REMARKS ON RUELLE OPERATOR AND INVARINAT LINE FIELDS PROBLEM

PETER M. MAKIENKO

Introduction and main statements

Let $R$ be a rational map with non-empty Fatou set. Assume that $J(R)$ supports an invariant non trivial conformal structure $\mu$. Let $f_\mu$ be its respective quasiconformal map. The main idea of this work is to find the conditions which allow to construct a quasiconformal map $h$ supported already by the Fatou set, so that $h$ and $f_\mu$ generate the same infinitesimal deformation of $R$ (see also [Mak]). This approach allows us to formulate conditions (in terms of Ruelle-Poincare series) which guarantee the absence of non trivial invariant conformal structures on the Julia set, see the theorems below. Also the necessary and sufficient conditions (in terms of convergence of sequences of measures) of existence of invariant conformal structures on $J(R)$ are obtained.

Main results.

Let $R$ be a rational map with non-empty Fatou set $F(R)$. Denote by $Pc(R)$ the postcritical set of $R$. In further, we always suppose that

1. all critical points are simple (that is if $R'(c) = 0$, then $R''(c) \neq 0$),
2. there exist no simple critical relations (that is $R$ has a simple critical relation iff there exist integers $n$ and $m$ and two different critical points $c_1$ and $c_2$, so that the following equality $R^n(c_1) = R^m(c_2)$ is hold)

Start again with a rational map $R$ and consider two actions $R_{n,m}^*$ and $R_{n,m}^{-1}$ on a function $\phi$ at point $z$ by the formulas

$$R_{n,m}^*(\phi) = \sum \phi(J_i)(J'_i)^n(J'_i)^m = \sum_{y \in R^{-1}(z)} \frac{\phi(y)}{(R'(y))^n(R'(y))^m},$$

$$R_{n,m}^{-1}(\phi) = \phi(R) \cdot (R')^n \cdot (R')^m,$$

where $n$ and $m$ are integers and $J_i, i = 1, ..., d$ are branches of the inverse map $R^{-1}$. Then we have

$$R_{n,m}^* \circ R_{n,m}^{-1}(\phi) = \text{deg}(R) \cdot \phi.$$

In other words the actions above are the natural action of $R$ on the spaces of the forms of the kind $\phi(z)D^m D\overline{z}^n$.

Definition.

1. The operator $R^* = R_{2,0}^*$ is called Ruelle operator of rational map $R$.
2. The operator $|R^*| = R_{1,1}^*$ is called modulus of the Ruelle operator
3. The operator $B_R = R_{-1,1}^*$ is called Beltrami operator of rational map $R$.

Operators $R^*$ and $|R^*|$ and its right inverse $R_* = \frac{R_{2,0}}{\text{deg}(R)}$ and $|R_*|(\phi) = \frac{|R_{2,0}|(\phi)}{\text{deg}(R)} = \frac{\phi(R)|R'|^2}{\text{deg}(R)}$ map the space $L_1(\mathbb{C})$ into itself with the unite norm. The operator $B_R$ maps the space $L_\infty(\mathbb{C})$ into itself evidently with the unite norm.

1991 Mathematics Subject Classification. Primary:37F45, Secondary:37F30.

This work has been partially supported by the Russian Fund of Basic Researches, Grant 99-01-01006.
Definition. Let \( R \in \mathbb{C}P^{d+1} \) be a rational map. The component of \( J \)-stability of \( R \) is the following space.

\[
qc_J(R) = \left\{ F \in \mathbb{C}P^{d+1} : \text{there are neighborhoods } U_R \text{ and } U_F \text{ of } J(R) \text{ and } J(F), \right.
\]
respectively and a quasiconformal homeomorphism \( h_F : U_R \to U_F \) such that

\[
F = h_F \circ R \circ h_F^{-1}. \]

Definition. The space of invariant conformal structures or Teichmüller space \( T(J(R)) \) on \( J(R) \) for a given rational map \( R \) is the following space

\[
T(J(R)) = \{ \text{Fix}(B_R)(L_\infty(J(R))) \},
\]
where \( \text{Fix}(B_R)(L_\infty(J(R))) \) is the space of fixed elements of Beltrami operator \( B_R : L_\infty(J(R)) \to L_\infty(J(R)) \). Due to D. Sullivan (see [S]) the dimension of \( T(J(R)) \) is bounded above by \( 2 \deg(R) - 2 \).

Definition. We will call a rational map \( K(z) \) convergent iff there exists a rational map \( R \in qc_J(K) \) such that:

1. there exists a point \( a \in \mathbb{C} \) with long orbit \( \geq 2\deg(R) \),
2. for any \( x \in \bigcup_n R^n(a) \) there exists a sequence of integers \( \{N_i\} \) such that the expression

\[
(*) \quad A_{N_i}(x, R, a) = \frac{1}{N_i} \sum_{j=0}^{N_i-1} \sum_{k} \left| (R^*)^j(\tau_x)(c_k) \right| \left| (R''(c_k)(R^{N_i-j-1})'(R(c_k))) \right|
\]

is uniformly bounded independently on \( i \) and where \( c_k \) are critical points for map \( R \).

Theorem A. Let \( R \) be a convergent map with simple critical points and no simple critical relations. Assume that Fatou set is non-empty and Lebesgue measure of postcritical set is zero. Then there is no non-trivial invariant conformal structures on \( J(R) \).

Definition. Ruelle-Poincaré series.

1. Backward Ruelle-Poincaré series.

\[
RS(x, R, a) = \sum_{n=0}^{\infty} (R^*)^n(\tau_a)(x),
\]

where \( \tau_a(z) = \frac{1}{z-a} \) and \( a \in \mathbb{C} \) is a parameter. The series

\[
S(x, R) = \sum_{n=0}^{\infty} |R^*|^n(1_C)(x)
\]

is called Backward Poincaré series.

2. Forward Ruelle-Poincaré series.

\[
RP(x, R) = \sum_{n=0}^{\infty} \frac{1}{(R^n)'(R(x))}. \]

The series

\[
P(x, R) = \sum_{n=0}^{\infty} \frac{1}{((R^n)'(R(x)))}
\]

is called forward Poincaré series. The series

\[
A(x, R, a) = \sum_{n=0}^{\infty} \frac{1}{(R^*)^j(a)(x - R^j(a))}
\]

is called modified Ruelle-Poincaré series.

Note that the Ruelle-Poincaré series are a kind of generalizations of the Poincaré series introduced by C. McMullen for rational maps (see [MM]).
Corollary A. Let \( R \) be a rational map with simple critical points and no simple critical relation. Then \( R \) is convergent map if for some \( a \in \mathbb{C} \) with \( \#\{j_i R^i(a)\} > 2\deg(R) - 1 \) the one of the following is true.

(1) Collet-Eckmann case. For any critical point \( c \) and an \( x \in \{\cup_i R^i(a)\} \) the series \( RS(c, R, x) \) and \( RP(c, R) \) are absolutely convergent.

(2) For any critical point \( c \) and \( x \in \{\cup_i R^i(a)\} \) one of the series \( RS(c, R, x) \) or \( RP(c, R) \) is absolutely convergent and the second one has uniformly bounded elements.

(3) Conjectural case. Both series diverge slow enough (like harmonic series).

As it will be shown below, our definition of Collet-Eckmann maps (see item 1 above) is a generalization of one given by Feliks Przytycki (see \([P]\)). Hence the item 1 of corollary A is reproof of Przytycki result in a weaker sense (we can not show that \( m(J(R)) = 0 \)).

The third case is conjectural in the sense that the case looks like a Log-Collet-Eckmann case (that is \(|(R^n)'(R(c)| \sim C \cdot n\)). It is not clear if it does exist a map with such behavior of its Ruelle-Poincare series.

The next proposition gives some formal relations between Ruelle-Poincare series.

Definition. We denote the Cauchy product of series \( A \) and \( B \) by \( A \otimes B \). Let us recall that if \( A = \sum_{i=1} a_i \) and \( B = \sum_{i=1} b_i \), then \( C = A \otimes B = \sum_{i=1} c_i \), where \( c_i = \sum_{j=1} a_j b_{i-j} \).

Then we have the following proposition.

Proposition A. Let \( R \) be a rational map with simple critical points. Let \( \infty \) be a fixed point for \( R \). Then there exist the following formal relations.

\[
RP(a, R) - 1 = \sum_i \lambda^i - \sum_i \frac{1}{R''(c_i)} RS(c_i, R, a) \otimes RP(c_i, R), \quad \text{where} \quad \lambda \text{ is the multiplier of } \infty
\]

\[
RS(x, R, a) = A(x, R, a) + \sum_k \frac{1}{R''(c_k)} A(c_k, R, a) \otimes RS(x, R, R(c_k)),
\]

where \( c_k \) are critical points of \( R \).

For polynomials of degree two this approach gives the following statement.

Theorem B. Let \( R(z) = z^2 + c \) and \( S_L = \sum_{i=0}^{L} \frac{1}{(R^n)'(c)} \). Assume that there exists a subsequence \( \{n_i\} \) of integers such that

(1) \( \lim_{i \to \infty} |(R^n)'(c)| = \infty \) and \( \lim_{i \to \infty} |S_{n_i}| > 0 \) or

(2) \( |(R^n)'(c)| \sim C = \text{Const} \) for \( i \to \infty \) and \( \lim_{i \to \infty} |S_{n_i}| = \infty \)

Then there exists no invariant conformal structures on its Julia set.

Finally, we give necessary and sufficient conditions for the existence of measurable invariant conformal structures on its Julia set in the case when the postcritical set has Lebesgue measure zero. Let \( U \) be a neighborhood of \( J(R) \). We call \( U \) -essential neighborhood if

(1) \( U \) does not contain disks centered at all attractive and superattractive points and

(2) \( R^{-1}(U) \subset U \).

Definition. Let us define the space \( H(U) \subset C(\overline{U}) \), where \( C(\overline{U}) \) is space of continuous functions and

(1) \( U \) is an essential neighborhood of \( J(R) \) and

(2) \( H(U) \) consists of \( h \in C(\overline{U}) \) such that \( \frac{dh}{R} \) (in the sense of distributions) belongs to \( L_\infty(U) \)

(3) \( H(U) \) inherits the topology of \( C(\overline{U}) \).

Measures \( \nu^i \).

(1) Let \( c_i \) and \( d_i \) be critical points and critical values, respectively. Then define \( \mu^i_n = \frac{\partial}{\partial x} \left((R^n)'(\frac{1}{R^i(d_i)}) \right) \)

(in sense of distributions). We will show below that \( (R^n)'(\frac{1}{R^i(d_i)}) = \sum_{j=0}^{n} \alpha_{j}^{i} R^i(d_j) \) and hence \( \mu^i_n = \sum_{j=0}^{n} \alpha_{j}^{i} \delta_{R^i(d_j)} \), where \( \delta_a \) denotes the delta measure with mass at the point \( a \).

(2) Define by \( \nu^i \) the average \( \frac{1}{t} \sum_{k=0}^{t-1} \mu^i_k \).
Remark 1. For $R(z) = z^2 + c$ we have
\[
\mu_0 = \delta_c, \mu_i = \frac{\delta_{R^i(c)} - \delta_c}{R(c)}, \mu_2 = \frac{\delta_{R^2(c)} - \delta_c}{(R^2)'(c)} - \frac{\mu_1}{R'(c)}, \mu_n = \frac{\delta_{R^n(c)} - \delta_c}{(R^n)'(c)} - \frac{\mu_1}{(R^{n-1})'(c)} - \cdots - \frac{\mu_{n-1}}{R'(c)}.
\]
Now define by $A$ the series $\sum_i \mu_i$, then by above we have the following formal equality
\[
A \otimes RP(0, R) = \sum_{i=0}^{\infty} \frac{\delta_{R^i(c)}}{(R^i)'(c)}
\]
In general the coefficients $a_i^j$ in definition above can be expressed as a combinations of the elements of Cauchy product $RP(c_i, R, d_i) \otimes RP(c_i, R)$ of Ruelle-Poincare series.

Theorem C. Let $R$ be a rational map with simple critical points and no simple critical relations. Assume that $F(R) \neq 0$ and $m(Pc(R)) = 0$, where $F(R)$ is the Fatou set and $m$ denote the Lebesque measure. Then $T(J(R)) = \emptyset$ if and only if there exist an essential neighborhood $U$ and a sequences of integers $\{l_k\}$ such that the measures $\{\nu_{l_k}\}$ converges in $\ast$-weak topology on $H(U)$ for any $i = 1, \ldots, 2\deg(R) - 2$.

Acknowledgement. I would like to thank to IMS at SUNY Stony Brook FIM ETH at Zurich and IM UNAM at Cuernavaca for its hospitality during the preparation of this paper.

Quadratic differentials for rational maps

Let $S_R$ be the Riemann surface associated with action of $R$ on its Fatou set, then (see [S]) $S_R$ is finite union $\sqcup_i S_i$ of punctured torii punctured spheres and foliated surfaces.

Let $A(S_R)$ be space of quadratic holomorphic integrable differentials on $S_R$ and if $S_R = \sqcup_i^N S_i$, then $A(S_R) = A(S_1) \times \cdots \times A(S_N)$, where $A(S_i)$ is the space of quadratic holomorphic integrable differentials on $S_i$.

Quadratic differentials for foliated surfaces. Due to Sullivan ([S]) a foliated surface $S$ is either unit disk or round ring with marked points and is equipped with a group $G_D$ of rotations. This group $G_D$ is everywhere dense subgroup in the group of all rotations of $S$ in the topology of uniform convergence on $S$. Hence for any $z \in S$ the closure of the orbit $G_D(z)$ presents a circle which is called leaf of invariant foliation. If the leaf $l$ contains a marked points $x$, then we call $l$ as critical leaf and denote it by $l_x$. With exception of one case the boundary $\partial S$ consists of critical leaves. This exception is the surface corresponding to the grand orbit of simply connected superattractive periodic component containing only one critical point. In this last case the surface $S$ does not contain critical leaves. Note, that in this case we will assume that the modulus of $S$ is not defined (see [S] for details).

Any quadratic absolutely integrable holomorphic differential $\phi$ has to be invariant under action the group $G_D$ for foliated surfaces. Hence only $\phi = 0$ is the absolutely integrable holomorphic function for $S$ with undefined modulus and therefore we have in this case $A(S) = \{0\}$.

After removing the critical leaves from $S$ we obtain the collection $\sqcup S_i \cup D$ of the rings $S_i$ and disk $D$ (in the case of Siegel disks). We call this decomposition as critical decomposition. For this decomposition we have
\[
\phi_{S_i} = h_i(z) \cdot dz^2, \quad \phi_D = h_0 \cdot dz^2
\]
where $h_i$ is holomorphic absolutely integrable on $S_i$ function and the same for $h_0$ on $D$. Easiest calculations show that $h_i(z) = \frac{c}{z^2}$ and $h_0 = 0$. From the discussion above we conclude that for a ring with $k$ critical leaves (two from it present the boundary of $S$) the dimension $\dim(A(S)) = k - 1$. 
Now let $S$ be ring with critical decomposition $\cup_{i=1}^{k} S_i$ and $\phi \in A(S)$ is a differential, then $\|\phi\| = 4\pi \sum_{i} |c_i| \text{mod}(S_i)$ where $\phi = \sum_{i} c_i \phi_{S_i} = \sum_{i} \frac{c_i}{z^2} \text{mod}(S_i)$ is modulus (or the extremal length of the family of curves connecting the boundary component of $S_i$) of the ring $S_i$.

We always assume here that the hyperbolic metric $\lambda$ on the foliated ring $S$ is the collection of complete hyperbolic metrics $\lambda_i$ on components of critical decomposition of $S$. For example if $\cup_i S_i$ is the critical decomposition of $S$, then the space $H(S)$ of harmonic differentials on $S$ consists of the elements

$$\lambda^{-2} \phi = \sum_{i} \frac{c_i \lambda_i^{-2}}{z^2},$$

where $\phi \in A(S)$.

The space of Teichmuller differentials $\text{td}(S)$ consists of the elements $\phi = \sum_{i} c_i \frac{z^2}{\pi} \phi_{S_i}$, where $\cup_i S_i$ is the critical decomposition of $S$.

Denote by $\Omega(R)$ the set

$$\overline{\mathbb{C}} \backslash \text{closure of grand orbits of all critical points of } R,$$

then $R$ acts on $\Omega(R)$ as unbranched autocovering. Let $D \subset \Omega(R)$ be a periodic component, then $D$ corresponds to either attractive or parabolic periodic domain. Let us fix the following notations.

1. $\Delta$ is unit disk,
2. $j_D$ is universal covering $\Delta \to D$,
3. let $R^k : D \to D$ be the first return map for $D$, then let $f$ be the lifting of $R^k$ onto $\Delta$ by $j_D$. Note that $f$ is a Möbius map,
4. $\Gamma$ is the group of the deck transformations of the covering $j_D$,
5. $G = \langle f, \Gamma \rangle$ is finitely generated Fuchsian group uniformizing the surface $S_D$ (i.e. $\Delta/G \cong S_D$),
6. For a given Fuchsian group $H$ let $A(H)$ be the space of all holomorphic functions on $\Delta$ such that $\phi(\gamma)(\gamma')^2 = \phi$ for all $\gamma \in H$ and for any $\phi \in A(H)$ and

$$\iint_{\Delta} |\phi| < \infty$$

here $\omega$ is a fixed fundamental domain for $H$. Further denote by $B(H)$ the following space

$$\{ \phi \text{ holomorphic functions on } \Delta, \phi(\gamma)(\gamma')^2 = \phi \text{ for all } \gamma \in H \text{ and }$$

$$\sup_{z \in \Delta} |\lambda^{-2} \phi| < \infty, \text{ where } \lambda \text{ is hyperbolic metric on } \Delta \},$$

with norm $\|\phi\| = \sup_{z \in \Delta} |\lambda^{-2} \phi|$ the space $B(H)$ presents a Banach space. Now let $S$ be a Riemann surface, then denote by $B(S)$ the following space

$$\{ \phi \text{ quadratic holomorphic differentials on } S \text{ and }$$

$$\sup_{z \in S} |\lambda^{-2} \phi| < \infty, \text{ where } \lambda \text{ is hyperbolic metric on } S \},$$

with norm $\|\phi\| = \sup_{z \in S} |\lambda^{-2} \phi|$ the space $B(S)$ presents a Banach space. If $\Delta/\Gamma \cong S$, then it exists an isometrical isomorphism $\Phi$ from $A(H)$ onto $A(S)$.

Now let $Y \subset \overline{\mathbb{C}}$ be an open subset. Then, as above, $A(Y)$ denotes the space of holomorphic functions on $Y$ absolutely integebrable over $Y$ and $B(Y)$ consists of holomorphic functions $\phi$ on $Y$ with the following norm

$$\|\phi\| = \sup_{z \in Y} |\Lambda_Y^{-2} \phi|,$$

where $\Lambda_Y$ is a metric so that its restriction over any component $D \subset Y$ satisfies

$$\Lambda_Y|_{D} = \lambda_D,$$

where $\lambda_D$ is Poincare metric on $D$. 

Lemma (Bers Duality Theorem).

1) The space $B(H)$ ($B(S)$) is isometrically isomorphic to the dual space $A^*(H)$ ($A^*(S)$) and this isomorphism is defined by the Peterson scalar product

$$\int \int \omega^{-2} \varphi \overline{\psi} (\int_S \lambda^{-2} \varphi \overline{\psi}),$$

where $\omega$ is a fundamental domain for $H$ and $\varphi \in A(H)$ ($\varphi \in A(S)$), $\psi \in B(H)$ ($\psi \in B(S)$).

2) If $H$ is finitely generated group or the surface $S$ is compact surface with finite number punctures or $S$ is foliated annuli. Then $A(\Gamma) = B(\Gamma)$ and $A(S) = B(S)$. Dimension $A(\Gamma)$ and $A(S)$ is finite and the Petersen scalar product becomes the inner scalar product.

3) Let $Y \subset \overline{C}$ be an open subset. Then as above, the spaces $A(Y)$ and $B(Y)$ are dual by the Peterson scalar product

$$\int_{Y} (\Lambda Y)^{-2} \varphi \overline{\psi}.$$

Proof. See [Kra].

Poincare $\Theta$-operator for rational maps. We construct this operator by the way which are similar to one in Kleinian group case.

1) Let $D \in \Omega(R)$ is corresponding to an attractive periodic domain.

Let $\Theta_H(\phi)$ be theta series of Poincare for the Fuchsian group $H$, that is

$$\Theta_H(\phi) = \sum_{\gamma \in H} \phi(\gamma)(\gamma')^2$$

for $\phi \in A(\Delta)$. This series defines the map from $A(\Delta)$ onto $A(H)$ and the kernel is the space

$$\text{closure of the linear span}\{\phi - \phi(\gamma)(\gamma')^2, \phi \in A(\Delta), \gamma \in H\}$$

In our case we have $G = \left< f, \Gamma \right>$. Let $G = \bigcup_i \Gamma g_i$, then by $\Theta_f$ we denote the relative $\Theta$-series i.e.

$$\Theta_f(\phi) = \sum_i \phi(g_i)(g_i')^2,$$

then it is clear that

$$\Theta_G = \Theta_f \circ \Theta_f$$

Finally we define the $\Theta$-series for our rational map $R$. Let $\Psi$ be isometrical isomorphism from $A(D)$ onto $A(\Gamma)$, then we set.

$$\Theta_D(\phi) = \Theta_f \circ \Psi(\phi), \text{ for } \phi \in A(D).$$

So we have $\Theta_D(A(D)) = A(G) \cong A(S_D)$.

Define $L(X)$ to be the grand or full orbit of the set $X \subset \overline{C}$. Now we construct the map $\Theta_{L(D)} : A(L(D)) \rightarrow A(S_D)$ by the following way. Let $\phi \in A(L(D))$ and $X_i$ be components of $L(D)$, then $\phi = \sum_i \phi|_{X_i}$. Let $\Theta_{X_i} : A(X_i) \rightarrow A(D)$ be $\Theta$-operator corresponding to the unbranched covering $R^{k_{X_i}} : X_i \rightarrow D$, (where $k_{X_i}$ is the minimal integer satisfying to the last property). Then we set

$$\Theta_{L(D)}(\phi) = \Theta_D \left( \sum_{X_i \in L(D)} \Theta_{X}(\phi|_{X_i}) \right)$$

2) The case of parabolic domains $D$ is similar to attractive one.

3) Let $D$ be a superattractive domain. This case corresponds to non-discrete groups. Therefore we need an additional information respect to this foliated case. Let us start with simple lemma about the Ruelle operator.
Lemma 2. Let $R$ be a rational map. Let $Y \subset \mathbb{C}$ be positive Lebesgue measure subset which is completely invariant under action of $R$, then the following is true.

1. $R^*: L_1(Y) \to L_1(Y)$ is linear surjection with unit norm. The operator
   \[ R_*(\phi) = \frac{\phi(R)(R')^2}{\deg(R)} \]
   is an isometry "into" and $R^* \circ R_* = I$, where $I$ is identity operator.

2. Beltrami operator
   \[ B_R(\phi) = \phi(R) \frac{R'}{R} : L_\infty(Y) \to L_\infty(Y) \]
   presents the dual operator to $R^*$. The operator $B_R$ is an isometry "into".

3. If $Y$ is an open set. Then $R^*: A(Y) \to A(Y)$ is surjection of unit norm and $R_*$ maps $A(Y)$ into itself too. Let $\sigma_R : B(Y) \to B(Y)$ be the dual operator respect to Peterson scalar product. Then
   \[ \sigma(R(\phi)) = \phi(R)(R')^2 \text{ and } \sigma = \deg(R)R_* \]

Proof. All items are immediate consequences of the definition of the operators.

Remark 3. If $R : X \to Y$ is a branched covering for a rational map $R$ and two domains $X, Y \subset \mathbb{C}$. Then $R^*: A(X) \to A(Y)$ is Poincare operator of the covering $R$.

The discussion above shows that the $\Theta$ operator (in attractive and parabolic cases) is invariant respect to Ruelle operator, that is $\Theta(R^*) = \Theta$ and hence $\text{ker}(\Theta) \supset (I - R^*)(A(\Omega))$.

Now let us continue the discussion on superattractive case. Let for simplicity $D$ be an invariant superattractive domain. Our aim is to prove the following theorem.

Theorem 4. Let $D \subset F(R)$ be invariant superattractive domain and $X = \text{expan}(P, R)$. Let $S$ be the foliated surface associated with $D$. Then the quotient space $A(X)/(I - R^*)(A(X))$ is isomorphic to the space $A(S)$.

Proof. In further we need the following basic facts about non-expansive operators.

Mean ergodicity lemma. Let $T$ be non-expansive ($\|T\| = 1$) linear endomorphism of a Banach space $B$ and let $\phi \in B$ be any element.

1. Assume that for Cesaro average $A_N(T, \phi) = \frac{1}{N} \sum_{i=0}^{N-1} T^i(\phi)$ there exists subsequence $\{n_i\}$ such that $A_{n_i}(T, \phi)$ weakly converges to an element $f \in B$, then $f$ is a fixed point for $T$ and $A_N(\phi)$ converges to $f$ strongly (i.e. by the norm). If $f = 0$ then $\phi \in (I - T)(B)$ and versa i.e. if $\phi \in (I - T)(B)$, then $A_n(T, \phi)$ tends to zero with respect to the norm.

2. The linear continuous operator $T$ on a norm space $B$ is called mean ergodic if and only if the Cesaro average $A_N(T, \phi)$ converges with respect to the norm for any element $\phi \in B$. In this case $B = \text{Fix} \times (I - T)(B)$ and $A_n(T, \phi)$ converges to projection $P : B \to \text{Fix}$, here $\text{Fix}$ is the space of fixed elements for $T$.

Proof. See the book of Krengel ([Kren]).

Now first consider the case of simply connectedness of $D$. Hence up to conformal changing of coordinates on $D$ we can think that $D = \Delta, X = \Delta^* = \Delta \setminus \{0\}$ and $R(z) = z^2$.

In this case we claim

Claim. $A(X) = (I - R^*)(A(X))$.

Proof of the claim. Let $F$ be the space of finite linear combinations of the functions of the following kind $\frac{1}{z-a}$, where either $a = 0$ or $|a| > 1$ or $a$ is repelling periodic point for $R(z)$. Then by Bers density theorem (see [Kra] or discussion below) we know that $F$ presents everywhere dense subset of $A(X)$.

By using direct calculations we have:

\[ R^\ast \left( \frac{1}{z-a} \right) = \frac{a}{2z(z-a^2)} = \frac{1}{R'(a)} \left( \frac{1}{z} - \frac{1}{z - R(a)} \right) \]
and hence by induction and the fact \((R^*)^n(\frac{1}{z}) = 0\), we obtain
\[
(R^*)^n \left( \frac{1}{z - a} \right) = \frac{1}{(R^a)'(a)} \left( \frac{1}{z} - \frac{1}{z - R^a(a)} \right).
\]
Therefore for any \(\phi \in F\) we have \((R^*)^n(\phi) \to 0\), for \(n \to \infty\) exponentially fast. Hence Cesaro averages \(A_N(R^*, \phi)\) strongly tends to 0 for any \(\phi \in A(X)\). By Mean ergodicity lemma above we complete this claim.

Let us again consider unit disk \(\Delta\) and the map \(R(z) = z^2\). Let \(b_1, ..., b_{k+1} \in \Delta\) be points such that \(|b_{k+1}| = |b_1|^2 < |b_k| < ... < |b_1| < 1\). Let \(S\) be the ring \(|b_1|^2 \leq |z| \leq |b_1|\) with \(\{b_i\}\) as marked points and let \(G\) be the group of rotation of \(S\). Then \((S, G)\) is foliated surface. Let \(X = \Delta \setminus \bigcup_i L(b_i)\), where like above \(L(b_i)\) means the grand orbit of \(b_i\). We claim that.

**Lemma 5.** There exists continuous surjection \(P : A(X) \to A(S)\) such that \(\ker(P) \supset (I - R^*)(A(X))\).

**Proof.** For simplicity assume that \(S\) has only two marked points \(x\) and \(R(x)\) on different components of \(\partial S\). Let for any integer \(i\) the rings \(S_i\), be component of \(L(S)\) such that \(S_0 = S\) and \(S_i = R(S_{i-1})\). Denote by \(W \in A(X)\) the subspace consisting of the elements \(\sum_i \frac{z^i}{S_i}\).

Then we claim.

**Claim.** There exists projection \(Q : A(X) \to W\) such that
1. \(Q = \lim_{n \to \infty} P_n\), where \(P_n = R^n \circ R^{-n}\) and \(\|Q\| = 1\).
2. \(Q \circ R^* = R^* \circ Q\) and \(R_n \circ Q = Q \circ R_n\).
3. \(R^n, R_n : W \to W\) present surjective isometries.

**Proof of the claim.** Such as \(\|P_n\| \leq 1\) and \(P_n(A(S_i)) \subset A(S_i)\) for any \(n\) and \(i\) it is enough to show the convergence of \(P_n\) on \(A(S_i)\) for a fixed \(i\). Again by Bers density theorem the linear span of the function \(\frac{1}{z-a}\) with \(a \notin S_i\) present everywhere dense subset of \(A(S_i)\).

For any \(a \in \mathbb{C} \setminus S_i\) we have
\[
P_n \left( \frac{1}{z - a} \right) = \frac{az^{2^n}}{z^2(z^{2^n} - a^{2^n})}
\]
and hence for big \(n > m\) obtain.
\[
\| (P_n - P_m) \left( \frac{1}{z - a} \right) \| = \int_{S_i} \left| \frac{1}{z} \right|^2 \left| \frac{1}{1 - \left| \frac{a}{z} \right|^2} - \frac{1}{1 - \left| \frac{a}{z} \right|^{2^n}} \right| dz^2 = *
\]
If \(|a| = \lambda \cdot \min_{z \in S_i} |z|\) with \(|\lambda| < 1\), then
\[
* \leq 2\lambda^{2^n} C \int_{S_i} \left| \frac{1}{z} \right|^2 \to 0 \text{ for } n > m \to \infty.
\]
If \(|a| = \lambda \cdot \max_{z \in S_i} |z|\) with \(|\lambda| > 1\), then
\[
* \leq \int_{S_i} \left| \frac{1}{z} \right|^2 \left| \left( \frac{a}{z} \right)^{-2m} - \left( \frac{a}{z} \right)^{-2n} \right| \to 0 \text{ when } n > m \to \infty.
\]
Let \(Q = \lim P_n\) in strong topology of \(L_1(S_i)\). Then we have
1) if \(|a| > \max_{z \in S_i} |z|\), then \(Q \left( \frac{1}{z - a} \right) = 0\) and
2) if \(|a| < \min_{z \in S_i} |z|\), then \(Q \left( \frac{1}{z - a} \right) = \frac{1}{z - a}\).
3) \(R^n \circ P_n = P_{n-1} \circ R^*\) and \(P_n \circ R_n = R_n \circ P_{n-1}\), hence \(R^*\) and \(R_n\) commute with \(P\).
4) By the construction \(\ker Q\) contains \(\ker P_n\) for all \(n\) and hence \(R^*_n \mid_W\) is an isomorphism.

Now let us consider the map \(T : W \to A(S_0) = A(S)\) defined by the formula
\[
T(\phi) = \sum_{-\infty}^{\infty} (R^*)^n(\phi).
\]
Lemma 6. $T$ is continuous non-expansive operator with ker$(T) \supset \overline{(I - R^*)(W)}$.

Proof. Let $\phi = \sum c_i z^{i-1} |S_i \in W$, then $T(\phi) = \frac{1}{z} \sum_{i=1}^{\infty} c_i \cdot 2^i$ and $\|\phi\| = \sum |c_i| \int_{S_i} \frac{1}{|z|} = 4\pi \sum |c_i| \text{mod}(S_i) = 4\pi \cdot \text{mod}(S_0) \sum 2^i |c_i|$. Hence $\|T(\phi)\| \leq \|\phi\|$.

The standard arguments imply that ker$(T) \supset \overline{(I - R^*)(W)}$, Lemma is complete.

By using lemmas above we conclude that the map $P = T \circ Q : A(X) \to A(S)$ is continuous surjection.

Now let us return to theorem 4. Let $h : U \subset D \to \Delta$ be conformal map with $h(y) = 0$ and $h'(y) = 1$, where $y \in U$ is the fixed superattractive point and $h$ conjugates $R$ with $z \to z^2$. Then the function $\alpha = \log |h|$ can be harmonically extended onto $L(D)$ to unique function, again denoted by $\alpha$. Let $c_i$ be critical points in $L(D) \setminus y$ ordered by values $\alpha(c_i)$. Then $h$ can be conformally extended on region $V = \{z, \alpha(z) < \alpha(c_1)\}$ and $h(V) = \Delta = \{|z| < r < 1\}$.

The ring $S_D = h(V) \setminus h(R(V))$ with critical decomposition presents the foliated surface associated with $L(D)$.

Let $Y = L(D) \setminus \{\cup_i L(c_i) \cup L(y)\}$ and $F = V \setminus \{\cup_i L(c_i) \cup L(y)\}$, then as in attractive case construct the operator $\Theta_Y : A(Y) \to A(F)$. If $h_a : A(F) \to A(X)$ is the injection generated by $h$, where $X$ is the set constructed by $S_0 = S_D$ like in lemma 5. Then we set

$$\Theta_{L(D)} = P \circ h_a \circ \Theta_Y : A(Y) \to A(S_D).$$

Theorem is proved.

Finally we set $\Theta(R) : A(\Omega) \to A(S_R)$ by

$$\Theta(R)(\phi) = (\Theta_{L(D_1)}, ..., \Theta_{L(D_k)}),$$

where $D_1 \subset F(R)$ are periodic components.

Space $A(R)$.

Now again consider the space $A(\Omega)$. Note that any function of the kind

$$\gamma_a(z) = \frac{a(a-1)}{z(z-1)(z-a)}$$

for $a \in \overline{C} \setminus \Omega$ belongs to $A(\Omega)$. Let us introduce the subspace $A(R) \subset A(\Omega)$ as follows. Let $S$ be the set

$$\{\cup_i \{L(c_i)\} \cup \{L(0, 1, \infty)\}\},$$

where $c_i$ are critical points. Then we set

$$A(R) = \text{linear span}\{\gamma_a(z), a \in S\}.$$

This space $A(R)$ is an linear space and we set on $A(R)$ two different topologies by the following norms $| \cdot |_1 = \int_{L(R)} | \cdot |$ and $| \cdot |_2 = \int_{\overline{C}} | \cdot |$. Denote by $A_i$ the spaces $\{A(R), | \cdot |_i\}$, respectively.

Remark 7. The space $A(R)$ serves a kind of connection between spaces $L_1(\Omega)$ and $L_1(J(R))$ and comparison of $\| \cdot \|_1$ and $\| \cdot \|_2$ topologies is basis for our discussion below.

Lemma 8. The operators $R^*, R_+$ and $\star R$ are continuous endomorphisms of both spaces $A_1$ and $A_2$.

Proof. It is sufficient to show that for any $\phi \in A(R)$ the functions $R^*(\phi)$ and $R_+(\phi)$ belong to $A(R)$ again.

Let $\phi = \gamma_a$. Then $R^*(\phi)$ and $R_+(\phi)$ are holomorphic everywhere except finite numbers of points belonging to set $S$ and hence are rational functions holomorphic on $\Omega$. Besides both $R^*(\phi)$ and $R_+(\phi)$ are integrable over $\overline{C}$ and hence belong to $A(R)$. Lemma is proved.

Then we have the following well-known result.

Lemma (Bers density theorem). $A_1$ is everywhere dense in $A(\Omega)$.

Proof. See for example book of I.Kra ([Kra]).

Lemma 9. Let $L$ be a continuous functional on $A_1$ invariant under action of $R^*$ (i.e. $L((R^*)(\phi)) = L(\phi)$). Then $L(\phi) = \int_{\Omega} \lambda^{-2}\psi\phi$, where $\lambda$ is hyperbolic metric on $\Omega$ and $\psi \in B(\Omega)$.

Proof. Bers density and Bers duality theorems complete the proof of this lemma.
Bers Isomorphism

Here we reproduce the Bers construction for the Beltrami differentials and Eichler cohomology group with corrections (which really often are evident) for the rational maps.

Consider the Beltrami action of $R$ on the space $L_\infty(\overline{C})$ i.e.

$$B_R(\phi)(z) = \phi(R)(z) \frac{\overline{R(z)}}{R(z)}.$$ 

So the subspace $\text{Fix}$ of fixed points for $B_R$ in $L_\infty(\overline{C})$ is indeed the space of the invariant Beltrami differentials for $R$ unit ball of which describes all quasiconformal deformations of $R$.

Now let $J_R$ be subspace of invariant Beltrami differentials supported by $J(R)$. Then it exists a continuous map $\Psi$ from $A(S_R) \times J_R$ into space $\text{Fix}$ of fixed points for $B_R$ by the following way. Let $\Theta^* : A^*(S_R) \to A^*(\Omega)$ be dual operator to $\Theta$--operator. Then the image $H(\Omega) = \Theta^*(A^*(S_R)) \subset B(\Omega) \subset \text{Fix}$ is called space of harmonic differentials and $\dim(H(\Omega)) = \dim(A^*(S_R)) = \dim(A(S_R))$. Let $\alpha : A(S_R) \to A^*(S_R)$ be isomorphism defined by Petersen scalar product. Then we can define

$$\Psi : A(S_R) \times J_R \to \text{Fix} \text{ by } \Psi(\phi,\mu) = (\Theta^* \circ \alpha(\phi),\mu).$$

Now normalize $R$ so that $0, 1, \infty$ are fixed points for $R$. Let $C$ be component of the subset of rational maps in $\mathbb{C}P^{2d+1}$ fixing the points $0, 1$ and $\infty$ containing $R$. By $H^1(R)$ we denote the tangent space to $C$ at the point $R$. Then $H^1(R)$ may be presented as follows. Let $R(z) = \frac{P}{Q}$, then

$$H^1(R) = \{z \frac{PQ_0 - QP_0}{Q_0^2}, \text{ where } Q(1) = P(1), \deg(Q) \leq \deg(R), \deg(P) \leq \deg(R) - 1\},$$

where $P, Q$ are polynomials and $\dim(H^1(R)) = 2d - 2$.

**Remark 10.** We use the notation $H^1(R)$ by the following reasons

1. The Weyl cohomology' construction for the action of $R$ (by the formula $\tilde{R}(f) = \frac{L(R)R^t}{R}$) on the space of all rational functions gives the space $H$ which is isomorphic to tangent space to $\mathbb{C}P^\infty$ at $R$ up to normalization. More precisely $H$ is equivalent to direct limit

$$H^1(R) \xrightarrow{j_1} H^1(R^2) \xrightarrow{j_2} H^1(R^3)...,$$

where $j_i$ are equivalent to the action $\tilde{R}$.

2. This construction for Kleinian group is called Eichler cohomologies.

Now follow Bers (see for example [Kra]) we introduce the Bers map $\beta$ from $J_R \times A(S_R)$ into $H^1(R)$.

Let $\mu \in L_\infty(\mathbb{C})$, then the function

$$F_\mu(z) = z(z-1) \int_{C} \frac{\mu d\xi d\overline{\xi}}{\xi(\xi-1)(\xi-z)},$$

is continuous on $\mathbb{C}$ and $|F(z)| = O|z^2|$ for $z \to \infty$.

$$\frac{\partial F_\mu}{\partial \overline{z}} = \mu.$$ 

in sense of distribution. $F_\mu$ is called potential for $\mu$.

Let us define the Bers map $\beta(t)$ for $t = (\mu, \phi) \in A(S_R) \times J_R$ by the formula

$$\beta(t = (\phi, \mu))(z) = F_{\Psi(\phi,\mu)}(R(z)) - R'(z)F_{\Psi(\phi,\mu)}(z).$$
Theorem 11.

1. $\beta$ is injective antilinear map,
2. $\beta(J_R \times A(S_R)) \subset H^1(R)$,
3. if $R$ is structurally stable, then $\beta$ is an isomorphism.

Proof. Fix $t \in J_R \times A(S_R)$, then on $\mathbb{C}\{\text{poles of } R\}$ the derivative $\frac{\partial \beta(t)(z)}{\partial z}$ (in sense of distribution) is zero, hence by Weyl lemma $\beta(t)(z)$ is holomorphic on $\mathbb{C}\{\text{poles of } R\}$. Further, poles of $R$ are at most than poles for $F$ and we conclude that $\beta(t)(z)$ is a rational function with zeros at 0 and 1 and simple pole at $\infty$.

Let $k$ be the norm $\|\Psi(t)\|_{L^\infty(\mathbb{C})}$. Consider the disk of Beltrami differentials $\mu_x(z) = x \Psi(t)(z)$, for $|x| < k$. Let $f_x$ quasiconformal maps normalized by $f_x(0, 1, \infty) = 0, 1, \infty$, respectively. Then for small $x$ we have

$$f_x(z) = z - z(z - 1) \int_\mathbb{C} \frac{x \Psi(t)(\xi) d\xi}{\xi(z - 1)(\xi - z)} + O(\|x \Psi(t)\|^2_{L^\infty(\mathbb{C})}),$$

and hence

$$\frac{\partial f_x(z)}{\partial x}_{|x=0} = -F_{\Psi(t)}(z).$$

But $R_x(z) = f_x \circ R \circ f_x^{-1}(z)$ are rational maps and so by differentiation respect to $x$ of equality above one have

$$\frac{\partial f_x}{\partial x}_{|x=0}(R(z)) - R'(z) \frac{\partial f_x}{\partial x}_{|x=0}(z) = \frac{\partial R_x}{\partial x}_{|x=0}(z) \in H^1(R).$$

Hence we have

$$F_{\Psi(t)}(R(z)) - R'(z)F_{\Psi(t)} = \frac{\partial R_x}{\partial x}_{|x=0}(z) \in H^1(R).$$

Now for finishing lemma it is enough to assume that $R$ does not have conformal centralizer and use the Sullivan result (see $[S]$) which particularly says that $\dim(T(R)) = \dim(T_R(c(R)))$, where $T_R(qc(R))$ is tangent space to $qc(R)$ in initial point and $T(R)$ is Teichmuller space of $R$. Besides we have $\dim(H^1(R)) = \dim(T_{qc(R)}(R))$. So theorem is proved.

Beltrami Differentials on Julia set

Here we consider in details the space $J_R$. Each element $\mu \in J_R$ defines an invariant respect to the Ruelle operator functional $L_\mu$ on the space $A(R)$ which is continuous in topology of the space $A_2$ (we recall that $A_i = (A(R), |.|_i)$). Continuity of $L_\mu$ in the topology of the space $A_1$ is crucial in the question of non-triviality of $\mu$. Indeed we have the following lemma.

Lemma 12. Let $\mu \in J_R$, then $\mu = 0$ if and only if $L_\mu$ is continuous functional on $A_1$.

Proof. Let $L_\mu$ is continuous on $A_1$. Then $L_\mu$ is continuous on $A(\Omega)$ by the density theorem. Then by lemma 9 the functional $L_\mu$ may be presented by the expression $L_\mu(\alpha) = \int_\mathbb{C} \alpha \lambda^{-2} \bar{v}$, for some $\psi \in A(S_R)$ and hence $F_\mu(a) = L_\mu(\gamma a) = \int_\mathbb{C} \gamma a \lambda^{-2} \bar{v} = F_{\lambda^{-2} \bar{v}}(a)$. In other words we have $\beta(t = (\mu, 0))(z) = \beta(0, \psi)(z)$ for $z \in \mathbb{S}$, but $\beta(t)(z)$ is rational map for any $t \in A(S_R) \times J_R$ and $\mathbb{S}$ is closed infinite set, hence we conclude

$$\beta(t = (\mu, 0)) = \beta(0, \psi)$$

that contradicts with injectivity of $\beta$. Lemma is proved.

Now we begin to consider the relations between continuity of $L_\mu$ for $\mu \in J_R$ and properties of the Ruelle operator $R^*: A_2 \rightarrow A_2$. Recall that operator $R^*$ acts as linear autosurjection of $L_1(J(R))$ with unit norm.
Proposition 12. Let \( R \) be a rational map with simple critical points. Assume that Lebesgue measure of \( P_c(R) \) is zero. Then \( J_R = \emptyset \) if and only if the Ruelle operator \( R^*: A_2 \to A_2 \) is mean ergodic.

Proof. If \( J_R = \emptyset \), then subspace \( (I - R^*)(A_2) \) is everywhere dense in \( A_2 \) and by the item (2) of Mean ergodicity lemma we complete proof.

Now suppose that \( R^* \) is mean ergodic on \( A_2 \). Let \( \mu \neq 0 \in J_R \), then by lemma 12 \( L_\mu \) is not continuous functional on \( A_1 \). But \( (I - R^*)(A_1) \subset ker(L_\mu) \).

Now we claim that

\[ \dim(A_1/(I - R^*)(A_1)) < \infty \]

Proof of the claim. We start with the following lemma. Let us consider an element \( \tau_a = \frac{1}{z-a} \) with \( a \in \mathbb{C} \). Now we show that

Lemma 14. For any integer \( n \) we have

\[ R^{*n}(\tau_a)(z) = \frac{1}{(R^n)'(a)(z - R^n(a))} - \sum_i \frac{b^n_i}{(a - c^n_i)(z - R^n(c^n_i))}, \]

where \( b^n_i \) are coefficients from the following decomposition \( \frac{1}{(R^n)'(z)} = \sum_i \frac{b^n_i}{z - c^n_i} \). The points \( c^n_i \) are critical points of \( R^n \) and \( p_n(z) \) is polynomial.

Proof of lemma. Let \( \phi \) be a differentiable function with compact support, then for any fixed \( n \) one obtains

\[
\begin{align*}
\int \phi \cdot R^{*n}((\tau_a)(z)) &= \int \phi \cdot \left( \frac{(R^n)'(a)(z - R^n(a))}{(R^n)'(a)} - \sum_i \frac{b^n_i}{(a - c^n_i)(z - R^n(c^n_i))} \right)
\end{align*}
\]

\[
\begin{align*}
+ \int \frac{(\phi \circ R^n) \cdot p_n(z)}{z - a} &= \sum_i \frac{b^n_i}{(a - c^n_i)} \left( \int \frac{(\phi \circ R^n)(z)}{z - a} - \int \frac{(\phi \circ R^n)(z)}{z - c^n_i} \right)
\end{align*}
\]

Finally,

\[
\begin{align*}
\int \phi \cdot R^{*n}((\tau_a)(z)) &= \int \phi \left( \frac{1}{(R^n)'(a)(z - R^n(a))} - \sum_i \frac{b^n_i}{(a - c^n_i)(z - R^n(c^n_i))} \right)
\end{align*}
\]

Hence Weyl lemma we have that function

\[
h_n(z) = R^{*n}(\tau_a)(z) - \frac{1}{(R^n)'(a)(z - R^n(a))} - \sum_i \frac{b^n_i}{(a - c^n_i)(z - R^n(c^n_i))}
\]

is integrable and holomorphic on \( \overline{\mathbb{C}} \) and hence \( h(z) = 0 \).

Remark 15. Assuming that \( z = \infty \) is fixed point for \( R \), we easily calculate that \( p_n(z) = \lambda^n \), where \( \lambda \) is multiplier of \( \infty \). For example, if \( F(R) \) has an attractive component, we always can think (by qc-surgery) that points \( z = \infty \) is a superattractive point.

Now by using lemma 14 and the fact \( \gamma_a = (a - 1)\tau_o - a\tau_i + \tau_a \) we can write

\[ R^*(\gamma_a) = \sum_i \omega_i \gamma_{R(c_i)}(z) + \omega_{\gamma_R(a)}(z) \]

So we have

\[ R^*(\gamma_a) = \gamma_a - (\gamma_a - R^*(\gamma_a)) = \sum_i \omega_i \gamma_{R(c_i)}(z) + \omega_{\gamma_R(a)}(z). \]

If \( a \in R^{-1}(0,1,\infty) \), then again the direct calculation shows that

\[ R^*(\gamma_a)(z) = \sum_i \alpha_i \gamma_{R(c_i)}(z). \]
We conclude that for any element $\phi \in \text{linear span}\{\gamma_a; a \in S \setminus \{\text{the forward orbits of critical values}\}\}$ it exists $n$ such that $R^n(\phi) \in \text{linear span}\{\gamma_a; a \in \{\text{forward orbits of critical points}\}\}$.

Now let $a = R^k(b)$, where $b$ is a critical value, then $\gamma_b \sim R^k(\gamma_b) = \sum_i \alpha_i \gamma_{b_i} + \alpha \gamma_a$ that means $\alpha \gamma_a \sim \gamma_b - \sum_i \alpha_i \gamma_{b_i}$, (here $\phi \sim \psi$ if $\phi - \psi \in (I - R^*)(A_1)$).

As result we obtain that the space $A_1/((I - R^*)(A_1))$ is isomorphic to a subspace in $X = \text{linear span}\{\gamma_{R(c_i)}\}$. Claim is proved.

**Remark 16.** Note that finiteness of $\text{dim}(A_1/((I - R^*)(A_1)))$ is purely algebraic fact does not depending on topology.

By the claim we conclude if $\ker(L_\mu)$ contains closure of $(I - R^*)(A_1)$ in $A_1$, then $L_\mu$ is continuous.

We claim that $\ker(L_\mu)$ contains the space $(I - R^*)(A_1)$.

**Proof of the claim.** Otherwise it exists an element $\phi \in (I - R^*)(A_1)$ such that $L_\mu(\phi) \neq 0$. By Mean ergodicity lemma we have that Cesaro averages $A_N(\phi)$ tends to zero in $A_1$ by the norm. Further by invariance of $\mu$ we have

$$L_\mu(A_N(\phi)) = \int \int_{J(R)} \mu A_N(\phi) = L_\mu(\phi).$$

Now using the mean ergodicity of $R^*$ we obtain convergence of $A_N(\phi)$ to an element $f \in L_1(J(R))$ by the norm and hence $L_\mu(\phi) = L_\mu(A_N(\phi)) = \int f$. Besides the functions $\{A_N(\phi)\}$ forms the normal family of holomorphic functions on $Y = \overline{C} \setminus \{\text{closure of forward orbit of } a \} \cup \{PC(R)\}$. Such that $\|A_N(\gamma_a)\|_{A_1} \to 0$ we obtain that $A_N(\phi)$ converges to zero uniformly on compacts in $Y$. Hence $f = 0$ on $Y$ and such as $m(C \setminus Y) = 0$ one has $L_\mu(\phi) = 0$. Contradiction. Claim is proved.

So $L_\mu$ is continuous functional on $A_1$ and by lemma 12 $\mu = 0$. Proposition is proved.

**Definition.** We call a rational map mean ergodic if and only if the Ruelle operator $R^* : A_2 \to A_2$ is mean ergodic.

Now we show that topologies $\| \cdot \|_1$ and $\| \cdot \|_2$ are "mutually disjoint.” Denote by $X_i$ the closure of the space $(I - R^*)(A(R))$ in spaces $A_1$ and $A_2$.

**Proposition 17.** Let $R$ be a rational map and $\text{dim}(A(S_R)) \geq 1$. Then the following conditions are equivalent.

1. the map $i = id : A_1 \to A_2$ maps weakly convergent sequences onto weakly convergent sequences.
2. $i(X_1) \supset X_2$.
3. the Lebesgue measure of Julia set is zero.

**Proof.** Condition (3) trivially implies the conditions (1) and (2).

Assume condition (1) is hold. then the dual map $i^* : A_2^* \to A_1^*$ is continuous in $\ast$-weak topologies on $A_1^*$ and $A_2^*$. Hence for any $\mu \in A_2^* \subset L_\infty(J)$ there exists an element $\nu \in A_1^* \subset L_\infty(F)$ such that $\nu = i^*(\mu)$ and

$$\int \int f \nu \gamma = \int \int f \nu \gamma.$$ 

Then for any $\gamma \in A(R)$ we have $\int \int \gamma (\mu - i^*(\mu)) = 0$. Let $F_\mu(z)$ and $F_\nu(z)$ be potentials. Then $F_{J(R)} = (F_\mu(z) - F_\nu(z)) |_{J(R)} = 0$ and if $m(J(R)) > 0$ we have $F_\nu = 0$ almost everywhere on $J(R)$, where $F_\nu$ in sense of distributions. Hence we deduce:

$$\mu - i^*(\mu) = 0$$

almost everywhere on $J(R)$. Since $F(R) \cap J(R) = \emptyset$ we have $\mu = 0$ almost everywhere and we conclude that $A_2^* = \{0\}$. Hence $A_2 = \{0\}$ and we obtain $m(J(R)) = 0$.

Now assume (2). Then conditions imply that any invariant continuous functional on $A_1$ generate an invariant line field on Julia set that contradicts to injectivity of the Bers map. Now using the fact that we can always think that $F(R)$ contains an attractive domain we conclude $m(J(R)) = 0$. Proposition is proved.
Proposition 18. Assume that $F(R) \neq \emptyset$ and $m(J(R)) > 0$ for the given rational map $R$. Then there exists no invariant line fields on Julia set if and only if $i^{-1}(X_2) \supset X_1$.

Proof. It is easy. If there exists no invariant line field, then $X_2 = A_2$. Now assume $i^{-1}(X_2) \supset X_1$, then existing of invariant line field contradicts to injectivity of the Bers map.

We finish this chapter with simple application of discussion above to polynomials of degree two.

Theorem B. Let $R(z) = z^2 + c$ and $S_L = \sum_{j=0}^{L} \frac{1}{(R^j(c))}$ Assume there exists a subsequence $\{n_i\}$ of integers such that

1. $\lim_{i \to \infty} |(R^{n_i})(c)| = \infty$ and $\lim_{i \to \infty} |S_{n_i}| > 0$ or
2. $|(R^{n_i})(c)| \sim C$ for $i \to \infty$ and $\lim_{i \to \infty} |S_{n_i}| = \infty$

Then there exists no invariant line field on Julia set.

Proof. Our aim is to show that under conditions $X_2 = A_2$. We know that $\dim \{A_2 / X_2\} = 1$. Hence any element $\gamma \in A_2$ has the following expression

$$\gamma(z) = A \frac{1}{z - c} + \gamma_1, \text{ where } \gamma_1 \in X_2.$$ 

Let us show that element $\gamma_c = \frac{1}{z - c}$ belongs to $X_2$. Let us define the sequences

$$\phi_0 = \gamma_c, \phi_i = \gamma_c + \sum_{j=1}^{n_i - 1} \frac{1}{(R^j(c))} \frac{1}{z - R^j(c)},$$

then

$$\phi_i - R^*(\phi_i) = \frac{1}{z - c} \left( \sum_{j=0}^{n_i} \frac{1}{(R^j(c))} \right) - \frac{1}{(R^{n_i})(c)} \left( \frac{1}{z - R^{n_i}(c)} \right).$$

Now assume that $\gamma_c \notin X_2$, then there exists a linear functional $f$ on $A_2$ such that $f(\gamma_c) \neq 0$ and $f|_{X_2} = 0$. Hence $f(R^{n_i}(\gamma)) = f(\gamma)$ and we calculate

$$0 = f(\phi_i - R^*(\phi_i)) = f(\gamma_c) \sum_{j=0}^{n_i} \frac{1}{(R^j(c))} - \frac{1}{(R^{n_i})(c)} f \left( \frac{1}{z - R^{n_i}(c)} \right).$$

By using condition we conclude that $f(\gamma_c) = 0$. Proposition 18 completes theorem B.

Convergent rational maps

We start with accumulation some facts (see books of I.Kra "Automorphic forms and Kleinian Group" I. N. Vekua "Generalized analytic function.")

Facts. Denote by $F_\mu(a)$ the following integral $\int_\mathbb{C} \mu(z) \tau_a(z)dzd\bar{z}$ where $\tau_a(z) = \frac{1}{z-a}$ for $a \in \mathbb{C}$ and $\mu \in L_\infty(J(R))$. Then

1. $F_\mu(a)$ is continuous function on $\mathbb{C}$ and $\frac{\partial F_\mu(a)}{\partial a} = \mu$ in sense of distributions.
2. $|F_\mu(a)| = O(1/ |z|)$ for big $z$, $\|F_\mu(a)\|_\infty \leq \|\mu\|_\infty M$, where $M$ does not depends on $\mu$ and $a \in \mathbb{C}$.
3. $|F_\mu(a_1) - F_\mu(a_2)| \leq \|\mu\|_\infty C|a_1 - a_2| |\ln|a_1 - a_2||$, where $C$ does not depends on $\mu$ and $a$.

Denote by $B$ the operator $\mu \rightarrow F_\mu(a) : L_\infty(J(R)) \rightarrow C(\mathbb{C})$ and by $X$ the image $B(L_\infty(J(R)))$. Now by $W$ denote the space $X$ with the following topology:

$$\phi_n \rightarrow 0 \text{ means } \|\phi_n\|_\infty \rightarrow 0 \text{ and } \frac{\partial \phi_n}{\partial z} \rightarrow 0 \text{ in } \ast\text{-weak topology of } L_\infty(J(R)).$$
Lemma 19.

1. $W$ is complete locally convex vector topological space.
2. $B$ is the compact operator mapping $L_\infty(J(R))$ onto $W$, that is $B$ maps bounded sets onto precompact sets. Here precompactness means any sequence contains "Cauchy" subsequence.
3. Any bounded set $U \subset W$ is precompact.

Proof. The first is evident.

2. Let $U \subset L_\infty(J(R))$ is bounded, then $U$ is precompact in *-weak topology of $L_\infty(J(R))$. Further from item 2 of Facts we have that $B(U)$ forms uniformly bounded and equicontinuous family of continuous functions that means $B(U)$ is precompact in topology of uniform convergence.

3. What that means boundedness in $W$? Particularly the set $V = \{ \frac{\partial \phi}{\partial z} \text{ in sense of distributions, for } \phi \in U \}$
forms bounded set in *-weak topology of $L_\infty(J(R))$ hence $V$ is bounded in the norm topology of $L_\infty(J(R))$.
We complete lemma by using the item 2 and the fact $\phi = B(\phi_\pi)$.

Let us define the operator $T$ on $X$ as follows

$$T(F_\mu(a)) = F_{BR(\mu)}(a) = \int \int \mathcal{C} B_R(\mu) \tau_a = \int \int \mathcal{C} \mu R^*(\tau_a(z)),$$

where $B_R$ is the Beltrami operator. Easily show that $T(\phi) = \phi(R(a)) - \sum \frac{b_i \phi(R(c_i))}{a - c_i}$, 

where $\sum \frac{b_i}{a - c_i} = \frac{1}{R(a)}$. For example for $R(z) = z^2 + c$ we have $T(\phi)(a) = \frac{\phi(R(a)) - \phi(c)}{R(a)}$.

Remark 20. From definition we see that

$$\{T^n(\phi), n = 0, 1, \ldots\}$$
forms bounded in $W$ set

Lemma 21. $T$ is continuous endomorphism of $W$.

Proof. Let $F_{\mu_i} \to 0$ in $W$, then $||\mu_i|| \leq C < \infty$ and hence $\{T(F_{\mu_i})\}$ forms precompact in $W$ family. Let $\psi_0$ be a limit point of this set, then 

$$\psi_0(a) = \lim_j T(F_{\mu_{ij}}) = \int \int \mathcal{C} \mu_{ij} R^*(\tau_a) \to 0(*) \text{ weak topology).}$$

So $\psi_0 = 0$.

Now we start with weak conditions implying the mean ergodicity of given rational map $R$.

Definition. We will say that a rational map $K(z)$ is strongly convergent if the space of $qc_J(K(z))$ contains a map $R$ for which there exists a point $d$ with

$$\text{card}(\cup_{n=0}^{\infty} R^n(d)) > 2d - 1 \text{ and } s_n(d) < \infty \text{ for all } n = 1, \ldots,$$

where

$$s_n(z) = \sum \frac{|b^n_i|}{|z - c^n_i|}(+)$$
and here $\sum_{z-c_i} b_n(z) + p_n(z) = \frac{1}{(R^n)'(z)}$.

Now and below we assume that $R(z)$ is the map from condition $\ast$ with point $z = \infty$ as superattractive point.

First note that

$s_n(R^m(a)) \leq s_{n+m}(a)(R^m)'(a)$.

Indeed

$$\frac{s_n(R^m(a))}{|(R^m)'(a)|} = \frac{1}{|(R^m)'(a)|} \sum \frac{|b_i|}{|R^m(a) - c_i|} = \sum \frac{|b_i|}{|a - c_i|} \sum \frac{|Q(a)b_i|}{|P(a) - Q(a)c_i|} = **$$

here $R(a) = \frac{P(a)}{Q(a)}$.

Lemma 22. Assume $s_n(a) \leq C < \infty$ for all $n$ for the given rational map $R$. Then Cesaro averages $A_N(\tau_a)$ converges with the $L_1(J)$–norm for any $x \in \cup_{i=0}^\infty R^i(a)$.

Proof. In notations of above we have

$$T(F_\mu)(y) = \int \int_{|\gamma_k|} \frac{B(\mu)}{z-y} \frac{dz}{\gamma_k} = \int \int_{|\gamma_k|} \mu R(\gamma_k) \frac{dz}{\gamma_k}$$

$$= \frac{F_\mu(R)(y)}{R(y)} \sum \frac{b_i F_\mu(R(c_i))}{y-c_i} = \sum \frac{b_i (F_\mu(R(y)) - F_\mu(R(c_i)))}{y-c_i}.$$

Now consider the sequence of functionals $l_i(F) = (A_i(T)(F))(a)$ on $W$. Under assumption we have

$$|l_i(F)| \leq 2 \frac{1}{i} \sum_{j=0}^{i-1} s_j(a) \sup_{w \in C} |F(w)|$$

and so the family functionals $\{l_i\}$ can be continued on the space $C(\bar{C})$ of continuous functions on $\bar{C}$ to family of uniformly bounded functionals. Therefore we can choose a subsequence $l_i$, converging pointwise to some continuous functional $l_0$ (Note that $l_0$ is the fixed point for the dual operator $T^*$ acting on dual $W^*$). Besides that means sequence $A_i(R^*)(\tau_a)$ weakly converges in $L_1(J)$ and hence by the standard ergodic arguments whole sequence $A_N(\tau_a)$ converges by the norm to a fixed for $R^*$ element.

By notation above we know that

$s_n(x) \leq C_x$, for any $x \in \cup R^n(a)$.

So by repeating of the arguments above we complete proof.

Now we prove main theorem of this chapter.

Theorem 23. Let condition $\ast$ holds for the rational map $R$, then $R$ is mean ergodic.

Proof. Proof of theorem we divide onto two steps.

The first step consists of proving theorem under additional assumption. Namely, Let $d_i, i = 1, ..., k$ be all critical values of $R$, now form $k$ families of functionals on $W$ like in lemma above $l_i^a (F) = A_n(T)(F)(d_i)$. Now assumption is: It exists a subsequence $\{n_m\} \subset \{n\}$ so that for all $1 \leq k$ subsequences $l_i^a$ are convergent pointwise on $W$.

We know that $A$ is the closure of the linear span of the family of functions $\frac{a(a-1)}{(z-a)^2}, a \in S$. So it is enough to show convergence $A_N(R^*)(\frac{a(a-1)}{(z-a)^2})$ for any fixed $a \in S$.

Now let $x \in J(R)$ be any periodic repulsive point, then we claim that the sequence of functionals $L_{n_m}(F) = A_{n_m}(T)(F)(x)$ converges pointwise on $W$. 
Proof of the claim. Without loss of generality we assume that $x$ is fixed point for $R$. Let us calculate

$$T(F)(a) = \frac{F(R(a))}{R'(a)} - \sum_i \frac{b_i F(R(c_i))}{a - c_i},$$

$$T^2(F)(a) = \frac{F(R^2(a))}{(R^2)'(a)} - \frac{1}{R'(a)} \sum_i \frac{b_i F(R(c_i))}{R(a) - c_i} - \sum_i \frac{b_i T(F)(R(c_i))}{a - c_i},$$

and

$$T^n(F)(a) = \frac{F(R^n(a))}{(R^n)'(a)} - \frac{1}{(R^n-1)'(a)} \sum_i \frac{b_i F(R(c_i))}{R^{n-1}(a) - c_i} - \frac{1}{(R^n-2)'(a)} \sum_i \frac{b_i T(F)(R(c_i))}{R^{n-2}(a) - c_i} - \cdots - \sum_i \frac{b_i T^{n-1}(F)(R(c_i))}{a - c_i}.$$ 

So for $a = x$ and $\lambda = R'(x)$ we conclude

$$T^n(F)(x) = \frac{F(x)}{\lambda^n} - \sum_i \frac{b_i}{x - c_i} \left( T^{n-1}(F)(R(c_i)) + \frac{T^{n-2}(F)(R(c_i))}{\lambda} + \cdots + \frac{F(R(c_i))}{\lambda^{n-1}} \right).$$

Continue calculation

$$L_m(F) = \frac{1}{m} \sum_{n=0}^{m-1} T^n(F)(x) = \frac{1}{m} \sum_{n=0}^{m-1} \frac{F(x)}{\lambda^n} - \sum_i \frac{b_i}{x - c_i} \left( \sum_{n=0}^{m-1} \sum_{j=0}^{n} T^j(F)(d_i) \frac{1}{\lambda^{n-j}} \right).$$

For $m \to \infty$ the first term tends to zero. Now let us calculate the second term

$$\sum_{n=0}^{m-1} \sum_{j=0}^{n} \frac{T^j(F)(d_i)}{\lambda^{n-j}} = F(d_i) \left( \frac{1}{\lambda} + \cdots + \frac{1}{\lambda^{m-1}} \right) +$$

$$+ T(F)(d_i) \left( \frac{1}{\lambda} + \cdots + \frac{1}{\lambda^{m-2}} \right) + \cdots + T^{m-1}(F)(d_i) \frac{1}{\lambda} =$$

$$= F(d_i) \frac{\lambda}{\lambda - 1} \left( 1 - \frac{1}{\lambda^{m-1}} \right) + T(F)(d_i) \frac{\lambda}{\lambda - 1} \left( 1 - \frac{1}{\lambda^{m-2}} \right) + \cdots + T^{m-1}(F)(d_i) \frac{1}{\lambda} =$$

$$= \frac{\lambda}{\lambda - 1} \left( \sum_{j=0}^{m-1} T^j(F)(d_i) - \sum_{j=0}^{m-1} \frac{T^j(F)(d_i)}{\lambda^{m-j}} \right).$$

Such as $|T^j(F)(d_i)| \leq M(F)$ for all $j$ and $i$ we have that for $m \to \infty$ the expression $\frac{1}{m} \sum_i \left( \frac{b_i}{x - c_i} \sum_{j=0}^{n-1} \frac{T^j(F)(d_i)}{\lambda^{n-j}} \right)$ tends to 0. So we conclude that $L_j$ converges if and only if the functionals $l_j$ converges. By using assumptions we conclude that sequences $L_{n,m}$ converges pointwise and it complete claim.

By claim we know that for any fixed $F \in W$ the sequence functions $A_{n,m}(T)(F)(a)$ converges on periodic points from Julia set. We know that family functions $A_{n,m}(T)(F)(a)$ forms bounded equicontinuous family functions. Now show that this family has unique limit point. Indeed let $F_1$ and $F_2$ be two different limit functions for our family, then by the claim $F_1(x) = F_2(x)$ for any repulsive periodic point and hence $F_1 = F_2$ on Julia set. Now remember that functions $F_i$ are holomorphic on Fatou set we obtain $F_1(z) = F_2(z)$ for all $z \in \mathbb{C}$.

Now by using the fact $\gamma_a = a r_1 - (a-1) r_0 + r_a$ we obtain that $A_{n,m}(R^\ast)(\gamma_a)$ converges weakly and hence $A_N(R^\ast)(\gamma_a)$ converges strongly for any $a \in \mathbb{C}$. This completes first step.
Second step Here we prove that *-condition implies additional conditions of the first step. Namely let us denote by $Y$ the subset of elements from $L_1(J(R))$ on which averages $A_N(R^*)$ are convergent. Note that $Y$ is closed space such as family $A_N(R^*)$ forms equicontinuous family of operators.

We claim that for any $d_i = R(c_i)$ the elements $\gamma_{d_i}$ belong to $Y$.

Proof of the claim. Otherwise it exists a continuous functional $L$ on $L_1(J(R))$ and $i_0$ so that $L(\gamma_{i_0}) \neq 0$ and $Y \subset \ker(L)$. Note that $L$ is invariant functional (i.e. $L(R^*(f)) = L(f)$) such as for any $f \in L_1(J(R))$ the element $f - R^*(f)$ belongs to $Y$. Let $\nu \in L_\infty(J(R))$ be the element corresponding to $L$, then $\nu$ is fixed point for Beltrami operator $B_R$ and hence the function $F_\nu(a) = \int \nu \tau$ is fixed point for operator $T$ i.e.

$$\frac{F_\nu(R(a))}{R'(a)} - \sum \frac{b_i F_\nu(d_i)}{a - c_i} = F_\nu(a).$$

Let $d$ be point from condition (*), then by lemma 22 and under assumption $F_\nu(x) = 0$ for any $x \in \bigcup_{i=0}^{\infty} R^i(d)$. Therefore meromorphic function $\Phi(a) = \sum \frac{b_i F_\nu(d_i)}{a - c_i}$ has big number ($> 2d - 2$) of zeros that immediately implies $\Phi(a) \equiv 0$. Function $F_\nu$ satisfy to equation

$$\frac{F_\nu(R(a))}{R'(a)} = F_\nu(a).$$

Finally we have that $F_\nu$ is zero on set of all repulsive periodic points hence on Julia set and hence everywhere because $F_\nu$ is holomorphic on Fatou set. Contradiction. Theorem is proved.

Theorem 24. Let a map $R$ be strongly convergent rational map. Assume that Lebesgue measure of $PC(R)$ is zero. Then $J(R)$ does not support non-trivial invariant measurable fields.

Proof. Theorem above and proposition 13 complete proof of theorem.

Corollary 25. Let rational map $R$ be as in theorem above. Assume in addition that $R$ is structurally stable, then $R$ is hyperbolic.

Proof. Theorem A and Sullivan result (see [MSS]) complete theorem.

Further we will give sufficient conditions on rational maps to be strongly ergodic. This conditions will be given in terms of Poincare series of rational map. We start now with the following calculations.

Lemma 26. Let $R$ be a rational map with no critical relations and simple critical points. Let $c$ be a critical point of $R$ and $d \in (R^k)^{-1}(c)$ be any point for some fixed $k$. Then for any fixed $m$ the coefficient $b$ corresponding to the item $\frac{1}{z - d}$ in expression $s_m(z)$ has the following type

$$b = \frac{1}{(R^m)'(d)} = \frac{1}{(R^m(c))(R^m-c-1)'(R(c))((R^k)'(d))^2}.$$

Proof. Proof consists of consideration of residue in the point $d$. So let $U$ be such neighborhood of $c$ so that

1. restriction $R|_U : U \rightarrow R(U)$ presents 2-to-1 branching covering and
2. $R(U)$ is the disc $\{z, |z - R(c)| = \epsilon\}$ for some arbitrary fixed $\epsilon$.
3. Let $l \subset R(U)$ be an arc going from $R(c)$ to $\partial R(U)$, then by $B_1$ and $B_2$ denote branches of $R^{-1}$ mapping $R(U) \setminus \{l\}$ into $U$ and
4. in $U$ the following decomposition is true $R(z) = R(c) + A(z - c)^2 + ...$

Now let $g$ be Jordan curve around point $d$ eventually mapping on $\partial R(U)$. Write $b = \frac{1}{2\pi i} \int_{\partial U} \frac{\partial z}{(R^m-c)'(z)}$. Under conditions there are no critical relations so it exists a branch $J$ of $(R^k)^{-1}$ such that $J(c) = d$ and we have

$$b = \frac{1}{2\pi i} \int_{\partial U} \frac{(J')^2(z)\partial z}{(R^{m-k})(z)}$$
and continue calculations

\[ b = \frac{1}{2\pi} \int_{\partial R(U)} \frac{(J')^2(B'_1(z))^2 + (J')^2(B'_2(z))^2}{(R_n - k - 1)'(z)} \partial z. \]

Now remember that \((B'_1(z))^2 = \frac{1}{(R'(J(z))R(z) - R_c(z) + \ldots)^2} = \frac{1}{4A^2(B_1(z) - c)^2 + \ldots}\) and use the fact

\[ 4A^2(B_1(z) - c)^2 = 4A(R(B_1(z)) - R(c) + \ldots) = 4A(z - R(c) + \ldots), \]

and for \(z \in \partial R(U)\) the members \(\ldots\) are equivalent to \(\epsilon\). The same calculations for \(B_2\) gives

\[ (B'_2(z))^2 = \frac{1}{4A(z - R(c) + \ldots)}. \]

So we have

\[ b = \frac{1}{2\pi} \int_{\partial R(U)} \frac{(J')^2(B_1(z))}{(4A(z - R(c) + \ldots)(R_n - k - 1)'(z)) + \frac{(J')^2(B_2(z))}{(4A(z - R(c) + \ldots)(R_n - k - 1)'(z))} \]

and now using arbitrariness of \(\epsilon\) obtain

\[ b = \frac{1}{2\pi} \int_{\partial R(U)} \frac{(J')^2(B_1(z)) + (J')^2(B_2(z))}{(4A(z - R(c))(R_n - k - 1)'(z))}. \]

Further note that numerator under integral forms holomorphic function in \(R(U)\) and \((R_n - k - 1)'(z)\) does not have zeros in \(R(U)\) (otherwise we do \(\epsilon\) smaller and \(A = \frac{R'(c)}{2}\)) we obtain

\[ b = \frac{1}{(R'(c))(R_n - k - 1)'(R(c))(R_n - k - 1)'(d))} = \frac{1}{(R'(c))(R_n - k - 1)'(d))}. \]

Lemma is proved

Follow by McMullen ([MM]) we recall the backward and forward Poincare series for the given rational map \(R\).

**Definition.** *Forward Poincare series* \(S(x, R)\)

\[ P(x, R) = \sum_{n=0}^{\infty} \frac{1}{|(R^n)'(R(x))|^2}. \]

*Backward Poincare series* \(P(x, R)\). Let \(|R^*| = R_{1,1}\) be the modulus of the Ruelle operator, then

\[ S(x, R) = \sum_{n=1}^{\infty} |R^*|^n(1\mathbf{1}_c)(x) = \sum_{n=1}^{\infty} \sum_{R^n(y) = x} \frac{1}{|(R^n)'(y)|^2}. \]

Let us again consider the function \(s_n(a) = \sum \frac{|h_i|}{|a - c_i|}\) and let \(B_n = \sum |h_i|\). Then by lemma above we have.

\[ B_n = \sum_{c \in \mathcal{C}_{R'}(R)} \frac{1}{|R'(c)|} \sum_{j=1}^{n-1} \frac{1}{|(R^n - j - 1)'(R(c))|} \sum_{R^n(y) = c} \frac{1}{|(R^j)'(y)|^2} \]

and hence we have the following formal equality

\[ \sum_{n=2}^{\infty} B_n = \sum_{c \in \mathcal{C}_{R'}(R)} \frac{1}{|R'(c)|} S(c, R) \otimes P(c, R), \]

we recall that \(\otimes\) means Cauchy product of series.
Corollary 27. Let \( R \) be a rational map with simple critical points and no simple critical relation. Then for any \( a \in F(R) \) the function \( s_n(a) \) is bounded in the following cases.

1. Collet-Eckmann case. For any critical point \( c \neq \infty \) the series \( S(c, R) \) and \( P(c, R) \) are bounded.
2. For any critical point \( c \neq \infty \) one of the series \( S(c, R) \) or \( P(c, R) \) are bounded and the second one has uniformly bounded elements.
3. Conjectural case. Both series diverge slow enough (like harmonic series).

Prof. The cases (1) and (2) are immediately follows from properties of Cauchy product. In last case also follows from properties of Cauchy product, such as Cauchy product of two harmonic series is divergent but has uniformly bounded elements. Let us again repeat that it is not clear does exists a rational map for which both Poincare series are equivalent to harmonic series.

Let us note that this corollary looks like corollary A with Poincare series.

Convergent maps. Now the our aim is to give a weaker condition on a rational map to be ergodic. Let us recall that the main goal of the discussion above is the estimating the norm of the operator \( T \) on space \( X \), that is to estimate the expression

\[
\sum_{i} \frac{b_i F(R(c_i))}{a - c_i}.
\]

Let us rewrite this expression by using lemma 26.

Let \( R \) be a rational map and \( c_i, i = 1, \ldots, 2\deg(R) - 2 \), and \( d_i, i = 1, \ldots, 2\deg(R) - 2 \), be critical points and critical values, respectively and let point \( z = \infty \) be superattractive point. Then by induction we have.

\[
\frac{1}{R'(z)} = \sum_i \frac{b_i}{z - c_i} = \sum_i \frac{1}{R''(c_i)} \frac{1}{z - c_i},
\]

\[
\frac{1}{(R^n)'(z)} = \sum_{i, k=0}^{n-1} \left( \sum_{y \in R^{n-k}(c_i)} \frac{1}{(R^n)'(y)} \frac{1}{z - y} \right).
\]

Hence we obtain

\[
\sum_{i} \frac{b_i^2 F(R(c_i))}{a - c_i^2} = \sum_i \sum_{k=0}^{n-1} F(R^{n-k-1}(d_i)) \left( \sum_{y \in R^{n-k}(c_i)} \frac{1}{(R^n)'(y)} \frac{1}{z - y} \right).
\]

Lemma 28.

\[
\sum_{y \in R^{-k}} \frac{1}{(R^n)'(y)} \frac{1}{a - y} = \frac{1}{R''(c_i)} \frac{1}{R^{n-k-1}'(d_i)} \sum_j \frac{(J_j)^2(c_i)}{a - J_j(c_i)} = \frac{1}{R''(c_i)} \frac{1}{R^{n-k-1}'(d_i)} (R^*)^k(-\tau_a)(c_i),
\]

where \( J_j \) are branches of \( R^{-k} \), \( \tau_a(z) = \frac{1}{z-a} \) and \( R^* \) is Ruelle operator.

Proof. Lemma 26 and equalities above complete this lemma.

Then we have the proposition.

Proposition A. Let \( R \) be rational map with simple critical points. Let \( \infty \) be a fixed point for \( R \). Then there exist the following formal relations.

\[
RP(a, R) - 1 = \sum_i \lambda_i - \sum_i \frac{1}{R''(c_i)} RS(c_i, R, a) \otimes RP(c_i, R), \quad \text{where } \lambda \text{ is multiplier of } \infty
\]

\[
RS(x, R, a) = A(x, R, a) - \sum_k \frac{1}{R''(c_k)} A(c_k, R, a) \otimes RS(x, R, c_k).
\]
where $c_k$ are critical points of $R$.

**Proof.** The first equality is immediate corollary of discussion above.

Let us show the second equality. By lemma 14 we can calculate.

$$
(R^*)^0 (\tau_a)(z) = \tau_a(z),
$$

and

$$
(R^*)^2 (\tau_a)(z) = \frac{1}{(R^2)'(a)(z - R^2(a))} - \frac{1}{R'(a)} \sum_i \frac{b_i}{(R(a) - c_i)(z - R(c_i))} - \sum_i \frac{b_i}{(a - c_i)} R^*(\frac{1}{z - R(c_i)})
$$

and by induction

$$
((R^*))^n (\tau_a)(z) = \frac{1}{(R^n)'(a)(z - R^n(a))} - \sum_i b_i \left( \frac{1}{(R^{n-1})'(a)(R^{n-1}(a) - c_i)(z - R(c_i))} + \sum \frac{1}{a - c_i} (R^{n-1})^*(\frac{1}{z - R(c_i)}) \right)
$$

hence summation respect to $n$ gives desired equality. Lemma is proved.

Concluding the discussion above we obtain the following expression.

$$
\frac{1}{(R^n)'(a)} = -\sum_i \sum_{k=0}^{n-1} \frac{1}{R''(c_i)} \frac{1}{(R^{n-k-1})'(c_i)} (R^*)^k(\tau_a)(c_i)
$$

and

$$
\sum_i \frac{b_i^n}{a - c_i} F(R^n(c_i)) = -\sum_i \sum_{k=0}^{n-1} \frac{F(R^n(c_i))}{R''(c_i)(R^{n-k-1})'(c_i)} (R^*)^k(\tau_a)(c_i).
$$

**Theorem 29.** Let $R$ be convergent map. Then $R$ is mean ergodic.

**Proof.** Assume $R$ satisfies itself to condition $*$ from definition of convergent map. Then by using arguments of theorem 23 and lemma 22 we have

$$
\left| \frac{1}{N_i} \sum_{j=0}^{N_i} T^j(F)(x) \right| \leq 2A_N(x, R) \| F \|_\infty.
$$

Hence sequences $\frac{1}{N_i} \sum_{j=0}^{N_i} T^j(F)(x)$ is convergent (up to passing to a subsequences) in $*-$weak topology on the space $W$ for any $x \in \bigcup_n R^n(a)$. Again by using arguments theorem 23 and Mean ergodicity lemma we conclude that $R$ is mean ergodic.

**Theorem A.** Let $R$ be convergent map. Assume Lebesgue measure of postcritical set is zero, then there is no non-trivial invariant line field on Julia set.

**Proof.** Theorem 29 and proposition 13 give desired conclusion.

In corollary 27 we use definition of Collet-Eckmann maps in Przytycki sense. That is for all critical points the both forward and backward Poincare series are absolutely convergent. Now we redefine these maps by the following way.

**Definition.** A rational map $R$ we will call Collet-Eckmann map if does exists a point $a \in \mathbb{C}$ with long orbit, $\#(\bigcup_n R^n(a)) > 2\deg(R) - 1$ such that

For any critical point $c$ and any $x \in \{ \bigcup_n R^n(a) \}$ the both Ruelle-Poincare series $RS(c, R, x)$ and $RP(c, R)$ are absolutely convergent.
Corollary A. Let $R$ be a rational map with simple critical points and no simple critical relation. Then $R$ is convergent map if for some $a \in \mathbb{C}$ with $\# \{J_i R^i(a)\} > 2\deg(R) - 1$ the one of the following is true.

1. Collet-Eckmann maps.
2. For any critical point $c$ and $x \in \{J_i R^i(a)\}$ one of the series $RS(c, R, x)$ and $RP(c, R)$ is absolutely convergent and second one has uniformly bounded elements.
3. Conjectural case. Both series diverge slow enough (like harmonic series).

Proof. Evidently the maps from all (1) - (3) cases are convergent.

Measures

Start again with rational map $R$. Consider an element $\gamma \in A(R)$ and Cesaro average sequence $A_N(R)(\gamma) = \frac{1}{N} \sum_{n=0}^{N-1} (R^*)^i(\gamma)$. Let $C(U)$ be the space of continuous functions defined on $\overline{U}$ for the fixed essential neighborhood $U$. Then any $\ast$-weak limit of $A_N(R)(\gamma)$ on $C(U)$ we call weak boundary of $\gamma$ respect to $R^*$ over $U$ and denote set of all limit measures by $\gamma(U, R)$.

Theorem 30. Let $R$ be a structurally stable rational map with non-empty Fatou set. Assume there exists non-zero weak boundary $\mu \in \gamma(U, R^*)$ for an element $\gamma \in A(R)$ and an essential neighborhood $U$. Then Lebesgue measure $m(J(R)) > 0$ and there exists a non-trivial invariant line field on $J(R)$.

Proof. Under assumptions there exists an essential $U$ and $\gamma \in A(R)$ and subsequence $N_i$ such that

1. $\int A_N(R)(\gamma) \phi d\gamma$ converges for any $\phi \in C(U)$ and
2. there exists $\psi \in C(U)$ such that $\lim_{i \to \infty} \int \psi A_N(R)(\gamma) \neq 0$.

By using density of space of compactly supported continuous function at the space $C(U)$ we can assume that $\psi$ has a compact support $D \subset U$. Continue $\psi$ on $\overline{U} \setminus D$ by zero we obtain $\lim_{i \to \infty} \int \psi A_N(R)(\gamma) \neq 0$. Hence the dual average $A_N(B_R) = \frac{1}{N} \sum_{n=0}^{N-1} (B_R)^n(\psi)$ has non-zero $\ast$-weak limit element in $\ast$-weak topology on $L_\infty(J(R))$. Let $\mu \in L_\infty(J(R))$ be this non-zero limit element, then $\mu$ is fixed for $B_R$ and $\mu = 0$ on $F(R)$ by construction. Hence $m(J(R)) > 0$ and $\mu$ defines desired invariant line field.

It is not clear when the inverse statement is true. But we suggest the following conjecture.

Conjecture. Let $R$ be a rational map with non-empty Fatou set, the $T(J(R)) = \emptyset$ if and only if weak boundaries $\gamma(U, R^*) = 0$ for all $\gamma \in A(R)$ and all essential neighborhood $U$.

In general the absence of invariant line fields on Julia set means the mean ergodicity of $R^*$ on $L_1(J(R))$ and so it is interesting to understand the conditions implying the mean ergodicity of $R^*$ from measure's point of view. To do this let us recall definition of the following objects:

1. $U$ is an essential neighborhood of $J(R)$ and
2. $H(U)$ consists of $h \in C(\overline{U})$ such that $\frac{\partial h}{\partial d}$ (in sense of distributions) belongs to $L_\infty(U)$
3. $H(U)$ inherits the topology of $C(\overline{U})$.

Measures $\nu_i$.

1. Let $c_i$ and $d_i$ be critical points and critical values, respectively. Then define $\mu_i = \frac{1}{2\deg(R^*)}((R^*)^n(\frac{1}{2\deg(R^*)})$ (in sense of distributions). We will show below that $(R^*)^n(\frac{1}{2\deg(R^*)}) = \sum_{j=0}^{n} \frac{\alpha_j^i}{z^j R^{(d_i)}}$ and hence $\mu_i = \sum_{j=0}^{n} \alpha_j^i \delta_R^{(d_i)}$, where $\delta_a$ denotes the delta measure with mass at the point $a$.
2. Define by $\nu_i$ the average $\frac{1}{2} \sum_{k=0}^{n} \mu_k$.

Theorem C. Let $R$ be a rational map with simple critical points and no critical relations. Assume that $F(R) \neq \emptyset$ and $m(Pc(R)) = 0$, where $Pc(R)$ is the postcritical set and $m$ is denote the Lebesgue measure. Then $T(J(R)) = \emptyset$ if and only if there exist an essential neighborhood $U$ and a sequences of integers $\{k_i\}$ such that the measures $\nu_{k_i}$ converges in $\ast$-weak topology on $H(U)$ for any $i = 1, \ldots, 2\deg(R) - 2$.

Proof. Suppose that $\nu_{k_i}$ converges in $\ast$-weak topology on $H(U)$ for all $i$ a subsequence $\{k_i\}$ and an essential neighborhood $U$. Then the sequence averages $A_N(R) = \frac{1}{N} \sum_{n=0}^{N-1} (R^*)^n(\frac{1}{2\deg(R^*)}) \in L_1(U)$ is convergent weakly, in case $m(J(R)) > 0$
that means $A_N(R)(\frac{1}{z-d_i})$ converges strongly in $L_1(J(R))$. By using arguments proposition 13 and theorem 23 (second step) there exist no invariant line fields.

Now assume there exist no invariant line fields on $J(R)$. Let us show that $\nu_l^i \to 0$ in $*$-weak topology on $H(U)$ for any essential neighborhood $U$. Otherwise there exist a sequence $\{l_k\}$ an essential neighborhood $U$ and a function $F \in H(U)$ such that

$$\lim_{k \to \infty} \int\int F\nu_l^{10} = \lim_{k \to \infty} \int\int F \frac{1}{l_k} \sum_{n=0}^{l_k-1} (R^*)^n \left( \frac{1}{z-d_{i0}} \right) \neq 0.$$ 

Again like in previous theorem we can think that $F$ is continuous differentiable function with compact support belonging to $U$. Hence the sequences $\frac{1}{l_k} \sum_{i=0}^{l_k-1} B_R(Fz)$ has non-zero limit element $\mu \in L_\infty(U)$ in $*$-weak topology on $L_\infty(U)$. By arguments of Mean ergodicity lemma $\mu$ is invariant. Contradiction. Lemma is proved.

**REFERENCES**

[Kra] I. Kra, *Automorphic forms and Kleinian groups* (1972), W.A.Benjamin, Inc, Massachusetts, 464.

[Kren] Krengel, Ulrich, *Ergodic theorems. With a supplement by Antoine Brunel.* (1985.), de Gruyter Studies in Mathematics, 6., Walter de Gruyter & Co., Berlin-New York., 357.

[MM] C. McMullen, *Hausdorff dimension and conformal dynamic II: Geometrically finite rational maps*, To appear, Comm. Math. Helv.

[Mak] P. Makienko, *On measurable field compatible with some rational functions*, Proceedings of conference ”Dynamical systems and related topics”, Japan, (1990.).

[MSS] R. Mane, P. Sad and D. Sullivan, *On the dynamic of rational maps*, Ann. Sci. Ec. Norm. Sup. 16 (1983), 193 – 217.

[MM] C. McMullen, *Holomorphic Functions and Moduli I*, MSRI Publications 10, Springer - Verlag, 1988, pp. 31 – 60.

[P] *Pitman Res. Notes Math.*, vol. Ser. 362, International Conference on Dynamical Systems, Montevideo, Longman, Harlow, 1996., 1995, pp. 167–181.

[S] D. Sullivan, *Quasiconformal homeomorphisms and dynamics I, II, III.*, Ann. of Math. 2 (1985), 401 – 418; Acta Math. 155 (1985), 243 – 260.

Permanent address: Institute for Applied Mathematics, 9 Shevchenko str., Khabarovsk, Russia and Instituto de Matematics, Av. de Universidad s/N., Col. Lomas de Chamilpa, C.P. 62210, Cuernavaca, Morelos, Mexico