Fair Allocation of Indivisible Chores: Beyond Additive Valuations*

Bo Li\(^{1}\) \quad Fangxiao Wang\(^{2}\) \quad Yu Zhou\(^{1}\)

\(^{1}\)Department of Computing, The Hong Kong Polytechnic University, Hong Kong, China  
comp-bo.li@polyu.edu.hk, csyzhou@comp.polyu.edu.hk

\(^{2}\)School of Information and Communication Engineering, UESTC, Chengdu, China  
2018011216019@std.uestc.edu.cn

April 29, 2022

Abstract

In this work, we study the maximin share (MMS) fair allocation of indivisible chores. For additive valuations, Huang and Lu [EC, 2021] designed an algorithm to compute a 11/9-approximate MMS fair allocation, and Feige et al. [WINE, 2021] proved that no algorithm can achieve better than 44/43 approximation. Beyond additive valuations, unlike the allocation of goods, very little is known. We first prove that for submodular valuations, in contrast to the allocation of goods where constant approximations are proved by Barman and Krishnamurthy [TEAC, 2020] and Ghodsi et al [AIJ, 2022], the best possible approximation ratio is \(n\). We then focus on two concrete settings where the valuations are combinatorial. In the first setting, agents need to use bins to pack a set of items where the items may have different sizes to different agents and the agents want to use as few bins as possible to pack the items assigned to her. In the second setting, each agent has a set of machines that can be used to process a set of items, and the objective is to minimize the makespan of processing the items assigned to her. For both settings, we design constant approximation algorithms, and show that if the fairness notion is changed to proportionality up to one/any item, the best approximation ratio is \(n\).

Key Words: Indivisible Chores, Maximin Share Fairness, Subadditive Valuations, Bin Packing, Job Scheduling.

1 Introduction

1.1 Background and related research

Since the introduction of maximin share (MMS) fairness by Budish (2011), MMS has been extensively used as a criterion to evaluate the fairness of an allocation when \(m\) indivisible items are allocated to \(n\) heterogeneous agents. MMS fairness is traditionally defined for the allocation of goods as

*The authors are ordered alphabetically. Part of this work was done when the second author was a research assistant with The Hong Kong Polytechnic University.
a relaxation of proportionality (PROP). A PROP allocation requires that the utility of each agent is no smaller than the average utility when all items are assigned to her. PROP is too demanding when the items are indivisible since such an allocation does not exist even when there is a single item and two agents: the agent who is not assigned the item has utility 0 which is arbitrarily smaller than her proportionality. Motivated by this strong impossibility result, Budish (2011) proposed to use MMS to replace proportionality. Informally, the maximin share (MMS) of an agent is the maximum value she can guarantee if she is to partition the items into \( n \) bundles but is the last one to pick a bundle. An allocation is MMS fair if every agent’s value is no smaller than her MMS.

For the allocation of goods, although MMS significantly weaken the fairness requirement, it is first shown by Kurokawa et al. (2018) that there are instances where no allocation is MMS fair to all agents. Accordingly, designing (efficient) algorithms to compute approximately MMS fair allocations steps into the center of the field of algorithmic fair division. Kurokawa et al. (2018) first proved there exists a \( 2/3 \)-approximate MMS fair allocation for additive valuations, and then Amanatidis et al. (2017) designed a polynomial time algorithm with the same approximation guarantee. Later, Ghodsi et al. (2021) improved the approximation ratio to \( 3/4 \), and Garg and Taki (2021) further improved it to \( 3/4 + 1/(12n) \). On the negative side, Feige et al. (2021) proved that no algorithm can ensure better than \( 39/40 \) approximation. Beyond additive valuations, Barman and Krishnamurthy (2020) initiated the study of approximate MMS fair allocation with submodular valuations, and proved that a \( 0.21 \)-approximate MMS fair allocation can be computed by the round-robin algorithm. Ghodsi et al. (2022) improved the approximation ratio to \( 1/3 \), and moreover, they gave constant and logarithmic approximation guarantees for XOS and subadditive valuations, respectively. The approximations for XOS and subadditive valuations are recently improved in Seddighin and Seddighin (2022).

As we have seen, the majority of efforts are devoted to indivisible goods, and the parallel problem of tasks (called chores where agents have costs or dis-utilities to complete the received items) has received less attention. Aziz et al. (2017) first pointed out this issue, and proved that the round-robin algorithm ensures \( 2 \) approximation for additive valuations for indivisible chores. Barman and Krishnamurthy (2020) and Huang and Lu (2021) respectively improved the approximation ratio to \( 4/3 \) and \( 11/9 \). Recently, Feige et al. (2021) proved that with additive valuations, no algorithm can beat the approximation of \( 44/43 \). However, very little is known beyond additive valuations.

Besides MMS fairness, proportionality up to one item (PROP1) and proportionality up to any item (PROPX) are also studied as relaxations of proportionality. For goods, PROP1 allocations always exist but PROPX allocations may not exist Barman and Krishnamurthy (2019); Brânzei and Sandomirskiy (2019); Aziz et al. (2020). For chores, both PROP1 and PROPX allocations exist and can be found efficiently Moulin (2018); Li et al. (2022). As far as we know, all the above works assume the valuations are additive, and combinatorial valuations have not been considered yet.

### 1.2 Main results

In this work, we aim at understanding the extent to which MMS fairness, as well as PROPX/PROP1, can be satisfied when the valuations are not additive. First, it is easy to see that allocating all items to a single agent achieves \( n \)-approximate proportionality and thus achieves \( n \)-approximate MMS fairness. Surprisingly, we show that \( n \)-approximation is the best possible even when the valuations are submodular. This result exhibits a significant difference between the allocations of chores and goods, since for goods with submodular (even XoS) valuations, we always have constant-approximate MMS fair allocations [Barman and Krishnamurthy, 2020; Ghodsi et al., 2022].

**Result 1** For any number \( n \) of agents, there is an instance with submodular valuations where no
allocation can be better than \(n\)-approximate MMS fair.

Due to this strong impossibility result, we turn to study two concrete settings where the agents have specific combinatorial valuations. The first setting deals with a bin packing problem, where the items have sizes and the sizes can be different to different agents. The agents have bins that can be used to pack the items allocated to them with the goal of using as few bins as possible. We call this valuation bin-packing. For bin-packing valuations, we show that no algorithm can be better than 2-approximation, and design a polynomial time algorithm that returns a 2-approximate MMS fair allocation. Moreover, the approximation can be improved to \(3/2\) when the maximin share gets large.

**Result 2** For bin-packing valuations, the best possible approximation of MMS fairness is 2, and an allocation matching this approximation ratio can be found in polynomial time.

The second setting deals with a job scheduling problem, where a set of jobs need to be processed by the agents. The agents are heterogeneous and thus each job may be of different lengths to different agents. Each agent controls a set of machines with possibly different speeds. Upon receiving a set of jobs, an agent’s value is determined by the corresponding minimum completion time when processing the jobs using her own machines (i.e., makespan). The corresponding valuation is called job-scheduling. As will be clear, job-scheduling is a more general setting than additive valuations, which uncovers new research directions for group-wise fairness.

**Result 3** For job-scheduling valuations, a 2-approximate MMS allocation can be found efficiently.

Finally, we consider PROPI and PROPX allocations, and show for both bin-packing and job-scheduling valuations, no algorithm can be better than \(n\)-approximation. Thus any allocation algorithm achieves this tight approximation.

## 2 Preliminaries

For any integer \(k \geq 1\), let \([k] = \{1, \ldots, k\}\). In a fair allocation instance, we have \(n\) agents denoted by \(N = [n]\) and \(m\) items denoted by \(M = [m]\). Each agent \(i\) has a valuation function over the items, \(v_i : 2^M \to \mathbb{R}^+ \cup \{0\}\). The items are tasks, and particularly, upon receiving \(S \subseteq M\), \(v_i(S)\) represents the effort or cost agent \(i\) needs to spend on completing the tasks in \(S\). The valuation functions are normalized and monotone, i.e., \(v_i(\emptyset) = 0\) and \(v_i(S_1) \leq v_i(S_2)\) for any \(S_1 \subseteq S_2 \subseteq M\). Note that no bounded approximation can be achieved for general valuation functions, and we provide one such example in the appendix. Thus we restrict our attention to the following three classes.

- **Subadditive**: Valuation function \(v_i\) is subadditive if for any \(S_1, S_2 \subseteq M\), we have
  \[
  v_i(S_1 \cup S_2) \leq v_i(S_1) + v_i(S_2).
  \]

- **Submodular**: Valuation function \(v_i\) is submodular if for any \(S_1 \subseteq S_2 \subseteq M\) and \(e \in M \setminus S_2\),
  \[
  v_i(S_2 \cup \{e\}) - v_i(S_2) \leq v_i(S_1 \cup \{e\}) - v_i(S_1).
  \]

- **Additive**: Valuation function \(v_i\) is additive if for any \(S \subseteq M\), we have
  \[
  v_i(S) = \sum_{j \in S} v_i(\{j\}).
  \]
It is widely known that any additive valuation function is also submodular, and any submodular valuation function is also subadditive.

An allocation $X = (X_1, \ldots, X_n)$ is an $n$-partition of the items where $X_i$ contains the items allocated to agent $i$ such that $X_i \cap X_j = \emptyset$ and $\bigcup_{i \in N} X_i = M$. For any set $S$ and integer $k$, let $\Pi_k(S)$ be the set of all $k$-partitions of $S$. Then the maximin share ($\text{MMS}_i$) of agent $i$ is defined as

$$\text{MMS}_i = \min_{X \in \Pi_n(M)} \max_{j \in N} v_i(X_j).$$

Note that the computation of $\text{MMS}_i$ is NP-hard even when the valuations are additive, which can be verified by a reduction from the Partition problem. Given an $n$-partition of $M$, $X = (X_1, \ldots, X_n)$, if $v_i(X_i) \leq \text{MMS}_i$ for all $i$, then $X$ is called an $\text{MMS}$-defining partition for agent $i$. Note that the original definition of $\text{MMS}_i$ for chores is defined with non-positive valuations, where the minimum valued bundle is maximized. In this work, to simplify the notions, we choose to use non-negative numbers (representing costs), and thus the definition is equivalently changed to be the maximum valued bundle is minimized. To be consistent with the literature, we still call it maximin share.

**Definition 1 ($\alpha$-MMS)** An allocation $X = (X_1, \ldots, X_n)$ is $\alpha$-approximate maximin share ($\alpha$-MMS) fair if $v_i(X_i) \leq \alpha \cdot \text{MMS}_i$ for any agent $i \in N$. The allocation is MMS fair if $\alpha = 1$.

Given the definition of MMS, we have the following simple property, whose proof is in the appendix.

**Lemma 1** For any agent $i$ with subadditive valuation $v_i(\cdot)$, we have the following bounds for $\text{MMS}_i$,

$$\text{MMS}_i \geq \max \left\{ \max_{j \in M} v_i(\{j\}), \frac{1}{n} \cdot v_i(M) \right\}.$$  

### 3 Submodular valuations

By Lemma 1, if the valuations are subadditive, allocating all items to a single agent ensures an approximation of $n$. We call this algorithm All-or-Nothing, which is somewhat the most unfair algorithm. Surprisingly, such an unfair algorithm achieves the optimal approximation ratio of MMS even if the valuations are submodular.

**Theorem 1** For any $n \geq 2$, there is an instance with submodular valuations for which no allocation is better than $n$-MMS.

**Proof** For any fixed $n \geq 2$, we construct the following instance with $n$ agents and $m = n^n$ items via an $n$-dimensional coordinate system. Let each item correspond to a point in the system and

$$M = \{(x_1, x_2, \ldots, x_n) \mid x_i \in [n] \text{ for all } i \in [n]\}.$$  

For each agent $i \in [n]$, we define $n$ covering planes $\{C_{il}\}_{l \in [n]}$ and for each $l \in [n]$,

$$C_{il} = \{(x_1, x_2, \ldots, x_n) \mid x_i = l \text{ and } x_j \in [n] \text{ for all } j \in [n] \setminus \{i\}\}.$$  

Note that $\{C_{il}\}_{l \in [n]}$ forms an exact cover of the points in $M$, i.e., $\bigcup_l C_{il} = M$ and $C_{il} \cap C_{iz} = \emptyset$ for all $l \neq z$. For any set of items $S \subseteq M$, $v_i(S)$ equals the minimum number of planes in $\{C_{il}\}_{l \in [n]}$
that can cover $S$. Therefore, $v_i(S) \in [n]$ for all $S$. We first show $v_i(\cdot)$ is submodular for every $i$. For any $S \subseteq T \subseteq M$ and any $e \in M \setminus T$, if $e$ is not in the same covering plane for any point in $T$, $e$ is not in the same covering plane for any point in $S$. Thus, $v_i(T \cup \{e\}) - v_i(T) = 1$ implies $v_i(S \cup \{e\}) - v_i(S) = 1$, and accordingly,

$$v_i(T \cup \{e\}) - v_i(T) \leq v_i(S \cup \{e\}) - v_i(S).$$

Since $\{C_{il}\}_{l \in [n]}$ is an exact cover of $M$, $MMS_i = 1$ where the MMS defining partition is simply $\{C_{il}\}_{l \in [n]}$. Then to prove the theorem, it suffices to show that for any allocation of $M$, there is at least one agent whose value is $n$. For the sake of contradiction, we assume there is an allocation $X = (X_1, \ldots, X_n)$ where every agent has value at most $n - 1$, which means for every agent $i \in [n]$, there exists a plane $C_{il}$ with $l_i \in [n]$ such that $X_i \cap C_{il_i} = \emptyset$. Consider the point $b = (l_1, l_2, \ldots, l_n)$. It is clear that $b \in C_{il_i}$ and thus $b \notin X_i$ for all $i$. Hence $b$ is not allocated to any agent, which is a contradiction with $X$ being an allocation and thus there must be an agent whose value is $n$. ■

4 Bin packing model

Although Theorem 1 shows that there is no algorithm that can be better than $n$-MMS for all submodular valuations, it does not wipe out the possibility of beating $n$-approximation for specific subadditive valuations. In this section and the next, we propose two models where agents have concrete combinatorial valuations and design constant-approximate MMS fair algorithms.

The first model encodes a bin packing problem, where the items have sizes and need to be packed into bins by the agents. Specifically, each item $j \in M$ has size $s_{i,j} \geq 0$ to agent $i$, and note that $j$ may be of different sizes to different agents. For a set of items $S$, $s_i(S) = \sum_{j \in S} s_{i,j}$. Each agent $i \in N$ has unlimited number of bins with the same capacity $c_i$, where it is assumed that $c_i = \max_{j \in M} s_{i,j}$. Given a set of items $S \subseteq M$, the value of agent $i$, $v_i(S)$, is determined by the minimum number of bins (with capacity $c_i$) that can pack all items in $S$, which involves solving a classic bin packing problem and thus the computation of $v_i(S)$ is NP-hard. Accordingly, $MMS_i$ is essentially the minimum number $k_i$ such that the items can be partitioned into $n$ bundles and the items in each bundle can be packed into no more than $k_i$ bins. We have the following simple lemma, whose proof is in the appendix.

**Lemma 2** $MMS_i \cdot c_i \geq \frac{s_i(M)}{n}$ for any $i \in N$

We first show the lower bound for the bin-packing model, whose proof is in the appendix.

**Theorem 2** For the bin-packing model, no algorithm performs better than 2-MMS.

Next, we design an algorithm to compute an allocation that matches the approximation 2, and thus is the best possible approximation algorithm. Note that although the computation of each $MMS_i$ and $v_i(S)$ is NP-hard, our algorithm runs in polynomial time without explicitly computing these values.

We first show that if we can solve the problem for the special case where all agents have the same order of items by ranking their sizes, we can also solve the general case. Similar lemmas have been widely used for designing algorithms to compute MMS fair allocations with additive valuations [Barman and Krishnamurthy (2020); Huang and Lu (2021)]. Although this lemma does not hold for arbitrary subadditive valuations, it holds for the bin-packing model.

We call a bin-packing instance identical ordering (IDO) if $s_{i,1} \geq s_{i,2} \geq \cdots \geq s_{i,n}$ holds for any $i \in N$. Note that although the agents have the same order of items by their sizes, the
cardinal numbers can still be significantly different. Given a general bin-packing instance $I = (N, M, \{v_i\}_{i \in N}, \{s_i\}_{i \in N})$, we can construct an IDO instance as follows. First, note that for every agent $i$, there exists a permutation $\sigma_i : M \rightarrow M$ such that for all $j, j' \in M$ with $j < j'$, $s_i,\sigma(i) \geq s_i,\sigma(j')$. Using these permutations, we can construct the IDO instance by setting $s_{i,j} = s_i,\sigma(j)$ for every $i \in N$ and $j \in M$. In other words, for every agent $i$, the size of the $j$-th item in the IDO instance is equal to the $j$-th largest size of items in the original instance. Note that the construction of the IDO instance runs in $O(nm \log m)$ time. Then we have the following lemma whose proof is in the appendix.

**Lemma 3** If there exists an allocation $\mathcal{A}' = (A'_1, ..., A'_n)$ in the IDO instance $I'$, such that $v_i(A'_i) \leq \alpha \cdot \text{MMS}_i$ for all $i \in N$, then there exists an allocation $\mathcal{A} = (A_1, A_2, ..., A_n)$ in the original instance $I$ such that $v_i(A_i) \leq \alpha \cdot \text{MMS}_i$ for all $i \in N$. Besides, $\mathcal{A}$ can be constructed in polynomial time.

Next, we present the main algorithm (Algorithm 1) that computes 2-MMS allocations for IDO bin-packing instances. The basic idea is similar to that of the bag filling algorithm, i.e., incrementally adding small items into a bag until the bag is large enough for all the agents, and allocating the bag to one of the agents who are the last to think the bag large enough. The major differences include the way we distinguish between small and large items, and the way we initialize and fill the bag. In the following, we prove the approximation of 2, and in the appendix, we show how to modify Algorithm 1 so that the approximation ratio improves to $(3 + \delta)/2$ when MMS values get large.

**Theorem 3** Algorithm 1 returns a 2-MMS allocation for bin-packing instances in polynomial time.

Before the proof of Theorem 3, we first introduce some notations and technical lemmas. In the bin-packing model, we say an item $j$ is large for an agent $i$ if the size of the item exceeds half the capacity of the agent’s bin, i.e., $s_{i,j} > 1/2 \cdot c_i$; Otherwise, we say item $j$ is small for agent $i$. Note that each bin can hold at most one large item, but may hold multiple small items. We denote $H$ as the items that are large for some agent, and $L_i$ as the items that are small for agent $i$. Formally,

$$
H = \{j \in M \mid \exists i \in N : s_{i,j} > \frac{c_i}{2}\}, \quad L_i = \{j \in M \mid s_{i,j} \leq \frac{c_i}{2}\}.
$$

Note that $H \cap L_i$ may not be empty since one item may be large for some agents but small for other agents. We also split all the items into $\lceil \frac{m}{n} \rceil$ sets, where the first set contains the $n$ largest items, the second set contains the next $n$ largest items, and so on. That is,

$$
G_1 = \{(i-1)n+1, ..., i \cdot n\} \quad i \in [1, \lceil \frac{m}{n} \rceil], \quad G_{\lceil \frac{m}{n} \rceil} = \{\lfloor \frac{m}{n} \rfloor \cdot n + 1, ..., m\}.
$$

Note that the sets $G$s are well-defined since the instances we consider here are identical ordering.

The algorithm consists of many rounds of bag initialization (Steps 5 to 9) and bag filling (Steps 11 through 15). In the bag initialization procedure, for each $G_i$ that contains large items for some agent, the largest item in it (which must be large for some agent) is put into the bag. In this way, for every agent, this initialized bag contains enough but not too much of her large items. Besides, for some agents, all the items in the bag are large, i.e., those who think the least item is large. In the bag filling procedure, whenever an agent who has small items thinks the bag not large enough, the agent will incrementally add her small items into the bag, until the bag is large enough for her or she uses up her small items. Each round of the algorithm ends when no agent will add items into the bag, i.e., each agent either thinks the bag large enough or has no small items. If this happens immediately after the bag is initialized, the bag is given to one of the agents who think all the items in the bag are large. Otherwise, the bag is given to the agent who adds the last item into the bag.
Lemma 4 The agent who is allocated the bag immediately after the bag is initialized can use no more than MMS bins to pack all the items in the bag.

Proof First, recall that in this case, all the items in the bag are large for the agent \(i\) who receives the bag. From the definition of maximin share, we know that agent \(i\) can use \(n \cdot \text{MMS}_i\) bins to pack all the items in \(M\). This implies that there are at most \(n \cdot \text{MMS}_i\) large items for agent \(i\). Therefore, according to the structure of the sets \(G_s\), we can know that at most \(\text{MMS}_i\) large items of agent \(i\) are put into the bag, which require no more than \(\text{MMS}_i\) bins to pack, thus completing the proof. ■

Algorithm 1: 2-MMS allocation for IDO bin-packing instance.

1 Input: An IDO instance over \(n\) agents and \(m\) indivisible chores with size \(s_{i,j}\) for each agent \(i \in N\) and chore \(j \in M\). The valuation \(v_i\) for each agent \(i\) is determined by the bin-packing problem, which is non-negative, monotone, and subadditive.

2 Output: An allocation \(A = (A_1, A_2, \cdots, A_n)\) such that \(v_i(A_i) \leq 2 \cdot \text{MMS}_i\) for every \(i \in N\).

3 Initialize \(R = M\).

4 while \(|N| > 1\) do

5 Initialize \(Q = N\), \(B = \emptyset\), \(k = \max\{i \in [1, \lceil \frac{m}{n} \rceil] \mid G_i \cap R \cap H \neq \emptyset\}\).

6 for \(i = 1\) to \(k\) do

7 Pick the largest item \(g\) in \(G_i \cap R\).

8 \(B \leftarrow B \cup \{g\}\), \(R \leftarrow R \setminus \{g\}\).

9 end

10 \(Q = \{i \in N \mid B = \emptyset \text{ or } \forall j \in B : s_{i,j} \geq \frac{2}{n}\}\).

11 while \(s_i(B) \leq \frac{s_i(M)}{n}\) for some agent \(i \in N\) and \(L_i \cap R \neq \emptyset\) do

12 Pick an item \(g\) in \(L_i \cap R\).

13 \(B \leftarrow B \cup \{g\}\), \(R \leftarrow R \setminus \{g\}\).

14 end

15 Pick an agent \(i\) in \(Q\).

16 \(A_i \leftarrow B\), \(N \leftarrow N \setminus \{i\}\).

17 end

18 Allocate the remaining items in \(R\) to the last agent.

Lemma 5 The agent who is allocated a bag during the bag filling procedure can use no more than 2MMS bins to pack all the items in the bag.

Proof Before proving the lemma, we first show that if the total size of items for an agent \(i\) is at most \(\frac{s_i(M)}{n}\), the agent can use no more than 2MMS bins to pack all the items. This is because in the worst case where only half capacity of each bin is taken up, the 2MMS bins can still hold items with total size of \(\text{MMS}_i \cdot c_i\), which is at least \(\frac{s_i(M)}{n}\) according to Lemma 2. This also implies at least one bin of the 2MMS bins is taken up no more than half of its capacity.

Now, we’re ready to prove Lemma 5. There are two cases when the agent \(i\) is allocated the bag \(B\). First, agent \(i\) thinks the bag large enough. Denote \(g\) as the last item agent \(i\) adds into the bag, we have \(s_i(B \setminus \{g\}) \leq \frac{s_i(M)}{n}\) and \(s_i(\{g\}) \leq \frac{2}{n}\). As we’ve shown, the items in \(B \setminus \{g\}\) as well as the item \(g\) can be packed into 2MMS bins. Second, agent \(i\) thinks the bag not large enough (i.e., \(s_i(B) \leq \frac{s_i(M)}{n}\)
while she has no small items. Clearly, she can also use 2MMS\textsubscript{i} bins to pack all the items, which completes the proof. ■

Now, we are ready to prove Theorem 3

**Proof of Theorem 3** Clearly, Algorithm [4] runs in polynomial time. For all the agents except the last one, Lemma [4] and Lemma [5] have proved that they can use no more than 2MMS bins to pack all the items allocated to them. To complete the proof, we now show that the last agent \( i \) can use no more than 2MMS\textsubscript{i} bins to pack all the remaining items.

Considering two cases, first, agent \( i \) thinks all the bags allocated to other agents are large enough, i.e., \( s_i(A_j) > \frac{s_i(M)}{n} \) for all \( j \in N \setminus i \). Denote the remaining items by \( M' \), then we have

\[
  s_i(M') = s_i(M) - \sum_{j \in N \setminus i} s_i(A_j) < \frac{s_i(M)}{n} - (n - 1) \frac{s_i(M)}{n} = \frac{s_i(M)}{n}.
\]

As proved in Lemma [5], agent \( i \) can use no more than 2MMS\textsubscript{i} bins to pack all the items in \( M' \). Second, at least one bag allocated to other agents is not large enough for agent \( i \), i.e., \( s_i(A_j) \leq \frac{s_i(M)}{n} \) for some \( j \in N \setminus i \). This implies that before \( A_j \) is allocated, there is already no small item for agent \( i \). Hence, all the remaining items are large for agent \( i \). As shown in the proof of Lemma [4], there are at most \( n \cdot 2\text{MMS} \) large items for agent \( i \). According to the structure of the sets \( G_s \), we know that at most \( \text{MMS} \) of these large items are left to the last round. Therefore, in this case, agent \( i \) can use no more than 2MMS\textsubscript{i} bins to pack all the remaining items. ■

## 5 Job scheduling model

Our second model considers a job scheduling environment, where the items are jobs that need to be processed by the agents. Similar to the bin packing model, each item \( j \in M \) has size \( s_{i,j} \geq 0 \) to agent \( i \in N \). For a set of items \( S \), \( s_i(S) = \sum_{j \in S} s_{i,j} \). And each agent \( i \in N \) exclusively controls a set of \( k_i \) machines \( P_i = [k_i] \) with possibly different speed \( \rho_{i,j} \) for \( j \in P_i \). Without loss of generality, we assume \( \rho_{i,1} \geq \rho_{i,2} \geq \cdots \geq \rho_{i,k_i} \). Upon receiving a set of items \( S \subseteq M \), agent \( i \)’s value \( v_i(S) \) is determined by the minimum competition time of processing \( S \) using her own machines \( P_i \) (i.e., makespan). Formally,

\[
  v_i(S) = \min_{(T_{1},\ldots,T_{k_i})} \max_{i \in [k_i]} \sum_{j \in T_i} s_{i,j} / \rho_{i,j}.
\]

Note that the computation of \( v_i(S) \) is NP-hard. Intuitively, the value of MMS\textsubscript{i} is obtained by partitioning the items into \( n \cdot k_i \) bundles, and allocating them to \( k_i \) different types of machines (with different speeds) where each type has \( n \) identical machines so that the makespan is minimized.\(^1\) Note that when each agent controls a single machine, i.e., \( k_i = 1 \) for all \( i \), the problem degenerates to the additive valuation case, and thus by Feige et al. (2021) no algorithm can be better than \( 44/43 \)-approximation.

\(^1\)An alternative way to explain the scheduling model is to view each agent \( i \) as a group of \( k_i \) small agents and the value of MMS\textsubscript{i} as the collective maximin share for these \( k_i \) small agents in the group. We believe this notion of collective maximin share is of independent interest as a groupwise fairness notion. We remark that this notion is different with the groupwise (and pairwise) maximin share defined in Barman et al. (2018) and Caragiannis et al. (2019), where the max-min value is defined for each single agent. In our definition, however, a set of agents share the same value for the items assigned to them.
Algorithm 2: 2-MMS allocation for IDO job-scheduling instance

1 Input: An IDO instance over $n$ agents and $m$ indivisible jobs with size $s_{i,j}$ for each agent $i \in N$ and job $j \in M$. The valuation $v_i$ for each agent $i$ is determined by the job scheduling problem, which is non-negative, monotone, and subadditive.

2 Output: An allocation $A = (A_1, A_2, ..., A_n)$ such that $v_i(A_i) \leq 2 \cdot \text{MMS}_i$ for every $i \in N$.

3 Initialize allocation $A = (A_1, ..., A_n)$ with $A_i = \emptyset$ for every $i \in N$.

4 while $M \neq \emptyset$ do

5 for $i = 1$ to $m$ do

6 Pick $g \in \arg \max_{g' \in M} s_{i,g'}$.

7 $A_i \leftarrow A_i \cup \{g\}$, $M \leftarrow M \setminus \{g\}$.

8 end

9 end

Next we focus on the upper bound and prove a result similar with Lemma 3 in the bin packing model. That is, it suffices for us to design algorithms for the IDO instances where $s_{i,1} \geq s_{i,2} \geq \cdots \geq s_{i,m}$ holds for any agent $i$, and the algorithms can be easily converted to handle the general case by the reduction similar to Algorithm 4 (presented in the appendix). Since the proof is almost the same as that of Lemma 3, we omit the details to avoid redundancy.

Now, we can focus on computing approximate MMS allocations for IDO instances of the job scheduling model. We allocate jobs among agents in a round-robin fashion (Algorithm 2): the agents take turns picking the largest job among the remaining jobs until no job is left. The algorithm is quite direct but surprisingly returns us a 2-MMS allocation. The technical contribution here is to show the returned allocation is indeed a 2-MMS allocation. That is, for each agent $i$, the makespan of processing the jobs in $A_i$ using her machines doesn’t exceed $2 \cdot \text{MMS}_i$.

Theorem 4 Algorithm 2 computes a 2-MMS allocation for every IDO job-scheduling instance in polynomial time.

To prove Theorem 4 we present an algorithm (Algorithm 3) for every agent to assign the jobs that Algorithm 2 allocates to her among her machines. Here, we consider the capacity of a machine instead of its speed. Specifically, for each machine with speed $\rho$, we set the capacity of the machine $c = \tau \cdot \rho$. The algorithm assigns as many large jobs as possible to the machines with larger capacities, while ensuring that the workload (i.e., total size of the jobs) on each machine is no more than twice its capacity. In this way, the completion time of each machine doesn’t exceed $2 \tau$. Besides, we have the following lemma.

Lemma 6 For every agent $i$, denote $A_i$ as the jobs that Algorithm 2 allocates to her. If $\tau \geq \text{MMS}_i$, Algorithm 3 assigns all the jobs in $A_i$ among her machines $P_i$.

Proof of Lemma 6 We will prove the lemma for an agent $i$. an analogous proof can be established for any other agent. We assume, for sake of contradiction that $A_i$ is not empty at the end of Algorithm 3. We will show that this assumption contradicts the definition of agent $i$’s maximin share.

For each machine $j$ in $P_i$, the capacity is set to $c_{i,j} = \tau \cdot \rho_{i,j}$, which satisfies $c_{i,j} \geq \text{MMS}_i \cdot \rho_{i,j}$. Besides, $c_{i,1} \geq c_{i,2} \geq \cdots \geq c_{i,k_i}$. We denote $T = (T_1, T_2, ..., T_{k_i})$ as the allocation returned by Algorithm 3 and $t_{k_i+1}^*$ as the largest job that is not allocated. We use $t_j^*$ to denote the largest job in $T_j$ for $j \in [1, k_i]$, then we can define $T' = (T_1', T_2', ..., T_{k_i}')$ where $T_j' = T_j \setminus t_j^* \cup t_j^*_{j+1}$ for $j \in [1, k_i]$. In
Algorithm 3: Job scheduling with threshold

1. **Input**: A set of jobs $A = \{1, 2, ..., t\}$ with sizes $s_1 \geq s_2 \geq \cdots \geq s_t$, and a set of machines $P = \{1, 2, ..., k\}$ with speeds $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_k$, and a threshold $\tau$.

2. **Output**: An allocation $T = (T_1, T_2, ..., T_k)$.

3. Initialize $T_i = \emptyset$, $c_i = \tau \cdot \rho_i$ for every $i \in P$, $g = 1$.

4. for $i = 1$ to $k$ do
   
   while $s(T_i) + s_g \leq 2c_i$ and $A \neq \emptyset$ do
     
     $T_i \leftarrow T_i \cup \{g\}$, $A \leftarrow A \setminus \{g\}$.
     
   $g \leftarrow g + 1$.

end

agent $i$’s MMS-defining partition, every job $g \in M$ must be assigned to one machine $j$. This implies that $s_{i,g} \leq \text{MMS}_i \cdot \rho_{i,j} \leq c_{i,1}$ for every $g \in M$. From Algorithm 3, we know that $s_i(T_1 \cup t_2^*) > 2c_{i,1}$, thus we have $s_i(T_1) = s_i(T_1 \setminus t_2^*) > c_{i,1}$.

We assume as our induction hypothesis that $s_{i,t_j^*} \leq c_{i,j}$ for every $j \in [1, l]$ where $l$ is less than $k_i$. Similar to the claim that $s_i(T_j^*') > c_{i,1}$, we have $s_i(T_j^*) > c_{i,j}$ for any $j \in [1, l]$. Summing up these $l$ inequalities, we have $\sum_{j=1}^{l} s_i(T_j^*) > \sum_{j=1}^{l} c_{i,j}$. Since Algorithm 2 allocates jobs to agents in a round-robin fashion, we know that for any job $g$ in $\cup_{j=1}^{l} T_j^*$, there are $n$ jobs right before $g$ in $M$ (i.e., jobs $g - n, g - n + 1, ..., g - 1$) that are not smaller than $g$. Therefore, denote $S$ as the set of jobs in $M$ that are before job $t_{l+1}^*$, we have

$$s_i(S) \geq \sum_{g \in \cup_{j=1}^{l} T_j^*} n \cdot s_{i,g} = n \cdot \sum_{j=1}^{l} s_i(T_j^*) > n \cdot \sum_{j=1}^{l} c_{i,j} \geq n \cdot \sum_{j=1}^{l} \text{MMS}_i \cdot \rho_{i,j}.$$  

By the pigeonhole principle, in agent $i$’s MMS-defining partition, at least one bundle contains a subset of $S$ with total size greater than $\sum_{j=1}^{l} \text{MMS}_i \cdot \rho_{i,j}$. Therefore, at least one job $g$ in this subset is assigned to a machine $j$ with $j \geq l + 1$. Otherwise, the completion time of one of the first $l$ machines must exceed MMS$_i$, which contradicts the definition of agent $i$’s maximin share. Hence, we have $s_{i,g} \leq \text{MMS}_i \cdot \rho_{i,j} \leq c_{i,l+1}$. Since any job in $S$ is not smaller than job $t_{l+1}^*$, we have $s_{i,t_{l+1}^*} \leq s_{i,g} \leq c_{i,l+1}$.

Therefore, we have $s_{i,t_{l+1}^*} \leq c_{i,j}$ for every $j \in [1, k_i]$ and $s_i(M) > n \cdot \sum_{j=1}^{k_i} \text{MMS}_i \cdot \rho_{i,j}$. By the pigeonhole principle, in agent $i$’s MMS-defining partition, the size of at least one bundle exceeds $\sum_{j=1}^{k_i} \text{MMS}_i \cdot \rho_{i,j}$. This contradicts with the definition of maximin share, since the completion time of processing the jobs in such a bundle must exceed MMS$_i$.

**Proof of Theorem 4**  Clearly, Algorithm 2 runs in polynomial time. For each agent $i$, we set the threshold $\tau = \text{MMS}_i$. Lemma 6 shows that all the jobs Algorithm 2 allocates to agent $i$ can be assigned to her machines. Besides, we know from Algorithm 3 that the completion time of each of agent $i$’s machines doesn’t exceed 2MMS$_i$. Therefore, $v_i(A_i) \leq 2\text{MMS}_i$ holds for every agent $i \in N$, which completes the proof.

**Remark**  Note that although the proof of Theorem 4 requires the value of each MMS$_i$, which is NP-hard to compute, Algorithm 2 runs in polynomial time, which means there exists a way for each
agent $i$ to complete the jobs allocated to her in $2 \cdot \text{MMS}_i$ time. Actually, we notice that it may not be necessary to obtain the exact value of maximin share for each agent. Instead, we can exploit a reasonable upper bound of each $\text{MMS}_i$. Similar ideas have been widely applied to designing efficient algorithms for fair allocation of indivisible goods [Barman and Krishnamurthy (2020); Garg and Taki (2021)]. Specifically, for each agent $i$, we start by setting the threshold to be less than $\text{MMS}_i$ (e.g., $\tau_{i,0} = \max_{g \in M} s_i(g)$). Then we (geometrically) increase the threshold by setting $\tau_i = (1 + \delta) \cdot \tau$ and rerun Algorithm 3 until no job remains unallocated at the end of Algorithm 3. Such an algorithm runs in polynomial time since Algorithm 3 is run for at most $\log_{(1+\delta)}(\frac{\tau_{i,0}}{\text{MMS}_i})$ times. Besides, it guarantees that each agent can complete the jobs allocated to her in $(2 + \delta) \cdot \text{MMS}_i$ time.

6 Proportionality up to one or any item

In this section, we discuss another relaxation for proportionality, “proportional up to one item” and “proportional up to any item”, which are also widely studied for additive valuations.

Definition 2 ($\alpha$-PROP1 and $\alpha$-PROPX) An allocation $X = (X_1, \ldots, X_n)$ is $\alpha$-approximate proportional up to one item ($\alpha$-PROP1) if $v_i(X_i \setminus \{g\}) \leq \alpha \cdot \frac{v_i(M)}{n}$ for all agents $i \in N$ and some item $g \in X_i$. It is $\alpha$-approximate proportional up to any item ($\alpha$-PROPX) if $v_i(X_i \setminus \{g\}) \leq \alpha \cdot \frac{v_i(M)}{n}$ for all agents $i \in N$ and any item $g \in X_i$. The allocation is PROP1 or PROPX if $\alpha = 1$.

It is easy to see that a PROPX allocation is also PROP1. Although exact PROPX and PROP1 allocation is guaranteed to exist for chores with additive valuation, when the valuation is subadditive, no algorithm can be better than $n$-PROP1 or $n$-PROPX. Consider an instance with $n$ agents and $n+1$ items. The valuation function is $v_i(S) = 1$ for all agents $i \in N$ and any non-empty subset $S \subseteq M$. Clearly, the valuation function is subadditive since $v_i(S) + v_i(T) \geq v_i(S \cup T)$ for any $S, T \subseteq M$. By the pigeonhole principle, at least one agent $i$ receives two or more items in any allocation of $M$. After removing any item $g \in X_i$, $X_i$ is still not empty. That is, $v_i(X_i \setminus \{g\}) = 1 = n \cdot \frac{v_i(M)}{n}$ for any $g \in X_i$. This example can be easily extended to the bin-packing and job-scheduling valuations, thus we have the following theorem, whose proof can be seen in the appendix.

Theorem 5 For the bin-packing model and the job-scheduling model, no algorithm performs better than $n$-PROP1 or $n$-PROPX.

7 Conclusion

In this work, we study the fair allocation problem of allocating indivisible chores when the valuations are subadditive and fairness is measured by MMS. We first show that no algorithm can ensure better than $n$-MMS even when the valuations are submodular. Then we considered two specific combinatorial valuations, namely, bin packing and job scheduling. For bin packing valuations, we show that one of the best possible approximation algorithms is bag filling which achieves 2-MMS. For job scheduling valuations, we show a 2-MMS allocation can be found efficiently, and inherit the lower bound of $44/43$ from additive valuations. Our work uncovers several future directions. One immediate direction is to design better approximation algorithms or lower bound instances for the job scheduling valuations. In this work, we restricted us on the case of related machines, it is intriguing to consider the general model of unrelated machines. As we mentioned, the notion of collective maximin
share fairness in the job scheduling model can be viewed as a group-wise fairness notion (for both goods and chores), which could be studied of independent interest. Finally, we can investigate other combinatorial valuations that can better characterize real-world problems.

References

G. Amanatidis, E. Markakis, A. Nikzad, and A. Saberi. Approximation algorithms for computing maximin share allocations. *ACM Trans. Algorithms*, 13(4):52:1–52:28, 2017.

G. Amanatidis, E. Markakis, and A. Ntokos. Multiple birds with one stone: Beating 1/2 for EFX and GMMS via envy cycle elimination. *Theor. Comput. Sci.*, 841:94–109, 2020.

G. Amanatidis, G. Birmpas, A. Filos-Ratsikas, and A. A. Voudouris. Fair division of indivisible goods: A survey. *CoRR*, abs/2202.07551, 2022.

H. Aziz, G. Rauchecker, G. Schryen, and T. Walsh. Algorithms for max-min share fair allocation of indivisible chores. In *AAAI*, pages 335–341. AAAI Press, 2017.

H. Aziz, H. Moulin, and F. Sandomirskiy. A polynomial-time algorithm for computing a pareto optimal and almost proportional allocation. *Oper. Res. Lett.*, 48(5):573–578, 2020.

H. Aziz, B. Li, H. Moulin, and X. Wu. Algorithmic fair allocation of indivisible items: A survey and new questions. *CoRR*, abs/2202.08713, 2022.

S. Barman and S. K. Krishnamurthy. On the proximity of markets with integral equilibria. In *AAAI*, pages 1748–1755, 2019.

S. Barman and S. K. Krishnamurthy. Approximation algorithms for maximin fair division. *ACM Trans. Economics and Comput.*, 8(1):5:1–5:28, 2020.

S. Barman, A. Biswas, S. K. K. Murthy, and Y. Narahari. Groupwise maximin fair allocation of indivisible goods. In *AAAI*, pages 917–924. AAAI Press, 2018.

U. Bhaskar, A. R. Sricharan, and R. Vaish. On approximate envy-freeness for indivisible chores and mixed resources. In *APPROX-RANDOM*, volume 207 of *LIPIcs*, pages 1:1–1:23. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021.

S. Brânzei and F. Sandomirskiy. Algorithms for competitive division of chores. *CoRR*, abs/1907.01766, 2019.

E. Budish. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *Journal of Political Economy*, 119(6):1061–1103, 2011.

I. Caragiannis, D. Kurokawa, H. Moulin, A. D. Procaccia, N. Shah, and J. Wang. The unreasonable fairness of maximum nash welfare. *ACM Trans. Economics and Comput.*, 7(3):12:1–12:32, 2019.

H. Chan, J. Chen, B. Li, and X. Wu. Maximin-aware allocations of indivisible goods. In *IJCAI*, pages 137–143. ijcai.org, 2019.

U. Feige, A. Sapir, and L. Tauber. A tight negative example for MMS fair allocations. In *WINE*, volume 13112 of *Lecture Notes in Computer Science*, pages 355–372. Springer, 2021.
J. Garg and S. Taki. An improved approximation algorithm for maximin shares. *Artif. Intell.*, 300:103547, 2021.

M. Ghodsi, M. T. Hajiaghayi, M. Seddighin, S. Seddighin, and H. Yami. Fair allocation of indivisible goods: Improvement. *Math. Oper. Res.*, 46(3):1038–1053, 2021.

M. Ghodsi, M. T. Hajiaghayi, M. Seddighin, S. Seddighin, and H. Yami. Fair allocation of indivisible goods: Beyond additive valuations. *Artif. Intell.*, 303:103633, 2022.

X. Huang and P. Lu. An algorithmic framework for approximating maximin share allocation of chores. In *EC*, pages 630–631. ACM, 2021.

D. Kurokawa, A. D. Procaccia, and J. Wang. Fair enough: Guaranteeing approximate maximin shares. *J. ACM*, 65(2):8:1–8:27, 2018.

B. Li, Y. Li, and X. Wu. Almost proportional allocations for indivisible chores. *TheWebConf*, 2022.

R. J. Lipton, E. Markakis, E. Mossel, and A. Saberi. On approximately fair allocations of indivisible goods. In *EC*, pages 125–131. ACM, 2004.

H. Moulin. Fair division in the age of internet. *Annu. Rev. Econ.*, 2018.

B. Plaut and T. Roughgarden. Almost envy-freeness with general valuations. *SIAM J. Discret. Math.*, 34(2):1039–1068, 2020.

M. Seddighin and S. Seddighin. Improved maximin guarantees for subadditive and fractionally subadditive fair allocation problem. 2022.

S. Zhou and X. Wu. Approximately EFX allocations for indivisible chores. *CoRR*, abs/2109.07313, 2021.
A  More Related Works

We refer the readers to Amanatidis et al. (2022); Aziz et al. (2022) for detailed surveys on fair division of indivisible items. Beyond proportionality, in another parallel line of research, envy-freeness and its relaxations, namely envy-free up to one item (EF1) and envy-free up one any item (EFX), are very widely studied. It is shown in Lipton et al. (2004) and Bhaskar et al. (2021) for goods and chores, respectively, an EF1 allocation exists for the monotone combinatorial valuations. However, the existence of EFX allocations is still unknown even with additive valuations. Therefore, approximation algorithms are proposed in Amanatidis et al. (2020); Zhou and Wu (2021) for additive valuations and in Plaut and Roughgarden (2020); Chan et al. (2019) for subadditive valuations.

B  Bin-packing model: a \((3 + \delta)/2\) approximation

In this section, we make a slight improvement on Algorithm 1 and prove that the new algorithm returns us \((3 + \delta)/2\)MMS allocations when MMS values get large. At Step [2] in Algorithm 1 instead of arbitrarily picking an item in \(L_i \cap R\), we pick the smallest item in it. Then we have the following theorem.

**Theorem 6** The modified Algorithm 1 computes a \((3 + \delta)/2\)-MMS allocation for all IDO bin-packing instances in polynomial time when MMS values get large.

Before proving Theorem 6, we introduce some notations and observations. If the total size of a set of items doesn’t exceed \(c_i\), we say these items are acceptable for the agent \(i\). If not, we still say they are passable for agent \(i\), as long as there exist small items for agent \(i\) among them and removing one small item makes the remaining items acceptable for the agent. Consider MMS, passable sets of items, first, the total size of these items is greater than \(\text{MMS}_i \cdot c_i\), which is at least \(s_i(M)n\) according to Lemma 2. Besides, agent \(i\) can use no more than \(\lceil \frac{3}{2} \cdot \text{MMS}_i \rceil\) bins to pack them. To achieve this, agent \(i\) first converts each passable set to an acceptable one by taking out the special small item. This gives her MMS\(_i\) acceptable sets which can be packed using MMS\(_i\) bins. Then, agent \(i\) can use at most \(\lceil \frac{3}{2} \cdot \text{MMS}_i \rceil\) to pack the MMS\(_i\) small items that are taken out.

Now, we are ready to prove Theorem 6.

**Proof of Theorem 6** To prove Theorem 6 we show that every agent \(i\) can use no more than \(\lceil \frac{3}{2} \cdot \text{MMS}_i \rceil\) bins to pack all the items in \(A_i\). For the agents who receive the initialized bags, we’ve shown in the proof of Lemma 4, that they can use MMS bins to pack the items. For each agent \(i\) who receives a bag in the bag filling procedure, we first show that there are at most MMS\(_i\) large items for her in the bag. According to the structure of Gs, we know that the initialized bag contains at most MMS\(_i\) large items for agent \(i\). Furthermore, after modifying the Step [2] in Algorithm 1 the items added by the agents before agent \(i\) cannot be large for agent \(i\). This is because when this happens, there remains no small item for agent \(i\), thus she won’t fill the bag and the bag won’t be allocated to her. Next, we show how agent \(i\) arranges the items in the bag into MMS\(_i\) passable sets, which can be packed using no more than \(\lceil \frac{3}{2} \cdot \text{MMS}_i \rceil\) bins. The agent first puts the large items into different sets, which creates at most MMS\(_i\) acceptable sets. Then, she makes these sets passable one by one by filling small items into them. If there are still some small items left, the agent will create a new set and make it passable by filling it with small items. This process is repeated until no small item is left. Recall that the size of the bag doesn’t exceed \(s_i(M)n\) before the last item is added. This implies that arranging all the items except the last one in the above way creates less than MMS\(_i\) passable sets. Otherwise, the total size of
these sets exceeds $s_i(M)/n$. Therefore, after filling the last item, agent $i$ can still get no more than $\text{MMS}_i$ passable sets.

For the last agent $i$ who receives the remaining items, there are two cases. In the first case, the total size of the remaining items is at most $s_i(M)/n$. Besides, from the proof of Theorem 3 we know that at most $\text{MMS}_i$ among them are large for the agent. As we’ve shown previously, these items can be arrange into less than $\text{MMS}_i$ passable sets and be packed in no more than $\left\lceil \frac{3}{2} \cdot \text{MMS}_i \right\rceil$ bins. In the second case, all the remaining items are large for agent $i$. Since there are at most $\text{MMS}_i$ such large items, the agent can use $\text{MMS}_i$ bins to pack them.

C  Missing examples and proofs

In this section, we provide the examples and proofs omitted in the main body of the paper.

An example of general valuation function  Note that there exist simpler examples, but we choose the following one because it represents a particular combinatorial structure – minimum spanning tree. Let $G = (V, E)$ be a graph shown in the left sub-figure of Figure 1, where the vertices $V$ are the items that are to be allocated, i.e., $M = V$. There are two agents $N = \{1, 2\}$ who possess different weights on the edges as shown in the middle and right sub-figures of Figure 1. The valuation functions are measured by the minimum spanning tree in their received subgraphs. Particularly, for any $S \subseteq V$, $v_i(S)$ equals the weight of the minimum spanning tree on $G[S]$ – the induced subgraph of $S$ in $G$ – under agent $i$’s weights. Thus, $\text{MMS}_i = 0$, for both $i = 1, 2$, where an MMS-defining partition for agent 1 is $\{v_1, v_2\}$ and $\{v_3, v_4\}$ and that for agent 2 is $\{v_1, v_4\}$ and $\{v_2, v_3\}$. However, it can be verified that no matter how the vertices are allocated to the agents, there is one agent whose value is at least 1, which implies that no bounded approximation is possible for general valuations.

![Figure 1: An instance with unbounded approximation ratio](image)

Proof of Lemma 1  The first claim is straightforward, since in any MMS-defining partition of agent $i$, the item with largest value must appear in one of the bundles, and by the monotonicity of the valuations, the value of this bundle is at least $\max_{j \in M} v_i(\{j\})$.

To see the second claim, let $X = (X_1, \ldots, X_n)$ be an MMS-defining partition for agent $i$, then $v_i(X_j) \leq \text{MMS}_i$ for all $j$. Summing up all these inequalities, we have

$$v_i(M) \leq \sum_{j \in N} v_i(X_j) \leq n \cdot \text{MMS}_i,$$

where the first inequality is by the subadditivity of the valuations.
Proof of Theorem 2 We first recall the impossibility instance given by Feige et al. (2021). In this instance there are three agents and nine items as arranged in a three by three matrix. The three agents’ valuations are shown in the matrices $V_1$, $V_2$ and $V_3$.

$$V_1 = \begin{pmatrix} 6 & 15 & 22 \\ 26 & 10 & 7 \\ 12 & 19 & 12 \end{pmatrix} \quad V_2 = \begin{pmatrix} 6 & 15 & 23 \\ 26 & 10 & 8 \\ 11 & 18 & 12 \end{pmatrix} \quad V_3 = \begin{pmatrix} 6 & 16 & 22 \\ 27 & 10 & 7 \\ 11 & 18 & 12 \end{pmatrix}$$

Feige et al. (2021) proved that for this instance the MMS value of every agent is 43, however, in any allocation, at least one of the three agents gets value no smaller than 44.

We can adapt this instance to the bin-packing model and obtain a lower bound of 2. In particular, we also have three agents and nine items. The numbers in matrices $V_1$, $V_2$ and $V_3$ are the sizes of the items to each agent 1, 2 and 3. Let the capacities of the bins of each agent be $c_i = 43$ for all $i$. Accordingly, we have MMS$_i = 1$. Since in any allocation, there is at least one agent who gets items with total size no smaller than 44, for this agent, she has to use 2 bins to pack the assigned items, which finishes the proof.

Proof of Lemma 2 From the definition of maximin share, we know that each agent $i$ can use $n \cdot \text{MMS}_i$ bins to pack all the items. This implies that the total capacity of $n \cdot \text{MMS}_i$ bins is no smaller than the total size of all the items. That is, $n \cdot \text{MMS}_i \cdot c_i \geq s_i(M)$, which completes the proof.

Proof of Lemma 3 We show that given $I$ and $\mathcal{A}'$, Algorithm 4 finds $\mathcal{A}$ in polynomial time. Clearly, Algorithm 4 runs in polynomial time. Now, we will show that Algorithm 4 finds the required allocation $\mathcal{A}$. Considering the $g$-th iteration of the second for-loop (Steps 5 to 8), suppose that item $k_g$ is allocated to agent $i$, we have $k \in A_i'$ and $k_g \in A_i$. Before $k_g$ is allocated, exactly $g - 1$ items have been allocated, since an item is removed from the set $R$ after it is allocated. Therefore, $k_g$ is among the top $g$ smallest items for agent $i$. Note that $g$ is the item with the exactly $g$-th smallest size for agent $i$, hence we have $s_i(k_g) \geq s_i(k_g)$ for any $g \in A_i'$ and $k_g \in A_i$. This implies that it requires no more bins for agent $i$ to pack items in $A_i$ than those in $A_i'$, i.e., $v_i(A_i) \leq v_i'(A_i')$. Note that the maximin share depends on the sizes of the items but not on the order, which means that the maximin share of agent $i$ in $I'$ is the same as her maximin share in $I$. Hence, the condition that $v_i'(A_i') \leq \alpha \cdot \text{MMS}_i$ gives us $v_i(A_i) \leq \alpha \cdot \text{MMS}_i$ for all $i \in N$.

**Algorithm 4:** $\alpha$-MMS Allocation for General Bin-Packing Instance

1. **Input:** Instance $I = (N, M, \{v_i\}_{i \in N}, \{s_i\}_{i \in N})$ with an allocation $A' = (A'_1, ..., A'_n)$ for the IDO instance $I' = (N, M, \{v'_i\}_{i \in N}, \{s'_i\}_{i \in M})$ such that $v'_i(A'_i) \geq \alpha \cdot \text{MMS}_i$ for all $i \in N$.
2. **Output:** An allocation $\mathcal{A} = (A_1, A_2, ..., A_n)$ such that $v_i(A_i) \geq \alpha \cdot \text{MMS}_i$ for all $i \in N$.
3. For all $i \in N$ and $g \in A'_i$ set $p_g := i$.
4. Initialize allocation $\mathcal{A} = (A_1, A_2, ..., A_n)$ with $A_i = \emptyset$ for all $i \in N$, and initialize $R \leftarrow M$.
5. for $g = m$ to 1 do
   6. Pick $k \in \arg \min_{g' \in R} \{s_{p_g, g'}\}$.
   7. Update $A_{p_g} \leftarrow A_{p_g} \cup \{k\}$ and $R \leftarrow R \setminus \{k\}$.
8. end

Proof of Theorem 5 For the bin-packing model, let’s consider an instance with $n$ agents and $n + 1$ items. The capacity of each agent’s bins is 1, i.e., $c_i = 1$ for any $i \in N$. Each item is very tiny so that every agent can pack all the items in just one bin, e.g., $s_{i,j} = \frac{1}{n+1}$ for any $i \in N$ and $j \in M$. Therefore, we have $v_i(M) = 1$ and $\text{PROP}_i = \frac{1}{n}$ for each agent $i \in N$. By the pigeonhole principle, at
least one agent $i$ receives two or more items in any allocation of $M$. After removing any item $g \in X_i$, agent $i$ stills needs one bin to pack the remaining items. Hence, we have $v_i(X_i \setminus \{g\}) = 1 = n \cdot \text{PROP}_i$ for any $g \in X_i$, which completes the proof.

For the job-scheduling model, we can consider an instance with $2n$ agents and $2n + 1$ jobs where each agent possesses $2n$ machines with the same speed of 1, and the size of each job is 1 for every agent. It can be easily seen that for every agent $i$, the maximum completion time of her machines is minimized when assigning two jobs to one machine and one job to each of the remaining $2n - 1$ machines. Therefore, we have $v_i(M) = 2$ and $\text{PROP}_i = \frac{2}{2n} = \frac{1}{n}$ for any $i \in N$. Similarly, by the pigeonhole principle, at least one agent $i$ receives two or more jobs in any allocation of $M$. This implies that $v_i(X_i \setminus \{g\}) = 1 = n \cdot \text{PROP}_i$, thus completing the proof.