On a Cuntz-Krieger functor

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Abstract

We construct a covariant functor from the topological torus bundles to the so-called Cuntz-Krieger algebras; the functor maps homeomorphic bundles into the stably isomorphic Cuntz-Krieger algebras. It is shown, that the $K$-theory of the Cuntz-Krieger algebra encodes torsion of the first homology group of the bundle. We illustrate the result by examples.

Key words and phrases: torus bundle, Cuntz-Krieger algebra

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1 Introduction

A. The Cuntz-Krieger algebras. Recall, that the Cuntz-Krieger algebra, $\mathcal{O}_A$, is a $C^*$-algebra generated by partial isometries $s_1, \ldots, s_n$; they act on the Hilbert space in such a way, that their support projections $Q_i = s_i^*s_i$ and their range projections $P_i = s_is_i^*$ are orthogonal and satisfy the relations

$$Q_i = \sum_{j=i}^n a_{ij}P_j,$$

for an $n \times n$ matrix $A = (a_{ij})$ consisting of 0's and 1's \[1\]. The notion is extendable to the matrices $A$ with the non-negative integer entries \textit{ibid.}, \textit{Remark 2.16}. It is known, that the $C^*$-algebra $\mathcal{O}_A$ is simple, whenever matrix $A$ is irreducible; the latter means that certain power of $A$ is a strictly

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positive integer matrix. If $\mathcal{K}$ is the $C^*$-algebra of compact operators on a Hilbert space, then the Cuntz-Krieger algebras $\mathcal{O}_A$, $\mathcal{O}_{A'}$ are said to be stably isomorphic, if $\mathcal{O}_A \otimes \mathcal{K} \cong \mathcal{O}_{A'} \otimes \mathcal{K}$, where $\cong$ is an isomorphism of the $C^*$-algebras. The $K$-theory of $\mathcal{O}_A$ was established in [1]; it was shown, that $K_0(\mathcal{O}_A) \cong \mathbb{Z} \oplus K_0(\mathcal{O}_A) = 0$, where $A$ is a transpose of the matrix $A$. It is easy to see, that if $\det (I - A^t) \neq 0$, then $K_0(\mathcal{O}_A)$ is a finite abelian group and $K_1(\mathcal{O}_A) = 0$; the two groups are invariants of the stable isomorphism.

B. The torus bundles. Let $T^n$ be a torus of dimension $n \geq 1$. Recall, that a torus bundle is an $(n + 1)$-dimensional manifold $M_{\alpha} = \{ T^n \times [0, 1] \mid (T^n, 0) = (\alpha(T^n), 1) \}$, where $\alpha : T^n \to T^n$ is an automorphism of $T^n$. It is known, that bundles $M_{\alpha}$ and $M_{\alpha'}$ are homeomorphic, whenever the automorphisms $\alpha$ and $\alpha'$ are conjugate, i.e. $\alpha' = \beta \circ \alpha \circ \beta^{-1}$ for an automorphism $\beta : T^n \to T^n$. Let $H_1(T^n; \mathbb{Z}) \cong \mathbb{Z}^n$ be the first homology of torus; consider the group $\text{Aut} (T^n)$ of (homotopy classes of) automorphisms of $T^n$. Any $\alpha \in \text{Aut} (T^n)$ induces a linear transformation of $H_1(T^n; \mathbb{Z})$, given by an invertible $n \times n$ matrix $A$ with the integer entries; conversely, each $A \in GL_n(\mathbb{Z})$ defines an automorphism $\alpha : T^n \to T^n$. In this matrix representation, the conjugate automorphisms $\alpha$ and $\alpha'$ define similar matrices $A, A' \in GL_n(\mathbb{Z})$, i.e. such that $A' = BAB^{-1}$ for a matrix $B \in GL_n(\mathbb{Z})$. Each class of matrices, similar to a matrix $A \in GL_n(\mathbb{Z})$ and such that $tr (A) \geq 0 (tr (A) \leq 0)$, contains a matrix with only the non-negative (non-positive) entries. We always assume, that our bundle $M_\alpha$ is given by a non-negative matrix $A$; the matrices with $tr (A) \leq 0$ can be reduced to this case by switching the sign (from negative to positive) in the respective non-positive representative.

C. The result. Denote by $\mathcal{M}$ a category of torus bundles (of fixed dimension), endowed with homeomorphisms between the bundles; denote by $\mathcal{A}$ a category of the Cuntz-Krieger algebras $\mathcal{O}_A$ with $\det (A) = \pm 1$, endowed with stable isomorphisms between the algebras. Consider a (Cuntz-Krieger) map, $F : \mathcal{M} \to \mathcal{A}$, which acts by the formula $M_\alpha \mapsto \mathcal{O}_A$. The following is true.

**Theorem 1** The map $F$ is a covariant functor, such that $H_1(M_\alpha; \mathbb{Z})$ and $\mathbb{Z} \oplus K_0(F(M_\alpha))$ are isomorphic abelian groups.

The article is organized as follows. Theorem [1] is proved in section 2. There is no formal section on the preliminaries; all necessary results and concepts
are introduced in passing. (We encourage the reader to consult [1]–[5] for the details.) In section 3, an application of theorem 1 is considered.

2 Proof of theorem 1

The idea of proof consists in a reduction of the conjugacy problem for the automorphisms of $T^n$ to the Cuntz-Krieger theorem on the flow equivalence of the subshifts of finite type (to be introduced in the next paragraph). There are no difficult parts in the proof, which is basically a series of observations. Moreover, theorem 1 follows from the results of P. M. Rodrigues and J. S. Ramos [3]. However, our accents are different and the proof is more direct (and shorter) than in the above cited work.

(i) The main reference to the subshifts of finite type (SFT) is [2]. A full Bernoulli $n$-shift is the set $X_n$ of bi-infinite sequences $x = \{x_k\}$, where $x_k$ is a symbol taken from a set $S$ of cardinality $n$. The set $X_n$ is endowed with the product topology, making $X_n$ a Cantor set. The shift homeomorphism $\sigma_n : X_n \to X_n$ is given by the formula

$\sigma_n(\ldots x_k x_{k+1} \ldots) = (\ldots x_k x_{k+1} x_{k+2} \ldots)$

The homeomorphism defines a (discrete) dynamical system $\{X_n, \sigma_n\}$ given by the iterations of $\sigma_n$.

Let $A$ be an $n \times n$ matrix, whose entries $a_{ij} := a(i,j)$ are 0 or 1. Consider a subset $X_A$ of $X_n$ consisting of the bi-infinite sequences, which satisfy the restriction $a(x_k, x_{k+1}) = 1$ for all $-\infty < k < \infty$. (It takes a moment to verify that $X_A$ is indeed a subset of $X_n$ and $X_A = X_n$, if and only if, all the entries of $A$ are 1’s.) By definition, $\sigma_A = \sigma_n | X_A$ and the pair $\{X_A, \sigma_A\}$ is called a SFT. A standard edge shift construction described in [2] allows to extend the notion of SFT to any matrix $A$ with the non-negative entries.

It is well known that the SFT’s $\{X_A, \sigma_A\}$ and $\{X_B, \sigma_B\}$ are topologically conjugate (as the dynamical systems), if and only if, the matrices $A$ and $B$ are strong shift equivalent (SSE), see [2] for the corresponding definition. The SSE of two matrices is a difficult algorithmic problem, which motivates the consideration of a weaker equivalence between the matrices called a shift equivalence (SE). Recall, that the matrices $A$ and $B$ are said to be shift equivalent (over $\mathbb{Z}^+$), when there exist non-negative matrices $R$ and $S$ and a positive integer $k$ (a lag), satisfying the equations $AR = RB, BS = SA, A^k = RS$ and $SR = B^k$. Finally, the SFT’s $\{X_A, \sigma_A\}$ and $\{X_B, \sigma_B\}$ (and the matrices $A$ and $B$) are said to be flow equivalent (FE), if the suspension flows of the SFT’s act on the topological spaces, which are homeomorphic.
under a homeomorphism that respects the orientation of the orbits of the suspension flow. We shall use the following implications:

\[ SSE \Rightarrow SE \Rightarrow FE. \]  

(The first implication is rather classical, while for the second we refer the reader to [2], p.456.)

We further restrict to the SFT’s given by the matrices with determinant \( \pm 1 \). In view of Corollary 2.13 of [5], the matrices \( A \) and \( B \) with \( \det (A) = \pm 1 \) and \( \det (B) = \pm 1 \) are SE (over \( \mathbb{Z}^+ \)), if and only if, matrices \( A \) and \( B \) are similar in \( GL_n(\mathbb{Z}) \).

Let now \( \alpha \) and \( \alpha' \) be a pair of conjugate automorphisms of \( T^n \). Since the corresponding matrices \( A \) and \( A' \) are similar in \( GL_n(\mathbb{Z}) \), one concludes that the SFT’s \( \{ X_A, \sigma_A \} \) and \( \{ X_{A'}, \sigma_{A'} \} \) are SE. In particular, the SFT’s \( \{ X_A, \sigma_A \} \) and \( \{ X_{A'}, \sigma_{A'} \} \) are FE.

One can now apply the known result due to Cuntz and Krieger; it says, that the \( C^* \)-algebra \( \mathcal{O}_A \otimes \mathcal{K} \) is an invariant of the flow equivalence of the irreducible SFT’s, see p. 252 of [1] and its proof in Sect. 4 of the same work. Thus, the map \( F \) sends the conjugate automorphisms of \( T^n \) into the stably isomorphic Cuntz-Krieger algebras, i.e. \( F \) is a functor.

Let us show that \( F \) is a covariant functor. Consider the following commutative diagram:

\[
\begin{array}{ccc}
A & \sim & A' = BAB^{-1} \\
F & & F \\
\mathcal{O}_A & \cong & \mathcal{O}_{BAB^{-1}},
\end{array}
\]

where \( A, B \in GL_n(\mathbb{Z}) \) and \( \mathcal{O}_A, \mathcal{O}_{BAB^{-1}} \in \mathcal{A} \). Let \( g_1, g_2 \) be the arrows (similarity of matrices) in the upper category and \( F(g_1), F(g_2) \) the corresponding arrows (stable isomorphisms) in the lower category. In view of the diagram, we have the following identities:

\[
F(g_1 g_2) = \mathcal{O}_{B_2 B_1 AB_1^{-1} B_2^{-1}} = \mathcal{O}_{B_2 (B_1 AB_1^{-1}) B_2^{-1}} = \mathcal{O}_{B_2 A' B_2^{-1}} = F(g_1) F(g_2),
\]

(4)

where \( F(g_1)(\mathcal{O}_A) = \mathcal{O}_{A'} \) and \( F(g_2)(\mathcal{O}_{A'}) = \mathcal{O}_{A''} \). Thus, \( F \) does not reverse
the arrows and is, therefore, a covariant functor. The first statement of theorem 1 is proved.

(ii) Let $M_\alpha$ be a torus bundle with a monodromy, given by the matrix $A \in GL_n(\mathbb{Z})$. It can be calculated, e.g., using the Leray spectral sequence for the fiber bundles, that $H_1(M_\alpha;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}^n/(A-I)\mathbb{Z}^n$. Comparing this calculation with the $K$-theory of the Cuntz-Krieger algebra, one concludes that $H_1(M_\alpha;\mathbb{Z}) \cong \mathbb{Z} \oplus K_0(\mathcal{O}_A)$, where $\mathcal{O}_A = F(M_\alpha)$. The second statement of theorem 1 follows. □

3 Examples

Consider the following (three-dimensional) torus bundles $M_\alpha^i$ ($i = 1, 2, 3$):\[A_1^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \quad n \in \mathbb{Z}, \quad K_0(\mathcal{O}_{A_1^n}) \cong \mathbb{Z} \oplus \mathbb{Z}_n,\]
\[A_2 = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}, \quad K_0(\mathcal{O}_{A_2}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2,\]
\[A_3 = \begin{pmatrix} 5 & 1 \\ 4 & 1 \end{pmatrix}, \quad K_0(\mathcal{O}_{A_3}) \cong \mathbb{Z}_4.\]

(Here the group $K_0(\mathcal{O}_{A_i})$ was calculated using a reduction of the matrix to its Smith normal form \[2\] \textsuperscript{.) Notice, that the Cuntz-Krieger invariant $K_0(\mathcal{O}_{A_1^n}) \cong \mathbb{Z} \oplus \mathbb{Z}_n$ is a complete topological invariant of the family of bundles $M_\alpha^n$; thus, such an invariant solves a classification problem for these bundles. For the bundles $M_\alpha^2$ and $M_\alpha^3$ the Alexander polynomial:
\[\Delta_{A_2}(t) = \Delta_{A_3}(t) = t^2 - 6t + 1.\]

Thus, the Alexander polynomial alone cannot distinguish between the bundles $M_\alpha^2$ and $M_\alpha^3$; however, since $K_0(\mathcal{O}_{A_2}) \not\cong K_0(\mathcal{O}_{A_3})$, theorem 1 says that the bundles $M_\alpha^2$ and $M_\alpha^3$ are topologically distinct.

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\footnote{It is interesting, that by Thurston’s Geometrization Theorem, the bundle $M_\alpha^1$ is a nilmanifold for any $n$, while bundles $M_\alpha^2$ and $M_\alpha^3$ are the solvmanifolds \[4\].}
References

[1] J. Cuntz and W. Krieger, A class of $C^*$-algebras and topological Markov chains, Invent. Math. 56 (1980), 251-268.

[2] D. Lind and B. Marcus, An Introduction to Symbolic Dynamics and Coding, Cambridge Univ. Press, 1995.

[3] P. M. Rodrigues and J. S. Ramos, Bowen-Franks groups as conjugacy invariants for $T^n$-automorphisms, Aequationes Math. 69 (2005), 231-249.

[4] W. P. Thurston, Three dimensional manifolds, Kleinian groups and hyperbolic geometry, Bull. Amer. Math. Soc. 6 (1982), 357-381.

[5] J. B. Wagoner, Strong shift equivalence theory and the shift equivalence problem, Bull. Amer. Math. Soc. 36 (1999), 271-296.

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