The concept of nuclear cluster forbiddenness

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Abstract

In nuclear cluster systems, a rigorous structural forbiddenness of virtual nuclear division into unexcited fragments is obtained. We re-analyze the concept of forbiddenness, introduced in [1] for the understanding of structural effects in nuclear cluster physics. We show that the concept is more involved than the one presented previously, where some errors were committed. Due to its importance, it is reanalyzed here. In the present contribution a simple way for the determination of forbiddenness is given, which may easily be extended to any number of clusters, though in this contribution we discuss only two-cluster systems, for illustrative reasons. A simple rule is obtained for the minimization of the forbiddenness, namely to start from a cluster system with a large $SU(3)$ irrep $(\lambda_c, \mu_c)$, but minimizing $(\lambda_c - \mu_c)$, i.e. the system has to be oblate. The rule can be easily implemented in structural studies, done up to now with an oversimplified definition of forbiddenness. The new method is applied to various systems of light clusters and to some decay channels of $^{236}$U and $^{252}$Cf.

1 Introduction

In a system of two light clusters, each nucleus can be well described within the $SU(3)$ model of the nucleus. The relation between the shell
and the cluster model was established by Wildermuth and Kanellopoulos [2], proving that within the harmonic oscillator the Hamiltonians of these two models can be related to each other exactly. For heavy clusters the pseudo-$SU(3)$ [3, 4] has to be applied, which takes into account effectively the spin-orbit interaction. The $SU(3)$ model is based on the harmonic oscillator as a mean field, plus residual interactions. In [5] a minimal condition is introduced for satisfying the Pauli exclusion principle: When the sum of the number of oscillation quanta of each cluster is determined, obtained by filling densely the nucleons into the oscillator states, a mismatch is obtained to the total number of oscillation quanta in the united nucleus. In order to observe minimally the Pauli exclusion principle, the missing oscillation quanta ($n_0$) are usually included in the relative motion of the two clusters. This constraint is known as the Wildermuth condition. In [6, 7] additional requirements are added, such that the final $SU(3)$ irreducible representations (irrep) also fulfill the Pauli exclusion principle. In [6, 7] light clusters were considered and the external product of the cluster $SU(3)$ irreps with the irrep of the relative motion always can be coupled to the ground state and excited states of the united nucleus, i.e., it suffices to add the missing $n_0$ quanta to the relative motion only.

It was proven in [1] that it is not always the case that the relative $SU(3)$ irrep, required by the Wildermuth condition can be coupled with the cluster irrep, $(\lambda_c, \mu_c)$ to the ground state of the united nucleus. A rigorous structural forbiddenness (in the sense that certain combinations of representations are forbidden) of virtual nuclear division into two unexcited fragments exists for all nuclei with masses $A_1, A_2 > 36$, and at $A_1, A_2 > 12$, for a wide range of nuclei [1]. The forbiddenness is universal and model independent.

In the literature there are known two apparently different definitions for cluster states: i) One definition requires a rigid molecular type separation while ii) others (including ourselves) speak both about localized and shell-model-like clusters. For a detailed review related to the first definition, see the report given in [8]. The second one is closely related to experimental observability: A state is called a cluster state when its wave function has an overlap with cluster wave functions [9], as for example in the case of $^{16}\text{O} \rightarrow ^{12}\text{C} + \alpha$. This definition is more general and includes the rigid definition of cluster states [17]. A detailed discussion is given in [10].

In order to understand basic processes, as fission and fusion or phenomena as cluster radioactivity [11, 12, 13], structural considera-
tions play a very important role in modern nuclear physics. For example, in [14] preferences of clusterizations in ternary fission processes were investigated. Hyperdeformed states in $^{36}$Ar were considered in [15], compared to experiments, and in [16] the clusterization in shape isomers of $^{58}$Ni. In all cases structural considerations lead to a fundamental understanding of these systems, which is of course not sufficient and energetic considerations [14, 17] are necessary as well as the penetration through the barrier [18].

Structural considerations play a role in understanding the connection between fission and cluster radioactivity, often seen as different processes. In [18] it was argued that they are the same, using the pseudo-$SU(3)$ model and determining the spectroscopic factors within an algebraic cluster model. Other applications are related to the understanding of structural aspects in radiation capture [19].

Most of these theoretical considerations are based on the shell model, which has been and still is the cornerstone of microscopic nuclear structure. Although it can be argued that some nuclear states as the Hoyle state [20] can not be described by a single $SU(3)$ irreducible representation [21] or that $SU(3)$ is strongly broken for heavy nuclei, giving the impression that $SU(3)$ is out of date, however this is not true. For heavy nuclei methods were developed as the pseudo-$SU(3)$ [3, 4] or the use of embedded (effective) $SU(3)$ representations [22, 23, 24], which was finally applied with great success in [25]. In all these extensions the language of the harmonic oscillator ($SU(3)$) was applied [9], which proved to be very successful for light nuclei and served as a starting point for different approximations for heavy nuclei. The considerations made in this contribution can be applied to all these extensions, with slight modifications. The $SU(3)$ language allows to understand many of features in modern nuclear physics experiments without recurring to complicated numerical routines, though ab-initio calculations [8, 21] may be necessary to understand the details. The report on modern nuclear physics [8] concentrates for a large part on the $SU(3)$ notation and simple structural considerations, showing that the harmonic oscillator helps to understand up-initio calculations. The theory, presented in [21], also used primarily the harmonic shell-model picture, including inter-shell excitations. The calculation is restricted to only a few $SU(3)$ irreps as band-heads of symplectic excitations, which start from $SU(3)$ bad heads in the valence shell and adds $2\hbar \omega$ excitations. This model has a lot in common with the Semimicroscopic Algebraic Cluster Model (SACM) [6, 7],
which also starts from $SU(3)$ irreps in the $0\hbar\omega$ shell and includes intershell excitations by adding relative oscillation quanta. Due to its starting point of a cluster picture, it is able to describe complicated cluster configurations.

In [1] it is observed that the coupling in $SU(3)$ of two light clusters, in their ground state, with the relative motion not always leads to the ground state of the united nucleus. Thus, some structural information is missing. Due to that, in [1] the concept of forbiddenness was introduced. The main idea is to allow the excitation of one or both clusters to higher shells, subtracting the number of excitation quanta from the relative motion. This increases the possibility to reach the ground state of the united nucleus. The minimal quanta to excite one of the clusters, needed to achieve this goal, is called the forbiddenness. This property was interpreted in Ref. [26] and, independently, by Bader and Kramer [27] as a consequence of the $SU(3)$ symmetry selection rules. In [1], a simple example was discussed in order to illustrate the calculation of the forbiddenness, though the system itself does not play an important role today: The analysis of the forbidden decay $A \rightarrow A_1 + A_2$, for the case of two spherical clusters ($^{40}$Ca) and a spherical united nucleus $^{80}$Zr, which is simple enough to understand the concept of forbiddenness. Smirnov et al. [1] obtained for the above cluster system the forbiddenness value of 20. However, we were unable to reproduce the results obtained in [1], which leads to the present contribution and in what follows we resume the prove why in [1] an error must have been committed: According to the Wildermuth condition, the minimal number of quanta to add is $n_0 = 60$, which is distributed between the relative oscillation quanta and the excitation of the clusters. When the number of relative quanta is $n_r$, the irrep of the cluster state has to be $(0, n_r)$ in order to couple to a total scalar irrep $(0,0)$ of $^{80}$Zr. The Young diagram for the irrep $(0, n_r)$ of the coupled cluster $^{40}$Ca+$^{40}$Ca has $2n_r$ intershell excitations and is given by $[40 + n_r, 40 + n_r, 40]$. From this, it is clear that adding the relative motion irrep $(n_r, 0)$, the total number is $3n_r = n_0 = 60$ and the number of $n_r\hbar\omega$ excitation in the cluster irrep has to be $2n_r = 40$, not 20 as reported previously in [1]. The above result demand a reconsideration of the concept of forbiddenness. Due to that, the presentation will tend to be more mathematical but the simple results, easy to implement in recent cluster calculations (see [14, 15, 16, 28, 29] and references therein) and their importance for current studies in nuclear physics justifies it.
Due to the importance of the concept of *forbiddenness* in modern nuclear structural analysis and the impossibility to reproduce the results in [1] an alternative definition was proposed in [28], which simply depends on the difference in the SU(3) irrep of the ground state of the united nucleus to the ones in the final coupling of the cluster system. This new definition was used in [16, 19, 30], proving its utility.

We consider it very important to present a consistent manner to determine easily the *forbiddenness*, which explains, according to [1] why certain cluster combinations are suppressed in fission and/or fusion processes. *The conclusions remain the same*: Structural considerations, coming solely from the coupling of nucleons to SU(3) irreps, are sufficient in order to understand the suppression and enhancements of cluster distributions in nuclear reactions. Of course, for the details one has to superimpose tunnel effects [18]. This conclusion was also obtained in [28, 29]. The concept of forbidness, as will be exposed in this contribution, can be applied to heavy nuclei too, using the extensions of the harmonic oscillator picture, as mentioned above. It has an important impact on super-heavy nuclei [31].

Due to the reasons explained above one has to reanalyze the concept of *forbiddenness*, which requires some group theory. Therefore, within the scope of this contribution we will concentrate on the *forbiddenness* and only indicate its use. It is out of the scope of this publication to do an explicit investigation neither on fusion and fission processes nor cluster radioactivity, though a couple of examples will be discussed at the end, constraint to the determination of the *forbiddenness*.

## 2 A new proposal for forbiddenness

In what follows, we will present a detailed corrected derivation of the *forbiddenness* and demonstrate that it depends on the *compactness* of the two clusters, before the relative motion is added. The *compactness* is defined by the eigenvalue of the second order Casimir operator of $SU_c(3)$, with respect to the irrep $(\lambda_c, \mu_c)$, and the difference $(\lambda_c - \mu_c)$. The larger the eigenvalue is but larger the difference of $\mu_c$ to $\lambda_c$ (oblate), the less oscillation quanta have to be invested in the excitation of the clusters. For the case of two spherical clusters and a spherical united nucleus, where the only irrep in the $0\hbar \omega$ shell is $(0,0)$, the *forbiddenness* is identical to include only inter-shell excitations in
the cluster system. However, especially when both clusters and the united nucleus are deformed, various values for the compactness are possible.

Let us first consider a two-cluster system, coupled to the cluster irrep \((\lambda_c, \mu_c)\). This irrep is obtained by coupling the ground state irreps \((\lambda_i, \mu_i)\) \((i = 1, 2)\) of the two clusters to \((\lambda_c, \mu_c)\). In contrast to the former assumption that each cluster has to be in its ground state, each one is allowed to be excited, including \(n\hbar\omega\) inter-shell excitations. We are not interested on how each cluster is excited but refer to the resulting cluster irrep as \((\lambda_c + \lambda_0, \mu_c + \mu_0)\), where \(\lambda_0\) and \(\mu_0\) are given in terms of the internal excitations of the two joined clusters. In what follows, an analytical expression for these internal excitations will be obtained.

The counting procedure starts with the product of the excited cluster irrep \((\lambda_c + \lambda_0, \mu_c + \mu_0)\) with the relative motion \((n_r, 0)\), in order to obtain the irrep of the parent nucleus \((\lambda, \mu)\). When the \(SU(3)\) irreps are converted to Young diagrams, the product reads

\[
[\lambda_c + \mu_c + a, \mu_c + b, c] \otimes [n_r, 0, 0] \rightarrow [\lambda_c + \mu_c + a + k_1, \mu_c + b + k_2, c + k_3],
\]

where

\[
\begin{align*}
n_0 &= n_c + n_r, & n_c &= a + b + c, & n_r &= k_1 + k_2 + k_3, \\
\lambda_0 &= a - b, & \mu_0 &= b - c
\end{align*}
\]

and \(n_0\) is the minimal number of oscillation quanta to be added, in order to guarantee the Pauli exclusion principle. These quanta are divided in the excitation \(n_c\) of the cluster irrep and the remaining relative oscillation quanta, denoted by \(n_r\). The forbiddenness is given by the excitation \(n_c\) of the cluster irrep and has to be as small as possible. We can start by setting \(c = 0\). The final \(SU(3)\) irrep is \((\lambda, \mu)\) and represents the final product of Young diagrams in \((1)\).

Using the rules for the direct product of an arbitrary Young diagram with a symmetric one \([32]\), we obtain the following additional equations and constraints:

\[\]
\[0 \leq k_2 \leq \lambda_c + \lambda_0\]
\[0 \leq k_3 \leq \mu_c + \mu_0\]
\[\lambda = \lambda_c + \lambda_0 + k_1 - k_2\]
\[\mu = \mu_c + \mu_0 + k_2 - k_3\]

(3)

“\(a\)” represents the number of boxes which are added in the first row of the cluster irrep (Young diagram: \([\lambda_c + \mu_c + a, \mu_c + b, 0]\)), “\(b\)” is the number of boxes which are added to the second row.

From the third and fourth equation in (3) we get for \(k_1\) and \(k_3\)

\[k_1 = \lambda - \lambda_c - \lambda_0 + k_2\]
\[k_3 = \mu_c - \mu + k_2 + \mu_0\]

(4)

from which we obtain

\[n_r = (\lambda - \lambda_c - \lambda_0 + k_2) + k_2 + (\mu_c + \mu_0 - \mu + k_2)\]
\[= (\lambda - \mu) - (\lambda_0 - \mu_0) - (\lambda_c - \mu_c) + 3k_2\]

(5)

With this, we get for the total number of quanta which have to be added according to the Wildermuth condition:

\[n_0 = n_r + n_c = n_r + (\lambda_0 + 2\mu_0)\]
\[= (\lambda - \mu) - (\lambda_c - \mu_c) + 3(\mu_0 + k_2)\]

(6)

Resolving for \((\mu_0 + k_2)\), we get

\[\mu_0 + k_2 = \frac{1}{3} [n_0 + (\mu - \lambda) - (\mu_c - \lambda_c)]\]

(7)

which is a fixed quantity, because all values on the right hand side of the equation are specified for a system considered.

The last equation is substituted into the second one in (4), giving

\[k_3 = \frac{1}{3} [n_0 - (\lambda + 2\mu) + (\lambda_c + 2\mu_c)]\]

(8)

which is also a fixed quantity!
Now, we use the first equation in (4), for $k_1$, and also use (7), for $k_2$, yielding

$$k_1 = \frac{1}{3} \left[ n_0 + (2\lambda + \mu) - (2\lambda_c + \mu_c) \right] - (\lambda_0 + \mu_0) . \quad (9)$$

Let us summarize the results for the $k_i$ in a slight different form:

$$k_1 + \lambda_0 + \mu_0 = \frac{1}{3} \left[ n_0 + (2\lambda + \mu) - (2\lambda_c + \mu_c) \right]$$
$$k_2 + \mu_0 = \frac{1}{3} \left[ n_0 + (\mu - \lambda) - (\mu_c - \lambda_c) \right]$$
$$k_3 = \frac{1}{3} \left[ n_0 - (\lambda + 2\mu) + (\lambda_c + 2\mu_c) \right] . \quad (10)$$

Using that $n_r = k_1 + k_2 + k_3$, $n_c = \lambda_0 + 2\mu_0$ and substituting only $k_1$ and $k_3$, we get

$$n_0 = n_c + n_r = \frac{1}{3} (2n_0 + \lambda - \mu - \lambda_c + \mu_c) + k_2 + \mu_0 . \quad (11)$$

For $(\mu_0 + k_2)$, we use (7) and resolve (11) for $n_c$, requiring that $n_c$ is minimal, giving

$$n_c = n_0 - n_r = \overbrace{\text{fixed}}^\text{max} \overbrace{(n_0 - k_3)} - \overbrace{(k_1 + k_2)}^\text{max} . \quad (12)$$

We joint $n_0$ and $k_3$ because these values are already fixed.

The $n_c$ is minimized, when $(k_1 + k_2)$ is maximized. Taking into account the first relation in (3) and the first equation in (4), according to which $k_2^{\text{max}} = \lambda_c + \lambda_0$ and $k_1^{\text{max}} = \lambda + k_2^{\text{max}} - (\lambda_c + \lambda_0) = \lambda$ and using (11), we obtain

$$\overbrace{(k_1 + k_2)}^\text{max} = \lambda + \lambda_c + \lambda_0$$
$$= \left\{ \frac{1}{3} \left[ n_0 + (2\lambda + \mu) - (2\lambda_c + \mu_c) \right] - (\lambda_0 + \mu_0) \right\}$$
$$+ \left\{ \frac{1}{3} \left[ n_0 + (\mu - \lambda) - (\mu_c - \lambda_c) \right] - \mu_0 \right\}$$
$$= \frac{1}{3} \left[ 2n_0 + (\lambda + 2\mu) - (\lambda_c + 2\mu_c) \right] - (\lambda_0 + 2\mu_0) \quad (13)$$
Resolving for \((\lambda_0 + \mu_0) = a^{\text{min}}\), we get for

\[
a^{\text{min}} = \max \left[ 0, (\lambda_0 + \mu_0) \right]
= \max \left[ 0, \frac{1}{3} \{ n_0 - (\lambda - \mu) - (2\lambda_c + \mu_c) \} \right]
\]  

having used \(\lambda_0 = a - b\) and \(\mu_0 = b\). When the expression depending on the irrep numbers is negative, one has to take the value zero.

Knowing \(a^{\text{min}}\), minimizing \(n_c = \lambda_0 + 2\mu_0 = a + b\) has been now reduced to minimize \(\mu_0 = b\). The \(b^{\text{min}}\) can be obtained considering that (see (14))

\[
\frac{1}{3} [n_0 - (\lambda + 2\mu) + (\lambda_c + 2\mu_c)] = k_3 \leq \mu_c + \mu_0
\]

or

\[
\mu_0 \geq - \mu_c + \frac{1}{3} [n_0 - (\lambda + 2\mu) + (\lambda_c + 2\mu_c)]
\]

Because \(\mu_0 = b\) and only positive values are allowed, we obtain

\[
b^{\text{min}} = \max \left[ 0, \frac{1}{3} \{ n_0 - (\lambda + 2\mu) + (\lambda_c - \mu_c) \} \right]
\]  

where again one considers the possibility that the expression in terms of \(n_0\) and the \(SU(3)\) irreps may be negative. In this case the value \(b^{\text{min}} = 0\) has to be chosen.

With this, we have minimized the forbiddenness

\[
n_c = (\lambda_0 + 2\mu_0)^{\text{min}} = (a)^{\text{min}} + (b)^{\text{min}}
= \max \left[ 0, \frac{1}{3} \{ n_0 - (\lambda - \mu) - (2\lambda_c + \mu_c) \} \right]
\]

\[
+ \max \left[ 0, \frac{1}{3} \{ n_0 - (\lambda + 2\mu) + (\lambda_c - \mu_c) \} \right]
\]

Eq. (18) is the main result of this contribution and can be interpreted as follows: The first term in (18) tells us, that in order to minimize \(n_c\), we have to maximize \((\lambda_c + 2\mu_c)\). The second term tells us that in addition the difference \((\lambda_c - \mu_c)\) has to be minimized. A
maximal \((\lambda_c + 2\mu_c)\) and a minimal \((\lambda_c - \mu_c)\) imply a large compact and oblate configuration of the two-cluster system.

One can achieve these conditions, now defined as compactness of the irrep \((\lambda, \mu)\), determining the whole product of \((\lambda_1, \mu_1) \otimes (\lambda_2, \mu_2)\) and searching for the irrep that corresponds to a large compact structure (large \((2\lambda_c + \mu_c)\) but with a maximal difference \((\mu_c - \lambda_c)\), leading to an oblate structure). For deformed clusters, there is also the possibility to excite it within the \(0\hbar \omega\), leading to other individual cluster irreps \((\lambda_1, \mu_1)\). One can take the whole \(0\hbar \omega\) space of each cluster and multiply them all, or even one can do it for the proton space and the neutron space for each cluster and then multiply the proton final space with the neutron final space.

In agreement to Smirnov’s definition, the forbiddenness is related to \(n_c\hbar \omega\) excitations of a cluster system before adding the relative oscillation quanta. But, the excitation might be distributed between the clusters and is a combination of these cluster excitations in \(0\hbar \omega\) and excitation of these clusters in \(n_c\hbar \omega\), with \(n_c > 0\).

With the derivation given above, we can address the example discussed in [1] and in the introduction, namely \(^{40}\text{Ca} + ^{40}\text{Ca} \rightarrow ^{80}\text{Zr}\). We have \((0, 0)\) for both \(^{40}\text{Ca}\) clusters, thus, \((\lambda_c, \mu_c) = (0, 0)\) is the only possibility. The irrep for \(^{80}\text{Zr}\) is \((0, 0)\) and \(n_0 = 60\) according to the Wildermuth condition, thus \(n_c = \frac{2n_0}{3} = 40\). There is no higher \(0\hbar \omega\) cluster irrep \((\lambda_c, \mu_c)\), because each cluster has \((0, 0)\), also when we consider protons and neutrons separately. Thus, in this example the solution is unique. As noted above, in [1] a forbiddenness of 20 is reported, which is incorrect.

The next, non-trivial example is \(^{20}\text{Ne} + ^{16}\text{O} \rightarrow ^{36}\text{Ar}\), when the clusters are in their ground state, we obtain the following list of the irreps for the cluster and total irrep:

\[
\begin{align*}
(\lambda, \mu)_{18}\text{Ar}_{18} &= (0, 8),
(\lambda_1, \mu_1)_{8}\text{O}_{8} &= (0, 0),
(\lambda_2, \mu_2)_{20}\text{Ne}_{20} &= (8, 0),
(\lambda_c, \mu_c) &= (8, 0),
(n_0, 0) &= (20, 0)
\end{align*}
\]

Taking the product of \((\lambda_c, \mu_c)\) with \((n_c, 0)\) the resulting irreps have the structure \((\lambda_c + a - b, \mu_c + b)\). Using the equation (18), we obtain for \(a = \lambda_0 + \mu_0 = 4, b = \mu_0 = 4\) and for the forbiddenness \(n_c = 8\). The \(k_i\) values are: \(k_1 = 0, k_2 = 8\) and \(k_3 = 4\).

A further, trivial example is \(^{12}\text{C} + ^{8}\text{Be} \rightarrow ^{20}\text{Ne}\), whose \(SU(3)\) irreps are \((\lambda_1, \mu_1)_{12}\text{C} = (0, 4), (\lambda_2, \mu_2)_{8}\text{Be} = (4, 0)\) and \((n_0, 0) = (8, 0)\),
respectively. In this case, $a^\text{min}$ and $b^\text{min}$ are both zero, thus, the forbiddenness is zero too, which is a known result.

In what follows, we present the results for $^{236}\text{U}$, decaying into two clusters as it happens in a fission process. Many of the clusters involved are not in the p or sd-shell, thus the $SU(3)$ model of Elliott [33] does not work due to the large spin-orbit interaction. One has to apply the pseudo-$SU(3)$ model [3, 4]. This model takes into account the spin-orbit interaction by renaming the spin and orbital angular momentum to pseudo-spin and pseudo-angular momentum. The space is divided into active nucleons and spectators. The spectators are all nucleons in the intruder orbital levels $j + \frac{1}{2}$, while the active nucleons are in the remaining orbitals. It would lead outside the scope of this presentation to give an explicit description of this model but rather refer to applications, like in [31, 34]. These references show in detail how to determine the pseudo-$SU(3)$ irreps for each cluster and the united nucleus. The coupling of the cluster irreps with the relative motion and all considerations made above stay the same.

Here, we will resume the main steps and present the results in Tables 1 and 2. In the system of the united nucleus the nucleons are filled into the Nilsson levels at its deformation $\epsilon_2 = 0.2$, taken from [35]. The nucleons in the active orbitals are filled into the levels of the pseudo-oscillator, which provides the pseudo-$SU(3)$ irrep, in the same way as has been done for light nuclei within the $SU(3)$ shell model [33]. This provides the pseudo-$SU(3)$ irrep for $^{236}\text{U}$. In a second step, the largest cluster is taken and its nucleons filled into the Nilsson orbital at the same deformation as the united nucleus. The number of nucleons in the active orbitals then determine the pseudo-$SU(3)$ irrep of the largest cluster. In a final step, the remaining nucleons, of the lightest cluster, are filled on top of the largest cluster, which again gives us the number of nucleons for the light cluster in the pseudo-oscillator. In this manner, the number of nucleons in the active orbitals of the light plus of the heavy cluster sum up to the nucleons in the active orbitals of the united nucleus.

This procedure is different to the pseudo-$SU(3)$ approach in [36, 37], where either different deformations were taken for the individual clusters and/or the light cluster was treated within the $SU(3)$ model for light clusters. Here, we decided to take another approach, noting that all nucleons move in the same mean field with the same deformation and that the number of nucleons in the active orbitals are determined by the united nucleus.
Table 1: A series of 2-cluster systems, all belonging to the nucleus $^{236}_{92}U_{144}$ are enumerated (first column). In the last column the forbiddenness $n_c$ for these systems is evaluated.

In Table 1 several decay channels $^{236}U \rightarrow X + Y$ are listed, whose forbiddenness was determined. The first column lists the case number, to which Table 2 refers to. This table shows the forbiddenness as obtained by our counting procedure. The pseudo-$SU(3)$ irreps for each cluster, for the cluster irrep $(\lambda_c, \mu_c)$ and the total irrep $(\lambda, \mu)$ are given in Table 2. For the cluster irrep we take the largest one as suggested by our counting procedure. This irrep is then coupled with $(n_c, 0)$ and the relative irrep $(n_r, 0)$ to the total irrep of the united nucleus $^{236}U$, which is $(54, 0)$. This irrep is obtained by first determining the irreps for the proton and neutron parts and then coupling them to the maximal irrep of the united nucleus. The number of protons in the active levels is 46 and for the neutrons it is 82.

Fig. 1 depicts the forbiddenness versus the nuclear mass of the lightest cluster. As a key feature we notice that when one cluster is light the forbiddenness is zero or very low, while it increases as
Table 2: List of pseudo-$SU(3)$ irreps. The first column refers to the number of the cluster system as listed in Table 1. The second and third columns list the irrep for the first and second cluster, respectively. The fourth column lists the cluster irreps, where we took always the largest one, according to the findings in the text. The fifth column lists again the forbiddenness, now written in terms of a pseudo-$SU(3)$ irrep, and the last column lists the number of oscillation quanta which have to be added according to the Wildermuth condition.

| No. | $(\lambda_1, \mu_1)$ | $(\lambda_2, \mu_2)$ | $(\lambda_c, \mu_c)$ | $n_c$ | $(n_0, 0)$ |
|-----|----------------------|----------------------|----------------------|-------|-------------|
| 1   | (0,0)                | (48,4)               | (48,4)               | 0     | (10,0)      |
| 2   | (0,4)                | (24,0)               | (24,4)               | 4     | (46,0)      |
| 3   | (0,2)                | (16,2)               | (16,4)               | 4     | (54,0)      |
| 4   | (4,2)                | (10,0)               | (6,6)                | 2     | (60,0)      |
| 5   | (4,0)                | (10,6)               | (8,6)                | 4     | (64,0)      |
| 6   | (4,0)                | (10,6)               | (8,6)                | 4     | (64,0)      |
| 7   | (8,0)                | (12,8)               | (4,16)               | 0     | (66,0)      |
| 8   | (8,2)                | (2,8)                | (10,10)              | 6     | (72,0)      |
| 9   | (10,0)               | (22,0)               | (32,0)               | 16    | (70,0)      |
| 10  | (0,0)                | (24,8)               | (24,8)               | 20    | (98,0)      |
| 11  | (16,4)               | (36,0)               | (52,4)               | 32    | (102,0)     |
| 12  | (0,6)                | (24,2)               | (24,8)               | 28    | (116,0)     |
| 13  | (12,0)               | (8,0)                | (20,0)               | 36    | (118,0)     |
Figure 1: Graphical representation of the results from Table 1. The forbiddenness is depicted versus the mass of the lightest cluster.

As a further example, we discuss the fission of $^{252}$Cf, which is of great interest (please, consult the web page of Ref. [38] which contains all possible links to recent experimental activities related to $^{252}$Cf). The procedure followed is the same as in $^{236}$U and the interpretation of the results is similar. In Table 3 a list of two-cluster systems is given, with the same united $^{252}$Cf nucleus. In the last column the forbiddenness is listed. Again, the heavier the lightest cluster is, the larger is $n_c$. For completeness, in Table 4 the pseudo-$SU(3)$ irreps for each cluster system is given. Finally, in Figure 2 the $n_c$ is plotted versus the mass number of the lightest cluster. As in the former case, the forbiddenness increases with the mass of the light cluster.

The introduction of forbiddenness will have another implication
Table 3: A series of 2-cluster systems, all belonging to the nucleus $^{252}\text{Cf}_{154}$, $(\lambda, \mu) = (56, 10)$, are enumerated (first column). In the last column the forbiddenness $n_c$ for these systems is evaluated.
Table 4: List of pseudo-$SU(3)$ irreps. The first column refers to the number of the cluster system as listed in Table 3. The second and third columns list the irrep for the first and second cluster, respectively. The fourth column lists the cluster irreps, where we took always the largest one, according to the findings in the text. The fifth column lists again the forbiddeness, now written in terms of a pseudo-$SU(3)$ irrep, and the last column lists the number of oscillation quanta which have to be Wildermuth condition.
Figure 2: Graphical representation of the results from Table 3. The forbiddenness is depicted versus the mass of the lightest cluster.
for the determination of spectroscopic factors within algebraic models: In [18, 39] a very effective parametrization of the spectroscopic factor within the SACM was given. In [39] this parametrization was applied to nuclei in the p- and sd-shell with only few percent deviations to exact calculations within the SU(3) model [40]. In [18] a simpler ansatz for the spectroscopic factor was used to deduce the spectroscopic factors for fission products in $^{236}\text{U}$. There it was shown that the process of preformation, followed by the penetration through the potential barrier, gives the same qualitative results as in hydrodynamic models, with following penetration through the potential barrier gives the same results as for fission using hydrodynamic models, which is stated differently in [13]. At the low mass cluster side the model also follows the linear scaling rule of Blendowske and Walliser [41], who suspected that this linear behavior has to be changed for heavy clusters in order to reproduce the observed half-lives. In [18] the algebraic spectroscopic factor, within the SACM, was constructed and in a natural way reproduced the required deviation. Due to the explicit appearance of microscopic information, via the pseudo-SU(3) irreps, shell effects are automatically included. The ansatz in [18] does not include the $n_c$ dependence yet.

In [18, 39] only dependencies of the spectroscopic factor on the number of $\pi$-bosons, and the different SU(3) irreps are given. With the introduction of an additional dependence on the forbiddenness, $n_c$, we hope to improve the description of the unification of the cluster radioactivity with the fission process, as intended in [18].

3 Conclusions

We have not only presented a consistent procedure to calculate the forbiddenness of a two-cluster system but also determined which cluster irrep has to be constructed in order to couple with the relative motion to the ground state of the united nucleus. It has been demonstrated that in general the two clusters within the united nucleus have to be in an excited state and not in their ground state. As the main results a simple rule easy to implement emerged, namely in order to reduce the forbiddenness one has to take the largest, most compact SU(3) cluster irrep (large $(2\lambda_c + \mu_c)$ but $\mu_c$ larger than $\lambda_c$).

The main idea of the procedure presented can be easily extended to the system of more than two clusters, increasing the number of
possibilities of cluster systems to be analyzed.

Also heavy cluster systems can be treated, where the pseudo-$SU(3)$ scheme has to be applied. As an example, we treated several decay channels of $^{236}\text{U}$ and $^{252}\text{Cf}$ and discussed the importance of the results in connection with the unification of cluster radioactivity with the fission process, i.e., that both follow the same underlying physics of first pre-formation and then penetration through the barrier.

There is also the possibility to improve the values for spectroscopic factors [18, 39], introducing an additional dependence on the forbiddenness $n_c$.

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