Data–based price discrimination: information theoretic limitations and a minimax optimal strategy

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Abstract

This paper studies the gap between the classical pricing theory and the data–based pricing theory. We focus on the problem of price discrimination with a continuum of buyer types based on a finite sample of observations. Our first set of results provides sharp lower bounds in the worst–case scenario for the discrepancy between any data–based pricing strategies and the theoretical optimal third–degree price discrimination (3PD) strategy (respectively, uniform pricing strategy) derived from the distribution (where the sample is drawn) ranging over a large class of distributions. Consequently, there is an inevitable gap between revenues based on any data–based pricing strategy and the revenue based on the theoretical optimal 3PD (respectively, uniform pricing) strategy. We then propose easy–to–implement data–based 3PD and uniform pricing strategies and show each strategy is minimax optimal in the sense that the gap between their respective revenue and the revenue based on the theoretical optimal 3PD (respectively, uniform pricing) strategy matches our worst–case lower bounds up to constant factors (that are independent of the sample size $n$). We show that 3PD strategies are revenue superior to uniform pricing strategies if and only if the sample size $n$ is large enough. In other words, if $n$ is below a threshold, uniform pricing strategies are revenue superior to 3PD strategies. We further provide upper bounds for the gaps between the welfare generated by our minimax optimal 3PD (respectively, uniform pricing) strategy and the welfare based on the theoretical optimal 3PD (respectively, uniform pricing) strategy.

Keywords: price discrimination, minimax optimality, empirical revenue maximization, information theory.

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1 Introduction

Uniform (or linear) pricing and third-degree price discrimination (3PD) are well-studied in classical economic theory. Uniform pricing refers to the situation where a seller sets the same price to a group of potential buyers, and 3PD refers to the situation where the seller observes the type of each buyer and sets (potentially) different prices for each type (Varian, 1989). In classical economic theory, the distribution from which a buyer’s valuation is drawn and the distribution of the buyer’s type is assumed to be known to the seller. In real life, the more plausible scenario is that the seller observes only an i.i.d. sample of valuations and type-defining characteristic values, \( \{ Y_i, X_i \}_{i=1}^n \), from a joint distribution, associated with a set of \( n \) buyers. Data on valuations and characteristics are often available from second-price auctions, in which buyers truthfully reveal their valuations. We study the revenue maximization problem where the seller engages in 3PD based on the finite sample \( \{ Y_i, X_i \}_{i=1}^n \).

In terms of generating revenue, the classical pricing theory states that 3PD is at least as good as uniform pricing when the joint distribution \( F_{Y,X} \) is known to the seller. If the seller only observes a finite sample of i.i.d. observations \( \{ Y_i, X_i \}_{i=1}^n \), she can choose a data-based uniform pricing strategy that ignores characteristic values \( \{ X_i \}_{i=1}^n \) and makes use of only valuations \( \{ Y_i \}_{i=1}^n \), or a 3PD strategy that makes use of both \( \{ X_i \}_{i=1}^n \) and \( \{ Y_i \}_{i=1}^n \). Relative to 3PD strategies, uniform pricing strategies charge buyers with different characteristic values the same price. This fact means that data-based uniform pricing strategies are poorer approximations of the theoretical optimal pricing strategy derived from \( F_{Y,X} \), but incur less variance (smaller “estimation” error) relative to data-based 3PD strategies. The trade-off between the quality of approximation and variance suggests that 3PD will be revenue superior to uniform pricing if and only if the sample size \( n \) is large enough.

The “only if” part is a mathematically challenging problem. To prove it, we first establish sharp lower bounds in the worst-case scenario for the discrepancy between any pricing strategies and the theoretical optimal 3PD (respectively, uniform pricing) strategy derived from the joint distribution \( F_{Y,X} \) (respectively, the marginal distribution \( F_Y \)) ranging over a large class of distributions. The 3PD problem is particularly difficult, and to establish the lower bounds, we convert the pricing problem into a multiple classification problem that tries to distinguish among \( M \) distributions, where \( M \) is a function of the sample size \( n \). Our proof is based on a delicate construction of conditional densities along with the Fano’s inequality from information theory and the Gilbert-Varshamov bound from coding theory: the key lies

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1 We assume that the cost is known to the seller. Therefore, the valuation can be treated as the net valuation.

2 By any pricing strategies, we mean any pricing function that takes the sample as inputs.
in finding a sufficiently large set of distributions (i.e., the cardinality $M$ is large enough) that are close to each other (small pairwise Kullback–Leibler divergence) but their optimal prices have a sufficiently large separation. Our information theoretic lower bounds reveal the fundamental and inevitable gap between revenues based on any pricing strategies and the revenue based on the theoretical optimal 3PD (respectively, uniform pricing) strategy.

To show the “if” part, we propose easy-to-implement data-based 3PD and uniform pricing strategies. We prove that these strategies are minimax optimal in the sense that the gap between their respective revenue and the revenue based on the theoretical optimal 3PD (respectively, uniform pricing) strategy matches the aforementioned lower bounds up to constant factors (that are independent of $n$). This result, together with our minimax lower bounds, provides the necessary and sufficient conditions on the sample size $n$ for 3PD to be revenue superior to uniform pricing.

Our proposed 3PD strategy is based on a “$K$–markets” approach, where we divide the buyers into $K(\equiv K_n)$ markets by splitting the characteristic space into $K$ equally spaced intervals. For each market $I_k, k = 1, \cdots, K$ (where $I_k$ is defined in Section 3), the seller charges the optimal price $\hat{p}_k$ from the empirical distribution of $\{Y_i : X_i \in I_k\}$. The gap between the revenue generated by the $K$–markets approach and the revenue based on the theoretical optimal 3PD strategy comes from two sources: one is related to the “variance” component or “estimation” error due to the randomness of the finite sample; the other is related to the approximation error, due to the fact that we set the same price for all characteristic values in the $k$th market. In analyzing the “variance” component, we evoke a peeling argument from the empirical processes theory; and in controlling the approximation error, we rely on general smoothness assumptions about $F_{Y,X}$. We also provide upper bounds for the gaps between the welfare generated by our minimax optimal 3PD (uniform pricing) strategy and the welfare based on the theoretical optimal 3PD (respectively, uniform pricing) strategy.

To our best knowledge, our work is the first to study data–based price discrimination by analyzing its information theoretic lower bounds, proposing a minimax optimal 3PD strategy, and providing necessary and sufficient conditions on the sample size $n$ for 3PD to be revenue superior to uniform pricing with a continuum of heterogeneous buyers.\(^3\) The closest work to ours is Huang et al. (2018), who provide the necessary and sufficient sample size for data–based uniform pricing strategies to obtain a given approximation accuracy.\(^4\)

\(^3\)We would like to point out that, when buyer types are discrete and finite, we can apply our analysis for uniform pricing to each type. This case is much simpler than our problem of interest, which involves a continuum of heterogeneous buyers.

\(^4\)There are other studies that derive the sample complexity of empirical revenue maximization with uniform pricing but are in an auction setting. See, for example, Cole and Roughgarden (2014);
The problem of data–based price discrimination is much more challenging than the problem of data–based uniform pricing.

**Notation.** For functions \( f(n) \) and \( g(n) \), write \( f(n) \gtrless g(n) \) to mean that \( f(n) \geq cg(n) \) for some constant \( c > 0 \). Similarly, write \( f(n) \lesssim g(n) \) to mean that \( f(n) \leq c'g(n) \) for some constant \( c' > 0 \), and write \( f(n) \asymp g(n) \) when \( f(n) \gtrsim g(n) \) and \( f(n) \lesssim g(n) \) hold simultaneously.

As a general rule for this paper, the various \( c \) and \( C \) constants denote positive universal constants that are independent of the sample size \( n \), and may vary from place to place. For functions, the \( L^2(\mathbb{P}) \) norm (\( L^2 \) as the short form) \( \|f - g\|_2 \equiv \left( \int_X [f(x) - g(x)]^2 \mathbb{P}(dx) \right)^{\frac{1}{2}} \).

## 2 Information theoretic limitation of pricing

### 2.1 The model

We begin by introducing the model setup. The seller is selling an item to a buyer. Let \( Y \) be the valuation (or willingness to pay) of the buyer, \( X \) the type–defining characteristic variable of the buyer. The joint distribution of \((Y, X)\) is denoted by \( F_{Y,X} \). Following the convention in the classical pricing literature, we assume that \((Y, X)\) is supported on \([0, 1] \times [0, 1] \). Given a characteristic value, the seller wants to set a price according to a mapping from the characteristics to a set of prices. We use \( D \) to denote the set of all pricing functions:

\[
\mathcal{D} \equiv \{ p : [0, 1] \rightarrow [0, 1], \text{measurable} \}.
\]

For a generic pricing scheme \( p \in \mathcal{D} \), the price depends on the characteristic value \( x \). This scheme falls in the realm of third–degree price discrimination (3PD). Uniform pricing can be viewed as a special case where the price is the same for all characteristic values. We use \( \mathcal{U} \) to denote the set of all uniform pricing functions:

\[
\mathcal{U} \equiv \{ p \in \mathcal{D} : p \text{ is a constant function} \}.
\]

To lighten the notation, we express \( p \in \mathcal{U} \) as a scalar rather than a function for the uniform pricing problem.

Let \( F_{Y|X} \) be the conditional distribution and \( f_X \) the marginal density function. Given a price \( p \in [0, 1] \) and characteristic value \( x \in [0, 1] \), there are \( 1 - F_{Y|X}(p|x) \) buyers whose

Dhangwatnotai et al. (2015); Guo et al. (2019). In addition, there are many less related previous papers, which all concern data–based uniform pricing in the context of auction design (Neeman, 2003; Segal, 2003; Goldberg et al., 2006; Chawla et al., 2014). These works do not provide information theoretic lower bounds.
valuation is higher than the price. The revenue generated from these buyers is

\[ r(p, x) \equiv (1 - F_{Y|X}(p|x))p, \]

and the expected revenue is

\[ R(p, F_{Y,X}) \equiv \int_0^1 r(p(x), x)f_X(x)dx. \]

In the special case where the pricing scheme is uniform (i.e., \( p \in U \)), the revenue only depends on the marginal distribution \( F_Y \):

\[ R(p, F_{Y,X}) = p \int_0^1 (1 - F_{Y|X}(p|x))f_X(x)dx \]
\[ = p \int_0^1 \mathbb{P}(Y \geq p|X = x)f_X(x)dx \]
\[ = p\mathbb{P}(Y \geq p) \]
\[ = p(1 - F_Y(p)), p \in U. \]

The (theoretical) optimal pricing scheme \( p^*_D \) is the one that maximizes the revenue:

\[ R(p^*_D(F_{Y,X}), F_{Y,X}) = \sup_{p \in D} R(p, F_{Y,X}). \]

In a similar fashion, we denote \( p^*_U \) as the (theoretical) optimal uniform pricing scheme such that

\[ R(p^*_U(F_{Y,X}), F_{Y,X}) = \sup_{p \in U} R(p, F_{Y,X}). \]

In terms of generating revenue, the classical pricing theory states that 3PD is at least as good as uniform pricing when the joint distribution \( F_{Y,X} \) is known to the seller. In this case, we can solve analytically for the optimal pricing schemes \( p^*_D(F_{Y,X}) \) and \( p^*_U(F_{Y,X}) \). Since \( U \) is contained in \( D \), \( p^*_D(F_{Y,X}) \) must achieve a (weakly) better revenue than \( p^*_U(F_{Y,X}) \). Intuitively, when \( Y \) is correlated with \( X \), \( p^*_D(F_{Y,X}) \) utilizes the information in \( X \). In practice, however, sellers are unlikely to know \( F_{Y,X} \); instead, the more plausible scenario is that the sellers observe only an i.i.d. sample of valuations and type–defining characteristic values. This scenario motivates us to establish a data–based pricing theory.

Suppose that a seller observes a random sample of data \( \equiv \{(Y_i, X_i), 1 \leq i \leq n\} \) drawn according to the joint distribution function \( F_{Y,X} \). The goal is to construct a pricing scheme based on the sample. A data–based pricing scheme maps the random sample into the space
of pricing schemes:

\[
\hat{p}_D : [0, 1]^n \times [0, 1]^n \rightarrow \mathcal{D}, \\
\hat{p}_U : [0, 1]^n \times [0, 1]^n \rightarrow \mathcal{U}.
\]

For any characteristic value \( x \in [0, 1] \), the corresponding price is \( \hat{p}_D(x; \text{data}) \) (\( \hat{p}_D \) for short), and \( \hat{p}_U(\text{data}) \) (\( \hat{p}_U \) for short) is the scalar data–based uniform pricing scheme.

The obvious difference between the theoretical pricing schemes and the data–based pricing schemes is that the theoretical pricing scheme requires the knowledge of the distribution \( F_{Y,X} \) while the data–based pricing scheme utilizes information only from a finite random sample. Due to this difference, the classical result that 3PD is revenue superior to uniform pricing may no longer hold when pricing decisions are made based on a sample.

We need a framework for assessing the quality of data–based pricing schemes. It makes little sense to consider a framework recommending data–based pricing schemes that are only good for a single distribution. For any fixed joint distribution \( F_{Y,X} \), there is always a trivial data–based pricing scheme: simply ignore the data and select the optimal pricing scheme given \( F_{Y,X} \). Such a pricing scheme is likely to perform poorly for other choices of the distribution of \( (Y, X) \). One framework that circumvents this issue is the minimax approach. Given a class \( \mathcal{F} \) of joint distributions, we consider the minimax difference in revenue for 3PD:

\[
\mathcal{R}_n^D(\mathcal{F}) \equiv \inf_{\hat{p}_D} \sup_{F_{Y,X} \in \mathcal{F}} \left( R(p^*_D, F_{Y,X}) - \mathbb{E}_{F_{Y,X}}[R(\hat{p}_D, F_{Y,X})] \right),
\]

where the infimum ranges over all possible pricing schemes and the expectation \( \mathbb{E}_{F_{Y,X}} \) is taken with respect to \( data \sim F_{Y,X} \). Similarly, for uniform pricing, we consider

\[
\mathcal{R}_n^U(\mathcal{F}) \equiv \inf_{\hat{p}_U} \sup_{F_{Y,X} \in \mathcal{F}} \left( R(p^*_U, F_{Y,X}) - \mathbb{E}_{F_{Y,X}}[R(\hat{p}_U, F_{Y,X})] \right),
\]

where the infimum ranges over all possible uniform pricing schemes.\(^5\) In this section, we derive a lower bound for \( \mathcal{R}_n^D(\mathcal{F}) \) and \( \mathcal{R}_n^U(\mathcal{F}) \), respectively. Our lower bounds are algorithm independent and reveal the fundamental and information theoretic limitation of pricing.

\(^5\)This perspective is related to “minimax regret” in a separate literature. The axiomatic foundation of the minimax regret decision criterion can be found in Brafman and Tennenholtz (2000); Stoye (2011).
2.2 Price discrimination

Two questions arise when studying the information theoretic limitation of pricing. The first question asks, at a given characteristic value $x_0 \in (0, 1)$, what is the lower bound on the quantity $|\tilde{p}_D(x_0; \text{data}) - p^*_D(x_0; F_{Y,X})|$ for the best-performing $\tilde{p}_D$ in the worst case distribution $F_{Y,X}$? The second question asks, what is the lower bound on $R(p^*_D, F_{Y,X}) - R(\tilde{p}_D(\text{data}), F_{Y,X})$ for the best-performing $\tilde{p}_D$ in the worst case distribution $F_{Y,X}$? These two problems require different tools to solve. The first problem is easier than the second problem.

2.2.1 Sketch of the proof for the price lower bound

In the first problem, the object of interest, $p^*_D(x_0; F_{Y,X})$, is a scalar. In this case, the information theoretic lower bound for the pricing problem can be reduced to that of a binary classification problem (also known as binary hypothesis testing in statistics). In a binary classification problem, we have two distributions $F_{Y,X}^1, F_{Y,X}^2 \in \mathcal{F}$ whose optimal prices are separated by some number $2\varepsilon$; that is,

$$|p^*_D(x_0; F_{Y,X}^j) - p^*_D(x_0; F_{Y,X}^{j'})| \geq 2\varepsilon, \quad j, j' \in \{1, 2\}. \quad (1)$$

A binary classification rule uses the data to decide whether the true distribution is $F_{Y,X}^j$ or $F_{Y,X}^{j'}$. To relate the binary classification problem to the pricing problem, note that, given any pricing function $\tilde{p}_D$, we can use it to distinguish between $F_{Y,X}^1$ and $F_{Y,X}^2$ in the following way. Define the binary classification rule

$$\psi(\text{data}) = \arg \min_{j \in \{1, 2\}} |p^*_D(x_0; F_{Y,X}^j) - \tilde{p}_D(x_0; \text{data})|.$$ 

We claim that when the underlying distribution is $F_{Y,X}^j$, the decision rule $\psi$ is correct if

$$|p^*_D(x_0; F_{Y,X}^j) - \tilde{p}_D(x_0; \text{data})| < \varepsilon. \quad (2)$$

To see this, note that by the triangle inequality, (1) and (2) guarantee that

$$|p^*_D(x_0; F_{Y,X}^{j'}) - \tilde{p}_D(x_0; \text{data})|$$

$$\geq |p^*_D(x_0; F_{Y,X}^{j'}) - p^*_D(x_0; F_{Y,X}^j)| - |p^*_D(x_0; F_{Y,X}^j) - \tilde{p}_D(x_0; \text{data})|$$

$$> 2\varepsilon - \varepsilon = \varepsilon, \text{ where } j' \neq j, j, j' \in \{1, 2\}.$$
This implies that

$$P_{F_{Y,X}^j}(\psi(data) \neq j) \leq P_{F_{Y,X}^j}(|p_D^*(x_0; F_{Y,X}^j) - \hat{p}_D(x_0; data)| \geq \varepsilon), \quad j = 1, 2.$$  

Therefore, we can upper bound the average probability of mistakes in the binary classification problem as

$$\frac{1}{2} P_{F_{Y,X}^1}(\psi(data) \neq 1) + \frac{1}{2} P_{F_{Y,X}^2}(\psi(data) \neq 2)$$

$$\leq \frac{1}{2} P_{F_{Y,X}^1}(|p_D^*(x_0; F_{Y,X}^1) - \hat{p}_D(x_0; data)| \geq \varepsilon) + \frac{1}{2} P_{F_{Y,X}^2}(|p_D^*(x_0; F_{Y,X}^2) - \hat{p}_D(x_0; data)| \geq \varepsilon)$$

$$\leq \sup_{F_{Y,X} \in \mathcal{F}} P_{F_{Y,X}}(|p_D^*(x_0; F_{Y,X}) - \hat{p}_D(x_0; data)| \geq \varepsilon).$$

By the Markov inequality, we have

$$\sup_{F_{Y,X} \in \mathcal{F}} \mathbb{E}|\hat{p}_D(x_0; data) - p_D^*(x_0; F_{Y,X})|$$

$$\geq \varepsilon \sup_{F_{Y,X} \in \mathcal{F}} \mathbb{P}(|\hat{p}_D(x_0; data) - p_D^*(x_0; F_{Y,X})| \geq \varepsilon)$$

$$\geq \varepsilon \left(\frac{1}{2} P_{F_{Y,X}^1}(\psi(data) \neq 1) + \frac{1}{2} P_{F_{Y,X}^2}(\psi(data) \neq 2)\right).$$

Finally, we take the infimum over all pricing scheme on the left–hand side (LHS), and the infimum over the induced set of binary decisions on the right–hand side (RHS). This leads to

$$\inf_{\hat{p}_D} \sup_{F_{Y,X} \in \mathcal{F}} \mathbb{E}|\hat{p}_D(x_0; data) - p_D^*(x_0; F_{Y,X})|$$

$$\geq \varepsilon \inf_{\psi} \left(\frac{1}{2} P_{F_{Y,X}^1}(\psi(data) \neq 1) + \frac{1}{2} P_{F_{Y,X}^2}(\psi(data) \neq 2)\right), \quad (3)$$

where the $\inf_{\psi}$ is taken over the set of all binary decisions. The RHS of the above inequality consists of two parts: (1) the separation between two optimal prices, $\varepsilon$, and (2) the average probability of making a mistake in distinguishing the two distributions. To obtain a meaningful bound, we want to find two distributions $F_{Y,X}^1$ and $F_{Y,X}^2$ that are close to each other (hard to distinguish) but their optimal prices are well–separated by a sufficiently large $\varepsilon$. We leave the details of the construction of such distributions to the proof of Theorem 2 given in Appendix A.
2.2.2 Sketch of the proof for the revenue lower bound

For the revenue problem, the object of interest concerns the entire pricing function $p^*_D(\cdot; F_{Y,X})$. Using Taylor expansion type of arguments, we relate the revenue difference to the minimax $L_2$-distance:

$$ R_n^D(\mathcal{F}_n) \gtrsim \inf_{\hat{p}} \sup_{F_{Y,X} \in \mathcal{F}_n} \mathbb{E}_{F_{Y,X}} \| \hat{p}_D(data) - p^*_D(F_{Y,X}) \|_2^2, $$

where

$$ \| \hat{p}_D(data) - p^*_D(F_{Y,X}) \|_2^2 = \int_0^1 |\hat{p}_D(x; data) - p^*_D(x; F_{Y,X})|^2 dx, $$

and the expectation $\mathbb{E}_{F_{Y,X}}$ is taken with respect to $data \sim F_{Y,X}$. Bounding the RHS of the above inequality is more complicated than the previous one. In particular, we consider a multiple classification problem that tries to distinguish among $M$ distributions, where $M$ is a function of the sample size $n$. Similar as before, we want the optimal prices of these $M$ distributions to be separated by some $\varepsilon$. Similar derivations show that the lower bound of the revenue problem can be reduced to that of a multiple classification problem:

$$ \inf_{\hat{p}} \sup_{F_{Y,X} \in \mathcal{F}_n} \mathbb{E}_{F_{Y,X}} \| \hat{p}_D(data) - p^*_D(F_{Y,X}) \|_2^2 \geq \varepsilon^2 \inf_{\psi} \frac{1}{M} \sum_{j=1}^M \mathbb{P}_{F_{Y,X}}(\psi(data) \neq j), \quad (4) $$

where the infimum $\inf_{\psi}$ is taken over the set of all multiple decisions (with $M$ choices). To proceed, we apply the Fano’s inequality from information theory (Cover and Thomas, 2005). Fano’s inequality gives a lower bound on the average probability of mistakes:

$$ \frac{1}{M} \sum_{j=1}^M \mathbb{P}_{F_{Y,X}}(\psi(data) \neq j) \geq 1 - \frac{\sum_{j,j'=1}^M \text{KL}(F_{Y,X}^j || F_{Y,X}^{j'})/M^2 + \log 2}{\log M}, \quad (5) $$

where $\text{KL}(\cdot || \cdot)$ denotes the Kullback–Leibler (KL) divergence between two distributions:

$$ \text{KL}(F_1 || F_2) \equiv \int f_1(y, x) \log \frac{f_1(y, x)}{f_2(y, x)} dydx. $$

To obtain a sharp bound based on the multiple classification problem, we want to find a large set of distributions (i.e., the cardinality $M$ of the set is large enough) that are close to each other (small pairwise KL divergence) but their optimal prices are well-separated by a sufficiently large $\varepsilon$. We leave the detailed proof to Appendix A. Our proof is based

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6We do not present the Fano’s inequality in its standard form as in Cover and Thomas (2005). Instead, we use a version from Wainwright (2019) that is more convenient for our purposes.
on a delicate construction of conditional densities along with an application of the Gilbert–Varshamov Lemma from coding theory; see, e.g., (Tsybakov, 2009).

2.2.3 The class of distributions of interest

In our construction of the conditional densities, the uniform distribution is used as the benchmark distribution. It is a widely assumed distribution in pricing models in the theory of industrial organization (IO). The conditional densities we construct for the lower bound proofs have joint distributions that fall in the class $\mathcal{F}_\kappa$ defined below. As a result, the minimax lower bounds we derive hold for the class $\mathcal{F}_\kappa$.

**Definition 1.** For $\kappa > 0$, let $\mathcal{F}_\kappa$ be the set of joint distributions that satisfy the following conditions.

(i) (Lipschitz continuity) There exists $L < \infty$ such that, for any $y, y', x \in [0, 1]$, the conditional density $f_{Y|X}$ satisfies

$$|f_{Y|X}(y|x) - f_{Y|X}(y'|x)| \leq L|y - y'|.$$

(ii) (Strong concavity) The revenue function $R(y|x) \equiv y(1 - F_{Y|X}(y|x))$ is strictly concave with second-order derivative

$$-2f_{Y|X}(y|x) - y\frac{\partial}{\partial y}f_{Y|X}(y|x) \leq -\kappa, \text{ a.e.} \quad (6)$$

(iii) (Interior solution) For each $x \in [0, 1]$, the optimal price is an interior solution; that is, $p^*_D(x; F_{Y,X}) \in (0, 1)$.

(iv) (Differentiability) The conditional distribution function $f_{Y|X}(y|x)$ is continuously differentiable in $(x, y)$ in a neighborhood of the curve $\{(x, p^*_D(x; F_{Y,X})) : x \in [0, 1]\}$.

(v) (Boundedness) The functions

$$\left|2f_{Y|X}(y|x) + y\frac{\partial}{\partial y}f_{Y|X}(y|x)\right| \text{ and } \left|\frac{\partial}{\partial x}F_{Y|X}(y|x) + y\frac{\partial}{\partial x}f_{Y|X}(y|x)\right|$$

are bounded a.e..

(vi) (Marginal density) The marginal density $f_X$ is bounded from above and bounded away from zero.
Assumption 1 requires the density function to be sufficiently smooth. The partial derivative \( \frac{\partial}{\partial y}f_{Y|X}(y|x) \) is well-defined almost everywhere because \( f_{Y|X} \) is Lipschitz continuous and hence absolutely continuous. Under Assumption 2, the optimal price is well-defined. Assumption 3 ensures that the first-order condition holds for the optimal price. Assumption 4 ensures that the optimal pricing function \( p^*_D(x; F_{Y,X}) \) is sufficiently smooth in \( x \). Assumption 5 requires the partial derivatives of the revenue to be bounded. Assumption 6 ensures that the characteristics do no take vanishing or dominating values.

2.2.4 Minimax lower bounds

Based on the intuition explained in Sections 2.2.1 and 2.2.2, we present the following theorem.

**Theorem 1** (Lower bounds for 3PD). For any \( F_\kappa \) with \( \kappa \in (0, 2) \), we have:

(i) the minimax difference in price at one point is bounded from below as

\[
\inf_{\tilde{p}_D} \sup_{F_{Y,X} \in F_\kappa} \mathbb{E}_{F_{Y,X}} |\tilde{p}_D(x_0; \text{data}) - p^*_D(x_0; F_{Y,X})| \geq n^{-1/4}, x_0 \in (0, 1);
\]

(ii) the minimax difference in revenue is bounded from below as

\[
\mathcal{R}_n^D(F_\kappa) \geq n^{-1/2}.
\]

**Remark.** By saying \( \kappa \in (0, 2) \) in Theorem 1, we allow \( R(y|x) \) associated with an \( f_{Y|X} \) to have a second derivative bounded from above by a number smaller than \(-2\). To motivate the use of \( \kappa \in (0, 2) \) in Theorem 1, suppose \( f_{Y|X} = f_Y \) (that is, valuation and characteristic are independent of each other) and \( f_Y \) is the uniform distribution on \([0, 1]\), \( U[0, 1] \). In this case, the revenue function equals \( R(y) = y(1 - y) \), which is twice-differentiable with second-order derivative \( R''(y) = -2 \) for any \( y \in [0, 1] \). In our proof for the lower bounds, \( U[0, 1] \) is used as the benchmark distribution. Ensuring \( F_\kappa \) to include the \( U[0, 1] \) and its perturbed distributions is important because \( U[0, 1] \) is a widely assumed distribution in pricing models in IO theory, as we mentioned earlier.

2.3 Uniform pricing

Recall that uniform pricing is a special case of price discrimination, where the price is set to be the same for each characteristic value. As we show in the theorem below, uniform pricing has a smaller information theoretic lower bound than 3PD.
Definition 2. For \( \kappa > 0 \), let \( \mathcal{F}_\kappa^U \) be the set of joint distributions such that the marginal distribution \( F_Y \) satisfies the following conditions.

(i) (Lipschitz continuity) There exists \( L < \infty \) such that, for any \( y, y' \in [0,1] \), the conditional density \( f_Y \) satisfies

\[
|f_Y(y) - f_Y(y')| \leq L|y - y'|.
\]

(ii) (Strong concavity) The revenue function \( R(y) \equiv y(1 - F_Y(y)) \) is strictly concave with second–order derivative

\[
-2f_Y(y) - y \frac{\partial}{\partial y} f_Y(y) \leq -\kappa, \text{ a.e.} \tag{7}
\]

(iii) (Interior solution) The optimal price is an interior solution; that is, \( p_U^*(F_Y) \in (0,1) \).

(iv) (Differentiability) The distribution function \( f_Y(y) \) is continuously differentiable in \( y \) in a neighborhood of \( p_U^*(F_Y) \).

(v) (Boundedness) The function \( \left| 2f_Y(y) + y \frac{d}{dy} f_Y(y) \right| \) is bounded a.e..

Remark. Definition 2 parallels Definition 1 where we replace the conditions on the conditional distribution \( F_Y|X \) by their counterparts on the marginal distribution \( F_Y \).

We have the following results for uniform pricing.

Theorem 2. For any \( \mathcal{F}_\kappa^U \) with \( \kappa \in (0,2) \), we have:

(i) the minimax difference in price is bounded from below as

\[
\inf_{p_U} \sup_{F_Y,X \in \mathcal{F}_\kappa^U} \mathbb{E}_{F_Y,X} |\hat{p}_U(data) - p_U^*(F_Y,X)| \gtrsim n^{-1/3};
\]

(ii) the minimax difference in revenue is bounded from below as

\[
\mathcal{R}_n(\mathcal{F}_\kappa^U) \gtrsim n^{-2/3}.
\]

3 Minimax optimal pricing strategies

In the previous section, we demonstrate the information theoretic limitations of pricing decisions: there is an inevitable gap in the worst–case scenario between revenues based on
any pricing strategy and the revenue based on the theoretical optimal 3PD (respectively, uniform pricing) strategy. In this section, we propose simple data–based pricing schemes and show that their performance bounds match the previous lower bounds up to constant factors (that are independent of \( n \)). This in turn shows that our lower bounds are tight, and our proposed pricing schemes are minimax optimal up to constant factors.

### 3.1 Price discrimination

We propose a “\( K \)–markets” approach for data–based price discrimination. We divide the individuals into \( K(\equiv K_n) \) markets by splitting the characteristic space \([0, 1]\) into \( K \) equally spaced intervals:

\[
I_k \equiv [(k - 1)/K, k/K), k = 1, \ldots, K.
\]

For each market \( I_k \), the seller charges the optimal price \( \hat{p}_k \) from the empirical distribution of \( \{Y_i : X_i \in I_k\} \),

\[
\hat{F}_k(\cdot) = \frac{1}{n} \sum_{i=1}^{n} 1\{Y_i > \cdot, X_i \in I_k\}.
\]

That is,

\[
\hat{p}_k \equiv \arg \max_p (1 - \hat{F}_k(p)).
\]

The resulting pricing function is a piece–wise function

\[
\hat{p}(x; data) = \hat{p}_k, x \in I_k.
\]

**Theorem 3.** Assume that \( K \leq n \) and \( F_{Y, X} \in \mathcal{F}_\kappa \), where \( \mathcal{F}_\kappa \) is given in Section 2.2.3.

(i) The proposed pricing scheme satisfies

\[
\mathbb{E}|\hat{p}_D(x_0; data) - p^*_D(x_0; F_{Y, X})| \lesssim (K/n)^{1/3} + 1/K, \quad x_0 \in (0, 1);
\]

moreover,

\[
(K/n)^{1/3} + 1/K \asymp n^{-1/4}
\]

when \( K \asymp n^{1/4} \).

(ii) The revenue generated by our proposed pricing scheme satisfies

\[
\mathbb{E}[R(p^*_D, F_{Y, X}) - R(\hat{p}_D, F_{Y, X})] \lesssim (K/n)^{2/3} + 1/K^2;
\]
moreover, 

$$(K/n)^{2/3} + 1/K^2 \asymp n^{-1/2}$$

when $K \asymp n^{1/4}$.

The errors above consist of two parts. The first part $(K/n)^{1/3}$ in the price difference (respectively, $(K/n)^{2/3}$ in the revenue difference) is the “variance” component or “estimation” error, which is due to the randomness of the finite sample, leaving $\hat{F}_k(p)$ different from its expectation. The second part $1/K$ in the price difference (respectively, $1/K^2$ in the revenue difference) is the approximation error due to the fact that we set the same price for all characteristic values in the market $I_k$.

### 3.2 Uniform pricing

A uniform pricing scheme charges the optimal price $\hat{p}_U$ from the empirical distribution of $\{Y_i\}_{i=1}^n$,

$$\hat{F}(\cdot) = \frac{1}{n} \sum_{i=1}^n 1\{Y_i > \cdot\}.$$ 

That is,

$$\hat{p}_U(data) \equiv \arg\max_{p \in [0,1]} p(1 - \hat{F}(p)).$$

We state this result in the following theorem.

**Theorem 4.** Assume that $F_{Y,X} \in \mathcal{F}_\kappa^U$, where $\mathcal{F}_\kappa^U$ is given in Section 2.3.

(i) The proposed pricing scheme satisfies

$$\mathbb{E}[\hat{p}_U(data) - p^*_U(F_{Y,X})] \lesssim n^{-1/3}.$$

(ii) The revenue generated by our proposed pricing scheme satisfies

$$\mathbb{E}[R(p^*_U(F_{Y,X}), F_{Y,X}) - R(\hat{p}_U(data), F_{Y,X})] \lesssim n^{-2/3}.$$

### 3.3 Welfare analysis

From the policy–maker perspective, it is also of interest to study the welfare achieved by the $K$–markets pricing strategy. In this section, we derive the rate at which the welfare achieved
by a data–based pricing scheme converges to its theoretical counterpart. In particular, the rates are the same as the ones derived for the price itself.

We assume that

- there is no production cost for the seller,
- and there is no utility for the seller if the item is not sold.

These assumptions are typically imposed in a benchmark model for auction and pricing settings. Under the two assumptions above, the welfare equals to the valuation of the item for the conditioning on the event of selling. For any pricing scheme $p \in D$, its welfare can be written as

$$W(p, F_{Y,X}) \equiv E[Y 1\{Y > p(X)\}].$$

**Theorem 5.**

(i) Take $K \asymp n^{-1/4}$ and assume that $F_{Y,X} \in F_{\kappa}$, where $F_{\kappa}$ is given in Section 2.2.3. Then

$$E|W(\hat{p}_D(data), F_{Y,X}) - W(p^*_D(F_{Y,X}), F_{Y,X})| \lesssim n^{-1/4}.$$ 

(ii) Assume that $F_{Y,X} \in F_{U\kappa}$, where $F_{U\kappa}$ is given in Section 2.3. Then

$$E|W(\hat{p}_U(data), F_{Y,X}) - W(p^*_U(F_{Y,X}), F_{Y,X})| \lesssim n^{-1/3}.$$ 

**4 Practical implementation**

In this section, we describe the implementation of the $K$–markets approach for data–based price discrimination.

- **STEP 1.** Choose $K = \lfloor n^{1/4} \rfloor$. For each market $k = 1, \cdots, K$, denote $data_k$ as the subset of observed $Y_i$’s whose corresponding characteristic value $X_i$ is in market $k$. Denote $n_k$ as the cardinality of $data_k$.

$$data_k \equiv \{Y_i : (Y_i, X_i) \in data, X_i \in [(k-1)/K, k/K)\},$$

$$n_k \equiv |data_k|.$$  

7The function $\lfloor z \rfloor$ denotes the largest integer less than $z$. 

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• **STEP 2.** For market $k$, find the price by solving the optimization problem

$$\hat{p}_k \equiv \arg \max_{Y_i \in \text{data}_k} \left(1 - \sum_{Y_i \in \text{data}_k} Y_i/n_k\right)Y_i.$$ 

For any $x$ in market $k$, set the price to $\hat{p}_k$.

**Remark.** For the price difference and revenue difference, the choice $K = \lfloor n^{1/4} \rfloor$ yields rates identical to those in Theorem 3.

The maximization in STEP 2 requires one to only find the maximum of $\{Y_i\}_{i \in \text{data}_k}$ which has $n_k$ entries. This means that the price is chosen from one of the observed valuations. The reason is that the empirical distribution function $\hat{F}_k$ is piecewise constant, and hence the objective function $p(1 - \hat{F}_k(p))$ is piecewise linear. Therefore, the optimizers must be located on the end points. This fact makes the $K$–markets approach easy to compute in practice.

When two characteristic values $x_1$ and $x_2$ are close to each other, the buyers with these characteristics values are more likely to be in the same market and face the same price. This makes the $K$–markets approach easy to implement in practice than, for example, the actual optimal pricing discrimination $p^*_D$. In general, $p^*_D$ would assign a distinct price for each characteristic value, which is impossible to carry out. From this perspective, the $K$–markets approach provides an easy–to–implement approximation of the theoretical optimal 3PD scheme and yet is the minimax optimal.

### 5 Economic implications

In this study, we have examined the performance of data–based price discrimination. Different from the classical pricing theory, where price discrimination is revenue superior to uniform pricing, an important implication of our results is that, data–based price discrimination can be inferior to data–based uniform pricing when the sample size is insufficient. More specifically, we can decompose the difference between the revenues generated respectively from data–based price discrimination and uniform pricing as follows:

$$\mathbb{E}[R(\hat{p}_D)] - \mathbb{E}[R(\hat{p}_U)] = \underbrace{\mathbb{E}[R(\hat{p}_D)] - R(p^*_D)}_{A_1} + \underbrace{R(p^*_D) - R(p^*_U)}_{A_2} + \underbrace{R(p^*_U) - \mathbb{E}[R(\hat{p}_U)]}_{A_3}, \quad (8)$$

where we have suppressed $F_{Y,X}$ and data in the notation. On the RHS, the first term $A_1$ is the gap between data–based and theoretical 3PD, $A_1 \asymp -n^{-1/2}$. The third term $A_3$ is the gap between theoretical and data–based uniform pricing, $A_3 \asymp n^{-2/3}$. The second term $A_2$ is the theoretical gap between price discrimination and uniform pricing. Note that $A_2$
is strictly positive unless the characteristic \( X \) does not provide any information about the valuation \( Y \) (e.g., \( Y \perp X \)).

If \( n \) is sufficiently large, then the dominating term in (8) is \( A_3 \). In this case, data–based 3PD is revenue superior to data–based uniform pricing. If \( n \) is below a threshold and \( A_2 \) is small, then the dominating term in (8) is \( A_1 \). In this case, data–based uniform pricing outperforms data–based 3PD.

### A Proofs for lower bounds

**Proof of Theorem 1.** For part (i), we use Lemma 4 to prove the lower bound. Define

\[
\omega_D(\varepsilon) \equiv \sup_{F_1, F_2 \in \mathcal{F}_n} \{ |p^*_D(x_0; F_1) - p^*_D(x_0; F_2)| : H(F_1 || F_2) \leq \varepsilon \}.
\]

Then by Lemma 4, we have

\[
\inf_{\hat{p}_D} \sup_{F_{Y,X} \in \mathcal{F}_n} \mathbb{E}_{F_{Y,X}} |\hat{p}_U(x_0; \text{data}) - p^*_U(x_0; F_{Y,X})| \geq \frac{1}{8} \omega_D \left( \frac{1}{(2\sqrt{n})} \right).
\]

Therefore, we only need to find a lower bound for \( \omega_U \). Based on the explanation in the main text, we want to construct two distributions that are hard to distinguish but their optimal prices are well–separated. We start by defining two perturbation functions. Let \( \phi_Y \) be defined as

\[
\phi_Y(t) \equiv \begin{cases} 
  t + 1, & t \in [-1, 0], \\
  -t + 1, & t \in [0, 2], \\
  t - 3, & t \in [2, 3], \\
  0, & \text{otherwise}.
\end{cases}
\]

Notice that \( \phi_Y \) is Lipschitz continuous on \( \mathbb{R} \). Let \( \phi_\infty \) be defined as

\[
\phi_X(t) \equiv \begin{cases} 
  e^{-(4t-1)^2/(1-(4t-1)^2)}, & t \in (0, 1/2), \\
  -e^{-(4t-3)^2/(1-(4t-3)^2)}, & t \in (1/2, 1), \\
  0, & \text{otherwise}.
\end{cases}
\]

Notice that \( \phi_X \) is infinitely differentiable on \( \mathbb{R} \). We plot the two perturbation functions in Figure 1.

Now we construct the two distributions. Let \( \delta \in (0, 1/4) \) be a small number (that depends
on \( n \) to be specified later. Let \( a \) be any number in the interval \((0, 4 - 2\kappa)\). Define the two conditional density functions of \( Y \) given \( X \) as

\[
\begin{align*}
    f_1(y|x) &\equiv 1, \\
    f_2(y|x) &\equiv 1 + a\phi_Y \left( \frac{y - 1/2}{\delta} \right) \phi_X \left( \frac{x - x_0}{\delta} + 1/4 \right). 
\end{align*}
\]

(10)

We let the marginal distribution \( f(x) \) of \( X \) be the uniform distribution on \([0, 1]\). Note that \( f_1(y|x), f_2(y|x), f_1(y,x) = f_1(y|x)f(x) \), and \( f_2(y,x) = f_2(y|x)f(x) \) are non-negative everywhere, with integrals over their respective entire spaces all equaling to 1.

The first task is to verify that the two distributions are indeed in the class \( \mathcal{F}_\kappa \). For \( \kappa \in (0, 2) \), the first distribution is in \( \mathcal{F}_\kappa \) by Lemma 2 and the fact that \( Y \) is independent of \( X \). Given any \( x \in [0, 1] \), we can treat the whole term \( a\phi_X((x - x_0)/\delta + 1/4) \) as the coefficient \( b \) in Lemma 3. Then the results of Lemma 3 applies since \( |\phi_X| \leq 1 \). In particular, the revenue function at \( x \) is twice–differentiable a.e., the absolute value of the second–order partial derivative with respect to \( y \) is bounded, and is also bounded from below by \( \kappa \). The optimal price is an interior solution and is in the interior of a region on which the revenue function is twice–differentiable. Lastly, the absolute value of the partial derivative of \( f_2(y|x) \) with respect to \( x \) is bounded. This ensures that the quantity \( |\frac{\partial}{\partial x} F_{Y|X}(y|x) + y\frac{\partial}{\partial x} F_{Y|X}(y|x)| \) is bounded.
Next, we want to derive the Hellinger distance between the two joint densities

\[ f_1(y, x) = 1, \]
\[ f_2(y, x) = 1 + a\delta \phi_Y \left( \frac{y - 1/2}{\delta} \right) \phi_X \left( \frac{x - x_0}{\delta} + 1/4 \right). \]

Let \( \Psi(t) \equiv \sqrt{1 + t} \). Its second-order derivative is bounded when \(|t| < 1/2\); that is,

\[ \sup_{|t|<1/2} |\Psi''(t)| < C. \]

We use \( H \) to denote the Hellinger distance:

\[ H(f_1\|f_2)^2 \equiv \int_0^1 \left( \sqrt{f_1(y)} - \sqrt{f_2(y)} \right)^2 dy. \]

The Hellinger distance can be bounded as

\[
H^2(f_1\|f_2)/2 = 1 - \int_0^1 \int_0^1 \Psi \left( a\delta \phi_Y \left( \frac{y - 1/2}{\delta} \right) \phi_X \left( \frac{x - x_0}{\delta} + 1/4 \right) \right) dxdy
\]
\[
= \int_0^1 \int_0^1 \Psi(0) - \Psi \left( a\delta \phi_Y \left( \frac{y - 1/2}{\delta} \right) \phi_X \left( \frac{x - x_0}{\delta} + 1/4 \right) \right) dxdy
\]
\[
\leq -a\Psi'(0) \int_0^1 \int_0^1 \phi_Y \left( \frac{y - 1/2}{\delta} \right) \phi_X \left( \frac{x - x_0}{\delta} + 1/4 \right) dxdy
\]
\[
+ a^2 C \delta^2 \int_0^1 \int_0^1 \phi_Y \left( \frac{y - 1/2}{\delta} \right)^2 \phi_X \left( \frac{x - x_0}{\delta} \right)^2 dxdy,
\]

where we have applied the second-order Taylor expansion to obtain the last inequality. By the change of variables \( u = (y - 1/2)/\delta \) and \( v = (x - x_0)/\delta + 1/4 \), for sufficiently small \( \delta > 0 \),

\[
\int_0^1 \int_0^1 \phi_Y \left( \frac{y - 1/2}{\delta} \right) \phi_X \left( \frac{x - x_0}{\delta} + 1/4 \right) dxdy = \delta^2 \int_1^3 \phi_Y (u) du \int_0^1 \phi_X (v) dv = 0, \quad (11)
\]

and

\[
\int_0^1 \int_0^1 \phi_Y \left( \frac{y - 1/2}{\delta} \right)^2 \phi_X \left( \frac{x - x_0}{\delta} + 1/4 \right)^2 dxdy = \delta^2 \int_1^3 \phi_Y (u)^2 du \int_0^1 \phi_X (v)^2 dv \leq C \delta^2.
\]

Therefore, the Hellinger distance is bounded as

\[ H^2(f_1\|f_2) \lesssim \delta^4. \]
Now we take $\delta$ such that $\delta^4 \asymp 1/n$. This ensures that $H^2(f_1\|f_2) \lesssim 1/n$. Then Lemma 4, we know that

$$
\inf_{\hat{p}_D} \sup_{F_{Y,X} \in \mathcal{F}_n} \mathbb{E}_{F_{Y,X}}[\hat{p}_D(x_0; \text{data}) - p^*_D(x_0; F_{Y,X})] \gtrsim n^{-1/4}, x_0 \in (0,1).
$$

For part (ii), we follow the explanation in the main text and use the Fano’s inequality to bound the probability of mistakes in the multiple classification problem. Before solving the revenue problem, we first study the lower bound for the $L_2$-distance of pricing functions. For two pricing functions $p_1$ and $p_2$, we define the $L_2$-distance as

$$
\|p_1 - p_2\|_2 \equiv \left( \int_0^1 |p_1(x) - p_2(x)|^2 dx \right)^{1/2}.
$$

In part (i), we defined the perturbation on the $X$ dimension at a fixed point $x_0$. Now we want to define a large set of perturbed distributions. Each of these distributions is perturbed in a small interval on the $X$ dimension. Let $m \geq 8$ be a large number (depending on $n$) that we specify later. Let $\alpha \in \{0,1\}^m$ be a vector of length $m$; that is,

$$
\alpha \equiv (\alpha_1, \ldots, \alpha_m), \text{ where } \alpha_j \in \{0,1\}, j = 1, \ldots, m.
$$

We construct a set of conditional density functions indexed by $\alpha$:

$$
f_{Y|X}^\alpha(y|x) \equiv 1 + \frac{a}{m} \sum_{j=1}^m \alpha_j \phi_Y(m(y - 1/2)) \phi_X(mx - (j - 1)).
$$

The marginal distribution of $X$ is taken to be the uniform distribution on $[0,1]$, that is, $f_X \equiv 1_{[0,1]}$. We denote the joint distribution by $f_{Y,X}^\alpha \equiv f_{Y|X}^\alpha f_X$.

We briefly describe this construction of the conditional density. The unit interval $[0,1]$ is divided equally into $m$ subintervals:

$$
I_j \equiv [(j - 1)/m, j/m], j = 1, \ldots, m.
$$

For $x \in I_j$, if $\alpha_j = 0$, then the conditional density is 1. If $\alpha_j = 1$, then the conditional density

$$
f_{Y|X}^\alpha(y|x) \equiv 1 + \frac{a}{m} \phi_Y(m(y - 1/2)) \phi_X(mx - (j - 1)), x \in I_j.
$$

By treating $1/m$ as the scalar $\delta$ in part (i), we can see that, for $m$ large enough, each $f_{Y,X}^\alpha$ hold when $\delta > 0$ is small enough, which in turns requires $n$ to be large enough.
belongs to the set $F_\kappa$.

From the set $\{f^\alpha_{Y,X} : \alpha \in \{0,1\}^m\}$, we want to pick out a large enough subset of distributions whose optimal price functions are well-separated. For this purpose, we use the Gilbert–Varshamov bound (Lemma 2.9, Chapter 2 Tsybakov, 2009). The Gilbert–Varshamov bound states that for $m \geq 8$, there exists a subset $\mathcal{A} \subset \{0,1\}^m$ with cardinality $M \equiv |\mathcal{A}| \geq 2^{m/8}$, and the pairwise rescaled Hamming distance between elements in this set is greater than $1/8$. That is,

$$\frac{1}{m} \sum_{j=1}^{m} 1\{\alpha_j \neq \alpha'_j\} \geq \frac{1}{8},$$

for any $\alpha, \alpha' \in \mathcal{A}$.

Applying the Gilbert–Varshamov bound, we can show that for $\alpha, \alpha' \in \mathcal{A}$, the optimal pricing functions of $f^\alpha_{Y,X}$ and $f^{\alpha'}_{Y,X}$ are well-separated. Let $p_\alpha$ be the pricing function associated with $f^\alpha_{Y,X}$; that is,

$$p_\alpha(x) \equiv \arg \max_{p \in [0,1]} p(1 - F^\alpha_{Y|X}(p|x)),$$

where $F^\alpha_{Y|X}(y|x)$ is the corresponding conditional cumulative distribution function. Note that $\alpha, \alpha' \in \mathcal{A}$ differ in at least $m/8$ positions. This means that $f^\alpha_{Y|X}$ and $f^{\alpha'}_{Y|X}$ differ in $m/8$ intervals. Suppose that $I_j$ is such an interval, where $\alpha_j = 0$ and $\alpha'_j = 1$. We restrict our attention to a subset of this interval:

$$\tilde{I}_j \equiv \left[ \frac{1}{6m} + \frac{j-1}{m}, \frac{1}{3m} + \frac{j-1}{m} \right] \subset I_j.$$

When $x \in \tilde{I}_j$, we have

$$mx - (j-1) \in [1/3, 1/6] \implies \phi_X(mx - (j-1)) \in [\phi_X(1/3), 1], \quad (12)$$

where $\phi_X(1/3) \approx 0.8825$. By Lemma 3 (where $b = a\phi_X(mx - (j-1))$, $\delta = 1/m$), the choice $a \in (0, 4 - 2\kappa)$, and the fact (12), if we fix $x \in \tilde{I}_j$, then $p_\alpha(x) = 1/2$ while

$$p_{\alpha'}(x) \leq 1/2 - \frac{c}{m} \phi_X(mx - (j-1)) \leq 1/2 - \frac{c\phi_X(1/3)}{m}, x \in \tilde{I}_j,$$

where $c > 0$ is a universal constant that does not depend on $n$. This implies that

$$|p_\alpha(x) - p_{\alpha'}(x)| \geq \frac{1}{m}, x \in \tilde{I}_j.$$

\footnotetext[9]{For example, $c$ can be equal to $a/8$ according to Lemma 3.}
Therefore, on the interval $I_j$, the separation between $p_\alpha$ and $p_\alpha'$ is lower bounded as

$$\int_{I_j} |p_\alpha(x) - p_\alpha'(x)|^2 dx \gtrsim \int_{I_j} \frac{1}{m^2} dx = \frac{1}{6m} \times \frac{1}{m^2} \gtrsim \frac{1}{m^3}.$$  

By the Gilbert–Varshamov bound, there are at least $m/8$ such intervals. Therefore, we can lower bound the total separation by

$$\|p_1 - p_2\|_2 \gtrsim \left( \frac{m}{8} \times \frac{1}{m^3} \right)^{1/2} \gtrsim \frac{1}{m}.$$  

Next, we want to compute the KL divergence between $f^\alpha_{Y,X}$ and $f^\alpha'_{Y,X}$. Note that the term $\phi_X(mx - (j - 1))$ is non–zero only when $x \in I_j$. The KL divergence can therefore be treated as a sum of $m$ integrals:

$$\text{KL}(f^\alpha_{Y,X} \| f^\alpha'_{Y,X}) = \int_0^1 \int_0^1 f^\alpha_{Y,X}(y,x) \log \frac{f^\alpha_{Y,X}}{f^\alpha'_{Y,X}} dy dx = \sum_{j=1}^m E_j,$$

where

$$E_j \equiv \int_{I_j} \int_0^1 \left( 1 + \frac{a}{m} \alpha_j \phi_Y(mx - (j - 1)) \phi_X(mx - (j - 1)) \right) \times \log \frac{1 + \frac{a}{m} \alpha_j \phi_Y(mx - (j - 1)) \phi_X(mx - (j - 1))}{1 + \frac{a}{m} \alpha_j' \phi_Y(mx - (j - 1)) \phi_X(mx - (j - 1))} dy dx.$$  

Notice that when $\alpha_j = \alpha_j'$, $E_j = 0$. Therefore, we only need to consider the $j$’s where $\alpha_j \neq \alpha_j'$. Denote $\Psi_1(t) = -\log(1 + t)$ and $\Psi_2(t) = (1 + t) \log(1 + t)$. Then we can write $E_j$ as

$$E_j = \begin{cases} f_{I_j} \int_0^1 \Psi_1 \left( \frac{a}{m} \phi_Y(mx - (j - 1)) \phi_X(mx - (j - 1)) \right) dy dx, & \text{if } \alpha_j = 0, \alpha_j' = 1, \\ f_{I_j} \int_0^1 \Psi_2 \left( \frac{a}{m} \phi_Y(mx - (j - 1)) \phi_X(mx - (j - 1)) \right) dy dx, & \text{if } \alpha_j = 1, \alpha_j' = 0. \end{cases}$$  

By the second–order Taylor expansion at zero, we have

$$\Psi_1(t) = -t + \frac{1}{2(1 + t')} t'^2,$$

for some $t'$ between 0 and $t$. When $|t| \leq 1/4$,\(^{10}\) we have

$$\Psi_1(t) \leq -t + C t'^2,$$

\(^{10}\)Later we show that $m \asymp n^{1/4}$ and as a result, $|t| \leq 1/4$ is guaranteed as long as $n$ is sufficiently large.
for some universal constant \( C > 0 \). Similarly, we can show that 

\[ \Psi_2(t) \leq t + Ct^2. \]

Applying these inequalities to \( E_j \), we have

\[
E_j \leq \pm \int_{I_j} \int_0^1 \frac{a}{m} \phi_Y (m(y - 1/2)) \phi_X (mx - (j - 1)) dydx \\
+ C \int_{I_j} \int_0^1 \frac{a^2}{m^2} \phi_Y^2 (m(y - 1/2)) \phi_X^2 (mx - (j - 1)) dydx.
\]

Similar to the derivation in Part (i), we know that the first term on the RHS is zero. For the second term, we can apply change of variables \( u = m(y - 1/2) \) and \( v = mx - (j - 1) \) and obtain that

\[
\int_{I_j} \int_0^1 \phi_Y^2 (m(y - 1/2)) \phi_X^2 (mx - (j - 1)) dydx \\
= \frac{1}{m^2} \int_0^1 \phi_Y^2 (v) dv \int_{-1}^{3} \phi_X^2 (u) du \leq \frac{C'}{m^2}
\]

for some universal constant \( C' > 0 \). Putting the results results together, we know that \( E_j \leq \frac{C'}{m^2} \) for all \( j \). Since there are \( m \) intervals, we can bound the KL divergence by

\[
\text{KL}(f_{Y,X}^* || f_{Y,X}') = \sum_{j=1}^{m} E_j \lesssim \frac{1}{m^3}.
\]

This is the KL distance for a single observation. For the entire data set with \( n \) i.i.d. observations, the KL divergence is upper bounded by \( Cn/m^3 \).

Lastly, we can summarize our results into the Fano inequality presented in Lemma 5. We have

\[
\inf \sup_{\hat{p} \in \mathcal{F}_{Y,X}} \mathbb{E} \| \hat{p}_D(data) - \hat{p}_D^*(F_{Y,X}) \|^2_2 \geq \frac{C_1}{m^2} \left( 1 - \frac{C_2 n/m^3 + \log 2}{\log 2^{m/8}} \right) \\
\geq \frac{C_1}{m^2} \left( 1 - \frac{C_2 n/m^3 + \log 2}{C_3 m} \right).
\]

By choosing \( m \simeq n^{1/4} \), we can make the factor \( \left( 1 - \frac{C_2 n/m^3 + \log 2}{C_3 m} \right) \) stay above, say, 1/2. Then we have

\[
\inf \sup_{\hat{p} \in \mathcal{F}_{Y,X}} \mathbb{E} \| \hat{p}_D(data) - \hat{p}_D^*(F_{Y,X}) \|^2_2 \gtrsim \frac{1}{m^2} \simeq n^{-1/2}.
\]
So far we have derived the lower bound for the $L_2$-distance of pricing. To work out the revenue problem, recall that the revenue achieved with price $p$ and characteristic $x$ is $r(p, x) = \max_p p(1 - F_Y|X(p|x))$. By Lemma 1, we have

$$r(p_D^*(x; F_{Y,X})) - r(\tilde{p}_D(x; data)) \geq \frac{K}{2} |p_D^*(x; F_{Y,X}) - \tilde{p}_D(x; data)|^2.$$  

Since $f_X$ is bounded away from zero, we have

$$\inf \sup_{\tilde{p}} \mathbb{E}[R(p^*) - R(\tilde{p})] = \inf \sup_{\tilde{p}} \int_0^1 (r(p^*(x), x) - r(\tilde{p}(x), x)) f_X(x) dx \geq \inf \sup_{p} \inf_{x \in [0,1]} f_X(x) \frac{K}{2} \int_0^1 |p_D^*(x; F_{Y,X}) - \tilde{p}_D(x; data)|^2 dx \gtrsim n^{-1/2}.$$  

Proof of Theorem 2. We use Lemma 4 to prove the lower bound. Define

$$\omega_U(\epsilon) \equiv \sup_{F_1, F_2 \in \mathcal{F}_n} \{ |p_U^*(F_1) - p_U^*(F_2)| : H(F_1 \| F_2) \leq \epsilon \}. $$

Then by Lemma 4, we have

$$\inf \sup_{\tilde{p}_U} \mathbb{E}_{F_{Y,X}} |\tilde{p}_U(data) - p_U^*(F_{Y,X})| \geq \frac{1}{8} \omega_U \left( 1/(2\sqrt{n}) \right).$$

Therefore, we only need to find a lower bound for $\omega_U$. The proof proceeds in three steps. In the first step, we construct two distributions and compute the separation between their optimal prices. The second step bounds the Hellinger distance between these two distributions. The third step summarizes.

**Step 1.** We construct two distribution functions. The first distribution is the uniform distribution on the unit interval $[0, 1]$. We denote this density function as

$$f_1(y) = 1_{[0,1]}(y).$$

The distribution function is $F_1(y) = y$ on the support $[0, 1]$. The revenue function under this distribution is $R_1(p) = p(1 - p)$. The optimal price is

$$p_1 = \arg \max_{p \in [0,1]} R_1(p) = \arg \max_{p \in [0,1]} p - p^2 = 1/2.$$
The second distribution function is a small twist of the uniform distribution. We use the same perturbation function $\phi_Y$ defined in (9).

We apply a small perturbation to the uniform density. Let $\delta > 0$ be a small number (that depends on $n$) specified later. Let $a \in (0, 4-2\kappa)$. The formula of the density $f_2$ is given by

$$f_2(y) \equiv 1 + a\delta \phi_Y \left( \frac{y - 1/2}{\delta} \right) = \begin{cases} 
1, & \text{if } y \in [0, 1/2 - \delta), \\
ay + 1 + \frac{a}{2} + a\delta, & \text{if } y \in [1/2 - \delta, 1/2), \\
-ay + 1 + \frac{a}{2} + a\delta, & \text{if } y \in [1/2, 1/2 + 2\delta), \\
ay + 1 - \frac{a}{2} - 3a\delta, & \text{if } y \in [1/2 + 2\delta, 1/2 + 3\delta), \\
1, & \text{if } y \in [1/2 + 3\delta, 1].
\end{cases}$$

We compare the two densities $f_1$ and $f_2$ in the following graph.

Denote the optimal price under $f_2$ by $p_2$. By Lemma 3(ii), we have

$$|p_2 - p_1| \geq a\delta/8$$

when $\delta$ is sufficiently small.

**Step 2.** We want to bound the Hellinger distance $H(F_1 || F_2)$. Define the function $\Psi(t) = \sqrt{1 + t}$. Its second–order derivative is bounded when $|t| < 1/2$; that is,

$$\sup_{|t| < 1/2} |\Psi''(t)| \leq \frac{\sqrt{2}}{2}.$$
Since $f_1(y) = 1$, we have

$$H(F_1 \| F_2)^2 / 2 = 1 - \int_0^1 \Psi \left( a \delta \phi_Y \left( \frac{y - 1/2}{\delta} \right) \right) dy$$

$$= \int_0^1 \Psi(0) - \Psi \left( a \delta \phi_Y \left( \frac{y - 1/2}{\delta} \right) \right) dy.$$ 

By the second–order Taylor expansion, we have

$$\Psi(0) - \Psi \left( a \delta \phi_Y \left( \frac{y - 1/2}{\delta} \right) \right) \leq -\Psi'(0) a \delta \phi_Y \left( \frac{y - 1/2}{\delta} \right) + \frac{\sqrt{2} \cdot a^2 \delta^2 \phi_Y^2 \left( \frac{y - 1/2}{\delta} \right)}{4}.$$

By the construction of $\phi_Y$, we have

$$\int_0^1 \phi_Y \left( \frac{y - 1/2}{\delta} \right) dy = 0.$$ 

By the change of variables $u = (y - 1/2)/\delta$, we have

$$\int_0^1 \phi_Y^2 \left( \frac{y - 1/2}{\delta} \right) dy = \delta \int_\mathbb{R} \phi_Y^2(u) du \leq 4\delta \int_{-1}^0 (x + 1)^2 dx = \frac{4}{3}\delta.$$

Combining these results together, we obtain a bound on the Hellinger distance

$$H(F_1 \| F_2)^2 \leq \frac{2\sqrt{2}}{3} a^2 \delta^3.$$ 

**Step 3.** By setting $\delta = (3/8\sqrt{2})^{1/3} a^{-2/3} n^{-1/3}$, we can ensure that $H(F_1 \| F_2) \leq 1/(2\sqrt{n})$.

Previously, we assumed that $a\delta \leq 1/2$ for the second–order Taylor expansion. This is true provided that $n \geq 3a/\sqrt{2}$. In this case, the separation between $p_1$ and $p_2$ is lower bounded as below:

$$|p_1 - p_2| \geq a\delta / 8 = \frac{1}{16} \left( \frac{3}{\sqrt{2}} \right)^{1/3} \left( \frac{a}{n} \right)^{1/3}.$$

By Lemma 4, we have

$$\inf_{\hat{p}_U} \sup_{F_{Y,X} \in F^U_Y} \mathbb{E}_{F_{Y,X}} |\hat{p}_U(data) - p_U^*(F_{Y,X})| \geq \frac{1}{16} \left( \frac{3}{\sqrt{2}} \right)^{1/3} \left( \frac{a}{n} \right)^{1/3}.$$
Lastly, we want to lower bound the revenue. By Lemma 1, we have

$$|R(p^*_U(F_{Y,X})) - R(p)| \geq \frac{\kappa}{2}(p^*_U(F_{Y,X}) - p)^2.$$ 

Then, by definition, the minimax revenue difference is

$$\mathcal{R}_n^U(\mathcal{F}_\kappa^U) = \inf_{\hat{p}_U} \sup_{F_{Y,X} \in \mathcal{F}_\kappa^U} \mathbb{E}_{F_{Y,X}}[R(\hat{p}_U(data), F_{Y,X}) - R(p^*_U(F_{Y,X}), F_{Y,X})] \geq \frac{\kappa}{2} \left( \frac{9}{2} \right)^{1/3} \left( \frac{a}{n} \right)^{2/3}.$$ 

\[\square\]

**B Proofs for upper bounds**

To facilitate the presentation, we first give the proof for Theorem 4.

*Proof of Theorem 4.* For simplicity, we omit writing $F_{Y,X}$ and data in the proof. Denote $\kappa' \equiv \inf_{p \in [0,1]} |R''(p)|/2 > 0$. By Taylor expansion, for any $p$,

$$R(p^*_U) - R(p) \geq \kappa(p - p^*_U)^2.$$ 

Denote $\hat{R}(p) \equiv p(1 - \hat{F}(p))$. Combining the inequality above with the basic inequality (i.e., $\hat{R}(\hat{p}_U) \geq \hat{R}(p^*_U)$), we have

$$\kappa(\hat{p}_U - p^*_U)^2 \leq R(p^*_U) - R(\hat{p}_U) \leq R(p^*_U) - \hat{R}(p^*_U) - (R(\hat{p}_U) - \hat{R}(\hat{p}_U)).$$ 

(13)

For $\delta \in (0, p^*_U]$, define

$$G_\delta \equiv \{y \mapsto p1\{y \geq p\} - p^*_U 1\{y \geq p^*\} : p \in [p^*_U - \delta, p^*_U + \delta] \}$$

and

$$G_\delta(y) \equiv \begin{cases} 
0, & \text{if } y < p^*_U - \delta, \\
p^*_U, & \text{if } p^*_U - \delta \leq y \leq p^*_U + \delta, \\
\delta, & \text{if } y > p^*_U + \delta.
\end{cases}$$
Then \( G_\delta \) is an envelope function of the class \( \mathcal{G}_\delta \). The \( L_2 \)-norm of \( G_\delta \) is bounded by

\[
\|G_\delta\|_{L_2} = \left( (p_U^*)^2 \mathbb{P}(Y \in [p_U^* - \delta, p_U^* + \delta]) + \delta^2 \mathbb{P}(Y > p_U^* + \delta) \right)^{1/2} \leq C\sqrt{\delta}.
\]

Since \( \mathcal{G}_\delta \) is a VC–subgraph class, we have

\[
\mathbb{E} \sup_{g \in \mathcal{G}_\delta} \left| \frac{1}{n} \sum_{i=1}^{n} g(Y_i) - \mathbb{E}g(Y) \right| \leq C\sqrt{\frac{\delta}{n}}.
\]

(14)

We derive the convergence rate of \( \hat{p} - p^* \) via a peeling argument. Consider the following decomposition

\[
\mathbb{P} \left( n^{1/3} |\hat{p}_U - p^*_U| > M \right) = \sum_{j=M+1}^{\infty} \mathbb{P} \left( n^{1/3} |\hat{p}_U - p^*_U| \in (j-1, j] \right).
\]

For any \( j \geq M + 1 \), we have

\[
\{ |\hat{p}_U - p^*_U| \in ((j-1)n^{-1/3}, jn^{-1/3}] \} = \{ |\hat{p}_U - p^*_U| > (j-1)n^{-1/3}, |\hat{p}_U - p^*_U| \leq jn^{-1/3} \} 
\subset \left\{ R(p^*_U) - \hat{R}(p^*_U) - (R(\hat{p}_U) - \hat{R}(\hat{p}_U)) \geq \kappa(j-1)^2n^{-2/3}, |\hat{p}_U - p^*_U| \leq jn^{-1/3} \right\} 
\subset \{ \Delta_{j,n} \geq \kappa(j-1)^2n^{-2/3} \},
\]

where the third line follows from (13), and \( \Delta_{j,n} \) in the last line is defined as

\[
\Delta_{j,n} \equiv \sup_{g \in \mathcal{G}_{jn^{-1/3}}} \left| \frac{1}{n} \sum_{i=1}^{n} g(Y_i) - \mathbb{E}g(Y) \right|.
\]

Therefore,

\[
\mathbb{P} \left( |\hat{p}_U - p^*_U| \in ((j-1)n^{-1/3}, jn^{-1/3}] \right) \leq \mathbb{P} \left( \Delta_{j,n} \geq \kappa(j-1)^2n^{-2/3} \right).
\]

To bound the probability on the RHS of the above inequality, we use the concentration inequality given by Theorem 7.3 in Bousquet (2003), which is a version of Talagrand’s (1996) inequality. The concentration inequality states that for all \( t > 0 \),

\[
\mathbb{P} \left( \Delta_{j,n} \geq \mathbb{E}\Delta_{j,n} + \sqrt{2t(\sigma^2 + 2\mathbb{E}\Delta_{j,n})/n + t/(3n)} \right) \leq \exp(-ct),
\]

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for some universal constant $c > 0$, where

$$\sigma^2 \equiv \sup_{g \in \mathcal{G}_{jn^{-1/3}}} \mathbb{E}g(Y_1)^2 \leq \|\mathcal{G}_{jn^{-1/3}}\|_{L_2}^2 \leq Cjn^{-1/3}.$$  

From (14), we have

$$\mathbb{E}\Delta_{j,n} \leq C\sqrt{jn^{-1/3}/n} = C\sqrt{jn^{-2/3}}.$$  

By setting $t = \kappa'j^2$, we have

$$\mathbb{E}\Delta_{j,n} + \sqrt{2t(\sigma^2 + 2\mathbb{E}\Delta_{j,n})/n + t/(3n)} \leq C'j^{3/2}n^{-2/3} \leq \kappa(j - 1)^2n^{-2/3},$$

when $j$ is large enough. Then we have

$$\mathbb{P}\left(\Delta_{j,n} \geq \kappa(j - 1)^2n^{-2/3}\right) \leq \mathbb{P}\left(\Delta_{j,n} \geq Cjn^{-2/3}\right) \leq \exp(-c\kappa'j^2), \text{ for } j \text{ large.}$$

To summarize, we have shown that

$$\mathbb{P}\left(n^{1/3}|\hat{p}_U - p^*_U| > M\right) \leq \sum_{j=M+1}^{\infty} \exp(-C_4j^2) \leq C_3\exp(-C_2M^2).$$

By integrating the tail probability, we have

$$\mathbb{E}|\hat{p}_U - p^*_U|^s \lesssim n^{-s/3}.$$  

For revenue, we use the second–order Taylor expansion and obtain that

$$\mathbb{E}[R(p^*_U) - R(\hat{p}_U)] \leq \sup_p |R''(p)|\mathbb{E}(\hat{p}_U - p^*_U)^2 \lesssim n^{-2/3}.$$  

Proof of Theorem 3. We introduce some notations. Let $\tilde{R}_k(p)$ denote the revenue collected
from the $k$th market by charging price $p$; that is,

$$
\tilde{R}_k(p) \equiv p \mathbb{P}(Y > p, X \in I_k)
= p \int_0^1 \int_{I_k} f_{Y|X}(y|x)f_X(x) dx dy.
$$

Denote $\hat{p}_k \equiv \arg \max_{p \in [0,1]} \tilde{R}_k(p)$ as the maximizer of $\tilde{R}_k$. The first– and second–order derivatives of $\tilde{R}_k$ are respectively

$$
\tilde{R}_k'(p) = \int_0^1 \int_{I_k} f_{Y|X}(y|x)f_X(x) dx dy - p \int_{I_k} f_{Y|X}(p|x)f_X(x) dx,
$$

$$
\tilde{R}_k''(p) = \int_{I_k} \left( -2f_{Y|X}(p|x) - p \frac{\partial}{\partial y} f_{Y|X}(p|x) \right) f_X(x) dx.
$$

By the Lipschitz continuity assumption, the second–order derivative $\tilde{R}_k''(p)$ exists for almost all $p \in [0,1]$. Recall that

$$
-2f_{Y|X}(p|x) - p \frac{\partial}{\partial y} f_{Y|X}(p|x) \leq -\kappa,
$$

and $f_X$ is bounded away from zero. Denote $2\kappa' \equiv \inf_{x \in [0,1]} f_X(x)$. Then

$$
\tilde{R}_k''(p) \leq -2\kappa' \int_{I_k} dx = -2\kappa'/K
$$

for almost all $p \in [0,1]$. By Lemma 1, we have

$$
\tilde{R}_k(\hat{p}_k) - \tilde{R}_k(p) = |\tilde{R}_k(\hat{p}_k) - \tilde{R}_k(p)| \geq \frac{\kappa'}{K}(\hat{p}_k - p)^2, p \in [0,1].
$$

(15)

Note that $\hat{p}_k$ is not the true optimal price under $F_{Y,X}$. We need to relate it to the true optimal price. For $x_0 \in [0,1]$, let $k(x_0)$ be such that $x_0 \in I_k$. Then by the triangle inequality, we can decompose the revenue difference into estimation error and approximation error:

$$
\mathbb{E}|\hat{p}_D(x_0; data) - p^*_D(x_0; F_{Y,X})| = \mathbb{E}|\hat{p}_k(x_0) - p^*_D(x_0; F_{Y,X})|
\leq \mathbb{E}|\hat{p}_k(x_0) - \hat{p}_k(x_0)| + |\hat{p}_k(x_0) - p^*_D(x_0; F_{Y,X})|.
$$

(16)

**Estimation error**. Denote $\hat{R}_k$ as the empirical counterpart of $\tilde{R}_k$; that is,

$$
\hat{R}_k(p) \equiv \frac{1}{n} \sum_{i=1}^n 1\{Y_i > p, X_i \in I_k\}.
$$
Recall that \( \hat{p}_k \) is the maximizer of \( \hat{R}_k \). The basic inequality (i.e., \( \hat{R}_k(\hat{p}_k) \geq \hat{R}_k(\hat{p}_k) \)) gives that

\[
\hat{R}_k(\hat{p}_k) - \hat{R}_k(\hat{p}) = \hat{R}_k(\hat{p}_k) - \hat{R}_k(\hat{p}) - \hat{R}_k(\hat{p}_k) + \hat{R}_k(\hat{p}_k) \\
\leq \hat{R}_k(\hat{p}_k) - \hat{R}_k(\hat{p}) - \hat{R}_k(\hat{p}_k) + \hat{R}_k(\hat{p}_k). \tag{17}
\]

Combining (15) and (17) yields

\[
\frac{k'}{K}(\hat{p}_k - \hat{p}_k)^2 \leq \hat{R}_k(\hat{p}_k) - \hat{R}_k(\hat{p}_k) - (\hat{R}_k(\hat{p}) - \hat{R}_k(\hat{p}_k)) \tag{18}
\]

We use a peeling argument to bound the difference \( \hat{p}_k - \hat{p}_k \). Consider the following function class

\[
\mathcal{G}_{k,\delta} \equiv \{(y, x) \mapsto (p1\{y \geq p\} - \hat{p}_k1\{y \geq \hat{p}_k\})1\{x \in I_k\} : p \in [\hat{p}_k - \delta, \hat{p}_k + \delta]\} \tag{19}
\]

where \( \delta \in (0, \bar{p}_k] \). The class \( \mathcal{G}_{k,\delta} \) has an envelope \( G_{\delta}(y, x) \) defined as

\[
G_{\delta}(y, x) \equiv \begin{cases} 
0, & \text{if } y < \hat{p}_k - \delta, \\
\hat{p}_k1\{x \in I_k\}, & \text{if } \hat{p}_k - \delta \leq y \leq \hat{p}_k + \delta, \\
\delta1\{x \in I_k\}, & \text{if } y > \hat{p}_k + \delta.
\end{cases}
\]

The (squared) \( L_2 \)-norm of \( G_{\delta} \) is bounded as

\[
\|G_{k,\delta}\|_{L_2}^2 = \hat{p}_k^2\mathbb{P}(Y \in [\hat{p}_k - \delta, \hat{p}_k + \delta], X \in I_k) + \delta^2\mathbb{P}(Y > \hat{p}_k + \delta, X \in I_k) \\
\leq \mathbb{P}(Y \in [\hat{p}_k - \delta, \hat{p}_k + \delta], X \in I_k) + \delta^2\mathbb{P}(X \in I_k) \\
\leq 2\delta \|f_{Y,X}\|_{\infty} + \frac{\delta^2}{K} \|f_X\|_{\infty} \lesssim \delta/K.
\]

By Lemma 6, the class \( \mathcal{G}_{k,\delta} \) is a VC–subgraph with VC dimension no greater than 2 for any \( \delta, k \) and \( K \). Then by Lemma 7, we have

\[
\mathbb{E} \sup_{g \in \mathcal{G}_{\delta}} \left| \frac{1}{n} \sum_{i=1}^{n} g(Y_i) - \mathbb{E}g(Y_i) \right| \lesssim \sqrt{\delta/(nK)}. \tag{20}
\]

Consider the following decomposition

\[
\mathbb{P} \left( (n/K)^{1/3}|\hat{p}_k - \hat{p}_k| > M \right) = \sum_{j=M+1}^{\infty} \mathbb{P} \left( (n/K)^{1/3}|\hat{p}_k - \hat{p}_k| \in (j-1, j] \right). \tag{21}
\]
To simplify notation, we define a localized empirical process \( \Delta_{j,n,K} \) by

\[
\Delta_{j,n,K} \equiv \sup_{g \in G_{j(K/n)^{1/3}}} \left| \frac{1}{n} \sum_{i=1}^{n} g(Y_i) - \mathbb{E}g(Y_i) \right|.
\]

Then, for any \( j \geq M + 1 \), we have

\[
\{(n/K)^{1/3} | \hat{p}_k - \bar{p}_k | \in (j - 1, j]\}
\]

\[
= \{|\hat{p}_k - \bar{p}_k | > (j - 1)(K/n)^{1/3}, |\hat{p}_k - \bar{p}_k | \leq j(K/n)^{1/3}\}
\]

\[
\subset \left\{ \hat{R}_k(\bar{p}_k) - \hat{R}_k(\hat{p}_k) - (\hat{R}_k(\hat{p}) - \hat{R}_k(\bar{p}_k)) \geq \kappa' (j - 1)^2 K^{-1/3} n^{-2/3}, |\hat{p}_k - \bar{p}_k | \leq j(K/n)^{1/3}\right\}
\]

\[
\subset \left\{ \Delta_{j,n,K} \geq \kappa' (j - 1)^2 K^{-1/3} n^{-2/3}\right\},
\]

where the third line follows from (18). Therefore, we can bound the summands in Equation (21) by using the tail probability of \( \Delta_{j,n,K} \):

\[
\mathbb{P} \left( (n/K)^{1/3} | \hat{p}_k - \bar{p}_k | \in (j - 1, j]\right) \leq \mathbb{P} \left( \Delta_{j,n,K} \geq \kappa' (j - 1)^2 K^{-1/3} n^{-2/3}\right).
\]

From (20), we know that

\[
\mathbb{E} \Delta_{j,n,K} \lesssim \sqrt{j(K/n)^{1/3} / (nK)} = \sqrt{jK^{-1/3} n^{-2/3}}.
\]

To bound the probability of \( \Delta_{j,n,K} \) deviating from its mean, we use the concentration inequality given by Theorem 7.3 in Bousquet (2003), which is a version of Talagrand’s (1996) inequality.

\[
\mathbb{P} \left( \Delta_{j,n,K} \geq \mathbb{E} \Delta_{j,n,K} + \sqrt{2t(\sigma^2 + 2\mathbb{E} \Delta_{j,n,K})/n + t/(3n)} \right) \leq \exp(-ct),
\]

for some universal constant \( c > 0 \), where

\[
\sigma^2 \equiv \sup_{g \in G_{j(K/n)^{1/3}}} \mathbb{E}g(Y_1)^2 \leq \|G_{j(K/n)^{1/3}}\|_{L_2}^2 \lesssim j(K/n)^{1/3} / K.
\]

Since \( K \leq n \), we have \( K^{-1/3} n^{-2/3} \geq 1/n \). For large enough \( j \), we have \( j^2 / 3 \leq (j - 1)^2 / 2 \).

Therefore, by setting \( t = \kappa' j^2 \), we have

\[
t/(3n) = \kappa' j^2 / (3n) \leq \frac{\kappa'}{2} (j - 1)^2 K^{-1/3} n^{-2/3},
\]

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and

\[ \mathbb{E}\Delta_{j,n} + \sqrt{2t(\sigma^2 + 2\mathbb{E}\Delta_{j,n})/n} \]

\[ \lesssim \sqrt{jK^{-1/3}n^{-2/3} + \sqrt{2\kappa'j^2(j(K/n)^{1/3}/K + 2\sqrt{jK^{-1/3}n^{-2/3}})/n}} \]

\[ \lesssim \frac{\sqrt{\kappa'}}{2}(j-1)^2K^{-1/3}n^{-2/3}, \]

when \( j \) is large enough. The above two inequalities together imply that

\[ \mathbb{E}\Delta_{j,n,K} + \sqrt{2t(\sigma^2 + 2\mathbb{E}\Delta_{j,n,K})/n + t/(3n)} \leq C\sqrt{\kappa'}\sqrt{1}(j-1)^2K^{-1/3}n^{-2/3}, \text{ for } t = \kappa'j^2. \]

To summarize, we have shown that

\[ \mathbb{P}\left((n/K)^{1/3}|\hat{p}_k - \tilde{p}_k| \in (j-1,j]\right) \leq \mathbb{P}\left(\Delta_{j,n,K} \geq \sqrt{\kappa'}(j-1)^2K^{-1/3}n^{-2/3}\right) \]

\[ \leq \mathbb{P}\left(\Delta_{j,n,K} \geq \mathbb{E}\Delta_{j,n,K} + \sqrt{2\kappa'j^2(\sigma^2 + 2\mathbb{E}\Delta_{j,n,K})/n + \kappa'j^2/(3n)}\right) \leq \exp(-c\kappa'j^2). \]

Going back to the peeling argument (21), we have

\[ \mathbb{P}\left((n/K)^{1/3}|\hat{p}_k - \tilde{p}_k| > M\right) \lesssim \sum_{j=M+1}^{\infty} \exp(-c\kappa'j^2) \]

\[ \lesssim \exp(-C_1\kappa'M^2). \]

Let \( n \) be sufficiently large such that the exponential term is dominated by the term \((K/n)^{s/3}\).

By integrating the tail probability, we have

\[ \mathbb{E}|\hat{p}_k - \tilde{p}_k|^s \lesssim (K/n)^{s/3}. \]

**Approximation error.** The second term \(|\hat{p}_{k(x_0)} - p^*_D(x_0; F_{Y,X})|\) in (16) is deterministic and can be controlled by using the smoothness conditions. By definition, \( p^*_D(x_0; F_{Y,X}) \) satisfies the first–order condition

\[ 0 = \frac{\partial}{\partial p}r(p^*_D(x; F_{Y,X}), x). \]

By the differentiability condition of \( F_\kappa, \frac{\partial}{\partial p}r(p, x) \) is continuously differentiable in \((p, x)\) in a neighborhood of \((p^*_D(x; F_{Y,X}), x)\). By the strong concavity, \( \frac{\partial^2}{\partial p^2}r(p^*_D(x; F_{Y,X}), x) \) is non–zero.
Then by the implicit function theorem, the function \( p_D^*(x; F_{Y,X}) \) is well-defined (uniquely determined by the first-order condition) and is differentiable. Its derivative is given as follows:

\[
\frac{d}{dx} p_D^*(x; F_{Y,X}) = -\frac{\partial^2}{\partial y \partial x} r(p_D^*(x; F_{Y,X}), x) \frac{\partial^2}{\partial y^2} r(p_D^*(x; F_{Y,X}), x).
\]

By the strong concavity, the absolute value of \( \frac{\partial^2}{\partial y \partial x} r(p, x) \) is bounded away from zero; also, \( |\frac{\partial}{\partial x} F_{Y|X}(y|x) + y \frac{\partial}{\partial x} f_{Y|X}(y|x) | \) are bounded a.e.. This implies that \( p_D^*(x; F_{Y,X}) \) is Lipschitz continuous on \([0, 1]\). We use \( L_1 \) to denote the Lipschitz constant. By applying Taylor expansion to the first-order condition of \( \tilde{p}_k \), we have

\[
0 = \int_{I_k} \frac{\partial}{\partial p} r(\tilde{p}_k, x) f_X(x) dx = \int_{I_k} \frac{\partial}{\partial p} r(p_D^*(x; F_{Y,X}), x) f_X(x) dx + \int_{I_k} \frac{\partial^2}{\partial p^2} r(\tilde{p}_k, x)(\tilde{p}_k - p_D^*(x; F_{Y,X})) f_X(x) dx,
\]

for some \( \tilde{p}(x) \) between \( \tilde{p}_k \) and \( p_D^*(x; F_{Y,X}) \). Rearranging terms shows that \( \tilde{p}_k \) is a weighted average of \( p_D^*(x; F_{Y,X}), x \in I_k \); that is,

\[
\tilde{p}_k = \frac{\int_{I_k} \frac{\partial}{\partial p} r(\tilde{p}(x), x) p_D^*(x; F_{Y,X}) f_X(x) dx}{\int_{I_k} \frac{\partial}{\partial p} r(\tilde{p}(x), x) f_X(x) dx}.
\]

This implies that there exists some \( x^* \in I_k \) such that \( \tilde{p}_k = p_D^*(x^*; F_{Y,X}) \). Since \( p_D^*(x; F_{Y,X}) \) is Lipschitz continuous, given \( x_0 \), we have

\[
|\tilde{p}_k - p_D^*(x_0; F_{Y,X})|^s = |p_D^*(x^*; F_{Y,X}) - p_D^*(x_0; F_{Y,X})|^s \leq L_1^s / K^s, \text{ for any } s \geq 1.
\]

Therefore, we obtain the following upper bound

\[
\mathbb{E}|\hat{p}_D(x_0; \text{data}) - p_D^*(x_0; F_{Y,X})| \lesssim (K/n)^{2/3} + 1/K.
\]

By choosing \( K \approx \sqrt{n} \), the above bound becomes \( n^{-1/4} \). This proves part (i) of the theorem.

For part (ii), we want to bound the revenue difference. Consider the following decomposition:

\[
R(p_D^*(F_{Y,X}), F_{Y,X}) - R(\hat{p}_D(\text{data}), F_{Y,X}) \leq R(p_D^*(F_{Y,X}), F_{Y,X}) - R(\tilde{p}(F_{Y,X}), F_{Y,X}) + |R(\tilde{p}(F_{Y,X}), F_{Y,X}) - R(\hat{p}_D(\text{data}), F_{Y,X})|.
\]

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The first term on the RHS is deterministic and can be bounded by using Lemma 1(iii) as follows:

\[
|R(p_D^*(F_{Y,X}), F_{Y,X}) - R(\tilde{p}(F_{Y,X}), F_{Y,X})| \\
\leq \int_0^1 |r(p_D^*(x; F_{Y,X}), x) - r(\tilde{p}(x; F_{Y,X}), x)| f_X(x) dx \\
= \sum_{k=1}^K \int_{I_k} |r(p_D^*(x; F_{Y,X}), x) - r(\tilde{p}_k, x)| f_X(x) dx \\
\leq \sum_{k=1}^K \int_{I_k} \frac{1}{2} |2f_{Y|X}(y|x) + y \frac{\partial}{\partial y} f_{Y|X}(y|x)| sup_{y,x} |p_D^*(x; F_{Y,X}), x) - \tilde{p}_k|^2 f_X(x) dx \\
\lesssim 1/K^2.
\]

where we have used the first–order condition of \( P_D^* \). For the second term, we have

\[
R(\tilde{p}(F_{Y,X}), F_{Y,X}) - R(\hat{p}_D(data), F_{Y,X}) = \sum_{k=1}^K \tilde{R}_k(\tilde{p}_k) - \tilde{R}_k(\hat{p}_k).
\]

This is because both \( \hat{p}(data) \) and \( \tilde{p}(F_{Y,X}) \) are constant within each \( I_k \). Their revenues on \( I_k \) are reduced to \( \tilde{R}_k \). Note that for every \( k \), \( \tilde{R}_k'(\tilde{p}_k) = 0 \), and

\[
|\tilde{R}_k''(p)| \leq \int_{I_k} \left| \frac{1}{2} 2f_{Y|X}(y|x) + y \frac{\partial}{\partial y} f_{Y|X}(y|x) \right| f_X(x) dx \\
\leq \frac{1}{K} sup_{y,x} \left( \frac{1}{2} 2f_{Y|X}(y|x) + y \frac{\partial}{\partial y} f_{Y|X}(y|x) \right)f_X(x).
\]

Then Lemma 1(iii) gives that

\[
\tilde{R}_k(\tilde{p}_k) - \tilde{R}_k(\hat{p}_k) \lesssim 1/K(\tilde{p}_k - \hat{p}_k)^2.
\]

Hence, we have

\[
\mathbb{E}|R(\tilde{p}(F_{Y,X}), F_{Y,X}) - R(\hat{p}_D(data), F_{Y,X})| \leq \sum_{k=1}^K \mathbb{E}|\tilde{R}_k(\tilde{p}_k) - \tilde{R}_k(\hat{p}_k)| \lesssim (K/n)^{2/3}.
\]

To summarize, we have shown that

\[
R(p_D^*(F_{Y,X}), F_{Y,X}) - R(\hat{p}_D(data), F_{Y,X}) \lesssim (K/n)^{2/3} + 1/K^2.
\]

By choosing \( K \approx n^{-1/4} \), the above bound becomes \( n^{-1/2} \). This proves part (ii) of the theorem. \( \square \)
Proof of Theorem 5. For part (i), notice that the welfare can be written as a double integral

$$W(p, F_{Y,X}) = \int_0^1 \int_0^{p(x)} yf_{Y|X}(y|x)dyf_X(x)dx.$$  

The function $yf_{Y|X}(y|x)$ is nonnegative and bounded for $y, x \in [0, 1]$. Then by the integral mean value theorem, we have

$$\mathbb{E}|W(\hat{p}_D(data), F_{Y,X}) - W(p_0^*(F_{Y,X}), F_{Y,X})|$$

$$= \mathbb{E}\left|\int_0^1 \int_{p_0^*(F_{Y,X})}^{\hat{p}_D(x; data)} yf_{Y|X}(y|x)dyf_X(x)dx\right|$$

$$\leq \sup_{y,x} |yf_{Y|X}(y|x)| \mathbb{E}\int_0^1 |\hat{p}_D(x; data) - p_0^*(x; F_{Y,X})|dx.$$  

The integral on the last line can be decomposed based on the $K$ markets:

$$\mathbb{E}\int_0^1 |\hat{p}_D(x; data) - p_0^*(x; F_{Y,X})|dx \leq \sum_{k=1}^K \int_{I_k} \mathbb{E}|\hat{p}_k(x; data) - \tilde{p}_k| + |\tilde{p}_k - p_0^*(x; F_{Y,X})|dx$$

$$\leq \sum_{k=1}^K \mathbb{E}|\hat{p}_k - \tilde{p}_k|/K + \sum_{k=1}^K \int_{I_k} |\tilde{p}_k - p_0^*(x; F_{Y,X})|dx$$

$$\lesssim (K/n)^{1/3} + 1/K \approx n^{-1/4},$$

where the last line follows from the proof of Theorem 3. For part (ii), since $p_0^*(F_{Y,X})$ is a scalar, the welfare can be simplified to

$$W(p_0^*(F_{Y,X}), F_{Y,X}) = \int_0^{p_0^*(F_{Y,X})} yf_Y(y)dy.$$  

Then we have

$$\mathbb{E}|W(\hat{p}_U(data), F_{Y,X}) - W(p_0^*(F_{Y,X}), F_{Y,X})| = \mathbb{E}\left|\int_{p_0^*(F_{Y,X})}^{\hat{p}_U(data)} yf_Y(y)dy\right|$$

$$\leq \sup_y |yf_Y(y)| \mathbb{E}|\hat{p}_U(data) - p_0^*(F_{Y,X})|$$

$$\lesssim n^{-1/3},$$

where have used Theorem 4 along with the fact that $yf_Y(y)$ is nonnegative and bounded for $y \in [0, 1]$.

\[\square\]
C Auxiliary Lemmas

**Lemma 1.** Let $f$ be a function on $[0, 1]$. Assume that $f$ is differentiable and its derivative $f'$ is Lipschitz continuous. Let $z^*$ be a point in $[0, 1]$ such that $f'(z^*) = 0$.

(i) The derivative $f'$ is a.e. differentiable on $[0, 1]$.

(ii) Assume that there exists $\kappa > 0$ such that $f''(z) \leq -\kappa$ for almost all $z \in [0, 1]$. Then, for any $z \in [0, 1]$, we have

$$|f(z) - f(z^*)| \geq \frac{\kappa}{2}(z - z^*)^2.$$  

(iii) Assume that there exists $\kappa > 0$ such that $|f''(z)| \leq \kappa$ for almost all $z \in [0, 1]$. Then, for any $z \in [0, 1]$, we have

$$|f(z) - f(z^*)| \leq \frac{\kappa}{2}(z - z^*)^2.$$  

**Proof of Lemma 1.** For part (i), notice that a Lipschitz continuous function is absolutely continuous. By Theorem 3.35 in Chapter 3 of Folland (1999), we know that $f'$ is differentiable a.e. with

$$f'(z_1) - f'(z_2) = \int_{z_2}^{z_1} f''(z)dz.$$  

For part (ii), we can apply the fundamental theorem of calculus twice and obtain that

$$f(z) - f(z^*) = \int_{z^*}^{z} f'(\tilde{z})d\tilde{z}$$  

$$= \int_{z^*}^{z} (f'(z_1) - f'(z^*))dz_1$$  

$$= \int_{z}^{z^*} \int_{z^*}^{z_1} f''(z_2)dz_2dz_1$$  

$$\leq -\kappa \int_{z}^{z^*} \int_{z^*}^{z_1} dz_2dz_1,$$

where in the second line we have used the assumption that $f'(z^*) = 0$, and in the last line we have used the assumption that $f''(z) \leq -\kappa$ for almost all $z \in [0, 1]$. The double integral in the last line is equal to

$$\int_{z}^{z^*} \int_{z^*}^{z_1} dz_2dz_1 = \int_{z}^{z^*} (z_1 - z^*)dz_1 = \frac{(z - z^*)^2}{2}.$$  

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Therefore, we have

$$|f(z) - f(z^*)| \geq \frac{\kappa}{2}(z - z^*)^2.$$  

Part (iii) can be proved analogously. \Halmos

**Lemma 2.** For the uniform distribution on $[0, 1]$, the revenue function $R(y) = y(1 - y)$. The revenue function is twice–differentiable with second-order derivative $R''(y) = -2, y \in [0, 1]$. The optimal price is $1/2$.

**Proof of Lemma 2.** The proof is straightforward. \Halmos

**Lemma 3.** Recall the perturbation function $\phi_Y$ defined in (9). Consider the following density function

$$f(y) \equiv 1 + b\delta \phi_Y \left( \frac{y - 1/2}{\delta} \right) = \begin{cases} 
1, & \text{if } y \in [0, 1/2 - \delta), \\
by + 1 - \frac{b}{2} + b\delta, & \text{if } y \in [1/2 - \delta, 1/2), \\
-by + 1 + \frac{b}{2} + b\delta, & \text{if } y \in [1/2, 1/2 + 2\delta), \\
by + 1 - \frac{b}{2} - 3b\delta, & \text{if } y \in [1/2 + 2\delta, 1/2 + 3\delta), \\
1, & \text{if } y \in [1/2 + 3\delta, 1], \\
0, & \text{otherwise}.
\end{cases}$$

Denote $F$ as the corresponding cumulative distribution function, $R(y) \equiv y(1 - F(y))$ the revenue function, and $p^* \equiv \arg \max_{y \in [0, 1]} R(y)$ the optimal price. If $\kappa \in (0, 2), |b| < 4 - 2\kappa$, and $\delta > 0$ is sufficiently small, then the following statements hold.

(i) The density $f$ is Lipschitz continuous.

(ii) The revenue function is twice–differentiable a.e.. The second–order derivative is bounded a.e. and satisfies that

$$-2f(y) - yf'(y) \geq -\kappa \text{ for almost all } y.$$  

(iii) For $b > 0$, the optimal price $p^* \in (1/2 - \delta, 1/2 - b\delta/8)$. For $b < 0$, the optimal price $p^* \in (1/2 - b\delta/8, 1/2 + 2\delta)$. For $b = 0$, the optimal price $p^* = 1/2$. In particular, $p^*$ is always an interior solution, and $f$ is always differentiable in a neighborhood of $p^*$.

**Proof of Lemma 3.** For reference, we plot here the perturbation function $\phi_Y$ and the perturbed density $f$. Part (i) is straightforward. The density $f$ is piecewise linear and hence
Lipschitz continuous with Lipschitz constant $b$. To verify the strong concavity in part (ii), note that the corresponding revenue function $R$ is continuously differentiable and twice-differentiable a.e. on the support $[0, 1]$. Its second-order derivative

$$R''(y) = -2f(y) - yf'(y) = \begin{cases} 
-2, & \text{if } y \in [0, 1/2 - \delta], \\
-3by - 2 + b - 2b\delta, & \text{if } y \in [1/2 - \delta, 1/2], \\
3by - 2 - b - 2b\delta, & \text{if } y \in [1/2, 1/2 + 2\delta], \\
-3by - 2 + b + 6b\delta, & \text{if } y \in [1/2 + 2\delta, 1/2 + 3\delta], \\
-2, & \text{if } y \in [1/2 + 3\delta, 1].
\end{cases}$$

We can see that $R''$ is piecewise linear and hence bounded a.e.. We further show that $R''$ is bounded away from zero by $\kappa$. On the intervals $[0, 1/2 - \delta]$ and $[1/2 + 3\delta, 1]$, we have $R''(y) = -2 < -\kappa$. We check the remaining three intervals one by one. On the interval $[1/2 - \delta, 1/2]$, the condition $|b| < 4 - 2\kappa$ ensures that

$$b \geq 0 \implies R''(y) \leq R''(1/2 - \delta) = -b/2 - 2 + b\delta \leq -\kappa,$$

$$b < 0 \implies R''(y) \leq R''(1/2) = -b/2 - 2 - 2b\delta \leq -\kappa,$$

when $\delta$ is sufficiently small. On the interval $[1/2, 1/2 + 2\delta]$, we have

$$b \geq 0 \implies R''(y) \leq R''(1/2 + 2\delta) = b/2 - 2 + 4b\delta \leq -\kappa,$$

$$b < 0 \implies R''(y) \leq R''(1/2) = b/2 - 2 - 2b\delta \leq -\kappa,$$
when $\delta$ is sufficiently small. On the interval $[1/2 + 2\delta, 1/2 + 3\delta]$, we have

$$b \geq 0 \implies R''(y) \leq R''(1/2 + 2\delta) = -b/2 - 2 < -\kappa,$$

$$b < 0 \implies R''(y) \leq R''(1/2 + 3\delta) = -b/2 - 2 - 3b\delta < -\kappa,$$

To summarize, we have shown that $R''(y) \leq -\kappa$ a.e. on $[0, 1]$ provided that $\delta > 0$ is sufficiently small.

For part (iii), we first consider the case $b > 0$. We only need to consider the interval $[1/2 - \delta, 1/2]$. The reason will become clear later. The cumulative distribution function

$$F(y) = \frac{b}{2} y^2 + \left(1 - \frac{b}{2} + b\delta\right) \delta y + \frac{b}{2} (1/2 - \delta)^2, y \in [1/2 - \delta, 1/2].$$

The revenue function

$$R(y) = -\frac{b}{2} y^3 - \left(1 - \frac{b}{2} + b\delta\right) y^2 + \left(1 - \frac{b}{2} (1/2 - \delta)^2\right) y, y \in [1/2 - \delta, 1/2].$$

The marginal revenue

$$R'(y) = -\frac{3b}{2} y^2 - (2 - b + 2b\delta) y + 1 - \frac{b}{2} (1/2 - \delta)^2, y \in [1/2 - \delta, 1/2].$$

We evaluate the marginal revenue at two points $1/2 - \delta$ and $1/2 - \frac{b\delta}{8}$. When $y = 1/2 - \delta$, the marginal revenue

$$R'(1/2 - \delta) = \delta > 0.$$ 

When $y = 1/2 - b\delta/8$, the marginal revenue

$$R'\left(1/2 - \frac{b\delta}{8}\right) \approx \frac{b(b - 4)}{16} \delta < 0,$$

where we have omitted higher order terms involving $\delta^2$. Therefore, $R'\left(1/2 - \frac{b\delta}{8}\right)$ is negative for sufficiently small $\delta$. Since the marginal revenue $R'$ is strictly decreasing on the entire domain $[0, 1]$, we know that the only zero of $R'$ (which is the optimal price $p^*$) is within the region $(1/2 - \delta, 1/2 - \frac{b\delta}{8})$. Within this region, the revenue is twice–differentiable everywhere.

Next, we consider the case $b < 0$. In this case, we only need to study the region $[1/2, 1/2+$
The cumulative distribution function

\[ F(y) = -\frac{b}{2}y^2 + \left(1 + \frac{b}{2} + b\delta\right) y + \frac{b}{2}\delta^2 - \frac{b}{8}, \quad y \in [1/2, 1/2 + 2\delta]. \]

The revenue function

\[ R(y) = y(1 - F(y)) = \frac{b}{2}y^3 - \left(1 + \frac{b}{2} + b\delta\right) y^2 + \left(1 + \frac{b}{8} - \frac{b}{2}\delta^2 + \frac{b}{2}\delta\right), \quad y \in [1/2, 1/2 + 2\delta]. \]

The marginal revenue

\[ R'(y) = \frac{3b}{2}y^2 - (2 + b + 2b\delta)y + \left(1 + \frac{b}{8} - \frac{b}{2}\delta^2 + \frac{b}{2}\delta\right), \quad y \in [1/2, 1/2 + 2\delta]. \]

We evaluate the marginal revenue at two points \(1/2 + \delta\) and \(1/2 - b\delta/8\). When \(y = 1/2 + \delta\), the marginal revenue

\[ R'(1/2 + \delta) \approx -2\delta < 0, \]

where we have omitted higher order terms involving \(\delta^2\). When \(y = 1/2 - b\delta/8\), the marginal revenue

\[ R'(1/2 - \frac{b\delta}{8}) \approx \frac{b(b + 4)}{16}\delta > 0, \]

where we have omitted higher order terms involving \(\delta^2\). Since the marginal revenue \(R'\) is strictly decreasing on the entire domain \([0, 1]\), we know that the only zero of \(R'\) (which is the optimal price \(p^*\)) is within the region \((1/2 - \frac{b\delta}{8}, 1/2 + \delta)\). Within this region, the revenue is twice–differentiable everywhere.

Lastly, when \(b = 0\), the density function is constant, and Lemma 2 shows that the optimal price is 1/2. Therefore, regardless of the sign of \(b\), the optimal price is always an interior solution, and is in the interior of a region on which the revenue function is twice–differentiable.

\[ \square \]

**Lemma 4.** Take \(x_0 \in [0, 1]\). Recall the following definition of \(\omega_D(\epsilon)\) and \(\omega_U(\epsilon)\):

\[ \omega_D(\epsilon) \equiv \sup_{F_1, F_2 \in \mathcal{F}_\kappa} \{ |p_D^*(x_0; F_1) - p_D^*(x_0; F_2)| : H(F_1 \parallel F_2) \leq \epsilon \}, \]

\[ \omega_U(\epsilon) \equiv \sup_{F_1, F_2 \in \mathcal{F}_\kappa} \{ |p_U^*(F_1) - p_U^*(F_2)| : H(F_1 \parallel F_2) \leq \epsilon \}. \]
Then

\[
\inf_{\tilde{D}} \sup_{F_{Y,X} \in \mathcal{F}_n} E_{F_{Y,X}} [\tilde{p}_D(x_0; \text{data}) - p_D^*(x_0; F_{Y,X})] \geq \frac{1}{8} \omega_D \left(1/(2\sqrt{n})\right),
\]

\[
\inf_{\tilde{U}} \sup_{F_{Y,X} \in \mathcal{F}_n} E_{F_{Y,X}} [\tilde{p}_U(data) - p_U^*(F_{Y,X})] \geq \frac{1}{8} \omega_U \left(1/(2\sqrt{n})\right).
\]

Proof of Lemma 4. By treating \(p_D^*(x_0; \cdot)\) and \(p_U^*(\cdot)\) as functionals, the desired results directly follow from Corollary 15.6 (Le Cam for functionals) in Chapter 15 of Wainwright (2019). □

Lemma 5. Let \(\{F_{Y,X}^j : 1 \leq j \leq M\} \subset \mathcal{F}_n\) be such that

\[
\|p_D^*(F_{Y,X}^j) - p_D^*(F_{Y,X}^{j'})\|_2 \geq 2\delta, j \neq j'.
\]

Then we have

\[
\inf_{\tilde{D}} \sup_{F_{Y,X} \in \mathcal{F}_n} E\|\tilde{p}_D(data) - p_D^*(F_{Y,X})\|_2^2 \geq 2\delta^2 \left(1 - \frac{\sum_{j,j'=1}^M KL(F_{Y,X}^j\|F_{Y,X}^{j'})/M^2 + \log 2}{\log M}\right)
\]

Proof of Lemma 5. The result follows from Proposition 15.12 (the Fano’s inequality) and inequality (15.34) (convexity of the KL divergence) in Chapter 15 of Wainwright (2019), where \(\Phi\) is taken to be the square function, \(\rho\) the \(L_2\)–distance, and \(\theta\) the functional \(p_D^*\). □

Lemma 6. Consider the following function class:

\[
\{(y, x) \mapsto (p\mathbf{1}\{y \geq p\} - \tilde{p}\mathbf{1}\{y \geq \tilde{p}\})\mathbf{1}\{x \in [k/K, (k+1)/K)\} : p \in [0, 1]\}.
\]

For any \(\tilde{p} \in [0, 1]\), \(K \geq 1\), and \(0 \leq k \leq K - 1\), the above class is a VC–subgraph with VC–dimension no greater than 2.

Proof of Lemma 6. By Lemma 2.6.22 in Chapter 2 of van der Vaart and Wellner (1996), the class

\[
\{(y, x) \mapsto p\mathbf{1}\{y \geq p\} : p \in [0, 1]\}
\]

is a VC–subgraph with VC–dimension no greater than 2.\(^{11}\) The function \(y, x) \mapsto \tilde{p}\mathbf{1}\{y \geq \tilde{p}\}\) is a fixed function that does not depend on the index \(p\). By the proof Lemma 2.6.18(v) in

\(^{11}\)In the original statement of the lemma, the VC dimension is no greater than 3. This is because the definition of VC dimension in van der Vaart and Wellner (1996) is the smallest number \(n\) for which no set of \(n\) points is shattered. The definition we use in this paper is the largest number \(n\) that some set of \(n\) points is shattered.
van der Vaart and Wellner (1996), the class
\[
\{(y,x) \mapsto p \mathbb{1}\{y \geq p\} - \tilde{p} \mathbb{1}\{y \geq \tilde{p}\} : p \in [0,1]\}
\]
is a VC–subgraph with VC–dimension no greater than 2. Lastly, we multiply each function in the class by an indicator \(1\{x \in [k/K, (k + 1)/K)\}\). This does not increase the VC–dimension.

**Lemma 7.** Let \(Z_1, \ldots, Z_n\) be an i.i.d. sequence of random variables from distribution \(P\). Let \(G\) be a class of VC–subgraph functions with VC–dimension \(v\) and envelope function \(G\). Assume that \(\|G\|_{L_2(P)} < \infty\). Then we have
\[
\mathbb{E}\sup_{g \in G} \left| \frac{1}{n} \sum_{i=1}^{n} g(Z_i) - \mathbb{E}g(Z_i) \right| \leq 8\sqrt{2} \frac{\|G\|_{L_2(P)}}{\sqrt{n}} \left( \log(2C) + \log(v) + (\log(16) + 3)v \right),
\]
for some universal constant \(C\).

**Proof of Lemma 7.** This is a well–known result in the literature. We include it here for completeness. Let \(N(G, L_2(Q), \tau)\) denote the covering number of \((G, L_2(Q))\). By Remark 3.5.5 in Chapter 3 of Giné and Nickl (2015), we know that
\[
\mathbb{E}\sup_{g \in G} \left| \frac{1}{n} \sum_{i=1}^{n} g(X_i) - \mathbb{E}g(X_i) \right| \leq 8\sqrt{2} \frac{\|G\|_{L_2(P)}}{\sqrt{n}} \int_0^1 \sup_Q \sqrt{\log 2N(G, L_2(Q), \tau\|G\|_{L_2(Q)})} d\tau,
\]
where the supremum is taken over all discrete probabilities with a finite number of atoms. By Theorem 2.6.7 in Chapter 2 of van der Vaart and Wellner (1996), we know that for any probability measure \(Q\),
\[
N(G, L_2(Q), \tau\|G\|_{L_2(Q)}) \leq C_v(16e)^v (1/\tau)^{2v},
\]
for some universal constant \(C\). Therefore,
\[
\int_0^1 \sup_Q \sqrt{\log 2N(G, L_2(Q), \tau\|G\|_{L_2(Q)})} d\tau \leq \log(2C) + \log(v) + (\log(16) + 3)v
\]
Then the desired result follows.

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