Mass Spectrum and Statistical Entropy of the BTZ black hole from Canonical Quantum Gravity

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ABSTRACT

In a recent publication we developed a canonical quantization program describing the gravitational collapse of a spherical dust cloud in 2+1 dimensions with a negative cosmological constant $-\Lambda \equiv -l^{-2} < 0$. In this paper we address the quantization of the Banados–Teitelboim–Zanelli (BTZ) black hole. We show that the mass function describing the black hole is made of two pieces, a constant non-vanishing boundary contribution and a discrete spectrum of the form $\mu_n = \frac{\hbar}{4}(n + \frac{1}{2})$. The discrete spectrum is obtained by applying the Wheeler–DeWitt equation with a particular choice of factor ordering and interpreted as giving the energy levels of the collapsed matter shells that form the black hole. Treating a black hole microstate as a particular distribution of shells among the levels, we determine the canonical entropy of the BTZ black hole. Comparison with the Bekenstein–Hawking entropy shows that the boundary energy is related to the central charge of the Virasoro algebra that generates the asymptotic symmetry group of the three-dimensional anti-de Sitter space $\text{AdS}_3$. This gives a connection between the Wheeler–DeWitt approach and the conformal field theory approach.

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I. INTRODUCTION

The four laws of black hole mechanics formalized by Bardeen, Carter, and Hawking \[1\] capture the macroscopic properties of black holes and provide compelling evidence that black holes are thermodynamic systems. Each of the four laws has an analogue in the laws of thermodynamics if one identifies the surface gravity, \( \kappa \), with some multiple of the temperature and the surface area of the horizon with the thermodynamic entropy.

Even before the formulation of the four laws, Bekenstein \[2, 3\] had proposed that black holes should possess an entropy given by some multiple of the horizon area since, according to the classical theory, there is no process by which the area of the horizon can decrease \[5\]. Bekenstein’s idea was later confirmed in an independent way by Hawking \[6\] who showed that an eternal black hole behaves as a heat reservoir whose temperature is determined precisely by its surface gravity according to \( T = \frac{\kappa}{2\pi} \). Hawking’s result is consistent with Bekenstein’s proposal if the multiple is taken to be \( \frac{1}{4} \). It leads to the now famous Bekenstein-Hawking formula for the entropy of the black hole:

\[
S_{\text{B-H}} = \frac{A}{4G\hbar}.
\]

For a solar mass black hole, this is on the order of \( 10^{77}k_B \), where \( k_B \) is Boltzmann’s constant, whereas the entropy of our sun is estimated to be about twenty orders of magnitude smaller. Thus, in collapsing to form a black hole, a matter cloud of one solar mass would gain entropy by a factor of \( 10^{20} \). Quantum gravity is expected to play a pivotal role in this increase of entropy \[7\].

Because a successful theory of quantum gravity should be able to determine the microstates of a black hole, it had long been hoped that obtaining the area law (1) by counting the states in the sense of Boltzmann would serve as a powerful test of a candidate theory and perhaps even to discriminate between different approaches. On the contrary, many apparently disparate models, from loop quantum gravity \[8, 9, 10, 11, 12\] to string theory \[13, 14, 15\] and the AdS/CFT correspondence \[16, 17, 18, 19, 20\], have claimed partial success in recovering (1). One could call this state of affairs an “embarrassment of riches” \[21\]. The states being counted differ in each approach and the true underlying quantum gravitational states remain shrouded in mystery. Moreover, often the actual physical degrees of freedom being counted are themselves poorly understood and there are differences in the details of the counting of states, in particular in the statistical principles that are used. For instance, in loop quantum gravity the black hole microstates are represented by punctures of a spin network on the horizon. The counting of these states yields the area law (1) provided that Boltzmann statistics (where the ‘particles’ are assumed distinguishable) and not Bose statistics (where the ‘particles’ are assumed indistinguishable) are implemented. On the other hand, in string theory the microstates of BPS black holes are dual to weak field D-brane states, which are counted using Bose statistics. This difference in the statistics is important for an understanding of the microscopic degrees of freedom and for the calculation of logarithmic corrections to (1), see \[22\].

Ever since its discovery, the Bañados–Teitelboim–Zanelli (BTZ) black hole \[23, 24\] has been an object of intense study both as a toy model for black-hole thermodynamics and as an
aid to understanding the thermodynamics of many higher-dimensional black holes of interest to string theory, which can often be understood in terms of the BTZ solution. Soon after its discovery, Carlip suggested [25] that the BTZ black hole entropy could be understood in terms of the degrees of freedom of a Wess–Zumino–Witten (WZW) theory on the horizon. Not long after, Strominger [26] proposed that because the asymptotic symmetry group of 2+1-dimensional gravity with a negative cosmological constant is generated by two copies of the Virasoro algebra [27], its degrees of freedom could be described by two conformal field theories (CFTs) at infinity with central charges

$$c_R = c_L = \frac{3l}{2G\hbar}. \quad (2)$$

Most approaches to date apply CFT techniques to compute the BTZ black hole entropy. Yet, more than a decade later the two most important open questions continue to be (a) what precisely are the degrees of freedom being counted and how are they related to the gravity theory that is being described, and (b) where do these degrees of freedom “lie” (if they can be localized at all), that is, whether they are asymptotic or whether they are excitations near the horizon. Recently, Carlip has proposed a way to unify the two ideas by imposing the existence of a horizon as an additional constraint on the initial values of the gravitational degrees of freedom [21, 28, 29]. The Hamiltonian and diffeomorphism constraints get modified by the “horizon constraint” so that they generate a new Poisson bracket algebra that gives rise to the desired central charge.

Although canonical quantization of Einstein’s general relativity cannot be expected to yield the final theory of quantum gravity, it is well adapted to address particular models with high symmetry [7]. It also has the virtue of ascribing a transparent meaning to the canonical variables and therefore to the degrees of freedom. In canonical quantum gravity, the entropy of a Schwarzschild black hole was understood in concrete terms as encoding the unavailability, in the final collapsed state, of the information contained in the original matter distribution [30, 31]. In this paper we address the BTZ black hole in 2+1 dimensions from a similar point of view. Using a program recently developed by us [32], we now determine its microstates. Under certain natural assumptions regarding factor ordering, we find that the energy spectrum coincides with that proposed in [26]. However, from the point of view of canonical gravity the meaning of the spectrum is clear: it represents the energy levels available to shells of matter that have undergone collapse and now reside within the horizon. The mass energy of the BTZ black hole is the sum of the contribution from all the collapsed shells and a boundary term.\textsuperscript{6} Explicitly counting the microstates yields an area law. The boundary term, which is arbitrary within the canonical theory, scales the entropy in much

\textsuperscript{6} In 3+1-dimensional collapse, the boundary contribution from the origin is generally set to zero because a non-vanishing mass function at the origin would represent a singular initial configuration corresponding to a point mass at the center. The situation is quite different in the 2+1-dimensional models we are considering. A non-vanishing contribution from the origin is essential to allow for an initial velocity profile that vanishes there. This does not lead to singular initial data and the boundary contribution does not have the interpretation of a point mass situated at the center.
the same way as the central charge in the AdS/CFT approach. Its value can be determined by comparison with the Bekenstein-Hawking entropy in (1). Thus one gets a physical picture of the states being counted and also determines the connection between the boundary energy and the central charge of the Virasoro algebra that generates the asymptotic AdS$_3$ symmetry.

This paper is organized as follows. In Section II we briefly describe the models and their quantization, focusing only upon those aspects that are immediately related to our present purpose. For details on the classical solutions we refer the reader to [33, 34], for a semiclassical study of the Hawking radiation to [35], and for more on the canonical quantization of these models we suggest [32], from where most of Section II is taken. In Section III we focus on the BTZ black hole, which is a special case of the models described in section I. Here, the stationary bound states describing the BTZ black hole and its mass spectrum are obtained. In Section IV we obtain the entropy and conclude with some brief comments in Section V.

II. THE MODELS

In [32] we considered the collapse of a spherical, inhomogeneous dust cloud in 2+1 dimensions with a negative cosmological constant $-\Lambda \equiv -l^{-2} < 0$. These models are represented by a solution of Einstein’s equations with pressureless dust described by the stress tensor $T_{\mu\nu} = \varepsilon U_{\mu} U_{\nu}$, where $\varepsilon(t, r)$ is the dust energy density. The solution is characterized by two arbitrary functions, the mass function $F(\rho) \equiv 4GM(\rho)$, representing the initial mass distribution and the energy function $E(\rho)$, representing the initial energy distribution within the cloud. In terms of these functions, the classical solution is given as

$$ds^2 = -d\tau^2 + \frac{(\partial_\rho R)^2}{2(E - F)}d\rho^2 + R^2 d\varphi^2,$$

where $\tau$ is the dust proper time and $\rho$ labels shells of curvature radius $R(\tau, \rho)$. The energy density $\varepsilon(\tau, \rho)$ is given by

$$8\pi G\varepsilon(\tau, \rho) = \frac{\partial_\rho F}{R(\partial_\rho R)},$$

and Einstein’s equations lead to

$$(\partial_\tau R)^2 = 2E - \Lambda R^2,$$

with solution

$$R(\tau, \rho) = \sqrt{\frac{2E}{\Lambda}} \sin \left(-\sqrt{\Lambda}\tau + \sin^{-1}\sqrt{\frac{\Lambda}{2E}} \rho \right).$$

Shells labeled by $\rho$ become singular when $R(\tau, \rho) = 0$.

There is the freedom to choose the physical radius of a shell at the initial time. If $R(0, \rho) = \rho$, the mass and energy functions may be given in terms of the initial energy density of the collapsing cloud according to

$$F(\rho) = 8\pi G \int_0^\rho \rho' \varepsilon(0, \rho')d\rho' + f_0,$$
\[ E(\rho) = [\partial_\tau R(0, \rho)]^2 + \Lambda \rho^2, \quad (7) \]

where \( f_0 \) is an arbitrary constant of integration. As \( \partial_\tau R(0, \rho) \) is the initial velocity distribution, the first term in the expression for \( E(\rho) \) represents twice the initial kinetic energy of the cloud.

The general circularly symmetric ADM metric may be embedded in the spacetime described by \( [32] \). In \[32\] we showed how, after a series of transformations, this leads to a canonical description in terms of a phase space consisting of \( \tau(r), R(r) \) and the mass density function, \( \Gamma(r) \), defined via \( F(r) \) according to
\[ F(r) = \frac{M_0}{2} + \int_0^r d\epsilon \Gamma(\epsilon), \quad (8) \]

where \( M_0 \) is an arbitrary constant, together with their canonical momenta, \( P_\tau(r), P_R(r) \) and \( P_\Gamma(r) \), respectively; here and in the rest of this section we set \( G = 1/8 \) for convenience. The boundary actions at \( r = 0 \) and \( r \to \infty \) can be absorbed into a single hypersurface action, and the effective canonical constraints of the gravity dust system are given as
\[ \mathcal{H}_\tau = \tau' P_\tau + R' P_R - \Gamma P_\Gamma' \approx 0 \]
\[ \mathcal{H} = P_\tau^2 + \mathcal{F} P_R^2 - \frac{\Gamma^2}{\mathcal{F}} \approx 0, \quad (9) \]

where \( \mathcal{F} = \Lambda R^2 - 2F \) and the prime represents derivation with respect to the ADM label coordinate \( r \). The quantity \( \mathcal{F} \) is vanishing at the horizon, passing from negative in the interior of the black hole to positive in the exterior. This means that the DeWitt supermetric changes sign across the horizon and it is worth reexamining how this comes about.

In \[32\] we start with a Hamiltonian constraint in which the gravitational part is hyperbolic everywhere and the matter part is only linear in momentum. The original canonical variables are sufficiently differentiable functions everywhere. However, they are not geometrically transparent and we perform a Kuchař (canonical) transformation \[36\] in the gravitational sector to a set of new variables, which are the mass function, \( F \), the radial coordinate, \( R \), and their conjugate momenta. Given in terms of the new variables, the Hamiltonian of the gravitational sector continues hyperbolic and is valid everywhere except at the horizon where it is degenerate and where the new canonical momenta generically suffer a jump. Introducing the new variable \( \Gamma \), as defined in \( [33] \), we arrive at yet another form of the gravitational Hamiltonian in which the kinetic term is of non-canonical form (containing the derivative of the momentum conjugate to \( \Gamma \)) and the matter part is still linear in the momentum. The advantage of this procedure is that it allows us to absorb the boundary terms into the hypersurface action. We then use the momentum constraint and square the Hamiltonian constraint of the gravity matter system to arrive at the final form \( (9) \), where the kinetic term is hyperbolic inside and elliptic outside the horizon. The new constraint is thus a consequence of canonical transformations, use of the momentum constraint, and squaring. This sequence of transformations does not preserve the hyperbolic form of the (gravitational part of the) original constraints, but the final constraints nevertheless are valid everywhere.
except at the horizon. We thus obtain the effective constraint system (9) and use it as a starting point for our discussion. In the quantum theory, the horizon will be treated as a boundary at which continuity and differentiability of the wave-functional are required.

Dirac’s quantization procedure may now be applied to turn the classical constraints into operator constraints on wave-functionals. According to it, the momenta are replaced by functional differential operators (we set $\hbar = 1$)

$$P_X = -i \frac{\delta}{\delta X(r)},$$

and one may write the quantum Hamiltonian constraint as

$$\hat{H}\Psi[\tau, R, \Gamma] = \left[ \frac{\delta^2}{\delta \tau^2} + \mathcal{F} \frac{\delta^2}{\delta R^2} + A\delta(0) \frac{\delta}{\delta R} + B\delta(0)^2 + \frac{\Gamma^2}{\mathcal{F}} \right] \Psi[\tau, R, \Gamma] = 0,$$

where $A(R, F)$ and $B(R, F)$ are smooth functions of $R$ and $F$ which encapsulate the factor ordering ambiguities. The divergent quantities $\delta(0)$ and $\delta(0)^2$ are introduced to indicate that the factor ordering problem can be dealt with only after a suitable regularization procedure has been implemented. We notice that the Hamiltonian constraint contains no functional derivative with respect to the mass density function. In fact the mass density appears merely as a multiplier of the potential term in the Wheeler-DeWitt equation. This indicates that $\Gamma(r)$, and hence the initial energy density distribution, $\varepsilon(0, r)$ may be externally specified. Once specified, $\Gamma(r)$ determines the quantum theory of a particular classical model.

The quantum momentum constraint, on the other hand,

$$\hat{H}_r\Psi[\tau, R, \Gamma] = \left[ \tau' \frac{\delta}{\delta \tau} + R' \frac{\delta}{\delta R} - \Gamma \left( \frac{\delta}{\delta \Gamma} \right)^\tau \right] \Psi[\tau, R, \Gamma] = 0,$$

requires no immediate regularization because it involves only first order functional derivatives. To describe a collapsing cloud with a smooth, non-vanishing matter density distribution over some label set of non-zero measure the Hamiltonian constraint was regularized on a lattice. The continuum limit of the wave-functional was taken to be of the form

$$\Psi[\tau, R, \Gamma] = \exp \left[ i \int dr \Gamma(r)\mathcal{W}(\tau(r), R(r), F(r)) \right].$$

It automatically obeys the momentum constraint provided that $\mathcal{W}(\tau, R, F)$ has no explicit dependence on the label coordinate $r$. We showed in [32] and [37] that, for the wave-functionals to be simultaneously factorizable on the lattice and to obey the momentum constraint in the continuum limit (as the lattice spacing is made to approach zero), they must satisfy not one but three equations, one of which is the Hamilton–Jacobi equation that was used in earlier studies [38] to describe Hawking radiation in the WKB approximation. The function $B(R, F)$ in (11) is forced to be identically vanishing and the remaining two equations together with hermiticity of the Hamiltonian constraint uniquely fixed $A(R, F)$, the measure and the wave-functionals. Hermiticity of the operator

$$\mathcal{F}P_R^2 = -\mathcal{F}(R, F) \frac{\partial^2}{\partial R^2} - A(R, F) \frac{\partial}{\partial R} - B(R, F)$$

(14)
on the lattice requires that

\[ A = |F| \partial_R \ln(\mu|F|), \]  

(15)

where \( \mu(R, F) \) is the measure that defines an inner product on the Hilbert space. Lattice regularization effectively turns the continuum (midi-superspace) problem into a countably infinite set of decoupled mini-superspace problems; the three equations mentioned earlier are required to ensure a sensible, diffeomorphism invariant continuum limit.

If the mass density function is distributional in character (here it is non-vanishing on a label set of measure zero to begin with), the wave-functional (13) is automatically a wave-function, or a countable product of wave-functions, and the functional differential equations become ordinary partial differential equations. The original midi-superspace problem then naturally collapses into a set of mini-superspace problems from the very beginning, and no further conditions must be met. The factor ordering ambiguities survive, of course, so there are not enough constraints to fully resolve the coefficients \( A(R, F) \) and \( B(R, F) \) together with the Hilbert space measure. This is the case of the BTZ black hole, which we discuss in the following section.

III. STATIONARY STATES REPRESENTING THE BTZ BLACK HOLE

As mentioned in the concluding paragraph of the previous section, the description of the eternal black hole differs fundamentally from our description of collapse in [32] because neither the coefficients \( A(R, F) \) and \( B(R, F) \) nor the Hilbert space measure are determined uniquely. They must be chosen by different physical arguments, which we present here.

The BTZ black hole with ADM mass parameter \( M \) is a special case of the solution (3). Reintroducing Newton’s constant, \( G \), which will facilitate comparison with other work, it is recovered when the mass function is taken to be constant, \( F = 4GM \), for \( \rho > 0 \), and the energy function is given by \( 2E = 1 + 8GM \), again as long as \( \rho > 0 \). The metric in (3) may then be brought to canonical (static) form,

\[ ds^2 = -\left( \Lambda R^2 - 8GM \right) dT^2 + \frac{dR^2}{\left( \Lambda R^2 - 8GM \right)} + R^2 d\varphi^2, \]  

(16)

by the transformations \( R = R(\tau, \rho) \) as given in (6) and

\[ T = \tau + \int dR \frac{\sqrt{1 + 8GM - \Lambda R^2}}{\Lambda R^2 - 8GM} \]  

(17)

for the Killing time, \( T \).

Within the framework of the canonical theory described in the previous section, the BTZ black hole should be described by a mass function [see (8)] of the general form

\[ F(r) = 4GM_0 + 4G\mu \Theta(r) = 4GM, \]  

(18)

where \( \mu \) represents the mass of a shell at \( r = 0 \) and where \( \Theta \) is the Heaviside function. Likewise, the energy function should be given by

\[ E(r) = \frac{1}{2} [1 + 8GM_0 + 8G\mu \Theta(r)]. \]  

(19)
The mass function in (18) yields a mass density that is the $\delta$–distribution (recall that $F'(r) = \Gamma(r)$)

$$\Gamma(r) = 4G\mu\delta(r),$$

(20)

and the wave-functional in (13) turns into the wave-function,

$$\Psi = e^{\frac{4G}{\mu} \int_{0}^{\infty} d\Gamma(r) W(\tau(\tau), R(\tau), F(\tau))} = e^{i\mu W(\tau, R, F)},$$

(21)

where $\tau = \tau(0)$, $R = R(0)$ and $F = F(0)$. The Wheeler–DeWitt equation becomes the Klein–Gordon equation describing a shell of mass $\mu$,

$$\left[ \frac{\partial^2}{\partial \tau^2} + F \frac{\partial^2}{\partial R^2} + A \frac{\partial}{\partial R} + B \right] e^{i\mu W(\tau, R, F)} = 0,$$

(22)

where we have absorbed the term $16\mu^2/F$ into $B$, which here renormalizes the potential.

As there are no further conditions, the choice of $A(R, F)$ and $B(R, F)$ for the BTZ black hole will necessarily be to some extent ad hoc. Because we are describing a single shell in this simple quantum mechanical model of the black hole, we demand that (22) be the free wave equation. This is, of course, an assumption, although a natural one. The free wave equation would be obtained if we could write (22) as

$$\gamma^{ab} \nabla_a \nabla_b \Psi = 0,$$

(23)

where $\gamma_{ab}$ is the DeWitt supermetric on the effective configuration space $(\tau, R)$ and $\nabla_a$ represents the covariant derivative on this space with respect to $\gamma_{ab}$. The supermetric is non-degenerate and can be read off directly from (9). It is found to be flat, while being positive definite when $F > 0$ and indefinite when $F < 0$,

$$\gamma_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{F} \end{pmatrix}.$$ 

(24)

The Wheeler–DeWitt equation (22) is therefore the free Klein–Gordon equation if we take $B(R, F) = 0$,

$$A(R, F) = |F| \partial_R \ln \sqrt{|F|}$$

(25)

and an inner product given by the integral

$$\langle \Psi_1, \Psi_2 \rangle = \int \frac{dR}{\sqrt{|F|}} \Psi_1^* \Psi_2,$$

(26)

since inserting (25) into (15) gives $\mu(R, F) = 1/\sqrt{|F|}$.

When $F \neq 0$, the supermetric can be brought to a manifestly flat form by the coordinate transformation

$$R_* = \pm \int \frac{dR}{\sqrt{|F|}}$$

(27)

and in terms of $R_*$, (23) reads

$$\left[ \frac{\partial^2}{\partial \tau^2} \pm \frac{\partial^2}{\partial R_*^2} \right] e^{i\mu W(\tau, R, F)} = 0.$$

(28)
where in both the above equations, the upper sign refers to the exterior, \( \mathcal{F} > 0 \), and the lower to the interior, \( \mathcal{F} < 0 \). We note that the equation is hyperbolic in the interior, but elliptic in the exterior. This signature change has been noted in other models [39] and comes about because \( \mathcal{F} \) passes from positive outside the horizon to zero on the horizon and negative inside. For our particular problem, it corresponds to the fact that the degrees of freedom live inside the horizon.

For the interior we find
\[
R_{*}^{\text{in}} = \frac{1}{\sqrt{\Lambda}} \sin^{-1} \sqrt{\frac{\Lambda R^2}{8GM}},
\]
so that the value of \( R_{*}^{\text{in}} \) on the horizon is \( \pi / (2\sqrt{\Lambda}) \). In the exterior
\[
R_{*}^{\text{out}} = \frac{1}{\sqrt{\Lambda}} \left[ \ln \left( \frac{R\sqrt{\Lambda} + \sqrt{\Lambda R^2 - 8GM}}{\sqrt{8GM}} \right) + \frac{\pi}{2} \right],
\]
where we have adjusted the integration constant so that \( R_{*} \) is continuous across the horizon. If we extend its range to \( (-\infty, \infty) \) so as to avoid any issues connected with a boundary at \( R_{*} = 0 \), the solutions to the wave equation may be given as
\[
\begin{align*}
\psi_{\text{in}}(\tau, R_{*}) &= A_{\pm} e^{-i\mu(\tau \pm R_{*})} & \mathcal{F} < 0 \quad & (\text{31}) \\
\psi_{\text{out}}(\tau, R_{*}) &= B_{\pm} e^{-i\mu(\tau \pm i R_{*})} & \mathcal{F} > 0.
\end{align*}
\]
The “interior” is now the interval \( (-\pi / 2\sqrt{\Lambda}, +\pi / 2\sqrt{\Lambda}) \) and our solutions are oscillatory only in the interior. For a continuous and differentiable wave-function, matching conditions across the horizon require that
\[
\mu_{j} = \frac{\hbar}{l} \left( j + \frac{1}{2} \right), \quad j \in \{0\} \cup \mathbb{N},
\]
where \( l = 1/\sqrt{\Lambda} \) is again the AdS length, and we have disallowed all negative values of \( \mu \). We also re-insert now \( \hbar \) in the expressions. Equation (32) gives the allowed masses of a collapsed dust shell.

We remark that a similar spectrum was proposed in [26] using arguments related to the asymptotic symmetry group of AdS\(_3\), which is generated by (two copies of) the Virasoro algebra. In our current framework the CFT states get the interpretation of excited mass shells. Worthy of note is the fact that their spectrum has here been derived from the dynamics of the interior of the hole.

IV. ENTROPY

To understand how the degeneracy that leads to the black hole entropy comes about, we propose the following point of view. We think of the black hole as a single shell given by the mass and energy functions of (18) and (19) respectively, with the well defined spectrum in (32). However, we know that this single shell is in fact the end-state of the collapse of many
shells. Suppose that $N$ such shells were to collapse to form the black hole. We assume that, regardless of their history, each can eventually occupy only the energy levels of (32) upon collapse into the single shell. Thus, the black hole entropy is just the number of possible distributions of $N$ identical objects between these levels. In this way we realize Bekenstein’s original ideas in [4]. To quote from this article: “It is then natural to introduce the concept of black-hole entropy as the measure of the inaccessibility of information (to an exterior observer) as to which particular internal configuration of the black hole is actually realized in a given case.”

A black hole microstate is therefore viewed as a particular distribution of shells among the available energy levels. In a particular microstate, let $N_j$ shells occupy level $j$. Taking into account the boundary contribution, $M_0$, the black hole’s total mass becomes expressed in terms of the distribution of shells as

$$M = M_0 + \frac{\hbar}{l} \sum_j \left( j + \frac{1}{2} \right) N_j,$$

and the BTZ solution in (16) is interpreted as an excitation by collapsed shells about some ground state solution. In the following section, we count the number of possible microstates. By requiring agreement between the entropy derived from this point of view and the Bekenstein–Hawking entropy we will then relate $M_0$ to the energy of the ground state.

Consider the canonical ensemble described by the partition function

$$Z(\beta) = \sum_{\{N_1, \ldots, N_j, \ldots\}} g(N_1, \ldots, N_j, \ldots) \exp \left[ -\beta \left( M_0 + \sum_j \mu_j N_j \right) \right],$$

where $N_j$ represents the number of shells excited to level $j$, with mass $\mu_j$, and $g(N_1, \ldots, N_j, \ldots)$ represents the degeneracy of states. Our strategy will be to compute the canonical entropy

$$S_{\text{can}} = [\beta M + \ln Z(\beta)]_{M=-\partial \ln Z/\partial \beta},$$

where $M$ is the average energy in the canonical ensemble, which we associate with the black-hole mass. Both the canonical and the microcanonical ensembles are well defined for the BTZ black hole [40]. We implement bosonic statistics, $g(N_1, \ldots, N_j, \ldots) = 1$. It is not difficult to see that Boltzmann statistics leads to the wrong dependence of the entropy on the mass, see Appendix B. Again, it is crucial to perform the counting of states in an appropriate way [22].

Interchanging the product with the sum and performing the sum over shells, we obtain

$$Z(\beta \hbar/2l) = e^{-\beta M_0} \prod_{j=0}^{\infty} \left[ 1 - e^{-\frac{\beta \hbar}{2l} (2j+1)} \right]^{-1}. $$

Using the remarkable identity [41]

$$Z_0(\xi) = \prod_{j=1}^{\infty} \left[ 1 - e^{-\xi j} \right]^{-1} = \sqrt{\frac{\xi}{2\pi}} e^{\frac{e^2}{4\xi}} Z_0(4\pi^2/\xi),$$

10
which follows from the Poisson summation formula, we show in Appendix A that
\[ Z(\beta \hbar /2l) = \frac{1}{\sqrt{2}} e^{-\left( \frac{\pi^2}{3} + \frac{\hbar}{3} \right)(\frac{8M_0}{\hbar} - \frac{1}{6})} [Z(4\pi^2 l/\beta \hbar)]^{-1}, \quad (38) \]
thus connecting the high temperature behavior of our system to its low temperature dynamics. Since the temperature of the BTZ black hole becomes large in the semi-classical limit of large mass (this is different from the four-dimensional case), we will be interested in the high-temperature limit, \( \beta \to 0 \); we must, therefore, determine the partition function \( Z(4\pi^2 l/\beta \hbar) \) in the low-temperature limit, that is, in the limit of infinite argument. The result depends on what we take to be the lowest energy state. If we take it to be the \( M = 0 \) state, described by the metric
\[ ds^2 = -\frac{R^2}{l^2}dT^2 + \frac{l^2}{R^2}dR^2 + R^2 d\varphi^2, \quad (39) \]
then we get \( Z(4\pi^2 l/\beta \hbar) \to 1 \) in the limit of infinite argument. From (38) we then find the desired result for large temperature,
\[ \ln Z(\beta \hbar /2l) \approx \frac{\pi^2 l}{\beta \hbar} \left( \frac{1}{6} - \frac{8M_0}{\hbar} \right). \quad (40) \]
Therefore, when the temperature is large, the average energy of the system is
\[ M = -\frac{\partial \ln Z}{\partial \beta} = \frac{\pi^2 l}{\beta^2 \hbar} \left( \frac{1}{6} - \frac{8M_0}{\hbar} \right). \quad (41) \]
and we must require that \( M_0 \leq \hbar/48l = \hbar \sqrt{\Lambda}/48 \) for a positive mass. When this condition on \( M_0 \) holds, inserting
\[ \beta \approx \pi \sqrt{\frac{l}{hM} \left( \frac{1}{6} - \frac{8M_0}{\hbar} \right)} \quad (42) \]
into (35) yields the canonical entropy
\[ S_{\text{can}} \approx 2\pi \sqrt{\left( 1 - \frac{48M_0}{\hbar} \right) \frac{1M}{6\hbar}}. \quad (43) \]
Using \( R_h = l\sqrt{8GM} \) for the horizon of the BTZ black hole, the Bekenstein–Hawking entropy in (1),
\[ S_{\text{B-H}} = \frac{\pi R_h}{2G\hbar} = \frac{\pi l}{\hbar} \sqrt{\frac{2M}{G}}, \quad (44) \]
coincides with the canonical entropy provided that \( M_0 \) takes the value
\[ M_0 = -\frac{1}{16G} + \frac{\hbar}{48l}. \quad (45) \]
This is compatible with our requirement that \( M_0 \leq \hbar/48l \). In the semi-classical regime the cosmological constant must be small in Planck units, \( l \gg G \); therefore, \( M_0 \) is negative but larger than \(-1/16G\).
More generally, one would assign the value $\Delta_0$ to the energy of the ground state; then, as $\beta \to 0$,
\[
Z(4\pi^2 l/\beta\hbar) \approx e^{-\frac{8\pi^2 l^2 \Delta_0}{\beta\hbar}}
\] (46)
and therefore, using (38),
\[
\ln Z(\beta/2l\hbar) \approx \pi^2 l \left[ \frac{1}{6} - \frac{8l}{\hbar} (M_0 - \Delta_0) \right].
\] (47)
All the previous formulae apply after shifting $M_0$ to $M_{\text{eff}} = M_0 - \Delta_0$. Thus the energy of the vacuum solution is determined by the value of $M_0$ according to
\[
\Delta_0 = M_0 + \frac{1}{16G} - \frac{\hbar}{48l}.
\] (48)
For example,
\[
M_0 = -\frac{3}{16G} + \frac{\hbar}{48l}
\] (49)
gives $\Delta_0 = -1/8G$, which corresponds to the choice of pure AdS$_3$,
\[
ds^2 = -\left( \frac{R^2}{l^2} + 1 \right) dT^2 + \left( \frac{R^2}{l^2} + 1 \right)^{-1} dR^2 + R^2 d\varphi^2,
\] (50)
for a ground state [24]. For $-1/8G < \Delta_0 < 0$, (16) describes naked singularities. Even so, except for the fact that they are unstable, there seems to be no impediment to choosing such solutions for the ground state.

V. DISCUSSION

Our approach to defining and computing the number of microstates of the BTZ black hole has been significantly different from those taken so far. While most approaches use the AdS/CFT correspondence, we apply only standard canonical techniques and the canonical variables that were adapted to the problem of gravitational collapse in [32]. The advantage of our approach is that the microstates have a transparent physical meaning. As such, we can identify the states being counted and make a connection with approaches that use CFT. In this section we will summarize our results and make this connection more precise.

The black hole is viewed as a stationary state made up of collapsed shells. We obtained the energy levels available to the shells and found that these are the same as the levels expected from an asymptotic CFT. These energy levels were obtained subject to a physical requirement concerning the factor ordering ambiguities that plague the canonical approach, that is, we required the wave equation to be the free (Klein–Gordon) equation on the configuration space determined by the Hamiltonian constraint. Because the matter density describing a black hole is distributional we know of no a priori determination of the factor ordering. This means that other factor orderings are possible, and therefore other energy spectra. We do not know how, or even if, different factor orderings would be connected. This problem does not arise for genuine collapse scenarios, where the matter distribution is
taken to be smooth over some interval of the ADM label coordinate, \( r \). There, the diffeomorphism constraint uniquely fixed the factor ordering as well as the measure on the Hilbert space of states. It is therefore possible that, once the quantum collapse process is more fully understood, this uniqueness will carry over to a unique description of the end state. This seems to be a worthy direction for future research.

The actual microstates of the black hole consist of distinguishable distributions of matter shells amongst the energy levels we have determined. The question of where the states lie, whether near the horizon or at infinity, is a misleading one in quantum mechanics. The canonical theory only distinguishes between two regions, the interior and the exterior, and the microstates of our model are shown to live in the interior of the hole, where the Wheeler–DeWitt equation is hyperbolic. The same picture holds for the Schwarzschild black hole in 3+1 dimensions, as argued in [31]. However, there is a key difference between the BTZ black hole and its Schwarzschild counterpart. The BTZ black hole has positive specific heat, whereas the Schwarzschild black hole has negative specific heat. Thus the temperature of the BTZ black hole increases with its mass and the semi-classical approximation is also the high temperature limit. The Schwarzschild black hole, on the other hand, has negative specific heat, implying that the semi-classical approximation is in fact the low temperature limit.

Bose statistics were essential to correctly counting the number of these microstates, cf. [22]. We exploited a well known property of a certain partition function to obtain the number of states in the high temperature (large mass) limit. Our methods are no different from those used in CFT [42, 43]. The entropy of the black hole depends on two parameters, viz., the energy, \( \Delta_0 \), of the vacuum solution and a constant, \( M_0 \), arising from the boundary action at the origin of coordinates. Our result in (43) compares directly with the answer obtained via the CFT approach [26],

\[
S_{\text{CFT}} = 4\pi \sqrt{\frac{c_{\text{eff}} LM}{6\hbar}}
\]

for the zero angular momentum case and allows us to identify the quantity

\[
\frac{1}{2} \left[ 1 - \frac{48l}{\hbar}(M_0 - \Delta_0) \right]
\]

with the central charge, \( c_{\text{eff}} \), of the effective CFT being used to describe the hole. To achieve agreement with the Bekenstein–Hawking entropy, we must choose \( M_0 - \Delta_0 \) according to [48]. This leads to

\[
c_{\text{eff}} = \frac{3l}{2G\hbar},
\]

which is the central charge of the Liouville theory induced at spatial infinity by 2+1-dimensional gravity [44]. On the other hand, Boltzmann statistics were employed in [31] to obtain the statistical entropy of the Schwarzschild black hole. It is desirable to have a deeper understanding of the relationship between the statistics and the presence or not of a cosmological constant as well as the dimensionality of the space-time [45]. We leave this important issue for future research.
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APPENDIX A

Here we show the details of the calculation that leads to (38). To do so, we will exploit the fact that the partition function

\[ Z_0(\xi) = \prod_{j=1}^{\infty} \left(1 - e^{-\xi j}\right)^{-1} = (1 - e^{-\xi})^{-1}(1 - e^{-2\xi})^{-1}(1 - e^{-3\xi})^{-1} \ldots \]  

has the following remarkable property, which is obtained using the Poisson summation formula:

\[ Z_0(\xi) = \sqrt{\frac{\xi}{2\pi}} e^{\frac{\pi^2}{\xi}} Z_0(4\pi^2/\xi). \]  

The first step is to write the partition function we are interested in \( (\xi = \beta \hbar/2l) \),

\[ Z(\xi) = e^{-2M_0 \xi/\hbar} \prod_{j=0}^{\infty} \left[1 - e^{-\xi(2j+1)}\right]^{-1} = e^{-2M_0 \xi/\hbar}(1 - e^{-\xi})^{-1}(1 - e^{-3\xi})^{-1}(1 - e^{-5\xi})^{-1} \ldots, \]

in terms of \( Z_0(\xi) \). In fact, it is clear that

\[ Z(\xi) = e^{-2M_0 \xi/\hbar} \frac{Z_0(\xi)}{Z_0(2\xi)}. \]

Now using the property \( (55) \) of \( Z_0(\xi) \), we find

\[ K(\xi) \equiv \frac{Z_0(\xi)}{Z_0(2\xi)} = \sqrt{\frac{\xi}{2\pi}} e^{\frac{\pi^2}{\xi}} \frac{Z_0(4\pi^2/\xi)}{Z_0(2\pi^2/\xi)} \]

\[ = \frac{1}{\sqrt{2}} e^{\frac{\pi^2}{\xi} + \frac{\xi}{2\pi}} [K(\pi^2/\xi)]^{-1}, \]

implying that

\[ Z(\xi) = \frac{1}{\sqrt{2}} e^{-2M_0 \xi/\hbar + \frac{\pi^2}{\xi} + \frac{\xi}{2\pi} - \frac{4\pi^2 M_0}{\xi \hbar}} [Z(\pi^2/\xi)]^{-1}, \]

which is \( (38) \).

APPENDIX B

If the shells obey Boltzmann statistics, then

\[ g(N_1, \ldots, N_j, \ldots) = \frac{N!}{N_1! \ldots N_j! \ldots} \]
gives the number of distinguishable rearrangements of a particular distribution of $N$ shells between the energy levels. In this case,

$$Z(\beta) = \sum_{\{N_1, \ldots, N_j, \ldots\}} \frac{N!}{N_1! \ldots N_j! \ldots} \exp \left[ -\beta \left( M_0 + \sum_j \mu_j N_j \right) \right], \quad (61)$$

and the sum is to be evaluated subject to the requirement that the total number of shells is fixed to be $N$. Thus we find

$$Z(\beta) = 2^{-N} e^{-\beta M_0} \sinh^{-N} \left( \frac{\beta \hbar}{2l} \right). \quad (62)$$

The average energy in the limit as $\beta \to 0$ is easily found to be $M \approx M_0 + N/\beta$. However, the dependence of the inverse temperature on the black hole mass is considerably different from (41); we now find the entropy to be

$$S \approx N + N \ln \left[ \frac{(M - M_0)l}{N\hbar} \right]. \quad (63)$$

Its maximum value of $S \approx (M - M_0)l/\hbar$ is achieved when the number of shells $N \approx (M - M_0)l/\hbar$. This answer can also be obtained directly in the microcanonical ensemble by counting the number of ways in which $Q$ quanta can be distributed among $N$ shells,

$$\Omega(Q, N) = \frac{(N + Q - 1)!}{(N - 1)!Q!}. \quad (64)$$

In the high temperature limit, and keeping in mind that $(M - M_0)l/\hbar = Q + N/2$, we obtain precisely (63). It does not agree with the Bekenstein–Hawking entropy in its dependence on the mass of the black hole.
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