AUSLANDER-REITEN TRANSLATIONS IN MONOMORPHISM CATEGORIES

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Abstract. We generalize Ringel and Schmidmeier’s theory on the Auslander-Reiten translation of the submodule category $S_2(A)$ to the monomorphism category $S_n(A)$. As in the case of $n = 2$, $S_n(A)$ has Auslander-Reiten sequences, and the Auslander-Reiten translation $\tau_S$ of $S_n(A)$ can be explicitly formulated via $\tau$ of $A$-mod. Furthermore, if $A$ is a selfinjective algebra, we study the periodicity of $\tau_S$ on the objects of $S_n(A)$, and of the Serre functor $F_S$ on the objects of the stable monomorphism category $S_n(A)$. In particular, $\tau_S^{m(n+1)}X \cong X$ for $X \in S_n(\Lambda(m,t))$; and $F_S^{m(n+1)}X \cong X$ for $X \in S_n(\Lambda(m,t))$, where $\Lambda(m,t)$, $m \geq 1$, $t \geq 2$, are the selfinjective Nakayama algebras.

Key words and phrases. monomorphism category, Auslander-Reiten translation, triangulated category, Serre functor

Introduction

Throughout this paper, $n \geq 2$ is an integer, $A$ an Artin algebra, and $A$-mod the category of finitely generated left $A$-modules. Let $S_n(A)$ denote the monomorphism category of $A$ (it is usually called the submodule category if $n = 2$).

The study of such a category goes back to G. Birkhoff [B], in which he initiates to classify the indecomposable objects of $S_2(\mathbb{Z}/(p^t))$ (see also [RW]). In [Ar], $S_n(R)$ is denoted by $C(n,R)$, where $R$ is a commutative uniserial ring; the complete list of $C(n,R)$ of finite type, and of the representation types of $C(n,k[x]/(x^t))$, are given by D. Simson [S] (see also [SW]). Recently, after the deep and systematic work of C. M. Ringel and M. Schmidmeier ([RS1] - [RS3]), the monomorphism category receives more attention. X. W. Chen [C] shows that $S_2(A)$ of a Frobenius abelian category $A$ is a Frobenius exact category. D. Kussin, H. Lenzing, and H. Meltzer [KLM] establish a surprising link between the stable submodule category with the singularity theory via weighted projective lines of type $(2,3,p)$. In [Z], $S_n(X)$ is studied for any full subcategory $X$ of $A$-mod, and it is proved that for a cotilting $A$-module $T$, there is a cotilting $T_n(A)$-module $m(T)$ such that $S_n(-T) = (-m(T))$, where $T_n(A) = \begin{pmatrix} A & A & \cdots & A \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & \ddots & A \end{pmatrix}_{n \times n}$ is the upper triangular matrix algebra.

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of $A$, and $\perp T$ is the left perpendicular category of $T$. As a consequence, for a Gorenstein algebra $A$, $\mathcal{S}_n(\perp A)$ is exactly the category of Gorenstein-projective $T_n(A)$-modules.

Ringel and Schmidmeier construct minimal monomorphisms, and then prove that $\mathcal{S}_2(A)$ is functorially finite in $T_2(A)$-mod. As a result, $\mathcal{S}_2(A)$ has Auslander-Reiten sequences. Surprisingly, the Auslander-Reiten translation $\tau_S$ of $\mathcal{S}_2(A)$ can be explicitly formulated as $\tau_S X \cong \text{Mimo} \tau \text{Cok}X$ for $X \in \mathcal{S}_2(A)$ ([RS2], Theorem 5.1), where $\tau$ is the Auslander-Reiten translation of $A$-mod. Applying this to selfinjective algebras, among others they get $\tau_S^g X \cong X$ for indecomposable nonprojective object $X \in \mathcal{S}_2(A)$, where $A$ is a commutative uniserial algebra.

A beautiful theory should have a general version. The aim of this paper is to generalize Ringel and Schmidmeier’s work on $\mathcal{S}_2(A)$ to $\mathcal{S}_n(A)$. As in the case of $n = 2$, $\mathcal{S}_n(A)$ has Auslander-Reiten sequences, and $\tau_S$ of $\mathcal{S}_n(A)$ can be formulated in the same form as above; these can be achieved by using the idea in [RS2]. For selfinjective algebras, Sections 3 and 4 of this paper contain new considerations. In order to express the higher power of $\tau_S$, we need the concept of a rotation of an object in $\text{Mor}_A(A\text{-mod})$, which is defined in [RS2] for $n = 2$. In the general case, such a suitable definition needs to be chosen from different possibilities, and difficulties need to be overcome to justify that it is well-defined. Also, the Octahedral Axiom is needed in computing the higher power of the rotations, which is the key step in studying the periodicity of $\tau_S$ and the Serre functor on the objects.

We outline this paper. In Section 1 we set up some basic properties of the categories $\text{Mor}_n(A)$, $\mathcal{S}_n(A)$ and $\mathcal{F}_n(A)$, and of the functors $m_i$, $p_i$, $\text{Ker}$, $\text{Cok}$, $\text{Mono}$, $\text{Epi}$; and the construction of $\text{Mimo}$. Section 2 is to transfer the Auslander-Reiten sequences of $\text{Mor}_n(A)$ to those of $\mathcal{S}_n(A)$; and to give a formula for $\tau_S$ of $\mathcal{S}_n(A)$ via $\tau$ of $A\text{-mod}$ (Theorem 2.4).

In Section 3, $A$ is a selfinjective algebra, and hence the stable category $A\text{-mod}$ is a triangulated category ([H]), and $\tau$ is a triangle functor of $A\text{-mod}$. Using the rotation and the Octahedral Axiom, we get a formula for $\tau_S^{j(n+1)} X \in \text{Mor}_n(A\text{-mod})$ for $X \in \mathcal{S}_n(A)$ and $j \geq 1$ (Theorem 3.4). This can be applied to the study of the periodicity of $\tau_S$ on objects. In particular, $\tau_S^{2m(n+1)} X \cong X$ for $X \in \mathcal{S}_n(\Lambda(m,t))$ (Corollary 3.6), where $\Lambda(m,t)$, $m \geq 1$, $t \geq 2$, are the selfinjective Nakayama algebras.

In Section 4, $A$ is a finite-dimensional self-injective algebra over a field. By [Z], $\mathcal{S}_n(A)$ is exactly the category of Gorenstein projective $T_n(A)$-modules, and hence the stable monomorphism category $\mathcal{S}_n(A)$ is a Hom-finite Krull-Schmidt triangulated category with Auslander-Reiten triangles. By Theorem I.2.4 of I. Reiten and M. Van den Bergh [RV], $\mathcal{S}_n(A)$ has a Serre functor $F_S$. We study the periodicity of $F_S$ on the objects of $\mathcal{S}_n(A)$ (Theorem 4.3). In particular, $F_S^{m(n+1)} X \cong X$ for $X \in \mathcal{S}_n(\Lambda(m,t))$ (Corollary 4.4).

In order to make the main clue clearer, we put the proofs of Lemmas 1.5 and 1.6 in Appendix 1.
Note that \( S_{n,2}, S_{2,3}, S_{2,4}, S_{2,5}, S_{3,3} \) and \( S_{4,3} \) are the only representation-finite cases among all \( S_{n,t} = S_n(k[x]/(x^t)), n \geq 2, t \geq 2 \) ([S], Theorems 5.2 and 5.5). The Auslander-Reiten quivers of \( S_{2,t} \) with \( t = 2, 3, 4, 5 \) are given in [RS3]. In Appendix 2, we give the remaining cases. We also include the AR quivers of \( S_n(A(2,2)) \) with \( n = 3 \) and 4.

**1. Basics of morphism categories**

We set up some basic properties of several categories and functors, which will be used throughout this paper.

1.1. An object of the morphism category \( \text{Mor}_n(A) \) is \( X_{(\phi_i)} = \left( \begin{array}{c} X_1 \\ \vdots \\ X_n \end{array} \right) \), where \( \phi_i : X_{i+1} \to X_i \) are \( A \)-maps for \( 1 \leq i \leq n - 1 \); and a morphism \( f = (f_i) : X_{(\phi_i)} \to Y_{(\psi_i)} \) is \( \left( \begin{array}{c} f_1 \\ \vdots \\ f_n \end{array} \right) \), where \( f_i : X_i \to Y_i \) are \( A \)-maps for \( 1 \leq i \leq n \), such that the following diagram commutes

\[
\begin{array}{cccccccc}
X_n & \phi_{n-1} & X_{n-1} & \cdots & X_2 & \phi_1 & X_1 \\
\downarrow f_n & & \downarrow f_{n-1} & & \downarrow f_2 & \downarrow f_1 & \\
Y_n & \psi_{n-1} & Y_{n-1} & \cdots & Y_2 & \psi_1 & Y_1.
\end{array}
\]

(1.1)

We call \( X_i \) the \( i \)-th branch of \( X_{(\phi_i)} \), and \( \phi_i \) the \( i \)-th morphism of \( X_{(\phi_i)} \). It is well-known that \( \text{Mor}_n(A) \) is equivalent to \( T_n(A)\text{-mod} \) (see e.g. [Z], 1.4). Let \( Z_{(\phi_i)} \overset{f}{\to} Y_{(\psi_i)} \overset{g}{\to} X_{(\phi_i)} \) be a sequence in \( \text{Mor}_n(A) \). Then it is exact at \( Y_{(\psi_i)} \) if and only if each sequence \( Z_i \overset{f_i}{\to} Y_i \overset{g_i}{\to} X_i \) in \( A\text{-mod} \) is exact at \( Y_i \) for each \( 1 \leq i \leq n \).

By definition, the monomorphism category \( \text{S}_n(A) \) is the full subcategory of \( \text{Mor}_n(A) \) consisting of the objects \( X_{(\phi_i)} \), where \( \phi_i : X_{i+1} \to X_i \) are monomorphisms for \( 1 \leq i \leq n - 1 \). Dually, the epimorphism category \( \text{F}_n(A) \) is the full subcategory of \( \text{Mor}_n(A) \) consisting of the objects \( X_{(\phi_i)} \), where \( \phi_i : X_{i+1} \to X_i \) are epimorphisms for \( 1 \leq i \leq n - 1 \). Since \( \text{S}_n(A) \) and \( \text{F}_n(A) \) are closed under direct summands and extensions, it follows that they are exact Krull-Schmidt categories, with the exact structure in \( \text{Mor}_n(A) \).

The kernel functor \( \text{Ker} : \text{Mor}_n(A) \to \text{S}_n(A) \) is given by

\[
\begin{pmatrix}
X_1 \\
X_2 \\
\vdots \\
X_{n-1} \\
X_n \\
\phi_1 \\
\phi_2 \\
\vdots \\
\phi_{n-1} \\
(\phi_i)
\end{pmatrix}
\mapsto
\begin{pmatrix}
X_n \\
\phi_1 \cdots \phi_{n-1} \\
\phi_2 \cdots \phi_{n-1} \\
\vdots \\
\phi_{n-1} \\
(\phi_i')
\end{pmatrix}
\]

where \( \phi_i' : \text{Ker}(\phi_1 \cdots \phi_{n-1}) \to X_n \), and \( \phi_i' : \text{Ker}(\phi_1 \cdots \phi_{n-1}) \to \text{Ker}(\phi_1 \cdots \phi_{n-1}) \), \( 2 \leq i \leq n - 1 \), are the canonical monomorphisms. For a morphism \( f : X \to Y \) in \( \text{Mor}_n(A) \), \( \text{Ker}f : \text{Ker}X \to \text{Ker}Y \) is naturally defined via a commutative diagram induced from
(1.1). We also need the cokernel functor $\text{Cok} : \text{Mor}_n(A) \to \mathcal{F}_n(A)$ given by

$$
\begin{pmatrix}
X_1 \\
X_2 \\
\vdots \\
X_{n-1} \\
X_n
\end{pmatrix}
\mapsto
\begin{pmatrix}
\text{Coker} \phi_1 \\
\text{Coker} (\phi_1 \phi_2) \\
\vdots \\
\text{Coker} (\phi_1 \cdots \phi_{n-1}) \\
\phi''_i
\end{pmatrix},
$$

where $\phi''_i : \text{Coker}(\phi_1 \cdots \phi_{i+1}) \to \text{Coker}(\phi_1 \cdots \phi_i), 1 \leq i \leq n - 2$, and $\phi''_{n-1} : X_1 \to \text{Coker}(\phi_1 \cdots \phi_{n-1})$ are the canonical epimorphisms. It is clear that the restriction of the kernel functor $\text{Ker} : \mathcal{F}_n(A) \to \mathcal{S}_n(A)$ is an equivalence with quasi-inverse the restriction of the cokernel functor $\text{Cok} : \mathcal{S}_n(A) \to \mathcal{F}_n(A)$.

1.2. For each $1 \leq i \leq n$, the functors $m_i : A\text{-mod} \to \mathcal{S}_n(A)$ and $p_i : A\text{-mod} \to \mathcal{F}_n(A)$ are defined as follows. For $M \in A\text{-mod}$,

$$(m_i(M))_j = \begin{cases} M, & 1 \leq j \leq i; \\ 0, & i + 1 \leq j \leq n; \end{cases} \\ (p_i(M))_j = \begin{cases} 0, & 1 \leq j \leq n - i; \\ M, & n - i + 1 \leq j \leq n. \end{cases}$$

The $j$-th morphism of $m_i(M)$ is $\text{id}_M$ if $1 \leq j < i$, and $0$ if $i \leq j \leq n - 1$; and the $j$-th morphism of $p_i(M)$ is $0$ if $1 \leq j < n - i + 1$, and $\text{id}_M$ if $n - i + 1 \leq j \leq n - 1$. For an $A$-map $f : M \to N$, define

$$m_i(f) = \begin{pmatrix} f \\
\vdots \\
\vdots \\
0
\end{pmatrix} : \begin{pmatrix} M \\
M \\
\vdots \\
0
\end{pmatrix} \to \begin{pmatrix} N \\
N \\
\vdots \\
N
\end{pmatrix}; \\ p_i(f) = \begin{pmatrix} 0 \\
\vdots \\
\vdots \\
0
\end{pmatrix} : \begin{pmatrix} 0 \\
0 \\
\vdots \\
0
\end{pmatrix} \to \begin{pmatrix} 0 \\
0 \\
\vdots \\
0
\end{pmatrix}.$$ 

We have

$$\text{Ker} p_i(M) = m_{n-i+1}(M), \quad \text{Cok} m_i(M) = p_{n-i+1}(M), \quad 1 \leq i \leq n. \quad (1.2)$$

**Lemma 1.1.** (i) If $P$ runs over all the indecomposable projective $A$-modules, then $m_1(P), \cdots, m_n(P)$ are the all indecomposable projective objects in $\text{Mor}_n(A)$.

(ii) If $I$ runs over all the indecomposable injective $A$-modules, then $p_1(I), \cdots, p_n(I)$ are the all indecomposable injective objects in $\text{Mor}_n(A)$.

(iii) The indecomposable projective objects in $\mathcal{S}_n(A)$ are exactly those in $\text{Mor}_n(A)$.

(iv) If $I$ runs over all the indecomposable injective $A$-modules, then $m_1(I), \cdots, m_n(I)$ are the all indecomposable injective objects in $\mathcal{S}_n(A)$.

(v) If $P$ runs over all the indecomposable projective $A$-modules, then $p_1(P), \cdots, p_n(P)$ are the all indecomposable projective objects in $\mathcal{F}_n(A)$.

(vi) The indecomposable injective objects in $\mathcal{F}_n(A)$ are exactly those in $\text{Mor}_n(A)$.

(vii) Let $\mathcal{N}_A$ and $\mathcal{N}$ be the Nakayama functor of $\text{Mor}_n(A)$ and of $A\text{-mod}$, respectively. Then for a projective $A$-module $P$, $\mathcal{N}_A m_i(P) = p_{n-i+1}(NP), 1 \leq i \leq n$. 

Proof. For convenience, we include a justification. (i) can be seen from the equivalence Mor_{n}(A) \cong T_{n}(A)\text{-mod.} For (ii), see e.g. Lemma 1.3(ii) in [Z]. (iii) follows from (i), and (vi) follows from (ii). Using the equivalence Ker : \mathcal{F}_{n}(A) \to \mathcal{S}_{n}(A) together with (vi) and (1.2), we see (iv). Using the equivalence Cok : \mathcal{S}_{n}(A) \to \mathcal{F}_{n}(A) together with (iii) and (1.2), we see (v). To see (vii), note that if P is indecomposable, then \mathcal{N}_{M_{i}}(P) is an indecomposable injective \mathcal{T}_{n}(A)\text{-module, hence by (ii) it is of the form }p_{j}(I).\text{ Thus}

\[
\begin{pmatrix}
0 \\
\vdots \\
soc(I) \\
0
\end{pmatrix} = soc(p_{j}(I)) = soc(N_{M_{i}}(P)) = top(m_{i}(P)) = \begin{pmatrix}
0 \\
\vdots \\
0 \\
top(P)
\end{pmatrix}.
\]

Thus \(n - j + 1 = i, soc(I) = top(P),\) which means \(N_{M_{i}}(P) = p_{n-i+1}(NP).\) \(\Box\)

1.3. Recall the functors \textbf{Mono} : Mor_{n}(A) \to \mathcal{S}_{n}(A) and \textbf{Epi} : Mor_{n}(A) \to \mathcal{F}_{n}(A). The first one is given by

\[
\begin{pmatrix}
X_{1} \\
X_{2} \\
\vdots \\
X_{n-1} \\
X_{n}
\end{pmatrix}
\mapsto
\begin{pmatrix}
X_{1} \\
\text{Im } \phi_{1} \\
\vdots \\
\text{Im } (\phi_{1}\cdots \phi_{n-2}) \\
\text{Im } (\phi_{1}\cdots \phi_{n-1})
\end{pmatrix},
\]

where \(\phi_{i}^{t} : \text{Im } \phi_{1} \hookrightarrow X_{1},\) and \(\phi_{i}^{t} : \text{Im } (\phi_{1}\cdots \phi_{i}) \hookrightarrow \text{Im } (\phi_{1}\cdots \phi_{i-1}),\ 2 \leq i \leq n - 1,\) are the canonical monomorphisms. The second one is given by

\[
\begin{pmatrix}
X_{1} \\
X_{2} \\
\vdots \\
X_{n-1} \\
X_{n}
\end{pmatrix}
\mapsto
\begin{pmatrix}
\text{Im } (\phi_{1}\cdots \phi_{n-1}) \\
\text{Im } (\phi_{2}\cdots \phi_{n-1}) \\
\vdots \\
\text{Im } (\phi_{n-1}) \\
X_{n}
\end{pmatrix},
\]

where \(\phi_{i}^{t} : \text{Im } (\phi_{i+1}\cdots \phi_{n-1}) \twoheadrightarrow \text{Im } (\phi_{i}\cdots \phi_{n-1}),\ 1 \leq i \leq n - 2,\) and \(\phi_{n-1}^{t} : X_{n} \twoheadrightarrow \text{Im } \phi_{n-1}\) are the canonical epimorphisms. Then

\[
\textbf{Epi} \cong \text{Cok Ker}, \quad \textbf{Mono} \cong \text{Ker Cok}, \tag{1.3}
\]

and hence \(\textbf{Mono}X \cong X\) for \(X \in \mathcal{S}_{n}(A),\) and \(\textbf{Epi}Y \cong Y\) for \(Y \in \mathcal{F}_{n}(A).\)

For an A-map \(f : X \to Y,\) denote the canonical A-maps \(X \to \text{Im } f\) and \(\text{Im } f \hookrightarrow Y\) by \(\tilde{f}\) and incl, respectively. The following lemma can be similarly proved as in [RS2] for \(n = 2.\)

Lemma 1.2. Let \(X = X_{\{\phi_{i}\}} \in \text{Mor}_{n}(A).\) Then

(i) The morphism \(\begin{pmatrix}
1_{X_{1}} \\
\phi_{1} \\
\vdots \\
\phi_{1}\cdots \phi_{n-1}
\end{pmatrix} : X \to \textbf{Mono}X\) is a left minimal approximation of \(X\) in \(\mathcal{S}_{n}(A).\)
Lemma 1.3.

(i) The morphism:

\[
\begin{pmatrix}
\begin{bmatrix}
incl \\ \vdots \\
incl \\ 1_X
\end{bmatrix}
\end{pmatrix} : \text{Epi} X \to X
\]
is a right minimal approximation of \( X \) in \( \mathcal{F}_n(A) \).

1.4. For \( X_{(\phi_i)} \in \text{Mor}_n(A) \), we define \( \text{Mimo} X_{(\phi_i)} \in \mathcal{S}_n(A) \) and \( \text{Mepi} X_{(\phi_i)} \in \mathcal{F}_n(A) \) as follows (see [Z]). For each \( 1 \leq i \leq n - 1 \), fix an injective envelope \( \varepsilon_i' : \text{Ker} \phi_i \to \text{I} \text{Ker} \phi_i \). Then we have an \( A \)-map \( \varepsilon_i : X_{i+1} \to \text{I} \text{Ker} \phi_i \), which is an extension of \( \varepsilon_i' \). Define \( \text{Mimo} X_{(\phi_i)} \) to be

\[
\begin{pmatrix}
X_1 \oplus \text{I} \text{Ker} \phi_1 \oplus \cdots \oplus \text{I} \text{Ker} \phi_{n-1} \\
X_2 \oplus \text{I} \text{Ker} \phi_2 \oplus \cdots \oplus \text{I} \text{Ker} \phi_{n-1} \\
\vdots \\
X_{n-1} \oplus \text{I} \text{Ker} \phi_{n-1} \\
X_n
\end{pmatrix}
\]

where \( \theta_i = \begin{pmatrix}
\phi_i & 0 & 0 & \cdots & 0 \\
\varepsilon_i & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}_{(n-i+1) \times (n-i)} \).

By construction \( \text{Mimo} X_{(\phi_i)} \in \mathcal{S}_n(A) \). Since \( e_1, \cdots, e_{n-1} \) are not unique, we need to verify that \( \text{Mimo} X_{(\phi_i)} \) is well-defined. This can be seen from Lemma 1.3(i) below.

The object \( \text{Mepi} X_{(\phi_i)} \) is dually defined. Namely, for each \( 1 \leq i \leq n - 1 \), fix a projective cover \( \pi'_i : \text{P} \text{Coker} \phi_i \to \text{Coker} \phi_i \), then we have an \( A \)-map \( \pi_i : \text{P} \text{Coker} \phi_i \to X_i \), which is a lift of \( \pi'_i \), and define \( \text{Mepi} X_{(\phi_i)} \in \mathcal{F}_n(A) \) to be

\[
\begin{pmatrix}
X_1 \\
X_2 \oplus \text{P} \text{Coker} \phi_1 \\
\vdots \\
X_{n-1} \oplus \text{P} \text{Coker} \phi_{n-2} \oplus \cdots \oplus \text{P} \text{Coker} \phi_1 \\
X_n \oplus \text{P} \text{Coker} \phi_{n-1} \oplus \cdots \oplus \text{P} \text{Coker} \phi_1
\end{pmatrix}
\]

where \( \sigma_i = \begin{pmatrix}
\phi_i & \pi_i & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1
\end{pmatrix}_{i \times (i+1)} \).

Remark. (i) \( \text{Mimo} X = X \) for \( X \in \mathcal{S}_n(A) \), and \( \text{Mepi} Y = Y \) for \( Y \in \mathcal{F}_n(A) \).

(ii) If each \( X_i \) has no nonzero injective direct summands, then \( \text{Mimo} X_{(\phi_i)} \) has no nonzero injective direct summands in \( \mathcal{S}_n(A) \). If each \( X_i \) has no nonzero projective direct summands, then \( \text{Mepi} X_{(\phi_i)} \) has no nonzero projective direct summands in \( \mathcal{F}_n(A) \). These can be seen from Lemma 1.1(iv) and (v), respectively.

Lemma 1.3. Let \( X \in \text{Mor}_n(A) \). Then

(i) The morphism:

\[
\begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix} : \text{Mimo} X \to X
\]
is a right minimal approximation of \( X \) in \( \mathcal{S}_n(A) \).

(ii) The morphism:

\[
\begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix} : X \to \text{Mepi} X
\]
is a left minimal approximation of \( X \) in \( \mathcal{F}_n(A) \).
For a proof of Lemma 1.3 we refer to [RS2] for $n = 2$, and to [Z] in general case. By Lemmas 1.2 and 1.3, and by Auslander and Smalø [AS], we get the following consequence

**Corollary 1.4.** The subcategories $S_n(A)$ and $F_n(A)$ are functorially finite in $\text{Mor}_n(A)$ and hence have Auslander-Reiten sequences.

This corollary is the starting point of this paper. From now on, denote by $\tau$, $\tau_M$, $\tau_S$ and $\tau_F$ the Auslander-Reiten translations of $A$-mod, $\text{Mor}_n(A)$, $S_n(A)$ and $F_n(A)$, respectively.

1.5. Let $A$-mod (resp. $A$-mod) denote the stable category of $A$-mod modulo projective $A$-modules (resp. injective $A$-modules). Then $\tau = \text{DT}r$ induces an equivalence $A$-mod $\rightarrow A$-mod with quasi-inverse $\tau^{-} = \text{Tr}D$ ([ARS], p.106). Let $\text{Mor}_n(A$-mod) denote the morphism category of $A$-mod. Namely, an object of $\text{Mor}_n(A$-mod) is $X_{(\phi_i)} = X(\phi_i) = (X_1, \ldots, X_n)$ with $\phi_i : X_{i+1} \rightarrow X_i$ in $A$-mod for $1 \leq i \leq n - 1$; and a morphism from $X_{(\phi_i)}$ to $Y_{(\psi_i)} = (Y_1, \ldots, Y_n)$, such that the corresponding version of (1.1) commutes in $A$-mod. Similarly, one has the morphism category $\text{Mor}_n(A$-mod), in which an object is denoted by $X_{(\phi_i)} = X(\phi_i)$.

The following two lemmas will be heavily used in Sections 2 and 3. In order to make the main clue clearer, we put their proofs in Appendix 1.

**Lemma 1.5.** Let $X_{(\phi_i)} \in \text{Mor}_n(A)$.

(i) Let $I_2, \ldots, I_n$ be injective $A$-modules such that $X'_{(\phi_i)} = \begin{pmatrix} X_1 \oplus I_2 \oplus \cdots \oplus I_n \\ \vdots \\ X_{n-1} \oplus I_n \\ X_n \end{pmatrix} \in S_n(A)$, where each $\phi_i$ is of the form $\begin{pmatrix} \phi_i & \ast & \cdots & \ast \\ \vdots & \vdots & \ddots & \vdots \\ \ast & \cdots & \ast & \ast \end{pmatrix} (n-i+1) \times (n-i)$. Then $X'_{(\phi_i)} \cong \text{Mimo}X_{(\phi_i)} \oplus J$, where $J$ is an injective object of $S_n(A)$.

(ii) Let $P_1, \ldots, P_{n-1}$ be projective $A$-modules such that $X''_{(\phi_i)} = \begin{pmatrix} X_1 \\ X_2 \oplus P_1 \\ \vdots \\ X_{n-1} \oplus P_{n-1} \oplus \cdots \oplus P_1 \end{pmatrix} \in F_n(A)$, where each $\phi_i$ is of the form $\begin{pmatrix} \phi_i & \ast & \cdots & \ast \\ \vdots & \vdots & \ddots & \vdots \\ \ast & \cdots & \ast & \ast \end{pmatrix} (n-i+1) \times (n-i)$. Then $X''_{(\phi_i)} \cong \text{Mepi}X_{(\phi_i)} \oplus L$, where $L$ is a projective object of $F_n(A)$.

**Lemma 1.6.** Let $X_{(\phi_i)}$, $Y_{(\psi_i)} \in \text{Mor}_n(A)$.

(i) If all branches $X_i$ and $Y_i$ have no nonzero injective direct summands, then $\text{Mimo}X_{(\phi_i)} \cong \text{Mimo}Y_{(\psi_i)}$ in $S_n(A)$ if and only if $X_{(\phi_i)} \cong Y_{(\psi_i)}$ in $\text{Mor}_n(A$-mod).
(ii) If all $X_i$ and $Y_i$ have no nonzero projective direct summands, then $\text{Mepi}X_{(\phi_i)} \cong \text{Mepi}Y_{(\psi_i)}$ in $\mathcal{F}_n(A)$ if and only if $X_{(\phi_i)} \cong Y_{(\psi_i)}$ in $\text{Mor}_n(A\text{-mod})$.

2. The Auslander-Reiten translation of $\mathcal{S}_n(A)$

In this section, we first transfer the Auslander-Reiten sequences of $\text{Mor}_n(A)$ to those of $\mathcal{S}_n(A)$ and $\mathcal{F}_n(A)$; and then give a formula of the Auslander-Reiten translation $\tau_S$ of $\mathcal{S}_n(A)$ via $\tau$ of $A\text{-mod}$. Results and methods in this section are generalizations of the corresponding ones in the case of $n = 2$, due to Ringel and Schmidmeier [RS2].

2.1. The following fact is crucial for later use.

**Lemma 2.1.** Let $0 \to X_{(\phi_i)} \stackrel{f}{\to} Y \stackrel{g}{\to} Z \to 0$ be an Auslander-Reiten sequence of $\text{Mor}_n(A)$.

(i) If $\text{Ker}Z$ is not projective, then $0 \to \text{Ker}X \stackrel{\text{Ker}f}{\to} \text{Ker}Y \stackrel{\text{Ker}g}{\to} \text{Ker}Z \to 0$ is either split exact, or an Auslander-Reiten sequence of $\mathcal{S}_n(A)$.

(ii) If $\text{Cok}X$ is not injective, then $0 \to \text{Cok}X \stackrel{\text{Cok}f}{\to} \text{Cok}Y \stackrel{\text{Cok}g}{\to} \text{Cok}Z \to 0$ is either split exact, or an Auslander-Reiten sequence of $\mathcal{F}_n(A)$.

**Proof.** We only prove (i). Put $g' = \text{Ker}g$ and $f' = \text{Ker}f$. By Snake Lemma, $0 \to \text{Ker}X \stackrel{f'}{\to} \text{Ker}Y \stackrel{g'}{\to} \text{Ker}Z$ is exact. Assume that $g'$ is not a split epimorphism. We claim that $g'$ is right almost split. Let $v : W \to \text{Ker}Z$ be a morphism in $\mathcal{S}_n(A)$ which is not a split epimorphism. Applying $\text{Cok}$, we get $t' = \text{Cok}v : \text{Cok}W \to \text{Cok} \text{Ker}Z = \text{Epi}Z$, which is not a split epimorphism, and hence the composition $t : \text{Cok}W \stackrel{t'}{\to} \text{Epi}Z \stackrel{\sigma}{\to} Z$ is not a split epimorphism. So, there is a morphism $s : \text{Cok}W \to Y$ such that $t = gs$. Applying $\text{Ker}$, we get $\text{Kert} = g'\text{Kers}$ with $\text{Kers} : W \to \text{Ker}Y$. Since $\text{Ker}\sigma = \text{id}_{\text{Ker}Z}$, we see $v = g'\text{Kers}$. This proves the claim.

Since $g'$ is right almost split and $\text{Ker}Z$ is not projective, it follows that $g'$ is epic, and hence $f'$ is not a split monomorphism. We claim that $f'$ is left almost split. For this, let $p : \text{Ker}X \to B$ be a morphism in $\mathcal{S}_n(A)$ which is not a split monomorphism. Take an injective envelope $(e_i) : (B_1/B_2, \cdots, B_{n-1}/B_n)_{(\pi_j)} \hookrightarrow (I_1, \cdots, I_{n-1})_{(\beta_j)}$ in $\text{Mor}_{n-1}(A)$. Put $B' = \left(\begin{array}{c} I_1 \\ \vdots \\ I_{n-1} \\ B_1 \end{array}\right)_{(\beta_j)}$ in $\text{Mor}_n(A)$, where $b_{n-1}'$ is the composition $B_1 \stackrel{e_{n-1}}{\to} B_1/B_n \hookrightarrow I_{n-1}$. By construction we get a morphism $e_{n-1} : \text{Cok}B \to B'$, and $\text{Cok}B = \text{Epi}B'$. Hence $\text{Ker}B' = B$.

Put $r' = (r_1') = \text{Cok}p : \text{Epi}X \to \text{Cok}B$. Then we have a morphism $\left(\begin{array}{c} e_1r_1' \\ \vdots \\ e_{n-1}r_{n-1}' \end{array}\right)$:
Clearly, \( r \) is not a split monomorphism (otherwise, \( \text{Epi} = r' : \text{Epi}X \to \text{Epi}B' = \text{Cok}B \) is a split monomorphism, and hence \( p = \text{Ker} r' \) is a split monomorphism). So there is a morphism \( h : Y \to B' \) such that \( r = hf : X \to B' \). Applying \( \text{Ker} \), we get \( \text{Ker} r' = (\text{Ker} h)f' \), where \( \text{Ker} h : \text{Ker} Y \to \text{Ker} B' = B \). Since \( \text{Ker} r = \text{Ker} \text{Epi} = \text{Ker} r' = \text{Ker} \text{Cok}p = p \), we get \( p = (\text{Ker} h)f' \). This proves the claim, and completes the proof. \( \blacksquare \)

**Proposition 2.2.** Let \( 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \) be an Auslander-Reiten sequence of \( \text{Mor}_n(A) \).

(i) If \( Z \in \mathcal{F}_n(A) \), and \( Z \) is not projective in \( \mathcal{F}_n(A) \), then

\[
0 \to \text{Epi}X \xrightarrow{\text{Epi}f} \text{Epi}Y \xrightarrow{\text{Epi}g} Z \to 0 \tag{2.1}
\]
is an Auslander-Reiten sequence of $\mathcal{F}_n(A)$; and

$$0 \to \text{Ker} X \xrightarrow{\text{Ker} f} \text{Ker} Y \xrightarrow{\text{Ker} g} \text{Ker} Z \to 0$$

(2.2)

is an Auslander-Reiten sequence of $\mathcal{S}_n(A)$.

(iii) If $X \in \mathcal{S}_n(A)$, and $X$ is not injective in $\mathcal{S}_n(A)$, then

$$0 \to X \xrightarrow{\text{Mono} f} \text{Mono} Y \xrightarrow{\text{Mono} g} \text{Mono} Z \to 0$$

is an Auslander-Reiten sequence of $\mathcal{S}_n(A)$; and

$$0 \to \text{Cok} X \xrightarrow{\text{Cok} f} \text{Cok} Y \xrightarrow{\text{Cok} g} \text{Cok} Z \to 0$$

is an Auslander-Reiten sequence of $\mathcal{F}_n(A)$.

**Proof.** We only show (i). Since $\text{Ker} : \mathcal{F}_n(A) \to \mathcal{S}_n(A)$ is an equivalence, and $Z$ is not projective in $\mathcal{F}_n(A)$, it follows that $\text{Ker} Z$ is not projective, and hence by Lemma 2.1 (2.2) is either split exact, or an Auslander-Reiten sequence in $\mathcal{S}_n(A)$. Applying the equivalence $\text{Cok} : \mathcal{S}_n(A) \to \mathcal{F}_n(A)$, and using $\text{Epi} \cong \text{Cok} \text{Ker}$ and $Z \in \mathcal{F}_n(A)$, we see that (2.1) is either split exact, or an Auslander-Reiten sequence in $\mathcal{F}_n(A)$. While $\text{Epi} Y \to Z$ is the composition of the canonical monomorphism $\text{Epi} Y \to Y$ and the right almost split morphism $Y \to Z$, so $\text{Epi} Y \to Z$ is not a split epimorphism, hence (2.1) is an Auslander-Reiten sequence in $\mathcal{F}_n(A)$, so is (2.2). \hfill \blacksquare

2.2. We have the following relationship between $\tau_S$ and $\tau_M$.

**Corollary 2.3.** (i) If $Z \in \mathcal{S}_n(A)$, then $\tau_S Z \cong \text{Ker} \tau_M \text{Cok} Z$, and $\tau^-_S Z \cong \text{Mono} \tau^-_M Z$.

(ii) If $Z \in \mathcal{F}_n(A)$, then $\tau_F Z \cong \text{Epi} \tau_M Z$, and $\tau^-_F Z \cong \text{Cok} \tau^-_M \text{Ker} Z$.

**Proof.** We only prove the first formula of (i). Assume that $Z$ is indecomposable. If $Z$ is projective, then $Z = m_i(P)$ by Lemma 1.1(iii), where $P$ is an indecomposable projective $A$-module. By the definition of $\tau_M$ and a direct computation, we have $\tau_M \text{Cok} Z = \tau_M \text{Cok} m_i(P) = \tau_M p_{n-i+1}(P) = \begin{pmatrix} \ast \\ \vdots \\ 0 \end{pmatrix}$, it follows that $\text{Ker} \tau_M \text{Cok} Z = 0 = \tau_S Z$.

Assume that $Z \in \mathcal{S}_n(A)$ is not projective. Since $\text{Cok} : \mathcal{S}_n(A) \to \mathcal{F}_n(A)$ is an equivalence, $\text{Cok} Z \in \mathcal{F}_n(A)$ is not projective. By Lemma 1.1(i) and (v), $\text{Cok} Z$ is an indecomposable nonprojective object in $\text{Mor}_n(A)$. Replacing $Z$ by $\text{Cok} Z$ in (2.2), we get the assertion by $\text{Ker} \text{Cok} Z \cong Z$. \hfill \blacksquare

2.3. Example. Let $k$ be a field, $A = k[x]/(x^2)$, and $S$ be the simple $A$-module. Denote by $i : S \hookrightarrow A$ and $\pi : A \to S$ the canonical $A$-maps. Then we have the Auslander-Reiten
sequence in $\text{Mor}_3(A)$

\[
0 \rightarrow \left( \begin{array}{c}
\frac{0}{S} \\ (i,0)
\end{array} \right) \rightarrow \left( \begin{array}{c}
\frac{0}{S} \\ (0,1)
\end{array} \right) \oplus \left( \frac{S}{A} \right) (\varphi_0,\pi) \rightarrow \left( \begin{array}{c}
\frac{0}{S} \\ (1,\pi)
\end{array} \right) \rightarrow 0.
\]

By (2.1), we get an Auslander-Reiten sequence in $\mathcal{F}_3(A)$

\[
0 \rightarrow \left( \frac{S}{S} \right)_{(1,0)} \rightarrow \left( \frac{0}{S} \right)_{(0,1)} \oplus \left( \frac{S}{A} \right)_{(1,\pi)} \rightarrow \left( \frac{S}{A} \right)_{(\pi,1)} \rightarrow 0.
\]

By (2.2), we get an Auslander-Reiten sequence in $\mathcal{S}_3(A)$

\[
0 \rightarrow \left( \frac{S}{S} \right)_{(0,1)} \rightarrow \left( \frac{S}{S} \right)_{(1,1)} \oplus \left( \frac{A}{S} \right)_{(0,\pi)} \rightarrow \left( \frac{A}{S} \right)_{(1,\pi)} \rightarrow 0.
\]

2.4. In Corollary 2.3(i), $\tau_S$ is formulated via $\tau_M$ of $\text{Mor}_n(A)$. However, $\tau_M$ is usually more complicated than $\tau$. The rest of this section is to give a formula of $\tau_S$ via $\tau$.

Before stating the main result, we need a notation. For $X_{(\phi_i)} \in \text{Mor}_n(A,\text{-mod})$, define

\[
\tau X_{(\phi_i)} = \left( \begin{array}{c}
\tau X_1 \\ \vdots \\ \tau X_n
\end{array} \right)_{(\phi_i)} \in \text{Mor}_n(A,\text{-mod}).
\]

Consider the full subcategory given by

\[
\{ Y_{(\psi_i)} = \left( \begin{array}{c}
\tau X_1 \\ \vdots \\ \tau X_n
\end{array} \right)_{(\psi_i)} \mid Y_{(\psi_i)} \cong \tau X_{(\phi_i)} \}.
\]

Any object in this full subcategory will be denoted by $\tau X_{(\phi_i)}$ (we emphasize that this convention will cause no confusions). So, for $X_{(\phi_i)} \in \text{Mor}_n(A)$ we have $\tau X_{(\phi_i)} \cong \tau X_{(\phi_i)}$.

By Lemma 1.6(i), Mimo $\tau X_{(\phi_i)}$ is a well-defined object in $\text{S}_n(A)$, and there are isomorphisms Mimo $\tau X_{(\phi_i)} \cong \tau X_{(\phi_i)} \cong \tau X_{(\phi_i)}$ in $\text{Mor}_n(A,\text{-mod})$. If $A$ is selfinjective, then $\text{Mor}_n(A,\text{-mod}) = \text{Mor}_n(A,\text{-mod})$, so the isomorphism above is read as follows, which is needed in the next section

\[
\text{Mimo } \tau X_{(\phi_i)} \cong \tau X_{(\phi_i)} \cong \tau X_{(\phi_i)}.
\]

Similarly, one has the convention $\tau^{-1} X_{(\phi_i)}$, and Mepi $\tau^{-1} X_{(\phi_i)} \in \mathcal{F}_n(A)$ is well-defined.

The following result is a generalization of Theorem 5.1 of Ringel and Schmidmeier [RS2].
Theorem 2.4. Let $X_{(\phi)} \in S_n(A)$. Then

(i) $\tau_S X_{(\phi)} \cong \text{Mimo } \tau \text{ Cok} X_{(\phi)}$.

(ii) $\tau^- S X_{(\phi)} \cong \text{Ker } \text{Mepi } \tau^- X_{(\phi)}$.

Proof. We only prove (i). Recall $\text{Cok} X_{(\phi)} = \left( \begin{array}{c} \text{Coker } \phi_1 \\ \text{Coker } (\phi_1 \phi_2) \\ \vdots \\ \text{Coker } (\phi_1 \cdots \phi_{n-1}) \\ X_1 \end{array} \right)$ $(\phi'_i)$.

Fix a minimal projective presentation $Q_n \xrightarrow{d_i} P \xrightarrow{e} X_1 \to 0$. Then we get the following commutative diagram with exact rows

\[
\begin{array}{ccccccc}
Q_n & \xrightarrow{d_n} & P & \xrightarrow{e} & X_1 & \xrightarrow{} & 0 \\
\downarrow{s_{n-1}} & \downarrow{d_{n-1}} & \downarrow{\phi_{n-1}'} & \downarrow{\phi_{n-1}} & \downarrow{} & \downarrow{} & \\
Q_{n-1} & \xrightarrow{\phi_{n-1}'} & \text{Coker } (\phi_1 \cdots \phi_{n-1}) & \xrightarrow{0} & \\
\vdots & \vdots & \vdots & \vdots & \downarrow{} & \downarrow{} & \\
Q_1 & \xrightarrow{d_1} & \phi_{1}' \cdots \phi_{n-1}' & \xrightarrow{\phi_1'} & \text{Coker } (\phi_1) & \xrightarrow{0} & \\
\end{array}
\]

where $Q_i \to \text{Ker } (\phi_1' \cdots \phi_{n-1}')$ is a projective cover, and $d_i$ is the composition $Q_i \to \text{Ker } (\phi_1' \cdots \phi_{n-1}') \to P$. Applying the Nakayama functor $N = D \text{Hom}_A(-, AA)$, we get the following commutative diagram

\[
\begin{array}{ccccccc}
0 & \xrightarrow{} & \tau X_1 & \xrightarrow{\sigma_n} & NQ_n & \xrightarrow{N\phi} & NP \\
\downarrow{\alpha_{n-1}} & \downarrow{} & \downarrow{} & \downarrow{\alpha_{n-1}} & \downarrow{N\phi_{n-1}} & \downarrow{N\phi_{n-1}} & \downarrow{} \\
0 & \xrightarrow{} & \tau \text{Coker } (\phi_1 \cdots \phi_{n-1}) & \xrightarrow{\sigma_{n-1}} & NQ_{n-1} & \xrightarrow{N\phi_{n-1}} & NP \\
\vdots & \vdots & \vdots & \vdots & \downarrow{N\phi_{n-2}} & \downarrow{} & \downarrow{} \\
0 & \xrightarrow{} & \tau \text{Coker } (\phi_1 \phi_2) & \xrightarrow{\sigma_2} & NQ_2 & \xrightarrow{N\phi_2} & NP \\
\downarrow{\alpha_1} & \downarrow{} & \downarrow{} & \downarrow{\alpha_1} & \downarrow{N\phi_1} & \downarrow{N\phi_1} & \downarrow{} \\
0 & \xrightarrow{} & \tau \text{Coker } \phi_1 & \xrightarrow{\sigma_1} & NQ_1 & \xrightarrow{N\phi_1} & NP. \\
\end{array}
\]

Step 1. By (2.4), we get a projective presentation

\[
\bigoplus_{i=1}^n m_i(Q_i) \xrightarrow{(d_1, \ldots, d_n)} m_n(P) \xrightarrow{(\phi_1' \cdots \phi_{n-1}') \phi_{1}' \cdots \phi_{n-1}'} \text{Cok} X_{(\phi)}, \xrightarrow{} 0
\]

(the exactness can be seen as follows: by (2.4) we have $\text{Im } d_n \subseteq \text{Im } d_{n-1} \subseteq \cdots \subseteq \text{Im } d_1$, and hence $Q_i \oplus \cdots \oplus Q_n \xrightarrow{(d_i, \ldots, d_n)} P \xrightarrow{\phi_1' \cdots \phi_{n-1}'} \text{Coker } (\phi_1 \cdots \phi_{n-1}) \to 0$ is exact). In order to
obtain a minimal projective presentation from (2.6), we have to split off a direct summand of \( \bigoplus_{i=1}^{n} m_i(Q_i) \). By Lemma 1.1(i), this direct summand is of the form \( \bigoplus_{i=1}^{n-1} m_i(Q_i') \) where \( Q_i' \) is a direct summand of \( Q_i \), \( 1 \leq i \leq n - 1 \), since \( \left( \begin{array}{c} \phi_1' \\ \vdots \\ \phi_{n-1}'e \end{array} \right) \) is already minimal and \( Q_n \xrightarrow{d_n} P \xrightarrow{\theta} X_1 \to 0 \) is already a minimal projective presentation. Applying the Nakayama functor \( \mathcal{N}_M \), we get the exact sequence

\[
0 \to \tau_M \text{Cok} X_{(\phi_i)} \oplus \mathcal{N}_M \left( \bigoplus_{i=1}^{n-1} m_i(Q_i') \right) \to \mathcal{N}_M \left( \bigoplus_{i=1}^{n} m_i(Q_i) \right) \to \mathcal{N}_M m_n(P).
\]

By Lemma 1.1(vii) this exact sequence can be written as

\[
0 \to \tau_M \text{Cok} X_{(\phi_i)} \oplus \bigoplus_{i=1}^{n-1} P_{n-i+1}(\mathcal{N}Q_i) \to \bigoplus_{i=1}^{n} P_{n-i+1}(\mathcal{N}Q_i) \xrightarrow{d} P_1(\mathcal{N}P) \tag{2.7}
\]

where \( d = \left( \begin{array}{c} \begin{array}{ccc} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{array} \end{array} \right) \).

**Step 2.** Write \( Y_{(\theta_i)} = \tau_M \text{Cok} X_{(\phi_i)} \oplus \bigoplus_{i=1}^{n-1} P_{n-i+1}(\mathcal{N}Q_i). \) By taking the \( i \)-th branches and the \( n \)-th branches of terms in (2.7), we get the following commutative diagram with exact rows

\[
\begin{array}{cccccccc}
0 & \to & Y_n & \xrightarrow{(a)} & \bigoplus_{j=1}^{i} \mathcal{N}Q_j & \oplus & \bigoplus_{j=i+1}^{n} \mathcal{N}Q_j & \xrightarrow{(N\mathcal{N}d_1, \cdots, N\mathcal{N}d_i, (N\mathcal{N}d_{i+1}, \cdots, N\mathcal{N}d_n))} & NP \\
\theta_1 \cdots \theta_{n-1} & \downarrow & & & & & & \downarrow & (1,0) \\
0 & \to & Y_i & \xrightarrow{\cong} & \bigoplus_{j=1}^{i} \mathcal{N}Q_j & \to & 0.
\end{array}
\]

In particular, \( Y_n = \text{Ker}(N\mathcal{N}d_1, \cdots, N\mathcal{N}d_n). \) The upper exact sequence means that

\[
Y_n \xrightarrow{-b} \bigoplus_{j=i+1}^{n} \mathcal{N}Q_j \xrightarrow{a} \bigoplus_{j=1}^{i} \mathcal{N}Q_j \xrightarrow{\cong} \mathcal{N}P
\]

is a pull back square, for each \( 1 \leq i \leq n - 1 \). It follows that

\[\text{Ker}(\theta_1 \cdots \theta_{n-1}) = \text{Ker} a = \text{Ker}(N\mathcal{N}d_{i+1}, \cdots, N\mathcal{N}d_n), \ 1 \leq i \leq n - 1,\]

and hence \( \text{Ker} Y_{(\theta_i)} = \left( \begin{array}{c} \text{Ker}(N\mathcal{N}d_{i+1}, \cdots, N\mathcal{N}d_n) \\ \vdots \\ \text{Ker}(N\mathcal{N}d_{n-1}, N\mathcal{N}d_n) \end{array} \right) \). We explicitly compute \( \text{Ker} Y_{(\theta_i)} \) below.
Step 3. By (2.5) we get the following commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \tau X_1 & \rightarrow & \gamma_n & \rightarrow & NQ_n & \rightarrow & NP \\
\beta_{n-1} & \downarrow & & & & & (0) & \downarrow & (N_{d_{n-1}}, N_{d_n}) \\
0 & \rightarrow & \tau \text{Coker}(\phi_1 \cdots \phi_{n-1}) \oplus NQ_n & \rightarrow & NQ_{n-1} \oplus NQ_n & \rightarrow & NP \\
\beta_{n-2} & \downarrow & & & & & (0) & \downarrow & (E_2) \\
\vdots & & & & & & \vdots & & \vdots \\
0 & \rightarrow & \tau \text{Coker}(\phi_1 \phi_2) \oplus \bigoplus_{i=3}^n NQ_i & \rightarrow & NQ_2 \oplus \bigoplus_{i=3}^n NQ_i & \rightarrow & NP \\
\beta_1 & \downarrow & & & & & (0) & \downarrow & (E_{n-1}) \\
0 & \rightarrow & \tau \text{Coker} \phi_1 \oplus \bigoplus_{i=2}^n NQ_i & \rightarrow & NQ_1 \oplus \bigoplus_{i=2}^n NQ_i & \rightarrow & NP \\
\end{array}
\]  

where \( E_i \) is the identity matrix, \( \gamma_i = \left( \begin{array}{c} \sigma_i \\ \sigma_{i+1} \\ \vdots \\ \sigma_n \\ 0 \\ -N(s_1) \\ \vdots \\ -N(s_{i-1}) \\ 0 \\ -N(s_{i+1}) \\ \vdots \\ -N(s_{n-1}) \end{array} \right) \) for \( 1 \leq i \leq n \), \( \beta_i = \left( \begin{array}{c} \alpha_i \\ \alpha_{i+1} \\ \vdots \\ \alpha_n \\ 0 \\ -N(s_1) \\ \vdots \\ -N(s_{i-1}) \\ 0 \\ -N(s_{i+1}) \\ \vdots \\ -N(s_{n-1}) \end{array} \right) \) for \( 1 \leq i \leq n-1 \). From (2.8) we see \( \text{Ker} Y(\theta_i) \cong \tau \text{Cok} X(\phi_i) \oplus J \), where \( J \) is an injective object in \( \mathcal{S}_n(A) \). Thus

\[
\text{Mimo} \tau \text{Cok} X(\phi_i) \oplus J \cong \text{Ker} Y(\theta_i) \cong \text{Ker} \tau \mathcal{M} \text{Cok} X(\phi_i) \oplus \text{Ker} \left( \bigoplus_{i=1}^{n-1} p_{n-i+1}(NQ'_i) \right)
\]

\[
\cong \tau_S X(\phi_i) \oplus \text{Ker} \left( \bigoplus_{i=1}^{n-1} p_{n-i+1}(NQ'_i) \right) \cong \text{Mimo} \tau \text{Cok} X(\phi_i)
\]

Since \( \text{Mimo} \tau \text{Cok} X(\phi_i) \) and \( \tau_S X(\phi_i) \) have no nonzero injective direct summands in \( \mathcal{S}_n(A) \) (cf. Remark (ii) in 1.4), and \( \mathcal{S}_n(A) \) is Krull-Schmidt, we get \( \tau_S X(\phi_i) \cong \text{Mimo} \tau \text{Cok} X(\phi_i) \).

2.5. Example. Let \( A, S, i, \) and \( \pi \) be as in 2.3. Then there are 6 indecomposable non-projective objects in \( \mathcal{S}_3(A) \). By Theorem 2.4 we have

\[
\tau_S \left( \begin{array}{c} A \\ 0 \end{array} \right) (0_i) = \text{Mimo} \tau \left( \begin{array}{c} S \\ A \end{array} \right) (1, \pi) = \text{Mimo} \left( \begin{array}{c} S \\ 0 \end{array} \right) (0,0) = \left( \begin{array}{c} S \\ 0 \end{array} \right) (0,0)
\]

\[
\tau_S \left( \begin{array}{c} S \\ 0 \end{array} \right) (0,0) = \text{Mimo} \tau \left( \begin{array}{c} S \\ S \end{array} \right) (1,1) = \text{Mimo} \left( \begin{array}{c} S \\ S \end{array} \right) (1,1) = \left( \begin{array}{c} S \\ S \end{array} \right) (1,1)
\]
The distinguished triangles of $A$ are exactly triangles isomorphic to those given by all nonzero injective direct summands.

We have to take up pages to justify that it is well-defined. The $n$ we get a formula for $n$. The definition of a rotation of an object in $\text{Mor}_A$ the short exact sequences in $\text{Mod-}A$. Throughout this section, $A$ is a selfinjective algebra. Then $A=\text{Mod-}A$ is a triangulated category with the suspension functor $\Omega^{-1}$ ([H], p.16), where $\Omega^{-1}$ is the cosyzygy of $A$.

The following result will be used in the next section, whose proof is omitted, since it is the same as the case of $n=2$ (see [RS2], Corollary 5.4).

**Corollary 2.5.** Every object in $\mathcal{S}_n(A)_I$ has the form $\text{Mimo}X$, where each $X_i$ has no nonzero injective direct summands.

The following result will be used in the next section, whose proof is omitted, since it is the same as the case of $n=2$ (see [RS2], Corollary 5.4).

**Corollary 2.6.** The canonical functor $W : \mathcal{S}_n(A)_I \to \text{Mor}_n(\text{Mod-}A)$ given by $X(\phi_i) \mapsto X(\phi_i)$ is dense, preserves indecomposables, and reflects isomorphisms.

**Theorem 2.7.** Let $X \in \mathcal{F}_n(A)$. Then $\tau_X X \cong \text{Cok} \text{Mimo} \tau X$, and $\tau_X \cong \text{Mepi} \tau^{-1} \ker X$.

**3. Applications to selfinjective algebras**

Throughout this section, $A$ is a selfinjective algebra. Then $A=\text{Mod-}A$ is a triangulated category with the suspension functor $\Omega^{-1}$ ([H], p.16), where $\Omega^{-1}$ is the cosyzygy of $A$. The distinguished triangles of $\text{Mod-}A$ are exactly triangles isomorphic to those given by all the short exact sequences in $\text{Mod-}A$. Note that $\Omega$ and $\mathcal{N}$ commute and $\tau \cong \Omega^2 \mathcal{N} \cong \mathcal{N} \Omega^2$ is an endo-equivalence of $\text{Mod-}A$ ([ARS], p.126), and that $\tau$ is a triangle functor.

A rotation of an object in $\text{Mor}_2(\text{Mod-}A)$ is introduced by Ringel and Schmidmeier [RS2]. The definition of a rotation of an object in $\text{Mor}_n(\text{Mod-}A)$ needs new considerations. We have to take up pages to justify that it is well-defined. Then we get a formula for
\[ \tau_S^j X \in \text{Mor}_n(A\text{-mod}) \text{ for } X \in S_n(A) \text{ and } j \geq 1. \] This is applied to the study of the periodicity of \( \tau_S \) on the objects of \( S_n(A) \). In particular, for the selfinjective Nakayama algebras \( \Lambda(m,t) \) we have \( \tau_S^{2m(n+1)} X \cong X \) for \( X \in S_n(\Lambda(m,t)) \).

3.1. Let \( X_{(\phi_i)} \in \text{Mor}_n(A\text{-mod}) \). Just choose \( \phi_i \) as representatives for the morphisms \( \phi_i \) in \( A\text{-mod} \). Let \( h_{i+1} : X_{i+1} \rightarrow I_{i+1} \) be an injective envelope with cokernel \( \Omega^{-1}X_{i+1} \), \( 1 \leq i \leq n-1 \). Taking pushout we get the following commutative diagram with exact rows

\[
\begin{array}{ccccccc}
0 & \longrightarrow & X_{i+1} & \xrightarrow{h_{i+1}} & I_{i+1} & \xrightarrow{\phi_1 \cdots \phi_i} & X_1 & \xrightarrow{g_{i+1}} & Y_{i+1}^1 & \xrightarrow{\psi_i} & 0 \\
\phi_1 \cdots \phi_i & \downarrow & \downarrow & \downarrow & j_{i+1} & & & \downarrow & & \\
0 & \longrightarrow & X_1 & \xrightarrow{g_{i+1}} & Y_{i+1}^1 & \longrightarrow & \Omega^{-1}X_{i+1} & 0.
\end{array}
\]

This gives the exact sequence

\[
0 \longrightarrow X_{i+1} \xrightarrow{(\phi_1 \cdots \phi_i)} X_1 \oplus I_{i+1} \xrightarrow{(g_{i+1}, -j_{i+1})} Y_{i+1}^1 \longrightarrow 0,
\]

and induces the following commutative diagram with exact rows, \( 1 \leq i \leq n-2 \)

\[
\begin{array}{ccccccc}
0 & \longrightarrow & X_{i+2} & \xrightarrow{(\phi_1 \cdots \phi_{i+1})} & X_1 \oplus I_{i+2} & \xrightarrow{(g_{i+2}, -j_{i+2})} & Y_{i+2}^1 & \longrightarrow & 0 \\
\phi_{i+1} & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_{i+1} & \xrightarrow{(\phi_1 \cdots \phi_i)} & X_1 \oplus I_{i+1} & \xrightarrow{(g_{i+1}, -j_{i+1})} & Y_{i+1}^1 & 0.
\end{array}
\]

By (3.2) we have \( g_{i+1} = \psi_i g_{i+2}, 1 \leq i \leq n-2 \). Put \( \psi_{n-1} = g_n \). Then \( g_{i+1} = \psi_i \cdots \psi_{n-1}, 1 \leq i \leq n-1 \). By the construction of a distinguished triangle in \( A\text{-mod} \), we get distinguished triangles from (3.1)

\[
X_{i+1} \xrightarrow{\phi_1 \cdots \phi_i} X_1 \rightarrow Y_{i+1}^1 \rightarrow \Omega^{-1}X_{i+1}, 1 \leq i \leq n-1,
\]

and by (3.2) we get the following commutative diagram, where the rows are distinguished triangles from (3.3)

\[
\begin{array}{ccccccc}
X_n & \longrightarrow & X_1 & \xrightarrow{\psi_{n-1}} & Y_n^1 & \longrightarrow & \Omega^{-1}X_n \\
\phi_{n-1} & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\
X_{n-1} & \longrightarrow & X_1 & \xrightarrow{\psi_{n-2}} & Y_{n-1}^1 & \longrightarrow & \Omega^{-1}X_{n-1} \\
\phi_{n-2} & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
X_2 & \xrightarrow{\phi_1} & X_1 & \xrightarrow{\psi_1} & Y_2^1 & \longrightarrow & \Omega^{-1}X_2.
\end{array}
\]

The rotation \( \text{Rot}_X(\phi_i) \) of \( X_{(\phi_i)} \) is defined to be

\[
(X_1 \xrightarrow{\psi_{n-1}} Y_n^1 \longrightarrow \cdots \longrightarrow Y_2^1) \in \text{Mor}_n(A\text{-mod})
\]
(here and in the following, for convenience we write the rotation in a row). We remark that $\text{Rot} X_{(\phi_i)}$ is well-defined: if $X_{(\phi_i)} \cong Y_{(\theta_i)}$ in $\text{Mor}_n(A\text{-mod})$ with all $X_i$ and $Y_i$ having no nonzero injective direct summands, then $\text{Mimo} X_{(\phi_i)} \cong \text{Mimo} Y_{(\theta_i)}$ by Lemma 1.6(i), and hence $\text{Rot} X_{(\phi_i)} \cong \text{Rot} Y_{(\theta_i)}$, by Lemma 3.1 below.

**Lemma 3.1.** Let $X_{(\phi_i)} \in \text{Mor}_n(A)$. Then $\text{Rot} X_{(\phi_i)} \cong \text{Cok Mimo} X_{(\phi_i)}$ in $\text{Mor}_n(A\text{-mod})$.

Before proving Lemma 3.1, for later convenience, we restate Claim 2 in §4 of [RS2] in the more explicit way we will use.

**Lemma 3.2.** Let $0 \to A \xrightarrow{f} B \oplus I \xrightarrow{\phi} C \to 0$ be an exact sequence with $I$ an injective $A$-module. Then there is an injective $A$-module $J$ such that $I = \text{I Ker} f \oplus J$, and that the following diagram with exact rows commutes

$$
\begin{array}{cccc}
0 & \to & A & \xrightarrow{f} & B \oplus \text{I Ker} f \oplus J \xrightarrow{\phi} C' \oplus J \to 0 \\
0 & \xrightarrow{h} & A & \xrightarrow{h'} & B \oplus \text{I Ker} f \oplus J & \to & C & \to & 0,
\end{array}
$$

where $h = (f_i', \phi_i: A \to \text{I Ker} f$ is an extension of the injective envelope $\text{Ker} f \hookrightarrow \text{I Ker} f$, $h': A \to J$ satisfies $h' \text{Ker} f = 0$, and $C' = \text{Coker} (f_i')$.

**Proof of Lemma 3.1.** We divide the proof into three steps.

**Step 1.** Recall $\text{Mimo} X_{(\phi_i)} = \left( \begin{array}{c} X_1 \oplus \bigoplus_{l=1}^{n-1} \text{I Ker} \phi_l \\ X_1 \oplus \bigoplus_{l=1}^{n-1} \text{I Ker} \phi_l \end{array} \right)_{(\phi_i)}$ with $\theta_i = \left( \begin{array}{c} \phi_i \\ e_i \\ 0 \\ \vdots \\ e_{i-1} \end{array} \right)_{(\phi_i)}$, $\text{Coker} X_{(\phi_i)} = \left( \begin{array}{c} \text{Coker}(\phi_1) \\ \vdots \\ \text{Coker}(\phi_n) \\ X_1 \oplus \bigoplus_{l=1}^{n-1} \text{I Ker} \phi_l \end{array} \right)_{(\phi_i)}$. $e_i: X_{i+1} \to \text{I Ker} \phi_i$ is an extension of the injective envelope $\text{Ker} \phi_i \hookrightarrow \text{I Ker} \phi_i$, and

$$
\text{Cok Mimo} X_{(\phi_i)} = \left( \begin{array}{c} \text{Coker}(\phi_1) \\ \vdots \\ \text{Coker}(\phi_{n-1}) \\ X_1 \oplus \bigoplus_{l=1}^{n-1} \text{I Ker} \phi_l \end{array} \right)_{(\phi_i)}.
$$

Since $\theta_1 \cdots \theta_i = \text{diag}(\alpha_i, E_{n-i-1}) : X_{i+1} \oplus \bigoplus_{l=1}^{n-1} \text{I Ker} \phi_l \to X_1 \oplus \bigoplus_{l=1}^{n-1} \text{I Ker} \phi_l$, where $\alpha_i = \left( \begin{array}{c} \phi_1 \cdots \phi_i \\ e_1 \phi_2 \cdots \phi_i \\ \vdots \\ e_{i-1} \phi_i \\ e_i \phi_i \end{array} \right)$, $X_{i+1} \to X_1 \oplus \bigoplus_{l=1}^{i} \text{I Ker} \phi_l$, and $E_{n-i-1}$ is the identity matrix, we get the
following commutative diagram with exact rows, $2 \leq i \leq n - 1$,}

\[
\begin{array}{ccc}
0 & \longrightarrow & X_{i+1} \\
\phi_i & \downarrow & \downarrow (E_i,0) \\
0 & \longrightarrow & X_i \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \longrightarrow & X_{i+1} \oplus \bigoplus_{l=1}^{i} \text{IKer}\phi_l \\
\phi_i & \downarrow & \downarrow \pi_i \\
0 & \longrightarrow & X_i \oplus \bigoplus_{l=1}^{i-1} \text{IKer}\phi_l \\
\end{array}
\]

\[
0 \longrightarrow \text{Coker}(\theta_1 \cdots \theta_i) \longrightarrow 0
\]

\[
0 \longrightarrow \text{Coker}(\theta_1 \cdots \theta_{i-1}) \longrightarrow 0
\]

with $\pi_{n-1} = \theta'_{n-1}$.

Applying Lemma 3.2 to the upper exact sequence of (3.5) for $1 \leq i \leq n - 1$, we get injective $A$-modules $J_{i+1}$ such that $\bigoplus_{l=1}^{i} \text{IKer}\phi_l = \text{IKer}(\phi_1 \cdots \phi_i) \oplus J_{i+1}$ and that the following diagram with exact rows commutes

\[
\begin{array}{ccc}
0 & \longrightarrow & X_{i+1} \oplus \text{IKer}(\phi_1 \cdots \phi_i) \oplus J_{i+1} \\
\phi_i & \downarrow & \downarrow \beta_i \\
0 & \longrightarrow & X_i \oplus \text{IKer}(\phi_1 \cdots \phi_i) \oplus J_i \\
\end{array}
\]

\[
0 \longrightarrow \text{Coker}(\theta_1 \cdots \theta_i) \longrightarrow 0
\]

\[
0 \longrightarrow \text{Coker}(\theta_1 \cdots \theta_{i-1}) \longrightarrow 0
\]

where $\alpha_i = \left( \phi_1 \cdots \phi_i \atop a_i \right)$, $a_i : X_{i+1} \rightarrow \text{IKer}(\phi_1 \cdots \phi_i)$ is an extension of the injective envelope $\text{Ker}(\phi_1 \cdots \phi_i) \hookrightarrow \text{IKer}(\phi_1 \cdots \phi_i)$, $d_i : X_{i+1} \rightarrow J_{i+1}$ satisfies $d_i \text{Ker}(\phi_1 \cdots \phi_i) = 0$, and $Z_{i+1} = \text{Coker}(\phi_1 \cdots \phi_i)$. Thus by (3.6) and (3.5) we get the following commutative diagram with exact rows for $2 \leq i \leq n - 1$ (where the two rows in the middle come from (3.5)):

\[
\begin{array}{ccc}
0 & \longrightarrow & X_{i+1} \oplus \text{IKer}(\phi_1 \cdots \phi_i) \oplus J_{i+1} \\
\phi_i & \downarrow & \downarrow (E_i,0) \\
0 & \longrightarrow & X_i \oplus \text{IKer}(\phi_1 \cdots \phi_{i-1}) \oplus J_i \\
\end{array}
\]

\[
0 \longrightarrow \text{Coker}(\theta_1 \cdots \theta_{i-1}) \longrightarrow 0
\]

\[
0 \longrightarrow \text{Coker}(\theta_1 \cdots \theta_i) \longrightarrow 0
\]

Taking the first and the last rows, we get the following commutative diagram with exact rows:

\[
\begin{array}{ccc}
0 & \longrightarrow & X_{i+1} \oplus \text{IKer}(\phi_1 \cdots \phi_i) \oplus J_{i+1} \\
\phi_i & \downarrow & \downarrow \beta_i \\
0 & \longrightarrow & X_i \oplus \text{IKer}(\phi_1 \cdots \phi_i) \oplus J_i \\
\end{array}
\]

\[
0 \longrightarrow \text{Coker}(\theta_1 \cdots \theta_{i-1}) \longrightarrow 0
\]

\[
0 \longrightarrow \text{Coker}(\theta_1 \cdots \theta_i) \longrightarrow 0
\]

\[
0 \longrightarrow \text{Coker}(\theta_1 \cdots \theta_{i-1}) \longrightarrow 0
\]

\[
0 \longrightarrow \text{Coker}(\theta_1 \cdots \theta_i) \longrightarrow 0
\]
where for later convenience we write

\[
\beta_{i-1}^{-1} \theta_i' \beta_i = (f_i^{-1})^*, \quad 2 \leq i \leq n - 1. \tag{3.7}
\]

Taking off \( J_i \) and \( J_{i+1} \), we get the following commutative diagram with exact rows, \( 2 \leq i \leq n - 1 \),

\[
\begin{array}{cccccccccc}
0 & \xrightarrow{\phi_i} & X_{i+1} & \xrightarrow{\left( \frac{\phi_1 \cdots \phi_i}{a_i} \right)} & X_1 \oplus \text{I Ker}(\phi_1 \cdots \phi_i) & \xrightarrow{(b_1^1, b_1^2)} & Z_{i+1} & \xrightarrow{f_{i-1}} & 0 \\
0 & \xrightarrow{\phi_i} & X_i & \xrightarrow{\left( \frac{\phi_1 \cdots \phi_{i-1}}{a_{i-1}} \right)} & X_1 \oplus \text{I Ker}(\phi_1 \cdots \phi_{i-1}) & \xrightarrow{(b_{i-1}^1, b_{i-1}^2)} & Z_i & \xrightarrow{f_{i-1}} & 0.
\end{array}
\tag{3.8}
\]

**Step 2.** Now we consider the rotation \( \text{Rot} X_{\langle \beta \rangle} \). Recall from the beginning of this subsection that \( h_{i+1} : X_{i+1} \to I_{i+1} \) is an injective envelope. For \( 1 \leq i \leq n - 1 \), applying Lemma 3.2 to (3.1), we get injective \( A \)-modules \( J'_{i+1} \) such that \( I_{i+1} = \text{I Ker}(\phi_1 \cdots \phi_i) \oplus J'_{i+1} \) and that the following diagram with exact rows commutes (cf. (3.6))

\[
\begin{array}{cccccccccc}
0 & \xrightarrow{\phi_i} & X_{i+1} & \xrightarrow{\left( \frac{\phi_1 \cdots \phi_i}{a_i} \right)} & X_1 \oplus \text{I Ker}(\phi_1 \cdots \phi_i) \oplus J'_{i+1} & \xrightarrow{(b_1^1, b_1^2, 0, 0, 1)} & Z_{i+1} \oplus J'_{i+1} & \xrightarrow{f_{i-1}} & 0 \\
0 & \xrightarrow{\phi_i} & X_i & \xrightarrow{\left( \frac{\phi_1 \cdots \phi_{i-1}}{a_{i-1}} \right)} & X_1 \oplus \text{I Ker}(\phi_1 \cdots \phi_{i-1}) \oplus J'_{i+1} & \xrightarrow{(g_{i+1}, -j_{i+1})} & Y^1_{i+1} & \xrightarrow{f_{i-1}} & 0,
\end{array}
\tag{3.9}
\]

where \( h_{i+1} = \left( \frac{a_i}{d'_i} \right) \), and \( d'_i : X_{i+1} \to J'_{i+1} \), satisfying \( d'_i \text{Ker}(\phi_1 \cdots \phi_i) = 0 \). Thus by (3.9) and (3.2) we get the following commutative diagram with exact rows for \( 2 \leq i \leq n - 1 \) (where the two rows in the middle come from (3.2)):

\[
\begin{array}{cccccccccc}
0 & \xrightarrow{\phi_i} & X_{i+1} & \xrightarrow{\left( \frac{\phi_1 \cdots \phi_i}{a_i} \right)} & X_1 \oplus \text{I Ker}(\phi_1 \cdots \phi_i) \oplus J'_{i+1} & \xrightarrow{(b_1^1, b_1^2, 0, 0, 1)} & Z_{i+1} \oplus J'_{i+1} & \xrightarrow{f_{i-1}} & 0 \\
0 & \xrightarrow{\phi_i} & X_{i+1} & \xrightarrow{\left( \frac{\phi_1 \cdots \phi_{i-1}}{a_{i-1}} \right)} & X_1 \oplus \text{I Ker}(\phi_1 \cdots \phi_{i-1}) \oplus J'_{i+1} & \xrightarrow{(g_i, -j_i)} & Y^1_i & \xrightarrow{f_{i-1}} & 0,\end{array}
\]

Taking the first and the last rows, we get the following commutative diagram with exact rows:
\[\begin{align*}
0 \rightarrow X_{i+1} & \xrightarrow{(\phi_1 \ldots \phi_i)} X_1 \oplus \text{IKer}(\phi_1 \cdots \phi_i) \oplus J'_{i+1} \xrightarrow{(b^1_i, b^2_i)} Z_{i+1} \oplus J'_{i+1} \rightarrow 0 \\
\phi_i \downarrow & \downarrow (\phi_1 \cdots \phi_{i-1}) \downarrow (\frac{1}{c^1_{i1}}, \frac{1}{c^2_{i2}}) \downarrow (b^1_{i-1}, b^2_{i-1}) \\
0 \rightarrow X_i & \xrightarrow{(\phi_1 \cdots \phi_{i-1})} X_1 \oplus \text{IKer}(\phi_1 \cdots \phi_{i-1}) \oplus J'_i \xrightarrow{(b^1_{i-1}, b^2_{i-1})} Z_i \oplus J'_i \rightarrow 0,
\end{align*}\]

where for later convenience we write

\[\beta^1_{i-1}'\psi^i_{i-1}'\beta^i_i = \left(\begin{array}{c}
\beta^1_{i-1}' \\
\psi^i_{i-1}' \\
\end{array}\right), \quad 2 \leq i \leq n - 1.\quad (3.10)\]

Taking off \(J'_i\) and \(J'_{i+1}\), we get the following commutative diagram with exact rows for \(2 \leq i \leq n - 1\):

\[\begin{align*}
0 \rightarrow X_{i+1} & \xrightarrow{(\phi_1 \cdots \phi_i)} X_1 \oplus \text{IKer}(\phi_1 \cdots \phi_i) \oplus Z_{i+1} \rightarrow 0 \\
\phi_i \downarrow & \downarrow \left(\begin{array}{c}
\frac{1}{c^1_{i1}} \\
\frac{1}{c^2_{i2}} \\
\end{array}\right) \\
0 \rightarrow X_i & \xrightarrow{(\phi_1 \cdots \phi_{i-1})} X_1 \oplus \text{IKer}(\phi_1 \cdots \phi_{i-1}) \oplus Z_i \rightarrow 0.
\end{align*}\]

Comparing the above diagram with (3.8), by a computation we easily see that \(f_i - f'_i\) factors through an injective \(A\)-module for each \(1 \leq i \leq n - 2\).

**Step 3.** Now we get the following diagram, where the first row can be considered as \(\text{Cok Mimo} X_{(\phi_i)}\) (we identify \(X_1 \oplus \bigoplus_{l=1}^{n-1} \text{IKer}(\phi_l)\) with \(X_1 \oplus \text{IKer}(\phi_1 \cdots \phi_{n-1}) \oplus J_n\); and identify \(X_1 \oplus I_n\) with \(X_1 \oplus \text{IKer}(\phi_1 \cdots \phi_{n-1}) \oplus J'_n\):

\[\begin{align*}
&X_1 \oplus \bigoplus_{l=1}^{n-1} \text{IKer}(\phi_l) \xrightarrow{\theta^0_{n-1}} \text{Coker}(\theta_1 \cdots \theta_{n-1}) \xrightarrow{\theta^0_{n-2}} \cdots \xrightarrow{\theta^0_1} \text{Coker}(\theta_1 \theta_2) \rightarrow \text{Coker} \theta_1 \\
&\downarrow \left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{array}\right) \\
&X_1 \oplus \text{IKer}(\phi_1 \cdots \phi_{n-1}) \oplus J_n \xrightarrow{\beta^{-1}_{n-1} \bigoplus J_n} \text{Coker}(\theta_1 \cdots \theta_{n-1}) \xrightarrow{\theta^0_{n-2}} \cdots \xrightarrow{\theta^0_1} \text{Coker}(\theta_1 \theta_2) \rightarrow \text{Coker} \theta_1 \\
&\downarrow \left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}\right)
\end{align*}\]

(Note that the squares in the first two rows commute in \(A\)-mod: the left square comes from (3.6); and the remaining commutative squares come from (3.7). Also, note that the squares in the last two rows commute in \(A\)-mod: the left square comes from (3.9); and the remaining commutative squares come from (3.10)). However, the squares in the middle may **not** commute in \(A\)-mod; and the point is that they commute in \(A\)-mod, as we explain below.)
Note that the left square in the middle commutes by a direct computation. Since $f_i - f'_i$ factors through an injective $A$-module, $1 \leq i \leq n - 2$, we realize that the remaining $n - 2$ squares in the middle commute in $A\text{-mod}$. It follows that the above diagram commutes in $A\text{-mod}$. It is clear that the vertical morphisms are isomorphisms in $A\text{-mod}$. Regarding the above diagram in $A\text{-mod}$, the first row is exactly $\text{Cok Mimo} \mathcal{X}(\phi_i)$, and the last row is exactly $\text{Rot} \mathcal{X}(\phi_i)$. Thus, $\text{Rot} \mathcal{X}(\phi_i) \cong \text{Cok Mimo} \mathcal{X}(\phi_i)$ in $\text{Mor}_n(A\text{-mod})$. This completes the proof.

3.2. Let $\mathcal{X}(\phi_i) \in \text{Mor}_n(A)$. For $1 \leq k < i < j \leq n$, by (3.3) and the Octahedral Axiom we get the following commutative diagram with first two rows and the last two columns being distinguished triangles in $A\text{-mod}$:

\[
\begin{array}{cccccc}
\Omega Y^j & \xrightarrow{\phi_i} & X_i & \xrightarrow{\phi_{i-1}} & Y^i_j \\
\Omega Y^k & \xrightarrow{\phi_i} & X_k & \xrightarrow{\phi_{i-1}} & Y^k_i \\
\end{array}
\]

(3.11)

For $1 \leq m \leq n$ we prove the following formula by induction

\[
\text{Rot}^m \mathcal{X}(\phi_i) = (\Omega^{-(m-2)}Y^m_{m-1} \rightarrow \Omega^{-(m-2)}Y^m_{m-2} \rightarrow \ldots \rightarrow \Omega^{-(m-2)}Y^1_m \rightarrow \Omega^{-(m-1)}X_m \rightarrow \\
\Omega^{-(m-1)}Y^m_n \rightarrow \Omega^{-(m-1)}Y^m_{n-1} \rightarrow \ldots \rightarrow \Omega^{-(m-1)}Y^m_{m+1})
\]

(3.12)

Convention about (3.12): $\Omega^{-(m-1)}X_m$ is the $(n - m + 1)$-st branch of $\text{Rot}^m \mathcal{X}(\phi_i)$.

By definition (3.12) holds for $m = 1$. Assume that it holds for $1 \leq m \leq n - 1$. Consider the following commutative diagram with rows of distinguished triangles

\[
\begin{array}{cccccccc}
\Omega^{-(m-2)}Y^m_{m-1} & \rightarrow & \Omega^{-(m-2)}Y^m_{m-1} & \rightarrow & \Omega^{-(m-1)}Y^m_{m-1} & \rightarrow & \Omega^{-(m-1)}Y^m_{m-1} \\
\vdots & & \vdots & & \vdots & & \vdots \\
\Omega^{-(m-2)}Y^1_m & \rightarrow & \Omega^{-(m-1)}Y^m_{m-1} & \rightarrow & \Omega^{-(m-1)}Y^m_{m-1} & \rightarrow & \Omega^{-(m-1)}Y^m_{m-1} \\
\vdots & & \vdots & & \vdots & & \vdots \\
\Omega^{-(m-1)}X_m & \rightarrow & \Omega^{-(m-1)}Y^m_{m-1} & \rightarrow & \Omega^{-(m-1)}Y^m_{m-1} & \rightarrow & \Omega^{-(m-1)}Y^m_{m-1} \\
\vdots & & \vdots & & \vdots & & \vdots \\
\Omega^{-(m-1)}Y^m_n & \rightarrow & \Omega^{-(m-1)}Y^m_{m+1} & \rightarrow & \Omega^{-(m+1)}Y^m_{m+1} & \rightarrow & \Omega^{-(m+1)}Y^m_{m+1} \\
\vdots & & \vdots & & \vdots & & \vdots \\
\Omega^{-(m-1)}Y^m_{m+2} & \rightarrow & \Omega^{-(m-1)}Y^m_{m+2} & \rightarrow & \Omega^{-(m+1)}Y^m_{m+2} & \rightarrow & \Omega^{-(m+1)}Y^m_{m+2} \\
\end{array}
\]
where the \( l \)-th row \((1 \leq l \leq m - 1)\) is from the fourth column of (3.11) by taking \( j = m + 1, i = m, k = m - l \), and then applying \( \Omega^{-(m - 1)} \); the \( m \)-th row is from the first row of (3.11) by taking \( j = m + 1, i = m \), and then applying \( \Omega^{-(m - 1)} \); the \( l \)-th row \((m + 1 \leq l \leq n - 1)\) is from the fourth column of (3.11) by taking \( j = n + m + 1 - l, i = m + 1, k = m \), and then applying \( \Omega^{-(m)} \). By the definition of the rotation (cf. (3.4)), this proves (3.12) for \( m + 1 \).

3.3. For \( X_{(\phi_i)} \in \text{Mor}_n(A\text{-mod}) \), define \( \Omega^{-1}X_{(\phi_i)} \) to be \( \begin{pmatrix} \Omega^{-1}X_1 \\ \vdots \\ \Omega^{-1}X_n \end{pmatrix}_{(\Omega^{-1}\phi_i)} \in \text{Mor}_n(A\text{-mod}). \)

**Lemma 3.3.** Let \( X_{(\phi_i)} \in \text{Mor}_n(A) \). Then \( \text{Rot}^{j(n+1)}X_{(\phi_i)} = \Omega^{-j(n-1)}X_{(\phi_i)}, \forall j \geq 1. \)

**Proof.** By taking \( m = n \) in (3.12), we get

\[
\text{Rot}^nX_{(\phi_i)} = (\Omega^{-(n-2)}Y_{n}^{n-1} \to \Omega^{-(n-2)}Y_{n}^{n-2} \to \cdots \to \Omega^{-(n-2)}Y_{1}^{1} \to \Omega^{-(n-1)}X_{n}).
\]

We have the following commutative diagram with rows being distinguished triangles

\[
\begin{array}{cccc}
\Omega^{-(n-2)}Y_{n}^{n-1} & \rightarrow & \Omega^{-(n-2)}X_{n} & \rightarrow \Omega^{-(n-1)}X_{n-1} & \rightarrow \Omega^{-(n-1)}Y_{n}^{n-1} \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{-(n-2)}Y_{n}^{n-2} & \rightarrow & \Omega^{-(n-1)}X_{n} & \rightarrow \Omega^{-(n-1)}X_{n-2} & \rightarrow \Omega^{-(n-1)}Y_{n}^{n-2} \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots \\
\Omega^{-(n-2)}Y_{n}^{2} & \rightarrow & \Omega^{-(n-1)}X_{n} & \rightarrow \Omega^{-(n-1)}X_{n-2} & \rightarrow \Omega^{-(n-1)}Y_{n}^{2} \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{-(n-2)}Y_{n}^{1} & \rightarrow & \Omega^{-(n-1)}X_{n} & \rightarrow \Omega^{-(n-1)}X_{n-1} & \rightarrow \Omega^{-(n-1)}Y_{1} \\
\end{array}
\]

where the \( l \)-th row \((1 \leq l \leq n - 1)\) is from the first row of (3.11) by taking \( j = n, i = n - l, \) and then applying \( \Omega^{-(n-1)} \) (note that \((-1)^{(n-1)} \) arises from applying \( \Omega^{-(n-1)} \)). Using (3.4) we get that \( \text{Rot}^{n+1}X_{(\phi_i)} \) is

\[
( \Omega^{-(n-1)}X_{n} \xrightarrow{(-1)^{(n-1)}\phi_{n-1}} \Omega^{-(n-1)}X_{n-1} \xrightarrow{(-1)^{(n-1)}\phi_{n-2}} \cdots \xrightarrow{(-1)^{(n-1)}\phi_{1}} \Omega^{-(n-1)}X_{1} )
\]

\[
\cong ( \Omega^{-(n-1)}X_{n} \xrightarrow{\Omega^{-(n-1)}\phi_{n-1}} \Omega^{-(n-1)}X_{n-1} \xrightarrow{\Omega^{-(n-1)}\phi_{n-2}} \cdots \xrightarrow{\Omega^{-(n-1)}\phi_{1}} \Omega^{-(n-1)}X_{1} )
\]

\[
= \Omega^{-(n-1)}X_{(\phi_i)}.
\]

From this and induction the assertion follows.

3.4. Since \( \tau \) is a triangle functor, by construction we see

\[
\text{Rot} \tau X_{(\phi_i)} \cong \tau \text{Rot} X_{(\phi_i)}. \quad (3.13)
\]

We have the following important result.
Theorem 3.4. Let $A$ be a selfinjective algebra, $X_{(\phi_i)} \in S_n(A)$. Then there are the following isomorphisms in $\text{Mor}_n(A\text{-mod})$

(i) $\tau^j_S X_{(\phi_i)} \cong \tau^j \text{Rot}^j X_{(\phi_i)}$ for $j \geq 1$. In particular, $\tau_S X_{(\phi_i)} \cong \tau \text{Cok} X_{(\phi_i)}$.

(ii) $\tau^{s(n+1)} S X_{(\phi_i)} \cong \tau^{s(n+1)} \Omega^{s(n-1)} X_{(\phi_i)}$, $\forall$ $s \geq 1$.

Proof. (i) First, we claim that there are the following isomorphisms in $\text{Mor}_n(A\text{-mod})$:

$(\text{Cok Mimo} \tau)^j Y_{(\psi_i)} \cong \tau^j \text{Rot}^j Y_{(\psi_i)}$, $\forall$ $Y_{(\psi_i)} \in \text{Mor}_n(A\text{-mod})$, $\forall$ $j \geq 1$. (3.14)

In fact, by Lemma 3.1 and induction we have

\[
(\text{Cok Mimo} \tau)^j Y_{(\psi_i)} \cong \tau^j \text{Rot}^j Y_{(\psi_i)} \quad \text{Lemmas 3.1, 3.3 and (2.3) Induction}
\]

Now, we have the following isomorphisms in $\text{Mor}_n(A\text{-mod})$:

$\tau^j_S X_{(\phi_i)} \cong \text{Mimo} \tau^j S X_{(\phi_i)}$ Theorem 2.4

\[
\cong \text{Mimo} \tau (\text{Cok Mimo} \tau)^j \text{Cok} X_{(\phi_i)} \quad \text{Lemma 3.1}
\]

\[
\cong \tau (\text{Cok Mimo} \tau)^j \text{Cok} X_{(\phi_i)} \quad \text{Lemma 3.3, (3.14)}
\]

\[
\cong \tau^j \text{Rot}^{j-1} \text{Cok} X_{(\phi_i)} \quad \text{Lemma 3.1}
\]

where we have used $\text{Mimo} X_{(\phi_i)} = X_{(\phi_i)}$ since $X_{(\phi_i)} \in S_n(A)$.

(ii) This follows from Lemma 3.3 and (i) by taking $j = s(n + 1)$. ■

3.5. Now we pass from $\text{Mor}_n(A\text{-mod})$ to $\text{Mor}_n(A\text{-mod})$. Before stating the main result, we need a notation. Let $X_{(\phi_i)} \in \text{Mor}_n(A)$. For positive integers $r$ and $t$, the object $\tau^r \Omega^{-t} X_{(\phi_i)} \in \text{Mor}_n(A\text{-mod})$ is already defined (cf. 2.4 and 3.3). As in 2.4, we consider
the full subcategory of \( \text{Mor}_n(A) \) given by

\[
\{ Y_{(\psi)} = \left( \begin{array}{c} \tau^r X_1 \\ \vdots \\ \tau^r X_n \end{array} \right) \}_{(\psi)} \in \text{Mor}_n(A) \mid Y_{(\psi)} \cong \tau^r \Omega^{-t} X_{(\phi_i)} \}.
\]

Any object in this subcategory will be denoted by \( \tau^r \Omega^{-t} X_{(\phi_i)} \) (we emphasize that this convention will cause no confusions). So, we have \( \tau^r \Omega^{-t} X_{(\phi_i)} \cong \tau^r \Omega^{-t} X_{(\phi_i)} \). By Lemma 1.6(i), \( \text{Mimo} \tau^r \Omega^{-t} X_{(\phi_i)} \in \mathcal{S}_n(A) \) is a well-defined object, and there are the following isomorphisms in \( \text{Mor}_n(A-\text{mod}) \)

\[
\text{Mimo} \tau^r \Omega^{-t} X_{(\phi_i)} \cong \tau^r \Omega^{-t} X_{(\phi_i)} \cong \tau^r \Omega^{-t} X_{(\phi_i)}.
\] (3.15)

**Theorem 3.5.** Let \( A \) be a selfinjective algebra, and \( X_{(\phi_i)} \in \mathcal{S}_n(A) \). Then we have

\[
\tau_s^{(n+1)} X_{(\phi_i)} \cong \text{Mimo} \tau_s^{(n+1)} \Omega^{-s(n-1)} X_{(\phi_i)}, \quad s \geq 1.
\] (3.16)

**Proof.** By Theorem 3.4(ii) we have

\[
\tau_s^{(n+1)} X_{(\phi_i)} \cong \tau_s^{(n+1)} \Omega^{-s(n-1)} X_{(\phi_i)} \cong \text{Mimo} \tau_s^{(n+1)} \Omega^{-s(n-1)} X_{(\phi_i)}.
\] Since \( \text{Mimo} \tau_s^{(n+1)} \Omega^{-s(n-1)} X_{(\phi_i)} \in \mathcal{S}_n(A) \) (cf. Remark (ii) in 1.4), the assertion follows from Corollary 2.6.

3.6. We apply Theorem 3.4 to the algebra \( \Lambda(m, t) \), which is defined below. Let \( \mathbb{Z}_m \) be the cyclic quiver with vertices indexed by the cyclic group \( \mathbb{Z}/m\mathbb{Z} \) of order \( m \), and with arrows \( a_i : i \rightarrow i + 1, \forall i \in \mathbb{Z}/m\mathbb{Z} \). Let \( k\mathbb{Z}_m \) be the path algebra of the quiver \( \mathbb{Z}_m \), \( J \) the ideal generated by all arrows, and \( \Lambda(m, t) := k\mathbb{Z}_m/J^t \) with \( m \geq 1, t \geq 2 \). Any connected selfinjective Nakayama algebra over an algebraically closed field is Morita equivalent to \( \Lambda(m, t) \), \( m \geq 1, t \geq 2 \). Note that \( \Lambda(m, t) \) is a Frobenius algebra of finite representation type, and that \( \Lambda(m, t) \) is symmetric if and only if if \( m \mid (t - 1) \). For the Auslander-Reiten sequences of \( \Lambda(m, t) \) see [ARS], p.197. In the stable category \( \Lambda(m, t) \)-mod, we have the following information on the orders of \( \tau \) and \( \Omega \) (see 5.1 in [CZ])

\[
o(\tau) = m; \quad o(\Omega) = \begin{cases} m, & t = 2; \\
\frac{2m}{(m,t)}, & t \geq 3,
\end{cases}
\] (3.17)

where \( (m, t) \) is the g.c.d of \( m \) and \( t \). By (3.16) and (3.17) we get the following

**Corollary 3.6.** For an indecomposable nonprojective object \( X_{(\phi_i)} \in \mathcal{S}_n(\Lambda(m, t)) \), \( m \geq 1, t \geq 2 \), there are the following isomorphisms:

(i) If \( n \) is odd, then \( \tau_s^{m(n+1)} X_{(\phi_i)} \cong X_{(\phi_i)} \);

(ii) If \( n \) is even, then \( \tau_s^{2m(n+1)} X_{(\phi_i)} \cong X_{(\phi_i)} \).
3.7. Example. Let $A = kQ/\langle \delta \alpha, \beta \gamma, \alpha \delta - \gamma \beta \rangle$, where $Q$ is the quiver $2 \bullet \xrightarrow{\alpha} 1 \bullet \xrightarrow{\beta} 3 \bullet$

Then $A$ is selfinjective with $\tau \cong \Omega^{-1}$ and $\Omega^6 \cong \text{id}$ on the object of $A$-$\text{mod}$. The Auslander-Reiten quiver of $A$ is

Let $X_{(\phi)}$ be an indecomposable nonprojective object in $S_n(A)$. By (3.16), for $s \geq 1$ we have $\tau_S^{s(n+1)} X_{(\phi)} \cong \text{Mimo} \tau^{s(n+1)} \Omega^{-s(n-1)} X_{(\phi)} \cong \text{Mimo} \Omega^{-2sn} X_{(\phi)}$ in $S_n(A)$. Then by Remark (i) in 1.4 we get

(i) if $n \equiv 0$, or $3 \pmod{6}$, then $\tau_S^{n+1} X_{(\phi)} \cong X_{(\phi)}$; and

(ii) if $n \equiv \pm 1$, or $\pm 2 \pmod{6}$, then $\tau_S^{3(n+1)} X_{(\phi)} \cong X_{(\phi)}$.

4. Serre functors of stable monomorphism categories

Throughout this section, $A$ is a finite-dimensional selfinjective algebra over a field. We study the periodicity of the Serre functor $F_S$ on the objects of the stable monomorphism category $\underline{S}_n(A)$. In particular, $F_S^{n(n+1)} X \cong X$ for $X \in \underline{S}_n(\Lambda(m,t))$.

4.1. Let $\mathcal{A}$ be a Hom-finite Krull-Schmidt triangulated $k$-category with suspension functor $[1]$. For the Auslander-Reiten triangles we refer to [H]. In an Auslander-Reiten triangle $X \rightarrow Y \rightarrow Z \rightarrow X[1]$, the indecomposable object $X$ is uniquely determined by $Z$. Write $X = \tau_A Z$, and extend the action of $\tau_A$ to arbitrary objects, and put $\tau_A 0 = 0$. In general, $\tau_A$ is not a functor. By Theorem I.2.4 of [RV], $\mathcal{A}$ has a Serre functor $F$ if and if $\mathcal{A}$ has Auslander-Reiten triangles; if this is the case, $F$ and $[1] \tau_A$ coincide on the objects of $\mathcal{A}$. If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is an Auslander-Reiten triangle, then so is $X[1] \xrightarrow{-f[1]} Y[1] \xrightarrow{-g[1]} Z[1] \xrightarrow{-h[1]} X[2]$. It follows that

$$FX \cong ([1] \tau_A)Z \cong (\tau_A [1])Z, \ \forall \ Z \in \mathcal{A}. \quad (4.1)$$

4.2. Since $A$ is self-injective, by Corollary 4.1(ii) of [Z], $\underline{S}_n(A)$ is exactly the category of Gorenstein projective $T_n(A)$-modules, hence it is a Frobenius category whose projective-injective objects are exactly all the projective $T_n(A)$-modules. Thus, the stable category $\underline{S}_n(A)$ of $\underline{S}_n(A)$ modulo projective objects is a Hom-finite Krull-Schmidt triangulated
category with suspension functor $\Omega_S^{-1} = \Omega_{S_n(A)}^{-1}$. Since $S_n(A)$ has Auslander-Reiten sequences, it follows that $S_n(A)$ has Auslander-Reiten triangles, and hence it has a Serre functor $F_S = F_{S_n(A)}$, which coincides with $\Omega_S^{-1} \tau_S$ on the objects of $S_n(A)$.

In order to make the following computation more clear, we denote by $Q : S_n(A) \to S_n(A)$ the natural functor. Then

$$\tilde{\tau}_S QZ = Q \tau_S Z = \tau_S Z, \forall Z \in S_n(A). \quad (4.2)$$

4.3. In 2.6 we have considered the canonical functor $W : S_n(A)_I \to \text{Mor}_n(A_{\text{-mod}})$ given by $X_{(\phi_i)} \mapsto X_{(\phi_i)}$.

**Lemma 4.1.** The functor $W$ induces a functor $\tilde{W} : S_n(A) \to \text{Mor}_n(A_{\text{-mod}})$ satisfying $\tilde{W} Q|_{S_n(A)_I} = W$; and $\tilde{W}$ reflects isomorphisms.

**Proof.** The definition of $\tilde{W}$ is clear by the requirement $\tilde{W} Q|_{S_n(A)_I} = W$. We need to check that it is well-defined. If $X_{(\phi_i)}$ and $Y_{(\psi_i)}$ are indecomposable and nonprojective in $S_n(A)$, and $X_{(\phi_i)} \cong Y_{(\psi_i)}$ in $\underline{S_n(A)}$, then $X_{(\phi_i)} \cong Y_{(\psi_i)}$ in $S_n(A)$, and hence $X_{(\phi_i)} \cong Y_{(\psi_i)}$ in $\text{Mor}_n(A_{\text{-mod}})$, i.e., $\tilde{W}$ is well-defined on objects. For any morphism $f = (f_i) : X_{(\phi_i)} \to Y_{(\psi_i)}$ in $S_n(A)$ which factors through a projective object of $S_n(A)$, by Lemma 1.1(iii), the morphism $(f_i) = 0$ in $\text{Mor}_n(A_{\text{-mod}})$. Thus $\tilde{W}$ is well-defined.

Assume that $\tilde{W} X \cong \tilde{W} Y$ in $\text{Mor}_n(A_{\text{-mod}})$ for $X, Y \in S_n(A)$. We may write $X = QX', Y = QY'$ with $X', Y' \in S_n(A)_I$. Then $WX' \cong WY'$ in $\text{Mor}_n(A_{\text{-mod}})$. By Corollary 2.6, $W$ reflects isomorphisms, thus $X' \cong Y'$ in $S_n(A)_I$, and hence $X \cong Y$ in $S_n(A)$.

**Lemma 4.2.** For $X = X_{(\phi_i)} \in S_n(A)$, we have the following isomorphism in $\text{Mor}_n(A_{\text{-mod}})$

$$\tilde{W} \Omega_S X \cong \Omega \tilde{W} X. \quad (4.3)$$

**Proof.** Let $0 \to \Omega_S X \to P \to X \to 0$ be an exact sequence in $S_n(A)$ with $P$ projective. Taking the $i$-th branches we see that $(\Omega_S X)_i = \Omega X_i \oplus P'_i$ for some projective $A$-module $P'_i$. It follows that $(\Omega_S X)_i = \Omega X_i$ in $A_{\text{-mod}}$. Write $\Omega_S X$ as $(\Omega_S X)(\psi_i)$. By the following commutative diagram with exact columns

$$
\begin{array}{cccccccc}
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
(\Omega_S X)_n & \rightarrow & (\Omega_S X)_{n-1} & \rightarrow & \cdots & \rightarrow & (\Omega_S X)_2 & \rightarrow \psi_1 (\Omega_S X)_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
P_n & \rightarrow & P_{n-1} & \rightarrow & \cdots & \rightarrow & P_2 & \rightarrow P_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
X_n & \rightarrow & X_{n-1} & \rightarrow & \cdots & \rightarrow & X_2 & \rightarrow \phi_1 X_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & = & 0 & = & 0 & = & 0 & = \\
\end{array}
$$

we see $\psi_i = \Omega \phi_i$, and hence the assertion follows.
4.4. For positive integers $a$ and $b$, let $[a, b]$ denote the l.c.m of $a$ and $b$. The main result of this section is

Theorem 4.3. Let $A$ be a selfinjective algebra, and $F_S$ be the Serre functor of $S_n(A)$. Then we have an isomorphism in $S_n(A)$ for $X(\phi_i) \in S_n(A)$ and for $s \geq 1$

$$F_S^{s(n+1)} X(\phi_i) \cong \text{Mimo} \tau^{s(n+1)} \Omega^{-2sn} X(\phi_i). \quad (4.4)$$

Moreover, if $d_1$ and $d_2$ are positive integers such that $\tau^{d_1} M \cong M$ and $\Omega^{d_2} M \cong M$ for each indecomposable nonprojective $A$-module $M$, then $F_S^{N(n+1)} X(\phi_i) \cong X(\phi_i)$, where $N = \frac{d_1}{(n+1,d_1)} \cdot \frac{d_2}{(2n,d_2)}$.

Proof. We have isomorphisms in $\text{Mor}_n(A\text{-mod})$ for $s \geq 1$:

\[
\begin{align*}
\widetilde{W} F_S^{s(n+1)} X(\phi_i) & \cong \widetilde{W} \Omega_S^{s(n+1)} \tilde{\tau}^{s(n+1)} X(\phi_i) \\
& \cong \Omega^{-s(n+1)} \tilde{W} \tilde{\tau}^{s(n+1)} X(\phi_i) \\
& \cong \Omega^{-s(n+1)} \tilde{W} Q \tau^{s(n+1)} X(\phi_i) \\
& \cong \Omega^{-s(n+1)} \tilde{W} Q \text{Mimo} \tau^{s(n+1)} \Omega^{-s(n+1)} X(\phi_i) \\
& \cong \Omega^{-s(n+1)} \tau^{s(n+1)} \Omega^{-s(n-1)} X(\phi_i) \\
& \cong \Omega^{-s(n+1)} \tau^{s(n+1)} \Omega^{-s(n-1)} X(\phi_i) \\
& \cong \tau^{s(n+1)} \Omega^{-2sn} X(\phi_i) \\
& \cong \text{Mimo} \tau^{s(n+1)} \Omega^{-2sn} X(\phi_i).
\end{align*}
\]

Now (4.4) follows from Lemma 4.1. Since $d_1|N(n+1)$, $d_2|(2Nn)$, taking $s = N$ in (4.4) we get $F_S^{N(n+1)} X(\phi_i) \cong \text{Mimo} \tau^{N(n+1)} \Omega^{-2Nn} X(\phi_i) \cong \text{Mimo} X(\phi_i) = X(\phi_i)$. ■

Note that the conditions on $\tau$ and $\Omega$ in Theorem 4.3 hold in particular for representation-finite selfinjective algebras.

4.5. Applying Theorem 4.3 to the selfinjective Nakayama algebras $\Lambda(m, t)$, we get

Corollary 4.4. Let $F_S$ be the Serre functor of $S_n(\Lambda(m, t))$ with $m \geq 1$, $t \geq 2$, and $X$ be an arbitrary object in $S_n(\Lambda(m, t))$. Then

(i) If $t = 2$, then $F_S^{N(n+1)} X \cong X$, where $N = \frac{m}{(m,n-1)}$.

(ii) If $t \geq 3$, then $F_S^{N(n+1)} X \cong X$, where $N = \frac{m}{(m,t,n+1)}$. 
Proof.  

(i) In this case $\tau = \Omega$, and $o(\tau) = m = o(\Omega)$, by (3.17). Put $N = \frac{m}{(m,n-1)}$. It follows from (4.1) that

$$F_S^{N(n+1)} X \cong \text{Mimo } \tau^{N(n+1)} \Omega^{-2Nn} X \cong \text{Mimo } \Omega^{-N(n-1)} X \cong \text{Mimo } X.$$

(ii) This follows from Theorem 4.3 by taking $d_1 = m$ and $d_2 = \frac{2m}{(m,t)}$. By a computation in elementary number theory, we get

$$\left[ \frac{d_1}{(n+1,d_1)}, \frac{d_2}{(2n,d_2)} \right] = \left[ \frac{m}{(n+1, m)}, \frac{2m}{(2n, \frac{2m}{(m,t)})} \right] = \left[ \frac{m}{(m,t,n+1)} \right].$$

5. Appendix 1: Proofs of Lemmas 1.5 and 1.6

We give proofs of Lemmas 1.5 and 1.6.

Lemma 5.1. Let $X_{(\phi_i)} \in \text{Mor}_n(A)$, $I_2, \ldots, I_n$ be injective $A$-modules such that $X'_{(\phi_i')} = X_{(\phi_i)}$.

\[
\begin{pmatrix}
X_1 \oplus \cdots \oplus I_n \\
\vdots \\
X_{n-1} \oplus I_n \\
X_n
\end{pmatrix}
\in \mathcal{S}_n(A), \text{ where } \phi_i' =
\begin{pmatrix}
\phi_i & 0 & \cdots & 0 \\
0 & \ddots & \cdots & \cdots \\
0 & \cdots & \ddots & \cdots \\
0 & \cdots & \cdots & 1
\end{pmatrix}_{(n-i+1) \times (n-i)}.
\]

Then $X'_{(\phi_i')} \cong \text{Mimo } X_{(\phi_i)} \oplus J$, where $J$ is an injective object of $\mathcal{S}_n(A)$. Moreover, $J = \bigoplus_{i=1}^{n-1} m_i(Q_i)$, where $Q_i$ is an injective $A$-module such that $Q_i \oplus \text{IKer} \phi_i \cong I_{i+1}$, $1 \leq i \leq n-1$.

Proof. It is clear that the morphism

$$X'_{(\phi_i')} \rightarrow X_{(\phi_i)}$$

is a right approximation of $X_{(\phi_i)}$ in $\mathcal{S}_n(A)$ (this can be proved as Lemma 1.3(i), see [Z], Lemma 2.3). By Lemma 1.3(i), there is an object $J \in \mathcal{S}_n(A)$ such that $X'_{(\phi_i')} \cong \text{Mimo } X_{(\phi_i)} \oplus J$. Comparing the branches we get $J_n = 0$ and

$$I_{i+1} \oplus \cdots \oplus I_n \cong \text{IKer} \phi_i \oplus \cdots \oplus \text{IKer} \phi_{n-1} \oplus J_i, \forall 1 \leq i \leq n-1.$$  

(5.1)

Put $Q_{n-1} = J_{n-1}$. From $I_n \cong \text{IKer} \phi_{n-1} \oplus J_{n-1}$ we see that $Q_{n-1}$ is an injective $A$-module. Since $J \in \mathcal{S}_n(A)$, $Q_{n-1}$ is a submodule of $J_{n-2}$, thus $J_{n-2} = Q_{n-2} \oplus Q_{n-1}$. By $I_{n-1} \oplus I_n \cong \text{IKer} \phi_{n-2} \oplus \text{IKer} \phi_{n-1} \oplus J_{n-2}$ in (5.1), we see $I_{n-1} \cong \text{IKer} \phi_{n-2} \oplus Q_{n-2}$. Repeating this process we see that $J_i$ is of the form $J_i = Q_i \oplus \cdots \oplus Q_{n-1}$ with $Q_i$ being injective $A$-modules, and $Q_i \oplus \text{IKer} \phi_i \cong I_{i+1}$, $1 \leq i \leq n-1$. Thus $J = \bigoplus_{i=1}^{n-1} m_i(Q_i)$ is an injective object of $\mathcal{S}_n(A)$.

Now we can prove Lemma 1.5 (cf. Claim 2 in §4 of [RS2] for the case of $n = 2$).

Proof of Lemma 1.5. We just prove (i). Since the $A$-map $\phi_i' : X_{i+1} \oplus I_{i+2} \oplus \cdots \oplus I_n \rightarrow X_i \oplus I_{i+1} \oplus I_{i+2} \oplus \cdots \oplus I_n$ is monic, the restriction of $\phi_i'$ on $I_{i+2} \oplus \cdots \oplus I_n$ is also monic,
and hence it is a split monomorphism. Hence \(X'_{(\phi'_i)}\) is isomorphic to

\[
X''_{(\phi''_i)} = \left( \begin{array}{c}
X_1 \oplus I_2 \oplus \cdots \oplus I_n \\
\vdots \\
X_{n-1} \oplus I_n \\
X_n 
\end{array} \right) \in S_n(A), \quad \text{where} \quad \phi''_i = \left( \begin{array}{cccc}
\phi_i & 0 & 0 & \cdots & 0 \\
* & 0 & 0 & \cdots & 0 \\
* & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
* & 0 & 0 & \cdots & 1 
\end{array} \right)_{(n-i+1) \times (n-i)}.
\]

Then the assertion follows from Lemma 5.1.

**Lemma 5.2.** Let \(X_{(f_i)}, X_{(g_i)} \in \text{Mor}_n(A)\) such that \(f_i - g_i\) factors through an injective \(A\)-module, \(1 \leq i \leq n - 1\), and \(h_i : X_i \to I_i\) be an injective envelope, \(2 \leq i \leq n\). Set

\[
X'_{(f'_i)} = \left( \begin{array}{c}
X_1 \oplus I_2 \oplus \cdots \oplus I_n \\
\vdots \\
X_{n-1} \oplus I_n \\
X_n 
\end{array} \right) \quad \text{where} \quad f'_i = \left( \begin{array}{cccc}
f_i & 0 & 0 & \cdots & 0 \\
h_{i+1} & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 
\end{array} \right)_{(n-i+1) \times (n-i)}
\]

and

\[
X'_{(g'_i)} = \left( \begin{array}{c}
X_1 \oplus I_2 \oplus \cdots \oplus I_n \\
\vdots \\
X_{n-1} \oplus I_n \\
X_n 
\end{array} \right) \quad \text{where} \quad g'_i = \left( \begin{array}{cccc}
g_i & 0 & 0 & \cdots & 0 \\
h_{i+1} & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 
\end{array} \right)_{(n-i+1) \times (n-i)}.
\]

Then \(X'_{(f'_i)} \cong X'_{(g'_i)}\) in \(S_n(A)\).

**Proof.** For \(1 \leq i \leq n - 1\), it is clear that \(f_i - g_i : X_{i+1} \to X_i\) factors through the injective envelope \(h_{i+1} : X_{i+1} \to I_{i+1}\), and hence there is an \(A\)-map \(u_i : I_{i+1} \to X_i\) such that \(g_i - f_i = u_i h_{i+1}\). The following commutative diagram shows \(X'_{(f'_i)} \cong X'_{(g'_i)}\).

The matrices \(\alpha_i = \left( \begin{array}{cccc}
1 & u_i & & \\
0 & 0 & h_{i+1} u_{i+1} & \\
\vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & h_{n-1} u_{n-1} 
\end{array} \right)_{(n-i+1) \times (n-i+1)}\) are defined in Lemma 5.1, where \(1 \leq i \leq n\).

**Lemma 5.3.** Let \(X_{(f_i)}, X_{(g_i)} \in \text{Mor}_n(A)\) such that \(f_i - g_i\) factors through an injective \(A\)-module, \(1 \leq i \leq n - 1\). If each \(X_i\) has no nonzero injective direct summands, then \(\text{Mimo} X_{(f_i)} \cong \text{Mimo} X_{(g_i)}\) in \(S_n(A)\).

**Proof.** Consider \(X'_{(f'_i)}\) and \(X'_{(g'_i)}\) defined in Lemma 5.2 which are isomorphic in \(S_n(A)\). By Lemma 5.1 there exist injective \(A\)-modules \(Q_{f,i} \oplus Q_{f,i} \cong I_{i+1} \cong \text{IKer} g_i \oplus Q_{g,i}\), \(1 \leq i \leq n - 1\), and

\[
\text{Mimo} X_{(f_i)} \oplus \bigoplus_{i=1}^{n-1} m_i(Q_{f,i}) \cong X'_{(f'_i)} \cong X'_{(g'_i)} \cong \text{Mimo} X_{(g_i)} \oplus \bigoplus_{i=1}^{n-1} m_i(Q_{g,i}).
\]
By Claim 3 in §4 of [RS2], we have IKer$f_i$ ≅ IKer$g_i$, 1 ≤ i ≤ n − 1. Thus $Q_{f,i}$ ≅ $Q_{g,i}$, and
\[ \bigoplus_{i=1}^{n-1} m_i(Q_{f,i}) \cong \bigoplus_{i=1}^{n-1} m_i(Q_{g,i}) , \]
from which the assertion follows. \[ \square \]

Now we can prove Lemma 1.6 (cf. Theorem 4.2 of [RS2] for the case of n = 2).

**Proof of Lemma 1.6** We only prove (i). If $\text{Mimo}X_{(\phi_i)} \cong \text{Mimo}Y_{(\psi_i)}$ in $S_n(A)$, then by the construction of $\text{Mimo}$, $X_{(\phi_i)} \cong Y_{(\psi_i)}$ in $\text{Mor}_n(A\text{-mod})$. Conversely, assume that $X_{(\phi_i)} \cong Y_{(\psi_i)}$ in $\text{Mor}_n(A\text{-mod})$. Since all $X_i$ and $Y_i$ have no nonzero injective direct summands, there are $A$-isomorphisms $x_i : X_i \to Y_i$, such that each $x_i\phi_i - \psi_i x_{i+1}$ factors through an injective $A$-module, 1 ≤ i ≤ n − 1. Then each $\phi_i - x_i^{-1}\psi_i x_{i+1}$ factors through an injective $A$-module, 1 ≤ i ≤ n − 1. By Lemma 5.3, $\text{Mimo}X_{(x_i^{-1}\psi_i x_{i+1})} \cong \text{Mimo}X_{(\phi_i)}$. Since $Y_{(\psi_i)} \cong X_{(x_i^{-1}\psi_i x_{i+1})}$, we get $\text{Mimo}Y_{(\psi_i)} \cong \text{Mimo}X_{(\phi_i)}$. \[ \square \]

6. **Appendix 2: Auslander-Reiten quivers of some monomorphism categories**

We include the Auslander-Reiten quivers of some representation-finite monomorphism categories.

6.1. By Simson [S2], $S_{n,2}$, $S_{2,3}$, $S_{2,4}$, $S_{2,5}$, $S_{3,3}$ and $S_{4,3}$ are the only representation-finite cases among all $S_{n,t} = S_n(k[x]/(x^t))$, n ≥ 2, t ≥ 2. In [RS3], Ringel and Schmidmeier give the Auslander-Reiten quivers of $S_{2,t}$, t = 2, 3, 4, 5. In the following we give the remaining cases, namely, $S_{n,2}$ (n ≥ 3), $S_{3,3}$, and $S_{4,3}$.

(i) There are n indecomposable projective objects and $\frac{n(n+1)}{2}$ indecomposable nonprojective objects in $S_{n,2}$. For the Auslander-Reiten quivers of $S_{3,2}$ see 2.5. The Auslander-Reiten quiver of $S_{4,2}$ is as follows, where $A = k[x]/(x^2)$ and $S$ is the simple $A$-module.

(ii) Let $A = k[x]/(x^3)$. Denote by $M$ and $S$ the two indecomposable nonprojective $A$-modules, where $S$ is simple. There are 3 indecomposable projective objects and 24
indecomposable nonprojective objects in $\mathcal{S}_{3,3}$, whose Auslander-Reiten quiver is as follows.

(iii) Let $A, M, S$ be as in (ii). There are 4 indecomposable projective objects and 80 indecomposable nonprojective objects in $\mathcal{S}_{4,3}$, whose Auslander-Reiten quiver looks like.

6.2. Consider the selfinjective Nakayama algebra $\Lambda = \Lambda(2,2)$, whose indecomposable modules are denoted by $S_1 = k \begin{array}{c} 1 \\ 0 \end{array}$, $S_2 = 0 \begin{array}{c} 1 \\ k \end{array}$, $P_1 = k \begin{array}{c} 1 \\ 0 \\ k \end{array}$, $P_2 = k \begin{array}{c} 0 \\ 1 \\ k \end{array}$. 
There are 4 indecomposable projective objects and 6 indecomposable nonprojective objects in $S_2(\Lambda)$, with the Auslander-Reiten quiver as follows.

There are 6 indecomposable projective objects and 12 indecomposable nonprojective objects in $S_3(\Lambda)$, with the Auslander-Reiten quiver as follows.

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**References**

[Ar] D. M. Arnold, Abelian groups and representations of finite partially ordered sets, Canad. Math. Soc. Books in Math., Springer-Verlag, New York, 2000.

[ARS] M. Auslander, I. Reiten, S. O. Smalø, Representation theory of Artin algebras, Cambridge Studies in Adv. Math. 36., Cambridge Univ. Press, 1995.

[AS] M. Auslander, S. O. Smalø, Almost split sequences in subcategories, J. Algebra 69(1981), 426-454.

[B] G. Birkhoff, Subgroups of abelian groups, Proc. Lond. Math. Soc. II, Ser. 38(1934), 385-401.

[C] X. W. Chen, Stable monomorphism category of Frobenius category, available in arXiv: math. RT 0911.1987, 2009.

[CZ] C. Cibils, P. Zhang, Calabi-Yau objects in triangulated categories, Trans. Amer. Math. Soc. 361(2009), 6501-6519.

[H] D. Happel, Triangulated categories in representation theory of finite dimensional algebras, Lond. Math. Soc. Lecture Notes Ser. 119, Cambridge Univ. Press, 1988.

[KLM] D. Kussin, H. Lenzing, H. Meltzer, Nilpotent operators and weighted projective lines, available in arXiv: math. RT 1002.3797.

[RV] I. Reiten, M. Van den Bergh, Noetherian hereditary abelian categories satisfying Serre duality, J. Amer. Math. Soc. 15(2)(2002), 295-366.

[RW] F. Richman, E. A. Walker, Subgroups of $p^n$-bounded groups, in: Abelian groups and modules, Trends Math., Birkhäuser, Basel, 1999, 55-73.
[RS1] C. M. Ringel, M. Schmidmeier, Submodules categories of wild representation type, J. Pure Appl. Algebra 205(2)(2006), 412-422.

[RS2] C. M. Ringel, M. Schmidmeier, The Auslander-Reiten translation in submodule categories, Trans. Amer. Math. Soc. 360(2)(2008), 691-716.

[RS3] C. M. Ringel, M. Schmidmeier, Invariant subspaces of nilpotent operators I, J. rein angew. Math. 614(2008), 1-52.

[S] D. Simson, Representation types of the category of subprojective representations of a finite poset over $K[t]/(t^n)$ and a solution of a Birkhoff type problem, J. Algebra 311(2007), 1-30.

[SW] D. Simson, M. Wojewodzki, An algorithmic solution of a Birkhoff type problem, Fundamenta Informaticae 83(2008), 389-410.

[Z] P. Zhang, Monomorphism categories, cotilting theory, and Gorenstein-projective modules, preprint (2009). Available in arXiv: math. RT 1101.3872, 2011.

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