Extended systems of Baxter Q-functions and fused flags I: simply-laced case

Simon Ekhammar\textsuperscript{a} Hongfei Shu\textsuperscript{b} Dmytro Volin\textsuperscript{a, b}

\textsuperscript{a}Department of Physics and Astronomy, Uppsala University, Box 516, SE-751 20 Uppsala, Sweden
\textsuperscript{b}Nordita, KTH Royal Institute of Technology and Stockholm University, Roslagstullsbacken 23, SE-106 91 Stockholm, Sweden
E-mail: simon.ekhammar@physics.uu.se, hongfei.shu@su.se, dmytro.volin@physics.uu.se

Abstract: The spectrum of integrable models is often encoded in terms of commuting functions of a spectral parameter that satisfy functional relations. We propose to describe this commutative algebra in a covariant way by means of the extended Q-system that comprise Q-vectors in each of the fundamental representations of the (Langlands dual of) the underlying symmetry algebra. These Q-vectors turn out to parameterise a collection of complete flags which are fused with one another in a particular way. We show that the fused flag is gauge equivalent to a finite-difference oper, explicit equivalence depends on (an arbitrary choice of) a Coxeter element.

The paper considers the case of simple Lie algebras with a simply-laced Dynkin diagram. For the $A_r$ series, the construction coincides with already known results in the literature. We apply the proposed formalism to the case of the $D_r$ series and the exceptional algebras $E_r$, $r = 6, 7, 8$. In particular, we solve Hirota bilinear equations in terms of Q-functions and give the explicit character solution of the extended Q-system in the $D_r$ case. We also show how to build up the extended Q-system of $D_r$ type starting either from vectors, by a procedure similar to the $A_r$ scenario which however constructs a fused flag of isotropic spaces, or from pure spinors, via fused Fierz relations.

Finally, for the case of rational, trigonometric, and elliptic spin chains, we propose an explicit ansatz for the analytic structure of Q-functions of the extended Q-system. We conjecture that the extended Q-system constrained in such a way is always in bijection with actual Bethe algebra of commuting transfer matrices of these models and moreover can be used to show that the Bethe algebra has a simple joint spectrum.
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1 Introduction

1.1 Concept of a Q-system

One of the most famous equations in theory of quantum integrable systems is Baxter’s TQ relation [Bax72]

\[ t q = \phi^- q^{++} + \phi^+ q^{--}. \]  

(1.1)

It appears in numerous studies. In some, like spin chain models, \( t \) and \( q \) have operatorial meaning of certain transfer matrices and in some, like TBA and ODE/IM correspondence, \( t \) and \( q \) are convenient functions of a spectral parameter. Eventually it is the analytic properties of \( t, q, \phi \) that decide which physical system the TQ relation describes, demanding concrete analytic properties often goes under the name of analytic Bethe Ansatz [Res83].

A good example to have in mind is XXZ-type models based on \( U_q(\hat{g}) \) (with \( g = \mathfrak{sl}_2 \) for the above TQ-relation). Then \( t, q, \phi \) are Laurent polynomials in multiplicative spectral parameter \( z \) while the spectral parameter shift is defined \( f^\pm(z) := f(q^{\pm 1/2}z) \). This example builds upon the affine Lie algebra \( \hat{g} \) whose Langlands dual will play an important role in our discussion. Another example is XXX-type models based on the Yangian \( \mathcal{Y}(g) \), where \( t, q, \phi \) are polynomials in additive spectral parameter \( u \), and then \( f^\pm := f(u \pm \hbar/2) \), with \( \hbar \in \mathbb{C} \). We will also use \( f[n] \) which means applying the shift \( n \) times. Let us emphasize that the combinatorial aspects of functional equations – the primary focus of the paper – do not depend on how shift of the spectral parameter is explicitly realised. In fact, one can go as far as to consider only a discrete set of points that are related to one another by translations \( f[n], n \in \mathbb{Z} \). For clarity of exposition though, and for a comfortable treatment of potential poles/zeros arising, it is good to assume meromorphic dependence of functions on the spectral parameter, at least in a large enough domain that allows applying the shift operation as many times as will be needed.

It will be convenient to absorb the source term \( \phi \) into the definition of the Baxter Q-function and T-function

\[ Q = \sigma q, \quad T = \sigma^{++} \sigma^{--} t, \]  

(1.2)

where \( \sigma^+ \sigma^- = 1/\phi \), and so the Baxter relation becomes

\[ T Q = Q^{++} + Q^{--}. \]  

(1.3)

Rescaling (1.2) is often called gauge transformation however we warn that gauge transformations we are going to speak about is a different operation.

The function \( Q \) is in many aspects a more fundamental object than \( T \). One can readily see it from (1.3): If \( Q_1, Q_2 \) are two independent solutions of (1.3) then \( T \) can be written as the determinant

\[ T = \begin{vmatrix} Q_1^{++} & Q_2^{++} \\ Q_1^{--} & Q_2^{--} \end{vmatrix}. \]  

(1.4)
provided that we normalised solutions to satisfy

\[ W(Q_1, Q_2) = 1 \quad \text{with} \quad W(Q_1, Q_2) := \begin{vmatrix} Q_1^+ & Q_2^+ \\ Q_1^- & Q_2^- \end{vmatrix}. \] (1.5)

This example readily shows that there is not one but two Q-functions \( Q_1, Q_2 \) obeying certain relations. In higher-rank cases there will be many more Q-functions. A collection of these functions, together with relations they obey and symmetry transformations they enjoy shall be called a Q-system\(^1\).

### 1.2 State of the art for \( \mathfrak{sl}_{r+1} \) Q-systems

Generalisation of the above discussion from \( \mathfrak{sl}_2 \) to \( \mathfrak{sl}_{r+1} \) is well understood by now, and it can be done in several conceptually different and yet deeply interrelated ways.

**Quantum characters** One possibility is to perceive (1.3) as a “quantum character” relation (q-character of [FR98]. Think about the Schur polynomial \( \chi_\Box = x + y \) for the defining representation of \( \mathfrak{sl}_2 \) and (1.3) written as \( T = \frac{Q^{++}}{Q^{--}} + \frac{Q^{--}}{Q^{++}} \) as its quantum generalisation. For higher ranks, the character of finite-dimensional representation of \( \mathfrak{sl}_{r+1} \) labelled by an integer partition \( \lambda \) is given by

\[ \chi_\lambda = \sum_T \prod_{(a,s) \in \lambda} x^{T_{a,s}}, \] (1.6)

where the sum runs over all semi-standard Young tableaux \( T \) of shape \( \lambda \) and the product runs over all boxes of \( \lambda \) parameterised using Cartesian coordinates \( (a, s) \). The quantum version of this character is the transfer matrix in representation \( \lambda \) which can be computed as [KS95]

\[ T_\lambda = \sum_T \prod_{(a,s) \in \lambda} \Lambda_{T_{a,s}}^{[2(a-s)]}, \] (1.7)

with

\[ \Lambda_{a}^{[a-1]} = \frac{Q_{-a}^{[a]} Q_{-(a-1)}^{[-1]}}{Q_{-a}^{[-1]} Q_{-(a-1)}^{[1]}}, \quad a = 1, \ldots, r + 1, \] (1.8)

where, in our normalisation choice (1.2), \( Q_{-0} = Q_{-(r+1)} = 1 \). By imposing that \( T_\lambda \) are regular functions (probably up to a well-controlled prefactor like \( \sigma^{++} \sigma^{--} \)), one should require that poles coming from denominators of (1.8) are cancelled out which results in the conventional nested Bethe ansatz equations

\[ Q_{-a}^{+(a-1)} Q_{-(a+1)}^{-2} = -1 \quad \text{at zeros of} \quad Q_{-a}. \] (1.9)

\(^1\)This is not to be confused with Q-systems that are a character limit of T-systems e.g. in [KNS11].
Here we come to the most simplistic way of introducing Baxter Q-functions by analytic Bethe Ansatz: these are functions of type $Q = \sigma q$, where $q$ is a polynomial at whose zeros the nested Bethe ansatz equations (1.9) should be satisfied, and $\sigma$ provides source terms in these equations. At generic point (of a parameter space describing some model), solutions to the nested Bethe equations are expected to correctly describe the spectrum of the model but at special points which are of relevance for applications one needs to be more careful, a more detailed discussion is present in Section 4.3.

Whereas existence of Baxter Q-functions and of the generating sequence (1.7) can be guessed from the Bethe equations, their to-date character interpretation if far beyond pure guesswork, Q-functions are constructed [BLZ99, BFL+11, KLT12, DKM03, FH15] as explicit characters in special infinite-dimensional representations of quantum algebra (of $U_q(\mathfrak{b})$ actually, or its Yangian counterpart). They have aforementioned analytic properties thus justifying the analytic Bethe Ansatz.

**Quantum spectral curve** Another way for introducing Q-functions is to generalise (1.3) as a finite-difference linear equation. For $\mathfrak{sl}_{r+1}$, this equation becomes of degree $r + 1$ [KLWZ97]

$$
\sum_{a=0}^{r+1} (-1)^a T_{a,1} D^{-2a} Q^{[2]} = 0 ,
$$

(1.10)

where $D f := f^+$. For spin chains, $T_{a,1}$ have interpretation of transfer matrices in the $a$‘th fundamental representation of $\mathfrak{sl}_{r+1}$ and in this case the above equation can be equivalently written as [Tal04, CF08]

$$
\det(1 - MD^{-2})Q^{[2]} = 0 ,
$$

(1.11)

where $M$ is the monodromy matrix of the model (which is the universal R-matrix in a particular representation) and $\det$ is a column-ordered determinant. The higher-rank Baxter relation written as (1.11) clearly suggests to interpret it as a quantisation of the classical spectral curve $\det(\lambda - M) = 0$ with $M$ being classical Lax matrix, in pretty much the same way as the Schrödinger equation is a quantisation of $p^2/2 + V - E = 0$. This idea was one of the key ingredients for the Sklyanin’s separation of variables program which he realised for $\mathfrak{sl}_2$ [Skl85, Skl91] and partially for $\mathfrak{sl}_3$ [Skl96] cases. For rational XXX-type $\mathfrak{sl}_{r+1}$ spin chains in arbitrary finite-dimensional representation, an SoV basis which features $Q$ as the wave function was built in [RV19, RV20].

Equation (1.11) has $r + 1$ independent solutions $Q_a$ and we normalise them to satisfy

$$
W(Q_1, \ldots, Q_{r+1}) = 1 ,
$$

(1.12)

\footnote{Based on the recipe of [MN18], an equivalent SoV basis, with $Q$ playing the role of wave function, can be easily constructed for XXZ case as well, cf. [MN19]. However, the proof of [RV19, RV20] that this SoV basis also diagonalises the higher-rank version of Sklyanin’s B-operator [Smir01, GLMS17] cannot be that easily generalised and hence interpretation of separated variables as quantisation of classical dynamical divisor is not yet justified beyond the rational case. This topic requires further investigation.}
where \[ W(Q_1, \ldots, Q_{r+1}) := \det_{1 \leq a, b \leq r+1} Q_a^{r+1-2b} . \]

They can be also used to formulate [KLWZ97] a quantum analog of Weyl-Jacobi determinant character formula \[ \chi_\lambda = \det_{1 \leq a, b \leq r+1} x_a^{\lambda_b+r+1-b} ; \]

\[ T_\lambda = W(Q_1^{[2\lambda_1]}, \ldots, Q_{r+1}^{[2\lambda_{r+1}]}). \] (1.13)

Clearly, (1.12) is specialisation of (1.13) to the trivial representation.

Being solutions of a linear equation, functions \( Q_a \) are defined ambiguously, up to linear \( sl_{r+1} \) transformations\(^3\). As we are dealing with finite-difference equations, linear transformations with periodic \( (f^+ = f) \) functions are also allowed which promotes this symmetry to the loop algebra \( \hat{sl}_{r+1} \). However, one should not confuse it with \( \hat{sl}_{r+1} \) used to construct the quantum algebra \( U_q(\hat{sl}_{r+1}) \). To start with, algebra of symmetries of the Baxter equation exists independently of which quantum algebra was used to construct the integrable model. Also, quantum algebra has generators that commute with Hamiltonians, in particular with Baxter operators \( Q_k \), whereas \( (Q_1, \ldots, Q_{r+1}) \) transform as a vector representation under action of \( \hat{sl}_{r+1} \) symmetry of the Baxter equation. Even in the XXZ case, we are nominally dealing with two different algebras, one is \( q \)-deformed and the other one is not. The parameter \( q \) appears instead as a period of elements of the non-deformed \( \hat{sl}_{r+1} \), \textit{i.e.} \( \hat{sl}_{r+1} \simeq sl_{r+1} \otimes \mathbb{C}[t, t^{-1}] \) with \( t = \exp(2\pi i q) \). Although the two mentioned \( \hat{sl}_{r+1} \) algebras are conceptually different, they are nevertheless related: they are Langlands dual of one another. To get a better feeling about this statement we remark that the underlying zero-level algebras share the same Weyl group (the permutation group \( S_{r+1} \)). Identification goes beyond formal isomorphism: On the level of quantum algebra the action of the Weyl group changes the representation in which the Baxter operator is computed (explicitly this can be seen in the Yangian construction of [BFL+11]) which can be literally mapped to taking a different solution of the Baxter equation.

**Weyl transform** Let us take another look on (1.5). It relates two \( Q \)-functions. One of them, say \( Q_1 \), solves conventional Bethe equations (1.9) and so we identify \( Q_1 \equiv Q_{r+1} \). On the other hand, by applying exactly the same logic as in derivation of (1.9) we see that \( Q_2 \) also solves Bethe equations of the same form. The preference of one set of equations over another may exist (if \textit{e.g.} \( Q_1 \) is a polynomial of lower degree than \( Q_2 \)) but definitely it is not meaningful for as long as we are mostly ignoring explicit analytic structure of \( Q \)-functions. To summarise, starting from \( Q_1 \) which satisfies (1.9), we use (1.5) to compute \( Q_2 \) which satisfies an equivalent of (1.9).

This generalises to the higher-rank case as follows: Starting from the functions \( Q_{-(a-1)}, Q_{-a}, Q_{-(a+1)} \), one introduces a new \( Q \)-function \( \bar{Q}_{-a} \) as the one satisfying

\[ W(Q_{-a}, \bar{Q}_{-a}) = Q_{-(a-1)} Q_{-(a+1)}. \] (1.14)

Equation (1.9) can be seen as a consequence of (1.14) using the following standard argument: shift (1.14) in two different directions, \( \bar{Q}_{-a}^\pm \bar{Q}_{-a} - \bar{Q}_{-a}^{[\pm 2]} Q_{-a} = \pm \bar{Q}_{-a}^{[\pm 2]} \bar{Q}_{-(a-1)} \bar{Q}_{-(a+1)} \),

\(^3\)We want to keep normalisation (1.12) intact.
evaluate at zeros of $Q\leftarrow a$ thus cancelling one term assuming regularity of $Q$-functions, and
divide the shifted equation in one direction by the shifted equation in the other direction.
$Q\leftarrow a$ and $Q\leftarrow a$ enter symmetrically (up to a sign) the Plücker-type relation (1.14) and
hence we can derive using the same procedure the Bethe equations (1.9) of exactly the
same form but with $Q\leftarrow a$ replaced everywhere \footnote{A small computation confirms the statement also for (1.9) at zeros of $Q\leftarrow (\pm 1)$, where $Q\leftarrow a$ enter as $Q\leftarrow a$.} with $\bar{Q}\leftarrow a$.

Transformation from $Q\leftarrow a$ to $\bar{Q}\leftarrow a$ appeared numerous times in the literature and has
several different names: beyond equator [PS99], reproduction procedure [MV02], bosonic
(as opposed to fermionic) duality of Bethe equations [GV08], Bäcklund-type transformation
[FKSZ20]. We shall refer to it under yet another name “Weyl transform” of $Q$-functions in
attempt to settle the name that reflects group-theoretical meaning of what is happening.
Indeed, for $\mathfrak{sl}_2$ case we can readily notice that the transform maps between two solution
of the Baxter equation. Weyl symmetry interpretation for higher rank cases shall become
clear later in the text \footnote{Since $\bar{Q}\leftarrow k + \alpha Q\leftarrow k$ is a solution of (1.14) for any $\alpha$, the transform gets true meaning of the Weyl group
action only after this symmetry is taken under control.}.

**Miura transform** Introduce now a suggestive notation
\begin{align*}
Q\leftarrow a & := Q_{1,2,\ldots,a-1,a}, \\
\bar{Q}\leftarrow a & := Q_{1,2,\ldots,a-1,a+1}
\end{align*}
(1.15)
which alludes to the orthonormal basis $\varepsilon_a$ used for parameterisation of the $\mathfrak{sl}_{r+1}$ root
lattice (such that simple roots are $\alpha_a = \varepsilon_a - \varepsilon_{a+1}$) and to the transition from $Q\leftarrow a$ to $\bar{Q}\leftarrow a$
corresponding to an action of the Weyl reflection that permutes $\varepsilon_a$ and $\varepsilon_{a+1}$.

By performing in total $(r+1)$ Weyl transforms in a special way, one can generate $(r+1)$
new $Q$-functions that contain, among others, $Q_1,\ldots,Q_{r+1}$ solving the Baxter equation
(1.10). Thus we say that $Q_1,\ldots,Q_{r+1}$ can be derived from $Q\leftarrow 1,\ldots,Q\leftarrow r$ and $Q\leftarrow (r+1) = 1$.
The reverse procedure is neatly organised in the following determinant formulae
\begin{equation}
Q\leftarrow a = W(Q_1,\ldots,Q_a), \quad a = 1,\ldots,r+1.
\end{equation}
(1.16)
To better understand the meaning of (1.16), it is instructive to rewrite the Baxter equation
(1.10) in a factorised form
\begin{equation}
(1 - \Lambda_{r+1}D^{-2}) \cdots (1 - \Lambda_2D^{-2})(1 - \Lambda_1D^{-2})Q_1 = 0
\end{equation}
(1.17)
which is known under the name of Miura transform [FR96]. The conditions specifying
factorisation are the following ones

\[
0 = (1 - \Lambda_1 D^{-2}) Q_1 , \quad (1.18a)
\]

\[
0 = (1 - \Lambda_2 D^{-2})(1 - \Lambda_1 D^{-2}) Q_2 , \quad (1.18b)
\]

\[
\ldots
\]

\[
0 = (1 - \Lambda_a D^{-2}) \ldots (1 - \Lambda_1 D^{-2}) Q_k . \quad (1.18c)
\]

The factorisation procedure hence reduces the symmetry algebra \( \hat{\mathfrak{sl}}_{r+1} \) to \( \hat{\mathfrak{b}} \), where \( \mathfrak{b} \) is the Borel subalgebra of \( \mathfrak{sl}_{r+1} \).

The reader is welcome to verify that \( \Lambda_a \) are precisely the ones given by (1.8) and then

\[
(1 - \Lambda_a D^{-2}) \frac{Q_{\ell-a}}{Q_{\ell-(a-1)}} = 0 . \quad (1.19)
\]

Alternatively, if \( Q_1, \ldots, Q_{r-a} \) are solutions of the Baxter equation which is an equation of degree \( r + 1 \), the functions \( \tilde{Q}_b = W(Q_{a_1}, \ldots, Q_{a_k}, Q_b) \), for \( b = r - a + 1, \ldots, r + 1 \) are solutions of the degree \( a + 1 \) equation

\[
(1 - \Lambda_{r+1} D^{-2}) \ldots (1 - \Lambda_{r-a+1} D^{-2}) \tilde{Q}_b = 0 . \quad (1.20)
\]

This can be viewed as a Bäcklund flow from \( \mathfrak{sl}_{r+1} \) to \( \mathfrak{sl}_{a+1} \) Q-systems [KLWZ97], but, in simplest possible terms, it is just the method of variation of constants.

**Extended Q-system on the Weyl orbit** Relation (1.16) suggests an immediate generalisation. For any multi-index \( A = a_1 \ldots a_k \) of no more than \( r + 1 \) distinct entries, one can define a Q-function \( Q_A \)

\[
Q_A = W(Q_{a_1}, \ldots, Q_{a_k}) . \quad (1.21)
\]

A collection of \( 2^{r+1} \) such Q-functions (with \( Q_\emptyset = Q_{\emptyset} = 1 \)) shall be called the extended Q-system or the Q-system on the Weyl orbit (these two names will become distinct for other Lie algebras) because \( Q_A \) with \( |A| = a \) constitute the orbit of \( Q_{r-a} \) under action of Weyl transforms in the sense of (1.15). They generalise (1.14) to a well-known [PS00] QQ-relation

\[
W[Q_{Aa}, Q_{Ab}] = Q_A Q_{Aab} . \quad (1.22)
\]

The Q-functions \( Q_A \) with \( |A| = a \) can be viewed as components of an \( a \)-form thus transforming under \( a \)'th fundamental representation of \( \hat{\mathfrak{sl}}_{r+1} \). We see that if we assemble all Q-functions whose zeros satisfy all possible variations, under Weyl transforms, of the Bethe equations, such a collection gets a covariant description as representations of the symmetry group of the Baxter equation. Although these Q-functions are definitely not functionally independent, the gained covariance has its own benefits. To illustrate some of them, let us
introduce contra-variant Hodge-dual Q-functions [GKLV15]

\[ Q^A := \frac{1}{|A|!} \epsilon^{A\bar{A}} Q_{\bar{A}}. \]  

(1.23)

Simultaneous usage of both \( Q_a \) and \( Q^a \) was recently used for computation of scalar products [GLMRV20]. One of the reasons for which this computation was possible is that \( Q^a \) satisfy the “conjugated” Baxter equation

\[ Q \sum_{a=0}^{r+1} (-1)^a T_{a,1} \overleftarrow{D}^{-2a} = 0, \]  

(1.24)

where \( \overleftarrow{D} = f^- \).

Furthermore, one can form singlets from Q-functions and their Hodge duals which, by inspection, provides us with a compact bilinear formula for transfer matrices \( T_\lambda \) with \( \lambda = (s^a) \) being a Young diagram of rectangular shape (i.e. a Kirillov-Reshetikhin module):

\[ T_{a,s} = \frac{1}{a!} Q^{(s^a)}_{A} (Q^A)_{\{-s^a\}}. \]  

(1.25)

Supersymmetric version of the extended Q-system that appeared in the works of Tsuboi [Tsu10] which, with further elaboration, was instrumental in solution of the AdS5/CFT4 spectral problem. First, generalisation of (1.25) allowed to solve [GKLT11] the T-system on T-hook [GKV09] and then to exploit this solution to formulate a finite set of nonlinear integral equations [GKLV12]. Then the integral equations were simplified further into the AdS/CFT quantum spectral curve [GKLV14, GKLV15] – a \( \mathfrak{psu}(2,2|4) \) Q-system supplemented with a certain Riemann-Hilbert problem fixing the analytic properties of the Q-functions (the latter can be viewed conceptually as an analytic Bethe Ansatz). Further analysis of this curve (using the extended Q-system and not the nested Bethe equations!) allowed getting explicit solutions, up to numbers, thus providing exact results for spectrum of \( N=4 \) SYM, see e.g. [GLMS16, MV18a, MV18b] and reviews [Gro17, LM20].

**Fused flags and opers**  The extended Q-system also has a natural geometric interpretation. Recall that \( Q_A \) with \( |A| = a \) are components of an exterior \( a \)-form in \( \mathbb{C}^N \) which we shall denote as \( Q_{(a)} \). Based on (1.21), this form is not arbitrary but such that it defines a hyperplane \( V_a \subset \mathbb{C}^{r+1}, V_a \simeq \mathbb{C}^a \). Note that the embedding \( V_a \subset \mathbb{C}^{r+1} \) parameterised by the Plücker coordinates \( Q_A \) naturally depends on the spectral parameter.

Furthermore (1.21) also informs us that \( V_a \) are embedded into one another in a special
The embeddings are non-degenerate meaning that \( V_{a-1}^{[n-1]} \) and \( V_{a+1}^{[n+1]} \) span \( V_a^{[n]} \). This follows from the normalisation condition (1.12). Note that the normalisation condition (1.12) is enforced by a suitable normalisation which potentially introduces poles into the Q-functions and non-degeneracy is allowed to fail at such poles.

A chain of embeddings, for instance

\[
0 \subset V_1 \subset V_{1}^{+} \subset V_{2}^{[2]} \subset \ldots \subset V_{r}^{[r]} \subset \mathbb{C}^{r+1},
\]

defines a maximal flag of \( \mathbb{C}^{r+1} \). Given this observation, we shall call a collection \( V_a \) obeying (1.26) a **fused flag**. “Fusion” alludes here to the fusion procedure [KR86, Che86] used to construct higher representations and which involves shifting spectral parameter by an integer.

The concept of a fused flag for \( \mathfrak{sl}_{r+1} \) system was described in [KLV16] though this name was not used there (it is a new name that we propose in this paper). A very similar geometric construction appeared later and independently from [KLV16] in [KSZ18], from study of this work we learned that the fused flag is gauge equivalent to a \( q \)-difference oper. We postpone a detailed discussion of this correspondence until Section 3.4. The \( q \)-difference oper of [KSZ18] in turn generalises the idea of a differential oper appearing for instance in [Fre03, Fre04] and which goes back to the work of Drinfeld and Sokolov [DS85].

### 1.3 State of the art for Q-systems based on other simple Lie algebras

For arbitrary simple Lie algebras, the Bethe equations are known [OW86, OWR87]. By taking the most simplistic point of view on Q-functions as \( Q = \Gamma q \), where zeros of polynomial \( q \) are Bethe roots, the Bethe equations can be written as

\[
\prod_b \frac{(Q_{(b),1})^{+C_{ab}}}{(Q_{(b),1})^{-C_{ab}}} = -1 \quad \text{at zeros of } q_1^{(a)},
\]

where the product runs over the nodes of the Dynkin diagram and \( C_{ab} \) is the symmetrised Cartan matrix. The functions \( Q_{(a),1} \) are analogs of \( Q_{-a} \), they shall be called Q-functions on the Dynking diagram. The choice of notation \( Q_{(a),1} \) is done for future convenience.
The role of $Q_{(a),1}$ in the character interpretation is very well understood by now (for untwisted quantum affine algebras at least). In particular, analogs of relations (1.7) were lifted to the Grothendieck ring of certain $U_q(b)$ representation category, where $b$ is the Borel sublagebra of $\hat{\mathfrak{g}}$ [FH15].

To construct an analog of the Baxter equation in the form (1.11), one could expect [CT06] that we should take a determinant of the same operator that is featured in (quantum) Knizhnik-Zamolodchikov equations [KZ84, FR92] and Baxter $Q$-functions are reconstructable from solutions of this equation. It seems that an explicit realisation of this idea was not done, at least we are not aware of one. At the same time the main interest is not the Baxter equation itself but rather its solutions. For what concerns analogs of $Q_a$ and, more generally, $Q$-systems on the Weyl orbit comprising analogs of $Q_A$, quite a few results on the subject indeed exist.

A rather systematic approach to building up ensembles of $Q$-functions came through the study of the ODE/IM correspondence. First results in the form directly relevant for us were given by Sun [Sun12] and then in the works of Mazoero, Raimondo, and Valeri [MRV16, MRV17]. There they introduced the concept of the $Q\bar{Q}$-system based on the following observation which we explain on the example of simply-laced Lie algebras. As will be reviewed in detail in Section 2.1, there exist spectral parameter dependent vectors $Q_{(a)}$ in the $a$'th fundamental representation of the Lie algebra $\mathfrak{g}$, and $a$ running through all nodes of the Dynkin diagram such that

\[(Q^+_{(a)} \wedge Q^-_{(a)})_{L(\omega_{\text{max}})} = \left( \bigotimes_{b,C} a_{bc} = -1 Q_{(b)} \right)_{L(\omega_{\text{max}})}, \tag{1.29}\]

where $(\ldots)_{L(\omega_{\text{max}})}$ means restriction to the irreducible representation with the weight $\omega_{\text{max}} = \sum_{b,C} a_{bc} \omega_b$, and $\omega_b$ are fundamental weights.

Choose $Q_{(a),1}$ – the highest-weight component of $Q_{(a)}$ and $Q_{(a),2}$ – the component of $Q_{(a)}$ corresponding to the first descendent (the descendent is unique for any fundamental representation and it has weight $\omega_{a} - \alpha_a$, where $\alpha_a$ is the $a$'th simple root). Then, by specialising (1.29) to the highest-weight component, one gets the QQ-relation

\[W(Q_{(a),1}, Q_{(a),2}) = \prod_{b,C} a_{bc} = -1 Q_{(b),1}. \tag{1.30}\]

So, clearly $Q_{(a),2}$ plays the role of $\bar{Q}_{-k}$ in (1.14), and the $Q\bar{Q}$-system can be defined as collection of $Q_{(a),i}$, for all $a$ and $i = 1, 2$ satisfying (1.30). From (1.30) one easily gets the Bethe equaitons (1.28) by the argument explained after (1.14).

Instead of restriction on the highest-weight component, one can consider also any component on the Weyl orbit of (1.29) producing equations

\[W(Q_{(a),\sigma(1)}, Q_{(a),\sigma(2)}) = \pm \prod_{b,C} a_{bc} = -1 Q_{(b),\sigma(1)}, \tag{1.31}\]

where $\sigma$ is an element of the Weyl group (a precise meaning of how it acts on indices shall
be clarified later). These equations are analogs of (1.22) for $\mathfrak{sl}_{r+1}$ case, and in the context of ODE/IM they were explicitly mentioned in [MR18]. A result equivalent to (1.31) was obtained well before its appearance in the context of ODE/IM: In [MV05] an equivalent of the $Q\bar{Q}$-system (1.30) was considered and all possible Weyl transforms in the sense as they were defined on page 5 were performed to arrive to the same collection of Q-functions.

Restriction to the simply-laced Lie algebra in the above discussion can be waved [MRV17]: in order to get Bethe equations for a Lie algebra $\mathfrak{g}$, one builds a QQ-system for an affine Lie algebra $\hat{L}\mathfrak{g}$, where $L$ is the Langlands dual. For simply-laced cases, $\hat{L}\mathfrak{g} = \hat{\mathfrak{g}}$. This is no longer the case for non-simply-laced cases, however the main complication is not in the absence of invariance but in the fact that the affine algebra $\hat{L}\mathfrak{g}$ is twisted. After results of [MRV17], this complication is not too conceptual but it requires an extra layer of notations to be introduced, we hence decided to focus on simply-laced cases only in this paper.

A few months ago, a paper by Frenkel, Koroteev, Sage, and Zeitlin appeared [FKSZ20] where the notion of the finite-difference oper for arbitrary simple Lie algebras was introduced. The authors of this paper linked the oper construction to the $Q\bar{Q}$-system of [MRV16, MRV17] and hence (in the simply-laced case) to the Bethe equations.

All of the above-mentioned works about Q-system were focused on analytic part of the story. Construction of Q-functions as explicit operators (not only the highest-weight ones, but on the whole Weyl orbit) was recently realised, though not in full generality, for D-type Yangians [Fra20].

1.4 The goal, results, and structure of the paper

Although many aspects of Q-systems for arbitrary Lie algebras were developed in the works [Sun12, MRV16, MRV17, FKSZ20], the authors of these works were quite focused on reproducing the Q-functions on the Dynkin diagram $Q_{(a)}$ and their descendents $Q_{(a),2}$, probably with the intention to get to the conventional nested Bethe equations. While a subset of Weyl transforms is performed in [FKSZ20] and the full Weyl orbit Q-system is present in [MV05, MR18], these observations were not used towards some further concrete advantage.

Furthermore, in contrast to the $A_r$ case, Q-functions on the Weyl orbit are not the only Q-functions that may appear. Indeed, fundamental irreps of $\hat{L}\mathfrak{g}$ for $\mathfrak{g} \neq A_r$ contain weight subspaces which are not in the Weyl orbit of the highest-weight vector. And, as already clear from (1.29), we need to include these subspaces into discussion to fully benefit from covariance of the Q-system under action of $\hat{L}\mathfrak{g}$. Departing from Q-functions on the Weyl orbit to all weight subspaces of the fundamental irreps, to our knowledge, was not attempted in the literature.

The main goal of our paper is to launch a more systematic study of the full extended Q-system which we define as collection of all components of vectors $Q^{(a)}$ that satisfy (1.29). The extended Q-system enjoys covariance with respect to $\hat{L}\mathfrak{g}$ action and we expect that,

---

We provide a slightly stronger statement than that of [MR18] for what concerns normalisations. We shall demonstrate that it is always possible to normalise bases of fundamental irreps that equality (1.31) holds up to a sign simultaneously for all $a$ and $\sigma$, and we provide a way to control the sign as well.
similarly to the advances of the \( \mathfrak{sl}_{r+1} \) case, such a covariant description will lead to numerous insights in studies of integrable systems and beyond. In this paper we assemble first few results in this direction:

First, in addition to (and as a consequence of) (1.29), the extended Q-system enjoys a variety of other relations which we call projection properties. Many of them can be interpreted as Plücker relations defining a fused flag – a new structure generalising (1.26) that we shall introduce. The fused flag can be identified with a gauge transformation of a finite-difference oper, this transformation depends on a choice of a Coxeter element while the fused flag itself does not.

Second, while intuition is strongly based on the corresponding linear problem under the ODE/IM correspondence, the obtained relations are pertinent to the Lie algebra alone. We show that all the obtained relations between Q-functions can be universally satisfied admitting \( r \) functions as a functional freedom.

Third, an explicit concise similar to (1.25) parameterisation of T-functions for Kirillov-Reshetikhin modules in terms of Q-functions of the extended Q-system is given. This in turn yields solution of the corresponding Hirota equations and hence of the Y-systems appearing in the context of Thermodynamic Bethe Ansatz studies. For \( D_n \) series, we also provide an explicit character solution of the Q-system which, by substitution to the ansatz for T-functions produces characters of the corresponding \( D_n \)-representations.

For the technical reasons explained above, we restrict only to the case of simply-laced Lie algebras, non-simply-laced cases are planned for the sequel to this paper [ESV].

The paper is organised as follows: In Section 2 we review, with some updates, findings of [Sun12, MRV16] and use them to study \( \mathfrak{sl}(4) \cong \mathfrak{so}(6) \) extended Q-system as the simplest concrete example. In Section 3 we give general definition of the extended Q-system, show its universality, introduce notion of the fused flag, show that the extended Q-system is a fused flag, and, finally, link the fused flag to the notion of oper. In Section 4 we solve Hirota equations and comment on character solution of the Q-system and analytic Bethe Ansatz. Finally, in Sections 5 and 6 we give explicit realisations of the mentioned general ideas in the cases of \( D_n \) and exceptional series, respectively.

2 Motivation from ODE/IM

Throughout the paper \( \mathfrak{g} \) denotes a simply-laced simple Lie algebra over \( \mathbb{C} \) of rank \( r \); \( \mathfrak{h}, \mathfrak{b}, \mathfrak{n} \) are, respectively, its Cartan, maximal solvable, and maximal nilpotent subalgebras such that \([\mathfrak{h}, \mathfrak{b}] = \mathfrak{n}\). The corresponding simply-connected Lie groups of \( \mathfrak{g}, \mathfrak{b}, \mathfrak{n} \) are \( \mathbb{G}, \mathbb{B}, \mathbb{N} \). The Lie group associated to \( \mathfrak{h} \) is the maximal torus \( T \). \( \alpha \in \Phi \) are roots of the algebra, \( \alpha_a, \ a = 1, \ldots, r \) are simple roots, the set of simple roots shall be denoted \( \Delta \). \( W \) is the Weyl group of the root system. The degree of the Coxeter element is the Coxeter number \( h \). We shall use a Chevalley basis, with \( E_{\alpha} \) associated to roots, \( h_\alpha = \alpha_\alpha^\vee \), and \([h_a, E_{\pm \alpha_a}] = C_{ab} E_{\pm \alpha_b}\). The fundamental weights \( \omega_a \) are introduced by \( \omega_a(h_b) = \delta_{ab} \). The nilpotent subalgebra \( \mathfrak{n} \) shall be considered as spanned by the raising operators \( E_\alpha, \alpha > 0 \).

\(^7\)Capitalised letters for Cartan generators \( H_\alpha \) are reserved for an orthogonal basis to be introduced later.
As we are dealing with simply-laced case, we shall not distinguish between the Cartan matrix $C_{ab}$ and the symmetrised Cartan matrix $A_{ab} = (\alpha_a, \alpha_b)$. Also, as the Langlands dual is isomorphic to the algebra itself, we will write $\hat{g}$ instead of $L\hat{g}$, although the Q-system is actually a representation of $L\hat{g}$. Likewise, we shall not distinguish between Coxeter and dual Coxeter numbers.

2.1 Main features of the linear problem

**Linear problem** Our main intuition is coming from the results of [Sun12, MRV17] obtained in the context of the ODE/IM correspondence. In this subsection we summarise certain of their findings.

Consider the following linear problem:

$$L_g(x, z, \lambda) \Psi = \left(\frac{d}{dx} + A_g\right) \Psi = 0,$$

where $A$ is the $g$-valued matrix defined by

$$A_g = \sum_{i=1}^{r} E_{\alpha_i} + (x^hM - z) E_{\alpha_0}, \quad E_{\alpha_0} = \lambda E_{-\theta},$$

with $\theta$ being the longest root and $M > 0$. Equation (2.1) is understood as parallel transport equation in certain representations of $g$ with $\Psi$ being a vector transforming under action of $g$.

We note that $E_{\alpha_a}$ for $a = 0, 1, \ldots, r$ are generators of the untwisted affine Kac-Moody algebra $\hat{g}$. As we focus on representations which are finite-dimensional, the central charge of $\hat{g}$ is zero and hence $\hat{g}$ is isomorphic to the loop algebra $g \otimes \mathbb{C}[t, t^{-1}]$ and $E_{\alpha_0} = t E_{-\theta}$; all representations are of evaluation type where $t$ assumes a numerical value denoted here as $\lambda$. Our convention is to keep a fixed sign of $E_{-\theta}$ for all representations, so any sign properties will be reflected in relative values of $\lambda$.

The linear problem describes the equations of KdV-type [DS85] and first time appeared in the context of ODE/IM correspondence in [Sun12]. In the case of $g = \mathfrak{sl}(2)$ and in the fundamental representation, (2.1) is a linearisation of the Schrödinger equation for a particle in the homogeneous potential $x^hM$ with $z$ playing the role of energy. It is from study of this equation [Vor83] that the idea of the ODE/IM correspondence emerged.

**Symanzik rotation** Let $\rho^\vee \in \mathfrak{h}$ be the co-Weyl vector. Its defining feature is $[\rho^\vee, E_{\alpha_i}] = E_{\alpha_i}$ for $i = 1, \ldots, r$, and then it follows that $[\rho^\vee, E_{\alpha_0}] = (1 - h) E_{\alpha_0}$. Using these properties, one can verify that the linear problem enjoys a “Renorm-Group” equation [Sib75, Suz00, Sun12]

$$q^{-\frac{k}{\alpha_1} \rho^\vee} L_g(q^{\frac{k}{\alpha_1}} x, q^{k} z, \lambda) q^{\frac{k}{\alpha_1} \rho^\vee} = q^{-\frac{k}{\alpha_1} \rho^\vee} L_g(x, z, e^{2\pi i k \lambda}),$$

(2.3)

The linear problem also describes the conformal limit of modified affine Toda equations [LZ10, IL14], where affine means the connection also includes $E_{\alpha_0}$.

9In [DDT07], this is called the “Symanzik rescaling”. 

– 13 –
where \( q = e^{2\pi i \left( \frac{M}{M+1} \right)} \) and \( k \in \mathbb{C} \). Then “the RG flow” of solution is given by

\[
\Psi^{[2k]}(x, z) = q^{-\frac{k}{M}} \rho^\vee \Psi(q^{\frac{k}{M}} x, q^k z), \tag{2.4}
\]

where by \( \Psi^{[2k]} \) we denote a solution of (2.1) with the rescaled coupling constant: \( \lambda \to e^{2\pi i k} \lambda \).

For \( k \in \mathbb{Z} \), transformation (2.3) is a symmetry of the equation and the Symanzik rotation (2.4) is a way to generate its new solutions. For consistency with other parts of the paper, we have chosen a convention that \( \Psi^{[2]} \) corresponds to the minimal non-trivial Symanzik rotation which is a symmetry of the equation. In the following “Symanzik rotation” will typically refer to \( \Psi^{[\pm 2]} \) and “half of the Symanzik rotation” – to \( \Psi^{\pm} \equiv \Psi^{[\pm 1]} \).

In the following and without loss of generality we set \( \lambda = 1 \).

**WKB analysis** To analyse large-\( x \) behaviour of the solutions of (2.1), one should be a bit careful as the term \( x hM \) which is naively dominant at large \( x \) is multiplied by a nilpotent operator. To rectify this issue, one performs a gauge transformaiton

\[
\mathcal{L} \to \tilde{\mathcal{L}} = p^\rho^\vee \mathcal{L} p^{-\rho^\vee} \tag{2.5}
\]

with \( p = (x^M - z)^{\frac{1}{M}} \). Then, using the action variable \( S = \int^x p(x')dx' \), the gauge-transformed linear operator reads

\[
\tilde{\mathcal{L}} = p \left[ \frac{d}{dS} + \Lambda + \ldots \right], \tag{2.6}
\]

where dots stand for the terms suppressed at large \( x \) \(^{11}\), and

\[
\Lambda = \sum_{i=1}^r E_{\alpha_i} + E_{\alpha_0}. \tag{2.7}
\]

In these terms, the further WKB analysis is straightforward. Let \( U_\mu \) be an eigenvector of \( \Lambda \) with eigenvalue \( \mu \). Then there exists a solution of (2.1) whose large-\( x \) behaviour is

\[
\Psi = e^{-\mu \int^x p(x')dx'} p^{-\rho^\vee} U_\mu + \ldots = e^{-\mu \frac{M+1}{M} x^{-M} \rho^\vee} U_\mu + \ldots. \tag{2.8}
\]

**Stokes phenomena** We shall say that (2.8) is considered in the direction \( k \), \( k \in \mathbb{R} \), if \( x = q^{\frac{1}{M}} |x| \) with \(|x| \gg 1\). Hence \( k \) has meaning of a phase in units of the Symanzik angle. If \( k = k_0 \) is such that \( \mu e^{2\pi i k_0} \) is real and positive then it is a direction of the fastest descent of (2.8).

There always exists a solution with asymptotics (2.8) in a direction of the fastest descent. Moreover, if \( \mu e^{2\pi i k_0} \) is larger then \( \Re(\mu e^{2\pi i k_0}) \) for \( \mu' \) – any other eigenvalue of \( \Lambda \)

---

\(^{10}\)From this definition of \( q \) it may appear that \( q \) is a root of unity. However, \( M \) is not restricted to be an integer as we do not need to unambiguously define \( \Psi \) on the entire complex plane of \( x \), see Section 2. To avoid (inessential) issues with definition of \( \Psi \), the reader can think that \( M \) is a large enough integer so that \( q^n \not= 1 \) for all finite integer \( n \) that are encountered in practice.

\(^{11}\)We always assume that \( M \) is large enough, \( M > \frac{1}{\pi - 1} \) would suffice for suppression of the dotted terms.
then this solution is defined uniquely up to a normalisation and it shall be called a Stokes solution or S-solution with eigenvalue \( \mu \). If \( \Psi \) is such a solution then \( \Psi^{[2h]} \) is another one, with \( k_0 \to k_0 - h \), and so to avoid ambiguity we take \( k_0 \) to be the one with the smallest absolute value.\(^{12}\) Stokes solution is the smallest (the fastest decreasing) solution among all solutions of (2.1) for certain range of directions (Stokes sector) \( k \in k_0 + [-\epsilon, \epsilon] \), where \( \epsilon > 0 \), often \( \epsilon = 1/2 \); also the leading large \( x \) asymptotic behaviour of the S-solution is given by (2.8) in the applicability cone \( k \in k_0 + [-h/2 - \epsilon, +h/2 + \epsilon] \).

Even if \( \Re(\mu' e^{2\pi i k_0}) \geq \mu e^{2\pi i k_0} \), it is possible to define a unique solution with asymptotics (2.8) if we demand that the cone of applicability of (2.8) is large enough (for each \( \mu' \), it should contain exactly one connected domain of directions \( k \) where \( \Re(\mu' e^{2\pi i k}) < \Re(\mu e^{2\pi i k}) \) is realised). The definition depends on a choice of the applicability cone. We shall call such a solution \( S^* \)-solution.\(^{13}\)

If \( \Psi \) is an \( S \) - or \( S^* \) -solution with eigenvalue \( \mu \) then \( \Psi^{[2]} \) is also a solution of the same class, with the rotated counter-clock-wise eigenvalue \( e^{-2\pi i} \mu \) and the applicability cone rotated by one unit of the Symanzik angle clock-wise.

Q-functions Baxter Q-vectors are defined as

\[
Q_{(a)}(z) = z^{-\frac{\nu}{\pi i}} \Psi_{(a)}(0, z),
\]

(2.9)

The definition is designed to have the property\(^{14}\)

\[
Q^{[n]}_{(a)}(z) = \frac{n!}{z^{n/2}} Q_{(a)}(q^{n/2} z) = z^{-\frac{\nu}{\pi i}} \Psi_{(a)}^{[n]}(0, z).
\]

(2.10)

Baxter Q-functions are defined as the components \( Q_{(a),i} \) of the expansion \( Q_{(a)} = \sum Q_{(a),i} e_{(a),i} \) w.r.t. some basis. In general discussion, \( i \) runs through the set \( \{1, 2, \ldots, \dim L(\omega_a)\} \). For explicit cases however, it can be convenient to use \( i \) as an index from a more descriptive set in which case we do not write \( (a) \) in the subscript, cf. (1.21).

The basis elements \( e_{(a),i} \) should diagonalise Cartan generators. Let \( e_{(a),i} \) is of the weight \( \gamma_i \). One agrees that \( \gamma_1 = \omega_a \) is the highest weight of the irrep, and \( \gamma_2 = \omega_a - \alpha_a \) is the only leading descendent from the highest weight. For \( i \) such that \( \gamma_i \) is on the Weyl orbit of the highest weight, there is a natural notation to use: \( e_{\sigma(i)} = e_j \), where \( \sigma \) is an element of the Weyl group such that \( \sigma \gamma_i = \gamma_j \).

We shall impose the following requirement that partially restricts normalisation of basis vectors: For each \( \gamma_i \) on the Weyl orbit of the highest-weight vector, choose one concrete \( \sigma_i \in W \) such that \( \gamma_i = \sigma_i \gamma_1 \). Then we require that

\[
e_{(a),i} = s_{\sigma_i} e_{(a),i},
\]

(2.11)

\(^{12}\) \( \Psi \) and \( \Psi^{[2h]} \) do not generically coincide as solutions, requirement that they do up to a rescaling for some integer \( n \) is a quantisation condition on \( z \) in the sense of a quantum mechanical problem. We do not impose it here.

\(^{13}\) For the most of the discussion, focusing only on \( S \)-solutions suffices. For an explicit non-trivial example featuring \( S^* \) solutions, see (6.8).

\(^{14}\) More generally, for any \( x_0, z_0 \), the definition \( Q_{(a)}(z) = \left( \frac{z}{z_0} \right)^{-\frac{\nu}{\pi i}} \Psi_{(a)}(\left( \frac{z}{z_0} \right)^{-\pi i/2} x_0, z) \) can be used.
where $s_{\sigma_i}$ is the standard representative of the Weyl group element $\sigma_i$. It is defined as follows: for $\sigma_a$ being a reflection w.r.t. the simple root $\alpha_a$, a standard representative is (see e.g. [FH04], appendix D.4)

$$s_a = e^{E_{\alpha_a}} e^{-E_{-\alpha_a}} e^{E_{-\alpha_a}} .$$

(2.12)

For element $\sigma$ of length $\ell$ and its minimal length representation $\sigma = \sigma_{a_1} \ldots \sigma_{a_\ell}$, its standard representative is $s_\sigma = s_{a_1} \ldots s_{a_\ell}$. This choice of representative has the property (Proposition 3.1.2 of [Ros15])

$$s_\sigma s_{\sigma'} = s_{\sigma \sigma'} \prod_{\beta} (-1)^{h_\beta} ,$$

(2.13)

where the product runs over such positive roots $\beta$ that $\sigma' \beta$ is a negative root and $\sigma \sigma' \beta$ is a positive root.

Partial cases of (2.13) are: $s_2^2 = (-1)^{H_{\alpha_a}}$ for every simple root $\alpha_a$, and $s_\sigma s_{\sigma'} = s_{\sigma \sigma'}$ if $\ell(\sigma) + \ell(\sigma') = \ell(\sigma \sigma')$.

From (2.13) it then follows that $e_{(a),\sigma(i)} = \pm s_\sigma e_i$ for any $\sigma \in W$ as long as normalisation $(2.11)$ is chosen. This also gives a concrete recipe to fix signs in $(1.31)$.

**Ψ- and QQ-systems** One of the main results of [Sun12, MRV16] is the equality

$$\left( \Psi_+^{(a)} \wedge \Psi_-^{(a)} \right)_{L(\omega_{\text{max}})} = \left( \bigotimes_{b, C_{ab} = -1} \Psi_{(b)} \right)_{L(\omega_{\text{max}})} .$$

(2.14)

This is called the Ψ-system\(^{15}\). The proof is the following: First, if needed, perform a half-Symanzik rotation of (2.14) to make both sides of the equation solving (2.1) in the irrep $L(\omega_{\text{max}})$. Then, by analysing the large-$x$ asymptotics along the line of fastest descent one deduces that both l.h.s. and r.h.s. of (2.14) have the same growth rate which moreover coincides with the growth rate of the S-solution of (2.1) in the irrep $L(\omega_{\text{max}})$ along this line. Hence both sides of (2.14) should be, up to normalisation, this S-solution, and it is easy to check that the coefficient of proportionality is non-zero. Normalisation of $\Psi_{(a)}$ can be fixed to get an equality in (2.14) for all $a$.

By evaluation (2.14) at $x = 0$, one gets the relation (1.29) between the Q-vectors and eventually to the QQ-system defined by (1.30).

As discussed in the introduction, our goal is not only to focus on the QQ-system relations (1.30) but to explore a variety of properties of the Q-vectors, especially focusing on their covariance with respect to action of $g$. We stress that for algebras different from $A_n$, the Q-vectors have components outside of the Weyl orbit of the highest weight, and hence our study goes beyond the Weyl orbit Q-system (1.31).

\(^{15}\)See also [DDM+07] where the Ψ-system was obtained on the level of pseudo-differential equations.
Dynkin labels | Dimension | Name (\$\mathfrak{so}(6)$ point of view)  
|-----------------|-----------|-----------------------------|
| [010]            | 6         | vector                     |
| [100], [001]    | 4, 4̄     | [co]-spinor                |
| [101]            | 15        | adjoint                    |
| [110], [011]    | 20, 20̄   | Weyl vector multiplet      |
| [111]            | 64        | symmetric traceless tensor |
| [020]            | [111]     | 64                         |
| [200], [002]    | 10, 10̄   | [anti]-self-dual 3-form     |

Table 1: List of certain $A_3$ irreps

![Figure 1: Eigenvalues of $\Lambda$ for irreps of $A_3$.](image)

2.2 Example of $sl(4) \simeq so(6)$ extended Q-systems

We consider first an explicit example of the $A_3$ Q-system to illustrate types of relations that we would like to explore in this paper. Because it is also the $D_3$ system, we shall use both $sl(4)$ and $so(6)$ notations in parallel, with the goal of future generalisation to $D_n$ series. The $so(6)$ Q-system with spinor notations was featured for the first time in the context of AdS$_4$/CFT$_3$ correspondence [BCF+17].

There are three fundamental representations: 4, 6, 4̄, and we use the following notations for Q-vectors $Q_{(1)} \equiv \psi$, $Q_{(2)} \equiv V$, $Q_{(3)} \equiv \eta$:

| $sl(4)$ | $so(6)$ |  |
|---------|---------|---|
| 4       | $Q_a$   | $\psi_\alpha$ |
| 6       | $Q_{ab}$ | $V_1 = \gamma_1^\alpha \gamma^\beta \psi_\alpha \psi_\beta$ |
| 4̄      | $Q^a = \frac{1}{6} \epsilon^{abcd} Q_{bcd}$ | $\eta^\alpha = C^{\alpha\dot{\alpha}} q_{\dot{\alpha}}$ |

\[
(Q_1, Q_2, Q_3, Q_4) = (\psi_1, \psi_2, \psi_3, \psi_4) \]
\[
V_1 = Q_{12}, V_2 = Q_{13}, V_3 = Q_{23} \quad (2.15) \]
\[
V_{-3} = Q_{14}, V_{-2} = Q_{24}, V_{-1} = Q_{34} \]

Here we have introduced standard spinor notation using dotted and un-dotted indices for the representations 4 resp 4̄ and the matrices $(\gamma_A)^\alpha_\beta, (\bar{\gamma}_A)^{\dot{\alpha}}_\dot{\beta}$ are 4 × 4 Dirac gamma-matrices satisfying $(\gamma_A \bar{\gamma}_A)^\alpha_\beta + (\bar{\gamma}_B \gamma_B)^\alpha_\beta = \delta^\alpha_\beta g_{AB}$. The charge conjugation matrix $C^{\alpha\dot{\alpha}}$ lowers and raises indices and is anti-diagonal, so is the metric $g_{AB}$. Explicit expressions

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16With exception of [020], [200], [002], these are precisely all the minimal irreps whose highest-weight subspace is invariant under a parabolic subgroup action.
for these objects are

\[
V_1^\gamma \alpha \beta \eta_\alpha^+ \eta_\beta^- = V_1 W[\eta_1, \eta_2] + V_2 W[\eta_1, \eta_3] + V_3 W[\eta_1, \eta_4]
\]

\[
+ V_4^{-3} W[\eta_2, \eta_3] + V_5^{-2} W[\eta_2, \eta_4] + V_6^{-1} W[\eta_3, \eta_4]
\]

\[
\psi_\alpha C^{\alpha \beta} \eta_\beta = -\psi_1 \eta_4 + \psi_2 \eta_3 - \psi_3 \eta_2 + \psi_4 \eta_1
\]

\[
V_i g^{ij} W_j = (V_1 W_{-1} - V_2 W_2 + V_3 W_{-3} + V_{-3} W_3 - V_{-2} W_2 + V_{-1} W_1)
\]

Note that the pfaffian of antisymmetric tensors are related to the inner product of vectors since

\[
\frac{1}{4} e^{abcd} Q_{ab} Q_{cd} = 2(Q_{12} Q_{34} - Q_{13} Q_{24} + Q_{14} Q_{23}) = V_i g^{ij} V_j . \tag{2.16}
\]

Having introduced notation the goal is to take tensor products of these irreps and use Q-vectors to construct functions of the spectral parameter in other representations of \(A_3\). The irreps that we shall encounter while performing this fusion procedure are given in Table 1.

To illustrate the use of the linear problem we shall focus on the example \(V^+ \otimes V^{[1-2n]}\). This tensor product decomposes as \(6 \otimes 6 = 20' \oplus 15 \oplus 1\) and we are interested in specifying what happens when \(V^+ \otimes V^{[1-2n]}\) are projected onto the irreducible representations. Consider then \(\Psi_{(2)}^+ \otimes \Psi_{(2)}^{[1-2n]}\) and study their Stokes behaviour:

| \(n\) | \(\mu\) | WKB applicability | \(20'\) | \(15\) | \(1\) |
|------|-------|-----------------|-------|------|------|
| 0    | 2(1 + i) | \(-\frac{1}{2} + [-2, 2]_+\) | S     | 0    | 0    |
| 1    | 2     | \(0 + [\frac{1}{2}, \frac{3}{2}]_+\) | S*    | S    | 0    |
| 2    | 0     | \(\frac{1}{2} + [-1, 1]_+\) | S*    | 1    | 1    |
| 3    | 2i    | \(1 + [-\frac{1}{2}, \frac{1}{2}]_+\) | NS    | NS   | \(T_{2,1}^{[-2]}\) |
| 4    | 2(1 + i) | \(\frac{3}{2} + [-0, 0]_+\) | NS    | NS   | \(T_{2,2}^{[-3]}\) |
| \(\geq 5\) | \(2(1 + i)\) | none | NS    | NS   | \(T_{2,n}^{[-1-2n]}\) |

(2.17)

For \(n \leq 4\), the asymptotic behaviour at infinity is given by (2.8), where

\[
\mu = \gamma^{\frac{1}{2}} (\mu_{(2)} + \gamma^{-n} \mu_{(2)}) \tag{2.18}
\]

with \(\gamma = e^{\frac{2\pi i}{6}} = i\) and \(\gamma^{\frac{1}{2}} \mu_{(2)} = 1 + i\) being the eigenvalue of \(\Lambda_6\) for the S-solution \(\Psi_{(2)}^+\).

This WKB approximation is valid in a certain cone of applicability, these cones are listed for various \(n\) in the table above where the notation \(\lceil a, b \rceil_+\) means “at least in the range \([a, b]_+\)”. The listed range in the table is obtained by intersection of the applicability cones for \(\Psi_{(2)}^{[1-2n]}\) which are \(n - \frac{1}{2} + [-2, 2]_+\).

Now we compare \(\mu\) with the eigenvalues of \(\Lambda_{20'}, \Lambda_{15}\). The cases when these eigenvalues match with an eigenvalue of \(\Lambda_L\) for \(L = 20'\) or \(15\) and applicability cones allow to decide that \((\Psi_{(2)}^+ \otimes \Psi_{(2)}^{[1-2n]})_L\) is an \(S/S^*\)-solution are marked in the table \(S, S^*\). In these cases we can unambiguously, up to normalisation, identify \((\Psi_{(2)}^+ \otimes \Psi_{(2)}^{[1-2n]})_L\) with a concrete solution.
\( \Psi \) of \((2.1)\) and hence one also knows that
\[
(V^+ \otimes V^{[1-2n]})_L \propto \Psi(x = 0).
\tag{2.19}
\]
The coefficient of proportionality may depend on the spectral parameter \( z \) but this dependence is fixed by analysing the prefactors of the large-\( x \) asymptotics of \((\Psi^+_2 \otimes \Psi^{[1-2n]}_2)_L\).

The identification \((2.19)\) becomes of value if we can realise \( \Psi(x = 0) \) as projection from some other tensor product, for instance \( n = 1 \) and \( L = 15 \) is precisely \( V^+ \wedge V^- \) which is the l.h.s. of \((1.29)\). Equalities between projections of different tensor product shall be called fusion relations.

The cases when one cannot identify \((\Psi^+_2 \otimes \Psi^{[1-2n]}_2)_L\) with an \( \mathfrak{s} \) or \( \mathfrak{s}^* \)-solution are denoted as NS. For instance, the issue with \( n = 3, 4 \) is that the applicability cone is inappropriate, notably it does not feature the fastest descent line for the corresponding \( \mu \).

The case of the trivial representation \( 1 \) is a bit special. Equation \((2.1)\) has then the unique solution which is constant in \( x \). Hence projection of \((\Psi^+_2 \otimes \Psi^{[1-2n]}_2)_L\) on the trivial representation will be always a constant in \( x \). However, it can have a non-trivial dependence on \( z \), and only in the case when the WKB analysis can be applied can we conclude the value of \((V^+ \otimes V^{[1-2n]}_1) \).

It may also happen that \( \mu \) is not an eigenvalue of \( \Lambda_L \) in an irrep \( L \). Then, if the WKB approximation is valid along the direction of the fastest descent and, for this direction \( \mu x^{M+1} > \Re(\mu' x^{M+1}) \) for all eigenvalues \( \mu' \) of \( \Lambda_L \) then \( \Psi^+_2 \otimes \Psi^{[1-2n]}_2 \) is sub-dominant compared to any solution of \((2.1)\) in the representation \( L \) which is only possible if \((\Psi^+_2 \otimes \Psi^{[1-2n]}_2)_L = 0 \) and hence \((V^+ \otimes V^{[1-2n]}_1) = 0 \). We call this a projection property.

There are three instances of the projective property in the \( 6 \otimes 6 \) example. One of them, \((V \otimes V)_{15} = V \wedge V = 0\), is obvious while the other two are more interesting:

\[
\epsilon^{abcd} Q_{ab} Q_{cd} = 0, \quad V^i g^{ij} V^{[2]}_j = 0, \tag{2.21a}
\]

\[
\epsilon^{abcd} Q^+_{ab} Q^-_{cd} = 0, \quad V^i g^{ij} V^-_j = 0. \tag{2.21b}
\]

Equations \((2.21a)\) are certain relations between Plücker coordinates of \( \mathfrak{s}(N = 4) \) and \( \mathfrak{s}(M = 6) \) flags. For the \( \mathfrak{s}(N) \) case, \((2.21a)\) is the famous Plücker quadric telling us that the two-form \( Q_{(2)} \) identifies a plane embedded into \( \mathbb{C}^N \). For the \( \mathfrak{s}(M) \) flag, this is an assertion that all lines embedded into \( \mathbb{C}^M \) are null. Equations \((2.21b)\) are relations that are featured by the fused flag.

Finally we note that certain combinations of Q-functions are of physical significance even if they cannot be studied using the WKB analysis of \((2.1)\). For instance, as indicated in \((2.17)\), singlets constructed from the vector representation should be interpreted, in appropriate explicit systems, as transfer matrices in the symmetric powers of the vector
In addition to (2.21), we list a couple of other projection relations

\[ V^{(a)} g^{ij} = T_{2,n-2} . \]  

The above study of \( 6 \otimes 6 \) features all the properties we wanted to demonstrate. In the same way other tensor products can be studied. Below we summarise most of the interesting relations featured by the A\(_3\) which can be grouped into three classes:

**Fusion relations** The main example is the QQ-relations (1.29): The WKB analysis of \( 4 \otimes 4 = 6 \oplus 10 \) implies \( \psi^{(1)} \wedge \psi^{(2)} \), \( \bar{4} \otimes 4 = 6 \oplus \bar{10} \) implies \( \psi^{(3)} \wedge \psi^{(3)} = \psi^{(2)} \), and the above-discussed \( 6 \otimes 6 \) compared with \( 4 \otimes \bar{4} = 15 \oplus 1 \) implies \( \psi^{(2)} \wedge \psi^{(2)} = (\psi^{(1)} \psi^{(3)})_{15} \). The corresponding QQ-relations written in components are

\[
W[Q_a, Q_b] = Q_{ab} ,
V_i = \psi_i^\alpha \gamma_i^\beta \psi_\beta^-, \tag{2.23a}
W[Q^a_i, Q^b_j] = \frac{1}{2} \epsilon^{abcd} Q_{cd} ,
V_i = \eta_i^\alpha \gamma_i^\beta \eta_\beta^- , \tag{2.23b}
W[Q_{ab}, Q_{cd}] = -\frac{1}{2} (Q_a Q_{bcd} - Q_b Q_{acd} - Q_c Q_{dab} + Q_d Q_{cab}) , \tag{2.23c}
W[V_i, V_j] = \gamma_i^\alpha \psi_\alpha \eta_j^- , \tag{2.23d}
\]

where \( \gamma_i^\alpha = \frac{1}{2} (\gamma_i^\alpha \gamma_j - \gamma_j^\alpha \gamma_i) \). In the \( \mathfrak{sl}(4) \) interpretation, these are mostly the relations (1.22), with exception of three equations featuring pairwise non-equal \( a, b, c, d \) in (2.23c). These three cases correspond to the projection of the adjoint representation 15 to the zero-weight space which is not on the Weyl orbit of the highest weight.

In the \( \mathfrak{so}(6) \) interpretation, equations (2.23) are instances of the fused Fierz relations. Another example of such a relation is

\[
\gamma_{(\pm),ij} \psi_\alpha \psi^- = V_{(\pm),ijk} \text{ 10 of } 4 \otimes 4 = 10 \text{ of } 6 \otimes 6 \otimes 6 , \tag{2.24a}
\gamma_{(\pm),ij} \psi_\alpha \eta^- = V_{(\pm),ijk} \text{ 10 of } 4 \otimes 4 = 10 \text{ of } 6 \otimes 6 \otimes 6 , \tag{2.24b}
\]

where \( V_\pm \) are the self-/antiself-dual projections of the 3-form \( V_{(3)} = V^{[2]} \wedge V \wedge V^{[-2]} \).

**Projection relations** In addition to (2.21), we list a couple of other projection relations

For \( n = 0, \pm 2 \) : 
\[ Q^{[n]} Q^a = 0 \text{ 4 } Q^{[n]} \psi^a = 0 \text{ 4 } \bar{4} = . . . + 1 + . . . \]  
For \( n = \pm 1 \) : 
\[ \epsilon^{abcd} Q^{[n]} Q_c = 0 \text{ 6 } \psi^{[n]} \gamma_i^\alpha \gamma_A^a V^A = 0 \text{ 6 } \bar{4} = . . . + 20 + . . . \]  
For \( n = \pm 1 \) : 
\[ Q^{[n]} Q^b = 0 \text{ 6 } \eta_i^\alpha \gamma_i^\alpha \gamma_A^a V^A = 0 \text{ 6 } \bar{4} = . . . + 20 + . . . \]  
Together with (2.21), these are all the relations establishing that \( Q^{(1)}, Q^{(2)}, Q^{(3)} \) are Plücker coordinates of a complete flag. Equation (2.25a) is also known as a pure spinor condition.
The fact that we can choose various values of \( n \) reflects that we are dealing with a fused flag, its general definition will be given in section 3.3.

**Quantisation relations** These are special instances of fusion relations when the target projection representation is trivial, an example is (2.20). All of them can be shown to be equivalent to (1.12) which itself can be derived by performing the WKB analysis of \( 4 \otimes 4 \otimes 4 \otimes 4 = \ldots + 1 + \ldots \). By applying analytic Bethe ansatz, see section 4.3, the quantisation relation\(^{17}\) become the Wronskian Bethe equations in terminology of [CLV20]. They correctly describe the spectrum of the model for any values of the parameters [MTV13, CLV20] and in this sense are superior to standard nested Bethe equations.

### 2.3 Spectrum of \( \Lambda \)

As the previous subsection demonstrated, the spectrum of \( \Lambda \) in various irreps contains valuable information for our study. We shall find this spectrum explicitly in this subsection following closely [FLO91, MRV16] and also explore some other related properties of \( \Lambda \).

First, from the Symanzik rotation (2.3), it is straightforward to see that

\[
\gamma^{\text{ad}} \rho \Lambda = \gamma \Lambda , \quad \gamma \equiv e^{2\pi i \h}. \tag{2.26}
\]

Hence, if \( \mu \) is an eigenvalue of \( \Lambda \) then \( \gamma \mu \) is an eigenvalue as well. All the eigenvalues are therefore located on concentric circles, each such circle contains a multiple of \( h \) of the eigenvalues, see Fig. 2.

\( \Lambda \) is not ad-nilpotent, hence it can be viewed as an element of a Cartan subalgebra \( \mathfrak{h}' \). Choose a simple root system in \( (\mathfrak{h}')^* \), and use the corresponding co-roots \( h'_a \in \mathfrak{h}' \), \( a = 1, \ldots, r \) as a basis in \( \mathfrak{h}' \). Our goal is to find expansion of \( \Lambda \) in this basis, \( \Lambda = \sum_{a=1}^r c_a h'_a \).

Then the spectrum of \( \Lambda \) directly follows from the spectrum of \( h'_a \) which is given by Dynkin weights.

\(^{17}\)For at least rational spin chains in the defining representation of \( \mathfrak{sl}(4) \). Section ... also covers other representations
Relation (2.26) implies that $\gamma^{ad_\rho} \rho^\vee$ is in the normaliser of the maximal torus whose Lie algebra is $\mathfrak{h}'$ and hence it realises action of the Weyl group on $\mathfrak{h}'$. Moreover, it is an element of degree $h$ and hence it should be a Coxeter element. Recall how Coxeter elements are defined. Weyl reflection w.r.t. the $a$'th simple root shall be denoted by $s_a$. Its action on $\mathfrak{h}'$ is given by

$$s_a(h'_b) = h'_b - C_{ba} h'_a.$$  \hfill (2.27)

A Coxeter element is defined as a product of all simple Weyl reflections $\gamma^{ad_\rho} \rho^\vee = \prod_{a=1}^{r} s_a$. Choice of an order in this product defines the Coxeter element which we consider. All Coxeter elements form one adjoint orbit of the Weyl group.

For a given Coxeter element, there exists the unique its eigenvector with eigenvalue $\gamma = e^{2\pi i h}$, and so $\Lambda$ should be, up to normalisation, this eigenvector. $\Re(\Lambda)$ and $\Im(\Lambda)$ span the Coxeter plane – the unique plane where the Coxeter element acts as the rotation by angle $\frac{2\pi}{h}$. Hence spectrum of $\Lambda$ is the projection of the irrep weight space to the Coxeter plane.

First, we find $\Lambda$ explicitly for a particularly simple choice of order in $\prod_{a=1}^{r} s_a$. Introduce a bipartition of the Dynkin diagram into “even” and “odd” nodes, such that no lines link even with even (or odd with odd). Our convention is that the fundamental representation of the smallest dimension is associated to an even node. Now consider a Coxeter element $s_{\text{odd}}s_{\text{even}}$. Here $s_{\text{odd}}$ is a product of the reflections of the odd simple roots, and $s_{\text{even}}$ the one of even simple roots. Using (2.27), one finds [BLM89]

$$s_{\text{odd}}(s_{\text{even}})h'_a = \sum_{a=1}^{r} I_{ab} h'_b,$$  \hfill (2.28)

where $I_{ab}$ is the incidence matrix of the Dynkin diagram.

Let $q = \sum_{a=1}^{r} \mu_a h'_a$ be an eigenvector of the Coxeter element with some eigenvalue $\tilde{\gamma}$,

$s_{\text{odd}}s_{\text{even}} = \tilde{\gamma} q$. Parameterise it in the form $q = \tilde{\gamma}^{1/2} q_{\text{odd}} + q_{\text{even}}$, where $q_{\text{even}} = \sum_{a \in \text{even}} \mu_a h'_a$ and $q_{\text{odd}} = \sum_{a \in \text{odd}} \mu_a h'_a$. Based on (2.27), one has

$$s_{\text{even}}(q_{\text{even}}) = -q_{\text{even}}, \quad s_{\text{odd}}(q_{\text{odd}}) = -q_{\text{odd}},$$  \hfill (2.29)

while using (2.28) we can write

$$s_{\text{even}}(q_{\text{odd}}) = (1 + \hat{I}) q_{\text{odd}}, \quad s_{\text{odd}}(q_{\text{even}}) = (1 + \hat{I}) q_{\text{even}},$$  \hfill (2.30)

where $\hat{I}$ is an operator with matrix entries $I_{ab}$ in the basis $h'_a$.

Using the mentioned properties, one derives from $s_{\text{odd}}s_{\text{even}} = \tilde{\gamma} q$ that

$$\hat{I}(q_{\text{odd}} - \tilde{\gamma}^{1/2} q_{\text{even}}) = (\tilde{\gamma}^{1/2} + \tilde{\gamma}^{-1/2})(q_{\text{even}} - \tilde{\gamma}^{1/2} q_{\text{odd}}).$$  \hfill (2.31)
Since the incidence matrix is of the graph with only links between nodes of different type, the above relation can be projected to

\[ \hat{I}_q^{\text{odd}} = (\bar{\gamma}_1/2 + \bar{\gamma} - 1/2)q_{\text{even}}, \]
\[ \hat{I}_q^{\text{even}} = (\bar{\gamma}_1/2 + \bar{\gamma} - 1/2)q_{\text{odd}} \]

implying that \( q_{\text{odd}} \pm q_{\text{even}} \) are eigenvectors of \( \hat{I} \) with eigenvalues \( \pm(\bar{\gamma}_1/2 + \bar{\gamma} - 1/2) \).

As the logic can be reversed, we conclude that all the eigenvalues of \( \hat{I} \) are of the form \( \pm(\bar{\gamma}_1/2 + \bar{\gamma} - 1/2) \) where \( \bar{\gamma} \) is an eigenvalue of the Coxeter element. Now we notice that \( I_{ab} \) is a matrix of Perron-Frobenius type. Given the established bijection with the eigenvalues of the Coxeter element, the maximal eigenvalue of \( I_{ab} \) is identified to be \( \gamma' = e^{2\pi i}/2 \). The corresponding eigenvector allows us then to construct \( \Lambda \):

**Lemma 2.1.** Let \( (\mu_1, \mu_2, \ldots, \mu_r) \) be the Perron-Frobenius eigenvector of \( I_{ab} \),

\[ \sum_{b=1}^r I_{ab} \mu_b = (\gamma' + \gamma' - 1/2) \mu_a . \quad (2.32) \]

Then, for a choice of Cartan subalgebra and simple roots such that \( \gamma_a^{\text{adj}} \) is the Coxeter element \( s_{\text{odd}}s_{\text{even}} \),

\[ \Lambda = \sum_{a=1}^r \gamma_a^{\text{adj}} \mu_a h'_a , \quad (2.33) \]

where \( \tilde{p}_a = 0 \) for even Dynkin nodes \( a \) and \( \tilde{p}_a = 1 \) for odd Dynkin nodes \( a \).

Now we would like to understand how \( \Lambda \) looks like for a different choice of a Coxeter element.

First let us design a way to label different Coxeter elements. We define the Coxeter height function as a function \( p : \{1, \ldots, r\} \to \mathbb{Z} \) satisfying the property \( p_a - p_b = \pm 1 \) if \( I_{ab} \neq 0 \). The Coxeter height functions that differ only by a translation, \( p_a \to p_a + n, n \in \mathbb{Z} \), shall be considered as equivalent. In view of the equivalence we will always assume that \( p_a \) is even if \( a \) is an even node.

**Lemma 2.2.** The Coxeter height functions (up to the equivalence) are in bijection with distinguished Coxeter elements by the following rule: For \( a, a' \) being two adjacent nodes of the Dynkin graph, \( s_a \) is to the left of \( s_{a'} \) in the product \( \prod_{a=1}^r s_a \) defining the Coxeter element if and only if \( p_a > p_{a'} \).

**Proof.** Think about the product \( \prod_{a=1}^r s_a \) with certain order of elements as a word \( s_{a_1} \ldots s_{a_r} \). We allow to exchange two neighboring letters in the word \( \ldots s_a s_{a'} \ldots \simeq \ldots s_{a'} s_a \ldots \) if \( a, a' \) are not adjacent nodes of the Dynkin graph. Indeed, Weyl reflections \( s_a, s_{a'} \) commute then. Two words shall be called equivalent if they can be obtained from one another by a sequence of these exchanges. All Coxeter elements belonging to the same equivalence class coincide. By induction in \( r \), one shows that the equivalence classes of Coxeter height functions are in bijection with the equivalence classes of words.

Finally, different equivalence classes should define different Coxeter elements because the corresponding Coxeter elements have different eigenvectors with eigenvalue \( \gamma \), as shall be demonstrated by the Lemma below.
Lemma 2.3. If \(s[p]\) is the Coxeter element with the height function \(p\) then

\[ s[p] \Lambda[p] = \gamma \Lambda[p], \quad \text{where} \quad \Lambda[p] = \sum_{a=1}^{r} \gamma^\frac{p_a}{2} \mu_a h'_a. \quad (2.34) \]

Proof. Without loss of generality assume that the minimal value of \(p\) is either 0 or 1, depending on whether it is realised at an even or an odd node. Suppose now that \(p_a\) is the maximal value of \(p\). Then \(p_a = p_a - 1\) for all nodes \(a\) adjacent to \(a_s\). On the one hand, we note that \(s_a\) is to the left of all \(s_a\), \(I_{aa_s} \neq 0\), in \(s[p]\). Then, by using commutativity with other elementary reflections, we can write \(s[p] = s_a, s'\) and hence

\[ s_a, s[p_1, \ldots, p_a, \ldots, p_r] s_a = s' s_a = s[p_1, \ldots, p_a - 2, \ldots, p_r]. \quad (2.35) \]

On the other hand, by acting with \(s_a\) on \(\Lambda[p]\), we find

\[
\begin{align*}
   s_a \Lambda[p_1, \cdots, p_a, \cdots, p_r] &= \gamma^\frac{p_a}{2} h'_a (-\mu_a + \sum_{a=1}^{r} \gamma^{-\frac{1}{2}} \mu_a I_{aa_a}) + \sum_{a \neq a_s} \gamma \frac{p_a}{2} \mu_a h'_a \\
   &= \Lambda[p_1, \cdots, p_a - 2, \cdots, p_r],
\end{align*}
\]

where (2.32) was used.

We thus could decrease \(p_a\) to \(p_a - 2\) by acting with the same reflection on both \(s[p]\) and \(\Lambda[p]\). We repeat this progress, decreasing at each step one \(p_a\) that has currently the maximal value of \(p\). If there are several nodes with the maximal value, they are not adjacent and hence we can decrease their value in any order. We continue until we obtain \(p = \tilde{p}\), \(s[\tilde{p}] = s_{\text{odd}} s_{\text{even}}\). We already established that \(\Lambda[\tilde{p}]\) is the desired eigenvector of \(s[\tilde{p}]\), cf. (2.33). It remains to reverse all the performed reflections \(s_{a_s}\) to prove the same for any \(p\) thus confirming (2.34).

In conclusion, we managed to find the explicit form for the originally defined by (2.7) \(\Lambda\) in a reference frame where \(\gamma^\rho\) is the Coxeter element \(s[p]\): \(\Lambda = \Lambda[p]\). Obviously, spectrum of \(\Lambda[p]\) is the same for any choice of the Coxeter height function \(p\).

Finally, let us also comment about interpretation of the eigenvectors of \(\Lambda\). Denote by \(U_{(a)}^{[p_a]}\) the eigenvector of \(\Lambda L(a)\) with the eigenvalue \(\gamma^{p_a/2} \mu_a\). If we are in a frame where \(\Lambda = \Lambda[p]\), we conclude that \(h'_b U_{(a)}^{[p_a]} = \delta_{ab}\) and hence \(U_{(a)}^{[p_a]}\) gets meaning of the highest-weight vector in the \(a\)'th fundamental representation. Importantly, we can make this conclusion simultaneously for all \(a\):

Lemma 2.4. For every Coxeter height function \(p\), there exists a Cartan subalgebra \(\mathfrak{h}'\) and a choice of simple roots such that \(U_{(1)}^{[p_1]}, \ldots, U_{(r)}^{[p_r]}\) are the highest-weight vectors in the corresponding fundamental representations of \(\mathfrak{g}\).

3 Extended Q-system

In this section we explore, using general formalism of representation theory, various relations between the Q-functions of the extended Q-system and their geometric interpretation.
Recall that the Q-functions of the extended Q-system are the components of the vectors $Q^{(a)}$ in the $a^{th}$ fundamental representations of the Lie algebra.

### 3.1 Relations between Q-functions

Generalising from the $\mathfrak{so}(6) \simeq \mathfrak{sl}(4)$ example, we organise all possible relations into three categories: fusion, quantisation, and projection.

Let $A = a_1, \ldots, a_{|A|}$ be an ordered set of cardinality $|A|$ comprising elements from $\{1, 2, \ldots, r\}$, possibly with repetitions. Choose also some integers $n_1, \ldots, n_{|A|}$ and construct the following function

$$\Psi = \bigotimes_{i=1}^{|A|} \Psi^{[n_i]}_{(a_i)} \quad (3.1)$$

which is naturally a vector in the representation

$$L := \bigotimes_{i=1}^{|A|} L(\omega_{a_i}) = \bigoplus_{\omega} L(\omega), \quad (3.2)$$

we also noted the decomposition of $L$ into irreps, the sum $\bigoplus_{\omega}$ may feature repetitions of $\omega$.

The largest $\omega$ that appears in this sum is $\omega_{\text{max}} = \sum_{i=1}^{|A|} \omega_{a_i}$.

We demand that $\Psi$ is a solution of (2.1) and hence restrict $n_i$ to be even if $a_i$ is an even node of the Dynkin diagram and odd if $a_i$ is an odd node.

The cone of applicability of $\Psi$ is $[\alpha, \beta]_+$ with $\alpha = \max(n_1, \ldots, n_{|A|}) - \frac{h}{2}$, and $\beta = \min(n_1, \ldots, n_{|A|}) + \frac{h}{2}$. In this cone, the large-$x$ approximation (2.8) of $\Psi$ follows from that of $\Psi^{[n_i]}_{(a_i)}$, and the associated eigenvalue is $\mu = \sum_{i=1}^{|A|} \gamma_n \mu_{(a_i)}$. For the statements below, it is important that this cone is non-empty, and in certain cases it also must be large enough.

#### Fusion relations

If $\mu$ is an eigenvalue of $\Lambda_{L(\omega)}$ in some irrep $L(\omega)$ appearing in the decomposition (3.2) then one can apply the WKB analysis for $\Psi$ restricted to this irrep: If for each $\mu'$ – eigenvalue of $\Lambda_{L(\omega)}$ different from $\mu$ – there exists a direction $k \in [\alpha, \beta]_+$ such that $\Re(\mu' e^{\frac{2\pi i}{h} k}) < \Re(\mu e^{\frac{2\pi i}{h} k})$ then $\Psi_{L(\omega)}$ is a solution of (2.1) of $S^*$-type. If directions $k$ for each $\mu'$ can be made equal then this a solution of $S$-type. In either case, it is fixed uniquely by its large-$x$ asymptotics.

Specialising to the Q-system, one writes

$$\left( \bigotimes_{i=1}^{|A|} Q^{[n_i]}_{(a_i)} \right)_{\Lambda_{L(\omega)}} = z^{-\frac{|A| \lambda'}{2\pi}} (\Psi(0))_{\Lambda_{L(\omega)}}. \quad (3.3)$$

If there is a different way to get the same $S$/$S^*$-solution, e.g. using a set $A'$ and associated integers $n'_i$, then from uniqueness of such a solution we derive the fusion relation

$$\left( \bigotimes_{i=1}^{|A|} Q^{[n_i]}_{(a_i)} \right)_{\Lambda_{L(\omega)}} \propto \left( \bigotimes_{i=1}^{|A'|} Q^{[n'_i]}_{(a'_i)} \right)_{\Lambda_{L(\omega)}}. \quad (3.4)$$
We stress that coefficient of proportionality does not depend on the spectral parameter. It is just a number. Indeed, it can be fixed from comparison of the large-\(x\) asymptotics (2.8) which does not depend on the spectral parameter.

Typical examples of the fused relations are the QQ-relations (1.29), relations featuring Wronskians e.g. (1.21), and the fused version of Fierz identities.

**Quantisation relations** This is a special instance of the fused relations, when \(L(\omega)\) is the trivial representation. In this case we do not need two different ways to realise the same \(S/S^*\)–solution but instead we can write

\[
\left( \bigotimes_{i=1}^{[A]} Q_{[n_i]}^{[\alpha_i]} \right) \propto 1. \quad (3.5)
\]

It is important that the cone of applicability is non-empty to fix the normalisation constant from the large-\(x\) asymptotics and in particular to conclude that it does not depend on the spectral parameter.

We call this type of relations as quantisation relations because they essentially constrain possible functional dependence of the Q-functions on the spectral parameter. And indeed, on the example of the rational \(\mathfrak{sl}(n)\) case, we know that the quantisation condition (1.12) selects a finite set of polynomials \(Q_a\) which are precisely all physical solutions at least in the case of spin chains in the defining representations. We observed a similar situation in the explicit computations that we performed for \(\mathfrak{so}(8)\) case.

There is one particular quantisation relation which we would like to mention explicitly. Let \(Q_{(\alpha)}\) be the Q-vector in the contra-gradient representation\(^\text{18}\) compared to the Q-vector \(Q_{(a)}\). Then we can always, by natural pairing, construct a singlet from these two Q-vectors. Abusing a bit terminology, we shall refer to it as a scalar product and denote by \(\langle \cdot, \cdot \rangle\).

The quantisation relation reads

\[
\langle Q_{[h/2]}^{[\alpha]}, Q_{[-h/2]}^{[\alpha]} \rangle = 1. \quad (3.6)
\]

Normalisation of the Q-functions is fixed by demanding equality in (1.29). Then we rescale the definition of the scalar product in a way that (3.6) holds.

The proof of (3.6) is simple using the WKB analysis. Eigenvalues of \(\Lambda\) are the same for a representation and its contra-gradient, and \(\gamma h/4 + \gamma^{-h/4} = 0\). Hence \(\langle \Psi_{[h/2]}^{[\alpha]}, \Psi_{[-h/2]}^{[\alpha]} \rangle\) has constant large-\(x\) asymptotics, the cone of applicability is non-zero.

**Projection relations** Finally, it may happen that \(\mu\) is not an eigenvalue of \(L(\omega)\) for a particular \(\omega\). In such a case, if for each \(\mu'\) – eigenvalue of \(\Lambda_{L(\omega)}\) – there exists a direction \(k \in [\alpha, \beta]_+\) such that \(\Re(\mu' e^{2\pi i k}) < \Re(\mu e^{2\pi i k})\) then \(\Psi_{L(\omega)} = 0\) which implies

\[
\left( \bigotimes_{i=1}^{[A]} Q_{[n_i]}^{[\alpha_i]} \right)_{\Lambda_{L(\omega)}} = 0. \quad (3.7)
\]

\(^\text{18}\) obtained by minus transposition of the representation matrices
The above equality is easy to verify if $n_i = 0$ for even Dynkin nodes and $n_i = \pm 1$ (same sign for all $i$) for odd Dynkin nodes, and $\omega < \omega_{\text{max}}$. Condition on $n_i$ can be further relaxed which is the subject of Section 3.3.

There is also an important projection property featured by the scalar product:

$$\langle Q_m^n, Q_{(a)} \rangle = 0,$$  \hspace{1cm} (3.8)

for $n = h - 2, h - 4, \ldots, 2 - h$. Its derivation is based on the fact $\gamma^{n/2} + 1 \neq 0$ that implies that the eigenvalue controlling large-$x$ asymptotics of $\langle \Psi_m^n, \Psi_{(a)} \rangle$ is non-zero, and the cone of applicability contains the fastest descent line for the mentioned $n$. Compared to the constant solution of trivial representation, this is a sub-dominant solution.

Projection relations have a natural geometric interpretation of generalised Plücker relations as we shall soon see.

3.2 Universality of the Q-system

The relations listed in the previous subsection are derived for very specific Q-functions that originate from solutions of the linear problem (2.1) according to (2.10). We will show now that all these relations can be systematically imposed on Q-functions which are not a-priori linked to some ODE/IM problem. So the relations should be actually based on representation theory of the Lie algebra alone, they are not an exclusive feature of (2.1).

To be specific, recall the terminology that we use: Ensemble of Q-functions of type $Q_{(a),\sigma(1)}$ is said to be a QQ-system on the Weyl orbit if these Q-functions satisfy (1.31). Ensemble of Q-vectors $Q_{(a)}$ is said to be an extended Q-system if they satisfy all the fusion, quantisation, and projection relations introduced in the subsection above.

**Theorem 1.** For any generic enough choice of the functions $Q_{(a),1}$, $a = 1, \ldots, r$, there exists the unique, up to symmetries, QQ-system on the Weyl orbit containing $Q_{(a),1}$.

**Theorem 2.** For any QQ-system on the Weyl orbit, there exists the unique extended Q-system containing this QQ-system as its part.

Let us explain what does “up to symmetries” mean. For $g$ – any G-valued periodic function of the spectral parameter – transformation $Q_{(a)} \rightarrow g Q_{(a)}$ is a symmetry of the Q-system. By the condition of Theorem 1, $Q_{(a),1}$ are fixed which restricts the symmetry to the unipotent radical $N = [B, B]$. Hence, when computing the Weyl orbit QQ-system, we should look for solutions modulo the transformations $Q_{(a)} \rightarrow g Q_{(a)}$ with periodic functions $g$ that take values in $N$. Once the Weyl orbit QQ-system is fixed, there is no residual symmetry left; Theorem 2 implies that further extension to the full extended Q-system has no ambiguities.

**Proof of Theorem 1** \(^{19}\). Let the total number of unknown Q-functions on the Weyl orbit be $\#_{\text{unkn}}$. Consider an explicit algorithm that selects and solves a subset of $\#_{\text{unkn}}$ equations from (1.31) to compute the unknown Q-functions. Namely, each equation (1.31) relates

\(^{19}\) Certain technical aspects of the proof will be better clarified in the sequel of this paper where we plan to present them in a unifying setting covering also non-simply-laced cases.
four functions (five in the case of the bifurcation node of the Dynkin diagram). In the algorithm, one considers an equation with all but one already computed Q-functions to compute the remaining one, and proceeds recursively. Existence of such a recursion to compute all the Q-functions is a consequence of the results in [MV05].

Most steps of the recursion are straightforward where we fix an unknown Q-function as

\[ Q = \text{A rational combination of already fixed Q-functions (probably with some shifts).} \]

However, precisely on \( \dim N \) occasions one encounters equation

\[ W(Q_a, Q_b) = \prod_{c \in \{c_1, c_2, \ldots \}} Q_c, \quad (3.9) \]

where \( Q_b \) is the unknown. It is solved as follows. We fix some large enough integer \( R \) and write solution as

\[ Q_{[2n+p]}^{[2n+p]} = Q_{[2n+p]}^{[2n+p]} \left( \sum_{-m \leq k < n} \left( \frac{\prod_{c \in \{c_1, c_2, \ldots \}} Q^+_c}{Q_a^2 Q_a} \right)^{[2k+p]} + \frac{Q_{[2R+p]}^{[-2R+p]}}{Q_a^2 Q_a^{[2R+p]}} \right), \quad (3.10) \]

where \( n \) is an integer \( n > -R \) and \( p = 0 \) or \( 1 \). The term \( \frac{Q_{[2R+p]}^{[-2R+p]}}{Q_a^2 Q_a^{[2R+p]}} \) should be viewed as an integration constant, we can set it to any value using the residual symmetry of the problem (this is “up to symmetries” part of the theorem).

We therefore see that all \( Q_{[m]}^{[m]} \) for each \( a \) and \( m \) shall be considered as independent variables that assume certain numerical values – the input to the system of equations (1.31). “Generic enough choice” of \( Q_{[a],1} \) means that the denominators in the encountered rational combinations do not vanish, \( i.e. \) that \( Q_{[a],1} \) take values in a Zariski-open set.

Consequently, any relation between Q-functions – \( e.g. \) yet unused equations from (1.31) – becomes of type

\[ \text{Rational function of } Q_{[a],1} = 0. \quad (3.11) \]

It suffices to show that this rational function vanishes on a dense set to conclude that it is identically zero. To this end consider a generalisation of (2.1):

\[ \left( \frac{d}{dx} + \sum_{i=1}^r f_i \left( x/z^{1/hM} \right) H_i + \sum_{i=1}^r g_i \left( x/z^{1/hM} \right) E_{\alpha_i} + h \left( x/z^{1/hM} \right) (x^{hM} - z) E_{\alpha_0} \right) \Psi = 0. \quad (3.12) \]

This generalisation with dexterously chosen \( f_i, g_i, h \) was for instance used in [MR18, MR20]
to describe excited states of the quantum $\hat{g}$-KdV model, see also [BLZ03, Fio05, Car19].

The generalised equation still enjoys symmetry (2.3). Hence, if its asymptotic large-$x$ behaviour, at least in the relevant directions, coincides with the one of the original equation (2.1), all the equations satisfied by the defined by (2.10) Q-vectors and as a consequence of the WKB analysis will hold. In particular (1.31) will hold. There are $2r+1$ functions $f_i, g_i, h$ which can be be used as a functional freedom to engineer Q-vectors. However, $r$ of them can be fixed using gauge transformations that do not spoil the structure of the equation and one can be absorbed by a reparameterisation of $x$. Hence the actual functional freedom to non-trivially modify the Q-system are $r$ functions which is precisely what we need to vary $Q_{(a),1}$ and form a dense set.

If Q-functions are holomorphic functions of the spectral parameter $z$ then there should exist such its values $z_0$ (forming a Zariski-open set in fact) that Q-functions are generic if evaluated at any point of a vicinity of $z_0$. Afterwards, Q-functions can be analyticly continued outside the mentioned vicinities and the analytic continuation may reveal poles or other singularities for instance branch cuts; a scenario with brunch cuts is realised in AdS/CFT integrable systems. A typical requirement to impose that Q-functions describing spectrum of a physical model have singularities only of special type and at special values of the spectral parameter, see Section 4.3 for explicit examples.

**Proof of Theorem 2.** For $A_r$ case, all the Q-functions are already on the Weyl orbits. For the other algebras, it suffices to present an algorithm to compute all the extended Q-functions from the Q-functions on the Weyl orbit. Then we can represent any fusion, quantisation, or projection relation in the form (3.11) and use the same argumentation based on (3.12) to conclude that any such relation is identically satisfied.

For $D_r$ algebras, all the Q-functions of the vector and both spinor representations are on the Weyl orbit. The other Q-functions can be computed via the Wronskian determinant (5.11).

For $E_6$, all the Q-functions of the two 27-dimensional representations are on the Weyl orbit. Explicit ways to compute the other Q-functions are presented in Section 6.1

For $E_7$, all the Q-functions of the 56-dimensional representation are on the Weyl orbit. Explicit ways to compute the other Q-functions are presented in Section 6.2.

For $E_8$, the smallest non-trivial representation is the adjoint representation. In this representation, zero weight vectors (Cartan subalgebra) are not on the Weyl orbit, but the Cartan subalgebra Q-functions can be computed using (6.20), see explanation that follows this equation. From Q-functions of the adjoint representation, all the other Q-functions are computable using (6.19).

Note that all the extended Q-functions are computed polynomially from the Weyl orbit Q-function, no divisions are encountered. Hence Theorem 2 does not require generic position assumptions.
3.3 Fused flag

Recall some basic facts about compact homogeneous spaces (see e.g. [FH04] par. 23.3). These spaces are of the form $G/P$, where $P$ is a parabolic subgroup. Parabolic subgroups are defined as the ones containing a Borel subgroup $B$. In the following $B$ is assumed to be fixed. The set of all $P$’s containing $B$ is partially ordered by inclusion:

$$B \equiv P_0 \subset \ldots \subset P_{a_1a_2a_3} \subset P_{a_1a_2} \subset P_a \subset P_0 \equiv G,$$

where the Lie algebra of $P_{a_1...a_k}$ is generated by the Cartan generators, the raising operators $E_{a_i}$ for all simple roots $a_i$, and by the lowering operators $E_{-a_i}$, such that $a \neq a_i$, $i = 1, \ldots, k$. In particular, the proper maximal parabolic subgroups of $G$ are $P_a$, $a = 1, \ldots, r$.

If $G = \text{SL}(n)$, $G/P_a$ is the Grassmannian manifold $\text{Gr}(a, n)$.

A concrete way to realise $G/P$ is by considering a representation whose highest-weight eigenspace is invariant under action of $P$. Then $G/P$ is the orbit of the highest-weight vector under action of $G$ in the representation space considered projectively (i.e. up to normalisations). In the case of $P_a$, the minimal such representation is the $a$’th fundamental representation. Let vectors of this representation have components $V_{(a),i}$, for $i = 1, 2, \ldots, \dim L(\omega_a)$.

We call $V_{(a),i}$ the extended Plücker coordinates\footnote{For $\text{GL}(n)$, these are normal Plücker coordinates. In the works [FZ98, GS87] the name “generalised Plücker coordinates” refers to $V_{(a),i}$ with $i$ being only on the Weyl orbit of the highest-weight vector. This orbit is also important for us, cf. (1.31). The generalised Plücker coordinates are used to identify the Bruhat cell to which a given point of $G/P_a$ belongs to but, in contrast to the extended coordinates, they are not sufficient to identify the point uniquely.} if they are the coordinates of the $G$-orbit of the highest-weight vector. They are projective coordinates

$$[V_{(a),1} : V_{(a),2} : \ldots : V_{(a),\dim L(\omega_a)}]$$

(3.14)

that define embedding of $G/P_a$ into $\mathbb{P}L(\omega_a)$.

Consider now the minimal parabolic subgroup which is the Borel subgroup $B$ itself. In this case, the compact homogeneous space $G/B$ is called the complete flag manifold (in the following, simply flag manifold). To describe this space, one considers the orbit of the highest weight vector in $L(\rho)$, where $\rho = \sum_{a=1}^{r} \omega_a$ is the Weyl vector. It is also practical to embed this orbit into a bigger representation $L(\omega_1) \otimes L(\omega_2) \otimes \ldots \otimes L(\omega_r)$ because the latter is naturally parameterised by the products $\prod_{a=1}^{r} V_{(a),i_a}$, for all tuples $i_1 \ldots i_r$. When we are on the highest-weight orbit, these products are in (projective) one-to-one correspondence with the sets of Plücker coordinates (3.14) and so we can use $V_{(a),i}$ for all $a$ and the corresponding all $i$ to parameterise flags. By the same logic, we can use components of $V_{(a)}$ for $a \in \{a_1, \ldots, a_k\}$ to parameterise partial flags – points of $G/P_{a_1...a_k}$.

Extended Plücker coordinates satisfy (generalisation of) the Plücker relations which can be obtained as follows. Consider some set $A$ composed from (possibly repeating) numbers $1, 2, \ldots, r$. Consider the decomposition into irreps of the following tensor product

$$\bigotimes_{a \in A} L(\omega_a) = L(\omega_{\max} = \sum_{a \in A} \omega_a) + \bigoplus_{\omega < \omega_{\max}} L(\omega).$$

(3.15)
Then, for $V_{(a)}$ being the Plücker coordinates of the maximal flag, it must hold

$$
\left( \bigotimes_{a \in A} V_a \right)_{L(\omega)} = 0 \quad \text{if} \quad \omega < \omega_{\text{max}}. 
$$

(3.16)

Indeed, this is obviously true for the highest-weight vector and therefore also true for any vector in the $G$-orbit.

The Plücker relations (3.16) form an ideal in $\mathbb{C}[V_{(a)},i]$. By the Hilbert basis theorem, one needs only finitely many of them to generate all the rest. The flag manifold can be also identified as all such $V_{(a),i}$ for which (3.16) hold.

**Fused flag** is defined as follows: Consider the embedding of the complete flag manifold $G/P$ also identified as all such

Then a fused flag is a set of maps $22 Q_{(a)} : \Sigma \to G/P_a$, where $\Sigma$ is the space of spectral parameter, such that

$$
Q^{[p_1]}_{(1)} \times Q^{[p_2]}_{(2)} \times \ldots \times Q^{[p_r]}_{(r)} \in G/B. 
$$

(3.18)

for any Coxeter height function $p$ defined on page 23. For instance (3.18) should hold for an alternating pattern $p_a = 0$, where $a$ are even nodes, $p_a = 1$, where $a$ are odd nodes; but also for e.g. increasing patterns like $(p_1,p_2,p_3) = (0,1,2)$ for the $A_3$ case.

**Lemma 3.1.** The maps $Q_{(a)} : \Sigma \to G/P_a$ define a fused flag if and only if $(Q_{(a)},Q^\pm_{(a'),b}) \in G/P_{aa'}$ for all adjacent nodes $a,a'$ of the Dynkin diagram.

**Proof.** The statement is proven by induction using Lemma 3.2

**Lemma 3.2.** Let $A,B$ are two non-intersecting sets of Dynkin diagram nodes and $c$ is the node not belonging to $A$ or $B$. Denote by $x$ a point in $G/P_A$, by $y$ a point in $G/P_B$, and by $z$ a point in $G/P_c$. If $(x,z)$ belongs to $G/P_{Ac}$ and $(y,z)$ belongs to $G/P_{Bc}$ then $(x,y,z)$ belongs to $G/P_{ABC}$.

**Proof.** Because all properties can be viewed as defined via polynomial equations (3.16), it is enough to prove the statement for a dense set of points $x,y,z$. We have $(x,z) = g_1 \cdot (x_0,z_0)$ and $(y,z) = g_2 \cdot (y_0,z_0)$, where $x_0,y_0,z_0$ are the points of the standard partial flags (corresponding to the highest-weight vectors) and $g_1,g_2$ some group elements. We know that $g_1 \cdot z_0 = g_2 \cdot z_0$ and hence $g_1^{-1} g_2 \in P_c$. A dense set of elements of $P_c$ can be represented as $\prod_{a \neq c} c_a E^{-\alpha a} b$, where $b \in B$ and $c_a$ are complex numbers. By ordering $\alpha$ such that $\alpha_a$ with $a \in A$ are to the right compared to $\alpha_a$ with $a \in B$ we conclude that $\pi_c = \pi_A \pi_B$ for $\pi_c \in P_c$ from this dense set and some $\pi_A \in P_A$, $\pi_B \in P_B$. Hence we can write $(x,y,z) = g(x_0,y_0,z_0)$ for $g = g_1 \pi_A = g_2 \pi_B^{-1}$. 

---

22By slightly abusing notation we identify the map with the corresponding Plücker coordinates $Q_{(a)}$ that depend on the spectral parameter. We do not identify them yet with $Q$-vectors of an extended $Q$-system.
It is clear, by direct pattern recognition, that the projection properties (3.7) are instances of the Plücker relations (3.16). The next statement establishes that all the Plücker relations are encoded into the $Q$-system:

**Theorem 3.** $Q(a)$ – the $Q$-vectors of an extended $Q$-system – are Plücker coordinates of a fused flag.

**Proof.** It is easy to establish using the WKB analysis that (3.7) holds in the case $n_i - n_i' = \pm 1$, same sign for all $i, i'$ such that $a_i$ is any even node and $a_{i'}$ is any odd node. This relation implies that $Q_{(a_i)}, Q_{(a_{i'})}^\pm$ are Plücker coordinates of $G/P_{a_i a_{i'}}$. Then use Lemma 3.1. \qed

A fused flag shall be called non-degenerate if, for all $a$ and any $k$ and $n_1, \ldots, n_k$, the Plücker vectors $Q_{(a)}^{[n_1]}, Q_{(a)}^{[n_2]}, \ldots, Q_{(a)}^{[n_k]}$ span a vector space of maximal possible dimension provided the fused flag condition is satisfied. In the explicit physical systems that were studied and where $Q$-functions holomorphically depend on the spectral parameter, the non-degeneracy holds for all but a finite set of spectral parameter values. These values of the spectral parameter are related to the inhomogeneity parameters of the spin chain, they are part of the input information about the system allowing to fix its spectrum.

An interesting question arising is whether being a non-degenerate fused flag implies all the other relations between $Q$-functions. Using the dense set argument of Theorem 1, we can give a positive answer if we can find an algorithm to generate all the $Q$-functions from $Q_{(a), 1}$ using the fused flag properties only. We can show that the fused flag condition implies (1.29) and hence we can reproduce the Weyl-orbit $QQ$-system from the fused flag. Equation (1.29) allows also computing the extended $Q$-system for all cases except for $E_8$ because the latter does not have an irrep with all components being on the Weyl orbit, and we are not aware how to derive the fusion property (6.20) using only fused flag properties. Hence, for the $E_8$ case, we are not certain whether each non-degenerate fused flag is an extended $Q$-system, however we conjecture that it is.

### 3.4 Opers

To define opers properly, one needs to work on the level of principal $G$-bundle over $\Sigma$ (space of spectral parameter). But to simplify exposition and to be on the same level of formalisation as the other parts of the paper, we shall work locally and in a certain gauge (equivalently, in a certain trivialisation).

There are two objects that enter definition of an oper. The first one is a connection. In our case it is a finite-difference connection which can be thought as a $G$-valued function $U(z)$. Informally, it is the Wilson line $U(z) = P\exp \int_0^z A(z') dz'$ (though $A$ itself does not need to be defined). The second object is a $z$-dependent complete flag which we shall denote as $F(z) \in G/B$.

The finite-difference oper condition can be formulated as follows [FKSZ20]: In a gauge where $F(z)$ is a standard flag (corresponding to the highest-weight vector in the sense of Plücker coordinates) at each point $z$, the connection $U(z)$ should be an element of the
Bruhat cell $B\sigma B$, where $\sigma$ is a representative of a Coxeter element of the Weyl group. Explicitly

$$U(z) = n(z) \prod_{\alpha \in \Delta} s_{\alpha} b(z),$$

(3.19)

where $b(z) \in B$, $n(z) \in \mathbb{N}$, $s_{\alpha}$ are representatives of Weyl reflections w.r.t. to simple roots, see (2.12), and the order in which $\prod_{\alpha \in \Delta}$ is taken corresponds to the choice of a Coxeter element.

Let us understand a geometric interpretation of the oper condition. Let explicitly the product over simple roots be

$$\prod_{\alpha \in \Delta} s_{\alpha} = s_{a_{a_1}} \ldots s_{a_{a_r}},$$

(3.20)

where $a_i$ is a permutation of $(1, 2, \ldots, r)$. For a given $k$, define sets $A_k = (a_1, \ldots, a_k)$, $B_k = (a_{k+1}, \ldots, a_r)$. Define correspondingly $s_{A_k} = s_{a_{a_1}} \ldots s_{a_{a_k}}$ and $s_{B_k} = s_{a_{a_{k+1}}} \ldots s_{a_{a_r}}$. Then represent the standard complete flag as $(x_k, y_k)$, where $x_k$ is the standard partial flag of $G/P_{A_k}$ and $y_k$ is the standard partial flag of $G/P_{B_k}$. We note that $U(z)x_k = n(z)s_{A_k}x_k$ since $x_k$ is invariant under action of $B$ and $s_{B_k}$. On the other hand, $n(z)s_{A_k}y_k = y_k$. Hence we conclude that

$$(U(z)x_k, y_k) \in G/B, \quad \text{for } k = 0, 1, \ldots, r.$$ 

(3.21)

That is we can parallel-transport using $U$ only a special subset of Plücker coordinates, $V_{(a_1)}, \ldots, V_{(a_k)}$, and still remain in the maximal flag.

The argumentation to derive (3.21) from (3.19) can be reversed if we require general position: all the points $(U(z)x_k, y_k)$ should be distinct. More precisely (3.21) in general position identifies the Bruhat cell which $U(z)$ belongs to, i.e. it asserts that $U(z)$ is of the form (3.19).

In a gauge where the flag $F(z)$ is standard, all information about the oper is concentrated in the connection $U(z)$. Let us now perform a gauge transformation to make the connection trivial $U(z) = \text{Id}$. In this gauge, all information is transferred to the flag $F(z)$.

Remarkably, the oper condition in this gauge can be rewritten as the one of a fused flag. Indeed, let $V_{(a)}$ be the Plücker coordinates of $F(z)$ in this new gauge. Parallel transport with respect to connection $U$ does not change them: $V_{(a)}^{\text{pt}}(qz) = V_{(a)}(z)$. On the other hand, $V_{(a)}(z)$ as functions of $z$ are non-trivial. Property (3.21), together with the obvious $(x_k, y_k) \in G/B$, becomes in the new gauge

$$(V_{(a_1)}^-, \ldots, V_{(a_k)}^-, V_{(a_{k+1})}^\pm, \ldots, V_{(a_r)}^\pm) \in G/B, \quad \text{for } k = 0, 1, \ldots, r.$$ 

(3.22)

Now, recall that one can assign the Coxeter height function $p$ to the Coxeter element (3.20),

\[\text{(3.20) It is some arbitrary order, not necessarily the order in which Dynkin diagram is conventionally labelled.} \]

\[\text{(3.24) We can potentially spoil some nice analytic structure in this way but we do not loose information.} \]
see Section 2.3. Using this function, identify
\[ V_{(a)} = Q_{(a)}^{[p_a]} . \] (3.23)

Condition (3.22) ensures that \( Q_{(a)} \) satisfy conditions of Lemma 3.1 and hence define a fused flag.

We hence see that a non-degenerate fused flag is an oper in a particular gauge. A fused flag can be also gauged (in a special way, procedure is analogous to that of [KLV16] for \( sl(n) \) case). A gauged non-degenerate fused flag, i.e. gauged extended Q-system, is hence an equivalent of a finite-difference oper. There is however an interesting caveat. The definition of an oper involves a choice of the Coxeter element, and then one has to separately show that different choices are gauge-equivalent. In contrast, the fused flag does not require to make this choice. This choice is being made only when we link the fused flag and the oper. Namely, one has to choose one particular \( p_a \) among all possibilities in (3.18) and declare that Plücker coordinates (3.23) are the ones that define \( F(u) \) of an oper.

ODE/IM provides us an interesting connection between the extended Q-systems and opers. Using the WKB asymptotics (2.8) we can expect that the following Wilson line in the \( x \)-plane\(^{25}\) connects \( \Psi \)-function at the origin and the infinity
\[ Q_{(a)}^{[p_a]}(z) = z^{-\frac{\rho^\vee}{2\pi}} \left( \lim_{x_0 \to \infty} Pe^{\int_{x_0}^{x_0} A(x',z)dx'} e^{-\frac{M\rho^\vee}{M+1}x_0} \right) U_{(a)}^{[p_a]} . \] (3.24)

Here, we remind, \( U_{(a)}^{[p_a]} \) are eigenvectors in \( \Lambda \) with eigenvalue \( \gamma^{p_a/2} \mu_a \), for \( a = 1, \ldots, r \). As explained in Section 2.3, they are highest-weight vectors in a basis where \( \Lambda \) belongs to a Cartan subalgebra \( h' \) and for a specific choice of simple roots. Hence, in this basis, they are Plücker coordinates of the standard flag.

Furthermore, using (2.3) and (2.26), we can compute that
\[ Q_{(a)}^{[p_a]}(qz) = z^{-\frac{\rho^\vee}{2\pi}} \left( \lim_{x_0 \to \infty} Pe^{\int_{x_0}^{x_0} A(x',z)dx'} e^{-\Lambda x_0^{M+1}} \right) \gamma^{\rho^\vee} U_{(a)}^{[p_a]} . \] (3.25)

Recall that \( \gamma^{\rho^\vee} \) is a Coxeter element in the same basis where \( U_{(a)}^{[p_a]} \) define the standard flag. We hence can view this Wilson line as a gauge transformation from a fused flag gauge (where connection \( U(z) \) is trivial) to the standard flag gauge (where \( U(z) \) is a Coxeter element).

This very plausible explanation has however a drawback. The limit \( x_0 \to \infty \) is ill-defined, in particular due to Stokes phenomena. At this moment, the best we can do is to declare \( \left( \lim_{x_0 \to \infty} Pe^{\int_{x_0}^{x_0} A(x',z)dx'} e^{-\Lambda x_0^{M+1}} \right) \) to be such a group element depending on \( z \) that (3.24) and (3.25) hold. It would be interesting to provide an intrinsic self-consistent definition of this Wilson line.

\(^{25}\)Not to confuse with informal Wilson line in the interpretation of \( U(z) \), these are entirely different objects!
To conclude this section, we mention other types of oper mentioned in the literature. The connection of (2.1) is an example of an affine oper. Miura oper [Fre03, Fre04] is an oper (finite-difference in our case) with an extra data allowing to select highest-weight functions $Q^{(a),1}$. Miura-Plücker oper [FKSZ20] is an oper with an extra data to select both $Q^{(a),1}$ and $Q^{(a),2}$ that satisfy the QQ-relation (1.30) [FH18].

4 Applications

4.1 Solving Hirota equation

In this subsection we provide a solution to the so-called Y- and T-system in terms of Q-functions. These systems were considered for all simple Lie algebras, see [KNS11] for a review and references therein, we focus on the simply-laced cases only.

Y-system appears in the context of thermodynamic Bethe Ansatz. It is a collection of $Y_{a,s}$, where $a$ run through the nodes of the Dynkin diagram and, depending on the model, $s \in \mathbb{Z}$ or $s \in \mathbb{Z}_{\geq 0}$. For simply-laced case, these functions satisfy the following condition

$$Y_{a,s}^+ Y_{a,s}^- = \frac{\prod_{b=1}^{r} (1 + Y_{b,s})^{I_{ab}}}{(1 + Y_{a,s-1})(1 + Y_{a,s+1})}. \quad (4.1)$$

Upon substitution $Y_{a,s} = \prod_b T_{b,s}^{I_{ab}}$, one obtains the Hirota equation (T-system)

$$T_{a,s}^+ T_{a,s}^- - T_{a,s+1} T_{a,s-1} = \prod_b T_{b,s}^{I_{ab}}. \quad (4.2)$$

Apart from appearing in TBA, T-functions have also interpretation as transfer matrices with auxiliary space being a Kirillov-Reshetikhin module labeled by $a, s$.

Similarity in structure of (4.2) and (1.29) is very suggestive. Using the $\mathfrak{sl}(n)$ solution (1.25) as a further insight, it is then not difficult to guess the following ansatz for T-functions

$$T_{a,s} = (Q^{[s]}_{(a)}, \tilde{Q}^{[-s]}_{(a)}) , \quad (4.3)$$

where $Q$ and $\tilde{Q}$ are two a-priori different Q-systems.

Here is a proof that this ansatz indeed solves (4.2):

$$T_{a,s}^+ T_{a,s}^- - T_{a,s+1} T_{a,s-1} = \left\langle \left( \bigotimes_{b} Q_{(b)}^{I_{ab}} \right)^{[s]} , \left( \bigotimes_{b} \tilde{Q}_{(b')}^{I_{ab}} \right)^{[-s]} \right\rangle_{L(\omega_{max})} = \left\langle \left( \bigotimes_{b} Q_{(b)}^{I_{ab}} \right)^{[s]} , \left( \bigotimes_{b} \tilde{Q}_{(b')}^{I_{ab}} \right)^{[-s]} \right\rangle_{L(\omega_{max})} = \prod_b T_{b,s}^{I_{ab}} , \quad (4.4)$$
where we used (1.29) and, notably, the following projection relations of the Q-functions

\[
\left(Q^+_\alpha(h) \land Q^-_{\alpha(h)}\right)_{L(\omega)} = 0, \quad \left(\prod_b Q^{I_{ab}}\right)_{L(\omega)} = 0, \quad \text{for all } \omega < \omega_{\text{max}} = \sum_b \omega_f^{I_{ab}}.
\]  

(4.5)

There are cases when the T-system has a boundary. For instance, one has \( s \geq 0 \) in the transfer matrix interpretation and moreover one fixes \( T_{a,0} = 1 \) since these functions have meaning of the transfer matrices in the trivial representation. In addition, Hirota equation should make sense for \( s = 0 \) if one sets \( T_{a,-1} = 0 \).

These features can be reproduced if we identify \( Q \) and \( \tilde{Q} \). After slight redefinitions, one sets

\[
T_{a,s} = \langle Q_{(\alpha)}^{s+\frac{b}{2}}, Q_{(\alpha)}^{-s-\frac{b}{2}} \rangle.
\]  

(4.6)

This ansatz solves (4.2) and it has the following additional properties: \( T_{a,0} = 1 \) which is the quantisation relation (3.6) and moreover \( T_{a,s} = 0 \) for \( s = -1, -2, \ldots, 1 - h \), this is the projection relation (3.8).

### 4.2 Character solution

Choose an element of the Cartan algebra \( H \) and consider the ansatz

\[
Q_{(\alpha)} = z^H A_{(\alpha)},
\]  

(4.7)

where \( A_{(\alpha)} \) are vectors that do not depend on the spectral parameter. For this ansatz \( Q_{(\alpha)}^{[2]} = q^H Q_{(\alpha)} \) and hence all equations on \( Q \)-functions reduce to polynomial equations on \( A_{(\alpha)} \). A good parameterisation for \( H \) is \( H = \sum_{i=1}^{r'} H_i \log x_i \) in which case the coefficients of these polynomial equations are Laurent polynomials in \( x_i \) with integer coefficients. It is pertinent to choose \( H_i \) as generators in the orthogonal basis, see e.g. [FKS20]. For \( \mathfrak{g}(r+1) \), this is actually the basis of \( \mathfrak{g}(r+1) \) meaning that \( r' = r + 1 \) and that the \( Q \)-vector should not be sensible to shifts \( H_i \rightarrow H_i + C \) which is achieved by setting \( \prod_i x_i = 1 \). For \( \mathfrak{so}(2r) \) case, the orthogonal basis is explicitly described in Section 5.

A solution of equations for \( A_{(\alpha)} \) exists always as we can conclude based on the following two facts: First, Theorems 1 and 2 ensure that the extended \( Q \)-system exists for any \( Q_{(\alpha),1} \) and hence for \( Q_{(\alpha),1} = z^{\omega_{\alpha}(H)} A_{(\alpha),1} \). Second, all the QQ-relations have multiplicative nature, cf. (3.4), implying that the analytic dependence of other Q-functions on \( z \) can be only of the form (4.7) if we start from \( Q_{(\alpha),1} = z^{\omega_{\alpha}(H)} A_{(\alpha),1} \). Furthermore, we see that the solution is unique if we fix values of \( A_{(\alpha),1} \). Indeed, while generically a solution is defined modulo symmetry \( Q_{(\alpha)} \rightarrow g Q_{(\alpha)} \), the only group elements commuting with \( z^H \) are elements of the maximal torus whose action amounts in rescaling of \( A_{(\alpha),1} \) assumed to be fixed.

For any \( G \)-invariant combination \( S(Q) \) of \( Q \)-functions one has \( S^{[2]} = q^H S = S \). Hence all such combinations are independent of \( z \). Furthermore, for any group element \( g \), \( S(Q) = S(gQ) \). On the other hand, \( gQ = z^{gH} g^{-1} g A_{(\alpha)} \) can be interpreted as the solution (4.7)
of the Q-system with $H \to g H g^{-1}$. We hence conclude that combinations $S(Q)$ are class functions of $q^H$ considered as a group element.

In particular, $T_{a,s}$ computed by (4.6) are class functions. Since they are $z$-independent and satisfy (4.2) they should be characters in the corresponding representations. While these are irreps in the case of $\mathfrak{sl}(r+1)$, these representations are typically reducible for the case of other Lie algebras, the reason is that they are actually irreps of the relevant quantum algebra.

We build the explicit character solution for $\mathfrak{so}(2r)$ series in Section 5.4. Explicit solution for $\mathfrak{sl}(r+1)$ is given for instance by (3.9) of [KLV16].

4.3 Analytic Bethe ansatz

Until now, we mostly avoided discussing the explicit analytic properties of Q-functions. Specifying these is precisely what defines the physical model we are dealing with. In this section we propose analytic structure of Q-function that is supposed to describe rational, trigonometric, and elliptic spin chains. The story is a fairly straightforward generalisation of what was done for the $A_n$ case.

To give a uniform presentation, we shall use additive spectral parameter $u$ and agree to relate it to $z$ by $e^{2\pi u} = z$, correspondingly $e^{2\pi \hbar} = q$. The spectrum of spin chains should be described by Bethe equations

$$\prod_{\ell=1}^{L} \frac{\phi(u_{a,k} - \theta_\ell + \frac{1}{2} m^{\ell}_a)}{\phi(u_{a,k} - \theta_\ell - \frac{1}{2} m^{\ell}_a)} = - \prod_{b} e^{2\pi \hbar h_{b}} C_{ab} \prod_{k'=1}^{M_b} \frac{\phi(u_{a,k} - u_{b,k'} + \frac{1}{2} C_{ab})}{\phi(u_{a,k} - u_{b,k'} - \frac{1}{2} C_{ab})},$$

(4.8)

where $\phi(u) = u$ for the rational case, $\phi(u) = \sinh(2\pi u)$ for the trigonometric case, and $\phi(u) = \sigma(u)$ for the elliptic case. $\partial^2_a \ln \sigma(u) = -\varphi(u)$. For the trigonometric case, it is assumed that $h$ is not rational and $q$ is not a root of unity. For the elliptic case $h$ is not commensurate with the periods 1, $\tau$. $C_{ab}$ is the Cartan matrix of the Lie algebra $\mathfrak{g}$. We note that in non-simply-laced case this should be the symmetrised Cartan matrix [OW86]; ODE/IM approach to reproduces such Bethe equations after twisting of the affine algebra [MRV17].

For the rational and the trigonometric cases, the physical meaning of the remaining parameters are: $[m^{\ell}_{1}, \ldots, m^{\ell}_{r}]$ are Dynkin labels of the representation assigned to the $\ell$th node (it is the quantum algebra irrep but generically it might be reducible as a representation of the Lie algebra $\mathfrak{g}$); $h_{b} := \omega_{b}(H)$, where $H$ is the same as in (4.7), specify twisted boundary conditions of the spin chain; and $\theta_\ell$ are inhomogeneities. For the elliptic case, there are certain restrictions on admissible values of $M_{b}$ since $\phi$ are not periodic functions on the torus, also physical models in the elliptic case were not built to the same level of generality as in the case of rational and trigonometric systems.

From the character solution (4.7), Bethe equations (1.9), and using experience with $A_{r}$ system, it is natural to guess the following ansatz for Q-functions

$$Q_{(a),i}(u) = N_{(a),i} \times A_{(a),i} \prod_{j=1}^{r'} \left[ x_{j}^{\frac{2\pi i}{x a} \gamma_{j}(H_{j})} \times \sigma_{a}(u) \times q_{(a),i}(u) \right].$$

(4.9)
It is split into four factors. The first factor is a number whose sole purpose is to adjust normalisation such that there is an equality sign in (1.29). It has no physical importance. The second factor is the character solution (4.7) $e^{2\pi u H} A_1$, we just wrote it in components. The aim of the third factor is to reproduce the l.h.s. (source term) in Bethe equations (1.9). We shall study it in a moment. Finally, the last factor is the Baxter polynomial

$$q_{(a),i}(u) = \prod_{k=1}^{M_{(a),i}} \phi(u - u_{(a),i,k})$$

(4.10)

which, in the rational case, was featured in the original works of Baxter. Zeros of $q_{(a),1}$, $u_{a,k} \equiv u_{(a),1,k}$ satisfy conventional Bethe equations (1.9). Zeros of $q_{(a),\sigma(1)}$, where $\sigma$ is an element of the Weyl group, satisfy Weyl-dual Bethe equations.

The dressing factor $\sigma_a$ does not depend on $i$ which reflects the fact that all the Weyl-dual Bethe equations have structurally the same source term. By recalling that the Bethe equations in terms of $Q$-functions are written as (1.28) and requiring that the l.h.s. of (4.9) is reproduced from $\sigma_a$ when we substitute the Ansatz (4.9) into (1.28), one gets the following equation on $\sigma$

$$\sum_b [C_{ab}]_D \log \sigma_b = - \sum_{\ell=1}^L [m_{ab}^\ell]_D \log \phi(u - \theta_\ell),$$

(4.11)

where we used the shift operator $D = e^{\frac{\hbar}{2} \theta_n}$ and notation for “D-deformed” numbers $[n]_D := D^{n-1} + D^{n-3} + \ldots + 1$ for $n > 0$, $[n]_D := -[-n]_D$ for $n < 0$. Equation (4.11) is formally solved by

$$\log \sigma_a = - \sum_b \sum_{\ell=1}^L \left([C]^{-1}_D \right)_{ab} [m_b^\ell]_D \log \phi(u - \theta_\ell),$$

(4.12)

one can also provide a precise meaning for this formal solution, cf. [Vol11],

We see that the dressing factors are, in a sense, the inverse deformed Cartan matrices describing interaction between Bethe roots and source terms which is reminiscent of integrable relativistic integrable models where the dressing factors are the same inverse deformed Cartan matrices describing interactions between particles [Zin98].

The twist factor does not have nice periodicity properties to be considered as a function on a cylinder or torus. To improve on this issue, we can perform the gauge transformation

$$Q_{(a)}^{gt} = e^{-2\pi \frac{u}{H}} Q_{(a)}, \quad U^{gt} = e^{-2\pi (u+\frac{\hbar}{H})} e^{2\pi u H} = e^{-2\pi \hbar H}.$$

(4.13)

In the new gauge, $Q$-functions have better analytic properties, and we get a non-trivial constant finite difference connection $U$.

The dressing factor is also not a particularly pleasant function of the spectral parameter. On a torus, we most likely won’t be able to define it at all, as this would be an object
with everywhere dense set of poles. However we should recall that Q-functions are projective coordinates meaning that dressing factor nearly cancels out from the map. Its main value is the position of poles and zeros that determine precisely where and how the fused flag fails to be non-degenerate. Consider for instance the QQ-relation (1.29). Explicitly in terms of $q$ and in the case $H = 0$ it becomes

$$
\left( q^+_{(a)} \wedge q^-_{(a)} \right)_{L(\omega_{\text{max}})} = J_a \times \left( \bigotimes_b q^b_{(b)} \right)_{L(\omega_{\text{max}})},
$$

$$
J_a = \prod_b \sigma^a_{b} \prod_{\ell=1}^{L} \prod_{k=-\frac{m_b-1}{2}}^{\frac{m_b-1}{2}} \phi(u - \theta_{\ell} + k \hbar). \quad (4.14)
$$

Remembering also the relation $\left( q^+_{(a)} \wedge q^-_{(a)} \right)_{L(\omega<\omega_{\text{max}})} = 0$ which holds always, we see that vectors $q^+_{(a)}$ and $q^-_{(a)}$ are collinear at $u = \theta_{\ell} + k \hbar$ with $k$ being in the specified by (4.14) range.

Curiously, while the extended Q-system must be a non-degenerate fused flag almost everywhere, the prescription of degeneration points is an essential ingredient for selecting physically relevant solutions.

**Completeness and faithfulness conjectures** Based on the results established for the $A_r$ spin chains in the vector representation [MTV13, CLV20] we conjecture that the extended Q-system is always the right object to correctly encode the spectrum of the corresponding integrable model. In contrast, we know already for the $A_r$ case that Bethe equations (1.9) have shortcomings.

The completeness conjecture is that the algebraic number of extended Q-systems that verify analytic structure (4.9) is equal to the dimension of the corresponding weight subspace of the Hilbert space. The number of Q-systems should be computed modulo residual symmetries. For generic twist $H$, only action of Cartan subalgebra $h \in g$ is a symmetry (it only changes normalisations and hence inessential), while for zero twist $H = 0$, rotation by any element of $g$ is a symmetry. The weight subspace is defined as a space of highest-weight vectors with respect to action of residual symmetry\footnote{if symmetry is only Cartan, then it is simply a space of vectors of given weight}, of weight with Dynkin labels $[d_1, \ldots, d_r]$ computed as

$$
|d_a| = \sum_{\ell=1}^{L} m^a_{\ell} - \sum_{b=1}^{r} C_{ab} M_b. \quad (4.15)
$$

We performed a verification of the completeness conjecture by an explicit computation for $\mathfrak{so}(8)$ rational spin chain, this result will be published separately [EV].

The faithfulness conjecture is that the Bethe algebra, a certain commutative algebra containing physical Hamiltonians, restricted to the weight subspace is isomorphic to the algebra of Q-functions of analytic form (4.9) and satisfying equations of the extended Q-system.
In rational and trigonometric cases the algebra of $Q$-functions can be considered as a polynomial quotient ring whose variables are coefficients of Baxter polynomials. At least in these cases, isomorphism to the Bethe algebra also implies that the spectrum of the Bethe algebra restricted to the weight subspace is simple.

5 $D_r$ series

We use the same notation as in section 2.2 for vectors, spinors and co-spinors

$$\langle Q_{(1)} \rangle_i = V_i, \quad \langle Q_{(r-1)} \rangle_{\alpha} = \psi_{\alpha}, \quad \langle Q_{(r)} \rangle_{\tilde{\alpha}} = \eta_{\tilde{\alpha}}. \quad (5.1)$$

The remaining fundamental representations are antisymmetric tensors and can be written using multi-index notation, $I = \{i_1, i_2, \ldots, i_k\}$, $V_I = V_{i_1 \ldots i_k}$.

When labeling vectors and spinors we shall have the orthogonal basis of $\mathfrak{so}(2r)$ in the back of our mind. This is a basis spanned by $r$-dimensional vectors $\{\varepsilon_i\}_1$ with inner product $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$. The simple roots of $\mathfrak{so}(2r)$ are expressed in this basis as

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad \alpha_r = \varepsilon_{r-1} + \varepsilon_r, \quad (5.2)$$

so that $\langle \alpha_i, \alpha_j \rangle = C_{ij}$ with $C_{ij}$ being the Cartan matrix of $\mathfrak{so}(2r)$. We can assign to each component $V_I$ Dynkin labels, for example $[10 \ldots 0]$ to $V_1$. Dynkin labels can be converted into the orthogonal basis using

$$[0 \ldots 010 \ldots 0] = \sum_{j=1}^i \varepsilon_j, \quad [0 \ldots 010] = \frac{1}{2} \left( \sum_{i=1}^{r-1} \varepsilon_i \right) - \varepsilon_r, \quad [0 \ldots 01] = \frac{1}{2} \sum_i \varepsilon_i. \quad (5.3)$$

For more information see any textbook on Lie algebras, e.g. [FH04]. For every component $V_i$ of the vector representation, there is another one with the negative orthogonal weight, we denote this component as $V_{-i}$. The index structure of the vector is then $V_{(1)} = (V_{1}, \ldots, V_r, V_{-r}, \ldots, V_{-1})$. The inner product of two vectors is in explicit index notation $g^{ij} V_i V_j = \sum_{r=1}^r g^{i r} V_r V_{-i}$, which means that $g_{ij}$ is an anti-diagonal matrix. When discussing the character ansatz for spinors, it will be convenient to use the notation $\zeta_I$, where $I = \{i_1, \ldots, i_k\}$ is a multi-index that keeps track of the minus signs for the spinor’s weight in the orthogonal basis. For example $\zeta_0 = \eta_1$ and $\zeta_r = \psi_1$, see (5.3). Notice that $\zeta_I$ describes $\eta_{\tilde{\alpha}}$ when the number of entries in $I$ is even and $\psi_\alpha$ when the number of entries in $I$ is odd.

In the following we would like to relate spinors and vectors for which we need the $2^{r-1} \times 2^{r-1}$ generalized Pauli matrices $(\gamma_i)_{\alpha}^\beta, (\bar{\gamma}_i)_{\alpha}^\beta$. They satisfy

$$(\gamma_i)_{\alpha}^\gamma (\bar{\gamma}_j)_{\gamma}^\beta + (\bar{\gamma}_j)_{\alpha}^\gamma (\gamma_i)_{\gamma}^\beta = g_{ij} \delta_{\alpha}^\beta. \quad (5.4)$$

These matrices can be packaged as $\Gamma_i = \begin{pmatrix} 0 & (\gamma_i)_{\alpha}^\beta \\ (\bar{\gamma}_i)_{\alpha}^\beta & 0 \end{pmatrix}$. We will write $\Gamma_{i_1 i_2 \ldots i_k} = \Gamma_{[i_1} \Gamma_{i_2} \ldots \Gamma_{i_k]}$ for the weighted antisymmetrisation of $k$ matrices. Multiplying with the
charge conjugation matrix $C$ will make $\Gamma$ either symmetric or antisymmetric: $(C \Gamma_{a_1 a_2 \ldots a_k})^T = \pm C \Gamma_{a_1 a_2 \ldots a_k}$. The index structure of $C$ is dimension dependent, we have two matrices $C_{\alpha \beta}, C_{\dot{\alpha} \dot{\beta}}$ for even rank and $C_{\alpha \dot{\beta}}$ for odd rank and use $C$ and its inverse to raise and lower indices.

For explicit computations it is possible to build $\Gamma$-matrices using a recursive algorithm. First introduce the three basic matrices

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.5}$$

From these matrices we can construct $\Gamma$-matrices as

$$\Gamma^i = \sigma^z \otimes \cdots \otimes \sigma^z \otimes \sigma^- \otimes 1 \otimes \ldots 1, \quad \Gamma^{-i} = \sigma^z \otimes \cdots \otimes \sigma^z \otimes \sigma^+ \otimes 1 \otimes \ldots 1. \tag{5.6}$$

They satisfy the Clifford algebra

$$\{\Gamma^i, \Gamma^j\} = \delta^{i,-j}, \tag{5.7}$$

defining $\gamma^{ij}$ to be a positive anti-diagonal matrix. The charge conjugation matrix can be constructed out of these matrices as

$$C = \prod_{i=1}^{r} (\Gamma^i - \Gamma^{-i}). \tag{5.8}$$

5.1 Isotropic spaces and fused flag

In the general discussion of Section 3 a geometrical structure for the extended QQ-system emerged, naturally connected to the manifold $G/B$. In this section we discuss the specific example of $\mathfrak{so}(2r)$ and how this structure relates to the parameterisation of $Q$-vectors. The structure of the system should be described using both vectors, tensors as well as spinors. For the discussion we recall that $\gamma = e^{\frac{2\pi i}{h}}$ where $h = 2r - 2$ is the Coxeter number of $\mathfrak{so}(2r)$.

We focus first on the tensor product of two vectors, the decomposition into irreps is $V_{(1)} \otimes V_{(1)} = \text{Sym}^2(V_{(1)}) + \wedge^2 V_{(1)} + 1$. Let $\mu_1$ be the maximal eigenvalue of (2.1) in the vector representation, the eigenvalue of $V^m \otimes V^{-m}$ is then $\gamma^{m/2} \mu_1 + \gamma^{-m/2} \mu_1$. We ignore the symmetric part. For $m = 1$ we get a fusion relation, see (3.4), relating $V^+_{(1)} \wedge V^-_{(1)}$ and $V_{(2)}$. In components this means

$$V_{ij} = W[V_i, V_j]. \tag{5.9}$$

The trivial representation has eigenvalue zero. Since $\mu_1 > 0$ it follows that until the factors of $\gamma^{m/2}$ and $\gamma^{-m/2}$ cancels with each other the projection to the singlet must vanish, this gives a set of projection relations and one quantisation condition:

$$V_i^{|m|} (V^i)^{|-m|} = 0, \quad m = 0, 1, \ldots, \frac{h - 1}{2}; \quad V_i^{|\frac{1}{2}|} (V^i)^{|-\frac{1}{2}|} = 1, \tag{5.10}$$
The WKB approximation breaks down for \( m > \frac{h}{2} \) but we can still define the T-functions, see Section 5.3.

Much like \( V_{ij} \) the remaining antisymmetric tensors can all be expressed using \( V_{(1)} \) as

\[
V_{i_1 \ldots i_k} = W[V_{i_1}, \ldots, V_{i_k}], \quad k \leq r.
\]  

(5.11)

For \( k = 1, \ldots, r - 2 \), these are components of \( Q \)-vectors in fundamental representations. It follows from the \( V_I \)'s explicit form that they satisfies standard QQ-relations

\[
W[V_{Ii}, V_{Ij}] = V_{Iij} V_I.
\]

(5.12)

Since \( V_{i_1 \ldots i_k} \) is written as the antisymmetrisation of \( k \)-vectors it describes a \( k \)-dimensional hyperplane. Furthermore, for \( k \leq r - 1 \) the hyperplane is spanned by vectors that all have vanishing inner product with each other. Such a hyperplane is, by definition, an isotropic hyperplane.

There is another way to view the extended QQ-system for \( \mathfrak{so}(2r) \) the complete flag \( G/B \) has the structure [FH04]

\[
G/B = \{ W_1 \subset W_2 \subset \cdots \subset W_{r-1} \subset \mathbb{C}^{2r}, (W_k, W_k) = 0 \},
\]

(5.13)

where \( W_k \) is a \( k \)-dimensional hyperplane and \( (W_k, W_k) = 0 \) means that it is isotropic. From the explicit expressions for \( V_{(a)} \) (5.11) and from the projection properties (5.10) it is seen that \( \{ V_{(p_1)}^{[r_1]}, V_{(p_2)}^{[r_2]}, \ldots, V_{(r-1)}^{[r_{r-1}]} \} \in G/B \) for all \( p_i \) such that \( p_i - p_{i+1} = \pm 1, \ i = 1, \ldots, r - 2 \). This is the structure of the fused flag for \( \mathfrak{so}(2r) \). In the next subsection we shall split \( V_{(r-1)} \) into fermions \( "V_{(r-1)} = \psi/\eta" \) and then, for the \( Q \)-functions to belong to the complete flag, the shifted fermions \( \psi^{[r-1]}, \eta^{[r-1]} \), \( p_r-1, p_r \) can be not equal as long as \( p_r - p_{r-2} = \pm 1 \) and \( p_{r-1} - p_{r-2} = \pm 1 \) are satisfied\(^\text{27}\).

Note also that the \( r \)-dimensional hyperplane defined by \( V_{(r)} \) is not isotropic, but \( V_{(r)} \) should be projected to self- and anti-self dual irreps. The projections \( V_{(r)\pm} \) defines isotropic hyperplanes.

### 5.2 Pure spinors and fused Fierz relations

There is another way to view the extended QQ-system for \( \mathfrak{so}(2r) \) by building it up from spinors. In this case we study the linear problem (2.1) for the tensor product between spinors

\[
\psi^{[m]} \otimes \psi^{[-m]}, \quad \psi^{[m]} \otimes \eta^{[-m]}, \quad \eta^{[m]} \otimes \eta^{[-m]}.
\]

(5.14)

As exemplified in the case of \( \mathfrak{su}(4) \simeq \mathfrak{so}(6) \) we compare the eigenvalues of this linear problem with those of tensors. To this end, compute the Perron-Frobenius vector of the incidence matrix: \( \mu_a = [a]_{\gamma/2}, \ a \leq r - 2; \ \mu_{r-1} = \mu_r = \frac{1}{2}[r-1]_{\gamma/2}. \) Then one can observe

\(^{27}\)\( \psi^+ \otimes \eta^- \) projected to the irrep \( L(\omega_r + \omega_{r-1}) \) (the same one \( V_{(r-1)} \) belongs to) is not equal to \( V_{(r-1)}, \)

however, curiously, it corresponds to an \( S^+ \)-solution of the linear problem (2.1) associated to an eigenvalue of \( \Lambda \) on the next to the maximal concentric circle of the Coxeter plane, and it is also uniquely defined. We can call it \( V_{(r-1)}^\ast \), it would be a new type of \( Q \)-function, such that \( \{ V_{(1)}^{[r_1]}, V_{(2)}^{[r_2]}, \ldots, V_{(r-2)}^{[r_{r-2}]}, V_{(r-1)}^{[r_{r-1}]} \} \in G/B. \)
the following relation

$$\mu_{r-1}(\gamma^{m/2} + \gamma^{-m/2}) = \mu_{r-1-m}$$  \hfill (5.15)

which is immediate to verify, but we need to use the explicit value of the Coxeter number implying $\gamma^{\pm r-1\over 2} = \pm i$.

Hence, we can observe the following pattern: if we want to project the tensor product of spinors to the fundamental representation $L(\omega_a)$, the shift in (5.14) should be $m = r - 1 - a$. If the shift is smaller than this value, projection to $L(\omega_a)$ vanishes. Graphically on the Dynkin diagram, this means that we start at the bifurcation node for $m = 1$ and walk away from the spinor nodes by increasing $m$. For an even rank, we demonstrate this observation by the diagram in Figure 3. The case off an odd rank is obtained by reshuffling $\psi \otimes \psi \iff \psi \otimes \eta$ on the diagram, we have for example $V(1) = \psi^{[r-2]}(1)\psi^{[-r+2]}$.

We summarize our findings for how to relate spinors to vectors using an explicit index notation. First we have projection relations for products of spinors of equal type

$$\gamma^{\alpha\beta} I^{[m]} \psi_{\alpha}^{[m]} \psi_{\beta}^{[-m]} = \bar{\gamma}^{\dot{\alpha}\dot{\beta}} I^{[m]} \eta_{\dot{\alpha}}^{[m]} \eta_{\dot{\beta}}^{[-m]} = 0, \quad m = 0, 1, \ldots, r - 2 - |I|,  \hfill (5.16)$$

and almost the same expression for products between spinors of different type

$$\gamma^{\dot{\alpha}\beta} I^{[m]} \eta_{\dot{\alpha}}^{[m]} \psi_{\beta}^{[-m]} = 0, \quad m = 0, 1, \ldots, r - 2 - |I|.  \hfill (5.17)$$

We note that these projection properties imply in particular that $\psi_{\alpha}$ and $\eta_{\dot{\alpha}}$ are pure spinors. That is, they satisfy

$$\langle \psi_{\alpha} \gamma^{\alpha\beta} \psi_{\beta} \rangle = 0, \quad \langle \eta_{\dot{\alpha}} \bar{\gamma}^{\dot{\alpha}\dot{\beta}} \eta_{\dot{\beta}} \rangle = 0, \quad |I| < r.  \hfill (5.18)$$

The reason that we encounter pure spinors is because they parameterise maximal isotropic spaces [Car66].

**Figure 3:** Illustration of fused Fierz identities showing the relation between tensor representations and fused products of spinors.
When \( m = r - 1 - |I| \) we find fusion relations, which we call fused Fierz identities, relating spinors and other fundamental representations:

\[
\gamma_f^{\alpha\beta} \psi_{\alpha}^{[r-1-|I|]} \psi_{\beta}^{[r+1+|I|]} = V_I, \quad \gamma_f^{\dot{\alpha}\dot{\beta}} \eta_{\dot{\alpha}}^{[r-1-|I|]} \eta_{\dot{\beta}}^{[r+1+|I|]} = V_I.
\] 

(5.19)

In Figure 3, we have also indicated further “A-type” nodes to show the effect of projecting to the non-fundamental representation \( V_{(r-1)} \) and \( V_{(r)}^{\pm} \) which featured in the discussion of section 5.1. The fusion relations between spinors and \( V_{(r-1)} \) is

\[
\gamma_{(r-1)}^{\dot{\alpha}\dot{\beta}} \eta_{\dot{\alpha}}^{[r-1+|I|]} \eta_{\dot{\beta}}^{[r+1+|I|]} = V_{(r-1)},
\] 

(5.20a)

and for \( V_{(r)}^{+} \) and \( V_{(r)}^{-} \)

\[
(\gamma_{(r)}^{+})^{\alpha\dot{\beta}} \eta_{\dot{\alpha}}^{[r-1+|I|]} \eta_{\dot{\beta}}^{[r+1+|I|]} = V_{(r)}^{+}, \quad (\gamma_{(r)}^{-})^{\alpha\dot{\beta}} \psi_{\alpha}^{[r-1+|I|]} \psi_{\beta}^{[r+1+|I|]} = V_{(r)}^{-}.
\]

(5.20b)

Finally there are quantisation conditions

\[
\psi^{\left[\frac{1}{2}\right]}_{\alpha} (\psi_{\alpha}^{[-\frac{1}{2}]} \right] = 1, \quad \eta^{\left[\frac{1}{2}\right]}_{\dot{\alpha}} (\eta_{\dot{\alpha}}^{[-\frac{1}{2}]} \right] = 1.
\]

(5.21)

Written here for even rank, the odd rank case amounts to contracting \( \psi \) with \( \eta \). Equality sign in equations (5.19), (5.20), (5.21) fixes unambiguously relative normalisation between spinors and vectors (up to one sign in spinors as they always enter in bilinear combinations). This in turn implies concrete signs in Weyl-orbit QQ-relations (1.31) which can be computed from explicit form of the gamma-matrices (5.6) (note that the highest-weight component of QQ-relations (1.30) also generically gets a sign).

5.3 T-functions

In section 4.1 we proposed that T-functions are to be constructed using inner products between Q-vectors and their contra-gradient representations (4.3). We list here the explicit expressions for \( so(2r) \).

For the vector representation and antisymmetric tensor representations the inner product is constructed using \( g_{ij} \) and the T-functions are

\[
T_{a,s} = \frac{1}{a!} \psi_{i_1...i_a}^{[r-1+s]} (V_{i_1...i_a}^{r-1+s})^{[-r+1-s]}, \quad 1 \leq a \leq r - 2.
\]

(5.22)

The off-set shift of \( \frac{h}{2} = r - 1 \) is determined from the projection properties (5.10) and the condition \( T_{a,0} = 1 \) is the quantisation condition.

For spinor representations the statement is slightly dimension dependent. For even \( r \) the inner product is between spinors of the same type and the T-functions becomes\(^{28}\)

\[
T_{r-1,s} = \psi_{\alpha}^{[r-1+s]} (\psi_{\alpha}^{[r-1+s]} = T_{r,s} = \eta_{\dot{\alpha}}^{[r-1+s]} (\eta_{\dot{\alpha}}^{[r-1+s]}).
\]

(5.23)

\(^{28}\)Assuming normalisation of \( C \).
while for odd $r$ we must contract the two different spinor representations with each other giving

$$T_{r-1,s} = \psi_\alpha^{[r-1+s]}(\eta^\alpha)^{[-r+1-s]}, \quad T_{r,s} = \eta_\alpha^{[r-1+s]}(\psi^\alpha)^{[-r+1-s]}.$$  \hspace{1cm} (5.24)

5.4 Character solution for $\mathfrak{so}(2r)$

As an example of the above formalism we consider the character solution for transfer matrices. We discuss first vectors, the character ansatz (4.7) becomes

$$V_i = A_i x_i^{-i \sigma(i)},$$  \hspace{1cm} (5.25)

where $\sigma(a)$ is the sign function written out explicitly to remind us that $x_i^{-i \sigma(-i)} = \frac{1}{x_i}$.

There exist a trick to quickly find the solution for the vector $T$-function. The main observation is that we have $r-1$ conditions coming from the projection conditions, $V_i^{[s]}(V_i)^{[-s]} = 0, s = 0, 1, \ldots, r - 2$, and one from the quantisation condition $V_i^{[r-1]}(V_i)^{[-r+1]} = 1$. Now writing out the inner product explicitly

$$V_i^{[s]}(V_i)^{[-s]} = \sum_{i=1}^{r} g^{i,-i} A_i A_{-i} (x_i^s + \frac{1}{x_i^s}) = \sum_{i=1}^{r} \tau_i (x_i^s + \frac{1}{x_i^s})$$  \hspace{1cm} (5.26)

we see that there are exactly $r$-functions, $\tau_i$, to fix. The solution follows by taking a determinant ansatz, inspired by Weyl's character formula, for the sum

$$V_i^{[s]}(V_i)^{[-s]} = \sum_{i=1}^{r} g^{i,-i} A_i A_{-i} (x_i^s + \frac{1}{x_i^s}) = \left| \begin{array}{ccc} x_1^1 + \frac{1}{x_1^1} & \cdots & \cdots \\ x_2^2 + \frac{1}{x_2^2} & \cdots & \cdots \\ \vdots & \ddots & \ddots \\ x_r^r + \frac{1}{x_r^r} & \cdots & \cdots \\ \end{array} \right|. \hspace{1cm} (5.27)$$

The vectors are now fixed by the projection properties to be $\mathbf{\bar{W}}_i \propto (x_1^{i-1} + \frac{1}{x_1^i}, \ldots, x_r^{i-1} + \frac{1}{x_r^i})$ and the quantisation condition fixes the overall normalisation. The final result is then

$$T_1,s = (V_i^{[s+r-1]})(V_i^{[1-s-r]}), \hspace{0.5cm} (5.28)$$

which we recognize as the character of a completely symmetric traceless tensors.

From (5.28) we find $\tau_i$ by expanding the determinant in minors:

$$\tau_i \equiv g^{i,-i} A_i A_{-i} = \prod_j \sqrt{\frac{x_j}{x_j - x_i}} \prod_j \sqrt{\frac{-x_j}{x_j - x_i}}.$$  \hspace{1cm} (5.29)

The role of the metric will be inessential in the following and we make the symmetric choice
\[ g^{i,-i} = 1. \] The expression for \( \tau_i \) suggests a natural normalisation to choose for \( A_{\pm i} \):

\[ A_{\pm i} = \prod_j' \frac{\sqrt{-x_j}}{x_j - x_i^\pm}. \tag{5.30} \]

The vector \( V_i \) can be used to construct all other antisymmetric tensors. To see the pattern, start with \( V_{ij} \) which is constructed from \( W[V_i, V_j] \):

\[ V_{ij} = A_i A_j \sqrt{x_j^{-\sigma(j)} x_j^{-\sigma(i)} (x_i^{\sigma(i)} - x_j^{\sigma(j)})} x_i^{-iu\sigma(i)} x_j^{-iu\sigma(j)}. \tag{5.31} \]

The result for \( V_{(k)} \) is a slight generalisation where the last term is replaced by a Vandermonde-like determinant:

\[ V_{i_1 \ldots i_k} = \left( \prod_{s=1}^k A_{i_s} \sqrt{x_{i_s}^{-\sum_{k=1}^{k-1}\sigma(i_s)}} x_{i_s}^{-iu\sigma(i_s)} \right) \det_{1 \leq i, s \leq k} (x_i^{-\sum_{k=1}^{k-1}\sigma(i_s)}). \tag{5.32} \]

This becomes particularly nice when all indices are positive

\[ V_I = A_I x^{-1} u^I, \quad A_I = \frac{1}{\Delta(x_I)} \prod_{j \in I, s \in I} \frac{\sqrt{-x_j}}{x_j - x_i} I > 0. \tag{5.33} \]

The rectangular T-functions \( T_{a,s} \) for \( a \neq r - 1, r \) follows from the inner product between antisymmetric tensors. When expanded out in the already introduced variables, these are

\[ T_{a,s} = \frac{1}{a!} \sum_{i_1, \ldots, i_a = -r}^r A_{i_1, i_2, \ldots, i_a} A_{-i_1, -i_2, \ldots, -i_a} \prod_{k=1}^a x_{i_k}^{\sigma(i_k)}. \tag{5.34} \]

Just as for the vector case we expect that these expressions can be rewritten in terms of characters. It is known that the character solution of \( D \)-type Hirota equations is [KNS11]

\[ T_{a,s} = \sum_{k_0 + k_{a+2} + \ldots + k_a = s} \chi(k_{a_0} \omega_{a_0} + k_{a+2} \omega_{a+2} + \ldots k_{a-2} \omega_{a-2} + k_a \omega_a), \tag{5.33} \]

If \( a \) is even, \( \omega_{a_0} = \omega_0 \) which is the trivial representation and for \( a \) odd \( \omega_{a_0} = \omega_1 \). The sum over \( so(2r) \) irreps appears for Kirillov-Reshetikhin modules and is a sign of the underlying Yangian structure [KR90]. Using Mathematica we have checked numerically that for small representations and rank (5.33) and (5.34) agrees but we have not proved in a explicit way that this is the case in general. However, since \( T_{a,s} \) for \( a = 2, \ldots, s - 2 \) follow unambiguously from \( T_{1,s} \) using Cherednik-Bazhanov-Reshetikhin formulae [Che87, BR90], equality between (5.33) and (5.34) is guaranteed.

Having specified the character solution for the analytic ansatrz we can also write down expressions for \( T \)-functions with non-trivial \( u \)-dependence. Let \( V_i = A_{i} x^{-iu\sigma(i)} \). Expanding out the inner product again allows us to write down the transfer matrices explicitly.
including twist as

\[ T_{1,s} = \sum_{i=1}^{r} \tau_i (x_i^{r-1+s} v_{i}^{r-1-s} v_{i}^{[r-1-s]} + x_i^{-r+1-s} v_{i}^{-r+1-s} v_{i}^{-[r-1-s]}). \]  

(5.35)

For \( a \leq r - 2 \), \( T_{a,s} \) is obtained in the same way from (5.33).

We turn now to spinors. The analytic ansatz is, using the spinor \( \zeta_I \) for notational purposes,

\[ \zeta_I = B_I \prod_{i=1}^{r} \sqrt{x_i^{r-1}}. \]  

(5.36)

We will attempt to find the explicit expressions by using the fusion relation between \( V_\pm^{(r)} \) and fused symmetric square of \( \psi_\alpha \) and \( \eta_\alpha \). Using the orthogonal basis we see from

\[ \gamma_{(r)}^{+(\alpha\beta)} \eta_\alpha \eta_\beta = V_\pm^{(r)}, \]  

(5.37)

that \( \eta_1^+ \eta_1^- \) and \( V_{12...r} \) must be proportional. The same conclusion holds for \( \psi_1^+ \psi_1^- \) and \( V_{12...r-1...r} \). To fix the exact proportionality we compute the sign factor using the gamma-matrix basis (5.6) with charge conjugation matrix \( C \). Then

\[ \eta_1^+ \eta_1^- = (-1)^{\frac{1}{2}r(r+1)} V_{12...r}, \quad \psi_1^+ \psi_1^- = (-1)^{\frac{1}{2}r(r-1)} V_{123...r+1...r}. \]  

(5.38a)

Using (5.32) and (5.33) the expressions for \( \eta_0 \) and \( \psi_0 \) are found to be

\[ \eta_0^+ \eta_0^- = (-1)^{\frac{1}{2}r(r+1)} \prod_{i=1}^{r} x_i^{-iu}, \quad \psi_0^+ \psi_0^- = (-1)^{\frac{1}{2}r(r+1)} \prod_{i=1}^{r-1} x_i^{-iu}. \]  

(5.39)

We see that the shifts simply cancel each other and will not play a part. To get the other spinors we perform Weyl reflections on (5.39) which acts by flipping signs of the vector indices in (5.33). This procedure is well behaved for both sides of the equations, that is, there cannot be any new terms appearing. The resulting expression is

\[ B_2^f = (-1)^{\frac{1}{2}r(r+1)} (-1)^{|I|} \prod_{i \in I} x_i^{r-1} \prod_{i < j \in I} \frac{(x_i - x_j)^2}{(x_i x_j - 1)^2}. \]  

(5.40)

6 Exceptional algebras

Study of exceptional cases emphasises strongly that fused flag is a non-trivially constrained system compared to an ordinary bundle with flag manifold in the fiber. To define locally a section of an ordinary bundle, we need as many functions as \( \dim G/B \), whereas local definition of a fused flag could use, in principle, only as many functions as the rank of the algebra. These \( r \) functions have a covariant description through Q-vectors subject to various relations. Representation theory of exceptional algebras is very rich producing many remarkable such relations. Below we list some of them, however, without doubt, it is
only a tip of an iceberg. Their further study is likely to unveil new combinatorial structures enriching our knowledge of standard representation theory.

The notations for Q-vectors follow the enumeration for the nodes of Dynkin diagrams shown in Fig 4. To explore possible relations, we used LieArt 2.0 package [FKS20] and the explicit knowledge of Λ-eigenvalues following from the results of Section 2.3.

To keep presentation short, we use the following convention. Expression of type

\[ Q_{(a)}^{[m_0]} \otimes Q_{(b)}^{[-m_0]} \otimes \ldots \rightarrow \text{r.h.s.} \tag{6.1} \]

for a fixed integer \( m_0 \) means that the r.h.s. is in an irrep of the Lie algebra and one gets an equality between l.h.s. and r.h.s. by restricting the l.h.s. to this irrep. If \( \approx \) stands instead of \( \rightarrow \) then this means that the l.h.s. is also an irrep.

The fusion relations (6.1) always come with the associated projection relations: If we consider \( Q_{a}^{[m]} \otimes Q_{b}^{[-m]} \otimes \ldots \) with \( 0 \leq m < m_0 \) then the restriction of this expression to the irrep of the r.h.s. is zero. We won’t write the projection relations explicitly.

6.1 \( E_6 \)

This is a Lie algebra of dimension 78, with Coxeter number of the associated Weyl group \( h = 12 \).

\( E_6 \) is the only exceptional algebra which has representations that are not the same as their contragredients. The contragradient representation is obtained by reflection of the Dynkin diagram and so \( L(\omega_1)^* = L(\omega_5) \), and \( L(\omega_2)^* = L(\omega_4) \). Hence computation of transfer matrices involves pairing of different Q-functions, for instance

\[ T_{1,s} = \langle Q_{(1)}^{[s+6]}, Q_{(5)}^{[-s-6]} \rangle , \quad T_{5,s} = \langle Q_{(5)}^{[s+6]}, Q_{(1)}^{[-s-6]} \rangle . \tag{6.2} \]

The 27-dimensional fundamental representation \( L(\omega_1) \) has all its components on the Weyl orbit of the highest weight. This representation and its conjugate are analogs of the vector representations for algebras from classical series, in particular in the sense that the Q-vectors at other nodes of the Dynkin diagram can be obtained using familiar Wronskian formulae (with no projections to irreps needed):

\[ Q_{(1)}^{+} \wedge Q_{(1)}^{-} = Q_{(2)} , \quad Q_{(1)}^{[2]} \wedge Q_{(1)} \wedge Q_{(1)}^{-2} = Q_{(3)} , \tag{6.3a} \]

\[ Q_{(5)}^{+} \wedge Q_{(5)}^{-} = Q_{(4)} , \quad Q_{(5)}^{[2]} \wedge Q_{(5)} \wedge Q_{(5)}^{-2} = Q_{(3)} . \tag{6.3b} \]
Hence, we can use an embedding of lines into planes intuition to describe (at least partial) flags, however these lines are special: tensor powers of $L(\omega_1)$ have several irreps

\begin{align}
L(\omega_1) \otimes L(\omega_1) &= L(\omega_{\text{max}} = 2\omega_1) + L(\omega_1) + L(\omega_2), \quad (6.4a) \\
L(\omega_1) \otimes L(\omega_1) \otimes L(\omega_1) &= L(\omega_{\text{max}} = 3\omega_1) + \ldots + L(0) + \ldots \quad (6.4b)
\end{align}

and projection to all of them, except for the maximal ones, of the corresponding $Q$-functions tensor products is zero. The projection relations are analogs of null-vector/pure spinor conditions. In the fused flag, these projections are paired with the following fusion properties:

\begin{align}
Q^{[\pm 6]}(1) \otimes Q^{[\mp 6]}(5) &\rightarrow 1, \quad (6.5a) \\
Q^{[4]}(1) \otimes Q^{[-4]}(1) &\rightarrow Q(1), \quad Q^{[8]}(1) \otimes Q^{[-8]}(1) \rightarrow Q(1), \quad (6.5b) \\
Q^{[4]}(5) \otimes Q^{[-4]}(5) &\rightarrow Q(5), \quad Q^{[8]}(5) \otimes Q^{[-8]}(5) \rightarrow Q(5). \quad (6.5c)
\end{align}

Furthermore, there is a Fierz-type relation to get $Q(6)$

\begin{equation}
Q^{[\pm 3]}(1) \otimes Q^{[\mp 3]}(5) \rightarrow Q(6). \quad (6.6)
\end{equation}

Representation $L(\omega_6)$ is the adjoint representation of $E_6$. Hence $L(\omega_6) \wedge L(\omega_6)$ is definitely reducible. Indeed, for any simple Lie algebra with commutation relations $[J^i, J^j] = f^{ij}_k J^k$, $L_{\text{adj}} \wedge L_{\text{adj}}$ contains $L_{\text{adj}}$ as an irrep spanned by $f^{ij}_k J^i \otimes J^j$, where raising/lowering of indices is done by the Killing form. If $h$ is the Coxeter number and $Q_{(\text{adj})}$ is the $Q$-vector built from the S-solution of (2.1) with the maximal positive eigenvalue of $\Lambda_{\text{adj}}$ then

\begin{equation}
Q^{[h/3]}_{(\text{adj})} \wedge Q^{[-h/3]}_{(\text{adj})} \rightarrow Q_{(\text{adj})}, \quad (6.7)
\end{equation}

the equation is only meaningful in the sense of $S$-solutions of (2.1) if $h/3$ is an even number.

The first example where the adjoint is a fundamental representation and $h/3$ is not even is $D_5$ whose Coxeter number is $h = 8$. For this case, $Q_{(\text{adj})} \equiv Q(2)$. This $Q$-function originates from $\Psi(2)$, where the half-rotated $\Psi^+_2$ are the $S$-solutions of (2.1) corresponding to the complex eigenvalues $\gamma^{\pm 1/2} \mu_2$. There is also an $S^*\Psi^+_2$ of (2.1) with real eigenvalue $\mu^*_2$ such that $\mu^*_2/\mu_2 = \sqrt{2 - \sqrt{2}}$. We found that

\begin{equation}
Q^{[3]}_2 \wedge Q^{[-3]}_2 \rightarrow Q^*_2 \quad \text{(example from $D_5$).} \quad (6.8)
\end{equation}

Returning back to $E_6$, one has $L(\omega_6) \wedge L(\omega_6) = L(\omega_3) \oplus L(\omega_5)$ and the corresponding

---

\footnote{Cube of $L(\omega_1)$ contains total 10 irreps. We show only two for simplicity.}
fusion relations are

\[ Q^+_{(6)} \wedge Q^-_{(6)} \to Q_{(3)} , \quad (6.9a) \]
\[ Q^4_{(6)} \wedge Q^{-4}_{(6)} \to Q_{(6)} . \quad (6.9b) \]

Symmetric power of the adjoint representation decomposes as \( S^2(L(\omega_6)) = L(0) \oplus L(\omega_1 + \omega_5) \), for the symmetric trace-less part one then derives

\[ Q^{[3]}_{(6)} \otimes Q^{-[3]}_{(6)} \to (Q^{(1)} \otimes Q^{(5)} )_{L(\omega_1 + \omega_5)} . \quad (6.10) \]

6.2 \( E_7 \)

This is a Lie algebra of dimension 133, with Coxeter number of the associated Weyl group \( h = 18 \).

Its “vector” representation \( 56 \equiv L(\omega_6) \) has an interesting property. Alongside with the standard quadratic invariant existing because \( 56 \) is its own contra-gradient, there exists also an independent quartic invariant. It is fully symmetric w.r.t. permutations of its entries. There is the only way to multiply four solutions of (2.1) in the irrep \( 56 \) such they have a non-trivial cone of applicability for the constant solution. We use this to conclude that, for the quartic invariant denoted as \( \langle \cdot , \cdot , \cdot , \cdot \rangle \), it should be

\[ \langle Q^{[9]}_{(6)}, Q^{[9]}_{(6)}, Q^{-[9]}_{(6)}, Q^{-[9]}_{(6)} \rangle = 1 . \quad (6.11) \]

The associated projection relations are of the form \( \langle Q^{[s_1]}_{(6)}, Q^{[s_2]}_{(6)}, Q^{[s_3]}_{(6)}, Q^{[s_4]}_{(6)} \rangle = 0 \) if \(-9 \leq s_i \leq 9 \) and \( s_i \) are different from those featured in (6.11). On the other hand, by leaving the cone of applicability, one constructs an entirely novel family of “quartic transfer matrices”:

\[ T^{[s]}_{\{s_1,s_2,s_3,s_4\}} = \langle Q^{[s_1]}_{(6)}, Q^{[s_2]}_{(6)}, Q^{[s_3]}_{(6)}, Q^{[s_4]}_{(6)} \rangle , \quad s = \sum_{i=1}^{4} s_i . \quad (6.12) \]

In the set \( \{s_1, s_2, s_3, s_4\} \), order of entries \( s_i \) is of no importance.

The quadratic invariant, similarly to symplectic case, is anti-symmetric in its entries, meaning that wedging the vector representation to get other fundamentals along the bottom line of Dynkin diagram will require the subsequent projection to the corresponding irrep:

\[ Q^+_{(6)} \wedge Q^-_{(6)} \to Q_{(3)} , \quad (6.13) \]
\[ Q^2_{(6)} \wedge Q^{-2}_{(6)} \to Q_{(4)} , \quad (6.14) \]
\[ Q^3_{(6)} \wedge Q^+_{(6)} \wedge Q^-_{(6)} \to Q_{(3)} . \quad (6.15) \]

Adjoint 133 = \( L(\omega_1) \) sits in the symmetric square of 56:

\[ Q^{[5]}_{(6)} \otimes Q^{-[5]}_{(6)} \to Q_{(1)} . \quad (6.16) \]

\[ ^{30} \text{For this conclusion we assumed that this invariant is non-zero for combinatorial reasons if we take this particular combinations of } Q\text{-functions and normalised it accordingly to get 1 on the r.h.s..} \]
We can then use
\[ Q^+_1 \wedge Q^-_1 \rightarrow Q_2, \]  
(6.17)
to generate the Q-vector in \( L(\omega_2) \) and a Fierz-type relation
\[ Q^{[3]}_1 \otimes Q^{[-4]}_1 \rightarrow Q_7, \]  
(6.18)
to get the Q-vector in \( L(\omega_7) \).

We list also several other fused relations which make more direct transitions between
Q-vectors in fundamental representations:
\[ Q^{[5]}_4 \otimes Q^{[-5]}_4 \rightarrow Q_2, \]  
(6.19a)
\[ Q^{[3]}_4 \otimes Q^{[-2]}_4 \rightarrow Q_3, \]  
(6.19b)
\[ Q^{[5]}_4 \otimes Q^{[-5]}_4 \rightarrow Q_3, \]  
(6.19c)
\[ Q^{[7]}_4 \otimes Q^{[-7]}_4 \rightarrow Q_5, \]  
(6.19d)
\[ Q^{[11]}_1 \otimes Q^{[-4]}_7 \rightarrow Q_6, \]  
(6.19e)
\[ Q^{[3]}_1 \otimes Q^{[-2]}_7 \rightarrow Q_4, \]  
(6.19f)
\[ Q^{[11]}_6 \otimes Q^{[-3]}_7 \rightarrow Q_1, \]  
(6.19g)

Besides, there are many fusion relations featuring \( S^* \)-solutions, like (6.8). We do not
present them here.

6.3 \( E_8 \)

This is a Lie algebra of dimension 248, with Coxeter number of the associated Weyl group
\( h = 30 \). In addition to the quadratic invariant, this algebra has octic invariant [CP07], and
hence one can introduce “octic transfer matrices”, similarly to (6.12).

A unique feature of \( E_8 \) is that it does not possess a vector representation. The minimal
nontrivial representation is the adjoint 248 = \( L(\omega_7) \). Its eight-dimensional zero-weight
subspace is not on the Weyl orbit of the highest weight. To generate it from the Weyl-orbit
components (which is important for the proof of Theorem 2) we use (6.7) which explicitly
becomes
\[ f_k^{ij} Q^{[10]}_{(7)i} Q^{[-10]}_{(7)j} = Q_{(7)k}, \]  
(6.20)
where \( f_k^{ij} \) are the structure constants of \( E_8 \). Because Cartan generators commute between
themselves, zero-weight components on the r.h.s. of (6.20) are obtained from products of
Weyl-orbit components on the l.h.s. of (6.20).

From the adjoint representation, all other Q-vectors can be obtained using for instance
the following fusion relations

\begin{align}
Q^{[1]}_{(7)} \otimes Q^{-[1]}_{(7)} & \to Q_{(6)}, \\
Q^{[6]}_{(7)} \otimes Q^{-[6]}_{(7)} & \to Q_{(1)}, \\
Q^{[7]}_{(6)} \otimes Q^{-[7]}_{(6)} & \to Q_{(5)}, \\
Q^{[6]}_{(6)} \otimes Q^{-[6]}_{(6)} & \to Q_{(2)}, \\
Q^{[2]}_{(6)} \otimes Q^{-[2]}_{(6)} & \to Q_{(4)}, \\
Q^{[7]}_{(1)} \otimes Q^{-[7]}_{(1)} & \to Q_{(8)}, \\
Q^{[1]}_{(8)} \otimes Q^{-[1]}_{(8)} & \to Q_{(3)}. 
\end{align}

(6.21a) \hspace{2cm} (6.21b) \hspace{2cm} (6.21c) \hspace{2cm} (6.21d) \hspace{2cm} (6.21e) \hspace{2cm} (6.21f) \hspace{2cm} (6.21g) \hspace{2cm} (6.21h)

7 Conclusions

In this work we introduced a concept of the extended Q-system for simply-laced Lie algebras and studied its most essential properties. Quite remarkably, Q-functions of this system form a fused flag which can be defined as follows: if \( Q^{(a)} \) is a vector of Plücker coordinates of the minimal flag \( G/P_{a} \) then \( \{ Q^{(a)}, Q^{(a')} \} \) are Plücker coordinates of the flag \( G/P_{aa'} \), simultaneously for both directions of shift if \( a, a' \) are adjacent nodes of the Dynkin diagram. By Lemma 3.1, \( \{ Q^{[p_{1}]}_{(r)}, \ldots, Q^{[p_{r}]}_{(r)} \} \) are Plücker coordinates of the complete flag \( G/B \) for any choice of the Coxeter height function \( p_{a} \). The fused flag is gauge equivalent to an oper, one should choose and fix arbitrary function \( p_{a} \) to define the equivalence.

It is instructive to compare the extended Q-system to other collections of Baxter Q-functions. The simplest option is to choose \( Q^{(a)}_{1} \) – the Q-functions along Dynkin diagram. Their zeros satisfy nested Bethe equations (1.28) and they, in general position, contain in principle all information about the spectrum of an integrable model. However, Bethe equations are often not the best system of equations to solve in practice, and working with \( Q^{(a)}_{1} \) lacks covariance which brings bogus complexity to various computations. Then, \( Q^{(a)}_{1} \) are often supplemented with their first descendants \( Q^{(a)}_{2} \). The obtained pairs of Q-functions form the QQ-system. The advantage over Bethe equations is a polynomial-type formulation of equations on the spectrum, they however have typically too many solutions. The next addition is the Q-system on the Weyl orbit, where all \( Q^{(a), \sigma(1)} \) are considered. It is likely that this system already features completeness, i.e. solutions of (1.31) with right analytic properties of \( Q^{(a), \sigma(1)} \) are in precise bijection with eigenspaces of the Bethe algebra. The extended Q-system is a special further extension of the Weyl-orbit Q-system which we demonstrated to be unique. The added value of this extension is covariance of the obtained Q-vectors under action of (the Langlands dual of) the symmetry algebra which enables concise derivation of various remarkable relations. For instance, expression for transfer matrices is given by simple bilinear combinations (4.6) which should be compared with expansion over Young tableaux (1.7) if only Q-functions on the Dynkin diagram are used. Of course, both expressions are eventually equivalent, the point is that (4.6) is a recipe.
to resum (1.7) in a particular universal way applicable at once for all Kirillov-Reshetikhin representations of the auxiliary space.

Whereas we derived relations of the extended Q-system using a particular linear problem (2.1) and the machinery of ODE/IM correspondence, we demonstrated that the Q-system is a universal concept. For any given \( Q_{(a),1} \) there is a unique, up to symmetries, extension to the extended Q-system. Hence formal functional freedom one can enjoy in principle is as many independent functions as the rank of the algebra. However, demanding that Q-functions belong to a certain analyticity class significantly restricts this freedom. In Section 4.3 we proposed an explicit ansatz for analytic structure of Q-functions describing rational, trigonometric, and elliptic spin chains and we conjecture that all Q-functions obeying this ansatz provide complete and faithful description of the commuting charges spectra. “Complete” means that the number of solutions of QQ-relations is the right one, “faithful” means that algebra of Q-functions is isomorphic to a maximal commutative algebra of charges acting on the physical Hilbert space of the spin chain. The key point of this conjecture is that all Q-functions should satisfy the ansatz and then statements are true always and not only in general position\(^{31}\), demanding analyticity only for \( Q_{(a),1} \), or only for \( Q_{(a),1} \) and \( Q_{(a),2} \) would be not always enough.

The message of our work can be also re-stated from the point of view of representation theory. In the case of quantum algebras, the representation theory is not developed to the same level as for instance in the case of Lie algebras. One of the problems is not sufficient understanding of character ring relations. We believe that the development is hindered due to attempts to express all objects through prefundamental representations corresponding to functions \( Q_{(a),1} \). We suggest that lifting up the extended Q-system to the representation theory level can become highly beneficial for better understanding of the character ring. Curiously enough, extended Q-functions and transfer matrices seems to be not the end of the story. We gave some examples of functions \( Q^* \), cf. (6.8), which are novel-type objects corresponding to \( S^* \)-solutions of the ODE/IM problem. They are neither members of the extended Q-system nor transfer matrices but nevertheless well-defined. It would be really interesting to understand how these objects are interpreted in terms of quantum characters.

This work was focused only on simply laced cases, we discussed explicitly \( D_n \) series in detail and also exceptional series though more schematically. Generalisation of the extended Q-system to the non-simply laced case may be done in a “naive” way, simply by choosing the corresponding Cartan matrix in the described constructions. However, this approach will lead to wrong Bethe equations. Correct Bethe equations arise when we consider twisted affine Lie algebra, we plan to address the question of how the extended Q-system and fused flag look like in this case in a future publication. Another interesting generalisation would be supersymmetric case, where we can get new insight in description of AdS\(_4\)/CFT\(_3\) quantum spectral curve and yet unknown quantum spectral curve of

\(^{31}\)In the proved case of rational \( \mathfrak{gl}(N) \) spin chains with nodes in the vector representation, the only requirement is that spin chain is in a cyclic representation of Yangian.
AdS$_3$/CFT$_2$.

**Note added** When the results of this work were ready and we were preparing the paper for publication, the paper by Ferrando, Frassek and Kazakov appeared [FFK20]. Their results considerably intersect with our results applied to the case of $D_n$ algebras. However, the methods and some of key messages of their and our work are quite different.

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