Finite-difference method on electrostatics field calculation in oil storage tank

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Abstract. This paper starts with the basic principle of Finite-Difference Method, combined with the practical application of the method, focusing on the difference scheme of Poisson equation and Laplace equation in 2D Electric Field and Axisymmetrical Field under the square grid partition. This study solves the Finite-Difference Method on electrostatics field calculation and introduces discretization on the Dirichlet Problem in oil storage tanker. The Finite-Difference Method gives the analytic solutions of the potential equations by changing the continuous field problems into discrete system. The numerical solutions of discrete points could infinitely approximate the real solution of the continuous field through discrete model.

1. Introduction
Of all the calculation methods of the electromagnetic field numerical analysis, the Finite-Difference Method, also named Grid Method, is applied the earliest for its simple and visual features. It involves aspects from linear field to nonlinear field, constant field to variable field. Finite-Difference Method is still one important numerical analysis method that could not be ignored even though the Finite Element Method, which is the combination of Finite-Difference Method and Calculus of Variations, is increasingly wide range of applications.

The Finite-Difference Method gives the analytic solutions of the potential equations by changing the continuous field problems into discrete system. The numerical solutions of discrete points could infinitely approximate the real solution of the continuous field through discrete model.

2. Concept of Finite-Difference Method

2.1. Basic concepts
We hypothesize one function \( f(x) \), whose independent variable \( x \) increases tiny \( \Delta x = h \), the corresponding increment of \( f(x) \) would be

\[
\Delta f(x) = f(x + h) - f(x)
\]

Which is named as Finite-Difference (First order difference) of function \( f(x) \) and is different from Differential. It is commonly known as Finite-Difference because it is limited Difference. However, the difference between Finite-Difference \( \Delta f \) and Differential \( df \) would be very tiny if \( h \) is tiny enough.

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According to the definition of Finite-Difference, a second-order central difference is always used in the calculation of Finite-Difference

\[ \Delta f(x) = f(x + \frac{h}{2}) - f(x - \frac{h}{2}) \]  \hspace{1cm} (2)

And the first order difference quotient which is the division of \( \Delta f \) by \( h \) would close to the first order derivative \( \frac{df}{dx} \)

\[ \frac{\Delta f(x)}{\Delta x} = \frac{f(x + h) - f(x)}{h} \]  \hspace{1cm} (3)

The first order difference is still the function of the independent variable \( x \), when calculating the Finite-Difference of the first order difference like the same method of equation (1), we could get \( \Delta^2 f(x) \) which named the second order difference quotient of primitive function \( f(x) \) and it would close to the second order derivative \( d^2 f \) as \( h \) goes to 0.

Obviously, the second order difference quotient closes to the two order derivative. So we could define higher order difference quotient and higher order derivative. But in here, we wouldn’t discuss the higher order difference quotient and higher order derivative, because they are not involve into the actual numerical calculation of Finite-Difference Method in the electromagnetic field.

2.2. Principle of Finite-Difference
The Finite-Difference Method is one numerical calculation on the basis of Difference Principle, which could transform the solving of boundary value problem into a corresponding group of finite difference equations by using the difference quotient instead of the partial derivative on the very discrete point. The numerical solution of the boundary value problem could get through the calculation of function value on every discrete point according to the group of finite difference equations.

The Finite-Difference Method, the following abbreviation is Difference Method, is one method of solving which would transform the continuous domain problem in the electromagnetic field into discrete system problem, which makes the numerical solution on every discrete point in the model of grid discretization system approximate infinitely to the real value in continuous field domain. It is an approximate calculation method and the computer could guarantee enough the calculation precision.

3. Mathematical model of electric field in oil tank

3.1. Basic equation of electric field
It is assumed that the oil charges uniform and there is no space charge in the above of oil in tank. Also we could consider the tank is good grounding according to actual situation, which means that the potential of the tank wall is 0. We should know that all of above assumptions could simplify the problem and would not affect the feasibility and reliability of the method.

After those appropriate assumptions, the electric field in oil tank could be treated as 2D field in non-uniform medium. It is difficult to solve the problem by Analytical method. But modern computer provides effective means in numerical calculation of electrostatic field. Its essence is to transform continuous field domain problems into discrete system problems, and makes the numerical solution on discrete point approximate infinitely to the real value in continuous field domain. So we study the electric field in oil tank by Finite-Difference Method.

\[ \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\frac{\rho}{\varepsilon} \]
\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \]  \hspace{1cm} (4)

\( \phi \) – Potential, V;
\( x, y \) – Coordinate, m;
\( \rho \) – Charge density, Cm\(^{-3}\);
\( \varepsilon \) – Dielectric constant, Fm\(^{-1}\),
\( \varepsilon_0 = \varepsilon_\infty \).

The potential function in oil tank meets the first boundary value problem which is named Dirichlet problem. If there is free charge density area, the equation \( \nabla^2 \varphi = -\frac{\rho}{\varepsilon} \) is named Poisson equation, and the oil in tank is belong to this situation; on the country, if there is not free charge density area, the equation \( \nabla^2 \varphi = 0 \) is named Laplasse equation, and the air space in tank is belong to this situation.

Given surface potential of every conductor, that is
\[
\varphi|_{\Gamma_i} = 0 \tag{5}
\]
\( \Gamma \) is outer boundary, the surface potential of tank is 0 because of static grounding. \( i = 1 \) is the air space; \( i = 2 \) is the oil space. In interface, it meets join condition, that is
\[
\begin{aligned}
E_{1t} &= E_{2t} \\
D_{1n} &= D_{2n} \\
\frac{\partial \varphi_1}{\partial n} |_{r_{12}} &= \frac{\partial \varphi_2}{\partial n} |_{r_{12}} \\
\varphi_1 |_{r_{12}} &= \varphi_2 |_{r_{12}}
\end{aligned} \tag{6}
\]

\( D \) – Electric displacement, Cm\(^{-3}\);
\( E \) – Electric field strength, Vm\(^{-1}\).

3.2. Numerical solution of Electrostatic Field

We adopt the Finite-Difference Method to solve the boundary problem of electrostatic field in tank. The distribution form of discrete points is governed by the grid division.

In order to export the solution of difference discretization from basic equations under irregular grid division, also meet the discussion of boundary conditions discretization, we start from the common situation of step distance ranging, as shown in figure 1. Assumed there are two groups of lines, which are no-equal but respectively parallel to \( x \) and \( y \) coordinate axis, to divide the field domain \( \Gamma \). We take a arbitrarily point \( o(x_0, y_0) \) which joints together with its adjacent node 1, 2, 3, 4 to combine to form a star type structure. The function of node \( o \) could be expressed as \( \varphi(x_0, y_0) \) or \( \varphi_0 \), and the function of node 1, 2, 3, 4 also could be expressed as \( \varphi_1, \varphi_2, \varphi_3, \varphi_4 \), and so on.

So we could get following equations through Finite-Difference calculating.

\[
\begin{aligned}
\frac{\partial^2 \varphi}{\partial x^2} \bigg|_0 &= 2 \frac{h_3 (\varphi_1 - \varphi_0) + h_1 (\varphi_3 - \varphi_0)}{h_1 (h_1 + h_3)} \tag{7} \\
\frac{\partial^2 \varphi}{\partial y^2} \bigg|_0 &= 2 \frac{h_4 (\varphi_2 - \varphi_0) + h_2 (\varphi_4 - \varphi_0)}{h_2 (h_2 + h_4)} \tag{8}
\end{aligned}
\]

After substituting equation (7) and (8) into Poisson equation in (4), we could get
\[
\begin{aligned}
\nabla^2 \varphi = 2 \frac{h_3 (\varphi_1 - \varphi_0) + h_1 (\varphi_3 - \varphi_0)}{h_1 (h_1 + h_3)} + 2 \frac{h_4 (\varphi_2 - \varphi_0) + h_2 (\varphi_4 - \varphi_0)}{h_2 (h_2 + h_4)} = P \\
\varphi_0 = \frac{h_3 h_4 \varphi_1}{(h_1 + h_3)(h_3 h_4 + h_2 h_4)} + \frac{h_1 h_4 \varphi_2}{(h_2 + h_4)(h_1 h_4 + h_2 h_4)} \\
&+ \frac{h_1 h_4 \varphi_3}{(h_1 + h_4)(h_1 h_3 + h_2 h_4)} + \frac{h_1 h_4 \varphi_4}{(h_2 + h_4)(h_1 h_3 + h_2 h_4)} - \frac{h_1 h_3 h_4 P}{2(h_1 h_3 + h_2 h_4)} \tag{10}
\end{aligned}
\]

The above two equations are suit to oil space in tank.

After substituting equation (7) and (8) into Laplasse equation in (4), we could get
The above two equations are suit to air space in tank.

\[ \nabla^2 \varphi = 2 \frac{h_3 (\varphi_1 - \varphi_0) + h_4 (\varphi_3 - \varphi_0)}{h_1 h_3 (h_1 + h_3)} + 2 \frac{h_4 (\varphi_2 - \varphi_0) + h_3 (\varphi_4 - \varphi_0)}{h_2 h_4 (h_2 + h_4)} = 0 \]  
(11)

\[ \varphi_0 = \frac{h_2 h_3 h_4 \varphi_1}{(h_1 + h_3)(h_1 h_3 + h_2 h_4)} + \frac{h_1 h_2 h_4 \varphi_3}{(h_1 + h_3)(h_1 h_3 + h_2 h_4)} + \frac{h_1 h_2 h_4 \varphi_2}{(h_2 + h_4)(h_1 h_3 + h_2 h_4)} + \frac{h_1 h_2 h_4 \varphi_4}{(h_2 + h_4)(h_1 h_3 + h_2 h_4)} \]  
(12)

The above two equations are suit to air space in tank.

**Figure 1.** The asymmetric star type structure.  
**Figure 2.** The interface of air-oil.

We could change (10) and (12) into followings:

\[ \varphi_0 = K_a \varphi_1 + K_5 \varphi_2 + K_3 \varphi_3 + K_4 \varphi_4 - KP \]  
(13)

\[ \varphi_0 = K_b \varphi_1 + K_5 \varphi_2 + K_3 \varphi_3 + K_4 \varphi_4 \]  
(14)

According to the air-oil interface, see figure 2, node o, 1, 3 are not on the same interface of same media. \( \varphi_a, \varphi_b \) is respectively expressed as the potential in media \( \epsilon_a \) and \( \epsilon_b \), and there is charge density only in \( \epsilon_a \). That is, potential \( \varphi_a \) meets Poisson Equation (13) and \( \varphi_b \) meets Laplasse Equation (14). If media \( \epsilon_b \) is changed into \( \epsilon_a \), Equation (13) on node o could be written as

\[ \varphi_a^o = K_1 \varphi_1^o + K_5 \varphi_2^o + K_3 \varphi_3^o + K_4 \varphi_4^o - KP \]  
(15)

And if media \( \epsilon_a \) is changed into \( \epsilon_b \), Equation (14) on node o could be written as

\[ \varphi_b^o = K_1 \varphi_1^o + K_5 \varphi_2^o + K_3 \varphi_3^o + K_4 \varphi_4^o \]  
(16)

Actually, \( \varphi_a^2 \) and \( \varphi_b^3 \) are not existent in oil tank, they are virtual potentials just only in order to export the calculation format of Finite-Difference Method on the boundary node. So they should be deleted from the equations (15) and (16), and we could get following from equation (6)

\[ \epsilon_a (\varphi_a^2 - \varphi_a^4) = \epsilon_b (\varphi_b^2 - \varphi_b^4) \), namely, \( \epsilon_a \varphi_a^2 + \epsilon_b \varphi_a^4 = \epsilon_a \varphi_b^2 + \epsilon_b \varphi_b^4 \]  
(17)

\[ \varphi_a^2 = \varphi_a^0, \varphi_a^4 = \varphi_b^0, \varphi_b^2 = \varphi_b^0, \varphi_b^4 = \varphi_b^0 \]  
(18)

Equation (15) multiplied by \( \epsilon_a \), and (16) multiplied by \( \epsilon_b \), then plus the two results together and be substituted into (17) and (18), we could get the Finite-Difference equation
\[
\varphi_0 = K_1 \varphi_1 + K_2 \varepsilon_b \frac{K_2 + K_4}{K_2 \varepsilon_b + K_4 \varepsilon_a} \varphi_2 + K_3 \varphi_3 \\
+ K_4 \varepsilon_a \frac{K_2 + K_4}{K_2 \varepsilon_b + K_4 \varepsilon_a} \varphi_4 - P \varepsilon_a \frac{KK_4}{K_2 \varepsilon_b + K_4 \varepsilon_a}
\]

Equation (13) (14) (19) would be expressed by Over-relaxation iterative method

\[
\varphi^{(n+1)}_{(i,j)} = \varphi^{(n)}_{(i,j)} + \alpha R^{(n)}_{(i,j)} = \varphi^{(n)}_{(i,j)} \\
+ \alpha \left( K_1 \varphi^{(n)}_{(i+1,j)} + K_2 \varphi^{(n)}_{(i,j+1)} + K_3 \varphi^{(n+1)}_{(i-1,j)} + K_4 \varphi^{(n+1)}_{(i,j-1)} - KP - \varphi^{(n)}_{(i,j)} \right)
\]

\[
\varphi^{(n+1)}_{(i,j)} = \varphi^{(n)}_{(i,j)} + \alpha R^{(n)}_{(i,j)} = \varphi^{(n)}_{(i,j)} \\
+ \alpha \left( K_1 \varphi^{(n)}_{(i+1,j)} + K_2 \varphi^{(n)}_{(i,j+1)} + K_3 \varphi^{(n+1)}_{(i-1,j)} + K_4 \varphi^{(n+1)}_{(i,j-1)} - \varphi^{(n)}_{(i,j)} \right)
\]

\[
\varphi^{(n+1)}_{(i,j)} = \varphi^{(n)}_{(i,j)} + \alpha R^{(n)}_{(i,j)} = \varphi^{(n)}_{(i,j)} \\
+ \alpha \left( K_1 \varphi^{(n)}_{(i+1,j)} + K_2 \varphi^{(n)}_{(i,j+1)} + K_3 \varphi^{(n+1)}_{(i-1,j)} + K_4 \varphi^{(n+1)}_{(i,j-1)} - \varphi^{(n)}_{(i,j)} \right)
\]

The coefficient \( \alpha \) is commonly named as accelerating convergence factor. Its value would determine the degree of Over-relaxation and then affect the convergence velocity of iterative solutions. Our actual calculating indicates that the convergence velocity is the fastest when \( \alpha \) is 1.8.

So the Linear equation group, whose coefficient is known, would be available according above given calculating equations. In the discrete model, the electric field strength would be

\[
\overrightarrow{E} = \frac{\varphi_1 - \varphi_2}{h_1 + h_3} \hat{i} + \frac{\varphi_2 - \varphi_4}{h_2 + h_4} \hat{j}
\]

Of course the following equation is suitable when by using field map

\[
E = \left| \frac{\Delta \varphi}{\Delta n} \right|
\]

3.3. Mathematical model of oil tank electric field

The only one difference of mathematical models of electric field strength numerical calculation between rail tanker and road tanker is that the length of cross axle may not be the same as that of the longitudinal axle in road tanker, while the two lengths of that of rail tanker are similar. And the one difference could only reflect in calculating programming. All the others are the same in both types of tankers. Related computer simulation programming would be discussed in following paper.

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