Effect of weak geometrical forcing on the stability of Taylor-vortex flow

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Abstract. Linear stability analysis of fully developed axisymmetric steady and spatially modulated Taylor-Couette flow is carried out in the narrow-gap limit. The inner cylinder is sinusoidally modulated and rotating, while the outer cylinder is straight and at rest. The modulation amplitude is assumed to be small, and the base steady flow is determined using a regular perturbation expansion of the flow field coupled to a variable-step finite-difference scheme. The disturbance flow equations are derived within the framework of Floquet theory and solved using a nonlinear two-point boundary-value approach. In contrast to unforced Taylor-Couette flow, only vortical base flow is possible in the forced case. It is found that the forcing tends to generally destabilize the base flow, especially around the critical point. Both the critical Taylor number and wavenumber are found to decrease essentially linearly with modulation amplitude.

1. Introduction
The flow between concentric cylinders, or Taylor-Couette flow (TCF), as described by Taylor [1], is one of the most investigated problems of the motion near rotating bodies [2,3] given the fundamental interest in vortex flow. The onset of vortices also occurs in geometries other than straight circular cylinders. In this connection, Wimmer [4], among others, has investigated experimentally the flow between two rotating spheres. The flow between two-coaxial cones [5] and a variety of cone-cylinder combinations [6] were also studied by Wimmer. In the past, a number of modifications to the straight cylindrical TCF system have been the subject of investigations, such as the flow around a cylinder with hourglass [7] and V-grooved [8] geometry. One of the simplest geometric perturbations of TCF is when one or two of the cylindrical boundaries are allowed to have an axially-periodic radius. Although this problem is of interest in several practical contexts, with particular relevance to mixing, and is obviously of fundamental significance, little work has been devoted to its investigation, especially at the theoretical level. Besides work on the effect of localized radius variation [9,10], the only work on extended (periodic or random) radius variation known to us is the experimental work by Koschmieder [11], Ikeda and Maxworthy [12], and Painter and co-workers [13,14]. More recently, Khayat and co-workers examined theoretically the effect of cylinder spatial modulation on the onset of vortex flow, covering both Newtonian [15,16] and viscoelastic [17] fluids. Their work, was, however, limited to steady state flow.

The current study considers a geometry consisting of two coaxial cylinders such that the outer smooth cylinder is stationary while the inner rotating cylinder has an axi-symmetric wavy surface.
This geometry can be considered as an extension of Wimmer’s cone-cylinder configuration [5], and is closely related to the flow in a hourglass geometry [7]. The fundamental significance of this problem cannot be overstated. In this connection, measurements in experimental dynamical systems, such as the well-studied Rayleigh-Benard system, continue to become more precise while key length scales, such as the layer height in convection, continue to shrink [18]. With these trends, investigators are beginning to find significant discrepancies between theory and experiment. One cause of these discrepancies may be the spatial variability in the system due to imperfections. Systems with spatial variability have been much less studied than those which are relatively close to ideal. However, in experiments on the analogue of Rayleigh-Benard convection in a fluid-filled porous layer, spatial inhomogeneity plays an important role in pattern selection [19,20]. Zimmermann et al. [21] have studied the effects of spatial variability theoretically. Several related issues have also been studied. These include the role of smoothly ramped [22,23] and periodic [24] spatial variations in the control parameter. Finally, it is important to observe that the spatial modulation of cylindrical flow is also closely analogous to the presence of end effects in TCF. In both cases the forcing makes itself felt at pre-critical values of the Taylor number, Ta, and the periodicity of the vortex pattern is dictated by the spacing of the cylinder ends and modulation wavenumber. It is well established that vortices adjacent to the end plates form far below Ta [2]. When the pre-critical Taylor number (Ta < Ta) is gradually increased, additional vortices form on top of each other until the vortices coming from the (symmetric) top and bottom plates meet in the middle of the column, at the critical Taylor number predicted by linear analysis.

In close connection with the present problem, Li and Khayat [15] examined theoretically steady modulated axisymmetric Taylor-vortex flow, with weak forcing. The study clarifies the origin of the discrepancy between two earlier experiments [11,12], and reports on the conditions for emergence and structure of low Taylor-number secondary flow as a result of cylinder modulation. Low-inertia vortex formation and pattern selection were examined. The modulation amplitude was assumed to be small, and a regular perturbation expansion was used to determine the flow field at small to moderately large Taylor numbers (below the critical threshold). It was found that the presence of a weak modulation leads unambiguously to the emergence of steady Taylor-vortex flow, even at vanishingly small Taylor number. The vortex structure was found to have the same periodicity as the forcing. It was also shown that, for any modulation amplitude, the forcing wavenumber that generates the most intense vortex flow for a given Taylor number varies monotonically with Ta, but always reaches the critical value predicted by linear stability analysis for straight cylinders, regardless of which cylinder is modulated. The present work addresses the important question regarding the stability of the steady states computed previously [15]. Although these states are expected to be stable at low Taylor number, their stability is simultaneously expected to be lost as Ta exceeds a critical value that depends on the spatial forcing. The stability picture is examined for small amplitude sinusoidal modulation of the inner cylinder wall using Floquet theory. The method is validated by first recovering stability results for the flow between straight cylinders.

2. Problem formulation
The problem is formulated in two parts. In the first part, the general equations and boundary conditions for steady spatially forced TCF are derived for small-amplitude modulation in the narrow gap limit. A regular perturbation expansion for the flow field is carried out after the equations are mapped onto the rectangular domain. The second part is devoted to the linear stability analysis of the base flow.

2.1. Governing equations
Consider the steady flow of an incompressible Newtonian fluid, of density \( \rho \) and kinematic viscosity \( \nu \), between two concentric infinite cylinders, as shown schematically in Fig. 1. The inner cylinder is assumed to be periodically modulated along the axial direction, \( Z \), and the outer cylinder is straight.
Figure 1. Schematic of modulated TCF, showing the inner modulated cylinder and the straight outer cylinder.

The inner cylinder is assumed to rotate at constant angular velocity, $\Omega_i$, while the outer cylinder is at rest. Let $\mathbf{U} = (U_R, U_\Theta, U_Z)^T$ be the velocity vector in cylindrical polar coordinates $(R, \Theta, Z)$, and $P$ the pressure. Let $T$ denote the time variable. The radii of the inner and outer cylinders are denoted by $R_i$ and $R_o$, respectively, with the inner-cylinder radius given by

$$R_i(Z) = \bar{R}_i + A_i F_i(Z),$$

(1)

where $\bar{R}_i$ is the mean radius of the inner cylinder; $A_i$ is the amplitude of modulation; and $F_i(Z)$ represents the shape of the modulation. The gap between the cylinders is given as $D(Z) = R_o - R_i$. The fluid is assumed to adhere to the cylinders, so that

$$U_R(r = R_i, Z, T) = U_Z(r = R_i, Z, T) = 0, \quad U_\Theta(r = R_i, Z, T) = R_i \Omega_i,$$

$$U_R(r = R_o, Z, T) = U_Z(r = R_o, Z, T) = U_\Theta(r = R_o, Z, T) = 0.$$  

(2)

The conservation equations are reduced to the narrow-gap limit by introducing dimensionless coordinates, $x$ and $z$, in the radial and axial directions, respectively, time $t$, pressure $p$, and velocity components $u, v, w$, as follows:
\[ x = \frac{2R - (R_i + R_o)}{2D}, \quad z = \frac{Z}{D}, \quad t = \frac{v}{D^2} T, \]  
\[ p = \frac{D^2}{\rho v^2} P, \quad u = \frac{D}{v} U_R, \quad v = \frac{1}{R_i \Omega_i} U_\theta, \quad w = \frac{D}{v} U_Z, \]  

where \( D = R_o - R_i \) is the mean gap width. Note that the ratio of mean gap width and inner cylinder radius is small (narrow gap limit). After non-dimensionalization, the conservation equations in the narrow-gap limit are obtained, namely

\[ u_x + w_z = 0 \]  
\[ u_t + uu_x + wu_z = Ta v^2 + u_{xx} + u_{zz} - p_x \]  
\[ v_t + uv_x + wv_z = v_{xx} + v_{zz} \]  
\[ w_t + uw_x + ww_z = w_{xx} + w_{zz} - p_z \]

where a subscript denotes partial differentiation. The Taylor number, \( Ta \), is defined in terms of the Reynolds number, \( Re \), and the average gap-to-radius ratio, \( \delta \). Thus,

\[ Ta = Re^2 \delta, \quad Re = \frac{R_i \Omega_i D}{v}, \quad \delta = \frac{D}{R_i}. \]  

The dimensionless physical domain is defined by \( (x, z) \in [-1/2 + \varepsilon f(z), 1/2] \times (-\infty, +\infty) \), where \( \varepsilon = \frac{A_i}{R_i} \) is a measure of the dimensionless modulation amplitude. Note that \( \varepsilon = \frac{A_i}{R_i} = \frac{A_i D}{D R_i} = \frac{A_i}{D} \delta. \) Thus, \( \frac{A_i}{D} = \frac{\varepsilon}{\delta} \), which is assumed to be small. In this case, \( \varepsilon \ll \delta \), and is taken as the small parameter in the problem.

Equations (4) are solved by first mapping the physical domain in the \( (x, z) \) plane onto the rectangular domain \( (\eta, \xi) \in [-1/2, 1/2] \times (-\infty, +\infty) \). In this case, the mapping is given by
\( \tau(x, z, t) = t \), \( \eta(x, z, t) = \left(\frac{1}{1 - \epsilon f}\right)\left(\frac{x - \epsilon f}{2}\right) \), \( \xi(x, z, t) = z \), \( \epsilon \in \left[-\frac{1}{2}, \frac{1}{2}\right] \). The governing equations in the mapped domain become

\begin{align*}
\eta_\eta + (1 - \epsilon f) \eta_\xi + \epsilon f \xi_\eta \bigg(\eta - \frac{1}{2}\bigg) \eta_\eta = 0, \\
u_\tau + \nu u_\eta + \nu w_\xi + \epsilon c f \xi_\xi \bigg(\eta - \frac{1}{2}\bigg) \nu_\eta - Tav^2 = c^2 \eta_\eta + u_\xi_\xi \\
\eta_\xi + \epsilon c \bigg(f_{\xi_\xi} + 2 \epsilon c f_{\xi_\xi}^2\bigg) \bigg(\eta - \frac{1}{2}\bigg) \eta_\xi + \epsilon c f_{\xi_\xi} \bigg(\eta - \frac{1}{2}\bigg) \eta_\xi + \eta_\xi_\eta = 0, \\
v_\tau + \nu v_\eta + \nu w_\xi + \epsilon c f \xi_\xi \bigg(\eta - \frac{1}{2}\bigg) \nu_\eta = c^2 \eta_\eta + \nu_\xi_\xi \\
\xi_\tau + \epsilon c \bigg(f_{\xi_\xi} + 2 \epsilon c f_{\xi_\xi}^2\bigg) \bigg(\eta - \frac{1}{2}\bigg) \xi_\eta = \epsilon c f_{\xi_\xi} \bigg(\eta - \frac{1}{2}\bigg) p_\eta \\
\eta_\eta + \xi_\eta_\eta + \epsilon c f \xi_\xi \bigg(\eta - \frac{1}{2}\bigg) \xi_\eta = \epsilon c f_{\xi_\xi} \bigg(\eta - \frac{1}{2}\bigg) p_\eta + \epsilon c f_{\xi_\xi} \bigg(\eta - \frac{1}{2}\bigg) \xi_\eta_\eta \\
w_\eta = 0, \\
\nu_\eta = 0, \\
\xi_\eta = -1.
\end{align*}

where \( \epsilon \ll 1 \). Equations (7) are the transient version for straight outer cylinder of the equations given in ref. [15]. The boundary conditions for Eqs. (7) are

\begin{align*}
&u \bigg(\eta = \pm \frac{1}{2}, \xi, \tau\bigg) = w \bigg(\eta = \pm \frac{1}{2}, \xi, \tau\bigg) = v \bigg(\eta = \frac{1}{2}, \xi, \tau\bigg) = 0, \\
&\nu \bigg(\eta = \pm \frac{1}{2}, \xi, \tau\bigg) = 1.
\end{align*}

In this work, only small amplitude modulation is examined, so that \( \epsilon \) is small (\( \epsilon \ll 1 \)). In this case, a regular perturbation expansion is used for the velocity and pressure. At this point, it is necessary to introduce explicitly the modulated wall profile \( f \). Various configurations may be easily incorporated in the general formulation above. For instance, both walls could be assumed to be modulated, and the modulation can be represented by a general Fourier series, as long as the wall profile is smooth. In this work, however, only the inner wall is assumed to be modulated in the form of a sine wave, such that

\[
f(\xi) = \sin(\alpha \xi),
\]
where $\alpha$ is the (dimensionless) wavenumber of the modulation. Thus, $Ta$, $\varepsilon$, and $\alpha$, are the only similarity parameters in the present problem.

2.2. Base flow
The steady base flow solution may be written as

$$u^s(\eta, \xi) = u^{s0}(\eta, \xi) + \varepsilon u^{sl}(\eta) \sin(\alpha \xi) + O(\varepsilon^2),$$

$$v^s(\eta, \xi) = v^{s0}(\eta, \xi) + \varepsilon v^{sl}(\eta) \sin(\alpha \xi) + O(\varepsilon^2),$$

$$w^s(\eta, \xi) = w^{s0}(\eta, \xi) + \varepsilon w^{sl}(\eta) \cos(\alpha \xi) + O(\varepsilon^2),$$

$$p^s(\eta, \xi) = p^{s0}(\eta, \xi) + \varepsilon p^{sl}(\eta) \sin(\alpha \xi) + O(\varepsilon^2).$$

(10)

where terms of $O(\varepsilon^2)$ and higher are neglected. Substitution of expressions (10) into Eqs. (7) leads to a hierarchy of equations for the unknown coefficients $u^i, v^i, w^i$ and $p^i (i = 0, 1)$, which must be solved to each order in $\varepsilon$. Thus, to leading order, one recovers the equations that correspond to the flow between two straight cylinders, with solution

$$u^{s0} = w^{s0} = 0, \quad v^{s0} = \frac{1}{2} - \eta, \quad p^{s0}_\eta = Ta \left( \frac{1}{2} - \eta \right)^2.$$  

(11)

The equations to $O(\varepsilon)$ become

$$u^{sl}_\eta = \alpha w^{sl},$$

(12a)

$$Ta \left( \frac{1}{2} - \eta \right)^2 - Ta (1 - 2\eta) v^{sl} = -p^{sl}_\eta + u^{sl}_\eta \eta - \alpha^2 u^{sl},$$

(12b)

$$-u^{sl} + \left( \frac{1}{2} - \eta \right) \alpha^2 = v^{sl}_\eta \eta - \alpha^2 v^{sl},$$

(12c)

$$-\alpha Ta \left( \frac{1}{2} - \eta \right)^3 = w^{sl}_\eta \eta - \alpha^2 w^{s1} - \alpha p^{sl}.$$  

(12d)

In this case, the system above is a set of non-homogenous ordinary differential equations, which together with the corresponding homogeneous boundary conditions
consist of a boundary-value problem of the two-point type. The problem is solved using a variable order, variable-step-size finite-difference scheme with deferred corrections. The basic discretization is the trapezoidal rule over a nonuniform mesh. The mesh is chosen adaptively, to make the local error approximately the same size everywhere. Higher-order discretization is obtained by deferred corrections. Global error estimates are produced to control the computation. The resulting nonlinear algebraic system is solved using Newton’s method with step control. The linearized system of equations is resolved by a special form of Gauss elimination that preserves the sparseness.

2.3. Linear stability analysis
Following linear stability theory, disturbances are imposed on the mean flow in the form

\[ \mathbf{u}(\eta, \xi, \tau) = \mathbf{u}^s(\eta, \xi) + \mathbf{u}'(\eta, \xi, \tau), \quad p(\eta, \xi, \tau) = p^s(\eta, \xi) + p'(\eta, \xi, \tau) \]  

where \( \mathbf{u}' = (u', v', w')^T \) and \( p' \) are infinitesimal disturbances in velocity and pressure, respectively.

Substitution of (14) into Eqs. (7), and neglecting quadratic terms in the disturbances, lead to the following linearized set of equations for the disturbance variables:

\[ cu'_{\eta} + w'_{\xi} + \varepsilon c f_{\xi} \left( \eta - \frac{1}{2} \right) w'_{\eta} = 0 . \]  

\[ u'_{\xi} + c \left( u'^{\xi} u'_{\eta} + u' u'_{\eta} \right) + w'^{\xi} u'_{\xi} + w u'^{\xi} + \varepsilon c f_{\xi} \left( \eta - \frac{1}{2} \right) \left( w'^{\xi} u'_{\xi} + w u'_{\xi} \right) = 2 T \varepsilon v' + c^2 u'_{\eta} \eta \]  

\[ + u'_{\xi \xi} + \varepsilon c \left( f_{\xi \xi} + 2 c f_{\xi} \right) \left( \eta - \frac{1}{2} \right) u'_{\eta} + 2 \varepsilon c f_{\xi} \left( \eta - \frac{1}{2} \right) u'_{\xi} + \left[ \varepsilon c f_{\xi} \left( \eta - \frac{1}{2} \right) \right]^2 u'_{\eta} = -c p'_{\eta} . \]  

\[ v'_{\xi} + c \left( u'^{\xi} v'_{\eta} + u' v'_{\eta} \right) + w'^{\xi} v'_{\xi} + w v'^{\xi} + \varepsilon c f_{\xi} \left( \eta - \frac{1}{2} \right) \left( w'^{\xi} v'_{\xi} + w v'_{\xi} \right) = c^2 v'_{\eta} + v'_{\xi \xi} \]  

\[ + \varepsilon c \left( f_{\xi \xi} + 2 c f_{\xi} \right) \left( \eta - \frac{1}{2} \right) v'_{\eta} + 2 \varepsilon c f_{\xi} \left( \eta - \frac{1}{2} \right) v'_{\xi} + \left[ \varepsilon c f_{\xi} \left( \eta - \frac{1}{2} \right) \right]^2 v'_{\eta} , \]  

\[ w'_{\xi} + c \left( u'^{\xi} w'_{\eta} + u' w'_{\eta} \right) + w'^{\xi} w'_{\xi} + w w'^{\xi} + \varepsilon c f_{\xi} \left( \eta - \frac{1}{2} \right) \left( w'^{\xi} w'_{\xi} + w w'_{\xi} \right) = c^2 w'_{\eta} + w'_{\xi \xi} \]  

\[ + \varepsilon c \left( f_{\xi \xi} + 2 c f_{\xi} \right) \left( \eta - \frac{1}{2} \right) w'_{\eta} + 2 \varepsilon c f_{\xi} \left( \eta - \frac{1}{2} \right) w'_{\xi} + \left[ \varepsilon c f_{\xi} \left( \eta - \frac{1}{2} \right) \right]^2 w'_{\eta} \]  

\[ - p'_{\xi} - \varepsilon c \left( \eta - \frac{1}{2} \right) f_{\xi \xi} p'_{\eta} . \]  

The above equations must be solved subject to the homogeneous conditions.
\[ u'(\eta = \pm \frac{1}{2}, \xi, \tau) = v'(\eta = \pm \frac{1}{2}, \xi, \tau) = w'(\eta = \pm \frac{1}{2}, \xi, \tau) = 0. \quad (16) \]

Since the base flow is periodic in \( \xi \) with periodicity \( \frac{2\pi}{\alpha} \), the disturbance equations are transformed into ordinary differential equations using Floquet theory [25]. Thus, the disturbance flow is considered as a superposition of the cylinder wall modulation, with wavenumber \( \alpha \), and perturbation wave with wavenumber \( \beta \):

\[
\begin{pmatrix}
    u' \\
    p'
\end{pmatrix}(\eta, \xi, \tau) = \sum_{n=-\infty}^{\infty} \begin{pmatrix}
    u^{(n)} \\
    p^{(n)}
\end{pmatrix}(\eta)e^{i[(n\alpha+\beta)\xi-\lambda\tau]},
\quad (17)
\]

where the \( \eta \)-dependent amplitude functions \( u^{(n)}, v^{(n)}, w^{(n)}, p^{(n)} \) need to be determined. In addition, \( \lambda \) is the complex eigenvalue whose imaginary part, \( \lambda_i \), describes the rate of growth of the disturbances, and its real part, \( \lambda_r \), describes the frequency of the disturbances. Thus, the disturbances are damped if \( \lambda_i < 0 \) and the base flow is stable, and they are amplified if \( \lambda_i > 0 \) reflecting unstable base flow.

Using expressions (10), (14) and (17), the linearized disturbance equations (15) reduce to the following equations, after the orthogonality property of the complex exponential function is used:

\[
\begin{align}
    u^{(n)}_{\eta} - \frac{\varepsilon}{2}(n\alpha+\beta)\left(w^{(n-1)} - w^{(n+1)}\right) \\
    + i(n\alpha+\beta)w^{(n)} + \frac{\varepsilon}{2}\alpha\left(\eta - \frac{1}{2}\right)(w^{(n-1)} + w^{(n+1)}) &= 0 \\
    -\lambda_i u^{(n)}_{\eta} - \frac{\varepsilon}{2}iu^{s1}\left(u^{(n-1)}_{\eta} - u^{(n+1)}_{\eta}\right) - \frac{\varepsilon}{2}iu^{s1}\left(u^{(n-1)} - u^{(n+1)}\right) \\
    + i\frac{\varepsilon}{2}(n\alpha+\beta)w^{s1}\left(u^{(n-1)} + u^{(n+1)}\right) + \frac{\varepsilon}{2}\alpha u^{s1}\left(w^{(n-1)} + w^{(n+1)}\right) &= 0
\end{align}
\quad (18a)

\[
\begin{align}
    = 2Tav^{s0}_{n}v^{(n)} - iTav^{sl}_{n}\left(v^{(n-1)} - v^{(n+1)}\right) + u^{(n)}_{\eta\eta} - i\varepsilon\left(u^{(n-1)}_{\eta\eta} - u^{(n+1)}_{\eta\eta}\right) \\
    - (n\alpha+\beta)^2 u^{(n)} + i\frac{\varepsilon}{2}\alpha^2\left(\eta - \frac{1}{2}\right)(u^{(n-1)} - u^{(n+1)}) \\
    + i\varepsilon\alpha\left(\eta - \frac{1}{2}\right)(n\alpha+\beta)\left(u^{(n-1)} + u^{(n+1)}\right) - p^{(n)}_{\eta} + i\frac{\varepsilon}{2}\left(p^{(n-1)}_{\eta} + p^{(n+1)}_{\eta}\right)
\end{align}
\quad (18b)
\[-\lambda iv^{(n)} - \frac{i\varepsilon}{2} u_{s1} \left(v^{(n-1)} - v^{(n+1)}\right) - \frac{\varepsilon}{2} v_{s1} \left(u^{(n-1)} - u^{(n+1)}\right) - u^{(n)} \]
\[+ i\frac{\varepsilon}{2} (n\alpha + \beta) w_{s1} \left(v^{(n-1)} + v^{(n+1)}\right) + \frac{\varepsilon}{2} \alpha v_{s1} \left(w^{(n-1)} + w^{(n+1)}\right) \]
\[= v^{(n)}_{\eta\eta} - i\varepsilon \left(v^{(n-1)}_{\eta\eta} - v^{(n+1)}_{\eta\eta}\right) - (n\alpha + \beta)^2 v^{(n)} + i\frac{\varepsilon}{2} \alpha^2 \left(\eta - \frac{1}{2}\right) \left(v^{(n-1)}_{\eta} - v^{(n+1)}_{\eta}\right) \]
\[+ i\varepsilon \alpha \left(\eta - \frac{1}{2}\right) (n\alpha + \beta) \left(v^{(n-1)}_{\eta} + v^{(n+1)}_{\eta}\right) \]  \hspace{1cm} (18c)

\[-\lambda iw^{(n)} - \frac{i\varepsilon}{2} u_{s1} \left(w^{(n-1)} - w^{(n+1)}\right) + \frac{\varepsilon}{2} w_{s1} \left(u^{(n-1)} + u^{(n+1)}\right) \]
\[+ i\frac{\varepsilon}{2} (n\alpha + \beta) w_{s1} \left(w^{(n-1)} + w^{(n+1)}\right) + \frac{\varepsilon}{2} \alpha w_{s1} \left(w^{(n-1)} - w^{(n+1)}\right) \]
\[= v^{(n)}_{\eta\eta} - i\varepsilon \left(w^{(n-1)}_{\eta\eta} - w^{(n+1)}_{\eta\eta}\right) - (n\alpha + \beta)^2 w^{(n)} + i\frac{\varepsilon}{2} \alpha^2 \left(\eta - \frac{1}{2}\right) \left(w^{(n-1)}_{\eta} - w^{(n+1)}_{\eta}\right) \]
\[+ i\varepsilon \alpha \left(\eta - \frac{1}{2}\right) (n\alpha + \beta) \left(w^{(n-1)}_{\eta} + w^{(n+1)}_{\eta}\right) - i (n\alpha + \beta) p^{(n)} \]
\[+ \frac{\varepsilon}{2} \alpha \left(\eta - \frac{1}{2}\right) \left(p^{(n-1)}_{\eta} + p^{(n+1)}_{\eta}\right) \]  \hspace{1cm} (18d)

where terms of \(O(\varepsilon^2)\) are neglected. Equations (18) are of the recursive type, and must be solved subject to
\[u^{(n)} \left(\eta = \pm \frac{1}{2}\right) = v^{(n)} \left(\eta = \pm \frac{1}{2}\right) = w^{(n)} \left(\eta = \pm \frac{1}{2}\right) = 0. \]  \hspace{1cm} (19)

Obviously, the homogenous system (18) along with the homogenous boundary conditions (19) is far more complicated than the classical stability problem corresponding to \(\varepsilon = 0\). However, the problem is reduced to an infinite set of linear homogenous ordinary differential equations that can be solved in a similar manner to the classical problem. The selection of the solution method is important and will be addressed next.

2.4. Solution of the eigenvalue problem
In this study a solution procedure is introduced as an alternative and relatively simpler, more direct and more efficient method than conventional methods. Regardless of the solution procedure, the system (18)-(19) must be truncated to comprise a finite number, N, of modes. In this case, the resulting 4(2N + 1) coupled homogenous complex differential equations must be solved. Note that terms of order \(\pm (N+1)\) are set to zero. Traditionally, a pseudo-spectral method (based on Chebyshev polynomials) is used to discretize the truncated finite-dimensional system resulting in an algebraic eigenvalue problem [26]. In addition to the complexity of the derivation of the algebraic system, the accuracy of this method depends strongly on the number of modes involved within the calculation. In this study, the eigenvalue problem is solved following the numerical technique introduced by Ache and Core [27] for the solution of the Orr-Sommerfeld equation. The resulting problem is of the two-point boundary-value type and is now nonlinear, since, in addition to the coupling between the

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perturbation and steady-state flows, both the frequency and draw ratio are part of the unknown variables. The problem is solved using the same method described above for the solution of system (12). Other than being relatively simple to implement, calculation of spurious eigenvalues is not required. Thus, the accuracy of the two-point boundary value numerical solution is higher than the spectral method, especially for high Taylor number computations.

The system in (18)-(19) can be transformed into an equivalent two-point boundary-value problem of first-order nonlinear system of \(12(2N + 1) + 2\) real equations, with the eigenvalue \(\lambda\) considered as unknown in the problem. Note that system (18) consists of 12 degrees of freedom. Thus,

\[
Z_\eta = A(\alpha, \beta, Ta, \epsilon, \lambda, \eta) \cdot Z, \tag{20a}
\]

\[
\lambda \eta = 0, \tag{20b}
\]

where \(Z = (Z_1, Z_2, \ldots, Z_M)\), with \(M = 12(2N+1)\), is an \(M\)-dimensional vector containing the unknown velocity and pressure variables. Consequently, \(A\) is an \(M \times M\) matrix, which is function of \(\eta\) and \(\lambda\), as well as the flow parameters and cylinder geometry. Given the arbitrariness of the eigenvectors, system (20) must satisfy, in addition to the homogeneous boundary conditions (19), an auxiliary non-homogeneous boundary condition that ensures a non-trivial solution. Following Ache and Cores [27] in the treatment of the classical Orr-Sommerfeld stability problem, the first derivative of the radial velocity is set equal to one at the inner cylinder wall, and is taken as the auxiliary boundary condition.

The first stage of the computation is to determine the marginal stability curve that corresponds to the flow between straight cylinders (\(\epsilon = 0\)). In particular, the computed critical eigenfunctions and eigenvalues for the loss of stability of TCF, namely the critical Taylor number, \(Ta_c\), and corresponding wavenumber, \(\beta_c\), are used as an initial iterate to solve system (20) at vanishingly small wall modulation amplitude and wavenumber (\(\epsilon = 0.0001\) and \(\alpha = 0.001\)). The solution is then implemented as the initial iterate for a larger \(\alpha\). In other words, the solution at \(\alpha_1\) is implemented as the initial iterate to find the solution at \(\alpha_{i+1} = \alpha_i + \Delta \alpha\) until \(\alpha\) reaches a certain value that corresponds to the critical conditions for the onset of instability. Applying the same approach, the critical Taylor number, \(Ta_{cr}\), and corresponding wavenumber, \(\beta_{cr}\), for the loss of stability of the modulated base flow, are calculated for various wall modulation amplitudes. Note that \(Ta_c = Ta_{cr}(\epsilon = 0, \alpha = 0)\) and \(\beta_c = \beta_{cr}(\epsilon = 0, \alpha = 0)\). Despite the lengthy computational time, the two-point boundary value method is easier to implement, and is believed to be more accurate since it avoids the computation of spurious modes as encountered by the spectral method.

3. Results and discussion

In this section, results based on the formulation and solution procedure above are presented. Steady flow has been reported exhaustively elsewhere [15], and therefore will only be briefly discussed here. The steady profiles for velocity and pressure are typically illustrated in Fig. 2 for a flow at \(Ta = 500\). In the figure, \(u, v, w\) and \(p\) are plotted against \(x\), at \(z = 0.5\) for a modulation amplitude \(\epsilon = 0.05\) and wavenumber \(\alpha = 2\). Although both \(u\) and \(w\) are of \(O(\epsilon)\), the perturbations in azimuthal flow and pressure are relatively small, as they are dominated by the mean flow. An obvious limit case of the modulated flow is the flow between straight cylinders (\(\epsilon = \alpha = 0\)), which will be taken as reference to validate the current solution methodology. In this limit, the critical Taylor number and corresponding wavenumber are computed (see below) and found to be \(Ta_c = 1695\) and \(\beta_c = 3.12\), respectively, which
are in good agreement with the values in the literature [2,3]. This critical threshold is also taken as the starting point in the computations for modulated flow.

Consider first the rate of growth of a disturbance, $\lambda_i$, which changes sign upon loss of stability of steady vortex flow. Variation of $\lambda_i$ with $Ta$ is shown in Fig. 3 for $\beta = 3.12$, $\alpha = 0.2$ and different wall amplitudes. For any wall modulation amplitude, $\lambda_i$ is found to increase monotonically with $Ta$. Although the increase is not exactly linear, the deviation from linear behavior is small and is not significantly affected by $\epsilon$. In addition, the slope $d\lambda_i/dTa$ is essentially independent of $\epsilon$ for any $Ta$, and is estimated to be equal to 0.086. The growth rate increases with $\epsilon$, for any $Ta$ value. Clearly, the threshold Taylor number, $Ta_c \equiv Ta(\lambda_i = 0)$ is smaller as $\epsilon$ increases: $Ta_c(\epsilon = 0) > Ta_c(\epsilon > 0)$. Thus, the modulation has a destabilizing effect. The jump between two successive $\lambda_i$ curves tends to increase slightly with $\epsilon$. Moreover, the drop in critical Taylor number appears to be almost linear with $\epsilon$. However, this is not the case in general, as will be seen next.
The influence of wall modulation on the overall stability picture is typically illustrated in Fig. 4. The neutral stability curves are plotted in the (Ta-β) plane for α = 0.2 and different values of ε. In the absence of modulation (ε = 0), the neutral stability curve for conventional Taylor-vortex flow is recovered. The neutral stability curves for ε > 0 appear all to be below that corresponding to ε = 0, for any β value. However, at small modulation amplitude, the drop in threshold Taylor number is not uniform (with respect to β). Indeed, the two curves corresponding to ε = 0 and ε = 0.02 show that there is a relatively significant drop in Ta for small β (roughly 15% at β = 2.2), but hardly any drop for β > 4. Thus, there is a significant coupling between the effects of wall amplitude and wall wavenumber. The lack of influence of the wall modulation on the critical Taylor number for large wavenumber suggests that when a large number of vortices are about to form, the size of the modulation becomes irrelevant; the small vortex still sees the wall modulation as having a large (infinite) wavelength.

The drop in Ta becomes more uniform for larger ε. Additional calculations show that the drop in Ta becomes essentially uniform when the modulation wavenumber is small (α < 0.05). This is typically illustrated in Fig. 5 for α = 0.001. The figure suggests a similarity in the marginal stability
curves; the curves are expected to collapse onto the $\varepsilon = 0$ curve. This apparent similarity can be further confirmed upon examining the governing equations in the limit $\alpha \to 0$. In this case, the perturbation expansion of the steady-state solution (10) reduces to:

$$u^s(\eta, \xi) = u^{s0}(\eta, \xi) + \varepsilon \alpha \xi \eta^{s1}(\eta) + O\left(\varepsilon^2\right),$$

$$v^s(\eta, \xi) = v^{s0}(\eta, \xi) + \varepsilon \alpha \xi \eta^{s1}(\eta) + O\left(\varepsilon^2\right),$$

$$w^s(\eta, \xi) = w^{s0}(\eta, \xi) + \varepsilon w^{s1}(\eta) + O\left(\varepsilon^2\right),$$

$$p^s(\eta, \xi) = p^{s0}(\eta, \xi) + \varepsilon \alpha \xi p^{s1}(\eta) + O\left(\varepsilon^2\right).$$

It is not difficult to see that to $O\left(\varepsilon^2\right)$, the base flow reduces to

$$u^s = w^s = 0, \quad v^s = v^{s0}, \quad p^s = p^{s0},$$

(22)

where the leading-order terms are still given by (11). Clearly, solution (21) indicates that the base modulated flow at $\alpha = 0$ is the same as that corresponding to the flow between straight cylinders. Noting that $c = (1 - \varepsilon \alpha \xi)^{-1}$, and letting $\chi = \eta / c$, the perturbation equations (15) become

$$u'_{\chi} + w'_{\xi} = 0,$$  

(23a)

$$u'_{\tau} = 2 T\alpha v^s + u'_{\chi \chi} + u'_{\xi \xi} - p'_{\chi},$$  

(23b)

$$v'_{\chi} + u'_{\chi} v^s + v'_{\xi \xi} = v'_{\chi \chi} + v'_{\xi \xi},$$  

(23c)

$$w'_{\tau} = w'_{\chi \chi} + w'_{\xi \xi} - p'_{\xi},$$  

(23d)

which must be solved subject to

$$u'\left(\chi = \pm \frac{1}{2c}, \xi, \tau\right) = v'\left(\chi = \pm \frac{1}{2c}, \xi, \tau\right) = w'\left(\chi = \pm \frac{1}{2c}, \xi, \tau\right) = 0.$$

(24)

Clearly, the linear stability problem (22)-(24) has exactly the same form as that corresponding to the flow between cylinders of dimensionless gap $1/c$. This gap decreases linearly with $\xi$ (assuming $\xi \geq 0$). The effective Taylor number in this case decreases with $\varepsilon$, making the flow unstable. This behavior is reflected in Fig. 5.
The influence of wall modulation on criticality is illustrated in Figs. 6 and 7, where the critical Taylor number and critical disturbance wavenumber are plotted against $\varepsilon$ for different $\alpha$, respectively. In this case, both $T_{ac}$ and $\beta_{cr}$ decrease monotonically with $\varepsilon$. The decrease is essentially linear, reflecting the similarity character as discussed above, particularly at small $\alpha$. This linear dependence on $\varepsilon$ is somewhat unexpected for large $\alpha$ given the trend shown in Fig. 4. Indeed, that figure shows clear the nonlinear dependence in general, especially at large $\beta$. It is also interesting to observe that $\beta_{cr}$ appears to decrease linearly with $\varepsilon$ regardless of the value of $\alpha$. Moreover, both critical numbers decrease more rapidly at smaller $\alpha$. This means that smaller modulation wavenumbers tend to be more destabilizing.

Finally, it is worth mentioning that analysis of the flow in the vicinity of $\alpha = 3.12$ would provide further insight into the interaction of the forced and natural modes. However, the current formulation and analysis are only valid for the forced flow in the pre-critical range ($T_a = T_a < 1690$). The reason for this rather restrictive limitation lies in the perturbation approach used for weak forcing. This is
essentially a linearisation process, which leads to resonance if $\alpha$ is set equal to the critical value of 3.12 and $Ta = 1690$. There is a solution singularity with the flow field behaving like $\frac{1}{|Ta - Ta_c|^\alpha}$ (a > 0) as $Ta$ approaches $Ta_c$. A fully nonlinear solution is needed to cross criticality.

4. Conclusion
The linear stability analysis of spatially modulated TCF is examined in this study. The inner cylinder wall is modulated sinusoidally and rotating, while the outer cylinder is straight and at rest. The steady base flow is obtained using a regular perturbation approach in terms of the small modulation amplitude or wavenumber. The weak forcing is consistent with the narrow-gap assumption. Floquet theory is used to solve the stability problem. The resulting eigenvalue problem is solved by converting it into a nonlinear two-point boundary value problem. The main advantages of the two-point boundary value numerical technique are its simplicity and high accuracy in comparison with other conventional methods, such as the spectral method. The validity of the procedure is assessed by recovering the stability picture of unforced Taylor-Couette flow; in that case, vortical flow emerges only when the Taylor number exceeds a critical value. In contrast, the spatially modulated flow is always vortical no matter how small $Ta$ is.

It is found that cylinder modulation tends to destabilize the base vortex flow. However, the decrease in Taylor number corresponding to the onset of instability is not the same for any disturbance wavenumber. The critical Taylor and wavenumbers tend to decrease linearly with modulation amplitude. It is shown that the results for the modulated flow are collapsible onto those of unforced Taylor-Couette for small modulation wavenumber. Finally, it is emphasized that the present analysis is restricted to the pre-critical range of Taylor number corresponding to unforced Taylor-vortex flow. The post-critical range is currently being examined in an effort to understand the competition between the geometrical forcing and natural disturbance resulting from inertia.

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