Nonlinear longitudinal oscillations of an elastic rod in the presence of the nonstationary source

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Abstract. The article studies the nonlinear longitudinal oscillations of an elastic rod in the presence of a non-stationary external source described by a nonlinear differential equation that admits the widest group of Lie transformations among the equations of the main model that describes this physical process. The model under study describes longitudinal oscillations at the presence of a very strong external influence, singular at the initial moment of time. We studied invariant submodels of this model which are described by exact solutions. We found their physical meaning. For some specific values of the parameters that determine these submodels the graphs of the distribution of the longitudinal displacement are constructed.

1. Introduction

Many mathematical models of physics and continuum mechanics are formulated in the form of linear and quasi-linear differential equations. Mathematical model is a description of the real scheme by mathematical language. The symmetry analysis of the equations of the models of physics and mechanics of continuous media is one of the most effective ways to obtain quantitative and qualitative characteristics of the physical processes [1, 2]. The main task of the symmetry analysis of differential equations is to study the set of the solutions of these equations. All algorithms of the symmetry analysis are the preparation for achieving this purpose. The modern concept of the symmetry analysis is understood as the fullest using of the group of transformations admitted by the equations of model primarily to obtain and research the exact solutions. Exact solutions allow us to describe the specific physical processes. Exact solutions can be used as test solutions in numerical calculations which perform in the studies of the real processes. Exact solutions allow us to assess the degree of adequacy of a given mathematical model to the real physical processes, after carrying out experiments corresponding to these decisions, and estimating the deviations that arise.

This paper is devoted to the study of the longitudinal oscillations of an elastic rod in the presence of the non-stationary source [3–5] that are described by nonlinear wave differential equation

\[ u_{tt} - [g(u)u]_x + f(t)u, \]  

(1)

where \( t \) is a time, \( x \) is a coordinate of the rod cross section, \( u=u(t,x) \) is a longitudinal displacement of the cross-section of the rod in a time \( t \), \( g=g(u)>0 \) defines a nature of the nonlinearity of its longitudinal oscillations, \( f=f(t) \) defines a nonstationary nature of the source.

The term \( f(t) \) means that the deformation of the rod occurs under the action of an unsteady external source with a force proportional to the longitudinal displacement. As an unsteady source,
which is considered in this model, there can be dynamic tension, dynamic compression, other external influences on the rod, causing only longitudinal displacements. That is, the physical stability of the rod takes a place. For a vertically standing column or pile, an external source can be, for example, natural seismic waves or waves generated after an underground blast, propagating along the column.

The functions \( f(t) \) and \( g(u) \) are arbitrary. They are determined empirically. We assume that a condition

\[
 f'(t) \neq 0 \quad (2)
\]

is satisfied. This condition means that the oscillations are nonlinear, and the source is non-stationary.

Using the method proposed in [6], a group classification of this equation was performed. As practice shows [1, 7], the most promising from the point of view of mathematical research and from the point of view of the most adequate description of the real process is the model that allows the widest group of Lie transformations. In the case of model (1), such a model is the model described by an equation

\[
 u_{tt} = \left( u^4 \frac{d}{dx} + \frac{\lambda}{t} u \right), \lambda = \text{const} \neq 0. \quad (3)
\]

This model is the main object of our study.

2. Invariant submodels

We will study some invariant submodels of the model, described by equation (3). These submodels are described by the invariant solutions of the equation (3). These solutions are substantially different solutions. They are not related by any point transforms. In order not to clutter up the paper with mathematical algorithms, we will not dwell on the algorithm for obtaining these solutions. The fact that they really are the solutions of the equation (3) is easily verified by their substitution into this equation.

We will not give the basic operators of the main Lie algebra of the transformations of this equation. Also, we will not indicate the subgroups with respect to which each of the solutions studied below is invariant.

1. The model (3) has an invariant submodel determined by the formula

\[
 u = c_1 t^{ \frac{\gamma - 2}{4} } x^3
\]

with parameter \( \gamma = \gamma_1 = -\frac{5+\sqrt{4\lambda+1}}{3} \) or \( \gamma = \gamma_2 = -\frac{5-\sqrt{4\lambda+1}}{3} \), where \( c_1 \) is arbitrary constant, \( \lambda > -\frac{1}{4} \) (\( \lambda \neq 0 \)).

When \( \lambda > -\frac{1}{4} \) there are two modes of invariant oscillation.

When \( \lambda = -\frac{1}{4} \) there is only one oscillation mode.

For \( \gamma = \gamma_1 \) and \( \lambda \in (0, -\frac{1}{4}) \) with growth of the time, a decrease in the longitudinal displacement occurs, and \( u \to 0 \) when \( t \to \infty \) for each cross-section \( x > 0 \).

For \( \gamma = \gamma_1 \) and \( \lambda > 0 \) with growth of the time, an increase in the longitudinal displacement occurs, and \( u \to \infty \) when \( t \to \infty \) for each cross-section \( x > 0 \). For \( \gamma = \gamma_2 \) and \( \lambda \in (-\frac{1}{4}, -\frac{1}{2}) \) with growth of the time, a decrease in the longitudinal displacement occurs, and \( u \to 0 \) when \( t \to \infty \) for each cross-section \( x > 0 \).

The displacement \( u \to \infty \) when \( x \to 0 \) at each moment of the time.

Let \( c_1 = 1, \lambda = 2 \). Figure 1 and Figure 2 show the change in longitudinal displacement \( u(t,x) \) over time for different rod cross-section for \( \gamma = \gamma_1 = -\frac{5}{3} \) and \( \gamma = \gamma_2 = -\frac{1}{3} \) respectively.

Figure 3 shows the change in longitudinal displacement \( u(t,x) \) over time for different rod cross-
section for \( c_1 = 1, \lambda = \frac{1}{4} \) and \( \gamma_1 = \gamma_2 = \frac{5}{3} \).

**Figure 1.** Distribution of the longitudinal displacement.

**Figure 2.** Distribution of the longitudinal displacement.
Figure 3. Distribution of the longitudinal displacement.

2. For \( \lambda < -\frac{1}{4} \) the model (3) has two following invariant submodels which are determined by the formulas:

\[
\begin{align*}
\text{u} & = c_2 t^3 \sin(b \ln t), \\
\text{u} & = c_3 t^3 \cos(b \ln t),
\end{align*}
\]

where \( c_2, c_3 \) are arbitrary constants, \( b = \frac{\sqrt{|4\lambda + 1|}}{2} \). With growth of the time, an increase in the longitudinal displacement occurs.

Let \( c_2 > 0, x > 0 \). The submodel (4) has a physical meaning only at each time interval

\[
\begin{bmatrix}
\exp\left(\frac{2n}{b}\right), \\
\exp\left(\frac{n+1}{b}\right)
\end{bmatrix}, (n=0, \pm 1, \pm 2, ...).
\]

Let \( c_3 > 0, x > 0 \). The submodel (5) has a physical meaning only at each time interval

\[
\begin{bmatrix}
\exp\left(\frac{4n-1}{2b}\right), \\
\exp\left(\frac{4n+1}{2b}\right)
\end{bmatrix}, (n=0, \pm 1, \pm 2, ...).
\]

Each of these submodels describes the nonlinear process with a displacement \( u \rightarrow \infty \) when \( x \rightarrow 0 \) at each moment of the time.

Let \( c_2 = c_3 = 1, \lambda = -\frac{5}{4} \).

Figure 4 and Figure 5 show the change in longitudinal displacement \( u(t,x) \) over time for different rod cross-section for (4) and (5) respectively.
Figure 4. Distribution of the longitudinal displacement when $t \in [1, \exp \pi], x \in [1, 6]$ for (4).

Figure 5. Distribution of the longitudinal displacement when $t \in \left[ \exp \left( -\frac{x}{2} \right), \exp \left( \frac{x}{2} \right) \right], x \in [0, 1]$ for (5).

3. Invariant submodel determined by the formula

$$u = \left( \frac{12}{\sqrt{13}} \right)^{\frac{3}{2}} t^{\frac{3}{2}} (x^2 + 1)^{\frac{1}{2}},$$

(6)
exists only for an external source with parameter $\lambda > \frac{3}{4}$. At an initial moment of the time $t = 0$ the longitudinal displacement is equal to zero.

With growth of the time, an increase in the longitudinal occurs. This submodel describes the nonlinear process with a displacement $u \to \infty$ when $t \to \infty$ for each cross-section of the rod.

Figure 6 shows the change in the longitudinal displacement $u(t,x)$ over time for different rod cross-section for $\lambda = 4$. In the point $x = 0$ of the displacement has a soliton growth. Therefore, the destruction of the rod will occur at this point.

![Figure 6. Distribution of the longitudinal displacement.](image-url)

4. For $\lambda > \frac{3}{4}$ the model (3) has also invariant submodel determined by the formula

$$u = \left(\frac{4\gamma-1}{3(4\gamma-3)}\right)^{\frac{3}{4}} \left(x^2 + \gamma\right)^{\frac{3}{2}}.$$

For $\gamma > \frac{1}{4}$ this submodel is not connected by the point transformation with the model (6). $\lambda > \frac{3}{4}$. At the initial moment of the time $t = 0$ the longitudinal displacement is equal to zero.

With growth of the time, an increase in the longitudinal occurs. This submodel describes the nonlinear process with a displacement $u \to \infty$ when $t \to \infty$ for each cross-section of the rod.

Let $\lambda = \gamma = 1$. Figure 7 shows the change in the longitudinal displacement $u(t,x)$ over time for different rod cross-section for $\lambda = 4$. In the point $x = 0$ of the displacement has a soliton growth. Therefore, the destruction of the rod will occur at this point.
3. Conclusion
The mechanical relevance of the obtained solutions is as follows: 1) these solutions describe special nonlinear longitudinal oscillations in an elastic rod of the presence of a very strong external influence at the initial moment of the time, 2) these solutions can be used as tests in numerical calculations, performed in the study of the nonlinear longitudinal oscillations in an elastic rod in the presence of a very strong external influence at the initial moment of the time. The obtained results also can be used to study nonlinear wave processes in the presence of the nonstationary external source in acoustics and electromagnetism.

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