Matrix Models of 2D Gravity and Induced QCD

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Abstract

I review some recent works on the Hermitean one-matrix and $d$-dimensional gauge-invariant matrix models. Special attention is paid to solving the models at large-$N$ by the loop equations. For the one-matrix model the main result concerns calculations of higher genera, while for the $d$-dimensional model the large-$N$ solution for a logarithmic potential is described. Some results on fermionic matrix models are briefly reviewed.

Talk at the Workshop on Quantum Field Theoretical Aspects of High Energy Physics, Kyffhaeuser, Germany, September 20–24, 1993

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1 Introduction

Matrix models are usually associated with discretized random surfaces (or string theory) in a $d \leq 1$-dimensional embedding space and, in particular, with 2D quantum gravity. The simplest Hermitian one-matrix model corresponds to pure 2D gravity while a chain of Hermitian matrices describes 2D gravity interacting with $d \leq 1$ matter. A natural multi-dimensional extension of this construction is associated with induced lattice gauge theories. The matrix models which describe induced QCD can be constructed in a similar way.

1.1 Random surfaces

The typical problems which reduce to matrix models are associated with a statistical ensemble of random surfaces whose partition function is defined generically as

$$Z_{RS} = \sum_S e^{-\sigma A(S)}.$$  

Here $A(S)$ is the area of the surface $S$ which can be either closed or open and $\sigma$ stands for the string tension.

There is a lot of examples of such systems in quantum field theory:

- Strings either fundamental (graviton) or secondary (hadrons).
- 3D Ising model (the boundary between different phases is two-dimensional).
- Lattice gauge theory at strong coupling.
- 1/N-expansion of QCD.
- 2D quantum gravity.

The last system is described by the Euclidean partition function

$$Z_{2D} = \int Dg e^{-\int d^2x \sqrt{g} (\frac{1}{16\pi G} R)} = \int Dg e^{-\int d^2x \sqrt{g} \Lambda + \chi}$$

where $\Lambda$ stands for the cosmological constant and $\chi$ is the Euler characteristics of the 2D world. The path integral in Eq. (1.2) is over all metrics $g_{\mu\nu}(x)$.

1.2 Dynamical triangulation

The idea of dynamical triangulation of random surfaces is to approximate the surface by a set of equilateral triangles. The coordination number (the number of triangles meeting at a vertex) is not necessarily equal to six which is associated with internal curvature of the surface. The partition function (1.2) is approximated by

$$Z_{DT} = \sum_g e^g (1-g) \sum_{T_g} e^{-\Lambda n_t}$$

where one splits the sum over all the triangles into the sum over the genera, $g$, and the sum over all possible triangulations, $T_g$, at fixed genus $g$. Remember that $g = 0$ for a sphere, $g = 1$ for a torus and $\chi = 2(1-g)$. In (1.3) $n_t$ stands for the number of triangles which is not fixed and is a dynamical variable.

The exponential suppression with $n_t$ provides the convergence of the sum over $T_g$ in (1.3) at least for large enough $\Lambda$. However, the sum can diverge for some values of $\Lambda$.
due to the entropy factor (the number of graphs). It is crucial for what follows that the total number of graphs of genus \( g \) with \( n \) triangles grows at large \( n \) as a power of \( n \): \[
\sum_{T_g} \delta(n_t - n) = e^{\Lambda_c n - b_g} \left( 1 + \mathcal{O}(n^{-1}) \right),
\]
where \( \Lambda_c \) does not depend on \( g \). For this reason the genus \( g \) contributions to the string susceptibility

\[
f = \frac{\partial^2}{\partial \Lambda^2} Z_{DT} \sim \sum_g e^{\frac{g}{2} (1-g) (\Lambda - \Lambda_c)^{-g}}, \quad \gamma_g = -b_g + 3
\]
simultaneously diverge as \( \Lambda \to \Lambda_c + 0 \). This is the point where the continuum limit is reached and the discrete partition function \( Z_{DT} \) approaches the continuum one \( Z_{2D} \).

A similar dynamical triangulation can be written \([1]\) for the partition function (1.1) of 2D surfaces embedded in a \( d \)-dimensional space. The case of 2D gravity is associated with \( d = 0 \).

### 1.3 Large-\( N \) matrix models

The partition function (1.3) can be represented as a matrix model. The dual graph for the set of triangles coincides with the graph in a \( d = 0 \) quantum field theory with a cubic interaction as is depicted in Fig. 1. The precise statement is that \( Z_{DT} \) equals to the partition function of the \( N \times N \) Hermitian one-matrix model

\[
Z_{1M} \equiv e^{N^2 F} = \int d\Phi e^{-N \text{tr} V(\Phi)}
\]

with \( N = \exp(1/G) \) and the cubic coupling constant \( t_3 = \exp(-\Lambda_c) \). The integration measure in Eq. (1.6) is

\[
d\Phi = \prod_{i>j} d \text{Re} \Phi_{ij} d \text{Im} \Phi_{ij} \prod_{i=1}^N d\Phi_{ii}
\]

and \( V(\Phi) = \frac{1}{2} \Phi^2 + t_3 \Phi^3 \) is a cubic potential.
The fact that $Z_{DT} = Z_{1M}$ can be proven analyzing the “fat-graph” expansion of (1.6) with the propagator

$$(2\pi)^{-N^2/2} \int d\Phi e^{-\frac{N}{2} \text{tr}(\Phi^2)}\Phi_{ij}\Phi_{kl} = \frac{1}{N} \delta_{il} \delta_{kj}$$

which leads for $\log Z_{1M}$ to the factor $N^{2-2g}$ associated with a graph of genus $g$.

The partition function (1.6) with the general potential

$$V(\Phi) = \sum_{j=0}^{\infty} t_j \Phi^j$$

is associated with a discretization by regular polygons with $j \geq 3$ vertices whose area is $j-2$ times the area of the equilateral triangle.

### 1.4 Double-scaling limit

A question arises how the system described by the partition function (1.6) can undergo a phase transition for $\Lambda \to \Lambda_c$ which is associated, as is discussed in Subsect. 1.2, with the continuum limit. While the system is at $d = 0$, a (third-order) phase transition of the Gross–Witten type is possible as $N \to \infty$ when the number of degrees of freedom becomes infinite. Therefore, the continuum limit is reached at $N \to \infty$ and $\Lambda \to \Lambda_c$.

The $N = \infty$ limit corresponds to planar diagrams or genus zero (the spherical approximation). Higher genera are suppressed as $N^{-2g}$.

One can utilize, however, the fact that $\gamma_g$, which is defined by Eq. (1.5), linearly depends on $g$:

$$\gamma_g = 2 + \frac{5}{2}(g - 1).$$

Therefore, the parameter of the genus expansion near the critical point is

$$G = \frac{1}{N^2(\Lambda - \Lambda_c)^{\frac{5}{2}}}$$

and can be made finite if $(\Lambda - \Lambda_c) \sim N^{-\frac{4}{5}}$ as $N \to \infty$. This special limit when the couplings reach critical values in a $N$-dependent way as $N \to \infty$ is called the double scaling limit. The double scaling limit of the Hermitean one-matrix model allowed to construct the genus expansion of 2D quantum gravity.

### 1.5 The Kazakov–Migdal model and induced QCD

A natural $d > 1$-dimensional extension of (1.6) is the Kazakov–Migdal model which is defined by the partition function

$$Z_{KM} = \int \prod_{x,\mu} dU_\mu(x) \prod_x d\Phi_x e^{\sum_x N \text{tr} \left( -V(\Phi_x) + \sum_{\mu=1}^{D} \Phi_x U_\mu(x)\Phi_{x+\mu\mu} U^\dagger_\mu(x) \right)}.$$  

Here the integration over the gauge field $U_\mu(x)$ is over the Haar measure on $SU(N)$ at each link of a $d$-dimensional lattice with $x$ labeling its sites. The model (1.12) obviously...
recovery the standard \( d \leq 1 \) matrix chain if the lattice is just a one-dimensional sequence of points for which the gauge field can be absorbed by a unitary transformation of \( \Phi \).

The large-\( N \) solution of the Kazakov–Migdal model in the strong coupling phase [13]–[16] is associated [17] with an unbroken extra \( Z(N) \) symmetry of the partition function [11,12] and, therefore, with infinite string tension (see [18] for a review). One can easily modify the model in order to have a phase transition (with decreasing the bare mass parameter) after which the \( Z(N) \) symmetry is broken in some sense and normal area law associated with finite string tension is restored. While the arguments [8] are based on the mean field analysis, they look quite reasonable because the phase transition occurs only for systems which are not asymptotically free.

The simplest model of this type is the adjoint fermion model which is defined by the partition function [8]

\[
Z_{AFM} = \int \prod_{x,\mu} dU_\mu(x) \prod_x d\Psi_x d\bar{\Psi}_x e^{-S_F[\Psi, \bar{\Psi}, U]} \tag{1.13}
\]

where \( \Psi_x \) and \( \bar{\Psi}_x \) are the \( N \times N \) matrices whose elements are independent anticommuting Grassmann variables, \( S_F[\Psi, \bar{\Psi}, U] \) is the lattice fermion action

\[
S_F = \sum_x N \text{tr} \left( V_F(\Psi_x \Psi_x) - \sum_{\mu=1}^D \left[ \overline{\Psi}_x P_\mu U_\mu(x) \Psi_{x+x_\mu} U_\mu^\dagger(x) + \overline{\Psi}_{x+x_\mu} P_\mu^+ U_\mu^\dagger(x) \Psi_x U_\mu(x) \right] \right) \tag{1.14}
\]

and \( P_\pm^\mu \) are the standard projectors.

## 2 Higher genera in one-matrix model

The Hermitian one-matrix model was first solved in genus zero in Ref. [19] by the method of the saddle-point integral equations for the spectral density. The more powerful orthogonal polynomial technique allowed to calculate the partition function up to genus two for the quartic interaction [20]. I review in this section the method of solving the Hermitian one-matrix model which is based on the loop equations and provides an algorithm for genus by genus calculations. The explicit results [21, 22] on calculation of the partition function with an arbitrary potential and all correlators are presented up to genus two.

### 2.1 Loop equation

All correlators of the Hermitian one-matrix model (1.6) can be obtained from the Laplace image of the Wilson loop

\[
W(\lambda) = \left\langle \frac{1}{N} \frac{1}{\lambda - \Phi} \right\rangle \tag{2.1}
\]

where the averaging is w.r.t. the same measure as in (1.6). As is explained in Ref. [2], \( W(\lambda) \) is associated with the sum over discretized open surfaces with one fixed boundary.

The correlator (2.1) can be obtained from the free energy, \( F \), by applying the loop insertion operator, \( d/V(\lambda) \):

\[
W(\lambda) = \frac{dF}{dV(\lambda)}, \quad \frac{d}{dV(\lambda)} \equiv -\sum_{j=0}^{\infty} \frac{1}{\lambda^{j+1}} \frac{\partial}{\partial t_j}. \tag{2.2}
\]
(\lambda) is determined by the loop equation (see [23, 24] for a review)

\[ \int_{C_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{\lambda - \omega} W(\omega) = W^2(\lambda) + \frac{1}{N^2 dV(\lambda)} W(\lambda) \quad (2.3) \]

where the contour \( C_1 \) encloses counterclockwise singularities of \( W(\omega) \). The contour integration acts as a projector picking up negative powers of \( \lambda \). The second term on the r.h.s. of the loop equation (2.3) is expressed via \( W(\lambda) \), so that Eq. (2.3) is closed and unambiguously determines \( W(\lambda) \) imposing the boundary condition \( \lambda W(\lambda) \to 1 \) as \( \lambda \to \infty \).

The genus expansion of \( W(\lambda) \) and of the free energy, \( F \), is defined by

\[ W(\lambda) = \sum_{g=0}^{\infty} \frac{1}{N^{2g}} W_g(\lambda), \quad F = \sum_{g=0}^{\infty} \frac{1}{N^{2g}} F_g \quad \text{with} \quad W_g(\lambda) = \frac{dF_g}{dV(\lambda)}. \quad (2.4) \]

### 2.2 Genus zero solution

To leading order in \( 1/N^2 \) one can disregard the second term on the r.h.s of Eq. (2.3) which reduces to the quadratic equation

\[ \int_{C_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{\lambda - \omega} W(\omega) = W^2(\lambda). \quad (2.5) \]

The one-cut solution to Eq. (2.5) reads [23]

\[ W_0(\lambda) = \int_{C_1} \frac{d\omega}{4\pi i} \frac{V'(\omega)}{\lambda - \omega} \sqrt{\frac{(\lambda - x)(\lambda - y)}{(\omega - x)(\omega - y)}} \quad (2.6) \]

where \( x \) and \( y \) are determined by

\[ \int_{C_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{\sqrt{(\omega - x)(\omega - y)}} = 0, \quad \int_{C_1} \frac{d\omega}{2\pi i} \frac{\omega V'(\omega)}{\sqrt{(\omega - x)(\omega - y)}} = 2. \quad (2.7) \]

The formulas (2.6), (2.7) solves Eq. (2.5) when \( V'(\omega) \) is polynomial (which is usually associated with the one-matrix model) or has singularities outside the cut \([y, x]\).

Doing the contour integral in (2.6) by taking the residues at \( \omega = \lambda \) and \( \omega = \infty \), one finds

\[ W_0(\lambda) = \frac{1}{2} \left\{ V'(\lambda) - M(\lambda) \sqrt{(\lambda - x)(\lambda - y)} \right\} \quad (2.8) \]

where \( M(\lambda) \) is a polynomial in \( \lambda \) of degree \( J-2 \) if \( V(\lambda) \) is that of degree \( J \). This form of the genus zero solution is convenient to determine the spectral density, \( \rho(\lambda) \), which describes the distribution of eigenvalues of the matrix \( \Phi \) at the large-\( N \) saddle point. Calculating the discontinuity of \( W(\lambda) \) across the cut \([y, x]\), one gets

\[ \rho(\lambda) \equiv \text{Im} W(\lambda) = \frac{1}{\pi} M(\lambda) \sqrt{(\lambda - y)(x - \lambda)} \quad \lambda \in [y, x]. \quad (2.9) \]

Eq. (2.7) guarantees that \( \int d\lambda \rho(\lambda) = 1 \).

The spectral density \( \rho(\lambda) \) given by (2.9) vanishes under normal circumstances as a square root at both ends of its support. The critical behavior emerges when some of the roots of \( M(\lambda) \) approach the end point \( x \) or \( y \).
2.3 The iterative procedure

The iterative procedure of solving the loop equation is based on the genus zero solution \((2.7)\). Inserting the genus expansion \((2.4)\) in Eq. \((2.3)\), one gets the following equation for \(W_g(\lambda)\) at \(g \geq 1\):

\[
\int_{C_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{\lambda - \omega} W_g(\omega) - 2 W_0(\lambda) W_g(\lambda) = \sum_{g' = 1}^{g-1} W_{g'}(\lambda) W_{g-g'}(\lambda) + \frac{d}{dV(\lambda)} W_{g-1}(\lambda), \tag{2.10}
\]

which expresses \(W_g(\lambda)\) entirely in terms of \(W_{g'}(\lambda)\) with \(g' < g\). This makes it possible to solve Eq. \((2.10)\) iteratively genus by genus.

The iterative procedure simplifies if one introduces, instead of the coupling constants \(t_j\), the moments \(M_k\) and \(J_k\) defined for \(k \geq 1\) by

\[
M_k = \int_{C_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{(\omega - x)^{k+1/2}(\omega - y)^{1/2}}, \quad J_k = \int_{C_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{(\omega - x)^{1/2}(\omega - y)^{k+1/2}}. \tag{2.11}
\]

These moments depend on the coupling constants \(t_j\)'s both explicitly and via \(x\) and \(y\) which are determined by Eq. \((2.7)\). Notice that \(M_k\) and \(J_k\) depend explicitly only on \(t_j\) with \(j \geq k + 1\).

The main motivation for introducing the moments \((2.11)\) is that \(W_g(\lambda)\) depends only on \(2 \times (3g - 1)\) lower moments \((2 \times (3g - 2)\) for \(F_g\) \([21, 22]\). This is in contrast to the \(t\)-dependence of \(W_g\) and \(F_g\) which always depend on the infinite set of \(t_j\)'s \((1 \leq j < \infty)\).

2.4 Genus one and two results

To find \(F_g\), one first solves Eq. \((2.10)\) for \(W_g(\lambda)\) and then uses the last equation in \((2.4)\). The result in genus one reads \([21]\)

\[
F_1 = -\frac{1}{24} \ln M_1 - \frac{1}{24} \ln J_1 - \frac{1}{6} \ln d, \tag{2.12}
\]

where \(d = x - y\).

An analogous calculation in genus two yields \([22]\)

\[
F_2 = -\frac{119}{7680 J_1^2 d^3} - \frac{119}{7680 M_1^2 d^4} + \frac{181 J_2}{480 J_1^3 d^2} - \frac{181 M_2}{480 M_1^3 d^3} + \frac{3 J_2}{64 J_1^2 M_1 d^3} - \frac{3 M_2}{64 M_1 M_1^2 d^3} - \frac{11 J_2^2 - 11 M_2}{40 J_1^4 d^2} - \frac{43 M_3}{192 J_1^5 d} + \frac{43 J_3}{192 M_1^5 d^2} + \frac{J_3 M_2}{64 J_1^2 M_1^2 d^2} - \frac{17}{128 J_1 M_1^2 d^3} + \frac{21 J_2 J_3}{160 J_1^3 d} - \frac{29 J_4 J_3}{128 J_1^2 d} + \frac{35 J_3^2}{384 J_1^5 d} - \frac{21 J_2^2}{160 M_1^3 d} + \frac{29 M_2 M_3}{128 M_1^2 d} - \frac{35 M_3}{384 M_1^3 d}. \tag{2.13}
\]

Some of these coefficients have an interpretation in terms of the intersection indices on moduli space \([23]\), the others are associated with characteristics of the discretized moduli space \([26]\).

Since \(F_g\) is known, the genus \(g\) contribution to any connected correlator

\[
\left< \frac{\text{tr}}{N} (\Phi^{i_1}) \ldots \frac{\text{tr}}{N} (\Phi^{i_s}) \right>_g = N^{2-2g} \frac{\partial}{\partial t_{i_1}} \ldots \frac{\partial}{\partial t_{i_s}} F_g \tag{2.14}
\]

can be calculated by the differentiation. It depends on at most \(2 \times (3g - 2 + s)\) lower moments \([21, 22]\).
To obtain explicit formulas, say, for the symmetric quartic potential when all $t_j = 0$ except $t_2$ and $t_4$, one should solve Eq. (2.7) for $x = -y$:

$$x^2 = -\frac{t_2}{3t_4} + \sqrt{\left(\frac{t_2}{3t_4}\right)^2 + \frac{4}{3t_4}},$$

(2.15)

and express the moments (2.11) via $t_2$ and $t_4$ which is given by algebraic formulas

$$M_1 = J_1 = 2t_2 + 6t_4 x^2, \quad M_2 = -J_2 = 8t_4 x, \quad M_3 = J_3 = 4t_4, \quad M_k = J_k = 0 \quad \text{for} \quad k \geq 4.$$  

(2.16)

The results for $F_1$ and $F_2$ in the case of the quartic potential are in agreement with those of Ref. [20].

3 Solving matrix models at $d > 1$

The Kazakov–Migdal model was originally studied at large-$N$ by the Riemann-Hilbert method [13]. The explicit solution for the quadratic potential [15] was reproduced [16] by the loop equations. The equivalence of the two methods was shown for an arbitrary potential in Ref. [27] where the relation to the Hermitean two-matrix model was utilized. This approach allowed to solve explicitly the Kazakov–Migdal model with a logarithmic potential [28] and the adjoint fermion model with the quadratic potential [29] in the strong coupling phase.

3.1 Loop equation for one-link correlator

Let us define for the Kazakov–Migdal model (1.12) the loop average and the one-link correlator, respectively, by

$$W(\lambda) = \left\langle \frac{\text{tr}}{\mathcal{N}} \left( \frac{1}{\lambda - \Phi_x} \right) \right\rangle, \quad G(\nu, \lambda) = \left\langle \frac{\text{tr}}{\mathcal{N}} \left( \frac{1}{\nu - \Phi_x} U_\mu(x) \frac{1}{\lambda - \Phi_{x+\mu}} U^\dagger_\mu(x) \right) \right\rangle. \quad (3.1)$$

The definition of $W(\lambda)$ is similar to Eq. (2.1) while $G(\nu, \lambda)$, which is symmetric in $\nu$ and $\lambda$ due to invariance of the Haar measure, $dU$, under the transformation $U \to U^\dagger$, is absent in the one-matrix model. Expanding $G(\nu, \lambda)$ in $1/\nu$, one gets

$$G(\nu, \lambda) = \frac{W(\lambda)}{\nu} + \sum_{n=1}^{\infty} \frac{G_n(\lambda)}{\nu^{n+1}}, \quad G_n(\lambda) = \left\langle \frac{\text{tr}}{\mathcal{N}} \left( \Phi^n_x U_\mu(x) \frac{1}{\lambda - \Phi_{x+\mu}} U^\dagger_\mu(x) \right) \right\rangle. \quad (3.2)$$

The correlator $G(\nu, \lambda)$ obeys the large-$N$ limit the following equation [27]

$$\int_{C_1} \frac{d\omega}{2\pi i} \frac{\nu'}{\nu - \omega} G(\omega, \lambda) = W(\nu) G(\nu, \lambda) + \lambda G(\nu, \lambda) - W(\nu), \quad (3.3)$$

where the contour $C_1$ encircles counterclockwise the cut (or cuts) of the function $G(\omega, \lambda)$ and

$$\nu'(\omega) \equiv \nu'(\omega) - (2d - 1) F'(\omega). \quad (3.4)$$

The function

$$F(\omega) = \sum_{n=0}^{\infty} F_n \omega^n, \quad F_0 = \frac{\text{tr}}{\mathcal{N}} \left( \Phi - \sum_{n=1}^{\infty} F_n \Phi^n \right) \quad (3.5)$$
is determined by the pair correlator of the gauge fields

\[
\frac{\int dU \ e^{N tr(\Phi U \Psi U^\dagger)}}{\int dU \ e^{N tr(\Phi U \Psi U^\dagger)}} = \sum_{n=1}^\infty F_n \frac{tr}{N} (t^a \Phi^n) \tag{3.6}
\]

where \( \Phi \) and \( \Psi \) play the role of external fields and \( t^a \) (\( a = 1, \ldots, N^2-1 \)) stand for the generators of the \( SU(N) \). Eq. (3.6) holds \([13, 14]\) at \( N = \infty \). The choice of \( F_0 \) which is not determined by Eq. (3.6) is a matter of convenience \([28]\).

Taking the \( 1/\lambda \) term of the expansion of Eq. (3.3) in \( \lambda \) and using Eqs. (3.4) and (3.6), one arrives at the equation for \( W(\nu) \) which coincides with Eq. (2.5) for the Hermitean one-matrix model where \( V' \) is substituted by \( \tilde{V}'(\lambda) = V'(\lambda) - F(\lambda) \). \( (3.7) \)

The potential \( \tilde{V}(\omega) \) is, generally speaking, non-polynomial and has singularities on the complex plane outside of the cut (or cuts) of \( W(\omega) \).

It is worth mentioning that Eq. (3.3) coincides with the loop equation for the Hermitean two-matrix model \([30]\). This is because at \( d = 1/2 \), which is associated with the Hermitean two-matrix model, the last term on the r.h.s. of Eq. (3.4) disappears and one gets just \( V(\omega) = V(\omega) \).

### 3.2 The master field equation

To analyze the model (1.12), let us consider the Hermitean two-matrix model with the potential

\[
V(\Phi) = \sum_{m=1}^\infty \frac{g_m}{m} \Phi^m. \tag{3.8}
\]

The solution for \( W(\lambda) \) versus \( V(\lambda) \) is determined by the equation \([27]\)

\[
\sum_{m \geq 1} g_m G_{m-1}(\lambda) = \lambda W(\lambda) - 1 \tag{3.9}
\]

which is just the \( 1/\nu \) term of the expansion of Eq. (3.3) in \( 1/\nu \).

The functions \( G_n(\lambda) \) are expressed via \( W(\lambda) \) using the recurrence relation

\[
G_{n+1}(\lambda) = \int_{C_1} \frac{d\omega \ V'(\omega)}{2\pi i \lambda - \omega} G_n(\omega) - W(\lambda) G_n(\lambda), \quad G_0(\lambda) = W(\lambda) \tag{3.10}
\]

which is obtained expanding Eq. (3.3) in \( 1/\lambda \). If \( V(\lambda) \) is a polynomial of degree \( J \), Eq. (3.9) contains \( W(\lambda) \) up to degree \( J \) and the solution is algebraic \([31]\).

As is proven in Ref. \([27]\):

i) Equations which appear from the next terms of the \( 1/\nu \)-expansion of Eq. (3.3) are automatically satisfied as a consequence of Eqs. (3.9) and (3.10).

ii) \( G(\nu, \lambda) \) is symmetric in \( \nu \) and \( \lambda \) for any solution of Eq. (3.9). The symmetry requirement can be used directly to determine \( W(\lambda) \) alternatively to Eq. (3.9).

Since the approach based on Eq. (3.3) is equivalent \([27]\) to that of Ref. \([13]\), \( G(\nu, \lambda) \) can be expressed via \( W(\lambda) \) as follows \([13, 14, 31]\)

\[
G(\nu, \lambda) = 1 - \exp \left\{ \pm \int_{C_1} \frac{d\omega}{2\pi i \nu - \omega} \log (\lambda - r(\omega)) \right\} \tag{3.11}
\]
where
\[ r_\pm(\lambda) = \frac{\mathcal{V}(\lambda) + F(\lambda)}{2} \pm i \pi \rho(\lambda) = \left\{ \begin{array}{ll} \mathcal{V}(\lambda) - W(\lambda) \\ F(\lambda) + W(\lambda) \end{array} \right. . \] (3.12)

The condition for the r.h.s. of Eq. (3.11) to be symmetric in \( \nu \) and \( \lambda \) is
\[ r_\pm(r_\pm(\lambda)) = \lambda. \] (3.13)

This relation in \( d = 1 \) was advocated in Ref. [32] studying the large-\( N \) asymptotics of the integral over the unitary group in (1.12) (the Itzykson–Zuber integral). Since \( G(\nu, \lambda) \) is symmetric in \( \nu \) and \( \lambda \), the master field equation [13]
\[ W(\lambda) = \pm \int_{C_1} \frac{d\omega}{2\pi i} \log (\lambda - r_\pm(\omega)), \] (3.14)
obtained as the \( 1/\nu \) term of Eq. (3.11), will be satisfied as a consequence of Eq. (3.13) which guarantees the symmetry.

All three equations (3.9), (3.13) and (3.14) seem to be equivalent. It is a matter of practical convenience which equation to solve. The only known explicit solutions exist for the quadratic potential [15] and the logarithmic potential [28].

### 3.3 Explicit solution for logarithmic potential

Let us choose the following potential of the Hermitean two-matrix model
\[ \mathcal{V}(\Phi) = -(ab + c) \log (b - \Phi) - a\Phi = \sum_{m=2}^{\infty} \frac{a b + c}{m} b^m \Phi^m + \frac{c}{b} \Phi \] (3.15)
where \( a \) and \( b \) are real. In the one-matrix case this potential is associated with the Penner model [33]. The quadratic potential is recovered in the limit
\[ a, b \to \infty, \quad \frac{a}{b} \sim 1, \quad c \sim 1 \quad \text{(quadratic potential)}. \] (3.16)

The solution to Eq. (3.3) for \( G(\nu, \lambda) \) versus \( W(\lambda) \) with \( \mathcal{V}(\omega) \) given by Eq. (3.15) is
\[ G(\nu, \lambda) = \frac{W(\nu) - \frac{(a+\lambda) W(\lambda) - 1}{b-\nu}}{\lambda + W(\nu) - \frac{a \nu + c}{b - \nu}}. \] (3.17)

\( W(\lambda) \) is determined by Eqs. (3.9), (3.10) which reduce to the quadratic equation for \( W(\nu) \) of the form of Eq. (2.5) for the Hermitean one-matrix model with the logarithmic potential
\[ \mathcal{V}(\Phi) = -(ab + c) \log (b - \Phi) + (ab + c + 1) \log (a + \Phi) - (a + b)\Phi. \] (3.18)

\( G(\nu, \lambda) \) given by Eq. (3.17) is indeed symmetric in \( \nu \) and \( \lambda \) providing Eq. (2.5) with \( V = \mathcal{V} \) is satisfied.

Since \( \mathcal{V}'(\lambda) \) is known, the function \( F(\lambda) \) can be determined from Eq. (3.7) to be
\[ F(\lambda) = \frac{b\lambda - c - 1}{a + \lambda}. \] (3.19)
Now Eq. (3.24) determines the potential

\[ V(\Phi) = -(ab + c) \log(b - \Phi) - (2d - 1)(ab + c + 1) \log(a + \Phi) + [(2d - 1)b - a]\Phi. \quad (3.20) \]

This formula recovers at \( d = 1/2 \) the potential (3.13) of the two-matrix model and at \( d = 0 \) the potential (3.18) of the associated one-matrix model.

In the Gaussian limit (3.16) all the formulas recover the ones for the solution found by Gross [15]. For this reason the one-cut solution (2.6), (2.7) with the potential (3.18) is always realized for \( W(\lambda) \) if \( a \) and \( b \) are large enough for the points \( b \) and \( -a \) to lie outside of the cut.

While the above solution for the logarithmic potential was originally obtained [28] solving Eq. (3.3), it is easy to show that it satisfies Eq. (3.13) and, therefore, the master field equation (3.14). Let us notice for this purpose that \( r^\pm(\lambda) \) given by Eq. (3.12) with our solution for \( V'(\lambda) \), \( F(\lambda) \) and \( W(\lambda) \) satisfy the following equation

\[ \mathcal{D}(r^\pm(\lambda), \lambda) = \mathcal{D}(\lambda, r^\mp(\lambda)) = 0 \quad (3.21) \]

with

\[ \mathcal{D}(\nu, \lambda) = -\lambda^2 \nu^2 + (b - a)\nu(\lambda + \nu) + ab(\lambda^2 + \nu^2) - \lambda\nu(a^2 + b^2 + 2c + 1) \\
+ (\lambda + \nu)(b - ac + bc) - c^2 - b^2 + (ab + c)[(a + b)W(b) - 1], \quad (3.22) \]

where the constant \( W(b) \) depends on the type of the solution of Eq. (2.5) (one-cut or more-than-one-cut solutions). Since this \( \mathcal{D}(\nu, \lambda) \) is symmetric in \( \nu \) and \( \lambda \), Eq. (3.21) implies [32] that Eq. (3.13) is satisfied.

According to Ref. [27], the pair correlator of \( U \) and \( U^\dagger \) for the potential (3.20) can be calculated taking the discontinuity of (3.17) both in \( \nu \) and \( \lambda \) across the cut (cuts). For the solution (3.17) one gets [28]

\[ C(\nu, \lambda) \equiv \frac{1}{\pi^2 \rho(\nu)\rho(\lambda)} \text{Disc}_\nu \text{Disc}_\lambda G(\nu, \lambda) = \frac{(a + \nu)(a + \lambda)}{\mathcal{D}(\nu, \lambda)}, \quad \nu, \lambda \in \text{cut} \quad (3.23) \]

with \( \mathcal{D}(\nu, \lambda) \) given by Eq. (3.22). In the Gaussian limit (3.16) when \( W(b) \to 1/b \), one recovers the result [27] for \( C(\nu, \lambda) \) in the case of the quadratic potential.

Since \( \rho(\nu) \) is real for \( \nu \in \text{cut} \), the roots of the denominator, \( r^\pm(\nu) \), are complex so that \( C(\nu, \lambda) \) has no singularities at the cut. The arguments of Ref. [34] suggest, therefore, that the Kazakov–Migdal model with the logarithmic potential (3.20) always remains in the phase with infinite string tension.

The potential (3.20) of the \( d \)-dimensional matrix model admits the “naive” continuum limit when

\[ \Phi = \varepsilon^\frac{d}{2} \phi, \quad V(\Phi) = \varepsilon^d v(\phi) \quad (3.24) \]

with \( \phi \) and \( v(\phi) \) being finite as \( \varepsilon \to 0 \). This continuum limit is reached providing

\[ a = b \sim \varepsilon^\frac{d}{2} - 2 \quad \text{as} \quad \varepsilon \to 0 \quad (3.25) \]

and results in a \( d \)-dimensional continuum quartic action [28]. The procedure of taking the “naive” continuum limit works for \( d < 4 \) where \( b \) given by Eq. (3.25) is divergent. This is precisely where the scalar theory with the quartic interaction is renormalizable. Thus, one can look at the logarithmic potential (3.20) as at a latticization of the quartic one for \( d < 4 \).
3.4 Fermionic matrix models

The correlators of arbitrary powers of $\bar{\Psi}_x \Psi_x$ at the same site $x$ are determined for the fermionic matrix model (1.13) by

$$W_0(\nu) = \left\langle \frac{\text{tr}}{N} \left( \frac{\nu}{\nu^2 - \bar{\Psi}_x \Psi_x} \right) \right\rangle. \quad (3.26)$$

The analogue of Eq. (2.3) for the fermionic one-matrix model (defined by (1.13) with $d = 0$) reads \cite{29}

$$\int_{C_1} \frac{d\omega}{4\pi i} \frac{V_F'(\omega)}{\lambda - \omega} W(\omega) = W^2(\lambda) - \frac{2}{\lambda} W(\lambda) + \frac{1}{N^2} \frac{\delta}{\delta V_F(\lambda)} W(\lambda). \quad (3.27)$$

This equation is identical to Eq. (2.3) for the Hermitian one-matrix model with the logarithmic potential

$$V(\Phi) = V_F(\Phi) + 2 \log \Phi \quad (3.28)$$

and $\Phi = \bar{\Psi} \Psi$. These two models are equivalent to all orders of the $1/N$-expansion. However, the genus expansion has now alternating signs and is convergent contrary to Ref. \cite{3}.

Let us define for the fermionic matrix model (1.13) on a $d$-dimensional lattice the odd-odd and even-even one-link correlators:

$$G(\nu, \lambda) = P^\pm_\mu \left\langle \frac{\text{tr}}{N} \left( \frac{1}{\nu^2 - \bar{\Psi}_x \Psi_x} U_\mu(x) \frac{1}{\lambda^2 - \bar{\Psi}_{x+\mu} \Psi_{x+\mu}} \bar{\Psi}_{x+\mu} U^\dagger_\mu(x) \right) \right\rangle,$$

$$W(\nu, \lambda) = \left\langle \frac{\text{tr}}{N} \left( \frac{\nu}{\nu^2 - \bar{\Psi}_x \Psi_x} U_\mu(x) \frac{\lambda}{\lambda^2 - \bar{\Psi}_{x+\mu} \Psi_{x+\mu}} U^\dagger_\mu(x) \right) \right\rangle, \quad (3.29)$$

where $+(-)$ are associated with the positive (negative) direction $\mu$.

The fermionic analogue of Eq. (3.3) consists of two equations \cite{29}

$$\int_{C_1} \frac{d\omega}{4\pi i} \frac{V_F'(\omega)}{\nu - \omega} G(\omega, \lambda) = W_0(\nu) G(\nu, \lambda) + \left[ \lambda W(\nu, \lambda) - W_0(\nu) \right],$$

$$\int_{C_1} \frac{d\omega}{4\pi i} \frac{V_F'(\omega)}{\nu - \omega} W(\omega, \lambda) = W_0(\nu) W(\nu, \lambda) - \frac{1}{\nu} W(\nu, \lambda) - \lambda G(\nu, \lambda). \quad (3.30)$$

The solution to Eq. (3.30) for the quadratic potential $V_F(\omega) = m \omega^2$ is

$$W_0(\lambda) = \frac{1}{2} \left[ \mu \lambda + \frac{2}{\lambda} \frac{1}{\lambda} \sqrt{\mu^2 \lambda^4 + 4} \right] \quad (3.31)$$

with

$$\mu = \frac{(D - 1)m + D \sqrt{m^2 + 4(2D - 1)}}{(2D - 1)}. \quad (3.32)$$

This solution agrees \cite{8} with the result \cite{35} for lattice QCD with fundamental fermions at vanishing plaquette term.
4 Conclusions

There is no problems to calculate higher orders of genus expansion in the Hermitean one-matrix model using the iterative method which is described in Sect. 2. It would be interesting to perform analogous calculations for the potential (3.28) which is equivalent to the fermionic model whose genus expansion is expected to be convergent since the integral over the Grassmann variables in (1.13) converges.

The method of solving the Kazakov–Migdal model which is described in Sect. 3 reduces it at large-$N$ to the Hermitean two-matrix model. This goes along with Ref. [30] where the conformal field theories in $d \leq 1$ are obtained from the two-matrix model. For $d > 1$ the potential $\mathcal{V}$ of the two-matrix model should be, presumably, non-polynomial rather than polynomial as for $d \leq 1$.

The approach of Sect. 3 works, however, only when singularities of $\mathcal{V}$ lie outside of the cut. This is not the case for the solution of the Riemann–Hilbert equations found by Migdal [13] which exhibits a non-trivial critical behavior. While this solution is not associated with induced QCD, it would be interesting to find out what physical system it corresponds to.

The discussed solution of the $d > 1$ models are associated with the strong coupling phase, i.e. the phase with infinite string tension. An open question is whether the described approach can be extended to the phase with area law where QCD is induced.

Acknowledgements

I thank the organizers for the kind hospitality at Kyffhaeuser.

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