Approximation properties for $p$-adic symplectic groups and lattices

Benben Liao*

September 17, 2015

Abstract

Let $G$ be the symplectic group $Sp_4$ over a non Archimedean local field of any characteristic. It is proved in this paper that for $p \in [1, 4/3) \cup (4, \infty]$ neither the group $G$ nor its lattices have the property of approximation by Schur multipliers on Schatten $p$ class $(AP_{Schur}^{pcb})$ of Lafforgue and de la Salle. As a consequence, for any lattice $\Gamma$ in $G$, the associated non-commutative $L^p$ space $L^p(L\Gamma)$ of its von Neumann algebra $L(\Gamma)$ fails the operator space approximation property (OAP) and completely bounded approximation property (CBAP) for $p \in [1, 4/3) \cup (4, \infty]$. Together with previous work [LdlS, HdL13a, HdL13b, dL], one can conclude that lattices in a higher rank algebraic group over any local field do not have the group approximation property (AP) of Haagerup and Kraus. It is also shown that on some lattice $\Gamma$ in $Sp_4$ over some local field, the constant function $1$ cannot be approximated by radial functions with bounded (not necessarily completely bounded) Fourier multiplier norms on $C^*_r(\Gamma)$, nor on $L^p(L\Gamma)$ for finite $p > 4$.

1 Introduction

Let $X$ be a Banach space. Recall that $X$ has the Banach space approximation property (AP), if there exist a net of finite rank operators $T_\alpha \in B(X)$, such that $\lim_{\alpha} \max_{x \in K} \| T_\alpha x - x \|_B = 0$, for any compact subset $K \subset X$. If furthermore $\sup_{\alpha} \| T_\alpha \|_{B(X)} < \infty$, we say that $X$ has bounded approximation property (BAP). BAP is stronger than AP by definition, and in [Gro] Grothendieck showed that for a reflexive Banach space, AP is equivalent to BAP.

*ENS de Lyon and Université de Franche-Comté, supported by ANR OSQPI
An operator space is a closed linear subspace of bounded linear operators on a Hilbert space. An operator space \( X \subset B(H) \) is said to have the operator space approximation property (OAP), if there exist a net of finite rank operators \( T_\alpha \in B(X) \), such that for any \( x \in K(\ell^2) \otimes_{\min} X \subseteq B(\ell^2 \otimes H) \), we have \( \lim_\alpha \| Id_{\ell^2} \otimes T_\alpha(x) - x \|_{B(\ell^2 \otimes H)} = 0 \). If moreover, the complete bounded norms of \( T_\alpha \) are uniformly bounded \( \sup_\alpha \| T_\alpha \|_{cb} < \infty \), then we say that \( X \) has the completely bounded approximation property (CBAP). For an operator space, OAP (resp. CBAP) implies AP (resp. BAP) for the underlying Banach space structure [BO].

Let \( \Gamma \) be a countable discrete group. Denote \( L(\Gamma) \) its group von Neumann algebra and \( L^p(L\Gamma) \) the associated non-commutative \( L^p \) space, \( p \in [1, \infty) \).

In [LdlS], it is shown that for a lattice \( \Gamma \) in \( SL_3(F) \), where \( F \) is any local field (e.g. \( \mathbb{R}, \mathbb{C}, \mathbb{Q}_p, \mathbb{F}_p((T)) \)), \( L^p(L\Gamma) \) does not have OAP for \( p \in [1, 4/3) \cup (4, \infty) \). The result is extended in [dL] to lattices in \( Sp(2, \mathbb{R}) \) (i.e. \( Sp_4(\mathbb{R}) \)) and \( p \in [1, 12/11) \cup (12, \infty) \), which is improved in [dLdlS] to \( p \in [1, 10/9) \cup (10, \infty) \). In this article, we show that \( L^p(L\Gamma) \) do not have OAP for lattices \( \Gamma \) in \( Sp_4(F) \) over any non Archimedean local field \( F \), and \( p \in (1, 4/3) \cup (4, \infty) \).

Following [LdlS], this is achieved by investigating the property of approximations by Schur multipliers on Schatten class \( S^p \) (\( AP_{\text{Schur}}^{\text{Schur}} \)) for a locally compact group (see Section 2). It enjoys many nice properties: having \( AP_{\text{Schur}}^{\text{Schur}} \) is equivalent for lattices and the ambient group; having \( AP_{\text{Schur}}^{\text{Schur}} \) is equivalent to having \( AP_{\text{cb}}^{\text{Schur}} \) where \( p' \) is conjugate to \( p \); for \( p = 1 \) or infinity, it is equivalent to weak amenability.

**Theorem 1.1** Let \( F \) be a non Archimedean local field of any characteristic. Then neither the symplectic group of 4 by 4 matrices \( Sp_4(F) \subseteq M_{4 \times 4}(F) \) nor any of its lattices have the \( AP_{\text{cb}}^{\text{Schur}} \) for \( p \in [1, 4/3) \cup (4, \infty) \).

**Corollary 1.2** Let \( F \) be a non Archimedean local field. Then for any lattice \( \Gamma \) in \( Sp_4(F) \), the associated non-commutative \( L^p \) space of its von Neumann algebra \( L(\Gamma) \) does not have the operator space approximation property (OAP) nor the completely bounded approximation property (CBAP) for \( p \in (1, 4/3) \cup (4, \infty) \).

**Remark.** Theorem 1.1 and Corollary 1.2 are analogues statements of Theorem D and Theorem A on \( SL_3 \) in [LdlS]. From the original results on \( SL_3 \) one does not see the difference between Archimedean local fields and non Archimedean local fields: the ranges of \( p \) obtained in [LdlS] for both cases are the same \((1, 4/3) \cup (4, \infty) \). Whereas for \( Sp_4 \),
the ranges \((1, 4/3) \cup (4, \infty)\) obtained in this paper for non Archimedean local fields are better than the ones \((1, 10/9) \cup (10, \infty)\) established for Archimedean local fields \([dLdS]\). It is unlikely that this is a genuine difference between local fields, but existed arguments \([dLdS]\) do not improve the ranges for \(Sp_4(\mathbb{R})\).

As for group approximation properties, recall that for a discrete group \(\Gamma\), weak amenability for \(\Gamma\) is equivalent to CBAP for \(C^*_r(\Gamma)\) \([Haag]\); approximation property of Haagerup and Kraus (AP) is equivalent to OAP of \(C^*_r(\Gamma)\) (Theorem 2.1 \([HK]\), see also \([BO]\)).

Theorem 1.1 together with \([LdlS, HdL13a, HdL13b]\), we conclude

**Corollary 1.3** Let \(k\) be a local field, and \(G\) be an almost simple algebraic \(k\)-group with \(k\)-split rank \(\geq 2\). Then non of the lattices in \(G(k)\) has the approximation property (AP) of Haagerup and Kraus \([HK]\).

We turn back to Grothendieck’s Banach space AP. P. Enflo constructed the first example of Banach space without AP \([Enf]\). Later, the natural example of bounded linear operators on a Hilbert space was shown to fail AP \([Szan]\). It is not known whether there exists countable group \(\Gamma\) such that \(C^*_r(\Gamma)\) or \(L^p(\Gamma)\) for some finite \(p\) fails AP (or even BAP).

Let \(\Gamma\) be the finitely generated group \(Sp_4(\mathbb{F}_q[T])\) of symplectic matrices over the ring of polynomials \(\mathbb{F}_q[T]\) where the coefficients are in the finite field \(\mathbb{F}_q\) and \(q\) is an odd prime power. It is a lattice in \(Sp_4(\mathbb{F}_q((T^{-1})))\). The following theorem rules out the possibilities of approximations by radial Fourier multipliers on \(C^*_r(\Gamma)\) and \(L^p(\Gamma)\).

We say that \(\ell : \Gamma \to \mathbb{R}_{\geq 0}\) is a length function if \(\ell(\gamma \gamma') \leq \ell(\gamma) + \ell(\gamma'), \gamma, \gamma' \in \Gamma\). A function is called \(\ell\)-radial if \(f(\gamma) = f(\gamma')\) whenever \(\ell(\gamma) = \ell(\gamma')\).

**Theorem 1.4** Let \(\Gamma\) be the finitely generated group above. There exists a length function \(\ell : \Gamma \to \mathbb{R}_{\geq 0}\) which is biLipschitz to the word length on \(\Gamma\), such that the constant function \(1 \in C(\Gamma)\) cannot be approximated point-wise by any family of \(\ell\)-radial (not necessarily completely bounded) Fourier multiplier \((f_\alpha)_{\alpha \in I} \subset \mathbb{C}^\Gamma\) on \(C^*_r(\Gamma)\) with

\[\sup_{\alpha \in I} \|m_{f_\alpha}\|_{MC^*_r(\Gamma)} < +\infty,\]

nor by Fourier multipliers on \(L^p(\Gamma)\) with

\[\sup_{\alpha \in I} \|m_{f_\alpha}\|_{ML^p(\Gamma)} < +\infty\]

for finite \(p > 4\).
As a by-product of the arguments, a similar statement on Schur multipliers on Schatten class is also obtained. Recall that in [LdlS], it is shown that for a non discrete group, completely bounded Schur multiplier norms and Schur multiplier norms are equal. Whereas, a conjecture of Pisier postulates that there exists a Schur multiplier on $S^p(\ell^2)$ which is not completely bounded for any finite $p \geq 1$.

**Theorem 1.5** Let $\Gamma$ be the finitely generated group defined above (as in Theorem 1.4). There exists a length function $\ell$ on $\Gamma$ that is biLipschitz to its word length, such that the following holds: for any $p \in (4, +\infty)$, $1 \in C(\Gamma)$ cannot be approximated point-wise by $\ell$-radial functions $f_\alpha \in C(\Gamma)$ such that their Schur multiplier norms are bounded (not necessarily completely bounded) uniformly

$$\sup_{\alpha \in I} \|m_{f_\alpha}\|_{MS^p(\ell^2\Gamma)} < +\infty.$$ 

The paper is organized as follows.

In Section 2, $AP^{Schur}_{pcb}$ is recalled, some simple facts about non-commutative $L^p$ spaces and quantitative versions of the theorems above are given (modulo important results in [LdlS]).

In Section 3, the proof of Theorem 1.1 is given. The proofs are different for cases when the characteristic of $F$ is 2 and when it is different from 2. Matrices constructed in [Laf10c, Liao13, Liao14] are used and some arguments treating $SL_3$ [LdlS] (in particular Lemma 4.9) can be adapted to the case of $Sp_4$.

In Section 4, the proof of Theorem 1.4 is given. The reason for restricting to radial functions is technical: the arguments only give estimates for spherical functions on the ambient group. The matrices used in the proof of Theorem 1.1 do not apply since they do not give rise to invariant operators. Instead, explicit functions constructed in [Laf10a, Liao14] are used in the proof (without using Lemma 4.9 [LdlS]).

Lastly in Section 5, Theorem 1.5 is proved.

**Acknowledgment:** I thank Vincent Lafforgue for his encouragement to study the problem of group approximation properties for $Sp_4$. I also thank Mikael de la Salle for numerous helpful discussions and valuable suggestions on several improvements and simplifications of the proofs.
2 Multipliers on Schatten classes and non commutative $L^p$ spaces

Let $p \in [1, \infty]$ and $H$ be a Hilbert space. For $p < \infty$, denote $S^p(H)$ the Schatten $p$ class on $H$, i.e. the subspace of bounded operators $T \in B(H)$ such that the trace $\text{Tr}([T]^p)$ is finite. It is a Banach space with respect to the norm $\|T\|_{S^p(H)} = \text{Tr}([T^p]^{1/p})$, $T \in S^p(H)$. For $p = \infty$, denote $S^{\infty}(H)$ the space of compact operators.

Let $X$ be a topological space with a fixed Borel measure. A continuous function $\varphi \in C(X \times X)$ is said to be a Schur multiplier on $S^p(L^2X)$, if for any operator $T \in S^p(L^2X) \cap S^2(L^2X)$ (being a dense subspace of $S^p(L^2X)$) with symbol $(T_{x,y})_{x,y \in X}$, the operator with symbol $(\varphi(x,y)T_{x,y})_{x,y \in X}$ is in $S^p(L^2(X))$ and

$$\|(\varphi(x,y)T_{x,y})_{x,y \in X}\|_{S^p(L^2X)} \leq C\|T\|_{S^p(L^2X)}$$

for some $C > 0$ - the smallest $C$ is denoted by $\|\varphi\|_{MS^p(L^2X)}$. If furthermore there exists some $C' > 0$ such that for any operators $(T_{x,y} \in B(H))_{x,y \in X} \in S^p(L^2X \otimes H)$

$$\|(\varphi(x,y)T_{x,y})_{x,y \in X}\|_{S^p(L^2X \otimes H)} \leq C'\|T\|_{S^p(L^2X \otimes H)},$$

where $H$ is a Hilbert space, then we say that $\varphi$ is a completely bounded Schur multiplier on $S^p(L^2X)$, and the smallest possible $C'$ is denoted by $\|\varphi\|_{cbMS^p(L^2X)}$.

Let $G$ be a locally compact group with a fixed Haar measure. A continuous function on the group $f \in C(G)$ gives rise to a continuous function $[(x, y) \mapsto f(x^{-1}y)] \in C(G \times G)$ (denoted by $\hat{f}$ in [LdIS] on its product $G \times G$, and if it is a Schur multiplier on $S^p(L^2G)$ then we denote it by $m_f$. With our notation we have

$$\|m_f\|_{MS^p(L^2G)} = \sup_{T \in S^p(L^2G), \|T\|_{S^p(L^2G)} \leq 1} \|(f(x^{-1}y)T_{x,y})_{x,y \in G}\|_{S^p(L^2G)},$$

and

$$\|m_f\|_{cbMS^p(L^2G)} = \sup \|(f(x^{-1}y)T_{x,y})_{x,y \in G}\|_{S^p(L^2G \otimes H)}$$

where $H$ is a Hilbert space and the supremum is taken over operators $(T_{x,y} \in B(H))_{x,y \in X} \in S^p(L^2G \otimes H)$ with $\|T\|_{S^p(L^2G \otimes H)} \leq 1$.

**Definition 2.1 ([LdIS])** Let $G$ be a locally compact topological group with a fixed Haar measure. Let $p \in [1, \infty]$ as above. Say that $G$ has the property of approximations by Schur multipliers on Schatten $p$ class $AP^{Schur}_{pcb}$, if there exist a net of functions $(f_\alpha)_{\alpha \in I}$ in the Fourier
algebra \( A(G) \) (being a subset of \( C_0(G) \)) which are completely bounded Schur multipliers on Schatten \( p \) class \( S^p(L^2G) \) with uniformly bounded norms

\[
\sup_{\alpha \in I} \| m_{f_{\alpha}} \|_{cbMS^p(L^2G)} < +\infty,
\]
such that the constant function 1 on \( G \) can be approximated by these functions \( (f_{\alpha})_{\alpha \in I} \) uniformly on compact sets.

Since \( S^2(L^2G) \) is the space of Hilbert-Schmidt operators and

\[
\| m_f \|_{cbMS^2(L^2G)} = \| f \|_{L^\infty(G)},
\]
\( G \) always has \( AP_{Schur}^2 \).

After [BF], completely bounded multipliers on the Fourier algebra \( A(G) \) coincide with that on compact operators on \( L^2(G) \):

\[
\| f \|_{M_0A(G)} = \| m_f \|_{cbMS^\infty(L^2G)}, \forall f \in C(G),
\]
we see that \( AP_{Schur}^\infty \) (or \( AP_{1cb}^Schur \), see [LdlS]) is equivalent to weak amenability for \( G \).

Let \( \Gamma \) be a countable discrete group. Denote \( L(\Gamma) \) its group von Neumann algebra, namely the bicommutant of the operators generated by the left regular representation \( \lambda \) of \( \Gamma \) on \( \ell^2\Gamma \)

\[
L(\Gamma) = \{ \lambda(\gamma), \gamma \in \Gamma \}'' \subset B(\ell^2\Gamma).
\]
\( L(\Gamma) \) is equiped with the natural faithful tracial state \( \tau(x) = \langle \delta_1, x\delta_1 \rangle, x \in L(\Gamma) \). For finite \( p \geq 1 \), denote \( L^p(\ell^2\Gamma) \) the non commutative \( L^p \) space associated to \( L(\Gamma) \), i.e. the Banach space of the completion of \( L(\Gamma) \) under the norm

\[
\| x \|_{L^p(\ell^2\Gamma)} = (\tau(|x|^p))^{1/p}, x \in L(\Gamma).
\]

The following statement is probably well-known to expert.

**Proposition 2.2** Let \( \Gamma \) be a countable discrete group and \( f \in C(\Gamma) \). We have

\[
\| f \|_{L^p(\ell^2\Gamma)} \leq \| f \|_{L^q(\ell^2\Gamma)}, 1 \leq p \leq q < \infty,
\]
and

\[
\lim_{p \to \infty} \| f \|_{L^p(\ell^2\Gamma)} = \| f \|_{C_0^\gamma(\Gamma)}.
\]

**Proof of Proposition 2.2**
Let $H$ be a Hilbert space and $x \in B(H)$ be a normal operator. Denote $\Omega \subseteq \mathbb{C}$ the Gelfand spectrum of the abelian $C^*$ algebra generated by $x$. For any unit vector $\xi \in H$, by Riesz theorem there exists a Borel probability $\mu$ on $\Omega$ such that

$$\langle \xi, F(x)\xi \rangle = \int_{\Omega} F(t) d\mu,$$

$\forall F \in C(\Omega)$. Now apply it to $x = |f|$, $F(x) = x^p$, $\xi = \delta_x \in \ell^2\Gamma$, and by the inequalities of means we get the results. □

**Proposition 2.3** Let $H$ be a finite group, $1 \leq p < \infty$. Then for any function $f \in C(H)$, we have

$$| \sum_{h \in H} f(h) | \leq |H|^{1/p} \| f \|_{L^p(L^1H)}.$$

**Proof of Proposition 2.3.** We first have

$$\| f^1_H \|_{L^1(L^1H)} = \langle \delta_1, f^1_H \delta_1 \rangle = \sum_{h \in H} f(h),$$

where $1_H$ denotes the constant function one on $H$.

By Holder inequality

$$\| f^1_H \|_{L^1} \leq \| f \|_{L^p} \| 1_H \|_{L^q},$$

where $1/p + 1/q = 1$ and $\langle \delta_1, |1_H|^q \delta_1 \rangle = |H|^{q-1}$. □

For a countable discrete group $\Gamma$ and $f \in C\Gamma$, we set $\| f \|_{L^\infty(\Gamma)} = \| f \|_{C_0^\infty(\Gamma)}$, and $L^\infty(\Gamma) = \Gamma$, and denote $\| m_f \|_{ML^p(\Gamma)}$ the Fourier multiplier norm of $f$ on $L^p(\Gamma)$, $p \in [1, \infty]$ : 

$$\| m_f \|_{ML^p(\Gamma)} = \sup_{\varphi \in C\Gamma, \| \varphi \|_{L^p(\Gamma)} \leq 1} \| [\gamma \mapsto f(\gamma) \varphi(\gamma)] \|_{L^p(\Gamma)}.$$

Now we turn to a quantitative version of Theorem 1.1 based on which the theorem and Corollaries 1.2 and 1.3 are direct consequences of results in [LdIS].

Let $F$ be a non Archimedean local field, $\mathcal{O} \subset F$ its ring of integer. Let $G = Sp_4(F)$, i.e. the matrices $A \in M_{4 \times 4}(F)$ satisfying $A^TJA = J$, where

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$
Theorem 2.4 \( G = \text{Sp}_4(F), K = \text{Sp}_4(O) \). Let \( p \in (4, +\infty] \). There exists a continuous function \( \phi_p \in C_0(G) \) vanishing at infinity, such that for any \( K \)-biinvariant continuous function \( f \in C(G) \), we have
\[
|f(g)| \leq \phi_p(g) \|mf\|_{MS_p(L^2(G))}.
\]

When \( p = \infty \), the analogue of such an inequality turns out to be important also in the proof of negation of AP for \( \text{Sp}_4(\mathbb{R}) \) (i.e. \( \text{Sp}(2, \mathbb{R}) \)) in [HdL13a] and property \((T^*)\) for \( \text{SL}_3(\mathbb{R}) \) and \( \text{Sp}_4(\mathbb{R}) \) in [HKdL].

For \( p = \infty \) and residue field of \( F \) has char different from 2, the statement is already known by [Laf10c].

Proof of Theorem 1.1 by Theorem 2.4: By Theorem 2.4, \( 1 \in C(G) \) cannot be approximated on compact sets by \( K \)-biinvariant functions \( f_\alpha \in C_0(G) \) with \( \sup_{\alpha \in I} \|mf_\alpha\|_{cbMS_p(L^2(G))} \) being finite. Since both right and left \( K \) actions preserve the norm \( \|mf\|_{cbMS_p(L^2(G))} \) (Proposition 4.2 in [LdlS]), we see that the statement extends to all functions \( f_\alpha \in C_0(G) \) and thus \( G \) does not have \( AP^{\text{Schur}}_{\text{pcb}} \) for \( p \in (4, \infty] \).

Now that \( AP^{\text{Schur}}_{\text{pcb}} \) is symmetric for a pair of conjugate numbers \( p, p' \in [1, \infty], 1/p + 1/p' = 1 \) (Proposition 2.3 in [LdlS]), we have that \( G \) fails \( AP^{\text{Schur}}_{\text{pcb}} \) for \( p \in [1, 4/3) \cup (4, \infty] \).

Since \( AP^{\text{Schur}}_{\text{pcb}} \) extends from any lattice to the ambient group (Theorem 2.5 [LdlS]), we conclude that none of the lattices in \( G \) has \( AP^{\text{Schur}}_{\text{pcb}} \) for \( p \in [1, 4/3) \cup (4, \infty] \).

Proof of Corollaries 1.2 and 1.3: Corollary 1.2 is a direct consequence of the following fact: for a discrete group \( \Gamma \), \( L^p(\Gamma) \) having OAP is stronger than \( \Gamma \) having \( AP^{\text{Schur}}_{\text{pcb}} \) for the same \( p \in (1, \infty) \) (Corollary 3.13 [LdlS]).

The Archimedean case of Corollary 1.3 is proved in [HdL13a, HdL13b]. For a non Archimedean local field \( F \) (and in fact any field), we know that any almost simple algebraic group of split rank \( \geq 2 \) contains a subgroup that is isomorphic to a quotient of \( SL_3(F) \) or \( \text{Sp}_4(F) \) by a finite normal subgroup [BT, Mar]. Since for a discrete group, having AP implies having \( AP^{\text{Schur}}_{\text{pcb}} \) for all \( p \in (1, \infty) \) (Corollary 3.12 [LdlS]), we conclude the proof by showing the following lemma.

Lemma 2.5 Let \( G \) be a locally compact group and \( N \subset G \) a finite normal subgroup. Let \( H \) be the quotient group \( H = G/N \). Let \( f \in C_c(H) \mapsto f \in C_c(G) \) be the embedding of linear spaces defined by \( f(g) = f(gN \in H) \). We have
\[
\|mf\|_{cbMS_p(L^2(H))} \leq \|mf\|_{cbMS_p(L^2(G))}.
\]

Now we prove the lemma. Let \( K \) be a Hilbert space.
Set
\[ s^* : B(L^2(G, K)) \to B(L^2(H \times N, K)) \]
\[(T_{x,y} \in B(K))_{x,y \in G} \mapsto (T_{s(z)n,s(w)m})(z,n),(w,m) \in H \times N,\]
where \( s : H \to G \) is any fixed section. It is an isometry on the subspace of Schatten class \( S^p(BL^2(G, K)) \) since \( s^* \) is induced from the isomorphism of the underlying Hilbert spaces.

We have \( s^*(mfT) = mf(s^*(T)) \), since by assumption \( N \) is a normal subgroup. \( \square \)

The following two theorems are also quantitative versions of Theorem 1.4 and Theorem 1.5 respectively.

**Theorem 2.6** Let \( F = \mathbb{F}_q[[\pi]] \) where \( q \) is an odd prime power, and let \( \Gamma \) be the lattice \( S_{p4}(\mathbb{F}_q[\pi^{-1}]) \) in \( G = S_{p4}(F) \). Let \( K = S_{p4}(O) \). For any \( p \in (4, \infty] \), there exists a function vanishing at infinity \( \phi_p \in C_0(\Gamma) \), such that for any function \( f \in \mathbb{C}(\Gamma) \cap K \mathcal{C}(G)^K \) (i.e. \( f \in \mathbb{C}(\Gamma) \) and \( f(\gamma) = f(\gamma') \) whenever \( \gamma \in \Gamma \cap K\gamma'K, \gamma' \in \Gamma \), we have
\[ |f(\gamma)| \leq \phi_p(\gamma) \|mf\|_{ML^p(\ell^2)} \]

For \( p = \infty \), the statement is a special case of Theorem 1.2 when \( s = 0 \) in [Liao14].

**Proof of Theorem 1.4 by Theorem 2.6:** Denote \( D(i, j) \) the diagonal matrix
\[ \begin{pmatrix} \pi^{-i} & & & \\ & \pi^{-j} & & \\ & & \pi^j & \\ & & & \pi^i \end{pmatrix} \]

Let \( \ell : \Gamma \to \mathbb{R}_{\geq 0} \) be the function defined by \( \ell(\gamma) = i \) if \( \gamma \in KD(i, j)K, i \geq j \geq 0 \), or equivalently \( \ell(\gamma) = \log_q \|\gamma\| = \log_q \max_{1 \leq \alpha, \beta \leq 4} |\gamma_{\alpha\beta}|_F \).
It is a length function since \( (KD(i, j)K)^{-1} = KD(i, j)K \) and \( \|g_1\|\|g_2\| \geq \|g_1g_2\| \). It is the length function induced from the Bruhat-Tits building associated to \( G \), and thus biLipschitz to the word length on \( \Gamma \) [LMR].

By definition \( \ell \)-radial functions are \( K \) biinvariant functions on \( \Gamma \), and this completes the proof. \( \square \)

**Theorem 2.7** Let \( F, \Gamma, G, K \) be as in Theorem 2.6. Then for any \( p \in (4, \infty] \), there exists a function \( \phi_p \in C_0(\Gamma) \) such that for any function \( f \in \mathbb{C}(\Gamma) \cap K \mathcal{C}(G)^K \) we have
\[ |f(\gamma)| \leq \phi_p(\gamma) \|mf\|_{MS^p(\ell^2)} \]

**Proof of Theorem 1.5 by Theorem 2.7:** One can take the same length function as in the proof of Theorem 1.4 by Theorem 2.6 \( \square \)
3 Proof of Theorem 2.4

Denote $D(i,j)$ the diagonal matrix

\[
\begin{pmatrix}
\pi^{-i} & 0 & \cdots & 0 \\
0 & \pi^{-j} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \pi^{j} \\
0 & 0 & \cdots & \pi^{i}
\end{pmatrix}
\]

The set $\Lambda = \{(i,j) \in \mathbb{N}^2, i \geq j \geq 0\}$ is in bijection with the double cosets $K \setminus G / K$ via $(i,j) \mapsto KD(i,j)K$.

**Proposition 3.1** Let $G, K, p$ be as in Theorem 2.4.

1. If the characteristic of $F$ is different from 2, denote $v_0 \in \mathbb{N}$ the valuation of $2 \in F$. Then we have for any $K$-biinvariant function $f \in C(G)$,

\[|f(D(i,j)) - f(D(i,j+1))| \leq 2q^{-\frac{1}{2}(i-j-v_0-1)(1-4/p)} \|m_f\|_{MS^p(L^2(G))},\]

where $(i,j) \in \Lambda$ and $i \geq 1, i - j \geq v_0 + 1$.

2. If the characteristic of $F$ is 2, then $\forall f \in C(G)$ $K$-biinvariant we have

\[|f(D(i,j)) - f(D(i,j+2))| \leq 2q^{-\frac{1}{2}(i-j-2)(1-4/p)} \|m_f\|_{MS^p(L^2(G))},\]

where $i \geq j + 2$.

**Proposition 3.2** Let $F$ be a non-Archimdean local field of any characteristic, and $G, K, p$ as in Theorem 2.4. Let $f$ be any $K$-biinvariant function on $G$. Then for any $(i,j) \in \Lambda$ with $j \geq 3$, we have

\[|f(D(i,j)) - f(D(i+1,j-1))| \leq 2q^2 \cdot q^{-(j-2)(1-3/p)} \|m_f\|_{MS^p(L^2(G))}.\]

Proof of Theorem 2.4 by Propositions 3.1 and 3.2 above:

It is similar to the proof of Theorem 3.1 by Proposition 3.2 in [Liao14], i.e. a zig-zag argument along the line $i = 3j$.

Proof of Proposition 3.2:

**Lemma 3.3** (lemma 4.9 in [LaS]) Let $m, n \in \mathbb{N}^*, k \in \{1, 2, \ldots, m\}$. Let $p > 2 + 2/n$. Let $H$ be a locally compact group, $\alpha, \beta : (\mathcal{O}/\pi^m \mathcal{O})^{n+1} \rightarrow H$ two maps. Let $f \in C_c(H)$ satisfy

\[f(\alpha(a_1, a_2, \ldots, a_n, b) \beta(x_1, x_2, \ldots, x_n, y)) = \lambda,\]

if $y = \sum_{i=1}^{n} a_i x_i + b + \pi^k \in \mathcal{O}/\pi^m \mathcal{O}$, and
\( f(\alpha(x_1, \ldots, x_n, b) \beta(x_1, x_2, \ldots, x_n, y)) = \mu, \)

if \( y = \sum_{i=1}^{n} a_i x_i + b + \pi^{k-1} \in O/\pi^m O. \)

Then

\[ |\lambda - \mu| \leq 2q^{-k\varepsilon} M_{f}(L^2(H)), \]

where \( \varepsilon = \frac{p}{2p} \left( p - \frac{2}{n} \right). \) In particular when \( n = 1, \)

\[ |\lambda - \mu| \leq 2q^{-\frac{k}{2}(1-4/p)} M_{f}(L^2(H)), \]

when \( n = 2 \)

\[ |\lambda - \mu| \leq 2q^{-k(1-3/p)} M_{f}(L^2(H)). \]

We first prove the case when \( \text{char}(F) \neq 2. \)

We first show that there exist \( k \geq i - j - v_0 - 1 \) and two maps \( \alpha, \beta : (O/\pi^k O)^2 \rightarrow G \) such that when \( y = ax + b, \) we have

\[ \alpha(a, b) \beta(x, y) \in KD(i, j)K, \]

and when \( y = ax + b + \pi^{k-1}, \)

\[ \alpha(a, b) \beta(x, y) \in KD(i, j + 1)K. \]

Indeed, one can set \( k = 2m - 2j - v_0 \) where \( m \) is the integral part of \( (i + j)/2, \) and \( \alpha, \beta \) equal to \( \beta^{-1}, \alpha \) respectively in the proof of proposition 3.2 in \[ \text{Liao13}, \]

\[ \alpha(a, b) = \begin{pmatrix} \pi^m \\ \pi^{i-m+j} \\ \pi^{i-m-j} \\ \pi^{-m} \end{pmatrix}, \]

\[ \begin{pmatrix} 1 & 0 & 1 & 1 \\ \sigma(a) & 1 & \sigma(a) & 0 & 1 \end{pmatrix}, \]

\[ \beta(x, y) = \begin{pmatrix} 1 \\ 0 \\ \sigma(x) \\ \sigma(x)^2 + 2\sigma(y) \end{pmatrix}, \]

\[ \begin{pmatrix} \pi^{-m+j} \\ \pi^{-m-j} \\ \pi^{-m+j} \\ \pi^{-m-j} \end{pmatrix}, \]

where \( x, y, a, b \in O/\pi^k O, \) and \( \sigma : O/\pi^k O \rightarrow O \) is a section. The computations in \[ \text{Liao13} \]

show that these matrices indeed satisfy our requirements. It is also possible to construct \( \alpha, \beta \) as variants of the matrices used in \[ \text{Laf10c}. \]

Now apply Lemma 3.3 to \( \alpha, \beta \) above, \( m = k, H = G, \) and \( \lambda = f(D(i, j)), \mu = f(D(i, j + 1)), \) we have

\[ |f(D(i, j)) - f(D(i, j + 1))| \leq 2q^{-i-j-2(1-4/p)} M_{f}(L^2(G)). \]
Liao13. \[ \text{we see that these matrices satisfy our requirements.} \]

By similar (or simpler) computations as in the proof of lemma 4.1 in Proposition 3.3 to obtain the desired inequality.

\[ \alpha(a, b) = \begin{pmatrix} \pi^m & \pi^{i-m+j} \\ \pi^{-i+m-j+1} \sigma(b) & \pi^{-i+m-j}(1 + \pi \sigma(a))^2 & \pi^{-i+m-j} \\ 0 & \pi^{-m+1} \sigma(b) & 0 & \pi^{-m} \end{pmatrix}, \]

\[ \beta(x, y) = \begin{pmatrix} \pi^{-m+j} \\ 0 \\ \pi^{-m+j}(\sigma(x) + \pi \sigma(y)) & \pi^{-m+j}(\sigma(x) + \pi \sigma(y)) & 0 & \pi^{-m+j} \end{pmatrix}. \]

By similar (or simpler) computations as in the proof of the previous proposition, we will construct appropriate matrices in \( G \) and apply Lemma 3.3 to obtain the desired inequality.

Now prove the estimate when \( Char(F) = 2 \).

There exist \( k \geq (i - j - 2)/2 \) and \( \alpha, \beta : (O/\pi^kO)^3 \to \mathcal{O} \) such that when \( y = ax + b, \)

\[ \alpha(a, b) \beta(x, y) \in KD(i, j)K, \]

and when \( y = ax + b + \pi^{k-1}, \)

\[ \alpha(a, b) \beta(x, y) \in KD(i, j + 2)K. \]

We still use the constructions from Liao13. Let \( k = m - j - 1 \) where \( m = \lfloor \frac{i + j}{2} \rfloor \), i.e. the biggest integer \( \leq (i + j)/2 \). Let \( x, y, a, b \in O/\pi^{m-j}O \), and \( \sigma : O/\pi^{m-j}O \to \mathcal{O} \) be a section, and set

\[ \|f\|_{MS^p(L^2(G))}. \]

\[ |f(D(i, j)) - f(D(i, j + 2))| \leq 2q^{-\frac{1}{4}(1-4/p)}\|m_f\|_{MS^p(L^2(G))}. \]

\[ \square \]

**Proof of Proposition 3.2** Similarly to the proof of the previous proposition, we will construct appropriate matrices in \( G \) and apply Lemma 3.3 to obtain the desired inequality.

When \( i + j \) is an even number, there exist \( k \geq j - 2 \) and matrices \( \alpha, \beta : (O/\pi^kO)^3 \to G \) such that \( \forall a_1, a_2, b, x_1, x_2, y \in O/\pi^kO \), if \( y = a_1x_1 + a_2x_2 + b \), then

\[ \alpha(a_1, a_2, b) \beta(x_1, x_2, y) \in KD(i, j)K, \]

and if \( y = a_1x_1 + a_2x_2 + b + \pi^{k-1}, \)

\[ \alpha(a_1, a_2, b) \beta(x_1, x_2, y) \in KD(i + 1, j - 1)K. \]

Indeed, removing the discretization \([\cdot]\) in \( \alpha, \beta \) for \( i + j \in 2\mathbb{N} \) and in \( \bar{\alpha}, \bar{\beta} \) for \( i + j \in 2\mathbb{N} - 1 \) in the proof of the second inequality of proposition 3.2 in Liao14 (which are improved constructions of the matrices used
in the proof of lemma 2.1 in [Lat10c]), we get a construction of α, β. More precisely, let \( k = m = \lceil (i + j) / 2 \rceil - 1 \), i.e. when \( i + j \in 2\mathbb{N}, m = (i + j) / 2 - 1 \), and when \( i + j \in 2\mathbb{N} + 1, m = (i + j - 1) / 2 - 1 \). Let \( \sigma : \mathcal{O}/\pi^m+1\mathcal{O} \to \mathcal{O} \) be any section. When \( i + j \) is even, set \( \alpha, \beta : (\mathcal{O}/\pi^m+1\mathcal{O})^3 \to G \) by

\[
\alpha(a_1, a_2, b) = \alpha_1(a_1, a_2, b) = \begin{pmatrix} 1 & -\pi^{-m-1}(1 + \pi\sigma(a_1)) & \pi^{-m-1}(1 + \pi\sigma(a_2)) & -\pi^{-2m}\sigma(b) \\ 0 & 1 & 0 & \pi^{-m-1}(1 + \pi\sigma(a_2)) \\ 0 & 0 & 1 & \pi^{-m-1}(1 + \pi\sigma(a_1)) \\ 0 & 0 & 0 & 1 \end{pmatrix}, (*)
\]

\[
\beta(x_1, x_2, y) = \begin{pmatrix} 1 & \pi^{-m}\sigma(x_2) & \pi^{-m}\sigma(x_1) & \pi^{-m-1}\pi^{-m}x_2 \sigma(x_1) + \pi^{-m}x_2 \sigma(x_1) + \pi^{-2m}\sigma(y) \\ 0 & 1 & 0 & \pi^{-m}\sigma(x_1) \\ 0 & 0 & 1 & -\pi^{-m}\sigma(x_2) \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

And when \( i + j \) is odd, set

\[
\alpha(a_1, a_2, b) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \pi^{-1} & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \alpha_1(a_1, a_2, b),
\]

where \( \alpha_1 \) is as defined in (*). Identical (after removing \([\cdot]\)) computations as in [Liao14] show that they satisfy required properties. Note that even though in [Liao14] the local field \( F \) is assumed to have characteristic different from 2, the constructions of \( \alpha, \alpha, \beta \) are valid for any characteristic.

Now apply Lemma 3.3 to \( k, \alpha, \beta \), and \( H = G, n = 2, \lambda = f(D(i, j)), \mu = f(D(i + 1, j - 1)), \) we have

\[
|f(D(i, j)) - f(D(i + 1, j - 1))| \leq 2q^{-(j-2)(1-3/p)}\|m_f\|_{MS^p(L^2(G))}.
\]

\[\square\]

4 Proof of Theorem 2.6

We adopt the notations \( F, \mathcal{O}, G, K, D(i, j), \Lambda \) as in Section 3. Note that the ring of integer \( \mathcal{O} \) is \( \mathbb{F}_q[[\pi]] \).

**Proposition 4.1** Let \( F, G, K, \Gamma, p \) be as in Theorem 2.6. Then for any function \( f \in K C(G)^K \) we have

\[
|f(D(i, j)) - f(D(i, j + 1))| \leq C_{q,p}q^{-(1/2-2/p)(i-j)}\|m_f\|_{ML^p(L^2)},
\]

13
and

$$|f(D(i, j)) - f(D(i + 1, j - 1))| \leq C_{q,p}q^{2(i+j)/p-j}\|m_f\|_{MLP(\Gamma)}.$$

**Proof of Theorem 2.6 using Proposition 4.1.** For any $p > 4$, there exists $n \in \mathbb{N}$ such that $2(1 + 1/n + 1)/p - 1 < 0$. A zig-zag argument near the line $i = (1 + 1/n)j$ will yield the estimate. □

**Proof of Proposition 4.1.**

**Lemma 4.2** For each $(i, j) \in \Lambda$, there exist two finite subgroups $H_{1,i,j}, H_{2,i,j} \subseteq \Gamma$ of cardinality $q^{2(i-j)+3}$ and $q^{2(i+j)+2}$ respectively, and two family of functions $h_{1,i,j}, h_{1,i,j+1} \in \mathbb{C}H_{1,i,j}, h_{2,i,j}, h_{2,i,j+1-1} \in \mathbb{C}H_{2,i,j}$ that are normalized characteristic functions of points in $KD(i,j)K \cap H_{1,i,j}$ and $KD(i,j)K \cap H_{2,i,j}$ respectively, such that

$$\|h_{1,i,j} - h_{1,i,j+1}\|_{C^\ast(H_{1,i,j})} \leq 2q^{-(i-j)/2},$$

and

$$\|h_{2,i,j} - h_{2,i,j+1}\|_{C^\ast(H_{2,i,j})} \leq 2q^2q^{-j}.$$

Now prove the first inequality. We set

$$H_{1,i,j} = \{\alpha(a, b, \varepsilon) = \begin{pmatrix} 1 & 0 & \pi^{-i}a & \pi^{-i}b \\ 1 & \pi^{-i} & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, a, b \in \mathbb{F}_q + \mathbb{F}_q\pi + ... + \mathbb{F}_q\pi^{i-j}, \varepsilon \in \mathbb{F}_q\},$$

and the following function

$$h_{1,i,j} = \sum_{a \in \mathbb{F}_q + ... + \mathbb{F}_q\pi^{i-j}} e_{\alpha(a, b, \varepsilon)} a^{t_{-j}} a^{(a^2) + \pi^{i-j}, 1},$$

where $t_{-j} : \mathbb{F}_q + ... + \mathbb{F}_q\pi^{2i-2j} \to \mathbb{F}_q + ... + \mathbb{F}_q\pi^{i-j}$ is the obvious truncation $t_{-j}(\sum k \geq 0 a_k\pi^k) = \sum_{0 \leq k \leq i-j} a_k\pi^k$.

Let $\chi \in \hat{H}_{1,i,j}$, and suppose $\chi_1, \chi_2$ are characters of $\mathbb{F}_q + ... + \mathbb{F}_q\pi^{i-j}$ and $\chi_3 \in \hat{\mathbb{F}}_q$ such that $\chi(a, b, \varepsilon) = \chi_1(a)\chi_2(b)\chi_3(\varepsilon)$. We have the following: if $\chi(h_{1,i,j} - h_{1,i,j+1}) \neq 0$, then there exists $\theta \in \mathbb{F}_q + ... + \mathbb{F}_q\pi^{i-j}$ such that $\chi_1(\alpha) = \chi_2(t_{-j}(\theta a))$. Indeed, if $k_\alpha, \alpha = 1, 2$ is the smallest integer $k$ such that $\chi_\alpha$ is trivial on $\mathbb{F}_q\pi^{i-j-k} + \mathbb{F}_q\pi^{i-j-k+1} + ... + \mathbb{F}_q\pi^{i-j}$ and non-trivial on $\mathbb{F}_q\pi^{i-j-k-1}$, then we have $k_1 \geq k_2$ unless $\chi(h_{1,i,j} - h_{1,i,j+1}) = 0$. The existence of $\theta$ follows from the fact that $\mathbb{F}_q[\pi]/\pi^{i-j+1}\mathbb{F}_q[\pi]$ is a local ring.

By a lemma on Gauss sum [Laf10a] (see also Lemma 4.3 [Liao14]) we have that

$$|\chi(h_{1,i,j} - h_{1,i,j+1})| \leq 2q^{-(i-j)/2}, \forall \chi \in \hat{H}_{1,i,j}.$$
This yields the first inequality. 

For the second inequality in the lemma, set

\[ H_{2,i,j} = \{ \beta(a, b, c) = \begin{pmatrix} 1 & \pi^{-m_a} & \pi^{-m_c} \\ 1 & 0 & \pi^{-m_b} \\ 1 & -\pi^{-m_b} & 1 \end{pmatrix}, a, b, c \in O \}, \]

where \( m = [(i + j)/2] \), and \([·]: \mathbb{F}_q((\pi)) \rightarrow \mathbb{F}_q[\pi^{-1}] \) is defined by taking the integral part \( \sum_i a_i \pi^{-i} = \sum_{i \geq 0} a_i \pi^{-i} \).

The constructions of \( h_{2,i,j} \) are identical to the explicit functions \( h_{2,i,j} \) used in the proof of Proposition 4.1 in [Liao14], namely

\[ \| f(D(i,j)) - f(D(i, j+1)) \| \leq q^{2(i-j)+3}/p \| m_f(h_{1,i,j} - h_{1,i,j+1}) \|_{L^p(L(H_{1,i,j}))}, \]

and again by Lemma 4.2 and Proposition 2.2 it is

\[ \leq q^{2(i-j)+3}/p \| m_f \|_{ML^p(L\Gamma)} \| h_{1,i,j} - h_{1,i,j+1} \|_{L^p(L(H_{1,i,j}))} \]

\[ \leq C_{\varphi,p} q^{(2i-2j)/2} \| m_f \|_{ML^p(L\Gamma)} \| h_{1,i,j} - h_{1,i,j+1} \|_{C^\gamma(H_{1,i,j})} \]

\[ \leq C_{\varphi,p} q^{(2i-2j)/2} \| m_f \|_{ML^p(L\Gamma)} q^{-i(1/2-2/p)}. \]

The second inequality is exactly the second inequality of Proposition 4.2 [Liao14].

\[ \square \]

5 Proof of Theorem 2.7

We adopt the notations \( F, O, G, K, D(i,j), \Lambda \) as in Section 3.

---

1 If we set

\[ h'_{1,i,j} = \mathbb{E}_{a,b,c \in O/\pi^iO} e_{h_1([\pi^{-i}a], \pi^{-i}b, \pi^{-i}c + \pi^{-j}(1+\pi c))}, \]

then we also have \( \| \Delta = h'_{1,i,j} - h'_{1,i,j+1} \|_{C^\gamma(H'_{1,i,j})} \leq 2q^{-(i-j)/2} \) for some finite abelian subgroup \( H'_{1,i,j} \) and the support of the spectrum of \( \Delta \) has cardinality \( \leq q^{2i-2j} \). Since what contributes in the \( L^p \) norm of the spectrum of \( \Delta \) is the measure of its support, this gives a second proof of the first estimate in Proposition 4.1.
Proposition 5.1 Let $F, G, K, \Gamma, p$ be as in Theorem 2.7. Then we have for any function $f \in C(\Gamma) \cap K C(G) K$

$$|f(D(i, j)) - f(D(i, j + 1))| \leq 2q^{-\frac{1}{2}(i-j-2)(1-4/p)}\|m f\|_{MSp(\ell^2)},$$

and

$$|f(D(i, j)) - f(D(i + 1, j - 1))| \leq 2q^{-(j-2)(1-3/p)}\|m f\|_{MSp(\ell^2)}.$$

We remark that the arguments in Section 4 yield the same (up to a constant) decaying factor $q^{(i-j)(1/2-2/p)}$ for the first inequality and a worse one $q^{2(i+j)/p-j} > q^{-(j-3/p)}$ for the second inequality. To be consistent a complete proof of the first inequality is also given below.

Proof of Proposition 5.1: The proof proceeds in a similar way as the proof of Propositions 3.1 and 3.2 - we will construct matrices satisfying required conditions and the apply Lemma 3.3.

We prove prove the first estimate.

There exist $\alpha, \beta : (\mathcal{O}/\pi^i \mathcal{O})^2 \to \Gamma$ such that for $y = ax + b$ we have

$$\alpha(a, b) \beta(x, y) \in KD(i, j) K \cap \Gamma$$

and for $y = ax + b + \pi^{-i-j}$

$$\alpha(a, b) \beta(x, y) \in KD(i, j + 1) K \cap \Gamma.$$

The construction of $\alpha, \beta$ are identical to $\alpha, \beta$ used in the first proof of Proposition 3.2 in [Liao14] (which are matrices in [Laf10 c] after discretization). More precisely, set

$$\alpha(a, b) = \begin{pmatrix} 1 & 0 & [\pi^{-i}\sigma(a)] & [\pi^{-i}\sigma(a^2 - b)] \\ 0 & 1 & \pi^{-i} & [\pi^{-i}\sigma(a)] \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\beta(x, y) = \begin{pmatrix} 1 & 0 & [\pi^{-i}\sigma(x/2)] & [\pi^{-i}\sigma(x^2/4 + y)] \\ 0 & 1 & 0 & [\pi^{-i}\sigma(x/2)] \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, a, b, x, y \in \mathcal{O}/\pi^i \mathcal{O}$$

where $[\cdot] : \mathbb{F}_q(\pi) \to \mathbb{F}_q[\pi^{-1}]$ the integral part of an element and $\sigma : \mathcal{O}/\pi^i \mathcal{O} \to \mathcal{O}$ is a section.

Apply Lemma 3.3 to $H = \Gamma, \alpha, \beta, k = i - j, m = i + 1, \lambda = f(D(i, j)), \mu = f(D(i, j + 1))$ we get

$$|f(D(i, j)) - f(D(i, j + 1))| \leq 2q^{-\frac{1}{2}(i-j-2)(1-4/p)}\|m f\|_{MSp(\ell^2)}.$$
There exist $k \geq j - 2$ and $\alpha, \beta : (O/\pi^kO)^3 \to G$ such that when $y = a_1x_1 + a_2x_2 + b, \forall a_1, a_2, b, x_1, x_2, y \in O/\pi^kO$ we have

$$\alpha(a_1, a_2, b)\beta(x_1, x_2, y) \in KD(i, j)K,$$

and when $y = a_1x_1 + a_2x_2 + b + \pi^{k-1}$,

$$\alpha(a_1, a_2, b)\beta(x_1, x_2, y) \in KD(i + 1, j - 1)K.$$

The constructions of $\alpha, \beta$ are identical to $\alpha, \beta$ in the proof of the second inequality of proposition 3.2 in [Liao14] when $i + j$ is an even number, and identical to $\tilde{\alpha}, \tilde{\beta}$ when $i + j$ is odd. They are already used in the proof of Proposition 3.2 which we omit here.

By applying Lemma 3.3 to $H = \Gamma, n = 2, \alpha, \beta, k, \lambda = f(D(i, j)), \mu = f(D(i + 1, j - 1))$ we have

$$|f(D(i, j)) - f(D(i + 1, j - 1))| \leq 2q^{-(j-2)(1-3/p)}\|m_f\|_{MS^p(\ell^2\Gamma)}.$$

□

References

[BT] A. Borel and J. Tits. Groupes Réductifs. Publ. Math. l’I.H.E.S. 27 (1965) 55-151.

[BF] M. Bozejko and G. Fendler. Herz-Schur multipliers and completely bounded multipliers of the Fourier algebra of a locally compact group, Boll. Un. Mat. Ital. A (6) 3 (1984), 297-302. MR 0753889.

[BO] N. P. Brown and N. Ozawa. C*-Algebras and Finite-Dimensional Approximations, Grad. Stud. Math. 88. Amer. Math. Soc., Providence, 2008. MR 2391387.

[Enf] P. Enflo. A counterexample to the approximation property in Banach spaces. Acta Math., 130 (1973), 309-317.

[Gro] A. Grothendieck. Produits tensoriels topologiques et Espaces nucléaires. Mem. Amer. Math. Soc., 1955 (1955), no. 16, 140 pp.

[Haag] U. Haagerup. Group $C^*$-algebras without the completely bounded approximation property. Preprint, 1986.

[HdL13a] U. Haagerup and T. de Laat. Simple Lie groups without the approximation property. Duke Math. J. Volume 162, Number 5 (2013), 925-964.
[HdL13b] U. Haagerup and T. de Laat. Simple Lie groups without the approximation property II. Trans. Amer. Math. Soc.

[HKdL] U. Haagerup, S. Knudby and T. de Laat. A complete characterization of connected Lie groups with the Approximation Property. http://arxiv.org/abs/1412.3033

[HK] U. Haagerup and J. Kraus. Approximation properties for group $C^*$-algebras and group von Neumann algebras. Trans. Amer. Math. Soc. 344 (1994), 667-699. MR 1220905

[LMR] A. Lubotzky, S. Mozes and M.S. Raghunathan. The word and Riemannian metrics on lattices of semisimple groups. Publications Mathématiques de l’Institut des Hautes Études Scientifiques, December 2000, Volume 91, Issue 1, pp 5-53.

[dL] T. de Laat. Approximation properties for noncommutative $L^p$-spaces associated with lattices in Lie groups. J. Funct. Anal. 264 (2013), no. 10, 2300-2322.

[dLdlS] T. de Laat and M. de la Salle. Strong property (T) for higher rank simple Lie groups. http://arxiv.org/abs/1403.6415

[Laf10a] V. Lafforgue. Strong property (T) and the Baum-Connes conjecture, unpublished notes. École thématique autour de la conjecture de Baum-Connes et du principe d’Oka en géométrie non-commutative, 2010, Département de Mathématiques d’Orsay, Université de Paris-Sud 11.

[Laf10b] V. Lafforgue. Propriété (T) renforcée et conjecture de Baum-Connes, Quanta of maths, 323-345, Clay Math. Proc., 11, Amer. Math. Soc., Providence, RI, 2010

[LdlS] V. Lafforgue, M. De la Salle. Noncommutative $L^p$-spaces without the completely bounded approximation property Duke Math. J. Volume 160, Number 1 (2011), 71-116.

[Laf10c] V. Lafforgue. Un analogue non archimédien d’un résultat de Haagerup et lien avec la propriété (T) renforcée (anciennne version de 2010). http://vlafforg.perso.math.cnrs.fr/haagerup-rem-2010.pdf

[Liao13] B. Liao. Strong Banach property (T) for simple algebraic groups of higher rank. J. Topol. Anal., 06, 75 (2014). DOI: 10.1142/S1793525314500010

[Liao14] B. Liao. About the difficulty to prove the Baum Connes conjecture without coefficient for a non-cocompact lattice in $Sp_4$ in a local field. Submitted. http://arxiv.org/abs/1411.6151

18
[Laf08] V. Lafforgue. Un renforcement de la propriété \((T)\). *Duke Math. J.*, 143(3):559–602, 2008.

[Laf09] V. Lafforgue. Propriété \((T)\) renforcée banachique et transformation de Fourier rapide. *Journal of Topology and Analysis*, Volume: 1, Issue: 3(2009) pp. 191-206.

[Mar] G. A. Margulis. Discrete Subgroups of Semisimple Lie Groups. Springer-Verlag, 1991.

[Szan] A. Szankowski. \(B(H)\) does nol have the approximation properly. *Acta Math.* 147 (1981), 89-108.