Curvatures of metrics induced by the isotropic almost complex structures

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Abstract
Isotropic almost complex structures induce a class of Riemannian metrics on tangent bundle of a Riemannian manifold. In this paper the curvature tensors of these metrics will be calculated.

Keywords: Isotropic almost complex structure, tangent bundle, curvature tensor.

1 Introduction
Let \((M, g)\) be a Riemannian manifold and \(\pi : TM \to M\) be its tangent bundle. Moreover, denote by \(X^h, X^v\) the horizontal and vertical lifts of the vector field \(X\) on \(M\). In [2], Aguilar defined a class of almost complex structures \(J_{\delta,\sigma}\) on \(TM\), namely isotropic almost complex structures with definition

\[
J_{\delta,\sigma}(X^h) = \alpha X^v + \sigma X^h, \quad J_{\delta,\sigma}(X^v) = -\sigma X^v - \delta X^h,
\]

for functions \(\alpha, \delta, \sigma : TM \to \mathbb{R}\) which are in relation \(\alpha\delta - \sigma^2 = 1\).

He studied the integrability of these structures and proved that there exists an integrable isotropic almost complex structure on the tangent bundle of a

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Riemannian manifold \((M, g)\) if and only if the sectional curvature of \((M, g)\) is constant.

Also, using the Liouville one-form, he defined a new class of Riemannian metrics \(g_{\delta, \sigma}\) which are generalization of Sasaki metric. These metrics for an almost complex structure \(J_{\delta, \sigma}\) is defined by

\[
g_{\delta, \sigma}(X^h, Y^h) = \alpha g(X, Y) \circ \pi, \\
g_{\delta, \sigma}(X^h, Y^v) = -\sigma g(X, Y) \circ \pi, \\
g_{\delta, \sigma}(X^v, Y^v) = \delta g(X, Y) \circ \pi.
\]

for vector fields \(X, Y\) on \(M\).

## 2 Curvatures of \(g_{\delta, 0}\)

The following theorem states the formulas of the Levi-Civita connection of \(g_{\delta, \sigma}\).

**Theorem 1** Let \(g_{\delta, \sigma}\) be a Riemannian metric on \(TM\) as before. Then the Levi-Civita connection \(\bar{\nabla}\) of \(g_{\delta, \sigma}\) at \((p, u) \in TM\) is given by

\[
\bar{\nabla}_X Y^h = (\nabla_X Y)^h - \frac{\sigma}{\alpha}(R(u, X)Y)^h + \frac{1}{2\alpha}X^h(\alpha)Y^h + \frac{1}{2\alpha}Y^h(\alpha)X^h \\
- \frac{\sigma}{\delta}(\nabla_X Y)^v - \frac{1}{2}(R(X, Y)u)^v - \frac{1}{2\delta}X^h(\sigma)Y^v \\
- \frac{1}{2\delta}Y^h(\sigma)X^v - \frac{1}{2}g(X, Y)\bar{\nabla}\alpha, \quad (5)
\]

\[
\bar{\nabla}_X Y^v = -\frac{\alpha}{\sigma}(\nabla_X Y)^h + \frac{\delta}{2\alpha}(R(u, Y)X)^h - \frac{1}{2\alpha}X^h(\sigma)Y^h \\
+ \frac{1}{2\alpha}Y^v(\alpha)X^h + (\nabla_X Y)^v + \frac{1}{2\delta}X^h(\delta)Y^v - \frac{1}{2\delta}Y^v(\delta)X^v \\
+ \frac{1}{2}g(X, Y)\bar{\nabla}\sigma, \quad (6)
\]

\[
\bar{\nabla}_X Y^h = \frac{\delta}{2\alpha}(R(u, X)Y)^h + \frac{1}{2\alpha}X^v(\alpha)Y^h - \frac{1}{2\alpha}Y^h(\sigma)X^h \\
- \frac{1}{2\delta}X^v(\sigma)Y^h + \frac{1}{2\delta}Y^h(\delta)X^v + \frac{1}{2}g(X, Y)\bar{\nabla}\sigma, \quad (7)
\]

\[
\bar{\nabla}_X Y^v = -\frac{1}{2\alpha}X^v(\sigma)Y^h - \frac{1}{2\alpha}Y^v(\sigma)X^h + \frac{1}{2}\delta X^v(\delta)Y^v \\
+ \frac{1}{2\delta}Y^v(\delta)X^v - \frac{1}{2}g(X, Y)\bar{\nabla}\delta. \quad (8)
\]
Proof. We just prove (5), the remaining ones are similar. Using Koszul formula, we have
\[2g_{\delta,\sigma}(\nabla_{X^h}Y^h, Z^h) = X^h g_{\delta,\sigma}(Y^h, Z^h) + Y^h g_{\delta,\sigma}(X^h, Z^h) - Z^h g_{\delta,\sigma}(X^h, Y^h)\]
\[+ g_{\delta,\sigma}([X^h, Y^h], Z^h) + g_{\delta,\sigma}([Z^h, X^h], Y^h)\]
\[- g_{\delta,\sigma}([Y^h, Z^h], X^h).\]

Using relations (14), (19) and (28) gives us
\[2g_{\delta,\sigma}(\nabla_{X^h}Y^h, Z^h) = g(X^h(\alpha)Y^h, Z^h) + g(Y^h(\alpha)X^h, Z^h) - Z^h g(X^h, Y^h)\]
\[+2g(R(X,Y)u, Z) + \sigma g([X,Y], Z)\]
\[+ \sigma g(R(X,Y)u, Y) - \sigma g(R(Y,Z)u, X).\]

Using the properties of the Levi-Civita connection of \(g\), we can get
\[2g_{\delta,\sigma}(\nabla_{X^h}Y^h, Z^h) = g(X^h(\alpha)Y^h, Z^h) + g(Y^h(\alpha)X^h, Z^h) - Z^h g(X^h, Y^h)\]
\[+2\sigma g(R(X,Y)u, Z) + \sigma g(R(X,Y)u, Y)\]
\[+ \sigma g(R(Y,Z)u, X).\]

Taking into account (19) and the Bianchi’s first identity, we have
\[2g_{\delta,\sigma}(\nabla_{X^h}Y^h, Z^h) = g_{\delta,\sigma}(\frac{1}{\alpha}X^h(\alpha)Y^h + \frac{1}{\alpha}Y^h(\alpha)X^h - g(X,Y)\nabla\alpha\]
\[+ 2(\nabla_{X^h}Y^h) - \frac{2\sigma}{\alpha}(R(u,X)Y^h, Z^h),\]

so the horizontal component of \(\nabla_{X^h}Y^h\) is
\[h(\nabla_{X^h}Y^h) = \frac{1}{2\delta}X^h(\sigma)Y^h + \frac{1}{2\delta}Y^h(\sigma)X^h - \frac{1}{2}g(X,Y)h(\nabla\alpha) + (\nabla_{X^h}Y^h)\]
\[\frac{\sigma}{\alpha}(R(u,X)Y^h, Z^h),\]
where \(\nabla\alpha = h(\nabla\alpha) + v(\nabla\alpha)\) is the splitting of the gradient vector field of \(\alpha\) with respect to \(g_{\delta,\sigma}\) to horizontal and vertical components, respectively. Similarly the vertical component of \(\nabla_{X^h}Y^h\) is
\[v(\nabla_{X^h}Y^h) = -\frac{1}{2\delta}X^h(\sigma)Y^v - \frac{1}{2\delta}Y^h(\sigma)X^v - \frac{1}{2}g(X,Y)v(\nabla\alpha)\]
\[- \frac{\sigma}{\delta}(R(u,X)Y^v) - \frac{1}{2}(R(X,Y)u)v.\]

Using the equation \((\nabla_{X^h}Y^h) = h(\nabla_{X^h}Y^h) + v(\nabla_{X^h}Y^h),\) the proof will be completed. ■

Hereafter, we put \(\sigma = 0\) and represent the metric \(g_{\delta,0}\) by \(\bar{g}\) and its Levi-Civita connection by \(\bar{\nabla}\).
Definition 2 Let \((M, g)\) be a Riemannian manifold and \(\nabla\) be the Levi-Civita connection of \(g\). Moreover, let \(C^\infty M\) be the set of all smooth functions on \(M\). The differential operator \(\Delta_g : C^\infty M \rightarrow C^\infty M\) given by

\[
\Delta_g(f) = \sum_{i=1}^{n} \{ \nabla E_i \nabla E_i(f) - \nabla \nabla E_i E_i(f) \},
\]

is called rough Laplacian on functions, where \(\{E_1, \ldots, E_n\}\) is a locally orthonormal frame on \(M\) and \(f \in C^\infty M\).

Note that in some books this operator is defined with a minus sign. Let \(\{E_1, \ldots, E_n\}\) be a locally orthonormal frame on \((M, g)\) around \(p \in M\) such that \(\nabla E_i E_j = 0\) at \(p\). Then, it is obvious that \(\{\sqrt{\alpha} E_1, \ldots, \sqrt{\alpha} E_n\}\) is a locally orthonormal frame on \((T^*M, \bar{g})\). The Laplacian of \(\alpha\) at \(p\) is calculated as follow,

Lemma 3 Using the above notations \(\Delta_{\bar{g}} \alpha\) is given by

\[
\Delta_{\bar{g}} \alpha(p) = \sum_{i=1}^{n} \left\{ \frac{1}{\alpha} E_i^h(E_i^h(\alpha)) + \alpha E_i^v(E_i^v(\alpha)) \right\} - \frac{1}{\alpha^2} E_i^h(\alpha) E_i^h(\alpha) - E_i^v(\alpha) E_i^v(\alpha)\} (p).
\]

Proof. Using the definition 2 gives us,

\[
\Delta_{\bar{g}} \alpha(p) = \sum_{i=1}^{n} \left\{ \frac{1}{\alpha} E_i^h(E_i^h(\alpha)) + \sqrt{\alpha} E_i^v(\sqrt{\alpha} E_i^v(\alpha)) \right\}
- \left\{ \nabla E_i^h(\alpha) - \nabla \nabla E_i^h(\alpha) \right\} (p)
- \sum_{i=1}^{n} \left\{ \frac{1}{\alpha} E_i^h(E_i^h(\alpha)) + \alpha E_i^v(E_i^v(\alpha)) \right\}
- \left\{ \nabla E_i^h(\alpha) - \nabla \nabla E_i^h(\alpha) \right\} (p).
\]

By putting the equations Levi-Civita in the equation (9), we get the result. ■

Let \(u = u^i \frac{\partial}{\partial x^i} \in T^*M\) be any vector in \(T^*M\). Then the following relations hold for every vector fields \(X = X^i \frac{\partial}{\partial x^i}, Y = Y^i \frac{\partial}{\partial x^i}\) and \(Z = Z^i \frac{\partial}{\partial x^i}\) on \(M\):

\[
X^h(u^i) = -\Gamma^i_{js} X^j u^s, \quad \text{and} \quad X^v(u^i) = X^i,
X^h g(Y, u) = g(\nabla_X Y, u), \quad \text{and} \quad X^v g(Y, u) = g(X, Y),
X^h g(Y, Z) = X g(Y, Z), \quad \text{and} \quad X^v g(Y, Z) = 0,
\]

and the following derivatives are given in a computable form by using the two
The curvatures of $\tilde{\bar{\bar{\gamma}}}$ can be calculated using the following formulas

\[
\tilde{\bar{\bar{R}}}(A, B)C = \tilde{\bar{\bar{\nabla}}}_A \tilde{\bar{\bar{\nabla}}}_B C - \tilde{\bar{\bar{\nabla}}}_B \tilde{\bar{\bar{\nabla}}}_A C - \tilde{\bar{\bar{\nabla}}}_{[A, B]} C,
\]
\[
\tilde{\bar{\bar{Q}}}(A) = \frac{1}{\alpha^2} \sum_{i=1}^{n} \tilde{\bar{\bar{R}}}(A, E^h_i)E^h_i + \alpha \tilde{\bar{\bar{R}}}(A, E^v_i)E^v_i,
\]

for the Riemannian curvature tensor and the Ricci operator, respectively, where \(\frac{1}{\sqrt{\alpha}}E^h_i, \ldots, \frac{1}{\sqrt{\alpha}}E^h_i, \sqrt{\alpha}E^v_i, \ldots, \sqrt{\alpha}E^v_i\) represents a locally orthonormal frame for \((T\tilde{\bar{\bar{m}}}, \tilde{\bar{\bar{g}}})\) and \(A, B\) and \(C\) are vector fields on \(T\tilde{\bar{\bar{m}}}\).

**Theorem 4** Let \((M, g)\) be a Riemannian manifold and \(\tilde{\bar{\bar{g}}}\) be a Riemannian metric induced by \(J_{\tilde{\bar{\bar{g}}}}\) on \(T\tilde{\bar{\bar{m}}}\). Denote by \(\tilde{\bar{\bar{\nabla}}}\) and \(\tilde{\bar{\bar{R}}}\) the Levi-Civita connection and the Riemannian curvature tensor of \((M, g)\), respectively. Then the Riemannian curvature tensors of \((T\tilde{\bar{\bar{m}}}, \tilde{\bar{\bar{g}}})\) are completely determined by

\[
\tilde{\bar{\bar{R}}}(X^h, Y^h)Z^h = (R(X, Y)Z)^h - \frac{1}{4\alpha^2}(R(u, Y, Z)u)X^h
\]
\[
+ \frac{1}{4\alpha^2}(R(u, R(X, Z)u)Y^h + \frac{1}{2\alpha^2}(R(u, R(X, Y))u)Z^h
\]
\[
+ \frac{1}{2}(\nabla_Y Z)^h(\alpha) + \frac{3}{4\alpha^2}Y^h(\alpha)Z^h(\alpha) - \frac{1}{2\alpha}Y^h(Z^h(\alpha))
\]
\[
- \frac{1}{4\alpha}(R(Y, Z)u)^v(\alpha)X^h + \left\{ \frac{1}{2\alpha}(\nabla_X Z)^h(\alpha)
\]
\[
- \frac{3}{4\alpha^2}X^h(\alpha)Z^h(\alpha) + \frac{1}{2\alpha}X^h(Z^h(\alpha))
\]
\[
+ \frac{1}{4\alpha}(R(X, Z)u)^v(\alpha)\}Y^h
\]
\[
+ \frac{1}{2}(\nabla_Z R)(X, Y)u)^v - \frac{1}{2\alpha}Y^h(\alpha)(R(X, Z)u)^v
\]
\[
+ \frac{1}{2\alpha}X^h(\alpha)(R(Y, Z)u)^v - \frac{1}{\alpha}Z^h(\alpha)(R(X, Y)u)^v
\]
\[
+ \left\{ \frac{1}{4\alpha}X^h(\alpha)g(Y, Z) - \frac{1}{4\alpha}Y^h(\alpha)g(X, Z)\} \nabla \alpha
\]
\[
+ \frac{1}{2}g(X, Z)\nabla_{Y^h} \nabla \alpha - \frac{1}{2}g(Y, Z)\nabla_{X^h} \nabla \alpha,
\]
\[
\bar{R}(X^h, Y^h) Z^v = \frac{1}{2\alpha^2}((\nabla X R)(u, Z) Y^h) - \frac{1}{2\alpha^2}((\nabla Y R)(u, Z) X^h) \\
+ \frac{1}{2\alpha^3} Y^h(\alpha)(R(u, Z) X^h) - \frac{1}{2\alpha^3} X^h(\alpha)(R(u, Z) Y^h) \\
+ \left\{ \frac{1}{4\alpha^2}(R(u, Z) Y^h(\alpha) + \frac{1}{2\alpha} (\nabla Y Z)^v(\alpha) \\
+ \frac{1}{4\alpha^2} Y^h(\alpha) Z_v(\alpha) - \frac{1}{2\alpha} Y^h(Z_v(\alpha)))\right\} X^h \\
+ \left\{ - \frac{1}{4\alpha^2}(R(u, Z) X^h(\alpha) - \frac{1}{2\alpha} (\nabla X Z)^v(\alpha) \\
- \frac{1}{4\alpha^2} X^h(\alpha) Z_v(\alpha) + \frac{1}{2\alpha} X^h(Z_v(\alpha)))\right\} Y^h \\
+ (R(X, Y) Z)^v + \frac{1}{4\alpha^2}(R(u, Z) R(X, Y) u)^v \\
- \frac{1}{4\alpha^2}(R(u, Z) R(X, Y) u)^v + \frac{1}{4\alpha^2}(R(X, Y) R(u, Z) u)^v \\
+ \frac{1}{\alpha} Z_v(\alpha)(R(Y, X) u)^v - \frac{1}{2\alpha} Z_v(\alpha)[X, Y]^v \\
+ \frac{1}{\alpha^2} R(X, Y, u, Z) \nabla \alpha, \\
\right\}
\]

(15)

\[
\bar{R}(X^h, Y^v) Z^h = \frac{1}{2\alpha^2}((\nabla X R)(u, Y) Z^h) - \frac{1}{\alpha^3} X^h(\alpha)(R(u, Y) Z)^h \\
- \frac{1}{2\alpha^3} Z^h(\alpha)(R(u, Y) X)^h + \left\{ \frac{1}{4\alpha^2}(R(u, Y) Z)^h(\alpha) \\
+ \frac{1}{4\alpha^2} Z^h(\alpha) Y_v(\alpha) - \frac{1}{2\alpha} Y_v(Z^h(\alpha)))\right\} X^h \\
+ \left\{ \frac{1}{2}(R(X, Z) Y)^v - \frac{1}{4\alpha^2}(R(X, R(u, Y) Z) u)^v \\
- \frac{1}{2\alpha} Y_v(\alpha)(R(X, Z) u)^v + \left\{ - \frac{1}{2\alpha} X^h(Z_v(\alpha)) \\
+ \frac{1}{2\alpha}(\nabla X Z)^h(\alpha) + \frac{5}{4\alpha^2} Z^h(\alpha) X^h(\alpha) \\
- \frac{1}{4\alpha}(R(X, Z) u)^v(\alpha)\right\} Y^v \\
+ \left\{ - \frac{1}{2\alpha^2} R(u, Y, Z, X) - \frac{1}{4\alpha} Y_v(\alpha) g(X, Z)\right\} \nabla \alpha \\
+ \frac{1}{2} g(X, Z) \nabla \nabla \alpha, \\
\right\}
\]

(16)
From the equality (8) which is the calculated formula for the Levi-Civita curvatures formula one gets

\[ R(X^v, Y^h)Z^v = \frac{1}{2\alpha^2} (R(X, Z)Y^h) + \frac{1}{4\alpha^2} (R(u, X)R(u, Z)Y^h) \]
\[ - \frac{1}{2\alpha^2} X^v(\alpha)(R(u, Z)Y^h) + \frac{1}{2\alpha^2} Z^v(\alpha)(R(u, X)Y^h) \]
\[ + \{ \frac{1}{4\alpha^2} X^v(\alpha)Z^v(\alpha) + \frac{1}{2\alpha} X^v(Z^v(\alpha)) \} Y^h \]
\[ \{ - \frac{1}{4\alpha^2} (R(u, Z)Y^h(\alpha) - \frac{1}{2\alpha}(\nabla_Y Z)^v(\alpha) \]
\[ - \frac{3}{4\alpha^2} Y^h(\alpha)Z^v(\alpha) + \frac{1}{2\alpha} Y^h(Z^v(\alpha)) \} X^v \]
\[ + \frac{3}{4\alpha^2} g(X, Z)Y^h(\alpha)\nabla_\alpha - \frac{1}{2\alpha^2} g(X, Z)\nabla_Y \nabla_\alpha, \]
\[ (17) \]

\[ R(X^v, Y^h)Z^h = \frac{1}{\alpha^2} (R(X, Y)Z^h) - \frac{1}{\alpha^2} X^v(\alpha)(R(u, Y)Z^h) \]
\[ + \frac{1}{\alpha^2} Y^v(\alpha)(R(u, X)Z^h) + \frac{1}{4\alpha^4} (R(u, X)R(u, Y)Z^h) \]
\[ - \frac{1}{4\alpha^2} (R(u, Y)R(u, X)Z^h) + \{ - \frac{1}{4\alpha^2} (R(u, Y)Z^h(\alpha) \]
\[ + \frac{1}{2\alpha} Y^v(Z^h(\alpha)) - \frac{3}{4\alpha^2} Y^v(\alpha)Z^h(\alpha) \} X^v \]
\[ + \{ \frac{1}{4\alpha^2} (R(u, X)Z^h(\alpha) - \frac{1}{2\alpha} X^v(Z^h(\alpha)) \]
\[ + \frac{3}{4\alpha^2} X^v(\alpha)Z^h(\alpha) \} Y^v, \]
\[ (18) \]

and

\[ R(X^v, Y^h)Z^v = \{ \frac{1}{2\alpha} Y^v(Z^v(\alpha)) - \frac{1}{4\alpha^2} Y^v(\alpha)Z^v(\alpha) \} X^v \]
\[ + \{ - \frac{1}{2\alpha} X^v(Z^v(\alpha)) + \frac{1}{4\alpha^2} X^v(\alpha)Z^v(\alpha) \} Y^v \]
\[ + \{ \frac{3}{4\alpha^2} Y^v(\alpha)g(X, Z) - \frac{3}{4\alpha^2} X^v(\alpha)g(Y, Z) \} \nabla_\alpha \]
\[ + \frac{1}{2\alpha^2} g(Y, Z)\nabla_X \nabla_\alpha - \frac{1}{2\alpha^2} g(X, Z)\nabla_Y \nabla_\alpha. \]
\[ (19) \]

**Proof.** We only prove the equation (16), the remaining ones are similar. Using curvature formula one gets

\[ R(X^h, Y^v)Z^h = \nabla_X^h \nabla_Y^v Z^h - \nabla_Y^v \nabla_X^h Z^h - \nabla_{[X^h, Y^v]} Z^h. \]
\[ (20) \]

From the equality (8) which is the calculated formula for the Levi-Civita connection of \( \bar{g} \), one can get,

\[ \nabla_X^h \nabla_Y^v Z^h = \nabla_X^h (\frac{1}{2\alpha^2} (R(u, Y)Z^h) + \frac{1}{2\alpha} Y^v(\alpha)Z^h - \frac{1}{2\alpha} Z^h(\alpha)Y^v). \]
\[ (21) \]
By taking account (1), (7) and (10) in (21), one can get,

\[
\bar{\nabla}_{X^h}\bar{\nabla}_{Y^v} Z^h = \frac{u^i}{2\alpha^2} (\nabla_X R(\partial_i, Y) Z)^h - \frac{u^i}{2\alpha^2} (R(\nabla_X \partial_i, Y) Z)^h \\
- \frac{3}{4\alpha^2} X^h(\alpha)(R(u, Y) Z)^h + \frac{1}{4\alpha^2} Z^h(\alpha)(R(u, Y) X)^h \\
- \frac{1}{4\alpha^2} (R(X, R(u, Y) Z) u)^v - \frac{1}{4\alpha} Y^v(\alpha)(R(X, Z) u)^v \\
+ \frac{1}{4\alpha^2} (R(u, Y) Z)^h(\alpha) X^h + \left\{ - \frac{1}{4\alpha^2} X^h(\alpha) Y^v(\alpha) \\
+ \frac{1}{2\alpha} X^h(Y^v(\alpha)) \right\} Z^h + \left\{ \frac{3}{4\alpha^2} X^h(\alpha) Z^h(\alpha) - \frac{1}{2\alpha} X^h(Z^h(\alpha)) \right\} Y^v \\
+ \frac{1}{2\alpha} Y^v(\alpha)(\nabla X Z)^h - \frac{1}{2\alpha} Z^h(\alpha)(\nabla X Y)^v \\
+ \left\{ - \frac{1}{4\alpha} Y^v(\alpha) g(X, Z) - \frac{1}{4\alpha^2} R(u, Y, Z, X) \right\} \bar{\nabla} \alpha.
\]  

(22)

Using the equation (1) in \(\bar{\nabla}_{X^h} Z^h\) and putting the result in \(\bar{\nabla}_{Y^v} \bar{\nabla}_{X^h} Z^h\) gives,

\[
\bar{\nabla}_{Y^v} \bar{\nabla}_{X^h} Z^h = \bar{\nabla}_{Y^v}(\nabla_X Z)^h - \frac{1}{2\alpha^2} Y^v(\alpha) X^h(\alpha) Z^h + \frac{1}{2\alpha} Y^v(X^h(\alpha)) Z^h \\
+ \frac{1}{2\alpha} X^h(\alpha) \bar{\nabla}_{Y^v} Z^h - \frac{1}{2\alpha^2} Y^v(\alpha) Z^h(\alpha) X^h + \frac{1}{2\alpha} Y^v(Z^h(\alpha)) X^h \\
+ \frac{1}{2\alpha} Z^h(\alpha) \bar{\nabla}_{Y^v} X^h - \frac{1}{2}(R(X, Z) Y)^v - \frac{u^i}{2} \bar{\nabla}_{Y^v}(R(X, Z) \partial_i)^v \\
- \frac{1}{2} g(X, Z) \bar{\nabla}_{Y^v} \bar{\nabla} \alpha.
\]  

(23)
By taking account (8) in (23), one can get,

\[ \nabla_{Y^h} \nabla_{X^h} Z^h = \frac{1}{2\alpha^2} (R(u, Y)^h) \nabla_{X^h} Z^h + \frac{1}{2\alpha} Y^v(\alpha)(\nabla_{X^h} Z^h) \\
- \frac{1}{2\alpha} (\nabla_{X^h} Z^h(\alpha) Y^v - \frac{1}{2\alpha^2} Y^v(\alpha) X^h Z^h) \\
+ \frac{1}{2\alpha} Y^v(X^h(\alpha)) Z^h + \frac{1}{4\alpha^3} X^h(\alpha)(R(u, Y)^h) \\
+ \frac{1}{4\alpha^2} X^h(\alpha) Y^v(\alpha) Z^h - \frac{1}{4\alpha^2} X^h(\alpha) Z^h(\alpha) Y^v \\
- \frac{1}{2\alpha^2} Y^v(\alpha) Z^h(\alpha) X^h + \frac{1}{2\alpha} Y^v(Z^h(\alpha)) X^h \] (24)

According to the equality (8), \( \nabla_{(\nabla_{X^h} Y^h)} Z^h \) can be written as follow,

\[ \nabla_{(\nabla_{X^h} Y^h)} Z^h = \frac{1}{2\alpha^2} (R(u, \nabla_{X^h} Y^h) Z^h + \frac{1}{2\alpha} (\nabla_{X^h} Y^h(\alpha)) Z^h \\
- \frac{1}{2\alpha} Z^h(\alpha)(\nabla_{X^h} Y^h)^v. \] (25)

By putting (22), (24) and (25) in (20) and after some calculations, the result can be achieved. 

**Theorem 5** Let \( \{E_1, \ldots, E_n\} \) be an orthonormal locally frame on \( M \). Then the Ricci operator \( \bar{Q} \) of \( \nabla \) is determined by

\[ \bar{Q}(X^h) = \frac{1}{\alpha} Q^h(X) + \left\{ \frac{1}{4\alpha^2} \|\nabla \alpha\|^2 - \frac{1}{2\alpha} \Delta \alpha \right\} X^h \\
+ \frac{1}{2\alpha} \left\{ h \nabla_{X^h} h \nabla \alpha - v \nabla_{X^h} v \nabla \alpha \right\} \\
+ \frac{1}{2\alpha^2} \nabla_{X^h} \nabla \alpha - \frac{2n+1}{4\alpha^2} X^h(\alpha) \nabla \alpha - \frac{1}{4\alpha^2} X^h(\alpha) h \nabla \alpha \\
+ \frac{1}{4\alpha^2} X^h(\alpha) v \nabla \alpha \] (26)

\[ + \sum_{i=1}^{n} \left\{ \frac{3}{4\alpha^3} (R(u, R(X, E_i) u) E_i)^h + \frac{1}{4\alpha^2} (R(X, E_i) u)^v(\alpha) E_i^h \\
- \frac{1}{4\alpha^3} (R(u, E_i) R(u, E_i) X)^h + \frac{1}{2\alpha} ((\nabla_{E_i} R)(X, E_i) u)^v \\
- \frac{3}{2\alpha^2} E^h_i(\alpha)(R(X, E_i) u)^v + \frac{1}{4\alpha^2} (R(u, E_i) X)^h(\alpha) E_i^v \right\}, \]
Using (14) and (17) gives us

\[
\begin{align*}
M & = \text{from the definition} \\
\end{align*}
\]

We shall give the proof of the equation (26). If \( X \) is a vector field on \( M \) then from the definition \( \tilde{Q}(X^h) \) we have

\[
\tilde{Q}(X^h) = \sum_{i=1}^{n} \left\{ \frac{1}{\alpha} \tilde{R}(X^h, E_i^h) E_i^h + \alpha \tilde{R}(X^h, E_i^v) E_i^v \right\}
\]

(28)

Using (13) and (17) gives us

\[
\begin{align*}
\tilde{Q}(X^h) & = \frac{1}{\alpha} \sum_{i=1}^{n} \left\{ (R(X, E_i) E_i) + \frac{1}{4\alpha^2} (R(u, R(X, E_i) u) E_i) \\
& \quad + \frac{1}{2\alpha^2} (R(u, R(X, E_i) u) E_i) + \frac{3}{4\alpha^2} E_i^h(\alpha) E_i^h(\alpha) - \frac{1}{2\alpha} E_i^h(\alpha) \right\} X^h \\
& \quad + \left\{ - \frac{3}{4\alpha^2} X^h(\alpha) E_i^h(\alpha) + \frac{1}{2\alpha} X^h(\alpha) E_i^h(\alpha) + \frac{1}{4\alpha} (R(X, E_i) u)^v(\alpha) \right\} E_i^h \\
& \quad + \frac{1}{2} ((\nabla E_1 R)(X, E_i) u)^v - \frac{1}{2\alpha} E_i^h(\alpha) (R(X, E_i) u)^v \\
& \quad - \frac{1}{\alpha} E_i^h(\alpha) (R(X, E_i) u)^v + \frac{1}{4\alpha} X^h(\alpha) g(E_i, E_i) \\
& \quad - \frac{1}{4\alpha} E_i^h(\alpha) g(X, E_i) \} \tilde{\nabla} \alpha + \frac{1}{2} g(X, E_i) \nabla \nabla \alpha - \frac{1}{2} g(E_i, E_i) \nabla \nabla \alpha \\
& \quad - \alpha \sum_{i=1}^{n} \left\{ \frac{1}{4\alpha^4} (R(u, e_i R(X, E_i) X)^h + \frac{1}{4\alpha^2} E_i^v(\alpha) E_i^v(\alpha) \\
& \quad + \frac{1}{2\alpha^2} E_i^v(\alpha) ) X^h + \left\{ - \frac{1}{4\alpha^2} (R(u, E_i) X)^h(\alpha) \\
& \quad - \frac{3}{4\alpha^2} X^h(\alpha) E_i^v(\alpha) + \frac{1}{2\alpha} X^h(\alpha) E_i^v(\alpha) \right\} E_i^v \\
& \quad + \frac{3}{4\alpha^3} g(E_i, E_i) X^h(\alpha) \nabla \alpha - \frac{1}{2\alpha^2} g(E_i, E_i) \nabla \nabla \alpha \right\}
\]
\end{align*}
\]

(29)
By setting the expressions

\[ \sum_{i=1}^{n} (R(X, E_i)E_i)^h = Q^h(X), \]
\[ \sum_{i=1}^{n} \frac{1}{\alpha} E_i^h(\alpha) E_i^h = h\bar{\nabla}\alpha, \]
\[ \sum_{i=1}^{n} X^h(E_i^h(\alpha)) E_i^h = X^h(\alpha)h\bar{\nabla}\alpha - \frac{1}{2}||h\bar{\nabla}\alpha||^2 X^h + a h\bar{\nabla}_{X^h}h\bar{\nabla}\alpha, \]
\[ \sum_{i=1}^{n} g(E_i, E_i) = n, \]
\[ \sum_{i=1}^{n} E_i^h(\alpha) g(X, E_i) = X^h(\alpha), \]
\[ \sum_{i=1}^{n} g(X, E_i) E_i^h = X^h, \]
\[ \sum_{i=1}^{n} \alpha E_i^h(\alpha) E_i^u = v\bar{\nabla}\alpha, \]

and

\[ \sum_{i=1}^{n} X^h(E_i^h(\alpha)) = -\frac{1}{2\alpha^2} X^h(\alpha)v\bar{\nabla}\alpha + \frac{1}{\alpha} v\bar{\nabla}_{X^h}v\bar{\nabla}\alpha, \]

in [29], we get the result; where \( \bar{\nabla}\alpha = h\bar{\nabla}\alpha + v\bar{\nabla}\alpha \) and \( \bar{\nabla}_{X^h}\bar{\nabla}\alpha = h\bar{\nabla}_{X^h}\bar{\nabla}\alpha + v\bar{\nabla}_{X^h}\bar{\nabla}\alpha \) are the decompositions of \( \bar{\nabla}\alpha \) and \( \bar{\nabla}_{X^h}\bar{\nabla}\alpha \) to the horizontal and vertical components, respectively. Moreover, \( ||h\bar{\nabla}\alpha|| \) is the norm of the horizontal part of \( \bar{\nabla}\alpha \) with respect to the metric \( \bar{g} \).

The following proposition calculates the sectional curvatures of \( \bar{g} \).

**Proposition 6** Suppose \( K \) represents the sectional curvature of \( \bar{g} \). Let \( X \) and \( Y \) be locally orthonormal vector fields on \( M \). Then \( K \) is given by

\[ K(X^h, Y^h) = \frac{1}{\alpha} K(X, Y) - \frac{3}{4\alpha^3}||R(X, Y)u||^2 + \frac{3}{4\alpha^3} X^h(\alpha)Y^h(\alpha) \]
\[ - \frac{1}{2\alpha^2} Y^h(\alpha) + \frac{1}{4\alpha^3} X^h(\alpha)X^h(\alpha) \]
\[ - \frac{1}{2\alpha^2} \bar{g}(\bar{\nabla}_{X^h}\bar{\nabla}\alpha, X^h), \]

(30)

\[ K(X^h, Y^v) = \frac{1}{4\alpha^3}||R(u, Y)X||^2 - \frac{1}{4\alpha} Y^v(\alpha)Y^v(\alpha) - \frac{1}{2} Y^v(Y^v(\alpha)) \]
\[ - \frac{3}{4\alpha^3} X^h(\alpha)X^h(\alpha) + \frac{1}{2\alpha^2} \bar{g}(\bar{\nabla}_{X^h}\bar{\nabla}\alpha, X^h), \]

and

\[ K(X^v, Y^v) = \frac{1}{2} Y^v(Y^v(\alpha)) - \frac{1}{4\alpha} Y^v(\alpha)Y^v(\alpha) - \frac{3}{4\alpha} X^v(\alpha)X^v(\alpha) \]
\[ + \frac{1}{2} \bar{g}(\bar{\nabla}_{X^v}\bar{\nabla}\alpha, X^v), \]

where \( K(X, Y) \) is the sectional curvature of \( g \) at the plane spanned by \{ \( X, Y \) \}. 

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Proof. We only prove (30). According to the definition of sectional curvature, we have
\[
\bar{K}(\frac{X^h}{\sqrt{\alpha}}, \frac{Y^h}{\sqrt{\alpha}}) = \frac{1}{\alpha^2} \bar{g}(\bar{R}(X^h, Y^h)Y^h, X^h).
\]

Using the equation (14) gives us
\[
\frac{1}{\alpha^2} \bar{g}(\bar{R}(X^h, Y^h)Y^h, X^h) = \frac{1}{\alpha^2} \{\bar{g}((R(X, Y)Y^h, X^h)
+ \frac{1}{4\alpha^2} \bar{g}((R(u, R(X, Y)u)Y^h, X^h)
+ \frac{1}{2\alpha^2} \bar{g}((R(u, R(X, Y)u)Y^h, X^h)
+ \{\frac{3}{4\alpha^2} Y^h(\alpha)Y^h(\alpha) - \frac{1}{2\alpha} Y^h(Y^h(\alpha))\bar{g}(X^h, X^h)
+ \frac{1}{4\alpha} X^h(\alpha)X^h(\alpha) - \frac{1}{2} \bar{g}(\bar{\nabla}_X^h, \bar{\nabla}_\alpha, X^h)\}.
\]

Setting \(\bar{g}((R(X, Y)Y^h, X^h) = \alpha K(X, Y)\) and using the symmetries of \(R\) give us
\[
\bar{K}(\frac{X^h}{\sqrt{\alpha}}, \frac{Y^h}{\sqrt{\alpha}}) = \frac{1}{\alpha} K(X, Y) - \frac{3}{4\alpha^3} ||R(X, Y)u||^2
+ \frac{3}{4\alpha^3} Y^h(\alpha)Y^h(\alpha) - \frac{1}{2\alpha^2} Y^h(Y^h(\alpha))
+ \frac{1}{4\alpha^3} X^h(\alpha)X^h(\alpha) - \frac{1}{2\alpha^2} \bar{g}(\bar{\nabla}_X^h, \bar{\nabla}_\alpha, X^h).
\]

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