The inverse spectral transform for the Dunajski hierarchy and some of its reductions: I. Cauchy problem and longtime behavior of solutions

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Abstract

In this paper we apply the formal inverse spectral transform for integrable dispersionless partial differential equations (PDEs) arising from the commutation condition of pairs of one-parameter families of vector fields, recently developed by S V Manakov and one of the authors, to one distinguished class of equations, the so-called Dunajski hierarchy. We concentrate, for concreteness, (i) on the system of PDEs characterizing a general anti-self-dual conformal structure in neutral signature, (ii) on its first commuting flow, and (iii) on some of their basic and novel reductions. We formally solve their Cauchy problem and we use it to construct the longtime behavior of solutions, showing, in particular, that unlike the case of soliton PDEs, different dispersionless PDEs belonging to the same hierarchy of commuting flows evolve in time in very different ways, exhibiting either a smooth dynamics or a gradient catastrophe at finite time.

Keywords: Dunajski hierarchy, inverse scattering transform, longtime behavior

1. Introduction

Waves propagating in weakly nonlinear and dispersive media are well described by integrable soliton equations, like the Korteweg–de Vries [27] equation and its integrable (2+1)
dimensional generalization, the Kadomtsev–Petviashvili [30] equation. The inverse spectral transform (IST), introduced by Gardner, Green, Kruskal and Miura [20], is the spectral method allowing one to solve the Cauchy problem for such PDEs, predicting that a localized disturbance evolves into a number of soliton pulses + radiation, and solitons arise as an exact balance between nonlinearity and dispersion [1, 2, 9, 64]. Soliton PDEs arise in hierarchies of commuting flows, and equations of the same hierarchy share the same multisoliton solution and similar behavior. It is known that, apart from exceptional cases, soliton PDEs do not generalize naturally to more than (2+1) dimensions; therefore, in the context of soliton equations, integrability is a property of low dimensional PDEs.

There is another important class of integrable PDEs, the so-called dispersionless PDEs (dPDEs), or PDEs of hydrodynamic type, including as distinguished multidimensional examples the dispersionless Kadomtsev-Petviashvili (dKP) equation [60, 67], describing weakly nonlinear and quasi one dimensional waves in Nature [48, 60, 67], the heavenly equation, arising from the the vacuum Einstein equations and the conformal anti-self-duality condition for the signature metric in canonical Plebánski form [55], the Dunajski equation [10], an integrable generalization of the heavenly equation in which only the anti-self-duality condition in kept, the dispersionless 2D Toda equation [8, 18, 66], whose elliptic and hyperbolic versions are both relevant, describing, for instance, integrable $H$-spaces (heavens) [8, 21], integrable Einstein–Weyl geometries [25, 26, 63], and playing a key role in the study of the ideal Hele–Shaw problem [36–38, 51, 61]; the Pavlov equation [11, 17, 53], arising in the study of integrable hydrodynamic chains, and the Manakov–Santini system [42], giving a local description of any Lorentzian Einstein–Weyl geometry [12], and including, as particular cases, the dKP and the Pavlov equations. dPDEs arise, more in general, in various problems of mathematical physics and are intensively studied in the recent literature (see, f.i., [6, 7, 11–17, 22, 28, 29, 31–35, 39, 52–54, 56–59, 62, 65, 66]). Since they arise from the condition of commutation $[L, M] = 0$ of pairs of one-parameter families of vector fields, implying the existence of common zero energy eigenfunctions:

$$\left[ \hat{L}, \hat{M} \right] = 0 \Rightarrow \hat{L}\psi = \hat{M}\psi = 0,$$

they can be in an arbitrary number of dimensions [65]. In addition, due to the lack of dispersion, these multidimensional PDEs may or may not exhibit a gradient catastrophe at finite time and, as we shall see in this paper, even in the same hierarchy there exist equations evolving in a smooth way or into a gradient catastrophe at finite time. Their integrability gives a unique chance to study analytically such a mechanism and, also with this motivation, a novel IST for vector fields, significantly different from that of soliton PDEs [1, 2, 64], has been recently constructed [40–42, 50], at a formal level, by Manakov and one of the authors of this paper (PMS), (i) to solve their Cauchy problem [40–43, 46], (ii) obtain the longtime behavior of solutions [44–46], (iii) construct distinguished classes of exact implicit solutions [44–47], see also [4], (iv) establish if, due to the lack of dispersion, the nonlinearity of the PDE is ‘strong enough’ to cause the gradient catastrophe of localized multidimensional disturbances, and (v) study analytically the breaking mechanisms [44, 48, 49]. It is important to mention that the main difficulty to make the above IST for vector fields rigorous is associated with the proof of existence of analytic eigenfunction, motivated by the small field limit, in the spectral parameter $\lambda$. A proof of existence, but only for $\text{Im} \lambda > c$, was found in [23] for the IST of the dKP equation; a complete proof exists so far only in the simplest case of the Pavlov equation, for which the whole IST has been recently made rigorous [24].

In this paper we concentrate on the Dunajski hierarchy, whose elegant structure was enveiled in [4, 5]. More precisely, in section 2 we consider some basic members of the
Dunajski hierarchy, corresponding to the same Lax operator $\hat{L}$: (i) the system of PDEs [4] characterizing a general anti-self-dual conformal structure in neutral signature [12], (ii) its first commuting flow, and (iii) some of their basic and novel reductions. In section 3 we construct the formal IST for the Dunajski hierarchy corresponding to the same Lax operator $\hat{L}$, and we use it to solve the Cauchy problem for localized (in $x$, $y$, $z$) initial data. While the direct and inverse problems are the same for all the equations of this hierarchy, being associated with the same Lax operator $\hat{L}$, the $t$-evolution of different equations of such hierarchy, ruled by different $\hat{M}$ operators, is considerably different, as we shall see in this paper, leading either to a smooth dynamics or to a gradient catastrophe of dKP type [44, 49]. In section 4 we construct the nonlinear Riemann–Hilbert (NRH) dressing for the above equations, connecting also the Riemann–Hilbert (RH) data to the initial data of the dPDEs. In section 5 we discuss the constraints on the spectral data corresponding to the reductions of the first commuting flows of the hierarchy. In section 6 we use the NRH problem to construct the longtime behavior of the solutions. In section 7 we make some concluding remarks and we discuss interesting open problems to be investigated in future works.

This paper is dedicated to the memory of S V Manakov.

2. The first members of the Dunajski hierarchy and their basic reductions

The Dunajski hierarchy is a basic example of hierarchy of integrable dPDEs, including the heavenly and the Manakov–Santini hierarchies as particular cases; see [4] for details and for an elegant characterization of it.

2.1. The first two commuting flows

The first and basic member of such hierarchy is the following system of three PDEs [4]

\begin{align}
  u_{xt} - u_{yz} - (u_x - v_y)u_{xy} + u_yu_{xx} - v_xu_{yy} &= f_x, \\
  v_{xt} - v_{yz} - (u_x - v_y)v_{xy} + u_yv_{xx} - v_xv_{yy} &= -f_y, \\
  f_{xt} - f_{yz} - (u_x - v_y)f_{xy} + u_yf_{xx} - v_xf_{yy} &= 0,
\end{align}

(2)

equivalent to the commutation $[\hat{L}, \hat{M}_1] = 0$ of the vector fields:

\begin{align}
  \hat{L} &= \partial_z + \lambda \partial_x + \hat{u}_x - f_x \partial_{\lambda}, \\
  \hat{M}_1 &= \partial_t + \lambda \partial_y + \hat{u}_y - f_y \partial_{\lambda},
\end{align}

(3)

where

\begin{equation}
\hat{u} = u \partial_x + v \partial_y.
\end{equation}

(4)

It was recently shown [12] that there exist local coordinates $(t, z, x, y)$ such that any anti-self-dual conformal structure in signature $(2, 2)$ is locally represented by the metric

\begin{equation}
g = dt^2dx + dzdy + u_z \, dt \, dz + v_z \, dz^2,
\end{equation}

(5)

where $u$ and $v$ satisfy equations (2).

Keeping $\hat{L}$ fixed and varying $\hat{M}$, one obtains a hierarchy of commuting flows. The first commuting flow reads
where the fields $\alpha, \beta, \gamma$ are defined in terms of $u, v, f$ through the equations

$$\alpha_x = -u_{yy} - u_x u_{yy} - v_y u_{yy} + v_y u_{xy} + u_y u_{xx} - f_y,$$

$$\beta_x = -v_{yz} - u_x v_{yy} - v_y v_{yy} + v_y v_{xy} + u_y v_{xx} + 2 f_x,$$

$$\gamma_x = f_{yz} + \hat{u}_y f_y - \hat{u}_y f_y.$$  \hfill (7)

Equations (6) and (7) arise as the commutation condition $[\hat{L}, \hat{M}] = 0$ of $\hat{L}$ with the vector field

$$\hat{M}_2 = \partial_z + \left( u_y \lambda + \alpha \right) \partial_x + \left( \lambda^2 + v_y \lambda + \beta \right) \partial_y + \left( -f_y \lambda + \gamma \right) \partial_z,$$  \hfill (8)

whose coefficients increase by one their degree as polynomials of $\lambda$, with respect to $\hat{M}_2$.

### 2.2. Some basic reductions

The Dunajski hierarchy admits interesting reductions (see, f.i. [4, 5]). The reductions of equations (2), (6) and (7) we consider in this paper are of two types. Arising from the commutation conditions $[\hat{L}, \hat{M}_1] = 0$ for one parameter families of vector fields, the first and natural reduction considered here is the divergence free condition for such vector fields (the condition $\vec{V} \cdot \vec{u} = 0$ for the vector field $\vec{X} = \partial_t + \vec{u} \cdot \vec{V}$, implying the constant volume condition for the associated dynamical systems. The second basic set of reductions explored in this work, giving rise to non-autonomous systems, are associated with the condition that the so-called Orlov eigenfunctions of $(\hat{L}, \hat{M}_1)$ be polynomial in the spectral parameter $\lambda$ (see section 5 for more details).

Beginning with the system (2), the divergence free constraint for $\hat{L}$ and $\hat{M}_1$:

$$u_x + v_y = 0, \quad \Rightarrow \quad u = \theta_y, \quad v = -\theta_x \hfill (9)$$

leads to the well known Dunajski equation [10]

$$\theta_{xt} - \theta_{yz} + \theta_{xx} \theta_{yy} - \theta_{yy} \theta_{xx} = f,$$

$$f_{xt} - f_{yz} + \theta_{yy} f_{xx} + \theta_{xx} f_{yy} - 2 \theta_{xy} f_{xy} = 0,$$  \hfill (10)

and to its vector fields Lax pair:

$$\hat{L} = \partial_t + \lambda \partial_x + \theta_{xy} \partial_y + \theta_{xy} \partial_y - f_x \partial_z,$$

$$\hat{M}_1 = \partial_t + \lambda \partial_x + \theta_{xy} \partial_y - \theta_{xy} \partial_y - f_x \partial_z.$$  \hfill (11)

reducing to the second heavenly equation

$$\theta_{xt} - \theta_{yz} + \theta_{xx} \theta_{yy} - \theta_{yy} \theta_{xx} = 0 \hfill (12)$$

for $f = 0$.

The second set of reductions of system (2) lead to non-autonomous and, to the best of our knowledge, novel integrable nonlinear PDEs.
The reduction
\[ u + zf = 0, \]  
leads to the system
\[ v_{x_{\alpha}} - v_{yz} + (zf_x + v_y)v_{xy} - zf_yv_{xx} - v_xv_{yy} = -f_x, \]
\[ f_{x_{\alpha}} - f_{yz} + (zf_x + v_y)f_{xy} - zf_yf_{xx} - v_xf_{yy} = 0. \]  
\( (14) \)

The reduction
\[ v + tf = 0, \]  
plays a similar role and does not give anything new, leading to the system
\[ u_{x_{\alpha}} - u_{yz} - (tf_{x} + u_x)u_{xy} + tf_x u_{yy} + u_yu_{xx} = f_y, \]
\[ f_{x_{\alpha}} - f_{yz} - (tf_{x} + u_x)f_{xy} + tf_x f_{yy} + u_yf_{xx} = 0, \]  
\( (16) \)
equivalent to \( (14) \) by the change of variables \( u \leftrightarrow v, \ z \leftrightarrow t, \ x \leftrightarrow y. \)

The combination of the reductions \( (9) \) and \( (13): \)
\[ \theta_{x_{\alpha}} + \theta_{yz} = + = 
\]  
leads to the following scalar non-autonomous generalization of the second heavenly equation:
\[ \theta_{x_{\alpha}} - \theta_{yz} + \theta_{xx} \theta_{yy} - z\theta^2_{xy} = 0, \]  
\( (18) \)

while the combination of the reductions \( (13) \) and \( (15) \)
\[ u + zf = v + tf = 0, \]  
leads to the scalar dPDE
\[ f_{x_{\alpha}} - f_{yz} + zf_{xy} + tf_x f_{yy} - zf_y f_{xx} - tf_y f_{xy} = 0. \]  
\( (20) \)

Concentrating now on equations \( (6) \) and \( (7), \) the divergence free constraint for the vector fields \( \tilde{L}, \tilde{M}_2, \) expressed now by the equations
\[ u_x + v_y = \alpha_x + (\beta - f)_y = 0 \Rightarrow u = \theta_x, \ v = -\theta_y, \ f = -\theta_x, \]  
\( (17) \)
leads to the system
\[ \left( \theta_{x_{\alpha}} - \left( \partial_z + \theta_{\alpha y} \partial_x - \theta_{\alpha x} \partial_y \right) \rho + \theta_{\alpha x} f \right)_x + \left( \partial_z + \theta_{\alpha y} \partial_x - \theta_{\alpha x} \partial_y \right) f = 0, \]
\[ f_{x_{\alpha}} + f_x f_y + \left( \partial_z + \theta_{\alpha y} \partial_x - \theta_{\alpha x} \partial_y \right) y + (f - \rho_x) f_{xy} + \rho_y f_{xx} = 0, \]  
\( (22) \)

where
\[ \rho_x + \theta_{\alpha y} + \theta_{\alpha y}^2 - \theta_{\alpha x} \theta_y + f = 0, \]
\[ \gamma_y = f_{yz} + 2\theta_{\alpha y} f_{xy} - \theta_{\alpha x} f_{yy} - \theta_{\alpha x} f_{xx}, \]  
\( (23) \)

reducing to the second member of the heavenly hierarchy
\[ \theta_{x_{\alpha}} = \left( \partial_z + \theta_{\alpha y} \partial_x - \theta_{\alpha x} \partial_y \right) \rho, \]
\[ \rho_x + \theta_{\alpha y} + \theta_{\alpha y}^2 - \theta_{\alpha x} \theta_y = 0 \]  
\( (24) \)

for \( f = 0 \) and, consequently, \( \gamma = 0. \)
The reduction (13) leads instead to the system
\[ v_{xt} = \left( \partial_x - z f_x f_x + v_x \partial_x \right) \beta - f_x v_y - z f_{xx} - \beta v_y, \]
\[ f_{xt} + f_y f_y + \left( \partial_x - z f_x f_x + v_x \partial_x \right) \gamma + \beta f_{xy} + z f_{xx} = 0, \] (25)
where \( \alpha = z \gamma \) and
\[ \beta_x = - \left( \partial_x - z f_x f_x + v_x \partial_x \right) v_x + \left( - z f_x f_x + v_x \partial_x \right) v_x + 2 f_x, \]
\[ \gamma_x = \left( \partial_x - z f_x f_x + v_x \partial_x \right) f_y - \left( - z f_x f_x + v_x \partial_x \right) f_x, \] (26)
and the combination of the above two reductions leads to the equations
\[ \theta_t - \theta_y = \theta^2 - z \theta_{xx} \theta_y, \]
\[ \theta_t + \theta_{yz} + z \theta_{xx} \theta_{yy} = 0, \] (27)
where (17) hold and \( \theta = z^{-1} \rho, \ \alpha = z \beta, \ \beta = - \theta_x - z \theta_x, \ \gamma = \beta_x. \)

At last, the (less obvious) differential reduction
\[ \left( 1 - 2t f_y \right) v_t = 2t \left( f_x + u_x f_x \right), \] (28)
leads to the system
\[ \left( 1 - 2t f_y \right) \left( u_{xt} + u_x f_y + u_y \beta + u_x \alpha + \gamma - \alpha_x - u_x \alpha_x \right) = 2t \left( f_x + u_x f_x \right) \alpha_y, \]
\[ \left( 1 - 2t f_y \right) \left( f_{xt} + f_x f_y + f_y \beta + f_x \alpha + \gamma_x + u_x \gamma_x \right) = -2t \left( f_x + u_x f_x \right) \gamma_x, \] (29)
where
\[ \left( 1 - 2t f_y \right) \left( \alpha_x + u_y \gamma + u_x \alpha_x - u_x u_{xx} + f_x \right) = 2t \left[ \left( \gamma + u_x f_x \right) u_{xy} - \left( f_x + u_x f_x \right) u_{xy} \right], \]
\[ \left( 1 - 2t f_y \right) \beta = 2 \beta + 2t \left( f_x + af_x \right), \]
\[ \left( 1 - 2t f_y \right) \left( \gamma_x - f_{yx} - u_x f_{xy} + u_y f_{xx} \right) = 2t \left[ \left( f_x + u_x f_x \right) f_{yy} - \left( \gamma + u_x f_x \right) f_{yy} \right]. \] (30)

The combination of reductions (13) and (28) leads to the dPDE
\[ \left( 1 - 2t f_y \right) \left( f_{xt} + f_x f_y + f_y \beta + \gamma_x + z f_{xx} \gamma_x - z f_x f_x \right) = 2t \left( z f_x^2 - f_x \right) \gamma_x, \] (31)
where \( \alpha = z \gamma \) and
\[ \left( 1 - 2t f_y \right) \beta = 2 f + 2t \left( f_x + z f_x \right), \]
\[ \left( 1 - 2t f_y \right) \left( \gamma_x - f_{yy} + z f_{xy} f_{xy} - z f_x f_{xx} \right) = 2t \left[ \left( f_x + z f_x^2 \right) f_{yy} - \left( \gamma - z f_x f_x \right) f_{yy} \right]. \] (32)
At last, the combination of reductions (21) and (28) leads to the PDEs
\[
\begin{align*}
2t_2 \left[ \theta_{xx} - \left( \theta_z + \theta_{yy} \theta_x - \theta_{xy} \theta_x \right) \rho + \theta_{xy} f \right] - \theta_z &= 0, \\
\left( 1 - 2t_2 f_x \right) \theta_{xx} + 2t_2 \left( f_x + \theta_{yy} f_x \right) &= 0, \\
\rho_x + \theta_{yz} + \theta_{yy}^2 - \theta_{x} \theta_{yy} + f &= 0,
\end{align*}
\] (33)
where (21) holds, and \( \gamma = (\theta_{yy} f)_{xx} - (\theta_{yy} f)_{x} - \theta_{yy} f/(2t_2) \).

The solution of the Cauchy problem of all the above reductions, in terms of a nonlinear RH problem, will be presented in section 5.

3. IST for the Dunajski hierarchy

In this section we apply the IST method introduced in [40–42] to construct the formal solution of the Cauchy problem for the systems (2) and (6)–(7) in \((3 + 1)\) dimensions and for their reductions, within the class of rapidly decreasing real potentials \( u, v, f \):
\[
\begin{align*}
u, f &\rightarrow 0, \quad \left( x^2 + y^2 + z^2 \right) \rightarrow +\infty, \\
u, f &\in \mathbb{R}, \quad (x, y, z) \in \mathbb{R}^3, \quad t_j > 0;
\end{align*}
\] (34)
here \( t_j, j = 1, 2 \) are interpreted as time variables and \( x, y, z \) as space variables, therefore the initial data in the IST are characteristic.

3.1. Basic eigenfunctions

A basic role in this IST theory is played by the real Jost eigenfunctions \( \tilde{\varphi}_\lambda(x, y, z; \lambda) \in \mathbb{R}^3 \) for \( \lambda \in \mathbb{R} \), the solutions of \( \tilde{L} \tilde{\varphi}_\lambda = 0 \) defined by the asymptotic:
\[
\tilde{\varphi}_\lambda(x, y, z; \lambda) = \begin{pmatrix} q_{\lambda x 0}(x, y, z; \lambda) \\ q_{\lambda x 1}(x, y, z; \lambda) \\ q_{\lambda x 2}(x, y, z; \lambda) \end{pmatrix} \rightarrow \tilde{\xi}, \quad \text{as } z \rightarrow \pm \infty,
\] (35)
where
\[
\tilde{\xi} \equiv \begin{pmatrix} \lambda \\ \xi \\ \gamma \end{pmatrix}, \quad \xi \equiv x - \lambda z.
\] (36)
These Jost eigenfunctions are intimately connected to the dynamical system (in the time variable \( z \)):
\[
\frac{d\tilde{\xi}}{dz} = \begin{pmatrix} -f_x(x, y, z) \\ \lambda + u_x(x, y, z) \\ v_x(x, y, z) \end{pmatrix},
\] (37)
for the unknown \( \tilde{z} (z) = (\lambda(z), x(z), y(z))^T \) in the following way.
Assuming that the potentials \( u, v, f \) be smooth and sufficiently localized functions of \( x, y, z \), it follows from ODE theory that the solution
\[
\begin{pmatrix}
A \left( z; x_0, y_0, z_0, \lambda_0 \right) \\
X \left( z; x_0, y_0, z_0, \lambda_0 \right) \\
Y \left( z; x_0, y_0, z_0, \lambda_0 \right)
\end{pmatrix}
\]
of (37), satisfying the initial condition \( \left( \lambda(z_0), x(z_0), y(z_0) \right)^T = \left( \lambda_0, x_0, y_0 \right)^T \), exists unique, and it is globally defined for real \( z \), with the asymptotic states \( \lambda_{\pm}, x_{\pm}, y_{\pm} \):
\[
\lambda \sim \lambda_{\pm}(x_0, y_0, z_0, \lambda_0), \quad z \to \pm \infty,
\]
\[
x \sim \pm \lambda_{\pm}(x_0, y_0, z_0, \lambda_0) + x_{\pm}(x_0, y_0, z_0, \lambda_0), \quad z \to \pm \infty,
\]
\[
y \sim y_{\pm}(x_0, y_0, z_0, \lambda_0), \quad z \to \pm \infty.
\]
(39)

These asymptotic states \( \lambda_{\pm}(x_0, y_0, z_0, \lambda_0), x_{\pm}(x_0, y_0, z_0, \lambda_0), y_{\pm}(x_0, y_0, z_0, \lambda_0) \) are constants of motion for the dynamical system (37) when the point \( (x_0, y_0, z_0, \lambda_0) \) moves along the trajectories; therefore they are solutions of the vector field equation
\[
\begin{pmatrix}
\lambda_{\pm}(x, y, z, \lambda) \\
x_{\pm}(x, y, z, \lambda) \\
y_{\pm}(x, y, z, \lambda)
\end{pmatrix}
\]
(40)
and, due to (35) and (39), they coincide with the real Jost eigenfunctions \( \hat{\phi}_{\pm}(x, y, z; \lambda) \):
\[
\phi_{\pm 0}(x, y, z; \lambda) = \lambda_{\pm}(x, y, z, \lambda),
\]
\[
\phi_{\pm 1}(x, y, z; \lambda) = x_{\pm}(x, y, z, \lambda),
\]
\[
\phi_{\pm 2}(x, y, z; \lambda) = y_{\pm}(x, y, z, \lambda).
\]
(41)

A crucial role in the IST for the vector field \( \hat{L} \) is also played by the analytic eigenfunctions \( \hat{\psi}_{\pm}(x, y, z; \lambda) \), the solutions of \( \hat{L}\hat{\psi}_{\pm} = 0 \) analytic, respectively, in the upper and lower halves of complex \( \lambda \) plane, satisfying the asymptotics
\[
\hat{\psi}_{\pm}(x, y, z; \lambda) \to \xi, \quad \left( x^2 + y^2 + z^2 \right) \to + \infty.
\]
(42)

They can be characterized by the following integral equations
\[
\int_{\mathbb{R}^3} \text{d}x dx' \text{d}y dy' dz' \hat{G}_{\pm}(x - x', y - y', z - z'; \lambda) G_{\pm}(x', y', z'; \lambda)
\]
\[
\left[ u_{\pm}(x', y', z') \partial_{x'} + v_{\pm}(x', y', z') \partial_{y'} - f_{\pm}(x', y', z') \partial_{z'} \right] \hat{\psi}_{\pm}(x', y', z'; \lambda)
\]
\[
= \xi, \quad \lambda \gg 1.
\]
(43)

for the analytic Green’s functions of the undressed operator \( (\partial_x + \lambda \partial_z) \):
\[
\hat{G}(x, y, z; \lambda) = \pm \frac{\delta(y)}{2 \pi \text{Im}(x - (\lambda + i \varepsilon)z)}.
\]
(44)

Consequently, for \( |\lambda| \gg 1 \):
\[
\hat{\psi}_{\pm}(x, y, z; \lambda) = \xi + \hat{Q}(x, y, z) + O(\lambda^{-2}),
\]
(45)
where

\[
\tilde{Q}(x, y, z) = \begin{pmatrix}
  f(x, y, z) \\
  -u(x, y, z) - zf(x, y, z) \\
  -v(x, y, z)
\end{pmatrix}.
\] (46)

We observe that the analytic Green’s functions (44) exhibit the following asymptotics for \( z \to \pm \infty \)

\[
G_{\xi}(x' - x, y' - y, z - z'; \lambda) \to \pm \frac{\delta(y)}{2\pi i [\xi - \xi' \mp \imath\epsilon]}, \quad z \to \pm \infty,
\] (47a)

\[
G_{\xi}(x' - x, y' - y, z - z'; \lambda) \to \pm \frac{\delta(y)}{2\pi i [\xi - \xi' \pm \imath\epsilon]}, \quad z \to \mp \infty,
\] (47b)

where \( \xi = x - \lambda z, \xi' = x' - \lambda z' \), entailing that the \( z \to +\infty \) asymptotics of \( \tilde{\psi}^\pm \) are analytic, respectively, in the lower and upper halves of the complex \( \xi \)-plane, while the \( z \to -\infty \) asymptotics of \( \tilde{\psi}^\pm \) are analytic, respectively, in the upper and lower halves of the complex \( \xi \)-plane.

3.2. Scattering and spectral data

The \( z \) (time) scattering problem for the ODE system (37) allows one to construct the scattering data \( \tilde{\sigma}(\xi) \), defined by the \( z \to +\infty \) limit of \( \tilde{q}_z \):

\[
\lim_{z \to +\infty} \tilde{q}_z(x, y, z; \lambda) = \tilde{S}(\xi) = \tilde{\xi} + \tilde{\sigma}(\xi).
\] (48)

The first part of the direct problem is the mapping from the real potentials \( f, u, \) and \( v \), functions of the three real variables \( (x, y, z) \), to the real scattering vector \( \tilde{\sigma} \) defined in (48), function of the real variables \( \xi = (\lambda, \xi, y) \).

Together with the above scattering data, one defines also spectral data in the following way. The ring property of the space of eigenfunctions allows one to express the analytic eigenfunctions in terms of the Jost eigenfunctions, used as a basis for such a space, if \( \lambda \in \mathbb{R} \):

\[
\tilde{\psi}^+(x, y, z; \lambda) = \tilde{K}^+(\tilde{q}_z(x, y, z; \lambda)) = \tilde{K}^-(\tilde{q}_z(x, y, z; \lambda)),
\] (49)

\[
\tilde{\psi}^-(x, y, z; \lambda) = \tilde{K}^-(\tilde{q}_z(x, y, z; \lambda)) = \tilde{K}^+(\tilde{q}_z(x, y, z; \lambda)),
\] (50)

where

\[
\tilde{K}_a^b(\xi) := \xi + \chi_0^b(\xi), \quad a, b = \pm.
\] (51)

The \( z \to -\infty \) and \( z \to +\infty \) limits of respectively the first and second equalities in (49) and
(50) imply

\[
\lim_{z \to -\infty} \tilde{\psi}^\pm = \xi + \chi_0^\pm(\xi), \quad \lim_{z \to +\infty} \tilde{\psi}^\pm = \xi + \chi_0^\pm(\xi).
\] (52)

Therefore the analyticity properties of \( \tilde{\psi}^\pm \) established above imply that \( \chi_0^\pm(\xi) \) and \( \chi_0^\pm(\xi) \) are analytic in \( \text{Im} \ \xi > 0 \) decaying at \( \xi \to \infty \) like \( O(\xi^{-1}) \), while \( \chi_0^\pm(\xi) \) and \( \chi_0^\pm(\xi) \) are analytic in \( \text{Im} \ \xi < 0 \) decaying at \( \xi \to \infty \) like \( O(\xi^{-1}) \). In addition, taking the \( z \to +\infty \) limit of the second of (49), one obtains the following equation:
that must be viewed as three ‘linear scalar RH problems in the variable $\xi$, with the given shift $\sigma \xi \vec{F}()$ for the unknowns $\vec{X}^+$ and $\vec{X}^-$ (see, f.i., [19] for the associated theory). Such a RH problems with a shift are equivalent to the following three linear Fredholm equations [19]

$$\chi^+_j(\tilde{\xi}) = \frac{1}{2\pi i} \int_\mathbb{R} K(\tilde{\xi}, \xi) \chi^+_j(\xi) d\xi^\prime + M_j(\xi) = 0, \quad j = 0, 1, 2, \quad \lambda \in \mathbb{R}, \quad (54)$$

where

$$K(\tilde{\xi}, \xi) = \frac{\partial S(\xi')/\partial \xi^\prime}{S(\xi')} - S(\xi)/(\xi' - \xi),$$

$$M_j(\xi) = -\frac{1}{2} \sigma_j(\xi) + \frac{1}{2\pi i} \int_\mathbb{R} \frac{\partial S(\xi')/\partial \xi^\prime}{S(\xi') - S(\xi)} \sigma_j(\xi') d\xi', \quad j = 0, 1, 2,$$

$$S(\xi) = \xi + \sigma_0(\xi), \quad S(\xi') = \xi' + \sigma_1(\xi'),$$

$$\tilde{\xi} = (\lambda, \xi, y), \quad \tilde{\xi}' = (\lambda, \xi', y).$$

We remark that, if $u, v, f \in \mathbb{R}$, then the vector fields are real for $\lambda \in \mathbb{R}$, and

$$\vec{q}_e \in \mathbb{R}^3, \quad \vec{u} = \vec{u}^\prime, \quad \lambda \in \mathbb{R},$$

$$\vec{\sigma} \in \mathbb{R}^3, \quad \vec{K}^+_a = \vec{K}^+_a, \quad \vec{K}^-_a = \vec{K}^-_a. \quad (55)$$

Summarizing, the direct problem consists of the following steps: (i) given the initial data $(u(x, y, z, 0), v(x, y, z, 0), f(x, y, z, 0))$, one constructs the scattering data $\vec{\sigma}(\vec{F}, 0)$ from the solution of the ODE system (37); (ii) known $\vec{\sigma}(\vec{F}, 0)$, one solves the RH problem with a shift (53) constructing the spectral data $\vec{X}^+$ and $\vec{X}^-$.  

### 3.3. Inverse problems

First inversion. Known the spectral datum $\vec{X}^+(\vec{F})$, an inverse problem can be constructed from the first of equations (49), observing that

$$\hat{P}^+_1(\vec{q}_e + \vec{F} + \vec{X}^+(\vec{q}_e)) = 0, \quad (56)$$

where

$$\hat{P}^+_1 g(\lambda) \equiv \pm \frac{1}{2\pi i} \int_\mathbb{R} g(\lambda') \frac{d\lambda'}{\lambda' - (\lambda \pm i\epsilon)}, \quad (57)$$

are the $(\pm)$ analyticity projectors to the upper and lower halves of the complex $\lambda$ plane. Since $\hat{P}^+_1 f(\lambda) = \pm \frac{1}{2} H_f(\lambda) + \frac{1}{2} f(\lambda)$, equation (56) is equivalent to the inverse problem formula

$$\vec{q}_e(x, y, z; \lambda) + H_1 \vec{X}^+(\vec{q}_e(x, y, z; \lambda)) + \mathfrak{R} \vec{X}^+(\vec{q}_e(x, y, z; \lambda)) = \vec{F}, \quad (58)$$

where $\mathfrak{R} \vec{X}^+$ and $\mathfrak{I} \vec{X}^+$ are the real and imaginary parts of $\vec{X}^+$, i.e., $\vec{X}^+ = \mathfrak{R} \vec{X}^+ + i \mathfrak{I} \vec{X}^+$, and $H_1 g(\lambda)$ is the Hilbert transform operator.
Equation (58) must be viewed as a vector nonlinear integral equation for the unknown Jost eigenfunctions $\tilde{\phi}$. Once $\tilde{\phi}$ is reconstructed from $\tilde{\chi}^+$ solving the nonlinear integral equation (58), $f$, $u$, $v$ are finally reconstructed from (see [24])

\[
\begin{align*}
f &= -\frac{1}{\pi} \int_{\mathbb{R}} \chi_{-1}^+ (\tilde{\phi}(x, y, z, \lambda)) d\lambda, \\
v &= \frac{1}{\pi} \int_{\mathbb{R}} \chi_{-3}^+ (\tilde{\phi}(x, y, z, \lambda)) d\lambda, \\
u &= -zf + \frac{1}{\pi} \int_{\mathbb{R}} \chi_{-2}^+ (\tilde{\phi}(x, y, z, \lambda)) d\lambda.
\end{align*}
\]

To obtain the reconstruction formulae (60), one subtracts equations (49) and (50), and then uses the reality constraints (55):

\[
\tilde{\psi}^+ - \tilde{\psi}^- = \chi^+ (\tilde{\phi}) - \chi^- (\tilde{\phi}) = 2i \chi^+ (\tilde{\phi}),
\]

(61) together with the analyticity properties of $\tilde{\psi}^\pm$, to get

\[
\tilde{\psi}^\pm = \tilde{\psi} + \frac{1}{\pi} \int_{\mathbb{R}} \chi_{-1}^+ (\tilde{\phi}(x, y, z, \lambda)) d\lambda.
\]

(62) Equations (60) follow then from (46).

Second inversion: the NRH problem. An alternative inversion is based on the NRH inverse problem on the real $\lambda$ axis:

\[
\tilde{\psi}^+ (\lambda) = \tilde{R} (\tilde{\psi}^- (\lambda)), \quad \lambda \in \mathbb{R}
\]

(63) or, in component form:

\[
\begin{pmatrix}
\psi_0^+ (\lambda) \\
\psi_1^+ (\lambda) \\
\psi_2^+ (\lambda)
\end{pmatrix} =
\begin{pmatrix}
\tilde{R}_0 (\psi_0^- (\lambda), \psi_1^- (\lambda), \psi_2^- (\lambda)) \\
\tilde{R}_1 (\psi_0^- (\lambda), \psi_1^- (\lambda), \psi_2^- (\lambda)) \\
\tilde{R}_2 (\psi_0^- (\lambda), \psi_1^- (\lambda), \psi_2^- (\lambda))
\end{pmatrix}, \quad \lambda \in \mathbb{R},
\]

(64) where the RH data $\tilde{R}(\tilde{\zeta})$ are constructed from the spectral data $\tilde{\chi}_0^+ (\tilde{\zeta})$ via algebraic manipulations, eliminating $\tilde{\phi}$ from the first of equations (49) and (50). The solutions $\tilde{\psi}^\pm (\lambda) = (\psi_0^\pm (\lambda), \psi_1^\pm (\lambda), \psi_2^\pm (\lambda))^T \in \mathbb{C}^3$ are analytic respectively in the upper and lower halves of the complex $\lambda$ plane, with the following normalization in the neighborhood of $\lambda = \infty$:

\[
\begin{align*}
\psi_0^\pm (\lambda) &= \lambda + O \left( \lambda^{-1} \right), \\
\psi_1^\pm (\lambda) &= -z \lambda + x + O \left( \lambda^{-1} \right), \\
\psi_2^\pm (\lambda) &= y + O \left( \lambda^{-1} \right), \quad |\lambda| \gg 1,
\end{align*}
\]

(65) and the potentials $u$, $v$, $f$, are reconstructed from (45):

\[
\begin{align*}
f &= \lim_{\lambda \to \infty} \lambda \left( \psi_0^+ (\lambda) - \lambda \right), \\
u &= -zf - \lim_{\lambda \to \infty} \lambda \left( \psi_2^+ (\lambda) - x + z \lambda \right),
\end{align*}
\]
\[ v = \lim_{\lambda \to \infty} \lambda \left( \psi_2^\pm(\lambda) - y \right) . \tag{66} \]

Such a nonlinear RH problem is uniquely solvable as long as the nonlinear mapping is sufficiently close to the identity. We refer to [47] for an algorithmic construction of special RH data giving rise to solvable nonlinear RH problems.

### 3.4. Evolution of the scattering data

In all formulas of the direct and inverse problems we have deliberately omitted the time variable, appearing just as a parameter. Now it is the time to introduce it, and, as we shall see, (i) the evolution of the data is given by explicit formulae (and this is the main justification for the introduction of the IST); (ii) such formulae imply a substantially different evolution for different members of the hierarchy; (iii) the \( t_1 \) and \( t_2 \) flows commute on the level of the data, and this is the simplest way to prove that the corresponding flows \((2), (6)\) and \((7)\) commute as well.

It turns out that, as \( f, u, v \) evolve in time according to \((2)\), and, respectively, \((6)\) and \((7)\), the time dependence of the three components of the data \( \vec{S}, \vec{K}_\alpha^\text{a} \) and \( \vec{R} \) are defined by the same PDEs:

\[
\begin{align*}
\left( \partial_{\tau} + \hat{x}^n \partial_{x} \right) D_f &= 0, \quad j = 0, 1, \\
\left( \partial_{\tau} + \hat{x}^n \partial_{x} \right) D_2 &= D_0^n, 
\end{align*}
\tag{67}
\]

whose explicit solutions read

\[
\begin{align*}
D_f(\vec{\phi}_0, \vec{\xi}_1, \vec{\xi}_2, t_n) &= D_f(\vec{\phi}_0, \vec{\xi}_1, \vec{\xi}_2 - t_n\vec{\phi}_0^n, 0), \quad j = 0, 1, \\
D_2(\vec{\phi}_0, \vec{\xi}_1, \vec{\xi}_2, t_n) &= D_2(\vec{\phi}_0, \vec{\xi}_1, \vec{\xi}_2 - t_n\vec{\phi}_0^n, 0) + t_n \left( D_0(\vec{\phi}_0, \vec{\xi}_1, \vec{\xi}_2 - t_n\vec{\phi}_0^n, 0) \right)^n, 
\end{align*}
\tag{68}
\]

where \( n = 1 \) and, respectively, \( n = 2 \). For an arbitrary \( n \in \mathbb{N}^+ \) in \((67)\) and \((68)\), one generates the time dependence of the spectral data of a sequence of commuting flows of the Dunajski hierarchy.

Equations \((67)\) follow from the observation that one constructs, from the Jost \( \vec{q} \) and analytic \( \vec{\psi}^\pm \) eigenfunctions of \( \hat{L} \), the common Jost \( \vec{\phi} \) and analytic \( \vec{\Psi}^\pm \) eigenfunctions of both \( \hat{L} \) and \( \hat{M}_n \) operators via the formulae:

\[
\begin{align*}
\phi_1(x, y, z; t_n; \lambda) &\equiv \varphi_{-0}(x, y, z; t_n; \lambda), \\
\phi_2(x, y, z; t_n; \lambda) &\equiv \varphi_{-1}(x, y, z; t_n; \lambda), \\
\phi_3(x, y, z; t_n; \lambda) &\equiv \varphi_{-2}(x, y, z; t_n; \lambda) - t_n \left( \varphi_{-0}(x, y, z; t_n; \lambda) \right)^n, \\
\Psi_0^\text{a}(x, y, z; t_n; \lambda) &\equiv \psi_0^\pm(x, y, z; t_n; \lambda), \\
\Psi_1^\text{a}(x, y, z; t_n; \lambda) &\equiv \psi_1^\pm(x, y, z; t_n; \lambda), \\
\Psi_2^\text{a}(x, y, z; t_n; \lambda) &\equiv \psi_2^\pm(x, y, z; t_n; \lambda) - t_n \left( \psi_0^\pm(x, y, z; t_n; \lambda) \right)^n, 
\end{align*}
\tag{69}
\]

using arguments developed, f.i., in [42].

It is straightforward to verify that the flows \((67)\) for different \( n \) commute.
4. The nonlinear RH dressing

Since the RH data \( \tilde{R}(\zeta, t_0) \), \( n = 1, 2 \) satisfy equations (67) and (68) and the analytic eigenfunctions \( \tilde{\Phi}^{\pm} \) of \( \tilde{L} \) are connected to the analytic eigenfunctions \( \tilde{\Psi}^{\pm} \) of \( \tilde{L}, \tilde{M}_n, n = 1, 2 \) through the equations (70), it follows that the NRH formulation of the inverse problem, corresponding to the analytic eigenfunctions of both operators \( \tilde{L}, \tilde{M}_n, n = 1, 2 \), reads as follows.

Consider the NRH problem

\[
\tilde{\Psi}^+(\lambda) = \tilde{R} \left( \tilde{\Psi}^-(\lambda) \right), \quad \lambda \in \mathbb{R}
\]

or, in component form:

\[
\begin{pmatrix}
\Psi_0^0(\lambda) \\
\Psi_1^0(\lambda) \\
\Psi_2^0(\lambda)
\end{pmatrix} = \begin{pmatrix}
R_0(\Psi_0^0(\lambda), \Psi_1^0(\lambda), \Psi_2^0(\lambda)) \\
R_1(\Psi_0^0(\lambda), \Psi_1^0(\lambda), \Psi_2^0(\lambda)) \\
R_2(\Psi_0^0(\lambda), \Psi_1^0(\lambda), \Psi_2^0(\lambda))
\end{pmatrix}, \quad \lambda \in \mathbb{R},
\]

(72)

where \( R_j(\zeta) = \tilde{R}_j(\zeta, 0) \), the solutions \( \tilde{\Phi}^{\pm}(\lambda) = (\Psi_0^\pm(\lambda), \Psi_1^\pm(\lambda), \Psi_2^\pm(\lambda))^\top \in \mathbb{C}^3 \) are analytic respectively in the upper and lower halves of the complex \( \lambda \) plane, with the following normalizations in a neighborhood of \( \lambda = \infty \):

\[
\Psi_\pm^{\pm}(\lambda) = \nu_j(\lambda) + O(\lambda^{-1}), \quad j = 0, 1, 2,
\]

(73)

where, respectively,

\[
\nu_0(\lambda) = \lambda, \quad \nu_1(\lambda) = x - \lambda z, \quad \nu_2(\lambda) = y - \lambda t_1,
\]

(74)

and

\[
\nu_0(\lambda) = \lambda, \quad \nu_1(\lambda) = x - \lambda z, \quad \nu_2(\lambda) = -\lambda^2 t_2 + y - 2 f t_2.
\]

(75)

Then \( \tilde{\Phi}^{\pm}(\lambda) \) are vector eigenfunctions, respectively, of the vector fields \( \tilde{L}, \tilde{M}_1 \): \( \tilde{L}\tilde{\Phi}^{\pm} = \tilde{M}_1\tilde{\Phi}^{\pm} = 0 \), and of the vector fields \( \tilde{L}, \tilde{M}_2 \): \( \tilde{L}\tilde{\Phi}^{\pm} = \tilde{M}_2\tilde{\Phi}^{\pm} = 0 \); furthermore, the fields \( f, u, v \), reconstructed by, respectively, the following formulae:

\[
f = \lim_{\lambda \to \infty} \lambda \left( \Psi_0^0(\lambda) - \lambda \right),
\]

\[
u = -t_1 f - \lim_{\lambda \to \infty} \lambda \left( \Psi_1^0(\lambda) - x + z \lambda \right),
\]

\[
\lambda \to \infty
\]

(76)

and

\[
f = \lim_{\lambda \to \infty} \lambda \left( \Psi_0^0(\lambda) - \lambda \right),
\]

\[
u = -t_1 f - \lim_{\lambda \to \infty} \lambda \left( \Psi_1^0(\lambda) - x + z \lambda \right),
\]

\[
\lambda \to \infty
\]

(77)

are solutions respectively of the nonlinear systems of PDEs (2), (6) and (7).

In addition, the first few terms of the \( \lambda \) large expansions of the above eigenfunctions of \( (\tilde{L}, \tilde{M}_1) \) and \( (\tilde{L}, \tilde{M}_2) \) read, respectively...
\[\Psi_0^a(\lambda) = \lambda + \beta \lambda^{-1} + \delta_0^{(2)} \lambda^{-2} + O(\lambda^{-3}),\]
\[\Psi_1^a(\lambda) = -z\lambda + x - (u + zf)\lambda^{-1} + \delta_1^{(2)} \lambda^{-2} + O(\lambda^{-3}),\]
\[\Psi_2^a(\lambda) = -t_1 \lambda + y - (v + t_1f)\lambda^{-1} + \delta_2^{(2)} \lambda^{-2} + O(\lambda^{-3}),\]  
(78)

where
\[
\delta_0^{(2)} = -\left(\partial_z + u_s \partial_s + v_t \partial_t\right)f, \\
\delta_1^{(2)} = \left(\partial_z + u_s \partial_s + v_t \partial_t\right)(u + zf), \\
\delta_2^{(2)} = \left(\partial_z + u_s \partial_s + v_t \partial_t\right)(v + t_1f),
\]
(79)

and
\[\Psi_0^b(\lambda) = \lambda + \beta \lambda^{-1} + \delta_0^{(2)} \lambda^{-2} + O(\lambda^{-3}),\]
\[\Psi_1^b(\lambda) = -z\lambda + x - (u + zf)\lambda^{-1} + \delta_1^{(2)} \lambda^{-2} + O(\lambda^{-3}),\]
\[\Psi_2^b(\lambda) = -t_2 \lambda^2 + y - 2ft_2 + \tilde\delta_2^{(1)} \lambda^{-1} + O(\lambda^{-2}),\] 
(80)

where
\[
\tilde\delta_2^{(1)} = 2t_2 \left(f_z + u_s f_s + v_t f_t\right) - v_s.
\]
(81)

The coefficients \(\delta_2^{(2)}\) and \(\tilde\delta_2^{(1)}\) satisfy also the compatible equations
\[
\left(\delta_2^{(2)}\right)_j = \left(\partial_{\xi_j} + u_s \partial_s + v_t \partial_t\right)(v + t_1f), \\
\left(\tilde\delta_2^{(1)}\right)_j = 2t_2 \left(f_z + u_s f_s + v_t f_t\right) - v_s.
\]
(82)

The proof of these results is standard, in the NRH dressing philosophy (see, f.i., [44] for details). The NRH dressing for the Dunajski hierarchy was presented in [4], but no connection to the initial data was given.

We remark that the NRH problem (71) is characterized by the following nonlinear integral equations in the spectral variable \(\lambda\):
\[\Psi_j^{-} = \nu_j + \frac{1}{2\pi i} \int_{C_j} \frac{d\lambda'}{\lambda' - (\lambda - ir)} R_j\left(\overline{\Psi_j^{-}}(\lambda')\right), \quad j = 0, 1, 2,\]
(83)

where \(R_j(\zeta) \equiv \overline{R}_j(\overline{\zeta}) - \zeta_j, \quad j = 0, 1, 2,\) and the reconstruction formulae for \(f\) and \(u\) in terms of the RH data read
\[f = -\frac{1}{2\pi i} \int_{C_0} R_0\left(\overline{\Psi^{-}}(\lambda)\right) d\lambda, \\
u = -zf + \frac{1}{2\pi i} \int_{C_1} R_1\left(\overline{\Psi^{-}}(\lambda)\right) d\lambda, \]
(84)

while that for \(v\) is, respectively
\[v = -t_1f + \frac{1}{2\pi i} \int_{C_2} R_2\left(\overline{\Psi^{-}}(\lambda)\right) d\lambda, \]
(85)
and
\[ v_z = \left(1 - 2t_2 f_z \right)^{-1} \left(2t_2 \left(f_z + u_x f_z \right) + \frac{1}{2\pi i} \int_{\mathbb{R}} R_z \left(\bar{\Psi}^- (\lambda) \right) d\lambda \right), \tag{86} \]
for equations (2), (6) and (7).

The reality constraint for the RH data is a nonlinear formula:
\[ \bar{R} \left( \bar{R} \left( \xi \right) \right) = \xi, \quad \forall \xi \in \mathbb{C}^3. \tag{87} \]
It is possible to show that, if \( \bar{R} \) satisfies (87), then
\[ \bar{\Psi}^- (\lambda) = \bar{\Psi}^+ (\lambda), \quad \lambda \in \mathbb{R} \tag{88} \]
and \( u, v, f \in \mathbb{R} \) (see, f.i., [44] for the proof).

We end our remarks noticing that, in the normalization (75) corresponding to the second equation of the Dunajski hierarchy, the dependence on the variable \( y \) is through the combination \( y f t \). Therefore the eigenfunctions \( \bar{\Psi}^- \) and, through the reconstruction formulae (84) and (86), also the solutions \( f, u, v \), depend on \( y \) through the combination \( y f t \). This is conceptually similar to the case of dKP [44], and provides the spectral mechanism for breaking of smooth and localized solutions of equations (6) and (7) at finite time \( t_2 \). Such a mechanism is absent in equation (2), for which a smooth evolution is expected.

5. Constrained spectral data and reductions

In this section we show how the eigenfunctions and the spectral data are constrained for the reductions of section 2.2 We have already investigated in sections 4 and 3 the reality reduction. We concentrate now on the divergence free constraint and on the constraint that the Orlov eigenfunctions are polynomial in \( \lambda \).

5.1. The divergence free constraint

Consider a basis of three independent eigenfunctions \( \psi_j, \ j = 0, 1, 2 \) of \( \hat{L} \) and construct the \( 3 \times 3 \) Jacobian matrix
\[ M = \begin{pmatrix} \frac{\partial (\psi_0, \psi_1, \psi_2)}{\partial (x_0, x_1, x_2)} \\ \frac{\partial (x_0, x_1, x_2)}{\partial (\lambda, x, y)} \end{pmatrix} \tag{89} \]
(of components \( M_{ij} = \partial \psi_i / \partial x_j, \ i, j = 0, 1, 2 \), and its determinant \( J (\bar{\psi}) = \det M \), where \( x_0 = \lambda, \ x_1 = x, \ x_2 = y \). Then the following identity holds true:
\[ \hat{L} J = - (u_x + v_y) J, \tag{90} \]
implying that, if \( \hat{L} \) is divergence free, i.e., if \( u, v \) satisfy the constraint (9), then \( J (\bar{\psi}) \) is also an eigenfunction of \( \hat{L} : \hat{L} J (\bar{\psi}) = 0 \).

In the direct problem philosophy, it is possible to show that, if \( u, v \) satisfy the divergence free constraint (9), then the spectral data \( \hat{S}, \hat{\xi}, \hat{R} \) satisfy the following constraint
\[ \det \begin{pmatrix} \frac{\partial (D_0, D_1, D_2)}{\partial (x_0, x_1, x_2)} \end{pmatrix} = 1. \tag{91} \]
To show it, consider as a basis the Jost solutions \( \hat{\phi}_j \). Then \( \hat{L} J (\hat{\phi}) = 0 \) and \( J (\hat{\phi}) \to 1 \) as \( z \to -\infty \). Since 1 is an exact eigenfunction, then \( J (\hat{\phi}) = 1 \). Analogously, \( J (\bar{\Psi}^+) \sim 1 \) as
λ ∼ ∞; therefore \( J(\vec{\Psi}^\pm) \) are analytic eigenfunctions going like 1 at \( \lambda \sim \infty \); then \( J(\vec{\Psi}^\pm) = 1 \). Evaluating \( J(\vec{\phi}) = 1 \) at \( z = \infty \), we infer the constraint (91) for the scattering data \( \vec{S} \). In addition, from equations (49) and (50), it follows that

\[
J(\vec{\Psi}^\pm) = \det \left( \frac{\partial \left( K_{\vec{\rho}}, K_{\vec{\gamma}}, K_{\vec{z}} \right)}{\partial (\zeta_0, \zeta_1, \zeta_2)} \right) J(\vec{\phi}),
\]

(92)
implying the constraint (91) for the spectral data \( \vec{\kappa} \). At last, from the NRH problem (72), it follows that

\[
J(\vec{\Psi}^+) = \det \left( \frac{\partial \left( R_{\vec{\alpha}}, R_{\vec{\beta}}, R_{\vec{\gamma}} \right)}{\partial (\zeta_0, \zeta_1, \zeta_2)} \right) J(\vec{\phi}^-),
\]

(93)
implying the constraint (91) for the RH data \( \vec{R} \).

Viceversa, in the inverse problem, if the RH data satisfy the constraint (91), i.e., the NRH problem (71) is a volume preserving mapping, then \( u, v \), reconstructed via (76), (77), satisfy the divergence free constraint (9). The proof is standard (see, f.i., [4]): if the RH data \( \vec{R}(\zeta) \) satisfy the volume preserving constraint (91), then, from (93), it follows that \( J(\vec{\Psi}^+) = J(\vec{\Psi}^-) = 1 \). As a consequence of it, the coefficient \(- (u_+ + u_-)\) of the \( O(\lambda^{-1}) \) expansion of \( J(\vec{\Psi}^+) \) must be zero, and the constraint (9) follows.

\[\square\]

5.2. Polynomial Orlov eigenfunctions

Let \( g(\lambda) \) be an eigenfunction of \( \hat{L} \) characterized by the formal expansion

\[ g(\lambda) \sim \sum_{n=-\infty}^N g_n \lambda^n, \quad |\lambda| \gg 1, \quad N > 0. \]

(94)

Then the coefficients \( g_n \) satisfy the recursion relations

\[
0 = g_{N+1},
0 = g_{N-1} + (\hat{\zeta} + \hat{\eta})g_N,
0 = g_{n-1} - (n+1)g_{n+1} + (\hat{\zeta} + \hat{\eta})g_n, \quad n < N.
\]

(95)

This three term recursion implies that, if \( g_{-1} = 0 \), then \( g_n = 0 \) for any \( n \leq -1 \). Consequently \( g_{-1} = 0 \) is an admissible constraint for the dPDEs associated with \( \hat{L} \), and corresponds to the existence of polynomial eigenfunctions \( g(\lambda) \).

Let us apply this mechanism to the Orlov-type eigenfunctions \( \vec{\Psi}_1^\pm \) and \( \vec{\Psi}_2^\pm \) of the first two members of the hierarchy. The reduction (13) is the \( g_{-1} = 0 \) condition for the eigenfunctions \( \vec{\Psi}_1^\pm \) (see (78) and (80)); therefore it corresponds to the following elementary constraints on the eigenfunctions and data:

\[ \vec{\Psi}_1^\pm = x - \lambda \zeta \quad \Leftrightarrow \quad R_1 = 0 \]

(96)

for both dynamics (2), (6) and (7). The reduction (15) is the \( g_{-1} = 0 \) condition for the eigenfunctions \( \vec{\Psi}_2^\pm \) for the dynamics (2), and it corresponds to

\[ \vec{\Psi}_2^\pm = y - \lambda \eta \quad \Leftrightarrow \quad R_2 = 0. \]

(97)

The \( g_{-1} = 0 \) condition for the eigenfunctions \( \vec{\Psi}_2^\pm \) for the dynamics (6) and (7) corresponds instead to the differential reduction (28) (see (80) and (81)); it corresponds to the constraints
Using the above constraints on the RH data, we can characterize the solutions of the reductions presented in section 2 in terms of NRH problems.

1. Consider the NRH problem (72) and let the RH data satisfy the volume preserving condition

\[
\frac{\partial}{\partial(\zeta^0, \zeta^1, \zeta^2)} \det \begin{pmatrix} R_0 \left( \zeta^0, \zeta^1, \zeta^2 \right), R_1 \left( \zeta^0, \zeta^1, \zeta^2 \right), R_2 \left( \zeta^0, \zeta^1, \zeta^2 \right) \end{pmatrix} = 1; \tag{99}
\]

then \( f \) and \( \theta \), defined by

\[
f = -\frac{1}{2\pi i} \int_{\mathbb{R}} R_0 \left( \Psi_0^{-} (\lambda), \Psi_1^{-} (\lambda), \Psi_2^{-} (\lambda) \right) d\lambda,
\]

\[
\theta = -zf + \frac{1}{2\pi i} \int_{\mathbb{R}} R_1 \left( \Psi_0^{-} (\lambda), \Psi_1^{-} (\lambda), \Psi_2^{-} (\lambda) \right) d\lambda, \tag{100}
\]

solve the Dunajski equation (10) [4] and equations (23) for, respectively, the normalizations (74) and (75).

If, in addition, \( R_0 = 0 \), then \( f = 0, \Psi_0^+ = \lambda, (72) \) becomes the 2-vector NRH problem

\[
\begin{align*}
\Psi_0^+ (\lambda) &= R_1 (\lambda, \Psi_1^+ (\lambda), \Psi_2^+ (\lambda)), \\
\Psi_2^+ (\lambda) &= R_2 (\lambda, \Psi_1^+ (\lambda), \Psi_2^+ (\lambda)),
\end{align*} \tag{101}
\]

the RH data satisfy the constraint

\[
\frac{\partial}{\partial(\zeta^0, \zeta^1, \zeta^2)} \det \begin{pmatrix} R_1 \left( \zeta^0, \zeta^1, \zeta^2 \right), R_2 \left( \zeta^0, \zeta^1, \zeta^2 \right) \end{pmatrix} = 1, \tag{102}
\]

and

\[
\theta = \frac{1}{2\pi i} \int_{\mathbb{R}} R_1 (\lambda, \Psi_1^+ (\lambda), \Psi_2^+ (\lambda)) d\lambda \tag{103}
\]

solves the heavenly equation (12) [41] and equations (24) for, respectively, the normalizations (74) and (75).

2. Let \( R_1 = 0 \); then \( u = -zf, \Psi_1^+ = x - \lambda z, (72) \) becomes the 2-vector NRH problem

\[
\begin{align*}
\Psi_0^+ (\lambda) &= R_0 (\Psi_0^- (\lambda), x - \lambda z, \Psi_2^- (\lambda)), \\
\Psi_2^+ (\lambda) &= R_2 (\Psi_0^- (\lambda), x - \lambda z, \Psi_2^- (\lambda)),
\end{align*} \tag{104}
\]

and \( f, v \), constructed via

\[
f = -\frac{1}{2\pi i} \int_{\mathbb{R}} R_0 \left( \Psi_0^- (\lambda), x - \lambda z, \Psi_2^- (\lambda) \right) d\lambda,
\]

\[
v = -tf + \frac{1}{2\pi i} \int_{\mathbb{R}} R_2 \left( \Psi_0^- (\lambda), x - \lambda z, \Psi_2^- (\lambda) \right) d\lambda \tag{105}
\]

and via

\[
f = -\frac{1}{2\pi i} \int_{\mathbb{R}} R_0 \left( \Psi_0^- (\lambda), x - \lambda z, \Psi_0^+ (\lambda) \right) d\lambda,
\]

\[
v = \left( 1 - 2zf \right)^{-1} \left( 2zf - zf^2 \right) + \frac{1}{2\pi i} \int_{\mathbb{R}} R_2 \left( \Psi_0^- (\lambda), x - \lambda z, \Psi_2^- (\lambda) \right) d\lambda \tag{106}
\]

solve equations (14) and (25) for, respectively, the normalizations (74) and (75).
If, in addition, the RH data satisfy the volume constraint

$$\det \left( \frac{\partial \left( R_0(\xi), R_2(\xi) \right)}{\partial (\zeta_0, \zeta_2)} \right) = 1,$$

then (17) are satisfied and

$$\bar{\zeta} = \frac{1}{2\pi i} \int R_0(\Psi^0_0(\lambda), x - \lambda \zeta, \Psi^2_2(\lambda)) d\lambda,$$

solves equations (18) and (27) for, respectively, the normalizations (74) and (75).

3. Let $R_2 = 0$; then $v = -if$, $\Psi^2_2 = \nu^{(1)}_2(\lambda)$ for the normalization (74), and $v$ satisfies the constraint (28) and $\Psi^2_2 = \nu^{(2)}_2(\lambda)$ for the normalization (75), where of course

$$\nu^{(1)}_2(\lambda) = y - \lambda t_1, \quad \nu^{(2)}_2(\lambda) = -\lambda^2 t_2 + y - 2 ft_2.$$

The NRH problem (72) becomes the 2-vector NRH problems

$$\Psi^0_0(\lambda) = R_0 \left( \Psi^0_0(\lambda), \Psi^1_1(\lambda), \nu^{(0)}_2(\lambda) \right),$$

$$\Psi^1_1(\lambda) = R_1 \left( \Psi^0_0(\lambda), \Psi^1_1(\lambda), \nu^{(0)}_2(\lambda) \right), \quad n = 1, 2$$

for the normalizations (74) and (75), and

$$f = -\frac{1}{2\pi i} \int R_0 \left( \Psi^0_0(\lambda), \Psi^1_1(\lambda), \nu^{(0)}_2(\lambda) \right) d\lambda,$$

$$u = -zf + \frac{1}{2\pi i} \int R_1 \left( \Psi^0_0(\lambda), \Psi^1_1(\lambda), \nu^{(0)}_2(\lambda) \right) d\lambda$$

are solutions of (16) and (29) for respectively $n = 1$ and $n = 2$.

If $R_1 = R_2 = 0$, then we have also $u = -zf$, $\Psi^1_1 = x - \lambda \tilde{f}$, and (72) becomes the scalar NRH problems

$$\Psi^0_0 = R_0 \left( \Psi^0_0, x - \lambda \tilde{f}, \nu^{(0)}_2(\lambda) \right), \quad n = 1, 2$$

for, respectively, the normalizations (74) and (75). At last,

$$f = -\frac{1}{2\pi i} \int R_0 \left( \Psi^0_0, x - \lambda \zeta, \nu^{(0)}_2(\lambda) \right) d\lambda, \quad n = 1, 2$$

solve respectively the scalar dPDE (20) and equation (31).

If $R_2 = 0$ and the RH data satisfy the volume constraint

$$\det \left( \frac{\partial \left( R_0(\xi), R_1(\xi) \right)}{\partial (\zeta_0, \zeta_1)} \right) = 1,$$

then $f$ and $\theta$, defined by

$$f = -\frac{1}{2\pi i} \int R_0 \left( \Psi^0_0(\lambda), \Psi^1_1(\lambda), \nu^{(2)}_2(\lambda) \right) d\lambda,$$

$$\theta = -zf + \frac{1}{2\pi i} \int R_1 \left( \Psi^0_0(\lambda), \Psi^1_1(\lambda), \nu^{(2)}_2(\lambda) \right) d\lambda,$$

solve equations (33).
6. The longtime behaviour of the solutions

As in the classical IST method, the two inverse problems for dPDEs are expressed in terms of integral equations in the spectral variable \( \lambda \), and the dependent and independent variables of the nonlinear dPDE appear there just as parameters. Therefore they are convenient tools to construct the longtime behavior of the solutions of the dPDEs, and this section is devoted to this goal, concentrating on the NRH problem.

We start with the nonlinear integral equations (83), characterizing the NRH problems associated respectively with the dynamics (2), (6) and (7), corresponding respectively to the normalizations (74) and (75), and we rewrite it in a more convenient form

\[
\Phi_j(\lambda) = \frac{1}{2\pi i} \int_R \frac{d\lambda'}{\lambda' - (\lambda - i\epsilon)} R_j(\bar{\nu} + \mathcal{F}), \quad j = 0, 1, 2
\]

in terms of the functions \( \Phi_j \), defined by

\[
\Phi_j(\lambda) = \Psi_j(\lambda) - \nu_j(\lambda), \quad j = 0, 1, 2.
\]

Consider first the case (74), and

\[
t_1 \gg 1, \quad x = v_1 t_1, \quad y = v_2 t_1 + \bar{y}, \quad z = v_3 t_1, \quad \bar{y}, v_1, v_2, v_3 = O(1).
\]

Then equations (116) read

\[
\Phi_j(\lambda) = \frac{1}{2\pi i} \int_R \frac{d\lambda'}{\lambda' - (\lambda - i\epsilon)} R_j(\bar{\lambda}' + \Phi_0(\bar{\lambda}'), - v_3(\bar{\lambda}' - v_1/v_3) t_1 + \Phi_1(\bar{\lambda}'),
\]

\[
- (\bar{\lambda}' - v_2) t_1 + \bar{y} + \Phi_2(\bar{\lambda}')), \quad j = 0, 1, 2.
\]

For \( t \gg 1 \), the decay of the integral is partially contrasted if \( v_2 = v_1/v_3 \) or, equivalently, on the surface

\[
y - \frac{x}{\bar{z}} t_1 = \bar{y}
\]

of the \((x, y, z)\) space, on which the second and third argument of \( R_j \) \((j = 0, 1, 2)\) in (119) have the same linear growth in \( t_1 \). On such a surface, since the main contribution to the integrals occurs when \( \lambda' \sim v_2 \), it is convenient to make the change of variables \( \mu' = (\bar{\lambda}' - v_2) t_1 \), obtaining, for \( t_1 \gg 1 \):

\[
\Phi_j(\lambda) \sim \frac{1}{2\pi i t_1} \int_R \frac{d\mu'}{v_2 + \frac{\mu'}{t_1} - (\lambda - i\epsilon)} R_j(v_2 + \Phi_0(v_2 + \frac{\mu'}{t_1}), - v_3 \mu' + \Phi_1(v_2 + \frac{\mu'}{t_1}),
\]

\[
- \mu' + \bar{y} + \Phi_2(v_2 + \frac{\mu'}{t_1})), \quad j = 0, 1, 2.
\]

If \( |\lambda - v_2| \gg O(t_1^{-1}) \), equation (121) implies that \( \Phi_j(\lambda) = O(t_1^{-1}) \):

\[
\Phi_j(\lambda) \sim - \frac{1}{2\pi i t_1(\lambda - v_2 - i\epsilon)} \int_R d\mu R_j(v_2 + \Phi_0(v_2 + \frac{\mu}{t_1}), - v_3 \mu + \Phi_1(v_2 + \frac{\mu}{t_1}),
\]

\[
- \mu + \bar{y} + \Phi_2(v_2 + \frac{\mu}{t_1})), \quad j = 0, 1, 2.
\]
while, for $\lambda - \nu_2 = \mu / t_1, \mu = O(1)$, then $\Phi_j(\lambda) = O(1)$ and
\[
\Phi_j(v_2 + \mu / t_1) \sim \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{d\mu'}{\mu' - (\mu - i\varepsilon)} \mathcal{R}_j \left( v_2 + \Phi_0 \left( v_2 + \frac{\mu'}{t_1} \right) - \nu_3 \mu' + \Phi_1 \left( v_2 + \frac{\mu'}{t_1} \right) \right) - \mu' + \tilde{y} + \Phi_2 \left( v_2 + \frac{\mu'}{t_1} \right), \quad j = 0, 1, 2. \tag{123}
\]
Equation (122) for $|\lambda| \gg 1$ and (76) finally imply the following result.

In the space-time regions (118) and (120), the longtime behavior of the solution $(f, u, v)$ of (2) reads
\[
f \sim \frac{1}{t_1} F \left( \frac{x}{z}, \frac{y}{z}, \frac{y - x}{z} \right) + o \left( t_1^{-1} \right),
\]
\[
u = -z F \left( \frac{x}{z}, \frac{y}{z}, \frac{y - x}{z} \right) + o(1),
\]
\[
\tilde{v} = F \left( \frac{x}{z}, \frac{y}{z}, \frac{y - x}{z} \right) + o(1), \quad \tag{124}
\]
where
\[
F(\xi, \zeta, \eta) := -\frac{1}{2\pi i} \int_{\mathbb{R}} d\mu R_0 \left( \xi + A_0(\mu; \xi, \zeta, \eta), -\zeta \mu + A_1(\mu; \xi, \zeta, \eta), \right.
\]
\[
\left. - \mu + \eta + A_2(\mu; \xi, \zeta, \eta) \right), \quad \tag{125}
\]
and the fields $A_j(\mu) = \Phi_j(v_2 + \mu / t_1), j = 0, 1, 2$ are defined by the nonlinear integral equations
\[
A_j(\mu; \xi, \zeta, \eta) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{d\mu'}{\mu' - (\mu - i\varepsilon)} \mathcal{R}_j \left( \xi + A_0(\mu'; \xi, \zeta, \eta), -\zeta \mu' + A_1(\mu'; \xi, \zeta, \eta), \right.
\]
\[
\left. - \mu' + \eta + A_2(\mu'; \xi, \zeta, \eta) \right), \quad j = 0, 1, 2. \tag{126}
\]
We observe that the longtime behavior of the solution of (2) has the same structure as the solution of the linearized version of equation (2), except from the fact that function $F$ is constructed through the solution of a nonlinear system of integral equations. In addition, the solution, concentrated asymptotically on the wave front (120), does not break.

The situation drastically changes if we consider the longtime behavior of solutions of (6) and (7) in the space regions
\[
t_2 \gg 1, \quad x = v_1 t_2, \quad y = v_2 t_2 + \tilde{y}, \quad z = v_3 t_2, \quad \tilde{y}, v_1, v_2, v_3 = O(1). \tag{127}
\]
Indeed, in this case, in the region (127) the integral equations (116) become
\[
\Phi_j(\lambda) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{d\lambda'}{\lambda' - (\lambda - i\varepsilon)} \mathcal{R}_j \left( \lambda' + \Phi_0(\lambda'), -\nu_3 (\lambda' - v_1 / v_3) t_2 + \Phi_1(\lambda'), \right.
\]
\[
\left. - \left( \lambda' v_2 - v_3 \right) t_2 + \tilde{y} - 2 t_2 + \Phi_2(\lambda') \right), \quad j = 0, 1, 2 \tag{128}
\]
and now the decay of the integral is partially contrasted if $v_2 > 0$ and $\pm \sqrt{v_2} = v_1 / v_3$; i.e., on the surface
of the \((x, y, z)\) space. On such a surface, let us consider the case \(\sqrt{v_2} = v_1/v_3\), it is convenient to make the change of variables \(y' = (\lambda' - \sqrt{v_2})t_2\) in (128), obtaining, for \(t_2 \gg 1\):

\[
\Phi_j(\lambda) \sim \frac{1}{2\pi i t_2} \int_R \frac{d\mu'}{\sqrt{v_2} + \frac{\mu'}{t_2}} \left(\sqrt{v_2} + \Phi_0 \left(\sqrt{v_2} + \frac{\mu'}{t_2}\right)\right) - \nu_3 \mu' + \Phi_1 \left(\sqrt{v_2} + \frac{\mu'}{t_2}\right) - 2\sqrt{v_2} \mu' + \tilde{y} - 2f t_2 + \Phi_2 \left(\sqrt{v_2} + \frac{\mu'}{t_2}\right), \quad j = 0, 1, 2.
\]

If \(|\lambda - \sqrt{v_2}| \gg O(t_2^{-1})\), the \(\Phi'_j\) are \(O(t_2^{-1})\) and equation (121) becomes

\[
\Phi_j(\lambda) \sim -\frac{1}{2\pi i t_2(\lambda - i\epsilon)} \int_R d\mu R_j \left(\sqrt{v_2} + \Phi_0 \left(\sqrt{v_2} + \frac{\mu}{t_2}\right)\right) - \nu_3 \mu
\]

\[+ \Phi_1 \left(\sqrt{v_2} + \frac{\mu}{t_2}\right) - 2\sqrt{v_2} \mu + \tilde{y} - 2f t_2 + \Phi_2 \left(\sqrt{v_2} + \frac{\mu}{t_2}\right), \quad j = 0, 1, 2,
\]

while, if \(\lambda - \sqrt{v_2} = \mu/t_2, \mu = O(1)\), the \(\Phi'_j\) are \(O(1)\):

\[
\Phi_j \left(\sqrt{v_2} + \frac{\mu}{t_2}\right) \sim \frac{1}{2\pi i} \int_R \frac{d\mu'}{\mu' - (\mu - i\epsilon)} R_j \left(\sqrt{v_2} + \Phi_0 \left(\sqrt{v_2} + \frac{\mu'}{t_2}\right)\right) - \nu_3 \mu'
\]

\[+ \Phi_1 \left(\sqrt{v_2} + \frac{\mu'}{t_2}\right) - 2\sqrt{v_2} \mu' + \tilde{y} - 2f t_2 + \Phi_2 \left(\sqrt{v_2} + \frac{\mu'}{t_2}\right), \quad j = 0, 1, 2.
\]

Equation (131) for \(|\lambda| \gg 1\) and (77) finally imply the following result.

In the space-time regions (127) and (129) the longtime behavior of the solution \((f, u, v)\) of (6) and (7) reads

\[
f \sim \frac{1}{t_2} F \left(\frac{x}{z}, \frac{z}{t_2}, y - \frac{x^2}{z^2} t_2 - 2f t_2\right) + o(t_2^{-1}),
\]

\[u = -\frac{z}{t_2} F \left(\frac{x}{z}, \frac{z}{t_2}, y - \frac{x^2}{z^2} t_2 - 2f t_2\right) + o(1),
\]

\[v_3 = \frac{2f z (f_z + u s_{t})}{1 - 2t f z} + o(1),
\]

where \(F\) is defined by

\[
F(\xi, \zeta, \eta) = \frac{1}{2\pi i} \int_R d\mu R_0 \left(\xi + \Lambda_0(\mu; \xi, \zeta, \eta), -\zeta \mu + \Lambda_1(\mu; \xi, \zeta, \eta),
\]

\[-2\zeta \mu + \eta + \Lambda_2(\mu; \xi, \zeta, \eta),
\]

\]

\[+ \Phi_2 \left(\sqrt{v_2} + \frac{\mu}{t_2}\right), \quad j = 0, 1, 2.
\]

\]\n
\[+ \Phi_2 \left(\sqrt{v_2} + \frac{\mu}{t_2}\right), \quad j = 0, 1, 2.
\]

\[+ \Phi_2 \left(\sqrt{v_2} + \frac{\mu}{t_2}\right), \quad j = 0, 1, 2.
\]
and the fields $A_j$, $j = 0, 1, 2$ are defined by the nonlinear integral equations

$$A_j(\mu; \xi, \zeta, \eta) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{du'}{u' - (\mu - i\nu)} R_j(\xi + A_0(\mu'; \xi, \zeta, \eta), -\zeta u' + A_1(\mu'; \xi, \zeta, \eta), -2\zeta u' + \eta + A_2(\mu'; \xi, \zeta, \eta)).$$

By considering the case $\sqrt{\nu_2} = -\nu_1/\nu_3$, we obtain the same result as above. Equations (133) show that, even in the longtime regime, localized solutions of equations (6) and (7) break on the wave front surface (129) in a way similar to the dKP breaking [44].

7. Concluding remarks

In this paper we have applied the formal IST for integrable dPDEs to the so-called Dunajski hierarchy. We concentrated, in particular, (i) on the system of PDEs characterizing a general anti-self-dual conformal structure in neutral signature, (ii) on its first commuting flow, and (iii) on some of their basic and novel reductions. We have formally solved their Cauchy problem and used it to construct the longtime behavior of solutions, showing, in particular, that unlike the case of soliton PDEs, different dPDEs belonging to the same hierarchy of commuting flows evolve in time in very different ways, exhibiting either a smooth dynamics or a gradient catastrophe at finite time.

In a subsequent paper, we plan (i) to investigate the analytical aspects of such a wave breaking, taking place, in the longtime regime, on the 3-dimensional surface (129); (ii) to construct, following [47], distinguished classes of implicit solutions of the Dunajski hierarchy and of its reductions, concentrating also on the class of reductions introduced in [5].

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References

[1] Ablowitz M J and Clarkson P A 1991 Solitons, Nonlinear Evolution Equations and Inverse Scattering (London Mathematical Society Lecture Note)
[2] Ablowitz M J and Segur H 1981 Solitons and the Inverse Scattering Transform (Philadelphia: SIAM)
[3] Bogdanov L V 2010 On a class of reductions of Manakov–Santini hierarchy connected with the interpolating system J. Phys. A: Math. Theor. 43 115206
[4] Bogdanov L V, Dryuma V S and Manakov S V 2007 Dunajski generalization of the second heavenly equation: dressing method and the hierarchy J. Phys. A: Math. Theor. 40 14383–93
[5] Bogdanov L V 2011 Interpolating differential reductions of multidimensional integrable hierarchies Theor. Math. Phys. 167 705–13
[6] Bogdanov L V and Konopelchenko B G 2005 On the $\bar{\partial}$-dressing method applicable to heavenly equation Phys. Lett. A 345 137–43
[7] Bogdanov L V and Konopelchenko B G 2013 Grassmannians Gr(N–1, N+1), closed differential N-forms and N-dimensional integrable systems J. Phys. A: Math. Theor. 46 085201
[8] Boyer C and Finley J D 1982 Killing vectors in self-dual, Euclidean Einstein spaces J. Math. Phys. 23 1126–8
[9] Calogero F and Degasperis A 1982 Spectral Transform and Solitons (Amsterdam: North-Holland)
[10] Dunajski M 2002 Anti-self-dual four manifolds with a parallel real spinor Proc. R. Soc. A 458 1205–22
[11] Dunajski M 2004 A class of Einstein–Weyl spaces associated to an integrable system of hydrodynamic type J. Geom. Phys. 51 126–37
[12] Dunajski M, Ferapontov E V and Kruglikov B 2014 On the Einstein–Weyl and conformal self-duality equations arXiv:1406.0018v2
[13] Dunajski M and Mason L J 2000 Hyper-Kähler hierarchies and their twistor theory Commun. Math. Phys. 213 641–72
[14] Dunajski M and Mason L J 2003 Twistor theory of hyper-Kähler metrics with hidden symmetries J. Math. Phys. 44 3430–54
[15] Dunajski M, Mason L J and Tod K P 2001 Einstein–Weyl geometry, the dKP equation and twistor theory J. Geom. Phys. 37 63–93
[16] Dunajski M and Tod K P 2002 Einstein–Weyl spaces and dispersionless Kadomtsev–Petviashvili equation from Painlevé I and II Phys. Lett. A 303 253–64
[17] Ferapontov E V and Khusnutdinova K R 2004 On integrability of (2+1)-dimensional quasilinear systems Commun. Math. Phys. 248 187–206
[18] Finley J D and Plebanski J F 1979 The classification of all $\mathbb{K}^*$ spaces admitting a Killing vector J. Math. Phys. 20 1938
[19] Gakhov F D 1990 Boundary Value Problems (New York: Dover)
[20] Gardner C S, Greene J M, Kruskal M D and Miura R M 1967 Method for solving the Korteweg–de Vries equation Phys. Rev. Lett. 19 1095–7
[21] Gegenberg J D and Das A 1984 Stationary Riemannian space-times with self-dual curvature Gen. Relativ. Gravit. 16 817–29
[22] Guit F, Manas M and Martinez Alonso L 2003 On twistor solutions of the dKP equation J. Phys. A: Math. Gen. 36 6457–72
[23] Grinevich P G and Santini P M 2013 Holomorphic eigenfunctions of the vector field associated with the dispersionless Kadomtsev–Petviashvili equation J. Differ. Equ. 255 1469–91
[24] Grinevich P G, Santini P M and Wu D 2013 The Cauchy problem for the Pavlov equation preprint arXiv:1310.5834v2
[25] Hitchin N J 1982 Complex manifolds and Einstein equations from Painlevé I and II J. Math. Phys. 23 415–29
[26] Hitchin N J 1982–83 The $\tau$-function of the universal W itham hierarchy, matrix models and topological field theories Commun. Pure Appl. Math. 47 437–75
Manakov S V and Santini P M 2008 On the solutions of the dKP equation: the nonlinear Riemann–Hilbert problem, longtime behaviour, implicit solutions and wave breaking J. Phys. A: Math. Theor. 41 055204
Manakov S V and Santini P M 2009 On the solutions of the second heavenly and Pavlov equations J. Phys. A: Math. Theor. 42 404013
Manakov S V and Santini P M 2009 The dispersionless 2D Toda equation: dressing, Cauchy problem, longtime behaviour, implicit solutions and wave breaking J. Phys. A: Math. Theor. 42 095203
Manakov S V and Santini P M 2011 Solvable vector nonlinear Riemann problems, exact implicit solutions of dispersionless PDEs and wave breaking J. Phys. A: Math. Theor. 44 345203
Manakov S V and Santini P M 2011 On the dispersionless Kadomtsev–Petviashvili equation in n +1 dimensions: exact solutions, the Cauchy problem for small initial data and wave breaking J. Phys. A: Math. Theor. 44 405203
Manakov S V and Santini P M 2012 Wave breaking in solutions of the dispersionless Kadomtsev–Petviashvili equation at finite time Theor. Math. Phys. 172 1118–26
Manakov S V and Santini P M 2014 Integrable dispersionless PDEs arising as commutation condition of pairs of vector fields Proc. Conf. PMNP 2013 J. Phys.: Conf. Ser. 482 012029
Mineev-Weinstein M, Wigmann P and Zabrodin A 2000 Integrable structure of interface dynamics Phys. Rev. Lett. 84 5106
Neyzi F, Nutku Y and Sheftel M B 2005 A multi-hamiltonian structure of the Plebanski’s second heavenly equation J. Phys. A: Math. Gen. 38 8473–85
Pavlov M V 2003 Integrable hydrodynamic chains J. Math. Phys. 44 4134–56
Penrose R 1976 Nonlinear gravitons and curved twistor theory Gen. Relativ. Gravit. 7 31–52
Plebanski F 1975 Some solutions of complex Einstein equations J. Math. Phys. 16 2395–402
Takasaki K 1991 Area preserving diffeomorphisms and nonlinear integrable systems Proc. Topological and Geometrical Methods in Field Theory (Turku, Finland, 1991) p 383
Takasaki K and Takebe T 1991 SDiff(2) Toda equation—hierarchy, tau function and symmetries Lett. Math. Phys. 23 205–14
Takasaki K and Takebe T 1992 SDiff(2) KP hierarchy Infinite Analysis Adv. Ser. Math. Phys. vol 16 ed A Tsuchiya et al (Singapore: World Scientific) pp 889–922 (part B)
Takasaki K and Takebe T 1995 Integrable hierarchies and dispersionless limit Rev. Math. Phys. 7 743
Timman R 1962 Unsteady motion in transonic flow Symposium Transsonicum (Aachen, 1962) ed K Oswatitsch (Berlin: Springer) pp 394–401
Wigmann P and Zabrodin A 2000 Conformal maps and integrable hierarchies Commun. Math. Phys. 213 523–38
Zakharov V E 1994 Dispersionless limit of integrable systems in 2+1 dimensions Singular Limits of Dispersive Waves ed N M Ercolani et al (New York: Plenum)
Ward R S 1990 Einstein–Weyl spaces and SU(∞) Toda fields Class. Quantum Grav. 7 95–98
Zakharov V E, Manakov S V, Novikov S P and Pitaevsky L P 1984 Theory of Solitons (New York: Plenum)
[65] Zakharov V E and Shabat A B 1979 Integration of nonlinear equations of mathematical physics by the method of inverse scattering: II. Funct. Anal. Appl. 13 166–74
[66] Zakharov V E 1982 Integrable systems in multidimensional spaces Lecture Notes in Physics. vol 153 (Berlin: Springer) pp 190–216
[67] Zobolotskaya E A and Kokhlov R V 1969 Quasi-plane waves in the nonlinear acoustics of confined beams Sov. Phys.- Acoust. 15 35–40