Jacobi polynomials and $SU(2, 2)$

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Abstract

A ladder structure of operators is presented for the Jacobi polynomials, $J_n^{(\alpha, \beta)}(x)$, with parameters $n$, $\alpha$ and $\beta$ integers, showing that they are related to the unitary irreducible representation of $SU(2, 2)$ with quadratic Casimir $C_{SU(2, 2)} = -3/2$. As they determine also a base of square-integrable functions, the universal enveloping algebra of $su(2, 2)$ is homomorphic to the space of linear operators acting on the $L^2$ functions defined on $(-1, +1) \times \mathbb{Z} \times \mathbb{Z}/2$.

Keywords: Special functions, Jacobi polynomials, Lie algebras, Square-integrable functions

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1 Introduction

Many attempts have been done to find a wide but not too inclusive class of functions that can be defined “special”, where special means something more that “useful”, the old definition of Turán [1] (see also Ref. [2]).

The actual main line of work for a possible unified theory of special functions is the Askey scheme that is based on the analytical theory of linear differential equations. It includes as special functions all those functions that are related with hypergeometric functions and their \( q \)-analogues [1, 3, 4]. In this approach particular interest is related to the employ of integral transforms, in such a way, that the general hypergeometric function \( \textstyle{{}_pF_q} \) can be derived from \( \textstyle{{}_0F_0} = e^x \) by means of successive applications of Laplace transform and its inverse. The use of general properties of this transform and, in particular, of the convolution theorem enables also to derive differential identities for the corresponding functions.

However, a possible alternative point of view of the theory of the basic classes of special functions was established by employing considerations that belong to a field of mathematics seemingly quite far from them: the theory of representations of Lie groups. This way was introduced by Wigner [5] and Talman [6] and later developed mainly by Miller [7] and Vilenkin and Klimyk [8–10]. In this line, previous papers by us, Refs. [11,12], shown a direct connection between some special functions and well defined Lie groups.

The starting point of our work has been the paradigmatic example of Hermite functions that are a basis on the Hilbert space of the square integrable functions defined on the configuration space \( \mathbb{R} \). Besides the configuration basis \( \{|x\rangle \}_{x \in \mathbb{R}} \), as well known in the algebraic discussion of the harmonic oscillator, a discrete basis \( \{|n\rangle \}_{n \in \mathbb{N}} \) related to the Weyl-Heisenberg group \( H(1) \) can be considered such that Hermite functions are the transition matrices from one basis to the other [11]. The relevant point is that this scheme can be generalized from the Hermite polynomials to all the orthogonal polynomials we have, up to now, considered: Legendre and Laguerre polynomials [11], Associated Legendre polynomials and Spherical Harmonics [12] and Jacobi polynomials as we show in this paper.

In the seminal work of 1948 Truesdell introduced indeed the idea that a sub-class of special functions, called by him “familiar special functions”, are defined by means of a set of formal properties [13]. In his spirit, we proposed in [11,12] a possible definition of a fundamental sub-class of special functions, we called “algebraic special functions” (ASF), that look to be strictly related to the hypergeometric functions and are constructed starting from algebraic properties. The experience of the cases studied in [11,12] leads us to consider the following set of assumptions:

1. A set of recurrence relations are defined on these ASF that can be associated to a set of ladder operators that span a Lie algebra.

2. These ASF support an irreducible representation of this algebra.
3. A Hilbert space can be constructed on these ASF where these ladder operators have the hermiticity properties appropriate for constructing a unitary irreducible representation (UIR) of the associated Lie group.

4. Second order differential equation that define the ASF can be reconstructed from all diagonal elements of the universal enveloping algebra (UEA) and, in particular, the second orden Casimirs of the subalgebras and of the whole algebra.

From these assumptions, we have that:

i) Applying the exponential map to ASF different sets of functions can be constructed. If the transformation is unitary another algebraically equivalent basis of the Hilbert space is obtained. When the transformations are not unitary, as in the case of coherent states, sets with different properties are found (like overcomplete sets).

ii) The ASF are also a basis of an appropriate set of $L^2$ functions (defined on real spaces) and of an appropriate Hilbert space functions (in complex spaces). This, combined with the above properties implies that the vector space of the operators operating on $L^2$ (or Hilbert) space functions is homomorphic to the UEA built on the algebra.

In [11] it has been shown that Hermite, Laguerre and Legendre polynomials are ASF such that the Hermite functions support a UIR of the Weyl-Heisenberg group $H(1)$ with Casimir $C = 0$, while Laguerre functions and Legendre polynomials are both bases for the UIR of $SU(1, 1)$ with $C = -1/4$. Since Hermite functions are a basis of square-integrable functions (or wave functions) defined on the real line, as well as Laguerre functions on the semi-line and Legendre polynomials on the finite interval [14], all operators of the universal enveloping algebra (UEA) are defined on the appropriate basis functions and the wave functions built on them. In other words, the spaces of linear operators acting on these $L^2$ or Hilbert spaces are homomorphic to the UEA (i.e., the algebra constructed on the monomials of the Lie algebra generators) of the corresponding Lie algebra. Of course, this implies that this is true also for the elements of the corresponding Lie group that are contained in the UEA.

All these properties have been shown to not be restricted to the above mentioned 1-rank algebras (and groups). Indeed in [12] Associated Legendre polynomials and Spherical Harmonics are shown to share the same properties. The underlying Lie group is in both these cases $SO(3, 2)$ that is of rank 2 like two, $l$ and $m$, are the label parameters of these functions.

Here we discuss a further confirmation of this scheme, now related to the Jacobi polynomials that also satisfy the required conditions to be considered ASF and share the same properties. Indeed they can be associated to well defined “algebraic Jacobi functions” that support a UIR of $SU(2, 2)$ i.e., a Lie group of rank 3 like three are the parameters, $n, \alpha$ and $\beta$, of the Jacobi polynomials $J^{(\alpha, \beta)}_n(x)$. 

The suggestion to consider the Jacobi polynomials as candidates to ASF is related to the well known relation between Jacobi polynomials with $\alpha = \beta$ and the Associated Legendre Polynomials [9]:

$$J_n^{(\alpha,\alpha)}(x) = (-2)^\alpha \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + 2\alpha + 1)} (1 - x^2)^{-\alpha/2} P_{n+\alpha}^\alpha(x)$$

(1.1)

that shows that the Jacobi polynomials with $\alpha = \beta$ are ASF related to the Lie group $SO(3, 2)$ [12]. Indeed, since the relation (1.1) can be read in terms of the notation introduced in [12] as

$$J_n^{\alpha,\alpha}(x) = (-1)^\alpha T_{n+\alpha}^{\alpha}(x),$$

(1.2)

where $T^m_l(x)$ are the Associated Legendre functions, the results obtained for the Associated Legendre Polynomials $P^m_l(x)$ are easily rewritten for the Jacobi polynomials with $\alpha = \beta$.

We present here the generalization of the ladder operators of the Jacobi polynomials for independent integer values of the labels $\alpha$ and $\beta$. Obviously as Jacobi polynomials depend from three parameters $n, \alpha$ and $\beta$ we have to look for an algebra of rank three. In the following it will be shown that this algebra exists and is $D_3$ in its real form $su(2,2)$.

Since the complete construction is complex it not will be presented here. However, because of the great interest of the Jacobi polynomials in many areas of mathematics and physics, a detailed description will be published elsewhere.

The paper is organized as follows. Section 2 is devoted to present the main properties of the algebraic Jacobi functions (AJF) relevant for our discussion. In section 3 we study the symmetries of the the AJF that keep invariant the parameter $l$ and change $m$ and/or $q$. We prove that these ladder operators determine a $su(2) \oplus su(2)$ algebra, that allows us to build up a family of UIR of the group $SU(2) \otimes SU(2)$ labelled by the parameters $(l, m)$ and $(l, q)$, i.e. $U^l \otimes U^l$, with $-l \leq m \leq l$, $-l \leq q \leq l$, $l - m \in \mathbb{N}$, $l - q \in \mathbb{N}$ and $2l \in \mathbb{N}$. In section 4 we construct, by means of four new sets of ladder operators that change the three parameters $l, m$ and $q$ in $\pm1/2$, generating each of them a $su(1,1)$ algebra to which infinitely many UIR’s of $SU(1,1)$ – supported by the AJF – are associated. The complete set of ladder operators span a $su(2,2)$ Lie algebra and the AJF generate a UIR of $SU(2,2)$ characterized by the eigenvalue of the quadratic Casimir $C_{SU(2,2)} = -3/2$. In section 6 the homomorphism between the space of the operators on the $L^2$ space and the UEA of $su(2,2)$ is discussed. Finally, in section 7, some conclusions and comments are included.
2 Algebraic Jacobi functions and their operatorial structure

The Jacobi polynomials are defined in terms of the hypergeometric functions \( \,\!_2F_1 \) and Pochhammer’s symbol \([15,16]\)

\[
J_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{n!} \,\!_2F_1 \left[ -n, 1 + \alpha + \beta + n; \alpha + 1; \frac{1-x}{2} \right]. \quad (2.1)
\]

As shown in the following, the set of parameters \( \{n, \alpha, \beta\} \) and the same Jacobi polynomials are not directly related to the algebraic structure. Thus, consistently with \([11,12]\), we define three new discrete variables and include an \( x \)-depending factor. Moreover, a peculiar property of the Jacobi polynomials shared with the Legendre polynomials \([11,12]\) is that the standard form of the matrix elements of the algebra is not connected to an orthonormal basis of the Hilbert space but to an orthogonal one where a residual weight is preserved (see eqs. (??) and (2.3)).

Hence, we first introduce in the space of parameters the change from the integers \( (n, \alpha, \beta) \) to \( (l,m,q) \) all together integers or half-integers

\[
l := n + \frac{\alpha + \beta}{2}, \quad m := \frac{\alpha + \beta}{2}, \quad q := \frac{\alpha - \beta}{2},
\]

or, equivalently,

\[
n = l - m, \quad \alpha = m + q, \quad \beta = m - q.
\]

Then, in a second and final step, we include a \( x \)-depending factor

\[
\sqrt{\frac{\Gamma(l + m + 1) \Gamma(l - m + 1)}{2^m \Gamma(l + q + 1) \Gamma(l - q + 1)}} \left( 1 - x \right)^{\frac{m+q}{2}} \left( 1 + x \right)^{\frac{m-q}{2}}.
\]

Thus, the fundamental objects of this paper, that we call “algebraic Jacobi functions” (AJF), have the final form

\[
J_l^{m,q}(x) := \sqrt{\frac{\Gamma(l + m + 1) \Gamma(l - m + 1)}{2^m \Gamma(l + q + 1) \Gamma(l - q + 1)}} \left( 1 - x \right)^{\frac{m+q}{2}} \left( 1 + x \right)^{\frac{m-q}{2}} J_{l-m}^{(m+q,m-q)}(x).
\]

These new objects reveal additional symmetries hidden inside the Jacobi polynomials (2.1) to be added to the well known \( J_n^{(\alpha,\beta)}(x) = (-1)^n J_n^{(\beta,\alpha)}(-x) \). For instance,

\[
\begin{align*}
J_l^{m,q}(x) &= J_l^{q,m}(x), \\
J_l^{m,q}(x) &= (-1)^{m+q} J_l^{m,-q}(x), \\
J_l^{m,q}(x) &= (-1)^{l-q} J_l^{m,q}(-x), \\
J_l^{m,q}(x) &= (-1)^{l-m} J_l^{m,-q}(-x).
\end{align*}
\]
Furthermore these functions verify the normalization relations for \( m \) and \( q \) fixed

\[
\int_{-1}^{1} J^m_n(x) (l + 1/2) J^m_n(x) = \delta_{ll'}, \quad \sum_{l=\sup(|m|,|q|)}^{\infty} J^m_n(x) (l + 1/2) J^m_n(y) = \delta(x - y).
\]

similar to the ones imposed in [11] to the Legendre polynomials and in [12] to the associated Legendre polynomials. Note again that they are orthogonal but, like in the SO(3,2) case, orthonormal only up to the factor \( l + 1/2 \).

The Jacobi equation

\[
\left[ (1 - x^2) \frac{d^2}{dx^2} - ((\alpha + \beta + 2) x + (\alpha - \beta)) \frac{d}{dx} + n(n + \alpha + \beta + 1) \right] J_n^{(\alpha,\beta)}(x) = 0
\]

rewritten in terms of these new functions \( J^m_n(x) \) and of the new parameters \( l, m \) and \( q \) becomes

\[
\left[ -(1 - x^2) \frac{d^2}{dx^2} + 2x \frac{d}{dx} + \frac{2mqx + m^2 + q^2}{1 - x^2} - l(l + 1) \right] J^m_n(x) = 0.
\]

It is worthy noticing the symmetry under the interchange \( m \leftrightarrow q \) in the expressions (2.2) and (2.4).

In the spirit of [11,12], the starting point for the construction of the algebra associated to these algebraic Jacobi functions is now the construction of the rising/lowering differential operators that allow to obtain from each AJF the contiguous ones differing by 1 or 1/2 in the value of the discrete variables \( l, m \) and \( q \). The fundamental limitation of this approach is that the problem has been considered from the point of view of differential equations where the indices are considered as parameters [7]. The dependence of the formulas from the indices in iterated applications must thus be introduced by hand. This problem has been taken into account in [11] where a consistent vector space framework (where the indices are related to discrete operators) was introduced to allow the iterated use of recurrence formulas by means of operators. The parameters involved are thus eigenvalues of certain discrete operators acting on the space of the AJF.

Thus, in order to display the complete operator structure on the set \( \{ J^m_n(x) \} \) we introduce, in consistency with the quantum theory approach, not only the operators \( X \) and \( D_x \) of the configuration space, such that

\[
X f(x) = x f(x), \quad D_x f(x) = f'(x), \quad [X, D_x] = -1,
\]

but also three other operators \( L, M \) and \( Q \) such that

\[
L J^m_n(x) = l J^m_n(x) \quad M J^m_n(x) = m J^m_n(x) \quad Q J^m_n(x) = q J^m_n(x),
\]

(2.5)
i.e. diagonal on the Jacobi functions and, thus, commuting between them

\[ [L, M] = [L, Q] = [M, Q] = 0. \]

Hence, when we shall consider the whole algebra, all of them will belong to the corresponding Cartan subalgebra. It will be proved later that the obtained representation of the corresponding group \((SU(2, 2))\) is unitary if the variables \(l, m, q\) are such that

\[ l \geq |m|, \quad l \geq |q|, \quad 2l, \quad l - m, \quad l - q \in \mathbb{N}. \]

3 \quad \textbf{\(SU_A(2) \otimes SU_B(2)\) for Jacobi functions with \(\Delta l = 0\)}

We have now to introduce the action of the ladder operators on the set of the Jacobi functions, \(\{J^{m,q}_l(x)\}\), as differential-difference relations.

We start from the differential-difference equations and the difference equations verified by the Jacobi functions, a complete list of which can be found in Refs. [15–17]. The procedure is laborious, and, as we said before, it will be reproduced in a more detailed way in a following paper. Here we only sketch the simplest examples in order to enlighten the procedure. Let us consider the equations (18.9.15) and (18.9.16) of Ref. [15]

\[
\frac{d}{dx} P^{(\alpha, \beta)}_n(x) = \frac{1}{2}(n + \alpha + \beta + 1) P^{(\alpha+1, \beta+1)}_{n+1}(x),
\]

\[
\frac{d}{dx} \left[(1-x)^\alpha (1+x)^\beta P^{(\alpha, \beta)}_n(x)\right] = -2(n+1)(1-x)^{\alpha-1}(1+x)^{\beta-1} P^{(\alpha-1, \beta-1)}_{n+1}(x)
\]

which are far to be symmetric. But if we rewrite them in terms of the algebraic Jacobi functions \(J^{m,q}_l(x)\) they allow to define the operators

\[
A_\pm := \pm \sqrt{1-X^2} D_x + \frac{1}{\sqrt{1-X^2}} (XM + Q) \quad (3.1)
\]

that act in the following way

\[
A_\pm J^{m,q}_l(x) = \sqrt{(l \pm m)(l \pm m + 1)} J^{m\pm 1,q}_l(x). \quad (3.2)
\]

The operators (3.1) are a generalization for \(q \neq 0\) of the operators \(J_\pm\) introduced in Ref. [12] for the Associated Legendre functions related by eq. (1.2) to the Jacobi functions with \(q = 0\). Moreover eqs. (3.2), that are independent from \(q\), coincide with eqs. (2.11) and (2.12) of Ref. [12].

Taking into account the action of the operators \(A_\pm\) and \(M\) on the Jacobi functions, eqs. (3.2) and (2.5), respectively, and defining \(A_3 := M\) it is easy to check that \(A_\pm\) and \(A_3\) close a \(su(2)\) algebra, denoted in the following \(su_A(2)\),

\[
[A_3, A_\pm] = \pm A_\pm \quad [A_+, A_-] = 2A_3.
\]
and commute with $L$ and $Q$

\[[L, A_{\pm}] = 0, \quad [Q, A_{\pm}] = 0, \quad [L, A_3] = 0, \quad [Q, A_3] = 0. \tag{3.3}\]

Note that from (3.2) and (2.5) the Jacobi functions \{$J_{m,q}^l(x)$\} such that $2l \in \mathbb{N}$, $l - m \in \mathbb{N}$ and $-l \leq m \leq l$ support the $(2l+1)$-dimensional UIR of the Lie group $SU_A(2)$ independently from the value of $q$.

Like in [12], starting from the differential realization (3.1) of the $A_{\pm}$ operators we recover the Jacobi differential equation (2.4) from the Casimir of $su_A(2)$ ($C_A$), i.e.,

\[[C_A - L(L + 1)] J_{m,q}^l(x) \equiv \left[ A_3^2 + \frac{1}{2} \{A_+, A_-\} - L(L + 1) \right] J_{m,q}^l(x) = 0. \tag{3.4}\]

Effectively, eq. (3.4) reproduces the operatorial form of (2.4), i.e.

\[-(1 - X^2)D_x^2 + 2XD_x + \frac{1}{1 - X^2}(2XMQ + M^2 + Q^2) - L(L + 1)]J_{m,q}^l(x) = 0, \tag{3.5}\]

On the other hand we can make use of the factorization method [18, 19], that relates second order differential equations to recurrence formulae written in terms of first order derivatives in such a way that the application of the first operator modifies the values of the parameters of the second one. Taking into account this fact, by means of iterate application of (3.2) we obtain that the two equations

\[[A_+ A_- - (L + M)(L - M + 1)] J_{m,q}^l(x) = 0, \tag{3.6}\]

reproduce the Jacobi equation (2.4) in operator form (3.5). Equations (3.4), (3.5) and (3.6) are particular cases of a general rule: the defining Jacobi equation can be recovered from the Casimir operator of any algebra and sub-algebra involved acting in $J_{m,q}^l(x)$ as well as from any diagonal product of ladder operators.

Now the symmetry under the interchange $m \leftrightarrow q$ in $J_{m,q}^l(x)$ exhibited in eqs. (2.2) and (2.4) allows to define two new operators $B_{\pm}$ from $A_{\pm}$ by means of the exchange

\[B_{\pm}(X, D_x, M, Q) = A_{\pm}(X, D_x, Q, M) \tag{3.7}\]

Thus,

\[B_{\pm} := \pm \sqrt{1 - X^2} D_x + \frac{1}{\sqrt{1 - X^2}} (XQ + M), \tag{3.8}\]

such that their action on the Jacobi functions is

\[B_{\pm} J_{m,n}^l(x) = \sqrt{(l \pm q)(l \pm q + 1)} J_{m,q \pm 1}^l(x). \tag{3.9}\]

Obviously also the operators $B_{\pm}$ and $B_3 := Q$ close a $su(2)$ algebra we denote $su_B(2)$

\[[B_3, B_{\pm}] = \pm B_{\pm} \quad [B_+, B_-] = 2B_3, \]
and the Jacobi functions \( \{ J_{m,q}^l(x) \} \) with \( 2l \in \mathbb{N}, \ l - q \in \mathbb{N} \) and \(-l \leq q \leq l\) close the \((2l + 1)\)-dimensional UIR of the Lie group \( SU(2)_B \) independently from the value of \( m \).

Moreover similarly to the commutation relations between operators \( A_\pm, A_3 \) and \( L,Q \) (3.3) we have now

\[
[L, B_\pm] = 0 \quad [M, B_\pm] = 0, \quad [L, B_3] = 0, \quad [M, B_3] = 0.
\]

Again we can recover the Jacobi equation (2.4) from the Casimir of the algebra \( su_B(2) \)

\[
[\mathcal{C}_B - L(L+1)] J_{m,q}^l(x) = \left[B_3^2 + \frac{1}{2} \{B_+, B_- \} - L(L+1) \right] J_{m,q}^l(x) = 0
\]
as well as from the expressions

\[
[B_+ B_- - (L + Q) (L - Q + 1)] J_{m,q}^l(x) = 0, \quad [B_- B_+ - (L - Q) (L + Q + 1)] J_{m,q}^l(x) = 0.
\]

A more complex algebraic scheme appears in common applications of the operators \( A_\pm \) and \( B_\pm \). As the operators \( A_\pm, A_3 \) commute with \( B_\pm, B_3 \), the algebraic structure is the direct sum of Lie algebras

\[
su_A(2) \oplus su_B(2).
\]

A new symmetry of the AJF emerges that moves \( m \) and \( q \) without changing \( l \). For \( l, m, q \) integer and half-integer formulae (3.2), (3.9) and (2.5) are the well known expressions for the infinitesimal generators of the group \( SU_A(2) \otimes SU_B(2) \). The Jacobi functions \( J_{m,q}^l(x) \) for fixed \( l \) and \(-l \leq m \leq l, \ -l \leq q \leq l\) determine a UIR of this group. As we mention previously, the structure of the Hilbert space of the AJF and the hermiticity of the generators will be discussed in Sect 6. In Fig. 1 the action of the operators \( A_\pm, B_\pm \) on the parameters \((l, m, q)\) that label the Jacobi functions corresponds to the plane \( \Delta l = 0 \).

### 4 Other ladder operators inside algebraic Jacobi functions and \( su(1, 1) \) representations

We mention before that many difference and differential-difference relations for Jacobi polynomials are known [15, 16]. Starting from them a \( su(2,2) \) Lie algebra can be constructed. It has fifteen infinitesimal generators, three of them are Cartan generators (for instance, \( L, M \) and \( Q \) or three independent linear combinations of them). We have four generators \((A_\pm (3.1) \) and \( B_\pm (3.8)) \) that commute with \( L \), hence we need eight
non-diagonal operators more. They are in differential form:

\[
C_+ := \frac{(1 + X)\sqrt{1 - X}}{\sqrt{2}} D_x - \frac{1}{\sqrt{2(1 - X)}} (X(L + 1) - (L + 1 + M + Q)),
\]

\[
C_- := -\frac{(1 + X)\sqrt{1 - X}}{\sqrt{2}} D_x - \frac{1}{\sqrt{2(1 - X)}} (XL - (L + M + Q)),
\]

\[
D_+ := -\frac{(1 - X)\sqrt{1 + X}}{\sqrt{2}} D_x + \frac{1}{\sqrt{2(1 + X)}} (X(L + 1) + (L + 1 + M - Q)),
\]

\[
D_- := +\frac{(1 - X)\sqrt{1 + X}}{\sqrt{2}} D_x + \frac{1}{\sqrt{2(1 + X)}} (XL + (L + M - Q)),
\]

\[
E_+ := -\frac{(1 - X)\sqrt{1 + X}}{\sqrt{2}} D_x + \frac{1}{\sqrt{2(1 + X)}} (X(L + 1) + (L + 1 - M + Q)),
\]

\[
E_- := +\frac{(1 - X)\sqrt{1 + X}}{\sqrt{2}} D_x + \frac{1}{\sqrt{2(1 + X)}} (XL + (L - M + Q)),
\]

\[
F_+ := -\frac{(1 + X)\sqrt{1 - X}}{\sqrt{2}} D_x + \frac{1}{\sqrt{2(1 - X)}} (X(L + 1) - (L + 1 - M - Q)),
\]

\[
F_- := +\frac{(1 + X)\sqrt{1 - X}}{\sqrt{2}} D_x + \frac{1}{\sqrt{2(1 - X)}} (XL - (L - M - Q)).
\]
All these differential operators act on the space \( \{ \mathcal{J}^m_l \} \) for \( l, m, q \) integer and half-integer such that \( l \geq |m|, |q| \). The explicit form of its action is:

\[
C_+ \mathcal{J}^m_l(x) = \sqrt{(l + m + 1)(l + q + 1)} \mathcal{J}^m_{l+1/2}(x),
\]

\[
C_- \mathcal{J}^m_l(x) = \sqrt{(l + m)(l + q)} \mathcal{J}^m_{l-1/2}(x),
\]

\[
D_+ \mathcal{J}^m_l(x) = \sqrt{(l + m + 1)(l - q + 1)} \mathcal{J}^m_{l+1/2}(x),
\]

\[
D_- \mathcal{J}^m_l(x) = \sqrt{(l + m)(l - q)} \mathcal{J}^m_{l-1/2}(x),
\]

\[
E_+ \mathcal{J}^m_l(x) = \sqrt{(l - m + 1)(l + q + 1)} \mathcal{J}^m_{l+1/2}(x),
\]

\[
E_- \mathcal{J}^m_l(x) = \sqrt{(l - m)(l + q)} \mathcal{J}^m_{l-1/2}(x),
\]

\[
F_+ \mathcal{J}^m_l(x) = \sqrt{(l - m + 1)(l - q + 1)} \mathcal{J}^m_{l+1/2}(x),
\]

\[
F_- \mathcal{J}^m_l(x) = \sqrt{(l - m)(l - q)} \mathcal{J}^m_{l-1/2}(x).
\]

From (4.2) (analogously to what happen with \( A_\pm \) and \( B_\pm \)) it is obvious that

\[
C_\pm^l = C_\mp, \quad D_\pm^l = D_\mp, \quad E_\pm^l = E_\mp, \quad F_\pm^l = F_\mp,
\]

i.e. all these rising/lowering operators have the hermiticity properties required by the representation to be unitary. Note that the operators defined in (4.1) change the parameters \( (l, m, q) \to (l \pm 1/2, m \pm 1/2, q \pm 1/2) \). More precisely, \( X_+ : l \to l + 1/2 \) and \( X_- : l \to l - 1/2 \). In Fig. 1 the action of these operators on the labels \( (l, m, q) \) corresponds to the planes \( \Delta l = \pm 1/2 \).

From the eqs. (4.1) the following relations among these new operators are easily stated

\[
D_\pm(X, D_x, M, Q) = C_\pm(-X, -D_x, M, -Q),
\]

\[
E_\pm(X, D_x, M, Q) = C_\pm(-X, -D_x, -M, Q),
\]

\[
F_\pm(X, D_x, M, Q) = -C_\pm(X, D_x, -M, -Q).
\]

Because of the symmetries (4.3) we can only discuss the operators \( C_\pm \). Along this section we will see that this symmetry of interchange between the ladder operators ((3.7) and (4.3)) can be identify with the Weyl symmetry acting on the roots of the simple Lie algebra that they span \( (su(2,2) \) in this case).

Taking thus into account, like in the previous cases, the action of the operators \( C_\pm \) and \( L, M, Q \) on the Jacobi functions, eqs. (4.1) and (2.5), respectively, we get that

\[
[C_+, C_-] = -2C_3, \quad [C_3, C_\pm] = \pm C_\pm
\]
where
\[ C_3 := L + \frac{1}{2}(M + Q) + \frac{1}{2}. \] 
(4.5)

Hence \( \langle C_\pm, C_3 \rangle \) close a \( su(1, 1) \) algebra that we will denote, as usual, \( su_C(1, 1) \).

As in the cases of the operators \( A_\pm \) and \( B_\pm \), we recover the Jacobi differential equation (2.4) up to a nonvanishing factor using the differential realization (4.1) of the operators \( C_\pm \) from the Casimir of \( su_C(1, 1) \)

\[ C C J_{m,q}^l (x) \equiv \left[ C_3^2 - \frac{1}{4} \{ C_+, C_- \} \right] J_{m,q}^l (x) = \frac{1}{4} [(m + q)^2 - 1] J_{m,q}^l (x). \] 
(4.6)

So
\[ \left[ C_C - \frac{(M + Q)^2 - 1}{4} \right] J_{m,q}^l (x) \equiv \left[ C_3^2 - \frac{1}{2} \{ C_+, C_- \} - \frac{1}{4} ((M + Q)^2 - 1) \right] J_{m,q}^l (x) = 0. \] 
(4.7)

Analogously the factorization method from a diagonal product of two ladder operators gives
\[ [C_+ C_- - (L + M) (L + Q)] J_{m,q}^l (x) = 0, \] 
\[ [C_- C_+ - (L + 1 + M) (L + 1 + Q)] J_{m,q}^l (x) = 0, \] 
(4.8)

and all the three eqs. (4.7) and (4.8) allow us to recover the Jacobi equation (3.5).

From (4.7) we see that since \((m + q) = 0, \pm 1, \pm 2, \pm 3, \cdots\) the IR of \( su(1, 1) \) with \( C_C = (m + q)^2 - 1)/4 = -1/4, 0, 3/4, 2, 15/4, \cdots\) are obtained. Moreover the unitarity of these IR comes from the fact that \( C_{\pm}^\dagger = C_{\pm} \) [20]. The spectrum of the operator \( C_3 \) is given by (4.5) and, since \( l \geq |m| \) and \( l \geq |q| \), has the eigenvalues \( \sup(|m|, |q|), \sup(|m|, |q|) + 1/2, \sup(|m|, |q|) + 1, \sup(|m|, |q|) + 1/2, \cdots \). Hence, the set of AJF supports many infinite dimensional UIR, of \( SU(1, 1) \) of the discrete series for \( SU_C(1, 1) \).

Similar results can be found for the other ladder operators \( D_\pm, E_\pm, F_\pm \) provide that we do the corresponding exchange displayed in (4.3) on all the eqs. (4.4), (4.6) (4.8), (4.7) and (4.8).

5 The complete symmetry group of \( \{ J_{m,q}^l (x) \} \): \( SU(2, 2) \)

If one represents the action of the twelve operators \( A_\pm, A_\pm, C_\pm, D_\pm, E_\pm, F_\pm \), that we have defined in previous sections, we obtain Fig. 1. To obtain the root system of the simple Lie algebra \( A_3 \equiv D_3 \) we have only simply to add three points in the origin corresponding to the elements \( L, M \) and \( Q \) of the Cartan subalgebra.

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5.1 Commutation relations

The Lie commutators of the generators $A_\pm, B_\pm, C_\pm, D_\pm, E_\pm, F_\pm, M, Q, L$ are

\[
\begin{align*}
[L, A_\pm] &= 0, & [L, M] &= 0, & [L, B_\pm] &= 0, & [L, Q] &= 0, \\
[L, C_\pm] &= \pm \frac{1}{2} C_\pm, & [L, D_\pm] &= \pm \frac{1}{2} D_\pm, & [L, E_\pm] &= \pm \frac{1}{2} E_\pm, & [L, F_\pm] &= \pm \frac{1}{2} F_\pm, \\
[M, B_\pm] &= 0, & [M, Q] &= 0, \\
[M, C_\pm] &= \pm \frac{1}{2} C_\pm, & [M, D_\pm] &= \pm \frac{1}{2} D_\pm, & [M, E_\pm] &= \mp \frac{1}{2} E_\pm, & [M, F_\pm] &= \mp \frac{1}{2} F_\pm, \\
[Q, A_\pm] &= 0, \\
[Q, C_\pm] &= \pm \frac{1}{2} C_\pm, & [Q, D_\pm] &= \mp \frac{1}{2} D_\pm, & [Q, E_\pm] &= \pm \frac{1}{2} E_\pm, & [Q, F_\pm] &= \mp \frac{1}{2} F_\pm, \\
[A_+, A_-] &= 2A_3, & [A_3, A_\pm] &= \pm A_\pm, & (A_3 = M), \\
[B_+, B_-] &= 2B_3, & [B_3, B_\pm] &= \pm B_\pm, & (B_3 = Q), \\
[C_+, C_-] &= -2C_3, & [C_3, C_\pm] &= \pm C_\pm, & (C_3 = L + \frac{1}{2}(M + Q) + \frac{1}{2}), \\
[D_+, D_-] &= -2D_3, & [D_3, D_\pm] &= \pm D_\pm, & (D_3 = L + \frac{1}{2}(M - Q) + \frac{1}{2}), \\
[E_+, E_-] &= -2E_3, & [E_3, E_\pm] &= \pm E_\pm, & (E_3 = L + \frac{1}{2}(-M + Q) + \frac{1}{2}), \\
[F_+, F_-] &= -2F_3, & [F_3, F_\pm] &= \pm F_\pm, & (F_3 = L - \frac{1}{2}(M + Q) + \frac{1}{2}), \\
[A_\pm, B_\pm] &= 0, & [A_\pm, B_\mp] &= 0, \\
[A_\pm, C_\mp] &= 0, & [A_\pm, D_\mp] &= 0, & [A_\mp, D_\mp] &= \mp F_\mp, \\
[A_\pm, E_\mp] &= \pm C_\mp, & [A_\pm, E_\pm] &= 0, & [A_\mp, F_\mp] &= 0, \\
[B_\pm, C_\mp] &= 0, & [B_\pm, C_\mp] &= \mp D_\mp, & [B_\pm, D_\mp] &= \pm C_\pm, \\
[B_\pm, E_\pm] &= 0, & [B_\pm, E_\pm] &= \mp F_\mp, & [B_\pm, F_\pm] &= 0, \\
[C_\pm, D_\pm] &= 0, & [C_\pm, D_\pm] &= \mp B_\pm, & [C_\pm, E_\pm] &= 0, & [C_\pm, E_\pm] &= \mp A_\pm, \\
[C_\pm, F_\pm] &= 0, & [C_\pm, F_\pm] &= 0, \\
[D_\pm, E_\pm] &= 0, & [D_\pm, E_\pm] &= 0, & [D_\pm, F_\pm] &= 0, & [D_\pm, F_\pm] &= \mp A_\pm, \\
[E_\pm, F_\pm] &= 0, & [E_\pm, F_\pm] &= \mp B_\pm.
\end{align*}
\]
5.2 Casimir of $su(2,2)$

The quadratic Casimir of $su(2,2)$ has the form

$$C_{su(2,2)} = \frac{1}{2} \left( \{ A_+, A_- \} + \{ B_+, B_- \} - \{ C_+, C_- \} - \{ D_+, D_- \} - \{ E_+, E_- \} - \{ F_+, F_- \} \right) + \frac{1}{2} \left( A_3^2 + B_3^2 + C_3^2 + D_3^2 + E_3^2 + F_3^2 \right)$$

$$= \frac{1}{2} \left( \{ A_+, A_- \} + \{ B_+, B_- \} - \{ C_+, C_- \} - \{ D_+, D_- \} - \{ E_+, E_- \} - \{ F_+, F_- \} \right) + 2L(L+1) + M^2 + Q^2 + \frac{1}{2}$$

$$\equiv -\frac{3}{2}$$

From it and taking into account the differential realization of the operators involved, (3.1), (3.8) and (4.1), we recover again the Jacobi equation (2.4).

Hence, the AJF support a UIR of the group $SU(2, 2)$ with the value $-3/2$ of the Casimir $C_{su(2,2)}$ (see Fig. 2). Also, as we have seen along the previous sections, the Jacobi equations is recovered form the Casimir of any of the 3-dimensional subalgebras of $su(2,2)$.

A peculiar property of the Algebraic Special Functions, that perhaps can be assumed as their definition, seems to be that, using the fundamental second order differential equation, all diagonal elements of the UEA can be found to be equivalent to it while all non-diagonal elements can be written as first order differential operators.
6 Operators on $L^2$ spaces and UEA

We have shown in the preceding sections that the algebraic Jacobi functions $J_{mq}^l$ with $l, m, q$ all together integer or half-integer are a basis of a IR of $su(2, 2)$. They are a basis of the $L^2$ functions defined on $E \times Z \times Z/2$, where $E = (-1, 1) \subset \mathbb{R}$ and

$$Z/2 := \{\cdots, -3/2, -1, -1/2, 0, 1/2, 1, 3/2, \cdots\}$$

is related to $m$ and $Z$ to $m - q$. Hence, the property that the vector space of linear operators acting on them are homomorphic to the UEA of $su(2, 2)$ is extended to these $L^2$ functions. The results presented in [11, 12] for the one-variable and two-variable square-integrable functions, respectively, can be extended to the Jacobi case.

Thus, let us consider the space of square functions defined on $E \times Z \times Z/2$, $L^2(E, Z, Z/2)$, which is the direct sum of the Hilbert spaces with $m$ and $q$ fixed, $L^2(E, m, q)$:

$$L^2(E, Z, Z/2) = \bigcup_{m-q \in Z} \bigcup_{q \in Z/2} L^2(E, m, q).$$

A basis for $L^2(E, Z, Z/2)$ is $\{|x, m, q\rangle\} (-1 < x < 1, m, q \in Z/2, m - q \in Z)$. Orthonormality and completeness relations are, respectively,

$$\langle x, m, q|x', m', q'\rangle = \delta(x - x') \delta_{m, m'} \delta_{q, q'}, \quad \sum_{m, q} \int_{-1}^{+1} dx \langle x, m, q\rangle \langle x, m, q\rangle = \mathcal{I}.$$

As the set $\{J_{mq}^l(x)\}$ satisfy eqs. (2.3) and (3.8) we can now define inside the Hilbert space a new basis $\{|l, m, q\rangle\}$ with $l, m, q \in Z/2$, $l \geq |m|$, $l \geq |q|$, $l - m \in \mathbb{N}$, $l - q \in \mathbb{N}$

$$|l, m, q\rangle := \int_{-1}^{+1} dx \langle x, m, q\rangle \sqrt{l + 1/2} J_{mq}^l(x) dx .$$

such that

$$\langle l, m, q|l', m', q'\rangle = \delta_{ll'} \delta_{m, m'} \delta_{q, q'}, \quad \sum_{l, m, q} |l, m, q\rangle \langle l, m, q| = \mathcal{I},$$

where the $J_{mq}^l(x)$ play the role of transition matrices:

$$J_{mq}^l(x) = \frac{1}{\sqrt{l + 1/2}} \langle x, m, q|l, m, q\rangle = \frac{1}{\sqrt{l + 1/2}} \langle l, m, q|x, m, q\rangle .$$

This transition matrix role of the algebraic Jacobi functions $\{J_{mq}^l(x)\}$ reflects the fact that the generators can be seen as differential operators on the variable space (3.1), (3.8) and (4.1) or algebraic operators in the spaces of labels (3.2), (3.9) and (4.2) allowing to make explicit the Lie algebra structure in contraposition to previous works [7–9].
In analogy with [11], an arbitrary vector $|f\rangle \in L^2(\mathbb{E}, \mathbb{Z}, \mathbb{Z})$ can be expressed as

$$|f\rangle = \sum_{m,q=-\infty}^{\infty} \int_{-1}^{+1} dx \ |x,m,q\rangle f^{m,q}(x) = \sum_{m,q=-\infty}^{\infty} \sum_{l=\sup(|m|,|q|)}^{\infty} |l,m\rangle f_{l}^{m},$$

where

$$f^{m,q}(x) := \langle x,m,q|f \rangle = \sum_{l=\sup(|m|,|q|)}^{\infty} J_{l}^{m,q}(x) f_{l}^{m,q},$$

$$f_{l}^{m,q} := \langle l,m,q|f \rangle = \int_{-1}^{+1} dx \ J_{l}^{m,q}(x) f^{m,q}(x).$$

Thus, all the $L^2$–functions defined on $(\mathbb{E}, \mathbb{Z}, \mathbb{Z})$ can be written as

$$\sum_{m=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \sum_{l=\sup(|m|,|q|)}^{\infty} J_{l}^{m,q}(x) f_{l}^{m,q}.$$

Hence, they support the UIR of $su(2,2)$. The space of all linear operators that act on $L^2(\mathbb{E}, \mathbb{Z}, \mathbb{Z})$ is thus homomorphic the UEA of $su(2,2)$.

7 Conclusions

The relevance of the ASF seems to be related to:

1. To the role of intertwining between second order differential equations and Lie algebras played by the algebraic special functions.

2. To the fact that ASF (in this paper AJF) are at the same time an irreducible representation of a Lie algebra (here $su(2,2)$) and a basis of $L^2$ (and wave) functions (here the $L^2$ functions are defined on $(\mathbb{E} \times \mathbb{Z} \times \mathbb{Z}/2)$). Thus they allow to state a homorphism between the UEA constructed on the Lie algebra and the vector space of the operators defined on the $L^2$ functions.

3. As the ASF are a basis of a unitary irreducible representation of the corresponding Lie group also, all sets obtained from them applying a whatever element of the Lie group are bases in the space of the $L^2$ functions.

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