Weak Bruhat interval modules of the 0-Hecke algebra

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Abstract
The purpose of this paper is to provide a unified method for dealing with various 0-Hecke modules constructed using tableaux so far. To do this, we assign a 0-Hecke module to each left weak Bruhat interval, called a weak Bruhat interval module. We prove that every indecomposable summand of the 0-Hecke modules categorifying dual immaculate quasisymmetric functions, extended Schur functions, quasisymmetric Schur functions, and Young row-strict quasisymmetric Schur functions is a weak Bruhat interval module. We further study embedding into the regular representation, induction product, restriction, and (anti-)involution twists of weak Bruhat interval modules.

Keywords 0-Hecke algebra · Representation · Weak Bruhat order · Quasisymmetric characteristic

Mathematics Subject Classification 20C08 · 05E10 · 05E05

1 Introduction

The 0-Hecke algebra $H_n(0)$ is a degenerate Hecke algebra obtained from the generic Hecke algebra $H_n(q)$ by specializing $q$ to 0. The representation theory of $H_n(0)$ is very compli-
cated, as can be inferred from the fact that it is not representation-finite for \( n > 3 \) (see [7, 8]). Nevertheless, it has attracted the attention of many mathematicians because of its close connection with quasi-symmetric functions. This link was discovered by Duchamp, Krob, Leclerc, and Thibon [9], who constructed an isomorphism called the \textit{quasisymmetric characteristic} between the Grothendieck ring associated to 0-Hecke algebras and the ring \( \text{QSym} \) of quasi-symmetric functions. In particular, since the mid-2010s, there have been many attempts to construct \( H_n(0) \)-modules categorifying important quasisymmetric functions using tableau models, rather than simply adding irreducible modules (for instance, see [1, 2, 21, 24, 26]).

The purpose of the present paper is to provide a method to treat these modules in a uniform manner. We start with the observation that every indecomposable direct summand of these modules has a basis isomorphic to a left weak Bruhat interval of \( \mathfrak{S}_n \) when it is equipped with the partial order \( \leq \) defined by

\[ T \leq T' \quad \text{if} \quad \pi_\sigma \cdot T = T' \quad \text{for some} \quad \sigma \in \mathfrak{S}_n. \]

This leads us to consider the \( H_n(0) \)-module \( B(\sigma, \rho) \) for each weak Bruhat interval \( [\sigma, \rho]_L \), called the \textit{weak Bruhat interval module associated to} \( [\sigma, \rho]_L \), whose underlying space is the \( \mathbb{C} \)-span of \( [\sigma, \rho]_L \) and whose action is given by

\[ \pi_i \cdot \gamma := \begin{cases} 
\gamma & \text{if } i \in \text{Des}_L(\gamma), \\
0 & \text{if } i \notin \text{Des}_L(\gamma) \text{ and } s_i \gamma \notin [\sigma, \rho]_L, \\
s_i \gamma & \text{if } i \notin \text{Des}_L(\gamma) \text{ and } s_i \gamma \in [\sigma, \rho]_L.
\end{cases} \]

In a similar point of view, we also consider the \( H_n(0) \)-module \( \overline{B}(\sigma, \rho) \) for each weak Bruhat interval \( [\sigma, \rho]_L \), called the \textit{negative weak Bruhat interval module associated to} \( [\sigma, \rho]_L \), whose underlying space is the \( \mathbb{C} \)-span of \( [\sigma, \rho]_L \) and whose action is given by

\[ \overline{\pi}_i \star \gamma := \begin{cases} 
-\gamma & \text{if } i \in \text{Des}_L(\gamma), \\
0 & \text{if } i \notin \text{Des}_L(\gamma) \text{ and } s_i \gamma \notin [\sigma, \rho]_L, \\
s_i \gamma & \text{if } i \notin \text{Des}_L(\gamma) \text{ and } s_i \gamma \in [\sigma, \rho]_L.
\end{cases} \]

Here \( \overline{\pi}_i = \pi_i - 1 \). It should be pointed out that Hivert, Novelli, and Thibon [13] introduced semi-combinatorial \( H_n(0) \)-modules associated to Yang-Baxter intervals \( [Y_\sigma(\tau), Y_\rho(\tau)] \) to study the representation theory of 0-Ariki-Koike-Shoji algebras, and our \( B(\sigma, \rho) \) and \( \overline{B}(\sigma, \rho) \) can also be recovered by the \( \tau = \text{id} \) and \( \tau = w_0 \) specialization of these modules, respectively. Here, \( w_0 \) is the longest element of \( \mathfrak{S}_n \).

The family of weak and negative weak Bruhat interval modules is very adequate to our purpose in that it contains many \( H_n(0) \)-modules of our interest such as projective indecomposable modules, irreducible modules, the specializations \( q = 0 \) of the Specht modules of \( H_n(q) \) in [8], and all indecomposable direct summands of the \( H_n(0) \)-modules in [1, 2, 21, 24, 26]. What is more appealing is that weak and negative weak Bruhat interval modules can be embedded into the regular representation of \( H_n(0) \) and they behave very nicely with respect to induction product, restriction, and (anti-)involution twists. Let \( \mathcal{B}_n \) be the full subcategory of the category mod \( H_n(0) \) of finite dimensional \( H_n(0) \)-modules whose objects are direct sums of weak and negative weak Bruhat interval modules up to isomorphism. From a categorical point of view, \( \mathcal{B}_n \) is a good subcategory in the sense that

- the Grothendieck group of \( \mathcal{B}_n \) is isomorphic to the Grothendieck group of mod \( H_n(0) \), and
- the subcategory \( \bigoplus_{n \geq 0} \mathcal{B}_n \) of \( \bigoplus_{n \geq 0} \) mod \( H_n(0) \) is closed under induction product, restriction, and (anti-)involution twists.
In Sect. 3, we study structural properties of weak Bruhat interval modules. In the first two subsections, we present background material for weak and negative weak Bruhat interval modules and then construct an $H_n(0)$-module isomorphism

$$\text{em} : B(\sigma, \rho) \to H_n(0)\pi_\sigma \overline{\pi}_\rho^{-1} w_0, \quad \gamma \mapsto \pi_\gamma \overline{\pi}_\rho^{-1} w_0 \quad \text{for } \gamma \in [\sigma, \rho]_L$$

(Theorem 1). As an immediate consequence of this isomorphism, we see that projective indecomposable modules and irreducible modules appear as weak Bruhat interval modules up to isomorphism. By a slight modification of $\text{em}$, we also see that the specializations $q = 0$ of the Specht modules of $H_n(q)$ are certain involution twists of weak Bruhat interval modules (Remark 1). Although not covered here, the isomorphism $\text{em}$ and its modification reveal interesting connections between certain $U_0(gl_N)$-modules and weak Bruhat interval modules (Sect. 5 (3)).

In the third subsection, we study restriction of weak Bruhat interval modules. As for induction product $\boxtimes$ of weak Bruhat interval modules, the following formula can be derived from [13]:

$$B(\sigma, \rho) \boxtimes B(\sigma', \rho') \cong B(\sigma \bullet \sigma', \rho \boxplus \rho')$$  \hspace{1cm} (1.1)$$

(see Lemma 4). We provide an explicit formula concerning restriction of weak Bruhat interval modules. Let \(\binom{m+n}{m}\) be the set of \(m\)-element subsets of \([m+n]\). Given \(\sigma, \rho \in \mathfrak{S}_{m+n}\), set

$$\mathcal{J}_{\sigma, \rho}^{(m)} := \left\{ J \in \binom{m+n}{m} \mid J = \gamma^{-1}([1,m]) \text{ for some } \gamma \in [\sigma, \rho]_L \right\}.$$  

With this notation, our restriction rule appears in the following form:

$$B(\sigma, \rho) \downarrow_{H_m(0) \otimes H_n(0)}^{H_{m+n}(0)} \cong \bigoplus_{J \in \mathcal{J}_{\sigma, \rho}^{(m)}} B((\sigma_J)_{\leq m}, (\rho_J)_{\leq m}) \otimes B((\sigma_J)_{> m}, (\rho_J)_{> m})$$  \hspace{1cm} (1.2)$$

(Theorem 2). For undefined notations $(\sigma_J)_{\leq m}$, $(\rho_J)_{\leq m}$, $(\sigma_J)_{> m}$, and $(\rho_J)_{> m}$, see (3.9) and (3.11). The elements in $\mathcal{J}_{\sigma, \rho}^{(m)}$ parametrize the direct summands appearing in the right hand side of (1.2). It would be nice to find an easy description of this set, but it is not available at the current stage. Combining (1.1) with (1.2) yields a Mackey formula for weak Bruhat interval modules. We prove that this is a natural lift of the Mackey formula due to Bergeron and Li [3], which works for elements of the Grothendieck ring of 0-Hecke algebras, to weak Bruhat interval modules (Theorem 3).

In the final subsection, we describe the (anti-)involution twists of weak Bruhat interval modules for the involutions $\phi$, $\theta$ and the anti-involution $\chi$ due to Fayers [10] and then demonstrate the patterns how these (anti-)involution twists act with respect to induction product and restriction. Here, $\phi(\pi_i) = \pi_{n-i}, \theta(\pi_i) = -\pi_i$, and $\chi(\pi_i) = \pi_i$ for $1 \leq i \leq n-1$. Given an $H_n(0)$-module $M$ and an (anti-)automorphism $\mu$, we denote by $\mu[M]$ the $\mu$-twist of $M$. The precise definition can be found in (3.16) and (3.17). We prove that

$$\phi[B(\sigma, \rho)] \cong B(\mu w_0, \rho w_0), \quad \theta[B(\sigma, \rho)] \cong \overline{B}(\sigma, \rho), \quad \text{and } \chi[B(\sigma, \rho)] \cong \overline{B}(\rho w_0, \sigma w_0).$$

Here, $\sigma w_0$ is the conjugation of $\sigma$ by $w_0$. Using these isomorphisms, we can also describe the twists for the compositions of $\phi$, $\theta$, and $\chi$. For the full list of (anti-)involution twists, see Tables 1 and 2. Next, we explain the patterns how the (anti-)involution twists act with respect
to induction product and restriction. Let $M$, $N$, and $L$ be weak Bruhat interval modules of $H_m(0)$, $H_n(0)$, and $H_{m+n}(0)$ respectively. We see that

$$\theta[M \boxtimes N] \cong \theta[M] \boxtimes \theta[N],$$

whereas

$$\phi[M \boxtimes N] \cong \phi[N] \boxtimes \phi[M], \quad \chi[M \boxtimes N] \cong \chi[N] \boxtimes \chi[M]$$

and

$$\theta[L \downarrow_{H_m(0) \boxtimes H_n(0)}] \cong \theta[L] \downarrow_{H_{m+n}(0)}, \quad \chi[L \downarrow_{H_m(0) \boxtimes H_n(0)}] \cong \chi[L] \downarrow_{H_{m+n}(0)},$$

whereas

$$\phi[L \downarrow_{H_m(0) \boxtimes H_n(0)}] \cong \phi[L] \downarrow_{H_{m+n}(0)}.$$

(Corollary 1).

Section 4 is devoted to show that every indecomposable direct summand arising from the $H_n(0)$-modules in $[1, 2, 21, 24, 26]$ are weak Bruhat interval modules up to isomorphism. We begin with reviewing the results in these papers. Let $\alpha$ be a composition of $n$.

- Tewari and van Willigenburg [24] construct an $H_n(0)$-module $S_\alpha$ by defining a 0-Hecke action on the set of standard reverse composition tableaux of shape $\alpha$. Its image under the quasisymmetric characteristic is the quasisymmetric Schur function attached to $\alpha$.
- Let $\sigma$ be a permutation in $S_{\ell(\alpha)}$, where $\ell(\alpha)$ is the length of $\alpha$. Tewari and van Willigenburg [26] construct an $H_n(0)$-module $S_\alpha^\sigma$ by defining a 0-Hecke action on the set $\text{SPCT}^\sigma(\alpha)$ of standard permuted composition tableaux of shape $\alpha$ and type $\sigma$. When $\sigma = \text{id}$, this module is equal to $S_\alpha$ in [24].
- Berg et al. [2] construct an indecomposable $H_n(0)$-module $V_\alpha$ by defining a 0-Hecke action on the set $\text{SIT}(\alpha)$ of standard immaculate tableaux of shape $\alpha$. Its image under the quasisymmetric characteristic is the dual immaculate quasisymmetric function attached to $\alpha$.
- Searles [21] constructs an indecomposable $H_n(0)$-module $X_\alpha$ by defining a 0-Hecke action on the set $\text{SET}(\alpha)$ of standard extended tableaux of shape $\alpha$. Its image under the quasisymmetric characteristic is the Young row-strict quasisymmetric Schur function attached to $\alpha$.
- Bardwell and Searles [1] construct an $H_n(0)$-module $R_\alpha$ by defining a 0-Hecke action on the set $\text{SYRT}(\alpha)$ of standard Young row-strict tableaux of shape $\alpha$. Its image under the quasisymmetric characteristic is the Young row-strict quasisymmetric Schur function attached to $\alpha$.

The modules $V_\alpha$ and $X_\alpha$ are indecomposable, whereas $S_\alpha^\sigma$, and $R_\alpha$ are not indecomposable in general. The problem of decomposing $S_\alpha^\sigma$ into indecomposables has been completely settled out by virtue of the papers [5, 15, 24, 26]. Indeed, $S_\alpha^\sigma$ has the decomposition of the form

$$S_\alpha^\sigma = \bigoplus_{E \in \mathcal{E}^\sigma(\alpha)} S_{\alpha, E}^\sigma,$$

where $\mathcal{E}^\sigma(\alpha)$ represents the set of equivalence classes of $\text{SPCT}^\sigma(\alpha)$ under a certain distinguished equivalence relation and $S_{\alpha, E}^\sigma$ is the indecomposable submodule of $S_\alpha^\sigma$ spanned by $E$.

To achieve our purpose, we first show that all of $\text{SIT}(\alpha)$, $\text{SET}(\alpha)$, and $E \in \mathcal{E}^\sigma(\alpha)$ have the source and sink, which are written as follows:

| source | $\mathcal{SIT}(\alpha)$ | $\mathcal{SET}(\alpha)$ | $E$ |
|--------|-------------------------|-------------------------|-----|
| sink   | $\mathcal{SIT}'(\alpha)$| $\mathcal{T}(\alpha)$   | $\tau_E$ |
|        | $\mathcal{SET}'(\alpha)$| $\mathcal{T}'(\alpha)$  | $\tau'_E$ |
For the definitions of source and sink, see Definition 2. Then, using the essential epimorphisms constructed in [6], we introduce two readings $\text{read}$ and $\text{read}^\prime$, where the former is defined on $\text{SIT}(\alpha)$ and $\text{SET}(\alpha)$ and the latter is defined on each class $E$. With this preparation, we prove that

$$
\mathcal{V}_\alpha \cong \mathbb{B}(\text{read}(\mathcal{R}_\alpha), \text{read}(\mathcal{R}'_\alpha)), \ X_\alpha \cong \mathbb{B}(\text{read}(\mathcal{T}_\alpha), \text{read}(\mathcal{T}'_\alpha)),
$$

and $S^\alpha_{\alpha',E} \cong \mathbb{B}(\text{read}(\tau_\alpha), \text{read}(\tau'_E))$.

(Theorem 5 and Theorem 6). It should be remarked that a reading on $E$ different from ours has already been introduced by Tewari and van Willigenburg [24]. They assign a reading word $(\mathbb{R}',\mathbb{R})$ to each standard reverse composition tableau $\tau$ and show that $(E, \preceq)$ is isomorphic to $([\mathbb{R}, \mathbb{R}']_E, \preceq)$ as graded posets. The weak Bruhat interval module $\mathbb{B}(\mathbb{C}, \mathbb{C}')$ on $E$ and $E'$, however, is not isomorphic to $S_\alpha$ in general (see Remark 4). Concerned with $R_\alpha$, we consider its permuted version instead of itself. We introduce new combinatorial objects, called permuted standard Young row-strict tableaux of shape $\alpha$ and type $\sigma$, and define a 0-Hecke action on them. The resulting module $R^\alpha_\sigma$ turns out to be isomorphic to the $\widehat{\omega}$-twist of $S^\alpha_\sigma$. This enables us to transport various properties of $S^\alpha_\sigma$ to $R^\alpha_\sigma$ in a functorial way.

In the final section, we provide some future directions to pursue.

2 Preliminaries

Given any integers $m$ and $n$, define $[m,n]$ to be the interval $\{ t \in \mathbb{Z} : m \leq t \leq n \}$ whenever $m \leq n$ and the empty set $\emptyset$ else. For simplicity, we set $[n] := [1,n]$. Unless otherwise stated, $n$ will denote a nonnegative integer throughout this paper.

2.1 Compositions and their diagrams

A composition $\alpha$ of $n$, denoted by $\alpha \models n$, is a finite ordered list of positive integers $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ satisfying $\sum_{i=1}^k \alpha_i = n$. We call $\alpha_i$ $(1 \leq i \leq k)$ a part of $\alpha$, $k := \ell(\alpha)$ the length of $\alpha$, and $n := |\alpha|$ the size of $\alpha$. And, we define the empty composition $\emptyset$ to be the unique composition of size and length 0.

Given $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \models n$ and $I = \{ i_1 < i_2 < \cdots < i_k \} \subset [n-1]$, let $\text{set}(\alpha) := \{ \alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1} \}$ and $\text{comp}(I) := (i_1, i_2 - i_1, i_3 - i_2, \ldots, n - i_k)$. The set of compositions of $n$ is in bijection with the set of subsets of $[n-1]$ under the correspondence $\alpha \mapsto \text{set}(\alpha)$ (or $I \mapsto \text{comp}(I)$). The reverse composition $\alpha^r$ of $\alpha$ is defined to be $(\alpha_k, \alpha_{k-1}, \ldots, \alpha_1)$, the complement $\alpha^c$ of $\alpha$ be the unique composition satisfying $\text{set}(\alpha^c) = [n-1] \setminus \text{set}(\alpha)$, and the conjugate $\alpha^t$ of $\alpha$ be the composition $(\alpha^r)^c = (\alpha^c)^r$.

2.2 Weak Bruhat orders on the symmetric group

Every element $\sigma$ of the symmetric group $\mathfrak{S}_n$ may be written as a word in $s_i := (i, i+1)$ with $1 \leq i \leq n-1$. A reduced expression for $\sigma$ is one of minimal length. The number of simple transpositions in any reduced expression for $\sigma$, denoted by $\ell(\sigma)$, is called the length of $\sigma$. Let

$$
\text{Des}_L(\sigma) := \{ i \in [n-1] \mid \ell(s_i\sigma) < \ell(\sigma) \} \quad \text{and} \quad \text{Des}_R(\sigma) := \{ i \in [n-1] \mid \ell(\sigma s_i) < \ell(\sigma) \}.
$$
It is well known that if $\sigma = w_1 w_2 \cdots w_n$ in one-line notation, then
\[
\text{Des}_L(\sigma) = \{i \in [n-1] \mid i \text{ is right of } i+1 \text{ in } w_1 w_2 \cdots w_n\} \quad \text{and} \quad \text{Des}_R(\sigma) = \{i \in [n-1] \mid w_i > w_{i+1}\}.
\] (2.1)

The left weak Bruhat order $\leq_L$ (resp. right weak Bruhat order $\leq_R$) on $\mathfrak{S}_n$ is the partial order on $\mathfrak{S}_n$ whose covering relation $\leq^c_L$ (resp. $\leq^c_R$) is defined as follows: $\sigma \leq^c_L s_i \sigma$ if and only if $i \notin \text{Des}_L(\sigma)$ (resp. $\sigma \leq^c_R s_i \sigma$ if and only if $i \notin \text{Des}_R(\sigma)$). Equivalently, for any $\sigma, \rho \in \mathfrak{S}_n$,
\[
\sigma \leq_L \rho \text{ if and only if } \rho = \gamma \sigma \text{ and } \ell(\rho) = \ell(\sigma) + \ell(\gamma) \text{ for some } \gamma \in \mathfrak{S}_n, \quad \text{and} \quad \sigma \leq_R \rho \text{ if and only if } \rho = \sigma \gamma \text{ and } \ell(\rho) = \ell(\sigma) + \ell(\gamma) \text{ for some } \gamma \in \mathfrak{S}_n.
\]

Although these two weak Bruhat orders are not identical, there exists a poset isomorphism
\[
(\mathfrak{S}_n, \leq_L) \to (\mathfrak{S}_n, \leq_R), \quad \sigma \mapsto \sigma^{-1}.
\]
To avoid redundant overlap, our statements will be restricted to the left weak Bruhat order.

Given $\sigma$ and $\rho \in \mathfrak{S}_n$, the closed interval $[\gamma \in \mathfrak{S}_n \mid \sigma \leq_L \gamma \leq_L \rho]$ is called the left weak Bruhat interval from $\sigma$ to $\rho$ and denoted by $[\sigma, \rho]_L$. It can be represented by the colored digraph whose vertices are given by $[\sigma, \rho]_L$ and $\{1, 2, \ldots, n-1\}$-colored arrows given by
\[
\gamma \xrightarrow{i} \gamma' \quad \text{if and only if} \quad \gamma \leq_L \gamma' \text{ and } s_i \gamma = \gamma'.
\]

Let us collect notations which will be used later. We use $w_0$ to denote the longest element in $\mathfrak{S}_n$. For $I \subseteq [n-1]$, let $\mathfrak{S}_I$ be the parabolic subgroup of $\mathfrak{S}_n$ generated by $\{s_i \mid i \in I\}$ and $w_0(I)$ the longest element in $\mathfrak{S}_I$. For $\alpha \models n$, let $w_0(\alpha) := w_0(\text{set}(\alpha))$. Finally, for $\sigma \in \mathfrak{S}_n$, we let $w_0(\sigma) := w_0 \sigma w_0$.

### 2.3 The 0-Hecke algebra and the quasisymmetric characteristic

The 0-Hecke algebra $H_n(0)$ is the associative $\mathbb{C}$-algebra with 1 generated by the elements $\pi_1, \pi_2, \ldots, \pi_{n-1}$ subject to the following relations:
\[
\pi_i^2 = \pi_i \quad \text{for } 1 \leq i \leq n-1, \\
\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1} \quad \text{for } 1 \leq i \leq n-2, \\
\pi_i \pi_j = \pi_j \pi_i \quad \text{if } |i - j| \geq 2.
\]

Frequently we use another set of generators $\{\overline{\pi}_i := \pi_i - 1 \mid 1 \leq i \leq n-1\}$.

For any reduced expression $s_{i_1} s_{i_2} \cdots s_{i_p}$ for $\sigma \in \mathfrak{S}_n$, let
\[
\pi_\sigma := \pi_{i_1} \pi_{i_2} \cdots \pi_{i_p} \quad \text{and} \quad \overline{\pi}_\sigma := \overline{\pi}_{i_1} \overline{\pi}_{i_2} \cdots \overline{\pi}_{i_p}.
\]

It is well known that these elements are independent of the choices of reduced expressions, and both $\{\pi_\sigma \mid \sigma \in \mathfrak{S}_n\}$ and $\{\overline{\pi}_\sigma \mid \sigma \in \mathfrak{S}_n\}$ are $\mathbb{C}$-bases for $H_n(0)$.

According to [20], there are $2^{n-1}$ pairwise inequivalent irreducible $H_n(0)$-modules and $2^{n-1}$ pairwise inequivalent projective indecomposable $H_n(0)$-modules, which are naturally indexed by compositions of $n$. For $\alpha \models n$, let $F_\alpha = \mathbb{C} v_\alpha$ and endow it with the $H_n(0)$-action as follows: for each $1 \leq i \leq n-1$,
\[
\pi_i \cdot v_\alpha = \begin{cases} 
0 & \text{if } i \in \text{set}(\alpha), \\
v_\alpha & \text{if } i \notin \text{set}(\alpha).
\end{cases}
\]
This module is the irreducible 1-dimensional $H_n(0)$-module corresponding to $\alpha$. And, the projective indecomposable $H_n(0)$-module corresponding to $\alpha$ is given by the submodule

$$\mathcal{P}_\alpha := H_n(0)\pi_{w_0(\alpha)} \mathcal{R}_{w_0(\alpha)}$$

of the regular representation of $H_n(0)$. It is known that $F_\alpha$ is isomorphic to $\mathcal{P}_\alpha / \text{rad } \mathcal{P}_\alpha$, where $\text{rad } \mathcal{P}_\alpha$ is the radical of $\mathcal{P}_\alpha$, the intersection of maximal submodules of $\mathcal{P}_\alpha$.

Let $\mathcal{R}(H_n(0))$ denote the $\mathbb{Z}$-span of the isomorphism classes of finite dimensional $H_n(0)$-modules. We denote by $[M]$ the isomorphism class corresponding to an $H_n(0)$-module $M$. The Grothendieck group $G_0(H_n(0))$ is the quotient of $\mathcal{R}(H_n(0))$ modulo the relations $[M] = [M'] + [M'']$ whenever there exists a short exact sequence $0 \to M' \to M \to M'' \to 0$. The irreducible $H_n(0)$-modules form a free $\mathbb{Z}$-basis for $G_0(H_n(0))$. Let

$$\mathcal{G} := \bigoplus_{n \geq 0} G_0(H_n(0)).$$

Let us review the beautiful connection between $\mathcal{G}$ and the ring $\text{QSym}$ of quasisymmetric functions. For the definition of quasisymmetric functions, see [23, Section 7.19].

For a composition $\alpha$, the fundamental quasisymmetric function $F_\alpha$, introduced in [11], is defined by

$$F_\emptyset = 1 \quad \text{and} \quad F_\alpha = \sum_{\substack{1 \leq i_1 \leq \cdots \leq i_k \\ i_j < i_{j+1} \text{ if } j \in \text{set}(\alpha)}} x_{i_1} \cdots x_{i_k} \quad \text{if } \alpha \neq \emptyset.$$

It is known that $\{F_\alpha \mid \alpha \text{ is a composition}\}$ forms a $\mathbb{Z}$-basis for $\text{QSym}$. When $M$ is an $H_m(0)$-module and $N$ is an $H_n(0)$-module, we write $M \boxtimes N$ for the induction product of $M$ and $N$, that is,

$$M \boxtimes N = M \otimes N \uparrow_{H_m(0) \otimes H_n(0)}^{H_{m+n}(0)}.$$

Here, $H_m(0) \otimes H_n(0)$ is viewed as the subalgebra of $H_{m+n}(0)$ generated by $\{\pi_i \mid i \in [m+n-1] \setminus \{m\}\}$. It was shown in [9] that, when $\mathcal{G}$ is equipped with this product as multiplication, the linear map

$$\text{ch} : \mathcal{G} \to \text{QSym}, \quad [F_\alpha] \mapsto F_\alpha,$$

called quasisymmetric characteristic, is a ring isomorphism. Indeed it is not only a ring isomorphism, but also a Hopf algebra isomorphism.

3 Weak Bruhat interval modules

In this section, we present background material for weak Bruhat interval modules and then investigate their structural properties extensively.

3.1 Definition and basic properties

Let us start with the definitions of weak and negative weak Bruhat interval modules.
Definition 1 Let $\sigma, \rho \in \mathcal{S}_n$.

1. The weak Bruhat interval module associated to $[\sigma, \rho]_L$, denoted by $\mathcal{B}([\sigma, \rho])$, is the $H_n(0)$-module with $\mathbb{C}[\sigma, \rho]_L$ as the underlying space and with the $H_n(0)$-action defined by

\[
\pi_i \cdot \gamma := \begin{cases} 
\gamma & \text{if } i \in \text{Des}_L(\gamma), \\
0 & \text{if } i \notin \text{Des}_L(\gamma) \text{ and } s_i \gamma \notin [\sigma, \rho]_L, \\
s_i \gamma & \text{if } i \notin \text{Des}_L(\gamma) \text{ and } s_i \gamma \in [\sigma, \rho]_L.
\end{cases}
\]

2. The negative weak Bruhat interval module associated to $[\sigma, \rho]_L$, denoted by $\overline{\mathcal{B}}([\sigma, \rho])$, is the $H_n(0)$-module with $\mathbb{C}[\sigma, \rho]_L$ as the underlying space and with the $H_n(0)$-action defined by

\[
\overline{\pi_i} \ast \gamma := \begin{cases} 
-\gamma & \text{if } i \in \text{Des}_L(\gamma), \\
0 & \text{if } i \notin \text{Des}_L(\gamma) \text{ and } s_i \gamma \notin [\sigma, \rho]_L, \\
s_i \gamma & \text{if } i \notin \text{Des}_L(\gamma) \text{ and } s_i \gamma \in [\sigma, \rho]_L.
\end{cases}
\]

Hivert, Novelli, and Thibon [13] introduced a semi-combinatorial $H_n(0)$-module associated to the interval $[Y_{\sigma}(\tau), Y_{\rho}(\tau)]$ of a Yang-Baxter basis $\{Y_{\gamma}(\tau)\}_{\gamma \in \mathcal{S}_n}$ for each $\tau \in \mathcal{S}_n$. We omit the proof of the well-definedness of $\mathcal{B}([\sigma, \rho])$ and $\overline{\mathcal{B}}([\sigma, \rho])$ since they can be recovered as the semi-combinatorial modules associated to $[Y_{\sigma}(\text{id}), Y_{\rho}(\text{id})]$ and $[Y_{\sigma}(w_0), Y_{\rho}(w_0)]$, respectively.

The following properties are almost straightforward. In particular, the last one can be obtained by mimicking [24, Section 5].

- Given $\sigma' \in [\sigma, \rho]_L$, the linear map $\iota : \mathcal{B}([\sigma', \rho]) \to \mathcal{B}([\sigma, \rho])$, sending $\gamma \mapsto \gamma$, is an injective $H_n(0)$-module homomorphism.
- Given $\rho' \in [\sigma, \rho]_L$, the linear map $\text{pr} : \mathcal{B}([\sigma, \rho]) \to \mathcal{B}([\sigma, \rho'])$, sending $\gamma \mapsto \gamma$ if $\gamma \in [\sigma, \rho']_L$ and $\gamma \mapsto 0$ else, is a surjective $H_n(0)$-module homomorphism.
- $\text{ch}([\mathcal{B}([\sigma, \rho])]) = \sum_{\gamma \in [\sigma, \rho]_L} F_{\text{comp}(\gamma)}$, where $\text{comp}(\gamma) := \text{comp}(\text{Des}_L(\gamma))$.

Finally, we briefly remark the classification of weak Bruhat interval modules up to isomorphism. As pointed out in Sect. 2.2, every weak Bruhat interval can be viewed as a colored digraph. One sees from (3.1) that $\mathcal{B}([\sigma, \rho]) \cong \mathcal{B}([\sigma', \rho'])$ if there is a descent-preserving (colored digraph) isomorphism between $[\sigma, \rho]_L$ and $[\sigma', \rho']_L$. For instance,

\[
f : [14325, 24315]_L \to [41352, 42351]_L, \quad \pi_{41325} \mapsto \pi_{41325}, \quad (\gamma \in \mathcal{S}_5)
\]

is such an isomorphism, thus $\mathcal{B}(14325, 24315) \cong \mathcal{B}(41352, 42351)$. On the other hand, it is not difficult to show that there is no descent-preserving isomorphism between $[14325, 24315]_L$ and $[45312, 45321]_L$, although they are isomorphic as colored digraphs. Indeed, $\mathcal{B}(14325, 24315)$ is indecomposable (Remark 4), whereas $\mathcal{B}(45312, 45321)$ can be decomposed into $\mathbb{C}45321 \oplus \mathbb{C}(45312 - 45321)$ as seen in Fig. 1. It would be very nice to characterize when a descent-preserving isomorphism between two intervals exists and ultimately to classify all weak Bruhat interval modules up to isomorphism.

3.2 Embedding weak Bruhat interval modules into the regular representation

The purpose of this subsection is to see that every weak Bruhat interval modules can be embedded into the regular representation. More precisely, we prove that $\mathcal{B}([\sigma, \rho])$ is isomorphic to $H_n(0) \pi_{\sigma} \overline{\pi}_{\rho^{-1}w_0}$. 
For ease of notation, we use $\pi_i$ to denote an arbitrary element in $\{\pi_i, \bar{\pi}_i\}$ for each $1 \leq i \leq n - 1$. The following relations among $\pi_i$'s and $\bar{\pi}_i$'s, which will be used significantly, can be easily verified:

\[
\begin{align*}
\pi_i \pi_{i+1} \bar{\pi}_i &= \bar{\pi}_{i+1} \pi_i \pi_{i+1}, \\
\pi_i \pi_{i+1} \bar{\pi}_{i+1} &= \bar{\pi}_{i+1} \pi_i \pi_{i+1} & (1 \leq i \leq n - 2), \\
\pi_i \bar{\pi}_{i+1} \pi_i &= \bar{\pi}_{i+1} \pi_i \pi_{i+1}, \\
\pi_i \bar{\pi}_{i+1} \bar{\pi}_i &= \bar{\pi}_{i+1} \pi_i \pi_{i+1} & (1 \leq i \leq n - 2), \\
\pi_i \pi_j &= \bar{\pi}_j \pi_i & (|i - j| \geq 2).
\end{align*}
\]  

(3.2)

One sees that $\pi_r \pi_{r+1} \pi_r, \pi_r \pi_r \pi_{r+1}, \pi_r \pi_r \pi_r$, and $\pi_{r+1} \pi_r \pi_{r+1}$ are missing in the list of (3.2). The following lemma tells us the reason for this.

**Lemma 1** For $\sigma, \rho \in S_n$ with $\ell(\sigma \rho) = \ell(\sigma) + \ell(\rho)$, let $s_{u_1}s_{u_2} \cdots s_{u_l}$ and $s_{v_1}s_{v_2} \cdots s_{v_m}$ be arbitrary reduced expressions of $\sigma$ and $\rho$, respectively. Let $\mathbf{w}$ be the word

\[
\pi_{u_1}\pi_{u_2} \cdots \pi_{u_l}\pi_{v_1}\pi_{v_2} \cdots \pi_{v_m}
\]

from the alphabet $\{\pi_i, \bar{\pi}_i \mid 1 \leq i \leq n - 1\}$, where $\pi_i, \bar{\pi}_i$ are viewed as letters not as elements of $H_\rho(0)$. Then every word $\mathbf{w} = w_1w_2 \cdots w_{l+m}$ obtained from $\mathbf{w}$ by applying only the following six braid relations and three commutation-relations

\[
\begin{align*}
\pi_i \pi_{i+1} \pi_i &= \pi_{i+1} \pi_i \pi_{i+1}, & (1 \leq i \leq n - 2), \\
\pi_i \pi_{i+1} \bar{\pi}_i &= \bar{\pi}_{i+1} \pi_i \pi_{i+1}, & (1 \leq i \leq n - 2), \\
\pi_i \pi_{i+1} \bar{\pi}_{i+1} &= \bar{\pi}_{i+1} \pi_i \pi_{i+1}, & (1 \leq i \leq n - 2), \\
\pi_{i+1} \pi_i \pi_{i+1} &= \pi_i \pi_{i+1} \pi_i, & (1 \leq i \leq n - 2), \\
\pi_i \pi_j &= \pi_j \pi_i & (|i - j| \geq 2), \\
\pi_i \bar{\pi}_j &= \bar{\pi}_j \pi_i & (|i - j| \geq 2),
\end{align*}
\]

(3.3)

contains no subwords $w_r w_{r+1} w_{r+2}$ $(1 \leq r \leq l + m - 2)$ of the form

\[
\pi_i \bar{\pi}_{i+1} \pi_i, \ \pi_i \pi_{i+1} \pi_i, \ \pi_i \pi_{i+1} \bar{\pi}_i, \ \pi_{i+1} \pi_i \bar{\pi}_{i+1}.
\]

(3.5)

**Proof** Let $k_\mathbf{w}$ denote the number of occurrences of the relations in (3.3) in the middle of obtaining $\mathbf{w}$ from $\mathbf{w}$ by applying the relations in both (3.3) and (3.4). We will prove our assertion using the mathematical induction on $k_\mathbf{w}$. When $k_\mathbf{w} = 0$, our assertion is obvious since $\mathbf{w}$ is obtained from $\mathbf{w}$ by applying the relations in (3.4) only.

Given a positive integer $\hat{k}$, assume that our assertion holds for all $\mathbf{w}$ whenever $k_\mathbf{w} < \hat{k}$. Now, let $k_\mathbf{w} = \hat{k}$, which means that there is a word $\mathbf{w}' = w_1' w_2' \cdots w_{l+m}'$ such that $k_{\mathbf{w}'} = \hat{k} - 1$ and $\mathbf{w}$ is obtained from $\mathbf{w}'$ by applying the relations in (3.4) and one of the relations in (3.3) only once. By the first paragraph, we have only to consider the case where $\mathbf{w}$ is obtained from $\mathbf{w}'$ by applying a relation in (3.3) to $w_{l_0-1}' w_{l_0}' w_{l_0+1}'$ for some $1 \leq t_0 < l + m$. Suppose that $\mathbf{w}$ contains a subword $w_{j_0-1} w_{j_0} w_{j_0+1}$ of the form (3.5). We observe that no words in (3.5) appear in the six relations in (3.3), so $j_0 \neq t_0$. In case where $|j_0 - t_0| > 2, w_{j_0-1} w_{j_0} w_{j_0+1}$ is of the form (3.5). But, this is absurd since $k_{\mathbf{w}'} < \hat{k}$. In case where $0 < |j_0 - t_0| \leq 2$, $\mathbf{w}$ has a consecutive subword of the form $\pi_{r_0} \pi_{r_0+1} \pi_{r_0+1}$ or $\pi_{r_0} \pi_{r_0+1} \pi_{r_0+1} \pi_{r_0+1}$ or $\pi_{r_0} \pi_{r_0+1} \pi_{r_0+1} \pi_{r_0+1}$. For $1 \leq t \leq l + m$, let $i_t$ be the subindex given by $w_t = \pi_{i_t}$. Using the notation $\pi_t$ instead of $\pi_i$ and $\bar{\pi}_i$, one can see that the relations in both (3.3) and (3.4) coincide with braid and commutation relations of $S_n$. It says that $s_{u_1}s_{u_2} \cdots s_{u_{l+m}}$ is obtained from $s_{u_1}s_{u_2} \cdots s_{u_l}s_{v_1}s_{v_2} \cdots s_{v_m}$ by applying
braid and commutation relations of $\mathfrak{S}_n$, thus is a reduced expression for $\sigma \rho$. This, however, cannot occur since $s_{i_1}s_{i_2}\cdots s_{i_l+m}$ contains a consecutive subword of the form $s_r s_{r+1} s_r s_{r+1}$ or $s_{r+1} s_r s_{r+1} s_r$. Consequently, $w$ contains no subwords of the form (3.5), so we are done. \( \square \)

Given a word $w = w_1 w_2 \cdots w_n$ from the alphabet $\{\pi_i, \pi_i \mid 1 \leq i \leq n-1\}$, we can naturally see it as an element of $H_n(0)$. For clarity, we write it as $\iota(w)$. If a word $w'$ is obtained from $w$ by applying the relations in (3.3) and (3.4), then $\iota(w) = \iota(w')$ by (3.2). Combining (3.2) with Lemma 1 yields the following lemma.

**Lemma 2** For $\sigma, \rho \in \mathfrak{S}_n$ with $\ell(\sigma \rho) = \ell(\sigma) + \ell(\rho)$, if $j_0 \in \text{Des}_R(\sigma \rho) \backslash \text{Des}_R(\rho)$, then there exists a word $w = w_1 w_2 \cdots w_{\ell(\sigma \rho)}$ from the alphabet $\{\pi_i, \pi_i \mid 1 \leq i \leq n-1\}$ such that $\iota(w) = \pi_{\sigma, \pi_{\rho}}$ and $w_{\ell(\sigma \rho)} = \pi_{j_0}$.

**Proof** Let $s_{u_1} s_{u_2} \cdots s_{u_l}$ be a reduced expression of $\sigma$ and $s_{v_1} s_{v_2} \cdots s_{v_m}$ a reduced expression of $\rho$. Since $j_0 \in \text{Des}_R(\sigma \rho) \backslash \text{Des}_R(\rho)$, by the exchange property of Coxeter groups, there exists $1 \leq r \leq l$ such that $\sigma \rho = s_{u_1} s_{u_2} \cdots \widehat{s_{u_r}} \cdots s_{u_l} s_{v_1} s_{v_2} \cdots s_{v_m} s_{j_0}$, where $s_{u_1} s_{u_2} \cdots \widehat{s_{u_r}} \cdots s_{u_l}$ denotes the permutation obtained from $s_{u_1} s_{u_2} \cdots s_{u_l}$ by removing $s_{u_r}$. Viewing $\pi_{\sigma, \pi_{\rho}}$ just as a word from the alphabet $\{\pi_i \mid 1 \leq i \leq n-1\}$, due to (3.2) and Lemma 1, the relations in (3.3) and (3.4) play the same role as the braid and commutation relations of $\mathfrak{S}_n$. This implies that $\pi_{\sigma, \pi_{\rho}} = \pi_{u_1} \pi_{u_2} \cdots \pi_{u_r} \cdots \pi_{v_1} \pi_{v_2} \cdots \pi_{v_m} \pi_{j_0}$. Let $m(\pi_i) = 0$ if $\pi_i = \pi_j$ and 1 if $\pi_i = \pi_j$. Note that, while applying the relations in (3.3) and (3.4) to $\pi_{\sigma, \pi_{\rho}}$, the values of $\pi_{\sigma, \pi_{\rho}}$’s under $m$ are staying unchanged as seen in the following figure:
Thus, we conclude that $\pi_\sigma \pi_\rho = \pi_{u_1} \pi_{u_2} \cdots \pi_{u_t} \pi_{v_1} \pi_{v_2} \cdots \pi_{v_m} \pi_{j_0}$. \qed

Given $h = \sum_{\sigma \in S_n} c_\sigma \pi_\sigma \in H_n(0)$, let us say that $\pi_\sigma$ appears at $h$ if $c_\sigma \neq 0$. The following lemma plays a key role in describing the $H_n(0)$-action on $H_n(0)\pi_\sigma \pi_\rho$.

**Lemma 3** For any $\sigma, \rho \in S_n$, the following hold.

1. $\pi_\sigma \pi_\rho$ is nonzero if and only if $\ell(\sigma \rho) = \ell(\sigma) + \ell(\rho)$.
2. If $\ell(\sigma \rho) = \ell(\sigma) + \ell(\rho)$, then $\sigma \rho$ is a unique element of maximal length in $\{\gamma \in S_n \mid \pi_\gamma \text{ appears at } \pi_\sigma \pi_\rho\}$.
3. $\ell(\sigma \rho) = \ell(\sigma) + \ell(\rho)$ if and only if $\sigma \preceq_L w_0 \rho^{-1}$.

**Proof** Let $s_{u_1}s_{u_2} \cdots s_{u_t}$ be a reduced expression of $\sigma$ and $s_{v_1}s_{v_2} \cdots s_{v_m}$ a reduced expression of $\rho$.

Let us prove the “if” part of (1) and (2) simultaneously. For each $\gamma \in S_n$, let $c_\gamma$ be the integer defined by

$$\pi_\sigma \pi_\rho = \pi_\sigma (\pi_{v_1} - 1)(\pi_{v_2} - 1) \cdots (\pi_{v_m} - 1) = \pi_\sigma \rho + \sum_{\gamma \in S_n} c_\gamma \pi_\gamma.$$

It is clear that $c_\gamma = 0$ for all $\gamma \in S_n$ with $\ell(\gamma) \geq \ell(\sigma \rho)$. Thus, $\sigma \rho$ is a unique element of maximal length in $\{\gamma \in S_n \mid \pi_\gamma \text{ appears at } \pi_\sigma \pi_\rho\}$ and $\pi_\sigma \pi_\rho \neq 0$.

For the “only if” part of (1), we prove that if $\ell(\sigma \rho) < \ell(\sigma) + \ell(\rho)$, then $\pi_\sigma \pi_\rho = 0$. Since $\ell(\sigma \rho) < \ell(\sigma) + \ell(\rho)$, there exists $1 \leq m' \leq m$ such that

$$\ell(\sigma s_{v_1}s_{v_2} \cdots s_{v_m'}) < \ell(\sigma s_{v_1}s_{v_2} \cdots s_{v_{m'-1}}),$$

that is, $v_{m'} \in \text{Des}\left(\sigma s_{v_1}s_{v_2} \cdots s_{v_{m'-1}}\right)$. Let $\rho' = s_{v_1}s_{v_2} \cdots s_{v_{m'-1}}$. Since $s_{v_1}s_{v_2} \cdots s_{v_m}$ is a reduced expression of $\rho$, we have that $v_{m'} \in \text{Des}\left(\sigma \rho' \setminus \text{Des}(\rho')\right)$. By Lemma 2, there exists a word $w = w_1w_2 \cdots w_{l+m'-1}$ such that $\pi_\sigma \pi_{\rho'} = \iota(w)$ and $w_{l+m'-1} = \pi_{v_{m'}}$. But, since $\pi_{i_{l+m'-1}} = 0$ for all $1 \leq i \leq n - 1$, this implies that

$$\pi_\sigma \pi_{\rho'} = \iota \pi_{\sigma \rho'} \pi_{v_{m'}} \pi_{v_{m'+1}} \cdots \pi_{v_m} = 0.$$

To prove (3), note that for any $\gamma \in S_n$, $\ell(w_0 \gamma) = \ell(w_0) - \ell(\gamma)$ and $\ell(\gamma^{-1}) = \ell(\gamma)$. This implies that $\ell(\sigma \rho) = \ell(\sigma) + \ell(\rho)$ if and only if $\ell(w_0) - \ell(\gamma) - \ell(\gamma^{-1}) = \ell(w_0) - \ell(\sigma)$. Now our assertion is obvious from $(w_0 \rho^{-1} \sigma^{-1})\sigma = w_0 \rho^{-1}$. \qed

Now we are ready to state the main theorem of this subsection.

**Theorem 1** Let $\sigma, \rho \in S_n$.

1. The set $I(\sigma, \rho) := \{\pi_\gamma \pi_\rho \mid \gamma \in [\sigma, w_0 \rho^{-1}]_L\}$ forms a $C$-basis for $H_n(0)\pi_\sigma \pi_\rho$.
2. For any $\pi_\gamma \pi_\rho \in I(\sigma, \rho)$ and $1 \leq i \leq n - 1$,

$$\pi_i \cdot (\pi_\gamma \pi_\rho) = \begin{cases} 
\pi_\gamma \pi_\rho & \text{if } i \in \text{Des}_L(\gamma), \\
\pi_{\gamma \sigma} \pi_\rho & \text{if } i \notin \text{Des}_L(\gamma).
\end{cases}$$

Moreover, if $i \notin \text{Des}_L(\gamma)$, then $\pi_{\gamma \sigma} \pi_\rho = 0$ if and only if $i \in \text{Des}_L(\gamma \rho)$.

3. The linear map $\em : B(\sigma, \rho) \to H_n(0)\pi_\sigma \pi_{\rho^{-1}w_0}$ defined by

$$\gamma \mapsto \pi_\gamma \pi_{\rho^{-1}w_0} \text{ for } \gamma \in [\sigma, \rho]_L$$

is an $H_n(0)$-module isomorphism.
Proof The assertion (2) follows from the definition of Des$_L$(γ) and Lemma 3 (1).

For the assertion (1), we first observe that $H_n(0)\pi_{\sigma}\pi_{\rho}$ is spanned by $\{\pi_{\gamma}\pi_{\rho} \mid \gamma \in [\sigma, w_0]_{\rho}\}$. But, Lemma 3 (1) and Lemma 3 (3) ensure that $\pi_{\gamma}\pi_{\rho} = 0$ unless $\gamma \leq_L w_0\rho^{-1}$.

To prove that $I(\sigma, \rho)$ is linearly independent, we suppose that $\sum_{\gamma \in [\sigma, w_0\rho^{-1}]_L} c_{\gamma}\pi_{\gamma}\pi_{\rho} = 0$, but not all coefficients are zero. Let $A := \{\gamma \in [\sigma, w_0\rho^{-1}]_L \mid c_{\gamma} \neq 0\}$ and choose a permutation $\gamma_0 \in A$ which is of maximal length in $A$. Again, Lemma 3 (1) and Lemma 3 (3) ensure that $\pi_{\gamma}\pi_{\rho}$ is nonzero for all $\gamma \leq_L w_0\rho^{-1}$. Combining this fact with Lemma 3 (2) shows that $\pi_{\gamma_0}\rho$ cannot appear at $\pi_{\gamma}\pi_{\rho}$ for any $\gamma' \in A \setminus \{\gamma_0\}$. It means that $\pi_{\gamma_0}\rho$ appears at $\sum_{\gamma \in [\sigma, w_0\rho^{-1}]_L} c_{\gamma}\pi_{\gamma}\pi_{\rho} = 0$, which is absurd.

Finally, let us prove the assertion (3). Suppose that $\gamma \in [\sigma, \rho]_L$ and $i \notin$ Des$_L$(γ). For our purpose, by virtue of (1) and (2), we have only to prove that $\pi_i\gamma \in [\sigma, \rho]_L$ is equivalent to $i \notin$ Des$_L$(γ$\rho^{-1}$w$_0$). Let us first show that $\pi_i\gamma \in [\sigma, \rho]_L$ implies that $i \notin$ Des$_L$(γ$\rho^{-1}$w$_0$).

Since $\pi_i\gamma \in [\sigma, \rho]_L$, there exists $\xi \in S_n$ such that

$$\rho = \xi s_i \gamma \quad \text{and} \quad \ell(\rho) = \ell(\xi) + \ell(s_i \gamma). \quad (3.6)$$

From the second equality in (3.6) together with the assumption $i \notin$ Des$_L$(γ), we deduce that $\ell(\xi) + \ell(s_i \gamma) = \ell(\xi) + \ell(s_i) + \ell(\gamma)$. Applying this to the first equality in (3.6) says that $i \notin$ Des$_R$($\xi$), equivalently, $i \in$ Des$_L$(γ$^{-1}$w$_0$) or equally $i \notin$ Des$_L$(s$_i$γ$^{-1}$w$_0$). Now, $i \notin$ Des$_L$(γ$\rho^{-1}$w$_0$) is obvious since γ$\rho^{-1}$w$_0 = s_i\xi^{-1}$w$_0$ by the first equality in (3.6).

Next, let us show that $i \notin$ Des$_L$(γ$\rho^{-1}$w$_0$) implies that $\pi_i\gamma \in [\sigma, \rho]_L$. The assumption $i \notin$ Des$_L$(γ$\rho^{-1}$w$_0$) says that there exists a permutation $\xi \in S_n$ such that $\xi(s_i \gamma)\rho^{-1}$w$_0 = w_0$ and $\ell(w_0) = \ell(\xi) + \ell(s_i \gamma) + \ell(\rho w_0)$, thus $s_i \gamma \leq_L \rho$.

In Sect. 2.3, we introduced projective indecomposable modules $P_\alpha$ and irreducible modules $F_\alpha$. To be precise, $P_\alpha = H_n(0)\pi_{w_0(\alpha^c)}\pi_{w_0(\alpha)}$ and $F_\alpha$ is isomorphic to top($P_\alpha$) and one-dimensional. By applying Theorem 1, we derive the following isomorphisms of $H_n(0)$-modules:

$$P_\alpha \cong B(w_0(\alpha^c), w_0w_0(\alpha^c)) \quad \text{and} \quad F_\alpha \cong B(w_0(\alpha^c), w_0(\alpha^c)).$$

Example 1 Since $\rho^{-1}w_0 = 52314 \in S_5$ with $\rho = 24315$, by Theorem 1, we have an $H_5(0)$-module isomorphism $B(14325, 24315) \rightarrow H_5(0)\pi_{14325\pi_{52314}}$. This is illustrated in the following figure:

![Diagram](image-url)

Remark 1 It is remarked in [8, Definition 4.3] that all the specializations $q = 0$ of the Specht modules of $H_n(q)$ are of the form $H_n(0)\pi_{\xi}\pi_{w_0(\alpha^c)}$ for some $\alpha \mid n$ and...
$\xi \in [w_0(\alpha), w_0w_0(\alpha^c)]_L$. On the other hand, following the way as in Theorem 1, we can deduce that the $\mathbb{C}$-linear map

$$\bar{\eta}_\gamma : \bar{B}(\sigma, \rho) \to H_n(0)\Pi_\sigma \Pi_\rho^{-1}w_0, \quad \gamma \mapsto \Pi_\gamma \Pi_\rho^{-1}w_0$$

for $\gamma \in [\sigma, \rho]_L$ is an $H_n(0)$-module isomorphism. Combining these results, we see that all the specializations $q = 0$ of the Specht modules of $H_n(q)$ appear as $\bar{B}(\xi, w_0w_0(\alpha^c))$ for some $\alpha \models n$ and $\xi \in [w_0(\alpha), w_0w_0(\alpha^c)]_L$.

### 3.3 Restriction and Mackey formula

Throughout this subsection, $m$ and $n$ denote positive integers.

Hivert, Novelli, and Thibon [13] presented a formula on induction product of semi-combinatorial modules associated with Yang-Baxter intervals. Using this formula, one can derive the following lemma.

**Lemma 4** (cf. [13, Theorem 3.8]) For any $\sigma, \rho \in S_m$ and $\sigma', \rho' \in S_n$, we have

$$B(\sigma, \rho) \boxtimes B(\sigma', \rho') \cong B(\sigma \bullet \sigma', \rho \circ \rho').$$

Here\footnote{Following the notation in [13], $\sigma \bullet \sigma' = \sigma \cdot \sigma'[m]$ and $\rho \circ \rho' = \rho[n] \cdot \rho'$. And, there is a typo in [13, Theorem 3.8], where $\beta = \beta'[m - k] \cdot \beta'$ should appear as $\beta = \beta'[n - k] \cdot \beta''$.},

$$(\sigma \bullet \sigma')(i) := \begin{cases} \sigma(i) & \text{if } 1 \leq i \leq m, \\ \sigma'(i-m) + m & \text{if } m+1 \leq i \leq m+n, \end{cases}$$

and

$$(\rho \circ \rho')(i) := \begin{cases} \rho(i) + n & \text{if } 1 \leq i \leq m, \\ \rho'(i-m) & \text{if } m+1 \leq i \leq m+n. \end{cases}$$

**Remark 2** For $\sigma \in S_m$ and $\sigma' \in S_n$, let $\sigma \uplus \sigma'$ be the set of permutations $\gamma \in S_{m+n}$ satisfying that $\gamma(1)\sigma(2) \cdots \sigma(m)$ and $(\sigma'(1)+m)(\sigma'(2)+m) \cdots (\sigma'(n)+m)$ are subwords of $\gamma(1)\gamma(2) \cdots \gamma(m+n)$ and $\sigma \uplus \sigma' := \{\gamma^{-1} | \gamma \in \sigma^{-1} \uplus \sigma'^{-1}\}$. For $X \subseteq S_m$ and $Y \subseteq S_n$, let

$$X \uplus Y := \bigcup_{\sigma \in X, \sigma' \in Y} \sigma \uplus \sigma'.$$

It is not difficult to show that $[\sigma \bullet \sigma', \rho \circ \rho']_L = [\sigma, \rho]_L \uplus [\sigma', \rho']_L$ for $\sigma, \rho \in S_m$ and $\sigma', \rho' \in S_n$. Therefore, Lemma 4 can be rewritten as $B(\sigma, \rho) \boxtimes B(\sigma', \rho') \cong B([\sigma, \rho]_L \uplus [\sigma', \rho']_L)$. In particular, $B(\sigma, \sigma) \boxtimes B(\rho, \rho) \cong B(\sigma \uplus \rho)$, which can be regarded as a lift of

$$F_\alpha F_\beta = \sum_{\gamma \in \sigma \uplus \rho} F_{\text{comp}(\text{Des}_R(\gamma))}.$$
In contrast to induction product, restriction of semi-combinatorial modules associated with Yang-Baxter intervals has not yet been well studied except for the simple and projective indecomposable modules (see [14]). The main purpose of this section is to provide an explicit restriction rule for weak Bruhat interval modules. To begin with, let us collect necessary notations. Let \( \{m+n\}_m \) be the set of \( m \)-element subsets of \([m+n]\), on which \( \mathcal{S}_{m+n} \) acts in the natural way, that is, \( \gamma' \cdot J := \gamma'(J) \). Given \( \sigma, \rho \in \mathcal{S}_{m+n} \) and \( J \in \{m+n\}_m \), let

\[
\mathcal{I}(\sigma, \rho; J) := \{ \gamma \in [\sigma, \rho]_L \mid \gamma^{-1}([1, m]) = J \}
\]

and

\[
\mathcal{I}^{(m)}_{\sigma, \rho} := \left\{ J \in \{m+n\}_m \mid \mathcal{I}(\sigma, \rho; J) \neq \emptyset \right\}. \tag{3.7}
\]

For instance, \( \mathcal{I}(2134, 4312; [2, 3]) = \{3124, 4123, 3214, 4213\} \). A simple calculation shows that \( \mathcal{I}^{(2)}_{2134, 4312} = \{[1, 2], [2, 3], [3, 4]\} \).

When \( J = \{j_1 < j_2 < \cdots < j_m\} \), write \([m+n] \setminus J\) as \( \{j^c_1 < j^c_2 < \cdots < j^c_n\} \). Let \( \text{perm}_J \) and \( \text{perm}^J \) be the permutations in \( \mathcal{S}_{m+n} \) given by

\[
\begin{align*}
\text{perm}_J(j_k) &= k, \\
\text{perm}_J(j^c_s) &= m + s,
\end{align*}
\]

and

\[
\begin{align*}
\text{perm}^J(j_k) &= m - k + 1, \\
\text{perm}^J(j^c_s) &= m + n - s + 1,
\end{align*}
\]

respectively. For instance, \( \text{perm}_{[2, 3]} = 3124 \) and \( \text{perm}^{(2, 3)} = 4213 \). Note that

\[
\mathcal{I}(\text{id}, w_0; J) = [\text{perm}_J, \text{perm}^J]_L \quad \text{for any } J \in \{m+n\}_m, \tag{3.8}
\]

which will be used in the proof of Lemma 5. For each \( J \in \{m+n\}_m \), let

\[
\sigma_J := \text{perm}_{\sigma, J} \sigma \quad \text{and} \quad \rho_J := \text{perm}^{(w_0, \rho)} J w_0 \rho. \tag{3.9}
\]

The following lemma shows that \( [\sigma, \rho]_L = \bigcup_{J \in \mathcal{I}^{(m)}_{\sigma, \rho}} [\sigma_J, \rho^J]_L \).

**Lemma 5** Let \( \sigma, \rho \in \mathcal{S}_{m+n} \). For \( J \in \mathcal{I}^{(m)}_{\sigma, \rho} \), we have

\[
\mathcal{I}(\sigma, \rho; J) = [\sigma_J, \rho^J]_L. \tag{3.10}
\]

**Proof** Let us fix \( J \in \mathcal{I}^{(m)}_{\sigma, \rho} \). It is well known that given \( \gamma \in \mathcal{S}_{m+n} \), if \( \sigma \leq L \rho \) and \( \sigma \gamma \leq L \rho \gamma \), then there exists an isomorphism \( f_\gamma : [\sigma, \rho]_L \to [\sigma \gamma, \rho \gamma]_L \) defined by \( \xi \mapsto \xi \gamma \) (for instance, see [4, Proposition 3.1.6]). By the definition of \( \mathcal{I}(\sigma, \rho; J) \) in (3.7), for all \( \gamma \in \mathcal{S}_{m+n} \), \( \xi \in \mathcal{I}(\sigma, \rho; J) \) and if only if \( \xi \gamma \in \mathcal{I}(\sigma \gamma, \rho \gamma; \gamma^{-1} \cdot J) \). Indeed, the restriction

\[
f_\gamma|_{\mathcal{I}(\sigma, \rho; J)}: \mathcal{I}(\sigma, \rho; J) \to \mathcal{I}(\sigma \gamma, \rho \gamma; \gamma^{-1} \cdot J) \tag{3.10}
\]

is an isomorphism of posets.

Since \( J \in \mathcal{I}^{(m)}_{\sigma, \rho} \), the set \( f_\sigma^{-1}(\mathcal{I}(\sigma, \rho; J)) = \mathcal{I}(\text{id}, \rho \sigma^{-1}; \sigma \cdot J) \) is nonempty. Combining (3.8) with the equality \( \mathcal{I}(\text{id}, \rho \sigma^{-1}; \sigma \cdot J) = [\text{id}, \rho \sigma^{-1}]_L \cap \mathcal{I}(\text{id}, w_0; \sigma \cdot J) \) yields that \( \text{perm}_{\sigma, J} \leq L \xi \) for any \( \xi \in \mathcal{I}(\text{id}, \rho \sigma^{-1}; \sigma \cdot J) \). Since \( [\text{id}, \xi]_L \subseteq [\text{id}, \rho \sigma^{-1}]_L \), \( \text{perm}_{\sigma, J} \) is contained in \([\text{id}, \rho \sigma^{-1}]_L \). Therefore, \( \text{perm}_{\sigma, J} \leq L \xi \) for all \( \xi \in \mathcal{I}(\text{id}, \rho \sigma^{-1}; \sigma \cdot J) = \mathcal{I}(\text{id}, w_0; \sigma \cdot J) \).

\[\square\]
[id, ρσ⁻¹]_L \cap \mathcal{I}(id, w_0; σ \cdot J). Via the isomorphism (3.10), we have σ\jmath \in \mathcal{I}(σ, ρ; J) and σ\jmath \preceq L \xi for all \xi \in \mathcal{I}(σ, ρ; J). In the same manner, one can prove that ρJ \in \mathcal{I}(σ, ρ; J) and \xi \preceq L ρJ for all \xi \in \mathcal{I}(σ, ρ; J).

By a careful reading of the proof of Lemma 5, one can derive that

\[ \mathcal{S}^{(m)}_{σ, ρ} = \left\{ J \in \binom{m+n}{m} \mid \text{perm}_J \in [id, ρσ⁻¹]_L \right\}. \]

**Example 2** For σ = 2134 and ρ = 4312,

\[ \mathcal{S}^{(2)}_{σ, ρ} = \{[1, 2], [2, 3], [3, 4]\}. \]

One can easily calculate that

\[ σ_{(1, 2)} = 2134, \quad σ_{(2, 3)} = 3124, \quad σ_{(3, 4)} = 4312 \]

and

\[ ρ_{(1, 2)} = 2134, \quad ρ_{(2, 3)} = 4213, \quad ρ_{(3, 4)} = 4312. \]

Thus, by Lemma 5 (2),

\[ [2134, 4312]_L = [2134, 2134]_L \cup [3124, 4213]_L \cup [4312, 4312]_L. \]

Figure 2 illustrates this partition.

Given γ ∈ \mathcal{G}_{m+n}, let \gamma⁻¹(1, m) = \{i_1 < i_2 < \cdots < i_m\} and \gamma⁻¹([m + 1, m + n]) = \{i'_1 < i'_2 < \cdots < i'_{n}\}. Let γ≤m ∈ \mathcal{G}_m and γ>m ∈ \mathcal{G}_n be the permutations given by

\[ γ≤m(j) = γ(i_j) \quad (1 \leq j \leq m) \quad \text{and} \quad γ>m(j) = γ(i'_j) - m \quad (1 \leq j \leq n). \]

(3.11)

For instance, if m = 3, n = 5, and γ = 58326147, then γ≤3 = 321 and γ>m = 25314.

**Theorem 2** For σ, ρ ∈ \mathcal{G}_{m+n}, we have

\[ B(σ, ρ) \downarrow_{H_m(0) \otimes H_n(0)} ^{H_{m+n}(0)} \cong \bigoplus_{J \in \mathcal{S}^{(m)}_{σ, ρ}} B((σ\jmath)_{≤m}, (ρJ)_{≤m}) \otimes B((σ\jmath)_{>m}, (ρJ)_{>m}). \]

**Proof** For each J ∈ \mathcal{S}^{(m)}_{σ, ρ}, we observe that \mathcal{I}(σ, ρ; J) \cup \{0\} is closed under the π_i-action for i ∈ [m + n - 1] \setminus [m]. This means, by virtue of Lemma 5, that \mathbb{C}[σ\jmath, ρJ]_L is an \[ H_m(0) \otimes H_n(0) \]-module. Since

\[ [σ, ρ]_L = \bigcup_{J \in \mathcal{S}^{(m)}_{σ, ρ}} [σ\jmath, ρJ]_L, \]

it follows that

\[ B(σ, ρ) \downarrow_{H_m(0) \otimes H_n(0)} ^{H_{m+n}(0)} \cong \bigoplus_{J \in \mathcal{S}^{(m)}_{σ, ρ}} \mathbb{C}[σ\jmath, ρJ]_L \quad (\text{as} H_m(0) \otimes H_n(0) \text{- modules}). \]

On the other hand, [σ\jmath, ρJ]_L is in bijection with \[ (σ\jmath)_{≤m}, (ρJ)_{≤m}]_L \times [(σ\jmath)_{>m}, (ρJ)_{>m}]_L \]

under γ ↦ (γ≤m, γ>m), which again induces an \[ H_m(0) \otimes H_n(0) \]-module isomorphism

\[ \mathbb{C}[σ\jmath, ρJ]_L \rightarrow B((σ\jmath)_{≤m}, (ρJ)_{≤m}) \otimes B((σ\jmath)_{>m}, (ρJ)_{>m}), \quad γ ↦ (γ≤m, γ>m), \]

as required. □
The difference between the Henderson and Li [3, Subsection 3.1 (5)] provide a Mackey formula working on the Grothendieck ring well illustrated in Fig. 2.

Example 3 Using Theorem 2, we derive that $B(2134, 4312) \downarrow_{H_2(0) \otimes H_2(0)}^{H_4(0)}$ is isomorphic to

$$(B(21, 21) \otimes B(12, 12)) \oplus (B(12, 21) \otimes B(12, 21)) \oplus (B(12, 12) \otimes B(21, 21)).$$

The difference between the $H_4(0)$-action and the $H_2(0) \otimes H_2(0)$-action on $B(2134, 4312)$ is well illustrated in Fig. 2.

Next, let us deal with a Mackey formula for weak Bruhat interval modules. Bergeron and Li [3, Subsection 3.1 (5)] provide a Mackey formula working on the Grothendieck ring $G = \bigoplus_{n \geq 0} G_0(H_n(0))$ of 0-Hecke algebras. It says that for any $H_m(0)$-module $M$, $H_n(0)$-module $N$, and $k \in [1, m + n - 1]$,

$$[(M \boxtimes N) \downarrow_{H_k(0) \otimes H_{m+n-k}}^{H_{m+n}(0)}] = \sum_{t+s=k \atop t \leq m, s \leq n} \left[ T_{t,s} \left( M \downarrow_{H_k(0) \otimes H_{m-t}(0)}^{H_{m+n}(0)} \otimes N \downarrow_{H_{t}(0) \otimes H_{n-s}(0)}^{H_{m}(0) \otimes H_{m-n-k}} \right) \uparrow_{H_k(0) \otimes H_{m+n-k}}^{H_{m+n}(0)} \right],$$

where $T_{t,s} : \text{mod} (H_t(0) \otimes H_{m-t}(0) \otimes H_k(0) \otimes H_{n-s}(0) \otimes H_{m+n-k}(0)) \rightarrow \text{mod} (H_t(0) \otimes H_{m-t}(0) \otimes H_{n-s}(0) \otimes H_{m+n-k}(0))$

is the functor sending $M_1 \otimes M_2 \otimes N_1 \otimes N_2 \mapsto M_1 \otimes N_1 \otimes M_2 \otimes N_2$. On the other hand, by combining Theorem 4 with Theorem 2, we can derive a formula working weak Bruhat interval modules:

$$B(\sigma, \rho) \boxtimes B(\sigma', \rho') \downarrow_{H_{m+n-k}(0)}^{H_{m+n}(0)}$$

$$\cong \bigoplus_{J \in \mathcal{S}(k)} B\left( ((\sigma \bullet \sigma')_f)_{\leq k}, ((\rho \bullet \rho')^f)_{\leq k} \right) \otimes B\left( ((\sigma \bullet \sigma')_f)_{> k}, ((\rho \bullet \rho')^f)_{> k} \right).$$

(3.12)

Although it is very naive, one can expect that Bergeron and Li’s Mackey formula lifts to our formula at least for weak Bruhat interval modules.

To prove our result, we need the notion of standardization. For $\sigma \in S_n$ and $1 \leq k' \leq k \leq n$, let $s(t; \sigma; [k', k])$ be a unique permutation in $S_{k' - k + 1}$ satisfying the condition that $s(t; \sigma; [k', k])(i) < s(t; \sigma; [k', k])(j)$ if and only if $\sigma(k' + i - 1) < \sigma(k' + j - 1)$ for all

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1 ≤ i, j ≤ k − k′ + 1. For instance, if σ = 25143 ∈ S₅, then st(σ; [2, 4]) = 312 ∈ S₃. This standardization preserves the left weak Bruhat order on Sₙ, in other words,

$$st(σ; [k', k]) ≤_L st(ρ; [k', k]) \text{ whenever } σ ≤_L ρ.$$  \hfill (3.13)

The following theorem shows that (3.12) is a natural lift of Bergeron and Li’s Mackey formula.

**Theorem 3** For σ, ρ ∈ Sₘ, σ′, ρ′ ∈ Sₙ, and 1 ≤ k < m + n,

$$(B(σ, ρ) ⊗ B(σ′, ρ′)) \downarrow H_{m+n+k}(0) \downarrow H_{k}(0) \downarrow H_{n+k}(0) \downarrow H_{m+k-n}(0)$$

$$\cong \bigoplus_{t+1=s+k} \bigoplus_{t ≤ m, s ≤ n} T_{s,t} \left( B(σ, ρ) \downarrow H_{m}(0) \downarrow H_{n}(0) \downarrow H_{m-n}(0) \right) \downarrow H_{t}(0) \downarrow H_{s}(0) \downarrow H_{m-n-t}(0) \downarrow H_{n-t}(0).$$

**Proof** Using Lemma 4 and Theorem 2, we derive that

$$\bigoplus_{t+1=s+k} \bigoplus_{t ≤ m, s ≤ n} T_{s,t} \left( B(σ, ρ) \downarrow H_{m}(0) \downarrow H_{n}(0) \downarrow H_{m-n}(0) \right) \downarrow H_{t}(0) \downarrow H_{s}(0) \downarrow H_{m-n-t}(0) \downarrow H_{n-t}(0)$$

$$\cong \bigoplus_{J_1, J_2} B \left( (σ_{J_1})_{≤ i} \bullet (σ'_{J_2})_{≤ s}, (ρ_{J_1})_{≤ i} \bigcirc (ρ'_{J_2})_{≤ s} \right) \downarrow B \left( (σ_{J_1})_{> i} \bullet (σ'_{J_2})_{> s}, (ρ_{J_1})_{> i} \bigcirc (ρ'_{J_2})_{> s} \right),$$

where the sum ranges over all pairs (J₁, J₂) in

$$\bigcup_{t+1=s+k} \bigcup_{t ≤ m, s ≤ n} J_{σ, ρ} \times J_{σ', ρ'}.$$  \hfill (3.13)

Let $$f : J_{σ, ρ} \times J_{σ', ρ'} \rightarrow \bigcup_{t+1=s+k} \bigcup_{t ≤ m, s ≤ n} J_{σ, ρ} \times J_{σ', ρ'}$$ be a map defined by

$$J \mapsto (J \cap [1, m], \{i \mid i + m ∈ J \cap [m + 1, m + n]\})$$

for all $$J ∈ J_{σ, ρ} \times J_{σ', ρ'}.$$ To begin with, let us verify that $$f$$ is a well-defined bijection.

First, we prove that $$f$$ is well-defined. Given $$J ∈ J_{σ, ρ} \times J_{σ', ρ'},$$ let

$$J_1 = J \cap [1, m], \quad J_2 = \{i \mid i + m ∈ J \cap [m + 1, m + n]\}, \quad t = |J_1|, \quad \text{and } s = |J_2|. \hfill (3.14)$$

Since $$J ∈ J_{σ, ρ} \times J_{σ', ρ'},$$ there exists a permutation $$δ ∈ [σ \bullet σ', ρ \bigcirc ρ']_L$$ such that $$δ(J) = [1, k].$$ Combining (3.13) with the definition of $$σ \bullet σ'$$ and $$ρ \bigcirc ρ'$$ yields that

$$σ ≤_L st(δ; [1, m]) ≤_L ρ \quad \text{and} \quad σ' ≤_L st(δ; [m + 1, m + n]) ≤_L ρ'.$$

Note that

$$st(δ; [1, m])(J_1) = [1, t] \quad \text{and} \quad st(δ; [m + 1, m + n])(J_2) = [1, s].$$

This tells us that $$J_1 ∈ J_{σ, ρ}^{(t)}$$ and $$J_2 ∈ J_{σ', ρ'}^{(s)},$$ thus $$f$$ is well-defined.

Next, we prove that $$f$$ is bijective by constructing its inverse. Let $$t′$$ and $$s′$$ be nonnegative integers satisfying $$t′ + s′ = k,$$ $$t′ ≤ m,$$ and $$s′ ≤ n.$$ Given $$K_1 ∈ J_{σ, ρ}^{(t′)}$$ and $$K_2 ∈ J_{σ', ρ'}^{(s′)},$$ consider the mapping $$(K_1, K_2) ↦ K := K_1 ∪ \{i \mid i - m ∈ K_2\}$$. Note that there exist permutations $$γ_1 ∈ [σ, ρ]_L$$ and $$γ_2 ∈ [σ', ρ']_L$$ such that

$$γ_1(K_1) = [1, t'] \quad \text{and} \quad γ_2(K_2) = [1, s'].$$

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Let $\delta \in \mathfrak{S}_{m+n}$ given by

$$
\delta(i) = \begin{cases}
\gamma_1(i) & \text{if } i \in K_1, \\
\gamma_1(i) + t' + s' & \text{if } i \in [1, m] \setminus K_1, \\
\gamma_2(i - m) + t' & \text{if } i \in K'_2, \\
\gamma_2(i - m) + m + s' & \text{if } i \in [m, m + 1, \ldots, m + n] \setminus K'_2,
\end{cases}
$$

where $K'_2 := \{ j \mid j - m \in K_2 \}$. Then $\delta \in \gamma_1 \sqcup \gamma_2$ and $\delta(K) = [1, k]$, and therefore $K \in \mathcal{R}^{(k)}_{\sigma \bullet \sigma', \rho \cdot \rho'}$. In addition, $f(K) = (K_1, K_2)$ by the definition of $K$. On the other hand, given $J \in \mathcal{R}^{(k)}_{\sigma \bullet \sigma', \rho \cdot \rho'}$, letting $(K_1, K_2) = f(J)$, one can easily see that $K_1 \cup \{ i \mid i - m \in K_2 \} = J$. So the inverse of $f$ is well-defined and thus $f$ is bijective.

For our assertion, it suffices to show that for each $J \in \mathcal{R}^{(k)}_{\sigma \bullet \sigma', \rho \cdot \rho'}$,  

$$
\mathcal{B}((\sigma \bullet \sigma') J) \subseteq \mathcal{B}((\sigma \bullet \sigma') J) \otimes \mathcal{B}((\rho \cdot \rho') J),
$$

where $J_1, J_2, t, s$ are defined as in (3.14). This isomorphism immediately follows from the four equalities:

$$
\begin{align*}
(1) \quad ((\sigma \bullet \sigma') J) \subseteq (\sigma J_1) \otimes (\sigma J_2) \subseteq, \\
(2) \quad ((\rho \cdot \rho') J) \subseteq (\rho J_1) \otimes (\rho J_2) \subseteq, \\
(3) \quad ((\sigma \bullet \sigma') J) \supseteq (\sigma J_1) \otimes (\sigma J_2) \supseteq, \\
(4) \quad ((\rho \cdot \rho') J) \supseteq (\rho J_1) \otimes (\rho J_2) \supseteq.
\end{align*}
$$

Let us prove the equality (1). Let $J = \{ j_1 < j_2 < \cdots < j_k \} \in \mathcal{R}^{(k)}_{\sigma \bullet \sigma', \rho \cdot \rho'}$. Then,

$$
((\sigma \bullet \sigma') J) \subseteq (\sigma J_1 (j_i) \quad \text{for } 1 \leq i \leq k.
$$

Assume that $j_i \leq m$ and $j_{i+1} > m$. Set

$$
J_1 = \{ j_1 < j_2 < \cdots < j_t \} \quad \text{and} \quad J_2 = \{ j_{i+1} - m < j_{i+2} - m < \cdots < j_{t+s} - m \}.
$$

Since

$$
(\sigma J_1) \subseteq (i) = \sigma J_1 (j_i) \quad \text{for } 1 \leq i \leq t,
$$

$$
(\sigma J_2) \subseteq (i) = \sigma J_2 (j_i - m) \quad \text{for } 1 \leq i \leq s,
$$

it follows that

$$
(\sigma J_1) \otimes (\sigma J_2) (i) = \begin{cases}
\sigma J_1 (j_i) & \text{if } 1 \leq i \leq t, \\
\sigma J_2 (j_i - m) + t & \text{if } t + 1 \leq i \leq t + s.
\end{cases}
$$

In case where $i \in [1, t]$, we have

$$
(\sigma \bullet \sigma') J (j_i) = \text{perm}_{\sigma \bullet \sigma'} (\sigma \bullet \sigma') (j_i) = \text{perm}_{\sigma, J_1} \sigma (j_i) = \sigma J_1 (j_i).
$$

In case where $i \in [t + 1, t + s]$, we have

$$
(\sigma \bullet \sigma') J (j_i) = \text{perm}_{\sigma \bullet \sigma'} (\sigma \bullet \sigma') (j_i) = \text{perm}_{\sigma, J_2} \sigma' (j_i - m) + t = \sigma J_2 (j_i - m) + t.
$$

Thus, the equality (1) holds.

Next, let us prove the equality (2). Under the same setting with the above paragraph, we have
In the proof of Theorem 3, we employ the fact that

\[(\rho \circ \rho')(J)_{\leq k}(i) = (\rho \circ \rho')(J_k)(j_i) \quad \text{for } 1 \leq i \leq k,\]

\[(\rho J_1)_{\leq t}(i) = \rho J_1(j_i) \quad \text{for } 1 \leq i \leq t,\]

\[(\rho' J_2)_{\leq s}(i) = \rho' J_2(j_{t+i} - m) \quad \text{for } 1 \leq i \leq s.\]

From the second and third equalities, we have

\[(\rho J_1)_{\leq t} = \rho J_1(j_i) + s \quad \text{for } 1 \leq i \leq t,\]

\[(\rho' J_2)_{\leq s} = \rho' J_2(j_{t+i} - m) \quad \text{for } t+1 \leq i \leq t+s.\]

In case where \(i \in [1, t]\), we have

\[(\rho \circ \rho')(J_k)(j_i) = \text{perm}_{(m+n)}(\rho \circ \rho') \cdot \text{perm}_{(m+n)}(\rho \circ \rho')(j_i) = \text{perm}_{(m+n)}(\rho \circ \rho')(j_i) + s = \rho J_1(j_i) + s.\]

In case where \(i \in [t+1, t+s]\), we have

\[(\rho \circ \rho')(J_k)(j_i) = \text{perm}_{(m+n)}(\rho \circ \rho') \cdot \text{perm}_{(m+n)}(\rho \circ \rho')(j_i) = \text{perm}_{(m+n)}(\rho \circ \rho')(j_i) - m = \rho' J_2(j_i - m).\]

Thus, the equality (2) holds.

Equalities (3) and (4) can be proven in a similar way with (1) and (2) respectively, so we omit the proofs.

\[\square\]

**Remark 3** For \(\sigma \in \mathfrak{S}_m\) and \(\rho \in \mathfrak{S}_n\), let \(\sigma \sqcup \rho\) be the set of permutations \(\gamma \in \mathfrak{S}_{m+n}\) satisfying that \(\sigma(1)\sigma(2)\cdots \sigma(m)\) and \((\rho(1) + 1)(\rho(2) + 2)\cdots (\rho(n) + m)\) are subwords of \(\gamma(1)\gamma(2)\cdots \gamma(m+n)\) and \(\sigma \sqcup \rho := \{\gamma^{-1} | \gamma \in \sigma^{-1} \sqcup \rho^{-1}\}\). For \(X \subseteq \mathfrak{S}_m\) and \(Y \subseteq \mathfrak{S}_n\), let

\[X \sqcup Y := \bigcup_{\sigma \in X, \rho \in Y} \sigma \sqcup \rho.\]

In the proof of Theorem 3, we employ the fact that \([\sigma \bullet \sigma', \rho \circ \rho']_L \subseteq [\sigma, \rho]_L \sqcup [\sigma', \rho']_L\) for \(\sigma, \rho \in \mathfrak{S}_m\) and \(\sigma', \rho' \in \mathfrak{S}_n\). In fact,

\[[\sigma \bullet \sigma', \rho \circ \rho']_L = [\sigma, \rho]_L \sqcup [\sigma', \rho']_L.\]

Therefore, by Theorem 4, we have that

\[B(\sigma, \rho) \boxtimes B(\sigma', \rho') \cong B([\sigma, \rho]_L \sqcup [\sigma', \rho']_L).\]

It is well known that the multiplicative rule for the fundamental quasisymmetric functions are described as follow:

\[F_\alpha F_\beta = \sum_{\gamma \in \sigma_\alpha \sqcup \rho_\beta} F_{\text{comp}(\text{Des}_R(\gamma))}.\]  \hfill \quad (3.15)

Here, \(\sigma \in \mathfrak{S}_{\ell(\alpha)}\) and \(\rho \in \mathfrak{S}_{\ell(\beta)}\) satisfying \(\text{Des}_R(\sigma) = \text{set}(\alpha)\) and \(\text{Des}_R(\rho) = \text{set}(\beta)\). Now, one can see that the multiplicative rule (3.15) lifts to the induction product

\[B(\sigma, \sigma) \boxtimes B(\rho, \rho) \cong B(\sigma \sqcup \rho).\]
3.4 (Anti-)automorphism twists of weak Bruhat interval modules

Let $\mu : B \to A$ be an isomorphism of associative algebras over $\mathbb{C}$. Given an $A$-module $M$, we define $\mu[M]$ by the $B$-module with the same underlying space as $M$ and with the action $\cdot_\mu$ twisted by $\mu$ in such a way that
\[
 b \cdot_\mu v := \mu(b) \cdot v \quad \text{for} \ a \in A \text{ and } v \in M.
\]

Let $\text{mod } A$ be the category of finite dimensional left $A$-modules. Any isomorphism $\mu : B \to A$ induces a covariant functor
\[
 T_\mu^+ : \text{mod } A \to \text{mod } B, \quad M \mapsto \mu[M],
\]
where $T_\mu^+(h) : \mu[N] \to \mu[M], m \mapsto h(m)$ for every $A$-module homomorphism $h : M \to N$. We call $T_\mu^+$ the $\mu$-twist.

Similarly, given an anti-isomorphism $\nu : B \to A$, we define $\nu[M]$ to be the $B$-module with $M^\ast$, the dual space of $M$, as the underlying space and with the action $\cdot^\nu$ defined by
\[
 (b \cdot^\nu \delta)(v) := \delta(v(b) \cdot v) \quad \text{for } b \in B, \delta \in M^\ast, \text{ and } v \in M.
\]

Any anti-isomorphism $\nu : B \to A$ induces a contravariant functor
\[
 T^-_\nu : \text{mod } A \to \text{mod } B, \quad M \mapsto \nu[M],
\]
where $T^-_\nu(h) : \nu[N] \to \nu[M], \delta \mapsto \delta \circ h$ for every $A$-module homomorphism $h : M \to N$. We call $T^-_\nu$ the $\nu$-twist. In [10], Fayers introduced the involutions $\phi$, $\theta$ and the anti-involution $\chi$ of $H_n(0)$ defined in the following manner:
\[
\begin{align*}
 \phi : H_n(0) &\to H_n(0), \quad \pi_i \mapsto \pi_{n-i} \quad \text{for } 1 \leq i \leq n - 1, \\
 \theta : H_n(0) &\to H_n(0), \quad \pi_i \mapsto -\pi_i \quad \text{for } 1 \leq i \leq n - 1, \\
 \chi : H_n(0) &\to H_n(0), \quad \pi_i \mapsto \pi_i \quad \text{for } 1 \leq i \leq n - 1.
\end{align*}
\]

These morphisms commute with each other. We study the (anti-)involution twists for $\phi$, $\theta$, $\chi$, and their compositions $\omega := \phi \circ \theta$, $\widehat{\phi} := \phi \circ \chi$, $\widehat{\theta} := \theta \circ \chi$, $\widehat{\omega} := \omega \circ \chi$.

The viewpoint of looking at (anti-) involutions as functors is quite useful for many reasons. The primary reason is that using the exactness of the corresponding functors, one can transport various structures of a given $H_n(0)$-module to their twists in a functorial way. An application in this direction can be found in Sect. 4.2.2. Additional reasons include that some well known functors appear in the context of our (anti-)involution twists. Given any anti-automorphism $\nu$ of $H_n(0)$, the standard duality $D : \text{mod } H_n(0) \to \text{mod } H_n(0)^{\text{op}}$ appears as $F_{\nu^{-1}} \circ T^-_\nu$, where $F_{\nu^{-1}} : \text{mod } H_n(0) \to \text{mod } H_n(0)^{\text{op}}$ is the functor induced by the inverse of $\nu : H_n(0) \to H_n(0)^{\text{op}}, x \mapsto \nu(x)$. In particular, $D \cong F_\chi \circ T^-_\chi$. The Nakayama functor $\nu$ is naturally isomorphic to $T^-_{\nu^*}$, which can be derived by combining [10, Proposition 4.2] with [22, Proposition IV.3.13]. To explain in more detail, the former reference implies $\phi$ is a Nakayama automorphism and the latter reference shows the relationship between Nakayama automorphisms and $\nu$. And, the $H_n(0)$-dual functor, $\text{Hom}_{H_n(0)}(-, H_n(0))$, is naturally isomorphic to $D \circ \nu$, and therefore is naturally isomorphic to $F_\chi \circ T^-_{\phi}$.

Now, let us focus on the main topic of this subsection, (anti-)involution twists of weak Bruhat interval modules. For irreducible modules and projective indecomposable modules, it was shown in [10, 14] that
Theorem 4 For \( \sigma, \rho \in \mathfrak{S}_n \), we have the following isomorphisms of \( H_n(0) \)-modules.

1. \( \phi[\mathcal{B}(\sigma, \rho)] \cong \mathcal{B}(\sigma^w, \rho^w) \).
2. \( \theta[\mathcal{B}(\sigma, \rho)] \cong \overline{\mathcal{B}}(\sigma, \rho) \).
3. \( \chi[\mathcal{B}(\sigma, \rho)] \cong \overline{\mathcal{B}}(\rho w_0, \sigma w_0) \). In particular, \( \chi[\mathcal{P}_\alpha] \cong \mathcal{P}_{\alpha^c} \).

Proof Consider the \( \mathbb{C} \)-linear isomorphisms defined by

\[
\begin{align*}
f_1 &: \phi[\mathcal{B}(\sigma, \rho)] \to \mathcal{B}(\sigma^w, \rho^w), \quad \gamma \mapsto \gamma^w, \\
f_2 &: \theta[\mathcal{B}(\sigma, \rho)] \to \overline{\mathcal{B}}(\sigma, \rho), \quad \gamma \mapsto (-1)^{\ell(\gamma^{-1})} \gamma, \\
f_3 &: \chi[\mathcal{B}(\sigma, \rho)] \to \overline{\mathcal{B}}(\rho w_0, \sigma w_0), \quad \gamma^* \mapsto \gamma w_0,
\end{align*}
\]

where \( \gamma \in [\sigma, \rho]_L \) and \( \gamma^* \) denotes the dual of \( \gamma \) with respect to the basis \([\sigma, \rho]_L \) for \( \mathcal{B}(\sigma, \rho) \).

Since it can be proven in a similar manner that these maps are \( H_n(0) \)-isomorphisms, we here only deal with (3).

Note that, for \( 1 \leq i \leq n - 1 \),

\[
\pi_i \cdot X \gamma^* = \begin{cases} 
-\gamma^* & \text{if } i \notin \text{Des}_L(\gamma), \\
(s_i \gamma)^* & \text{if } i \in \text{Des}_L(\gamma) \text{ and } s_i \gamma \in [\sigma, \rho]_L, \\
0 & \text{if } i \in \text{Des}_L(\gamma) \text{ and } s_i \gamma \notin [\sigma, \rho]_L.
\end{cases}
\]

which yields that

\[
f_3(\pi_i \cdot X \gamma^*) = \begin{cases} 
-\gamma w_0 & \text{if } i \notin \text{Des}_L(\gamma), \\
s_i \gamma w_0 & \text{if } i \in \text{Des}_L(\gamma) \text{ and } s_i \gamma \in [\sigma, \rho]_L, \\
0 & \text{if } i \in \text{Des}_L(\gamma) \text{ and } s_i \gamma \notin [\sigma, \rho]_L.
\end{cases}
\]

On the other hand,

\[
\pi_i \cdot f_3(\gamma^*) = \pi_i \cdot \gamma w_0
\]

\[
= \begin{cases} 
-\gamma w_0 & \text{if } i \in \text{Des}_L(\gamma w_0), \\
s_i \gamma w_0 & \text{if } i \notin \text{Des}_L(\gamma w_0) \text{ and } s_i \gamma w_0 \in [\rho w_0, \sigma w_0]_L, \\
0 & \text{if } i \notin \text{Des}_L(\gamma w_0) \text{ and } s_i \gamma w_0 \notin [\rho w_0, \sigma w_0]_L.
\end{cases}
\]

It immediately follows from (2.1) that \( i \notin \text{Des}_L(\gamma w_0) \) if and only if \( i \in \text{Des}_L(\gamma w_0) \). Moreover, it is trivial that \( s_i \gamma \in [\sigma, \rho]_L \) if and only if \( s_i \gamma w_0 \in [\rho w_0, \sigma w_0]_L \). Thus, we verified \( \chi[\mathcal{B}(\sigma, \rho)] \cong \overline{\mathcal{B}}(\rho w_0, \sigma w_0) \). And, combining the equality \( w_0(\alpha)^w = w_0(\alpha^c) \) with (3) yields \( \chi[\mathcal{P}_\alpha] \cong \mathcal{P}_{\alpha^c} \).

As seen in Table 1, various (anti-)involution twists can be obtained from Theorem 4 by composing \( \theta, \phi \), and \( \chi \). For the reader’s understanding, we deal with irreducible modules and projective indecomposable modules in a separate table (see Table 2).

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Table 1 (Anti-)involution twists of weak Bruhat interval modules

| φ-twist | θ-twist | ω-twist | χ-twist | φ-twist | θ-twist | ω-twist |
|---------|---------|---------|---------|---------|---------|---------|
| $\mathcal{B}(\sigma, \rho)$ | $\overline{\mathcal{B}}(\sigma, \rho)$ | $\overline{\mathcal{B}}(\sigma^w, \rho^w)$ | $\overline{\mathcal{B}}(\sigma^w, \rho^w)$ | $\overline{\mathcal{B}}(\sigma^w, \rho^w)$ | $\overline{\mathcal{B}}(\sigma^w, \rho^w)$ | $\overline{\mathcal{B}}(\sigma^w, \rho^w)$ |

Table 2 (Anti-)involution twists of $F_\alpha$ and $P_\alpha$

| φ-twist | θ-twist | ω-twist | χ-twist | φ-twist | θ-twist | ω-twist |
|---------|---------|---------|---------|---------|---------|---------|
| $F_\alpha$ | $F_\alpha^t$ | $F_\alpha^c$ | $F_\alpha^t$ | $F_\alpha$ | $F_\alpha^t$ | $F_\alpha^c$ |
| $P_\alpha$ | $P_\alpha^t$ | $P_\alpha^c$ | $P_\alpha^t$ | $P_\alpha$ | $P_\alpha^t$ | $P_\alpha^c$ |

Example 4 By Theorem 4, we have

$$\phi[\mathcal{B}(2134, 4123)] \cong \mathcal{B}(1243, 2341),$$
$$\theta[\mathcal{B}(2134, 4123)] \cong \overline{\mathcal{B}}(2134, 4123),$$
and
$$\chi[\mathcal{B}(2134, 4123)] \cong \overline{\mathcal{B}}(3214, 4312).$$

We illustrate these $H_n(0)$-modules as follows:

For an (anti-)automorphism $\zeta : H_m+n(0) \to H_m+n(0)$ and an $H_m(0) \otimes H_n(0)$-module $M$, we simply write $\zeta[M]$. The subsequent corollary shows that (anti-) involution twists behave nicely with respect to induction product and restriction.

Corollary 1 Let $M$, $N$, and $L$ be weak Bruhat interval modules of $H_m(0)$, $H_n(0)$, and $H_{m+n}(0)$, respectively. Then we have following isomorphisms of modules.

(A1) $\phi[M \boxtimes N] \cong \phi[N] \boxtimes \phi[M]$

(A2) $\theta[M \boxtimes N] \cong \theta[M] \boxtimes \theta[N]$

(A3) $\chi[M \boxtimes N] \cong \chi[N] \boxtimes \chi[M]$

(B1) $\phi[L \downarrow_{H_m+n(0)} H_n(0) \otimes H_m(0)] \cong \phi[L] \downarrow_{H_m+n(0)} H_n(0) \otimes H_m(0)$

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(B2) \( \theta[L \downarrow H_{m+n}(0)] \cong \theta[L] \downarrow H_{m+n}(0) \)
(B3) \( \chi[L \downarrow H_{m+n}(0)] \cong \chi[L] \downarrow H_{m+n}(0) \)

Proof (A1), (A2), (B1), (B2), and (B3) are straightforward from the definitions of \( \phi \), \( \theta \), and \( \chi \).

For (A3), let \( M = B(\sigma_1, \rho_1) \) and \( N = B(\sigma_2, \rho_2) \). Then, by Theorems 4 and 4 (3),

\[
\chi[M \boxtimes N] \cong B(\rho_1 \circ \rho_2, \sigma_1 \circ \sigma_2, w_0^{(m+n)})
\]

and

\[
\chi[N] \boxtimes \chi[M] \cong B(\rho_2 w_0^{(n)}, \sigma_2 w_0^{(n)} \circ B(\rho_1 w_0^{(m)}, \sigma_1 w_0^{(m)}))
\cong B(\rho_2 w_0^{(n)} \circ \rho_1 w_0^{(m)}, \sigma_2 w_0^{(m)} \circ \sigma_1 w_0^{(m)}).
\]

Here, the notation \( w_0^{(k)} \) denotes the longest element in \( S_k \) and the last isomorphism follows from Theorem 4 and (A2). Therefore, the assertion follows from the fact that the one-line notation of \( \sigma w_0 \) is obtained by reversing \( \sigma \) for any permutation \( \sigma \).

\[ \square \]

4 Various 0-Hecke modules constructed using tableaux

Suppose we have a family of quasisymmetric functions that can be expanded in the basis of the fundamental quasisymmetric functions with positive coefficients. The correspondence (2.2) tells us that each of them appears as the image of the isomorphism classes of certain \( H_n(0) \)-modules. Among them, it would be very nice to find or construct one which is nontrivial, in other words, not a direct sum of irreducible modules and has a combinatorial model that can be handled well. Since the mid-2010s, some \( H_n(0) \)-modules have been constructed in line with this philosophy, more precisely, in [1, 2, 21, 24]. In this section, we show that all of them are equipped with the structure of weak Bruhat interval modules.

To deal with these modules, we need the notion of source and sink.

Definition 2 Let \( B \) be a basis for an \( H_n(0) \)-module such that \( B \cup \{0\} \) is closed under the action of \( \{\pi_i \mid 1 \leq i \leq n - 1\} \).

(1) An element \( x_0 \in B \) is called a source of \( B \) if, for each \( x \in B \), there exists \( \sigma \in S_n \) such that \( \pi_\sigma \cdot x_0 = x \).

(2) An element \( x'_0 \in B \) is called a sink of \( B \) if, for each \( x \in B \), there exists \( \sigma \in S_n \) such that \( \pi_\sigma \cdot x = x'_0 \).

Following the way as in [24], one can see that there are at most one source and sink in \( B \). In case where \( B \) is the basis \( [\sigma, \rho]_L \) for \( B(\sigma, \rho) \), \( \sigma \) is the source and \( \rho \) is the sink.

Hereafter, \( \alpha \) denotes a composition of \( n \). To introduce the tableaux in our concern, we need to define the composition diagram \( \circ \circ \alpha \) of shape \( \alpha \). It is a left-justified array of \( n \) boxes where the \( i \)th row from the top has \( \alpha_i \) boxes for \( 1 \leq i \leq k \). For a filling \( \tau \) of \( \circ \circ \alpha \), we denote by \( \tau_{i,j} \) the entry in the \( i \)th row from the top and \( j \)th column from the left.

4.1 Standard immaculate tableaux, standard extended tableaux, and their \( H_n(0) \)-modules

We begin with introducing the definition of standard immaculate tableaux and standard extended tableaux.
Definition 3 ([2, 21]) Let \( \alpha \) be a composition of \( n \).

1. A **standard immaculate tableau of shape** \( \alpha \) is a filling \( T \) of the composition diagram \( cd(\alpha) \) with \( \{1, 2, \ldots, n\} \) such that the entries are all distinct, the entries in each row increase from left to right, and the entries in the first column increase from top to bottom.

2. A **standard extended tableau of shape** \( \alpha \) is a filling \( T \) of the composition diagram \( cd(\alpha) \) with \( \{1, 2, \ldots, n\} \) such that the entries are all distinct, the entries in each row increase from left to right, and the entries in each column increase from top to bottom.

We remark that our standard extended tableaux are slightly different from those of Searles [21]. In fact, the former can be obtained by flipping the latter horizontally.

Denote by \( SIT(\alpha) \) the set of all standard immaculate tableaux of shape \( \alpha \) and by \( SET(\alpha) \) the set of all standard extended tableaux of shape \( \alpha \).

Berge et al. [2] define a 0-Hecke action on \( SIT(\alpha) \) and denote the resulting module by \( V_\alpha \). And, Searles [21] define a 0-Hecke action on \( SET(\alpha) \) and denote the resulting module by \( X_\alpha \). By the construction of \( V_\alpha \) and \( X_\alpha \), it is clear that \( SIT(\alpha) \) and \( SET(\alpha) \) are bases for \( V_\alpha \) and \( X_\alpha \), respectively.

It is not difficult to show that both \( SIT(\alpha) \) and \( SET(\alpha) \) have a unique source and a unique sink. Denote the source of \( SIT(\alpha) \) by \( T_\alpha \) and the source of \( SET(\alpha) \) by \( T'_\alpha \). They are obtained by filling \( cd(\alpha) \) with entries \( 1, 2, \ldots, n \) from left to right and from top to bottom. Denote the sink of \( SIT(\alpha) \) by \( T'_\alpha \) and the sink of \( SET(\alpha) \) by \( T'_{\alpha} \). In contrast of \( T_\alpha \) and \( T'_\alpha \), \( T'_\alpha \) and \( T'_{\alpha} \) have to be constructed separately. The former \( T'_\alpha \) is obtained from \( cd(\alpha) \) in the following steps:

1. Fill the first column with entries \( 1, 2, \ldots, \ell(\alpha) \) from top to bottom.
2. Fill the remaining boxes with entries \( \ell(\alpha) + 1, \ell(\alpha) + 2, \ldots, n \) from left to right from bottom to top.

On the other hand, the latter \( T'_{\alpha} \) is obtained by filling \( cd(\alpha) \) with the entries \( 1, 2, \ldots, n \) from top to bottom and from left to right.

Definition 4 For a filling \( T \) of a composition diagram, \( \text{read}(T) \) is defined to be the word obtained from \( T \) by reading the entries from right to left starting with the top row.

With this definition, we can state the following theorem.

Theorem 5 For any \( \alpha \models n \), we have the \( H_n(0) \)-module isomorphisms

\[
V_\alpha \cong B(\text{read}(T_\alpha), \text{read}(T'_\alpha)) \quad \text{and} \quad X_\alpha \cong B(\text{read}(T'_\alpha), \text{read}(T'_\alpha)).
\]

(4.1)

Here, the words in the parentheses are being viewed as permutations in one-line notation.

Proof To prove the first isomorphism in (4.1), we need the \( H_n(0) \)-module homomorphisms

\[
\theta[\mathbf{P}_\alpha^c] \xrightarrow{w} \mathcal{P}_\alpha \quad \text{and} \quad \theta[\mathbf{P}_\alpha^c] \xrightarrow{\Phi} \mathcal{V}_\alpha.
\]

(4.2)

where

1. the notation \( \mathbf{P}_\alpha^c \) denotes the projective indecomposable module spanned by the standard ribbon tableaux of shape \( \alpha^c \) in [14, Subsection 3.2],
2. the first homomorphism \( w \) is an isomorphism given in [14, Theorem 3.3 and Proposition 5.1], which is given by reading standard ribbon tableaux from left to right starting with the bottom row, and
3. the second homomorphism \( \Phi \) is an essential epimorphism given in [6, Theorem 3.2].
Composing \( w^{-1} \) with \( \Phi \) yields a surjective \( H_n(0) \)-module homomorphism \( \tilde{\Phi} : \mathcal{P}_\alpha \to \mathcal{V}_\alpha \).
In view of the definition of \( w \) and \( \Phi \), one can see that

\[
\tilde{\Phi}(\pi_{\gamma}\mathcal{T}_{w_0(\alpha)}) = \begin{cases} 
\mathcal{T} & \text{if } \text{read}(\mathcal{T}) = \gamma \text{ for some } \mathcal{T} \in \text{SIT}(\alpha), \\
0 & \text{else.}
\end{cases}
\]

Composing \( \tilde{\Phi} \) with the isomorphism

\[
\text{em} : \text{B}(w_0(\alpha^c), w_0 w_0(\alpha)) \to \mathcal{P}_\alpha \quad (\text{in Theorem 3.5 (3)}),
\]
we finally have the surjective \( H_n(0) \)-module homomorphism

\[
\tilde{\Phi} \circ \text{em} : \text{B}(w_0(\alpha^c), w_0 w_0(\alpha)) \to \mathcal{V}_\alpha.
\]

Next, let us consider the projection

\[
\text{pr} : \text{B}(w_0(\alpha^c), w_0 w_0(\alpha)) \to \text{B}(\text{read}(\mathcal{T}_\alpha), \text{read}(\mathcal{T}_\alpha')),
\]

\[
\gamma \mapsto \begin{cases} 
\gamma & \text{if } \gamma \in [\text{read}(\mathcal{T}_\alpha), \text{read}(\mathcal{T}_\alpha')]_L, \\
0 & \text{else.}
\end{cases}
\]

By the definition of \( \tilde{\Phi} \), one sees that \( \text{read}(\mathcal{T}_\alpha) = w_0(\alpha^c) \) and \( \text{read}(\mathcal{T}_\alpha') \leq_L w_0 w_0(\alpha) \). This implies that \( \text{pr} \) is a surjective \( H_n(0) \)-module homomorphism.

For our purpose, we have only to show \( \ker(\tilde{\Phi} \circ \text{em}) = \ker(\text{pr}) \). From the definition of the \( H_n(0) \)-action on \( \mathcal{V}_\alpha \), it follows that \( \text{read}(\mathcal{T}) \in [\text{read}(\mathcal{T}_\alpha), \text{read}(\mathcal{T}_\alpha')]_L \) for any \( \mathcal{T} \in \text{SIT}(\alpha) \) and therefore \( \ker(\tilde{\Phi} \circ \text{em}) \supseteq \ker(\text{pr}) \). On the other hand, using the fact that the 0-Hecke action on \( \text{SIT}(\alpha) \) satisfies the braid relations, one can show that every \( \gamma \in [\text{read}(\mathcal{T}_\alpha), \text{read}(\mathcal{T}_\alpha')]_L \) appears as \( \text{read}(\mathcal{T}) \) for some \( \mathcal{T} \in \text{SIT}(\alpha) \). This says that \( \ker(\tilde{\Phi} \circ \text{em}) \cong \ker(\text{pr}) \), so we are done.

The second isomorphism in (4.1) can be obtained by replacing \( \Phi \) with \( \Gamma \circ \Phi \) in (4.2), where \( \Gamma : \mathcal{V}_\alpha \to X_\alpha \) is the surjection in [6, Subsection 3.2].

### 4.2 Standard permuted composition tableaux and their \( H_n(0) \)-modules

We begin with introducing the definition of standard permuted composition tableaux. In this subsection, \( \sigma \) denotes a permutation in \( \mathfrak{S}_{\ell(\alpha)} \).

**Definition 5** [26] Given \( \alpha \models n \) and \( \sigma \in \mathfrak{S}_{\ell(\alpha)} \), a **standard permuted composition tableau** (SPCT) of shape \( \alpha \) and type \( \sigma \) is a filling \( \tau \) of \( \mathfrak{c}\alpha(\alpha) \) with entries in \( \{1, 2, \ldots, n\} \) such that the following conditions hold:

1. The entries are all distinct.
2. The standardization of the word obtained by reading the first column from top to bottom is \( \sigma \).
3. The entries in each rows decrease from left to right.
4. If \( i < j \) and \( \tau_{i,k} > \tau_{j,k+1} \), then \( (i, k + 1) \in \mathfrak{c}\alpha(\alpha) \) and \( \tau_{i,k+1} > \tau_{j,k+1} \).

Denote by \( \text{SPCT}^\sigma(\alpha) \) the set of all standard permuted composition tableaux of shape \( \alpha \) and type \( \sigma \). Tewari and van Willigenburg define a 0-Hecke action on \( \text{SPCT}^\sigma(\alpha) \) and denote the resulting module by \( S^\sigma_\alpha \). Contrary to \( \mathcal{V}_\alpha \) and \( X_\alpha \), this module is not indecomposable in general. In the following, we briefly explain how to decompose \( S^\sigma_\alpha \) into indecomposables.

For \( \tau, \tau' \in \text{SPCT}^\sigma(\alpha) \), define \( \tau \sim \tau' \) if for each positive integer \( k \), the relative order of the entries in the \( k \)th column of \( \tau \) is equal to that of \( \tau' \). This relation is an equivalence relation.
on SPCT$^\sigma(\alpha)$. Let $E^\sigma(\alpha)$ be the set of all equivalence classes under $\sim$. Every class in $E^\sigma(\alpha)$ is closed under the 0-Hecke action, which gives rise to the decomposition

$$S^\sigma = \bigoplus_{E \in E^\sigma(\alpha)} S^\sigma_{\alpha,E},$$

where $S^\sigma_{\alpha,E}$ is the $H_n(0)$-module spanned by $E$. All the results in the above can be found in [26]. The decomposition was improved in [5] by showing that every direct summand is indecomposable.

### 4.2.1 Weak Bruhat interval module structure of $S^\sigma_{\alpha,E}$

We show that $S^\sigma_{\alpha,E}$ is isomorphic to a weak Bruhat interval module. To do this, we need a special reading of standard permuted composition tableaux. Let us introduce the necessary notations and the results. Given $\tau \in SPCT^\sigma(\alpha)$, let

$$\text{Des}(\tau) := \{1 \leq i \leq n - 1 \mid i + 1 \text{ lies weakly right of } i \text{ in } \tau\}.$$ 

It should be noticed that the set $\text{Des}(\tau)$ plays the same role as the complement of $\text{Des}_L(\gamma)$ since $\pi \cdot \gamma \neq \gamma$ if and only if $i \notin \text{Des}_L(\gamma)$ for $\gamma \in [\sigma, \rho]_L$.

Every equivalence class $E$ has a unique source and a unique sink. Denote the source by $\tau_E$ and the sink by $\tau'_E$. Let $m_E$ be the number of elements in $\text{Des}(\tau_E)$ and set

$$\text{Des}(\tau_E) = \{d_1 < d_2 < \ldots < d_{m_E}\}, \quad d_0 := 0, \quad \text{and} \quad d_{m_E+1} := n.$$ 

For $1 \leq j \leq m_E + 1$, let $H_j$ be the horizontal strip occupied by the boxes with entries from $d_{j-1} + 1$ to $d_j$ in $\tau_E$. For each $\tau \in E$, let $\tau(H_j)$ be the subfilling of $\tau$ occupied by $H_j$ in $\mathcal{C}(\alpha)$.

**Definition 6** For $\tau \in E$ and $1 \leq j \leq m_E + 1$, let $w(j)(\tau)$ be the word obtained by reading $\tau(H_j)$ from left to right. The reading word, $\text{read}(\tau)$, of $\tau$ is defined to be the word $w^{(1)}(\tau) w^{(2)}(\tau) \ldots w^{(m_E+1)}(\tau)$.

**Example 5** Let $E_0 = \{\begin{array}{c} \begin{array}{c} 4 \ 3 \\ 2 \\ 5 \\ 1 \end{array} \end{array} \ , \begin{array}{c} \begin{array}{c} 4 \ 3 \\ 2 \\ 5 \\ 1 \end{array} \end{array}\} \in \mathcal{E}^d((3, 2))$. We have

$$\tau_{E_0} = \begin{array}{c} \begin{array}{c} 4 \ 3 \\ 2 \\ 5 \\ 1 \end{array} \end{array}, \quad \text{Des}(\tau_{E_0}) = \{1, 4\}, \quad \text{and} \quad \begin{array}{c} \begin{array}{c} H_2 \\ H_1 \end{array} \end{array}.$$ 

Therefore,

$$\text{read} \left( \begin{array}{c} \begin{array}{c} 4 \ 3 \\ 2 \\ 5 \\ 1 \end{array} \end{array} \right) = 14325 \quad \text{and} \quad \text{read} \left( \begin{array}{c} \begin{array}{c} 4 \ 3 \\ 2 \\ 5 \\ 1 \end{array} \end{array} \right) = 24315.$$ 

Next, we introduce the notion of generalized compositions. A generalized composition $\alpha$ of $n$ is defined to be a formal sum $\beta^{(1)} \oplus \beta^{(2)} \oplus \cdots \oplus \beta^{(k)}$, where $\beta^{(i)} \models n_i$ for positive integers $n_i$'s with $n_1 + n_2 + \cdots + n_k = n$. Given $\alpha = \beta^{(1)} \oplus \beta^{(2)} \oplus \cdots \oplus \beta^{(k)}$, let $\alpha^c := \beta^{(1)} \oplus \beta^{(2)} \oplus \cdots \oplus \beta^{(k)}$. Springer
\((\beta^{(1)})^e \oplus (\beta^{(2)})^e \oplus \cdots \oplus (\beta^{(k)})^e\) and \(w_0(\alpha) := w_0(\beta^{(1)} \cdot \beta^{(2)} \cdot \cdots \cdot \beta^{(k)})\), where \(\beta^{(i)} \cdot \beta^{(i+1)}\) is the concatenation of \(\beta^{(i)}\) and \(\beta^{(i+1)}\) for \(1 \leq i \leq k - 1\). Let \(P_\alpha := H_n(0)\pi_{w_0(\alpha)} \pi_w(\alpha)\).

Choi, Kim, Nam, and Oh [6] found the projective cover of \(S_{\alpha, E}^\sigma\) by constructing an essential epimorphism \(\eta : \theta[P_{\alpha(\sigma; E)}] \to S_{\alpha, E}^\sigma\), where \(\alpha(\sigma; E)\) is a generalized composition defined by using the source of \(E\) in a suitable manner and \(P_{\alpha(\sigma; E)}\) is the projective module spanned by standard ribbon tableaux of shape \(\alpha(\sigma; E)^c\). From now on, we simply write \(\alpha_E\) for \(\alpha(\sigma; E)\) since \(E\) contains information on \(\alpha\) and \(\sigma\). For the details, see [6, Subsection 2.3 and Section 5].

Now, we are ready to prove the following theorem.

**Theorem 6** Let \(\alpha \vdash n\) and \(\sigma \in \mathcal{S}_{\ell(\alpha)}\). For each \(E \in \mathcal{E}_\alpha(\alpha)\),
\[
S_{\alpha, E}^\sigma \cong B(\text{read}(\tau_E), \text{read}(\tau'_E)).
\]

**Proof** To prove our assertion, we need the \(H_n(0)\)-module homomorphisms
\[
\theta[P_{\alpha_E}] \xrightarrow{w} P_{\alpha_E}^c \quad \text{and} \quad \theta[P_{\alpha_E}] \xrightarrow{\eta} S_{\alpha, E}^\sigma.
\]
where

- the notation \(P_{\alpha_E}\) denotes the projective module spanned by the generalized ribbon tableaux of shape \(\alpha_E\),
- the first homomorphism \(w\) is an isomorphism given in [14, Theorem 3.3 and Proposition 5.1], which is given by reading standard ribbon tableaux from left to right starting with the bottom row, and
- the second homomorphism \(\eta\) is an essential epimorphism given in [6, Theorem 5.3].

Composing \(w^{-1}\) with \(\eta\) yields a surjective \(H_n(0)\)-module homomorphism \(\tilde{\eta} : P_{\alpha_E}^c \to S_{\alpha, E}^\sigma\).

In view of the definition of \(w\) and \(\eta\), one can see that
\[
\tilde{\eta}(\pi_{\tau} \pi_{w_0(\alpha_E^c)}) = \begin{cases} \tau & \text{if} \text{read}(\tau) = \gamma \text{ for some } \tau \in E, \\ 0 & \text{otherwise}. \end{cases}
\]

Composing \(\tilde{\eta}\) with the isomorphism \(\text{em} : B(w_0(\alpha_E), w_0 w_0(\alpha_E^c)) \to P_{\alpha_E}^c \quad \text{(in Theorem 3.5 (3))}\)
we finally have the surjective \(H_n(0)\)-module homomorphism
\[
\tilde{\eta} \circ \text{em} : B(w_0(\alpha_E), w_0 w_0(\alpha_E^c)) \to S_{\alpha, E}^\sigma.
\]
Next, let us consider the projection
\[
\text{pr} : B(w_0(\alpha_E), w_0 w_0(\alpha_E^c)) \to B(\text{read}(\tau_E), \text{read}(\tau'_E)),
\]
\[
\gamma \mapsto \begin{cases} \gamma & \text{if } \gamma \in [\text{read}(\tau_E), \text{read}(\tau'_E)]_L, \\ 0 & \text{else}. \end{cases}
\]

By the definition of \(\tilde{\eta}\), \(\text{read}(\tau_E) = w_0(\alpha_E)\) and \(\text{read}(\tau'_E) \leq_L w_0 w_0(\alpha_E^c)\) and therefore \(\text{pr}\) is a surjective \(H_n(0)\)-module homomorphism.

For our purpose, we have only to show that \(\ker(\tilde{\eta} \circ \text{em}) = \ker(\text{pr})\). From the definition of the \(H_n(0)\)-action on \(S_{\alpha, E}^\sigma\), it follows that \(\text{read}(\tau) \in [\text{read}(\tau_E), \text{read}(\tau'_E)]_L\) for any \(\tau \in E\) and therefore \(\ker(\tilde{\eta} \circ \text{em}) \supseteq \ker(\text{pr})\). On the other hand, using the fact that the 0-Hecke action on \(E\) satisfies the braid relations, one can show that every \(\gamma \in [\text{read}(\tau_E), \text{read}(\tau'_E)]_L\) appears as \(\text{read}(\tau)\) for some \(\tau \in E\). This says that \(\ker(\tilde{\eta} \circ \text{em}) \subseteq \ker(\text{pr})\), so we are done. 

\(\square\)
Remark 4 In case where $\sigma = \text{id}$, a different reading from ours in Definition 6 has already been introduced in [24, Definition 4.1]. More precisely, for each $\tau \in \text{SPCT}^\text{id}(\alpha)$, they define a reading word $\text{col}_\tau$, called the column word of $\tau$. They also introduce a partial order $\preceq_{\alpha}$ on $\text{SPCT}^\text{id}(\alpha)$ and prove that $(E, \preceq_{\alpha})$ is a graded poset isomorphic to $((\text{col}_{\tau_E}, \text{col}_{\tau'_E})_{\text{L}}, \preceq_{\text{L}})$ (see [24, Theorem 6.18]). In view of Theorem 5, one may expect that $S_{\alpha, E}^{\text{id}}$ is isomorphic to $B(\text{col}_{\tau_E}, \text{col}_{\tau'_E})$. This, however, turns out to be false. For instance, let $E_0$ be the equivalence class given in Example 5. Then $\text{col}_{\tau_{E_0}} = 45312$, $\text{col}_{\tau'_{E_0}} = 45321$, and $B(45312, 45321)$ is not indecomposable as seen in Fig. 1. Therefore, $S_{\alpha, E_0}^{\text{id}}$ is not isomorphic to $B(45312, 45321)$.

4.2.2 Involution twists of $S_{\alpha}^{\sigma}$

Standard Young row-strict tableaux were first introduced in [18] as a combinatorial model for the Young row-strict quasisymmetric Schur functions $R_{\alpha}$. Recently, Bardwell and Searles [1] succeeded in constructing an $H_n(0)$-module $R_{\alpha}$ whose quasisymmetric characteristic image equals $R_{\alpha}$. It is constructed by defining a 0-Hecke action on the set of standard Young row-strict tableaux of shape $\alpha$. We here introduce permuted standard Young row-strict tableaux which turn out to be very useful in describing the $H_n(0)$-module $\widehat{\omega}[S_{\alpha}^{\sigma}]$, where $\widehat{\omega} = \phi \circ \theta \circ \chi$.

Definition 7 Given $\alpha \vdash n$ and $\sigma \in \mathfrak{S}_{\ell(\alpha)}$, a standard permuted Young row-strict composition tableau $T$ (SPYRT) of shape $\alpha$ and type $\sigma$ is a filling of $c\mathfrak{c}(\alpha^T)$ with entries $\{1, 2, \ldots, n\}$ such that the following conditions hold:

(1) The entries are all distinct.
(2) The standardization of the word obtained by reading the first column from bottom to top is $\sigma$.
(3) The entries in each row are increasing from left to right.
(4) If $i < j$ and $T_{i,k} < T_{j,k+1}$, then $(i, k + 1) \in c\mathfrak{c}(\alpha^T)$ and $T_{i,k+1} < T_{j,k+1}$.

Denote by $\text{SPYRT}^\sigma(\alpha)$ the set of all SPYRTs of shape $\alpha$ and type $\sigma$. Let $R_{\alpha}^{\sigma}$ be the $\mathbb{C}$-span of $\text{SPYRT}^\sigma(\alpha)$. Define

$$
\pi_i \cdot T := \begin{cases} 
T & \text{if } i + 1 \text{ is weakly left of } i \text{ in } T, \\
0 & \text{if } i + 1 \text{ is right-adjacent to } i \text{ in } T, \\
s_i \cdot T & \text{otherwise}
\end{cases}
$$

for $1 \leq i \leq n - 1$ and $T \in \text{SPYRT}^\sigma(\alpha)$. Here, $s_i \cdot T$ is obtained from $T$ by swapping $i$ and $i + 1$.

We claim that (4.3) defines an $H_n(0)$-action on $R_{\alpha}^{\sigma}$. For $T \in \text{SPYRT}^\sigma(\alpha)$, let $\tau_T$ be the filling of $c\mathfrak{c}(\alpha^T)$ defined by $(\tau_T)_{i,j} = n + 1 - T_{i,j}$. Define a $\mathbb{C}$-linear isomorphism $\mathcal{W} : R_{\alpha}^{\sigma} \to \widehat{\omega}[S_{\alpha^T^{\sigma_0}}]$ by letting

$$
\mathcal{W}(T) = (-1)^{\text{rank}(\tau_T)} \tau_T^{\sigma_0} \quad \text{for } T \in \text{SPYRT}^\sigma(\alpha),
$$

then extending it by linearity. Here, $\tau_T^{\sigma_0}$ is the dual of $\tau_T$ with respect to the basis $\text{SPCT}^{\sigma_0}(\alpha^T)$ for $S_{\alpha^T^{\sigma_0}}$ and $\text{rank}(\tau_T) := \min\{\ell(y) \mid y \in \mathfrak{c}(E) : y \cdot \tau_{E^T} = \tau_T\}$, where $E$ is the equivalence class containing $\tau_T$. One can verify that

$$
\mathcal{W}(\pi_i \cdot T) = \pi_i \cdot \mathcal{W}(T) \quad \text{for all } 1 \leq i \leq n - 1,
$$

which proves our claim. In particular, when $\sigma = \text{id}$, our $R_{\alpha}^{\sigma}$ is exactly same to $R_{\alpha}$ due to Bardwell and Searles. To summarize, we state the following proposition.
Proposition 1  For each $\alpha \models n$ and $\sigma \in \mathcal{S}_\ell(\alpha)$, (4.3) defines an $H_n(0)$-action on $R^\sigma_{\alpha}$. Moreover, $\mathcal{W} : R^\sigma_{\alpha} \to \hat{\omega}[S^\sigma_{\alpha}]$ is an $H_n(0)$-module isomorphism.

Remark 5  In the combinatorial aspect, our SPYRTs are precisely the standard permuted Young composition tableaux (SPYCT) in [6, Definition 4.4]. But, they should be distinguished as well as their images under the quasisymmetric characteristic when $\rho$ is the isomorphism $\hat{\omega} : S^\sigma_{\alpha} \to \hat{\omega}[S^\sigma_{\alpha}]$. For each $E \in \mathcal{S}^\sigma_{\alpha}(\alpha')$, let $R^\sigma_{\alpha, E} := \mathcal{W}^{-1}(\hat{\omega}[S^\sigma_{\alpha}])$. Combining Proposition 1 with the results in [5, 6, 26], we have the following corollary.

Corollary 2  For each $E \in \mathcal{S}^\sigma_{\alpha}(\alpha')$, the following hold:

1. $R^\sigma_{\alpha, E}$ is indecomposable. In particular, $R^\sigma_{\alpha} = \bigoplus_{E \in \mathcal{S}^\sigma_{\alpha}(\alpha)} R^\sigma_{\alpha, E}$ is a decomposition of $R^\sigma_{\alpha}$ into indecomposables.
2. The injective hull of $R^\sigma_{\alpha, E}$ is $P_{\alpha E} (= P_{\alpha \sigma(\alpha') \sigma(\alpha)} E)$.
3. $R^\sigma_{\alpha, E} \cong B(w_0 \text{read}(\tau_0), w_0 \text{read}(\tau_0))$.

Proof  (1) Since $T_{\hat{\omega}} : \text{mod } H_n(0) \to \text{mod } H_n(0)$ is an isomorphism, it preserves direct sum. Therefore, the assertion can be obtained by combining Proposition 1 with [5, Theorem 3.1].

(2) As seen in the proof of Theorem 6, $\tilde{\eta} : P_{\alpha E} \to S_{\alpha^\sigma, E}$ is an essential epimorphism, thus $P_{\alpha E}$ is the projective cover of $S_{\alpha^\sigma, E}$. It is clear that $T_{\hat{\omega}}$ is a contravariant exact functor. So, taking $T_{\hat{\omega}}$ on $\tilde{\eta}$ yields that $\hat{\omega}[P_{\alpha E}]$ is the injective hull of $\hat{\omega}[S^\sigma_{\alpha^\sigma, E}] \cong R^\sigma_{\alpha, E}$. Now the assertion follows from the isomorphism $\hat{\omega}[P_{\alpha E}] \cong P_{\alpha E}$, which is due to Table 2.

(3) The assertion follows from Theorem 4 with Theorem 6.

The three commutative diagrams in Fig. 3 show various (anti-)involution twists of $S^\sigma_{\alpha}$ as well as their images under the quasisymmetric characteristic when $\sigma = \text{id}$. In the first and second diagram, the functors assigned to parallel arrows are all same and the arrows in red are being used to indicate that the domain and codomain have the same image under the quasisymmetric characteristic. In the last diagram, $S^\sigma_{\alpha}$ is the Young quasisymmetric Schur function in [17, Definition 5.2.1], $\mathcal{R}S_{\alpha}$ is the row-strict quasisymmetric Schur function in [19, Definition 3.2], and $\psi, \rho$ are automorphisms of QSym defined by $\psi(F_{\alpha}) = F_{\alpha^c}$ and $\rho(F_{\alpha}) = F_{\alpha'}$.

Remark 6  (1) Let $k$ be a positive integer and $\omega := \psi \circ \rho$. It was stated in [19, Theorem 5.1] that

$$\omega(S_{\alpha}(x_1, x_2, \ldots, x_k)) = \mathcal{R}S_{\alpha}(x_k, x_{k-1}, \ldots, x_1).$$
On the other hand, the third diagram in Fig. 3 shows that \( \omega(S_{\alpha}) = R_{\alpha} \), thus

\[
\omega(S_{\alpha}(x_1, x_2, \ldots, x_k)) = R_{\alpha}(x_1, x_2, \ldots, x_k).
\]

As a consequence, we derive that

\[
R_S\alpha(x_k, x_{k-1}, \ldots, x_1) = R_{\alpha}(x_1, x_2, \ldots, x_k).
\]

We add a remark that in some literature such as [17, Subsection 5.2] and [25, Remark 4.4], the identity \( \omega(S_{\alpha}) = R_{\alpha} \) is incorrectly stated as \( \omega(S_{\alpha}) = RS_{\alpha} \).

(2) One can also observe \( \omega(\tilde{S}_{\alpha}) = R_{\alpha} \) in [18, Theorem 12]. This identity, however, should appear as \( \omega(\tilde{S}_{\alpha}) = RS_{\alpha} \) by the third diagram in Fig. 3. Within the best understanding of the authors, this error seems to have occurred for the reason that the descent sets of Young composition tableaux and that of standard Young row-strict composition tableaux are defined in a different manner. The proof of [18, Theorem 12], with a small modification, can be used to verify \( \omega(\tilde{S}_{\alpha}) = RS_{\alpha} \).

5 Further avenues

(1) We have studied the structure of weak Bruhat interval modules so far. However, there are still many unsolved fundamental problems including the following:

- Classify all weak Bruhat interval modules up to isomorphism.
- Given an interval \([\sigma, \rho]_L\), decompose \( B(\sigma, \rho) \) into indecomposables.
- Given an interval \([\sigma, \rho]_L\), find the projective cover and the injective hull of \( B(\sigma, \rho) \).

(2) In [24, Section 9 and 10], Tewari and van Willigenburg provide a restriction rule for \( S_{\alpha} \downarrow \nabla_{\alpha} \) and ask if there is a reciprocal induction rule for \( S_{\alpha} \uparrow \nabla_{\alpha} \) with respect to the restriction rule. By combining Theorem 4 with Theorem 6, we successfully decompose \( S_{\alpha} \uparrow \nabla_{\alpha} \) into weak Bruhat interval modules. But, at the moment, we do not know if it can be expressed as a direct sum of \( S_{\beta} \)'s. We expect that a better understanding of the weak Bruhat interval modules appearing in the decomposition would be of great help in solving this problem. In line with this philosophy, it is interesting to find or characterize all intervals \([\sigma, \rho]_L\) such that \( B(\sigma, \rho) \) is isomorphic to \( V_{\alpha}, X_{\alpha}, \) or \( S_{\sigma, E, \alpha} \).

(3) Let \( n \) and \( N \) be arbitrary positive integers and \( V := \mathbb{C}^N \). Using the fact that the left \( U_0(gl_N) \)-action on \( V^\otimes n \) commutes with the right \( H_n(0) \)-action on \( V^\otimes n \), Krob and Thibon [16] construct \( U_0(gl_N) \)-modules

\[
D_{\alpha} := V^\otimes n \cdot \pi_{w_0(\alpha^c)} w_0 \pi_{w_0(\alpha^t)} \quad \text{and} \quad N_{\alpha} := V^\otimes n \cdot \pi_{w_0(\alpha)} \pi_{w_0(\alpha^c)}
\]

for every composition \( \alpha \) of \( n \). Then they prove that, as \( \alpha \) ranges over the set of nonempty compositions, \( D_{\alpha} \)'s form a complete family of irreducible polynomial \( U_0(gl_N) \)-modules and \( N_{\alpha} \)'s a complete family of indecomposable polynomial \( U_0(gl_N) \)-modules which arise as a direct summand of \( V^\otimes n \) for some \( n > 0 \). They also realize \( F_{\alpha} \) and \( P_{\alpha} \) as the left ideals of \( H_n(0) \)

\[
F_{\alpha} \cong H_n(0) \cdot \pi_{w_0(\alpha^c)} w_0 \pi_{w_0(\alpha^t)} \quad \text{and} \quad P_{\alpha} \cong H_n(0) \cdot \pi_{w_0(\alpha)} \pi_{w_0(\alpha^c)}.
\]
Hence, by replacing $H_n(0)$ by $V \otimes n$ in (5.1), one obtains the $U_0(gl_N)$-modules $D_\alpha$ and $N_\alpha$ from the $H_n(0)$-modules $F_\alpha$ and $P_\alpha$, respectively. This relationship seems to work well at the character level as well. In this regard, Hivert [12] shows that the Weyl character of $D_\alpha$ is equal to the quasisymmetric polynomial $F_\alpha(x_1, x_2, \ldots, x_N, 0, 0, \ldots)$, where $F_\alpha(x_1, x_2, \ldots) = \text{ch}(\mathcal{F}_\alpha)$.

In the present paper, we study intensively weak Bruhat interval modules, which are of the form $H_n(0) \pi_\sigma \pi_\rho$ or $H_n(0) \pi_\sigma \pi_\rho$ up to isomorphism (Theorem 1). Hence, it would be very meaningful to investigate how our results about weak Bruhat interval modules are reflected on the corresponding $U_0(gl_N)$-modules, in other words, the $U_0(gl_N)$-modules of the form $V \otimes n \cdot \pi_\sigma \pi_\rho$ and $V \otimes n \cdot \pi_\sigma \pi_\rho$ for $\sigma, \rho \in \mathfrak{S}_n$.

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