Preorders, Partial Semigroups, and Quantales

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Abstract. It is known that each powerset quantale is embeddable into some relational unital quantale whose underlying set is the powerset of some preorder. An aim of this paper is to understand the relational embedding as a relationship between quantales and preorders. For that, this paper introduces the notion of weak preorders, a functor from the category of weak preorders to the category of partial semigroups, and a functor from the category of partial semigroups to the category of quantales and lax homomorphisms. By using these two functors, this paper shows a correspondence among four classes of weak preorders (including the class of ordinary preorders), four classes of partial semigroups, and four classes of quantales. As a corollary of the correspondence, we can understand the relational embedding map as a natural transformation between functors onto certain category of quantales.

1 Introduction

A unital quantale is defined to be a complete join semilattice together with a monoid structure satisfying the distributive laws. It was introduced by Conway under the name S-algebras \cite{Conway}. A relational example of unital quantale is a powerset $\wp(A \times A)$, which is the set of all binary relations on a set $A$ and whose monoid structure is given by relational composition and the identity relation. We call it a relational quantale. A relational quantales play an important role in computer science, for example, it is a model for the semantics of non-deterministic while-programs \cite{ackermann, pous}.

In the paper \cite{Fur}, the relational representation theorem for powerset quantales is shown, where a powerset quantale is defined to be a unital quantale whose complete join semilattice part is isomorphic to the powerset of some set. The relational representation is given as an embedding (i.e., injective unital homomorphism) $\eta$ from a powerset quantale $\wp(A)$ to some relational...
quantale $\varphi(A \times A)$. The paper [5] includes this embedding $\eta$ in the special case where $A$ is a free monoid.

$(\varphi(N), \subseteq, \cup, [+] , \{0\})$ is the leading example of powerset quantale, where $N$ is the set of all natural numbers (including 0) and $S \ [+] S' \stackrel{\text{def}}{=} \{n + n' \mid n \in S, n' \in S'\}$. When the relational representation theorem [4] is applied to $\varphi(N)$, the embedding $\eta: \varphi(N) \rightarrow \varphi(N \times N)$ embeds a subset $S$ into the left and right reversed relation of $\{(m, n) \mid n - m \in S\}$. The subtraction $-$ for natural numbers plays an important role.

In general, the representation theorem helps to understand the algebra, not only that, but it is meaningful in the field of computer science, because it suggests the possibility of representing infinite entities with finite data. Embedding from power set quantale into binary relations also shows the possibility that infinite entities can be represented by finite data. For example, as an application example of the above $\eta: \varphi(N) \rightarrow \varphi(N \times N)$, the finite set $\{2\}$ is representing $\eta(\{2\})$, i.e., the infinite binary relation (or the infinite directed graph) $\{(m, m + 2) \mid m \in N\}$. In addition, the multiplicity in the quantale can replace the relational composition operation. This shows the same effect as the correspondence between linear mapping and its matrix representation. Analyzing quantales’ representation theorem is meaningful in the field of computer science.

The definition of relational quantale is extendable for the powerset $\varphi(\leq)$ of each preorder $\leq$, where a preorder is regarded as a subset of $A \times A$ satisfying reflexivity and transitivity. An aim of this paper is to understand the relational embedding $\eta$ as a relationship between quantales and preorders.

Since direct construction of a relationship between quantales and preorders is not very obvious, this paper introduces the notion of weak preorder and partial semigroup as a relaxation of preorder and semigroup, respectively. We also introduce the notion of lax homomorphism between (not unital) quantales as a relaxation of ordinary homomorphism between quantales. And, we give a functor $\text{comp}$ from the category $\text{WPreOrd}$ of weak preorders to the category $\text{PSG}$ of partial semigroups, and a contravariant functor $\varphi$ from $\text{PSG}$ to the category $\text{Qt}_{\text{lax}}$ of quantales and lax homomorphisms. Moreover, we also define three categories (including the category $\text{UQt}$ of unital quantales and unital homomorphisms) as restrictions of $\text{Qt}_{\text{lax}}$ in a step-by-step manner. By proving the following six pullbacks in $\text{Cat}$, we show the relationship among orders, semigroups, and quantales.

Recall that a pullback of $F: B \rightarrow A$ and $G: C \rightarrow A$ in $\text{Cat}$ is isomorphic to the category $B \times_A C$ which consists of pairs of $b \in B$ and $c \in C$ satisfying $Fb = Gc$ with projections $\pi: B \times_A C \rightarrow B$ and $\pi': B \times_A C \rightarrow C$, since for functors $H: D \rightarrow B$ and $K: D \rightarrow C$ satisfying $F \circ H = G \circ K$, the functor $L(d) = (Hd, Kd)$ is the unique functor $L: D \rightarrow B \times_A C$ satisfying $\pi \circ L = H$ and $\pi' \circ L = K$.

By using these functors, we understand the relational embedding map $\eta$ as a natural transformation.

This paper is organized as follows. Section 2 defines the notions of weak preorder, partial semigroup, and lax homomorphism between quantales. In Sect. 3,
we restrict arrows between quantales to ordinary homomorphisms and give the corresponding classes of arrows between weak preorders or partial semigroups. Section 4 extends quantales to unital quantales and give the corresponding extension of weak preorders or partial semigroups. In Sect. 5, we introduce sufficient classes to induce the relational embedding. Section 6 summarizes this work and discusses about future work.

2 Weak Preorders and Partial Semigroups

In this section, we define the notions of weak preorder, partial semigroup, and lax homomorphism between quantales. By using the three notions, we give three categories and two functors among them.

First, we recall the definition of quantale [6–8] and define the notion of lax homomorphism between quantales.

Definition 1 ((non unital) quantale and lax homomorphism). A quantale is defined to be a tuple \((Q, \leq, \lor, \otimes)\) such that

1. \((Q, \otimes)\) is a semigroup (i.e., a binary function \(\otimes\) on \(Q\) that is associative),
2. \((Q, \leq, \lor)\) is a complete join semilattice (i.e., a partially ordered set \((Q, \leq)\) has the least upper bound \(\lor S\) for arbitrary subset \(S\) of \(Q\)),
3. \((\lor S) \otimes q = \lor \{s \otimes q | s \in S\}\) for each element \(q\) and each subset \(S\) of \(Q\), and
4. \(q \otimes (\lor S) = \lor \{q \otimes s | s \in S\}\) for each element \(q\) and each subset \(S\) of \(Q\).

For two quantales \((Q, \leq, \lor, \otimes), (Q', \leq', \lor', \otimes')\), a lax homomorphism from \((Q, \leq, \lor, \otimes)\) to \((Q', \leq', \lor', \otimes')\) is defined to be a map \(f: Q \to Q'\) such that
1. \( f(\bigvee S) = \bigvee \{ f(s) \mid s \in S \} \) for each subset \( S \) of \( Q \) (join-preserving), and
2. \( f(q_1) \circ f(q_2) \leq f(q_1 \circ q_2) \) for each elements \( q_1, q_2 \) of \( Q \) (closed map).

Lax homomorphisms between quantales are closed under composition as maps. The identity map is a lax homomorphism on a quantale. Therefore, we can define the category whose objects are quantales and whose arrows are lax homomorphisms between them and we write \( \mathcal{Q}_{\text{lax}} \) for it.

As the sufficient structure to construct \( \circ \) in the powerset case \( Q = \wp(X) \), we define the notion of partial semigroup.

**Definition 2 (partial semigroup and homomorphism).** A partial semigroup is defined to be a tuple \((X, \cdot)\) such that

1. \( X \) is a set,
2. \( \cdot \) is a partial binary function on \( X \) (i.e., \( x \cdot y \) may be undefined),
3. \( x \cdot y \) and \((x \cdot y) \cdot z \) are defined if and only if \( y \cdot z \) and \( x \cdot (y \cdot z) \) are defined for \( x, y, z \in X \), and
4. \((x \cdot y) \cdot z = x \cdot (y \cdot z)\) if they are defined.

For partial semigroups \((X, \cdot), (X', \cdot')\), a homomorphism from \((X, \cdot)\) to \((X', \cdot')\) is defined to be a map \( f: X \rightarrow X' \) such that

1. \( f(x) \cdot f(y) \) is defined for \( x, y \in X \) such that \( x \cdot y \) is defined, and
2. \( f(x) \cdot f(y) = f(x \cdot y) \) if they are defined.

Homomorphisms between partial semigroups are closed under composition as maps. The identity map is a homomorphism on a partial semigroup. We write \( \mathcal{PSG} \) for the category whose objects are partial semigroups and whose arrows are homomorphisms between them.

**Example 1.** For each set \( S \), \((X, \cdot)\) is a partial semigroup, where \( x \cdot y \overset{\text{def}}{=} x \) if \( x = y \) and \( x \cdot y \) is undefined otherwise.

**Example 2.** For a set \( A \), \((A \times A, \cdot)\) is a partial semigroup, where \((a, b); (c, d) \overset{\text{def}}{=} (a, d)\) if \( b = c \) and \((a, b); (c, d)\) is undefined otherwise.

Next, we present the construction of a powerset (not unital) quantale from a partial semigroup. Here, we denote by the binary map \([\cdot]\) on \( \wp(X) \) so that \( S_1 [\cdot] S_2 \overset{\text{def}}{=} \{ s_1 \cdot s_2 \mid s_1 \in S_1, s_2 \in S_2, s_1 \cdot s_2 \) is defined\} for a partial semigroup \((X, \cdot)\).

**Theorem 1.** The following data form a functor \( \varphi: \mathcal{PSG}^{\text{op}} \rightarrow \mathcal{Q}_{\text{lax}} \).

- For an object \((X, \cdot)\), \( \varphi(X, \cdot) \overset{\text{def}}{=} (\varphi(X), \subseteq, \bigcup, [\cdot]) \)
- For an arrow \( f: (X, \cdot) \rightarrow (X', \cdot') \), \( \varphi(f): \varphi(X', \cdot') \rightarrow \varphi(X, \cdot) \) is a map \( \varphi(f)(S') = \{ x \in X \mid f(x) \in S' \} \).

**Proof.** Take an arrow \( f: (X, \cdot) \rightarrow (X', \cdot') \) in \( \mathcal{PSG} \). \( \varphi(f) \) preserves \( \bigcup \). Take \( S'_1, S'_2 \subseteq \varphi(X') \) and \( x \in \varphi(f)(S'_1) [\cdot] \varphi(f)(S'_2) \). There are \( s_1, s_2 \in X \) satisfying \( f(s_1) \in S'_1, f(s_2) \in S'_2 \), and \( x = s_1 \cdot s_2 \). They satisfy \( f(x) = f(s_1 \cdot s_2) = \ldots \)
We write \( f(s_1) \preceq f(s_2) \in S'_1 [\preceq] S'_2 \) and then \( x \in \varphi(f)(S'_1 [\preceq] S'_2) \). Therefore, we have shown \( \varphi(f)(S'_1 [\preceq] \varphi(f)(S'_2) \subseteq \varphi(f)(S'_1 [\preceq] S'_2) \). \( \varphi \) preserves composition and identities. \( \square \)

A transitive relation \( R \) on a set \( X \) forms a partial semigroup \((R,;)\) with the same : as Example 2. However, we define weak preorders, as a stronger notion than transitive relations and a weaker notion than preorders.

**Definition 3 (weak preorder).** A binary relation \( R \subseteq X \times X \) on \( X \) is called a weak preorder on \( X \), if \( R \) is transitive and \( R \) satisfies \( \forall x \in X. \exists y \in X.(x, y) \in R \) or \( (y, x) \in R \). A weak preordered set is defined to be a tuple \((X, R)\) of a set \( X \) and a weak preorder \( R \) on \( X \).

**Example 3.** Let \( \mathbb{N} \) be the set of all natural numbers (including 0). We write \( m <_\mathbb{N} n \) when \( m \) is less than \( n \) as natural numbers and write \( m \leq_\mathbb{N} n \) when \( m <_\mathbb{N} n \) or \( m = n \). We also regard \( <_\mathbb{N} \) as the set of \((m, n)\) satisfying \( m <_\mathbb{N} n \) and regard \( \leq_\mathbb{N} \) as the set of \((m, n)\) satisfying \( m \leq_\mathbb{N} n \).

\( \leq_\mathbb{N} \) is a preorder on \( \mathbb{N} \). On the other hand, \( <_\mathbb{N} \) is not a preorder, but a weak preorder on \( \mathbb{N} \).

**Definition 4.** For weak preordered sets \((X, R)\), \((X', R')\), a monotone map from \((X, R)\) to \((X', R')\) is defined to be a map \( f: X \to X' \) such that \((x, y) \in R \) implies \((f(x), f(y)) \in R'\) for each \( x, y \in X \).

Monotone maps between weak preordered sets are closed under composition as maps. The identity map is a monotone map on a weak preordered set. Therefore, weak preordered sets and monotone maps between them form a category. We write \( \text{WPreOrd} \) for it.

Next, we present the construction of a partial semigroup from a weak preordered set.

**Theorem 2.** The following data form a functor \( \text{comp}: \text{WPreOrd} \to \text{PSG} \).

- For an object \((X, R)\), \( \text{comp}(X, R) \overset{\text{def}}{=} (R,;) \) where \((w, x);(y, z)\) is defined and equal to \((w, z)\) if \( x = y \).
- For an arrow \( f: (X, R) \to (X', R') \), \( \text{comp}(f): (R,;) \to (R',;) \) sends \((x, y)\) to \((f(x), f(y))\).

**Proof.** Take an arrow \( f: (X, R) \to (X', R') \) in \( \text{WPreOrd} \). Take \((x, y) \in R \). By monotonicity of \( f \), \( \text{comp}(f)(x, y) = (f(x), f(y)) \in R' \). Take \((w, x), (y, z) \in R \) such that \((w, x);(y, z)\) is defined. Then, \( x = y \) and \((w, x);(y, z) = (w, z)\). \( \text{comp}(f)(w, x);\text{comp}(f)(y, z) = (f(w), f(x));(f(x), f(z)) \) is defined and equal to \((f(w), f(z)) = \text{comp}(f)(w, z)\). \( \text{comp} \) preserves composition and identities. \( \square \)

**Example 4.** We define the arrow \( \text{plus1}: (\mathbb{N}, <_\mathbb{N}) \to (\mathbb{N}, \leq_\mathbb{N}) \) in \( \text{WPreOrd} \) by \( \text{plus1}(n) \overset{\text{def}}{=} n + 1 \).

\( \text{comp(plus1)} \) is the arrow from \( \text{comp}(\mathbb{N}, <_\mathbb{N}) = (<_\mathbb{N},;) \) to \( \text{comp}(\mathbb{N}, \leq_\mathbb{N}) = (\leq_\mathbb{N},;) \) in \( \text{PSG} \) such that \( \text{comp(plus1)}(m, n) = (m + 1, n + 1) \) for \((m, n)\) satisfying \( m <_\mathbb{N} n \).
In this section, we recall the category of quantales and ordinary homomorphisms of quantales. It is a subcategory of $\text{Qt}_{\text{lax}}$ in Sect. 2. We study which maps between partial semigroups correspond to homomorphisms between quantales, and which maps between weak preorders correspond to those maps between partial semigroups.

**Definition 5 (homomorphism between quantales).** For quantales $(Q, \leq, \lor, \circ)$, $(Q', \leq', \lor', \circ')$, a homomorphism from $(Q, \leq, \lor, \circ)$ to $(Q', \leq', \lor', \circ')$ is defined to be a map $f: Q \to Q'$ satisfying

1. $f(\lor S) = \lor'\{f(s) \mid s \in S\}$ for each subset $S$ of $Q$, and
2. $f(q_1 \circ q_2) = f(q_1) \circ' f(q_2)$ for each elements $q_1, q_2$ of $Q$.

We write $\text{Qt}$ for the subcategory of $\text{Qt}_{\text{lax}}$, whose arrows are homomorphisms.

Next, we introduce the additional condition for homomorphisms between partial semigroups which corresponds to homomorphisms between quantales. We call it the dividing condition.

**Definition 6 (dividing map between partial semigroups).** A partial semigroup homomorphism $f: (X, \cdot) \to (X', \cdot')$ is called dividing, if for each $x', y' \in X'$ and $z \in X$ satisfying $x' \cdot y' = f(z)$, there exist $x, y \in X$ such that $f(x) = x'$, $f(y) = y'$, and $x \cdot y = z$.

Dividing maps between partial semigroups are closed under composition as maps. The identity map is dividing. We write $\text{PSG}_{\text{div}}$ for the subcategory of $\text{PSG}$, whose arrows are only dividing maps.

Next, we show that dividing maps between partial semigroups correspond to homomorphisms between quantales.

**Theorem 3.** For an arrow $f: (X, \cdot) \to (X', \cdot')$ in $\text{PSG}$, the following statements are equivalent.

1. $f$ is an arrow $f: (X, \cdot) \to (X', \cdot')$ in $\text{PSG}_{\text{div}}$.
2. $\varphi(f)$ is an arrow $\varphi(f): \varphi(X', \cdot') \to \varphi(X, \cdot)$ in $\text{Qt}$.

**Proof.** (1 $\Rightarrow$ 2) Assume $f: (X, \cdot) \to (X', \cdot')$ in $\text{PSG}_{\text{div}}$. Take $S_1', S_2' \in \varphi(X')$ and $x \in \varphi(f)(S_1' \cdot S_2')$. There exist $x_1' \in S_1'$, $x_2' \in S_2'$ satisfying $f(x) = x_1' \cdot x_2'$. Since $f \in \text{PSG}_{\text{div}}$, there exist $x_1, x_2 \in X$ satisfying $f(x_1) = x_1'$, $f(x_2) = x_2'$, and $x = x_1 \cdot x_2$. Therefore, $x = x_1 \cdot x_2 \in \{x_1\} \cdot \{x_2\} \subseteq \varphi(f)(S_1') \cdot \varphi(f)(S_2')$. 



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Corollary 1. Let $\varphi: \text{PSG}_{\text{div}} \to \text{Qt}^{\text{op}}_{\text{lax}}$ be the restriction of $\varphi: \text{PSG} \to \text{Qt}^{\text{op}}_{\text{lax}}$. The forgetful functor $\text{forget}: \text{PSG}_{\text{div}} \to \text{PSG}$ and $\varphi: \text{PSG}_{\text{div}} \to \text{Qt}^{\text{op}}_{\text{lax}}$ is a pullback of $\varphi: \text{PSG} \to \text{Qt}^{\text{op}}_{\text{lax}}$ and the forgetful functor $\text{forget}: \text{Qt}^{\text{op}} \to \text{Qt}^{\text{op}}_{\text{lax}}$ in $\text{Cat}$ (Fig. 1).

Next, we introduce the additional condition for monotone maps between weak preorders which corresponds to dividing maps between partial semigroups. The condition is the intermediate value property.

Definition 7 (the intermediate value property). For weak preordered sets $(X, R)$, $(X', R')$, a monotone map $f$ from $(X, R)$ to $(X', R')$ is said to satisfy the intermediate value property, if for $x, y, z$ s.t. $(x, y) \in R$, $(f(x), z') \in R'$ and $(z', f(y)) \in R'$, there exists $z \in X$ s.t. $f(z) = z'$, $(x, z) \in R$, and $(z, y) \in R$.

Monotone maps satisfying the intermediate value property between weak preordered sets are closed under composition as maps. The identity map on a weak preordered set satisfies the intermediate value property. We write $\text{WPreOrd}_{\text{int}}$ for the subcategory of $\text{WPreOrd}$, whose arrows are only arrows satisfying the intermediate value property.

Next, we show that monotone maps satisfying the intermediate value property correspond to dividing maps between partial semigroups.

Theorem 4. For $f: (X, R) \to (X', R')$ in $\text{WPreOrd}$, the following statements are equivalent.

1. $f$ is an arrow $f: (X, R) \to (X', R')$ in $\text{WPreOrd}_{\text{int}}$.
2. $\text{comp}(f)$ is an arrow $\text{comp}(f): \text{comp}(X, R) \to \text{comp}(X', R')$ in $\text{PSG}_{\text{div}}$.

Proof. (1 $\Rightarrow$ 2) Assume $f: (X, R) \to (X', R')$ in $\text{WPreOrd}_{\text{int}}$. Take $(w', x')$, $(y', z') \in R'$ and $(w, z) \in R$ satisfying $(w', x'); (y', z') = \text{comp}(f)(w, z)$. Then, $x' = y'$, $w' = w$, and $f(z) = z'$. By the intermediate value property of $f$, there exists $x \in X$ such that $f(x) = x'$, $(w, x) \in R$, and $(x, z) \in R$. They satisfy $\text{comp}(f)(w, x) = (f(w), f(x)) = (w', x')$, $\text{comp}(f)(x, z) = (f(x), f(z)) = (x', z') = (y', z')$, and $(w, x); (x, z) = (w, z)$.

(2 $\Rightarrow$ 1) Assume $\text{comp}(f): \text{comp}(X, R) \to \text{comp}(X', R')$ in $\text{PSG}_{\text{div}}$. Take $(x, y) \in R$ and $z' \in X'$ satisfying $(f(x), z') \in R'$ and $(z', f(y)) \in R'$. They satisfy $\text{comp}(f)(x, y) = (f(x), f(y)) = (f(x), z') \in R'$. Since $\text{comp}(f)$ is dividing, there exist $(x, z), (z, y) \in R$ such that $\text{comp}(f)(x, z) = (f(x), z')$, $\text{comp}(f)(z, y) = (z', f(y))$, and $(x, z); (z, y) = (x, y)$. Therefore, $f(z) = z'$.

Corollary 2. Let $\text{comp}: \text{WPreOrd}_{\text{int}} \to \text{PSG}_{\text{div}}$ be the restriction of $\text{comp}: \text{WPreOrd} \to \text{PSG}$. The forgetful functor $\text{forget}: \text{WPreOrd}_{\text{int}} \to \text{WPreOrd}$ and $\text{comp}: \text{WPreOrd}_{\text{int}} \to \text{PSG}_{\text{div}}$ is a pullback of $\text{comp}: \text{WPreOrd} \to \text{PSG}$ and the forgetful functor $\text{forget}: \text{PSG}_{\text{div}} \to \text{PSG}$ in $\text{Cat}$ (Fig. 1).
Example 5. In \( \text{WPreOrd}_{\text{int}} \), the map \( \text{plus1}(n) \) \( \text{def} = n + 1 \) is an arrow from \( (\mathbb{N}, <_{\mathbb{N}}) \) to \( (\mathbb{N}, <_{\mathbb{N}}) \), but not to \( (\mathbb{N}, \leq_{\mathbb{N}}) \), since \( 1 \leq_{\mathbb{N}} 1 \leq_{\mathbb{N}} 2 \) but not \( 0 <_{\mathbb{N}} 0 \).

For the same reason, the map \( \text{comp}(\text{plus1})(m, n) = (m + 1, n + 1) \) is a dividing map from \( (\mathbb{N}, ;) \) to \( (\mathbb{N}, ;) \), but not to \( (\leq_{\mathbb{N}}, ;) \).

Similarly, the map \( \varphi(\text{comp}(\text{plus1}))(S) = \{(m, n) \mid m <_{\mathbb{N}} n, (m + 1, n + 1) \in S\} \) is a quantale homomorphism from \( (\varphi(<_{\mathbb{N}}), \subseteq, \cup, [:]) \) to \( (\varphi(<_{\mathbb{N}}), \subseteq, \cup, [:]) \), but not to \( (\varphi(\leq_{\mathbb{N}}), \subseteq, \cup, [:]) \).

4 Preorders and Unital Partial Semigroups

In this section, we recall the definition of unital quantale. We show which weak preorders and which partial semigroups correspond to unital quantales. We also study the correspondence among their maps.

Definition 8 (unital quantale). A unital quantale is defined to be a tuple \((Q, \leq, \lor, 1)\) such that

1. \((Q, \leq, \lor, 1)\) is a quantale, and
2. \(1 \in Q\) satisfies for each \(q \in Q\), \(1 \lor q = q = q \lor 1\) (1 is called the unit of \(\lor\)).

For unital quantales \((Q, \leq, \lor, 1)\), \((Q', \leq', \lor', 1')\), a unital homomorphism from \((Q, \leq, \lor, 1)\) to \((Q', \leq', \lor', 1')\) is defined to be a homomorphism from \((Q, \leq, \lor)\) to \((Q', \leq', \lor')\) satisfying \(f(1) = 1'\).

Unital homomorphisms between unital quantales are closed under composition as maps. The identity map is a unital homomorphism on a unital quantale. We write \(\text{UQt}\) for the category whose objects are unital quantales and whose arrows are unital homomorphisms between them.

Next, we introduce the additional structure for a partial semigroup which corresponds to the unit of the multiplication of a quantale. We call the structure a unital subset.

Definition 9 (unital subset). A unital subset of a partial semigroup \((X, \cdot)\) is defined to be a subset \(U \subseteq X\) such that

1. if \(u \in U\) and \(u \cdot x\) is defined, then \(u \cdot x = x\),
2. if \(u \in U\) and \(x \cdot u\) is defined, then \(x \cdot u = x\),
3. for any \(x \in X\), there exists \(u \in U\) such that \(u \cdot x\) is defined, and
4. for any \(x \in X\), there exists \(u \in U\) such that \(x \cdot u\) is defined.

Lemma 1 (uniqueness of unital subset). For each partial semigroup \((X, \cdot)\), if \(U\) and \(U'\) are unital subsets of \((X, \cdot)\), then \(U = U'\).

Proof. Assume that \((X, \cdot)\) is a partial semigroup and that \(U\) and \(U'\) are unital subsets of \((X, \cdot)\). Take an element \(u\) of \(U\). Since \(U'\) is a unital subset of \((X, \cdot)\), there exists \(u' \in U'\) such that \(u' \cdot u\) is defined and \(u' \cdot u = u\). Since \(U\) is also a unital subset of \((X, \cdot)\) and \(u \in U\), \(u' \cdot u\) is equal to \(u' \in U'\). Therefore, \(U \subseteq U'\). The converse is proven, similarly. \(\square\)
Definition 10 (unital partial semigroup and dividing map). A unital partial semigroup is defined to be a tuple $(X, \cdot, U)$ such that

1. $(X, \cdot)$ is a partial semigroup, and
2. $U$ is the unital subset of $(X, \cdot)$.

For unital partial semigroups $(X, \cdot, U), (X', \cdot', U')$, a dividing map from $(X, \cdot, U)$ to $(X', \cdot', U')$ is defined to be an arrow $f: (X, \cdot) \to (X', \cdot') \in \text{PSG}_{\text{div}}$ such that for any $x \in X$, it satisfies $x \in U$ if and only if $f(x) \in U'$.

Note that a unital partial semigroup is not equal to a partial monoid, which is a base structure of an effect algebra [9], since the unital subset of a unital partial semigroup is not always singleton.

Dividing maps between unital partial semigroups are closed under composition as maps. The identity map is dividing on a unital partial semigroup. We write $\text{UPSG}_{\text{div}}$ whose objects are unital partial semigroups and whose arrows are dividing maps between them.

Example 6. The set of all arrows of a small category forms the unital partial semigroup whose partial binary operator is the composition of arrows and whose unital subset is the set of identities. On the other hand, a unital partial semigroup $\mathbf{X}$ is a base structure of an effect algebra [9], since the unital subset of a unital partial semigroup, when $0 \cdot 1 = 1 \cdot 0 = 0, 1 \cdot 1 = 1$ and $0 \cdot 0$ is undefined. To regard $\{1\}$ as the set of all identities, however, the category can have only one object, and then $0 \cdot 0$ must be defined. Therefore, a unital partial semigroup is not equal to a poloid [10]. If $x \cdot y$ and $y \cdot z$ are defined in a poloid, then $(x \cdot y) \cdot z$ must be defined. In this unital partial semigroup $\{(0, 1), \cdot, \{1\}\}$, however, $0 \cdot 1$ and $1 \cdot 0$ are defined, but $(0 \cdot 1) \cdot 0$ is undefined.

Next, we show that a unital partial semigroup corresponds to a unital quantale.

Theorem 5. For a partial semigroup $(X, \cdot)$ and $U \subseteq X$, the following statements are equivalent.

1. $(X, \cdot, U)$ is a unital partial semigroup.
2. $(\varphi(X), \subseteq, \bigcup, \llbracket, U \rrbracket, U)$ is a unital quantale.

Proof. $(1 \Rightarrow 2)$ Assume that $(X, \cdot, U)$ is a unital partial semigroup. Take $S \subseteq X$. The condition 1 of Definition 9 implies $U \llbracket \cdot \rrbracket S \subseteq S$, the condition 2 implies $S \llbracket U \rrbracket \subseteq S$, the conditions 1,3 imply $S \subseteq U \llbracket \cdot \rrbracket S$, and the conditions 2,4 imply $S \subseteq S \llbracket \cdot \rrbracket U$. Therefore, $U$ is the unit of $\llbracket \cdot \rrbracket$.

$(2 \Rightarrow 1)$ Assume that $(\varphi(X), \subseteq, \bigcup, \llbracket, U \rrbracket, U)$ is a unital quantale. If $u \in U$ and $u \cdot x$ is defined, then $u \cdot x = x$, since $U \llbracket \cdot \rrbracket \{x\} \subseteq \{x\}$. If $u \in U$ and $x \cdot u$ is defined, then $x \cdot u = x$, since $\{x\} \llbracket U \rrbracket \subseteq \{x\}$. For any $x \in X$, there exists $u \in U$ such that $u \cdot x$ is defined, since $\{x\} \subseteq U \llbracket \cdot \rrbracket \{x\}$. For any $x \in X$, there exists $u \in U$ such that $x \cdot u$ is defined, since $\{x\} \subseteq \{x\} \llbracket U \rrbracket$. Therefore, $U$ is the unital subset of $(X, \cdot)$. □
Next, we show that a dividing map between unital partial semigroups corresponds to a unital homomorphism between unital quantales.

**Theorem 6.** For unital partial semigroups $(X,\cdot,U),(X',\cdot',U')$ and $f: (X,\cdot) \rightarrow (X',\cdot')$ in $\text{PSG}_{\text{div}}$, the following statements are equivalent.

1. $f$ is an arrow $f: (X,\cdot,U) \rightarrow (X',\cdot',U')$ in $\text{UPSG}_{\text{div}}$.
2. $\varphi(f)$ is an arrow $\varphi(f): (\varphi(X'),\subseteq_\varphi,\llbracket\cdot\rrbracket,U') \rightarrow (\varphi(X),\subseteq,\llbracket\cdot\rrbracket,U)$ in $\text{UQt}$.

**Proof.** The condition $x \in U \iff f(x) \in U'$ is equivalent to $\varphi(f)(U') = U$. □

**Corollary 3.** Let $\varphi: \text{UPSG}_{\text{div}} \rightarrow \text{UQt}_{\text{op}}$ be the extension of $\varphi: \text{PSG}_{\text{div}} \rightarrow \text{Qt}_{\text{op}}$ by $\varphi(X,\cdot,U) = (\varphi(X),\subseteq_\varphi,\llbracket\cdot\rrbracket,U)$. $\varphi: \text{UPSG}_{\text{div}} \rightarrow \text{UQt}_{\text{op}}$ and the forgetful functor $\text{forget}: \text{UPSG}_{\text{div}} \rightarrow \text{PSG}_{\text{div}}$ is a pullback of $\varphi: \text{PSG}_{\text{div}} \rightarrow \text{Qt}_{\text{op}}$ and the forgetful functor $\text{forget}: \text{UQt}_{\text{op}} \rightarrow \text{Qt}_{\text{op}}$ in $\text{Cat}$ (Fig. 1).

Next, we show that a preordered set which corresponds to a unital partial semigroup. A preordered set is defined to be a tuple $(X,\leq)$ of a set $X$ and a preorder $\leq$ on $X$, that is to say, $\leq$ is reflexive and transitive. We write $x \leq y$ for $(x,y) \in \leq$.

**Theorem 7.** A weak preordered set $(X,R)$ is a preordered set, if and only if $
abla_X^\text{def} = \{(x,x) \mid x \in X\}$ is the unital subset of $\text{comp}(X,R) = (R,\cdot)$.

**Proof.** $(\implies)$ Assume that $(X,\leq)$ is a preordered set. Since $\leq$ is reflexive, the set $\nabla_X = \{(x,x) \mid x \in X\}$ is a subset of $\leq$. If $(x,x) \in \nabla_X$ and $(x,x);(y,z)$ is defined, then $x = y$ and $(x,x);(y,z) = (y,z)$. For any $x \leq y$, $(x,x);(x,y)$ is defined and $(x,x) \in \nabla_X$. The remaining condition is proven similarly.

$(\impliedby)$ Take $x \in X$. There is $y \in X$ such that $(x,y) \in R$ or $(y,x) \in R$. When $(x,y) \in R$, there is $(u,u') \in \nabla_X$ such that $(u,u');(x,y)$ is defined and equal to $(x,y)$. Then, $u = x = u'$. When $(y,x) \in R$, there is $(u,u') \in \nabla_X$ such that $(y,x);(u,u')$ is defined and equal to $(y,x)$. Then, $u = x = u'$. In both cases, $(x,x)$ is equal to $(u,u') \in \nabla_X \subseteq R$. Therefore, $R$ is reflexive. □

Next, we introduce the condition for maps between preordered sets corresponds to dividing maps between unital partial semigroups. The condition is monotone, satisfying the intermediate value property, and Id-reflecting.

**Definition 11 (Id-reflecting map between preordered sets).** For preordered sets $(X,\leq),(X',\leq')$, an Id-reflecting map from $(X,\leq)$ to $(X',\leq')$ is defined to be a monotone map $f: X \rightarrow X'$ such that if $x \leq y$ and $f(x) = f(y)$, then $x = y$.

Id-reflecting maps satisfying the intermediate value property between preordered sets are closed under composition as maps. The identity map on a preordered set is an Id-reflecting map satisfying the intermediate value property. Therefore, we write $\text{PreOrd}_{\text{int,idref}}$ for the category whose objects are preordered sets and whose arrows are Id-reflecting maps satisfying the intermediate value property between them.

Next, we show that Id-reflecting maps satisfying the intermediate value property correspond to dividing maps between unital partial semigroups.
Theorem 8. For \((X, \leq), (X', \leq')\) in \(\text{PreOrd}_{\text{int}, \text{idref}}\), \(f: (X, \leq) \to (X', \leq')\) in \(\text{WPreOrd}_{\text{int}}\), the following statements are equivalent.

1. \(f\) is an arrow \(f: (X, \leq) \to (X', \leq')\) in \(\text{PreOrd}_{\text{int}, \text{idref}}\).
2. \(\text{comp}(f)\) is an arrow \(\text{comp}(f): (\leq, ;, \Delta_X) \to (\leq', ;, \Delta_{X'})\) in \(\text{UPSG}_{\text{div}}\).

Proof. (1 \(\Rightarrow\) 2) Assume that \(f\) is Id-reflecting and \(x \leq y\). If \((x, y) \in \Delta_X\), then \(x = y\) and \(\text{comp}(f)(x, y) = (f(x), f(y)) = (f(x), f(x)) \in \Delta_{X'}\). Conversely, if \(\text{comp}(f)(x, y) = (f(x), f(y)) \in \Delta_{X'}\), then \(x = y\) and \((x, y) \in \Delta_X\), since \(f(x) = f(y)\) and \(f\) is Id-reflecting. Therefore, we have \((x, y) \in \Delta_X\) if and only if \(\text{comp}(f)(x, y) \in \Delta_{X'}\).

(2 \(\Rightarrow\) 1) Assume that \(f\) satisfies \((x, y) \in \Delta_X \iff \text{comp}(f)(x, y) \in \Delta_{X'}\) for any \(x \leq y\). Take \(x \leq y\) such that \(f(x) = f(y)\). They satisfy \(\text{comp}(f)(x, y) = (f(x), f(y)) = (f(x), f(x)) \in \Delta_{X'}\). Since \(f\) is a dividing maps between unital partial semigroups, \(x, y\) satisfy \((x, y) \in \Delta_X\) and \(x = y\). Therefore, \(f\) is Id-reflecting.

\(\square\)

Corollary 4. Let \(\text{comp}: \text{PreOrd}_{\text{int}, \text{idref}} \to \text{UPSG}_{\text{div}}\) be the extension of \(\text{comp}: \text{WPreOrd}_{\text{int}} \to \text{PSG}_{\text{div}}\) by \(\text{comp}(X, \leq) = (\leq, ;, \Delta_X)\). The forgetful functor \(\text{forget}: \text{PreOrd}_{\text{int}, \text{idref}} \to \text{WPreOrd}_{\text{int}}\) and \(\text{comp}: \text{PreOrd}_{\text{int}, \text{idref}} \to \text{UPSG}_{\text{div}}\) is a pullback of \(\text{comp}: \text{WPreOrd}_{\text{int}} \to \text{PSG}_{\text{div}}\) and the forgetful functor \(\text{forget}: \text{UPSG}_{\text{div}} \to \text{PSG}_{\text{div}}\) in \(\text{Cat}\) (Fig. 1).

Example 7. \((N, <_N)\) is an object of \(\text{WPreOrd}_{\text{int}}\), but not of \(\text{PreOrd}_{\text{int}, \text{idref}}\), since \(<_N\) is not reflexive. For the same reason, \(\text{comp}(N, <_N) = (<_N, ;)\) has no unital subset.

The map \(\text{plus1}(n) \overset{\text{def}}{=} n + 1\) is Id-reflecting on \((N, \leq_N)\) in \(\text{PreOrd}_{\text{int}, \text{idref}}\), since \(m \leq n\) and \(m + 1 = n + 1\) imply \(m = n\). For the same reason, the map \(\text{comp}(\text{plus1})(m, n) = (m + 1, n + 1)\) is a dividing map on \((\leq_N, ;, \Delta_N)\) in \(\text{UPSG}_{\text{div}}\).

On the other hand, the map \(\text{zero}(n) \overset{\text{def}}{=} 0\) is not Id-reflecting on \((N, \leq_N)\) in \(\text{PreOrd}_{\text{int}, \text{idref}}\), since \(0 \leq 1\) and \(\text{zero}(0) = \text{zero}(1)\) but not \(0 = 1\). For the same reason, the map \(\text{comp}(\text{zero})(m, n) = (0, 0)\) is not a dividing map on \((\leq_N, ;, \Delta_N)\) in \(\text{UPSG}_{\text{div}}\).

5 Partial Semigroups with Partial Subtraction and Relational Embedding of Quantales

In this section, we define a subcategory of \(\text{UPSG}_{\text{div}}\) and the corresponding subcategory of \(\text{PreOrd}_{\text{int}, \text{idref}}\). Between them, we give a functor \(\text{suff}\) in the converse direction of \(\text{comp}\). Moreover, we give the natural transformation \(- \circ \text{suff} \to \text{Id}\) which induces the relational embedding maps mentioned in Sect. 1.

Definition 12 (diagonal unital partial semigroup). A partial semigroup \((X, \cdot)\) is called diagonal, if
1. if \( w \cdot x = y \cdot z \), then there exists \( v \) such that \( w = y \cdot v \) or \( y = w \cdot v \), and
2. if \( w \cdot x = w \cdot y \), then \( x = y \).

A unital partial semigroup \( (X, \cdot, U) \) is called diagonal, if \( (X, \cdot) \) is diagonal. We write \( \text{DUPSG}_{\text{div}} \) for the subcategory of \( \text{UPSG}_{\text{div}} \) whose objects are only diagonal unital partial semigroups.

**Definition 13** (interval-total preorder). A preorder \( \leq \) on \( X \) is called interval-total, if \( x \leq y \leq z \) and \( x \leq y' \leq z \) imply \( y \leq y' \) or \( y' \leq y \) for each \( x, y, y', z \in X \). A preordered set \( (X, \leq) \) is called an interval-totaly preordered set, if \( \leq \) is interval-total. We write \( \text{ITPreOrd}_{\text{int, idref}} \) for the subcategory of \( \text{PreOrd}_{\text{int, idref}} \) whose objects are only interval-totally preordered sets.

**Theorem 9.** A preordered set \( (X, \leq) \) is an interval-totally preordered set, if and only if the unital partial semigroup \( \text{comp}(X, \leq) = (\leq, \cdot; \Delta_X) \) is diagonal.

**Proof.** (\( \implies \))

(1) Assume that \( w, x, y, z \in \leq \) satisfy \( w; x = y; z \). There exists \( a, b, b', c \in X \) such that \( w = (a, b), x = (b, c), y = (a, b'), \) and \( z = (b', c) \), that is to say, \( a \leq b \leq c \) and \( a \leq b' \leq c \). Since \( \leq \) is interval-total, \( b \leq b' \) or \( b' \leq b \). When \( b \leq b' \), \( v = (b, b') \in \leq \) satisfies \( y = (a, b') = (a, b); (b, b') = w; v \). When \( b' \leq b \), \( v = (b', b) \in \leq \) satisfies \( w = (a, b) = (a, b'); (b', b) = y; v \).

(2) Assume that \( w, x, y \in \leq \) satisfy \( w; x = w; y \). There exists \( a, b, c \in X \) such that \( w = (a, b), x = (b, c), \) and \( y = (b, c) \). Therefore, \( x = y \).

(\( \impliedby \)) Assume that \( \leq \) is diagonal. Assume that \( x, y, y', z \in X \) satisfy \( x \leq y \leq z \) and \( x \leq y' \leq z \). Then, we have that \( (x, y); (y, z) = (x, z) = (x, y'); (y', z) \). Since \( \leq \) is diagonal, there exists \( v \leq w \) such that \( (x, y') \); \( (v, w) = (x, y) \) or \( (v, w) = (x, y') \). If \( (x, y'); (v, w) = (x, y), \) then \( y' = v \leq w = y \). On the other hand, if \( (x, y); (v, w) = (x, y'), \) then \( y = v \leq w = y' \). Therefore, \( \leq \) is interval-total.

**Corollary 5.** Let \( \text{comp} : \text{ITPreOrd}_{\text{int, idref}} \to \text{DUPSG}_{\text{div}} \) be the restriction of \( \text{comp} : \text{PreOrd}_{\text{int, idref}} \to \text{UPSG}_{\text{div}} \).

The forgetful functor \( \text{forget} : \text{ITPreOrd}_{\text{int, idref}} \to \text{PreOrd}_{\text{int, idref}} \) and \( \text{comp} : \text{ITPreOrd}_{\text{int, idref}} \to \text{DUPSG}_{\text{div}} \) is a pullback of \( \text{comp} : \text{PreOrd}_{\text{int, idref}} \to \text{UPSG}_{\text{div}} \) and the forgetful functor \( \text{forget} : \text{DUPSG}_{\text{div}} \to \text{UPSG}_{\text{div}} \) in \( \text{Cat} \) (Fig. 1).

Next, we give a functor \( \text{suff} \) in the converse direction of \( \text{comp} \).

**Theorem 10.** The following data form a functor \( \text{suff} \) from \( \text{DUPSG}_{\text{div}} \) to \( \text{ITPreOrd}_{\text{int, idref}} \).

- For an object \( (X, \cdot, U) \), \( \text{suff}(X, \cdot, U) \) \( \overset{\text{def}}{=} (X, \leq) \)
  where \( x \leq y \iff \exists z \in X. x \cdot z = y \).
- For an arrow \( f : (X, \cdot, U) \to (X', \cdot', U') \), \( \text{suff}(f) : (X, \leq) \to (X', \leq) \) is \( f \).
Proof. (object part) Assume that \((X, \cdot, U)\) is a diagonal unital partial semigroup. Since \((X, \cdot, U)\) is a unital partial semigroup, \(\leq\) is transitive and reflexive. If \(x \leq y \leq z\) and \(x \leq y' \leq z\), then there exist \(v, w\) such that \(y \cdot v = z = y' \cdot w\). Since \((X, \cdot, U)\) is diagonal, there exists \(u\) such that \(y = y' \cdot u\) or \(y' = y \cdot u\), that is to say, \(y' \leq y\) or \(y \leq y'\). Therefore, \(\leq\) is interval-total.

(arrow part) Take \(f: (X, \cdot, U) \to (X', \cdot', U')\) in \(\text{DUPSG}_{\text{div}}\). \(f\) is monotone as \(f: (X, \leq) \to (X', \leq')\), since \(x \cdot z = y\) implies \(f(y) = f(x \cdot z) = f(x) \cdot' f(z)\).

We show that \(\text{suff}(f) = f\) is Id-reflecting. Assume that \(x \leq y\) and \(f(x) = f(y)\). There exist \(z \in X\) such that \(x \cdot z = y\). Therefore, \(f(x) = f(y) = f(x \cdot z) = f(x) \cdot' f(z)\). Since \((X', \cdot', U')\) is a unital partial semigroup, there exists \(u' \in U'\) such that \(f(x) \cdot' u' = f(x) = f(x) \cdot' f(z)\). Since \((X', \cdot', U')\) is diagonal, \(u', f(z)\) satisfy \(u' = f(z)\). Since \(f\) is a dividing map between unital partial semigroups, \(f(z) = u' \in U'\) implies \(z \in U\) and \(x = x \cdot z = y\). Therefore, \(f\) is Id-reflecting.

We show that \(f\) satisfies the intermediate value property. Assume that \(x \leq y\), \(f(x) \leq z'\), and \(z' \leq f(y)\). There exist \(u \in X, v', w' \in X'\) such that \(x \cdot u = y\), \(f(x) \cdot' v' = z'\), and \(z' \cdot' w' = f(y)\). Since \(f\) is dividing, there exist \(w, z \in X\) such that \(f(w) = w'\), \(f(z) = z'\), and \(z \cdot w = y\). Therefore, \(z \leq y\). Since \(f\) is dividing and \(f(z) = z' = f(x) \cdot' v'\), there exist \(v, t \in X\) such that \(f(t) = f(x)\), \(f(v) = v'\), and \(z = t \cdot v\). Since \((X, \cdot, U)\) is diagonal and \(x \cdot u = y = z \cdot w = (t \cdot v) \cdot w = t \cdot (v \cdot w)\), \(x, t\) satisfy \(x \leq t\) or \(t \leq x\). Since \(f\) is Id-reflecting and \(f(t) = f(x)\), \(x\) satisfies \(t = x\) and \(z = t \cdot v = x \cdot v\), that is to say, \(x \leq z\). Therefore, \(f\) satisfies the intermediate value property.

\(\text{suff}(f)\) preserves composition and identities. \(\square\)

We call the following operation \(\div\) the partial subtraction on a diagonal unital partial semigroup.

**Theorem 11.** The following data form a natural transformation \(\div: \text{comp} \circ \text{suff} \to \text{Id}: \text{DUPSG}_{\text{div}} \to \text{DUPSG}_{\text{div}}\):

- For an object \((X, \cdot, U)\), its component \(\div: (\leq, \cdot, \Delta_X) \to (\cdot, \cdot, U)\) sends \(y \leq x\) to \(x \div y \overset{\text{def}}{=} z\) such that \(y \cdot z = x\).

Proof. (well-definedness) For \(y \leq x\), there exists \(z \in X\) such that \(y \cdot z = x\). Since \((X, \cdot, U)\) is diagonal, if \(y \cdot z = y \cdot z'\) then \(z = z'\).

(homomorphism of \(\text{PSG}\)) Assume that \((z, y); (x, w)\) is defined, \(z \leq y\), and \(x \leq w\). Then, \(z \leq y = x = w\) and \(w\) is representable as \(w = x \cdot (w' - x) = (z \cdot (x - z)) \cdot (w' - x)\). Therefore, \((x - z) \cdot (w' - x) = z \cdot ((x - z) \cdot (w' - x))\) are also defined and \(z \cdot ((x - z) \cdot (w' - x)) = (z \cdot (x - z)) \cdot (w' - x) = w\). Therefore, \((x - z) \cdot (w' - x) = w\).

(dividing) Assume that \(x \leq w\) and \(w' = z \cdot y\). Then, \(x \cdot (w' - w)\) is defined and equal to \(w\). Therefore, \(x \cdot (z, y)\), \(x \cdot z\), and \((x \cdot z) \cdot y\) are also defined. Since \((x \cdot z) \cdot y = x \cdot (z \cdot y) = x \cdot (w' - x) = w\), they satisfy \(x \cdot z \leq w\) and \(w' = x \cdot z\).

Since \(x \cdot z\) is defined, obviously \(x \leq x \cdot z\) and \((x \cdot z) \div x = z\). \((x' \cdot z, w)\) is defined and equal to \((x, w)\).

(unital) For \(y \leq x\), we show \((y, x) \in \Delta_X \iff x \div y \in U\). Assume \((y, x) \in \Delta_X\). Then, \(x = y\). There exists \(u \in U\) satisfying \(x \cdot u = x\). Since \((X, \cdot, U)\) is diagonal and \(x \cdot (x' - x) = x \cdot u = u \in U\). Conversely,
assume \(x^-y \in U. y \cdot (x^-y)\) is defined and equal to \(x\). By the definition of unital subset, \(y \cdot (x^-y) = y\). Therefore, \(x = y \cdot (x^-y) = y\) and \((x, y) \in \triangle_X\).

(naturality) Assume \(f\) is an arrow \(f: (X, \cdot, U) \to (X', \cdot', U')\) in DUPSG\(_{\text{div}}\). For \(y \leq x\), \(f(y) \cdot (f(x^-y)) = f(x) = f(y) \cdot (x^-y)\). Since \((X', U')\) is diagonal, we have \(f(x^-y) = f(x) - f(y)\). \(\square\)

Example 8. \((\mathbb{N}, +, 0)\) is a diagonal unital partial semigroup. Then, the interval-totally preordered set \(\text{suff}(\mathbb{N}, +, 0)\) is equal to \((\mathbb{N}, \leq_{\mathbb{N}})\), since \(x \leq_{\mathbb{N}} y \Leftrightarrow \exists z \in \mathbb{N}. x + z = y\). Therefore, \(\text{comp}(\text{suff}(\mathbb{N}, +, 0)) = (\leq_{\mathbb{N}}, ;)\) is also a diagonal unital partial semigroup. The partial subtraction \(\cdot: (\leq_{\mathbb{N}}, ;) \to (\mathbb{N}, +, 0)\) sends \(x \leq_{\mathbb{N}} y\) to \(y - x\).

Definition 14 (diagonal powerset quantale). A diagonal powerset quantale is a unital quantale \((Q, \leq, \lor, \odot, 1)\) such that

1. \((Q, \leq, \lor) = (\varphi(X), \subseteq, \bigcup)\) for some set \(X\),
2. for any \(x, y \in X\), \(\{x\} \odot \{y\}\) is singleton or empty,
3. if \(\{w\} \odot \{x\} = \{y\} \odot \{z\} \neq \emptyset\), then there exists \(v\) such that \(\{w\} = \{y\} \odot \{v\}\) or \(\{y\} = \{w\} \odot \{v\}\), and
4. if \(\{w\} \odot \{x\} = \{w\} \odot \{y\} \neq \emptyset\), then \(x = y\).

We define the subcategory \(\text{DPQt}\) of \(\text{UQt}\), whose object is a diagonal powerset quantale.

Theorem 12. For a unital partial semigroup \((X, \cdot, U)\), the following statements are equivalent.

1. \((X, \cdot, U)\) is diagonal.
2. \(\varphi(X, \cdot, U) = (\varphi(X), \subseteq, \bigcup, [\cdot], U)\) is a diagonal powerset quantale.

Proof. Let \((X, \cdot, U)\) be a diagonal unital partial semigroup. For any \(x, y \in X\), \(\{x\} [\cdot] \{y\}\) is \(\{x \cdot y\}\) if \(x \cdot y\) is defined and the emptyset otherwise.

For any \(w, x, y, z \in X\) satisfying \(\{w\} [\cdot] \{x\} = \{y\} [\cdot] \{z\} \neq \emptyset\), \(w \cdot x\) and \(y \cdot z\) are defined and equivalent. Since \((X, \cdot, U)\) is a diagonal unital partial semigroup, there exists \(v\) such that \(w = y \cdot v\) or \(y = w \cdot v\). Therefore, \(v\) satisfies \(\{w\} = \{y\} [\cdot] \{v\}\) or \(\{y\} = \{w\} [\cdot] \{v\}\).

For any \(w, x, y \in X\) satisfying \(\{w\} [\cdot] \{x\} = \{w\} [\cdot] \{y\} \neq \emptyset\), \(w \cdot x\) and \(w \cdot y\) are defined and equivalent. Since \((X, \cdot, U)\) is a diagonal unital partial semigroup, \(x = y\). \(\square\)

Corollary 6. Let \(\varphi: \text{DUPSG}_{\text{div}} \to \text{DPQt}^{\text{op}}\) be the restriction of \(\varphi: \text{UPSG}_{\text{div}} \to \text{UQt}^{\text{op}}\).

The forgetful functor \(\text{forget}: \text{DUPSG}_{\text{div}} \to \text{UPSG}_{\text{div}}\) and \(\varphi:\text{DUPSG}_{\text{div}} \to \text{DPQt}^{\text{op}}\) is a pullback of \(\varphi: \text{UPSG}_{\text{div}} \to \text{UQt}^{\text{op}}\) and the forgetful functor \(\text{forget}: \text{DPQt}^{\text{op}} \to \text{UQt}^{\text{op}}\) in \(\text{Cat}\) (Fig. 1).

Theorem 13. For any diagonal powerset quantale \((Q, \leq, \lor, \odot, 1)\), there exists a diagonal unital partial semigroup \((X, \cdot, U)\) satisfying \(\varphi(X, \cdot, U) = (Q, \leq, \lor, \odot, 1)\).
Proof. Let \((Q, \leq, \lor, 1)\) be a diagonal power set quantale. There exists a set \(X\) satisfying \((Q, \leq, \lor) = (\varphi(X), \subseteq, \cup)\). Let \(x \cdot y\) be the element of \(\{x\} \circ \{y\}\) if \(\{x\} \circ \{y\}\) is singleton and undefined otherwise. Assume that \(x \cdot y\) and \((x \cdot y) \cdot z\) are defined. There exist \(u, v \in X\) satisfying \(\{x\} \circ \{y\} = \{u\} \cup \{v\}\). Since \(\{x\} \circ \{y\} \circ \{z\} = \{u\} \cup \{v\} = \{v\} \neq \emptyset\), the distributivity of \(\circ\) implies \(\{y\} \circ \{z\} \neq \emptyset\). Therefore, \(y\cdot z\) and \(x \cdot (y \cdot z)\) are defined.

For any \(S_1, S_2 \subseteq X\), the distributivity of \(\circ\) also implies \(S_1 \circ S_2 = \bigcup \{\{x\} \circ \{y\} \mid x \in S_1, y \in S_2\} = \{x \cdot y \mid x \in S_1, y \in S_2\text{ is defined}\}\). Therefore, \((X, \cdot)\) is the partial semigroup satisfying \(\varphi(X, \cdot) = (\varphi(X), \subseteq, \cup, \circ)\). By Theorem 5, \((X, \cdot, 1)\) is the unital partial semigroup satisfying \(\varphi(X, \cdot, 1) = (Q, \leq, \lor, \circ, 1)\). By Theorem 12, \((X, \cdot, 1)\) is diagonal, since \((\varphi(X), \subseteq, \cup, \cdot, 1)\) is a diagonal power set quantale.

\[\square\]

Theorem 14. For any diagonal power set quantale \((Q, \leq, \lor, 1)\), there exist a diagonal unital partial semigroup \((X, \cdot, U)\) and an injective arrow from \((Q, \leq, \lor, 1)\) to the relational quantale \((\varphi(\leq, \cdot, U), \subseteq, \cup, \cdot, 1, \Delta_X)\) in DPQt.

Proof. By Theorem 11 and Corollary 6, there is a natural transformation \(\varphi(\cdot): \varphi \circ \text{comp} \circ \text{suff} \to \varphi: \text{DUPSG}_{\text{div}}^{\text{op}} \to \text{DPQt}\). By Theorem 13, for any object \((Q, \leq, \lor, 1) \in \text{DPQt}\), there exists an object \((X, \cdot, U) \in \text{DUPSG}_{\text{div}}^{\text{op}}\) satisfying \(\varphi(X, \cdot, U) = (Q, \leq, \lor, 1)\). Therefore, the component \(\varphi(\cdot)(X, \cdot, U)\) is an arrow from \((Q, \leq, \lor, 1)\) to \(\varphi(\text{comp}(\text{suff}(X, \cdot, U)))\) in DPQt. By the definitions of \(\varphi, \text{comp},\) and \(\text{suff}\), the power set quantale \((Q, \leq, \lor, 1)\) is equal to \((\varphi(X), \subseteq, \cup, \cdot, U)\) and \(\varphi(\text{comp}(\text{suff}(X, \cdot, U)))\) is the relational quantale \((\varphi(\leq, \cdot, U), \subseteq, \cup, \cdot, 1, \Delta_X)\). The component \(\varphi(\cdot)(X, \cdot, U)\) is the map from \(\varphi(X)\) to \(\varphi(\leq, \cdot, U)\) such that \(\varphi(\cdot)(X, \cdot, U)(S) = \{(x, y) \mid y \in \{x\} \circ S\}\). Each subset \(S \subseteq X\) is represented as follows.

\[S = 1 \circ S = (\bigcup \{\{x\} \mid x \in 1\}) \circ S = \bigcup \{\{x\} \circ S \mid x \in 1\}\]
\[= \{y \mid \exists x \in 1, y \in \{x\} \circ S\}\]
\[\varphi(\cdot)(X, \cdot, U)\text{ is injective, since } \varphi(\cdot)(X, \cdot, U)(S) \subseteq \varphi(\cdot)(X, \cdot, U)(S')\text{ implies } S \subseteq S'.\]

\[\square\]

Example 9. For a set \(X\), \((\varphi(X), \subseteq, \cup, \cap, X)\) is a diagonal power set quantale.

Example 10. For a group \((G, \cdot, 1, -1)\), \((\varphi(G), \subseteq, \cup, [\cdot], \{1\})\) is a diagonal power set quantale.

Example 11. For a set \(A\), we write \(A^*\) for the set of all finite sequences on \(A\). \((\varphi(A^*), \subseteq, \cup, [\cdot], \{\epsilon\})\) is a diagonal power set quantale, where \(\epsilon\) is the empty sequence. \((\varphi(N), \subseteq, \cup, [\cdot], \{0\})\) is a special case.

Example 12. For a set \(A\), \((\varphi(A \times A), \subseteq, \cup, \Delta_A)\) is a diagonal power set quantale, where \(R; Q \overset{\text{def}}{=} \{(a, b) \mid \exists c \in A, (a, c) \in R, (c, b) \in Q\}\) and \(\Delta_A \overset{\text{def}}{=} \{(a, a) \mid a \in A\}\).
6 Conclusion

This paper has introduced the notions of weak preorder, partial semigroup, lax homomorphism between quantales. And, we have shown three correspondences by proving the six pullbacks in Fig. 1.

1. The correspondence among a monotone map satisfying the intermediate value property between weak preordered sets, a dividing map between partial semigroups, and a homomorphism between quantales
2. The correspondence among a preordered set, a unital partial semigroup, and a unital quantales (including the correspondence among maps for them)
3. The correspondence among an interval-totally preordered set, a diagonal unital partial semigroup, and a diagonal powerset quantale (including the correspondence among maps for them)

We also have shown that each diagonal powerset quantale has the relational embedding which is the image of partial subtraction \( \dot{-} \) by the functor \( \mathcal{P} \). It is a future work to generalize our result for any (possibly not diagonal) powerset quantale and to extend it to a Stone duality [11].

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