Canonical Chern-Simons Theory and the Braid Group on a Riemann Surface∗

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Abstract

We examine the problem of determining which representations of the braid group on a Riemann surface are carried by the wave function of a quantized Abelian Chern-Simons theory interacting with non-dynamical matter. We generalize the quantization of Chern-Simons theory to the case where the coefficient of the Chern-Simons term, $k$, is rational, the Riemann surface has arbitrary genus and the total matter charge is non-vanishing. We find an explicit solution of the Schrödinger equation. We find that the wave functions carry a representation of the braid group as well as a projective representation of the discrete group of large gauge transformations. We find a fundamental constraint which relates the charges of the particles, $q_i$, the coefficient $k$ and the genus of the manifold, $g$. 

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1 Introduction

It is by now well established that particles confined to a two dimensional space can have fractional statistics. Interest in such particles, which are called anyons, is partially motivated by their physical effects such as their conjectured role in the fractionally quantized Hall effect \[2\] or high temperature superconductivity \[3\], and partially by the fact that the description of anyons uses interesting mathematical structures. Anyons are a generalization of ordinary bosons or fermions where the wave functions of many identical particles, instead of being symmetric or antisymmetric, carry a representation of the braid group on their two dimensional configuration space \[4\]. The braid group on a two-dimensional space is an infinite, discrete, non-Abelian group and has many potentially interesting representations (see, for example \[10\]).

Anyons are sometimes described mathematically by coupling the currents of particles to the gauge field of a Chern-Simons theory. This coupling has been argued to produce fractional statistics both for the case where particles are excitations of a dynamical quantum field \[5\] and when the matter is non-dynamical classical point particles \[6\]. The representation of the braid group which arises (and therefore also the fractional statistics) can be either Abelian or non-Abelian. The former case arises from the quantization of Abelian Chern-Simons theory. It is also know that non-Abelian statistics can arise from either non-Abelian Chern-Simons theory or else Abelian Chern-Simons theory on a manifold whose fundamental group is non-trivial.

For an Abelian Chern-Simons theory, the action is

\[
S = -\frac{k}{4\pi} \int A dA + \int A_{\mu} j^\mu d^3 x \tag{1.1}
\]

where

\[
 j^\mu(x) = \sum_{i=1}^{n} q_i \int d\tau \frac{d r^\mu_i}{d\tau} \delta^3(x - r_i(\tau)) \tag{1.2}
\]

and \(r^\mu_i(\tau)\) is the trajectory and \(q_i\) the charge of the \(i\)'th particle. (If particles are identical, then their charges should be equal.)

In this paper we shall examine the question of which representations of the braid group on a Riemann surface are obtained from the wave functions of an Abelian Chern-Simons theory in the most general case where the constant \(k\) is a rational number, the Riemann surface has arbitrary genus \(g\) and the total charge of the particles is non-zero. We shall construct the wave functions of the quantum theory with action (1.1) explicitly and find that, depending on the coefficient \(k\) and the genus of the configuration space, the wave function carries certain multi-dimensional, in general non-Abelian representations of the braid group.

The wave function of Abelian Chern-Simons theory coupled to classical point particles on the plane was found by Dunne, Jackiw and Trugenberger \[16\]. In this case the Chern-Simons theory has no physical degrees of freedom, the Hilbert space is one-dimensional and the only quantum state is given by a single unimodular complex number. For a trajectory of \(n\) particles with positions \(z_i(t), \ i = 1, \ldots, n, \ t \in [0, 1]\) which is periodic up to a permutation,
$z_i(1) = z_{P(i)}(0)$, the phase of the wave function changes by the well-known factor

\[ q_i q_j \frac{1}{k} \sum_{i < j} \int_0^1 dt \frac{d}{dt} \text{Im} \ln (z_i(t) - z_j(t)) \]

which counts the changes of relative angles of positions of the particles. This can be interpreted as the wave function carrying a one-dimensional unitary representation of the braid group of order $n$ on the plane. The element of the braid group which generates an exchange of particles is represented by

\[ \sigma = e^{i \pi q^2} \]  

i.e., when two identical particles are interchanged, the wave function changes by the phase (1.3) (or some power of $\sigma$, depending on the exchange path). This yields a representation of the braid group on the plane.

Because of the Gauss’ law constraint (see ahead (3.4))

\[ \vec{\nabla} \times \vec{A} = \frac{2\pi k}{j} \delta^0(x) \]

the case when the two-dimensional space is compact is somewhat more complicated than that of the plane. In order to have an assembly of identical particles, it is necessary to have non-zero total charge. If we have non-zero total charge, Gauss’ law requires a non-zero total magnetic flux, which on a compact manifold means that the gauge connection $A$ is not a function but a section of a line bundle. This requires some modifications of the Chern-Simons action which we shall discuss in detail in Section 2 of this paper.

In previous literature, this complication has been avoided by considering more than one kind of particles so that their total charge adds to zero. In that case, Bos and Nair [9] solved the Schrödinger equation for Abelian Chern-Simons theory coupled to particles when the space is a Riemann surface of genus $g$ and when $k$, the coefficient of the Chern-Simons term, is an integer. They found that the wave functions carry a representation of the braid group on the Riemann surface. They found that the Hilbert space is $k^g$ dimensional and the wave functions are conveniently represented by a set of theta functions. In a previous work, [12] we found a generalization of their quantization to the case where $k$ is a rational number. We found that the wave functions were essentially composed of theta functions with rational indices and discussed their properties.

In that work we found that the correct geometrical description of Chern-Simons theory on a Riemann surface necessarily introduces a framing of particle trajectories. Framing is a standard part of the study of the relationship between the Chern-Simons theory and knot polynomials in the Lagrangian path-integral approach which was first introduced by Witten [8]. Variants of framing (such as the point splitting discussed by Bos and Nair [9]) have also appeared in literature on the Hamiltonian approach to quantizing Chern-Simons theory. Here, we shall find that our geometrical approach to framing plays an important role in the consistency relations between the parameters $k$, $g$ and the values of the charges of particles $q_i$.

We shall start from Abelian Chern-Simons theory coupled to the current due to a gas of identical charged particles (1.1). In Section 2 we discuss the definition of the gauge field in the case where the total charge (and therefore total magnetic flux) is not zero. We demonstrate
how to define the gauge field on patches and extract the physical degrees of freedom from its non-trivial cohomology classes and the complex structure of the Riemann surface. In section 3, we quantize the theory and discuss the solutions of the Schrödinger equation. In section 4, we use invariance under large gauge transformations and modular transformations to show that the Hilbert space is finite dimensional. Finally, in section 5, we will show that wave functions are functionals of the trajectories of particles in such a way that they carry a pure $\theta$-statistics (see subsection 1.1) representation of the braid group.

In the following subsection we shall give a brief review of the braid group on a Riemann surface.

1.1 The Braid Group on a Riemann Surface

Let us consider $n$ identical particles, with coordinates $X = (x_1, \ldots, x_n)$, on a two dimensional manifold $\mathcal{M}$. The positions of particles at a given time is given by the set of $n$ coordinate functions

$$X(t) \equiv (x_1(t), \ldots, x_n(t))$$

The classical configuration space of this system is $\mathcal{M}^n$. Since the particles are identical, and if we assume that they cannot occupy the same positions (Pauli principle), the quantum configuration space is obtained by subtracting the diagonal sub-space (where the positions of two or more particles coincide)

$$\Delta = \{ X \in \mathcal{M}^n | x_i = x_j, \text{ for any } i \neq j \}$$

and then factoring the remaining space by the permutation group $S_n$ to obtain

$$Q_n(\mathcal{M}) = \frac{\mathcal{M}^n - \Delta}{S_n}$$

A trajectory of the particles which is periodic up to a permutation of the positions is a closed loop on $Q_n(\mathcal{M})$. The allowed statistics are given by the representation of the fundamental group, $\Pi_1(Q_n(\mathcal{M})) = B_n(\mathcal{M})$, which is the braid group.

An element of $B_n$, called a braid, is a periodic trajectory, (i.e. $X(t), t \in [0, 1]$, such that $x_i(1) = x_{P(i)}(0)$ where $P(i)$ is a permutation). It can be represented pictorially by $n$ strings, each string depicting the trajectory of one particle. Since the particles are not allowed to pass through each other, the strings do not intersect.

The composition law of two braid elements for this group corresponds to attaching the beginning of the second braid to the end of the first braid, on the common configuration $X$, to form one new braid. The identity element is $n$ non-braiding strings. It can also be shown that the inverse of a braid exists. (It corresponds to applying the inverse of each generator in inverse order of the original braid, as defined below.)

Let us study the case $R^2$. It can be shown [10] that we can represent an arbitrary braid in terms of $n - 1$ generators $\sigma_i$, that represent the exchange of the strings which we label $i$ and $i + 1$. The string $i$ can go around the string $i + 1$ by going either in front or behind it; we have to choose one of this move (similar to the right hand rule) to represent $\sigma_i$. The other move correspond to $\sigma_i^{-1}$, since we do find that $\sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1$. 


The generators are subject to the relations
\[ \sigma_i \sigma_j = \sigma_j \sigma_i \quad |i - j| \geq 2 \]  
\[ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad 1 \leq i \leq n - 2 \]  
It is straightforward to verify that these relations correspond to identical braids as we would visualize them in three dimensions.

The braid group on an arbitrary Riemann surface \( \mathcal{M} \) of genus \( g \) has more generators. In fact, by taking the string 1, we can associate to each homology generator \( a_l \) and \( b_l \) of \( \mathcal{M} \), see ahead (2.6), a corresponding braid group generator, that we will call \( \alpha_l \) and \( \beta_l \). Now in addition to the relations (1.4) and (1.5), there are a number of additional structure relations as follow:

\[ [\alpha_l, \beta_l] = 0 \quad 2 \leq i \leq n - 1; \ 1 \leq l \leq g \]
\[ \sigma_1 \alpha_p \sigma_1 \alpha_l = \alpha_l \sigma_1 \alpha_p \sigma_1 \quad p \geq l; \ 1 \leq l, p \leq g \]
\[ \sigma_1 \beta_l \sigma_1 \beta_l = \beta_l \sigma_1 \beta_l \sigma_1 \quad 1 \leq l \leq g \]
\[ \sigma_1 \beta_p \sigma_1 \beta_l = \alpha_l \sigma_1 \beta_p \sigma_1 \quad p \geq l; \ 1 \leq l, p \leq g \]
\[ \sigma_1 \alpha_p \sigma_1 \beta_l = \beta_l \sigma_1 \alpha_p \sigma_1 \quad p > l; \ 1 \leq l, p \leq g \]
\[ \sigma_1^{-1} \alpha_l \sigma_1 \beta_l = \beta_l \sigma_1 \alpha_l \sigma_1 \quad 1 \leq l \leq g \]  
These relations are necessary so that topologically equivalent braids are represented by identical elements of the braid group. There is one additional relation which follows from the fact that there always exists a trajectory of a particle which encircles all other particles and traces all homology generators of \( \mathcal{M} \) and which is equivalent to a trivial loop on \( Q_n(\mathcal{M}) \).

This leads to the relation
\[ \sigma_1 \cdots \sigma_{n-1} \cdots \sigma_1 \beta_g \beta_{g-1} \cdots \beta_1 (\alpha_1^{-1} \beta_1^{-1} \alpha_1) \cdots (\alpha_g^{-1} \beta_g^{-1} \alpha_g) = 1 \]  
These generators and relations constitute a presentation of the general abstract braid group. In most cases, we are interested in representations of this group, even finite dimensional ones.

The representations which follow from Abelian Chern-Simons theory are the so-called pure \( \theta \)-statistics representations where the generator of an interchange of neighboring particles is represented by a phase, times a unit matrix, as the \( \sigma \) in (1.3). In these particular type of representations, the generators for particle exchanges \( \sigma_i \) and those for transport around handles satisfy a far less restrictive set of relations due to the Abelian structure of these \( \sigma_i \). They satisfy the relations (1.4) trivially while the relations (1.5) tell us that the \( \sigma_i \) are equal, which we will called \( \sigma \). The remaining relations (1.6) becomes
\[ [\sigma, \alpha_l] = [\sigma, \beta^l] = [\alpha_l, \alpha_m] = [\beta^l, \beta^m] = 0 \]
\[ [\alpha_l, \beta^m] = 0 \quad \text{for} \quad l \neq m \]
\[ \alpha_l \cdot \beta^l = \sigma^2 \beta^l \cdot \alpha_l \]  
and the global constraint (1.7) for closed manifold is
\[ \sigma^{2(n+g-1)} = 1 \]

We will show that the wave functions of a Chern-Simons action coupled to charges gives pure \( \theta \)-statistic representations of the braid group on a Riemann surface, as these charges form braids in space-time.
The decomposition of the gauge field

Our space will be an orientable 2-dimensional Riemann surface, \( \mathcal{M} \), of genus \( g \). While our space-time will be a 3-dimensional manifold formed as the Riemann surface \( \mathcal{M} \) times a real line for the time direction. In other words, the space-time metric is \( g_{00} = 1 \), \( g_{01} = g_{02} = 0 \) and the remaining components form the metric on \( \mathcal{M} \). Since we have to consider the case of a non-zero total flux on \( \mathcal{M} \), the representation of this type of gauge field can be done only on a set of patches covering \( \mathcal{M} \). Let us consider the set of patches \( U^i \) as a good cover of the manifold \( \mathcal{M} \). We have a field \( A^{(i)} \) on each patch \( U^i \), with the transition functions defined on the intersection of any two patches \( U^i \cap U^j \) given by

\[
A^{(i)} - A^{(j)} = d\chi^{(ij)} \tag{2.1}
\]

where \( \chi^{(ij)} = -\chi^{(ji)} \) by definition. On triple intersection \( U^i \cap U^j \cap U^k \) we can use (2.1) to find the relation

\[
\chi^{(ij)} + \chi^{(jk)} + \chi^{(ki)} = c^{(ijk)} = \text{constant} \tag{2.2}
\]

The set of constants \( c^{(ijk)} \) are related to the total flux by [15]

\[
F_0 = \int_{\mathcal{M}} dA = \sum_i \int_{V^i} dA^{(i)} = \sum_{ij} \int_{U^{ij}} d\chi^{(ij)} = \sum_{P^{ijk}} c^{(ijk)} \tag{2.3}
\]

where \( V^i \subset U^i \) and it is bounded by a line, \( V^{ij} \), dividing the intersection \( U^i \cap U^j \). On the triple intersection, we let the three lines \( V^{ij}, V^{jk} \) and \( V^{ki} \) meet at one point \( P^{ijk} \).

Before quantizing, we will decompose the degrees of freedom of \( A \) in its various components. To separate the effect of the non-zero total flux (2.3) we will break it in two parts. First a fixed field \( A_p \) with a total flux \( F_0 \) on \( \mathcal{M} \) localized at a reference point \( z_0 \). This is an "imaginary" field without a direct physical meaning, its purpose is to take care of the total flux. This will be the field that has to be defined on patches, as explained above. The second field, \( A_r \), is the remaining degree of freedom of \( A \) on \( \mathcal{M} \), a globally well defined 1-form. So we have

\[
A = A_p + A_r \tag{2.4}
\]

We decompose \( A_r \) (without the \( A_0 dt \) part) into its exact, coexact and harmonic parts. More precisely, the Hodge decomposition of \( A_r \), on \( \mathcal{M} \), is given by (\( d \) and \( * \) act on \( \mathcal{M} \) in this paper)

\[
A_r = d\left( \frac{1}{\Box} \ast d^* A_r \right) + \ast d\left( \frac{1}{\Box} \ast dA_r \right) + \frac{2\pi i}{\tilde{k}} \sum_{l=1}^{g} (\bar{\gamma}_l \omega^l - \gamma_l \bar{\omega}^l) \tag{2.5}
\]

where \( 1/\Box' \) is the inverse of the laplacian (\( \Box \)) acting on 0-forms where the prime means that the zero modes are removed. With our decomposition (2.4), \( dA_r \) do not have a zero mode. Also we will set the zero mode of \( \ast d^* A_r = \vec{\nabla} \cdot \vec{A}_r \) to zero, using a time independent gauge transformation.

The first homology and cohomology group of \( \mathcal{M} \) tell us the number of additional degrees of freedom of \( A \) there are on \( \mathcal{M} \) compared to a plane, and how to take them into account. The zero modes for both \( d \) and \( \ast d \) (or for \( \Box \)) acting on a one-form are spanned by the set of Abelian differentials, \( \omega^l \), on \( \mathcal{M} \), called the holomorphic (function of \( z \)) harmonic forms.
We can represent the homology of $\mathcal{M}$ in terms of generators $a_l$ and its conjugate generators $b_l$, $l = 1, \ldots, g$. The intersection numbers of these generators are given by

$$\nu(a_l, a_m) = \nu(b_l, b_m) = 0, \ \nu(a_l, b_m) = - \nu(b_m, a_l) = \delta^{lm}$$

where $\nu(C_1, C_2)$ is the signed intersection number (number of right-handed minus number of left handed crossings) of the oriented curves $C_1$ and $C_2$. The holomorphic harmonic one-forms $\omega^l$ have the standard normalization

$$\int_{a_l} \omega^m = \delta^{lm}, \ \int_{b_l} \omega^m = \Omega^{lm}$$

The matrix $\Omega^{lm}$ is symmetric and its imaginary part is positive definite. This actually defines a metric in the space of holomorphic harmonic forms

$$i \int_{\mathcal{M}} \omega^l \wedge \bar{\omega}^m = 2 \text{Im}(\Omega^{lm}) = G^{lm}, \ G_{lm}G^{mn} = \delta^{ln}$$

We will use $G_{lm}$ and $G^{lm}$ to lower or raise indices when needed and use Einstein summation convention over repeated indices.

Any linear relation, with integer coefficients, of $a_l$ and $b_l$ that satisfy (2.6) is another valid basis for the homology generators. These transformations form a symmetry of the Chern-Simons theory and comprise the modular group, $Sp(2g, \mathbb{Z})$:

$$\begin{pmatrix} a \\ b \end{pmatrix} \rightarrow S \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{where} \quad S = \begin{pmatrix} D & C \\ B & A \end{pmatrix}$$

with $SES^{\top} = E$ and $E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The $g \times g$ matrices $A, B, C, D$ have integer entries.

We can define

$$\xi = - \frac{k}{2\pi} \frac{1}{\Box'} d^* A_r, \ \ F_r = * d A_r$$

So we then have the complete decomposition of the gauge field, with the $A_0 dt$ part,

$$A_r = A_0 dt - \frac{2\pi}{k} d\xi + * d(\frac{1}{\Box'} F_r) + 2\pi i (\bar{\gamma}_l \omega^l - \gamma_l \bar{\omega}^l)$$

Similarly we can write the current $j = j^\mu \frac{\partial}{\partial x^\mu}$ into a one-form $j = j_\mu dx^\mu = j_0 dt + \tilde{j}$, using the metric. We can use again the Hodge decomposition of $* \tilde{j}$ on $\mathcal{M}$

$$* \tilde{j} = - d\chi + * d\psi + i(j_\mu \bar{\omega}^\mu - \bar{j}_l \omega^l)$$

The continuity equation, using the 3-dimensional star operator $*$,

$$d^* j = (\hat{\nabla} \cdot \tilde{j}) d^3 x = \frac{\partial j_0}{\partial t} d^3 x + d^* \tilde{j} \wedge dt = 0$$

can be used to solve for $\psi$

$$\psi = - \frac{1}{\Box'} \frac{\partial j_0}{\partial t}$$
We shall consider a set of point charges moving on $\mathcal{M}$, with trajectories $z_i(t)$ and charge $q_i$, where $z_i(t) \neq z_j(t)$ for $i \neq j$. The current is represented by

$$j_0(z,t) = \sum_{i} q_i \delta(z - z_i(t)), \quad \tilde{j}(z,t) = \sum_{i} q_i \delta(z - z_i(t)) \frac{1}{2}(\dot{z}_i(t)d\bar{z} + \dot{\bar{z}}_i(t)dz) \quad (2.11)$$

Integrating (2.11) with the harmonic forms $\omega^j$, we find the topological components of the current in (2.10)

$$j^l(t) = \sum_{i} q_i \dot{z}_i(t)\omega^j(z_i(t)) \quad \tilde{j}^l(t) = \sum_{i} q_i \dot{\bar{z}}_i(t)\tilde{\omega}^j(\tilde{z}_i(t)) \quad (2.12)$$

This is just telling us that integrating the topological currents $j^l(t)$ over time is equivalent to a sum of the integral of the harmonic forms $\omega^j$ over each charge trajectory.

To solve (2.13) for $\chi$, it is best to use complex notation

$$R = \psi + i\chi = R(z, \bar{z})$$

where we find $*d\chi + d\psi = \partial_z \bar{R}dz + \partial_{\bar{z}} R d\bar{z}$. From (2.10), (2.11) and using (2.12) we find

$$\partial_z \bar{R} + \bar{j} \omega^l(z) = \frac{1}{2} \sum_{i} q_i \dot{z}_i \delta(z - z_i(t))$$

$$\partial_{\bar{z}} R + j \bar{\omega}^l(\bar{z}) = \frac{1}{2} \sum_{i} q_i \dot{\bar{z}}_i \delta(z - z_i(t)) \quad (2.13)$$

To solve (2.13) for $R$, we will need the prime form

$$E(z, w) = (h(z)h(w))^{-\frac{1}{2}} \cdot \Theta \left( \frac{1}{2} \frac{1}{2} \right) \left( \int_z^w \omega |\Omega \right)$$

where $h(z) = \frac{\partial}{\partial \sigma} \Theta \left( \frac{1}{2} \frac{1}{2} \right) \left( u |\Omega \right)_{u=0} \cdot \omega^l(z)$. The prime form is antisymmetric in the variables $z$ and $w$ and behaves like $z - w$ when $z \approx w$ (the $h(z)$ which appear in the denominator are for normalization). \footnote{This formalism can also be extended to include the sphere, where there are no harmonic 1-forms at all (the space of cohomology generators has dimension zero) by properly defining the prime form. We use stereographic projection to map the sphere into the complex plane and use $E(z, w) = z - w$ as the definition of the prime form.}

The theta functions [14] are defined by

$$\Theta \left( \frac{\alpha}{\beta} \right) (z|\Omega) = \sum_{m} e^{i\pi(n_1+\alpha_i)\Omega^{lm}(n_m+\alpha_m)+2\pi i(n_1+\alpha_i)(z^l+\beta^l)} \quad (2.14)$$

where $\alpha, \beta \in [0, 1]$, and have the following property

$$\Theta \left( \frac{\alpha}{\beta} \right) (z^m + s^m + \Omega^{ml}t_l|\Omega) = e^{2\pi i\alpha s^l - i\pi l_m t_l - 2\pi i t_m (z^m + \beta^m)} \Theta \left( \frac{\alpha}{\beta} \right) (z|\Omega)$$
for integer–valued vectors $s^m$ and $t_i$. For a non-integer constant $c$

$$\Theta\left(\frac{\alpha}{\beta}\right)(z^m + c\Omega^{ml}t_l|\Omega) = e^{-i\pi c^2 t_m \Omega^{ml} t_l - 2\pi i c t_m (z^m + \beta^m)} \Theta\left(\frac{\alpha - ct}{\beta}\right)(z|\Omega)$$

The solution of (2.13) is

$$R = \frac{\partial}{\partial t}\left[\frac{1}{2\pi} \sum_i q_i \log\left(\frac{E(z, z_i(t))}{E(z_0, z_i(t))}\right)\right] - j(t) \int_{z_0}^{z} (\omega^t - \omega^l)$$

(2.15)

where we have chosen $R$ such that $R(z_0, z_0) = 0$ for an arbitrary point $z_0$, which we choose to be the same as the $z_0$ in the definition of $A_p$. (We can choose $z_0 = \infty$ for genus zero). The important fact about $R$ is that it is a single-valued function. If we move $z$ around any of the homology cycles, $R$ returns to its original value. In fact, this is also true for windings of $z_0$, an important relation since it is only a reference point. So

$$\chi = \frac{\partial}{\partial t}\left[\frac{1}{2\pi} \sum_i q_i \text{Im} \log\left(\frac{E(z, z_i(t))}{E(z_0, z_i(t))}\right)\right] + \frac{i}{2} \left[\left\langle j(t) + \bar{j}(t)\right\rangle \int_{z_0}^{z} (\omega^t - \omega^l)\right]$$

(2.16)

The action (1.4) is written for a trivial U(1) bundle over $\mathcal{M}$, that is for zero total flux. Every integral of the gauge field, which is invariant under a gauge transformation of $A$, can be extended uniquely into an integral using the $A^{(i)}$, defined on the set of patches, that is patch independent by adding appropriate terms. We will represent our set of 3-dimensional $V$ where the one-form $W$ is defined by $*j = dW$ locally, but since $\int_{\mathcal{M}} *j = Q$, this can be done only on patches where $W^{(i)} = W^{(j)} = d\chi^{(ij)}$, in the same way as we did for the gauge field $A$. The bracket $\langle ... \rangle$ mean put the indices in increasing order (with appropriate sign) and set the repeated index according to position, see [17]. It will be useful to do the same decomposition of $j$ as we did for $A$, by having $j = j_p + j_r$, where $j_p$ is a term corresponding to a single particle of charge $Q$ at the reference point $z_0$.

The complicated expression (2.17) for the action ensure that the total expression is independent of the triangulation of the manifold used for the evaluation of each integral. For example, if we change the patches $V^i$, the integrand in the first term will change by a total derivative, leading to a correction term integrated over the boundaries of the $V^i$, that is
the $V^{ij}$. But the second term, in return, will change in such a way as to cancel the change generated by this first term, leaving the total action invariant. A similar effect can be found for the other terms in (2.17).

Using (2.4), the decomposition of $j$ and performing several integrations by parts gives

$$S = -\frac{k}{4\pi} \int_{\mathcal{M} \otimes \mathbb{R}} A_r dA_r + \frac{k}{2\pi} \int_{\mathcal{M} \otimes \mathbb{R}} A_r dA_p + \int_{\mathcal{M} \otimes \mathbb{R}} A_r (\ast j_r + \ast j_p) + \int_{\mathcal{M} \otimes \mathbb{R}} W_r dA_p$$

$$+ \left[ -\frac{k}{4\pi} \int A_p dA_p + \int A_p \ast j_p \right] + \text{Surface terms} \quad (2.18)$$

The terms in brackets, involving $A_p$ and $j_p$, has to be performed using the extended decomposition (2.17), by replacing $A$ with $A_p$. For our case, we extend the triangulation of $\mathcal{M}$ trivially through the time direction. The surface terms, appearing at the time boundaries ($t = 0$ and $t = t_f$), are not important for the quantum theory or the braid group representation that we will find later on. They can’t be avoided since the action is not invariant under gauge transformations at the time boundaries. Thus there is no terms to cancel the triangulation dependent terms. This will not be a problem since under a periodic configuration we are effectively working on $\mathcal{M} \times S^1$, so there is no surface term, or alternatively the surface terms are equal and cancel each other. Also, surface terms do not affect the dynamics or quantization of the system. We also represent $A_p$ such that $dA_p = F_0 \delta(z - z_0) d^2 x$, which implies that $z_0$ must stay within one patch at all time. And similarly for $j_p$ since it is equal to $Q \delta(z - z_0) dt$. After a quick calculation, we find that the terms inside the brackets are all zero, except for the integral, $\oint_c <\epsilon^{ijk} W^k>$, which is defined modulo $c^{(ijk)}Q$ (for periodic motion). This is because $W$ is defined on patches also, due to the total charge $Q$. At the quantum level, we are left with a phase $e^{i(c^{ijk})Q}$, but since the $c^{(ijk)}$ are arbitrary except for the constraint (2.3), the real ambiguity is $e^{iQF_0}$. Actually, the integral $\int A \ast j$ is equal to $\sum_i q_i \int_{C_i} A$, the Wilson line integral for a set of charges $q_i$ following the curves $C_i$. In this case for each of these Wilson line integrals, corresponding to the charge $q_i$, we find a phase $e^{iq_i F_0}$ instead. To solve these ambiguities we impose these phases to be equal to unity as constraints on our system.

On the other hand, if in addition to the gauge field $A$, we had a second independent Abelian gauge field, say $\Gamma$, then a similar phase ambiguity, $e^{ih_i \chi E}$, would arise. Here $h_i$ will be the charge attached to the particle $i$ corresponding to this new field, and $\chi E = \int_{\mathcal{M}} d\Gamma$ is the total flux. The important fact, now, is that the phase ambiguity from both gauge fields would appear at the same time, thus we would have to impose the constraint

$$e^{iq_i F_0 - ih_i \chi E} = 1 \quad (2.19)$$

to obtain a consistent quantum theory (the minus sign has been added to simplify the notation later on). At this stage, the new field $\Gamma$ seems artificial, but it turns out that it is necessary to introduce such a field for Chern-Simons theory. In fact, it correspond to a connection on the tangent space of $\mathcal{M}$. We will need it because for each charge trajectory we will attach a framing (a unit vector on $\mathcal{M}$). Such a framing has to be defined in relation to the basis of the tangent space, so $\Gamma$ does not have to be the associated metric connection, but it will enjoy the same global properties. It is well known that $\chi E = 4\pi(1 - g)$, known as the Euler class of $\mathcal{M}$. Note that we will assume that the field $\Gamma$ does not have any flux.
around the particles (an effect similar to cosmic string), this would lead to a gravitational change in the statistic of these particles. The charges $q_i$ will be equal to $q_i^2/2k$, this will appear quite naturally in the next section. Like we did for the field $A$, we will concentrate all the flux, $\chi_E$, of $\Gamma$ around the point $z_0$. This will allow us to assume a constant framing on $M$, except when we cross the point $z_0$, in which case the constraint (2.19) will be used to fix any phase ambiguity.

The term $\int_{M \otimes \mathcal{R}} \omega \cdot d\mathbf{A}_\nu = F_0 \int_{\mathcal{R}} \omega_{\nu 0}(z_0) \cdot \mathbf{d}t$, but a simple calculation shows that $\omega_{\nu 0} = -\chi$. Since $\chi(z_0) = 0$, we set it up this way by definition, this term vanishes. If we had not used our freedom in the definition of $\chi$ to set it up this way, we would have to take care of its effects on the hamiltonian and ultimately the wave function.

3 Quantization

Now we are ready to solve for the action. By putting (2.9) and (2.10) back into (2.18) we find

$$S = \frac{1}{2} \int (\xi \dot{F}_r - \dot{\xi} F_r) d^3x + i\pi k \int (\gamma^I \dot{\gamma}_I - \dot{\gamma}^I \gamma_I) dt + \int A_0 (j_0 - \frac{k}{2\pi} F) d^3x$$

$$- \int \left( \frac{2\pi}{k} \xi \frac{\partial j_0}{\partial t} + F_r \chi \right) d^3x + 2\pi i \int (j_I \dot{\gamma}_I - \dot{j}_I \gamma_I) dt + \text{Surface terms}$$

(3.1)

From this we obtain the equal-time commutation relations of the quantum theory

$$[\xi(z), F_r(w)] = -iP \delta(z - w) \quad \text{or} \quad F_r(z) = iP \frac{\delta}{\delta \xi(z)}$$

(3.2)

and

$$[\gamma^I, \bar{\gamma}_m] = -\frac{1}{2\pi k} G_{lm} \quad \text{or} \quad \bar{\gamma}_l = \frac{1}{2\pi k} G_{lm} \frac{\partial}{\partial \gamma_m} = \frac{1}{2\pi k} \frac{\partial}{\partial \gamma_l}$$

(3.3)

The projection operator, $P$, in (3.2) changes the delta function to $\delta(z - w) - 1/\text{Area}(M)$, this is needed since $F_r$ does not have a zero mode ($\int_M F_r d^2x = 0$). The functional derivative must also be defined using this projection operator. With this holomorphic polarization [9] it is convenient to use the following measure in $\gamma$ space

$$(\Psi_1 | \Psi_2) = \int e^{-2\pi k \gamma^m G_{ml} \gamma^l} \Psi_1^*(\bar{\gamma}) \Psi_2(\gamma) |G|^{-1} \prod_m d\gamma^m d\bar{\gamma}^m$$

where $|G| = \det(G_{mn})$. With this measure, we find that $\gamma^l = \bar{\gamma}$ as it should be.

$A_0$ is a Lagrange multiplier which enforces the Gauss' law constraint

$$F(z) - 2\pi k j_0(z) = iP \frac{\delta}{\delta \xi(z)} + F_0 \delta(z - z_0) - \frac{2\pi}{k} j_{r0}(z) + \frac{2\pi}{k} Q \delta(z - z_0) \approx 0$$

from which we extract $F_0 = 2\pi k Q$. Since $F_0$ and $Q$ are not quantum variables, this is a strong equality. Thus leaving

$$F_r(z) - 2\pi k j_{r0}(z) = iP \frac{\delta}{\delta \xi(z)} - \frac{2\pi}{k} j_{r0}(z) \approx 0$$

(3.4)
Under a modular transformation, the basis $\gamma^I$, $\tilde{\gamma}^I$ will be transformed accordingly. This will not change the choice of polarization, since the modular transformations do not mix $\gamma$ and $\tilde{\gamma}$.

From (3.1), (3.2) and (3.3), we find that the hamiltonian, in the $A_0 = 0$ gauge, can be separated into two commuting parts (where we used $\frac{\partial j_0}{\partial t} = \frac{\partial j_r}{\partial t}$)

$$H = -\int_M A^* \tilde{j} = H_0 + H_T$$

where

$$H_0 = \int_M \left(\frac{2\pi}{k} \xi \frac{\partial j_r}{\partial t} + i\chi P \frac{\delta}{\delta\xi}\right) d^2x$$

(3.5)

while the additional part that takes care of the topology is

$$H_T = i(2\pi \bar{j} \gamma^I - \frac{1}{k} j^I \frac{\partial}{\partial \gamma^I})$$

(3.6)

To solve the Schrödinger equation, we will use the fact that the hamiltonian separates, thus writing the wave function as

$$\Psi(\xi, \gamma, t) = \Psi_0(\xi, t) \Psi_T(\gamma, t)$$

with the Gauss’ law constraint (3.4)

$$(iP \frac{\delta}{\delta\xi} - \frac{2\pi}{k} j_\alpha ) \Psi_0(\xi, t) = 0$$

which is solved by

$$\Psi_0(\xi, t) = \exp\left[-\frac{2\pi i}{k} \int_M \xi(z)j_\alpha(z, t)d^2x\right]\Psi_c(t)$$

(3.7)

Note that in (3.7) there is a term $-Q\xi(z_0)$ out of the integral, this shows the presence of an ”imaginary” charge at $z_0$, with a flux $F_0$.

The first Schrödinger equation is

$$i \frac{\partial \Psi_0(\xi, t)}{\partial t} = H_0 \Psi_0(\xi, t) = \left[\int_M \left(\frac{2\pi}{k} \xi \frac{\partial j_r}{\partial t} + i\chi P \frac{\delta}{\delta\xi}\right) d^2x\right] \Psi_0(\xi, t)$$

which has the solution [8]

$$\Psi_c(t) = \exp \left[-\frac{2\pi i}{k} \int_0^t \int_M \chi(z, t')j_\alpha(z, t')d^2x dt'\right]$$

(3.8)

For a system of point charges, the use of (2.16) with (3.7) and (3.8), allows us to write $\Psi_0$ as

$$\Psi_0(\xi, t) = \exp \left[-\frac{2\pi i}{k} \sum_i q_i \xi(z_i(t)) - Q\xi(z_0)) + \frac{i}{2k} \sum_{ij} q_i q_j \int_0^t dt' \delta_{ij}(t) + \Phi(t)\right]$$

(3.9)
where

\[ \Phi(t) = \frac{\pi}{k} \left[ \int_0^t j_i(t') dt' \int_0^{t'} j_j(t'') dt'' - \int_0^t \bar{j}_i(t') dt' \int_0^t \bar{j}_j(t'') dt'' \right] \]

\[ + \frac{\pi}{2k} \left[ \int_0^t \bar{j}_i(t') dt' \int_0^t \bar{j}_j(t') dt' - \int_0^t j_i(t') dt' \int_0^t j_j(t') dt' \right] \]

and

\[ \theta_{ij}(t) = \text{Im} \log \left[ \frac{E(z_i(t), z_j(t))}{E(z_i(0), z_j(0))} \right] \]

\[ + \text{Im} \left[ \int_{z_i(0)}^{z_i(t)} \omega^j \int_{z_j(0)}^{z_j(t)} (\omega_l + \bar{\omega}_l) + \int_{z_i(0)}^{z_i(t)} \omega^j \int_{z_j(0)}^{z_j(t)} (\omega_l + \bar{\omega}_l) \right] \]

is a multi-valued function defined using the prime form. We will need the phase (3.10) for the topological part of the wave function. The function \( \theta_{ij}(t) \) is the angle function for particle \( i \) and \( j \).

For \( i = j \), we find a self-linking term of the form \( \text{Im} \log(z_i - z_i) = \text{Im} \log(0) \) which is an undetermined expression, although not a divergent one. One way to solve the problem is to choose a framing

\[ z_i(t) = z_j(t) + \epsilon f_i(t) \] (3.12)

which leads to the replacement of \( E(z_i(t), z_i(t)) \) by \( f_i(t) \). This correspond to a small shift in the position of the charges in \( j_\pi \), but not in \( \chi \). In effect, this leads to a small violation of the continuity equation. Alternatively, we can view this term as the additional gauge field \( \Gamma \) introduced in the last section. With the framing (3.12), we find that

\[ \int \frac{q_i^2}{2k} \theta_{ii} dt = h_i \int_{C_i} \Gamma \]

where \( C_i \) is the trajectory of \( q_i \) on \( \mathcal{M} \), representing a coupling of the particles, of charges \( h_i \), to an Abelian gauge field \( \Gamma \), as claimed in the last section. We also recover these charges as \( h_i = \frac{q_i^2}{2k} \), which actually are the conformal weights of the underlying two dimensional conformal field theory [8].

The angle function (3.11) depends on \( z_0 \), but it should be invariant if we move \( z_0 \) either infinitesimally or around an homology cycle. For a small displacement there is no change unless one of the charge trajectories, \( z_i(t) \), passing by \( z_0 \) from one side is now going from the other side. Looking at the denominator of \( \theta_{ij} \), we see that this will change \( \Psi_0 \) by \( e^{i2\pi q_i Q} \), while looking at the numerator, we find a phase \( e^{i2\pi q_i^2(g-1)} \) due to the flux \( \chi_E \) of \( \Gamma \). Or alternatively, the framing of \( z_i \) is subject to a rotation of \( \chi_E/2\pi = 2(1 - g) \) turns as we go around \( \mathcal{M} \), an effect that we concentrated around \( z_0 \) here. The total phase shift is

\[ e^{i2\pi q_i (Q + q_i (g-1))} = 1 \] (3.13)

This is equal to one by imposing the constraint (2.19), with the use of the Gauss’ law constraint \( F_0 = \frac{2\pi}{k} Q \) and our choice of \( \chi_E \). The equation (3.13) will represent a fundamental constraint that has to be satisfied by all charges if we want a consistent solution to Chern-Simons theory.
Looking at (3.11) shows that we can write \( \sum_{ij} q_i q_j \text{Im} \log[E(z_i(t), z_0) E(z_0, z_j(t))]^{-1} \) as \( \sum_i q_i (-Q) \text{Im} \log[E(z_i(t), z_0)] + \sum_j (-Q) q_j \text{Im} \log[E(z_0, z_j(t))] \), thus representing an additional charge \(-Q\) at \(z_0\). The constraint (3.13) is indicating that this is indeed an "imaginary" charge and that it should not be seen by any real charge. For the displacement of \(z_0\) around an homology cycle, we find that the angle function changes only by a constant, thanks to the second term in (3.11), which will cancel out when we take the difference in (3.9). This point is actually more complicated; we will come back to it later on. So the wave function (3.9), with the angle function (3.11), accurately forms a representation of the braid group on a plane \([8, 16]\), or \(\sigma\) is one of the generator of the full braid group (1.8)-(1.9). We will cover the full braid group in more detail later on.

Now, the topological part of the hamiltonian is used to find the part of the wave function affected by the currents going around the non-trivial loops of \(\mathcal{M}\). The Schrödinger equation for (3.6) is

\[
i \frac{\partial \Psi_T(\gamma, t)}{\partial t} = H_T \Psi_T(\gamma, t) = i \left( 2\pi j_l \gamma_l - \frac{1}{k} \frac{\partial}{\partial \gamma_l} \right) \Psi_T(\gamma, t)
\]

which has the solution

\[
\Psi_T(\gamma, t) = \exp \left[ 2\pi j_l \int_0^t \tilde{j}_l(t') dt' - \frac{2\pi}{k} \int_0^t \tilde{j}_l(t') dt' \int_0^{t'} \tilde{j}_l(t'') dt'' \right] \tilde{\Psi}_T(\gamma, t)
\]

(3.14)

Note that with the phase (3.10), the double integral above will turn into \( \int_0^t \tilde{j}_l(t') dt' \cdot \int_0^t \tilde{j}_l(t) dt' \), a topological expression.

The remaining equation for \( \tilde{\Psi}_T(\gamma, t) \)

\[
\frac{\partial \tilde{\Psi}_T(\gamma, t)}{\partial t} = -\frac{1}{k} j_l \frac{\partial \tilde{\Psi}(\gamma, t)}{\partial \gamma_l}
\]

(3.15)

is easily solved in the form

\[
\tilde{\Psi}_T(\gamma, t) = \tilde{\Psi}_T(\gamma_l - \frac{1}{k} \int_0^t j_l(t') dt')
\]

(3.16)

4 Large gauge transformations

The wave function (3.16) is not arbitrary, but must satisfy the invariance of the action (1.1) under large gauge transformations, when there is no current around. So let us set \( j^\mu = 0 \) for this section and find the condition on \( \tilde{\Psi}_T \).

In general, the large \(U(1)\) gauge transformations are given by the set of single-valued gauge functions, with \(s^m\) and \(t_m\) integer-valued vectors,

\[
U_{s,t}(z) = \exp (2\pi i (t_m \eta^m(z) - s^m \tilde{\eta}_m(z)))
\]

where

\[
\eta^m(z) = i \int_{z_0}^z (\Omega^m \omega_l - \Omega^m \tilde{\omega}_l), \quad \tilde{\eta}_m(z) = -i \int_{z_0}^z (\omega_m - \tilde{\omega}_m)
\]
If we change the endpoint of integration by \( z \to z + a_i u^i + b^m v_m \) with \( u, v \) integer and \( a, b \) defined in \((2.6)\), we find \( \eta^m \to \eta^m + u^m, \tilde{\eta}_m \to \tilde{\eta}_m + v_m \) and \( U_{s,t} \to U_{s,t} e^{2\pi i (t_m u^m - s^m v_m)} = U_{s,t} \).

The transformation of the gauge field \((2.5)\) under \( U_{s,t} \) is given by

\[
\gamma^m \to \gamma^m + s^m + \Omega^{ml} t_l, \quad \tilde{\gamma}^m \to \tilde{\gamma}^m + s^m + \tilde{\Omega}^{ml} t_l \quad (4.1)
\]

The classical operator that produces the transformation \((4.1)\)

\[
c_{s,t}(\gamma, \tilde{\gamma}) = \exp \left[ (s^m + \Omega^{ml} t_l) \frac{\partial}{\partial \gamma^m} + (s^m + \tilde{\Omega}^{ml} t_l) \frac{\partial}{\partial \tilde{\gamma}^m} \right]
\]

must be transformed into the proper quantum operator acting on the wave function \( \tilde{\Psi}_T \). By using the commutation \((3.3)\) to replace \( \frac{\partial}{\partial \gamma} \) by \(-2\pi i k \gamma^m\) we find the operators \( C_{s,t} \) which implement the large gauge transformations \((3.1)\)

\[
C_{s,t}(\gamma) = \exp \left[ -2\pi k (s^m + \Omega^{ml} t_l) \gamma^m - \pi k (s^m + \tilde{\Omega}^{ml} t_l) G_{mn} (s^n + \Omega^{nl} t_l) \right] e^{(s^m + \tilde{\Omega}^{ml} t_l) \frac{\partial}{\partial \gamma^m}} \quad (4.2)
\]

The quantum operators \( C_{s,t} \) do not commute among themselves for non-integer \( k \). From now on we will set \( k = \frac{k_1}{k_2} \) for integer \( k_1 \) and \( k_2 \). Now, in contrast with their classical counterparts, the operators \( C_{s,t} \) satisfy the clock algebra

\[
C_{s_1,t_1} C_{s_2,t_2} = e^{-2\pi i k (s_1^m t_{m2} - s_2^m t_{m1})} C_{s_2,t_2} C_{s_1,t_1} \quad (4.3)
\]

Their action on the wave function is

\[
C_{s,t}(\gamma) \tilde{\Psi}_T(\gamma^m) = \exp \left[ -2\pi k (s^m + \Omega^{ml} t_l) \gamma^m - \pi k (s^m + \tilde{\Omega}^{ml} t_l) G_{mn} (s^n + \Omega^{nl} t_l) \right] \tilde{\Psi}_T(\gamma^m + s^m + \Omega^{ml} t_l) \quad (4.4)
\]

On the other hand \( C_{k_1,k_2} \) commutes with everything and must be represented only by phases \( e^{i\phi_{s,t}} \). This implies, using \((4.4)\),

\[
\tilde{\Psi}_T(\gamma^m + k_2 (s^m + \Omega^{ml} t_l)) = \exp \left[ -i \phi_{s,t} + 2\pi k_1 (s^m + \tilde{\Omega}^{ml} t_l) \gamma^m \right] + \pi k_1 k_2 (s^m + \tilde{\Omega}^{ml} t_l) G_{mn} (s^n + \Omega^{nl} t_l) \tilde{\Psi}_T(\gamma^m) \quad (4.5)
\]

The only functions that are doubly (semi-)periodic are combinations of the theta functions \((2.14)\). After some algebra, we find that the set of functions

\[
\Psi_{p,r} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) (\gamma | \Omega) = e^{x k \gamma^m \gamma_m} \Theta \left( \frac{\alpha + k_1 p + k_2 r}{k_1 k_2} \frac{\beta}{\Omega} \right) (k_1 | k_1 k_2 \Omega) \quad (4.6)
\]

where \( p = 1, 2, \ldots, k_2 \) and \( r = 1, 2, \ldots, k_1 \) with \( \alpha, \beta \in [0,1] \) solve the above algebraic conditions \((4.5)\). Their inner product is given by

\[
(\Psi_{p_1,r_1} | \Psi_{p_2,r_2}) = \int_P e^{-2\pi x k \gamma^m G_{ml} \gamma_l} \Psi_{p_1,r_1} (\gamma) \bar{\Psi}_{p_2,r_2} (\gamma) G^{-1} \prod_m d\gamma^m d\gamma^m \quad (4.7)
\]
\[ = |G|^{-\frac{1}{2}} \delta_{p_1, p_2} \delta_{r_1, r_2} \]

The integrand is completely invariant under the translation \((4.1)\), thus we restrict the integration to one of the plaquettes \(P (\gamma^m = u^m + \Omega^m v_t \text{ with } u, v \in [0, 1])\), the phase space of the \(\gamma\)’s.

Under a large gauge transformation
\[
C_{s, t} \Psi_{p, r} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) (\gamma) = \exp \left[ 2\pi i k p_m s^m + i \pi k s^m t_m + \frac{2\pi i}{k_2} (\alpha m_s^m - \beta m_t^m) \right] \Psi_{p + t, r} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) (\gamma)
\]
\[ = \sum_{p'} [C_{s, t}]_{p, p'} \Psi_{p', r} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) (\gamma) \quad (4.8)\]

The matrix \([C_{s, t}]_{p, p'}\) forms a \((k_2)^g\) dimensional representation of the algebra \((4.3)\) of large gauge transformations.

The parameters \(\alpha\) and \(\beta\) appear as free parameters, but in fact they may be fixed such that we obtain a modular invariant wave function. The modular transformation \((2.8)\) on our set of functions \((4.6)\) is
\[
\Psi_{p, r} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) (\gamma | \Omega) \rightarrow |C\Omega + D|^2 e^{-i\pi \phi} \Psi_{p, r} \left( \begin{array}{c} \alpha' \\ \beta' \end{array} \right) (\gamma' | \Omega')
\]
where \(\gamma' = (C\Omega + D)^{-1}\gamma\), \(\Omega' = (A\Omega + B)(C\Omega + D)^{-1}\) and \(\phi\) is a phase that will not concern us here (and \(G'_{lm} = [(C\Omega + D)^{-1}]_{tr} G_{rs} [(C\Omega + D)^{-1}]_{sm}\)). Most important are the new variables
\[
\alpha' = D\alpha - C\beta - \frac{k_1 k_2}{2} (CD^\top)_d, \quad \beta' = -B\alpha + A\beta - \frac{k_1 k_2}{2} (AB^\top)_d
\]
where \((M)_d\) mean \([M]_{dd}\), the diagonal elements.

A set of modular invariant wave functions \([10, 11, 13]\) can exist only when \(k_1 k_2\) is even, where we set \(\alpha = \beta = 0\) (and also \(\phi = 0\)). In the case of odd \(k_1 k_2\), we can set \(\alpha, \beta\) to either 0 or \(\frac{1}{2}\), which amount to the addition of a spin structure on the wave functions. This will increase the number of functions by \(4^g\) which will now transform non trivially under modular transformations.

### 5 The braid group on a Riemann surface and Chern-Simons statistics

Considering a set of point charges leads to the set of wave functions
\[
\Psi_{p, r} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) (\xi, \gamma, t | \Omega) = \exp \left[ \pi k^m \gamma^m m + 2\pi \gamma^m \int_0^t (\bar{j}_m - j_m) dt' - \frac{2\pi i}{k} \left( \sum_i q_i \xi(z_i(t)) - Q\xi(z_0) \right) \right.
\]
\[ + \frac{i}{2k} \sum_{ij} q_i q_j (\theta_{ij}(t) - \theta_{ij}(0)) + \frac{\pi}{2k} \int_0^t (\bar{j}_m - j_m) dt' \cdot \int_0^t (\bar{j}^m - j^m) dt' \right]
\]
\[ \cdot \Theta \left( \frac{a + k_1 p + k_2 r}{k_1 k_2 / \beta} \right) (k_1 \gamma^m - k_2 \int_0^t j^m \, dt') | k_1 k_2 \Omega \right) \]  

(5.1)

The wave function depends on charge positions through the integrals over the topological components of the current \( j^m, \bar{j}^m \), and through the function \( \theta_{ij}(t) - \theta_{ij}(0) \). Consider for a moment motions of the particles which are closed curves, and are homologically trivial. We focus first on the integrals over \( j^m, \bar{j}^m \). If, for example a single particle moves in a circle, we find that the integral of these topological currents vanishes, we conclude that these currents contribute nothing additional to the phase of the wave function under these kinds of motions. The function \( \theta_{ij}(t) - \theta_{ij}(0) \) must be treated differently here, because it has singularities when particles coincide, and thus, while motions that encircle no other particles may be easily integrated to get zero, this is not true when other particles are enclosed by one of the particle paths, and the result is non-zero in this case, in fact it is \( 2\pi \) (with appropriate sign depending on the loop orientation). Nevertheless, this function is still independent of the particular shape of the particle path. Actually the definition of \( \theta_{ij} \) in term of the prime form \( E(z, w) \) is just the generalization to an arbitrary Riemann surface of the well known angle function on the plane, that is as the angle of the line joining the particle \( i \) and \( j \) compare to a fixed axis of reference, determined by \( z_0 \) here. Thus, we may conclude that under the permutation of two identical particles of charge \( q \), the wave functions defined here acquire the phase

\[ \sigma = e^{i \bar{\Psi} q^2} \]  

(5.2)

For homologically non-trivial motions of a single particle on \( M \), the current integral \( \int_0^t j^i(t') \, dt' \) will in general change as \( \int_0^t j^i(t') \, dt' \rightarrow \int_0^t j^i(t') \, dt' + s^i + \Omega_{im} t_m \), where \( s^i \) and \( t_m \) are integer-valued vectors whose entries denote the number of windings of the particle around each homological cycle. However, now, for multi-particle non-braiding paths, there is no contribution coming from \( \theta_{ij} \). Thus, for closed path on \( M \), the wave functions become

\[ \Psi_{p,r}(t) = \exp \left[ -\frac{2\pi i}{k} r_m s^m - \frac{2\pi i}{k} \sum_i q_i ((\alpha - k_2 \alpha_0)_m s^m_i - (\beta - k_2 \beta_0)_m t_m) - i J \right] \]

\[ \cdot \Psi_{p,r+i}(0) = \sum_{r'} [B_{s,t}]_{r,r'} \Psi_{p,r'}(0) \]  

(5.3)

with \( \sum_i q_i \int_{z_0}^{z_i(0)} \omega^i = \alpha_0^i + \Omega_{im} \beta_{0m} \) and where \( J = \sum_i \frac{q_i^2}{2k} (f_i(t) - f_i(0)) \) is the self-linking term. The matrices \( (5.3) \) satisfy the cocycle relation

\[ B_{s_1,t_1} B_{s_2,t_2} = e^{-\frac{2\pi i}{k} (s_1^m t_{m2} - s_2^m t_{m1})} B_{s_2,t_2} B_{s_1,t_1} \]  

(5.4)

This cocycle has to be contrasted with the large gauge transformations cocycle \( (L.3) \). They are very similar except that \( k \) is now \( \frac{1}{k} \) and the operator act on the wave function \( \Psi_{p,r} \) on the other index. In this sense, these two cocycles play a dual role on the wave function.

The self-linking contribution, \( J \), in \( (5.3) \), plays an important role here. For homologically trivial closed particle trajectories, we find \( J = 0 \) if the path does not enclose \( z_0 \) in the patch that we are working on, since we choose to put all the flux of \( \Gamma \) around \( z_0 \). Otherwise, we find a contribution \( \frac{q_i^2}{2k} \chi_E = 2\pi k_2 (1 - g) \) to \( J \), for the particle \( i \). This can be illustrated by checking for independence of the braiding \( (5.3) \) on \( z_0 \). In the definition of the angle function
\[\theta_{ij} \text{ in (3.11), we argue that by moving } z_0 \text{ along an homology cycle, the angle function is changed by a constant that should cancel out in (5.3). Now the function (5.3) changes by } e^{\frac{i\pi}{2}Qq}e^{\frac{i\pi}{2}q^2(g-1)} \text{ to an integer power. Fortunately this is one, being our fundamental consistency condition (3.13). The first phase come from the shift in } \alpha_0 \text{ and } \beta_0, \text{ while the second phase come from the fact that each charge trajectory is being crossed by } z_0, \text{ which produces a shift in } J.\]

To study the permuted (identical particles) braid group, we will consider \( n \) particles of charge \( q \), so \( Q = nq \). The representation of the braid group is characterized by its generators, the permutation phase \( \sigma \) in (5.2), and the braid matrices \( B_{s,t} \) in (5.3). These generators are the result of the action of elements of the permuted braid group on the particles which form the external sources in our theory. In fact, let the integer vectors \( \hat{s}^l, \hat{t}_m \) denote vectors that are 0 in all entries except for the \( l \)th and \( m \)th, respectively, and 1 at the remaining position. Then with the identifications \( \alpha_l = B_{\hat{s}^l,0}, \beta_m = B_{0,\hat{t}_m} \), it is easy to check that we recover all of the necessary relations of the braid group on the Riemann surface, given in (1.8). In particular, we recover the global constraint (1.9), this is just our fundamental constraint (3.13), using (5.2), applied to this case.

6 Conclusion

We have quantized Abelian Chern-Simons theory coupled to arbitrary external sources on an arbitrary Riemann surface, and solved the theory. We find that the presence of non-trivial spatial topology introduces extra dimensionality to the Hilbert space separately for the large gauge transformations and the braid group. We find a fundamental constraint (3.13), relating the charges, \( k \), and \( g \) such that we recover a consistent topological field theory representing a general (with some identical and non-identical particles) braid group on \( \mathcal{M} \). In particular, we recover the permuted braid group on \( \mathcal{M} \).

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