Efficient Numerical Scheme for the Solution of Tenth Order Boundary Value Problems by the Haar Wavelet Method

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Abstract: In this paper, an accurate and fast algorithm is developed for the solution of tenth order boundary value problems. The Haar wavelet collocation method is applied to both linear and nonlinear boundary value problems. In this technique, the tenth order derivative in boundary value problem is approximated using Haar functions and the process of integration is used to obtain the expression of lower order derivatives and approximate solution for the unknown function. Three linear and two nonlinear examples are taken from literature for checking validation and the convergence of the proposed technique. The maximum absolute and root mean square errors are compared with the exact solution at different collocation and Gauss points. The experimental rate of convergence using different number of collocation points is also calculated, which is nearly equal to 2.

Keywords: boundary value problems; Gauss elimination method; collocation method; Haar wavelet

1. Introduction

Boundary value problems (BVPs) with higher order arise in the field of astrophysics, and the narrow convecting layers bounded by stable layers, which are believed to surround A-type stars, may be modeled by tenth order BVPs. Many researchers studied higher order BVPs because of mathematical importance and their uses in various fields of applied sciences. These equations are modeled in physics of stability problems in hydrodynamics [1]. In the presence of magnetic field and in the direction of gravity, when an infinite horizontal layer of fluid is heated from below, subject to the axis of rotation, an instability takes place inside. When this phenomena of instability take place as ordinary convection, then it is modeled by tenth-order BVP [2].

Several researchers worked on the solution of higher order BVPs. Bishop et al. modeled the phenomena of torsional vibration of beams as eight-order BVP [3]. Siddiqi and Akram [4,5] introduced Nonic spline and non-polynomial spline method for the numerical solution linear BVPs models of eighth order. In [4,5], Siddiqi also proved that the convergence of the aforementioned techniques are of second order. Wazwaz [6] established an efficient technique by using the Adomian decomposition technique to solve some special eight order BVPs numerically. Recently, a numerical technique based on polynomial splines of degree six was introduced by Siddiqi and Twizell in [7] for the numerical solution for some
special types of eight order BVPs. However, it was found that, at the points adjacent to the boundary, the method diverges. Furthermore, in [8,9] Twizell et al. observed the same problems by solving some higher order BVPs numerically. It was investigated that the results diverge due to the utilization of test functions with a lower order in the aforesaid technique. In [7], the authors applied differential quadrature methods (DQM), which uses the higher order test functions in the entire domain. However, the shortcoming of the DQM method is that it dealt with only second order BVPs. For higher-order BVPs, a \( \delta \)-point method should be utilized [8]. The details of improvement and applications of the DQM technique can be seen in [7,8]. The current improvement in the DQM leads to the establishment of generalized (DQM) [10,11]. Theoretically, the generalized DQM can be utilized for any higher order BVPs coupled with the conventional \( \delta \)-point method. In structure mechanics, the generalized DQM is used for the numerical solution of fourth and sixth order BVPs [11–13]. Furthermore, in fluid mechanics, it is utilized for the numerical solution of linear Onsager and nonlinear Blasius problems with third and sixth orders, respectively [10]. The initial value problems of second, third, and fourth orders are also solved by using the aforesaid method [11]. The generalized DQM is extended by Liu et al. in [14] for the numerical solution of tenth-order BVPs. In [15], a reproducing kernel method is introduced by Geng et al. for the solution linear tenth order BVPs. A variational iterative method is utilized by Siddiqi et al. for the numerical solution of tenth order BVPs in [16]. Akram et al. developed a numerical technique based on 11th degree spline for the linear special case of BVPs [17]. A tenth degree spline was utilized by Twizell et al. for the numerical solution of tenth order BVPs and faced some problems in getting results near boundaries of the interval in [18]. A new numerical technique was introduced by Boutayeb et al. for the solution of 8th, 10th, and 12th order BVPs arises in thermal instability [19].

In this article, we developed the Haar wavelet collocation method (HWCM) for the numerical simulation of 10th-order BVPs of ordinary differential equations. The following nonlinear problem of order 10 will be considered in this article:

\[
\dot{u}^{(10)}(t) = F(t, u, u^{(1)}, u^{(2)}, u^{(3)}, u^{(4)}, u^{(5)}, u^{(6)}, u^{(7)}, u^{(8)}, u^{(9)}), \quad 0 \leq x \leq 1.
\]  

where \( F \) is the known function, while, in the case of linear, the following general form is considered:

\[
u^{(10)}(t) + a_1(t)u^{(9)}(t) + a_2(t)u^{(8)}(t) + a_3(t)u^{(7)}(t) + a_4(t)u^{(6)}(t) + a_5(t)u^{(5)}(t) + a_6(t)u^{(4)}(t) + a_7(t)u^{(3)}(t) + a_8(t)u^{(2)}(t) + a_9(t)u^{(1)}(t) + a_{10}(t)u(t) = h(t), \quad t \in [0,1],
\]

with the following BCs:

\[
\begin{align*}
    u(0) = \alpha_0 \quad & u(1) = \beta_0, \\
    u^{(2)}(0) = \alpha_1 \quad & u^{(2)}(1) = \beta_1, \\
    u^{(4)}(0) = \alpha_2 \quad & u^{(4)}(1) = \beta_2, \\
    u^{(6)}(0) = \alpha_3 \quad & u^{(6)}(1) = \beta_3, \\
    u^{(8)}(0) = \alpha_4 \quad & u^{(8)}(1) = \beta_4,
\end{align*}
\]

where \( a_i(t), \quad i = 1,2,...,10 \), and \( h(t) \) are known functions.

The main contributions are:

- We developed HWCM for the numerical solution of linear and nonlinear tenth order BVPs.
- We examine the effectiveness of the HWCM on some examples and compare our results with other methods.

The paper is organized as: Haar functions are defined in Section 2. Numerical HWCM for the solution of both linear and nonlinear tenth order BVPs are given in Section 3. In Section 4, some problems from literature are given for validation of HWCM. Conclusions are given in Section 5.
2. Haar Wavelet

The Haar functions are piecewise constant functions having three values 1, −1, and 0. The Haar scaling function on interval \( [\gamma_1, \gamma_2) \) is given by [20]

\[
h_1(t) = \begin{cases} 
1 & \text{for } t \in [\gamma_1, \gamma_2), \\
0 & \text{otherwise.}
\end{cases}
\]  

(4)

The mother wavelet on \( [\gamma_1, \gamma_2) \) is

\[
h_2(t) = \begin{cases} 
1 & \text{for } t \in [\gamma_1, \frac{\gamma_1 + \gamma_2}{2}), \\
-1 & \text{for } t \in \left[\frac{\gamma_1 + \gamma_2}{2}, \gamma_2), \\
0 & \text{otherwise.}
\end{cases}
\]  

(5)

The other functions in the Haar family on subintervals \( [\rho_1, \rho_2) \) can be generated by the dilation and translation except the scaling function

\[
h_i(t) = \begin{cases} 
1 & \text{for } t \in [\rho_1, \rho_2), \\
-1 & \text{for } t \in [\rho_2, \rho_3), \\
0 & \text{otherwise,}
\end{cases}
\]  

(6)

where

\[
\rho_1 = \gamma_1 + (\gamma_2 - \gamma_1) \frac{\zeta}{d'}, \\
\rho_2 = \gamma_1 + (\gamma_2 - \gamma_1) \frac{\zeta + 0.5}{d'}, \\
\rho_3 = \gamma_1 + (\gamma_2 - \gamma_1) \frac{\zeta + 1}{d'},
\]

where integer \( d = 2^r \), where \( r = 0, 1, \ldots, r' \) and let the integer \( \zeta = 0, 1, \ldots, d-1 \). The number \( i \) can be obtained as \( i = d + \zeta + 1 \). In the interval \( [0,1] \), \( \rho_1, \rho_2, \) and \( \rho_3 \) are defined as:

\[
\rho_1 = \frac{\zeta}{d'}, \quad \rho_2 = \frac{\zeta + 0.5}{d'}, \quad \rho_3 = \frac{\zeta + 1}{d'}.
\]  

(7)

Any member of \( L^2[0,1) \), space of square integrable function is expressed as:

\[
u(t) = \sum_{k=1}^{\infty} \lambda_k h_k(t).
\]  

(8)

This series is truncated at finite \( N \) terms for approximation purposes, i.e.,

\[
u(t) \approx \sum_{k=1}^{N} \lambda_k h_k(t).
\]

We use the symbol

\[
p_i(t) = \int_{0}^{t} h_i(t) dx.
\]  

(9)
and the value of the above integral is calculated by a definition of \( h_i \) and is given by

\[
p_{i,1}(t) = \begin{cases} 
  t - \rho_1 & \text{for } t \in [\rho_1, \rho_2), \\
  \rho_3 - t & \text{for } t \in [\rho_2, \rho_3), \\
  0 & \text{elsewhere.}
\end{cases}
\] (10)

Thus, the value of \( p_{i,2} \) is

\[
p_{i,2}(t) = \int_0^t p_{i,1}(s)ds.
\]

By simplifying this integral, we have

\[
p_{i,2}(t) = \begin{cases} 
  \frac{1}{2} (t - \rho_1)^2 & \text{if } t \in [\rho_1, \rho_2), \\
  \frac{1}{4m^2} - \frac{1}{2} (\rho_3 - t)^2 & \text{if } t \in [\rho_2, \rho_3), \\
  \frac{1}{4m^2} & \text{if } t \in [\rho_3, 1), \\
  0 & \text{elsewhere.}
\end{cases}
\] (11)

Similarly, the value of \( p_{i,3} \) is given by

\[
p_{i,3}(t) = \int_0^t p_{i,2}(s)ds.
\]

By simplifying this integral, we obtain

\[
p_{i,3}(t) = \begin{cases} 
  \frac{1}{6} (t - \rho_1)^3 & \text{if } t \in [\rho_1, \rho_2), \\
  \frac{1}{4m^2} (t - \rho_2) - \frac{1}{6} (\rho_3 - t)^2 & \text{if } t \in [\rho_2, \rho_3), \\
  \frac{1}{4m^2} (t - \rho_2) & \text{if } t \in [\rho_3, 1), \\
  0 & \text{elsewhere.}
\end{cases}
\] (12)

By simplifying this integral, we obtain

\[
p_{i,4}(t) = \begin{cases} 
  \frac{1}{24} (t - \rho_1)^4 & \text{if } t \in [\rho_1, \rho_2), \\
  \frac{1}{8m^2} (t - \rho_2)^2 - \frac{1}{24} (\rho_3 - t)^4 + \frac{1}{192m^2} (\rho_3 - t)^3 & \text{if } t \in [\rho_2, \rho_3), \\
  \frac{1}{8m^2} (t - \rho_2)^2 + \frac{1}{192m^2} & \text{if } t \in [\rho_3, 1), \\
  0 & \text{elsewhere.}
\end{cases}
\] (13)

Generally,

\[
p_{i,n}(t) = \int_0^t p_{i,n-1}(s)ds.
\]

Thus, \( p_{i,n}(t) \) is obtained as [7],

\[
p_{i,n}(t) = \begin{cases} 
  0 & \text{for } t \in [0, \rho_1), \\
  \frac{(t - \rho_1)^n}{n!} & \text{for } t \in [\rho_1, \rho_2), \\
  \frac{[(t - \rho_1)^n - 2(\rho_1 - \rho_2)^n]}{n!} & \text{for } t \in [\rho_2, \rho_3), \\
  \frac{1}{n!} [(t - \rho_1)^n - 2(\rho_1 - \rho_2)^n + (t - \rho_3)^n] & \text{for } t \in [\rho_3, 1). 
\end{cases}
\] (15)
For HWCM, the interval \([a, \beta]\) is discretized using formula:

\[
t_m = a + (\beta - a) \frac{m - 1/2}{2M} \quad m = 1, 2, 3, 4, \ldots, 2M.
\]  \hfill (16)

Equation (16) is known as collocation point (CP). Gauss points (GPs) are also termed as integration points because numerical integration is carried out at these points. These points are represented as:

\[
G_j = h \left( \frac{j - 1}{2} + \frac{3 - \sqrt{3}}{6} \right), \quad G_{j+1} = h \left( \frac{j - 1}{2} + \frac{3 + \sqrt{3}}{6} \right), \quad j = 1, 2, 3, \ldots, N - 1.
\]

Some of the recent work using HWCM in literature can be seen in the references [20–24]. Haar wavelets are constant functions having jump discontinuity, and its derivatives of all orders vanishes. In this manner, HWCM can not be applied directly for a solution of tenth order BVPs. To overcome this weakness, we approximate the highest order derivatives involved in the BVPs and we utilized a method of integration to obtain an expression for unknown function and lower order derivatives.

3. Haar Wavelet Collocation Method

In this section, HWCM is developed for a solution of eight-order both linear and nonlinear BVPs. We developed HWCM for interval \([0, 1]\). We use the notation \(\Theta = \sum_{i=1}^{N}\). Let \(u^{(10)}(t) \in L_2[0, 1]\); then,

\[
u^{(10)}(t) = \Theta a_i h_i(t).
\]  \hfill (17)

Integrating Equation (17) from 0 to \(t\) and using BCs, we obtain the values of lower order derivatives and the value of an unknown function, which are given below:

\[
u^{(9)}(t) = u^{(9)}(0) + \Theta a_i p_{i,1}(t),
\]  \hfill (18)

\[
u^{(8)}(t) = a_4 + tu^{(9)}(0) + \Theta a_i p_{i,2}(t),
\]  \hfill (19)

\[
u^{(7)}(t) = u^{(7)}(0) + ta_4 + \frac{t^2}{2} u^{(9)}(0) + \Theta a_i p_{i,3}(t),
\]  \hfill (20)

\[
u^{(6)}(t) = a_3 + tu^{(7)}(0) + \frac{t^2}{2} a_4 + \frac{t^3}{6} u^{(9)}(0) + \Theta a_i p_{i,4}(t),
\]  \hfill (21)

\[
u^{(5)}(t) = u^{(5)}(0) + ta_3 + \frac{t^2}{2} u^{(7)}(0) + \frac{t^3}{6} a_4 + \frac{t^4}{24} u^{(9)}(0) + \Theta a_i p_{i,5}(t),
\]  \hfill (22)

\[
u^{(4)}(t) = a_2 + tu^{(5)}(0) + \frac{t^2}{2} a_3 + \frac{t^3}{6} u^{(7)}(0) + \frac{t^4}{24} a_4 + \frac{t^5}{120} u^{(9)}(0) + \Theta a_i p_{i,6}(t),
\]  \hfill (23)

\[
u^{(3)}(t) = u^{(3)}(0) + ta_2 + \frac{t^2}{2} u^{(5)}(0) + \frac{t^3}{6} a_3 + \frac{t^4}{24} u^{(7)}(0) + \frac{t^5}{120} a_4 + \frac{t^6}{720} u^{(9)}(0) + \Theta a_i p_{i,7}(t),
\]  \hfill (24)

\[
u^{(2)}(t) = a_1 + tu^{(3)}(0) + \frac{t^2}{2} a_2 + \frac{t^3}{6} u^{(5)}(0) + \frac{t^4}{24} a_3 + \frac{t^5}{120} u^{(7)}(0) + \frac{t^6}{720} a_4 + \frac{t^7}{5040} u^{(9)}(0)
\] \hfill (25)\begin{align*}
&+ \Theta a_i p_{i,8}(t),
\end{align*}

\[
u^{(1)}(t) = u^{(1)}(0) + ta_1 + \frac{t^2}{2} u^{(3)}(0) + \frac{t^3}{6} a_2 + \frac{t^4}{24} u^{(5)}(0) + \frac{t^5}{120} a_3 + \frac{t^6}{720} u^{(7)}(0) + \frac{t^7}{5040} a_4 + \\
&\frac{t^8}{40320} u^{(9)}(0) + \Theta a_i p_{i,9}(t),
\]  \hfill (26)
and

\[
    u(t) = a_0 + tu^{(1)}(0) + \frac{t^2}{2}a_1 + \frac{t^3}{6}u^{(3)}(0) + \frac{t^4}{24}a_2 + \frac{t^5}{120}u^{(5)}(0) + \frac{t^6}{720}a_3 + \frac{t^7}{5040}u^{(7)}(0) + \frac{t^8}{40320}a_4 + \frac{t^9}{362880}u^{(9)}(0) + \Theta_4 p_1 u(t).
\]

(27)

Now, to find the unknown \( u^{(1)}(0) \), \( u^{(3)}(0) \), \( u^{(5)}(0) \), \( u^{(7)}(0) \), and \( u^{(9)}(0) \) in Equation (27), we integrate Equations (18), (20), (22), (24), and (26) each from 0 to 1 and using BCs to obtain the following five equations:

\[
    \beta_4 - a_4 = u^{(9)}(0) + \Theta_4 c_{i,1},
\]

(28)

\[
    \beta_3 - \alpha_3 = u^{(7)}(0) + \frac{1}{2}a_4 + \frac{1}{6}u^{(9)}(0) + \Theta_4 c_{i,3},
\]

(29)

\[
    \beta_2 - \alpha_2 = u^{(5)}(0) + \frac{1}{2}a_4 + \frac{1}{6}u^{(7)}(0) + \frac{1}{24}a_4 + \frac{1}{120}u^{(9)}(0) + \Theta_4 c_{i,5},
\]

(30)

\[
    \beta_1 - \alpha_1 = u^{(3)}(0) + \frac{1}{2}a_4 + \frac{1}{6}u^{(5)}(0) + \frac{1}{24}a_4 + \frac{1}{120}u^{(7)}(0) + \frac{1}{720}a_4 + \frac{1}{5040}u^{(9)}(0) + \Theta_4 c_{i,7},
\]

(31)

\[
    \beta_0 - \alpha_0 = u^{(1)}(0) + \frac{1}{2}a_4 + \frac{1}{6}u^{(3)}(0) + \frac{1}{24}a_4 + \frac{1}{120}u^{(5)}(0) + \frac{1}{720}a_4 + \frac{1}{5040}u^{(7)}(0) + \frac{1}{40320}a_4 + \frac{1}{362880}u^{(9)}(0) + \Theta_4 c_{i,9}.
\]

(32)

By solving these Equations (28)–(32), we get these unknown \( u^{(9)}(0) \), \( u^{(7)}(0) \), \( u^{(5)}(0) \), \( u^{(3)}(0) \), and \( u^{(1)}(0) \), respectively:

\[
    u^{(9)}(0) = \beta_4 - a_4 - \Theta_4 c_{i,1},
\]

(33)

\[
    u^{(7)}(0) = \beta_3 - \alpha_3 - \frac{1}{3}a_4 - \frac{1}{6}\beta_4 + \frac{1}{6}\Theta_4 c_{i,1} - \Theta_4 c_{i,3},
\]

(34)

\[
    u^{(5)}(0) = \beta_2 - \alpha_2 - \frac{1}{3}a_4 - \frac{1}{3}\beta_3 + \frac{1}{45}a_4 + \frac{7}{360}\beta_4 - \frac{7}{360}\Theta_4 c_{i,1} + \frac{1}{6}\Theta_4 c_{i,3} - \Theta_4 c_{i,5},
\]

(35)

\[
    u^{(3)}(0) = \beta_1 - \alpha_1 - \frac{1}{3}a_4 - \frac{1}{3}\beta_2 + \frac{1}{45}a_3 + \frac{7}{360}\beta_3 - \frac{2}{945}a_4 - \frac{31}{15120}\beta_4 + \frac{31}{15120}\Theta_4 c_{i,1},
\]

(36)

and

\[
    u^{(1)}(0) = \beta_0 - \alpha_0 - \frac{1}{3}a_4 - \frac{1}{6}\beta_1 + \frac{1}{45}a_2 + \frac{7}{360}\beta_2 - \frac{2}{945}a_3 - \frac{31}{15120}\beta_3 + \frac{1}{45}a_4 + \frac{127}{604800}\beta_4 - \frac{127}{604800}\Theta_4 c_{i,1} - \frac{31}{15120}\Theta_4 c_{i,3} - \frac{7}{360}\Theta_4 c_{i,5} + \frac{1}{6}\Theta_4 c_{i,7} - \Theta_4 c_{i,9}.
\]

(37)

Next, we substitute the above values of Equations (33)–(37) in Equations (18)–(27), we have

\[
    u^{(9)}(t) = \beta_4 - a_4 - \Theta_4 c_{i,1} + \Theta_4 p_1(t),
\]

(38)

\[
    u^{(8)}(t) = a_4 + t(\beta_4 - a_4 - \Theta_4 c_{i,1} + \Theta_4 p_1(t)),
\]

(39)

\[
    u^{(7)}(t) = \left( \beta_3 - \alpha_3 - \frac{1}{3}a_4 - \frac{1}{6}\beta_4 + \frac{1}{6}\Theta_4 c_{i,1} - \Theta_4 c_{i,3} + t\Theta_4 c_{i,1} \right) + \frac{t^2}{2}(\beta_4 - a_4 - \Theta_4 c_{i,1}) + \Theta_4 p_1(t),
\]

(40)

\[
    u^{(6)}(t) = a_3 + t\left( \beta_3 - \alpha_3 - \frac{1}{3}a_4 - \frac{1}{6}\beta_4 + \frac{1}{6}\Theta_4 c_{i,1} - \Theta_4 c_{i,3} + t\Theta_4 c_{i,1} \right) + \frac{t^2}{2}a_4 + \Theta_4 p_1(t),
\]

(41)
\[ u^{(5)}(t) = \left( \beta_2 - \alpha_2 - \frac{1}{3} \alpha_3 - \frac{1}{6} \beta_3 + \frac{1}{45} \alpha_4 + \frac{7}{360} \beta_4 - \frac{7}{360} \Theta a_c i_{1,3} - \frac{7}{360} \Theta a_c i_{1,5} \right) + t \alpha_3 + \frac{t^2}{2} \left( \beta_3 - \alpha_3 - \frac{1}{3} \alpha_4 - \frac{1}{6} \beta_4 + \frac{1}{6} \Theta a_c i_{1,1} - \frac{1}{6} \Theta a_c i_{1,3} + t \alpha_4 \right) + \frac{t^3}{6} \alpha_4 + \frac{t^4}{24} \left( \beta_4 - \alpha_4 - \Theta a_c i_{1,4} \right) + \Theta a_i p_{i,5}(t), \]

\[ u^{(4)}(t) = a_2 + t \left( \beta_2 - \alpha_2 - \frac{1}{3} \alpha_3 - \frac{1}{6} \beta_3 + \frac{1}{45} \alpha_4 + \frac{7}{360} \beta_4 - \frac{7}{360} \Theta a_c i_{1,1} - \frac{7}{360} \Theta a_c i_{1,3} - \Theta a_c i_{1,5} \right) + \frac{t^2}{2} \alpha_3 + \frac{t^3}{6} \left( \beta_3 - \alpha_3 - \frac{1}{3} \alpha_4 - \frac{1}{6} \beta_4 + \frac{1}{6} \Theta a_c i_{1,1} - \Theta a_c i_{1,3} + t \alpha_4 \right) + \frac{t^4}{24} \alpha_4 \]

\[ + \frac{t^5}{120} \left( \beta_4 - \alpha_4 - \Theta a_c i_{1,1} \right) + \Theta a_i p_{i,4}(t), \]

\[ u^{(3)}(t) = \left( \beta_1 - \alpha_1 - \frac{1}{3} \alpha_2 - \frac{1}{6} \beta_2 + \frac{1}{45} \alpha_3 + \frac{7}{360} \beta_3 - \frac{2}{945} \alpha_4 - \frac{31}{15120} \beta_4 + \frac{31}{15120} \Theta a_c i_{1,1} \right) - \frac{7}{360} \Theta a_c i_{1,3} + \frac{1}{6} \Theta a_c i_{1,5} - \Theta a_c i_{1,7} + t \alpha_2 + \frac{t^2}{2} \left( \beta_2 - \alpha_2 - \frac{1}{3} \alpha_3 - \frac{1}{6} \beta_3 + \frac{1}{45} \alpha_4 \right) + \frac{7}{360} \beta_4 - \frac{7}{360} \Theta a_c i_{1,1} + \frac{1}{6} \Theta a_c i_{1,3} - \Theta a_c i_{1,5} \]

\[ + \frac{t^3}{6} \alpha_3 + \frac{t^4}{24} \alpha_3 + \frac{t^5}{120} \left( \beta_3 - \alpha_3 - \frac{1}{3} \alpha_4 - \frac{1}{6} \beta_4 \right) + \frac{1}{6} \Theta a_c i_{1,1} - \Theta a_c i_{1,3} + t \alpha_4 \right) + \frac{t^5}{120} \alpha_4 + \frac{t^6}{720} \left( \beta_4 - \alpha_4 - \Theta a_c i_{1,1} \right) + \Theta a_i p_{i,7}(t), \]

\[ u^{(2)}(t) = a_1 + t \left( \beta_1 - \alpha_1 - \frac{1}{3} \alpha_2 - \frac{1}{6} \beta_2 + \frac{1}{45} \alpha_3 + \frac{7}{360} \beta_3 - \frac{2}{945} \alpha_4 - \frac{31}{15120} \beta_4 + \frac{31}{15120} \Theta a_c i_{1,1} \right) - \frac{7}{360} \Theta a_c i_{1,3} + \frac{1}{6} \Theta a_c i_{1,5} - \Theta a_c i_{1,7} + t \alpha_2 + \frac{t^2}{2} \alpha_2 + \frac{t^3}{6} \left( \beta_2 - \alpha_2 - \frac{1}{3} \alpha_3 - \frac{1}{6} \beta_3 + \frac{1}{45} \alpha_4 \right) + \frac{7}{360} \beta_4 - \frac{7}{360} \Theta a_c i_{1,1} + \frac{1}{6} \Theta a_c i_{1,3} - \Theta a_c i_{1,5} \]

\[ + \frac{t^4}{24} \alpha_3 + \frac{t^5}{120} \left( \beta_3 - \alpha_3 - \frac{1}{3} \alpha_4 - \frac{1}{6} \beta_4 \right) + \frac{1}{6} \Theta a_c i_{1,1} - \Theta a_c i_{1,3} + t \alpha_4 \right) + \frac{t^5}{720} \alpha_4 + \frac{t^6}{5040} \left( \beta_4 - \alpha_4 - \Theta a_c i_{1,1} \right) + \Theta a_i p_{i,8}(t), \]

\[ u^{(1)}(t) = \left( \beta_0 - \alpha_0 - \frac{1}{3} \alpha_1 - \frac{1}{6} \beta_1 + \frac{1}{45} \alpha_2 + \frac{1}{360} \beta_2 - \frac{2}{945} \alpha_3 - \frac{31}{15120} \beta_3 + \frac{31}{15120} \Theta a_c i_{1,1} \right) - \frac{127}{604800} \Theta a_c i_{1,1} + \frac{31}{15120} \Theta a_c i_{1,3} - \frac{7}{360} \sum_{i=1}^{N} a_i c_i 5 + \frac{1}{6} \sum_{i=1}^{N} a_i c_i 7 - \Theta a_c i_{1,9} + t \alpha_1 \right) + \frac{t^2}{2} \left( \beta_1 - \alpha_1 - \frac{1}{3} \alpha_2 - \frac{1}{6} \beta_2 + \frac{1}{45} \alpha_3 + \frac{7}{360} \beta_3 - \frac{2}{945} \alpha_4 - \frac{31}{15120} \beta_4 + \frac{31}{15120} \Theta a_c i_{1,1} \right) - \frac{7}{360} \Theta a_c i_{1,3} + \frac{1}{6} \Theta a_c i_{1,5} - \Theta a_c i_{1,7} \right) + \frac{t^3}{6} \alpha_2 + \frac{t^4}{24} \left( \beta_2 - \alpha_2 - \frac{1}{3} \alpha_3 - \frac{1}{6} \beta_3 + \frac{1}{45} \alpha_4 \right) + \frac{7}{360} \beta_4 - \frac{7}{360} \Theta a_c i_{1,1} + \frac{1}{6} \Theta a_c i_{1,3} - \Theta a_c i_{1,5} \right) + \frac{t^5}{120} \alpha_3 + \frac{t^6}{720} \left( \beta_3 - \alpha_3 - \frac{1}{3} \alpha_4 \right) + \frac{1}{6} \beta_4 + \frac{1}{6} \Theta a_c i_{1,1} - \Theta a_c i_{1,3} + t \alpha_4 \right) + \frac{t^7}{5040} \alpha_4 + \frac{t^8}{40320} \left( \beta_4 - \alpha_4 - \Theta a_c i_{1,1} \right) + \Theta a_i p_{i,9}(t), \]
and

\[ u(t) = a_0 + t \left( \beta_0 - a_0 - \frac{1}{3} \alpha_1 - \frac{1}{6} \beta_1 + \frac{1}{45} \alpha_2 + \frac{7}{360} \beta_2 - \frac{2}{945} \alpha_3 - \frac{31}{15120} \beta_3 + \frac{1}{4725} \alpha_4 + \frac{127}{604800} \beta_4 \right. \]

\begin{align*}
&\left. - \frac{127}{604800} \Theta a_{c_1,j} + \frac{31}{15120} \Theta a_{c_3,j} - \frac{7}{360} \Theta a_{c_1,i} + \frac{1}{6} \Theta a_{c_3,i} - \Theta a_{c_1,i} + \frac{1}{6} \Theta a_{c_3,i} - \Theta a_{c_1,j} + \frac{1}{2} \alpha_1 \right) + t^3 \beta_1 \\
&+ \frac{t^3}{6} \left( \beta_1 - a_1 - \frac{1}{3} \alpha_2 - \frac{1}{6} \beta_2 + \frac{7}{360} \beta_3 - \frac{2}{945} \alpha_4 - \frac{31}{15120} \beta_4 + \frac{31}{15120} \Theta a_{c_1,i} \\
&- \frac{7}{360} \Theta a_{c_1,i} + \frac{1}{6} \Theta a_{c_3,i} - \Theta a_{c_1,j} \right) + t^4 \alpha_2 + \frac{t^5}{120} \left( \beta_2 - a_2 - \frac{1}{3} \alpha_3 - \frac{1}{6} \beta_3 + \frac{1}{45} \alpha_4 \right) + \frac{7}{360} \beta_4 \\
&+ \frac{7}{360} \Theta a_{c_1,i} + \frac{1}{6} \Theta a_{c_3,i} - \Theta a_{c_1,j} \right) + t^6 \beta_3 - \frac{1}{3} \alpha_4 \right) + \frac{1}{6} \Theta a_{c_1,i} - \Theta a_{c_3,i} + \frac{1}{6} \Theta a_{c_3,i} - \Theta a_{c_1,j} \right) + t^8 \alpha_4 + \frac{t^9}{362880} \left( \beta_4 - a_4 - \Theta a_{c_1,i} \right) \\
&+ \Theta a_{p_{1,0}}(t). \end{align*}

3.1. Linear Case

For a linear case substituting the values from Equations (17) and (38)-(47) to Equation (2) and the discrete CPs from Equation (16), simplification, we obtain:

\[ \Theta a_i \left( h_i(t) + \left( -a_i(t) - t_i a_2(t) - \frac{1}{6} a_3(t) \right) - \frac{t^2}{2} a_3(t) + \frac{t^3}{6} a_4(t) - \frac{7}{360} a_5(t) + \frac{t^2}{12} a_6(t) \right. \]

\begin{align*}
&\left. - \frac{t^4}{24} a_5(t) - \frac{7 t_i}{360} a_6(t) + \frac{t^3}{360} a_6(t) - \frac{t^5}{120} a_6(t) + \frac{31}{15120} \sigma_7(t) - \frac{7 t_i}{720} a_7(t) + \frac{t^4}{144} a_7(t) - \frac{t^6}{720} a_7(t) \right) \\
&+ \frac{31 t_i}{15120} a_8(t) - \frac{t^3}{2160} a_8(t) + \frac{t^5}{720} a_8(t) - \frac{t^7}{5040} a_8(t) - \frac{127}{604800} a_9(t) + \frac{31 t_i}{30240} a_9(t) - \frac{7 t_i}{8640} a_9(t) \\
&+ \frac{t^6}{4320} a_9(t) - \frac{t^8}{40320} a_9(t) - \frac{127 t_i}{604800} a_{10}(t) + \frac{31 t_i}{90720} a_{10}(t) - \frac{7 t_i}{30240} a_{10}(t) \\
&- \frac{t^8}{362880} a_{10}(t) \right) c_{1,1} \\
&+ \left( -a_5(t) - t_i a_6(t) + \frac{1}{6} a_5(t) \right) - \frac{t^2}{2} a_5(t) + \frac{t^3}{6} a_6(t) - \frac{7}{360} a_6(t) + \frac{t^2}{12} a_7(t) - \frac{t^4}{24} a_7(t) \right) \\
&- \frac{7 t_i}{360} a_8(t) + \frac{t^3}{360} a_8(t) - \frac{t^5}{120} a_8(t) + \frac{31}{15120} \sigma_9(t) - \frac{7 t_i}{720} a_9(t) + \frac{t^4}{144} a_9(t) - \frac{t^6}{720} a_9(t) \right) \\
&+ \frac{31 t_i}{15120} a_{10}(t) - \frac{t^3}{2160} a_{10}(t) + \frac{t^5}{720} a_{10}(t) - \frac{t^7}{5040} a_{10}(t) \right) c_{1,3} \\
&+ \left( -a_5(t) - t_i a_6(t) - \frac{1}{6} a_7(t) + \frac{t^2}{6} a_8(t) - \frac{7}{360} a_8(t) + \frac{t^2}{12} a_9(t) - \frac{t^4}{24} a_9(t) \right) \\
&- \frac{7 t_i}{360} a_{10}(t) + \frac{t^3}{360} a_{10}(t) - \frac{t^5}{1200} a_{10}(t) \right) c_{1,5} \\
&+ \left( -a_7(t) - t_i a_8(t) + \frac{1}{6} a_9(t) - \frac{7}{360} a_9(t) + \frac{t^2}{6} a_{10}(t) - \frac{t^4}{6} a_{10}(t) \right) c_{1,7} \\
&+ \left( -a_9(t) - t_i a_{10}(t) \right) c_{1,9} + a_1(t) + a_{1,1}(t) + a_2(t) + a_3(t) + a_4(t) + a_5(t) + a_{1,4}(t) + a_6(t) + a_7(t) + a_8(t) + a_9(t) \\
&+ a_{10}(t) + a_{10}(t) + a_{10}(t) + a_{10}(t) + a_{10}(t) + a_{10}(t) \right) c_{1,5}.
The Gauss elimination technique is utilized for the solution of this $N \times N$ linear system. The solution gives the values of unknown coefficients $a_i$'s. The solution at discrete CPs is obtained by using these coefficients $a_i$'s in Equation (47).
3.2. Nonlinear Case

For nonlinear tenth-order BVP substituting the values of \( u(t), u'(t), u''(t), u'''(t), u^{(4)}(t), u^{(5)}(t), u^{(6)}(t), u^{(7)}(t), u^{(8)}(t), u^{(9)}(t) \) and \( u^{(10)}(t) \) in (1) and the discrete CPs, thus simplification leads the following nonlinear system of equations:

\[
F(t, u, u'(t), u''(t), u'''(t), u^{(4)}(t), u^{(5)}(t), u^{(6)}(t), u^{(7)}(t), u^{(8)}(t), u^{(9)}(t)) =
\]

\[
\Theta a_i h_i(t_j) - f(t_j),
\]

\[
\alpha_0 + t_j \left( \beta_0 - \alpha_0 - \frac{1}{3} \alpha_1 - \frac{1}{6} \beta_1 + \frac{1}{45} \alpha_2 + \frac{7}{360} \beta_2 - \frac{2}{945} \alpha_3 - \frac{31}{15120} \beta_3 + \frac{1}{4725} \beta_4 + \frac{1}{604800} \beta_4 \right)
\]

\[
- \frac{127}{604800} \Theta a_i c_{i,1} + \frac{31}{15120} \Theta a_i c_{i,3} - \frac{7}{360} \Theta a_i c_{i,5} + \frac{1}{6} \Theta a_i c_{i,7} - \Theta a_i c_{i,9}
\]

\[
+ \frac{1}{6} \Theta a_i c_{i,1} - \Theta a_i c_{i,3} + t_j \alpha_4 \right) + \frac{t^5_j}{40320} \alpha_4 + \frac{t^9_j}{362880} \left( \beta_4 - \alpha_4 - \Theta a_i c_{i,1} + \Theta a_i p_{1,10}(t_j),
\]

\[
\left( \beta_0 - \alpha_0 - \frac{1}{3} \alpha_1 - \frac{1}{6} \beta_1 + \frac{1}{45} \alpha_2 + \frac{7}{360} \beta_2 - \frac{2}{945} \alpha_3 - \frac{31}{15120} \beta_3 + \frac{1}{4725} \beta_4 + \frac{1}{604800} \beta_4 \right)
\]

\[
- \frac{127}{604800} \Theta a_i c_{i,1} + \frac{31}{15120} \Theta a_i c_{i,3} - \frac{7}{360} \Theta a_i c_{i,5} + \frac{1}{6} \Theta a_i c_{i,7} - \Theta a_i c_{i,9}
\]

\[
+ \frac{1}{6} \Theta a_i c_{i,1} - \Theta a_i c_{i,3} + t_j \alpha_4 \right) + \frac{t^5_j}{40320} \alpha_4 + \frac{t^9_j}{362880} \left( \beta_4 - \alpha_4 - \Theta a_i c_{i,1} + \Theta a_i p_{1,10}(t_j),
\]

\[
\left( \beta_0 - \alpha_0 - \frac{1}{3} \alpha_1 - \frac{1}{6} \beta_1 + \frac{1}{45} \alpha_2 + \frac{7}{360} \beta_2 - \frac{2}{945} \alpha_3 - \frac{31}{15120} \beta_3 + \frac{1}{4725} \beta_4 + \frac{1}{604800} \beta_4 \right)
\]

\[
- \frac{127}{604800} \Theta a_i c_{i,1} + \frac{31}{15120} \Theta a_i c_{i,3} - \frac{7}{360} \Theta a_i c_{i,5} + \frac{1}{6} \Theta a_i c_{i,7} - \Theta a_i c_{i,9}
\]

\[
+ \frac{1}{6} \Theta a_i c_{i,1} - \Theta a_i c_{i,3} + t_j \alpha_4 \right) + \frac{t^5_j}{40320} \alpha_4 + \frac{t^9_j}{362880} \left( \beta_4 - \alpha_4 - \Theta a_i c_{i,1} + \Theta a_i p_{1,10}(t_j),
\]

\[
+ \Theta a_i p_{1,9}(t_j),
\]

with \( \Theta a_i, \Theta b_i, \Theta c_i, \Theta d_i, \Theta e_i, \Theta f_i, \Theta g_i, \Theta h_i, \Theta i_i, \Theta j_i \) being the parameters of the system.
\[ a_1 + t_j \left( \beta_1 - a_1 - \frac{1}{3} \beta_2 - \frac{1}{6} \beta_3 + \frac{1}{45} \alpha_3 + \frac{7}{360} \beta_3 - \frac{2}{945} \alpha_4 - \frac{31}{15120} \beta_4 + \frac{31}{15120} \Theta_{a_i c_{i,1}} \right) \]
\[ - \frac{7}{360} \Theta_{a_i c_{i,3}} + \frac{1}{6} \Theta_{a_i c_{i,5}} - \Theta_{a_i c_{i,7}} \right) + \frac{t^2_j}{2} \alpha_2 + \frac{t^3_j}{6} \left( \beta_2 - a_2 - \frac{1}{3} \beta_3 - \frac{1}{6} \beta_3 + \frac{1}{45} \alpha_4 \right) \]
\[ + \frac{7}{360} \beta_4 - \frac{7}{360} \Theta_{a_i c_{i,1}} + \frac{1}{6} \sum_{i=1}^{N} a_i c_{i,3} - \frac{N}{120} a_i c_{i,5} \right) + \frac{t^4_j}{24} \alpha_3 + \frac{t^5_j}{120} \left( \beta_3 - a_3 - \frac{1}{3} \beta_4 - \frac{1}{6} \beta_4 \right) \]
\[ + \frac{1}{6} \Theta_{a_i c_{i,1}} + \Theta_{a_i c_{i,3}} + t_j a_4 \right) + \frac{t^6_j}{6} \alpha_4 + \frac{t^7_j}{720} \left( \beta_4 - a_4 - \Theta_{a_i c_{i,1}} + \Theta_{a_i p_{1,8}}(t_j) \right), \]
\[ \left( \beta_1 - a_1 - \frac{1}{3} \beta_2 - \frac{1}{6} \beta_3 + \frac{1}{45} \alpha_3 + \frac{7}{360} \beta_3 - \frac{2}{945} \alpha_4 - \frac{31}{15120} \beta_4 + \frac{31}{15120} \Theta_{a_i c_{i,1}} \right) \]
\[ - \frac{7}{360} \Theta_{a_i c_{i,3}} + \frac{1}{6} \Theta_{a_i c_{i,5}} - \Theta_{a_i c_{i,7}} \right) + t_j a_2 + \frac{t^2_j}{2} \left( \beta_2 - a_2 - \frac{1}{3} \beta_3 - \frac{1}{6} \beta_3 + \frac{1}{45} \alpha_4 \right) \]
\[ + \frac{7}{360} \beta_4 - \frac{7}{360} \Theta_{a_i c_{i,1}} + \frac{1}{6} \Theta_{a_i c_{i,3}} - \Theta_{a_i c_{i,5}} \right) + \frac{t^3_j}{6} \alpha_3 + \frac{t^4_j}{24} \left( \beta_3 - a_3 - \frac{1}{3} \beta_4 - \frac{1}{6} \beta_4 \right) \]
\[ + \frac{1}{6} \Theta_{a_i c_{i,1}} - \Theta_{a_i c_{i,3}} + t_j a_4 \right) + \frac{t^5_j}{120} \alpha_4 + \frac{t^6_j}{720} \left( \beta_4 - a_4 - \Theta_{a_i c_{i,1}} + \Theta_{a_i p_{1,7}}(t_j) \right), \]
\[ a_2 + t_j \left( \beta_2 - a_2 - \frac{1}{3} \beta_3 - \frac{1}{6} \beta_3 + \frac{1}{45} \alpha_3 + \frac{7}{360} \beta_3 - \frac{2}{945} \alpha_4 - \frac{31}{15120} \beta_4 + \frac{31}{15120} \Theta_{a_i c_{i,1}} \right) \]
\[ + \frac{1}{6} \Theta_{a_i c_{i,3}} - \Theta_{a_i c_{i,5}} - \Theta_{a_i c_{i,7}} \right) + \sum_{i=1}^{N} a_i p_{1,6}(t_j), \]
\[ \left( \beta_2 - a_2 - \frac{1}{3} \beta_3 - \frac{1}{6} \beta_3 + \frac{1}{45} \alpha_3 + \frac{7}{360} \beta_4 - \frac{7}{360} \Theta_{a_i c_{i,1}} + \frac{1}{6} \Theta_{a_i c_{i,3}} - \Theta_{a_i c_{i,5}} \right) \]
\[ + t_j a_3 + \frac{t^2_j}{2} \left( \beta_3 - a_3 - \frac{1}{3} \beta_4 + \frac{1}{6} \Theta_{a_i c_{i,1}} - \Theta_{a_i c_{i,3}} + t_j a_4 \right) + \frac{t^3_j}{6} \alpha_4 \]
\[ + \frac{t^4_j}{24} \left( \beta_4 - a_4 - \Theta_{a_i c_{i,1}} + \Theta_{a_i p_{1,5}}(t_j) \right), \]
\[ a_3 + t_j \left( \beta_3 - a_3 - \frac{1}{3} \beta_4 + \frac{1}{6} \Theta_{a_i c_{i,1}} - \Theta_{a_i c_{i,3}} + t_j a_4 \right) + \frac{t^2_j}{2} \alpha_4 \]
\[ + \frac{t^3_j}{6} \left( \beta_4 - a_4 - \Theta_{a_i c_{i,1}} + \Theta_{a_i p_{1,4}}(t_j) \right), \]
\[ \frac{t^4_j}{24} \left( \beta_4 - a_4 - \Theta_{a_i c_{i,1}} \right) + \Theta_{a_i p_{1,3}}(t_j), \]
\[ a_4 + t_j \left( \beta_4 - a_4 - \Theta_{a_i c_{i,1}} + \Theta_{a_i p_{1,2}}(t_j) \right) + \beta a_4 - \Theta_{a_i c_{i,1}} + \Theta_{a_i p_{1,1}}(t_j). \]
The Jacobian is:

\[
\frac{\partial}{\partial a_n} (F_j(a_1, a_2, \ldots, a_N))
\]

\[
h_m(t_j) = \left(-a_1(t_j) - t_j a_2(t_j) + \frac{1}{6} a_3(t_j) - \frac{r_1^2}{2} a_3(t_j) + \frac{t_j^2}{6} a_3(t_j) - \frac{t_j^3}{360} a_5(t_j) + \frac{t_j^2}{12} a_7(t_j) - \frac{r_1^4}{25} a_5(t_j) + \frac{r_1^3}{360} a_6(t_j) - \frac{r_1^2}{120} a_6(t_j) + \frac{31}{15120} a_8(t_j) - \frac{r_1^3}{720} a_7(t_j) + \frac{t_j^4}{144} a_8(t_j) - \frac{r_1^5}{720} a_7(t_j) + \frac{311}{15120} a_9(t_j) - \frac{r_1^2}{127} a_8(t_j) + \frac{r_1^3}{5040} a_9(t_j) - \frac{127}{604800} a_8(t_j) + \frac{311^2}{50240} a_9(t_j) - \frac{71}{8640} a_9(t_j) + \frac{r_1^6}{4320} a_{10}(t_j) - \frac{r_1^3}{40320} a_{10}(t_j) - \frac{127}{90720} a_{10}(t_j) + \frac{311^3}{30240} a_{10}(t_j) - \frac{r_1^3}{362880} a_{10}(t_j) \right) e_{n,1}
\]

\[
- \left(-a_3(t_j) - t_j a_4(t_j) + \frac{1}{6} a_5(t_j) - \frac{r_1^2}{2} a_5(t_j) + \frac{t_j^2}{6} a_5(t_j) - \frac{t_j^3}{12} a_6(t_j) - \frac{r_1^4}{24} a_6(t_j) - \frac{r_1^2}{120} a_6(t_j) + \frac{311}{362880} a_9(t_j) - \frac{r_1^3}{720} a_8(t_j) + \frac{t_j^4}{144} a_9(t_j) - \frac{r_1^5}{720} a_8(t_j) + \frac{311^2}{30240} a_9(t_j) - \frac{71}{8640} a_9(t_j) + \frac{r_1^6}{4320} a_{10}(t_j) - \frac{r_1^3}{40320} a_{10}(t_j) - \frac{127}{90720} a_{10}(t_j) + \frac{311^3}{30240} a_{10}(t_j) - \frac{r_1^3}{362880} a_{10}(t_j) \right) e_{n,2}
\]

Broyden’s technique is utilized for the solution of this \( N \times N \) nonlinear system of algebraic equations. The solution gives the values of unknown Haar coefficients \( a_i \)'s. The approximate solution at discrete CPs can be easily obtained using these unknown coefficients \( a_i \)'s in Equation (47).

4. Numerical Examples

In this section, some examples are given to show the performance of the HWCM. Three linear and two nonlinear tenth order BVPs are tested by using the proposed HWCM. If \( u_{ap} \) represents the approximate and \( u_{ex} \) represents the exact solution at CPs and GPs, respectively, then maximum absolute errors \( E_{cp} \) and \( E_{gp} \) are defined as:

\[
E_{cp} = \max |u_{exc} - u_{apc}|,
\]

\[
E_{gp} = \max |u_{exg} - u_{apg}|.
\]

The root mean square root errors \( M_{cp} \) and \( M_{gp} \) at CPs and GPs are defined as:

\[
M_{cp} = \sqrt{\frac{1}{N} \left( \Theta |u_{exc} - u_{apc}|^2 \right)}
\]

\[
M_{gp} = \sqrt{\frac{1}{N} \left( \Theta |u_{exg} - u_{apg}|^2 \right)}.
\]
The convergence rate at CPs is denoted by $R_{cp}$ and is:

$$R_{cp} = \frac{\log[\frac{u_{apc}(N/2)}{u_{apc}(N)}]}{\log 2}.$$  \hfill (50)

**Problem 1.** Consider the following tenth order BVP [25]

$$
\begin{align*}
&u^{(10)}(t) = -(80 + 19t + t^2)e^t, \quad t \in [0, 1], \\
&u(0) = 0, \; u(1) = 0, \; u^{(2)}(0) = 0, \; u^{(2)}(1) = -4e, \; u^{(4)}(0) = -8, \\
&u^{(4)}(1) = -16e, \; u^{(6)}(0) = -24, \; u^{(6)}(1) = -36e, \; u^{(8)}(0) = -48, \; u^{(8)}(1) = -64e.
\end{align*}
$$  \hfill (51)

The analytical solution is

$$u(t) = t(1-t)e^t.$$  \hfill (52)

The maximum absolute values of errors of cubic spline techniques [1] are $3.328 \times 10^{-7}$, reproducing kernel Hilbert space technique [25] is $2.682 \times 10^{-8}$, while the results of our method is decreased up to order $10^{-14}$.

From Table 1, it is observed that HWCM results are better than other methods.

The comparison of approximate and exact solution is given in Figure 1.

![Figure 1. Comparison of numerical and analytical solution for 32 CPs of Problem 1.](image)

| $J$ | $N = 2J+1$ | $E_{cp}$ | $R_{cp}$ | $E_{gp}$ | $M_{cp}$ | $M_{gp}$ |
|-----|-------------|----------|----------|----------|----------|----------|
| 1   | 4           | $4.278027 \times 10^{-7}$ | —        | $8.066078 \times 10^{-10}$ | $2.141112 \times 10^{-7}$ | $4.034141 \times 10^{-10}$ |
| 2   | 8           | $2.088323 \times 10^{-7}$ | 0.0346   | $4.199775 \times 10^{-10}$ | $7.664428 \times 10^{-8}$ | $1.493509 \times 10^{-10}$ |
| 3   | 16          | $7.091844 \times 10^{-8}$ | 1.5581   | $1.39787 \times 10^{-10}$ | $2.107698 \times 10^{-9}$ | $4.023501 \times 10^{-11}$ |
| 4   | 32          | $2.053994 \times 10^{-8}$ | 1.7877   | $3.954403 \times 10^{-11}$ | $5.378154 \times 10^{-9}$ | $1.020752 \times 10^{-11}$ |
| 5   | 64          | $5.519342 \times 10^{-9}$ | 1.8959   | $1.052874 \times 10^{-10}$ | $1.352397 \times 10^{-9}$ | $2.559945 \times 10^{-12}$ |
| 6   | 128         | $1.430068 \times 10^{-9}$ | 1.9484   | $2.714107 \times 10^{-12}$ | $3.85924 \times 10^{-10}$ | $6.404723 \times 10^{-13}$ |
| 7   | 256         | $3.639369 \times 10^{-10}$ | 1.9743   | $6.888379 \times 10^{-13}$ | $8.467896 \times 10^{-11}$ | $1.601455 \times 10^{-13}$ |
| 8   | 512         | $9.179565 \times 10^{-11}$ | 1.9872   | $1.735279 \times 10^{-13}$ | $2.117167 \times 10^{-11}$ | $4.003981 \times 10^{-14}$ |
| 9   | 1024        | $2.305088 \times 10^{-11}$ | 1.9936   | $4.363176 \times 10^{-14}$ | $5.293039 \times 10^{-12}$ | $1.001098 \times 10^{-14}$ |
Problem 2. Next, we have tenth order BVP \cite{26}
\[
\begin{align*}
&u^{(10)}(t) = -e^t \left( 89 + 21t + t^2 - t^3 \right) + tu(t), \quad -1 \leq t \leq 1, \\
&u(-1) = u(1) = 0, \quad u^{(1)}(-1) = \frac{2}{7}, \quad u^{(1)}(1) = -2e, \quad u^{(2)}(-1) = \frac{2}{7}, \\
&u^{(2)}(1) = -2e, \quad u^{(3)}(-1) = 0, \quad u^{(3)}(1) = -12e, \quad u^{(4)}(-1) = \frac{4}{7}, \quad u^{(4)}(1) = -20e.
\end{align*}
\]

The analytical solution is
\[
u(t) = \left(1 - t^2\right)e^t. \tag{54}
\]

The comparison of approximate and exact solution is given in Figure 2.

The maximum absolute errors are shown in Table 2. It is observed that the errors in absolute values are better than those of the homotopy analysis technique \cite{26}, Geng and Li \cite{15}, Siddiqi et al. \cite{16}, Siddiqi and Twizell \cite{17}, Siddiqi and Akram \cite{18}, Lamnii et al. \cite{27}, and Farajeyan and Maleki \cite{28}, as shown in Table 3.

| J | N = 2^{J+1} | E_{cp} | R_{cp} | E_{gp} | M_{cp} | M_{gp} |
|---|------------|-------|-------|-------|-------|-------|
| 1 | 4          | 4.706274 × 10^{-7} | ——   | 8.873970 × 10^{-10} | 2.355446 × 10^{-7} | 4.438198 × 10^{-10} |
| 2 | 8          | 2.297826 × 10^{-7} | 1.0343 | 4.534035 × 10^{-10} | 8.433378 × 10^{-8} | 1.643689 × 10^{-10} |
| 3 | 16         | 7.803594 × 10^{-8} | 1.5581 | 1.524194 × 10^{-10} | 2.312742 × 10^{-8} | 4.430672 × 10^{-11} |
| 4 | 32         | 2.260144 × 10^{-8} | 1.7877 | 4.352585 × 10^{-10} | 5.918073 × 10^{-9} | 1.123549 × 10^{-11} |
| 5 | 64         | 6.073282 × 10^{-9} | 1.8959 | 1.1558917 × 10^{-11} | 1.488169 × 10^{-9} | 2.817759 × 10^{-11} |
| 6 | 128        | 1.573592 × 10^{-9} | 1.9484 | 2.987166 × 10^{-12} | 3.725854 × 10^{-10} | 7.049689 × 10^{-13} |
| 7 | 256        | 4.040617 × 10^{-10} | 1.9743 | 7.580603 × 10^{-13} | 9.318073 × 10^{-11} | 1.762671 × 10^{-13} |
| 8 | 512        | 1.010802 × 10^{-10} | 1.9872 | 1.907363 × 10^{-13} | 2.329721 × 10^{-11} | 4.406964 × 10^{-14} |
| 9 | 1024       | 2.536421 × 10^{-11} | 1.9936 | 4.818368 × 10^{-14} | 5.824435 × 10^{-12} | 1.103406 × 10^{-14} |
Problem 3. Consider the linear tenth order BVP [29]

\[
\begin{aligned}
\begin{cases}
  \frac{d^{10}u}{dt^{10}} + u(t) = -8e^t + u^{(2)}(t), & t \in [0,1], \\
  u(0) = 1, & u(1) = 0, \quad u(0)(0) = 0, \quad u^{(1)}(1) = -e, \quad u^{(2)}(0) = -1, \\
  u^{(1)}(1) = -2e, & u^{(3)}(0) = -2, \quad u^{(3)}(1) = -3e, \quad u^{(4)}(0) = -3, \quad u^{(4)}(1) = -4e.
\end{cases}
\end{aligned}
\]

The exact solution is

\[ u(t) = (1 - t) e^t. \]

The numerical results for different number of CPs and GPs are given in Table 4. The R_{cp} convergence rate is also calculated which is nearly equal to 2. The comparison of approximate of and exact solution is given in Figure 3.

| J | N = 2^{J+1} | E_{cp} | R_{cp} | E_{gp} | M_{cp} | M_{gp} |
|---|-------------|--------|--------|--------|--------|--------|
| 1 | 4           | 4.282469 × 10^{-8} | ——      | 8.078904 × 10^{-11} | 2.143337 × 10^{-8} | 4.040556 × 10^{-11} |
| 2 | 8           | 2.095035 × 10^{-8} | 1.0315  | 4.142597 × 10^{-11} | 7.689499 × 10^{-9}  | 1.501796 × 10^{-11} |
| 3 | 16          | 7.117493 × 10^{-9} | 1.5975  | 1.394107 × 10^{-11} | 2.109728 × 10^{-9}  | 4.052754 × 10^{-12} |
| 4 | 32          | 2.061499 × 10^{-9} | 1.7877  | 3.982148 × 10^{-12} | 5.399189 × 10^{-10} | 1.028014 × 10^{-12} |
| 5 | 64          | 5.539396 × 10^{-10} | 1.8959  | 1.060374 × 10^{-12} | 1.357729 × 10^{-10} | 2.578336 × 10^{-13} |
| 6 | 128         | 1.435236 × 10^{-10} | 1.9484  | 2.735589 × 10^{-13} | 3.399299 × 10^{-11} | 6.450834 × 10^{-14} |
| 7 | 256         | 3.652479 × 10^{-11} | 1.9743  | 6.938894 × 10^{-14} | 8.501362 × 10^{-12} | 1.611802 × 10^{-14} |
| 8 | 512         | 9.212566 × 10^{-12} | 1.9872  | 1.743050 × 10^{-14} | 2.125535 × 10^{-12} | 4.026448 × 10^{-15} |
| 9 | 1024        | 2.513387 × 10^{-12} | 1.9936  | 4.440892 × 10^{-15} | 5.313967 × 10^{-13} | 1.017865 × 10^{-15} |

![Figure 3. Comparison of numerical and analytical solutions for 32 CPs of Problem 3.](image-url)
Problem 4. Consider the nonlinear tenth order BVP [29]:

\[
\begin{align*}
  u^{(10)}(t) &= e^{-t}u^2(t), \quad t \in [0, 1], \\
  u(0) &= u^{(2)}(0) = u^{(4)}(0) = u^{(6)}(0) = u^{(8)}(0) = 1, \\
  u(1) &= u^{(2)}(1) = u^{(4)}(1) = u^{(6)}(1) = u^{(8)}(1) = e.
\end{align*}
\] (57)

The analytical solution is

\[ u(t) = e^t. \] (58)

The comparison of exact solutions with an approximate solution obtained using HWCM at CPs is shown in Figure 4. Table 5 shows the values of \(E_{cp}, R_{cp}, E_{gp}, M_{cp}\), and \(M_{gp}\) for different numbers of CPs and GPs.

### Table 5. \(E_{cp}, R_{cp}, E_{gp}, M_{cp}\) and \(M_{gp}\) for Problem 4.

| \(J\) | \(N = 2^J+1\) | \(E_{cp}\) | \(R_{cp}\) | \(E_{gp}\) | \(M_{cp}\) | \(M_{gp}\) |
|------|-------------|---------|--------|--------|--------|--------|
| 1    | 4           | 4.248266 \times 10^{-9} | 0.018031 \times 10^{-12} | 1.216221 \times 10^{-9} | 4.010113 \times 10^{-12} |
| 2    | 8           | 2.082389 \times 10^{-9} | 0.258603 \times 10^{-12} | 7.643666 \times 10^{-10} | 1.495812 \times 10^{-12} |
| 3    | 16          | 7.077143 \times 10^{-10} | 0.899999 \times 10^{-13} | 2.098077 \times 10^{-9} | 4.041318 \times 10^{-13} |
| 4    | 32          | 2.049893 \times 10^{-10} | 0.971225 \times 10^{-13} | 7.649992 \times 10^{-11} | 1.025474 \times 10^{-13} |
| 5    | 64          | 5.508127 \times 10^{-11} | 1.04712 \times 10^{-13} | 1.350423 \times 10^{-10} | 2.570024 \times 10^{-14} |
| 6    | 128         | 1.427125 \times 10^{-11} | 1.054712 \times 10^{-13} | 3.81028 \times 10^{-12} | 6.443958 \times 10^{-15} |
| 7    | 256         | 3.631761 \times 10^{-12} | 1.9744 \times 10^{-15} | 7.105427 \times 10^{-13} | 1.645147 \times 10^{-15} |

Problem 5. Consider the nonlinear tenth order BVP [26]:

\[
\begin{align*}
  u^{(10)}(t) &= \frac{14175}{4} (t + u(t) + 1)^{11}, \quad t \in [0, 1], \\
  u(0) &= 1, \quad u(1) = 0, \quad u^{(1)}(0) = -\frac{1}{2}, \quad u^{(2)}(0) = \frac{1}{2}, \quad u^{(3)}(0) = \frac{3}{4}, \\
  u^{(4)}(0) &= \frac{3}{2}, \quad u^{(5)}(1) = 1, \quad u^{(6)}(1) = 4, \quad u^{(7)}(1) = 12, \quad u^{(8)}(1) = 48.
\end{align*}
\] (59)

The analytical solution is

\[ u(t) = \frac{2}{2 - t} - t - 1. \] (60)
The maximum absolute error, root mean square error, and experimental rates of convergence for \( N = 4, 8, 16, \ldots, 256 \) numbers of CPs for this example have been calculated, as shown in Table 6. Both the maximum absolute error and root mean square error are decreased by taking more CPs and GPs. The comparison of exact solution with approximate solution at the 32 CPs are shown in Figure 5.

![Figure 5. Comparison of numerical and analytical solutions for 32 CPs of Problem 5.](image)

**Table 6.** \( E_{cp}, R_{cp}, E_{gp}, M_{cp} \) and \( M_{gp} \) for Problem 5.

| \( J \) | \( N = 2^{J+1} \) | \( E_{cp} \) | \( R_{cp} \) | \( E_{gp} \) | \( M_{cp} \) | \( M_{gp} \) |
|---|---|---|---|---|---|---|
| 1 | 4 | 1.376894 \times 10^{-4} | --- | 2.335245 \times 10^{-7} | 6.890091 \times 10^{-5} | 1.167937 \times 10^{-7} |
| 2 | 8 | 6.439662 \times 10^{-5} | 1.0964 | 1.004889 \times 10^{-7} | 2.349402 \times 10^{-5} | 9.364978 \times 10^{-9} |
| 3 | 16 | 2.199108 \times 10^{-5} | 1.5501 | 3.235740 \times 10^{-8} | 6.401687 \times 10^{-6} |
| 4 | 32 | 6.424091 \times 10^{-6} | 1.7754 | 9.156188 \times 10^{-9} | 1.636086 \times 10^{-6} | 2.346317 \times 10^{-9} |
| 5 | 64 | 1.734789 \times 10^{-6} | 1.8887 | 2.434131 \times 10^{-9} | 4.110933 \times 10^{-7} | 5.866305 \times 10^{-10} |
| 6 | 128 | 4.493562 \times 10^{-7} | 1.9488 | 6.274573 \times 10^{-10} | 1.026679 \times 10^{-7} | 1.466562 \times 10^{-10} |
| 7 | 256 | 1.129015 \times 10^{-7} | 1.9928 | 1.592808 \times 10^{-10} | 2.546429 \times 10^{-8} | 3.666386 \times 10^{-11} |

5. Conclusions

In this paper, the HWCM is utilized to find the solution of linear and nonlinear tenth order BVPs. The maximum absolute errors for distant number of discrete CPs and GPs are shown for each example in tables. The convergence rate is also calculated, which is approximately equal to two. The maximum absolute errors of the present method are compared with the homotopy analysis technique [26], Geng and Li [15], Siddiqi et al. [16], Siddiqi and Twizell [17], Siddiqi and Akram [18], Lamnii et al. [27], and Farajeyan and Maleki [28]. The results show that the HWCM is better than other techniques available in the literature. MATLAB software is used for all computational work. Thus, as to get more precision, there is a need for more CPs, which is the main disadvantage to this procedure because of the choice of more CP results expanding the computational cost due to the inversion of the \( N \times N \) matrix. Our future work addresses overcoming this limitation of this study. In addition, we plan to apply this numerical method for the solution of high order BVPs, two-dimensional, and three-dimensional BVPs.

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