Immersed Spheres and Finite Type for Donaldson Invariants

Wojciech Wieczorek

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1 Introduction

Not that long time ago Donaldson invariants were the main tool in studying the four dimensional manifolds. Even though these invariants have been overshadowed by the arrival of the Seiberg-Witten invariants, they still are of interest to both mathematicians and physicists. For a simply connected smooth 4-manifold with \( b_+ > 1 \) one can define a linear function

\[
D_w : A(X) = \text{Sym}^* (H_0(X) \oplus H_2(X)) \to \mathbb{R}
\]

for any \( w \in H^2(X, \mathbb{Z}) \). By studying universal relations generated by the presence of embedded surfaces Kronheimer and Mrówka \cite{Kronheimer-Mrowka} have shown that these invariants satisfy certain structure equations. To make these relations even more compact, they have introduced a special class of manifolds.

Definition 1.1 A manifold \( X \) is of \( w \)-simple type for some \( w \) if \( D_w((x^2 - 4)z) = 0 \) for any \( z \in A(X) \) and where \( x \) is a generator of \( H_0(X) \).

For manifolds with this property one can combine invariants of various degree into a function \( D_w(\alpha) = D ((1 + \frac{x}{2}) \exp(\alpha)) \). This function \( D \) has a very nice structure:

Theorem 1.2 (Kronheimer, Mrówka \cite{Kronheimer-Mrowka}) For a simply connected 4–manifold \( X \) of simple type there are finitely many basic classes \( K_1, \ldots, K_s \in H^2(X, \mathbb{Z}) \) such that:

1. \( K_i \equiv w_2(X) \pmod{2} \).

2. There are rational numbers \( a_1, \ldots, a_s \) such that

\[
D_w = \exp(Q_X/2) \sum_{i=1}^{s} a_i \sinh K_i \quad \text{if } b_+ \equiv 1 \pmod{4}, \text{ or } (1.1)
\]

\[
D_w = \exp(Q_X/2) \sum_{i=1}^{s} a_i \cosh K_i \quad \text{if } b_+ \equiv 3 \pmod{4}. \quad (1.2)
\]

where \( Q_X \) denotes the intersection form on \( X \).
Soon after the discovery of this theorem, Fintushel and Stern in [4] have shown that identical theorem can be proven by studying embedded spheres. Moreover they proved that if a simply connected manifold is \(w\)-simple type for one \(w \in H^2(X)\) then it is of simple type for any other \(w'\).

It is still an open question whether there are simply connected manifolds with \(b_+ > 1\) that are not of simple type. The most important result that does not use the assumption that a manifold is of simple type is Fintushel-Stern blowup formula ([5] and Theorem 2.1), which is a special kind of structure equation for manifolds containing embedded spheres with self-intersection \(-1\).

In an attempt to understand the structure equation in general case Kronheimer and Mrówka in non-published paper [6] have defined a manifold \(X\) to be of finite type \(r\) if it satisfies the condition \(D_w((x^2 - 4)^rz) = 0\) for some non-negative number \(r\). The main conjecture of their informal announcement is that all simply connected manifolds with \(b_+ > 1\) are of finite type.

Recently Muñoz using Fukaya-Floer homology of \(\Sigma \times S^1\) (Theorem 7.6 in [10]) proved that when \(X\) contains an embedded surface \(\Sigma\) with genus \(g\) and \(\Sigma \cdot \Sigma = 0\), then \(X\) is of finite type with \[r = \left\lceil \frac{2g + 2}{4} \right\rceil\]

In this paper instead of embedded surfaces we study immersed spheres. First we focus on the relations between Donaldson invariants involving embedded spheres. We show how to write these relations in a compact form. Next we move to studying immersed spheres. Our main result is the following:

**Theorem 1.3** Let \(\alpha\) be an immersed sphere with \(p\) positive double points and any self-intersection \(a\). Assume that there exist a cohomology class \(w\) such that \(w \cdot \alpha = 1 \pmod{2}\). Let \(z\) be a product of classes perpendicular to \(\alpha\). For every \(s = 0, 1, \ldots, p\) define the numbers:

\[r = r(p, s) = \left\lfloor \frac{p + 1 - s}{2} \right\rfloor\]

and

\[k = k(a, s) = s - \left\lfloor \frac{a + 1}{2} \right\rfloor - 1\]

\[k_0 = \begin{cases} k & \text{if } a \text{ is even} \\ k + 1 & \text{if } a \text{ is odd} \end{cases}\]

Then for every such \(s\) there are the following structure equations:

\[D_w\left( (x^2 - 4)^r \cosh(t\alpha)z \right) = D_w\left( B^{-a} \frac{(x^2 - 4)^rz}{(2 - xq)^s} \sum_{i=0}^{k} q^i Q' \cdot c_i(\alpha) \right)\]

and

\[D_w\left( (x^2 - 4)^r \sinh(t\alpha)z \right) = D_w\left( B^{-a} \frac{(x^2 - 4)^rz}{(2 - xq)^s} \sum_{i=0}^{k_0} q^i Q \cdot d_i(\alpha) \right)\]

where \(c_i(\alpha)\) and \(d_i(\alpha)\) are polynomials of degree \(2i\) and respectively \(2i + 1\) on \(\alpha\) (and some powers of \(x\)). In the above \(B\) and \(S\) are the functions defined by Fintushel and Stern in their blowup formula [4], and we define \(Q = (B/S)\) and \(q = Q^2\).
As a consequence of this theorem we show that every simply connected manifold containing an immersed sphere with \( p \) positive double points and non-negative self intersection \( a \) is of finite type with

\[ r = \left\lfloor \frac{2p + 2 - a}{4} \right\rfloor \]

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2 The structure equations for embedded spheres

The relations between various Donaldson invariants \( D_w(\sigma^n) \) when a homology class \( \sigma \) is represented by an embedded sphere had been studied in [4] as well as in [11]. In this section we first review the basic definitions, after which we state the main result for \( U(2) \)-invariants (Theorem 2.1). Since the proof of this Theorem is only a slight modification of our previous result from [11], we include that proof in the Appendix. In this section, as a Corollary, we will show how to find the coefficients of the general structure equations in terms of some elliptic functions.

To define the \( U(2) \)-Donaldson invariants for a simply connected 4-manifold with \( b_+ \geq 3 \) odd we consider the compactified moduli space \( \mathcal{M}_{k,w}(X) \) of anti-self–dual (ASD) connections on an \( U(2) \)-bundle \( P \) over \( X \) with \( c_2(P) = k \in H^4(X, \mathbb{Z}) \) and first Chern class \( w \in H^2(X, \mathbb{Z}) \). Set

\[ 2d = \dim \mathcal{M}_{k,w}(X) = 8k - 4w^2 - 3(b_+ + 1) \]

Then there is a universal \( SO(3) \) fibration \( P \) over \( \mathcal{M}_{k,w}(X) \) which gives rise to the homomorphism

\[ \mu : H_i(X) \to H^{4-i}(\mathcal{M}_{k,w}(X)) \]

given by \( \mu(\sigma) = -\frac{1}{2}p_1(P)/\sigma \). This allows one to define Donaldson invariants as linear maps \( D_{d,w} : A^d(X) = \text{Sym}^d(H_0(X) \oplus H_2(X)) \to \mathbb{R} \) where the elements of \( H_i(X) \) have the degree \( \frac{1}{2}(4 - i) \) and \( A^d(X) \) is the set of elements of \( A(X) = \text{Sym}_w(H_0(X) \oplus H_2(X)) \) having degree \( d \). The function \( D_{d,w} \) assigns to the generator \( x \in H_0(X) \) and the classes \( \sigma_1, \ldots, \sigma_r \in H_2(X) \) the number

\[ D_{d,w}(\sigma_1 \cdots \sigma_r \cdot x^s) = \langle \mu(\sigma_1) \cdots \mu(\sigma_r) \cdot \mu(x)^s, [\mathcal{M}_k(X)] \rangle \]

For simply connected manifolds with \( b_+ = 1 \) the same construction can be performed, except that now the Donaldson invariant depends on metric on \( X \). The details of the above constructions can be found in [3].

One combines the Donaldson invariants into a formal power series

\[ D_w(\exp(t\sigma)) = \sum D_w(\sigma^n) \frac{t^n}{n!} \]

With the notation set above we can state the structure theorem for embedded spheres:

**Theorem 2.1** Let \( X \) be a simply connected smooth 4-manifold that contains an embedded 2-sphere \( \sigma \) with self-intersection \( \sigma \cdot \sigma = s \). Let \( z \) denote an arbitrary product of classes \( z_i \) that are
perpendicular to $\sigma$ and let $\varepsilon = w \cdot \sigma \pmod{2}$ for some given cohomology class $w$. Then there are universal functions $C_i = C_i(t, x; \varepsilon)$ such that when $s = -2k$:

$$D_w(\exp(t\sigma)z) = \begin{cases} D_w \left( (C_0 + C_1\sigma + \cdots + \sigma^{2k-1} + C_{2k}\sigma^{2k})z \right) & \text{when } \varepsilon = 0 \\ D_w \left( (C_0 + C_1\sigma + \cdots + C_{2k-1}\sigma^{2k-1})z \right) & \text{when } \varepsilon = 1 \end{cases}$$

and when $s = -(2k + 1)$:

$$D_w(\exp(t\sigma)z) = \begin{cases} D_w \left( (C_0 + C_1\sigma + \cdots + C_{2k}\sigma^{2k})z \right) & \text{when } \varepsilon = 0 \\ D_w \left( (C_0 + C_1\sigma + \cdots + \sigma^{2k} + C_{2k}\sigma^{2k+1})z \right) & \text{when } \varepsilon = 1 \end{cases}$$

In the above the $\hat{\sigma}^k$ means that the corresponding term does not appear in the formula.

For brevity we shall omit the class $z$ in structure equations that will follow, reminding about the condition $z \cdot \sigma = 0$ only when necessary.

The first structure equation for embedded spheres that did not use the simple type condition was Fintushel-Stern blowup formula. In [5] they proved that when $\sigma$ is a class represented by an embedded sphere with self-intersection $-1$, then for every $k$ there exist a function $B_k(x)$ such that:

$$D_w(\sigma^k) = D_w(B_k(x)) \quad (2.1)$$

for every $w \in H^2(X)$ such that $w \cdot \sigma = 0 \pmod{2}$.

The functions $B_k$ of (2.1) when combined in the power series $B(x, t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} B_k(x)$ satisfy:

$$B(x, t) = \exp \left( -\frac{t^2x}{6} \right) \sigma_3(t) \quad (2.2)$$

where $\sigma_3$ is a particular quasiperiodic Weierstrass sigma-function associated to the $\wp$–function $y$, which satisfies the differential equation

$$(y')^2 = 4y^3 - g_2y - g_3$$

with $g_i$'s given by:

$$g_2 = 4 \left( \frac{x^2}{3} - 1 \right), \quad g_3 = \frac{8x^3 - 36x}{27}$$

(for details on elliptic functions see for example [2]). Using the power series notation we can rewrite (2.1) as

$$D_w(\exp(t\sigma)) = D_w(B(t, x)) \quad (2.3)$$

This formula is called the blowup formula (since after performing the blowup on the manifold $X$, the exceptional divisor in $X \# \overline{CP^2}$ is represented by an embedded sphere with self-intersection $-1$). There is also a variant of the blowup formula for the twisted Donaldson invariants, that originally was written as:

$$D_{w+\sigma}(\exp(t\sigma)) = D_w(S(t, x)) \quad (2.4)$$
where \( S(t, x) = e^{-t^2x^2/6}\sigma(t) \), and \( \sigma(t) \) is the standard Weierstrass sigma–function. To make our next formulas fit into a single pattern we use the fact that \( D_{w+\sigma}(\sigma) = D_w \) and write (2.4) as:

\[
D_w'(\exp(t\sigma)) = D_w'(\sigma \cdot S(t, x))
\]

for \( w' = w + \sigma \).

It turns out that we can express the functions \( C_i(t, x, \varepsilon) \) of Theorem 2.1 in terms of \( B \) and \( S \) defined above. For convenience we define

\[
\Delta = S'B - SB'
\]

\[
Q = S/B
\]

\[
q = Q^2
\]

With this notation we have:

**Theorem 2.2** Let \( \sigma \) be a homology class represented by an embedded sphere and set \( n = -\sigma \cdot \sigma \).

Like before, let \( \varepsilon = w \cdot \sigma \) (mod 2). Then modulo the kernel of \( D_w(.) \) we have:

\[
\det \begin{bmatrix}
\cosh(t\sigma) & 1, \sigma^2, \ldots \\
[S^{2i}B^{n-2i}] & W_c(n, \varepsilon)
\end{bmatrix}
= \det \begin{bmatrix}
\sinh(t\sigma) & [\sigma, \sigma^3, \ldots] \\
[S^{2i+1}B^{n-2i-3}] & W_o(n, \varepsilon)
\end{bmatrix} = 0
\]

when \( \varepsilon = 0 \), and:

\[
\det \begin{bmatrix}
\cosh(t\sigma) & 1, \sigma^2, \ldots \\
[S^{2i}\Delta B^{n-2i-2}] & W_c(n, \varepsilon)
\end{bmatrix}
= \det \begin{bmatrix}
\sinh(t\sigma) & [\sigma, \sigma^3, \ldots] \\
[S^{2i+1}B^{n-2i-1}] & W_o(n, \varepsilon)
\end{bmatrix} = 0
\]

when \( \varepsilon = 1 \). The symbol \( [S^{2i+1}B^{n-2i-1}] \) denotes the column vector consisting of the entries indicated in brackets, where \( i \) varies from \( i = 1 \) to the maximum possible for which the power over \( B \) is non-negative. Similarly \( [1, \sigma^2, \ldots] \) denotes a row vector. The \( W_c(n, \varepsilon) \) and \( W_o(n, \varepsilon) \) denote the Wronskian containing even and correspondingly odd derivatives of functions appearing in the first column of each matrix.

**Proof:** It turns out that all one needs to establish formulas like the ones above is:

1. The general structure equation as described in Theorem 2.1.
2. The blowup formulas (2.3) and (2.5).

We shall describe this procedure in the case when \( \sigma^2 = -2k \) and \( \varepsilon = 0 \):

In this case there are \( 2k \) functions \( C_i(t, x) \) of Theorem 2.1 to be found: the \( (k+1) \) coefficients with even powers of \( \sigma \) and \( (k-1) \) coefficients with odd powers of \( \sigma \). The structure equation is valid for any sphere of self-intersection \( -2k \) and any perpendicular class \( z \), so we take any manifold \( X \) with nontrivial Donaldson invariant and blow it up \( 2k \) times. Let \( e_1, e_2, \ldots, e_{2k} \) denote the exceptional divisors, and let \( \sigma = e_1 + e_2 + \cdots + e_{2k} \). Also let \( w \) be a cohomology class such that \( w \cdot e_i = 0 \) for every \( i \).
In order to find the first \((k+1)\) functions \(C_i\) we apply Theorem 2.1 for \(w' = w + e_1 + e_2 + \cdots + e_{2i}\) where \(i = 0, 1, \ldots, k\). Note that for such \(w'\) and for any odd number \(s\), \(D_w(\sigma^s) = 0\). Thus we get \((k+1)\) equations of the form:

\[
D_w'(\exp(t\sigma)) = D_w(S^{2i}B^{2k-2i}) = D_w' \left( (C_0 + C_2\sigma^2 + \cdots + C_{2k}\sigma^{2k})z \right)
\]

For each \(r\) the number \(D_w(\sigma^{2r})\) can be computed from the blowup formula. In fact if we expand

\[
S^{2i}B^{2k-2i} = \sum_{j=0}^{\infty} m_j(x) t^j
\]

then \(D_w'(\sigma^{2r}) = D_w(m_{2i}(x))\) which explains the appearance of the Wronskian in Theorem 2.2.

To find the remaining \((k-1)\) functions we consider \(\sigma\) as before and take as an orthogonal class \(z = e_1 - e_2\). Multiplying \(D_w(\exp(te))\) by a class \(e\) acts as differentiation, i.e.

\[
D_w(e \cdot \exp(te)) = D_w(B'(t, x)) \\
D_{w+e}(e \cdot \exp(te)) = D_w(S'(t, x))
\]

To ensure that we get a nontrivial relation, the class \(w'\) must include precisely one of \(\{e_1, e_2\}\), e.g. \(w' = w + e_2 + e_3 + \cdots + e_{2i+1}\) for \(i = 1, 2, \ldots, (k-1)\). Similarly like before we notice that for any even \(n\)

\[
D_w'(\sigma^n e_1) = D_w'(\sigma^n e_2) = 0
\]

Then, plugging these in we get:

\[
D_w'(\exp(t\sigma)(e_1 - e_2)) = D_w(S^{2i+1}B^{n-2i-3}(B'S - S'B)) = D_w' \left( (C_1\sigma + C_3\sigma^3 + \cdots + C_{2k-3}\sigma^{2k-3})(e_1 - e_2) \right)
\]

We can arrange these equations in the Wronskian in a similar way as before.

The proof of the other two cases is analogous. \(\square\)

One can show that the functions \(B\) and \(S\) satisfy: \(S(t, x) = t + O(t^3)\) and \(B(t, x) = 1 + O(t^4)\). Thus for any \(r\) and \(n\):

\[
B'(t, x)S^n(t, x) = t^n + O(t^{n+2}) \quad (2.6)
\]

This shows that all Wronskians \(W(n, \varepsilon)\) appearing in Theorem 2.2 are upper diagonal matrices with determinant equal to one. Thus, for example, the first case of this Theorem may be rephrased as:

**Corollary 2.3** Let \(X\) be a 4-manifold containing an embedded sphere \(\sigma\) with self-intersection \(-2k\). Let \(w \cdot \sigma = 0\). Then there are polynomials \(c_i(\sigma, x)\) of degree \(2i\) in \(\sigma\), such that:

\[
D_w(\cosh(t\sigma)) = D_w \left( \sum_{i=0}^{k} c_i(\sigma, x)S^{2i}B^{2k-2i} \right)
\]
Proof: Expand the corresponding matrix in Theorem 2.2 by the first column. □

Once we have the procedure of finding relations for embedded spheres, we can ask a computer to find particular polynomials $c_i(\sigma)$ (we shall skip in our notation the dependence of these polynomials on $x$.) Next theorem lists several of such formulas, the last of them being of particular importance:

**Corollary 2.4** Let $\sigma$ be a homology class represented by embedded sphere and $f, w \in H^2(X, \mathbb{Z})$ such that $f \cdot \sigma = 1 \pmod{2}$ and $w \cdot \sigma = 0 \pmod{2}$.

If $\sigma \cdot \sigma = -2$:

$$D_w(\exp(t\sigma)) = D_w(B^2 + \sigma^2 \frac{1}{2} S^2)$$

$$D_f(\exp(t\sigma)) = D_f(\Delta + \sigma BS)$$

If $\sigma \cdot \sigma = -3$:

$$D_w(\exp(t\sigma)) = D_w(B^2 + \sigma S \Delta + \sigma^2 \frac{1}{2} BS^2)$$

$$D_f(\exp(t\sigma)) = D_f(B \Delta + \sigma S (B^2 + \frac{x}{3!} S^2) + \sigma^3 \frac{1}{3!} S^3)$$

If $\sigma \cdot \sigma = -4$:

$$D_w(\exp(t\sigma)) = D_w((B^4 + \frac{x}{3} S^4) + \sigma SB \Delta + \sigma^2 \frac{S^2}{2} (B^2 + \frac{x}{3} S^2) + \sigma^4 \frac{1}{4!} S^4)$$

$$D_f(\exp(t\sigma)) = D_f(\Delta \left(B^2 + \frac{x}{2} S^2\right) + \sigma SB (B^2 + \frac{x}{3!} S^2) + \sigma^2 \frac{1}{2} S^2 \Delta + \sigma^3 \frac{1}{3!} S^3 B)$$

**Note 2.5** The formula for $-2$ sphere was first proved by R. Brussee and also has been known to other authors. The formula for $-3$ sphere and the class $w$ was proven by the author in [12]. At the time of writing that article we could not handle formulas for spheres of self-intersection $-4$ or lower.

From the relation for $-4$ spheres we get the following formulas:

**Lemma 2.6 (Double angle formulas)** The functions $B$ and $S$ used in the blowup formula satisfy:

$$D_w(B(2t)) = D_w(B^4 - S^4)$$

$$D_w(S(2t)) = 2D_w(\Delta BS)$$

Also we have:

$$D_w(\Delta^2) = D_w(B^4 - x S^2 B^2 + S^4)$$

**Proof:** Let $\hat{X}$ denote $X$ blown up at one point and $e$ an exceptional class of $\hat{X}$. For any natural number $n$ we have $D_X(x^n) = D_{\hat{X}}(x^n)$, thus it is sufficient to prove the Lemma for $\hat{X}$. The class $2e$ is represented by embedded sphere with self-intersection $-4$. Note that $f \cdot 2e \equiv 0 \pmod{2}$ when $f = e$. Thus we can plug into the first formula for $-4$ sphere $\sigma = 2e$ with $\sigma \cdot \sigma = 0$ to get the “double angle formula” for the function $B(2t)$, and then set $f = e$ to get the formula for $S(2t)$. In the last identity we blew up $X$ four times and used $\sigma = e_1 + \cdots + e_4, f = e_1 + e_3$ and the perpendicular class $z = (e_1 - e_2)(e_3 - e_4)$. □
3 Auxiliary lemmas

In computations involving the functions $B$ and $S$ it is crucial to understand their quotient $Q = S/B$. According to [2] we have:

\[ Q(t) = \frac{\sigma(t)}{\sigma_3(t)} = \frac{1}{\sqrt{e_1 - e_3}} \text{sn} \left( \sqrt{e_1 - e_3} \cdot t, k \right) \]  

where:

\[ e_1 = \frac{x}{6} + \frac{\sqrt{x^2 - 4}}{2}, \quad e_2 = \frac{x}{6} - \frac{\sqrt{x^2 - 4}}{2}, \quad e_3 = -\frac{x}{3} \]

d and:

\[ k^2 = \frac{e_2 - e_3}{e_1 - e_3} = \frac{x - \sqrt{x^2 - 4}}{x + \sqrt{x^2 - 4}} = \frac{4}{\left( \frac{x + \sqrt{x^2 - 4}}{2} \right)^2} \]

With the above we can write (3.1) as:

\[ Q(t) = \sqrt{k} \cdot \text{sn} \left( \frac{t}{\sqrt{k}}, k \right) \]

The function $y = \text{sn}(t)$ satisfies the differential equation:

\[ \left( \frac{dy}{dt} \right)^2 = (1 - y^2)(1 - k^2 y^2) \]

From this by simple computations we get:

\[ (Q'(t))^2 = (1 - \frac{1}{k} \cdot Q^2)(1 - k \cdot Q^2) \]

\[ = 1 - x \cdot Q^2 + Q^4 \]

We could use this formula to prove the last statement in Lemma 2.4. In fact this shows that we can remove $D_w(.)$ from its statement.

We need one more statement about the functions $q^i$, which is immediate from studying the Wronskian of these functions:

**Lemma 3.1** The functions $q^i$ are linearly independent.

For further computations we shall need some more elementary observations.

**Lemma 3.2** Let $\sigma = \alpha + 2e_1 + 2e_2 + \cdots + 2e_k$ for some class $\alpha$ that is perpendicular to all exceptional divisors $e_j$. Let us fix a number $m \in \{0, 1, \ldots, k\}$ and let $w$ be such that $w \cdot e_j = 0$ for every $j$. Then for $w' = w + \sum_{i=0}^{m} e_m$ we have:

\[ D_{w'}(\cosh(t\alpha) + \sum_{i=0}^{k} 2e_j)) = \begin{cases} 
D_w(\cosh(t\alpha) S(2t)^i B(2t)^{k-i}) & \text{when } m \text{ is even.} \\
D_w(\sinh(t\alpha) S(2t)^i B(2t)^{k-i}) & \text{when } m \text{ is odd.} 
\end{cases} \]  

and

\[ D_{w'}(\sinh(t\alpha) + \sum_{i=0}^{k} 2e_j)) = \begin{cases} 
D_w(\cosh(t\alpha) S(2t)^i B(2t)^{k-i}) & \text{when } m \text{ is odd.} \\
D_w(\sinh(t\alpha) S(2t)^i B(2t)^{k-i}) & \text{when } m \text{ is even.} 
\end{cases} \]
Proof: The result follows immediately from the relations:
\[
\begin{align*}
\cosh(\alpha + \beta) &= \cosh \alpha \cdot \cosh \beta + \sinh \alpha \cdot \sinh \beta \\
\sinh(\alpha + \beta) &= \cosh \alpha \cdot \sinh \beta + \sinh \alpha \cdot \cosh \beta
\end{align*}
\]
combined with the blowup formulas (2.3), (2.4). □

Definition 3.3 We shall call a polynomial \(a_0 + a_1 x + \cdots + a_n x^n\) a doubly monic of degree \(n\) if both \(a_0\) and \(a_n\) are equal to \(\pm 1\).

Lemma 3.4 Let \(f = f(q)\) be a doubly monic polynomial of the degree \(d\). Let \(g\) be any polynomial of degree \(k + d\). If
\[
P = \frac{1}{f} \cdot g
\]
is a polynomial, then its degree does not exceed \(k\).

Proof: By long division, we get
\[
g = f_1 \cdot f + f_2
\]
Then a direct check of the coefficients of degree \(k + d + 1, k + d + 2, \ldots\) in:
\[
P \cdot f = g = f_1 \cdot f + f_2
\]
proves the result. Notice that \(f\) needs to be monic for this conclusion to hold, and its first coefficient needs to be \(\pm 1\) for the formal inverse \(1/f\) to make sense. □

Lemma 3.5 Let \(f\) and \(g\) be two doubly monic polynomials, and let \(w_1\) and \(w_2\) be some other two polynomials satisfying:
\[
\begin{align*}
\begin{align*}
f \cdot w_1 &= g \cdot w_2 \\
f \cdot \phi_1 + g \cdot \phi_2 &= C
\end{align*}
\end{align*}
\]}

for some polynomials \(\phi_1, \phi_2\) and a constant \(C\). Then
\[
w_1 \cdot \phi_2 + w_2 \cdot \phi_1 = w_1 \cdot \frac{C}{g}
\]

Proof: By direct computations:
\[
w_1 \cdot \phi_2 + w_2 \cdot \phi_1 = w_1 (\phi_2 + (f/g) \cdot \phi_1) = \frac{w_1}{g} (g \phi_2 + f \phi_1) = C \frac{w_1}{g}
\]
The structure equation immersed spheres

Now we are ready to prove our main result, which was stated in the introduction:

**Theorem 1.3** Let $\alpha$ be an immersed sphere with $p$ positive double points and any self-intersection $a$. Assume that there exist a cohomology class $w$ such that $w \cdot \alpha = 1 \pmod{2}$. Let $z$ be a product of classes perpendicular to $\alpha$. For every $s = 0, 1, \ldots, p$ define the numbers:

$$r = r(p, s) = \left\lfloor \frac{p + 1 - s}{2} \right\rfloor$$

and

$$k = k(a, s) = s - \left\lfloor \frac{a + 1}{2} \right\rfloor - 1$$

$$k_0 = \begin{cases} k & \text{if } a \text{ is even} \\ k + 1 & \text{if } a \text{ is odd} \end{cases}$$

Then for every such $s$ there are the following structure equations:

$$D_w \left((x^2 - 4)^r \cosh(t\alpha) z\right) = D_w \left(B^{-a} \frac{(x^2 - 4)^r z}{(2 - xq)^s} \cdot \sum_{i=0}^{k} q^i Q' \cdot c_i(\alpha)\right)$$

and

$$D_w \left((x^2 - 4)^r \sinh(t\alpha) z\right) = D_w \left(B^{-a} \frac{(x^2 - 4)^r z}{(2 - xq)^s} \cdot \sum_{i=0}^{k_0} q^i Q \cdot d_i(\alpha)\right)$$

where $c_i(\alpha)$ and $d_i(\alpha)$ are polynomials of degree $2i$ and respectively $2i + 1$ on $\alpha$ (and some powers of $x$).

![Diagram showing the relation between $p$, $s$, and $r$ in Theorem 1.3](image)

**Proof:** We shall prove the Theorem by induction on $p$.

For $p = 0$ there is only one possible $s = 0$, for which the conclusion of the Theorem is the structure equation (2.1) with $r = 0$.

Thus we can assume that the Theorem is true for all $p' < p$. The existence of the next $p + 1$ structure equations we shall prove in the following three steps:
1. The \((p - 1, s)\) - structure equation implies the \((p, s + 1)\) structure equation.

2. The \((p - 1, 0)\) - structure equation implies the \((p, 0)\) when \(p\) is odd.

3. The \((p - 2, 0)\) - structure equation implies the \((p, 0)\) when \(p\) is even.

**Step 1.** First let us fix \(s \in \{0, 1, \ldots, p - 1\}\). Let \(\beta = \alpha + 2e\) be an immersed sphere with \(p - 1\) positive double points obtained from \(\alpha\) by blowing up one of its positive double points. Notice that \(r(p - 1, s) = r(p, s + 1)\), which we shall denote simply as \(r\). Denote \(k = k(a, s + 1)\). Then for \(b = \beta \cdot \beta = a - 4\) we have:

\[
k(b, s) = s - b/2 - 1 = s - a/2 + 1 = k + 1
\]

Similarly

\[
k_0(b, s) = k_0 + 1
\]

with \(k_0 = k_0(a, s + 1)\). Let \(w \in H^2(X)\) be a cohomology class such that \(w \cdot e = 0 \pmod{2}\) (and of course \(w \cdot \alpha = 1 \pmod{2}\)). Then from the \((p - 1, s)\) structure equation we have:

\[
D_w((x^2 - 4)^r \cosh(t\beta)) = D_w((x^2 - 4)^r \cosh(t\alpha)B(2t)) \quad \text{by} \quad (3.2) \text{ and } (2.3)
\]

\[
= D_w \left( \frac{B^{4-a}}{(2-xq)} \sum_{i=0}^{k+1} c_i(\alpha) \cdot q^i Q' \right) \quad \text{by the inductive assumption}
\]

which gives us:

\[
D_w((x^2 - 4)^r \cosh(t\alpha)(1 - q^2)) = D_w \left( \frac{B^{4-a}}{(2-xq)} \sum_{i=0}^{k+1} c_i(\alpha) \cdot q^i Q' \right) \quad (4.1)
\]

Note that \((w + e) \cdot (\alpha + 2e) = w \cdot \alpha \pmod{2}\), thus similarly like before we obtain:

\[
D_{w+e}((x^2 - 4)^r \sinh(t\beta)) = D_w((x^2 - 4)^r \cosh(t\alpha)S(2t)) \quad \text{by} \quad (3.3) \text{ and } (2.4)
\]

\[
= D_w \left( \frac{B^{4-a}}{(2-xq)} \sum_{i=0}^{k_0+1} d_i(\alpha) \cdot q^i Q \right) \quad \text{by the inductive assumption}
\]

which after using \(S(2t) = 2QQ'\), dividing both sides by \(Q\) and finally multiplying by \(Q'\) gives:

\[
D_w((x^2 - 4)^r \cosh(t\alpha)(1 - xq + q^2)) = D_w \left( \frac{B^{4-a}}{(2-xq)} \sum_{i=0}^{k_0+1} d_i(\alpha) \cdot q^i Q' \right) \quad (4.2)
\]

**Note 4.1** Keep in mind that the functions \(c_i(\alpha)\) and \(d_i(\alpha)\) are not quite the same as in the statement of the Theorem. In fact in this case one should use

\[
D_w(\tilde{c}_i(\alpha)) = D_w(c_i(\beta)) = D_w(c_i(\alpha + 2e))
\]

In an attempt to simplify the notation we skip the extra “tildes”.

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Step 2. Let $\alpha$ be an immersed sphere obtained by blowing up one of the positive double points of $\alpha$. Let us denote $k = k(a, 0)$, so then $k(b, 0) = k + 2$.

Similarly like before we get the relations:

$$D_w((x^2 - 4)^r \cosh(t\alpha)(1 - q^2)) = D_w \left( \frac{B^{-a}}{2 - xq} \sum_{i=0}^{k+2} c_i(\alpha) \cdot q^i Q' \right)$$

$$D_w((x^2 - 4)^r \cosh(t\alpha)(1 - xq + q^2)) = D_w \left( \frac{B^{-a}}{2 - xq} \sum_{i=0}^{k+2} d_i(\alpha) \cdot q^i Q' \right)$$

from which we get:

$$D_w \left( (1 - q^2) \sum_{i=0}^{k+2} d_i(\alpha) \cdot q^i \right) = D_w \left( (1 - qx + q^2) \sum_{i=0}^{k+2} c_i(\alpha) \cdot q^i \right)$$

Direct check verifies the following identity:

$$(1 - q^2) \cdot (-2 - qx + x^2) + (1 - qx + q^2) \cdot (-2 - qx) = (x^2 - 4)$$

Using this we get that $(x^2 - 4)^{r+1} \cosh(t\alpha)$ modulo the kernel of $D_w(\cdot)$ is equal to:

$$(x^2 - 4)^r \cosh(t\alpha) \left[ (1 - q^2) \cdot (-2 - qx + x^2) + (1 - qx + q^2) \cdot (-2 - qx) \right]$$

by (4.8)

$$= B^{-a} \left( \sum_{i=0}^{k+2} c_i(\alpha) \cdot q^i \right) \cdot (2 + qx - x^2) + \left( \sum_{i=0}^{k+2} d_i(\alpha) \cdot q^i \right) \cdot (2 + qx)$$

by (4.3) and (4.6)

$$= \left( \sum_{i=0}^{k+2} c_i(\alpha) \cdot q^i \right) \cdot \frac{B^{-a}(x^2 - 4)}{(1 - q^2)}$$
In the last step we used Lemma 3.5 for \( f = (1 - q^2) \) and \( g = (1 - qx + q^2) \) and relations (4.6) and (4.8) as our assumptions. Now applying Lemma 3.4 for the function \( 1 - q^2 \) concludes the inductive step in this case.

**Step 3.** As before set \( k = (a, 0) \) and \( r = r(p, 0) \). For \( p \) even we have:

\[
r(p - 2, 0) = r - 1
\]

The \( p = 0 \) is our inductive assumption, thus we can assume that \( p \geq 2 \). Let \( \beta = \alpha + 2e_1 + 2e_2 \) be an immersed sphere obtained from \( \alpha \) by blowing up two of its double points. Like above let \( b = \beta \cdot \beta \).

\[
k(b, 0) = s - (a - 8)/2 - 1 = k + 4
\]

From \((p - 2, 0)\) structure equation applied to \( w, w + e_1 \) and \( w + e_1 + e_2 \) we get the following formulas:

\[
D_w((x^2 - 4)^{-1} \cosh(t\alpha)(1 - q^2)^2) = D_w \left( B^{-\alpha} \sum_{i=0}^{k+4} a_i(\alpha) \cdot q^i Q' \right) \quad (4.9)
\]

\[
D_w((x^2 - 4)^{-1} \cosh(t\alpha)(1 - xq + q^2)(1 - q^2)) = D_w \left( B^{-\alpha} \sum_{i=0}^{k+4} b_i(\alpha) \cdot q^i Q' \right) \quad (4.10)
\]

\[
D_w((x^2 - 4)^{-1} \cosh(t\alpha)q(1 - qx + q^2)) = D_w \left( B^{-\alpha} \sum_{i=0}^{k+4} c_i(\alpha) \cdot q^i Q' \right) \quad (4.11)
\]

The relations (4.10) and (4.11) yield the following relation, modulo the kernel of \( D_w(.) \):

\[
q \cdot \left( \sum_{i=0}^{k_0+4} b_i(\alpha) \cdot q^i \right) = (1 - q^2) \cdot \left( \sum_{i=0}^{k_0+4} c_i(\alpha) \cdot q^i \right)
\]

By comparing the free term on both sides we obtain \( c_0 = 0 \), thus:

\[
D_w((x^2 - 4)^{-1} \cosh(t\alpha)(1 - qx + q^2)) = D_w \left( \sum_{i=0}^{k+3} d_i(\alpha) \cdot q^i Q' \right) \quad (4.12)
\]

where \( d_i(\alpha) = c_{i-1}(\alpha) \). Putting together (4.9) and (4.11) we get that:

\[
\left( \sum_{i=0}^{k+4} a_i(\alpha) \right) (1 - qx + q^2) = \left( \sum_{i=0}^{k+3} d_i(\alpha) \right) (1 - q^2)^2 \quad (4.13)
\]

modulo the kernel of \( D_w(.) \). One can check directly that:

\[
(1 - q^2)^2 \cdot (x^2 - 1 - qx) + (1 - qx + q^2) \cdot (-3 - 2qx + q^2 + q^3x) = (x^2 - 4) \quad (4.14)
\]

From which, similarly like in the previous step, we conclude that \((x^2 - 4)^r \cosh(t\alpha)\) is equal to:

\[
(x^2 - 4)^{-1} \cosh(t\alpha) [ (1 - q^2)^2 \cdot (x^2 - 1 - qx) + (1 - qx + q^2) \cdot (-3 - 2qx + q^2 + q^3x) ]
\]

by (4.14)

\[
= (x^2 - 4)^{-1} \left[ (x^2 - 1 - qx) \cdot \left( \sum_{i=0}^{k+4} a_i(\alpha) \right) + (-3 - 2qx + q^2 + q^3x) \cdot \left( \sum_{i=0}^{k+3} d_i(\alpha) \right) \right]
\]

by (4.9),(4.12)

\[
= (x^2 - 4)^{-1} \left( \sum_{i=0}^{k+4} a_i(\alpha) \right)
\]

by Lemma 3.5
modulo the kernel of $D_w(.)$. Now we can use Lemma 3.4 for the polynomial $(1 - q^2)^2$ to conclude the induction. \ \Box

**Theorem 4.2** Let $X$ be a 4-dimensional manifold containing an immersed sphere $\alpha$ with $p$ positive double points and self-intersection $a \geq 0$. We also assume that there exist a homology class $f$ such that $f \cdot \alpha = 1 \pmod{2}$. Then

$$D_w((x^2 - 4)^r) = 0$$

for $r = \left\lfloor \frac{2p + 2 - a}{4} \right\rfloor$.

**Proof:** Assume first that $a$ is even. Then set

$$s = \begin{cases} a/2 & \text{if } a/2 \leq p \\ p & \text{otherwise} \end{cases}$$

With the above definition $r = \left\lfloor \frac{2p + 2 - a}{4} \right\rfloor$ and $k(a, s) \leq -1$. As a result the $(p, s)$ structure equation of Theorem 1.3 has the form:

$$D_w\left((x^2 - 4)^r \cosh(t\alpha)\right) = 0$$

The coefficient at the free term $t^0$ gives the desired result.

When $a$ is odd, we first blow up $\alpha$ at one of its regular points, getting an immersed sphere with even self-intersection $b = a - 1$. The result follows from the observation that for $a$ odd

$$\left\lfloor \frac{2p + 2 - a}{4} \right\rfloor = \left\lfloor \frac{2p + 2 - (a - 1)}{4} \right\rfloor$$

\ \Box

**A Appendix: The proof of Theorem 2.1**

**A.1 Review of the gluing technique**

Let $X$ be an arbitrary smooth 4-manifold with $b_+ > 1$ that contains an embedded sphere with self-intersection $-p$. In order to prove Theorem 2.1 we write the manifold $X$ as the sum $Y \cup \cup_{L(p,1)} N$, where $N$ is the tubular neighborhood of the sphere, and $L = L(p,1)$ is the lens space, equal to the boundary of $N$.

In $X$ we can identify a set isometric to $(-\epsilon, \epsilon) \times L$ and take a sequence of metrics on $X$ that stretches the length of $(-\epsilon, \epsilon)$ to infinity. In the limit we get two manifolds with cylindrical end isometric to $R \times L$. We shall denote these manifolds by the same letters $Y$ and $N$. If one considers the anti-self-dual connections on $U(2)$ bundles over cylindrical end manifolds, then
the restrictions of these connections to the bundles over \( \{t \} \times L \) as \( t \to \infty \) approach some flat connection. More precisely, let

\[
\mathcal{M}^x_{k,w}(Y) = ([\text{ASD connections}] \times U(2)) / \mathcal{G}
\]

denote the space of based connections. Note that since the center \( Z(U(2)) = S^1 \) of \( U(2) \) acts trivially on the space of connections, the fibration \( \mathcal{M}^x_{k,w}(Y) \to \mathcal{M}_{k,w}(Y) \) is in fact an \( SO(3) \)-fibration.

It has been proved in [9] that there exists a smooth boundary map \( \partial^\rho : \mathcal{M}^x_{k,w}(Y) \to \mathcal{R}(L) \) from the based moduli space to the representation variety \( \mathcal{R}(L) = \text{Hom}(\pi_1(L), U(2))/S^1 \). The representation variety \( \mathcal{R} \) can be identified with gauge equivalence classes of based flat connections.

Let \( \chi = \text{Map}(\pi_1(L), G)/\sim \) be the representation variety of \( L \). Here \( \sim \) is the conjugation relation, i.e. \( \alpha \sim \beta \) if and only if there is some \( g \in G \) such that \( \alpha = g\beta g^{-1} \). The map \( \partial^\rho \) descends to \( \partial : \mathcal{M}_{k,w}(Y) \to \chi(L) \).

Fix \( k \) and \( w \) in \( \mathcal{M}_{k,w}(X) \). With the help of the boundary map \( \partial^\rho \) we can define fibered product:

\[
U^m_{k,N} = \left( \mathcal{M}^x_{k_Y,w_Y}(Y,m) \times_\mathcal{R} \mathcal{M}^x_{k_N,w_N}(N,m) \right) / SO(3)
\]

where \( m \in \chi(L) \) and \( (k_Y, w_Y) \), \( (k_N, w_N) \) describe Chern classes of the restriction of the bundles to \( Y \) and \( N \) correspondingly. (So for example \( k = k_N + k_Y \), thus since \( k \) is fixed, we can drop \( k_Y \) in the notation above). The space \( \mathcal{M}^x_{k_Y,w_Y}(Y,m) \) denotes the space of those connections on cylindrical end manifold \( Y \) that are mapped onto \( m \in \chi \) under the map \( \partial \).

Let \( X_l \) denote a manifold obtained from \( X \) by stretching the cylinder \((-\epsilon, \epsilon) \times L\) to the length \( l \). Then we have the following theorem:

**Theorem A.1** (see Theorem 4.3.1 of [8] or [11]) For \( l \) large enough there exist a subset \( Q \subset \mathbb{Z} \left[ \frac{1}{p} \right] \), a subset \( \chi_0 \subset \chi(L) \) and a smooth map

\[
\prod_{(m,k,N) \in \chi_0 \times Q} U^m_{k,N,m} \xrightarrow{\rho} \mathcal{M}_{k,w}(X_l)
\]

such that

1. \( \rho \) is diffeomorphism onto its image.

2. If the term \( \sigma^a x^b \) appears in one of formulas of Theorem 2.1 and \( \dim \mathcal{M}_{k,w}(X) = 2a + 4b \), then the support of the form \( \mu^a(\sigma)\mu^b(x) \) is in the image of \( \rho \).

**A.2 Flat \( U(2) \) connections**

In [11] we proved Theorem 2.1 for \( SU(2) \) connections. The main difference between \( SU(2) \) connections and the \( U(2) \) connections is their representation variety.

First let us refine the notion of the character variety. Since we study connections on \( U(2) \) bundle with a fixed first Chern class \( w \), then it makes sense to define \( \chi_w \) to be the set of equivalence classes of those flat connections that “live” on the bundle with fixed first Chern class \( w \). (In order
to simplify the notation we do not distinguish the class \( w \in H^2(X) \) and its restriction to \( H^2(L) \).
Similarly define \( R_w \).

Every matrix of \( U(2) \) can be represented in the form:
\[
\begin{pmatrix}
a & b \\
-\overline{b}\phi & \overline{a}\phi
\end{pmatrix}
\]
where \(|a|^2 + |b|^2 = 1\) and \(|\phi|^2 = 1\). Every element \( g \in U(2) \) is conjugate (in fact by some \( SU(2) \) matrix) to a diagonal matrix
\[
\begin{pmatrix}
\alpha & 0 \\
0 & \beta
\end{pmatrix}
\]
where \(|\alpha|^2 = |\beta|^2 = 1\). Since \( \pi_1(L) = \mathbb{Z}_p = H^2(L) \), we can identify \( w \) with \( \xi^a \) for some \( a \in \mathbb{Z}_p \) and \( \xi \) a fixed primitive root of unity.

Let \( \text{diag}(k, l) \) denote the matrix \( \begin{pmatrix} \xi^k & 0 \\ 0 & \xi^l \end{pmatrix} \). The relation:
\[
\begin{pmatrix} \xi^k & 0 \\ 0 & \xi^l \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi^l & 0 \\ 0 & \xi^k \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1}
\]
shows that the matrices \( \text{diag}(k, l) \) and \( \text{diag}(l, k) \) are conjugate. Then:
\[
\chi_w = \{ \text{diag}(k, l) \mid k + l = a \}/\sim
\]
where \((k, l) \sim (l, k)\). There is a one-to-one correspondence between elements of \( \chi_w \) and the set
\[
\{ i \in \mathbb{Z}_{2p} \mid i = a \pmod{2} \}/\pm
\]
In this correspondence the matrix \( \text{diag}(k, l) \in \chi_w \) is send to \((k - l) \in \mathbb{Z}_{2p}\).

In the process of gluing it is also important to know what is the inverse image of each component of \( \chi_w \) in the fibration \( R_w \rightarrow \chi_w \). This fiber \( F_m \) is the stabilizer of \( m \in \chi_w \) divided by the central \( S^1 \).

When \( \alpha \neq \beta \) then the stabilizer is the subgroup of diagonal matrices, which can be identified with \( S^1 \times S^1 \). In case when \( \alpha = \beta \) then the stabilizer is the whole group \( U(2) \). Thus, after dividing by \( S^1 \) we have - like in the case of \( SU(2) \) connections - precisely two kinds of fibers: \( SO(3) \) and \( S^1 \). In \([1]\) we called the elements that have an \( SO(3) \) as the fiber to be the trivial elements of \( \chi \).

We can summarize this in the following lemma:

**Lemma A.2** Let \( \chi_w \) denote the character variety of gauge equivalence classes of flat \( U(2) \) connections on the bundle \( E \) over \( L = L(p, 1) \) with first Chern class \( w \). We assume that the bundle \( E \) extends to a tubular neighborhood of embedded sphere \( \sigma \) with self-intersection \( p \). Then if \( p = -2k \), we have:
\[
\chi_w = \begin{cases}
\{0, 2, \ldots, 2k\} & \text{if } w \cdot \sigma = 0 \pmod{2} \\
\{1, 3, \ldots, 2k - 1\} & \text{if } w \cdot \sigma = 1 \pmod{2}
\end{cases}
\]
and when \( p = -(2k + 1) \), then:

\[
\chi_w = \begin{cases} 
\{0, 2, \ldots, 2k\} & \text{if } w \cdot \sigma \equiv 0 \pmod{2} \\
\{1, 3, \ldots, 2k + 1\} & \text{if } w \cdot \sigma \equiv 1 \pmod{2}
\end{cases}
\]

In the above the bold face numbers indicate the trivial elements of the character variety.

### A.3 Completion of the proof of Theorem 2.1

Now that we understand the character variety for \( U(2) \) connections, the proof of Theorem 2.1 is a slight modification of the corresponding structure theorem for \( SO(3) \) connections (Theorem 1.3 [11]). We shall describe the main points of the proof, referring for the details to [11].

Let \( \mathcal{M}_{k,w}(\mathbb{R} \times L, [m, m']) \) denote the moduli space of ASD connections on the \( U(2) \) bundle over \( \mathbb{R} \times L \) with second Chern class \( k \) and whose \( \pm \infty \) limits are correspondingly \( m, m' \in \chi_w \). Similarly let \( \mathcal{M}_{k,w}(N, m) \) denote the moduli space of ASD connections on cylindrical end manifold \( N \), whose \( \infty \) limits are equal to \( m \in \chi_w \). Then we have the following:

**Theorem A.3** The dimension of \( \mathcal{M}_{k,w}(\mathbb{R} \times L, [m, m']) \) is equal to

\[
\dim \mathcal{M}_{k,w}(\mathbb{R} \times L, [m, m']) = 8k + 2(m' - m) - \frac{2((m')^2 - m^2)}{p} - s(m) \quad (1.2)
\]

where \( s(m) \) is the dimension of the gluing parameter and is equal to:

\[
s(m) = \begin{cases} 
1 & \text{if } m \text{ is non-trivial element of } \chi_w \\
3 & \text{otherwise}
\end{cases}
\]

The dimension of \( \mathcal{M}_{k,w}(N, m) \) is:

\[
\dim \mathcal{M}_{k,w}(N, m) = 8k - 3 + 2m - \frac{2m^2}{p} \quad (1.3)
\]

**Theorem A.4** For fixed \( m \) and \( m' \) let \( k \) denote the minimal amount of energy for which the moduli space \( \mathcal{M}_{k,w}(\mathbb{R} \times L, [m, m']) \) is non-empty. Then:

\[
k = \begin{cases} 
\frac{(m')^2 - m^2}{4} & \text{if } m' > m \\
\frac{(m')^2 - m^2}{4} + p(m - m') & \text{otherwise}
\end{cases}
\]

In other words, the minimal dimension for which the moduli space \( \mathcal{M}_{k,w}(\mathbb{R} \times L, [m, m']) \) is non-empty is

\[
2(m' - m) - s(m)
\]
Let us assume that $\sigma \cdot \sigma = -2a$ and we want to prove that there exist polynomials $C_i = C_i(x)$ such that:

$$D_w(\sigma^n) = \begin{cases} D_w \left((C_0 + C_1 \sigma + \cdots + \sigma^{2a-1} + C_{2k} \sigma^{2a})z\right) & \text{when } w \cdot \sigma = 0 \pmod{2} \\ D_w \left((C_0 + C_1 \sigma + \cdots + C_{2a-1} \sigma^{2a-1})z\right) & \text{when } w \cdot \sigma = 1 \pmod{2} \end{cases}$$

Then similarly like in [11] define the set

$$\mathcal{J}_n = \{(m, k) \mid m \in \chi_w, k \in \mathbb{Z}_{[p]} \cdot \mathcal{M}_{k,w}(N, m) \neq \emptyset, \quad \text{and } 0 < \dim \mathcal{M}_{k,w}(N, m) + s(m) \leq 2n\}$$

For each $n$ this is a finite set, on which we define a partial order by saying that $(m_1, k_1) \leq (m_2, k_2)$ if there is a nonempty moduli space $\mathcal{M}_{k_2 - k_1, w}(R \times L, [m_1, m_2])$. Thus for example when $\sigma \cdot \sigma = -6$ the set $\mathcal{J}_{10}$ is

\[
\begin{array}{c}
2 \\
4 \\
6 \\
4 \\
6 \\
2 \\
\end{array} \quad \quad \quad \quad \quad \quad \begin{array}{c}
1 \\
3 \\
5 \\
3 \\
5 \\
1 \\
\end{array}
\]

$\mathcal{J}_{10}$ when $w \cdot \sigma = 0 \pmod{2}$ \quad $\mathcal{J}_{10}$ when $w \cdot \sigma = 1 \pmod{2}$

In the above diagram we put only the $m$ coordinate of each pair $(m, k)$. In order to obtain the remaining $k$ coordinate we assign to each edge of the diagram the minimal energy defined in Theorem [A.3]. Then $k$ coordinate of each vertex is the sum of those energies along any path from the top of the diagram to that vertex. Each vertex of $(k, m) \in \mathcal{J}$ represents one of the open set $U_{k,m}^w$ defined in (1.1), which cover the support of $\mu(\sigma)$. As we proved in [11] the $\mu(\sigma)^n$ in each set $U_{k,m}^w$ can be computed in terms of a polynomial $p_{n,m}(\mu(x), \mu(\sigma))$ whose degree in $\mu(\sigma)$ is at most $2m$.

For trivial a element $m$ of the character variety the dimension of the gluing parameter $s(m)$ is by two bigger than $s(m)$ for all other elements. This is the reason for which we can delete the $\sigma^{2a-1}$ term in our relation. \[\square\]

References

[1] David M. Austin, *SO(3) Instantons on $L(p, q) \times R$*, J.Diff. Geom. 32 (1990), 383–413.

[2] N. Akheizer, “Elements of the Theory of Elliptic Functions”, translated by H. McFaden, A.M.S. Translations of Math. Monographs 79 (1990).

[3] S. Donaldson and P. Kronheimer: “The Geometry of Four Manifolds”, Oxford Mathematical Monographs, Oxford University Press, Oxford, 1990.
[4] R. Fintushel and R. Stern, *Donaldson invariants of 4- manifolds with simple type*, J. Diff. Geom. **42** (1995), no. 3, 577–633.

[5] R. Fintushel and R. Stern, *The blowup formula for Donaldson invariants*, Annals of Math. (2) **143** (1996), no. 3, 529–546.

[6] P. Kronheimer and T. Mrówka, *Embedded surfaces and the structure of Donaldson’s polynomial invariants*, J. Diff. Geom. **41**, (1995), no. 3, 573–734.

[7] P. Kronheimer and T. Mrówka, *The structure of Donaldson’s invariants for four-manifolds not of simple type*, www/math.harvard.edu/~kronheim.

[8] J. Morgan and T. Mrówka, *On the gluing theorem for instantons on manifolds containing long neck*, preprint.

[9] J. Morgan, T. Mrówka and D. Ruberman, “The L² Moduli Space and Vanishing Theorem for Donaldson Polynomial Invariants”, International Press, Boston, 1994.

[10] V. Muñoz, Fukaya-Floer homology of Σ × S¹ and applications, [math.DG/9804081](#).

[11] W. Wieczorek, *The Donaldson invariant and embedded 2-spheres*, J. reine. angew. Math. **489** (1997), 15–51.

[12] W. Wieczorek, *Basic classes and embedded spheres*, Topology Appl. **88** (1998), 67–78.