Semilinear substructural logics with the finite embeddability property

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Abstract. Three semilinear substructural logics $\text{HpsUL}^*_\omega$, $\text{UL}_\omega$ and $\text{IUL}_\omega$ are constructed. Then the completeness of $\text{UL}_\omega$ and $\text{IUL}_\omega$ with respect to classes of finite $\text{UL}$ and $\text{IUL}$-algebras, respectively, is proved. Algebraically, non-integral $\text{UL}_\omega$ and $\text{IUL}_\omega$-algebras have the finite embeddability property, which gives a characterization for finite $\text{UL}$ and $\text{IUL}$-algebras.

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1. Introduction

The finite embeddability property (FEP), or actually, the finite model property (FMP), as shown in [11], fails for some known non-integral semilinear substructural logics including Metcalfe and Montagna’s uninorm logic $\text{UL}$ and involutive uninorm logic $\text{IUL}$ [8], and a suitable extension $\text{HpsUL}^*$ [10] of Metcalfe, Olivetti and Gabbay’s pseudo-uninorm logic $\text{HpsUL}$ [7]. This shows that $\text{UL}$, $\text{IUL}$ and $\text{HpsUL}^*$ are incomplete with respect to the corresponding classes of finite algebras.

A natural problem is whether we can construct logics which are complete with respect to finite $\text{UL}$, $\text{IUL}$ and $\text{HpsUL}^*$-algebras. Algebraically, our motivation is how to characterize the variety generated by its finite members when a class of algebras does not enjoy the FEP (or FMP).

In this paper, we construct three schematic extensions $\text{UL}_\omega$, $\text{IUL}_\omega$ and $\text{HpsUL}_\omega^*$ by adding one simple axiom

$$(\text{FIN}) \vdash (\varphi \vDash e) \iff ((\varphi \otimes \varphi) \vDash e)$$

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to UL, IUL and HpsUL∗, respectively, where ϕ ↔ ψ is defined to be \((ϕ\backslash ψ)\land (ψ\backslash ϕ)\). Then we prove that ULω and IULω are complete with respect to classes of finite UL and IUL-algebras, respectively. Algebraically, non-integral ULω and IULω-algebras have the finite embeddability property, which gives a characterization for finite UL and IUL-algebras.

Classes of ULω and IULω-algebras are non-integral varieties which usually, as pointed out in [5], do not enjoy the FEP. We prove the FEP for ULω and IULω-algebras by Blok and Alten’s construction [1, 2]. But in proving the finiteness of Blok and Alten’s construction in Lemma 4.7 we have not used Dickson’s lemma [1, 3] or Higman’s finite basis theorem [2, 4] but used a specific property of ULω and IULω-algebras which is given in Lemma 2.4.

Since almost all proofs are done algebraically, as suggested by the referee, Hilbert-style systems for the logics under consideration have not been given. For details on UL and IUL, we refer to [8]. For details on HpsUL∗, we refer to [7, 10]. In addition, we are unable to prove the FEP for HpsUL∗ω and left it as an open problem. In the paper, Z+ denote the set of positive integers, N = Z+ ∪ {0}.

2. HpsUL∗ω-algebras, ULω-algebras and IULω-algebras

Definition 2.1. [6, 7, 9] An HpsUL-algebra is a bounded semilinear residuated lattice \( A = \langle A, \land, \lor, \cdot, /, e, f, \perp, \top \rangle \) with universe \( A \), binary operations \( \land, \lor, \cdot, / \), and constants \( e, f, \perp, \top \) such that:

(i) \( \langle A, \land, \lor, \cdot, / \rangle \) is a bounded lattice with top element \( \top \) and bottom element \( \perp \);
(ii) \( \langle A, \cdot, e \rangle \) is a monoid;
(iii) \( \forall x, y, z \in A, x \cdot y \leq z \iff x \leq z / y \iff y \leq x / z \);
(iv) \( \forall x, y, u, v \in A, (\lambda_u((x \lor y) / x)) \lor (\rho_v((x \lor y) / y)) = e \), where, for any \( a, b \in A \), \( \lambda_a(b) := (a \backslash (b \cdot a)) \land e \), \( \rho_a(b) := ((a \cdot b) / a) \land e \).

We use the convention that \( \cdot \) binds stronger than other binary operations and we shall often omit \( \cdot \). For example, we will thus write \( xy \) instead of \( x \cdot y \). We also define \( x^0 = e \) and \( x^{n+1} = x^n \).

Definition 2.2. [7, 8, 10] Let \( A = \langle A, \land, \lor, \cdot, /, e, f, \perp, \top \rangle \) be an HpsUL-algebra. Then

(i) \( A \) is an HpsUL-chain if it is linearly ordered.
(ii) \( A \) is an HpsUL∗-algebra if the following weak commutativity (Wcm) holds for all \( x, y \in A \):

\[
x y \leq e \text{ implies } y x \leq e.
\]

(iii) \( A \) is an UL-algebra if \( xy = yx \) for all \( x, y \in A \).
(iv) \( A \) is an IUL-algebra if it is an UL-algebra such that \( \neg \neg x = x \) for all \( x \in A \).
Lemma 2.5. Let $\mathcal{A}$ be an $\mathsf{HpsUL}_\omega^*$-algebra (UL or $\mathsf{IUL}_\omega$-algebra) if it is an $\mathsf{HpsUL}^*$-algebra (UL or $\mathsf{IUL}$-algebra) such that the following identity (Fin)

$$x\e = x^2\e$$

holds for all $x \in A$.

Theorem 2.3. \cite{7,8,9} Let $L \in \{ \mathsf{HpsUL}^*, \mathsf{UL}, \mathsf{IUL}, \mathsf{HpsUL}_\omega^*, \mathsf{UL}_\omega, \mathsf{IUL}_\omega \}$. Then

(i) Each $L$-algebra has a subdirect representation with $L$-chains;

(ii) each finite $L$-algebra has a subdirect representation with finitely many finite $L$-chains.

Lemma 2.4. Let $\mathcal{A}$ be an $\mathsf{HpsUL}_\omega^*$-algebra. Then

(i) $xy \leq e$ iff $xy^2 \leq e$ for any $x, y \in A$;

(ii) $x_1^{k_1}\cdots x_n^{k_n} \leq e$ iff $x_1^{l_1}\cdots x_n^{l_n} \leq e$ for any $x_1, \ldots, x_n \in A$, $k_1, \ldots, k_n$, $l_1, \ldots, l_n \in \mathbb{Z}_+$.

Proof. (i) Let $xy \leq e$ then $yx \leq e$ by (Wcm). Thus $x \leq y\e$. Hence $x \leq y^2\e$ by (Fin). Hence $y^2x \leq e$. Therefore $xy^2 \leq e$ by (Wcm). The sufficiency part of (i) is proved in the same way. (ii) is immediate from (i). \hfill $\square$

Lemma 2.5. Let $\mathcal{A}$ be an $\mathsf{HpsUL}_\omega^*$-chain. Then

(i) $st > u$ iff $s \leq s\tu$;

(ii) $su < tu$ implies $s > t$;

(iii) $stu = u$ implies $tu = u$.

Proof. (i) and (ii) are clear. Only (iii) is proved as follows. If $st \leq e$ then $tst \leq e$ and $sts \leq e$ by Lemma 2.4 and (Wcm). Thus $tu = tstu \leq u$ and $u = ststu \leq tu$. Hence $tu \leq u$ and $u \leq tu$. Therefore $tu = u$. The case of $st > e$ is proved in the same way. Thus $tu = u$. \hfill $\square$

Lemma 2.6. (i) Each finite $\mathsf{HpsUL}^*$-chain is an $\mathsf{HpsUL}_\omega^*$-chain;

(ii) Each finite $\mathsf{HpsUL}^*$-algebra is an $\mathsf{HpsUL}_\omega^*$-algebra.

Proof. (i) Let $\mathcal{A}$ be a finite $\mathsf{HpsUL}^*$-chain. We prove that $x\e = x^2\e$ for all $x$ in $A$. Since $\mathcal{A}$ is finite, there is a positive integer $n$ such that $x^n = x^{n-1}$ for all $x \in A$.

Suppose that $x\e > x^2\e$ then $x^2(x\e) > e$. Let $z = x\e$ then $xz \not\leq e < x^2z$. Thus $xz \leq e < xz^2$ by (Wcm).

If $x^k = x^{k-1}$ and $xz \leq e < xz^2$ for any $k \geq 3$. Then $x^{k-1} \leq x^{k-2} \leq xz^k$. Thus $z^{k-1} = x^{k-2}$ by $x^k = x^{k-1}$. Hence $x^{k-1} = x^{k-2}$ by Lemma 2.5 (iii).

Since $x^n = x^{n-1}$ and $xz \leq e < xz^2$, then $x^{n-1} = x^{n-2}, \ldots, x^2 = x$ by repeatedly applying the property above. Thus $xz = x^2$, a contradiction and hence $x\e \leq x^2\e$. Similarly, we can prove that $x^2\e \leq x\e$. Thus $x\e = x^2\e$.

(ii) follows from (i) and Theorem 2.3 (ii). \hfill $\square$

Clearly, Lemmas 2.4 ~ 2.6 hold for all $\mathsf{UL}_\omega$ and $\mathsf{IUL}_\omega$-algebras.
3. Blok and Alten’s construction for HpsUL\(^+_\omega\), UL\(_\omega\), IUL\(_\omega\)-algebras

Definition 3.1. Given an ordered algebra \( \mathcal{A} = \langle A,\{ f^A_i : i \in I\},\leq^A \rangle \) (of any type), with \( \leq^A \) a (partial) order on \( A \), and any non-empty subset \( B \subseteq A \), the partial subalgebra \( \mathcal{B} \) of \( \mathcal{A} \) with domain \( B \) is the ordered partial algebra \( \mathcal{B} = \langle B,\{ f^B_i : i \in I\},\leq^B \rangle \), where \( a \leq^B b \) iff \( a \leq^A b \) for all \( a,b \in B \), and for each \( i \in I, f_i \) \( k \)-ary, \( b_1,\ldots,b_k \in B \),

\[
f^B_i(b_1,\ldots,b_k) = \begin{cases} f^A_i(b_1,\ldots,b_k) & \text{if } f^A_i(b_1,\ldots,b_k) \in B, \\ \text{undefined} & \text{if } f^A_i(b_1,\ldots,b_k) \notin B. \end{cases}
\]

Definition 3.2. A partial embedding of an ordered partial algebra \( \mathcal{B} \) into an ordered algebra \( \mathcal{A} \) is a 1-1 map \( \iota : B \to A \) such that (i) \( a \leq^B b \) iff \( \iota(a) \leq^A \iota(b) \) for all \( a,b \in B \); (ii) \( \iota(f^B_i(b_1,\ldots,b_k)) = f^A_i(\iota(b_1),\ldots,\iota(b_k)) \) if \( f^B_i(b_1,\ldots,b_k) \) is defined for some operation \( f_i \) and \( b_1,\ldots,b_k \in B \) where \( f^A_i \) denotes the realization of \( f_i \) in \( \mathcal{A} \).

Definition 3.3. A class \( \mathcal{K} \) of ordered algebras of the same type has the finite embeddability property (FEP for short) if every finite partial subalgebra \( \mathcal{B} \) of any algebra \( \mathcal{A} \in \mathcal{K} \) can be partially embedded into some finite member of \( \mathcal{K} \).

Lemma 3.4. Let \( \mathcal{K} \) be a variety and \( \mathcal{K}_{si} \) be the class of all subdirectly irreducible members of \( \mathcal{K} \). Then \( \mathcal{K} \) has the FEP if \( \mathcal{K}_{si} \) has the FEP.

Proof. See [3] Lemma 20]. \( \square \)

Definition 3.5. Let \( \mathcal{A} = \langle A,\cdot,\setminus,\wedge,\vee,e,f,\perp,\top \rangle \) be an HpsUL\(^+_\omega\)-chain and \( \mathcal{B} = \langle B,\cdot,\setminus,\wedge,\vee,e,f,\perp,\top \rangle \) be a partial subalgebra of \( \mathcal{A} \) such that \( \{ e,f,\perp,\top \} \subseteq B \). Let \( \mathcal{M} = \langle M,\cdot,\wedge,\vee,e,f,\perp,\top \rangle \) be the linearly ordered monoid of \( \{ A,\cdot,\setminus,\wedge,\vee,e,f,\perp,\top \} \) generated by \( B \).

Let \( a_1,\ldots,a_n \in M \) and let \( \delta_1,\ldots,\delta_n \in \{ l,r \} \) \((l \text{ and } r \text{ stand for “left” and “right”, respectively})\). We will write \( a^\delta \) to denote the sequence \( a_1^{\delta_1} \ldots a_n^{\delta_n} \); we will use \( \varepsilon \) to denote the empty sequence and we denote by \( M^{l,r} \) the set of all possible \( a^\delta \), that is,

\[
M^{l,r} = \{ a_1^{\delta_1} \ldots a_n^{\delta_n} \mid n < \omega; a_1,\ldots,a_n \in M; \delta_1,\ldots,\delta_n \in \{ l,r \} \}.
\]

Clearly any two elements of \( M^{l,r} \) can be concatenated to form a new element of \( M^{l,r} \). The sequence \( a^\delta \) is to be understood as a unary polynomial operating on \( M \), defined inductively as follows: For each \( c \in M \), set \( \varepsilon(c) = c \) and, for \( a^\delta \in M^{l,r} \) and \( b \in M \), set \( a^\delta b \cdot (c) = a^\delta (b \cdot c) \) and \( a^\delta b^r \cdot (c) = a^\delta (c \cdot b) \).

For each \( a^\delta \in M^{l,r} \) and \( b \in B \), define

\[
(a^\delta)^{-1}(b) = \{ c \in M \mid a^\delta (c) \leq b \}, \quad (b) = \{ c \in M \mid c \leq b \},
\]

\[
\bar{D} = \{ (a^\delta)^{-1}(b) \mid a^\delta \in M^{l,r}, b \in B \}, \quad D = \{ \bigcap X \mid X \subseteq \bar{D} \}.
\]

For \( X \subseteq M \), define

\[
C(X) = \bigcap \{ (a^\delta)^{-1}(b) \in \bar{D} \mid X \subseteq (a^\delta)^{-1}(b) \}.
\]
For $X, Y \subseteq M$ and $X_i \subseteq M, i \in I$, define

$$XY = \{ ab \mid a \in X, b \in Y \}; Xa = X \{ a \}; X . ^D Y = C(XY),$$

$$X \mid i D Y = \{ a \in M \mid X a \subseteq Y \}; Y \setminus D X = \{ a \in M \mid a X \subseteq Y \},$$

$$\bigvee_{i \in I} X_i = C(\bigcup_{i \in I} X_i); \bigwedge_{i \in I} X_i = \bigcap_{i \in I} X_i; \sim X = X \setminus D (f),$$

$$\downarrow D = \{ 1 \}; \uparrow D = \{ \top \} = M; e^D = (e); f^D = (f).$$

When $A$ is an $\text{HpsUL}_\omega^*$-chain, all $a^\delta \in M^{l \omega}$ have the form $a_1^1 a_2^r$ by the associativity of $\text{HpsUL}_\omega^*$. Then $(a_1^1 a_2^r)^{-1} (b) = \{ c \in M \mid a_1 c a_2 \leq b \}$.

When $A$ is an $\text{UL}_\omega$-chain, we need not $M^{l \omega}$ to define $(a^\delta)^{-1} (b)$, and simplify it as $(a \mapsto b) = \{ c \in M \mid ac \leq b \}$ for all $a \in M, b \in B$.

**Lemma 3.6.** If $A$ is an $\text{HpsUL}_\omega^*$-chain. Then the following properties hold.

1. $\{ 1^D, \uparrow^D, e^D, f^D \} \subseteq D$ and $C(X) = X$ for all $X \subseteq D$;
2. $X \subseteq C(X), C(X) \subseteq C(Y)$ if $X \subseteq Y$ and $C(C(X)) = C(X)$ for all $X, Y \subseteq M$;
3. $(X \setminus D Y) \setminus D Z = (X \setminus D Z) \wedge D (Y \setminus D Z)$;
4. If $X \subseteq M$ and $Y_i \subseteq M$ for $i \in I$, then $X \setminus D (\bigcap_{i \in I} Y_i) = \bigcap_{i \in I} (X \setminus D Y_i)$ and $(\bigcap_{i \in I} Y_i)^D = \bigcap_{i \in I} (Y_i)^D$;
5. If $X \subseteq M$ and $Y \subseteq D$ then $X \setminus D Y \subseteq D$ and $Y \setminus D X \subseteq D$;
6. $X . ^D e^D = e^D . ^D X = X (X . ^D Y) . ^D Z = X . ^D Y \setminus D Z = C(X Y Z)$ for all $X, Y, Z \subseteq D$ and $X . ^D e^D = e^D . ^D X \subseteq D$ iff $Y . ^D X \subseteq D$ for all $X, Y, Z \subseteq D$;
7. $X . ^D Y \subseteq Z$ iff $Y \subseteq Z \setminus D Y$ for all $X, Y, Z \subseteq D$;
8. $X \setminus D (Y \setminus D Z) = (Y \setminus D X) \setminus D Z$ for all $X, Y \subseteq M$ and $Z \subseteq D$;
9. $e^D = (\lambda \iota ((X \setminus D Y) \setminus D X)) \setminus D (\rho \nu ((X \setminus D Y) \setminus D X))$ for all $X, Y, U, V \subseteq D$;
10. $X \sim X = \sim X \subseteq M$;
11. If $a, b \in B$ and $a \setminus b \in B$ then $(a \mid b) = (a) \setminus D (b)$, where, (10) and (11) are valid if $A$ is an $\text{UL}_\omega$ (or $\text{IUL}_\omega$)-chain.

Proof. See [1] Section 5] and [2] Section 2].

**Lemma 3.7.** Let $A$ be an $\text{HpsUL}_\omega^*$-chain. Then $X . ^D Y \subseteq (e)$ iff $X . ^D Y . ^D Y \subseteq (e)$ for all $X, Y \subseteq D$.

Proof. Let $X . ^D Y \subseteq (e)$. Then $C(X Y) \subseteq (e)$. Thus $X Y \subseteq (e)$. Hence $x y \leq e$ for all $x \in X, y \in Y$. Let $x \in X, y, y' \in Y$ then $xy \leq e$ and $xy' \leq e$. Thus $yx \leq e$ by (Wcm). Then $y' y x \leq e$. Hence $y' y x \leq e$ by (Wcm). Thus $y' y x \leq e$ by Lemma 2.4. Therefore $y' y x \leq e$ by (Wcm). Thus $X Y Y \subseteq (e)$.

Then $C(X Y Y) \subseteq (e)$ by Lemma 3.6(2) and $C((e)) = (e)$. Hence $X . ^D Y . ^D Y \subseteq (e)$ by Lemma 3.6(6).

Let $X . ^D Y . ^D Y \subseteq (e)$. Then $C(X Y Y) \subseteq (e)$ by Lemma 3.6(6). Thus $X Y Y \subseteq (e)$. Let $x \in X, y \in Y$ then $xy \leq e$. Thus $xy \leq e$ by Lemma 2.4. Hence $X Y \subseteq (e)$. Therefore $C(X Y) \subseteq (e)$ by Lemma 3.6(2) and $C((e)) = (e)$. Then $X . ^D Y \subseteq (e)$.

**Lemma 3.8.** (i) $D = \{ D, \setminus D, \setminus D, ^D, \setminus D, \setminus D, \wedge D, e^D, f^D, \downarrow D, \uparrow D \}$ is an $\text{HpsUL}_\omega^*$-algebra if $A$ is an $\text{HpsUL}_\omega^*$-chain;
(ii) \( D = \{ D, D, \wedge_D, \vee_D, e_D, \perp_D, \top_D \} \) is an \( \textbf{UL}_\omega \)-algebra if \( A \) is an \( \textbf{UL}_\omega \)

chain;

(iii) \( D = \{ D, \vee_D, \wedge_D, \perp_D, \top_D \} \) is a complete lattice;

(iv) \( (\wedge_i D X_i) \wedge_D Y = \vee_i D (X_i \wedge_D Y) \) and \( (\vee_i D X_i) \wedge_D Y = \wedge_i D (X_i \wedge_D Y) \).

Proof. (i) and (ii) are immediate from Lemma 3.6(6),(7), (9) and Lemma 3.7. (iii) is clear. (iv) follows from (i), (ii) and (iii).

Lemma 3.9. Let \( A \) be a linearly ordered \( \textbf{IUL}_\omega \)-algebra and \( B \) be a partial subalgebra of \( A \) such that \( \{ e, f, \perp, \top \} \subseteq B \) and \( -b \in B \) for all \( b \in B \). Then

(i) \( \{ b \} = \sim \sim (b) \);

(ii) \( (a \mapsto b) = \sim \sim (a \mapsto b) \) for all \( (a \mapsto b) \in \bar{D} \);

(iii) \( X = \sim \sim X \) for all \( X \in D \);

(iv) \( D = \{ D, D, \to_D, \vee_D, \wedge_D, e_D, f_D, \perp_D, \top_D \} \) is an \( \textbf{IUL}_\omega \)-algebra.

Proof. (i) Let \( b \in B \) then \( \sim (b) = (\sim (b)) \wedge_D (f) = (b) \wedge_D (f) = (b) \) by Lemma 3.6(11) and \( b, f, -b \in B \). Thus \( \sim (b) = (b) \).

(ii) Let \( (a \mapsto b) \in \bar{D} \). Then \( (a \mapsto b) = \{ a \} \wedge_D (b) = \{ a \} \wedge_D \sim (b) = \{ a \} \wedge_D (\sim (b) \wedge_D (f)) = (\sim (b) \to_D \{ a \}) \wedge_D (f) = \sim \sim (\sim (b) \to_D \{ a \}) \) by (i) and Lemma 3.6(8). Thus \( (a \mapsto b) = \sim \sim (\sim (b) \to_D \{ a \}) = \sim \sim (\sim (b) \wedge_D \{ a \}) = (a \mapsto b) \).

(iii) Let \( X \in D \). Then we can write \( X = \bigcap_i (a_i \wedge_i b_i) = \bigwedge_i D (a_i \mapsto b_i) \). Then

\[
\sim \sim X = (X \wedge_D (f)) \wedge_D (f) = (\bigwedge_i D (a_i \mapsto b_i) \wedge_D (f)) \wedge_D (f) = (\bigvee_i D (a_i \mapsto b_i) \wedge_D (f)) \wedge_D (f) = \bigwedge_i D (a_i \mapsto b_i) \wedge_D (f) = \bigwedge_i D (a_i \mapsto b_i) = X.
\]

by (ii) and Lemma 3.8(iv).

(iv) is immediate from (iii) and Lemma 3.8(ii).

Lemma 3.10. The map \( \iota : B \to D \), which sends \( a \) to \( (a) = \{ x \in M : x \leq a \} \) for \( a \in B \), is an partial embedding of the partial subalgebra \( B \) of \( A \) into \( D \). Moreover, \( \iota(e) = e_D \), \( \iota(f) = f_D \), \( \iota(\perp) = \perp_D \), \( \iota(\top) = \top_D \) and \( \iota \) preserves all meets and joins that exist in \( B \).

Proof. It is proved by a procedure similar to that of [21, Lemma 2.6].

4. Finite embeddability property and decidability

In this section we show that \( \textbf{UL}_\omega \) and \( \textbf{IUL}_\omega \) have the finite embeddability property and are hence decidable.
We sometimes write $p_1 \cdots p_k$ by $\prod_{i=1}^{k} p_i$ for simplicity. We denote the $i$-th component of $\alpha = (m_1, \ldots, m_k) \in \mathbb{N}^k$ by $\alpha(i)$, i.e., $\alpha(i) = m_i$ for all $1 \leq i \leq k$.

**Definition 4.1.** A subsequence index is a mapping $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\alpha \leq \sigma(n) < \sigma(n + 1)$ for all $n$ in $\mathbb{N}$. 

**Remark 4.2.** There is a correspondence between the set of subsequences of a sequence and the set of subsequence indexes, i.e., (i) Let $\{ \alpha_n \}$ be a sequence and $\sigma$ be a subsequence index then $\{ \alpha_{\sigma(n)} \}$ is a subsequence of $\{ \alpha_n \}$; (ii) There is a subsequence index $\sigma$ for each subsequence $\{ \alpha_n \}$ of $\{ \alpha_n \}$ such that $\sigma(l) = n_i$ for all $l$ in $\mathbb{N}$.

**Definition 4.3.** Let $k \in \mathbb{N}$ and $\{ \alpha_n \}$ be a sequence in $\mathbb{N}^k$. $\{ \alpha_n \}$ is an $\Omega$-sequence if $\{ \alpha_n(i) \}$ is an infinite constant chain or an infinite strictly ascending chain for all $1 \leq i \leq k$. A subsequence index $\sigma$ is an $\Omega$-subsequence index of $\{ \alpha_n \}$ if $\{ \alpha_{\sigma(n)} \}$ is an $\Omega$-subsequence of $\{ \alpha_n \}$.

**Lemma 4.4.** (i) Let $\{ \alpha_n \}$ be a sequence in $\mathbb{N}$ then there exists a subsequence index $\sigma$ such that $\{ \alpha_{\sigma(n)} \}$ is an $\Omega$-subsequence of $\{ \alpha_n \}$; (ii) The composition $\sigma_1 \circ \sigma_2$ of two subsequence indexes $\sigma_1$ and $\sigma_2$ of $\{ \alpha_n \}$ is a subsequence index of $\{ \alpha_n \}$; (iii) $\sigma_2 \circ \sigma_1$ is an $\Omega$-subsequence index of $\{ \alpha_n \}$ if $\sigma_2$ is an $\Omega$-subsequence index of $\{ \alpha_n \}$ and $\sigma_1$ is a subsequence index of $\{ \alpha_n \}$.

**Proof.** (i) If $\{ \alpha_n \}$ is bounded, it contains an infinite constant subsequence. Otherwise it contains an infinite strictly ascending subsequence. Then it contains an $\Omega$-subsequence.

(ii) That is to say, the subsequence of any subsequence of $\{ \alpha_n \}$ is a subsequence of $\{ \alpha_n \}$.

(iii) That is to say, the subsequence $\{ \alpha_{\sigma_2 \circ \sigma_1(n)} \}$ of the $\Omega$-subsequence $\{ \alpha_{\sigma_2(n)} \}$ is an $\Omega$-subsequence.

\[ \square \]

**Lemma 4.5.** Let $\{ \alpha_n \}$ be a sequence in $\mathbb{N}^k$. Then there exists a subsequence index $\sigma$ such that $\{ \alpha_{\sigma(n)} \}$ is an $\Omega$-subsequence of $\{ \alpha_n \}$.

**Proof.** Since $\{ \alpha_n(1) \}$ is a sequence in $\mathbb{N}$, then, by Lemma 4.4 (i), there exists a subsequence index $\sigma_1$ such that $\{ \alpha_{\sigma_1(n)}(1) \}$ is an $\Omega$-subsequence of $\{ \alpha_n(1) \}$. Thus $\{ \alpha_{\sigma_1(n)} \}$ is a subsequence of $\{ \alpha_n \}$ such that $\{ \alpha_{\sigma_1(n)}(1) \}$ is an $\Omega$-subsequence of $\{ \alpha_n(1) \}$. Note that $\{ \alpha_{\sigma_1(n)} \}$ is also a sequence in $\mathbb{N}^k$.

Similarly, $\{ \alpha_{\sigma_1(n)}(2) \}$ is a sequence in $\mathbb{N}$, then, by Lemma 4.4 (i), there exists a subsequence index $\sigma_2$ such that $\{ \alpha_{\sigma_1(\sigma_2(n))}(2) \}$ is an $\Omega$-subsequence of $\{ \alpha_{\sigma_1(n)}(2) \}$. Since $\{ \alpha_{\sigma_1(\sigma_2(n))}(1) \}$ is a subsequence of $\{ \alpha_{\sigma_1(n)}(1) \}$, then it is also an $\Omega$-subsequence of $\{ \alpha_n(1) \}$ by Lemma 4.4 (iii).

Sequentially, we construct subsequence indexes $\sigma_1, \sigma_2, \ldots, \sigma_k$ such that $\{ \alpha_{\sigma_1(n)}(1) \}, \{ \alpha_{\sigma_1 \circ \sigma_2(n)}(2) \}, \{ \alpha_{\sigma_1 \circ \cdots \circ \sigma_k(n)}(k) \}$ are $\Omega$-subsequences by Lemma 4.4 (i) and (iii). Let $\sigma = \sigma_1 \circ \cdots \circ \sigma_k$ then $\{ \alpha_{\sigma(n)} \}$ is an $\Omega$-subsequence of $\{ \alpha_n \}$ by Lemma 4.4 (iii). \[ \square \]
By Lemma 4.5, there is a subsequence index \( \sigma \) of or \( n \) such that there is a sequence \( A. \) Proof. (i) follows directly from the fact that these elements of \( D \) are all down-sets of \( A. \)

(ii) Suppose that there is an infinite strictly ascending sequence

\[
\left\{ \left( \prod_{i=1}^{k} p_i^{\beta_n(i)} \rightarrow p \right) \right\} \text{ in } M \Rightarrow p \text{ under set inclusion, i.e., } \beta_n \in \mathbb{N}^k \text{ and }
\]

\[
\left( \prod_{i=1}^{k} p_i^{\beta_n(i)} \rightarrow p \right) \subset \left( \prod_{i=1}^{k} p_i^{\beta_{n+1}(i)} \rightarrow p \right)
\]

for all \( n \in \mathbb{Z}_+. \) By Lemma 4.5 there is a subsequence index \( \tau \) such that \( \{ \beta_{\tau(n)} \} \) is a \( \Omega \)-subsequence of \( \{ \beta_n \}. \) Then

\[
\left( \prod_{i=1}^{k} p_i^{\beta_{\tau(n)}(i)} \rightarrow p \right) \subset \left( \prod_{i=1}^{k} p_i^{\beta_{\tau(n)+1}(i)} \rightarrow p \right)
\]

for all \( n \in \mathbb{Z}_+ \), where for each \( 1 \leq i \leq k, \)

\[
\beta_{\tau(1)}(i) = \beta_{\tau(2)}(i) = \cdots = \beta_{\tau(n)}(i) = \cdots
\]

or

\[
\beta_{\tau(1)}(i) < \beta_{\tau(2)}(i) < \cdots < \beta_{\tau(n)}(i) < \cdots. \quad (4.1)
\]

Thus there is a sequence \( \{ \alpha_n \} \) in \( \mathbb{N}^k \) such that

\[
\prod_{i=1}^{k} p_i^{\alpha_n(i)} \in \left( \prod_{i=1}^{k} p_i^{\beta_{\tau(n+1)}(i)} \rightarrow p \right) \text{ and } \prod_{i=1}^{k} p_i^{\alpha_n(i)} \notin \left( \prod_{i=1}^{k} p_i^{\beta_{\tau(n)}(i)} \rightarrow p \right).
\]

Thus for all \( n \) in \( \mathbb{Z}_+ \),

\[
\prod_{i=1}^{k} p_i^{\beta_{\tau(n)+1}(i)} \prod_{i=1}^{k} p_i^{\alpha_n(i)} \leq p < \prod_{i=1}^{k} p_i^{\beta_{\tau(n)}(i)} \prod_{i=1}^{k} p_i^{\alpha_n(i)}.
\]

By Lemma 4.5 there is a subsequence index \( \sigma \) such that \( \{ \alpha_{\sigma(n)} \} \) is a \( \Omega \)-subsequence of \( \{ \alpha_n \}. \) Then for each \( 1 \leq i \leq k, \)

\[
\alpha_{\sigma(1)}(i) = \alpha_{\sigma(2)}(i) = \cdots = \alpha_{\sigma(n)}(i) = \cdots
\]

or

\[
\alpha_{\sigma(1)}(i) < \alpha_{\sigma(2)}(i) < \cdots < \alpha_{\sigma(n)}(i) < \cdots. \quad (4.2)
\]

Then for all \( n \) in \( \mathbb{Z}_+ \),

\[
\prod_{i=1}^{k} p_i^{\beta_{\sigma(n)+1}(i)} \prod_{i=1}^{k} p_i^{\alpha_{\sigma(n)}(i)} \leq p < \prod_{i=1}^{k} p_i^{\beta_{\sigma(n)}(i)} \prod_{i=1}^{k} p_i^{\alpha_{\sigma(n)}(i)}. \quad (4.3)
\]

Then by letting \( n = 1, 3 \) in (4.3),

\[
\prod_{i=1}^{k} p_i^{\beta_{\sigma(1)+1}(i)} \prod_{i=1}^{k} p_i^{\alpha_{\sigma(1)}(i)} \leq p < \prod_{i=1}^{k} p_i^{\beta_{\sigma(1)}(i)} \prod_{i=1}^{k} p_i^{\alpha_{\sigma(1)}(i)},
\]
\[
\prod_{i=1}^{k} p_i^{\beta_{\tau(\sigma(3)+1)}(i)} \prod_{i=1}^{k} p_i^{\alpha_{\sigma(3)}(i)} \leq p < \prod_{i=1}^{k} p_i^{\beta_{\tau(\sigma(3))}(i)} \prod_{i=1}^{k} p_i^{\alpha_{\sigma(3)}(i)}.
\]
Thus
\[
\prod_{i=1}^{k} p_i^{\beta_{\tau(\sigma(1)+1)}(i)} \prod_{i=1}^{k} p_i^{\alpha_{\sigma(1)}(i)} < \prod_{i=1}^{k} p_i^{\beta_{\tau(\sigma(3))}(i)} \prod_{i=1}^{k} p_i^{\alpha_{\sigma(3)}(i)}
\]
and
\[
\prod_{i=1}^{k} p_i^{\beta_{\tau(\sigma(3)+1)}(i)} \prod_{i=1}^{k} p_i^{\alpha_{\sigma(3)}(i)} < \prod_{i=1}^{k} p_i^{\beta_{\tau(\sigma(3))}(i)} \prod_{i=1}^{k} p_i^{\alpha_{\sigma(1)}(i)}. \tag{4.4}
\]
Since
\[
\sigma(1) < \sigma(1) + 1 \leq \sigma(2) < \sigma(3) \quad \text{and} \quad \sigma(1) < \sigma(3) + 1,
\]
then by (4.1) and (4.2), for all \(1 \leq i \leq k\),
\[
\beta_{\tau(\sigma(3))}(i) - \beta_{\tau(\sigma(1)+1)}(i) \geq 0, \tag{4.5}
\]
\[
\beta_{\tau(\sigma(3)+1)}(i) - \beta_{\tau(\sigma(1))}(i) \geq 0, \tag{4.6}
\]
\[
\alpha_{\sigma(3)}(i) - \alpha_{\sigma(1)}(i) \geq 0. \tag{4.7}
\]
Hence
\[
e < \prod_{i=1}^{k} p_i^{\beta_{\tau(\sigma(3))}(i)-\beta_{\tau(\sigma(1)+1)}(i)} \prod_{i=1}^{k} p_i^{\alpha_{\sigma(3)}(i)-\alpha_{\sigma(1)}(i)}
\]
and
\[
\prod_{i=1}^{k} p_i^{\beta_{\tau(\sigma(3)+1)}(i)-\beta_{\tau(\sigma(1))}(i)} \prod_{i=1}^{k} p_i^{\alpha_{\sigma(3)}(i)-\alpha_{\sigma(1)}(i)} < e \tag{4.8}
\]
by (4.4) and Lemma 2.5(ii).
On the other hand, for all \(1 \leq i \leq k\),
\[
\beta_{\tau(\sigma(3))}(i) - \beta_{\tau(\sigma(1)+1)}(i) > 0
\]
iff
\[
\beta_{\tau(\sigma(3)+1)}(i) - \beta_{\tau(\sigma(1))}(i) > 0 \tag{4.9}
\]
by (4.1) and \(\sigma(1) < \sigma(1) + 1 < \sigma(3) < \sigma(3) + 1\).
Then
\[
\beta_{\tau(\sigma(3))}(i) - \beta_{\tau(\sigma(1)+1)}(i) + \alpha_{\sigma(3)}(i) - \alpha_{\sigma(1)}(i) > 0
\]
iff
\[
\beta_{\tau(\sigma(3)+1)}(i) - \beta_{\tau(\sigma(1))}(i) + \alpha_{\sigma(3)}(i) - \alpha_{\sigma(1)}(i) > 0. \tag{4.10}
\]
The necessity part of (4.10) is proved as follows and, the sufficiency part is by a similar procedure and omitted.
Let
\[
\beta_{\tau(\sigma(3))}(i) - \beta_{\tau(\sigma(1)+1)}(i) + \alpha_{\sigma(3)}(i) - \alpha_{\sigma(1)}(i) > 0.
\]
Then by (4.5) and (4.7),
\[
\beta_{\tau(\sigma(3))}(i) - \beta_{\tau(\sigma(1)+1)}(i) > 0
\]
or
\[
\alpha_{\sigma(3)}(i) - \alpha_{\sigma(1)}(i) > 0.
\]
If $\beta_{\tau(3)}(i) - \beta_{\tau(1)(1)+1}(i) > 0$ then by (4.9),
$$\beta_{\tau(3)(1)+1}(i) - \beta_{\tau(1)(1)}(i) > 0$$
and thus by (4.7),
$$\beta_{\tau(3)(1)+1}(i) - \beta_{\tau(1)(1)}(i) + \alpha_{\sigma(3)}(i) - \alpha_{\sigma(1)}(i) > 0.$$ If $\alpha_{\sigma(3)}(i) - \alpha_{\sigma(1)}(i) > 0$ then by (4.6),
$$\beta_{\tau(3)(1)+1}(i) - \beta_{\tau(1)(1)}(i) + \alpha_{\sigma(3)}(i) - \alpha_{\sigma(1)}(i) > 0.$$ Hence
$$\prod_{i=1}^{k} p_i^{\beta_{\tau(\sigma(3))}(i)-\beta_{\tau(\sigma(1)+1)}(i)} \prod_{i=1}^{k} p_i^\alpha_{\sigma(3)}(i)-\alpha_{\sigma(1)}(i) > e$$
iff
$$\prod_{i=1}^{k} p_i^{\beta_{\tau(\sigma(3))}(i)-\beta_{\tau(\sigma(1))}(i)} \prod_{i=1}^{k} p_i^\alpha_{\sigma(3)}(i)-\alpha_{\sigma(1)}(i) > e$$
by (4.10) and Lemma 2.4(ii), which contradicts with (4.8). Hence there is no infinite strictly ascending sequence in $M \Rightarrow p$. Similarly, we can prove that there is no infinite strictly descending sequence in $M \Rightarrow p$. Thus $M \Rightarrow p$ is finite by (i).

Lemma 4.8. If $B$ is a finite partial subalgebra of $A$ then the algebra $D$ is finite.

Proof. It is immediate from Lemma 4.7.

Theorem 4.9. The varieties of $\text{UL}_{\omega}$-algebras and $\text{IUL}_{\omega}$-algebras have the FEP.

Proof. It is immediate from Lemmas $3.8\sim 3.10$, Lemma 4.8.

Corollary 4.10. The universal theories of $\text{UL}_{\omega}$-algebras and $\text{IUL}_{\omega}$-algebras are decidable.

Theorem 4.11. Let $L \in \{\text{UL, IUL}\}$ and $L_{\omega} \in \{\text{UL}_{\omega}, \text{IUL}_{\omega}\}$. For any formula $\varphi$ in $L$, the following statements are equivalent:

(i) $\Gamma \vdash_{L_{\omega}} \varphi$;
(ii) $\Gamma \models_{A} \varphi$ for every $L_{\omega}$-algebra $A$;
(iii) $\Gamma \models_{A} \varphi$ for every $L_{\omega}$-chain $A$;
(iv) $\Gamma \models_{A} \varphi$ for every finite $L_{\omega}$-algebra $A$;
(v) $\Gamma \models_{A} \varphi$ for every finite $L$-algebra $A$.

Proof. (i) is equivalent to (ii) by a canonical procedure. (iii) implies (ii) by Theorem 2.3 (i). Clearly, (ii) implies (iv). Then (iii) implies (iv). (iv) implies (iii) by Theorem 4.9 (iv) is equivalent to (v) by Lemma 2.6 (ii).

Theorem 4.11 shows that, as was expected, axiomatic systems $\text{UL}_{\omega}$ and $\text{IUL}_{\omega}$ are complete with respect to finite $\text{UL}$ and $\text{IUL}$-algebras, respectively. In other words, $\text{UL}_{\omega}$ and $\text{IUL}_{\omega}$ are logics for the classes of finite $\text{UL}$ and $\text{IUL}$-algebras, respectively.
5. Concluding remarks

The suitability of Blok and Alten’s Construction for $\textbf{UL}_\omega$, $\textbf{IUL}_\omega$-algebras mainly depends on that elements of the monoid $M$ generated by $\{p_1, \ldots, p_k\}$ has the form $\prod_{i=1}^{k} p_i^{a(i)}$. It seems difficult to extend the proof of Lemma 4.7 to $\textbf{HpsUL}_\omega^*$-algebras.

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