EVERY 2-SEGAL SPACE IS UNITAL

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Introduction

2-Segal spaces were introduced by Dyckerhoff and Kapranov [1] for applications in representation theory, homological algebra, and geometry, motivated in particular by Waldhausen’s $S$-construction and Hall algebras. A 2-Segal space is a simplicial space $X$ such that for every triangulation $T$ of every convex plane $n$-gon (for $n \geq 2$), we have $X_n \simeq \lim_{t \in T} X(t)$. Independently, a little later, Gálvez-Carrillo, Kock, and Tonks [2] introduced the notion of decomposition space for applications in combinatorics, in connection with Möbius inversion. A decomposition space is a simplicial space $X: \Delta^{op} \to S$ for which all pushouts of active maps along inert maps in $\Delta$ are sent to pullbacks in $S$. Here, the inert maps in $\Delta$ are generated by the outer coface maps, while the active maps are generated by the codegeneracy and inner coface maps. The condition satisfied by $X$ with respect to pushouts of outer coface maps against inner ones is precisely equivalent to the 2-Segal condition. For Dyckerhoff and Kapranov, the condition for pushouts of outer cofaces against codegeneracies is a further axiom which they call unitality [1, Definition 2.5.2]. Thus, decomposition spaces are the same thing as unital 2-Segal spaces. While the 2-Segal axiom is expressly the condition required in order to induce a (co)associative (co)multiplication on the linear span of $X_1$, the unitality condition ensures that this (co)multiplication is (co)unital, which is an important property in many applications.

The present note shows that the unitality condition is actually automatic, by proving:

**Theorem.** Every 2-Segal space is unital.

This result is unexpected. Firstly, it cannot be derived by the standard tricks with pullback squares; secondly, it is not so common in mathematics for (co)associativity to imply (co)unitality.

1 Definitions and theorem

In order to cover all flavours of 2-Segal space that appear in the literature, we give two versions of the result: one for 2-Segal objects in an $\infty$-category with finite limits and one for 2-Segal objects in a Quillen model category. In the remainder of this section, $\mathcal{C}$ will denote either an $\infty$-category with finite limits or a Quillen model category; in the latter case, “pullback” will mean homotopy pullback.

**Definition.** (cf. [1], [2]) A simplicial object $X: \Delta^{op} \to \mathcal{C}$ is called 2-Segal when the commuting squares that express the simplicial identities between inner and outer face maps of $X$ are pullback squares. More precisely, for all $0 < i < n$ we have pullbacks:

$$
\begin{array}{c}
& X_{n+1} \xrightarrow{d_{i+1}} X_n \\
\phi \downarrow & \downarrow d_0 & \downarrow d_0 & \downarrow d_0 \\
X_n \xrightarrow{d_i} X_{n-1} & & & \\
& X_{n+1} \xrightarrow{d_i} X_n \\
\phi \downarrow & \downarrow d_{n+1} & \downarrow d_n & \downarrow d_n \\
X_n \xrightarrow{d_i} X_{n-1} & & & \\
\end{array}
$$

(1)
We say that $X$ is **upper 2-Segal** when only squares as to the left are required to be pullbacks, and **lower 2-Segal** when this is only required for squares as to the right.\(^1\)

**Definition.** A 2-Segal space $X$ is called **unital** if the following two squares are pullbacks:

\[
\begin{array}{ccc}
X_1 & \xrightarrow{s_1} & X_2 \\
\downarrow d_0 & & \downarrow d_0 \\
X_0 & \xrightarrow{s_0} & X_1 \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
X_1 & \xrightarrow{s_0} & X_2 \\
\downarrow d_1 & & \downarrow d_2 \\
X_0 & \xrightarrow{s_0} & X_1 \\
\end{array}
\]

We call an upper 2-Segal space **upper unital** when only the pullback on the left is required, and call a lower 2-Segal space **lower unital** when only the pullback on the right is required.

**Theorem.** Every 2-Segal space is unital. More precisely, every upper 2-Segal space is upper unital, and every lower 2-Segal space is lower unital.

By symmetry, it is enough to prove the theorem for upper 2-Segal spaces. We do this separately for the cases where $\mathcal{C}$ is an $\infty$-category (Proposition 2.1) and where $\mathcal{C}$ is a model category (Proposition 3.1).

### 2 $\infty$-categorical proof

Throughout this section, $\mathcal{C}$ denotes an $\infty$-category with finite limits in the sense of Lurie [3]. We write $X : \Delta^\op \to \mathcal{C}$ to denote an object in the $\infty$-category $\text{Fun}(\Delta^\op, \mathcal{C})$.

**Proposition 2.1.** If $X : \Delta^\op \to \mathcal{C}$ is upper 2-Segal, then it is also upper unital.

**Proof.** Let $\Delta^i$ denote the category of finite ordinals with top element and top-preserving monotone maps, so that $(\Delta^i)^{\op}$-diagrams are split augmented cosimplicial objects. Precomposing $X$ by a suitable functor $\Delta^i \times \Delta^i \to \Delta$

\[
(i, [n]) \mapsto [i+n],
\]

we obtain the following diagram in $\mathcal{C}$:

\[
\begin{array}{ccc}
X_1 & \xrightarrow{s_1} & X_2 \\
\downarrow d_0 & & \downarrow d_0 \\
X_0 & \xrightarrow{s_0} & X_1 \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
X_1 & \xrightarrow{s_0} & X_2 \\
\downarrow d_1 & & \downarrow d_2 \\
X_0 & \xrightarrow{s_0} & X_1 \\
\end{array}
\]

In each of the two rows, the solid arrows form a cosimplicial diagram in $\mathcal{C}$; the dashed arrow endows this with an augmentation; and the dotted arrows provide the augmented cosimplicial object with a *splitting*. Just as any split fork in an ordinary category is an equaliser, so any split augmented cosimplicial object in an $\infty$-category is a limit; for a proof see [3, Lemma 6.1.3.16]. In other words, for each row of (3) the dashed arrow exhibits the leftmost entry as the limit of the rest of the row.

The vertical maps $d_0$ in (3) constitute a natural transformation between augmented cosimplicial diagrams. Because $X$ is upper 2-Segal, each of the **solid** naturality squares so obtained is a pullback. For the left-pointing squares, this is immediate from the definition of upper 2-Segality. For the right-pointing squares, we note that every degeneracy map $s_i$ is a section of some inner face map, and apply upper 2-Segality along with the standard cancellation properties of pullbacks (cf. [2, Proposition 3.5]). This shows that the $d_0$'s constitute a **cartesian** natural transformation between the solid parts of (3). Applying the following lemma with $D = \Delta$ shows that we also have a cartesian natural transformation on the dashed parts. Therefore, the leftmost square is also a pullback as required. \(\square\)

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\(^1\)For our purposes, splitting into upper 2-Segal and lower 2-Segal is just for economy; in the theory of higher Segal spaces [4] ($k$-Segal spaces for $k > 2$), the distinction between upper and lower becomes an essential aspect.
Lemma 2.2. Let $D$ be an $\infty$-category with a terminal object $1$ and $\mathcal{C}$ an $\infty$-category with finite limits. Suppose we have a cartesian natural transformation $u$ as to the left in:

\[
\begin{array}{ccc}
D & \xrightarrow{Y} & \mathcal{C} \\
\downarrow{u} & & \downarrow{\pi} \\
X & \xrightarrow{\pi} & \mathcal{C}
\end{array}
\]

If $Y$ and $X$ are limit cones for $Y$ and $X$, as indicated to the right, then the induced natural transformation $\pi$ extending $u$ is again cartesian.

Proof. The limit of $Y : D \to \mathcal{C}$ can equally well be computed in $\mathcal{C}/Y_1$ (since the forgetful functor $\mathcal{C}/Y_1 \to \mathcal{C}$ preserves and detects connected limits), and similarly the limit of $X : D \to \mathcal{C}$ can be computed in $\mathcal{C}/X_1$. The two functors are compared by $u^* : \mathcal{C}/X_1 \to \mathcal{C}/Y_1$: since $u$ is cartesian, we have $Y \simeq u^* \circ X$.

But $u^*$ preserves limits, and therefore

\[
\lim Y \simeq \lim u^* \circ X \simeq u^* \lim X.
\]

This shows that the outer rectangle in

\[
\begin{array}{ccc}
\lim Y & \rightarrow & Yd \rightarrow Y1 \\
\downarrow & & \downarrow \\
\lim X & \rightarrow &Xd \rightarrow X1
\end{array}
\]

is a pullback. For any $d \in D$, the right-hand square is a pullback by assumption, and so we conclude that the left-hand square is a pullback, as desired.

3 Model-categorical proof

Throughout this section, $\mathcal{C}$ denotes an arbitrary Quillen model category, and $X : \Delta^{op} \to \mathcal{C}$ a (strict) simplicial diagram.

Proposition 3.1. If $X : \Delta^{op} \to \mathcal{C}$ is upper 2-Segal, then it is also upper unital.

Proof. Given an arbitrary $X : \Delta^{op} \to \mathcal{C}$, we may form its Reedy-fibrant replacement $X \xrightarrow{\sim} X'$. Since Reedy weak equivalences are levelwise, and homotopy pullbacks are stable under levelwise weak equivalence, a commuting square built from face and degeneracy maps of $X$ will be a pullback if and only if the corresponding square for $X'$ is a pullback. We can therefore assume without loss of generality that $X : \Delta^{op} \to \mathcal{C}$ is Reedy fibrant as well as upper 2-Segal.

With these assumptions, we must prove that the left square of (2) is homotopy cartesian. Since $X$ is Reedy fibrant, it is sufficient to show that the comparison map $(d_0, s_1)$ into the pullback

\[
\begin{array}{ccc}
X_1 & \xrightarrow{s_1} & X_2 \\
\downarrow{d_0} & & \downarrow{d_0} \\
X_0 & \xrightarrow{s_0} & X_1
\end{array}
\]

is a weak equivalence; we show that it is in fact a deformation retract. The map $d_1 \pi_2 : P \to X_1$ provides a retraction for $(d_0, s_1)$, and composing these two maps the other way around gives:

\[
(d_0, s_1)d_1 \pi_2 = (d_0 d_1 \pi_2, s_1 d_1 \pi_2) = (d_0 d_0 \pi_2, s_1 d_1 \pi_2) = (d_0 s_0 \pi_1, s_1 d_1 \pi_2) = (\pi_1, s_1 d_1 \pi_2) : P \to P.
\]

So it suffices to construct a left homotopy $\alpha : (\pi_1, \pi_2) \xrightarrow{\sim} (\pi_1, s_1 d_1 \pi_2) : P \to P$. To do so, let $\text{Cyl}(P)$ be a good cylinder object, i.e., a factorisation

\[
P \sqcup P \xrightarrow{i} \text{Cyl}(P) \xrightarrow{p} P
\]

3
of the canonical fold map, where the first part is a cofibration and the second a weak equivalence. Observing that \(d_2s_1\pi_2 = \pi_2 = d_2s_2\pi_2\) and that
\[
d_0s_1\pi_2 = s_0d_0\pi_2 = s_0s_0\pi_1 = s_1d_0\pi_2 = d_0s_2\pi_2,
\]
we therefore have a commuting diagram around the outside of
\[
P \sqcup P \xrightarrow{(s_1s_2, s_2\pi_2)} X_3 \xrightarrow{(d_0, d_2)} X_2d_1 \times_{d_0} X_2.
\]
The left side is a cofibration by construction. We claim that the right side is a trivial fibration. Since \(X\) is Reedy fibrant, this will be true if \((d_0, d_2)\) exhibits \(X_3\) as the homotopy pullback of \(d_1 : X_2 \to X_1\) along \(d_0 : X_2 \to X_1\); but this is so since \(X\) is upper 2-Segal. Thus, there is a diagonal filler \(k\) as indicated.

Note \(k\) defines a left homotopy \(s_1\pi_2 \sim s_2\pi_2 : P \to X_3\); since \(\pi_2 = d_1s_1\pi_2\), it follows that \(d_1k\) is a left homotopy \(\pi_2 \sim d_1s_2\pi_2 : P \to X_2\). On the other hand, \(\pi_1p\) is a homotopy \(\pi_1 \sim \pi_1 : P \to X_0\), and since
\[
s_0\pi_1p = d_0\pi_2p = d_0d_1s_1\pi_2p = d_0d_0s_1\pi_2p = d_0d_0k = d_0d_1k : \text{Cyl}(P) \to X_1
\]
we see that \(\alpha = (\pi_1p, d_1k) : \text{Cyl}(P) \to P\) is a well-defined map into the pullback \(P = X_0s_0 \times_{d_0} X_2\). This gives the desired left homotopy \(\alpha : (\pi_1, \pi_2) \sim (\pi_1, d_1s_2\pi_2) : P \to P\).

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