Online Convex Optimization with Classical and Quantum Evaluation Oracles

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Abstract

As a fundamental tool in AI, convex optimization has been a significant research field for many years, and the same goes for its online version. Recently, general convex optimization problem has been accelerated with the help of quantum computing, and the technique of online convex optimization has been used for accelerating the online quantum state learning problem, thus we want to study whether online convex optimization (OCO) model can also be benefited from quantum computing. In this paper, we consider the OCO model, which can be described as a $T$ round iterative game between the player and the adversary. A key factor for measuring the performance of an OCO algorithm $A$ is the regret denoted by regret$_T(A)$, and it is said to perform well if its regret is sublinear as a function of $T$. Another factor is the computational cost (e.g., query complexity) of the algorithm. We give a quantum algorithm for the online convex optimization model with only zeroth-order oracle available, which can achieve $O(\sqrt{T})$ and $O(\log T)$ regret for general convex loss functions and $\alpha$-strong loss functions respectively, where only $O(1)$ queries are needed in each round. Our results show that the zeroth-order quantum oracle is as powerful as the classical first-order oracle, and show potential advantages of quantum computing over classical computing in the OCO model where only zeroth-order oracle available.

1 Introduction

Convex optimization is a basic foundation of artificial intelligence, particularly in the field of machine learning. While many efficient algorithms have been developed [4, 7], people still hunger for more efficient solutions in the era of big data. Recently, since quantum computing shows advantages over classical computing [11, 12, 13, 28], people seek to employ quantum computing techniques to accelerate the convex optimization process [32, 8].

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However, while many studies focus on improving offline convex optimization with quantum computing techniques, few work considers applying the quantum computing methods to the problem of online convex optimization (OCO).

Online convex optimization is an important framework in online learning, particularly useful in sequential decision making problems, such as online routing [3], portfolio selection, and recommendation systems (See [15] for more information). In online convex optimization, an algorithm sequentially makes predictions and get a corresponding convex loss. A natural goal of the algorithm is to make its cumulative loss as less as possible.

In online learning, an algorithm is usually evaluated in two ways. First, we measure its performance in terms of the regret, which is defined as the difference between the total loss of the algorithm and that of the best fixed solution in hindsight. The other metric is the computational cost of the algorithm. It could be the time complexity, query complexity, or sample complexity per round, according to different settings.

In online learning, it usually assumes that there is a gradient oracle (first-order oracle) in each round such that the algorithm can access the gradient of the loss function directly and the gradient descent method can be applied. However, in this paper, we consider the setting that only evaluation oracles (zeroth-order oracles) are available, that is, the algorithm can only access to the value of the loss function at a given point. We argue that this setting is more practical since in many real-world applications the gradient is not always available. In this setting, we allow that the algorithm can query the oracle for multiple times in one round for model update and only the first query is counted as loss accumulation. This assumption is reasonable since in many applications the system can interacts with a user for multiple rounds in a session or a period of short time. Since the evaluation of a loss function is usually the most time consuming part [29], we measure the computational cost of an algorithm in terms of the query complexity, specifically, the total number of queries the algorithm accesses the oracle.

In this paper, we consider two types of zeroth-order oracles. If the oracle is classical, we give an algorithm by using the most well-known gradient estimated method (finite difference method), and ensure that it can guarantee an $O(\sqrt{T})$ regret. $O(n)$ queries are needed in each round, where $n$ is the number of dimensions of the data. Then, we show that only $O(1)$ queries are sufficient in each round if the oracle is a quantum oracle, and the $O(\sqrt{T})$ regret can also be guaranteed, by giving a quantum algorithm. Furthermore, we show that the quantum algorithm can guarantee $O(\log T)$ regret when the convex loss functions are $\alpha$-strong. Our results show that the zeroth-order quantum oracle is as powerful as the classical first-order oracle, and show potential advantages of quantum computing over classical computing in the OCO model where only zeroth-order oracle is available.

The technical difficulty under this setting is that we need to ensure the sublinear regret to keep the algorithm performing well, in the presence of estimation errors caused by the zeroth-order oracle. The property of convexity is used differently from the setting where the first-order oracle is available.

This paper is organized as follow: we state the problem setting, our result and related work in Section 1; Section 2 is for the online convex optimization with classical evaluation oracles; Section 3 is for the online convex optimization with quantum evaluation oracles. Proofs are placed at Appendix A except the proofs of our main theorems.
1.1 Problem Setting and Our Results

The online convex optimization model can be described as a $T$-round iterative game between the player and the adversary (or the environment). At every iteration $t$, the player generates a prediction $x_t$ in a convex set $K \subseteq \mathbb{R}^n$. After committing the choice, the player suffers the loss $f_t(x_t)$ and gets some feedback information about the convex loss function $f_t$, which is chosen by the adversary.

Let $A$ be an algorithm chosen by the online player, which generates the prediction $x_t$ based on the game history $x_1, x_2, \ldots, x_{t-1}$ and feedback information from the past loss functions. We measure its performance in terms of the regret, which is defined as

$$\text{regret}_T(A) = \sum_{t=1}^{T} f_t(x_t) - \min_{x^* \in K} \sum_{t=1}^{T} f_t(x^*).$$

Note that in the definition of regret, the OCO model uses the best fixed solution in hindsight as the reference in each round.

An algorithm performs well if its regret is sublinear as a function of $T$, since this implies that on the average the algorithm performs as well as the best fixed strategy in hindsight [15]. Thus our goal is to find such a well-performing algorithm $A$.

In this paper, we assume that only the evaluation oracles to the loss functions are available for model update. Specifically, the algorithm is allowed to access the evaluation oracle for multiple times after committing the prediction for getting feedback in each round. Here we consider two types of oracles, the classical oracle and the quantum oracle. In a classical evaluation oracle $O_f$ to the loss function $f$, queried with a vector $x \in K$, the oracle outputs $O_f(x) = f(x)$. Similarly, in a quantum evaluation oracle $Q_f$ to the loss function $f$, queried with a quantum state $|x\rangle |q\rangle$, the oracle outputs the quantum state $|x\rangle |q + f(x)\rangle$ in the way of numerical representation.

In addition, as usually in online convex optimization, we also make the following assumptions: The loss functions are $G$-Lipschitz continuous, that is, $|f_t(x) - f_t(y)| \leq G\|y - x\|$, $\forall x, y \in K$; the feasible set $K$ is bounded and its diameter has an upper bound $D$, that is, $\forall x, y \in K, \|x - y\|_{\infty} \leq D$. $K, D, G$ are known to the player.

In this paper, we first give a straightforward classical algorithm for the case with zeroth-order classical oracles that can guarantee an $O(\sqrt{T})$ regret and its query complexity is $O(n)$ in each round in Section 2.

Then we show that $O(1)$ queries are sufficient if the oracle is a zeroth-order quantum oracle, and the $O(\sqrt{T})$ regret can also be guaranteed, as the following result.

**Result 1.** (Informal version of Theorem 2) In online convex optimization problems with quantum evaluation oracles, there exists a quantum algorithm that can achieve the regret bound $O(DG\sqrt{T}/\delta)$, with probability $1 - \delta$, and its query complexity is $O(1)$ in each round.

Further, we show that, for $\alpha$-strong loss functions, the $O(\log T)$ regret can be guaranteed by the quantum algorithm with the same query complexity.

**Result 2.** (Informal version of Theorem 3) In online convex optimization problems with quantum evaluation oracles and $\alpha$-strong convex loss functions, there exists a quantum
algorithm that can achieve the regret bound $O(DG^2 \log T/\delta)$, with probability $1 - \delta$, and its query complexity is $O(1)$ in each round.

Our results show that:

- The zeroth-order quantum oracle is as powerful as the classical first-order oracle because they both achieve the same regret bound for both convex loss functions and $\alpha$-strong loss functions, in $O(T)$ queries;

- The quantum computing potentially outperforms classical computing in the zeroth-order OCO model because, to the best we know, the classical algorithm needs $O(n)$ queries each round to achieve $O(\sqrt{T})$ regret, where $n$ is the number of dimensions of the data, while the quantum algorithm needs only $O(1)$ queries each round.

1.2 Related work

People have sought to employ the quantum computing techniques to accelerate the optimization process for a long time. They seized every opportunity to improve the optimization efficiency, thus powerful technology like quantum computing is no exception. They tried to apply quantum computing to discrete optimization first [2, 9, 10, 11], using quantum techniques such as Grover’s algorithm or quantum walks. Then they turned their attention to continuous optimization. In the last few years, some significant quantum improvements were achieved in polynomial optimization[26], and semidefinite optimization [30, 5, 20, 21, 31], which are all special cases of convex optimization. In the last two years, general convex optimization were accelerated eventually [32, 8], where the query complexity of membership oracle $O(n)$ and evaluation oracle $O(n)$ were concerned. Quantum lower bound $\tilde{\Omega}(\sqrt{n})$ was shown which promoted people to find better quantum algorithm to solve the general convex optimization problem. In this year, quantum gradient descent for linear systems and least squares was proposed[22], which significantly reduces the dependence on the number of iterations in time complexity for quantum iterative methods and the problem of it is still convex.

We can see that many studies focus on improving offline convex optimization with quantum computing techniques. How about online convex optimization? We have already known that the quantum state learning problem can be benefited from the technology of online convex optimization[33, 1], thus we want to study whether online convex optimization (OCO) model can also be benefited from quantum computing.

Online convex optimization is impressive in the last decade since it was first proposed [34]. Further study of the efficient algorithms and application were carried out later [16]. The relation between online learning and OCO was claimed in [27], and the systematic introduction of its technique can be seen in [14, 15]. Few work considers applying the quantum computing methods to OCO. Thus, in this paper, we focus on the general OCO model and show potential advantages of a quantum oracle over a classical oracle at least in terms of the query complexity.

At last, we wonder whether there exist quantum algorithms for OCO with constraints after noticing [18, 24]. We leave them as open problems.
2 Online convex optimization with classical evaluation oracles

This section aims to prove Theorem 1. We first give a classical OCO algorithm using the evaluation oracle, which is stated in Algorithm 1, and analyze the query complexity. Then, we analyze its performance and show how we choose those appropriate parameters in Algorithm 1 to ensure that it performs well.

Here we give the classical OCO algorithm. By combining online gradient descent with finite difference method, and introducing randomness following the idea of [23], we get the following:

Algorithm 1 $A_1$

Require: convex set $K$, total round $T$, initial point $x_1 \in K$, step sizes $\{\eta_t\}$, parameters $r_1, r_2$

Ensure: $x_2, x_3, \ldots, x_T$

1: for $t = 1$ to $T$ do
2: play $x_t$, get the oracle of loss function $O_{f_t}$
3: Sample $z \in B_{\infty}(x_t, r_1)$
4: for $j = 1$ to $n$ do
5: $\nabla_{(r_2)}^{(r_2)} f_t(z) = \frac{O_{f_t}(z+r_2 e_j)-O_{f_t}(z-r_2 e_j)}{2r_2}$
6: end for
7: $\tilde{\nabla} f_t(x_t) = \left(\nabla_{1}^{(r_2)} f(z), \nabla_{2}^{(r_2)} f(z), \ldots, \nabla_{n}^{(r_2)} f(z)\right)$
8: update $x_{t+1} = \Pi_{K}(x_t - \eta_t \tilde{\nabla} f_t(x_t))$
9: end for

where the projection operation in Step 8 is defined as $\Pi_{K}(y) \triangleq \arg \min_{x \in K} \|x-y\|; B_{\infty}(x, r)$ is the ball in $L_{\infty}$ norm with radius $r$ and center $x$.

Remark: In general, we can hardly know the round number $T$ of game beforehand, so we use the doubling tricks, a well known technique, to handle this situation, only with an additional $\sqrt{2}$ coefficient in the regret.

Now we analyze the query complexity of Algorithm 1. In each round, it needs to call the oracle twice to compute each partial derivative, so totally $2n$ times for compute the gradient. Thus, $O(n)$ times for each round.

Next, we show that Algorithm 1 guarantees $O(DG\sqrt{T})$ regret for all $T \geq 1$ under the setting of our paper, which means that it performs as well as online gradient descent with gradient oracle [15].

To prove this, we need some additional technical lemmas. We first give the error bound of gradient evaluation in each round (Lemma 1), and then we give the subgradient bound in each round (Lemma 2). Based on Lemma 1 and Lemma 2, we obtain the subgradient bound for all $T$ rounds (Lemma 3). At last, we prove the regret bound, which gives Theorem 1. Note that we omit the subscript $t$ in the statement of the lemmas as they hold for each round.

The evaluating error of gradient of $f_t$ at point $z$ in each round of Algorithm 1 can
be bounded. By using the similar technique of analyzing the finite difference method, we get Lemma 1.

**Lemma 1.** In each round of Algorithm 1, if \( r_1 \geq r_2 > 0 \), and \( f : B_\infty(x, r_1 + r_2) \to \mathbb{R} \) is convex with Lipschitz parameter \( G \), then

\[
\mathbb{E}_{z \in B_\infty(x, r_1)} \| g - \tilde{\nabla} f(x) \|_1 \leq \frac{nGr_2}{2r_1},
\]

where \( g \) is the gradient of \( f \) on point \( z \).

See Lemma 10 and Lemma 11 of [32] for the proof of Lemma 1.

The evaluating error of subgradient of \( f \) at point \( x \) in each round can be bounded. By using convexity and simple equivalence transformation, we get Lemma 2. See Appendix A for the proof of Lemma 2.

**Lemma 2.** In each round of Algorithm 1, if \( r_1 \geq r_2 > 0 \), \( z \in B_\infty(x, r_1) \) and \( f : K \to \mathbb{R} \) is convex with Lipschitz parameter \( G \), where \( K \) is a convex set, then for any \( y \in K \), \( \tilde{\nabla} f(x) \), the gradient of each round, satisfy

\[
f(y) \geq f(x) + \tilde{\nabla} f(x)^T (y - x) - \frac{nGr_2 \| y - x \|_\infty}{2r_1},
\]

where \( g \) is the gradient of \( f \) on point \( z \).

Combining Lemma 2 with Lemma 1, we get Lemma 3.

**Lemma 3.** If \( r_1 \geq r_2 > 0 \), and \( f : K \to \mathbb{R} \) is convex with Lipschitz parameter \( G \), where \( K \) is a convex set with diameter \( D \), then for any \( y \in K \), with probability \( 1 - \rho \), the gradient of each round, \( \tilde{\nabla} f(x) \), satisfies

\[
f(y) \geq f(x) + \tilde{\nabla} f(x)^T (y - x) - \frac{nGr_2 D}{2\rho r_1} - 2G \sqrt{n} r_1.
\]

Note that the Inequality (3) is required to hold for all \( T \) rounds, then by union bound, the probability that the Algorithm 1 fails to satisfy Inequality (3) at least one round is less than \( T \rho \), which means the probability that the Algorithm 1 succeeds for all \( T \) round is greater than \( 1 - T \rho \). See Appendix A for the proof of Lemma 3.

At last we give the regret bound.

**Theorem 1.** In online convex optimization problems with classical evaluation oracles, algorithm \( A_1 \) with parameter \( \eta_t = \frac{D}{G \sqrt{T}}, r_1 = \frac{1}{\sqrt{n} T}, r_2 = \frac{1}{T \sqrt{n}} \) can achieve the regret bound \( O(DG \sqrt{T} / \delta) \), with probability \( 1 - \delta \), and its query complexity is \( O(n) \) in each round.

See Appendix A for the proof of Theorem 1.
3 Online convex optimization with quantum evaluation oracle

This section aims to prove Result 1 and 2. We assume that the convex loss functions are $\beta$-smooth in Subsection 3.1 as the quantum gradient technology is quite suitable for this situation, give a quantum OCO algorithm (Algorithm 2) using the evaluation oracle, and analyze the query complexity of it. Then in Subsection 3.2, we extend the algorithm to non $\beta$-smooth version by showing that the loss function is $\beta$-smooth in a small region with high probability by bounded the trace of the Hessian matrix. We also analyze its performance and show how we choose those appropriate parameters in Algorithm 2 to ensure that it achieves the sub-linear regret bound, which gives the Result 1. Finally in Subsection 3.3, we show that for $\alpha$-strong convex loss function, the $O(\log T)$ regret can be guaranteed by Algorithm 2, which gives the Result 2.

3.1 $\beta$-smooth convex loss functions

Here we give the quantum OCO algorithm for $\beta$-smooth convex loss functions. By combining the idea of online gradient descent with quantum gradient estimation [19], and introducing randomness following the idea of [23], we get the Algorithm 2.

Fundamental knowledge of quantum computing can be found in [25], where the Dirac notation can be found in chapter 2, Hadamard transform and quantum inverse Fourier transformation can be found in chapter 4.2 and 5.1. The circuit of quantum gradient evaluating method in each round is shown in Figure 1.

Note that the quantum circuit of $Q_F$ in step 6 is constructed after the sampling of $z$ by using $Q_f$ twice; $e$ in step 6 is the $n$-dimensional all 1’s vector; the last register and the operation of addition modulo 2 in step 7 are used for implementing the common technique in quantum algorithm known as phase kickback; step 8 is known as uncompute trick; the projection operation is defined as $\prod_K(y) \triangleq \arg\min_{x\in K} \|x-y\|;

$B_\infty(x, r)$ is the ball in $L_\infty$ norm with radius $r$ and center $x$.

Now we analyze the query complexity of Algorithm 2. In each round, it needs to call the oracle twice to construct $Q_F$, and twice to perform the uncompute step $Q_F^{-1}$, so totally 4 times for compute the gradient. Thus, $O(1)$ times for each rounds.

The evaluating error of the gradient can be bounded (See Lemma 4). In this subsection, we suppose the loss function is $\beta$-smooth.

**Lemma 4.** In each round of Algorithm 2, if $r_1 \geq r_2 > 0$, and $f : B_\infty(x, r_1 + r_2) \rightarrow \mathbb{R}$ is $\beta$-smooth convex function with Lipschitz parameter $G$, $g$ is the gradient of $f$ on point $z$, then

$$\Pr\left[\|g - \nabla f(x)\|_1 > 96\pi n^2 \beta r_2 \right] < \frac{1}{3}, \forall i \in [n]$$

(4)

See Appendix A for the proof of Lemma 4.

We can repeat this process constant times to get nearly 100% probability of success.
Figure 1: Quantum gradient estimation. Quantum circuit is a general model for describing quantum algorithms. The meaning of quantum circuits is similar to that of classic circuits except that the quantum gates need to be reversible transformations and measurements are needed to get classical information (the rightmost gates).

Algorithm 2 $A_2$

Require: convex set $\mathcal{K}$, total round $T$, initial point $x_1 \in \mathcal{K}$, step sizes $\{\eta_t\}$, parameters $r_1, r_2$

Ensure: $x_2, x_3, \ldots, x_T$

1: for $t = 1$ to $T$ do
2: play $x_t$, get the oracle of loss function $Q_f$
3: Sample $z \in B_\infty(x_t, r_1)$
4: Prepare the initial state: $n$ b-qubits registers $|0^\otimes b, 0^\otimes b, \ldots, 0^\otimes b\rangle$ where $b = \log_2 \frac{6G}{2n \pi r_2}$. Prepare $1$ c-qubits register $|0^\otimes c\rangle$ where $c = \log_2 \frac{4G}{2n r_2^2} - 1$.
5: Perform Hadamard transform to the first $n$ register
6: Perform the quantum query oracle $Q_F$, where $F(u) = \frac{2^b}{2Gr_2} [f(z + \frac{2^b}{2Gr_2}(u - \frac{2^b}{2Gr_2})) - f(z)]$ to the first $n + 1$ register, and the result is stored in the $(n + 1)$th register.
7: Perform the addition modulo $2^c$ operation to the last two registers.
8: Perform the inverse evaluating oracle $O_{-1}^F$ to the first $n + 1$ register.
9: Perform the quantum inverse Fourier transformation to the first $n$ registers respectively.
10: Measure the first $n$ registers in computation bases respectively to get $m_1, m_2, \ldots, m_n$.
11: $\widetilde{\nabla} f_t(x_t) = \frac{2G}{2^c}(m_1 - \frac{2^b}{2}, m_2 - \frac{2^b}{2}, \ldots, m_n - \frac{2^b}{2})^T$
12: update $x_{t+1} = \prod_{K}(x_t - \eta_t \widetilde{\nabla} f_t(x_t))$
13: end for
3.2 Non $\beta$-smooth convex loss functions

In this subsection, we show that Algorithm 2 guarantees $O(DG\sqrt{T})$ regret for all $T \geq 1$ under the setting of our paper, which means that it performs as well as online gradient descent with gradient oracle [15]. To prove this, we need to show that the loss function is $\beta$-smooth in a small region with high probability (Lemma 5) for the further analysis of the error bound of quantum gradient estimation (Lemma 4). Combine with Lemma 2 we give the subgradient bound for all $T$ rounds (Lemma 6). At last we prove the regret bound, which gives Theorem 1. Note that we omit the subscript $t$ in the statement of the lemmas as they hold for each round.

Firstly, we show that the non $\beta$-smooth loss functions are still $\beta$-smooth in a small region with high probability by bounded the trace of the Hessian matrix (See Lemma 5). Note that we can use the mollification of $f$, a infinitely differentiable convex function with the same Lipschitz parameter of $f$, to approximate $f$ with the approximated error much less than the evaluating error, by choosing appropriate width of mollifier [8, 17].

Lemma 5. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a twice differentialble $G$-Lipschitz function. Then for any $r_1 > 0$,

$$
\Pr_{z \in B_{\infty}(x, r_1)} \left[ \exists y \in B_{\infty}(z, r_2), \text{Tr}\{\nabla^2 f(y)\} \geq \frac{nG}{pr_1} \right] \leq (1 + (2r_2)^n)p
$$

See Lemma 2.5 and 2.6 of [8] for the proof.

Combine Lemma 4 with Lemma 2 and 5, we have the subgradient bound of Algorithm 2 (See Lemma 6).

Lemma 6. If $r_1 \geq r_2 > 0$, and $f : K \rightarrow \mathbb{R}$ is convex with Lipschitz parameter $G$, where $K$ is a convex set with diameter $D$, $g$ is the gradient of $f$ on point $z$, then for any $y \in K$, with probability $1 - \rho$, $\tilde{\nabla} f(x)$, the gradient of each round, with parameter $r_1, r_2$, satisfy

$$
f(y) \geq f(x) + \tilde{\nabla} f(x)^T (y - x) - \frac{96\pi n^3 G r_2 D}{pr_1} - 2G\sqrt{n} r_1
$$

Note that the Inequality (6) is required to hold for all $T$ rounds, then by union bound, the probability that the Algorithm 2 fails to satisfy Inequality (6) at least one round is less than $T \rho$, which means the probability that the Algorithm 2 succeeds for all $T$ round is greater than $1 - T \rho$. See Appendix A for the proof of Lemma 6.

At last, we show how to choose those appropriate step sizes and parameters, and prove the regret bound, which gives Theorem 2 (Result 1).

Theorem 2. In online convex optimization problems with quantum evaluation oracles, the algorithm $A_2$ with parameter $\eta_t = \frac{D}{G\sqrt{t}}, r_1 = \frac{1}{\sqrt{t}}, r_2 = \frac{1}{96\pi \sqrt{t}}$, can achieve the regret bound $O(DG\sqrt{T}/\delta)$, with probability $1 - \delta$, and its query complexity is $O(1)$ in each round.
Proof. Let \( x^* \in \arg \min_{x \in \mathcal{K}} \sum_{t=1}^{T} f_t(x) \). By Lemma 6, for the fixed \( y = x^* \), with probability \( 1 - \delta \) (where \( \delta = T \rho \)) we have

\[
f_t(x_t) - f_t(x^*) \leq \nabla f_t(x_t)^T (x_t - x^*) + \frac{96T \pi n^3 Gr_2 D}{\delta r_1} + 2G \sqrt{nr_1} \tag{7}
\]

By the update rule for \( x_t + 1 \) and the Pythagorean theorem

\[
\|x_{t+1} - x^*\|^2 = \|\prod_{t \in \mathcal{K}} (x_t - \eta_t \nabla f_t(x_t)) - x^*\|^2 \\
\leq \|x_t - \eta_t \nabla f_t(x_t) - x^*\|^2 \\
= \|x_t - x^*\|^2 + \eta_t^2 \|\nabla f_t(x_t)\|^2 \\
- 2\eta_t \nabla f_t(x_t)^T (x_t - x^*) \tag{8}
\]

Hence

\[
\nabla f_t(x_t)^T (x_t - x^*) \leq \frac{\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2}{2\eta_t} + \frac{\eta_t \|\nabla f_t(x_t)\|^2}{2} \tag{9}
\]

Substitute inequation 9 into inequation 7 and summing inequation 7 from \( t = 1 \) to \( T \), we have

\[
\sum_{t=1}^{T} (f_t(x_t) - f_t(x^*)) \\
\leq \sum_{t=1}^{T} (\nabla f_t(x_t)^T (x_t - x^*) + \frac{96T \pi n^3 Gr_2 D}{\delta r_1} + 2G \sqrt{nr_1}) \\
\leq \sum_{t=1}^{T} \left( \frac{\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2}{2\eta_t} + \frac{\eta_t \|\nabla f_t(x_t)\|^2}{2} + \frac{96T \pi n^3 Gr_2 D}{\delta r_1} + 2G \sqrt{nr_1} \right) \tag{10}
\]
We deal with this inequality term-by-term. For the first term, define \( \frac{1}{\eta_0} := 0 \),

\[
\frac{1}{2} \sum_{t=1}^{T} \left( \|x_t - x^*\|_2^2 - \|x_{t+1} - x^*\|_2^2 \right) \leq \frac{1}{2} \sum_{t=1}^{T} \left( \frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right)
\]

For the second term,

\[
\sum_{t=1}^{T} \eta_t \|\bar{\nabla} f_t(x_t)\|_2^2 \leq \sum_{t=1}^{T} \eta_t \|\bar{\nabla} f_t(x_t) - g + g\|_2^2 \leq \sum_{t=1}^{T} \eta_t \left( \|\bar{\nabla} f_t(x_t) - g\|_1 + G \right)^2 \leq \sum_{t=1}^{T} \frac{\eta_t \left( 96\pi n^3 G^2 \mathcal{D} \sqrt{r^*} + G \right)^2}{2}
\]

Setting \( \eta_t = \frac{D}{G \sqrt{t}} \) (where \( \frac{1}{\eta_0} := 0 \), \( r_1 = \frac{1}{\sqrt{tn}} \), \( r_2 = \frac{1}{96T \pi \sqrt{r^* n^3}} \), we have

\[
\sum_{t=1}^{T} (f_t(x_t) - f_t(x^*)) \leq \frac{DG \sqrt{T}}{2} + \sum_{t=1}^{T} \frac{D(G + 1)^2}{2G \sqrt{T}} + \sum_{t=1}^{T} \frac{DG}{2 \delta \sqrt{T}} + \sum_{t=1}^{T} \frac{2G}{\sqrt{T}}
\]

\[
\leq \frac{DG \sqrt{T}}{2} + \sum_{t=1}^{T} \frac{D(G + 1)^2}{2G \sqrt{T}} + \frac{DG \sqrt{T}}{2 \delta} + 2G \sqrt{T}
\]

\[
\leq \frac{DG \sqrt{T}}{2} + \sum_{t=1}^{T} \frac{D(G + 1)^2}{2G \sqrt{T}} + \frac{DG \sqrt{T}}{2 \delta} + 2G \sqrt{T}
\]

\[
= O(DG \sqrt{T}/\delta)
\]
which give the theorem.

3.3 α-strong convex loss functions

At this subsection, we show that for α-strong convex loss function, the $O(\log T)$ regret can be guaranteed by Algorithm 2. The key trick is to use strong convexity at the calculated point $z$ instead of the estimated point $x$. After simple equivalence transformation, we get the subgradient bound (Lemma 7). See Appendix A for the proof.

**Lemma 7.** In each round of Algorithm 2, if $r_1 \geq r_2 > 0$, $z \in B_{\infty}(x, r_1)$ and $f : K \to \mathbb{R}$ is α-strong convex with Lipschitz parameter $G$, where $K$ is a convex set, then for any $y \in K$, with probability $1 - \rho$, $\tilde{\nabla} f(x)$, the gradient of each round, satisfy

$$f(y) \geq f(x) + \tilde{\nabla} f(x)^T (y - x) - \frac{96\pi n^3 G r_2 D}{\rho r_1} - (2G\sqrt{n} + \alpha nD) r_1 + \frac{\alpha}{2} \|y - x\|^2,$$  

(14)

where $g$ is the gradient of $f$ on point $z$.

At last, we show how to choose those appropriate step sizes and parameters for Algorithm 2 to achieve the $O(\log T)$ regret, which gives Theorem 3 (Result 2).

**Theorem 3.** In online convex optimization problems with quantum evaluation oracles and α-strong convex loss functions, the algorithm $A_2$ with parameter $\eta_t = \frac{1}{\alpha t}, r_1 = \frac{1}{n}, r_2 = \frac{96 T \pi n^3}{\alpha}$, can achieve the regret bound $O(DG^2 \log T/\delta)$, with probability $1 - \delta$, and its query complexity is $O(1)$ in each round.

**Proof.** Let $x^* \in \arg \min_{x \in K} \sum_{t=1}^T f_t(x)$. By Lemma 7, for the fixed $y = x^*$, with probability $1 - \delta$ (where $\delta = T \rho$) we have

$$f_t(x_t) - f_t(x^*) \leq \tilde{\nabla} f_t(x_t)^T (x_t - x^*) + \frac{96T \pi n^3 G r_2 D}{\delta r_1} + (2G\sqrt{n} + \alpha nD) r_1 - \frac{\alpha}{2} \|x_t - x^*\|^2$$  

(15)

By the update rule for $x_{t+1}$ and the Pythagorean theorem

$$\|x_{t+1} - x^*\|^2 = \prod_{K} (x_t - \eta_t \tilde{\nabla} f_t(x_t)) - x^* \|^2$$

$$\leq \|x_t - \eta_t \tilde{\nabla} f_t(x_t) - x^*\|^2$$

$$= \|x_t - x^*\|^2 + \eta_t^2 \|\tilde{\nabla} f_t(x_t)\|^2$$

$$- 2\eta_t \tilde{\nabla} f_t(x_t)^T (x_t - x^*)$$  

(16)

Hence

$$\tilde{\nabla} f_t(x_t)^T (x_t - x^*) \leq \|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2$$

$$+ \frac{\eta_t \|\tilde{\nabla} f_t(x_t)\|^2}{2}$$  

(17)

12
Substitute inequation 17 into inequation 15 and summing inequation 15 from $t = 1$ to $T$, we have

$$
\sum_{t=1}^{T} (f_t(x_t) - f_t(x^*))
\leq \sum_{t=1}^{T} \left( \nabla f_t(x_t)^T (x_t - x^*) + \frac{96T\pi n^3 Gr_2 D}{\delta r_1} \right. \\
+ (2G\sqrt{n} + \alpha n D) r_1 - \frac{\alpha}{2} \|x_t - x^*\|^2 \\
\left. \right) \\
\leq \sum_{t=1}^{T} \left( \frac{\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2}{2\eta_t} + \frac{\eta_t \|\nabla f_t(x_t)\|^2}{2} \right. \\
+ \frac{96T\pi n^3 Gr_2 D}{\delta r_1} \left. \right) \\
+ (2G\sqrt{n} + \alpha n D) r_1 - \frac{\alpha}{2} \|x_t - x^*\|^2 \\
\tag{18}
$$

For the first term and the last term, define $\frac{1}{\eta_0} := 0$,

$$
\sum_{t=1}^{T} \left( \frac{\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2}{2\eta_t} - \frac{\alpha}{2} \|x_t - x^*\|^2 \right)
\leq \frac{1}{2} \sum_{t=1}^{T} \left( \frac{\|x_t - x^*\|^2}{\eta_t} - \frac{\alpha}{2} \|x_t - x^*\|^2 \right)
\leq \frac{D^2}{2} \sum_{t=1}^{T} \left( \frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} - \alpha \right)
= \frac{D^2}{2} \left( \frac{1}{\eta T} - \alpha T \right) \\
\tag{19}
$$

The handing of the second term are the same as Equation (12). Setting $\eta_t = \frac{1}{\alpha t}$ (where $\frac{1}{\eta_0} := 0$), $r_1 = \frac{1}{t^n}$, $r_2 = \frac{1}{96T\pi n T}$, we have

$$
\sum_{t=1}^{T} (f_t(x_t) - f_t(x^*))
\leq \frac{D^2}{2} (\alpha T - \alpha T) + \sum_{t=1}^{T} \frac{(4G + G)^2}{2\alpha t} + \sum_{t=1}^{T} \frac{DG}{2\delta t} \\
+ \sum_{t=1}^{T} \left( \frac{2G}{\sqrt{nt}} + \frac{\alpha D}{t} \right) \\
\leq \frac{G^2 (1 + 1)^2}{2\alpha} \log T + \frac{DG \log T}{2\delta} + \frac{2G}{\sqrt{n}} + \alpha D \log T \\
= O(DG^2 \log T/\delta) \\
\tag{20}
$$

which give the theorem. □
So far, we have completed all the proofs that the zeroth-order quantum oracle achieves the same regret level as first-order classical algorithm for both general convex loss functions and $\alpha$-strong loss functions, and both of them need $O(T)$ queries in total.

4 Conclusion

In this paper, we study online convex optimization with evaluation oracles instead of gradient oracles, i.e., with zeroth-order oracles instead of first-order oracles. In summary, we can conclude that the zeroth-order quantum oracle is as powerful as the classical first-order oracle because they both achieve the same regret bound, in the same query complexity, for both general convex loss functions and $\alpha$-strong loss functions. Further, our results show that the quantum computing potentially outperforms classical computing in the zeroth-order OCO model because, to the best we know, the classical algorithm using finite difference method, the most popular method of gradient estimation, needs $O(n)$ queries each round to achieve $O(\sqrt{T})$ regret, where $n$ is the number of dimensions of the data, while the quantum algorithm needs only $O(1)$ queries each round.

In future work, the lower bound of query complexity of the classical zeroth-order oracle is still needed to be proved to show the quantum advantage rigorously, we leave it as an open problem. On the other hand, since the query complexity of zeroth-order quantum oracle has been achieved $O(T)$ and the regret bound has been achieved the lower bound of this model, the only room for improvement is to decrease the dependence of the accuracy $\delta$ in the regret. We still expect the improvement. Further, it will be interesting to discuss some special setting of online convex optimization such as Bandit Convex optimization and projection-free algorithms, we will consider to make use of the quantum acceleration in these fields in the future.

A proof of lemmas and theorems

At this appendix, we give the proof or the reference of lemmas which mentioned in the text.

Lemma 1: See [32] Lemma 10 and Lemma 11.

Lemma 2:
\textbf{Proof.} For any $y \in \mathbb{R}^n$, by convexity and simple equivalence transformation,

\begin{align*}
f(y) & \geq f(z) + \langle g, y - z \rangle \\
& = f(z) + \langle g, y - z \rangle + (\nabla f(x)^T (y - x) \\
& \quad - \nabla f(x)^T (y - x) + (f(x) - f(z)) \\
& = f(x) + \nabla f(x)^T (y - x) + (g - \nabla f(x))^T (y - x) \\
& \quad + (f(z) - f(x)) + g^T (x - z) \\
& \geq f(x) + \nabla f(x)^T (y - x) - \|g - \nabla f(x)\|_1 \|y - x\|_\infty \\
& \quad - G \|z - x\|_2 + \|g\|_2 \|x - z\|_2 \\
& \geq f(x) + \nabla f(x)^T (y - x) - \|g - \nabla f(x)\|_1 \|y - x\|_\infty \\
& \quad - 2G \sqrt{n} r_1. \\
\end{align*}

(21)

\textbf{Lemma 3:}

\textbf{Proof.} By Lemma 1 and Markov’s inequality, we have

\[ \Pr[\|g - \nabla f(x)\|_1 \leq \frac{nG r_2}{2 G \rho}] \geq 1 - \rho, \] 

(22)

combining with Lemma 2, we have

\[ f(y) \geq f(x) + \nabla f(x)^T (y - x) - \frac{nG r_2 D}{2 \rho r_1} - 2G \sqrt{n} r_1. \]

(23)

succeed with probability $1 - \rho$. \hfill \Box

\textbf{Theorem 1:}

\textbf{Proof.} Let $x^* \in \arg \min_{x \in K} \sum_{t=1}^T f_t(x)$. By Lemma 3, for the fixed $y = x^*$, with probability $1 - \delta$ (where $\delta = T \rho$) we have

\[ f_t(x_t) - f_t(x^*) \leq \nabla f_t(x_t)^T (x_t - x^*) + \frac{T nG r_2 D}{2 \rho r_1} + 2G \sqrt{n} r_1 \]

(24)

By the update rule for $x_{t+1}$ and the Pythagorean theorem

\[ \|x_{t+1} - x^*\|^2 = \left\| \prod_{k \in K} (x_t - \eta_t \nabla f_k(x_t)) - x^* \right\|^2 \]

\[ \leq \|x_t - \eta_t \nabla f_k(x_t) - x^*\|^2 \]

\[ = \|x_t - x^*\|^2 + \eta_t^2 \|\nabla f_k(x_t)\|^2 \]

\[ - 2\eta_t \nabla f_k(x_t)^T (x_t - x^*) \]

(25)
\[
\n\n\begin{align*}
\tilde{\nabla} f_t(x_t)^T(x_t - x^*) & \leq \frac{\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2}{2\eta_t} \\
& \quad + \eta_t \|\tilde{\nabla} f_t(x_t)\|^2 \\
& \quad + \frac{TnGr_2D}{2\delta r_1} + 2G\sqrt{nr_1}.
\end{align*}
\]

(26)

Substituting (26) into (24) and summing (24) from \(t = 1\) to \(T\), we have

\[
\sum_{t=1}^{T} (f_t(x_t) - f_t(x^*)) \leq \sum_{t=1}^{T} \left( \tilde{\nabla} f_t(x_t)^T(x_t - x^*) + \frac{TnGr_2D}{2\delta r_1} + 2G\sqrt{nr_1} \right)
\]

\[
\leq \sum_{t=1}^{T} \left( \frac{\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2}{2\eta_t} + \eta_t \|\tilde{\nabla} f_t(x_t)\|^2 \right)
\]

\[
+ \frac{TnGr_2D}{2\delta r_1} + 2G\sqrt{nr_1}.
\]

(27)

The handling of the first term and the second term are the same as equation (11) (12).

Hence, setting \(\eta_t = \frac{D}{G\sqrt{t}}\) (where \(\eta_0 := 0\), \(r_1 = \frac{1}{\sqrt{tn}}, r_2 = \frac{1}{2T\sqrt{tn}}\), we have

\[
\sum_{t=1}^{T} (f_t(x_t) - f_t(x^*)) \leq \frac{1}{2} DG\sqrt{T} + \sum_{t=1}^{T} \frac{DG(1 + 1)^2}{2\delta} + \sum_{t=1}^{T} \frac{2G}{\sqrt{t}}
\]

\[
\leq \frac{1}{2} DG\sqrt{T} + \sum_{t=1}^{T} \frac{DG(1 + 1)^2}{2\delta} + \frac{DG\sqrt{T}}{2\delta} + 2G\sqrt{T}
\]

\[
\leq \frac{1}{2} DG\sqrt{T} + \sum_{t=1}^{T} \frac{DG(1 + 1)^2}{2\delta} + \frac{DG\sqrt{T}}{2\delta} + 2G\sqrt{T}
\]

\[
\leq \frac{1}{2} DG\sqrt{T} + \frac{DG\sqrt{T}}{2\delta} + \frac{DG\sqrt{T}}{2\delta} + 2G\sqrt{T}
\]

\[
= O(DG\sqrt{T}/\delta).
\]

(28)

Lemma 4:

Proof. The states after step 4 will be:

\[
\frac{1}{\sqrt{2^n}} \sum_{a \in \{0,1,\ldots,2^n-1\}} e^{\frac{2\pi i a}{2^n}} |0^{\otimes b}, 0^{\otimes b}, \ldots, 0^{\otimes b}\rangle |0^{\otimes c}\rangle |a\rangle.
\]

(29)
After step 5:

\[
\frac{1}{\sqrt{2^{bn+c}}} \sum_{u_1, u_2, \ldots, u_n \in \{0,1,\ldots,2^{n}-1\}} \sum_{a \in \{0,1,\ldots,2^{c}-1\}} e^{\frac{2\pi i a}{2^n}} |u_1, u_2, \ldots, u_n\rangle \langle 0^c | a\rangle.
\]  

(30)

After step 6:

\[
\frac{1}{\sqrt{2^{bn+c}}} \sum_{u_1, u_2, \ldots, u_n \in \{0,1,\ldots,2^{b}-1\}} \sum_{a \in \{0,1,\ldots,2^{c}-1\}} e^{\frac{2\pi i a}{2^n}} |u_1, u_2, \ldots, u_n\rangle \langle F(u) | a\rangle.
\]  

(31)

After step 7:

\[
\frac{1}{\sqrt{2^{bn+c}}} \sum_{u_1, u_2, \ldots, u_n \in \{0,1,\ldots,2^{b}-1\}} \sum_{a \in \{0,1,\ldots,2^{c}-1\}} e^{2\pi i F(u) + \frac{2\pi i a}{2^n}} |u_1, u_2, \ldots, u_n\rangle \langle F(u) | a\rangle.
\]  

(32)

After step 8:

\[
\frac{1}{\sqrt{2^{bn+c}}} \sum_{u_1, u_2, \ldots, u_n \in \{0,1,\ldots,2^{b}-1\}} \sum_{a \in \{0,1,\ldots,2^{c}-1\}} e^{2\pi i F(u)} e^{\frac{2\pi i a}{2^n}} |u_1, u_2, \ldots, u_n\rangle \langle 0^c | a\rangle.
\]  

(33)

We omit the last two registers on the rest of the proof because they have done their job and will keep unchanged on the rest of the proof. Which leave:

\[
\frac{1}{\sqrt{2^n}} \sum_{u_1, u_2, \ldots, u_n \in \{0,1,\ldots,2^{b}-1\}} e^{2\pi i F(u)} |u_1, u_2, \ldots, u_n\rangle.
\]  

(34)

And then we simply relabel the state, change \( u \rightarrow v = u - \frac{2^b}{2} \):

\[
\frac{1}{\sqrt{2^n}} \sum_{v_1, v_2, \ldots, v_n \in \{-2^{b-1}, -2^{b-1}+1, \ldots, 2^{b-1}\}} e^{2\pi i F(v)} |v\rangle.
\]  

(35)

We denote formula 35 as \( |\phi\rangle \). Consider the idealized state

\[
|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{v_1, v_2, \ldots, v_n \in \{-2^{b-1}, -2^{b-1}+1, \ldots, 2^{b-1}\}} e^{\frac{2\pi i g_i}{2G}} |v\rangle.
\]  

(36)

After step 9, from the analysis of phase estimation [6]:

\[
\Pr \left[ \left| \frac{N g_i}{2G} - m_i \right| < e \right] < \frac{1}{2(e-1)} \forall i \in [n],
\]  

(37)
let \( e = 4 \) we have

\[
\Pr \left[ \left| \frac{Nq_i}{2G} - m_i \right| > 4 \right] < \frac{1}{6}, \forall i \in [n], \tag{38}
\]

Note that the difference in the probabilities of measurement on \( \phi \) and \( \psi \) can be bounded by the trace distance between them:

\[
\| \langle \phi \rangle - \langle \psi \rangle \|_1 = 2\sqrt{1 - |\langle \phi |\psi \rangle|^2} \leq 2\| \phi - \psi \| \tag{39}
\]

Since \( f \) is \( \beta \)-smooth, we have

\[
F(v) \leq \frac{2b}{2Gr_2} [f(z + \frac{r_2v}{N}) - f(z)] + \frac{1}{2^{c+1}}
\leq \frac{2b}{2Gr_2} \frac{r_2g \cdot v + \beta r_2 v^2}{2^{b+2}} + \frac{1}{2^{c+1}}
\leq \frac{g \cdot v}{2G} + \frac{2b \beta r_2 n}{4G} + \frac{1}{2^{c+1}}. \tag{40}
\]

Then,

\[
\| \langle \phi \rangle - \langle \psi \rangle \|^2 = \frac{1}{2^{2m}} \sum_v |e^{2\pi i F(v)} - e^{2\pi i g \cdot v}|^2
\leq \frac{1}{2^{2m}} \sum_v |2\pi i F(v) - \frac{2\pi i g \cdot v}{2G}|^2
\leq \frac{1}{2^{2m}} \sum_v 4\pi^2 \left( \frac{2b \beta r_2 n}{4G} + \frac{1}{2^{c+1}} \right)^2. \tag{41}
\]

Set \( b = \log_2 \frac{G}{12\pi n \beta r_2} \), \( c = \log_2 \frac{4G}{2^3 \pi n \beta r_2} - 1 \), we have

\[
\| \langle \phi \rangle - \langle \psi \rangle \|^2 \leq \frac{1}{144}, \tag{42}
\]

which implies \( \| \langle \phi \rangle - \langle \psi \rangle \|_1 \leq \frac{1}{4} \). Therefore, by union bound,

\[
\Pr \left[ \left| \frac{2g_i}{2G} - m_i \right| > 4 \right] < \frac{1}{3}, \forall i \in [n], \tag{43}
\]

further,

\[
\Pr \left[ \left| g_i - \nabla_i f(x) \right| > \frac{8G}{2b} \right] < \frac{1}{3}, \forall i \in [n], \tag{44}
\]

as \( b = \log_2 \frac{G}{12\pi n \beta r_2} \)

\[
\Pr \left[ \left| g_i - \nabla_i f(x) \right| > 96\pi n \beta r_2 \right] < \frac{1}{3}, \forall i \in [n] \tag{45}
\]

Hence

\[
\Pr \left[ \| g - \nabla f(x) \|_1 > 96\pi n \beta r_2 \right] < \frac{1}{3}, \forall i \in [n] \tag{46}
\]
Proof. By lemma 4, we have
\[ \|g - \nabla f(x)\|_1 \leq 96\pi n^2 \beta r_2, \]  
(47)
succeeds with probability \( \frac{2}{3} \). Repeat constant times to get nearly 100% probability.

By lemma 5, we have \( \beta = \frac{\rho G}{r^2} \) succeeds with probability \( 1 - \rho \).

Combining with Lemma 2, we have
\[ f(y) \geq f(x) + \nabla f(x)^T (y - x) - \frac{96\pi n^3 Gr_2 D}{\rho r_1} - 2G\sqrt{n}r_1, \]  
(48)
success with probability \( 1 - \rho \).

Lemma 7:
Proof. For any \( y \in \mathbb{R}^n \), by strong convexity,
\[ f(y) \geq f(z) + \langle g, y - z \rangle + \frac{\alpha}{2} \| y - z \|^2, \]  
(49)
for the last term,
\[ \frac{\alpha}{2} \| y - z \|^2 = \frac{\alpha}{2} \| (y - x) - (z - x) \|^2 \]
\[ \geq \frac{\alpha}{2} (\| y - x \| - \| z - x \|)^2 \]
\[ = \frac{\alpha}{2} (\| y - x \|^2 + \| z - x \|^2 - 2\| y - x \| \| z - x \|) \]
\[ \geq \frac{\alpha}{2} (\| y - x \|^2 - 2\sqrt{n} \| y - x \|_\infty \| z - x \|) \]
\[ \geq \frac{\alpha}{2} (\| y - x \|^2 - 2\sqrt{n} D\sqrt{n}r_1) \]
\[ = \frac{\alpha}{2} (\| y - x \|^2 - 2nDr_1) \]  
(50)
For other terms, by the same technique of Lemma 2,
\[ f(y) \geq f(z) + \langle g, y - z \rangle + \frac{\alpha}{2} \| y - z \|^2 \]
\[ \geq f(x) + \nabla f(x)^T (y - x) - \| g - \nabla f(x) \|_1 \| y - x \|_\infty \]
\[ - 2G\sqrt{n}r_1 + \frac{\alpha}{2} \| y - z \|^2 \]
\[ \geq f(x) + \nabla f(x)^T (y - x) - \| g - \nabla f(x) \|_1 \| y - x \|_\infty \]
\[ - 2G\sqrt{n}r_1 + \frac{\alpha}{2} (\| y - x \|^2 - 2nDr_1) \]
\[ \geq f(x) + \nabla f(x)^T (y - x) - \| g - \nabla f(x) \|_1 D \]
\[ - 2G\sqrt{n} + \alpha nD \]  
(51)
By lemma 4, we have
\[ \|g - \nabla f(x)\|_1 \leq 96\pi n^2 \beta r_2, \] (52)
succeeds with probability \( \frac{2}{3} \). Repeat constant times to get nearly 100% probability.

By lemma 5, we have \( \beta = \frac{G}{\rho r_1} \) succeeds with probability \( 1 - \rho \), which we have,
\[
\begin{align*}
f(y) &\geq f(x) + \nabla f(x)^T (y - x) - \|g - \nabla f(x)\|_1 D \\
&\quad - (2G\sqrt{n} + \alpha nD) r_1 + \frac{\alpha}{2} \|y - x\|^2 \\
&\geq f(x) + \nabla f(x)^T (y - x) - \frac{96\pi n^2 G r_2 D}{\rho r_1} \\
&\quad - (2G\sqrt{n} + \alpha nD) r_1 + \frac{\alpha}{2} \|y - x\|^2,
\end{align*}
\] (53)
succeeds with probability \( 1 - \rho \), which give the lemma. \( \square \)

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