THE DIRAC-HARDY AND DIRAC-SOBOLEV INEQUALITIES
IN $L^1$

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Abstract. Dirac-Sobolev and Dirac-Hardy inequalities in $L^1$ are established in which the $L^p$ spaces which feature in the classical Sobolev and Hardy inequalities are replaced by weak $L^p$ spaces. Counter examples to the analogues of the classical inequalities are shown to be provided by zero modes for appropriate Pauli operators constructed by Loss and Yau.

Key words: Dirac-Sobolev inequalities, Dirac-Hardy inequalities, zero modes, Sobolev inequalities, Hardy inequalities

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1. INTRODUCTION

Let $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ be the triple of $2 \times 2$ Pauli matrices
\begin{equation}
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{equation}
and set
\[ p := -i \nabla, \quad \sigma \cdot p = -i \sum_{j=1}^{3} \sigma_j \frac{\partial}{\partial x_j}. \]

By the Dirac-Sobolev inequality we mean the following: that when $1 \leq p < 3$, $p^* = 3p/(3-p)$, and for all $f \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)$, the space of $\mathbb{C}^2$-valued functions whose components lie in $C_0^\infty(\mathbb{R}^3)$,
\begin{equation}
\left( \int_{\mathbb{R}^3} |f(x)|_{p^*}^{p^*} \, dx \right)^{1/p^*} \leq C(p) \left( \int_{\mathbb{R}^3} |(\sigma \cdot p)f(x)|_p^p \, dx \right)^{1/p}
\end{equation}
where for $a = (a_1, a_2) \in \mathbb{C}^2$, $|a|^p_p = |a_1|^p + |a_2|^p$. It is shown by Ichinose and Saitô in [3] (see “Note added in proof” at end of paper) that for $1 < p < \infty$, \[ \int_{\mathbb{R}^3} |f(x)|_{p^*}^{p^*} \, dx \]
there are positive constants $c_1(p), c_2(p)$ which are such that
\[
c_1(p) \int_{\mathbb{R}^3} |\mathbf{p} f(x)|_p^p \, dx \leq \int_{\mathbb{R}^3} |(\boldsymbol{\sigma} \cdot \mathbf{p}) f(x)|_p^p \, dx \leq c_2(p) \int_{\mathbb{R}^3} |\mathbf{p} f(x)|_p^p \, dx, \tag{1.3}
\]
and hence for $1 < p < 3$, (1.2) is a consequence of the Sobolev inequality
\[
\left( \int_{\mathbb{R}^3} |f(x)|_p^{p'} \, dx \right)^{1/p'} \leq C(p) \left( \int_{\mathbb{R}^3} |\mathbf{p} f(x)|_p^p \, dx \right)^{1/p}. \tag{1.4}
\]

On defining the Dirac-Sobolev space $H_{D,0}^{1,p}(\mathbb{R}^3, \mathbb{C}^2)$ to be the completion of $C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)$ with respect to the norm
\[
\|f\|_{D,1,p} := \left\{ \int_{\mathbb{R}^3} \|f(x)\|_p^p + |(\boldsymbol{\sigma} \cdot \mathbf{p}) f(x)|_p^p \, dx \right\}^{1/p},
\]
then (1.3) proves that $H_{D,0}^{1,p}(\mathbb{R}^3, \mathbb{C}^2)$ is isomorphic to $H_0^{1,p}(\mathbb{R}^3, \mathbb{C}^2)$ if $1 < p < \infty$, where $H_0^{1,p}(\mathbb{R}^3, \mathbb{C}^2)$ denotes the Sobolev space defined to be the completion of $C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)$ with respect to the norm
\[
\|f\|_{S,1,p} := \left\{ \int_{\mathbb{R}^3} \|f(x)\|_p^p + |\mathbf{p} f(x)|_p^p \, dx \right\}^{1/p}.
\]

However, as $p \to 1$, $c_1(p) \to 0$ and so (1.3) only implies that $H_0^{1,1}(\mathbb{R}^3, \mathbb{C}^2)$ is continuously embedded in $H_{D,0}^{1,1}(\mathbb{R}^3, \mathbb{C}^2)$. In fact Ichinose and Saitô go on to prove that the embedding $H_0^{1,1}(\mathbb{R}^3, \mathbb{C}^2) \hookrightarrow H_{D,0}^{1,1}(\mathbb{R}^3, \mathbb{C}^2)$ is indeed strict. Hence, in the case $p = 1$, (1.2) is not a consequence of the analogous Sobolev inequality. We prove that the $p = 1$ case of (1.2) is untrue. We demonstrate this with a function used by Loss and Yau in [5] to prove the existence of zero modes of a Pauli operator $(\boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A}))^2$ (or equivalently, of the Weyl-Dirac operator $\boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A})$) with some appropriate magnetic potential $\mathbf{A}$. A result of Saitô and Umeda in [6] on the growth properties of zero modes of Pauli operators indicates that zero modes have quite generally the properties we need of the counter-example. We prove in Theorem [2.1]
\[
\|f\|_{3/2,\infty} \leq C_1 \int_{\mathbb{R}^3} |(\boldsymbol{\sigma} \cdot \mathbf{p}) f(x)| \, dx, \tag{1.5}
\]
where $|\cdot| = |\cdot|_1$ and for any $q > 0$,
\[
\|f\|_{q,\infty} := \sup_{t > 0} t^q \mu(\{x \in \mathbb{R}^3 : |f(x)| > t\}), \tag{1.6}
\]
$\mu$ denoting Lebesgue measure. We recall that $\|f\|_{q,\infty} < \infty$ if and only if $f$ belongs to the weak-$L^q$ space $L^{q,\infty}(\mathbb{R}^3, \mathbb{C}^2)$. Moreover, $\|\cdot\|_{q,\infty}$ is not a norm on $L^{q,\infty}(\mathbb{R}^3, \mathbb{C}^2)$ but for $q > 1$ it is equivalent to a norm; see [2], Section 3.4.
Analogous questions arise for the Dirac-Hardy inequality
\[
\int_{\mathbb{R}^3} \frac{|f(x)|^p}{|x|^p} \, dx \leq C(p) \int_{\mathbb{R}^3} |(\sigma \cdot p)f(x)|^p \, dx \tag{1.7}
\]
and similar answers are obtained. The inequality is true for \(1 < p < \infty\) by (1.3), but not for \(p = 1\) in which case we prove that
\[
\| |f| / |x| \|_{1,\infty} \leq C_2 \int_{\mathbb{R}^3} |(\sigma \cdot p)f(x)| \, dx. \tag{1.8}
\]

The plan of the paper is as follows. In Section 2 we shall prove the results concerning the Dirac-Sobolev and Dirac-Hardy inequalities discussed above. We shall give estimates of the optimal constant \(C(p)\) in the Dirac-Sobolev inequality (1.2) for \(1 < p < 3\) in Section 3 and show that \(C(p) \to \infty\) as \(p \downarrow 1\). In order to check if the results in Section 2 are dimension related, we investigate higher dimensional analogues in Section 4. A weak Hölder-type inequality is given in an Appendix.

2. THE WEAK DIRAC-SOBOLEV AND DIRAC-HARDY INEQUALITIES

To show that the inequality (1.2) does not hold, we shall prove that a counter-example is provided by a zero mode for an appropriate Pauli (or Weyl-Dirac) operator constructed by Loss-Yau in [5]. This is the \(C^2\)-valued function
\[
\psi(x) = \frac{1}{(1 + r^2)^{3/2}}(I + ix \cdot \sigma) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad r = |x|, \tag{2.1}
\]
where \(I\) is the \(2 \times 2\) identity matrix. In view of the anti-commutation relation \(\sigma_j \sigma_k + \sigma_k \sigma_j = 2 \delta_{jk} I\), it follows that
\[
|\psi(x)| = \frac{1}{1 + r^2}. \tag{2.2}
\]
Also, \(\psi\) satisfies the Loss-Yau equation
\[
(\sigma \cdot p)\psi(x) = \frac{3}{1 + r^2} \psi(x). \tag{2.3}
\]
Let \(\chi_n \in C_0^\infty(\mathbb{R})\) be such that
\[
\chi_n(r) = \begin{cases} 
1, & r \leq n \\
0, & r \geq n + 2, 
\end{cases} \quad |\chi'_n(r)| \leq 1. \tag{2.4}
\]
Then \( \psi_n := \chi_n \psi \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^2) \) and we see that
\[
\|(\sigma \cdot p) \psi_n\|_{L^1(\mathbb{R}^3, \mathbb{C}^2)} = \|\chi_n (\sigma \cdot p) \psi - i \chi_n'(\sigma \cdot \frac{x}{r}) \psi\|_{L^1(\mathbb{R}^3, \mathbb{C}^2)}
\leq 4\pi \left( \int_0^{n+2} \frac{3}{1 + r^2} \, dr + \int_n^{n+2} \, dr \right)
\leq C_0,
\]
for some constant positive \( C_0 \), independent of \( n \).

Now suppose that the case \( p = 1 \) of the inequality (1.2) is true. Then it would follow from (2.5) that
\[
C_0 \geq \|\psi_n\|_{L^{3/2}(\mathbb{R}^3, \mathbb{C}^2)}
\geq \left( \int_{|x| \leq n} |\psi(x)|^{3/2} \, dx \right)^{2/3}
\geq \text{const.} (\log n)^{2/3},
\]
and hence a contradiction.

The properties of the zero mode \( \psi \), defined by (2.1), which lead to the inequality (1.2) being contradicted when \( p = 1 \) are that \((\sigma \cdot p) \psi \in L^1(\mathbb{R}^3, \mathbb{C}^2)\) and \( \psi(x) \asymp r^{-2} \) at infinity (i.e., \( r^2 \psi(x) \) goes to a constant vector in \( \mathbb{C}^2 \) as \( r \to \infty \)). It was shown in Saitō-Umeda [6] that these two properties are satisfied by the zero modes of any Weyl-Dirac operator
\[
\mathbb{D}_A = \sigma \cdot (p + A(x))
\]
whose magnetic potential \( A = (A_1, A_2, A_3) \) is such that
\[
A_j \text{ is measurable, } |A_j(x)| \leq C(1 + r)^{-\rho}, \quad \rho > 1
\]
for \( j = 1, 2, 3 \).

As was mentioned in the Introduction, what is true is the following

**Theorem 2.1.** There exists a positive constant \( C_1 \) such that
\[
\|f\|_{L^{3/2, \infty}(\mathbb{R}^3, \mathbb{C}^2)} \leq C_1 \| (\sigma \cdot p) f \|_{L^1(\mathbb{R}^3, \mathbb{C}^2)}
\]
for all \( f \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^2) \).

**Proof.** Let \( g = (\sigma \cdot p) f \). Since \( (\sigma \cdot p)^2 = -\Delta \) and the fundamental solution of \(-\Delta\) in \( \mathbb{R}^3 \) is convolution with \((1/4\pi | \cdot |)\), it follows that \( (\sigma \cdot p) \) has fundamental
solution with kernel \( (\sigma \cdot \mathbf{p}) (1/4\pi | \cdot |) \) and hence that
\[
f(\mathbf{x}) = -\frac{i}{4\pi} \int_{\mathbb{R}^3} \frac{[(\sigma \cdot \nabla)|x - y|^{-1}]g(y)}{|x - y|} dy
= \frac{i}{4\pi} \int_{\mathbb{R}^3} \frac{\sigma \cdot (x - y)}{|x - y|^3} g(y) dy.
\]
(2.10)

Note that this also follows from the more general result in Saitô-Umeda [7, Theorem 4.2]. Consequently
\[
|f(\mathbf{x})| \leq \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x - y|^2} |g(y)| dy
= \frac{1}{4\pi} I_1(|g|)(\mathbf{x}),
\]
(2.11)

where \( I_1(|g|) \) is the 3-dimensional Riesz potential of \(|g|\); see Edmunds and Evans [2, Section 3.5] for the terminology and properties we need. In view of [2, Remark 3.5.7(i)], we see that the Riesz potential \( I_1 \) is of weak type \((1, 3/2; 3, \infty)\). In particular, \( I_1 \) is of weak type \((1, 3/2)\) (cf. [2, Theorem 3.5.13], [8, Theorem 1, pp.119 - 120]), which means that there exists a positive constant \( C \) such that for all \( u \in L^1(\mathbb{R}^3) \)
\[
\|I_1(u)\|_{L^{3/2,\infty}(\mathbb{R}^3)} \leq C\|u\|_{L^1(\mathbb{R}^3)}.
\]
(2.12)

The inequality (2.9) follows.

It is evident that the two properties of the zero mode \( \psi \) defined by (2.1) also leads to a contradiction of the inequality (1.7). What is now true is the following:

**Theorem 2.2.** For all \( f \in C^\infty_0(\mathbb{R}^3, \mathbb{C}^2) \)
\[
\|f/|\cdot|\|_{L^{1,\infty}(\mathbb{R}^3, \mathbb{C}^2)} \leq C_2 \| (\sigma \cdot \mathbf{p}) f \|_{L^1(\mathbb{R}^3, \mathbb{C}^2)},
\]
(2.13)

where \( C_2 \leq (9\pi)^{1/3} C_1 \) and \( C_1 \) is the optimal constant in (2.9).

**Proof.** On applying the weak Hölder inequality in the Appendix with \( p = 3/2 \) and \( q = 3 \), and noting that \( \|1/|\cdot||_{3,\infty} = (4\pi/3)^{1/3} \), we get
\[
\|f/|\cdot|\|_{1,\infty} \leq 2^{2/3} \pi^{1/3} \|f\|_{3/2,\infty}.
\]
(2.14)

Hence the theorem follows from (2.9).

\[ \square \]

3. **Estimate of the optimal constants**

In this section, we estimate the optimal constant \( C(p) \) in the inequality (1.2) for \( 1 < p < 3 \), and show that \( C(p) \to \infty \) as \( p \downarrow 1 \).
Let $\psi$ be the Loss-Yau zero mode defined by (2.1). It does not lie in $C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)$ but is in $H_{D,0}^{1,p}(\mathbb{R}^3, \mathbb{C}^2)$. Hence the optimal constant $C(p)$ must satisfy the inequality
\begin{equation}
C(p) \geq \|\psi\|_{L^{p^*}(\mathbb{R}^3, \mathbb{C}^2)} / \|((\sigma \cdot p)\psi\|_{L^p(\mathbb{R}^3, \mathbb{C}^2)}, \tag{3.1}
\end{equation}
where $p^* = 3p/(3-p)$. On passing to polar coordinates, we have
\begin{align*}
\|\psi\|_{p^*}^p &= 4\pi \int_0^\infty (1 + r^2)^{-p^*}r^2 dr \\
&\geq 4\pi \left\{ \int_0^1 2^{-p^*}r^2 dr + \int_1^\infty (2r^2)^{-p^*}r^2 dr \right\} \tag{3.2}
&= 4\pi 2^{-p^*} 3^{-2} \frac{2p}{p-1}.
\end{align*}
On the other hand, by (2.3), we see that
\begin{align*}
\|((\sigma \cdot p)\psi\|_p^p &= \int_{\mathbb{R}^3} \frac{3^p}{(1 + r^2)^{2p}} dx \\
&= 4\pi 3^p \int_0^\infty (1 + r^2)^{-2p}r^2 dr \\
&\leq 4\pi 3^p \left\{ \int_0^1 r^2 dr + \int_1^\infty r^{-4p+2} dr \right\} \\
&= \pi 2^4 3^{p-1} \frac{p}{4p-3}.
\end{align*}
Combining (3.1) with (3.2) and (3.3), we obtain
\begin{equation}
C(p) \geq \pi^{-1/3} 2^{-1/p} 3^{-1/3-1/p} p^{3/2} \frac{p^{-1/3}(4p-3)^{1/p}}{(p-1)^{1/3}}. \tag{3.4}
\end{equation}
It is evident that the right hand side of (3.4) goes to $\infty$ as $p \downarrow 1$.

We recall that for $p > 1$, the optimal constant $\tilde{C}(p)$ in the Sobolev inequality (1.4) is
\begin{equation}
\tilde{C}(p) = \pi^{-1/2} 3^{-1/p} \left( \frac{p-1}{3-p} \right)^{(p-1)/p} \left\{ \frac{\Gamma(5/2)\Gamma(3)}{\Gamma(3/p)\Gamma(4-3/p)} \right\}^{1/3}
\end{equation}
which tends to $\tilde{C}(1)$, the optimal constant in the case $p = 1$, as $p \to 1$.

4. The weak Dirac-Sobolev and weak Dirac-Hardy inequalities in $m$ dimensions

Let $\gamma_1, \ldots, \gamma_m$ be Hermitian $\ell \times \ell$ matrices satisfying the anti-commutation relations
\begin{equation}
\gamma_j \gamma_k + \gamma_k \gamma_j = 2\delta_{jk}I, \tag{4.1}
\end{equation}
where $I$ denotes the $\ell \times \ell$ identity matrix. For example, we can take $\ell = 2^{m-2}$ and construct the matrices by the following iterative procedure. To indicate the dependence on $m$, write the matrices as $\gamma_1^{(m)}, \ldots, \gamma_m^{(m)}$. For $m = 3$, $\ell = 2$ and they are given by the Pauli matrices in (1.1). Given matrices $\gamma_1^{(m)}, \ldots, \gamma_m^{(m)}$ we define

$$
\gamma_j^{(m+1)} = \begin{pmatrix} 0 & \gamma_j^{(m)} \\ \gamma_j^{(m)} & 0 \end{pmatrix}, \quad j = 1, \ldots, m,
\gamma_m^{(m+1)} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.
$$

(4.2)

The $m$-dimensional analogue of the inequality (1.2) for $p = 1$ is

$$
\left( \int_{\mathbb{R}^m} |f(x)|^{m/(m-1)} \, dx \right)^{(m-1)/m} \leq C \int_{\mathbb{R}^m} |(\gamma \cdot p) f(x)| \, dx
$$

(4.3)

for $f \in C^\infty_0(\mathbb{R}^m, \mathbb{C}^\ell)$, where

$$
\gamma \cdot p = -i \sum_{j=1}^{m} \gamma_j \frac{\partial}{\partial x_j}, \quad p = -i \nabla.
$$

To show that (4.3) does not hold we introduce an $m$-dimensional analogue of the Loss-Yau zero mode, namely

$$
\phi_0 = t(1, 0, \ldots, 0) \in \mathbb{C}^\ell.
$$

(4.4)

It follows from the anti-commutation relations (4.1) that

$$
|\psi(x)| = \frac{1}{(1 + r^2)^{m/2}},
$$

(4.5)

and that $\psi$ satisfies the $m$-dimensional analogue of the Loss-Yau equation (2.3), namely,

$$
(\gamma \cdot p)\psi(x) = \frac{m}{1 + r^2} \psi(x).
$$

(4.6)

Let $\chi_n \in C^\infty_0(\mathbb{R})$ be the same function as in (2.4), and put $\psi_n := \chi_n \psi \in C^\infty_0(\mathbb{R}^m, \mathbb{C}^\ell)$. As in to (2.5), we see that

$$
\| (\gamma \cdot p)\psi_n \|_{L^1(\mathbb{R}^m, \mathbb{C}^\ell)} = \| \chi_n(\gamma \cdot p) \psi - i \chi_n (\gamma \cdot \frac{x}{r}) \psi \|_{L^1(\mathbb{R}^m, \mathbb{C}^\ell)}
\leq S_m \left( \int_0^{n+2} \frac{m}{1 + r^2} \, dr + \int_n^{n+2} \, dr \right)
\leq C_0,
$$

(4.7)

for some positive constant $C_0$, independent of $n$. Here $S_m$ is the surface area of the unit sphere in $\mathbb{R}^m$. If the inequality (4.3) is true then it would follow
from (4.3) and (4.7) that
\[ C_0 \geq \|\psi_n\|_{L^{m/(m-1)}(\mathbb{R}^m, C^\ell)} \]
\[ \geq \left( \int_{|x| \leq n} |\psi(x)|^{m/(m-1)} \, dx \right)^{(m-1)/m} \]
\[ \geq \text{const.} \,(\log n)^{(m-1)/m}. \] \hspace{1cm} (4.8)
which is a contradiction. Therefore the inequality (4.3) does not hold. Instead, what is true is the following inequality.

**Theorem 4.1.** There exists a positive constant \( C_{1,m} \) such that
\[ \|f\|_{L^{m/(m-1),\infty}(\mathbb{R}^m, C^\ell)} \leq C_{1,m} \|\gamma \cdot p\|_{L^1(\mathbb{R}^m, C^\ell)} \] \hspace{1cm} (4.9)
for all \( f \in C^\infty_0(\mathbb{R}^m, C^\ell). \)

**Proof.** Let \( f \in C^\infty_0(\mathbb{R}^m, C^\ell), \) and define \( g = (\gamma \cdot p)f. \) Since \((\gamma \cdot p)^2 = -\Delta I, \) we have that \((-\Delta)f = (\gamma \cdot p)g. \) By Stein [8, p.118, (7)],
\[ J_2(-\Delta)u = u, \quad u \in C^\infty_0(\mathbb{R}^m, C), \] \hspace{1cm} (4.10)
where
\[ J_2(u) = \frac{\Gamma((m-2)/2)}{4\pi^{m/2}} I_2(u), \quad I_2(u)(x) = \int_{\mathbb{R}^m} \frac{1}{|x-y|^{m-2}} u(y) \, dy. \] \hspace{1cm} (4.11)
It follows that
\[ f(x) = J_2(-\Delta)f(x) \]
\[ = \frac{\Gamma((m-2)/2)}{4\pi^{m/2}} \int_{\mathbb{R}^m} \frac{1}{|x-y|^{m-2}} (\gamma \cdot p)g(y) \, dy. \] \hspace{1cm} (4.12)
On integration by parts, this yields
\[ f(x) = \frac{\Gamma((m-2)/2)}{4\pi^{m/2}} \int_{\mathbb{R}^m} \frac{i \gamma \cdot (x-y)}{|x-y|^m} g(y) \, dy. \] \hspace{1cm} (4.13)
Then it follows that
\[ |f(x)| \leq \frac{\Gamma((m-2)/2)}{4\pi^{m/2}} \int_{\mathbb{R}^m} \frac{1}{|x-y|^{m-1}} |g(y)| \, dy \]
\[ = \frac{\Gamma((m-2)/2)}{4\pi^{m/2}} 2\pi^{(m+2)/2} \frac{I_1(|g|)(x)}{\Gamma((m-1)/2)}. \] \hspace{1cm} (4.14)
Here \( I_1(|g|) \) is the \( m \)-dimensional Riesz potential of \(|g|; \) see [2, Section 3.5].
In view of [2, Remark 3.5.7(i)], we see that the Riesz potential \( I_1 \) is of weak type \((1, m/(m-1); m, \infty), \) in particular, of weak type \((1, m/(m-1)); \) (cf. [2]
Theorem 3.5.13], [8, Theorem 1, pp.119 - 120], which means that there exists a positive constant $\theta$ such that for all $u \in L^1(\mathbb{R}^m)$

$$\|I_1(u)\|_{L^m/(m-1,\infty}(\mathbb{R}^m) \leq \theta \|u\|_{L^1(\mathbb{R}^m)}.$$  \hspace{1cm} (4.15)

The inequality (4.9) follows. \hfill \Box

The $m$-dimensional Hardy inequality for $L^1$ is

$$\int_{\mathbb{R}^m} \frac{|u(x)|}{|x|} \, dx \leq (m - 1)^{-1} \int_{\mathbb{R}^m} |p \, u(x)| \, dx, \quad u \in C_0^\infty(\mathbb{R}^m). \hspace{1cm} (4.16)$$

It can be proved as in the 3-dimensional case that its natural analogue

$$\int_{\mathbb{R}^m} \frac{|f(x)|}{|x|} \, dx \leq C \int_{\mathbb{R}^m} |(\gamma \cdot p)f(x)| \, dx, \quad f \in C_0^\infty(\mathbb{R}^m, \mathbb{C}^t) \hspace{1cm} (4.17)$$

is not true.

**Theorem 4.2.** For all $f \in C_0^\infty(\mathbb{R}^m, \mathbb{C}^t)$

$$\|f/\cdot\|_{L^1,\infty(\mathbb{R}^m,\mathbb{C}^t)} \leq C_{2,m} \|\gamma \cdot p\|_{L^1,\infty(\mathbb{R}^m,\mathbb{C}^t)}, \hspace{1cm} (4.18)$$

where

$$C_{2,m} \leq C_{1,m} \frac{\pi^{1/2} m}{\Gamma((m+2)/2)^{1/m}(m-1)^{1-1/m}}.$$  \hspace{1cm} (4.19)

and $C_{1,m}$ is the optimal constant in Theorem 4.1.

**Proof.** It is easy to see that $\|1/\cdot\|_{m,\infty} = (\omega_m)^{1/m}$, where $\omega_m$ denotes the volume of the $m$-dimensional unit ball, and is given by

$$\omega_m = \frac{\pi^{m/2}}{\Gamma((m+2)/2)}.$$  \hspace{1cm} (4.20)

On applying the weak Hölder inequality in the Appendix with $p = m/(m-1)$ and $q = m$, we get

$$\|f/\cdot\|_{1,\infty} \leq \left( (m-1)^{1/m} + (m-1)^{(m-1)/m} \right) \omega_m^{1/m} \|f\|_{m/(m-1,\infty)}. \hspace{1cm} (4.19)$$

The theorem follows on combining this inequality with (4.9). \hfill \Box

5. Appendix

The proofs of Theorems 2.2 and 4.2 are consequences of the following Hölder - type inequality in weak $L^p$ spaces, which we have been unable to find in the literature.
Theorem 5.1 (Weak Hölder inequality). Let \( p > 1, q > 1 \) and \( p^{-1} + q^{-1} = 1 \). If \( f \in L^{p,\infty}(\mathbb{R}^d) \) and \( g \in L^{q,\infty}(\mathbb{R}^d) \), then \( fg \in L^{1,\infty} \) and
\[
\|fg\|_{1,\infty} \leq \left( \frac{q}{p} \right)^{1/q} + \left( \frac{p}{q} \right)^{1/p} \|f\|_{p,\infty} \|g\|_{q,\infty}.
\] (5.1)

Proof. Let \( \varepsilon > 0 \) be arbitrary, and set
\[
A = \{ x \in \mathbb{R}^d : \varepsilon |f(x)| > t^{1/p} \}
\]
\[
B = \{ x \in \mathbb{R}^d : \frac{1}{\varepsilon} |g(x)| > t^{1/q} \}
\]
\[
E = \{ x \in \mathbb{R}^d : |f(x)g(x)| > t \}.
\]
Since
\[
|f(x)g(x)| \leq p^{-1} (\varepsilon |f(x)|)^p + q^{-1} \left( \frac{1}{\varepsilon} |g(x)| \right)^q,
\] (5.2)
we have
\[
E \subset A \cup B,
\] (5.3)
which implies that
\[
t\mu(E) \leq t\mu(\{ x : \varepsilon |f(x)| > t^{1/p} \}) + t\mu(\{ x : \frac{1}{\varepsilon} |g(x)| > t^{1/q} \}).
\] (5.4)

With
\[
s := \frac{t^{1/p}}{\varepsilon}, \quad r := \varepsilon t^{1/q},
\] (5.5)

it follows from (5.4) that
\[
t\mu(\{ x : |f(x)g(x)| > t \})
\leq \varepsilon^p s^p \mu(\{ x : |f(x)| > s \}) + \varepsilon^{-q} r^q \mu(\{ x : |g(x)| > r \})
\leq \varepsilon^p \|f\|_{p,\infty}^p + \varepsilon^{-q} \|g\|_{q,\infty}^q.
\] (5.6)
The minimum value of this is the expression on the right-hand side of (5.1), this being attained when \( \varepsilon = (q\|g\|_{q,\infty}^q/p\|f\|_{p,\infty}^p)^{1/pq} \). \qed

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Note added in proof. We are grateful to the referee for the comment that the first inequality in (1.3) is not a direct consequence of Ichinose and Saitō [3, Theorem 1.3(ii)], but can be established as follows. Since $(\sigma \cdot p)^2 = -\Delta$, one has
\[-i\partial_j(\sigma \cdot p)^{-1} = \{-i\partial_j/\sqrt{-\Delta}\}{(\sigma \cdot p)/\sqrt{-\Delta}} = \sum_{k=1}^{3} \sigma_k R_j R_k\]
where $R_j = -i\partial_j/\sqrt{-\Delta}$, $j = 1, 2, 3$, are the Riesz transforms. Since $R_j$ is a pseudo-differential operator with symbol $\xi_j/|\xi|$, it is bounded on $L^p$ by the Calderón-Zygmund theorem (see [8]). The first inequality in (1.3) therefore follows.

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