The Numerical Stability of Regularized Barycentric Interpolation Formulae for Interpolation and Extrapolation

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Abstract

The $\ell_2$- and $\ell_1$-regularized modified Lagrange interpolation formulae over $[-1, 1]$ are deduced in this paper. This paper mainly analyzes the numerical characteristics of regularized barycentric interpolation formulae, which are presented in [2, C. An and H.-N. Wu, 2019], and regularized modified Lagrange interpolation formulae for both interpolation and extrapolation. Regularized barycentric interpolation formulae can be carried out in $O(N)$ operations based on existed algorithms [30, H. Wang, D. Huybrechs and S. Vandewalle, Math. Comp., 2014], and regularized modified Lagrange interpolation formulae can be realized in an algorithm of $O(N \log N)$ operations. For interpolation, the regularized modified Lagrange interpolation formulae are blessed with backward stability and forward stability, whereas the regularized barycentric interpolation formulae are only provided with forward stability. For extrapolation, the regularized barycentric interpolation formulae meet loss of accuracy outside $[-1, 1]$, but the regularized modified Lagrange interpolation formulae still work. Consistent results for extrapolation are also verified outside the Chebfun ellipse (a special Bernstein ellipse) in the complex plain. Finally, we illustrate that regularized interpolation formulae perform better than classical interpolation formulae without regularization in noise reduction.

Keywords. barycentric formula, polynomial interpolation, regularization, numerical stability, rounding error analysis, Bernstein ellipse, extrapolation

AMS subject classifications. 41A05, 65D05, 65D15, 65G50

1 Introduction

It is well known that the Lagrange interpolation formula is one of the most fundamental polynomial approximation scheme, as introduced in every numerical analysis textbook [25]. We thus have what is called Lagrange interpolation formula. Let $\{x_0, x_1, \ldots, x_N\} \subset [-1, 1]$ be a set of distinct interpolation nodes. Polynomial $p_N(x)$ of degree $N$ that interpolates the function $f(x)$ at nodes is defined by

$$p_N(x) = \sum_{j=0}^{N} f(x_j) \ell_j(x), \quad (1.1)$$

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where \( \ell_j(x) = \prod_{k \neq j} \frac{x-x_k}{x_j-x_k} \), \( j = 0, 1, \ldots, N \) are the Lagrange fundamental polynomials. Apart from the high cost operation in typical algorithm, the Lagrange form of interpolating polynomial eq. (1.1) meets numerical instability on equispaced nodes [20] and requires recalculation for each Lagrange fundamental polynomial. As pointed out in [3], there is merit in manipulating the Lagrange polynomial through the formula of barycentric interpolation. The barycentric formula has many attractive properties, such as numerical stability and high efficiency, see [3, 18, 31, 30] and the references therein. There are two variants of the barycentric formula. The modified Lagrange interpolation originates with Jacobi in 1825 [19], which is also called the first barycentric formula [21]:

\[
P_N^{\text{mdf}}(x) = \ell(x) \sum_{j=0}^{N} \frac{\Omega_j}{x-x_j} f(x_j),
\]

where \( \ell(x) = (x-x_0)(x-x_1) \cdots (x-x_N) \), and \( \Omega_j, j = 0, 1, \ldots, N \), are the so-called barycentric weights (\( \Omega_j \) is denoted by \( \lambda_j \) in many texts) which are defined as

\[
\Omega_j = \frac{1}{\prod_{k \neq j} (x_j-x_k)}, \quad j = 0, 1, \ldots, N. \tag{1.2}
\]

The barycentric formula, which is also called the second barycentric formula [21], is in the form of

\[
P_N^{\text{bary}}(x) = \frac{\sum_{j=0}^{N} \frac{\Omega_j}{x-x_j} f(x_j)}{\sum_{j=0}^{N} \Omega_j},
\]

where barycentric weight \( \Omega_j \) is defined as eq. (1.2), which initially appears in Taylor’s article [27], and the term “barycentric” seems to appear firstly in Dypuy’s article [7], both in the 1940s. Higham proves that the modified Lagrange formula is backward stable (which is also forward stable since backward stability implies forward stability [17]) and the barycentric formula is forward stable for the well distribution interpolation nodes [18]. The study of barycentric weights for roots and extrema of the classical polynomials is well developed, see [3, 22, 23, 30, 31] and references therein. Both modified Lagrange interpolation formula and barycentric formula have advantages and disadvantages [18, 28, 34].

In practical problems, one might collect data with noise and perturbations. In our previous paper, the \( \ell_2 \)- and \( \ell_1 \)-regularized barycentric interpolation formulae denoise the approximation function with noisy data over the interval \([-1, 1]\) successfully. In this paper, we deduce the \( \ell_2 \)- and \( \ell_1 \)-regularized modified Lagrange interpolation formulae, see eq. (2.6) and eq. (2.7), respectively. It is desirable that these two regularized modified Lagrange interpolation formulae could be applied to realize denoising and extrapolation. One of our motivation is to analyze the numerical stability of these four regularized barycentric formulae, for interpolation and extrapolation.

The rest of this paper is organized as follows. With some preliminaries, regularized modified Lagrange interpolation formulae are deduced in Section 2. Lower time complexity and high efficiency reveal the practicality of these regularized formulae. In Section 3, we define condition numbers on regularized barycentric interpolation polynomials. Then we study the stability and error analysis of these four regularized barycentric interpolation formulae, in terms of floating point arithmetic model [17]. Based on Bernstein ellipse [32, 36], we consider extrapolation of our four regularized interpolation formulae in Section 4. Section 5 gives several numerical experiments to illustrate the effectiveness and usefulness of the regularized modified Lagrange interpolation formulae. Then we conclude this paper with some remarks.
All numerical results in this paper are carried out by using MATLAB R2017A on a desktop (8.00 GB RAM, Intel(R) Processor 5Y70 at 1.10 GHz and 1.30 GHz) with Windows 10 operating system.

2 Regularized barycentric interpolation and regularized modified Lagrange interpolation

We first review the regularized barycentric interpolation formulae given in [2] and deduce the regularized modified Lagrange interpolation.

2.1 Regularized barycentric interpolation

For details the reader may refer to [2] or appendix A. The \( \ell_2 \)-regularized barycentric interpolation formula is expressed as

\[
p_N^{\ell_2-\text{bary}}(x) = \frac{\sum_{j=0}^{N} \frac{\Omega_j}{x-x_j} f(x_j)}{(1 + \lambda \mu_0^2) \sum_{j=0}^{N} \frac{\Omega_j}{x-x_j}}, \tag{2.1}
\]

where \( \Omega_j \) is defined by eq. (1.2).

We also obtain the \( \ell_1 \)-regularized barycentric interpolation formula:

\[
p_N^{\ell_1-\text{bary}}(x) = \frac{\sum_{j=0}^{N} \frac{\Omega_j}{x-x_j} \left( f(x_j) + \sum_{\ell=0}^{N} c_\ell \Phi_\ell(x_j) \right)}{\sum_{j=0}^{N} \frac{\Omega_j}{x-x_j}}, \tag{2.2}
\]

where \( \Omega_j \) is defined by eq. (1.2) and coefficients \( c_\ell \) is defined by

\[
c_\ell = \frac{S_{\mu_\ell}}{2} \left( 2 \sum_{j=0}^{N} \omega_j \Phi_\ell(x_j) f(x_j) \right) - \sum_{j=0}^{N} \omega_j \Phi_\ell(x_j) f(x_j), \quad \ell = 0, 1, \ldots, N. \tag{2.3}
\]

Remark. When \( \lambda = 0 \), both \( \ell_2 \)- and \( \ell_1 \)-regularized barycentric interpolation return to classical barycentric interpolation.

2.2 Regularized modified Lagrange interpolation

We then convert to the modified Lagrange interpolation

\[
p_N^{\text{mdf}}(x) = \ell(x) \sum_{j=0}^{N} \frac{\Omega_j}{x-x_j} f(x_j),
\]

which can be traced back to Jacobi’s doctoral dissertation in 1825 [19]. For regularized modified Lagrange interpolation, which is not mentioned in our previous work, we reverse the derivation from modified Lagrange interpolation to barycentric interpolation in [3].

Let \( \ell(x) = (x-x_0)(x-x_1) \cdots (x-x_N) \). \tag{2.4}
Since when \( f(x) \equiv 1 \),
\[
1 = \sum_{j=0}^{N} \ell_j(x) = \ell(x) \sum_{j=0}^{N} \frac{\Omega_j}{x-x_j}, \tag{2.5}
\]
where
\[
\ell_j(x) = \frac{\prod_{k=0, k \neq j}^{N} (x-x_k)}{\prod_{k=0, k \neq j}^{N} (x_j-x_k)}, \quad j = 0, 1, \ldots, N,
\]
are Lagrange polynomials, then we obtain the \( \ell_2 \)-regularized modified Lagrange interpolation formula
\[
p_{\ell_2}^{mdf}(x) = \ell(x) \sum_{j=0}^{N} \frac{\Omega_j}{x-x_j} \frac{f(x_j)}{1 + \lambda \mu_0^2}, \tag{2.6}
\]
and the \( \ell_1 \)-regularized modified Lagrange interpolation formula
\[
p_{\ell_1}^{mdf}(x) = \ell(x) \sum_{j=0}^{N} \frac{\Omega_j}{x-x_j} \left( f(x_j) + \sum_{\ell=0}^{N} c_{\ell} \tilde{\Phi}_{\ell}(x_j) \right), \tag{2.7}
\]
where \( \Omega_j \) is defined by eq. (1.2) and \( c_{\ell} \) is defined by eq. (2.3).

### 2.3 Misconvergence

The convergence rate for smooth function in classical barycentric interpolation are discussed in [3, 35], showing the fast convergence of classical barycentric interpolation for smooth functions. Moreover, classical orthogonal polynomial interpolation may be blessed with superconvergence properties [37]. However, we must address that such an excellent property does not hold for regularized barycentric interpolation as the interpolation conditions \( f(x_j) = p_N(x_j) \) for \( j = 0, 1, \ldots, N \) do not hold. The substance of regularized barycentric interpolation is preprocess of sampling values \( f(x_j), \ j = 0, 1, \ldots, N \). Preprocessed sampling values, \( f(x_j)/(1 + \lambda \mu_0^2) \) by \( \ell_2 \)-regularization and \( f(x_j) + \sum_{\ell=0}^{N} c_{\ell} \tilde{\Phi}_{\ell}(x_j) \) by \( \ell_1 \)-regularization for \( j = 0, 1, \ldots, N \), return to classical barycentric interpolation scheme. Thus
\[
p_{\ell_2}^{bary}(x_j) = \frac{f(x_j)}{1 + \lambda \mu_0^2} \neq f(x_j) \quad \forall j = 0, 1, \ldots, N, \tag{2.8}
\]
and
\[
p_{\ell_1}^{bary}(x_j) = f(x_j) + \sum_{\ell=0}^{N} c_{\ell} \tilde{\Phi}_{\ell}(x_j) \neq f(x_j) \quad \forall j = 0, 1, \ldots, N, \tag{2.9}
\]
where “\( \neq \)” holds except for the cases of 1) \( \lambda = 0 \); or 2) \( \mu_\ell = 0 \) for all \( \ell = 0, 1, \ldots, N \).

Actually, the regularized barycentric interpolation generates the interpolant of sampling values of the “preprocessed” function \( f^{pre} \) as follows
\[
f^{\ell_2-pre}(x) = \frac{f(x)}{1 + \lambda \mu_0^2}, \tag{2.10}
\]
and
\[
f^{\ell_1-pre}(x) = f(x) + \sum_{\ell=0}^{N} c_{\ell} \tilde{\Phi}_{\ell}(x), \tag{2.11}
\]
rather than the original function \( f \). And so does the regularized modified Lagrange interpolation.
2.4 Time complexity and fast computation

With the definition of $\Omega_j$, say eq. (1.2), classical barycentric interpolation formula and modified Lagrange interpolation formula both requires $O(N^2)$ operations. However, the reader may note that it requires only $O(N)$ for evaluating $p_N(x)$ once the barycentric weights are known [3]. For example, for Chebyshev points of the first kind, the barycentric weights are given by [16, p. 249]

$$
\Omega_{CH1}^j = (-1)^j \sin \left( \frac{(2j+1)\pi}{2N+2} \right), \quad j = 0, 1, \ldots, N,
$$

for Chebyshev points of the second kind, the barycentric weights are given by [22]

$$
\Omega_{CH2}^j = (-1)^j \delta_j, \quad \delta_j = \begin{cases} 
1/2, & j = 0 \text{ or } j = N \\
1, & \text{otherwise}
\end{cases}, \quad j = 0, 1, \ldots, N,
$$

for Legendre points, the barycentric weights are given by [31]

$$
\Omega_{Leg}^j = (-1)^j \sqrt{(1-x_j^2)} \omega_j, \quad j = 0, 1, \ldots, N,
$$

where $\omega_j$ denotes the Gauss quadrature weight at $x_j$, and generally, for Jacobi points, i.e., the roots of the Jacobi polynomial $P^{(\alpha,\beta)}_{N+1}(x)$, the barycentric weights are given by [30]

$$
\Omega_j^{(\alpha,\beta)} = C_N^{(\alpha,\beta)} (-1)^j \sqrt{(1-x_j^2)} \omega_j, \quad j = 0, 1, \ldots, N,
$$

where $\omega_j$ still denotes the Gauss quadrature weight at $x_j$, and for the coefficient $C_N^{(\alpha,\beta)}$ the reader may refer to [30] (2.24) and (2.25)].

The computation of the Gauss points $x_j$ and the corresponding Gauss quadrature weight $\omega_j$ has been explored for several decades because of the expensive computation of $O(N^2)$ operations in terms of three-term recurrence relation satisfied by the orthogonal polynomials [12]. CHEerfully, recent fast algorithms [4, 10, 11, 13] cost only $O(N)$ operations to obtain $x_j$ and $\omega_j$. Therefore, the barycentric interpolation formula and modified Lagrange interpolation formula can be carried out in $O(N)$ operations.

Then we immediately obtain the time complexity of $\ell_2-$regularized barycentric interpolation formula and $\ell_2-$regularized modified Lagrange interpolation formula.

**Theorem 2.1** Suppose Gauss quadrature weights $\omega_j$ can be computed in $O(N)$ operations, then $\ell_2-$regularized barycentric interpolation formula and $\ell_2-$regularized modified Lagrange interpolation formula can be carried out in $O(N)$ operations.

**Proof.** Since classical barycentric interpolation and modified Lagrange interpolation requires only $O(N)$ operations and there only exists a factor $1/(1 + \lambda \mu_2)$ between classical formulae and $\ell_2-$regularized formulae, then both eq. (2.1) and eq. (2.2) can be carried out in $O(N)$ operations. □

However, $\ell_1-$regularized formulae cannot be carried out in $O(N)$ operations except for the case that $f^{1-\text{pre}}(x_j)$ are known.

**Theorem 2.2** Suppose Gauss quadrature weights $\omega_j$ can be computed in $O(N)$ operations, then $\ell_1-$regularized barycentric interpolation formula and $\ell_1-$regularized modified Lagrange interpolation formula can be carried out in $O(N^2)$ operations. Moreover, if $\sum_{j=0}^{N} \omega_j \hat{f}(x_j) f(x_j)$ can be computed by Fast Fourier Transform (FFT), which costs only $O(N \log N)$, then both formulae can be carried out in $O(N \log N)$ operations.
Proof. The computation of $O(N^2)$ stems from the $\ell_1$-regularized pre-process of $f$: the computation of $\sum_{\ell=0}^N c_\ell \check{\Phi}_\ell(x_j)$, where

$$c_\ell = S_{\mu_{\ell}} \left( \frac{2 \sum_{j=0}^N \omega_j \check{\Phi}_\ell(x_j) f(x_j)}{2} - \sum_{j=0}^N \omega_j \check{\Phi}_\ell(x_j) f(x_j) \right),$$

requires $O(N^2)$ operations. However, the computation of $\sum_{j=0}^N \omega_j \check{\Phi}_\ell(x_j) f(x_j)$ can be carried out in $O(N \log N)$ operations if it is computed by FFT.

Remark. For example, if the points are the Chebyshev points of the first kind, i.e., the roof of the Chebyshev polynomials of the first kind $T_{N+1}(x)$, let $\{\omega_j\}_{j=2^{N+1}}$ be a set of zeros’s, FFT can be used to evaluate $\sum_{j=0}^N \omega_j \check{\Phi}_\ell(x_j) f(x_j)$ in the form of

$$\sum_{j=0}^N \omega_j \check{\Phi}_\ell(x_j) f(x_j) = \text{Re} \left( \frac{e^{2\pi i j \ell / (N+1)}}{\|T_\ell(x)\|_{L_2}} \sum_{j=0}^{2^{N+1}} \omega_j f(x_j) e^{2\pi i j / (N+1)} \right), \quad j = 0, 1, \ldots, N,$$

which requires $O(N \log N)$ operations. For other points, for instance Legendre points, one may use inverse discrete Legendre Transform (IDLT) to obtain expansion coefficients with fast algorithm [15]; one can also use fast algorithms that convert between Legendre and Chebyshev coefficients, such as [14].

3 The numerical stability for interpolation

In this section, we start with introductory backgrounds and preliminaries, then we discuss the stability and error analysis of the regularized barycentric interpolation formulae eq. (2.1) and eq. (2.2), and regularized modified Lagrange interpolation formulae eq. (2.6) and eq. (2.7), respectively.

3.1 Preliminaries

In our error analysis we use the standard tools, backward stability and forward stability [17]. An algorithm $y = f(x)$ is called backward stable if the backward error $\Delta x$ is relatively small, i.e., for a relatively small perturbation $\Delta x$ with respect to any $x$ there exists that computed $\tilde{y} = \hat{f}(x) = f(x + \Delta x)$. The algorithm is called forward stable if the error $|y - \tilde{y}|$ or relative error $|y - \tilde{y}|/|y|$ are relatively small.

We adopt the standard model of floating point arithmetic [17] Section 2.2,

$$fl(x \circ y) = (x \circ y)(1 + \delta)^{\pm 1}, \quad |\delta| \leq u, \quad \circ \in +, -, *, /,$$

to carry out our rounding error analysis, where $u$ is the unit roundoff or machine precision with order $10^{-8}$ or $10^{-16}$ in single and double precision computer arithmetic, respectively.

We mimic Higham’s work on the numerical stability of classical barycentric interpolation [18]. Thus we also employ the notation of relative error counter introduced by Stewart [24],

$$\langle k \rangle = \prod_{i=0}^k (1 + \delta_i)^{\rho_i}, \quad \rho_i = \pm 1, \quad |\delta_i| \leq u,$$

and $\langle k \rangle_j$ denotes $k$ rounding errors depending on $j$. The following lemma is also charished.
Lemma 3.1 ([17] Lemma 3.1) If $|\delta_i| \leq \epsilon$ and $\rho_i = \pm 1$ for $i = 1, 2, \ldots, k$, and $ku \leq 1$, then
\[
(k) = 1 + \theta_k,
\]
where
\[
|\theta_k| \leq \frac{ku}{1 - ku} =: \gamma_k.
\]
Besides, we also succeed the condition number derive by Higham for polynomials interpolation:

**Definition 3.1** ([18]) The condition number of $p_N$ at $x$ with respect to $f$ is, for $p_N(x) \neq 0$,
\[
\text{cond}(x, N, f) = \limsup_{\epsilon \to 0} \left\{ \left| \frac{p_N(x; f) - p_N(x; f + \Delta f)}{\epsilon p_N(x; f)} \right| : |\Delta f| \leq \epsilon |f| \right\},
\]
where $p_N(x) = \sum_{j=0}^{N} f(x_j) \ell_j(x)$.

In this paper, we are interested with $p_N^{\ell_2}$ and $p_N^{\ell_1}$. Hence, we define their condition numbers as the following.

**Definition 3.2** For regularized interpolants (no matter for barycentric interpolants or modified Lagrange interpolants), the condition numbers of $p_N^{\ell_2}$ and $p_N^{\ell_1}$ at $x$ with respect to $f$ is, for $p_N^{\ell_2} \neq 0$ and $p_N^{\ell_1} \neq 0$,
\[
\text{cond}^{\ell_2}(x, N, f) = \limsup_{\epsilon \to 0} \left\{ \left| \frac{p_N^{\ell_2}(x; f) - p_N^{\ell_2}(x; f + \Delta f)}{\epsilon p_N^{\ell_2}(x; f)} \right| : |\Delta f| \leq \epsilon |f| \right\},
\]
where $p_N^{\ell_2}(x) = \sum_{j=0}^{N} \frac{f(x_j)}{1 + \lambda \mu_0^2} \ell_j(x)$; and
\[
\text{cond}^{\ell_1}(x, N, f) = \limsup_{\epsilon \to 0} \left\{ \left| \frac{p_N^{\ell_1}(x; f) - p_N^{\ell_1}(x; f + \Delta f)}{\epsilon p_N^{\ell_1}(x; f)} \right| : |\Delta f| \leq \epsilon |f| \right\},
\]
where $p_N^{\ell_1}(x) = \sum_{j=0}^{N} \left( f(x_j) + \sum_{\ell=0}^{N} c_\ell \tilde{\Phi}_\ell(x_j) \right) \ell_j(x)$.

**Remark.** The reader should note that
\[
\text{cond}^{\ell_2}(x, N, f) = \text{cond}(x, N, f^{\ell_2-\text{pre}}) \quad \text{and} \quad \text{cond}^{\ell_1}(x, N, f) = \text{cond}(x, N, f^{\ell_1-\text{pre}}).
\]

Higham [18] also shows
\[
\text{cond}(x, N, f) = \frac{\sum_{j=0}^{N} |\ell_j(x)| f(x_j)|}{|p_N(x)|} = \frac{\sum_{j=0}^{N} |\Omega_j f(x_j)|}{\sum_{j=0}^{N} |\Omega_j f(x_j)|} \frac{x}{x - x_j},
\]
where the last equality is due to $\ell_j(x) = \ell(x) \frac{\Omega_j}{x - x_j}$. With the same trick, we have
\[
\text{cond}^{\ell_2}(x, N, f) = \frac{\sum_{j=0}^{N} |\ell_j(x)| f^{\ell_2-\text{pre}}(x_j)|}{|p_N^{\ell_2}(x)|} = \frac{\sum_{j=0}^{N} |\Omega_j f^{\ell_2-\text{pre}}(x_j)|}{\sum_{j=0}^{N} |\Omega_j f^{\ell_2-\text{pre}}(x_j)|} \frac{x}{x - x_j}.
\]
and
\[ \text{cond}^\ell_i(x, N, f) = \sum_{j=0}^{N} \left| \frac{p_{\Omega}^\ell_i(x) f_{\ell_{\Omega}}^i(x) - p_{\Omega}^\ell_i(x) f_{\ell_{\Omega}}^i(x + \Delta f)}{|p_{\Omega}^\ell_i(x)|} \right| = \frac{\sum_{j=0}^{N} \left| \frac{\Omega_j f_{\ell_{\Omega}}^i(x) - f_{\ell_{\Omega}}^i(x + \Delta f)}{|x - x_j|} \right|}{\sum_{j=0}^{N} \left| \frac{\Omega_j f_{\ell_{\Omega}}^i(x) - f_{\ell_{\Omega}}^i(x + \Delta f)}{|x - x_j|} \right|}. \]

Another lemma should be noted here.

**Lemma 3.2** (IS, Second part of Lemma 2.2) For any \( \Delta f \) with \( |\Delta f| \leq \epsilon |f| \) we have
\[ \frac{|p_N(x; f) - p_N(x; f + \Delta f)|}{|p_N(x; f)|} \leq \text{cond}(x, N, f) \epsilon. \]

Similarly,

**Lemma 3.3** For any \( \Delta f \) with \( |\Delta f| \leq \epsilon |f| \) we have
\[ \frac{|p_N^\ell_i(x; f) - p_N^\ell_i(x; f + \Delta f)|}{|p_N^\ell_i(x; f)|} \leq \text{cond}^\ell_i(x, N, f) \epsilon, \]
and
\[ \frac{|p_N^\ell_i(x; f) - p_N^\ell_i(x; f + \Delta f)|}{|p_N^\ell_i(x; f)|} \leq \text{cond}^\ell_i(x, N, f) \epsilon, \]
where \( p_N^\ell_i(x; f) = p_N^\ell_i(x) \) and \( p_N^\ell_i(x; f) = p_N^\ell_i(x) \).

**Proof.** From
\[ p_N^\ell_i(x; f) - p_N^\ell_i(x; f + \Delta f) = \sum_{j=0}^{N} \frac{\Delta f(x_j)}{1 + \lambda_\mu} \ell_j(x), \]
and
\[ p_N^\ell_i(x; f) - p_N^\ell_i(x; f + \Delta f) = \sum_{j=0}^{N} \left( \Delta f(x_j) + \sum_{\ell=0}^{N} \tilde{c}_\ell \tilde{\Phi}_\ell(x_j) \right) \ell_j(x), \]
where \( \tilde{c}_\ell = \mathcal{S}_{\lambda_\mu} \left( 2 \sum_{j=0}^{N} \omega_j \tilde{\Phi}_\ell(x_j) \Delta f(x_j) \right) / 2 - \sum_{j=0}^{N} \omega_j \tilde{\Phi}_\ell(x_j) \Delta f(x_j), \) and \( |\Delta f| \leq \epsilon |f| \), both inequalities are immediately obtained. Equalities are attained for \( \Delta f(x_j) = \epsilon \text{sign}(\ell_j(x_j)) f(x_j) \).

In this paper, we use the notation “hat” \( \wedge \) to denote computed factors, rather than exact values. In rounding error analysis, we respectively consider different special point distributions, including Chebyshev points of the first kind, Chebyshev points of the second kind and Legendre points (all in \([-1, 1]\)), rather than a general discussion. Equally spaced points lead to instability of classical barycentric interpolation, and do not meet the restriction of regularized least squares approximation (points set \( \mathcal{X}_{N+1} \) is a set of Gauss quadrature points), which is the origin of regularized barycentric interpolation. Hence we ignore the equispaced point distribution in this paper. Finally, we assume all numbers, such as \( x_j, f(x_j) \) and \( x \), are floating point numbers in this paper.

**Lemma 3.4** (IS, Lemma 3.1) Let \( \ell(x) \) be defined as eq. (2.4) and \( \Omega_j \) be defined as eq. (1.2). The computed \( \hat{\ell}(x) \) satisfies
\[ \hat{\ell}(x) = \ell(x) (2N + 1), \]
and the computed \( \hat{\Omega}_j \) satisfies
\[ \hat{\Omega}_j = \Omega_j (2N). \]
Here the computation of $\Omega_j$ is based on its uniform expression eq. (1.2); however, there exist explicit expressions for different certain point distribution (c.f. section 2.4). Thus the error bounds could be smaller when certain point distribution is considered. For example,

$$\hat{\Omega}_j = \Omega_j \langle 7 \rangle \quad \text{and} \quad \hat{\Omega}_j = \Omega_j \langle 1 \rangle$$

for Chebyshev point of the first kind distribution and Chebyshev point of the second kind distribution, respectively.

**Lemma 3.5** Assume $c_\ell$ are precalculated, the computed preprocessed function values $f(x_j)$ satisfy

$$\hat{f}^{\ell_2-\text{pre}}(x_j) = f^{\ell_2-\text{pre}}(x_j) \langle 4 \rangle_j,$$

$$\hat{f}^{\ell_1-\text{pre}}(x_j) = f^{\ell_1-\text{pre}}(x_j) \langle N + 2 + \eta \rangle_j,$$

where $\eta \in \mathcal{O}(N)$.

**Proof.** For $\ell_2$—regularized preprocessed function,

$$fl \left( \frac{f(x_j)}{1 + \lambda \mu_0^2} \right) = \frac{f(x_j)}{1 + \lambda \mu_0^2} \langle 1 \rangle_j = \frac{f(x_j)}{1 + \lambda \mu_0^2} \langle 4 \rangle_j.$$

The evaluation of the orthogonal polynomial in all the points $x_j$ requires $\mathcal{O}(N^2)$ operations based, for example, on recurrence relations for the polynomials [9]. Thus for each point $x_j$ we have $\hat{\Phi}_\ell(x_j) = \tilde{\Phi}_\ell(x_j) \langle \eta \rangle_j$ where $\eta \in \mathcal{O}(N)$. Then for $\ell_1$ case, we have

$$fl \left( f(x_j) + \sum_{\ell=0}^{N} c_\ell \tilde{\Phi}_\ell(x_j) \right) = \left( f(x_j) + \sum_{\ell=0}^{N} c_\ell \tilde{\Phi}_\ell(x_j) \langle \eta \rangle_j \langle 1 \rangle_j \langle N \rangle_j \right) \langle 1 \rangle_j$$

$$= f(x_j) + \sum_{\ell=0}^{N} c_\ell \tilde{\Phi}_\ell(x_j) \langle N + 2 + \eta \rangle_j.$$

Hence the lemma proves. \(\square\)

### 3.2 Regularized barycentric interpolation

We find that

$$\hat{P}_N^{\ell_2-\text{bary}}(x) = \sum_{j=0}^{N} \frac{\Omega_j \langle 2 \rangle_j}{x - x_j} f^{\ell_2-\text{pre}}(x_j) \langle 4 \rangle_j \langle N + 3 \rangle_j \langle 1 \rangle_j$$

$$= \sum_{j=0}^{N} \frac{\Omega_j \langle 2 \rangle_j}{x - x_j} \langle N + 2 \rangle_j$$

$$= \sum_{j=0}^{N} \frac{\Omega_j}{x - x_j} \langle 3N + 2 \rangle_j.$$
and

\[ \hat{p}_{N}^{\ell_1-\text{bary}}(x) = \frac{\sum_{j=0}^{N} \Omega_j (2N)_j f^{\ell_1-\text{pre}}(x_j) \langle N + 2 + \eta \rangle_j (N + 3)_j}{\sum_{j=0}^{N} \Omega_j (2N)_j \langle N + 2 \rangle_j} \langle 1 \rangle_j \]

\[ = \frac{\sum_{j=0}^{N} \Omega_j (2N)_j f^{\ell_1-\text{pre}}(x_j) \langle 4N + 6 + \eta \rangle_j}{\sum_{j=0}^{N} \Omega_j (x - x_j) (3N + 2)_j} \]

There does not provide any useful information on the backward error of regularized barycentric interpolation. But it leads to a forward relative error bound.

**Theorem 3.1** The computed \( \hat{p}_{N}^{\ell_2-\text{bary}} \) and \( \hat{p}_{N}^{\ell_1-\text{bary}} \) satisfy

\[
\frac{|p_{N}^{\ell_2-\text{bary}}(x) - \hat{p}_{N}^{\ell_2-\text{bary}}(x)|}{|p_{N}^{\ell_2-\text{bary}}(x)|} \leq (3N + 8)u \frac{\sum_{j=0}^{N} \Omega_j (x - x_j) f^{\ell_2-\text{pre}}(x_j)}{\sum_{j=0}^{N} \Omega_j (x - x_j) f^{\ell_2-\text{pre}}(x_j)} + (3N + 2)u \sum_{j=0}^{N} \frac{\Omega_j}{x - x_j} + O(u^2) \tag{3.1}
\]

\[
= (3N + 8)u \text{cond}^{\ell_2}(x, N, f) + (3N + 2)u \text{cond}(x, N, 1) + O(u^2),
\]

and

\[
\frac{|p_{N}^{\ell_1-\text{bary}}(x) - \hat{p}_{N}^{\ell_1-\text{bary}}(x)|}{|p_{N}^{\ell_1-\text{bary}}(x)|} \leq (4N + 6 + \eta)u \frac{\sum_{j=0}^{N} \Omega_j (x - x_j) f^{\ell_1-\text{pre}}(x_j)}{\sum_{j=0}^{N} \Omega_j (x - x_j) f^{\ell_1-\text{pre}}(x_j)} + (3N + 2)u \sum_{j=0}^{N} \frac{\Omega_j}{x - x_j} + O(u^2) \tag{3.2}
\]

\[
= (4N + 6 + \eta)u \text{cond}^{\ell_1}(x, N, f) + (3N + 2)u \text{cond}(x, N, 1) + O(u^2),
\]

respectively.

The forward error bounds are determined by both the unit roundoff \( u(\approx 10^{-16} \text{ or } 10^{-8}) \) and the condition numbers. Regularized barycentric interpolation formulae become significantly large when \( \text{cond}(x, N, 1) \gg \text{cond}^{\ell_2}(x, N, f) \) and \( \text{cond}(x, N, 1) \gg \text{cond}^{\ell_1}(x, N, f) \). However, due to the restriction of regularized least squares approximation model that point distribution must be a Gauss quadrature points set, the bad cases of \( \text{cond}(x, N, 1) \gg \text{cond}^{\ell_2}(x, N, f) \) and \( \text{cond}(x, N, 1) \gg \text{cond}^{\ell_1}(x, N, f) \) would not come. Higham offers an insight into the bad case in
terms of the Lebesgue constant $\Lambda_N$ [18]. One of definitions of the Lebesgue constant is given in [28]:

$$\Lambda_N = \sup_{x \in [-1,1]} \sum_{j=0}^{N} |\ell_j(x)|.$$ 

With notice of $\text{cond}(x, N, 1) = \sum_{j=0}^{N} |\ell_j(x)|$, the terms $\text{cond}(x, N, f^{\text{pre}})$ in eq. (3.1) and eq. (3.2) can be bounded by $\Lambda_N$. In the special point distribution, the Lebesgue constant grows slowly as $N$ grows, see [5], which confirms the fact that the forward stability of regularized barycentric interpolation is guaranteed by merit point distribution.

We see that although there is no evidence supporting the backward stability of regularized barycentric interpolation, it is forward stable with a relatively small error bound. This coincides with the classical barycentric interpolation in “good” point distribution discussed by Higham [18].

### 3.3 Regularized modified Lagrange interpolation

Similar with Section 3.2, we work with the numerical stability of regularized modified Lagrange interpolation.

**Theorem 3.2** The computed $\hat{p}^{\ell_2-\text{mdf}}_N$ and $\hat{p}^{\ell_1-\text{mdf}}_N$ satisfy

$$\hat{p}^{\ell_2-\text{mdf}}_N(x) = \ell(x) \sum_{j=0}^{N} \frac{\Omega_j}{x-x_j} f^{\ell_2-\text{pre}}(x_j) \langle 5N + 9 \rangle_j,$$

and

$$\hat{p}^{\ell_1-\text{mdf}}_N(x) = \ell(x) \sum_{j=0}^{N} \frac{\Omega_j}{x-x_j} f^{\ell_1-\text{pre}}(x_j) \langle 6N + 7 + \eta \rangle_j,$$

respectively.

**Proof.** We have

$$\hat{p}^{\ell-i-\text{mdf}}_N(x) = \ell(x) \langle 1 + \sum_{j=0}^{N} \frac{\Omega_j}{x-x_j} f^{\ell-i-\text{pre}}(x_j) \langle N \rangle_j \rangle_i, \quad i = 1, 2.$$ 

Following Lemma 3.4 and Lemma 3.5 we obtain

$$\hat{p}^{\ell_2-\text{mdf}}_N(x) = \ell(x) \langle 2N + 2 \rangle \sum_{j=0}^{N} \frac{\hat{\Omega}_j}{x-x_j} f^{\ell_2-\text{pre}}(x_j) \langle 4 \rangle_j \langle 3 \rangle_j \langle N \rangle_j ,$$

and

$$\hat{p}^{\ell_1-\text{mdf}}_N(x) = \ell(x) \langle 2N + 2 \rangle \sum_{j=0}^{N} \frac{\hat{\Omega}_j}{x-x_j} f^{\ell_1-\text{pre}}(x_j) \langle N + 2 + \eta \rangle_j \langle 3 \rangle_j \langle N \rangle_j ,$$

which leads to the desired results. □

This shows the backward stability of regularized barycentric interpolation: for a relatively small perturbation $\Delta f$ in data $f = [f(x_0), f(x_1), \ldots, f(x_N)]^T$, we have

$$\hat{p}^{\text{mdf}}_N(x; f) = p^{\text{mdf}}_N(x; f + \Delta f).$$

Similar to the forward stability of regularized barycentric interpolation formulae, we obtain the forward error bounds for regularized modified Lagrange interpolation formulae.
Corollary 3.1  Together with Lemma \[\text{Lemma 2.1}\] and Theorem \[\text{Theorem 2.2}\], we see the forward stability of regularized modified Lagrange interpolation immediately:

\[
\left| \frac{p_{\mathcal{N}}^{\ell_2-\text{mdf}}(x) - p_{\mathcal{N}}^{\ell_2-\text{mdf}}(x)}{p_{\mathcal{N}}^{\ell_2-\text{mdf}}(x)} \right| \leq \gamma_{5N+9} \text{cond}^{\ell_2}(x, N, f),
\]

and

\[
\left| \frac{p_{\mathcal{N}}^{\ell_1-\text{mdf}}(x) - p_{\mathcal{N}}^{\ell_1-\text{mdf}}(x)}{p_{\mathcal{N}}^{\ell_1-\text{mdf}}(x)} \right| \leq \gamma_{6N+7+\eta} \text{cond}^{\ell_1}(x, N, f).
\]

The forward error bounds are also determined by both the unit roundoff \(u \approx 10^{-16}\) or \(10^{-8}\) and the condition numbers.

### 3.4 Numerical implication

Let the points set \(x_N \in [1, 1]\) be the set of 501 Chebyshev points of the first kind over the interval \([-1, 1]\). Set the regularization parameter for \(\ell_2\)-regularized barycentric interpolation be \(\lambda_2 = 10^{-0.5}\), and set that for \(\ell_1\)-regularized barycentric interpolation be \(\lambda_1 = 10^{-1.5}\). Note that since regularized formulae generates interpolants of the processed function rather than the original function, we use

\[
\frac{|f(x) - p_{\mathcal{N}}(x)|}{|f(x)|}, \quad \frac{|f^{\ell_2-\text{pre}}(x) - p_{\mathcal{N}}^{\ell_2}(x)|}{|f^{\ell_2-\text{pre}}(x)|} \quad \text{and} \quad \frac{|f^{\ell_1-\text{pre}}(x) - p_{\mathcal{N}}^{\ell_1}(x)|}{|f^{\ell_1-\text{pre}}(x)|}
\]

to illustrate the numerical stability, respectively. Figure 1 shows the forward errors for Lagrange interpolation formula (\(\text{“Lag”}\)) and Lagrange interpolation formula with preprocessed values (\(\text{“\ell_2-Lag” and “\ell_1-Lag”}\)), least squares approximation (\(\text{“LS”}\)) and regularized least squares approximation (\(\text{“\ell_2-LS” and “\ell_1-LS”}\)) under the condition of \(L = N\), modified Lagrange interpolation formula (\(\text{“mdf”}\)) and regularized modified Lagrange interpolation formulae (\(\text{“\ell_2- mdf” and “\ell_1-mdf”}\)), and barycentric interpolation formula (\(\text{“bary”}\)) and regularized barycentric interpolation formulae (\(\text{“\ell_2-bary” and “\ell_1-bary”}\)), respectively. Each color denotes a class of formulae, including three Lagrange interpolation formulae (blue), three Least squares minimizer under the condition of \(L = N\) (red), three modified Lagrange interpolation formulae (purple) and three barycentric interpolation formulae (yellow). The results confirm the instability of Lagrange interpolation formula. They also illustrates that the weighted discrete least squares approximation under the condition of \(L = N\) is numerically stable. Moreover, we can see that regularized barycentric interpolation, which is efficient in denoising shown in [2], and the regularized modified Lagrange interpolation, are both numerically stable. The numerical stability of classical modified Lagrange interpolation and barycentric interpolation is claimed by Higham in 2004 [13]. Recall the forward error bounds eq. (3.1), eq. (3.2), eq. (3.3) and eq. (3.4), the bounds are also related to \(\text{cond}(x, N, f)\), i.e., the property of \(f\). As the length limit, the reader can find two more examples (test functions) in Appendix B which show differences in the scale of forward error based on different functions.

### 4 The numerical stability for extrapolation

Both regularized barycentric interpolation formulae eq. (2.1) and eq. (2.2), and regularized modified Lagrange interpolation formulae eq. (2.6) and eq. (2.7), are shown forward stable for interpolating a function on special distributions of point over \([-1, 1]\), such as Chebyshev points and Legendre points. We show, in the meantime, regularized modified Lagrange interpolation is blessed with backward stability. But evidence does not accumulate to support the backward stability of regularized barycentric interpolation.
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Figure 1: Relative forward errors in computed \( p_{\text{mdf}}(x) \), \( p_{\ell_2}^{\text{mdf}}(x) \), \( p_{\ell_1}^{\text{mdf}}(x) \), \( p_{\text{bary}}(x) \), \( p_{\ell_2}^{\text{bary}}(x) \), \( p_{\ell_1}^{\text{bary}}(x) \), Lagrange interpolants and least squares approximation minimizers for 501 Chebyshev points of the first kind in increasing order.
4.1 Instability of barycentric formulae and underflow/overflow problem of modified Lagrange formulae

As $x$ moves away from $[-1, 1]$, the instability of regularized barycentric formula is a consequence of cancellation [34]. Historically speaking, barycentric interpolation is a descendant of modified Lagrange interpolation; however, we directly present novel regularized barycentric interpolation first in our previous work, and we give regularized modified Lagrange interpolation by reversing the steps from modified Lagrange formula to barycentric formula in this paper. So the accuracy of regularized barycentric interpolation actually depends on the accuracy of eq. (2.5), which is the most critical step from modified Lagrange formula to barycentric formula. eq. (2.5) can be rewritten as

$$\sum_{j=0}^{N} \frac{\Omega_j}{x - x_j} = \frac{1}{\ell(x)}, \quad (4.1)$$

where $\Omega_j$ is defined by eq. (1.2) and $\ell(x)$ is defined by eq. (2.4). When $x$ moves away from $[-1, 1]$, the right term $1/\ell(x)$ shrinks rapidly, which results in loss of accuracy outside $[-1, 1]$. Although it seems that regularized modified Lagrange interpolation is better than regularized barycentric interpolation in terms of stability, there exists a critical disadvantage of it: it is not scale-invariant, which will lead to underflow or overflow on a computer in standard IEEE double precision arithmetic for $N$ bigger than about 1000 (the digit 1000 is given by Trefethen [28]). As $N \to \infty$, the scale of the barycentric weights $\Omega_j$ as defined by eq. (1.2) will grow or decay exponentially at the rate $2^{-N}$. In other words, it has size approximately $2^N$. The troubles should be handled by rescaling $[-1,1]$ to $[-2,2]$ or by computing products by addition of logarithms [28, Chapter 5]. Different from regularized modified Lagrange interpolation eq. (2.6) and eq. (2.7), this kind of inaccuracy still remains in regularized barycentric formula but appears in both the numerator and the denominator of eq. (2.1) and eq. (2.2), so the inaccuracies can be cancelled out. Table 1 illustrates the overflow of $\ell_1$-regularized modified Lagrange interpolation formula when extrapolating, also shows the digit of $N$, which gives rise to overflow, would be smaller than that of interpolation.

| $N$   | $p_N^{q_{-\text{bary}}}(-2)$ | $p_N^{q_{-\text{mdf}}}(-2)$ | $f(-2)$   |
|-------|-------------------------------|-------------------------------|------------|
| 10    | -4.7916854774702e+04          | -4.7916854775306e+04          | 1.3371553969333e-01 |
| 110   | 4.6413502109704e-01           | 1.3164905532140e+45           | 1.3371553969333e-01 |
| 210   | 1.6365360303413e+00           | -1.2762107352562e+103         | 1.3371553969333e-01 |
| 310   | -1.8584392014519e+00          | 1.6276018424718e+160          | 1.3371553969333e-01 |
| 410   | 4.8846487424111e-01           | -1.5906469788448e+217         | 1.3371553969333e-01 |
| 510   | -1.456993918331e+00           | 3.3232306181876e+274          | 1.3371553969333e-01 |
| 610   | 4.2724522791279e-01           | Inf                           | 1.3371553969333e-01 |
| 710   | -1.7716262975778e+00          | Inf                           | 1.3371553969333e-01 |
| 810   | 1.236025981429e+01            | Inf                           | 1.3371553969333e-01 |
| 910   | -2.2736608381903e+00          | Inf                           | 1.3371553969333e-01 |
| 1010  | -5.8487548549234e-01          | Inf                           | 1.3371553969333e-01 |

4.2 Chebfun ellipse

We are also interested in numerical analytic continuation in the complex plane. Then here comes a surprising but similar instability [33]. Geometrically speaking, the instability in the
complex $x$–plane is the effect that barycentric interpolation formula leads to entirely wrong results outside an ellipse enclosing $[-1, 1]$. The addressed ellipse is a Bernstein ellipse,

**Definition 4.1** ([28, 32, 36]) Bernstein ellipse is defined as

$$\mathcal{E}_\rho = \left\{ z \in \mathbb{C} \mid z = \frac{1}{2}(w + w^{-1}), \ w = \rho e^{i\theta}, \ \rho \geq 1, \ 0 \leq \rho \leq 2\pi \right\},$$

where $\mathbb{C}$ is the complex plain. The ellipse $\mathcal{E}_\rho$ has the foci at $\pm 1$ and the major and minor semi-axes are

$$a = \frac{1}{2}(\rho + \rho^{-1}), \quad b = \frac{1}{2}(\rho - \rho^{-1}),$$

respectively.

**Remark.** The ellipse passed through $z = (w + w^{-1})/2$ for some $w$ with $|w| = \rho$.

Specially, Trefethen calls this kind of Bernstein ellipse *Chebfun ellipse* [28, Chapter 8], which is a Bernstein ellipse with parameter $\rho$ satisfying

$$\rho^N = u^{-1},$$

where $N$ is the degree of interpolant or extrapolant and $u$ is the unit roundoff. For the relation eq. (4.2) the reader may consult [34]. In real segment $[-1, 1]$, cancellation prevents regularized barycentric interpolation from underflow or overflow. The left side of eq. (4.1) has size approximately $2^N$. For $x \in [-1, 1]$, $\ell(x)$ is approximately of size $2^{-N}$, thus cancellation is not a problem for eq. (4.1). However, when $x$ moves away from $[-1, 1]$, $\ell(x)$ grows to order approximately $2^{-N} \rho^N$ [34]. Thus eq. (4.1) relies on the cancellation of magnitude $\rho^N$. The motivation of eq. (4.2) is that the construction of interpolant of $f$ to machine precision coincides with the fact that $f$ is analytic and bounded inside the ellipse. $\rho^N$ with size $u^{-1}$ or larger leads to the loss of accuracy [34]. If $|x|$ grows, such that $\rho^N$ becomes larger than $u^{-1}$, then the left side of eq. (4.1) fails to decrease and then the computed interpolants from regularized barycentric formulae eq. (2.1) and eq. (2.2) will fail to increase. Figure 2 and Figure 3 illustrate the instability of regularized barycentric interpolation outside the Chebfun ellipse.

![Figure 2: $\ell_2$–regularized extrapolant in 42 Chebyshev points of the first kind in $[-1, 1]$ to $f(x) = \exp(x) \sin(65x)$ in the complex $x$–plane. To highlight the polynomial structure, the quantity plotted is $\text{sign}(\text{Re}(x)) \log(1+|\text{Re}(x)|)$, which is inspired by the same idea in [34]. The left picture (regularized barycentric formula) is entirely wrong outside ellipse with seemingly random values being there.](image)
5 Numerical experiments for contaminated data

Regularized barycentric interpolation and regularized modified Lagrange interpolation are both special cases of regularized least squares approximation when the interpolation conditions achieve. Thus the merit property of regularized least squares approximation, namely noise-reduction ability, is also inherited by regularized barycentric interpolation and regularized modified Lagrange interpolation. Then we focus on problems involving contaminated data in this section.

Figure 4 illustrates the noise-reduction ability of regularized formulae, with shape reserved. Specially, \( \ell_1 \)-regularized formulae are better than \( \ell_2 \)-regularized formulae, which coincides with the numerical experiments in our previous work [2]. The interpolants of classical barycentric interpolation, modified Lagrange interpolation, \( \ell_1 \)-regularized barycentric interpolation, \( \ell_1 \)-regularized modified Lagrange interpolation, \( \ell_2 \)-regularized barycentric interpolation and \( \ell_2 \)-regularized modified Lagrange interpolation are titled as bary, mdf, \( \ell_1 \) bary, \( \ell_1 \) mdf, \( \ell_2 \) bary and \( \ell_2 \) mdf, respectively. And these titles are still used in Figure 6 and Figure 7. Here \( \lambda = 10^{-0.5} \) and

\[
\mu_\ell = \frac{1}{F(\ell/L)}, \quad \ell = 0, 1, \ldots, L,
\]

where the filter function \( F \) is defined as [1]

\[
F(x) = \begin{cases} 
1, & x \in [0, 1/2] \\
\sin^2 \pi x, & x \in [1/2, 1] \\
0, & x \in [1, +\infty].
\end{cases}
\]

For another sight, let \( \lambda = 10^{-1.5} \) and \( \{\mu_\ell\}_{\ell=0}^{N} \) be the same as above. Figure 5 repeats Figure 4 but for \( f(x) = \sin(10x) \), illustrating the power of regularization.

Let \( \lambda = 10^{-0.5} \) and all \( \mu_\ell \) be 1. Figure 6 and Figure 7 show the noise-reduction ability of regularized formulae.

6 Final remarks

In this paper, the interpolation nodes are only allow to be zeros of Jacobi polynomials over \([-1, 1]\). We introduce the \( \ell_2 \)- and \( \ell_1 \)-regularized modified Lagrange interpolation formulae. Together with \( \ell_2 \)- and \( \ell_1 \)-regularized barycentric interpolation formulae, we present lower complexity and fast computation operability under some spectral computation conditions. These four regularized interpolation formulae are numerically stable in terms of floating arithmetic
Figure 4: Six Interpolants in 501 Chebyshev points of the first kind in $[-1,1]$ to function $f(x) = |x| + |x|^2 + |x|^3 + |x|^4 + |x|^5$ with 15 dB Gauss white noise added. Comparing between the interpolants (blue) and exact function (original function, red), regularized formulae win. Note that the contaminated function is not plotted.

Figure 5: Six Interpolants in 501 Chebyshev points of the first kind in $[-1,1]$ to function $f(x) = \sin(10x)$ with 15 dB Gauss white noise added. Comparing between the interpolants (blue) and exact function (original function, red), regularized formulae win. Note that the contaminated function is not plotted.
Figure 6: Exact data: repetition of Fig. 2 but for interpolating Runge function $f(x) = 1/(1 + 25x^2)$ on 42 Chebyshev points of the first kind in $[-1, 1]$.

Figure 7: Contaminated data: repetition of Fig. 2 but for interpolating noisy Runge function $f(x) = 1/(1 + 25x^2)$ (with 5dB Gauss white noise added) on 42 Chebyshev points of the first kind in $[-1, 1]$. The red ellipse detects a place caused by contaminated data.
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Moreover, our four regularized interpolation formulae are implemented by the Chebfun package [29]. We also investigate the extrapolation property of these four regularized formulae. The result is in accordance with [34]. That is, the regularized modified Lagrange interpolation formulae are stable in the case of extrapolation. Numerical experiments also show the regularized modified Lagrange formula is able to recover function with noisy data, under proper choice of regularization parameter. [21]

A Appendix A: Towards regularized barycentric interpolation formulae

In [2] we have considered the \(\ell_2\)-regularized approximation problem

\[
\min_{p \in \mathbb{P}_L} \left\{ \sum_{j=0}^{N} \omega_j (p(x_j) - f(x_j))^2 + \lambda \sum_{j=0}^{N} (R_L p(x_j))^2 \right\}, \quad \lambda > 0, \tag{A.1}
\]

and the \(\ell_1\)-regularized approximation problem appearing in the similar form of

\[
\min_{p \in \mathbb{P}_L} \left\{ \sum_{j=0}^{N} \omega_j (p(x_j) - f(x_j))^2 + \lambda \sum_{j=0}^{N} |R_L p(x_j)| \right\}, \quad \lambda > 0, \tag{A.2}
\]

where \(f\) is a given continuous function with values (possibly noisy) taken at \(X_{N+1} = \{x_j\}_{j=0}^{N}\); \(\mathbb{P}_L\) is a linear space of polynomials of degree at most \(L\) with \(L \leq 2N + 1\); \(\mathbf{w} = [\omega_0, \omega_1, \ldots, \omega_N]^T\) is a vector of positive Gauss quadrature weights [9]; the regularization operator \(R_L : \mathbb{P}_L \to \mathbb{P}_L\) is a linear operator; \(\lambda > 0\) is the regularization parameter; and nonnegative parameters \(\{\mu_L\}_{L=0}^{\infty}\) are the penalty parameters. With the basis for \(\mathbb{P}_L\) given as a class of normalized orthogonal polynomials \(\tilde{\Phi}_L\), the minimizer \(p(x)\) can be expressed as

\[
p_{L,N+1}(x) = \sum_{\ell=0}^{L} \beta_{\ell} \tilde{\Phi}_\ell(x),
\]

where \(\{\beta_{\ell}\}_{\ell=0}^{L}\) are the coefficients. As to point distribution, it is merited and desirable by choosing \(X_{N+1}\) to be Gauss (quadrature) points set [9, 26], which is a points set satisfies Gauss quadrature rule. Consider the special case that \(R_L\) act as \(R_L p(x) = \sum_{\ell=0}^{L} \mu_{\ell} \tilde{\Phi}_\ell(x)\). With the aid of Gauss quadrature, the minimizer polynomials can be expressed in explicit form without solving a linear algebra or an optimization problem. The minimizers for problems eq. (A.1) and eq. (A.2) are

\[
p_{L,N+1}^\ell(x) = \sum_{j=0}^{N} \omega_j \frac{\tilde{\Phi}_\ell(x_j) f(x_j)}{1 + \lambda \mu_\ell^2}, \tag{A.3}
\]

and

\[
p_{L,N+1}^\ell(x) = \sum_{\ell=0}^{L} S_{\lambda \mu_{\ell}} \left( \frac{2 \sum_{j=0}^{N} \omega_j \tilde{\Phi}_\ell(x_j) f(x_j)}{2} \right) \tilde{\Phi}_\ell(x), \tag{A.4}
\]

respectively, where \(S_k(a)\) is the soft threshold operator [6] defined by

\[S_k(a) = \max(0, a - k) + \min(0, a + k).\]

For details of the minimizers the reader may see [2, Section 2].
From the orthogonality of $\tilde{\Phi}_\ell(x)$, $\ell = 0, 1, \ldots, N$, and the definition of Gauss quadrature rule \([9]\), we have

$$\sum_{j=0}^{N} \omega_j \sum_{\ell=0}^{N} \frac{\tilde{\Phi}_\ell(x_j)\tilde{\Phi}_\ell(x)}{1 + \lambda \mu_\ell^2} = \frac{N}{\ell} \left( \sum_{j=0}^{N} \omega_j \tilde{\Phi}_\ell(x_j) \cdot 1 \right) \frac{\tilde{\Phi}_\ell(x)}{1 + \lambda \mu_\ell^2}$$

$$= \|\tilde{\Phi}_0(x)\|_{L^2} \frac{\tilde{\Phi}_0(x)}{1 + \lambda \mu_0^2}$$

$$= \frac{1}{1 + \lambda \mu_0^2}.$$

Then the $\ell_2$–regularized minimizer eq. (A.3) can be expressed as

$$p_{L,N+1}^\ell(x) = \frac{\sum_{j=0}^{N} \left( \omega_j \sum_{\ell=0}^{N} \tilde{\Phi}_\ell(x_j)\tilde{\Phi}_\ell(x) \right) f(x_j)}{(1 + \lambda \mu_0^2) \sum_{j=0}^{N} \omega_j \sum_{\ell=0}^{N} \tilde{\Phi}_\ell(x_j)\tilde{\Phi}_\ell(x)}.$$

Without loss of generality, suppose $\mu_\ell = 1$ for $\ell \geq N + 1$, then $\left\{ \frac{\tilde{\Phi}_\ell(x)}{\sqrt{1 + \lambda \mu_\ell^2}} \right\}_{\ell \in \mathbb{N}}$ is a sequence of orthogonal polynomials. By Christoffel-Darboux formula \([9]\) Section 1.3.3, we have

$$\sum_{\ell=0}^{N} \frac{\tilde{\Phi}_\ell(x_j)\tilde{\Phi}_\ell(x)}{1 + \lambda \mu_\ell^2} = \frac{k_N}{h_N k_{N+1}} \frac{\tilde{\Phi}_{N+1}(x_j)\tilde{\Phi}_{N+1}(x)}{\sqrt{1 + \lambda \mu_{N+1}^2}} \frac{\tilde{\Phi}_N(x_j)\tilde{\Phi}_N(x)}{\sqrt{1 + \lambda \mu_N^2}}$$

$$= \frac{k_N}{h_N k_{N+1}} \frac{\tilde{\Phi}_{N+1}(x_j)\tilde{\Phi}_N(x_j)}{\sqrt{1 + \lambda \mu_{N+1}^2} \sqrt{1 + \lambda \mu_N^2}(x - x_j)}.$$

where $k_\ell$ and $h_\ell$ denote the leading coefficient and $L_2$ norm of $\frac{\tilde{\Phi}_\ell(x)}{\sqrt{1 + \lambda \mu_\ell^2}}$, respectively. Combine this with eq. (A.5) and cancel $\frac{k_N}{h_N k_{N+1}} \frac{\tilde{\Phi}_{N+1}(x_j)}{\sqrt{1 + \lambda \mu_{N+1}^2}}$ from both numerator and denominator. Hence we obtain the $\ell_2$–regularized barycentric interpolation formula:

$$p_{L,N+1}^{\ell_2-\text{bary}}(x) = \frac{\sum_{j=0}^{N} \frac{\Omega_j}{x - x_j} f(x_j)}{(1 + \lambda \mu_0^2) \sum_{j=0}^{N} \frac{\Omega_j}{x - x_j}},$$

where $\Omega_j = \omega_j \tilde{\Phi}_N(x_j)$ is the corresponding barycentric weight at $x_j$. The relation between barycentric weights $\Omega_j$ and Gauss quadrature weights $\omega_j$ is revealed by Wang, Huybrechs and Vandewalle \([30]\).

Then we deduce the $\ell_1$–regularized barycentric interpolation formula. Since $\ell_1$–regularized minimizer can be expressed as the sum of two terms:

$$p_{L,N+1}^{\ell_1}(x) = \sum_{\ell=0}^{N} \left( \sum_{j=0}^{N} \omega_j \tilde{\Phi}_\ell(x_j) f(x_j) \right) \tilde{\Phi}_\ell(x) + \sum_{\ell=0}^{N} c_\ell \tilde{\Phi}_\ell(x),$$

(A.6)
where
\[ c_\ell = \frac{S_{\lambda \mu} \left( 2 \sum_{j=0}^{N} \omega_j \tilde{\varphi}_\ell(x_j) f(x_j) \right)}{2} - \sum_{j=0}^{N} \omega_j \tilde{\varphi}_\ell(x_j) f(x_j), \quad \ell = 0, 1, \ldots, N. \]

The first term in eq. (A.6) can be written as barycentric form directly by letting \( \lambda = 0 \) in \( \ell_2 \)-regularized barycentric least squares formula. Let the basis \( \{ \tilde{\varphi}_\ell \}_{\ell=0}^{N} \) transform into Lagrange polynomials \( \{ \ell_j(x) \}_{j=0}^{N} \). By the basis-transformation relation between orthogonal polynomials and Lagrange polynomials [8], we have
\[ \sum_{\ell=0}^{N} c_\ell \tilde{\varphi}_\ell(x) = \sum_{j=0}^{N} \left( \sum_{\ell=0}^{N} c_\ell \tilde{\varphi}_\ell(x_j) \right) \ell_j(x). \tag{A.7} \]

With the same procedure of obtaining barycentric formula from classical Lagrange interpolation formula in [3], we have
\[ \sum_{\ell=0}^{N} c_\ell \tilde{\varphi}_\ell(x) = \frac{\sum_{j=0}^{N} \Omega_j \left( \sum_{\ell=0}^{N} c_\ell \tilde{\varphi}_\ell(x_j) \right)}{\sum_{j=0}^{N} \Omega_j \frac{x - x_j}{x - x_j}}. \tag{A.8} \]

Together with eq. (A.6) and eq. (A.8), we obtain the \( \ell_1 \)-regularized barycentric interpolation formula:
\[ p_1^{\ell_1 \text{-bary}}(x) = \frac{\sum_{j=0}^{N} \Omega_j \left( f(x_j) + \sum_{\ell=0}^{N} c_\ell \tilde{\varphi}_\ell(x_j) \right)}{\sum_{j=0}^{N} \Omega_j \frac{x - x_j}{x - x_j}}. \]

B  Additional numerical examples of numerical stability for interpolation

Figure 8 and Figure 9 are two more examples for illustrating the numerical stability of regularized interpolation formulae.

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Figure 8: Repetition of Figure 11 but for function $f(x) = \sin(20\pi x) - x$ on $[-1, 1]$.

Figure 9: Repetition of Figure 11 but for function $f(x) = \exp(x)\sin(15x)$ on $[-1, 1]$.
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