Singularities of plurisubharmonic functions and positive closed currents

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Abstract

This is a survey of results, both classical and recent, on behaviour of plurisubharmonic functions near their $-\infty$-points, together with the related topics for positive closed currents.

Key words: plurisubharmonic function, positive closed current, Lelong number, directional Lelong number, generalized Lelong number, Monge-Ampère operator

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The history of plurisubharmonic functions starts in 1942 due to P. Lelong [61] and K. Oka [78]. A model example of a plurisubharmonic function is logarithm of modulus of a holomorphic mapping $f$, and the behaviour of the mappings near their zeros (or at least of $|f|$) corresponds to the behaviour of plurisubharmonic functions near the points where they take the value $-\infty$ (points of their singularities).

Being considered as elements of a distribution space, functions $\log |f|$ serve as potentials for the zero sets of $f$. To this end, the machinery of currents (due to de Rham) was developed by Lelong for complex spaces, and central role here is played by closed positive currents. In 1957, Lelong proved that the trace measure of any closed positive current has density at every point of its support [63]. The main objects of his study were integration currents over analytic varieties, and later R. Thie showed that for this case the densities coincide with the multiplicities of the varieties (and he also called these values Lelong numbers) [97]. The notion has turned out to be of great importance. In particular, it provides us with a powerful link between analytical and geometrical objects of modern complex analysis. See Lelong’s view of the subject in [68], [?]. A collection of his relevant papers is presented in [71].

Further developments in the field rest mainly on technique of Monge-Ampère operators, the key contribution being made by J.P. Demailly. Among various applications we mention those to algebraic geometry and number theory (e.g., [13], [23], [72]).

Here we present a survey on the theory of Lelong numbers and related topics (and not on the theory of plurisubharmonic functions and positive closed currents). For more aspects of pluripotential theory, see [6], [15], [52], [104], and for its backgrounds, [64], [54],
An excellent reference is a book by Demailly [30]. The theory of positive closed currents was surveyed in [96]; a systematic treatment of the general theory of currents is contained in [37]. The main source for the present notes was the paper [29] (which is actually a part of the book [30]). Closely connected to the subject is the notion of pluricomplex Green functions; it is left however beyond the scope of the survey, a brief presentation being given in [52].

Section 1 introduces Lelong numbers of plurisubharmonic functions as characteristics of their behaviour at logarithmic singularity points. It leads to the notion of Lelong number for positive closed currents (Section 2). In Section 3, generalized Lelong numbers due to Demailly are studied. Section 4 deals with analyticity theorems for Lelong numbers. In Section 5, structural formulas for positive closed currents are obtained. Finally, Lelong numbers for Monge-Ampère currents are evaluated in Section 6 by means of technique of local indicators.

Most of the results of Sections 1 and 2 are classical (see, e.g., [64], [72], [39], [43]), however some recent developments (mainly due to C.O. Kiselman) are included as well. The exposition of Sections 3 – 5 follows the lines of [29], with a few modifications. Section 6 is based on [73], [82], [84].

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1 Lelong numbers for plurisubharmonic functions

The most transparent is the case of integration currents over analytic varieties \( \{ f(z) = 0 \} \) of codimension 1, and in fact it was studied by Lelong already in 1950 ([62]). The main idea is to reduce the problem to asymptotic behaviour of the functions \( \log |f| \). The same approach works for arbitrary positive closed currents of bidegree \((1,1)\), and such currents
have local potentials – plurisubharmonic functions. So we start with the notion of Lelong number for plurisubharmonic functions.

1.1 Plurisubharmonic functions

Throughout the exposition, Ω is a domain in \(\mathbb{C}^n\), \(n > 1\), and \(u\) is a plurisubharmonic (psh) function in \(\Omega\), i.e., an upper semicontinuous function whose restriction to each complex line \(L\) is subharmonic in \(\Omega \cap L\).

The class of all psh functions in \(\Omega\) will be denoted by \(PSH(\Omega)\). Any psh function is locally integrable, and the topology of \(PSH(\Omega)\) is generated by \(L^1_{loc}\)-convergence or, equivalently, by the weak convergence on compactly supported continuous (or smooth) functions on \(\Omega\).

Plurisubharmonicity of an upper semicontinuous function \(u\) is characterized by positive semidefiniteness of its Levi form

\[
\sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \eta_j \bar{\eta}_k \geq 0, \quad \forall \eta \in \mathbb{C}^n,
\]

the derivatives being understood in the sense of distributions.

For general theory of psh functions, see, e.g., [43], [54], [64], [87].

Plurisubharmonicity can be viewed as convexity with respect to the complex structure, however in contrast with convex functions on real affine spaces, psh functions need not be continuous and, which is more important, they main attain \(-\infty\) values. And we will be interested mainly in the behaviour of psh functions \(u\) near their singularity points, that is, the points \(x\) where \(u(x) = -\infty\). Note that for the functions \(\log |f|\), such points form analytic varieties.

Some more terminology. A set \(E \subset \Omega\) is called pluripolar in \(\Omega\) if there exists a psh function \(u \neq -\infty\) in \(\Omega\), such that \(u|_E = -\infty\). It was shown by B. Josefson [44] (see also [8]) that if \(E\) is pluripolar in a neighbourhood of every its point then its pluripolar in \(\Omega\). A set \(E \subset \Omega\) is called completely pluripolar if there exists \(u \in PSH(\Omega), u \neq -\infty\), such that \(u^{-1}(-\infty) = E\).

Most of our considerations are local, however sometimes we will have to specify a domain \(\Omega\) to be pseudoconvex. This means that there exists a function \(u \in PSH(\Omega)\) such that \(u(z) \to +\infty\) as \(z \to \partial \Omega\).

1.2 Standard characteristics of local behaviour

We use the following notation:

\(B_r(x) = \{x \in \mathbb{C}^n : |x| < r\}\), \(S_r(x) = \partial B_r(x)\), \(B_r := B_r(0)\), \(S_r := S_r(0)\);

\(\tau_p = \pi^p / p!\) is the \(2p\)-volume of the unit ball in \(\mathbb{C}^p\);

\(\omega_p = 2\pi^p / (p - 1)!\) is the \((2p - 1)\)-volume of the unit sphere in \(\mathbb{C}^p\).
Let \( u \in PSH(\Omega) \). For \( x \in \Omega \) and \( t < \log \text{dist}(x, \partial \Omega) \), define
\[
\Lambda(u, x, t) = \sup \{ u(z) : z \in B_{e^t}(x) \},
\]
\[
\lambda(u, x, t) = \omega_n^{-1} \int_{S_1} u(x + ze^t) \, dS_1(z).
\]
With \( t \) fixed, the both functions are continuous and plurisubharmonic in \( x \). With \( x \) fixed, they are convex and increasing in \( t \),
\[
u(u, x) = u(x) \leq \lambda(u, x, t) \leq \Lambda(u, x, t),
\]
and \( \lambda(u, x, t), \Lambda(u, x, t) \to u(x) \) as \( t \to -\infty \). Moreover, \( \lambda(u, x, t)/\Lambda(u, x, t) \to 1 \) as \( t \to -\infty \) if \( u(x) = -\infty \) (a consequence of Harnack’s inequality). For more relations between these functions, see Sections 1.6 and 1.7.

### 1.3 Definition of Lelong number for plurisubharmonic functions

Since psh functions are locally integrable, it is possible to apply the machinery of differential operators.

Let \( \Delta = 4 \sum_k \partial^2 / \partial z_k \partial \bar{z}_k \) be the Laplace operator, then \( \Delta u \) is a positive measure on \( \Omega \) (which is, up to a constant factor, the Riesz measure of \( u \) considered as a subharmonic function in \( \mathbb{R}^{2n} \)). Denote
\[
\sigma_u(x, r) = \frac{1}{2\pi} \int_{B_r(x)} \Delta u.
\]
Green’s formula implies

**Proposition 1.3.1**

\[
\frac{\sigma_u(x, r)}{\tau_{n-1} r^{2n-2}} = \frac{\partial \lambda(u, x, \log r)}{\partial \log r},
\]
\( \partial / \partial \log r \) being understood as the left derivative.

Since \( \lambda(u, x, t) \) is convex and increasing, the right-hand side of (1.1) is increasing in \( r \), so is its left-hand side and hence there exists the limit
\[
\lim_{r \to 0} \frac{\sigma_u(x, r)}{\tau_{n-1} r^{2n-2}} =: \nu(u, x),
\]
the Lelong number of \( u \) at \( x \). In other words, the Lelong number of \( u \) is the \((2n - 2)\)-dimensional density of its Riesz measure at \( x \).

This point of view on Lelong numbers will be developed in Section 2. And here we concentrate on representations of \( \nu(u, x) \) in terms of the asymptotic behaviour of \( u \) near \( x \). Consideration of the right-hand side of (1.1) and the equivalence between \( \lambda(u, x, t) \) and \( \Lambda(u, x, t) \) gives us
Theorem 1.3.2 [5], [46]

\[ \nu(u, x) = \lim_{t \to -\infty} \frac{\lambda(u, x, t)}{t} = \lim_{t \to -\infty} \frac{\Lambda(u, x, t)}{t}. \]  

(1.3)

As follows from this result, \( \nu(u, x) > 0 \) is possible only when \( u(x) = -\infty \), and the converse is not true. Namely, if the Lelong number of a psh function \( u \) is strictly positive, then \( u \) has a logarithmic singularity at \( x \):

Corollary 1.3.3

\[ \nu(u, x) = \sup \{ \nu > 0 : u(z) \leq \nu \log |z - x| + O(1), \ z \to x \}. \]

Evidently, for finite collections of psh functions \( u_k \),

\[ \nu(\sum_k u_k, x) = \sum_k \nu(u_k, x) \]

and

\[ \nu(\max_k u_k, x) = \min_k \nu(u_k, x). \]

Next result is more difficult, and it is a particular case of Theorem 1.5.3 or 2.5.1.

Theorem 1.3.4 [90] The Lelong number \( \nu(u, x) \) is independent of the choice of local coordinates.

1.4 Examples

The following relations can be easily derived from Theorem 1.3.2.

(a) if \( u(z) = \log |z| \), then \( \nu(u, 0) = 1 \).

(b) let \( u(z) = \log |f(z)| \) and \( f : \Omega \to \mathbb{C} \) be a holomorphic function, \( f(x) = 0 \); then \( \nu(u, x) = m \), the multiplicity (vanishing order) of \( f \) at \( x \) (the least degree of a monomial in the Taylor expansion of \( f \) near \( x \)).

(c) if \( u(z) = \log |f(z)| = \frac{1}{2} \log \sum_k |f_k|^2 \) and \( f = (f_1, \ldots, f_m) : \Omega \to \mathbb{C}^m \) is a holomorphic mapping, \( f(x) = 0 \), then

\[ \nu(u, x) = \min_k m_k, \]

where \( m_k \) are the multiplicities of the zeros of the components \( f_k \) of the mapping \( f \) at \( x \).
1.5 Lelong numbers of slices and pull-backs

Fix \( x \in \Omega \). Given \( y \in \mathbb{C}^n \), \( L \) is the complex line through \( x \) and \( y \), and \( u_y \) is the restriction of \( u \) to \( \Omega \cap L \) (the slice of \( u \) on \( L \)):

\[
u(u,0) = \int_{S_1} \nu(u_y,0) \, dS_1(y).
\] (1.5)

**Theorem 1.5.1** \( \nu(u_y,0) \geq \nu(u,x) \) for all \( y \in \mathbb{C}^n \), and \( \nu(u_y,0) = \nu(u,x) \) \( \forall y \in \mathbb{C}^n \setminus A \), \( A \) being a pluripolar subset of \( \mathbb{C}^n \).

The first statement is evident in view of (1.3), and the second can be derived then from the relation

\[
u(u,x) = \nu(u_y,0) \quad \text{for all } y \in \mathbb{C}^n \setminus A.
\]

On the exceptional set \( A \), the values \( \nu(u_y,0) \) can behave as bad as possible:

**Proposition 1.5.2** [18] For any countable \( G_\delta \)-subset \( \{L_j\} \) of the Riemann sphere and any sequence \( c_j > c > 0 \) there exists a psh function \( u \) in the unit ball of \( \mathbb{C}^2 \) such that \( \nu(u_y,0) = c_j \) and \( \nu(u,0) = c \).

Theorem 1.5.1 remains true when considering slicing of the functions by \( p \)-dimensional planes [47].

Let now \( f \) be a holomorphic mapping \( \Omega' \to \Omega \) with \( f(x') = x \), and \( f^*u \) be the pull-back of a function \( u \in PSH(\Omega) \), that is, \( f^*u(z) = u(f(z)) \).

**Theorem 1.5.3** [47] \( \nu(f^*u, x') \geq \nu(u, x) \). Moreover, strict inequality here occurs only for \( f \) belonging to a polar set in the Frechet space of holomorphic mappings on a neighbourhood of \( x' \).

A relation in the opposite direction is given by

**Theorem 1.5.4** [35], [53] If \( f(U) \) has non-empty interior for every neighbourhood \( U \) of \( x \), then there exists a constant \( C \), independent of \( u \), such that \( \nu(f^*u, x') \leq C \nu(u, x) \) for any function \( u \) plurisubharmonic in a neighbourhood of \( x \). No such bound is possible if \( f(U) \) has no interior points for some neighbourhood \( U \).

The proof can be derived, for example, from Proposition 1.10.1 below.

Characteristics for singularities of pull-backs of psh functions under blow-ups of points and subvarieties of \( \Omega \), *microlocal Lelong numbers*, were defined and studied in [1], see also [60].
1.6 Attenuating the singularities

The following construction was proposed by C.O. Kiselman [46], [51]. Let $u, q \in PSH(\Omega)$, and $q(x) \geq -\log \operatorname{dist}(x, \partial \Omega)$. For any $x \in \Omega$ and $\alpha > 0$, define

$$u_{\alpha, q}(x) = \inf \{ \lambda(u, x, t) - \alpha t : t < -q(x) \}.$$ 

It turns out to be a plurisubharmonic function in $x$ (by the minimum principle of Kiselman [45]) and concave in $\alpha$. Moreover, its singularities are close to those of $u$:

**Theorem 1.6.1** [46], [51] If $\nu(q, x) = 0$ then $\nu(u_{\alpha, q}, x) = \max\{\nu(u, x) - \alpha, 0\}$.

The same is true when using the function $\Lambda(u, x, t)$ instead of $\lambda(u, x, t)$, however the corresponding function $u_{\alpha, q}$ may be different. Generally, the functions $\Lambda(u, x, t)$ and $\lambda(u, x, t)$ can be related as follows.

**Proposition 1.6.2** [67]

$$\limsup_{t \to -\infty} [\Lambda(u, x, t) - \lambda(u, x, t)] \leq c_n \nu(u, x) \quad (1.6)$$

with $c_1 = 0$ and $c_n = \sum_{1 \leq k \leq 2n-2} k^{-1}$ for $n > 1$.

Moreover, if $n > 1$ and $\nu(u, x) > 0$, the relation $\lim_{t \to -\infty} [\Lambda(u, x, t) - \lambda(u, x, t)] = 0$ is not true in general; this limit may even not exist.

1.7 Principal parts

Given a function $u \in PSH(\Omega)$, let $u_y$ be its slice (1.4) along the line through $x \in \Omega$ and $y \in \mathbb{C}^n$. Consider

$$\tilde{u}_x(y) = \limsup_{\zeta \to 0} [u_y(\zeta) - \nu(u, x) \log |\zeta|].$$

The function $u$ is said to have a principal part at $x$ if $\tilde{u}_x \not\equiv -\infty$, and the function $\tilde{u}_x$ is called the principal part of $u$ at $x$ [66].

**Theorem 1.7.1** [66] A function $u$ has a principal part at $x$ if and only if and only if there exists $a_{u,x} = \lim_{t \to -\infty} [\Lambda(u, x, t) - \nu(u, x) t] > -\infty$.

Note that if $u$ has a principal part then there exist the limit in (1.6). Besides, the function $\tilde{u}_0(y) - a_{u,0}$ is the maximal element of the limit set $LS(u)$ of $u$ as defined in Section 2.6.
1.8 Directional Lelong numbers

The preceding considerations are based on comparing psh functions with convex functions of a real variable. More detailed information can be obtained by comparing them with convex functions in $\mathbb{R}^n$.

For $x \in \Omega$, $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, one can consider the polydisk characteristics

$$\lambda(u, x, a) := (2\pi)^{-n} \int_{[0,2\pi]^n} u(x_k + e^{a_k+i\theta_k}) d\theta,$$

$$\Lambda(u, x, a) := \sup \{u(z) : z \in T_a(x)\},$$

where

$$T_a(x) = \{z : |z_k - x_k| = e^{a_k}, 1 \leq k \leq n\}.$$

These functions are convex in $a$ and increasing in each $a_k$, $u(x) \leq \lambda(u, x, a) \leq \Lambda(u, x, a)$, and $\lambda(u, x, a), \Lambda(u, x, a) \to u(x)$ as $a_k \to -\infty$, $1 \leq k \leq n$.

So, there exist the limits

$$\lim_{t \to -\infty} \frac{\lambda(u, x, ta)}{t} = \lim_{t \to -\infty} \frac{\Lambda(u, x, ta)}{t} =: \nu(u, x, a)$$

(1.7)

for any $a \in \mathbb{R}^n_+$, and the value $\nu(u, x, a)$ is called the directional (or refined) Lelong number due to Kiselman [49], [51].

If $1 = (1, \ldots, 1)$ then $\nu(u, x) = \nu(u, x, 1)$; besides,

$$\min_j a_j \nu(u, x) \leq \nu(u, x, a) \leq \max_j a_j \nu(u, x).$$

(1.8)

**Example.** Let $u(z) = \log |f(z)|$ with a holomorphic function $f$, $f(x) = 0$. In a neighborhood of $x$ the function has the form

$$f(z) = \sum_{J \in \omega_x} c_J (z - x)^J, \quad c_J \neq 0$$

($\omega_x \subset \mathbb{Z}^n_+$). Then ([70])

$$\nu(u, x, a) = \min \{\langle a, J \rangle : J \in \omega_x\}.$$ 

(1.9)

Note that when $f$ is a polynomial, the right-hand side of (1.9) is known as the index of the polynomial with respect to the weight $a$ [59].

Directional Lelong numbers appear naturally when studying monomial transforms $f : z \mapsto (z^{M_1}, \ldots, z^{M_m})$, $M_j$ being the $j$-th row of a matrix $M$ with nonnegative integer entries, $\det M \neq 0$: $\nu(f^* u, 0) = \nu(u, 0, 1M^*)$. More generally, for any direction $a$, we have $\nu(f^* u, 0, a) = \nu(u, 0, aM^*)$, so the monomial transforms affect the directional numbers linearly [51].
The procedure of attenuating singularities (Section 1.6) can be applied as well to the directional Lelong numbers by considering the function

\[ u_{a,\alpha,q}(x) = \inf_{t < q(x)} (\Lambda(u, x, ta) - \alpha t). \]

**Proposition 1.8.1** [51] If \( \nu(q, x) = 0 \) then \( \nu(u_{a,\alpha,q}, x, a) = \max\{\nu(u, x, a) - \alpha, 0\} \).

### 1.9 Partial Lelong numbers

When taking the mean values with respect to some of the variables \( z_k \), it produces the partial Lelong numbers [65], [51], [98]. Namely, for an ordered \( p \)-tiple \( J = (j_1, \ldots, j_p) \) from \( \{1, \ldots, n\} \) and a direction \( a' \in \mathbb{R}_+^p \),

\[ \nu_J(u, x, a') := \lim_{t \to -\infty} t^{-1} \lambda_J(u, x, ta'), \quad a' \in \mathbb{R}_+^p, \quad (1.10) \]

\( \lambda_J(u, x, a') \) being the mean value of \( u \) over the set

\[ T_{a'}(x) = \{ z \in \mathbb{C}^n : |z_{j_m} - x_{j_m}| = \exp a'_{j_m} (1 \leq m \leq p), \ z_k = x_k (k \not\in J) \}, \quad a' \in \mathbb{R}^p. \]

It was shown in [98] that if \( u \) satisfies certain regularity conditions near \( x \), then \( \nu(u, x, a) \to \nu_J(u, x, a') \) as \( a_k \to +\infty, \ k \not\in J \), and \( a_j = a'_j, \ j \in J \). In the general situation the limit exists although it can be strictly less than the corresponding partial Lelong number, even if \( u \) is assumed to be locally bounded outside \( x \) and multicircled around \( x \) (i.e., \( u(x + \zeta) = u(|x_{1} + \zeta_{1}|, \ldots, |x_{n} + \zeta_{n}|) \) for any \( \zeta \) near \( 0 \)). Indeed, for

\[ u(z_1, z_2) = \max \{\log |z_1|, -|\log |z_2||^{1/2}\}, \]

\( \nu(u, 0, a) = 0 \) for all \( a \in \mathbb{R}_+^n \), while \( \nu_1(u, 0, a') = a' \).

### 1.10 Sublevel sets, integrability index, and multiplier ideals

The rate of approaching \( u(z) \) its \(-\infty\) value can be characterized by behaviour of the volumes of its sublevel sets

\[ A_u(t) = \{ y : u(y) < t \} \]

as \( t \to -\infty \).

Another way is to study local integrability of \( e^{-u/\gamma} \) for \( \gamma > 0 \) (note that since \( u \) may be discontinuous, even integrability of \( e^{-u} \) near \( x \) with \( u(x) > -\infty \) is far from being evident). The value

\[ I(u, x) = \inf \{ \gamma > 0 : e^{-u/\gamma} \in L^2_{loc}(x) \} \quad (1.11) \]

is called the integrability index, or Arnold multiplicity, of \( u \) at \( x \).

The both characteristics turn out to be closely related.
Theorem 1.10.1 [31], [53] $I(u, x)$ is the infimum of $\gamma > 0$ such that $e^{-2t/\gamma} \text{Vol } A_u(t) \cap U$ is bounded as $t \to -\infty$ for some neighbourhood $U$ of the point $x$.

In other words, volumes of the sublevel sets have exponential decay with the rate controlled by the integrability index. Uniform relations of such type for families of psh functions were obtained in [105]. Besides, Proposition 3.1.1 implies that the size of the sublevel sets can be also measured in terms of plurisubharmonic capacity (3.5):

Proposition 1.10.2 Given sets $U \subset \subset V \subset \subset \Omega$, there exists a constant $C$ such that for any negative function $u \in \text{PSH}(\Omega)$ and $t < 0$,

$$\text{Cap}(A_u(t) \cap U, \Omega) \leq C \frac{||u||_{L^1(V)}}{|t|}.$$ 

Integrability indices can be estimated in terms of Lelong numbers [92]:

$$\frac{1}{n} \nu(u, x) \leq I(u, x) \leq \nu(u, x),$$

the extremal situation being realized for $u = \log |z_1|$ and $u = \log |z|$. A more refined relation is given by

Theorem 1.10.3 [51] $\sup \{ \nu(u, x, a) : \sum_j a_j = 1 \} \leq I(u, x) \leq \nu(u, x)$. Moreover, if $u(z) = u(|z_1 - x_1|, \ldots, |z_n - x_n|)$ near $x$, then $I(u, x) = \sup \{ \nu(u, x, a) : \sum_j a_j = 1 \}$.

An analytic object representing the singularities of $u$ is the multiplier ideal sheaf $J(u)$ consisting of germs of holomorphic functions $f$ such that $|f|e^{-u} \in L^2_{\text{loc}}$. It is a coherent analytic sheaf [76]. Moreover, if $U \subset \subset \Omega$ is pseudoconvex, then the restriction of $J(u)$ to $U$ is generated as an $O_U$-module by a Hilbert basis $\{ \sigma_l \}$ of the Hilbert space $H_u(U)$ of holomorphic functions $f$ on $U$ such that $|f|e^{-u} \in L^2(U)$ [32].

An application of this notion will be given in Section 4.4.

2 Lelong numbers for positive closed currents

Up to this moment we developed an approach to Lelong numbers based on asymptotic properties of plurisubharmonic functions. More information can be obtained by considering them as densities of the Riesz measures. To do this, the measures should be viewed as the trace measures of the corresponding positive closed currents of degree $(1, 1)$. And this can be extended to currents of higher degrees. One of the motivations for such an extension is as follows. When $u = \log |f|$ and $f : \Omega \to \mathbb{C}$, the Lelong number of $u$ at a point $x$ is just the multiplicity of the zero of $f$ at $x$, and it is not the case for holomorphic mappings $f$ to $\mathbb{C}^m$ when $m > 1$. As will be seen in Section 5, multiplicities of holomorphic mappings can be characterized as Lelong numbers of certain currents of higher degrees.
So we pass to Lelong numbers for positive closed currents, starting with recalling some basic notions of the theory of currents. The subject of Sections 2.1 – 2.5 is treated, e.g., in [43], [54], [64], [72]. More information on Lelong numbers of the currents will be presented in Sections 4 and 5.

2.1 Positive closed currents

Let \( D_{p,q}(\Omega) \) be the space of all smooth compactly supported differential forms \( \phi \) of bidegree \((p,q)\) on \( \Omega \):

\[
\phi = \sum_{|I|=p, |J|=q} \phi_{IJ} dz_I \wedge d\bar{z}_J, \quad \phi_{IJ} \in D(\Omega),
\]

with the topology of \( C^\infty \)-convergence.

Differentiation of forms: the operator \( \partial : D_{p,q}(\Omega) \to D_{p+1,q}(\Omega) \) is defined by

\[
\partial \phi = \sum_{I,J} \sum_{1 \leq k \leq n} \frac{\partial \phi_{IJ}}{\partial z_k} dz_k \wedge d z_I \wedge d \bar{z}_J,
\]

and \( \bar{\partial} : D_{p,q}(\Omega) \to D_{p,q+1}(\Omega) \) is given by

\[
\bar{\partial} \phi = \sum_{I,J} \sum_{1 \leq k \leq n} \frac{\partial \phi_{IJ}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_I \wedge d \bar{z}_J.
\]

The currents of bidimension \((p,q)\) (bidegree \((n-p,n-q)\)) are elements of the dual space \( D'_{p,q}(\Omega) \) (continuous linear functionals on \( D_{p,q}(\Omega) \)). Any current \( T \in D'_{p,q}(\Omega) \) has a representation

\[
T = \sum_{|I|=n-p, |J|=n-q} T_{IJ} dz_I \wedge d \bar{z}_J, \quad T_{IJ} \in D'(\Omega).
\]

The action of \( T \) on \( \phi \) will be written as \( \langle T, \phi \rangle \) or \( \int T \wedge \phi \).

The topology on \( D'_{p,q}(\Omega) \) (referred to as the weak topology of currents):

\[
T \rightarrow T \iff \langle T_j, \phi \rangle \to \langle T, \phi \rangle \quad \forall \phi \in D_{p,q}(\Omega).
\]

Differentiation of currents:

\[
\langle \partial T, \phi \rangle := (-1)^{p+q+1} \langle T, \partial \phi \rangle, \quad \langle \bar{\partial} T, \phi \rangle := (-1)^{p+q+1} \langle T, \bar{\partial} \phi \rangle.
\]

The operators

\[
d = \partial + \bar{\partial}, \quad d^c = \frac{\partial - \bar{\partial}}{2\pi i}
\]

are real, and \( dd^c = \frac{i}{\pi} \partial \bar{\partial} \). (There is no general convention on normalizing the operator \( d^c \), some authors use \( d^c = i(\bar{\partial} - \partial) \); we prefer the above one to avoid extra factors \((2\pi)^p\) in the sequel.)
A current $T \in \mathcal{D}_{p,p}'(\Omega)$ is called positive ($T \geq 0$) if $\langle T, \phi \rangle \geq 0$ for every differential form $\phi = i\alpha_1 \wedge \bar{\alpha}_1 \wedge \ldots \wedge i\alpha_p \wedge \bar{\alpha}_p$ with $\alpha_k \in \mathcal{D}_{1,0}(\Omega)$. (In the literature, such currents are sometimes called weakly positive). The coefficients $T_{IJ}$ of such a current $T$ are Borel measures on $\Omega$. Therefore, the action of a positive current $T = \sum T_{IJ}dz^I \wedge d\bar{z}^J$ can be continuously extended to the space of compactly supported forms $\phi$ with continuous coefficients $\phi_{KL}$ and

$$\|T\|_E = \sum |T_{JK}|_E, \quad |T_{JK}|_E$$

is the total variation of the measure $T_{JK}$ on $E$ and

$$\|\phi\| = \sup_{K,L,x} |\phi_{KL}(x)|.$$

Denote by

$$\beta := \sum_{1 \leq k \leq n} dz_k \wedge d\bar{z}_k = \frac{\pi}{2} dd^c |z|^2$$

the standard Kähler form on $\mathbb{C}^n$, so

$$\beta_p := \frac{1}{p!} \beta^p$$

is the $p$-dimensional volume element. Then for every positive current $T \in \mathcal{D}_{p,p}'(\Omega)$,

$$\|T\|_E \leq c_n |T \wedge \beta_p|_E.$$

A current $T$ is called closed if $dT = 0$. When $T \in \mathcal{D}_{p,p}'(\Omega)$, this is equivalent to saying that $\partial T = 0$ or $\bar{\partial} T = 0$.

$\mathcal{D}_{p}^+(\Omega)$ will denote the cone of all positive closed currents from $\mathcal{D}_{p,p}'(\Omega)$.

An important tool in the theory of positive closed currents is the following Skoda-El Mir extension theorem.

**Theorem 2.1.1** [95], [34], [89] Let $E$ be a closed complete pluripolar set in $\Omega$ and $T \in \mathcal{D}_{p}^+(\Omega \setminus E)$ whose coefficients $T_{IJ}$ have locally finite mass near $E$. Consider the current $\tilde{T} = \sum \tilde{T}_{I,J}dz^I \wedge d\bar{z}^J$ with the measures $\tilde{T}_{I,J}(A) := T_{I,J}(A \setminus E)$ for all Borel $A \subset \Omega$. Then $\tilde{T} \in \mathcal{D}_{p}^+(\Omega)$.

(The current $\tilde{T}$ is called the simple, or trivial, extension of $T$, and it was first introduced by Lelong when studying integration over analytic varieties, see Example 3 in Section 2.2).

If $f : \Omega \to \Omega' \subset \mathbb{C}^n$ is a holomorphic mapping such that its restriction to the support of a current $t \in \mathcal{D}_{p}^+(\Omega)$ is proper, then the direct image (or push-forward) $f_\ast T$ of $T$ is defined by the relation $\langle f_\ast T, \phi \rangle = \langle T, f^\ast \phi \rangle$. If $T \in \mathcal{D}_{p}^+(\Omega)$ then $f_\ast T \in \mathcal{D}_{p}^+(\Omega')$.

If a holomorphic mapping $f : \Omega \to \Omega'$ has constant maximal rank on $\Omega$, then for any current $T \in \mathcal{D}_{p}^+(\Omega')$ its inverse image (or pull-back) $f^\ast T$ is defined as $\langle f^\ast T, \phi \rangle = \langle T, f_\ast \phi \rangle$.

The inverse image of $T$ can be also defined for any surjective holomorphic mapping $f$ with $\dim f^{-1}(x) = 0$ for every $x \in \Omega'$ (see [74]).
2.2 Examples of currents

The standard examples are as follows.

1) Currents generated by psh functions:

\[ u \in PSH(\Omega) \iff \left( \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right) \geq 0 \iff dd^c u \in \mathcal{D}^+_{n-1}(\Omega). \]

Furthermore, if \( T \in \mathcal{D}^+_{n-1}(\Omega) \) then for any \( x \in \Omega \) there is a neighbourhood \( U_x \) and a function \( u_x \in PSH(U_x) \) such that \( T = dd^c u_x \) in \( U_x \).

2) For a complex manifold of dimension \( p \), the current \([M]\) of integration over \( M \) is defined as

\[ \langle [M], \phi \rangle = \int_M \phi. \]

Then \([M] \in \mathcal{D}^+_p(\Omega)\) (that it is closed, follows from Stokes’ theorem).

3) Integration currents over analytic varieties. Let \( A \) be an analytic variety, i.e., locally \( A = \{ z : f_\alpha(z) = 0, \ \alpha \in A \} \), and \( \text{Reg} A \) be the set of its regular points (where \( A \) is locally a manifold). If \( A \) is of pure dimension \( p \), define

\[ \langle [A], \phi \rangle := \int_{\text{Reg} A} \phi. \]

Then \([A] \in \mathcal{D}^+_p(\Omega)\). (Non-trivial part is that \([A]\) is closed; this fundamental result is due to P. Lelong [63], and it can be seen today as a consequence of Theorem 2.1.1.)

4) Holomorphic chains \( T = \sum \alpha_k[A_k] \in \mathcal{D}^+_p(\Omega) \), where \( \alpha_k \in \mathbb{Z}_+ \) and \( A_k \) are analytic varieties of pure dimension \( p \). When \( p = n-1 \), the holomorphic chains represent positive, or effective, divisors.

2.3 Lelong numbers for currents

For \( T \in \mathcal{D}^+_p(\Omega) \), \( \sigma_T := T \wedge \beta_p \in \mathcal{D}^+_0 \) is the trace measure of \( T \). (If \( T = dd^c u \) then \( \sigma_T \) is just the Riesz measure of \( u \).

Denote \( \sigma_T(x, r) = \sigma_T(B_r(x)) \). It can be also represented in the following form.

**Proposition 2.3.1**

\[ \sigma_T(x, r) = \tau_p r^{2p} \int_{B_r(x)} T \wedge (dd^c \log |z-x|)^p. \]

Therefore,

\[ \nu(T, x, r) := \frac{\sigma_T(x, r)}{\tau_p r^{2p}} \downarrow \nu(T, x), \]

the Lelong number of the current \( T \in \mathcal{D}^+_p(\Omega) \) at \( x \). So,

\[ \nu(T, x) = \lim_{r \to 0} \frac{1}{\tau_p r^{2p}} \int_{B_r(x)} T \wedge \beta_p = \lim_{r \to 0} \int_{B_r(x)} T \wedge (dd^c \log |z-x|)^p. \quad (2.1) \]
The Lelong number of a current $T \in \mathcal{D}^+_p(\Omega)$ can be viewed as the $2p$-dimensional density of its trace measure $\sigma_T$ or, equivalently, as the mass charged at $x$ by its “projective” trace measure $T \wedge (dd^c \log |\cdot - x|)^p$.

**Corollary 2.3.2** $\sigma_T(x, r) \geq \tau_p r^{2p} \nu(T, x)$.

### 2.4 Special cases

Lelong numbers of the model examples of currents presented in Section 2.2 are as follows.

1) If $T = dd^c u$ with a plurisubharmonic function $u$, then $\nu([A], x) = \nu([M], x)$. This follows easily from the original definition (1.2) of Lelong numbers of psh functions, since $\sigma_u = \sigma_T$. For the uniformity, it should be written $\nu(dd^c u, x)$ instead of $\nu(u, x)$, however we prefer to keep the original notation for the Lelong numbers of functions, both for the sake of brevity and since this is their standard notation.

2) The situation with manifolds is also easy: For any complex manifold $M$, $\nu([M], x) = 1$ at all points $x \in M$ (and of course $\nu([M], x) = 0$ for $x$ outside $M$).

3) Much more difficult is the result about the Lelong numbers of analytic varieties (Thie’s theorem). Let $A \subset \Omega$ be an analytic variety of pure dimension $p$, then $[A] \in \mathcal{D}^+_p(\Omega)$. For any fixed $x \in A$ there is a neighbourhood $U$ and local coordinates $(z', z'') \in \mathbb{C}^p \times \mathbb{C}^{n-p}$ such that $A \cap U \subset \{(z', z'') : |z''| \leq C|z'|\}, \ C > 0$. In these coordinates, $x = 0$. Consider the projection $\pi : A \cap (U' \times U'') \to U'$; it gives a ramified covering of $U'$. The number of sheets of the covering, $m_x$, is called the multiplicity of $A$ at $x$.

**Theorem 2.4.1** [97] For an analytic variety $A$ of dimension $p$, $\nu([A], x)$ equals the multiplicity $m_x$ of $A$ at $x$.

For any Borel $D \subset \subset \Omega$, the $2p$-dimensional volume of $A \cap D$ is precisely $\sigma_{[A]}(D)$. Therefore, we have

**Corollary 2.4.2** (Volume estimation) If $K \subset \subset A$ and $r_0 < \text{dist}(K, \partial \Omega)$, then

$$\tau_p r^{2p} m_x \leq \text{Vol}_{2p} A \cap B_r(x) \leq C(r_0, K, A) r^{2p} \quad \forall r < r_0, \ \forall x \in K.$$

### 2.5 Stability of Lelong numbers

Lelong numbers of currents, just as of psh functions, have the following invariance property.

**Theorem 2.5.1** [90] The Lelong number $\nu(T, x)$ is independent of the choice of local coordinates.
Another important relation concerns Lelong numbers of slices of positive closed currents. Given a current \( T \in \mathcal{D}^+_p(\Omega) \), one can define its slices \( T|_{x+S} \) by complex planes \( x+S \), where \( x \in \Omega \) and \( S \in G(q,n) \), \( q \geq n-p \), \( G(q,n) \) being the Grassmanian of \( q \)-dimensional complex linear subspaces of \( \mathbb{C}^n \). For any \( x \in \Omega \) and almost all (with respect to the Haar measure) \( S \in G(q,n) \), the current \( T|_{x+S} := T \wedge [x+S] \in \mathcal{D}^+_{p+q-n}(\Omega) \) is well defined and supported on \( x+S \). For details of the slicing theory, see [37].

**Theorem 2.5.2** [90] Let \( T \in \mathcal{D}^+_p(\Omega) \) and \( q \geq n-p \). Then for every \( x \in \text{supp} T \) and almost all \( S \in G(q,n) \), we have \( \nu(T,x) = \nu(T|_{x+S},x) \).

The proof follows along the same lines as that of Theorem 1.5.1, however technically is much more complicated. In particular, the role of (1.5) is played now by the Crofton formula

\[
\int_{G(p,n)} [S] d\kappa(S) = (dd^c \log |z|)^{n-p}
\]

(\( d\kappa \) is the Haar measure on \( G(p,n) \)).

### 2.6 Tangents to positive closed currents

Let \( h_r(z) = z/r \), \( r > 0 \). Given a current \( T \in \mathcal{D}^+_p(\Omega) \), consider its push-forwards \( h_r*T \) (we suppose that \( 0 \in \Omega \)). If \( T \) is the integration current over an analytic variety, then the limit

\[
\lim_{r \to 0} h_r*T =: tc(T)
\]

exists and is called the tangent cone to \( T \) at 0, see [42]. In particular, when \( T = dd^c \log |f(z)| \) with \( f \) a holomorphic function near the origin, \( tc(T) = dd^c \log |P_m| \), where \( P_m \) is the homogeneous term of \( f \) of minimal degree.

A problem of existence of the tangent cones for other positive closed currents was formulated by Harvey. It was shown by Kiselman to be answered in negative even in the situation of currents of bidegree (1, 1). Given a psh function \( u \) on a neighbourhood of the origin, consider the family

\[
\tag{2.2}
u_r(z) := h_{r*}u(z) - \sup_{B_r} u = u(rz) - \sup_{B_r} u.
\]

It is relatively compact in \( L^1_{loc}(\mathbb{C}^n) \) for \( r < r_0 \). Denote by \( LS(u) \) the limit set of this family as \( r \to 0 \), i.e., the set of all partial limits of sequences \( u_{r_j}, r_j \to 0 \). Evidently, the current \( dd^c u \) has the tangent cone if and only if \( LS(u) \) consists of a unique element. For example, \( LS(\log |f|) = \log |P_m| - \sup_{B_r} P_m \).

It can be checked that \( LS(u) \) is formed by functions \( g \in PSH(\mathbb{C}^n) \) satisfying \( g(tz) = \nu(u,0) \log |t| + g(z) \) for any \( t \in \mathbb{C} \) (in particular, \( \nu(g,0) = \nu(u,0) \forall g \in LS(u) \)). When \( n = 1 \), \( LS(u) \) consists of the only function \( \nu(u,0) \log |z| \) for every \( u \) (pluri)subharmonic near the origin of \( \mathbb{C} \), and it is not the case in higher dimensions:
Theorem 2.6.1 [50] Given any closed and connected subset $M$ of $\text{PSH}(\mathbb{C}^n)$, $n > 1$, which consists of functions $g$ satisfying $\sup_B g < 0$ and $g(tz) = C \log |t| + g(z)$ for all $t \in \mathbb{C}$, there exists a psh function $u$ with $\text{LS}(u) = M$ (and, consequently, $\nu(u,0) = C$).

The corresponding limit set $\text{LS}(T)$ for $T \in \mathcal{D}_p^+(\Omega)$, $p \leq n - 1$, is formed by currents $S$ with $\nu(S,0) = \nu(T,0)$ and $S \wedge (dd^c \log |z|)^p = 0$ on $\mathbb{C}^n \setminus \{0\}$ [22]. Theorem 2.6.1 was extended to currents of arbitrary degree in [9].

It turns out that the existence of a tangent cone depends on the rate at which the projective masses $\nu(T,0,r)$ approach the Lelong number of $T$:

**Theorem 2.6.2 [11]** Let $T \in \mathcal{D}_p^+(\Omega)$. If the function $n_T(r) := r^{-1}[\nu(T,0,r) - \nu(T,0)]$ is integrable at $0$, then $T$ has tangent cone at the origin.

Moreover, the condition is sharp: it is possible to construct a current $T$ of bidegree $(1,1)$ with no tangent cone and such that the integral of the function $n_T$ has the divergence rate at 0 as small as one likes [11].

The set of points where a current $T \in \mathcal{D}_{n-1}^+$ has no tangent cones was studied in [10].

### 2.7 Pluripositive currents and other generalizations

A real current $T \in \mathcal{D}_p(\Omega)$ is called **pluripositive** if $dd^c T \geq 0$. It was shown in [95] that the notion of Lelong number can be extended to all negative pluripositive currents. The idea was developed in [89]; in particular, it was applied to positive pluripositive currents. For other types of Lelong numbers for such currents, see [3]. Extensions of pluripositive currents across sets of small Hausdorff dimension (analogs of 2.1.1) were studied in [89], [21].

Being a local characteristic, the notion of Lelong number extends easily to positive closed currents on complex manifolds. For currents on singular complex spaces, see [24]. Currents on almost complex manifolds were treated in [41].

### 3 Generalized Lelong numbers due to Demailly

An important notion of **generalized Lelong numbers with respect to psh weights** was introduced and studied by J.P. Demailly [22] – [25]. The idea is to replace the projective trace measure $T \wedge (dd^c \log |\cdot - x|)^p$ by $T \wedge (dd^c \varphi)^p$ with quite general psh functions $\varphi$ (weights) with singularity at $x$. Classical and directional Lelong numbers are particular cases of these ones, with specified weight functions. Moreover, the technique of generalized Lelong numbers allows to give more simple and natural proofs for deep results concerning standard Lelong numbers.

The notion is based on machinery of complex Monge-Ampère operators, and eventually Monge-Ampère currents are one of the main objects in investigation of singularities of psh...
functions. We start with a quick overview of basic results on Monge-Ampère operators, referring to [6], [15], [29], [55], [56] for a more detailed introduction to the theory. The presentation of generalized Lelong numbers follows Demailly’s paper [29].

3.1 Monge-Ampère operators

By complex Monge-Ampère operator of a plurisubharmonic function \( u \) we mean the operator

\[
(\ddc u)^n
\]

or, more generally, for plurisubharmonic functions \( u_1, \ldots, u_p, p \leq n \), the wedge product

\[
(\ddc u_1 \wedge \ldots \wedge \ddc u_p).
\]  

For \( u \) smooth,

\[
(\ddc u)^n = \left( \frac{2}{\pi} \right)^n n! \det \left( \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right) \beta_n
\]

(and the corresponding mixed determinant for (3.1) with smooth \( u_j \)).

The problem is that such a wedge product cannot be defined for arbitrary plurisubharmonic functions, see [91], [48]. Using a Chern-Levine-Nirenberg estimate for the complex Monge-Ampère operator [19], it was shown by Bedford and Taylor [7] that \( (\ddc u)^k \) can be defined inductively as

\[
(\ddc u)^k = \ddc [u(\ddc u)^{k-1}]
\]

for continuous and even for all locally bounded psh functions \( u \). More generally, for any current \( T \in \mathcal{D}^+_p(\Omega) \) and a function \( u \in PSH(\Omega) \cap L^\infty(\Omega) \), the current \( uT \) is well defined, has locally bounded mass, and \( \ddc u \wedge T := \ddc (uT) \in \mathcal{D}^+_{p-1} \).

The following refined version of the Chern-Levine-Nirenberg estimate is due to Cegrell.

**Proposition 3.1.1** [15] For any \( K \subset \subset \Omega \) there is a constant \( C_{K,\Omega} \) such that for every \( v \in PSH(\Omega) \) and \( u_j \in PSH(\Omega) \cap L^\infty(\Omega) \), \( 1 \leq j \leq p \leq n \),

\[
\|v \ddc u_1 \wedge \ldots \wedge \ddc u_p\|_K \leq C_{K,\Omega} \|v\|_{L^1(\Omega)} \|u_1\|_{L^\infty(\Omega)} \ldots \|u_p\|_{L^\infty(\Omega)}.
\]

So the obstacles for the definition of the operator arise from the singularity sets of plurisubharmonic functions. For the Monge-Ampère operator be well defined, either the singularity set has to be “small” or the function must not decrease too rapidly to \(-\infty\). The latter situation was studied in [6], Theorem 4.3. However having in mind applications to holomorphic mappings, one needs to make restrictions to the singularity sets themselves (since the decay of \( \log |f| \) is the strongest possible). The following result is due to Fornaess and Sibony.
Let \( T \) be a positive closed current of bidimension \((p,p)\), and the unbounded loci \( L_j \) of plurisubharmonic functions \( u_j, 1 \leq j \leq q \leq p \), satisfy
\[
\mathcal{H}_{2(p-m+1)}(L_{j_1} \cap \ldots \cap L_{j_m} \cap \text{supp} \, T) = 0
\]
(3.2)
for all choices of indices \( j_1 < \ldots < j_m \), \( m = 1, \ldots, q \), \( \mathcal{H}_{2(p-m+1)} \) being the \( 2(p-m+1) \)-dimensional Hausdorff measure. Then the currents
\[
\ddc u_1 \wedge \ddc u_2 \wedge \ldots \wedge \ddc u_q \wedge T := \ddc (u_1 \ddc u_2 \wedge \ldots \wedge \ddc u_q \wedge T)
\]
are well defined and have locally finite mass.

(The result with zero \((2p-2m+1)\)-dimensional Hausdorff measure was obtained earlier by Demailly [29].)

Therefore, if \( u_1, \ldots, u_q \in PSH(\Omega) \cap L^\infty_{\text{loc}}(\Omega \setminus K), K \subset \subset \Omega \), then \( \ddc u_1 \wedge \ldots \wedge \ddc u_q \wedge T \geq 0 \) \((q \leq p)\) and in particular, \((\ddc u)^q \in \mathcal{D}^+_{n-k}(\Omega)\), is well defined for all \( q \leq n \). A wider (though less explicitly presented) class of functions \( u \) with well-defined Monge-Ampère operator \((\ddc u)^q\) was introduced in [16], [17].

Another problem is that the Monge-Ampère operators are not continuous with respect to the weak convergence of plurisubharmonic functions (first mentioned in [14]). However they are continuous under decreasing limits:

\textbf{Theorem 3.1.3} [29] Let functions \( u_j \) and a positive closed current \( T \) satisfy the conditions of Theorem 3.1.2. If plurisubharmonic functions \( u^s_j \) decrease to \( u_j \) as \( s \to \infty \), then
\[
\ddc u^s_1 \wedge \ddc u^s_2 \wedge \ldots \wedge \ddc u^s_q \wedge T \to \ddc u_1 \wedge \ddc u_2 \wedge \ldots \wedge \ddc u_q \wedge T
\]
in the weak topology of currents.

In particular, the result is true for functions locally bounded outside a compact subset of \( \Omega \).

When functions \( u_j \) have the form \( u_j = \log |f_j| \) with holomorphic \( f_j : \Omega \to \mathbb{C} \), condition (3.2) with \( T = 1 \) means
\[
\dim Z_{j_1} \cap \ldots \cap Z_{j_m} \leq n - m, \quad m = 1, 2, \ldots, q,
\]
(3.3)
and for a function \( u = \log \sum_j |f_j|^2 =: \log |f|^2 \), the operator \((\ddc u)^q\) is well defined if
\[
\dim Z_f \leq n - q,
\]
(3.4)
\( Z_f = Z_1 \cap \ldots \cap Z_q \) being the zero set of the mapping \( f = (f_1, \ldots, f_q) \).

For this specific situation, the following convergence result is known.
Theorem 3.1.4 [88], [79], [80] Let a sequence of holomorphic mappings \( f^s : \Omega \to \mathbb{C}^q \), \( q \leq n \), converge to a mapping \( f \) uniformly on compact subsets of \( \Omega \). If the zero set \( Z_f \) of \( f \) satisfies (3.4) then the currents \( (dd^c \log |f^s|)^p \), \( p \leq q \), are well defined on each subset \( \omega \subset \subset \Omega \) for \( s \geq s_0(\omega) \), and \( (dd^c \log |f^s|)^p \to (dd^c \log |f|)^p \), \( p \leq q \).

For locally bounded psh functions, the Monge-Ampère operators are continuous also for increasing sequences [8]. More general conditions can be given in terms of convergence with respect to the Bedford-Taylor capacity

\[
Cap (E, \Omega) := \sup \{ \int_E (dd^c u)^n : u \in PSH(\Omega), -1 \leq u < 0 \}. \tag{3.5}
\]

A sequence of functions \( u_s \) is said to converge in capacity to \( u \) on \( \Omega \) if for any \( \epsilon > 0 \) and any Borel set \( K \subset \subset \Omega \),

\[
\lim_{s \to \infty} Cap (K \cap \{ z : |u(z) - u_s(z)| > \epsilon \}, \Omega) = 0.
\]

Theorem 3.1.5 [99] Let a sequence of locally uniformly bounded plurisubharmonic functions \( \{u_j^s\}_{s=1}^\infty \) converge in capacity to \( u_j \) as \( s \to \infty \), \( 1 \leq j \leq q \), then

\[
dd^c u_1^s \wedge dd^c u_2^s \wedge \ldots \wedge dd^c u_q^s \to dd^c u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_q
\]
in the weak topology of currents.

The result remains true if \( u_j^s \) are assumed to be uniformly locally bounded outside a finite number of fixed points \( x_1, \ldots, x_m \) of \( \Omega \), so they may have singularities at these points [100].

Some other sufficient conditions for the convergence are given in [38], [16], [17], [101].

3.2 Definition of generalized Lelong numbers

Given \( \varphi \in PSH(\Omega) \) and \( r \in \mathbb{R} \), denote

\[
B_r(\varphi) = \{ z : \varphi(z) < r \},
\]

\[
S_r(\varphi) = \{ z : \varphi(z) = r \}.
\]

A psh function \( \varphi \) is semiexhaustive if \( B_R(\varphi) \subset \subset \Omega \) for some \( R \in \mathbb{R} \). In particular, \( \varphi \in L^\infty_{loc}(\Omega \setminus B_R(\varphi)) \) and thus \( (dd^c \varphi)^k \) is well defined for all \( k \leq n \). If, in addition, \( \varphi \) is such that \( e^\varphi \) is continuous on \( \Omega \), it is called a psh weight on \( \Omega \).

Let \( T \in D_p^+(\Omega) \). Define

\[
\nu(T, \varphi, r) = \int_{B_r(\varphi)} T \wedge (dd^c \varphi)^p
\]
and
\[ \nu(T, \varphi) = \lim_{r \to -\infty} \nu(T, \varphi, r), \]
the generalized Lelong number, or the Lelong-Demailly number, with respect to the weight \( \varphi \) ([22], [25]).

**Examples.**
1) \( \varphi(z) = \log |z - x| \Rightarrow B_r(\varphi) = B_{e^r}(x), \nu(T, \varphi, r) = \nu(T, x, e^r) \) and \( \nu(T, \varphi) = \nu(T, x) \).
2) the "directional" weights
\[ \varphi(z) = \varphi_{a,x}(z) := \sup_k a_k^{-1} \log |z_k - x_k|, \quad a_k > 0, \] (3.6)
generate the directional Lelong numbers with respect to \((a_1, \ldots, a_n)\) (to be shown in Section 3.3).

The following useful formula can be derived by means of Stokes’ theorem.

**Proposition 3.2.1** [29] For any convex increasing function \( \gamma : \mathbb{R} \to \mathbb{R} \),
\[ \nu(T, \gamma \circ \varphi, \gamma(r)) = \gamma'(r)^p \nu(T, \varphi, r), \]
\( \gamma' \) being understood as the left derivative. In particular,
\[ \nu(T, \varphi, r) = e^{-2pr} \int_{B_r(\varphi)} T \wedge \left( \frac{1}{2} dd^c e^{2\varphi} \right)^p. \]
(Actually, we need not suppose \( \gamma \) to be convex and increasing, the only requirement being \( \gamma \circ \varphi \) to be a psh weight [20]).

### 3.3 Lelong-Jensen-Demailly formula

For the classical Lelong number of a psh function, Theorem 1.3.2 serves as a bridge between its definition as the density of the associated measure (i.e., as \( \nu(dd^c u, x) \)) and as an asymptotic characteristic of the function itself (as given in Corollary 1.3.3). A similar relation for the generalized Lelong numbers exists, too.

Let \( \varphi \) be a psh weight in \( \Omega, \varphi_r = \max \{ \varphi, r \} \). The swept out Monge-Ampère measure on \( S_r(\varphi) \) is defined as
\[ \mu^\varphi_r = (dd^c \varphi_r)^n \mid_{S_r(\varphi)}. \]
It is a positive measure with the total mass \( \mu^\varphi_r(S_r(\varphi)) = (dd^c \varphi)^n(B_r(\varphi)) \). If \( (dd^c \varphi)^n = 0 \) on \( \Omega \setminus \varphi^{-1}(-\infty) \), then \( \mu^\varphi_r = (dd^c \varphi_r)^n \).

**Example.** For \( \varphi = \log |z - x| \), \( \mu^\varphi_r \) is the normalized Lebesgue measure on \( S_{e^r}(x) \).

More generally, if \( \varphi \) is smooth near \( S_r(\varphi) \) and \( d\varphi \neq 0 \) near \( S_r(\varphi) \), then
\[ \mu^\varphi_r = (dd^c \varphi)^{n-1} \wedge d\varphi \mid_{S_r(\varphi)}. \]
Theorem 3.3.1 (Lelong-Jensen-Demailly formula) [23], [24] Any \( u \in \text{PSH}(\Omega) \) is \( \mu_r \)-integrable for \(-\infty < r < R\), and

\[
\mu_r^\varphi(u) - \int_{B_r(\varphi)} u(dd^c \varphi)^n = \int_{-\infty}^r \nu(dd^c u, \varphi, t) \, dt.
\]

Example. For \( \varphi(z) = \log |z-x| \), this becomes

\[
u(dd^c u, \varphi) = \lim_{r \to -\infty} \frac{\mu_r^\varphi(u)}{r}.
\]

3.4 Semicontinuity properties
The following results [29] are useful when studying families of currents or weights.

Theorem 3.4.1 If currents \( T_k \in D^+_p(\Omega) \) converge to a current \( T \), then

\[
\limsup_{k \to \infty} \nu(T_k, \varphi) \leq \nu(T, \varphi).
\]

Theorem 3.4.2 If psh weights \( \varphi_k \) and \( \varphi \) are such that \( \exp\{\varphi_k\} \to \exp\{\varphi\} \) uniformly on compact subsets of \( \Omega \), then

\[
\limsup_{k \to \infty} \nu(T, \varphi_k) \leq \nu(T, \varphi).
\]
3.5 Comparison theorems

The first comparison theorem describes variation of the generalized Lelong numbers with respect to the weights. For a psh weight $\varphi$, denote $L(\varphi) = \varphi^{-1}(-\infty)$.

**Theorem 3.5.1** [24], [29] Let $T \in D^+_p(\Omega)$ and $\varphi$ and $\psi$ be psh weights such that

$$\limsup \frac{\psi(z)}{\varphi(z)} = l < \infty \quad \text{as } z \to L(\varphi), \ z \in \text{supp } T,$$

then $\nu(T, \psi) \leq l^p \nu(T, \varphi)$.

The second comparison theorem indicates dependence of the generalized Lelong numbers on the currents.

**Theorem 3.5.2** [24], [29] Let $T \in D^+_p(\Omega)$ and $u_k$, $v_k \in \text{PSH}(\Omega)$, $1 \leq k \leq q$, be such that the currents $\ddc u_1 \wedge \ldots \wedge \ddc u_q \wedge T$ and $\ddc v_1 \wedge \ldots \wedge \ddc v_q \wedge T$ are well defined (see Theorem 3.1.2), $v_k = -\infty$ on $\text{supp } T \cap L(\varphi)$ and

$$\limsup \frac{u_k(z)}{v_k(z)} = l_k < \infty \quad \text{as } z \to L(\varphi), \ z \in \text{supp } T \setminus v_{-1}(\varphi).$$

Then $\nu(\ddc u_1 \wedge \ldots \wedge \ddc u_q \wedge T, \varphi) \leq l_1 \ldots l_q \nu(\ddc v_1 \wedge \ldots \wedge \ddc v_q \wedge T, \varphi)$.

The above results make it possible to obtain relatively simple proofs for the statements of Theorems 2.4.1 – 2.5.2. Besides, Theorem 3.5.2 shows that the residual Monge-Ampère mass $(\ddc u)^n(x)$ of $u \in \text{PSH}(\Omega) \cap L^\infty(\Omega \setminus \{x\})$ is a function of asymptotic behaviour of $u$ near $x$. This leads to the notion of standard singularities at $x$ as equivalence classes of psh functions with respect to their asymptotics [104]. In particular, we have that the functions $\max_j \log |z_j|^{a_j}$ and $\log \sum_j |z_j|^{a_j}$ (with $a_1, \ldots, a_n > 0$ being fixed) represent the same singularity at 0 and

$$(\ddc \max_j |z_j|^{a_j})^n = (\ddc \log \sum_j |z_j|^{a_j})^n = a_1 \ldots a_n \delta_0. \quad (3.7)$$

Another application is the following result comparing the Lelong number of a wedge product with the Lelong numbers of the factors.

**Corollary 3.5.3** [29] If $\ddc u_1 \wedge \ldots \wedge \ddc u_q$ is well defined, then

$$\nu(\ddc u_1 \wedge \ldots \wedge \ddc u_q \wedge T, x) \geq \nu(u_1, x) \ldots \nu(u_q, x) \nu(T, x).$$

A remarkable relation between standard and generalized Lelong numbers is given by
Theorem 3.5.4 [29] For $T \in D^+_p(\Omega)$ and a psh weight $\varphi$ with $\varphi^{-1}(-\infty) = x$,

$$\nu(T, \varphi) \geq \nu(T, x) \nu((dd^c \varphi)^p, x).$$

When $p = 1$ or $p = n - 1$, this can be deduced from the comparison theorems. But for $1 < p < n - 1$ they would give us a more rough inequality $\nu(T, \varphi) \geq \nu(T, x)[\nu(\varphi, x)]^p$. Instead, the proof in this case uses the relation (assuming $x = 0$)

$$\nu(T, \varphi) \geq \int_{U_n} \nu(T, \varphi \circ g) \, d\kappa(g),$$

$d\kappa$ being the Haar measure on the unitary group $U_n$, which can be derived from the deep Theorem 4.6.1.

4 Analyticity theorems for upperlevel sets

Plurisubharmonicity assumes no a priori analyticity. Nevertheless, analytic varieties appear from any psh function (moreover, from any positive closed current) with singularities.

4.1 Upperlevel sets for Lelong numbers

Let $T \in D^+_p(\Omega)$ and

$$E_c(T) := \{ x \in \Omega : \nu(T, x) \geq c \}, \quad c > 0,$$

be the upperlevel sets for the Lelong numbers of $T$. Since $\nu(T, x)$ is lower semicontinuous, $E_c(T)$ is closed. Furthermore, it has locally finite $\mathcal{H}_{2p}$ Hausdorff measure (this follows from Corollary 2.3.2).

Section 4 is devoted to the following fundamental result:

Theorem 4.1.1 Siu [90] $E_c(T)$ is an analytic variety of dimension $\leq p$.

In other words, $\nu(T, x)$ is lower semicontinuous with respect to the analytic Zariski topology (in which closed sets = analytic varieties).

The original Siu’s proof [90] (1974) (developing results from [13] and [92]) takes about 100 pages. A considerable simplification was made by Lelong (1977) [65] who reduced the problem to that for psh function. In 1979 Kiselman applied the attenuating singularities technique (Subsection 1.6) to get a more simple proof of Siu’s theorem for the classical Lelong numbers [46] and, in 1986, for the directional numbers [49], [51]. His ideas were used by Demailly to prove the theorem for the generalized Lelong numbers [25] (1987). Perhaps, the shortest known proof was proposed by Demailly in 1992 [28]. It is based on his approximation theorem for psh functions.

All the proofs rest heavily on $L^2$ estimates for the $\bar{\partial}$ operator. We sketch below the proof of Theorem 4.1.1 based on Demailly’s approximation theorem, as well as Kiselman-Demailly’s proof for generalized Lelong numbers.
4.2 Reduction to plurisubharmonic functions

The initial observation is as follows. Let \( A = \{ z : f_1(z) = \ldots = f_N(z) = 0 \} \), then the function \( v = \frac{1}{2} \log \sum_k |f_k|^2 \) has the property

\[
\nu([A], x) > 0 \iff x \in A \iff \nu(u, x) > 0.
\]

Actually, it is possible to construct a psh function whose Lelong numbers just coincide with those of \( A \) (and more generally, with the Lelong numbers of an arbitrary given positive closed current).

**Theorem 4.2.1** [92], [93], [65] Let \( \Omega \) be a pseudoconvex domain. Given a current \( T \in D^+_p(\Omega) \), there exists a function \( u \in \text{PSH}(\Omega) \) such that \( \nu(T, x) = \nu(u, x) \) for every \( x \).

The proof uses technique of canonical potentials

\[
U_j(z) = -\omega^{-1}_p \int |z - \zeta|^{-2p} \eta_j(\zeta) \, d\sigma_T(\zeta)
\]

with \( \eta_j \) a non-negative, smooth function supported in \( \Omega \), and \( \eta_j \equiv 1 \) on a neighbourhood of \( \Omega_j \subset \subset \Omega \). The function \( U_j \) is subharmonic in \( \mathbb{R}^{2n} \), and

\[
\sigma_{U_j}(x, r) = \frac{1}{2\pi} \int_{B_r(x)} \Delta U_j = [1 + o(1)] \frac{\tau_{n-1} r^{2n-2} \nu(T, x) + o(r^{2n-2})}{r \to 0},
\]

so

\[
\lim_{r \to 0} \frac{\sigma_{U_j}(x, r)}{\tau_{n-1} r^{2n-2}} = \nu(T, x) \quad \forall x \in \Omega_j.
\]

One can show that \( dd^c U_j \geq -N_j \, dd^c |z|^2 \), so \( u_j(z) := U_j(z) + N_j |z|^2 + M_j \in \text{PSH}(\Omega) \) and \( \nu(T, x) = \nu(u_j, x) \) for all \( x \in \Omega_j \). Exhausting \( \Omega \) by \( \Omega_j \) we get the desired function \( u \).

4.3 \( L^2 \)-extension theorems

A bridge between plurisubharmonicity and analyticity is based on the Hörmander type results on solutions for the \( \bar{\partial} \)-problem. In particular, the following two theorems have great importance in studying singularities of plurisubharmonic functions.

**Theorem 4.3.1** (Hörmander-Bombieri-Skoda) [13], [92], [94] If \( u \) is plurisubharmonic on a pseudoconvex domain \( \Omega \) and \( e^{-u} \in L^2_{\text{loc}}(x) \) for some \( x \in \Omega \), then there exists a holomorphic function \( f \) on \( \Omega \) such that

\[
\int |f|^2 e^{-2u} (1 + |x|^2)^{-m-\epsilon} \beta_p < \infty
\]

and \( f(x) = 1 \).
Theorem 4.3.2 (Ohsawa-Takegoshi) [77] Let $Y$ be an affine linear $p$-dimensional subspace of $\mathbb{C}^n$, $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$, and $u \in PSH(\Omega)$. Then any function $h \in Hol(Y \cap \Omega)$ with
\[
\int_{Y \cap \Omega} |h|^2 e^{-u} \beta_p < \infty
\]
can be extended to a function $f \in Hol(\Omega)$ and
\[
\int_\Omega |f|^2 e^{-u} \beta_n \leq A(p, n, \text{diam } \Omega) \int_{Y \cap \Omega} |h|^2 e^{-u} \beta_p.
\]

4.4 Approximation theorem of Demailly

Let $\Omega$ be a bounded pseudoconvex domain, $u \in PSH(\Omega)$. Consider the Hilbert space
\[
H_m := H_m,u(\Omega) = \{ f \in Hol(\Omega) : \int_\Omega |f|^2 e^{-2mu} \beta_n < \infty \},
\]
see Section 1.10. Let $\{ \sigma^{(m)}_l \}_l$ be an orthonormal basis of $H_m$,
\[
u_m(z) := \frac{1}{2m} \log \sum_l |\sigma^{(m)}_l(z)|^2 \in PSH(\Omega).
\]
Note that $\nu_m(z) = \frac{1}{m} \sup \{ \log |f(z)| : \|f\|_m < 1 \}$.

Theorem 4.4.1 [28], [31] There are constants $C_1, C_2 > 0$ such that for any $z \in \Omega$ and every $r < \text{dist} (z, \partial \Omega)$,
\[
u(z) - \frac{C_1}{m} \leq \nu_m(z) \leq \sup_{\zeta \in B_r(z)} u(\zeta) + \frac{1}{m} \log \frac{C_2}{r^m}.
\]
In particular, $\nu_m \rightarrow u$ pointwise and in $L^1_{\text{loc}}(\Omega)$, and
\[
u(u, x) - \frac{n}{m} \leq \nu(u_m, x) \leq \nu(u, x) \quad \forall x \in \Omega.
\]
Besides, the integrability indices of $u$ and $u_m$ are related as
\[
I(u, x) - \frac{1}{n} \leq I(u_m, x) \leq I(u, x).
\]
The proof uses Theorem 4.3.2 (more exactly, its particular case of a one-point set $Y$).

It is worth mentioning that the functions $u_m$ from Theorem 4.4.1 control not only classical Lelong numbers of $u$ but also all its directional ones [84] (see also [36]):
\[
u(u, x, a) - m^{-1} \sum_j a_j \leq \nu(u_m, x, a) \leq \nu(u, x, a) \quad \forall x \in \Omega, \forall a \in \mathbb{R}^n_+.
\]
4.5 Proof of Siu’s theorem

By (4.1),

\[ E_c(T) = E_c(u) = \bigcap_{m \geq m_0} E_{c-n/m}(u_m). \]

Any set \( E_a(u_m) \) is analytic since

\[ x \in E_{c-n/m}(u_m) \iff \frac{\partial^\alpha}{\partial z^\alpha} \sigma^{(m)}_l(x) = 0 \quad \forall \alpha : |\alpha| < cm - n, \]

and so is \( E_c(T) \).

4.6 Siu’s theorem for Lelong-Demailly numbers

Due to relations (4.2), Siu’s theorem for directional Lelong numbers can be proved exactly as for the classical ones in Section 4.5. Another its proof was given by Kiselman (see [51]) by means of the attenuating singularities technique. It can be seen as a particular case of the following Demailly’s result on generalized Lelong numbers which actually is a development of Kiselman’s approach.

Let \( X \) be a Stein manifold (e.g. a pseudoconvex domain in \( C^m \)), and \( \varphi \) be a semiexhaustive psh function on \( \Omega \times X \). The function \( \varphi_x(z) := \varphi(z, x) \) is a psh weight on \( \Omega \). In this setting, \( E_c = E_c(T, \varphi) = \{ x \in X : \nu(T, \varphi_x) \geq c \} \).

**Theorem 4.6.1** [29] If \( \exp \varphi \in C(\Omega \times X) \) and is locally Hölder with respect to \( x \), then \( E_c \) is analytic in \( X \).

**Scheme of the proof.**

1) Construction of a family of psh potentials \( u_a(x), a \geq 0 \), whose behaviour is determined by \( \nu(T, \varphi_x) \) (a refined version of theorem 4.2.1).

2) Let \( N_{a,b} = \{ x \in X : \exp\{-u_a/b\} \notin L^2_{loc}(x) \} \), then \( E_c = \bigcap N_{a,b} \) where the intersection is taken over all \( a < c \) and \( b < (c - a)\gamma/m \) (\( \gamma \) is the Hölder exponent of \( \varphi \)).

3) For any psh function \( u \), the set

\[ NI_u = \{ x \in X : e^{-u} \notin L^2_{loc}(x) \} \]

is analytic subset of \( X \). This fact follows from Theorem 4.3.1.

4) Since \( N_{a,b} = NI_{u_{a/b}} \), the conclusion follows.

Partial Lelong numbers (1.10) cannot be viewed as a particular case of Lelong-Demailly numbers, and actually their upperlevel sets need not be analytic. Some sufficient conditions for the analyticity were given in [98], see also [51].
4.7 Semicontinuity theorems for integrability indices

Let $I(u, x)$ be the integrability index of $u$ at $x$ (1.11), and $IE_c(u) = \{ x : I(u, x) \geq c \}$ its upperlevel set. Theorem 4.3.1 implies Zariski’s semicontinuity of the map $x \mapsto I(u, x)$:

**Theorem 4.7.1** $IE_c(u)$ is an analytic variety for all $c > 0$.

The technique of Demailly’s approximation theorem makes it possible to show that the map $u \mapsto I(u, x)$ is upper-semicontinuous with respect to the weak convergence of plurisubharmonic functions:

**Theorem 4.7.2** [31] If $\gamma > I(u, x)$ and $u_j \to u$, then $\exp\{-u_j/\gamma\} \to \exp\{-u/\gamma\}$ in $L^2_{\text{loc}}$.

5 Structure of positive closed currents

Importance of Siu’s fundamental result (Theorem 4.1.1) becomes clear by means of structural formulas for closed positive currents. The proofs can be found in [29].

5.1 Siu’s decomposition formula

Let $T \in D^+_p(\Omega)$, $A$ be an irreducible analytic variety of dimension $p$. Define the value $\nu(T, A) := \inf \{ \nu(T, x) : x \in A \}$. We have $\nu(T, A) = \nu(T, x)$ $\forall x \in A \setminus A'$, $A'$ being a proper analytic subset of $A$. So, $\nu(T, A)$ is the generic Lelong number of $T$ along $A$.

**Proposition 5.1.1** $\chi_A T = \nu(T, A) [A]$ with $\chi_A$ the indicator function of the set $A$.

Note that $\chi_A T \in D^+_p(\Omega)$ in view of Theorem 2.1.1.

**Theorem 5.1.2** (Siu’s formula) [90] For any current $T \in D^+_p(\Omega)$, there is a unique decomposition

$$T = \sum_j \lambda_j [A_j] + R,$$

where $\lambda_j > 0$, $[A_j]$ are the integration currents over irreducible $p$-dimensional varieties $A_j$, and $R \in D^+_p(\Omega)$ is such that $\dim E_c(R) < p$ for every $c > 0$. Here $\lambda_j$ are generic Lelong numbers of $T$ along the varieties $A_j$.

Some components of lower dimension can actually occur in $E_c(R)$, however $\chi_A R = 0$ for any $p$-dimensional variety $A$. And Siu’s formula cannot be applied to $R$ since it is of bidimension $(p, p)$.  

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5.2 King-Demailly formula

Let \( f = (f_1, \ldots, f_q) : \Omega \to \mathbb{C}^q \) be a holomorphic mapping, \( A_f = f^{-1}(0) \) be its zero set,
\[
    u = \log |f| = \frac{1}{2} \log \sum_k |f_k|^2.
\]
If \( \dim A_f \leq n - l \), then \( (dd^c u)^l \) is well defined. Here \( \dim A = \max_j \dim A_j \) and \( A_j \) are the irreducible components of \( A_f \). Note also that \( A_f = A_{f_1} \cap \ldots \cap A_{f_q} \) where \( A_{f_k} \) are the zero sets of the components \( f_k \) of \( f \) (of codimension \( \leq 1 \)).

When dealing with holomorphic mappings, it is convenient to consider the corresponding holomorphic chains, i.e., the currents
\[
    Z_f = \sum_j m_j [A_j]
\]
where the summation is made only for \((n - p)\)-dimensional components \( A_j \) of the variety \( A \) and \( m_j \in \mathbb{Z}_+ \) are the generic multiplicities of \( f \) at \( A_j \).

The following result (originally established by J. King for \( q = p \leq n \)) represents holomorphic chains as singular parts of Monge-Ampère currents.

**Theorem 5.2.1** \([40], [29], [75]\) If the zero set \( A_f \) of a holomorphic mapping \( f : \Omega \to \mathbb{C}^q \) has codimension \( p \), then the currents \( (dd^c \log |f|)^l \) and \( \log |f| (dd^c \log |f|)^l \) with \( l < p \) have locally integrable coefficients, and
\[
    (dd^c \log |f|)^p = Z_f + R \tag{5.1}
\]
where \( Z_f \) is the corresponding holomorphic chain and the current \( R \in \mathcal{D}^{n-p}_+(\Omega) \) is such that \( \chi_{A_f} R = 0 \), \( \nu(R, x) \) are nonnegative integers and \( \{ x : \nu(R, x) > 0 \} \) is an analytic set of codimension at least \( p + 1 \).

**Particular cases.**
1) \( p = q = 1 \): no condition on \( A \) is required, \( R = 0 \), so we have the Lelong-Poincaré equation \( dd^c \log |f| = Z_f \).
2) \( p = q \leq n \): \( R = 0 \), which gives King’s formula \([40]\)
\[
    (dd^c \log |f|)^p = \sum_j m_j [A_j] = Z_f
\]
(since \( A \) has no components of dimension \( < n - q \)).

Another formula for the case \( p = q \leq n \) represents \( Z_f \) as the wedge product of the divisors of the components \( f_k \) of the mapping \( f \). Denote \( u_k = \log |f_k|, \) then \( dd^c u_k = Z_{f_k} \).

**Theorem 5.2.2** \([29]\) If the zero sets \( A_{f_k} \) satisfy condition (3.3), then
\[
    dd^c u_1 \wedge \ldots \wedge dd^c u_q = Z_{f_1} \wedge \ldots \wedge Z_{f_q} = Z_f.
\]
5.3 Lelong numbers of direct and inverse images

Let $T \in \mathcal{D}^+(\Omega)$ and a holomorphic mapping $f : \Omega \to \Omega'$ be such that its restriction to the support of $T$ is proper. Then its direct image $f_* T \in \mathcal{D}^+(\Omega')$ and $\nu(f_* T, \varphi) = \nu(T, \varphi \circ f)$. The problem is to evaluate this value in terms of characteristics of $T$ and $f$, at least in the case of the classical Lelong numbers.

Let $x \in \Omega$ and $y = f(x)$. If the codimension of the fiber $f^{-1}$ at $x$ is at least $p$, one can define the \textit{multiplicity} of $f$ at $x$ as

$$\mu_p(f, x) := \nu((dd^c \log |f - y|)^p, x).$$ (5.2)

It is the multiplicity of the restriction of $f$ to a generic $p$-dimensional plane through $x$ (and thus a non-negative integer).

**Theorem 5.3.1** [29] Let $T$ and $f$ be as above. Let further $I(y)$ be the set of points $z \in \text{supp } T \cap f^{-1}(y)$ such that $z$ coincides with its connected component in $\text{supp } T \cap f^{-1}(y)$ and codim$_z f^{-1}(y) \geq p$. Then

$$\nu(f_* T, y) \geq \sum_{z \in I(y)} \mu_p(f, z) \nu(T, z).$$

For finite holomorphic mappings, it is also possible to get an upper bound. Set

$$\bar{\mu}_p(f, z) = \inf_G \frac{[\sigma(G \circ f, z)]^p}{\mu_p(G, 0)},$$

where $G$ runs over all germs of maps $(\Omega, z) \to (\mathbb{C}^n, 0)$ such that $G \circ f$ is finite, and $\sigma(H, z)$ is the \textit{Lojasiewicz exponent} of a mapping $H$ at $z$ (the infimum of $\alpha > 0$ such that $|\zeta - z|^\alpha/|H(\zeta)|$ is bounded near $z$).

**Theorem 5.3.2** [29] Let $f$ be a proper and finite holomorphic mapping and $T \in \mathcal{D}^+_p(\Omega)$. Then

$$\nu(f_* T, y) \leq \sum_{z \in \text{supp } T \cap f^{-1}(y)} \bar{\mu}_p(f, z) \nu(T, z).$$

As an application, it gives the following relation for the Lelong numbers with respect to the directional weights $\varphi_{a,x}$ (3.6). Let $a_{j_1} \leq \ldots \leq a_{j_n}$, then

$$\frac{\nu(T, x)}{a_{j_{n-p+1}} \ldots a_{j_n}} \leq \nu(T, \varphi_{a,x}) \leq \frac{\nu(T, x)}{a_{j_1} \ldots a_{j_p}}$$ (5.3)

which is an extension of relation (1.8) between directional and classical Lelong numbers to currents of higher degrees.
Relations for the inverse images were studied in [74]. Let $f : \Omega \to \Omega' \subset \mathbb{C}^m$ be a surjective, proper and finite holomorphic mapping. Then $\nu(f^*T, x) \geq \nu(T, y)$. For the opposite direction, Theorem 5.3.1 implies the bound

$$\sum_{z \in f^{-1}(y)} \mu_p(f, z) \nu(f^*T, z) \leq s \nu(T, y),$$

where $s$ is the degree of the ramified covering $f$.

If $f$ is assumed to be only open and surjective, then

$$\nu(f^*T, x) \leq \mu_m(f, x) \nu(T, y).$$

5.4 Propagation of singularities

Siu’s theorem shows that once a psh function has logarithmic singularity on a massive subset of an analytic variety, it must have them on the whole variety. It gives an idea of constructing upper bounded psh functions without (sub)extensions to larger domains.

**Proposition 5.4.1** [85] For every bounded convex domain $\Omega$ there exists a negative function $u \in PSH(\Omega)$ with the following property. If $\omega$ is an open subset of $\Omega$ and $v \in PSH(\omega)$ satisfies $v \leq u$ in $\omega$, then $v$ cannot be extended to a function which is plurisubharmonic in a domain strictly larger than $\Omega$.

The obstacle for an extension here are upperlevel sets for the Lelong numbers which are constructed to have no extension outside the domain. On the other hand, for any negative psh function $f$ in the unit ball, any $\epsilon \in (0, 1/n)$ and $r < 1$ there exists a psh function $v$ of logarithmic growth in $\mathbb{C}^n$ such that $v(z) \leq -|v(z)|^{1/n - \epsilon}$ on $|z| < r$ [33] (a refined version of Josefson’s theorem mentioned in Section 1.1).

Another application concerns unbounded maximal psh functions. A function $u \in PSH(\Omega)$ is called maximal on $\Omega$ if for any domain $\omega \subset \Omega$ the relation $v \leq u$ in $\Omega \setminus \omega$ for $v \in PSH(\Omega)$ implies $v \leq u$ in the whole $\Omega$. For example, the function $\log |z_1|$ is maximal on $\mathbb{C}^n$, $n > 1$. Another example is $\log |f|$ with $f : \Omega \to \mathbb{C}^n$ such that its Jacobian determinant $J_f$ is identically zero. If $J_f \neq 0$, $\log |f|$ is maximal outside the zero set $A_f$ of $f$. Moreover, if the zero set contains components of positive dimension, $\log |f|$ can be maximal on a neighbourhood of some its points.

**Theorem 5.4.2** [81] For any holomorphic mapping $f : \Omega \to \mathbb{C}^n$ there exists a discrete set $CI_f \subset A_f$ such that $\log |f|$ is maximal on a neighbourhood of every point $z \in \Omega \setminus CI_f$. The set $CI_f$ is the complete indeterminance locus for $f$ considered as a meromorphic mapping to $\mathbb{P}^{n-1}$.

So, $\log |f|$ is locally maximal on $\Omega \setminus CI_f$. It is not known if it implies its maximality there (for bounded psh functions, the maximality is a local property due to the characterization $(ddc u)^n = 0$).
6 Evaluation of residual Monge-Ampère masses

Here we will study the problem of evaluation of the Lelong numbers of the Monge-Ampère currents \((dd^c u)^m\). Even for \(u = \log |f|\) with \(f : \Omega \to \mathbb{C}^m, m > 1\), no explicit formulas for computation of these values are available, and only bounds for them are known (e.g., via the multiplicities of the components of the mapping or in terms of the corresponding Newton polyhedra).

No upper bound of \(\nu((dd^c u)^m, x)\) in terms of \(\nu(u, x)\) is possible. Nevertheless, it seems to be unknown if there exist a psh function with zero Lelong number and nonzero residual Monge-Ampère mass.

As to lower estimates, a standard bound is
\[
\nu((dd^c u)^m, x) \geq [\nu(u, x)]^m
\]
(cf. Corollary 3.5.3). More precise relations can be obtained by means of more refined characteristics of local behaviour of a function, e.g., directional Lelong numbers. To this end, we consider a notion of local indicator, see [73].

6.1 Definition and properties of local indicators

Let \(0 \in \Omega, u \in PSH(\Omega), u \leq 0\), and let \(\nu(u, o, a)\) be its directional Lelong numbers (1.7). Take \(t \in \mathbb{R}^n_- := -\mathbb{R}^n_+\); the function
\[
\psi_u(t) = -\nu(u, 0, -t)
\]
is non-positive, convex in \(t\) and increasing in each \(t_k\), so
\[
\Psi_u(z) := \psi_u(\log |z_1|, \ldots, \log |z_n|)
\]
is plurisubharmonic in \(\{z : 0 < |z_k| < 1, 1 \leq k \leq n\}\) and thus extends to a (unique) psh function in the unit polydisk \(D\), which is called the local indicator of \(u\) at \(0\). (This notion was introduced in [73], however plurisubharmonicity of \(\lambda(u, x, \text{Re}z)\) was observed already in [46].)

It is easily checked that
\[
\psi_u(ct) = c\psi_u(t) \quad \forall c > 0,
\]
which implies \((dd^c \Psi_u)^n = 0\) on \(D \setminus \{z : z_1 \cdot \ldots \cdot z_n = 0\}\).

Besides, \(\Psi_{\psi_u} = \Psi_u\), which means that \(\Psi_u\) has the same directional Lelong numbers as the function \(u\).

Theorem 6.1.1 [73] For any function \(u\) psh in a neighborhood of \(0\),
\[
u(u, 0, a)
\]

near the origin.
Examples.
1) For $u(z) = \log |z|$, $\Psi_u(z) = \sup_k \log |z_k|$.
2) If $u(z) = \varphi_{a,0}(z)$ (3.6), then $\Psi_u = \varphi_{a,0}$, so the directional weights are their own indicators.
3) Let $u = \log |f|$, $f : \Omega \to \mathbb{C}^m$, consider the set

$$\omega_0 = \{J \in \mathbb{Z}_+^n : \sum_j \left| \frac{\partial^f f_j}{\partial z^J}(0) \right| \neq 0 \}. \quad (6.3)$$

As follows from (1.9), $\Psi_u(z) = \sup \{\log |z^J| : J \in \omega_0\}$.

As was mentioned in Section 2.6, a psh function need not have a unique tangent. At the same time, instead of the family (2.2) one may consider the collection

$$\tilde{u}_m(z) = m^{-1} u(z_1^m, \ldots, z_n^m), \quad m = 1, 2, \ldots.$$ 

Theorem 6.1.2 [82] $\tilde{u}_m \to \Psi_u$ in $L^1_{\text{loc}}(D)$.

So, local indicators can be viewed as logarithmic tangents to psh functions.

6.2 Reduction to indicators

Let $\varphi$ be a psh weight such that $\varphi^{-1}(\infty) = 0 \in \Omega$.

In view of (6.2), Theorem 3.5.1 implies

Theorem 6.2.1 [73] If $dd^c u_1 \wedge \ldots \wedge dd^c u_q$ is well defined near the origin (see Theorem 3.1.2), then

$$\nu(dd^c u_1 \wedge \ldots \wedge dd^c u_q, \varphi) \geq \nu(dd^c \Psi_{u_1} \wedge \ldots \wedge dd^c \Psi_{u_q}, \varphi) \geq \nu(dd^c \Psi_{u_1} \wedge \ldots \wedge dd^c \Psi_{u_q}, \Psi_{\varphi}).$$

For $u \in L^\infty_{\text{loc}}(\Omega \setminus \{0\})$, the operator $(dd^c u)^n$ is well defined, and the value

$$\mathcal{R}_u := \nu((dd^c u)^n, 0)$$

is called the residual measure of $(dd^c u)^n$ at 0. In this situation, $(dd^c \Psi_u)^n = 0$ on $D \setminus \{0\}$, so that

$$(dd^c \Psi_u)^n = \tau_u \delta_0$$

with

$$\tau_u = \mathcal{R}_{\Psi_u}$$

which will be called the Newton number of $u$ at 0 (the reason for using the name is clarified in the section 6.6).

Corollary 6.2.2 [73] If $u \in PSH(\Omega) \cap L^\infty_{\text{loc}}(\Omega \setminus \{0\})$, then $\mathcal{R}_u \geq \tau_u$. 

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Similarly, for an $n$-tuple $(u)$ of psh function $u_1, \ldots, u_n$, the residual Monge-Ampère mass

$$\mathcal{R}_u := (dd^c u_1 \wedge \ldots \wedge dd^c u_n)(0)$$

has the bound

$$\mathcal{R}_u \geq \tau(u),$$

where

$$\tau(u) = \mathcal{R}_{\Psi_u} = (dd^c \Psi_{u_1} \wedge \ldots \wedge dd^c \Psi_{u_n})(0).$$

To make all this reasonable, one has to look for good bounds for the Newton numbers.

### 6.3 Bounds in terms of directional Lelong numbers

Let $\Psi(z) = \Psi(|z_1|, \ldots, |z_n|) \in PSH(D) \cap L^\infty_{loc}(D \setminus 0)$, $\Psi < 0$ in $D$, and let its convex image $\psi(t) := \Psi(\exp(t_1), \ldots, \exp(t_n))$ be homogeneous: $\psi(ct) = c\psi(t), \forall c > 0$. Such a function will be called an (abstract) indicator. Note that our assumptions mean that $\psi$ is the restriction to $\mathbb{R}^n$ of the support function of a convex subset of $\mathbb{R}^n_+$. For a slightly different way to introduce the abstract indicators, see [103], [104].

We have $(dd^c \Psi)^n = \tau \delta_0$. It is easy to see that for all $z, \zeta \in D$, there is the inequality

$$\Psi(\zeta) \leq |\Psi(z)| \Phi_z(\zeta),$$

where

$$\Phi_z(\zeta) = \sup_k \frac{\log |\zeta_k|}{\log |z_k|}.$$

Therefore, in view of Comparison Theorem 3.5.2 and equation (3.7), we have for an $n$-tuple $(\Psi)$ of indicators $\Psi_1, \ldots, \Psi_n$ the bound

$$\mathcal{R}_u \geq |\Psi_1(z) \ldots \Psi_n(z)|(dd^c \Phi_z)^n = \prod_k \left| \frac{\Psi_k(z)}{\log |z_k|} \right|$$

for all $z \in D$ with $z_1 \ldots z_n \neq 0$, and for a $q$-tuple of indicators $\Psi_1, \ldots, \Psi_q$, $q < n$, and a psh weight $\varphi$,

$$\nu(dd^c u_1 \wedge \ldots \wedge dd^c u_q, \varphi) \geq |\Psi_1(z) \ldots \Psi_q(z)| \nu((dd^c \Phi_z)^q, \varphi).$$

Thus (5.3) and Theorem 6.2.1 with $\varphi(z) = \log |z|$ imply the following result.

**Theorem 6.3.1** [82] In the conditions of Theorem 6.2.1, for any $q \leq n$,

$$\nu(dd^c u_1 \wedge \ldots \wedge dd^c u_q, 0) \geq \frac{\nu(u_1, 0, a) \ldots \nu(u_q, 0, a)}{a_{j_{q+1}} \ldots a_{j_n}} \forall a \in \mathbb{R}^n_+,$$

where $a_{j_1} \leq \ldots \leq a_{j_n}$. In particular, if $u \in PSH(\Omega) \cap L^\infty_{loc}(\Omega \setminus 0)$, then

$$\mathcal{R}_u \geq \tau_u \geq \frac{[\nu(u, 0, a)]^n}{a_1 \ldots a_n} \forall a \in \mathbb{R}^n_+.$$
6.4 Geometric interpretation: volumes

More sharp bounds can be obtained by precise calculation of the Monge-Ampère masses of the indicators.

Let \( U(z) = U(|z_1|, \ldots, |z_n|) \in PSH(D) \cap L^\infty_{\text{loc}}(D) \), and \( h(t) := U(\exp(t_1), \ldots, \exp(t_n)) \) be its convex image in \( \mathbb{R}^n_+ \). Then

\[
(dd^c U)^n = n!(2\pi)^{-n} \mathcal{MA}_R[h] d\theta, \quad (z_k = \exp\{t_j + i\theta_j\})
\]

\( \mathcal{MA}_R \) being the real Monge-Ampère operator. For \( h \) smooth,

\[
\mathcal{MA}_R[h] = \det \left( \frac{\partial^2 h}{\partial t_j \partial t_k} \right) dt,
\]

and it extends, as a measure-valued operator, to all convex functions \( h \). Furthermore,

\[
\int_F \mathcal{MA}_R[h] = \text{Vol} \, G_h(F), \quad F \subset \subset \mathbb{R}^n_+,
\]

with

\[
G_h(F) = \bigcup_{t^0 \in F} \{a \in \mathbb{R}^n_+ : h(t) \geq h(t^0) + \langle a, t - t^0 \rangle \ \forall t \in \mathbb{R}^n_+ \}
\]

the gradient image of \( F \) for the surface \( \xi = h(t) \). The terminology comes from the smooth situation because then we have \( \mathcal{MA}_R[h] \) equal the Jacobian determinant \( J_{\nabla h} \) for the gradient mapping \( \nabla h \), \( G_h(F) = \nabla h(F) \) and the equation (6.4) is just the coordinate change formula.

For a thorough treatment of the real Monge-Ampère operator, see [86].

So, for any \( n \)-circled set \( E \subset \subset D \) and \( E^* = \{t \in \mathbb{R}^n_+ : (\exp t_1, \ldots, \exp t_n) \in E\} \),

\[
(dd^c U)^n(E) = n! \text{Vol} \, G_h(E^*).
\]

Let now \( U = \max\{\Psi, -1\} \) with \( \Psi \) an indicator. If \( \Psi \in L^\infty_{\text{loc}}(D \setminus \{0\}) \), then the current \( (dd^c U)^n \) is supported by the set \( E_\Psi = \{z : \Psi(z) = -1\} \subset \subset D \setminus \{0\} \), and \( E^*_\Psi = \mathbb{R}^n_+ \setminus B_\Psi \), where

\[
B_\Psi = \{a \in \mathbb{R}^n_+ : \langle a, t \rangle \leq \psi(t) \ \forall t \in \mathbb{R}^n_+ \}
\]

(6.5)

(as before, \( \psi \) is the convex image of \( \Psi \)). Note that the function \( \psi \) is just the restriction of the support function of the convex set \( B_\Psi \) to \( \mathbb{R}^n_+ : \psi(t) = \sup \{\langle a, t \rangle : a \in B_\Psi\} \).

For any convex and complete subset \( B \) of \( \mathbb{R}^n_+ \) (the latter means that \( a + \mathbb{R}^n_+ \subset B \) for each \( a \in B \)) we put

\[
\text{Covol} \, B = \text{Vol} \, (\mathbb{R}^n_+ \setminus B),
\]

the covolume of \( B \).
Theorem 6.4.1 [82] (see also [26]) If an indicator $\Psi \in L^\infty_{\text{loc}}(D \setminus \{0\})$, then its Monge-Ampère mass is

$$\mathcal{R}_\Psi = n! \text{Covol } B_\Psi,$$

where $B_\Psi$ is defined in (6.5).

When $\Psi = \Psi_u$, the set

$$B_\Psi = B_u = \{a \in \mathbb{R}^n_+ : \nu(u, 0, a) \leq \langle a, b \rangle \forall b \in \mathbb{R}^n_+\}.$$

Thus Corollary 6.2.2 gives us

Theorem 6.4.2 If $u$ has isolated singularity at 0, then

$$\mathcal{R}_u \geq \tau_u = n! \text{Covol } B_u \tag{6.6}.$$ 

Note that the value

$$\sup_a \frac{[\nu(u, 0, a)]^n}{a_1 \ldots a_n}$$

from Theorem 6.3.1 is the supremum over the volumes of all simplices contained in the set $\mathbb{R}^n_+ \setminus B_u$, and $[\nu(u, 0)]^n$ is the volume of the symmetric simplex $\{a \in \mathbb{R}^n_+ : \sum a_j \leq \nu(u, 0)\} \subset \mathbb{R}^n_+ \setminus B_u$.

To compute the mass of the corresponding mixed Monge-Ampère operators of indicators, we consider a (unique) form $\text{Covol}(B_1, \ldots, B_n)$ on $n$-tuples of complete convex subsets $B_1, \ldots, B_n$ of $\mathbb{R}^n_+$ which is multilinear with respect to Minkowsky’s addition and such that for every convex complete $B$ with bounded complement in $\mathbb{R}^n_+$ we have $\text{Covol}(B, \ldots, B) = \text{Covol } B$. The form can be shown to be well defined on all $n$-tuples $B_1, \ldots, B_n$ such that $\mathbb{R}^n_+ \setminus \bigcup_j B_j$ is bounded.

Theorem 6.4.3 If plurisubharmonic functions $u_1, \ldots, u_n$ have 0 as an isolated point of $\cap_k L(u_n)$, then

$$(dd^c u_1 \wedge \ldots \wedge dd^c u_n)(0) \geq n! \text{Covol}(B_{u_1}, \ldots, B_{u_n}).$$

Taking $u_1 = u$ and $u_2 = \ldots = u_n = \phi$, we get an estimate for the Lelong-Demailly numbers $\nu(u, \phi)$. Another its form can be derived by means of the Lelong-Jenson-Demailly formula (see Theorem 3.3.1). Let $F$ be a subset of the convex set

$$L^\phi = \{t \in \mathbb{R}^n_\phi : \nu(\phi, 0, -t) \geq 1\},$$

consider the sets

$$\Gamma_F^\phi = \{a \in \mathbb{R}^n_+ : \sup_{t \in F} \langle a, t \rangle = \sup_{t \in L^\phi} \langle a, t \rangle = -1\},$$

and

$$\Theta_F^\phi = \{\gamma a : 0 \leq \gamma \leq 1, a \in \Gamma_F^\phi\}.$$
Theorem 6.4.4 [84] If \( \varphi^{-1}(\infty) = 0 \), then
\[
\nu(u, \varphi) \geq n! \int_{E^{\varphi}} \nu(u, 0, -t) d\gamma^{\varphi}(t),
\]
where the measure \( \gamma^{\varphi} \) on the set \( E^{\varphi} \) of extreme points of \( L^{\varphi} \) is given by the relation
\[
\gamma^{\varphi}(F) = \text{Vol} \Theta_{E}^{\varphi}
\]
for compact subsets \( F \) of \( E^{\varphi} \).

6.5 Functions with multicircled singularities

We will say that a plurisubharmonic function \( u \) on a domain \( \Omega \subset \mathbb{C}^n \) has \textit{multicircled singularity} at a point \( x \in \Omega \) if there exists a multicircled plurisubharmonic function \( g \) (i.e., \( g(z) = g(|z_1|, \ldots, |z_n|) \)) in a neighbourhood of the origin such that
\[
\exists \lim_{z \to x} \frac{u(z)}{g(z-x)} = 1.
\]
(6.8)

It is easy to see that \( u \) has multicircled singularity at \( x \) if and only if it satisfies relation (6.8) with \( g \) equal to some its ”circularization”, say to
\[
g(z) = (2\pi)^{-n} \int_{[0,2\pi]^n} u(x_1 + z_1 e^{i\theta_1}, \ldots, x_n + z_n e^{i\theta_n}) d\theta
\]
or to its maximum on the same set.

One can expect that such a regular behaviour implies nice relations for the above characteristics of the singularity, and this is really the case. As follows from Theorem 1.10.3,
\[
I(u, x) = \sup \{\nu(u, x, a) : \sum a_j = 1\}
\]
if \( u \) has multicircled singularity at \( x \). Furthermore, the residual mass \( R_u \) for such a function \( u \) (assumed to be locally bounded outside the origin) equals its Newton number \( \tau_u \) [83], [84]. Moreover, it can be estimated from above in terms of the limit values of the directional Lelong numbers [83]: if \( \nu_j' \) is the limit of the values \( \nu(u, 0, a) \) as \( a_j \to 1 \), \( a_k \to +\infty \), \( k \neq j \), then
\[
R_u = \tau_u \leq \nu_1' \ldots \nu_n'.
\]
It implies a bound in terms of the corresponding partial Lelong numbers \( \nu_j \) (which can be strictly greater than \( \nu_j' \) even for multicircled functions, see Section 1.9).

Finally, if a weight \( \varphi \) has multicircled singularity, then there is also equality in (6.7), provided the \( -\infty \) set of \( u \) does not contain lines parallel to the coordinate axes and passing through \( x \) [84]. And it fails to be true, for example, for \( u(z_1, z_2) = \log |z_1| \) and \( \varphi(z_1, z_2) = \max\{-|\log |z_1||^{1/2}, \log |z_2|\} + \log |z| \).
6.6 Applications to holomorphic mappings: Newton polyhedra

For functions $u = \log |f|$ with holomorphic $f : \Omega \to \mathbb{C}^n$ such that $f^{-1}(0) = 0$, the residual Monge-Ampère mass $R_u$ of $(dd^c u)^n$ at 0 is just the multiplicity $m_f$ of $f$ at the origin.

Theorem 6.3.1 with $a_1 = \ldots = a_n$ gives us the bound

$$m_f \geq m_{f_1} \cdots m_{f_n}$$

via the multiplicities of the components of the mapping $f$ (a local variant of Bezout’s theorem), and – with integer $a_k$ – the relation

$$m_f \geq \frac{m_{f_1, a_1} \cdots m_{f_n, a_n}}{a_1 \cdots a_n}$$

with the multiplicities of the weighted initial homogeneous polynomial terms of the functions $f_{j,a}(z) = f_j(z_1^{a_1}, \ldots, z_n^{a_n})$ (Tsikh-Yuzhakov [102], see also [2]).

In view of (1.9), $B_u$ is the convex hull of the set $\omega_0$ defined in (6.3), so it is the Newton polyhedron for the mapping $f$ at 0, and $\tau_u = N_f$ is the Newton number of $f$, see [2], [4], [58]. This particular case of Theorem 6.4.2 recovers the result $m_f \geq N_f$ obtained by Kouchnirenko (1975) by means of analytic and algebraic techniques.

The corresponding specification of Theorem 6.4.3 gives a Kouchnirenko-Bernstein’s type result from [2], Theorem 22.10.

Note that for holomorphic mappings $f : \Omega \to \mathbb{C}^q$ with codim$_0 f^{-1}(0) = q < n$, Theorem 6.4.3 gives the bound

$$m_f \geq n! \text{Covol}(B_{u_1}, \ldots, B_{u_q}, B_1, \ldots, B_1),$$

where $u_j = \log |f_j|$, $B_{u_j}$ are the Newton polyhedra of the functions $f_j$ at 0, and $B_1 = \{a \in \mathbb{R}^n_+ : \sum a_k \geq 1\}$.

Concerning Theorem 6.4.4, it is worth noticing that if a weight $\varphi$ has the form $\varphi = \log |g|$ with a holomorphic mapping $g$, then the measure $\gamma^\varphi$ has finite support and is determined by the $(n-1)$-dimensional faces of the Newton polyhedron of $g$.

So, methods of pluripotential theory are quite powerful to produce, in a simple and unified way, efficient bounds for multiplicities of holomorphic mappings.

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