INCLUSION OF $\Lambda BV(p)$ SPACES IN THE CLASSES $H_q^2$

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Abstract. A characterization of the inclusion of Waterman-Shiba classes into classes of functions with given integral modulus of continuity is given. This corrects and extends an earlier result of a paper from 2005.

1. Preliminaries

Let $\Lambda = (\lambda_i)$ be a nondecreasing sequence of positive numbers such that $\sum \frac{1}{\lambda_i} = +\infty$ and let $p$ be a number greater than or equal to 1. A function $f : [a, b] \to \mathbb{R}$ is said to be of bounded $p$-\(\Lambda\)-variation on a not necessarily closed subinterval $P \subset [a, b]$ if

$$V(f) := \sup \left( \sum_{i=1}^{n} \left| f(I_i) \right|^p \lambda_i \right)^{\frac{1}{p}} < +\infty,$$

where the supremum is taken over all finite families $\{I_i\}_{i=1}^{n}$ of nonoverlapping subintervals of $P$ and where $f(I_i) := f(\sup I_i) - f(\inf I_i)$ is the change of the function $f$ over the interval $I_i$. The symbol $\Lambda BV^p$ denotes the linear space of all functions of bounded $p$-\(\Lambda\)-variation with domain $[0, 1]$. The Waterman-Shiba class $\Lambda BV^p$ was introduced in 1980 by M. Shiba in [9]. When $p = 1$, $\Lambda BV^1$ is the well-known Waterman class $\Lambda BV$. Some of the basic properties of functions of class $\Lambda BV^p$ were discussed by R. G. Vyas in [12] recently. More results concerned with the Waterman-Shiba classes and their applications can be found in [1], [2], [4], [6], [7], [8], [10] and [11]. $\Lambda BV^p$ equipped with the norm $||f||_{\Lambda, p} := |f(0)| + V(f)$ is a Banach space.

Functions in a Waterman-Shiba class $\Lambda BV^p$ are regulated [11, Thm. 2], hence integrable, and thus it makes sense to consider their integral modulus of continuity

$$\omega_q(\delta, f) := \sup_{0 \leq \gamma \leq \delta} \left( \int_{0}^{1-\gamma} |f(t + \gamma) - f(t)|^q \right)^{\frac{1}{q}} dt,$$

Date: May 3, 2014.

2000 Mathematics Subject Classification. Primary 26A15; Secondary 26A45.

Key words and phrases. generalized bounded variation, modulus of variation, Banach space.
for $0 \leq \delta \leq 1$. However, if $f$ is defined on $\mathbb{R}$ instead of on $[0, 1]$ and if $f$ is 1-periodic, it is convenient to modify the definition and put

$$\omega_q(\delta, f) := \sup_{0 \leq \gamma \leq \delta} \left( \int_0^1 |f(t + \gamma) - f(t)|^q \right)^{\frac{1}{q}} dt,$$

since the difference between the two definitions is then nonessential in all applications of the concept. A function $\omega : [0, 1] \to \mathbb{R}$ is said to be a modulus of continuity if it is nondecreasing, continuous and $\omega(0) = 0$. If $\omega$ is a modulus of continuity, then $H_q^\omega$ denotes the class of functions $f \in L^q[0, 1]$ for which $\omega_q(\delta, f) = O(\omega(\delta))$ as $\delta \to 0^+$. 

In [4], a necessary and sufficient condition for the inclusion $\Lambda BV^{(p)} \subset H^\omega_1$, is given. Also, Wang [13] by using an interesting method found a necessary and sufficient condition for the embedding $H_q^\omega \subset \Lambda BV$. Here, we give a necessary and sufficient condition for the inclusion of $\Lambda BV^{(p)}$ in $H^\omega_q$.

2. MAIN RESULT

In [3], it was claimed that the following is true.

**Theorem 2.1.** For $q \in [1, \infty)$, the inclusion $\Lambda BV \subset H^\omega_q$ holds if and only if

$$\limsup_{n \to \infty} \frac{1}{\omega(1/n)n^{\frac{1}{q}}} \max_{1 \leq k \leq n} \frac{k^{\frac{1}{q}}}{\left( \sum_{i=1}^{k}\frac{1}{\lambda_i} \right)^{\frac{1}{q}}} < +\infty.$$  

The proof of sufficiency of the condition came up immediately from [5] and so, the main part of [3] concerns the proof of necessity. The theorem itself is correct but, unfortunately, there is a major mistake regarding the existence of some subsequences, which ensures that the proof of [3] is incorrect. To understand this, take $\omega(x) = x^{1/p}$. Then we can choose $\gamma_k = 2^k, \gamma'_k = n'_k = k$. If we take $s'_k = k$, then the condition for case (a) in [3] is satisfied, since $m(x) \leq 2^x$ by definition. But for subsequences $r_k$ of $s'_k$, relation (3) in [3], $\omega(\frac{1}{2^r}) \cdot 2^{r_k/p} \leq 4^{-k}$, is not true.

Our main result provides a characterization of the embedding of a generalization of $\Lambda BV$, Waterman-Shiba classes, into classes of functions with given integral modulus of continuity. Thus, by considering $p = 1$, the correctness of [3] Thm. 1] can be verified.

**Theorem 2.2.** For $p, q \in [1, \infty)$, the inclusion $\Lambda BV^{(p)} \subset H^\omega_q$ holds if and only if

$$\limsup_{n \to \infty} \left\{ \frac{1}{\omega(1/n)n^{\frac{1}{q}}} \max_{1 \leq k \leq n} \frac{k^{\frac{1}{q}}}{\left( \sum_{i=1}^{k}\frac{1}{\lambda_i} \right)^{\frac{1}{q}}} \right\} < +\infty.$$
Proof. To observe that equation (2) is a sufficiency condition for the inclusion \( \Lambda BV^{(p)} \subset H^q_{\omega} \), we prove an inequality which gives us the sufficiency:

\[
\omega\left(\frac{1}{n}, f\right)_q \leq V(f) \left\{ \frac{1}{n} \max_{1 \leq k \leq n} \frac{k}{(\sum_{j=1}^{k} 1/\lambda_j)^q}\right\}^{\frac{1}{q}}.
\]

Kuprikov [5] obtained Lemma 2.3 and Corollary 2.4 where \( q \geq 1 \).

Lemma 2.3. Let \( q \geq 1 \) and suppose \( F(x) = \sum_{i=1}^{n} x_i^q \) takes its maximum value under the following conditions

\[
\left(\sum_{i=1}^{n} \frac{x_i}{\lambda_i}\right) \leq 1,
\]

\[x_1 \geq x_2 \geq x_3 \geq ... \geq x_n \geq 0,
\]

then, \( x = (x_1, x_2, ..., x_n) \) satisfy

\[(3) \quad x_1 = x_2 = ... = x_k = \frac{1}{\sum_{j=1}^{k} 1/\lambda_j} > x_{k+1} = x_{k+2} = ... = x_n = 0,
\]

for some \( k \) \((1 \leq k \leq n)\).

Corollary 2.4. The maximum value of \( F(x) \), under the conditions of Lemma 2.3 is

\[
\max_{1 \leq k \leq n} \frac{k}{(\sum_{j=1}^{k} 1/\lambda_j)^q}.
\]

Lemma 2.5. Suppose \( 0 < q < 1 \) and the conditions of Lemma 2.3 hold, then

\[
\sum_{i=1}^{n} x_i^q = \frac{n}{(\sum_{j=1}^{n} 1/\lambda_j)^q} = \max_{1 \leq k \leq n} \frac{k}{(\sum_{j=1}^{k} 1/\lambda_j)^q}.
\]

Proof. Hölder inequality yields

\[
\sum_{i=1}^{n} x_i^q \leq n^{1-q} \sum_{i=1}^{n} x_i \leq \frac{n}{(\sum_{j=1}^{n} 1/\lambda_j)^q}.
\]

Thus \( F(x) \) takes its maximum when \( x_i = \frac{1}{\sum_{j=1}^{n} 1/\lambda_j} \) for \( 1 \leq j \leq n \). □

Now, we return to the proof of inequality:

\[
\omega_q\left(\frac{1}{n}, f\right)_q = \sup_{0 < h \leq \frac{1}{n}} \int_{0}^{1} |f(x + h) - f(x)|^q dx
\]

\[
= \sup_{0 < h \leq \frac{1}{n}} \int_{0}^{\frac{1}{n}} \sum_{k=1}^{n} |f(x + \frac{k-1}{n} + h) - f(x + \frac{k-1}{n})|^q dx.
\]
For \( h \leq \frac{1}{n} \) and fixed \( x \), denote \( x_k := |f(x + \frac{k-1}{n} + h) - f(x + \frac{k-1}{n})|^p \).

We reorder \( x_k \) such that

\[
x_1 \geq x_2 \geq \ldots \geq x_n \geq 0,
\]

\[
\left( \sum_{k=1}^{n} \frac{x_k}{\lambda_k} \right)^{\frac{1}{p}} \leq V(f).
\]

Therefore, by replacing \( q \) by \( q/p \) in Lemma 2.3, Lemma 2.5 and Corollary 2.4, we get

\[
\omega_q(\frac{1}{n}, f)^q = \sup_{0 < h \leq \frac{1}{n}} \int_0^{\frac{1}{n}} \sum_{k=1}^{n} x_k^p dx 
\]

\[
\leq \int_0^{\frac{1}{n}} V^q(f) \max_{1 \leq k \leq n} \frac{k}{(\sum_{i=1}^{k} 1/\lambda_i)^{\frac{1}{p}}} dx
\]

\[
= \frac{1}{n} V^q(f) \max_{1 \leq k \leq n} \frac{k}{(\sum_{i=1}^{k} 1/\lambda_i)^{\frac{1}{p}}}.
\]

**Necessity.** Suppose (2) doesn’t hold, that is, there are sequences \( n_k \) and \( m_k \) such that

(4) \( n_k \geq 2^{k+2} \),

(5) \( m_k \leq n_k \),

(6) \( \omega(\frac{1}{n_k}, f)^q n_k^q \cdot \left( \sum_{i=1}^{m_k} \frac{1}{\lambda_i} \right)^{\frac{1}{q}} < \frac{1}{24k} \),

where

\[
\max_{1 \leq \rho \leq n_k} \frac{\rho}{(\sum_{i=1}^{\rho} 1/\lambda_i)^{\frac{1}{p}}} = \frac{m_k}{(\sum_{i=1}^{m_k} 1/\lambda_i)^{\frac{1}{p}}}.
\]

Denote

(7) \( \Phi_k := \frac{1}{\sum_{i=1}^{m_k} 1/\lambda_i} \).

Consider

\[
g_k(y) := \begin{cases} 2^{-k} \Phi_k^{1/p}, & y \in \left[ \frac{1}{2^k}, \frac{1}{2^k} + \frac{1}{n_k} \right); 1 \leq j \leq N_k, \\ 0 & \text{otherwise}, \end{cases}
\]

where

(8) \( s_k = \max\{j \in \mathbb{N} : 2j \leq \frac{n_k}{2^k} + 1\} \),

and

(9) \( N_k = \min\{m_k, s_k\} \).
Hence, applying the fact $2(s_k + 1) \geq \frac{n_k}{2^{s_k}} + 1$ and (11), we have

\[
\frac{2s_k - 1}{n_k} \geq 2^{-k-1}.
\]

The functions $g_k$ have disjoint support. Thus $g := \sum_{k=1}^{\infty} g_k$ is a well-defined function on $[0, 1]$. Since

\[
\|g\| \leq \sum_{k=1}^{\infty} \|g_k\|
\]

\[
= \sum_{k=1}^{\infty} \left( \sum_{j=1}^{2N_k} \left| 2^{-k} \Phi_k^{1/p} \right| \frac{1}{\lambda_j} \right)^{1/p}
\]

\[
\leq \sum_{k=1}^{\infty} 2^{-k} \left( \sum_{j=1}^{2N_k} \left| \Phi_k \right| \frac{1}{\lambda_j} \right)^{1/p}
\]

\[
\leq \sum_{k=1}^{\infty} 2^{-k} \left( \sum_{j=1}^{m_k} \left| \Phi_k \right| \frac{1}{\lambda_j} \right)^{1/p}
\]

\[
\leq \sum_{k=1}^{\infty} 2^{-k} \left( \frac{\sum_{j=1}^{m_k} 1}{\sum_{j=1}^{m_k} 1/\lambda_j} \right)^{1/p}
\]

we observe that $g \in \Lambda BV(p)$.

If $N_k = m_k$, then

\[
\frac{(2N_k - 1)}{n_k} \cdot \left| \Phi_k \right|^\frac{q}{p} \geq \frac{1}{(\frac{n_k}{m_k}) \left( \sum_{i=1}^{m_k} 1/\lambda_i \right)^\frac{q}{p}}
\]

and if $N_k = s_k$ then

\[
\frac{(2N_k - 1)}{n_k} \cdot \left| \Phi_k \right|^\frac{q}{p} \geq 2^{-k-1} \cdot \frac{m_k}{n_k \left( \sum_{i=1}^{m_k} 1/\lambda_i \right)^\frac{q}{p}}.
\]
Since \( |g(x + \frac{1}{n_k}) - g(x)| = 2^{-k} \Phi_k^{1/p} \) for \( x \in [\frac{1}{2^k}, \frac{1}{2^k} + \frac{2N_k-1}{n_k}] \), we get

\[
\omega_q(\frac{1}{n_k}, g)^q = \sup_{0 < \gamma \leq \frac{1}{n_k}} \int_0^1 |g(x + \gamma) - g(x)|^q \, dx
\]

\[
\geq \int_0^1 |g(x + \frac{1}{n_k}) - g(x)|^q \, dx
\]

\[
\geq \int_{\frac{1}{2^k}}^{\frac{1}{2^k} + \frac{2N_k-1}{n_k}} |g(x + \frac{1}{n_k}) - g(x)|^q \, dx
\]

\[
= \frac{2N_k - 1}{n_k} - 2^{-kq} |\Phi_k|_p^2
\]

\[
\geq \frac{1}{2^{k+qk+1}} \cdot \frac{m_k}{n_k(\sum_{i=1}^{m_k} 1/\lambda_i)^p}
\]

and finally

\[
\frac{\omega_q(\frac{1}{n_k}, g)}{\omega(\frac{1}{n_k})} \geq \left( \frac{1}{2^{k+qk+1}} \right)^q \cdot \frac{1}{\omega(\frac{1}{n_k})} \cdot \left( \frac{1}{\left( \frac{m_k}{n_k} \right)^q (\sum_{i=1}^{m_k} 1/\lambda_i)^p} \right)
\]

\[
\geq 2^k \rightarrow +\infty,
\]

which shows that \( g \not\in H^q_\omega \). \( \square \)

Condition (2) simplifies when \( p \geq q \). Thus we have the following Corollary.

**Corollary 2.6.** For \( p, q \in [1, \infty) (p \geq q) \), the inclusion \( \Lambda BV(p) \subset H^q_\omega \)

holds if and only if

\[
\limsup_{n \rightarrow \infty} \left\{ \frac{1}{\omega(n/\sum_{i=1}^{n} 1/\lambda_i)^p} \right\} < +\infty.
\]

3. ACKNOWLEDGMENTS

The author is so grateful to Professor Hjalmar Rosengren for valuable comments, helpful discussions and for reviewing earlier drafts very carefully. The author also thanks Peter Hegarty for pointing out a mistake in an earlier version.
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