Robust optimization modelling with applications to industry and environmental problems

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Abstract.
Robust Optimization (RO) modeling is one of the existing methodology for handling data uncertainty in optimization problem. The main challenge in this RO methodology is how and when we can reformulate the robust counterpart of uncertain problems as a computationally tractable optimization problem or at least approximate the robust counterpart by a tractable problem. Due to its definition the robust counterpart highly depends on how we choose the uncertainty set. As a consequence we can meet this challenge only if this set is chosen in a suitable way. The development on RO grows fast, since 2004, a new approach of RO called Adjustable Robust Optimization (ARO) is introduced to handle uncertain problems when the decision variables must be decided as a "wait and see" decision variables. Different than the classic Robust Optimization (RO) that models decision variables as "here and now". In ARO, the uncertain problems can be considered as a multistage decision problem, thus decision variables involved are now become the wait and see decision variables. In this paper we present the applications of both RO and ARO. We present briefly all results to strengthen the importance of RO and ARO in many real life problems.

1. Introduction: Optimization under Uncertainty
Optimization under uncertainty refers to the branch of optimization where there are uncertainties involved in the data. The importance of this class of problem is in many models the uncertainty is ignored and a representative nominal value of the data is used. However, the decision variables cannot be perfectly forecasted but instead should be considered random or uncertain.

The general formulation of uncertain optimization problem as presented in the following term.

\[
\min_{x,t} \{ t : f_0(x, \zeta) - t \leq 0, f_i(x, \zeta) \in K_i, i = 1, \ldots, m \}
\]  

where \((x, t) \in \mathbb{R}^{(n+1)}\) is the vector of decision variables, \(\zeta \in \mathbb{R}^d\) is the vector of problem’s data, \(f_0(x, \zeta) : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}, f_i(x, \zeta) : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^{k_i}, i \leq i \leq m\), are given function and \(K_i \subset \mathbb{R}^{k_i}\) are given nonempty set. This means that the data vector \(\zeta\) is not known exactly at the time when the solution has to be determined.

2. Robust Optimization
In this section we present a brief discussion about the theory of Robust Optimization (RO) as proposed by [1] and discussed in [7], [8], also discussed in [4]. Let \(c \in \mathbb{R}^n, b \in \mathbb{R}^m\) and
\( A \in \mathbb{R}^m \times \mathbb{R}^n \) be the parameters of the canonical linear programming problem

\[
\min_x \left\{ c^T x : Ax \leq b \right\} \tag{LP}
\]

The main idea of Robust Optimization is we are interested in a version of (LP) problem in which the data \((c,A,b)\) are uncertain, but are known to reside in an uncertain set \(U\). In this case the (LP) is not a single deterministic problem, but a family of problems, one for each realisation \((c,A,b) \in U\):

\[
\min_{x \in \mathbb{R}^n} \left\{ c^T x : Ax \leq b \right\}_{(c,A,b) \in U} \tag{ULP}
\]

We assume that the decision environment is such that

(i) Decision is offline: the entire decision vector \(x\) is to be fixed prior to knowing which value the actual parameters take "here-and-now-decision".

(ii) In the dynamic (ULP) case only part of the variables \((x_1, \ldots, x_n)\) need to be determined off-line. The rest may be determined after some of the uncertain parameters become known ("wait and see" decision).

(iii) The prior information on the data \((c,A,b)\) is crude and is captured by a compact uncertainty set \(U\).

(iv) The inequality constraints \(Ax \leq b\) are hard constraints, i.e., they must all be satisfied whenever the uncertain parameters reside in \(U\).

Under the above conditions the robust optimization approach converts the uncertain family of the problems (ULP) into the following single deterministic problem, which we call Robust Counterpart:

\[
\pi^* = \min_{x \in \mathbb{R}^n} \left\{ c^T x : Ax \leq b, (c,A,b) \in U \right\} \tag{RC}
\]

A vector \(x^*\) is the called a robust optimal solution if for all realizations \((c,A,b) \in U\), \(x^*\) is feasible and the value of objective function is guaranteed to be at most \(\pi^*\). Problem (RC) can be written equivalently as a problem with a linear certain objective function and only uncertain constraints as follows:

\[
\min t \tag{2}
\]

\[
\text{s.t } c^T x - t \leq 0 \tag{3}
\]

\[
a_i^T x - b_i \leq 0, i = 1, \ldots, m, \forall (c,A,b) \in U \tag{4}
\]

2.1. The Treatment of Robust Linear Optimization (RLO)

We cite from [7] that for the treatment of Robust Linear Optimization (RLO) we will assume without loss of generality that,

(i) Firstly, the objective \(c^T x\) is certain. If there is uncertainty in the objective, we can reformulate the problem such that this uncertainty appears in a constraint see (2).

(ii) Secondly, the right-hand-side \(b\) is certain. If \(b\) is uncertain, we can introduce an extra variable \(x_{n+1}\) and change the problem into

\[
\min c^T x \tag{5}
\]

\[
\text{s.t } a_i^T x - b_i x_{n+1} \leq 0, i = 1, \ldots, m, \forall (A,b) \in U \tag{6}
\]

\[
x_{n+1} = 1, i = \ldots, m, \forall (A,b) \in U \tag{7}
\]

This assumption is made to make the notation for the uncertainty region and the resulting robust counterpart formulations easier.
(iii) Third, the robustness with respect to $U$ can be formulated constraint-wise and the last, the uncertainty set $U$ is closed and convex.

Notes that because this inequality on $x$ is of the semiinfinite type (it should be satisfied for an infinite number of values of the parameters $A$ and $b$), it cannot directly be efficiently solved by standard optimization methods. The challenge is to find for which types of uncertainty sets problem (RC) can be reformulated into a tractable optimization problem.

2.2. Types of uncertainty sets

The challenge is to find $U$ for which types of uncertainty sets problem can be reformulated into a tractable optimization problem. Since the robustness with respect to $U$ can be formulated constraint-wise, thus we can reformulate each constraint which involves the uncertain data. Refers to [7], consider the canonical robust semi-infinite constraint

$$a^T x - b \leq 0, \forall (a, b) \in U. \quad (8)$$

Here $a$ is a vector in $R^n$ and $b$ which are the general representatives of $a_i$ and $b_i$.

Similarly, $U$ stands for $U_i$. Its convenient to describe the uncertainty parameters $a$ and $b$ and the uncertainty set $U$ in terms of a primitive factor $\zeta \in R^L$. Namely,

$$a = \bar{a} + Q\zeta, \quad b = \bar{b} + q^T \zeta \quad (9)$$

where $\bar{a} \in R^n, Q \in R^{n \times L}, \bar{b} \in R$ and $q \in R^L$ and

$$U = \left\{ \begin{pmatrix} a = \bar{a} + Q\zeta \\ b = \bar{b} + q^T \zeta \end{pmatrix} : \zeta \in Z \right\} \quad (10)$$

where $Z \subset R^L$ is the uncertainty set for the primitive factors. The fixed vector $\bar{a}$ and the scalar $\bar{b}$ will thereafter be called nominal. In the view of representation (10) the alternative formulation of (8). Replacing $a$ and $b$ by their expressions in $\zeta$, we have

$$(\bar{a}^T x - \bar{b}) + (Q^T x - q)^T \zeta \leq 0, \forall \zeta \in Z. \quad (11)$$

2.2.1. Case 1: Using Interval or Box Uncertainty Set

In general, a tractable robust counterpart formulation for RLO of (8) with interval (or box) uncertainty regions can be stated as the following formulation.

$$(\bar{a} + P\zeta)^T x \leq b, \forall \zeta : \|\zeta\|_\infty \leq \mu. \quad (12)$$

This problem is a semi-infinite programming problem.

2.2.2. Case 2: Using Ellipsoidal Uncertainty Set

Refers to [7], in the case of ellipsoidal uncertainty, the robust counterpart of (8) becomes

$$(\bar{a} + P\zeta)^T x \leq b, \forall \zeta : \|\zeta\|_2 \leq \mu. \quad (13)$$

It can easily verify that

$$\max_{\zeta : \|\zeta\|_2 \leq \mu} (P^T x)^T = \mu \|P^T x\|_2. \quad (14)$$
Hence \( x \) satisfies (13) if and only if
\[
\bar{a}^T x + \mu \left\| P^T x \right\|_2 \leq b. \tag{15}
\]

Now the constraint does not have the semi-infinite structure as the original one in (13). This is a conic quadratic programming problem. Again as in the case of box uncertainty, in the final robust counterpart an extra safety term \( \mu \left\| P^T x \right\|_2 \) is added to account for uncertainty and it depends on the value of \( x \).

2.3. Case 3: Using Polyhedral Uncertainty Set
In case of the uncertainty set is polyhedral, i.e., a system of linear inequalities
\[
(\bar{a} + P\zeta)^T x \leq b, \forall \zeta : d - D\zeta \geq 0. \tag{16}
\]
This is equivalent with
\[
\bar{a}^T x + \max_{\zeta : d - D\zeta \geq 0} (P^T x)\zeta \leq b. \tag{17}
\]
For this formulation, the robust counterpart cannot be obtained explicitly. Thus, refers to [7] the robust counterpart formulation can be obtained via its dual formulation.

Consider that by the duality theory we have that
\[
\max \left\{ (P^T x)\zeta : d - D\zeta \geq 0 \right\} = \min \left\{ d^T y : D^T y = P^T x, y \geq 0 \right\}. \tag{18}
\]
Hence \( x \) satisfies (17) if and only if \( x \) satisfy
\[
\bar{a}^T x + \min_{y} \left\{ d^T y : D^T y = P^T x, y \geq 0 \right\} \leq b. \tag{19}
\]
Notice that if this constraint is satisfied for a feasible \( y \), i.e., that satisfies \( D^T y = P^T x, y \geq 0 \) then the constraint is certainly satisfied for the minimum over \( y \). Hence we can delete the min. Thus, \( x \) satisfies (17) if and only if there exists a \( y \) such that \((x, y)\) satisfies
\[
\bar{a}^T x + d^T y \leq b, D^T y = P^T x, y \geq 0. \tag{20}
\]
This result is very tractable since it is a system of linear (in) equalities.

3. Adjustable Robust Optimization
The development on RO grows fast, since 2004, a new approach of RO called Adjustable Robust Optimization (ARO) is introduced to handle uncertain problems when the decision variables must be decided as a "wait and see" decision variables. Different than the classic Robust Optimization (RO) that models decision variables as "here and now". In ARO, the uncertain problems can be considered as a multistage decision problem, thus decision variables involved are now become the wait and see decision variables. Adjustable robust counterpart handles uncertain problem when the variable has to be decided after the uncertainty parameter values (or part of it) has been revealed. In other words, in such case we can postpone part of decision until we have obtained more of full information on (some of the ) uncertain parameters. Considering the Adjustable Robust Counterpart of one constraint of a multi-stage problem:
\[
(\bar{a} + P\zeta)^T x + d^T y \leq b, \forall \zeta \in Z. \tag{21}
\]
where \( x \in R^{n_1} \) is non-adjustable, \( y \in R^{n_2} \) is adjustable. This situation is called a fixed recourse situation, since the coefficients for the adjustable variable \( y \) are certain.
(i) The optimization variable $y = y(\zeta)$ called a decision rule is in fact a vector function. It has been shown that in general solving such problem is NP-hard. Therefore, $y(\zeta)$ is restricted to a given class of function.

(ii) However it appears that the linear decision rule already shows good performance in practice.

4. Applications of RO and ARO
Some results of the author on developing model for handling uncertain data, has been published in [2], [3], [4], [5], [6], [9],[12],[11] and [13].

In [2], a discussion on Robust Shortest Path Problems by means of Robust Linear Optimization is presented. Modeling Robust Design Problems via Conic Optimization is discussed in [3]. We discussed Handling Optimization under Uncertainty Problem Using Robust Counterpart Methodology in [4].

Some application topics in environmental issues such as vaccination and water supply is discussed. We have the results of developing the Robust Optimization to the Uncertain Vaccination Strategy Problem (see [5]). A recent results about Determining Robust Counterpart of Spatial Optimization Model for Water Supply Allocation can be seen in [6].

To engineering application we focus in developing the robust optimization model based Model Predictive Control using box uncertainty set (see [9]). Also a topic on robust electrical network topology design by conic optimization is discuss in [12].

For application of ARO, we have work on the problem of determining the adjustable robust counterpart of water distribution problem in the case of Waduk Jatiluhur (see [13]) and also the adjustable robust counterpart for transportation problem with discount factor (see [11]).

5. Conclusion
The fast development on RO has brought us to a new era of ARO. Some applications has been discuss in many papers. The future research will be the ARO version with binary integer variables.

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