Smeared Hairs and Black Holes in Three-Dimensional de Sitter Spacetime

Mu-In Park

Research Institute of Physics and Chemistry, Chonbuk National University, Chonju 561-756, Korea

Abstract

It is known that there is no three-dimensional analog of de Sitter black holes. I show that the analog does exist when non-Gaussian (i.e., ring-type) smearings of point matter hairs are considered. This provides a new way of constructing black hole solutions from hairs. I find that the obtained black hole solutions are quite different from the usual large black holes in that there are i) large to small black hole transitions which may be considered as inverse Hawking-Page transitions and ii) soliton-like (i.e., non-perturbative) behaviors. For Gaussian smearing, there is no black hole but a gravastar solution exists.

PACS numbers: 04.60.Kz, 04.60.-m, 04.70.Dy

* E-mail address: muinpark@yahoo.com
I. INTRODUCTION

Over the years black hole solutions have played a key role in recent developments in theoretical physics due to its unique window into quantum gravity. In particular, the lower-dimensional black holes, like BTZ black hole [1], has been crucial in understanding the holographic description of asymptotically anti-de Sitter (AdS) spacetimes [2], other than as just a simpler analog of its higher dimensional counterpart.

In asymptotically de Sitter (dS) spacetimes, however, the lower-dimensional analog of black holes is not known, even though there are solutions with the cosmological horizon [3, 4]. It would be desirable to have available a lower-dimensional analog which could exhibit the key features of its higher-dimensional counterparts. This would be important in the holographic description of dS spacetimes also [4, 5, 6], as BTZ black hole did in AdS. On the other hand, the conventional black holes are perturbative solution in that their horizon sizes scale with (some positive powers of) the Newton’s constant $G$.

There have been numerous works on the emergence of a new (inner) horizon in the black hole spacetimes when the Gaussian but anisotropic, i.e., $p_r \neq p_\phi$, smearing of point-matter distributions with energy-momentum $T^\mu_\nu = \text{diag}(-\rho, p_r, p_\phi, ...)$, in a “self-consistent” way [7]. However, the Gaussianity needs not be required always, though beyond the Gaussianity has not been studied much so far: As a deformation of $\delta$-function source, it is enough that the distribution has a sharp peak at the origin as the characteristic size shrinks to zero, maintaining the integration of the distribution function to be finite so that it can be normalized to unity always. Actually, the necessity of this kind of non-Gaussian regularization of the $\delta$-function has been noted earlier, in a study of quantum gravity by DeWitt [8], though its physical origin was not understood. More recently, Myung and Yoon have constructed a deformed AdS$_3$ black hole, based on Rayleigh distribution which is one of the non-Gaussian distributions [9]. It is the purpose of this paper to report that the three-dimensional analog of dS black holes does exist when non-Gaussian smearings of point matter hairs are considered, generally.

In this paper, I consider the general form of matter distributions, which include the Gaussian, Rayleigh, and Maxwell-Boltzmann distributions with moments $n = 0, 1, 2$, respectively. It is shown that there are black hole solutions for all the higher-moment matter distributions, except the conventional Gaussian one. This provides a new way of constructing black hole solutions from hairs. It is found that the strong energy condition is satisfied near the black holes so that the usual area (increasing) theorem is guaranteed, like the conventional black holes in vacuum. By demanding the area law, following the Bekenstein’s argument, and the first law of thermodynamics, the black hole mass is identified. However, I find that the obtained black hole solutions are quite different from those of the conventional (large) black holes in thermodynamical properties in that there are large to small black hole transitions which may be considered as inverse Hawking-Page transitions. The black holes show also soliton-like (i.e., non-perturbative) behaviors. For the Gaussian distribution, there is no black hole but a gravastar solution exists, instead.

1 For an early treatment on the dS black holes in higher dimensions, see [10].
II. DE SITTER BLACK HOLES AND GRAVASTARS FROM SMEARED MATTERS

The three-dimensional Einstein gravity with a positive cosmological constant $\Lambda = +1/L^2$ is described by the action on a manifold $\mathcal{M}$ (omitting some boundary terms)

$$I_g = \frac{1}{16\pi G} \int_{\mathcal{M}} d^3x \sqrt{-g} (R - 2\Lambda) + I_{\text{matter}},$$

(1)

where $I_{\text{matter}}$ is a matter action whose (microscopic) details are not important in this paper. The equations of motion for the metric are given by

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \frac{1}{L^2} g^{\mu\nu} = 8\pi G T_{\mu\nu}^{\text{matter}},$$

(2)

with the matter’s energy-momentum tensor

$$T_{\mu\nu}^{\text{matter}} = -\frac{2}{\sqrt{-g}} \frac{\delta I_{\text{matter}}}{\delta g_{\mu\nu}}.$$  

(3)

In order to solve the equation (2), I take the static metric ansatz

$$ds^2 = -N^2(r)dt^2 + N^{-2}(r)dr^2 + r^2d\phi^2.$$  

(4)

Here note that $g_{tt} = -g_{rr}^{-1}$ like the usual vacuum solution [3, 4] but this may not be true for arbitrary $T_{\mu\nu}$. So, I am considering the specific matter configurations which do not deform (4). By considering matter’s energy-momentum

$$T^\mu_\nu = \text{diag}(-\rho, p_r, p_\phi),$$

(5)

the Einstein equation (2) reads

$$\frac{(N^2)'}{2r} = -\Lambda - 8\pi G \rho,$$

(6)

$$\frac{(N^2)'}{2r} = -\Lambda + 8\pi G p_r,$$

(7)

$$\frac{(N^2)''}{2r} = -\Lambda + 8\pi G p_\phi,$$

(8)

where prime (') denotes the derivative with respect to the radial coordinate $r$. The solutions of $N^2, \rho, p_r, p_\phi$ are given by

$$N^2 = -\Lambda r^2 - 16\pi G \int_0^r \rho r dr,$$

(9)

$$p_r = -\rho,$$

(10)

$$p_\phi = -(r \rho)' ,$$

(11)

where I have set $N(0) = 0$ in order to be agreed with the vacuum de Sitter solution for $\rho = 0$. Eqs. (10) and (11) show that a non-vanishing radial pressure $p_r$ and an-isotropic tangential pressure $p_\phi = -\rho - r \rho'$ for an arbitrary matter distribution with $\rho' \neq 0$ are needed,
respectively. Hence, the ansatz (4), together with (5), determines the metric and matter’s pressures completely, in terms of the matter density $\rho$.

Now, let me introduce the matter density as

$$\rho = A r^n e^{-r^2/L^2},$$

where $L$ is a characteristic length scale of the matter distribution and $A$ is a normalization constant. One can obtain the Gaussian distribution for $n = 0$, and non-Gaussian (i.e., ring-type) distributions for higher moments, i.e., Rayleigh for $n = 1$, Maxwell-Boltzmann for $n = 2$, etc. Then, by plugging (12) into (9), one can easily find

$$N^2 = -\Lambda r^2 - 8\pi G A \gamma \left(\frac{n}{2} + 1, \frac{r^2}{L^2}\right)$$

$$= -\Lambda r^2 - 8\pi G A \left[ \Gamma \left(\frac{n}{2} + 1\right) - \Gamma \left(\frac{n}{2} + 1, \frac{r^2}{L^2}\right) \right],$$

where $\gamma(n/2 + 1, x^2) = 0^\infty \frac{t^{n/2}e^{-t}}{t^n}dt$, $\Gamma(n/2 + 1, x^2) = 0^\infty \frac{t^{n/2}e^{-t}}{t^n}dt - \Gamma(n/2 + 1)$ are the incomplete lower and upper Gamma functions, respectively.

What we know is that the Einstein equation is uniquely solved by the $dS_3$ solution [3, 4]

$$N^2_{dS_3} = -\frac{r^2}{l^2} + 8G m$$

in the vacuum, i.e., $L \to 0$, limit and there is a cosmological horizon at $r_+ = l\sqrt{8Gm}$ with the de Sitter mass $m$. From this boundary condition, which distinguishes the matter distribution for the smearing of point sources with the classical matter distributions, one obtains

$$A = -\frac{m}{\pi \Gamma \left(\frac{n}{2} + 1\right)}$$

and then (13) reads

$$N^2 = -\frac{r^2}{l^2} + 8G m \left[ 1 - \frac{\Gamma \left(\frac{n}{2} + 1, \frac{r^2}{L^2}\right)}{\Gamma \left(\frac{n}{2} + 1\right)} \right]$$

with $\Lambda = 1/l^2$. It is easy to see that this reduces to the $dS_3$ metric with (14) from the fact that the last term in [ ] vanishes in the $L \to 0$ limit. In the presence of the smeared matters the cosmological horizon of the (vacuum) $dS_3$ solution (14) would be shifted also. Now, in order to see whether there exists a black hole horizon, I need to know whether $N^2 = 0$ has interior roots, other than the cosmological horizons. The (black hole and cosmological) horizons, if exist, satisfy

$$\hat{r}_+^2 = 8Gm l^2 \left[ 1 - \frac{\Gamma \left(\frac{n}{2} + 1, \frac{\hat{r}_+^2}{L^2}\right)}{\Gamma \left(\frac{n}{2} + 1\right)} \right]$$

In the flat space with a vanishing cosmological constant, i.e., $l \to \infty$, one rather solves $N^2 = (1 - 4G\bar{m})^2 \left[ 1 - \Gamma \left(\frac{n}{2} + 1, \frac{\hat{r}_+^2}{L^2}\right)/\Gamma \left(\frac{n}{2} + 1\right) \right] = 0$ with a particle’s mass $\bar{m}$. But it is easy to see that the horizon does not occur since $\Gamma \left(\frac{n}{2} + 1, \frac{\hat{r}_+^2}{L^2}\right) < \Gamma \left(\frac{n}{2} + 1\right)$ always for $r > 0$. 


but it is hard to solve this analytically.

An easier way to find the existence of the (interior) black hole horizons is to consider the, so-called, *Nariai* limit, where the black hole horizon \( \hat{r}_- \) and the cosmological horizon \( \hat{r}_+ \) meet, i.e., \( \hat{r}_- = \hat{r}_+ \equiv \hat{r}_{\text{Nar}} \), at which \( (N^2)' = 0 \) as well as \( N^2 = 0 \). If there exists a positive solution for the *Nariai radius* \( \hat{r}_{\text{Nar}} \), the black hole horizon \( \hat{r}_- \) also exist, as well as the cosmological horizon \( \hat{r}_+ \), since \( \hat{r}_- \leq \hat{r}_{\text{Nar}} \leq \hat{r}_+ \). From \( (N^2)' = 0 \), one has

\[
r = \frac{8Gml^2}{\Gamma \left( \frac{n}{2} + 1 \right) L} e^{-\frac{\hat{r}_{\text{Nar}}^2}{L^2}}.
\]  

(18)

By plugging this into the left hand side of (17), with an identification of \( r = \hat{r}_\pm = \hat{r}_{\text{Nar}} \), one has an algebraic equation

\[
f(\hat{r}_{\text{Nar}}; L) \equiv \left( \frac{\hat{r}_{\text{Nar}}}{L} \right)^{n+2} e^{-\frac{\hat{r}_{\text{Nar}}^2}{L^2}} - \gamma \left( \frac{n}{2} + 1, \frac{\hat{r}_{\text{Nar}}^2}{L^2} \right) = 0,
\]

(19)

which does not depend on the dimensionful parameters of \( G, m, \) and \( l^2 \), but depends only on the dimensionless parameter \( x_{\text{Nar}} \equiv \hat{r}_{\text{Nar}}/L \), for each \( n \). So, the existence problem of the \( dS_3 \) black holes reduces to a purely mathematical problem of finding the number \( x_{\text{Nar}(n)} \) and let me call this *Nariai number*, for convenience. The explicit formula for the Nariai numbers is not known but one can find them numerically by plotting \( f(\hat{r}_{\text{Nar}}; L) \) as in Fig.1, which reads \( x_{\text{Nar}(0)} = 0, x_{\text{Nar}(1)} \approx 0.96786, x_{\text{Nar}(2)} \approx 1.33914, x_{\text{Nar}(3)} \approx 1.61363, x_{\text{Nar}(4)} \approx 1.83947, \) etc.

This result shows that there exist (three-dimensional) de Sitter black holes only for the non-Gaussian distributions, i.e., \( n \geq 1 \). On the other hand, for the Gaussian distributions the solution correspond to a “gravastar”, a compact self-gravitating object without horizons [11].

### III. PROPERTIES OF DE SITTER BLACK HOLES

In order to study the physical properties of de Sitter black holes, I need to know the detailed form of the horizon \( \hat{r}_- \). However, since (17) can not be solved analytically, I consider the perturbative solution near the origin, by demanding \( \hat{r}_- \) is very small; this is a reasonable assumption for small \( L \) since \( \hat{r}_- \) should be disappeared as \( L \) goes to zero.

By expanding (16) near \( r \approx 0 \), neglecting higher- order terms

\[
N^2 = -\frac{r^2}{l^2} \left[ 1 - \frac{16Gml^2}{(n+2)\Gamma \left( \frac{n}{2} + 1 \right) L^{n+2}} \frac{y^n}{L^{n+2}} + O \left( \frac{y^{n+2}}{L^{n+2}} \right) \right],
\]

(20)

one finds the black hole horizon as

\[
\hat{r}_- \approx \left( \frac{(n+2)\Gamma \left( \frac{n}{2} + 1 \right)}{16Gml^2} \right)^{\frac{1}{n}} L^{1+\frac{n}{2}}.
\]

(21)

The black hole horizon is proportional to (positive powers of) \( L \) and so becomes very small as \( L \to 0 \), which provides a validity criteria of the approximation. In contrast to the (outer)
FIG. 1: Plots of $f(\hat{r}_{\text{Nar}}; L)$ vs. $x_{\text{Nar}} = \hat{r}_{\text{Nar}}/L$ for various $n$. The Nariai numbers $x_{\text{Nar}}$ are the intersection points of $f(\hat{r}_{\text{Nar}}; L) = 0$, other than the trivial one at $x_{\text{Nar}} = 0$. Nariai number increases indefinitely as $n$ is increased. Numerically, it reads $x_{\text{Nar}}(0) \approx 0$, $x_{\text{Nar}}(1) \approx 0.96786$, $x_{\text{Nar}}(2) \approx 1.33914$, $x_{\text{Nar}}(3) \approx 1.61363$, $x_{\text{Nar}}(4) \approx 1.83947$, etc. (left to right).

cosmological horizon, the size of black hole is “inversely” proportional to the initial $dS_3$ mass $m$ and $l$. Furthermore, the black holes are soliton-like (i.e., non-perturbative) objects since their (horizon) sizes dominate in weaker couplings of the Newton’s constant $G$.

The Hawking temperature is obtained as

$$T_H = \frac{\hbar \kappa}{2\pi} \left| \frac{\hbar \hat{r}_\pm}{2\pi l^2} \left| 1 - \frac{x_\pm^{n+2}}{\gamma \left( \frac{n}{2} + 1, x_\pm^2 \right)} \right| \right|_{(22)}$$

with the (positive)$^3$ surface gravity function $\kappa = |\partial N^2/\partial r|$ and $x_\pm = \hat{r}_\pm/L$. In deriving (22), I have expressed the initial $dS_3$ mass $m$ in terms of $\hat{r}_\pm$, from (17). Note that the temperature (22) is an exact form in $\hat{r}_\pm$, though we do not know the exact algebraic form of $\hat{r}_\pm$ in terms of $m$ and $n$. Some of the plots for $n \approx 4$ are shown in Fig.2 (solid lines). For the Gaussian case ($n = 0$), there is no zero-temperature cusp: This is due to the absence of a black hole horizon $\hat{r}_-$ and (22) is the Hawking temperature for the cosmological horizon $\hat{r}_+$ only. For the non-Gaussian cases ($n \geq 1$), there are two branches of temperature curves with a zero-temperature cusp at the intersections: The left-hand side curves represent the black hole temperatures for the horizon $\hat{r}_-$ and the right-hand side curves represent those of cosmological horizons $\hat{r}_+$. The systems are in thermal equilibrium with zero temperature

$^3$ In dS space, there is a subtlety in defining the temperature, associated with the definition of the mass. But here I take the usual convention with the positive surface gravity and temperature [12, 13].
FIG. 2: Plots of Hawking temperature vs. $x = \hat{r}_+/L$ for $n = 0 \sim 4$ (bottom to top in the small (left) branches and top to bottom in the large (right) branches). The solid lines denote the exact formula (22) and the cross lines denote the small black hole approximations in (23). The dotted line denotes the vacuum $dS_3$ (i.e., $\rho = p = 0$). I have chosen $\hbar/2\pi l^2 = L \equiv 1$.

at the Nariai limit, like the usual higher-dimensional $dS$ black hole systems [12]. But it is remarkable that there exists an upper bound of the black hole temperature, which will be discussed further below. On the other hand, the temperature of the cosmological horizon becomes lower, due to the existence of zero temperature at the Nariai limit.

In order to study the black hole thermodynamics, let me consider the small black hole case, i.e., small $\hat{r}_-$, again which reduces the Hawking temperature as

$$T_H = \frac{n\hbar}{4\pi l^2} \hat{r}_- \left[ 1 - \frac{2(n+2)}{n(n+4)} \left( \frac{\hat{r}_-}{L} \right)^2 + \mathcal{O} \left( \frac{(\hat{r}_-)^4}{L^4} \right) \right].$$

(23)

Note that the temperature vanishes for the Gaussian ($n = 0$) case, as expected, and those for non-Gaussian ($n \geq 1$) cases are plotted in Fig.2 (cross lines), in comparison with the exact curves (solid lines). This shows that there are good agreements for the first several non-Gaussianities. From this approximate temperature formula (23), one can find the approximate formula for “small” Nariai numbers $x_{Nar} = \hat{r}_-/L$ where $T_H$ vanishes:

$$x_{Nar(n)} \simeq \sqrt{\frac{n(n+4)}{2(n+2)}}.$$

(24)

4 In AdS or flat cases, this corresponds to the extremal black holes [9, 14].

5 This is in sharply contrast to the $G(E)UP$ modifications though the existence of the minimal sizes are the same [13].
TABLE I: Comparisons of numerical values of the Nariai numbers $x_{\text{Nar}(n)}$ and their first approximations of (24). The values in the brackets denote the discrepancies with the numerical values.

| n | $x_{\text{Nar}(n)}$, Numerical | Approximation |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0.96786 | 0.91287 (-5.7 %) |
| 2 | 1.33914 | 1.22475 (-8.5 %) |
| 3 | 1.61363 | 1.44914 (-10.2 %) |
| 4 | 1.83947 | 1.63299 (-11.2 %) |

The comparisons with the numerical results are shown in Table 1. Note that the discrepancy increase as $x_{\text{Nar}(n)}$ increases as it would be.

For the black hole entropy, there is no canonical derivation, like the Euclidean action approach. But it does not seem that these black holes violate the Bekenstein’s area law, i.e., entropy being “linearly” proportional to the horizon area \[16\], since the Hawking’s area (increasing) theorem is guaranteed due to the strong energy conditions, as will be discussed in detail below. In other words, I demand that the black hole entropy as

$$S \equiv \alpha \frac{2 \pi \hat{r}_-}{4 \hbar G}. \tag{25}$$

But here, the (positive) coefficient $\alpha$ is not fixed to 1 as in the Bekenstein-Hawking entropy for the conventional large black holes \[17\], similarly to the so-called “small black holes” in higher curvature supergravities \[18\]; this would be determined from other independent analysis.

Demanding the first law of thermodynamics $dM = T_H dS = \frac{4 \pi \alpha}{4 \hbar G} d\hat{r}_- T_H$ yields the black hole mass, if monotonically increasing in $\hat{r}_-$, as

$$M(\hat{r}_-) = \frac{2 \pi \alpha}{4 \hbar G} \int_0^{\hat{r}_-} d\hat{r}_- T_H(\hat{r}_-)$$

$$= \frac{\alpha L^2}{4 G L^2} \int_0^{\hat{r}_- / L} d x \left| 1 - x^{n+2} \frac{e^{-x^2}}{\gamma \left( \frac{n}{2} + 1, x^2 \right)} \right|,$$  \tag{26}

where I have set $M(0) = 0$ in order to be agreed with the conventional vacuum $dS_3$ solution without black holes, i.e., $\hat{r}_- = 0$. The analytic integration of (26) is not available. But for the small black hole approximation, (26) reads

$$M(\hat{r}_-) = \frac{\alpha}{16 G L^2} \hat{r}_-^2 \left[ 1 - \frac{n + 2}{n(n + 4)} \left( \frac{\hat{r}_-}{L} \right)^2 + O \left( \left( \frac{\hat{r}_-}{L} \right)^4 \right) \right]. \tag{27}$$

In this approximate formula, the mass has maxima at the Nariai radius where the temperature vanishes by definition, from $dM/d\hat{r}_- = (2 \pi \alpha / G h) T_H = 0$. Since $\hat{r}_-$ is less than or equal to the Nariai radius, the black hole mass is always monotonically increasing in the allowed regions, consistently with the assumption for (26).

The heat capacity $C = dM/dT_H$ is given by

$$C = \frac{\alpha \pi L}{2 G} T_H \left( \frac{dT_H}{dx} \right)^{-1}, \tag{28}$$
\[ \frac{dT_H}{dx} = \frac{\hbar L}{2\pi L^2} \frac{1}{\gamma (n/2 + 1, x^2)} \left[ \frac{\gamma (n/2 + 1, x^2)^2}{(n/2 + x^2)} - (n + 3 - 2x^2)x^{n+2}e^{-x^2} \gamma (n/2 + 1, x^2) + x^{2n+3}e^{-2x^2} \right] \]

with \( x \equiv \hat{r}_c/L \). There is an infinite discontinuity\(^6\) in the heat capacity \( C \sim \epsilon |T - T_c|^{-1/2} \) [\( \epsilon \equiv \text{sign}(\hat{r}_c - \hat{r}) \)] at the location of the maximum temperature \( T_c \) and the critical horizon radius \( \hat{r}_c \) since \( T - T_c \sim (\hat{r} - \hat{r}_c)^2 \). The critical location is obtained approximately as

\[ x_c \equiv \hat{r}_c/L = \sqrt{\frac{n(n+4)}{6(n+2)}} \tag{29} \]

from

\[ \frac{dT_H}{dx} = \frac{n\hbar L}{4\pi L^2} \left[ 1 - \frac{6(n+2)}{n(n+4)}x^2 + \mathcal{O}(x^4) \right] \tag{30} \]

for the small black holes. As for the black holes radiate, \( T_H \) decreases (\( C > 0 \)) for the smaller black holes with \( \hat{r}_- < \hat{r}_c \) (i.e., \( x < x_c \)) but \( T_H \) increases (\( C < 0 \)) for the larger black holes with \( \hat{r}_- > \hat{r}_c \) (i.e., \( x > x_c \)). So, there are transitions between the locally thermodynamically stable (\( C > 0 \)) and unstable (\( C < 0 \)) phases but it is peculiar that the smaller ones are more stable, in contrast to the three-dimensional Kerr-de Sitter\(^4\) or higher-dimensional \( AdS \) black holes in \textit{vacuum}\(^20\).

Generally, the “local” thermodynamic instability does not necessarily imply the “global” instability, i.e., unstable to decay into \textit{globally} favored states via quantum tunneling\(^{21}\). In the canonical ensemble with a fixed temperature, the global (in)stability is governed by the (Helmholtz) free energy \( F = M - T_H S \) in such a way that the free energy is minimized, globally. The numerical plots of the free energy vs. Hawking temperature are shown in Fig. 3. It is important to note that there are two, upper and lower, branches for the large and small black holes, respectively, with \( \Delta F \equiv F_{\text{small}} - F_{\text{large}} < 0 \) at the same temperature and they meet exactly at the point where \( T_H \) is maximum\(^7\). This means that the region of local (in)stability coincides “exactly” with that of global (in)stability and the global transitions via tunnelings from large to small black holes corresponds to “inverse” Hawking-Page transitions\(^{20,8}\), with tunneling amplitudes \( \Gamma \sim Ae^{-\Delta I_E} \) and \( \Delta I_E \approx -\Delta F/T_H > 0 \) (\( A \) is some determinant and \( I_E \) is the \textit{on-shell} Euclidean action). There is no lower bound in the size of small black holes but rather an upper bound at the Nariai limit. As one increases the size of the small black hole, i.e., increasing the characteristic scale \( L \), there are more smearings of the matter hairs \textit{outside} the (inner, black-hole) horizon until the critical size \( \hat{r}_c \) when the maximum temperature \( T_c \) is reached. Further increase of the size beyond the critical size produces tunneling decays into the smaller ones with the same temperature, by

\(^6\) The rotating black \( M2 \)-branes have the same critical exponent but its heat capacity is always positive\(^{19}\).

\(^7\) This shows a second-order phase transition since \( F \) and \( dF/dT = -S \) are continuous but only \( d^2F/dT^2 = -C/T \) is discontinuous at the critical point.

\(^8\) In charged \( AdS \) black holes, similar phenomena can happen, for a range of temperature, in the canonical ensemble with a (fixed) charge \( q < q_{\text{crit}} \)^{22}. (See the branches 1 and 2 in Fig. 5.) Recently, this has been argued in the higher-dimensional regular black holes\(^{14,23}\), where the lower bounds of the horizons exist at the extremal black holes, but the status of the ADM mass or the first law of thermodynamics for the regular black holes remains unclear.
FIG. 3: Plots of free energy $F$ vs. Hawking temperature for $n = 1 \sim 4$ (left to right) ($h/2\pi l^2 = L \equiv 1$). For each curve with a given $n$, there are two (upper and lower) branches which meet at the maximum temperature $T_c$. The upper (lower) branch represents the large (small) black hole, giving $F_{\text{small}} - F_{\text{large}} < 0$.

re-absorbing the excess matter hairs on the one hand and more rapid Hawking radiations due to $C < 0$ on the other hand. I note also that the transitions from small black holes with higher moments ($n'$) into large or other small black holes with lower moments ($n$) are also possible, i.e., $F_{\text{large, small}(n)} - F_{\text{small}(n')} < 0 \ (n < n')$, when the temperatures have overlapping regions.

IV. ENERGY CONDITIONS

From the normalization of (15), one finds the energy density of smeared (point) matters

as

$$\rho = -\frac{m}{\pi \Gamma \left(\frac{u}{2} + 1\right)} \frac{r^n}{L^{n+2}} e^{-r^2/L^2}. \quad (31)$$

For the initially $dS_3$ space, one has a positive mass $m > 0$, the matter’s energy density $\rho$ is negative, and so the weak and dominant energy conditions are violated in the whole space.

This can be also written as $\rho = -\frac{m}{2\pi L} \frac{\Gamma(n+1)}{\Gamma(\frac{n}{2}+1)} \delta_n(r)$ with the non-Gaussian smearing of $\delta$-function, $\delta_n(r) = \frac{2^n}{\Gamma(\frac{n}{2}) L^{n+1}} e^{-r^2/L^2}$, which satisfies $\lim_{L \rightarrow 0} \delta_n(r) = \delta(r)$. Sometimes, it has been said that the black holes “degenerate” to a conical singularity at the origin, by Bousso, et. al. [24]. Their idea is “materialized” in my construction.
However, less restrictive but more important conditions in the black hole dynamics, like the strong or null energy condition can be satisfied.

To see this, let me consider \( \rho + p_i \), \( \rho + \sum_i p_i \) \( (i = r, \phi) \) which are obtained as

\[
\begin{align*}
\rho + p_\phi &= -r \rho' = \frac{m}{\pi \Gamma \left( \frac{n}{2} + 1 \right) L^{n+2}} \left( n - \frac{2r^2}{L^2} \right) r^n e^{-r^2/L^2}, \\
\rho + \sum_i p_i &= p_\phi = \frac{m}{\pi \Gamma \left( \frac{n}{2} + 1 \right) L^{n+2}} \left( n + 1 - \frac{2r^2}{L^2} \right) r^n e^{-r^2/L^2},
\end{align*}
\]

where I have used \( \rho + p_r = 0 \) from (10), (11), and (31). Then one finds that \( \rho + p_\phi \geq 0 \) for \( r \leq x_{sec} \equiv \sqrt{\frac{\gamma}{2}} \), whereas \( \rho + \sum_i p_i \geq 0 \) for \( \frac{r}{L} \leq \sqrt{\frac{n+1}{2}} \). Hence, it is found that the strong energy condition (SEC), which includes the null energy condition (NEC) \([25]\) is satisfied when \( \frac{r}{L} \leq x_{sec} \equiv \sqrt{\frac{\gamma}{2}} \). In order that the black hole has the required behaviors like the increasing horizon area for the accretion of the surrounding matters, I need to require \( \frac{\dot{r}}{L} < x_{sec} \). This yields

\[
\frac{8GmL^2}{L^2} > \frac{\frac{n}{2} \Gamma \left( \frac{n}{2} + 1 \right)}{\gamma \left( \frac{n}{2} + 1, \frac{n}{2} \right)}
\]

from \( N^2 |_{x_{sec}} > 0 \). For a good approximation, up to \( n \approx 30 \), the right hand side of (31) converges as ‘2.345 + 1.08n’ and so (31) can be written approximately as

\[
m \geq \frac{L^2}{8G\lambda^2} (2.345 + 1.08n).
\]

Moreover, for small black holes, it is easy to see that \( x_{sec} \), which is actually the maximum point of the density \( \rho \), i.e., \( \rho' |_{x_{sec}} = 0 \), is smaller than the Nariai radius, i.e., \( x_{sec} = \sqrt{\frac{n}{2}} \approx x_{Nar} \approx \sqrt{\frac{\frac{n+1}{2}}{\frac{n}{2} + 1}} \), which implies that there is a region where the SEC and NEC are violated, in between \( x_{sec} \) and \( x_{Nar} \). So, in this case, the cosmological horizon would not satisfy the Hawking’s area (increasing) theorem. However, the black hole would satisfy the area theorem well. For the gravastar case \( (n = 0) \), there is no black hole horizon and the condition (34) is trivially satisfied, which means that the SEC and WEC are trivially satisfied as far as \( \frac{r}{L} \leq x_{sec} \).

V. DISCUSSION

I have studied the \( dS_3 \) black holes for the non-Gaussian and gravastars for Gaussian smears of point-matter sources or hairs. I have studied the particular class of metrics satisfying \( g_{tt} = -g_{rr}^{-1} \) for simplicity. It would be a challenging problem to generalize this construction to include the “rotating” black holes and its associated horizon smearings which have been perturbatively studied in non-commutative BTZ black hole showing the interior black hole inside the inner BTZ black hole horizon, as well \([26]\). On the other hand, it seems that the smeared sources can be realized in the non-commutative solitons \([27, 28]^{10}\).

---

\[^{10}\text{I thank H. S. Yang for pointing out this possibility.}\]
It would be quite interesting to study the gravitating non-commutative solitons and their associated black hole spacetimes.

From the obtained $dS_3$ black holes, it is straightforward to construct four-dimensional (A)dS black strings which share the thermodynamical properties of $dS_3$ black holes [29]. According to Gubser-Mitra conjecture [30], the large black strings with the heat capacity $C_{b.s.} \sim C < 0$ would then be unstable under gravitational perturbations. This system provides an interesting test of the conjecture in four dimensions. Moreover, this provides a challenging problem of the final states of black string instability, i.e., whether it be the thinner black strings which are favored by the lower values of free energies or the four-dimensional Schwarzschild-de Sitter black hole, or in between them.

I have shown that finding the Nariai radius involves an interesting, purely mathematical, algebraic equation which is solved by the Nariai numbers $x_{Nar(n)}$. I have found an approximate formula of $x_{Nar(n)}$ for small black holes. It would be a challenging mathematical problem to find an improved or exact formula.

Acknowledgments

This work was supported by the Korea Research Foundation Grant funded by Korea Government(MOEHRD) (KRF-2007-359-C00011).

[1] M. Banados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. 69, 1849 (1992).
[2] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, Phys. Rept. 323, 183 (2000) and references therein.
[3] S. Deser and R. Jackiw, Ann. Phys. 153, 405 (1984).
[4] M.-I. Park, Phys. Lett. B 440, 275 (1998).
[5] A. Strominger, JHEP 10, 034 (2001).
[6] V. Balasubramanian, J. de Boer, and D. Minic, Phys. Rev. D 65, 123508 (2002).
[7] I. Dymnikova, Gen. Rel. Grav. 24, 235 (1992); P. Nicolini, A. Smailagic, and E. Spallucci, Phys. Lett. B 632, 547 (2006); for recent reviews, see S. Ansoldi, arXiv:0802.0330 [gr-qc] and P. Nicolini, arXiv:0807.1939 [hep-th].
[8] B. S. DeWitt, Phys. Rev. 160, 1113 (1967).
[9] Y. S. Myung and M. Yoon, arXiv:08010.0078 [gr-qc].
[10] I. Dymnikova and B. Soltystek, Gen. Rel. Grav. 24, 235 (1998).
[11] P. O. Mazur and E. Mottola, gr-qc/0109035.
[12] G. W. Gibbons and S. W. Hawking, Phys. Rev. D 15, 2738 (1977).
[13] M.-I. Park, Class. Quantum Grav. 25, 135003 (2008).
[14] Y.-S. Myung, Y.-W. Kim, and Y.-J. Park, JHEP 0702, 012 (2007); W. Kim, E. J. Son, and M. Yoon, JHEP 0804, 042 (2008).
[15] M.-I. Park, Phys. Lett. B 659, 698 (2008).
[16] J. D. Bekenstein, Phys. Rev. D 7, 2333 (1973).
[17] S. W. Hawking, Phys. Rev. Lett. 26, 1344 (1971).
[18] R. G. Cai, C. M. Chen, K. i. Maeda, N. Ohta and D. W. Pang, Phys. Rev. D 77, 064030 (2008).
[19] R.-G. Cai and K.-S. Soh, Mod. Phys. Lett. A 14, 1895 (1999).
[20] S. W. Hawking and D. N. Page, Commun. Math. Phys. 87, 577 (1983).
[21] T. Prestidge, Phys. Rev. D61, 084002 (2000).
[22] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, Phys. Rev. D 60, 064018 (1999).
[23] Y.-S. Myung, Y.-W. Kim, and Y.-J. Park, Phys. Lett. B 656, 221 (2007).
[24] R. Bousso, A. Maloney, and A. Strominger, Phys. Rev. D 65, 104039 (2002).
[25] E. Poisson, “A Relativist’s Toolkit: The Mathematics of Black-Hole Mechanics” (Cambridge University Press, Cambridge, England, 2004).
[26] H. C. Kim, M. I. Park, C. Rim, and J. H. Yee, JHEP 0810, 060 (2008).
[27] R. Gopakumar, S. Minwalla and A. Strominger, JHEP 0005, 020 (2000).
[28] I. Cho and Y. Lee, arXiv:0809.3580 [hep-th].
[29] M. I. Park, in preparation.
[30] S. S. Gubser and I. Mitra, arXiv:hep-th/0009126.