On the geometry of the Clairin theory of conditional symmetries for higher-order systems of PDEs with applications

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Abstract
This work presents a geometrical formulation of the Clairin theory of conditional symmetries for higher-order systems of partial differential equations (PDEs). We devise methods to obtain Lie algebras of conditional symmetries from known ones, and unnecessary previous assumptions of the theory are removed. As a byproduct, new insights into other types of conditional symmetries arise. Then, we apply the so-called PDE Lie systems to the derivation and analysis of Lie algebras of conditional symmetries. In particular, we develop a method to obtain solutions of a higher-order system of PDEs via the solutions and geometric properties of a PDE Lie system, whose form gives a Lie algebra of conditional symmetries of Clairin type. Our methods are illustrated with physically relevant examples such as nonlinear wave equations, Gauss–Codazzi equations for minimal soliton surfaces, and generalised Liouville equations.

Keywords: characteristic function; conditional symmetry, contact form, Gauss–Codazzi equation, jet bundle, Lie point symmetry, nonlinear wave equation, minimal surface, PDE Lie system

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1. Introduction

Over the last two centuries, Lie’s theory of symmetries of PDEs [24] has been the subject of extensive research in mathematics and physics [24, 29, 39, 43]. Over those years, the development of this theory has led to significant progress in classifying and solving differential equations, yielding many new interesting results (see e.g. [22] and references therein). A number of attempts to generalise this subject and to develop its applications can be found in the literature [4, 9, 14, 15, 18, 22, 25, 26, 28, 40, 43, 44].
Of particular interest from a physical viewpoint has been the development of the theory of conditional symmetries which evolved in the process of extending Lie’s classical theory of symmetries of PDEs [4, 10, 18, 19]. This approach consists essentially in augmenting the original system of PDEs with certain first-order differential constraints (DCs) for which a symmetry criterion is applied. As a result we obtain an overdetermined system of PDEs admitting, in some cases, a larger class of Lie point symmetries than the original system. Consequently, this approach enables us to construct new classes of solutions of the original system. The problem of the determination of such classes of explicit solutions was first solved by M. Clairin [10] in 1903 by subjecting the original system of PDEs with several DCs. Since then several generalisations of conditional symmetries has been formulated by many authors [4, 18, 19, 28, 30, 44].

In the hereafter called Clairin theory of conditional symmetries [10, 18, 19], the DCs are given by an integrable (in the sense of fulfilling the zero curvature condition [11, 21]) first-order system of PDEs in normal form compatible (in a sense to be explained later) with the initial system of PDEs. Although there exist more general conditional symmetry methods based on adding DCs that need not give rise to first-order systems of PDEs in normal form [14, 23, 28, 30, 33, 44], the Clairin theory allows us to study the solutions of the DCs through many techniques. For instance, one can use different types of Lie systems [18, 19] or a naturally related Abelian Lie algebra of Lie point symmetries, which generates new particular solutions of the initial system of PDEs from known ones satisfying the given DCs.

In the Clairin theory, the DCs are determined by the so-called characteristics [29] of the elements of a Lie algebra of vector fields $L$, a so-called Lie algebra of conditional symmetries, consisting of Lie point symmetries of the restriction of the initial system of PDEs to solutions satisfying the added DCs. The crux of the Clairin theory is to find $L$ [18, 19, 28]. To simplify the task, $L$ is assumed to be Abelian and to admit a basis of a particular type [18, 19]. Then, $L$ can be obtained by solving a nonlinear system of PDEs [19, 28].

The Clairin method, which is mainly described in coordinates (cf. [10, 18, 19]), lacks an intrinsic geometric formulation. Additionally, there exists to our knowledge no detailed geometric formalism for studying higher-order systems of PDEs in this context, as works were so far focused on the formalism for first-order systems of PDEs [10, 19] or on applying conditional symmetry techniques to higher-order systems of PDEs without a detailed analysis [10, 18, 19].

Hence, our first aim is to present a geometric Clairin theory of conditional symmetries for higher-order systems of PDEs. This allows us to avoid most previous assumptions on $L$, to clarify some of their geometric features [18, 19], and, as a byproduct, to provide new insights into other conditional symmetries and related structures [28, 44]. Our second aim is to develop methods to obtain conditional symmetries through the so-called PDE Lie systems and to apply them to study physical systems [7].

Geometrically, the Clairin theory of conditional symmetries for higher-order systems of PDEs goes as follows. An $n$-order system of PDEs whose dependent and independent variables are functions on $U$ and $X$, respectively, amounts to a subset $S_\Delta$ of the $n$-th order jet bundle, $J^n$, of the bundle $\pi_{-1}^0 : (x, u) \in X \times U \mapsto x \in X$ [29]. Lie algebras of conditional
symmetries of $\mathcal{S}_\Delta$ are given by a Lie algebra $L$ of vector fields on $J^0 = X \times U$ defining a submanifold $\mathcal{S}_L^\eta \subset J^n$, the called characteristic system of $L$, given by the common zeroes of the total differentials [29] of the characteristics of the elements of $L$ in such a way that $L$ consists of Lie point symmetries of the system of PDEs related to $\mathcal{S}_L^\eta \cap \mathcal{S}_\Delta$. The Clairin theory requires $\mathcal{S}_L^\eta$ to be related to a first-order system of PDEs in normal form satisfying the so-called zero curvature condition [11, 21]. If $L$ is Abelian and admits a basis of a particular type [18, 19], then $L$ can be then derived by solving a nonlinear system of PDEs [18, 19, 28].

Apart from providing a careful geometric Clairin theory for conditional symmetries of higher-order systems, let us describe other new contributions of our work.

First, by describing the characteristics of vector fields via the contact forms on $J^n$ [29, 39], the characteristic system $\mathcal{S}_L^\eta$ for a linear space $L$ of vector fields on $J^n$ is defined in Definition 3.1 or, equivalently, in Definition 3.2 in an intrinsic geometrical way. This geometrises and generalises several types of Lie algebras of conditional vector fields [18, 19, 44].

Theorems 4.3 and 4.5 characterise when $\mathcal{S}_L^\eta$ can be described as a system of PDEs in normal form. If $\mathcal{S}_L^\eta$ is assumed to be a system of PDEs in normal form, then Theorem 5.3 provides necessary and sufficient conditions to ensure that the prolongations to $J^n$ (see [29]) of the vector fields of $L$ are tangent to $\mathcal{S}_L^\eta$. This fulfills results in the previous literature, where only necessary conditions are detailed [18, 19, 28]. Remarkably, $L$ need not be a Lie algebra, as assumed previously [18, 19]. Since our results state that $L$ must span a distribution $\mathcal{D}_L$ of dimension $q$ projecting onto $TX$, one obtains that this condition, in coordinates, amounts to the existence of a non-degenerate $p \times p$ matrix $\xi$ appearing in the Clairin formalism [18, 19].

Subsequently, Proposition 5.5 details the necessary and sufficient conditions on $L$ to ensure that $\mathcal{S}_L^\eta$, related to system in normal form, is locally solvable [29]. Corollary 6.6 shows that the standard conditions found in the literature [28] ensure that $\mathcal{S}_L^\eta$ is locally solvable. Remarkably, if $n > 1$, then Theorem 4.5 and Proposition 5.5 ensure that if $\mathcal{S}_L^\eta$ is a system of PDEs in normal form, then $\mathcal{S}_L^\eta$ is locally solvable, which also implies that the projection of $\mathcal{S}_L^\eta$ to $J^1$ satisfies the zero curvature condition [11, 21]. This implies that the solutions of $\mathcal{S}_\Delta \cap \mathcal{S}_L^\eta$ are the same as those of the system of PDEs given by $\mathcal{S}_\Delta \cap \mathcal{S}_L^\eta$ (where $\mathcal{S}_L^\eta$ is considered as a subspace of $J^n$ in the natural way [40]). This is the usual approach appearing in the literature (see [18, 19] and Section 9).

The Clairin approach to conditional symmetries assumes that $L$ has a basis of a particular form [10, 18, 19]. Section 6 characterises geometrically when $L$ admits such a basis. Such Lie algebras are here called normal PDE Lie algebras. Next, it is studied when a linear space of vector fields $L'$ is such that $\mathcal{S}_{L'}^\eta = \mathcal{S}_L^\eta$. If $L'$ is a Lie algebra, then it is called a renormalizable Lie algebra.

Next, we survey the geometric properties of conditional Lie symmetries and raises certain technical questions frequently overlooked in the literature. We show that every normal PDE Lie algebra of conditional symmetries $L$ is such that every renormalizable Lie algebra of vector fields $L'$ satisfying that $\mathcal{S}_{L'}^\eta = \mathcal{S}_L^\eta$ is also a Lie algebra of conditional Lie symmetries of the same system of PDEs. This allows us to obtain new Lie algebras of conditional symmetries from known ones.

Finally, we study the differential equations characterising normal PDE Lie algebras of
conditional symmetries. To solve them, we extend the methods in [18, 19] and provide conditions to ensure that normal PDE Lie algebras can be derived via the so-called PDE Lie systems [6, 27]. PDE Lie systems are first-order systems of PDEs in normal form admitting a superposition rule, i.e. a function allowing us to obtain their general solutions in terms of a generic family of particular solutions and some constants [6, 27]. These systems of PDEs have attracted much attention lately and their special structure allows for the development of methods to study their solutions [5, 7].

Our use of PDE Lie systems to obtain Lie algebras of conditional symmetries is more general than the approach used in [18, 19], where only particular types of PDE Lie systems or standard Lie systems appear. Moreover, we provide assumptions on higher-order systems of PDEs $S_\Delta$ that enable us to construct a PDE Lie system describing some of its conditional symmetries (in the Clairin context) and whose solutions are solutions of $S_\Delta$.

As applications, PDE Lie systems are employed to study nonlinear wave equations [19] and minimal surfaces for Gauss–Codazzi equations [37]. The fact that PDE Lie systems related to solvable Vessiot–Guldberg Lie algebras can be solved is employed to obtain minimal surfaces of Gauss–Codazzi equations. (cf. [6, 7, 27]). Finally, some solutions to generalised Liouville equations [36] are provided.

This paper is organized as follows. Section 2 describes the basic geometric tools employed in the work. Section 3 is concerned with the definition of a characteristic system for a linear space of vector fields on $X \times U$. Section 4 establishes when $S^n_\Delta$ amounts to an $n$-order systems of PDEs in normal form. Section 5 analyses certain geometric properties of characteristic systems. Meanwhile, Section 6 focuses on the generality of a certain type of Lie algebras appearing in the Clairin theory of conditional symmetries. Section 7 studies Lie algebras of Clairin conditional symmetries. The use of PDE Lie systems to obtain conditional symmetries is developed in Section 8. In Section 9 several applications of our results are presented. Finally, Section 10 summarises our new results, and acknowledgements are detailed in Section 11.

2. Geometric preliminaries on jet bundles and systems of PDEs

This section provides the notions of the theory of jet bundles and the notation to be employed in this work. This allows us to make our work more self-contained and easier to follow. To highlight the main points of our presentation and unless otherwise stated, we assume that all structures will be smooth and globally defined.

Let $X$ and $U$ be $p$-dimensional and $q$-dimensional manifolds, respectively. We define $J^0 = X \times U$ and $J^{-1} = X$, whereas $J^n$ stands for the $n$-order jet bundle of the trivial bundle $\pi_{-1} : X \times U \to X$ given by the projection onto $X$. Let $\{x^1, \ldots, x^p\}$ and $\{u^1, \ldots, u^q\}$ be coordinate systems on $X$ and $U$, respectively. Such coordinate systems induce coordinates $u^u_K$ on $J^n$, where $K = (k_1, \ldots, k_p)$ is a multi-index with $k_1, \ldots, k_p \in \mathbb{N} \cup \{0\}$ and $|K| = \sum_{i=1}^p k_i \leq n$. Then, $\{x^i, u^a, u^u_K\}$, with $1 \leq |K| \leq n$, becomes a local coordinate system of $J^n$. We write $u^u_{K,i}$, with $i = 1, \ldots, p$, for the coordinate $u^u_{K'}$, where $k'_i = k_i + 1$ and $k_j = k'_j$ for the indexes $j \neq i$. 4
If $n, m$ are integers and $n \geq m \geq -1$, then $J^{n}$ is a bundle over $J^{m}$ relative to the natural projection $\pi_{m}^{n} : J^{n} \to J^{m}$ and we write $\Gamma(\pi_{m}^{n})$ for its space of sections.

We denote by $j^{n}_{x} \sigma = (x, u, u^{(n)})$ an arbitrary point of $J^{n}$. Given a section $\sigma \in \Gamma(\pi_{0}^{0})$, say $\sigma(x) = (x, u(x))$, its prolongation to $J^{n}$ is the section $j^{n}_{x} \sigma \in \Gamma(\pi_{n}^{0})$ of the form $j^{n}_{x} \sigma(x) = (x^{r}, u^{(r)}(x), \frac{\partial}{\partial x^{1}, \ldots, \partial x^{r}}(x))$ for $1 \leq r = \left| k_{1}, \ldots, k_{p} \right| \leq n$. Sections $\sigma^{(n)} \in \Gamma(\pi_{n}^{0})$ of the form $\sigma^{(n)} = j^{n}_{x} \sigma$ for $\sigma \in \Gamma(\pi_{0}^{0})$ are called holonomic.

The main geometric structure on $J^{n}$, the so-called Cartan distribution $\mathcal{C}^{n}$, is the smallest distribution on $J^{n}$ tangent to all prolongations $j^{n}_{x} \sigma$ for an arbitrary $\sigma \in \Gamma(\pi_{0}^{0})$ [29]. In coordinates, the Cartan distribution $\mathcal{C}^{n}$ is spanned by the vector fields

$$D_{i} = \frac{\partial}{\partial x^{i}} + \sum_{|K| \leq n-1} \sum_{\alpha=1}^{q} u_{K,i}^{\alpha} \frac{\partial}{\partial u_{K}^{\alpha}}, \quad V_{\alpha K'} = \frac{\partial}{\partial u_{K'}^{\alpha}}, \quad i = 1, \ldots, p, \quad \alpha = 1, \ldots, q, \quad |K'| = n,$$

where $D_{i}$ is called the total derivative relative to $x^{i}$ in $J^{n}$. The Cartan distribution is not involutive [29]. We write $D_{K} = D_{K_{1}}^{1} \circ \cdots \circ D_{K_{p}}^{p}$ for a multi-index $K$ and $D_{K} = \text{Id}$ when $|K| = 0$.

A one-form $\theta$ on $J^{n}$ is a contact form if $j^{n}_{x} \sigma^{*} \theta = 0$ for every $\sigma \in \Gamma(\pi_{0}^{0})$. Hence, contact forms allow us to determine when a section $\sigma^{(n)} \in \Gamma(\pi_{n}^{0})$ is holonomic. We write $\mathcal{CF}(J^{n})$ for the space of contact forms on $J^{n}$. In particular, the one-forms

$$\theta_{K}^{\alpha} = du_{K}^{\alpha} - \sum_{i=1}^{p} u_{K,i}^{\alpha} dx^{i}, \quad \alpha = 1, \ldots, q, \quad 0 \leq |K| \leq n - 1, \quad (2.1)$$

are called the basic contact forms on $J^{n}$ relative to the coordinate system $\{x^{i}, u^{\alpha}, u_{K}^{\alpha}\}$, with $1 \leq |K| \leq n$. Contact forms on $J^{n}$ can be written as linear combinations with functions on $J^{n}$ of basic contact forms. In this sense, the basic contact forms (2.1) form a basis of $\mathcal{CF}(J^{n})$.

A vector field $Y$ on $J^{0}$ can be written in local coordinates as

$$Y = \sum_{i=1}^{p} \xi^{i} \frac{\partial}{\partial x^{i}} + \sum_{\alpha=1}^{q} \varphi^{\alpha} \frac{\partial}{\partial u^{\alpha}}, \quad (2.2)$$

where $\xi^{i}, \varphi^{\alpha}$ are univocally defined functions on $J^{0}$.

The prolongation to $J^{n}$ of $Y$ is the only vector field, $j^{n}_{x} Y$, on $J^{n}$ leaving the space of vector fields taking values in $\mathcal{C}^{n}$ invariant (relative to the Lie bracket of vector fields) and projecting onto $Y$ via $\pi_{0}^{n} : J^{n} \to J^{0}$. Consequently, if $\theta$ is a contact form on $J^{n}$, then the Lie derivative $\mathcal{L}_{j^{n}_{x} Y} \theta$ is also a contact form. The expression of $j^{n}_{x} Y$ in coordinates\footnote{To provide an expression easy to deal with, it is assumed that functions and total derivatives are defined on $J^{n+1}$. Despite that, $j^{n}_{x} Y$ is a well-defined vector field on $J^{n}$. This trick is standard in the literature [29, p.110].} reads [29, 39]

$$j^{n}_{x} Y = Y + \sum_{1 \leq |K| \leq n} \sum_{\alpha=1}^{q} \psi_{K}^{\alpha} \frac{\partial}{\partial u_{K}^{\alpha}}, \quad \psi_{Y K}^{\alpha} = D_{K} Q_{Y}^{\alpha} + \sum_{i=1}^{p} \xi^{i} u_{K,i}^{\alpha}, \quad Q_{Y}^{\alpha} = \varphi^{\alpha} - \sum_{i=1}^{p} \xi^{i} u_{i}^{\alpha}, \quad (2.3)$$
where the functions \( Q_\alpha^\alpha \) are the so-called *characteristics* of the vector field \( Y \). Geometrically,
\[
Q_\alpha^\alpha = \iota_Z Y^\alpha, \quad \alpha = 1, \ldots, q,
\]
where \( \iota_Z Y \) stands for the contraction of the one-form \( Y \) with the vector field \( Z \).

First-order systems of PDEs in \( p \) independent and \( q \) dependent variables are defined in the form
\[
\Delta^\mu (j^\alpha_x \sigma) = 0, \quad \mu = 1, \ldots, s, \tag{2.4}
\]
where \( \Delta^\mu : J^n \to \mathbb{R} \) for \( \mu = 1, \ldots, s \), are certain functions. A particular solution of (2.4) is a map \( u(x) \) from \( X \) to \( U \) whose associated section \( \sigma(x) = (x, u(x)) \in \Gamma(\pi_0) \) is such that its prolongation to \( J^n \) satisfies (2.4).

The system of PDEs (2.4) determines a region \( S_\Delta \subset J^n \) where all the functions \( \Delta^\mu \), with \( \mu = 1, \ldots, s \), vanish simultaneously. As standardly in the literature \[29\], it is hereafter assumed that the system (2.4) has *maximal rank*, i.e. the functions \( \Delta^\mu \) are functionally independent and \( S_\Delta \) can be considered as a submanifold of \( J^n \) (cf. \[29\], p. 158). A system of PDEs (2.4) is *locally solvable* if for each \( j^\alpha_x \sigma \in S_\Delta \) there exists a solution \( u(x) \) of the system (2.4) such that \( j^\alpha_x \sigma \), with \( \sigma(x) = (x, u(x)) \), belongs to \( j^\alpha \sigma(x) \) \[29\], p. 158.

A large family of differential equations are locally solvable and have maximal rank. To understand the generality of the locally solvability assumption, we provide the following proposition characterising local solvability of first-order systems of PDEs in normal form. Similar results, with less details, can be found in \[20, 21\].

**Proposition 2.1.** A first-order system of PDEs in normal form
\[
\frac{\partial u^\alpha}{\partial x^i} = \phi^\alpha_i(x, u), \quad i = 1, \ldots, p, \quad \alpha = 1, \ldots, q, \tag{2.5}
\]
is locally solvable if and only if it satisfies the zero curvature condition. In turn, the zero curvature condition amounts to the fact that the Lie algebra of vector fields spanned by
\[
Z_j = \frac{\partial}{\partial x^j} + \sum_{\alpha=1}^q \phi_j^\alpha \frac{\partial}{\partial u^\alpha}, \quad j = 1, \ldots, p. \tag{2.6}
\]
is Abelian.

**Proof.** Geometrically, the system (2.5) amounts to a submanifold \( S_\Delta \) given by points of the form \((x^i, u^\alpha, \phi^\alpha_i(x, u))\) of \( J^1 \). The system (2.5) has a particular solution passing through each point \((x_0^i, u_0^\alpha, \phi^\alpha_i(x_0, u_0))\) in \( S_\Delta \) if and only if the system (2.5) admits a particular solution for each initial condition \((x_0, u_0) \in X \times U \). The system (2.5) satisfies the zero curvature condition if and only if it admits a particular solution for every \((x_0, u_0) \in X \times U \). This finishes the first part of the proof.

Moreover, observe that the zero curvature condition for (2.5) reads
\[
\phi_{j,k}^\alpha - \phi_{k,j}^\alpha + \sum_{\beta=1}^q (\phi_{j,\alpha}^\beta \phi_k^\beta - \phi_{k,\alpha}^\beta \phi_j^\beta) = 0, \quad j, k = 1, \ldots, p, \quad \alpha = 1, \ldots, q, \quad (2.7)
\]
while

$$[Z_k, Z_j] = \sum_{\alpha=1}^{q} \left( \phi_{j,k}^\alpha - \phi_{k,j}^\alpha + \sum_{\beta=1}^{q} (\phi_{j,u}^\alpha \phi_{k}^\beta - \phi_{k,u}^\alpha \phi_{j}^\beta) \right) \frac{\partial}{\partial u^\alpha}. $$

This shows that the zero curvature condition for (2.5) is satisfied if and only if the vector fields (2.6) span an Abelian Lie algebra.

A vector field $Y$ on $J^0$ is a Lie point symmetry of the system of PDEs (2.4) if

$$ (j^n Y) \Delta^\mu |_{S^\Delta} = 0, \quad \mu = 1, \ldots, s. \quad (2.8) $$

As the prolongation to $J^n$ of the Lie bracket of two vector fields on $X \times U$ is the Lie bracket of their prolongations to $J^n$ [29], the Lie bracket of two Lie point symmetries of $\Delta$ is a new Lie point symmetry. Thus, Lie point symmetries form a Lie algebra $V$, which, when finite-dimensional, locally defines a Lie group action of $G$ on $J^n$. This Lie group action transforms solutions of (2.4) into solutions of the same equation and allows for the reduction of the initial systems of PDEs [39].

3. A definition of characteristic systems for linear spaces of vector fields

The description of Lie algebras of conditional symmetries for systems of PDEs is based on the geometry of the characteristic system, namely a submanifold of $J^n$ to which we restrict the study of solutions of our initial system of PDEs [18, 19, 31, 44]. As shown in this section, the literature on conditional symmetries lacks a purely geometric definition of characteristic systems and the study of characteristics systems for higher-order PDEs is scarcely treated [18, 19, 28, 44]. This section aims to fill this gap. Moreover, our definition is not reduced to characteristic systems for Lie algebras of vector fields [28, 44], which will allow us to dismiss in next sections certain unnecessary technical conditions on characteristic systems given in previous works [18, 19, 44].

Definition 3.1. Let $L$ be a finite-dimensional linear space of vector fields on $X \times U$. The characteristic system of $L$ in $J^n$ is the subset of $J^n$ given by

$$ S^n_L = \{ j^n_x \sigma \in J^n : [j^n_x Y] (j^n_x \sigma) = 0, \forall Y \in L, \forall \theta \in CF(J^n) \}, $$

where $CF(J^n)$ is the space of contact forms on $J^n$.

Let us describe Definition 3.1 in coordinates to compare it with other definitions [18, 19, 28, 44] and to justify the term ‘characteristic system’.

If $L$ admits a basis (as a linear space) $Y_j = \sum_{k=1}^{p} \xi_j^k \partial/\partial x^k + \sum_{\alpha=1}^{q} \varphi_j^\alpha \partial/\partial x^\alpha$, with $j = 1, \ldots, l$, then the basis $\theta^\alpha_K$ of $CF(J^n)$, with $\alpha = 1, \ldots, q$ and $|K| \leq n-1$, allows us, by using (2.3), to write that

$$ t^n_{j,y} \theta^\alpha = \varphi^\alpha_j - \sum_{i=1}^{p} u^\alpha_i \xi^i_j, \quad t^n_{x,y} \theta^\alpha_K = \psi^\alpha_{y,K} - \sum_{i=1}^{p} u^\alpha_{K,i} \xi^i_j = D_K \left( \varphi^\alpha_j - \sum_{i=1}^{p} u^\alpha_i \xi^i_j \right), $$
for \( \alpha = 1, \ldots, q, j = 1, \ldots, l, \) and \( 1 \leq |K| \leq n - 1. \) Hence, the functions \( \iota^\alpha_\theta \), with \( \alpha = 1, \ldots, q \) are all the compositions of up to \( n - 1 \) total derivatives of the characteristics \( Q^\alpha_j \) of \( Y_j \). Then, \( J^n \sigma \in S^n_L \) if and only if \( [\iota^\alpha_\theta \iota^K]\sigma = 0 \) for \( j = 1, \ldots, l \) and the basis of basic contact forms \( \theta^K_\alpha \) on \( J^n \) with \( \alpha = 1, \ldots, q \) and \( 0 \leq |K| \leq n - 1. \) Consequently, \( S^n_L \) is the subset of \( J^n \) where vanish all total derivatives up to order \( n - 1 \) of the characteristics of a basis of vector fields of \( L \).

In the theory of conditional symmetries, a characteristic system in \( J^n \) is mostly defined to be the subset of zeros of the characteristics (and their total derivatives up to order \( n - 1 \)) of a Lie algebra of vector fields \([18, 19, 28, 31, 44]\). Hence, Definition 3.1 retrieves the standard definition when \( L \) is a Lie algebra. Our definition is purely geometrical as it does not rely in the use of characteristics of a basis of \( L \) and it does not need to give their coordinate expressions as in previous works.

Some definitions of characteristic systems demand additional technical conditions on the elements of \( L \) (see \([44]\)). The reasons to assume these conditions will be explained in following sections. Meanwhile, Olver and Roseneau proposed a very general definition of differential constraints, the so-called side conditions, for studying higher-order systems of PDE \([28, 30]\). This generalization covers Definition 3.1 as a particular case. Notwithstanding, Olver and Roseneau’s definition is not linked to characteristics of vector fields due to its generality \([28, pg. 15]\), which makes it unappropriate to study conditional symmetries, which are strongly related to characteristics.

Note that since \( S^n_L \subset J^n \) can be understood as the \( n \)-order system of PDEs determined by the characteristics and their successive total differentials up to order \( n - 1 \) of the elements of a linear space of vector fields \( L \), it makes sense to call \( S^n_L \) a characteristic system.

As the Cartan distribution \( C^n \) is the intersection of all the kernels of contact forms on \( J^n \), the definition of \( S^n_L \) can be rewritten in the following dual equivalent manner.

**Definition 3.2.** Let \( L \) be a finite-dimensional linear space of vector fields on \( X \times U \). The characteristic system of \( L \) in \( J^n \) is the subset \( S^n_L \subset J^n \) where the prolongations to \( J^n \) of the vector fields of \( L \) take values in the Cartan distribution \( C^n \).

### 4. On characteristic systems and sections of jet bundles

In the Clairin approach to the theory of conditional symmetries \([10, 18, 19]\), characteristic systems are employed to study systems of PDEs in normal form. This section studies this relation in geometric terms.

Note that \( S^1_L \) does not need to be a submanifold of \( J^1 \). For instance, if \( X = \mathbb{R}, U = \mathbb{R} \) and \( L = (x_1(u_2^2 - u_1)\partial_{u_1}), \) then \( S^1_L \) for is not a submanifold of \( J^1 \). Nevertheless, \( S^n_L \) will be a submanifold in all cases of relevance due to the nature of the Clairin theory of conditional symmetries \([10, 18, 19]\). Following results will explain the geometrical meaning of such assumptions and which of them can be dismissed. Let us start by a useful lemma.
Lemma 4.1. Let \( Y_1, \ldots, Y_l \) be a basis of a linear space \( L \) of vector fields on \( X \times U \). If \( f^1, \ldots, f^l \in C^\infty(X \times U) \), then one has the following equality at points of \( S^n_L \):

\[
j^n \left( \sum_{j=1}^l f^j Y^*_j \right) = \sum_{j=1}^l f^j j^n Y_j, \quad \alpha = 1, \ldots, q, \tag{4.1}
\]

where the functions \( f^1, \ldots, f^l \) on the right-hand side are considered as functions on \( J^n \) in the natural way, namely as functions on \( J^n \) depending only on \( x \) and \( u \) (see [40] for further details).

Proof. In coordinates we write

\[
Y_j = \sum_{i=1}^p \xi^i_j \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \varphi^\alpha_j \frac{\partial}{\partial u^\alpha}. \tag{4.2}
\]

To simplify notation, we write \( \bar{Y} = \sum_{j=1}^l f^j Y_j \). In view of (2.3), one has that \( \bar{Q}^\alpha = \sum_{j=1}^l f^j Q^\alpha_{Y_j} \). Since all total derivatives \( D_K \) of order \( |K| \leq n - 1 \) of the characteristics of the \( Y_1, \ldots, Y_l \) vanish on \( S^n_L \), we obtain that \( D_K \bar{Q}^\alpha = \sum_{j=1}^l f^j D_K Q^\alpha_{Y_j} \) for \( |K| \leq n - 1 \) and \( \alpha = 1, \ldots, q \). In view of (2.3), one has that \( \bar{\psi}^\alpha_{Y,K} = \sum_{j=1}^l f^j \psi^\alpha_{Y_j,K} \) for \( \alpha = 1, \ldots, q, |K| \leq n - 1 \), and the formula (4.1) follows easily from the expression (2.3) for the prolongations to \( J^n \) of \( Y_1, \ldots, Y_l \).

We have already explained that \( S^n_L \) amounts to an \( n \)-order system of PDEs. We are now concerned with establishing when such a system is in normal form. We prove the following rather simple fact to stress that \( S^n_L \) is a section of \( J^n \) if and only if its associated \( n \)-order system of PDEs can be written in normal form.

Proposition 4.2. A characteristic system \( S^n_L \) amounts to an \( n \)-order system of PDEs in normal form if and only if \( S^n_L \) is a section of \( J^n \).

Proof. If \( S^n_L \) is a section of \( J^n \), then every coordinate \( u^\alpha_K \) of points in \( S^n_L \) can be written as a function \( u^\alpha_K = \phi^\alpha_K(x, u) \). Therefore, \( S^n_L \) can be written as the subset of \( J^n \) where the series of conditions \( u^\alpha_K - \phi^\alpha_K(x, u) = 0 \), with \( \alpha = 1, \ldots, q \) and \( |K| \leq n - 1 \) are obeyed. Such conditions amount to an \( n \)-order system of PDEs in normal form. The converse statement is immediate.

Let us characterise through \( L \) when \( S^n_L \) amounts to an \( n \)-order system of PDEs in normal.

Theorem 4.3. Let \( L \) be a linear space of vector fields on \( X \times U \). Then, \( S^1_L \) is a section of the bundle \( \pi_0^1 : J^1 \to J^0 \) if and only if the vector fields of \( L \) span a regular distribution \( D^L \) of rank \( p \) and \( D^L \) projects onto \( TX \) under \( \pi_{0,1}^1 \).

Proof. The linear space \( L \) admits a basis \( Y_1, \ldots, Y_l \) of the form (4.2).
Let us prove first the converse part of our theorem. If $D^L$ is regular of rank $p$ and its projection onto $X$ is $TX$ (relative to $\pi^0_{-1}$), then $Y_1, \ldots, Y_l$ span $D^L_{(x,u)} \subset T(X \times U)$ and

$$Y_j^H(x, u) = \sum_{i=1}^p \xi^i_j(x, u) \frac{\partial}{\partial x^i}, \quad j = 1, \ldots, l = \dim L,$$

span $T_x X$ for every $u \in U$. Consequently, we can assume without loss of generality that the first $Y_1, \ldots, Y_p$ from $Y_1, \ldots, Y_l$ are such that $Y_1(x, u), \ldots, Y_p(x, u)$ are linearly independent and span $D^L$ at a point $(x, u) \in X \times U$. If $\xi$ is the $p \times p$ matrix with coefficients $\xi^i_j$ of $Y_1, \ldots, Y_p$, then $\det \xi \neq 0$ on an open subset of $(x, u)$ and $\xi$ has an inverse $\xi^{-1}$. The space $S^1_L$ is given by the zeroes of the functions $\iota_j^1, Y_j^\theta = \phi^\alpha_j(x, u) - \sum_{i=1}^p \xi^i_j(x, u) u^\alpha_i$, with $j = 1, \ldots, p$. Since $Y_1, \ldots, Y_p$ span the distribution $D^L$ and Lemma 4.1, the $S^2_L$ is locally given by the zeroes of the functions $\iota_j^1, Y_j^\theta = \phi^\alpha_j(x, u) - \sum_{i=1}^p \xi^i_j(x, u) u^\alpha_i$, with $j = 1, \ldots, p$. This shows that

$$u^\alpha_k = \sum_{i=1}^q (\xi^{-1})^i_k(x, u) \phi^\alpha_j(x, u), \quad k = 1, \ldots, p, \quad \alpha = 1, \ldots, q. \quad (4.3)$$

Thus, the value of $u^\alpha_k$ is determined for every $(x, u)$ univocally in terms of the $\phi^\alpha_j$ and $\xi$. Hence, $S^1_L$ is locally a section of $\pi^1_{-1}$. The smoothness of the vector fields $Y_1, \ldots, Y_p$ ensures that $S^1_L$ is locally a smooth section of $\pi^1_{-1}$. As the same can be applied to any $(x, u)$, one easily finds that $S^1_L$ is a global section of $\pi^1_{-1}$.

Let us prove the direct part of this theorem. If $S^1_L$ is a smooth section of $\pi^1_{-1}$, then the points $(x^i, u^\alpha, u^\beta_k)$ of $S^1_L$ are such that for every $x, u$, the values of the possible $u^\beta_k$ are univocally defined by the conditions

$$\phi^\alpha_j(x, u) - \sum_{i=1}^p \xi^i_j(x, u) u^\alpha_i = 0, \quad j = 1, \ldots, l.$$

The values of $u^\beta_k$ for every $x, u$ are univocally determined if and only if the extended and the standard matrix of coefficients of the previous system have rank $p$. The matrix of coefficients has rank $p$ if and only if the projection of the vector fields of $L$ onto $X$ span a distribution of rank $p$, namely $TX$. As the extended matrix has also rank $p$, the vector fields $Y_1, \ldots, Y_l$ span also a regular distribution of rank $p$ on $J^0$.

**Example 4.1.** Theorem 4.3 shows that $S^1_L$ may be a section of $\pi^1_{-1}$ even if $L$ is not nor a Lie algebra neither its elements span an integrable distribution, being these conditions used standardly in other works [18, 19]. For instance, consider $U = \mathbb{R}$, $X = \mathbb{R}^2$ and $L = (\partial/\partial x + \partial/\partial u, \partial/\partial t + \partial/\partial u, x\partial/\partial u)$, which is not a Lie algebra. Then $S^1_L$ is given by the zeroes of the conditions $u_x - 1 = 0, u_t - x = 0$, which amount to a section on $\pi^1_{-1}$.

**Example 4.2.** Theorem 4.3 states that $S^1_L$ may be a section of $\pi^1_{-1}$ when $L$ is a Lie algebra of dimension different from $\dim X$. This case does not appear in the Clairain approach, where $\dim L = \dim X$, (see [10, 18, 19]). To illustrate such a new possibility, consider for $U = \mathbb{R}$ and $X = \mathbb{R}^2$. The Lie algebra $L = (\partial/\partial x + \partial/\partial u, \partial/\partial t + \partial/\partial u, x(\partial/\partial t + \partial/\partial u))$ is a three-dimensional Lie algebra giving rise to a section on $\pi^1_{-1}$ given by the equations $u_x - 1 = 0, u_t - 1 = 0, xu_t - x = 0$. 10
Let us now extend Proposition 4.3 to \( J^n \). This will lead to impose that the distribution spanned by the elements of \( L \) is involutive in cases relevant to us, namely where we are studying systems of PDEs and therefore \( \dim X > 1 \). Before obtaining this result, we need the following lemma.

**Lemma 4.4.** If \( L \) gives rise to a section \( S^1_L \) of \( \pi_0^1 \), then the first-order system of PDEs (2.5) associated with \( S^1_L \) is such that the vector fields (2.6) span the same distribution as \( L \).

**Proof.** Let \( Y_1, \ldots, Y_l \) be a basis of \( L \) given by (4.2). In view of Theorem 4.3, the distribution generated by the elements of \( L \) has rank \( p \) and there exists \( p \) vector fields among \( Y_1, \ldots, Y_l \), let us say without loss of generality that these are \( Y_1, \ldots, Y_p \), whose coefficients allow us to determine a first-order system of PDEs given by (4.3). Then, the vector fields

\[
\frac{\partial}{\partial x^j} + \sum_{\alpha=1}^q \sum_{i=1}^p (\xi^{-1})^i_k(x,u)\varphi_{ij}^\alpha \frac{\partial}{\partial u^\alpha} = \sum_{k=1}^p (\xi^{-1})^k_j Y_k, \quad j = 1, \ldots, p
\]

span the same distribution as the vector fields of \( L \). \( \Box \)

**Theorem 4.5.** Let \( L \) be a linear space of vector fields on \( X \times U \). Then, \( S^n_L \), with \( n > 1 \), is a smooth section of the bundle \( \pi^n_0 : J^n \to J^0 \) if and only if the vector fields of \( L \) span an involutive distribution \( D^L \) of rank \( p \) projecting onto \( TX \) under \( \pi^0_{n-1} \).

**Proof.** Let us prove the converse part of the theorem by induction relative to \( n \). Assume that \( Y_1, \ldots, Y_l \) given by (4.2) is a basis of the linear space \( L \). In view of Theorem 4.3 and the considered assumptions, \( S^1_L \) is a section of \( \pi^1_0 \) and then \( u^a_j = \phi^a_j(x,u) \) for certain functions \( \phi^a_j \) on \( S^1_L \), with \( \alpha = 1, \ldots, q \) and \( j = 1, \ldots, p \). Let us prove that if the \( u^0_o \), where \( O \) is any multi-index with \( |O| \leq h \leq n - 2 \) for a natural number \( h \), can be written as functions depending on \( x, u \) only on \( S^1_L \), then the \( u^0_o \) for \( |K| = h + 1 \) do also. Recall that one can ensure on \( S^1_L \) that

\[
0 = u_{j^\alpha}Y_j = D_O \left( \varphi^\alpha - \sum_{i=1}^p \xi^i_j u^a_i \right), \quad \alpha = 1, \ldots, q, \quad j = 1, \ldots, l, \quad |O| \leq h. \tag{4.4}
\]

By our induction hypothesis, the coordinates \( u^0_o \) of points of \( S^n_L \) for \( |O| \leq h \) can be written as functions of \( x, u \) only, namely \( u^0_o = u^0_o(x,u) \). Hence, rewritting the right-hand side of (4.4), we obtain that

\[
0 = -\sum_{i=1}^p u^0_{i,j} \xi^i_j(x,u) + F^0_{ij}(x,u), \tag{4.5}
\]

for certain functions \( F^0_{ij}(x,u) \) with \( \alpha = 1, \ldots, q, j = 1, \ldots, l \), and \( |O| \leq h \) that gather all the terms of \( D_O(\varphi^\alpha - \sum_{i=1}^p \xi^i_j u^a_i) \) with derivatives of the \( u^a \) up to order \( h \). Since \( S^1_L \) is a section of \( \pi^1_0 \), Theorem 4.3 ensures that there exists \( p \) elements of the basis of \( L \), which are assumed without loss of generality to be the first \( p \) ones, such that their coordinate functions \( \xi^i_j \), for \( i, j = 1, \ldots, p \), are the entries of an invertible matrix \( \xi \). Then, the expressions (4.5) show...
ensures that the vector fields of points of \( S^i_L \) are determined by a system of algebraic equations depending only on \( x, u \). Hence, if the system (4.5) admits a solution, then it is unique.

Let us prove that the system (4.5) is compatible. If \( \dim X = 1 \), this is obvious as the expressions (4.5) determine each derivative of \( u^a \) in terms of the unique independent variable uniquely. If \( \dim X > 1 \), recall that \( u^a_{i,j} = u^a_{j,i} \) with \( i \neq j \) but the expressions for \( u_{i,j} \) and \( u_{j,i} \) obtained from (4.5) may be different values at a point \((x, u)\). To prove that system (4.5) is compatible, recall that the elements of \( L \) span an involutive distribution. Hence, Lemma 4.4 ensures that the vector fields \( Z_j \) span an involutive distribution also and the associated first-order system of PDEs in normal form induced by \( L \), let us say (2.5), admits a local solution \( u(x) \) for every initial condition \((x_0, u_0)\). On the sections \( j^\sigma \), for \( \sigma(x) = (x, u(x)) \), the characteristics of all \( Y_1, \ldots, Y_l \), and their total derivatives vanish identically. Hence, the pull-back of expressions (4.5) relative to \( j^\sigma \) are zero and the system (4.5) is compatible. By the induction hypothesis, \( S^0_L \) is a section relative to \( \pi^0_0 \).

Let us prove the direct part of the theorem. If \( S^0_L \) is a section of \( \pi^0_0 \), then it is immediate that \( S^1_L \) is a section of \( \pi^1_0 \) and, using Theorem 4.3, we obtain that \( D_L \) has rank \( p \) and projects onto \( TX \) under \( \pi^0_{\otimes} \). It is left to prove that \( D_L \) is involutive. If \( \dim X = 1 \), the result is immediate. If \( \dim X > 1 \), we can use the same arguments as in the proof of Theorem 4.3 to obtain that the first-order system of PDEs associated with \( S^1_L \) reads (4.3). In view of Lemma 4.1, this system is given by the zeros of the characteristics \( \sum_{i=1}^q (\xi^{-1})^i_k \varphi^\alpha_i - u^\alpha_k = \sum_{j=1}^p (\xi^{-1})^j_k \varphi^\alpha_j + \theta^\alpha = \sum_{j=1}^p \sum_{i=1}^q (\xi^{-1})^j_k \varphi^\alpha_j \theta^\alpha = 0 \) for \( k = 1, \ldots, p \) and \( \alpha = 1, \ldots, q \). Since \( S^1_L \) is a section, one has that the equations \( D_{ij} \theta^\alpha \) and \( D_{ij} \varphi^\alpha \), with \( i \neq j \), must lead to a unique solution to \( u^\alpha_{i,j} \) for every \( x, u \). In terms of the Lemma 4.1, one has that \( D_1 (\sum_{j=1}^p (\xi^{-1})^j_k \varphi^\alpha_j - u^\alpha_k) = D_k (\sum_{j=1}^p (\xi^{-1})^j_k \varphi^\alpha_j - u^\alpha_j) \), which implies that the vector fields

\[
\frac{\partial}{\partial x^j} + \sum_{\alpha=1}^q \sum_{j=1}^p (\xi^{-1})^j_k (x, u) \varphi_j^\beta \frac{\partial}{\partial u^\alpha}, \quad j = 1, \ldots, p
\]

commute among themselves. In view of Lemma 4.4, the vector fields of \( L \) span are a linear combination with functions of \( C^\infty(U \times X) \), it follows that they span an involutive distribution.

\[\square\]

**Remark 4.6.** The proof of Theorem 4.5 shows that the equations (4.5) determine the value each coordinate \( u_{K}^\alpha \), with \( |K| \geq 1 \), of points of a section \( S^1_L \) of \( \pi^0_0 \) with \( n = 2, 3, 4, \ldots \), in terms of the \( u^1_0, \ldots, u^p_0 \) with \( |O| < |K| \). Thus, two integrable sections \( S^1_L \) and \( S^1_{L'} \), with \( n > 1 \), sharing the same projection onto \( J^1 \) are the same. Even more, solutions to the system of PDEs \( S^0_L \), with \( n > 1 \), must be holonomic sections. Additionally, Proposition 5.1 will permit us to determine the necessary and sufficient conditions on \( L \) and \( L' \) to ensure that their characteristic systems are sections of \( \pi^0_0 \) such that \( S^0_L = S^0_{L'} \).

### 5. On the geometry of characteristic systems

It is frequently assumed in the Clairin theory of conditional symmetries that \( L \) is an Abelian Lie algebra of dimension \( \dim X \) admitting a basis of a particular type \([10, 18, 19]\).
This ensures that $S^n_L$ is a section of $\pi_0^n$ amounting to an integrable system of PDEs and the vector fields of $L$ are tangent to $S^n_L$. This section provides necessary and sufficient conditions on $L$ to ensure previous results. More particularly, we will find that the conditions appearing in \cite{18, 19} can be significantly relaxed.

**Proposition 5.1.** Let $L$ and $L'$ be linear spaces of vector fields on $X \times U$ whose characteristic systems in $J^n$ are sections of $\pi_0^n$. Then, $S^n_L = S^n_{L'}$ if and only if the vector fields of $L$ and $L'$ span the same distribution.

**Proof.** Let us prove the direct part. Since $S^n_L$ is a section of $\pi_0^n$, then the projection of $S^n_L = S^n_{L'}$, to $J^1$ via $\pi^1_0$ give rise to a section of $\pi^1_0$. Hence, $L$ and $L'$ determine the same first-order system of PDEs in normal form, let us say $(Q)$, respectively, coincide with the one generated by $(2.6)$. Hence, $D^L = D^{L'}$.

Conversely, if $L$ and $L'$ span the same distribution and since $S^n_L$ and $S^n_{L'}$ are sections of $\pi_0^n$, then the ranks of $D^L$, $D^{L'}$ are in virtue of Theorems 4.3 and 4.5 equal to $p$ and there exists a family of functions $\xi^i$ on $J^0$ giving rise to an invertible $p \times p$ matrix $\xi$ mapping $p$ elements of $L$ spanning $D^L$ onto $p$ elements of $L'$ spanning $D^{L'}$. In consequence, the characteristics of the selected elements of $L$ and $L'$, let us say $Q^i_j$ and $(Q')^i_j$, with $\alpha = 1, \ldots, q, j = 1, \ldots, p$, satisfy that $Q^i_j = \sum_{k=1}^p \xi^k_j (Q')^i_k$, with $i, j = 1, \ldots, p$ and $\alpha = 1, \ldots, q$. From this, it follows that $D_K Q^i_j = \sum_{k=1}^p \xi^k_j D_K (Q')^i_k$ on $S^n_L$, for every multi-index $K$ such that $|K| \leq n - 1$. In view of Lemma 4.1, the same can be extended to arbitrary elements of $L$. Hence, $S^n_L \subset S^n_{L'}$. Since $\xi$ can be inverted and repeating mutatis mutandis the above procedure considering that $(Q')^i_j = \sum_{k=1}^p (\xi^{-1})^k_j Q^i_k$, one has that $S^n_L \subset S^n_{L'}$ and $S^n_L = S^n_{L'}$. \hfill $\square$

**Lemma 5.2.** If the vector fields $Z_1, \ldots, Z_p$ given by (2.6) span an involutive distribution on $J^1$, then $L = \langle Z_1, \ldots, Z_p \rangle$ is an Abelian Lie algebra.

**Proof.** By the involutivity assumption on the distribution spanned by the elements of $L$, we have that

\[
[Z_j, Z_k] = \sum_{m=1}^p f^m_{jk} Z_m, \quad j, k = 1, \ldots, p,
\]

for certain functions $f^m_{jk} \in C^\infty (X \times U)$ with $j, k, m = 1, \ldots, p$. Since the left-hand side projects onto zero relative to $\pi^{-1}_{n-1}$, the right-hand side does also. Hence,

\[
\sum_{m=1}^p f^m_{jk}(x, u) \frac{\partial}{\partial x^m} = 0
\]

for every $(x, u) \in X \times U$. Therefore, $f^m_{jk} = 0$ for all possible indexes $j, k, m$ and the vector fields $Z_1, \ldots, Z_p$ are in involution. \hfill $\square$

**Theorem 5.3.** Let $S^n_L$ be a section of $\pi_0^n$. The prolongations to $J^n$ of the elements of $L$ are tangent to $S^n_L$ if and only if the distribution $D^L$ spanned by the elements of $L$ is involutive.
Proof. Assume first that $D^L$ is involutive. The prolongations to $J^n$ of the elements of $L$ are tangent to $S^n_L$ if and only if the functions $i_Z \theta$, where $Y$ is any element of $L$ and $\theta$ is any contact form on $J^n$, are first-integrals of any $j^n Z$ with $Z \in L$. Now,

$$j^n Z(i_Z \theta) = i_Z [L^n_Z \theta] + i_Z [Z,Y] \theta. \quad (5.1)$$

Let us analyse the both right-hand terms on $S^n_L$. First, as $j^n Z$ is a prolongation to $J^n$ of $Z$, one has that $L^n_Z \theta$ is a contact form and, by the definition of $S^n_L$, one has

$$i_Z [L^n_Z \theta]|_{S^n_L} = 0. \quad (5.2)$$

Second, by assumption $S^n_L$ is a section of $\pi^n_0$ and Theorem 4.5 states that $D^L$ has order $p$. Since $D^L$ is assumed to be involutive, we have $[Z,Y] = \sum_{k=1}^p f_k Y_k$ for functions $f^1, \ldots, f^p \in C^\infty(J^0)$ and some elements $Y_1, \ldots, Y_p$ with $Y_1 \wedge \ldots \wedge Y_p \neq 0$ chosen from a basis of $L$. In view of Lemma 4.1, one gets that

$$i_Z [Z,Y] \theta |_{S^n_L} = i_Z [\sum_{k=1}^p f_k Y_k] \theta |_{S^n_L} = 0. \quad (5.3)$$

Using (5.2) and (5.3) in (5.1), we get that $j^n Z [i_Z \theta]|_{S^n_L} = 0$. Consequently, the prolongations to $J^n$ of the elements of $L$ are tangent to $S^n_L$.

We now prove the converse by contradiction. Assume that $D^L$ is not involutive and the prolongations to $J^n$ of the elements of $L$ are tangent to $S^n_L$. Then, there exist $Z, Y \in L$ such that $[Z,Y]$ does not take values in $D^L$. Thus, the elements of the linear space $L' = L \oplus ([Z,Y])$ span a distribution $D^{L'}$ of rank $p+1$. Since by assumption $j^n Z [i_Z \theta]|_{S^n_L} = 0$ for any contact form on $J^n$, equality (5.1) shows that $i_Z [Z,Y] \theta |_{S^n_L} = 0$, and $S^n_L$ contains $S^n_L$, namely $S^n_L \subset S^n_L$. Since $L \subset L'$, one has that $S^n_L \subset S^n_L$ and therefore $S^n_L = S^n_L$. Since $S^n_L$ is a section, Theorems 4.5 and 4.3 state that $D^{L'}$ has rank $p$. This is a contradiction and $D^L$ must be involutive. \hfill \Box

The theory of conditional symmetries focuses on the case where $L$ is a Lie algebra. Then, $D^L$ is involutive [32] and, consequently, an immediate consequence of Theorem 5.3 is the following trivial corollary. As we do not assume $S^n_L$ to be a section of $\pi^n_0$, the corollary can be applied to general conditional symmetries that need not be of Clairain type [28].

Corollary 5.4. If $L$ is a Lie algebra of vector fields on $X \times U$, then the prolongations to $J^n$ of the elements of $L$ are tangent to $S^n_L$.

Any section of $\pi^n_0$, say $\sigma : (x,u) \in X \times U \mapsto (x^i, u^\alpha, \phi_1^\alpha(x,u), \ldots, \phi_{K_j-1}^\alpha(x,u), \phi_{K_j}^\alpha(x,u)) \in J^n$, where $|K_j| = j$ for $j = 2, \ldots, n$, leads to an $n$-order system of PDEs in normal form

$$\frac{\partial u^\alpha}{\partial x^i} = \phi_1^\alpha(x,u), \ldots, \frac{\partial^j u^\alpha}{\partial x^{K_j}} = \phi_{K_j}^\alpha(x,u), \quad \alpha = 1, \ldots, q, \quad i = 1, \ldots, p, \quad 1 < j \leq n. \quad (5.4)$$

and vice versa. This fact is frequently employed in the theory of conditional symmetries, where $S^n_L$ is constructed in such a way that it is a section of $\pi^n_0$ and, consequently, amounts to a system of PDEs in normal form (cf. [18, 19, 31]).

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From now on we assume that all characteristic systems $S^n_L$ are sections of $\pi^0_n$. Theorem 4.5 ensures then that the distribution $\mathcal{D}^L$ spanned by the elements of $L$ has rank $p$ and its projection to $X$ (via $\pi^0_{n-1}$) is $TX$. Our main aim until the end of this section is to show that the system of PDEs related to $S^n_L$ is locally solvable if and only if $\mathcal{D}^L$ is an involutive distribution.

**Proposition 5.5.** Let $L$ be a linear space of vector fields whose $S^n_L$ is a section of $\pi^0_n$. Then, $S^n_L$ is locally solvable if and only if the vector fields of $L$ span an involutive distribution $\mathcal{D}^L$.

**Proof.** Assume first that $\mathcal{D}^L$ is involutive. Theorem 5.3 ensures that $\mathcal{D}^L$ is tangent to $S^n_L$. Given any point $j^0_n\sigma \in S^n_L$, there exists a unique integral submanifold $\sigma^{(n)}$ of $\mathcal{D}^L$ passing through this point. Since $\mathcal{D}^L$ projects onto $TX$, this can be considered as a section of $\pi^0_n$. As the elements of $\mathcal{D}^L$ are tangent to $S^n_L$, then $\sigma^{(n)}$ is contained in $S^n_L$. Due to Remark 4.6, the section $\sigma^{(n)}$ is holonomic. Hence, it gives rise to a solution of $S^n_L$, which becomes a locally solvable system of PDEs.

The projection of $S^n_L$ to $J^1$ is given by the points of $J^1$ satisfying that the characteristics of elements of $L$ vanish. This is the condition characterising $S^1_L$. Hence, the projection of $S^n_L$ to $J^1$ is $S^1_L$. If $S^n_L$ is locally solvable, then its projection to $J^1$ via $\pi^0_n$, namely $S^1_L$, is also locally solvable and, in view of Proposition 2.1, the zero curvature condition for this system is satisfied. The vector fields $Z_1, \ldots, Z_p$ of the form (2.6) describing $S^1_L$ and those of $L$ span the same distribution in virtue of Lemma 4.4. Then, one has that $\mathcal{D}^L$ is involutive.

The above proposition justifies the following definition.

**Definition 5.6.** A section $\hat{\sigma}$ of $\pi^0_n$ is integrable when its associated system of PDEs is locally solvable.

Finally, let us give a rather immediate consequence of Proposition 5.5 that it will allow us to simplify the application of the Clairin theory of conditional symmetries.

**Corollary 5.7.** If $S^n_L$ is integrable, then every initial condition admits a unique particular solution. Moreover, the solutions of $S^n_L$ and $S^1_L$ are the same.

6. Generalising a standard assumption in the theory of conditional symmetries

The Clairin theory of conditional symmetries [10] mainly focuses on the case where $L$, which is aimed at describing Lie algebras of conditional symmetries of a particular type, is a $p$-dimensional Lie algebra $L$ of vector fields admitting a basis of a particular type [10, 18, 19]. As far as we know no work deals with more general types of $L$. This section is aimed at characterising this type of Lie algebras. We will also define a larger family of Lie algebras whose properties will make them more useful to study Lie algebras of conditional symmetries, as witnessed in following sections.

**Proposition 6.1.** A Lie algebra $L$ of vector fields on $X \times U$ admits a basis of the form (2.6) if and only if $L$ is Abelian, $p$-dimensional, and the distribution $\mathcal{D}^L$ generated by the elements of $L$ projects onto $TX$ via $\pi^0_{n-1}$. 

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Proof. Let us prove the direct part of the proposition. If L admits a basis $Z_1, \ldots, Z_p$ given by (2.6), then $Z_1, \ldots, Z_p$, and therefore all the elements of L, are projectable onto X. The projections of $Z_1, \ldots, Z_p$ are $\partial/\partial x^k$, with $k = 1, \ldots, p$, and therefore $D^L$ projects onto $TX$ via $\pi_{0-1}^0$.

Let us now prove the converse. If L is a Lie algebra of vector fields projectable onto X, then the elements of a basis $X_1, \ldots, X_p$ of L are also projectable. Since L is Abelian, the projections span an Abelian Lie algebra. Since $D^L$ projects onto $TX$ under $\pi_{0-1}^0$, the projections $\pi_{0-1}^0 X_j$, with $j = 1, \ldots, p$, span de distribution $TX$ and commute among themselves. The Frobenius Theorem ensures that there exists a coordinate system $x^1, \ldots, x^p$ on X such that

$$\pi_{0-1}^0 X_j = \frac{\partial}{\partial x^j}, \quad j = 1, \ldots, p.$$  

Adding to the coordinates $x^1, \ldots, x^p$ a new set of coordinates to form a coordinate system on $X \times U$, we obtain that the vector fields $X_1, \ldots, X_p$ can be written in the form

$$X_j = \frac{\partial}{\partial x^j} + \sum_{\alpha=1}^q \phi_j^\alpha \frac{\partial}{\partial u^\alpha}, \quad j = 1, \ldots, p,$$

for certain functions $\phi_j^\alpha \in C^\infty(J^0)$, with $\alpha = 1, \ldots, q$ and $j = 1, \ldots, p$. This finishes the converse part of our proposition. 

Due to its appearance in the literature (cf. [18, 19, 28]) and in this work, the Lie algebras studied in the above proposition deserve a special denomination.

Definition 6.2. A normal PDE Lie algebra is a $p$-dimensional Abelian Lie algebra of vector fields on $X \times U$ whose projections onto X span $TX$.

We aim to show in following sections that one can enlarge significatively the Clairin theory of conditional symmetries by considering Lie algebras $L$ of conditional symmetries giving a section $S^n_L$ of $\pi_{0-1}^n$ such that there exists normal PDE Lie algebra $L'$ satisfying that $S^n_L = S^n_{L'}$. The following lemma is a key to accomplish this goal.

Lemma 6.3. Let $L$ be a linear space of vector fields on $X \times U$ whose $S^n_L$ is an integrable section of $\pi_{0-1}^n$ and such that the distribution $D^L$ has rank $p$ and projects onto $TX$ under $\pi_{0-1}^0$. Then, there exists a normal PDE Lie algebra $L'$ such that $S^n_L = S^n_{L'}$.

Proof. Under the considered assumptions on $D^L$, one has that L admits a family $Y_1, \ldots, Y_p$ of elements spanning $D^L$ such that $Y_1 \wedge \ldots \wedge Y_p \neq 0$. Assume that the vector fields $Y_1, \ldots, Y_p$ take the form (4.2). Since $D^L$ projects onto $TX$ onto X (under $\pi_{0-1}^0$), one has that the functions $\xi^j_k$ for $j, k = 1, \ldots, p$ are the coefficients of an invertible $p \times p$ matrix $\xi$. Since $\xi$ is invertible, one can define a new family of linearly independent vector fields

$$Z_k = \sum_{j=1}^p (\xi^{-1})^j_k Y_j = \frac{\partial}{\partial x^k} + \sum_{j=1}^p \sum_{\alpha=1}^q (\xi^{-1})^j_k \phi_j^\alpha \frac{\partial}{\partial u^\alpha}, \quad k = 1, \ldots, p.$$  

(6.1)
Since $\mathcal{D}^L$ coincides with the distribution spanned by the elements $L' = \langle Z_1, \ldots, Z_p \rangle$, Lemma 5.2 shows that $L'$ is an involutive Lie algebra and $L'$ becomes a normal PDE Lie algebra. Since $\mathcal{D}^L = \mathcal{D}'$, Proposition 5.1 ensures that $S^0_L = S^n_L$. \hfill \blacksquare

Lemma 6.3 and other results obtained in following sections will allow to extend findings concerning normal PDE Lie algebras $L'$ of conditional symmetries to more general Lie algebras $L$. This justifies to introduce the following definition.

**Definition 6.4.** A renormalizable linear space of vector fields $L$ is a linear space of vector fields on $X \times U$ whose characteristic system is equal to the characteristic system of a normal PDE Lie algebra $L'$. Then, $L'$ is called an associated normal PDE Lie algebra of $L$.

The following proposition determines straightforwardly in terms of the distribution $\mathcal{D}^L$ when $L$ is a renormalizable linear space of vector fields.

**Proposition 6.5.** A linear space of vector fields $L$ is renormalizable if and only if the distribution $\mathcal{D}^L$ is involutive of rank $p$ and its projection, via $\pi^0_{-1}$, is $TX$.

**Proof.** Assume that $L$ is renormalizable. By Definition 6.4 and Proposition 5.1, one has that $L$ and the normal PDE Lie algebra $L'$ span the same distribution $\mathcal{D}$. In view of Proposition 6.1, the distribution $\mathcal{D}$ is $p$-dimensional and projects onto $TX$ via $\pi^0_{-1}$.

Let us prove the inverse. If $\mathcal{D}^L$ projects onto a distribution $TX$, then there must exist $p$ vectors on $T_{(x,u)}(X \times U)$, for arbitrary $(x, u) \in X \times U$, projecting onto $\partial/\partial x^k$ for $k = 1, \ldots, p$, respectively. Therefore, $\mathcal{D}^L$ admits $p$ vector fields of the form

$$Z_k = \frac{\partial}{\partial x^k} + \sum_{\alpha=1}^q \psi^k_\alpha(x, u) \frac{\partial}{\partial u^\alpha}, \quad k = 1, \ldots, p.$$  

Since $\mathcal{D}^L$ has rank $p$, then the elements of $L' = \langle Z_1, \ldots, Z_p \rangle$ span the distribution $\mathcal{D}^L$ and since $\mathcal{D}^L$ is involutive, one has in view of Lemma 5.2 that $L'$ is $p$-dimensional and Abelian. In consequence, $L$ is renormalizable. \hfill \blacksquare

The following corollary is an immediate consequence of previous results that will be useful in remaining sections of our work.

**Corollary 6.6.** If $L$ is a renormalizable family of vector fields, then $S^n_L$ is an integral section of $\pi^0_0$, the prolongations of the elements of $L$ to $J^n$ span the tangent space to $S^n_L$, and $S^n_L$ is a locally solvable differential equation.

**7. Conditional symmetries**

This section introduces the theory of conditional symmetries with special enfasis on the Clairin approach. Next, we use the properties of the submanifolds $S^n_L$ obtained in previous sections to investigate Lie algebra of conditional symmetries of a system of PDEs $S_\Delta$. Recall that our aim is to restrict the analysis of the Lie algebras of Lie symmetries of $S_\Delta$ to determining the Lie algebras of symmetries of a system of PDEs described by $S^n_L \cap S_\Delta$
where $S^n_L$ must be an integrable section of $\pi^n_0$, namely a locally solvable $n$-order system of PDEs.

In general, $S^n_L \cap S_\Delta$ does not need to be a manifold and it can even be empty, e.g. the problem on $X = \mathbb{R}^2, U = \mathbb{R}$ given by

$$\Delta^1 = u_x, \quad \Delta^2 = u_t, \quad L = \langle \partial_x + \partial_u, \partial_t + u\partial_u \rangle,$$

gives rise to an empty set $S^1_L \cap S_\Delta$. Hence, the corresponding system of PDEs has no solutions. This problem is not commented in applications, which are indeed focused on those cases where $S^n_L \cap S_\Delta$ is a manifold and it admits particular solutions of physical relevance [8, 10, 18, 19].

We hereafter assume that $S^n_L$ and $S_\Delta$ are non-vacuously transversal, i.e. $S^n_L \cap S_\Delta$ is not empty and at every point $j^n_x \sigma \in S^n_L \cap S_\Delta$ one has that $T_{j^n_x \sigma} S^n_L + T_{j^n_x \sigma} S_\Delta = T_{j^n_x \sigma} J^n$. This turns $S^n_L \cap S_\Delta$ into a submanifold of $J^n$ [1]. As illustrated by examples in this work and other previous ones [18, 19], this assumption seems to be reasonable. In particular, $\dim S_\Delta = \dim J^n - s$ and $\dim S^n_L = q$ with $s \leq q$ for the cases of interest in the Clairin theory of conditional symmetries.

The following notion of a Lie algebra of conditional symmetries represents an intrinsic formulation of the definition given in [19, Definition 2.2].

**Definition 7.1.** A Lie algebra of conditional symmetries of an $n$-order system of PDEs $S_\Delta \subset J^n$ is a Lie algebra $L$ of vector fields on $X \times U$ such that $j^n Z$ is tangent to $S_\Delta \cap S^n_L$ for every $Z \in L$.

As we did not assume that $D^L$ projects onto $TX$ or its rank is $p$, the previous definition does not need to give rise to a section $S^n_L$. As a consequence, this definition is more general than the Clairin approach and it covers more general Lie algebras of conditional symmetries [18, 19, 44]. Relevantly, our definition is purely geometrical and it does not rely on coordinates expressions of characteristics or basis of $L$ [44].

Let us comment on the definition of a Lie algebra of conditional symmetries and other related notions.

We say that a vector field $Y$ is a conditional Lie symmetry of the system of PDEs given by $S_\Delta$ if $L = \langle Y \rangle$ is a Lie algebra of conditional symmetries of $S_\Delta$. This amounts to saying that $Y$ be a nonclassical infinitesimal symmetry in the terminology employed in [31].

Note that if $Y = \sum_{i=1}^p \xi^i \partial/\partial x^i + \sum_{\alpha=1}^q \varphi^\alpha \partial/\partial u^\alpha$ on $X \times N$, then

$$\langle j^n Y \rangle \Delta^\mu = 0, \quad \mu = 1, \ldots, s,$$

and Corollary 5.4 ensures that $j^n Y$ is also tangent to $S^n_L$. Consequently, $j^n Y$ is tangent to $S^n_L \cap S_\Delta$ and $Y$ becomes a conditional Lie symmetry of $S_\Delta$. Therefore, Lie point symmetries of $S_\Delta$ induce a conditional Lie symmetry. Nevertheless, a conditional Lie symmetry does not necessarily give rise to a Lie point symmetry. In particular, there exist systems of PDEs that do not admit any Lie point symmetry while admitting conditional Lie symmetries (cf. [25] in references therein).
Many Lie algebras of conditional symmetries in the literature \[18, 19\] admit a basis of the form (2.6). Nevertheless, we want to study the theory of Lie algebras of conditional symmetries given by renormalizable Lie algebras of vector fields. To start with, the following theorem shows that a renormalizable PDE Lie algebra of vector fields consists of conditional Lie symmetries if and only if an associated normal PDE Lie algebra is a Lie algebra of conditional Lie symmetries.

**Theorem 7.2.** A renormalizable Lie algebra of vector fields \(L\) is a Lie algebra of conditional symmetries of \(S_\Delta\) if and only if an associated normal Lie algebra of vector fields \(L'\) is a Lie algebra of conditional symmetries.

**Proof.** Let us prove the direct part of the theorem. By definition, if \(L\) is a renormalizable PDE Lie algebra of vector fields, then there exists a normal PDE Lie algebra \(L'\) such that \(S^n_L = S^n_{L'}\). Let \(Y_1, \ldots, Y_l\) be a basis of \(L\) and let \(Z_1, \ldots, Z_p\) be a basis of \(L'\). Hence, to prove that \(L\) is a Lie algebra of conditional Lie symmetries if and only if an \(L'\) is, it is enough to show that

\[
(j^n Z_k)\Delta|_{S^n_L \cap S_\Delta} = 0, \quad k = 1, \ldots, p \iff (j^n Y_k)\Delta|_{S^n_L \cap S_\Delta} = 0, \quad k = 1, \ldots, l,
\]

Since \(L'\) is a normal PDE Lie algebra associated with \(L\), they span in virtue of Proposition 5.1 the same distribution, namely \(D^L = D^{L'}\), and \(Y_j = \sum_{k=1}^{p} \xi_{j}^{k} Z_k\) for \(j = 1, \ldots, l\) and some functions \(\xi_{j}^{k} \in C^\infty(J^0)\) with \(k = 1, \ldots, p, j = 1, \ldots, l\). Lemma 4.1 shows that \(j^n Y_j = \sum_{k=1}^{p} \xi_{j}^{k} j^n Z_k\) on \(S^n_L\). Hence, the vector fields \(j^n Y_j\) vanish on \(\Delta^n\) over \(S^n_L \cap S_\Delta\) if the \(Z_1, \ldots, Z_p\) do. The converse is analogous. \(\square\)

Consequently, the obtention of conditional symmetries for a renormalizable PDE Lie algebra reduces to studying a normal PDE Lie algebra of conditional symmetries. Conversely, the knowledge of a normal PDE Lie systems algebra allows us to generate many other renormalizable PDE families of vector fields. This fact has been overlooked in the literature \[18, 19, 28\] and it will be exemplified in Section 9, where we apply our techniques to physically motivated examples. In fact, this is a generalization of the cases studied in above-mentioned works.

Let us know study the determination of normal PDE Lie algebras of conditional symmetries. In this case, the system of PDEs associated with \(S^n_L\) is given, in an appropriate coordinate system where a basis of \(L\) takes the form (2.6), by

\[
\frac{\partial u^{\alpha}}{\partial x^i} = \phi_{i}^{\alpha}(x, u), \quad i = 1, \ldots, p, \quad \alpha = 1, \ldots, q, \quad (7.1)
\]

and all the total derivatives of this system with respect to the \(D_K\) with \(|K| \leq n - 1\). Recall that in view of Proposition 2.1 the zero curvature condition for (7.1), which in coordinates read (2.7), amounts to the fact that \(L\) is Abelian. From a practical point of view, this condition and the fact that \(S^n_L\) is locally solvable (due to Corollary 6.6) ensure that every solution to (7.1) gives rise to a solution of \(S^n_L\) and vice versa. For this reason, the space of solutions of (7.1) and the \(n\)-order system of PDEs \(S^n_L\) is the same and one can restrict
oneself to write down (7.1) in applications. This is the approach accomplished in works on conditional symmetries [10, 18, 19]. The solutions to the system $S_\Delta \cap S_L^n$ are therefore given by the system of PDEs of the form

$$\Delta^\mu(x^i, u^\alpha, u_{K\mu}^\alpha) = 0, \quad \frac{\partial u^\alpha}{\partial x^i} = \phi_i^\alpha(x, u), \quad i = 1, \ldots, p, \quad \alpha = 1, \ldots, q, \quad \mu = 1, \ldots, s. \quad (7.2)$$

If $L$ is a Lie algebra of conditional symmetries of $S_\Delta$, then one has the additional condition

$$j^n Z \Delta^\mu|_{S_\Delta \cap S_L} = 0, \quad \mu = 1, \ldots, s, \quad \forall Z \in L.$$

The description of these conditions in coordinates are frequently called the invariant surface conditions (see [2]).

Consequently, the obtenation of a normal Lie algebra of conditional symmetries requires solving of a nonlinear system of PDEs, which is in general much more complicated than the standard system of linear PDEs employed to obtain classical Lie point symmetries [39]. Nevertheless, recall that this method is applied indeed when standard Lie point symmetries do not provide enough information about the system of PDEs under study. Moreover, the conditional symmetry approach may offer Lie symmetries of particular families of solutions of the original system of PDEs, which may not be related to Lie symmetries of the general solutions [18], and the DCs can be used to obtain particular solutions of the initial system of PDEs, which is a topic to be studied in Section 2.1.

The following theorem allows us to simplify the system of PDEs necessary to obtain Lie algebras of conditional symmetries of $S_\Delta$ in the Clairin approach.

**Theorem 7.3.** Let $L$ be a renormalizable Lie algebra of vector fields and let $L'$ be an associated normal PDE Lie family with a basis $Z_k = \partial_k + \sum_{\alpha=1}^q \phi_k^\alpha \partial_\alpha$, where $k = 1, \ldots, p$. Consider the $n$-order system of PDEs, $S_\Delta$, described by the zeroes of the functions $\Delta^\mu \in C^\infty(J^n)$, with $\mu = 1, \ldots, s$. If $\Delta^\mu(x^i, u^\alpha, D_K \phi_i^\alpha) = 0$, with $\mu = 1, \ldots, s$ and $|K| \leq n - 1$, then $L$ is a Lie algebra of conditional Lie symmetries of $S_\Delta$.

**Proof.** Theorem 7.2 shows that $L$ is a Lie algebra of conditional symmetries of $S_\Delta$ if and only if $L' = \langle Z_1, \ldots, Z_p \rangle$ is a normal PDE Lie algebra of conditional symmetries of $S_\Delta$.

Let us prove that $L'$ is a Lie algebra of conditional symmetries if satisfies the conditions of our theorem. Take a point $j^n \sigma \in S_L^n \cap S_\Delta$. Since $L'$ is a normal PDE Lie algebra, Corollary 6.6 states that $S_L^n$ is a locally solvable $n$-order system of PDEs, every $j^n Z$, with $Z \in L'$, is tangent to $S_L^n$, and the $j^n Z$ is tangent to $S_L^n$. Since $j^n Z$ is the prolongation to $J^n$ of a vector field on $X \times U$, one has that the one-parametric group of diffeomorphisms $\phi_i^{(n)}$ of $j^n Z$ maps holonomic sections on $J^n$ into holonomic sections. The locally solvability of $S_L^n$ allows us to ensure that there exists a solution $u(x)$ of $S_L^n$ such that $j^n \sigma$, with $\sigma(x) = (x, u(x))$, passes through $j^n \sigma$. Since $j^n Z$, for $Z \in L'$ is tangent to $S_L^n$, one has that $\phi_i^{(n)} j^n \sigma = j^n \sigma, \quad \sigma_t(x) = (x, u_t(x))$ and the $u_t(x)$ are solutions to $S_L^n$. Hence, we can write

$$j^n \sigma_t = (x^i, u_t^\alpha(x), D_K \phi_i^\alpha(x, u_t(x))), \quad |K| \leq n - 1.$$
By assumptions of our present, \( \Delta^\mu(j^n \sigma_i) = 0 \) for \( \mu = 1, \ldots, s \). Hence

\[
0 = \left. \frac{d}{dt} \right|_{t=0} \Delta^\mu(j^n \sigma_i) = j^n Z \Delta^\mu(j^n \sigma) = 0, \quad \mu = 1, \ldots, s.
\]

Consequently, \( j^n Z \) is tangent to the intersection \( S_{L'} \cap S_\Delta \) and \( L \) becomes a Lie algebra of conditional symmetries.

8. Conditional symmetries and PDE Lie systems

The study of conditional symmetries demands solving a nonlinear system of PDEs (7.2), which is more complicated than the linear one appearing in the determination of classical Lie point symmetries [39]. As commented previously, the solution of (7.2) is still justified by the fact that we assume that Lie point symmetries are not enough to study the properties of the system of PDEs, \( S_\Delta \), under study.

In this work, we improve the approach initiated in [19] and suggest appropriate assumptions allowing for the description of conditional Lie symmetries through the so-called PDE Lie systems, which provides many techniques to obtain the solutions of such nonlinear PDEs like reduction and integration methods (cf. [5, 7, 34]).

Consider a family of functions on the coordinates of \( u \) that is closed relative to linear combinations, products and derivatives in terms of the coordinates of \( u \). For instance, consider the families

- \( A_1 = \{ \text{Polynomials in the variables } u^\alpha \} \).
- \( A_2 = \{ \text{Polynomials in the variables } e^u^\alpha \} \).
- \( A_3 = \{ \text{Linear combinations in the functions } 1, \cos(nu^\alpha), \sin(nu^\alpha), n \in \mathbb{N} \} \).

Let us consider the family \( A_1 \). Our methods can be accomplished similarly for \( A_2, A_3 \) or other sets of families satisfying commented properties. Assume now that the functions \( \phi_{j\alpha}^a \), which depend on the dependent and independent coordinates, admit a polynomial expansion in the dependent coordinates of \( u \) with coefficients given by functions depending on the independent variables, namely

\[
\phi_{j\alpha}^a = \sum_{\tilde{K}} a_{j\tilde{K}}^a(x) u^{\tilde{K}}, \quad j = 1, \ldots, p \quad \alpha = 1, \ldots, q.
\] (8.1)

where the tilded multi-indexes match the form \( \tilde{K} = (\tilde{k}_1, \ldots, \tilde{k}_q) \) with \( \tilde{k}_i \in \mathbb{N} \cup \{0\} \), we define \( u^{\tilde{K}} = (u^1)^{\tilde{k}_1} \cdots (u^q)^{\tilde{k}_q} \), the \( a_{j\tilde{K}}^a(x) \) are certain \( x \)-dependent functions, and the sum in (8.1) is over arbitrary tilded multi-indexes \( \tilde{K} \). In this case, the system determining conditional symmetries (7.1) reads

\[
\frac{\partial u^\alpha}{\partial x^j} = \sum_{\tilde{K}} a_{j\tilde{K}}^a(x) u^{\tilde{K}}, \quad j = 1, \ldots, p, \quad \alpha = 1, \ldots, q.
\] (8.2)
Instead of assuming $\phi_j^\alpha$ to be a polynomial of second-order degree as in [19] or making assumptions on the reductions of this system of differential equations as in [18], we propose to consider that equation (8.2) gives rise to a general polynomial PDE Lie system [16], namely it can be written in the form of an integrable system (in the sense of satisfying the zero curvature condition)

$$\frac{\partial u^\alpha}{\partial x^j} = \sum_{\beta=1}^r b_j^{\alpha\beta}(x)X_\beta, \quad j = 1, \ldots, p, \quad \alpha = 1, \ldots, q, \quad (8.3)$$

for certain $x$-dependent functions $b_j^{\alpha\beta}(x)$ and vector fields $X_1, \ldots, X_r$ spanning a finite-dimensional Lie algebra of vector fields. The Lie algebra spanned by $X_1, \ldots, X_r$ is called a Vessiot–Guldberg Lie algebra of the PDE Lie system [5, 7]. The Vessiot–Guldberg Lie algebra shows when the PDE Lie system can be straightforwardly integrated [34] or allows for the construction of superposition rules, which enables ones to obtain the general solution of the PDE Lie system from a finite set of particular solutions and a set of constants [7, 34].

If we restrict the expansion of $\phi_j^\alpha$ to the case where its multi-indices satisfy $|K| \leq 2$, then the right-hand side of (8.2) becomes a second-order polynomial, which suggests us to impose conditions on the $a_{jK}^\alpha(x)$ to match the form of the so-called partial differential matrix Riccati equations [5, 6, 13], namely an integrable first-order system of PDEs (in the sense of satisfying the zero curvature condition) of the form

$$\frac{\partial u}{\partial x_j} = A_j(x) + B_j(x)u + uC_j(x) + u^TD_j(x)u, \quad j = 1, \ldots, p, \quad (8.4)$$

where $u$ is considered to be a vertical vector $(u^1, \ldots, u^q) \in \mathbb{R}^q$, the $A_j(x)$ belong to $\mathbb{R}^q$, and $B_j(x), C_j(x), D_j(x)$ are $q \times q$ matrices for every $j = 1, \ldots, p$. In this case, the Vessiot–Guldberg Lie algebra, $V_{MR2}$, is isomorphic to $\mathfrak{sl}(q+1, \mathbb{R})$ (cf. [17, 43]). Partial differential matrix Riccati equations appear, for instance, in the study of the Wess-Zumino-Novikov-Witten (WZNW) equations [5]. They also appear in the study of Bäcklund transformations of relevant systems of PDEs [19].

The previous conditions on the coefficients $a_{jK}^\alpha$ are quite general. If $q = 1$ all PDE Lie systems are such that a change of variables may map the Lie algebra spanned by the $X_\alpha$ onto $(\partial/\partial u, u\partial/\partial u, u^2\partial/\partial u)$ (see [24]). When $q = 2$, not all PDE Lie systems can be mapped by means of a change of variables on $U$ onto a partial differential matrix Riccati equation (cf. [17, Table 4]). For instance, no change of variables on $U$ can map a generic PDE Lie system with a Vessiot–Guldberg Lie algebra of vector fields $V$ into a matrix Riccati equation when $\dim V > \dim V_{MR2}$. The existence of such Lie algebras $V$ appears, for instance, in the classification of Lie algebras of vector fields in the plane [16, 43]. In particular, one can assume that the $\phi_j^\alpha$ are polynomial of any order (cf. [16] or [17, Table 4]). For instance, one may consider the PDE Lie system

$$\frac{\partial u^1}{\partial x^j} = b_j^1(x) + c_j^1(x)u^1 + e_j^1(x)(u^1)^2,$$

$$\frac{\partial u^2}{\partial x^j} = b_j^2(x) + c_j^2(x)u^1 + d_j^2(x)u^2 + e_j^2(x)ru^1u^2 + f_j^{(1)}(x)(u^1)^2 + \ldots + f_j^{(3)}(x)(u^1)^3,$$
with \( j = 1, \ldots, p \) and where the \( x \)-dependent functions are chosen so that the system satisfies the zero curvature condition. This PDE Lie system is related to a Vessiot–Gulddberg Lie algebra of type \( I_{20} \) (cf. [17]). Consequently, there exists no difeomorphism on \( \mathbb{R}^2 \) on the plane mapping this PDE Lie system into a subcase of (8.4) because, for instance, the dimension of \( I_{20} \) is larger than the dimension of \( V_{\text{MR}2} \). Obviously, studying PDE Lie systems more general than partial differential matrix Riccati equations offers new possibilities for the study of conditional symmetries of PDEs.

Once the form of the system (8.2) has been established, it is convenient to look for methods to determine Lie algebras of conditional symmetries. In particular, we first focus on the case when the Lie algebra of conditional symmetries is spanned by

\[
Z_k = \frac{\partial}{\partial x_k} + \sum_{\alpha=1}^q \phi_k^\alpha \frac{\partial}{\partial u^\alpha}, \quad k = 1, \ldots, p.
\]

As already commented, this amounts to the fact that the Lie algebra must be Abelian. Recall also that this latter fact ensures that the differential constraints (7.1) give rise to an integrable first-order system of PDEs that therefore admits solutions for every initial condition \((x_0, u_0)\). In coordinates, this condition amounts to the system of PDEs (2.7) and the substitution of the expansion (8.1) into the \( n \)-order system of PDEs \( \Delta^\mu(x, u, u_K) = 0 \) leads to a series of conditions

\[
\Delta^\mu(x, u^\alpha, D_K \phi_j^\alpha) = 0, \quad \mu = 1, \ldots, s.
\]

More specifically, the substitution of (8.1) in the zero curvature condition (2.7) of the system (8.2) gives

\[
a_{jK,k}^\alpha u^\tilde{K} + \sum_{\beta=1}^q \sum_{\tilde{K}} a_{jK,k}^\alpha u^{\tilde{K}-1\beta} \tilde{K}_\beta \sum_{\tilde{\alpha}} a_{k\tilde{\alpha}}^\beta u^\tilde{\alpha} - a_{kK,j}^\alpha u^\tilde{K} - \sum_{\beta=1}^q \sum_{\tilde{K}} a_{k\tilde{K}}^\alpha u^{\tilde{K}-1\beta} \tilde{K}_\beta \sum_{\tilde{\alpha}} a_{\tilde{\alpha}}^\beta u^\tilde{\alpha} = 0,
\]

(8.5)

where \( \tilde{\alpha} \) is an arbitrary multi-index for the polynomials on \( u \). Taking into account that the above expressions must be zero for every \( u \), we get that the coefficients of the expansion in different polynomials in the variables \( u^\alpha \) of (8.1) must vanish. This in turn can be considered as a first-order system of PDEs on the coefficients. This can be summed up by writing that

\[
E^r(x, a_{jK}^\alpha, a_{jK,k}^\alpha) = 0
\]

for certain \( r \) functions \( E^r \). Additionally, one has to recall that the \( a_{jK}^\alpha \) must still satisfy that

\[
\Delta^\mu \left( x^j, u^\alpha, D_K \left( \sum_{\tilde{K}} a_{jK}^\alpha(x) u^\tilde{K} \right) \right) = 0, \quad \mu = 1, \ldots, s.
\]

(8.6)
At this point, we assume that $\Delta^\mu$ are polynomials with $x$-dependent coefficients of the functions of $A_1$ and their total derivatives of arbitrary order. In view of our assumptions for $A_1$, this implies that (8.6) can be written as a polynomial of the form

$$\sum_{\tilde{O}} f_{\tilde{O}}(x, D_K a_j^{\alpha_k}) u_{\tilde{O}} = 0.$$  

By assuming that the coefficients $a_j^{\alpha_k}$ are solutions to the family of differential equations $f_{\tilde{O}}(x, D_K a_j^{\alpha_k}) = 0$, for all the multi-indexes $\tilde{O}$, and (8.5), one obtains that all solutions to the PDE Lie system (8.3) become a solution of the higher-order system of PDEs under analysis, namely $\Delta = 0$. Hence, all solutions of our PDE Lie system (8.3) become solution of the initial system of PDEs under investigation. Moreover, the form of the PDE Lie system gives rise to Lie symmetries of the PDE Lie system, which implies that they can be employed to solve it.

9. Applications

This section is aimed at illustrating the results given in previous parts of the work through three different physical models: a nonlinear wave-equation, a Gauss-Codazzi equation for minimal surfaces, and a generalised Liouville equation.

9.1. Nonlinear wave-equation

Let us study the conditional symmetries of a nonlinear wave-equation [19, 38]

$$u_{x_1 x_2} = b(u), \quad (9.1)$$

where $b(u)$ is, so far, an underdetermined function depending on $u$. Our study will fulfill several theoretical and practical details not explained in [19]. The nonlinear wave-equation (9.1) is a PDE on the second-order jet bundle $J^2$ relative to $X = \mathbb{R}^2$ and $U = \mathbb{R}$.

Consider a normal PDE Lie algebra $L$ of conditional symmetries generated by the vector fields

$$Z_1 = \frac{\partial}{\partial x_1} + \psi(x_1, x_2, u) \frac{\partial}{\partial u}, \quad Z_2 = \frac{\partial}{\partial x_2} + \varphi(x_1, x_2, u) \frac{\partial}{\partial u},$$

where $\varphi$ and $\psi$ are for the time being some functions on $X \times U$ to be determined. In any case, $\mathcal{D}^L$ is a projectable regular distribution of rank two. Therefore, the characteristic system in $J^1$ is given by the conditions

$$u_{x_1} = \psi(x_1, x_2, u), \quad u_{x_2} = \varphi(x_1, x_2, u) \quad (9.2)$$

and their total derivatives

$$u_{x_1 x_2} = D_{x_2} \psi(x_1, x_2, u), \quad u_{x_2 x_2} = D_{x_2} \varphi(x_1, x_2, u), \quad u_{x_1 x_1} = D_{x_1} \psi(x_1, x_2, u), \quad (9.3)$$

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give rise to the characteristic system $S_L^2 \subset J^2$. In view of Proposition 2.1, the condition $[Z_1, Z_2] = 0$ amounts to the zero curvature condition for (9.2), which in turn ensures that $u_{x^2 x^1} = D_{x^1} \varphi(x_1, x_2, u) = D_{x^2} \psi(x_1, x_2, u) = u_{x_1 x_2}$, which yields that $S_L^2 \subset J^2$ is a section of $\pi_0^2$ in virtue of Theorem 4.5.

In the Clairain approach to Lie algebras of conditional symmetries, it is assumed that $\varphi$ and $\psi$ are such that the first-order system of PDEs (9.2) is integrable. In view of Proposition 2.1 and Theorem 4.5, system $S_L^2$ is locally solvable. In view of Corollary 5.7, the solutions to the system (9.1), (9.2), (9.3) are the same as the solutions to (9.1) and (9.2). This is why works dealing with applications of conditional symmetries in the Clairain approach restrict themselves to studying $S_L^1$ instead of $S_L^n$ (cf. [18, 19]).

In order to use PDE Lie systems to study the conditional symmetries of (9.1), consider the expansion of $\psi$ and $\varphi$ up to second order in the dependent variable $u$. Then,

$$\psi = a_2(x_1, x_2)u^2 + a_1(x_1, x_2)u + a_0(x_1, x_2), \quad \varphi = b_2(x_1, x_2)u^2 + b_1(x_1, x_2)u + b_0(x_1, x_2)$$

(9.4)

for certain functions $a_2(x_1, x_2), a_1(x_1, x_2), a_0(x_1, x_2), b_2(x_1, x_2), b_1(x_1, x_2), b_0(x_1, x_2)$ whose particular form must be determined to ensure that $L$ is a Lie algebra of conditional symmetries.

In view of (9.1) and (9.2), we get that $D_{x^2} \psi = D_{x^1} \varphi = b(u)$. Therefore, one can restrict oneself to the particular case $b(u) = c_0 + c_1 u + c_2 u^2 + c_3 u^3$ for certain constants $c_0, c_1, c_2, c_3 \in \mathbb{R}$ (other more general cases could be also consider). Note that therefore the system of PDEs (9.1) satisfies the conditions of the formalism given in Section 8 for $A_1$ and we can try to obtain solutions to this equation by means of solutions to a PDE Lie system of the form (9.2).

By substituting the expansion (9.4) and the differential constraints (9.2) in the wave-equation (9.1), one obtains

$$2a_2 b_2 u^3 + (2b_2 a_1 + b_1 a_2 + \partial_{x_1} b_2)u^2 + (2b_2 a_0 + b_1 a_1 + \partial_{x_1} b_1) u + a_0 b_1 + \partial_{x_1} b_0 = c_3 u^3 + c_2 u^2 + c_1 u + c_0$$

To assume that all solutions of (9.2) are solutions to (9.1), we assume that the coefficients in $u$ are all zero.

Then, the zero curvature condition for (9.2) amounts to the fact that

$$2b_2 a_1 + b_1 a_2 + \partial_{x_1} b_2 = 2a_2 b_1 + a_1 b_2 + \partial_{x_2} a_2, \quad 2b_2 a_0 + b_1 a_1 + \partial_{x_1} b_1 = 2a_2 b_0 + a_1 b_1 + \partial_{x_2} a_1, \quad a_0 b_1 + \partial_{x_1} b_0 = a_1 b_0 + \partial_{x_2} a_0.$$

In view of previous equalities, we obtain

$$2a_2 b_2 = c_3, \quad 2b_2 a_1 + b_1 a_2 + \partial_{x_1} b_2 = c_2, \quad 2b_2 a_0 + b_1 a_1 + \partial_{x_1} b_1 = c_1, \quad a_0 b_1 + \partial_{x_1} b_0 = c_0, \quad 2a_2 b_1 + a_1 b_2 + \partial_{x_2} a_2 = c_2, \quad 2a_2 b_0 + a_1 b_1 + \partial_{x_2} a_1 = c_1, \quad a_1 b_0 + \partial_{x_2} a_0 = c_0.$$

(9.5)

If the coefficients of the expansion (9.4) satisfy the above conditions, Theorem 7.3 ensures that $Z_1$ and $Z_2$ are conditional Lie symmetries of (9.1). It is not necessary therefore consider the extension of the vector fields $Z_1, Z_2$ to $J^2$ and to restrict to $S_L^2 \cap S_\Delta$ to verify whether the functions $\varphi$ and $\psi$ give rise to conditional symmetries.
Let us provide a simple case of conditional symmetries for (9.1). A particular solution to the equations (9.5), e.g. \( a_2 = b_2 = 1, a_0 = b_0 = a_1 = b_1 = 0 \), gives rise to
\[
Z_1 = \partial x_1 + u^2 \partial u, \quad Z_2 = \partial x_2 + u^2 \partial u
\]
span a Lie algebra of conditional symmetries of
\[
u_{x_1 x_2} = 2u^3.
\]
Moreover, all solutions to
\[
\frac{\partial u}{\partial x_1} = u^2, \quad \frac{\partial u}{\partial x_2} = u^2,
\]
namely \( u = -1/(x_1 + x_2 + \lambda) \) with \( \lambda \in \mathbb{R} \), are particular solutions to (9.6).

We can obtain a new Lie algebra of conditional Lie symmetries by obtaining a renormalizable PDE family of vector fields \( L' \) whose \( S_{L'}^n \) will match \( S_L^n \), for instance,
\[
Y_1 = \partial x_1 + u^2 \partial u, \quad Y_2 = e^{x_2/u}(\partial x_2 + u^2 \partial u).
\]
In fact, one gets that \([Y_1, Y_2] = -Y_2\). It is worth noting that the new vector fields can be interesting if they leave invariant a geometric structure on \( J^0 \) whereas the vector fields \( Z_1, Z_2 \) do not (see [3] for examples of this).

Note that we could have repeated the whole above formalism by using the set of functions \( A^3 \). In that case, we would have added differential constraints given by a PDE Lie system with a Vessiot–Guldberg Lie algebra given by
\[
V = \left\langle \cos(u) \frac{\partial}{\partial u}, \sin(u) \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle.
\]
In turn, this gives rise to studying the sine-Gordon equation \( u_{xt} = \frac{1}{2} \sin(2u) \), which admits pseudo-spherical surfaces and gives rise to Bäcklund transformations [28, 35, 41]. Other set of functions \( \mathcal{A} \) would have given rise to Vessiot–Guldberg Lie algebras
\[
V = \left\langle e^u \frac{\partial}{\partial u}, e^{-u} \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle, \quad V = \left\langle \text{ch}(u) \frac{\partial}{\partial u}, \text{sh}(u) \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle
\]
and new results and Lie algebras conditional symmetries could be derived for the obtained types of Sinh-Gordon equations [35].

9.2. Gauss-Codazzi equations

Let us now apply our formalism to the study of a class of Gauss–Codazzi equations [37], which take the form
\[
\partial \bar{\partial} u + \frac{1}{2} H^2 e^u - 2|Q|^2 e^{-u} = 0, \quad \bar{\partial} Q = \frac{1}{2}(\partial H)e^u,
\]
where \( \partial = (\partial/\partial x_1 - i\partial/\partial x_2)/2, \bar{\partial} = (\partial/\partial x_1 + i\partial/\partial x_2)/2, u = u(z, \bar{z}) \) and \( H = H(z, \bar{z}) \) are real functions, and \( Q = Q(z, \bar{z}) \) is a complex valued function. We will focus on the first
part of the Gauss–Codazzi equations. Although it is a complex differential equation due to the appearance of $\partial$ and $\bar{\partial}$, it is simple to see that it can be considered as a real differential equation on the bundle $J^2$ relative to $U = \mathbb{R}$ and $X = \mathbb{R}^2$.

In order to apply the conditional symmetry theory, we consider the differential constraints

$$\partial u = \eta_0 + \eta_1 e^{-u/2} + \eta_2 e^{u/2}, \quad \bar{\partial} u = \bar{\eta}_0 + \bar{\eta}_1 e^{-u/2} + \bar{\eta}_2 e^{u/2}, \quad (9.8)$$

whose zero curvature condition amounts to

$$\bar{\partial} \eta_0 - \partial \bar{\eta}_0 - \eta_1 \bar{\eta}_2 + \bar{\eta}_1 \eta_2 = 0, \quad \bar{\partial} \eta_1 - \partial \bar{\eta}_1 + \frac{\eta_0 \bar{\eta}_1 - \bar{\eta}_0 \eta_1}{2} = 0, \quad \bar{\partial} \eta_2 - \partial \bar{\eta}_2 + \frac{\eta_2 \bar{\eta}_0 - \bar{\eta}_2 \eta_0}{2} = 0. \quad (9.9)$$

The system of PDEs (9.8) can easily be rewritten as a PDE Lie system of the form

$$\frac{\partial u}{\partial x_1} = \text{Re}(\eta_0) + \text{Re}(\eta_1) e^{-u/2} + \text{Re}(\eta_2) e^{u/2}, \quad \frac{\partial u}{\partial x_2} = \text{Im}(\bar{\eta}_0) + \text{Im}(\bar{\eta}_1) e^{-u/2} + \text{Im}(\bar{\eta}_2) e^{u/2},$$

related to a Vessiot–Guldberg Lie algebra $V = \langle \partial/\partial u, e^{-u/2}\partial/\partial u, e^{u/2}\partial/\partial u \rangle$. This allows us to study two-dimensional Lie algebras of conditional symmetries spanned by the vector fields

$$\frac{\partial}{\partial x_1} + (\text{Re}(\eta_0) + \text{Re}(\eta_1) e^{-u/2} + \text{Re}(\eta_2) e^{u/2}) \frac{\partial}{\partial u}, \quad \frac{\partial}{\partial x_2} + (\text{Im}(\bar{\eta}_0) + \text{Im}(\bar{\eta}_1) e^{-u/2} + \text{Im}(\bar{\eta}_2) e^{u/2}) \frac{\partial}{\partial u}. \quad (9.10)$$

To verify whether the vector fields related to our differential constraints give rise to conditional Lie symmetries, we have to substitute (9.8) in (9.7). Then, we obtain

$$\bar{\partial} \eta_0 + \frac{\eta_2 \bar{\eta}_1 - \eta_1 \bar{\eta}_2}{2} + \left( \bar{\partial} \eta_1 - \frac{\eta_0 \bar{\eta}_1}{2} \right) e^{-u/2} + \left( \bar{\partial} \eta_2 + \frac{\bar{\eta}_0 \eta_2}{2} \right) e^{u/2} + \frac{H^2 + \eta_2 \bar{\eta}_2}{2} e^u - \eta_1 \bar{\eta}_1 + 4 \vert Q \vert^2 e^{-u} = 0. \quad (9.11)$$

Let us assume that all coefficients accompanying the different exponentials of the variable $u$ are zero. This ensures that all solutions to (9.8) give rise to solutions of the Gauss–Codazzi equations. In view of Theorem 7.3, this also ensures that the vector fields (9.10) span a Lie algebra of conditional symmetries. Then, we obtain

$$\bar{\partial} \eta_0 + \frac{\eta_2 \bar{\eta}_1 - \eta_1 \bar{\eta}_2}{2} = 0, \quad (9.12)$$

$$\bar{\partial} \eta_1 - \frac{\eta_0 \bar{\eta}_1}{2} = 0, \quad (9.13)$$

$$\bar{\partial} \eta_2 + \frac{\bar{\eta}_0 \eta_2}{2} = 0, \quad (9.14)$$

$$H^2 + \eta_2 \bar{\eta}_2 = 0, \quad (9.15)$$

$$\eta_1 \bar{\eta}_1 + 4 \vert Q \vert^2 = 0. \quad (9.16)$$

Since $H$ is a real-valued function, equation (9.15) implies that $\eta_2 = H = 0$ and the Gauss–Codazzi equations (9.7) shows that $Q = Q(z)$. It is immediate to verify that the latter
fact along with (9.12)-(9.14) allow us to ensure that the compatibility conditions (9.9) are satisfied.

Let us now obtain the conditional symmetries of this system. Since \( \eta_2 = 0 \), one has from (9.12) that \( \eta_0 = \eta_0(z) \). The equations (9.12) can easily be solved then to obtain that

\[
\eta_1 = \alpha(z)e^{\frac{1}{2}\int \overline{\eta}_0dz}
\]

for an arbitrary function \( \alpha = \alpha(z) \). Substituting this in (9.16), we get

\[
|Q|^2 = \frac{1}{4}|\alpha(z)|^2 \left| e^{\frac{1}{2}\int \overline{\eta}_0dz} \right|^2 = -\frac{1}{4} \left| \alpha(z)e^{\frac{1}{2}\int \eta_0dz} \right|^2,
\]

which is acceptable as \( Q = Q(z) \) is a holomorphic function. Consequently, the initial Gauss–Codazzi equation reduces under previous assumptions to

\[
\frac{\partial \bar{\eta}}{\partial u} + \frac{1}{2} \left| \alpha(z)e^{1/2\int \eta_0dz} \right|^2 e^u = 0,
\]

whereas

\[
\frac{\partial u}{\partial x_1} = \text{Re}(\eta_0(z)) + \text{Re}(\alpha(z)e^{\int \overline{\eta}_0dz})e^{-u/2}, \quad \frac{\partial u}{\partial x_2} = \text{Im}(\eta_0(z)) + \text{Im}(\alpha(z)e^{\int \overline{\eta}_0dz})e^{-u/2},
\]

(9.17)

where we recall that \( z = x_1 + ix_2 \). The latter is a PDE Lie system related to a solvable Vessiot–Guldberg Lie algebra \( V_2 = \langle \partial/\partial u, e^{-u/2}\partial/\partial u \rangle \). Consequently, it is solvable (cf. [7]). In fact, a simple method to solve it goes as follows. Let us rewrite (9.17) in terms of the variable \( w = e^{u/2} \) as

\[
\frac{\partial w}{\partial x_1} = \text{Re}(\eta_0(z))w/2 + \text{Re}(\alpha(z)e^{\int \overline{\eta}_0dz})/2, \quad \frac{\partial w}{\partial x_2} = \text{Im}(\eta_0(z))w/2 + \text{Im}(\alpha(z)e^{\int \overline{\eta}_0dz}).
\]

(9.18)

The homogeneous part of the system reads

\[
\frac{\partial w_H}{\partial x_1} = \text{Re}(\eta_0(z))w_H/2, \quad \frac{\partial w_H}{\partial x_2} = \text{Im}(\eta_0(z))w_H/2
\]

(9.19)

and a particular solution reads \( w_H = e^{\frac{1}{2}\int (\text{Re}(\eta_0(z))dx_1 + \text{Im}(\eta_0(z))dx_2)} \). Substituting \( w = w_Hw_N \) in (9.17), we obtain that

\[
w_N = \int \frac{1}{2w_H} \text{Re}(\alpha(z)e^{\frac{1}{2}\int \overline{\eta}_0dz})dx_1 + \int \frac{1}{2w_H} \text{Im}(\alpha(z)e^{\frac{1}{2}\int \eta_0dz})dx_2
\]

and the final solution for (9.17) follows immediately. It is remarkable that \( H = 0 \) implies that the obtained surfaces are minimal [12, 42].
9.3. A generalised Liouville equation

Finally, let us study the generalised Liouville equation in $\mathbb{R}^{n+1}$ introduced by Santini [36]. Let $(x_0 = t, x)$ be a general point in $\mathbb{R} \times \mathbb{R}^n$ and let $\nabla$ be the gradient operator in $\mathbb{R}^n$ relative to the Euclidean metric $\langle \cdot | \cdot \rangle$ on $\mathbb{R}^n$. Then, generalised Liouville equation is given by

$$\frac{\partial}{\partial t} \left( \nabla^2 u - \frac{1}{2} \langle \nabla u | \nabla u \rangle \right) = 0, \quad (9.20)$$

where $u = u(t, x)$ is a function on $\mathbb{R}^{n+1}$.

We make use of the set of functions $A_2$ and we assume that the conditional symmetry is given by

$$\frac{\partial u}{\partial x_j} = A_0^j(t, x) + A_+^j(t, x)e^{u/2} + A_-^j(t, x)e^{-u/2}, \quad j = 0, \ldots, n. \quad (9.21)$$

The coefficients $A_0^j(t, x), A_+^j(t, x), A_-^j(t, x)$, with $\alpha = 1, \ldots, n$, give rise to three $t$-dependent vector fields $A^0_-, A^+_, A^-$ on $\mathbb{R}^n$, respectively. Using this and substituting the differential constraints (9.21) in (9.20), we obtain the following equality:

$$\frac{\partial}{\partial t} \left[ \nabla A^0_0 - \frac{1}{2} |A^0_0|^2 + \left( \nabla A^+ - \frac{1}{2} \langle A^+ | A^0_0 \rangle \right) e^{u/2} + \left( \nabla A^- - \frac{3}{2} \langle A^- | A^0_0 \rangle \right) e^{-u/2} - |A^-|^2 e^{-u} \right] = 0.$$

Therefore, the particular conditions

$$\frac{\partial}{\partial t} \left[ \nabla A^0_0 - \frac{1}{2} |A^0_0|^2 \right] = 0, \quad \nabla A^+ = \frac{1}{2} \langle A^+ | A^0_0 \rangle = 0, \quad A^- = 0, \quad (9.22)$$

ensure that every solution of the DCs (9.21) gives rise to a particular solution to (9.20). Using that $A^- = 0$ and assuming $A_0^0 = 0$, the zero curvature condition for (9.21) reads

$$\frac{\partial A_0^0}{\partial x_j} - \frac{\partial A_j^0}{\partial x_0} = 0, \quad \frac{\partial A_+^0}{\partial x_j} - \frac{\partial A_j^+}{\partial x_0} + \frac{1}{2} \left( A^+_k A^0_j - A^0_k A^+_j \right) = 0, \quad 0 \leq k < j \leq n. \quad (9.23)$$

The first condition implies that there exists a function $u_p \in C^\infty(\mathbb{R}^{n+1})$ such that

$$A_j^0 = \frac{\partial u_p}{\partial x_j}, \quad j = 0, \ldots, n. \quad (9.24)$$

In particular, $A^0 = \nabla u_p$ and substituting this in the first equality of (9.22), one obtains that

$$\nabla^2 u_p - \frac{1}{2} |\nabla u_p|^2 = \lambda(x)$$

and $u_p$ becomes a particular solution to (9.20). Using (9.24) in the second compatibility condition in (9.23), we obtain that

$$\frac{\partial}{\partial x_j} \left( A^+_k e^{u_p/2} \right) - \frac{\partial}{\partial x_k} \left( A^+_j e^{u_p/2} \right) = 0, \quad 0 \leq k < j \leq n.$$
Therefore,
\[ \exists w \in C^\infty(\mathbb{R}^{n+1}), \quad A^+_j = e^{-u_p/2} \frac{\partial w}{\partial x_j}, \quad 0 \leq j \leq n. \]
In particular, \( A^+ = e^{-u_p/2} \nabla w \). Using \( A^+ \) in the second expression of (9.23), we obtain
\[ \nabla^2 w - \langle \nabla u_p | \nabla w \rangle = 0. \quad (9.25) \]
Under previous assumptions, the DCs (9.21) reduce to
\[ \frac{\partial u}{\partial x_j} = \frac{\partial u_p}{\partial x_j} + \frac{\partial w}{\partial x_j} e^{(u-u_p)/2}, \quad j = 0, \ldots, n. \quad (9.26) \]
As this is a PDE Lie system related to the solvable Lie algebra \( \langle \partial/\partial u, e^{u/2} \partial/\partial u \rangle \), its general solution can be obtained [34]. In fact, it stems from (9.26) that
\[ \frac{\partial}{\partial x_j} \left( e^{-(u-u_p)/2} - w \right) = 0, \quad j = 0, \ldots, n. \]
and
\[ u(t, x) = u_p(t, x) - 2 \log(w(t, x) + \lambda), \quad \lambda \in \mathbb{R}, \quad (9.27) \]
is the general solution to the system of equations given by (9.20) and (9.26). This can be viewed as a generalised type of auto-Bäcklund transformation mapping solutions of generalised Liouville equation satisfying the DC (9.21) into new solutions of the same equations. Additionally, this can also be seen as a so-called \textit{t-dependent superposition rule} for a system of PDEs, which was recently introduced in [5].

Let us detail a several particular solutions. If \( n = 2 \) and \( u_p = h(t) \) for any \( t \)-dependent function \( h(t) \), then equation (9.25) reduces to \( \nabla^2 w = 0 \) and \( w \) becomes any \textit{harmonic function} on \( \mathbb{R}^2 \), e.g. the real \( \text{ref}(x_1 + ix_2, t) \) and imaginary parts \( \text{im}f(x_1 + ix_2, t) \) of a \( t \)-dependent holomorphic function \( f(x_1 + ix_2, t) \). In this case,
\[ u = h(t) - 2 \ln(\lambda + \text{ref}(x_1 + ix_2, t) + \text{im}f(x_1 + ix_2, t)). \]
The above procedure gives a new approach to [18], explains many details not given in there, and avoids several of its \textit{ad-hoc} constructions, e.g. our approach does not need any \textit{ad-hoc} transformation. Let us finally to explain how to obtain a particular solution proposed in [18] without a detailed explanation.

Every function \( u_p(t, x) = f(t) + g(x) \), for arbitrary functions \( f(t) \) and \( g(x) \), is a particular solution to (9.20). In particular, if we assume \( \omega(x) = \sum_{j=1}^n \omega_j(x_j) \), then
\[ u_p(t, x) = \ln|g(t)| + \sum_{j=1}^n \ln \left[ \frac{d\omega_j}{dx_j}(x_j) \right] \]
is a particular solution to (9.20) and \( \omega(x) \) is a solution to (9.25). According to (9.27), we obtain a solution with the freedom of \( n+1 \) arbitrary functions of one variable
\[ u(t, x) = \ln \left[ \frac{g(t) \prod_{j=1}^n \frac{d\omega_j}{dx_j}(x_j)}{\left[ g(t) + \sum_{j=1}^n \omega_j(x_j) + \lambda \right]^2} \right], \quad \lambda \in \mathbb{R} \]
to the system of PDEs given by (9.20) and (9.21). This retrieves the multi-mode particular solution given in [18] without a detailed explanation.

10. Conclusions

This work has provided a geometric approach to the theory of conditional symmetries for higher-order systems of PDEs. Many of the hypotheses employed in this theory have been analysed and their geometrical meaning has been explained. In certain cases, it was stated that standard assumptions can be relaxed. Then, the role of PDE Lie systems in the description of conditional symmetries has been analysed.

After the study of the Clairin theory of conditional symmetries, we believe that we should continue the analysis of other conditional symmetries and to consider the case of conditional symmetries given by the characteristics of non-Lie point symmetries. It seems to us that the basics of the geometric formalism developed in this work should work still to investigate these more general cases, but also important changes and results must be found to find a full theory. For instance, the whole theory of higher-order systems of PDEs in normal form does not apply and the use of PDE Lie systems is no longer available.

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References

[1] R. Abraham, J.E. Marsden, and T. Ratiu, Manifolds, Tensor Analysis, and Applications, Applied Mathematical Sciences Vol. 75, Springer-Verlag, New York, 1988.
[2] R.L. Anderson et al., CRC Handbook of Lie group analysis of differential equations Vol. 3. New trends in theoretical developments and computational methods, CRC Press, Boca Raton, 1996.
[3] A. Ballesteros, A. Blasco, F.J. Herranz, J. de Lucas, and C. Sardón, Lie-Hamilton systems on the plane: properties, classification and applications, J. Differential Equations 258, 2873–2907 (2015).
[4] G.W. Bluman and J.D. Cole, The general similarity solution of the heat equation, J. Math. Mech. 18, 1025–1042 (1969).
[5] J.F. Cariñena, J. Grabowski, and J. de Lucas, Quasi-Lie schemes for PDEs, arXiv:1712.02238v2.
[6] J.F. Cariñena, J. Grabowski, and G. Marmo. Superposition rules, Lie Theorem, and partial differential equations, Rep. Math. Phys. 60, 237–258 (2007).
[7] J.F. Cariñena and J. de Lucas, Lie systems: theory, generalisations, and applications, Dissertationes Math. 479, 1–162 (2011).
[8] R. Cherniha and V. Davydovych, Lie and conditional symmetries of the three-component diffusive Lotka–Volterra system, J. Phys. A 46, 185204 (2013).
[9] G. Cicogna and G. Gaeta, Partial Lie-point symmetries of differential equations, J. Phys. A 34, 491–512 (2001).
[10] M. J. Clairin, Sur quelques equations aux derivees partielles du second ordre, Ann. Fac. Sci. Toulouse Sci. Mat. Phys. 5, 437–458 (1903).
[11] M. Dunajski, Solitons, instantons, and twistors, Oxford Graduate Texts in Mathematics Vol. 19, Oxford University Press, Oxford, 2010.
[12] N. Enneper, Nachr. Königl. Gesell. Wissensch. Georg-Augustus, Univ. Göttingen 12, 258 (1868).
[13] L.A. Ferreira, J.F. Gomes, A.V. Razumov, M.V. Saveliev, and A.H. Zimerman, Riccati-Type equations, generalised WZNW Equations, and multidimensional Toda systems, Comm. Math. Phys. 203, 649–666 (1999).
[14] W. Fuschchych, Conditional symmetries of the equations of mathematical physics, in: Modern group analysis: advanced analytical and computational methods in mathematical physics, Kluwer Acad. Publ., Dordrecht, 1993, pp. 231–239.
[15] G. Gaeta and P. Morando, On the geometry of \( \lambda \)-symmetries and PDE reduction, J. Phys. A 37, 6955–6975 (2004).
[16] A. Gonzalez-Lopez, N. Kamran, and P.J. Olver, Lie algebras of vector fields in the real plane, Proc. London Math. Soc. 64, 339–368 (1992).
[17] A.M. Grundland and J. de Lucas, A Lie systems approach to the Riccati hierarchy and partial differential equations, J. Differential Equations 263, 299–337 (2017).
[18] A.M. Grundland, L. Martina, and G. Rideau, Partial differential equations with differential constraints, in: Advances in mathematical sciences: CRM’s 25 years. CRM Proc. Lecture Notes, 11, Amer. Math. Soc., Providence, 1997, pp. 135–154.
[19] A.M. Grundland and G. Rideau, Conditional symmetries for 1st order systems of PDES in the context of the Clairin method, in: Modern group theor. methods phys., Kluwer Acad. Publ., Dordrecht, 1995, pp. 167–178.
[20] C. Gu, H. Hu, and Z. Hesheng, Darboux transformations in integrable systems, Theory and their applications to geometry, Mathematical Physics Studies Vol. 26, Springer, Dordrecht, 2005.
[21] M.A. Guest, From quantum cohomology to integrable systems, Oxford Graduate Texts in Mathematics Vol. 15, Oxford University Press, Oxford, 2008.
[22] N.H. Ibragimov, M. Torrisi, and A. Valenti (eds.), Modern Group Analysis: Advanced analytical and computational methods in mathematical physics, Vol. I, II, III, Springer, Dordrecht, 1993.
[23] D. Levi and P. Winternitz, Nonclassical symmetry reduction: example of the Boussinesq equation, J. Phys. A 22, 2915–2924 (1989).
[24] S. Lie, Theorie der Transformationsgruppen I, Math. Ann. 16, 441–528 (1880).
[25] C. Muriel and J.L. Romero, \( C^\infty \)-symmetries and reduction of equations without Lie point symmetries, J. Lie Theory 13, 167–188 (2003).
[26] C. Muriel and J.L. Romero, Integrating factors and \( \lambda \)-symmetries, J. Nonlinear Math. Phys. 15, 300–309 (2008).
[27] A. Odzijewicz and A.M. Grundland, The superposition principle for the Lie type first-order PDEs, Rep. Math. Phys. 45, 293–306 (2000).
[28] P.J. Olver, Symmetry and explicit solutions of partial differential equations, Appl. Numer. Math. 10, 307–324 (1992).
[29] P.J. Olver, Applications of Lie groups to differential equations, Springer-Verlag, New York, 1993.
[30] P.J. Olver and P. Rosenau, Group invariant solutions of differential equations, SIAM J. Appl. Math. 47, 263–278 (1987).
[31] P.J. Olver and E.M. Vorob’ev, Nonclassical and conditional symmetries, in: CRC Handbook of Lie group analysis vol. 3, CRC press, London, 1995.
[32] R.S. Palais, A global formulation of the Lie theory of transformation groups, Mem. Amer. Math. Soc. 22, 1–123 (1957).
[33] E. Pucci and G. Saccomandi, On the weak symmetry groups of partial differential equations, J. Math. Anal. Appl. 163, 588–598 (1992).
[34] A. Ramos, *Lie systems and applications in Physics and Control Theory*, PhD Thesis, arXiv:1106.3775.

[35] C. Rogers C and W.K. Schief, *Bäcklund and Darboux Transformations. Geometry and Modern Applications in Soliton Theory*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2000.

[36] P.M. Santini, *Linear theories, hidden variables and integrables nonlinear equations*, Phys. Lett. A 212, 43–49 (1996).

[37] A. Schwartz, *The Gauss-Codazzi-Ricci equations in Riemannian manifolds*, J. Math. Phys. Mass. Inst. Tech. 20, 30–79 (1941).

[38] W.F. Shadwick, *The Bäcklund problem for the equation \( z_{x_1x_2} = f(z) \)*, J. Math Phys. 19, 2312–2317 (1978).

[39] H. Stephani, *Differential equations. Their solution using symmetries*, Cambridge University Press, Cambridge, 1989.

[40] A.M. Vinogradov and I.S. Krasil’shchik, *Symmetries and conservation laws for differential equations of mathematical physics*, Trans. Math. Monographs Vol. 182, American Mathematical Society, Providence, 1999.

[41] E.M. Vorob’ev, *Reduction and quotient equations for differential equations with symmetries*, Acta Appl. Math. 23, 1–24 (1991).

[42] K. Weierstrass, *Fortsetzung der Untersuchung über die Minimalflächen*, Mathematische Werke 3, 219–248 (1866).

[43] P. Winternitz, *Lie groups and solutions of nonlinear differential equations*, in: *Nonlinear Phenomena*, Lecture Notes in Phys. 189, Springer-Verlag, Berlin, 1983, pp. 263–305.

[44] S. Zidowicz, *Conditional symmetries and the direct reduction of partial differential equations*, in: *Modern group analysis: advanced analytical and computational methods in mathematical physics (Acireale, 1992)*, Kluwer Acad. Publ., Dordrecht, 1993, pp. 387–393.