Sharp Large Deviations for Hyperbolic Flows

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Abstract. For hyperbolic flows \( \varphi_t \), we examine the Gibbs measure of points \( w \) for which
\[
\int_0^T G(\varphi_tw)dt - aT \in (-e^{-\epsilon n}, e^{-\epsilon n})
\]
as \( n \to \infty \) and \( T \geq n \), provided \( \epsilon > 0 \) is sufficiently small. This is similar to local central limit theorems. The fact that the interval \((-e^{-\epsilon n}, e^{-\epsilon n})\) is exponentially shrinking as \( n \to \infty \) leads to several difficulties. Under some geometric assumptions, we establish a sharp large deviation result with leading term \( C(a)\epsilon_n e^{\gamma(a)T} \) and rate function \( \gamma(a) \leq 0 \). The proof is based on the spectral estimates for the iterations of the Ruelle operators with two complex parameters and on a new Tauberian theorem for sequence of functions \( g_n(t) \) having an asymptotic as \( n \to \infty \) and \( t \geq n \).

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1. Introduction

Let \( \varphi_t : M \rightarrow M \) be a \( C^2 \) weak mixing Axiom A flow on a compact Riemannian manifold \( M \), and let \( \Lambda \) be a basic set for \( \varphi_t \). The restriction of the flow on \( \Lambda \) is a hyperbolic flow [11]. For any \( x \in M \) let \( W^s_\epsilon(x), W^u_\epsilon(x) \) be the local stable and unstable manifolds through \( x \), respectively (see [2,7,11]). It follows from the hyperbolicity of \( \Lambda \) that if \( \epsilon_0 > 0 \) is sufficiently small, there exists \( \epsilon_1 > 0 \) such that if \( x, y \in \Lambda \) and \( d(x, y) < \epsilon_1 \), then \( W^s_\epsilon_0(x) \) and \( \varphi_{[-\epsilon_0, \epsilon_0]}(W^u_\epsilon_0(y)) \) intersect at exactly one point \( [x, y] \in \Lambda \) (cf. [7]). That is, there exists a unique \( t \in [-\epsilon_0, \epsilon_0] \) such that \( \varphi_t([x, y]) \in W^u_\epsilon_0(y) \). Setting \( \Delta(x, y) = t \), defines the so-called temporal distance function. Here and throughout the whole paper we denote by \( d(\cdot, \cdot) \) the distance on \( M \) determined by the Riemannian metric.

Let \( R = \{ R_i \}_{i=1}^k \) be a fixed (pseudo) Markov family of pseudo-rectangles \( R_i = [U_i, S_i] = \{ [x, y] : x \in U_i, y \in S_i \} \) (see Sect. 2). Set \( R = \cup_{i=1}^k R_i, U = \)
Consider the Poincaré map $P : R \rightarrow R$, defined by $P(x) = \varphi_{\tau(x)}(x) \in R$, where $\tau(x) > 0$ is the smallest positive time with $\varphi_{\tau(x)}(x) \in R$ (first return time function). The shift map $\sigma : R \rightarrow U$ is given by $\sigma = \pi_U \circ P$, where $\pi_U : R \rightarrow U$ is the projection along stable leaves.

Define a $(k \times k)$ matrix $A = \{A(i,j)\}^{k}_{i,j=1}$ by

$$A(i,j) = \begin{cases} 1 & \text{if } P(\text{Int } R_i) \cap \text{Int } R_j \neq \emptyset, \\ 0 & \text{otherwise}. \end{cases}$$

Following [2], it is possible to construct a Markov family $\mathcal{R}$ so that $A$ is irreducible and aperiodic.

Consider the suspension space $R^\tau = \{(x,t) \in R \times \mathbb{R} : 0 \leq t \leq \tau(x)\}/\sim$,

where by $\sim$ we identify the points $(x, \tau(x))$ and $(P(x,0))$. The suspension flow on $R^\tau$ is defined by $\varphi_t^\tau(x,s) = (x,s+t)$ taking into account the identification $\sim$. For a Hölder continuous function $f$ on $R$, the topological pressure $\text{Pr}_P(f)$ with respect to $P$ is defined by

$$\text{Pr}_P(f) = \sup_{m \in \mathcal{M}_P} \left\{ h(P,m) + \int f dm \right\},$$

where $\mathcal{M}_P$ denotes the space of all $P$-invariant Borel probability measures and $h(P,m)$ is the entropy of $P$ with respect to $m$. We say that $u$ and $v$ are cohomologous, and we denote this by $u \sim v$ if there exists a continuous function $w$ such that $u = v + w \circ P - w$. The flow $\varphi_t$ on $\Lambda$ is naturally related to the suspension flow $\varphi_t^\tau$ on $R^\tau$. There exists a natural semi-conjugacy projection $\pi(x,t) : R^\tau \rightarrow \Lambda$ which is one-to-one on a residual set (see [2]) such that $\pi(x,t) \circ \varphi_s^\tau = \varphi_s \circ \pi(x,t)$. For $z \in R$ set

$$\tau^n(z) := \tau(z) + \tau(\pi(z)) + \cdots + \tau(\pi^{n-1}(z)).$$

Notice that since $\tau(x)$ is constant along stable leaves for $x = \pi_U(z)$ we have

$$\tau^n(z) = \tau^n(x) = \tau(x) + \tau(\sigma(x)) + \cdots + \tau(\sigma^n(x)).$$

Denote by $\widehat{U}$ (or $\widehat{R}$) the set of those $x \in U$ (resp. $x \in R$) such that $P^m(x)$ does not belong to the boundary of any rectangle $R_i$ for all $m \in \mathbb{Z}$. In a similar way define $\widehat{R}^\tau$. It is well known (see [1]) that $\widehat{U}$ (resp. $\widehat{R}$) is a residual subset of $U$ (resp. $R$) and has full measure with respect to any Gibbs measure on $U$ (resp. $R$). Clearly in general $\tau$ is not continuous on $U$; however, $\tau$ is essentially Hölder on $U$, i.e. there exist constants $L > 0$ and $\alpha > 0$ such that $|\tau(x) - \tau(y)| \leq L \cdot d(x,y)\alpha$ whenever $x,y \in U_i$ and $\sigma(x),\sigma(y) \in U_j$ for some $i,j$. The same applies to $\sigma : U \rightarrow U$ and to $P : R \rightarrow R$. Throughout we will mainly work with the restrictions of $\tau$ and $\sigma$ to $\widehat{U}$ and also with the restrictions of $\tau$ and $P$ to $\widehat{R}$.

Consider the space $C^\alpha(\widehat{R}^\tau)$ of all $\alpha$-Hölder functions on $\widehat{R}^\tau$ with norm $\|w\|_\alpha = |w|_\alpha + \|w\|_\infty$. We should stress that throughout the whole paper the Hölder norms $|w|_\alpha$ for functions on $\widehat{U}$, $\widehat{R}$ or $\widehat{R}^\tau$ are always determined with respect to distance $d(\cdot,\cdot)$ on $M$ determined by the Riemannian metric.
For \( F \in C^\alpha(\hat{R}^\tau) \), define the function \( f : R \rightarrow \mathbb{R} \) by
\[
  f(z) = \int_0^{\tau(z)} F(z, t) \, dt, \quad z \in R.
\]
Here and in the following, we use the notation \( F(z, t) = F(\sigma_t^\tau(z, 0)) \). Given a function \( G \in C(\hat{R}^\tau) \), we define
\[
  G^T(w) = \int_0^T G(\varphi_t^\tau(w)) \, dt, \quad w \in R^\tau.
\]

Throughout the paper, we assume that \( F, G \in C^\alpha(\hat{R}^\tau) \). Let \( a = \int_{R^\tau} Gdm_{F+tG} \), where \( m_{F+tG} \) is the equilibrium state of \( F + tG \) for some \( t \in \mathbb{R} \). More precisely, for a function \( A \) on \( \hat{R}^\tau \) the equilibrium state \( m_A \) of \( A \) is a \( \sigma_t^\tau \)-invariant probability measure on \( R^\tau \) such that
\[
  \Pr_{\varphi_t^\tau}(A) = \sup_{m \in \mathcal{M}_\tau} \{ h(\varphi_t^\tau, m) + \int A(w) \, dm(w) \},
\]
h(\( \varphi_t^\tau, m \)) being the entropy of \( \varphi_t^\tau \) with respect to \( m \) and \( \mathcal{M}_\tau \) the space of all \( \varphi_t^\tau \) invariant Borel probability measures on \( R^\tau \). The supremum above is given by a measure \( m_A \) called equilibrium state of \( A \). Moreover, if \( G \) is not cohomologous to a constant, we have
\[
  0 < \sigma^2_m(G) = \lim_{T \rightarrow \infty} \frac{1}{T} \left( \int_0^T (G \circ \sigma_t^\tau) dt - T \int G \, dm \right)^2 < \infty
\]
and \( \frac{d^2\Pr(F+tG)}{dt^2} = \sigma^2_{m_{F+tG}}(G) \), \( m_H \) being the equilibrium state of \( H \).

Introduce the rate function
\[
  \gamma(a) := \inf_{t \in \mathbb{R}} \{ \Pr(F + tG) - \Pr(F) - ta \} = \Pr(F + \xi(a)G) - \Pr(F) - \xi(a)a,
\]
where \( \xi(a) \) is the unique real number such that
\[
  \frac{d\Pr(F + tG)}{dt} \bigg|_{t=\xi(a)} = \int G \, dm_{F+\xi(a)G} = a
\]
and \( \Pr(A) = \Pr_{\sigma_t^\tau}(A) \) is the pressure of \( A \) with respect to the flow \( \sigma_t^\tau \) on \( R^\tau \).

For simplicity of the notations, we will write \( \Pr(A) \) instead of \( \Pr_{\varphi_t^\tau}(A) \). Let
\[
  \beta(t) = \Pr(F + tG) - \Pr(F).
\]
Clearly,
\[
  \gamma'(a) = \beta'(\xi(a))\xi'(a) - \xi'(a)a - \xi(a) = -\xi(a),
\]
and
\[
  \xi'(a) = \frac{1}{\beta''(\xi(a))} = \frac{1}{\sigma^2_{m_{F+\xi(a)G}}(G)}.
\]
Consequently, \( \gamma'(a) = 0 \) if and only if \( \xi(a) = 0 \) which is equivalent to \( \int G \, dm_F = a \). Thus, \( \gamma(a) \) is a nonpositive concave function and \( \gamma(a) = 0 \) if only if \( \int G \, dm_F = a \).
In this paper, we continue the analysis of sharp large deviations in [12,14]. Our purpose is to improve the results in [14] and to study for a fixed $q \geq 0$
the asymptotic of \[ m_F \left\{ w \in R^\tau : \forall T \geq n - q \text{ we have } \int_0^T G(\varphi_t^\tau (w))dt - aT \in \left(-e^{-\epsilon n}, e^{-\epsilon n}\right) \right\} \]
as $n \to \infty$. Here, $0 < \epsilon \leq \mu_0/8$ is a small constant, where $\mu_0 > 0$ defined in Sect. 4 is related to the meromorphic continuation of the function $Z(s, \omega, a)$ across the line $\text{Res} = \gamma(a)$.

Recall the subset $\hat{U}$ of $\hat{R}$ defined earlier. We will regard $\hat{U}$ as a subset of $\hat{R}^\tau$ using the identification $x \leftrightarrow (x, 0)$ for $x \in R$. Now we introduce two definitions of independence.

**Definition 1.1.** Two functions $f_1, f_2 \in C^\alpha(\hat{U})$ are called $\sigma$-independent if whenever there are constants $t_1, t_2 \in R$ such that $t_1 f_1 + t_2 f_2$ is cohomologous to a function in $C(\hat{U} : 2\pi\mathbb{Z})$, we have $t_1 = t_2 = 0$.

For a function $G \in C^\alpha(\hat{R}^\tau)$, consider the skew product flow $S^G_t$ on $S^1 \times R^\tau$ defined by

\[ S^G_t(e^{2\pi i\alpha}, y) = \left(e^{2\pi i(\alpha + G^t(y))}, \varphi^\tau_t(y)\right). \]

**Definition 1.2.** ([10]) Let $G \in C^\alpha(\hat{R}^\tau)$. Then, $G$ and $\varphi^\tau_t$ are flow independent if the following condition is satisfied. If $t_0, t_1 \in R$ are constants such that the skew product flow $S^H_t$ with $H = t_0 + t_1 G$ is not topologically transitive, then $t_0 = t_1 = 0$.

Following the result in [10], if $G$ and $\varphi^\tau_t$ are flow independent, then the flow $\varphi^\tau_t$ is topologically weak-mixing and the function $G$ is not cohomologous to a constant function. On the other hand, if $G$ and $\varphi^\tau_t$ are flow independent, then $g(x) = \int_0^{\tau(x)} G(x, t)dt, x \in U$ and $\tau(x)$ are $\sigma$-independent.

Introduce the set \[ \Gamma_G = \left\{ \int Gdm_{F + tG} : t \in R \right\}. \]

Our first result is the following

**Theorem 1.3.** Assume that the standing assumptions stated in Sect. 2 are satisfied. Let $G : \Lambda \longrightarrow (0, \infty)$ be a Hölder function for which there exists a Markov family $\mathcal{R} = \{R_i\}_{i=1}^k$ for the flow $\varphi_t$ on $\Lambda$ such that $G$ is constant on the stable leaves of all “rectangular boxes”

\[ B_i = \{ \varphi_t(x) : x \in R_i, 0 \leq t \leq \tau(x)\}, \]

$i = 1, \ldots, k$. Assume in addition that $G$ and $\varphi^\tau_t$ are flow independent. Let $q \geq 0$ be fixed, let $0 < \mu_0 < 1$ be the constant in Proposition 4.2, and let $\epsilon_n = e^{-\epsilon n}, 0 < \epsilon \leq \mu_0/8$. Then, for any compact set $J \subset \Gamma_G$ and $0 < \eta \ll 1$
there exists $n_0(\eta) \in \mathbb{N}$ such that for $a \in J$, $n \geq n_0(\eta) + q$ and $T \geq n_0(\eta) - q$ we have

$$\frac{\sqrt{2} \epsilon_n C(a)}{\sqrt{\pi T \beta''(\xi(a))}} e^{\gamma(a) T (1 - \eta)}$$

$$\leq m_F \left\{ w \in \mathbb{R}^T : \int_0^T G(\varphi^+_t(w)) \, dt - aT \in (-\epsilon n, e^{-\epsilon n}) \right\}$$

$$\leq \frac{\sqrt{2} \epsilon_n C(a)}{\sqrt{\pi T \beta''(\xi(a))}} e^{\gamma(a) T (1 + \eta)},$$

(1.1)

where $C(a) > 0$ is a constant defined in Sect. 6.

The above theorem says that for $T \geq n - q$ and $n \to \infty$ we have

$$m_F \left\{ w \in \mathbb{R}^T : \int_0^T G(\varphi^+_t(w)) \, dt - aT \in (-\epsilon n, e^{-\epsilon n}) \right\}$$

$$\sim \frac{\sqrt{2} \epsilon_n C(a)}{\sqrt{\pi T \beta''(\xi(a))}} e^{\gamma(a) T}.$$  

(1.2)

Notice that the proof of Theorem 1.3 works if we assume that $G$ is not cohomologous to a constant and $\tau(x)$ and $g(x) = \int_0^{\tau(x)} G(x, t) \, dt$ are $\sigma-$ independent. As it was mentioned above, these properties are satisfied if $G$ and $\varphi^+_t$ are flow independent.

Theorem 1.3 is an improvement of a result in [14], where the asymptotic of

$$\mu_f \left\{ x \in U : \int_0^{\tau^n(x)} G(\varphi^+_t(x, 0)) \, dt - a \tau^n(x) \in (-\epsilon n, e^{-\epsilon n}) \right\}, \ n \to \infty$$

has been investigated, $\mu_f$ being the equilibrium state of $f(x)$. When we replace $\tau^n(x)$ by $T$, we have to study two limits: $n \to \infty$ and $T \to \infty$, and the condition $T \geq n - q$ is natural. It is easier to study the case when $n$ is fixed and we take $T \geq T(\eta, n)$ to arrange (1.1). However, assuming $n \geq n_0(\eta)$, for fixed $0 < \eta \ll 1$ we could have $\lim_{n \to \infty} T(\eta, n) = \infty$ and we cannot arrange an uniformity with respect to $n$. Thus, our result is much more precise and to prove it we follow a strategy based on Tauberian theorems with two parameters $n, T$ examining the asymptotic of a sequence of functions $g_n(T)$. We discuss briefly this approach below.

Our second result concerns the function

$$\zeta(T; a) = m_F \left\{ w \in \mathbb{R}^T : \int_0^T G(\varphi^+_t(w)) \, dt - aT \in (-e^{-\epsilon T}, e^{-\epsilon T}) \right\}.$$  

Applying Theorem 1.3, we prove the following
Theorem 1.4. Under the assumptions of Theorem 1.3, for $a \in J$ and any $0 < \eta \ll 1$ there exists $n_0(\eta) \in \mathbb{N}$ such that for $T \geq n_0(\eta) + 1$ we have

$$
\frac{\sqrt{2}e^{-\epsilon T}C(a)}{\sqrt{\pi T}\beta''(\xi(a))} e^{\gamma(a)T}(1 - \eta) \leq \zeta(T; a) \leq \frac{\sqrt{2}e^{-\epsilon T}C(a)}{\sqrt{\pi T}\beta''(\xi(a))} e^{\gamma(a)T}(1 + \eta).
$$

(1.3)

It is possible to obtain a slightly better result assuming one can generalise Theorem 1.3 for sequences $n_k \to \infty$ instead of a sequence of integers $n \to \infty$; however, we are not going to discuss such generalisations.

The results of the type discussed above are known as local central limit theorem (LCLT) (see [4] for recent results and references). In particular, LCLTs in a very general setting are studied in [4] and asymptotics of the form

$$
m_F \left\{ w \in R^\tau : \int_0^T G(\varphi_t^\tau(w)) dt - T \int G dm_F - c\sqrt{T} \in I \right\}
$$

are proved when $I$ is a bounded interval in $\mathbb{R}$, $g(c)$ is a Gaussian density and $\text{Leb}(I)$ is the Lebesgue measure of $I$. The case considered in the present paper, where we deal with exponentially shrinking intervals $I_n \subset \mathbb{R}$ and we want to have an asymptotic as $n \to \infty$ and $T \to \infty$, is more difficult. Large deviations for Anosov flows have been examined by Waddington [20], where for the measure

$$
m_F \left\{ w \in R^\tau : \int_0^T G(\varphi_t^\tau(w)) dt - aT \in [c, d] \right\}
$$

an asymptotic similar to (1.2) was obtained with leading term having the form

$$
\int_c^d e^{-\xi(a)t}dt \frac{C(a)}{\sqrt{2\pi T}\beta''(\xi(a))} e^{\gamma(a)T}.
$$

In [20], there are several points presented without proofs. In the exposition in [20] the case when $G$ is constant along stable leaves is treated, while in the general case, no argument is provided. This gap is essential since for large deviations, applying a reduction based on Proposition 2.2 (see Sect. 2), new terms appear in the analysis of the Laplace transform when we work with the iterations of Ruelle operators. A second gap is related to Proposition 6(ii) in [20] which is also presented without proof. This proposition concerns a Tauberian theorem for a nonnegative function which is not monotonic. In general, without a slowly decreasing condition (see Section 10 in [9]), or without some condition on the growth of the derivative, it is not known if the result is true.

In the analysis of the large deviations in the case when a fixed interval $[c, d]$ is replaced by an exponentially shrinking interval $(e^{-\epsilon n}, e^{-\epsilon n})$, several additional difficulties appear. In two previous papers [12, 14], some partial cases have been treated, but in these papers, we have considered only limits $n \to \infty$.
(see also [16], where the case of an interval \((-n^{-\kappa}, n^{-\kappa})\) with suitable \(\kappa > 1\) has been studied). In this paper, we deal with two limits \(n \to \infty, T \geq n - q\). Our approach is based on spectral estimates for the iterations of a Ruelle operator

\[ \mathcal{L}_{s, \omega, a} = \mathcal{L}_{f - s \sigma + (\xi(a) + i \omega)(g - a \sigma)}, \quad s \in \mathbb{C}, \quad \omega \in \mathbb{R} \]

with two complex parameters \(s\) and \(\omega\) which may have modulus going to \(\infty\). We exploit the estimates obtained in [13,14] (see Theorem 2.1) in order to obtain an analytic continuation of the Laplace transform \(F_n(s)\) of the function \(g_n(T)\) defined below for \(|\text{Im} s| \geq M\) and \(|\omega| \geq \epsilon_0\). We establish the existence of an analytic continuation of \(F_n(s)\) across the line \(\text{Re} \gamma(a) \leq \gamma(a) - \mu_0 \leq \text{Re}, \quad |\text{Im} s| \geq M \gg 1\). This continuation and the corresponding estimates play a crucial role in the new type Tauberian theorems concerning double limits \(n \to \infty, T \to \infty\). These Tauberian theorems are of independent interest.

For convenience of the reader, we explain briefly the idea of the proof of Theorem 1.3. Let \(\chi(t) \in C^\infty_0(\mathbb{R} : \mathbb{R}^+)\) be a nonnegative cut-off function and let

\[ G^T(w) = \int_0^T G(\varphi_t(w))dt, \quad w \in R^\tau. \]

Set \(\chi_n(t) = \chi\left(\frac{t}{\epsilon_n}\right)\). We study the function

\[ g_n(T) := \epsilon_n e^{-\gamma(a)T + T} \int_{R^\tau} \chi\left(\frac{G^T - aT}{\epsilon_n}\right)(w)dm_F(w) \]

\[ = \frac{\epsilon_n^2}{2\pi} e^{-\gamma(a)T + T} \int_{R^\tau} \int_{R} e^{i\omega(G^T - aT)(w)} \hat{\chi}(\epsilon_n \omega) d\omega dm_F(w), \]

where \(\hat{\chi}_n(\omega) = \epsilon_n \hat{\chi}(\epsilon_n \omega)\) is the Fourier transform of \(\chi_n(t)\). We extend this function as zero for \(T < 0\) and examine the Laplace transform

\[ F_n(s) = \frac{\epsilon_n^2}{2\pi} \int_{R} \int_0^\infty e^{-sT - \gamma(a)T + T} \left( \int_{R^\tau} e^{i\omega(G^T - aT)(w)} dm_F(w) \right) dT \hat{\chi}(\epsilon_n \omega) d\omega. \]

Our purpose is to prove that for fixed \(q \geq 0\), as \(n \to \infty, T \geq n - q\), we have

\[ g_n(T) \sim \frac{\sqrt{2C(a)} \epsilon_n^2}{\sqrt{\pi \beta''(\xi(a))T}} e^{T}, \quad (1.4) \]

which yields the asymptotic (1.2). The factor \(\epsilon_n\) in \(g_n(T)\) is involved to have an independent of \(n\) bound for the derivative \(g'_n(T)\) (see Proposition 5.6 and Lemma 5.7 in Sect. 5). Let

\[ Z(s, \omega, a) = \int_0^\infty e^{-(s+\gamma(a)-1)T} \left( \int_{R^\tau} e^{i\omega(G^T - aT)w} dm_F(w) \right) dT. \]

For fixed \(a\), the function \(Z(s, \omega, a)\) depends on two complex parameters \(s \in \mathbb{C}\) and \(i \omega\). Moreover, in \(Z(s, \omega, a)\) we have no integration with respect to \(\omega\). First, we show that this function is analytic for \(\text{Re} s > \gamma(a)\). Second, we prove that in a small neighbourhood of \((\gamma(a), 0) \in \mathbb{C}^2\) this function has a pole \(s(\omega, a)\) with residue \(C(\omega, a) > 0\). To establish an analytic continuation across the line \(\text{Re} s = \gamma(a)\) for \(|s - \gamma(a)| \geq \epsilon_0 > 0\), first we reduce the integration on \(R\), and
then by using the hypothesis that $G$ is constant along stable leaves, we reduce once more the integration on $U$ and write (see Sect. 3)

$$Z(s, \omega, a) = \int_U B_2(s, \omega, a, u) \sum_{m=0}^{\infty} \left( \mathcal{L}_{f-s\tau+i\omega(g-a\tau)}^m B_1(s, \omega, a, \cdot) \right)(u) h(u) \, dv(u),$$

$\mathcal{L}_{f-s\tau+i\omega(g-a\tau)}$ being the Ruelle operator related to $f-s\tau+i\omega(g-a\tau)$, where $f(x), g(x), x \in U$, are determined by $F, G$, respectively, and the measure $\nu(u)$ on $U$ is determined by $f(u)$. We exploit the estimates for the iterations $\mathcal{L}_{f-s\tau+i\omega(g-a\tau)}^m$ obtained in [14] (see Theorem 2.1 in Sect. 2) to obtain a meromorphic continuation of $Z(s, \omega, a)$ across $\text{Res} = \gamma(a)$.

The integration with respect to $\omega$ is not involved in the definition of $Z(s, \omega, a)$. Taking the integration in a small interval $[-\epsilon_0, \epsilon_0]$, writing $\hat{\chi}(\epsilon_n \omega) = \hat{\chi}(0) + \hat{\chi}'(0) \epsilon_n \omega + \mathcal{O}(\epsilon_n^2 \omega^2)$, and repeating the calculus in [8, 20], we get

$$\epsilon_n^2 \int_{-\epsilon_0}^{\epsilon_0} \frac{\hat{\chi}(\epsilon_n \omega)}{s-1-s(\omega, a)} \, d\omega = \frac{C(a) \hat{\chi}(0) \epsilon_n^2}{\sqrt{2}\beta^\gamma(\xi(a))(s-1)} + \text{smoother terms.}$$

The leading term becomes $\frac{A_n}{\sqrt{s-1}}$ with $A_n = \mathcal{O}(\epsilon_n^2) = \mathcal{O}(e^{-2\epsilon n})$, and this difficulty corresponds to the type of Tauberian results proved in Sect. 5, where the remainder of the asymptotic has order $\mathcal{O}(\epsilon_n^2)$. The integral with respect to $\omega$ over $\mathbb{R} \setminus [-\epsilon_0, \epsilon_0]$ yields analytic functions; however, we need precise estimates on their growth as $|\text{Im}s| \to \infty$ independent on $n$ in order to apply Proposition 5.6. For this purpose, applying the results of Sect. 4, we are going to estimate the integral

$$\epsilon_n^2 \int_{|\omega| \geq M} \frac{1 + |\text{Im}s|^{\nu} + |\omega|^{\nu}}{|\hat{\chi}(\epsilon_n \omega)|} \, d\omega$$

with $0 < \nu < 1$, uniformly with respect to $n$. Here, the presence of the factor $\epsilon_n^2$ is crucial, and our choice of $\epsilon_n$ in the definition of $g_n(T)$ is once more very convenient. To check the hypothesis of Proposition 5.6, we use in an essential way the analytic continuation of $Z(s, \omega, a)$ and its corresponding estimates.

The plan of the paper is as follows: In Sect. 2, we introduce some definitions and our standing assumptions. A Sinai’s lemma for the suspended flow is stated; it is proved in Appendix. In Sect. 3, a representation of the Laplace transform $Z(s, \omega, a)$ of the function $g_n(T)$ is obtained. Section 4 is devoted to the analysis of the meromorphic continuation of $Z(s, \omega, a)$ based, as mentioned above, on the results in [14]. In Sect. 5, two Tauberian theorems are proved for a sequence of nonnegative functions $g_n(T)$. The novelty here is that these functions have Laplace transforms having singularities $\frac{A_n}{\sqrt{s-1}}$ with $0 < e^{-\mu n} \leq A_n \leq C_1$ and $0 < \mu \leq \mu_0/4$ and we have a double limit $n \to \infty$, $t \to \infty$. Theorems 1.3 and 1.4 are proved in Sect. 6.

By using Proposition 2.2, we can study also the general case when the function $G$ is not constant on stable leaves. A part of Proposition 4.2 in Sect. 4 concerning the analytic continuation for $|\text{Im}s| \geq M$ can be established. However, some new difficulties appear in the description of the singularities of
There exist nonintegrability condition about the flow on $\Lambda$: basic set $\Lambda$ if there exists a constant $\epsilon$ of the present paper would apply without change under much more general assumptions. Estimates should be established in much higher generality. So, we expect that the methods Ruelle transfer operators (see [14]). However, recent developments in [19] suggest that similar open problem.

2. Definitions and Standing Assumptions

2.1. Standing Assumptions

Throughout we use the notation and assumptions from the beginning of Sect. 1. In particular, $\varphi_t : M \longrightarrow M$ is a $C^2$ weak mixing Axiom A flow and $\Lambda$ is a basic set for $\varphi_t$. As in [13, 14], we will work under the following rather general nonintegrability condition about the flow on $\Lambda$:

(LNIC): There exist $z_0 \in \Lambda$, $\epsilon_0 > 0$ and $\theta_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$, any $\hat{z} \in \Lambda \cap W^u(z_0)$ and any tangent vector $\eta \in E^u(\hat{z})$ to $\Lambda$ at $\hat{z}$ with $||\eta|| = 1$ there exist $\tilde{z} \in \Lambda \cap W^u(\hat{z})$, $\tilde{y}_1, \tilde{y}_2 \in \Lambda \cap W^s(\tilde{z})$ with $\tilde{y}_1 \neq \tilde{y}_2$, $\delta = \delta(\tilde{z}, \tilde{y}_1, \tilde{y}_2) > 0$ and $\epsilon' = \epsilon'(\tilde{z}, \tilde{y}_1, \tilde{y}_2) \in (0, \epsilon]$ such that

$$|\Delta(\exp_v^u(v), \pi_{\tilde{y}_1}(z)) - \Delta(\exp_v^u(v), \pi_{\tilde{y}_2}(z))| \geq \delta \|v\|$$

for all $z \in W^s_{\epsilon'}(\tilde{z}) \cap \Lambda$ and $v \in E^u(z; \epsilon')$ with $\exp_v^u(v) \in \Lambda$ and $\langle v, \eta_z \rangle \geq \theta_0$, where $\eta_z$ is the parallel translate of $\eta$ along the geodesic in $W^s_{\epsilon'}(z_0)$ from $\tilde{z}$ to $z$.

Next, given $x \in \Lambda$, $T > 0$ and $\delta \in (0, \epsilon]$ set

$$B^u_T(x, \delta) = \{y \in W^u(\hat{z}) : d(\varphi_t(x), \varphi_t(y)) \leq \delta, \ 0 \leq t \leq T\}.$$ 

We will say that $\varphi_t$ has a regular distortion along unstable manifolds over the basic set $\Lambda$ if there exists a constant $\epsilon_0 > 0$ with the following properties:

(a) For any $0 < \delta \leq \epsilon \leq \epsilon_0$ there exists a constant $R = R(\delta, \epsilon) > 0$ such that

$$\text{diam}(\Lambda \cap B^u_T(z, \epsilon)) \leq R \text{diam}(\Lambda \cap B^u_T(z, \delta))$$

for any $z \in \Lambda$ and any $T > 0$.

(b) For any $\epsilon \in (0, \epsilon_0]$ and any $\rho \in (0, 1)$ there exists $\delta \in (0, \epsilon]$ such that for any $z \in \Lambda$ and any $T > 0$ we have $\text{diam}(\Lambda \cap B^u_T(z, \delta)) \leq \rho \text{diam}(\Lambda \cap B^u_T(z, \epsilon))$.

Standing Assumptions:

(A) $\varphi_t$ has Lipschitz local holonomy maps over $\Lambda$,

(B) the local nonintegrability condition (LNIC) holds for $\varphi_t$ on $\Lambda$,

(C) $\varphi_t$ has a regular distortion along unstable manifolds over the basic set $\Lambda$.

In this paper, we will work under the above Standing Assumptions. A large class of examples satisfying the conditions (A)–(C) is provided by imposing the following pinching condition:

(P): There exist constants $C > 0$ and $\beta \geq \alpha > 0$ such that for every $x \in M$ we have

$$\frac{1}{C} e^{\alpha \epsilon \cdot \epsilon t} \|u\| \leq \|d\varphi_t(x) \cdot u\| \leq C e^{\beta \epsilon \cdot \epsilon t} \|u\|, \ \ u \in E^u(x), \ t > 0$$

1These assumptions are needed to ensure the validity of certain strong spectral estimates for Ruelle transfer operators (see [14]). However, recent developments in [19] suggest that similar estimates should be established in much higher generality. So, we expect that the methods of the present paper would apply without change under much more general assumptions.
for some constants $\alpha_x,\beta_x > 0$ with $\alpha \leq \alpha_x \leq \beta_x \leq \beta$ and $2\alpha_x - \beta_x \geq \alpha$ for all $x \in M$.

It is well known that (P) holds for geodesic flows on manifolds of strictly negative sectional curvature satisfying the so-called $1/4$-pinching condition. (P) always holds when $\dim(M) = 3$.

**Simplifying Assumptions:** $\varphi_t$ is a $C^2$ contact Anosov flow satisfying the condition (P).

It follows from the results in [18], that the pinching condition (P) implies that $\varphi_t$ has Lipschitz local holonomy maps and regular distortion along unstable manifolds and moreover:

**The Simplifying Assumptions imply the Standing Assumptions.**

Throughout this paper we work under the Standing Assumptions.

### 2.2. Some Definitions and Ruelle Transfer Operators

As in Sect. 1, let $\mathcal{R} = \{R_i\}_{i=1}^k$ be a Markov family for $\varphi_t$ over $\Lambda$ consisting of rectangles $R_i = [U_i,S_i]$, where $U_i$ (resp. $S_i$) are (admissible) subsets of $W^u(z_i) \cap \Lambda$ (resp. $W^s(z_i) \cap \Lambda$) for some $\epsilon > 0$ and $z_i \in \Lambda$. We will use the set-up and some arguments from [13,17]. As in these papers, fix a (pseudo) Markov family $\mathcal{R} = \{R_i\}_{i=1}^k$ of pseudo-rectangles

$$R_i = [U_i,S_i] = \{[x,y] : x \in U_i, y \in S_i\}.$$  

Set

$$R = \bigcup_{i=1}^k R_i \ , \ U = \bigcup_{i=1}^k U_i.$$  

Consider the Poincaré map $P : R \rightarrow R$, defined by $P(x) = \varphi_{\tau(x)}(x) \in R$, where $\tau(x) > 0$ is the smallest positive time with $\varphi_{\tau(x)}(x) \in R$ (first return time function). The shift map $\sigma : U \rightarrow U$ is given by $\sigma = \pi_U \circ P$, where $\pi_U : R \rightarrow U$ is the projection along stable leaves.

Recall the subsets $\hat{U}$ of $U$ and $\hat{R}$ of $R$ introduced in Sect. 1. Throughout $\alpha > 0$ will be fixed constant such that $\tau \in C^\alpha(\hat{U})$. We assume in Theorem 1.3 that $F,G \in C^\alpha(\hat{R}^\tau)$. If $\tau \in C^\alpha(\hat{U})$ for some $0 < \hat{\alpha} < 1$, we may take $\alpha = \min\{\alpha, \hat{\alpha}\}$, to arrange that $F,G,\tau$ are in the Hölder spaces with the same $\alpha$. For simplicity of the notations, we assume in the following that this is arranged. Since the local stable (and unstable) holonomy maps are uniformly Hölder ([5,6]), we may assume $\alpha$ is chosen so that

$$d([x,y],[x',y]) \leq C(d(x,x'))^\alpha, \ x,x',y \in R_i \ , \ i = 1, \ldots, k. \quad (2.1)$$

The hyperbolicity of the flow implies the existence of constants $c_0 \in (0,1]$ and $\gamma_1 > \gamma > 1$ such that

$$c_0 \gamma^m (d(x,y))^{1/\alpha} \leq d(\mathcal{P}^m(x),\mathcal{P}^m(y)) \leq \frac{\gamma^m}{c_0} (d(x,y))^{\alpha} \quad (2.2)$$

for all $x,y \in R$ such that $\mathcal{P}^i(x),\mathcal{P}^i(y)$ belong to the same $R_i$ for all $j = 0,1,\ldots,m$. Moreover, we choose the constants so that if $y \in W^s(x)$ for some $x,y \in R_i$, then

$$d(\mathcal{P}^n(x),\mathcal{P}^n(y)) \leq \frac{c_0}{\gamma^n}, \quad n \geq 0. \quad (2.3)$$
Fix a constant $\alpha' \in (0, \alpha]$ so that
\[ \rho = \frac{\gamma^{\alpha'}}{\gamma} < 1. \]

Let $\|h\|_\infty$ denote the standard sup norm of $h$ on $U$. For $|b| \geq 1$, and $\beta > 0$, as in [3], define the norm
\[ \|h\|_{\beta,b} = \|h\|_\infty + \frac{|h\|_\beta}{|b|} \]
on the space $C^\beta(\hat{U})$ of $\beta$-Hölder functions on $\hat{U}$.

As in [13, 14], in this paper we will frequently use Ruelle transfer operators of the form
\[ L_{f-s\tau+zg}v(y) = \sum_{\sigma y = y} e^{f(x)-s\tau(x)+zg(x)}v(x), s, z \in \mathbb{C}, y \in U, \]
depending on two complex parameters $s$ and $z$. The following theorem was proved in [14].

**Theorem 2.1.** Let $\varphi_t : M \to M$ satisfy the standing assumptions over the basic set $\Lambda$, and let $\alpha > \beta > 0$. Let $\mathcal{R} = \{R_i\}_{i=1}^k$ be a Markov family for $\varphi_t$ over $\Lambda$ as above. Then, for any real-valued functions $f, g \in C^\alpha(\hat{U})$ and any constants $\nu > 0$ and $B > 0$ there exist constants $0 < \rho < 1$, $a_0 > 0$, $b_0 \geq 1$ and $C = C(B, \nu) > 0$ such that if $a, c \in \mathbb{R}$ satisfy $|a|, |c| \leq a_0$ then
\[ \|L_{f-(a+ib)\tau+(c+iw)g}h\|_{\beta,b} \leq Ce^{mPr_\sigma(f)} \rho^m \|h\|_{\beta,b} \]
for all $h \in C^\beta(U)$, all integers $m \geq 1$ and all $b, w \in \mathbb{R}$ with $|b| \geq b_0$ and $|w| \leq B |b|$.

### 2.3. Sinai’s Lemma for Suspension Flows

Given functions $G$ and $\tilde{G}$ on $\hat{R}$, define
\[ g(x) = \int_0^{\tau(x)} G(x,t) \, dt, \quad \tilde{g}(x) = \int_0^{\tau(x)} \tilde{G}(x,t) \, dt \]
for all $x \in \hat{R}$. Here, $G(x,t) = G(\pi(x,t)) = G(\sigma^t_0(x,0))$. It is easy to see that if $G \in C^\alpha(\hat{R})$, then $g \in C^\alpha(\hat{R})$.

Consider the function (defined as in the proof of Proposition 1.2 in [11])
\[ p(x) = \sum_{n=0}^{\infty} \left[ g(\mathcal{P}^n(x)) - g(\mathcal{P}^n(\pi_U(x))) \right], \quad x \in \hat{R}. \]

Since $x, \tilde{x} = \pi_U(x)$ belong to the same stable leaf in $\hat{R}$, (2.3) implies
\[ d(\mathcal{P}^n(x), \mathcal{P}^n(\pi_U(x))) \leq c_0 \gamma^{-n} \]
for all $n \geq 0$. Thus, the series in (2.5) is convergent. Now define
\[ \tilde{G}(x,t) = G(\pi_U(x), t) \]
\[ + \sum_{n=0}^{\infty} \left( G(\mathcal{P}^{n+1}(\pi_U(x))), t \tau(\mathcal{P}^{n+1}(x))/\tau(x) \right) \\
- G(\mathcal{P}^n(\pi_U(\mathcal{P}(x))), t \tau(\mathcal{P}^n(x))/\tau(x)) \]

for \( x \in \hat{R} \) and \( 0 \leq t < \tau(x) \). Notice that since \( \tau(x) \) is constant on stable leaves, we have

\[ \tau(\mathcal{P}^{n+1}(x)) = \tau(\mathcal{P}^n(\pi_U(\mathcal{P}(x)))). \]

The following is the analogue of the well-known Sinai’s lemma (see, for example, Proposition 1.2 in [11]) for suspension flows.

**Proposition 2.2.** (a) The function \( p \) defined above belongs to \( C^\beta(\hat{R}) \) for some \( \beta > 0 \) and

\[ g(x) = \tilde{g}(x) + p(x) - p(\mathcal{P}(x)) \]

for all \( x \in \hat{R} \). Moreover, \( \tilde{G} \) is constant on stable leaves of \( \hat{R}^\tau \) and is \( \beta \)-Hölder, where

\[ \beta = \alpha^2 \alpha'/2 > 0. \]

(b) The function

\[ P(x, t) = \sum_{n=0}^{\infty} [G(\mathcal{P}^n(x), t \tau(\mathcal{P}^n(x))/\tau(x)) - G(\mathcal{P}^n(\pi_U(x)), t \tau(\mathcal{P}^n(x))/\tau(x))] \]

(2.8)

(\( x \in \hat{R}, 0 \leq t < \tau(x) \)) is also \( \beta \)-Hölder on \( \hat{R}^\tau \),

\[ p(x) = \int_0^{\tau(x)} P(x, t) \, dt, \quad x \in \hat{R}, \]

and

\[ G(x, t) = \tilde{G}(x, t) + P(x, t) - P(\mathcal{P}(x), t \tau(\mathcal{P}(x))/\tau(x)) \]

(2.10)

for all \( x \in \hat{R} \) and \( 0 \leq t < \tau(x) \).

We prove Proposition 2.2 in “Appendix”.

**Remark 2.3.** (a) It follows from the definition of \( g \) that it is \( \alpha \)-Hölder with \( |g|_\alpha \leq C |G|_\alpha \). Then, (2.5) and \( d(\mathcal{P}^n(x), \mathcal{P}^n(\pi_U(x)) \leq c_0 / \gamma^n \), which follows from (2.3), imply

\[ |p(x)| \leq \sum_{n=0}^{\infty} C |G|_\alpha c_0 / \gamma^n \leq C |G|_\alpha \]

for all \( x \in R \), so \( |p|_\infty \leq C |G|_\alpha \). Similarly, \( |P|_\infty \leq C |G|_\alpha \).

(b) Given \( y \in R_i \) for some \( i \), consider the function \( w_y(x) = h([x, y]) \) on \( R_i \). Now (2.1) implies

\[ |w_y(x) - w_y(x')| \leq |h|_\alpha (d([x, y], [x', y]))^{\alpha} \leq C |h|_\alpha (d(x, x'))^{\alpha^2}. \]
Thus, \( w_y \in C_{\alpha^2} \) and \(|w_y|_{\alpha^2} \leq C|h|_\alpha \leq C|g|_\alpha \). This can be written as
\[
|p([\cdot, y])|_{\alpha^2} \leq C|G|_\alpha, \quad y \in R_i, \ i = 1, \ldots, k.
\]
By (2.7), \( \beta < \alpha^2 \), so the above is also true with \( \alpha^2 \) replaced by \( \beta \). With (a) this gives \( \|h([\cdot, y])\|_{\alpha^2} \leq C|G|_\alpha \) and so
\[
\|p([\cdot, y])\|_\beta \leq C|G|_\alpha, \quad y \in R_i, \ i = 1, \ldots, k.
\]
(2.11)

c (Following Ch. 1 in [2]; in particular sections 1B and 1C)

Let \( \tilde{f} : R \rightarrow \mathbb{R} \) depend only on \( x \in U \), i.e. \( \tilde{f}(x) = \tilde{f}(x') \) whenever \( \pi_U(x) = \pi_U(x') \). Then, we can regard \( \tilde{f} \) as a function on \( U \), \( \tilde{f} \in C^\beta(U) \), and by the Ruelle–Perron–Frobenius theorem, there exist (unique) positive function \( h_0 \in C^\beta(U) \) and a probability measure \( \nu \) on \( U \) such that \( (\mathcal{L}_f)^*\nu = \nu \) and \( \int_U h_0 \, d\nu = 1 \). Then
\[
dm_f(x) = h_0(x) \, \nu(x)
\]
(2.12)
is a \( \sigma \)-invariant probability measure on \( U \), called the Gibbs measure determined by \( \tilde{f} \). It gives rise to a \( \mathcal{P} \)-invariant probability measure \( m \) on \( R \) as follows. Given a continuous real valued function \( w \) on \( R \), define
\[
w^*(x) = \min \{w(y) : y \in W^s_R(x)\}, \ x \in R.
\]
Then, \( w^* \in C(R) \) and \( w^* \) is constant on stable leaves of \( R \), so it can be considered as a function in \( C(U) \). Define
\[
\int_R w(z) \, dm(z) = \int_U w^*(x) \, dm_{\tilde{f}}(x).
\]
As in section 1C in [2], one checks that this defines a \( \mathcal{P} \)-invariant probability measure \( m \) on \( R \). Usually, one denotes \( m = dm_{\tilde{f}} \) and calls this the Gibbs measure determined by \( \tilde{f} \) on \( R \). The above definition shows that if \( w \in C(R) \) depends only on \( x \in U \), i.e. \( w(x) = w(x') \) whenever \( \pi_U(x) = \pi_U(x') \), then we have
\[
\int_R w(z) \, dm(z) = \int_U w(x) \, dm_{\tilde{f}}(x).
\]
(d) To estimate \( \|e^{p([\cdot, y])}h_0\|_\beta \), first we have
\[
|e^{p([\cdot, y])}h_0|_\infty \leq e^{|G|_\alpha} |h_0|_\infty \leq Ce^{|G|_\alpha}.
\]
Using (a),
\[
|e^{p([\cdot, y])}|_\beta \leq e^{|G|_\alpha} |p([\cdot, y])|_\beta \leq C|G|_\alpha e^{|G|_\alpha}.
\]
This implies
\[
|e^{p([\cdot, y])}h_0|_\beta \leq |e^{p([\cdot, y])}|_\infty |h_0|_\beta + |e^{p([\cdot, y])}|_\beta |h_0|_\infty \leq C e^{|G|_\alpha} + C|G|_\alpha e^{|G|_\alpha} \leq C|G|_\alpha e^{|G|_\alpha}.
\]
Combining the above estimates yields
\[
\|e^{p([\cdot, y])}h_0\|_\beta \leq C|G|_\alpha e^{|G|_\alpha}
\]
and this estimate is uniform in \( y \in R \).
2.4. Application of Proposition 2.2

By Proposition 2.2, there exist functions \( \tilde{F}(w,t) \) and \( Y(w,t) \) such that

\[
F(w,t) = \tilde{F}(w,t) + Y(w,t) - Y\left( \mathcal{P}(w), \frac{t\tau(\mathcal{P}(w))}{\tau(w)} \right),
\]

where \( \tilde{F}(w,t) \) is constant on stable leaves. Let

\[
\tilde{f}(w) = \int_{0}^{\tau(w)} \tilde{F}(w,t) dt, \quad y(w) = \int_{0}^{\tau(w)} Y(w,t) dt.
\]

By a change of variables \( t \frac{\tau(P(w))}{\tau(w)} = s \), one deduces

\[
\int_{0}^{\tau(w)} Y\left( \mathcal{P}(w), \frac{t\tau(\mathcal{P}(w))}{\tau(w)} \right) dt = y(\mathcal{P}(w)).
\]

Now for the equilibrium state \( m_F \), one obtains

\[
m_F = \frac{1}{\int \tau(u) d\mu_{\tilde{f}}} \left( \mu_{\tilde{f}}(w) \times l \right)
\]

since

\[
\mu_f(w) = \mu_{\tilde{f}}(w) + y(w) - y(\mathcal{P}(w)) = \mu_{\tilde{f}}(w).
\]

Here, \( \mu_q \) denote the equilibrium state of \( q \) which is a probability measure invariant with respect to \( \mathcal{P} \) and \( l \) is the Lebesgue measure on \( \mathbb{R} \). Obviously, \( \tilde{f} \) depends only on \( x = \pi_U(w) \in U \). Moreover, adding a constant, we preserve \( m_F \) and can arrange \( \text{Pr}(F) = 0 \). Since

\[
\text{Pr}_{\mathcal{P}}(f(w) - \text{Pr}(F)\tau) = 0,
\]

this implies

\[
0 = \text{Pr}_{\mathcal{P}}(\tilde{f}(w) + y(w) - y(\mathcal{P}(w)) = \text{Pr}_{\mathcal{P}}(\tilde{f}(w)).
\]

We may express the pressure by

\[
\text{Pr}_{\mathcal{P}}(\tilde{f}) = \lim_{n \to \infty} \log \sum_{\mathcal{P}^n w = w} e^{\tilde{f}^n(w)}.
\]

Since \( \tilde{f}(w) \) depends only on \( x = \pi_U w \) and \( \mathcal{P}^n w = w \) implies \( \sigma^n x = x \), one deduces \( \text{Pr}_{\mathcal{P}}(\tilde{f}(x)) = \text{Pr}_{\sigma}(\tilde{f}(x)) = 0 \). Therefore, the Ruelle operator \( \mathcal{L}_{\tilde{f}} \) has 1 as an eigenvalue with eigenfunction \( h(x) > 0 \). Moreover, we have \( d\mu_{\tilde{f}}(x) = h(x)d\nu(x) \) and \( \nu(x) \) is a \( \sigma \)-invariant measure on \( U \) which can be considered as a \( \mathcal{P} \)-invariant measure on \( R \) as we have mentioned above. For this measure, we have

\[
(\mathcal{L}_{\tilde{f}}^*)^n \nu(x) = \nu(x), \quad n \geq 1.
\]
3. Representation of the Function $\mathbf{Z}(s, \omega, a)$

Let $\chi \in C_0^\infty(\mathbb{R}; \mathbb{R}^+)$ be a fixed cut-off function. Set $q_n(t) = e^{\xi(t)} \chi_n(t)$, where

$$
\chi_n(t) = \chi\left(\frac{t}{\epsilon_n}\right), \quad \epsilon_n = e^{-\epsilon_n}, \quad 0 < \epsilon \leq \mu_0/8,
$$

$\mu_0 > 0$ being the constant introduced in Proposition 4.2 in Sect. 4.

The Fourier transform of $\chi_n$ satisfies $\hat{\chi}_n(\omega) = \epsilon_n \hat{\chi}(\epsilon_n \omega)$. Given a continuous function $Q$ on $R^r$, consider

$$
Q^T(\bar{w}) = \int_0^T Q(\varphi^T_t(\bar{w})) dt, \quad \bar{w} \in R^r.
$$

Notice that we have

$$
\text{Pr}_p(q - \text{Pr}(Q)\tau) = 0
$$

if $q(w) = \int_0^{\tau(w)} Q(w, t) dt$, $w \in R$ (see [11]).

To establish Theorem 1.3, we will study the nonnegative function

$$
\rho_n(T) := \int_{R^r} q_n((G^T - aT)(y)) dm_F(y).
$$

We have

$$
\rho_n(T) = \int_{R^r} q_n((G - a)^T(y)) dm_F(y) = \int_{R^r} e^{\xi(G - a)^T(y)} \chi_n((G - a)^T(y)) dm_F(y)
$$

$$
= \frac{1}{2\pi} \int_{R^r} \int_{\mathbb{R}} e^{\xi(G - a)^T(y)} e^{i\omega(G - a)^T(y)} \hat{\chi}_n(\omega) dm_F(y) d\omega
$$

$$
= \frac{1}{2\pi} \int_{\mathbb{R}} \left( \int_{R^r} e^{z(G - a)^T(y)} dm_F(y) \right) \hat{\chi}_n(\omega) d\omega,
$$

where $z = \xi(a) + i\omega$.

Define $\Gamma_z(T)$ for $T \geq 0$ by

$$
\Gamma_z(T) = \int_{R^r} e^{z(G - a)^T(y)} dm_F(y)
$$

and $\Gamma_z(T) = 0$ for $T < 0$. Our purpose is to study the Laplace transform

$$
Z(s, \omega, a) = \int_0^\infty e^{-sT} \Gamma_z(T) dT, \quad s \in \mathbb{C}, \quad \omega \in \mathbb{R}. \quad (3.1)
$$

Since for large $M_0 > 0$ we have $|\Gamma_z(T)| \leq C e^{C_1 T}, \quad T > M_0$, with some constant $C_1 > 0$, the transformation $Z(s, \omega, a)$ exists for Re $s > C_1$ uniformly with respect to $\omega \in \mathbb{R}$. In Sect. 4, we will show that $Z(s, \omega, a)$ has an analytic extension to

$$
\{s \in \mathbb{C} : \gamma(a) - \mu_0 \leq \text{Re}s, \omega \in \mathbb{R}\} \backslash \{(s(\omega, a) : |\omega| \leq \epsilon_0)\},
$$

with $\epsilon_0 > 0$ and $\mu_0 > 0$ sufficiently small. Here, $s(\omega, a)$ is a simple pole described in Sect. 4. We use the notation $Z(s, \omega, a)$ since the Laplace transform depends on $s \in \mathbb{C}, \omega \in \mathbb{R}$ and $a$. 
we deduce that for fixed $Z$ (depending on $T$), we will denote $\tilde{f}(x)$ again by $f(x)$ and $\mu_f$ by $\mu_f$. Moreover, in the following we assume that $G$ is constant on the stable foliations in $R^\tau$, so $g(x)$ depends only on $x \in U$.

Given a Hölder function $Q(w)$ on $R^\tau$, we have

$$\int_{R^\tau} Q(w)\,dm_F(w) = \frac{\int_{R} \int_{0}^{\tau(x)} Q(x,\eta)\,d\eta\,d\mu_q(x)}{\int_{R} d\mu},$$

where $\mu_q$ is the equilibrium state of $q(w) = \int_{0}^{\tau(x)} Q(w, t)\,dt$. Therefore, setting $Q = z(G - a)$, we obtain

$$Z(s, w, a) = \frac{1}{\int_{R} d\mu} \int_{0}^{\infty} e^{-(s+az)T} \left( \int_{R} \int_{0}^{\tau(x)} e^{zG_T(x, \eta)}\,d\eta\,d\mu(x) \right) \,dT$$

$$= \frac{1}{\int_{R} d\mu} \int_{0}^{\infty} e^{-(s+az)T} \left( \int_{U} \int_{0}^{\tau(x)} e^{zG_T(x, \eta)}\,d\eta\,dh(x)\,d\nu(x) \right) \,dT.$$

Here, we interpret the integral on $R$ as an integral on $U$ as we mentioned in Remark 1(c) in Sect. 2. Given $T > 0, x \in U, 0 \leq \eta \leq \tau(x)$, there exists a unique choice of $n \geq 0$ and $0 < v < T(\sigma^n x)$ so that $T + \eta = v + \tau^n(x)$. Notice that when $x \in U$ changes the integer $n$ may change but since

$$T - \tau(\sigma^n x) \leq \tau^n(x) = T + \eta - v \leq T + \tau(x),$$

we deduce that for fixed $T, T \geq T_0$, and all $x \in U$ there is only a finite number (depending on $T$) of possible choices for $n$.

For $T + \eta = \tau^n(x) + v$, one applies the formula

$$e^{G_T+\eta(x,0)} = \sum_{n=0}^{\infty} \int_{0}^{\tau(\sigma^n x)} e^{G_{\nu+\tau^n(x)}(x,0)}\delta(\eta + T - v - \tau^n(x))\,d\nu,$$

where for fixed $x \in U, T \geq T_0$ only one term in the infinite sum is not vanishing (see [15,20] for a similar argument). Then, we may transform the integral

$$\frac{1}{\int_{R} d\mu} \int_{0}^{\infty} e^{-(s+az)T} \left( \int_{U} \int_{0}^{\tau(x)} e^{zG_T+\eta(x,0)-zG^n(x,0)}\,d\eta\,dh(x)\,d\nu(x) \right) \,dT$$

in the above expression for $Z(s, \omega, a)$, as in Section 4 in [20] to obtain the representation

$$Z(s, \omega, a) = \frac{1}{\int_{R} d\mu} \sum_{n=0}^{\infty} \int_{U} e^{-(s+(\xi(a)+i\omega)a)}\tau^n(x) + (\xi(a)+i\omega)g^n(x)$$
\[ \times B_1(s, \omega, a, \sigma^n(x)) B_2(s, \omega, a, x) h(x) d\nu(x), \]

where

\[
B_1(s, \omega, a, x) = \int_0^{\tau(x)} \exp (- (s + az) v + zG^n(x, 0)) \, dv,
\]

\[
B_2(s, \omega, a, x) = \int_0^{\tau(x)} \exp ((s + az) \eta - zG^n(x, 0)) \, d\eta.
\]

We apply (2.13) and then use the adjoint of the Ruelle operator \( L^* \), noting that

\[
\left[ L^n_f \left( e^{-(s+\xi(a)+i\omega)a\tau^n+\xi(a)+i\omega)(g\tau^n)} \right) \right](y) = \left[ L^n_{f-s\tau+(\xi(a)+i\omega)(g-a\tau)} \right](y).
\]

Therefore, we conclude that

\[
Z(s, \omega, a) = \frac{1}{\int \tau d\mu} \sum_{n=0}^{\infty} \int_U B_1(s, \omega, a, y) \times \left( \left[ L^n_{f-s\tau+(\xi(a)+i\omega)(g-a\tau)} \right][h(.), B_2(s, \omega, a, .)] \right)(y) d\nu(y).
\]

(3.2)

4. Meromorphic Extension of \( Z(s, \omega, a) \)

We assume \( f(x) \) and \( g(x), x \in U \), fixed as in Sect. 3. Introduce the Ruelle operator

\[ L_{s,\omega,a} = L_{f-s\tau+(\xi(a)+i\omega)(g-a\tau)}, s \in \mathbb{C}, \omega \in \mathbb{R}. \]

It is easy to see that for \( s = \gamma(a) \) and \( \omega = 0 \) we have

\[ \Pr_{\sigma}(f + \xi(a)(g - a\tau) - \gamma(a)\tau) = 0. \]

Indeed,

\[ \gamma(a) = \Pr (F + \xi(a)G) - \xi(a)a \]

and

\[ \Pr_{\sigma}(f + \xi(a)g - \Pr (F + \xi(a)G)\tau) = 0. \]

Notice that there is an unique number \( t \) such that

\[ \Pr_{\sigma}(f + \xi(a)(g - a\tau) - t\tau) = 0. \]

Since \( f, g, \tau \) depend only on \( x \in U \), as in Sect. 2.4 we deduce that

\[ \Pr_{\sigma}(f + \xi(a)(g - a\tau) - \gamma(a)\tau) = 0. \]

Below we will write simply \( \Pr \) instead of \( \Pr_{\sigma} \) if there are no confusions.

Set \( f_a = f + \xi(a)(g - a\tau) \) and consider the Ruelle operator \( L_{s,\omega,a} \), where \( s = \gamma(a) + q + ib, q \in \mathbb{R}, b \in \mathbb{R}, \omega \in \mathbb{R}. \) Let

\[ p(s, w, a) = f_a - s\tau + i\omega(g - a\tau), s = \gamma(a) + q + ib. \]
Since $\Pr p(\gamma(a), 0, a) = 0$, by a standard argument we may define $\Pr p(s, \omega, a)$ for $(s, \omega)$ in a small neighbourhood of $(\gamma(a), 0)$ in $\mathbb{C}^2$ (see [11]). Recall the following result proved in [20].

**Proposition 4.1.** (Proposition 4, [20]) Let $G^\alpha(R^\tau)$ be a function such that $G$ and $\omega^\tau$ are flow independent. Assume that $G$ is constant on stable leaves. Then,

(i) The function $Z(s, \omega, a)$ is analytic for $(s, \omega) \in \{s \in \mathbb{C} : \text{Res} > \gamma(a)\} \times \mathbb{R}$.

(ii) There exists an open neighbourhood $W$ of $(\gamma(a), 0)$ in $\mathbb{C}^2$ such that for $(s, \omega) \in W$ we have

$$Z(s, \omega, a) = \frac{B_3(s, \omega, a)}{1 - \exp\left(\Pr p(s, \omega, a)\right)} + J(s, \omega, a), \quad (4.1)$$

where

$$B_3(s, \omega, a) = \frac{1}{\tau d\mu} \int_R B_1(s, \omega, a, \cdot) h_{p(s, \omega, a)}(x) d\nu(x) \int h B_2(s, \omega, a, \cdot) d\nu_{p(s, \omega, a)}, \quad (4.2)$$

and $J(s, \omega, a)$ is analytic for $(s, \omega) \in W$. Here, $h_{p(s, \omega, a)}(x) > 0$ is the eigenfunction corresponding to the eigenvalue $e^{\Pr p(s, \omega, a)}$ of $L_{p(s, \omega, a)}$ and similarly the measure $\nu_{p(s, \omega, a)}$ is determined by the eigenvalue $e^{\Pr p(s, \omega, a)}$ of the dual operator $L^*_{p(s, \omega, a)}$.

(iii) $Z(s, \omega, a)$ is analytic for $(s, \omega)$ in an open neighbourhood $V_1$ of $\{s : \text{Res} = \gamma(a), s \neq \gamma(a)\} \times \{0\}$.

(iv) For each $\omega \in \mathbb{R} \setminus \{0\}$, $Z(s, \omega, a)$ is analytic for $(s, \omega)$ in an open neighbourhood $V_2$ of $\{s : \text{Res} = \gamma(a)\} \times \{\omega\}$.

For our analysis, we need to estimate the norms

$$\|B_1(s, \omega, a, x)\|_\infty, \|B_2(s, \omega, a, \cdot)\|_\beta.$$

The norm $\|B_2(s, \omega, a, \cdot)\|_\infty$ is easily estimated uniformly with respect to $\omega \in \mathbb{R}$, since

$$\left|\exp\left((s + a(\xi(a) + i\omega))\eta - (\xi(a) + i\omega)G^\eta(x, 0)\right)\right| \leq \exp\left(||\text{Res}| + a(\xi(a))|\eta + |\xi(a)||G^\eta(x, 0)\right)$$

and

$$\|B_2(s, \omega, a, x)\|_\infty \leq \exp\left(||\text{Res}| + a(\xi(a))|\eta_1 + |\xi(a)||\max_{x \in U} G(x, 0)\eta_1\right),$$

where $\eta_1 = \max_{x \in U} \tau(x)$. Similarly, one treats the norm $\|B_1(s, \omega, a, \cdot)\|_\infty$. For the norm $\|B_2(s, \omega, a, \cdot)\|_\beta$, we apply the following elementary estimate. Let

$$k(x) = \int_0^{\tau(x)} e^{(s + az)} e^{K^\eta(x, 0)} d\eta, \ K \in \mathcal{C}^\beta(R^\tau).$$

Then,

$$|k(x) - k(y)| = \left|\int_0^{\tau(x)} e^{(s + az)} e^{K^\eta(x, 0)} d\eta - \int_0^{\tau(y)} e^{(s + az)} e^{K^\eta(y, 0)} d\eta\right|$$
\[
\int_0^{\tau(x)} e^{\left|\text{Res} + a\xi(a)\right|\eta} \left| e^{K^\eta(x,0)} - e^{K^\eta(y,0)} \right| d\eta \\
+ \left| \int_{\tau(x)}^{\tau(y)} e^{(s+az)\eta} e^{K^\eta(y,0)} d\eta \right|.
\]

The second term in the right-hand side is estimated by

\[
\exp\left( \max_{0 \leq \eta \leq \kappa_1} |\text{Re}K^\eta| + |\text{Res} + a\xi(a)|\kappa_1 \right) |\tau(x) - \tau(y)|.
\]

For the first term on the right, we use the inequality

\[
|e^{z_1} - e^{z_2}| = \left| \int_{z_1}^{z_2} e^u du \right| \leq e^{\left|\text{Re}z_1\right| + \left|\text{Re}z_2\right| |z_1 - z_2|},
\]

and we obtain

\[
\int_0^{\tau(x)} e^{\left|\text{Res} + a\xi(a)\right|\eta} \left| e^{K^\eta(x,0)} - e^{K^\eta(y,0)} \right| d\eta \\
\leq \exp\left( 2\|\text{Re}K\|_\infty + |\text{Res} + a\xi(a)|\kappa_1 \right) \\
\times \int_0^{\tau(x)} |K^\eta(x,0) - K^\eta(y,0)| d\eta \\
\leq \exp\left( 2\|\text{Re}K\|_\infty + |\text{Res} + a\xi(a)|\kappa_1 \right) \int_0^{\tau(x)} \int_0^{\eta} |K(\sigma_t^\tau(x,0)) \\
- K(\sigma_t^\tau(y,0))| dt d\eta
\]

which yields an estimate for \(|B_2(s, \omega, a, \cdot)|_{\beta}\) uniformly with respect to \(\omega \in \mathbb{R}\).

For \((s, \omega) = (\gamma(a), 0)\), one has a maximal real eigenvalue 1 of \(L(\gamma(a), 0, a)\) and the rest of the spectrum is contained in a disk of radius \(0 < r < 1\). By perturbation theory, there exists an unique eigenvalue with maximal modulus of \(L_{s, \omega, a}\) given by

\[
\lambda_{s, \omega, a} = \exp\left( \Pr(p(s, \omega, a)) \right),
\]

defined for \((s, \omega) \in W\). We get

\[
\frac{\partial \lambda_{s, \omega, a}}{\partial s} \bigg|_{(s, \omega, a) = (\gamma(a), 0, a)} = - \int \tau d\mu_{f_a} < 0,
\]

where \(\mu_{f_a}\) is the equilibrium state of \(f_a\). By the implicit function theorem (see Lemma 3 in [20]) for small \(\epsilon_1 > 0\), we may determine \(s = s(\omega, a), |\omega| \leq \epsilon_0\), from the equation \(\lambda_{s, \omega, a} = 1\) so that

\[
\lambda_{s(\omega, a), \omega, a} = 1, s(0, a) = \gamma(a).
\]

Therefore,

\[
\frac{s - s(\omega, a)}{1 - \exp\left( \Pr(p(s, \omega, a)) \right)} = \left( \int \tau d\nu_{f_a - s(\omega, a)\tau + i\omega g} \right)^{-1} + O(s - s(\omega, a)).
\]
This shows that we have a pole at \( s = s(\omega, a) \) and taking the residue at \( s(\omega, a) \), the singular term in (4.1) becomes

\[
\left( \frac{B_3(s(\omega, a), \omega, a)}{\int \tau d\nu_{f_a-s(\omega, a)\tau+\omega g}} \right) \frac{1}{s - s(\omega, a)}.
\]

Now we will show that \( Z(s, \omega, a) \) has an meromorphic continuation across the line \( \text{Res} = \gamma(a) \). First note that for \( a = \int_U G(y) dm_{F + \xi(a)} G \) with a Hölder function \( G \geq g_0 > 0 \) on \( U \) one has

\[
g_0 \leq a \leq \max_{y \in U} G(y) = m.
\]

Let \( 0 < \eta < g_0/2 \) be a fixed number. We will apply the spectral estimates for the operator \( L_{s,\omega,a} \) given in Theorem 2.1 in Sect. 2 (see [14]). It is possible to write \( L_{s,\omega,a} \) in two different forms

\[
L_{1,s,\omega,a} = L_{h_a-(\text{Res} - \gamma(a))\tau - i\text{Im}\omega(g-a\tau)},
\]

\[
L_{2,s,\omega,a} = L_{h_a-(\text{Res} - \gamma(a))\tau - i(\text{Im}\omega + aw)\tau + \omega g},
\]

where \( h_a = f_a - \gamma(a)\tau \) and \( \text{Pr}_\sigma(h_a) = 0 \). In the operators \( L_{k,s,\omega,a}, k = 1,2 \), we have different factors \(-i\text{Im}\omega\) and \(-i(\text{Im}\omega + aw)\) in front of \( \tau \). Applying Theorem 2.1, we can find \( a_0 > 0 \) and constants \( 0 < \rho < 1, M > 0 \) such that for \( |\text{Res} - \gamma(a)| \leq a_0 \) and any \( \nu > 0 \) we have

\[
||L_{1,s,\omega,a}h||_{\beta,\text{Im}\omega} \leq C(\nu, B_1)\rho^m|\text{Im}\omega|^{\nu}\|h\|_{\beta,\text{Im}\omega}
\]

for \( |\text{Im}\omega| \geq M, |\omega| \leq B_1|\text{Im}\omega| \),

\[
||L_{2,s,\omega,a}h||_{\beta,|\text{Im}\omega + aw|} \leq D(\nu, B_2)\rho^m|\text{Im}\omega + aw|^{\nu}\|h\|_{\beta,|\text{Im}\omega + aw|}
\]

for \( |\text{Im}\omega + aw| \geq M, |\omega| \leq B_2|\text{Im}\omega + aw| \). Let us remark that we can take the same constants \( a_0, \rho \) and \( M \) in both estimates above, since if we have constants

\[
a_k > 0, 0 < \rho_k < 1, M_k > 0, k = 1,2
\]

for the operators \( L_{k,s,\omega,a} \), we can choose

\[
a_0 = \min\{a_1, a_2\}, \rho = \max\{\rho_1, \rho_2\}, M = \max\{M_1, M_2\}.
\]

On the other hand, the constants \( C(\nu, B_1) \) and \( D(\nu, B_2) \) depend on \( (\nu, B_1) \) and \( (\nu, B_2) \), respectively.

**Proposition 4.2.** Assume the assumptions of Theorem 1.3 fulfilled. Then, for any Hölder continuous functions \( F, G \in C^\alpha(\hat{R}^+) \) there exist \( \mu_0 > 0 \) and \( \epsilon_0 > 0 \) such that the function \( Z(s, \omega, a) \) admits a meromorphic continuation for

\[
(s, \omega) \in \{ (s, \omega) \in \mathbb{C}^2 : \text{Res} \geq \gamma(a) - \mu_0, \omega \in \mathbb{R} \}
\]

with only one simple pole at \( s(\omega, a), |\omega| \leq \epsilon_0 \). The pole \( s(\omega, a) \) is determined as the root of the equation \( \text{Pr}(f_a - s\tau + i\omega(g-a\tau)) = 0 \) with respect to \( s \) for \( |\omega| \leq \epsilon_0 \). Moreover, there exist constants \( \eta > 0, M > 0 \) such that for any \( \nu > 0 \) if \( |\text{Im}\omega| \geq M \) or \( |\omega| \geq \frac{1}{\eta}M \), we have the estimate

\[
|Z(s, \omega, a)| \leq B_\nu(|\text{Im}\omega|^{\nu} + |\omega|^{\nu}), \text{Res} \geq \gamma(a) - \mu_0,
\]
uniformity with respect to \(a \in J\) in a compact interval \(J \subset \Gamma_G\) with a constant \(B_\nu > 0\) independent on \(s, \omega\) and \(a \in J\).

**Proof.** We suppose below that \(|\text{Res} - \gamma(a)| \leq a_0\), since for \(\text{Res} > \gamma(a) + a_0\) the result follows from Proposition 4.1, (i). Consider three cases.

**Case 1.** \((\text{Im} s, \omega) \in D_M = \{ |\text{Im} z| \leq M, |\omega| \leq \frac{1}{\eta} M \}\).

For \((\text{Im} s, \omega) \in B_{\epsilon_0} = \{ |\text{Im} z| < \epsilon_0, |\omega| < \epsilon_0 \}\) the result follows from Proposition 4.1, (ii). So assume that \((\text{Im} s, \omega) \in D_M \setminus B_{\epsilon_0}\).

In this situation, we may apply the statement (iii) of Proposition 4.1. For reader’s convenience, we present a proof. Let \((s_0, w_0)\) with \((\text{Im} s_0, \omega_0) \in D_M \setminus B_{\epsilon_0}\) be fixed. Assume first that \(\text{Imp}(s_0, \omega_0, a)\) is cohomologous to \(c + 2\pi Q\) with an integer-valued function \(Q \in C(U; \mathbb{Z})\) and a constant \(c \in [0, 2\pi]\). Then, we define the pressure \(\text{Pr}(p(s_0, \omega_0, a)) = \text{Pr}(f_a) + c\) and we extend the pressure in a small neighbourhood of \((s_0, \omega_0)\). Since \(G\) and \(\sigma^*_t\) are flow independent, the functions \(g\) and \(\tau\) are \(\sigma\)-independent. If we have \(c = 0\), from the fact that \(\text{Im} s_0 \tau + \omega_0 \theta\) is cohomologous to a function in \(C(U; 2\pi \mathbb{Z})\), we deduce \(\text{Im} s_0 = \omega_0 = 0\) which is impossible. Thus, we have \(c \neq 0\). Consequently, the operator \(\mathcal{L}_{s_0, \omega_0, a}\) has an eigenvalue \(e^c\). Then, there exists a neighbourhood \(U_2\) of \((s_0, \omega_0)\) such that for \((s, \omega) \in U_2\) we have \(\text{Pr}(p(s, \omega, a)) \neq 0\) and for \((s, \omega) \in U_2\) we have an analytic extension of \(Z(s, \omega, a)\) given by

\[
Z(s, \omega, a) = \left[ \frac{B_4(s, \omega, a)}{1 - e^{\text{Pr}(p(s, \omega, a))}} + J_2(s, \omega, a) \right]
\]

with a function \(J_2(s, \omega, a)\) analytic with respect to \(s\) for \((s, \omega) \in U_2\). Second, let \(\text{Imp}(s_0, \omega_0, a)\) be not cohomologous to \(c + 2\pi Q\). Then, the spectral radius of \(\mathcal{L}_{s_0, \omega_0, a}\) is strictly less than 1, and this will be the case for \((s, \omega)\) is a small neighbourhood \(U_3\) of \((s_0, \omega_0)\). Therefore, it is easy to see that the series in (3.2) is absolutely convergent, and we obtain again an analytic extension. Covering the compact set \(D_M \setminus B_{\epsilon_0}\) by a finite number of neighbourhoods, we may choose \(\mu_0 > 0\) small so that for \(\gamma(a) - \mu_0 \leq \text{Res} \leq \gamma(a), (\text{Im} s, \omega) \in D_M \setminus B_{\epsilon_0}\), we have an analytic extension of \(Z(s, \omega, a)\) in (4.5).

**Case 2.** \(|\text{Im} s| \leq M, |\omega| > \frac{1}{\eta} M > \frac{2}{\eta_0} M\).

Notice that \(\frac{2M}{\eta_0} \geq \frac{2M}{a}\). We consider the operator \(\mathcal{L}_{2, s, \omega, a}\) and observe that

\[
|\text{Im} s + a\omega| \geq a|\omega| - |\text{Im} s| \geq M
\]

and also

\[
|\text{Im} s + a\omega| \geq \frac{a}{2} |\omega| + \frac{a}{2} |\omega| - |\text{Im} s| \geq \frac{a}{2} |\omega|.
\]

Hence, \(|\omega| \leq \frac{2}{a} |\text{Im} s + a\omega| \leq \frac{2}{\eta_0} |\text{Im} s + a\omega|\).

To apply the estimate (4.4) with \(B_2 = \frac{2}{\eta_0}\) to the series in (3.2), we must estimate the norm

\[
\|hB_2(s, \omega, a, \cdot)\|_{\beta, \text{Im} s + a\omega}
\]
uniformly with respect to $\omega \in \mathbb{R}$. The norm $\|B_2(s, \omega, a, .)\|_\beta$ has been estimated above. Next, one gets
\[ \left| \frac{1}{\text{Im}s + a\omega} \right| \leq \frac{1}{M} \]
and we deduce the needed estimate.

Now, the series in (3.2) is absolutely convergent, and we obtain an analytic extension of $Z(s, w, a)$ for $|\text{Res} - \gamma(a)| \leq a_0$, $|\text{Im}s| \geq M$, $|\omega| > \frac{1}{\eta}M$ as well as the estimate
\[ |Z(s, \omega, a)| \leq C_{\nu,B_2}|\text{Im}s + a\omega|^\nu \leq C_{\nu,B_2,M}(1 + |\omega|)^\nu. \quad (4.7) \]
Decreasing, if is necessary, $\mu_0$ we obtain an analytic extension in (4.5).

**Case 3.** $|\text{Im}s| > M$.

We consider two subcases:

**Subcase 3a.** $|\omega| \leq \frac{1}{\eta}|\text{Im}s|$. We work with the operator $L_{1,s,\omega,a}$. For every $\nu > 0$ with $b = \text{Im}s$, $B_1 = \frac{1}{\eta}$ and $|\text{Im}s| \geq M$, one obtains from (4.3) the spectral estimates
\[ \|L_{s,\omega,a}^m h\|_{\beta,b} \leq C(\nu,\eta)\rho^m|\text{Im}s|^\nu\|h\|_{\beta,b}, \quad m \in \mathbb{N}. \quad (4.8) \]
The series (3.2) is absolutely convergent, and we deduce
\[ |Z(s, \omega, a)| \leq \dot{C}(\nu,\eta)|\text{Im}s|^\nu. \]

**Subcase 3b.** $|\omega| > \frac{1}{\eta}|\text{Im}s|$. We work now with the operator $L_{2,s,\omega,a}$. In this case
\[ |\text{Im}s + a\omega| \geq a|\omega| - |\text{Im}s| \geq (a - \eta)|\omega| \geq \frac{1}{2}g_0|\omega|. \]
Hence,
\[ |\omega| \leq \frac{2}{g_0} |\text{Im}s + a\omega|. \]
On the other hand, for $|\omega| > \frac{1}{\eta}|\text{Im}s| \geq \frac{1}{\eta}M$ we have
\[ |\text{Im}s + a\omega| \geq \frac{g_0}{2}|\omega| \geq \frac{g_0}{2\eta}M \geq M, \]
because $2\eta < g_0$. Therefore, for any $\nu > 0$ we can use (4.4) with $B_2 = \frac{2}{g_0}$ and obtain
\[
\|L_{s,\omega,a}^m h\|_{\beta,\text{Im}s+a\omega} \leq D(\nu, g_0)\rho^m|\text{Im}s + a\omega|^\nu\|h\|_{\beta,\text{Im}s+a\omega} \\
\leq D(\nu, g_0, M)\rho^m(1 + |\omega|)^\nu\|h\|_{\beta,\text{Im}s+a\omega}, \quad m \in \mathbb{N}. \quad (4.9)
\]
These estimates lead again to (4.7), and the proof is complete. □
5. Tauberian Theorem

In this section, we prove a Tauberian theorem for a sequence of functions \( \{g_n(t)\}_{n \in \mathbb{N}} \) similar to that in [8] (see also Proposition 6 (i) in [20]). The novelty here is that the leading terms contain a factor \( A_n \geq e^{-\mu n} \) with \( \mu > 0 \) which can converge to 0 exponentially fast and the remainders must be smaller than this leading term. Moreover, we have two limits \( n \to \infty \) and \( t \to \infty \), and this creates new difficulties. Under some assumptions on the Laplace transform of \( g_n(t) \), stronger that those in [8], we obtain an asymptotic for \( t \geq n - q \) and \( n \to \infty \).

**Proposition 5.1.** Let \( g_n(t), n \in \mathbb{N}, \) be monotonic nondecreasing functions defined for \( t \in [0, \infty) \) such that \( g_n(0) = 0, n \in \mathbb{N} \). Assume that for any \( n \in \mathbb{N} \) the Laplace transform

\[
F_n(s) = \int_0^\infty e^{-st}g_n(t)dt
\]

is analytic for \( \text{Res} > 1 \). Assume that there exist \( \mu_0 > 0 \) and \( M > 0, C_0 > 0, C_1 > 0, \delta_0 > 0 \) such that \( F_n(s) \) has a representation

\[
F_n(s) = \frac{A_n}{\sqrt{s-1}} + A_nK_n(s) + L_n(s),
\]

where \( C_0e^{-\mu n} \leq A_n \leq C_1, 0 < \mu \leq \mu_0/4, \forall n \in \mathbb{N}, K_n(s) \) are analytic functions for \( \text{Res} > 1 \) and for \( t \in \mathbb{R} \) and \( s = 1 + \delta + it, 0 < \delta \ll 1, K_n(s) \) has a limit \( k_n(1 + it) \in W_{1,1}^{1,1}(\mathbb{R}) \) (functions which are locally integrable and have locally integrable derivatives) almost everywhere on \( \mathbb{R} \) as \( \delta \downarrow 0 \) and \( |k_n(1 + \delta + it)| \leq k_0(t) \), where \( k_0(t) \) is locally integrable, while \( L_n(s) \) has an analytic continuation for \( 1 - \mu_0 \leq \text{Res} \leq 1 + \delta_0 \). Moreover, assume that the functions \( K_n(s) \) have analytic continuations to \( 1 - \mu_0 \leq \text{Res} \leq 1 + \delta_0 \), \( |\text{Im}s| \geq M \), and for every compact set \( D \subset \mathbb{R} \) uniformly with respect to \( n \), we have

\[
\|k'_n(1 + it)\|_{L^1(D)} \leq C(D), \forall n \in \mathbb{N}.
\]

Next assume that for any \( 0 < \nu < 1 \) with a constant \( B(\nu) > 0 \) independent on \( n \) we have for any \( n \in \mathbb{N} \) the estimates

\[
\left| \frac{d^k}{ds^k} L_n(s) \right| \leq B(\nu)(1 + |\text{Im}s|^\nu), 1 - \mu_0 \leq \text{Res} \leq 1 + \delta_0, k = 0,1, \tag{5.2}
\]

\[
\left| \frac{d^k}{ds^k} K_n(s) \right| \leq B(\nu)(1 + |\text{Im}s|^\nu), 1 - \mu_0 \leq \text{Res} \leq 1 + \delta_0, k = 0,1, |\text{Im}s| \geq M. \tag{5.3}
\]

Then, for fixed \( q \geq 0 \) and for any \( 0 < \eta \ll 1 \) there exists \( n_0(\eta) \in \mathbb{N} \) such that for \( t \geq n - q \) and \( n \geq n_0(\eta) + q \) we have

\[
\frac{A_n e^t}{\sqrt{\pi t}}(1 - \eta) \leq g_n(t) \leq \frac{A_n e^t}{\sqrt{\pi t}}(1 + \eta). \tag{5.4}
\]

**Remark 5.2.** Notice that if the estimates (5.2), (5.3) are satisfied for \( \mu_0 > 0 \) and \( k = 0 \), then fixing numbers \( 0 < \mu_1 < \mu_0, 0 < \delta_1 < \delta_0 \), we obtain the estimates (5.2) (resp.(5.3)) for \( 1 - \mu_1 \leq \text{Re} \leq 1 + \delta_1 \), (resp. for \( 1 - \mu_1 \leq \text{Res} \leq 1 + \delta_1 \)).
$1 + \delta_1, \text{Im}s \geq M$ with another constants $B_1(\nu)$ by applying the Cauchy formula for the first derivative of the analytic functions $K_n(s)$ and $L_n(s)$. In the proof, we use only the estimates with $\nu = 1/3$, but if we change the relation between $\mu, \mu_0$ and $t \geq \beta(n)$, we need estimates with different $\nu$. In Sect. 6 we establish (5.2), (5.3).

**Proof.** We follow the proof in [8] with a more precise analysis concerning the dependence of $n$. If we replace the function $g_n(t)$ by $\tilde{g}_n(t) = g_n(t)\sqrt{t}$, one obtains

$$F_n(s) = \int_0^\infty \frac{1}{\sqrt{t}} e^{-st} \tilde{g}_n(t) dt.$$  

For simplicity of the notations below, we will denote $\tilde{g}_n(x)$ by $g_n(x)$. Now introduce the function $H_n(x) = g_n(x)e^{-x}$ and for $s = 1 + \epsilon + it$ define

$$K_{\epsilon,n}(t) = F_n(s) - \frac{A_n}{\sqrt{s - 1}}.$$  

After a change of variable $x = v^2$, one obtains

$$K_{\epsilon,n}(t) = 2 \int_0^\infty e^{-(s-1)v^2} \left( H_n(v^2) - \frac{A_n}{\sqrt{\pi}} \right) dv.$$  

Consequently,

$$K_{\epsilon,n}(t) = \lim_{\xi \to \infty} \frac{2}{\xi} \int_0^\xi \left( H_n(v^2) - \frac{A_n}{\sqrt{\pi}} \right) e^{-(s-1)v^2} dv$$

and for fixed $n$ and fixed $\epsilon$ this limit is uniform for $|t| \leq 2\lambda$. We multiply $K_{\epsilon,n}(t)$ by $\sqrt{y}e^{ity} \left( 1 - \frac{|t|}{2\lambda} \right)$ and integrate over $t$ in $[-2\lambda, 2\lambda]$. Thus

$$\int_{-2\lambda}^{2\lambda} \sqrt{y} \left( 1 - \frac{|t|}{2\lambda} \right) K_{\epsilon,n}(t)e^{ity} dt$$

$$= \lim_{\xi \to \infty} \frac{2}{\xi} \int_{-2\lambda}^{2\lambda} \sqrt{y}e^{ity} \left( 1 - \frac{|t|}{2\lambda} \right) \left( \int_0^\xi \left( H_n(v^2) - \frac{A_n}{\sqrt{\pi}} \right) e^{-\epsilon v^2 - itv^2} dv \right) dt.$$  

As in [8], we interchange the limit $\xi \to \infty$ and the integration and write the right-hand side of the last term as

$$2 \int_0^\infty \left( H_n(v^2) - \frac{A_n}{\sqrt{\pi}} \right) e^{-\epsilon v^2} \left( \int_{-2\lambda}^{2\lambda} \sqrt{y}e^{i\lambda(y-v^2)}u \left( 1 - \frac{|u|}{2} \right) du \right) dv.$$  

We change the variable $v = \sqrt{y - \frac{w}{\lambda}}$ and the last term becomes

$$\int_{-\infty}^{\lambda y} \left( H_n \left( y - \frac{w}{\lambda} \right) - \frac{A_n}{\sqrt{\pi}} \right) e^{-\epsilon(y-w/\lambda)} \sqrt{y} \left( \int_{-2\lambda}^{2\lambda} \left( 1 - \frac{|u|}{2} \right) e^{iuw} du \right) \frac{dw}{\sqrt{y - \frac{w}{\lambda}}}$$

$$= 2 \int_{-\infty}^{\lambda y} H_n \left( y - \frac{w}{\lambda} \right) e^{-\epsilon(y-w/\lambda)} \frac{\sin^2 w}{w^2} \frac{\sqrt{y}}{\sqrt{y - \frac{w}{\lambda}}} dw$$

$$- \frac{2A_n}{\sqrt{\pi}} \int_{-\infty}^{\lambda y} \frac{\sin^2 w}{w^2} \sqrt{y - \frac{w}{\lambda}} e^{-\epsilon(y-w/\lambda)} dw.$$  

Now, as in [8], we take the limit $\epsilon \searrow 0$ and set $K_{0,n}(t) = \lim_{\epsilon \searrow 0} K_{\epsilon,n}(t)$. By the Lebesgue convergence theorem and Sub-Lemma 4.5 in [8], we obtain

$$
\lim_{y \to \infty} \frac{1}{2} \int_{-2\lambda}^{2\lambda} K_{0,n}(t)(1 - |t|/2\lambda) \sqrt{y}e^{it\varphi}dy + A_n \sqrt{\pi}
$$

By using an integration by parts and the fact that $\eta > \omega$, the equality $L_{\infty} \leq 1$ and fixed $\lambda > 1$ if we wish to arrange the inequality

$$
\eta \geq y \geq 1 + \frac{1}{\lambda} - \frac{1}{\lambda} \lambda
$$

we must take $y \geq Y(\eta, \lambda, n)$ and we may have $Y(\eta, \lambda, n) \to \infty$ as $n \to \infty, \lambda \to \infty$. By using the representation (5.1) and the estimates (5.2), (5.3), we will prove a more precise result.

**Lemma 5.3.** Let $q \geq 0$ be fixed and let $y \geq n - q, \lambda = \lambda_n = e^{1/\mu_n}$. Then, for any $\eta > 0$ there exists $n_0(\eta) \in \mathbb{N}$ such that for all $n \geq n_0(\eta) + q$ we have (5.6).

**Proof.** First, we treat the integral

$$
J_n(y) = \int_{-2\lambda}^{2\lambda} \mathcal{L}(1 + it)(1 - |t|/2\lambda) \sqrt{y}e^{it\varphi}dt,
$$

where $L_n(1 + it) = \lim_{\delta \searrow 0} L_n(1 + \delta + it)$. We write this integral as follows:

$$
J_n(y) = -\int_{\gamma_1} iL_n(s)(1 - s - 1/2\lambda_n) \sqrt{y}e^{(s-1)y}ds - \int_{\gamma_2} iL_n(s)(1 + s - 1/2\lambda_n) \sqrt{y}e^{(s-1)y}ds,
$$

where

$$
\gamma_1 = \{s \in \mathbb{C} : s = 1 + it, 0 \leq t \leq 2\lambda_n\}, \quad \gamma_2 = \{s \in \mathbb{C} : s = 1 + it, -2\lambda_n \leq t \leq 0\}.
$$

Since $L_n(s)$ has an analytic continuation for $1 - \mu_0 \leq \text{Re} s \leq 1 + \delta_0$, we have the equality

$$
\int_{\gamma_1} + \int_{\omega_{1,1}} + \int_{\omega_{1,2}} + \int_{\omega_{1,3}} = 0,
$$

where the function under integration is $-iL_n(s)(1 - s - 1/2\lambda_n) \sqrt{y}e^{(s-1)y}$ and

$$
\omega_{1,1} = \{s \in \mathbb{C} : s = z + 2i\lambda_n, 1 - \mu_0 \leq z \leq 1\},
$$

$$
\omega_{1,2} = \{s \in \mathbb{C} : s = 1 - \mu_0 + it, 0 \leq t \leq 2\lambda_n\},
$$

$$
\omega_{1,3} = \{s \in \mathbb{R} : 1 - \mu_0 \leq s \leq 1\}.
$$
with suitable orientation (see Fig. 1). The integral over $\omega_{1,2}$ has the form

$$-e^{-\mu_0 y} \sqrt{y} \int_0^{2\lambda_n} L_n(1 - \mu_0 + it) \left(1 + \frac{\mu_0 - it}{2i\lambda_n}\right) e^{ity} dt.$$ 

We integrate by parts and one obtains

$$\frac{e^{-\mu_0 y}}{i\sqrt{y}} \int_0^{2\lambda_n} \frac{d}{dt} \left[L_n(1 - \mu_0 + it) \left(1 + \frac{\mu_0 - it}{2i\lambda_n}\right)\right] e^{ity} dt$$

$$-e^{-\mu_0 y} \left(\frac{\mu_0}{2i\lambda_n} L_n(1 - \mu_0 + 2i\lambda_n) e^{2i\lambda_n y} - \left(1 + \frac{\mu_0}{2i\lambda_n}\right) L_n(1 - \mu_0)\right).$$

The function $L_n(1+it)$ has an analytic continuation $L_n(z+it)$ for $1-\mu_0 \leq z \leq 1+\delta_0$ and for any $0 < \nu < 1$ and any $n$ we have the estimate

$$\left|\frac{d}{dt} L_n(z+it)\right| \leq B(\nu)(1 + |t|^\nu), \quad 1 - \mu_0 \leq z \leq 1 + \delta_0.$$ 

Thus, for $y \geq n - q$ and large $n$ one gets

$$\left|\frac{e^{-\mu_0 y}}{i\sqrt{y}} \int_0^{2\lambda_n} \frac{d}{dt} \left[L_n(1 - \mu_0 + it) \left(1 + \frac{\mu_0 - it}{2i\lambda_n}\right)\right] e^{ity} dt\right|$$

$$\leq C_{1,2}(\nu) \frac{e^{-\mu_0 y}}{\sqrt{y}} \lambda_n^{1+\nu} \leq C_{1,2}(\nu) e^{-\mu_0 (n-q)} e^{\frac{1}{2} (1+\nu) \mu_0 n}$$

$$= C_{1,2}(\nu) e^{\mu_0 q} e^{(-\frac{1}{2}+\frac{1}{2}\nu)\mu_0 n}.$$ 

We choose $\nu = 1/3$, and for the last term of the above inequality, one obtains a bound $O(e^{-\frac{1}{2}\mu_0 n})$. Since $A_n \geq C_0 e^{-\frac{5}{6} n}$, we obtain a term $A_n o(n)$. The boundary terms are easily estimated, and we get

$$\left|\int_{\omega_{1,2}}\right| \leq C_{1,2} e^{-\frac{1}{4}\mu_0 n}. \quad (5.7)$$
Passing to the integral over \( \omega_{1,1} \), notice that for \( s \in \omega_{1,1} \), we have
\[
1 - \left( \frac{s - 1}{2i\lambda_n} \right) = \frac{1 - z}{2i\lambda_n}, \quad |e^{(s-1)y}| \leq e^{(z-1)y} \leq 1, \quad \text{Res} = z.
\]
We integrate by parts with respect to \( z \) and deduce
\[
|\int_{\omega_{1,1}}| \leq A_{1,1}(\nu) \frac{e^{-\mu_n y}}{\sqrt{y}} \frac{(1 + \lambda_n^2)}{\lambda_n} + \frac{1}{2\sqrt{y}\lambda_n} \int_{1-\mu_0}^{1} \left| \frac{d}{dz} [(1 - z)L_n(z + 2i\lambda_n)] \right| dz.
\]
Therefore, applying (5.2) for the second term in the right-hand side, one obtains
\[
|\int_{\omega_{1,1}}| \leq C_{1,1}(\nu) \frac{e^{-\frac{1}{2}(1-\nu)\mu_n y}}{\sqrt{n}} \leq C_{1,1}e^{-\frac{1}{2}\mu_n y}, \quad (5.8)
\]
choosing \( \nu = 1/3 \). Before treating the integral over \( \omega_{1,3} \), consider the equality
\[
\int_{\gamma_2} + \int_{\omega_{2,1}} + \int_{\omega_{2,2}} + \int_{\omega_{2,3}} = 0,
\]
where the function under integration is \(-iL_n(s)\left(1 + \frac{s-1}{2i\lambda_n}\right)e^{(s-1)y}\) and
\[
\omega_{2,1} = \{s \in \mathbb{C} : s = z - 2i\lambda_n, 1 - \mu_0 \leq z \leq 1\},
\omega_{2,2} = \{s \in \mathbb{C} : s = 1 - \mu_0 + it, -2\lambda_n \leq t \leq 0\}, \quad \omega_{2,3} = \{s \in \mathbb{R} : 1 - \mu_0 \leq s \leq 1\}
\]
with suitable orientation (see Fig. 1). In particular, the curves \( \omega_{1,3} \) and \( \omega_{2,3} \) coincide, but they have inverse orientations. The analysis of \( \int_{\omega_{2,2}} \) is completely similar and one obtains (5.7). For \( s \in \omega_{2,1} \) we have
\[
1 + \left( \frac{s - 1}{2i\lambda_n} \right) = \frac{z - 1}{2i\lambda_n}, \quad |e^{(s-1)y}| \leq e^{(z-1)y} \leq 1
\]
and as above one has (5.8). Now, we take the sum of the integrals over \( \omega_{1,3} \) and \( \omega_{2,3} \), and we are going to estimate the integral
\[
\frac{1}{\lambda_n} \int_{1-\mu_0}^{1} (z - 1)\sqrt{y}L_n(z)e^{(z-1)y}dz.
\]
We integrate by parts and the analysis is reduced to the integral
\[
\frac{1}{\sqrt{y}\lambda_n} \int_{1-\mu_0}^{1} e^{(z-1)y} \frac{d}{dz} \left((z - 1)L_n(z)\right) dz
\]
which can be estimated by \( C_{1,3}e^{-\frac{1}{2}\mu_n y} \).
Next, we pass to the analysis of the integral
\[
I_n(y) = \int_{-2\lambda_n}^{2\lambda_n} k_n(1 + it)\left(1 - \frac{|t|}{2\lambda_n}\right)\sqrt{y}e^{ity}dt,
\]
where
\[
k_n(1 + it) = \lim_{\delta \to 0} K_n(1 + \delta + it).
\]
Our purpose is to show that for \( y \geq n - q \) and \( \lambda_n = e^{-\frac{t}{2 \mu_0 n}} \) for any \( 0 < \eta \ll 1 \) there exists \( n_0(\eta) \) such that for \( n \geq n_0(\eta) + q \) we have \( |I_n(y)| < \eta \). We integrate by parts with respect to \( t \) and deduce

\[
I_n(y) = -\frac{1}{i \sqrt{y}} \int_{-2\lambda_n}^{2\lambda_n} \frac{d}{dt} \left[ \left( 1 - \frac{|t|}{2\lambda_n} \right) k_n(1 + it) \right] e^{ity} dt
\]

\[
= \frac{i}{\sqrt{y}} \left( \int_{-M-1}^{M+1} + \int_{-2\lambda_n}^{-M-1} + \int_{M+1}^{2\lambda_n} \right).
\]

The integral over \([-M - 1, M + 1]\) can be estimated taking \( n \) large by using the factor \( \frac{1}{\sqrt{y}} \) and the fact that by hypothesis

\[
\|k_n(1 + it)\|_{L^1([-M-1,M+1])} + \|k_n'(1 + it)\|_{L^1([-M-1,M+1])} \leq C(M), \forall n \in \mathbb{N}.
\]

For the other two integrals, we apply the argument used above exploiting the analytic continuation of \( k_n(s) \) for \( 1 - \mu_0 \leq \text{Re}s \leq 1, |\text{Im}s| \geq M \). We treat below only the integral over \([M + 1, 2\lambda_n]\), the analysis of the other one is very similar. We have

\[
\frac{1}{\sqrt{y}} \int_{M+1}^{2\lambda_n} \frac{d}{dt} \left[ \left( 1 - \frac{t}{2\lambda_n} \right) k_n(1 + it) \right] e^{ity} dt
\]

\[
= -\frac{1}{2\lambda_n \sqrt{y}} \int_{M+1}^{2\lambda_n} k_n(1 + it) e^{ity} dt
\]

\[
+ \frac{1}{\sqrt{y}} \int_{M+1}^{2\lambda_n} \left( 1 - \frac{t}{2\lambda_n} \right) \frac{d}{dt} k_n(1 + it) dt.
\]

We write the term on the right-hand side as follows:

\[
\frac{i}{2\lambda_n \sqrt{y}} \int_{\beta_1} k_n(s) e^{(s-1)y} ds - \frac{1}{\sqrt{y}} \int_{\beta_1} \left( 1 - \frac{s - 1}{2i\lambda_n} \right) \frac{d}{ds} (k_n(s)) ds,
\]

where

\[
\beta_1 = \{ s \in \mathbb{C} : s = 1 + it, M + 1 \leq t \leq 2\lambda_n \}.
\]
The integral is equal to a sum of three integrals over the curves
\[\beta_{1,1} = \{ s \in \mathbb{C} : s = z + 2i\lambda_n, 1 - \mu_0 \leq z \leq 1 \},\]
\[\beta_{1,2} = \{ s \in \mathbb{C} : s = 1 - \mu_0 + it, M + 1 \leq t \leq 2\lambda_n \},\]
\[\beta_{1,3} = \{ s \in \mathbb{C} : s = z + i(M + 1), 1 - \mu_0 \leq z \leq 1 \}\]
with suitable orientation (see Fig. 2). For the integral over \(\beta_{1,2}\), we obtain an estimate \(O\left(e^{-\mu_0 y^{1+\nu} \sqrt{y}}\right) = O\left(e^{-\frac{1}{3} \mu_0 n}\right)\), choosing \(\nu = 1/3\), while for that over \(\beta_{1,1}\) one deduces an estimate \(O(\lambda_n^{-1+\nu} \sqrt{y})\) which yields the same bound. To investigate the integral over \(\beta_{1,3}\), we use the factor \(\frac{1}{\sqrt{y}}\) and the fact that \(M\) is fixed. This completes the proof of Lemma 5.3. \(\square\)

**Lemma 5.4.** For any \(0 < \eta \ll 1\), there exists \(n_0(\eta) \in \mathbb{N}\) such that for \(y \geq 1\), \(\lambda_n = e^{\frac{1}{3} \mu_0 n}\) we have
\[
\left| \int_{-\infty}^{\lambda_n y} \frac{\sqrt{y}}{\sqrt{y} - w/\lambda_n} \frac{\sin^2 w}{w^2} \frac{\sqrt{y}}{\sqrt{y} - w/\lambda_n} dw - \pi \right| < \eta, n \geq n_0(\eta). \tag{5.9}
\]

The proof is a repetition of the proof of Sub-Lemma 4.5 in [8], and we leave the details to reader.

Combining Lemmas 5.3 and 5.4, one obtains from the equality (5.5) that for fixed \(q \geq 0\) and any \(0 < \eta \ll 1\) there exists \(n_0(\eta) \in \mathbb{N}\) such that for \(y \geq n - q\) and \(n \geq n_0(\eta) + q\) we have
\[
A_n\sqrt{\pi}(1 - \eta) \leq \int_{-\infty}^{\lambda_n y} H_n\left(y - \frac{w}{\lambda_n}\right) \frac{\sin^2 w}{w^2} \frac{\sqrt{y}}{\sqrt{y} - w/\lambda_n} dw < A_n\sqrt{\pi}(1 + \eta). \tag{5.10}
\]

Passing to the function \(H_n(y)\), we have the following

**Lemma 5.5.** Let \(q \geq 0\) be fixed. For any \(0 < \eta \ll 1\), there exists \(n_0(\eta) \in \mathbb{N}\) such that for \(y \geq n - q\) and \(n \geq n_0(\eta) + q\) we have
\[
\frac{A_n}{\sqrt{\pi}}(1 - \eta) \leq H_n(y) \leq \frac{A_n}{\sqrt{\pi}}(1 + \eta). \tag{5.11}
\]

**Proof.** Since \(H_n(y)\) is nonnegative, from (5.10) for \(y \geq n - q\), \(n \geq n_0(\eta) + q\) it follows that
\[
\int_{-\sqrt{\lambda_n}}^{\sqrt{\lambda_n}} H_n\left(y - \frac{w}{\lambda_n}\right) \frac{\sin^2 w}{w^2} \frac{\sqrt{y}}{\sqrt{y} - w/\lambda_n} dw \leq A_n\sqrt{\pi}(1 + \eta). \tag{5.12}
\]
By the monotonicity of \(g_n(w)\) for \(w \in [-\sqrt{\lambda_n}, \sqrt{\lambda_n}]\), we deduce
\[
H_n\left(y - \frac{1}{\sqrt{\lambda_n}}\right) \geq H_n\left(y - \frac{1}{\sqrt{\lambda_n}}\right) \exp\left(-\frac{2}{\sqrt{\lambda_n}}\right),
\]
hence,
\[
H_n\left(y - \frac{1}{\sqrt{\lambda_n}}\right) \exp\left(-\frac{2}{\sqrt{\lambda_n}}\right) \int_{-\sqrt{\lambda_n}}^{\sqrt{\lambda_n}} \frac{\sin^2 w}{w^2} \frac{\sqrt{y}}{\sqrt{y} - w/\lambda_n} dw \leq A_n\sqrt{\pi}(1 + \eta)
\]
and

\[ H_n \left( y - \frac{1}{\sqrt{\lambda_n}} \right) \leq \frac{A_n \sqrt{\pi} (1 + \eta)e^{\frac{2}{\sqrt{\lambda_n}}} \int_{\sqrt{\lambda_n}}^{\sqrt{\lambda_n} - \sqrt{\lambda_n}} \frac{\sin^2 w}{w^2} \sqrt{\frac{y - w}{\lambda_n}} \, dw}{\int_{-\sqrt{\lambda_n}}^{\sqrt{\lambda_n}} \frac{\sin^2 w}{w^2} \sqrt{\frac{y - w}{\lambda_n}} \, dw}. \]

As in Lemma 5.4, for large \( n \) we can arrange

\[ \left| \int_{\sqrt{\lambda_n}}^{\sqrt{\lambda_n} - \sqrt{\lambda_n}} \frac{\sin^2 w}{w^2} \sqrt{\frac{y - w}{\lambda_n}} \, dw - \pi \right| < \eta. \]

The above inequalities imply

\[ H_n \left( y - \frac{1}{\sqrt{\lambda_n}} \right) \leq \frac{A_n \sqrt{\pi} \left( (1 + \eta)e^{\frac{2}{\sqrt{\lambda_n}}} \right) \left( 1 - \frac{\eta}{\pi} \right)^{-1} \leq \frac{A_n \sqrt{\pi} (1 + C_3 \eta)}{\int_{-\sqrt{\lambda_n}}^{\sqrt{\lambda_n}} \frac{\sin^2 w}{w^2} \sqrt{\frac{y - w}{\lambda_n}} \, dw}. \]

with \( C_3 > 0 \) independent on \( n \). Replacing \( y \) by \( y + \frac{1}{\sqrt{\lambda_n}} \), we obtain for large \( n \) the upper bound in (5.11).

Next we pass to the analysis of the lower bound. Applying the upper bound obtained above and the monotonicity of \( H_n(y) \), from the left-hand side inequality in (5.10), we obtain

\[ A_n \sqrt{\pi} (1 - \eta) \leq A_n \sqrt{\pi} \int_{-\infty}^{-\sqrt{\lambda_n}} \frac{1}{w^2} \, dw + \int_{-\sqrt{\lambda_n}}^{\sqrt{\lambda_n}} H_n \left( y + \frac{1}{\sqrt{\lambda_n}} \right) \frac{\sin^2 w}{w^2} \sqrt{\frac{y - w}{\lambda_n}} \, dw + A_n \sqrt{\pi} (1 + \eta) \int_{\sqrt{\lambda_n}}^{\lambda_n y} \frac{\sin^2 w}{w^2} \sqrt{\frac{y - w}{\lambda_n}} \, dw. \]

Clearly, for large \( n \)

\[ \int_{-\infty}^{-\sqrt{\lambda_n}} \frac{1}{w^2} \, dw = \frac{1}{\sqrt{\lambda_n}} < \eta. \]

For the third term on the right-hand side for large \( n \), one gets

\[ \int_{\sqrt{\lambda_n}}^{\lambda_n y} = \int_{\sqrt{\lambda_n}}^{\lambda_n y/2} + \int_{\lambda_n y/2}^{\lambda_n y} \leq \frac{1}{\sqrt{2}} \int_{\sqrt{\lambda_n}}^{\lambda_n y/2} \frac{\sin^2 w}{w^2} \, dw + \frac{4\sqrt{y}}{\lambda_n y^2} \int_{\lambda_n y/2}^{\lambda_n y} \frac{1}{\sqrt{y - w/\lambda_n}} \, dw \]

\[ \leq \frac{1}{\sqrt{2}} \int_{\sqrt{\lambda_n}}^{\lambda_n y/2} \frac{\sin^2 w}{w^2} \, dw + \frac{4\sqrt{2}}{\lambda_n y} < \eta. \]

Consequently,

\[ A_n \sqrt{\pi} (1 - \eta) - 2A_n \sqrt{\pi} (1 + \eta) \eta \]

\[ \leq H_n \left( y + \frac{1}{\sqrt{\lambda_n}} \right) \frac{\sin^2 w}{w^2} \sqrt{\frac{y - w}{\lambda_n}} \, dw. \]
and for large \( n \) with a constant \( C_4 > 0 \) independent on \( n \), we obtain
\[
\frac{A_n \sqrt{\pi}(1 - C_4 \eta)e^{-\frac{A_n}{\sqrt{\lambda_n}}}}{\pi + \eta} \leq \frac{A_n \sqrt{\pi}(1 - C_4 \eta)e^{-\frac{A_n}{\sqrt{\lambda_n}}}}{\int_{\sqrt{\lambda_n}} \sin^2 w \frac{u}{w} \sqrt{y - w/\lambda_n} \, dw} \leq H\left(y + \frac{1}{\sqrt{\lambda_n}}\right).
\]
This estimate implies a lower bound for \( y \geq n + \frac{1}{\sqrt{\lambda_n}} - q \). Changing \( q \) by \( q + 1 \), we obtain a lower bound for \( y \geq n - q \).

\[\square\]

Obviously Proposition 5.1 follows from Lemma 5.5.

For functions which are not monotonic, we prove the following

**Proposition 5.6.** Let \( g_n(t) \in C^1([0, \infty[; \mathbb{R}^+) \), \( n \in \mathbb{N} \), be nonnegative functions such that
\[
\max_{0 \leq t \leq 1} g_n(t) \leq B_0, \ |g_n'(t)| \leq B_1 \frac{e^t}{\sqrt{t}}, \ t \geq 1, \ \forall n \in \mathbb{N}, \tag{5.13}
\]
with constants \( B_0 > 0, B_1 > 0 \) independent of \( n \). Assume that for any \( n \in \mathbb{N} \) the Laplace transforms
\[
F_n(s) = \int_0^\infty e^{-st} g_n(t) \, dt
\]
are analytic for \( \text{Re} s > 1 \) and have the same properties as in Proposition 5.1. Then, for any fixed \( q \geq 0 \) and any \( 0 < \eta \ll 1 \) there exists \( n_0(\eta) \in \mathbb{N} \) such that for \( t \geq n - q \) and \( n \geq n_0(\eta) + q \) we have
\[
\frac{A_n e^t}{\sqrt{\pi t}}(1 - \eta) < g_n(t) < \frac{A_n e^t}{\sqrt{\pi t}}(1 + \eta). \tag{5.14}
\]

**Proof.** Replacing the function \( g_n(t) \) by \( \tilde{g}_n(t) = g_n(t)\sqrt{t} \), we can assume that
\[
F_n(s) = \int_0^\infty \frac{e^{-st}}{\sqrt{t}} \tilde{g}_n(t) \, dt
\]
has the representation (5.1). On the other hand,
\[
g_n(t) = \int_1^t g_n'(\sigma) \, d\sigma + g_n(1) \leq B_0 + B_1 \int_1^t \frac{e^\sigma}{\sqrt{\sigma}} \, d\sigma < B_0 + B_1 e^t, \ t \geq 1, \ \forall n \in \mathbb{N}.
\]
Therefore,
\[
|\tilde{g}_n'(t)| = \left|\frac{g_n(t)}{2\sqrt{t}} + \sqrt{t} g_n'(t)\right| \leq B_3 e^t, \ t \geq 1, \ \forall n \in \mathbb{N} \tag{5.15}
\]
with a constant \( B_3 > 0 \) independent of \( n \). Below, we denote \( \tilde{g}_n(t) \) again by \( g_n(t) \), and we assume that \( |g_n'(t)| \leq B_3 e^t \) for \( t \geq 1 \). We will repeat a part of the proof of Proposition 5.1, and for simplicity, we use the notations of this proof. Set \( H_n(y) = g_n(y)e^{-y} \) and for \( s = 1 + \epsilon + it \) define
\[
K_{\epsilon,n}(t) = F_n(s) - \frac{A_n}{\sqrt{s - 1}} = A_n K_n(s) + L_n(s).
\]
Since for \( F_n(s), K_n(s), L_n(s) \) we have the same assumptions as in Proposition 5.1, we can apply Lemma 5.3. Thus, for \( y \geq n - q, \lambda_n = e^{\frac{1}{2} \mu_0 n} \) and \( n \geq n_0(\eta) + q \), we obtain the estimate (5.10).
Because $H_n(y) \geq 0$ to get an upper bound, we use the inequality
\[
\int_{-\sqrt{\lambda_n}}^{\sqrt{\lambda_n}} H_n\left(y - \frac{w}{\lambda_n}\right) \sin^2 \frac{w}{\lambda_n} \frac{\sqrt{y}}{\sqrt{y - w/\lambda_n}} \, dw < A_n \sqrt{\pi}(1 + \eta). \tag{5.16}
\]
Clearly, for $-\sqrt{\lambda_n} \leq w \leq \sqrt{\lambda_n}$ one has
\[
H_n\left(y - \frac{w}{\lambda_n}\right) \geq g_n\left(y - \frac{w}{\lambda_n}\right) e^{-y - \frac{1}{\sqrt{\lambda_n}}} = H_n(y) e^{-\frac{1}{\sqrt{\lambda_n}}} + \left(g_n\left(y - \frac{w}{\lambda_n}\right) - g_n(y)\right) e^{-y - \frac{1}{\sqrt{\lambda_n}}}.
\]
By Taylor expansion, we write
\[
g_n\left(y - \frac{w}{\lambda_n}\right) - g_n(y) = -\frac{w}{\lambda_n} g'_n\left(y - \frac{\theta w}{\lambda_n}\right), \quad 0 < \theta < 1,
\]
hence, for $-\sqrt{\lambda_n} \leq w \leq \sqrt{\lambda_n}$, we deduce
\[
|g_n\left(y - \frac{w}{\lambda_n}\right) - g_n(y)| \leq \frac{B_3}{\sqrt{\lambda_n}} e^{y + \frac{1}{\sqrt{\lambda_n}}},
\]
where we have used the estimate (5.15). Thus, we obtain
\[
H_n\left(y - \frac{w}{\lambda_n}\right) \geq H_n(y) e^{-\frac{1}{\sqrt{\lambda_n}}} - \frac{B_3}{\sqrt{\lambda_n}}
\]
and for $y \geq n - q$ and large $n$ we conclude that
\[
H_n(y) \leq \frac{A_n \sqrt{\pi}(1 + \eta)e^{\frac{1}{\sqrt{\lambda_n}}}}{\int_{-\sqrt{\lambda_n}}^{\sqrt{\lambda_n}} \sin^2 \frac{w}{\lambda_n} \frac{\sqrt{y}}{\sqrt{y - w/\lambda_n}} \, dw} + \frac{B_3}{\sqrt{\lambda_n}} e^{\frac{1}{\sqrt{\lambda_n}}}
\leq \frac{A_n \sqrt{\pi}(1 + \eta)^2}{\pi - \eta} + A_n \eta \leq \frac{A_n \sqrt{\pi}}{\pi} (1 + C_5 \eta)
\]
with $C_5 > 0$ independent of $n$. This implies the upper bound in (5.14).

Passing to the lower bound, we repeat the argument of Lemma 5.5 and we are going to find an upper bound for
\[
\int_{-\sqrt{\lambda_n}}^{\sqrt{\lambda_n}} H_n\left(y - \frac{w}{\sqrt{\lambda_n}}\right) \sin^2 \frac{w}{\lambda_n} \frac{\sqrt{y}}{\sqrt{y - w/\lambda_n}} \, dw.
\]
As above, by Taylor expansion for $-\sqrt{\lambda_n} \leq w \leq \sqrt{\lambda_n}$ and large $n$, one deduces
\[
H_n\left(y - \frac{w}{\sqrt{\lambda_n}}\right) \leq g_n\left(y - \frac{w}{\sqrt{\lambda_n}}\right) e^{-y + \frac{1}{\sqrt{\lambda_n}}}
\leq H_n(y) e^{\frac{1}{\sqrt{\lambda_n}}} + \frac{B_3}{\sqrt{\lambda_n}} e^{\frac{2}{\sqrt{\lambda_n}}},
\]
and we obtain easily the lower bound in (5.14). This completes the proof. \[\square\]

As a preparation for the proof of the estimate (5.13), we establish the following
Lemma 5.7. Let \( \omega(t) \in C^1([0, \infty[) \) be a nonnegative function such that for a fixed \( 0 < \nu_0 \leq \frac{\mu_0}{4} \) we have the estimate
\[
|\omega'(t)| \leq D_1 \frac{e^{(1+\nu_0)t}}{\sqrt{t}}, \forall t \geq t_0 > 1.
\] (5.17)

Assume that the Laplace transform
\[
\Omega(s) = \int_0^\infty e^{-st} \omega(t)dt
\]
is analytic for \( \text{Re} s > 1 \). Assume that there exist \( A > 0, \delta_0 > 0, M > 0 \) such that \( \Omega(s) \) has a representation
\[
\Omega(s) = \frac{A}{\sqrt{s-1}} + K(s),
\] (5.18)
where \( K(s) \) is a function which is analytic for \( \text{Re} s > 1 \) and \( K(s) \) satisfies the same assumptions as \( K_1(s) \) in Proposition 5.1. Then, there exists a constant \( D_0 > 0 \) such that
\[
\omega(t) \leq D_0 \frac{e^t}{\sqrt{t}}, \forall t \geq t_0.
\] (5.19)

Proof. We follow the proof of Proposition 5.6. First, we replace the function \( \omega(t) \) by \( \tilde{\omega}(t) = \sqrt{t} \omega(t) \) and for the Laplace transform
\[
\int_0^\infty e^{-st} \tilde{\omega}(t)dt
\]
we have the representation (5.1) with \( K(s) \) instead of \( K_n(s) \), \( L_n(s) = 0 \) and \( |\tilde{\omega}'(t)| \leq C_2e^{(1+\nu_0)t} \). We denote below \( \tilde{\omega}(t) \) by \( \omega(t) \). We set \( H(y) = \frac{\omega(y)}{e^{y}}, K_0(t) = \lim_{\epsilon \to 0} K(1 + \epsilon + it) \) and prove the following

Lemma 5.8. There exist \( y_0 > 1 \) and \( A_1 > 0 \) independent of \( y \) such that for \( y \geq y_0, \lambda(y) = \exp(\frac{1}{2} \mu_0 y) \) we have
\[
\left| \int_{-2\lambda(y)}^{2\lambda(y)} K_0(t) \left( 1 - \frac{|t|}{2\lambda(y)} \right) \sqrt{y} e^{it} dt \right| \leq A_1.
\]

The proof is a repetition of that of Lemma 5.3, and we omit the details.

We return to the proof of Lemma 5.7. For \( y \geq y_0 \) and \( \lambda(y) = \exp(\frac{1}{2} \mu_0 y) \), we deduce
\[
\int_{-\sqrt{\lambda(y)}}^{\sqrt{\lambda(y)}} H(y - \frac{w}{\lambda(y)}) \frac{\sin^2 \frac{w}{\sqrt{y-w/\lambda(y)}}}{w^2} \sqrt{y} \frac{\sqrt{y}}{w/\lambda(y)} dw < A \sqrt{\pi} + A_1.
\]

It is important to estimate for \( -\sqrt{\lambda(y)} \leq w \leq \sqrt{\lambda(y)} \) and \( 0 < \theta < 1 \) the term
\[
\left| \omega\left( y - \frac{w}{\lambda(y)} \right) - \omega(y) \right| = \left| \frac{w}{\lambda(y)} \omega'(y - \frac{\theta w}{\lambda(y)}) \right| \leq D_1 e^{(y + \frac{1+\nu_0}{\sqrt{\lambda(y)}})} e^{(\nu_0 - \frac{1}{2} \mu_0) y}.
\]
Since $0 < \nu_0 \leq \mu_0/4$, for large $y$ we may bound the right-hand side of the last inequality by $c e^y$ with a small constant $c > 0$ independent on $y$ and $D_1$. Consequently, as in the proof of Proposition 5.6, one obtains

$$H\left(y - \frac{w}{\lambda(y)}\right) \geq H(y)e^{-\frac{1}{\sqrt{\lambda(y)}}} - B_3$$

with a constant $B_3 > 0$ independent on $y$ and $D_1$ and we complete the proof as in Proposition 5.6. \hfill \Box

**Remark 5.9.** Notice that our proof shows that the constant $D_0 > 0$ can be chosen independently of $D_1$ by taking $t \geq t_1 > t_0$ and $t_1$ sufficiently large in (5.19).

### 6. Asymptotic of $\rho_n(T)$

In the section, we use the notations of the previous sections. It is more convenient to study the function

$$g_n(T) = \epsilon_n e^{(-\gamma(a)+1)T} \rho_n(T) = \epsilon_n e^{(-\gamma(a)+1)T} \int_{\mathbb{R}^\tau} q_n(G^T - aT)(y) dm_F(y)$$

$$= \frac{\epsilon_n}{2\pi} e^{(-\gamma(a)+1)T} \int_{\mathbb{R}^\tau} \left( \int_{\mathbb{R}^\tau} e^{z(G-a)^T(y)} dm_F(y) \right) \hat{\chi}(\epsilon_n \omega) \, d\omega.$$ 

Recall that $\epsilon_n = e^{-\epsilon_n}$, $z = \xi(a) + i \omega$, $a \in J \subset \Gamma_C$, $q_n(t) = e^{\xi(a)t} \chi(t/\epsilon_n)$. We take $0 < \epsilon \leq \mu_0/8$, $\mu_0$ being the constant in Proposition 4.2. We extend $g_n(T)$ as 0 for $T < 0$ and we wish to apply Proposition 5.6 for $g_n(T)$. We are going to check the assumptions of Proposition 5.6, where

$$A_n = \frac{C(a)\hat{\chi}(0)}{\sqrt{2\beta'(\xi(a))}} \epsilon_n^2, \forall n \in \mathbb{N},$$

with $C(a) > 0$ determined below. Clearly, $g_n(T)$ is a nonnegative function. The Laplace transform of $g_n(T)$ becomes

$$F_n(s) = \frac{\epsilon_n^2}{2\pi} \int_0^\infty e^{-(s+\gamma(a)-1)T} \int_{\mathbb{R}^\tau} \left( \int_{\mathbb{R}^\tau} e^{z(G-a)^T(y)} dm_F(y) \right) \hat{\chi}(\epsilon_n \omega) \, d\omega \, dT$$

and by Proposition 4.1 (i) for $\text{Re} \sigma > 1$ the function $F_n(s)$ is analytic. Next we write the integral with respect to $\omega$ as a sum

$$\int_{\mathbb{R}^\tau} (...) = \int_{|\omega| \leq \epsilon_0} + \int_{\epsilon_0 < |\omega| \leq M} + \int_{|\omega| \geq M},$$

where $\epsilon_0 > 0$ and $M > 0$ are the constants introduced in the proof of Proposition 4.2. The corresponding decomposition of $F_n(s)$ will be a sum $F_n(s) = F_{1,n}(s) + F_{2,n}(s) + F_{3,n}(s)$. Notice that the factor $\hat{\chi}(\epsilon_n \omega)$ is not involved in the integration with respect to $y$ and $T$ and in the analysis of $F_{1,n}(s)$, $k = 1, 2$ we will have a coefficient $\epsilon_n^2$ implying the factor $A_n = \mathcal{O}(\epsilon_n^2)$. Moreover, in the Laplace transform $Z(s, \omega, a)$ we must replace $s$ by $s + \gamma(a) - 1$. According to
Proposition 4.2, the functions $F_{2,n}(s)$ are analytic for $1 - \mu_0 \leq \text{Res}, \mu_0 > 0$ and 

$$\lim_{\delta \searrow 0} F_{2,n}(1 + \delta + it) = \epsilon_n^2 f_{2,n}(1 + it)$$

with $f_{2,n}(1 + it) \in W^{1,1}_{loc}(\mathbb{R})$. For $|t| \leq M$ the functions $|f_{2,n}(1 + it)|$, $|f_{2,n}^2(1 + it)|$ are clearly uniformly bounded with respect to $n$, while for $|t| \geq M$, we have an analytic continuation $f_2(s)$ for $1 - \mu_0 \leq \text{Res} \leq 1 + \delta_0$. In the latter case, we apply the estimate (4.6) with $0 < \nu < 1$ uniformly with respect to $n$ (see the case 2 in the proof of Proposition 4.2). Since 

$$\int_{|\epsilon| \leq |\omega| \leq M} |\hat{\chi}(\epsilon_\omega)| d\omega \leq M \sup_{\xi \in \mathbb{R}} |\hat{\chi}(\xi)|, \forall n \in \mathbb{N},$$

with a constant $C_2(\nu) > 0$ independent of $n$, we obtain 

$$|f_{2,n}(s)| \leq C_2(\nu)(1 + |\text{Im}|\nu). \quad (6.1)$$

Thus, the term $F_{2,n}(s)$ contributes to $A_n K_n(s)$ in (5.1).

Passing to the analysis of $F_{3,n}(s)$, we apply the same argument based on the estimate (4.6). According to Proposition 4.2, $F_{3,n}(s)$ is analytic for $1 - \mu_0 \leq \text{Res} \leq 1 + \delta_0$. In this case, we have an infinite integral with respect to $\omega$ and we will exploit the factor $\epsilon_n^2$ to estimate it. By using (4.6) with $0 < \nu < 1$, we must treat 

$$\epsilon_n^2 \int_{|\omega| \geq M} \left(1 + |\text{Im}|\nu + |\omega|^\nu\right)|\hat{\chi}(\epsilon_\omega)| d\omega.$$ 

The Fourier transform of $\chi \in C_0^\infty(\mathbb{R})$ satisfies 

$$|\hat{\chi}(\epsilon_\omega)| \leq D_2(1 + |\epsilon_\omega|)^{-2}$$

with a constant $D_2 > 0$ independent on $\epsilon_n$ and $\omega$. By a change of variable $\epsilon_\omega = \xi$, we get a convergent integral with respect to $\xi$ and with a constant $C_3(\nu) > 0$ independent on $n$ we deduce the estimate 

$$|F_{3,n}(t)| \leq C_3(\nu)\epsilon_n^{1-\nu}(1 + |\text{Im}|\nu) \leq C_3(\nu)(1 + |\text{Im}|\nu). \quad (6.2)$$

Therefore, $F_{3,n}(s)$ contributes to the term $L_n(s)$ in (5.1), and we cannot get a coefficient $A_n$.

It remains to study the behaviour of $F_{1,n}(s)$. Here, there are no problems with the convergence of the integral with respect to $\omega$, and we gain the factor $\epsilon_n^2$. For $\epsilon_0 \leq |\text{Im}|$ the Laplace transform $F_{1,n}(s)$ has no singularities and as above we obtain a contribution to $A_n K_n(s)$ in (5.1). Let $U_2 = \{(s, \omega) : s \in \mathbb{C}, \omega \in \mathbb{R}, |s - 1| \leq \mu_0, |\omega| \leq \epsilon_0\}$. Recall that for $(s, \omega) \in U_2$ the function $Z(s + \gamma(a) - 1, \omega, a)$ (independent on $n$) has a pole $s(\omega, a)$ and 

$$Z(s + \gamma(a) - 1, \omega, a) = \left(\frac{B_3(s(\omega, a), \omega, a)}{\int \tau d\nu f_n(s(\omega, a) - 1 + i\omega \tau)}\right) \frac{1}{s + \gamma(a) - 1 - s(\omega, a)} + J_5(s, \omega, a)$$

with a function $J_5(s, \omega, a)$ analytic in $s$ and real analytic in $\omega$. Here, we may repeat without any change the argument in Section 5 in [20]. Applying the
Morse lemma to the function $\text{Res}(\omega, a)$, there exists a function $y = y(\omega, a)$ defined for $|\omega| \leq \epsilon_0$ such that

$$\text{Res}(\omega, a) = \gamma(a) - y^2.$$ 

Therefore, the analysis is reduced to the integral

$$\frac{\epsilon_n^2}{2\pi} \int_{-\epsilon_0}^{\epsilon_0} B_3(\gamma(\omega) - y^2(\omega, a) + iq(\omega, a), \omega, a) \frac{1}{s - 1 + y^2(\omega, a) - iq(\omega, a)} \hat{\chi}(\epsilon_n \omega) d\omega,$$

where (see Lemma 3 in [20]) $q(\omega, a) = \text{Im}(s(\omega, a))$ is such that $q(0, a) = \frac{\partial q}{\partial \omega}(0, a) = \frac{\partial^2 q}{\partial \omega^2}(0, a) = 0$, and

$$\frac{\partial^2}{\partial \omega^2} \text{Res}(0, a) = -\sigma^2(m_F + \xi(a)G) = -\beta''(\xi(a)) < 0$$

with $\beta(\xi)$ and $\xi(a)$ introduced in Sect. 1.

Next the analysis follows that in Section 3 in [8] without any change. After a change of variable $\omega = \omega(y, a)$, the integral has the representation

$$\frac{\epsilon_n^2 C(a)}{\pi} \frac{\hat{\chi}(0)}{\sqrt{2\beta''(\xi(a))}} \left[ \int_{-\epsilon_0}^{\epsilon_0} \frac{1}{s - 1 + y^2} dy \right. + \left. \int_{-\epsilon_0}^{\epsilon_0} \frac{iQ(y, a)}{(s - 1 + y^2 - iQ(y, a))(s - 1 + y^2)} dy \right] + \frac{\epsilon_n^2}{2\pi} \frac{1}{\sqrt{2\beta''(\xi(a))}} \int_{-\epsilon_0}^{\epsilon_0} \frac{P(y)}{s - 1 + y^2 - iQ(y, a)} dy,$$

where

$$C(a) = \frac{1}{(\int \tau d\mu)^2} B_3(\gamma(a), 0, a),$$

$P(y)$ is a complex-valued function such that $P(0) = 0$ and $Q(y, a)$ is real-valued odd function with respect to $y$ such that $Q(0, a) = Q'(0, a) = Q''(0, a) = 0$. Here, $B_3(\gamma(a), 0, a)$ is given by (4.2) and we have used that

$$\frac{\partial y}{\partial \omega}(0, a) = \left( \sqrt{-\frac{1}{2}} \frac{\partial^2}{\partial \omega^2} \text{Res}(\omega, a) \right) \bigg|_{\omega=0} = \frac{\beta''(\xi(a))}{\sqrt{2}},$$

combined with the Taylor expansions for the functions $\text{Res}(\omega, a)$, $\text{Im}(\omega, a)$ and $\hat{\chi}(\epsilon_n \omega)$ around $\omega = 0$. In particular,

$$\hat{\chi}(\epsilon_n \omega(y, a)) = \hat{\chi}(0) + \epsilon_n \omega(y, a) \hat{\chi}'(0) + O(\epsilon_n^2 \omega^2(y, a)).$$

The first term in the above representation yields the singularity $\frac{A_n}{\sqrt{s-1}}$ plus more regular terms, where

$$A_n = \frac{C(a) \hat{\chi}(0)}{\sqrt{2\beta''(\xi(a))} \epsilon_n^2}.$$
The dependence on $n$ is caused by the coefficient $\epsilon_n^2$ involved in $A_n$. Finally, we obtain

$$F_{1,n}(s) = \frac{A_n}{\sqrt{s-1}} + A_n \omega_{1,n}(s)$$

and for $|\text{Im}s| \leq \epsilon_0$ we have uniform with respect to $n$ bounds for the $L^1(-\epsilon_0, \epsilon_0)$ norms of $f_{1,n}(1 + i t)$ and $f'_{1,n}(1 + i t)$. Summing the analysis of $F_{k,n}(s)$, $k = 1, 2, 3$, we get the representation (5.1).

Our purpose is to apply Proposition 5.6 for the functions $g_n(t)$. To do this, we need to estimate the derivative

$$g'_n(t) = \epsilon_n(-\gamma(a) + 1)e^{(-\gamma(a)+1)t} \int_{R^\tau} q_n(G^t - at)(y)dm_F(y)$$

$$+ \epsilon_n e^{(-\gamma(a)+1)t} \xi(a) \int_{R^\tau} q_n(G^t - at)(y)(G(\sigma^t_\tau(y)) - a)dm_F(y)$$

$$+ e^{(-\gamma(a)+1)t} \int_{R^\tau} e^{\xi(a)(G^t(y)-at)} \chi'(\frac{G^t - at}{\epsilon_n})(y)(G(\sigma^t_\tau(y)) - a)dm_F(y).$$

(6.3)

We can choose a function $0 \leq \psi(t) \in C_0^\infty(\mathbb{R})$ such that $\psi(t) = M_1 > 0$ for $t \in \text{supp} (\frac{t}{\epsilon_n})$, where the constant $M_1$ (independent of $n$) is chosen so that

$$\max_{t \in \mathbb{R}} \chi(t) + \max_{t \in \mathbb{R}} |\chi'(t)| < M_1.$$

Then, for every fixed compact $J \in \Gamma_G$ and $a \in J$ there exists a constant $C(J) > 0$ independent of $n$ and $a$ such that

$$|g'_n(t)| \leq C(J)e^{(-\gamma(a)+1)t} \int_{R^\tau} e^{\xi(a)(G^t(y)-at)} \psi(G^t - at)(y)dm_F(y) = C(J)\Psi(t).$$

Notice that $C(J)$ depends on $\xi(a), a$ and the maximum of $G(w)$, but $C(J)$ is independent of $M_1$. The problem is reduced to an estimate of $\Psi(t)$. For the nonnegative function $\Psi(t)$, we wish to apply Lemma 5.7. Consider the Laplace transform

$$Y(s) = \int_{0}^{\infty} e^{-st} \Psi(t)dt$$

for $\text{Re}s > 1$ and its limit as $\text{Re}s \searrow 1$. The analysis is completely the same as that of $F_n(s)$, where the function $\tilde{\chi}_n(w)$ must be replaced by $\tilde{\psi}(w)$ which is independent on $n$. Therefore, with some constant $A > 0$ one deduces the representation

$$Y(s) = \frac{A}{\sqrt{s-1}} + P(s)$$

with a function $P(s)$ having the properties of $K(s)$ mentioned in Lemma 5.7. To satisfy the condition (5.17), first as above we obtain an upper bound

$$|\Psi'(t)| \leq C_1(J)e^{(-\gamma(a)+1)t} \int_{R^\tau} e^{\xi(a)(G^t(y)-at)} \psi_1(G^t - at)(y)dm_F(y)$$

$$= C_1(J)\Psi_1(t),$$

(6.4)
where \( 0 \leq \psi_1(t) \in C_0^\infty(\mathbb{R}) \) is such that for \( t \in \text{supp} \psi(t) \) we have
\[
\psi_1(t) \geq \max_{t \in \mathbb{R}} \psi(t) + \max_{t \in \mathbb{R}} |\psi'(t)|.
\]

We repeat this procedure once more and with a function \( 0 \leq \psi_2(t) \in C_0^\infty(\mathbb{R}) \) and constant \( C_2(J) \) one arranges the bound
\[
|\Psi_1(t)| \leq C_2(J) e^{-(\gamma(a)+1)t} \int_{R^r} e^{\xi(a)(G^t - at)} \psi_2(G^t - at)(y) \, dm_F(y)
\]
\[
= C_2(J) \Psi_2(t).
\]

Now the analysis in Sect. 4 shows that the Laplace transforms
\[
\int_0^\infty e^{-st} \Psi_k(t) \, dt, \quad k = 1, 2,
\]
are analytic for \( \text{Re} s > 1 \). Therefore, the integral
\[
\int_0^\infty e^{-st} \Psi_1(t) \, dt
\]
is absolutely convergent for \( \text{Re} s > 1 \), and for \( s = 1 + \delta > 1 \) we have
\[
\int_0^\infty e^{-st} \Psi_1(t) \, dt = \left[ e^{-st} \Psi_1(t) \right]_0^\infty + s \int_0^\infty e^{-st} \Psi_1(t) \, dt,
\]
hence for every \( \delta > 0 \) we have \( \lim_{t \to \infty} e^{-(1+\delta)t} \Psi_1(t) = 0 \). This estimate combined with (6.4) implies a bound
\[
|\Psi'(t)| \leq C_\delta C_1(J) \frac{e^{(1+\delta)t}}{\sqrt{t}}, \quad t \geq 1
\]
and we are in position to apply Lemma 5.7 for the function \( \Psi(t) \). The statement of Lemma 5.7 yields the estimate (5.19) for \( \Psi(t) \) with a constant \( D_0 > 0 \) independent of \( n, C_\delta, C_1(J), C_2(J) \), and we obtain
\[
|g'_n(t)| \leq D_0 C(J) \frac{e^t}{\sqrt{t}}, \quad \forall t \geq t_0, \quad \forall n \in \mathbb{N}.
\]
(6.5)

Following Remark 5.9, we may take \( t_0 > 1 \) large to guarantee the independence of \( D_0 \).

On the other hand, it is clear that we may estimate \( \max_{0 \leq t \leq 1} g_n(t) \) uniformly with respect to \( n \) and \( a \in J \). Thus, we can apply Proposition 5.6 for \( g_n(t) \). We cancel the coefficients \( \epsilon_n \) and \( \epsilon^t \) in the estimates for \( g_n(t) \) and one concludes that for fixed \( q \geq 0, \quad T \geq n - q \) and any \( 0 < \eta \ll 1 \) there exists \( n_0(\eta) \in \mathbb{N} \) such that for \( n \geq n_0(\eta) + q \) we have
\[
\frac{\epsilon_n C(a) \int \chi(t) \, dt}{\sqrt{2\pi T \beta''(\xi(a))}} e^{\gamma(a)T(1 - \eta)} \leq \rho_n(T) \leq \frac{\epsilon_n C(a) \int \chi(t) \, dt}{\sqrt{2\pi T \beta''(\xi(a))}} e^{\gamma(a)T(1 + \eta)}.
\]
(6.6)

It is easy to see that if one examines the function
\[
\tilde{\rho}_n(T) = \int_U \chi \left( \frac{G^T - aT}{\epsilon_n} \right) \, dm_F(y),
\]
then for fixed $q \geq 0$ and any $0 < \eta \ll 1$ there exists $n_0(y) \in \mathbb{N}$ such that for $n \geq n_0(\eta) + q$ we get

$$\epsilon_n C(a) \int \sqrt{2\pi T} \beta''(\xi(a)) e^{\gamma(a)T} (1 - \eta) \leq \hat{\rho}_n(T) \leq \epsilon_n C(a) \int \sqrt{2\pi T} \beta''(\xi(a)) e^{\gamma(a)T} (1 + \eta),$$

(6.7)

since the Fourier transform $\epsilon_n \hat{\chi}(\epsilon_n \omega)$ must be replaced by the Fourier transform $\epsilon_n \hat{\chi}(\epsilon_n (\omega - i\xi(a)))$ and

$$\int e^{-\epsilon_n \xi(a) t} \chi(t) dt = \int \chi(t) dt + O(\epsilon_n).$$

Approximating the characteristic function $1_{[-1,1]}(t)$ by cut-off functions $\chi_{\pm}(t) \in C_0^\infty(\mathbb{R}; [0,1])$ such that

$$\chi_{-}(t) \leq 1_{[-1,1]}(t) \leq \chi_{+}(t),$$

we obtain for $T \geq n - q$, $n \geq n_0(\eta) + q$ the estimates (1.1) and this proves Theorem 1.3.

It is worth noting that the derivatives $\chi'_{\pm}(t)$ may increase for $-1 - \delta \leq t \leq -1$ and $1 \leq t \leq 1 + \delta$, but this is not important for the estimate (6.5) since we may arrange $D_0$ to be independent of the derivatives $\chi'_{\pm}(t)$ choosing $t \geq t_1$. This reflects in the choice of $n_0(\eta)$ in (6.5). The same observation holds for the derivative $\chi'_{\pm}(t)$ in $-1 \leq t \leq 1 - \delta$ and $1 - \delta \leq t \leq 1$.

Now it is easy to pass to the analysis of the intervals $\left( -\frac{e^{-\epsilon T}}{T}, \frac{e^{-\epsilon T}}{T} \right)$. In fact, let $q = 1$, $0 < \eta \ll 1$ and $n_0(\eta) \in \mathbb{N}$ be as above. Let $T \geq n_0(\eta) + 1$ and let $N(\eta) \geq n_0(\eta) + 1$ be chosen so that $N(\eta) \leq T \leq N(\eta) + 1$. Obviously, we have $\left( -e^{-\epsilon (N(\eta) + 1)}, e^{-\epsilon (N(\eta) + 1)} \right) \subset \left( -e^{-\epsilon T}, e^{-\epsilon T} \right) \subset \left( -e^{-\epsilon N(\eta)}, e^{-\epsilon N(\eta)} \right)$. Now we may examine

$$\zeta(T; a) = m_F \left\{ w \in R^T : \int_0^T G(\sigma^T_I(w)) dt - aT \in \left( -e^{-\epsilon T}, e^{-\epsilon T} \right) \right\}.$$

For $T \geq n_0(\eta) + 1$, we deduce the estimates

$$\frac{2e^{-\epsilon T} e^{-\epsilon C(a)}}{\sqrt{2\pi T \beta''(\xi(a))}} e^{\gamma(a)T} (1 - \eta) \leq \frac{2e^{-\epsilon (N(\eta) + 1)} C(a)}{\sqrt{2\pi T \beta''(\xi(a))}} e^{\gamma(a)T} (1 - \eta) \leq \zeta(T; a),$$

(6.8)

$$\zeta(T; a) \leq \frac{2e^{-\epsilon N(\eta)} C(a)}{\sqrt{2\pi T \beta''(\xi(a))}} e^{\gamma(a)T} (1 + \eta) \leq \frac{2e^{-\epsilon T} C(a)}{\sqrt{2\pi T \beta''(\xi(a))}} e^{\gamma(a)T} (1 + \eta).$$

(6.9)

To obtain (6.8), we exploit $T \geq [N(\eta) + 1] - 1$, while for (6.9) we use $T \geq N(\eta)$. These estimates prove the statement of Theorem 1.4.
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Appendix

Proof of Proposition 2.2. In what follows, we will denote global generic constants by $C > 0$ and $c > 0$.

(a) We will first show that $\tilde{G}$ is constant on stable leaves of $R^\tau$. Let $\xi, \eta \in R^\tau$ be on the same stable leaf of $R^\tau$. Thus, $\xi = \pi(x, t), \eta = \pi(y, t)$ for some $x, y \in R_i$ with $\pi_U(x) = \pi_U(y) = z \in U_i$ and some $t \in [0, \tau(x))$. Then $\tau(y) = \tau(x) = \tau(z)$. Moreover, $s = \tau(P^{n+1}(x)) = \tau(P^{n+1}(y))$ and also $s = \tau(P^n(P(y)))$. Finally, $\pi_U(P(x)) = \pi_U(P(y))$. Using all these in the definition of $\tilde{G}$, gives $\tilde{G}(x, t) = \tilde{G}(y, t)$.

To prove (2.6), write

$$h(x) - h(P(x)) = \sum_{n=0}^{\infty} [g(P^n(x)) - g(\pi_U(P(x)))]
- \sum_{n=0}^{\infty} [g(P^n(P(x))) - g(\pi_U(P(x)))]
= g(x) - g(\pi_U(x)) + \sum_{n=1}^{\infty} [g(P^n(x)) - g(\pi_U(P(x)))]
- \sum_{n=0}^{\infty} [g(P^n(P(x))) - g(\pi_U(P(x)))]
= g(x) - g(\pi_U(x)) + \sum_{n=0}^{\infty} [g(P^{n+1}(x)) - g(\pi_U(P(x)))]
- \sum_{n=0}^{\infty} [g(P^n(P(x))) - g(\pi_U(P(x)))]
= g(x) - \left[ g(\pi_U(x)) + \sum_{n=0}^{\infty} (g(P^{n+1}(\pi_U(x))) - g(\pi_U(P(x)))) \right].$$

(A.1)

Next, for every $x \in \hat{R}$, using the change of variable $t \mapsto s = t \tau(P^{n+1}(x))/\tau(x)$ in some of the integrals below, we get

$$\tilde{g}(x) = \int_0^{\tau(x)} \tilde{G}(x, t) \, dt = \int_0^{\tau(x)} G(\pi_U(x), t) \, dt$$
$$+ \sum_{n=0}^{\infty} \left[ \int_0^{\tau(x)} G(\pi_U(P(x))), t \tau(P^{n+1}(x))/\tau(x) \right] dt.$$
This and (A.1) imply $h(x) - h(\mathcal{P}(x)) = g(x) - \tilde{g}(x)$, thus proving (2.6).

It remains to prove that $\tilde{G}$ and $G$ are $\beta$-Hölder on $\hat{R}_t$ and $\hat{R}$, respectively, for some $\beta > 0$.

Let $x \neq y$ belong to some $R_i \cap \hat{R}$ and let $0 \leq t < \tau(x)$ and $0 \leq t' < \tau(y)$. We may assume $\tau(x) \leq \tau(y)$. We have to estimate $|\tilde{G}(x, t) - \tilde{G}(y, t')|$. We will first estimate $|G(x, t) - \tilde{G}(y, t)|$.

Set $\tilde{x} = \pi_U(x)$ and $\tilde{y} = \pi_U(y)$ and $s = t \tau(\mathcal{P}^{n+1}(x))/\tau(x)$. Then, $d(\tilde{x}, \tilde{y}) \leq C(d(x, y))^\alpha$ for some global constant $C > 0$.

Let $2m$ (or $2m + 1$) be the maximal positive integer so that $\mathcal{P}^j(x), \mathcal{P}^j(y)$ belong to the same rectangle $R_{ij}$, for $j = 0, 1, \ldots, 2m - 1$. Then, by (2.2),

$$c \leq d(\mathcal{P}^{2m}(x), \mathcal{P}^{2m}(y)) \leq (c/c_0)^{2m}(d(x, y))^\alpha,$$

so $d(x, y) \geq \frac{c}{\gamma_{1/2m}}$, and therefore

$$(d(x, y))^{\alpha\gamma'/2} \geq \frac{c}{\gamma_1^{\alpha\gamma'/2}} = \frac{c}{(\rho \gamma)^m} > \frac{c}{\gamma^m}.$$  \hspace{1cm} (A.2)

For any integer $n = 0, 1, \ldots, m - 1$, using (2.2) and the latter, we get

$$d(\mathcal{P}^{n+1}(\tilde{x}), \mathcal{P}^{n+1}(\tilde{y})) \leq d(\mathcal{P}^{2m}(\tilde{x}), \mathcal{P}^{2m}(\tilde{y})) \leq C / \gamma^{2m-n}$$

$$< \frac{C}{\gamma^{m-n}} \cdot \frac{1}{\gamma^m} \leq \frac{C}{\gamma^m} \cdot (d(x, y))^{\alpha\gamma'/2}. \hspace{1cm} (A.3)$$

We will now estimate $|\tilde{G}(x, t) - \tilde{G}(y, t)|$. We have

$$|\tilde{G}(x, t) - \tilde{G}(y, t)| \leq |G(\tilde{x}, t) - G(\tilde{y}, t)| + \sum_{n=0}^{m-1} |G(\mathcal{P}^{n+1}(\tilde{x}), s) - G(\mathcal{P}^{n+1}(\tilde{y}), s)|$$

$$+ \sum_{n=0}^{m-1} |G(\mathcal{P}^n(\pi_U(\mathcal{P}(x))), s) - G(\mathcal{P}^n(\pi_U(\mathcal{P}(x))), s)|$$

$$+ \sum_{n=m}^{\infty} |G(\mathcal{P}^{n+1}(\pi_U(x)), s) - G(\mathcal{P}^n(\pi_U(\mathcal{P}(x))), s)|$$

$$+ \sum_{n=m}^{\infty} |G(\mathcal{P}^{n+1}(\pi_U(y)), s) - G(\mathcal{P}^n(\pi_U(\mathcal{P}(y))), s)|$$

$$= I + II + III + IV + V.$$
Clearly, \( I \leq |G|_\alpha (d(\tilde{x}, \tilde{y}))^\alpha \leq C |G|_\alpha (d(x, y))^{\alpha^2} \).

Next,

\[
II = \sum_{n=0}^{m-1} |G(\mathcal{P}^{n+1}(\tilde{x})), s) - G(\mathcal{P}^{n+1}(\tilde{y}), s)| \\
\leq \sum_{n=0}^{m-1} |G|_\alpha \left( \frac{C}{\gamma_{m-n}} \cdot (d(x, y))^{\alpha\alpha'/2} \right) ^\alpha \leq C |G|_\alpha (d(x, y))^{\beta}.
\]

Similarly, \( III \leq C |G|_\alpha (d(x, y))^{\beta} \).

Since \( \mathcal{P}(\pi_U(x)) \) and \( \pi_U(\mathcal{P}(x)) \) are on the same stable leaf of some rectangle, it follows from (2.3) that

\[
|G(\mathcal{P}^{n+1}(\pi_U(x))), s) - G(\mathcal{P}^n(\pi_U(\mathcal{P}(x))), s)| \\
\leq |G|_\alpha (d(\mathcal{P}^{n+1}(\pi_U(x))), \mathcal{P}^n(\pi_U(\mathcal{P}(x)))) \alpha \leq C |G|_\alpha \frac{1}{\gamma_{\alpha m}}.
\]

This and (A.2) yield

\[
IV \leq C |G|_\alpha \sum_{n=m}^{\infty} \frac{1}{\gamma_{\alpha n}} \leq C |G|_\alpha \frac{1}{\gamma_{\alpha m}} \leq C |G|_\alpha (d(x, y))^{\alpha^2\alpha'/2} C |G|_\alpha (d(x, y))^{\beta}.
\]

In a similar way, we obtain \( V \leq C |G|_\alpha (d(x, y))^{\beta} \). Thus,

\[
|\tilde{G}(x, t) - \tilde{G}(y, t)| \leq C |G|_\alpha (d(x, y))^{\beta}.
\]

An estimate of the form \( |\tilde{G}(y, t) - \tilde{G}(y, t')| \leq C |G|_\alpha |t - t'|^{\beta} \) can be obtained rather easily, and we leave the details to the reader. This proves that \( \tilde{G} \) is \( \beta \)-Hölder and \( |\tilde{G}|_\beta \leq |G|_\alpha \).

The proof that \( h \) is also \( \beta \)-Hölder is very similar to the above; in fact, it is easier. We leave the details to the reader.

(b) The proof that \( H \) is \( \beta \)-Hölder is very similar to the proof above that \( \tilde{G} \) is \( \beta \)-Hölder. We leave the details to the reader.

To establish (2.9), replace \( H(x, t) \) in the integral in (2.9) by the right-hand side of (2.8) and use the change of variable \( t \mapsto s = t \tau(\mathcal{P}^n(x))/\tau(x) \). What we obtain in this way is the right-hand side of (2.5). This proves (2.9).

Finally, to establish (2.10), we write:

\[
H(x, t) - H(\mathcal{P}(x), t \tau(\mathcal{P}(x))/\tau(x)) \\
= \sum_{n=0}^{\infty} \left[ G(\mathcal{P}^n(x), t \tau(\mathcal{P}^n(x))/\tau(x)) - G(\mathcal{P}^n(\pi_U(x)), t \tau(\mathcal{P}^n(x))/\tau(x)) \right] \\
- \sum_{n=0}^{\infty} \left[ G(\mathcal{P}^{n+1}(x), t \tau(\mathcal{P}^{n+1}(x))/\tau(x)) - G(\mathcal{P}^n(\pi_U(\mathcal{P}(x))), t \tau(\mathcal{P}^{n+1}(x))/\tau(x)) \right] \\
= G(x, t) - G(\pi_U(x), t) \\
+ \sum_{n=1}^{\infty} \left[ G(\mathcal{P}^n(x), t \tau(\mathcal{P}^n(x))/\tau(x)) - G(\mathcal{P}^n(\pi_U(x)), t \tau(\mathcal{P}^n(x))/\tau(x)) \right] \\
- \sum_{n=0}^{\infty} \left[ G(\mathcal{P}^{n+1}(x), t \tau(\mathcal{P}^{n+1}(x))/\tau(x)) - G(\mathcal{P}^n(\pi_U(\mathcal{P}(x))), t \tau(\mathcal{P}^{n+1}(x))/\tau(x)) \right]
\]
\[ G(x,t) - G(\pi_U(x),t) \\
+ \sum_{n=0}^{\infty} \left[ G(P^{n+1}(x), t \tau(P^{n+1}(x))/\tau(x)) - G(P^{n+1}(\pi_U(x)), t \tau(P^{n+1}(x))/\tau(x)) \right] \\
- \sum_{n=0}^{\infty} \left[ G(P^{n+1}(x), t \tau(P^{n+1}(x))/\tau(x)) - G(P^n(\pi_U(P(x))), t \tau(P^{n+1}(x))/\tau(x)) \right] \\
= G(x,t) - (G(\pi_U(x),t) + \sum_{n=0}^{\infty} \left[ G(P^{n+1}(\pi_U(x)), t \tau(P^{n+1}(x))/\tau(x)) \\
- G(P^n(\pi_U(P(x))), t \tau(P^{n+1}(x))/\tau(x)) \right] ) \\
= G(x,t) - \tilde{G}(x,t). \]

This proves (2.10). \( \square \)

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