COMPLEX B-SPLINE COLLOCATION METHOD FOR SOLVING WEAKLY SINGULAR VOLterra INTEGRAL EQUATIONS OF THE SECOND KIND

M. RAMEZANI, H. JAFARI, S.J. JOHNSTON, AND D. BALEANU

Received 18 December, 2014

Abstract. In this paper we propose a new collocation type method for solving Volterra integral equations of the second kind with weakly singular kernels. In this method we use the complex B-spline basics in collocation method for solving Volterra integral. We compare the results obtained by this method with exact solution. A few numerical examples are presented to demonstrate the effectiveness of the proposed method.

2010 Mathematics Subject Classification: 65R20; 45J05
Keywords: Volterra integral equation, complex B-spline, collocation method

1. INTRODUCTION

In this paper we consider the Volterra integral equation with the second kind weakly singular kernel, namely

\[ u(t) = f(t) + \int_0^t k(t,s)u(s)\,ds, \quad t \in (a,b], \tag{1.1} \]

where \( k(t,s) \) and \( f(t) \) are known and \( u(t) \) is unknown. The function \( k(t,s) \) is called a polar kernel if

\[ k(t,s) = \frac{g(t,s)}{(t-s)^\alpha}, \quad \alpha \in (0,1), \]

where \( g \) is bounded on \( s \), \( g(t,t) \neq 0 \) and for all

\[ t, s \in C[a,b]; g(t,s) \in C([a,b] \times [a,b]). \]

We rewrite the equation (1.1) in the following operator form:

\[ (I - K)u = f. \]

where the operator \( K \) is assumed to be compact on a Banach Space \( X \) to \( X \).

During the past few decades, this equation has been used to study various problems of mathematical chemistry and physics, such as reactions including stereology, heat conduction with mixed boundary conditions [10], crystal growth, electrochemistry,
super fluidity and the radiation of heat [7], electrochemistry, semi-conductors, scattering theory, seismology, heat conduction, metallurgy, fluid flow, chemical reactions and population dynamics [12, 13, 27], the particle transport problems of astrophysics, potential theory and Dirichlet problem, electrostatic and radiative heat transfer problems and in some engineering fields [29, 30], astronomy, optics, computational electromagnetic, quantum mechanics, seismology image processing [1, 6].

In addition a method with exponential order convergence rate has been developed by Riley [23] for Volterra integral equations of the form

\[ u(t) - \int_a^t (t-s)^{p-1} k(t,s)u(s)ds = f(t), \quad a \leq t \leq b, \quad (1.2) \]

where the kernel is also assumed to be weakly singular and the solution \( u \) is generally not differentiable at \( t = a \). In [16], the equation (1.2) has been solved with fractional B-spline basics.

In most of the cases, it is difficult to obtain analytical solution of integral equations, therefore many numerical methods such as collocation method with different basics [2,19,20], orthogonal bases and wavelets [17,21], Galerkin methods have been developed to solve equation (1.1) [4–6, 14].

Recently, many different basic functions have been utilized to estimate the result of integral equations, such as modified quadrature [24], optimal homotopy asymptotic method [15], Tau approximate method [18].

Spline functions are very efficient and useful in signal processing, mathematical and computer graphics [8, 9, 22, 25, 26]. In [3], Blu and Unser gave an extension of B-splines to fractional orders and later in [11], Forster et. al. gave an extension to complex power.

In this paper, we solve equation (1.1) by using complex B-spline to obtain approximate solution. The paper is organized as follows: In Section 2, we recall some basic definitions and theorems of complex B-splines and its properties. Section 3 is devoted to the solution of weakly singular integral equation of second kind using collocation methods with complex B-spline basics. In Section 4, by considering numerical examples reported in our work, the accuracy of the proposed scheme is demonstrated.

2. **Complex B-splines**

In this section we state some definitions and theorems [11, 20] that will be used later in our work.

**Definition 1.** The inner product \( \int f(x)g(x)dx \) between two complex \( L^2 \) functions \( f, g \) is denoted by \( (f,g) \), and the associated Euclidean norm is written as \( \| \cdot \|_2 \).

**Definition 2.** The Riemann zeta function is defined as \( \xi(s) = \sum_{n \geq 1} n^{-s} \) for all real \( s > 1 \).
Definition 3. The basic functions for Schoenberg’s polynomial splines with uniform knots [3, 20] are defined as

\[ \beta^n(x) = \frac{1}{n!} \sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} (x-j)_+^n \quad x \in \mathbb{R}, n \in \mathbb{N}, \]

where

\[ (x-j)_+^n = \begin{cases} (x-j)^n & \text{if } x > j, \\ 0 & \text{if } x \leq j. \end{cases} \]

Definition 4. \( x_+^\zeta \) denotes the truncated power function of complex degree \( \zeta \) with knot zero:

\[ x_+^\zeta = \begin{cases} x^{\Re \zeta} e^{i \Im \zeta \ln x}, & x > 0, \\ 0, & \text{elsewhere.} \end{cases} \]

Definition 5. The complex \( B \)-spline \( \beta^\zeta \) of complex degree \( \zeta \) is defined in \( L^2(\mathbb{R}) \) via its Plancherel transform as

\[ \hat{\beta}^\zeta(\omega) = \left( \frac{1-e^{-i\omega}}{i\omega} \right)^{z+1}, \quad (2.1) \]

where \( \zeta = \alpha + i \gamma \) with parameters \( \alpha, \gamma \in \mathbb{R} \) and \( \alpha > -\frac{1}{2} \).

Theorem 1 (cf. [11]). The complex \( B \)-spline \( \beta^\zeta \) is well-defined, uniformly continuous and belongs to the space \( L^2(\mathbb{R}) \).

Theorem 2 (cf. [11]). The time domain representation of the complex \( B \)-spline \( \beta^\zeta \) is given by

\[ \beta^\zeta(x) = \frac{1}{\Gamma(z+1)} \sum_{k \geq 0} (-1)^k \binom{z+1}{k} (x-k)_+^{\zeta}. \quad (2.2) \]

This equation is valid pointwise for all \( x \in \mathbb{R} \) and \( L^2(\mathbb{R}) \).

The complex \( B \)-splines generate dyadic multiresolution analysis; i.e. they generate a sequence of spaces:

\[ \{0\} \subset \ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots \subset L^2(\mathbb{R}) \]

with the following properties:

1. \( \cap_i V_i = \{0\} \) and \( \cup_i V_i = L^2(\mathbb{R}) \).
2. \( f(\bullet) \in V_i \) if and only if \( f(2^{-i} \bullet) \in V_0 \).
3. \( f(\bullet) \in V_0 \) if and only if \( f(\bullet - k) \in V_0 \) for all \( k \in \mathbb{Z} \).
4. there exists a function \( \varphi \in V_0 \), called a scaling function, such that \( \varphi(\bullet - k)_{k \in \mathbb{Z}} \) forms an orthonormal basis of \( V_0 \).

\( V_i \) is the complex \( B \)-spline of order \( \zeta \in \mathbb{C} \) with knot points \( k.2^i, k \in \mathbb{Z} \).
Theorem 3 (cf. [11]). Let \( \text{Re}\ z > 0 \). Then the spaces
\[
V_i = \text{span}\{ \beta^z(\frac{x - 2^i k}{2^i}) \}_{L^2(\mathbb{R})}, \quad i \in \mathbb{Z}
\] (2.3)
form a dyadic multiresolution analysis with scaling function \( \beta^z \).

The complex \( B \)-spline spaces at scale \( a \) is defined as
\[
S^z_a = \{ s_a : \exists c \in \ell^2, s_a(x) = \sum_{k \in \mathbb{Z}} c(k) \beta^z(\frac{x}{a} - k) \}. \quad (2.4)
\]
Then given an arbitrary function \( f \in L^2(\mathbb{R}) \), we determine its least-squares approximation in \( S^z_a \) by applying the following orthogonal projection operator
\[
P_a f = \sum_{k \in \mathbb{Z}} (f, \frac{1}{a} \hat{\beta}^z(\cdot - k)) \beta^z(\frac{\cdot}{a} - k). \quad (2.5)
\]
This defines a projector because the functions \( \beta^z \) and \( \hat{\beta}^z \) are biorthonormal [11], where \( \hat{\beta}^z \in S^z_a \) is the dual \( B \)-spline whose Fourier transform is
\[
\hat{\beta}^z(\bullet) = \frac{\hat{\beta}^z(\bullet)}{\sqrt{\sum_{k \in \mathbb{Z}} |\hat{\beta}^z(\bullet + 2\pi k)|^2}}. \quad (2.6)
\]

Theorem 4 (cf. [28]). The complex \( B \)-splines have a fractional order of approximation \( \alpha + 1 \). Specifically, the least-squares approximation error is bounded by:
\[
\forall f \in W^{\alpha+1}_2, \| f - P_a f \|_2 \leq \sqrt{\frac{2\pi (\alpha + 2) - \frac{1}{2}}{\pi^{\alpha+1}}} \| D^{\alpha+1} \|_2 a^{\alpha+1}. \quad (2.7)
\]

3. THE COMPLEX \( B \)-SPLINE COLLOCATION METHOD

To solve approximately the integral equation equation (1.1), we assume that \( K \) is compact on a Banach space \( X \) to \( X \). We choose a finite dimensional family of functions \( \tilde{u}(x) \) which is close to the exact solution \( u(x) \). In practice, we choose a sequence of dimensional subspaces \( X_n \subset X, n \geq 1 \), with \( X_n \) having dimension \( d_n \). Let \( X_n \) have a basis \( \{ \varphi_1, \ldots, \varphi_d \} \) with \( d = d_n \) for notational simplicity. We seek \( u_n(x) \in X_n \), which can be written as
\[
u_n(x) = \sum_{j=0}^d c_j \varphi_j(x), \quad x \in D. \quad (3.1)
\]
This is substituted into equation (1.1), and coefficients \( \{ c_1, \ldots, c_d \} \) are determined by forcing the equation to be exact in some sense. For later use, we introduce
\[
r_n(x) = u_n(x) - \int_D k(x, s) u_n(s) ds - f(x),
\]
We pick distinct node \(x_1, \ldots, x_d \in D\), and require
\[
r_n(x_i) = 0, \quad i = 1, \ldots, d.
\]  
This leads to determining \(\{c_1, \ldots, c_d\}\) as the solution of the linear system
\[
\sum_{j=1}^{d} c_j \varphi_j(x_i) - \int_D k(x_i, s) \varphi_j(s) ds = f(x_i), \quad i = 1, \ldots, d.
\]  
In following we show that this method can be used to solve equation (1.1). In this regard we give the following Lemma 1 and Theorem 5.

**Lemma 1** (cf. [2]). Let \(X\) be a Banach space and \(P_n\) be a family of bounded projections on \(X\) with
\[
P_nu \rightarrow u \quad \text{as} \quad n \rightarrow \infty, u \in X
\]
and \(K : X \rightarrow X\) be compact. Then
\[
\|K - P_n K\| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

**Theorem 5** (cf. [11]). If \(G \in \mathbb{R}\) be an integral equation with a weakly singular kernel then it is a compact operator on \(C(G)\), where \(C(G)\) is space of continuous real or complex valued functions on compact subsets \(G \in \mathbb{R}\).

**Theorem 6.** Equation (1.1) can be solved with collocation method by using complex B-spline basis.

**Proof.** If we introduce equation (3.1) to projection operator \(P_n\) that maps \(X\) onto \(X_n\), define \(P_n u(x)\) to be that element of \(X_n\) that interpolates \(X\) at the nodes \(\{x_1, \ldots, x_d\}\). This means writing
\[
P_n u(x) = \sum_{j=1}^{d} c_j \varphi_j(x)
\]
with the coefficients \(\{c_j\}\) determined by solving the linear system
\[
\sum_{j=1}^{d} c_j \varphi_j(x_i) = u(x_i), \quad i = 1, \ldots, d
\]
Then this linear system has a unique solution if
\[
det[\varphi_j(x_i)] \neq 0.
\]
From Theorem 1, complex B-spline basis belong to \(L^2(\mathbb{R})\) and with the help Theorem 2 this method is convergent. Then in view of Lemma 1 and Theorem 5 we can use collocation method for these type of integral equations. \(\square\)
Now we can use collocation methods for solving weakly singular integral equation of second kind with complex B-spline basis.

In equation (1.1), let $X = L^2(\mathbb{R})$ and $V_n = X_n$. Then if $u(x) \in L^2(\mathbb{R})$ and $u_n(x) \in V_n$, where

$$u_n(x) = \sum_{j \in \mathbb{Z}} c(j) \beta^2(2^n x - j), \quad j \in \mathbb{Z},$$

with $0 \leq t \leq b$ and $n \in \mathbb{N}$ then we have

$$u_n^{2n}(x) = \sum_{j=1-2^n}^{b} c(j) \beta^2(2^n x - j), \quad b \in \mathbb{R}, \quad (3.5)$$

with nodes $x_i = \frac{b}{2^n}$. Then

$$r_n^{2n}(x_i) = \sum_{j=1-2^n}^{b} c_j \{u(x_i) - \int_{D} k(x_i,s)u(s)ds\} - f(x_i) = 0 \quad i = 0, \ldots, b. \quad (3.6)$$

We define the absolute error

$$E_n^{2n}(u(x)) = \|u(x) - u_n^{2n}(x)\|_2 = \left(\int_{0}^{b} |u(x) - u_n^{2n}(x)|^2 dx\right)^{\frac{1}{2}}. \quad (3.7)$$

and note that when $n \to \infty$ and $d \to \infty$ then $u_n^{2n}(x) \to u(x)$.

Using Theorem 1 and Lemma 1, the relatively error is defined as

$$e_n = \max_{0 \leq i \leq 2n} \left| \frac{E_n^{2n}(u(x_i))}{\max_{0 \leq x \leq b} |u(x)|} \right|. \quad (3.8)$$

4. ILLUSTRATIVE EXAMPLES

In order to show better the theoretical results of the previous sections, we now consider the numerical solution of the equation (1.1), with various choices of $f(x)$ for $x \in [0, 1] = D$. By using equation (3.6), we obtain $\{c_1, \ldots, c_d\}$. Then in view of (3.7) and (3.8) at several points of interval $D$ we obtain the absolute and the relative errors.

**Example 1.** Let $b = 1$, $g(t,s) = ts$ and $f(x) = x(1-x) + \frac{16}{105}x^2(7 - 6x)$ with the exact solution $u(x) = x(1-x)$. Table 1 shows the absolute errors obtained by the knot points $x_i = \frac{i}{n}; i = 0, \ldots, 2^n$ with $z = 0.5 + i$.

In Figure 1, the horizontal axis represents the $n$ index’s $V_n$ the vertical axis represents the relative error $e_n$ intentional, as can be seen, by increasing the index of $n$, the relative error decreases.

From Table 1 we see that the maximum error occurs at point $x = 1$. We now show the relative error for different interval of $z$ this point $(x = 1)$ in Tables 3, 4 and 5.
TABLE 1. Absolute Errors

| $x$ | $E_0^{20}(u(x))$ | $E_1^{21}(u(x))$ | $E_2^{22}(u(x))$ | $E_3^{23}(u(x))$ | $E_4^{24}(u(x))$ |
|-----|------------------|------------------|------------------|------------------|------------------|
| 0   | 0                | 0                | 0                | 0                | 0                |
| 0.25| 0.180375         | 0.119014         | 0.0013166        | 0.0008720        | 0.0005532        |
| 0.5 | 0.1676936        | 0.126606         | 0.019329         | 0.013126         | 0.00670011       |
| 0.75| 0.1676936        | 0.126606         | 0.019329         | 0.0131267        | 0.00670011       |
| 1   | 0.0720055        | 0.0578148        | 0.0425467        | 0.0263254        | 0.0111375        |

TABLE 2. Relatively Errors

| $n$ | 0 | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|---|
| $e_n$ | 0.837556 | 0.506424 | 0.1701868 | 0.10530168 | 0.04455 |

![Figure 1. Points relative error in spaces $V_n$.](image)

TABLE 3. The relative error for $|z| < 1$

| $z$ | 0.1+0.01i | 0.5+0.1i | 0.2+0.9i |
|-----|-----------|----------|----------|
| $E_4^{2}\bar{u}(x)$ | 0.00913271 | 0.00301154 | 0.01920683 |

TABLE 4. The relative error for $1 \leq |z| < 2$

| $z$ | 1+i | 1+0.5i | 1+0.9i |
|-----|-----|-------|-------|
| $E_4^{2}\bar{u}(x)$ | 0.00745215 | 0.00248121 | 0.00842375 |
We note that when \( 1 \leq |z| < 2 \), the error is minimum.

**Example 2.** Let \( b = 1 \), \( g(x,s) = 1 \) and \( f(x) = \frac{1}{2} \pi x + \sqrt{x} \) with the exact solution \( u(x) = \sqrt{x} \) and \( z = 0.5 + 0.5i \). Table 6 shows the absolute error obtained by the knot points \( x_i = \frac{i}{2n} \), \( i = 0, ..., 2^n \) with \( z = 0.5 + 0.5i \).

**Table 6. The absolute errors**

| x    | \( E_0^2(u(x)) \) | \( E_1^2(u(x)) \) | \( E_2^2(u(x)) \) | \( E_3^2(u(x)) \) | \( E_4^2(u(x)) \) |
|------|-------------------|-------------------|-------------------|-------------------|-------------------|
| 0    | 0                 | 0                 | 0                 | 0                 | 0                 |
| 0.25 | 0.290563          | 0.126959          | 0.0413826         | 0.0199285         | 0.000803727       |
| 0.5  | 0.168809          | 0.0702198         | 0.0333131         | 0.0130709         | 0.0044490         |
| 0.75 | 0.0356813         | 0.0508203         | 0.00297352        | 0.00907127        | 0.00297352        |
| 1    | 0.1156            | 0.0540076         | 0.0205252         | 0.0067599         | 0.00221725        |

**Table 7. The relatively error**

| \( m \) | 0 | 1 | 2 | 3 | 4 |
|--------|---|---|---|---|---|
| \( e_m \) | 0.290563 | 0.1269598 | 0.0413826 | 0.0199285 | 0.00803727 |

In Figure 2, the horizontal axis represents the \( n \) index’s \( V_n \) and vertical axis represents the relative error \( e_n \) is intentional, as can be seen by increasing the index of \( n \) relative error decreases.

From Table 6 we see that the maximum error occurs at point \( x = 0.5 \). We now show the relative error for different interval of \( z \) this point \( (x = 0.5) \) in Tables 8, 9 and 10.

**Table 8. The relative error for \( |z| < 1 \)**

| \( z \) | \( 0.01+0.1i \) | \( 0.5+0.01i \) | \( 0.9+0.1i \) |
|--------|-----------------|-----------------|-----------------|
| \( E_4^2(u(x)) \) | 0.0156468 | 0.000083763 | 0.00115488 |

Here we note that when \( |z| < 1 \), the error is minimum.
FIGURE 2. Points relative error in spaces \( V_n \).

**Table 9. The relative error for \( 1 \leq |z| < 2 \)**

| \( z \)        | 1.0+i   | 0.5+i   | 0.9+i   |
|----------------|---------|---------|---------|
| \( E^2_n(u(x)) \) | 0.00112826 | 0.019533 | 0.00526669 |

**Table 10. The relative error for \( 2 \leq |z| < 3 \)**

| \( z \)        | 1.5+i   | 2.0+i   | 2.5+i   |
|----------------|---------|---------|---------|
| \( E^2_n(u(x)) \) | 0.022596 | 0.0175306 | 0.0224039 |

**Example 3.** Let \( a = -1, b = 1, g(x,s) = 1 \) and \( f(x) = \sqrt{x+1} + \frac{1}{3}(x+1)^{3/2}(3 \ln (x+1)^2 - 16 + \ln(4096)) \) with the exact solution \( u(x) = \sqrt{x+1} \).

Table 11 shows the absolute error obtained by the knot points \( x_i = \frac{i}{2\pi} - 1, \quad i = 0, \ldots, 2^n \) with \( z = 1 + 0.5i \).

**Table 11. The absolute errors**

| \( x \) | \( E^2_n(u(x)) \) | \( E^2_n(u(x)) \) | \( E^2_n(u(x)) \) | \( E^2_n(u(x)) \) | \( E^2_n(u(x)) \) |
|---------|----------------|----------------|----------------|----------------|----------------|
| -1.0    | 0.016592       | 0.0056849      | 0.0026678      | 0.0012071      | 0.00052264     |
| 0       | 0.016534       | 0.0077106      | 0.0034793      | 0.0015144      | 0.00063639     |
| 0.5     | 0.007966       | 0.00063781     | 0.0030579      | 0.0013167      | 0.00060392     |
| 1       | 0.019279       | 0.00068       | 0.0004755      | 0.0002275      | 0.0001357      |
In Figure 3, the horizontal axis represents the $n$ index's $V_n$, axis represents the relative error $e_n$ is intentional, as can be seen by increasing the index of $n$ relative error decreases.

From Table 12 see that the maximum error occurs at point $x = 0.5$. Now show the relative error for different interval of $z$ this point($x = 0.5$) in Tables 13, 14 and 15.

**Table 12. The relative errors**

| $n$ | 0    | 1    | 2    | 3    | 4    |
|-----|------|------|------|------|------|
| $e_n$ | 0.0117325 | 0.0054522 | 0.0024602 | 0.0010708 | 0.00044999 |

We see that when $|z| < 1$, the error is minimum.
Table 15. The relative error for $2 \leq |z| < 3$

| $z$ | $1+i$ | $2+0.5i$ | $2+i$ |
|-----|-------|---------|-------|
| $E_{2}^{*}(u(x))$ | 0.0298086 | 0.0398539 | 0.0419062 |

5. Conclusion

In this paper, we proposed an efficient algorithm for solving Volterra integral equations of second kind with weakly singular kernels by collocation type method. We used complex B-spline basics as basic functions in the collocation method. This approach gives better solution with respect to ordinary B-spline basics function. We presented three numerical examples which demonstrated That our proposal method is very attractive. Mathematica has been used in this paper for computation.

References

[1] V. M. Aleksandrov and E. V. Kovalenko, “Mathematical method in the displacement problem,” Inzh. Zh. Mekh. Tverd. Tela, vol. 2, pp. 77 – 89, 1984.
[2] K. Atkinson, The Numerical Solution of Integral Equations of the Second Kind, ser. Cambridge Monographs on Applied and Computational Mathematics. Cambridge University Press, 1997. doi: 10.1017/CBO9780511626340.
[3] T. Blu and M. Unser, “Quantitative Fourier analysis of approximation techniques: Part I – Interpolators and projectors,” IEEE Transactions on Signal Processing, vol. 47, no. 10, pp. 2783–2795, October 1999.
[4] H. Brunner, “The numerical solution of weakly singular Volterra integral equations by collocation on graded meshes,” Mathematics of Computation, vol. 45, no. 172, pp. 417–437, 1985, doi: 10.1090/S0025-5718-1985-0804933-3.
[5] H. Brunner, Collocation methods for Volterra integral and Related Functional Differential Equations, ser. Cambridge Monographs on Applied and Computational Mathematics. Cambridge University Press, 2004.
[6] H. Brunner, “On the numerical solution of first-kind Volterra integral equations with highly oscillatory kernels,” Isaac Newton Institute, HOP, pp. 13–17, 2010.
[7] Z. Chen and W. Jiang, “Piecewise homotopy perturbation method for solving linear and nonlinear weakly singular VIE of second kind,” Applied Mathematics and Computation, vol. 217, no. 19, pp. 7790 – 7798, 2011, doi: 10.1016/j.amc.2011.02.086.
[8] C. K. Chui, Multivariate Splines, ser. CBMS-NSF Regional Conference Series in Applied Mathematics. Society for Industrial and Applied Mathematics, 1988, vol. 2, doi: 10.1137/1.9781611970173.
[9] C. de Boor, K. Höllig, and S. Riemenschneider, Box Splines, ser. Applied Mathematical Sciences. Springer, 1993, vol. 98.
[10] T. Diogo, N. B. Franco, and P. Lima, “High order product integration methods for a Volterra integral equation with logarithmic singular kernel,” Communications on Pure and Applied Analysis, vol. 3, no. 2, pp. 217–235, 2004, doi: 10.3934/cpaa.2004.3.217.
[11] B. Forster, T. Blu, and M. Unser, “Complex B-splines,” Applied and Computational Harmonic Analysis, vol. 20, no. 2, pp. 261 – 282, 2006, Computational Harmonic Analysis – Part 3, doi: 10.1016/j.acha.2005.07.003.
[12] R. Gorenflo, *Computation of rough solutions of Abel integral equations*, ser. Preprint / A: Mathematik. Freie Univ., Fachbereich Mathematik, 1986, vol. 235.

[13] R. Gorenflo and S. Vessella, *Abel Integral Equations*, ser. Lecture Notes in Mathematics. Springer-Verlag, 1991. doi: 10.1007/BFb0084665.

[14] I. G. Graham, “Galerkin methods for second kind integral equations with singularities,” *Mathematics of Computation*, vol. 39, no. 160, pp. 519–533, 1982, doi: 10.1090/S0025-5718-1982-0669644-3.

[15] M. S. Hashmi, N. Khan, and S. Iqbal, “Numerical solutions of weakly singular Volterra integral equations using the optimal homotopy asymptotic method,” *Computers & Mathematics with Applications*, vol. 64, no. 6, pp. 1567 – 1574, 2012, doi: 10.1016/j.camwa.2011.12.084.

[16] H. Jafari, C. M. Khalique, M. Ramezani, and H. Tajadodi, “Numerical solution of fractional differential equations by using fractional B-spline,” *Central European Journal of Physics*, vol. 11, no. 10, pp. 1372–1376, 2013, doi: 10.2478/s11534-013-0222-4.

[17] Z. H. Jiang and W. Schaufelberger, *Block pulse functions and their applications in control systems*, ser. Lecture Notes in Control and Information Sciences. Springer-Verlag, 1992, vol. 179, doi: 10.1007/BFb0009162.

[18] S. Karimi Vanani and F. Soleymani, “Tau approximate solution of weakly singular Volterra integral equations,” *Mathematical and Computer Modelling*, vol. 57, no. 3-4, pp. 494 – 502, 2013, doi: 10.1016/j.mcm.2012.07.004.

[19] R. Kress, *Linear Integral Equations*, ser. Applied Mathematical Sciences. Springer, 1999, vol. 82.

[20] P. Kythe and P. Puri, *Computational Methods for Linear Integral Equations*. Birkhäuser, 2002. doi: 10.1007/978-1-4612-0101-4.

[21] K. Maleknejad and M. Hadizadeh, “A new computational method for Volterra-Fredholm integral equations,” *Computers & Mathematics with Applications*, vol. 37, no. 9, pp. 1 – 8, 1999, doi: 10.1016/S0898-1221(99)00107-8.

[22] G. Nürnberger, *Approximation by spline functions*. Springer-Verlag, 1989. doi: 10.1002/zamm.19920720227.

[23] B. V. Riley, “The numerical solution of Volterra integral equations with nonsmooth solutions based on sinc approximation,” *Applied Numerical Mathematics*, vol. 9, no. 3-5, pp. 249 – 257, 1992, doi: 10.1016/0168-9274(92)90019-A.

[24] J. Saberi Najafi and M. Heidari, “A new modified quadrature method for solving linear weakly singular integral equations,” *World Journal of Modeling and Simulation*, vol. 10, no. 1, pp. 69–78, February 2014.

[25] W. Schempf, *Complex contour integral representation of cardinal spline functions*, ser. Contemporary Mathematics. American Mathematical Society, 1982.

[26] I. Schoenberg, “Contributions to the problem of approximation of equidistant data by analytic functions,” in *I. J. Schoenberg Selected Papers*, ser. Contemporary Mathematicians, C. de Boor, Ed. Birkhäuser Boston, 1988, pp. 3–57, doi: 10.1007/978-1-4899-0433-1.

[27] H. J. J. Te Riele, “Collocation methods for weakly singular second-kind Volterra integral equations with non-smooth solution,” *IMA Journal of Numerical Analysis*, vol. 2, no. 3, pp. 437–449, 1982, doi: 10.1093/imanum/2.4.437.

[28] M. Unser and T. Blu, “Fractional splines and wavelets,” *SIAM Rev.*, vol. 42, no. 1, pp. 43–67, 2000, doi: 10.1137/S0036144598349435.

[29] W. Wang, “Mechanical algorithm for solving the second kind of Volterra integral equation,” *Applied Mathematics and Computation*, vol. 173, no. 2, pp. 1149 – 1162, 2006, doi: 10.1016/j.amc.2005.04.060.

[30] A. M. Wazwaz and S. A. Khuri, “A reliable technique for solving the weakly singular second-kind Volterra-type integral equations,” *Applied Mathematics and Computation*, vol. 80, no. 2-3, pp. 287 – 299, 1996, doi: 10.1016/0096-3003(95)00279-0.
Authors’ addresses

M. Ramezani
M. Ramezani, Young Researchers and Elite Club, Parand Branch, Islamic Azad University, Tehran, Iran
E-mail address: mr_63_90@yahoo.com

H. Jafari
Department of Mathematics, University of Mazandaran, Babolsar, Iran
Current address: Department of Mathematical Sciences, University of South Africa, PO Box 392, UNISA 0003, Johannesburg, South Africa
E-mail address: jafarh@unisa.ac.za

S.J. Johnston
S.J. Johnston, Department of Mathematical Sciences, University of South Africa, PO Box 392, UNISA 0003, Johannesburg, South Africa
E-mail address: johnssj@unisa.ac.za

D. Baleanu
D. Baleanu, Department of Mathematics and Computer Science, Çankaya University, Ankara, Turkey
Current address: Institute of Space Sciences, PO Box MG-23, R 76900, Magurele-Bucharest, Romania
E-mail address: dumitru@cankaya.edu.tr