Motion of four-dimensional rigid body around a fixed point: an elementary approach. I.

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Abstract

The goal of this note is to give the explicit solution of Euler-Frahm equations for the Manakov four-dimensional case by elementary means. For this, we use some results from the original papers by Schottky [Sch 1891], Kötter [Koe 1892], Weber [We 1878], and Caspary [Ca 1893]. We hope that such approach will be useful for the solution of the problem of $n$-dimensional top.

1. The equations of motion for a rigid body in a four-dimensional Euclidean space with a fixed point coinciding with the center of mass (and also for the $n$-dimensional case) are the generalization of famous Euler’s equations. They were found first by Frahm [Fr 1874] and they have the form

$$\dot{l}_{ij} = \sum_{k=1}^{4} (l_{ik} \omega_{kj} - \omega_{ik} l_{kj}), \quad \omega_{ij} = c_{ij} l_{ij}, \quad l_{ij} = -l_{ji}, \quad i, j = 1, \ldots, 4. \quad (1)$$

Here $c_{ij} = I_{ij}^{-1}$, the dot denotes the derivative with respect to time $t$, and $l_{ik}, \omega_{jk},$ and $I_{ik}$ are components of angular momentum, angular velocity and principal momenta of inertia tensors, respectively.

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1The problem of generalization of Euler’s equations was posed by Cayley [Ca 1846].
In this paper we consider completely integrable Manakov’s case [Ma 1977], when quantities $c_{ij}$ have the form

$$c_{ij} = \frac{b_i - b_j}{a_i - a_j}. \quad (2)$$

In a number of papers (see [AM 1982], [Ha 1983], [AM 1988], and references therein) so called method of linearization on the Jacobian of a spectral curve defined by the characteristic polynomial of one of the matrix in the Lax pair was used. However, as it was mention in [AM 1988], ”this approach has remained unsatisfactory; indeed (i) finding such families of Lax pairs often requires just as much ingenuity and luck as to actually solve the problem; (ii) it often conceals the actual geometry of the problem”.

So, in the present note we return to the original Schottky–Kötter approach [Sch 1891], [Koe 1892]. In our opinion, this elementary and natural approach is more adequate for the problem under consideration. We hope that it will be useful also for the more complicated problem of $n$-dimensional top at $n > 4$.

Let us remind that in the paper [Sch 1891] the problem under consideration was reduced to the Clebsch problem [Cl 1871] of the motion of a rigid body in an ideal fluid³. For the special cases, the last problem was integrated explicitly by Weber [We 1878] and by Kötter [Koe 1892].

However, the Clebsch problem is related not to $so(4)$ Lie algebra but to the $e(3)$ Lie algebra – the Lie algebra of motion of the three-dimensional Euclidean space. Hence, it is important to extend the Schottky–Kötter approach to give the solution in $so(4)$ covariant form. Here we give such a solution using the elementary means⁴.

2. Note first at all that equations (1) are Hamiltonian with respect to the Poisson structure for the $so(4)$ Lie algebra – the Lie algebra of rotations of the four-dimensional Euclidean space,

$$\{l_{ij}, l_{km}\} = l_{im} \delta_{jk} - l_{ik} \delta_{jm} + l_{jk} \delta_{im} - l_{jm} \delta_{ik}. \quad (3)$$

²Note that for the ”physical” rigid body $c_{ij} = I_j^{-1}$, $I_{ij} = I_i + I_j$. In this paper we consider a general integrable case when quantities $c_{ij}$ and $I_{ij}$ are arbitrary.
³This result was rediscovered one century later in the paper [Bo 1986].
⁴A special $so(4)$ case with tensor $l_{jk}$ of rank 2 was integrated explicitly by Moser [Mo 1980].
The Hamiltonian is given by the formula

$$H = \frac{1}{2} \sum_{j<k} c_{jk} l^2_{jk},$$  \hspace{1cm} (4)$$

where quantities $c_{ij}$ are given by formula (2), and equations (1) may be written in the form

$$\dot{l}_{jk} = \{H, l_{jk}\}.$$  \hspace{1cm} (5)$$

Let us remind that equations (1) have four integrals of motion

$$H_0 = l_{12} l_{34} + l_{23} l_{14} + l_{31} l_{24} = h_0,$$  \hspace{1cm} (6)$$

$$H_1 = \sum_{j<k} l^2_{jk} = h_1, \quad H_2 = \sum (a_j + a_k) l^2_{jk} = h_2, \quad H_3 = \sum a_j a_k l^2_{jk} = h_3. \hspace{1cm} (7)$$

Note that $H_0$ and $H_1$ are the Casimir functions of $so(4)$-Poisson structure, and the manifold $\mathcal{M}_h$ defined by equations (6) – (7) is an affine part of two-dimensional Abelian manifold (see Appendix by Mumford to the paper [AM 1982])\textsuperscript{5}. Then formula (5) defines Hamiltonian vector field on $\mathcal{M}_h$.

The main result of this note is the following one: by elementary means, it is shown that the dynamical variables $l_{jk}(t)$ are expressed in terms of Abelian functions $f_{j4}(u_1, u_2)$, $f_{kl}(u_1, u_2)$, $f_0(u_1, u_2)$, and $g(u_1, u_2)$ related to genus two algebraic curve

$$y^2 = \prod_{j=0}^{4} (x - d_j), \quad d_0 = 0, \quad d_4 = d_1 d_2 d_3,$$  \hspace{1cm} (8)$$

with arguments depending linearly on time.

**Theorem.** Solution of equations (1) has the form

$$m_j = l_{kl} = g(u_1, u_2) (\alpha_j f_{kl}(u_1, u_2) + \beta_j f_{j4}(u_1, u_2)), \hspace{1cm} (9)$$

$$n_j = l_{j4} = g(u_1, u_2) (\gamma_j f_{kl}(u_1, u_2) + \delta_j f_{j4}(u_1, u_2)). \hspace{1cm} (10)$$

Here $(j, k, l)$ is a cyclic permutation of $(1, 2, 3)$, $\alpha_j$, $\beta_j$, $\gamma_j$, $\delta_j$, and $d_j$ are algebraic functions of integrals of motion and quantities $a_j$ and $b_k$. Explicit expressions for them are given by (24)–(26), (34), (35), (41), and (44).

\textsuperscript{5}I am grateful to A. N. Tyurin for the explanation of algebraic geometry related to this Appendix.
Proof. The key problem is the "uniformization" of the manifold \( \mathcal{M}_h \), i.e., finding of the "good" coordinates on it. The proof consists of several steps.

A. Following Kötter [Koe 1892] and using the linear change of variables \( m_j \) and \( n_j \) to new variables \( \xi_j \) and \( \eta_j \), we transform equation (7) to the more appropriate form:

\[
\begin{align*}
3 \sum_{j=1}^{3} (\xi_j^2 + \eta_j^2) &= 0, \\
3 \sum_{j=1}^{3} \xi_j \eta_j &= 0, \\
3 \sum_{j=1}^{3} (d_j \xi_j^2 + d_j^{-1} \eta_j^2) &= 0.
\end{align*}
\]

(11)

For this, following Schottky [Sch 1891], let us introduce the three-dimensional vector \( \mathbf{l}(s) \) depending on parameter \( s \):

\[
\mathbf{l}(s) = (l_1(s), l_2(s), l_3(s)), \quad l_j(s) = \sqrt{s_{j4}} m_j + \sqrt{s_{kl}} n_j,
\]

(12)

where

\[
m_j = l_{kl}, \quad n_j = l_{j4}, \quad s_{jk} = (s - a_j)(s - a_k),
\]

(13)

and \( \{j, k, l\} \) is a cyclic permutation of \( \{1, 2, 3\} \). It is easy to check that the function

\[
f(s) = \mathbf{l}(s)^2 = \sum_{j=1}^{3} l_j(s) l_j(s)
\]

(14)

does not depend on time. So, it is the generating function of integrals of motion

\[
f(s) = h_1 s^2 - h_2 s + h_3 + 2 h_0 \sqrt{G(s)}, \quad G(s) = \prod_{j=1}^{4} (s - a_j).
\]

(15)

From formulae (12) and (14) it is easy to get the Lax representation\(^6\)

\[
\dot{L}(s) = [L(s), M(s)],
\]

(16)

where \( L(s) \) and \( M(s) \) are antisymmetric matrices of the third order corresponding to vectors \( \mathbf{l}(s) \) and \( \mathbf{m}(s) \),

\[
\mathbf{m}(s) = (m_1(s), m_2(s), m_3(s)), \quad m_j(s) = \sqrt{s_{kl}} m_j + \sqrt{s_{j4}} n_j,
\]

(17)

\(^6\) However, this representation does not need for the proof of Theorem. For the generalization of such representation for the \( n \)-dimensional case see [Fe 2000].
\[
L(s) = \begin{pmatrix}
0 & l_3 & -l_2 \\
-l_3 & 0 & l_1 \\
l_2 & -l_1 & 0
\end{pmatrix},
\quad
M(s) = \begin{pmatrix}
0 & m_3 & -m_2 \\
-m_3 & 0 & m_1 \\
m_2 & -m_1 & 0
\end{pmatrix}.
\] (18)

The equation \( f(s) = 0 \) is equivalent to the algebraic equation of fourth degree \( F(s) = \prod_{j=1}^{4} (s - s_j) = 0 \), where

\[
F(s) = \left[ (h_1 s^2 - h_2 s + h_3)^2 - 4 h_0^2 G(s) \right] / (h_1^2 - 4 h_0^2).
\] (19)

This equation has four roots \( s_1, s_2, s_3, \) and \( s_4 \) that, in general, are complex ones. To them correspond four complex vectors

\[
l^{(p)} = l^{(p)}(s_p) / \sqrt{F'(s_p)}, \quad p = 1, 2, 3, 4,
\] (20)

(here \( F'(s) \) is the derivative of \( F(s) \)) but only two of them, for example \( l^{(1)} \) and \( l^{(2)} \), are linearly independent, and

\[
\left( l^{(p)} \right)^2 = \sum_{k=1}^{3} \left( l^{(p)}_k \right)^2 = 0, \quad p = 1, 2, 3, 4;
\]

\[
\sum_{p=1}^{4} \left( l^{(p)}_k \right)^2 = 0, \quad k = 1, 2, 3.
\] (21)

Let us introduce also the vectors \( \xi \) and \( \eta \) by the formulae \(^7\)

\[
\xi_j = l^{(1)}_j + i l^{(2)}_j, \quad \eta_j = l^{(1)}_j - i l^{(2)}_j.
\] (22)

Using (12) and (22) we may express \( m_j \) and \( n_j \) in terms of \( \xi_j \) and \( \eta_j \)

\[
m_j = \alpha_j \xi_j + \beta_j \eta_j, \quad n_j = \gamma_j \xi_j + \delta_j \eta_j,
\] (23)

where

\[
\alpha_j = \sqrt{s_{kl}^{(2)} / F'(s_2) - i \sqrt{s_{kl}^{(1)} / F'(s_1)}} / \Delta_j^{(3)}, \quad \beta_j = \sqrt{s_{kl}^{(2)} / F'(s_2) + i \sqrt{s_{kl}^{(1)} / F'(s_1)}} / \Delta_j^{(3)},
\]

\[
\gamma_j = \sqrt{s_{j4}^{(2)} / F'(s_2) - i \sqrt{s_{j4}^{(1)} / F'(s_1)}} / \Delta_j^{(3)}, \quad \delta_j = \sqrt{s_{j4}^{(2)} / F'(s_2) + i \sqrt{s_{j4}^{(1)} / F'(s_1)}} / \Delta_j^{(3)}.
\] (24)

\(^7\) As it was noted by Yu. N. Fedorov, there is relation of these vectors to the problem of geodesics on two-dimensional ellipsoid with half-axes \( \sqrt{a_j}, j=1,2,3 \). Namely, \( \xi \) may be considered as a tangent vector to geodesics and \( i \eta \) as a normal vector to this geodesics.
Here \((j, k, l)\) and \((p, q, r)\) are cyclic permutations of \((1, 2, 3)\). Now it is easy to check that equations (7) take the form of three Kötter’s quadrics (11), where

\[
\sqrt{d_j} = \frac{\Delta_j^{(1)} - i\Delta_j^{(2)}}{\Delta_j^{(3)}} , \quad \frac{1}{\sqrt{d_j}} = -\frac{\Delta_j^{(1)} + i\Delta_j^{(2)}}{\Delta_j^{(3)}} . \tag{26}
\]

B. Following [Koe 1892], let us show that the manifold defined by equations (11) may be ”uniformized” by means of the Weierstrass Wurzelfunctionen related to the hyper-elliptic curve (8) that are defined as

\[
P_j(z_1, z_2) = \sqrt{(z_1 - d_j)(z_2 - d_j)} , \quad j, k = 0, 1, 2, 3, 4, \tag{27}
\]

\[
P_{jk}(z_1, z_2) = \frac{P_j P_k}{(z_1 - z_2)} \left( \sqrt{R(z_1)} (z_1 - d_j)(z_1 - d_k) - \sqrt{R(z_2)} (z_2 - d_j)(z_2 - d_k) \right) . \tag{28}
\]

These sixteen functions \(P_j(z_1, z_2)\) and \(P_{jk}(z_1, z_2)\) satisfy a lot of identities. All of them may be obtained from definitions (27) and (28) (for details, see [We 1878], [Koe 1892], and [Ca 1893]). Here we give only few of them which are useful for us:

\[
\sum_{j=1}^{3} c_j \left( \frac{P_{kl}^2}{(s - d_k)(s - d_l)} + \frac{P_{j4}^2}{(s - d_j)(s - d_4)} \right) = \frac{s}{\prod_{j=1}^{4}(s - d_j)} , \tag{29}
\]

\[
\sum_{j=1}^{3} \tilde{c}_j P_{j4}^2 = d_4 , \quad \sum_{j=1}^{3} d_j \tilde{c}_j P_{kl}^2 = P_0^2 , \tag{30}
\]

\[
\sum_{j=1}^{3} c_j P_{j4} P_{kl} = 0 , \quad \sum_{j=1}^{3} \tilde{c}_j P_{j4} P_{kl} = -P_0 , \tag{31}
\]

\[
\sum_{j=1}^{3} c_j \left( d_j^{-1} P_{j4}^2 + d_j P_{kl}^2 \right) = 0 , \tag{32}
\]

\[\text{See also modern survey [BEL 1997].}\]
where
\[ \tilde{c}_j = \frac{1}{(d_j - d_k)(d_j - d_l)}, \quad c_j = \frac{d_j - d_4}{(d_j - d_k)(d_j - d_l)}. \]  
(33)

It is known (see [We 1878]) that \( P_j(z_1, z_2) \) and \( P_{jk}(z_1, z_2) \) up to the factors are the ratio of the theta functions with half-integer theta characteristics 9
\[ P_j(z_1, z_2) = f_j(u_1, u_2) = \frac{\theta_j(u_1, u_2)}{\theta(u_1, u_2)}, \quad P_{kl}(z_1, z_2) = f_{kl}(u_1, u_2) = \frac{\theta_{kl}(u_1, u_2)}{\theta(u_1, u_2)}, \]  
(34)

\[ \theta_{23}(u_1, u_2) = \theta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}(u_1, u_2), \quad \theta_{31}(u_1, u_2) = \theta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}(u_1, u_2), \]
\[ \theta_{12}(u_1, u_2) = \theta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}(u_1, u_2), \quad \theta_{14}(u_1, u_2) = \theta \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}(u_1, u_2), \]
\[ \theta_{24}(u_1, u_2) = \theta \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}(u_1, u_2), \quad \theta_{34}(u_1, u_2) = \theta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}(u_1, u_2), \]
\[ \theta_0(u_1, u_2) = \theta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}(u_1, u_2), \quad \theta(u_1, u_2) = \theta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}(u_1, u_2). \]  
(35)

Here
\[ \theta(u_1, u_2) = \sum_{n_1, n_2} \exp\{i\pi(n_1(2u_1 + n_1\tau_{11} + n_2\tau_{12}) + n_2(2u_2 + n_1\tau_{21} + n_2\tau_{22}))\}, \]  
(36)

and \( \tau_{jk} \) are elements of period matrix.

The comparison (11) with (29)–(33) shows that
\[ \xi_j = \sqrt{c_j g} P_{kl}, \quad \eta_j = \sqrt{c_j g} P_{j4}, \]  
(37)

where \( g \) is an unknown function.

C. The rest part of the proof is the uniformization of equation (6).

Let us substitute expressions (23) for \( m_j \) and \( n_j \) into equation (6). Then by using of (24) and (25) we transform it to the form
\[ H_0 = \sum_{j=1}^{3} (A_j (\xi_j^2 - \eta_j^2) + B_j \xi_j \eta_j) = h_0. \]  
(38)

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9 We give here just one series of such expressions. Relative other series, see [Koe 1892].
where
\[ A_j = \alpha + \beta d_j + \gamma d_j^{-1}, \quad B_j = \delta \left( d_j + d_j^{-1} \right). \] (39)

Here \( \alpha, \beta, \gamma, \) and \( \delta \) are algebraic functions of \( h_0, h_1, h_2, h_3, a_j, \) and \( b_j. \)

This sum may be calculated by using of (24), (29)–(33). The result is
\[ H_0 = \frac{(1 - \varepsilon P_0)^2}{4 \varepsilon d_4} g^2 = h_0, \] (40)

where
\[ \varepsilon = \frac{\sqrt{d_4} \left( \sqrt{(s_3 - s_1)(s_2 - s_4)} - \sqrt{(s_2 - s_3)(s_1 - s_4)} \right)}{\sqrt{(s_1 - s_2)(s_3 - s_4)}}. \] (41)

From this we obtain
\[ g = (1 - \varepsilon f_0)^{-1}, \quad \xi_j = g \sqrt{c_j} f_{kl}, \quad \eta_j = g \sqrt{c_j} f_{j4}. \] (42)

The fact of linear dependence of arguments \( u_1 \) and \( u_2 \) on time \( t \) follows as from the algebraic geometrical approach [AM 1982] as from the old Kötter approach [Koe 1892].

This completes the proof of Theorem.

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