The cognitive process of students understanding quadratic equations

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Abstract. Algebra is mandatory learning in junior high school. That is a continuation of arithmetic with the bridge being variable. Students often have difficulty in understanding it, especially quadratic equations. The purpose of this study is how students' cognitive processes in understanding the principles of quadratic equations. It was an exploratory study conducted on research subjects. The subjects were junior high school students selected based on their ability to understand concepts and principles about quadratic equations. That's done through tests about understanding quadratic equations. There were 16 selected research subjects. They were interviewed in-depth by researchers using interview guides. Interview data were analyzed qualitatively based on genetic decomposition. The results of this study are that students can use square, rectangular and unit square cards to determine the principle of factorization of quadratic equations. Students can generalize until completing a perfect quadratic equation. The conclusion is that the cognitive process of students who learn to use learning media based on square and rectangular able to reach a high level (trans-level).

1. Introduction
Algebra is mandatory learning in junior high school. It is a continuation of arithmetic with the bridge being variable. Algebra discusses the relationships between variables. Students often have difficulty in understanding it, especially quadratic equations. The results showed inadequate student knowledge about the definition of the concept of quadratic equations, limited student concept pictures and the lack of student knowledge to understand quadratic equations [1]. Students make mistakes in the process of solving quadratic equations. That is a result of their weakness in mastering the prerequisite material, such as algebra, fractions, negative numbers, and algebra expansion [2]. In understanding quadratic equations, students make procedural mistakes. That is about fractions, algebraic processes, and conceptual understanding of algebraic conventions [3]. There are most students experiencing difficulties and errors in solving quadratic equations. The student's mistake was an understanding of algebra, fractions, integers, rules of the solution of quadratic equations, calculation and simplification of algebra [4]. To be able to overcome the difficulties and mistakes
of students in understanding these quadratic equations, an in-depth analysis needs to be done. Therefore, students must know and understand the formal definition of quadratic equations. It is important to understand conceptual relations beyond symbolic calculations in solving quadratic equations [1]. Students make mistakes in solving quadratic equations, especially in the process of carrying out operations to reach solutions and find possible values of substitute variables. They lack understanding of integers, linear equations, and basic mathematical properties [4]. A study revealed that students lack algebraic competence which results in difficulty in determining the solution of a quadratic equation [5]. Therefore, we want to know about how the cognitive processes of students in understanding the principles of quadratic equations?

Tracing the cognitive processes of students about quadratic equations needs to be done, because the findings from the study show that students make mistakes and difficulties in solving mathematical problems involving quadratic equations. The difficulty is in terms of determining the purpose of the problem, given the appropriate mathematical concepts and principles. Also, the difficulty of finding the correct strategy for solving problems related to quadratic equations [6]. Students' basic knowledge about algebra needs to be improved so that they can solve problems related to algebra, such as quadratic equations [7]. The quadratic equation is a polynomial equation that has a second order. The general form is $ax^2 + bx + c = 0$, where $a \neq 0$, $a$, $b$, $c$ are real numbers, and $x$ is a variable. To solve quadratic equations, students can use various methods, such as factoring, quadratic formulas, completing quadratic, geometric and graphical [1].

The teacher needs to give emphasis to students in factorizing quadratic expressions. In a study, there were students who could not apply the correct factorization method, so using the trial and error method. Students still lack understanding of the factorization principle of quadratic equations [7]. In classroom learning the teacher should present typical quadratic examples, as well as various types of quadratic equations. Also, discuss the differences between quadratic examples and non-examples. The teacher shows the importance of connecting polynomials and quadratic equations and suggests that polynomials and quadratic equations should be taught in relation to each other [1]. Therefore, the cognitive schemes of students in understanding quadratic equations can be achieved carefully. It will be stored in long-term memory in the form of a maturity scheme [8,9]. That can be achieved through a cognitive process. The cognitive processes include several stages: first, knowing. That includes facts, procedures, and concepts that students need to know. The next process is applying. It focuses on students' ability to apply knowledge and conceptual understanding to solve problems or answer questions. The third process is reasoning. At this stage, the solution is beyond routine problems. It covers foreign situations, complex contexts and multi-step problems [10].

The cognition is someone's belief about something that is obtained from the process of thinking about someone or something. The process carried out is gaining knowledge and manipulating knowledge through the activities of remembering, analysing, understanding, assessing, reasoning, imagining and speaking [11]. Cognitive processes are stages to associate information received through the senses of the human body with information that has been stored in long-term memory. The information is processed in working memory which functions as a place for processing information. This processing capability is limited by working memory capacity and time factors. The next process is the implementation of the selected actions [12]. Actions taken include cognitive processes and physical processes with members of the human body. Actions can also be in the form of passive actions, i.e. to continue the work that has been done before [13].

To examine cognitive processes, genetic decomposition analysis can be used. The genetic decomposition is a description of the actions, objects, processes, schemes, and their relationships that individuals have for this concept [9,14]. Instruction tries to make students build these actions, processes, objects, and schemes, and arrange them in a coherent scheme to be used in trying to learn mathematics [15-17]. There are students who are at the Trans Level. They can build new schemes through mental activities, namely actions, processes, objects, and retrieval of the previous schema. It can form a mature scheme [8]. It is a cognitive process of students in learning mathematics. Collection of mental activities explicitly identified and
described as a result of task-based interviews of students' thought processes [18]. Students can explore their thinking through mathematical problem-solving. They must start with something close to their mind and culture. That is a culture in mathematics (ie ethnomathematics) [19,20]. Realistic problem-based learning enhances students' ability to understand mathematics [21,22]. Thus, we are interested in knowing the cognitive processes of students in understanding quadratic equations, based on something contextual.

2. Methods
This research is an exploratory research conducted on research subjects. The subject was junior high school students in Bengkulu City, Indonesia. They are chosen based on their ability to understand concepts and principles about quadratic equations. That's done through tests about understanding quadratic equations. The research subjects were 16 selected students. They were interviewed in depth by the researchers by utilizing interview guides. That is a task-based interview. The assignment given to students is to solve quadratic equations. The aim is to find out the students' cognitive processes in discovering the factoring principle. Assignments given to research subjects are:

Assignments for Students:
Using the size of the cardboard, please solve the quadratic equation:

(1) \( x^2 + 3x + 2 = 0 \)
(2) \( 3x^2 + 5x - 2 = 0 \)

Students are asked to make new rectangles based on squares, squares and rectangles according to terms of quadratic equations. Square, rectangle and square units are made using cardboard as shown in Figure 1.

![Figure 1. Square model \( x^2 \) (a), Rectangle \( x \) (b), Unit square 1 (c).](image)

Models of Squares (Figure 1 (a)), Rectangles (Figure 1 (b)) and Unit squares (Figure 1 (c)) are the basic models used by research subjects to think and determine other rectangles that correspond to terms of quadratic equation. All this process is recorded with audio-visual. All mental and kinesthetic activities of the subject can be fully documented. It is then encrypted, reduced and verified until it can be concluded. Collection of these activities is a genetic decomposition [23]. Interview data were analyzed qualitatively based on genetic decomposition.

3. Results and discussion
Research subjects complete their tasks by utilizing models made of cut cardboard. Those are square pieces with area \( x^2 \), rectangles with area \( x \), and unit movements with area 1. By utilizing these models, there are
those who are able to factor the given quadratic equation and some are only to compile a new rectangle, and some are not able at all. Based on the results of the completion of the task, a circular diagram of the ability of students to utilize the model to solve the given quadratic equation can be presented (see Figure 2).

![Figure 2. Presentation based on students' ability to complete assignments.](image)

Based on Figure 2, there are 50% of students are able to complete the task until they are able to generalize the completion of the given quadratic equation. There are 31% of students who are able only in part, and there are 19% of students who are unable to complete assignments. This shows that most students have the ability to complete the task. They are able to use extents models to solve quadratic equations. Therefore, we will analyze further only for students who are able to complete their assignments to be able to generalize. We choose just two subjects which we present here (i.e. AT, and JK).

### 3.1. AT case

AT is able to complete the task for $x^2 + 3x + 2 = 0$ quadratic equations. He can use the model appropriately. Some interview excerpts showed good activity while completing the task. The following is an interview with him (R: Researcher).

R : Please explain the results of your settlement ... for the quadratic equation $x^2 + 3x + 2 = 0$?

AT.01 : Yes ... I used one square, three rectangles, and two square units. (See Figure 3)

![Figure 3. Model selection by AT.](image)
R : What do you do next?
AT.02 : Based on my choice, .... Then I rearrange it to become a new rectangle ...
AT.03 : The new rectangle that I got ... is a rectangle with width (x + 1) and Length is (x + 2) ...
R : Ok ... continue ... your argument ...
AT.04 : I can calculate the square area that I just mentioned ...
R : How wide ..
AT.05 : Alright ... The area is (x + 2) (x + 1) ...
R : What can you say from the results?
AT.06 : ... means \(x^2 + 3x + 2 = (x + 2)(x + 1)\) ... and because \(x^2 + 3x + 2 = 0\), I can conclude that \((x + 2)(x + 1) = 0\) (See Figure 4).

![Figure 4. Student cognitive processes in utilizing models.](image)

R : Fine .... How do you solve it ...
AT.07 : Please help ... it's rather difficult next ...
R : Try to remember again, for example, given 2 real numbers a, b with \(ab = 0\), what is the conclusion?
AT.08 : ... Oh ... yes ... a = 0 or b = 0 ...
R : Try applying it to \((x + 2)(x + 1) = 0\) ...
AT.09 : That means ... \((x + 2) = 0\) or \((x + 1) = 0\) ...
R : How to solve it?
AT.10 : By using linear equation solving ... i.e. \(x = -2\) or \(x = -1\) ... (See Figure 5).
Based on interviews with AT and referring to Figure 3 through Figure 5, genetic decomposition analysis was applied. The result is AT performs the following actions. He performed kinesthetic performance, making squares and rectangles with cardboard, one square with area x2, 3 rectangles with width 1 and length x, 2 units square. AT carries out interiorization as a process whose stages that are AT arranges rectangular and rectangular parts into new rectangles. The subject produces a rectangle with width \((x + 1)\) and length \((x + 2)\). He did the encapsulation to produce an object. That is a rectangular area of size \((x + 1)(x + 2)\). He stated that \(x^2 + 3x + 2 = (x + 1)(x + 2)\). The object is \(x^2 + 3x + 2 = 0\) then \((x + 1)(x + 2) = 0\). AT is given a trigger so that it is able to reach a mature scheme. He recalled the previous scheme of multiplying two real numbers \(a, b\) with the statement "If \(ab = 0\) then \(a = 0\) or \(b = 0\)." Finally, the subject concluded that \((x + 1)(x + 2) = 0\), where \(x + 1\) and \(x + 2\) are factors of \(x^2 + 3x + 2 = 0\), which means \(x + 1 = 0\) or \(x + 2 = 0\). He is also able to recall linear equations, and the subject states that \(x = -1\) or \(x = -2\).

AT subject performs cognitive processes in understanding the principles of quadratic equations. The initial process he did start from the action by utilizing a model of a square and rectangular area. Then, the subject arranges another rectangle based on action, so that the object is obtained as factors of quadratic equations. That is the product of length and width. He obtained a mature scheme. That is about the principle of factoring quadratic equations.

3.2. JK case
JK has a good ability to solve quadratic equations \(3x^2 + 5x - 2 = 0\). He also solved them according to orders. JK makes good use of square and rectangular models. JK's cognitive process is clearly illustrated through interviews and paper and pencil completion. The following are excerpts of the interview.

R : Please explain the completion of your task?
JK.01 : I chose three squares of \(x^2\) each ... that is to represent \(3x^2\) (see Figure 6).
Figure 6. Three squares of $3x^2$ area.

R : What next?
JK.02 : I also took five rectangles whose width was each $x$, so that they could represent $5x$ ...
JK.03 : I arranged it together with $3x2$ so I got a new flat shape which was $3x2 + 5x$ wide ... (See Figure 7).

Figure 7. Three square and five square lengths with area $3x^2 + 5x$.

JK.04 : Three squares and five rectangles with an area of $3x^2 + 5x$ have not formed a rectangle...
JK.05 : To form a new rectangle, I add one rectangle with an area of $x$ and by reducing it to that size on an area of $3x^2 + 5x$, I also reduce it by a unit square of $2$.
R : OK ...
JK.06 : I got the new rectangle with an area of $3x^2 + 5x - 2$.
JK.07 : The area of the rectangle is $(x + 2) (3x-1)$, meaning that $3x^2 + 5x - 2 = (x + 2) (3x-1)$ ... (see Figure 8).
R : What next ...
JK.08 : Because $3x^2 + 5x - 2 = 0$, means that $(x + 2) (3x-1) = 0$. 
Figure 8. Rectangular area of $3x^2+5x-2$.

JK.09: Based on a quadratic equation $3x^2 + 5x - 2 = 0$, then $(x + 2) (3x-1) = 0$. ... by using the multiplication of two real numbers equal to zero, then $(x + 2) = 0$ or $(3x-1) = 0$ ...

JK.10: I wrote on paper that $(x + 2) = 0$ or $(3x-1) = 0$ ... which means $x = -2$ or $3x = 1$, it shows that $x = -2$ or $x = 1/3$ ... and finished (See Figure 9).

Figure 9. The solution of $3x^2 + 5x - 2$ in paper and pencil.

Figure 6-9 is a cognitive process of students that can be seen visually. JK performs physical activity through the preparation of square displays by utilizing square, rectangle, and paper and pencil models. His mental activity was also illustrated through interview footage. Genetic decomposition analysis is applied to uncover
the cognitive processes of students in completing quadratic equations. JK did the action of choosing 3 squares with an area of $x^2$ each, five square lengths with each area of $x$, and 2 identical white squares by covering an area of 2 (or say -2). Interiorization is carried out by Sinswa as a process of compiling a new rectangle. The rectangle has a width $(x + 2)$ and a length $(3x-1)$. JK reaches the stage of encapsulation as an object about the area of the rectangle whose size is $(x + 2)(3x-1)$. JK made a statement that $3x^2 + 5x-2 = (x + 2)(3x-1)$. He reaches an object that is, because $x^2 + 3x + 2 = 0$, then $(x + 2)(3x-1) = 0$. JK is able to conclude that if $(x + 2)(3x-1) = 0$, then $x + 2$ and $3x-1$ are factors of the quadratic equation. Therefore, $x + 2 = 0$ or $3x-1 = 0$. JK is able to solve it by saying that $x = -2$ or $x = 1/3$.

JK is a subject that has a good cognitive process. That is a process of understanding quadratic equations. In line with AT, JK also began the process with an action that utilizes square and rectangular models. He arranged a new rectangle through actions, processes and objects. Encapsulation is a factorization of a quadratic equation. Square area is the product of the quadratic equation factors. That is the product of the length and width of the rectangle formed. JK reached a mature scheme. He reaches the principle of factoring quadratic equations correctly.

The results of this study are in line with Widada et al. that students are able to do process activities about the properties of functions at a certain interval. He is able to encapsulate into an object and the thematization reaches a mature scheme [9]. It is students who reach the trans level [16,17]. A student who experiences an imbalance by a problem situation will try to re-synchronize through assimilation of the situation to the existing schema and available to him. That can be done through four activities namely action, process, object, and scheme [15]. Trans Level students build a structure that was developed at the previous level and provides a coherent scheme in the scope of the scheme or not. To understand the concept of function, students at that level can build various systems of transformation of functions together with operations in a mathematical structure [23]. Thus, students' cognitive processes in understanding the principle of quadratic equations can be traced through their genetic decomposition. That is the mental construction of the action-process-object-scheme. Students use the cardboard area model to achieve an understanding of factorization of quadratic equations.

4. Conclusions
The conclusion of this research is the cognitive process of students in understanding the principles of quadratic equations, starting from the action by utilizing the area of a square and rectangle, by the process of arranging other rectangles based on action, so that the objects obtained are the factors of quadratic equations. That is the product of length and width. Finally, we get a scheme about the principle of factoring quadratic equations. Suggestion: Mathematics teachers are expected to be able to explore students' cognitive processes before conveying mathematical concepts/principles. That is to develop meaningful learning plans.

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