A model of anaerobic digestion for biogas production using Abel equations

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Abstract: We consider a nonlinear mathematical model for the study of anaerobic digestion processes. We decompose the original system of nonlinear ordinary differential equations into subsystems. For these subsystems we prove existence of lower and upper solutions in reverse order for one of the variables. The upper and lower solutions are constructed in analytical form. Furthermore, the upper solutions of subsystem for feeding bacteria are related with solutions of Abel equations of the first kind. Using numerical and theoretical arguments we examine how to obtain upper and lower solutions approximated to the numerical solution of the system. In this work we establish special techniques of lower-upper solution, which includes reverse order for non monotone systems, in contrast to the techniques used by H.L. Smith and P. Waltman on their monograph [15].

Keywords: Anaerobic digestion; Abel equation; biogas; Cauchy problem; lower-upper solution; reverse order

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1 Introduction

Biogas is a generic name for a combustible gas mixture produced in the decomposition of organic substances due to anaerobic microbiological process (methane fermentation). Biogas, the end-product of anaerobic digestion mainly consisting of

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CH$_4$(60 – 70%) and CO$_2$(30 – 40%), provides considerable potential as a versatile carrier of renewable energy, not solely because of the wide range of substrates that can be used for the anaerobic digestion process [14]. The anaerobic digestion has been used to convert biomass into biogas microalgae [14, 10], as well in wastewater treatment [10, 7]. Biogas is widely used as a combustible fuel in Germany, Denmark, China, United States and other developed countries. It is used for home consumption and public transportation.

To model the process of anaerobic digestion, see [7], we start considering the following nonlinear system of ordinary differential equations:

**Biomass balance**

\[
\frac{dX_1}{dt} = \left( \mu_1(S_1) - \alpha D \right) X_1 \quad \text{(acidogenesis)}
\]

\[
\frac{dX_2}{dt} = \left( \mu_2(S_2) - \alpha D \right) X_2 \quad \text{(methanogenesis)}
\]

**Substrate balance**

\[
\frac{dS_1}{dt} = D \left( S_{1in} - S_1 \right) - k_1 \mu_1(S_1) X_1
\]

\[
\frac{dS_2}{dt} = D \left( S_{2in} - S_2 \right) + k_2 \mu_1(S_1) X_1 - k_3 \mu_2(S_2) X_2
\]

**Alkalinity balance**

\[
\frac{dA}{dt} = D \left( A_{in} - A \right)
\]

**Carbon exchange rate**

\[
\frac{dC}{dt} = D \left( C_{in} - C \right) + k_4 \mu_1(S_1) X_1 + k_5 \mu_2(S_2) X_2
\]

\[
- K_L\left[ C + S_2 - A - K_H P_C \right]
\]

The product $K_H P_C = B$ determines concentration of oxygen dissolved in $C$.

**Net rate of methane production**

\[
\frac{dF_M}{dt} = k_6 \mu_2(S_2) X_2
\]

with methane concentration $F_M$.

Nonlinear kinetic behavior, occurs due to the reaction rates, which are given by

\[
\mu_1(S_1) = \mu_{1\text{max}} \frac{S_1}{S_1 + K_{S_1}} \quad \text{- Monod type kinetic}
\]

\[
\mu_2(S_2) = \mu_{2\text{max}} \frac{S_2}{K_{S_2} + S_2} \quad \text{- Hal-dane type kinetic}
\]

represent bacterial growth rates associated with two bioprocesses.
In this case the variables are:

\[ S_1 := \text{Organic substrate concentration} \ [\text{g/l}] \]
\[ X_1 := \text{Concentration of acidogenic bacteria} \ [\text{g/l}] \]
\[ S_2 := \text{Volatile fatty acids concentration} \ [\text{mmol/l}] \]
\[ X_2 := \text{Concentration of methanogenic bacteria} \ [\text{g/l}] \]
\[ A := \text{Concentration of alkalinity} \ [\text{mmol/l}] \]
\[ C := \text{Total inorganic carbon concentration} \ [\text{mmol/l}] \]
\[ F_M := \text{Methane concentration} \ [\text{mmol/l \ d}^{-1}] \].

With \( \mu_{1\text{max}} > 0, \mu_{2\text{max}} > 0, K_{S_1} > 0, K_{S_2} > 0, K_{I_2} > 0, A_{in} > 0, B > 0, C_{in} > 0, K_{L_a} > 0, S_{1in} > 0, S_{2in} > 0, \) and \( k_i > 0, i = 1, 2, 3, 4, 5, 6. \), are positive constants. The parameter \( \alpha \) \((0 \leq \alpha \leq 1)\) therefore reflects this process of heterogeneity: \( \alpha = 0 \) corresponds to an ideal fixed-bed reactor, whereas \( \alpha = 1 \) corresponds to an ideal continuously stirred tank reactor [7].

We consider nonlinear equation

\[ u'(t) = f(t, u(t)), \quad t \in I = [0, T], \quad T > 0 \]  \hspace{1cm} (9)

satisfying the following condition proposed in [9],

\[ g(u(0), u(T)) = 0, \]  \hspace{1cm} (10)

where \( f : I \times \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R}^2 \to \mathbb{R} \) are continuous functions. If \( g(x, y) = x - c \) with \( c \in \mathbb{R} \), then (10) is the initial condition

\[ u(0) = c. \]

**Definition 1** (See [9]). We say that

- A function \( \beta_1 \in C^1(I) \) is lower-solution of (9) whether

\[ \beta_1'(t) \leq f(t, \beta_1(t)), \quad t \in I \]

- A function \( \beta_2 \in C^1(I) \) is upper solution of (9) whether

\[ \beta_2'(t) \geq f(t, \beta_2(t)), \quad t \in I \]

with either condition

\[ \beta_1(t) \leq \beta_2(t), \quad t \in I \]  \hspace{1cm} (11)

or

\[ \beta_2(t) \leq \beta_1(t), \quad t \in I \]  \hspace{1cm} (12)

For \( u, v \in C(I), \ u \leq v \) we define the set

\[ [u, v] = \{ \forall w \in C(I) \ : u(t) \leq w(t) \leq v(t), \text{ with } t \in I \}. \]
**Definition 2** (See [9]). We say that $\beta_1, \beta_2 \in C^1(I)$ are coupled lower and upper solutions for the problem (9) - (10) in reverse order whether $\beta_1$ is a lower solution and $\beta_2$ a upper solution for the equation (9), where the conditions

\[
\beta_2(t) \leq \beta_1(t), \quad t \in I,
\]

\[
\max\{g(\beta_1(0), \beta_1(T), g(\beta_2(0), \beta_1(T))\} \leq 0 \quad \text{and}
\]

\[
0 \leq \min\{g(\beta_2(0), \beta_2(T)), g(\beta_1(0), \beta_2(T))\} \quad \text{hold}.
\]

**Theorem 1** (See [9]). Assume that $\beta_1, \beta_2$ are coupled lower and upper solutions in reverse order for the problem (9) - (10). In addition, suppose that the functions

\[
h_{\beta_1}(x) := g(x, \beta_1(T)), \quad h_{\beta_2}(x) := g(x, \beta_2(T))
\]

are monotonic (either nonincreasing or nondecreasing) in $[\beta_2(0), \beta_1(0)]$. Then there exists at least one solution of the problem (1) - (3) in $[\beta_2, \beta_1]$. Reverse order of lower and upper solution for systems is not previously considered. In contrast to the H.L. Smith and P. Waltman monograph [15], we operate with non monotone functions in the right part of original system. It requires development the special techniques of lower-upper solution including reverse order for non monotone systems.

The goal of this work is the construction of positive lower and upper solutions of system (1) - (8) in reverse order with respect a part of the variables with initial conditions on the given observation interval. We decompose original system into subsystems and we construct lower and upper solutions for every one.

For $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ and $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, we write for each $i = 1, \ldots, n$

$u \leq v$ if $u_i \leq v_i$.

First we study a subsystem made up by equations (1) and (3). It corresponds two equations:

\[
\begin{align*}
    u' &= f(t, u, v) \quad \text{on } I \\
    v' &= g(t, u, v), \quad \text{on } I,
\end{align*}
\]

where $I = [a, b]$, $u = X_1$, $v = Y_1$. We observe that the definition itself of lower-upper solutions of (13) depends heavily on the monotonicity properties on $f$ and $g$. Thus, following notation of C.V. Pao in [12] and new results in [8], we can classify (13) according to their relative monotonicity as follows:

1. **Quasi-monotone systems**: $f$ is nondecreasing in $v$ and $g$ in $u$, or $f$ is nonincreasing in $v$ and $g$ in $u$.

2. **Mixed quasi-monotone systems**: $f$ is nondecreasing in $v$ and $g$ is nonincreasing in $u$ or viceversa.

3. **Nonquasi-monotone systems**: either $f$ or $g$ has not monotony properties in $v$ or $u$ respectively.
Case 1 implies the existence of lower \((u_*, v_*)\) and upper \((u^*, v^*)\) solution with ordering in \(I\)

\[ u_* \leq u \leq u^*, \quad v_* \leq v \leq v^* \]

It is impossible to enforce quasi-monotonicity by a simple transformation for Case 2. Case 3 requires some regularity conditions imposed on \(f\) and \(g\). We employ Muller’s theorem for arbitrary systems and it generalization to elliptic systems

\[ \Delta u + f(t, u) = 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega, \]

where \(u = (u_1, u_2, \ldots, u_n), f = (f_1, \ldots, f_n)\) are \(n\)-vectors; \(\Omega\) is an open bounded subset of \(\mathbb{R}^m\) with smooth boundary \(\partial \Omega\), and \(f(t, u)\) is uniformly Hölder continuous in \(t\) and Lipschitz continuous in \(u\), see [11].

**Theorem 2** (Muller’s Theorem, see [16]). Let \(f(t, y) : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n\) be continuous and locally Lipschitzian in \(y\). Let \(u, v, w : J = [\xi, \xi + a] \to \mathbb{R}^n\) be differentiable, \(v \leq w\) in \(J\), \(v(\xi) \leq y(\xi) \leq w(\xi), u' = f(t, u)\), and

\[ v'_i \leq f_i(x, z) \quad \text{for all} \quad z \in \mathbb{R}^n \quad \text{such that} \]

\[ v(x) \leq z \leq w(x), \quad z_i = v_i(x), \]

\[ w'_i \geq f_i(x, z) \quad \text{for all} \quad z \in \mathbb{R}^n \quad \text{such that} \]

\[ v(x) \leq z \leq w(x), \quad z_i = w_i(x), \]

for \(i = 1, \ldots, n\). Then \(v \leq u \leq w\) in \(J\).

The second subsystem contains four equations (1), (2), (3), (4):

\[ u' = F(t, u) \tag{14} \]

where \(u \triangleq (u_1, u_2, u_3, u_4)\) and \(F \triangleq (F_1, F_2, F_3, F_4)\) are vectors with:

**I.** \(u_1 = X_1, u_2 = S_1, u_3 = X_2, u_4 = S_2\) and \(x\) is lower \((X_{10}, S_{10})\) or upper \((X_{11}, S_{11})\) solutions of the system (13) or in the second case, four equations (2), (4), (5), (7).

**II.** \(u_1 = X_2, u_2 = S_2, u_3 = C, u_4 = A\) and \(x\) is lower \((X_{10}, S_{10})\) or upper \((X_{11}, S_{11})\) solutions of the system (13).

The system (14) we reduce to subsystem with two equations for (2), (4) \(u_1 = X_2, u_2 = S_2\) and \(x\) is lower-upper solution of the system (13). General subsystem includes seven equations (1)-(8) with variables \(u_1 = X_1, u_2 = S_1, u_3 = X_2, u_4 = S_2, u_5 = A, u_6 = C, u_7 = F_M\). In the above-quoted systems, the usual order \((X_{10} \leq X_{11})\) for the lower and upper solutions is considered. In the opposite case \(S_1 \leq S_{10}\), the situation is quite different.

The outline of the paper is as follows. In Section 2, we consider a submodel with dynamics of biomass and substrate in acidogenesis. First we give new definition on lower-upper solutions in reverse order with respect to a part of the variables
and after we give definition of semitrivial solutions for subsystem of two differential equations. In this part the Abel differential equation of the first kind plays a key role, which appear as an upper solution of subsystem for feeding bacteria. We proof Theorem 3 on existence of upper and lower solutions with direct order in variable $X_1$ and with reverse order in variable $S_1$ with conditions of mixed quasi-monotonicity. Section ?? contains results on upper and lower solutions of submodel of biomass and substrate in metanogenesis. This is more complicated subsystem with five differential equations. We decompose this system in two subsystems via definition of upper-lower solution. Results are preliminar, because the system does not possess property of quasi monotonicity. Despite this, we show existence of upper and lower solutions for $X_1$ with ordinary order and for variable $S_2$, which can change order of upper-lower solution depending on restrictions on the parameters.

Definition 3. A trivial solution of the system (1)-(8) has the form

$$E_1(0, S_{1in}), \ E_2(0, S_{1in}, 0, S_{2in}), \ E_3(0, S_{2in}), \ E_4(0, S_{1in}, 0, S_{2in}, A_{in}), \ E_5(A_{in})$$

(15)

where $X_1 = 0$, $X_2 = 0$, $S_1 = S_{1in}$, $S_2 = S_{2in}$, $A = A_{in}$, $C = C_{in}$, $F_M = 0$.

We do not consider other possible trivial solutions in this paper. A similar approach can be seen in [6].

2 Lower and upper solutions for initial value problem (16)

We are interested in the solution of the nonlinear subsystem with initial conditions, which models dynamics of biomass and substrate in acidogenesis.

$$\begin{align*}
\frac{dX_1}{dt} &= \left(\mu_{1,\max} \frac{S_1}{S_1 + K_{S_1}} - \alpha D\right) X_1 \triangleq F(t, X_1(t), S_1(t)), \\
\frac{dS_1}{dt} &= D (S_{1in} - S_1) - k_1 \mu_{1,\max} \frac{S_1}{S_1 + K_{S_1}} X_1 \triangleq G(t, X_1(t), S_1(t)), \\
X_1(0) &= c_1, \\
S_1(0) &= c_2.
\end{align*}$$

(16)

where $t \in [0, T] = I$, with $T > 0$ and $F, G$ are functions of $C^0(I) = C(I)$ class, i.e., continuous functions in $I$.

Theorem 3. Assume $F$ and $G$ satisfy a local Lipschitz condition in $X_1, S_1$ mixed quasi-monotone properties: $\frac{\partial F}{\partial S_1} > 0$, $\frac{\partial G}{\partial X_1} < 0$ and the existence a pair $(X_{10}, X_{1}^0)$, $(S_{10}, S_{1}^0)$ of lower-upper, upper-lower solutions of de system (16). Then there exists at least one solution $(X_1, S_1)$ of (16) such that

$$X_{10} \leq X_1(t) \leq X_{1}^0, \quad S_{10} \leq S_1(t) \leq S_{1}^0, \quad t \in I.$$
Proof. We introduce the space

$$K \equiv [X_{10}, X_1^0] \times [S_1^0, S_{10}] \subset E \equiv C(I) \times C(I).$$

$K$ is a bounded closed convex set in $E$. Let $\lambda > 0$ and consider the modified problems

$$\begin{cases} 
\dot{X}_1(t) + \lambda X_1 = F^*(t, X_1(t), S_1(t)), & t \in I \\
X_1(0) = c_2 
\end{cases} \quad (17)$$

$$\begin{cases} 
\dot{S}_1(t) + \lambda S_1 = G^*(t, X_1(t), S_1(t)), & t \in I \\
S_1(0) = c_1 
\end{cases} \quad (18)$$

and

$$F^*(t, X_1, S_1) = \begin{cases} 
F(t, X_1^0, S_1) + \lambda X_1^0, & \text{if } X_1^0 < X_1 \\
F(t, X_1, S_1) + \lambda X_1, & \text{if } X_{10} \leq X_1 \leq X_1^0 \\
F(t, X_{10}, S_1) + \lambda X_{10}, & \text{if } X_1 < X_{10}. 
\end{cases}$$

and

$$G^*(t, X_1, S_1) = \begin{cases} 
G(t, X_1, S_1^0) - \lambda S_1^0, & \text{if } S_1^0 < S_1 \\
G(t, X_1, S_1) - \lambda S_1, & \text{if } S_1^0 \leq S_1 \leq S_{10} \\
G(t, X_{10}, S_1) - \lambda S_{10}, & \text{if } S_{10} < S_1. 
\end{cases}$$

and

$$p(t, x) = \max \left\{ X_{10}, \min \left\{ x, X_1^0 \right\} \right\}.$$ $$p_{\text{reverse}}(t, x) = \max \left\{ S_1^0, \min \left\{ x, S_{10} \right\} \right\}.$$ 

Note that if $(X_1, S_1)$ is a solution of (19a)-(19e) between $X_{10}$ and $X_1^0$, and also between $S_1^0$ and $S_{10}$, then $(X_1, S_1)$ is a solution of (16).

Let us consider the problem (19a)-(19e). We define the mappings

$$L : C(I) \rightarrow C_0(I) \times \mathbb{R}$$

$$N : C(I) \rightarrow C_0(I) \times \mathbb{R}$$

by

$$[L X_1](t) = \left( X_1(t) - X_1(0) + \lambda \int_0^t X_1(s) ds, X_1(0) \right)$$

and

$$[N X_1](t) = \left( \int_0^t F^* \left( t, X_1(t, x), p(0, x) \right) \right).$$

Clearly $N$ is continuous and compact (by the Arzelá-Ascoli theorem). On the other hand, solving (17) is equivalent to find a fixed point of $L^{-1}N : C(I) \rightarrow C(I)$. Now, Schauder’s fixed point theorem guarantees the existence of least a fixed point since $L^{-1}N$ is continuous and compact. Its remains to show that $X_1$ satisfies

$$X_{10}(t) \leq X_1(t) \leq X_1^0(t), \quad t \in [0, T].$$
Assume that $X_{10} - X_{1}^{0}$ attains a positive maximum on $[0, T]$ at $s_0$. Let $s_0 \in (0, T]$. Then there exists $\tau \in (0, s_0)$ such that

$$0 \leq X_{10}(t) - X_{1}^{0}(t) \leq X_{10}(s_0) - X_{1}^{0}(s_0), \quad \text{for all } t \in [\tau, s_0].$$

This yields a contradiction, since

$$X_{1}^{0}(s_0) - X_{1}^{0}(\tau) \leq \int_{\tau}^{s_0} \left[ F(s, X_{1}^{0}(s)) - \lambda(X_{10}(s) - X_{1}^{0}(s)) \right] ds \leq \int_{\tau}^{s_0} (X_{1}^{0})'(s) ds = X_{1}^{0}(s_0) - X_{1}^{0}(\tau).$$

Case $s_0$ is proved with condition of mixed quasi monotony in $F$ and Lipschitz conditions in $X_{11}$. Consequently $X_{1}(t) \leq X_{1}^{0}(t)$ for all $t \in [0, T]$. Similar, one can show that $X_{10}(t) \leq X_{1}(t)$, on $I$. The proof of the Theorem 3 with reverse order of upper-lower solutions in variable $S_{1}$ ($S_{1}^{0} \leq S_{1}$) for modified problem (18) with $p_{\text{reverse}}(t, x)$ is analogous to the proof of the first part of Theorem 3 with normal order in variable $X_{1}$ ($X_{10} \leq X_{1}$) for modified problem (17).

It is possible to define the concept of lower-upper solution in reverse order with respect to variable $S_{1}$ for subsystem (16) as follows.

**Definition 4 (Lower-upper solution).** A pair $[(X_{10}, S_{10}), (X_{1}^{0}, S_{1}^{0})]$ is called

(a) Lower-upper solution of the problem (16), whether the following conditions are satisfied

\begin{align}
(X_{10}, S_{10}) & \in C^{1}(I), \quad (X_{1}^{0}, S_{1}^{0}) \in C^{1}(I), \quad t \in I \\
\dot{X}_{10} - F(t, X_{10}, S_{1}) & \leq 0 \quad \text{(lower)} \quad \text{(19a)} \\
\dot{X}_{1}^{0} - F(t, X_{1}^{0}, S_{1}) & \geq 0 \quad \text{in } I, \quad \forall S_{1} \in [S_{1}^{0}, S_{10}], \quad \text{(upper)} \quad \text{(19b)} \\
\dot{S}_{10} - G(t, X_{10}, S_{10}) & \leq 0 \quad \text{(reverse order)} \quad \text{(19c)} \\
\dot{S}_{1}^{0} - G(t, X_{1}, S_{1}^{0}) & \geq 0 \quad \text{in } I, \quad \forall X_{1} \in [X_{10}, X_{1}^{0}], \quad \text{(reverse order)} \quad \text{(19d)}
\end{align}

and with initial conditions

$$X_{10}(0) \leq c_2 \leq X_{1}^{0}(0), \quad S_{1}^{0}(0) \leq c_1 \leq S_{10}(0); \quad \text{(19e)}$$

(b) Lower-lower solution of the problem (16), whether the following conditions are satisfied

\begin{align}
\dot{X}_{10} - F(t, X_{10}, S_{10}) & \leq 0 \quad \text{in } I \\
\dot{S}_{0} - G(t, X_{10}, S_{10}) & \leq 0 \quad \text{in } I 
\end{align}

and with initial conditions $S_{1}^{0}(0) \leq c_1 \leq S_{10}(0); \quad \text{(19f)}$
(c) Upper-upper solution of the problem (16), whether the following conditions are satisfied
\[
\begin{align*}
\dot{X}_1^0 &= F(t, X_1^0, S_1^0) \geq 0 \text{ in } I \\
S_1^0 &= G(t, X_1^0, S_1^0) \geq 0 \text{ in } I
\end{align*}
\]
with \(X_{10} \leq X_1^0, \quad S_1^0 \leq S_{10}\) in \(I\).

For \(u, v \in C(I)\), let \(u \leq v\) we define the set
\[
[u, v] = \{w \in C(I) : u(t) \leq w(t) \leq v(t), \text{ with } t \in I\}.
\]

**Definition 5.** If \(\Psi(t, t_{S_1}, \mu_{1\text{max}}, K_{S_1}, D, \alpha)\) is a solution of the ordinary differential equation
\[
\dot{X}_1 = \left(\mu_{1\text{max}} \frac{t_{S_1}}{t_{S_1} + K_{S_1}} - \alpha D\right) X_1
\]
then the function \(\Psi(t, t_{S_1}, \mu_{1\text{max}}, K_{S_1}, D, \alpha)\) is called semitrivial solutions of the problem (16), and if \(\Psi_1(t, t_{X_1}, D, S_{1\text{in}}, k_1, \mu_{1\text{max}}, K_{S_1}, \alpha)\) is a solution of the following ordinary differential equation
\[
\dot{S}_1 = D(S_{1\text{in}} - S_1) + k_1 \mu_{1\text{max}} \frac{S_1}{S_1 + K_{S_1}} t_{X_1}.
\]
then the function \(\Psi_1(t, t_{X_1}, D, S_{1\text{in}}, k_1, \mu_{1\text{max}}, K_{S_1})\) is called semitrivial solutions of the problem (16).

Here \(t_{S_1}, i = 1, 2, 3\) and \(t_{X_1}, i = 1, 2, 3\) are respectively, the indicators of semitrivial solutions \(\Psi(t, t_{S_1}, \mu_{1\text{max}}, K_{S_1}, D, \alpha)\) and
\(\Psi_1(t, t_{X_1}, D, S_{1\text{in}}, k_1, \mu_{1\text{max}}, K_{S_1})\) defined by the following way:

- If \(S_1 = S_{1\text{in}}\), then \(t_{S_1} = t_{S_{1\text{in}}}\);
- If \(S_1 = S_1^0\), then \(t_{S_{12}} = S_1^0\) be upper solution of the problem (23);
- If \(S_1 = S_{10}\), then \(t_{S_{13}} = S_{10}\) be lower solution of the problem (23);
- If \(X_1 = 0\), then \(t_{X_{11}} = 0\);
- If \(X_1 = X_1^0\), then \(t_{X_{12}} = X_1^0\) be upper solution of the problem (22);
- If \(X_1 = X_{10}\), then \(t_{X_{13}} = X_{10}\) be lower solution of the problem (22).

From Definition 5 we obtain 6 types of scalar ordinary differential equations for semitrivial solutions (16), which include the following two:
\[
\begin{align*}
\dot{X}_1 &= \left(\mu_{1\text{max}} \frac{S_{1\text{in}}}{S_{1\text{in}} + K_{S_1}} - \alpha D\right) X_1 \quad (24a) \\
\dot{S}_1 &= D(S_{1\text{in}} - S_1) \quad (24b)
\end{align*}
\]
for \((X_1, S_1)\) with ordering of lower and upper solution
\[
X_{10}(t_{X_{13}}) \leq X_1^0(t_{X_{12}}), \quad S_1^0(t_{S_{12}}) \leq S_{10}(t_{S_{13}}).
\]

**Lemma 1.** If \(u_\ast\) a lower-solution of \(u' = f(t, u)\), then \(u_\ast + \epsilon\) is upper-solutions of \(u' = f(t, u), \forall \epsilon > 0\).
Proof. Since \( u_* \) is lower-solution, we have that \( u_* < u \). Making \( u^* = u_* + \epsilon \) then by hypothesis has \( u_* = u^* - \epsilon < u, \forall \epsilon > 0 \), then \( u^* > u, \forall \epsilon > 0 \). \( \square \)

**Lemma 2.** Let \( u^* \) a upper-solution of \( u' = f(t, u) \), then \( u^* - \epsilon \) is lower-solutions of \( u' = f(t, u), \forall \epsilon > 0 \).

**Proof.** The proof is analogous to lemma (1). \( \square \)

### 2.1 Semitrivial solutions \( S_1 \) and \( X_1 \)

System (16) has trivial solutions defined in (19d). Considering system (16) in the trivial solution \( E_1(0, S_{1in}) \), we search analytical solution of linear equation (19d)

\[
\frac{dS_1}{dt} - D(S_{1in} - S_1) = 0,
\]

\[
S_1 = S_{1in} - [S_{1in} - S_1(0)] \exp(-Dt).
\]  

(25)

When \( S_{1in} > S_1(0) \) graph of solution at \( t \to \infty \) decreases asymptotically to a value \( S_{1in} \). And for \( S_{1in} < S_1(0) \) graph of solution at \( t \to \infty \) asymptotically increases to a value \( S_{1in} \). However, we will proof the method of upper and lower solutions. We take \( \epsilon > 0 \), then let the solution

\[
S_1 = S_{1in} - [S_{1in} - S_1(0)] \exp(-Dt) + \epsilon.
\]

Thus

\[
\frac{dS_1}{dt} - D(S_{1in} - S_1) \geq 0.
\]

\[
D(S_{1in} - S_1) \exp(-Dt) - D(S_{1in} - S_1) \exp(-Dt) + D\epsilon \geq 0.
\]

Then \( D\epsilon \geq 0 \) for \( D > 0 \) and solution \( S_1 = S_1^0 \) is a upper solution of (24b). If \( \epsilon < 0 \), then \( S_{10} \) is a lower solution of (24b).
| Parameter | Value | Units | SD |
|-----------|-------|-------|----|
| $S_1(0)$  | 5     | [g/l] |    |
| $D$       | 0.395 | [d$^{-1}$] | 0.135 |
| $S_{1in}$ | 10    | [g/l] | 6.4 |

Table 1: Parameter for simulation (from [7].) SD = standard deviation

The solution of initial value problem of linear equation (24a) with condition $X_1(0)$ is

$$X_1 = X_1(0) \exp \left( \mu_{1max} \frac{S_{1in}}{S_{1in} + K_{S1}} - \alpha D \right) t$$

$$X_1 = X_1(0) \exp (m \cdot t) \quad \text{with} \quad m = \frac{\mu_{1max} S_{1in}}{S_{1in} + K_{S1}} - \alpha D.$$  

We take $\Gamma > 0$ therefore, the solution

$$X^0_1 = X_1 + \Gamma \quad \text{and} \quad X^0_{10} = X_1 - \Gamma$$

are an upper solution and lower solutions of (24a) respectively. Figure 2 shows that

Figure 2: Trend graph to the $X_1 \to \infty$ as $t \to \infty$, with $m > 0$. The values of the parameters were taken from Table 1, with initial condition $X_1(0) = 0, 5$.

if $m > 0$, then the solution $X_1 \to \infty$, this is not representative in the process.

Figure 3 shows that if $m < 0$, then the solution $X_1 \to 0$. This behavior is known as condition washout.

Now we consider upper-lower solutions on the part nonlinear of (19c)-(19d). From definition 2.1 to $S_1$

$$\dot{S}_{10} - D (S_{1in} - S_{10}) + k_1 \mu_{1max} \frac{S_{10}}{S_{10} + K_{S1}} \leq 0 \quad \text{(lower)}$$

$$\dot{S}_1^0 - D (S_{1in} - S_1^0) + k_1 \mu_{1max} \frac{S_1^0}{S_1^0 + K_{S1}} \geq 0 \quad \text{(upper)}$$
Figure 3: Trend graph to the $X_1 \to 0$ as $t \to \infty$, with $m < 0$. The values of the parameters were taken from Table 1, with initial condition $X_1(0) = 0, 5$.

We analyze the case where $X_{10} \neq 0$, to upper-solution of $S_1$, then

$$\Rightarrow \frac{dS_1^0}{dt} = D(S_{1in} - S_1^0) - \left[ \frac{k_1 \mu_{1max} S_1^0}{S_1^0 + K_{S_1}} \right] X_{10}$$

taking $X_1^0$ from (26):

$$\frac{dS_1^0}{dt} = D(S_{1in} - S_1^0) - \left[ \frac{k_1 \mu_{1max} S_1^0}{S_1^0 + K_{S_1}} \right] [X_1(0) \exp(mt) - \Gamma]$$

we have $V = (S_1^0 + K_{S_1})^{-1}$ is obtained Abel equation the first kind

$$\frac{dV}{dt} = DV + [-D(S_{in} + K_{S_1}) + k_1 \mu_{1max}X_1(0) \exp(mt) - k_1 \mu_{1max} \Gamma] V^2$$

$$+ [-k_1 \mu_{1max} X_1(0) K_{S_1} \exp(mt) + k_1 \mu_{1max} \Gamma] V^3,$$

(28)

which is solved numerically to represent graphically the solution $S_1^0$.

The difficult question is searching solutions to Abel’s differential equation of first kind (28). The general solutions to Abel’s equation (28) is an open problem. There are not many results on the construction of analytical solutions of Abel’s equation (28) in evident form. We considered the work [13] where is suggested a new construction method of analytical solutions of Abel’s equation of first kind (28) in evident form. Figure 4 shows that the upper solution $S_1^0$ has a reverse order. In the process this represents the acidogenic bacteria $X_1$ substrate feed $S_1$, over time.

Now we look for a upper-solution semi-trivial $X_1^0$ from the equations (19b) y (25). From definition 2.1 to $X_1^0, S_1^0$

$$\dot{X}_1^0 - \left( \mu_{1max} \frac{S_1^0}{S_1^0 + K_{S_1}} - \alpha D \right) X_1^0 \geq 0 \quad \text{ (super)}$$

Taking (22) and (25) on (19b), with $c_8 = S_{1in} - S_1(0)$ we obtain

$$\Rightarrow \dot{X}_1^0 - \left( \mu_{1max} \frac{S_{1in} - c_8 \exp(-Dt)}{S_{1in} - c_8 \exp(-Dt) + K_{S_1}} - \alpha D \right) X_1^0 = 0$$

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Figure 4: The upper-solution $S_1^0$ trend 0 as $t \to \infty$

then

$$X_1^0 = \frac{c_9 \left[ c_8 \exp(-Dt) (S_{1in} + Ks_1 - c_8 \exp(-Dt)) \right]^{\mu_1_{\max} S_{1in}}}{\left[ S_{1in} + Ks_1 - c_8 \exp(-Dt) \right]^{\mu_1_{\max}}} \exp(-\alphaDt)$$

and graphic representation is

Figure 5: The graph of the solution $X_1$ trend to 0 as $t \to \infty$

Figure 5 shows that the upper solution $X_1^0$ presents condition washout.
3 Lower and upper solutions for initial value problem (29)

Now we are interested in the solution of the nonlinear subsystem with initial conditions, which models dynamics of biomass and substrate in methanogenesis.

\[
\begin{aligned}
\frac{dX_2}{dt} &= \left( \mu_{2_{max}} \frac{S_2}{S_2^2 + S_2 + K_{S_2}} - \alpha D \right) X_2 \triangleq F_1 \\
\frac{dS_2}{dt} &= D (S_{2_{in}} - S_2) + k_2 \left( \frac{\mu_{1_{max}} S_1}{S_1 + K_{S_1}} \right) X_1 - k_3 \left( \frac{\mu_{2_{max}} S_2}{S_2^2 + S_2 + K_{S_2}} \right) X_2 \triangleq F_2 \\
\frac{dA}{dt} &= D (A_{in} - A) \triangleq F_3 \\
\frac{dC}{dt} &= D (C_{in} - C) + \left( k_4 \mu_{1_{max}} S_1 X_1 \right) + \left( \frac{k_5 \mu_{2_{max}} S_2 X_2}{S_2^2 + S_2 + K_{S_2}} \right) - K_{Iw} [C + S_2 - A - B] \triangleq F_4 \\
\frac{dF_M}{dt} &= k_6 \left( \frac{\mu_{2_{max}} S_2}{S_2^2 + S_2 + K_{S_2}} \right) X_2 \triangleq F_5 \\
X_2(0) &= c_3, \quad S_2(0) = c_4, \quad A(0) = c_5, \quad C(0) = c_6, \quad F_M(0) = c_7.
\end{aligned}
\]

In the subsystem (29), we use the trivial solutions or solutions semitrivials to \((S_1, X_1)\).

**Definition 6.** A pair \([(X_{20}, S_{20}^0, A_0, C_0^0), (X_{20}^0, S_{20}, A_0^0, C_0)]\) is called

(a) Lower-upper solutions from problem (29) to \(X_2\), if satisfies the following

\[
(X_{20}, S_{20}^0) \in C^1(I), \quad (X_{20}^0, S_{20}) \in C^1(I)
\]

\[
\dot{X}_{20} - \left( \mu_{2_{max}} \frac{S_2}{S_2^2 + S_2 + K_{S_2}} - \alpha D \right) X_{20} \leq 0.
\]

\[
F_2(X_{20}, S_{20}) \leq 0 \text{ in } I \quad \forall S_2 \in [S_{20}^0, S_{20}]
\]

\[
\dot{X}_{20}^0 - \left( \mu_{2_{max}} \frac{S_2}{S_2^2 + S_2 + K_{S_2}} - \alpha D \right) X_{20}^0 \geq 0
\]

\[
F_2(X_{20}^0, S_{20}) \geq 0 \text{ in } I \quad \forall S_2 \in [S_{20}^0, S_{20}].
\]

(b) Upper-lower-upper solutions of the problem (29) with reverse order to \(S_2\), if satisfies

\[
S_{20}^0 - D (S_{2_{in}} - S_{20}^0) - k_2 \left( \frac{\mu_{1_{max}} S_{10}}{S_{10} + K_{S_1}} \right) X_{10} + k_3 \left( \frac{\mu_{2_{max}} S_{20}^0}{S_{20}^2 + S_{20}^0 + K_{S_2}} \right) X_{20}^0 \geq 0
\]
\[ F_1(X_{10}, S_{10}, X_2^0, S_2^0) \geq 0 \text{ in } I \]
\[ \forall X_1 \in [X_{10}, X_1^0] \quad \forall S_1 \in [S_1^0, S_{10}] \]

(c) Upper-upper-lower solution of the problem (29) with reverse order to \( S_2 \), if satisfies
\[
\dot{S}_2^0 - D (S_{2in} - S_2^0) - k_2 \left( \frac{\mu_{1max} S_1^0}{S_1^0 + K_{S_1}} \right) X_1^0 \\
+ k_3 \left( \frac{\mu_{2max} S_2^0}{(S_{2in})^2 + S_2^0 + K_{S_2}} \right) X_20 \geq 0
\] (31)

(d) Lower-upper-lower solutions of the problem (29) with order reverse to \( S_2 \), if satisfies
\[
\dot{S}_20 - D (S_{2in} - S_20) - k_2 \left( \frac{\mu_{1max} S_1^0}{S_1^0 + K_{S_1}} \right) X_1^0 \\
+ k_3 \left( \frac{\mu_{2max} S_20}{(S_{2in})^2 + S_20 + K_{S_2}} \right) X_20 \leq 0
\] (32)

(e) Upper solution of the problem (29) with to \( A \), if satisfies
\[
\dot{A}^0 - F_3(A^0) \geq 0 \text{ in } I \\
F_3(A^0) \geq 0 \text{ in } I \quad \forall A \in [A_0, A^0].
\] (33)

(f) Lower solution of the problem (29) with to \( A \), if satisfies
\[
\dot{A}_0 - F_3(A_0) \leq 0 \text{ in } I \\
F_3(A_0) \leq 0 \text{ in } I \quad \forall A \in [A_0, A^0].
\] (34)

(g) Upper-upper-lower solution of the problem (29) to \( C \), if satisfies
\[
\dot{C}^0 - D (C_{in} - C^0) + k_4 \left( \frac{\mu_{1max} S_1^0}{S_1^0 + K_{S_1}} \right) X_1^0 \\
+ \left( \frac{k_5 \mu_{2max} S_20}{(S_{2in})^2 + S_20 + K_{S_2}} \right) X_20 - K_{Lm} [C^0 + S_{20} - A^0 - B] \geq 0
\]
\[ F_4(X_1^0, S_1^0, X_20, S_{20}, A^0, C^0) \geq 0 \text{ in } I \\
\forall X_1 \in [X_{10}, X_1^0] \quad \forall S_1 \in [S_1^0, S_{10}] \\
\forall X_2 \in [X_{20}, X_2^0] \quad \forall S_2 \in [S_2^0, S_{20}] \quad \forall A \in [A_0, A^0].
\]
(h) Upper-lower-lower solution of the problem (29) to C, if satisfies
\[
\dot{C}_0 - D (C_{in} - C_0) + k_4 \left( \frac{\mu_{1 \max} S_0^1}{S_1^1 + K S_1} \right) X_1^0 \\
+ \left( \frac{k_5 \mu_{2 \max} S_0^2}{(S_0^2)^2 + S_2^0 + K S_2^0} \right) X_2^0 - K L_a [C_0 + S_2^0 - A_0 - B] \leq 0
\]
\[
F_4(X_1^0, S_1^0, X_2^0, S_2^0, A_0, C_0) \geq 0 \text{ in I}
\]
\[
\forall X_1 \in [X_{10}, X_{11}^0] \ \forall S_1 \in [S_{10}^0, S_{11}]
\]
\[
\forall X_2 \in [X_{20}, X_{21}^0] \ \forall S_2 \in [S_{20}^0, S_{21}] \ \forall A \in [A_0, A^0].
\]

(i) Upper solution of the problem (29) to \( F_M \), if satisfies
\[
\dot{F}_M^0 - k_6 \left( \mu_{2 \max} \frac{S_2^0}{(S_0^2)^2 + S_2^0 + K S_2^0} \right) X_2^0 \geq 0
\]
\[
F_5(X_2^0, S_2^0) \geq 0 \text{ in I} \ \forall X_2 \in [X_{20}, X_{21}^0] \ \forall S_2 \in [S_{20}^0, S_{21}].
\]

(j) Lower solution of the problem (29) to \( F_M \), if satisfies
\[
\dot{F}_M^0 - k_6 \left( \mu_{2 \max} \frac{S_2^0}{(S_0^2)^2 + S_2^0 + K S_2^0} \right) X_2^0 \leq 0
\]
\[
F_5(X_2^0, S_2^0) \leq 0 \text{ in I} \ \forall X_2 \in [X_{20}, X_{21}^0] \ \forall S_2 \in [S_{20}^0, S_{21}].
\]

with \( X_{10} \leq X_{11}^0, \ S_{10}^0 \leq S_{11}, \ X_{20} \leq X_{21}^0, \ S_{20}^0 \leq S_{21}, \ A_0 \leq A^0, \ C_0 \leq C^0, \) in I.

**Definition 7.** If \( \Psi(t_t, t_{S_{21}}, \mu_{2 \max}, K_{12}, K_{S_2}, D, \alpha) \) is a solution of ODE
\[
\dot{X}_2 = \left( \mu_{2 \max} \frac{t_{S_{21}}}{(S_0^2)^2 + t_{S_{21}} + K S_2} - \alpha D \right) X_2
\]
then the function \( \Psi(t, t_{S_{21}}, \mu_{2 \max}, K_{12}, K_{S_2}, D, \alpha) \) is called a *semilocal solutions* of the problem (29), and if \( \Psi_1(t, t_{X_1}, t_{X_2}, D, S_{21}, k_2, \mu_{1 \max}, K_{S_1}, k_3, \mu_{2 \max}, K_{12}, K_{S_2}) \) is a solution of the following ODE
\[
\dot{S}_2 = D (S_{21} - S_2) + \left( \frac{k_2 \mu_{1 \max} S_{11}^1}{S_1^1 + K S_1} \right) t_{X_1} - \left( \frac{k_3 \mu_{2 \max} S_2}{(S_2^1)^2 + S_2 + K S_2} \right) t_{X_2},
\]
then the function
\[
\Psi_1(t, t_{X_1}, t_{X_2}, D, S_{21}, k_2, \mu_{1 \max}, K_{S_1}, k_3, \mu_{2 \max}, K_{12}, K_{S_2})
\]
is called a *semilocal solutions* of the problem (29).
Here $t_{S_{2i}}, i = 1, 2, 3$ and $t_{X_{2i}}, i = 1, 2, 3$ are respectively, the indicators of semitrivial solutions $\Psi(t, t_{S_{2i}}, \mu_{2\text{max}}, K_{I_2}, K_{S_2}, D, \alpha)$ and $\Psi_1(t, t_{X_{2i}}, t_{X_{2i}}, D, S_{2in}, k_2, \mu_{1\text{max}}, K_{S_1}, k_3, \mu_{2\text{max}}, K_{I_2}, K_{S_2})$ defined by the following way:

If $S_{2i} = S_{2in}$, then $t_{S_{2i}} = S_{2in}$;

If $S_{2i} = S_0^{i}$, then $t_{S_{2i}} = S_0^{i}$ be upper solution of the problem (36);

If $S_{2i} = S_{20}$, then $t_{S_{2i}} = S_{20}$ be lower solution of the problem (36);

If $X_{2i} = 0$ then $t_{X_{2i}} = 0$;

If $X_{2i} = X_0^{i}$, then $t_{X_{2i}} = X_0^{i}$ be upper solution of the problem (35);

If $X_{2i} = X_{20}$, then $t_{X_{2i}} = X_0$ be lower solution of the problem (35).

From Definition 3.2, we obtain 6 types of scalar ODE for semitrivial solutions (29) which include the following two:

$$\dot{S}_2 = D(S_{2in} - S_2)$$ (37)

$$\dot{X}_2 = \left(\mu_{2\text{max}} \frac{S_2}{[S_{2in}^2 + S_2 + K_{S_2}]^{K_{I_2}}} - \alpha D\right) X_2$$ (38)

for $(X_2, S_2)$ with ordering of lower and upper solution

$$X_{20}(t_{X_{23}}) \leq X_0^i(t_{X_{22}}), \quad S_0^i(t_{S_{22}}) \leq S_{20}(t_{S_{23}}).$$

### 3.1 Semitrivial solutions to $S_2$ and $X_2$

System (29) has trivial solutions defined in (15). Considering system (29) with trivial solution $E_2(0, S_{1in}, 0, S_{2in})$, we can easily solve for the solution of initial value problem of linear equation (37) with condition $S_2(0)$

$$\frac{dS_2}{dt} - D(S_{2in} - S_2) = 0,$n

$$S_2 = S_{2in} - [S_{2in} - S_2(0)]\exp(-D t).$$

When $S_{2in} > S_2(0)$ graph of solution at $t \to \infty$ decreases asymptotically to a value $S_{2in}$. And for $S_{2in} < S_2(0)$ graph of solution at $t \to \infty$ asymptotically increases to a value $S_{2in}$.

However, we wish to demonstrate the method of upper and lower solutions. We take $\delta > 0$, then let the solution

$$S_2 = S_{2in} - [S_{2in} - S_2(0)]\exp(-D t) + \delta.$$ (39)

Thus

$$\frac{dS_2}{dt} - D(S_{2in} - S_2) \geq 0.$$ 

Then $D\delta \geq 0$ for $D > 0$ and solution $S_2 = S_0^{i}$ is a upper solution of (37). If $\delta < 0$, then $S_{20}$ is a lower solution of (37).

Figure 6 shows that the solution of $S_2$ tends to the value $S_{2in}$ as $t \to \infty$. 

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Figure 6: Trend graph to the solution $S_2$ as $t \to \infty$

| Parameter | Value | Units    | SD |
|-----------|-------|----------|----|
| $S_2(0)$  | 0     | [mmol /l] |    |
| $S_{2in}$ | 50    | [mmol /l] | 25 |

Table 2: Parameter for simulation (from [7].) The other parameters are taken from Table 1.

Solution of initial value problem of linear equation (35) with condition $X_2(0)$ is

$$X_2 = X_2(0) \exp \left( \frac{S_{2in}}{K_s^2} - \alpha D \right) t$$

$$X_2 = X_2(0) \exp (n \, t) \quad \text{with} \quad n = \frac{\mu_{2max} S_{2in}}{[S_{2in}]^2 + S_{2in} + K_{S_2}} - \alpha D$$

We take $\Lambda > 0$ therefore, the solution

$$X_2^0 = X_2 + \Lambda \quad \text{and} \quad X_{20} = X_2 - \Lambda$$

are an upper-lower solutions of (35) respectively, and their graphical trends are the same as Figure 2 and Figure (3) respectively.

Now we consider upper-lower solutions on the part nonlinear of (6)-(34). We assume $(X_2 \equiv 0)$ with \( (X_1 \neq 0) \) or $(S_1 \neq 0)$ and

$$S_1 = \frac{1}{\alpha t} - K_{S_1}$$

$$X_1 = X_1(0) \exp (-\alpha D \, t)$$

then

$$\frac{dS_2}{dt} = D(S_{2in} - S_2) + \omega_1 \exp (-\alpha D \, t) - \omega_2 \, t \exp (-\alpha D \, t)$$

with

$$\omega_1 = k_2 \mu_{2max} X_1(0), \quad \omega_2 = \omega_1 K_{S_1} \alpha$$
which have analytical solution

\[ S_2 = S_{2n} + \left[ \frac{\omega_1}{D(1-\alpha)} + \frac{\omega_2}{[D(1-\alpha)]^2} \right] \exp(-\alpha D t) + \left[ k - \frac{\omega_2}{D(1-\alpha)} t \right] \exp(-D t) \]

with integration constant

\[ k = S_2(0) - S_{2n} - \frac{\omega_1}{D(1-\alpha)} - \frac{\omega_2}{[D(1-\alpha)]^2} \].

Verification the order to \( S_2 \) semitrivial (lower)

\[ S_2 = S_{2n} - [S_{2n} - S_2(0)] \exp(-D t) - \xi \]

via substitution in

\[ \frac{dS_{20}}{dt} = D(S_{2n} - S_{20}) \]

gives

\[ 0 \geq -D \xi \]

a direct order. Verification the order to \( S_2 \) nontrivial (upper)

\[ S_2^0 = S_{2n} + \left[ \frac{\omega_1}{D(1-\alpha)} - \frac{\omega_2}{[D(1-\alpha)]^2} \right] \exp(-\alpha D t) + \left[ k - \frac{\omega_2}{D(1-\alpha)} t \right] \exp(-D t) + \xi \]

gives

\[ [-\alpha D m + D m - \omega_1 + \omega_2 t] \exp(-\alpha D t) - \frac{\omega_2}{D(1-\alpha)} \exp(-D t) + D \xi \geq 0 \]

with

\[ m = \frac{\omega_1}{D(1-\alpha)} - \frac{\omega_2}{[D(1-\alpha)]^2} \]

We control sign of inequalities

\[ D \xi \geq [-\alpha D m + D m - \omega_1 + \omega_2 t] \exp(-\alpha D t) - \frac{\omega_2}{D(1-\alpha)} \exp(-D t) \geq 0 \]

\[ \frac{\omega_2}{D(1-\alpha)} \exp(-Dt) \geq [-\alpha Dm + Dm - \omega_1 + \omega_2 t] \exp(-\alpha Dt) \]

\( D, \omega_2 \) and \((1 - \alpha)\) are positive and

\[ \exp(-D(1-\alpha)t) \geq -1 + D(1-\alpha)t \]
Figure 7: Conditions for direct order of $S_2$ as $t \to \infty$. The simulation parameters are same of Table 1 and Table 2.

**Proposition 1.** If inequality

$$\exp (-D(1-\alpha)t) \geq -1 + D(1-\alpha)t \ \forall \alpha, \forall D,$$

is satisfied then $S_2$ has **direct order**.

Figure 7 shows a rapid growth and later a moderate decrease, this means than at the start of the process acidogenic bacteria are not fed, then consume the substrate is stabilized the consumption. Analogously

**Proposition 2.** If inequality

$$\exp (-D(1-\alpha)t) < -1 + D(1-\alpha)t \ \forall \alpha, \forall D,$$

holds then $S_2$ has **reverse order**.

Figure 8: Conditions to reverse order of $S_2$ as $t \to \infty$

Figure 8 shows the trend similar to Figure 7.
| Parameter | Value | Units | SD |
|-----------|-------|-------|----|
| $K_{I_2}$ | 256   | [mmol/l] | 320 |
| $K_{S_2}$ | 22.98 | [mmol/l] | 13.7 |

Table 3: Parameter for simulation from [7], Table 1 and Table 2.

Now we look for a semitrivial solution to $X_2^0$ from the definition (30). The upper solution to $X_2$ with $S_2 = S_{2in} - [S_{2in} - S_2(0)] \exp (-D t)$ and $S_{2in} - S_2(0) = c_8$

$$\Rightarrow \quad X_2^0 - \left( \mu_{2max} \frac{S_2}{K_{I_2}^2 + S_2 + K_{S_2}} - \alpha D \right) X_2^0 \geq 0$$

then upper solution is:

$$X_2^0 \geq \left( \frac{c_9 \left[ \left( \frac{K_{I_2}^2}{x} + S_{2in} - c_8 \exp (-D t)^2 + \rho^2 \right) \right]^\beta}{\left[ c_8 \exp (-D t)^\beta \right]} \right) \ast \exp \left\{ - \left[ (\eta_1 + \eta_2) \arctan \left( \frac{K_{I_2}^2 + S_{2in} - c_8 \exp (-D t)}{\rho} \right) + \alpha D t \right]\right\}$$

with

$$\beta = \frac{\mu_{2max} K_{I_2} S_{2in}}{2 D \nu}; \quad \delta = 2 \beta; \quad \rho^2 = \left[ K_{I_2} K_{S_2} - \frac{(K_{I_2})^2}{4} \right];$$

$$\eta_1 = \frac{\mu_{2max} (K_{I_2})^2 K_{S_2}}{2 \rho \nu}; \quad \eta_2 = \frac{\mu_{2max} (K_{I_2})^2 K_{S_2}}{D \rho \nu}; \quad \nu = S_{2in}^2 + K_{I_2} S_{2in} + K_{I_2} K_{S_2}$$

$$c_9 = \frac{X_2(0)(c_8)^\delta}{\left[ \left( \frac{K_{I_2}^2}{x} + S_{2in} - c_8 \right)^2 + \rho^2 \right]^\beta} \exp \left\{ \left( \eta_1 + \eta_2 \right) \arctan \left( \frac{K_{I_2}^2 + S_{2in} - c_8 \exp (-D t)}{\rho} \right) \right\}.$$ 

Figure 9 shows that over time the dynamics of methanogenic bacteria stabilizes. The simulation parameters are those of Table 1 and Table 2.

Now we study the existence of semitrivial solutions to alkalinity ($A$). We consider the trivial solution $E_5(A_{in})$ from system (29), then

$$\frac{dA}{dt} = D \left( A_{in} - A \right) \rightarrow A(t) = A_{in} - [A_{in} - A(0)] \exp (-D t) \quad (41)$$

When $A_{in} > A(0)$ graph of solution at $t \to \infty$ increases asymptotically to a value $A_{in}$. And for $A_{in} < A(0)$ graph of solution at $t \to \infty$ asymptotically decreases to a value $A_{in}$, its graph is

In the Figure 10 it can be seen that the alkalinity over time tends to stabilize. It is noteworthy that the alkalinity is directly affected by the pH, so if the pH changes the alkalinity also change.

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Figure 9: Trend graph to the $X_2$ as $t \to \infty$

Figure 10: Trend of the alkalinity as $t \to \infty$

| Parameter | Value | Units | SD |
|-----------|-------|-------|----|
| $A(0)$   | 87.5  | [mmol/l] | 12.5 |
| $D$      | 0.395 | [day$^{-1}$] | 0.135 |
| $A_{in}$ | 87.5  | [mmol/l] | 12.5 |

Table 4: Parameter for simulation (from [7].)

3.2 Existence of semitrivial solutions to $C$

Considering the trivial solution $E_4(S_{1in}, 0, S_{2in}, 0, A_{in})$ from system (29), we look for a upper solutions $C^0$

$$
\dot{C}^0 = -(D + K_{La})C^0 + (D C_{in} - K_{La} S_{2in} + K_{La} A_{in} + K_{La} B)
$$

with $w_1 = D + K_{La}$, $w_2 = D C_{in} - K_{La} S_{2in} + K_{La} A_{in} + K_{La} B$, $c_{10} = C(0) - \frac{w_2}{w_1}$, then

$$
C^0 = \frac{w_2}{w_1} + c_{10} \exp(-w_1 t).
$$

Figure 11 shows that the total carbon tends to stabilize over time.

22
From (29), we take the analytical solution of $A$. Now look for an upper solution to $C^0$ with $X_{10} \equiv 0 \text{ and } A(t) = A_{in} - [A_{in} - A(0)] \exp(-D \ t)$

$$
\frac{dC^0}{dt} = -(D + K_{La}) C^0 + k_5 \left( \mu_{2max} \frac{(S_2^0)^2}{K_{12}} + S_2^0 + K_{S_2} \right) X_{20} - K_{La} S_2^0 + K_{La} A^0 + (D \ C_{in} + B)
$$

The other simulation parameters were taken from the Table 2, Table 3 and Table 4 with $267 \leq k_5 \leq 420,$ $w_1$ y $w_2$ above, $w_3 = w_2 - K_{La} A_{in},$ $w_4 = w_3 + K_{La} A_{in},$ $w_5 = K_{La} (A_{in} - A(0))$ and $c_{11} = C(0) + \frac{w_5}{w_1 - D} - \frac{w_3}{w_1}$.

From Definition 3.1 we can construct upper-lower analytical solutions for $F_M$ variable, which represents the methane flow rate, and semi-trivials solutions or trivial solutions to $(S_2, X_2)$ are used.

## 4 Conclusions

In this paper the existence of lower and upper solutions of digestion anaerobic model was presented with the corresponding existence theorem. It is also considered the reverse order of lower and upper solutions for variable $S_1$. The connection between the upper solution of substrate variables with solutions of Abel equation was found.
It showed the existence of upper and lower solutions for variable $S_2$, which can change order of upper-lower solution depending on restrictions on the parameters. We are well aware that this is only the first steps in the complete study of the problem. The next step is to consider the V.M. Matrosov comparison principle to explore the global stability of solutions and a complete study of their solution spaces. In this way, we would need explicit solutions of the Abel equations related with the upper solution of substrate variables, which can be treated with Galoisan techniques (see [1, 2, 3, 5]). In [4] we studied the integrability of such Abel equations in the framework of differential Galois theory.

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