Remarks on a Decrumpling Model of the Universe

J.A.S. Lima$^{1,2}$, M. Mohazzab$^{1,3}$

1) Physics Department, Brown University, Providence RI. 02912, USA.  
2) Departamento de Fisica, Universidade Federal do Rio Grande do Norte,  
59072-970, C.P. 1641 Natal, Rio Grande do Norte, Brazil.  
3) Institute for Studies in Theoretical Physics and Mathematics, P.O.Box  
5746, Tehran 19395, Iran.  
& Physics Dept. Alzahra University Tehran 19834, Iran

Abstract

It is argued that when the dimension of space is a constant integer the full set of Einstein’s field equations has more information than the spatial components of Einstein’s equation plus the energy conservation law. Applying the former approach to decrumpling FRW cosmology recently proposed, it is shown that the spacetime singularity cannot be avoided and that turning points are absent. This result is in contrast to the decrumpling nonsingular spacetime model with turning points previously obtained using the latter approach.

$^a$e-mail: limajas@het.brown.edu  
$^b$e-mail: mohazzab@het.brown.edu
1 Introduction

In the standard model of cosmology, the spacetime is usually assumed to be a differentiable manifold even at the very early stages of the evolution of the universe. However, many studies on the underlying structure of the spacetime suggest a more complex nature e.g., foam like[1] or even a fractal structure[2, 3].

Recently, a cosmological model has been proposed assuming that the basic building blocks of the spacetime has a cellular structure[4]. In the early universe this cellular space is crumpled so that a fractal (and therefore non-integer) dimension can naturally be defined for the whole space. In such a work the dimension of the space starts from a large (but not infinity) number when the universe is a minimum in its size. The expansion of the universe causes the wrinkled space to decrumple while the dimension of space decreases very fast to the observational value.

The Lagrangian equations of the model lead to an oscillatory universe (a universe with turning points) which may solve the horizon and the big bang singularity problems. Later on, this scenario was extended to the class of multidimensional cosmological models [5] where extra factor spaces play the role of matter fields. In this multidimensional cosmological model, an inflationary solution was found together with the prediction that the universe starts from a nonsingular spacetime.

In the approach of reference [4] the leading equations are a generalization of the space components of the Einstein field equations(EFE) to $D$ variable dimensions plus the generalized equation expressing the energy conservation law for a self-gravitating fluid.

In this letter we argue that the use of the space components of EFE plus the energy conservation law lead to results with less information than the use of the full set of equations. As we known, the time(00) component of the EFE is a constraint which can be easily implemented in the most simple cases, as happens for the class of Friedmann-Robertson-Walker(FRW) spacetimes. However, for more complex situations, like in the case of a decrumpling universe which is endowed with a fractal structure, such approaches give rise to completely different physical properties. At first sight, one may argue
that the discrepancy may arise from the fact that the generalization of the EFE to noninteger dimensions is not uniquely defined. However, we believe that the puzzle may in principle be solved if the constraint equation is further implemented when the energy conservation law has ab initio been considered.

In what follows, we first make explicit the difference between the two schemes in the case of the standard FRW model, and then an extension to the case of a decrumpling universe it will be discussed.

2 The Problem

For the sake of simplicity, let us consider the 3-dim spatially flat FRW line element:

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)$$  \hspace{1cm} (2.1)

where $a(t)$ is the scale factor. In this background, the non-trivial EFE for a comoving perfect fluid and the energy conservation law ($u_\alpha T^{\alpha \beta}_{\beta} = 0$), which is contained in the EFE, may be written as (in our units $8\pi G = c = 1$)

$$\rho = 3\left(\frac{\dot{a}}{a}\right)^2$$  \hspace{1cm} (2.2)

$$p = -2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2$$  \hspace{1cm} (2.3)

$$\dot{\rho} + 3(\rho + p)\frac{\dot{a}}{a} = 0$$  \hspace{1cm} (2.4)

where an overdot means time derivative and $\rho$ and $p$ are the energy density and pressure, respectively. From Bianchi identities, we know that (2.4) can be obtained of (2.2) and (2.3), by just eliminating the second derivative of the scale factor. As usual, it will be assumed that the material medium obeys the barotropic equation of state

$$p = (\gamma - 1)\rho$$  \hspace{1cm} (2.5)

where $0 \leq \gamma \leq 2$ is the “adiabatic index”. Combining the above equation with the EFE (2.2) and (2.3), we get the FRW differential equation describing
the evolution of the scale factor

\[ a\ddot{a} + \left( \frac{3\gamma - 2}{2} \right) \dot{a}^2 = 0 \]  \hspace{1cm} (2.6)

On the other hand, using (2.5), the energy conservation law (2.4) yields for the energy density \( \rho = \frac{B}{a^\gamma} \) and for the pressure \( p = (\gamma - 1) \frac{B}{a^{\gamma+1}} \), where \( B \) must be positive due to the weak energy condition (\( \rho > 0 \)). Now, inserting this value of \( p \) into (2.3) one obtains a quite different equation of motion

\[ a\ddot{a} + \frac{a^2}{2} + \frac{(\gamma - 1)Ba^{2-3\gamma}}{2} = 0 \]  \hspace{1cm} (2.7)

which reduces to (2.6) only in the case of a dust-filled universe (\( \gamma = 1 \)).

As one may check, the above equations (2.6) and (2.7) have, respectively, the following first integrals:

\[ \dot{a}^2 = Aa^{2-3\gamma} \]  \hspace{1cm} (2.8)

and

\[ \dot{a}^2 = \frac{F}{a} + \frac{Ba^{2-3\gamma}}{3} \]  \hspace{1cm} (2.9)

where \( A \) and \( F \) are two \( \gamma \)-dependent constants. As expected, if \( \gamma = 1 \) equations (2.8) and (2.9) become identical up to trivial identifications. Note also from (2.2) that \( A \) (like \( B \)) is positive definite, whereas in the approach leading to (2.9) the sign of \( F \) is arbitrary. If \( F \) is negative, for instance, (2.9) has a turning point i.e. a specific value of the scale factor, say \( a^* \), for which \( \dot{a}(a^*) = 0 \) and the motion is inverted. How this fictitious turning point is avoided? Only if one uses the constraint given by the (00) component of the EFE. As easily seen, such ambiguity is completely solved by using (2.2) since it implies that \( F = 0 \) with (2.3) reducing to (2.8) for \( B = 3A \). It thus follows that only the full set of EFE fix the unique physical solution for a given problem. Hence, if one uses the energy conservation law in a non trivial situation e.g., for anisotropic or inhomogeneous models, the constraint equation need to be further satisfied. Although quite familiar for cosmologists working with exact solutions, this result is apparently not well known, and potentially, it may generate paradoxes when new ingredients are added to the FRW geometries. This point will now be exemplified for the case of a decrumpling universe.
3 New setup of the equations of motion

In [4], using the space component of Einstein equation together with the energy conservation equation yields an oscillatory universe. Unlike the result of the previous section, the reason for finding two turning points is not the lack of knowledge (or not using) the time component of the Einstein equation, because the existence of the turning point does not depend on any constant.

Here we would like to change the approach of [4], by considering the time component of the Einstein equation instead of the energy conservation equation.

Let us now consider the D-dimensional FRW flat line element:

$$ds^2 = dt^2 - a^2(t)\delta_{ij}dx^idx^j,$$

where $i,j = 1, 2, ..., D$.

The generalized Lagrangian is given by [4]

$$\frac{L}{a_{D_0}^D} = -\frac{D(D-1)}{2\kappa}\left(\frac{\dot{a}}{a}\right)^2\left(\frac{a}{a_0}\right)^D + \left( -\frac{\dot{\rho}}{2} + \frac{\hat{\rho}Da^2}{2}\right) =: \mathcal{L} ,$$

(3.11)

where

$$\dot{\rho} := \rho\left(\frac{a}{a_0}\right)^D ,$$

(3.12)

and

$$\dot{\rho} := pa^{-2}\left(\frac{a}{a_0}\right)^D ,$$

(3.13)

with the constraint

$$\left(\frac{a}{\delta}\right)^D = e^C .$$

(3.14)

In the above equations, $a_0$ and $D_0$ are the present values for the scale factor and the dimension of the universe, respectively, $\delta$ is a fundamental length and $C$ is a dimensionless constant, which could be determined from observations. As shown in the Ref.[4], in the limit $C \to \infty$ the standard D-dimensional FRW model is recovered.

From (3.11) one obtains the following equation of motion:

$$(D - 1)\left\{\frac{\ddot{a}}{a} + \left[\frac{D^2}{2D_0} - 1 - \frac{D(2D - 1)}{2C(D - 1)}\right]\left(\frac{\dot{a}}{a}\right)^2\right\} + \kappa p\left(1 - \frac{D}{2C}\right) = 0 ,$$

(3.15)
which is equivalent to the spacelike components of the Einstein equations. Naturally, (3.15) is also the analogous to (2.3) in the 3-dimensional formulation. As remarked earlier, at this point we do not insert the expression for the pressure obtained by combining the energy conservation law with the equation of state as has been done in Ref. [4]. In order to obtain the alternative differential equation we notice that the (00)-component of the Einstein equation can be written as \((D \neq 1)\)

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{2\kappa \rho}{D(D - 1)} .
\] (3.16)

To get (3.16), we may simply variate the Lagrangian with respect to a lapse function. It is clear that this lapse function does not interfere either with the variation with respect to the scale factor or with the dimension.

Therefore, the equation (3.16) is the same generalization of the (00)-component of the EFE in \(D\) dimensions. We stress that in the case of integer dimension, the above equations ((3.13) and (3.16)) also contain the energy conservation law.

Now, combining (3.15), (3.16) and the constraint (2.5) we get the following evolution equation for the scale factor

\[
\frac{\ddot{a}}{a} + \left( \frac{D^2}{2D_0} - 1 - \frac{D(2D - 1)}{2C(D - 1)} \right) + \frac{1}{2} \left( \gamma(D) - 1 \right) D(1 - \frac{D}{2C}) \left( \frac{\dot{a}}{a} \right)^2 = 0 ,
\] (3.17)

where \(\gamma(D)\) is the “adiabatic index” for \(D\) spatial dimensions. It is clear that in the limit \(C \to \infty\), we recover the FRW equation (2.6). The above equation should be compared with (II-6) of Ref. [4].

The first integral of (3.17) is

\[
\left( \frac{\dot{a}}{a} \right)^2 = E e^{2C \int \frac{dD}{D^2} + \int \frac{f(D)}{D^2}} ,
\] (3.18)

where the function \(f(D)\) reads

\[
f(D) = \frac{D^2}{2D_0} - 1 - \frac{D(2D - 1)}{2C(D - 1)} + \frac{1}{2} \left( \gamma(D) - 1 \right) D(1 - \frac{D}{2C}) .
\] (3.19)

Inserting (3.19) into (3.18) and assuming a radiation dominated universe (i.e. \(\gamma(D) = 1 + \frac{1}{D}\)), we find
\[
\left( \frac{\dot{a}}{a} \right)^2 = E \frac{e^{\frac{\omega D}{C}} - \frac{\dot{\omega}}{C}}{D^2(D - 1)} \quad ,
\]
(3.20)

and combining this result with the constraint expression (3.14) it follows that
\[
\dot{D}^2 = D^2 E \frac{e^{\frac{\omega D}{C}} - \frac{\dot{\omega}}{C}}{C^2(D - 1)} \quad .
\]
(3.21)

The above equations should be compared with the equivalent ones of Ref.[4], which were obtained using the energy conservation law. As one may check, in the FRW limit, equation (3.21) yields \( \dot{D} = 0 \), whereas (3.20) reduces to (2.8) as it should be expected. Here as there, the nonexistence of turning points may easily be determined by examining the qualitative behavior of the above coupled first integrals. In this connection, we remark that (3.21) may be interpreted as the energy equation of a point mass system having one degree of freedom whose potential energy is given by
\[
V(D) = -D^2 e^{\frac{\omega D}{C}} \frac{\dot{\omega}}{C} \quad ,
\]
(3.22)

and the total energy \( H = \frac{1}{2} \dot{D}^2 + V(D) \) is identically zero. This potential (3.22) is singular at \( D = 1 \) and tends to \(-\infty\) either if \( D \to 1 \) or \( D \to \infty \). Since it is always negative there is no points where the “velocity” \( \dot{D} = 0 \), or equivalently there is no points at which the potential term equals the total energy. Similarly, using the constraint equation (3.14), one may see the scale factor present the same behavior. Therefore, in contrast with Ref.[4], the flat decrumpling universe in this approach does not have any turning points. If the universe starts from a large dimension (which corresponds to a small radius), while it expands indefinitely its dimension decreases, however, since there is no turning point(s), the dimension approaches continuously to its singular value \( D = 1 \).

4 Conclusion

We have shown by an explicit example that the use of the space components of the Einstein field equation plus the energy conservation law has, in
principle, less information than the full set of Einstein’s field equations. In this way, if the constraint equation has not been considered, the two calculations schemes lead, in general, to completely different physical properties. In the original decrumpling model, the result is a nonsingular and oscillatory universe, whereas in the approach followed here the universe expands and its dimension decreases continuously even to less than 3, approaching asymptotically to $D = 1$ when the scale factor is infinitely large.

In sum, the decrumpling universe worked out here is singular and evolves either in size or in the number of dimensions with no turning points.

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