Derived intersections over the Hochschild cochain complex

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February 26, 2018

Abstract

The paper generalizes a result of Behrend-Fantechi and Baranovsky-Ginzburg to the 1-shifted cotangent bundle \( T^*X[1] \) of a smooth scheme \( X \) over \( \mathbb{C} \). We show how one can obtain twisted cotangent bundles as derived intersections of Lagrangians in \( T^*X[1] \), moreover and we show that the derived intersection of the quantized Lagrangians coincide with the canonical quantization of the twisted cotangent bundles.

1. Introduction

1.1. Let \( S \) be a Poisson manifold over a field \( k \) of characteristic 0 and let \( P \) be a Poisson bivector on \( S \). Consider a flat \( k[\hbar]/(\hbar^2) \)-deformation, \( \mathcal{A} \) of \( \mathcal{O}_S \) such that its commutator induces the Poisson bracket given by \( P \). Let \( Y \) and \( Z \) be two smooth coisotropic subvarieties of \( S \) and assume that we have a flat \( k[\hbar]/(\hbar^2) \)-deformation, \( \mathcal{B} \) of \( \mathcal{O}_Y \) to a right \( \mathcal{A} \)-module and a flat deformation, \( \mathcal{C} \) of \( \mathcal{O}_Z \) to a left \( \mathcal{A} \)-module. Associated to this data, greatly inspired by the work of Behrend and Fantechi (\[3\]), Baranovsky and Ginzburg (\[2\]) constructed a non-trivial second order differential operator

\[
\delta : \text{Tor}^\mathcal{O}_S(\mathcal{O}_Y, \mathcal{O}_Z) \to \text{Tor}^\mathcal{O}_{S+1}(\mathcal{O}_Y, \mathcal{O}_Z)
\]

(here \( \text{Tor} \) denotes the sheaf-Tor functor) which squares to 0. This construction makes the algebra \( \text{Tor}^\mathcal{O}_S(\mathcal{O}_Y, \mathcal{O}_Z) \) a Batalin-Vilkoviski (BV for short) algebra.

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1.2. Let us explain the construction in the simplest case, in the case of the cotangent bundle of a smooth scheme, $S = T^*X$ (where we consider the natural symplectic structure on $S$). In this case the canonical quantization of $\mathcal{O}_S$ is convergent, at $\hbar = 1$, we have the sheaf of differential operators, $D_X$. Assume further that $X$ is a Calabi-Yau variety over $k$ and let $f$ be a global regular function on $X$. Consider the zero section, $Y = X: i : X \to T^*X$ and the graph of $df$, $Z$ inside $T^*X$. The Tor sheaves $\text{Tor}^\bullet_{T^*X}(\mathcal{O}_X, \mathcal{O}_Z)$ are given by the cohomology of the derived tensor product

$$\mathcal{O}_X \otimes^L_{T^*X} \mathcal{O}_Z.$$ 

As a complex (over $X$) it is quasi-isomorphic to the dual of the twisted de Rham complex

$$0 \to \mathcal{O}_X \xrightarrow{\wedge df} \Omega^1_X \xrightarrow{\wedge df} \Omega^2_X \to \cdots$$

Now, consider the ring of differential operators, $D_X$. The algebra $\mathcal{O}_Z$ has a natural left $D_X$-module structure. Moreover, given a volume form on $X$, we have $\mathcal{O}_X \cong \omega_X$, and since $\omega_X$ has a natural right $D_X$-module structure, we can regard $\mathcal{O}_X$ as a right $D_X$-module. The derived tensor product

$$\omega_X \otimes^R_{D_X} \mathcal{O}_Z$$

has a natural BV-structure. As a complex (over $X$), this derived tensor product is quasi-isomorphic to the dual of the twisted de Rham complex

$$0 \to \mathcal{O}_X \xrightarrow{d+\wedge df} \Omega^1_X \xrightarrow{d+\wedge df} \Omega^2_X \to \cdots$$

(here we use again the CY structure of $X$). We see that the BV differential is basically given by the de Rham differential.

1.3. We can organize our spaces in the following diagram

$$\begin{array}{ccc}
W & \longrightarrow & Y \\
\downarrow & & \downarrow \\
Z & \underset{df}{\longrightarrow} & T^*X
\end{array}$$

where $W$ is the derived scheme given by the derived intersection of the zero section, $Y$ and the graph of $df$, $Z$ inside $T^*X$. The cotangent bundle has a canonical symplectic structure, the two Lagrangians, $Y$ and $Z$ have canonical Lagrangian structures (in the sense of [9]). Therefore, their derived intersection, $W$ is equipped with a $-1$-shifted symplectic structure ([9]). In the Costello-Li framework (see, for instance, [6]), the quantizations of $\mathcal{O}_W$
(where $W$ is equipped with a $-1$-shifted symplectic structure) is a BV-algebra structure on the sheaf $\mathcal{O}_W[[\hbar]]$ which is, in this case, exactly the BV-algebra structure constructed above.

Summarizing the above discussion we see that the quantization of the derived intersection of Lagrangians is the derived intersection of the quantized Lagrangians.

1.4. The purpose of the current paper is to generalize the above picture when the ambient space is the $1$-shifted cotangent bundle $T^*X[1]$ (we assume that $X$ is a smooth scheme over $\mathbb{C}$.) We consider two special closed subschemes: the zero section $Y = X \to T^*X[1]$ and the graph of $\alpha$, $Z$, where $\alpha \in H^1(X, \Omega^1_X)$. These subschemes can be equipped with Lagrangian structures whenever $d\alpha \in H^1(X, \Omega^2_{cl,X}) = 0$. (Here $\Omega^2_{cl,X}$ denotes the sheaf of closed 2-forms on $X$.) We see that their derived intersection, $W$ is $0$-shifted (again, by PTVV [9]), moreover one can show that $W$ is a twisted cotangent bundle over $X$ (corresponding to $\alpha$).

1.5. In the PTVV framework the quantization of an $n$-shifted symplectic derived stack, $S$ is a deformation of $\mathcal{O}_S$ over $\mathbb{C}[[\hbar]]$ as a sheaf of $E_{n+1}$-algebras. In the case of $S = T^*X[1]$ with its natural $1$-shifted symplectic structure, the canonical quantization of $\mathcal{O}_{T^*X[1]}$ is convergent, at $\hbar = 1$, we have the sheaf of the Hochschild cochains of $X$. The two Lagrangian subschemes, $X$ and $Z$ are endowed with Lagrangian structures, the quantization of these Lagrangians are left or right modules over this $E_2$-algebra.

We follow Safronov [11] to equip the derived intersection of the two modules over our $E_2$-algebra with an associate structure. We consider a specific model of the sheaf of Hochschild cochains which is equipped with a brace algebra structure ([7]). We equip the sheaves $\mathcal{O}_X$ (and $\mathcal{O}_Z$) with a natural structure of a left (and of a right) brace module over this brace algebra. Given a brace algebra, the derived tensor product of a left and a right brace module is endowed with an associative structure ([11]).

We are ready to state our main result (for a precise statement, see Theorem 4.3).
1.6. Theorem. The derived intersection (in the above sense) of $\mathcal{O}_X$ and $\mathcal{O}_Z$ over the Hochschild cochain complex is the canonical quantization of the twisted cotangent bundle, $W$.

1.7. Remark: Given a symplectic 1-shifted derived Artin stack $S$ and two Lagrangians $X \to S$ and $Y \to S$, it is expected that the quantization of the derived intersection $W = X \times_S Y$ is given by the derived tensor product of the quantized Lagrangians (if exists) over the quantization of $S$ (see [10] and [11]). The current paper gives an explicit example.

1.8. Acknowledgment. The author expresses his thanks to Jonathan Block, Damien Calaque, Tony Pantev, Jon Pridham, Pavel Safronov, Junwu Tu and Shilin Yu for useful conversations.

2. Twisted differential operators

In this section we review the notions of the rings of twisted differential operators and twisted cotangent bundles. Our references are [4] and [8]. We also show how to obtain the twisted cotangent bundles as derived intersections of Lagrangians inside the shifted cotangent bundle $T^*X[1]$.

2.1. We begin with the definition of a ring of twisted differential operators.

2.2. Definition. A sheaf of rings, $D$ of twisted differential operators (TDO for short) on $X$ is a positively filtered sheaf of associative algebras, whose associated graded, $\text{gr} D$ is isomorphic to the symmetric algebra of the tangent bundle:

$$\text{gr} D \cong \text{Sym} T_X.$$

2.3. In particular, we have a short exact sequence

$$0 \to \mathcal{O}_X \to D^1 \to T_X \to 0$$

where $D^1$ denotes the first filtered piece of the TDO $D$. The TDO is completely determined by the so-called Atiyah-algebra structure on $D^1$. The set of TDO’s can be classified as follows.

2.4. Theorem. ([7], Lemma A.1.6) The set of TDO’s is in one-to-one correspondence between elements of the cohomology group $H^1(X, \Omega_{\geq 1}^X)$ (where $\Omega_{\geq 1}^X := \Omega_X^1 \xrightarrow{d} \Omega_X^2$, where $d$ is the de Rham differential).
2.5. Let us briefly explain the above result. The exact triangle
\[
\Omega^2_{\text{cl},X}[-1] \rightarrow \Omega_X^2 \rightarrow \Omega_X^1 \rightarrow \Omega^2_{\text{cl},X}
\]
induces a long exact sequence on cohomology:
\[
\cdots \rightarrow H^0(X, \Omega^2_{\text{cl},X}) \rightarrow H^1(X, \Omega_X^2) \rightarrow H^1(X, \Omega_X^1) \rightarrow \cdots
\]
The image of the class of TDO inside \(H^1(X, \Omega_X^1)\) is the class of the extension
\[
0 \rightarrow \mathcal{O}_X \rightarrow D^1 \rightarrow T_X \rightarrow 0
\]
where we forget the Atiyah-algebra structure on the first filtered piece \(D^1\) of \(D\). Moreover given an extension
\[
0 \rightarrow \mathcal{O}_X \rightarrow D^1 \rightarrow T_X \rightarrow 0
\]
the Atiyah-algebra structures on \(D^1\) form a torsor over the image of \(H^0(X, \Omega^2_{\text{cl},X})\) inside \(H^1(X, \Omega_X^1)\).

2.6. Twisted differential operators can be obtained as deformation quantizations of the structure sheaves of the twisted cotangent bundles.

2.7. Definition. A twisted cotangent bundle \(W\) on \(X\) is an \(\Omega^1_X\)-torsor \(\pi : W \rightarrow X\) such that
- The space \(W\) is endowed with a symplectic form \(\omega\),
- The fibers of \(\pi\) are Lagrangian with respect to this symplectic form,
- For any 1-form \(\nu\) on \(X\), \(t^*_\nu(\omega) = \pi^*(d\nu) + \omega\) where \(t_\nu : W \rightarrow W\) is the torsor-action corresponding to \(\nu\).

2.8. Theorem. ([4], Lemma A.1.10) The set of twisted cotangent bundles is in one-to-one correspondence between elements of the cohomology group \(H^1(X, \Omega_X^1)\).
2.9. Let us explain the above result as well. Again, we consider the long exact sequence on cohomology

\[ \cdots \to H^0(X, \Omega^2_{\text{cl},X}) \to H^1(X, \Omega^1_X) \to H^1(X, \Omega_X^1) \to \cdots \]

The image of the class of the twisted cotangent bundle in \( H^1(X, \Omega^1_X) \) classifies \( W \) (as a space) up to isomorphism as follows. Since \( W \to X \) is an \( \Omega^1_X \)-torsor, there is a covering \( \{ U_i \to X \} \) such that the restriction of \( W \) to each \( U_i \) is isomorphic to the cotangent bundle \( T^*U_i \). The gluing data gives rise to the class in \( H^1(X, \Omega^1_X) \).

Once the class in \( H^1(X, \Omega^1_X) \) is fixed, the set of twisted cotangent bundle structure on \( W \) (the set of possible symplectic structures satisfying the required properties) forms a torsor over the image of \( H^0(X, \Omega^2_{\text{cl},X}) \) inside \( H^1(X, \Omega^1_X) \).

2.10. For a twisted cotangent bundle the pushforward of the structure sheaf, \( \pi_* \mathcal{O}_W \) carries a Poisson-algebra structure. The canonical quantization is convergent, at \( \hbar = 1 \), we have the TDO corresponding to the class of the twisted cotangent bundle inside \( H^1(X, \Omega^1_X) \).

2.11. Now, we show how to obtain the twisted cotangent bundles (with the symplectic structure) as derived intersections of Lagrangians inside the shifted cotangent bundle \( T^*X[1] \). Consider the two closed smooth subschemes of \( T^*X[1] \) given by the zero section \( X \to T^*X[1] \) and the graph of \( \alpha \in H^1(X, \Omega^1_X) \) which we denote by \( Z \). We remark that as a scheme \( Z \) is abstractly isomorphic to \( X \), but they are not isomorphic as subschemes of \( T^*X[1] \).

The tangent complex of \( T^*X[1] \) (over \( X \)) is isomorphic to \( T_X \oplus \Omega^1_X[1] \), the cotangent complex is isomorphic to \( T_X[-1] \oplus \Omega^1_X \). The natural 1-symplectic structure on \( T^*X[1] \) is \( \text{id}[1] : T_{T^*X[1]} \to L_{T^*X[1]}[1] \).

We recall the definition of a Lagrangian structure in the sense of [9].

2.12. Definition. A smooth subscheme \( i : Y \to T^*X[1] \) is equipped with a Lagrangian structure inside \( T^*X[1] \) if

- the pullback form \( i^*\text{id}[1] \in \Omega^2_{\text{cl},Y}[1] \) is homotopic to 0, and
- the homotopy induces a quasi-isomorphism between the tangent complex of \( i \) and the cotangent bundle of \( Y \): \( T_i \to \Omega^1_Y \).

The Lagrangian structure is the homotopy between \( i^*\text{id}[1] \) and 0.
2.13. Proposition. The graph of $Z$ is equipped with a Lagrangian structure if and only if $d\alpha \in H^1(X, \Omega^2_{cl,X})$ is 0, where $d : \Omega^1_X \to \Omega^2_{cl,X}$ is the de-Rham differential.

Proof. It is easy to see that the pullback form $i^* id[1] \in H^1(Z, \Omega^2_{cl,X})$ is the class of $d\alpha \in H^1(X, \Omega^2_{cl,X})$ (using the isomorphism $Z \cong X$). The morphism $T_i \to \Omega^1_Z$ fits into a diagram

$$
\begin{array}{ccc}
T_Z & \longrightarrow & T_{T^*X[1]} \\
\downarrow & & \downarrow \\
T_i[1] & \longrightarrow & \Omega^1_Z[1]
\end{array}
$$

where the map $T_{T^*X[1]} \to L_{T^*X[1]}[1]$ is given by the symplectic structure and $T_i \to T_Z \to T_{T^*X[1]} \to T_i[1]$ is the exact triangle of tangent complexes. The map $T_i \to \Omega^1_Z$ is homotopic to the identity, hence it is a quasi-isomorphism.

\[\square\]

2.14. Fix an $\alpha \in H^1(X, \Omega^1_X)$ so that $Z$ can be equipped with a Lagrangian structure. The possible Lagrangian structures are parametrized by the first homotopy group $\pi_1(A^2_{cl}(X,1))$ of the space of shifted closed 2-forms on $X$. This homotopy group is isomorphic to $H^0(X, \Omega^2_{cl,X})$.

2.15. Proposition. The set of isomorphism classes of twisted cotangent bundles $W \to X$ are in one-to-one correspondence with the set of $Z$’s with Lagrangian structures modulo exact 2-forms.

Proof. We organize our spaces as follows.

$$
\begin{array}{ccc}
W & \longrightarrow & X \\
\downarrow & & \downarrow \alpha \\
Z & \longrightarrow & T^*X[1]
\end{array}
$$

(Here $W$ denotes the derived intersection of $X$ and $Z$ inside $T^*X[1]$.) Locally $\alpha$ is trivial, therefore the derived intersection of the zero section and the graph of $\alpha$ is locally isomorphic to the derived self-intersection of the zero section, which is the cotangent bundle, $T^*X$. The class $\alpha$ specifies the gluing data of these cotangent bundles, and we see that for a given $\alpha$, the derived intersection, $W \to X$, is isomorphic to the twisted cotangent bundle (as a space) corresponding to $\alpha$. 

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The zero section has a canonical Lagrangian structure. Given a Lagrangian structure on $Z$, the derived intersection of $X$ and $Z$ is endowed with a symplectic structure. Recall that Lagrangian structures on $Z$ are one-to-one correspondence with $H^0(X, \Omega^2_{cl, X})$. Translating the Lagrangian structure on $Z$ by an exact 2-form gives rise to isomorphic twisted cotangent bundles, therefore we obtain a map from the set of $Z$'s with Lagrangian structures modulo exact 2-forms to the set of isomorphism classes of twisted cotangent bundles. This map is clearly an isomorphism.

3. Polydifferential operators

In this section we define the complex $\text{Diff}(\mathcal{O}^*_X, \mathcal{O}_X)$, which is a global model for the Hochschild cochain complex equipped with its brace algebra structure. Similarly, for every $\mathcal{O}_X$-bimodule, $P$, we define the complex $\text{Diff}(\mathcal{O}^*_X, P)$, which is a global model of the Hochschild cochain complex with coefficients in $P$ equipped with its brace module structure (see [5], [13] for more details). We also show how $\text{Diff}(\mathcal{O}_X, D)$ is the convergent quantization of $\mathcal{O}_Z$ at $\hbar = 1$ (see [10] for more details).

We begin with the definition of (poly)differential operators.

3.1. Definition. Let $P$ be an $\mathcal{O}_X$-bimodule, and $A : \mathcal{O}_X \to P$ a $k$-linear map. Given a sequence of functions $f_0, f_1, ..., f_n \in \mathcal{O}_X$, define a sequence of $k$-linear maps $A_m : \mathcal{O}_X \to P$ given by $A_{-1} = A$ and, $A_n := f_n A_{n-1} - A_{n-1} f_n$. We say that $A$ is a differential operator of order at most $N$ if for every point $x \in X$ and every section $s \in \mathcal{O}_X$ defined at $x$, there exists a neighborhood $U$ of $x$ and $N \geq 0$ such that for any open subset $V \subset U$ and any choice of functions $f_0, f_1, ..., f_N$ on $V$, so that $A_N(s|_V)$ vanishes.

The differential operators $\mathcal{O}_X \to P$ form a sheaf that we denote by $\text{Diff}_X(\mathcal{O}_X, P)$. In the case of $P = \mathcal{O}_X$ we denote the sheaf of differential operators by $D_X := \text{Diff}_X(\mathcal{O}_X, \mathcal{O}_X)$.

3.2. Remark: The sheaf of differential operators, $D_X$ is a TDO.

3.3. Definition. A $k$-polylinear map $A : \mathcal{O}_X \times ... \times \mathcal{O}_X \to P$ (of $n$ arguments) is a polydifferential operator of (poly)order at most $(N_1, ..., N_n)$ if it is a differential operator of order at most $N_j$ in the $j$-th argument whenever the remaining $n - 1$ arguments are fixed.

The polydifferential operators $\mathcal{O}_X \times ... \times \mathcal{O}_X \to P$ of $i$ arguments form a sheaf, which we denote by $\text{Diff}(\mathcal{O}^*_X, P)$. 

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3.4. We can identify the sheaf $\Diff(e^i_X, P)$ with the tensor product

$$D_X \otimes e_x \cdots \otimes e_x D_X \otimes e_x P$$

where the number of the $D_X$ terms is $i$ as follows. The map

$$D_X \otimes e_x \cdots \otimes e_x D_X \otimes e_x P \to \Diff(e^i_X, P)$$

given by

$$A_1 \otimes \cdots \otimes A_i \otimes p \mapsto A_1(-)A_2(-)\cdots(-)A_i(-)p$$

(for local sections $A_i \in D_X$ and $p \in P$) is clearly an isomorphism. (Here we use the natural $e_X$-bimodule structure on $D_X$.) We denote the isomorphism $\Diff(e_X, P) \cong D_X \otimes P$ by $i_P$.

3.5. The sheaves of polydifferential operators form a natural complex $\Diff(e^i_X, P)$ whose $i$-th term is $\Diff(e^i_X, P)$ and the differential $d$ is given by

$$dA(g_1, ..., g_{i+1}) = g_1A(g_2, ..., g_{i+1}) - A(g_1g_2, g_3, ..., g_{i+1}) + A(g_1, g_2g_3, ..., g_{i+1}) +
\sum_{j=1}^{i+1} (-1)^j A(g_1, ..., g_jg_{i+1}) + (-1)^{i+1} A(g_1, ..., g_i) g_{i+1}$$

where $A : e_X \times \cdots \times e_X \to P$ is a polydifferential operator of $i$ arguments and $\{g_1, ..., g_{i+1}\}$ is a local section of $e_X \times \cdots \times e_X$ (of $i + 1$ arguments).

3.6. Let $A \in \Diff(e^i_X, e_X)$ and $B \in \Diff(e^{i+j}_X, e_X)$. We define $A \cdot B \in \Diff(e^{i+j}_X, e_X)$ as the differential operator mapping $a_1, ..., a_{i+j}$ to

$$(-1)^j A(a_1, ..., a_i) \cdot B(a_{i+1}, ..., a_{i+j}).$$

This product endows the complex $\Diff(e^i_X, e_X)$ a differential graded algebra structure.

3.7. Similarly, we see that if $P$ is an $e_X$-bimodule, then the complex $\Diff(e^i_X, P)$ has a differential graded bimodule structure over $\Diff(e^i_X, e_X)$. Moreover, if the $e_X$-bimodule, $P$ has an associative algebra structure, then $\Diff(e^i_X, P)$ has a differential graded algebra structure defined parallel to the differential graded algebra structure on $\Diff(e^i_X, e_X)$.

3.8. The complex, $\Diff(e^i_X, e_X)$ is equipped with a brace algebra structure as follows. Let $A \in \Diff(e^i_X, e_X)$ and $A_i \in \Diff(e^{i+l}_X, e_X)$ (for $l = 1, ..., m$). The brace operations $A[A_{i_1}, ..., A_{i_m}]$ are defined as operations of degree $-m$, i.e. $A[A_{i_1}, ..., A_{i_m}] \in \Diff(e^n_X, e_X)$, where $n = i + \sum_{l=1}^{m} i_l - m$. Explicitly, $A[A_{i_1}, ..., A_{i_m}]$ maps $n$ sections of $e_X$, $a_1, ..., a_n$ to

$$\sum_{0 \leq i_1 \leq \cdots \leq i_m \leq n} (-1)^\varepsilon A(a_1, ..., a_{i_1}, A_1(a_{i_1+1}, ...), ..., a_{i_m}, A_m(a_{i_m+1}, ...), ..., a_n)$$

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where $\epsilon := \sum_{i=1}^{m} i_i (j_i - 1)$.

Similarly, if $P$ is an $\mathcal{O}_X$-bimodule, then $\text{Diff}(\mathcal{O}_X^*, P)$ has a module structure over the brace algebra $\text{Diff}(\mathcal{O}_X^*, \mathcal{O}_X)$ as follows. Let $B \in \text{Diff}(\mathcal{O}_X^*, P)$ and $A_l \in \text{Diff}(\mathcal{O}_X^{\mathcal{H}_l}, \mathcal{O}_X)$ (for $l = 1, \ldots, m$). The brace operations $B\{A_1, \ldots, A_m\}$ are defined as operations of degree $-m$, i.e. $B\{A_1, \ldots, A_m\} \in \text{Diff}(\mathcal{O}_X, P)$, where $n = i + \sum_{i=1}^{m} j_i - m$. Explicitly, $B\{A_1, \ldots, A_m\}$ maps $n$ sections of $\mathcal{O}_X$, $a_1, \ldots, a_n$ to

$$
\sum_{0 \leq i_1 \leq \cdots \leq i_m \leq n} (-1)^e B(a_1, \ldots, a_{i_1}, A_1(a_{i_1+1}, \ldots), \ldots, a_{i_m}, A_m(a_{i_m+1}, \ldots), \ldots, a_n)
$$

where $e := \sum_{i=1}^{m} i_i (j_i - 1)$.

3.9. A right $\text{Diff}(\mathcal{O}_X^*, \mathcal{O}_X)$ brace module structure on a complex is equivalent to a left $\text{Diff}(\mathcal{O}_X^*, \mathcal{O}_X)^\text{op}$ brace module structure. We equip the complex $\text{Diff}(\mathcal{O}_X^*, \mathcal{O}_X)^\text{op}$ with a right $\text{Diff}(\mathcal{O}_X^*, \mathcal{O}_X)$ structure, coming from the left $\text{Diff}(\mathcal{O}_X^*, \mathcal{O}_X)$ brace module structure on $\text{Diff}(\mathcal{O}_X^*, \mathcal{O}_X^{\mathcal{H}})$.

3.10. Let $P = D$ be a TDO. Then the complex $\text{Diff}(\mathcal{O}_X^*, D)$ starts as

$$D \xrightarrow{d_D} \text{Diff}(\mathcal{O}_X, D) \to \ldots$$

where the differential acts as

$$d_D(\partial) = (x \mapsto \partial x - x \partial).$$

Clearly, the kernel of this map is the centralizer of $\mathcal{O}_X$ inside $D$, which is $\mathcal{O}_X$. Hence, the zeroth cohomology sheaf of the complex $\text{Diff}(\mathcal{O}_X^*, D_X)$ is $\mathcal{O}_X$. Moreover, all other cohomology sheaves vanish (see [12] for instance), hence we obtain the following isomorphism of algebras.

3.11. Proposition. We have

$$\text{Diff}(\mathcal{O}_X^*, D) = \mathcal{O}_X$$

(and similarly

$$\text{Diff}(\mathcal{O}_X^*, D_X)^\text{op} = \mathcal{O}_X).$$

4. Derived intersection of the quantizied Lagrangians

In this section we prove our main theorem, Theorem 1.6. We begin with a simple but key lemma.
4.1. Lemma. Let D be a TDO. The map

$$\varphi : D \to \text{Diff}(O_X, D) \cong D_X \otimes O_X$$

given by

$$\partial \mapsto i_D(d_D \partial) + 1 \otimes \partial$$

is an algebra homomorphism.

Proof. Clearly this map is a group homomorphism with respect to the additive structure. We only need to prove that $$\varphi(\partial_1 \partial_2) = \varphi(\partial_1) \varphi(\partial_2)$$ for local sections $$\partial_1, \partial_2 \in D$$. Recall that $$d_D(\partial) = (x \mapsto \partial x - x \partial)$$, therefore

$$d_D(\partial_1 \partial_2) = (x \mapsto \partial_1 \partial_2 x - x \partial_1 \partial_2) = (x \mapsto (\partial_1 x - x \partial_1) \partial_2 + \partial_1 (\partial_2 x - x \partial_2)).$$

The first term, $$(x \mapsto (\partial_1 x - x \partial_1) \partial_2)$$, is equal to $$i_D(d_D \partial_1) \cdot (1 \otimes \partial_2)$$.

Let $$\varphi(\partial_2) = \sum t_i \otimes s_i \in D_X \otimes D$$. We have

$$(x \mapsto \partial_1 (\partial_2 x - x \partial_2)) = (x \mapsto \partial_1 \left( \sum t_i(x) s_i \right)) = (x \mapsto \sum \partial_1(t_i(x)) s_i) =$$

$$(x \mapsto \sum (\partial_1(t_i(x)) - t_i(x) \partial_1 + t_i(x) \partial_1)s_i)$$

We see that

$$i_D(x \mapsto \sum (\partial_1(t_i(x)) - t_i(x) \partial_1)s_i) = i_D(d_D \partial_1)i_D(d_D \partial_2)$$

and

$$i_D(x \mapsto \sum t_i(x) \partial_1 s_i) = 1 \otimes \partial_1 \cdot i_D(d_D \partial_2).$$

We are done. 

4.2. Before we prove our main theorem we recall how Safronov ([11]) defined the associative structure on the derived intersection. Let A be a brace algebra and N (and M resp.) be a right (and left resp.) brace module over A. The bar complex $$T_*(A[1])$$ is a dg coalgebra. Gerstenhaber and Voronov [7] defined a multiplicative structure on the bar complex which makes $$T_*(A[1])$$ into a bialgebra. Similarly, the one-sided bar complexes $$N \otimes T_*(A[1])$$ and $$T_*(A[1]) \otimes M$$ carry a natural structure of a dg algebra compatible with the $$T_*(A[1])$$ comodule structure. Therefore, the cotensor product

$$N \otimes T_*(A[1]) \otimes_{T_*(A[1])} T_*(A[1]) \otimes M$$

carries a natural structure of a dg algebra. This cotensor product is quasi-isomorphic to the derived tensor product $$N \otimes^L_A M$$, hence we obtain a dg algebra structure on the derived tensor product.

We are ready to prove our main theorem.
4.3. Theorem. The derived tensor product

\[ \text{Diff}(\mathcal{O}_X, D_X^{\text{op}})^{\text{op}} \otimes_{\text{Diff}(\mathcal{O}_X, \mathcal{O}_X)} \text{Diff}(\mathcal{O}_X, \mathcal{O}_X) \]

is isomorphic to \( D \) as associative algebras.

Proof. First of all, since \( D_X \cong \text{Diff}(\mathcal{O}_X, D_X^{\text{op}}) \) and \( D \cong \text{Diff}(\mathcal{O}_X, \mathcal{O}_X) \) as algebras, we obtain a morphism

\[ \psi: D_X \otimes D \to \text{Diff}(\mathcal{O}_X, D_X^{\text{op}})^{\text{op}} \otimes_{\text{Diff}(\mathcal{O}_X, \mathcal{O}_X)} \text{Diff}(\mathcal{O}_X, \mathcal{O}_X) \]

It is clear from the definition of the dg algebra structure on the complex

\[ \text{Diff}(\mathcal{O}_X, D_X^{\text{op}})^{\text{op}} \otimes_{\text{Diff}(\mathcal{O}_X, \mathcal{O}_X)} \text{Diff}(\mathcal{O}_X, \mathcal{O}_X) \]

that \( \psi \) is a dg algebra morphism. Therefore, we obtain a map of dg algebras

\[ \chi: D \to D_X \otimes D \to \text{Diff}(\mathcal{O}_X, D_X^{\text{op}})^{\text{op}} \otimes_{\text{Diff}(\mathcal{O}_X, \mathcal{O}_X)} \text{Diff}(\mathcal{O}_X, \mathcal{O}_X) \]

Let \( d \) denote the differential of the complex

\[ \text{Diff}(\mathcal{O}_X, D_X^{\text{op}})^{\text{op}} \otimes_{\text{Diff}(\mathcal{O}_X, \mathcal{O}_X)} \text{Diff}(\mathcal{O}_X, \mathcal{O}_X) \]

We claim that \( d(\psi(\delta)) = 0 \) for every \( \delta \in D \). Since, \( \psi \) is a dg-algebra map, it is enough to show that \( d(\psi(\delta)) = 0 \) for the generators of \( D \). For \( D^0 \subset D \), we have \( \varphi(D^0) = 0 \). For \( \delta \in D^1 \subset D \) we have

\[ \varphi(\delta) = \bar{\delta} \otimes 1 + 1 \otimes \delta \in D_X \otimes D_X \]

where \( \bar{\delta} \) denotes the image of \( \delta \) under the natural map \( D^1 \to T_X \). Therefore,

\[ d(\chi(\delta)) = -1 \otimes \bar{\delta} \otimes 1 + 1 \otimes 1 \otimes \bar{\delta} \otimes 1 = 0 \in D_X \otimes D_X \otimes_D X \otimes D_X \]

where the negative sign appears in the first term because of the opposite algebra structure on \( \text{Diff}(\mathcal{O}_X, D_X^{\text{op}})^{\text{op}} \).

Finally, we show that \( \chi \) is a quasi-isomorphism. By Proposition 3.11, we know that \( \text{Diff}(\mathcal{O}_X, D_X^{\text{op}})^{\text{op}} \) is quasi-isomorphic to \( \mathcal{O}_X \), moreover the action of \( \text{Diff}(\mathcal{O}_X, \mathcal{O}_X) \) on \( \text{Diff}(\mathcal{O}_X, D_X^{\text{op}})^{\text{op}} \) becomes the natural action of \( \text{Diff}(\mathcal{O}_X, \mathcal{O}_X) \) on \( \mathcal{O}_X \) and \( \text{Diff}(\mathcal{O}_X, \mathcal{O}_X) \) just acts trivially if \( i > 0 \). Therefore, we have

\[ \text{Diff}(\mathcal{O}_X, D_X^{\text{op}})^{\text{op}} \otimes_{\text{Diff}(\mathcal{O}_X, \mathcal{O}_X)} \text{Diff}(\mathcal{O}_X, \mathcal{O}_X) \cong \mathcal{O}_X \otimes \mathcal{O}_X \]

Moreover, we see that the image of \( \chi(\delta) \) ends in \( \delta \) (for every \( \delta \in D \)). \( \Box \)
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