LOCAL DONALDSON-THOMAS INVARIANTS OF BLOWUPS OF SURFACES

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Abstract. Using the degeneration formula for Donaldson-Thomas invariants, we proved a formula for the change of Donaldson-Thomas invariants of local surfaces under blowing up along points.

1. Introduction

Given a smooth projective Calabi-Yau 3-fold $X$, the moduli space of stable sheaves on $X$ has virtual dimension zero. Donaldson and Thomas [D-T] defined the holomorphic Casson invariant of $X$ which essentially counts the number of stable bundles on $X$. However, the moduli space has positive dimension and is singular in general. Making use of virtual cycle technique (see [B-F] and [L-T]), Thomas in [Thomas] showed that one can define a virtual moduli cycle for some $X$ including Calabi-Yau and Fano 3-folds. As a consequence, one can define Donaldson-type invariants of $X$ which are deformation invariant. Donaldson-Thomas invariants provide a new vehicle to study the geometry and other aspects of higher-dimensional varieties. It is important to understand these invariants.

It is well-known [MNOPT1, MNOPT2] that there is a correspondence between Donaldson-Thomas invariants and Gromov-Witten invariants. Both invariants are deformation independent. On the side of Gromov-Witten invariants, Li and Ruan in [L-R] first established the degeneration formula of Gromov-Witten invariants in symplectic geometry. J. Li proved an algebraic geometry version of this degeneration formula. In [Hu3, Hu2], the author studied the change of Gromov-Witten invariants under the blowup. The author [Hu3] also studies the change of local Gromov-Witten invariants of Fano surfaces under the blowup. In the birational geometry of 3-folds, we have blowups and flops which are semistable degenerations. In [HL] the authors studied how Donaldson-Thomas invariants change under the blowup at a point, some flops and extremal transitions.

Local del Pezzo surface used to play an important role in physics. Local de Pezzo surfaces are usually associated to phase transitions in the Kähler moduli space of various string, M-theory, and F-theory compactifications. Non-toric del Pezzo surfaces seem to be related to exotic physics in four, five and...
six dimensions such as nontrivial fixed points of the renormalization group without lagrangian description and strongly interacting noncritical strings. There is also a relation between non-toric del Pezzo surfaces and string junctions in F-theory [KMV, LMW]. Certain problems of physical interest such as counting of BPS states reduce to questions related to topological strings on local del Pezzo surfaces. In this paper, we will use the degeneration formula for Donaldson-Thomas invariants to study the change of the Donaldson-Thomas invariants of the local surface under the blowup.

Let $S$ be a smooth surface and $K_S$ its canonical bundle. Denote by $Y_S = \mathbb{P}(K_S \oplus O)$ the projective bundle completion of the total space of the canonical bundle $K_S$. The Donaldson-Thomas theory of $Y_S$ is well-defined in every rank. Let $\gamma_i \in H^*(Y_S)$, $i = 1, \cdots, r$. Denote by $\tilde{\tau}_k(\gamma_i)$ the associated descendent fields in Donaldson-Thomas theory, which is defined in [MNOP2]. For $\beta \in H_2(Y_S, \mathbb{Z})$ and an integer $n \in \mathbb{Z}$, denote by $\langle \tilde{\tau}_k(\gamma_1), \cdots, \tilde{\tau}_k(\gamma_r) \rangle^{Y_S}_n, \beta$ the descendent Donaldson-Thomas invariant of $Y_S$.

Denote by $\tilde{S}$ the blown-up surface of $S$ at a smooth point $p_0 \in S$. Let $\beta \in H_2(S, \mathbb{Z})$ and $p!(\beta) = PDp^*PD(\beta) \in H_2(\tilde{S}, \mathbb{Z})$. In [Hu3], we use the degeneration formula to study the change of local Gromov-Witten invariants under the blowup of the Fano surfaces. Similarly, we observed that the Donaldson-Thomas invariants of $Y_S$ of degree $\beta$ is equal to the Donaldson-Thomas invariants of $\tilde{Y}_S$ of degree $p!(\beta)$.

We can find a sequence of birational threefolds all of whose invariants are equal. In fact, the birational threefolds are the projective completion $Y_S$ of $K_S$, the blow-up $\tilde{Y}_S$ of $Y_S$ along the fiber over $p_0$, the projective completion $Y_{\tilde{S}}$ of $\tilde{K}_{\tilde{S}}$ and $Z$, a threefold dominating the last two, obtained by blowing them up along a specific section of the exceptional divisor in $\tilde{S}$. For each pair of spaces, a degeneration is constructed with the goal of comparing absolute invariants of one with relative invariants of the other. Then we prove that the virtual dimension of one of the moduli spaces of relative stable maps appearing in the degeneration formula is negative as soon as there are nontrivial contacts with the relative divisors. Next a second application of the degeneration formula compares such relative invariants with the absolute invariants of the same space. This sequence of comparing results implies the following theorem:

**Theorem 1.1.** Suppose that $S$ is a smooth surface and $\tilde{S}$ is the blown-up surface of $S$ at a smooth point $p$. Let $\beta \in H_2(S, \mathbb{Z})$. Then we have

\begin{equation}
Z'_{DT}(S; q)_\beta = Z'_{DT}(\tilde{S}; q)_{p!(\beta)},
\end{equation}

where $p : \tilde{S} \rightarrow S$ is the natural projection of the blowup.

**Remark 1.2.** Theorem 1.1 makes it possible to compute the Donaldson-Thomas invariants of local nontoric del Pezzo surfaces $\tilde{\mathbb{P}}^2_r$, $4 \leq r \leq 8$, from the Donaldson-Thomas invariants of toric del Pezzo surfaces $\mathbb{P}^2_r$, $1 \leq r \leq 3$. 

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2. Preliminaries

In this section, we shall discuss the basic materials on Donaldson-Thomas invariants studied by Maulik, Nekrasov, Okounkov and Pandharipande. For the details, one can consult [D-T, L-R, MNOP1, MNOP2, Thomas].

Let $X$ be a smooth projective 3-fold and $I$ be an ideal sheaf on $X$. Assume the sub-scheme $Y$ defined by $I$ has dimension $\leq 1$. Here $Y$ is allowed to have embedded points on the curve components. Therefore we have the exact sequence

$$0 \rightarrow I \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Y \rightarrow 0.$$ 

The 1-dimensional components, with multiplicities taken into consideration, determine a homology class $[Y] \in H_2(X, \mathbb{Z})$.

Let $I_n(X, \beta)$ denote the moduli space of ideal sheaves $I$ satisfying $\chi(\mathcal{O}_Y) = n$, $[Y] = \beta \in H_2(X, \mathbb{Z})$.

$I_n(X, \beta)$ is projective and is a fine moduli space. From the deformation theory, one can compute the virtual dimension of $I_n(X, \beta)$ to obtain the following result

**Lemma 2.1.** The virtual dimension of $I_n(X, \beta)$, denoted by $\text{vdim}$, equals

$$\int_{\beta} c_1(T_X).$$

Note that the actual dimension of the moduli space $I_n(X, \beta)$ is usually larger than the virtual dimension.

Let $\mathcal{F}$ be the universal family over $I_n(X, \beta) \times X$ and $\pi_i$ be the projection of $I_n(X, \beta) \times X$ to the $i$-th factor. For a cohomology class $\gamma \in H^l(X, \mathbb{Z})$, consider the operator

$$ch_{k+2}(\gamma) : H_*(I_n(X, \beta), \mathbb{Q}) \rightarrow H_{*-2k+2-l}(I_n(X, \beta), \mathbb{Q}),$$

$$ch_{k+2}(\gamma)(\xi) = \pi_{1*}(ch_{k+2}(\mathcal{F}) \cdot \pi_2^*(\gamma) \cap \pi_1^*(\xi)).$$

Descendent fields in Donaldson-Thomas theory are defined in [MNOP2], denoted by $\tilde{\tau}_k(\gamma)$, which correspond to the operations $(-1)^{k+1}ch_{k+2}(\gamma)$. The descendent invariants are defined by

$$\langle \tilde{\tau}_{k_1}(\gamma_{l_1}) \cdots \tilde{\tau}_{k_r}(\gamma_{l_r}) \rangle_{n,\beta} = \int_{[I_n(X, \beta)]^{\text{vir}}} \prod_{i=1}^{r} (-1)^{k_i+1}ch_{k_i+2}(\gamma_{l_i}),$$

where the latter integral is the push-forward to a point of the class

$$(-1)^{k_1+1}ch_{k_1+2}(\gamma_{l_1}) \circ \cdots \circ (-1)^{k_r+1}ch_{k_r+2}(\gamma_{l_r})([I_n(X, \beta)]^{\text{vir}}).$$
The Donaldson-Thomas partition function with descendent insertions is defined by
\[ Z_{DT}(X; q | \prod_{i=1}^{r} \tilde{\tau}_k(\gamma_i)) = \sum_{n \in \mathbb{Z}} < \prod_{i=1}^{r} \tilde{\tau}_k(\gamma_i) >_{n, \beta} q^n. \]

The degree 0 moduli space \( I_n(X, 0) \) is isomorphic to the Hilbert scheme of \( n \) points on \( X \). The degree 0 partition function is \( Z_{DT}(X; q) \).

The reduced partition function is obtained by formally removing the degree 0 contributions,
\[ Z'_{DT}(X; q | \prod_{i=1}^{r} \tilde{\tau}_k(\gamma_i)) = \frac{Z_{DT}(X; q | \prod_{i=1}^{r} \tilde{\tau}_k(\gamma_i))}{Z_{DT}(X; q)_{0}}. \]

Relative Donaldson-Thomas invariants are also defined in [MNOP2]. Let \( S \) be a smooth divisor in \( X \). An ideal sheaf \( I \) is said to be relative to \( S \) if the morphism \( I \otimes O_S \rightarrow O_X \otimes O_S \) is injective. A proper moduli space \( I_n(X/S, \beta) \) of relative ideal sheaves can be constructed by considering the ideal sheaves relative to the expended pair \((X[k], S[k])\). For details, one can read [Li2] and [MNOP2].

Let \( Y \) be the subscheme defined by \( I \). The scheme theoretic intersection \( Y \cap S \) is an element in the Hilbert scheme of points on \( S \) with length \( |Y \cdot S| \).

If we use \( \text{Hilb}(S, k) \) to denote the Hilbert scheme of points of length \( k \) on \( S \), we have a map
\[ \epsilon : I_n(X/S, \beta) \rightarrow \text{Hilb}(S, \beta \cdot [S]). \]

The sheaves of \( \mathbb{C} \) on \( X \) have a basis via the representation of the Heisenberg algebra on the cohomologies of the Hilbert schemes.

Following Nakajima in [Nakajima], let \( \eta \) be a cohomology weighted partition with respect to a basis of \( H^\ast(S, \mathbb{Q}) \). Let \( \eta = \{ \eta_1, \ldots, \eta_s \} \) be a partition whose corresponding cohomology classes are \( \delta_1, \ldots, \delta_s \), let
\[ C_\eta = \frac{1}{\mathfrak{z}(\eta)} P_{\delta_1}[\eta_1] \cdots P_{\delta_s}[\eta_s] \cdot 1 \in H^\ast(\text{Hilb}(S, |\eta|), \mathbb{Q}), \]
where
\[ \mathfrak{z}(\eta) = \prod_i |\eta_i| \cdot |\text{Aut}(\eta)|, \]
and \( |\eta| = \sum_j \eta_j \). The Nakajima basis \( \{ C_\eta \}_{|\eta| = k} \) of the cohomology of \( \text{Hilb}(S, k) \) is the set.

We can choose a basis of \( H^\ast(S) \) so that it is self dual with respect to the Poincaré pairing, i.e., for any \( i, \delta_i^* = \delta_j \) for some \( j \). To each weighted
partition $\eta$, we define the dual partition $\eta^\vee$ such that $\eta^\vee_i = \eta_i$ and the corresponding cohomology class to $\eta^\vee$ is $\delta_i^\ast$. Then we have
\[
\int_{\text{Hilb}(S,k)} C_\eta \cup C_\nu = \frac{(-1)^{k-\ell(\eta)}}{\delta(\eta)} \delta_{\nu,\eta^\vee},
\]
see [Nakajima].

The descendent invariants in the relative Donaldson-Thomason theory are defined by
\[
< \tilde{\tau}_{k_1}(\gamma_{l_1}) \cdots \tilde{\tau}_{k_r}(\gamma_{l_r}) | \eta >_{n, \beta} = \int_{[\text{In}(X/S,\beta)]^{\text{vir}}} \prod_{i=1}^r (-1)^{k_i+1} \text{ch}_{k_i+2}(\gamma_{l_i}) \cap c^*(C_\eta),
\]

Define the associated partition function by
\[
Z_{DT}(X/S; q | \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_{l_i}))_{\beta, \eta} = \sum_{n \in \mathbb{Z}} < \tilde{\tau}_{k_1}(\gamma_{l_1}) | \eta >_{n, \beta} q^n.
\]

The reduced partition function is obtained by formally removing the degree 0 contributions,
\[
Z'_{DT}(X/S; q | \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_{l_i}))_{\beta, \eta} = \frac{Z_{DT}(X/S; q | \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_{l_i}))_{\beta, \eta}}{Z_{DT}(X/S; q)_{0}}.
\]

Since this is the main tool employed in this paper, so in the remaining of the section, we shall review some notations in the degeneration formula, see [Li2] for the details.

Let $\pi: \mathcal{X} \to C$ be a smooth 4-fold over a smooth irreducible curve $C$ with a marked point denoted by 0 such that $\mathcal{X}_t = \pi^{-1}(t) \cong X$ for $t \neq 0$ and $\mathcal{X}_0$ is a union of two smooth 3-folds $X_1$ and $X_2$ intersecting transversely along a smooth surface $S$. We write $\mathcal{X}_0 = X_1 \cup_S X_2$. Assume that $C$ is contractible and $S$ is simply-connected.

Consider the natural maps
\[
i_t: X = \mathcal{X}_t \to \mathcal{X}, \quad i_0: \mathcal{X}_0 \to \mathcal{X},
\]
and the gluing map
\[
g = (j_1, j_2): X_1 \coprod X_2 \to \mathcal{X}_0.
\]

We have
\[
H_2(X) \xrightarrow{i_{0*}} H_2(\mathcal{X}) \xrightarrow{i_{\ast}} H_2(\mathcal{X}_0) \xrightarrow{g_*} H_2(X_1) \oplus H_2(X_2),
\]
where $i_{0*}$ is an isomorphism since there exists a deformation retract from $\mathcal{X}$ to $\mathcal{X}_0$ (see [Clemens]) and $g_*$ is surjective from Mayer-Vietoris sequence. For $\beta \in H_2(X)$, there exist $\beta_1 \in H_2(X_1)$ and $\beta_2 \in H_2(X_2)$ such that
\[
i_t(\beta) = i_{0*}(j_1*(\beta_1) + j_2*(\beta_2)).
\]

For simplicity, we write $\beta = \beta_1 + \beta_2$ instead.
Lemma 2.2. With the assumption as above, given $\beta = \beta_1 + \beta_2$. Let $d = \int_{\beta_1} c_1(X)$ and $d_i = \int_{\beta_i} c_1(X_i)$, $i = 1, 2$. Then

$$d = d_1 + d_2 - 2\int_{\beta_1} [S] - \int_{\beta_1} [S] = \int_{\beta_2} [S].$$

Proof. The formulae come from the adjunction formulae $K_{X_i} = K_{X_i}|_{X_i}$ and $K_{X_i} = (K_X + X_i)|_{X_i}$ for $i = 1, 2$, and $X_1 \cdot (X_1 + X_2) = X_1 \cdot X_0 = 0$. □

Similarly for cohomology, we have the maps

$$H^k(X_\ell) \xrightarrow{i^\ast_\ell} H^k(X) \xrightarrow{j^\ast_\ell} H^k(X_0) \xrightarrow{\tau^\ast} H^k(X_1) \oplus H^k(X_2),$$

where $i^\ast_\ell$ is an isomorphism. Take $\alpha \in H^k(X)$ and let $\alpha(t) = i^\ast_\ell \alpha$.

There is a degeneration formula which takes the form

$$Z'_{DT}(X_\ell; q) = \sum_{\eta} Z'_{DT}(X_1/S; q) \prod_{i=1}^r \tilde{\tau}_0(j_\ell^\ast \gamma_t(0)))_{\beta_1, \eta} \frac{(-1)^{\eta} \cdot \ell(\eta) \cdot \eta}{\eta^{\eta} \cdot \eta^{\eta}} \cdot Z'_{DT}(X_2/S; q) \prod_{i=1}^r \tilde{\tau}_0(j_\ell^\ast \gamma_t(0)))_{\beta_2, \eta} \cdot \eta^{\eta},$$

where the sum is over the splittings $\beta_1 + \beta_2 = \beta$, and cohomology weighted partitions $\eta$. $\gamma_t$'s are cohomology classes on $X$. There is a compatibility condition

$$|\eta| = \beta_1 \cdot [S] = \beta_2 \cdot [S].$$

For details, one can see [Li1, Li2, MNOP2].

3. Projective completion

In this section, we describe how to obtain $Y_S$ from $Y_S$ by the degenerations. This makes it possible to find some relations between the local Donaldson-Thomas invariants of $\tilde{S}$ and $S$.

Let $S$ be a smooth surface and $Y_S = \mathbb{P}(K_S \oplus \mathcal{O})$ the projective completion of its canonical bundle $K_S$. Consider the blowup $p : \tilde{S} \rightarrow S$ of $S$ at a smooth point $p_0$ and denote by $E$ the exceptional divisor in $\tilde{S}$. Since $Y_S$ is the bundle $\mathbb{P}(K_S \oplus \mathcal{O})$ over $S$, one can pull this bundle back to $\tilde{S}$ using the projection $p$. It is easy to see that the pullback bundle is the same thing as blowing up the fiber over $p_0$. Denote by $\tilde{Y}_S$ the blowup of $Y_S$ along the fiber $F_{p_0} \cong \mathbb{P}^1$ over $p_0$, and the exceptional divisor in $\tilde{Y}_S$ is denoted by $D_1 := E \times \mathbb{P}^1 = \mathbb{P}_{p_1}(O \oplus \mathcal{O})$. In $\tilde{Y}_S$, take a section, $\sigma$, corresponding to $O \rightarrow O \oplus K_S$, of the exceptional divisor $D_1$ over $E$ and blow it up. Denote by $Z$ the blown-up manifold, then $Z$ has a natural projection $\pi$ to $\tilde{S}$ given by the composition of the blowup projection $Z \rightarrow \tilde{Y}_S$ and the bundle projection $\tilde{Y}_S \rightarrow \tilde{S}$. It is easy to see that the fiber $\pi^{-1}(E)$ has two normal crossing components: $D_1 \cong F_0$ and $D_2 \cong F_1$ intersecting along a section $\sigma$ with the normal bundle $N_{\sigma|F_0} \cong O$ and $N_{\sigma|F_1} \cong O(-1)$ respectively.
Next, we consider the projective completion $Y_{\tilde{S}}$. Since the restriction $K_{\tilde{S}}|_E$ of the canonical bundle $K_{\tilde{S}}$ to the exceptional divisor $E$ in $\tilde{S}$ is isomorphic to $\mathcal{O}(-1)$, so we can pick up a section, $\sigma_1$, of the restriction of $Y_{\tilde{S}}$ to $E$ satisfying $\sigma_1^2 = -1$. Then we blow this section $\sigma_1$ up, and it is easy to know that the blown-up manifold is $Z$. Here we illustrate the sequence of birational maps by Figure 1.

Let $\mathbb{P}^2_r$ be the blowup of $\mathbb{P}^2$ at $r$ points. Pick one more point $p$ and blow it up, then we obtain $\mathbb{P}^2_{r+1}$ with the map $p : \mathbb{P}^2_{r+1} \to \mathbb{P}^2_r$ and denote by $E$ the exceptional divisor in $\mathbb{P}^2_{r+1}$. It is well-known that for $0 \leq r \leq 3$, $\mathbb{P}^2_r$ is toric, but for $4 \leq r \leq 8$, $\mathbb{P}^2_r$ is non-toric. In [MNOP1], via the localization technique, the authors computed the local Donaldson-Thomas invariants of toric surfaces, in particular, their method is valid for del Pezzo surfaces $\mathbb{P}^2_r$ with $0 \leq r \leq 3$. As opposed to toric del Pezzo surfaces, one can not directly use localization with respect to a torus action because there is no torus action on a generic del Pezzo surface $\mathbb{P}^2_r$, $4 \leq r \leq 8$. Our Theorem 1.1 implies that for some degrees, we could compute the local Donaldson-Thomas invariants of non-toric surfaces $\mathbb{P}^2_r$ with $4 \leq r \leq 8$ from the local Donaldson-Thomas invariants of $\mathbb{P}^2_r$ with $0 \leq r \leq 3$.

4. Main theorems

Using the notation as before, we have
Lemma 4.1. Suppose that $S$ is a smooth surface. Let $\tilde{Y}_S$ be the blowup of $Y_S$ along the fiber over $p_0 \in S$. Then for any $\beta \in H_2(S; \mathbb{Z})$, we have

$$Z_{DT}(Y_S; q)_\beta = Z'_{DT}(\tilde{Y}_S/D_1; q)_{pl(\beta), 0},$$

where $D_1 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}) \cong \mathbb{P}^1 \times \mathbb{P}^1$ is the exceptional divisor in $\tilde{Y}_S$, $p : \tilde{S} \longrightarrow S$ is the blowup of $S$ at $p_0$ and $pl(\beta) = PDp^*PD(\beta)$.

Proof. Let $X$ be the blow up of $Y_S \times \mathbb{C}$ along $F_{p_0} \times \{0\}$, where $F_{p_0}$ is the fiber of $Y_S$ over $p_0$ and let $\pi$ be the natural projection from $X$ to $\mathbb{C}$. It is a semistable degeneration of $Y_S$ with the central fiber $X_0$ being a union of $X_1 = \tilde{Y}_S$ and $X_2 \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}) \cong \mathbb{P}^2 \times \mathbb{P}^1$ with the common divisor $D_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$.

By the degeneration formula (4), we may express the absolute Donaldson-Thomas invariants of $Y_S$ in terms of the relative Donaldson-Thomas invariants of $(X_1, D_1)$ and $(X_2, D_1)$ as follows:

$$Z_{DT}(Y_S; q)_\beta = \sum_{\eta, \beta_1 + \beta_2 = \beta} Z'_{DT}(Y_S/D_1; q)_{\beta_1, \eta}$$

$$\times \frac{(-1)^{|\eta| - \ell(\eta)} \delta(\eta)}{q^{\eta}} Z'_{DT}(X_2/D_1; q)_{\beta_2, \eta'},$$

where the summation runs over the splittings $\beta_1 + \beta_2 = \beta$ and the cohomology weighted partitions $\eta$.

Now we need to compute the summands in the right hand side of the degeneration formula. For this we have the following claim:

Claim: There are only terms with $\beta_2 = 0$.

In fact, if $|\eta| \neq 0$, then $\beta_2 \neq 0$ because $\beta_2 \cdot D_1 = |\eta|$. By Lemma 2.1 we have

$$C_1(X_1) : \beta_1 = vdimI_n(X_1/D_1; \beta_1) = \deg \epsilon_1^*(C_\eta),$$

where $C_1(X_1)$ denotes the first Chern class of $X_1$ and $\epsilon_1 : I_n(X_1/D_1, \beta_1) \rightarrow \text{Hilb}(D_1, |\eta|)$ is the canonical intersection map.

Let $V$ be a complex rank $r$ vector bundle over a complex manifold $M$, and $\pi : \mathbb{P}(V) \longrightarrow M$ be the corresponding projective bundle. Let $\xi_V$ be the first Chern class of the tautological bundle in $\mathbb{P}(V)$. A simple calculation shows

$$C_1(\mathbb{P}(V)) = \pi^*C_1(M) + \pi^*C_1(V) - r\xi_V.$$

Applying (7) to $X_2$, we obtain

$$C_1(X_2) = \pi^*\mathcal{O}_{\mathbb{P}^1}(2) - 3\xi,$$

where $\xi$ is the first Chern class of the tautological bundle in $X_2$. Since the homology class $\beta_2$ may be decomposed into the sum of the base class $\beta_2^{\mathbb{P}^1}$ and the fiber class $\beta_2^f$, so we have

$$C_1(X_2) : \beta_2 = vdimI_n(X_2/D_1, \beta_2)$$

$$= \pi^*\mathcal{O}_{\mathbb{P}^1}(2) \cdot \beta_2^{\mathbb{P}^1} - 3\xi \cdot \beta_2^f \geq 3|\eta|.$$
In the last inequality, we use the fact that $-\xi$ is the infinite section, so $-\xi \cdot \beta_1 = |\eta|$ and $\pi^*\mathcal{O}_{P^1}(2)\beta_1^2 = \mathcal{O}_{P^1}(2) \cdot \pi_*\beta_1^2 \geq 0$. Since $\beta \in H_2(S, \mathbb{Z})$, from (7), we have
\[ c_1(Y_S) \cdot \beta = \text{vdim} I_n(X, \beta) = 0. \]

From Lemma 4.1, we have
\[ C_1(Y_S) \cdot \beta = c_1(X_1) \cdot \beta_1 + C_1(X_2) \cdot \beta_2 - 2|\eta|. \]

Therefore, we obtain
\[ \deg C_\eta + |\eta| > 0. \]

This is a contradiction. Therefore $|\eta| = 0$. So the claim is proved.

Thus $\beta_2 \cdot D_1 = 0$. Since $D_1$ is the hyperplane in $X_2 \cong \mathbb{P}^3$, we must have $\beta_2 = 0$. Also we have $\beta_1 = p(\beta)$.

By the degeneration formula, we have
\[ Z'_{DT}(Y_S; q)_{\beta} = Z'_{DT}(\tilde{Y}_S/D_1; q)p(\beta). \]

This proves the lemma. \hfill \Box

**Lemma 4.2.** Under the assumption of Lemma 4.1, then for $\beta \in H_2(S; \mathbb{Z})$, we have
\[ Z'_{DT}(\tilde{Y}_S; q)p(\beta) = Z'_{DT}(\tilde{Y}_S/D_1; q)p(\beta). \]

**Proof.** Let $\mathcal{X}$ be the blow up of $\tilde{Y}_S \times \mathbb{C}$ along $D_1 \times \{0\}$. Let $\pi : \mathcal{X} \rightarrow \mathbb{C}$ be the natural projection. Thus we get a semi-stable degeneration of $\tilde{Y}_S$ whose central fiber is a union of $X_1 \cong \tilde{Y}_S$ and $X_2 = \mathbb{P}P_1(N_{D_1} \oplus \mathcal{O})$, where the normal bundle of the divisor $D_1$ is $N_{D_1} = \mathcal{O}(-1, -1)$.

By the degeneration formula (1), we may express the absolute Donaldson-Thomas invariants of $\tilde{Y}_S$ in terms of the relative Donaldson-Thomas invariants of $(X_1, D_1)$ and $(X_2, D_1)$ as follows:
\[ Z'_{DT}(\tilde{Y}_S; q)p(\beta) = \sum_{\eta, \beta_1 + \beta_2 = \beta} Z'_{DT}(\tilde{Y}_S/D_1; q)_{\beta_1, \eta} \times \frac{(-1)^{|\eta|-\ell(\eta)}3(\eta)}{q^{|\eta|}} Z'_{DT}(X_2/D_1; q)_{\beta_2, \eta^\vee}, \]

where the summation runs over the splittings $\beta_1 + \beta_2 = \beta$ and the cohomology weighted partitions $\eta$.

Similar to the proof of Lemma 4.1, we need to prove that there are only terms with $\beta_2 = 0$ in the right hand side of the degeneration formula.

In fact, note that $X_2 = \mathbb{P}P_1(N_{D_1} \oplus \mathcal{O})$ and $D_1 = \mathbb{P}P_1(\mathcal{O} \oplus \mathcal{O})$. Denote by $F_{p_0} \cong \mathbb{P}^1$ the fiber of $Y_S$ at the point $p_0$. Applying (7) to $X_2$ and $D_1$, we obtain
\[ C_1(X_2) = \pi^*C_1(D_1) + \pi^*C_1(N_{D_1}) - 2\xi = \pi^*C_1(F_{p_0}) + \pi^*C_1(N_{F_{p_0}}Y_S) - 2\xi_1 + \pi^*C_1(N_{D_1}) - 2\xi, \]

where $\xi_1$ and $\xi$ are the first Chern classes of the tautological bundles in $\mathbb{P}(N_{F_{p_0}}Y_S)$ and $\mathbb{P}(N_{D_1} \oplus \mathcal{O})$ respectively. Here we denote the Chern class
and its pullback by the same symbol. It is well-known that the normal bundle to $D_1$ in $\tilde{Y}_S$ is just the tautological line bundle on $D_1 \cong \mathbb{P}(N_{F_{po}}|_{Y_S})$. Therefore $C_1(N_{D_1}) = \xi_1$. So we have

$$C_1(X_2) = \pi^*C_1(F_{po}) - \xi_1 - 2\xi,$$

where $-\xi$ is the infinite section which has positive intersections with the effective curve classes.

Note that $X_2$ is a projective bundle over $D_1$ with fiber $\mathbb{P}^1$. Let $L$ be the class of a line in the fiber $\mathbb{P}^1$ and $e$ be the class of a line in the fiber $\mathbb{P}^1$ in $D_1 = \mathbb{P}(N_{F_{po}}|_{Y_S})$. Denote by $\beta_2^{F_{po}}$ the homology class of the projection in $F_{po}$ of the curve component. Denote by $\beta_2^L$ the difference of $\beta_2$ and $\beta_2^{F_{po}}$, i.e. $\beta_2^L = \beta_2 - \beta_2^{F_{po}}$. Then it is easy to know $\beta_2^L = aL + be$. Since $(-\xi) \cdot \beta_2 = |\eta|$ and $(-\xi) \cdot \beta_2 = a = |\eta|$. On the other hand, since all curves of class $\beta_2$ come from the curve of class $p!(\beta)$ by the degeneration and the degeneration only happens away from the divisor $D_1$, from $p!(\beta) \cdot D_1 = 0$, we have $D_1 \cdot \beta_2 = 0$. Thus we have $D_1 \cdot \beta_2 = a - b = 0$. Therefore, we have $a = b = |\eta|$. So we have $\beta_2^L = |\eta|(L + e)$. Since $C_1(F_{po}) + C_1(N_{F_{po}}|_{Y_S}) = C_1(F_{po}) \geq 0$, we have

$$C_1(X_2) \cdot \beta_2 \geq 4|\eta|.$$

From Lemma 2.2, we have

$$C_1(\tilde{Y}_S) \cdot p!(\beta) = C_1(X_1) \cdot \beta_1 + C_1(X_2) \cdot \beta_2 - 2|\eta|.$$

Therefore,

$$\deg C_\eta + 2|\eta| > 0.$$

This is a contradiction. Thus $|\eta| = 0$.

Therefore, from the discussion above, we have $\beta_2 = \beta_2^{F_{po}}$. So $C_1(X_2) \cdot \beta_2 = C_1(F_{po}) \cdot \beta_2^{F_{po}}$. Thus $C_1(X_2) \cdot \beta_2 > 0$ if $\beta_2^{F_{po}} \neq 0$. Furthermore, if $\beta_2 = \beta_2^{F_{po}} \neq 0$, then, by definition, we have

$$Z'_{DT}(X_2/D_1; q)_{\beta_2, 0} = 0.$$

Therefore, we have proved that there are only terms with $\beta_2 = 0$ in the right hand side of (S).

By the degeneration formula, we have

$$Z'_{DT}(\tilde{Y}_S/q)_{p!(\beta)} = Z'_{DT}(\tilde{Y}_S/D_1; q)_{p!(\beta), 0}.$$

This proves the lemma.

Summarizing Lemma 4.1 and 4.2, we have

**Theorem 4.3.**

$$Z'_{DT}(\tilde{Y}_S/q)_{\beta} = Z'_{DT}(\tilde{Y}_S/q)_{p!(\beta)}.$$

Next, we want to compare the Donaldson-Thomas invariants $Z'_{DT}(\tilde{Y}_S/q)_{p!(\beta)}$ of $\tilde{Y}_S$ to the Donaldson-Thomas invariants of $Z$. In fact, we have
where $\xi$ is the first Chern class of the tautological line bundle over $\tilde{X}$. It is easy to see that

$$C_1(X_2) \cdot \beta_2 \geq 3|\eta|.$$  

From Lemma 2.2, we have

$$C_1(\tilde{Y}_S) \cdot p!(\beta) = C_1(Z) \cdot \beta_1 + C_1(X_2) \cdot \beta_2 - 2|\eta|.$$  

Therefore,

$$C_1(\tilde{Y}_S) \cdot p!(\beta) \geq \deg C_\eta + |\eta| > 0.$$  

Since $C_1(\tilde{Y}_S) \cdot p!(\beta) = 0$, so this is a contradiction. Thus $|\eta| = 0$.

The same argument as in the proof of Lemma 4.2 shows that $\beta_2 = 0$.

Therefore, by the degeneration formula, we have

$$Z'_{DT}(\tilde{Y}_S; q)_{p!(\beta)} = Z'_{DT}(Z; q)_{p!(\beta)}.$$  

Now it remains to prove

$$Z'_{DT}(Z; q)_{p!(\beta)} = Z''_{DT}(Z/\mathbb{F}_1; q)_{p!(\beta)}.$$  

To prove this, we degenerate $Z$ along the exceptional divisor $\mathbb{F}_1$. Then we obtain two smooth 3-folds

$$X_1 = Z,$$  

$$X_2 = \mathbb{F}_1(\mathbb{N}_1 \oplus \mathcal{O}),$$
intersecting along the exceptional divisor $F_1$ in $Z$ and the infinite section of the $\mathbb{P}^1$-bundle $X_2$.

Applying the degeneration formula to $Z'_{\text{DT}}(Z; q)p!p(\beta)$, we have

$$Z'_{\text{DT}}(Z; q)p!p(\beta) = \sum_{\eta, \beta_1 + \beta_2 = \beta} Z'_{\text{DT}}(Z/F_1; q)_{\beta_1, \eta}$$

$$\times \frac{(-1)^{|\eta| - \ell(\eta)} z(\eta)}{q^{|\eta|}} Z_{\text{DT}}(X_2/F_1; q)_{\beta_2, \eta}$$

where the summation runs over the splittings $\beta_1 + \beta_2 = p!(\beta)$ and the cohomology weighted partitions $\eta$.

Note that $X_2 = \mathbb{P}_{F_1}(N_{F_1} \oplus O)$ and $F_1 = \mathbb{P}_{\sigma}(O(-1) \oplus O)$. Applying (7) to $X_2$ and $F_1$, we obtain

$$C_1(X_2) = \pi^*C_1(F_1) + \pi^*C_1(N_{F_1}) - 2\xi$$

$$= \pi^*C_1(O_{\sigma}(1)) - \xi_1 - 2\xi,$$

where $\xi_1$ and $\xi$ are the first Chern classes of the tautological bundles in $\mathbb{P}_{\sigma}(O(-1) \oplus O)$ and $\mathbb{P}(N_{F_1} \oplus O)$ respectively. Here we denote the Chern class and its pullback by the same symbol. The same calculation as in the proof of Lemma 4.2 shows that

$$C_1(X_2) \cdot \beta_2 \geq 4|\eta|.$$

From Lemma 2.2 we have

$$C_1(Z) \cdot p!(\beta) = C_1(X_1) \cdot \beta_1 + C_1(X_2) \cdot \beta_2 - 2|\eta|.$$ 

Therefore, if $|\eta| \neq 0$, then

$$\deg C_\eta + 2|\eta| > 0.$$ 

This is a contradiction because $C_1(Z) \cdot p!(\beta) = 0$. Thus $|\eta| = 0$.

The same argument shows that $\beta_2 = \beta_\sigma$, i.e. the class of a curve in $\sigma$. So $C_1(X_2) \cdot \beta_2 = C_1(O_{\sigma}(1))(\beta_\sigma)$. Thus $C_1(X_2) \cdot \beta_2 > 0$ if $\beta_\sigma \neq 0$. Furthermore, if $\beta_2 = \beta_\sigma \neq 0$, then, by definition, we have

$$Z'_{\text{DT}}(X_2/D_1; q)_{\beta_2, \emptyset} = 0.$$ 

Therefore, we have proved that there are only terms with $\beta_2 = 0$ in the right hand side of (11).

By the degeneration formula, we have

$$Z'_{\text{DT}}(Z; q)p!p(\beta) = Z'_{\text{DT}}(Z/F_1; q)p!p(\beta), \emptyset.$$ 

This proves the lemma.

Finally, we want to prove the following theorem

**Theorem 4.5.**

$$Z'_{\text{DT}}(Y_\beta; q)p!p(\beta) = Z'_{\text{DT}}(Z; q)p!p(\beta).$$
Proof. Take a section $\sigma_1 \cong \mathbb{P}^1$ of $Y_{\tilde{S}}|_E = \mathbb{F}_1$ such that $\sigma_1^2 = -1$. Then we degenerate $Y_{\tilde{S}}$ along the section $\sigma_1$ and obtain two 3-folds, see Section 3,

$$X_1 = Z, \quad X_2 = \mathbb{P}_{\sigma_1}(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}),$$

with the common divisor $F_0 = \mathbb{P}_{\sigma_1}(\mathcal{O}(-1) \oplus \mathcal{O}(-1))$.

Applying the degeneration formula to $Z'_\mathrm{DT}(Y_{\tilde{S}}; q)_{p!(\beta)}$, we have

$$Z'_\mathrm{DT}(Y_{\tilde{S}}; q)_{p!(\beta)} = \sum_{\eta, \beta_1 + \beta_2 = \beta} Z'_\mathrm{DT}(Z/\mathbb{F}_0; q)_{\beta_1, \eta}$$

$$\times \frac{(-1)^{|\eta| - \ell(\eta)} \delta(\eta)}{q^{|\eta|}} Z'_\mathrm{DT}(X_2/\mathbb{F}_0; q)_{\beta_2, \eta}$$

where the summation runs over the splittings $\beta_1 + \beta_2 = p!(\beta)$ and the cohomology weighted partitions $\eta$.

Similar to the proof of Lemma 4.1, we need to prove that the summand with nonzero contribution in the right hand side of (12) must have the trivial partition $\eta = \emptyset$.

Note that $X_2 = \mathbb{P}_{\sigma_1}(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O})$. From (7), it is easy to know

$$C_1(X_2) = \pi^* C_1(\sigma_1) + \pi^* C_1(\mathcal{O}(-1) \oplus \mathcal{O}(-1)) - 3\xi = -3\xi,$$

where $\xi$ is the first Chern class of the tautological line bundle over $X_2$.

Therefore, we have

$$C_1(X_2) \cdot \beta_2 = 3|\eta|.$$

From Lemma 2.2, we have

$$C_1(Y_{\tilde{S}}) \cdot p!(\beta) = C_1(X_1) \cdot \beta_1 + C_1(X_2) \cdot \beta_2 - 2|\eta|.$$}

Therefore, if $|\eta| \neq 0$, then

$$C_1(Y_{\tilde{S}}) \cdot p!(\beta) = \deg C_\eta + |\eta| > 0.$$}

This is a contradiction because $C_1(Y_{\tilde{S}}) \cdot p!(\beta) = 0$. This means that the summand with nonzero contribution in the right hand side of (12) must have $\eta = \emptyset$.

Since the section $\sigma_1$ and the old exceptional divisor $E$ have the same homology class in $Y_{\tilde{S}}$, so from (2) and $\eta = \emptyset$, we have $\beta_2 = 0$.

Therefore, by the degeneration formula, we have

$$Z'_\mathrm{DT}(Y_{\tilde{S}}; q)_{p!(\beta)} = Z'_\mathrm{DT}(Z/\mathbb{F}_0; q)_{p!(\beta), \emptyset}.$$

Now it remains to prove

$$Z'_\mathrm{DT}(Z; q)_{p!(\beta)} = Z'_\mathrm{DT}(Z/\mathbb{F}_0; q)_{p!(\beta), \emptyset}.$$}

To prove this, we degenerate $Z$ along the exceptional divisor $\mathbb{F}_0$. Then we obtain two 3-folds

$$X_1 = Z, \quad X_2 = \mathbb{P}_{\mathbb{F}_0}(N_{\mathbb{F}_0} \oplus \mathcal{O}).$$
Note that $X_2 = \mathbb{P}_{F_0}(N_{F_0} \oplus O)$. Applying (4) to $X_2$ and $F_0$, we have
\begin{align*}
C_1(X_2) &= \pi^*C_1(F_0) + \pi^*C_1(N_{F_0}) - 2\xi \\
&= \pi^*C_1(\sigma_1) + \pi^*C_1(O(-1) \oplus O(-1)) - 2\xi_1 + \pi^*C_1(N_{F_0}) - 2\xi \\
&= -\xi_1 - 2\xi,
\end{align*}
where $\xi_1$ and $\xi$ are the first Chern classes of the tautological bundles in $\mathbb{P}_{\sigma_1}(O(-1) \oplus O(-1))$ and $\mathbb{P}(N_{F_0} \oplus O)$ respectively. The same calculation as in the proof of Lemma 4.2 shows that
\[ C_1(X_2) \cdot \beta_2 = 4|\eta|. \]

From Lemma 2.2, we have
\[ C_1(Z) \cdot p!(\beta) = C_1(X_1) \cdot \beta_1 + C_1(X_2) \cdot \beta_2 - 2|\eta| = \deg C_\eta + 2|\eta| > 0. \]
This is a contradiction because $C_1(Z) \cdot p!(\beta) = 0$. Thus $|\eta| = 0$.

Since $F_0 \cdot p!(\beta) = 0$, the same argument as above shows that $\beta_2 = 0$. As before, this implies (13). This comletes the proof of the theorem. \qed

**Remark 4.6.** From Theorem 4.4 and 4.5, it is easy to know that Theorem 1.1 holds.

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