CONCORDANCE INVARIANTS AND THE TURAEV GENUS

HONGTAEK JUNG, SUNGKYUNG KANG, AND SEUNGWON KIM

Abstract. We show that the differences between various concordance invariants of knots, including Rasmussen’s $s$-invariant and its generalizations $s_n$-invariants, give lower bounds to the Turaev genus of knots. Using the fact that our bounds are nontrivial for some quasi-alternating knots, we show the additivity of Turaev genus for a certain class of knots. This leads us to the first example of an infinite family of quasi-alternating knots with Turaev genus exactly $g$ for any fixed positive integer $g$, solving a question of Champanerkar-Kofman.

1. Introduction

A link diagram is alternating if its crossings alternate overpass and underpass when we travel along the diagram. An alternating link is a link that has an alternating link diagram.

Alternating links have many interesting properties, with one of them being that their minimal crossing number is realized by their alternating diagrams. This property was originally conjectured by Tait in 1880’s and proved by several mathematicians using the Jones polynomial [Kau87, Mur87, Thi87]. Turaev [Tur87] gave a new proof of Tait’s conjecture by introducing the Turaev surface, which comes from the two extremal Kauffman states.

It turns out that the Turaev surface has its own interesting properties. For instance, it is a Heegaard surface of $S^3$, and the link has an alternating projection which gives a disk decomposition of the Turaev surface. Moreover, the Turaev genus, the minimal genus among all possible Turaev surfaces of a given knot, can be considered as a distance between a link and the set of alternating links. There are many previous works about computing Turaev genus [CKS07, Low08, Abe09, DL11, DL18], but no general method to compute its exact value is known yet.

In this paper, we find an infinite family of new lower bounds of the Turaev genus. To state our result, we need to introduce a class $DL$ of knot concordance invariants. One can find its definition in Section 3. We just remark here that this class $DL$ contains the following well-known invariants.

1. Rasmussen’s $s$-invariant;
2. $\frac{s_n}{1-n}$ for $n \geq 2$ where $s_n$ is the $s_n$ link invariant with a suitable normalization;
3. $-\sigma$, the negative of the knot signature;
4. $2v_0 - \ell + 1$ where $v_0$ is a slice-torus link invariant in the sense of Cavallo-Collari and $\ell$ is the number of link components. In particular, $2\tau - \ell + 1$, where $\tau$ is the Ozsváth-Szabó $\tau$-invariant.

Then our theorem on the lower bound to $g_T$ reads

Theorem 1. Let $\mu, \nu$ be in $DL$. Then for any knot $K$, we have

$$\frac{1}{2} |\mu(K) - \nu(K)| \leq g_T(K).$$

Note that our lower bound covers the previous result of Dasbach-Lowrance [DL11].

One can consider the Turaev surface as a Jones polynomial theoretic, or more generally, Khovanov homology theoretic way to generalize alternating links. Then one can ask what is the Heegaard Floer homology theoretic generalization of alternating links.

2010 Mathematics Subject Classification. Primary 57M25; secondary 57M27.

Key words and phrases. Quasi-alternating knots; $sl_n$ homology; Turaev genus.
In Heegaard Floer homology, alternating links are very simple objects, since the branched double cover \( \Sigma(K) \) of an alternating knot \( K \) has the simplest possible \( \hat{HF}(\Sigma(K)) \). In other words, \( \Sigma(K) \) is an \( L \)-space. This result leads one to define a natural generalization of alternating links; we say that a link is quasi-alternating if it admits a diagram so that its two possible resolutions at a crossing are again quasi-alternating and the sum of the determinants of two resolutions is the same as the determinant of the given link. Note that for any quasi-alternating knot \( K \), the branched double cover \( \Sigma(K) \) is an \( L \)-space.

All alternating links are quasi-alternating. However, distinguishing quasi-alternating knots from alternating ones is a hard problem. One natural way to do that is to consider the Turaev genus of a quasi-alternating knot and see whether it is nontrivial. However, all known lower bounds to the Turaev genus vanish for quasi-alternating knots. Champanerkar and Kofman \[CK09, CK14\] asked whether the Turaev genus of quasi-alternating knots can take any non-negative integral value.

As a corollary of Theorem 1, we answer the question of Champanerkar-Kofman in the affirmative way. More precisely, we will prove the following theorem.

**Theorem 2.** For each integer \( g > 0 \), there are infinitely many quasi-alternating knots whose Turaev genus equals \( g \).

**Remark 1.** Previously, the largest known Turaev genus of quasi-alternating knots was two. The Turaev genus two quasi-alternating knots can be identified by combining the work of Dasbach and Lowrance \[DL18\] and the work of Slavik Jablan \[Jab14\]. They are \( 11n95, 12n253, 12n254, 12n280, 12n323, 12n356, 12n375, 12n452, 12n706, 12n729, 12n811, 12n873 \).

We finish this section by giving a sketch of the proof of Theorem 2 assuming Theorem 1. As a consequence of Theorem 1, one can show the additivity of \( g_T \) for a certain class of knots.

**Corollary 3.** Let \( K \) be a knot such that \( s(K) + \limsup_{n \to \infty} \frac{s_n(K)}{n} \geq 2g_T(K) \). Then

\[
s(K) + \limsup_{n \to \infty} \frac{s_n(K)}{n} = 2g_T(K).
\]

In particular, if \( K \) and \( L \) are knots with the prescribed property, then

\[
g_T(L \# K) = g_T(L) + g_T(K).
\]

We then argue that the \((p,q)\) pretzel knots \( K_{p,q} = P(2p + 1, -2q - 1, 2) \) with \( p \geq q \) enjoy the assumption in Corollary 3. Due to \[CK09\] Theorem 3.2, these knots \( K_{p,q} \) are known to be quasi-alternating. Since being quasi-alternating is invariant under taking the connected sum, we may deduce Theorem 2 by setting \( K = \#^g K_{p,q} \).

**Acknowledgements:** The authors would like to thank Lukas Lewark, Adam Lowrance, Peter Feller, and Ilya Kofman for helpful conversations. The authors owe Remark 1 to Adam Lowrance.

This work was supported by Institute for Basic Science (IBS-R003-D1).

2. **Preliminaries**

In this section, we review our main objects of interest, quasi-alternating knots and Turaev surfaces.

2.1. **Quasi-alternating knots.** Quasi-alternating links were first introduced by Ozsváth and Szabó \[OS05\] to generalize alternating links in terms of Heegaard Floer theory.

**Definition 4.** The set of quasi-alternating links \( Q \) is the smallest set of links such that:

1. \( \{\text{unknot}\} \subseteq Q \);
2. if a link \( L \) contains a crossing \( c \) so that the two smoothings \( L_0 \) and \( L_1 \) of \( L \) at \( c \) are in \( Q \) and \( \det(L) = \det(L_0) + \det(L_1) \), then \( L \in Q \).

We remark that the connected sum of two quasi-alternating knots is again quasi-alternating.

Ozsváth and Szabó \[OS05\] showed that the double branched cover of \( S^3 \) along a quasi alternating link is always an \( L \)-space, which is a rational homology 3-sphere with the simplest Heegaard Floer homology. In fact, this \( L \)-space is interesting object since it is related to the taut foliation and left-orderability of fundamental group of 3-manifolds, which are fundamental objects in 3-dimensional topology.
2.2. Turaev surfaces and Turaev genus. Let \( D \subset S^2 \) be a link diagram of a link \( L \). For each crossing \( \times \) we can obtain the \( A \)-smoothing \( \times \) or the \( B \)-smoothing \( \approx \) as in the Kauffman bracket. If we smooth every crossing, then we get a set of simple loops on \( S^2 \) called a state.

There are two extreme cases of states. Suppose that we take the \( A \)-smoothing for each crossing. The resulting state is called the all-\( A \) state and is denoted as \( s_A \). The other extreme is to take the \( B \)-smoothing for each crossing, which yields the all-\( B \) state, denoted as \( s_B \).

The Turaev surface is a closed orientable surface obtained in the following way: first, we push \( s_A \) to above \( D \) and \( s_B \) below \( D \). Then we make a cobordism from \( s_A \) to \( s_B \) as in Figure 2.2 by adding saddles for each crossing as in Figure 2.1. The Turaev surface \( F(D) \) is then obtained by capping each boundary component of such cobordism off with a disk. The genus of \( F(D) \) is called the Turaev genus of a diagram \( D \), denoted by \( g_T(D) \). The minimal value of \( g_T(D) \) over all possible diagrams \( D \) of \( L \) is called the Turaev genus \( g_T(L) \) of a link \( L \).

![Figure 2.1. A local model of a Turaev surface near a saddle](KK19).

![Figure 2.2. A Turaev surface, before capping-off](KK19).

We record some properties of the Turaev genus \( g_T \) that are relevant to our discussion.
- If \( K \) is alternating, we have \( g_T(K) = 0 \).
- For given two knots \( K_1, K_2 \), we have \( g_T(K_1 \natural K_2) \leq g_T(K_1) + g_T(K_2) \) where \( \natural \) denotes the connected sum.

Note that these properties make Turaev genus an alternating distance - a distance which measures how far a given link is from being alternating.

3. Main results

This section is devoted to proving our main results Theorem 1 and Theorem 2.

**Definition 5.** An oriented link invariant \( \nu \) which is invariant under overall orientation reversal is called a **DL invariant** if it satisfies the following conditions.

1. Suppose that an oriented link \( L \) is obtained from a knot \( K \) by performing \( n \)-oriented band surgeries. Then we have
   \[
   |\nu(K) - \nu(L)| \leq n.
   \]
2. For any positive (resp. negative) diagram \( D \) of a non-split positive (resp. negative) link \( L \),
   \[
   s_B(D) - n_-(D) - 1 \leq \nu(L) \leq 1 + n_+(D) - s_A(D),
   \]
where \( s_A(D) \) and \( s_B(D) \) denote the number of components in the all-A and all-B resolutions of \( D \), respectively, and \( n_+(D) \) denotes the number of positive (or negative) crossings in \( D \).
We denote by $DL$ the set of DL invariants.

Dasbach and Lowrance implicitly proved the following lemma in [DL11] to get the lower bounds of the Turaev genus.

**Lemma 6 (Dasbach-Lowrance).** The concordance invariants $-\sigma$ and $s$ are DL invariants. Here, we are taking the extension of $s$ to links as defined by Beliakova and Wehrli in [BW08]. Also, $2\tau$ satisfies the inequality in condition (2) in Definition 5 for knots.

The fact that $-\sigma$ and $s$ are DL invariants can be generalized to the setting of slice-torus invariants. Cavallo and Collari defined the notion of slice-torus link invariants in [CC18]. We can prove that any slice-torus link invariant naturally induces a DL invariant.

**Lemma 7.** Let $\nu_0$ be a slice-torus link invariant in the sense of Cavallo-Collari. Then the oriented link invariant $\nu = 2\nu_0 - \ell + 1$, where $\ell$ denotes the number of components, is a DL invariant.

**Proof.** From the Cavallo-Collari axioms [CC18], we know that if $L_1$ and $L_2$ are related by an oriented band move and $\ell(L_1) = \ell(L_2) - 1$, then $\nu_0(L_2) - 1 \leq \nu_0(L_1) \leq \nu_0(L_2)$. Hence $|\nu(L_1) - \nu(L_2)| \leq 1$, so we see that $\nu$ satisfies axiom (1) of DL invariants. To prove axiom (2), let $L$ be a non-split positive (negative) link and $D$ be its positive (negative) diagram. Since the value of Cavallo-Collari slice-torus link invariants coincide for all positive links and negative links, we only have to prove that the $s$-invariant satisfies the desired inequality, which was already given by the fact that $s$ is a DL invariant. □

**Corollary 8.** The normalized $sl_n$ link invariants $\frac{s_n}{n}$ are DL invariants.

**Proof.** $\frac{s_n + (\ell - 1)(n-1)}{2(n-1)}$ is a Cavallo-Collari slice-torus invariant for each $n$, as mentioned in [CC18, Example 2.4]. □

Now we are ready to prove that the Dasbach-Lowrance inequality also holds for all DL invariants.

**Lemma 9.** Let $K$ be a knot, $D$ be its diagram, and $\nu$ be a DL invariant. Then we have

$$s_B(D) - n_-(D) - 1 \leq \nu(K) \leq 1 + n_+(D) - s_A(D).$$

**Proof.** Note that $A$-smoothing is the oriented resolution for positive crossings. Consider the link diagram we get by $A$-smoothing all positive crossings of $D$ by $D_-$. Then we can form a graph $\Gamma$ whose vertices are connected components of $D_-$ and edges are positive crossings of $D$. Choose a spanning tree $T^+$ of $\Gamma$. Then we can $A$-smooth all positive crossings except those which form an edge of $T^+$ to get a new link diagram $D^{\text{conn}}$. This process is drawn for a case of knot 6_2 in Figure 3.1.

![Figure 3.1](image)

**Figure 3.1.** Left, a diagram of the knot 6_2 with two positive crossings. Center-left, the negative diagram one gets by $A$-smoothing all positive crossings. Center-right, the graph $\Gamma$, where the red edge denotes the spanning tree $T^+$. Right, the diagram $D^{\text{conn}}$. We claim that $D^{\text{conn}}$ is isotopic to a negative diagram. To prove this, we choose a component $D_0$ of $D_-$, which corresponds to a leaf of the tree $T^+$. Such a component must be innermost, which means that the crossings of $D$ that correspond to edges of $T^+$ do not lie on any of the bounded components of $\mathbb{R}^2 \setminus D_0$. 
The component $D_0$ is then connected to exactly one positive crossing in $D^\text{conn}_-$, so we can simply untwist to remove that crossing because $D_0$ is innermost. Repeating this process inductively gives us a negative diagram $D'$ which is isotopic to $D^\text{conn}_-$.

Since $D'$ is negative and connected, we have $n_+(D') = 0$, and the link $K^-$ represented by $D'$ is non-split. Also, by the construction of $D'$, the all-$A$ smoothing of $D'$ is isotopic to the smoothing of $D$ one gets by $A$-smoothing all crossings except those arising as edges of $T^+$ and then $B$-smoothing the rest. Hence we have $s_A(D') = s_A(D) - \sharp E(T^+)$, where $\sharp E(T^+)$ denotes the number of edges in $T^+$.

Now we can prove the given inequality. Note that the link $K^-$ is formed by performing $n_+(D) - \sharp E(T)$ oriented band surgeries on $K$. Thus there exists an oriented cobordism of Euler characteristic $-n_+(D) + \sharp E(T^+)$ between $K$ and $K^-$, so we have the inequality

$$|\nu(K) - \nu(K^-)| \leq n_+(D) - \sharp E(T^+).$$

Furthermore, since $K^-$ is a non-split negative link and $D'$ is a negative diagram representing it, we also have

$$\nu(K^-) \leq 1 + n_+(D') - s_A(D') = 1 - s_A(D) + \sharp E(T^+).$$

Therefore we get

$$\nu(K) \leq \nu(K^-) + n_+(D) - \sharp E(T^+) \leq 1 + n_+(D) - s_A(D),$$

which proves the upper bound.

To prove the lower bound, we simply dualize our arguments. Instead of $A$-smoothing positive crossings of $D$, we can $B$-smooth its negative crossings to construct a positive link diagram $D_+$. Then we can apply the spanning tree argument to construct a tree $T^-$ and a connected positive diagram $D^\text{conn}_+$.

Denote the non-split link represented by the diagram $D^\text{conn}_+$ as $L^+$. Then $L^+$ is obtained from $K$ by performing $n_-(D) - \sharp E(T^-)$ oriented band surgeries. Therefore we get

$$\nu(K) \geq \nu(L^+) - n_-(D) + \sharp E(T^-) \geq 1 - n_-(D) + s_B(D),$$

completing the proof.

**Proof of Theorem 7.** This follows from Lemma 9 and the fact that

$$1 + n_+(D) - s_A(D) - (s_B(D) - n_-(D) - 1) = g_T(D).$$

**Proof of Theorem 3.** For any positive integers $p, q$ satisfying $p \geq q$, we consider the pretzel knot $K_{p,q} = P(2p + 1, -2q - 1, 2)$. It is proven in Theorem 3.2 of [CK09] that such knots are always quasi-alternating, so its $s$-invariant is determined by its signature. More specifically, we have

$$s(K_{p,q}) = \sigma(K_{p,q}) = 2(p - q).$$

On the other hand, Lewark [Lew14] showed that the value $s_{\|}(K_{p,q})$ can be either $2(p-q)-2$ or $2(p-q)-2+\frac{2}{n-1}$. So, by Theorem 7 we see that the following inequality holds for all integers $g \geq 1$ and $n \geq 2$, where $g_k K_{p,q}$ denotes the connected sum of $g$ copies of $K_{p,q}$:

$$g - \frac{g}{n-1} \leq g_T(g_k K_{p,q}).$$

Taking the limit $n \to \infty$ on both sides gives $g \leq g_T(g_k K_{p,q})$.

From the direct application of Turaev surface algorithm to such pretzel knot diagrams, their Turaev genus are all at most one. By the subadditivity of Turaev genus, we get

$$g_T(g_k K_{p,q}) \leq g \cdot g_T(K_{p,q}) \leq g.$$ 

Therefore we get $g_T(g_k K_{p,q}) = g$. Since quasi-alternating knots are closed under connected sum, we obtained an infinite family of quasi-alternating knots with Turaev genus exactly $g$.
Remark 2. One can show that all DL invariants take the same value for homogeneous knots, simply by mimicking the proof of \cite[Theorem 5]{Lew14}. Since alternating knots are homogeneous, our lower bounds for the Turaev genus vanish for alternating knots. In contrast, quasi-alternating knots are not always homogeneous, and for those knots, the $s_n$ invariants can behave differently from $s$, $\tau$, or $\sigma$. This is why our lower bounds can be nonzero for some quasi-alternating knots.

References

\begin{itemize}
  \item [Abe09] Tetsuya Abe, \emph{The turaev genus of an adequate knot}, Topology and its Applications 156 (2009), no. 17, 2704–2712.
  \item [BW08] Anna Beliakova and Stephan Wehrli, \emph{Categorification of the colored jones polynomial and Rasmussen invariant of links}, Canadian Journal of Mathematics 60 (2008), no. 6, 1240–1266.
  \item [CC18] Alberto Cavallo and Carlo Collari, \emph{Slice-torus concordance invariants and whitehead doubles of links}, Canadian Journal of Mathematics (2018), 1–40.
  \item [CK09] Abhijit Champanerkar and Ilya Kofman, \emph{Twisting quasi-alternating links}, Proceedings of the American Mathematical Society 137 (2009), no. 7, 2451–2458.
  \item [CK14] Abhijit Champanerkar and Ilya Kofman, \emph{A survey on the turaev genus of knots}, Acta Mathematica Vietnamica 39 (2014), no. 4, 497–514.
  \item [CKS07] Abhijit Champanerkar, Ilya Kofman, and Neal Stoltzfus, \emph{Graphs on surfaces and khovanov homology}, Algebraic & Geometric Topology 7 (2007), no. 3, 1531–1540.
  \item [DL11] Oliver T Dasbach and Adam M Lowrance, \emph{Turaev genus, knot signature, and the knot homology concordance invariants}, Proceedings of the American Mathematical Society (2011), 2631–2645.
  \item [DL18] Oliver T. Dasbach and Adam M. Lowrance, \emph{Invariants for Turaev genus one links}, Comm. Anal. Geom. 26 (2018), no. 5, 1103–1126. MR 3900481
  \item [Jab14] Slavik Jablan, \emph{Tables of quasi-alternating knots with at most 12 crossings}, arXiv preprint arXiv:1404.4965 (2014).
  \item [Kau87] Louis H Kauffman, \emph{State models and the jones polynomial}, Topology 26 (1987), no. 3, 395–407.
  \item [KK19] Seungwon Kim and Ilya Kofman, \emph{Turaev surfaces}, arXiv preprint arXiv:1901.09995 (2019).
  \item [Lew14] Lukas Lewark, \emph{Rasmussen’s spectral sequences and the $sl_N$-concordance invariants}, Advances in mathematics 260 (2014), 59–83.
  \item [Low08] Adam Lowrance, \emph{On knot floer width and turaev genus}, Algebraic & Geometric Topology 8 (2008), no. 2, 1141–1162.
  \item [Mur87] Kunio Murasugi, \emph{Jones polynomials and classical conjectures in knot theory}, Topology 26 (1987), no. 2, 187–194.
  \item [OS05] Peter Ozsváth and Zoltán Szabó, \emph{On the heegaard floer homology of branched double-covers}, Advances in Mathematics 194 (2005), no. 1, 1–33.
  \item [Thi87] Morwen B Thistlethwaite, \emph{A spanning tree expansion of the jones polynomial}, Topology 26 (1987), no. 3, 297–309.
  \item [Tur87] V. G. Turaev, \emph{A simple proof of the Murasugi and Kauffman theorems on alternating links}, Enseign. Math. (2) 33 (1987), no. 3-4, 203–225. MR 925987
\end{itemize}