Instability of solitary waves on Euler’s elastica

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Stability of solitary waves in a thin inextensible and unshearable rod of infinite length is studied. Solitary-wave profile of the elastica of such a rod without torsion has the form of a planar loop and its speed depends on a tension in the rod. The linear instability of a solitary-wave profile subject to perturbations escaping from the plane of the loop is established for a certain range of solitary-wave speeds. It is done using the properties of the Evans function, an analytic function on the right complex half-plane, that has zeroes if and only if there exist the unstable modes of the linearization around a solitary-wave solution. The result follows from comparison of the behaviour of the Evans function in some neighbourhood of the origin with its asymptotic at infinity. The explicit computation of the leading coefficient of the Taylor series of the Evans function near the origin is performed by means of the symbolic computer language.

Keywords: Euler’s elastica, solitary wave, orbital stability, unstable eigenfunction, Evans function

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1. Introduction

Dynamics of inextensible and unshearable thin rods of the infinite length described in the framework of the Bernoulli-Euler beam model is governed by the following equations (in dimensionless form)

\[ \tau_{tt}^i = (p\tau^t)\xi\xi + \tau_{\xi\xi}^i - \tau_{\xi\xi\xi\xi}^i, \]
\[ \tau^i_1 = 1, \]
\[ \tau_1 \to 1, \quad \tau_2, 3 \to 0, \quad \xi \to \pm\infty, \quad (1.1) \]

where \( \xi \) is the dimensionless arc-length along the elastica, \( \tau_i = \tau^i, \) \( i = 1, 2, 3 \) are the components of the unit vector tangent to the elastica and \( p = P - 1, \) \( P \) being the absolute value of tension in the rod; summation is assumed under repeating indices and subscripts \( t \) and \( \xi \) denote differentiation with respect to these variables. Hereafter, the different notations for the same components are given both with lower as well as upper indices; it is done for the sake of convenience (when a component already has the subscripts denoting differentiation or some other) and also in the case when the rule about summation under repeating indices is assumed.

The classification of the forms of elastica was for the first time presented by Euler, who derived the ordinary differential equation of elastica (Love [19]). For the
elastica of the infinite length its profile admits the configuration described by the
solitary-wave solution of (1.1), having the form of the planar loop (fig.1):

\[ p = -p^0 = -6(1 - c^2) \text{sech}^2 \sqrt{1 - c^2} \zeta, \quad \tau_3 = 0, \quad \tau_1 = \tau_1^0 = 1 - 2 \text{sech}^2 \sqrt{1 - c^2} \zeta, \]
\[ \tau_2 = \tau_2^0 = -2 \text{sech}^2 \sqrt{1 - c^2} \zeta \sinh \sqrt{1 - c^2} \zeta, \quad \zeta = \xi - ct, \quad 0 \leq c < 1. \] (1.2)

The formula (1.2) gives the particular localized solution, where the tension \( P \) is
distributed in the rod in a following way: it increases from the minimal value at
the top of the loop (for \( 1 - c^2 > \frac{1}{6} \), for example, this value is negative, that
corresponds to the compression of the rod at the place of the loop’s localization)
to the maximal value at both infinities.

The system of equations (1.1) is a hamiltonian one and it can be written as

\[ \tau_i^t = \frac{\partial}{\partial \xi} \frac{\delta H}{\delta v_i}, \quad v_i^i = \frac{\partial}{\partial \xi} \frac{\delta H}{\delta \tau_i}, \quad \tau^i \tau_i = 1, \]

where

\[ H = \int_{-\infty}^{\infty} \left[ (v_i v^i + \tau_i \tau_i^1 + P(\tau_i \tau_i^1 - 1)) \right] d\xi. \]

Along with the hamiltonian \( H \) one has two more conserved quantities:

\[ Q = \int_{-\infty}^{\infty} (\tau_i^1 - \tau_i^1) v_i d\xi, \quad R = \int_{-\infty}^{\infty} (\tau_i - 1) d\xi, \quad \tau^1_\infty = 1, \quad \tau^2_\infty = 0. \]

In this paper we prove linear instability of certain solitary waves with respect
to transverse perturbations \( \delta \tau_3 \neq 0 \). It is shown that there exists exponentially
growing with time eigenfunction being the solution of the linearization about a
solitary wave when its dimensionless speed does not exceed \( 1/\sqrt{2} \). The existence
of the unstable eigenfunction is expected to imply the physical instability, i.e. the
decay of the loop under the transverse perturbations. The opposite result about
the stability, when the loop is not destroyed, takes place for plane perturbations
\( \delta \tau_3 = 0, \delta \tau_{1,2} \neq 0 \). Mathematically this result is expressed by the Theorem 1.1
below.
Let us denote \( \phi_c = \{ \tau_1^0 - 1, \tau_2^0, 0, v_1^0, v_2^0, 0 \}^T \) (\( \tau_1^0 = -c(\tau_1^0 - 1), v_2^0 = -c\tau_2^0 \)), \( w(t) = \{ \tau_1, \tau_2, \tau_3, v_1, v_2, v_3 \}^T \) and let \( X = H^1(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}) \) be a Hilbert space, \( \| \cdot \| \) is the norm in \( X \). In Beliaev and Il'ichev [6] using the ideas from Grillakis et al. [12] the following theorem is proved relating to the nonlinear stability of solitary waves (1.2) subject to perturbations of \( \tau_{1,2}^0 \) only.

**Theorem 1.1.** Let \( \tau_3 \equiv 0 \) and for any given \( w_0 \in X \) near \( \phi_c \) in \( X \), \( \| w_0 - \phi_c \| < \gamma \), there exist \( T = T(\gamma) > 0 \) and a vector function \( w(t) \in C([0, T), X) \) (continuous with values in \( X \)) such that \( w(0) = w_0 = \{ \tau_1^0, 0, v_1^0, 0 \}^T \), \( l = 1, 2 \), and for all \( 0 \leq t \leq T \), \( \tau_{1,2}^0 \tau_0 = \tau_1 \tau_1^1 = 1 \) and \( H(w) = H(w_0), Q(w) = Q(w_0), R(w) = R(w_0) \). Then, for all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( \| w_0 - \phi_c \| < \delta \), then

\[
\sup_{t > 0} \inf_{\omega \in \mathbb{R}} \| w(t) - \phi_c(\cdot + \omega) \| < \varepsilon.
\]

The theorem was proved by demonstrating that the set of translates of a solitary wave gives a local minimum to the invariant functional

\[
F(w) = H(w) + cQ(w) - R(w)
\]

on the closed submanifold \( \tau_1 \tau_1^1 = 1 \) in \( X \). This, in its turn, follows from the fact that the ‘linearized hamiltonian’ \( H(\phi_c) = \delta^2 F(\phi_c)/\delta^2 w \) has exactly one zero eigenvalue and positive spectrum bounded away from zero. The analysis of Beliaev and Il'ichev [6] is extended in Dichmann et al. [9] to the case when the rotational kinetic energy (which is small compared to the total energy for thin rods) is not neglected and the same stability result for subsonic solitary waves was established. For \( \delta \tau_3 = \tau \neq 0 \), however, the operator \( H(\phi_c) \) has an extra negative eigenvalue and the sufficient conditions for the stability of bound states as they appear in Grillakis et al. [12] are violated. The system of equations (1.1) is also invariant under group of rotations (see sec. 6). However, the stability theory given in Grillakis et al. [13] for hamiltonian systems with several symmetries is not applicable in the case under consideration by the reason that solitary wave orbit is in fact the set of translates of a solitary wave, and therefore the conserved quantity associated with the rotational symmetry, being nonlocal, identically equals zero at a solitary wave.

The linearized equations (1.1) for \( \delta \tau_{1,2} \) and \( \tau \) decouple and we analyze here only the equation for \( \tau \) since the stability result in the planar case is known. A solitary wave is called linearly unstable if there exist solutions of the linearized equation of the form

\[
\tau = \tau(\xi, t) = e^{\lambda t} w(\lambda, \zeta)
\]

with \( w(\lambda, \zeta) \) decaying exponentially as \( \zeta \to \pm \infty \) and \( \text{Re} \lambda > 0 \). The function \( w(\lambda, \zeta) \) obeys the eigenvalue problem

\[
(\lambda - c \frac{d}{d\zeta})^2 w = \frac{d^2}{d\zeta^2} w - \frac{d^4}{d\zeta^4} w - \frac{d^2}{d\zeta^2}(p^0 w).
\]

The equation (1.4) has the same form as the corresponding equation for the stability problem discussed in Alexander and Sachs [2]. That is why, the theoretical results of this paper are also valid for our analysis. Yet, the coefficient \( p^0 \) in (1.4) does not coincide as a function of \( \zeta \) with the corresponding coefficient in Alexander and
We get the instability result by means of the Evans function. The Evans function $D(\lambda)$ is constructed as an analytic on the right complex half-plane function of the spectral parameter; $D(\lambda)$ has zeroes if and only if there exist the unstable modes of the linearization around a solitary-wave solution. The use of an Evans function to get instability result for solitary waves appears in Pego and Weinstein [20]. In parabolic problems, the ideas of Evans [10] were further developed by Jones [15] and Alexander et al. [3] (see also Kapitula [16] and references therein). Alexander and Sachs [2] extended the technique of Pego and Weinstein [20] to the case of the Boussinesq-type equations where two decaying modes are present in the model. The application of the Evans function method for instability problems in various fields of hydrodynamics and physics can be found in Swinton and Eglin [22], Pego et al. [21], Alexander et al. [4], Gardner and Zumburin [11], Kapitula [17], Kapitula and Sandstede [18], Afendikov and Bridges [1], Bridges et al. [8].

The Evans function is real on the real axis and tends to unity as $|\lambda| \to \infty$. Our instability result follows from comparison of the behaviour of the Evans function in some neighbourhood of the origin with its asymptotic at infinity. We show that in a small neighbourhood of the origin one has $D(\lambda) < 0$ for real positive $\lambda$. More precisely, the following theorem holds.

**Theorem 1.2.** The Evans function $D(\lambda)$ constructed for the eigenfunction $w(\lambda, \zeta)$ from (1.3) is analytic in a neighbourhood of the origin and its converging Taylor series in this neighbourhood reads

$$D(\lambda) = -\frac{1 - 2c^2}{4(1 - c^2)^3} \lambda^2 + \sum_{n=3}^{\infty} e_n(c)\lambda^n.$$  

(1.5)

It follows then, that for $c^2 < 1/2$ the function $D(\lambda)$ has a zero somewhere at $\lambda = \lambda_0$ on the positive real axis, that, in its turn, implies the existence of the unstable eigenfunction $w(\lambda_0, \zeta)$ from (1.3).

In section 2 we give the formulation and linear stability problem. In section 3 we discuss analytic properties of the solutions of the linearized system as $\lambda$ varies in a zero neighbourhood. In section 4 the Evans function is introduced and its behaviour for large $|\lambda|$ is described. In section 5 we describe computations (made by means of the symbolic language Mathematica 4.0) of the leading coefficient of the Taylor series of $D(\lambda)$ in a neighbourhood of the origin. The explicit form of key expressions in our computations are listed in Appendix A. In section 6 we discuss our conclusions.

## 2. Formulation and linear stability problem

We consider a thin inextensible and unshearable elastic rod of infinite length initially coinciding with the $x_1$ axis of a Cartesian coordinate system. The total energy of such a rod is the sum of the kinetic energy and the bending energy, the torsion energy is neglected. The respective energy densities $K$ and $W$ are given by the expressions

$$K = \frac{1}{2} \rho S x_t^i x_{it}, \quad W = \frac{1}{2} I E x_\xi^i x_{\xi t},$$
where \(x_i, \ i = 1, 2, 3\) are the coordinates of the points on a neutral line (elastica) of the rod, \(\rho\) is the density of the rod, \(S\) is the area of the cross section of the rod, \(\rho I\) is the moment of inertia of the cross section of the rod about the line, orthogonal to the principal plane of bending \(x_1x_2\), \(E\) is the Young module, \(\xi\) is the arc-length along the elastica. The elastica is given by the equations \(x^i = x^i(\xi, t)\). For thin rods the rotational part of the kinetic energy is small in comparison with the kinetic energy of points of the elastica \([6]\) and we neglect it here.

The equations of motion can be obtained by taking the variations of the lagrangian \(\Lambda\),

\[
\Lambda = \frac{1}{2} \int_{t_0}^{t} \int_{-\infty}^{\infty} (\rho S x^i_{tt} - I E x^i_{\xi\xi\xi\xi}) d\xi dt
\]

under the condition of inextensibility

\[
x_{i\xi} x^i_{\xi} = 1.
\]

These equations read

\[
\rho S x^i_{tt} = (P x^i_{\xi})\xi - I E x^i_{\xi\xi\xi\xi}, \quad x_{i\xi} x^i_{\xi} = 1,
\]

where \(P(\xi, t) = p(\xi, t) + p_\infty\) is the Lagrange multiplier, \(p \to 0\) as \(\xi \to \pm\infty\). Making the scaling transformations in (2.1)

\[
p \to p_\infty p, \quad \xi \to \sqrt{IE/p_\infty} \xi, \quad t \to \sqrt{\rho SIE/p_\infty^2} t,
\]

and preserving the old notations one gets (1.1) with \(\tau_i = x^i_{\xi}\).

Linearizing the equations (1.1) about the solitary-wave solution (1.2) we get the following equation for the perturbation \(\delta\tau_3 = \tau\) of the third component of the tangent vector (1.2)

\[
\tau_{tt} = -(p^0 \tau)_{\xi\xi} + \tau_{\xi\xi} - \tau_{\xi\xi\xi\xi}.
\]

We seek the solution of (2.2) in the form (1.3). After substitution of (1.3) into the equation (2.2) it transforms into the equation (1.4). The last equation, in its turn, can be written in the matrix form

\[
y' = M(\lambda, \zeta)y,
\]

\[
y = \{y_1, y_2, y_3, y_4\}^T, \quad y_1 = w, \quad y_2 = w', \quad y_3 = w'', \quad y_4 = w''',
\]

where prime denotes differentiation with respect to \(\zeta\), and

\[
M(\lambda, \zeta) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-\lambda^2 - p^0 & 2\lambda c - 2p^0 & 1 - c^2 - p^0 & 0
\end{pmatrix}.
\]

The adjoint equation reads

\[
(\lambda + c \frac{d}{d\zeta})^2 w^* = \frac{d^2}{d\zeta^2} w^* - \frac{d^4}{d\zeta^4} w^* - p^0 \frac{d^2}{d\zeta^2} w^*,
\]

(2.4)
or in the matrix form
\[
z' = -z\mathcal{M}(\lambda, \zeta),
\]

\[
z = \{z_1, z_2, z_3, z_4\}, \quad z_4 = w^*, \quad z_3 = -w''', \quad z_2 = w''' - (1 - c^2 - p^0)w^*,
\]
\[
z_1 = -w'''' + (1 - c^2 - p^0)w''' - (2\lambda c - p^0')w^*.
\]

\section{3. Analytic properties of the solutions of the linearized system for \(\lambda\) in the zero neighbourhood}

In this section we enumerate the results of Alexander and Sachs [2] concerning the analytic properties of the solutions of systems having the form of (2.3), (2.5) when \(\lambda\) locates close to zero.

Let us denote \(\mathcal{M}_\infty(\lambda) = \lim_{\zeta \to \pm \infty} \mathcal{M}(\lambda, \zeta)\), \(\mu_\alpha(\lambda) (\alpha = 1, 2, 3, 4)\) the eigenvalues of \(\mathcal{M}_\infty(\lambda)\), \(r_\alpha(\lambda)\) the right and \(l_\alpha(\lambda)\) the left eigenvectors of the matrix \(\mathcal{M}_\infty(\lambda)\).

The matrix \(\mathcal{M}_\infty(\lambda)\) is given by the same expression as the corresponding matrix in Alexander and Sachs [2]. For \(\lambda = 0\) the vectors labeled by the subscript 4 represent the generalized eigenvectors of \(\mathcal{M}_\infty(\lambda)\) (see formulas (3.9), (3.10) in Alexander and Sachs [2]). The eigenvectors are normalized so, that the first (for right eigenvectors) and last (for left eigenvectors) components are unity.

The characteristic equation \(\det [\mathcal{M}_\infty(\lambda) - \mu \mathcal{E}] = 0\), where \(\mathcal{E}\) denotes the unit matrix, has the form
\[
\mu^4 - (1 - c^2)\mu^2 - 2\mu c\lambda + \lambda^2 = 0.
\]

In Alexander and Sachs [2] the following lemma is proved.

**Lemma 3.1.** For \(\text{Re } \lambda \neq 0\) the equation (3.1) has two roots in each complex half-plane.

Following [2] we denote \(\mu_1(\lambda)\), \(\mu_3(\lambda)\) the roots, lying in the left complex half-plane for \(\text{Re } \lambda > 0\) (\(|\text{Re } \mu_1(\lambda)| > |\text{Re } \mu_3(\lambda)|\) for \(\lambda\) in a zero neighbourhood). In the neighborhood of the origin the roots \(\mu_k(\lambda)\), \(k = 1, 3\) have the asymptotic form [2]
\[
\begin{align*}
\mu_1(\lambda) &= -\sqrt{1 - c^2} + \frac{c\lambda}{1 - c^2} + \frac{1 + 2c^2}{2(1 - c^2)^{5/2}}\lambda^2 + O(\lambda^3), \\
\mu_3(\lambda) &= -\frac{\lambda}{1 - c} + O(\lambda^3).
\end{align*}
\]

The eigenvalues \(\mu_{1,3}(\lambda)\) and the eigenvectors \(r_{1,3}(\lambda)\), \(l_{1,3}(\lambda)\) may lose the analyticity property in the points where the eigenvalues are not simple any more or, in other words, when the corresponding roots of equation (3.1) are multiple. The roots of (3.1) become multiple when the resultant of the polynomial in the left hand side of (3.1) and its derivative equals zero. The resultant is
\[
16\lambda^2[(1 - c^2)^3 + (-8 - 20c^2 + c^4)\lambda^2 + 16\lambda^4],
\]
Figure 2. Curves $\lambda$ versus $c$ where the eigenvalues are multiple. Along the branch $\lambda = \lambda_3$ the eigenvalues $\mu_2$ and $\mu_4$ coincide; along the branch $\lambda = \lambda_5$ the eigenvalues $\mu_1$ and $\mu_3$ coincide; along the branch $\lambda = \lambda_6$ the eigenvalues $\mu_3$ and $\mu_2$ coincide and along the branch $\lambda = \lambda_4$ the eigenvalues $\mu_1$ and $\mu_4$ coincide.

and it equals zero when $\lambda_{1,2} = 0$, and

$$
\lambda_{3,4} = \pm \frac{\sqrt{8 + 20c^2 - c^4 + c(8 + c^2)^{3/2}}}{4\sqrt{2}},
$$

$$
\lambda_{5,6} = \pm \frac{\sqrt{8 + 20c^2 - c^4 - c(8 + c^2)^{3/2}}}{4\sqrt{2}}.
$$

The nonzero branches (3.3) are pictured in Fig.2. It is seen that for nonzero $\lambda$ the points where the eigenvalues are multiple are separated from zero when $0 \leq c < 1$. It is also seen that when $\lambda$ crosses the imaginary axis the eigenvalues $\mu_3$ and $\mu_4$, coinciding with zero at $\lambda = 0$, escaping from their complex half-planes.

At $\lambda = 0$ one has $\mu_3(0) = \mu_4(0) = 0$. Yet, the kernel of $M_\infty(\lambda) - \mu_3(\lambda)E$ spanned by $r_3(\lambda)$ ($l_3(\lambda)$) is uniformly one dimensional and the vectors themselves are continuous at $\lambda = 0$ (see formulas (3.9), (3.10) in Alexander and Sachs [2]). Hence, in a small enough neighbourhood of the origin, not including the points given by (3.3), the normalized vectors $r_{1,3}(\lambda)$ and $l_{1,3}(\lambda)$ are analytic.

It follows from general theory of the ordinary differential equations that there exist solutions of (2.3), (2.5), satisfying

$$
\lim_{\zeta \to \infty} e^{-\mu_k(\lambda)}y_k(\lambda, \zeta) = r_k(\lambda),
$$

$$
\lim_{\zeta \to -\infty} e^{\mu_k(\lambda)}z_k(\lambda, \zeta) = l_k(\lambda), \quad k = 1, 3.
$$

Analyticity of $r_{1,3}(\lambda)$ $l_{1,3}(\lambda)$ implies the following theorem, proved in Pego and Weinstein [20] and generalized for the two mode case in Alexander and Sachs [2].

**Theorem 3.2.** Solutions $y_{1,3}(\lambda, \zeta)$, $z_{1,3}(\lambda, \zeta)$ are analytic in some neighborhood of $\lambda = 0$. 
4. Evans function

Consider the vector fields \( y^\wedge(\lambda, \zeta), z^\wedge(\lambda, \zeta) \) with the components
\[
g_{\alpha\wedge,\beta} = y_{1\alpha} y_{3\beta} - y_{1\beta} y_{3\alpha}, \quad z_{\alpha\wedge,\beta} = z_{1\alpha} z_{3\beta} - z_{1\beta} z_{3\alpha}, \quad \alpha > \beta, \quad \alpha, \beta = 1, 2, 3, 4, \tag{4.1}
\]
where \( y_{\alpha\beta}, z_{\alpha\beta} \) are the components of the vectors \( y_k, z_k \), correspondingly. We number the pairs \( \alpha \wedge \beta \) by the following way: \( 1 \wedge 2 \rightarrow 1, 1 \wedge 3 \rightarrow 2, 1 \wedge 4 \rightarrow 3, 2 \wedge 3 \rightarrow 4, 2 \wedge 4 \rightarrow 5, 3 \wedge 4 \rightarrow 6 \). The vectors \( y^\wedge(\lambda, \zeta), z^\wedge(\lambda, \zeta) \) obey the equations
\[
y^\wedge' = M^\wedge(\lambda, \zeta)y^\wedge, \quad z^\wedge' = -z^\wedge M^\wedge(\lambda, \zeta), \tag{4.2}
\]
where
\[
M^\wedge(\lambda, \zeta) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
2\lambda - 2p^{0'} & 1 - e^2 - p^0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\lambda^2 + p^{0''} & 0 & 0 & 1 - e^2 - p^0 & 0 & 1 \\
0 & \lambda^2 + p^{0''} & 0 & -2\lambda - 2p^{0'} & 0 & 0
\end{pmatrix}.
\]
The eigenvalues of \( M_\infty^\wedge(\lambda) = M^\wedge(\lambda, \pm \infty) \) are given by \( \mu_\alpha(\lambda) + \mu_\beta(\lambda) \). The eigenvalue \( \mu^\wedge(\lambda) = \mu_1(\lambda) + \mu_3(\lambda) \) with the minimal real part is simple in the right complex half-plane \( \text{Re} \lambda > 0 \). Therefore, the right \( r^\wedge(\lambda) \) and left \( l^\wedge(\lambda) \) eigenvectors of the matrix \( M_\infty^\wedge(\lambda) \) associated with this eigenvalue are analytic everywhere in the right complex \( \lambda \)-half-plane. This, in its turn, implies the analyticity of the vector-functions \( y^\wedge(\lambda, \zeta) \) and \( z^\wedge(\lambda, \zeta) \) everywhere in the right complex \( \lambda \)-plane. The components of the vectors \( r^\wedge(\lambda) \) and \( l^\wedge(\lambda) \) are given via the components of the vectors \( r_{1,3}(\lambda) \) and \( l_{1,3}(\lambda) \) by the expressions similar to (4.1) everywhere where the last vectors are well defined, i.e. in some neighbourhood of the origin of the complex \( \lambda \)-plane. The vector-solutions \( y^\wedge \) and \( z^\wedge \) satisfy
\[
\lim_{\zeta \to \infty} e^{-\mu^\wedge(\lambda)} y^\wedge(\lambda, \zeta) = r^\wedge(\lambda),
\]
\[
\lim_{\zeta \to -\infty} e^{\mu^\wedge(\lambda)} z^\wedge(\lambda, \zeta) = l^\wedge(\lambda).
\]

The analytic in the complex half-plane \( \text{Re} \lambda > 0 \) Evans function \( \hat{D}(\lambda) \) is defined as follows:
\[
\hat{D}(\lambda) = z^\wedge \cdot y^\wedge = \det \begin{pmatrix}
z_1(\lambda, \zeta) \cdot y_1(\lambda, \zeta) & z_1(\lambda, \zeta) \cdot y_3(\lambda, \zeta) \\
z_3(\lambda, \zeta) \cdot y_1(\lambda, \zeta) & z_3(\lambda, \zeta) \cdot y_3(\lambda, \zeta)
\end{pmatrix}. \tag{4.3}
\]
The last equality in (4.3) has the sense where the determinant in the right hand side is well defined, in particular, in the neighborhood of the origin, where, according to Theorem 3.2, the vector-functions \( y_k(\lambda, \zeta), z_k(\lambda, \zeta) \), \( k = 1, 3 \) are analytic. From (4.2) it follows that \( \hat{D}(\lambda) \) is independent on \( \zeta \). It also follows that the function \( \hat{D}(\lambda) \) is analytic in the neighborhood of the origin of the complex \( \lambda \)-plane.

**Theorem 4.1.** ([2], [3]). For \( \text{Re} \lambda > 0 \) the function \( \hat{D}(\lambda) \) has zeroes if and only if there is a solution of (1.4), which decays exponentially as \( \zeta \to \pm \infty \).
The asymptotic behaviour of the Evans function $\hat{D}(\lambda)$ as $\lambda \to \infty$ is governed by the following

**Theorem 4.2.** $\hat{D}(\lambda) \to \lambda^\wedge (\lambda) \cdot r^\wedge (\lambda)$ as $|\lambda| \to \infty$.

To prove Theorem 4.2 it is sufficient to verify the conditions of proposition 1.17 in Pego and Weinstein [20] (we note that in [20] the vectors corresponding to $r^\wedge (\lambda)$ and $\lambda^\wedge (\lambda)$ are normalized so that $\lambda^\wedge (\lambda) \cdot r^\wedge (\lambda) = 1$, and their Evans function has asymptotic value unity). It has to be checked that

- $M_\infty^\wedge (\lambda)$ is diagonalizable for large $\lambda$;
- $F(\lambda, \zeta) = W(\lambda)R(\zeta)V(\lambda) \to 0$ as $|\lambda| \to \infty$, where $V(\lambda)$ is a matrix of right eigenvectors, $W(\lambda)$ is its inverse and $R(\zeta) = M^\wedge (\lambda, \zeta) - M_\infty^\wedge (\lambda)$.

The validity of the first condition is given by the expression $\Delta = \hat{\Delta}$, where

$$\hat{\Delta} = -16\lambda^2(c_4^2 - 4\lambda^2)^2$$

is the determinant of $\hat{\mathcal{V}}$. Hence, both $\mathcal{V}$ and $\hat{\mathcal{V}}$ are invertible for large $|\lambda|$ and $\mathcal{F} = \hat{\mathcal{F}}(1 + O(|\lambda|^{-1}))$, where $\hat{\mathcal{F}} = \mathcal{W}\mathcal{R}\hat{\mathcal{V}}$ and $\mathcal{W} = \hat{\mathcal{V}}^{-1}$. The direct computation shows that $\hat{\mathcal{F}} = \Delta^{-1}\mathcal{F}$, where the components of the matrix $\mathcal{N}(\lambda, \zeta)$ are polynomials in $\sqrt{\lambda}$ and the maximal degree of these polynomials is 11. Therefore, $\mathcal{F}(\lambda, \zeta) \to 0$ as $|\lambda| \to \infty$ and the proposition 1.17 of Pego and Weinstein [20] may be used in our case.
5. Computation of the leading coefficient of the Taylor series of $D(\lambda)$

Let us denote $Y_k$ the first, and $Z_k$ the last component of the vectors $y_k$ and $z_k$, obeying at the same time (1.4), (2.4), respectively. From (3.4) and (3.2) one has

$$Y_1 = e^{-\sqrt{1-c^2} \zeta} \left( 1 + \frac{c \zeta}{1-c^2} \lambda + \left[ \frac{c^2 \zeta^2}{2(1-c^2)^2} + \frac{1 + 2c^2}{2(1-c^2)^{5/2}} \right] \lambda^2 \right) + O(\lambda^3),$$

$$Y_3(\lambda, \zeta) = 1 - \zeta \left( 1 + \frac{c \zeta^2}{2(1-c^2)^2} \lambda^2 + O(\lambda^3) \right); \quad (5.1)$$

as $\zeta \to \infty$, and

$$Z_1 = e^{\sqrt{1-c^2} \zeta} \left( 1 - \frac{c \zeta}{1-c^2} \lambda + \left[ \frac{c^2 \zeta^2}{2(1-c^2)^2} - \frac{1 + 2c^2}{2(1-c^2)^{5/2}} \right] \lambda^2 \right) + O(\lambda^3),$$

$$Z_3(\lambda, \zeta) = 1 + \zeta \left( 1 - \frac{c \zeta^2}{2(1-c^2)^2} \lambda^2 + O(\lambda^3) \right), \quad (5.2)$$

as $\zeta \to -\infty$.

According to Theorem 3.2 we look for solutions of the equations (1.4), (2.4) in the neighbourhood of $\lambda = 0$ in the form of expansions

$$Y_k(\lambda) = Y_{k0} + \lambda Y_{k1} + \frac{1}{2} \lambda^2 Y_{k2} + O(\lambda^3),$$

$$Z_k(\lambda) = Z_{k0} + \lambda Z_{k1} + \frac{1}{2} \lambda^2 Z_{k2} + O(\lambda^3).$$

The coefficients of the above expansions obey the equations

$$L Y_{k0} = 0, \quad (5.3)$$

$$L^* Z_{k0} = 0, \quad (5.4)$$

$$L Y_{k1} = 2c Y'_{k0}, \quad (5.5)$$

$$L^* Z_{k1} = -2c Z'_{k0}, \quad (5.6)$$

$$L Y_{k2} = -2 Y_{k0} + 4c Y'_{k1}, \quad (5.7)$$

$$L^* Z_{k2} = -2 Z_{k0} - 4c Z'_{k1}, \quad (5.8)$$

where the differential operators $L$ and $L^*$ are defined as follows:

$$L Y = Y''' - (1-c^2) Y'' + (p^0 Y)'', \quad L^* Z = Z''' - (1-c^2) Z'' + p^0 Z''.$$

The equations (5.3-5.8) have to be solved taking into account the conditions (5.1), (5.2).
For $\lambda = 0$ the complete bases of solutions of the equations (1.4) and (2.4), which coincide with the equations (5.3) and (5.4), are determined explicitly. Putting $\lambda = 0$ in (1.4) and integrating it twice we get

$$L_0 w = \frac{d^2 w}{d\zeta^2} - (1 - c^2)w + p^0 w = a + b\zeta. \tag{5.9}$$

The solving of the equation (5.3) is equivalent to solving of the second order equation (5.9) with arbitrary constants $a$ and $b$. When $a = b = 0$ the particular solutions are

$$w_1 = \tau_2^0, \quad w_2 = w_1 \int \tau_2^0 d\zeta.$$

For some $a \neq 0$, $b = 0$ the solution of (5.9) is given by

$$w_3 = -w_1 \int w_2 d\zeta + w_2 \int w_1 d\zeta.$$

For $a = 0$ and some $b \neq 0$ the solution of (5.9) is

$$w_4 = -w_1 \int \zeta w_2 d\zeta + w_2 \int \zeta w_1 d\zeta.$$

The solutions $w_1, w_2, w_3, w_4$ constitute the basis in the space of solutions of the equation (5.3). By the appropriate normalization of $w_1$ and $w_3$ one gets $Y_{10}$ and $Y_{30}$ correspondingly, such that according to (5.1) $Y_{10} \to e^{-s\zeta}$ and $Y_{30} \to 1$ as $\zeta \to \infty$:

$$Y_{10} = \frac{e^{s\zeta} - e^{-s\zeta}}{(e^{s\zeta} + e^{-s\zeta})^2}, \quad Y_{30} = \frac{1 - 6e^{2s\zeta} + e^{4s\zeta}}{(1 + e^{2s\zeta})^2},$$

where $s = \sqrt{1 - c^2}$ hereafter.

Normalizing $w_2$ one gets $Y_{20}$, having asymptotic $Y_{20} \to e^{s\zeta}$ as $\zeta \to \infty$. Linear combination of $w_4$, $Y_{03}$ and $Y_{20}$ gives $Y_{40}$ asymptotic to $\zeta$ as $\zeta \to \infty$.

Putting $\lambda = 0$ in (2.4) we get the equation (5.4). Following [2] we denote $w^{*''} = \phi$. Then $\phi = w_1$ and $\phi = w_2$ are two solutions of (5.9) with $a = b = 0$. Therefore,

$$w_1^* = \int \left( \int \tau_2^0 d\zeta \right) d\zeta, \quad w_2^* = \int \left( \int w_2 d\zeta \right) d\zeta$$

satisfy the equation (5.4). Normalizing $w_1^*$ to get the asymptotic (5.2) we obtain

$$Z_{10} = \arctan e^{s\zeta},$$

$Z_{10} \to e^{s\zeta}$ as $\zeta \to \infty$. The solution $Z_{20}$ asymptotic to $e^{-s\zeta}$ as $\zeta \to \infty$ is got by normalizing $w_2^*$. The solutions $Z_{30} = 1$ and $Z_{40} = \zeta$ complete the basis in the space of solutions of the equation (5.4).

The equations (5.5), (5.7) have the form $LY = W$. Integrating the last equation twice one gets

$$L_0 Y = \hat{W}, \quad \hat{W} = \int (\int W d\zeta) d\zeta. \tag{5.10}$$
The particular solution of (5.10) is given by
\[
Y = -Y_{10} \int \hat{W} Y_{20} \, d\zeta + Y_{20} \int \hat{W} Y_{10} \, d\zeta. \tag{5.11}
\]

The symbolic language \textit{Mathematica} 4.0 was employed for construction of the solutions of the form (5.11) in the explicit form. To get the asymptotics (5.1) we modify the solutions of (5.5), (5.7) in the form (5.11) by adding a linear combination of the solutions \( Y_{c0} \) of the homogeneous equation (5.3). For example,
\[
Y_{11} = \frac{ce^{s\zeta}(e^{2s\zeta} - s\zeta - 2)}{s^3(1 + e^{2s\zeta})^2}, \quad Y_{11} \rightarrow \frac{c}{1 - e^s} s^3 e^{-s\zeta}, \quad \zeta \rightarrow \infty.
\]
To get \( Y_{11} \) we add \( c Y_{10} \) to the solution of (5.5) in the form (5.11), where the constant \( c \) is determined by (5.1).

The equations (5.6), (5.8) have the form \( L^* Z = \hat{W} \). Putting \( \phi = \zeta^* \) we get the equation of the form (5.10) for the function \( \phi \). Using the formula (5.11) to obtain the particular solution, we come back to \( Z \) integrating \( \phi \) twice. Again, to get the asymptotics (5.1) we modify the solutions of (5.6), (5.8) by adding a linear combination of the solutions \( Z_{c0} \) of the homogeneous equation. For example,
\[
Z_{11} = -\frac{c \arctan(e^{s\zeta})}{s^3} - \frac{c \zeta \arctan(e^{s\zeta})}{s^2} + \frac{ic \text{PolyLog}[2, -ie^{s\zeta}]}{2s^3}, \quad Z_{11} \rightarrow \frac{c}{1 - e^s} s^3 e^{-s\zeta} \zeta, \quad \zeta \rightarrow -\infty
\]
\[
Z_{31} = \frac{\zeta}{1 - c}, \quad Z_{32} = \frac{s^2(1 + c)\zeta^2 - 4(1 + c) \ln(1 + e^{2s\zeta})}{s^4(1 - c)},
\]
\[
Z_{11} \rightarrow -\frac{c}{1 - e^s} s^3 e^{s\zeta}, \quad Z_{32} \rightarrow \frac{\zeta^2}{(1 - c)^2}, \quad \zeta \rightarrow \infty
\]
where \( \text{PolyLog}[2, y] = \int_0^1 t^{-1} \ln(1 - t) \, dt, \) so that
\[
i \text{PolyLog}[2, -ie^{s\zeta}] - i \text{PolyLog}[2, ie^{s\zeta}] \rightarrow 2 \pi e^{s\zeta}, \quad \zeta \rightarrow -\infty,
\]
\[
i \text{PolyLog}[2, -ie^{s\zeta}] - i \text{PolyLog}[2, ie^{s\zeta}] \rightarrow \pi s\zeta + 4e^{-s\zeta}, \quad \zeta \rightarrow \infty.
\]
To get \( Z_{11} \) and \( Z_{32} \) we modify the solution of (5.6), (5.8), obtained with the help of (5.11) and further integration, by adding \( \gamma_1 Z_{10} \) and \( \gamma_2 Z_{40} \), respectively, for the constants \( \gamma_{1,2} \) to be determined from (5.2).

The expressions for the other solutions of (5.3)-(5.8) are rather long and they are given in Appendix A.

The coefficients at the powers of \( \lambda \) for the expansion of the vector functions \( y_{1,3, \lambda, \zeta}, z_{1,3, \lambda, \zeta} \) are uniquely determined by the corresponding functions \( Y_{ik}, Z_{ik}, \) \( i = 1, 3, \) \( k = 0, 1, 2 \) (see the expressions at the bottom of (2.3) and (2.5)).

To get the coefficients of the expansion series in a zero neighbourhood for \( \hat{D}(\lambda) \) the formula (4.3) was used. We compute
\[
\hat{D}(\lambda) = \det \begin{pmatrix}
    a_{11} + b_{11} \lambda + d_{11} \lambda^2 + O(\lambda^3) & a_{12} + b_{12} \lambda + d_{12} \lambda^2 + O(\lambda^3) \\
    a_{21} + b_{21} \lambda + d_{21} \lambda^2 + O(\lambda^3) & a_{22} + b_{22} \lambda + d_{22} \lambda^2 + O(\lambda^3)
\end{pmatrix},
\]
where
\[
a_{mn} = 0, \quad m, n = 1, 2, \quad b_{11} = b_{21} = d_{21} = 0, \quad b_{12} = \frac{\pi s^2}{2(1 + c)}, \quad b_{22} = 2,
\]
\[
d_{11} = \frac{1 - 2e^s}{2s^3}, \quad d_{12} = -\frac{2 + 3c}{2(1 + c)s}, \quad d_{22} = \frac{4(1 + c)}{(1 - c)s}.
\]
Since $a_{mn} = 0$ the first nonzero coefficient of the expansion of $\hat{D}(\lambda)$ is that one at the third power of $\lambda$ and, moreover, this coefficient is completely determined by the coefficients at the first and second powers of $\lambda$ of the elements of the determinant. Finally, we have

$$\hat{D}(\lambda) = \frac{1 - 2c^2}{s^3} \lambda^3 + O(\lambda^4)$$

in some neighbourhood $U_0$ of the origin in the $\lambda$-plane.

According to Theorem 4.2 the normalized Evans function $D(\lambda)$, $D(\lambda) \to 1$ as $|\lambda| \to \infty$, is given by

$$D(\lambda) = \frac{\hat{D}(\lambda)}{I^\wedge(\lambda) \cdot r^\wedge(\lambda)}.$$ 

In a zero neighborhood one has

$$\hat{D}(\lambda) = -\frac{1}{I^\wedge \cdot r^\wedge} \int_{-\infty}^{\infty} z^\wedge(\lambda, \zeta) \cdot \hat{M}^\wedge(\lambda, \zeta) \cdot y^\wedge(\lambda, \zeta) \, d\zeta,$$ (5.12)

where dot denotes differentiation with respect to $\lambda$. The straightforward computation gives

$$\hat{D}(\lambda) = -2 \frac{1 - 2c^2}{4s^6} \lambda + O(\lambda^2),$$

that is in full agreement with (1.5). We list the expression for the coefficient at $\lambda^2$ of the integrand in (5.12) in Appendix A.

6. Conclusion and Discussion

In this paper we established that for $c^2 < 1/2$ the leading coefficient of the Taylor series of the normalized Evans function $D(\lambda)$ in a neighbourhood of zero is negative and thus, $D(\lambda)$ has to vanish somewhere on the positive real axis. It follows then from Theorem 4.1, that there exist the unstable eigenvalue and associated with it unstable eigenfunction given by (1.3). This in its turn, implies the exponential instability of the loop solitary wave in the prescribed velocity range.

In all cases known to us, when the equation under analysis has a hamiltonian structure and an invariant momentum associated with translational invariance of the problem, there is the link between the convexity of the momentum and the sign of the leading Taylor coefficient (Pego and Weinstein [20], Alexander and Sachs [2], Bridges and Derks [7]). In these cases the existence of the unstable eigenfunction is caused by the translational symmetry. In the case under consideration the
momentum associated with translational invariance $T(\omega)w(\zeta) = \exp(\omega \partial_\zeta)w(\zeta) = w(\zeta + \omega)$, $\omega \in \mathbb{R}$, is $Q = Q(w)$ (see Introduction). One has
\[ \frac{dQ(\phi_c)}{dc} = -\frac{8}{s^3} < 0, \]
i.e. the momentum is concave for all $c \in [0,1)$. Yet, the lead coefficient in the expansion of $D(\lambda)$ changes its sign at $c = 1/\sqrt{2}$ and, therefore, the above rule about the link is violated. The reason of this violation is that the other symmetry than the translational one is responsible for the existence of the unstable eigenfunction in our case, namely the rotational symmetry around the $x_1$ axis:
\[ G(\varphi)w = \exp(A\varphi)w, \quad \varphi \in S^1, \]
where $A$ is the block diagonal $6 \times 6$ matrix with the blocks
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{pmatrix}.
\]

The question about stability of the loop solitary wave for $c \in [1/\sqrt{2},1)$ is still open, however it looks plausible that the impulse of the moving loop for this range of speeds is large enough to stabilize it and the loop manifests at least marginal stability. This conjecture is based on the analogy with a different problem of stability of solitary impulses in a composite medium. The linearized problem about the solitary impulse in the composite is exactly the same as the linear problem (2.2) of the paper, and it is known from the rigorous nonlinear stability analysis (and also from numerical evidence) that solitary impulses in question are orbitally stable. It may mean that there is no unstable eigenfunction for (2.2) for $c > 1/\sqrt{2}$.

Let us explain this analogy in more detail. The system of governing equations, describing propagation of quasi-transverse elastic waves in some model of composite with nonlinear anisotropic elastic matrix has the form (see e.g. Il’ichev [14])
\[ \frac{\partial u_i}{\partial t} - \frac{\partial v_i}{\partial x} = 0, \quad \rho_0 \frac{\partial v_i}{\partial t} - \frac{\partial}{\partial x} \left( \frac{\partial \Phi}{\partial u_i} \right) + m \frac{\partial^3 u_i}{\partial x^3} = 0, \quad i = 1, 2. \] (6.1)
Here $\rho_0$ is the average density, $u_i, v_i$ - gradients of displacements and particle velocities in the wave front, respectively, $\Phi$ is the elastic potential, given by the expression
\[ \Phi = \frac{1}{2} f(u_1^2 + u_2^2) + \frac{1}{2} g(u_2^2 - u_1^2) - \frac{1}{4} \kappa (u_1^2 + u_2^2)^2. \]
The constants $f > 0, g > 0, \kappa$ and $m > 0$ characterize elastic modules, anisotropy, nonlinearity and dispersion, respectively. There is one-parametric anisotropic family of solitary wave solutions of (6.1) for $\kappa > 0$
\[ u_1^0 = \pm \sqrt{2\rho_0 \kappa^{-1}(\mu_1 - V^2)} \text{sech} \sqrt{\rho_0 m^{-1}(\mu_1 - V^2)}(x - V t), \quad u_2^0 = 0, \] (6.2)
where $\mu_1 = (f - g)/\rho_0$ and $V$ is the dimensional speed [14]. The solitary wave orbit is stable for $\mu_1/2 < V^2 < \mu_1$ ([14], Theorems 4, 5). The results of numerical calculations reported in Bakholdin et al [5] confirm this theoretical result, and also give the evidence that solitary wave family (6.2) is unstable for $0 < V^2 < \mu_1/2$. 


The equations (6.1), being linearized about the solitary wave (6.2) for small perturbations $\delta u_1 = u_1 - u_1^0$, are transformed into one linear equation

$$\frac{\partial^2 \delta u_1}{\partial x^2} = \mu_1 \frac{\partial^2 \delta u_1}{\partial x^2} - \frac{3}{\rho_0} (u_1^0)^2 \frac{\partial^2 \delta u_1}{\partial x^2} - \frac{m}{\rho_0} \frac{\partial^4 \delta u_1}{\partial x^4}. \quad (6.3)$$

Next, define the dimensionless speed $c \in [0, 1)$ and variables $\xi, t, \tau$ by

$$c = \frac{V}{\sqrt{p_1}}, \quad x = \sqrt{\frac{m}{\mu_1 \rho_0}} \xi, \quad t^* = \sqrt{\frac{m}{\rho_0 \mu_1^2}} t, \quad \delta u_1 = \sqrt{\frac{\mu_1 \rho_0}{\kappa}}. $$

Now, in the dimensionless variables, we have that solitary pulses in the composite are nonlinearly orbitally stable for $c^2 > 1/2$ (Theorems 4, 5 of [14]). In these variables the linearized problem (6.3) has exactly the same form as the basic linear problem for the loop stability (2.2) with $p^0$ given by (1.2). This result strongly prompts that the linearized problem (2.2) can have no unstable eigenfunctions for $c^2 > 1/2$, i.e. that our loop solitary wave is at least marginally stable inside this range of speeds.

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**Appendix A. Mathematica expressions for $Y_{31}, Y_{12}, Y_{32}, Z_{12}$ and for the integrand in (5.12)**

In the expressions below the notation $x$ instead of $\zeta$ is used.

\[
Y_{31} = - (e^{-s x} (32 c e^s x + 32 c^2 e^s x - 96 c e^3 s x - 96 c^2 e^3 s x + 2 e^s x s^2 - 
2 e^5 s x s^2 + 2 e^6 s x s^2 - e^2 s x \pi s^2 - 17 e^4 s x \pi s^2 + e^6 s x \pi s^2 + 8 c e^s x s x + 
8 c^2 e^s x s x - 48 c e^3 s x s x - 48 c^2 e^3 s x s x + 8 c e^5 s x s x + 
8 c^2 e^5 s x s x + 4 e^s x s^3 x - 24 e^3 s x s^3 x + 4 e^5 s x s^3 x - 
12 e^2 s x \pi s^3 x + 12 e^4 s x \pi s^3 x - 2 s^2 \arctan[e^s x] + 
18 e^2 s x s^2 \arctan[e^s x] + 18 e^4 s x s^2 \arctan[e^s x] - 
2 e^6 s x s^2 \arctan[e^s x] + 12 i e^s x s^2 \text{PolyLog}[2, -i e^s x] - 
12 i e^4 s x s^2 \text{PolyLog}[2, -i e^s x] - 12 i e^2 s x s^2 \text{PolyLog}[2, i e^s x] + 
12 i e^4 s x s^2 \text{PolyLog}[2, i e^s x]/(4 (1 + e^2 s x)^2 s^3);)
\]

\[
Y_{12} = \frac{1}{2 (1 + e^2 s x)^2 s^6} (e^{-s x} (c^2 - 17 c e^2 s x - s^2 + 13 e^2 s x s^2 - 
4 e^4 s x s^2 + 2 e^6 s x s^2 - 12 e^3 s x \pi s^2 + 2 e^5 s x \pi s^2 - 
14 c^2 e^2 s x s x + 6 c^2 e^4 s x s x - 2 e^2 s x s^3 x + 2 e^4 s x s^3 x - 
2 c^2 e^2 s x s^2 s^2 + 2 e^4 s x s^2 s^2 - 4 e^8 s x s^2 \arctan[e^s x] + 
24 e^3 s x s^2 \arctan[e^s x] - 4 e^5 s x s^2 \arctan[e^s x] - 
8 e^2 s x s^2 \log[1 + e^{-2 s x}] + 8 e^4 s x s^2 \log[1 + e^{-2 s x}])};
\]
\[ Y_{32} = -\frac{1}{2 (1 + c) (1 + e^{2 s x})^2 s^6} (e^{-s x} \left(-128 c^2 e^{s x} - 128 c^3 e^{s x} + ight. \]
\[ 256 c^2 e^{3 s x} + 256 c^3 e^{3 s x} - 2 e^{s x} s^2 + 6 c e^{s x} s^2 - 52 e^{3 s x} s^2 - \]
\[ 124 c e^{3 s x} s^2 + 6 e^{5 s x} s^2 + 14 c e^{5 s x} s^2 - 2 \pi s^2 + c \pi s^2 + \]
\[ 2 e^{s x} \pi s^2 - 10 c e^{s x} \pi s^2 + 34 e^{s x} \pi s^2 + 62 c e^{s x} \pi s^2 - \]
\[ 2 e^{s x} \pi s^2 - 5 c e^{6 s x} \pi s^2 - 64 c e^{6 s x} s x - 64 c e^{s x} s x + \]
\[ 192 c^2 e^{3 s x} s x + 192 c^3 e^{3 s x} s x - 18 e^{s x} s^3 x - 50 c e^{s x} s^3 x + \]
\[ 96 e^{3 s x} s^3 x + 192 c e^{3 s x} s^3 x - 14 e^{5 s x} s^3 x - 14 c e^{5 s x} s^3 x + \]
\[ c \pi s^3 x + 24 e^{2 s x} \pi s^3 x + 11 c e^{2 s x} \pi s^3 x - 24 c e^{s x} \pi s^3 x - \]
\[ 53 c e^{4 s x} \pi s^3 x + c e^{6 s x} \pi s^3 x - 8 c e^{s x} s^2 x^2 - 8 c e^{3 s x} s^2 x^2 + \]
\[ 48 c^2 e^{3 s x} s^2 x^2 + 48 c^3 e^{3 s x} s^2 x^2 - 8 c^2 e^{5 s x} s^2 x^2 - 8 c^3 e^{5 s x} s^2 x^2 - \]
\[ 2 e^{s x} s^4 x^2 - 6 e^{s x} s^4 x^2 + 12 e^{3 s x} s^4 x^2 + 36 c e^{s x} s^4 x^2 - \]
\[ 2 e^{5 s x} s^4 x^2 - 6 e^{5 s x} s^4 x^2 - 12 c e^{s x} \pi s^2 x^2 + 12 c e^{s x} \pi s^2 x^2 + \]
\[ 4 s^2 \arctan[e^{s x}] - 2 c e^{s x} \arctan[e^{s x}] - 100 e^{s x} s^2 \arctan[e^{s x}] + \]
\[ 22 c e^{s x} s^2 \arctan[e^{s x}] + 28 c e^{s x} s^2 \arctan[e^{s x}] - \]
\[ 94 c e^{4 s x} s^2 \arctan[e^{s x}] + 4 e^{6 s x} s^2 \arctan[e^{s x}] + \]
\[ 10 c e^{6 s x} s^2 \arctan[e^{s x}] + 2 s^3 x \arctan[e^{s x}] + \]
\[ 2 c s^3 x \arctan[e^{s x}] - 18 e^{2 s x} s^3 x \arctan[e^{s x}] - \]
\[ 18 c e^{s x} s^3 x \arctan[e^{s x}] - 18 e^{4 s x} s^3 x \arctan[e^{s x}] - \]
\[ 18 c e^{4 s x} s^3 x \arctan[e^{s x}] + 2 e^{6 s x} s^3 x \arctan[e^{s x}] + \]
\[ 2 c e^{s x} \arctan[e^{s x}] - 16 c e^{s x} s^2 \log[1 + e^{-2 s x}] + \]
\[ 96 c e^{3 s x} s^2 \log[1 + e^{-2 s x}] - 16 c e^{5 s x} s^2 \log[1 + e^{-2 s x}] + \]
\[ 8 e^{s x} s^2 \log[1 + e^{-2 s x}] + 8 c e^{s x} s^2 \log[1 + e^{-2 s x}] - \]
\[ 48 e^{3 s x} s^2 \log[1 + e^{-2 s x}] - 48 c e^{3 s x} s^2 \log[1 + e^{-2 s x}] + \]
\[ 8 e^{5 s x} s^2 \log[1 + e^{-2 s x}] + 8 c e^{5 s x} s^2 \log[1 + e^{-2 s x}] - \]
\[ i s^2 \text{PolyLog}[2, -i e^{s x}] - 2 i c s^2 \text{PolyLog}[2, -i e^{s x}] - 15 i e^{s x} s^2 + \]
\[ \text{PolyLog}[2, -i e^{s x}] + 6 i c e^{2 s x} s^2 \text{PolyLog}[2, -i e^{s x}] + \]
\[ 33 i e^{4 s x} s^3 \text{PolyLog}[2, -i e^{s x}] + 54 i c e^{4 s x} s^3 \text{PolyLog}[2, -i e^{s x}] + \]
\[ i e^{6 s x} s^2 \text{PolyLog}[2, -i e^{s x}] - 2 i c e^{6 s x} s^2 \text{PolyLog}[2, -i e^{s x}] - \]
\[ 12 i e^{2 s x} s^3 x \text{PolyLog}[2, -i e^{s x}] - 12 i c e^{2 s x} s^3 x \text{PolyLog}[2, -i e^{s x}] + \]
\[ 12 i e^{4 s x} s^3 x \text{PolyLog}[2, -i e^{s x}] + i s^2 \text{PolyLog}[2, -i e^{s x}] + \]
\[ 2 i c s^2 \text{PolyLog}[2, i e^{s x}] + 15 i e^{2 s x} s^2 \text{PolyLog}[2, i e^{s x}] - \]
\[ 6 i c e^{2 s x} s^2 \text{PolyLog}[2, i e^{s x}] - 33 i e^{4 s x} s^2 \text{PolyLog}[2, i e^{s x}] - \]
\[ 54 i c e^{4 s x} s^2 \text{PolyLog}[2, i e^{s x}] + i e^{6 s x} s^2 \text{PolyLog}[2, i e^{s x}] + \]
\[ 2 i c e^{6 s x} s^2 \text{PolyLog}[2, i e^{s x}] + 12 i e^{2 s x} s^3 x \text{PolyLog}[2, i e^{s x}] + \]
\[ 12 i c e^{4 s x} s^3 x \text{PolyLog}[2, i e^{s x}] - \]
\[ 12 i e^{4 s x} s^3 x \text{PolyLog}[2, i e^{s x}] - 12 i c e^{4 s x} s^3 x \text{PolyLog}[2, i e^{s x}] + \]
\[ x \text{PolyLog}[2, i e^{s x}] + 24 i e^{2 s x} s^2 \text{PolyLog}[3, -i e^{s x}] + \]
\[ 48 i e^{2 s x} s^2 \text{PolyLog}[3, -i e^{s x}] - 24 i e^{4 s x} s^2 \text{PolyLog}[3, -i e^{s x}] - \]
\[ 24 i e^{2 s x} s^2 \text{PolyLog}[3, i e^{s x}] - \]
\[ 48 i c e^{2 s x} s^2 \text{PolyLog}[3, i e^{s x}] + 24 i e^{4 s x} s^2 \text{PolyLog}[3, i e^{s x}] + \]
\[ 48 i c e^{4 s x} s^2 \text{PolyLog}[3, i e^{s x})];\]
\[ Z_{12} = -\frac{1}{2} s^6 \left( -c^2 e^s x + e^{s x} s^2 + 7 c^2 \arctan[e^s x] + 13 s^2 \arctan[e^s x] + 2 c^2 s x \arctan[e^s x] + 2 s^3 x \arctan[e^s x] - 2 c^2 s^2 x^2 \arctan[e^s x] - 16 s^2 \arctan[e^s x] \log[2] - i c^2 \text{PolyLog}[2, -i e^s x] - 5 i s^2 \text{PolyLog}[2, -i e^s x] + 2 i c^2 s x \text{PolyLog}[2, -i e^s x] + i c^2 \text{PolyLog}[2, i e^s x] + 5 i s^2 \text{PolyLog}[2, i e^s x] - 2 i c^2 s x \text{PolyLog}[2, i e^s x] + 8 i s^2 \text{PolyLog}\left[2, \frac{1}{2} - \frac{1}{2} i e^s x\right] - 8 i s^2 \text{PolyLog}\left[2, \frac{1}{2} + \frac{1}{2} i e^s x\right] \right) - 2 i c^2 \text{PolyLog}[3, -i e^s x] - 2 i s^2 \text{PolyLog}[3, -i e^s x] + 2 i c^2 \text{PolyLog}[3, i e^s x] + 2 i s^2 \text{PolyLog}[3, i e^s x] \right); 

\text{Integrand} = (e^s x \times ( -16 c^3 e^s x + 16 c^5 e^s x + 16 c^3 e^s x - 16 c^5 e^s x + 16 c e^s x s^2 - 16 c^3 e^s x s^2 - 16 c^3 e^s x s^2 + 16 c^3 e^s x s^2 + 2 c \pi s^2 - 2 c^2 \pi s^2 - 12 c e^s x \pi s^2 + 12 c^3 e^s x \pi s^2 + 2 c e^s x \pi s^2 + 2 c e^s x \pi s^2 - 4 c^3 e^s x \pi s^2 + \pi s^4 - c \pi s^4 - e^4 s x \pi s^4 + c e^4 s x \pi s^4 - c^2 \pi s^3 x + c^3 \pi s^3 x + c^2 e^s x \pi s^3 x - c \pi s^5 x + 6 c e^s x \pi s^5 x - c e^s x \pi s^5 x - 8 c s^2 \arctan[e^s x] - 8 c e^s x \arctan[e^s x] + 12 c^2 e^s x s^2 \arctan[e^s x] + 8 c e^s x s^2 \arctan[e^s x] = 8 c e^s x s^2 \arctan[e^s x] - 24 c^2 e^s x s^2 \arctan[e^s x] - 8 c^3 e^s x s^2 \arctan[e^s x] - 4 s^2 \arctan[e^s x] + 4 c^4 s \pi s^4 \arctan[e^s x] + 2 i c^2 s^2 \text{PolyLog}[2, -i e^s x] - 12 i c^2 e^s x s^2 \text{PolyLog}[2, -i e^s x] + 2 i c^2 e^s x s^2 \text{PolyLog}[2, -i e^s x] - 2 i c^2 s^2 \text{PolyLog}[2, i e^s x] + 12 i c^2 e^s x s^2 \text{PolyLog}[2, i e^s x] - 2 i c^2 e^s x s^2 \text{PolyLog}[2, i e^s x]) / ((-1 + c)(1 + c)(1 + c e^s x)^3 s^2); 

The rest of solutions of (5.3)-(5.8) with their asymptotics are given in sec. 5. The listed functions have the asymptotics

\[ Y_{12} \rightarrow \left( \frac{c^2 x^2}{(1 - c^2)^2} + \frac{1 + 2c^2}{(1 - c^2)^{5/2}} x \right) e^{-sx}, \]
\[ Y_{31} \rightarrow -\frac{x}{1 - c}, \quad Y_{32} \rightarrow \frac{x^2}{(1 - c)^2}, \text{ as } x \rightarrow \infty, \]

and

\[ Z_{12} \rightarrow \left( \frac{c^2 x^2}{(1 - c^2)^2} - \frac{1 + 2c^2}{(1 - c^2)^{5/2}} x \right) e^{sx}, \text{ as } x \rightarrow -\infty. \]
The function $P(x) = \text{PolyLog}[3, i e^x]$ above is defined via
\[
\frac{dP(x)}{dx} = \text{PolyLog}[2, i e^x],
\]
and has the asymptotic behaviour
\[
\text{Im } P(x) \to -e^x, \text{ as } x \to -\infty; \quad \text{Im } P(x) \to -\frac{\pi}{4}x^2, \text{ as } x \to \infty.
\]

References

[1] A. L. Afendikov and T. J. Bridges, Instability of the Hocking–Stewartson pulse and its implications for three-dimensional Poiseuille flow, Proc. R. Soc. Lond. A 457 (2001), 257-272.

[2] J. C. Alexander and R. Sachs, Linear instability of solitary waves of a Boussinesq-type equation: A computer assisted computation, Nonlin. World 2 (1995), 471-507.

[3] J. C. Alexander, R. Gardner and C. Jones, A topological invariant arising in the stability analysis of travelling waves, J. Reine Angew. Math. 410 (1990), 167-212.

[4] J. C. Alexander, M. G. Grillakis, C. K. R. T. Jones and B. Sandstede, Stability of pulses on optical fibers with phase-sensitive amplifiers, Z. angew. Math. Phys. 48 (1997), 175-192.

[5] I. Bakholdin, A. Il’ichev and V. Tomashpol’skii, Stability, instability and interaction of solitary pulses in a composite medium, Eur. J. Mech. A/Solids 21 (2002), 333-346.

[6] A. Beliaev and A. Il’ichev, Conditional stability of solitary waves propagating in elastic rods, Physica D 90 (1996), 107-118.

[7] T. J. Bridges and G. Derks, The symplectic Evans matrix, and the instability of solitary waves and fronts with symmetry, Arch. Rat. Mech. Anal. 156 (2001), 1-87.

[8] T. J. Bridges, G. Derks and G. Gottwald, Stability and instability of solitary waves of the fifth-order KdV equation: a numerical framework, Physica D, 172 (2002), 190-216.

[9] D. J. Dichmann, J. H. Maddocks and R. L. Pego, Hamiltonian dynamics of an elastica and the stability of solitary waves, Arch. Rat. Mech. Anal. 135 (1996), 347-396.

[10] J. W. Evans, Nerve axon equations, III: Stability of the nerve impulse, Indiana Univ. Math. J. 22 (1972), 577-594.

[11] R. A. Gardner and K. Zumbrun, The gap lemma and geometric criteria for instability of viscous shock profiles, Comm. Pure Appl. Math. 41 (1988), 797-855.

[12] M. Grillakis, J. Shatah and W. Strauss, Stability theory of solitary waves in the presence of symmetry I, J. Funct. Anal. 74 (1987), 160-197.

[13] M. Grillakis, J. Shatah and W. Strauss, Stability theory of solitary waves in the presence of symmetry II, J. Funct. Anal. 94 (1990), 308-348.

[14] A. Il’ichev, Stability of solitary waves in nonlinear composite media, Physica D 150 (2001), 261-277.

[15] C. K. R. T. Jones, 1984 Stability of the travelling wave solution of the FitzHugh-Nagumo system, Trans. Amer. Math. Soc. 286 (1984), 431-469.

[16] T. Kapitula, The Evans function and generalized Melnikov integrals, SIAM J. Math. Anal. 30 (1998), 273-297.

[17] T. Kapitula, Stability criterion for bright solitary waves of the perturbed cubic-quintic Schrödinger equation, Physica D 116 (1998), 95-120.

[18] T. Kapitula and B. Sandstede, Stability for bright solitary wave solutions to perturbed nonlinear Schrödinger equations, Physica D 124 (1998), 58-103.

[19] A. E. H. Love, A treatise on the mathematical theory of elasticity Dover 1944.
[20] R. L. Pego and M. I. Weinstein, Eigenvalues, and instabilities of solitary waves, *Phil Trans. R. Soc. Lond. A* **340** (1992), 47-94.

[21] R. L. Pego, P. Smereka and M. I. Weinstein, Oscillatory instability of travelling waves for KdV–Burgers equation, *Physica D* **67** (1993), 45-65.

[22] J. Swinton and J. Eglin, Stability of travelling pulse to a laser equation, *Phys. Lett. A* **145** (1990), 428-433.