On Anomaly-free Supergravity as an Effective String Theory

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Abstract

The equations of motion of anomaly-free supergravity are shown to fulfil (to all orders in \(\alpha'\)) a differential condition corresponding to the one relating the Weyl anomaly coefficients for a non-linear sigma model representing a (heterotic) string propagating in a non-trivial background. This supports the possibility that anomaly-free supergravity could provide the complete massless effective theory for the heterotic string.
When studying the propagation of strings in background fields, the conditions for conformal invariance of the non-linear sigma model coincide, as is well known, to the lowest order with the equations of motion for 10-dimensional supergravity, coupled to Yang-Mills in the case of the heterotic string. See, for instance, [1] for an introductory review. Imposing conformal invariance also at quantum level thus gives “stringy” corrections to these equations, with $\alpha'$ the sigma-model loop expansion parameter. The dilaton equation turns out to play the role of the central charge of the corresponding Virasoro algebra [2, 3], and must thus be independent of position in spacetime. This is indeed true since it has been shown that the Weyl anomaly coefficients satisfy a “Bianchi identity” of the form

$$D^\mu \beta^{\mu(\theta)} = \partial_\nu \beta^{(\theta)}$$

(1)

once the other equation(s) are imposed. Hence, if the l.h.s. vanishes, so does the derivative of the dilaton equation, $\beta^{(\theta)}$, and it must then itself be a constant. This identity was later proven to all orders in $\alpha'$ for the bosonic sigma model [4, 5, 6], and a supersymmetric version of it was made in [7].

A given sigma model corresponds to a string theory only if there exists a Virasoro algebra, which requires a condition like (1). Another way to regard this equation is to observe that it is a direct consequence of D-dimensional general covariance of the effective action for the supergravity theory [3, 5]. We can now turn the argument around: Given a set of dynamical equations for a (super)gravity theory, are they derivable from an action, and is this also the effective theory corresponding to a string propagating in a non-trivial background? As long as we do not succeed in deriving the background action by integrating out the string coordinates, this question can only be answered by an explicit calculation, order by order in $\alpha'$. However, the Bianchi identity (1) provides us with a necessary condition that our candidate must satisfy.

The so-called anomaly-free supergravity (AFS) is a model of supergravity coupled to a Yang-Mills theory in ten dimensions containing (implicit) corrections to all orders in $\alpha'$. It is obtained by imposing constraints on the superspace Bianchi identities, with the Lorentz Chern-Simons term added in the way required for gauge and gravitational anomaly cancellations [8], and then solving for the physical fields. Thanks to the observation made by Bonora, Pasti and Tonin [9] that the new $O(\alpha')$ terms do not add any new irreducible representations of $SO(1,9)$ to the equations, these can indeed be solved explicitly, albeit after very cumbersome calculations [10, 11, 12]. The solution consists of a set of equations of motion, supersymmetry transformations, and $x$-space Bianchi identities. They contain, as already mentioned, implicit corrections to all orders in $\alpha'$. To express these equations in physical fields only, we need to solve the equation relating $H_{\mu\nu\rho}$ to the torsion order by order, and we thus obtain corrections to all orders. The lowest order(s) coincide (after field redefinitions) with the effective theory for the heterotic string.
Unfortunately, AFS is not unique, so it need not be the heterotic string effective theory. The relation between the torsion and the physical fields is a differential equation, and different theories might emerge as a result of different boundary conditions. It is also, at least in principle, possible to add non-minimal terms by keeping more $SO(1,9)$ representations non-zero in the constraints imposed. Both these mechanisms have been suggested [11, 13] to account for the $\zeta(3) R^4$ corrections [14], and string loop terms should somehow turn up this way too. Strong restrictions on possible non-minimal terms must be provided by the fact that AFS and the heterotic sigma model have in common a non-trivial instanton solution [15].

The purpose of this letter is to strengthen the case that AFS is indeed closely related to the full effective theory by showing that its bosonic equations of motion satisfy an equation corresponding to (1) to all orders in $\alpha'$.

In [12] the bosonic parts of the equations of motion of AFS are given explicitly. We get the dilaton equation

$$\beta \equiv \Box \phi + \frac{1}{6} T^2 + \frac{4}{3} (\partial \phi)^2 - 2 e^{-\frac{4}{9} \phi} \text{tr} F^2 - \gamma_1 e^{-\frac{4}{9} \phi} W = 0,$$

and Einstein’s equations can be written

$$\beta_{\mu\nu} \equiv R_{\mu\nu} + \frac{2}{3} D(\partial_\nu \phi) + \frac{8}{9} \partial_\mu \phi \partial_\nu \phi + g_{\mu\nu} \left( \frac{1}{3} \Box \phi + \frac{4}{9} (\partial \phi)^2 \right) - 8 e^{-\frac{4}{9} \phi} \text{tr} (F_{\mu\rho} F_{\nu}^{\rho}) - \gamma_1 e^{-\frac{4}{9} \phi} \left[ W_{\mu\nu} + \frac{1}{2} \left( \frac{2}{27} + h_1 \right) g_{\mu\nu} \Box T^2 \right] = 0. \tag{3}$$

Here

$$W = -R_{\mu\nu} R^{\mu\nu} + \frac{1}{2} R_{\mu\rho\nu\sigma} R^{\mu\rho\nu\sigma}$$

$$+ \frac{1}{2} \left( - \frac{1}{9} + 3h_1 \right) \Box T^2 - \frac{2}{3} D_{[\mu} T_{\nu_1 \nu_3]} D^{\mu} T^{\nu_1 \nu_3} - 4 T^{\mu_1 \mu_2 \rho \sigma T_{\nu_1 \nu_2}} \rho D_{\mu_1} T_{\mu_2 \nu_1 \nu_2}$$

$$- 2 (T_{\mu} T_{\nu}) (T^\mu T^\nu) + 4 (T_{\mu} T_{\nu} T^\mu T^\nu) + \frac{1}{3} \left( \frac{2}{27} + h_1 \right) (T^2)^2, \tag{4}$$

and

$$W_{\mu\nu} = -\Box R_{\mu\nu} - 4 R_{\mu\rho} R_{\nu}^{\rho} + 2 R_{\rho_1 \rho_2 \sigma \mu} R^{\rho_1 \rho_2 \sigma \nu}$$

$$+ \left( \frac{2}{27} + h_1 \right) D(\partial_\nu T^2) - \frac{8}{3} g_{\rho \sigma} D_{[\mu} T_{\nu_1 \nu_3]} D^{[\sigma T^{\rho_1 \rho_3]} + 2 \left( \frac{2}{27} + h_1 \right) R_{\mu\nu} T^2$$

$$- 8 T^{\rho_1 \sigma} T_{\sigma}^{\rho_2 \rho_3} D_{[\mu} T_{\nu_1 \nu_3]} - 8 T_{\nu}^{\rho_1 \sigma} T_{\sigma}^{\rho_2 \rho_3} D_{[\mu} T_{\nu_1 \nu_3]}$$

$$+ 8 T_{\mu\rho\sigma} T_{\nu\sigma}^{\tau} (T^\mu T^\tau) + 16 (T_{\mu} T_{\rho} T_{\nu} T^\rho). \tag{5}$$

The curvature tensor is defined as

$$R_{\mu\rho\sigma} \tau = \partial_{[\mu} \tilde{\Gamma}_{\nu]}^{\rho} \tau - \tilde{\Gamma}_{[\mu}^{\nu] \partial_{\rho} \tilde{\Gamma}_{\nu]}^\tau, \tag{6}$$
where $\tilde{\Gamma} = \Gamma + T$, and the torsion is
\[
T_{\mu\nu\rho} = -3H_{\mu\nu\rho}e^{-\frac{4}{3}\phi} - 2\gamma_1 e^{-\frac{4}{3}\phi}W_{\mu\nu\rho},
\] (7)
with
\[
W_{\mu\nu\rho} = \frac{1}{2} \square T_{\mu\nu\rho} + 3T_{\mu}^{\hspace{1em} \rho_1 \rho_2} R_{\nu\rho_1 \sigma_1 \sigma_2} + 3T_{\mu\nu}^{\hspace{1em} \sigma} R_{\rho\sigma} - 4(T_{\mu}^{\hspace{1em} \nu} T_{\nu \rho}) - \left(\frac{2}{2T} + h_1\right) T^2 T_{\mu\nu\rho}.
\] (8)

The use of the notation “$\beta$” and “$\beta_{\mu\nu}$” does not imply that the above expressions are $\beta$-functions or Weyl anomaly coefficients. Since AFS only exists as an on-shell theory, all we know is that they are linear combinations of the equations of motion, as we will see below, when we study the lowest-order action.

We employ the shorthand notation $T_{\mu}^{\hspace{1em} \nu} T_{\nu \rho} = T_{\mu}^{\rho_1 \rho_2} T_{\rho_1 \rho_2}$ and similarly for the higher “traces” and $T^2 = T_{\mu_1 \mu_3} T_{\mu_1 \mu_3}$. The coefficient $\gamma_1$ introduced in [11] is proportional to $\alpha'$ once AFS is interpreted as an effective string theory, and $h_1$ is an arbitrary constant, which can be removed by a redefinition of $\phi$ in terms of $T^2$. We also have the trace of (3)
\[
\beta_{\mu}^{\hspace{1em} \mu} = R - \frac{2}{3}T^2,
\] (9)
which we regard as a constraint, and the remaining equations of motion are
\[
D^\mu F_{\mu\nu} = 0
\] (10)
\[
D^\mu T_{\mu\nu\rho} = 0.
\] (11)

The AFS version of (1) can rather easily be found by trial and error, but a nicer way is to use [3] for the action with $\gamma_1 = 0$. This action can be obtained from equation (10.4) of [11], and is
\[
S = \int d^{10}x \sqrt{g} e^{\frac{4}{3}\phi} \left(R + \frac{3}{2} e^{-\frac{8}{3}\phi} H^2 - 4 e^{-\frac{4}{3}\phi} \text{tr} F^2\right).
\] (12)

Here we use an $R$ depending only on the metric, and have expressed $T$ in $H$. We then find that (the zero superscript of course denotes $\gamma_1 = 0$)
\[
\frac{e^{-\frac{4}{3}\phi}}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}} = \beta_\nu^{\mu(0)} - \frac{g_{\mu\nu}}{2} (\beta_\rho^{(0)\rho} + 2\beta^{(0)}_\rho)
\] (13)
and
\[
\frac{e^{-\frac{4}{3}\phi}}{\sqrt{g}} \frac{\delta S}{\delta \phi} = \frac{4}{3} \beta_\rho^{(0)\rho}.
\] (14)

The “Bianchi identity” can be written
\[
D^\mu \frac{\delta S}{\delta g^{\mu\nu}} + \frac{1}{2} \partial_\nu \phi \frac{\delta S}{\delta \phi} = 0,
\] (15)
which is exactly the condition for invariance of the action under \( \delta x^\mu = V^\mu \), \( V^\mu \) being an arbitrary vector field, and if we insert (13) and (14), directly imposing the constraint \( \beta_\rho^{(0)} = 0 \), we find

\[
D^\mu \left( e^{\frac{4}{3} \phi \beta^{(0)}_\mu} \right) - \partial_\nu \left( e^{\frac{4}{3} \phi \beta^{(0)}} \right) = 0. \tag{16}
\]

We will now try to prove this equation extended to all orders in \( \gamma_1 \). To do this we will need various identities. From (6) we get the symmetries of the curvature

\[
R_{\mu\nu\rho\sigma} - R_{\rho\sigma\mu\nu} = D_{[\mu} T_{\nu\rho\sigma]} - D_{[\mu} T_{\sigma]\nu\rho} \tag{17}
\]

\[
R_{\mu[v_1..v_3]} = \frac{1}{2} \left( D_{\mu} T_{v_1..v_3} - D_{[v_1} T_{v_2 v_3] \mu} \right) + 2 T_{[v_1 v_2} \rho T_{v_3] \mu \rho} \tag{18}
\]

\[
R_{[v_1..v_3] \mu} = D_{[v_1} T_{v_2 v_3] \mu} + 2 T_{[v_1 v_2} \rho T_{v_3] \mu \rho} \tag{19}
\]

the Bianchi identity

\[
D_{[\mu} R_{\nu] \rho | \sigma} = 2 T_{[\mu \nu} ^ \lambda R_{\rho | \lambda \sigma} \tag{20}
\]

and its contractions

\[
2 D_{[\mu} R_{\nu] \rho | \sigma} + D_\sigma R_{\mu \rho | \sigma} = -2 T_{[\mu \nu} ^ \sigma R_{\rho | \sigma} + 4 T_{\sigma [\mu \nu} ^ \tau R_{\rho | \tau \sigma} \tag{21}
\]

\[
D^\mu R_{\mu \nu} = \frac{1}{12} \partial_\nu T^2 + \frac{1}{2} T_{\mu \nu \lambda} D_{\lambda} T_{\mu \nu \lambda}. \tag{22}
\]

Furthermore, for the gauge field and the torsion, the Bianchi identities are

\[
D_{[\mu} F_{\nu] \rho | \sigma} = 2 T_{[\mu \nu} ^ \sigma F_{\rho | \sigma} \tag{23}
\]

\[
D_{[\mu} T_{\nu \rho \sigma]} = -\frac{4}{3} \partial_{[\mu} \phi T_{\nu \rho \sigma]} - 3 T_{[\mu \nu} ^ \tau T_{\rho \sigma] \tau} + 12 e^{-\frac{4}{3} \phi} \text{tr}(F_{[\mu \nu} F_{\rho \sigma]}) + \frac{1}{3} \gamma_1 e^{-\frac{4}{3} \phi} \left[ -2 D_{[\mu} W_{\nu \rho \sigma]} - 6 T_{[\mu \nu} ^ \tau W_{\rho \sigma] \tau} + 3 R_{[\mu \nu} ^ \tau \lambda R_{\rho \sigma] \tau \lambda} \right]. \tag{24}
\]

Equation (24) is obtained by combining the Bianchi identity for \( H \) from (12) with (7). We also have the Ricci identity (for an arbitrary covariant vector, \( V_\rho \))

\[
[D_\mu, D_\nu] V_\rho = -2 R_{\mu \rho} ^ \sigma V_\sigma - 2 T_{\mu \nu} ^ \sigma D_\tau V_\rho, \tag{25}
\]

from which we immediately derive the useful commutator

\[
[D_\mu, \Box] V_\nu = -2 \left[ D^\rho R_{\mu \rho} ^ \sigma V_\sigma + 2 R_{\mu \rho} ^ \sigma D_\tau V_\sigma - R_{\mu} ^ \rho D_\rho V_\nu \right]. \tag{26}
\]

After a lengthy calculation, the details of which are of little interest, we can now prove

\[
D^\mu \left( e^{\frac{4}{3} \phi \beta_\mu} \right) - \partial_\nu \left( e^{\frac{4}{3} \phi \beta} \right) = 0. \tag{27}
\]
The lowest order in $\gamma_1$ is easy. Using our identities and the equations of motion for $F$ and $T$, everything vanishes except a higher-order contribution from (24). We are then left with the $O(\gamma_1)$ part

$$D^\mu W_{\mu\nu} + \frac{1}{2} \left( \frac{2}{27} + h_1 \right) \partial_\nu \Box T^2 - \partial_\nu W$$

$$- \frac{4}{3} T^{\mu_1\mu_3} D_{[\nu} W_{\mu_1\mu_3]} - 2 T^{\mu_1\mu_3} T_{\nu \mu_1} \phi W_{\mu_3\mu_3 \rho} + 2 T^{\mu_1\mu_3} R_{\nu \mu_1}^{\rho \sigma} R_{\mu_2 \mu_3 \rho \sigma}. \quad (28)$$

Written out explicitly, this is a very long expression containing terms of the form $D \Box R, RDR, R^2 T, T D^5 T, D T D^2 T, D R T, T^2 D^2 T, R T^3, T(D T)^2,$ and $T^3 D T$. ($T^5$ terms with one free index cannot exist for symmetry reasons.) A systematic elimination using (9), (11), (17) – (22), and the Ricci identity leaves us with only terms of the two last types, which separately cancel.

We have thus proven that the equations of motion derived in the AFS scheme do indeed fulfil the differential condition to all orders in $\alpha'$. The crucial point is that we need not use (24) in the $O(\gamma_1)$ part of the calculation, so we never need to go higher than the first order in $\gamma_1$ explicitly. Hence AFS satisfies a necessary condition for the existence of an action from which its equations of motion can be derived, and of a string theory, the Weyl anomaly conditions of which are the equations of motion of AFS. This might also give some useful hints for the derivation of the full AFS Lagrangian (cf. [16, 17]). However, the fact that $H$ and $T$ are related via a differential constraint might still make this a very difficult task. If the full AFS theory including the fermions is also derivable from an action, (27) should still hold, and the full theory might then be an effective theory for the heterotic string with a non-trivial fermionic background. It would be rather interesting to have this, since very little work has been done with such sigma models; see however [2, 18].

The question is now which parts of string theory can be accounted for by our effective theory. We do not know whether string loop effects are also contained in AFS, but we can argue that they are unlikely to occur in the minimal model: To obtain the sigma model action we have to make the rescaling $g_{\mu \nu} \rightarrow e^{-4\phi} g_{\mu \nu}$, remembering that the “fundamental” torsion is the one with two covariant and one contravariant indices, so that $T_{\mu \nu \rho}$ has to be rescaled as $g_{\mu \nu}$. The lowest-order action then has the form $S = \int d^{10}x \sqrt{g} e^{-4\phi} \mathcal{L}$, where $\mathcal{L}$ does not contain any exponentials, and these can also be divided out from the equations of motion and the relation between $H$ and $T$. Adding higher-order terms and solving for $H$ should then introduce no terms with a different power of $e^\phi$, that is the string coupling constant, anywhere. Nevertheless, unless supersymmetry is explicitly broken by string loops there must exist a set of constraints, perhaps very complicated, extending minimal AFS to a model also containing these effects.

Could minimal AFS then be the full effective theory corresponding to string tree level? We already know that there is trouble with the $\zeta(3) R^4$ term, since no transcendental coefficients turn up naturally. As was mentioned already in
the introduction two possible scenarios have been proposed; it is an effect of the choice of boundary conditions for the solution of (7), or it is non-minimal [11, 13]. An argument in favour of the latter suggestion is that this term can be separately supersymmetrized, at least if an (off-shell) superfield exists [19]. However, in a very recent paper by de Roo et al. [20] it is argued that the supersymmetrization clashes with gauge-invariance if only physical fields are present, while there is no such problem with the similar $R^4$ term coming from string loops [21]. It is then very tempting to believe that what we have is the full effective theory corresponding to the tree level heterotic string, the apparent contradiction found in [20] perhaps being resolved by the addition of higher-order corrections to the supersymmetry transformations, and that loop effects would be accounted for by the addition of non-minimal terms. This should clearly be examined by calculating the corrections predicted by AFS to cubic order in $\alpha'$ and comparing to known results, see for instance [20, 22, 23] and references therein. The reason why this has not already been done is probably the difficulty expected in handling (7). In view of the rather strong indications that AFS has to be taken seriously as an effective string theory, compactifications and other classical solutions should also be studied via AFS. This could be an easier problem, since it might be unnecessary to solve for $H$ explicitly. Some attempts in this direction have already been made in the second reference of [12] and in [13].

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