Binary Decision Diagrams
for Affine Approximation

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Abstract. Selman and Kautz’s work on “knowledge compilation” established how approximation (strengthening and/or weakening) of a propositional knowledge-base can be used to speed up query processing, at the expense of completeness. In this classical approach, querying uses Horn over- and under-approximations of a given knowledge-base, which is represented as a propositional formula in conjunctive normal form (CNF). Along with the class of Horn functions, one could imagine other Boolean function classes that might serve the same purpose, owing to attractive deduction-computational properties similar to those of the Horn functions. Indeed, Zanuttini has suggested that the class of affine Boolean functions could be useful in knowledge compilation and has presented an affine approximation algorithm. Since CNF is awkward for presenting affine functions, Zanuttini considers both a sets-of-models representation and the use of modulo 2 congruence equations. In this paper, we propose an algorithm based on reduced ordered binary decision diagrams (ROBDDs). This leads to a representation which is more compact than the sets of models and, once we have established some useful properties of affine Boolean functions, a more efficient algorithm.

1 Introduction

A recurrent theme in artificial intelligence is the efficient use of (propositional) knowledge-bases. A promising approach, which was initially proposed by Selman and Kautz [11], is to query (and perform deductions from) upper and lower approximations, commonly called envelopes and cores respectively, of a given knowledge-base. By choosing approximations that allow more efficient inference, it is often possible to quickly determine that the envelope of the given knowledge-base entails the query, and therefore so does the full knowledge-base, avoiding the costly inference from the full knowledge-base. When this fails, it may be possible to quickly show that the query is not entailed by the core, and therefore not entailed by the full knowledge-base. Only when both of these fail must the full knowledge-base be used for inference.

It is usually assumed that Boolean functions are represented in clausal form, and that approximations are Horn [11, 5], as inference from Horn knowledge-bases is exponentially more efficient than from unrestricted knowledge-bases. However, it has been noted that there are other well-understood classes that
have computational properties that include some of the attractive properties of the Horn class.

Zanuttini [12, 13] discusses the use of other classes of Boolean functions for approximation and points out that affine approximations have certain advantages over Horn approximations, most notably the fact that they do not blow out in size. This is certainly the case when affine functions are represented in the form of modulo-2 congruence equations. The more general sets-of-models representation is also considered by Zanuttini. In this paper, we consider another general representation, namely the well-known Reduced Ordered Binary Decision Diagrams (ROBDDs). We prove some important properties of affine functions represented as ROBDDs, and present a new ROBDD algorithm for deriving affine envelopes.

The balance of the paper proceeds as follows. In Section 2 we recapitulate the definition of the Boolean affine class, and we establish some of their important properties. We also briefly introduce ROBDDs, but mainly to fix our notation, as we assume that the reader is familiar with Boolean functions and their representation as decision diagrams. Section 3 recalls the model-based affine envelope algorithm, and develops our own ROBDD-based algorithm, along with a correctness proof. Section 4 describes our testing methodology, including our algorithm for generating random ROBDDs, and presents our results. Section 5 discusses related work and applications, and concludes.

2 Propositional Classes, Approximation and ROBDDs

We use ROBDDs [1, 2] to represent Boolean functions. Horiyama and Ibaraki [6] have recommended ROBDDs as suitable for implementing knowledge bases. Our choice of ROBDDs as a data structure was not so much influenced by that recommendation, as by the convenience of working with a canonical representation for Boolean functions, and one that lends itself to inductive reasoning and recursive problem solving. Additionally, ROBDD-based inference is fast, and in particular, checking whether a valuation is a model of an $n$-place function given by an ROBDD requires a path traversal of length no more than $n$.

ROBDD algorithms for approximation are of interest in their own right and some find applications in dataflow analysis [8]. From this aspect, this paper continues earlier work by Schachte and Søndergaard [8, 9] who gave algorithms for finding monotone, Krom, and Horn envelopes. Here we introduce an ROBDD algorithm for affine envelopes, which is new.

2.1 Boolean functions

Let $B = \{0, 1\}$ and let $V$ be a denumerable set of variables. A valuation $\mu : V \rightarrow B$ is a (total) assignment of truth values to the variables in $V$. Let $I = V \rightarrow B$ denote the set of $V$-valuations. A partial valuation $\mu : V \rightarrow B \cup \{\perp\}$ assigns truth values to some variables in $V$, and $\perp$ to others. Let $I_p = V \rightarrow B \cup \{\perp\}$. 
We use the notation \( \mu[x \mapsto i] \), where \( x \in \mathcal{V} \) and \( i \in \mathcal{B} \), to denote the valuation \( \mu \) updated to map \( x \) to \( i \), that is,

\[
\mu[x \mapsto i](v) = \begin{cases} 
i \\
\mu(v) 
\end{cases} \text{ if } v = x
\]

A Boolean function over \( \mathcal{V} \) is a function \( \varphi : \mathcal{I} \rightarrow \mathcal{B} \). We let \( \mathcal{B} \) denote the set of all Boolean functions over \( \mathcal{V} \). The ordering on \( \mathcal{B} \) is the usual: \( x \leq y \) iff \( x = \varnothing \lor y = 1 \). \( \mathcal{B} \) is ordered pointwise, so that the ordering relation corresponds exactly to classical entailment, \( \models \). It is convenient to overload the symbols for truth and falsehood. Thus we let \( 1 \) denote the largest element of \( \mathcal{B} \) (that is, \( \lambda \mu.1 \)) as well as of \( \mathcal{B} \). Similarly \( 0 \) denotes the smallest element of \( \mathcal{B} \) (that is, \( \lambda \mu.0 \)) as well as of \( \mathcal{B} \). A valuation \( \mu \) is a model for \( \varphi \), denoted \( \mu \models \varphi \), if \( \varphi(\mu) = 1 \).

We let \( \text{models}(\varphi) \) denote the set of models of \( \varphi \). Conversely, the unique Boolean function that has exactly the set \( \mathcal{M} \) as models is denoted \( \text{fn}(\mathcal{M}) \). A Boolean function \( \varphi \) is said to be independent of a variable \( x \) when for all valuations \( \mu \),

\[
\mu[x \mapsto 0] \models \varphi \text{ iff } \mu[x \mapsto 1] \models \varphi.
\]

In the context of an ordered set of \( k \) variables of interest, \( x_1, \ldots, x_k \), we may identify with \( \mu \) the binary sequence \( \text{bits}(\mu) \) of length \( k \):

\[
\mu(x_1), \ldots, \mu(x_k)
\]

which we will write simply as a bit-string of length \( k \). Similarly we may think of, and write, the set of valuations \( \mathcal{M} \) as a set of bit-strings:

\[
\text{bits}(\mathcal{M}) = \{ \text{bits}(\mu) \mid \mu \in \mathcal{M} \}
\]

As it hardly creates confusion, we shall present valuations variously as functions or bitstrings. We denote the zero valuation, which maps \( x_i \) to \( \varnothing \) for all \( 1 \leq i \leq k \), by \( 0 \).

We use the Boolean connectives \( \neg \) (negation), \( \wedge \) (conjunction), \( \lor \) (disjunction) and \( + \) (exclusive or, or “xor”). These connectives operate on Boolean functions, that is, on elements of \( \mathcal{B} \). Traditionally they are overloaded to also operate on truth values, that is, elements of \( \mathcal{B} \). However, we deviate at this point, as the distinction between xor and its “bit-wise” analogue will be critical in what follows. Hence we denote the \( \mathcal{B} \) (bit) version by \( \oplus \). We extend this to valuations and bit-strings in the natural way:

\[
(\mu_1 \oplus \mu_2)(x) = \mu_1(x) \oplus \mu_2(x)
\]

and we let \( \oplus_3 \) denote the “xor of three” operation \( \lambda \mu_1 \mu_2 \mu_3. \mu_1 \oplus \mu_2 \oplus \mu_3 \). We follow Zanuttini [12] in further overloading ‘\( \oplus \)’ and using the notation

\[
M_\mu = \mu \oplus M = \{ \mu \oplus \mu' \mid \mu' \in \mathcal{M} \}
\]

We read \( M_\mu \) as “\( M \) translated by \( \mu \)”. Note that for any set \( \mathcal{M} \), the function \( \lambda \mu. M_\mu \) is an involution: \( (M_\mu)_\mu = \mathcal{M} \).

A final overloading results in the following definition. For \( \varphi \in \mathcal{B} \), and \( \mu \in \mathcal{I} \), let \( \varphi \oplus \mu = \text{fn}(\mathcal{M}_\mu) \) where \( \mathcal{M} = \text{models}(\varphi) \).
2.2 The affine class

An affine function is one whose set of models is closed under pointwise application of $\oplus_3$ [10]. Affine functions have a number of attractive properties, as we shall see. Syntactically, a Boolean function is affine iff it can be written as a conjunction of affine equations

$$c_1x_1 + c_2x_2 + \ldots + c_kx_k = c_0$$

where $c_i \in \{0, 1\}$ for all $i \in \{0, \ldots, k\}$.

1 This is well known, but for completeness we prove it below.

The affine class contains $1$ and is closed under conjunction. Hence the concept of a (unique) affine envelope is well defined, and the operation of taking the affine envelope is an upper closure operator [8]. For convenience, let us introduce a name for this operator:

**Definition 1.** Let $\varphi$ be a Boolean function. The affine envelope, $\text{aff}(\varphi)$, of $\varphi$ is defined:

$$\text{aff}(\varphi) = \bigwedge \{ \psi \mid \varphi \models \psi \text{ and } \psi \text{ is affine} \}$$

There are numerous other classes of interest, including isotone, antitone, Krom, Horn, $k$-Horn [4], and $k$-quasi-Horn functions, for which the concept of an envelope is well-defined, as they form upper closure operators [9].

Zanuttini [12] exploits the close connection between vector spaces and the sets of models of affine functions. For our purposes we call a set $S$ of bistrings a vector space iff $0 \in S$ and $S$ is closed under $\oplus$. The next proposition simplifies the task of doing model-closure under $\oplus_3$.

**Proposition 1 ([12]).** A non-empty set of models $M$ is closed under $\oplus_3$ iff $M_\mu$ is a vector space, where $\mu$ is any element of $M$.

**Proof:** Let $\mu$ be an arbitrary element of $M$. Clearly $M_\mu$ contains $0$, so the right-hand side of the claim amounts to $M_\mu$ being closed under $\oplus$.

For the 'if' direction, assume $M_\mu$ is closed under $\oplus$ and consider $\mu_1, \mu_2, \mu_3 \in M$. Since $\mu \oplus \mu_2$ and $\mu \oplus \mu_3$ are in $M_\mu$, so is $\mu_2 \oplus \mu_3$. And since furthermore $\mu \oplus \mu_1$ is in $M_\mu$, so is $\mu \oplus \mu_1 \oplus \mu_2 \oplus \mu_3$. Hence $\mu_1 \oplus \mu_2 \oplus \mu_3$ is in $M$.

For the 'only if' direction, assume $M$ is closed under $\oplus_3$, and consider $\mu_1, \mu_2 \in M_\mu$. All of $\mu, \mu \oplus \mu_1$ and $\mu \oplus \mu_2$ are in $M$, and so $\mu \oplus (\mu \oplus \mu_1) \oplus (\mu \oplus \mu_2) = \mu \oplus \mu_1 \oplus \mu_2 \in M_\mu$. Hence $\mu_1 \oplus \mu_2 \in M_\mu$.

1 In some circles, such as cryptography/coding community, the term “affine” is used only for a function that is 0 or 1, or can be written $c_1x_1 + c_2x_2 + \ldots + c_kx_k + c_0$ (the latter is what Post [7] called an “alternating” function). The resulting set of “affine” functions is not closed under conjunction.

2 Popular classes such as unate functions and renamable Horn are not closed under conjunction and therefore do not have well-defined concepts of (unique) envelopes. For example, $x \rightarrow y$ and $x \leftarrow y$ both are unate, while $x \leftrightarrow y$ is not, so the “unate envelope” of the latter is not well-defined.
Proposition 2. A Boolean function is affine iff it can be written as a conjunction of equations

\[ c_1 x_1 + c_2 x_2 + \ldots + c_k x_k = c_0 \]

where \( c_i \in \{0, 1\} \) for all \( i \in \{0, \ldots, k\} \).

Proof: Assume the Boolean function \( \varphi \) is given as a conjunction of equations of the indicated form and let \( \mu_1, \mu_2 \) and \( \mu_3 \) be models. That is, for each equation we have

\[
\begin{align*}
    c_1 \mu_1(x_1) + c_2 \mu_1(x_2) + \ldots + c_k \mu_1(x_k) &= c_0 \\
    c_1 \mu_2(x_1) + c_2 \mu_2(x_2) + \ldots + c_k \mu_2(x_k) &= c_0 \\
    c_1 \mu_3(x_1) + c_2 \mu_3(x_2) + \ldots + c_k \mu_3(x_k) &= c_0
\end{align*}
\]

Adding left-hand sides and adding right-hand sides, making use of the fact that ‘·’ distributes over ‘+’, we get

\[
\begin{align*}
    c_1 \mu(x_1) + c_2 \mu(x_2) + \ldots + c_k \mu(x_k) &= c_0 + c_0 + c_0 = c_0
\end{align*}
\]

where \( \mu = \mu_1 \oplus \mu_2 \oplus \mu_3 \). As \( \mu \) thus satisfies each equation, \( \mu \) is a model of \( \varphi \). This establishes the ‘if’ direction.

For the ‘only if’ part, note that by Proposition 1, we obtain a vector space \( M_\mu \) from any non-empty set \( M \) closed under \( \oplus_3 \) by translating each element of \( M \) by \( \mu \in M \). A basis for \( M_\mu \) can be formed by taking one vector at a time from \( M_\mu \) and adding it to the basis if it is linearly independent of the existing basis vectors. From this basis, a set of linear equations

\[
\begin{align*}
    a_{11} x_1 &+ \cdots + a_{1k} x_k = 0 \\
    a_{21} x_1 &+ \cdots + a_{2k} x_k = 0 \\
    \vdots & \vdots \\
    a_{j1} x_1 &+ \cdots + a_{jk} x_k = 0
\end{align*}
\]

can be computed that have exactly \( M_\mu \) as their set of models (a method is provided by Zanuttini [12], in the proof of his Proposition 3). Each function \( f_i = \lambda x_1, \ldots, x_k. a_{i1} x_1 \oplus \cdots \oplus a_{ik} x_k \) is linear, so for \( \nu \in M_\mu \), \( f_i(\nu \oplus \mu) = f_i(\nu) + f_i(\mu) = f_i(\mu) \). Hence \( M \) can be described by the set of affine equations

\[
\begin{align*}
    a_{11} x_1 &+ \cdots + a_{1k} x_k = f_1(\mu) \\
    a_{21} x_1 &+ \cdots + a_{2k} x_k = f_2(\mu) \\
    \vdots & \vdots \\
    a_{j1} x_1 &+ \cdots + a_{jk} x_k = f_j(\mu)
\end{align*}
\]

as desired.  

It follows from the syntactic characterisation that the number of models possessed by an affine function is either 0 or a power of 2. Other properties will now be established that are used in the justification of the affine envelope algorithm.
of Section 3. The first property is that if a Boolean function \( \varphi \) has two models that differ for exactly one variable \( v \), then its affine envelope will be independent of \( v \). To state this precisely we introduce a concept of a “characteristic” valuation for a variable.

**Definition 2.** In the context of a set of variables \( V \), let \( v \in V \). The characteristic valuation for \( v \) is defined by

\[
\chi_v(x) = \begin{cases} 
1 & \text{if } x = v \\
0 & \text{otherwise}
\end{cases}
\]

Note that \( \mu \oplus \chi_v \) is the valuation which agrees with \( \mu \) for all variables except \( v \). Moreover, if \( \mu \models \varphi \), then both of \( \mu \) and \( \mu \oplus \chi_v \) are models of \( \exists v(\varphi) \).

**Proposition 3.** Let \( \varphi \) be a Boolean function whose set of models forms a vector space, and assume that for some valuation \( \mu \) and some variable \( v \), \( \mu \) and \( \mu \oplus \chi_v \) both satisfy \( \varphi \). Then \( \varphi \) is independent of \( v \).

*Proof:* The set \( M \) of models contains at least two elements, and since it is closed under \( \oplus, \chi_v \) is a model. Hence for every model \( \nu \) of \( \varphi \), \( \nu \oplus \chi_v \) is another model. It follows that \( \varphi \) is independent of \( v \). \( \blacksquare \)

**Proposition 4.** Let \( \varphi \) be a Boolean function. If, for some valuation \( \mu \) and some variable \( v \), \( \mu \) and \( \mu \oplus \chi_v \) both satisfy \( \varphi \), then \( \text{aff}(\varphi) = \exists v(\text{aff}(\varphi)) \).

*Proof:* Let \( \mu \) be a model of \( \varphi \), with \( \mu \oplus \chi_v \) also a model. For every model \( \nu \) of \( \varphi \), we have that \( \nu \oplus \mu \oplus (\mu \oplus \chi_v) \) satisfies \( \text{aff}(\varphi) \), that is, \( \nu \oplus \chi_v \models \text{aff}(\varphi) \). Now since both \( \nu \) and \( \nu \oplus \chi_v \) satisfy \( \text{aff}(\varphi) \), it follows that \( \exists v(\text{aff}(\varphi)) \) cannot have a model that is not already a model of \( \text{aff}(\varphi) \) (and the converse holds trivially). Hence \( \text{aff}(\varphi) = \exists v(\text{aff}(\varphi)) \). \( \blacksquare \)

**Proposition 5.** For all Boolean functions \( \varphi \), \( \text{aff}(\exists v(\varphi)) = \exists v(\text{aff}(\varphi)) \).

*Proof:* We need to show that the models of \( \text{aff}(\exists v(\varphi)) \) are exactly the models of \( \exists v(\text{aff}(\varphi)) \). Clearly \( \text{aff}(\exists v(\varphi)) \) is \( \emptyset \) iff \( \varphi \) is \( \emptyset \) iff \( \exists v(\text{aff}(\varphi)) \) is \( \emptyset \). So we can assume that \( \text{aff}(\exists v(\varphi)) \) is satisfiable—let \( \mu \models \text{aff}(\exists v(\varphi)) \). Then, for some positive odd number \( k \),

\[
\mu = \mu_1 \oplus \mu_2 \oplus \cdots \oplus \mu_k
\]

with \( \mu_1, \ldots, \mu_k \) being different models of \( \exists v(\varphi) \). These \( k \) models can be partitioned into two sets, according as they satisfy \( \varphi \); let

\[
M = \{ \mu_i \mid 1 \leq i \leq k, \mu_i \models \varphi \} \quad M' = \{ \mu_i \mid 1 \leq i \leq k, \mu_i \not\models \varphi \}
\]

Then both \( M \) and \( M' \) consist entirely of models of \( \varphi \). Hence, depending on the parity of \( M \)’s cardinality, either \( \mu \) or \( \mu \oplus \chi_v \) is a model of \( \text{aff}(\varphi) \) (or both are). In either case, \( \mu \models \exists v(\text{aff}(\varphi)) \).

Conversely, let \( \mu \models \exists v(\text{aff}(\varphi)) \). Then either \( \mu \) or \( \mu \oplus \chi_v \) is a model of \( \text{aff}(\varphi) \) (or both are). Hence \( \mu \) (or \( \mu \oplus \chi_v \) as the case may be) can be written as a sum of \( k \) models \( \mu_1, \ldots, \mu_k \) (\( k \) odd) of \( \varphi \). It follows that both \( \mu_1 \oplus \mu_2 \oplus \cdots \oplus \mu_k \) and \( \mu_1 \oplus \mu_2 \oplus \cdots \oplus \mu_k \oplus \chi_v \) are models of \( \exists v(\varphi) \). Hence \( \mu \models \exists v(\text{aff}(\varphi)) \). \( \blacksquare \)
2.3 ROBDDs

We briefly recall the essentials of ROBDDs [3]. Let the set $V$ of propositional variables be equipped with a total ordering $\prec$. Binary decision diagrams (BDDs) are defined inductively as follows:

- 0 is a BDD.
- 1 is a BDD.
- If $x \in V$ and $R_1$ and $R_2$ are BDDs then $\text{ite}(x, R_1, R_2)$ is a BDD.

Let $R = \text{ite}(x, R_1, R_2)$. We say a BDD $R'$ appears in $R$ if $R' = R$ or $R'$ appears in $R_1$ or $R_2$. We define $\text{vars}(R) = \{v \mid \text{ite}(v, \bot, \bot) \text{ appears in } R\}$. The meaning of a BDD is given as follows.

\[
\begin{align*}
[0] &= 0 \\
[1] &= 1 \\
\text{ite}(x, R_1, R_2) &= (x \land [R_1]) \lor (\bar{x} \land [R_2])
\end{align*}
\]

A BDD is an \textit{Ordered binary decision diagram} (OBDD) iff it is 0 or 1 or if it is $\text{ite}(x, R_1, R_2)$, $R_1$ and $R_2$ are OBDDs, and $\forall x' \in \text{vars}(R_1) \cup \text{vars}(R_2) : x \prec x'$.

An OBDD $R$ is a \textit{Reduced Ordered Binary Decision Diagram} (ROBDD) [2, 3] iff for all BDDs $R_1$ and $R_2$ appearing in $R$, $R_1 = R_2$ when $[R_1] = [R_2]$. Practical implementations [1] use a function $\text{mknd}(x, R_1, R_2)$ to create all ROBDD nodes as follows:

1. If $R_1 = R_2$, return $R_1$ instead of a new node, as $[\text{ite}(x, R_1, R_2)] = [R_1]$.
2. If an identical ROBDD was previously built, return that one instead of a new one; this is accomplished by keeping a hash table, called the \textit{unique table}, of all previously created nodes.
3. Otherwise, return $\text{ite}(x, R_1, R_2)$.

This ensures that ROBDDs are strongly canonical: a shallow equality test is sufficient to determine whether two ROBDDs represent the same Boolean function.

Figure 2 shows some example ROBDDs. The ROBDD in Figure 2(a) denotes the function which has five models: $\{00011, 00110, 01001, 01101, 10101\}$. In general we depict the ROBDD $\text{ite}(x, R_1, R_2)$ as a directed acyclic graph rooted in $x$, with a solid arc from $x$ to the dag for $R_1$ and a dashed line from $x$ to the dag for $R_2$. However, to avoid unnecessary clutter, we omit the node (sink) for 0 and all arcs leading to that sink.

As a typical example of an ROBDD algorithm, Algorithm 1 generates the disjunction of two given ROBDDs. This operation will be used by the affine approximation algorithm presented in Section 3.

Algorithm 2 is used to extract a model from an ROBDD. For an unsatisfiable ROBDD (that is, 0) we return $\bot$. Although presented here in recursive fashion, it is better implemented in an iterative manner whereby we traverse through the ROBDD, one pointer moving down the “else” branch at each node, a second pointer trailing immediately behind. If a 1 sink is found, we return the path traversed thus far and note that any further variables which we are yet to
Algorithm 1 The “or” operator for ROBDDs

\[
\begin{align*}
or(1, \_)& = 1 \\
or(0, \_)& = 0 \\
or(\_, 1)& = 1 \\
or(\_, 0)& = 0 \\
or(\text{ite}(x, T, E), \text{ite}(x', T', E'))& \\
|x < x'| &= \text{mknd}(x, \text{or}(T, \text{ite}(x', T', E')), \text{or}(E, \text{ite}(x', T', E')))) \\
|x' < x| &= \text{mknd}(x', \text{or}(\text{ite}(x, T, T'), \text{or}(\text{ite}(x, T, E), E')))) \\
\text{otherwise}& = \text{mknd}(x, \text{or}(T, T'), \text{or}(E, E'))
\end{align*}
\]

Algorithm 2 get\_model algorithm for ROBDDs

\[
\begin{align*}
\text{get\_model}(0) &= \bot \\
\text{get\_model}(1) &= \lambda v. \bot \\
\text{get\_model}(\text{ite}(x, T, E))& = \\
& \quad \text{let } \mu = \text{get\_model}(T) \\
& \quad \text{in} \\
& \quad \text{if } \mu = \bot \text{ then} \\
& \quad \quad \text{get\_model}(E)[x \mapsto 0] \\
& \quad \text{else } \mu[x \mapsto 1]
\end{align*}
\]

encounter may be assigned any value. If a 0 sink is found, we use the trailing pointer to step up a level, follow the “then” branch for one step and continue searching for a model by following “else” branches. This method relies on the fact that ROBDDs are “reduced”, so that if no 1 sink can be reached from a node, then the node itself is the 0 sink.

We shall later use the following obvious corollary of Proposition 3:

**Corollary 1.** Let ROBDD \( R \) represent a function whose set of models form a vector space. Then every path from \( R \)'s root node to the 1-sink contains the same sequence of variables, namely \( \text{vars}(R) \) listed in variable order.

It is important to take advantage of fan-in to create efficient ROBDD algorithms. Often some ROBDD nodes will appear multiple times in a given ROBDD, and algorithms that traverse that ROBDD will meet these nodes multiple times. Many algorithms can avoid repeated work by keeping a cache of previously seen inputs and their corresponding outputs, called a *computed table*, see Brace et al. [1] for details.

### 3 Finding Affine Envelopes for ROBDDs

Zanuttini [12] gives an algorithm, here presented as Algorithm 3, for finding the affine envelope, assuming a Boolean function \( \varphi \) is represented as a set of models. This algorithm is justified by Proposition 1.
Algorithm 3 The sets-of-models based affine envelope algorithm

Input: The set $M$ of models for function $\varphi$.
Output: $\text{aff} (M)$ — the set of models of $\varphi$’s affine envelope.

if $M = \emptyset$ then
  return $M$
end if

$N \leftarrow \emptyset$
choose $\mu \in M$

$New \leftarrow M_\mu$

repeat
  $N \leftarrow N \cup New$
  $New \leftarrow \{ \mu_1 \oplus \mu_2 \mid \mu_1, \mu_2 \in N \} \setminus N$
until $New = \emptyset$

return $N_\mu$

\[ M = \begin{pmatrix}
01011 \\
01100 \\
10111 \\
11001
\end{pmatrix} \quad M_\mu = \begin{pmatrix}
00111 \\
00000 \\
11011 \\
10101
\end{pmatrix} \quad N_\mu = \text{aff} (M) = \begin{pmatrix}
01011 \\
01100 \\
10111 \\
11001
\end{pmatrix} \quad N = \begin{pmatrix}
01011 \\
01100 \\
10111 \\
11001
\end{pmatrix} \quad N_\mu = \begin{pmatrix}
11001 \\
10010 \\
01110 \\
01001
\end{pmatrix}
\]

$\mu = 01100$

Fig. 1: Steps in Algorithm 3

Example 1. To see Algorithm 3 in action, assume that $\varphi$ has four models, $M = \{01011, 01100, 10111, 11001\}$, and refer to Figure 1. We randomly pick $\mu = 01100$ and obtain $M_\mu$ as shown. The first round of completion under $\oplus$ adds three bit-strings: $\{11001, 10010, 01110\}$, and another round adds 01001 to produce $N$. Finally, “adding back” $\mu = 01100$ yields the affine envelope $N_\mu = \text{aff} (M)$.

We are interested in developing an algorithm for ROBDDs. We can improve on Algorithm 3 and at the same time make it more suitable for ROBDD manipulation. The idea is to build the result $N$ step by step, by picking the models $\nu$ of $M_\mu$ one at a time and computing $N := N \cup N_\nu$ at each step. We can start from $N = \{0\}$, as 0 has to be in $M_\mu$. This leads to Algorithm 4.

This formulation is well suited to ROBDDs, as the operation $N_\nu$, that is, taking the xor of a model $\nu$ with each model of the ROBDD $N$ can be implemented by traversing $N$ and, for each $\nu$-node with $\nu(\nu) = 1$, swapping that node’s children. And we can do better, utilising two observations.

First, during its construction, there is no need to traverse the ROBDD $N$ for each individual model $\nu$. A full traversal of $N$ will find all its models systematically, eliminating a need to remove them one by one.
Algorithm 4 A variant of Algorithm 3

Input: The set $M$ of models for function $\varphi$.

Output: $\text{aff}(M)$ — the set of models of $\varphi$’s affine envelope.

if $M = \emptyset$ then
    return $M$
end if

$N \leftarrow \{0\}$

choose $\mu \in M$

$R \leftarrow M \setminus \{0\}$

for all $\nu \in R$ do
    $N \leftarrow N \cup N_\nu$
end for

return $N_\mu$

Second, the ROBDD being constructed can be simplified aggressively during its construction, by utilising Propositions 4 and 5. Namely, as we traverse ROBDD $R$ systematically, many paths from the root to the 1-sink will be found that do not contain every variable in $\text{vars}(R)$. Each such path corresponds to a model set of cardinality $2^k$, $k$ being the number of “skipped” variables. Proposition 4 tells us that, eventually, the affine envelope will be independent of all such “skipped” variables, and Proposition 5 guarantees that variable elimination can be interspersed arbitrarily with the process of “xoring” models, that is, we can eliminate variables aggressively.

This leads to Algorithm 5. The algorithm combines several operations in an effort to amortise their cost. We present it in Haskell style, using pattern matching and guarded equations. In what follows we step through the details of the algorithm.

The $\text{to\_aff}$ function finds an initial model $\mu$ of $R$, before translating $R$, through the call to $\text{translate}$. This initial call to $\text{translate}$ has the effect of “xor-ing” $\mu$ with all of the models of $R$. Once translated, the xor closure is taken, before translating again using the initial model $\mu$ to obtain the affine closure.

$\text{translate}$ is responsible for computing the xor of a model with an ROBDD. Its operation relies on the observation that for a given node $v$ in the ROBDD, if $\mu(v) = 1$, then the operation is equivalent to exchanging the “then” and “else” branches of $v$.

$\text{xor\_close}$ is used to compute the xor-closure of an ROBDD $R$. The third argument passed to $\text{trav}$ is an accumulator in which the result is constructed. As in Algorithm 4, we know that $0$ will be a model of the result, so we initialise the accumulator as (the ROBDD for) $\bigwedge \{ \bar{v} \mid v \in \text{vars}(R) \}$.

$\text{trav}$ implements a recursive traversal of the ROBDD, and when a model is found in $\mu$, we “extend” the affine envelope to include the newly found model. Namely, $\text{extend}(R, S, \mu)$ produces (the ROBDD for) $R \lor S_\mu$. Note that once a model is found during the traversal, $\text{trav}$ checks if $\mu$ is already present within the xor-closure, and if it is not, invokes $\text{extend}$ accordingly. This simple check avoids making unnecessary calls to $\text{extend}$.
Algorithm 5 Affine envelopes for ROBDDs

Input: An ROBDD $R$.

Output: The affine envelope of $R$.

$\text{to}_\text{aff}(0) = 0$
$\text{to}_\text{aff}(R) = \text{let } \mu = \text{get}_\text{model}(R) \text{ in translate(xor}_\text{close}(\text{translate}(R, \mu)), \mu)$

$\text{translate}(0, \_) = 0$
$\text{translate}(1, \_) = 1$
$\text{translate}(\text{ite}(x, T, E), \mu)$
  $| (\mu(x) = 0) = \text{cons}(x, \text{translate}(T, \mu), \text{translate}(E, \mu), \mu)$
  $| (\mu(x) = 1) = \text{cons}(x, \text{translate}(E, \mu), \text{translate}(T, \mu), \mu)$

$\text{xor}_\text{close}(R) = \text{trav}(R, \lambda v. \bot, V\{\bar{v} \mid v \in \text{vars}(R)\})$

$\text{trav}(0, \_, S) = S$
$\text{trav}(1, \mu, S)$
  $| (\mu \models S) = S$
  $| \text{otherwise} = \text{extend}(S, S, \mu)$
$\text{trav}(\text{ite}(x, T, E), \mu, S) = \text{trav}(T, \mu[x \mapsto 1], \text{trav}(E, \mu[x \mapsto 0], S))$

$\text{cons}(x, T, E, \mu)$
  $| (\mu(x) = \bot) = \text{or}(T, E)$
  $| \text{otherwise} = \text{mknd}(x, T, E)$

$\text{extend}(1, \_, \_) = 1$
$\text{extend}(\_, 1, \_) = 1$
$\text{extend}(0, S, \mu) = \text{translate}(S, \mu)$
$\text{extend}(\text{ite}(x, T, E), 0, \mu) = \text{cons}(x, \text{extend}(T, 0, \mu), \text{extend}(E, 0, \mu), \mu)$
$\text{extend}(\text{ite}(x, T, E), \text{ite}(x, T', E'), \mu)$
  $| (\mu(x) = 1) = \text{mknd}(x, \text{extend}(T, E', \mu), \text{extend}(E, T', \mu))$
  $| \text{otherwise} = \text{cons}(x, \text{extend}(T, T', \mu), \text{extend}(E, E', \mu), \mu)$

The $\text{cons}$ function represents a special case of $\text{mknd}$. It takes an additional argument in $\mu$ and uses it to determine whether to restrict away the corresponding node being constructed. The correctness of $\text{cons}$ rests on Propositions 4 and 5, which guarantee that affine approximation can be interspersed with variable elimination, so that the latter can be performed aggressively.

Finally, once a model is found during a traversal, $\text{extend}$ is used to build up the affine closure of the ROBDD. In the context of the initial call $\text{extend}(S, S, \mu)$, Corollary 1 ensures that the pattern of the last equation for $\text{extend}$ is sufficient: If neither argument is a sink, the two will have the same root variable.

Example 2. Consider the ROBDD $R$ (shown again in Figure 2(a)), whose set of models is $\{00011, 00110, 01001, 01101, 10101\}$. Picking $\mu = 00011$ and translating gives $R_\mu$, shown in Figure 2(b). This ROBDD represents a set of vectors $\{00000, 00101, 01010, 01110, 10110\}$ which is to be extended to a vector space.
Fig. 2: (a): An example ROBDD $R$; note that all our ROBDD diagrams leave out the $\emptyset$-sink and all arcs to it. (b): The translated version $R_\mu$. (c): The vector space $S$ that has been extended to cover 00101.

Fig. 3: (a): The vector space $S$ after being extended to cover 01X10. (b): $S$ after extending to cover 10110. (c): $S$ translated to give the affine closure of $R$.

The algorithm now builds up $S$, the xor-closure of $R_\mu$, by taking one vector $v$ at a time from $R_\mu$ and extending $S$ to a vector space that includes $v$. $S$ begins as the zero vector.

The first step of the algorithm just adds 00101 to the existing zero vector (Figure 2(c)). The next step comes across the vector 01X10 (which actually represents two valuations) and existentially quantifies away the variable $x$ (Figure 3(a)). Note that the variable $z$ also disappears: this is due to the extension required to include 01X10 that adds enough valuations such that $z$ is “covered” by the vector space.
Algorithm 6 Generation of random Boolean functions as ROBDDs

Input: The number \( n \) of variables in the random function, 
\( \text{pr} \) a calibrator set so that the probability 
of a valuation being a model is \( 2^{-\text{pr}} \). 

Output: A random Boolean function represented as an ROBDD.

\[
\text{gen}_{\text{rand}}\text{bdd}(n, \text{pr}) = \text{rand}\text{bdd}(0, n - 1, \text{pr}) \\
\text{rand}\text{bdd}(m, n, \text{pr}) \\
| (m = n) = \text{mknd}(m, \text{rand}_{\text{sink}}, \text{rand}_{\text{sink}}) \\
| \text{otherwise} = \text{mknd}(m, T, E) \\
\quad \text{where} \\
\quad T = \text{if } (m > n - \text{pr}) \wedge \text{cointoss()} \text{ then } \text{rand}\text{bdd}(m + 1, n, \text{pr}) \text{ else } 0 \\
\quad E = \text{if } (m > n - \text{pr}) \wedge \text{cointoss()} \text{ then } \text{rand}\text{bdd}(m + 1, n, \text{pr}) \text{ else } 0 \\
\text{rand}_{\text{sink}} = \text{if } \text{cointoss()} \text{ then } 0 \text{ else } 1
\]

\text{cointoss()} \text{ returns } 1 \text{ or } 0 \text{ with equal probability.}

Extending to cover 10110 simply requires every model to be copied, with \( v \) mapped to 1 (Figure 3(b)). Finally, translating back by \( \mu \) produces \( A \), the affine closure of \( R \), shown in Figure 3(c).

4 Experimental Evaluation

To evaluate Algorithms 3 and 5 we generated random Boolean functions using Algorithm 6. We generated random Boolean functions of \( n \) variables, with an additional parameter to control the density of the generated function, that is, to set the likelihood of a random valuation being a model. For Algorithm 3 we extracted models from the generated ROBDDs, so that both algorithms were tested on identical Boolean functions.

\( \text{gen}_{\text{rand}}\text{bdd}(n, \text{pr}) \) builds, as an ROBDD \( R \), a random Boolean function with the property that the likelihood of an arbitrary valuation satisfying \( R \) is \( 2^{-\text{pr}} \). It invokes \( \text{rand}\text{bdd}(0, n - 1, \text{pr}) \). This recursive algorithm builds a ROBDD of \((n - \text{pr})\) variables and at depth \((n - \text{pr})\), a random choice is made as to whether to continue generating the random function or to simply join the branch with a 0 sink. If the choice is to continue, then the algorithm recursively applies \( \text{rand}\text{bdd}(m + 1, n, \text{pr}) \) to the branch.

By building a “complete” ROBDD of \((n - \text{pr})\) variables, we were able to distribute the number of models for a given number of variables. In this way, we were able to compare the various algorithms for differing model distributions.

Table 1 shows the average time (in milliseconds) taken by each of the algorithms over 10,000 repetitions with the probability 1/1024 of a valuation being a model. Timing data were collected on a machine running Solaris 9, with two Intel Xeon CPUs running at 2.8GHz and 4GB of memory. Only one CPU was
### Table 1: Average time in milliseconds to compute one affine envelope

| Variables | Algorithm 3 | Algorithm 5 |
|-----------|------------|-------------|
| 12        | 0.021      | 0.017       |
| 15        | 5.991      | 0.272       |
| 18        | —          | 0.407       |
| 21        | —          | 1.710       |
| 24        | —          | 14.967      |

used and tests were run under minimal load on the system. Our implementation of Algorithm 3 uses sorted arrays of bitstrings (so that search for models is logarithmic). As the number of models grows exponentially with the number of variables, it is not surprising that memory consumption exceeded available space, so we were unable to collect timing data for more than 15 variables.

### 5 Conclusion

Approximation and the generation of envelopes for Boolean formulas is used extensively in the querying of knowledge bases. Previous research has focused on the use of Horn approximations represented in conjunctive normal form (CNF). In this paper, following the suggestion of Zanuttini, we instead focused on the class of affine functions, using an approximation algorithm suggested by Zanuttini [12]. Our initial implementation using a naive sets-of-models (as arrays of bitstrings) representation was disappointing, as even for functions with very few models, the affine envelope often has very many models (in fact, the affine envelope of very many functions is $1$), so storing sets of models as an array becomes prohibitive even for functions over rather few variables.

ROBDDs have proved to be an appropriate representation for many applications of Boolean functions. Functions with very many models, as well as very few, have compact ROBDD representations. Thus we have developed a new affine envelope algorithm using ROBDDs. Our approach is based on the same principle as Zanuttini’s, but takes advantage of some useful characteristics of ROBDDs. In particular, Propositions 4 and 5 allow us to project away variables aggressively, often significantly reducing the sizes of the representations being manipulated earlier than would happen otherwise.

Zanuttini [12] suggests an affine envelope algorithm using modulo 2 congruence equations as output, and proves a polynomial complexity bound. However, we preferred to use ROBDDs. As a functionally complete representation for Boolean functions, ROBDDs allow the same representation for input and output, keeping the algorithms simple. For example, the algorithm for evaluating whether one ROBDD entails another is very straightforward, whereas evaluating whether a set of congruence equations entails a Boolean function in some other representation would be more complicated. It also means that systems which repeatedly construct an affine approximation, then manipulate it as a
general Boolean function, and then approximate this again, can operate without having to repeatedly convert between different representations. Importantly for our purposes, computing envelopes as ROBDDs permits us to use the same representation for approximation to many different Boolean classes.

Further research in this area includes implementing Zanuttini’s suggested modulo 2 congruence equations representation and comparing to our ROBDD implementation. This also includes evaluating the cost of determining whether a set of congruence equations entail a given general Boolean function. We also will compare affine approximation to approximation to other classes for information loss to evaluate whether affine functions really are as suitable for knowledge-base approximations as Horn or other functions.

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