Possibility of using NURBS for surface plotting by survey data

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Abstract. Different methods of surface plotting were discussed in this article. Constructing the surface with the help of the Delaunay triangulation algorithm is described. The TIN-surfaces (triangles irregular net) method is used in the entire CAD software. This type of surfaces is plotting by results of laser scanning and stadia surveying. Possibility of using spline surfaces (NURBS) for surface plotting is studied. For a defined number of points by Mathcad software, the curvilinear function that described two-dimensional spline surfaces was calculated and plotted.

1. Introduction
Methods of mathematical modeling are widely used in researches of the patterns in mining [5]. The ground surface is constructed in the form of a network of irregular triangles (TIN-surfaces) by software. This net is constructed by Delaunay triangulation algorithm.

A function of constructing the surface with the help of the Delaunay triangulation algorithm is used practically in all modern programs (AutoCAD, Micromine, Trimble Business Office, RiscanPro, Cyclone MODEL, RapidForm, Microstation etc.). These programs work with “point clouds” as results of tachymetry and laser-scanning survey [3,4,7,8].

The same programs can plot another type of surfaces: NURBS, they are very smooth and visual [9]. But NURBS are not used for plotting the soil surface by data of surveys.

In this research, these two ways of surfaces plotting were considered. And an attempt to analyze the feasibility of replacing the traditional method of surface constructing using the algorithms of irrational functions by the example of NURBS was made.

2. Materials and methods
Research was carried out by theoretical mathematical methods using Mathcad software.

PTC Mathcad is one’s systems of the equation solver that allows one to solve any number of equations with unknown variables simply and easily through the use of the software's for solving block feature.

- Solving complex systems of equations without performing linear algebra or matrix manipulations.
- Solving problems with confidence, knowing that the appropriate algorithm is automatically selected.
- Setting up problems in the natural math notation — equations are not hidden in definitions of vectors and matrices or in solver definitions [12].

3. Main part
The triangulation is the graph, all internal regions of which are triangles. It is said that the
triangulation satisfies the Delaunay condition, if there are no triangulation points inserting the circle that is circumscribed around any triangle (Figure 1).

![Figure 1. An example of Delaunay triangulation](image)

At present, a significant number of different algorithms for constructing the Delaunay triangulation are known [13]. The algorithm of dynamic caching and the algorithm of layer-by-layer concentration are strongly recommended, because the number of operations performed on the average is equal to the number of points with a high ease of implementation. They refer to iterative algorithms, they are based on the very simple idea of sequentially adding points to a partially constructed Delaunay triangulation.

Different data structures, consisting of triangulation objects (knots, edges and triangles) that provide Delaunay triangulation construction, occupy 40 to 90 bytes of computer memory (with an 8-byte coordinate representation).

So one can conclude that the use of this method of surface construction is convenient:
- using in the construction of directly existing points;
- simplicity of construction algorithms (and as a consequence, high process speed);
- relatively small file size (increase of the speed of the software);
- the model makes it possible to use the variable density of the initial points depending on the changes in the relief (one can create an effective and accurate model of the surface).

At the same time, more than thirty years ago, a mathematical algorithm was developed that makes it possible to accurately represent a surface of arbitrary shape. First, these are Bezier surfaces. Pierre Bézier, an engineer of the automobile company Renault in the 60’s of the 21st century developed on the basis of Hermitian curves a constructively defined curve, the form of which can be controlled in intermediate, so-called control points. He used them for computer designing of automobile bodies. Bézier curve always leaves the first control point, touching the first segment of the broken line connecting all control points, and ends at the last control point, touching the last segment. Moreover, any point of the curve always remains inside the convex closure of the set of control points.

The same results, regardless of Bézier, were obtained by Paul de Casteljou, an engineer at Citroën, who developed a recursive way of defining curves, named after him (de Casteljou’s algorithm). The Bezier curve is a parametric curve given by the expression:

\[
B(t) = \sum_{i=1}^{n} p_i b_{i,n}(t), \quad 0 \leq t \leq 1, \tag{1}
\]

\(p_i, i \in 1:n\) are support vertices in the space of a curve, called poles, and \(b_{i,n}(t)\) is Bernstein basis polynomials:
\[ b_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}, \quad i \in 1:n. \] (2)

The segments joining the poles are called the characteristic polygon.

In connection with the fact that the equation of the curve includes the calculation of factorials, the curve points calculation is irrational.

The algorithm for plotting the Bezier curve is the following:
- At each step, parameter \( t \) takes on a value on segment \([0,1]\).
- The sides of the characteristic polygon are segmented by points in relation to the \( t \) parameter.
- Obtained points are joined by intervals.
- Intervals are also segmented by points in relation to the \( t \) parameter.
- And so long as there is a single point that belongs to the Bezier, the curve will be plotted.

For example, let us consider the construction of a second-order curve point given by characteristic polygons \( p_0, p_1, p_2 \). Let us take some arbitrary value of the \( t \) parameter from interval \([0, 1]\). The sides are segmented in relation to \( t \), resulting in points \( a_1, a_2, a_3 \). Intervals \( a_1a_2, a_2a_3 \) are also segmented in relation to the \( t \) parameter, points \( b_1, b_2 \) are obtained. Segment \( b_1b_2 \) is divided in relation to the \( t \) parameter, point \( c_1 \) is obtained that lies on the required curve.

One can go from curves to Bezier surfaces in two ways. In the first, the so-called generating Bezier curves are introduced, which have the same parametrization. For each value of the parameter by curves points, in turn, a Bezier curve is constructed. Moving along the guide curves, let us obtain a surface, which is called the Bezier surface on a quadrilateral.

Another method uses the generalization of Bernstein polynomials in the case of two variables. The surface that is given by such polynomial is called the Bezier surface on the triangle.

An algorithm for plotting Bezier surfaces: Bezier curves are constructed in the \( u \)-direction, the resulting points are considered as poles for constructing Bezier curves in the \( v \)-direction [11].

Bezier curves and surfaces have some properties that significantly limit their scope: with their help, it is impossible to accurately represent the conic sections (for example, the arc of a circle), their algebraic degree grows together with the number of control points, and it makes the numerical calculations very difficult.

The method or decreasing the algebraic degree of a complex curve is enough to construct a curve consisting of smoothly conjugate segments, each of which has a bounded algebraic degree. Such curves are called splines. B-splines are a generalization of Bezier curves and surfaces. B-splines allow one similarly to define the shape of the curve using control points, but the algebraic degree of the B-spline does not depend on the number of control points [2,10].

The B-spline equation is similar to the Bezier curve, but the conjugating functions are not Bernstein polynomials, but are determined recursively, depending on the value of the parameter. The area of specifying the B-spline parameter is divided into knots, which correspond to the points of conjugation of algebraic curves of a given degree.

Models created on the basis of non-uniform rational B-Spline or NURBS can consist of a single surface, and a set of pieces (patches). But in any case, a smooth surface is guaranteed. The shape of the created surface is defined and controlled by isoparametric curves, the shape of which, in turn, determines the control vertices. Thus, the shape of the surface is edited by manipulating control vertices. The NURBS curves and surfaces have a clear geometric interpretation, which is especially useful for designers with good knowledge of geometry. NURBS have a huge set of tools that can be used to create and analyze these objects. Among the problems of approximation of functions an important place is occupied by the problem of replenishing the function with respect to the initial data given on a finite set of points [13]. In the course of this work, an attempt to analyze the feasibility of replacing the traditional method of surface constructing by the survey data of using the algorithms of irrational functions by the example of NURBS was made. To do this, the spline surface was “manually” constructed for this, passing through a limited set of points obtained by the laser scanner survey. So, first one needs to determine the function by which the surface will be built. For this, let us
consider the concept of interpolation.

The function is known only at the nodes of some mesh \( x_i, i = 1, N \), that is, given by table \{ \{ x_i, f_i = f(x_i) \}, i = 1, N \}. Let us restore this unknown function.

Then one can construct function \( \varphi(x; a) \) that depends on parameter vector \( a = (a_1, \ldots, a_N)^T \), so that:

\[
\varphi(x; a_1, \ldots, a_N) = f_i, \quad i = 1, N.
\]

The problem of constructing function \( \varphi(x) \equiv f(x) \) is called the interpolation problem, and the method of selecting parameters \( a \) in the equations system is called interpolation (Lagrange interpolation) [6].

Traditionally, linear interpolation is considered when the function is linearly dependent on parameters, that is, it can be represented as a so-called generalized polynomial:

\[
\varphi(x; a_1, \ldots, a_N) = \sum_{k=1}^{N} a_k \varphi_k(x),
\]

where \( \varphi_k(x), \ k = 1, N \) is a set of some given functions.

Then, to find parameters \( a \), a system of linear equations is solved:

\[
\sum_{k=1}^{N} a_k \varphi_k(x_i) = f_i, \quad i = 1, N.
\]

Obviously, the selection of a successful form of formula dependence is an art, for the true solution of the problem is not known, but it is advisable to choose a sufficiently smooth function, a spline function.

Solving this problem, according to these points \{ \{ x_k, \ k = 1, N \} \}, an interpolating curve with the least curvature \( \varphi(x) \) is plotted.

Veniamin Ashkenazy in his work "Spline surfaces. Fundamentals of the theory and computational algorithms" obtained the relations that led to the algebraic definition of the spline function as a piecewise-polynomial function that satisfies the conditions of interpolation at nodes and" glued together "by the continuity conditions of the function and its derivatives [1].

That is, in the intervals between each pair of neighboring points (interpolation nodes) \( x_{i-1} \) and \( x_i \), function \( \varphi(x) \) is a polynomial of the third degree:

\[
\varphi(x) = a_i + b_i(x - x_{i-1}) + c_i(x - x_{i-1})^2 + d_i(x - x_{i-1})^3,
\]

\[ x_{i-1} \leq x \leq x_i, \ i = 2, N. \]

Within the framework of the algebraic approach, multidimensional splines are most often constructed from one-dimensional ones by means of a tensor product or a mixing. It has two drawbacks: the restriction of areas and grids of rectangular nodes, the use of unnatural functional spaces and norms.

For arbitrarily placed nodes in the multidimensional problem of recovering a function, the initial variational method is more useful. Let us consider the mathematical foundations and computational aspects of interpolation by a spline surface [1].

To derive the spline surface equation, one will consider spline surface \( \varphi(x, y) \) based on interpolation data \( x_i, y_i, f_i, i = 1, N \) as a mathematical model of an elastic thin plate bent under the influence of external forces applied at points \( (x_i, y_i), i = 1, N \).

Considering the equation of the total free energy of a curved elastic plate, let us obtain a relation describing a two-dimensional spline surface:
\[
\varphi(x, y) = \sum_{i=1}^{N} C_i \left[ (x - x_i)^2 + (y - y_i)^2 \right] \ln\left[ (x - x_i)^2 + (y - y_i)^2 \right] + A x + B y + D .
\]

The coefficients \( C_1, C_2, \ldots, C_N, A, B, D \) are unknown and must be determined from equations \( \varphi(x_i, y_i) = f_i, \quad i = 1, N \) and \( \sum_{i=1}^{N} C_i = 0, \quad \sum_{i=1}^{N} C_i x_i = 0, \quad \sum_{i=1}^{N} C_i y_i = 0 \). These equations are sufficient to find unknown quantities, and one can use the method of solving a system of linear algebraic equations to solve them.

To calculate these coefficients, two matrices were constructed on the basis of the coordinates of twelve real points for solving a system of linear equations:

\[
\begin{align*}
\varphi_1(x_1, y_1) &= C_1 \left[ (x_1 - x_1)^2 + (y_1 - y_1)^2 \right] x \\
& \times \ln\left[ (x_1 - x_1)^2 + (y_1 - y_1)^2 \right] + A x_1 + B y_1 + D; \\
\varphi_2(x_2, y_2) &= C_1 \left[ (x_2 - x_1)^2 + (y_2 - y_1)^2 \right] x \\
& \times \ln\left[ (x_2 - x_1)^2 + (y_2 - y_1)^2 \right] + C_2 \left[ (x_2 - x_2)^2 + (y_2 - y_2)^2 \right] x \\
& \times \ln\left[ (x_2 - x_2)^2 + (y_2 - y_2)^2 \right] + A x_2 + B y_2 + D; \\
& \vdots \\
\varphi_N(x_N, y_N) &= C_1 \left[ (x_N - x_1)^2 + (y_N - y_1)^2 \right] x \\
& \times \ln\left[ (x_N - x_1)^2 + (y_N - y_1)^2 \right] + C_2 \left[ (x_N - x_2)^2 + (y_N - y_2)^2 \right] x \\
& \times \ln\left[ (x_N - x_2)^2 + (y_N - y_2)^2 \right] + A x_N + B y_N + D; \\
0 &= C_1 + C_2 + \ldots + C_N; \\
0 &= C_1 x_1 + C_2 x_2 + \ldots + C_N x_N; \\
0 &= C_1 y_1 + C_2 y_2 + \ldots + C_N y_N.
\end{align*}
\]

As a result, the coefficients for the equation, describing the spline surface interpolating the values of these points, were obtained.

By the function equation, a surface was plotted (Figure 2).
4. Conclusion
The resulting surface does not pass directly through the survey points, but uses them as interpolation nodes. In this case, a significant coarsening of the final results occurs. The constructed surface degenerates into a plane in the immediate vicinity of points because of the small number of points chosen to calculate the function coefficients. To construct a more intuitive surface, it is necessary to interpolate by means of function (3) and solve \((N+3)\) linear equations again.

A comparison of the two methods of constructing the surface showed that the Delaunay triangulation method is optimal.

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