Simplifying operators by polynomials

Olavi Nevanlinna

June 9, 2022

Abstract

We collect, organise known results and add some new ones of the following nature: if $A$ is a bounded operator in a Hilbert or Banach space, does there exist a nonconstant polynomial $p(z)$ such that $p(A)$ is "simpler", "nicer" than $A$.

For example $p(A)$ could be compact or normal even when $A$ is not; then one says that $A$ is polynomially compact or polynomially normal. Using multicentric calculus to represent scalar functions $\varphi(z)$ as functions of $p(z)$ one can then apply functional calculus available for $p(A)$ to represent $\varphi(A)$ even so that functional calculus could not be formulated directly for $A$.

We consider inclusion chains of increasing generality as for example: finite rank $\prec$ compact $\prec$ Riesz $\prec$ almost algebraic $\prec$ quasialgebraic $\prec$ biquasitriangular $\prec$ quasitriangular $\prec$ bounded. We also discuss whether such classes are stable under addition, typically from a subclass. For example, the sum of an almost algebraic and a compact operator need not be polynomially almost algebraic, while the sum of a polynomially almost algebraic operator with a finite rank operator is always polynomially almost algebraic, etc.

Block $2 \times 2$ triangular operators are considered as a special case, mostly in form that if the diagonal blocks have a property, does the whole operator share the similar polynomial property.

Keywords: polynomially compact, polynomially normal, polynomially Riesz, block triangular operators, multicentric calculus, functional calculus

MSC (2020): 47-02, 47A55, 47A60, 47B99
Preface

This is an attempt to collect and organise results on classes of bounded operators for which functional calculus based on multicentric calculus would be particularly effective. So, what is this I begin to call multicentric calculus?

Some dozen years ago I decided to have a look at representing functions in the complex plane using "several centers" rather than just one, the origin. Such a representation should be useful e.g. in functional calculus when the spectrum of an operator has a few condensed clusters.

If $\Lambda = \{\lambda_i\}_{i=1}^d$ denotes the centers, we take the geometric average of $|z - \lambda_i|$ to measure the closeness of $z$ to $\Lambda$:

$$\text{dist}(z, \Lambda) = \prod_{i=1}^d |z - \lambda_i|^{1/d}.\$$

This immediately suggests to consider a "change of variable" $w = p(z)$ with $p(z) = \prod_{i=1}^d (z - \lambda_i)$, as then sets satisfying $|p(z)| \leq \rho$ are mapped to discs $|w| \leq \rho$. In fact, already Jacobi\textsuperscript{1} had considered expanding functions in the form

$$\varphi(z) \sim \sum_{n=0}^\infty c_n(z)p(z)^n$$

where the coefficients $c_n$ were polynomials of degree less than that of $p$. The idea that there would be a function of a new variable $w$, for which the power series would converge in a disc was not visible, nor in the later works on Jacobi series of this type.

What I wanted was to have $w = p(z)$ as a new global variable and for that purpose a scalar function

$$\varphi : z \mapsto \varphi(z) \in \mathbb{C}$$

is represented with a vector valued function

$$f : w \mapsto f(w) \in \mathbb{C}^d.$$ 

One can view this point of departure as just a simple rearrangement of terms in the expansions above, but the scenery changes: there is, to put it mildly, a lot of mathematics on functions defined in discs. Create $f$ from $\varphi$, apply analysis to $f$ on a disc and transform the results back to original domain and to $\varphi$. For example, $f$ is holomorphic if and only if $\varphi$ is; for continuous $f$ one can define a product and Banach algebra such that $\varphi$ shows up as the Gelfand transform of $f$.

In this survey the intention is to collect results on operator classes in which $B = p(A)$ would have a property allowing $f(B)$ to be defined using some functional calculus, although $\varphi(A)$ could not be directly defined or composed. The results of this nature, either deep or some of them really simple, are scattered in the literature and this attempt most likely is far from comprehensive.

Part of the purpose of making this available is to prompt comments and pointers to known results which I have missed or overlooked.

Karjalohja 26.3.2022

Olavi Nevanlinna

---

\textsuperscript{1}C. G. J. Jacobi, Über Reihenentwicklungen, welche nach den Potenzen eines gegebenen Polynoms fortschreiten, und zu Coeffizienten Polynome eines niedereren Grades haben, J. Reine Angew. Math. 53 (1856), 103 - 126
Content

1. Introduction
   1.1 Multicentric calculus: polynomials as new variables
   1.2 Summary of inclusion relations

2. Basic operator classes
   2.1 Algebraic, almost algebraic, quasialgebraic operators
   2.2 Excursion into meromorphic operator valued functions
   2.3 Compact and Riesz operators
   2.4 Observations on the polynomial classes

3. Special results in Hilbert spaces
   3.1 Special classes
   3.2 Polynomially normal and polynomially unitary operators

4. Block triangular operators
   4.1 Notation and spectrum
   4.2 Perturbation results
   4.3 $M_C$ polynomially almost algebraic, compact and Riesz

References

Notation and definitions
1 Introduction

1.1 Multicentric calculus: polynomials as new variables

In this paper we consider the following problem: given a bounded operator or matrix $A$, does there exist a (monic) polynomial $p$, possibly of low order, such that $p(A)$ would be "nicer", for example small in norm, diagonalizable, compact or normal etc, such that a suitable functional calculus could be applied to it even if it would not be available for $A$ directly.

In multicentric calculus [37] we represent scalar functions $\varphi : z \mapsto \varphi(z) \in \mathbb{C}$ using functions $f : w \mapsto f(w) \in \mathbb{C}^d$. Without going into details how $f$ is created from $\varphi$, if a scalar function $\varphi$ is given for which $\varphi(A)$ should be defined, we may represent $\varphi$ in multicentric form

$$
\varphi(z) = \sum_{j=1}^{d} \delta_j(z)f_j(p(z))
$$

and then obtain

$$
\varphi(A) = \sum_{j=1}^{d} \delta_j(A)f_j(p(A)).
$$

Here $\delta_j$ is the Lagrange polynomial $\delta_j(z) = \prod_{k \neq j} \frac{z - \lambda_k}{\lambda_j - \lambda_k}$ and hence $\delta_j(A)$ is always well defined for any bounded operator, while the terms $f_j(p(A))$ would be defined and computed with suitable functional calculus. The approach was introduced in [37] with roots in [36] and further developments in [38], [4], [39], [40], [2] and [3].

We shall ask questions such as whether there exists a polynomial $p$ such that $p(A)$ becomes compact, normal, unitary, etc. Much of the answers are known but scattered in the literature. Perhaps the most obvious one is that of an operator being algebraic: if there exists a nontrivial polynomial such that $p(A) = 0$, then $A$ is algebraic and its minimal polynomial is the unique monic polynomial of smallest degree at which happens. We begin with listing related definitions, survey what is known and formulate some new results and pose some questions. In particular we consider block operators of the form

$$
M = \begin{pmatrix} A & C \\ B & \end{pmatrix}
$$

which provide a rich class of operators at which we can demonstrate the concepts. For example, if $A$ and $B$ share a property, does there exist a polynomial such that $p(M)$ would have that property, too.

1.2 Summary of inclusion relations

We sum here up some of the basic inclusions and related perturbations. The notation is explained in the next section. Many of these are obvious, most of these are known but we find it useful to collect and present them all here for easy consideration, while the next section contains more details and proofs or references when needed.

The basic chains are as follows

$$
\mathcal{F} \subset \mathcal{A} \subset AA \subset PA A \subset QA \subset B,
$$

\footnote{and there is a list of symbols at the end of the paper}
\[ F \subset K \subset R \subset \mathbb{A} \]  
(1.5)

In addition, we have e.g. \( N \subset A \) and \( N \subset QN \subset R \). All these inclusions hold in all Banach spaces and there are separable Hilbert spaces where all inclusions are proper. On the other hand, when we write for example \( R \not\subset PK \), we mean that there is a Banach space \( X \) and an operator \( R \in R(X) \) such that for all polynomials \( p \) and compact operators \( K \in K(X) \) there holds \( R \neq p(K) \).

Consider next the "stability" of these classes under summation. That is, we ask whether the sum of any two operators from these classes belongs to the class.

\[ QA + K = QA \quad \text{while} \quad N + N \not\subset QA \]  
(1.6)

\[ AA + F = AA, \quad PA A + F = PA A \quad \text{while} \quad AA + K \not\subset PAA \]  
(1.7)

\[ A + F = A, \quad A + K \subset PK \quad \text{while} \quad A + K \not\subset AA \]  
(1.8)

\[ R + K = R, \quad PR + K = PR \quad \text{while} \quad R \not\subset PK. \]  
(1.9)

Some of these claims take a different form if we only allow sums between commuting operators. However, we shall not discuss that here.

In separable Hilbert spaces one can formulate further classes, in particular by combining properties of the operator and its adjoint together. In particular we have

\[ QA \subset BiQT \subset QT \subset B, \]  
(1.10)

and

\[ N_{\text{orm}} \subset N_{\text{orm}} + K \subset QD \subset BiQT. \]  
(1.11)

One can also ask how an arbitrary operator in a class can be approximated using operators which are lower in the chain. For example \( \text{cl} \mathcal{F} = K \) and

\[ \text{cl} N \subset \text{cl} (N + K) \subset \text{cl} A = BiQT. \]  
(1.12)

Here again both inclusions are proper. All these claims are discussed below, definitions and references given.

## 2 Basic operator classes

### 2.1 Algebraic, almost algebraic, quasialgebraic operators

We consider only bounded operators, and write for example \( A \in B(X) \) for bounded operators in a Banach space \( X \).

**Definition 2.1.** If \( A \in B(X) \) is such that there exists a nontrivial polynomial \( p \) such that \( p(A) \) has a property \( \mathcal{X} \), then we say that \( A \) is polynomially \( \mathcal{X} \) and denote it by \( A \in \mathcal{P}\mathcal{X} \). We say that \( A \) is polynomially \( \mathcal{X} \) of degree \( d \) if \( d \) is the smallest degree of such a polynomial.

**Remark 2.2.** Some authors have denoted \( A \in \text{Poly}^{-1}(\mathcal{X}) \) for the same purpose.

**Remark 2.3.** Warning. A similar expression is sometimes used also in the case where \( A \) has a property and this property is shared with \( p(A) \) for all polynomials. Notice that these are very different concepts. For example, there are hyponormal operators \( A \) such that \( A^2 \) is not hyponormal, and \( A \) is called polynomially hyponormal if \( p(A) \) is hyponormal for all polynomials \( p \), [12].
Example 2.4. In infinite dimensional spaces the identity operator $I$ is not compact. But with $p(z) = z - 1$ we have $p(I) = 0$ and thus $I$ is polynomially compact. More generally, all algebraic operators are polynomially compact as they vanish at their characteristic polynomials.

Example 2.5. The 0 operator with $p(z) = z + 1$ satisfies $p(0) = I$ so it would be polynomially unitary. However, in order to make the concept useful we shall require in the unitary case that $p$ is of the form $p(z) = zq(z)$. Thus, invertible algebraic operators are polynomially unitary. In fact, the inverse of $A$ is a polynomial $q(A)$ and we have $Aq(A) = I$ which is unitary.

Notation Let $X$ be a Banach space and $H$ likewise a Hilbert space. Then we denote by $F(X)$ operators with finite rank, by $K(X)$ the compact operators, by $A(X)$ the algebraic operators, by $Q(A)(X)$ the quasialgebraic operators and by $AA(X)$ the almost algebraic ones. With $N(X)$ we denote the nilpotent operators, with $QN(X)$ the quasinilpotent ones. In Hilbert spaces we have $N_{\text{norm}}$ the normal operators, $U$ for unitary ones and $QT$ for quasitriangular ones. When the space is clear from the context, we simply write $B$, $K$, $A$ etc. With $H = \mathbb{C}^n$ we write $B(H) = M_n(\mathbb{C})$.

Recall the following definitions.

**Definition 2.6.** A bounded operator $A$ is algebraic, $A \in A$, if there exists a nontrivial polynomial $p$ such that $p(A) = 0$.

It is almost algebraic, $A \in AA$, if there exists a sequence $\{a_j\}$ of complex numbers such that if we put

$$p_j(z) = z^j + a_1 z^{j-1} + \cdots + a_j$$

(2.1)

then as $j \to \infty$

$$\|p_j(A)\|^{1/j} \to 0.$$  

(2.2)

Finally, $A$ is quasialgebraic, $A \in QA$, if

$$\inf \|p(A)\|^{1/\deg(p)} = 0,$$

(2.3)

where the infimum is over all monic polynomials.

**Proposition 2.7.** We have in all Banach spaces inclusions

$$F \subset A \subset AA \subset QA \subset B.$$  

(2.4)

All inclusions are proper in separable infinite dimensional Hilbert spaces.

**Proof.** When $X$ is finite dimensional $F = B$ while for example in infinite dimensional separable Hilbert spaces all inclusions are proper. In fact, in infinite dimensional spaces the identity $I$ is algebraic but not of finite rank. If $\{e_j\}$ is a sequence of orthonormal vectors and $\{\lambda_j\}$ a sequence of nonzero complex numbers converging to 0 then the associated diagonal operator $K : e_j \mapsto \lambda_j e_j$ is compact and thus in $AA$ but is not algebraic. As $K$ is almost algebraic it is also quasialgebraic and so is automatically also $1 + K$, which however, is not almost algebraic. Finally, for example the forward shift $e_j \mapsto e_{j+1}$ has large spectrum and is not quasialgebraic as the following theorem shows.

By a theorem by Halmos, $A$ is quasialgebraic if and only if the logarithmic capacity of its spectrum vanishes [23]. Let $\text{cap}(\sigma(A))$ denote the logarithmic capacity of the spectrum.
Theorem 2.8. (P. Halmos [23]) If $A \in \mathcal{B}(X)$, then
\[
\inf \|p(A)\|^{1/\deg(p)} = \text{cap}(\sigma(A)),
\]  
(2.5)
where the infimum is over all monic polynomials $p$.

If $A \in \mathcal{B}(X)$ can be approximated fast enough with algebraic operators, then $A$ is always quasialgebraic.

Definition 2.9. Given $A \in \mathcal{B}(X)$ let
\[
\alpha_j(A) = \inf \|A - A_j\|
\]
where the infimum is over all algebraic operators $A_j$ of degree at most $j$.

Then the following holds.

Proposition 2.10. If $A$ is a bounded operator in a Banach space such that
\[
\lim_{j \to \infty} \alpha_j(A)^{1/j} = 0,
\]
then $A$ is quasialgebraic.

Proof. See Theorem 5.10.4 in [33].

In separable Hilbert spaces $\alpha_j(A) \to 0$ if and only if $A \in \mathcal{B}i\mathcal{Q}T$, see below Theorem 3.3 by Voiculescu.

The sum of two algebraic operators need not be quasialgebraic,
\[
A + A \not\subset QA.
\]  
(2.6)

Example 2.11. Let $Ae_k = e_{k+1}$ for even $k$ while $Be_k = e_{k+1}$ for odd $k$ and with $A^2 = B^2 = 0$ so that $A + B = S$ is the forward shift. Then both $A, B$ are are in particular Riesz operators and we see that the capacity is not invariant under Riesz perturbation as the capacity of nilpotent operators vanishes while that of the unilateral shift equals 1.

Perturbation with a compact operator, however, leaves the capacity invariant.

Theorem 2.12. (Stirling [47]). Let $A \in \mathcal{B}(X)$ and $K \in \mathcal{K}(X)$, then
\[
\text{cap}(\sigma(A + K)) = \text{cap}(\sigma(A)).
\]  
(2.7)

Hence in particular the sum of a quasialgebraic and a compact operator is always quasialgebraic,
\[
QA + K \subset QA.
\]  
(2.8)

Further,
\[
A + \mathcal{F} \subset A
\]  
(2.9)
and
\[
\mathcal{A}A + \mathcal{F} \subset \mathcal{A}A
\]  
(2.10)

Here (2.9) is a simple fact. Let $A \in \mathcal{A}$ be given with minimal polynomial $p$, then with $B \in \mathcal{F}$
\[
p(A + B) = p(A) + C = C
\]
where $C$ is of finite rank. If $\deg(p) = d$ and $\text{rank}(B) = q$, then there is a polynomial $p_1$ of degree at most $q + 1$ such that $p_1 \circ p(A + B) = p_1(C) = 0$. In particular

$$\deg(A + B) \leq \deg(A) \text{rank}(B) + 1. \quad (2.11)$$

The inclusion (2.10) follows from the characterization of $A_A$ as those with meromorphic resolvents, Theorem 5.7.2 in [33] and from a general perturbation result of operator valued meromorphic functions by finite rank functions. Theorem 6.1 in [35] covers the Hilbert space using an exact identity which as such is limited to Hilbert spaces, but we derive below a quantitative perturbation bound holding in all Banach spaces.

**Definition 2.13.** A vector valued function $F: z \mapsto F(z) \in B(X)$ is called meromorphic for $|z| < R$ if it is holomorphic except at singularities, and all singularities are poles: for each singularity $z_0$ with $|z_0| < R$ there exists an integer $n_0 < \infty$ such that

$$z \mapsto (z - z_0)^{n_0}F(z)$$

is holomorphic near $z_0$. The smallest such nonnegative $n_0$ is the multiplicity of the pole.

In this connection it is convenient to deal the resolvents in the "Fredholm form" $z \mapsto (1 - zA)^{-1}$ rather than in $\lambda \mapsto (\lambda - A)^{-1} = \frac{1}{\lambda}(1 - \frac{1}{\lambda}A)^{-1}$.

**Theorem 2.14.** (Theorem 5.7.2, [33]). A bounded operator $A$ in a Banach space is almost algebraic if and only if $z \mapsto (1 - zA)^{-1}$ is meromorphic for all $z \in \mathbb{C}$.

**Remark 2.15.** Some authors, including [14], [52], have defined operators to be meromorphic, or, of meromorphic type, if the resolvent $(\lambda - A)^{-1}$ is meromorphic for $\lambda \neq 0$. Denoting these operators by $\mathcal{M}$ we thus have $A_A = \mathcal{M}$. Observe that in general for bounded operators the function $(1 - zA)^{-1}$ is analytic for $|z| < 1/\rho(A)$ and there always exists a largest $R \leq \infty$, such that it is meromorphic for $|z| < R$. Much of our discussion is independent of whether $R$ is finite or not.

### 2.2 Excursion into meromorphic operator valued functions

Consider now an operator valued function $F: z \mapsto F(z) \in B(X)$ which we assume to be meromorphic for $|z| < R \leq \infty$ and for convenience, normalized as $F(0) = 1$. We denote for $r < R$

$$m_\infty(r, F) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ ||F(re^{i\theta})|| d\theta, \quad (2.12)$$

and if $\{b_j\}$ denote the poles, listed with multiplicities, then the logarithmic average of poles smaller than $r$ in modulus is

$$N_\infty(r, F) = \sum \log^+ \frac{r}{|b_j|} \quad (2.13)$$

Finally, the tool to measure the growth of $F$ as a meromorphic function is the sum of these:

$$T_\infty(r, F) = m_\infty(r, F) + N_\infty(r, F). \quad (2.14)$$
Now $A \in AA$ if and only if $T_\infty(r, (1 - zA)^{-1}) < \infty$ for all $r < \infty$ while $A \in A$ if and only if the resolvent is rational which happens when

$$T_\infty(r, (1 - zA)^{-1}) = \mathcal{O}(\log r) \text{ as } r \to \infty.$$  

(Corollary 3.1, [35]). We shall first refer the key properties of the low rank perturbation theory on [34], [35]. In order to deal with finite rank perturbations we need to measure not only the norm of the function but all its singular values and thus we restrict the discussion for a while to $A \in B(H)$. Denote

$$\sigma_j(A) = \inf_{\text{rank}(B) < j} \|A - B\|$$

and set

$$s(A) = \sum \log^+ \sigma_j(A),$$

(2.15)

which we call the total logarithmic size of $A$, [34, 35]. Without going into details, we replace in the definition of $T_\infty$ the $\log^+$ of the norm by the total logarithmic size and obtain another growth function $T_1(r, F)$ for which always $T_\infty \leq T_1$. However, with functions $F$ of the form $z \mapsto 1 + G(z)$ where $G(z)$ is of finite rank and $G(0) = 0$ we have an exact inversion formula

$$T_1(r, F) = T_1(r, F^{-1})$$

(2.16)

which allows us to do perturbation theory with low rank operators. In fact, assume that $A \in B(H)$ is such that $(1 - zA)^{-1}$ is meromorphic for $|z| < R \leq \infty$ and $B$ is an operator of rank $q$. Then we estimate as follows

$$T_\infty(r, (1 - z(A + B))^{-1}) \leq T_\infty(r, (1 - zA)^{-1}) + T_\infty(r, (1 - (1 - zA)^{-1}zB)^{-1})$$

using the fact that $T_\infty$ is submultiplicative. Then we replace in the second term on the right $T_\infty$ by $T_1$ and use the inversion identity (2.16) to get

$$T_\infty(r, (1 - (1 - zA)^{-1}zB)^{-1}) \leq T_1(r, (1 - (1 - zA)^{-1}zB)).$$

In order to bound the term now on the right hand side, notice that if $a, b$ are positive, then

$$\log(1 + ab) \leq \log^+ a + \log^+ b + \log 2.$$  

For $j > q$ we have $\sigma_j(1 - (1 - zA)^{-1}zB) \leq 1$ while for $j \leq q$ we estimate

$$\sigma_j(1 - (1 - zA)^{-1}zB) \leq 1 + \|(1 - zA)^{-1}zB\|$$

which gives

$$s(1 - (1 - zA)^{-1}zB) \leq q \log^+ \|(1 - zA)^{-1}\| + \log^+ \|zB\| + \log 2).$$

Combining the estimates we may formulate a special case of Theorem 4.1 of [34], or Theorem 6.1 [35].

**Theorem 2.16.** Let $A \in B(H)$ be such that $(1 - zA)^{-1}$ is meromorphic for $|z| < R \leq \infty$. Let $B$ be a finite rank perturbation of $A$. Then $(1 - z(A + B))^{-1}$ is also meromorphic for $|z| < R$ and the following quantitative estimate holds for $r < R$

$$T_\infty(r, (1 - (1 - zA)^{-1}zB)^{-1}) \leq (\text{rank}(B) + 1)T_\infty(r, (1 - zA)^{-1}) + \text{rank}(B)(\log^+ \|rB\| + \log 2).$$
Example 2.17. Let $I$ denote the identity in $\ell_2(\mathbb{N})$ and $K = \text{diag}\{\alpha_j\}$ where $0 \leq \alpha_j < 1$, and $\alpha_j \to 0$. Then $K$ is compact and we can view $I-K$ as a compact perturbation of the identity. Clearly $z \mapsto (I-zI)^{-1}$ is meromorphic in the whole plane and
\[
T_\infty(r, (I-zI)^{-1}) = \log^+ r
\]
while
\[
T_\infty(r, (I-z(I-K)^{-1})) = \sum \log^+ |1 - \alpha_j|r + \mathcal{O}(1).
\]
Thus, if $K$ is of finite rank, then $(I-zK)^{-1}$ is meromorphic in the whole plane, otherwise it is meromorphic only for $r < 1$. Hence, in general we have
\[
\mathcal{A} + \mathcal{K} \not\subset \mathcal{AA}.
\]  
(2.17)

Remark 2.18. Notice that this proves (2.10) in the case of Hilbert space operators: if $A \in \mathcal{A}$, then $T_\infty(r, (1-zA)^{-1}) < \infty$ for all $r < \infty$ and hence, the same holds for $T_\infty(r, (1-(1-zA)^{-1}zB)^{-1})$.  

Remark 2.19. When $A \in \mathcal{A}$ with deg$(A) = d$ then $T_\infty(r, (1-zA)^{-1} = d \log^+ r + \mathcal{O}(1)$ so the estimation above would yield deg$(A+B) \leq (\text{rank}(B) + 1)\text{deg}(A) + \text{rank}(B)$. However, splitting in estimating $\|(1-zA)^{-1}zB\|$ as $\|(1-zA)^{-1}\| \|zB\|$ then implies (2.11), since $z(1-zA)^{-1}$ is a rational function of degree $d$, as is easy to verify.

We shall now consider $T_\infty(r, (1-z(A+B))^{-1})$ where $A, B$ are almost algebraic and of finite rank, respectively, but in a general Banach space $X$. In Banach spaces we need to work without the inversion identity, but we can still derive a bound which implies $\mathcal{AA} + \mathcal{F} \subset \mathcal{AA}$. Observe that we can decompose a finite rank operator as a sum of rank-1 operators and hence it suffices to show that a rank-1 perturbation of an almost algebraic operator stays almost algebraic, which follows from the next theorem with $R = \infty$. We denote the dual of $X$ by $X^*$ and the functionals by $b^* \in X^*$. When needed, we write $<x, b^*>$ for the dual pair. Thus, the rank-1 operator $ab^*$ in a Banach space maps $x \mapsto <x, b^*>$ for the dual pair.

Theorem 2.20. Let $X$ be a Banach space and $A \in \mathcal{B}(X)$ be such that $(1-zA)^{-1}$ is meromorphic for $|z| < R \leq \infty$. Let $a \in X$ and $b^* \in X^*$ be given. Then also $(1-z(A+ab^*))^{-1}$ is meromorphic for $|z| < R$ and the following estimate holds
\[
T_\infty(r, (1-z(A+ab^*))^{-1}) \leq 2 \left(T_\infty(r, (1-zA)^{-1}) + \log^+ r + \log^+ (\|a\|\|b^*\|) + \log 2 \right).
\]

Proof. Denoting $B = ab^*$ we can begin in the same way by estimating
\[
T_\infty(r, (1-z(A+B))^{-1}) \leq T_\infty(r, (1-zA)^{-1}) + T_\infty(r, (1-(1-zA)^{-1}zB)^{-1})
\]
but now we need to bound the second term on the right without the inversion identity. To that end notice that $(1-zA)^{-1}zab^*$ is a rank-1 operator and we may write
\[
(1-(1-zA)^{-1}zab^*)^{-1} = 1 + \frac{z}{\varphi(z)}ab^*
\]
where $\varphi(z) = 1-zb^*(1-zA)^{-1}$ is a meromorphic scalar valued function such that $\varphi(0) = 1$. In particular we have
\[
T(r, 1/\varphi) = T(r, \varphi) \leq T(r, zb^*(1-zA)^{-1}) + \log 2.
\]  
(2.18)
But \( |zb^*(1 - zA)^{-1}a| \leq \|(1 - zA)^{-1}\| |z| \|a\|\|b\| \) and we hence have
\[
T(r, zb^*(1 - zA)^{-1}a) \leq T_\infty(r, (1 - zA)^{-1}) + \log^+ r + \log^+ (\|a\|\|b\|). \tag{2.19}
\]
Combining we have
\[
T_\infty(r, z\varphi ab^*) \\
\leq T_\infty(r, \frac{1}{\varphi}) + T_\infty(r, zab^*) + \log 2 \\
\leq T_\infty(r, (1 - zA)^{-1}) + \log^+ r + \log^+ (\|a\|\|b\| + \log 2 \\
+ \log^+ r + \log^+ \|ab^*\| + \log 2
\]
and so
\[
T_\infty(r, (1 - z(A + ab^*)^{-1})) \leq 2(T_\infty(r, (1 - zA)^{-1}) + \log^+ r + \log^+ (\|a\|\|b\| + \log 2).
\]

### 2.3 Compact and Riesz operators

We collect first some properties and characterizations of Riesz operators.

**Definition 2.21.** A bounded operator \( R \) in a Banach space \( X \) is called a Riesz operator if every nonzero spectral point is a pole with a finite dimensional invariant subspace. We denote the set of Riesz operators in \( X \) by \( \mathcal{R}(X) \).

The following characterization holds.

**Theorem 2.22.** (Ruston [46]) A bounded operator \( R \) in a Banach space is a Riesz operator if and only if
\[
\lim_{n \to \infty} \inf \|R^n - K\|^{1/n} = 0 \tag{2.20}
\]
where the infimum is taken over all compact operators \( K \in \mathcal{K} \).

In separable Hilbert spaces Riesz operators are sums of compact and quasinilpotent ones. This is due to West [51] and here is a minor sharpening:

**Proposition 2.23.** [10] Let \( R \) be a Riesz operator in a separable infinite dimensional Hilbert space. Then there exist a compact \( K \) and a quasinilpotent \( Q \) such that \( R = K + Q \) such that the commutator \([K, Q]\) is quasinilpotent as well.

**Remark 2.24.** Suppose all nonzero poles \( \lambda_j \) of the resolvent of an almost algebraic operator \( A \) are simple so that \((\lambda - \lambda_j)(\lambda - A)^{-1}\) is holomorphic near \( \lambda_j \). If \( P_j \) denotes the spectral projection onto the invariant subspace related to the eigenvalue \( \lambda_j \) then one can set
\[
B = \sum_{j=1}^{\infty} \lambda_j P_j
\]
for which the resolvent can be written, since \( P_j P_k = \delta_{jk} P_j \), as
\[
(\lambda - B)^{-1} = \frac{1}{\lambda} + \sum_{j=1}^{\infty} \frac{1}{\lambda - \lambda_j} - \frac{1}{\lambda} P_j.
\]
In particular $\sigma(A) = \sigma(B)$. A. E. Taylor discusses this and shows that if $C = A - B$ then $C$ is quasinilpotent and commutes with $B$, [49]. In [14] these are generalized to allow higher order poles. We shall not use these in the following and leave the details out. In dealing with Mittag-Leffler type expressions a challenge is to obtain knowledge on the size of projections $P_j$ and the sensitivity of the expression under perturbations.

We shall consider splittings of almost algebraic operators into sums of two operators where the first one is algebraic and the second one is small in the norm so that the spectrum is near origin. The possibility of splitting is based on the following result. In order to state it, put for $\theta > 1$

\[
C(\theta) = \sqrt{\theta} + 1 + \log \frac{4\epsilon \sqrt{\theta} (\sqrt{\theta} + 1)}{\sqrt{\theta} - 1}.
\] (2.21)

For example, $C(4) < 7.2$. The following is Corollary 7.5 in [35]. Denote by $n(\rho, A)$ the number of poles of the resolvent larger than $\rho$ in absolute value and counted with multiplicities.

**Theorem 2.25.** Assume that $A \in \mathcal{B}(X)$ is such that $(1 - zA)^{-1}$ is meromorphic for $|z| < R \leq \infty$. Let $\theta > 1$ be fixed. Then for any $r > 0$ such that $\theta r < R$, there exists $\rho$ such that

\[
\frac{1}{r} \leq \rho \leq \sqrt{\theta}
\]

so that

\[
\log ||P_\rho|| \leq C(\theta) T_\infty(\theta r, (1 - zA)^{-1})
\] (2.22)

holds, where

\[
P_\rho = \frac{1}{2\pi i} \int_{|\lambda|=\rho} (\lambda - A)^{-1} d\lambda.
\]

Moreover $n(\rho, A)$ satisfies

\[
n(\rho, A) < \frac{1}{\log \theta} T_\infty(\theta r, (1 - zA)^{-1}).
\] (2.23)

Thus, there is a radius $\rho$ whose exact value remains unknown, but it is known to be within an interval, such that based on the growth function we get a quantitative bound for the projection and for the degree of the algebraic part. Notice that as the growth function $T_\infty$ of the resolvent is robust under low rank updates of the operator, the bound holds essentially unchanged - although the radius may have moved within the interval.

With $R = \infty$ we can formulate the following corollary.

**Corollary 2.26.** Let $A \in \mathcal{A}$ and choose $\epsilon > 0$ and $\theta > 1$. Then there exists $\rho$, satisfying $\epsilon \leq \rho \leq \sqrt{\theta} \epsilon$, such that with $B = (1 - P_\rho)A$ and $E = P_\rho A$ we have $A = B + E$ where $B$ is algebraic and $E$ such that $\sigma(E) \subset \{ \lambda : |\lambda| < \rho \}$. Here $P_\rho$ can be bounded with $r = 1/\epsilon$ by (2.22), while the degree of $B$ satisfies $\deg(B) = n(\rho, A)$ and can be bounded by (2.23).

**Example 2.27.** We recall Example 1.5 in [35]. Denote by $V^2$ the quasinilpotent solution operator in $L_2[0, 1]$ solving $u'' = f$ with initial conditions $u(0) = 1, u'(0) = 0$:

\[
V^2 f(t) = \int_0^t (t - s) f(s) ds.
\]
Denoting further by $B$ the solution operator of the same equation with boundary conditions $u(0) = u(1) = 0$ we have a negative selfadjoint operator with eigenvalues $\lambda_j = -1/(\pi j)^2$. Now $V^2$ is a rank-1 perturbation of $B$. If we denote by $A = \alpha B + (1 - \alpha)V^2$ their resolvents grow with speed

$$T_{\infty}(r, (1 - zA)^{-1}) \sim \sqrt{r}$$

as $r \to \infty$. With $V^2$ all growth is seen thru $m_\infty$ while with $B$ all growth is in $N_\infty$, the sum of $m_\infty$ and $N_\infty$ staying essentially constant along the homotopy. Thus for all operators along the homotopy

$$\|P_\rho\| \leq e^{O(1/\sqrt{\rho})}$$

and the maximum number of eigenvalues larger than $\rho$ is bounded by $O(1/\sqrt{\rho})$ which in the self-adjoint case is obvious. Near the self-adjoint end the projections are nearly orthogonal but as the operators become increasingly "nonnormal" when approaching $V^2$ the norms of the projections grow fast and the eigenvalues get compressed near the origin.

Consider next the chain (2.4) with $\mathcal{K} \subset \mathcal{R}$ replacing $A$:

$$\mathcal{F} \subset \mathcal{K} \subset \mathcal{R} \subset \mathcal{A}A \subset \mathcal{Q}A \subset B.$$  \quad (2.24)

**Proposition 2.28.** All inclusions in (2.24) are proper in infinite dimensional separable Hilbert spaces.

**Proof.** First of all, if $A$ is any bounded invertible algebraic operator, then it cannot be a Riesz operator which implies the third inclusion to be proper. The first inclusion is trivially proper: let $H = \ell_2(\mathbb{N})$ and put $K : e_j \mapsto \frac{1}{j}e_j$. Then clearly $K \in \mathcal{K} \setminus \mathcal{F}$. Let further $N : e_{2j-1} \mapsto e_{2j}$ while $e_{2j} \mapsto 0$, so that $N^2 = 0$. Then $R = K + N$ is a Riesz operator but not compact. It is a Riesz operator by the previous theorem as $R^2$ is compact. However, $R$ is not compact because the sequence

$$Re_{2j-1} = \frac{1}{2j-1} e_{2j-1} + e_{2j}$$

does not contain any convergent subsequence. Notice additionally that $KN - NK \neq 0$. In [22] the authors show additionally that this $R$ cannot be decomposed into a sum of compact $C$ and quasinilpotent $Q$ so that they would commute. It has been shown later, that any Riesz operator can be represented as a sum of compact $C$ and quasinilpotent $Q$ such that the commutator $CQ - QC$ is also quasinilpotent, [10]. \hfill \square

We may now continue listing the perturbation inclusions. Along this chain we have clearly

$$\mathcal{F} + \mathcal{F} \subset \mathcal{F}, \mathcal{K} + \mathcal{K} \subset \mathcal{K} \quad \text{and} \quad \mathcal{R} + \mathcal{K} \subset \mathcal{R}$$

while the Example 2.11 shows that

$$\mathcal{R} + \mathcal{R} \nsubseteq \mathcal{Q}A.$$

On the other hand, Theorem 2.12 implies that

$$\mathcal{Q}A + \mathcal{K} \subset \mathcal{Q}A.$$
Here QA cannot be replaced by AA, as AA is no longer invariant under compact perturbations. In fact $1 + K \not\subset AA$. Likewise, in $AA + F \subset AA$ we cannot replace $F$ by $K$. To see these, consider $1 + K$ with $K : e_j \mapsto \frac{1}{j} e_j$ which is compact but the resolvent of $1 + K$ has poles accumulating at $\lambda = 1$ and so, $1 + K \not\in AA$. Note however that $1 + K$ is polynomially compact and as such quasialgebraic. We shall concentrate in polynomial classes in then next section.

### 2.4 Observations on the polynomial classes

To begin with, let us observe the following simple relations. When we write inclusions, it means that the inclusions hold for operators in all Banach spaces. But when we write $X \not\subset Y$ it means that there exists a Banach space (e.g. $\ell_2(\mathbb{N})$) where this happens.

**Proposition 2.29.** We have $P F = P A = A$ and $P Q A = Q A$.

*Proof.* These follow immediately from the definitions.

**Proposition 2.30.** We have $P A A \subset Q A \not\subset P A A$.

*Proof.* If $A \in P A A$, then the spectrum $\sigma(A)$ has only a finite number of accumulation points and $\text{cap}(\sigma(A)) = 0$ so that $A \in Q A$. (For $P A A$ see the characterisation in Theorem 2.34 below). To see that the inclusion is proper, take a bounded sequence $\{Q_j\}$ of quasinilpotent operators which are not nilpotent e.g. in $\ell_2(\mathbb{N})$ and let $A$ to be the direct sum of $\frac{1}{j} + Q_j$. Then $\sigma(A) = \{1/j\}_{j=1}^{\infty} \cup \{0\}$ and $A$ is quasialgebraic but for any nontrivial polynomial $p$ the resolvent of $p(A)$ has singularities outside origin which are not poles.

**Proposition 2.31.** We have $P F \not\subset F$, $P K \not\subset K$, $P R \not\subset R$, and $P A A \not\subset AA$.

*Proof.* Let

$$M = \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$$

denote the operator in $\ell_2 \oplus \ell_2$ where $S$ denotes the forward shift. Then $M$ is not compact but as $M^2 = 0$ the two first claims follow. As $M$ is nilpotent, it is a Riesz operator but if we consider $1 + M$ then it is not Riesz but clearly polynomially Riesz with polynomial $p(\lambda) = \lambda - 1$, which simultaneously implies the claim on almost algebraic operators.

We can still add a few relations, not covered by the previous ones.

**Proposition 2.32.** We have in all Banach spaces

$$P A A + F \subset P A A, \quad P K + K \subset K, \quad P R + K \subset P R,$$

while there exist an infinite dimensional separable Hilbert space such that

$$R \not\subset P K, \quad AA + K \not\subset P A A, \text{ and } AA \not\subset P R.$$
Proof. Let $p$ be such that $p(A)$ is almost algebraic and $F$ of finite rank. Then $p(A + F) = p(A) + G$ where $G$ is of finite rank and (2.25) follows from (2.10). The two other claims in (2.25) follow similarly.

Let now $A_j = \frac{j}{2^m}$ denote the scalar multiplication in $\ell_2$ and let $A = \oplus A_j$. Then all nonzero spectral points corresponds to poles of the resolvent and $A \in AA$. On the other hand, with any nontrivial polynomial $p$ there are eigenvalues $1/j$ such that $p(1/j) \neq 0$ and for such the invariant subspace is infinite dimensional. Hence $A \not\in \mathcal{PR}$.

Consider now the middle claim of (2.26). We shall define two diagonal operators $A \in AA$ and $K \in K$ such that for all integers $m \geq 0$ the point $1/(m + 1)$ is an accumulation point of $\sigma(A + K)$. Thus, for any nontrivial polynomial $p$ the spectrum of $p(A + K)$ contains an infinite amount of accumulation points and hence $A + K$ is not in $\mathcal{P}AA$. In fact, for $K$ we choose simply the diagonal operator $K$ mapping $e_j \mapsto \frac{1}{j}e_j$. In order to define $A$, notice that every positive integer $j$ can uniquely be given by a pair $(m, n)$ in the form

$$j = 2^m(1 + 2n).$$

We denote the $j^{th}$ coordinate vector by $e_{(m, n)}$ and can define another, noncompact, almost algebraic diagonal operator by

$$A : e_{(m, n)} \mapsto \frac{1}{m + 1} e_{(m, n)}.$$

Hence $A + K$ is again a diagonal operator such that

$$(A + K) : e_{(m, n)} \mapsto \left[\frac{1}{m + 1} + \frac{1}{2^m(1 + 2n)}\right] e_{(m, n)},$$

which shows that $1/(m + 1)$ is an accumulation point of eigenvalues for every $m \geq 0$, completing the example.

Finally, $\mathcal{R} \not\subset \mathcal{PK}$ is due to [26]. A Riesz operator in a separable Hilbert space is polynomially compact if and only if there is a positive integer $n$ such that $R^n$ is compact, Lemma 2 [26]. Foias and Pearcy [19], [43] have an example of a quasinilpotent operator $T$ in $\ell_2(\mathbb{N})$ such that $T^n$ is not compact for any $n \geq 1$. The operator is a weighted backward shift $Te_1 = 0$ while $Te_{j+1} = \omega_j e_j$ where the weight sequence $\{\omega_j\}$ begins

$$\{2^{-1}, 2^{-4}, 2^{-1}, 2^{-16}, 2^{-1}, 2^{-4}, 2^{-1}, 2^{-64}, 2^{-1}, 2^{-4}, \cdots \}.$$

\[ \square \]

Remark 2.33. Recall that in $X_{AH}$, the infinite dimensional Banach space constructed by Argyros and Haydon [8], all bounded operators are sums of a multiple of identity plus a compact one. Hence $\mathcal{PK} = \mathcal{B}$ and only the inclusions $\mathcal{F} \subset A \subset \mathcal{PK}$ and $\mathcal{F} \subset K \subset \mathcal{PK}$ are proper.

We begin with a structure theorem for polynomially almost algebraic operators, as polynomially Riesz and polynomially compact ones are subclasses of these.

Theorem 2.34. A bounded operator $A$ in a Banach space $X$ is polynomially almost algebraic if and only if there exists a decomposition $X = X_0 \oplus \cdots \oplus X_d$ such that $AX_i \subset X_i$ and denoting by $A_i$ the restriction of $A$ to $X_i$, $A_0$ is algebraic while for $i = 1, \ldots, d$, there exist points $\lambda_i$ such that each $A_i - \lambda_i$ is almost algebraic in $X_i$. 

15
Proof. Assume \( p(A) \) is almost algebraic and denote by \( \lambda_1, \cdots, \lambda_d \) the roots of \( p(\lambda) = 0 \). We need to show that there exist invariant subspaces \( X_i \) for \( A \) such that each restriction \( A_i - \lambda_i \) is almost algebraic in \( X_i \).

Thus, \((w - p(A))^{-1}\) is meromorphic for \( w \neq 0 \). Thus there exists a small neighbourhood \( V \) of 0 such that \( \partial V \) does not contain any singularities of \((w - p(A))^{-1}\) and such that \( p^{-1}(\sigma(p(A))) \) splits into different components \( D_j \), each containing one root \( \lambda_j \) of \( p \). We may also assume that the closures \( \overline{D_j} \) do not intersect. Then put \( D_0 = \mathbb{C} \setminus \bigcup \overline{D_j} \). Let \( P_j \) to denote, for \( j = 0, \ldots, d \), the spectral projection of \( A \) wrt the spectrum inside \( D_j \) and put \( X_j = P_j X \). Then \( A \) restricted to \( X_0 \) is algebraic, as \( D_0 \) contains only a finite number of poles while, when restricted to \( X_j \) for \( j \neq 0 \), all singularities are inside \( D_j \) with \( \lambda_j \) as the only possible accumulation point. Thus \( A - \lambda_j \) restricted to \( X_j \) is almost algebraic.

In the other direction, by assumption, all singularities of \((z - A)^{-1}\) are poles in \( \mathbb{C} \setminus \{\lambda_1, \cdots, \lambda_d\} \). Consider \( w \mapsto (w - p(A))^{-1} \) where \( p(z) = \prod (z - \lambda_j) \). We should conclude that all nonzero singularities are poles. This can be based on the Cauchy integral

\[
(w - p(A))^{-1} = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{w - p(\lambda)(\lambda - A)^{-1}}d\lambda, \tag{2.27}
\]

where \( \gamma \) surrounds the spectrum of \( A \) and \( w \neq p(\lambda) \) for all \( \lambda \) inside and on \( \gamma \). In fact, fix a nonzero \( w_0 \) in \( \sigma(p(A)) \). Now follow the proof of Theorem 5.9.2, [33].

Remark 2.35. This is essentially a reformulation of Theorem 5.9.2 in [33]. In [16] this has appeared as well.

The structure theorems for polynomially compact and polynomially Riesz operators can be formulated by specifying in Theorem 5.9.2.1 the operators \( A_i - \lambda_i \) as compact and Riesz, respectively. The structure theorem for polynomially compact operators is due to Gilfeather [20]. Similar result for Riesz operators in Banach spaces is contained in [52].

Here is the original formulation of Gilfeather.

**Theorem 2.36.** (Gilfeather [20]) Let \( A \) be a polynomially compact operator with minimal polynomial \( p(z) = (z - \lambda_1)^{n_1} \cdots (z - \lambda_k)^{n_k} \). Then the Banach space \( X \) is decomposed into the direct sum \( X = X_1 \oplus \cdots \oplus X_k \) and \( A = A_1 \oplus \cdots \oplus A_k \) where \( A_i \) is the restriction of \( X \) to \( X_i \). The operators \( (A_i - \lambda_i)^{n_i} \) are all compact. The spectrum of \( A \) consists of countably many points with \( \{\lambda_1, \ldots, \lambda_k\} \) as the only possible limit points and such that all but possibly \( \{\lambda_1, \ldots, \lambda_k\} \) are eigenvalues with finite dimensional generalized eigenspaces. Each point \( \lambda_i \in \{\lambda_1, \ldots, \lambda_k\} \) is either the limit of eigenvalues of \( A \) or else \( A_i - \lambda_i \) is quasinilpotent with \( X_i \) infinite dimensional.

If \( A \) is algebraic and \( K \) is compact, then \( T = A + K \) is polynomially compact. In fact, if \( p \) is the minimal polynomial of \( A \), then \( p(T) = p(A) + C = C \) where \( C \) is compact. Catherine L. Olsen [42] showed 1971 that in separable Hilbert spaces the decomposition is always possible.

**Theorem 2.37.** (Olsen [42]) Each polynomially compact \( A \) in a separable Hilbert space is the sum of an algebraic operator plus a compact one.

The decomposition holds in fact in all Hilbert spaces, [32], in the same way as West decomposition.
Using Corollary 2.26 we can have quantitative bounds for decomposed parts of polynomially compact operators. In fact, if \( B = p(A) \) is in the Schatten class \( S_p \) then

\[
T_\infty(r, (1 - zB)^{-1}) \leq \frac{k + 1}{p} \|B\|^p r^p + k \log(1 + r\|B\|)
\]

where \( k \) is a nonnegative integer such that \( k < p \leq k + 1 \) and \( \|\cdot\|_p \) denotes the Schatten norm, see Theorem 6.5 in [35]. The following formulation is for all polynomially almost algebraic operators.

If \( A \in PAA \) and \( p \) is the minimal polynomial such that \( p(A) \) is almost algebraic, denote by \( \lambda_j \) for \( j = 1, \cdots, d \) its roots. Let then \( \rho_0 \) be small enough so that each \( \gamma_\rho = \{\lambda : |p(\lambda)| = \rho\} \) consists of \( d \) components \( \gamma^j_\rho \), each surrounding one root of \( p \), when \( \rho < \rho_0 \). Then denote the contour of integration in defining the spectral projection \( P_\rho \):

\[
P_\rho = \frac{1}{2\pi i} \int_{\gamma_\rho} (\lambda - A)^{-1} d\lambda.
\]

(2.28)

and further

\[
P_{\rho,j} = \frac{1}{2\pi i} \int_{\gamma^j_\rho} (\lambda - A)^{-1} d\lambda
\]

(2.29)

so that \( P_\rho = \sum_{j=1}^d P_{\rho,j} \). We conclude that, based on the growth of \( T(r, (1 - w p(A))^{-1}) \) that there exists a \( \rho \) which allows us to obtain a bound for these projections. We use again the factorization \( p(\lambda) - p(A) = (\lambda - A)q(\lambda, A) \) and denote

\[
C_0 = \left\| \frac{1}{2\pi \rho} \int_{\gamma_\rho} \frac{q(\lambda, A)}{p(\lambda)} d\lambda \right\|
\]

and

\[
C_j = \left\| \frac{1}{2\pi \rho} \int_{\gamma^j_\rho} \frac{q(\lambda, A)}{p(\lambda)} d\lambda \right\|
\]

Then we have the following.

**Theorem 2.38.** Let \( A \) be polynomially almost algebraic and \( p, \varepsilon, \theta, \rho_0 \) as above. Then there exists \( \rho \) satisfying \( \varepsilon \leq \rho \leq \sqrt{\theta} \varepsilon \) such that \( P_{\rho,j} \) and \( P_\rho \) in (2.29) and (2.28) satisfy

\[
\|P_{\rho,j}\| \leq C_j M \text{ and } \|P_\rho\| \leq C_0 M
\]

where

\[
\log M = C(\theta) T_\infty(\theta/\varepsilon, (1 - w p(A))^{-1})
\]

With \( X_0 = (1 - P_\rho)X \) and \( X_j = P_{\rho,j}X \) for \( j = 1, \cdots, d \) the space gets split into invariant subspaces, \( X = X_0 \oplus \cdots \oplus X_d \), such that \( A \) restricted to \( X_0 \) is algebraic with \( \deg A_0 \leq d \) \( n(p, p(A)) \) while \( A - \lambda_j \) restricted to \( X_j \) is almost algebraic. Here \( C(\theta) \) satisfy (2.21) and

\[
n(p, p(A)) < \frac{1}{\log \theta} T_\infty(\theta/\varepsilon, (1 - w p(A))^{-1})
\]

17
Proof. This follows from writing with \( w = 1 / p(\lambda) \)

\[
(\lambda - A)^{-1} = q(\lambda, A)(p(\lambda) - p(A))^{-1} = \frac{q(\lambda, A)}{p(\lambda)} (1 - w p(A))^{-1}
\]

and returning to the earlier discussion. Note that the number of poles \( w_k \) of \((1 - w p(A))^{-1}\) satisfying \(|w_k| > \rho\) is given by \( n(\rho, p(A)) \) and \( p^{-1}(w_k) \) contains at most \( d \) points.

Example 2.39. Elastic Neumann-Poincaré operators give examples of "real life" polynomially compact operators. In short, in bounded domains in dimensions 2 and 3 with smooth boundaries the operators are polynomially compact but not compact. If the boundary has corners, continuous spectrum appears. For details, see Section 6 of the survey article [1].

\[
\square
\]

3 Special results in Hilbert spaces

3.1 Special classes

In this section we assume that the space is a separable Hilbert space. The first observation to be made is that the classes \( \mathcal{X} \) we have discussed above are such that in Hilbert spaces \( A \in \mathcal{X} \) typically imply \( A^* \in \mathcal{X} \). We start with a class where this does not hold.

Definition 3.1. A bounded operator \( A \) in a separable Hilbert space is quasitriangular, if there exists an increasing sequence \( \{P_n\} \) of finite rank orthogonal projections converging pointwise to the identity such that

\[
\lim_{n \to \infty} \| (1 - P_n) A P_n \| = 0.
\]

We denote then \( A \in QT \). It is biquasitriangular, \( A \in BiQT \), if both \( A \) and \( A^* \) are quasitriangular.

Quasitriangularity was introduced by Halmos in [25], where he in particular proved that

\[
QT + K \subset QT.
\]

He later proved, that \( QA \subset QT \). Since \( A^* \in QA \) if and only if \( A \in QA \) we have in fact

\[
QA \subset BiQT,
\]

and this is again proper.

Another proper subclass of \( BiQT \) is provided by normal operators and their perturbations. We denote the normal operators by \( \mathcal{N}_{orm} \) and these are in particular quasidiagonal.

Definition 3.2. A bounded operator \( A \) in a separable Hilbert space is quasidiagonal, if there exists an increasing sequence \( \{P_n\} \) of finite rank orthogonal projections converging pointwise to the identity such that

\[
\lim_{n \to \infty} \| A P_n - P_n A \| = 0.
\]

We denote then \( A \in QD \).
The following chain holds
\[ N_{orm} \subset N_{orm} + K \subset Q^D \subset BiQT. \]  
(3.3)

**Theorem 3.3.** (Voiculescu [49])

*In separable Hilbert spaces the norm closure of algebraic operators equals the biquasitriangular ones:*

\[ \text{cl} A = BiQT. \]

Observe that the set of algebraic operators of uniformly bounded degree is, on the other hand, closed. See more e.g. [27].

An important subclass of \( Q^T \) consists of the so called thin operators, that is sums, of scalar and compact operators. Douglas and Pearcy showed that a bounded operator \( A \) is thin iff

\[ Q(A) = \limsup_P \| (I - P)AP \| = 0, \]

where \( P \) runs over all increasing sequences of finite rank orthogonal projections, while \( A \) being quasitriangular can be written likewise as

\[ q(A) = \liminf_P \| (I - P)AP \| = 0. \]

These were considered in 1970’s and classifying nonquasitriangular operators the ratio of \( q(A)/Q(A) \) turned out important; on these developments see e.g. [18]. Observe that it is essential that one requires the sequence of projections to be ordered. In fact for every bounded \( A \) and any \( n \) and \( \varepsilon \) there exists a rank-\( n \) orthogonal projector \( P_n \) such that \( \| (I - P_n)AP_n \| < \varepsilon \), [25].

To end this list of particular classes, we still mention the following results.

**Theorem 3.4.** (Apostol, Voiculescu, [7])

*In separable Hilbert spaces every quasinilpotent bounded operator is a norm limit of nilpotent ones: \( QN \subset \text{cl} N \).*

**Theorem 3.5.** (Apostol, Foias [5])

*An operator \( T \in BiQT \) if and only if it is unitarily similar to a block operator
\[
\begin{pmatrix}
A & C \\
D & B
\end{pmatrix}
\]
where \( A \) and \( B \) are block diagonal (direct sums of finite size operators) and at least one of \( C \) and \( D \) is compact.*

**Definition 3.6.** *If \( A \in BiQT \) has connected spectrum and essential spectrum with \( 0 \in \sigma_e(A) \), then we denote it as \( A \in C \).*

This class is interesting as every operator in \( B \setminus C \) has a nontrivial hyperinvariant subspace.

**Theorem 3.7.** (Apostol, Foias, Voiculescu [6])

*In separable Hilbert spaces we have \( \text{cl} N = C \).*

### 3.2 Polynomially normal operators

In [21] Gilfeather asks what operators \( A \) satisfy equations of the form \( f(A) = N \) where \( N \) is a normal operator. In particular, if \( f \) is a polynomial it leads to a classification of polynomially normal operators. Kittaneh [30] discusses this further and it contains the following formulation of Gilfeather’s theorem.
Theorem 3.8. Let $A$ be a bounded operator in a separable infinite dimensional Hilbert space and assume there exists a nontrivial polynomial $p$ such that $p(A)$ is normal. Then there exist reducing subspaces $\{H_n\}_{n=0}^\infty$ for $A$ such that $H = \bigoplus_{n=0}^\infty H_n$, $A_0 = A|_{H_0}$ is algebraic, and $A_n = A|_{H_n}$ is similar to a normal operator for $n > 0$.

For further results we refer to [30].

Remark 3.9. In finite dimensional spaces similarity to diagonal matrices is an important subset of matrices for which e.g. a functional calculus can be naturally defined pointwise: If $A = SDS^{-1}$ with $D = \text{diag}(d_i)$, then defining $f(D) = \text{diag}(f(d_i))$ one can set $f(A) = Sf(D)S^{-1}$. Of course, using characteristic polynomials all matrices are "polynomially diagonalisable". However, in practise the natural question is about a simplifying polynomial with smallest degree such that $p(A)$ is diagonalisable. This is easy to describe by assuming a Jordan form of $A$ to be known. In fact, if $J$ denotes a $k \times k$ matrix

$$J = \begin{pmatrix}
\lambda & 1 \\
 & \ddots & 1 \\
& & \ddots & 1 \\
& & & \lambda \\
\end{pmatrix}$$

then we want $p(J) = p(\lambda)I$ which requires that $p^{(\nu)}(\lambda) = 0$ for $\nu = 1, \ldots, k-1$. Thus, if the minimal polynomial of $A$ is

$$m_A(z) = \prod (z - \lambda_j)^{n_j}$$

then with

$$s_A(z) = \int_0^z (\zeta - \lambda_j)^{n_j-1} d\zeta + c$$

the matrix $s_A(A)$ is diagonalisable. A related Banach algebra and functional calculus was discussed in [39].

Remark 3.10. Observe that if $A$ is polynomially normal, $p(A)^*p(A) = p(A)p(A)^*$, then also $p(A) - p(0)$ is normal and we could restrict our attention to polynomials of the form $p(z) = zq(z)$. This is a natural requirement in particular when considering polynomially unitary operators.

Normal operators $A$ have the property that $p(A)$ is normal for all polynomials $p$. If $A$ is self-adjoint, then $p(A)$ is self-adjoint if all coefficients of $p$ are real. This is in contrast with the unitary case as pointed out in Lemma 3.15 below. We close this by a simple example of a self-adjoint operator which perturbed by a diagonal one is no longer self-adjoint but still polynomially self-adjoint.

Example 3.11. Let $S$ denote the unitary shift operator in $\ell_2(\mathbb{Z})$ so that $S + S^*$ is self-adjoint. Let $D$ denote the diagonal operator mapping $e_j \mapsto (-1)^je_j$ and set

$$A = S + S^* + iD$$

which is normal. Take $p(\lambda) = \lambda^2 + 1$ for which $p(iD) = 0$. Then $p(A) = S^2 + (S^*)^2$ is selfadjoint.
3.3 Polynomially unitary operators

Let $A$ be an algebraic operator and $p$ such that $p(A) = 0$. Then $(p + 1)(A) = I$ is unitary. However, it is more useful to consider polynomials of the form $p(z) = zq(z)$, as then it is more naturally related to e.g. solving linear equations. In fact, consider solving $Ax = b$ and assume that there is a known polynomial $q$ such that $Aq(A)$ is unitary. Then $q(A)^{-1}A^{-1} = q(A)^*A^*$ and we have an explicit expression for the inverse:

$$A^{-1} = q(A)q(A)^*A^*.$$

**Definition 3.12.** We say that $A$ is polynomially unitary if there exists $q$ such that $Aq(A)$ is unitary and denote $A \in \mathcal{P}_0U$.

We write $\mathcal{P}_0$ to make the restriction that the polynomial must vanish at origin. Then we can formulate the following

**Proposition 3.13.** An algebraic operator $A$ is polynomially unitary if and only if it is invertible.

**Proof.** Indeed, if $Aq(A)$ is unitary, $A$ must be invertible. On the other hand, if $A$ is invertible and algebraic, then there exists a polynomial $q$ such that $q(A) = A^{-1}$. But then $Aq(A) = I$ is unitary.

In [29] there is a discussion on how close $Aq(A)$ can be to unitary.

**Proposition 3.14.** (Theorem 2.5 in [29]) Let $A$ be an $n \times n$ complex matrix. Then for every eigenvalue of $A$ and for any polynomial $q$

$$| |q(\lambda)| - 1 | \leq \min_{U \in U} \| Aq(A) - U \| = \max_{\| x \| = 1} | \| Aq(A)x \| - 1 | = 1.$$  

(3.4)

For numerical solution of $Ax = b$ it is then of interest to consider the minimum of the right hand side of (3.4) over all $q$ of a given degree and how fast that would decay with increasing the degree. We refer to [29] for further discussion on this direction.

Suppose now that $A$ is invertible (in an arbitrary Banach space) then there exists a sequence of polynomials $q_j$ such that $Q_j := Aq_j(A) \rightarrow I$ if and only if the spectrum $\sigma(A)$ does not separate 0 from $\infty$, see [33]. Note, however, that there are unitary operators like the unitary shift for which for all nonzero polynomials $q$ we have $\|Sq(S) - I\| > 1$ as by maximum principle max$_{|x| = 1} |zq(z) - 1| > 1$.

We shall now consider in more detail some examples related to special cases where $\sigma(A) = \mathbb{T}$, the unit circle. This means that we may restrict our considerations to the polynomials $z \mapsto z^n$, which follows from the following simple facts.

**Lemma 3.15.** Let $q$ be a polynomial of degree $n$ such that $|q(z)| = 1$ on $\mathbb{T}$. Then $q(z) = \alpha z^n$ where $|\alpha| = 1$.

**Proof.** Write $q(z) = \alpha_n z^n + lower$ with $\alpha_n \neq 0$. Then

$$q(e^{i\theta})\overline{q}(e^{-i\theta}) = \sum_j |\alpha_j|^2 + \cdots + 2Re \{ [\alpha_n\overline{\alpha} + \alpha_{n-1}\overline{\alpha_0}] e^{i(n-1)\theta} \} + 2Re \{ \alpha_n\overline{\alpha_0} e^{in\theta} \}.$$

Since this has to be identically 1, all nonconstant terms have to vanish. Beginning from the last term we obtain $\alpha_0 = 0$ and then recursively $\alpha_j = 0$ for all $j < n$. Thus $q(z) = \alpha_n z^n.$ \qed
Lemma 3.16. If \( q(z) = z^n + \alpha_{n-1}z^{n-1} + \cdots + \alpha_0 \), then
\[
|q(e^{i\theta})| \leq 1
\]
implies \( q(z) = z^n \).

Proof. We have
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} |q(e^{i\theta})|^2 d\theta = 1 + \sum_{j=0}^{n-1} |\alpha_j|^2.
\]

The following theorem is due to Sz.-Nagy [48].

Theorem 3.17. A bounded operator is similar to unitary if and only if it is invertible and the set \( \{A^n\}_{n \in \mathbb{Z}} \) is bounded.

If an operator \( A \) is similar to a normal one, then it satisfies a Linear Resolvent Growth ("LRG") condition with some constant \( C \):
\[
\| (\lambda - A)^{-1} \| \leq \frac{C}{\text{dist}(\lambda, \sigma(A))} \quad \text{for } \lambda \notin \sigma(A). \tag{3.5}
\]

The backward shift \( T \) is a contraction satisfying LRG but is not normal as \( T^* T \neq TT^* \).

Benamara and Nikolski have the following result.

Theorem 3.18. [9] Let \( A = U + F \), where \( U \) is unitary and \( F \) of finite rank, be a contraction \( \|A\| \leq 1 \). Then \( A \) is similar to a normal operator if and only if \( A \) satisfies (3.5) and \( D \not\subset \sigma(A) \).

On the sharpness on this, see [31]. Further, Nikolski and Treil have the following.

Theorem 3.19. [41] Let \( U \) be unitary such that its spectrum contains a nontrivial absolutely continuous part. Then there exists a rank-1 perturbation \( ba^* \) such that the operator \( A = U + ba^* \) satisfies (3.5), \( \sigma(A) \subset \mathbb{T} \) but \( A \) is not similar to a unitary operator.

We shall now go through a list of simple examples.

Our first example deals with a diagonal unitary operator perturbed with a nilpotent rank-1 operator. Depending on whether \( \varphi/2\pi \) is rational or not, the operator is polynomially unitary.

Example 3.20. Let \( A = \sum_{j=1}^{\infty} \lambda_j e_j e_j^* + (\lambda_2 - \lambda_1) e_1 e_2^* \). Assume then that \( \text{cl}\{\lambda_j\} = \mathbb{T} \) so that the spectrum of \( A \) is the unit circle \( \mathbb{T} \). Hence, we may consider \( z \mapsto z^n \). We have
\[
A^n = D^n + (\lambda_2^n - \lambda_1^n)e_1 e_2^*
\]
where the off-diagonal term measures the distance from \( A^n \) to be unitary. Thus we may set e.g. \( \lambda_1 = 1 \) while \( \lambda_2 = e^{i\varphi} \). Now we have \( |\lambda_2^n - 1| = 0 \) for some \( n \) if and only if \( \varphi/2\pi \) is rational. In particular, if \( \lambda_2 = -1 \), then \( A^2 \) is unitary and \( A \) is similar to unitary if we write \( A = B \oplus D_3 \) with
\[
B = \begin{pmatrix} 1 & -2 \\ -1 \end{pmatrix}
\]
and $D_3 = \text{diag}(\lambda_3, \lambda_4,...)$. Then $TAT^{-1}$ is diagonal, unitary where

$$T = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \oplus I.$$  

Assume now that $\varphi/2\pi$ is not rational. Then $Aq(A)$ is not unitary for any polynomial $q$ as

$$\|Aq(A)(Aq(A))^* - I\| > 0$$

but still

$$\inf_n \|A^n(A^n)^* - I\| = 0.$$

The next example presents an operator $A$ which is not normal but $A^2$ is unitary.

**Example 3.21.** Let

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and with $\{e^{i\theta_j}\}$ dense on $\mathbb{T}$ we set

$$A = \bigoplus_{j=1}^{\infty} e^{i\theta_j} B.$$  

With $I$ denoting the 2-dimensional identity we have, since $B^2 = I$,

$$A^2 = \bigoplus_{j=1}^{\infty} e^{2i\theta_j} I$$

and hence $A^2$ is unitary.

**Example 3.22.** Let $\{\lambda_j\}$ de dense in $\mathbb{D}$ and if $B$ is as in the previous example, then set

$$A = \bigoplus_{j=1}^{\infty} \lambda_j B.$$  

Then again $A^2$ is normal, and $A$ has large spectrum: $\sigma(A) = \mathbb{T}$.

**Example 3.23.** In this example we meet an operator $A \in l_2(\mathbb{N})$ with the following properties

(i) $A$ is similar to unitary

(ii) $\|A^n\| = 2$ for all $n \neq 0$.

(iii) $\sigma(A) = \mathbb{T}$.

Denote by $C_n$ the circulant unitary matrix in $\mathbb{C}^n$ such that $c_{i+1, i} = c_{1, n} = 1$ and let $D_n(r)$ be the diagonal matrix such that $d_{i,i} = r^{-1}$ $(i < n)$ while $d_{nn} = r^{n-1}$. Then $A_n(r) = C_n D_n(r)$ is given in its polar form, with eigenvalues at the roots of unities. In particular

$$\|A_n(r)^k\| = r^{-k} \text{ for } k < n$$

I must have seen this one somewhere but cannot find the reference.
while $A_n(r)^n = I$. Choose $r = r_n = 2^\frac{r}{n}$ so that $\|A_n(r)^{n-1}\| = 2$. Finally set $A = \bigoplus_{n=2}^\infty A_n(r_n)$. Now this one has the properties asked for, as we have likewise, for $k \leq n$ 
\[ \|A_n(r)^{-k}\| = r^{k-n} \]
and hence, $\|A^{-k}\| = 2$, $k > 0$. Summarizing we have $\|A^n\| = 2$ for $n \neq 0$.

The next example shows an operator $T$ such that $p(T)$ is unitary.

**Example 3.24.** Let $S$ denote the unitary shift in $\ell_2(\mathbb{Z})$ and $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)$. Note that we may set e.g. $\lambda_2 = 0$ to obtain $p$ of the form $zq(z)$. Define $T$ as follows

- $e_k \mapsto e_{k+1} + \lambda_1 e_k$ for $k$ odd
- $e_k \mapsto e_{k+1} + \lambda_2 e_k$ for $k$ even.

Denoting by $D$ the diagonal operator with $\lambda_1$ and $\lambda_2$ alternating on the diagonal we have $p(D) = 0$ and $T = S + D$. Now we have

\[ p(T) = S^2. \]

In fact, suppose $k$ is odd. Then

\[ T^2 e_k = T(e_{k+1} + \lambda_1 e_k) = e_{k+2} + (\lambda_1 + \lambda_2)e_{k+1} + \lambda_1^2 e_k \]
\[ - (\lambda_1 + \lambda_2) T e_k = - (\lambda_1 + \lambda_2)e_{k+1} - (\lambda_1 + \lambda_2)\lambda_1 e_k \]

and we obtain

\[ p(T)e_k = (T^2 - (\lambda_1 + \lambda_2)T + \lambda_1 \lambda_2)e_k = e_{k+2}. \]

With $k$ even the computation is analogous. Further, if $p(\lambda) - p(z) = (\lambda - z)p[\lambda, z] = (\lambda - z)(\lambda + z - \lambda_1 - \lambda_2)$ then

\[ (\lambda - T)^{-1} = p[\lambda, T](p(\lambda) - p(T))^{-1} = p[\lambda, T](\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1 \lambda_2 - S^2)^{-1}. \]

Notice that this holds as such if we denote by $S$ the forward shift in $\ell_2(\mathbb{N})$. By spectral mapping theorem the spectrum of $T$ is the lemniscate $\{ \lambda : |p(\lambda)| = 1 \}$ (or together with the inside if $S$ not invertible).

We may modify the operator $T$ as follows. Let $T = D + \rho S$ where $\rho > 0$. Then the same calculation gives

\[ p(T) = (\rho S)^2. \]

In particular we may have the spectrum of $T$ to equal the lemniscate, with any level $\rho$.

**Example 3.25.** Now let $S$ be again the unitary shift and set $T = S + \alpha e_0 e_0^*$. By Weyl’s theorem $\sigma(T) \subset \sigma(S) \cup \sigma_p(T)$. So, assume $|\lambda| \neq 1$. We have

\[ (\lambda - T)^{-1} = (1 - \alpha f_0 e_0^*)^{-1}(\lambda - S)^{-1} \]

where

\[ f_0 = (\lambda - S)^{-1} e_0. \]

Since

\[ (1 - \alpha f_0 e_0^*)^{-1} = 1 + \frac{\alpha}{1 - \alpha f_0 e_0^*} f_0 e_0^* \]

24
\( \lambda \) is an eigenvalue iff \( \alpha e_0 f_0 = 1 \) and \( f_0 \) is likewise an eigenvector, if

\[
(\lambda - T) f_0 = e_0 - \alpha e_0 e_0^* f_0 = 0.
\]

When \(|\lambda| > 1\) we have

\[
f_0 = (\lambda - S)^{-1} e_0 = \lambda^{-1} e_0 + \lambda^{-2} e_1 + \cdots
\]

and thus \( \alpha e_0^* f_0 = 1 \) iff \( \alpha = \lambda \).

When \(|\lambda| < 1\) we have

\[
f_0 = -S^{-1}(1 - \lambda S^{-1})^{-1} e_0 = -e_0 - \lambda e_{-1} - \cdots
\]

and thus \( \alpha e_0^* f_0 = 0 \).

Thus

\[
\sigma(T) = \sigma(S) \cup \{ \alpha \}
\]

if \(|\alpha| > 1\), while otherwise \( \sigma(T) = \sigma(S) \).

Notice that again the claim stays the same if we replace \( S \) by the forward shift with the unit disc as the spectrum.

**Example 3.26.** Now \( T = S + \alpha e_0 e_0^* \) with \( k \geq 1 \). For \(|\lambda| > 1\) the condition now is \( \alpha e_0^* f_0 = \alpha \lambda^{-k-1} = 1 \) while for \(|\lambda| < 1\) we have \( \alpha e_0^* f_0 = 0 \). Thus

\[
\sigma(T) = \sigma(S) \cup \{ \lambda_1, \ldots, \lambda_{k+1} \}
\]

for \(|\alpha| > 1\), where \( \lambda_j \) denote the \( k+1 \) roots of \( \lambda^{k+1} = \alpha \), while for \(|\alpha| < 1\) we have

\[
\sigma(T) = \sigma(S).
\]

**Example 3.27.** Consider now \( T = S + \alpha e_0 e_0^* \) with \( k \geq 1 \). The condition for \(|\lambda| > 1\) now reads

\[
\alpha e_0^* f_0 = \alpha e_0^* (\lambda^{-1} e_0 + \lambda^{-2} e_2 + \cdots) = 0
\]

and \( (\lambda - T) \) is invertible for \(|\lambda| > 1\). On the other hand, for \(|\lambda| < 1\) the condition takes the form

\[
\alpha e_0^* f_0 = \alpha e_0^* (-e_{-1} - \cdots - \lambda^{k-1} e_{-k-1} - \cdots) = -\alpha \lambda^{k-1} = 1
\]

Hence, with \( k = 1 \) the operator \( T \) is invertible except when \( \alpha = -1 \) and then \( T \) actually splits into a sum of forward and backward shifts with the open disc consisting of eigenvalues.

For \( k > 1 \) we obtain eigenvalues \( \{ \lambda_1, \ldots, \lambda_{k-1} \} \) where \( \lambda_j \) are the roots of \( \lambda^{k-1} = 1/\alpha \). Thus we summarize with \( k = 1 \):

\[
\sigma(T) = \sigma(S) \text{ for } \alpha \neq -1
\]

\[
\sigma(T) = \sigma(S) \cup \mathbb{D}, \text{ for } \alpha = -1
\]

while for \( k \geq 2 \)

\[
\sigma(T) = \sigma(S), \text{ for } |\alpha| \geq 1
\]

\[
\sigma(T) = \sigma(S) \cup \{ \lambda_1, \ldots, \lambda_{k-1} \}, \text{ for } |\alpha| < 1.
\]

25
4 Block triangular operators

4.1 Notation and spectrum

Let $X$ and $Y$ be Banach spaces. We consider triangular operators of the form

$$M_C = \begin{pmatrix} A & C \\ B & \end{pmatrix}$$

(4.1)

where $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$. Thus $M_C$ is always a bounded operator in $\mathcal{B}(X \oplus Y)$. When $X$ and $Y$ are Hilbert spaces and $(x, y) \in X \oplus Y$, we norm $\|(x, y)\|^2 = \|x\|^2 + \|y\|^2$ while in general e.g. $\|(x, y)\| = \max\{\|x\|, \|y\|\}$.

To motivate the special interest in these block operators, notice that if we write

$$M_C = \begin{pmatrix} A & C \\ B & \end{pmatrix} + \begin{pmatrix} 0 & C \\ 0 & \end{pmatrix} = M_0 + N$$

then $M_C$ can be thought as a perturbation of the block diagonal operator by a nilpotent one, as $N^2 = 0$.

Similarly to the discussion in the previous section we first list properties on $M_0$ which are preserved under addition of a corner element $C$. And then we ask - if the original property is not preserved, whether there would exist a $p$ such that $p(M_C)$ would again share the original property.

We need to know the relation between the spectrum of $M_C$ and those of $A$ and $B$. Clearly $\sigma(M_0) = \sigma(A) \cup \sigma(B)$.

**Lemma 4.1.** We have always

$$\sigma(M_C) \subset \sigma(M_0)$$

(4.2)

and

$$\overline{\sigma(M_C)} = \overline{\sigma(M_0)}.$$  

(4.3)

**Proof.** Take $\lambda \notin \sigma(M_0)$. Then $(\lambda - M_C)$ is invertible. In fact we may set

$$X = \begin{pmatrix} (\lambda - A)^{-1} & (\lambda - A)^{-1}C(\lambda - B)^{-1} \\ (\lambda - B)^{-1} & \end{pmatrix}$$

and multiply $X(\lambda - M_C) = 1$. Thus we have

$$(\lambda - M_C)^{-1} = \begin{pmatrix} (\lambda - A)^{-1} & (\lambda - A)^{-1}C(\lambda - B)^{-1} \\ (\lambda - B)^{-1} & \end{pmatrix}.$$  

(4.4)

Choose then an arbitrary boundary point $\lambda_0$ of $\overline{\sigma(M_0)}$. Then $\|(\lambda - M_0)^{-1}\| \to \infty$ as $\lambda \to \lambda_0$ from outside of $\overline{\sigma(M_0)}$. Then necessarily $\|(\lambda - M_C)^{-1}\| \to \infty$ as well. Thus, $\lambda_0$ is a also a boundary point of $\overline{\sigma(M_C)}$.

We shall be mainly interested in classes where spectra do not have interior points. Then in particular $\sigma(M_C) = \sigma(A) \cup \sigma(B)$. However, in general the inclusion in (4.2) can be proper.
Example 4.2. ([17]) Let $H = l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$ and $A = S$ the forward shift, $B = S^*$ the backward shift and $C : \{\eta_j\} \mapsto \{\eta_1, 0, 0, \ldots\}$. Then $M_C = \begin{pmatrix} S & C \\ S^* & \end{pmatrix}$ is unitary and $\sigma(M_C) = \partial \mathbb{D}$ while $\sigma(M_0) = \mathbb{D}$ so that $\sigma(M_C) \subseteq \sigma(M_0)$. In particular $M_C$ is invertible $M_C^{-1} = M^*_C = \begin{pmatrix} S^* \\ C^* \\ S \end{pmatrix}$. However, $M_C^{-1}$ is not upper block triangular and hence not in the closed subalgebra generated by the nonnegative powers of $M_C$.

Notice that if the Sylvester equation
\[ AX - XB = C \] (4.5)
has a solution $X$, then we have
\[ \begin{pmatrix} I & X \\ I & \end{pmatrix} \begin{pmatrix} A & C \\ B & I \end{pmatrix} \begin{pmatrix} I & -X \\ I & \end{pmatrix} = \begin{pmatrix} A & B \\ \end{pmatrix} \] (4.6)
and in particular, $M_C$ and $M_0$ have the same spectrum. Further, since
\[ \begin{pmatrix} A & C \\ B & I \end{pmatrix} = \begin{pmatrix} I & C \\ B & I \end{pmatrix} \begin{pmatrix} A & \end{pmatrix} \] (4.7)
we see that if $M_C$ is invertible, then $A$ is left invertible and $B$ invertible from right.

Definition 4.3. The approximate defect spectrum $\sigma_\delta(A)$ is the set
\[ \sigma_\delta(A) = \{ \lambda \in \mathbb{C} : \lambda - A \text{ is not onto} \} \].

The approximate point spectrum $\sigma_a(A)$ is the set of $\lambda \in \sigma(A)$ for which there exists a sequence $\{x_n\}$ of unit vectors such that $\|Ax_n - \lambda x_n\| \to 0$.

Theorem 4.4. (Davis and Rosenthal [13]) If\[ \sigma_\delta(A) \cap \sigma_a(B) = \emptyset \] (4.8)
holds, then the Sylvester equation (4.5) has a solution for every $C$. If $A$ and $B$ act in Hilbert spaces, then (4.8) is also necessary.

In Hilbert spaces the following holds (Corollary 3.4 in [28]):

Theorem 4.5. Let $A \in \mathcal{B}(H_1), B \in \mathcal{B}(H_2)$. Then for every $C \in \mathcal{B}(H_2, H_1)$ we have $\sigma(M_C) = \sigma(M_0)$ if one of the following conditions hold

(i) $\sigma(A) \cap \sigma(B)$ has empty interior
(ii) $A^*$ or $B$ has SVEP.
4.2 Perturbation results

**Proposition 4.6.** $M_C$ is of finite rank if and only if $A, B, C$ are of finite rank.

*Proof.* This is obvious. □

We have $\text{rank } M_C \leq \text{rank } A + \text{rank } B + \text{rank } C$.

**Proposition 4.7.** $M_C$ is compact if and only if all $A, B, C$ are compact.

*Proof.* This follows from considering bounded sequences \{(x_n, y_n)\} and asking for a convergent subsequence. □

**Proposition 4.8.** $M_C$ is algebraic if and only if $A$ and $B$ are algebraic.

*Proof.* If $p(M_C) = 0$ then clearly $p(A) = 0$ and $p(B) = 0$. Suppose $p(A) = 0$ and $q(B) = 0$. Then $(pq)(A) = 0$ and $(pq)(B) = 0$ so that

$$(pq)(M_C) = \begin{pmatrix} 0 & (pq)[A, B](C) \\ 0 & 0 \end{pmatrix}.$$ □

In particular $\deg M_C \leq 2(\deg A + \deg B)$.

**Proposition 4.9.** $M_C$ is almost algebraic if and only if $A$ and $B$ are almost algebraic.

*Proof.* This follows from the characterization of the resolvent being meromorphic in $1/\lambda$. We have

$$(\lambda - M_C)^{-1} = \begin{pmatrix} (\lambda - A)^{-1} & (\lambda - A)^{-1}C(\lambda - B)^{-1} \\ (\lambda - B)^{-1} \end{pmatrix}$$ (4.9)

and hence the resolvent of $M_C$ has the same singularities as the resolvents of $A$ and $B$ together. □

From

$$(I - zM_C)^{-1} = (I - zM_0)^{-1} + \begin{pmatrix} 0 & z(I - zA)^{-1}C(I - zB)^{-1} \\ 0 & 0 \end{pmatrix}$$

we obtain

$$T_\infty(r, (I - zM_C)^{-1}) \leq 2 \left( T_\infty(r, (I - zA)^{-1}) + T_\infty(r, (I - zB)^{-1}) + \log^+ r + O(1) \right).$$

**Proposition 4.10.** $M_C$ is a Riesz operator if and only if both $A$ and $B$ are Riesz operators.

*Proof.* As the resolvents of $A$ and $B$ jointly have the same poles as the resolvent of $M_C$ the necessity follows from the block structure.

In the other direction, if $\lambda_0 \neq 0$ is a pole of either $A$ or $B$ but not both, the claim follows again from (4.4). In fact, this is seen from example from the spectral projection

$$P = \frac{1}{2\pi i} \int_\gamma (\lambda - M_C)^{-1} d\lambda$$ (4.10)
with \( \gamma = \{ \lambda : |\lambda - \lambda_0| = \varepsilon \}, \varepsilon > 0 \) small enough so that all other spectral points stay outside of \( \gamma \), where the block form makes it straightforward to see that the rank of the projection is finite.

Suppose therefore that \( A - \lambda_0 \) and \( B - \lambda_0 \) both have finite dimensional null spaces, say dimension \( m \) and \( n \). It is instructive to decompose \( A = A_m \oplus A_\infty \) and \( B = B_n \oplus B_\infty \) where \( A_m \) and \( B_n \) are the restrictions onto the finite dimensional invariant subspaces, respectively and \( A_\infty \) and \( B_\infty \) likewise operate in the complementary invariant subspaces. Then decompose \( C \) into block form

\[
C = \begin{pmatrix}
C_{m,n} & C_{m,\infty} \\
C_{\infty,n} & C_{\infty,\infty}
\end{pmatrix}
\]

where \( C_{m,n} \) can be thought as an \( m \times n \)-matrix, while \( C_{m,\infty} \) and \( C_{\infty,n} \) are finite rank operators as well. Writing the resolvent into block form we have

\[
(\lambda - A)^{-1}C(\lambda - B)^{-1}
\]

\[
= \begin{pmatrix}
(\lambda - A_m)^{-1}C_{m,n}(\lambda - B_m)^{-1} & (\lambda - A_m)^{-1}C_{m,\infty}(\lambda - B_\infty)^{-1} \\
(\lambda - A_\infty)^{-1}C_{\infty,n}(\lambda - B_n)^{-1} & (\lambda - A_\infty)^{-1}C_{\infty,\infty}(\lambda - B_\infty)^{-1}
\end{pmatrix}
\]

Here the lower right hand corner is holomorphic near \( \lambda_0 \) while the other three blocks are finite rank valued functions and the claim follows.

In Example 2.11 the sum of two nilpotent operators sum up to the shift operator. The structure of \( M_C = M_0 + N \) prevents this type of phenomenon to happen:

**Proposition 4.11.** \( M_C \) is quasinilpotent if and only if both \( A \) and \( B \) are quasinilpotent.

**Proof.** We have always \( \sigma(M_C) \subset \sigma(A) \cup \sigma(B) \) and hence \( M_C \) is quasinilpotent. Reversely, if \( M_C \) is quasinilpotent, then \( M_0 \) is quasinilpotent as well, by Lemma 4.1.

**Proposition 4.12.** \( M_C \) is quasialgebraic if and only if \( A \) and \( B \) are quasialgebraic.

**Proof.** Assume first that \( A \) and \( B \) are quasialgebraic so that both \( \text{cap}(\sigma(A)) \) and \( \text{cap}(\sigma(B)) \) vanish. While the capacity is not in general subadditive, sets of (logarithmic) capacity zero are polar and polar sets are countably subadditive [45]. Hence

\[
\text{cap}(\sigma(A) \cup \sigma(B)) = 0
\]

and \( M_0 \) is quasialgebraic, too. We know that \( \sigma(M_C) \subset \sigma(M_0) \) and hence \( M_C \) is quasialgebraic as the capacity is a monotonous set function. On the other hand, if \( M_C \) is quasialgebraic, then its spectrum is totally disconnect and \( \hat{\sigma}(M_C) = \sigma(M_C) \). By Lemma 4.1 we obtain

\[
\text{cap}(\hat{\sigma}(A) \cup \hat{\sigma}(B)) = \text{cap}(\hat{\sigma}(M_C)) = 0
\]

and so both \( A \) and \( B \) must be quasialgebraic.

In Example 4.2 we have \( M_C \) unitary while \( M_0 \) is not normal.

**Proposition 4.13.** If \( M_C \) is normal and \( C \neq 0 \), then \( M_0 \) is not normal.
Proof. If $MC$ is normal then $C^*C + B^*B = BB^*$ and thus $M_0$ is normal only if $C^*C = 0$.

All normal operators are quasitriangular. In Example 4.2 $M_0 = S \oplus S^*$ and it is known [25] that $S$ is not quasitriangular. Further, $S \oplus 0 = 0$ is not quasitriangular but $S \oplus M$ is, where $M$ is diagonal with the closed unit disc as the spectrum [44]. Whether $S \oplus S^*$ is, was a question in [25]. Using a result in [15] we may formulate the following:

**Proposition 4.14.** If $A$ and $B$ are quasitriangular, then so is $MC$.

### 4.3 $MC$ polynomially almost algebraic, compact and Riesz

**Proposition 4.15.** $MC \in \mathcal{PAA}$ if and only if both $A \in \mathcal{PAA}$ and $B \in \mathcal{PAA}$.

**Proof.** If $p(MC)$ is almost algebraic, then by Proposition 4.9 both $p(A)$ and $p(B)$ are almost algebraic. In the other direction, assume that $p(A)$ and $q(B)$ are almost algebraic. Here we need to conclude that both $(pq)(A)$ and $(pq)(B)$ are then polynomially almost algebraic. Then so is $(pq)(MC)$ again by Proposition 4.9. Consider $(pq)(A)$ as $(pq)(B)$ is similar. As $p(A)$ is polynomially almost algebraic, it means that $(\lambda - A)^{-1}$ is meromorphic except at a finite set of points $\lambda_1, \ldots, \lambda_m$ and $p$ vanishes at these points. But then $pq$ is another nontrivial polynomial which also vanishes at these points, and all we need to conclude that all nonzero singularities of $(z - (pq)(A))^{-1}$ are poles. In the proof of Theorem 5.9.2, [33] this has been carried out for the minimal polynomial but the discussion holds as such for any monic polynomial which vanish at $\lambda_1, \ldots, \lambda_m$.

Consider next polynomial compactness. While $MC$ is compact only if all $A, B, C$ are compact, notice that

$$M_C^2 = \begin{pmatrix} A^2 & AC + CB \\ B^2 & \end{pmatrix}$$

is compact when $A^2, B^2$ and $AC + CB$ are. This happens in particular when $A$ and $B$ are compact. This allows us to formulate

**Proposition 4.16.** $MC \in \mathcal{PK}$ if and only if both $A \in \mathcal{PK}$ and $B \in \mathcal{PK}$.

**Proof.** Let $p$ and $q$ be polynomials such that $p(A) \in \mathcal{K}$ and $q(B) \in \mathcal{K}$. Then both $(pq)(A)$ and $(pq)(B)$ are compact. Thus

$$(pq)(MC) = \begin{pmatrix} (pq)(A) & (pq)[A, B]C \\ (pq)(B) & \end{pmatrix}$$

has compact diagonal blocks and hence $(pq)^2(MC)$ is compact.

On the other hand, if $p(MC)$ is compact in then both diagonal blocks are compact and thus $A$ and $B$ are polynomially compact.

For Riesz operators we restrict the discussion to Hilbert spaces, where the Olsen’s characterization Theorem 2.37 can be used.
Proposition 4.17. Let $A$ and $B$ operate in separable Hilbert spaces. Then $M_C \in \mathcal{PR}$ if and only if both $A \in \mathcal{PR}$ and $B \in \mathcal{PR}$.

Proof. We may assume that $A = G + K$ and $B = H + L$ where $G$ and $H$ are algebraic and $K$ and $L$ are compact. Let $p$ and $q$ be such that $p(G) = q(H) = 0$ so that $p(A)$ and $q(B)$ are compact. But then $(pq)(A)$ and $(pq)(B)$ are compact and we conclude again that $(pq)^2(M_C)$ is compact. By Theorem 2.22 $(pq)(M_C)$ is then Riesz. 

Proposition 4.18. If $M_0 \in \mathcal{N}_{\text{orm}}$, then $M_C \in \mathcal{PN}_{\text{orm}}$ if and only if there exists a nontrivial $p$ such that $p(M_0) = p(M_C)$.

Proof. If $0 \neq p(M_C) \in \mathcal{N}_{\text{orm}}$ then by Proposition 4.13 $p(B) \notin \mathcal{N}_{\text{orm}}$ and further $p(M_0) \notin \mathcal{N}_{\text{orm}}$ which contradicts $M_0 \in \mathcal{N}_{\text{orm}}$.

References

[1] K. Ando, H. Kang, Y. Miyanishi, and M. Putinar, Rev. Roumaine Math. Pures Appl. 66 (2021), 3-4. 545-575
[2] Diana Andrei, Multicentric holomorphic calculus for n-tuples of commuting operators, Adv. Oper. Theory, Vol. 4, Number 2 (2019), 447-461
[3] Diana Andrei, Olavi Nevanlinna, Tiina Vesansen, Rational functions as new variables, arXiv:2104.11088 [math.CV] (April 2021)
[4] Apetrei, Diana, Nevanlinna, Olavi: Multicentric calculus and the Riesz projection, Journal of Numerical Analysis and Approximation Theory. 44 (2), 2016, p. 127-145.
[5] C. Apostol and C. Foias, On the distance to biquasitriangular operators, Rev. Roum. Math. Pures Appl. 20 (1975) 261-265
[6] C. Apostol, C. Foias, D. Voiculescu, On the norm-closure of nilpotents. II, Rev. Roumaine Math. Pures Appl. 19 (1974), 549-557
[7] C. Apostol, D. Voiculescu, On a problem of Halmos Rev. Roumaine Math. Pures Appl. 19 (1974), 283-284
[8] Spiros A. Argyros and Richard G. Haydon, A hereditarily indecomposable $L_{\infty}$ space that solves the scalar-plus-compact problem, Acta Math. 206 (2011), no. 1, 1 - 54, Doi:10.1007/s11511-011-0058-y. MR2784662 (2012e:46031)
[9] N.E. Benamara and N.K. Nikolski, Resolvent tests for similarity to a normal operator, Proc. London. Math. Soc., 78 (1999), no.3, pp. 585 - 626.
[10] C. K. Chui, P. W. Smith, J. D. Ward, A note on Riesz operators, Proc. Amer. Math. Soc. Vol. 60, Oct. 1976, 92-94
[11] John B. Conway, Domingo A. Herrero and Bernard B. Morrel, Completing the Riesz-Dunford functional calculus, Memoirs of AMS, November 1989, Vol. 82, Number 417
[12] Raúl Curto, Mihai Putinar, Polynomially Hyponormal Operators, Operator Theory: Advances and Applications, Vol. 207, 195-207. Springer 2010
[13] C. Davis and P. Rosenthal, Solving linear operator equations, Canad. J. Math. XXVI
[14] John Derr, Angus E.Taylor: Operators of meromorphic type with multiple poles of the resolvent, Pacific J. Math. 12 (1962), no. 1, 85–111
[15] R. G. Douglas, Carl Pearcy, A note on quasitriangular operators Duke Math. J. 37(1): 177-188 (March 1970). DOI: 10.1215/S0012-7094-70-03724-5
[16] B.P. Duggal, Dragan S. Djordjević, Robin E. Harte and Snežana Č. Živković-Zlatanović, Polynomially meromorphic operators, Mathematical Proceedings of the Royal Irish Academy 116A (2016), 71 - 86 ; [http://dx.doi.org/10.3318/PRIA.2016.116.07]
[17] H.K. Du and J. Pan, Perturbation of spectrums of $2 \times 2$ operator matrices, Proc. Amer.Math. Soc. 121(1994), 761-776. MR 94i:47004
[18] L.A.Fialkow, A Note on Non-Quasitriangular Operators II, Indiana Math. J. 23. No.3 (1973), 213-220
[19] C. Foias and C. Pearcy, A model for quasinilpotent operators, Michigan Math. J. 21 (1974), 399-404
[20] F. Gilfeather, The structure and asymptotic behavior of polynomially compact operators, Proc. Amer. Math. Soc. 25 (1970) 127-134
[21] Frank Gilfeather, Operator valued roots of Abelian analytic functions, Pacific J. Math. Vol. 55, No.1, (1974) 127-148
[22] T. A. Gillespie and T. T. West, A characterisation and two examples of Riesz operators, Glasgow Math. J. 9 (1968), 106-110.
[23] P. R. Halmos, Capacity in Banach Algebras, Indiana University Mathematics Journal Vol. 20, No. 9 (March, 1971), pp. 855-863
[24] P. R. Halmos, Invariant subspaces of polynomially compact operators, Pacific J. Math. Vol.16, No.3, 1966, 433 - 438
[25] P. R. Halmos, Quasitriangular operators, Acta Sci. Math. (Szeged) 29 (1968), 283-293. MR 38 2627.
[26] Young Min Han, Sang Hoon Lee, Woo Young Lee, On the structure of polynomially compact operators, Math. Z. Vol. 232 257-263 (1999)
[27] Domingo A. Herrero, Most quasitriangular operators are triangular, most biquasitriangular operators are bitriangular, J. Operator Theory, 20, (1988), 251-267
[28] Junjie Huang, Aichun Liua, Alatancang Chen: Spectra of $2 \times 2$ Upper Triangular Operator Matrices, Filomat 30:13 (2016), 3587 3599 DOI 10.2298/FIL1613587H, [http://www.pmf.ni.ac.rs/filomat]
[29] Marko Huhtanen, Olavi Nevanlinna, Polynomials and lemniscates of indefiniteness, Numer. Math. (2016) 133: 233 - 253, DOI 10.1007/s00211-015-0745-2
[30] Fuad Kittaneh, On the structure of polynomially normal operators, Bull.Austral. Math.Soc. Vol 30, (1984), 11 -18.
[31] S.Kupin and S.Trei1, Linear resolvent growth of weak contraction does not imply its similarity to a normal operator, Illinois Journal of Mathematics Volume 45, Number 1, Spring 2001, 229 - 242
[32] Matjaž Konvalinka, Integr. equ. oper. theory, Vol. 52 (2) (2005), 271-284
[33] Olavi Nevanlinna, Convergence of Iterations for Linear Equations, Birkhäuser, (1993)
[34] Olavi Nevanlinna, Growth of operator valued meromorphic functions, Ann. Acad. Sci.Fenn. Math. Vol. 25, 2000, 3-30
[35] O. Nevanlinna, Meromorphic Functions and Linear Algebra, AMS Fields Institute Monograph 18 (2003)
[36] O. Nevanlinna, Computing the spectrum and representing the resolvent, Numerical Functional Analysis and Optimization, 30 (9 - 10):1025 - 1047, 2009
[37] O. Nevanlinna, Multicentric Holomorphic Calculus, Computational Methods and Function Theory, June 2012, Vol. 12, Issue 1, 45 - 65.
[38] O. Nevanlinna, Lemniscates and K-spectral sets, J. Funct. Anal. 262, (2012), 1728 - 1741.
[39] O. Nevanlinna, Polynomial as a New Variable - a Banach Algebra with Functional Calculus, Oper. and Matrices 10 (3) (2016) 567 - 592
[40] O. Nevanlinna, Sylvester equations and polynomial separation of spectra, Oper. and Matrices 13, (3) (2019), 867- 885
[41] N. Nikolski, Sergei Treil, Linear resolvent growth of rank one perturbation of a unitary operator does not imply its similarity to a normal operator, December 2002, Journal d'Analyse Mathématique 87(1):415 - 431 DOI: 10.1007/BF02868483
[42] Catherine L. Olsen A Structure Theorem for Polynomially Compact Operators, American Journal of Mathematics, Vol. 93, No. 3 (Jul., 1971), pp. 686-698
[43] C.M. Pearcy, Some recent developments in operator theory, CBMS 36, Providence:AMS, 1978.
[44] C. Pearcy, N. Salinas, Can. J. Math., Vol. XXVI, No. 1, 1974, pp. 115-120
[45] Th. Ransford, Potential Theory in the Complex Plane, London Math. Soc. Student Texts 28, Cambridge Univ. Press, 1995
[46] A.F. Ruston, Operators with a Fredholm theory, J. London Math. Soc. 29 (1954) pp. 318 - 326
[47] David S.G. Stirling, The Capacity of Elements of Banach Algebras, Doctor of Philosophy Thesis, University of Edinburgh, 1972
[48] B. Sz.-Nagy, On uniformly bounded linear transformations in Hilbert Space, Acta Sci. Math., 11 (1947), 152-157.
[49] A. E. Taylor, Mittag-Leffler expansions and spectral theory, Pacific J. Math., 10 (1960), 1049-1066
[50] D. Voiculescu, Norm-limits of algebraic operators, Rev. Roumaine Math. Pures et Appl. 19 (1974), 371-378.
[51] T. T. West, The decomposition of Riesz operators, Proc. London Math. Soc. (3) 16 (1966), 737-752
[52] Snežana Ć. Živković-Zlatanović, Dragan S. Djordjević, Robin E. Harte, Bhagwati P. Duggal, Filomat 28:1 (2014), 197 205 DOI 10.2298/FIL1401197Z

**Notation and definitions**

- $A$ algebraic operators, Def 2.6
- $AA$ almost algebraic, Def 2.6
- $B$ bounded operators
- $B_{iQT}$ biquasitriangular, Def 3.1
- $C$ see Def 3.6 ($= \text{cl}\mathcal{N}$)
- $F$ finite rank
\( K \) compact
\( M \) meromorphic, of meromorphic type (= \( \mathcal{A}A \))
\( N \) nilpotent
\( \mathcal{N}_{orm} \) normal operators
\( \mathcal{U} \) unitary operators
\( \mathcal{P}\mathcal{A}\mathcal{A} \) polynomially almost algebraic, Def 2.1 and Def 2.6
\( \mathcal{P}\mathcal{K} \) polynomially compact
\( \mathcal{P}\mathcal{N}_{orm} \) polynomially normal
\( \mathcal{P}\mathcal{R} \) polynomially Riesz
\( \mathcal{R} \) Riesz, Def 2.21
\( \mathcal{Q}\mathcal{A} \) quasialgebraic, Def 2.6
\( \mathcal{Q}\mathcal{D} \) quasidiagonal, Def 3.2
\( \mathcal{Q}\mathcal{N} \) quasinilpotent,
\( \mathcal{Q}\mathcal{T} \) quasitriangular, Def 3.1

\( \sigma(A) \) spectrum of \( A \)
\( \sigma_{\delta}(A) \) approximate defect spectrum, Def 4.3
\( \sigma_{\alpha}(A) \) approximate point spectrum, Def 4.3
\( \sigma_{j}(A) \) \( j \)th singular value of \( A \)
\( s(A) \) total logarithmic size (2.15)
\( \alpha_{j}(A) \) distance to algebraic operators of degree \( j \), Def 2.9
\( \hat{K} \) polynomially convex hull of a compact set \( K \)
\( m_{A}(z) \) minimal polynomial of \( A \)
\( s_{A}(z) \) simplifying polynomial of \( A \), Remark 3.9
\( m_{\infty}(r, F) \) (2.12)
\( N_{\infty}(r, F) \) (2.13)
\( T_{\infty}(r, F) \) (2.14)
\( T_{1}(r, F) \) (2.16)