Gravitational field of a slowly rotating black hole with a phantom global monopole

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Abstract
We present a slowly rotating black hole solution with a phantom global monopole by solving Einstein’s field equation and find that the presence of the global monopole changes the black hole structure. The metric coefficient $g_{t\phi}$ contains the hypergeometric function of the polar coordinate $r$, which is more complex than that in the usual slowly rotating black hole. The energy scale of symmetry breaking $\eta$ affects the black hole horizon and the deficit solid angle. In particular, the solid angle is surplus rather than deficit for the black hole with the phantom global monopole. We also study the effects of the global monopole on the angular velocity of the horizon $\Omega_H$, Kepler’s third law, the innermost stable circular orbit and the radiative efficiency $\epsilon$ in the thin accretion disc model. Our results also show that for a phantom black hole, the radiative efficiency $\epsilon$ is positive only in the case $\eta \leq \eta_c$. The threshold value $\eta_c$ increases with the rotation parameter $a$.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Phantom field is a special kind of dark energy with the negative kinetic energy [1], which is applied extensively in cosmology to explain the accelerating expansion of the current Universe [2–7]. Compared with other dark energy models, the phantom field is more interesting because the negative kinetic energy results in that the equation of state of the phantom field is less than $-1$ and then the null energy condition is violated. Although the phantom field owns such exotic properties, it is still not excluded by recent precise observational data [8], which encourages many people to focus on investigating the phantom field from various aspects of physics.
The phantom field also exhibits some peculiar properties in the black hole physics. Babichev [9] found that the mass of a black hole decreases after it absorbs the phantom dark energy. This means that the cosmic censorship conjecture could be challenged severely by a fact that the charge of a Reissner–Nordström-like black hole absorbing the phantom energy will be larger than its mass. We studied the wave dynamics of the phantom scalar perturbation in the Schwarzschild black hole spacetime and found that in the late-time evolution the phantom scalar perturbation grows with an exponential rate rather than decays as the usual scalar perturbations [10, 11]. Moreover, we also find that the phantom scalar emission enhances the Hawking radiation of a black hole [12]. Recently, some black hole solutions describing gravity coupled to phantom scalar fields or phantom Maxwell fields have been found in [13–20]. The thermodynamics and the possible phase transitions occurred in these phantom black hole spacetimes are also studied in [16, 17]. Moreover, the gravitational collapse of a charged scalar field and the path of a photon propagating in such kind of backgrounds are investigated in [18, 19]. Bolokhov et al also study the regular electrically and magnetically charged black hole with a phantom scalar in [20]. These investigations could help us to get a deep understanding about dark energy and black hole physics.

The global monopole is one of the topological defects which could be formed during phase transitions in the evolution of the early Universe. The metric describing a static black hole with a global monopole was obtained by Barriola and Vilenkin [21], which arises from the breaking of the global SO(3) symmetry of a triplet scalar field in a Schwarzschild background. Due to the presence of the global monopole, the black hole owns the different topological structure from that of the Schwarzschild black hole. The physical properties of the black hole with a global monopole have been studied extensively in recent years [22–25].

The main purpose of this paper is to study the gravitational field of the phantom global monopole arising from a triplet scalar field with negative kinetic energy and to see how the energy scale of symmetry breaking $\eta$ influences the black hole structure, Kepler’s third law, the innermost stable circular orbit (ISCO) and the radiative efficiency $\epsilon$ in the thin accretion disc model. Moreover, we will explore how it differs from that in the black hole with the ordinary global monopole.

The paper is organized as follows. In the following section, we will construct a static and spherical symmetric black hole solution of a phantom global monopole from a triplet scalar field with negative kinetic energy, and then study the effect of the parameter $\eta$ on the black hole. In section 3, we obtain a slowly rotating black hole solution with the phantom global monopole by solving Einstein’s field equation and find that the presence of the global monopole makes the metric coefficient of black hole more complex. In section 4, we will focus on investigating the effects of the parameter $\eta$ on Kepler’s third law, the ISCO and the radiative efficiency $\epsilon$ in the thin accretion disc model. We end the paper with a summary.

2. A static and spherical symmetric black hole with the phantom global monopole

Let us now first study a static and spherical symmetric black hole with the phantom global monopole formed by spontaneous symmetry breaking of a triplet of phantom scalar fields with a global symmetry group $O(3)$. The action giving rise to the phantom global monopole is

$$S = \int \sqrt{-g} \, d^4x \left[ R - \frac{\xi}{2} \partial^\mu \psi^a \partial_\mu \psi^a - \frac{\lambda}{4} (\psi^a \psi^a - \eta^2)^2 \right],$$

where $\psi^a$ is a triplet of scalar field with $i = 1, 2, 3$, $\eta$ is the energy scale of symmetry breaking and $\lambda$ is a constant. The coupling constant $\xi$ in the kinetic term with the value $\xi = 1$ corresponds to the case of the ordinary global monopole originating from the scalar field with
the positive kinetic energy [21]. As the coupling constant $\xi = -1$, the kinetic energy of the scalar field is negative and then the phantom global monopole is formed.

Following in [21], we can take ansatz describing a monopole as
\[ \psi^a = \frac{\eta f(r)}{r} \xi^a, \tag{2} \]
where $\xi^a \xi^a = r^2$. Equipping with a general static and spherical symmetric metric
\[ ds^2 = -B(r) dt^2 + A(r) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\phi^2, \tag{3} \]
one can find that the field equations for $\psi^a$ can be reduced to a single equation for $f(r)$
\[ \frac{\xi f'}{A} + \left[ \frac{2}{Ar} + \frac{1}{2B} \left( \frac{B}{A} \right) \right] \xi f'' - \frac{2 \xi f}{r^2} - \lambda \eta^2 f (f^2 - 1) = 0, \tag{4} \]
and the nonzero components of the energy–momentum tensor can be expressed as
\[ T^r_0 = T^\phi_0 = \frac{\xi \eta^2 f^2}{2A} + \frac{\xi \lambda}{4} \eta^2 (f^2 - 1)^2, \tag{5} \]
\[ T^\phi_\phi = \frac{\xi \eta^2 f^2}{2A} + \frac{\xi \lambda}{4} \eta^2 (f^2 - 1)^2. \tag{6} \]

Similarly, as in [21], one can take an approximation $f(r) = 1$ outside the core due to a fact that $f(r)$ grows linearly when $r < (\eta \sqrt{\lambda})^{-1}$ and it tends exponentially to unity as soon as $r > (\eta \sqrt{\lambda})^{-1}$. With this approximation, we can obtain a solution of the Einstein equations
\[ B = A^{-1} = 1 - 8\pi \xi \eta^2 - \frac{2M}{r}, \tag{8} \]
where $M$ is an integrate constant. Obviously, the radius of the event horizon is $r_H = 2M/(1 - 8\pi \xi \eta^2)$. Here, we must point out that it is possible that the radius of the event horizon $r_H$ is larger than the monopole’s core $\delta \sim (\eta \sqrt{\lambda})^{-1}$ if the coupling constant $\lambda \gg \frac{(1 - 8\pi \xi \eta^2)^2}{M^2}$. In this case, the metric (3) can describe the geometry near the horizon in the spacetime with the global monopole. With the increase of the energy scale of symmetry breaking $\eta$, one can find that the radius of the event horizon $r_H$ increases for a Schwarzschild black hole with the ordinary global monopole ($S_+$), but decreases for a Schwarzschild black hole with the phantom global monopole ($S_-$). Thus, comparing with the system $S_+$, one can find that the system $S_-$ possesses the higher Hawking temperature and the lower entropy. In the low energy limit, the luminosity of the Hawking radiation of a spherical symmetric black hole can be approximated as $L = \frac{2\pi^2 r_H^4}{15} \propto (1 - 8\pi \xi \eta^2)^4 / r_H^4$, which tells us that the energy scale of symmetry breaking $\eta$ enhances the Hawking radiation for the system $S_+$, but it decreases the Hawking radiation in the system $S_-$. The property of the phantom field enhancing the Hawking emission of a Kerr black hole is also found in [12].

Introducing the transformations
\[ t \rightarrow (1 - 8\pi \xi \eta^2)^{1/2} t, \quad r \rightarrow (1 - 8\pi \xi \eta^2)^{1/2} r, \quad M \rightarrow (1 - 8\pi \xi \eta^2)^{3/2} M, \tag{9} \]
one can rewrite the metric (3) with the functions (8) as
\[ ds^2 = -\left( 1 - \frac{2M}{r} \right) dt^2 + \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + (1 - 8\pi \xi \eta^2)^2 (d\theta^2 + \sin^2 \theta \, d\phi^2). \tag{10} \]
It is clear that there exists a deficit solid angle $(1 - 8\pi \eta^2)$ for the system $S_-$. However, for the system $S_-$ (i.e., $\xi = -1$), one can find that the solid angle becomes $(1 + 8\pi \eta^2)$, which is surplus rather than deficit. This implies that the topological properties of a spacetime with the phantom global monopole are different from that of a spacetime with an ordinary global monopole.
3. A slowly rotating black hole with the phantom global monopole

In this section, we first obtain a slowly rotating black hole solution with the phantom global monopole by solving Einstein’s field equation. Then, we will study the properties of the black hole spacetime.

From the static and spherical symmetric solution (3) with the metric function (8), we can assume that the metric has a form for a slowly rotating black hole with the phantom global monopole:

\[ ds^2 = -U(r) dt^2 + \frac{1}{U(r)} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right), \tag{11} \]

where \( a \) is the rotation parameter associated with its angular momentum. Moreover, we assume that in the slowly rotating spacetime (11) the ansatz describing a monopole is

\[ \psi^a = \frac{\eta f(r) x^i}{r}. \tag{12} \]

It is easy to find that the field equations for \( \psi^i \) can also be reduced to a single equation for \( f(r) \)

\[ \xi f'' U(r) + \left[ \frac{2U(r)}{r} + U(r)^2 \right] \xi f' - \frac{2\xi f}{r^2} - \lambda \eta^2 f(f^2 - 1) = 0, \tag{13} \]

which is similar to that in the static and spherical symmetric spacetime (3). It is not surprising since the triplet scalar field \( \psi^i \) does not depend on the time coordinate \( t \).

Inserting the metric (11) and the triplet scalar field (12) into Einstein’s field equation, we find that the non-vanishing components of the field equation can be expanded to the first order in the angular momentum parameter \( a \) as

\[
\begin{align*}
\text{tt} & : \frac{U(r)}{r^2} [U'(r)r + U(r) - 1] + \frac{8\pi U(r)}{U(r)} \left[ \left( \frac{\eta^2 f'^2 U(r)}{2} + \eta^2 f^2 \right) \xi + \frac{\lambda}{4} \eta^4 (f^2 - 1) \right] \\
& \quad = 0 + \mathcal{O}(a^2), \\
\text{rr} & : \frac{1}{U(r)r^2} [U'(r)r + U(r) - 1] + \frac{8\pi U(r)}{U(r)} \left[ \left( \frac{\eta^2 f'^2 U(r)}{2} + \eta^2 f^2 \right) \xi + \frac{\lambda}{4} \eta^4 (f^2 - 1) \right] \\
& \quad = 0 + \mathcal{O}(a^2), \\
\text{θθ} & : -\frac{r}{2} [U''(r)r + 2U'(r)] + 8\pi r^2 \left[ \frac{\xi \eta^2 f'^2 U(r)}{2} + \frac{\lambda}{4} \eta^4 (f^2 - 1) \right] = 0 + \mathcal{O}(a^2), \\
\text{ϕϕ} & : -\frac{r}{2} \sin^2 \theta [U''(r)r + 2U'(r)] + 8\pi r^2 \left[ \frac{\xi \eta^2 f'^2 U(r)}{2} + \frac{\lambda}{4} \eta^4 (f^2 - 1) \right] \sin^2 \theta \\
& \quad = 0 + \mathcal{O}(a^2), \\
\text{tϕ} & : \frac{1}{2r^2} \left\{ r^2 U(r) \frac{\partial^2 F(r, \theta)}{\partial r^2} - F(r, \theta) \left[ r^2 U''(r) + 2rU'(r) + 2U(r) \right] + 2F(r, \theta) \right. \\
& \quad + \frac{\partial^2 F(r, \theta)}{\partial \theta^2} - \frac{\partial F(r, \theta)}{\partial \theta} \cot \theta \left. \right\} - 8\pi F(r, \theta) \left[ \left( \frac{\eta^2 f'^2 U(r)}{2} + \frac{\eta^2 f^2}{r^2} \right) \xi \\
& \quad + \frac{\lambda}{4} \eta^4 (f^2 - 1) \right] = 0 + \mathcal{O}(a^2). \tag{18} \end{align*}
\]
Figure 1. The dependence of $h(r)$ on the parameter $\eta$. The left and right panels are for the systems $SR_+$ and $SR_-$, respectively. Here we set $M = 1$.

Solving the Einstein equations (14)–(17) with the approximation $f(r) = 1$ outside the core, we can obtain the metric coefficient

$$U(r) = 1 - 8\pi \xi \eta^2 - \frac{2M}{r}. \tag{19}$$

Separating $F(r, \theta) = h(r) \Theta(\theta)$, we can obtain the equation for the angular part

$$\frac{d^2 \Theta(\theta)}{d\theta^2} - \frac{d\Theta(\theta)}{d\theta} \cot \theta = \lambda_0 \Theta(\theta). \tag{20}$$

In order that the coefficient $g_{\phi\phi}$ can be reduced to that in the usual slowly rotating black hole without the global monopole, here we set $\lambda_0 = -2$ and then find that $\Theta(\theta) = \sin^2 \theta$ in this case. Thus, the radial part of equation (18) becomes

$$r^2 U(r) \frac{d^2 h(r)}{dr^2} - 2h(r)U(r) - 16\pi \xi \eta^2 h(r) = 0. \tag{21}$$

Substituting equation (19) into the above radial equation, we obtain (see in the appendix)

$$h(r) = \left[ \frac{2M}{(1 - b)r} \right]^{\frac{1}{2}} \left[ \frac{1}{2} \left( \frac{9 - b}{1 - b} - 3 \right), \frac{1}{2} \left( \frac{9 - b}{1 - b} + 3 \right) \right], \tag{22}$$

where $b = 8\pi \xi \eta^2$ and $\, _2F_1(a_1, b_1, c_1; x)$ is the hypergeometric function. As a usual slowly rotating black hole, the horizon of the black hole (11) is given by the zero of the function $U(r) = (g_{rr})^{-1}$, i.e., $r_H = \frac{2M}{b}$, which is the same as that in the static and spherical symmetric case (3). The main reason is that we here expand the metric only to the first order in the angular momentum parameter $a$.

From equation (22), it is obvious to see that due to the presence of the global monopole the metric coefficient $g_{\phi\phi}$ becomes more complicated in the slowly rotating black hole (11). The dependence of $h(r)$ on the parameter $\eta$ is shown in figure 1, which tells us that for the slowly rotating black hole with the ordinary global monopole ($SR_+$) the function $h(r)$ increases with the parameter $\eta$ near the horizon, but decreases at the far field region with the larger value.
of $r$. For a slowly rotating black hole with the phantom global monopole ($SR_-$), the behavior of $h(r)$ is just the opposite. In a word, the dependence of the function $h(r)$ on the \( \eta \) near the horizon is different than in the far field region in these two global monopole cases. As the parameter $b \to 0$, one can find that $h(r) \to \frac{2M}{r}$, which recovers that of a usual slowly rotating black hole without the global monopole. When the rotation parameter $a$ vanishes, one can obtain the previous solution of a static and spherical symmetric black hole with the phantom global monopole (3).

For a rotating black hole, one of the important quantities is the angular velocity of the horizon $\Omega_H$, which affects the region where the super-radiance occurs in the black hole background. In the spacetime of a slowly rotating black hole with the global monopole, the angular velocity of the horizon $\Omega_H$ can be expressed as

\[
\Omega_H = -\frac{g_{\phi\phi}}{g_{tt}} \bigg|_{r=r_H} = \frac{a(1-b)^2}{2r_H^2} \frac{\Gamma \left[ \frac{3b-1}{1-b} + 1 \right]}{4\Gamma \left[ \frac{1}{2} \left( \sqrt{\frac{3b-1}{1-b} - 1} \right) \right] \Gamma \left[ \frac{1}{2} \left( \frac{3b-1}{1-b} + 5 \right) \right]},
\]

which depends on the energy scale of symmetry breaking $\eta$. We plot the change of the angular velocity $\Omega_H$ with the parameter $\eta$ in figure 2. For the system $SR_+$, we find that the angular velocity $\Omega_H$ first decreases slowly and then increases rapidly with the increase of $\eta$. There exists a minimum for $\Omega_H$ at where $\eta = 0.1725$ (i.e., $b = 0.7477$). This means that there exist a minimum region for the occurrence of the super-radiance in the black hole with the ordinary global monopole for fixed $a$. For the system $SR_-$, $\Omega_H$ increases monotonically with the energy scale of symmetry breaking $\eta$.

Making the same transformations (9) as in section 2, one find that the area of the horizon becomes $A_H = br_H^2$. It means that for the system $SR_-$, the solid angle is surplus rather than deficit, which is similar to that in the static and spherical symmetric case (3).

4. Kepler’s third law and the ISCO

In this section, we will focus on how the symmetry breaking scale $\eta$ of the global monopole affects Kepler’s third law and the ISCO in this slowly rotating black hole.

![Figure 2](image-url)
In the stationary and axially symmetric spacetime, one can find that the time-like geodesics takes the form
\[ \dot{t} = \frac{E g_{\phi \phi} - L_z g_{\theta \theta}}{g''_{\theta \phi} + g_{\phi \phi}}, \]
\[ \dot{\phi} = \frac{E g_{\phi \phi} + L_z g_{\theta \theta}}{g''_{\theta \phi} + g_{\phi \phi}}, \]
\[ g_{rr} \dot{r}^2 + g_{\theta \theta} \dot{\theta}^2 = V_{\text{eff}}(r, \theta; E, L_z), \]
with the effective potential
\[ V_{\text{eff}}(r) = \frac{E^2 g_{\phi \phi} - 2EL_z g_{\phi \theta} - L_z^2 g_{\theta \theta}}{g''_{\theta \phi} + g_{\phi \phi}} - 1, \]
where the overhead dot stands for a derivative with respect to the affine parameter. The constants \( E \) and \( L_z \) correspond to the conserved energy and the (z-component of) orbital angular momentum of the particle, respectively.

For simplicity, we set the orbits on the equatorial plane. With the restriction that \( \theta = \pi/2 \), one can find that for a stable circular orbit in the equatorial plane, the effective potential \( V_{\text{eff}}(r) \) must obey
\[ V_{\text{eff}}(r) = 0, \quad \frac{dV_{\text{eff}}(r)}{dr} = 0. \]
Solving the above equations, one can obtain
\[ E = \frac{g_{\theta \theta} + g_{\phi \phi} \Omega}{\sqrt{g''_{\theta \phi} + g_{\phi \phi} \Omega^2}}, \]
\[ L_z = -\frac{g_{\theta \theta} + g_{\phi \phi} \Omega}{\sqrt{g''_{\theta \phi} + g_{\phi \phi} \Omega^2}}, \]
\[ \Omega = \frac{d\phi}{dr} = \frac{g_{\phi \phi} + \sqrt{(g_{\phi \phi})^2 + g_{\theta \theta} g_{\phi \phi}}}{g_{\phi \phi} \cdot g_{\phi \phi}} \]
where \( \Omega \) is the angular velocity of particle moving in the orbits. From equation (29), one can obtain Kepler’s third law in the slowly rotating black hole spacetime with the global monopole
\[ T^2 = \frac{4\pi^2 R^3}{M} \left[ 1 + \frac{4a}{(1 - b)^2 \left( \frac{9 - b}{1 - b} + 1 \right) M^{1/2} R^{3/2} \left[ \frac{2M}{(1 - b)R} \right]^{1/2} \left( \sqrt{\frac{9 - b}{1 - b}} - 1 \right) \right] \]
\[ \times F_1 \left[ \frac{1}{2} \left( \sqrt{\frac{9 - b}{1 - b}} - 1 \right), \frac{1}{2} \left( \sqrt{\frac{9 - b}{1 - b}} + 3 \right), \frac{9 - b}{1 - b} + 1, \frac{2M}{(1 - b)R} \right] \]
\[ + bM F_1 \left[ \frac{1}{2} \left( \sqrt{\frac{9 - b}{1 - b}} - 1 \right), \frac{1}{2} \left( \sqrt{\frac{9 - b}{1 - b}} + 5 \right), \frac{9 - b}{1 - b} + 2, \frac{2M}{(1 - b)R} \right] \]
\[ + O(a^2) \],
where \( T \) is the orbital period and \( R \) is the radius of the circular orbit. The later terms on the right-hand side are the correction from the \( a \) and the symmetry breaking scale \( \eta \) of the global monopole. From equation (30), we find that for fixed \( R \), the \( \eta \) affects the orbital period \( T \) only in the case with the nonzero rotation parameter \( a \). In figure 3, we present the change of the
corrected term $\Delta T^2 = \frac{4a^2r^2M}{M} - 1$ with the parameter $\eta$ in this spacetime. It is shown that the absolute value $|\Delta T^2|$ increases with the scale $\eta$ for the system $SR_+$, but decreases with $\eta$ in the system $SR_-$. Moreover, we also find that the presence of the global monopole makes the orbital period $T$ increases for a prograde particle (i.e., $a > 0$) and decreases for a retrograde one (i.e., $a < 0$).

The ISCO of the particle around the black hole is given by the condition $V_{\text{eff}, rr} = 0$. For a slow rotating black hole with the global monopole (11), we obtain

$$r_{\text{ISCO}} = \frac{6M}{1 - b} - \frac{4a\sqrt{23 - 2^2\sqrt{\frac{3}{7}}} }{(1 - b)^{5/2}} \left( \frac{\sqrt{\frac{9 - b}{1 - b}} + 1}{\sqrt{\frac{9 - b}{1 - b}} + 2} \right) \left\{ 9\left[ \sqrt{(9 - b)(10 - 3b)} \right] 

+ (3b^2 - 25b + 30) \right\} _2 F_1 \left[ \frac{1}{2} \left( \frac{9 - b}{1 - b} - 3 \right), \frac{1}{2} \left( \frac{9 - b}{1 - b} + 3 \right), \sqrt{\frac{9 - b}{1 - b}} \right]

\times _2 F_1 \left[ \frac{1}{2} \left( \frac{9 - b}{1 - b} - 1 \right), \frac{1}{2} \left( \frac{9 - b}{1 - b} + 5 \right), \frac{9 - b}{1 - b} + 2, \frac{1}{3} \right]

\times + b \left[ \sqrt{(9 - b)(1 - b)} + (1 + b) \right] _2 F_1 \left[ \frac{1}{2} \left( \frac{9 - b}{1 - b} + 1 \right),

\frac{1}{2} \left( \frac{9 - b}{1 - b} + 7 \right), \sqrt{\frac{9 - b}{1 - b}} + 3, \frac{1}{3} \right] \right\} + O(a^2), \quad (31)

Obviously, the ISCO radius $r_{\text{ISCO}}$ decreases with the rotation parameter $a$, which is similar to that in the usual Kerr black hole. In figure 4, we set $M = 1$ and present the variety of the ISCO radius $r_{\text{ISCO}}$ with the parameter $\eta$ in the slowly rotating black hole spacetime with the global monopole. It is shown that the ISCO radius $r_{\text{ISCO}}$ increases with the scale $\eta$ for the system...
Figure 4. The change of the ISCO radius $r_{\text{ISCO}}$ with the parameters $\eta$ and $a$. The dashed and solid lines are for the systems $\text{SR}_+$ and $\text{SR}_-$, respectively. The thin and thick lines correspond to the cases with $a = -0.2$ and $a = 0.2$. Here, we set $M = 1$.

Let us now compute the effect of the symmetry breaking scale $\eta$ on the radiative efficiency $\epsilon$ in the thin accretion disc model, which is defined by

$$\epsilon = 1 - E(r_{\text{ISCO}}). \quad (32)$$

This quantity corresponds to the maximum fraction of energy being radiated when a test particle accretes into a central black hole. For a Schwarzschild black hole and an extremal Kerr black hole, $\epsilon \sim 0.06$ and $\epsilon \sim 0.42$, respectively. For a slowly rotating black hole with the global monopole, the radiative efficiency $\epsilon$ can be expressed as

$$\epsilon = 1 - \frac{2\sqrt{2}}{3}\sqrt{1 - b} + \frac{a^{3-2-\frac{1}{3}\sqrt{9}}}{M\left(\frac{9-b}{1-b}+1\right)}\left[\sqrt{(9-b)}\left(\sqrt{(9-b)} + \sqrt{1-b}\right) + \frac{2b}{3}\right]$$

$$\times_2 F_1\left[\frac{1}{2}\left(\sqrt{\frac{9-b}{1-b}}-3\right), \frac{1}{2}\left(\sqrt{\frac{9-b}{1-b}} + 3\right), \sqrt{\frac{9-b}{1-b}} + 1, \frac{1}{3}\right] + \frac{2b}{3}$$

$$\times_2 F_1\left[\frac{1}{2}\left(\sqrt{\frac{9-b}{1-b}}-1\right), \frac{1}{2}\left(\sqrt{\frac{9-b}{1-b}} + 5\right), \sqrt{\frac{9-b}{1-b}} + 2, \frac{1}{3}\right] + O(a^2). \quad (33)$$

It is clear that the radiative efficiency $\epsilon$ increases with the rotation parameter $a$ in these two global monopole black holes. The effect of the symmetry breaking scale $\eta$ on the radiative efficiency $\epsilon$ is shown in figure 5.

It tells us that with the scale $\eta$ the radiative efficiency $\epsilon$ increases for the system $\text{SR}_+$, but decreases for the system $\text{SR}_-$. Moreover, we also note that for the phantom black hole the radiative efficiency $\epsilon$ is positive only for the case $\eta \leq \eta_c$. This could be explained by a fact that for a phantom black hole with $\eta \geq \eta_c$, its capability of capturing particle could become so weak that the accreted matter around it is very dilute, which could lead to the fact that the
radiation cannot be generated because of the lack of the enough stress and dynamical friction in the accretion disc model. The critical value $\eta_c$ can be approximated as

$$\eta_c = 0.0705 + 0.0227 \frac{a}{M} + O(a^2).$$

(34)

It means that the critical value $\eta_c$ increases with the rotation parameter $a$, which is also shown in figure 6.

5. Summary

In this paper, we present firstly a four-dimensional spherical symmetric black hole with the phantom global monopole and find that the scale of symmetry breaking $\eta$ affects the radius of the black hole horizon and the deficit solid angle. For the system $SR_-$, the solid angle is surplus rather than deficit as in the systems $SR_+$. We also obtain a slowly rotating black hole solution.

Figure 5. The change of the radiative efficiency $\epsilon$ with the parameters $\eta$ and $a$. The dashed and solid lines are for the systems $SR_+$ and $SR_-$, respectively. The thin and thick lines correspond to the cases with $a = -0.2$ and $a = 0.2$. Here, we set $M = 1$.

Figure 6. The change of the critical value $\eta_c$ with the rotation parameter $a$ for the system $SR_-$. Here, we set $M = 1$. 
with the global monopole by solving Einstein’s field equation. We find that the presence of the
global monopole makes the metric coefficient $g_{tt}$ contains the hypergeometric function of the
polar coordinate $r$, which is more complex than that in the usual slowly rotating black hole.
We study the property of the angular velocity of the horizon $\Omega_{H}$, which is connected with the
region where the super-radiance occurs in the black hole background. With the increase of $\eta$, the
angular velocity $\Omega_{H}$ first decreases slowly and then increases rapidly for the system $SR_{+}$,
but increases monotonically with $\eta$ for the system $SR_{-}$.

We also analyze the effects of the scale of symmetry breaking $\eta$ on Kepler’s third law, the
ISCO and the radiative efficiency $\epsilon$ in the thin accretion disc model. Our results show that only
in the case with the nonzero rotation parameter $a$ the orbital period $T$ depends on the scale
of symmetry breaking $\eta$ for the fixed orbital radius $R$. The presence of the global monopole
makes the orbital period $T$ increases for a prograde particle (i.e., $a > 0$) and decreases for a
retrograde one (i.e., $a < 0$). The absolute value of the corrected term to Kepler’s third law
(i.e., $|\Delta T^2|$), the ISCO radius $r_{ISCO}$ and the radiative efficiency $\epsilon$ in the thin accretion disc
model increase with the scale $\eta$ for the system $SR_{+}$, but decrease with $\eta$ in the system $SR_{-}$.
Moreover, we also find that for a phantom black hole the radiative efficiency $\epsilon$ is positive only
for the case $\eta \leq \eta_c$. The threshold value $\eta_c$ increases with the rotation parameter $a$.

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Appendix. The form of $h(r)$

Here, we present the form of $h(r)$ by solving the radial equation (21). Defining $z = \frac{2M}{(1-b)\nu}$, the
radial equation (21) can be expressed as

$$z(1-z)\frac{d^2 h(z)}{dz^2} + 2(1-z)\frac{dh(z)}{dz} + 2\left[\frac{1 - \frac{b}{(1-b)z}}{1 - b}\right] h(z) = 0. \quad (A.1)$$

Employing the transformation $h(z) = z^\alpha(1-z)^\beta H(z)$, we can write equation (A.1) into the
standard form of the hypergeometric equation

$$z(1-z)\frac{d^2 H(z)}{dz^2} + [c - (1 + a_1 + b_1)z] \frac{dH(z)}{dz} - a_1 b_1 H(z) = 0, \quad (A.2)$$

with

$$c_1 = 2 + 2\alpha, \quad a_1 = \alpha + \beta + 2, \quad b_1 = \alpha + \beta - 1. \quad (A.3)$$

Because of the constraint from the coefficient of $H(z)$, the power coefficients $\alpha$ and $\beta$ must
satisfy the second-order algebraic equations

$$\beta = 0, \quad \alpha(\alpha + 1) - \frac{2}{1-b} = 0. \quad (A.4)$$
Considering the asymptotical behavior \( h(r) \) at spatial infinite \( r \to \infty \), we choose \( \alpha = \frac{1}{2} \left[ \sqrt{\frac{9}{1-b} - 1} \right] \). Then, the function \( h(r) \) has the form

\[
h(r) = \left[ \frac{2M}{(1-b)r} \right]^2 \left[ \frac{2M}{(1-b)^{1-b}} \right] \left( \frac{\sqrt{9}-b}{1-b} - 3 \right)^{\frac{1}{2}} \left( \frac{\sqrt{9}-b}{1-b} + 3 \right)^{\frac{1}{2}} \left( \frac{\sqrt{9}-b}{1-b} + 1 \right)^{\frac{1}{2}} \left( \frac{\sqrt{9}-b}{1-b} + 2M \right)\]

(A.5)

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