Asymptotic behavior of global entropy solutions for nonstrictly hyperbolic systems with linear damping

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Abstract

In this paper we investigate the large time behavior of the global weak entropy solutions to the symmetric Keyfitz-Kranzer system with linear damping. It is proved that as \( t \to \infty \) the entropy solutions tend to zero in the \( L^p \) norm.

1 Introduction

In this paper we consider the Cauchy problem to the symmetric system of Keyfitz-Kranzer type with linear damping

\[
\begin{aligned}
  u_t + (u\phi(r))_x + au &= 0, \\
  v_t + (v\phi(r))_x + bv &= 0.
\end{aligned}
\]  \hspace{1cm} (1.1)

with initial data

\[
\begin{aligned}
  u(x, 0) &= u_0(x), \\
  v(x, 0) &= v_0(x).
\end{aligned}
\]  \hspace{1cm} (1.2)

This system models of propagation of forward longitudinal and transverse waves of elastic string which moves in a plane, see [1], [3]. General source term for the system (2.8) was considered in [6]. The damping in the system (2.8) represents external forces proportional to velocity, and this term can be produce lost of total energy of system. Consider the scalar case, by example

\[
\begin{aligned}
  u_t + au_x + bu &= 0, \\
  u(x, 0) &= u_0(x).
\end{aligned}
\]  \hspace{1cm} (1.3)

From the integral representation of (1.3) it is easy to find the following solution

\[
\begin{aligned}
  u(x, t) &= u_0(x - at)e^{-bt}.
\end{aligned}
\]  \hspace{1cm} (1.4)
The term $bu$ produce a dissipative effect in the solutions, i.e, the solutions tends to zero when $t \to \infty$. We are looking for condition under wich the terms $a$, $b$ have a dissipative efect in the solutions of (2.2).

Let $r(x, t) = \sqrt{u(x, t)^2 + v(x, t)^2}$ be, we are going to show the following main theorem.

**Theorem 1.1.** If the initial data $(u_0(x), v_0(x)) \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ then the Cauchy problem (2.2) has a weak entropy solutions satisfying

$$\|u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)} < M$$

Moreover $r(u, v)$ converges to zero in $L^p$ with exponential time decay, i.e.

$$\|r(x, t)\|_{L^p(\mathbb{R})} \leq K e^{-Mt}\|r(x, 0)\|_{L^p(\mathbb{R})}$$

### 2 Preliminars

We start with some preliminaries about the general systems of conservation laws, see [2] chapter 5. Let $f : \Omega \to \mathbb{R}^n$ be a smooth vector field. Consider Cauchy problem for the system

$$\begin{cases}
  u_t + f(u)_x = g(u), \\
  u(x, 0) = u_0(x).
\end{cases} \quad (2.1)$$

When $g(u) = 0$ the system (2.1) is called homogeneous system of conservation laws, if $g(u) \neq 0$ the system (2.1) is called inhomogeneous system or balance system of conservation laws. We shall work also with the parabolic perturbation to the system (2.1), namely

$$\begin{cases}
  u_t + f(u)_x = \epsilon u_{xx} + g(u), \\
  u(x, 0) = u_0(x).
\end{cases} \quad (2.2)$$

Denote by $A(u) = Df(u)$ the Jacobian matrix of partial derivates of $f$.

**Definition 2.1.** The system (2.1) is strictly hyperbolic if for every $u \in \Omega$, the matrix $A(u)$ has $n$ real distinct eigenvalues $\lambda_1(u) < \cdots < \lambda_n(u)$.

Let $r_i(u)$ the correspond eigenvetor to $\lambda_i(u)$, then

**Definition 2.2.** We say that the $i$-th characteristic field is genuinely non-linear if

$$\nabla \lambda_i(u) \cdot r_i(0) \neq 0,$$

If instead

$$\nabla \lambda_i(u) \cdot r_i(0) = 0,$$  

we say that the $i$-th characteristic field is linearly degenerate.

For the following definitions see [3], [7]

**Definition 2.3.** A k-Riemann invariant is a smooth function $w_k : \mathbb{R}^n \to \mathbb{R}$, such that

$$\nabla w_k(u) \cdot r_k(u) = 0$$

(2.5)
**Definition 2.4.** A pair of function $\eta, q : \mathbb{R}^n \rightarrow \mathbb{R}$ is called an entropy-entropy flux pair if it satisfies
\[
\nabla \eta(u) A(u) = \nabla q(u),
\]
if $\eta(u)$ is a convex function then the pair $(\eta, q)$ is called convex entropy-entropy flux pair.

**Definition 2.5.** A bounded measurable function $u(x, t)$ is an entropy (or admissible) solution for the Cauchy problem (2.1), if it satisfies the following inequality
\[
\eta(u)_t + q(u)_x + \nabla \eta(u) g(u) \leq 0.
\]
in the distributional sense, where $(\eta, q)$ is any convex entropy-entropy flux pair.

We consider the general system of Keyfitz-Kranzer system
\[
\begin{cases}
  u_t + (u\phi(u, v))_x = 0, \\
v_t + (v\phi(u, u))_x = 0,
\end{cases}
\]
to get some general observations about this type of systems. Making $F(u, v) = (u\phi(u, v), v\phi(u, v))$ in (2.8), we have that the eigenvalues and eigenvector of the Jacobian’s matrix $Df$ are given by
\[
\begin{align*}
  \lambda_1(u, v) &= \phi(u, v) \\
  \lambda_2(u, v) &= \phi(u, v) + (u, v) \cdot \nabla \phi(u, v) \\
  r_1 &= (1, -\frac{\phi_u}{\phi_v}) \\
r_2 &= (1, \frac{\phi_v}{\phi_u}).
\end{align*}
\]
From (2.9), (2.10) we have that $\nabla \phi \cdot r_1 = 0$, and $\nabla Z(u, v) \cdot r_2 = 0$, where $Z(u, v) = \frac{u}{v}$, then the Riemann invariants are given by
\[
\begin{align*}
  W(u, v) &= \phi(v), \\
  Z(u, v) &= \frac{u}{v}.
\end{align*}
\]

**Lemma 2.6.** The system (2.8) is always linear degenerate in the first characteristic field. If $(u, v)\nabla \phi(u, v) \neq 0$, then the system (2.8) is strictly hyperbolic and non linear degenerate in the second characteristic field, moreover
\[
\nabla \lambda_2(u, v) \cdot r_2 = 2\frac{(u, v)\nabla \phi(u, v) + (u, v)H(\phi)(u, v)^T}{u}
\]
where $H$ represents the Hessian matrix.

**Lemma 2.7.** Let $\eta(u, v) \in C^1(\mathbb{R}_+)$ a Lipschitz function in a neighborhood of the origin, $q(u, v) = \psi(u, v) + \eta(u, v)\phi(u, v)$ be a function, such that $\psi$ satisfies
\[
\nabla \psi(u, v) = ((u, v) \cdot \nabla \eta(u, v) - \eta(u, v)) \nabla \phi(u, v).
\]

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Then the pair
\[(n(u,v), q(u,v)) \tag{2.15}\]
is an entropy-entropy flux pair for the system \[(2.8).\]
Moreover if \(\eta(u,v)\) is a convex function, then the pair \[(2.15)\]
is a convex entropy-entropy flux pair.

### 3 Global existence of weak entropy solutions and asymptotic behavior

We consider the parabolic regularization of the system \[(2.8),\]
\[
\begin{align*}
u_t + (u \phi(r))_x + au &= \epsilon u_{xx}, \\
v_t + (v \phi(r))_x + bv &= \epsilon v_{xx},
\end{align*}\]
\[(3.1)\]
with initial data
\[
\begin{align*}
u(x,0) &= u_0 * j, \\
v(x,0) &= v_0 * j,
\end{align*}\]
\[(3.2)\]
where \(j\) is a mollifier. In this case \(\phi(u,v) = \phi(r)\), with \(r = \sqrt{u^2 + v^2}\).
By \[(2.9)\] the eigenvectors and eigenvalues are given by
\[
\begin{align*}
\lambda_1(u,v) &= \phi(r) \\
\lambda_2(u,v) &= \phi(r) + r \phi'(r)
\end{align*}\]
\[(3.3)\]
\[(3.4)\]
The following conditions will be necessary in our next discussion
\[C_1 \lim_{r \to 0} r \phi(r) = 0, \quad r \phi'(r) \neq 0\]
\[C_2 \quad a > b\]
The condition \(C_1\) guarantees the strictly hyperbolicity to the system \[(3.2),\]
while condition \(C_2\) ensures the existence of a positive invariant region. Now we consider the following subset of \(\mathbb{R}\)
\[
\Sigma = \{(u,v) : \phi(r) \leq C_0, 0 < C_1 \leq \frac{u}{v} \leq C_2\}.
\]
We affirm that \(\Sigma\) is an invariant region. Let \(h(u,v) = (au,bv)\) be, if \((\overline{r}, \overline{u}) \in \gamma_1\) where \(\gamma_1\) is the level curve of \(Z = \phi(r)\) we have that
\[
(\nabla W \cdot h)(\overline{r}, \overline{u}) = (a+b)r \phi'(r) > 0
\]
and if \((\overline{r}, \overline{u}) \in \gamma_2\) where \(\gamma_2\) is the level curve of \(Z\) we have that
\[
(\nabla Z \cdot h)(\overline{r}, \overline{u}) = (a-b)\alpha_i > 0
\]
with \(i = 1, 2,\), then by the Theorem 14.7 of \[6,\] \(\Sigma\) is an invariant region for the system \[(6.1)\]. It is easy to verify that \((au,bv)\) satisfies the condition \(H_1 \cdots H_5\) in \[6,\] thus we have the following Lemma.

**Proposition 3.1.** If \((u_0,v_0) \in \Sigma\) and the \(C\) conditions holds, then the Cauchy problem \[(6.1),\]
\[(6.2)\]
has a global weak entropy solution.
Now for the global behavior of solutions, using ideas of the author in [4], we construct the following entropy-entropy flux pairs

\[ n(r) = r^m, \ m \leq 2. \]

From (2.14) we have

\[ q(r) = (m - 1) \int_0^r s^m \phi'(s) ds + r^m \phi(r), \]

Integrating by parts we have that

\[ q(r) = (m - 1) \int_0^r s^m \phi'(s) ds + r^m \phi(r), \]

integrating by parts we have

\[ q(r) = m\phi(r) - m(m - 1) \int_0^r s^{m-1} \phi(s) ds. \]

Let \( M = \sup_{(u,v) \in [0,\|u\|_{L^\infty}] \times [0,\|u\|_{L^\infty}]} \{ \phi(r) \}, \) then we have that

\[ |q(r)| \leq 2mMr^m. \] (3.5)

Multiplying in (2.8) by \( \nabla \eta \) we have that

\[ \eta(r)_t + q(r)_x \leq -3mMr^m \] (3.6)

Now we choose \( h(x) \in C^2(\mathbb{R}) \) a function such that \(|h'(x)| \leq 1, |h''(x)| \leq 1\) and \( h(x) = |x| \) for \(|x| \geq 1\) and set \( k(x) = e^{-h(x)} \), then \( k'(x) \leq k(x) \).

Multiplying by \( k(x) \) in (3.6), and integrating over \( x \) we have

\[ \frac{d}{dt} \int_{-\infty}^{\infty} \eta(r)g(x) \leq \int_{-\infty}^{\infty} q(r)k'(x) dx - 3mM \int_{-\infty}^{\infty} r^m dx \] (3.7)

by the inequality (3.5) we have

\[ \frac{d}{dt} \int_{-\infty}^{\infty} \eta(r)k(x) dx \leq -mM \int_{-\infty}^{\infty} r^m k(x) dx. \] (3.8)

If \( \psi(t) = \int_{-\infty}^{\infty} \eta(r)k(x) dx \) we have

\[ \frac{d}{dt} \psi(t) \leq -mM \psi(t), \]

by Gronwall’s inequality we have

\[ \psi(t) \leq e^{-mMt} \psi(0). \]

Thus we have

\[ \left( \int_{-\infty}^{\infty} r^m(t)k(x) dx \right)^{\frac{m}{m+1}} \leq e^{-Mt} \left( \int_{-\infty}^{\infty} r^m(0)k(x) dx \right)^{\frac{m}{m+1}} \] (3.9)

Passing to limit \( m \to \infty \) in (3.9) we have the inequality (1.5)
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