Some Applications of Group Theoretic Rips Constructions to the Classification of von Neumann Algebras

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Abstract

In this paper we study various von Neumann algebraic rigidity aspects for the property (T) groups that arise via the Rips construction developed by Belegradek and Osin in geometric group theory [BO06]. Specifically, developing a new interplay between Popa's deformation/rigidity theory [Po07] and geometric group theory methods we show that several algebraic features of these groups are completely recognizable from the von Neumann algebraic structure. In particular, we obtain new infinite families of pairwise non-isomorphic property (T) group factors thereby providing positive evidence towards Connes’ Rigidity Conjecture.

In addition, we use the Rips construction to build examples of property (T) II$_1$ factors which possess maximal von Neumann subalgebras without property (T) which answers a question raised in an earlier version of [JS19] by Y. Jiang and A. Skalski.

1 Introduction

The von Neumann algebra $L(G)$ associated to a countable discrete group $G$ is called the group von Neumann algebra and it is defined as the bicommutant of the left regular representation of $G$ computed inside the algebra of all bounded linear operators on the Hilbert space of the square summable functions on $G$. $L(G)$ is a II$_1$ factor (has trivial center) precisely when all nontrivial conjugacy classes of $G$ are infinite (icc), this being the most interesting for study [MvN43]. The classification of group factors is a central research theme revolving around the following fundamental question: What aspects of the group $G$ are remembered by $L(G)$? This is a difficult topic as algebraic group properties usually do not survive after passage to the von Neumann algebra regime. Perhaps the best illustration of this phenomenon is Connes’ celebrated result asserting that all amenable icc groups give isomorphic factors, [Co76]. Hence genuinely different groups such as the group of all finite permutations of the positive integers, the lamplighter group, or the wreath product of the integers with itself give rise to isomorphic factors. Ergo, basic algebraic group constructions such as direct products, semidirect products, extensions, inductive limits or classical algebraic invariants such as torsion, rank, or generators and relations in general cannot be recognized from the von Neumann algebraic structure. In this case the only information on $G$ retained by the von Neumann algebra is amenability.

When $G$ is non-amenable the situation is far more complex and an unprecedented progress has been achieved through the emergence of Popa’s deformation/rigidity theory [Po07, Va10b, Io12, Io18]. Using this completely new conceptual framework it was shown that various algebraic/analytic properties of groups and their representations can be completely recovered from their von Neumann algebras, [OP03, OP07, IPV10, BV12, CdSS15, DHI16, CI17, CU18]. In this direction an impressive milestone was Ioana, Popa and Vaes’s discovery of the first examples of groups $G$ that can be completely

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reconstructed from $\mathcal{L}(G)$, i.e. $W^*$-superrigid groups. Additional examples were found subsequently in [PV10]. It is also worth noting that the general strategies used in establishing these results share a common essential ingredient—the ability to first reconstruct from $\mathcal{L}(G)$ specific given algebraic features of $G$. For instance, in the examples covered in [PV10, BV12, B13], the first step was to show that whenever $\mathcal{L}(G) \cong \mathcal{L}(H)$ then the mystery group $H$ admits a generalized wreath product decomposition exactly as $G$ does; also in the case of [CI17] Theorem A again the main step was to show that $H$ admits an amalgamated free product splitting exactly as $G$. These aspects motivate a fairly broad and independent study on this topic—the quest of identifying a comprehensive list of algebraic features of groups which completely pass to the von Neumann algebraic structure. While a couple of works have already appeared in this direction [CdSS15, CI17, CU18] we are still far away from having a satisfactory overview of these properties and a great deal of work remains to be done.

A striking conjecture of Connes predicts that all icc property (T) groups are $W^*$-superrigid. Despite the fact that this conjecture motivated to great effect a significant portion of the main developments in Popa’s deformation/rigidity theory [Po03, Po04, Io11, IPV10], no example of a property (T) $W^*$-superrigid group is currently known. The first hard evidence towards Connes’ conjecture was found by Cowling and Haagerup in [CH89], where it was shown that uniform lattices in $Sp(n,1)$ give rise to non-isomorphic factors for different values of $n \geq 2$. Moreover, for any collection $\{G_n\}_n$ of uniform lattices in $Sp(n,1)$, $n \geq 2$, the group algebras $\{\mathcal{L}(\times_{i=1}^n G_i)\}_n$ are pairwise non-isomorphic. Later on, using a completely different approach, Ozawa and Popa [OP03] obtained a far-reaching generalization of this result by showing that for any collection $\{G_n\}_n$ of hyperbolic, property (T) groups (e.g. uniform lattices in $Sp(n,1)$, $n \geq 2$, [CH89] the group algebras $\{\mathcal{L}(\times_{i=1}^n G_i)\}_n$ are pairwise non-isomorphic. However, little is known beyond these two classes of examples. Moreover, the current literature offers an extremely limited account on what algebraic features that occur in a property (T) group are completely recognizable at the von Neumann algebraic level. For instance, besides the preservation of the Cowling-Haagerup constant [CH89], the amenability of normalizers of infinite amenable subgroups in property (T) groups is currently known. The first hard evidence towards Connes’ conjecture was found in Popa’s deformation/rigidity theory [Po03, Po04, Io11, IPV10], no example of a property (T) group that survive the passage to the von Neumann algebraic regime. Any success in this direction will potentially hint at what group theoretic methods to pursue in order to address Connes’ conjecture.

In this paper we make new progress on this study by showing that many algebraic aspects of the Rips constructions developed in geometric group theory by Belegradek and Osin [BO06] are entirely recoverable from the von Neumann algebraic structure. To properly introduce the result we briefly describe their construction. Using the prior Dehn filling results from [Osi06], Belegradek and Osin showed in [BO06] Theorem] that for every finitely generated group $Q$ one can find a property (T) group $N$ such that $Q$ can be realized as a finite index subgroup of $Out(N)$. This canonically gives rise to an action $Q \rtimes \sigma N$ by automorphisms such that the corresponding semidirect product group $N \rtimes \sigma Q$ is hyperbolic relative to $\{Q\}$. Throughout the document the semidirect products $N \rtimes \sigma Q$ will be termed Belegradek-Osin’s group Rips constructions. When $Q$ is torsion free then one can pick $N$ to be torsion free as well and hence both $N$ and $N \rtimes \sigma Q$ are icc groups. Also when $Q$ has property (T) then $N \rtimes \sigma Q$ has property (T). Under all these assumptions we will denote by $Rip_T(Q)$ the class of these Rips construction groups $N \rtimes \sigma Q$.

The first main result of our paper concerns a fairly large class of canonical fiber products of groups in $Rip_T(Q)$. Specifically, consider any two groups $N_1 \rtimes \sigma_1 Q, N_2 \rtimes \sigma_2 Q \in Rip_T(Q)$ and form the canonical fiber product $G = (N_1 \times N_2) \rtimes \sigma Q$ where $\sigma = (\sigma_1, \sigma_2)$ is the diagonal action. Notice that since property (T) is closed under extensions [BDHV00] Section 1.7] it follows that $G$ has property (T). Developing new interplay between geometric group theoretic methods [Rip82, DG01, Osi06, BO06] and deformation/rigidity methods [Io11, IPV10, CdSS15, CdSS17, CI17, CU18], for a fairly large family of groups
we show that the semidirect product feature of $G$ is an algebraic property completely recoverable from the von Neumann algebraic regime. In addition, we also have a complete reconstruction of the acting group $Q$. The precise statement is the following

**Theorem A** (Theorem 5.1). Let $Q = Q_1 \times Q_2$, where $Q_i$ are icc, biexact, weakly amenable, property (T), torsion free, residually finite groups. For $i = 1, 2$ let $N_i \rtimes \tau_i Q_i \in \mathcal{R}(Q)$ and denote by $\Gamma = (N_1 \times N_2) \rtimes \tau Q$ the semidirect product associated with the diagonal action $\sigma = \sigma_1 \times \sigma_2 : Q \sim N_1 \times N_2$. Denote by $\mathcal{M} = \mathcal{L}(\Gamma)$ be the corresponding II$_1$ factor. Assume that $\Lambda$ is any arbitrary group and $\Theta : \mathcal{L}(\Gamma) \to \mathcal{L}(\Lambda)$ is any $*$-iso- morphism. Then there exist group actions by automorphisms $H \sim \Lambda Q_i$ such that $\Lambda = (K_1 \times K_2) \rtimes_{\tau} H$ where $\tau = \tau_1 \times \tau_2 : H \sim K_1 \times K_2$ is the diagonal action. Moreover one can find a multiplicative character $\eta : Q \to \mathbb{T}$, a group isomorphism $\delta : Q \to H$ and unitary $w \in \mathcal{L}(\Lambda)$ and $*$-isomorphisms $\Theta_i : \mathcal{L}(N_i) \to \mathcal{L}(K_i)$ such that for all $x_i \in \mathcal{L}(N_i)$ and $g \in Q$ we have

$$\Theta((x_1 \otimes x_2)u_g) = \eta(g)w((\Theta_1(x_1) \otimes \Theta_2(x_2))v_{\delta(g)})w^*.$$  

Here $\{u_g : g \in Q\}$ and $\{v_{\eta_i} : h \in H\}$ are the canonical unitaries implementing the actions of $Q \sim \mathcal{L}(N_1) \otimes \mathcal{L}(N_2)$ and $H \sim \mathcal{L}(K_1) \otimes \mathcal{L}(K_2)$, respectively.

There are countably infinitely many groups that are residually finite, torsion free, hyperbolic, and have property (T). A concrete such family is $\{\Lambda_k \mid k \geq 2\}$ where $\Lambda_k < Sp(k,1)$ is a uniform lattice. It is well known the $\Lambda_k$’s are residually finite [Mal40], (virtually) torsion free [Sel60], hyperbolic [Gr87], Example B], have property (T) (see for instance [BdhHV00] Theorem 1.5.3) and are pairwise non-isomorphic [CH89]. However there are infinitely many pairwise non-isomorphic such lattices even in the same Lie group. To see this fix $k \geq 2$ together with a torsion free, uniform lattice $\Gamma < Sp(k,1)$. Since $\Gamma$ is residually finite there is a sequence of normal, finite index, proper subgroups $\ldots \triangleleft \Gamma_{n+1} \triangleleft \Gamma_n \triangleleft \ldots \triangleleft \Gamma_1 \triangleleft \Gamma$ such that $\cap_n \Gamma_n = 1$. Being subgroups, $\Gamma_n$ are clearly residually finite and torsion free. Moreover, the finite index condition implies that all $\Gamma_n$’s are hyperbolic and have property (T). As the $\Gamma_n$’s are co-hopfian [Lr76] and for every $n \neq m$ we have $\Gamma_n \neq \Gamma_m$ then $\Gamma_n \neq \Gamma_m$. Therefore the class $\{\Gamma_n \mid n \in \mathbb{N}\}$ satifies our conditions. Finally we note that, since every hyperbolic group is finitely presented and there are only countably many such groups, one cannot built examples of larger cardinality than the ones presented above.

In conclusion, Theorem A provides explicit examples of infinitely many pairwise non-isomorphic group II$_1$ factors with property (T). Moreover these groups are quite from the previously classes [CH89, OP03] as they give rise to factors that are non-solid ($\mathcal{L}(\Gamma)$ contains two commuting non-a- menable subfactors $\mathcal{L}(N_1)$ and $\mathcal{L}(N_2)$), are tensor indecomposable (ID20, Lemma 2.3) and do not admit Cartan subalgebras (Corollary 7.2). Moreover, using Margulis normal subgroup theorem the factors covered by Theorem A are non-isomorphic to any factor arising from any irreducible lattices in a higher-rank semisimple Lie group (see remarks after the proof of Theorem 5.1). We also mention that Theorem A or its strong rigidity version Theorem 6.1 (see also Corollary 6.2) below) provides examples of infinite families of finite index subgroups $\Gamma_n \triangleleft \Gamma$ in a given icc property (T) group $\Gamma$ such that the corresponding group factors $\mathcal{L}(\Gamma_n)$ and $\mathcal{L}(\Gamma_m)$ are nonisomorphic for $n \neq m$. As $\Gamma_n$’s are measure equivalent this provides new counterexamples to D. Shlyakhtenko’s question, asking whether measure equivalence of icc groups implies isomorphism of the corresponding group factors (see [Po09, Page 18]), which are very different in nature from the ones obtained in [CI09, CdSS15]. We summarize this discussion in the next corollary.

**Corollary B** (Corollary 6.2 below). Assume the same notations as in Theorem A.

1) Let $Q_1, Q_2$ be uniform lattices in $Sp(n,1)$ with $n \geq 2$ and let $Q = Q_1 \times Q_2$. Also let $\ldots \triangleleft Q_1^\vee \triangleleft \ldots \triangleleft Q_2^\vee \triangleleft Q^\vee$ be an infinite family of finite index subgroups and denote by $Q_\delta = Q_1^\vee \times Q_2$. Then consider $N_1 \rtimes_{\tau_1} Q_1, N_2 \rtimes_{\tau_2} Q_2 \in \mathcal{R}(Q)$ and let $\Gamma = (N_1 \times N_2) \rtimes_{\tau_1 \times \tau_2} Q$. Inside $\Gamma$ consider the finite index subgroups $\Gamma_\delta = (N_1 \times N_2) \rtimes_{\tau_1 \times \tau_2} Q_\delta$. Then the family $\{\mathcal{L}(\Gamma) \mid s \in I\}$ consists of pairwise non-isomorphic finite index subfactors of $\mathcal{L}(\Gamma)$.
2) Let $\Gamma, \Gamma_n$ be as above. Then $\Gamma_n$ is measure equivalent to $\Gamma$ for all $n \in \mathbb{N}$, but $\mathcal{L}(\Gamma_n)$ is not isomorphic to $\mathcal{L}(\Gamma_m)$ for $n \neq m$.

From a different perspective our theorem can be also seen as a von Neumann algebraic superrigidity result regarding conjugacy of actions on noncommutative von Neumann algebras. Notice that very little is known in this direction as most of the known superrigidity results concern algebras arising from actions of groups on probability spaces.

In certain ways one can view Theorem A as a first step towards providing an example of a property (T) superrigid group. While the acting group $Q$ can be completely recovered, as well as certain aspects of the action $Q \sim N_1 \times N_2$ (e.g. trivial stabilizers) only the product feature of the "core" $\mathcal{L}(N_1 \times N_2)$ can be reconstructed at this point. While the reconstruction of $N_1$ and $N_2$ seems to be out of reach momentarily, we believe that a deeper understanding of the Rips construction, along with new von Neumann algebraic techniques are necessary to tackle this problem. We also remark that in a subsequent article it was shown that the group factors as in Theorem A have trivial fundamental group (see CDHK20 Theorem B).

Besides the aforementioned rigidity results we also investigate applications of group Rips constructions to the study of maximal von Neumann algebras. If $\mathcal{M}$ is a von Neumann algebra then a von Neumann subalgebra $\mathcal{N} \subset \mathcal{M}$ is called maximal if there is no intermediate von Neumann subalgebra $\mathcal{P}$ so that $\mathcal{N} \subset \mathcal{P} \subset \mathcal{M}$. Understanding the structure of maximal subalgebras of a given von Neumann algebra is a rather difficult problem that is intimately related with the very classification of these objects. Despite a series of remarkable earlier successes on the study of maximal amenable subalgebras initiated by Popa [Po83] and continued more recently [Sh06, CFRW08, H014, BC14, BC15, Su18, CD19, JS19], significantly less is known for the arbitrary maximal ones. For instance Ge’s question [Ge03, Section 3, Question 2] on the existence of non-amenable factors that posses maximal factors that are hyperfinite was settled in the affirmative only very recently by Y. Jiang and A. Skalski in [JS19]. In fact in their work Jiang-Skalski proposed a more systematic approach towards the study of maximal von Neumann subalgebras within various categories such as the von Neumann algebras with Haagerup’s property or with property (T) of Kazhdan. Their investigation also naturally led to several interesting open problems, [JS19, Section 5].

In this paper we explain how in a setting similar with [JS19] the Belgradek-Osin’s group Rips construction techniques and Ol’shanski’s type monster groups can be used in conjunction with Galois correspondence results for II$_1$ factors à la Choda [Ch78] to produce many maximal von Neumann subalgebras arising from group/subgroup situation. In particular, through this mix of results we are able to construct many examples of II$_1$ factors with property (T) that have maximal von Neumann subalgebras without property (T), thereby answering Problem 5.5 in the first version of the paper [JS19] (see Theorem 4.4). More specifically, using the Ol’shanski’s small cancellation techniques in the setting of lacunary hyperbolic groups [OO97] we explain how one can construct a property (T) monster group $Q$ whose maximal subgroups are all isomorphic to a given rank one group $Q_m$ (see Section 2.3). Then if one considers the Belgradek-Osin Rips construction $N \times Q$ corresponding to $Q$ then using a Galois correspondence (Theorem 4.2) one can show the following

**Theorem C.** (Theorem 4.4) For every maximal rank one subgroup $Q_m < Q$ consider the subgroup $N \times Q_m < N \times Q$. Then $\mathcal{L}(N \times Q_m) \subset \mathcal{L}(N \times Q)$ is a maximal von Neumann subalgebra.

Note that since $N$ and $Q$ have property (T) then so does $N \times Q$ and therefore the corresponding II$_1$ factor $\mathcal{L}(N \times Q)$ has property (T) by [CJ85]. However since $N \times Q_m$ surjects onto the infinite abelian group $Q_m$ then it does not have property (T) and hence $\mathcal{L}(N \times Q_m)$ does not have property (T) either. Another solution to the problem of finding maximal subalgebras without property (T) inside factors with property (T) was also obtained independently by Y. Jiang and A. Skalski in the most recent version of their paper. Their beautiful solution has a different flavor from ours; even though the Galois correspondence theorem à la Choda is a common ingredient in both of the proofs. Hence we refer the reader to [JS19, Theorem 4.8] for another solution to the aforementioned problem.

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1any group that is isomorphic to a subgroup of $(\mathbb{Q}, +)$ is called rank one
2 Preliminaries

2.1 Notations and Terminology

We denote by \( \mathbb{N} \) and \( \mathbb{Z} \) the set of natural numbers and the integers, respectively. For any \( k \in \mathbb{N} \) we denote by \( \mathbb{N}_k \) the integers \{1, 2, ..., \( k \)\).

All von Neumann algebras in this document will be denoted by calligraphic letters e.g. \( \mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N} \), etc. Given a von Neumann algebra \( \mathcal{M} \) we will denote by \( \mathcal{U}(\mathcal{M}) \) its unitary group, by \( \mathcal{P}(\mathcal{M}) \) the set of all its nonzero projections, and by \( \mathcal{Z}(\mathcal{M}) \) its center. We also denote by \( (\mathcal{M}), \) its unit ball. All algebras inclusions \( \mathcal{N} \subseteq \mathcal{M} \) are assumed unital unless otherwise specified. Given an inclusion \( \mathcal{N} \subseteq \mathcal{M} \) of von Neumann algebras we denote by \( \mathcal{N}' \cap \mathcal{M} \) the relative commutant of \( \mathcal{N} \) in \( \mathcal{M} \), i.e. the subalgebra of all \( x \in \mathcal{M} \) such that \( xy = yx \) for all \( y \in \mathcal{N} \). We also consider the one-sided quasinormalizer \( \mathcal{Q}_{\mathcal{M}}(\mathcal{N}) \) (the semigroup of all \( x \in \mathcal{M} \) for which there exist \( x_1, x_2, ..., x_n \in \mathcal{M} \) such that \( \mathcal{N} x \subseteq \sum_i x_i \mathcal{N} \)) and the quasinormalizer \( \mathcal{Q}_{\mathcal{M}}(\mathcal{N}) \) (the set of all \( x \in \mathcal{M} \) for which there exist \( x_1, x_2, ..., x_n \in \mathcal{M} \) such that \( \mathcal{N} x \subseteq \sum_i x_i \mathcal{N} \) and \( x \mathcal{N} \subseteq \sum_i \mathcal{N} x_i \)) and we notice that \( \mathcal{N} \subseteq \mathcal{Q}_{\mathcal{M}}(\mathcal{N}) \subseteq \mathcal{Q}_{\mathcal{M}}(\mathcal{N}) \subseteq \mathcal{Q}_{\mathcal{M}}(\mathcal{N}) \).

All von Neumann algebras considered in this article will be tracial, i.e. endowed with a unital, faithful, normal linear functional \( \tau : \mathcal{M} \to \mathcal{C} \) satisfying \( \tau(xy) = \tau(yx) \) for all \( x, y \in \mathcal{M} \). This induces a norm on \( \mathcal{M} \) by the formula \( |x|_2 = \tau(x^* x)^{1/2} \) for all \( x \in \mathcal{M} \). The \( \| \cdot \|_2 \)-completion of \( \mathcal{M} \) will be denoted by \( L^2(\mathcal{M}) \). For any von Neumann subalgebra \( \mathcal{N} \subseteq \mathcal{M} \) we denote by \( E_\mathcal{N} : \mathcal{M} \to \mathcal{N} \) the \( \tau \)-preserving conditional expectation onto \( \mathcal{N} \).

For a countable group \( G \) we denote by \( \{ u_g | g \in \mathcal{G} \} \in \mathcal{U}(L^2(\mathcal{G})) \) its left regular representation given by \( u_g(\delta_h) = \delta_{gh} \), where \( \delta_h : G \to \mathcal{C} \) is the Dirac function at \( \{h\} \). The weak operatorial closure of the left regular representation of \( \mathcal{G} \) in \( \mathcal{U}(L^2(\mathcal{G})) \) is the group von Neumann algebra and will be denoted by \( \mathcal{L}(\mathcal{G}) \). \( \mathcal{L}(\mathcal{G}) \) is a II_1 factor precisely when \( G \) has infinite non-trivial conjugacy classes (icc). If \( \mathcal{M} \) is a tracial von Neumann algebra and \( G \acts \mathcal{M} \) is a trace preserving action we denote by \( \mathcal{M} \times G \) the corresponding cross product von Neumann algebra [VN37]. For any subset \( K \subseteq \mathcal{G} \) we denote by \( P_{\mathcal{M}K} \) the orthogonal projection from \( L^2(\mathcal{M} \times \mathcal{G}) \) onto the closed linear span of \( \{u_g x | x \in \mathcal{M}, g \in K \} \). When \( \mathcal{M} \) is trivial we will denote this simply by \( P_K \).

Finally, for any groups \( G \) and \( \mathcal{N} \) and an action \( G \acts \mathcal{N} \) we denote by \( \mathcal{N} \rtimes G \) the corresponding semidirect product group.

2.2 Popa’s Intertwining Techniques

Over more than fifteen years ago, Sorin Popa has introduced in [Po03] Theorem 2.1 and Corollary 2.3 a powerful analytic criterion for identifying intertwiners between arbitrary subalgebras of tracial von Neumann algebras. Now this is known in the literature as Popa’s intertwining-by-bimodules technique and has played a key role in the classification of von Neumann algebras program via Popa’s deformation/rigidity theory.

**Theorem 2.1.** [Po03] Let \( (\mathcal{M}, \tau) \) be a separable tracial von Neumann algebra and let \( \mathcal{P}, \mathcal{Q} \subseteq \mathcal{M} \) be (not necessarily unital) von Neumann subalgebras. Then the following are equivalent:
1. There exists \( p \in \mathcal{P}(\mathcal{P}), q \in \mathcal{P}(\mathcal{Q}), a \ast\)-homomorphism \( \theta : p\mathcal{P}p \to q\mathcal{Q}q \) and a partial isometry \( 0 \neq v \in q\mathcal{M}p \) such that \( \theta(x)v = vx \), for all \( x \in p\mathcal{P}p \).

2. For any group \( G \subseteq \mathcal{U}(\mathcal{P}) \) such that \( \mathcal{G}'' = \mathcal{P} \) there is no sequence \( (u_n)_n \subset G \) satisfying \( \|E_{\mathcal{Q}}(xu_ny)\|_2 \to 0 \), for all \( x, y \in \mathcal{M} \).

3. There exists finitely many \( x, y \in \mathcal{M} \) and \( C > 0 \) such that \( \sum_i \|E_{\mathcal{Q}}(x_iu_y)\|_2^2 \geq C \) for all \( u \in \mathcal{U}(\mathcal{P}) \).

If one of the three equivalent conditions from Theorem 2.1 holds then we say that \( \mathcal{P} \) embeds into \( \mathcal{Q} \) inside \( \mathcal{M} \), and write \( \mathcal{P} \prec_\mathcal{M} \mathcal{Q} \). If \( \mathcal{P} \) moreover have that \( \mathcal{P} \mathcal{P}' \prec_\mathcal{M} \mathcal{Q} \), for any projection \( 0 \neq p' \in \mathcal{P}' \cap 1_p\mathcal{M}1_p \) (equivalently, for any projection \( 0 \neq p' \in \mathcal{P}' \cap 1_p\mathcal{M}1_p \)), then we write \( \mathcal{P} \prec_\mathcal{M}^{1_p} \mathcal{Q} \).

For further use we record the following result which controls the intertwiners in algebras arising from normal subgroups. Its proof is essentially contained in [Po03] Theorem 3.1 so it will be left to the reader.

**Lemma 2.2 (Popa [Po03])**. Assume that \( H \leq G \) is an almost malnormal subgroup and let \( G \to \mathcal{N} \) be a trace preserving action on a tracial von Neumann algebra \( \mathcal{N} \). Let \( \mathcal{P} \subset \mathcal{N} \times H \) be a von Neumann algebra such that \( \mathcal{P} \prec_\mathcal{M}^\ast \mathcal{N} \times H \). Then for all elements \( x, x_1, x_2, \ldots, x_l \in \mathcal{D} \times G \) satisfying \( \mathcal{P}x \subseteq \sum_{i=1}^{l} x_i \mathcal{P} \) we must have that \( x \in \mathcal{N} \times H \).

We continue with the following intertwining result for group algebras which is a generalization of some previous results obtained under normality assumptions [DH16]. For reader’s convenience we also include a brief proof.

**Lemma 2.3**. Assume that \( H_1, H_2 \leq G \) are groups, let \( G \to \mathcal{N} \) be a trace preserving action on a tracial von Neumann algebra \( \mathcal{N} \) and denote by \( \mathcal{M} = \mathcal{N} \rtimes G \) the corresponding crossed product. Also assume that \( \mathcal{A} \prec_\mathcal{M}^\ast \mathcal{N} \times H_1 \) is a von Neumann algebra such that \( \mathcal{A} \prec_\mathcal{M} \mathcal{N} \times H_2 \). Then one can find \( h \in \mathcal{N} \) such that \( \mathcal{A} \prec_\mathcal{M} \mathcal{N} \times (H_1 \rtimes hH_2h^{-1}) \).

**Proof**. Since \( \mathcal{A} \prec_\mathcal{M} \mathcal{N} \times H_1 \) then by [Va10a, Lemma 2.6] for every \( \varepsilon > 0 \) there exists a finite subset \( S \subseteq G \) such that \( \|P_{SH_1}(x) - x\|_2 \leq \varepsilon \) for all \( x \in (\mathcal{A})_1 \). Here for every \( K \subseteq G \) we denote by \( P_K \) the orthogonal projection from \( L^2(\mathcal{M}) \) onto the closure of the linear span of \( \mathcal{N}u_S^g \) with \( g \in K \). Also since \( \mathcal{A} \prec_\mathcal{M} \mathcal{N} \times H_2 \) then by Popa’s intertwining techniques there exist a scalar \( 0 < \delta < 1 \) and a finite subset \( T \subseteq G \) so that \( \|P_{TH_2}(x)\|_2 \geq \delta \), for all \( x \in (\mathcal{A})_1 \). Thus, using this in combination with the previous inequality, for every \( x \in \mathcal{U}(\mathcal{A}) \) and every \( \varepsilon > 0 \), there are finite subsets \( S, T \subseteq G \) so that \( \|P_{TH_2} \circ P_{SH_1}(x)\|_2 \geq \delta - \varepsilon \). Since there exist finite subsets \( \mathcal{R}, S \subseteq G \) such that \( TH_2 \cap SH_1 \subseteq \bigcup_{r \in R} H_2 \cap rH_1^{-1} \) we further get that \( \|P_{\bigcup_{r \in R} H_2 \cap rH_1^{-1}}(x)\|_2 \geq \delta - \varepsilon \). Then choosing \( \varepsilon > 0 \) sufficiently small and using Popa’s intertwining techniques together with a diagonalization argument (see proof of [PP05, Theorem 4.3]) one can find \( r \in R \) so that \( \mathcal{A} \prec_\mathcal{M} \mathcal{N} \times (H_2 \rtimes rH_1^{-1}) \), as desired.

In the sequel we need the following three intertwining lemmas, which establish that under certain conditions, intertwining in a larger algebra implies that the intertwining happens in a “smaller subalgebra”.

**Lemma 2.4**. Let \( A, B \subseteq \mathcal{N} \subseteq \mathcal{M} \) be von Neumann algebras so that \( \mathcal{N}(\mathcal{M})'' = \mathcal{M} \). If \( B \prec_\mathcal{M} A \) then \( B \prec_\mathcal{N} A \).

**Proof**. Since \( B \prec_\mathcal{M} A \) then by Theorem 2.1 one can find \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in \mathcal{M} \) and \( c > 0 \) such that \( \sum_{i=1}^n \|E_A(x_iy_i)\|_2^2 \geq c \), for all \( b \in \mathcal{U}(B) \). Since \( \mathcal{N}(\mathcal{M})'' = \mathcal{M} \) then using basic \( \| \cdot \|_2 \)-approximation for \( x_1 \) and \( y_1 \) and shrinking \( c > 0 \) if necessary one can find \( g_1, g_2, \ldots, g_i, h_1, h_2, \ldots, h_i \in \mathcal{N}(\mathcal{M}) \) and \( c' > 0 \) such that for all \( b \in \mathcal{U}(B) \) we have

\[
\sum_{i=1}^n \|E_A(g_ih_i)\|_2^2 \geq c' > 0.
\] (2.2.1)

Using normalization we see that \( E_A(g_1h_1) = E_{\mathcal{N}}(g_1h_1) = g_1E_A(h_1g_1)g_1^* \). This combined with (2.2.1) and \( A \subseteq \mathcal{N} \) give \( 0 < c' \leq \sum_{i=1}^n \|E_A(b_ih_i)\|_2^2 = \sum_{i=1}^n \|E_A \circ E_{\mathcal{N}}(h_i)g_i\|_2^2 \geq \sum_{i=1}^n \|E_A(bE_{\mathcal{N}}(h_i)g_i)\|_2^2 \) for all \( b \in \mathcal{U}(B) \). Since \( E_{\mathcal{N}}(h_i) \in \mathcal{N} \) then using Theorem 2.1 this clearly shows that \( B \prec_\mathcal{N} A \).

\[ \text{6} \]
Lemma 2.5. Let $Q$ be a group and denote by $d(Q) = \{(q, q') | q \in Q\}$. Let $A$ be a tracial von Neumann algebra and assume $(Q \times Q) \cong^\sigma A$ is a trace preserving action. Let $B \subseteq A$ be a regular von Neumann subalgebra which is invariant under the action $\sigma$. Let $D \subseteq A \rtimes_r d(Q)$ be a subalgebra such that $D \subseteq A \rtimes_r (Q \times Q) \rtimes_r d(Q)$. Then $D \subseteq A \rtimes_r d(Q) \rtimes_r d(Q)$.

Proof. Denote by $M := A \rtimes_r (Q \times Q), N := A \rtimes_r d(Q)$, and $P := B \rtimes_r d(Q)$. Thus $P \subseteq N \subseteq M$ and with these notations we establish the following

Claim 1. Let $(v_n) \subseteq \mathcal{W}(N)$ be a sequence such that $\lim_{n \to \infty} \|E_P(av_n b)\|_2 = 0$ for all $a, b \in P$. Then

$$\lim_{n \to \infty} \|E_P(xv_n y)\|_2 = 0 \text{ for all } x, y \in M. \quad (2.2.2)$$

Proof of Claim 1. Notice that $(Q \times Q) = (Q \times 1) \times_{\rho_i} d(Q)$, where $d(Q) \cong^\rho (Q \times 1)$ is the action by conjugation. Therefore using basic $\| \cdot \|_2$-approximations and the $P$-bimodularity of the conditional expectation $E_P$ it suffices to show $(2.2.2)$ only for $x = (u_\bar{g} \otimes 1)c$ and $y = d(u_h \otimes 1)$ for all $g, h \in Q$ and $c, d \in A$. Under these assumptions we see that

$$E_P((u_\bar{g} \otimes 1)cv_n d(u_h \otimes 1)) = E_P \circ P_{(u_\bar{g} \otimes 1)M(u_\bar{g} \otimes 1)}((u_\bar{g} \otimes 1)cv_n d(u_h \otimes 1))
= P_{B(d(Q) \cap (g, 1)d(Q)(h, 1))}((u_\bar{g} \otimes 1)cv_n d(u_h \otimes 1)). \quad (2.2.3)$$

Here, and throughout the proof for every set $S \subseteq Q \times Q$ we denoted by $P_{BS}$ the orthogonal projection onto the closed subspace $\overline{\text{span}}\{Bu_g | g \in S\}$.

To this end observe there exists an element $s \in S$ such that $d(Q) \cap (g, 1)d(Q)(h, 1) \subseteq [d(Q) \cap (g, 1)d(Q)(g^{-1}, 1)]d(s)$. Moreover, a basic computation shows that $d(Q) \cap (g, 1)d(Q)(g^{-1}, 1) = d(C_Q(g))$, where $C_Q(g)$ is the centralizer of $g$ in $Q$. Hence altogether we have that $d(Q) \cap (g, 1)d(Q)(h, 1) \subseteq d(C_Q(g))d(s)$. Combining this with $(2.2.3)$ and using the fact that $u_\bar{g} \otimes 1$ normalizes $B \times d(C_Q(g))$ we see that

$$\|E_P((u_\bar{g} \otimes 1)cv_n d(u_h \otimes 1))\|_2 \leq \|P_B(d(C_Q(g))d(s))((u_\bar{g} \otimes 1)cv_n d(u_h \otimes 1))\|_2
= \|E_{Bd(C_Q(g))}((u_\bar{g} \otimes 1)cv_n d(u_{hs^{-1}} \otimes u_{s^{-1}}))\|_2
= \|E_{Bd(C_Q(g))}(cv_n d(u_{hs^{-1}} \otimes u_{s^{-1}}))\|_2
= \|E_{Bd(C_Q(g))}(cv_n dE_N(u_{hs^{-1}} \otimes u_{s^{-1}}))\|_2
= \|E_{Bd(C_Q(g))}(cv_n d)\|_2
\leq \|E_P(cv_n d)\|_2. \quad (2.2.4)$$

Letting $n \to \infty$ in $(2.2.4)$ and using the hypothesis assumption, the claim obtains.

To show our lemma assume by contradiction that $D \not\subseteq_N P$. By Theorem 2.1 there is a sequence $(v_n) \subseteq D \subseteq N$ so that $\lim_{n \to \infty} \|E_P(av_n b)\|_2 = 0$ for all $a, b \in N$. Using Claim 1 we get $\lim_{n \to \infty} \|E_P(xv_n y)\|_2 = 0$ for all $x, y \in M$ which by Theorem 2.1 again implies that $D \not\subseteq_N P$, a contradiction.

Lemma 2.6. Let $C \subseteq B$ and $N \subseteq M$ be inclusions of tracial von Neumann algebras. If $A \subseteq N \otimes B$ is a von Neumann subalgebra such that $A \otimes_{\text{alg}} B \subseteq C$ then $A \otimes_{\text{alg}} (N \otimes B) \subseteq C$.

Proof. By Theorem 2.1 one can find $x_i, y_i \in M \otimes B, i = 1, k$ and a scalar $c > 0$ such that

$$\sum_{i=1}^n \|E_{M \otimes C}(x_i y_i)\|_2 \geq c \text{ for all } d \in \mathcal{U}(A). \quad (2.2.5)$$

Using $\| \cdot \|_2$-approximations of $x_i$ and $y_i$ by finite linear combinations of elements in $M \otimes 1 \otimes B$ together with the $M \otimes 1$-bimodularity of $E_{M \otimes C}$ after increasing $k$ and shrinking $c > 0$ if necessary, in $(2.2.5)$ we can assume wlog that $x_i, y_i \in 1 \otimes B$. However, since $A \subseteq N \otimes B$ then in this situation we have $E_{M \otimes C}(x_i y_i) = E_{M \otimes C} \circ E_{N \otimes B}(x_i y_i) = E_{N \otimes B}(x_i y_i)$. Thus $(2.2.5)$ combined with Theorem 2.1 give $A \otimes_{N \otimes B} (N \otimes C)$, as desired.
In the sequel we need the following (minimal) technical variation of [CI17, Lemma 2.6]. The proof is essentially the same with the one presented in [CI17] and we leave the details to the reader.

**Lemma 2.7** (Lemma 2.6 in [CI17]). Let $\mathcal{P}, \mathcal{Q} \subseteq \mathcal{M}$ be inclusions of tracial von Neumann algebras. Assume that $\mathcal{M}(\lambda^{-1})^{-1}(\mathcal{P}) = \mathcal{P}$ and $\mathcal{Q}$ is a II$_1$ factor. Suppose there is a projection $z \in \mathcal{Z}(\mathcal{P})$ such that $\mathcal{P} z \triangleleft \mathcal{Q}$ and a projection $p \in \mathcal{P} z$ such that $p \mathcal{P} p = p \mathcal{Q} p$. Then one can find a unitary $u \in \mathcal{M}$ such that $u \mathcal{P} z u^* = r \mathcal{Q} r$ where $r = u z u^* \in \mathcal{Z}(\mathcal{Q})$.

The next lemma is a mild generalization of [IPV10, Proposition 7.1], using the same techniques (see also the proof of [KV15, Lemma 2.3]).

**Lemma 2.8.** Let $\Lambda$ be an icc group, and let $\mathcal{M} = \mathcal{L}(\Lambda)$. Consider the comultiplication map $\Delta : \mathcal{M} \to \mathcal{M} \otimes \mathcal{M}$ given by $\Delta(v_\lambda) = v_\lambda \otimes v_\lambda$ for all $\lambda \in \Lambda$. Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{M}$ be a (unital) $*$-subalgebras such that $\Delta(\mathcal{A}) \subseteq \mathcal{M} \otimes \mathcal{B}$. Then there exists a subgroup $\Sigma < \Lambda$ such that $\mathcal{A} \subseteq \mathcal{L}(\Sigma) \subseteq \mathcal{B}$. In particular, if $\mathcal{A} = \mathcal{B}$, then $\mathcal{A} = \mathcal{L}(\Sigma)$.

**Proof.** Let $\Sigma = \{s \in \Lambda : v_s \in \mathcal{B}\}$. Since $\mathcal{B}$ is a unital $*$-subalgebra, $\Sigma$ is a subgroup, and clearly $\mathcal{L}(\Sigma) \subseteq \mathcal{B}$. We argue that $\mathcal{A} \subseteq \mathcal{L}(\Sigma)$.

Fix $a \in \mathcal{A}$, and let $a = \sum \lambda a_\lambda v_\lambda$ be its Fourier decomposition. Let $I = \{s \in \Lambda : a_s \neq 0\}$. Fix $s \in I$, and consider the normal linear functional $\omega$ on $\mathcal{M}$ given by $\omega(x) = a_s \tau(x v_s^*)$. Note that $(\omega \otimes 1)(a) = |a_s|^2 \otimes v_s$. Since $\Delta(\mathcal{A}) \subseteq \mathcal{M} \otimes \mathcal{B}$, we have that $(\omega \otimes 1)(\Delta(a)) \subseteq \mathcal{C} \otimes \mathcal{B}$. Thus, $v_s \in \mathcal{B} = s \in \Sigma$. Since this holds for all $s \in I$, we get that $a \in \mathcal{L}(\Sigma)$, and hence we are done.

We end this section with the following elementary result. We are grateful to the referee for suggesting a (much) shorter proof than the one we originally had, which used [CD18, Proposition 2.3].

**Lemma 2.9.** Let $\mathcal{M}$ be a tracial von Neumann algebra and let $\mathcal{N}$ be a type II$_1$ factor, with $\mathcal{N} \subseteq \mathcal{M}$ a unital inclusion. If there is $p \in \mathcal{P}(\mathcal{N})$ so that $p \mathcal{N} p = p \mathcal{M} p$ then $\mathcal{N} = \mathcal{M}$.

**Proof.** Shrinking $p$ if necessary we can assume $\tau(p) = 1/n$. Let $v_1, ..., v_n \in \mathcal{N}$ be partial isometries such that $v_i v_i^* = p$, for all $i$, and $\sum_{i=1}^n v_i^* v_i = 1$. Fix $x \in \mathcal{M}$. Since for every $1 \leq i, j \leq n$ we have $v_i x v_j^* \in p \mathcal{M} p = p \mathcal{N} p$ we get $x = \sum_{i,j=1}^n v_i^* (v_i x v_j^*) v_j \in \mathcal{N}$, as desired.

### 2.3 Small Cancellation Techniques

In this section, we recollect some geometric group theoretic preliminaries that will be used throughout this paper. We refer the reader to the book [Oz91] and the papers [Oz93, OOS07] for more details related to the small cancellation techniques. We also refer the reader to the book [LS77] for details concerning van Kampen diagrams.

#### 2.3.1 van Kampen Diagrams

Given a word $W$ over the alphabet set $S$, we denote its length by $|W|$. We also write $W \equiv V$ to express the letter-for-letter equality for words $W, V$.

Let $G$ be a group generated by a set of alphabets $S$. A van Kampen diagram $\Delta$ over a presentation

$$G = \langle S | R \rangle$$

(2.3.1)

is a finite, oriented, connected, planar 2-complex endowed with a labeling function $\text{Lab} : E(\Delta) \to S^{|1|}$, where $E(\Delta)$ denotes the set of oriented edges of $\Delta$, such that $\text{Lab}(e^{-1}) = (\text{Lab}(e))^{-1}$. Given a cell $\Pi$ of $\Delta$, $\partial \Pi$ denotes its boundary. Similarly, $\partial \Delta$ denotes the boundary of $\Delta$. The labels of $\partial \Delta$ and $\partial \Pi$ are defined up to cyclic permutations. We also stipulate that the label for any cell $\Pi$ of $\Delta$ is equal to (up to a cyclic permutation) $R^{|1|}$, where $R \in R$.

Using the van Kampen lemma ([LS77, Chapter 5, Theorem 1.1]), a word $W$ over the alphabet set $S$ represents the identity element in the group given by the presentation (2.3.1) if and only if there exists a connected, simply-connected planar diagram $\Delta$ over (2.3.1) satisfying $\text{Lab}(\partial \Delta) \equiv W$. 
2.3.2 Small Cancellation over Hyperbolic Groups

Let $G = \langle X \rangle$ be a finitely generated group and $X$ be a finite generating set for $G$. Recall that the Cayley graph $\Gamma(G,X)$ of a group $G$ with respect to the set of generators $X$ is an oriented labeled 1-complex with the vertex set $V(\Gamma(G,X)) = G$ and the edge set $E(\Gamma(G,X)) = G \times X^{\pm 1}$. An edge $e = (g,a)$ goes from the vertex $g$ to the vertex $ga$ and has label $a$. Given a combinatorial path $p$ in the Cayley graph $\Gamma(G,X)$, the length $|p|$ is the number of edges in $p$. The word length $|g|$ of an element $g \in G$ with respect to the generating set $X$ is defined to be the length of a shortest word in $X$ representing $g$ in the group $G$; i.e., $|g| = \min_{h \in G} \|h\|$.

The formula $d(f, g) = |g^{-1}f|$ defines a metric on the group $G$. The metric on the Cayley graph $\Gamma(G,X)$ is the natural extension of this metric. A word $W$ is called a $(\lambda,c)$-quasi geodesic in $\Gamma(G,X)$ for some $\lambda > 0, c \geq 0$ if $\lambda |W| - c \leq |W| \leq \lambda |W| + c$. A word $W$ is called a geodesic if it is a $(1,0)$-quasi geodesic. A word $W$ in the alphabet $X^{\pm 1}$ is called a $(\lambda,c)$-quasi geodesic (respectively, geodesic) in $G$ if any path in the Cayley graph $\Gamma(G,X)$ labeled by $W$ is $(\lambda,c)$-quasi geodesic (respectively, geodesic).

Throughout this section, $\mathcal{R}$ denotes a symmetric set of words (i.e., it is closed under taking cyclic shifts and inverses of words; and all the words are cyclically reduced) from $X^*$. A common initial sub-word of any two distinct words in $\mathcal{R}$ is called a piece. We say that $\mathcal{R}$ satisfies the $C'(\mu)$ condition if any piece contained (as a sub-word) in a word $R \in \mathcal{R}$ has length smaller than $\mu |R|$. 

**Definition 2.10.** [O93] Section 4] A subword $U$ of a word $R \in \mathcal{R}$ is called an $e$-piece of the word $R$, for $e \geq 0$, if there exists a word $R' \in \mathcal{R}$ satisfying the following conditions:

1. $R = UV$ and $R' \equiv U'V'$ for some $U', V' \in \mathcal{R};$
2. $U' =_G YUZ$ for some $Y, Z \in X^*$ where $|Y|, |Z| \leq e$;
3. $YRY^{-1} \neq_G R'$.

We say that the system $\mathcal{R}$ satisfies the $C(\lambda,c,e,\mu,\rho)$-condition for some $\lambda \geq 1, c \geq 0, e \geq 0, \mu > 0, \rho > 0$ if:

a) $|R| \geq \rho$ for any $R \in \mathcal{R};$

b) Any word $R \in \mathcal{R}$ is a $(\lambda,c)$-quasi geodesic;

c) For any $e$-piece $U$ of any word $R \in \mathcal{R}$, the inequalities $|U|, |U'| < \mu |R|$ hold.

In practice, we will need some slight modifications of the above definition [O93] Section 4].

**Definition 2.11.** A subword $U$ of a word $R \in \mathcal{R}$ is called an $e'$-piece of the word $R$, for $e' \geq 0$, if:

1. $R = UVU'V'$ for some $V, U', V' \in X^*$,
2. $U' =_G YUZ$ for some words $Y, Z \in X^*$ where $|Y|, |Z| \leq e$.

We say that the system $\mathcal{R}$ satisfies the $C'(\lambda,c,e,\mu,\rho)$-condition for some $\lambda \geq 1, c \geq 0, e \geq 0, \mu > 0, \rho > 0$ if:

a) $\mathcal{R}$ satisfies the $C(\lambda,c,e,\mu,\rho)$ condition, and

b) Every $e'$-piece $U$ of $R$ satisfies $||U'|| < \mu ||R||$, where $U'$ is as above.

Let $G$ be a group defined by

$$G = \langle X|O \rangle,$$

where $O$ is the set of all relators (not just the defining relations) of $G$. Given a symmetrized set of words $\mathcal{R}$ in the alphabet set $X$, we consider the quotient group

$$H = \langle G|\mathcal{R} \rangle = \langle G|O \cup \mathcal{R} \rangle.$$

A cell over a van Kampen diagram over $\mathcal{R}$ is called an $\mathcal{R}$-cell (respectively, an $O$-cell) if its boundary label is a word from $\mathcal{R}$ (respectively, $O$). We always consider a van Kampen diagram over $\mathcal{R}$ up to some elementary transformations. For example we do not distinguish diagrams if one can be obtained from other by joining two distinct $O$-cells having a common edge or by inverse transformations ([O93] Section 5)].
3 Some Examples of Ol’shanskii’s Monster Groups in the Context of Lacunary Hyperbolic Groups

In this section, we collect some group theoretic results needed for our main theorems in Sections 4 and 5. Readers who are mainly interested in the results in Section 5 may skip ahead to Subsection 3.3. The results in Subsections 3.1 and 3.2 shall be required for our main results in Section 4.

In order to derive our main results on the study of maximal von Neumann algebras (i.e. Theorem 4.4) we need to construct a new monster-like group in the same spirit with Ol’shanskii’s famous examples from [O180]. Specifically, generalizing the geometric methods from [O93] to the context of lacunary hyperbolic groups [OOS07] and using techniques developed by the third author from [K19] we construct a group $G$ such that every maximal subgroup of $G$ is isomorphic to a subgroup of $\mathbb{Q}$, the group of rational numbers. While in our approach we explain in detail how these results are used, the main emphasis will be on the new aspects of these techniques. Therefore we recommend the interested reader to consult beforehand the aforementioned results [O193, K19].

3.1 Elementary subgroups

In this section, using methods developed in [O93], we construct a group $G$ whose maximal (proper) subgroups are rank 1 abelian groups, see Theorem 3.12. More specifically, we study “special limits” of hyperbolic groups, called lacunary hyperbolic groups, as introduced in [OOS07].

Definition 3.1. Let $\alpha : G \to H$ be a group homomorphism and $G = \langle A \rangle, H = \langle B \rangle$. The injectivity radius $r_A(\alpha)$ is the radius of largest ball centered at identity of $G$ in the Cayley graph of $G$ with respect to $A$ on which the restriction of $\alpha$ is injective.

Definition 3.2. [OOS07, Theorem 1.2] A finitely generated group $G$ is called lacunary hyperbolic group if $G$ is the direct limit of a sequence of hyperbolic groups and epimorphisms;

$$G_1 \xrightarrow{\eta_1} G_2 \xrightarrow{\eta_2} \ldots G_i \xrightarrow{\eta_i} G_{i+1} \xrightarrow{\eta_{i+1}} G_{i+2} \xrightarrow{\eta_{i+2}} \ldots$$

(3.1.1)

where $G_i$ is generated by a finite set $S_i$ and $\eta_i(S_i) = S_{i+1}$. Also $G_i$’s are $\delta_i$-hyperbolic where $\delta_i = o(r_{S_i}(\eta_i))$, where $r_{S_i}(\eta_i)$ is injective radius of $\eta_i$ w.r.t. $S_i$.

Fix $\omega$ a nonprincipal ultrafilter. An asymptotic cone $\text{Cone}^\omega(X, \varepsilon, d)$ of a metric space $(X, \text{dist})$ where $\varepsilon = \{e_i\}, e_i \in X$ for all $i$ and $d = \{d_i\}$ is an unbounded sequence of nondecreasing positive real numbers, is the $\omega$-limit of the spaces $(X, \text{dist}_d)$. The sequence $d = \{d_i\}$ is called a scaling sequence. Following [OOS07, Theorem 3.3], $G$ being lacunary hyperbolic group, is equivalent to the existence of a scaling sequence $d = \{d_i\}$ such that the asymptotic cone $\text{Cone}^\omega(\Gamma(G, X), \varepsilon, d)$ associated with the Cayley graph $\Gamma(G, X)$ for a finite generating set $X$ of $G$ with $\varepsilon = \{\text{identity}\}$ is an $\mathbb{R}$-tree for any nonprincipal ultrafilter $\omega$. For more details on asymptotic cones and their connection with lacunary hyperbolic groups we refer the reader to [OOS07, Section 2.3, Section 3.1].

Our construction relies heavily on the notion of elementary subgroups. For the readers’ convenience, we collect below some preliminaries regarding elementary subgroups.

Definition 3.3. A group $E$ is called elementary if it is virtually cyclic. Let $G$ be a hyperbolic group and $g \in G$ be an infinite order element. Then the elementary subgroup containing $g$ is defined as

$$E(g) := \{x \in G | x^{-1}g^n x = g^{+n} \text{ for some } n = n(x) \in \mathbb{N}\}.$$ 

For further use we need the following result describing in depth the structure of elementary subgroups.

Lemma 3.4. 1) [O93] If $E$ is a torsion free elementary group then $E$ is cyclic.
2) [O93, Lemma 1.16] Let $E$ be an infinite elementary group. Then $E$ contains normal subgroups $T \triangleleft E^+ \triangleleft E$.
such that \([ E : E^+ ] \leq 2\), \(T\) is finite and \(E^+/T \cong \mathbb{Z}\). If \(E \neq E^+\) then \(E/T \cong D_{\infty}\) (infinite dihedral group). For a hyperbolic group \(G\), \(E(g)\) is unique maximal elementary subgroup of \(G\) containing the infinite order element \(g \in G\).

In the context of lacunary hyperbolic groups we need to introduce the following definition which generalizes Definition 3.3.

**Definition 3.5.** Let \(G\) be a lacunary hyperbolic group and let \(g \in G\) be an infinite order element. We define \(E^L(g) := \{ x \in G | x^g = g^n, \text{for some } n = n(x) \in \mathbb{N}\}\).

For future reference we now recall the following structural result regrading torsion elements in a \(\delta\)-hyperbolic group.

**Theorem 3.6.** \([Gr87, 2.2.B]\) Let \(g \in G\) be a torsion element in a \(\delta\)-hyperbolic group \(G\). Then \(g\) is conjugate to an element \(h\) in \(G\) such that \(|h|_G \leq 4\delta + 1\).

The following elementary lemma will be used in the proof of Theorem 3.8. For convenience we include a short proof.

**Lemma 3.7.** If \(G\) is a torsion free lacunary hyperbolic group, then one can choose \(G_i\) to be torsion free such that \(G = \lim_{i \to \infty} G_i\).

**Proof.** Fix a presentation \(G = \langle S | R \rangle\). By [OOS97] Theorem 3.3, one can choose \(G_i := \langle S | R_{c(i)} \rangle\), where \(\{c(n)\}_n\) is a strictly increasing sequence such that \(R_{c(i)}\) consists of labels of all cycles in the ball of radius \(d_i\) (corresponding to the scaling sequence \(\{d_i\}_i\) of the lacunary hyperbolic group) around the identity in \(\Gamma(G, S)\). Let \(r_i\) be the injectivity radius of the quotient map \(\phi_i : G_i \rightarrow G_{i+1}\). The lacunary hyperbolic condition implies that \(\lim_{i \to \infty} \frac{x_i}{r_i} = 0\), where \(x_i\) is the hyperbolic constant for the group \(G_i\) for all \(i\). Choose \(i_0\) such that for all \(j \geq i_0\) we have \(r_j > 9\delta_j\). We will show the \(G_j\)'s are torsion free for all \(j \geq i_0\), which proves the lemma.

Fix any \(j \geq i_0\). Assume by contradiction that \(g \in G_j \setminus \{1\}\) is a torsion element. By Theorem 3.6 there is an element \(h \in G_j \setminus \{1\}\) such that \(h\) is conjugate to \(g\) and \(|h|_{G_j} \leq 4\delta_j + 1\). Thus \(h\) is a torsion element of \(G_j\).

Since \(|h|_{G_j} \leq 4\delta_j + 1 < r_j\), then \(h\) is a non trivial element of \(G_k\) for all \(k \geq j\). Thus \(h\) is a non trivial torsion element in the limit group \(G\), which is a contradiction!  

The next result generalizes Lemma 3.4 and provides a complete description of the structure of elementary subgroups of a torsion free lacunary hyperbolic group. This result can be deduced from the main theorem of [K19]. For readers’ convenience, we include a short proof.

**Theorem 3.8.** Let \(G\) be a torsion free lacunary hyperbolic group and let \(g \in G\) be an infinite order element. Then \(E^L(g)\) is an abelian group of rank 1 (i.e. \(E^L(g)\) embeds in \((\mathbb{Q}, +))\).

**Proof.** From the definition (3.1.1) of lacunary hyperbolic group it follows that \(E^L(g) = \lim E_i(g)\) for every \(e \neq g \in G\), where \(E_i(g)\) is the elementary subgroup containing the element \(g\) in the hyperbolic group \(G_i\) when viewing \(g \in G_i\). Since \(G\) is torsion free one can choose \(G_i\) to be torsion free by Lemma 3.7. By Lemma 3.4 part 1) we get that \(E_i(g)\) is cyclic for all \(i\). Observe that every surjective homomorphism between hyperbolic groups takes elementary subgroups into elementary subgroups, in particular \(E_i(g)\) maps into \(E_{i+1}(g)\). We now get the group \(E^L(g) = \lim E_i(g)\) as an inductive limit of cyclic groups, which proves the theorem.  

**Remark.** Let \(G\) be a torsion free lacunary hyperbolic group and let \(e \neq g \in G\). Note that \(C_{G}(g) \leq E^L(g)\), where \(C_{G}(g)\) is the centralizer of \(g\) in \(G\).
3.2 Maximal Subgroups

Let $G_0 = \langle X \rangle$ be a torsion free $\delta$-hyperbolic group with respect to $X$, where $X = \{x_1, x_2, \ldots, x_n\}$ is a finite generating set. Without loss of generality we assume that $E(x_i) \cap E(x_j) = \{e\}$ for $i \neq j$. We define a linear order on $X$ by $x_i^{-1} < x_j^{-1} < x_i < x_j$, whenever $i < j$. Let $F'(X)$ denote the set of all non empty reduced words on $X$. Note that the order on $X$ induces the lexicographic order on $F'(X)$. Let $F'(X) = \{w_1, w_2, \ldots\}$ be an enumeration with $w_i < w_j$ for $i < j$. Observe that $w_1 = x_1$ and $w_2 = x_2$. We now consider the set $S := F'(X) \times F'(X) \setminus \{(w, w) \mid w \in F'(X)\}$ and enumerate the elements of $S$ as $S = \{(u_1, v_1), (u_2, v_2), \ldots\}$.

Our next goal is to construct the following chain,

$$G_0 \xrightarrow{\beta_0} K_1 \rightarrow G_1' \rightarrow G_1 \xrightarrow{\beta_1} K_2 \rightarrow G_2' \rightarrow G_2 \rightarrow \cdots$$ (3.2.1)

where $K_i, G_i, G_i'$ are hyperbolic for all $i$ and $\eta_i = \gamma_i \circ \alpha_i \circ \beta_i; i \geq 1$, satisfies the conditions in (3.1.1).

Let $L$ be a rank 1 abelian group. Then $L$ can be written as $L = \cup_{s=0}^{\infty} L_s$, where $L_i = \langle L_i \rangle_\infty$ and $L_i = \langle L_i \rangle_{m_i+1}$ for some $m_i \in \mathbb{N}$. Here $\langle L_i \rangle_\infty$ denotes the infinite cyclic group generated by the infinite order element $\{L_i\}$.

Since $G_0$ is non-elementary, there exists a smallest index $j_i \geq i$ such that $\rho_{j_i} \notin E(L_i)$. For $m \in \mathbb{N}$, define

$$H_{i+1}^k := H_{i+1}^{k-1} \ast_{u_i \in \gamma_i((k+1)_i)} \langle L_i \rangle_\infty$$ where $H_{i+1}^0 = G_i$ and $H_{i+1}^k = \langle L_i \rangle_{k+1}$ for $k = 1, 2, \ldots, j_i$.

(3.2.2)

For $i \geq 0$ let $K_{i+1}$ be $H_{i+1}^{j_i}$. Note that $K_{i+1}$ is hyperbolic as $H_{i+1}^{j_i}$ is hyperbolic for all $k$ by [MO98, Theorem 3]. Choose $c_i, c_i' \in G_i$ such that $c_i, c_i' \notin E(U_i)$ for all $1 \leq k \leq j_i$ and $c_i, c_i' \notin E(V_i)$. One can find such $c_i$ and $c_i'$ since there are infinitely many elements in a non elementary hyperbolic group which are pairwise non commensurable, [O93, Lemma 3.8]. Let $V_i := \{L_i \mid 1 \leq k \leq j_i\}$. Denote by

$$R_k := \langle L_i \rangle_{j_i}^{n_{1,k}} c_i^{n_{2,k}} c_i' c_i^{n_{3,k}} c_i' \cdots c_i^{n_{s,k}} c_i'$$ (3.2.3)

where $n_{s,k}$, for $1 \leq k \leq j_i$ are defined as follows:

$$n_{1,k} = 2^{k-1} n_{1,s}, s_k = n_{1,k-1} \text{ and } n_{s,k} = n_{1,k} + (s-1).$$

We also denote by $R_i$ be the set of all cyclic shifts of $\{R_k^{-1} : 1 \leq k \leq j_i\}$.

**Lemma 3.9.** [Dar17, Lemma 5.1] There exists a constant $K$ such that the set of words $R_i$ defined above by (3.2.3) are $(\lambda, \gamma)$ quasi geodesic in $\Gamma(G, X)$, provided $n_{1,1} \geq K, \epsilon \notin E(L_i)$, and $c_i' \notin E(L_i)$.

We now denote by $\bar{R}_{i+1}$ to be the set of words $R_i$, defined as above, with $n_{1,k} \geq K$.

**Lemma 3.10.** [Dar17, Lemma 5.2] For any given constant $\epsilon_i \geq 0, \mu_i > 0, \rho_i > 0$, the system of words $\bar{R}_{i+1}$ (defined above) satisfies the $C'(\lambda_{ij}, \epsilon_i, \mu_i, \rho_i)$ condition over $K_{i+1}$.

By construction there is a natural embedding $\beta_i : G_i \rightarrow K_{i+1}$. Let $G'_{i+1} := \langle K_{i+1}[\bar{R}_{i+1}] \rangle$ (where we are using the notations as in Subsection 2.3.1). The factor group $G'_{i+1}$ is hyperbolic by [O93, Lemma 7.2]. Now consider the natural quotient map $\alpha_{i+1} : K_{i+1} \rightarrow G'_{i+1}$. Since $\alpha_{i+1} \circ \beta_i$ takes generators of $G_i$ to generators of $G'_{i+1}$, the map $\alpha_{i+1} \circ \beta_i$ is surjective.

Consider the following set

$$Z_i := \{x \in X \mid x \notin E(U_i)\}.$$

Let $Z_i := G'_{i+1} / \langle \mathcal{R}(Z_i, u_j, v_j, \lambda_i, c_i, \epsilon_i, \mu_i, \rho_i) \rangle$ and let $\gamma_{i+1} : G'_{i+1} \rightarrow G_{i+1}$ be the quotient map. Here $\mathcal{R}(Z_i, u_j, v_j, \lambda_i, c_i, \epsilon_i, \mu_i, \rho_i) = \langle \gamma_{i+1} \rangle$ is the set of all conjugates and the cyclic shifts of some relations where we identify the elements of $Z_i$ with words of the form $\{R_{i}^{-1} : 1 \leq k \leq j_i\}$ generated by $u_j$ and $v_j$. Since the relators $\mathcal{R}_i$
are generic we have added all the parameters to indicate these relations satisfy the small cancellation conditions with the parameters and their dependency to the specific set of words. One can choose the powers of \( u_i \) and \( v_i \) such that the small cancellation condition is satisfied by Lemma 3.9 and Lemma 3.10. For more details on how to choose these words we refer the reader to [O93, Section 5], [Dar17, Section 5.4]. Thus it follows that the group \( G_{i+1} \) is hyperbolic by [O93, Lemma 7.2] as one can choose parameters \( \lambda_i, G_i, \rho_i, p_i \) such that \( R(Z_i, u_j, v_j, \lambda_i, \rho_i, p_i) \) satisfies the \( C'(\lambda_i, \rho_i, \mu_i, \rho_i) \) small cancellation condition 2.11 and the map \( \gamma_i+1 \) takes generating set to generating set. In particular, \( \eta_{i+1} := \gamma_i+1 \circ \alpha_i+1 \circ \beta_i \) is a surjective homomorphism which takes the generating set of \( G_i \) to the generating set of \( G_{i+1} \). Let \( G_L := \lim G_i \). From its definition, it follows that \( G_{i+1} \) is the group generated by \( u_j \) and \( v_j \).

We summarize the above discussion in the following statement.

**Lemma 3.11.** The above construction satisfies the following properties:

1. \( G_i \) is non elementary hyperbolic group for all \( i \);
2. Either \( u_i \in E(v_i) \) or the group generated by \( \{u_i, v_i\} \) in \( G_{i+1} \) is equal to all of \( G_{i+1} \);
3. For each element \( x \in X, E(x) = \langle y \rangle \) in \( G_i \), where \( x = y^{m_1 m_2 \cdots m_i} \). The exponent \( m_i \)'s are described as follows: Being a rank 1 abelian group \( L \) can be written as \( L = \bigcup_{i=0}^{\infty} L_i \), where \( L_i = \langle g_i \rangle^1 \) and \( g_i = g_i^{m_i+1} \) for some \( m_i+1 \) \( \in \mathbb{N} \);
4. \( G^L := \lim G_i \) may be chosen to have property (T).

**Proof.** Part 1. follows from [O93, Lemma 7.2]. To see part 2. notice that by definition if \( j_i > i \) then \( v_i \in E(u_i) \) in \( G_i \). Otherwise if \( j_i = i \) then \( v_i \notin E(u_i) \) in \( G_i \) and \( G_i \) is the group generated by \( \{u_i, v_i\} \). Part 3. follows immediately from the fact that \( x \) is not a proper power in \( G_0 \). Finally, for part 4. notice that we may start the above construction with \( G_0 \) being a property (T) group. Then \( G_1 \) has property (T), as \( G_0 \) surjects onto \( G_1 \). By induction, each of the groups \( G_i \) in the above construction have property (T). Hence \( G^L \) has property (T).

We are now ready to prove the main theorem of this section.

**Theorem 3.12.** For any subgroup \( Q_m \) of \( \langle Q, * \rangle \) there exists a non elementary torsion free lacunary hyperbolic group \( G \) such that all maximal subgroups of \( G \) are isomorphic to \( Q_m \). Moreover, we may choose \( G \) to have property (T).

**Proof.** In the above construction let \( L = Q_m, G = G^Q_m \) and take \( d = m_1 m_2 \cdots m_i \) in 3.2.2, where \( L_i = \langle g_i \rangle_{\infty} \) and \( g_i = g_i^{m_i} \) for some \( m_i+1 \) \( \in \mathbb{N} \) and \( Q_m = \bigcup_{i=1}^{\infty} L_i \). One can choose sparse enough parameters to satisfy the injectivity radius condition in Definition 3.11 which in turn will ensure that \( G \) is lacunary hyperbolic. The above construction also guarantees that \( E^L(g) = Q_m \) for all \( g \in G \setminus \{1\} \). Suppose \( P \notin G \) is a maximal subgroup of \( G \). As \( P \) is a proper subgroup, \( P \) is abelian by part 2. of Lemma 3.11. Now let \( e \neq h \in G \). Note that As \( P \) is abelian, \( P \) is contained in the centralizer of \( h \). Now from Definition 3.2 it follows that \( g \in P \leq E^L(g) \langle \neq Q_m \rangle \). By maximality of \( P \) we get that \( P \neq Q_m \). Thus, all maximal subgroups of \( G \) are isomorphic to \( Q_m \) and hence any proper subgroup of \( G \) is isomorphic to a subgroup of \( Q_m \).

The moreover part follows from part 4) of Theorem 3.11.

We end this section with the following well known counterexamples to von Neumann’s conjecture.

**Corollary 3.13 ([O93], [O80]).** For every non-cyclic torsion free hyperbolic group \( \Gamma \), there exists a non abelian torsion free quotient \( \hat{\Gamma} \) such that all proper subgroups of \( \hat{\Gamma} \) are infinite cyclic.

**Proof.** Take \( Q_m = \mathbb{Z} \) in Theorem 3.12.
3.3 Belegradek-Osin’s Rips Construction in Group Theory

Rips constructions emerged in geometric group theory with the work of Rips from [Rip82] and represent a rich source of examples for various pathological properties in group theory. This type of construction was used effectively to study automorphisms of property (T) groups. In this direction Ollivier-Wise [OW04] were able to construct property (T) groups whose automorphism group contain any given countable group. This answered an important older question of P. de la Harpe and A. Valette about finiteness of outer automorphism groups of property (T) groups. Using the small cancellation methods developed in [Os06] and [AMO07], Belegradek and Osin discovered the following version of the Rips construction in the context of relatively hyperbolic groups:

**Theorem 3.14.** [BO06] Let $H$ be a non-elementary hyperbolic group, $Q$ be a finitely generated group and $S$ a subgroup of $Q$. Suppose $Q$ is finitely presented with respect to $S$. Then there exists a short exact sequence

$$1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1,$$

and an embedding $i : Q \rightarrow G$ such that

1. $N$ is isomorphic to a quotient of $H$.
2. $G$ is hyperbolic relative to the proper subgroup $i(S)$.
3. $i \circ \epsilon = Id$.
4. If $H$ and $Q$ are torsion free then so is $G$.
5. The canonical map $\phi : Q \rightarrow \text{Out}(N)$ is injective and $[\text{Out}(N) : \phi(Q)] < \infty$.

This construction is extremely important for our work. We are particularly interested in the case when $H$ is torsion free and has property (T) and $Q = S$ and it is torsion free. In this situation Theorem 3.14 implies that $G$ is admits a semidirect product decomposition $G = N \rtimes Q$ and it is hyperbolic relative to $\langle Q \rangle$. Notice that the finite conjugacy radical $FC(N)$ of $N$ is invariant under the action of $Q$ and hence $FC(N)$ is an amenable normal subgroup $G$. Since $G$ is relative hyperbolic it follows that $FC(N)$ is finite and hence it is trivial as $G$ is torsion free; in particular $N$ is an icc group. Since $G$ is hyperbolic relative to $Q$ it follows that the stabilizer of any $n \in N$ in $Q$ under the action $Q/\sigma N$ is trivial.

We now introduce the following classes of groups that shall play an extremely important role throughout the rest of the paper.

**Definition 3.15.** We denote by $\text{Rip}(Q)$ the class of all semidirect product $G = N \rtimes Q$ satisfying the properties of Theorem 3.14, where $Q = S$, $Q$ and $H$ are torsion free and $H$ has property (T).

Moreover, when $Q$ has property (T), we denote the class $\text{Rip}(Q)$ by $\text{Rip}_T(Q)$.

Since property (T) is closed under extensions it follows that all groups in $\text{Rip}_T(Q)$ have property (T). Our rigidity results in Section 5 concern this class of groups.

In the second part of this section we recall a powerful method from geometric group theory, termed Dehn filling. We are interested specifically in the group theoretic Dehn filling constructions developed by D. Osin and his collaborators in [Os06, DGO11]. The next result, which is due to Osin, is a technical variation of [Os06, Theorem 1.1] and [DGO11, Theorem 7.9] and plays a key role to derive some of our main rigidity theorems in Section 5 (see Theorems 5.2 and 5.3). For its proof the reader may consult [CIK13, Corollary 5.1].

**Theorem 3.16** (Osin). Let $H \leq G$ be infinite groups where $H$ is finitely generated and residually finite. Suppose that $G$ is hyperbolic relative to $\langle H \rangle$. Then there exist a non-elementary hyperbolic group $K$ and an epimorphism $\delta : G \twoheadrightarrow K$ such that $R = \ker(\delta)$ is isomorphic to a non-trivial (possible infinite) free product $R = \ast_{g \in T} R_0^g$, where $T \subset G$ is a subset and $R_0^g = gR_0g^{-1}$ for a finite index normal subgroup $R_0 \triangleleft H$. 

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We end this section with an application of Theorem 3.16. The result describes the structure of the normal subgroups \( N \) of \( N \times Q \in Rip_T(Q) \). Namely, combining Theorems 3.16 and 3.14 we show that these groups are free-by-hyperbolic. This result will be essential to the proof of Theorem 5.1.

**Proposition 3.17.** Let \( G = N \times Q \in Rip_T(Q) \) and assume that \( Q \) is an infinite residually finite group. Then \( N \) is a \( F_{n+1} \)-by-(non elementary, hyperbolic property (T)) group where \( n \in \mathbb{N} \cup \{ \infty \} \).

**Proof.** Since \( G \) is hyperbolic relative to \( \{ Q \} \) and \( Q \) is residually finite then by Theorem 3.16 there is a non-elementary hyperbolic group \( K \) and an epimorphism \( \delta : G \to K \) such that \( L = ker(\delta) \) is isomorphic to a non-trivial free product \( L = \ast_{g \in T} Q_0^g \), where \( T \subset G \) is a subset and \( Q_0 \triangleleft Q \) is finite index, normal subgroup. Since \( G = N \times Q \) and \( Q_0 \) is normal in \( Q \) one can assume without any loss of generality that \( T \subset N \). Next we show that \( N \cap L \) is infinite. If it would be finite, as \( G \) is icc, it follows that \( N \cap L = 1 \). As \( N \) and \( L \) are normal in \( G \) then the commutator satisfies \( [N, L] \leq N \cap L = 1 \) and hence \( L \leq C_G(N) \). To describe this centralizer fix \( g = nq \in C_G(N) \) where \( n \in N \), \( q \in Q \). Thus for all \( m \in N \) we have \( nmQ = mQnQ \) and hence \( nGm(m) = mnQ \) for all \( x \in L \). Therefore \( \sigma_q = ad(n) \) and by part 5. in Theorem 3.14 we must have that \( q = 1 \). This further implies that \( m \in Z(N) = 1 \) and hence \( C_G(N) = 1 \); in particular, \( L = 1 \) which is a contradiction. In conclusion \( N \cap L < N \) is an infinite normal subgroup. Using the isomorphism theorem we see that \( N/(N \cap L) \cong (NL)/L \). Also from the free product description of \( L \) we see that \( N \times Q_0 \leq NL \) and hence \( [G : NL] < \infty \). In particular \( (NL)/L \) is a free group with free basis \( X \). In particular, for every \( i \in I \) the previous relation implies that \( Q_0^{g_i} \leq N \) and writing \( g_i = n_i q_i \) for some \( n_i \in N \), \( q_i \in Q \) we see that \( Q_0^{g_i} \leq N \). As \( Q_0^{Q} \leq Q \) we conclude that \( Q_0^{g_i} \leq N \cap Q = 1 \) and hence \( Q_i = 1 \). Thus \( N \cap L = F(X) \) and since \( G \) is icc and \( N \cap L \) is normal in \( G \) we see that \( |X| \geq 2 \), which finishes the proof.

\[ \square \]

4 **Maximal von Neumann Subalgebras Arising from Groups Rips Construction**

If \( \mathcal{M} \) is a von Neumann algebra then a von Neumann subalgebra \( \mathcal{N} \subset \mathcal{M} \) is called maximal if there is no intermediate von Neumann subalgebra \( \mathcal{P} \) so that \( \mathcal{N} \subset \mathcal{P} \subset \mathcal{M} \). Understanding the structure of maximal subalgebras of a given von Neumann algebra is a rather difficult problem that plays a key role in the very classification of these objects. Despite a series of earlier remarkable successes on the study of maximal amenable subalgebras initiated by Popa [Po83] and continued more recently [Sh06, CFRW08, Ho14, BC14, BC15, Su18, CD19, JS19], much less is known for the maximal ones. For instance Ge’s question [Ge03, Section 3, Question 2] on the existence of non-amenable factors that possess maximal factors which are amenable was settled in the affirmative only very recently in the work of Y. Jiang and A. Skalski, [JS19]. We also remark that the study of maximal (or by duality minimal) intermediate subfactors has recently led to the discovery of a rigidity phenomenon for the intermediate subfactor lattice in the case of irreducible finite index subfactors [BDLR19].

In this section we make new progress in this direction by describing several concrete collections of maximal subalgebras in the von Neumann algebras arising from the groups \( Rip(Q) \) introduced in the previous subsection (see Theorem 4.1 below). In particular, these examples allow construction of property (T) von Neumann algebras which have maximal von Neumann subalgebras without property (T). This answers a question raised by Y. Jiang and A. Skalski in [JS19, Problem 5.5] in the first version of their paper. Our arguments rely on the usage of Galois correspondence results for von Neumann algebras à la Choda [Ch78] and the classification of maximal subgroups in the monster-type
groups provided in Theorem 3.12. We remark that in the second version of their paper [JS19] Theorem 4.8, Y. Jiang and A. Skalski independently obtained a different solution, using different techniques.

First we need a couple of basic lemmas concerning automorphisms of groups. For the reader’s convenience we include short proofs.

**Lemma 4.1.** Let $N$ be a group, let $\text{Id} \neq \alpha \in \text{Aut}(N)$ and denote by $N_1 = \{ n \in N | \alpha(n) = n \}$ its fixed point subgroup. Then the following hold:

1. Either $[N : N_1] = \infty$ or there is a subgroup $N_0 \leq N_1 \leq N$ that is normal in $N$ with $[N : N_0] < \infty$ and such that the induced automorphism $\tilde{\alpha} \in \text{Aut}(N/C_N(N_0))$ given by $\tilde{\alpha}(nC_N(N_0)) = \alpha(n)C_N(N_0)$ is the identity map; in particular, when $N$ is icc we always have $[N : N_1] = \infty$.

2. Either $[N : N_1] = \infty$, or $\alpha$ has finite order in $\text{Aut}(N)$, or there is a $k \in \mathbb{N}$ and a subgroup $N_0 \leq N_1 \leq N$ that is normal in $N$ with $[N : N_0] < \infty$ and such that the induced automorphism $\tilde{\alpha} \in \text{Aut}(N/Z(N_0))$ given by $\tilde{\alpha}(nZ(N_0)) = \alpha(n)Z(N_0)$ has order $k$; in particular, when all finite index subgroups of $N$ have trivial center we either have $[N : N_1] = \infty$, or, $\alpha$ has finite order.

**Proof.** 1. Assume that $2 \leq [N : N_1] < \infty$. Then $N_0 := \cap_{n \in N} hN_1 h^{-1} \leq N_1$ is a finite index normal subgroup of $N$. Notice that the centralizer $C_N(N_0)$ is also normal in $N$. Let $n \in N$ and $n_0 \in N_0$. As $N_0$ is normal we have $n_0 n^{-1} \in N_0 \leq N_1$ and hence $n_0 n^{-1} = \alpha(n_0 n^{-1}) = \alpha(n) n_0 (n^{-1})$. This implies that $n_0 n^{-1} \alpha(n) n_0 = n^{-1} \alpha(n)$ and hence $n^{-1} \alpha(n) \in C_N(N_0)$. Since $\alpha$ acts identically on $N_0$ one can see that $\alpha(C_N(N_0)) = C_N(N_0)$. Thus one can define an automorphism $\tilde{\alpha} : N/C_N(N_0) \to N/C_N(N_0)$ by letting $\tilde{\alpha}(nC_N(N_0)) = \alpha(n)C_N(N_0)$. However the previous relations show that $\tilde{\alpha}$ is the identity map, as desired. For the remaining part of the statement, we notice that if $[N : N_1] < \infty$ and $N$ is icc then the centralizer $C_N(N_0)$ is trivial and hence $\alpha = Id$, which is a contradiction.

2. Assume $[N : N_1] < \infty$ and $\alpha$ has infinite order in $\text{Aut}(N)$. Also for each $i \geq 2$ denote by $N_i = \{ n \in N | \alpha^i(n) = n \}$. Then notice that $N_1 \leq N_i \leq N_{i+1} \leq N$. Since $[N : N_1] < \infty$ there is $s \in \mathbb{N}$ so that, either $N_s = N_i$ for all $i \geq s$, or $N_s = N$. If $N_s = N$ then $\alpha^s = Id$, contradicting the infinite order assumption on $\alpha$. Now assume that $N_s = N_{s+1}$. For every $n \in N_{s+1}$ we have $\alpha^s(n) = \alpha^{s+1}(n)$ and thus $\alpha(n) = n$ which is equivalent to $n \in N_1$. This shows that $N_1 = N_{s+1}$ and combining with the above we conclude that $N_1 = N_i$ for all $i$.

As $[N : N_1] < \infty$ then $N_0 := \cap_{n \in N} hN_1 h^{-1} \leq N_1$ is a finite index normal subgroup of $N$. $\alpha$ induces an automorphism $\tilde{\alpha}$ on the quotient group $N/N_0$ by $\tilde{\alpha}(nN_0) = \alpha(n)N_0$ for all $n \in N$. Since $[N : N_0] < \infty$ there is $k \in \mathbb{N}$ such that $\tilde{\alpha}^k = Id$ on $N/N_0$. Thus for every $n \in N$ we have $n^{-1} \alpha^k(n) \in N_0$.

Let $n \in N$ and $n_0 \in N_0$. By normality we have $n n_0 n^{-1} \in N_0 \leq N_1$ and hence $n n_0 n^{-1} = \tilde{\alpha}^k(n) n_0 k (n^{-1})$. This implies that $n_0 n^{-1} \alpha^k(n) n_0 = n^{-1} \alpha^k(n)$ and hence $n^{-1} \alpha^k(n) \in Z(N_0)$. Since $N_0$ is normal in $N$, so is $Z(N_0)$. Since $\alpha$ leaves $Z(N_0)$ invariant, the map $\tilde{\alpha} : N/Z(N_0) \to N/Z(N_0)$ given by $\tilde{\alpha}(nZ(N_0)) = \alpha(n)Z(N_0)$ is an automorphism. The previous relations show that it has order $k$.

Using this we will see that, in the case of icc groups, outer group actions $Q \simeq N$ by automorphisms lift to outer actions $Q \simeq \mathcal{L}(N)$ at the von Neumann algebra level. More precisely we have the following

**Lemma 4.2.** Let $N$ be an icc group and let $Q$ be a group together with an outer action $Q \simeq^\sigma N$. Then $\mathcal{L}(N)^\sigma \cap \mathcal{L}(N \rtimes_v Q) = \mathbb{C}$.

**Proof.** To get $\mathcal{L}(N)^\sigma \cap \mathcal{L}(N \rtimes_v Q) = \mathbb{C}$ it suffices to show that for all $\varphi \in (N \rtimes_v Q) \setminus \{ e \}$ the $N$-conjugacy orbit $O_N(\varphi) = \{ n \varphi n^{-1} : n \in N \}$ is infinite. Suppose by contradiction there is $h = n_0 q_0 \in (N \rtimes Q) \setminus \{ e \}$ with $n_0 \in N$ and $q_0 \in Q$ such that $|O_N(h)| < \infty$. Hence there exists a finite index subgroup $N_1 \leq N$ such that $n h n^{-1} = h$ for all $n \in N_1$. This entails that $n_0 q_0 h n^{-1} = n_0 q_0$ and thus $n_0 q_0 n_0 q_0 h n^{-1} = \text{ad}(n_0) \circ \sigma_{q_0}(n)$ for all $n \in N_1$. Also, since $N$ is icc, we have that $q_0 \neq e$. Let $\alpha = \text{ad}(n_0) \circ \sigma_{q_0}$. Since $Q \simeq N$ is outer it follows that $\text{Id} \neq \alpha \in \text{Aut}(N)$. Since $N$ is icc and $[N : N_1] < \infty$ then the first part in Lemma 4.1 leads to a contradiction.
With these results at hand we are now ready to deduce the main result of the section.

**Notation 4.3.** Fix any rank one group \( Q_m \). Consider the lacunary hyperbolic groups \( Q \) from Theorem 3.12 where the maximal rank one subgroups of \( Q \) are isomorphic to \( Q_m \). Also let \( N \times Q \in \text{Rip}(Q) \) be the semidirect product obtained via the Rips construction together with the subgroups \( N \times Q_m \subset N \times Q \). Throughout this section we will consider the corresponding von Neumann algebras \( \mathcal{M}_m := \mathcal{L}(N \times Q_m) \subset \mathcal{L}(N \times Q) := \mathcal{M} \).

Assuming Notation 4.3 we now show the following

**Theorem 4.4.** \( \mathcal{M}_m \) is a maximal von Neumann algebra of \( \mathcal{M} \). In particular, when \( N \times Q \in \text{Rip}(Q) \) then \( \mathcal{M}_m \) is a non-property (T) maximal von Neumann subalgebra of a property (T) von Neumann algebra \( \mathcal{M} \).

**Proof.** Fix \( P \) be any intermediate subalgebra \( \mathcal{M}_m \subset P \subset \mathcal{M} \). Since \( \mathcal{M}_m \subset \mathcal{M} \) is spatially isomorphic to the crossed product inclusion \( \mathcal{L}(N) \times Q_m \subset \mathcal{L}(N) \times Q \) we have \( \mathcal{L}(N) \times Q_m \subset P \subset \mathcal{L}(N) \times Q \). By Lemma 4.2 we have that \( (\mathcal{L}(N) \times Q_m)^\prime \cap (\mathcal{L}(N) \times Q) \subset (\mathcal{L}(N) \times Q)^\prime \cap (\mathcal{L}(N) \times Q) = C \). In particular, \( P \) is a factor. Moreover, by the Galois correspondence theorem [Ch78] (see also [CD19, Corollary 3.8]) there is a subgroup \( Q_m \subset K \subset Q \) so that \( P = \mathcal{L}(N) \times K \). Since by construction, \( Q_m \) is a maximal subgroup of \( Q \), we must have that \( K = Q_m \) or \( Q \). Thus we get that \( P = \mathcal{M}_m \) or \( \mathcal{M} \) and the conclusion follows.

For the remaining part note that \( \mathcal{M} \) has property (T) by [C85]. Also, since \( N \times Q_m \) surjects onto an infinite abelian group then it does not have property (T). Thus by [C85] again \( \mathcal{M}_m = \mathcal{L}(N \times Q_m) \) does not have property (T) either.

As pointed out at the beginning of the section, the above theorem provides a positive answer to [S19] Problem 5.5. Another solution to the problem of finding maximal subalgebras without property (T) inside factors with property (T) was also obtained independently by Y. Jiang and A. Skalski in the most recent version of their paper. Their beautiful solution has a different flavor from ours; even though the Galois correspondence theorem a la Choda is a common ingredient in both of the proofs. Hence we refer the reader to [S19, Theorem 4.8] for another solution to the aforementioned problem. Also note that while the algebras \( \mathcal{M}_m \) do not have property (T) they are also non-amenable. In connection with this it would be very interesting if one could find an example of a property (T) II\(_1\) factor which have maximal hyperfinite subfactors. This is essentially Ge’s question but for property (T) factors.

In the final part of the section we show that whenever \( Q_i \) is not isomorphic to \( Q_k \) then the resulting maximal von Neumann subalgebras \( \mathcal{M}_{m_i} \) and \( \mathcal{M}_{m_k} \) are non-isomorphic. In fact we have the following more precise statement

**Theorem 4.5.** Assume that \( Q_i, Q_k \subset \langle Q, + \rangle \) and let \( \Theta : \mathcal{M}_1 \to \mathcal{M}_k \) be a \(*\)-isomorphism. Then there exists a unitary \( u \in \mathcal{U}(\mathcal{M}_k) \) such that \( \text{ad}(u) \circ \Theta : \mathcal{L}(N_1) \to \mathcal{L}(N_2) \) is a \(*\)-isomorphism. Moreover there exist a group isomorphism \( \delta : Q_i \to Q_k \) and a \( 1\)-cocycle \( r : \mathcal{L}(N_1) \to \mathcal{U}(\mathcal{L}(N_2)) \) such that for all \( a \in \mathcal{L}(N_1) \) and \( g \in Q_i \) we have \( \text{ad}(u) \circ \Theta(a u_g) = \text{ad}(u) \circ \Theta(a) v_{\delta(g)} r_{\delta(g)} \). In particular, we have \( \text{ad}(u) \circ \Theta \circ \alpha_g = \text{ad}(r_{\delta(g)}) \circ \beta_{\delta(g)} \circ \text{ad}(u) \circ \Theta \).

**Proof.** Identify \( \mathcal{M}_1 = \mathcal{L}(N_1) \times Q_i \) and \( \mathcal{M}_k = \mathcal{L}(N_2) \times Q_k \) and let \( \Theta : \mathcal{M}_1 \times Q_i \to \mathcal{L}(N_2) \times Q_k \) be the \(*\)-isomorphism. Notice that since \( \mathcal{L}(N_1) \) has property (T) and \( Q_i \) is amenable then by [Po01] we have that \( \Theta(\mathcal{L}(N_1)) \cong \mathcal{M}_k \). Also by Lemma 4.2 we note that \( \mathcal{L}(N) \) is a regular irreducible subfactor of \( \mathcal{M}_k \), i.e. \( \Theta(\mathcal{L}(N_1))' \cap \mathcal{M}_k = \Theta(\mathcal{L}(N_1))' \cap \mathcal{M}_k = C_1 \). Similarly, \( \mathcal{L}(N_2) \) is a regular irreducible subfactor of \( \mathcal{M}_k \) satisfying \( \mathcal{L}(N_2) \times Q_k \). Thus by the proof of [IP15, Lemma 8.4], since \( Q_i \)'s are torsion free, one can find a unitary \( u \in \mathcal{U}(k) \) such that \( \text{ad}(u) \circ \Theta(\mathcal{L}(N_1)) = \mathcal{L}(N_2) \). So replacing \( \Theta \) with \( \text{ad}(u) \circ \Theta \) we can assume that \( \Theta(\mathcal{L}(N_1)) = \mathcal{L}(N_2) \). Hence for every \( g \in Q_i \) we have that \( \Theta(\alpha_g(x)) \Theta(u_g) = \Theta(u_g) \Theta(x) \) for all \( x \in \mathcal{L}(N_1) \). Consider the Fourier decomposition of \( \Theta(u_g) = \sum_h \chi_h n_h \beta_h \) where \( n_h \in \mathcal{L}(N_2) \). Using the previous relations we get that \( \Theta(\alpha_g(x)) n_h = n_h \beta_h \Theta(x) \) for all \( h \in Q_k \) and \( x \in \mathcal{L}(N_2) \). Thus \( n_h \beta_h \in \mathcal{L}(N_2)' \cap \mathcal{M}_k = C_1 \) and hence there exist unitary \( t_h \in \mathcal{L}(N_2) \) and scalar \( s_h \in C \) so that \( n_h = s_h t_h \). Assume there exist \( h \neq h_2 \in Q_k \) so that \( s_{h_1} s_{h_2} \neq 0 \). This implies that \( \Theta(\alpha_g(x)) = t_{h_1} \beta_{h_1} \Theta(x) t_{h_2}^* \beta_{h_2} \Theta(x) t_{h_2}^* \) for all \( x \in \mathcal{L}(N_2) \). Thus \( \beta_{h_1}(t_{h_2}^* t_{h_2}) \v_{h_1}^{-1} \v_{h_2}^* = \v_{h_2} t_{h_1} \beta_{h_1} \Theta(x) t_{h_2}^* \beta_{h_2} \Theta(x) t_{h_2}^* \v_{h_2} \in \mathcal{L}(N_2)' \cap \mathcal{M}_k = C_1 \). Thus
5 Von Neumann Algebraic Rigidity Aspects for Groups Arising via
Rips Constructions

An impressive milestone in the classification of von Neumann algebras was the emergence over
the past decade of the first examples of groups \( G \) that can be completely reconstructed from their
von Neumann algebras \( \mathcal{L}(G) \), i.e. \( W^* \)-superrigid groups [IPVI10, BV12, CI17]. The strategies used in
establishing these results share a common key ingredient, namely, the ability to first reconstruct from
\( \mathcal{L}(G) \) various algebraic feature of \( G \) such as its (generalized) wreath product decomposition in [IPVI10,
BV12], and respectively, its amalgam splitting in [CI17] Theorem A. This naturally leads to a broad
and independent study, specifically identifying canonical group algebraic features of a group that pass
to its von Neumann algebra. While several works have emerged recently in this direction [CdSS15,
CI17, CU18] the surface has been only scratched and still a great deal of work remains to be done.

A difficult conjecture of Connes predicts that all icc property (T) groups are \( W^* \)-superrigid. Unfortunately,
not a single example of such group is known at this time. Moreover, in the current literature there is an almost complete lack of examples of algebraic features occurring in a property (T) group
that are recognizable at the von Neumann algebraic level. In this section we make progress on this
problem for property (T) groups that appear as certain fiber products of Belegradek-Osin Rips type
constructions. Specifically, we have the following result:

**Theorem 5.1.** Let \( Q = Q_1 \times Q_2 \), where \( Q_i \) are icc, torsion free, biexact, property (T), weakly amenable, residually
finite groups. For \( i = 1, 2 \) let \( N_i \times_{\sigma_i} Q \in \text{Ripp}(Q) \) and denote by \( \Gamma = (N_1 \times N_2) \times_{\sigma} Q \) the semidirect product
associated with the diagonal action \( \sigma = \sigma_1 \times \sigma_2 : Q \to N_1 \times N_2 \). Denote by \( \mathcal{M} = \mathcal{L}(\Gamma) \) the corresponding \( II_1 \)
factor. Assume that \( \Gamma \) is arbitrary group and \( \Theta : \mathcal{L}(\Gamma) \to \mathcal{L}(\Lambda) \) is any \( * \)-isomorphism. Then there exist
group actions by automorphisms \( H \times_{\tau} K_i \) such that \( \Lambda = (K_1 \times K_2) \times_{\tau} H \) where \( \tau = \tau_1 \times \tau_2 : H \to K_1 \times K_2 \) is the
diagonal action. Moreover one can find a multiplicative character \( \eta : Q \to \mathbb{T} \), a group isomorphism \( \delta : Q \to H \),
a unitary \( w \in \mathcal{L}(\Lambda) \), and \( * \)-isomorphisms \( \Theta_i : \mathcal{L}(N_i) \to \mathcal{L}(K_i) \) such that for all \( x_i \in \mathcal{L}(N_i) \) and \( g \in Q \) we have

\[
\Theta((x_1 \otimes x_2)u_g) = \eta(g)w((\Theta(x_1) \otimes \Theta(x_2))v_{\delta(g)})w^*. \tag{5.0.1}
\]

Here \( \{u_g | g \in Q\} \) and \( \{v_h | h \in H\} \) are the canonical unitaries implementing the actions of \( Q \to \mathcal{L}(N_1) \otimes \mathcal{L}(N_2) \)
and \( H \to \mathcal{L}(K_1) \otimes \mathcal{L}(K_2) \), respectively.

From a different perspective our theorem can be also seen as a von Neumann algebraic superrigid-
ity result regarding conjugacy of actions on noncommutative von Neumann algebras. Notice that very
little is known in this direction as well, as most of the known superrigidity results concern algebras
arising from actions of groups on probability spaces.
We continue with a series of preliminary results that are essential to derive the proof of Theorem 5.1 at the end of the section. First we present a location result for commuting diffuse property (T) subalgebras inside a von Neumann algebra arising from products of relative hyperbolic groups.

**Theorem 5.2.** For any $i = 1, n$ let $H_i < G_i$ be an inclusion of infinite groups such that $H_i$ is residually finite and $G_i$ is hyperbolic relative to $H_i$. Denote by $H = H_1 \times \ldots \times H_n < G_1 \times \ldots \times G_n = G$ the corresponding direct product inclusion. Let $\mathcal{N}_i, \mathcal{N}_j \subseteq \mathcal{L}(G)$ be two commuting von Neumann subalgebras with property (T). Then for every $i \in 1, n$ there exists $k \in 1, 2$ such that $\mathcal{N}_k \prec \mathcal{L}(\hat{G}_i \times H_j)$, where $\hat{G}_i := \times_{j \neq i} G_j$.

**Proof.** Fix $i \in 1, n$. Since $H_i$ is residually finite then using Theorem 5.1 there is a short exact sequence

$$1 \to \ker(\pi_i) \to G_i \xrightarrow{\pi_i} F_i \to 1,$$

where $F_i$ is a non-elementary hyperbolic group and $\ker(\pi_i) = \langle H^0_i \rangle = \star_{i \in T}(H^0_i)$, for some subset $T \subseteq G_i$ and a finite index normal subgroup $H^0_i \triangleleft H_i$.

Following [CJ13, Notation 3.3] we now consider the von Neumann algebraic embedding corresponding to $\pi_i$, i.e. $\Pi_i : \mathcal{L}(G) \to \mathcal{L}(G) \otimes \mathcal{L}(F_i)$ given by $\Pi_i(\pi_i) = u_{\pi_i} \otimes \pi_i$ for all $g \in G_i$; here $u_{\pi_i}$ are the canonical unitaries of $\mathcal{L}(G)$ and $\pi_i$ are the canonical unitaries of $\mathcal{L}(F_i)$. From the hypothesis we have that $\Pi_i(\mathcal{N}_1), \Pi_i(\mathcal{N}_2) \subseteq \mathcal{L}(G) \otimes \mathcal{L}(F_i) = \mathcal{M}_i$ are commuting property (T) subalgebras. Fix $A \subseteq \Pi_i(\mathcal{N}_1)$ any diffuse amenable von Neumann subalgebra. Using [PV12, Theorem 1.4] we have either: a) $A <_\mathcal{M}_i \mathcal{L}(G) \otimes 1$, or b) $\Pi_i(\mathcal{N}_2)$ is amenable relative to $\mathcal{L}(G) \otimes 1$ inside $\mathcal{M}_i$.

Since the $\mathcal{N}_k$'s have property (T) then so do the $\Pi_i(\mathcal{N}_k)$'s. Thus using part b) above we get that $\Pi_i(\mathcal{N}_k) \prec_\mathcal{M}_i \mathcal{L}(G) \otimes 1$. On the other hand, if case a) above were to hold for all $A$'s then by [BO08, Corollary F.14] we would get that $\Pi_i(\mathcal{N}_1) \prec_\mathcal{M}_i \mathcal{L}(G) \otimes 1$. Therefore we can always assume that $\Pi_i(\mathcal{N}_k) \prec_\mathcal{M}_i \mathcal{L}(G) \otimes 1$ for $k = 1$ or $2$.

Due to symmetry we only treat $k = 1$. Using [CJ13, Proposition 3.4] we get that $\mathcal{N}_1 \prec \mathcal{L}(\ker(\Pi_i)) = \mathcal{L}(\hat{G}_i \times \ker(\pi_i))$. Thus there exist nonzero projections $p \in \mathcal{N}_1, q \in \mathcal{L}(\hat{G}_i \times \ker(\pi_i))$, nonzero partial isometry $v \in \mathcal{M}$ and a $*$-isomorphism $\phi : p\mathcal{N}_1 p \to B := \phi(p\mathcal{N}_1 p) \subset q\mathcal{L}(\hat{G}_i \times \ker(\pi_i))q$ on the image such that

$$\phi(x)v = vx \text{ for all } x \in p\mathcal{N}_1 p. \quad (5.0.2)$$

Also notice that since $\mathcal{N}_1$ has property (T) then so does $p\mathcal{N}_1 p$ and therefore $B \subseteq q\mathcal{L}(\hat{G}_i \times \ker(\pi_i))q$ is a property (T) subalgebra. Since $\ker(\pi_i) = \star_{i \in T}(H^0_i)$ then by further conjugating $q$ in the factor $\mathcal{L}(\hat{G}_i \times \ker(\pi_i))$ we can assume that there exists a unitary $u \in \mathcal{L}(\hat{G}_i \times \ker(\pi_i))$ and a projection $q_0 \in \mathcal{L}(\hat{G}_i)$ such that $B \subseteq u(q_0\mathcal{L}(\hat{G}_i)q_0)\mathcal{L}(\ker(\pi_i))u^*$. Using property (T) of $B$ and [IP05, Theorem] we further conclude that there is $t_0 \in T$ such that $B \subseteq u(q_0\mathcal{L}(\hat{G}_i)q_0)\mathcal{L}(\ker(\pi_i))u^*$. Composing this intertwining with $\phi$ we finally conclude that $\mathcal{N}_1 \prec_\mathcal{M} \mathcal{L}(\hat{G}_i \times H_k)$, as desired. 

**Theorem 5.3.** Under the same assumptions as in Theorem 5.2 for every $k \in 1, n$ one of the following must hold

1) there exists $i \in 1, 2$ such that $\mathcal{N}_i \prec_\mathcal{M} \mathcal{L}(\hat{G}_k)$;

2) $\mathcal{N}_1 \vee \mathcal{N}_2 \prec_\mathcal{M} \mathcal{L}(\hat{G}_k \times H_k)$.

**Proof.** From Theorem 5.2 there exists $i \in 1, 2$ such that $\mathcal{N}_i \prec_\mathcal{M} \mathcal{L}(\hat{G}_k \times H_k)$. For convenience assume that $i = 1$. Thus there exist nonzero projections $p \in \mathcal{N}_1, q \in \mathcal{L}(\hat{G}_k \times H_k)$, nonzero partial isometry $v \in \mathcal{M}$ and a $*$-isomorphism $\phi : p\mathcal{N}_1 p \to B := \phi(p\mathcal{N}_1 p) \subset q\mathcal{L}(\hat{G}_k \times H_k)q$ on the image such that

$$\phi(x)v = vx \text{ for all } x \in p\mathcal{N}_1 p. \quad (5.0.3)$$

Notice that $q \geq vv^* \in B' \cap q\mathcal{M}q$ and $p \geq vv^* \in p\mathcal{N}_1 p' \cap p\mathcal{M}p$. Also we can pick $x$ such that $s(E_{\mathcal{L}(\hat{G}_k \times H_k)}(vv^*)) = q$. Next we assume that $B \prec_\mathcal{L}(\hat{G}_k \times H_k)$. Thus there exist nonzero projections $p' \in B, q' \in \mathcal{L}(\hat{G}_k)$,
nonzero partial isometry \( w \in \mathcal{L}(\hat{G}_k \times H_k) p' \) and a \( * \)-isomorphism \( \psi : p' B p' \to \mathcal{Q}(\hat{G}_k) \mathcal{Q}' \) on the image such that
\[
\psi(x)w = wx \text{ for all } x \in p' B p'.
\] (5.0.4)

Notice that \( q \geq p' \geq w^*w \in (p' B p')' \cap p' \mathcal{M} p' \) and \( q' \geq w^*w \in (\psi p' B p')' \cap \mathcal{Q}' \mathcal{M} q' \). Using (5.0.3) and (5.0.4) we see that
\[
\psi(\phi(x))wvw = w\psi(x)vw = wxw \text{ for all } x \in p_0 N_i p_0,
\] (5.0.5)
where \( p_0 \in N_i \) is a projection picked so that \( \phi(p_0) = p' \). Also we note that if \( 0 = wv \) then \( 0 = w^*w \) and hence \( 0 = E_{\mathcal{L}(\hat{G}_k \times H_k)}(w^*w^*) = wE_{\mathcal{L}(\hat{G}_k \times H_k)}(w^*) \). This further implies that \( 0 = w(\psi E_{\mathcal{L}(\hat{G}_k \times H_k)}(w^*)) = wz = wz \) which is a contradiction. Thus \( wv \neq 0 \) and taking the polar decomposition of \( wv \) we see that (5.0.5) gives 1).

Next we assume that \( B \neq \mathcal{L}(\hat{G}_k \times H_k) \mathcal{L}(\hat{G}_k) \). Since \( G_k \) is hyperbolic relative to \( H_k \) then by Lemma 2.2 we have that for all \( x, x_1 x_2, \ldots, x_j \in M \) such that \( B x \subseteq \sum_{i=1}^j x i \mathcal{B} \) we must have that \( x \in \mathcal{L}(\hat{G}_k \times H_k) \). Hence in particular we have that \( \psi w^* \in B' \cap \mathcal{Q} \mathcal{M} q \subseteq \mathcal{L}(\hat{G}_k \times H_k) \) and thus relation (5.0.3) implies that \( B w^* \mathcal{Q} = v N_i \mathcal{Q}^* \subseteq \mathcal{L}(\hat{G}_k \times H_k) \). Also for every \( c \in N_{i+1} \) we can see that
\[
B \psi w^* = B w^* v^* = v N_i \mathcal{Q} \mathcal{Q}^* = w \psi v N_i \mathcal{Q} \mathcal{Q}^* = \psi w^* v^* = w \psi v^* = w v^* B = \psi v^* B.
\] (5.0.6)
Therefore by Lemma 2.2 again we have that \( \psi v^* \in \mathcal{L}(\hat{G}_k \times H_k) \) and hence \( v N_{i+1} v^* \subseteq \mathcal{L}(\hat{G}_k \times H_k) \). Thus \( v N_i \mathcal{Q}^* v^* = v^* v N_i v^* = v N_i v^* \subseteq \mathcal{L}(\hat{G}_k \times H_k) \), which by Popa’s intertwining techniques implies that \( N_1 \mathcal{Q} \mathcal{Q}^* N_2 \subseteq \mathcal{L}(\hat{G}_k \times H_k) \), i.e. 2) holds.

We now proceed towards proving the main result of this chapter. To simplify the exposition we first introduce a notation that will be used throughout the section.

**Notation 5.4.** Denote by \( Q = Q_1 \times Q_2 \) where \( Q_i \) are infinite, residually finite, biequivalence (T), icc groups. Then consider \( N_i = N_i \times Q \in \mathcal{R} \mathcal{P} \mathcal{R}(Q) \) and consider the semi-direct product \( \Gamma = (N_1 \times N_2) \rtimes \sigma Q \) arising from the diagonal action \( \sigma = \sigma_1 \times \sigma_2 : Q \to \text{Aut} N_1 \times N_2 \), i.e. \( \sigma_q (n_1, n_2) = ((\sigma_1)_q (n_1), (\sigma_2)_q (n_2)) \) for all \( (n_1, n_2) \in N_1 \times N_2 \). For further use we observe that \( \Gamma \) is the fiber product \( \Gamma_1 \times \mathcal{Q} \Gamma_2 \) and thus embeds into \( \Gamma_1 \times \Gamma_2 \) where \( \Gamma \) embeds diagonally into \( \mathcal{Q} \times \mathcal{Q} \). Over the next proofs when we refer to this copy we will often denote it by \( d(Q) \). Also notice that \( \Gamma \) is a property (T) group as it arises from an extension of property (T) groups. Furthermore, \( \Gamma_1, \Gamma_2 \in \mathcal{R} \mathcal{P} \mathcal{R}(Q) \) easily implies that \( \Gamma \) is an icc group.

For future use, use also recall the notion of the comultiplication studied in [PV10, 601]. Let \( \Gamma \) be a group as above, and assume that \( \Lambda \) is a group such that \( \mathcal{L}(\Gamma) = \mathcal{L}(\Lambda) = \mathcal{M} \). Then the “comultiplication along \( \Lambda \)” \( \Delta : \mathcal{M} \to \mathcal{M} \mathcal{O} \mathcal{M} \) be defined by \( \Delta(\mathcal{V}_\Lambda) = \mathcal{V}_\Lambda \otimes \mathcal{V}_\Lambda \), for all \( \Lambda \in \mathcal{L} \).

**Theorem 5.5.** Let \( \Gamma \) be a group as in Notation 5.4 and assume that \( \Lambda \) is a group such that \( \mathcal{L}(\Gamma) = \mathcal{L}(\Lambda) = \mathcal{M} \). Let \( \Delta : \mathcal{M} \to \mathcal{M} \mathcal{O} \mathcal{M} \) be comultiplication along \( \Lambda \) as in Notation 5.4. Then the following hold:

3) for all \( j \in \Omega, \Pi \) there is \( i \in \Omega, \Pi \) such that \( \Delta(\mathcal{L}(N_i)) \otimes _{\mathcal{M} \mathcal{O} \mathcal{M}} \mathcal{M} \mathcal{O} \mathcal{M} \mathcal{L}(N_j) \), and

4) \( a) \) for all \( j \in \Omega, \Pi \) there is \( i \in \Omega, \Pi \) such that \( \Delta(\mathcal{L}(Q_i)) \otimes _{\mathcal{M} \mathcal{O} \mathcal{M}} \mathcal{M} \mathcal{O} \mathcal{M} \mathcal{L}(N_j) \) or

\( b) \) \( \Delta(\mathcal{L}(Q_j)) \otimes _{\mathcal{M} \mathcal{O} \mathcal{M}} \mathcal{M} \mathcal{O} \mathcal{M} \mathcal{L}(Q_i) \), moreover in this case for every \( j \in \Omega, \Pi \) there is \( i \in \Omega, \Pi \) such that \( \Delta(\mathcal{L}(Q_i)) \otimes _{\mathcal{M} \mathcal{O} \mathcal{M}} \mathcal{M} \mathcal{O} \mathcal{M} \mathcal{L}(Q_j) \),

**Proof.** Let \( \tilde{\mathcal{M}} = \mathcal{L}(\Gamma_1 \times \Gamma_2) \). Since \( \Gamma \gt \Gamma_1 \times \Gamma_2 \) we notice the following inclusions \( \Delta(\mathcal{L}(N_1)) \subseteq \mathcal{M} \mathcal{O} \mathcal{M} \mathcal{L}(\Gamma_1 \times \Gamma_2) \subseteq \mathcal{L}(\Gamma_1 \times \Gamma_2) \). Since \( \Gamma_1 \) is hyperbolic relative to \( Q \) then using Theorem 5.3 we have either

5) there exists \( i \in 1, 2 \) such that \( \Delta(\mathcal{L}(N_i)) \otimes _{\mathcal{M} \mathcal{O} \mathcal{M}} \mathcal{M} \mathcal{O} \mathcal{L}(\Gamma_1) \), or

6) \( \Delta(\mathcal{L}(N_1 \times N_2)) \otimes _{\mathcal{M} \mathcal{O} \mathcal{M}} \mathcal{L}(\Gamma_1 \times \Gamma_2) \)
Assume 5) holds. Since \(\Delta(\mathcal{L}(N_1)) \subset M \hat{\otimes} L(\Gamma)\) then by Lemma 2.3 there is an \(h \in \Gamma_1 \times \Gamma_2 \times \Gamma_1 \times \Gamma_2\) so that

\[
\Delta(\mathcal{L}(N_1)) < M \hat{\otimes} L(\Gamma) \times (\Gamma \times h(\Gamma_1 \times \Gamma_2 \times \Gamma_1 \times \Gamma_2)^{-1}) = \Delta(\Gamma \times (\Gamma \times \Gamma_1 \times \Gamma_2 \times \Gamma_2)^{-1}) = \Delta(\mathcal{L}((N_1 \times N_2) \times Q) \cap (N_1 \times Q \times 1)) = M \hat{\otimes} L(\Gamma_1). \]

Note that since \(\Delta(\mathcal{L}(N_1))\) is regular in \(M \hat{\otimes} M\), using Lemma 2.4 we get that \(\Delta(\mathcal{L}(N_1)) < M \hat{\otimes} L(\Gamma)\), thereby establishing 3).

Assume 6) holds. Since \(\Delta(\mathcal{L}(N_1 \times N_2)) \subset \Delta(\Gamma \times \Gamma)\) then by Lemma 2.3 there is an \(h \in \Gamma_1 \times \Gamma_2 \times \Gamma_1 \times \Gamma_2\) such that \(\Delta(\mathcal{L}(N_1 \times N_2)) < \mathcal{L}(\Gamma \times (\Gamma \times h(\Gamma_1 \times \Gamma_2 \times \Gamma_1 \times \Gamma_2)^{-1})) = L(\mathcal{L}((N_1 \times N_2) \times (Q \cap (N_1 \times Q \times 1))) = M \hat{\otimes} L(\Gamma_1). \)

In conclusion, there exist a *-isomorphism on its image \(\phi: p\Delta(\mathcal{L}(N_1 \times N_2)) \to B := \phi(p\Delta(\mathcal{L}(N_1 \times N_2)))\) for all \(\theta \in qM \hat{\otimes} M\) such that

\[
\phi(x)v = vx \quad \text{for all } x \in p\Delta(\mathcal{L}(N_1 \times N_2)). \tag{5.0.7}
\]

Next assume that 3) doesn’t hold. Thus proceeding as in the first part of the proof of Theorem 5.3 we get

\[
B \nsubseteq M \hat{\otimes} L(\mathcal{L}(N_1 \times d(Q))) =: M_1. \tag{5.0.8}
\]

Next we observe the following inclusions

\[
M_1 \times_{B_{\mathcal{L}(\mathcal{L}(N_1 \times d(Q)))}} d(Q) = M \hat{\otimes} L(\mathcal{L}(N_1 \times d(Q))) \subset M \hat{\otimes} L(\mathcal{L}(N_1 \times d(Q))) = M_1 \times_{B_{\mathcal{L}(\mathcal{L}(N_1 \times d(Q)))}} d(Q).
\]

Also since \(Q\) is malnormal in \(N_2 \times Q\) it follows from Lemma 2.2 that \(\mathcal{L}(\mathcal{L}(N_1 \times d(Q))) \subset M \hat{\otimes} L(\mathcal{L}(N_1 \times d(Q)))\) and hence \(B\mathcal{L}(\mathcal{L}(N_1 \times d(Q))) \subset M \hat{\otimes} L(\mathcal{L}(N_1 \times d(Q)))\). Pick \(u \in \mathcal{L}(\mathcal{L}(N_1 \times d(Q)))\) and using (5.0.7) we see that there exist \(n_1, n_2, \ldots, n_s \in p(M \hat{\otimes} M)\) satisfying

\[
B\mathcal{L}(\mathcal{L}(N_1 \times d(Q))) = \mathcal{L}(\mathcal{L}(N_1 \times d(Q))) = \mathcal{L}(\mathcal{L}(N_1 \times d(Q))).
\]

Then by Lemma 2.2 again we must have that \(\mathcal{L}(\mathcal{L}(N_1 \times d(Q))) \subset M \hat{\otimes} L(\mathcal{L}(N_1 \times d(Q)))\). Hence we have shown that

\[
\mathcal{L}(\mathcal{L}(N_1 \times d(Q))) = \mathcal{L}(\mathcal{L}(N_1 \times d(Q))).
\]

Since \(\mathcal{L}(\mathcal{L}(N_1 \times d(Q))) \subset \mathcal{L}(\mathcal{L}(N_1 \times d(Q)))\) and (5.0.11) further implies that

\[
\mathcal{L}(\mathcal{L}(N_1 \times d(Q))) \subset \mathcal{L}(\mathcal{L}(N_1 \times d(Q))).
\]

Here for every inclusion of von Neumann algebras \(R \subseteq T\) and projection \(p \in R\) we used the formula

\[
\mathcal{L}(\mathcal{L}(N_1 \times d(Q))) = \mathcal{L}(\mathcal{L}(N_1 \times d(Q))).
\]

Again we notice that \(\mathcal{L}(\mathcal{L}(Q_1)) \subset \mathcal{L}(\mathcal{L}(M)) \subset \mathcal{L}(\mathcal{L}(\Gamma)) \subset \mathcal{L}(\Gamma_1 \times \Gamma_2 \times \Gamma_1 \times \Gamma_2). \)

Using Theorem 5.3 we must have that either

7) \(\Delta(\mathcal{L}(Q_1)) \subset \mathcal{L}(\mathcal{L}(\Gamma)) \subset \mathcal{L}(\mathcal{L}(\Gamma)).\) or
8) $\Delta(L(Q)) <_M M \otimes L(G_1 \times Q).

Proceeding exactly as in the previous case, and using Lemma 2.4 we see that 7) implies $\Delta(L(Q_1)) <_M M \otimes L(N_1)$ which in turn gives 4a). Also proceeding as in the previous case, and using Lemma 2.5 we see that 8) implies

$$\Delta(L(d(Q)) <_M M \otimes L(N_1 \times d(Q)).$$

(5.0.13)

To show the part 4b) we will exploit (5.0.13). Notice that there exist nonzero projections $r \in \Delta(L(Q))$, $t \in M \otimes L(N_1 \times d(Q))$, nonzero partial isometry $w \in r(M \otimes M)t$ and $*$-isomorphism onto its image $\phi: r \Delta(L(Q)) \to C := \phi(r \Delta(L(Q)))r \in (M \otimes L(N_1 \times d(Q)))t$ such that

$$\phi(x) = wx \text{ for all } x \in r \Delta(L(Q))r.$$  

(5.0.14)

Since $L(Q)$ is a factor we can assume without loss of generality that $r = \Delta(r_1 \otimes r_2)$ where $r_i \in L(Q_i)$. Hence $C = \phi(\Delta(r_1 \otimes r_2)) = \phi(\Delta(r_1 \otimes r_2)) \otimes \phi(r_1 \otimes r_2) = C_1 \vee C_2$ where we denoted by $C_i = \phi(\Delta(r_i \otimes r_1))r_i \in t(M \otimes L(N_1 \times d(Q)))t$. Notice that $C_i$’s are commuting property (T) subfactors of $M \otimes L(N_1 \times d(Q))$. Since $N_1 \times Q$ is hyperbolic relative to $Q$ and seeing $C_1 \vee C_2 \subseteq M \otimes L(N_1 \times d(Q)) \subseteq L(G_1 \times G_2 \times (N_1 \times d(Q)))$ then by applying Theorem 5.3 we have that there exits $i \in 1, 2$ such that

9) $C_1 <_M M \otimes L(N_1 \times d(Q)) L(G_1 \times G_2)$ or

10) $C_1 \vee C_2 <_M M \otimes L(N_1 \times d(Q)) L(G_1 \times G_2 \times d(Q)).$

Since $C_1 \subseteq M \otimes M$ then 9) and Lemma 2.6 imply that $C_1 <_M M \otimes 1$ which by [Go11] Lemma 9.2 further implies that $C_1$ is atomic, which is a contradiction. Thus we must have 10). However since $C_1 \vee C_2 \subseteq M \otimes M$ then 10) and Lemma 2.6 give that $C_1 \vee C_2 <_M M \otimes L(d(Q))$ and composing this intertwining with $\phi$ (as done in the proof of the first case in Theorem 5.3) we get that $\Delta(L(Q)) <_M M \otimes L(d(Q))$. Now we show the moreover part. So in particular the above intertwining shows that we can assume from the beginning that $C = C_1 \vee C_2 \subseteq t(M \otimes L(d(Q)))t$. Since $Q_1$ are biexact, weakly amenable then by applying [PV12] Theorem 1.4) we must have that either $C_1 \subseteq M \otimes L(d(Q))$ or $C_2 \subseteq M \otimes L(d(Q))$ or $C_1 \vee C_2$ is amenable relative to $M \otimes L(d(Q))$ inside $M \otimes M$. However since $C_1 \vee C_2$ has property (T) the last case above still entails that $C_1 \vee C_2 <_M M \otimes L(d(Q))$ which completes the proof. 

Theorem 5.6. Let $\Gamma$ be a group as in Notation 5.4 and assume that $\Lambda$ is a group such that $L(\Gamma) = L(\Lambda) = M$. Let $\Delta: M \rightarrow M \otimes M$ be the “comultiplication along $\Lambda$” as in the Notation 5.4. Also assume for every $j \in 1, 2$ there is $i \in 1, 2$ such that either $\Delta(L(Q)) \leq M \otimes L(Q)$ or $\Delta(L(Q)) \leq M \otimes L(N_1)$. Then one can find subgroups $\Phi_1, \Phi_2 \subseteq \Phi \subseteq \Lambda$ such that

1. $\Phi_1, \Phi_2$ are infinite, commuting, property (T), finite-by-icc groups;
2. $[\Phi: \Phi_1 \Phi_2] < \infty$ and $Q \text{N}^{(1)}(\Phi) = \Phi$;
3. there exist $\mu \in U(M), z \in \mathcal{P}(Z(L(\Phi))), h = \mu z \mu^* \in \mathcal{P}(L(Q))$ such that

$$\mu L(\Phi) z \mu^* = h L(Q) h.$$  

(5.0.15)

Proof. For the proof we use an approach based upon the methods developed in [CdSS15, Ch17, Cu18]. For the reader’s convenience we include all the details.

Since the relative commutants $L(Q) \cap M$ and $L(N_1) \cap M$ are non-amenable then in both cases using [DHI16] Theorem 4.1) (see also [Io11] Theorem 3.1] and [CdSS15] Theorem 3.3), one can find a subgroup $\Sigma < \Lambda$ with $C_\Lambda(\Sigma)$ non-amenable such that $L(Q) \leq M \otimes L(\Sigma)$. Thus there are $0 \neq p \in \mathcal{P}(L(Q_1)), 0 \neq f \in \mathcal{P}(L(\Sigma))$, a partial isometry $0 \neq \nu \in f \mathcal{M} p$ and a $*$-isomorphism onto its image $\phi: p L(Q_1) p \rightarrow B := \phi(p L(Q_1) p) \subseteq f L(\Sigma) f$ so that

$$\phi(x) = vx, \text{ for all } x \in p L(Q_1) p.$$  

(5.0.16)
Moreover, replacing $w^* \in B' \cap fMf$ and $v^* \in (pL(Q_1))' \cap pMp = L(Q_2)p$. Then (5.0.16) implies that $Bw^* = vL(Q_1)v^*$, where $u_1 \in \mathcal{V}(M)$ extends $v$. Passing to relative commutants we get $w^*(B' \cap fMf)v^* = u_1v^*(pL(Q_1))' \cap pMp)p_v^* = u_1v^*(pL(Q_2))v^*$. These relations further imply $w^*(B \vee B' \cap fMf)v^* = Bw^* \vee v^{*}(B' \cap fMf)v^* \subseteq u_1L(Q_1)u_1^*$. As $L(Q)$ is a factor, there is a new $u_2 \in \mathcal{V}(M)$ with
\[(B \vee B' \cap fMf)z_2 \subseteq u_2L(Q)u_2^*. \tag{5.0.17}\]
Here $z_2$ is the central support of $w^* \in B \vee B' \cap fMf$ and hence $z_2 \in Z(B' \cap fMf)$ and $w^* \leq z_2 \leq f$.
Let $\Omega = C_\Lambda(\Sigma)$ and notice that $L(\Omega)z_2 \subseteq ((fL(\Sigma)f)' \cap fMf)z_2 \subseteq (B' \cap fMf)z_2 \subseteq u_2L(Q)u_2^*$. Since $Q$ is malnormal in $\Gamma$ and $z_2 \in L(\Omega)')' \cap fMf$ we further have $z_2L(\Omega)f \vee ((L(\Omega)f)' \cap fMf))z_2 \subseteq u_2L(Q)u_2^*$. Again since $L(Q)$ is a factor there is $\eta \in \mathcal{V}(M)$ so that
\[(L(\Omega)f \vee ((L(\Omega)f)(\cap fMf)))z \subseteq \eta^*L(Q)\eta, \tag{5.0.18}\]
where $z$ is the central support of $z_2L(\Omega) \cap fMf$ and hence $z \leq f$. Now since $fL(\Sigma)f \subseteq (L(\Omega)f)' \cap fMf$ then by (5.0.18) we get $\eta \eta^*L(\Omega)f \subseteq \eta^*L(Q)\eta$ and hence
\[\eta \eta^*L(\Omega)f \subseteq \eta^*L(Q)\eta. \tag{5.0.19}\]
Since $w^* \leq z \in (fL(\Sigma)f)' \cap fMf$ and $B$ is a factor then the map $\psi': pL(Q)p \rightarrow Bw^* \subseteq fL(\Sigma)fz$ given by $\phi'(x) = \eta \eta^*L(\Omega)f \subseteq \eta^*L(Q)\eta$ is a $\star$-isomorphism that satisfies $\phi'(x)y = yx$, for any $x \in pL(Q_1)p$, where $0 \neq y = \eta \eta^*L(\Omega)f \subseteq \eta^*L(Q)\eta$ is a partial isometry. Hence, $L(Q_1) \subseteq \mathcal{U}u^*fL(\Sigma)fzu$. Since $Q$ is malnormal in $\Gamma$, it follows that $L(Q_1) < \mathcal{L}(\Sigma)$. Since $\psi'(x)y = yx$, for any $x \in pL(Q_1)p$, and $\star$-isomorphism onto its image $\psi : aL(Q_1)a \rightarrow DO := \psi(aL(Q_1)a) \subseteq \eta \eta^*L(\Omega)f \subseteq \eta^*L(Q)\eta$ satisfying the following properties:

4) the inclusion $D \vee (D' \cap \eta \eta L(Q)(\Sigma)qz)\subseteq \eta \eta L(\Omega)qz$ has finite index;
5) there is a partial isometry $0 \neq w \in L(Q)$ such that $\psi(x)w = wx$ for all $x \in aL(Q_1)a$.

Now observe the algebras $D, D' \cap \eta \eta L(Q)(\Sigma)qz$ and $\eta \eta L(Q)qz$ are mutually commuting. Also the prior relations show that $D$ and $\eta \eta L(Q)qz$ has no amenable direct summand. Since $Q_1$ and $Q_2$ are bi-exact it follows that $D' \cap \eta \eta L(Q)(\Sigma)qz$ must be purely atomic. Therefore, one can find $0 \neq e \in \mathcal{P}(Z(D' \cap u^*qL(\Sigma)qz))$ such that after cutting down by $q$ the containment in 4) and replacing $D$ by $De$ one can assume that

4') $D \subseteq \eta \eta L(\Omega)qz$ is a finite index inclusion of non-amenable II_1 factors.

Moreover, replacing $w$ by $e$ and $\psi(x)$ by $\psi(x)e$ in the intertwining in 5) still holds.

Notice that 5) implies $ww^* \in D' \cap rL(Q)r, w^*w \in aL(Q_1)a \cap aL(Q)a = Ca \cap L(Q_2)$. Thus there exists $0 \neq b \in \mathcal{P}(L(Q_2))$ such that $w^*w = a \otimes b$. Pick $c \in \mathcal{V}(L(Q))$ such that $w = c(a \otimes b)$ then 5) gives that
\[Dw^* = wL(Q_1)w^* = c(aL(Q_1)a \otimes Cb)c^*. \tag{5.0.20}\]
Let $\Xi = QN_{L(\Sigma)}$. Then using (5.0.20) and 4') above we see that
\[c(a \otimes b)Q(L)(\Sigma)(a \otimes b)c^* = w^* \eta \eta QN_{L(\Sigma)}L(\Sigma)qz^*ww^* = w^* \eta \eta Q(\Xi)qz^*ww^* \tag{5.0.21}\]
and also
\[c(Ca \otimes bL(Q_2)b)c^* = (c(aL(Q_1)a \otimes Cb)c^*)' \cap c(a \otimes b)Q(L)(\Sigma)(a \otimes b)c^* \]
\[= (Dw^*)' \cap w^* \eta \eta Q(\Xi)qz^*ww^* \]
\[= w^* (D' \cap \eta \eta Q(\Xi)qz^*)ww^*. \tag{5.0.22}\]
Using $4')$ and [Po02] Lemma 3.1 we also have that
\[ D \vee (\eta qz\Sigma zq^*)' \cap \eta qz\Sigma zq^* \subseteq D \vee D' \cap \eta qz\Sigma zq^*, \quad (5.0.23) \]
where the symbol $\subseteq$ above means inclusion of finite index.

Relation $(5.0.24)$ also shows that
\[ D \vee (\eta qz\Sigma zq^*)' \cap \eta qz\Sigma zq^* \subseteq \eta qz\Sigma(vC\Lambda(\Sigma))zq^* \subseteq \eta qz\Sigma zq^*. \quad (5.0.24) \]

Here $vC\Lambda(\Sigma) = \{ \lambda \in \Lambda : |\lambda^2| < \infty \}$ is the virtual centralizer of $\Sigma$ in $\Lambda$.

Relation $(5.0.25)$ also shows that $\eta qz\Sigma zq^* \subseteq \eta qz\Sigma zq^*$ and using the finite index condition in $(5.0.23)$ we have $\eta qz\Sigma zq^* \subseteq \eta qz\Sigma zq^*$. Thus, by $(5.0.25)$ we further have $\eta qz\Sigma zq^* \subseteq \eta qz\Sigma zq^* \subseteq \eta qz\Sigma(vC\Lambda(\Sigma))zq^*$ and since $\Sigma(vC\Lambda(\Sigma)) \leq \Phi$ and $[\Phi : \Xi] < \infty$ then using [CH17] Lemma 2.6 we get that $[\Phi : \Sigma(vC\Lambda(\Sigma))] < \infty$.

Combining $(5.0.21)$, $(5.0.26)$ and $(5.0.21)$ we notice that
\[ \text{c}(a \odot b)(\mathcal{Q})(a \odot b)c^* = \text{c}(a \odot b)(\mathcal{Q})(a \odot b)c^* = \text{c}(a \odot b)(\mathcal{Q})(a \odot b)c^*. \quad (5.0.26) \]

As $Q$ has property (T) then by [CH17] Lemma 2.13 so is $\Phi$ and $\Xi$ and hence $\Sigma vC\Lambda(\Sigma)$ as well. Let $\{O_n\}_n$ be an enumeration of all the orbits in $\Lambda$ under conjugation by $\Sigma$. Denote by $\Omega_n = \{O_1, ..., O_n\}$. Clearly $\Omega_n \leq \Omega_{n+1}$ and $\Sigma$ normalizes $\Omega_n$ for all $n$. Notice that $\Omega_n \Sigma \leq \Omega_{n+1} \Sigma$ for all $n$ and in fact $\Omega_n \Sigma$ and $\Sigma(vC\Lambda(\Sigma))$ since $\Sigma(vC\Lambda(\Sigma))$ has property (T) there exists $n_0$ such that $\Omega_{n_0} \Sigma = (\Sigma(vC\Lambda(\Sigma)))$. In particular, there is a finite index subgroup $\Sigma' \leq \Sigma$ such that $[\Sigma', \Omega_{n_0}] = 1$ and hence $\Sigma', \Omega_{n_0} \leq \Sigma(vC\Lambda(\Sigma)) \leq \Phi$ are commuting subgroups. Moreover, if $r_1$ is the central support of $\eta qz\Sigma$ then by $(5.0.26)$ we also have that $\eta qz\Sigma \subseteq \eta qz\Xi \subseteq \eta qz\Sigma zq^* r_1$ for some unitary $\eta$.

Now since the $\Omega_i$'s are biaequotient the same argument from [CdS15] shows that the finite conjugacy radical of $\Phi$ is finite. Hence $\Phi$ is a finite-by-icc group and this canonically implies that $\Phi_1 := \Sigma'$ and $\Phi_2 := \Omega_{n_0}$ are also finite-by-icc. As $\Phi$ has property (T) then so do the $\Phi_i$'s. Altogether, the above arguments and $(5.0.26)$ show that there exist subgroups $\Phi_1, \Phi_2 \leq \Phi$ satisfying the following properties:

1) $\Phi_1, \Phi_2$ are infinite, commuting, property (T), finite-by-icc groups;
2) $[\Phi : \Phi_1\Phi_2] < \infty$ and $QN^1(\Lambda)(\Phi) = \Phi$;
3) there exist $\mu \in \mathcal{Q}(\mathcal{M}), d \in \mathcal{P}(\mathcal{L}(\Phi)), h = \mu d^* \mu^* \in \mathcal{P}(\mathcal{L}(Q))$ such that
\[ \mu d^* \mu^* \in \mathcal{P}(\mathcal{L}(Q))h. \quad (5.0.27) \]

In the last part of the proof we show that after replacing $d$ with its central support in $\mathcal{L}(Q)$, all the required relations in the statement still hold. Since $\mathcal{L}(Q)$ is a factor then using $(5.0.27)$ one can find $\xi \in \mathcal{Q}(\mathcal{M})$ such that $\mathcal{L}(\Phi)\xi^* \subseteq \mathcal{L}(Q)$ where $t$ is the central support of $d$ in $\mathcal{L}(Q)$. Hence $\mathcal{L}(\Phi)\xi^* \subseteq \mathcal{L}(Q)$ where $r_0 = \mathcal{L}(Q) r_0$, where $r_0 = \mathcal{L}(Q) r_0$. Fix $c_0 \leq t$ and $f_0 \leq d$ projections in the factor $\mathcal{L}(\Phi)\xi^*$ such that $\tau(f_0) \geq \tau(c_0)$. From $(5.0.27)$ we have $\mu c_0 \mathcal{L}(\Phi) f_0 \mu^* = \mathcal{L}(Q) l$ and $\mathcal{L}(\Phi) e_0 \xi^* \subseteq \mathcal{L}(Q) r_0$, where $r_0 = \mathcal{L}(Q) r_0$. Let $\xi \in \mathcal{L}(Q)$ be a unitary such that $r_0 \leq \xi c_0 \xi^*$. Thus $\mathcal{L}(\Phi) e_0 \xi^* \subseteq \mathcal{L}(Q) r_0 \subseteq \xi c_0 \mathcal{L}(Q) l \xi^* = \xi c_0 \mathcal{L}(Q) l \xi^* \subseteq \mathcal{L}(\Phi)\xi^*$ and hence
\[ \mu^* \xi c_0 \mathcal{L}(\Phi) e_0 \xi c_0 \mathcal{L}(\Phi) f_0 \mu^* \xi^* \xi^* \xi = \mathcal{L}(\Phi) \mu^* \xi c_0 \xi. \quad (5.0.28) \]
Next let \( e_0 + p_1 + p_2 + \ldots + p_s = t \) where \( p_i \in \mathcal{L}(\Phi) \) \( i \) are mutually orthogonal projection such that \( e_0 \) is von Neumann equivalent (in \( \mathcal{L}(\Phi) \)) to \( p_i \) for all \( i \in 1, s-1 \) and \( p_0 \) is von Neumann subequivalent to \( e_0 \).

Now let \( u_i \) be unitaries in \( \mathcal{L}(\Phi) \) such that \( u_i p_i u_i^* = e_0 \) for all \( i \in 1, s-1 \) and \( u_s p_s u_s^* = z' \leq e_0 \). Combining this with \[5.0.28\] we get

\[
\begin{align*}
\mu^* \xi^*_0 \xi_0 \mathcal{L}(\Phi) t_i & = \mu^* \xi^*_0 \xi_0 \mathcal{L}(\Phi) p_i = \mu^* \xi^*_0 \xi_0 \mathcal{L}(\Phi) u_i^* e_0 u_i = \mu^* \xi^*_0 \xi_0 \mathcal{L}(\Phi) e_0 u_i \\
& \subseteq \mathcal{L}(\Phi) \mu^* \xi^*_0 \xi_0 + \sum_{i=1}^s \mathcal{L}(\Phi) \mu^* \xi^*_0 \xi_0 u_i.
\end{align*}
\]

In particular, this relation shows that \( \mu^* \xi^*_0 \xi_0 e_0 \in QN^{(1)}_{\mathcal{L}(\Lambda)}(\mathcal{L}(\Phi)) \) and since \( QN^{(1)}_{\mathcal{L}(\Lambda)}(\mathcal{L}(\Phi))^\prime = \mathcal{L}(\Phi) \) by \( 2 \) then we conclude that \( \mu^* \xi^*_0 \xi_0 \in \mathcal{L}(\Phi). \) Thus using this together with \[5.0.28\] one can check that

\[
\begin{align*}
\xi e_0 \mathcal{L}(\Phi) e_0 \xi^* & = \xi \mathcal{L}(\Phi) e_0 \xi^* \mu(\mu^* \xi^*_0 \xi_0 \mathcal{L}(\Phi) e_0 \xi^*_0 \xi_0 \mu) \mu^* \xi^*_0 \xi_0 \xi_0 \\
& = \xi \mathcal{L}(\Phi) e_0 \xi^* \mu(\mu^* \xi^*_0 \xi_0 \mathcal{L}(\Phi) f_0 \mu^* \xi^*_0 \xi_0 \xi^* \\
& = \xi \mathcal{L}(\Phi) e_0 \xi^* \mathcal{L}(Q) \xi^*_0 \xi_0 = r_0 \mathcal{L}(Q) r_0.
\end{align*}
\]

In conclusion we have proved that \( \xi \mathcal{L}(\Phi) t \xi^* \subseteq r_2 \mathcal{L}(Q) r_2 \) and for all \( e_0 \leq t \) and \( f_0 \leq d \) projections in the factor \( \mathcal{L}(\Phi) \) such that \( \tau(f_0) \geq \tau(e_0) \) we have \( \xi e_0 \mathcal{L}(\Phi) e_0 \xi^* = r_0 \mathcal{L}(Q) r_0 \) where \( r_0 \leq r_2 = \xi \xi^*. \) By Lemma \[5.0.29\] this clearly implies that \( \xi \mathcal{L}(\Phi) t \xi^* = r_2 \mathcal{L}(Q) r_2 \) which finishes the proof.

\[\Box\]

**Lemma 5.7.** Let \( \Gamma \) be a group as in Note \[5.0.4\] and assume that \( \Lambda \) is a group such that \( \mathcal{L}(\Gamma) = \mathcal{L}(\Lambda) = M. \) Also assume there exists a subgroup \( \Phi \subset \Lambda, \) a unitary \( \mu \in \mathcal{U}(M) \) and projections \( z \in \mathcal{Z}(\mathcal{L}(\Phi)) \), \( r = \mu z^* \mu \in \mathcal{L}(Q) \) such that

\[
\mathcal{L}(\Phi) z \mu^* = r \mathcal{L}(Q) r.
\]

For every \( \lambda \in \Lambda \cap \Phi \) so that \( |\Phi \cap \Phi^\lambda| = \infty \) we have \( z u_\lambda z = 0. \) In particular, there is \( \lambda_0 \in \Lambda \cap \Phi \) so that \( |\Phi \cap \Phi^{\lambda_0}| < \infty. \)

**Proof.** Notice that since \( Q < \Gamma = (N_1 \times N_2) < Q \) is almost malnormal then we have the following property: for every sequence \( \mathcal{L}(Q) \ni x_n \to 0 \) weakly and every \( x, y \in M \) such that \( E_{\mathcal{L}(Q)}(x) = E_{\mathcal{L}(Q)}(y) = 0 \) we have

\[
\|E_{\mathcal{L}(Q)}(x x_n y)\|_2 \to 0, \quad \text{as } k \to \infty.
\]

Using basic approximations and the \( \mathcal{L}(Q)\)-bimodularity of the expectation we see that it suffices to check \[5.0.30\] only for elements of the form \( x = u_n \) and \( y = u_m \) where \( n, m \in (N_1 \times N_2) \setminus \{1\}. \) Consider the Fourier decomposition \( x_n = \sum_{h \in Q} \tau(x_h u_{h^{-1}}) u_h \) and notice that

\[
\begin{align*}
\|E_{\mathcal{L}(Q)}(x x_n y)\|_2 & = \|\sum_{h \in Q} \tau(x_h u_{h^{-1}}) \delta_{n h m, Q} u_{n h m}\|_2^2 \\
& = \|\sum_{h \in Q} \tau(x_h u_{h^{-1}}) \delta_{n h m, Q} u_{n h m} u_{n h m}^{-1}\|_2^2 = \sum_{h \in Q, \sigma_h(m) = n^{-1}} |\tau(x_h u_{h^{-1}})|^2.
\end{align*}
\]

Since the action \( Q < N_i \) has finite stabilizers one can easily see that the set \( \{ h \in Q : \sigma_h(m) = n^{-1} \} \) is finite and since \( x_n \to 0 \) weakly then \( \sum_{h \in Q, \sigma_h(m) = n^{-1}} |\tau(x_h u_{h^{-1}})|^2 \to 0 \) as \( k \to \infty \) which concludes the proof of \[5.0.30.\] Using the conditional expectation formula for compression we see that \[5.0.30\] implies that for every sequence \( \mathcal{L}(Q) \ni x_n \to 0 \) weakly and every \( x, y \in \mathcal{L}(M) \) so that \( E_{\mathcal{L}(Q)}(x) = E_{\mathcal{L}(Q)}(y) = 0 \) we have \( \|E_{\mathcal{L}(Q)}(x x_n y)\|_2 \to 0, \) as \( k \to \infty. \) Thus using the formula \[5.0.29\] we get that
for all $\mu \mathcal{L}(\Phi)z\mu^* \ni x_n \to 0$ weakly and every $x, y \in \mu z M z \mu^*$ so that $E_{\mu \mathcal{L}(\Phi)}(x) = E_{\mu \mathcal{L}(\Phi)}(y) = 0$ we have $\|E_{\mu \mathcal{L}(\Phi)}(x)\| \to 0$, as $k \to \infty$. This entails that for all $\mathcal{L}(\Phi)z \ni x_n \to 0$ weakly and every $x, y \in z M z$ satisfying $E_{\mathcal{L}(\Phi)}(x) = E_{\mathcal{L}(\Phi)}(y) = 0$ we have

$$\|E_{\mathcal{L}(\Phi)}(x)\| \to 0, \text{ as } k \to \infty. \quad (5.0.32)$$

Fix $\lambda \in \Lambda \setminus \Phi$ so that $|\Phi \cap \Phi^\lambda| = \infty$. Hence there are infinite sequences $\lambda_k, \omega_n \in \Lambda$ so that $\lambda\omega_k\lambda^{-1} = \lambda_k$ for all integers $k$. Since $\lambda \in \Lambda \setminus \Phi$ then $E_{\mathcal{L}(\Phi)}(u_{\lambda} z) = E_{\mathcal{L}(\Phi)}(z u_{\lambda^{-1}}) = 0$. Also we have that $u_{\omega_k} z \to 0$ weakly as $k \to \infty$. Using these calculations we have that

$$\|E_{\mathcal{L}(\Phi)}(z u_{\lambda} z u_{\lambda^{-1}})\| = \|E_{\mathcal{L}(\Phi)}(u_{\lambda} z u_{\lambda^{-1}})\| = \|E_{\mathcal{L}(\Phi)}(u_{\lambda} z u_{\lambda^{-1}})\| \to 0 \text{ as } k \to \infty. \quad (5.0.33)$$

Also using (5.0.33) the last quantity above converges to 0 as $k \to \infty$ and hence $E_{\mathcal{L}(\Phi)}(z u_{\lambda} z u_{\lambda^{-1}}) = 0$ which entails that $z u_{\lambda} z = 0$, as desired. For the remaining part notice first that since $[\Gamma : \mathcal{Q}] = \infty$ then (5.0.29) implies that $[\Lambda : \Phi] = \infty$. Assume by contradiction that for all $\lambda \in \Lambda \setminus \Phi$ we have $z u_{\lambda} z = 0$. As $[\Lambda : \Phi] = \infty$ then for every positive integer $l$ one can construct inductively $\lambda_i \in \Lambda \setminus \Phi$ with $i \in \mathbb{N}$ such that $\lambda_i \lambda_j^{-1} \in \Lambda \setminus \Phi$ for all $i > j$ such that $i, j \in \mathbb{N}$. But this implies that $0 = z u_{\lambda_i} z u_{\lambda_j} z u_{\lambda_j} = z u_{\lambda_i} u_{\lambda_j} z u_{\lambda_j}$ and hence $u_{\lambda_i} z u_{\lambda_j}$ are mutually orthogonal projections when $i = \mathbb{N}$. This is obviously false when $l$ sufficiently large.

\[\Box\]

**Theorem 5.8.** Assume the same conditions as in Theorem 5.6. Then one can find subgroups $\Phi_1, \Phi_2 \leq \Phi \leq \Lambda$ so that

1. $\Phi_1, \Phi_2$ are infinite, icc, property (T) groups so that $\Phi = \Phi_1 \times \Phi_2$;
2. $Q N^\lambda(\Phi) = \Phi$;
3. There exists $\mu \in U(\mathcal{M})$ such that $\mu \mathcal{L}(\Phi)\mu^* = \mathcal{L}(Q)$.

**Proof.** From Theorem 5.6 there exist subgroups $\Phi_1, \Phi_2 \leq \Phi \leq \Lambda$ such that

1. $\Phi_1, \Phi_2$ are, infinite, commuting, finite-by-icc, property (T) groups so that $[\Phi : \Phi_1 \Phi_2] < \infty$;
2. $Q N^\lambda(\Phi) = \Phi$;
3. There exist $\mu \in U(\mathcal{M})$ and $z \in P(\mathcal{Z}(\mathcal{L}(\Phi)))$ with $h = \mu z\mu^* \in P(\mathcal{L}(Q))$ satisfying
   \[\mu \mathcal{L}(\Phi)z\mu^* = h\mathcal{L}(Q)h. \quad (5.0.34)\]

Next we show that in (5.0.34) we can pick $z \in \mathcal{Z}(\mathcal{L}(\Phi))$ maximal with the property that for every projection $t \in \mathcal{Z}(\mathcal{L}(\Phi)z^*)$ we have

$$L(\Phi_1)t \ltimes \mathcal{M} \mathcal{L}(Q) \text{ for } i = 1, 2. \quad (5.0.35)$$

To see this let $z \in \mathcal{F}$ be a maximal family of mutually orthogonal (minimal) projections $z_i \in \mathcal{Z}(\mathcal{L}(\Phi))$ such that $\mathcal{L}(\Phi)z_i \ltimes \mathcal{M} \mathcal{L}(Q)$. Note that since $\Phi$ has finite conjugacy radical it follows that $\mathcal{F}$ is actually finite. Next let $z \leq \sum z_i := a \in \mathcal{Z}(\mathcal{L}(\Phi))$ and we briefly argue that $\mathcal{L}(\Phi)a \ltimes \mathcal{M} \mathcal{L}(Q)$. Indeed since $\mathcal{L}(\Phi)a' \cap a Ma = a(\mathcal{L}(\Phi)' \cap a \mathcal{M})a = \mathcal{Z}(\mathcal{L}(\Phi))a$ and the latter is finite dimensional then for every $r \in (\mathcal{L}(\Phi)a)' \cap a \mathcal{M}a$ there is $z_i \in \mathcal{F}$ such that $rz_i = z_i \neq 0$. Since $\mathcal{L}(\Phi)p_i \ltimes \mathcal{M} \mathcal{L}(Q)$ and then $\mathcal{L}(\Phi)p \ltimes \mathcal{M} \mathcal{L}(Q)$ as desired. Thus applying Lemma 2.7 after perturbing $\mu$ to a new unitary we get $\mu \mathcal{L}(\Phi)at\mu^* = h_0 \mathcal{L}(Q)h_0$. Finally, we show (5.0.35). Assume by contradiction there is $t_0 \in \mathcal{Z}(\mathcal{L}(\Phi)z^*)$ so that $\mathcal{L}(\Phi)t_0 \ltimes \mathcal{M} \mathcal{L}(Q)$ for some $i = 1, 2$. Thus there exist projections $r \in \mathcal{L}(\Phi)t_0, q \in \mathcal{L}(Q)$, a partial isometry $w \in \mathcal{M}$ and a $*$-isomorphism on the image $\phi : r\mathcal{L}(\Phi)r \to B := \phi(r\mathcal{L}(\Phi)r) \leq q\mathcal{L}(Q)q$
such that $\phi(x)w = wx$. Notice that $w^*w \in t_0(\mathcal{L}(\Phi))' \cap M_{t_0}$ and $ww^* \in B' \cap qMq$. But since $Q < \Gamma$ is
malnormal it follows that $B' \cap qMq \subseteq q\mathcal{L}(Q)q$ and hence $ww^* \in q\mathcal{L}(Q)q$. Using this in combination
with previous relations we get that $wr(\mathcal{L}(\Phi))rw^* = Bwv^* \subseteq \mathcal{L}(Q)$ and extending $w$ to a unitary $u$ we
have that $u\mathcal{L}(\Phi)ru^* \subseteq \mathcal{L}(Q)$. Since $\mathcal{L}(Q)$ is a factor we can further perturb the unitary $u$ so that
$u\mathcal{L}(\Phi)r_0u^* \subseteq \mathcal{L}(Q)$ where $r < r_0 < t_0$ is the central support of $r$ in $\mathcal{L}(\Phi)_{t_0}$. Using malnormality of $Q$
again we further get $r_0(\mathcal{L}(\Phi)) \vee \mathcal{L}(\Phi)' \cap M_{r_0}u^* \subseteq \mathcal{L}(Q)$ and perturbing $u$ we can further assume that
$(\mathcal{L}(\Phi)) \vee \mathcal{L}(\Phi)' \cap M_{r_0}u \subseteq \mathcal{L}(Q)$ where $r_0 \leq s_0$ is the central support of $r_0$ in $\mathcal{L}(\Phi) \vee \mathcal{L}(\Phi)' \cap M$. In
particular, $u(\mathcal{L}(\Phi))_{s_0}u^* \subseteq \mathcal{L}(Q)$ and hence $\mathcal{L}(\Phi)_{s_0} \subseteq u^*\mathcal{L}(Q)u$. Since $r < r_0 \leq s_0$ and $r < t_0$ the previous
containment implies that there is a minimal projection $s' \in (\mathcal{L}(\Phi)a^i)$ so that $(\mathcal{L}(\Phi))s' \subseteq \mathcal{L}(Q)$ which
contradicts the maximality assumption on $\mathcal{F}$. Finally replacing $z$ with $a$ in our statement, etc our claim
follows.

Next fix $t \in \mathcal{Z}(\Phi)z^\perp$. Since $\mathcal{L}(\Phi_1)t$ and $\mathcal{L}(\Phi_2)t$ are commuting property (T) von Neumann
algebras then using the same arguments as in the first part of the proof of Theorem 5.3 there are two possibilities: either i) there exists $j \in 1,2$ such that $\mathcal{L}(\Phi_j)t \prec \mathcal{M} L(N_2)$ or ii) $\mathcal{L}(\Phi)t \prec \mathcal{M} L(N_2 \times Q)$. Next we briefly argue ii) is impossible. Indeed, assuming ii), Theorem 5.2 for $n = 1$ would imply the existence of $j \in 1,2$ so that $\mathcal{L}(\Phi_j)t \prec \mathcal{M} L(Q)$ which obviously contradicts the choice of $z$. Thus we have i) and passing to the relative commutants we have that $\mathcal{L}(N_1) \prec \mathcal{L}(\Phi_1)'t \cap \mathcal{M} t \mathcal{M} = t(\mathcal{L}(\Phi_1)' \cap M)$. Using the relationships between the $\Phi_j$'s we see that $t(\mathcal{L}(\Phi_1)' \cap M)t \cap t(\mathcal{L}(\Phi_2)' \cap M)t \subseteq \mathcal{L}(\Phi_1)(v\mathcal{C}_A(\Phi_j)))t \subseteq t(\mathcal{L}(\Phi))t$. In conclusion, we have

$$\mathcal{L}(N_1) \prec \mathcal{M} t(\mathcal{L}(\Phi))t,$$
(5.0.36)

Let $A = \{ \lambda \in \Lambda : |\Phi \cap \Phi| = \infty \}$ and $B = \{ \lambda \in \Lambda : |\Phi \cap \Phi| = \infty \}$. Note that $A \cup B = \Lambda$ and $A \neq \emptyset$. Since $N_1$ is infinite then for every $\lambda \in A$ we have that $\mathcal{L}(N_1) \prec \mathcal{M} \mathcal{L}(\Phi \cap \Phi)^\perp z^\perp$. Thus using (5.0.36) together with the same argument from the proof of [PV08, Theorem 6.16], working under $z^\perp$, we get $z^\perp E_{\Phi}(u_\lambda z^\perp x z^\perp) = 0$ for all $x \in M$. This further implies that $z^\perp u_\lambda z^\perp = 0$ for all $\lambda \in A$ and hence $u_\lambda z^\perp u_{\lambda+1} \subseteq z$.

On the other hand by Lemma 5.7 for all $\lambda \in B$ we get $zu_\lambda z = 0$ and hence $u_\lambda zu_{\lambda+1} \subseteq z^\perp$. So if $B \neq \emptyset$ we obviously have equality in the previous two relations, i.e. $u_\lambda z u_{\lambda+1} = z^\perp$ for all $\lambda \in B$ and $u_\lambda z^\perp u_{\lambda+1} = z$ for all $\lambda \in A$. These further imply there exist $a_0 \in A$ and $b_0 \in B$ such that $A = a_0 C_\Lambda(z^\perp)$ and $B = b_0 C_\Lambda(z)$; here $C_\Lambda(z) \subseteq \Lambda$ is the subgroup of all elements of $\Lambda$ that commute with $z$ and similarly for $C_\Lambda(z^\perp)$. Thus $A = A \cup B = a_0 C_\Lambda(z^\perp) \cup b_0 C_\Lambda(z)$. Thus we can assume, without loss of generality, that $[A : C_\Lambda(z)] < \infty$. But since $\Lambda$ is icc this implies that $z = 1$. The rest of the statement follows.

**Theorem 5.9.** In the Theorem 5.3 we cannot have case 4a).

**Proof.** Assume by contradiction that for all $j \in 1,2$ there is $i \in 1,2$ such that $\Delta(\mathcal{L}(N_i)j) \prec \mathcal{M} \mathcal{M} \mathcal{L}(N_j)$. Using [DHI16, Theorem 4.1] and the property (T) on $N_j$ one can find a subgroup $\Sigma < \Lambda$ such that $\mathcal{L}(Q_j) \prec \mathcal{M} \mathcal{L}(\Sigma)$ and $\mathcal{L}(N_j) \prec \mathcal{M} \mathcal{L}(C_\Lambda(\Sigma))$. Since $\mu \mathcal{L}(\Phi)^* = \mathcal{L}(Q)$ and $Q_i$ are bisection then by the product
rigidity results in [CdSS15] one can assume that there is a unitary $u \in \mathcal{L}(Q)$ such that $u \mathcal{L}(Q_1)^{*} u = \mathcal{L}(\Phi_1)^{*}$ and $u \mathcal{L}(Q_2) u^* = \mathcal{L}(\Phi_2)^{*}$. Thus we get that $\mathcal{L}(\Phi_1) \prec \mathcal{M} \mathcal{L}(\Sigma)$ and hence $[\Phi_1 : \Sigma^{1-1} \cap \Phi_1] \Vert \infty$. So working with $g^{\Sigma_1}$ instead of $\Sigma$ we can assume that $[\Phi_1 : \Sigma \cap \Phi_1] < \infty$. In particular $\Sigma \cap \Phi_1$ is
infinite and since $\Phi$ is almost malnormal in $\Lambda$ it follows that $C_\Lambda(\Sigma \cap \Phi_1) < \Phi$. Thus we have that
$\mathcal{L}(N_1) \prec \mathcal{M} \mathcal{L}(C_\Lambda(\Sigma)) \subseteq \mathcal{L}(C_\Lambda(\Sigma \cap \Phi_1)) \subset \mathcal{L}(\Phi) = \mu^* \mathcal{L}(Q)u$ which is obviously a contradiction.

**Theorem 5.10.** Let $\Gamma$ be a group as in Notation 5.4 and assume that $\Lambda$ is a group such that $\mathcal{L}(\Gamma) = \mathcal{L}(\Lambda) = \mathcal{M}$. Let $\Delta : \mathcal{M} \to \mathcal{M} \mathcal{M} \mathcal{M}$ be the comultiplication along $\Lambda$ as in Notation 5.4. Then the following hold:

i) $\Delta(\mathcal{L}(N_1)), \Delta(\mathcal{L}(N_2)), \Delta(\mathcal{L}(N_1 \times N_2)) \prec \mathcal{M} \mathcal{M} \mathcal{M} \mathcal{L}(N_1 \times N_2)$, and

ii) there is a unitary $u \in \mathcal{M} \mathcal{M} \mathcal{M}$ such that $u \Delta(\mathcal{L}(Q))u^* \subseteq \mathcal{L}(Q) \mathcal{M} \mathcal{M} \mathcal{L}(Q)$.

**Proof.** First we show i). From Theorem 5.5 we have that for all $j \in 1,2$ there is $j_i \in 1,2$ such that
$\Delta(\mathcal{L}(N_{j_i})) \prec \mathcal{M} \mathcal{M} \mathcal{M} \mathcal{L}(N_j)$. Notice that since $N_{j_i} \prec \mathcal{M} \mathcal{M} \mathcal{M} \mathcal{L}(N_j)$.
then by [DH16] Lemma 2.4 part (3) we actually have $\Delta(\mathcal{L}(N_i)) \triangleleft M_{\mathfrak{A} M}$. Notice that for all $i \neq k$ we have $j_i \neq j_k$. Otherwise we would have $\Delta(\mathcal{L}(N_i)) \triangleleft M_{\mathfrak{A} M} \mathfrak{M} \mathcal{L}(N_1)$ and $\Delta(\mathcal{L}(N_i)) \triangleleft M_{\mathfrak{A} M} \mathfrak{M} \mathcal{L}(N_2)$ which by [DH16] Lemma 2.8 (2) would imply that $\Delta(\mathcal{L}(N_i)) \triangleleft M_{\mathfrak{A} M} \mathfrak{M} \mathcal{L}(N_1 \cap N_2) = M \otimes 1$ which is a contradiction. Furthermore using the same arguments as in [IS16] Lemma 2.6 we have that $\Delta(\mathcal{L}(N_1 \times N_2)) \triangleleft M_{\mathfrak{A} M} \mathfrak{M} \mathcal{L}(N_1 \times N_2)$. Then working on the left side of the tensor we get that $\Delta(\mathcal{L}(N_1 \times N_2)) \triangleleft M_{\mathfrak{A} M} \mathfrak{M} \mathcal{L}(N_1 \times N_2)$.

Finally, notice that part ii) is a direct consequence of Theorem 5.8.

5.1 Proof of Theorem 5.1

Proof. We divide the proof into separate parts to improve the exposition.

Reconstruction of the Acting Group $\mathfrak{A}$

To accomplish this we will use the notion of height for elements in group von Neumann algebras as introduced in [IPV10, Io11]. From the previous theorem recall that $u \Delta(\mathcal{L}(Q)) u^* \in \mathcal{L}(Q) \otimes \mathcal{L}(Q)$. Let $A = u \Delta(\mathcal{L}(N_1)) u^*$. Next we claim that

$$h_{Q \times Q}(u \Delta(\mathcal{L}(Q)) u^*) > 0. \quad (5.1.1)$$

For every $x, y \in \mathcal{L}(Q) \otimes \mathcal{L}(Q)$ and every $a \in A \otimes A$ supported on a finite set $F \subset N = N_1 \times N_2$ we have that

$$\| E_{A \otimes A}(xy) \|_2^2 = \| \sum_{\alpha, \beta} \tau(x_{\alpha, \beta}) \tau(y_{\alpha, \beta}) E_{A \otimes A}(u_{\alpha, \beta} u_{\alpha, \beta}^*) \|_2^2 \quad (5.1.2)$$

$$= \| \sum_{\alpha, \beta} \tau(x_{\alpha, \beta}) \tau(y_{\alpha, \beta}) E_{A \otimes A}(s_{\alpha, \beta}^2) \|_2^2$$

$$= \| \sum_{\alpha} \tau(x_{\alpha, -}) \tau(y_{\alpha, -}) \tau(a_{\alpha, -}) \|_2^2$$

$$\leq h_{Q \times Q}^2(x) \sum_{\alpha} \left( \sum_{\beta} \| \tau(y_{\alpha, -}) \| \| \tau(a_{\alpha, -}) \| \right)^2 \quad (5.1.3)$$

This estimate leads to the following property: for every finite sets $K, S \subset Q$, every $a \in \text{span}\{ A \otimes A u_g : g \in K \}$ and all $\varepsilon > 0$ there exist a scalar $C > 0$ and a finite set $F \subset N^2$ such that for all $x, y \in \mathcal{L}(Q) \otimes \mathcal{L}(Q)$ we have

$$\| P_{\Sigma_{i \in S}} A \otimes A u_i (xyy) \|_2^2 \leq |K||S|C h_{Q \times Q}^2(x) \| y \|_2^2 \| a \|_2 \max_{\alpha \in F} \{ |q \in Q : \sigma_q(r^{-1}) \in F| \} + \varepsilon \| x \|_\infty \| y \|_\infty \quad (5.1.4)$$

Note this follows directly from (5.1.2) after we decompose the $a$ and the projection $P_{\Sigma_{i \in S}} A \otimes A u_i$.

Next we use (5.1.3) to prove our claim. Fix $\varepsilon > 0$. Since $\Delta(A) \triangleleft M \otimes 1$, $1 \otimes M$ then by Theorem 2.4 one can find a finite subset $F_0 \subset N^2 \setminus ((N_1 \times 1) \cup (1 \times N))$ such that $a_{F_0} \in A \otimes A$ is supported on $F_0$ and $\| a - a_{F_0} \|_2 \leq \varepsilon$. Since $\Delta(A) \triangleleft A \otimes A$ there is a finite $S \subset Q \times Q$ such that

$$\| P_{\Sigma_{i \in S}} A \otimes A u_i (a) - a \|_2 \leq \varepsilon \quad \text{for all } a \in \Delta(A). \quad (5.1.4)$$
Assume by contradiction \((5.1.1)\) doesn’t hold. Thus there is a sequence \(t_n \in Q\) such that \(h_{Q \times Q}(t_n) = h_{Q \times Q}(u\Delta(t_n)u^*) \to 0\) as \(n \to \infty\). As \(t_n\) normalizes \(\Delta(A)\) then one can see that

\[
1 - \varepsilon = \|t_n a t_n^*\|^2 - \varepsilon \leq \|P_{\Sigma \times A} \circ A \circ t_n a t_n^*\|^2 + \varepsilon = \|P_{\Sigma \times A} \circ A \circ t_n a t_n^*\|^2 + \varepsilon \leq \|F_0\|C(h_{Q \times Q}(t_n))\|a t_n\|^2 + \varepsilon \leq \|F_0\|C(h_{Q \times Q}(t_n))\max_{\{q \in Q : \sigma_{q-1}(r^{-1}) \in F_0\}} + \varepsilon\|t_n\|^2.
\]

(5.1.5)

Since the stabilizer sizes are uniformly bounded we get a contradiction if \(\varepsilon > 0\) is arbitrary small.

Now we notice that the height condition together with Theorem \[5.8\] and \([\text{CU18}]\) Lemmas 2.4,2.5 already imply that \(h_{Q}(\mu \Phi \mu^*) > 0\) and by \([\text{IPVI10}]\) Theorem 3.1 there is a unitary \(\mu_0 \in M\) such that \(\mathbb{T} \mu_0 \Phi \mu_0^* = \mathbb{T} Q\).

Reconstruction of a Core Subgroup and its Product Feature

From Theorem \[5.10\] we have that \(\Delta(L(N_1 \times N_2)) <_{M_{\Sigma \times M}} \mathbb{L}(N_1 \times N_2) \otimes \mathbb{L}(N_1 \times N_2)\). Proceeding exactly as in the proof of \([\text{CU18}]\) Claim 4.5 we can show that \(\Delta(A) \subseteq A \otimes A\), where \(A = u \mathbb{L}(N_1 \times N_2) u^*\). By Lemma \[2.8\] there exists a subgroup \(\Sigma < L\) such that \(A = \mathbb{L}(\Sigma)\). The last part of the proof of \([\text{CU18}]\) Theorem 5.2] shows that \(L = \Sigma \times \Phi\). In order to reconstruct the product feature of \(\Sigma\), we need a couple more results.

Claim 2. For every \(i = 1,2\) there exists \(j = 1,2\) such that

\[
\Delta(L(N_i)) < \mathbb{L}(N_1 \times N_2) \otimes \mathbb{L}(N_1).
\]

(5.1.6)

Proof of Claim. We prove this only for \(i = 1\) as the other case is similar. We also notice that since \(N_{\Sigma \times M}(\Delta(L(N_1)))'' \subseteq \Delta(M)\) and \(\Delta(M)'' \cap M = \mathbb{C}1\) then to establish \((5.1.6)\) we only need to show that \(\Delta(L(N_1)) < \mathbb{L}(N_1 \times N_2) \otimes \mathbb{L}(N_1)\). From above we have \(\Delta(L(N_1 \times N_2)) <_{M_{\Sigma \times M}} \mathbb{L}(N_1 \times N_2) \otimes \mathbb{L}(N_1 \times N_2)\). Hence there exist nonzero projections \(a_1, a_2 \in \Delta(L(N_1))\) and \(b \in \mathbb{L}(N_1 \times N_2) \otimes \mathbb{L}(N_1 \times N_2)\), a partial isometry \(v \in M_{\Sigma \times M}\) and an \(*\)-isomorphism on the image \(\Psi: a_1 \otimes a_2\Delta(L(N_1 \times N_2))a_1 \otimes a_2 \rightarrow \Psi(a_1 \otimes a_2\Delta(L(N_1 \times N_2))a_1 \otimes a_2) = R \subseteq b \in \mathbb{L}(N_1 \times N_2) \otimes \mathbb{L}(N_1 \times N_2)\) such that \(\Psi(x) = vx\) for all \(x \in \mathbb{L}(N_1 \times N_2) \otimes \mathbb{L}(N_1 \times N_2)\).

Denote by \(D_1 := \Psi(a_1 \Delta(L(N_1)))a_1 \subseteq b \mathbb{L}(N_1 \times N_2) \otimes \mathbb{L}(N_1 \times N_2)b\) and notice that \(D_1\) and \(D_2\) are commuting property (T) diffuse subfactors. Since the group \(N_2\) is \((\mathbb{F}_\infty)\)-by-(non-elementary hyperbolic group) then by \([\text{CIIK13}, \text{CK15}]\) it follows that there is \(j = 1,2\) such that \(D_j < \mathbb{L}(N_1 \times N_2) \otimes \mathbb{L}(N_1 \times N_2)\) \(\mathbb{L}(N_1 \times N_2) \otimes \mathbb{L}(\mathbb{F}_\infty)\). Since \(\mathbb{F}_\infty\) has Haagerup’s property and \(D_j\) has property (T) this further implies that \(D_j < \mathbb{L}(N_1 \times N_2) \otimes \mathbb{L}(N_1 \times N_2)\) \(\mathbb{L}(N_1 \times N_2) \otimes \mathbb{L}(N_1)\). Composing this intertwining with \(\Psi\) we get \(\Delta(L(N_j)) < \mathbb{L}(N_1 \times N_2) \otimes \mathbb{L}(N_1)\), as desired.

Also, we note that \(j_1 \neq j_2\). Otherwise we would have that \(\Delta(L(N_j)) < \mathbb{L}(N_1 \times N_2) \otimes \mathbb{L}(N_1) \cap \mathbb{L}(N_2) = \mathbb{L}(N_1 \times N_2) \otimes \mathbb{1}\), which obviously contradicts \([\text{IPVI10}]\) Proposition 7.2.1.

Let \(A = u \mathbb{L}(N_1) u^*\). Thus, we get that \(\Delta(A) < \mathbb{L}(N_1 \times N_2) \otimes \mathbb{L}(N_1)\) for some \(i = 1,2\). This implies that for every \(\varepsilon > 0\), there exists a finite set \(S \subseteq u^* Q u\), containing \(e\), such that \(\|d - P_{\Sigma \times M}(d)\|_2 \leq \varepsilon\) for all \(d \in \Delta(A)\). However, \(\Delta(A)\) is invariant under the action of \(u^* Q u\), and hence arguing exactly as in \([\text{CU18}]\) Claim 4.5 we get that \(\Delta(A) \subseteq (\mathbb{L}(\Sigma) \otimes u \mathbb{L}(N_1) u^*)\). We now separate the argument into two different cases:

Case 1: \(i = 1\).

In this case, \(\Delta(A) \subseteq \mathbb{L}(\Sigma) \otimes A\). Thus by Lemma \[2.8\] we get that there exists a subgroup \(\Sigma_0 < \Sigma\) with \(A = \mathbb{L}(\Sigma_0)\). Now, \(A' \cap \mathbb{L}(\Sigma) = u \mathbb{L}(N_2) u^*\). Thus, \(\mathbb{L}(\Sigma_0)' \cap \mathbb{L}(\Sigma) = u \mathbb{L}(N_2) u^*\). Note that \(\Sigma' \cap \mathbb{L}(\Sigma_0) = \Sigma\) are both icc property (T) groups. This implies that \(\mathbb{L}(\Sigma_0)' \cap \mathbb{L}(\Sigma) = \mathbb{L}(v \mathbb{C}_\Sigma(\Sigma_0))\), where \(v \mathbb{C}_\Sigma(\Sigma_0)\) denotes the virtual centralizer of \(\Sigma_0\) in \(\Sigma\). Proceeding as in \([\text{CdSS17}]\) we can show that \(\Sigma' = \Sigma_0 \times \Sigma_1\).
Case II: $i = 2$.

Let $B = uL(N_2)u^*$. In this case, $\Delta(A) \subseteq L(\Sigma)@B$. However, Lemma [2.8] then implies that $A \subseteq B$, which is absurd, as $L(N_1)$ and $L(N_2)$ are orthogonal algebras. Hence this case is impossible and we are done.

Remarks. 1) There are several immediate consequences of the Theorem [5.1]. For instance one can easily see the von Neumann algebras covered by this theorem are non-isomorphic with the ones arising from any irreducible lattice in higher rank Lie group. Indeed, if $\Lambda$ is any such lattice satisfying $L(\Gamma) \cong L(\Lambda)$, then Theorem [5.1] would imply that $\Lambda$ must contain an infinite normal subgroup of infinite index which contradicts Margulis’ normal subgroup theorem.

2) While it well known there are uncountable many non-isomorphic group II$_1$ factors with property (T) [Po07] little is known about producing concrete examples of such families. In fact the only currently known infinite families of pairwise non-isomorphic property (T) groups factors are $\{L(G_n)|n \geq 2\}$ for $G_n$ uniform lattices in $Sp(n,1)$ [CH89] and $\{L(G_1 \times G_2 \times \cdots \times G_k)|k \geq 1\}$ where $G_k$ is any icc property (T) hyperbolic group [OP03]. Theorem [5.1] makes new progress in this direction by providing a new explicit infinite family of icc property (T) groups which gives rise to pairwise non-isomorphic II$_1$ factors. For instance, in the statement one can simply $Q_i$ to vary in any infinite family of non-isomorphic uniform lattices in $Sp(n,1)$ for any $n \neq 2$. Unlike the other families ours consists of factors which are not solid, do not admit tensor decompositions [CdSS17], and do not have Cartan subalgebras, [CIK13].

3) We notice that Theorem [5.1] still holds if instead of $\Gamma = (N_1 \times N_2) \times (Q_1 \times Q_2)$ one considers any finite index subgroup of $\Gamma$ of the form $\Gamma' = (N_1 \times N_2) \times (Q_1' \times Q_2')$, where $Q_1' \leq Q_1$ and $Q_2' \leq Q_2$ are arbitrary finite index subgroups. One can verify these groups still enjoy all the algebraic/geometric properties used in the proof of Theorem [5.1] (including the fact that $N_1 \times Q_1'$ is hyperbolic relative to $Q_1'$ and $N_1 \times Q_2'$ is hyperbolic relative to $Q_2'$) and hence all the von Neumann algebraic arguments in the proof of Theorem [5.1] apply verbatim. The details are left to the reader.

4) The group factors considered in Theorem [5.1] have trivial fundamental group by [CDHK20] Theorem B.

6 Concrete Examples of Infinitely Many Pairwise Non-isomorphic Group II$_1$ Factors with Property (T)

In this section we present several applications of our main techniques to the structural study of property (T) group factors. An earlier result of Popa [Po07] shows that the map $\Gamma \mapsto L(\Gamma)$ is at most countable to one. Since there are uncountably many icc property (T) groups, this obviously implies the existence of uncountably many group property (T) factors which are pairwise non-isomorphic. However, currently there are still no explicit constructions of such families in the literature. In this section we make new progress in this direction by showing that the canonical fiber product of Belegradek-Osin Rips construction groups can be successfully used to provide possibly the first such examples (Corollary [6.4]). In addition, our methods also yields other interesting consequences. For instance, they can be used to provide an infinite series of finite index subfactors of a given property (T) II$_1$ factor that are pairwise non-isomorphic which is also a novelty in the area (Corollary [6.2]). This further gives infinitely many examples of icc, property (T) group factors $\Gamma_n$ measure equivalent to a fixed group $\Gamma$, such that $L(\Gamma_n)$ are pairwise mutually nonsimiorphic. The first examples of group measure equivalent groups $\Gamma$ and $\Lambda$ giving rise to nonisomorphic group von Neumann algebrastructure were given in [CI09], thereby answering a question of D. Shlyakhtenko. Note that the examples in [CI09] didn’t have property (T).

The following theorem is the main von Neumann algebraic result of the section. Some of the arguments used in the proof are very similar to the ones used in the proof of Theorem [5.1] and thus we shall just refer the reader to the previous section for these. However, we will include all the details on the new aspects of the proof.
Theorem 6.1. Let $Q_1, Q_2, P_1, P_2$ be icc, torsion free, residually finite property (T) groups. Let $Q = Q_1 \times Q_2$ and $P = P_1 \times P_2$. Assume that $N_1 \equiv Q, N_2 \equiv Q \in \text{Rip}_T(Q)$ and $M_1 \equiv P, M_2 \equiv P \in \text{Rip}_T(P)$. Assume that $\Theta : \mathcal{L}((N_1 \times N_2) \times Q) \rightarrow \mathcal{L}((M_1 \times M_2) \times P)$ is an $*$-isomorphism.

Then one can find a $*$-isomorphism, $\Theta_1 : \mathcal{L}(N_1) \rightarrow \mathcal{L}(M_1)$, a group isomorphism $\delta : Q \rightarrow P$, a multiplicative character $\eta : Q \rightarrow \mathbb{T}$, and a unitary $u \in \mathcal{U}(\mathcal{L}((M_1 \times M_2) \times P))$ such that for all $\gamma \in Q, x_i \in N_i$ we have that

$$\Theta((x_1 \otimes x_2)u_\gamma) = \eta(\gamma)u(\Theta_1(x_1) \otimes \Theta_2(x_2)v_\delta(\gamma))u^*.$$ 

Proof. Let $\mathcal{M} = \mathcal{L}((M_1 \times M_2) \times P)$, $\Gamma_i = N_i \times Q$ and let $\tilde{\mathcal{M}} = \mathcal{L}(\Gamma_1 \times \Gamma_2)$. Note that $\Theta(\mathcal{L}(N_1))$ and $\Theta(\mathcal{L}(N_2))$ are commuting property (T) subfactors of $\mathcal{L}((M_1 \times M_2) \times P)$. Hence by Theorem 5.3 we have that either

1) exists $i \in \{1, 2\}$ such that $\Theta(\mathcal{L}(N_i)) <_{\mathcal{M}_1} \mathcal{L}(\Gamma_1)$ or

2) $\Theta(\mathcal{L}(N_1 \times N_2)) <_{\mathcal{M}} \mathcal{L}(\Gamma_1 \times P)$.

Assume 1) holds. Then proceeding exactly as in the first part of proof of Theorem 5.5 we have that $\Theta(\mathcal{L}(N_1)) <_{\mathcal{M}} \mathcal{L}(M_1)$. As $\mathcal{L}(M_1)$ is regular in $\mathcal{M}$, we conclude using Lemma 2.4 that $\Theta(\mathcal{L}(N_1)) <_{\mathcal{M}} \mathcal{L}(M_1)$.

Assume 2). Then the same argument as in the second part of the proof of Theorem 5.5 we have that $\Theta((N_1 \times N_2)) <_{\mathcal{M}} \mathcal{L}(M_1 \times \text{diag}(P))$. Thus if $\Theta(\mathcal{L}(N_1)) \equiv \mathcal{L}(M_1)$ for all $i = 1, 2$, then the same argument as in the last part of Theorem 5.5 will lead to a contradiction.

In conclusion, we have shown that for all $i = 1, 2$ there exists $j \in \{1, 2\}$ such that $\Theta(\mathcal{L}(N_i)) <_{\mathcal{M}} \mathcal{L}(M_i)$. As $\Theta(\mathcal{L}(N_j))$ is regular in $\mathcal{M}$, we actually have that $\Theta(\mathcal{L}(N_j)) <_{\mathcal{M}} \mathcal{L}(M_j)$. Notice that in particular this forces different $i$'s to give rise to different $j$'s. Indeed, otherwise we would have that $\Theta(\mathcal{L}(N_1)) <_{\mathcal{M}} \mathcal{L}(M_1)$ and $\Theta(\mathcal{L}(N_2)) <_{\mathcal{M}} \mathcal{L}(M_2)$. Then by [DH16] Lemma 2.6], this would imply that $\Theta(\mathcal{L}(N_j)) <_{\mathcal{M}} \mathcal{L}(M_1 \cap M_2) = \mathcal{C}$, which is obviously a contradiction. Therefore we get that either

4a) $\Theta(\mathcal{L}(N_1)) <_{\mathcal{M}} \mathcal{L}(M_1)$ and $\Theta(\mathcal{L}(N_2)) <_{\mathcal{M}} \mathcal{L}(M_2)$, or

4b) $\Theta(\mathcal{L}(N_1)) <_{\mathcal{M}} \mathcal{L}(M_2)$ and $\Theta(\mathcal{L}(N_2)) <_{\mathcal{M}} \mathcal{L}(M_1)$.

Note that both cases imply that $\Theta(\mathcal{L}(N_1)), \Theta(\mathcal{L}(N_2)) <_{\mathcal{M}} \mathcal{L}(M_1 \times M_2)$. Using [Is16] Lemma 2.6, we further get that

$$\Theta(\mathcal{L}(N_1 \times N_2)) <_{\mathcal{M}} \mathcal{L}(M_1 \times M_2).$$

(6.0.1)

Proceeding in a similar manner, we also have the reverse intertwining $\mathcal{L}(M_1 \times M_2) <_{\mathcal{M}} \Theta(\mathcal{L}(N_1 \times N_2))$. Since $\mathcal{L}(M_1 \times M_2), \mathcal{L}(N_1 \times N_2)$ are irreducible, regular subfactors of $\mathcal{M}$, by [PP03] Lemma 8.4 one can find $u \in \mathcal{U}(\mathcal{M})$ such that

$$u \mathcal{L}(M_1 \times M_2) u^* = \Theta(\mathcal{L}(N_1 \times N_2)).$$

(6.0.2)

Note that $\Theta(\mathcal{L}(Q_1)), \Theta(\mathcal{L}(Q_2))$ are commuting property (T) subfactors of $\mathcal{L}((M_1 \times M_2) \times P)$. Proceeding exactly as in the first part of the proof, we conclude that either $\Theta(\mathcal{L}(Q_i)) <_{\mathcal{M}} \mathcal{L}(\Gamma_1)$ or $\Theta(\mathcal{L}(Q_i \times Q_2)) <_{\mathcal{M}} \mathcal{L}(\Gamma_1 \times P)$. As before, this further implies that either

7) $\Theta(\mathcal{L}(Q_i)) <_{\mathcal{M}} \mathcal{L}(M_1)$, or

8) $\Theta(\mathcal{L}(Q_1 \times Q_2)) <_{\mathcal{M}} \mathcal{L}(M_1 \times \text{diag}(P))$.

Assume 7). Since by (6.0.2) we also have $\mathcal{L}(M_1) <_{\mathcal{M}} \Theta(\mathcal{L}(N_1 \times N_2))$ and hence by [Va07] Lemma 3.7] we conclude that $\Theta(\mathcal{L}(Q_i)) <_{\mathcal{M}} \mathcal{L}(N_1 \times N_2))$. However, this implies that $Q_i$ is finite, which is a contradiction.

Hence, we must have 8). Proceeding as in the end of proof of Theorem 5.5 we conclude that $\Theta(\mathcal{L}(Q)) <_{\mathcal{M}} \mathcal{L}(P)$. Thus there exists $\Psi : \mathcal{L}(P) \rightarrow \mathcal{L}(P)$ such that $\Psi(x) = x\Psi$ for all $x \in \mathcal{L}(P)$. Also note that $\Psi = \Psi' \circ \Psi''$ such that $\Psi(x) = x\Psi''$ for all $x \in \mathcal{L}(P)$. Since $\Psi \equiv \Psi'' \circ \Psi'' \circ \Psi''$ is diffuse and $P \equiv (M_1 \times M_2) \times P$ is a malnormal subgroup, we have that $\mathcal{L}(\mathcal{N}_Q \mathcal{M}) \equiv \mathcal{L}(P)$.
Proof. 1) Assume that 

\[ \nu \nu^* \in qL(P)q \text{ and hence } \nu \nu^* = R \nu \nu^* \subseteq qL(P)q. \]

Extending \( \nu \) to a unitary \( \nu_0 \) in \( \mathcal{M} \) we have that \( \nu_0 \nu \nu(\mathcal{L}(Q)) \nu_0^* \subseteq L(P) \). As \( \mathcal{L}(P) \) and \( \mathcal{L}(Q) \) are factors, after perturbing \( \nu_0 \) to a new unitary we may assume that

9) \( \nu_0 \nu(\mathcal{L}(Q)) \nu_0^* \subseteq L(P) \).

In a similar manner we have that there exists \( \nu_0 \in \mathcal{M} \)

10) \( \nu_0 \mathcal{L}(P) \nu_0^* \subseteq \Theta(\mathcal{L}(Q)). \)

Conditions 9) and 10) together imply that \( \nu_0 \mathcal{L}(P) \nu_0^* \subseteq \Theta(\mathcal{L}(Q)) \subseteq \nu_0^* \mathcal{L}(P) \nu_0 \). In particular, \( \nu_0 \mathcal{L}(P) \nu_0^* \subseteq \mathcal{L}(P) \). Since \( P \) is malnormal in \( (M_1 \times M_2) \times P \) we have that \( \nu_0 \mathcal{L}(P) \nu_0^* \subseteq \mathcal{L}(P) \). Hence \( \nu_0 \mathcal{L}(P) \nu_0^* = \nu_0^* \mathcal{L}(P) \nu_0 \). Combining this with the above relations we get that

11) \( \nu_0 \mathcal{L}(P) \nu_0^* = \Theta(\mathcal{L}(Q)). \)

Since the action \( Q \cong (N_1 \times N_2) \) has trivial stabilizers, using conditions 11) and 6), arguing as in the proof of Theorem 5.3, we get that \( h_{(\nu_0 \mathcal{L}(P) \nu_0^*)} (\Theta(Q)) > 0 \). By [IPV10] Theorem 3.3 we get that there exists \( \nu_1 \in \mathcal{M} \), and isomorphism \( \delta : Q \to P \) such that

\[ \Theta(\nu_1) = \nu_1 \nu (\mathcal{L}(Q)) \nu_1^* \]

for all \( g \in Q \).

Finally, this together with relation 4), proceeding exactly as in the proof of Theorem 3.4, implies the desired conclusion.

The previous theorem can be used to provide an infinite series of finite index subfactors of a given property (T) II\(_1\) factor that are pairwise non-isomorphic.

Corollary 6.2. 1) Let \( Q_1, Q_2 \) be uniform lattices in \( Sp(n, 1) \) with \( n \geq 2 \) and let \( Q := Q_1 \times Q_2 \). Also let \( \cdots \leq Q^3_1 \leq Q^2_1 \leq Q_1 \) be an infinite family of finite index subgroups and denote by \( Q_s := Q^s_1 \times Q_2 \leq Q \). Then consider \( N_1 \times c_1 Q, N_2 \times c_2 Q \in \mathcal{R}p_{T}(Q) \) and let \( \Gamma = (N_1 \times N_2) \times c_1, c_2 Q \). Inside \( \Gamma \) consider the finite index subgroups \( \Gamma_s := (N_1 \times N_2) \times c_1, c_2 Q_s \). Then the family \( \{ \mathcal{L}(\Gamma_s) | s \in I \} \) consists of pairwise non-isomorphic finite index subfactors of \( \mathcal{L}(\Gamma) \).

2) Let \( \Gamma, \Gamma_n \) be as above. Then \( \Gamma_n \) is measure equivalent to \( \Gamma \) for all \( n \in \mathbb{N} \), but \( \mathcal{L}(\Gamma_n) \) is not isomorphic to \( \mathcal{L}(\Gamma_m) \) for \( n \neq m \).

Proof. 1) Assume that \( \mathcal{L}(\Gamma_s) \equiv \mathcal{L}(\Gamma_1) \). Notice that \( Q_2, Q^3_1, Q^1_1 \) are torsion free, residually finite property (T) groups. Thus applying Theorem 6.2, we get in particular that \( Q_s \equiv Q_1 \). However since \( Q_2, Q^3_1, \) and \( Q^1_1 \) are icc hyperbolic, this further implies that \( Q^3_1 \equiv Q^1_1 \). However by [PR76] or the co-hopfian property of one ended hyperbolic groups this implies that \( s = l \) and the proof follows.

2) As \( [\Gamma : \Gamma_n] < \infty \), \( \Gamma_n \) is measure equivalent to \( \Gamma \), and hence \( \Gamma_n \) is measure equivalent to \( \Gamma_m \) for all \( n, m \in \mathbb{N} \). The rest follows from part 1.

Notation Denote by \( ST \) denote the family of all icc, torsion free, residually finite property (T) groups.

For further use we record the following elementary result. Its proof is left to the reader.

Proposition 6.3. Fix \( Q \) to be an icc, torsion free, residually finite, hyperbolic property (T) group. For instance, \( Q \) can be chosen to be a uniform lattice in \( Sp(n, 1) \) for \( n \geq 2 \). Then the family \( ST' = \{ G \times Q : G \in ST \} \) consists of pairwise non-isomorphic groups.

Finally, we present the main application of this section:

Corollary 6.4. Let \( \{ Q_i \}_{i \in \mathcal{I}} \) be an infinite family of pairwise nonisomorphic groups in \( ST' \). Consider the semidirect products \( N_i \times c_1 Q_i, N_i \times c_2 Q_i \in \mathcal{R}p_{T}(Q_i) \) for every \( i \in \mathcal{I} \). Consider the canonical semidirect product \( \Gamma_i = (N_i \times N_j) \times c_1, c_2 Q_i \) corresponding to the diagonal action \( c_1 \times c_2 \). Then \( \{ \mathcal{L}(\Gamma_i) | i \in \mathcal{I} \} \) is an infinite family of pairwise nonisomorphic group II\(_1\) factors with property (T).
However, using [CIK13, Proposition 3.5] this further entails that and using [CIK13, Proposition 3.6] we can find a projection 0

[CIK13, Section 3] consider the In particular there are infinite groups

Proof. This follows directly from Theorem 6.1 and Proposition 6.3

The authors strongly believe the family \( ST \) consists of uncountably many pairwise nonisomorphic groups. In this scenario, Corollary 6.4 would provide an explicit family of uncountably many non-isomorphic property (T) group von Neumann algebras. However, we were unable to find in the literature a reference for whether \( ST \) contains uncountably many nonisomorphic groups. Therefore we leave the following as an open question.

Open Problem. Find examples of uncountably many non-isomorphic icc property (T) groups \( G \) that give non-stably isomorphic \( II_1 \) factors \( \mathcal{L}(G) \).

7 Cartan-rigidity for von Neumann algebras of groups in \( \text{Rip}(Q) \)

In this last section we classify the Cartan subalgebras in \( II_1 \) factors associated with the groups in class \( \text{Rip}(Q) \) and their free ergodic pmp actions on probability spaces (see Theorem 7.1 and Corollary 7.2). Our proofs rely in an essential way on the methods introduced in [PV12] and [CIK13] as well as on the group theoretic Dehn filling discussed in Section 3.3. For convenience we include detailed proofs.

First we establish the following general intertwining result regarding crossed product algebras arising from groups in \( \text{Rip}(Q) \).

Theorem 7.1. Let \( Q = Q_1 \times Q_2 \) where \( Q_i \) are residually finite groups. For every \( i = 1, 2 \) let \( \Gamma_i = N_i \rtimes \rho_i \)

Q \in \text{Rip}(Q) and denote by \( \Gamma = (N_1 \times N_2) \rtimes \sigma Q \) the semidirect product associated with the diagonal action \( \sigma = (\sigma_1, \sigma_2) : Q \to \text{Aut}(N_1 \times N_2) \). Let \( P \) be a von Neumann algebra together with an action \( \Gamma \sim P \) and denote by \( M = P \rtimes \Gamma \). Let \( p \in M \) be a projection and let \( A \subset MP \) be a masa whose normalizer \( \mathcal{N}_{pMP}(A)^I \leq pMP \) has finite index. Then \( A \prec M P \).

Proof. Since \( \Gamma_i = N_i \rtimes Q \) is hyperbolic relative to a residually finite group \( Q_i \), then by Theorem 3.16 there exist a non-elementary hyperbolic group \( H_i \), a subset \( T_i \subseteq N_i \) with \( |T_i| \geq 2 \) and a normal subgroup \( K_i \triangleleft \langle T_i \rangle \) of finite index such that we have a short exact sequence

\[ 1 \to \ast_{t \in T_i} R_i^t \to \Gamma_i \overset{\varepsilon_i}{\to} H_i \to 1. \]

In particular there are infinite groups \( K_1, K_2 \) so that \( \ast_{t \in T_i} R_i^t = K_1 * K_2 \).

Denote by \( \pi_i : \Gamma \to \Gamma_i \) the canonical projection given by \( \pi_i((n_1, n_2)q) = n_i q, \) for all \( (n_1, n_2)q \in (N_1 \times N_2) \triangleleft Q = \Gamma \). Then for every \( i = 1, 2 \) consider the epimorphism \( \rho_i = \varepsilon_i \circ \pi_i : \Gamma \to H_i \). Following [CIK13 Section 3] consider the \( * \)-embedding \( \Delta^{\rho_i} : M \to \mathcal{M} \rtimes \mathcal{L}(H_i) := \mathcal{M}_i \) given by \( \Delta^{\rho_i}(xu_g) = xu_g \otimes v_{\rho_i(g)} \) for all \( x \in M, g \in G \). Here \( (u_g)_{g \in G} \) and \( (v_h)_{h \in H_i} \) are the canonical group unitaries in \( \mathcal{P} \rtimes \Gamma \) and \( \mathcal{L}(H_i) \), respectively. As \( A \) is amenable, [PV12 Theorem 1.4] implies either a) \( \Delta^{\rho_i}(A) \prec \mathcal{M}_i, \mathcal{M} \otimes 1 \) or b) the normalizer \( \mathcal{N}_{p,MP}(A)^I \) is amenable relative to \( \mathcal{M} \otimes 1 \) inside \( \mathcal{M}_i \). Assume b) holds. As \( \mathcal{N}_{p,MP}(A)^I \leq pMP \) has finite index it follows that \( \Delta^{\rho_i}(pMP) \) is amenable relative to \( \mathcal{M} \otimes 1 \) inside \( \mathcal{M}_i \).

However, using [CIK13 Proposition 3.5] this further entails that \( H_i \) is amenable, a contradiction. Thus a) must hold and using [CIK13 Proposition 3.4] we get that \( A \prec \mathcal{M} \rtimes \ker(\rho_i) \). Let \( \mathcal{N} = \mathcal{P} \rtimes \ker(\rho_i) \) and using [CIK13 Proposition 3.6] we can find a projection \( 0 \neq q \in \mathcal{N}, \) a masa \( B \subset q\mathcal{N}q \) with \( \mathcal{Q} = \mathcal{N}_{q\mathcal{N}q}(B)^I \leq q\mathcal{N}q \) has finite index. In addition one can find projections \( 0 \neq p_0 \in A, 0 \neq q_0 \in B \cap pMP \) and a unitary \( u \in M \) such that \( u(Ap_0)u^* = Bp_0 \).

To end this observe the restriction homomorphism \( \pi_i : \ker(\rho_i) \to K_1 \rtimes K_2 \) is an epimorphism with \( \ker(\pi_i) = N_i \). As before, consider the \( * \)-embedding \( \Delta^{\pi_i} : \mathcal{N} \to \mathcal{N} \rtimes \mathcal{L}(K_1 \rtimes K_2) \) given by given by \( \Delta^{\pi_i}(xu_g) = xu_g \otimes v_{\pi_i(g)} \) for all \( x \in \mathcal{P}, g \in \ker(\rho_i) \). Denote by \( \mathcal{N}_i := \mathcal{N} \rtimes \mathcal{L}(\ker(\rho_i)) \). Also fix \( 0 \neq z \in \mathcal{Z}(q^I \cap q\mathcal{N}q) \). Since \( \Delta^{\pi_i}(Bz) \subset \mathcal{N} \rtimes \mathcal{L}(K_1 \rtimes K_2) \) is amenable then using [Ol12 Va13] one of the following must hold: c) \( \Delta^{\pi_i}(Qz) \) is amenable relative to \( \mathcal{N} \otimes 1 \) inside \( \mathcal{N}_i \); d) \( \Delta^{\pi_i}(Qz) \prec \mathcal{N}_i \rtimes \mathcal{L}(K_j) \) for some \( j = 1, 2 \); e) \( \Delta^{\pi_i}(Qz) \) is amenable relative to \( \mathcal{N} \otimes 1 \).

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Assume c) holds. As $Q \subseteq qNq$ is finite index so is $Qz \subseteq zNz$ and [CIK13 Lemma 2.4] implies that $zNz \ll Qz$ and using [OP07 Proposition 2.3 (3)] we get that $\Delta_{\pi_i}(zNz)$ is amenable relative $N \otimes 1$ inside $N_i$. Thus [CIK13 Proposition 3.5] implies that $K_1 \ast K_2$ is amenable, a contradiction. Assume d) holds. By [CIK13 Proposition 3.4] we have that $Qz \cong P \times (\pi_i)^{-1}(K_j)$ and using [DHI16 Lemma 2.4 (3)] one can find a projection $0 \neq r \in Z(Qz \cap zNz)$ such that $Qr \ll P \times (\pi_i)^{-1}(K_j)$. Since $Qz \subseteq zNz$ is finite index then so is $Qr \subseteq rNr$ and thus $rNr \ll N$. Therefore using [DHI16 Lemma 2.4(1)] (or [Val07 Remark 3.7]) we conclude that $N \ll P \times (\pi_i)^{-1}(K_j)$. However this implies that $\pi^{-1}(K_j) \subseteq \ker(\rho_i)$ is finite index, a contradiction. Hence e) must hold and using [CIK13 Proposition 3.4] we further get that $Bz \ll P \times N_i$. This combined with the prior paragraph clearly implies that $A \ll P \times N_i$.

Since all the arguments above still work and the same conclusion holds if one replaces $A$ by $Aa$ for any projection $0 \neq a \in A$ one actually has $A \ll P \times N_i$. Since this holds for all $i = 1, 2$, using [DHI16 Lemma 2.8(2)] one concludes that $A \ll P$, as desired.

**Corollary 7.2.** Let $\Gamma$ be a group as in the previous theorem and let $\Gamma \curvearrowright X$ be a free ergodic pmp action on a probability space. Then the following hold:

1. The crossed product $L^\infty(X) \rtimes \Gamma$ has unique Cartan subalgebra;
2. The group von Neumann algebra $L(\Gamma)$ has no Cartan subalgebra.

**Proof.** 1. Let $A \subset L^\infty(X) \rtimes \Gamma =: \mathcal{M}$ be a Cartan subalgebra. By Theorem 7.1 we have that $A \ll \mathcal{M} L^\infty(X)$ and since $L^\infty(X) \subseteq \mathcal{M}$ is Cartan then [Po01] Theorem gives the conclusion. 2. If $A \subset L(\Gamma)$ is a Cartan subalgebra then Theorem 7.1 implies that $A \ll C_1$ which contradicts that $A$ is diffuse. \qed

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