = \cap
\end{align*}
\]

and again then all wires would separate and the theory would be trivial:

\[
\begin{align*}
\cup & = \cup \\
\cap & = \cap \\
\cup & = \cap
\end{align*}
\]

Consequently, non-trivial causal subtheories won’t have cups and caps.
**Examples 2.16.** For the quantum case the cups and caps realise the Choi-Jamiołkowski isomorphism. That is, the cup and cap are (non-normalised) maximally entangled states and effects respectively. More formally, the cup for a $d$-dimensional system $A$ is given by the super-normalised state written in Dirac notation as:

$$ \bigcup A \sim \sum_{ij=1}^d |ii\rangle\langle jj| $$

similarly the cap is given by the same matrix but interpreted as a POVM element. For general C*-algebras, e.g. $A = \bigoplus_k A_k$ where $A_k$ are $d_k$-dimensional quantum systems, we can write the cup as:

$$ \bigcup A = \bigoplus_k A_k \sim \bigoplus_k \sum_{ij=1}^{d_k} |ii\rangle\langle jj| = \sum_{ij=1}^{d_1} |ii\rangle\langle jj| + \sum_{ij=d_1+1}^{d_1+d_2} |ii\rangle\langle jj| + \cdots $$

the cap is again given by the same matrix interpreted as a POVM element. In the classical case this reduces to:

$$ \sum_k |kk\rangle\langle kk| $$

which corresponds to the super-normalised perfectly correlated state (or effect) as is expected.

Cups and caps provide one way to swap inputs and outputs, and hence, an operation of ‘time-reversing’:

In the case of quantum theory this corresponds to the transpose, but quantum theory has a second manner of doing so, namely the adjoint, a.k.a. dagger [2, 81], which turns Dirac kets into Dirac bras and vice versa. The difference between the transpose and the dagger is the conjugate, and hence witnesses the non-trivial involution on the number field of complex numbers. Hence having a dagger besides cups and caps is a truly fundamental structure.

**Definition 2.17 (Dagger).** A dagger is a reflection of processes:

$$ \begin{array}{c}
B \\
\text{\_\_}\text{\_}\text{\_}\text{\_}
\end{array} \quad \leftrightarrow \quad \begin{array}{c}
\dagger
A \\
\text{\_\_}\text{\_}\text{\_}\text{\_}
\end{array} $$

(Where an asymmetry of the box shape has been introduced to make the reflection clear.) In particular, it reflects the entire diagram structure:

Just like how a Dirac bra can be seen as a ‘test’ for the corresponding ket, we would our general process-theoretic dagger to do the same, that is, the dagger of a state can be interpreted as a test for that state [79]. This sharpness assumption then gives a clear conceptual meaning to the dagger (which is obviously complementary to the conceptual meaning of cups and caps). Formally, we will assume this condition for every testable family of states, which in diagrammatic terms means Eq. (6) below. Moreover, if $S$ is maximal, then $S^T$ should be causal as it is a test where no other outcomes are possible.

Also, while in quantum theory testing is a non-trivial structure, due to the invasive nature of measurements, in classical theory it is trivial, so we expect the dagger not to add any new structure on classical processes, and hence to be symmetry which is already present due to the cups and caps.
**Definition 2.18** (Sharp dagger). A dagger is *sharp* if for all testable (cf. Def. 2.13) causal state preparations $S$, its dagger $S^\dagger$ tests it, i.e.:

\[
\begin{array}{c}
S \\
S
\end{array}
= \quad \quad (6)
\]

In addition, if $S$ is maximal then $S^\dagger$ needs to be causal:

\[
\begin{array}{c}
S \\
 \quad \quad \rightarrow
\end{array}
= \quad \quad (7)
\]

and for classical processes this test-structure should be trivial:

\[
\begin{array}{c}
\quad \quad \rightarrow
\end{array}
= \quad \quad (7)
\]

**Examples 2.19.** For quantum theory generally (including the fully quantum and classical cases) the sharp-dagger is provided by the Hermitian adjoint.

**Remark 2.20.** This ‘sharpening’ of the dagger is essentially the same as the definition given in [79], the main distinction being that here we consider general state preparation procedures rather than just individual states.

### 2.4 Classical interface

Let us first, for simplicity, introduce a shorthand notation for certain diagrams that will be used in this section, that is:

\[
\begin{array}{c}
C \\
B  \chi \quad \quad X \quad \quad A  \chi \\
D
\end{array}
:= \quad \quad (8)
\]

We now formalise how we access general systems by means of a ‘classical interface’.

**Definition 2.21** (Classical interface). A classical interface comprises of three parts:

i. a classical subtheory,

ii. classically controlled processes, and

iii. tomography tests.

The first of these is self-explanatory, the process theory must have classical systems with classical processes between them as described in Ex. 2.3. The other two parts of the interface introduce the interactions between this subtheory and the rest of the process theory.

Firstly, ‘classically controlled processes’ formalise the idea that we can choose which process out of a family to implement, possibly using randomness (e.g. by rolling a die).
Definition 2.22 (Classically controlled processes).

For any set of processes \( \{ f_i \} \) there is a controlled process \( F \) such that:

\[
\forall i \quad f_i = F
\]  

(8)

For example, if we let \( \{ f_i \} \) be a set of states then this provides the controlled state preparations considered in Sec. 2.1.

Secondly, ‘tomography tests’ formalise how classical systems can be used to characterise processes by the probabilities obtained in experiments.

Definition 2.23 (Tests for finite process tomography). For any pairs of systems \((A, B)\) there exists a controlled test \( \tau \) such that:

\[
\begin{array}{c}
B \leftarrow A \\
\tau
\end{array} = \begin{array}{c}
B \leftarrow A \\
g
\end{array} \iff \begin{array}{c}
B \leftarrow A \\
\tau
\end{array} = \begin{array}{c}
B \leftarrow A \\
g
\end{array}
\]  

(9)

For example, if we let \( B \) be the trivial system such that \( \tau \) is doing state tomography, then this provides us with the measurements discussed in Sec. 2.1.

Remark 2.24. Note that the definition of classical theory that we are using automatically implies that \( n \) and \( m \) are finite. This means that is never necessary to perform an infinite number of distinct experiments to characterise a process. Consequently, the set of probabilities needed to describe a process is finite. Note that in practice it is never possible to perform an infinite number of experiments so we should at least expect this to hold for our best effective theory of nature.

It is common to assume that tomography can be performed locally. This expresses the idea that: although we know the world to be non-local (in the sense of \[14\]), there are still no holistic degrees of freedom, and that the description of two distinct regions of space can be formulated entirely in terms of their individual properties and the correlations between them. One can consider that the ability to be characterised locally is really the defining feature of what we mean by a system, it is something that we can isolate and study in its own right independent of the rest of the world.

Definition 2.25 (Local tomography). Locality implies that \( \tau \) in the definition of process tomography can be chosen such that it factorises over parallel composition:

\[
\begin{array}{c}
A \\
\tau \\
C
\end{array} = \begin{array}{c}
\alpha \\
\cdots \\
\beta
\end{array} \quad \begin{array}{c}
\gamma \\
\cdots \\
\delta
\end{array}
\]  

(10)

where \( n = n_\alpha \cdots n_\beta \) and \( m = m_\gamma \cdots m_\delta \).

Example 2.26. Quantum theory, as mentioned earlier, has a classical subtheory and moreover has all possible classically controlled processes and suitable tests for finite, local, tomography. In particular, local tomography \[14, 15\] has been used in many reconstructions of quantum theory, e.g. \[24, 55\], this can be extended to the full process-theoretic description of quantum theory by viewing the general C*-algebras as restricted quantum systems (in the sense of \[11, 32, 57\]).
3 The postulates

The first postulate provides the basic framework that we will use for the reconstruction. This allows one to describe essentially any conceivable physical theory, in particular, as we show later, encompassing the generalised probabilistic theory framework which has recently served as the basis for other reconstructions such as [52, 42, 23, 51, 73, 74, 11, 24].

**Postulate 1** (The theory is a process theory). As defined in Def. 2.1.

The second postulate provides a more operational layer to our theory, describing how we interact with the world via classical inputs and outputs.

**Postulate 2** (There is a finite local classical interface). As defined in Def. 2.21 where tomography can be done locally as defined in 2.25.

The next two postulates are based on standard compositional tools from CQM.

**Postulate 3** (The theory has cups and caps). As defined in Def. 2.15.

**Postulate 4** (There is a sharp dagger). As defined in Def. 2.18.

3.1 Symmetric purification

The final postulate symmetrises the ‘standard’ purification postulate introduced in [22], which states that for all states $\rho$ there exists a pure bipartite state $\psi$ such that:

$$
\begin{array}{c}
\phi \\
\end{array} =
\begin{array}{c}
\psi \\
\end{array}
$$

where this purification is ‘essentially unique’ [24], that is, if there are two such purifications $\psi$ and $\phi$ with the same purifying system, then they are related via a reversible transformation $R$:

$$
\begin{array}{c}
\phi \\
\end{array} =
\begin{array}{c}
\psi \\
\end{array} =
\begin{array}{c}
\phi \\
\end{array} =
\begin{array}{c}
\psi \\
\end{array}
\Rightarrow
\begin{array}{c}
\psi \\
\end{array} =
\begin{array}{c}
R \\
\end{array}
$$

This notion of purification is problematic for us for two reasons. Firstly, it is not compatible with the classical interface as classical theory does not satisfy this postulate. Secondly, it is formulated specifically in terms of states: we aim to treat all processes on an equal footing viewing them as more fundamental entities, states being special instances thereof. In this section we therefore introduce the postulate of symmetric purification which resolves these issues. Symmetric purification stipulates that every process arises from a pure process by ‘discarding’ a system to both the future and the past.

To produce a time-symmetric version of purification we need to have a notion of ‘discarding’ systems in the past. We can think of the standard discarding effect as an operational way to describe a scenario where we have no knowledge about, control over, or interaction with the future of a system. However, we can also imagine a scenario where we have no information about, no control over, and no interaction with the past of a system, to represent this we use the time reverse (i.e. dagger) of the discarding (to the future) map:

$$
\begin{array}{c}
\phi \\
\end{array} =
\begin{array}{c}
\psi \\
\end{array} =
\begin{array}{c}
A \\
\end{array} =
\begin{array}{c}
R \\
\end{array}
$$

In the cases of quantum and classical theory, the dagger of the discarding map is the unnormalised maximally mixed state. One may worry that this is not actually a valid state (i.e. is not normalised/causal), however, discarding to the past is not something that we can ‘do in the lab’ and so should not correspond to a state that we can prepare.

**Postulate 5** (Symmetric purification). Every process $f : A \rightarrow B$ can be dilated to a pure process $F : A \otimes B \rightarrow B \otimes A$ as follows:

$$
\begin{array}{c}
B \\
\end{array} =
\begin{array}{c}
F \\
\end{array}
$$
where we call $F$ a purification of $f$. Moreover, purification are essentially unique, that is, any two purifications $F, G : A \otimes C \to B \otimes D$ are connected by a diagram $R$:

$$
\begin{align*}
\begin{array}{c}
\begin{array}{c}
F
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A \otimes C
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
B \otimes D
\end{array}
\end{array}
\end{align*}
\Rightarrow \\
\begin{align*}
\begin{array}{c}
\begin{array}{c}
G
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A \otimes C
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
B \otimes D
\end{array}
\end{array}
\end{align*}
\begin{array}{c}
\begin{array}{c}
R
\end{array}
\end{array}
\end{align*}

\tag{12}
$$

where the `backwards leak' is provided by the time-reverse (i.e. dagger) of a leak. Moreover $R$ must be `bi-causal':

$$
\begin{array}{c}
\begin{array}{c}
R
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
= \\
\begin{array}{c}
\begin{array}{c}
A \otimes C
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
B \otimes D
\end{array}
\end{array}
\end{array}
\end{array}
\end{align*}

\tag{13}
$$

Symmetric purification expresses the requirement that all processes of the theory are fundamentally pure, and the apparent lack of purity should arise from lacking information about and control over the past and/or future of some environment systems.

As it is formulated, symmetric purification provides a common ground to explain the lack of purity in all systems in the process-theoretic description of quantum theory, whether they be fully quantum, fully classical or more general hybrid and composite systems. For a proof of this fact see App C.1.

**Remark 3.1.** Quantum theory actually satisfy a stronger form of symmetric purification where we can demand also that $F = F^\dagger$, i.e. the purified process is invariant under time-reversal. This strengthening however, whilst an interesting feature of quantum theory, is not necessary for our reconstruction.

## 4 The reconstruction

A high level overview of the structure of the reconstruction is provided in Fig. showing which postulates are necessary for each lemma and how these are combined to reconstruct quantum theory.

To begin reconstructing quantum theory we first demonstrate how we can obtain diagrammatic sums from our postulates. These are formally defined in App. but intuitively this provides another way to represent the branching structure of probabilistic processes, and so, is represented by a summation operation which distributes over diagrams. In particular, we can then show that the state spaces have the structure of convex cones bringing us close to the structure that is found in the GPT framework [52, 13].

**Lemma 4.1.** In a theory with classically controlled processes (Def. 2.22) we can define a sum of processes.

**Proof.** We define the sum of any finite set of processes as:

$$
\sum_i \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
B
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
j_i
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
F
\end{array}
\end{array}
\end{align*}

\tag{14}
$$

where $F$ is the classically controlled process satisfying:

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
B
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
j_i
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
F
\end{array}
\end{array}
\end{align*}

\tag{15}
$$

15
Figure 1: Flowchart outlining the structure of the reconstruction where the green rectangles correspond to postulates and the blue ellipses to the lemmas and theorems constituting the proof. In the top section we obtain something similar to the framework of generalised probabilistic theory. In the middle section we reconstruct the process-theoretic description of quantum theory consisting of hybrid quantum-classical systems. Finally, in the bottom section we give two equivalent routes to restrict to the fully quantum systems.
Given this definition we need to check that this is indeed a valid summation, i.e. that it satisfies the conditions of Def. A.1. The key step is to show that:

\[
\sum_i f_i \chi = \sum_i \chi B A D C \sum_i \chi \chi B A D C (16)
\]

holds for any diagram \( \chi \). To check this first note that the LHS of this is defined through Eq. (14) as:

\[
\sum_i f_i \chi = \sum_i \chi B A D C (16)
\]

On the other hand, to understand the RHS let us first define:

\[
\begin{aligned}
\sum_i f_i \chi &= \sum_i g_i = \sum_i G \\
\end{aligned}
\]

where we are using classical control to define \( G \) such that it satisfies:

\[
\begin{aligned}
g_i &= G \\
\end{aligned}
\]

It is then simple to see that:

\[
\forall i \quad G = g_i = f_i \chi = F \chi
\]

which using local tomography of classical theory implies that:

\[
G = F \chi
\]

We therefore find that:

\[
\sum_i f_i \chi = \sum_i F \chi = \sum_i G
\]
and so, Eq. 22 is satisfied for all diagrams $\chi$, and hence, the sums are free to move around diagrams. The other constraints on the sum (e.g. commutativity etc.) are immediately inherited from the equivalent property of the classical sum.

We are, as mentioned earlier, working with the full non-causal theory so as to realise the full symmetries of the theory. However, we now show that if we restrict to the causal classical interface then the whole theory must be causal, and hence, we characterise the causal subtheory.

**Proposition 4.2.** If a theory has a causal classical interface (i.e. has a classical interface (Post. 2 where the classical processes are restricted to the causal subtheory), then it is a causal process theory (Def. 2.5).

*Proof adapted from Lem. 7 in [22].* Firstly, note that scalars are always classical (as the trivial system $I$ must also be the trivial system for the classical sub-theory) and hence, the only causal scalar is 1. Therefore,

$$\forall e, s \quad \frac{\chi}{1} = 1.$$  

(17)

In other words, we know that all effects $e$ in the theory are deterministic. To prove that the theory is causal it just remains to check that there is a unique effect for each system. We check for uniqueness via tomography on effects: finite tomography (Def. 2.23) implies that there is some test $\tau$ such that:

$$\tau e_1 = \tau e_2 \iff e_2 = e_1$$

Now, note that as we have restricted to the causal classical processes the only classical effect is the discarding map and so we find:

$$E = \sum_i e_i$$

(18)

where the last equality is given by decomposing the classical discarding map into a sum over basis elements. Eq. (18) is true for any effects $e$ and $E$, but now let us pick a specific choice of $E$ which exists due to Classical Control (Def. 2.22):

$$E = \sum_i e_i$$

Now, by decomposing the classical identity as a sum over rank-1 projectors and using the definition of $E$ we obtain:

$$\tau \sum_i e_i \tau = \sum_i \tau e_i \tau$$

Therefore, by using Eq. (18) we find that:

$$\sum_i e_i = \sum_i e_i + \sum_{i \neq i} e_i$$
and so:

\[
\tau = \tau
\]

There is nothing special about \( \tilde{i} \) so we could equally prove this for any \( i \) just by considering different \( E \), therefore, we obtain:

\[
\tau = \tau
\]

and so, by the definition of finite tomography this means that \( e = \tilde{e} \). As these could be arbitrary effects we therefore have a unique effect for each system, which, by Eq. [17] must be deterministic. Hence, by Def. [2.5] the theory is causal.

Given this notion of summation and characterisation of the causal subtheory, we can prove some basic properties regarding the state spaces of systems and maps between them.

**Lemma 4.3.** In a theory with a classical interface (Post. [2]), the states form a finite dimensional pointed convex cone. Processes then induce positive linear maps between these cones. The causal states are defined by an intersecting hyperplane and causal processes preserve this hyperplane.

**Proof.** Firstly note that the scalars in the theory are going to be non-negative real numbers (scalars are just processes \( s : I \rightarrow I \) as there is a classical subtheory then \( I \) must be classical, hence, these scalars are classical, i.e. the non-negative real numbers). It is then clear that the state space of a given system \( A \) is a convex cone \( C_A \) as it is closed under linear combinations with non-negative real coefficients, i.e.:

\[
\sum_i r_i s_i := \sum_i r_i s_i
\]

is a valid state.

Allowing the coefficients of linear combination to be negative, the cone extends naturally to a real ordered vector space, spanned and ordered by the cone itself. By construction the cone is then full dimensional (i.e. spans the vector space) and it is simple to show that it is pointed (i.e. the zero-vector is in the cone and is the unique vector for which \( v \) and \(-v\) are in the cone). Moreover the cones are finite dimensional: this immediately follows from finite tomography as it implies that a state is characterised by a finite number of real numbers.

Thanks to distributivity of sums (Eq. [22]), any process \( f : A \rightarrow B \) induces a positive linear map between the vector space spanned by the states of system \( A \) and the vector space spanned by the states of system \( B \). This is defined as:

\[
\sum_i r_i s_i
\]

where linearity follows immediately as a special case of Eq. [22] as:

\[
\sum_i r_i (f \circ s_i)
\]

Hence, as it maps the convex cone of states of \( A \) to the convex cone of states of \( B \) the induced linear map is positive. Moreover, the map is *completely positive*, that is, it maps bipartite states to bipartite states.
An immediate corollary is that effects are linear functionals on the cone of states. Therefore \( \sum \circ s = 1 \) defines a hyperplane. This hyperplane intersects the cone, but not the origin: first consider \( \sum \circ s = 0 \) and note that \( \sum \circ s = 0 \implies \forall e \circ e = 0 \implies s = 0 \). Therefore, for any state \( s \neq 0 \) there is some scalar \( r_s \) such that \( \sum \circ (r_s s) = 1 \) hence the hyperplane intersects the cone. \( \square \)

**Remark 4.4.** The systems therefore have the same structure as found in a GPT [13, 52] apart from the fact that we have made no mention of the necessity of the convex sets to be closed. For related work connecting these frameworks see, for example, [95, 50, 86, 7, 93, 92].

**Remark 4.5.** Note that many of the above results can easily be extended to arbitrary processes, for example, the set of processes from \( A \) to \( B \) will generally form a finite dimensional proper cone with a convex subset of causal processes defined by a set of linear equality constraints.

We now consider how various key features of the quantum state space arise from our axioms. To begin with we show that the state-cones are homogeneous.

**Definition 4.6 (Homogeneous cone).** A convex cone \( C \) is homogeneous if for every pair of vectors \( s_1, s_2 \) internal to \( C \), there exists a cone automorphism \( T \) such that \( T(s_1) = s_2 \).

**Lemma 4.7.** If in addition to the finite classical interface (Post. 2) the theory satisfies symmetric purification (Post. 5) then the state cone is homogeneous (Def. 4.6).

**Proof.** We must show that, for any pair of internal states \( s_1 \) and \( s_2 \) there exists a cone automorphism \( T \) such that \( T(s_1) = s_2 \). As \( T \) is reversible this is equivalent to the statement that, there is a cone automorphism between a particular chosen internal state and any other. For this proof we take the particular internal state to be the ‘discarding’ state. We proceed in two steps, firstly, we show that there is a process that maps the discarding state to any other internal state, then secondly, we show that this map is surjective and so —as it is linear by Lem. 4.1— is a cone automorphism.

The first part is a simple corollary of symmetric purification. Consider an arbitrary internal state \( s \), then its purification \( S \) provides a map from the discarding state to \( s \):

\[
\begin{array}{ccc}
\text{Discarding} & \xrightarrow{\sigma} & A_B \\
A \xrightarrow{S} & \phantom{A} & \text{internal} \\
\end{array}
\]

For the second part, we adapt [24 Proposition 7] to show that \( S \) is surjective. Note that as \( s \) is internal, then any state \( a \) is in some decomposition of \( s \):

\[
\begin{array}{ccc}
\text{Discarding} & \xrightarrow{\sigma} & B \\
\text{internal} & \xrightarrow{p} & A \\
\end{array}
\]

where \( p \in (0, 1) \). Therefore we can construct the following dilation \( \sigma \) of \( s \) where \( \text{Discarding} \) and \( \text{internal} \) are causal, perfectly distinguishable states of some system \( A_B^2 \)

\[
\begin{array}{ccc}
\text{Discarding} & \xrightarrow{\sigma} & B \\
\text{internal} & \xrightarrow{p} & A \\
\end{array}
\]

This has a symmetric purification \( \Sigma \), which is moreover a purification of \( s \), hence we can construct two purifications of \( s \) with the same input and output systems:

\[
\begin{array}{ccc}
\Sigma & \xrightarrow{\sigma} & s \\
A & \xrightarrow{B} & A_B \\
\end{array}
\]

\[
\begin{array}{ccc}
\Sigma & \xrightarrow{\sigma} & s \\
A & \xrightarrow{B} & A_B \\
\end{array}
\]

Note that we can take \( B \) to be a classical bit to ensure that such states exist.
where $N_B := \frac{\tilde{\Pi} \circ \Pi^B}{\Pi}$. Then, by the definition of symmetric purification, these are related by some $R$ as:

\[
\Sigma = \frac{1}{N_B} S R = \frac{1}{N_B} R := S
\]

In the second step we use that $S$ is pure and so, by the definition of pure processes, we can replace the leak afterwards with a leak before. It is then clear that there is a state, $\alpha$, that is mapped to $p \alpha$ by $S$:

\[
\begin{align*}
\frac{S}{\alpha} := & \begin{array}{c}
\Sigma \\
\alpha
\end{array} = \begin{array}{c}
\Sigma \\
\alpha
\end{array} = \begin{array}{c}
\Sigma \\
\alpha
\end{array} = \begin{array}{c}
p \\
\alpha
\end{array}
\end{align*}
\]

Hence, $\frac{1}{T} \alpha$ is mapped to $a$. As $a$ is an arbitrary state $S$ is surjective, and hence is a cone automorphism. We therefore have homogeneity of the state cone.

Secondly, we show that the state spaces satisfy spectrality.

**Definition 4.8 (Spectrality).** Any (causal) state can be written as

\[
\begin{array}{c}
p \\
\alpha
\end{array} = \begin{array}{c}
S \\
\alpha
\end{array}
\]

where $p$ is a (causal) classical state and $S$ is a maximal testable state preparation (Def. 2.13).

**Lemma 4.9.** If, in addition to the finite classical interface (Post. 2) and the Homogeneity from Lem. 4.7, the theory has a sharp-dagger (Post. 4) then the state-cone is spectral (Def. 4.8).

**Proof.** Firstly note that, for any reversible transformation $T$ that:

\[
\begin{align*}
\begin{array}{c}
S \\
\alpha
\end{array} & \text{pure} \Rightarrow \begin{array}{c}
T \\
S
\end{array} \text{pure.}
\end{align*}
\]

To see this we must consider an arbitrary dilation, $D$, of this process:

\[
\begin{align*}
\begin{array}{c}
T \\
S
\end{array} = \begin{array}{c}
D
\end{array}
\end{align*}
\]

Any such dilation will provide a dilation of the original process by composing it with $T^{-1}$, that is:

\[
\begin{align*}
\begin{array}{c}
S \\
\alpha
\end{array} = \begin{array}{c}
T^{-1} \circ T \\
S
\end{array} = \begin{array}{c}
T^{-1} \circ D \\
S
\end{array}
\end{align*}
\]
As this original process is, by assumption, pure we therefore have:

\[ S = DT - 1 \]

and hence, by composing this with \( T \) we find that this arbitrary dilation \( D \), can be rewritten as a leak before or a leak after the dilated process, i.e.:

\[ S = Dl = lT \]

Therefore, the conditions necessary such that the process is pure are satisfied.

Now to obtain spectrality, note that every system \( A \) must have a (generally non-unique) maximal testable state preparation \( S: n \to A \) (Def. 2.13) although it could be trivial (i.e. \( n = 0 \)). For such a maximal testable state preparation, by the definition of the sharp dagger (Def. 4), we have:

\[ \dagger S = \dagger \]

and so taking the dagger of this equation we find that:

\[ S = 1 \]

Using this together with homogeneity, means that for any internal state \( t \), there exists a reversible map \( T \) such that:

\[ \dagger V = \dagger T \]

We then define a classical state and effect, \( p \) and \( \bar{p} \), by:

\[ \dagger p := \dagger T \]

and

\[ \dagger \bar{p} = \dagger \]

respectively.

Therefore we have:

\[ \dagger V = \dagger T \]

\[ := \dagger \]
We can then define a perfectly distinguishing measurement by:

\[
\begin{align*}
M &:= \begin{array}{c}
\cdots \\
\quad S \\
\quad \tau \\
\end{array}
\end{align*}
\]

One may worry that \( T^{-1} \) is not guaranteed to be a physical transformation, however, regardless, this measurement is well defined as each of the effects that make it up must be physical (as \( T \) induces an automorphism on the effect cone so does \( T^{-1} \)). Hence, this measurement can then be defined as a classically controlled process. It is then simple to check that the pair \((\tau, M)\) satisfy the conditions for the sharp dagger. Firstly that \( \tau \) is causal:

\[
\begin{align*}
\tilde{T} \tau &\Rightarrow \tilde{T} \tilde{\tau} = \tau \\
S &\Rightarrow \tilde{S} \tilde{\beta} = \beta \\
\end{align*}
\]

and secondly that \( M \) perfectly distinguishes the states of \( \tau \):

\[
\begin{align*}
M &\Rightarrow \begin{array}{c}
\cdots \\
\quad S \\
\quad T \\
\quad \tau \\
\end{array}
\end{align*}
\]

Therefore the definition of the sharp dagger implies that \( \tau \) is perfectly distinguished by \( \tau^\dagger \):

\[
\begin{align*}
\tilde{T} \tilde{\tau} &\Rightarrow \tilde{T} \tilde{\tau} = \tau \\
\end{align*}
\]

Moreover \( \tau \) is maximal as \( S \) was maximal and so:

\[
\begin{align*}
\tilde{T} \tilde{\tau} &\Rightarrow \tilde{T} \tilde{\tau} = \tilde{T} \tilde{\tau} \\
\end{align*}
\]

Therefore,

\[
\begin{align*}
\tilde{T} \tilde{\tau} &\Rightarrow \tilde{T} \tilde{\tau} = \tilde{T} \tilde{\tau} \\
\end{align*}
\]

satisfies the conditions of spectrality.

So far we have proved spectrality of the interior of the cone, we will now show that this can be extended to the full cone, and moreover, the entire vector space in which the cone lives. To do so we can adapt Corollary 21 from \[23\]. Note that any vector can be written as the difference of two internal states which can each be spectrally decomposed:

\[
\begin{align*}
\tilde{T} \tilde{\tau} &\Rightarrow \tilde{T} \tilde{\tau} = \tilde{T} \tilde{\tau} \\
\end{align*}
\]

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\[
\begin{align*}
\tilde{T} \tilde{\tau} &\Rightarrow \tilde{T} \tilde{\tau} = \tilde{T} \tilde{\tau} \\
\end{align*}
\]

satisfies the conditions of spectrality.
Then define,

\[ R := \max_i \left\{ \frac{\triangle \tau_1}{\triangle \tau_2} \right\} + \epsilon, \]

where \( \epsilon > 0 \). Then

\[ \triangle \tau_1 + R \triangle \tau_2 = \frac{\triangle \tau_1}{\triangle \tau_2} - \frac{\triangle \tau_2}{\triangle \tau_1} + R \frac{\triangle \tau_2}{\triangle \tau_1} \]

\[ = \frac{\triangle \tau_1}{\triangle \tau_2} - \frac{\triangle \tau_2}{\triangle \tau_1} + R \frac{\triangle \tau_2}{\triangle \tau_1} \]

\[ = \frac{\triangle \tau_1}{\triangle \tau_2} + \frac{\triangle \tau_2}{\triangle \tau_1} \]

is a state as all of the elements of the classical vectors are strictly positive thanks to the definition of \( R \), and moreover, is internal as there is a decomposition of the state as \( s + \epsilon \frac{1}{\epsilon} \) where \( s \) is a vector in the cone. Therefore, as this is an internal state it has a spectral decomposition:

\[ \triangle \tau_1 + R \triangle \tau_2 = \frac{\triangle \tau_3}{\triangle \tau_2} \]

and so we can write:

\[ \triangle \tau_1 = \frac{\triangle \tau_3}{\triangle \tau_2} - R \frac{\triangle \tau_2}{\triangle \tau_1} \]

\[ = \frac{\triangle \tau_3}{\triangle \tau_2} - R \frac{\triangle \tau_2}{\triangle \tau_1} \]

\[ = \frac{\triangle \tau_3}{\triangle \tau_2} - R \frac{\triangle \tau_2}{\triangle \tau_1} \]

\[ := \frac{\triangle \tau_3}{\triangle \tau_2} \]

We therefore have spectral decompositions for arbitrary vectors. \( \square \)

Finally, we show that the state-cones are strongly self dual.

**Definition 4.10** (Strongly self dual cone). A convex cone \( C \) is strongly self dual if there exists an inner product on the vector space spanned by \( C \), \( \langle \cdot, \cdot \rangle \) such that,

\[ x \in C \iff \langle x, c \rangle \geq 0 \quad \forall \ c \in C. \]

**Lemma 4.11.** If, in addition to the finite classical interface (Post. 2) and the Spectrality from Lem. 4.9 the theory has a sharp-dagger (Post. 4) then the state-cone is strongly self dual (Def. 4.10).

**Proof.** First we will show that the sharp-dagger provides an inner product defined as:

\[ \langle s_1, s_2 \rangle := \frac{s_1}{s_2} \quad \text{(19)} \]
and secondly we show that the state cone is strongly self dual with respect to this inner product. To show that this is a valid inner product, firstly, we check that this is symmetric:

$$\langle s_1, s_2 \rangle = s_1^\dagger s_2 = s_2^\dagger s_1 = \langle s_2, s_1 \rangle$$

Secondly that it is linear follows immediately from linearity of effects given by Lem. 4.3:

$$\langle s_1, \alpha s_2 + \beta s_3 \rangle = s_2^\dagger \alpha s_2 + s_3^\dagger \beta s_3 = \alpha \langle s_1, s_2 \rangle + \beta \langle s_1, s_3 \rangle$$

Finally, positivity follows easily from spectrality of arbitrary vectors:

$$\langle v, v \rangle = r^\dagger r \geq 0$$

where equality implies that $r = 0$ and hence $v = 0$. Eq. 19 therefore defines a valid inner product.

Note that if all elements of $r$ are strictly positive we have an internal state, if they are non-negative then we have a state and if any are negative then the vector cannot be a state as it would give a negative probability for some effect. It is then simple to check strong self-duality. Firstly, if $s$ is an element of the state cone $C$ then $\langle s, c \rangle \geq 0$ for all $c \in C$ as $\langle s, \cdot \rangle = s^\dagger$ is an effect and so evaluates to a positive real number on the cone of states. Conversely, if $v \not\in C$ then there is a negative coefficient in the spectral decomposition; without loss of generality we label this element $i$. There then exists some $c \in C$ such that $\langle c, v \rangle < 0$, that is:

$$\langle c, v \rangle = \tau^\dagger v = \tau < 0$$

The state cone is therefore strongly self dual with respect to the inner product defined by the sharp dagger.

These properties, in particular homogeneity and strong self duality, are well known to get us close to quantum theory, specifically, by using the Koecher-Vinberg theorem \[66, 90\] we get that the state cones correspond to Euclidean Jordan Algebras.

**Theorem 4.12** (Koecher-Vinberg theorem). There is a one-to-one correspondence between Euclidean Jordan Algebras and symmetric cones i.e. convex cones that are closed, pointed, homogeneous, and self-dual.

**Lemma 4.13.** Systems in our theory correspond to finite dimensional Euclidean Jordan Algebras.

**Proof.** By using the Koecher-Vinberg theorem above we must simply demonstrate that our state-cones are indeed symmetric cone and hence also correspond to EJAs.

First note that given a cone $C$ in a vector space $V$ we define the dual cone by

$$C' := \{v|v \in V \text{ s.t. } \langle v, u \rangle \geq 0 \text{ } \forall u \in C\}$$

This implies that the dual cone is closed as it is the intersection of closed half spaces, one for each $u \in C$ defined by $\langle v, u \rangle \geq 0$. Hence, strong self-duality implies that the state cone must be closed too. This therefore follows for our systems from Lem. 4.11 Lem. 4.1 then implies that the cones are pointed, finite dimensional and Lem. 4.7 that they are homogeneous. Hence the state spaces are finite dimensional symmetric cones. It therefore immediately follows that each system in our theory corresponds to a finite dimensional EJA.
There is a well known classification result for finite dimensional EJAs [61]: they correspond to
direct sums of five types of simple EJAs. There are two important properties of each of these, their
\textit{rank}, which corresponds to the number of states in a maximal testable state preparation, and their
\textit{dimension}, the minimal number of effects necessary for state tomography.

i. \( \mathbf{C}_n \), the algebra of self adjoint \( n \times n \) complex matrices. These have rank \( n \) and dimension \( n^2 \).

ii. \( \mathbf{R}_n \), the algebra of self-adjoint \( n \times n \) real matrices. These have rank \( n \) and dimension \( n^2(n+1)/2 \).

iii. \( \mathbf{H}_n \), the algebra of self-adjoint \( n \times n \) quaternionic matrices. These have rank \( n \) and dimension \( n(2n-1) \).

iv. \( \mathbf{O}_3 \), the algebra of self adjoint \( 3 \times 3 \) octonionic matrices. This has rank 3 and dimension \( 3^3 \).

v. \( \text{Spin}_K \), the spin factors. These have rank 2 and dimension \( K \).

Note that for the spin factors in the case of \( K = 3 \) coincides with \( \mathbf{R}_2 \), \( K = 4 \) with \( \mathbb{C}_2 \) and \( K = 6 \) with \( \mathbf{H}_2 \). Given this classification we are in a position to ask which of these EJAs is compatible with
our compositional structure.

\textbf{Theorem 4.14.} Given that state-cones correspond to EJAs, then local tomography (Def. 2.25) and
the existence of cups and caps (Post. 3) restrict us to Quantum Theory (Ex. 2.4).

\textbf{Proof.} We start by making two assumptions that need to be checked. Firstly that if a Euclidean
Jordan Algebra \( A \) is simple, then the composite \( A \otimes A \) is also simple. Secondly, that \( \text{Rank}(A \otimes A) = \text{Rank}(A)^2 \).

Firstly, we consider the composite \( \mathbf{H}_n \otimes \mathbf{H}_n \) this must be a simple-EJA with \( \text{Rank} = n^2 \) and
\( \text{Dim} = n^2(2n-1)^2 \), it is straightforward to check that this does not exist for \( n > 1 \). Therefore, the
quaternionic case is ruled out. Next we turn to the real case, \( \mathbf{R}_n \otimes \mathbf{R}_n \) and see that this requires a
simple-EJA with \( \text{Rank} = n^2 \) and \( \text{Dim} = n^2(n+1)^2/4 \) we already know that the quaternionic case is ruled
out, and it is again straightforward to rule out the other options as well. Considering the octonionic
case we have for \( \mathbf{O}_3 \otimes \mathbf{O}_3 \) that \( \text{Rank} = 3 \) and \( \text{Dim} = 3^2 \) again having ruled out the quaternionic and
real cases then it is simple to check that this is not satisfied by the complex case for which if \( \text{Rank} = 3 \)
then \( \text{Dim} = 3^4 \) and rule out the spin factors as they all have \( \text{Rank} = 2 \). Finally we consider the spin
factors \( \text{Spin}_K \otimes \text{Spin}_K \) which requires that \( \text{Rank} = 4 \) and \( \text{Dim} = K^2 \) we only have the complex case
to check now which implies that \( K = 4 \) is the situation when \( \text{Spin}_4 = \mathbb{C}_2 \) i.e. the cone is
the Bloch Ball. Hence, given our two assumptions, it is only the complex case which have valid self
composition. We can then check that the standard quantum tensor product i.e. \( C_n \otimes C_m := C_{nm} \) is
the only choice consistent with our constraints as it has \( \text{Rank} = nm \) and \( \text{Dim} = n^2m^2 \) as required.

To extend this result to non-simple cases we note that \( \otimes \) is bilinear and so distributes over \( \oplus \). Therefore our above result rules out any EJA with a non-complex component in its decomposition.

This leaves only the C*-algebras as valid systems, with their standard tensor product.

Now let us check the two assumptions we made. We intend to show that if \( A \otimes A \) is not simple,
then \( A \) is not simple either. Consider \( A \otimes A = B \oplus C \), then there is a decomposition of the identity
into orthogonal projectors onto \( B \) and \( C \):

\[
\begin{array}{cc}
\vdots & \vdots \\
\vdots & \vdots \\
\end{array}
= 
\begin{array}{cc}
P_B \\
P_C \\
\end{array}
\]

where \( P_I \circ P_J = \delta_{I,J} P_I \) for \( I, J \in \{B, C\} \).

We can use this to define a leak \( \Delta \) for system \( A \) by:

\[
\begin{array}{c}
\Delta \\
\end{array}
= 
\begin{array}{c}
P_B \\
P_C \\
\end{array}
\]

It is straightforward to check that this is indeed a leak, and moreover, using the orthogonality of $P_B$ and $P_C$ we can show that:

\[ \Delta = \Delta \]

(21)

Now consider the effect of the leak on pure states $\chi$, the definition of purity immediately implies that:

\[ \rho_\chi = p_x \rho_x \chi + (1 - p_\chi) \rho_c \chi \]

Now by considering this along with Eq. 21 implies that:

\[ p_x \rho_x \chi = p_x \chi \quad \text{and} \quad (1 - p_\chi) \rho_c \chi = p_\chi \chi \]

are orthogonal projectors, $p_i^\chi \circ p_j^\chi = \delta_{ij} p_i^\chi$ where $i, j \in \{b, c\}$ and therefore:

\[ A = b \oplus c \]

provides a decomposition of $A$ unless either $p_b^\chi$ or $p_c^\chi$ is zero. As $A$ is assumed to be simple this must be the case, so for each $\chi$ either

\[ P_B = 0 \quad \text{or} \quad P_C = 0 \]

The same argument can be made for the other input. Therefore, without loss of generality, consider some $\psi$ such that:

\[ P_B = 0 \]

this means that

\[ \forall \chi \quad P_B = 0 \]

and therefore we must have

\[ \forall \chi \quad P_C \neq 0 \]

otherwise applying identity to the state $\chi$ would give 0. However, as either $P_B$ or $P_C$ must evaluate to zero on any given state this then implies that

\[ \forall \chi \quad P_B = 0 \]

27
which then, by local tomography, implies that

\[ P_B = 0 \]

We therefore have \( A \otimes A = C \oplus B = C \) and so \( A \otimes A \) is not decomposable. This contradicts our initial assumption, so if \( A \) is simple then so is \( A \otimes A \).

Now let us consider the second assumption. It is clear that \( \text{Rank}(A \otimes B) \geq \text{Rank}(A) \text{Rank}(B) \) as given a set of pure and perfectly distinguishable states for \( A \) and for \( B \) we can form a set of the composite as just all possible product of these. We then know that (using tomographic locality):

\[ \overrightarrow{I_{A \otimes B}} = \overrightarrow{I_A} \overrightarrow{I_B} \]

and as the discarding map is an internal effect we have a spectral decomposition of this effect given by the composite of the spectral decomposition of the two individual discarding maps. Hence, \( \text{Rank}(A \otimes B) \leq \text{Rank}(A) \text{Rank}(B) \), the conjunction of these two inequalities then gives the required result.

This completes the reconstruction as we have demonstrated that our postulates are equivalent to systems being finite-dimensional C*-algebras and processes being completely positive trace preserving maps between them: precisely the process-theoretic description of Quantum Theory (Ex. 2.4) that we were aiming for.

5 Singling out fully quantum and classical systems

In the previous section we reconstruct the full process-theoretic description of quantum theory. However, it is still of interest to ask what distinguishes the fully quantum and classical subtheories. We now introduce two novel ways to do this and moreover demonstrate that they are equivalent for any process theory with cups and caps.

**Proposition 5.1** (Purity of cups (and/or caps) restricts to quantum systems).

*Proof.* In Ex. 2.16 we defined the cup for C*-algebraic systems, it is simple to see that this has the following dilation:

\[ \bigcup_k A_k = \bigoplus_k A_k \overrightarrow{A_k} = \bigoplus_k A_k \overrightarrow{A_k} = \bigcup_k A_k \overrightarrow{A_k} \]

where \( \{k\} \) are some set of perfectly distinguishable states. Considering the definition of purity for states (Ex. 2.12) we find that for the cup to be pure any dilation must separate. This can only be the case if \( k \) takes only a single value, hence it is pure if and only if the C*-algebra has only a single branch, that is, when it is a fully quantum system.

In other words, it is really the existence of correlations mediated by pure states (and/or effects) that is the distinguishing feature of quantum theory. Any other apparently odd feature of quantum theory should, in principle, be able to be traced back to this.

**Proposition 5.2** (Triviality of leaks restricts to quantum systems).

*Proof.* For quantum theory all leaks are trivial, and for any other C*-algebra we can construct a non trivial leak by leaking the ‘which branch’ information, i.e.:

\[ \bigcup_k A_k = \bigoplus_k A_k \overrightarrow{A_k} = \bigoplus_k A_k \overrightarrow{A_k} \overrightarrow{A_k} = \bigcup_k A_k \overrightarrow{A_k} \overrightarrow{A_k} \]

is only trivial when \( k \) takes a single value and the system is fully quantum.
This can be seen as a generalised no-broadcasting theorem, or, that information gain always causes disturbance of a quantum system. This does not seem to be a particularly surprising feature of nature. At least, as soon as one takes the view that measurements should be described by an interaction between systems, then even going back to Newton it is not surprising that this should have an effect on the system being measured.

The fact that we have these two ways to characterise quantum systems is also not surprising as it can generally be shown that:

**Proposition 5.3.** For any process theory with cups and caps, purity of the cup (or cap) is equivalent to the triviality of all leaks.

**Proof.** Firstly note that there is a one-to-one correspondence between the leaks for a system and dilations of the cup. That is, any leak, \( l \), will define a dilation of the cup, \( L \):

\[
L := \begin{array}{c}
\text{cup} \\
\text{dilation}
\end{array}
\]

Moreover, any dilation, \( S \), of the cup will define a leak, \( s \) as:

\[
\begin{array}{c}
\text{cup} \\
S
\end{array} := \begin{array}{c}
\text{dilation} \\
\text{leak}
\end{array}
\]

Then it is clear, that if the leak is trivial then the corresponding dilation factorises and vice versa:

\[
\begin{array}{c}
\text{cup} \\
S
\end{array} = \begin{array}{c}
\text{dilation} \\
\text{factorises}
\end{array} \iff \begin{array}{c}
\text{leak} \\
\text{trivial}
\end{array} = \begin{array}{c}
\text{dilation} \\
\text{factorises}
\end{array}
\]

Therefore, triviality of all leaks is equivalent to factorisation of all dilations of the cup, which is precisely what it means for the cup to be pure (see Ex. 2.12). The proof for the cap rather than cup proceeds similarly.

Similarly to singling out the quantum systems one could ask how the classical systems can be distinguished. Intuitively, these can be seen as those that are maximally leaking, and which have the minimally pure cups and caps. All of the other systems therefore have intermediate strength of leaks and intermediate purity of cups and caps. To formalise this notion, we would have to introduce a measure of purity (for example \( \text{tr}(\rho^2) \)) and a measure of strength of leaks. (for example the quality of a leak [80]). It is interesting therefore to note that fully quantum systems are much easier to single out from this perspective—as it does not require such measures to be introduced—and so, the fully quantum theory appears to be, in some sense, the more fundamental, or natural, subtheory.

### 6 Related work

#### 6.1 Purification

In [80] we showed that leaks must be taken into account when defining purity for general processes in general process theories. In particular, in the case of classical theory, this refined definition is essential for purity of the identity process. Given this notion of purity we can then define the symmetric purification postulate which holds for both quantum and classical theory.

It is therefore clear that the essential component of the standard purification postulate in distinguishing quantum and classical theory is the purity of the cup. Indeed, we can show that the conjunction of symmetric purification and the existence of a pure cup implies the standard purification postulate.

**Proposition 6.1.** Any theory with symmetric purifications and pure cups satisfies the standard purification postulate.
Proof. Firstly we note that given a symmetric purification $F$ of any process $f$, we can bend the ‘discarded’ input into an output using a cup:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
B \\
A
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
F \\
A \\
B
\end{array}
\end{array}
\end{array}
\]

such that $F$ provides a standard purification for $f$. Next we must check the essential uniqueness holds. Note that any purification can be considered as a symmetric purification where the input system is trivial:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
B \\
A
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
F \\
A \\
C
\end{array}
\end{array}
\end{array}
\]

Therefore, if we have two such purifications $F$ and $G$ then the essential uniqueness condition of symmetric purification implies that:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
B \\
A
\end{array} \\
\begin{array}{c}
\begin{array}{c}
F \\
A \\
C
\end{array}
\end{array}
\end{array}
\end{array}
\]

where $R$ is a pure causal process, hence is reversible (as is proved in [50]). This is precisely the standard essential uniqueness condition for purification. 

An alternative way of viewing this is that the pure cups and caps available for fully quantum systems allow us to arbitrarily interchange uncertainty about the past and uncertainty about the future. In contrast, in classical theory past uncertainty and future uncertainty play distinct roles.

6.2 GPTs

Most recent reconstructions of quantum theory have been in the generalised probabilistic theory (GPT) framework. These typically describe systems as finite dimensional, regular, closed cones, along with an intersecting hyperplane specifying the normalised states. Transformations are then described as linear maps and, in particular, effects are linear functionals on the state space. In our reconstruction we see, in Lem. 4.1, that we obtain the essence of this structure from our classical interface. In fact our classical interface can be interpreted as representing the probabilistic layer of the GPT framework in a process theoretic manner.

This classical interface however, does not provide all of the structure of the GPT framework. Specifically, we do not obtain that the state space is closed. Closure is—from an operational perspective— a very natural assumption, as, up to any finite error, a state space should be indistinguishable from its closure. It is however not clear yet how this property should be understood from the process theoretic perspective.

Despite the lack of closure, we are still able to use some tools commonly found in the GPT literature—in particular the Koecher-Vinberg theorem. To do this we have focused on the states of the theory. However, our postulates apply equally well to arbitrary processes, and so, many of our results can immediately be generalised. We have omitted such generalisations in an attempt to keep the mathematics of the paper to a minimum and present only the results that directly pertain to the reconstruction.

6.3 Reconstructions

There have been several other reconstructions of quantum theory from various different perspectives, many results of which have been adapted for this work. It is therefore worth highlighting the features of this particular reconstruction which distinguish it from the others.
Firstly, the postulates that we impose are diagrammatic, moreover, they do not pick out states as being special, instead applying equally well to all processes. As such, they fit with the spirit of Categorical Quantum Mechanics and the process-theoretic understanding of the world, that is, as being about processes and composition.

Secondly, the reconstruction is relatively simple. In particular, the structure of the reconstruction is clear (see Fig. 1) allowing for a high-level view of how the different postulates relate to each other and how they are used in each step. This should make it simpler to understand how relaxing or altering any given postulate will lead to new theories. In contrast, in many other reconstructions it is difficult to know precisely which postulates, and which of the assumptions of the framework, are necessary to obtain each result.

Finally, we reconstruct the full process-theoretic description of Quantum Theory. Rather than aiming to reconstruct quantum theory in its standard presentation we instead first reconstruct the process theory of C*-algebras and CP maps. This allows for a unified way to describe both quantum and classical systems. We argue that this is how quantum theory should be described, and, moreover, this perspective allows us to clearly see what it is that separates the fully quantum systems from the others—specifically, the triviality of leaks, or equivalently, purity of the cups. In contrast, the majority of other reconstructions take transitivity as a postulate\(^3\). Transitivity is the property that there is a reversible transformation between any pair of pure states. This does not hold for general C*-algebras and so most other reconstructions immediately rule out the full C*-algebraic theory.

7 Future work

Whilst the postulates of this reconstruction are diagrammatic the proof ultimately relies on standard linear algebraic techniques. Moreover, whilst the postulates are defined at the level of processes we often only use them in the context of states. Both of these are against the spirit of process theories, as such, it would be interesting in future work to try to make the proof of the reconstruction process-theoretic along with postulates. Indeed, as discussed earlier, it seems plausible that by using the full strength of the postulates that there may be a much simpler and direct way to go about the reconstruction. Another more recent categorical reconstruction of quantum theory [87, 88] may help providing a route to this.

As with all reconstructions, we assume the existence of a classical interface for the theory representing how we interact with the world, for example, deciding the experiments to perform, and obtaining classical data as outputs. However, at some level we expect this classical interface to be an emergent feature of quantum theory, and so it would be interesting to see if we can move beyond this probabilistic approach. Can we instead express everything in diagrammatic terms and find some replacement for the classical interface? For example, can we capture the copiability and deletability of classical data via ‘spiders’ [40, 39, 36].

A second issue with the classical interface (as it is defined here) is that it immediately limits the scope of the reconstruction to finite dimensional quantum theory. It would be interesting to see whether our axioms could reconstruct quantum theory even if we removed this limitation, that is, if we removed the ‘finite’ part of finite tomography. If not, it would be interesting to explore these alternative theories and how close they are to quantum theory. One potential approach to this would be to follow the work of [48, 19] which demonstrates how to extend various diagrammatic features—such as cups and caps—to the infinite dimensional setting.

There are many recent results in quantum foundations [83, 68, 67, 69, 25, 26, 27, 28, 10] which use the standard notion of purification in the derivation but where the result is also valid in classical theory. It therefore seems plausible that the same results could be obtained using our notion of symmetric purification, and so, apply to a wider range of theories. Another recent research direction in quantum foundations is in formulating quantum theory in a causally neutral [70, 76] or time symmetric [75, 3] way or with indefinite causal order [53, 5, 64, 17, 20]. This notion of purification may be more applicable in such situations as it does not distinguish between input and output systems.

We presented the cups and caps of postulate [5] as a relaxation of a basic process-theoretic constraints on compositionality. That is, by freely allowing inputs to be connected to inputs and outputs to be connected to outputs. One could equally ask what structure do we obtain if we relax the

\(^3\)Or have it as a direct corollary of a stronger postulate such as purification or strong symmetry.
other constraints on compositionality. In particular, relaxing the constraint that only pairs of inputs and outputs are connected means that we need some sort of notion of ‘multi-wires’. This naturally gives the structure of ‘spiders’ described in [36]. Moreover, allowing for different system types to be connected leads to many other typical quantum processes. It is the subject of ongoing work to investigate how much of the structure of quantum theory can be described in such terms, and, how to reconcile such a perspective with causality and the classical interface.

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A Diagrammatic sums

Definition A.1 (Diagrammatic summation). A sum is an associative, commutative, binary operation with a zero element on processes with the same inputs and outputs, i.e. given a set of processes \( f_i : A \rightarrow B \) there is a process \( \sum f_i : A \rightarrow B \). Moreover it distributes over diagrams, that is for all \( \chi, C \) and \( D \):

\[
\sum_i f_i \chi = \chi \sum_i f_i , \quad (22)
\]

where \( \chi \) is shorthand for a diagram of that shape, i.e.:

\[
\begin{align*}
\chi := & \quad \begin{array}{c}
C \\
D
\end{array} \\
\begin{array}{c}
A_1 \\
B_1
\end{array} \\
\begin{array}{c}
E_x \\
x_x
\end{array}
\end{align*} \quad , \quad (23)
\]

Example A.2 (Quantum). We can also now discuss sums of processes within quantum theory. This is given by the standard sum of CP maps, such that if we have processes \( f, g : A \rightarrow B \) then we can construct a new process \( f + g : A \rightarrow B \). Linearity of CP maps and bilinearity of the tensor product ensures that these distribute over diagrams.

In general such a sum will not exist in the causal subtheory, at least, in the probabilistic theories that we are interested in here. The structure of diagrammatic summation however gives us some insights into the structure of the causal sub-theory. For example, in a theory where the scalars are
non-negative real numbers, if \( f \) and \( g \) are causal processes, then:

\[
p \begin{array}{c} f \end{array} \quad \text{and} \quad (1 - p) \begin{array}{c} g \end{array}
\]

with \( p \in (0, 1) \), are not. However, their sum is a causal process as

\[
p \begin{array}{c} f \end{array} + (1 - p) \begin{array}{c} g \end{array} = p \begin{array}{c} f \end{array} + (1 - p) \begin{array}{c} g \end{array} = (p + (1 - p)) \begin{array}{c} \cdot \end{array}
\]

That is, convex combinations of causal processes are themselves causal. Introducing sums in such theories therefore ‘reveals’ the convex structure of the subcausal theory. This is a common theme in this paper. Many compositional structures that we use, are defined in the full theory, but, still have an important impact on the causal sub-theory. In particular, this will be the case for postulates 3 and 4 as is discussed in detail in section 2.3.

B Classification of leaks and pure processes

We now classify the leaks and pure processes for Quantum Theory. First however, recall the representation of a finite dimensional C*-algebra \( \mathcal{A} \) from Ex. 2.4

\[
\mathcal{A} = \bigoplus_i A_i \quad \text{where} \quad A_i = \mathcal{B}(\mathcal{H}_i)
\]

and recall that processes in the processes theory are completely positive maps between these C*- algebras. We now introduce a few processes and basic results that will be used in this appendix.

1. Given a C*-algebra \( \mathcal{A} = \bigoplus_i A_i \) there is a decomposition of the identity into orthogonal projectors

\[
\begin{array}{c} \mathcal{A} \end{array} = \sum_i \begin{array}{c} \cdot \end{array}
\]

2. such that each of these projectors splits through an irreducible C*-algebra, that is

\[
\begin{array}{c} \mathcal{A} \end{array} = \bigoplus_i A_i \quad \text{where} \quad \begin{array}{c} \bigoplus \end{array} \begin{array}{c} A_i \end{array} \bigoplus \begin{array}{c} A_i \end{array} = \begin{array}{c} A_i \end{array}
\]

3. this decomposition of the identity provides us with a matrix representation of processes with input \( \mathcal{A} = \bigoplus_{i=1}^n A_i \) and output \( \mathcal{B} = \bigoplus_{j=1}^m B_j \) as follows

\[
\begin{array}{c} \mathcal{B} \end{array} = \sum_{ij} \begin{array}{c} \cdot \end{array} \begin{array}{c} f_{ij} \end{array} := \sum_{ij} f_{ij} \sim (f_{ij})_{i=1,\ldots,n}^{j=1,\ldots,m}
\]
4. where each $f_{ij}$ defines a map between the irreducible C*-algebras $A_i$ and $B_j$

\[
\begin{array}{c}
\text{B}_j \\
\text{f}_{ij} \\
\text{A}_i
\end{array}
\]

5. Note the discarding map for the C*-algebra and for the irreducible components are related by

\[
\overline{\downarrow A_i} = \overline{\downarrow A_i}
\]

6. Finally, note that,

\[
F = 0 \iff F = 0
\]

Recalling the definition of pure processes (Def. 2.11) we see that to understand what the pure processes are we must understand what the leaks are for these systems. That is, what leaks do we have for a C*-algebra? For fully quantum systems all leaks are trivial, that is, any leak separates:

\[
\begin{array}{c}
\text{L} \\
\text{A}
\end{array}
\]

however, in classical theory, and more general C*-algebras the leaks are more interesting as we will now demonstrate.

**Proposition B.1.** Given a C*-algebra, $\mathcal{A} = \bigoplus A_i$ any leak can be written as:

\[
\left(\begin{array}{c}
\text{L} \\
\text{A}
\end{array}\right) = \sum_i \left(\begin{array}{c}
\text{L} \\
\text{A}
\end{array}\right)_{A_i}
\]

**Proof.** Note first that any leak for $\mathcal{A}$ defines a leak for each of the $A_i$ by pre- and post- composing with the relevant projector, hence, as the $A_i$ are fully quantum systems these leaks must be constant:

\[
\begin{array}{c}
\sum_i \left(\begin{array}{c}
\text{L} \\
\text{A}
\end{array}\right)_{A_i} = \sum_i \left(\begin{array}{c}
\text{L} \\
\text{A}
\end{array}\right)_{A_i}
\end{array}
\]

We therefore have

\[
\begin{array}{c}
\sum_{ij} \left(\begin{array}{c}
\text{L} \\
\text{A}
\end{array}\right)_{A_i} = \sum_i \left(\begin{array}{c}
\text{L} \\
\text{A}
\end{array}\right)_{A_i} + \sum_{i\neq j} \left(\begin{array}{c}
\text{L} \\
\text{A}
\end{array}\right)_{A_i}
\end{array}
\]

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However, by decomposing the identity as a sum of the projectors and using the defining equation of a leak we find

\[
\sum_i \begin{array}{c}
\hline
1
\hline
\end{array}
= \begin{array}{c}
\hline
\hline
\hline
\hline
1
\hline
\end{array}
= \sum_i \begin{array}{c}
\hline
1
\hline
\end{array} + \sum_{i \neq j} \begin{array}{c}
\hline
\hline
\hline
\hline
1
\hline
\end{array}
\]

and hence if \( i \neq j \)

\[
\begin{array}{c}
\hline
1
\hline
\end{array}
= 0 \quad \text{and so} \quad \begin{array}{c}
\hline
\hline
\hline
\hline
1
\hline
\end{array} = 0
\]

Combining this with equation 24 provides us with the result. Using classical control then allows us to write this in the form

\[
\begin{array}{c}
\hline
A
\hline
\end{array}
\begin{array}{c}
\hline
L
\hline
\end{array}
\begin{array}{c}
\hline
i
\hline
\end{array}
\]

where \( \begin{array}{c}
\hline
L
\hline
\end{array} := \begin{array}{c}
\hline
\hline
\hline
\hline
0
\hline
\end{array} \)

which completes the proof.

We are now in a position to understand what the pure processes are for Quantum Theory.

**Proposition B.2.** Pure processes in Quantum Theory are processes whose matrix representation has i) pure processes as matrix elements, and ii) at most a single non-zero process in each row and column.

*Proof.* Consider a pure process \( f : A \to B \) where \( A = \bigoplus_i A_i \) and \( B = \bigoplus_j B_j \), and its matrix representation \( f = \sum_{ij} f_{ij} \). Now given the leak of system \( B \) which leaks the ‘which branch’ information to a classical system

\[
\begin{array}{c}
\hline
B
\hline
\end{array} := \sum_j \begin{array}{c}
\hline
\hline
\hline
\hline
0
\hline
\end{array}
\]

we find that

\[
\begin{array}{c}
\hline
f
\hline
\end{array} = \begin{array}{c}
\hline
\hline
\hline
\hline
0
\hline
\end{array} = \begin{array}{c}
\hline
\hline
\hline
\hline
0
\hline
\end{array} = \begin{array}{c}
\hline
\hline
\hline
\hline
1
\hline
\end{array} = \sum_{ij} \begin{array}{c}
\hline
\hline
\hline
\hline
f_{ij}l_{ij}
\hline
\end{array} = \sum_{ij} \begin{array}{c}
\hline
\hline
\hline
\hline
f_{ij}l_{ij}
\hline
\end{array}
\]

Hence we find that \( f_{ij}l_{ij} = f_{ij} \) and so either \( f_{ij} = 0 \) or \( l_{ij} = 1 \). However, as we know that \( l \) is causal this means that \( \sum_j l_{ij} = 1 \) for all \( i \) and so for each \( i \) there is only a single value of \( j \) such that \( l_{ij} = 1 \). Therefore for that \( i \) all other values of \( j \) must result in \( f_{ij} = 0 \). That is, the matrix representation of \( f \) has at most a single non-zero element in each row. We can make an equivalent argument starting with the leak at the bottom pushing it through to the top, in this case we find that there is at most one non-zero element in each column.
Now note that every dilation $F$ of $f$ must be given by some leak, which means that

$$
\sum_{ij} F = f = \sum_{ij} F
$$

therefore

$$
\sum_{ij} F = f
$$

and so every dilation of each of the $f_{ij}$ must separate. In other words, the $f_{ij}$ are pure quantum processes.

To summarise, from the ‘commutativity’ conditions we find that a pure process $f$ maps each branch of $A$ to at most one branch of $B$ and vice versa, and, from the ‘all dilations are leaks’ condition we find that these maps between branches are themselves pure.

\[\square\]

\section{Symmetric purification in quantum theory}

Given the characterisation of the pure processes in App. B we can show that Quantum Theory satisfies the symmetric purification postulate.

\textbf{Theorem C.1.} Completely positive maps between C*-algebras have symmetric purifications.

\textit{Proof.} First let us consider the quantum case. We know that any process $f : A \to B$ can be purified to a process $F : A \to B \otimes C$ where $C = A \otimes B$, and so it is simple to see that it can also be purified in a symmetric way $F : A \otimes B \to B \otimes A$:

Where we have used the fact that the caps are pure for fully quantum systems and that the composite of pure processes is pure. It therefore just remains to check that any two such purifications are related in the correct way. Consider two such purifications $F$ and $G$, then we can use the cup—as it is pure for fully quantum systems—to define two more purifications as:

these must be related by a reversible, and hence causal, transformation $r$:
Using the caps for a second time therefore means that:

\[
F = G_r := G_R (25)
\]

It is simple to check that causality of \( r \) implies that \( R \) satisfies Eq. (13) as required.

Next we turn to the general C*-algebraic case. Consider a process \( f : A \rightarrow B \) where \( A = \bigoplus_i A_i \) and \( B = \bigoplus_j B_j \) with \( \{ A_i \} \) and \( \{ B_j \} \) irreducible C*-algebras, i.e. fully quantum systems. It is simple to check that this does have a symmetric purification by noting that we can define a dilation of a process \( f \) by symmetrically purifying the quantum maps of matrix representation

\[
f = \sum_{ij} f_{ij} = \sum_{ij} F_{ij} = \sum_{ij} F_{ij} := F
\]

It is then simple to check that this dilation is moreover pure and hence a symmetric purification of \( f \). Like for the quantum case, the more interesting part to check is Eq. (12).

Given some process \( f \) with two symmetric purifications \( F \) and \( G \), i.e.

\[
F = f = G
\]

let us define:

\[
F := X_{ij} \quad \text{and} \quad G := Y_{ij}
\]

Noting that these are purifications of the same fully quantum process, that is

\[
X_{ij} = f = Y_{ij}
\]

and so using the above result for fully quantum systems we obtain:

\[
X_{ij} = Y_{ij} R_{ij}
\]
and therefore:

\[
F = \sum_{ij} X_{ij} = \sum_{ij} Y_{ij} R_{ij} := \sum_{ij} Y_{ij} R = \sum_{ij} Y_{ij} R
\]

where in the last step we have used classical control to construct \( R \) and defined the forwards and backwards leaks as:

\[
\begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\end{array}
= \sum_j \begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\end{array}
\text{ and }
\begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\end{array}
= \sum_i \begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\end{array}
\text{ respectively.}
\end{array}
\]

That \( R \) satisfies Eq. 13 follows directly from the quantum case.

\[\square\]

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