The $S$-global dimensions of commutative rings

Xiaolei Zhang$^a$, Wei Qi$^b$
Department of Basic Courses, Chengdu Aeronautic Polytechnic, Chengdu 610100, China
b. School of Mathematical Sciences, Sichuan Normal University, Chengdu 610068, China
E-mail: zxlrghj@163.com

Abstract

Let $R$ be a commutative ring with identity and $S$ a multiplicative subset of $R$. First, we introduce and study the $S$-projective dimensions and $S$-injective dimensions of $R$-modules, and then explore the $S$-global dimension $S$-gl.dim($R$) of a commutative ring $R$ which is defined to be the supremum of $S$-projective dimensions of all $R$-modules. Finally, we investigated the $S$-global dimension of factor rings and polynomial rings.

Key Words: $S$-projective dimensions, $S$-injective dimensions, $S$-global dimensions, polynomial rings.

2010 Mathematics Subject Classification: 13D05, 13D07.

Throughout this article, $R$ always is a commutative ring with identity and $S$ always is a multiplicative subset of $R$, that is, $1 \in S$ and $s_1 s_2 \in S$ for any $s_1 \in S, s_2 \in S$. In 2002, Anderson and Dumitrescu [1] defined $S$-Noetherian rings $R$ for which any ideal of $R$ is $S$-finite. Recall from [1] that an $R$-module $M$ is called $S$-finite provided that $sM \subseteq F$ for some $s \in S$ and some finitely generated submodule $F$ of $M$. An $R$-module $T$ is called uniformly $S$-torsion if $sT = 0$ for some $s \in S$ (see [12]). So an $R$-module $M$ is $S$-finite if and only if $M/F$ is uniformly $S$-torsion for some finitely generated submodule $F$ of $M$. The idea derived from uniformly $S$-torsion modules is deserved to be further investigated.

In [14], the author of this paper introduced the class of $S$-projective modules $P$ for which the functor $\text{Hom}_R(P, -)$ preserves $S$-exact sequences. The class of $S$-projective modules can be seen as a “uniform” generalization of that of projective modules, since an $R$-module $P$ is $S$-projective if and only if $\text{Ext}_R^1(P, M)$ is uniformly $S$-torsion for any $R$-module $M$ (see [14, Theorem 2.5]). The class of $S$-projective modules owns the following $S$-hereditary property: let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an $S$-exact sequence, if $B$ and $C$ are $S$-projective so is $A$ (see [14, Proposition 2.8]). So it is worth to study the $S$-analogue of projective dimensions of $R$-modules.
Similarly, By the discussion of $S$-injective modules in [9], we can study the $S$-analogue of injective dimensions of $R$-modules. Together these, an $S$-analogue of global dimensions of commutative rings can also be introduced and studied.

In this article, we define the $S$-projective dimension $S\text{-}pd_R(M)$ (resp., $S$-injective dimension $S\text{-}id_R(M)$) of an $R$-module $M$ to be the length of the shortest $S$-projective (resp., $S$-injective) $S$-resolution of $M$. We characterize $S$-projective dimensions (resp., $S$-injective) of $R$-modules using the uniform torsion property of the "Ext" functors in Proposition 2.4 (resp., Proposition 2.5). Besides, we obtain local characterizations of projective dimensions and injective dimensions of $R$-modules in Corollary 3.3. The $S$-global dimension $S\text{-}gl.dim(R)$ of a commutative ring $R$ is defined to be the supremum of $S$-projective dimensions of all $R$-modules. We find that $S$-global dimensions of commutative rings is also the supremum of $S$-injective dimensions of all $R$-modules. A new characterization of global dimensions is given in Corollary 3.3. $S$-semisimple rings are firstly introduced in [14] for which any free $R$-module is $S$-semisimple. By [12, Theorem 3.11], a ring $R$ is $S$-semisimple if and only if all $R$-modules are $S$-projective (resp., $S$-injective). So $S$-semisimple are exactly commutative rings with $S$-global dimension equal to 0 (see Corollary 3.4). In the final section, we investigate the $S$-global dimensions of factor rings and polynomial rings and show that $S\text{-}gl.dim(R[x]) = S\text{-}gl.dim(R) + 1$ (see Theorem 4.6).

1. Preliminaries

Recall from [12], an $R$-module $T$ is called a uniformly $S$-torsion module provided that there exists an element $s \in S$ such that $sT = 0$. An $R$-sequence $M \xrightarrow{f} N \xrightarrow{g} L$ is called $S$-exact (at $N$) provided that there is an element $s \in S$ such that $s\text{Ker}(g) \subseteq \text{Im}(f)$ and $s\text{Im}(f) \subseteq \text{Ker}(g)$. We say a long $R$-sequence $\ldots \to A_{n-1} \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \to \ldots$ is $S$-exact, if for any $n$ there is an element $s \in S$ such that $s\text{Ker}(f_{n+1}) \subseteq \text{Im}(f_n)$ and $s\text{Im}(f_n) \subseteq \text{Ker}(f_{n+1})$. An $S$-exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is called a short $S$-exact sequence. Let $\xi : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an $S$-short exact sequence. Then $\xi$ is said to be $S$-split provided that there is $s \in S$ and $R$-homomorphism $f' : B \to A$ such that $f'(f(a)) = sa$ for any $a \in A$, that is, $f' \circ f = s\text{Id}_A$ (see [14, Definition 2.1]).

An $R$-homomorphism $f : M \to N$ is an $S$-monomorphism (resp., $S$-epimorphism, $S$-isomorphism) provided $0 \to M \xrightarrow{f} N$ (resp., $M \xrightarrow{f} N \to 0$, $0 \to M \xrightarrow{f} N$ (resp., $M \xrightarrow{f} N \to 0$) is $S$-exact. It is easy to verify an $R$-homomorphism $f : M \to N$ is an $S$-monomorphism (resp., $S$-epimorphism, $S$-isomorphism) if and only if $\text{Ker}(f)$ (resp., $\text{Coker}(f)$, both $\text{Ker}(f)$ and $\text{Coker}(f)$) is a uniformly $S$-torsion module. Let $R$ be a ring and $S$ a multiplicative subset of $R$. Suppose $M$ and $N$ are $R$-modules. We say $M$ is
Lemma 1.1. \cite[Proposition 1.1]{13} Let $R$ be a ring and $S$ a multiplicative subset of $R$. Suppose there is an $S$-isomorphism $f : M \to N$ for $R$-modules $M$ and $N$. Then there is an $S$-isomorphism $g : N \to M$. Moreover, there is $s \in S$ such that $f \circ g = s\text{Id}_N$ and $g \circ f = s\text{Id}_M$.

The following result says that a short $S$-exact sequence induces a long $S$-exact sequence by the functor ‘$\text{Ext}$’ as the classical case.

Lemma 1.2. Let $R$ be a ring and $S$ a multiplicative subset of $R$. Let $L$, $M$ and $N$ be $R$-modules. If $f : M \to N$ is an $S$-isomorphism, then $\text{Ext}^n_R(L, f) : \text{Ext}^n_R(L, M) \to \text{Ext}^n_R(L, N)$ and $\text{Ext}^n_R(f, L) : \text{Ext}^n_R(N, L) \to \text{Ext}^n_R(M, L)$ are all $S$-isomorphisms for any $n \geq 0$.

Proof. We only show $\text{Ext}^n_R(L, f) : \text{Ext}^n_R(L, M) \to \text{Ext}^n_R(L, N)$ is an $S$-isomorphism for any $n \geq 0$ since the other one is similar. Consider the exact sequences: $0 \to \text{Ker}(f) \to M \xrightarrow{\pi_{\text{im}(f)}} \text{Im}(f) \to 0$ and $0 \to \text{Im}(f) \xrightarrow{i_{\text{im}(f)}} N \to \text{Coker}(f) \to 0$ with $\text{Ker}(f)$ and $\text{Coker}(f)$ uniformly $S$-torsion. Then there are long exact sequences

\[
\text{Ext}^n_R(L, \text{Ker}(f)) \to \text{Ext}^n_R(L, M) \xrightarrow{\text{Ext}^n_R(L, \pi_{\text{im}(f)})} \text{Ext}^n_R(L, \text{Im}(f)) \to \text{Ext}^{n+1}_R(L, \text{Ker}(f))
\]

and

\[
\text{Ext}^{n-1}_R(L, \text{Coker}(f)) \to \text{Ext}^n_R(L, \text{Im}(f)) \xrightarrow{\text{Ext}^n_R(L, i_{\text{im}(f)})} \text{Ext}^n_R(L, N) \to \text{Ext}^n_R(L, \text{Coker}(f)).
\]

Since $\text{Ext}^n_R(L, \text{Ker}(f))$, $\text{Ext}^{n+1}_R(L, \text{Ker}(f))$, $\text{Ext}^{n-1}_R(L, \text{Coker}(f))$ and $\text{Ext}^n_R(L, \text{Coker}(f))$ are all uniformly $S$-torsion by \cite[Lemma 4.2]{9}, we have

\[
\text{Ext}^n_R(L, f) : \text{Ext}^n_R(L, M) \xrightarrow{\text{Ext}^n_R(L, \pi_{\text{im}(f)})} \text{Ext}^n_R(L, \text{Im}(f)) \xrightarrow{\text{Ext}^n_R(L, i_{\text{im}(f)})} \text{Ext}^n_R(L, N)
\]

is an $S$-isomorphism. \hfill \Box

Theorem 1.3. Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $M$ and $N$ $R$-modules. Suppose $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an $S$-exact sequence of $R$-modules. Then for any $n \geq 1$ there is an $R$-homomorphism $\delta_n : \text{Ext}^{n-1}_R(M, C) \to \text{Ext}^n_R(M, A)$ such that the induced sequences

\[
0 \to \text{Hom}_R(M, A) \to \text{Hom}_R(M, B) \to \text{Hom}_R(M, C) \to \text{Ext}^1_R(M, A) \to \cdots \to \text{Ext}^{n-1}_R(M, B) \to \text{Ext}^{n-1}_R(M, C) \xrightarrow{\delta_n} \text{Ext}^n_R(M, A) \to \text{Ext}^n_R(M, B) \to \cdots
\]

and
0 \to \text{Hom}_R(C, N) \to \text{Hom}_R(B, N) \to \text{Hom}_R(A, N) \to \text{Ext}^1_R(C, N) \to \cdots \to \\
\text{Ext}^{n-1}_R(B, N) \to \text{Ext}^{n-1}_R(A, N) \xrightarrow{\delta_n} \text{Ext}^n_R(C, N) \to \text{Ext}^n_R(B, N) \to \cdots

are \ S\text{-exact}.

Proof. We only show the first sequence is \( S \)-exact since the other one is similar. Since the sequence \( 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \) is \( S \)-exact at \( B \). There is an exact sequence \( 0 \to \text{Ker}(g) \xrightarrow{i_{\text{Ker}(g)}} B \xrightarrow{\pi_{\text{Im}(g)}} \text{Im}(g) \to 0. \) So there is a long exact sequence of \( R \)-modules:

\[
0 \to \text{Hom}_R(M, \text{Ker}(g)) \to \text{Hom}_R(M, B) \to \text{Hom}_R(M, \text{Im}(g)) \to \\
\text{Ext}^1_R(M, \text{Ker}(g)) \to \cdots \to \text{Ext}^{n-1}_R(M, B) \to \text{Ext}^{n-1}_R(M, \text{Im}(g)) \xrightarrow{\delta_n} \\
\text{Ext}^n_R(M, \text{Ker}(g)) \to \text{Ext}^n_R(M, B) \to \cdots
\]

Note that there are \( S \)-isomorphisms \( t_1 : A \to \text{Ker}(g), t'_1 : \text{Ker}(g) \to A, t_2 : \text{Im}(g) \to C \) and \( t'_2 : C \to \text{Im}(g) \) by Lemma \[1.1\]. So, by Lemma \[1.2\], \( \text{Ext}^n_R(M, t'_1) : \text{Ext}^n_R(M, \text{Ker}(g)) \to \text{Ext}^n_R(M, A) \) and \( \text{Ext}^n_R(M, t'_2) : \text{Ext}^n_R(M, C) \to \text{Ext}^n_R(M, \text{Im}(g)) \) are \( S \)-isomorphisms for any \( n \geq 0 \). Setting \( \delta_n = \text{Ext}^n_R(M, t'_1) \circ \delta_n' \circ \text{Ext}^n_R(M, t'_2) \), we have an \( S \)-exact sequence

\[
0 \to \text{Hom}_R(M, A) \to \text{Hom}_R(M, B) \to \text{Hom}_R(M, C) \to \text{Ext}^1_R(M, A) \to \cdots \to \\
\text{Ext}^{n-1}_R(M, B) \to \text{Ext}^{n-1}_R(M, C) \xrightarrow{\delta_n} \text{Ext}^n_R(M, A) \to \text{Ext}^n_R(M, B) \to \cdots
\]

\[\Box\]

Recall from \[14\], Definition 3.1] that an \( R \)-module \( P \) is called \( S \)-projective provided that the induced sequence

\[
0 \to \text{Hom}_R(P, A) \to \text{Hom}_R(P, B) \to \text{Hom}_R(P, C) \to 0
\]

is \( S \)-exact for any \( S \)-exact sequence \( 0 \to A \to B \to C \to 0 \). And recall from \[9\] Definition 4.1] that an \( R \)-module \( E \) is called \( S \)-injective provided that the induced sequence

\[
0 \to \text{Hom}_R(C, E) \to \text{Hom}_R(B, E) \to \text{Hom}_R(A, E) \to 0
\]

is \( S \)-exact for any \( S \)-exact sequence \( 0 \to A \to B \to C \to 0 \). Following from \[12\] Theorem 3.2], an \( R \)-module \( P \) is projective if and only if \( \text{Ext}^1_R(P, M) \) is uniformly \( S \)-torsion for any \( R \)-module \( M \), if and only if every \( S \)-short exact sequence \( 0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0 \) is \( S \)-split. Similarly, an \( R \)-module \( E \) is \( S \)-injective if and only if \( \text{Ext}^1_R(M, E) \) is uniformly \( S \)-torsion for any \( R \)-module \( M \), if and only every \( S \)-short exact sequence \( 0 \to E \xrightarrow{i} A \xrightarrow{\delta} B \to 0 \) is \( S \)-split by \[9\] Theorem 4.3] and \[14\] Proposition 2.3]. Following from Theorem \[1.3\] we have the following result.
Corollary 1.4. Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $M$ and $N$ $R$-modules. Suppose $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an $S$-exact sequence of $R$-modules.

1. If $B$ is $S$-projective, then $\text{Ext}^n_R(C, N)$ is $S$-isomorphic to $\text{Ext}^n_R(A, N)$ for any $n \geq 0$.
2. If $B$ is $S$-injective, then $\text{Ext}^n_R(M, A)$ is $S$-isomorphic to $\text{Ext}^n_R(M, C)$ for any $n \geq 0$.

2. On the $S$-Projective Dimensions and $S$-Injective Dimensions of Modules

In this section we mainly introduced the the $S$-versions of projective dimensions and injective dimensions of $R$-modules.

Definition 2.1. Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $M$ an $R$-module. We write $S$-$pd_R(M) \leq n$ ($S$-$pd$ abbreviates $S$-projective dimension) if there exists an $S$-exact sequence of $R$-modules

$$0 \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0$$

where each $F_i$ is $S$-projective for $i = 0, \ldots, n$. The $S$-exact sequence $(\diamond)$ is said to be an $S$-projective $S$-resolution of length $n$ of $M$. If such finite $S$-projective $S$-resolution does not exist, then we say $S$-$pd_R(M) = \infty$; otherwise, define $S$-$pd_R(M) = n$ if $n$ is the length of the shortest $S$-projective $S$-resolution of $M$.

Definition 2.2. Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $M$ an $R$-module. We write $S$-$id_R(M) \leq n$ ($S$-$id$ abbreviates $S$-injective dimension) if there exists an $S$-exact sequence of $R$-modules

$$0 \to M \to E_0 \to E_1 \to \ldots \to E_{n-1} \to E_n \to 0$$

where each $E_i$ is $S$-injective for $i = 0, \ldots, n$. The $S$-exact sequence $(\ast)$ is said to be an $S$-injective $S$-resolution of length $n$ of $M$. If such finite $S$-injective $S$-resolution does not exist, then we say $S$-$id_R(M) = \infty$; otherwise, define $S$-$id_R(M) = n$ if $n$ is the length of the shortest $S$-injective $S$-resolution of $M$.

Trivially, $S$-$pd_R(M) \leq pd_R(M)$ and $S$-$id_R(M) \leq id_R(M)$. And if $S$ is composed of units, then $S$-$pd_R(M) = pd_R(M)$. It is also obvious that an $R$-module $M$ is $S$-projective if and only if $S$-$pd_R(M) = 0$, and is $S$-injective if and only if $S$-$id_R(M) = 0$.

Lemma 2.3. Let $R$ be a ring, $S$ a multiplicative subset of $R$. If $A$ is $S$-isomorphic to $B$, then $S$-$pd_R(A) = S$-$pd_R(B)$ and $S$-$id_R(A) = S$-$id_R(B)$.
Proof. We only prove $S$-pd$_R(A) = S$-pd$_R(B)$ as the $S$-injective dimension is similar. Let $f : A \to B$ be an $S$-isomorphism. If $\ldots \to P_n \to \ldots \to P_1 \to P_0 \xrightarrow{f} A \to 0$ is an $S$-projective resolution of $A$, then $\ldots \to P_n \to \ldots \to P_1 \to P_0 \xrightarrow{f \circ g} B \to 0$ is an $S$-projective resolution of $B$. So $S$-pd$_R(A) \geq S$-pd$_R(B)$. Similarly we have $S$-pd$_R(B) \geq S$-pd$_R(A)$ by Proposition 1.1. □

Proposition 2.4. Let $R$ be a ring and $S$ a multiplicative subset of $R$. The following statements are equivalent for an $R$-module $M$:

1. $S$-pd$_R(M) \leq n$;
2. Ext$_R^{n+k}(M, N)$ is uniformly $S$-torsion for all $R$-modules $N$ and all $k > 0$;
3. Ext$_R^{n+k}(M, N)$ is uniformly $S$-torsion for all $R$-modules $N$;
4. if $0 \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0$ is an $S$-exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are $S$-projective $R$-modules, then $F_n$ is $S$-projective;
5. if $0 \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0$ is an $S$-exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are $S$-projective $R$-modules, then $F_n$ is $S$-projective;
6. if $0 \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0$ is an exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are $S$-projective $R$-modules, then $F_n$ is $S$-projective;
7. if $0 \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0$ is an exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are $S$-projective $R$-modules, then $F_n$ is $S$-projective;
8. there exists an $S$-exact sequence $0 \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0$, where $F_0, F_1, \ldots, F_{n-1}$ are $S$-projective $R$-modules and $F_n$ is $S$-projective;
9. there exists an exact sequence $0 \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0$, where $F_0, F_1, \ldots, F_{n-1}$ are $S$-projective $R$-modules and $F_n$ is $S$-projective;
10. there exists an exact sequence $0 \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0$, where $F_0, F_1, \ldots, F_n$ are $S$-projective $R$-modules.

Proof. (1) $\Rightarrow$ (2): We prove (2) by induction on $n$. For the case $n = 0$, we have $M$ is $S$-projective, then (2) holds by [14] Theorem 2.5. If $n > 0$, then there is an $S$-exact sequence $0 \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0$, where each $F_i$ is $S$-projective for $i = 0, \ldots, n$. Set $K_0 = \text{Ker}(F_0 \to M)$ and $L_0 = \text{Im}(F_1 \to F_0)$. Then both $0 \to K_0 \to F_0 \to M \to 0$ and $0 \to F_n \to F_{n-1} \to \ldots \to F_1 \to L_0 \to 0$ are $S$-exact. Since $S$-pd$_R(L_0) \leq n-1$ and $L_0$ is $S$-isomorphic to $K_0$, $S$-pd$_R(K_0) \leq n-1$ by Lemma [2.3]. By induction, Ext$_R^{n-1+k}(K_0, N)$ is uniformly $S$-torsion for all $R$-modules $N$ and all $k > 0$. It follows from Corollary [1.4] that Ext$_R^{n+k}(M, N)$ is uniformly $S$-torsion.

(2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (7) and (4) $\Rightarrow$ (6) $\Rightarrow$ (7): Trivial.

(3) $\Rightarrow$ (4): Let $0 \to F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} F_{n-2} \ldots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$ be an $S$-exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are $S$-projective. Then $F_n$ is $S$-projective if and only if Ext$_R(F_n, N)$ is uniformly $S$-torsion for all $R$-modules $N$, if and only if

...
\[ \text{Ext}^2_R(\text{Im}(d^{n-1}), N) \text{ is uniformly } S\text{-torsion for all } R\text{-modules } N. \] Iterating these steps, we can show \( F_n \) is \( S\)-projective if and only if \( \text{Ext}^{n+1}_R(M, N) \) is uniformly \( S\)-torsion for all \( R\)-modules \( N \).

(9) \( \Rightarrow \) (10) \( \Rightarrow \) (1) and (9) \( \Rightarrow \) (8) \( \Rightarrow \) (1): Trivial.

(7) \( \Rightarrow \) (9): Let ... \( P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0 \) be a projective resolution of \( M \). Set \( F_n = \text{Ker}(d^{n-1}) \). Then we have an exact sequence \( 0 \rightarrow F_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0 \). By (7), \( F_n \) is \( S\)-projective. So (9) holds.

\[ \square \]

Similarly, we have the following result.

**Proposition 2.5.** Let \( R \) be a ring and \( S \) a multiplicative subset of \( R \). The following statements are equivalent for an \( R\)-module \( M \):

1. \( S\text{-id}_R(M) \leq n; \)
2. \( \text{Ext}^{n+k}_R(N, M) \text{ is uniformly } S\text{-torsion for all } R\text{-modules } N \text{ and all } k > 0; \)
3. \( \text{Ext}^{n+1}_R(N, M) \text{ is uniformly } S\text{-torsion for all } R\text{-modules } N; \)
4. if \( 0 \rightarrow M \rightarrow E_0 \rightarrow \cdots \rightarrow E_{n-1} \rightarrow E_n \rightarrow 0 \) is an \( S\)-exact sequence, where \( E_0, E_1, \ldots, E_{n-1} \) are \( S\)-injective \( R\)-modules, then \( F_n \) is \( S\)-injective;
5. if \( 0 \rightarrow M \rightarrow E_0 \rightarrow \cdots \rightarrow E_{n-1} \rightarrow E_n \rightarrow 0 \) is an \( S\)-exact sequence, where \( E_0, E_1, \ldots, E_{n-1} \) are \( S\)-injective \( R\)-modules, then \( E_n \) is \( S\)-injective;
6. if \( 0 \rightarrow M \rightarrow E_0 \rightarrow \cdots \rightarrow E_{n-1} \rightarrow E_n \rightarrow 0 \) is an exact sequence, where \( E_0, E_1, \ldots, E_{n-1} \) are \( S\)-injective \( R\)-modules, then \( E_n \) is \( S\)-injective;
7. if \( 0 \rightarrow M \rightarrow E_0 \rightarrow \cdots \rightarrow E_{n-1} \rightarrow E_n \rightarrow 0 \) is an exact sequence, where \( E_0, E_1, \ldots, E_{n-1} \) are \( S\)-injective \( R\)-modules, then \( E_n \) is \( S\)-injective;
8. there exists an \( S\)-exact sequence \( 0 \rightarrow M \rightarrow E_0 \rightarrow \cdots \rightarrow E_{n-1} \rightarrow E_n \rightarrow 0 \), where \( E_0, E_1, \ldots, E_{n-1} \) are injective \( R\)-modules and \( E_n \) is \( S\)-injective;
9. there exists an exact sequence \( 0 \rightarrow M \rightarrow E_0 \rightarrow \cdots \rightarrow E_{n-1} \rightarrow E_n \rightarrow 0 \), where \( E_0, E_1, \ldots, E_{n-1} \) are injective \( R\)-modules and \( E_n \) is \( S\)-injective;
10. there exists an exact sequence \( 0 \rightarrow M \rightarrow E_0 \rightarrow \cdots \rightarrow E_{n-1} \rightarrow E_n \rightarrow 0 \), where \( E_0, E_1, \ldots, E_n \) are \( S\)-injective \( R\)-modules.

**Corollary 2.6.** Let \( R \) be a ring and \( S' \subseteq S \) multiplicative subsets of \( R \). Suppose \( M \) is an \( R\)-module, then \( S\text{-pd}_R(M) \leq S'\text{-pd}_R(M) \) and \( S\text{-id}_R(M) \leq S'\text{-id}_R(M) \).

**Proof.** Suppose \( S' \subseteq S \) are multiplicative subsets of \( R \). Let \( M \) and \( N \) be \( R\)-modules. If \( \text{Ext}^{n+1}_R(M, N) \) is uniformly \( S'\)-torsion, then \( \text{Ext}^{n+1}_R(M, N) \) is uniformly \( S\)-torsion. The result follows by Proposition 2.4. \[ \square \]

7
Proposition 2.7. Let $R$ be a ring and $S$ a multiplicative subset of $R$. Let $0 \to A \to B \to C \to 0$ be an $S$-exact sequence of $R$-modules. Then the following assertions hold.

1. $S$-$pd_R(C) \leq 1 + \max\{S$-$pd_R(A), S$-$pd_R(B)\}$.
2. If $S$-$pd_R(B) < S$-$pd_R(C)$, then $S$-$pd_R(A) = S$-$pd_R(C) - 1 > S$-$pd_R(B)$.
3. $S$-$id_R(A) \leq 1 + \max\{S$-$id_R(B), S$-$id_R(C)\}$.
4. If $S$-$id_R(B) < S$-$id_R(A)$, then $S$-$id_R(C) = S$-$id_R(A) - 1 > S$-$id_R(B)$.

Proof. The proof is similar with that of the classical case (see [11, Theorem 3.5.6] and [111 Theorem 3.5.13]). So we omit it. \qed

Proposition 2.8. Let $0 \to A \to B \to C \to 0$ be an $S$-split $S$-exact sequence of $R$-modules. Then the following assertions hold.

1. $S$-$pd_R(B) = \max\{S$-$pd_R(A), S$-$pd_R(C)\}$.
2. $S$-$id_R(B) = \max\{S$-$id_R(A), S$-$id_R(C)\}$.

Proof. We only show the first assertion since the other one is similar. Since the $S$-projective dimensions of $R$-modules are invariant under $S$-isomorphisms by Lemma 2.3, we may assume $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an $S$-split exact sequence. So there exists $R$-homomorphisms $f' : B \to A$ and $g' : C \to B$ such that $f' \circ f = s_1\text{Id}_A$ and $g \circ g' = s_2\text{Id}_C$ for some $s_1, s_2 \in S$. To prove (1), we just need to show that $0 \to \text{Ext}_R^n(M, A) \xrightarrow{\text{Ext}_R^n(M, f')} \text{Ext}_R^n(M, B) \xrightarrow{\text{Ext}_R^n(M, g)} \text{Ext}_R^n(M, C) \to 0$ is an $S$-exact sequence for any $R$-module $M$. Since the composition map $\text{Ext}_R^n(M, f') \circ \text{Ext}_R^n(M, f) : \text{Ext}_R^n(M, A) \to \text{Ext}_R^n(M, A)$ is equal to $\text{Ext}_R^n(M, s_1\text{Id}_A)$ which is just the multiplication map by $s_1$, we have $\text{Ext}_R^n(M, f)$ is an $S$-split $S$-monomorphism. Similarly, $\text{Ext}_R^n(M, g)$ is an $S$-split $S$-epimorphism. \qed

Let $p$ be a prime ideal of $R$ and $M$ an $R$-module. Denote $p$-$pd_R(M)$ (resp., $p$-$id_R(M)$) to be $(R-p)$-$pd_R(M)$ (resp., $(R-p)$-$id_R(M)$) briefly. The next result gives a new local characterization of projective dimension and injective dimension of an $R$-module.

Proposition 2.9. Let $R$ be a ring and $M$ an $R$-module. Then

$$pd_R(M) = \sup\{p$-$pd_R(M) | p \in \text{Spec}(R)\} = \sup\{m$-$pd_R(M) | m \in \text{Max}(R)\}.$$ and

$$id_R(M) = \sup\{p$-$id_R(M) | p \in \text{Spec}(R)\} = \sup\{m$-$id_R(M) | m \in \text{Max}(R)\}.$$ 

Proof. We only show the first equation since the other one is similar. Trivially, $\sup\{m$-$pd_R(M) | m \in \text{Max}(R)\} \leq \sup\{p$-$pd_R(M) | p \in \text{Spec}(R)\} \leq pd_R(M)$. Suppose
sup\{m-pd_R(M) \mid m \in \text{Max}(R)\} = n. For any \( R \)-module \( N \), there exists an element \( s^m \in R - m \) such that \( s^m \text{Ext}_R^{n+1}(M, N) = 0 \) by Proposition 2.4. Since the ideal generated by all \( s^m \) is \( R \), we have \( \text{Ext}_R^{n+1}(M, N) = 0 \) for all \( R \)-modules \( N \). So \( pd_R(M) \leq n \). Suppose \( \sup\{m-pd_R(M) \mid m \in \text{Max}(R)\} = \infty \). Then for any \( n \geq 0 \), there exists a maximal ideal \( m \) and an element \( s^m \in R - m \) such that \( s^m \text{Ext}_R^{n+1}(M, N) \neq 0 \) for some \( R \)-module \( N \). So for any \( n \geq 0 \), we have \( \text{Ext}_R^{n+1}(M, N) \neq 0 \) for some \( R \)-module \( N \). Thus \( pd_R(M) = \infty \). So the equalities hold. \( \square \)

3. On the \( S \)-global dimensions of rings

Recall that the global dimension \( \text{gl.dim}(R) \) of a ring \( R \) is the supremum of projective dimensions of all \( R \)-modules (see [11, Definition 3.5.17]). Now, we introduce the \( S \)-analogue of global dimensions of rings \( R \) for a multiplicative subset \( S \) of \( R \).

**Definition 3.1.** The \( S \)-global dimension of a ring \( R \) is defined by

\[ S-\text{gl.dim}(R) = \sup\{S-pd_R(M) \mid M \text{ is an } R\text{-module}\} \]

Obviously, \( S-\text{gl.dim}(R) \leq \text{gl.dim}(R) \) for any multiplicative subset \( S \) of \( R \). And if \( S \) is composed of units, then \( S-\text{gl.dim}(R) = \text{gl.dim}(R) \). The next result characterizes the \( S \)-global dimension of a ring \( R \).

**Proposition 3.2.** Let \( R \) be a ring and \( S \) a multiplicative subset of \( R \). The following statements are equivalent for \( R \):

1. \( S-\text{gl.dim}(R) \leq n \);
2. \( S-pd_R(M) \leq n \) for all \( R \)-modules \( M \);
3. \( \text{Ext}_R^{n+k}(M, N) \) is uniformly \( S \)-torsion for all \( R \)-modules \( M, N \) and all \( k > 0 \);
4. \( \text{Ext}_R^{n+1}(M, N) \) is uniformly \( S \)-torsion for all \( R \)-modules \( M, N \);
5. \( S-\text{id}_R(M) \leq n \) for all \( R \)-modules \( M \).

**Proof.** (1) \( \Rightarrow \) (2) and (3) \( \Rightarrow \) (4): Trivial

(2) \( \Rightarrow \) (3) and (5) \( \Rightarrow \) (3): Follows from Proposition 2.4

(4) \( \Rightarrow \) (2): Let \( M \) be an \( R \)-module and \( 0 \rightarrow F_n \rightarrow \ldots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0 \) an exact sequence, where \( F_0, F_1, \ldots, F^{n-1} \) are projective \( R \)-modules. To complete the proof, it suffices, by Proposition 2.4 to prove that \( F_n \) is \( S \)-projective. Let \( N \) be an \( R \)-module. Thus \( S-pd_R(N) \leq n \) by (4). It follows from Corollary 1.4 that \( \text{Ext}_R^1(N, F_n) \cong \text{Ext}_R^{n+1}(N, M) \) is uniformly \( S \)-torsion. Thus \( F_n \) is \( S \)-projective.

(4) \( \Rightarrow \) (5): Let \( M \) be an \( R \)-module and \( 0 \rightarrow M \rightarrow E_0 \rightarrow \ldots \rightarrow E_{n-1} \rightarrow E_n \rightarrow 0 \) an exact sequence with \( E_0, E_1, \ldots, E_{n-1} \) are injective \( R \)-modules. By dimension shifting, we have \( \text{Ext}_R^{n+1}(M, N) \cong \text{Ext}_R^1(E_n, N) \). So \( \text{Ext}_R^1(E_n, N) \) is uniformly \( S \)-torsion for any \( R \)-module \( N \). Thus \( E_n \) is \( S \)-injective by [9, Theorem 4.3]. Consequently, \( S-\text{id}_R(M) \leq n \) by Theorem 2.5. \( \square \)
Consequently, we have $S\text{-gl.dim}(R) = \sup\{S-pd_R(M) | M \text{ is an } R\text{-module}\} = \sup\{S-id_R(M) | M \text{ is an } R\text{-module}\}$.

Let $p$ be a prime ideal of a ring $R$ and $p\text{-gl.dim}(R)$ denote $(R - p)\text{-gl.dim}(R)$ briefly. By Proposition 2.9, we have a new local characterization of global dimensions of commutative rings.

**Corollary 3.3.** Let $R$ be a ring. Then

$$gl.dim(R) = \sup\{p\text{-gl.dim}(R) | p \in \text{Spec}(R)\} = \sup\{m\text{-dim}(R) | m \in \text{Max}(R)\}.$$

Recall from [14] that an $R$-module $M$ is called $S$-semisimple provided that any $S$-short exact sequence $0 \to A \to M \to C \to 0$ is $S$-split. And $R$ is called an $S$-semisimple ring provided that any free $R$-module is $S$-semisimple. Thus by [14] Theorem 3.5, the following result holds.

**Corollary 3.4.** Let $R$ be a ring and $S$ a multiplicative subset of $R$. The following assertions are equivalent:

1. $R$ is an $S$-semisimple ring;
2. every $R$-module is $S$-semisimple;
3. every $R$-module is $S$-projective;
4. every $R$-module is $S$-injective;
5. $R$ is uniformly $S$-Noetherian and $S$-von Neumann regular;
6. there exists an element $s \in S$ such that for any ideal $I$ of $R$ there is an $R$-homomorphism $f_I : R \to I$ satisfying $f_I(i) = si$ for any $i \in I$.
7. $S\text{-gl.dim}(R) = 0$.

The following example shows that the global dimension of rings and the $S$-global dimension of rings can are be wildly different.

**Example 3.5.** Let $T = \mathbb{Z}_2 \times \mathbb{Z}_2$ be a semi-simple ring and $s = (1, 0) \in T$. Let $R = T[x]/\langle sx, x^2 \rangle$ with $x$ the indeterminate and $S = \{1, s\}$ be a multiplicative subset of $R$. Then $S\text{-gl.dim}(R) = 0$ by [14] Theorem 3.5. Since $R$ is a non-reduced noetherian ring, $gl.dim(R) = \infty$ by [4] Corollary 4.2.4.

4. $S$-GLOBAL DIMENSIONS OF FACTOR RINGS AND POLYNOMIAL RINGS

In this section, we mainly consider the $S$-global dimensions of factor rings and polynomial rings. Firstly, we give an inequality of $S$-global dimensions for ring homomorphisms. Let $\theta : R \to T$ be a ring homomorphism. Suppose $S$ is a multiplicative subset of $R$, then $\theta(S) = \{\theta(s) | s \in S\}$ is a multiplicative subset of $T$.

**Proposition 4.1.** Let $\theta : R \to T$ be a ring homomorphism, $S$ a multiplicative subset of $R$. Suppose $M$ is an $T$-module. Then
Proposition 4.2. Let \( R \) be a ring, \( S \) a multiplicative subset of \( R \). Then \( \pi : R \to R/I \) is a ring epimorphism and \( \pi(S) := S/I = \{ s + I \in R/I | s \in S \} \) is naturally a multiplicative subset of \( R/I \).

Let \( R \) be a ring, \( I \) an ideal of \( R \) and \( S \) a multiplicative subset of \( R \). Then \( \pi : R \to R/I \) is a ring epimorphism and \( \pi(S) := S/I = \{ s + I \in R/I | s \in S \} \) is naturally a multiplicative subset of \( R/I \).

**Proposition 4.2.** Let \( R \) be a ring, \( S \) a multiplicative subset of \( R \). Let \( a \) be a non-zero-divisor in \( R \) which does not divide any element in \( S \). Written \( \overline{R} = R/aR \) and \( \overline{S} = \{ s + aR \in \overline{R} | s \in S \} \). Then the following assertions hold.

1. Let \( M \) be a nonzero \( \overline{R} \)-module. If \( \overline{S} \)-\( \text{pd}_{\overline{R}}(M) \) < \( \infty \), then
\[
\overline{S} \text{-pd}_{\overline{R}}(M) = \overline{S} \text{-pd}_{\overline{R}}(M) + 1.
\]

2. If \( \overline{S} \text{-gl.dim}(\overline{R}) \) < \( \infty \), then
\[
\overline{S} \text{-gl.dim}(R) \geq \overline{S} \text{-gl.dim}(\overline{R}) + 1.
\]

**Proof.** (1) Set \( \overline{S} \text{-pd}_{\overline{R}}(M) = n \). Since \( a \) is a non-zero-divisor which does not divide any element in \( S \), then the exact sequence \( 0 \to aR \to R \to R/aR \to 0 \) does not \( S \)-split. Thus \( S \text{-pd}_{\overline{R}}(R) = 1 \). By Proposition 4.11 we have \( S \text{-pd}_{\overline{R}}(M) \leq \overline{S} \text{-pd}_{\overline{R}}(M) + 1 = n + 1 \). Since \( \overline{S} \text{-pd}_{\overline{R}}(M) = n \), then there is an injective \( \overline{R} \)-module \( C \) such that \( \text{Ext}^{n}_{\overline{R}}(M, C) \) is not uniformly \( \overline{S} \)-torsion. By [11, Theorem 2.4.22], there is an injective \( R \)-module \( E \) such that \( 0 \to C \to E \to E \to 0 \) is exact. By [11, Proposition 3.8.12(4)], \( \text{Ext}^{n+1}_{\overline{R}}(M, E) \cong \text{Ext}^{n}_{\overline{R}}(M, C) \). Thus \( \text{Ext}^{n+1}_{\overline{R}}(M, E) \) is not uniformly \( S \)-torsion. So \( S \text{-pd}_{\overline{R}}(M) = \overline{S} \text{-pd}_{\overline{R}}(M) + 1 \).

(2) Let \( n = \overline{S} \text{-gl.dim}(\overline{R}) \). Then there is a nonzero \( \overline{R} \)-module \( M \) such that \( \overline{S} \text{-pd}_{\overline{R}}(M) = n \). Thus \( S \text{-pd}_{\overline{R}}(M) = n + 1 \) by (1). So \( S \text{-gl.dim}(R) \geq \overline{S} \text{-gl.dim}(\overline{R}) + 1 \).
Lemma 4.3. Let $R$ be a ring and $M$ an $R$-module. $R[x]$ denotes the polynomial ring with one indeterminate, where all coefficients are in $R$. Set $M[x] = M \otimes_R R[x]$, then $M[x]$ can be seen as an $R[x]$-module naturally. It is well-known $\text{gl.dim}(R[x]) = \text{gl.dim}(R)$ (see [13] Theorem 3.8.23]). In this section, we give a $S$-analogue of this result. Let $S$ be a multiplicative subset of $R$, then $S$ is a multiplicative subset of $R[x]$ naturally.

Proof. Suppose $P$ is an $S$-projective $R[x]$-module. Then there exists a free $R[x]$-module $F$ and a $S$-split $R[x]$-short exact sequence $0 \to K \to F \xrightarrow{s} P \to 0$. Thus we have an $R[x]$-homomorphism $\pi' : P \to F$ such that $\pi \circ \pi' = s \text{Id}_P$ for some $s \in S$. Note that $\pi'$ is also an $R$-homomorphism. So $0 \to K \to F \xrightarrow{s} P \to 0$ is also $S$-split over $R$. Note that $F$ is also a free $R$-module. So $P$ is $S$-projective over $R$ by [14] Proposition 2.8].

□

Proposition 4.4. Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $M$ an $R$-module. Then $\text{S-pd}_{R[x]}(M[x]) = S\text{-pd}_{R}(M)$.

Proof. Assume that $S\text{-pd}_{R}(M) \leq n$. Then $M$ has an $S$-projective resolution over $R$:

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0.$$  

Since $R[x]$ is free over $R$, $R[x]$ is an $S$-flat $R$-module by [14] Proposition 2.7. Thus the natural sequence

$$0 \to P_n[x] \to \cdots \to P_1[x] \to P_0[x] \to M[x] \to 0$$

is $S$-exact over $R[x]$. Consequently, $S\text{-pd}_{R[x]}(M[x]) \leq n$ by Proposition 2.4.

Let $0 \to F_n \to \cdots \to F_1 \to F_0 \to M[x] \to 0$ be an exact sequence with each $F_i$ $S$-projective over $R[x]$ ($1 \leq i \leq n$). Then it is also $S$-projective resolution of $M[x]$ over $R$ by Lemma 4.3. Thus $\text{Ext}^{n+1}_R(M[x], N)$ is uniformly $S$-torsion for any $R$-module $N$ by Proposition 2.4. It follows that $s\text{Ext}^{n+1}_R(M[x], N) = s \prod_{i=1}^{\infty} \text{Ext}^{n+1}_R(M, N) = 0$. Thus $\text{Ext}^{n+1}_R(M, N)$ is uniformly $S$-torsion. Consequently, $S\text{-pd}_{R}(M) \leq S\text{-pd}_{R[x]}(M[x])$ by Proposition 2.4 again.

Let $M$ be an $R[x]$-module then $M$ can be naturally viewed as an $R$-module. Define $\psi : M[x] \to M$ by

$$\psi(\sum_{i=0}^{n} x^i \otimes m_i) = \sum_{i=0}^{n} x^i m_i, \quad m_i \in M.$$
And define $\varphi : M[x] \to M[x]$ by
$$
\varphi \left( \sum_{i=0}^{n} x^i \otimes m_i \right) = \sum_{i=0}^{n} x^{i+1} \otimes m_i - \sum_{i=0}^{n} x^i \otimes xm_i, \quad m_i \in M.
$$

**Lemma 4.5.** [11, Theorem 3.8.22] Let $R$ be a ring, $S$ a multiplicative subset of $R$. For any $R[x]$-module $M$,
$$
0 \to M[x] \xrightarrow{\varphi} M[x] \xrightarrow{\psi} M \to 0
$$
is exact.

**Theorem 4.6.** Let $R$ be a ring, $S$ a multiplicative subset of $R$. Then $S$-gl.dim$(R[x]) = S$-gl.dim$(R) + 1$.

**Proof.** Let $M$ be an $R[x]$-module. Then, by Lemma 4.5 there is an exact sequence over $R[x]$:
$$
0 \to M[x] \to M[x] \to M \to 0.
$$

By Proposition 2.7, Proposition 4.1 and Proposition 4.4,
$$
S$-pd$_R(M) \leq S$-pd$_{R[x]}(M) \leq 1 + S$-pd$_{R[x]}(M[x]) = 1 + S$-pd$_R(M)
$$
(*).

Thus if $S$-gl.dim$(R) < \infty$, then $S$-gl.dim$(R[x]) < \infty$.

Conversely, if $S$-gl.dim$(R[x]) < \infty$, then for any $R$-module $M$, $S$-pd$_R(M) = S$-pd$_{R[x]}(M[x]) < \infty$ by Proposition 4.4. Therefore we have $S$-gl.dim$(R) < \infty$ if and only if $S$-gl.dim$(R[x]) < \infty$. Now we assume that both of these are finite. Then $S$-gl.dim$(R[x]) \leq S$-gl.dim$(R) + 1$ by (*). Since $R \cong R[x]/xR[x]$, $S$-gl.dim$(R[x]) \leq S$-gl.dim$(R) + 1$ by Proposition 4.2. Consequently, we have $S$-gl.dim$(R[x]) = S$-gl.dim$(R) + 1$. \hfill \Box

**Corollary 4.7.** Let $R$ be a ring, $S$ a multiplicative subset of $R$. Then for any $n \geq 1$ we have
$$
S$-gl.dim$(R[x_1, \ldots, x_n]) = S$-gl.dim$(R) + n.
$$

**Acknowledgement.**
The author was supported by the Natural Science Foundation of Chengdu Aeronautic Polytechnic (No. 062026) and the National Natural Science Foundation of China (No. 12061001).

**References**
[1] D. D. Anderson, T. Dumitrescu, $S$-Noetherian rings, Commun. Algebra 30 (2002), 4407-4416.
[2] D. Bennis, M. El Hajoui, On $S$-coherence, J. Korean Math. Soc. 55 (2018), no. 6, 1499-1512.
[3] L. Fuchs, L. Salce, Modules over Non-Noetherian Domains, Providence, AMS, 2001.
[4] S. Glaz, *Commutative Coherent Rings*, Lecture Notes in Mathematics, vol. 1371, Springer-Verlag, Berlin, 1989.

[5] J. W. Lim, *A Note on S-Noetherian Domains*, Kyungpook Math. J. 55, (2015), 507-514.

[6] J. W. Lim, D. Y. Oh, *S-Noetherian properties on amalgamated algebras along an ideal*, J. Pure Appl. Algebra 218, (2014), 2099-2123.

[7] I. Kaplansky. *Commutative Rings*, Allyn and Bacon, Boston, 1970.

[8] H. Kim, M. O. Kim, J. W. Lim, *On S-strong Mori domains*, J. Algebra 416, (2014): 314-332.

[9] W. Qi, H. Kim, F. G. Wang, M. Z. Chen, W. Zhao, *Uniformly S-Noetherian rings*, submitted.

[10] B. Stenström, *Rings of Quotients*, Die Grundlehren Der Mathematischen Wissenschaften, Springer-Verlag, 1975.

[11] F. G. Wang, H. Kim, *Foundations of Commutative Rings and Their Modules*, Singapore, Springer, 2016.

[12] X. L. Zhang, *Characterizing S-flat modules and S-von Neumann regular rings by uniformity*, arxiv.org/abs/2105.07941v1.

[13] X. L. Zhang, *The S-weak global dimension of commutative rings*, https://arxiv.org/abs/2106.00535.

[14] X. L. Zhang, *Characterizing S-projective modules and S-semisimple rings by uniformity*, https://arxiv.org/abs/2106.10441.