Zitterbewegung (ZB, the trembling motion) of free relativistic electrons in a magnetic field is calculated. It is shown that the motion of an electron wave packet has intraband frequency components, corresponding to the classical cyclotron motion, and several interband frequency components corresponding to the Zitterbewegung. For a two-dimensional situation, the presence of a magnetic field makes the ZB motion stationary, i.e. not decaying in time.

The purpose of our work is twofold. First, we calculate the ZB motion for a magnetic field in the presence of an external magnetic field. The parameters of DE simulated in Ref. \[6\], the effects of the simulated magnetic field on ZB are sufficiently strong to be observable.

The Hamiltonian for a relativistic electron in a magnetic field is

\[ \hat{H} = c_0 \hat{\pi}_x + c_0 \hat{\pi}_y + c_0 \hat{\pi}_z + \beta mc^2, \]

where \( \hat{\pi}_x, \hat{\pi}_y, \hat{\pi}_z \) is the generalized momentum, \( q \) is the electron charge, \( \alpha_0 \) and \( \beta \) are the Dirac matrices in the standard notation. Taking a magnetic field \( B \parallel z \) we choose the vector potential \( A = (−By, 0, 0) \) and look for eigenfunctions in the form

\[ \Psi(r) = e^{ik_x x + ik_z z} \Phi(y). \]

Introducing the magnetic radius \( L = \sqrt{\hbar/eB} \) and \( \xi = y/L - k_x L \), we have \( y = \xi L + k_x L^2, eBy/\hbar = y/L^2, \) and \( \partial/\partial y = (1/L)\partial/\partial \xi \). Defining the standard raising and lowering operators for the harmonic oscillator \( \hat{a}_y = (\xi + \partial/\partial \xi)/\sqrt{2} \) and \( \hat{a}_y^+ = (\xi - \partial/\partial \xi)/\sqrt{2} \) one has \( \{\hat{a}_y, \hat{a}_y^+\} = 1 \) and \( \xi = (\hat{a}_y + \hat{a}_y^+)/\sqrt{2} \). Now the Hamiltonian reads

\[ \hat{H} = \begin{pmatrix} mc^2 \hat{I} & \hat{H} \\ \hat{H} & -mc^2 \hat{I} \end{pmatrix}, \]

where \( \hat{I} \) is the 2x2 identity matrix, and

\[ \hat{H} = -\hbar \omega \begin{pmatrix} c p_z & \hat{a}_y \\ \hat{a}_y^+ & -c p_z \end{pmatrix}, \]

with \( \omega = \sqrt{2c}/L \). An eigenstate \( |n\rangle \) of \( \hat{H} \) is characterized by five quantum numbers: \( n, k_x, k_z, \epsilon, s \), where \( n \) is the Landau level number, \( k_x \) and \( k_z \) are the wave vector components, \( \epsilon = \pm 1 \) labels the positive and negative energy branches, and \( s = \pm 1 \) is the spin index. In the representation of Johnson and Lippmann \[8\], the state \( |n\rangle \) is

\[ |n\rangle = \frac{e^{ik_x x + ik_z z}}{N_{nk_z}} \begin{pmatrix} s_0 (\epsilon E_n k_z + mc^2) |n - 1\rangle \\ s_1 (\epsilon E_n k_z + mc^2) |n\rangle \\ (s_0 h k_z - s_1 \omega_n)(n - 1) \\ -(s_0 \omega_n + s_1 h k_z c) |n\rangle \end{pmatrix}, \]

where \( s_0 = \epsilon \hbar c k_z + mc^2 \) and \( s_1 = \epsilon \hbar c k_z + mc^2 \).
where \( s_u = (s + 1)/2 \) and \( s_l = (s - 1)/2 \) are the projection operators on the states \( s = \pm 1 \), respectively. The frequency is \( \omega_n = \omega \sqrt{n} \), the energy is

\[
E_{nk} = \sqrt{(mc^2)^2 + (\hbar \omega_n)^2 + (\hbar k_z c)^2},
\]

and the norm is \( N_{nk} = (2E_{nk}^2 + 2mc^2E_{nkz})^{1/2} \). In this representation the energy \( E_{nk} \) does not depend explicitly on \( s \). The harmonic oscillator states \( \langle r|n \rangle \) have the standard form.

In order to calculate the average electron position we introduce four-component operators \( \mathcal{A} = \text{diag}(a_y, a_y, a_y, a_y) \) and \( \mathcal{A}^+ = \text{diag}(a_y^+, a_y^+, a_y^+, a_y^+) \). Now we define \( \mathcal{Y} = L(\mathcal{A} + \mathcal{A}^+)/\sqrt{2} \) and \( \mathcal{X} = L(\mathcal{A} - \mathcal{A}^+)/i\sqrt{2} \) in analogy to the position operators \( \hat{y} \) and \( \hat{x} \). We use averaging of \( \mathcal{A} \) and \( \mathcal{A}^+ \) operators in the Heisenberg picture \( \langle \hat{A}(t) \rangle = e^{i\mathcal{Y}t/\hbar}\langle \mathcal{A}(0)e^{-i\mathcal{X}t/\hbar} \rangle \) over a wave packet \( f(r) \). Inserting two unity operators \( 1 = \sum_n |n\rangle \langle n| \) one obtains

\[
\langle \hat{A}(t) \rangle = \sum_{n,n'} \langle n|e^{i\epsilon E_{nk} t/\hbar}\langle n|\mathcal{A}|n'\rangle e^{-i\epsilon E_{nk} t/\hbar}\langle n'|f \rangle, \tag{7}
\]

and similarly for \( \langle \hat{A}^+(t) \rangle \). The wave packet is assumed to be in the form \( f(r) = f_x(z)f_y(x,y) \), where \( i = 1, 2, 3, 4 \) labels the packet components. Below we limit our calculations to a packet with the second nonzero component. The summation over \( n,n',c ',s,s' \) denotes summations over the wave packet. In this limit the ZB part

\[
\langle \hat{A}(t) \rangle = \sum_{n,n'} \int_{-\infty}^{\infty} dk_x dk'_x g^2_{k,z}(k_x)g^2_{z,k'_x}(k'_x) \times
\]

\[
\sum_{c',s,s'} e^{i(\epsilon E_{nk} - \epsilon' E_{n'k'} + j \mathcal{Y} )/\hbar}\chi_{n,k}\chi_{n',k'} \times
\]

\[
\sum_{n,s} \int_{-\infty}^{\infty} dk_x dk'_x s_i(s'_{j} F^s_{n,k}(k_x)F'^s_{n',k'}(k'_x)|n\rangle|\mathcal{A}|n'\rangle, \tag{8}
\]

where \( \chi_{n,k} = (\epsilon E_{nk} + mc^2)/N_{nk} \), and

\[
F^s_{n,k}(k_x) = \frac{1}{2\sqrt{Lc}} \int_{-\infty}^{\infty} g^s_{z,k}(k_x, y)e^{-i\xi^s_{2}H_n(\xi)}dy, \tag{9}
\]

in which

\[
g^s_{z,k}(k_x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^s_{z,k}(y, x)e^{ik_x x}dx, \tag{10}
\]

and

\[
g^s_{z,k}(k_x) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} f^s_{z,k}(x)e^{ik_x x}dz. \tag{11}
\]

The selection rules for \( \langle n|\mathcal{A}|n'\rangle \) are \( n' = n + 1, k'_x \neq k_x \), \( k'_z \neq k_z \), while for \( \langle n|\mathcal{A}^+|n'\rangle \) they are \( n' = n - 1, k'_x \neq k_x \), \( k'_z \neq k_z \). After some manipulation we finally obtain

\[
\langle \hat{A}(t) \rangle = \frac{1}{2} \sum_{n} \sqrt{n + 1} \langle \hat{I}_{2,n,n+1}^s \rangle (I^+_c + I^-_c + iI^+_s + iI^-_s), \tag{12}
\]

\[
\langle \hat{A}^+(t) \rangle = \frac{1}{2} \sum_{n} \sqrt{n + 1} \langle \hat{I}_{2,n,n+1}^s \rangle (I^+_c - I^-_c - iI^+_s - iI^-_s). \tag{13}
\]

FIG. 1: Calculated motion of the electron wave packet in vacuum in the presence of a gigantic magnetic field \( B = 2 \times 10^9 \text{T} \) according to 3+1 Dirac equation for two packet widths. Both interband and intraband frequencies are present in the spectrum. Time is given in \( \tau \equiv \hbar /mc^2 \), positions are given in Compton wavelengths \( \lambda_c = \hbar /mc \).

where

\[
I^\pm_c = \int_{-\infty}^{\infty} \left[ 1 \pm \frac{E_{nk}}{E_{n+1,k+1}} \right] |g^2_z(k_z)|^2 \times \cos \left[ (E_{n+1,k+1} \pm E_{nk})t/\hbar \right] dk_z, \tag{14}
\]

\[
I^\pm_s = mc^2 \int_{-\infty}^{\infty} \left[ 1 \pm \frac{E_{nk}}{E_{n+1,k+1}} \right] |g^2_z(k_z)|^2 \times \sin \left[ (E_{n+1,k+1} \pm E_{nk})t/\hbar \right] dk_z, \tag{15}
\]

and

\[
U^{i,j}_{m,n} = \int_{-\infty}^{\infty} F_{m,k}^{s*}(k_x)F_{n,k}(k_x)dk_z, \tag{16}
\]

for \( i, j = 1, 2, 3, 4 \). We perform specific calculations taking \( f(r) \) in the form of an ellipsoidal Gaussian packet \( (0, f(r), 0, 0) \) characterized by three widths \( d_x, d_y, d_z \) and having a non-zero momentum \( \hbar k_0 = h(k_0 x, 0, 0) \). The quantities \( g_{xy}(k_x, y), F_i(k_x), \) and \( U^i_{m,n} \) can be obtained analytically, see Ref. 10.

In the non-relativistic limit: \( \hbar \omega, cp_x \ll mc^2 \), equations (12) - (16) reduce to the cyclotron motion with the frequency \( \omega_c = (E_{n+1,k_z} - E_{nk_z})/\hbar z \approx eB/m \) and the radius \( r = k_0 L^2 = mv/eB \). In this limit the ZB part of the motion has the amplitude several orders of magnitude smaller than \( \lambda_c \). In the opposite relativistic limit, in which \( \hbar \omega, cp_z \ll mc^2 \), both the cyclotron and ZB components have comparable amplitudes. In Fig. 1 we plot the average position in \( x \) and \( y \) directions for an electron wave packet in a gigantic magnetic field \( B = 2 \times 10^9 \text{T} \) for two packet parameters. It is seen that the ZB oscillations consist of several frequencies. This is the main effect of the magnetic field, which quantizes both positive and negative electron energies into Landau levels. In larger time scale the oscillations in 3+1 space go through decays and revivals, but finally disappear.
The involved DE must have at least 2+1 dimensions since a magnetic field \( B \parallel z \) couples the electron motion in \( x \) and \( y \) directions. The above equations, derived for the 3+1 DE, can be reduced to the 2+1 case by setting \( g_j(k_z)^2 = \delta(k_z) \). The main difference between 3+1 and 2+1 cases is that for the 3+1 case the ZB of the wave packet has a transient character (it decays in time), whereas in the 2+1 case it has a permanent character going through decays and revivals. This difference, due to the integration over \( k_z \) in the 3+1 case, is in agreement with general predictions of Lock \[11\]. However, a slow decay of ZB in time will occur also in the 2+1 case since the trembling electron will emit radiation. This is possible because a Gaussian wave packet is not an eigenstate of the Dirac Hamiltonian. Also a broadening of Landau levels due to external perturbations results in a transient character of ZB, c.f. \[12\].

The main experimental problem in investigating the above ZB phenomenon in an external magnetic field is the fact that for free relativistic electrons the basic ZB (interband) frequency corresponds to the energy \( h\omega_c \approx 1 \text{ MeV} \), whereas the cyclotron frequency for a magnetic field of 100 T is \( h\omega_c \approx 0.1 \text{ eV} \), so that the magnetic effects in ZB are very small. However, as mentioned above, it is possible by now to simulate the Dirac equation changing thereby its basic parameters. This gives a possibility to modify drastically the ratio \( h\omega_c/2mc^2 \) making its value much more advantageous. To simulate the DE for an electron in a magnetic field, we transform it first to the much more advantageous. To simulate the DE for an electron in a magnetic field of 100 T is possible by now to simulate the Dirac equation changing thereby its basic parameters. This gives a possibility to modify drastically the ratio \( h\omega_c/2mc^2 \) making its value much more advantageous. To simulate the DE for an electron in a magnetic field, we transform it first to the form \( H = c \sum \alpha\psi_i + \delta mc^2 \), using the unitary operator \( \tilde{\Omega}(\eta - \pi) \). Here \( \Omega \) and \( \tilde{\Omega} \) are the coupling strengths and \( \eta \) is trap’s frequency in the \( z \) direction and \( k \) is the wave vector of the driving field in a trap. The subscripts in parenthesis of Eq. (18) denote states involved in a given transition. The JC interaction gives \( \tilde{a}_y \) in \( \tilde{H}_{12}^z \) and \( \tilde{a}_y^+ \) in \( \tilde{H}_2^1 \) elements of the Hamiltonian \( \tilde{H}^z \) in Eq. (17), while AJC gives \( \tilde{a}_y \) in \( \tilde{H}_{21}^z \) and \( \tilde{a}_y^+ \) in \( \tilde{H}_1^2 \) elements. The simulation of 3+1 DE by Eq. (18) can be realized with 12 pairs of laser excitations. If one omits the \( p_z \) interaction, which corresponds to the 2+1 DE, one needs 8 pairs of laser excitations. The simulated magnetic field can be found from the correspondence (see Ref. \[14\]): \( \tilde{a}_y = \sqrt{2}L(\partial/\partial y) = 2\Delta(\partial/\partial y) \), which gives \( \Delta \equiv L/\sqrt{2} \). Since \( c \equiv 2\eta\Delta\Omega \) and \( mc^2 \equiv \hbar\Omega \), we have

\[
\kappa = \frac{\hbar eB}{m(2mc^2)} = \left(\frac{\eta\Omega}{\Omega}\right)^2.
\]

Therefore, by adjusting frequencies \( \Omega \) and \( \tilde{\Omega} \) one can simulate different regimes of the ratio \( \kappa = \hbar\omega_c/2mc^2 \). In Fig. 2 we show the calculated ZB for three values of \( \kappa \): 16.65, 1.05, 0.116. It is seen that, as \( \kappa \) gets larger

\[
\begin{align*}
\text{FIG. 2: Calculated motion of the electron wave packet as described by 2+1 DE, simulated by trapped } & \text{ions for three values of the effective rest energy } \Omega. \text{ Trap parameters: } \\
& \eta = 0.06, \Omega = 2\pi \times 68 \text{ kHz}, \Delta = 96 A, \kappa_0 = \Delta^{-1}, \kappa_1 = \Delta^{-1/2}, d_0 = 0.9d_w. \text{ Simulations correspond to ratios } \kappa = \hbar\omega_c/2mc^2 = 16.65(a), 1.05(b), 0.116(c), \\
& \text{respectively. Positons are given in } L = \sqrt{2}\Delta.
\end{align*}
\]
(i.e. the field intensity increases or the effective gap decreases), the frequency spectrum of ZB becomes richer. This means that more interband and intraband frequency components contribute to the spectrum. Both interband and interband frequencies correspond to the selection rules \( n' = n \pm 1 \) so that, for example, one deals with ZB (interband) energies between Landau levels \( n = 0 \) to \( n' = 1 \), and \( n = 1 \) to \( n' = 0 \), as the strongest contributions. For high magnetic fields the interband and intraband components are comparable, so one can legitimately talk about the ZB. We believe that ZB oscillations of the type shown in Fig. 2a, based on the 2+1 DE for \( \kappa = \hbar c \omega_{c} / 2mc^{2} \gg 1 \), are the best candidate for an observation of the simulated trembling motion in the presence of a magnetic field. The calculated spectra shown in Fig. 2 use the simulated parameters already realized experimentally, see [6]. Notice the tremendous differences of the time and position scales between the results for free electrons in a vacuum shown in Fig. 1 and the simulated ones in Fig. 2.

The anisotropy of ZB with respect to the \( \langle x(t) \rangle \) and \( \langle y(t) \rangle \) components, seen in Figs. 1 and 2 is due to the initial conditions, namely \( k_{0x} \neq 0 \) and \( k_{0y} = 0 \). A similar anisotropy was predicted in the zero-gap situation in graphene [10].

We emphasize that the Dirac equation (1) and our resulting calculation, as well as Eq. (17) and its simulation in Eq. (18), represent the ‘empty’ Dirac Hamiltonian which does not take into account the ‘Fermi sea’ of electrons with negative energies in a vacuum. This one-electron model corresponds to the original considerations of Schrödinger’s. On the other hand, the filled states of electrons with negative energies may affect the phenomenon of ZB, see [3, 17].

Finally, one should recognize that the experiment of Gerritsma et al. [6] simulates not only the 1+1 Dirac equation for free relativistic electrons in a vacuum but also the two-band \( k, p \) model for electrons in narrow-gap semiconductors and the ZB resulting from this description [18, 19]. In fact, the results of Ref. [6] look remarkably similar to our predictions 19.

In summary, we calculated the trembling motion of relativistic electrons in a vacuum in the presence of a magnetic field for 3+1 and 2+1 spaces. In contrast to the no-field case, the presence of a magnetic field results in many interband frequencies contributing to the trembling motion. In the 2+1 case the ZB oscillations of the electron wave packet are stationary, i.e. they do no decay in time. We indicate how to simulate the Dirac electron in a magnetic field and the resulting ZB using trapped ions and laser excitations. We show that, for the parameters of DE simulated very recently by Gerritsma et al. [6], the effect of a magnetic field on the trembling motion should be clearly observable.

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