ON THE PROOF OF THE DIPOLE APPROXIMATION

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Abstract. The dipole approximation is widely used to study the interaction of atoms and Lasers. Employing it essentially means to replace the Laser’s vector potential $A(r, t)$ in the Hamiltonian by $A(0, t)$. Heuristically this is justified under usual experimental conditions, because the Laser varies only slowly on atomic length scales. We make this heuristics rigorous by proving the dipole approximation in the limit in which the Laser’s length scale becomes infinite compared to the atomic length scale. Our results apply to $N$-body Hamiltonians.

1. Introduction

The interaction of atoms with Lasers is governed by the time dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} U_{\lambda}(t, t_0) = \left[ \frac{1}{2m}(-i\hbar \nabla - eA_{\lambda}(r, t))^2 + V(r) \right] U_{\lambda}(t, t_0),$$

where $V$ is the atomic binding potential and $A_{\lambda}$ is the vector potential that describes the Laser with wave length $\lambda$ in Coulomb gauge ($\nabla \cdot A_{\lambda} = 0$). However, in the mathematical as well as the physical literature the analysis of atoms interacting with Lasers is very often based on

$$i\hbar \frac{\partial}{\partial t} U_D(t, t_0) = \left[ -\frac{\hbar^2}{2m}\Delta + V(r) - eE(0, t) \cdot \hat{r} \right] U_D(t, t_0),$$

where $E = -\frac{i}{\hbar} \partial_t A_{\lambda}$, rather than eq. (1).

Heuristically, one arrives at eq. (2) by applying the dipole approximation to eq. (1), which we will explain now using the example of a Coulomb potential $V(r) = -e^2/r$ interacting with a continuous wave Laser described by the electric field

$$E(r, t) = E \cos \left( \frac{2\pi}{\lambda} \hat{k} \cdot \hat{r} - \omega t \right) \hat{\varepsilon}.$$

Here $E$ denotes the electric field strength, $\hat{k}$ the normalized vector pointing in propagation direction and $\hat{\varepsilon}$ the normalized vector pointing in polarization direction. For $E$ to satisfy the sourceless Maxwell equations, we further need $\hat{k} \cdot \hat{\varepsilon} = 0$ and $\omega = 2\pi c/\lambda$. In natural units eq. (1) then reads

$$i\hbar \frac{\partial}{\partial t} U_{\lambda}(t, t_0) = \left[ -\nabla - e\frac{a_0}{\lambda} E \sin \left( \frac{2\pi a_0}{\lambda} \hat{k} \cdot \hat{r} - \tau \omega t \right) \right]^2 - \frac{2}{r} U_{\lambda}(t, t_0),$$

where $a_0$ is the characteristic length scale of an atom (Bohr radius) and $\tau = 2ma_0^2/\hbar$ is the characteristic time scale. Defining the characteristic velocity by $v = a_0/\tau$, the Dipole approximation of eq. (4) is obtained by taking the scaling limits $a_0/\lambda \to 0$ and $v/c \to 0$ in such a way that $\omega = 2\pi c/\lambda$ remains constant. Performing these limits on eq. (4), we obtain

$$i\hbar \frac{\partial}{\partial t} U_{\infty}(t, t_0) = \left[ -\nabla + e\frac{a_0}{\lambda} E \sin \left( \tau \omega t \right) \right]^2 - \frac{2}{r} U_{\infty}(t, t_0),$$

which upon gauge transformation yields eq. (2) in natural units.

The purpose of this paper is to prove that in the scaling limits $a_0/\lambda \to 0$ and $v/c \to 0$ with $\omega$ kept constant the dipole approximation is exact, in the sense that the time evolution generated

Date: 28.08.2013.
2010 Mathematics Subject Classification. 81Q10, 35Q41.
Key words and phrases. Dipole approximation, Schrödinger operator, Ionization, Laser, N-body.
by eq. (1) is the same as the one generated by eq. (2), up to gauge equivalence. In the rest of the paper we will use units, where \( h = e = 1 \) and \( m = 1/2 \). Our main result is the following

**Theorem 1.** Assume that the potential \( V \in L^2_{\text{loc}}(\mathbb{R}^n) \) is infinitesimally \( -\Delta \)-bounded and that \( A_\lambda(r,t) = \frac{1}{\lambda}a(\frac{r}{\lambda}, \omega t) \), where \( a \in C^2(\mathbb{R}^{n+1}) \) is independent of \( \lambda, \omega, c \) and satisfies \( \nabla \cdot a(r,t) = 0 \) as well as \( \|\partial_t^j a^i(-t)\|_\infty \leq C \) for some \( C < \infty \) uniformly in \( t, i = 1, \ldots, n \) and \( j = 0, 1, 2 \). Then

1. the operators

\[
H_\lambda(t) = \left( -i\nabla - \frac{1}{\lambda}A_\lambda(r,t) \right)^2 + V(r) \quad \text{and} \quad H_\infty(t) = \left( -i\nabla - \frac{1}{\lambda}a(0, \omega t) \right)^2 + V(r).
\]

with common domain \( \mathcal{D}(H_\lambda(t)) = \mathcal{D}(H_\infty(t)) = W^{2,2}(\mathbb{R}^n) \) are selfadjoint and generate unitary evolution operators \( (U_\lambda(t, t_0))_{t_0 \leq t} \) and \( (U_\infty(t, t_0))_{t_0 \leq t} \), respectively. \( U_\lambda(t, t_0) \) and \( U_\infty(t, t_0) \) are strongly continuous in \( t \) as well as \( t_0 \) and leave \( W^{2,2}(\mathbb{R}^n) \) invariant.

2. Further, for every \( \psi \in L^2(\mathbb{R}^n) \) and \( 0 < t_0 < t < \infty \),

\[
\|(U_\lambda(t, t_0) - U_\infty(t, t_0))\psi\| \to 0 \quad \text{as} \quad \lambda \to \infty \quad \text{and} \quad c \to \infty
\]

such that \( \omega \) remains constant.

There are two difficulties in proving this theorem. Firstly, the Hamiltonian is time dependent. Time independent Hamiltonians \( H \) generate time evolution operators \( U(t, t_0) \), which are given by \( e^{-iH(t-t_0)} \). So if we have a series of Hamiltonians \( \{H_n\}_{n=0}^\infty \) with \( H_n \to H \) as \( n \to \infty \) in strong resolvent sense, we know that the time evolution operators \( e^{-iH_n(t-t_0)} \) converge to \( e^{-iH(t-t_0)} \). This is a consequence of the spectral theorem (see [3] Theorem VIII.21 for details). In contrast, time evolution operators generated by time dependent Hamiltonians that do not commute at different instances of time can not be expressed as functions of the Hamiltonian. Hence, the spectral theorem is not applicable to the time evolution operator. To show that \( U_\lambda(t, t_0) \to U_\infty(t, t_0) \) as \( \lambda \to \infty \) and \( c \to \infty \) with \( \omega \) constant, we will therefore use Cook’s argument. This allows us to express the difference between time evolution operators in terms of the difference between their respective generators:

\[
U_\lambda(t, t_0) - U_\infty(t, t_0)
\]

\[
= -i \int_{t_0}^t U_\lambda(t, s)(H_\lambda(s) - H_\infty(s))U_\infty(s, t_0) \, ds
\]

\[
= -i \int_{t_0}^t U_\lambda(t, s)(2i(\frac{1}{\lambda}A_\lambda(r, s) - \frac{1}{\lambda}a(0, \omega s)) \cdot \nabla + (\frac{1}{c^2}A_\lambda(r, s)^2 - \frac{1}{\omega^2}a(0, \omega s)^2))U_\infty(s, t_0) \, ds
\]

\[
= -i \int_{t_0}^t U_\lambda(t, s)(\frac{2}{c^2}(a(\frac{r}{\lambda}, \omega s) - a(0, \omega s)) \cdot \nabla + \frac{1}{\omega^2}(a(\frac{r}{\lambda}, \omega s)^2 - a(0, \omega s)^2))U_\infty(s, t_0) \, ds.
\]

Since \( a(\frac{r}{\lambda}, \omega s) - a(0, \omega s) \) and \( a(\frac{r}{\lambda}, \omega s)^2 - a(0, \omega s)^2 \) tend pointwise to zero as \( \lambda \to \infty \), the main task is now to show that the theorem of dominated convergence applies to eq. (10). The second and main difficulty in proving the theorem is seen from eq. (10): we need control over \( \nabla U_\infty(s, t_0) \). If the first order term did not appear in eq. (10), we would only deal with bounded operators, in which case the application of the dominated convergence theorem is trivial. However, since \( \nabla \) is an unbounded operator the application of the dominated convergence theorem needs careful justification.

We want to stress that physically, time dependent vector potentials are very important, because they describe Lasers. In fact proofs of ionization such as [2] [10] [3], which rely on the time dependence of the Laser field and make use of the dipole approximation, have been the main motivation for this work. Ionization also has been studied in the framework of non-relativistic QED (Pauli equation coupled to the second quantized vector potential), see e.g. [3]. In this paper the authors show that the ionization probability given by formal time-dependent perturbation theory is rigorously justified. As the vector potential only enters the ionization probability via \( A(0, t) \) their result also justifies the dipole approximation, but in a weaker sense than our Theorem 1. The use of the dipole approximation in non-relativistic QED dates back at least to a paper of
Pauli and Fierz [7] and in [1] this use is justified regarding the Hamiltonians. Here we justify the dipole approximation directly for the time evolution.

The conditions on the vector potential in Theorem [1] are very general. They are fulfilled e.g. by continuous wave Lasers and also by Laser pulses (in this case \( \lambda \) is the central wave length), to mention two important examples. To see that, we discuss these cases in more detail.

**Example 1. Continuous wave Laser**

From eq. (3), we see that the vector potential for a continuous wave Laser in \( \mathbb{R}^3 \) is given by

\[
A_\lambda(r, t) = c \frac{E}{\omega} \sin \left( \frac{2\pi}{\lambda} k \cdot r - \omega t \right),
\]

(11)

and

\[
a(r, t) = E \sin \left( 2\pi \hat{k} \cdot r - t \right),
\]

(12)

which evidently satisfies the assumptions of Theorem [1].

**Example 2. Laser pulses**

An example for a Laser pulse with Gaussian shape in \( \mathbb{R}^3 \) is

\[
E(r, t) = E e^{-\left( \frac{4\pi k}{\omega} r - \omega t \right)^2} \cos \left( \frac{2\pi}{\lambda} k \cdot r - \omega t \right).
\]

(13)

The parameters have the same physical meaning as for the continuous wave Laser and as before we need \( \hat{k} \cdot \hat{\varepsilon} = 0 \) and \( \omega = 2\pi c/\lambda \) for \( E \) to satisfy the sourceless Maxwell equations. Now we have

\[
A_\lambda(r, t) = -c \int_{-\infty}^{t} E(r, s) \, ds = \frac{c}{\omega} a(s, \omega t) \quad \text{with}
\]

(14)

\[
a(r, t) = -\int_{-\infty}^{t} E e^{-\left(2\pi \hat{k} \cdot r - s\right)^2} \cos(2\pi \hat{k} \cdot r - s) \hat{\varepsilon} \, ds.
\]

(15)

and since

\[
a'(r, t) = E \int_{\infty}^{2\pi \hat{k} \cdot r - t} e^{-s^2} \cos(s) \hat{\varepsilon} \, ds,
\]

(16)

it is evident that \( \partial_t^j a \in C^2(\mathbb{R}^{3+1})^3 \) for \( j = 0, 1, 2 \) and

\[
\nabla \cdot a(r, t) = 2\pi E e^{-\left(2\pi \hat{k} \cdot r - t\right)^2} \cos(2\pi \hat{k} \cdot r - t) \hat{k} \cdot \hat{\varepsilon} = 0.
\]

(17)

Moreover, \( ||\partial_t^j a'(r, t)||_{\infty} \) is bounded uniformly in \( i, j \) and \( t \), so that the vector potential of our Laser pulse with Gaussian shape satisfies the conditions of Theorem [1].

In [6] the dipole approximation is used to prove ionization for a two-body Schrödinger equation. Theorem [2] applies to \( N \)-body Schrödinger equations, too. Let us illustrate that with the following

**Example 3. \( N \)-body Hamiltonian**

An atom with \( N \) electrons interacting with a Laser described by the vector potential \( A_\lambda \) is described by the Hamiltonian

\[
H_\lambda^N(t) = \sum_{k=1}^{N} \left( -i \nabla_k - \frac{1}{\varepsilon} A_\lambda(r_k, t) \right)^2 - \sum_{k=1}^{N} \frac{2N}{|r_k|} + \sum_{k<l}^{N} \frac{2}{|r_k - r_l|}.
\]

(18)

\( H_\lambda^N(t) \) can be rewritten in the form given in Theorem [1] via the definitions

\[
A_\lambda(r, t) \equiv (A_\lambda(r_1, t), A_\lambda(r_2, t), \ldots, A_\lambda(r_N, t))^4, \quad V(r) \equiv -\sum_{k=1}^{N} \frac{2N}{|r_k|} + \sum_{k<l}^{N} \frac{2}{|r_k - r_l|},
\]

(19)

\[
\nabla \equiv (\nabla_1, \nabla_2, \ldots, \nabla_N)^4, \quad r \equiv (r_1, r_2, \ldots, r_N)^4.
\]

(20)

Clearly \( A_\lambda \) satisfies the assumptions of Theorem [1] if \( A_\lambda \) does and [3] Theorem X.16] shows that \( V \in L^2_{\text{loc}}(\mathbb{R}^{3N}) \) is infinitesimally \(-\Delta\)-bounded, where \(-\Delta\) denotes the Laplacian on \( \mathbb{R}^{3N} \).
2. Proof

Proof. (Theorem 1) Assertion one is in fact well known, but we include these results for completeness. Since \( V \) is infinitesimally \(-\Delta\)-bounded and \( \|a(\cdot, t)\|_\infty < C \) uniformly in \( i \) and \( t \),

\[
W_\lambda(r, t) := \frac{2}{\pi} A_\lambda(r, t) \cdot \nabla + \frac{1}{\pi} A_\lambda(r, t)^2 + V(r)
\]

(21)

\[
= \frac{2}{\pi} a_\lambda(x, \omega t) \cdot \nabla + \frac{1}{\omega} a_\lambda(x, \omega t)^2 + V(r) \quad \text{and}
\]

(22)

\[
W_\infty(r, t) := \frac{2}{\pi} a(0, \omega t) \cdot \nabla + \frac{1}{\omega} a(0, \omega t)^2 + V(r)
\]

(23)

satisfy

\[
\|W_{\lambda/\infty}(t)\psi\|^2 \leq C \left( \|\psi\|^2 + \sum_{i=1}^n \|\partial_i \psi\|^2 + \|V\psi\|^2 \right) \leq \|\psi\|^2 - \Delta \|\psi\|^2 + C \|\psi\|^2
\]

(24)

for every \( \psi \in W^{2,2}(\mathbb{R}^n) \) and suitable constants \( C, C_\varepsilon \). This implies selfadjointness of \( H_\lambda(t) \) and \( H_\infty(t) \) on \( \mathcal{D}(H_\lambda(t)) = \mathcal{D}(H_\infty(t)) = W^{2,2}(\mathbb{R}^n) \). The existence of the unitary evolution operators, their strong continuity and the fact that they leave the domain invariant follow from Theorem X.70 in [3]. Lemma 1 below proves that the assumptions of Theorem X.70 are fulfilled.

Now, we prove assertion two. In view of eq. (10), we note that the differences \( a_\lambda(x, \omega s) - a(0, \omega s)^2 \) and \( a_\lambda(x, \omega s) - a(0, \omega s) \) converge pointwise to zero as \( \lambda \to \infty \) and \( c \to \infty \) such that \( \omega = 2\pi c/\lambda \) remains constant. We will make use of this by employing the theorem of dominated convergence: Due to eq. (10), we have

\[
\|(U_\lambda(t, t_0) - U_\infty(t, t_0))\psi\|
\]

(25)

\[
\leq \frac{2}{\omega} \int_{t_0}^t \|(a_\lambda(x, \omega s) - a(0, \omega s)) \cdot \nabla U_\infty(s, t_0)\psi\| \, ds
\]

(26)

\[
+ \frac{1}{\omega^2} \int_{t_0}^t \|(a_\lambda(x, \omega s)^2 - a(0, \omega s)^2)U_\infty(s, t_0)\psi\| \, ds.
\]

(27)

Assertion two is proven once we have shown that we can pull the combined limit \( \lambda \to \infty \) and \( c \to \infty \) with \( \omega = 2\pi c/\lambda \) kept fixed into the \( s \)-integral and into the \( t \)-integral due to the norm \( \|\cdot\| \). Note that eqs. (26) and (27) depend on \( c \) only through \( \omega \), which is why the limit \( c \to \infty \) only needs to be taken in order to keep \( \omega \) fixed.

Consider eq. (26) first. By assumption \( \|a(\cdot, \omega s)^2 - a(0, \omega s)^2\|_\infty \leq C \). Therefore, we get an integrable dominating function for the \( s \)-integral from

\[
\|(a_\lambda(x, \omega s)^2 - a(0, \omega s)^2)U_\infty(s, t_0)\psi\| \leq C
\]

(28)

and similarly for the \( t \)-integral, from

\[
|(a_\lambda(x, \omega s)^2 - a(0, \omega s)^2)U_\infty(s, t_0)\psi(r)|^2 \leq C |U_\infty(s, t_0)\psi(r)|^2.
\]

(29)

So the theorem of dominated convergence applies to eq. (27).

For eq. (26), we have

\[
\|(a_\lambda(x, \omega s) - a(0, \omega s)) \cdot \nabla U_\infty(s, t_0)\psi\| \leq C \sum_{i=1}^n \|\partial_i U_\infty(s, t_0)\psi\| \quad \text{and}
\]

(30)

\[
|a_\lambda(x, \omega s) - a(0, \omega s)) \cdot \nabla U_\infty(s, t_0)\psi(r)|^2 \leq Cn \sum_{i=1}^n |\partial_i U_\infty(s, t_0)\psi(r)|^2,
\]

(31)

using the assumption that \( \|a(\cdot, t)\|_\infty \) is bounded uniformly in \( i \) and \( t \). It remains to control \( \partial_i U_\infty(s, t_0)\psi \). For this purpose, we will use a side result in the proof of Theorem X.70 in [3], which states that if \( P(s) \) denotes the generator of the unitary group \( U(s, t_0) \) with \( 0 \in \rho(P(s)) \) for all \( s \), then \( P(s)U(s, t_0)P(t_0)^{-1} \) is bounded. We bring \( \partial_i U_\infty(s, t_0) \) in this form by observing that

\[
\sum_{i=1}^n \|\partial_i U_\infty(s, t_0)\psi\| \leq \sum_{i=1}^n (1 + \|\partial_i U_\infty(s, t_0)\psi\|)^2 \leq C \|U_\infty(s, t_0)\psi\|^2_{W^{2,2}(\mathbb{R}^n)}.
\]

(32)
because $U_\infty(s, t_0)$ leaves $W^{2,2}(\mathbb{R}^n)$ invariant. Using Lemma 2, we can now show that $\|U_\infty(s, t_0)\psi\|_{W^{2,2}(\mathbb{R}^n)}$ is bounded by $C(\|\psi\|^2 + \|H_\infty(s)U_\infty(s, t_0)\psi\|^2)$, which is exactly what we need. However, Theorem X.70 in \[8\] requires $0 \in \rho(P(s))$. Clearly, this does not hold if $P(s) = H_\infty(s)$, but Lemma 1 shows that $H_\infty(s) + \alpha$ fulfills this requirement as long as $\alpha \in \mathbb{R}$ is big enough. Therefore, we write

$$\|U_\infty(s, t_0)\psi\|_{W^{2,2}(\mathbb{R}^n)} = \|U_\infty(s, t_0)\psi\|_{W^{2,2}(\mathbb{R}^n)} = \|U_\infty(s, t_0)\psi\|_{W^{2,2}(\mathbb{R}^n)},$$

where $\tilde{U}_\infty(s, t_0)$ is generated by $H_\infty(s) + \alpha$. Due to Lemma 2 there is a constant $C$ such that

$$\|\tilde{U}_\infty(s, t_0)\psi\|^2_{W^{2,2}(\mathbb{R}^n)} \leq C(\|\psi\|^2 + \|H_\infty(s) + \alpha\tilde{U}_\infty(s, t_0)\psi\|^2)$$

(34)

$$= C(\|\psi\|^2 + \|H_\infty(s) + \alpha\tilde{U}_\infty(s, t_0)(H_\infty(t_0) + \alpha)^{-1}(H_\infty(t_0) + \alpha)\psi\|^2).$$

Choosing $P(s) = H_\infty(s) + \alpha$ and $U(s, t_0) = \tilde{U}_\infty(s, t_0)$ in Theorem X.70 in \[8\] we then obtain

$$\sum_{i=1}^{n} \|\partial_i U_\infty(s, t_0)\psi\| \leq C(\|\psi\|^2 + C'(H_\infty(t_0) + \alpha)\psi)^2 < \infty.$$

(36)

Thereby we get an integrable dominating function for the $s$-integral in eq. (20) from

$$\|(a(\nabla, \omega \cdot s) - a(0, \omega \cdot s)) \cdot \nabla U_\infty(s, t_0)\psi\| \leq C$$

and for the $r$-integral due to $\|\cdot\|$, we can directly use eq. (31).

Having proven that we can use the theorem of dominated convergence in eqs. (20) and (27), we get

$$\|(U_\lambda(t, t_0) - U_\infty(t, t_0))\psi\| \to 0$$

for all $\psi \in W^{2,2}(\mathbb{R}^n)$ in the limit $\lambda \to \infty$ and $c \to \infty$ such that $\omega$ is kept constant.

To extend the assertion to $\psi \in L^2(\mathbb{R}^n)$, we observe that for every $\psi \in L^2(\mathbb{R}^n)$ there exists $\psi_k \in W^{2,2}(\mathbb{R}^n)$ such that $\|\psi - \psi_k\| \leq 1/k$. Using the triangle inequality and the fact that the evolution operators $U_\lambda(t, t_0)$ and $U_\infty(t, t_0)$ are unitary operators we conclude

$$\|(U_\lambda(t, t_0) - U_\infty(t, t_0))\psi\| \leq \|(U_\lambda(t, t_0) - U_\infty(t, t_0))\psi_k\| + \|(U_\lambda(t, t_0) - U_\infty(t, t_0))(\psi - \psi_k)\|$$

(39)

$$\leq \|U_\lambda(t, t_0) - U_\infty(t, t_0)\|\psi_k\| + 2\|\psi - \psi_k\|$$

(40)

$$\leq \|U_\lambda(t, t_0) - U_\infty(t, t_0)\psi_k\| + \frac{2}{k}.$$

This shows that for every $k \in \mathbb{N}$ we have $\|(U_\lambda(t, t_0) - U_\infty(t, t_0))\psi\| \leq 2/k$ as $\lambda \to \infty$ and $c \to \infty$ such that $\omega$ remains constant.

**Lemma 1.** For large enough $\alpha \in \mathbb{R}$, $H_\infty(t) + \alpha$ satisfies the assumptions of \[8\] Theorem X.70: Let $a, b \in \mathbb{R}$ and $P(t) \equiv H_\infty(t) + \alpha$. For each $t \in [a, b]$, $P(t)$ is the generator of a unitary group on $L^2(\mathbb{R}^n)$ and $0 \in \rho(P(t))$. Define $C(t, s) = P(t)P(s)^{-1} - 1$ and assume further that

(a) $\mathcal{D}(P(t)) \equiv \mathcal{D}$ is independent of $t$ and dense in $L^2(\mathbb{R}^n)$,

(b) For each $\psi \in L^2(\mathbb{R}^n)$, $(t-s)^{-1}C(t, s)\psi$ is uniformly strongly continuous and bounded with bound $M$ uniformly in $s, t \in [a, b]$ for $t \neq s$,

(c) For each $\psi \in L^2(\mathbb{R}^n)$, $C(t)\psi \equiv \lim_{\lambda \to \infty}(t-s)^{-1}C(t, s)\psi$ exists uniformly for $t \in [a, b]$ and $C(t)$ is bounded and strongly continuous in $t$.

**Proof.** Since $P(t)$ is selfadjoint for all $\alpha \in \mathbb{R}$ and each $t \in [a, b]$, it is a generator of a unitary group. Clearly, $0 \in \rho(P(t))$ for all $t$ when $\alpha$ is large enough. That $\mathcal{D}(P(t))$ is independent of $t$ follows from assertion one in Theorem [4].
Now we will prove condition (b). To see that \((t-s)^{-1}C(t,s)\) is uniformly bounded in \(s\) and \(t\), we write

\[
(t-s)^{-1}C(t,s) = (t-s)^{-1}(P(t) - P(s))P(s)^{-1}
\]

where we have used the Taylor expansion of \(A(0,t)\) and \(A(0,t)^2\) in \(t\). Uniform boundedness of the second term in eq. (44) follows from the boundedness of \(A(0,t)\) as well as its derivatives and the uniform boundedness of \(P(s)^{-1}\). To prove the latter, observe that

\[
P(s)^{-1} = (H_\infty(t) + \alpha)^{-1} = (-\Delta + W(s) + \alpha)^{-1}
\]

\[
= [(1 + W(s)(-\Delta + \alpha)^{-1})(-\Delta + \alpha)]^{-1}
\]

\[
= (-\Delta + \alpha)^{-1} [1 + W(s)(-\Delta + \alpha)^{-1}]^{-1},
\]

where \(W(s) \equiv 2iA(0,s) \cdot \nabla + A(0,s)^2 + V\). Due to the fact that \(V\) is infinitesimally \(-\Delta\)-bounded and the boundedness of \(A(0,s)\), we have

\[
\|W(s)(-\Delta + \alpha)^{-1}\psi\| \leq \varepsilon \| -\Delta(-\Delta + \alpha)^{-1}\psi\| + C_\varepsilon\|(-\Delta + \alpha)^{-1}\psi\|
\]

\[
\leq \varepsilon \|(1 - \alpha(-\Delta + \alpha)^{-1})\psi\| + \frac{C_\varepsilon}{\alpha}\|\psi\|
\]

\[
\leq (2\varepsilon + \frac{C_\varepsilon}{\alpha})\|\psi\|
\]

for all \(\psi \in L^2(\mathbb{R}^n)\). For \(\alpha\) large enough \((2\varepsilon + \frac{C_\varepsilon}{\alpha}) < 1\) and hence

\[
P(s)^{-1} = (-\Delta + \alpha)^{-1} \sum_{n=0}^\infty \left[ -W(s)(-\Delta + \alpha)^{-1}\right]^n.
\]

This implies uniform boundedness of \(P(s)^{-1}\). Uniform boundedness of the first term in eq. (44) follows from the boundedness of \(A(0,t)\) as well as its derivatives and the estimate

\[
\|\partial_i P(s)^{-1}\psi\| \leq \|P(s)^{-1}\psi\| + \| -\Delta P(s)^{-1}\psi\|
\]

\[
= \|P(s)^{-1}\psi\| + \| -\Delta(-\Delta + \alpha)^{-1} [1 + W(s)(-\Delta + \alpha)^{-1}]^{-1}\psi\|
\]

\[
= \|P(s)^{-1}\psi\| + \| (1 - \alpha(-\Delta + \alpha)^{-1}) [1 + W(s)(-\Delta + \alpha)^{-1}]^{-1}\psi\|
\]

\[
\leq (\frac{1}{2} + \varepsilon) \sum_{n=0}^\infty (2\varepsilon + \frac{C_\varepsilon}{\alpha})^n\|\psi\|
\]

\[
\leq C\|\psi\|
\]

which holds for all \(i\). The strong continuity of \((t-s)^{-1}C(t,s)\) in \(t\) is immediately evident from eq. (44) and the fact that \(P(s)^{-1}\) as well as \(\partial_i P(s)^{-1}\) are uniformly bounded. Strong continuity in \(s\) follows from eq. (44) and the strong continuity of \(P(s)^{-1}\). The latter can be seen from eq. (47) and the fact that \(\sum_{n=0}^\infty \left[ -W(s)(-\Delta + \alpha)^{-1}\right]^n\psi\| = q^n\|\psi\|\| with q < 1, so that

\[
\lim_{s \to \infty} \sum_{n=0}^\infty \left[ -W(s)(-\Delta + \alpha)^{-1}\right]^n\psi = \sum_{n=0}^\infty \lim_{s \to \infty} \left[ -W(s)(-\Delta + \alpha)^{-1}\right]^n\psi.
\]

Next we will prove condition (c). Due to eq. (44) and the strong continuity of the right hand side we get

\[
C(t)\psi = \lim_{s \to \infty} (t-s)^{-1}C(t,s)\psi = 2i(A(0,t) \cdot \nabla + A(0,t)^2)P(t)^{-1}\psi.
\]
Uniform boundedness and strong continuity of $C(t)$ now follow from same arguments as used for $(t-s)^{-1}C(t,s)$.

**Lemma 2.** Let $P(t) = H_\infty(t) + \alpha$ with $\alpha, t \in \mathbb{R}$. Then the graph norm $\| \cdot \|_{P(t)} \equiv \| \cdot \| + \|P(t) \cdot \|$ of $P(t)$ and $\| \cdot \|_{W^2,2(\mathbb{R}^n)}$ are equivalent.

**Proof.** The proof is standard, we include it only for convenience of the reader. Due to assertion one in Theorem 1, $P(t)$ is selfadjoint on $W^{2,2}(\mathbb{R}^n)$. Hence, $W^{2,2}(\mathbb{R}^n)$ is closed not only under the Sobolev norm, but also under the graph norm of $P(t)$. Define the map

$$T : (W^{2,2}(\mathbb{R}^n), \| \cdot \|_{W^2,2(\mathbb{R}^n)}) \to (W^{2,2}(\mathbb{R}^n), \| \cdot \|_{P(t)})$$

$$\psi \mapsto \psi.$$  

Clearly $T$ is bijective and eq. (24) implies $\| T^{-1} \cdot \|_{W^2,2(\mathbb{R}^n)} \leq C \| \cdot \|_{P(t)}$ for some $C$. By the inverse mapping theorem we then know that $T^{-1}$ is continuous and thereby bounded. Thus, for some $D$

$$\| \cdot \|_{W^2,2(\mathbb{R}^n)} \leq D \| \cdot \|_{W^2,2(\mathbb{R}^n)} \leq D \| \cdot \|_{P(t)}$$

We thank Volker Bach for helpful remarks and pointing out references [1, 7].

**Acknowledgments**

We thank Volker Bach for helpful remarks and pointing out references [1, 7].

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