Elementary proof of reducedness of Hilbert schemes of points in higher dimensions

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The criterion for an affine primary algebra over the field to be integral, is proven. Using this criterion we give a simple proof that Hilbert scheme of 0-dimensional subschemes of length $l$ of nonsingular $d$-dimensional algebraic variety is reduced for all $d$ and $l$.

Bibliography: 10 items.

1 Introduction

In the present article we work with algebraic schemes of finite type over a field $k$ of zero characteristic. This field is assumed to be algebraically closed. A variety is a reduced Noetherian separated scheme of finite type over an algebraically closed field. A curve is a variety understood as described if it has dimension 1; a surface is a variety of dimension 2. Hilbert scheme $\text{Hilb}_P$ whose points correspond to closed subschemes of the scheme $P$ is a convenient and oftenly used tool in algebro-geometric constructions as a simplest version of (fine) moduli space. Following [1] we recall its definition.

Let $f : X \to S$ be a projective morphism, $\mathcal{L}$ invertible $\mathcal{O}_X$-sheaf which is ample relative to $S$, $S$ is locally Noetherian, and $p(t) \in \mathbb{Q}[t]$ is a polynomial with rational coefficients. Define $\mathfrak{Hilb}_{X/S}(S')$ as a set of $S'$-flat subschemes $Z \subset X \times_S S'$ such that the fibre $Z_s = Z \times_{S'} s$ over each closed point $s \in S'$ has Hilbert polynomial equal to $p(t)$, i.e. $\chi((\mathcal{O}_Z \otimes \mathcal{L}^t)|_{f^{-1}(s)}) = p(t)$. The symbol $\chi(\cdot)$ denotes the (sheaf) Euler – Poincaré characteristic. This defines a subfunctor $\mathfrak{Hilb}_{X/S}^{p(t)} \subset \mathfrak{Hilb}_{X/S}$ and a decomposition $\mathfrak{Hilb}_{X/S} = \bigsqcup_{p(t) \in \mathbb{Q}[t]} \mathfrak{Hilb}_{X/S}^{p(t)}$. Now cite the classical A. Grothendieck’s theorem.
Theorem 1. [2, theorem 3.2] Let $X 	o S$ be a projective morphism of schemes and $S$ be Noetherian. Then for any polynomial $p(t) \in \mathbb{Q}[t]$ the functor $\mathbf{fHilb}^{p(t)}_{X/S}$ is representable by a scheme $\text{Hilb}^{p(t)}(X/S)$ which is projective over $S$. Hence the functor $\mathbf{fHilb}_{X/S}$ is also representable, and its representing scheme is a disjoint union of schemes $\text{Hilb}^{p(t)}(X/S)$.

The scheme $\text{Hilb}^{p(t)}(X/S)$ is referred to as the Hilbert scheme of the scheme $X$ over $S$ with Hilbert polynomial $p(t)$.

For our purposes we restrict ourselves by an "absolute" case when $S = \text{Spec } k$.

In general $\text{Hilb}^{p(t)}{P}$ can have rather complicated structure. In particular it can be nonreduced and can consist of several connected components. This depends of the polynomial $p(t)$ and of the structure of the scheme $P$. The simplest case is a constant Hilbert polynomial $p(t) = t = h^0(O_Z)$. The natural number $l$ is called usually a length of zero-dimensional subscheme $Z$. If $Z$ is such that $O_Z \cong \bigoplus_{i=1}^{r} k_{x_i}$ where $k_{x_i}$ is a skyscraper sheaf whose nonzero fibre in the point $x_i$ is isomorphic to its residue field (and, by algebraic closedness of the field $k$, to the field $k$ itself), then the length of the subscheme $Z$ equals the number of points. In general case the sheaf $O_Z$ is isomorphic to the direct sum $\bigoplus_{i=1}^{r} A_i$ of Artinian local $k$-algebras $A_i$ and $l = \sum_{i=1}^{r} \dim_k A_i$.

In the case when $P$ is a surface schemes $\text{Hilb}^1P$ have been studied in details during 70ies – 90ies of the past century (J. Briançon, J. Fogarty [4], R. Hartshorne, A. Iarrobino [5]). If $P$ is nonsingular surface then $\text{Hilb}^1P$ is nonsingular projective variety of dimension 2l – cf. [4]. Also in [4] it is proven that if $P$ is 3-dimensional variety then the scheme $\text{Hilb}^3P$ contains singular points. Consequently all schemes $\text{Hilb}^P$ for $l \geq 4$ are singular. Also the conjecture that $\text{Hilb}^P \mathbb{P}^N$ are reduced and irreducible for all $n$ and $N$ is formulated. There is a classical result of R. Hartshorne: if $P$ is relative projective space over the connected scheme $S$ then Hilbert schemes $\text{Hilb}^{p(t)}P$ are connected provided they are nonempty (cf. [1]). This result was reproven in the different way in 1996 by K. Pardue [6] for usual ("absolute") projective space over a field and for a projective space in zero characteristic in 2004 by I. Peeva and M. Stillman [7]. Let $P$ be a nonsingular irreducible variety of dimension $d$. By the best knowledge of the author it is not known till now about reducedness of schemes $\text{Hilb}^{P}$ for $l \geq 4$, $d \geq 3$.

Take a closed point of the scheme $\text{Hilb}^1P$. Let it correspond to the closed subscheme $i : Z \to P$. Denote by $I_Z$ the sheaf of ideals $\ker(O_P \to i_*O_Z)$. The Zariski tangent space to the scheme $\text{Hilb}^1P$ at the point $Z$ is isomorphic to the $k$-vector space $\text{Hom}_{O_Z}(I_Z, O_P/I_Z)$:

$$T_Z\text{Hilb}^1P \cong \text{Hom}_{O_Z}(I_Z, O_P/I_Z).$$

Its dimension equals $ld$ if the point $Z$ of the scheme $\text{Hilb}^1P$ is general enough and corresponds to a subscheme $Z$ which consists of $l$ distinct reduced points. A closed point $i : Z \to P$ of the scheme $\text{Hilb}^1P$ is called singular if $\dim T_z > ld$.

Questions on the dimension of the tangent space and on reducedness of the scheme are local. Then we can replace an arbitrary nonsingular $d$-dimensional variety by local neighborhood of its closed point. This replacement and the introduction of coordinates (in usual sense) are based on the result we recall following [8].

Let $(A, m, K)$ be a local ring with maximal ideal $m$ and residue field $K = A/m$. We need particular case of the theorem [8, Theorem 29.7].
Theorem 2. A complete regular local ring of characteristic 0 is formal power series a ring over a field.

Since the base field $k$ is assumed to be algebraically closed then it is isomorphic to the residue field $K$.

The cited theorem yields that it is enough to prove reducedness of the scheme $\text{Hilb}^1 \text{Spec} k[[x_1, \ldots, x_d]]$ with number of indeterminates $d$ equal to the dimension of the variety $P$ of interest, since by to this theorem local neighborhoods of all simple points of $d$-dimensional varieties over algebraically closed field of characteristic 0 are isomorphic. For convenience of computations we fix standard homomorphism of rings $k[x_1, \ldots, x_d] \hookrightarrow k[[x_1, \ldots, x_d]]$ (inclusion into the localization) and consider the scheme $\text{Hilb}^1 \mathbb{A}^d$ for $\mathbb{A}^d = \text{Spec} k[x_1, \ldots, x_d]$.

If $\text{Nil} \, \mathcal{O}_X$ is a nilradical of the structure sheaf of some scheme $X$ then its support $\text{Supp} \, \text{Nil} \, \mathcal{O}_X$ is a closed subscheme in $X$. In particular this means that if we are given a flat 1-parameter family of subschemes $Z \to \text{Spec} D$ where $D$ is an integral $k$-algebra of Krull dimension equal to 1 and $\text{Supp} \, \text{Nil} \, \mathcal{O}_{\text{Hilb}^1 P}$ contains an image of open subset $U \subset \text{Spec} D$, then $\text{Supp} \, \text{Nil} \, \mathcal{O}_{\text{Hilb}^1 P}$ contains an image of the whole of the curve $\text{Spec} D$. This reasoning will be applied several times and we replace it by the short formulation: reduced point has reduced generalization. Similarly, nonsingular point has nonsingular generalization.

The main result of the present note is contained in the following theorem.

Theorem 3. Hilbert scheme $\text{Hilb}^1 P$ of $l$-point subschemes of the nonsingular $d$-dimensional variety $P$ is reduced. Since it is separated and has a finite type, then it is singular 1d-dimensional variety.

The article is organized as follows. In sect.2 we prove irreducibility of the scheme $\text{Hilb}^1 P$ if the variety $P$ is irreducible. In sect.3 bases of tangent spaces at most special points of Hilbert schemes $\text{Hilb}^{d+1} \mathbb{A}^d$ are written down. Also we describe the deformations of the most special subscheme $Z$ which continue basis tangent directions. In sect.4 we prove that the computations done imply reducedness of Hilbert schemes of $d+1$-point subschemes of $d$-dimensional variety $P$. At last, in sect.5 is shown that the reasonings done imply reducedness of schemes $\text{Hilb}^1 P$ for all values $l$ and $\text{dim} P = d$.

2 Irreducibility

In this section we prove irreducibility of the scheme $\text{Hilb}^1 P$ in the case when the variety $P$ is irreducible. The connectivity of this scheme is proven in just cited classical work by R. Hartshorne but connectivity does not implies irreducibility.

To prove irreducibility of the scheme of interest it is enough to confirm ourselves that all subschemes of length $l$ supported at one point belong to the closure of the open subset in $\text{Hilb}^1 P$ formed by unions of $l$ reduced points.

Let $Z$ be a subscheme supported at one point $p$ on the variety $P$; choose a local isomorphism $\mathcal{O}_{P, p} \cong k[[x_1, \ldots, x_d]]$ so that $\text{Supp} \, Z = \mathfrak{m} = (x_1, \ldots, x_d)$.

We construct the sequence of 1-parameter flat families $Z_0(\alpha), Z_1(\alpha), \ldots, Z_q(\alpha), \alpha \in \mathbb{P}^1$, such that $Z_0(\infty) = Z$ and for $\alpha \neq \infty$ $Z_q(\alpha) = \{p_1(\alpha), \ldots, p_l(\alpha)\}$ is a collection of $d$ distinct points.
Let the subscheme $Z$ be defined by the ideal $I \subset k[[x_1, \ldots, x_d]]$. Fix a natural ordering of indeterminates $x_1 < x_2 < \cdots < x_d$, relate a lexicographic ordering of monomials to it in the ring $k[[x_1, \ldots, x_d]]$. Take a reduced Gröbner basis $f_1, f_2, \ldots, f_m$ in $I$ with respect to this ordering. Then according to [9] ch. 3 §1, theorem 2, §2, theorem 3], one (the last) of polynomials $f_1, f_2, \ldots, f_m$ contains dependence of $x_1$ only. Note that all transformations done for computations of Gröbner bases do not change scheme structure; then in cited theorems of elimination theory from [9] the term "variety" can be replaced by the term "scheme". Since the initial subscheme $Z$ has a support at the point $m = (x_1, \ldots, x_d)$ and the subscheme defined by the ideal $(f_m) \subset k[x_1]$ contains Zariski closure for the image of the subscheme $Z \subset \text{Spec } k[x_1, \ldots, x_d]$ under projection to the line $\text{Spec } k[x_1] \subset \text{Spec } k[x_1, \ldots, x_d]$, then $f_m = x_1^l$ for appropriate $l' \leq l$.

Now consider the family of ideals $I(\alpha) = (f_1, \ldots, f_{m-1}, x_d - \alpha f_m)$ where $\alpha \in \mathbb{P}^1$. For $\alpha = \infty$ we obtain the ideal $I$ of the initial subscheme $Z$. Ideals corresponding to $\alpha \neq \infty$, are taken to the ideal $I(0) = (f_1, \ldots, f_{m-1}, x_d)$ by corresponding automorphism of the ring $k[[x_1, \ldots, x_d]]$ such that $x_i \mapsto x_i$ for $1 \leq i \leq d - 1$, $x_d \mapsto x_d + \alpha f_m$. From this we conclude that all ideals $I(\alpha)$ for $\alpha \neq \infty$ define subschemes of equal lengths; let it equal $l_0$. The family $I(\alpha)$ defines the family of subschemes $Z(\alpha) \subset \text{Spec } k[x_1, \ldots, x_d] \times \mathbb{P}^1$ and a morphism of the open subset of the base $\mu : \mathbb{P}^1 \setminus \infty \to \text{Hilb}^{l_0}\text{Spec } k[x_1, \ldots, x_d]$. Consider a standard immersion of the affine space $\text{Spec } k[x_1, \ldots, x_d]$ as affine coordinate chart to the projective space $\mathbb{P}^d$ with homogeneous coordinates $(X_0 : X_1 : \cdots : X_d)$ such that $X_0 \neq 0$, $x_i = X_i/X_0$, $i > 0$. It induces the immersion of Hilbert schemes $\text{Hilb}^{l_0}\text{Spec } k[x_1, \ldots, x_d] \subset \text{Hilb}^{l_0}\mathbb{P}^d$ and the composite morphism $\mu : \mathbb{P}^1 \setminus \infty \to \text{Hilb}^{l_0}\text{Spec } k[x_1, \ldots, x_d] \subset \text{Hilb}^{l_0}\mathbb{P}^d$. Then there is a closed subscheme $Z(\alpha) \subset \mathbb{P}^d \times \mathbb{P}^1 \to \mathbb{P}^1$. Since all closed fibres of the family $Z(\alpha)$ in points $\alpha \in \mathbb{P}^1 \setminus \infty$ have constant length which equals $l_0$, then the family $Z(\alpha)$ is flat over $\mathbb{P}^1 \setminus \infty$. According to [10] ch. III, Proposition 9.8, there exist a unique closed subscheme in $\mathbb{P}^d \times \mathbb{P}^1$ which is scheme-theoretic closure of the subscheme $Z(\alpha)|_{\alpha \neq \infty}$ and it is flat over $\mathbb{P}^1$. But required scheme-theoretic closure is precisely $Z(\alpha)$. This proves that the scheme $Z(\alpha)$ is flat over $\mathbb{P}^1$ and hence all its closed fibres have equal lengths $l = l_0$.

From this we conclude that the scheme $Z$ belongs to the closure of the locally closed subset whose points correspond to subschemes with one-point support on nonsingular hypersurfaces in $\text{Spec } k[x_1, \ldots, x_d]$. Hypersurfaces are defined by equations of the form $x_d - \alpha f_m = 0$ where $f_m$ is a polynomial in variable $x_1$. Every such hypersurface is isomorphic to the hyperplane $x_d = 0$, and we come to the analogous problem in the space of lower dimension for a subscheme defined by the ideal $I' = (f_1, \ldots, f_{m-1}) \subset \text{Spec } k[x_1, \ldots, x_{d-1}]$. Choosing reduced Gröbner basis in it and continuing the process we conclude that the initial subscheme $Z \subset \text{Spec } k[x_1, \ldots, x_d]$ belongs to the closure of locally closed subset formed by subschemes each of which lies on a smooth curve $C$ defined by the ideal $(g_1, \ldots, g_s)$. Under an appropriate choice of coordinate system on this curve subschemes of locally closed subset can be defined by ideals of the form $(x_1^l)$. Obviously, the point corresponding to the subscheme of the described form belongs to the closure of the set of points corresponding to reduced $l$-point schemes.
3 Basis vectors of the tangent space and their continuing homomorphisms

Since $P$ is a scheme of finite type over a field (nonsingular variety) then $\text{Hilb}^IP$ is also of finite type over the field. The question about presence or absence of nilpotents in the structure sheaf of the Hilbert scheme allows us to reduce the consideration of general $P$ of dimension $d$ to the consideration of $\mathbb{A}^d = \text{Spec} k[x_1, \ldots, x_d]$ at the neighborhood of the point $m = (x_1, \ldots, x_d)$. We consider the point in the Hilbert scheme corresponding to the subscheme $Z \subset \mathbb{A}^d$ defined by the ideal

$$I = (x_1, \ldots, x_d)^2$$

$$= (x_1^2, x_1x_2, \ldots, x_1x_d, x_2^2, x_2x_3, \ldots, x_2x_d, \ldots, x_i^2, x_ix_{i+1}, \ldots, x_d^2).$$

Obviously, its length equals $l = d + 1$.

This allows to write down explicitly the basis vectors $v_{ijm} : I \to k[x_1, \ldots, x_d]/I$ of Zariski tangent space:

$$v_{ijm} : x_r x_s \mapsto \begin{cases} x_m \text{ for } (r, s) = (i, j), & i \leq j, \quad r \leq s. \\ 0 \text{ otherwise} \end{cases}$$

Totally there are $r = d^2(d+1)/2 = d^2l/2$ of linearly independent vectors. One-parameter families of $(d+1)$-subschemes with germs constituting a basis of Zariski tangent space in the point of interest, can be chosen, for example, as follows (they are subdivided into two groups by geometrical meaning):

**Group 1:** reattachment of points.

$$i = 1, \ldots, d \quad (x_i(x_i - \alpha), x_ix_j, j \neq i, x_ix_j, t \neq i)$$

$$i = 1, \ldots, d, \ j \neq i \quad (x_i(x_i - \alpha), (x_i - \alpha)x_j, x_s x_t, (s, t) \neq (i, j), (s, t) \neq (j, i)).$$

**Group 2:** subschemes on quadrics.

$$i, j = 1, \ldots, d, \ s \neq i, \ s \neq j \quad (x_i x_j - \nu x_s, x_q x_t, (q, t) \neq (i, j), (q, t) \neq (j, i)).$$

**Example 1.** Set $d = 3, \ l = 4$. The point of the interest in the scheme $\text{Hilb}^I \mathbb{A}^3$ corresponds to the ideal $m^2 = (x^2, xy, y^2, yz, z^2, zx)$. The dimension of the tangent space to the scheme $\text{Hilb}^I \mathbb{A}^3$ equals 18, as well as in the general point it equals 12. We enumerate 1-parameter families of 4-subschemas such that germs of these families in the point of interest constitute a basis of the tangent space.

1. $(x(x - \alpha), xy, y^2, yz, z^2, zx)$
2. $(x^2, xy, y(y - \beta), yz, z^2, zx)$
3. $(x^2, xy, y^2, yz, z(z - \gamma), zx)$
4. $(x(x - \alpha), xy, y^2, yz, z^2, z(x - \alpha))$
5. $(x^2, (y - \beta), y(y - \beta), yz, z^2, zx)$
6. $(x^2, xy, y^2, y(z - \gamma), z(z - \gamma), zx))$
7. $(x(x - \alpha), (x - \alpha)y, y^2, yz, z^2, zx)$
8. $(x^2, xy, y(y - \beta), (y - \beta)z, z^2, zx)$
9. $(x^2, xy, y^2, yz, z(z - \gamma), (z - \gamma)x))$
10. $(x^2 - \mu y, xy, y^2, yz, z^2, zx)$
11. $(x^2, xy, y^2 - \nu y, yz, z^2, zx)$
12. $(x^2, xy, y^2, yz, z^2 - \sigma x, zx)$
13. $(x^2 - \pi x, xy, y^2, yz, z^2, zx)$
14. $(x^2, xy, y^2 - \rho y, yz, z^2, zx)$
15. $(x^2, xy, y^2, yz, z^2 - \tau y, zx)$
16. $(x^2, xy - \eta y, y^2, yz, z^2, zx)$
17. $(x^2, xy, y^2, z^2 - \xi x, z^2, zx)$
18. $(x^2, xy, y^2, yz, z^2, zx - \zeta y)$. 


4 Reducedness

Let \( A \) be a commutative ring, \( \text{Nil} A \) its nilradical, \( A_{\text{red}} \) the quotient ring \( A/\text{Nil} A \) which is called the reduction of \( A \). If \( I \subset A \) is an ideal then set by the definition \( I_{\text{red}} := I/(I \cap \text{Nil} A) \). It is clear that this is an ideal in \( A_{\text{red}} \). Now prove the following simple lemma.

**Lemma 1.** \((m^2)_{\text{red}} = (m_{\text{red}})^2\)

**Proof.** It is clear that \((m_{\text{red}})^2 \subset (m^2)_{\text{red}}\). Take an element \( x \in (m^2)_{\text{red}} \) and choose any of preimages \( \overline{x} \in A \) for \( x; \overline{x} \in m^2 \). Then \( \overline{x} = \overline{y} \overline{z}, \overline{y}, \overline{z} \in m \). Denoting images \( \overline{y} \) and \( \overline{z} \) in the reduction by \( y \) and \( z \) correspondingly we obtain \( x = yz \), where \( y, z \in m_{\text{red}} \).}

According to the lemma we omit brackets in the notation of the square of the maximal ideal of the reduction: \( m_{\text{red}}^2 := (m_{\text{red}})^2 = (m^2)_{\text{red}} \).

Since we work in the category of schemes of finite type over a field \( k \) then for our purposes it is enough to consider commutative associative algebras of finite type over the field \( k \) instead of arbitrary associative commutative rings with unity. Otherwise speaking, the class of rings of our interest are quotient algebras of polynomial rings \( k[x_1, \ldots, x_N] \) in appropriate number of indeterminates over the field \( k \). These \( k \)-algebras are called for brevity affine algebras.

Let \( A \) be an affine algebra, \( \varphi : k[x_1, \ldots, x_N] \to A \) the corresponding ring epimorphism, \( I = \ker \varphi \) its kernel, \( I = \bigcap_i q_i \) the primary decomposition of the kernel. Here \( q_i \) is a primary ideal for any \( i \). Then the examination of the reducedness of the scheme \( \text{Spec} A \) reduces to the consideration of its Zariski irreducible components \( \text{Spec} A_i \) for algebras of the special form \( A_i = k[x_1, \ldots, x_N]/q_i \). Such \( k \)-algebras are called primary. It is clear that for \( A = k[x_1, \ldots, x_N]/q \) we have \( A_{\text{red}} = k[x_1, \ldots, x_N]/\sqrt{q} \).

**Proposition 1.** The primary \( k \)-algebra \( A \) is integral if and only if

\[
\dim_k m/m^2 = \dim_k m_{\text{red}}/m_{\text{red}}^2.
\]

**Proof.** Consider the exact diagram of \( A \)-modules

\[
\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \text{Nil} A/(m^2 \cap \text{Nil} A) & \to & m/m^2 & \to & m_{\text{red}}/m_{\text{red}}^2 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \text{Nil} A & \to & m & \to & m_{\text{red}} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & m^2 \cap \text{Nil} A & \to & m^2 & \to & m_{\text{red}}^2 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 \\
\end{array}
\]
The equality \( \dim_k m/m^2 = \dim_k m_{\text{red}}/m_{\text{red}}^2 \) in the formulation of the proposition means that \( m/m^2 = m_{\text{red}}/m_{\text{red}}^2 \). This implies \( m^2 \cap \text{Nil} A = \text{Nil} A \) and \( \text{Nil} A \subset m^2 \).

Let \( \eta \in \text{Nil} A \) be a nilpotent element of index \( \iota > 1 \). Since \( \text{Nil} A \subset m^2 \) then \( \eta = \eta_1 \xi_1 \) for certain \( \eta_1, \xi_1 \in m \). Since \( A \) is affine algebra then choose any preimages \( \pi_1, \xi_1 \in k[x_1, \ldots, x_N] \) for \( \eta_1, \xi_1 \) under the epimorphism \( k[x_1, \ldots, x_N] \to A \) with kernel \( q \subset k[x_1, \ldots, x_N] \) for a primary ideal \( q \subset k[x_1, \ldots, x_N] \). Then \( \pi_1 \xi_1 \in q \), and by primarity of the ideal \( q \) we conclude that \( \eta_1, \xi_1 \) are nilpotent in \( A \). If one of them belongs to \( m \setminus m^2 \) then \( \dim_k m/m^2 > \dim_k m_{\text{red}}/m_{\text{red}}^2 \), and the contradiction completes the proof. Let as before \( \eta_1, \xi_1 \in m^2 \), and we can apply the reasoning described above, say, to \( \eta_1 = \eta_2 \xi_2 \), etc. We come to strictly ascending chain of principal ideals \( (\eta) \subset (\eta_1) \subset (\eta_2) \subset \ldots \). The ascending chain condition for ideals in \( k[x_1, \ldots, x_N] \) yields that there is a nilpotent belonging to \( m \setminus m^2 \), and then \( \dim_k m/m^2 > \dim_k m_{\text{red}}/m_{\text{red}}^2 \), what contradicts the condition of the proposition.

Hence \( \text{Nil} A = 0 \), then \( q \) is prime, \( q = \sqrt{\mathfrak{n}} \) and \( A \cong k[x_1, \ldots, x_N]/\sqrt{\mathfrak{n}} \). The opposite implication is obvious and the proof is complete. \( \square \)

Since Zariski tangent space to the scheme \( \text{Spec} A \) in its closed point \( m \) is given by the vector space \( (m/m^2)^r \), then the equality \( \dim_k (m/m^2) = r \) means existence of \( r \) \( k \)-linearly independent homomorphisms of \( k \)-algebras \( v_i : A \to k[\varepsilon]/(\varepsilon^2) \). Let the homomorphism \( v : A \to k[\varepsilon]/(\varepsilon^2) \) factors as \( v : A \xrightarrow{\phi} D \to k[\varepsilon]/(\varepsilon^2) \) where \( D \) is integral \( k \)-algebra. For convenience of computations we can assume integral domain \( D \) to have Krull dimension equal to \( 1 \). Then \( \text{Nil} A \subset \ker \phi \), and \( \phi \) and \( v \) factor through \( A_{\text{red}} \). If all basis vectors \( v_i, i = 1, \ldots, r \) have this property then we are in the realm of the proposition \( \square \) and \( A \) is integral. The opposite is obviously true.

Geometrically this means the following: there are \( r \) curves through the point \( m \) of the scheme \( \text{Spec} A \) such that their tangent vectors are linearly independent.

**Example 2.** The requirement of primarity of the ring \( A \) is not superfluous. For example the scheme \( \text{Spec} k[x, y]/(x^2y) \) is nonreduced at the point \( (x, y) \), but there is a basis of tangent space which consists of 2 homomorphisms \( v_i : k[x, y]/(x^2y) \to k[\varepsilon]/(\varepsilon^2) \), \( i = 1, 2 \), defined by correspondences \( v_1 : x \mapsto \varepsilon, v_1 : y \mapsto 0 \) and \( v_2 : x \mapsto 0, v_2 : y \mapsto \varepsilon \). These homomorphisms factor through obvious homomorphisms \( \phi_1 : k[x, y]/(x^2y) \to k[x] \) and \( \phi_2 : k[x, y]/(x^2y) \to k[y] \) respectively.

Performing primary decomposition we come to two components \( A_1 = k[x, y]/(y) = k[x] \) \( A_2 = k[x, y]/(x^2) \) where only one is reduced. The same conclusion is provided by our criterion.

## 5 Proof for all schemes \( \text{Hilb}^l \mathbb{A}^d \)

It is known that schemes \( \text{Hilb}^l \mathbb{A}^{d-1} \) are reduced for all \( l \geq 2 \). Show that this implies that any scheme \( \text{Hilb}^l \mathbb{A}^d \) and hence any \( \text{Hilb}^l P \) is reduced where \( P \) is \( d \)-dimensional nonsingular variety.

If \( d > l - 1 \) then reducedness of the scheme \( \text{Hilb}^l \mathbb{A}^d \) follows from specializations. For the proof it is necessary to construct a deformation of the most special point of the scheme \( \text{Hilb}^{d+l} \mathbb{A}^d \) to disjoint union of an appropriate subscheme \( Z' \in \text{Hilb}^{d+l} \mathbb{A}^d \) supported at a point, and \( d - l + 1 \) reduced points. It is enough to consider the case when \( Z' \) is a
subscheme of the most special form. Under appropriate choice of coordinate system it is defined by the ideal \( I' = (x_i x_j, 1 \leq i \leq j \leq l - 1, x_i, \ldots, x_d). \) The desired deformation is defined by the ideal \((x_i x_j, 1 \leq i \leq j \leq l - 1, x_s(x_t - \alpha t), 1 \leq s \leq d, l \leq t \leq d, s \leq t).\) Since reduced point has reduced generization then the proof for \( d > l - 1 \) is complete.

Now let \( d < l - 1. \) Fix the immersion of the plane \( \mathbb{A}^d \hookrightarrow \mathbb{A}^{l-1} \) by vanishing of \( l - d - 1 \) last coordinates: \( x_{d+1} = \cdots = x_{l-1} = 0, \) and the induced immersion of the Hilbert scheme \( \text{Hilb}^1 \mathbb{A}^d \hookrightarrow \text{Hilb}^1 \mathbb{A}^{l-1}. \) Since we prove reducedness it is enough to consider the most special point of the scheme \( \text{Hilb}^1 \mathbb{A}^d. \) Let the point \( Z \) of the scheme \( \text{Hilb}^1 \mathbb{A}^d \) correspond to the ideal \( I \) with generators \( f_1, \ldots, f_s. \) Then the image of this point in the scheme \( \text{Hilb}^1 \mathbb{A}^{l-1} \) corresponds to the subscheme with the ideal \( I' = (f_1, \ldots, f_s, x_{d+1}, \ldots, x_{l-1}). \) Since the scheme \( \text{Hilb}^1 \mathbb{A}^{l-1} \) is reduced then there exists a collection of curves on the scheme \( \text{Hilb}^1 \mathbb{A}^{l-1} \) through the point \( Z \) with tangent directions forming a basis of tangent space \( T_Z \text{Hilb}^1 \mathbb{A}^{l-1}. \) These tangent directions are enumerate by the correspondences

\[
f_i \mapsto \overline{f}_j, \quad \overline{f}_j \in k[x_1, \ldots, x_d]/I, \quad f_r \mapsto 0, \quad r \neq i, \quad x_q \mapsto 0, \quad d + 1 \leq q \leq l - 1,
\]

and

\[
f_i \mapsto 0, \quad 1 \leq i \leq s, \quad x_t \mapsto \overline{f}_j, \quad x_q \mapsto 0, \quad q \neq t.
\]

Correspondences of the second group generate 1-parameter families of the form \((f_1, \ldots, f_s, x_{d+1}, \ldots, x_{l-1}, x_t - \alpha \overline{f}_j, x_{l+1}, \ldots, x_{l-1}).\) Since the scheme \( \text{Hilb}^1 \mathbb{A}^{l-1} \) is reduced, there exist \( \dim T_Z \text{Hilb}^1 \mathbb{A}^{l-1} \) curves in it with linearly independent tangent directions i.e. 1-parameter subfamilies of the family

\[
(f_1 - \sum_j \alpha_{1j} \overline{f}_j, \ldots, f_u - \sum_j \alpha_{uj} \overline{f}_j, \ldots, f_s - \sum_j \alpha_{sj} \overline{f}_j, x_{d+1} - \sum_j \beta_{d+1,j} \overline{f}_j, \ldots, x_t - \sum_j \beta_{t,j} \overline{f}_j, \ldots, x_{l-1} - \sum_j \beta_{l-1,j} \overline{f}_j).
\]

In such expression of deformations some of \( \alpha_{uj} \) and of \( \beta_{t,j} \) can equal 0 in all families; it depends on the structure of the ideal \( I. \) The automorphism of the ring \( k[x_1, \ldots, x_{l-1}] \) defined by the correspondence \( x_i \mapsto x_i \) for \( i \leq d, \) \( x_t \mapsto x_t + \sum_j \beta_{t,j} \overline{f}_j \) for \( d + 1 \leq t \leq l - 1 \) takes this family to the family

\[
(f_1 - \sum_j \alpha_{1j} \overline{f}_j, \ldots, f_u - \sum_j \alpha_{uj} \overline{f}_j, \ldots, f_s - \sum_j \alpha_{sj} \overline{f}_j, x_{d+1}, \ldots, x_t, \ldots, x_{l-1}).
\]

We come to the projection to \( \text{Hilb}^1 \mathbb{A}^d \) for all families chosen. The number of independent tangent directions in the image of the projection equals to the dimension \( \dim T_Z \text{Hilb}^1 \mathbb{A}^d. \) Considering \( \dim T_Z \text{Hilb}^1 \mathbb{A}^{l-1} \) curves with linearly independent tangent directions provided by the family \((5.1), \) and applying the projection described, we come to \( \dim T_Z \text{Hilb}^1 \mathbb{A}^d \) curves in \( \text{Hilb}^1 \mathbb{A}^d \) with linearly independent directions. This proves reducedness of the scheme \( \text{Hilb}^1 \mathbb{A}^d \) in its point \( Z, \) and hence everywhere.

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