A COMPARISON PRINCIPLE FOR STOCHASTIC INTEGRO-DIFFERENTIAL EQUATIONS

KONSTANTINOS DAREIOTIS† AND ISTVAN GYŐNGY*

Abstract. A comparison principle for stochastic integro-differential equations driven by Lévy processes is proved. This result is obtained via an extension of an Itô formula from [11] for the square of the norm of the positive part of $L_2$-valued, continuous semimartingales, to the case of discontinuous semimartingales.

1. Introduction

Our goal is to prove a comparison principle for stochastic integro-differential equations (SIDEs) driven by Lévy processes. For this, first we present an Itô’s formula for the square of the $L_2$-norm of the positive part of (possibly) discontinuous semimartingales with values in $L_2$-spaces. Our formula extends an Itô formula from [11] proved for continuous semimartingales. In [11] Itô’s formulas for the square of $L_2$-norm of certain convex functions $r(u)$ of continuous semimartingales $u = u_t$ with values in $L_2$-spaces are obtained, and the important special case, $r(u) = (u)^+ = \max(u, 0)$, is then applied to prove a maximum principle for linear stochastic partial differential equations (SPDEs). In section 2 of the present paper we extend the Itô formula corresponding to the positive part function, to the case of discontinuous semimartingales with values in $L_2$ spaces. Then, in section 3 we apply our formula to prove comparison theorems for solutions of SIDEs driven by Lévy processes.

The comparison principles are powerful tools and play important roles in PDE theory. Comparison theorems for SPDEs are known in various generalities in the literature. To the best of our knowledge, the first results on comparison of solutions of SPDEs appear in [10] and [4]. Recent results appear in [11], [2], [1] and [3]. In [1] and [2] quasi linear SPDEs, and in [3] quasi-linear SPDEs with obstacle are considered, and the $p$-th moments of the positive part of the supremum norm of the solutions are also estimated. In the above publications, SPDEs driven by Wiener processes, or cylindrical Wiener processes are considered.

Key words and phrases. comparison principle, Itô’s formula, SPDEs.
Our main result, theorems 3.1 and 3.2, are comparison theorems for two classes of quasilinear SIDEs, linear versions of which, arise in non-linear filtering. We apply our result to investigate the solvability of a class of SPDEs driven by Lévy processes in another paper.

In conclusion we introduce some basic notation of the paper. Let \((\Omega, \mathcal{F}, P)\) be a probability space equipped with a right-continuous filtration \((\mathcal{F}_t)_{t \geq 0}\), such that \(\mathcal{F}_0\) contains all \(P\)-zero sets. We consider a \(\sigma\)-finite measure space, \((Z, \mathcal{Z}, \nu)\) and a quasi left-continuous, adapted point process \((p_t)_{t \in [0,T]}\) in \(Z\), for a finite \(T > 0\). Let \(N(dt,dz)\) be the random measure on \([0,T] \times Z\), corresponding to the point processes \(p\).

We assume that its compensator is \(dt \nu(dz)\) and we use the notation \(\tilde{N}(dt,dz) = N(dt,dz) - dt \nu(dz)\).

We also consider a sequence of independent real valued \(\mathcal{F}_t\)-Wiener processes \(\{w^k_t\}_{k=1}^\infty\).

If \(X\) is a topological space then \(\mathcal{B}(X)\) is the Borel \(\sigma\)-algebra on \(X\). The notation \(\mathcal{P}\) is used for the predictable \(\sigma\)-algebra on \(\Omega \times [0,T]\). If \(X\) is a normed linear space then \(|x|_X\) denotes the norm of \(x \in X\). If \(X^*\) is the dual of \(X\) then \(\langle x, x^* \rangle\) denotes the action of \(x^* \in X^*\) on \(x \in X\). If \(X\) is a Hilbert space we write \((\cdot, \cdot)\) for the inner product in \(X\). The notation \(Q\) stands for the whole space \(\mathbb{R}^d\) or for a bounded Lipschitz domain in \(\mathbb{R}^d\). As usual we denote by \(W^k_p(Q)\) the space of functions \(u \in L_p(Q)\), such that the distributional derivatives of \(k\)-th order lie in \(L_p(Q)\). We set \(H^1(Q) := W^1_2(Q)\) and we write \(H^1_0(Q)\) for the closure of \(C_c^\infty(Q)\) in \(H^1(Q)\) under the norm

\[
|u|_{H^1} = \left( \sum_{i=1}^d |D_i u|_{L^2} + |u|_{L^2} \right)^{1/2}.
\]

We will use the notation \(H^{-1}(Q)\) for the dual of \(H^1_0(Q)\). Finally, we note that the summation convention is used with respect to repeated integer-valued indices throughout the paper.

2. Itô’s formula for the square of the norm of the positive part

We are interested in an Itô’s formula for \(|u_t^+|^2_{L_2(Q)}\), where \(u_t\) is an \(H^{-1}(Q)\)-valued semimartingale taking values in \(H^1_0(Q)\) for \(dP \times dt\) almost every \((\omega, t) \in \Omega \times [0,T]\). Our approach to obtain it is similar to that in [11]. To state the formula we set

\[
V := H^1_0(Q), \quad H := L_2(Q), \quad V^* := H^{-1}(Q),
\]
and we consider processes

\[ v : \Omega \times [0, T] \to V, \quad v^* : \Omega \times [0, T] \to V^*, \quad h^k : \Omega \times [0, T] \to H, \]

\[ K : \Omega \times [0, T] \times Z \to H, \]

for integers \( k \geq 1 \), where \( v \) is progressively measurable, \( v^* \) and \( h^k \) are \( \mathcal{F}_t \)-adapted, measurable in \((\omega, t)\), and \( K \) is \( \mathcal{P} \times Z \) measurable. We consider also \( \psi \), an \( \mathcal{F}_0 \)-measurable random variable in \( H \). We make the following assumption.

**Assumption 2.1.**

i) Almost surely

\[
\int_{(0, T]} \left( |v_t|^2_V + |v^*_t|^2_{V^*} + \sum_k |h^k_t|^2_H + \int_Z |K_t(z)|^2_H \nu(dz) \right) dt < \infty,
\]

ii) for each \( \phi \in V \) we have for \( dP \times dt \)-almost every \((\omega, t)\),

\[
(v_t, \phi) = (\psi, \phi) + \int_{(0, t]} \langle v^*_s, \phi \rangle ds + \int_{(0, t]} \langle h^k_s, \phi \rangle dw^k_s
\]

\[ + \int_{(0, t]} \int_Z (K_s(z), \phi) \tilde{N}(ds, dz), \]

where \( (\cdot, \cdot) \) is the inner product in \( H \).

**Theorem 2.1.** Suppose that Assumption 2.1 is satisfied. Then there exists a set \( \tilde{\Omega} \subset \Omega \) of probability one, and an \( H \)-valued strongly càdlàg adapted process \( u_\cdot \) such that \( u_t = v_t \) for \( dP \times dt \)-almost every \((\omega, t)\).

Moreover for \( \omega \in \tilde{\Omega} \), \( t \in [0, T] \) we have

i) \( u_t = \psi + \int_{(0, t]} v^*_s ds + \int_{(0, t]} h^k_s dw^k_s + \int_{(0, t]} \int_Z K_s(z) \tilde{N}(ds, dz), \quad (2.1) \)

ii) \( |u^+_t|_H^2 = |\psi^+_t|_H^2 + 2 \int_{(0, t]} \langle v^*_s, u^+_s \rangle ds + 2 \int_{(0, t]} \langle h^k_s, u^+_s \rangle dw^k_s \)

\[ + 2 \int_{(0, t]} \int_Z (K_s(z), u^+_s) \tilde{N}(dz, ds) + \int_{(0, t]} \sum_k |I_{u^+_s > 0} h^k_s|_H^2 ds \]

\[ + \int_{(0, t]} \int_Z (u^-_s + K_s(z))^+ |^2_H - |u^-_s|_H^2 - 2(K_s(z), u^-_s)_H N(dz, ds). \]

To prove Theorem 2.1 we need two lemmas.

**Lemma 2.2.** Let \((X, \Sigma, \mu)\) be a measure space, and let \( u_n, u \in L^1(X) \).

Suppose that \( u_n \to u \) in \( L^1(X) \). Then there exists a subsequence \( u_{n(k)} \) and a function \( v \in L^1(X) \) such that for all \( x \in X \), we have \( |u_{n(k)}(x)| \leq v(x) \).
Proof. There exists a subsequence $u_{n(k)}$ such that

$$|u_{n(k)} - u|_{L^1(X)} \leq 1/2^k.$$ 

Set $v(x) = |u(x)| + \sum_k |u_{n(k)}(x) - u(x)|$. Then $v$ has the desired properties. \hfill $\square$

The next lemma is from [2].

**Lemma 2.3.** Let $Q$ be a bounded Lipschitz domain in $\mathbb{R}^d$. Take $\phi_n \in C_c^\infty(Q)$, $n \in \mathbb{N}$, with

i) $0 \leq \phi_n \leq 1$

ii) $\phi_n = 1$ on $\{ x \in Q, r(x) \geq 1/n \}$

iii) $\phi_n = 0$ on $\{ x \in Q, r(x) \leq 1/(2n) \}$

iv) $|\phi_n(x)| \leq Cn$,

where $C$ is a constant and $r(x) = \text{dist}(x, \partial Q)$. Then $\phi_n v \to v$ in $H^1_0(Q)$ for all $v \in H^1_0(Q)$, and for some constant $C$ we have

$$\sup_n |\phi_n v|_{H^1_0} \leq C|v|_{H^1_0}, \quad \forall v \in H^1_0(Q).$$

**Remark 2.1.** One can easily see the existence of a sequence $(\phi_n)_{n \in \mathbb{N}}$ satisfying the conditions of the previous lemma. Then note that $\phi_n^2$ also satisfies i)-iv). Hence, $\phi_n^2 v \to v$ in $H^1_0(Q)$, for all $v \in H^1_0(Q)$, and for some constant $C$ we have

$$\sup_n |\phi_n^2 v|_{H^1_0} \leq C|v|_{H^1_0}, \quad \forall v \in H^1_0(Q).$$

We introduce now the functions $\alpha_\delta(r)$, $\beta_\delta(r)$ and $\gamma_\delta(r)$ on $\mathbb{R}$, for $\delta > 0$, given by

$$a_\delta(r) = \begin{cases} 
1 & \text{if } r > \delta \\
\frac{r}{\delta} & \text{if } 0 \leq r \leq \delta \\
0 & \text{if } r < 0
\end{cases}$$

$$\beta_\delta(r) = \int_0^r a_\delta(s)ds, \quad \gamma_\delta(r) = \int_0^r \beta_\delta(s)ds.$$

For all $r \in \mathbb{R}$ we have $\alpha_\delta(r) \to I_{r>0}$, $\beta_\delta(r) \to r^+$ and $\gamma_\delta(r) \to (r^+)/2$ as $\delta \to 0$. Also, for all $r, r_1, r_2$ and $\delta$, the following inequalities hold

$$|\alpha_\delta(r)| \leq 1, \quad |\beta_\delta(r)| \leq |r|, \quad |\gamma_\delta(r)| \leq \frac{r^2}{2},$$

$$|\gamma_\delta(r_1 + r_2) - \gamma_\delta(r_1) - \beta_\delta(r_1) r_2| \leq |r_2|^2. \quad \text{(2.2)}$$

We are now ready to prove Theorem 2.1.
Proof of Theorem 2.1. We only prove ii) since the rest of the assertions are proved in [6]. Notice that by using a standard localization argument we can assume that

$$\sup_{t \leq T} \left| u_t \right|^2_H + \left| \psi \right|^2_H$$

$$+ \int_{[0,T]} \left( \left| u_t \right|_V^2 + \left| v_t^* \right|_{V^*}^2 + \sum_k \left| h_t^k \right|_H^2 + \int_Z |K_t(z)|_H^2 \nu(dz) \right) dt \leq C,$$

for some $C > 0$. First we prove the statement when $Q = \mathbb{R}^d$. We have that equality (2.1) is satisfied if and only if, almost surely, for all $v \in V$ and $t$ we have

$$(u_t, v)_H = (u_0, v)_H + \int_{[0,t]} \langle v^*_s, v \rangle ds + \int_{[0,t]} (h^k_s, v)_H dw^k_s$$

$$+ \int_{[0,t]} \int_Z (K_s(z), v)_H \tilde{N}(ds, dz). \tag{2.3}$$

Let $\phi$ be a mollifier with compact support and set $\phi_\epsilon(x) := \epsilon^{-d} \phi(x/\epsilon)$. For fixed $x$, the function $\phi_\epsilon(x - \cdot)$ is in $V$, so we can plug it in (2.3) instead of $v$, to get that almost surely, for all $t \in [0, T]$,

$$u_t^\epsilon(x) = u_0^\epsilon(x) + \int_{[0,t]} v_s^{\epsilon*}(x) ds + \int_{[0,t]} h_s^{\epsilon k}(x) dw^k_s$$

$$+ \int_{[0,t]} \int_Z K_s^\epsilon(z, x) \tilde{N}(ds, dz),$$

where for $g \in V^*$ we have used the notation $g^\epsilon(x) := \langle g, \phi_\epsilon(x - \cdot) \rangle$. Note that $u_0^\epsilon$ is $\mathcal{F}_0 \times \mathcal{B}(\mathbb{R}^d)$ measurable. Also $u^\epsilon, v^{\epsilon*}$ and $h^{\epsilon k}$ are jointly measurable in $(t, \omega, x)$, $\mathcal{F}_t \times \mathcal{B}(\mathbb{R}^d)$ measurable for each $t$, and $K^\epsilon$ is $\mathcal{P} \times \mathcal{Z} \times \mathcal{B}(\mathbb{R}^d)$ measurable. It is also easy to see that there exists a constant $C_\epsilon$, depending on $\epsilon$, such that for all $t, \omega, x, z$

$$|u_t^\epsilon(x)| \leq C_\epsilon \|u_t\|_H, \quad |u_0^\epsilon(x)| \leq C_\epsilon \|u_0\|_H, \quad |v_t^{\epsilon*}|_H \leq C_\epsilon \|v_t^*\|_{V^*},$$

$$|v_t^{\epsilon*}(x)| \leq C_\epsilon \|v_t^*\|_{V^*}, \quad |h_t^{\epsilon k}(x)| \leq C_\epsilon \|h_t^k\|_H,$$

$$|K_t^\epsilon(x, z)| \leq C_\epsilon \|K_t(z)\|_H.$$ 

One can also check that for a constant $C$, for all $\epsilon$

$$|u_t^\epsilon|_H \leq C \|u_t\|_H, \quad |u_0^\epsilon|_H \leq C \|u_0\|_H, \quad |K_t^\epsilon(z)|_H \leq C \|K_t(z)\|_H$$

$$|h_t^{\epsilon k}|_H \leq C \|h_t^k\|_H, \quad |v_t^{\epsilon*}|_{V^*} \leq C \|v_t^*\|_{V^*}, \quad |u_t^\epsilon|_V \leq C \|u_t\|_V.$$
Now let $\alpha_\delta, \beta_\delta, \gamma_\delta$ be as before, and fix $x$. By Itô’s formula (see for example [9] or [13]), we have that for each $x$ there exists a set $\Omega_x$ of full probability, such that for all $\omega \in \Omega_x$ and $t \in [0,T]$ we have
\[
\gamma_\delta(u_t^\ast(x)) = \gamma_\delta(u_0^\ast(x)) + \int_{(0,t]} \beta_\delta(u_s^\ast(x))v_s^{\ast\ast}(x)ds \\
+ \sum_k \int_{(0,t]} \beta_\delta(u_s^\ast(x))h_s^{k\ast}(x)dw_s^k + \frac{1}{2} \sum_k \int_{(0,t]} \alpha_\delta(u_s^\ast(x))|h_s^{k\ast}(x)|^2ds \\
+ \int_{(0,t]} \int_Z \beta_\delta(u_{s-}^\ast(x))K_s^\ast(z,x)\tilde{N}(ds,dz) \\
+ \int_{(0,t]} \int_Z \gamma_\delta(u_s^\ast(x) + K_s^\ast(z,x)) \\
- \gamma_\delta(u_{s-}^\ast(x)) - \beta_\delta(u_{s-}^\ast(x))K_s^\ast(z,x)N(ds, dz) .
\] (2.4)
One can redefine the stochastic integrals such that (2.4) holds for all $(\omega, t, x)$. Integrating (2.4) over $\mathbb{R}^d$, taking appropriate versions of the stochastic integrals and using the Fubini and the stochastic Fubini theorems we get for each $t \in [0,T]$,
\[
\int_{\mathbb{R}^d} \gamma_\delta(u_t^\ast(x))dx = \int_{\mathbb{R}^d} \gamma_\delta(u_0^\ast(x))dx + \int_{(0,t]} \int_{\mathbb{R}^d} \beta_\delta(u_s^\ast(x))v_s^{\ast\ast}(x)dxds \\
\int_{(0,t]} \int_{\mathbb{R}^d} \beta_\delta(u_s^\ast(x))h_s^{k\ast}(x)dwdx_s^k + \frac{1}{2} \sum_k \int_{(0,t]} \int_{\mathbb{R}^d} \alpha_\delta(u_s^\ast(x))|h_s^{k\ast}(x)|^2dxds \\
+ \int_{(0,t]} \int_{Z} \int_{\mathbb{R}^d} \beta_\delta(u_{s-}^\ast(x))K_s^\ast(z,x)\tilde{N}(ds,dz) \\
+ \int_{(0,t]} \int_{Z} \int_{\mathbb{R}^d} \gamma_\delta(u_s^\ast(x) + K_s^\ast(z,x)) \\
- \gamma_\delta(u_{s-}^\ast(x)) - \beta_\delta(u_{s-}^\ast(x))K_s^\ast(z,x)N(ds, dz) (a.s.) .
\] (2.5)
For a stochastic Fubini theorem we refer to [12]. By the right-continuity of each of the terms we see that (2.5) holds almost surely, for all $t \in [0,T]$. We claim that letting a sequence $\epsilon_t \to 0$ we get
\[
\int_{\mathbb{R}^d} \gamma_\delta(u_t(x))dx = \int_{\mathbb{R}^d} \gamma_\delta(u_0(x))dx + \int_{(0,t]} \langle \beta_\delta(u_s(\cdot)), v_s^\ast \rangle ds \\
\int_{(0,t]} \int_{\mathbb{R}^d} \beta_\delta(u_s(x))h_s^k(x)dwdx_s^k + \frac{1}{2} \sum_k \int_{(0,t]} \int_{\mathbb{R}^d} \alpha_\delta(u_s(x))|h_s^k(x)|^2dxds \\
+ \int_{(0,t]} \int_{Z} \int_{\mathbb{R}^d} \beta_\delta(u_{s-}(x))K_s(z,x)dx\tilde{N}(ds,dz)
\]
We show that each of the terms in (2.5) converges to the corresponding ones in (2.6). For the left-hand side, we show that for any sequence $\epsilon_k \rightarrow 0$, there exists a subsequence of $\epsilon_k$, such that the convergence takes place. To this end, let $\epsilon_k \rightarrow 0$ and fix $(\omega, t)$. Since $u^{\epsilon_k}_t \rightarrow u_t$ in $L^2(\mathbb{R}^d)$, by the equality $(a^2 - b^2) = (a - b)(a + b)$ we see that $(u^{\epsilon_k}_t)^2 \rightarrow u^2_t$ in $L^1(\mathbb{R}^d)$. By Lemma 2.2, there exist $g \in L^1(\mathbb{R}^d)$ and a subsequence, denoted again $\epsilon_k$, such that $|u^{\epsilon_k}_t(x)|^2 \leq g(x)$ for all $x$. We have $\gamma_\delta(u^{\epsilon_k}_t(x)) \rightarrow \gamma_\delta(u_t(x))$ for almost every $x$, and since $|\gamma_\delta(u^{\epsilon_k}_t(x))| \leq (u^{\epsilon_k}_t(x))^2 \leq g(x)$, we obtain $\int_{\mathbb{R}^d} \gamma_\delta(u^{\epsilon_k}_t(x)) dx \rightarrow \int_{\mathbb{R}^d} \gamma_\delta(u_t(x)) dx,$ by Lebesgue’s dominated convergence theorem. To see the convergence of the second term in the right-hand side of (2.5) we fix $(s, \omega)$ such that $u_s \in V$. Taking into account the well-known fact that there exist $f^i_s \in L^2(\mathbb{R}^d)$ for $i = 1, \ldots, d$ such that $v^* = f^0_s + D_i f^i_s,$ it is not difficult to see that $v^{\epsilon\ast}_s = f^{\epsilon\ast}_s + D_i f^{\epsilon\ast}_s.$ Therefore $|v^*_s - v^{\ast\epsilon}_s|_V \leq \left( \sum_{i=0}^n |f^{\epsilon\ast}_s - f^i_s|_H \right)^{1/2} \rightarrow 0.$ It is also straightforward to check that $|\beta_\delta(u^*_s) - \beta_\delta(u_s)|_V \rightarrow 0.$ Hence we conclude $\int_{\mathbb{R}^d} \beta_\delta(u^*_s(x))v^{\ast\epsilon}_s(x) dx = \langle v^{\ast\epsilon}_s, \beta_\delta(u^*_s) \rangle = \langle v^*_s, \beta_\delta(u_s) \rangle.$ Notice now that for each $\epsilon$ for almost every $(\omega, s)$ we have $\left| \int_{\mathbb{R}^d} \beta_\delta(u^*_s(x))v^{\ast\epsilon}_s(x) dx \right| \leq C(|u_s|_V^2 + |v^*_s|_{\ast V}^2).$
Therefore, almost surely
\[ \int_{(0,t]} \int_{\mathbb{R}^d} \beta_\delta(u^*_s(x))v^*_s(x)dxds \to \int_{(0,t]} \langle v^*_s, \beta_\delta(u_s) \rangle ds \quad \text{for all } t. \]

For the sum of the stochastic integrals against the Wiener processes we just note that for almost all \((\omega, s)\)
\[ \sum_k \left| \int_{\mathbb{R}^d} \beta_\delta(u^*_s(x))h^{k_\epsilon}_s(x)dx - \int_{\mathbb{R}^d} \beta_\delta(u_s(x))h^k_s(x)dx \right|^2 \to 0 \]
and
\[ \sum_k \left| \int_{\mathbb{R}^d} \beta_\delta(u^*_s(x))h^{k_\epsilon}_s(x)dx - \int_{\mathbb{R}^d} \beta_\delta(u_s(x))h^k_s(x)dx \right|^2 \leq 4 \sup_{t \leq T} |u_t|_{L^2}^2 \sum_k |h^k_s|_{L^2}. \]

Hence,
\[ \int_{(0,T]} \sum_k \left| \int_{\mathbb{R}^d} \beta_\delta(u^*_s(x))h^{k_\epsilon}_s(x)dx - \int_{\mathbb{R}^d} \beta_\delta(u_s(x))h^k_s(x)dx \right|^2 ds \to 0 \]
almost surely, which implies that the sum of the stochastic integrals converges in probability, uniformly in \(t\), and there exists a sequence \(\epsilon_t\) such that the convergence happens almost surely. Note that for each \(k\) we have
\[ \left| \int_{\mathbb{R}^d} \alpha_\delta(u^*_s(x)) |h^{k_\epsilon}_s(x)|^2 - \alpha_\delta(u_s(x)) |h^k_s(x)|^2 dx \right| \leq \int_{\mathbb{R}^d} |(h^{k_\epsilon}_s(x))^2 - (h^k_s(x))^2| dx \]
\[ + \int_{\mathbb{R}^d} |h^k_s(x)|^2 |\alpha_\delta(u^*_s(x)) - \alpha_\delta(u_s(x))| dx \to 0. \]

Moreover,
\[ \left| \int_{\mathbb{R}^d} \alpha_\delta(u^*_s(x)) |h^{k_\epsilon}_s(x)|^2 dx \right| \leq |h^k_s|_{H}^2, \]
where the right-hand side is almost surely integrable on \([0,T]\). Hence the convergence of the fourth term in the right-hand side of (2.5) follows. Using the inequalities in (2.2), similar arguments show the convergence of the last two terms. We conclude that almost surely (2.6) holds for all \(t \in [0,T]\).

Now by letting \(\delta \to 0\) in (2.6), using arguments similar to the previous ones, and keeping in mind the inequalities (2.2) and the fact that for all \(v \in V\)
\[ |\beta_\delta(v) - v^+|_V \to 0, \quad |\beta_\delta(v)|_V \leq |v|_V, \]
we can finish the proof of the theorem for \(Q = \mathbb{R}^d\).
We reduce the case of a bounded Lipschitz domain $Q$ to that of the whole space by using the sequence $\phi_n$ from Lemma 2.3. Remember that $\phi_n$ has compact support in $Q$. Thus for a function $\eta$ on $Q$ we denote by $\phi_n \eta$, not only the function defined on $Q$ by the multiplication of $\phi_n$ and $\eta$, but also its extension to zero outside of $Q$. Notice that when $u$ satisfies (2.1) on $Q$, then $\phi_n u$ satisfies
\[
\phi_n u_t = \phi_n u_0 + \int_{(0,t]} \phi_n v_s^* ds + \int_{(0,t]} \phi_n h_s^k dw_s^k
\]
+ $\int_{(0,t]} \phi_n K_s(z) \tilde{N}(ds, dz)$
on the whole $\mathbb{R}^d$, where the functional $\phi_n v^*$ is defined by
\[
\langle \phi_n v^*_s, g \rangle := \langle v^*_s, \phi_n g \rangle_Q
\]for $g \in H^1(\mathbb{R}^d)$. The notation $\langle \cdot, \cdot \rangle_Q$ means the duality product between $H^1_0(Q)$ and $H^{-1}(Q)$. Notice that $\langle v^*_s, \phi_n g \rangle_Q$ is well defined, since the restriction of $\phi_n g$ to $Q$ belongs to $H^1_0(Q)$. Then by the result in the case of the whole space, we have
\[
\int_Q \phi_n^2 |u|^2 dx = \int_Q |\phi_n u_0^+|^2 dx + 2 \int_{(0,t]} \langle v^*_s, \phi_n u_s^+ \rangle_Q ds
\]
+ $2 \int_{(0,t]} \int_Q \phi_n^2 h_s^k u_s^+ dw_s^k + \int_{(0,t]} \int_Q \sum_k [I_{\{\phi_n u_s > 0\}} \phi_n h_s^k]^2 dx ds$
+ $\int_{(0,t]} \int_Z \int_Q 2K_s(z) \phi_n^2 u_s^+ dx \tilde{N}(ds, dz)$
+ $\int_{(0,t]} \int_Z \int_Q |\phi_n (u_s + K_s(z))^+|^2 - |\phi_n u_s^+|^2 - 2K_s(z) \phi_n^2 u_s^+ dx N(dz, ds)$,
since $\phi_n$ is supported in $Q$. It is now easy to take $n \to \infty$ here to finish the proof of the theorem. We only note that for the second term on the right-hand side we have by Lemma 2.3 and Remark 2.1
\[
\langle v^*_s, \phi_n^2 u_s^+ \rangle \to \langle v^*_s, u_s^+ \rangle \quad \text{for all } \omega, s,
\]
and for a constant $C$,
\[
\langle v^*_s, \phi_n^2 u_s^+ \rangle \leq C |v^*_s|_{V} |u_s|_V \quad \text{for all } n.
\]
3. Comparison Theorems

In this section we present our comparison theorems for two types of equations. We consider another measurable space \((\mathcal{F}, \mathcal{F})\), a quasi left-continuous, adapted point process \((\bar{p}_t)_{t \in [0,T]}\) in \(\mathcal{F}\), and two \(\sigma\)-finite measures \(\pi^{(1)}, \pi^{(2)}\) on \(\mathcal{F}\). Let \(M(dt, d\zeta)\) be the corresponding random measure on \([0,T] \times \mathcal{F}\). We assume that its compensator is \(dt \pi^{(2)}(d\zeta)\) and we write
\[
\tilde{M}(dt, d\zeta) = M(dt, d\zeta) - dt \pi^{(2)}(d\zeta). 
\]
The first equation is
\[
du_t(x) = \{L_t u_t(x) + f_t(x, u_t(x), \nabla u_t(x))\} dt 
+ G^k_t(u)(x) dw^k_t + \int_Z g_t(x, z, u_{t-}(x)) \tilde{N}(dt, dz),
\]
for \((t, x) \in [0, T] \times Q\), with initial condition\
\[
u_0(x) = \psi(x), \quad x \in Q, 
\]
where
\[
L_t u(x) = D_i (a^{ij}_t(x) D_j u(x)) + \mathcal{I}^{(1)}_t u(x), \quad D_i = \frac{\partial}{\partial x^i}, \quad i = 1, \ldots, d, 
\]
\[
\mathcal{I}^{(1)}_t u(x) = \int_{\mathcal{F}} [u(x + c_t(x, \zeta)) - u(x) - c_t(x, \zeta) \cdot \nabla u(x)] m_t(x, \zeta) \pi^{(1)}(d\zeta), 
\]
\[
G^k_t(u)(x) = \phi^{ik}_t(x) D_i u_t(x) + \sigma^k_t(x, u_t(x)).
\]

We make the following assumptions. Let \(K > 0\) denote a constant.

**Assumption 3.1.**

i) The coefficients \(a^{ij}_t\), are real-valued \(\mathcal{P} \times \mathcal{B}(Q)\) measurable functions on \(\Omega \times [0, T] \times Q\) and are bounded by \(K\) for every \(i, j = 1, \ldots, d\). The coefficient \(\phi^i = (\phi^{ik})_{k=1}^{\infty}\) is an \(l_2\)-valued \(\mathcal{P} \times \mathcal{B}(Q)\)-measurable function such that
\[
\sum_i \sum_k |\phi^{ik}_t(x)|^2 \leq K \text{ for all } \omega, t \text{ and } x, 
\]
ii) \(f\) is a real valued \(\mathcal{P} \times \mathcal{B}(Q) \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^d)\)-measurable function on \(\Omega \times [0, T] \times Q \times \mathbb{R} \times \mathbb{R}^d\), and \(\sigma = (\sigma^k)_{k=1}^{\infty}\) is a \(\mathcal{P} \times \mathcal{B}(Q) \times \mathcal{B}(\mathbb{R})\)-measurable function on \(\Omega \times [0, T] \times Q \times \mathbb{R}\), with values in \(l_2\). The function \(g\) is defined on \(\Omega \times [0, T] \times Q \times Z \times \mathbb{R}\) with values in \(\mathbb{R}\) and it is \(\mathcal{P} \times \mathcal{B}(Q) \times \mathcal{Z} \times \mathcal{B}(\mathbb{R})\)-measurable. We assume that there exists \(h \in L_2(\Omega \times [0, T] \times Q)\), predictable as a process with values in \(L_2(Q)\) such that for all \(\omega, t, x, z, r, r'\)
Assumption 3.2. For all $E_{\xi} = (\omega, t, x) \times (\rho, z)$:

\[ |f_t(x, r, r')|^2 + \sum_{k} |\sigma_t^k(x, r)|^2 + \int_{\mathbb{Z}} |g_t(x, z, r)|^2 \nu(dz) \leq K|\rho|^2 + K|r'|^2 + |h_t(x)|^2, \]

i) $\psi$ is an $\mathcal{F}_0$-measurable random variable in $L_2(Q)$ and we have $E|\psi|^2_{L_2} < \infty$.

iii) $\psi$ is an $\mathcal{F}_0$-measurable random variable in $L_2(Q)$ and we have $E|\psi|^2_{L_2} < \infty$.

iv) there exists a constant $\kappa > 0$ such that for all $\omega, t, x$ and for all $\xi = (\xi_1, ..., \xi_d) \in Q$ we have

\[ a_{ij}^k(\omega, t, x, \xi, \eta) = \frac{1}{2} \phi_t^i(x) \phi_t^j(x) \xi_i \xi_j \geq \kappa |\xi|^2, \]

v) for all $\omega, t, x, z, r_1, r_2$,

\[ \sum_{k} |\sigma_t^k(x, r_1) - \sigma_t^k(x, r_2)|^2 \leq K|r_1 - r_2|^2 \]

vi) For each $\omega, t, x, r', f_t(x, r, r')$ is continuous in $r$.

Assumption 3.3. The function $r + g_t(x, z, r)$ is non-decreasing in $r$ for all $\omega, t, x, z$.

Assumption 3.4. The function $c : \Omega \times [0, T] \times \mathbb{R}^d \times F \to \mathbb{R}^d$ is $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{F}$-measurable and there exists a measurable function $c^{(0)} : F \to \mathbb{R}^d$ such that

(i) $|c_t(x, \zeta)| \leq |c^{(0)}(\zeta)|$, for all $\omega, t, x, \zeta$,

(ii) $\int_F |c^{(0)}(\zeta)|^2 \pi(\zeta) d\zeta \leq K$,

(iv) $|c_t(x, \zeta) - c_t(y, \zeta)| \leq c^{(0)}(\zeta) |x - y|$, for all $\omega, t, x, y, \zeta$.

Assumption 3.5. The function $m : \Omega \times [0, T] \times \mathbb{R}^d \times F \to \mathbb{R}$ is $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{F}$-measurable and satisfies the following,

(i) $0 \leq m_t(x, \zeta) \leq K$, for all $\omega, t, x, \zeta$.

(ii) $|m_t(x, \zeta) - m_t(y, \zeta)| \leq K|x - y|$, for all $\omega, t, x, y, \zeta$.

Assumption 3.6. The functions $c^l_t(x, \zeta)$, $l = 1, ..., d$, for each $\omega, t, \zeta$ are twice continuously differentiable in $x$, and

(i) $|D_c^l_t(x, \zeta)| \leq K$, $|D_{ij}^l c^l_t(x, \zeta)| \leq K$ for all $i, j, l = 1, ..., d$,

(ii) $K^{-1} \leq |\det(\mathbb{I} + \theta \nabla c^l_t(x, \zeta))| \leq K$ for all $\omega, t, x, \zeta$ and $\theta \in [0, 1]$. 
Under these assumptions, we see that $I$ maps $\Omega \times [0, T] \times H_0^1(Q)$ into $H^{-1}(Q)$ and is $\mathcal{P} \times \mathcal{B}(H_0^1(Q))$ measurable. For fixed $(\omega, t)$, its action on $u \in H_0^1(Q)$ is understood in the following way. First suppose that $Q$ is the whole $\mathbb{R}^d$ and denote by $T_{\theta,t,\zeta}$ the mapping $x \mapsto x + \theta c_t(x, \zeta)$, and by $J_{\theta,t,\zeta}$ its inverse. For $u \in C^\infty_c(\mathbb{R}^d)$ (even for $u \in W_2^2(\mathbb{R}^d)$) one can easily see that $I(t)u(x)$ is a function in $L^2(\mathbb{R}^d)$. Then for $v \in H^1(\mathbb{R}^d)$ we have

$$\langle I(t)u, v \rangle = \langle I(t)u, v \rangle = \int_0^1 (\theta - 1) \int \int_{\mathbb{R}^d} D_i u(x + \theta c_t(x, \zeta)) D_j (q_{ij}(t, x, \zeta, \theta) v(x)) dx \pi^{(1)}(d\zeta) d\theta,$$

(3.2)

where the last equality is obtained by Taylor Formula and integration by parts, and $q_{ij}$ is given by

$$q_{ij}(t, x, \zeta, \theta) := \sum_{l=1}^d c^l_i(x, \zeta) c^l_j(x, \zeta) m_l(x, \zeta) D_l J_{\theta,t,\zeta}(T_{\theta,t,\zeta}(x)).$$

It follows immediately that for a constant $N = N(d, K)$,

$$|\langle I(t)u, v \rangle| \leq N |u|_{H^1(\mathbb{R}^d)} |v|_{H^1(\mathbb{R}^d)},$$

showing that $I(t)$ extends uniquely to a bounded linear operator from $H^1$ to $H^{-1}$, and in fact is given by (3.2). In case $Q$ is a bounded Lipschitz domain, one can define the action of $I(t)$ on $u \in H_0^1(Q)$, again by (3.2), where $u$ and $v$ this time are extended to zero outside of $Q$. For further study of these operators we refer to [5].

**Definition 3.1.** A strongly càdlàg adapted process $u$ with values in $L_2(Q)$ is called a solution of equation (3.1) if

i) $u_t \in H_0^1(Q)$ for $dP \times dt$ almost every $(\omega, t) \in \Omega \times [0, T]$

ii) $E \int_{(0,T)} |u_t|^2_{H_0^1} dt < \infty$

iii) for all $v \in H_0^1(Q)$ we have almost surely, for all $t \in [0, T]$

$$(u_t, v) = (\psi, v) + \int_{[0,T]} \left\{ - (a_{ij} D_i u_s D_j v) + (f_s(u_s, \nabla u_s), v) + \langle I_s^{(1)} u_s, v \rangle \right\} ds$$

$$+ \int_{[0,t]} \{ (\phi^{ik} D_i u_s, v) + (\sigma^{ik}_s(u_s), v) \} dw^k_s + \int_{(0,t]} \int_{Z} (g_s(z, u_{-s}) v) \tilde{N}(dz, ds),$$

where $(\cdot, \cdot)$ is the inner product in $L^2(Q)$.

We are now ready to state our result for equation (3.1).
Theorem 3.1. Suppose that Assumptions 3.1, 3.3-3.6 hold. Let \( u \) and \( v \) satisfy the equations

\[
\begin{align*}
\frac{du_t}{dt}(x) &= \left\{ L_t u_t(x) + f_t(x, u_t(x), \nabla u_t(x)) + \int_Z g_t(x, z, u_t(x)) \tilde{N}(dt, dz) \right\} dt \\
&\quad + G^k_t(u(x))dw_t^k + \int_Z g_t(x, z, u_t(x)) \tilde{N}(dt, dz), \\
\quad u_0(x) = \psi(x),
\end{align*}
\]

\[
\begin{align*}
\frac{dv_t}{dt}(x) &= \left\{ L_t v_t(x) + F_t(x, v_t(x), \nabla v_t(x)) + \int_Z g_t(x, z, v_t(x)) \tilde{N}(dt, dz) \right\} dt \\
&\quad + G^k_t(v(x))dw_t^k + \int_Z g_t(x, z, v_t(x)) \tilde{N}(dt, dz), \\
\quad v_0(x) = \Psi(x).
\end{align*}
\]

Suppose that either \( f \) or \( F \) satisfy Assumption 3.2. Let \( f \leq F \) and \( \psi \leq \Psi \). Then almost surely, for all \( t \in [0, T] \) we have \( u_t(x) \leq v_t(x) \) for almost every \( x \in Q \).

Remark 3.1. Assumption 3.3 cannot be omitted in Theorem 3.1. Consider for example the SDE

\[
u_t = 1 - \int_{(0,t]} 2u_{s-} d\tilde{N}_s,
\]

where \( N_t \) is a Poisson process with intensity one. Let \( \tau \) be the time that the first jump of \( N \) occurs. Then \( P(\tau \leq T) > 0 \). Since \( u_t = e^{-2t} \) on \([0, \tau)\), one can see that on the set \( \{ \tau \leq T \} \) we have \( u(\tau) = e^{-2\tau} < 0 \).

The second equation that we will deal with is

\[
\begin{align*}
\frac{du_t}{dt}(x) &= \left\{ L_t u_t(x) + f_t(x, u_t(x), \nabla u_t(x)) \right\} dt \\
&\quad + G^k_t(u_t(x))dw_t^k + \int_{F} S_{t,\zeta} u_{t-}(x) M(ds, d\zeta) \\
\quad u_0(x) = \psi(x), \quad x \in \mathbb{R}^d,
\end{align*}
\]

for \((t, x) \in [0, T] \times \mathbb{R}^d\), with initial condition

\[
u_0(x) = \psi(x), \quad x \in \mathbb{R}^d,
\]
where
\[ L_t u(x) = L_t u(x) + I^{(2)}_t u(x) + J_t u(x) - K_t u(x) \]
\[ I^{(2)}_t u(x) = \int_F [u(x + b_t(\zeta)) - u(x) - b_t(\zeta) \cdot \nabla u(x)] \pi^{(2)}(d\zeta), \]
\[ J_t u(x) = \int_F (u(x + b_t(\zeta)) - u(x)) \pi_t(x,\zeta) \eta(2)(d\zeta), \]
\[ K_t u(x) = \int_F u(x + b_t(\zeta)) - u(x) \pi^{(2)}(d\zeta), \]
\[ S_{t,\zeta} u(x) = \lambda_t(x + b_t(\zeta),\zeta) u_t(x + b_t(\zeta)) - \lambda_t(x,\zeta) u_t(x) \]
\[ + (\lambda_t(x,\zeta) - 1) u_t(x) \]

Obviously, if we ask later for some of the previous assumptions to hold for equation (3.3), we mean with \( g \equiv 0 \).

**Assumption 3.7.** The function \( b : \Omega \times [0, T] \times F \rightarrow \mathbb{R}^d \) is \( \mathcal{P} \times \mathcal{F} \)-measurable and there exists positive function \( b^{(0)}(0) \) on \( F \), such that for all \( \omega, t \) and \( \zeta \) we have
\[ |b_t(\zeta)| \leq b^{(0)}(\zeta), \int_F b^{(0)}(\zeta) \eta(2)(d\zeta) \leq K. \]

The function \( \lambda : \Omega \times [0, T] \times \mathbb{R}^d \times F \rightarrow (0, \infty) \) is \( \mathcal{P} \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{F} \)-measurable, is continuously differentiable in \( x \) and for all \( \omega, t, x \) and \( \zeta \)
\[ |\lambda_t(x,\zeta)| + |\nabla \lambda_t(x,\zeta)| \leq K, \quad |1 - \lambda_t(x,\zeta)| \leq b^{(0)}(\zeta). \]

As before, the mapping \( I^{(2)} : \Omega \times [0, T] \times H^1(\mathbb{R}^d) \rightarrow H^{-1}(\mathbb{R}^d) \) is well defined and it is appropriately measurable. In the same way, one can check that \( J \) and \( K \) are mappings defined on \( \Omega \times [0, T] \times H^1(\mathbb{R}^d) \) taking values in \( L_2(\mathbb{R}^d) \) and they are \( \mathcal{P} \times \mathcal{B}(H^1(\mathbb{R}^d)) \)-measurable.

The solution of equation (3.3) is understood in the same sense as the one of equation (3.1).

**Theorem 3.2.** Suppose that Assumptions 3.1, 3.3-3.6 and 3.7 hold. Let \( u \) and \( v \) satisfy the equations
\[ du_t(x) = \{ L_t u_t(x) + f_t(x, u_t(x), \nabla u_t(x)) \} dt \]
\[ + G^k_t(u) dw^k_t + \int_F S_{t,\zeta} u(x) M(ds,d\zeta), \]
\[ u_0(x) = \psi(x), \]
\[ dv_t(x) = \{ \mathcal{L}_t v_t(x) + F_t(x, v_t(x), \nabla v_t(x)) \} dt + G^k_t(v(x)) dw^k_t + \int_F S_t, \zeta v(x) \tilde{M}(ds, d\zeta), \]

\[ v_0(x) = \Psi(x), \]

respectively. Suppose that either \( f \) or \( F \) satisfy Assumption 3.2. Let \( f \leq F \) and \( \psi \leq \Psi \). Then almost surely, for all \( t \in [0, T] \) we have \( u_t(x) \leq v_t(x) \) for almost every \( x \in \mathbb{R}^d \).

**Remark 3.2.** Under Assumptions 3.1-3.6, there exists a unique solution of equation (3.1). Also, under Assumptions 3.1-3.7 there exists a unique solution of equation (3.3). This is due to [7].

### 4. Auxiliary Facts

In this section we give some results that we will need for the proof of Theorems 3.1 and 3.2. The following is well known.

**Lemma 4.1.** Let \( u \in W^1_p(Q) \). Let \( u^n \in W^1_p(Q) \) such that \( |u_n - u|_{W^1_p} \to 0 \). Then we have that \( |u^n + u - 2u|_{W^1_p} \to 0 \).

For the next three Lemmas, we assume that Assumptions 3.4, 3.5, 3.6 and 3.7 hold.

**Lemma 4.2.** For any \( u \in C^\infty_c(\mathbb{R}^d) \), \( (\omega, t) \in \Omega \times \mathbb{R}^+ \), and \( \varepsilon > 0 \) we have

\[ \int_{\mathbb{R}^d} I_i^{(1)}(u^+)^2 dx \leq \varepsilon |u^+|_{H^1(\mathbb{R}^d)}^2 + N(\varepsilon) |u^+|_{L^2(\mathbb{R}^d)}^2, \]

for \( i = 1, 2 \), where the constant \( N(\varepsilon) \) depends only on \( \varepsilon, K \) and \( d \).

**Proof.** It suffices to prove the Lemma for \( i = 1 \). We have

\[ \int_{\mathbb{R}^d} I_t^{(1)}(u^+)^2 dx = \int_{\mathbb{R}^d} I_t^{(1, \delta)}(u^+)^2 dx + \int_{\mathbb{R}^d} I_t^{(1, \delta^c)}(u^+)^2 dx \]

where the operators \( I_t^{(1, \delta)} \) and \( I_t^{(1, \delta^c)} \) are defined as \( I_t^{(1)} \) with the only difference being that integration happens only on \( F_\delta := \{ \zeta : c(\zeta) < \delta \} \) and \( F_\delta^c \) respectively. The first term on the right hand side is equal to

\[ \int_0^1 (1 - \theta) \int_{F_\delta} \int_{\mathbb{R}^d} D_{ij}(u^+)^2(x + \theta c_t(x, \zeta)) \times c_i^t(x, \zeta) c_j^t(x, \zeta) m_t(x, \zeta) d\pi_1(d\zeta) d\theta \]

\[ = E_1(t, \delta) + E_2(t, \delta), \]
where
\[
E_1(t, \delta) = \int_0^1 (1 - \theta) \int_{F_3} \int_{\mathbb{R}^d} 2D_i u^+(x + \theta c_t(x, \zeta)) D_j u^+(x + \theta c_t(x, \zeta))
\]
\[\times c^j_t(x, \zeta) c^i_t(x, \zeta) m_t(x, \zeta) d\pi^{(1)}(d\zeta) d\theta,\]
\[
E_2(t, \delta) = \int_0^1 (1 - \theta) \int_{F_3} \int_{\mathbb{R}^d} 2u^+(x + \theta c_t(x, \zeta)) D_{ij} u(x + \theta c_t(x, \zeta))
\]
\[\times c^j_t(x, \zeta) c^i_t(x, \zeta) m_t(x, \zeta) d\pi^{(1)}(d\zeta) d\theta.\]

Using Assumptions 3.4, 3.5 and 3.6, we see after a change of variables that
\[
|E_1(t, \delta)| \leq C(\delta) C|u^+|_{H^1(\mathbb{R}^d)}^2,
\]
where \(C(\delta) = \int_{F_3} |c^{(0)}(\zeta)|^2 \pi(d\zeta)\) and \(C\) is a constant depending only on \(K\) and \(d\). For \(E_2\) we have
\[
E_2(t, \delta) = \int_0^1 (1 - \theta) \int_{F_3} \int_{\mathbb{R}^d} 2D_j (D_i u(x + \theta c_t(x, \zeta))
\]
\[\times q_{ij}(t, x, \zeta, \theta) u^+(x + \theta c_t(x, \zeta)) d\pi^{(1)}(d\zeta) d\theta.
\]
By integration by parts and using the Assumptions 3.4, 3.5 and 3.6 again we see that
\[
|E_2(t, \delta)| \leq C(\delta) C|u^+|_{H^1(\mathbb{R}^d)}^2.
\]

For the second term in (4.4), by Young’s inequality and Assumptions 3.5, 3.4 we have
\[
\int_{\mathbb{R}^d} \mathcal{I}^{(1, \delta)}_t (u ^ +)^2(x) \, dx \leq \gamma |u|_{H^1(\mathbb{R}^d)}^2 + C(\gamma)|u|_{L^2(\mathbb{R}^d)}^2,
\]
for all \(\gamma > 0\), where \(C(\gamma)\) depends only on \(\gamma, K\) and \(d\). Putting all these estimates together, and choosing \(\delta\) and \(\gamma\) sufficiently small proves the lemma. \(\square\)

**Lemma 4.3.** For any \(u \in H^1_0(Q)\), \((\omega, t) \in \Omega \times \mathbb{R}^+\) and \(\varepsilon > 0\) we have
\[
2 \langle \mathcal{I}^{(1)}_t u, u^+ \rangle \leq \varepsilon |u^+|^2_{H^1_0(Q)} + N(\varepsilon)|u^+|^2_{L^2_\omega(Q)}, \tag{4.5}
\]
for \(i = 1, 2\), where the constant \(N(\varepsilon)\) depends only on \(\varepsilon\) and \(K\) and \(d\).

**Proof.** It suffices to prove the lemma for \(i = 1\) and \(Q = \mathbb{R}^d\). Suppose first that \(u \in C_c^\infty(\mathbb{R}^d)\). Then
\[
2 \langle \mathcal{I}^{(1)}_t u, u^+ \rangle = 2 \langle \mathcal{I}^{(1)}_t u, u^+ \rangle
\]
\[
= \int_{\mathbb{R}^d} \int_F [(u^+)^2(x + c_t(x, \zeta)) - (u^+)^2(x) - 2c_t(x, \zeta) \cdot \nabla u(x) u^+(x)]
\]
\[\times m_t(x, \zeta) \pi^{(1)}(d\zeta) dx.
\]
\[-\int_{\mathbb{R}^d} \int_F [(u^+)^2(x + c_t(x, \zeta)) - 2u^+(x)u(x + c_t(x, \zeta)) + (u^+)^2(x)]
\times m_t(x, \zeta) \pi^{(1)}(d\zeta) dx
\leq \int_{\mathbb{R}^d} \mathcal{I}_t^{(1)}(u^+)^2(x) dx - \int_{\mathbb{R}^d} \int_F [(u^+)^2(x + c_t(x, \zeta)) - 2u^+(x)u^+(x + c_t(x, \zeta)) + (u^+)^2(x)] m_t(x, \zeta) \pi^{(1)}(d\zeta) dx \leq \int_{\mathbb{R}^d} \mathcal{I}_t^{(1)}(u^+)^2(x) dx
\leq \varepsilon |u^+|^2_{H^1(\mathbb{R}^d)} + N(\varepsilon) |u^+|^2_{L^2(\mathbb{R}^d)},\]

by Lemma 4.2. Since \(\mathcal{I}_t^{(1)}\) is a continuous linear operator from \(H^1(\mathbb{R}^d)\) into \(H^{-1}(\mathbb{R}^d)\), for each \((\omega, t)\), we see that inequality (4.5) remains valid for all \(u \in H^1(\mathbb{R}^d)\) by virtue of Lemma 4.1.

For \(u \in H^1(\mathbb{R}^d)\) we set
\[
\mu_t(u) := \int_F \int_{\mathbb{R}^d} [(\lambda_t(x + b_t(\zeta), \zeta)u(x + b_t(\zeta)))^+ - u^+(x)]^2 dx \pi^{(2)}(d\zeta)
- 2u^+(x)[\lambda_t(x + b_t(\zeta), \zeta)u(x + b_t(\zeta)) - u(x)]dx \pi^{(2)}(d\zeta),
\]
\[
\rho_t(u) := 2\langle \mathcal{I}_t^{(2)}u, u^+ \rangle + 2\langle \mathcal{J}_t u, u^+ \rangle - 2\langle \mathcal{K}_t u, u^+ \rangle + \mu_t(u). \tag{4.6}
\]

Using the simple inequality \(|[x + y]^+|^2 - [x^+]^2 - 2|x|^2 \leq |y|^2\), and Assumption 3.7 one can see that \(\mu_t(u)\) is continuous in \(u \in H^1(\mathbb{R}^d)\).

**Lemma 4.4.** For any \(u \in H^1(\mathbb{R}^d)\), \((\omega, t) \in \Omega \times \mathbb{R}^+\) and \(\varepsilon > 0\) we have
\[
\rho_t(u) \leq \varepsilon |u^+|^2_{H^1(\mathbb{R}^d)} + N(\varepsilon) |u^+|^2_{L^2(\mathbb{R}^d)}. \tag{4.7}
\]

**Proof.** Clearly it suffices to show (4.7) for \(u \in C^\infty_c(\mathbb{R}^d)\). A simple calculation shows that
\[
\rho_t(u) = 2 \int_F \int_{\mathbb{R}^d} (1 - \lambda_t(x, \zeta))[u^+(x)]^2 dx \pi^{(2)}(d\zeta)
+ \int_F \int_{\mathbb{R}^d} [(\lambda_t(x + b_t(\zeta), \zeta)u(x + b_t(\zeta)))^+ - u^+(x)]^2
- 2\nabla u(x) \cdot b_t(\zeta)u^+(x) dx \pi^{(2)}(d\zeta).
\]

By Assumption 3.7 we see that
\[
\rho_t \leq 2K|u^+|^2_{L^2(\mathbb{R}^d)} + \int_F \int_{\mathbb{R}^d} (\lambda_t(x + b_t(\zeta), \zeta))^2 [u^+(x + b_t(\zeta))]^2 - [u^+(x)]^2
- 2\nabla u(x) \cdot b_t(\zeta)u^+(x) dx \pi^{(2)}(d\zeta)
= 2K|u^+|^2_{L^2(\mathbb{R}^d)} + \int_{\mathbb{R}^d} \mathcal{I}_t^{(2)}(u^+)^2(x) dx
\]
for a local martingale by Assumption 3.7 and Lemma 4.2. This finishes the proof. □

A

Without loss of generality we can assume that

Proof of Theorem 3.1.

Assumption 3.2 is satisfied by only on

\( \varepsilon > 0 \), there exist \( \varepsilon > 0 \), such that

One can easily see that for every \( \varepsilon > 0 \), there exist \( C(\varepsilon) > 0 \) depending only on \( \varepsilon \), \( K \) and \( d \), such that

\[ A_s^{(1)} \leq (\varepsilon + \varepsilon)|h_s^+|_{L^2(Q)}^2 + C(\varepsilon)|h_s^+|_{L^2(Q)}^2. \]
For $A^{(2)}$, we have the following. Since $f \leq F$, using the second part of Assumption 3.2 and Young’s inequality we have

$$A^{(2)}_t \leq \int_Q (f_s(x, u_s, \nabla u_s) - f_s(x, v_s, \nabla u_s)) h^+_s(x) \, dx$$

$$+ \frac{K}{\varepsilon} |h^+_s|_{L^2(Q)}^2 + \varepsilon |\nabla h^+_s|_{L^2(Q)}^2.$$

By Assumption 3.3 we obtain

$$A^{(3)}_t = \int_Q \int_Z I_{h_s > 0} |g_s(x, z, u_s) - g_s(x, z, v_s)|^2 \nu(dz) \, dx.$$

Hence, combining these estimates, using Assumption 3.2 and Lemma 4.3 we have a constant $C$ such that, almost surely

$$|h^+_t|_{L^2(Q)}^2 \leq C \int_{(0,t]} |h^+_s|_{L^2(Q)}^2 \, ds + m_t \quad \text{for all } t \in [0, T].$$

By a standard localization argument and Fatou’s lemma we get

$$E|h^+_t|_{L^2(Q)}^2 \leq C \int_{(0,t]} E|h^+_s|_{L^2(Q)}^2 \, ds < \infty \quad \text{for all } t \in [0, T],$$

and the result follows by Gronwall’s lemma.

Proof of Theorem 3.2. We assume again that Assumption 3.2 is satisfied by $f$. For the difference $h = u - v$ we have

$$h_t = h_0 + \int_{(0,t]} \{\mathcal{L}_s h_s + f_s(u_s, \nabla u_s) - F_s(v_s, \nabla v_s)\} \, ds$$

$$+ \int_{(0,t]} \{\phi^{ki}_s D_i h_s + \sigma^k_s(u_s) - \sigma^k_s(v_s)\} \, dw^k_s + \int_{(0,t]} \int_F S_{s,\zeta} h_s - \tilde{M}(ds, d\zeta)$$

By Theorem 2.1 we have

$$|h^+_t|_{L^2} \geq \int_{(0,t]} A^{(1)}_s + A^{(2)}_s + \rho_s(h_s) + \langle \mathcal{I}^{(1)}_s h_s, h^+_s \rangle \, ds + m_t$$

for a local martingale $m_t$. Here $A^{(1)}, A^{(2)}$ are as before, and $\rho_s(h_s)$ is given in (4.6). By using the same arguments as in the previous proof, this time also using Lemma 4.4 we bring the proof to an end.


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†School of Mathematics, University of Edinburgh, King’s Buildings, Edinburgh, EH9 3JZ, UK, e-mail: A.C.Dareiotis@sms.ed.ac.uk

*School of Mathematics, University of Edinburgh, King’s Buildings, Edinburgh, EH9 3JZ, UK, e-mail: I.Gyongy@ed.ac.uk