Equation of state and phase transition in AdS/QCD with dynamical background

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Abstract

We construct an improved soft-wall AdS/QCD model with a cubic coupling term of the dilaton and the bulk scalar field. The background fields in this model are solved from the Einstein-dilaton system with a nontrivial dilaton potential, which is shown to be able to reproduce the equation of state from lattice QCD with two flavors. The chiral transition behaviors are investigated in two typical cases of this model. We find that the crossover behavior of chiral transition can be realized for both cases in such an improved soft-wall AdS/QCD model with dynamical background. Our study provides a possibility to address the deconfining and chiral phase transitions simultaneously in the bottom-up holographic framework.
I. INTRODUCTION

As well known, quantum chromodynamics (QCD) is the fundamental theory of strong interaction. Thanks to asymptotic freedom, in the ultra-violet (UV) region, we can make use of the perturbative QCD to study various high-energy physical processes related to strong interaction. However, the strong coupling character in the infrared (IR) region makes the perturbative method invalid to look into the low-energy nonperturbative properties of QCD. The physics on quark confinement and chiral symmetry breaking, which are two essential features of low-energy QCD, has been attracting a great deal of interest since many years ago, and QCD phase transition is just a good and proper subject to study these low-energy properties of QCD [1]. With the increase of temperature, the QCD matters will undergo a crossover transition from the hadronic state to the quark-gluon plasma (QGP) state, along with the deconfining process of the partonic degrees of freedom and the restoration of chiral symmetry [2–4].

Many nonperturbative methods have been developed to study the QCD phase transition and the issues of low-energy hadron physics [5–8]. As a powerful method, lattice QCD has been widely used to tackle the low-energy QCD problems from the first principle. However, there still exist limitations for this method, e.g., in the case of nonzero chemical potential. In recent decades, the anti-de Sitter/conformal field theory (AdS/CFT) correspondence generates great interest in the study of low-energy QCD by using the conjectured duality between a weakly coupled gravity theory in asymptotic AdS_{5} spacetime and a strongly coupled gauge field theory on the boundary [9–11]. Large amounts of research works have been done in this field, following either the top-down approach or the bottom-up approach [12–56].

Holographic studies in the top-down approach have shown that the simplest nonsupersymmetric deformation of AdS/CFT with nontrivial dilaton profiles can reproduce the confining properties of QCD [57, 58], and also realize the pattern of chiral symmetry breaking with quarks introduced by the D7-brane probes [59–63]. However, it is not clear how to realize the crossover transition indicated from the lattice results in the top-down framework. Actually, AdS/CFT per se is inadequate to provide a complete characterization for the QCD thermodynamics because it is a semi-classical method that demoted from the type IIB string theory in the low-energy approximation and the large N limit. The string-loop corrections must be incorporated in order to give an adequate account for thermal QCD. Nevertheless, we assume that the main features of these issues considered here can be captured by this semi-classical gauge/gravity duality.

As we know, the deconfinement in the pure gauge theory corresponds to a Hawking-Page type phase transition between a thermal AdS space and a black hole configuration in AdS/CFT [64–66]. However, many works from the bottom-up approach have shown that one can use a bulk gravity system with nontrivial dilaton profiles to characterize the equation of state and the deconfining behaviors of QCD [67–82], and moreover, unlike that in the pure gauge theory, the crossover transition in these bottom-up models is only related to the black hole solution solved from the Einstein-dilaton(-Maxwell) system, which seems contrary to the usual wisdom that the black-hole solution is dual to the deconfined phase at high temperature. As we cannot expect to make use of two distinct bulk geometries to
generate a smooth crossover transition in AdS/QCD [83], it seems that the only possible way out of this dilemma is to assume the black hole description to be still valid around the intermediate transition region [67]. We will take this viewpoint in this work, with the caveat that this is only a tentative treatment in the phenomenological sense.

In the bottom-up approach, the soft-wall AdS/QCD model provides a concise framework to address the issues on chiral transition [18]. However, it has been shown that the original soft-wall model lacks spontaneous chiral symmetry breaking [18, 25]. The chiral transition for the two-flavor case has been considered in a modified soft-wall AdS/QCD model, where the second-order chiral phase transition in the chiral limit and the crossover transition with finite quark masses were first realized in the holographic framework [84, 85]. In Ref. [86], we proposed an improved soft-wall model which can generate both the correct chiral transition behaviors and the light meson spectra in a consistent way. The generalizations to the 2 + 1 flavor case have been considered in Ref. [87–89], and the quark-mass phase diagram consistent with the standard scenario can be reproduced. The case of finite chemical potential has also been investigated [90, 91], and the chiral phase diagram containing a critical end point can be obtained from the improved soft-wall AdS/QCD model with 2 + 1 flavors [91].

It should be noted that the AdS-Schwarzschild black hole has been used to characterize the thermodynamical properties of QCD at zero chemical potential in the above studies on chiral transition. However, such an AdS black hole solution is dual to a conformal gauge theory, which cannot generate the QCD equation of state without breaking the conformal invariance [67]. As stated above, one resorts to the Einstein-dilaton system with a nontrivial dilaton profile to realize this aim. Thus, we may wonder whether the correct chiral transition behaviors can still be attained from a soft-wall model with dynamical background solved from the Einstein-dilaton system. In this work, we will consider this issue and try to combine the description of chiral transition with that of the equation of state that signifies the deconfinement in a unified holographic framework.

This paper is organized as follows. In Sec. II, we consider an Einstein-dilaton system with a nontrivial dilaton potential, which can generate the equation of state matching with the lattice results in the two-flavor case. The vacuum expectation value (VEV) of the Polyakov loop will also be computed in this background system, and will be compared with the lattice data. In Sec. III, we propose an improved soft-wall AdS/QCD model with a cubic coupling term of the dilaton and the bulk scalar field and the dynamical background considered in the last section. We remark that only the two-flavor case will be addressed in our work. As will be seen, the crossover transition of the chiral condensate with the temperature $T$ can be realized in this model, and the parameter dependence of chiral transition will be investigated. In Sec. IV we give a brief summary of our work and conclude with some remarks.

II. QCD EQUATION OF STATE FROM HOLOGRAPHY

A. The Einstein-dilaton system

In the previous works, we proposed an improved soft-wall AdS/QCD model with a running bulk scalar mass $m_5^2(z)$, which gives quite a good characterization for the chiral transition in both the two-flavor and the 2 + 1 flavor case [86, 89]. However, the AdS-Schwarzschild black hole presumed in this model cannot describe the thermodynamical behaviors of QCD
equation of state and other equilibrium quantities which show obvious violation of conformal invariance [67]. In order to acquire these basic features of thermal QCD, we need to construct a proper gravity background other than the AdS-type black hole to break the conformal invariance of the dual gauge theory. The minimal action of such a background system is given in the string frame as

\[ S_g = \frac{1}{2\kappa^2} \int d^5x \sqrt{-g} e^{-2\phi} \left[ R + 4(\partial \phi)^2 - V(\phi) \right], \]  

(1)

where \( \kappa^2 = 8\pi G_5 \), and a dilaton field \( \phi \) has been introduced to produce relevant deformations of the dual conformal field theory. The dilaton \( \phi(z) \) is assumed to depend only on the radial coordinate \( z \). The key point of this model is to find a particular form of the dilaton potential \( V(\phi) \) with necessary ingredients to describe the QCD thermodynamics, such as the equation of state.

The metric of the bulk geometry in the string frame can be written as

\[ ds^2 = \frac{L^2 e^{2A_s(z)}}{z^2} \left( -f(z)dt^2 + dx^i dx^i + \frac{dz^2}{f(z)} \right) \]  

(2)

with the asymptotic structure of AdS\(_5\) spacetime at \( z \to 0 \) to guarantee the UV conformal behavior of the dual gauge theory on the boundary. We take the AdS radius \( L = 1 \) for convenience. To simplify the calculation, we will work in the Einstein frame with the metric ansatz

\[ ds^2 = \frac{L^2 e^{2A_E(z)}}{z^2} \left( -f(z)dt^2 + dx^i dx^i + \frac{dz^2}{f(z)} \right). \]  

(3)

The warp factors in the two frames are related by \( A_s = A_E + \frac{2}{3}\phi \), in terms of which the background action in the Einstein frame can be obtained from the string-frame action (1) as

\[ S_g = \frac{1}{2\kappa^2} \int d^5x \sqrt{-g_E} \left[ R_E - \frac{4}{3}(\partial \phi)^2 - V_E(\phi) \right] \]  

(4)

with \( V_E(\phi) \equiv e^{4\phi} V(\phi) \) (the subscript \( E \) denotes the Einstein frame).

**B. The EOM with a nontrivial dilaton potential**

The independent Einstein equations can be derived by the variation of the action (4) with respect to the metric \( g_{MN} \),

\[ f'' + 3A'_E f' - \frac{3}{z} f' = 0, \]  

(5)

\[ A''_E + \frac{2}{z} A'_E - A'^2_E + \frac{4}{9} \phi'^2 = 0. \]  

(6)

The equation of motion (EOM) of the dilaton \( \phi \) in the Einstein frame can also be derived as

\[ \phi'' + \left( 3A'_E + \frac{f'}{f} - \frac{3}{z} \right) \phi' - \frac{3 e^{2A_E} \partial_\phi V_E(\phi)}{8z^2 f} = 0. \]  

(7)
Given the dilaton potential $V_E(\phi)$, the numerical solution of the background fields $A_E$, $f$ and $\phi$ can be solved from the coupled differential equations (5), (6) and (7).

Although there are few constraints on the form of the dilaton potential from the top-down approach of AdS/QCD, it has been shown that a proper $V_E(\phi)$ can be constructed from bottom up to describe the equation of state of the strongly coupled QGP [67, 68]. Near the boundary, the bulk geometry should approach the AdS$_5$ spacetime that corresponds to a UV fixed point of the dual gauge theory. This requires that the dilaton potential at UV has the following asymptotic form:

$$V_c(\phi_c \to 0) \simeq -\frac{12}{L^2} + \frac{1}{2} m^2 \phi_c^2 + \mathcal{O}(\phi_c^4)$$

with the rescaled dilaton field defined by $\phi_c = \sqrt{\frac{8}{3}} \phi$, in terms of which the action (4) can be recast into the canonical form

$$S_g = \frac{1}{2 \kappa_5^2} \int d^5x \sqrt{-g_E} \left[ R_E - \frac{1}{2} (\partial \phi_c)^2 - V_c(\phi_c) \right]$$

with $V_c(\phi_c) = V_E(\phi)$. As argued in Ref. [68], the dilaton potential at IR takes an exponential form $V_c(\phi_c) \sim V_0 e^{\gamma \phi_c}$ with $V_0 < 0$ and $\gamma > 0$ in order to yield the Chamblin-Reall solution, whose adiabatic generalization links the QCD equation of state to the specific form of $V_c(\phi_c)$.

According to AdS/CFT, the mass squared of $\phi_c$ is related to the scaling dimension $\Delta$ of the dual operator on the boundary by $m^2 L^2 = \Delta(\Delta - 4)$ [16]. We only consider the case of $2 < \Delta < 4$, which corresponds to the relevant deformations satisfying the Breitenlohner-Freedman (BF) bound [67, 71]. It is usually assumed that the dilaton field $\phi_c$ is dual to the gauge-invariant dimension-4 glueball operator $\text{tr} F_{\mu\nu}^2$, yet other possibilities such as a dimension-2 gluon mass operator have also been considered [28]. In view of the fact that the supergravity approximation might be invalid in the UV region, it is natural for us to match QCD at some intermediate semi-hard scale, where the scaling dimension of $\text{tr} F_{\mu\nu}^2$ would have a smaller value than 4 [73]. In this work, we will take $\Delta = 3$, which has been shown to be able to mimick the equation of state from lattice QCD with 2 + 1 flavors [72, 73]. Following the studies in Ref. [67], we choose a relatively simple dilaton potential which satisfies the required UV and IR asymptotics,

$$V_c(\phi_c) = \frac{1}{L^2} \left(-12 \cosh \gamma \phi_c + b_2 \phi_c^2 + b_4 \phi_c^4\right),$$

where $\gamma$ and $b_2$ are constrained by the UV asymptotic form (8) as

$$b_2 = 6 \gamma^2 + \frac{\Delta(\Delta - 4)}{2} = 6 \gamma^2 - \frac{3}{2}.$$  

The dilaton potential $V_E(\phi)$ has the form

$$V_E(\phi) = V_c(\phi_c) = V_c(\sqrt{8/3} \phi).$$

We will see that the Einstein-dilaton system given above can also be used to mimick the two-flavor lattice results of the QCD equation of state, by which the dilaton potential $V_E(\phi)$ and the background geometry can be reconstructed for the two-flavor case.
C. Equation of state

Now we come to the equation of state in the Einstein-dilaton system with the given form of dilaton potential (10). First note that the background geometry has an event horizon at \( z = z_h \) which is determined by \( f(z_h) = 0 \). In terms of the metric ansatz (3), the Hawking temperature \( T \) of the black hole is given by

\[
T = \frac{|f'(z_h)|}{4\pi},
\]

and the entropy density \( s \) is related to the area of the horizon,

\[
s = \frac{e^{3A_E(z_h)}}{4G_5 z_h^3}.
\]

Thus we can compute the speed of sound \( c_s \) by the formula

\[
c_s^2 = \frac{d\log T}{d\log s}.
\]

Moreover, the pressure \( p \) can be obtained from the thermodynamic relation \( s = \frac{\partial p}{\partial T} \) as

\[
p = -\int_{\infty}^{z_h} s(\bar{z}_h) T'(\bar{z}_h)d\bar{z}_h.
\]

The energy density \( \varepsilon = -p + sT \) and the trace anomaly \( \varepsilon - 3p \) can also be computed. Then we can study the temperature dependence of the equation of state in such an Einstein-dilaton system. As we constrain ourselves to the two-flavor case, the equation of state from lattice QCD with two flavors will be used to construct the dilaton potential \( V_{E}(\phi) \).

Instead of implementing the numerical procedure elucidated in Ref. [68], we will directly solve the background fields from Eqs. (5), (6) and (7). To simplify the computation, note that Eq. (5) can be integrated into a first-order differential equation

\[
f' + f_c e^{-3A_E} z^3 = 0,
\]

where \( f_c \) is the integral constant. In view of \( \Delta = 3 \), the UV asymptotic forms of the background fields at \( z \to 0 \) can be obtained as

\[
A_E(z) = -\frac{2p_1^2}{27} z^2 + \cdots,
\]

\[
f(z) = 1 - \frac{f_c}{4} z^4 + \cdots,
\]

\[
\phi(z) = p_1 z + p_3 z^3 + \frac{4p_1^3}{9} \left(12b_4 - 6\gamma^4 + 1\right) z^3 \log z + \cdots
\]

with three independent parameters \( p_1, p_3 \) and \( f_c \). As we have \( f(z_h) = 0 \), to guarantee the regular behavior of \( \phi(z) \) near the horizon, Eq. (7) must satisfy a natural boundary condition at \( z = z_h \),

\[
\left[ f' \phi' - \frac{3e^{2A_E}}{8z^2} \partial_\phi V_E(\phi) \right]_{z=z_h} = 0.
\]
FIG. 1. The profile of the temperature $T$ as a function of $z_h$ with the given parameter values.

![Temperature Profile](image1.png)

FIG. 2. The model results of the entropy density $s/T^3$ (left panel) and the speed of sound squared $c_s^2$ (right panel) as functions of $T$ obtained from the Einstein-dilaton system.

![Entropy and Speed of Sound](image2.png)

With the UV asymptotic form (18) and the IR boundary condition (19), the background fields $f$, $A_E$ and $\phi$ can be solved numerically from Eqs. (6), (7) and (17). We find that the dilaton potential (10) with $\gamma = 0.55$, $b_2 = 0.315$ and $b_4 = -0.125$ can well reproduce the two-flavor lattice QCD results of the equation of state. Note that $\gamma$ and $b_2$ are related by the formula (11). The parameter $p_1 = 0.675$ is also fitted by the lattice results, and the 5D Newton constant is just taken as $G_5 = 1$ in our consideration. We show the profile of the temperature $T$ as a function of $z_h$ in Fig. 1. The temperature dependences of the entropy density $s/T^3$ and the speed of sound squared $c_s^2$ are shown in Fig. 2, while in Fig. 3 we compare the numerical results of the pressure $3p/T^4$ and the energy density $\varepsilon/T^4$ in units of $T^4$ with the lattice interpolation results for the B-mass ensemble considered in Ref. [92]. In Fig. 4, we present the model result of the trace anomaly $(\varepsilon - 3p)/T^4$, which is also compared with the lattice interpolation result.

We can see that the Einstein-dilaton system with a nontrivial dilaton potential can generate the crossover behavior of the equation of state that matches up well with the lattice results. The crossover transition requires that the black hole solution of the gravity system should be the most stable one in the temperature range we concerned, which is different from the holographic picture of the pure gauge theory with a first-order phase transition emerging from the competition of the thermal gas solution and the black hole solution.
FIG. 3. The model results of the pressure $3p/T^4$ (left panel) and the energy density $\varepsilon/T^4$ (right panel) as functions of $T$ compared with the lattice interpolation results of two-flavor QCD denoted by the red band [92].

FIG. 4. The model result of the trace anomaly $(\varepsilon - 3p)/T^4$ as a function of $T$ compared with the lattice interpolation results of two-flavor QCD denoted by the red band [92].

D. Polyakov loop

The deconfining phase transition in thermal QCD is characterized by the VEV of the Polyakov loop which is defined as

$$ L(T) = \frac{1}{N_c} \text{tr} P \exp \left[ ig \int_0^{1/T} d\tau \hat{A}_0 \right], \quad (20) $$

where $\hat{A}_0$ is the time component of the non-Abelian gauge field operator, the symbol $P$ denotes path ordering and the trace is over the fundamental representation of $SU(N_c)$.

The VEV of the Polyakov loop in AdS/CFT is schematically given by the world-sheet path integral

$$ \langle L \rangle = \int DX e^{-S_w}, \quad (21) $$

where $X$ is a set of world-sheet fields and $S_w$ is the classical world-sheet action [70, 71]. In principle, $\langle L \rangle$ can be evaluated approximately in terms of the minimal surface of the string world-sheet with given boundary conditions. In the low-energy and large $N_c$ limit, we have $\langle L \rangle \sim e^{-S_{NG}}$ with the Nambu-Goto action

$$ S_{NG} = \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{\det(g_{\mu\nu}^S \partial_a X^\mu \partial_b X^\nu)}, \quad (22) $$
FIG. 5. The model result of the VEV of Polyakov loop $\langle L \rangle$ as a function of $T$ compared with the lattice data of the renormalized Polyakov loop for the B-mass ensemble denoted by the colored points with error bars \[92\].

where $\alpha'$ denotes the string tension, $g_{\mu \nu}^S$ is the string-frame metric and $X^\mu = X^\mu(\tau, \sigma)$ is the embedding of the world-sheet in the bulk spacetime. The regularized minimal world-sheet area takes the form

$$S_R = \frac{g_p}{\pi T} \int^\infty \epsilon d\epsilon \frac{e^{2A_S}}{z^2}$$

with $g_p = \frac{L^2}{2\alpha'}$ \[70\]. Subtracting the UV divergent terms and letting $\epsilon \to 0$, the renormalized world-sheet area can be obtained as

$$S_0 = S'_0 + c_p = \frac{g_p}{\pi T} \int^\infty d\epsilon \left[ \frac{e^{2A_S}}{z^2} - \left( \frac{1}{z^2} + \frac{4p_1}{3z} \right) \right] + \frac{g_p}{\pi T} \left( \frac{4p_1}{3} \log z_h - \frac{1}{z_h} \right) + c_p,$$

where $c_p$ is a scheme-dependent normalization constant. Thus the VEV of the Polyakov loop can be written as

$$\langle L \rangle = w e^{-S_0} = e^{-S'_0 + c'_p}$$

with a weight factor $w$ and the constant $c'_p = \ln w - c_p$.

By choosing the parameter values $g_p = 0.29$ and $c'_p = 0.16$, the formula (25) can be used to describe the two-flavor lattice data of the renormalized Polyakov loop for the B-mass ensemble \[92\], as shown in Fig. 5.

III. CHIRAL TRANSITION IN AN IMPROVED SOFT-WALL MODEL WITH DYNAMICAL BACKGROUND

Our previous studies have shown that the chiral transition at zero baryon chemical potential can be characterized by an improved soft-wall AdS/QCD model in the AdS-Schwarzschild black hole background \[86, 89\]. However, this black hole solution cannot describe the QCD equation of state due to the conformal invariance of the dual gauge theory. Our main aim of this work is to combine the advantages of the improved soft-wall model in the description of chiral transition with a background system which can reproduce the deconfinement properties of QCD. As a first attempt, we investigate the possible ways to realize the chiral transition behaviors in the two-flavor case based on an improved soft-wall model (as the flavor part) with the dynamical background solved from the above Einstein-dilaton system.
A. The flavor action

We first outline the improved soft-wall AdS/QCD model with two flavors which is proposed in Ref. [86]. The bulk action relevant to the chiral transition in this model is the scalar sector,

$$S_{\text{isw}} = - \int d^5x \sqrt{-g} e^{-\Phi} \text{Tr} \left\{ |\partial X|^2 + V_X(X) \right\},$$

(26)

where the dilaton takes the form $\Phi(z) = \mu_0^2 z^2$ to produce the linear Regge spectra of light mesons, and the scalar potential is

$$V_X(X) = m^2_5 z^2 |X|^2 + \lambda |X|^4$$

(27)

with a running bulk mass $m^2_5(z) = -3 - \mu_0^2 z^2$. The constant term of $m^2_5(z)$ is determined by the mass-dimension relation $m^2_5 L^2 = \Delta(\Delta - 4)$ for a bulk scalar field [15, 16], while the $z$-squared term is motivated by the phenomenology of meson spectrum and the quark mass anomalous dimension [86].

In the holographic framework, a natural mechanism to produce such a $z$-dependent term of $m^2_5(z)$ is to introduce a coupling between the dilaton and the bulk scalar field. As we can see, without changing the results of the improved soft-wall model, the scalar potential can be recast into another form

$$V_X(X, \Phi) = m^2_5 |X|^2 - \lambda_1 |X|^2 + \lambda_2 |X|^4,$$

(28)

where $m^2_5 = -3$, and a cubic coupling term of $\Phi$ and $X$ has been introduced. The effects of similar couplings on the low-energy hadron properties have also been considered in the previous studies [28]. Here we propose such a change of $V_X$ from (27) to (28) with the aim to describe the chiral transition behaviors for the two-flavor case. Thus the flavor action that will be addressed in this work is

$$S_X = - \int d^5x \sqrt{-g} e^{-\Phi} \text{Tr} \left\{ |\partial X|^2 + V_X(X, \Phi) \right\}.$$  

(29)

Unlike the previous studies, the metric and the dilaton in the flavor action (29) will be solved from the Einstein-dilaton system (6), (7) and (17), which has been shown to be able to reproduce the two-flavor lattice results of equation of state. Note that the flavor action (29) is assumed to be in the string frame with the metric ansatz (2). For a more general consideration, we also assume that the dilaton field $\Phi$ in (29) is related to the one in the Einstein-dilaton system by $\Phi = k \phi$ with $k$ being a constant. The probe approximation neglecting the backreaction effect of the flavor sector on the background system will be adopted in this work, as in the most studies on AdS/QCD with fixed background.

B. The EOM of the scalar VEV

According to AdS/CFT, the VEV of the bulk scalar field in the two-flavor case can be written as $\langle X \rangle = \frac{\chi(z)}{2} I_2$ with $I_2$ denoting the $2 \times 2$ identity matrix, and the chiral condensate...
is incorporated in the UV expansion of the scalar VEV $\chi(z)$ \cite{16}. To address the issue on chiral transition, we only need to consider the action of the scalar VEV,

$$
S_\chi = -\int d^5x \sqrt{-g} e^{-\Phi} \left( \frac{1}{2} g^{zz}(\partial_z \chi)^2 + V(\chi, \Phi) \right)
$$

(30)

with

$$
V(\chi, \Phi) = \text{Tr} V_X(\langle X \rangle, \Phi) = \frac{1}{2}(m^2_5 - \lambda_1 \Phi)\chi^2 + \frac{\lambda_2}{8} \chi^4.
$$

(31)

In terms of the metric ansatz (2), the EOM of $\chi(z)$ can be derived from the action (30) as

$$
\chi''(z) + \left( 3A_S'(z) - \Phi'(z) + \frac{f'(z)}{f(z)} - \frac{3}{z} \right) \chi'(z) - \frac{e^{2A_S} \partial_\chi V(\chi, \Phi)}{z^2 f(z)} = 0.
$$

(32)

The UV asymptotic form of $\chi(z)$ at $z \to 0$ can be obtained from Eq. (32) as

$$
\chi(z) = m_q \zeta z + (6 - k + k\lambda_1)m_q p_1 \zeta z^2 + \frac{\sigma}{\zeta} z^3 \\
+ \frac{1}{3} \left[ m_q \zeta p_1^2 \left( 30k - 3k^2 - 23k\lambda_1 - \frac{3}{2} k^2 \lambda_1^2 \right) \\
+ \frac{9}{2} k^2 \lambda_1 - \frac{224}{3} \right] + \frac{3}{4} m_q \zeta^3 \lambda_2 \right] z^3 \log z + \cdots,
$$

(33)

where $m_q$ is the current quark mass, $\sigma$ is the chiral condensate, and $\zeta = \frac{\sqrt{N_c}}{2\pi}$ is a normalization constant \cite{20}. As in Eq. (7), a natural boundary condition at horizon $z_h$ follows from the regular condition of $\chi(z)$ near $z_h$,

$$
\left[ f' \chi' - \frac{e^{2A_S}}{z^2} \partial_\chi V(\chi, \Phi) \right]_{z=z_h} = 0.
$$

(34)

C. Chiral transition

To study the chiral transition properties in the improved soft-wall model with dynamical background, we need to solve the scalar VEV $\chi(z)$ numerically from Eq. (32) with the UV asymptotic form (33) and the boundary condition (34). The chiral condensate can then be extracted from the UV expansion of $\chi(z)$. We take the same set of parameter values that has been used to fit the two-flavor lattice results of equation of state (see Sec. IIIC), and the quark mass $m_q = 5$ MeV. The quartic coupling constant will be fixed as $\lambda_2 = 1$.

We consider two typical cases corresponding to $k = 1$ ($\Phi = \phi$) and $k = 2$ ($\Phi = 2\phi$). In each case, we study the temperature dependence of chiral condensate normalized by $\sigma_0 = \sigma(T = 0)$ for a set of values of $\lambda_1$, which needs to be negative for a sensible behavior of chiral transition. The model results of the normalized chiral condensate $\sigma/\sigma_0$ as a function of $T$ are shown in Fig. 6 and 7 from which we can see that the crossover transition can be realized qualitatively in this improved soft-wall model with a dynamical background, and there is a decreasing tendency for the transition temperature with the decrease of $\lambda_1$. Nevertheless, for both cases, an obvious bump will emerge near the transition region when
FIG. 6. The normalized chiral condensate \( \sigma/\sigma_0 \) as a function of \( T \) for \( \lambda_1 = 0, -1, -1.2, -1.4 \) in the case of \( k = 1 \).

FIG. 7. The normalized chiral condensate \( \sigma/\sigma_0 \) as a function of \( T \) for \( \lambda_1 = 0, -0.2, -0.4, -0.5 \) in the case of \( k = 2 \).

\(|\lambda_1|\) is large enough. We also find that a smaller transition temperature \( T_\chi \sim 240 \text{ MeV} \) can be obtained with a reasonable chiral transition behavior when \( \lambda_1 = -0.4 \) in the case of \( k = 2 \), yet this is still too large compared with the lattice result \( T_\chi \sim 193 \text{ MeV} \) obtained in Ref. [92].

**IV. CONCLUSION AND DISCUSSION**

We considered an improved soft-wall AdS/QCD model with a cubic coupling term of the dilaton and the bulk scalar field in a dynamical background, which is solved from the Einstein-dilaton system with a nontrivial dilaton potential. Such an Einstein-dilaton system has been used to reproduce the equation of state from lattice QCD with two flavors. Then the chiral transition behaviors were investigated in the improved soft-wall model based on the solved bulk background. We selected two typical cases with \( k = 1 \) and \( k = 2 \) in our model, and for both cases the crossover behavior of chiral transition can be generated, as seen from Fig. 6 and 7. Nevertheless, the chiral transition temperature \( T_\chi \) obtained from the model is much larger than the lattice result. Although \( T_\chi \) decreases with the increase of \(|\lambda_1|\), an obvious bump will emerge near the transition region when \(|\lambda_1|\) is large enough. As the probe approximation has been used in our study, we expect that this bump might be disappeared if the backreaction of the flavor part were considered.
In this work, the scaling dimension of the dual operator $\text{tr} F_{\mu\nu}^2$ of the dilaton has been taken as $\Delta = 3$, following Refs. [72, 73]. We remark that this is not the unique one that can do the work, and different values of $\Delta$ have been adopted for the realization of the crossover transition of the equation of state with slightly different forms of $V_E(\phi)$ [67, 71]. As mentioned above, the holographic model of the background system is assumed to match with QCD at some semi-hard scale, but the low-energy behaviors of thermal QCD are insensitive to such a scale. As the specific form of $V_E(\phi)$ and the scaling dimension $\Delta$ cannot be uniquely determined by the bottom-up holographic model, we are content with a phenomenological description for the QCD equation of state. The main point of our work is that we have built an improved soft-wall AdS/QCD model under a solved dynamical background, which provides a possibility in the holographic framework to address the deconfining and chiral phase transition simultaneously.

To clarify the phase structure in this improved soft-wall AdS/QCD model, we need to consider the backreaction of the flavor part to the background system. The correlation between the deconfining and chiral phase transitions can then be studied in such an improved soft-wall model with a dynamical background. The case of finite chemical potential can also be considered by introducing a $U(1)$ gauge field in order to study the properties of QCD phase diagram.

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