THE SYMPLECTIC DISPLACEMENT ENERGY

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Abstract. We define the symplectic displacement energy of a non-empty open subset of a compact symplectic manifold as the infimum of the Hofer-like norm \[ \mathcal{H} \] of symplectic diffeomorphisms that displace this set. We show that this energy (like the usual displacement energy defined using Hamiltonian diffeomorphisms) is a strictly positive number. As a consequence we prove a result justifying the introduction of the notion of strong symplectic homeomorphisms [3].

1. Statement of results

In [9], Hofer defined a norm \[ \| \cdot \|_H \] on the group \( \text{Ham}(M, \omega) \) of compactly supported Hamiltonian diffeomorphisms of a symplectic manifold \((M, \omega)\).

For any bounded open set \( A \subset M \), he introduced the notion of the displacement energy \( e(A) \) of \( A \):

\[
e(A) = \inf \{ \| \phi \|_H \mid \phi \in \text{Ham}(M, \omega), \phi(A) \cap A = \emptyset \}.
\]

The displacement energy is defined to be \( +\infty \) if no compactly supported Hamiltonian diffeomorphism displaces \( A \).

Eliashberg and Polterovich [7] proved the following result.

Theorem 1. For any non-empty open subset \( A \) of \( M \), \( e(A) \) is a strictly positive number.

It is easy to see that if \( A \) and \( B \) are non-empty open subsets of \( M \) such that \( A \subset B \), then \( e(A) \leq e(B) \), and that \( e \) is a symplectic invariant. That is,

\[
e(f(A)) = e(A)
\]

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for all \( f \in \text{Symp}(M, \omega) = \{ \phi \in \text{Diff}(M) \mid \phi^* \omega = \omega \} \). This follows from the fact that \( \| f \circ \phi \circ f^{-1} \|_H = \| \phi \|_H \).

In [4], a Hofer-like metric \( \| \cdot \|_{HL} \) was constructed on the group \( \text{Symp}_0(M, \omega) \) of all symplectic diffeomorphisms of a compact symplectic manifold \((M, \omega)\) that are isotopic to the identity. It was proved recently by Buss and Leclercq [6] that the restriction of \( \| \cdot \|_{HL} \) to \( \text{Ham}(M, \omega) \) is a metric equivalent to the Hofer metric.

Let us now propose the following.

**Definition 2.** The **symplectic displacement energy** \( e_s(A) \) of a bounded open set \( A \) is defined to be:

\[
e_s(A) = \inf \{ \| \phi \|_{HL} \mid \phi \in \text{Symp}_0(M, \omega), \phi(A) \cap A = \emptyset \}.
\]

Clearly, if \( A \) and \( B \) are non-empty open subsets of \( M \) such that \( A \subset B \), then \( e_s(A) \leq e_s(B) \).

The goal of this note is to prove the following result.

**Theorem 3.** For any closed symplectic manifold \((M, \omega)\), the symplectic displacement energy of any non-empty open subset \( A \subset M \) satisfies \( e_s(A) > 0 \).

2. The Hofer metric \( \| \cdot \|_H \) and the Hofer-like metric \( \| \cdot \|_{HL} \)

2.1. Let \( \text{Iso}(M, \omega) \) be the set of all compactly supported symplectic isotopies of a symplectic manifold \((M, \omega)\). A compactly supported symplectic isotopy \( \Phi \in \text{Iso}(M, \omega) \) is a smooth map \( \Phi : M \times [0, 1] \to M \) such that for all \( t \), if we denote by \( \phi_t(x) = \Phi(x, t) \), then \( \phi_t \) is a symplectic diffeomorphism with compact support and \( \phi_0 = \text{id} \). Isotopies \( \Phi = \{ \phi_t \} \) are in one-to-one correspondence with families of smooth vector fields \( \{ \dot{\phi}_t \} \) defined by

\[
\dot{\phi}_t(x) = \frac{d\phi_t}{dt}(\phi_t^{-1}(x)).
\]

If \( \Phi \in \text{Iso}(M, \omega) \), then the one-form \( i(\dot{\phi}_t)\omega \) such that

\[
i(\dot{\phi}_t)\omega(X) = \omega(\dot{\phi}_t, X)
\]

for all vector fields \( X \) is closed. For any isotopy \( \Phi \), we denote by \( \phi_1 \) its time-one map.
If there exists a smooth family $F = F(x, t)$ of functions on $M \times [0, 1]$ with compact supports such that $i(\dot{\phi}_t)\omega = dF_t$, then the isotopy $\Phi$ is called a Hamiltonian isotopy and will be denoted by $\Phi_F$. We define the group $\text{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms as the set of time-one maps of Hamiltonian isotopies.

For each $\Phi = \{\phi_t\} \in \text{Iso}(M, \omega)$, the mapping

$$\Phi \mapsto \left[ \int_0^1 (i(\dot{\phi}_t)\omega) dt \right],$$

where $[\alpha]$ denotes the cohomology class of a closed form $\alpha$, induces a well defined map $\tilde{S}$ from the universal cover of $\text{Symp}_0(M, \omega)$ to the first de Rham cohomology group $H^1(M, \mathbb{R})$. This map is called the Calabi invariant (or the flux). It is a surjective group homomorphism. Let $\Gamma \subset H^1(M, \mathbb{R})$ be the image by $\tilde{S}$ of the fundamental group of $\text{Symp}_0(M, \omega)$. We then get a surjective homomorphism

$$S : \text{Symp}_0(M, \omega) \to H^1(M, \mathbb{R})/\Gamma.$$ 

The kernel of this homomorphism is the group $\text{Ham}(M, \omega)$ $[1, 2]$.  

2.2. Hofer $[9]$ defined the length $l_H$ of a Hamiltonian isotopy $\Phi_F$ as

$$l_H(\Phi_F) = \int_0^1 (\text{osc } F(x, t)) \, dt,$$

where the oscillation of a function $f : M \to \mathbb{R}$ is

$$\text{osc } (f) = \max_{x \in M} (f(x)) - \min_{x \in M} (f(x)).$$

For $\phi \in \text{Ham}(M, \omega)$, the Hofer norm of $\phi$ is

$$\|\phi\|_H = \inf \{l_H(\Phi_F) \},$$

where the infimum is taken over all Hamiltonian isotopies $\Phi_F$ with time-one map equal to $\phi$, i.e. $\phi_{F,1} = \phi$.

The Hofer distance $d_H(\phi, \psi)$ between two Hamiltonian diffeomorphisms $\phi$ and $\psi$ is

$$d_H(\phi, \psi) = \|\phi \circ \psi^{-1}\|_H.$$ 

This distance is bi-invariant. This was the main ingredient used to prove Theorem [1].
2.3. Now let \((M, \omega)\) be a compact symplectic manifold without boundary, on which we fix a Riemannian metric \(g\). For each \(\Phi = \{\phi_t\} \in \text{Iso}(M, \omega)\), we consider the Hodge decomposition \[13\] of the 1-form \(i(\dot{\phi}_t)\omega\) as

\[ i(\dot{\phi}_t)\omega = \mathcal{H}_t + du_t, \]

where \(\mathcal{H}_t\) is a harmonic 1-form. One knows that \(\mathcal{H}_t\) and \(u_t\) are unique and depend smoothly on \(t\).

For \(\Phi \in \text{Iso}(M, \omega)\), define

\[ l_0(\Phi) = \int_0^1 (|\mathcal{H}_t| + \text{osc} (u(x, t))) \, dt, \]

where \(|\mathcal{H}_t|\) is a Euclidean norm on the finite dimensional vector space of harmonic 1-forms. We let

\[ l(\phi) = \frac{1}{2}(l_0(\Phi) + l_0(\Phi^{-1})), \]

where \(\Phi^{-1} = \{\phi_t^{-1}\}\).

For each \(\phi \in \text{Symp}_0(M, \omega)\), let

\[ \|\phi\|_{HL} = \inf\{l(\phi)\}, \]

where the infimum is taken over all symplectic isotopies \(\Phi\) with \(\phi_1 = \phi\).

The following result was proved in \[4\].

**Theorem 4.** For any closed symplectic manifold \((M, \omega)\), \(\|\cdot\|_{HL}\) is a norm on \(\text{Symp}_0(M, \omega)\).

**Remark 5.** The norm \(\|\cdot\|_{HL}\) depends on the choice of the Riemannian metric \(g\) on \(M\) and the choice of the Euclidean norm \(|\cdot|\) on the space of harmonic 1-forms. However, different choices for \(g\) and \(|\cdot|\) yield equivalent metrics. See Section 3 of \[4\] for more details.

### 3. Proof of the main result

We will closely follow the proof given by Polterovich of Theorem 2.4.A in \[11\] that \(e(A) > 0\). We will use without any change Proposition 1.5.B, which states:

*For any non-empty open subset \(A\) of \(M\), there exists a pair of Hamiltonian diffeomorphisms \(\phi\) and \(\psi\) that are supported in \(A\) and whose commutator \([\phi, \psi] = \psi^{-1} \circ \phi^{-1} \circ \psi \circ \phi\) is not equal to the identity.*
For the sake of completeness we provide the following alternate proof of Polterovich’s Proposition 1.5.B based on the transitivity lemmas in [2] (pages 29 and 109). (For a proof of $k$-fold transitivity for symplectomorphisms see [5].)

**Proof.** Let $U$ be an open subset of $A$ such that $U \subset A$. Pick three distinct points $a, b, c \in U$. By the transitivity lemma of $\text{Ham}(M, \omega)$, there exist $\phi, \psi \in \text{Ham}(M, \omega)$ such that $\phi(a) = b$ and $\psi(b) = c$. Moreover, we can choose $\phi$ and $\psi$ so that $\text{supp}(\phi) = V$ and $\text{supp}(\psi) = W$ are small tubular neighborhoods of distinct paths in $U$ joining $a$ to $b$ and $b$ to $c$ respectively, and we can assume that $c \in U \setminus V$.

Then $(\psi^{-1} \phi^{-1} \psi \phi)(a) = (\psi^{-1} \phi^{-1})(c) = \psi^{-1}(c) = b$. Hence $[\phi, \psi] \neq \text{id}$. □

We will say that a map $h$ displaces $A$ if $h(A) \cap A = \emptyset$. We note the following fact.

**Proposition 6.** If $h$ displaces $A$, then for any homeomorphism $\phi$ with $\text{supp}(\phi) \subset A$ the maps $\phi^{-1} \circ h \circ \phi$ and $\phi \circ h \circ \phi^{-1}$ also displace $A$.

**Proof.** Assume $h(A) \cap A = \emptyset$ but $(\phi^{-1} \circ h \circ \phi)(A) \cap A \neq \emptyset$. Then there exist $x, y \in A$ such that $x = (\phi^{-1} \circ h \circ \phi)(y)$. Hence, $

\phi(x) = h(\phi(y))

$ where $\phi(x)$ and $\phi(y)$ are in $A$ since $\text{supp}(\phi) \subset A$. Therefore $\phi(x) \in h(A) \cap A$, contradicting the assumption that $h(A) \cap A = \emptyset$.

The proof that $\phi \circ h \circ \phi^{-1}$ displaces $A$ is similar, since $\text{supp}(\phi) \subset A$ implies that $\text{supp}(\phi^{-1}) \subset A$. □
Proof of Theorem 3 continued. Let us denote by $D(A)$ the set of all $h \in \text{Symp}_0(M,\omega)$ that displace $A$. Following the proof of Theorem 2.4.A in [11] we assume there exists $h \in D(A) \neq \emptyset$. Otherwise, we are done since $\epsilon_s(A) = +\infty$. Now let $\phi$ and $\psi$ be as in Polterovich’s Proposition 1.5.B and consider the commutator

$$\theta = [h, \phi^{-1}] = \phi \circ h^{-1} \circ \phi^{-1} \circ h,$$

which is contained in $\text{Ham}(M,\omega)$ because commutators are in the kernel of the Calabi invariant. Now, if $x \in A$ then $h(x) \not\in A$. Hence,

$$\theta(x) = (\phi \circ h^{-1})(\phi^{-1}(h(x))) = \phi(h^{-1}(h(x))) \quad \text{since supp} \, (\phi^{-1}) \subset A = \phi(x),$$

and we see that $\theta|_A = \phi|_A$. Similarly, for $x \in A$ we have $\phi^{-1}(x) \in A$, and hence $h(\phi^{-1}(x)) \not\in A$ since $h(A) \cap A = \emptyset$. Thus,

$$\theta^{-1}(x) = h^{-1}(\phi(h(\phi^{-1}(x)))) = h^{-1}(h(\phi^{-1}(x))) \quad \text{since supp} \, (\phi) \subset A = \phi^{-1}(x),$$

and we see that $\theta^{-1}|_A = \phi^{-1}|_A$. Thus, $(\phi^{-1} \circ \psi \circ \phi)(x) = (\theta^{-1} \circ \psi \circ \theta)(x)$ for all $x \in A$ since supp $(\psi) \subset A$.

Now, if $x \not\in A$ and $\theta(x) \in A$ we would have $x = \theta^{-1}(\theta(x)) = \phi^{-1}(\theta(x)) \in A$ since supp $(\phi^{-1}) \subset A$, a contradiction. Hence, for $x \not\in A$ we have $\theta(x) \not\in A$ and

$$(\phi^{-1} \circ \psi \circ \phi)(x) = x = (\theta^{-1} \circ \psi \circ \theta)(x)$$

since both $\phi$ and $\psi$ have support in $A$. Therefore, $\phi^{-1} \circ \psi \circ \phi = \theta^{-1} \circ \psi \circ \theta$, and we have

$$[\phi, \psi] = [\theta, \psi].$$

Because both $\theta$ and $\psi$ are in $\text{Ham}(M,\omega)$ and the Hofer norm is conjugation invariant, we have

$$\| [\theta, \psi] \|_H = \| \psi^{-1} \circ \theta^{-1} \circ \psi \circ \theta \|_H \leq \| \psi^{-1} \circ \theta^{-1} \circ \psi \|_H + \| \theta \|_H \leq 2\| \theta \|_H.$$
By Buss and Leclercq’s theorem \cite{6} there is constant $\lambda$ such that

$$\|\theta\|_H \leq \lambda \|\theta\|_{HL}.$$  

Using the triangle inequality and the symmetry of $\|\cdot\|_{HL}$ we have

$$[[\theta,\psi]]_H \leq 2\lambda (\|\phi \circ h \circ \phi^{-1}\|_{HL} + \|h\|_{HL}).$$

However, we do not know if $\|h\|_{HL} = \|\phi \circ h \circ \phi^{-1}\|_{HL}$, which is a key step used for $\|\cdot\|_H$ in Eliashberg and Polterovich’s proof that $e(A) > 0$. We therefore consider two cases:

(i) $\|\phi \circ h \circ \phi^{-1}\|_{HL} \leq \|h\|_{HL}$ and
(ii) $\|h\|_{HL} \leq \|\phi \circ h \circ \phi^{-1}\|_{HL}.$

In the first case we estimate the right hand side of inequality (1) by

$$[[\phi,\psi]]_H = [[\theta,\psi]]_H \leq 4\lambda \|h\|_{HL},$$

whereas in the second case, (1) is controlled by

$$[[\phi,\psi]]_H = [[\theta,\psi]]_H \leq 4\lambda \|\phi \circ h \circ \phi^{-1}\|_{HL}.$$

Consider the following sets $X$ and $Y$ of real numbers

$$X = \{\|\phi \circ h \circ \phi^{-1}\|_{HL} \mid h \in D(A)\}$$
and

$$Y = \{\|h\|_{HL} \mid h \in D(A)\}.$$  

We show that $X = Y$. If $a \in X$, then $a = \|\phi \circ h \circ \phi^{-1}\|_{HL}$ with $h \in D(A)$. But by Proposition \cite{6}, $\phi \circ h \circ \phi^{-1} \in D(A)$, meaning that $a \in Y$. Conversely, if $b \in Y$ then $b = \|h\|_{HL}$, with $h \in D(A)$. Hence $b = \|\phi \circ (\phi^{-1} \circ h \circ \phi) \circ \phi^{-1}\|_{HL}$ with $\phi^{-1} \circ h \circ \phi \in D(A)$ by Proposition \cite{6}. This means that $b \in X$.

Therefore, the fact that $X = Y$ combined with inequalities (2) and (3) yields

$$\inf X = \inf Y = e_s(A) \geq \frac{1}{4\lambda} [[\phi,\psi]]_H > 0.$$  

This completes the proof of Theorem 3. \qed

**Remark 7.** The proof of Theorem 4 relies on the bi-invariance of the distance $d_H$. It is quite remarkable that the proof of Theorem 3 did not rely on any invariance property of $d_{HL}$.
4. Examples

A harmonic 1-parameter group is an isotopy $\Phi = \{\phi_t\}$ generated by the vector field $V_H$ defined by $i(V_H)\omega = H$, where $H$ is a harmonic 1-form. It is immediate from the definitions that

$$l_0(\Phi) = l_0(\Phi^{-1}) = |H|$$

where $|\cdot|$ is a Euclidean norm on the space of harmonic 1-forms. Hence $l(\Phi) = |H|$. Therefore, if $\phi_1$ is the time one map of $\Phi$ we have

$$\|\phi_1\|_{HL} \leq |H|.$$ 

For instance, take the torus $T^{2n}$ with coordinates $(\theta_1, \ldots, \theta_{2n})$ and the flat Riemannian metric. Then all the 1-forms $d\theta_i$ are harmonic. Given $v = (a_1, \ldots, a_n, b_1, \ldots, b_n) \in \mathbb{R}^{2n}$, the translation $x \mapsto x + v$ on $\mathbb{R}^{2n}$ induces a rotation $\rho_v$ on $T^{2n}$, which is a symplectic diffeomorphism. Moreover, $x \mapsto x + tv$ on $\mathbb{R}^{2n}$ induces a harmonic 1-parameter group $\{\rho^t_v\}$ on $T^{2n}$.

Taking the 1-forms $d\theta_i$ for $i = 1, \ldots, 2n$ as basis for the space of harmonic 1-forms and using the standard symplectic form

$$\omega = \sum_{j=1}^{n} d\theta_j \wedge d\theta_{j+n}$$
on $T^{2n}$ we have

$$i(\dot{\rho^t_v})\omega = \sum_{j=1}^{n} (b_j d\theta_j - a_j d\theta_{j+n}).$$

Thus,

$$l(\{\rho^t_v\}) = |(b_1, \ldots, b_n, -a_1, \ldots, -a_n)|$$

where $|\cdot|$ is a Euclidean norm on the space of harmonic 1-forms, and we see that

$$\|\rho_v\|_{HL} \leq |v|$$

if we use $|v| = |a_1| + \cdots + |a_n| + |b_1| + \cdots + |b_n|$ as the Euclidean norm on both $\mathbb{R}^{2n}$ and the space of harmonic 1-forms.

Consider the torus $T^2$ as the rectangle:

$$\{(p, q) \mid 0 \leq p \leq 2 \text{ and } 0 \leq q \leq 1\} \subset \mathbb{R}^2,$$
with opposite sides identified. For any \( r < 1 \), let
\[
A_0(r) = \{(x, y) \mid 0 \leq x, y < r\},
\]
and \( A(r) \) the corresponding subset in \( T^2 \). If \( v = (a_1, 0) \) with \( r \leq a_1 \leq 2 - r \), then the rotation \( \rho_v \) induced by the translation \( (p, q) \mapsto (p + a_1, q) \) displaces \( A(r) \). Therefore, using the norm \( |v| = |a_1| + |b_1| \) we have
\[
\|\rho_a\|_{HL} \leq l(\{\rho_a^t\}) = a_1.
\]
Since this holds for all \( r \leq a_1 \) we have,
\[
es(A(r)) \leq r.
\]

**Remark 8.** Observe that the above computation depended on the choice of the flat Riemannian metric and the choice of the Euclidean norm on the space of harmonic 1-forms. However, for any other Riemannian metric \( g \), a 1-form \( \alpha \) that is harmonic with respect to \( g \) has a Hodge decomposition \( \alpha = \mathcal{H} + d\mu \) with respect to the flat metric. Mapping \( \alpha \mapsto \mathcal{H} \) gives an explicit isomorphism between the space of harmonic 1-forms with respect to \( g \) and the space of harmonic 1-forms with respect to the flat metric. Thus, if we fix a Euclidean metric on the space of harmonic 1-forms with respect to the flat metric, then the above isomorphism induces a Euclidean metric on the space of harmonic 1-forms for any Riemannian metric \( g \).

5. **Application**

The following result is an immediate consequence of the positivity of the symplectic displacement energy of non-empty open sets. For two isotopies \( \Phi \) and \( \Psi \) denote by \( \Phi^{-1} \circ \Psi \) the isotopy given at time \( t \) by
\[
(\Phi^{-1} \circ \Psi)_t = \phi_t^{-1} \circ \psi_t.
\]

**Theorem 9.** Let \( \Phi_n \) be a sequence of symplectic isotopies and let \( \Psi \) be another symplectic isotopy. Suppose that the sequence of time-one maps \( \phi_{n,1} \) of the isotopies \( \Phi_n \) converges uniformly to a homeomorphism \( \phi \), and \( l(\Phi_n^{-1} \circ \Psi) \to 0 \) as \( n \to \infty \), then \( \phi = \psi_1 \).

This theorem can be viewed as a motivation for the following.
Definition 10. A homeomorphism $h$ of a compact symplectic manifold is called a strong symplectic homeomorphism if there exist a sequence $\Phi_n$ of symplectic isotopies such that $\phi_{n,1}$ converges uniformly to $h$, and $l(\Phi_n)$ is a Cauchy sequence.

Proof of Theorem 9. Suppose $\phi \neq \psi_1$, i.e. $\phi^{-1} \circ \psi_1 \neq \text{id}$. Then there exists a small open ball $B$ such that $(\phi^{-1} \circ \psi_1)(B) \cap B = \emptyset$. Since $\phi_{n,1}$ converges uniformly to $\phi$, $((\phi_{n,1})^{-1} \circ \psi_1)(B) \cap B = \emptyset$ for $n$ large enough. Therefore, the symplectic energy $e_s(B)$ of $B$ satisfies

$$e_s(B) \leq \|(\phi_{n,1})^{-1} \circ \psi_1\|_{HL} \leq l(\Phi_n^{-1} \circ \Psi).$$

The later tends to zero, which contradicts the positivity of $e_s(B)$. □

Remark 11. This theorem was first proved by Hofer and Zehnder for $M = \mathbb{R}^{2n}$ [8], and then by Oh-Müller in [10] for Hamiltonian isotopies using the same lines as above, and very recently by Tchuiaga [12], using the $L^\infty$ version of the Hofer-like norm.

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