On Certain Degenerate Whittaker Models for Cuspidal Representations of GL_{k,n}(\mathbb{F}_q)

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Abstract

Let $\pi$ be an irreducible cuspidal representation of GL_{k,n}(\mathbb{F}_q).$ Assume that $\pi = \pi_\theta$, corresponds to a regular character $\theta$ of $\mathbb{F}_q^\times$. We consider the twisted Jacquet module of $\pi$ with respect to a non-degenerate character of the unipotent radical corresponding to the partition $(n^k)$ of $kn$. We show that, as a GL_{n}(\mathbb{F}_q)-representation, this Jacquet module is isomorphic to $\pi\theta|_{\mathbb{F}_n^\times} \otimes \text{St}^{k-1}$, where St is the Steinberg representation of GL_{n}(\mathbb{F}_q). This generalizes a theorem of D. Prasad, who considered the case $k = 2$. We prove and rely heavily on a formidable identity involving $q$-hypergeometric series and linear algebra.

1 Introduction

Let $\mathbb{F} := \mathbb{F}_q$ be a finite field of size $q$. We will fix a nontrivial character $\psi_0$ of $\mathbb{F}$. Denote by $\mathbb{F}_m := \mathbb{F}_q^m$ the unique degree $m$ field extension of $\mathbb{F}$. Let $k$ be a positive integer. Denote the diagonal subgroup of $(\text{GL}_\ell(\mathbb{F}))^r$ by

$$\Delta^r(\text{GL}_\ell(\mathbb{F})) := \{(g, \ldots, g) \in (\text{GL}_\ell(\mathbb{F}))^r \mid g \in \text{GL}_\ell(\mathbb{F})\}.$$ (1.1)

For a partition $\rho = (k_1, k_2, \ldots, k_s)$ of $\ell$, consider the corresponding standard parabolic subgroup $P_\rho$ of GL_{\ell}(\mathbb{F}), and $M_\rho, N_\rho$ be the corresponding Levi part and unipotent radical.

We begin by describing a theorem of Prasad [Pra00, Thm. 1]. Here we consider GL_{2n}(\mathbb{F}), and denote $P' = P_{n,n}, M' = M_{n,n}$ and $N' = N_{n,n}$. Consider the following character $\psi'$ of $N'$,

$$\psi'(\begin{pmatrix} I_n & X \\ 0 & I_n \end{pmatrix}) := \psi_0(\text{tr}(X)).$$

Let $\pi'$ be an irreducible representation of GL_{2n}(\mathbb{F}) acting on a space $V_{\pi'}$. Define

$$V_{\pi'_{\psi', \psi'}} := \{v \in V_{\pi'} \mid \pi'(u)v = \psi'(u)v, \forall u \in N'\}.$$
This is the ($N', \psi'$)-isotypic subspace of $V_{\pi'}$. We know that $V_{\pi', \psi'}$ is the image of the canonical projection of $V_{\pi'}$ on $V_{\pi', \psi'}$, given by

$$P_{N', \psi'}(v) = \frac{1}{|N'|} \sum_{u \in N'} \psi'(u) \pi'(u)v.$$

Since $\operatorname{tr}(gXg^{-1}) = \operatorname{tr}(X)$ for all $g \in \text{GL}_n(\mathbb{F})$, and by identifying $\Delta^2(\text{GL}_n(\mathbb{F}))$ with $\text{GL}_n(\mathbb{F})$, it follows that $V_{\pi', \psi'}$ is a representation space for $\text{GL}_n(\mathbb{F})$. The space $V_{\pi', \psi'}$ is referred to as the twisted Jacquet module of the space $V_{\pi'}$ with respect to $(N', \psi')$. Prasad proved the following theorem.

**Theorem 1.1.** [Pra00, Thm. 1] Let $\pi'$ be an irreducible cuspidal representation of $\text{GL}_{2n}(\mathbb{F})$ obtained from a character $\theta$ of $\mathbb{F}_n^*$. Then

$$\pi'_{N', \psi'} \cong \text{Ind}_{\mathbb{F}_n^*}^{\text{GL}_{2n}(\mathbb{F})} \theta |_{\mathbb{F}_n^*} .$$

Prasad proved this theorem by an explicit calculation of the characters of $\pi'_{N', \psi'}$, and of the induced representation $\text{Ind}_{\mathbb{F}_n^*}^{\text{GL}_{2n}(\mathbb{F})} \theta |_{\mathbb{F}_n^*}$. At any element of $\text{GL}_n(\mathbb{F})$ the characters are the same. Therefore, the two representations are equivalent.

Fix $k \geq 1$. Let $\rho = (n^k)$ be the partition of $kn$ consisting of $k$ parts of size $n$. In this paper we denote $G := \text{GL}_{kn}(\mathbb{F})$, $P = P_n$, $M = M_n$, and $N = N_n$. We have the Levi decomposition $P = M \ltimes N$. We write $U \in N$ in the form

$$U = \begin{pmatrix}
I_n & X_{1,1} & X_{1,2} & \cdots & X_{1,k-2} & X_{1,k-1} \\
0 & I_n & X_{2,2} & \cdots & X_{2,k-2} & X_{2,k-1} \\
0 & 0 & I_n & \cdots & X_{3,k-2} & X_{3,k-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & I_n & X_{k-1,k-1} \\
0 & 0 & 0 & \cdots & 0 & I_n
\end{pmatrix},$$

(1.3)

where the matrices $X_{i,j}$ ($1 \leq i \leq j \leq k - 1$) are elements of $M_n(\mathbb{F})$.

**Definition 1.2.** A character $\psi : N \to \mathbb{C}^*$ is said to be non-degenerate if it has the form

$$\psi(U) := \psi_0 \left( \operatorname{tr} \left( \sum_{i=1}^{k-1} A_iX_{i,i} \right) \right) = \prod_{i=1}^{k-1} \psi_0 \left( \operatorname{tr} (A_iX_{i,i}) \right),$$

where the matrices $A_i$ are invertible.

Let $\psi : N \to \mathbb{C}^*$ be a non-degenerate character. Let $\pi$ be an irreducible representation of $G$, acting on a space $V_{\pi}$. We will denote by $V_{\pi, N, \psi}$ the largest subspace of $V_{\pi}$, on which $N$ operates through $\psi$, i.e.

$$V_{\pi, N, \psi} = \{ v \in V_{\pi} \mid \pi(U)v = \psi(U)v, \forall U \in N \} .$$

(1.4)

This is the $(N, \psi)$-isotypic subspace of $V_{\pi}$. Same as before, $V_{\pi, N, \psi}$ is the image of the canonical projection of $V_{\pi}$ on $V_{\pi, N, \psi}$ given by

$$P_{k, N, \psi}(v) = \frac{1}{|N|} \sum_{U \in N} \psi(U) \pi(U)v .$$

(1.5)
Since $M$ normalizes $N$, it acts on the characters of $N$ as follows. If $m \in M$, then for all $u \in N$
\[
(m \cdot \psi)(U) = \psi(m^{-1}Um).
\]
We have, for $m \in M$,
\[
\pi(m)V_{\pi_k,N,\psi} = V_{\pi_k,N,m \cdot \psi}.
\]
Let us compute the stabilizer of $\psi$ in $M$. We write
\[
m = \left( \begin{array}{cccc}
B_1 & 0 & \cdots & 0 \\
0 & B_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_k
\end{array} \right), \quad \forall 1 \leq i \leq k : B_i \in \text{GL}_n(F),
\]
Then
\[
(m \cdot \psi)(U) = \psi_0 \left( \text{tr} \left( \sum_{i=1}^{k-1} A_i B_i^{-1} X_{i,i} B_{i+1} \right) \right).
\]
Thus, $m \cdot \psi = \psi$ iff $B_i = B_{i+1}$ for all $1 \leq i \leq k - 1$. In other words,
\[
\text{stab}_M \psi = \Delta^k(\text{GL}_n(F)) \cong \text{GL}_n(F).
\]
Therefore, $V_{\pi_k,N,\psi}$ is a $\text{GL}_n(F)$-module. We denote by $\pi_{k,N,\psi}$ the resulting representation of $\text{GL}_n(F)$ in $V_{\pi_k,N,\psi}$. It is easy to see that by conjugation with an element in the Levi part, we may simply take all the $A_i$ to be the identity matrix. The corresponding twisted Jacquet modules are isomorphic. In the rest of the paper we assume $A_i = I_n$ and fix $\psi(U) := \psi_0 \left( \text{tr} \left( \sum_{i=1}^{k-1} X_{i,i} \right) \right)$.

The goal of this paper is to calculate the character of $\pi_{k,N,\psi}$, and to describe it by known representations, for an irreducible, cuspidal representation $\pi = \pi_\theta$ of $\text{GL}_{kn}(F)$, associated to a regular character $\theta$ of $F^*_{kn}$. The paper generalizes Prasad’s result for the case $k = 2$. The methods used in this paper are generalizations of the methods used by the second author in his thesis [Haz16] for the case $k = 3$. From the character calculation, done in Theorem 3, we were able to describe in Theorem 4 $\pi_{k,N,\psi}$ by the representations $\text{Ind}_{\text{GL}_n(F)}^{\text{GL}_{kn}(F)}(\theta | F^*_{\ell})$, where $\ell | n$. Furthermore, we give a compact description of $\pi_{k,N,\psi}$ by the Steinberg representation in the following theorem.

**Theorem 1.** Let $k \geq 1$. Let $\pi_\theta$ be an irreducible cuspidal representation of $\text{GL}_{kn}(F)$ obtained from a character $\theta$ of $F^*_{kn}$. Then
\[
\pi_{k,N,\psi} \cong \pi_\theta | F^*_{\ell} \otimes \text{St}^{k-1}, \quad (1.6)
\]
where $\pi_\theta | F^*_{\ell}$ is the irreducible cuspidal representation of $\text{GL}_n(F)$ obtained from $\theta | F^*_{\ell}$ and $\text{St}$ is the Steinberg representation of $\text{GL}_n(F)$.

Note that for $n = 1$, Theorem 1 gives $\pi_{k,N,\psi} \cong \theta | F^*$, which also follows from Gel’fand-Graev [GG62] in case of $\text{GL}_k(F)$ (cf. [Car93, Ch. 8.1]).

We are currently investigating the analogue construction for $F$ a non-Archimedean local field.
1.1 Structure of paper

In §3 we calculate the dimension of \( \pi_{k,N,\psi} \). Green’s formula allows us to express the dimension as rather complicated sum. We used tools from \( q \)-hypergeometric series and linear algebra to show that this sum admits the following compact form.

**Theorem 2.** Let \( k \geq 2 \). We have

\[
\dim (\pi_{k,N,\psi}) = q^{(k-2)n(n-1)}/q^n - 1.
\]

We denote the character of \( \pi_{k,N,\psi} \) by \( \Theta_{k,N,\psi} \). In §4 we compute \( \Theta_{k,N,\psi} \), which apart from the tools used in Theorem 2 requires understanding of some conjugacy classes of \( \text{GL}_n(F) \).

**Theorem 3.** Let \( k \geq 2 \). Let \( g = s \cdot u \) be the Jordan decomposition of an element \( g \) in \( \text{GL}_n(F) \).

(I) If the semisimple part \( s \) does not come from \( F_n \), then

\[
\Theta_{k,N,\psi}(g) = 0.
\]

(II) If \( u \neq I_n \), then

\[
\Theta_{k,N,\psi}(g) = 0.
\]

(III) Assume that \( u = I_n \) and that the semisimple element \( s \) comes from \( F_d \subseteq F_n \) and \( d \mid n \) is minimal. Let \( \lambda \) be an eigenvalue of \( s \) which generates \( F_d \) over \( F \). Then,

\[
\Theta_{k,N,\psi}(s) = (-1)^k n^{(d'-1)} \cdot \sum_{i=0}^{d-1} \theta(\lambda^i) \cdot |\text{GL}_{d'}(F_d)|/q^{n-1},
\]

where \( d' = n/d \).

For any \( \ell \) dividing \( n \) and any \( k \geq 2 \), let

\[
ak_{k,n,\ell}(q) = \frac{q^{\ell} - 1}{q^n - 1} \sum_{m: \ell \mid m \mid n} \mu\left(\frac{m}{\ell}\right)(-1)^k \frac{n}{m} q^{k(n-\frac{m}{\ell})} q^{(k-2)\frac{d'}{d}((\frac{n}{m} - \frac{n}{\ell}) - 1)}.
\]

(1.7)

where \( \mu \) is the Möbius function. When \( k = 2 \), it is easily shown (see Lemma 2.10) that

\[
a_{2,n,\ell}(q) = \delta_{\ell,n}.
\]

(1.8)

If \( k > 2 \) we show in Lemma 2.10 that \( a_{k,n,\ell}(q) \in \mathbb{N}_{>0} \), except when \( k \) is odd, \( n \) is even and \( 2 \not \mid \frac{n}{\ell} \), in which case \( -a_{k,n,\ell}(q) \in \mathbb{N}_{>0} \). In §5 we conclude from Theorem 3 and Lemma 2.10 the following decomposition of representations.

**Theorem 4.** Let \( k \geq 2 \).

(I) If \( k \) is even or \( n \) is odd, we have

\[
\pi_{k,N,\psi} \cong \bigoplus_{\ell \mid n} \; a_{k,n,\ell}(q) \cdot \text{Ind}_{F_{d'}^\ell}^{\text{GL}_n(F)} \theta |_{\mathbb{Z}_{\ell}}.
\]

(1.9)
(II) If $k$ is odd and $n$ is even, we have

$$\pi_{k,N,\psi} \bigoplus_{e \mid n, d \mid \frac{k}{e}} (-a_{k,n,e}(q)) \cdot \text{Ind}_{F_e^\star}^{F_m^\star} \theta \big| F_e^\star \cong \bigoplus_{e \mid n, d \mid \frac{k}{e}} a_{k,n,e}(q) \cdot \text{Ind}_{F_e^\star}^{F_m^\star} \theta \big| F_e^\star.$$  

(1.10)

In §6 we deduce Theorem 1 from Theorem 3.

2 Preliminaries

2.1 Cuspidal representations

We review the irreducible cuspidal representations of $GL_m^*(F)$ as in S. I. Gel’fand [Gel75, §6] (originally in J. A. Green [Gre55]). Irreducible cuspidal representations of $GL_m^*(F)$ are obtained via the process of parabolic induction, are associated to regular characters of $F_m^\star$. A multiplicative character $\theta$ of $F_m^\star$ is called regular if, under the action of the Galois group of $F_m$ over $F$, the orbit of $\theta$ consists of $m$ distinct characters of $F_m^\star$.

We denote the irreducible cuspidal representation of $GL_m^*(F)$ associated to a regular character $\theta$ of $F_m^\star$ by $\pi_{\theta}$ and the character of the representation $\pi_{\theta}$ by $\Theta_{\theta}$.

Given $a \in F_m$, consider the map $m_a : F_m \to F_m$, defined by $m_a(x) = ax$. The map $a \mapsto m_a$ is an injective homomorphism of algebras $F_m \hookrightarrow \text{End}(F_m)$. This way, every element of $F_m^\star$ gives rise to a well-defined conjugacy class in $GL_m(F)$. The elements in the conjugacy classes in $GL_m(F)$, which are so obtained from elements of $F_m^\star$, are said to come from $F_m^\star$.

We summarize the information about the character $\Theta_{\theta}$ in the following theorem. We refer to the paper of S. I. Gel’fand [Gel75, §6] for the statement of this theorem in this explicit form, which is originally due to Green [Gre55, Thm. 14] (See also the paper of Springer and Zelevinsky [SZ84]) The theorem is quoted as it appears in [Pra00, Thm. 2].

**Theorem 2.1** (Green [Gre55]). Let $\Theta_{\theta}$ be the character of a cuspidal representation $\pi_{\theta}$ of $GL_m^*(F)$ associated to a regular character $\theta$ of $F_m^\star$. Let $g = s \cdot u$ be the Jordan decomposition of an element $g$ in $GL_m(F)$ ($s$ is a semisimple element, $u$ is unipotent and $s, u$ commute). If $\Theta_{\theta}(g) \neq 0$, then the semisimple element $s$ must come from $F_m^\star$. Suppose that $s$ comes from $F_m^\star$. Let $\lambda$ be an eigenvalue of $s$ in $F_m^\star$, and let $t = \dim_{F_m^\star} \ker(g - \lambda I)$. Then

$$\Theta_{\theta}(s \cdot u) = (-1)^{m-1} \sum_{c=0}^{d-1} \theta(\lambda^c) \left(1 - q^c(1 - (q^d)^c) \cdots (1 - (q^d)^{d-1}) \right)$$

(2.1)

where $q^d$ is the cardinality of the field generated by $\lambda$ over $F$, and the summation is over the various distinct Galois conjugates of $\lambda$.

**Corollary 2.2.** The value $\Theta_{\theta}(g)$ is determined by the eigenvalue of $g$ and the number of Jordan blocks of $g$, which, in turn, is determined by $\dim_{F_m^\star} \ker(g - \lambda I)$.  

5
2.2 Characters induced from subfields

The following lemma summarizes the information about the character of \( \text{Ind}_{F'_{\bar{\ell}}}^{GL_n(F)}(\theta | F'_{\bar{\ell}}) \), where \( \ell | n \) and \( \theta \) is a character of \( F^*_n \).

**Lemma 2.3.** Let \( \theta \) be a character of \( F^*_n \). Suppose that \( s \in GL_n(F) \) comes from \( F_d \subseteq F_{\ell} \) \((d | \ell \) is minimal\). Then, the character \( \Theta_{\text{Ind}_{\ell}} \) of \( \text{Ind}_{F'_{\bar{\ell}}}^{GL_n(F)}(\theta | F'_{\bar{\ell}}) \) at \( s \) is given by

\[
\Theta_{\text{Ind}_{\ell}}(s) = \frac{1}{q^\ell - 1} \sum_{g \in GL_n(F)} \theta(g^{-1}s) \quad (2.2)
\]

\[
= \frac{|GL_{d'}(F_d)|}{q^\ell - 1} \left[ d' - 1 \sum_{i=0}^{d'-1} \theta(\lambda^i) \right], \quad (2.3)
\]

where \( d' = n/d \). The last sum is over the different Galois conjugates of \( s \), thought of as an element of \( F_d \). The value of the character \( \Theta_{\text{Ind}_{\ell}} \) at an element of \( GL_n(F) \) which does not come from \( F'_{\bar{\ell}} \) is zero.

**Remark 2.4.** Recall that in (2.2) \( F'_{\bar{\ell}} \) is considered a subgroup of \( GL_n(F) \) by the injective map \( a \mapsto [m_a] \), where \([m_a]\) is the representing matrix of \( m_a \) with respect to a fixed basis of \( F_n \) over \( F \). Note that the choice of basis for \([m_a]\) does not affect the values of \( \Theta_{\text{Ind}_{\ell}} \).

2.3 On some conjugacy classes of \( GL_n(F) \)

2.3.1 Analogue of Jordan form

Let \( g \in GL_n(F) \) and \( g = s \cdot u \) its Jordan decomposition. Assume that \( s \) comes from \( F_d \subseteq F_{\ell} \) \((d | n \) is minimal\). Let \( \lambda \in F_{\bar{\ell}}^* \) be an eigenvalue of \( s \), which generates the field \( F_d \) over \( F \). Denote by \( f \) the characteristic polynomial of \( \lambda \) \((\text{of degree } d) \), and by \( L_f \in GL_d(F) \) the companion matrix of \( f \). For \( \ell \geq 1 \) we denote

\[
L_{f,\ell} = \begin{pmatrix} L_f & I_d & & \\ L_f & L_f & & \\ & & \ddots & I_d \\ & & & L_f \end{pmatrix} \in GL_{d \cdot \ell}(F). \quad (2.4)
\]

This is an analogue of a Jordan block. As in [Gel75, Gre55], there exists \( \rho = (\ell_1, \ldots, \ell_r) \), a partition of \( \frac{n}{d} \), \( \ell_1 \geq \ell_2 \geq \ldots \geq \ell_r \), \( \frac{n}{d} = \sum_{i=1}^{r} \ell_i \), such that \( g \) is conjugate to

\[
L_{\rho}(f) := \begin{pmatrix} L_{f,\ell_1} & & & \\ & L_{f,\ell_2} & & \\ & & \ddots & \\ & & & L_{f,\ell_r} \end{pmatrix}, \quad (2.5)
\]

i.e. there exists \( R \in GL_n(F) \) such that

\[
R^{-1} g R = L_{\rho}(f). \quad (2.6)
\]

Notice that in case \( u = I_n \) \((g \text{ is semisimple})\), we have \( \rho = (1^{n/d}) \) and there exists \( R \in GL_n(F) \) such that \( R^{-1} g R \) is a diagonal block matrix with \( d' = n/d \).
times \( L_f \) on the diagonal. Otherwise, \( \ell_1 > 1 \) and, in particular, there exists \( R \in \text{GL}_n(\mathbb{F}) \) such that the upper \( 2d \times 2d \) left corner of \( R^{-1} g R \) is
\[
\begin{pmatrix}
L_f & I_d \\
L_f & I_d
\end{pmatrix}.
\]
(2.7)

Now, \( s \) (and so \( g \)) has \( d \) different eigenvalues obtained by applying the Frobenius automorphism \( \sigma \), which generates the Galois group \( \text{Gal}(\mathbb{F}_d/\mathbb{F}) \), namely
\[
\{\lambda, \sigma(\lambda), \ldots, \sigma^{d-1}(\lambda)\} = \{\lambda, \lambda^q, \ldots, \lambda^{q^{d-1}}\},
\]
al of multiplicity \( d' = n/d \) in the characteristic polynomial of \( s \). Let \( 0 \neq v_0 \in \mathbb{F}_d^d \) satisfy \( L_f \cdot v_0 = \lambda v_0 \). So \( L_f \cdot \sigma^i(v_0) = \lambda^q^i \sigma^i(v_0) \), for \( 0 \leq i \leq d - 1 \). Hence, \( B = \{v_0, \sigma(v_0), \ldots, \sigma^{d-1}(v_0)\} \subseteq \mathbb{F}_d^d \) is linearly independent over \( \mathbb{F}_d \), since its elements are eigenvectors of \( L_f \) for different eigenvalues. Let \( T \in \text{GL}_d(\mathbb{F}_d) \) be the diagonalizing matrix of \( L_f \) obtained by \( B \), i.e.
\[
T^{-1} L_f T = D,
\]
(2.8)
where
\[
D := \text{diag} \left( \lambda, \ldots, \lambda^{q^{d-1}} \right).
\]
(2.9)
Denote by \( \Delta^d(T) \) the diagonal block matrix with \( d' \) times \( T \) on the diagonal. Explicitly, the columns of \( \Delta^d(T) \) are the vectors of the basis
\[
C = \{v_0(i,j)\}_{0 \leq j \leq d'-1}^{0 \leq i \leq d-1},
\]
(2.10)
whose \((j \cdot d + i)\)-th vector is given by
\[
v_0(i,j) = \begin{pmatrix}
\sigma^i(v_0) \\
0_{n-(j+1) \cdot d}
\end{pmatrix} \in \mathbb{F}_d^n,
\]
(2.11)
where \( 0 \leq i \leq d - 1 \) and \( 0 \leq j \leq d' - 1 \). Thus, in case \( u = I_n \)
\[
\Delta^d(T^{-1}) R^{-1} g R \Delta^d(T) = \begin{pmatrix}
D & & \\
& \ddots & \\
& & D
\end{pmatrix}.
\]
(2.12)
Otherwise
\[
\Delta^d(T^{-1}) R^{-1} g R \Delta^d(T) = \begin{pmatrix}
D & I_d & & \\
D & & D & * \\
& \ddots & & \ddots \\
& & & D
\end{pmatrix}.
\]
(2.13)
We denote
\[
g_\rho := g_{\rho,R} = \Delta^d(T^{-1}) R^{-1} g R \Delta^d(T).
\]
(2.14)
The matrix \( g_\rho \) is sometimes referred to as an analogue of the Jordan form of \( g \) [Gel75, §0].
2.3.2 Conjugating an arbitrary matrix

Let $A \in M_n(\mathbb{F})$. We use the notation of §2.3.1. In particular, we have a fixed $g \in \text{GL}_n(\mathbb{F})$ and corresponding $R$ and $T$ as defined in (2.6) and (2.8). We will study the following conjugation

$$A_p := A_p R = \Delta_d^T (T^{-1}) R^{-1} A R \Delta_d^T (T) \in M_n(\mathbb{F}_d).$$

(2.15)

Since $R \in \text{GL}_n(\mathbb{F})$, $A \mapsto R^{-1} A R$ is an isomorphism. Hence, there exists a unique $A_R$ such that $A = R A R^{-1}$ so we write $A_p = \Delta_d^T (T^{-1}) A R \Delta_d^T (T)$.

Let $B \in M_n(\mathbb{F}_d)$. Let us represent the vectors $B \cdot v_0(0, m)$, for any $0 \leq m \leq d' - 1$, as a linear combination of the basis $C$ given in (2.10):

$$B \cdot v_0(0, m) = \sum_{0 \leq i \leq d - 1} \sum_{0 \leq j \leq d' - 1} a_{m,i,j} \cdot v_0(i, j), \quad a_{m,i,j} \in \mathbb{F}_d. \quad (2.16)$$

A necessary and sufficient condition for $B \in M_n(\mathbb{F})$ is that for all $0 \leq m \leq d' - 1$, $0 \leq r \leq d - 1$,

$$B \cdot v_0(r, m) = \sum_{0 \leq i \leq d - 1} \sum_{0 \leq j \leq d' - 1} \sigma^r(a_{m,i,j}) \cdot v_0(i + r \pmod{d}, j). \quad (2.17)$$

By taking $B = A_R \in M_n(\mathbb{F})$, we get that (2.17) holds for $A_R$. Therefore, $[A_R]_C = A_p$ is a $d' \times d'$ matrix with entries from $M_d(F_d)$. For $0 \leq m, j \leq d' - 1$, the $m$-th row and $j$-th column of $A_p$, denoted by $A_{m,j}$, is given by

$$A_{m,j} = (\sigma^r(a_{m,i-r \pmod{d},j})_{0 \leq i, r \leq d - 1}, \quad (2.18)$$

i.e. $A_{m,j} \in M_d(\mathbb{F}_d)$ and for $0 \leq i, r \leq d - 1$, the $i$-th row and $r$-th column of $A_{m,j}$ is $\sigma^r(a_{m,i-r \pmod{d},j})$. We proved,

**Lemma 2.5.** In the above notations, the map $A \mapsto A_p$ induces an isomorphism $M_n(\mathbb{F}) \to M_{n \times d'}(\mathbb{F}_d) \cong [M_{d \times d'}(\mathbb{F}_d)]^{d'}$, viewed as $\mathbb{F}$-vector space. It is given by

$$A \mapsto \left( \begin{array}{cccc}
(a_{0,0}) & \cdots & (a_{0,d'}-1) \\
\vdots & \ddots & \vdots \\
(a_{d'-1,0}) & \cdots & (a_{d'-1,d'}-1)
\end{array} \right), \quad (2.19)$$

where the $(m \cdot d + i)\text{-th row and } j\text{-th column of the image of } A \text{ is } a_{m,i,j} \in \mathbb{F}_d$, for $0 \leq m, j \leq d' - 1 \text{ and } 0 \leq i \leq d - 1$.

2.3.3 Trace under conjugation

For $g \in \text{GL}_n(\mathbb{F})$ and $A \in M_n(\mathbb{F})$ we will be interested in $\text{tr}(g^{-1} A)$. We use the notation of §2.3.1 and §2.3.2. By (2.14), we have

$$\text{tr}(g^{-1} A) = \text{tr}(g^{-1} A_p).$$

The inverse of an analogue of a Jordan block of order $d \cdot \ell$, is given by

$$
\begin{pmatrix}
D & I_d \\
\vdots & \ddots \\
& I_d \\
& & D
\end{pmatrix}^{-1} = \begin{cases}
(-1)^{j-i} D^{-j+i+1}, & i \leq j \\
0, & i > j
\end{cases}, \quad (2.20)
\]
for \(0 \leq i, j \leq \ell\), where the LHS of (2.20) denotes the block in the \(i\)-th row and \(j\)-th column. Each block is in \(M_d(\mathbb{F}_q)\). Therefore, in case \(g\) is not semisimple, we have that \(g^{-1}\) is an upper triangular block matrix, with \(D^{-1}\) appearing \(d'\) times on the diagonal and at least one signed negative power of \(D\) appearing in a block above the diagonal. Hence,

\[
\text{tr} \left( g^{-1} A_\rho \right) = \sum_{m=0}^{d'-1} \text{tr} \left( D^{-1} A_{m,m} + D^{-2} \alpha_m (g, D^{-1}, A_\rho) \right)
\]

\[
= \text{tr} \left( \sum_{m=0}^{d'-1} D^{-1} A_{m,m} \right) + \sum_{m=0}^{d'-1} \text{tr} \left( D^{-2} \alpha_m (g, D^{-1}, A_\rho) \right),
\]

(2.21)

where \(\alpha_m (g, D^{-1}, A_\rho)\), for \(0 \leq m \leq d' - 1\). Notice, that in case \(g\) is semisimple, then \(\alpha_m (g, D^{-1}, A_\rho) = 0\) for all \(0 \leq m \leq d' - 1\). Otherwise, for \(0 \leq m \leq d' - 1\), \(D^{-2} \alpha_m (g, D^{-1}, A_\rho)\) equals to a sum of terms of the form \((-1)^m D^{-m} A_{\ell,m}\), where \(m < \ell \leq d' - 1\).

By (2.18) we have

\[
D^{-1} A_{m,m} = \left( (\lambda^{-1})^{q^r} \sigma^r (a_{m,i-r \text{ (mod } d);m}) \right)_{1 \leq i, r \leq d'-1}.
\]

(2.22)

So the first sum in the RHS of (2.21) becomes

\[
\sum_{m=0}^{d'-1} \sum_{r=0}^{d'-1} (\lambda^{-1})^{q^r} \sigma^r (a_{m,0,m}) = \sum_{r=0}^{d'-1} \sigma^r \left( \lambda^{-1} \sum_{m=0}^{d'-1} a_{m,0,m} \right) = \text{Tr}_{\mathbb{F}_d/F} \left( \lambda^{-1} \sum_{m=0}^{d'-1} a_{m,0,m} \right).
\]

On the other hand, for each \(0 \leq m \leq d' - 1\), the term \(\text{tr} \left( D^{-2} \alpha_m (g, D^{-1}, A_\rho) \right)\) in (2.21) does not depend on the elements \(a_{\ell,0,m}\), where \(\ell = m\) (only on \(\lambda\) and on \(a_{\ell,0,m}\) where \(\ell > m\)). Thus, we proved

**Lemma 2.6.** In the above notations,

\[
\text{tr} \left( g^{-1} A \right) = \text{Tr}_{\mathbb{F}_d/F} \left( \lambda^{-1} \sum_{m=0}^{d'-1} a_{m,0,m} \right) + \sum_{m=0}^{d'-1} \text{tr} \left( D^{-2} \alpha_m (g, D^{-1}, A_\rho) \right),
\]

(2.23)

and each summand \(\text{tr} \left( D^{-2} \alpha_m (g, D^{-1}, A_\rho) \right)\) is independent of \(a_{m,0,m}\) appearing in the first summand, for all \(0 \leq m \leq d' - 1\).

In case \(g = s\) is semisimple we have

\[
\text{tr} \left( g^{-1} A \right) = \text{Tr}_{\mathbb{F}_d/F} \left( \lambda^{-1} \sum_{m=0}^{d'-1} a_{m,0,m} \right).
\]

(2.24)

### 2.4 \(q\)-hypergeometric identity

In order to calculate the dimension of \(\pi_{k,N,\psi}\), we need a combinatorial identity related to ranks of triangular block matrices. Before we present the identity, we prove a lemma that will be needed in its proof. The lemma is a special case of
a $q$-analogue of the Chu-Vandermonde identity, phrased in a manner that will be useful for us. We recall the definition of the $q$-Pochhammer symbol:

\[(a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i). \tag{2.25} \]

**Lemma 2.7.** Let $R_q(n, m, r)$ be the number of $n \times m$ matrices of rank $r$ over a finite field of size $q$ ($n, m$ may be $0$, by the convention that the empty matrix has rank $0$). Let $a$ be an integer greater or equal to $n + m$. Then

\[
\sum_{r \geq 0} R_q(n, m, r)(q; q)_{a-r} = q^{nm} \left( \frac{q^n (q; q)_{a-n} (q; q)_{a-m}}{(q; q)_{a-n-m}} \right). \tag{2.26} 
\]

**Proof.** We start by stating a $q$-analogue of the Chu-Vandermonde identity [GR04, Eq. (1.5.2)]:

\[
\sum_{r \geq 0} \frac{(q^{-i}; q)_r (b; q)_r}{(c; q)_r (q; q)_r} \left( \frac{aq^i}{b} \right)^r = \frac{(c/b; q)_i}{(c; q)_i}, \tag{2.27}
\]

where $i$ is a non-negative integer. Note that the LHS of (2.27) is terminating, since the terms corresponding to $r > i$ vanish. The identity (2.27) is valid whenever $b \neq 0$ and $c \notin \{q^{-1}, \ldots, q^{-(i-1)}\}$. Choosing $i = n$, $b = q^{-m}$, $c = q^{-a}$, we obtain

\[
\sum_{r \geq 0} \frac{(q^{-n}; q)_r (q^{-m}; q)_r}{(q^{-a}; q)_r (q; q)_r} q^{r(n+m-a)r} = \frac{(q^{n-a}; q)_n}{(q^{-a}; q)_n}. \tag{2.28}
\]

We have the following formula for $R_q(n, m, r)$ by Landsberg [Lan93]:

\[
R_q(n, m, r) = (1)^r(q^{-n}; q)_r (q^{-m}; q)_r q^{(n+m)r-(r)} \frac{(q^{n-a}; q)_n}{(q^{-a}; q)_n}. \tag{2.29}
\]

By expressing the $r$-th summand of (2.28) as

\[
\frac{(-1)^r(q^{-n}; q)_r (q^{-m}; q)_r q^{(n+m)r-(r)} (q^{-a}+r)^{(-1)} (q^{-a}; q)_r}{(q^{-a}; q)_r}, \tag{2.30}
\]

we obtain that

\[
\sum_{r \geq 0} R_q(n, m, r) q^{-ar+(r)} \frac{(q^{-a}; q)_r}{(q^{-a}; q)_r} q^r = \frac{(q^{n-a}; q)_n}{(q^{-a}; q)_n}. \tag{2.31}
\]

The proof is concluded by applying to (2.31) the simple identity

\[
(q^{-x}; q)_y = (1)^y q^{y-x} \frac{(q; q)_{x+y}}{(q; q)_{x-y}}, \tag{2.32}
\]

with $(x, y) \in \{(a, n), (a - m, n), (a, r)\}$. \qed
We now state our main combinatorial identity needed for computing the dimension. Let $k$ be a positive integer. We define the following family of functions.

$$f_{k,q}(a; n_1, \ldots, n_k, m_1, \ldots, m_k) = \sum_A (q; q)_{a-rkA},$$

where $\{n_i\}_{i=1}^k, \{m_j\}_{j=1}^k$ are sequences of non-negative integers, $a$ is an integer such that

$$a \geq \max\{\sum_{j=1}^i n_j + \sum_{j=i}^k m_j \mid 1 \leq i \leq k\}$$

and the sum is over all matrices $A \in M_{\sum_{i=1}^k n_i \times \sum_{j=1}^k m_j}(F)$ of the form

$$A = \begin{pmatrix} Y_{1,1} & Y_{1,2} & \cdots & Y_{1,k} \\ 0 & Y_{2,2} & \cdots & Y_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Y_{k,k} \end{pmatrix},$$

where $Y_{i,j} \in M_{n_i \times m_j}(F)$ for all $1 \leq i \leq j \leq k$.

**Proposition 2.8.** Let $k \geq 1$. For any sequences of non-negative integers, $\{n_i\}_{i=1}^k$ and $\{m_j\}_{j=1}^k$, and for any integer $a$ satisfying (2.34), we have

$$f_{k,q}(a; n_1, \ldots, n_k, m_1, \ldots, m_k) = q^{\sum_{1 \leq i \leq j \leq k} n_i m_j} \prod_{i=0}^{k} (q; q)_{a - \sum_{j=i}^{k-i} n_j - \sum_{j=k-i+1}^{k} m_j},$$

(2.36)

Proof. We will use the following notation:

$$I_{m,n} = \begin{pmatrix} I_m & 0 \\ 0 & 0_{n-m} \end{pmatrix}, \quad (m \leq n).$$

(2.37)

We prove the proposition by induction on $k$. Let $k=1$. Then

$$f_{1,q}(a; n, m) = \sum_{A \in M_{n \times m}(F)} (q; q)_{a-rkA} = \sum_{r \geq 0} R_q(n, m, r) (q; q)_{a-r}.$$  

(2.38)

By Lemma 2.7 we find that

$$f_1(a; n, m) = q^{am}(q; q)_{a-n}(q; q)_{a-m},$$

(2.39)

as needed. We now perform the induction step, i.e. assume that (2.36) holds for $k-1$ in place of $k$, and prove it for $k$. We split the sum defining $f_{k,q}(a; n_1, \ldots, n_k, m_1, \ldots, m_k)$ as follows:

$$f_{k,q}(a; n_1, \ldots, n_k, m_1, \ldots, m_k) = \sum_{Y_{i,i} \in M_{n_i \times m_i}(F)} \sum_{1 \leq i \leq k} (q; q)_{a-rkA},$$

(2.40)
In the inner sum of (2.40) the ranks of $Y_{i,t}$ are fixed for all $1 \leq i \leq k$, so we set $r_i = \text{rk}(Y_{i,t})$. There exist invertible matrices $E_i, C_i$ such that $Y_{i,t} = E_i Y_{i,t} E_i^T$, for all $1 \leq i \leq k$. So, one can write $A$ in the inner sum of (2.40) as $\text{diag}(E_1, \ldots, E_k) \cdot \tilde{A} \cdot \text{diag}(C_1, \ldots, C_k)$, where

$$
\tilde{A} = \begin{pmatrix}
I_{r_1,n} & \tilde{Y}_{1,2} & \cdots & \tilde{Y}_{1,k} \\
0 & I_{r_2,n} & \cdots & \tilde{Y}_{2,k} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & I_{r_k,n}
\end{pmatrix}
$$

(2.41)
and $\tilde{Y}_{i,j} = E_i^{-1} Y_{i,j} C_j^{-1}$ for all $1 \leq i < j \leq k$. Together with the fact that rank is invariant under elementary operations, (2.40) becomes

$$
f_{k,q}(a; m_1, \ldots, m_k) = \sum_{\forall \ell \leq k; r_i = 1}^{k} \prod_{r_i \geq 1} R_q(n_i, m_i, r_i) \sum_{\tilde{A}} (q,q)_{a-rk\tilde{A}},
$$

(2.42)
where the inner sum is over matrices $\tilde{A}$ of the form (2.41). We can use Gaussian elimination operations on $Y_{i,j}$ for all $1 \leq i < j \leq k$ (which do not affect the rank of $\tilde{A}$) as follows: the first $r_i$ rows of each $\tilde{Y}_{i,j}$ are being canceled by the pivot elements in $I_{r_i,n}$ (using elementary row operations) and the first $r_j$ columns of each $\tilde{Y}_{i,j}$ are being canceled by the pivot elements in $I_{r_j,n}$ (using elementary column operations). Formally, the composition of these elementary operations maps the sequence of matrices $\{\tilde{Y}_{i,j}\}_{1 \leq i < j \leq k}$ $\mathbb{F}$-linearly to a sequence of matrices $Z_{i,j} \in M_{(n-r_i) \times (n-r_j)}(\mathbb{F})$. This linear map is a projection by construction. Its kernel is of size $q^{\sum_{i=1}^{k-1} r_i \sum_{i=1}^{r_i} m_i + \sum_{i=2}^{k} r_i \sum_{i=1}^{r_i} (n_i-r_i)}$. The dimension of the kernel corresponds to the number of elements which we canceled. Equation (2.42) becomes

$$
f_{k,q}(a; m_1, \ldots, m_k) = \sum_{\forall \ell \leq k; r_i = 1}^{k} \prod_{r_i \geq 1} R_q(n_i, m_i, r_i) q^{\sum_{i=1}^{k-1} r_i \sum_{i=1}^{r_i} m_i + \sum_{i=2}^{k} r_i \sum_{i=1}^{r_i} (n_i-r_i)}
\cdot \sum_{\tilde{A}} (q,q)_{a-rk\tilde{A}},
$$

(2.44)
where the inner sum is over matrices of the form

$$
\tilde{A} = \begin{pmatrix}
I_{r_1,n} & \tilde{Y}_{1,2} & \cdots & \tilde{Y}_{1,k} \\
0 & I_{r_2,n} & \cdots & \tilde{Y}_{2,k} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & I_{r_k,n}
\end{pmatrix}
$$

(2.45)
and \( \hat{Y}_{i,j} \) are of the form defined in (2.43). Note that \( \text{rk}_j \hat{A} = \sum_{j=1}^k r_j + \text{rk} Z \), where

\[
Z = \begin{pmatrix} Z_{1,2} & \cdots & Z_{1,k} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Z_{k-1,k} \end{pmatrix}.
\]  

Hence, from (2.44) we obtain the following recursive relation:

\[
f_{k,q}(a; n_1, \ldots, n_k) = \sum_{\forall 1 \leq i \leq k: \atop r_i \geq 0} R_q(n_i, m_i, r_i) \sum_{t=1}^{k-1} r_t \sum_{t=1}^{k} m_t + \sum_{t=1}^{k} r_t \sum_{t=1}^{k-1} (n_t-r_t) \cdot f_{k-1,q}(a - \sum_{j=1}^{k} r_j, n_1-r_1, \ldots, n_{k-1}-r_{k-1}, r_k).
\]

Plugging the induction assumption in (2.47) we get that \( f_{k,q}(a; n_1, \ldots, n_k) \) equals

\[
\sum_{\forall 1 \leq i \leq k: \atop r_i \geq 0}^{k} \prod_{i=1}^{k} R_q(n_i, m_i, r_i) q^{\sum_{i=1}^{k} r_i + \sum_{t=1}^{k} m_t + \sum_{t=1}^{k} r_t + \sum_{t=1}^{k-1} (n_t-r_t)} \prod_{i=1}^{k-1} \left( \sum_{j=1}^{k} (q;q)_{a-\sum_{j=1}^{k} r_j} \cdot q^{(n_j-r_j)} \right) \prod_{j=k+1}^{k} (q;q)_{a-\sum_{j=1}^{k} r_j} \cdot q^{(n_k-r_k)}.
\]

Rearranging (2.48), we see that the sum over \( r_1, \ldots, r_k \) may be written as a product over \( k \) sums, where the \( i \)-th sum is over \( r_i \):

\[
f_{k,q}(a; n_1, \ldots, n_k) = \sum_{q^{\sum_{i=1}^{k} n_i m_{j+1}}}^{k-1} \prod_{i=1}^{k-1} (q;q)_{a-\sum_{j=1}^{k} n_j} \cdot \prod_{j=1}^{k} \left( \sum_{r_i \geq 0} R_q(n_i, m_i, r_i) (q;q)_{a-\sum_{j=1}^{k} n_j - \sum_{j=1}^{k} m_{j+1}} \right).
\]

Using Lemma 2.7 we substitute each inner sum of (2.49) with

\[
q^{n_i, m_i} (q;q)_{a-\sum_{j=1}^{k} n_j - \sum_{j=1}^{k} m_{j+1}} (q;q)_{a-\sum_{j=1}^{k-1} n_j - \sum_{j=1}^{k-1} m_{j+1}}.
\]

and by simplifying we complete the induction step and obtain the desired identity. \( \square \)
Remark 2.9. Solomon [Sol90] proved a relation between the following two quantities: the number of placements of \( k \) non-attacking rooks on a \( n \times n \) chessboard, counted with certain weights depending on \( q \), and the number of matrices in \( M_{n \times n}(\mathbb{F}) \) of rank \( k \). Haglund generalized Solomon’s result to any ”Ferrers’ board” [Hag98, Thm. 1], which means that the number of matrices of the form (2.35) over \( \mathbb{F} \) of rank \( k \) is related to the \( q \)-rook polynomial \( R_k(B, q) \), where \( B \) is a certain Ferrers’ board associated with (2.35). For the definition of a Ferrers’ board and \( R_k(B, q) \), see the introduction to the paper by Garsia and Remmel [GR86]. In particular, Proposition 2.8 may be deduced from a result of Garcia and Remmel on \( q \)-rook polynomials, see [Hag98, Cor. 2]. Our proof of Proposition 2.8 is direct and so we believe it is more accessible. More importantly, the ideas used in the proof reappear in the proofs of Theorem 2 and Theorem 3.

2.5 Arithmetic properties of certain polynomials

For any \( d \) dividing \( n \) and any \( k \geq 2 \), let

\[
a_{k,n,d}(x) = \frac{x^d - 1}{x^n - 1} \sum_{m \in [n]} \frac{\mu \left( \frac{m}{d} \right)}{d} (-1)^{k(n - \frac{m}{d})} x^{(k-2) \frac{m}{d} (\frac{m}{d} + 1)} \in \mathbb{Q}(x),
\]

(2.51)

where \( \mu : \mathbb{N}_0 \to \mathbb{C} \) is the Möbius function, defined by

\[
\mu(n) = \begin{cases} 
0 & \text{if } p^2 \mid n \text{ for some prime } p, \\
(-1)^n & \text{if } n = p_1 p_2 \ldots p_m, \text{ } p_i \text{ are distinct primes.}
\end{cases}
\]

(2.52)

We recall the following properties of \( \mu \) [IR90, Ch. 2].

- The divisor sum \( \sum_{d \mid n} \mu(d) \) is given by

\[
\sum_{d \mid n} \mu(d) = \delta_{1,n}.
\]

(2.53)

- The Möbius function is multiplicative.

Lemma 2.10. Let \( k \geq 2 \). The following hold.

(I) For any \( d \mid n \), \( a_{k,n,d}(x) \) is a polynomial in \( \mathbb{Z}[x] \). Furthermore, in case \( d \notin \{n, \frac{n}{2} \} \), \( a_{k,n,d}(x) \) is divisible by \( x^d - 1 \). In the remaining cases we have

\[
a_{k,n,d}(x) = \begin{cases} 
(-1)^{k(n-1)} x^{\frac{(k-2)n}{2}} \frac{1}{x^{\frac{n}{2}} + 1} & \text{if } d = n, \\
\frac{(-1)^{k(n-1)}}{x^{\frac{(k-2)n}{2}} + (-1)^{k+1}} & \text{if } d = \frac{n}{2
\end{cases}
\]

(2.54)

(II) If \( k > 2 \) we have \( \deg a_{n,d}(x) = \frac{(n(k-2)-2d)(n-d)}{2d} \), and \( a_{n,d} \) has leading coefficient \( (-1)^{k(n-\frac{n}{2})} \). If \( k = 2 \), we have \( a_{n,d} = \delta_{n,d} \).

(III) Assume \( k > 2 \). For any prime power \( q \), \( a_{k,n,d}(q) \) is a non-zero integer. Its sign equals the sign of \( (-1)^{k(n-\frac{n}{2})} \), i.e. it is a positive integer unless \( k \) is odd, \( n \) is even and \( 2 \nmid \frac{n}{2} \).
Proof. We begin by proving the first part of the lemma. If \( d \in \{ n, \frac{n}{2} \} \), a short calculation reveals that (2.54) holds. From now on we assume that \( d \notin \{ n, \frac{n}{2} \} \).

We shall show that

\[
x^n - 1 \mid \sum_{m \cdot d \mid m \mid n} \mu \left( \frac{m}{d} \right) (-1)^{k(n - \frac{m}{d})}x^{(k-2)\frac{m}{d}+1}
\]

(2.55)

in \( \mathbb{Q}[x] \), which implies that \( a_{k,n,d}(x) \) is a polynomial divisible by \( x^d - 1 \). Gauss’s lemma, applied to (2.55), implies that \( a_{k,n,d}(x) \in \mathbb{Z}[x] \). We now prove (2.55).

Let \( z \) be a root of unity of order dividing \( n \). Assume first that \( n \) is odd or that \( k \) is even. Then for all \( m \mid n \) we have

\[
z^{(k-2)\frac{m}{d}} (\frac{m}{d} - 1) = (z^n)^{(k-2)\frac{m}{d}+1} = 1.
\]

(2.56)

Hence, using (2.53),

\[
\sum_{m \cdot d \mid m \mid n} \mu \left( \frac{m}{d} \right) (-1)^{k(n - \frac{m}{d})}z^{(k-2)\frac{m}{d}+1} = \sum_{m \cdot d \mid m \mid n} \mu \left( \frac{m}{d} \right) = \sum_{a \mid \frac{m}{d}} \mu(a) = \delta_{d,n} = 0.
\]

(2.57)

Now we assume instead that \( n \) is even and \( k \) is odd. We are led to consider two cases.

- If \( z = -1 \) then for all \( m \mid n \) we have,

\[
z^{(k-2)\frac{m}{d}} (\frac{m}{d} - 1) = (-1)^\frac{m}{d}.
\]

(2.58)

Hence, using (2.53),

\[
\sum_{m \cdot d \mid m \mid n} \mu \left( \frac{m}{d} \right) (-1)^{k(n - \frac{m}{d})}z^{(k-2)\frac{m}{d}+1} = \sum_{m \cdot d \mid m \mid n} \mu \left( \frac{m}{d} \right) = -\sum_{a \mid \frac{m}{d}} \mu(a) = -\delta_{d,n} = 0.
\]

(2.59)

- If \( z = 1 \) then for all \( m \mid n \) we have,

\[
z^{(k-2)\frac{m}{d}} (\frac{m}{d} - 1) = 1.
\]

(2.60)

Hence,

\[
\sum_{m \cdot d \mid m \mid n} \mu \left( \frac{m}{d} \right) (-1)^{k(n - \frac{m}{d})}z^{(k-2)\frac{m}{d}+1} = \sum_{m \cdot d \mid m \mid n} \mu \left( \frac{m}{d} \right) (-1)^\frac{m}{d} = \sum_{a \mid \frac{m}{d}} \mu(a)
\]

\[
\begin{cases}
0 - \sum_{a \mid \frac{m}{d}} \mu(a) & \text{if } 2 \nmid \frac{m}{d}

\sum_{a \mid \frac{m}{d}} \mu(a) - \sum_{a \mid 2^i a} \mu(2 \cdot \frac{m}{d}) & \text{if } 2 \mid \frac{m}{d}, 4 \mid \frac{m}{d}

\sum_{a \mid \frac{m}{d}} \mu(a) - \sum_{a \mid 2^i a} \mu(4 \cdot \frac{m}{d}) & \text{if } 4 \mid \frac{m}{d}

-\delta_{d,n} & \text{if } 2 \nmid \frac{m}{d}

\delta_{d,n} - \mu(2)\delta_{2d,n} & \text{if } 2 \mid \frac{m}{d}, 4 \mid \frac{m}{d}

\delta_{2d,n} & \text{if } 4 \mid \frac{m}{d}
\end{cases}
\]

(2.61)
Equations (2.57), (2.59) and (2.61) show that the RHS of (2.55) vanishes on each root of the separable polynomial $x^n - 1$, which establishes (2.55). This concludes the proof of the first part of the lemma.

The second part of the lemma for $k > 2$ follows by noticing that the numerator of $a_{k,n,d}(x)$ has degree $d + (k - 2)\frac{n}{2}(\frac{n}{2} - 1)$ (arising from the term corresponding to $m = d$) and leading coefficient equal to $(-1)^{k(n - \frac{n}{2})}$, while the denominator of $a_{k,n,d}(x)$ has degree $n$ and leading coefficient equal to 1.

When $k = 2$, all terms in the sum in (2.51) are constants, and we have

$$a_{2,n,d}(x) = \frac{x^d - 1}{x^n - 1} \sum_{m : d|n} \mu\left(\frac{m}{d}\right) = \frac{x^d - 1}{x^n - 1} \delta_{n,d} = \delta_{n,d}. \tag{2.62}$$

We now turn to the third part of the lemma. Since $a_{k,n,d}(x)$ has integer coefficients, $a_{k,n,d}(q)$ is an integer. We now determine its sign when $k > 2$, and in particular show that it is non-zero.

Since $q^d - 1$, $q^n - 1$, $q^\frac{n}{2}$ are positive, we deal with the expression

$$\tilde{a}_{k,n,d}(q) := \frac{q^n - 1}{q^d - 1} (q^2)^{k-2} \cdot a_{k,n,d}(q)$$

$$= \sum_{m : d|n} \mu\left(\frac{m}{d}\right) (-1)^{\frac{n}{2} - \frac{m}{2}} (q^{(k-2)\frac{n}{2}})^{\frac{m}{2}}$$

$$= \sum_{a|\frac{n}{2}} \mu(a) (-1)^{\frac{n}{2} - \frac{a}{2}} (q^{(k-2)\frac{n}{2}})^{\frac{a}{2}}, \tag{2.63}$$

whose sign is the same as the sign of $a_{k,n,d}(q)$. If $d = n$ then

$$(-1)^{\frac{n}{2} - \frac{n}{2}} \tilde{a}_{k,n,d}(q) = q^{(k-2)\frac{n}{2}} > 0. \tag{2.64}$$

If $d = \frac{n}{2}$ then

$$(-1)^{\frac{n}{2} - \frac{n}{2}} \tilde{a}_{k,n,d}(q) = (q^{(k-2)\frac{n}{2}})^2 + (-1)^{k+1} q^{(k-2)\frac{n}{2}} > 0. \tag{2.65}$$

If $\frac{n}{2} \geq 3$, we set $t = q^{(k-2)\frac{n}{2}}$. Then, $t \geq 2^\frac{n}{2} > 2$ and

$$(-1)^{\frac{n}{2} - \frac{n}{2}} \tilde{a}_{k,n,d}(q) \geq (q^{(k-2)\frac{n}{2}})^{\frac{t}{2}} - \sum_{1 \leq i \leq \frac{t}{2}} (q^{(k-2)\frac{n}{2}})^i \geq (q^{(k-2)\frac{n}{2}})^{\frac{t}{2}} \geq \frac{(q^{(k-2)\frac{n}{2}})^{\frac{t}{2}}}{1 - q^{-(k-2)\frac{n}{2}}}$$

$$= (q^{(k-2)\frac{n}{2}})^{\frac{t}{2}} \left(1 + \frac{1}{1 - q^{-(k-2)\frac{n}{2}}}ight)$$

$$\geq (q^{(k-2)\frac{n}{2}})^{\frac{t}{2}} \left(1 + \frac{1}{1 - q^{-(k-2)\frac{n}{2}}}ight)$$

$$= \frac{(q^{(k-2)\frac{n}{2}})^{\frac{t}{2}}}{1 - q^{-(k-2)\frac{n}{2}}} (t^\frac{t}{2} (t - 1) - 1) > 0. \tag{2.66}$$

Remark 2.11. The polynomials $a_{k,n,q}(x)$ may be expressed using the necklace polynomials (see Moreau [Mor72]), defined by

$$M_n(x) = \frac{1}{n} \sum_{d|n} \mu(d)x^\frac{n}{d}. \tag{2.67}$$
Indeed, 
\[ a_{k,n,d}(x) = \frac{x^d - 1}{x^n - 1} \cdot \left( \frac{(-1)^n}{x^n} \right)^{k-2} \cdot \frac{1}{2} \left( -x^d \right)^{k-2}. \]  
(2.68)

3 Calculation of the Dimension of \( \pi_{k,N,\psi} \)

Here we prove Theorem 2. Given \( U \in N \), we write it in the notation of (1.3). From (1.5),

\[ \dim (\pi_{k,N,\psi}) = \frac{1}{|N|} \sum_{U \in N} \Theta_\theta(U) \overline{\psi}(U) = \frac{1}{q(1)^2} \sum_{U \in N} \Theta_\theta(U) \overline{\psi}(U). \]  
(3.1)

By Corollary 2.2, the value \( \Theta_\theta(U) \) is determined by \( \dim_{F_\psi} \ker(U - I) \) which is in turn determined by \( \text{rank}_{F_{\psi}}(U - I) \). Therefore, we will start by splitting the sum in (3.1) by the \( X_{i,i}, 1 \leq i \leq k - 1 \).

\[ \dim (\pi_{k,N,\psi}) = \frac{1}{q(1)^2} \sum_{X_{i,i} \in M_n(F)} \sum_{X_{1,1} \in M_n(F)} \Theta_\theta(U) \overline{\psi}(U). \]  
(3.2)

The character \( \psi(U) = \psi(X_{1,1}, \ldots, X_{k-1,k-1}) \) is determined by the traces of \( X_{i,i}, 1 \leq i \leq k - 1 \). Hence,

\[ \dim (\pi_{k,N,\psi}) = \frac{1}{q(1)^2} \sum_{X_{i,i} \in M_n(F)} \overline{\psi}(U) \sum_{X_{i,j} \in M_n(F)} \Theta_\theta(U). \]  
(3.3)

In the inner sum of (3.3) set \( r_i = \text{rk}(X_{i,i}) \) for \( 1 \leq i \leq k - 1 \). There exist invertible matrices \( E_i, C_i+1 \) such that \( X_{i,i} = E_i r_{i,n} C_{i+1} \). So, one can write \( U \) in the inner sum of (3.3) as \( I_{kn} \) plus

\[ \text{diag} (E_1, \ldots, E_{k-1}, I_n) \begin{pmatrix} 0 & I_{r_{i,n}} & \cdots & \tilde{X}_{1,k-2} & \tilde{X}_{1,k-1} \\ 0 & 0 & \cdots & \tilde{X}_{2,k-2} & \tilde{X}_{2,k-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I_{r_{k-1,n}} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \text{diag} (I_n, C_2, \ldots, C_k), \]  
(3.4)

where \( \tilde{X}_{i,j} = E_i^{-1} X_{i,j} C_{j+1}^{-1} \) for all \( 1 \leq i < j \leq k - 1 \). Together with the fact that rank is invariant under elementary operations, we now have

\[ \dim (\pi_{k,N,\psi}) = \frac{1}{q(1)^2} \sum_{X_{i,i} \in M_n(F)} \overline{\psi}(U) \sum_{X_{i,j} \in M_n(F)} \Theta_\theta \begin{pmatrix} 0 & I_{r_{i,n}} & \cdots & \tilde{X}_{1,k-2} & \tilde{X}_{1,k-1} \\ 0 & 0 & \cdots & \tilde{X}_{2,k-2} & \tilde{X}_{2,k-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I_{r_{k-1,n}} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \]  
(3.5)
As in the proof of Proposition 2.8, we can use Gaussian elimination operations on \( \hat{X}_{i,j} \) for all \( 1 \leq i < j \leq k - 1 \) (which do not affect the rank nor dimension of the kernel of the matrix minus \( I_{kn} \), and the number of Jordan blocks is not affected as well) in such a way that the sequence of matrices \( \{ \hat{X}_{i,j} \}_{1 \leq i < j \leq k - 1} \) is mapped \( \mathbb{F} \)-linearly to a sequence of matrices

\[
\{ \hat{X}_{i,j} = \begin{pmatrix} 0 & 0 \\ 0 & Y_{i,j} \end{pmatrix} \}_{1 \leq i < j \leq k - 1}, \tag{3.6}
\]

where \( Y_{i,j} \in M_{(n-r_i) \times (n-r_j)}(\mathbb{F}) \). The kernel of this mapping is of size \( q^{\sum_{i=1}^{k-2} r_i(k-i-1)n + \sum_{i=2}^{k-1} r_i \sum_{j=1}^{i-1} (n-r_j)} \). The dimension of the kernel corresponds to the number of elements which we cancel. Equation (3.5) becomes

\[
\dim (\pi_{k,N,\psi}) = \frac{1}{q^{|G|} n^2} \sum_{X_{i,j} \in M_{n}(\mathbb{F})} \sum_{1 \leq i \leq k-1} \tilde{\psi}(U) q^{\sum_{i=1}^{k-2} r_i(k-i-1)n + \sum_{i=2}^{k-1} r_i \sum_{j=1}^{i-1} (n-r_j)} 
\cdot \sum_{Y_{i,j} \in M_{(n-r_i) \times (n-r_j)}(\mathbb{F})} \Theta_\theta(g), \tag{3.7}
\]

where

\[
g = I_{kn} + \begin{pmatrix} 0 & I_{r_1,n} & \cdots & \hat{X}_{1,k-2} & \hat{X}_{1,k-1} \\ 0 & 0 & \cdots & \hat{X}_{2,k-2} & \hat{X}_{2,k-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I_{r_{k-1},n} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{3.8}
\]

According to the character formula (2.1), we can calculate \( \Theta_\theta(g) \). In this case \( m = kn, g = s \cdot u \) where \( s = I_{kn}, \lambda = 1 \) and

\[
t = \dim \ker(g - I) = kn - \text{rk}(g - I) = kn - \sum_{i=1}^{k-1} r_i - \text{rk}A,
\]

where

\[
A = \begin{pmatrix} Y_{1,2} & \cdots & Y_{1,k-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Y_{k-2,k-1} \end{pmatrix}, \quad 1 \leq i < j \leq k - 1. \tag{3.9}
\]

So,

\[
\Theta_\theta(g) = (1)^{kn-1}(1 - q)(1 - q^2) \cdots (1 - q^{kn-\sum_{i=1}^{k-2} r_i - \text{rk}A-1}) 
= (1)^{kn-1}(q; q)_{kn-\sum_{i=1}^{k-2} r_i - \text{rk}A-1}. \tag{3.10}
\]

Equation (3.7) can now be written as

\[
\dim (\pi_{k,N,\psi}) = \frac{1}{q^{|G|} n^2} \sum_{X_{i,j} \in M_{n}(\mathbb{F})} \sum_{1 \leq i \leq k-1} \tilde{\psi}(U) q^{\sum_{i=1}^{k-2} r_i(k-i-1)n + \sum_{i=2}^{k-1} r_i \sum_{j=1}^{i-1} (n-r_j)} 
\cdot (1)^{kn-1} \sum_{A} (q; q)_{kn-\sum_{i=1}^{k-2} r_i - \text{rk}A-1}, \tag{3.11}
\]

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where the inner sum is over all matrices of the form (3.9) and by the definition (2.33) it is equal to

\[ f_{k-2,q} \left( kn - \sum_{i=1}^{k-1} r_i - 1; n-r_1, \ldots, n-r_{k-2}, n-r_{k-1} \right). \] (3.12)

By applying Proposition 2.8 we replace the inner sum in (3.11) by

\[ \sum_{q^1 \leq \cdots \leq q^{k-2}} (n-r_1) (n-r_{j+1}) \prod_{i=0}^{k-2} (q; q)_i \frac{\prod_{i=1}^{k-2} (q; q)_{k-1} - \sum_{j=1}^{k-2} (n-r_j) - \sum_{j=1}^{k-2} (n-r_j+1)}{\prod_{i=1}^{k-2} (q; q)_{k-1} - \sum_{j=1}^{k-2} (n-r_j)} \] (3.13)

which equals

\[ q^1 \leq \cdots \leq q^{k-2} \frac{\prod_{i=1}^{k-2} (q; q)_{2n-1-r_i}}{((q; q)_{n-1})^{(k-2)}}. \] (3.14)

Now (3.11) becomes

\[ \dim (\pi_{k,N,\psi}) = \frac{(-1)^{kn-1}}{(q; q)_{n-1}^{(k-2)}} q^{(k-1)n^2} \sum_{X_i, i \in M_n(F)} \prod_{i=1}^{k-1} \overline{\psi_0}(\text{tr}(X_i)) (q; q)_{2n-1-r_i}. \] (3.15)

Changing the order of sum and product in (3.15) we get that

\[ \dim (\pi_{k,N,\psi}) = \frac{(-1)^{kn-1}}{(q; q)_{n-1}^{(k-2)}} q^{(k-1)n^2} \prod_{i=1}^{k-1} \sum_{X_i, i \in M_n(F)} \overline{\psi_0}(\text{tr}(X_i)) (q; q)_{2n-1-r_i}. \] (3.16)

From Section 5 of [Pra00], each inner sum in (3.16) is equal to

\[ \sum_{X_i, i \in M_n(F)} \overline{\psi_0}(\text{tr}(X_i)) (q; q)_{2n-1-r_i} = (-1)^n \cdot q^{n^2} \cdot q^{(2)}(q; q)_{n-1}. \] (3.17)

Plugging (3.17) in (3.16), we obtain

\[ \dim (\pi_{k,N,\psi}) = q^{(k-1)(2)} (-1)^{n-1} (q; q)_{n-1} = q^{(k-2)(2)} \frac{|GL_n(F)|}{q^n - 1}, \] (3.18)

as needed.

\[ \square \]

4 Calculation of the Character \( \Theta_{k,N,\psi} \)

In this section we prove Theorem 3. Namely, we calculate \( \Theta_{k,N,\psi} \). From now on we will use the following notations:
where $U$ (and so $X_{i,j}$) were defined in (1.3). Note that $h_{I_n,U} = U$. We also define

$$
\Delta^r(g) = \text{diag}(g, \ldots, g) \in \Delta^r(\text{GL}_m(F)), \quad g \in \text{GL}_m(F).
$$

By definition,

$$
\Theta_{k,N,\psi}(g) = \text{tr}(\pi_{k,N,\psi}(g)) = \text{tr} \left( \pi(\Delta^k(g)|V_{k,N,\psi}) \right) = \text{tr} \left( \pi(\Delta^k(g)) \circ P_{k,N,\psi} \right).
$$

Substituting (1.5) into (4.3) we have

$$
\Theta_{k,N,\psi}(g) = \frac{1}{q(\frac{k}{2})n^2} \sum_{U \in N} \text{tr} \left( \pi(\Delta^k(g)\cdot U) \right) \psi(U).
$$

Now we perform the change of variables

$$
X_{i,j} \mapsto g^{-1}X_{i,j}, \quad 1 \leq i \leq j \leq k - 1
$$

in (4.4) and obtain

$$
\Theta_{k,N,\psi}(g) = \frac{1}{q(\frac{k}{2})n^2} \sum_{U \in N} \Theta_{\phi}(h_{g,U}) \psi \left( g^{-1}X_{1,1}, \ldots, g^{-1}X_{k-1,k-1} \right).
$$

In parts §4.1, §4.2 and §4.3 we prove parts (I), (II) and (III) of Theorem 3, respectively.

### 4.1 Character at $g = s \cdot u$ such that the semisimple part $s$ does not come from $\mathbb{F}_n$

Let $g = s \cdot u$. Assume that the semisimple part $s$ does not come from $\mathbb{F}_n$. The semisimple part of $h_{g,U}$ is $\Delta^k(s)$, which also does not come from $\mathbb{F}_n$. By Theorem 2.1, we have $\Theta_{\phi}(h_{g,U}) = 0$. Hence, by (4.6) $\Theta_{k,N,\psi}(g) = 0$. \hfill \Box

### 4.2 Character calculation at a non-semisimple element

Assume that $s$ comes from $\mathbb{F}_d \subseteq \mathbb{F}_n$ and $d \mid n$ is minimal. In addition, $d < n$ since $g$ is not semisimple. Let $\lambda \in \mathbb{F}_d^*$ be an eigenvalue of $s$ which generates the field $\mathbb{F}_d$ over $\mathbb{F}$. We use the notations of §2.3. Thus, there exist $R \in \text{GL}_n(F)$ and $\rho$ partition of $n/d$ such that $R^{-1}gR = L_{\rho}(f)$ and there exists $\Delta^d(T) \in \text{GL}_n(\mathbb{F}_d)$ such that

$$
g_{\rho} = \Delta^d\left( T^{-1} \right) R^{-1}gR\Delta^d(T), \quad (4.7)
$$
the analogue of the Jordan form of \( g \). Recall that by Lemma 2.5, the map 

\[ A \mapsto A_\rho := A_{\rho,R} = \Delta^d \left( T^{-1} \right) R^{-1} A R \Delta^d \left( T \right) \]

induces an isomorphism. By the notation of \( \S 2.3.2 \) we have for each 

\[ X_{a,b}, \quad \forall 1 \leq a \leq b \leq k - 1, \quad (4.8) \]

the corresponding isomorphism of Lemma 2.5

\[ X_{a,b} \mapsto \begin{pmatrix} (x^{(a,b)}_{0,0,0})_{0 \leq i \leq d-1, 0 \leq j \leq d'-1} \\ \vdots \\ (x^{(a,b)}_{d-1,d-1})_{0 \leq i \leq d-1, 0 \leq j \leq d'-1} \end{pmatrix}. \]

Note that

\[ \Delta^k \left( \Delta^d \left( T^{-1} \right) \right) \Delta^k \left( R^{-1} \right) h_{g,U} \Delta^k \left( R \right) \Delta^k \left( \Delta^d \left( T \right) \right) = h_{g,U_\rho}, \quad (4.9) \]

where \( U_\rho \) is the element of \( N \) with \( (X_{a,b})_\rho \) instead of \( X_{a,b} \). From (4.9) we obtain

\[ \text{rk} \left( h_{g-\lambda I_a,U} \right) = \text{rk} \left( h_{g-\lambda I_a,U_\rho} \right). \quad (4.10) \]

We prove that \( \text{rk} \left( h_{g-\lambda I_a,U} \right) \) (which by Corollary 2.2 determines the value of \( \Theta_\rho \left( h_{g,U} \right) \)) is independent of \( x^{(k-1,k-1)}_{1,0,1} \in \mathbb{F}_d \). The matrix \( h_{g_\rho-\lambda I_a,U_\rho} \) has the form

\[ h_{g_\rho-\lambda I_a,U_\rho} = \begin{pmatrix} g_\rho - \lambda I_n & (X_{1,1})_\rho & \cdots & (X_{1,k-2})_\rho & (X_{1,k-1})_\rho \\ 0 & g_\rho - \lambda I_n & \cdots & (X_{2,k-2})_\rho & (X_{2,k-1})_\rho \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & g_\rho - \lambda I_n & (X_{k-1,k-1})_\rho \\ 0 & 0 & \cdots & 0 & g_\rho - \lambda I_n \end{pmatrix}. \quad (4.11) \]

Consider the boxed block in (4.11). The \( 2d \times 2d \) upper left block of the boxed matrix \( g_\rho - \lambda I_n \) has the form

\[ \begin{pmatrix} 0 & \lambda^q - \lambda & 1 \\ \ddots & \ddots & \ddots & 1 \\ \lambda^{q-1} - \lambda & \lambda^q - \lambda & \lambda^{q-1} - \lambda \end{pmatrix}. \quad (4.12) \]

Let \( Z := X_{k-1,k-1}, Z_\rho := (X_{k-1,k-1})_\rho \) and \( z_{m,i,j} := x^{(k-1,k-1)}_{m,i,j} \). One can eliminate the \((d + 1)\text{-}\)th column in \( Z_\rho \) by the boxed 1 from (4.12), i.e.

all
the elements \( z_{m,i} \) \( 0 \leq i \leq d-1 \). In particular, \( z_{1,0,1} = x_{1,0,1}^{(k-1,k-1)} \) is eliminated.

Now, by Lemma 2.6, (4.6) can be written as

\[
\Theta_{k,N,\psi}(g) = \frac{1}{q(2)^{n^2}} \sum_{U \in N} \Theta_{\theta}(h_{g,U}) \cdot \prod_{i=1}^{k-2} \psi_0(\theta X_{i,i})
\]

\[
\cdot \psi_0 \left( \text{Tr}_{F_d/F} \left( \lambda^{-1} \cdot \sum_{m=0}^{d'-1} z_{m,0,m} \right) + \text{tr} (D^{-2} \alpha (g,D^{-1},Z_\rho)) \right).
\]

By Lemma 2.5, going over \( Z \in M_n(F) \) is equivalent to going over \( (z_{m,i,j}) \) \( 0 \leq i \leq d-1 \), \( 0 \leq j,m \leq d'-1 \), \( z_{m,i,j} \in F_d \). We have just shown that \( \Theta_{\theta}(h_{g,U}) \) is independent of \( z_{1,0,1} \), and by Lemma 2.6 \( \text{tr} (D^{-2} \alpha (g,D^{-1},Z_\rho)) \) in (4.13) is also independent of \( z_{1,0,1} \). Thus, we may write (4.13) as the following double sum, where the inner sum is over \( z_{1,0,1} \) and the outer sum is over the rest of the coordinates of \( U \):

\[
\Theta_{k,N,\psi}(g) = \frac{1}{q(2)^{n^2}} \sum_{X_{i,j} \in N, (i,j) \neq (k-1,k-1)} \Theta_{\theta}(h_{g,U}) \cdot \prod_{i=1}^{k-2} \psi_0(\theta X_{i,i})
\]

\[
\cdot \psi_0 \left( \text{tr} (D^{-2} \alpha (g,D^{-1},Z_\rho)) \right) \cdot \psi_0 \left( \text{Tr}_{F_d/F} \left( \lambda^{-1} \cdot \sum_{0 \leq m \leq d'-1} z_{m,0,m} \right) \right)
\]

\[
\cdot \sum_{z_{1,0,1} \in F_d} \psi_0 \left( \text{Tr}_{F_d/F} (\lambda^{-1} \cdot z_{1,0,1}) \right).
\]

Since \( \psi_0 \circ \text{Tr}_{F_d/F} \) is a nontrivial character, we have

\[
\sum_{z_{1,0,1} \in F_d} \psi_0 \left( \text{Tr}_{F_d/F} (\lambda^{-1} \cdot z_{1,0,1}) \right) = 0.
\]

Thus, \( \Theta_{k,N,\psi}(g) = 0. \)

\[ \square \]

### 4.3 Character calculation at a semisimple element

Here we will use (4.6) to calculate the value of \( \Theta_{k,N,\psi}(g) \) for \( g = s \) where \( s \) is semisimple element which comes from a subfield of \( F_n \) (\( u = I_n \)). Again, we use the notations of §2.3. Thus, there exist \( R \in \text{GL}_n(\mathbb{F}) \), \( \rho \) partition of \( n/d \) and \( \Delta^{d^2}(T) \in \text{GL}_n(F_d) \) such that

\[
s_\rho = \Delta^{d^2} (T^{-1}) R^{-1} sR \Delta^{d^2} (T),
\]

the analogue of the Jordan form of \( s \). We also use the notations of §2.3.2, and in particular define \( (X_{a,b})_\rho \) as in §4.2.

Let \( \lambda \in F_n^* \) be an eigenvalue of \( s \). If \( \lambda \in F_n^* \) then \( s = \lambda I \), and we have by (4.6)

\[
\Theta_{k,N,\psi}(\lambda I) = \frac{1}{q(2)^{n^2}} \sum_{U \in N} \Theta_{\theta}(h_{\lambda I,\psi}) \psi \left( \lambda^{-1} X_{1,1}, \ldots, \lambda^{-1} X_{k-1,k-1} \right).
\]

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By the change of variables $X_{i,j} \mapsto \lambda X_{i,j},$
we get
\[
\Theta_{k,N,\psi} (\lambda I) = \frac{1}{q(2)n^2} \sum_{U \in \mathcal{N}} \Theta_\theta (\lambda h_{I,X,Y,Z}) \bar{\psi} (X_{1,1}, \ldots, X_{k-1,k-1}).
\]

By Theorem 2.1, we have $\Theta_\theta (\lambda \cdot h_{I,U}) = \theta (\lambda) \Theta_\theta (h_{I,U}),$ and so
\[
\Theta_{k,N,\psi} (\lambda I) = \theta (\lambda) \Theta_{k,N,\psi} (I) = \theta (\lambda) \dim (\pi_{k,N,\psi}). \tag{4.18}
\]

By Theorem 2, this proves the case $\lambda \in \mathbb{F}^*.$

If $\lambda \in \mathbb{F}_d^* \subseteq \mathbb{F}_n^*$ is an eigenvalue of $s$ and $1 < d \mid n$ is such that $\mathbb{F}_d$ is generated by $\lambda$ over $\mathbb{F},$ we have by (4.6)
\[
\Theta_{k,N,\psi} (s) = \frac{1}{q(2)n^2} \sum_{U \in \mathcal{N}} \Theta_\theta (h_{s,U}) \bar{\psi} (s^{-1} X_{1,1}, \ldots, s^{-1} X_{k-1,k-1}). \tag{4.19}
\]

In order to compute $\Theta_\theta (h_{s,U}),$ we need to find conditions for $X_{i,j},$ such that $h_{s,U}$ will have a fixed number of Jordan blocks. This is equivalent to saying that $h_{s,U} - \lambda M_{kN}$ will have a given kernel dimension, or a given rank. Rank and trace are invariant under conjugation, so let us denote by $h_{s,\Upsilon}$ the matrix $h_{s,U}$ conjugated by $\Delta^k (R \Delta^k (T^{-1})).$ We have a matrix in $\text{GL}_{kn}(\mathbb{F}_d)$ and our goal is to find out how many matrices of the form
\[
h_{s,\Upsilon} := \Delta^k (\Delta^k (T^{-1})) \Delta^k (R) \Delta^k (R) \Delta^k (T^{-1} h_{s,U} \Delta^k (R) \Delta^k (T^{-1})),
\]
where $U$ varies, have a given rank $\ell.$

First, notice that by the invariance of rank under elementary row and column operations on $h_{s,\Upsilon} - \lambda M_{kN},$ we can use the nonzero elements on the diagonal of $s,\Upsilon - \lambda I_n$ to cancel the corresponding elements of $(X_{a,b})_\rho.$ These elementary operations map the sequence of matrices $\{(X_{a,b})_\rho\}_{1 \leq a \leq b \leq k-1}$ $\mathbb{F}_d$-linearly to the sequence
\[
\left\{ (\hat{X}_{a,b})_\rho = \begin{pmatrix} x_{0,0;0}^{(a,b)} & \cdots & x_{1;0,0;0}^{(a,b)} \\
\vdots & \ddots & \vdots \\
 x_{0,0;0}^{(a,b)} & \cdots & x_{1;0,0;0}^{(a,b)} 
\end{pmatrix} \in M_{d'} (\mathbb{F}_d) \right\}_{1 \leq a \leq b \leq k-1}. \tag{4.20}
\]

The dimension of the kernel of this map is $\binom{k}{2} (n - d') d',$ corresponding to the number of elements we canceled. Hence, the number of matrices $h_{s,\Upsilon} - \lambda M_{kN}$ of rank $\ell$ is $\binom{k}{2} (n - d') d'$ times the number of matrices of the form
\[
A := \begin{pmatrix} (\hat{X}_{1,1})_\rho & \cdots & (\hat{X}_{1,k-2})_\rho & (\hat{X}_{1,k-1})_\rho \\
0 & \cdots & (\hat{X}_{2,k-2})_\rho & (\hat{X}_{2,k-1})_\rho \\
0 & \cdots & 0 & (\hat{X}_{k-1,k-1})_\rho 
\end{pmatrix} \in M_{(k-1)d'} (\mathbb{F}_d). \tag{4.21}
\]
of rank \( \ell - k(n - d') \). According to the character formula (2.1), we can calculate \( \Theta_{\theta}(h_{s,U}) \). In this case \( m = kn \), \( g = h_{s,U} \) and

\[
t = \dim \ker(h_{s,U} - I) = kn - \rk(h_{s,U} - I) = kn - k(n - d') - \rk A = kd' - \rk A.
\]

Thus

\[
\Theta_{\theta}(h_{s,U}) = (-1)^{k-1} \left[ \sum_{i=0}^{d-1} \theta(\lambda^q) \right] \left( 1 - q^d \right) \left( 1 - (q^d)^2 \right) \cdots \left( 1 - (q^d)^{kd' - \rk A - 1} \right)
\]

(4.22)

Now, by (4.22) and Lemma 2.6, (4.19) can be written as

\[
\Theta_{k,N,\psi}(s) = \frac{(-1)^{k-1} (q^d)^{(k-n-d')d'}}{q^d (k-n)^2} \left[ \sum_{i=0}^{d-1} \theta(\lambda^q) \right] \sum_{A} (q^d; q^d)_{kd' - \rk A - 1}
\]

\[
\cdot \prod_{i=1}^{k-1} \psi_0 \left( \Tr_{F_d/F} \left( \lambda^{-1} \cdot \sum_{m=0}^{d-1} x_m^{(i,j)} \right) \right),
\]

(4.23)

where the sum is over matrices \( A \) as in (4.21). By the character formula (2.1), the RHS of (4.23) is \((-1)^{k(n-d')} \left[ \sum_{i=0}^{d-1} \theta(\lambda^q) \right] \) times the RHS of (3.3), when one replaces \( n \) with \( d' \), \( q \) with \( q^d \) and \( \psi_0 \) with

\[
\psi_0 : F_d \rightarrow \mathbb{C}^*, \quad \psi'_0(x) = \psi_0 \left( \Tr_{F_d/F} \left( \lambda^{-1} x \right) \right).
\]

(4.24)

Thus, the RHS of (4.23) is equal to \( \dim(\pi_{k,N,\psi}) \) (which was calculated in Theorem 1) after the substitution of \( n, q, \psi_0 \) with the relevant values. Hence,

\[
\Theta_{k,N,\psi}(s) = (-1)^{k(n-d')} \left[ \sum_{i=0}^{d-1} \theta(\lambda^q) \right] (q^d)^{(k-n-d')d'} \frac{[\GL_d(F_d)]}{q^n - 1},
\]

(4.25)

as desired.

\[
5 \text{ Proof of Theorem 4}
\]

Notice first that by part (III) of Lemma 2.10, the coefficients in both (1.9) and (1.10) are positive integers, unless \( k = 2 \) in which case they may also be zero.

Representations of a finite group are equivalent if the corresponding characters coincide. Hence, both parts of the theorem are equivalent to

\[
\forall g \in \GL_n(F) : \Theta_{k,N,\psi}(g) = \sum_{\ell \in \mathbb{C}^d} a_{k,n,\ell}(g) \cdot \Theta_{\Ind_{\ell}}(g).
\]

(5.1)

We prove now (5.1) for any \( g \in \GL_n(F) \). If \( g \) is not semisimple or does not come from \( S_n \) then the LHS of (5.1) is zero by parts (I) and (II) of Theorem 3. The RHS of (5.1) is also zero on such elements by Lemma 2.3.
Let \( g \) be a semisimple element, which comes from \( F_d \subseteq F_n \) and \( d \mid n \) is minimal. Let \( \lambda \) be an eigenvalue of \( s \), which generates \( F_d \) over \( F \). For such \( g \), part (III) of Theorem 3 and Lemma 2.3 imply that (5.1) is equivalent to

\[
(-1)^k (n-d') \left[ \sum_{\ell | d | n} \theta(\lambda^\ell) \right] q^{k-2} \frac{[\mathbb{GL}_d(F_d)]}{q^n - 1} = \sum_{\ell \mid d \mid n} a_{k,n,\ell}(q) \frac{[\mathbb{GL}_d(F_d)]}{d' - 1} \left[ \sum_{\ell=0}^{d-1} \theta(\lambda^\ell) \right],
\]

where \( d' = n/d \). Proving the following identity will establish (5.2):

\[
\frac{(-1)^k (n-d') q^{k-2} \frac{[\mathbb{GL}_d(F_d)]}{q^n - 1}}{\ell | d | n} = \sum_{\ell \mid d \mid n} a_{k,n,\ell}(q). \tag{5.3}
\]

Using (1.7), the RHS of (5.3) is

\[
\sum_{\ell | d | n} \sum_{\ell | m | n} \mu\left( \frac{m}{\ell} \right) \frac{(-1)^k (n-\ell) q^{(k-2)\frac{d'}{d} (\frac{\ell}{d}) - 1}}{q^n - 1}. \tag{5.4}
\]

We simplify (5.4) using (2.53):

\[
\sum_{\ell | d | n} \sum_{\ell | m | n} \mu\left( \frac{m}{\ell} \right) \frac{(-1)^k (n-\ell) q^{(k-2)\frac{d'}{d} (\frac{\ell}{d}) - 1}}{q^n - 1} = \sum_{m | d | n} \frac{(-1)^k (n-m) q^{(k-2)\frac{d'}{d} (\frac{m}{d}) - 1}}{q^n - 1} \sum_{\ell | d | m} \mu\left( \frac{m}{\ell} \right) \tag{5.5}
\]

\[
= \sum_{m | d | n} \frac{(-1)^k (n-m) q^{(k-2)\frac{d'}{d} (\frac{m}{d}) - 1}}{q^n - 1} \delta_{d,m},
\]

which is the LHS of (5.3). Hence the proof is completed.

6 Proof of Theorem 1

Representations of a finite group are equivalent if the corresponding characters coincide. Hence, the theorem is equivalent to

\[
\forall g \in \mathbb{GL}_n(F) : \quad \Theta_{k,N,\psi}(g) = \Theta_{\Theta_{|G_n^1}}(g) \cdot \text{St}^{k-1}(g), \tag{6.1}
\]

where we use the notation St also for the character of the Steinberg representation. We prove now (6.1) for any \( g \in \mathbb{GL}_n(F) \).

We first prove (6.1) for \( k = 1 \). Note that \( N = \{ I_n \} \) and so

\[
\pi_{\gamma_1,N,\varphi}(g) = \{ v \in V_{\pi_\gamma} \mid \pi(I_n)v = v \} = V_{\pi_\gamma}. \tag{6.2}
\]

Hence \( \pi_{1,N,\varphi}(g) = \pi_\varphi(g) \) as needed.

Now assume \( k \geq 2 \). If the semisimple part \( s \) of \( g \) does not come from \( F_n \), or \( g \) is not semisimple, then \( \Theta_{k,N,\psi}(g) = 0 \) by Theorem 3. From Theorem 2.1, we have \( \Theta_{\Theta_{|G_n^1}}(g) = 0 \). Hence, (6.1) is proved in that case.
Otherwise, \( g = s \) is a semisimple element which comes from \( \mathbb{F}_d \subseteq \mathbb{F}_n \) and \( d \mid n \) is minimal. We begin by calculating the character value \( \text{St}(g) \). For any prime \( p \), let \( m_p \) be the \( p \)-part of \( m \). By [Car93, Thm. 6.5.9],

\[
\text{St}(g) = \varepsilon_{\text{GL}_n(\mathbb{F})} \varepsilon_{\text{C}(g)^p} \left| \text{C}(g)^F \right|_{\text{char}(\mathbb{F})},
\]

where \( \varepsilon_G \) is \((-1)^{\text{r}}\) to the power of the \( \mathbb{F} \)-rank of \( G \), \( C(g) \) is the centralizer of \( g \) in \( \text{GL}_n(\mathbb{F}) \), and \( C(g)^p \) is its identity component and \( C(g)^F \) is the subgroup of \( \mathbb{F} \)-rational points in \( C(g) \). The \( \mathbb{F} \)-rank of \( \text{GL}_n(\mathbb{F}) \) is \( n \). Let \( \rho = (1^{\frac{n}{d}}) \), a partition of \( d' = \frac{n}{d} \) and let \( f \) be the characteristic polynomial of \( s \). By \$2.3.1\), the centralizer \( C(g)^F \) is isomorphic to \( C(L_{f,p})^F \), which in turn is isomorphic to \( \text{GL}_{d'}(\mathbb{F}_d) \) (cf. [Gre55, Lem. 2.4] and the discussion preceding it). Thus, \( \varepsilon_{C(g)^p} = \varepsilon_{\text{GL}_{d'}(\mathbb{F})} \equiv (-1)^{d'} \) and

\[
\left| C(g)^F \right| = q^{\sum_{i=1}^{d'} d(i-1)} \prod_{k=1}^{d'} (q^{dk} - 1), \quad \left| C(g)^F \right|_{\text{char}(\mathbb{F})} = q^{\frac{n(d'-1)}{2}}. \tag{6.4}
\]

The discussion shows that

\[
\text{St}(g) = (-1)^{n-d'} q^{\frac{n(d'-1)}{2}}. \tag{6.5}
\]

By Theorem 2.1,

\[
\Theta_{\theta|_{\mathbb{F}_x}}(g) = (-1)^{n-1} \left[ \sum_{\alpha=0}^{d-1} \theta(\lambda^{q^\alpha}) \right] (1 - q^d)(1 - (q^d)^2) \cdots (1 - (q^d)^{d'-1})
\]

\[
= (-1)^{n-d'} \left[ \sum_{\alpha=0}^{d-1} \theta(\lambda^{q^\alpha}) \right] (q^d - 1)(q^{2d} - 1) \cdots (q^{n-d} - 1) \frac{q^n - 1}{q^n - 1} \tag{6.6}
\]

\[
= (-1)^{n-d'} \left[ \sum_{\alpha=0}^{d-1} \theta(\lambda^{q^\alpha}) \right] \frac{|\text{GL}_{d'}(\mathbb{F}_d)|}{(q^n - 1)q^{\frac{n(d'-1)}{2}}},
\]

where \( \lambda \) is an eigenvalue of \( g \). By Theorem 3

\[
\Theta_{k,N,\psi}(g) = (-1)^{k(n-d')} q^{(k-2)\frac{n(d'-1)}{2}} \left[ \sum_{i=0}^{d-1} \theta(\lambda^{q^i}) \right] \frac{|\text{GL}_{d'}(\mathbb{F}_d)|}{q^n - 1}. \tag{6.7}
\]

Multiplying (6.6) by (6.5) raised to the \((k-1)\)-th power, we get (6.7) as needed.

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