Hydrodynamic signatures of low-lying excitations of a quantum vortex

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We shed new light on the low-lying excitations of a quantum vortex in a quasi-two-dimensional Bose-Einstein Condensate (quasi-2D BEC) by introducing a class of hydrodynamic densities of states which express the interactions of an impurity with the excitations of the quantum fluid. We establish that the impurity becomes sensitive to the remnant kelvon mode of the vortex and to the phase fluctuations of the BEC via its Nambu-Goldstone mode, and show that the vortex renders non-trivial features to the spectrum of excitations of a BEC. The kelvon mode is further shown to allow for external control of the state of an impurity localized in the vortex core by action of a pinning potential. These findings offer fresh insights into the polaron physics of impurity-BEC systems and the quantum treatment of vortex dynamics.

I. INTRODUCTION

Bose-Einstein condensates (BEC) feature a variety of so-called topological defects emerging from the interactions among its constituent atoms. Vortices are a notable example of these: occurring in three-dimensional and in quasi-two-dimensional (quasi-2D) configurations, their vorticity is topologically quantized [1]; such a vortex cannot decay dissipatively; instead, a multiple-charge vortex is dynamically unstable and eventually splits into single-charge vortices as a consequence of quantum fluctuations [2]. The latter, however, may be energetically unstable: a kelvon mode (i.e., a transversal excitation of the vortex filament) [1] can precipitate a single-charge vortex off its axis, which can then spiral out towards the edges of the system. This dynamic has been studied theoretically from a number of perspectives [3–11] following the first experiments on BEC dynamics [12–15], in response to the challenges of taming this phenomenon. In fact, much of the research effort was devoted to finding mechanisms of external stabilization [8, 16]. In this work, we consider one such mechanism: the application of a pinning potential. We show, in particular, how this mechanism of stabilization may comprise a novel mechanism of control over an impurity bound in the core of a vortex.

The physics of a BEC in the presence of impurities (e.g., heterogeneous atomic species) has aroused interest in both theory and experiment over the past decade [17–24]. Of particular interest are cases when the BEC hosts one or multiple vortices, motivated by the possibility of impurities becoming bounded in their core or hopping across the Abrikosov lattice [25–27]. Such a problem of an impurity interacting with particles constituting a medium is familiar of polaron physics, a branch of quantum many-body physics that comprises a body of notable problems and results [28, 29].

In this work, we provide a detailed investigation of the quantum excitations of a quasi-2D BEC supporting a single-charge, on-axis vortex (vortex-BEC). We compute quantities of ubiquitous importance in the description of many-body systems: the local density of states (LDOS) and the density of states (DOS). We make use of both the LDOS and DOS to analyze the low-lying spectrum of excitations, focusing on those that correctly account for the leading-order coupling of an impurity with the excitations of the vortex-BEC, which we refer to as the hydrodynamic densities of states. Via this analysis, we show that an impurity may be exceptionally sensitive to the remnant kelvon mode [30], a.k.a. the lowest core-localized state (LCLS) [8], and to the Nambu-Goldstone mode produced by the spontaneous symmetry-breaking of the phase degree-of-freedom of a BEC [31, 32].

The present paper is organized as follows: In Sec. II A, we review the Bogoliubov formalism applied to a vortex-BEC. We then compute the relevant densities of states in Sec. II B, where we introduce the hydrodynamic DOS, followed by a description of the physical setup in Sec. II C. Numerical results are shown in Sec. III, where we highlight the novel features of vortices by drawing comparisons with the homogeneous (zero vorticity) case. Finally, we point out the capacity of the LCLS and of the Nambu-Goldstone mode to interact with impurities and the non-trivial features of the spectrum of excitations, as discussed in Sec. IV.

II. FORMULATION

A. Bogoliubov formalism for a vortex-BEC

We begin by considering the field-quantized Hamiltonian

$$\hat{H}_{\text{BEC}} = \int d^2r \hat{\Phi}^\dagger(r) \left[ \frac{-\hbar^2}{2M} \nabla^2 - \mu + V(r) \right] \hat{\Phi}(r) + \frac{1}{2} \int d^2r \hat{\Phi}^\dagger(r) \hat{\Phi}^\dagger(r) \hat{\Phi}(r) \hat{\Phi}(r),$$

(1)

describing the field of a bosonic species of mass $M$ interacting by an effective contact potential of strength $g > 0$. Atomic annihilation and creation operators $\hat{\Psi}$ and $\hat{\Psi}^\dagger$ satisfy the commutation relation $[\hat{\Psi}(r), \hat{\Psi}^\dagger(r')] = \delta(r - r')$ and $\hat{N} = \int d^2r \hat{\Psi}^\dagger(r) \hat{\Psi}(r)$ is the number operator, whose
The external potential $V$ is comprised of two contributions: a highly anisotropic trapping potential $V_{tr}$, rendering the system quasi-2D; a small pinning potential $V_p$, introduced here to energetically stabilize the vortex. (These potentials are detailed in Sec. II C.)

The Bogoliubov approach suffices in the case of a weakly interacting BEC [33]. We thus decompose the field as

$$\left(\hat{\phi}(r) \quad \hat{\phi}^\dagger(r)\right) = \sqrt{n_0} \left(\frac{\hat{\phi}(r)}{\Phi_0(r)} \quad \frac{\hat{\phi}^\dagger(r)}{\Phi_0^\dagger(r)}\right),$$  

where $\Phi_0$ is a BEC wavefunction ($\overline{\Phi_0}$ its complex-conjugate), on top of which $\hat{\phi}(r)$ ($\hat{\phi}^\dagger(r)$) annihilates (creates) quantum excitations. Here, $n_0 = N_0/A$, with $N_0$ the number of BEC atoms and $A$ the area covered by the quasi-2D BEC cloud. The mean-field $\Phi_0$ satisfies the time-independent Gross-Pitaevskii (GP) equation,

$$\left(h - \mu + n_0 g |\Phi_0|^2\right) \Phi_0 = 0,$$

with $h = -(\hbar^2/2M)\nabla^2 + V$, subject to the boundary condition consistent with the external potential $V$ and to the normalization condition

$$\langle \hat{N} \rangle_0 = n_0 \int d^2r |\Phi_0(r)|^2 = N_0,$$

where $\langle \ldots \rangle_0$ represents the expected value in the many-body state of the BEC, which can be used to determine $\mu$.

We choose, at this point, the natural microscopic units to be the coupling energy $n_0 g$ and the coherence length $\xi = \hbar/\sqrt{2Mn_0 g}$, so that quantities in the Hamiltonian (1) become rescaled as $r \mapsto \xi r$, $\hat{\Phi} \mapsto \xi^{-1} \hat{\Phi}$, $\mu \mapsto (n_0 g) \mu$, $h \mapsto (n_0 g) h$ and $\hat{H} \mapsto (n_0 \xi^2) (n_0 g) \hat{H}$; substituting for (2) and by virtue of (3), we have

$$\hat{H} = F_0 + \frac{1}{n_0 \xi^2} \hat{H}_B + \ldots$$

where $F_0 = \int d^2r \overline{\Phi_0} \left(\hbar - \mu + \frac{1}{2} |\Phi_0|^2\right) \Phi_0$ is the classical free energy of the BEC and

$$\hat{H}_B = \frac{1}{2} \int d^2r \left(\frac{\hat{\phi}^\dagger}{\Phi_0^\dagger} \sigma_3 \hat{H}_B \left(\frac{\hat{\phi}^\dagger}{\Phi_0^\dagger}\right) \right)$$

is the Bogoliubov excitation Hamiltonian, with

$$\hat{H}_B = \sigma_3 \left(\hbar - \mu + 2 |\Phi_0|^2 - \frac{\Phi_0^2}{\overline{\Phi_0^2}} \hbar - \mu + 2 |\Phi_0|^2\right)$$

the Bogoliubov operator, where $\sigma_i$ ($i = 1, 2, 3$) denote Pauli matrices. Higher-order contributions of Eq. (5) are neglected within the Bogoliubov approximation; this requires $\langle \hat{N} \rangle_0 \approx \langle \hat{N} \rangle$, which holds provided $k_B T \ll \mu$ and $n_0 \xi^2 \gg 1$, i.e. in the limit of both weak thermal and quantum depletions. We then expand the fluctuation operators as

$$\left(\frac{\hat{\phi}(r)}{\hat{\phi}^\dagger(r)}\right) = \sum_{\lambda=0}^\infty (X_\lambda(r) \hat{b}_\lambda + \sigma_1 X_\lambda^\dagger(r) \hat{b}_\lambda^\dagger),$$

in a complete basis of complex vector-valued functions $X_\lambda(r) = (u_\lambda(r), v_\lambda(r))^T (X_\lambda^\dagger$ its complex-conjugate), whose components $u_\lambda$ and $v_\lambda$ are the particle and hole components, respectively. By requiring that each $\hat{b}_\lambda$ and $\hat{b}_\lambda^\dagger$ inherit bosonic commutation relations, the diagonalization of (6) becomes an eigenproblem for the $X_\lambda$ as

$$\hat{H}_B X_\lambda = \omega_\lambda X_\lambda.$$

The operator $\hat{H}_B$ is non-hermitian and, in general, admits complex eigenvalues [2]; the Bogoliubov Hamiltonian explored in the present work possesses a fully real spectrum [5], however, and the bi-orthonormality relations

$$\langle X_\lambda, X'_{\lambda'} \rangle = \int d^2r X_{\lambda}^\dagger \sigma_3 X_{\lambda'} = \delta_{\lambda,\lambda'},$$

$$\langle \sigma_1 X_\lambda, X'_{\lambda'} \rangle = \int d^2r (\sigma_1 X_{\lambda})^\dagger \sigma_3 X_{\lambda'} = 0,$$

hold for every normalized eigenstate, where we introduce the bilinear product $\langle \cdot, \cdot \rangle$; the diagonalized form of (6) thus reads

$$\hat{H}_B = \frac{1}{2} \alpha \hat{\phi}^2 + \sum_{\lambda \neq 0} \omega_\lambda \hat{b}_\lambda^\dagger \hat{b}_\lambda,$$

where the first term accounts for the dynamics of the phase degree-of-freedom of the BEC and completes the basis used in Eq. (8) [34]. We explain the structure and importance of this term by observing that the variation of Eq. (4), i.e., $\delta \langle \hat{N} \rangle_0 = \delta N_0$, yields the condition

$$\int \frac{d^2r}{A} \left(\overline{\Phi_0} \Theta_0 + \Phi_0 \overline{\Theta_0}\right) = 1,$$

where we introduce the adjoint wave function

$$\Theta_0(r) = \left(N_0 \frac{\partial \mu}{\partial N_0}\right) \frac{d \Phi_0}{d \mu}(r).$$

It is because Eq. (3) holds that $\Phi_0$ and $\Theta_0$ comprise the zero-mode of $\hat{H}_B$ (the Nambu-Goldstone mode of the BEC phase [31]) and its adjoint, respectively, as

$$\hat{H}_B \left(\frac{\Phi_0}{\overline{\Phi_0}}\right) = 0, \quad \hat{H}_B \left(\frac{\Theta_0}{\overline{\Theta_0}}\right) = \alpha \left(\frac{\Phi_0}{\overline{\Phi_0}}\right).$$

The second of these equations is obtained by taking the derivative $d/d\mu$ of the first and then multiplying it by
\( \rho \) with the spatial density

\[ \rho(r) = |\alpha(r)|^2 \]

acting on \( X_0 \) as the Nambu-Goldstone (NG) mode henceforth. The basis \( (8) \) thus becomes complete. (The zero-mode pertaining to the position of the vortex core, though related to the present discussion, is present only when the vortex is off-axis \([36]\).)

Making the potential \( V \) be isotropic in the plane and assuming a vortex to be on the axis of a disk-shaped quasi-2D BEC, a vortex solution of the GP Eq. (3) can be written in polar coordinates as

\[ \Phi_\nu(r, \varphi) = e^{i\nu \varphi} \Phi_\nu(r), \tag{16} \]

where \( \nu \) is an integer, the quantization of the vorticity, and \( \Phi_\nu \) satisfies a reduced form of Eq. (3) in the radial coordinate \([37]\). Inserting (16) in Eq. (7), it becomes apparent that solutions of Eq. (9) can be separated, in polar coordinates, as

\[ X_\lambda(r, \varphi) = e^{i\varphi(m+\nu \sigma_3)} X_{m,n}(r), \tag{17} \]

where \( m \) is an integer, the angular momentum of the excitation, and \( n \), the number of nodes in the radial coordinate, is the quantum number associated to the non-zero eigenstates of the reduced Bogoliubov operator

\[ \sigma_3 \mathcal{H}_m = \left( \begin{array}{cc} h - \mu + 2\phi^2 \nu & \phi^2 \nu \\ \phi^2 \nu & h - \mu + 2\phi^2 \nu \end{array} \right) + \frac{1}{r^2} (m + \nu \sigma_3)^2, \tag{18} \]

acting on \( X_{m,n}(r) = (\alpha_{m,n}(r), \beta_{m,n}(r))^T \). The label \( \lambda \neq 0 \) is, thus, explicitly identified with the pair of quantum numbers \((m, n)\) of solutions of the eigenproblem

\[ \mathcal{H}_m X_{m,n} = \omega_{m,n} X_{m,n}. \tag{19} \]

Note that, by Eqs. (14) and (15), the NG mode has angular momentum \( m = 0 \).

### B. Densities of states

The local density of states (LDOS) is typically defined in terms of the imaginary part of the associated Green’s function \([38]\). In particular, we can consider an angular momentum-resolved LDOS (AM-LDOS), given by

\[ D^{(S)}_m(r; \omega) = \sum_n \varrho^{(S)}_{m,n}(r) \delta(\omega - \omega_{m,n}), \tag{20} \]

with the spatial density \( \varrho^{(S)}_{m,n} \) given by

\[ \varrho^{(S)}_{m,n}(r) = |u_{m,n}(r)|^2 - |v_{m,n}(r)|^2. \tag{21} \]

These may be interpreted as the AM-LDOS seen by a test particle of the species comprising the BEC and, thus, interacting coherently with the system, that is, via both density and exchange interactions. A test particle of a different species (i.e., an atomic impurity) is found to be sensitive to AM-LDOS of a distinct form:

\[ D^{(H)}_m(r; \omega) = \sum_n \varrho^{(H)}_{m,n}(r) \delta(\omega - \omega_{m,n}), \tag{22} \]

with the spatial density \( \varrho^{(H)}_{m,n} \) given by

\[ \varrho^{(H)}_{m,n}(r) = |\Phi_0(r) u_{m,n}(r) + \phi_0(r) v_{m,n}(r)|^2, \tag{23} \]

which is due, physically, to the interactions with the bosonic system being exclusively density interactions. The derivation of this AM-LDOS is illustrated in Appendix \((A)\). We shall refer to the quantities in, and derived from, Eqs. (20) and (22) as the symplectic and the hydrodynamic densities of states, respectively. We refer to the first as *symplectic* because the Hamiltonian \( \mathcal{H}_B \), Eq. (7) (from which the orthogonality condition Eq. (10) and the \( \varrho^{(S)}_{m,n} \) follow) is diagonalized by a symplectic transformation \([35]\). We coin the second *hydrodynamic* densities due to the \( \varrho^{(H)}_{m,n} \) being identical to the density degree of freedom used in the hydrodynamic formalism for the excitations of a BEC \([33]\).

Associated to the LDOS is the density of states of states (DOS), which provides a measure for counting excitations of a many-body system \([33]\). The angular momentum-resolved DOS (AM-DOS) are obtained by integrating in space each of the AM-LDOS, yielding

\[ D^{(O)}_m(\omega) = \sum_n c^{(O)}_{m,n} \delta(\omega - \omega_{m,n}), \tag{24} \]

where \( O = S, H \), with weights \( c^{(S)}_{m,n} = 1 \) for the symplectic densities of states, by virtue of Eq. (10), and

\[ c^{(H)}_{m,n} = \int d^2r \varrho^{(H)}_{m,n}(r) \tag{25} \]

for the hydrodynamic density of states. The (total) LDOS and DOS are recovered upon summing over all \( m \), i.e.,

\[ D^{(O)}(\omega) = \sum_m D^{(O)}_m(\omega) \tag{26} \]

for the DOS; as in Eq. (24), this is for \( O = S, H \).

For comparison with numerical results, we consider a large, homogeneous (i.e., vorticity zero and no boundary effects) quasi-2D BEC, in which case excitations have the Bogoliubov spectrum \( \omega_B(k) = \sqrt{k^2(2\mu + k^2)} \), with \( k \) the mode’s wave-vector, or momentum, and \( k = ||k|| \). The symplectic and the hydrodynamic DOS of excitations of the homogeneous quasi-2D BEC can then be explicitly
computed in the continuum approximation of momentum space, i.e., approximating the sum over modes by an integral over $k$, yielding

$$ D_B^{(S)}(\omega) = \frac{A}{4\pi} \frac{\omega}{\sqrt{\mu^2 + \omega^2}} = \frac{A}{4\pi} \frac{\omega}{\mu} - \mathcal{O}(\omega^3), \quad (27) $$

$$ D_B^{(H)}(\omega) = \frac{\mu A}{4\pi} \left(1 - \frac{\mu}{\sqrt{\mu^2 + \omega^2}}\right) = \frac{A \omega^2}{8\pi} \frac{1}{\mu} - \mathcal{O}(\omega^4), \quad (28) $$

with $A = A/\xi^2$ the area covered by the BEC in natural units.

C. Vortex nucleation and external potentials

We envision the on-axis vortex to be nucleated by phase imprinting via Laguerre-Gauss beams [39], that is, without imposing any laboratory-frame rotation on the fluid.

We consider the anisotropy of the trapping potential $V_{tr}$ to be produced by a tight harmonic potential in the $z$-direction of energy $\hbar \omega_z \gg \mu$, yielding the effective interaction strength $g = 2\sqrt{2\pi} \hbar^2 a/(M l_z)$, where $a > 0$ is the $s$-wave scattering length and $l_z = \sqrt{\hbar/(M \omega_z)}$ is the characteristic length of the harmonic potential.

The in-plane radial trap is a box potential of radius $\mathcal{R} \gg \xi$, making Eq. (3) subject to the boundary condition $\Phi_0 (\mathcal{R}, \varphi) = 0$. In natural units, the radius $\mathcal{R} = R \xi$ is given by $R = \sqrt{N/(\pi n_0 \xi^2)}$, where $n_0 \xi^2 = l_z/(4\sqrt{2\pi} a)$ is independent of the in-plane radial size ($R$ is related to a dimensionless coupling strength $\epsilon$ and waist length $w$). For the pinning potential, we consider $V_p(r) = \epsilon \exp(-r^2/w^2)$, i.e., a Gaussian beam with maximum optical potential $\epsilon$ and waist length $w$; we shall refer to pinning configurations in terms of the ordered pair $(\epsilon, w)$.

For the numerical calculations, we consider experiments with $^7$Li BECs [42] wherein positive values of $a$ of tens of nanometers are accessible via Feshbach resonances, while typical trapping frequencies of hundreds of kiloHertz yield $l_z$ of hundreds of nanometers [43], resulting $n_0 \xi^2 \gg 1$ and, thus, guaranteeing a regime of negligible quantum depletion consistent with the Bogoliubov approximation. Moreover, in these conditions, a BEC of up to $N \approx 10^5$ atoms may yield up to $R \approx 200$, while experiments with box potentials up to $R \approx 70\mu m$ [44] yield, in laboratory units, $n_0 g/\hbar \approx 30\, kHz$, or $n_0 g/k_B \approx 200 nK$, and $\xi \approx 300 nm$.

III. RESULTS

We give an account of a numerical analysis of the quantities presented in Sec. II B. We begin by briefly explaining the numerical methods used and the motivation for the inclusion of a pinning potential. Then, we present and discuss results for the symplectic and the hydrodynamic DOS, Eq. (26) for $O = S, H$, where we will encounter details that motivate looking into the low-lying states of definite angular momentum. Using the AM-DOS and the AM-LDOS, we make the physical origin of the hydrodynamic signatures clear. We follow up with a detailed account of the dependence of low-energy modes on the pinning potential showing, in particular, the sensitivity of the LCLS and of its hydrodynamic weight to this perturbation. Furthermore, provide a scaling analysis whence we identify non-trivial features of the low-lying spectrum of excitations.

A. Numerical vortex solutions

Combining Eq. (3) and (16) yields the reduced radial equation

$$ \left( h + \frac{\nu^2}{\zeta^2} - \mu + \phi_c^2(r) \right) \phi_\nu(r) = 0. \quad (29) $$

We generate numerical solutions of (29) for $\nu = 1$ using a combination of imaginary-time evolution and an $R$-asymptotic approximation, as outlined in Appendix B; results are presented in Fig. 1(a). We then use these to obtain numerical solutions of Eq. (19) using a discretization-based solver.

Single-charge (i.e., $|\nu| = 1$) vortices are dynamically stable, meaning that the associated Bogoliubov operator (7) possesses only real eigenvalues. However, due to the negative energy of the LCLS, they are energetically unstable, meaning that the spectrum of $H_B$, Eq. (12), has a negative eigenvalue and, therefore, that the mean-field (29) is energetically unstable; the existence of the LCLS has long been recognized and known to trigger the vortex’s spiraling-out motion [3–11]. In Fig. 1 we show features of the BEC wave function and its adjoint near the vortex core of the BEC and adjoint wave functions, as well as the Bogoliubov and hydrodynamic spatial densities of the LCLS.

B. Total densities of states

Figure 2 shows plots of the symplectic and the hydrodynamic DOS for multiple values of $R$ and fixed $(\epsilon, w)$. We have chosen to represent the Dirac delta function in Eqs. (20) by a Lorentz distribution,

$$ \delta(\omega) \sim \frac{1}{\pi} \frac{\delta \omega}{\omega^2 + \delta \omega^2}, $$

with the width $\delta \omega = \mathcal{O}(R^{-2})$; this width is of the energy scale of a single particle in a rigid wall potential of size $R$ and yields the Dirac delta in the limit $R \to \infty$. We see that, apart from noise, the numerical results are in
good agreement with the analytical results in Eqs. (27) and (28) for the homogenous BEC.

Both $D_B^{(S)}$ and $D_B^{(H)}$ deviate from the numerical result only in the low-energy region of the spectrum, around $\omega \sim 0$, as shown in the insets of Figs. 2. This deviation is twofold: i) the low-lying hydrodynamic DOS is dominated by a peak at energy $\omega_{-1,0}$ of the LCLS (or remnant kelvonic mode) [inset of Fig. 2(b)]; ii) the symplectic DOS shows that the energy $\omega_{0,0} = \omega_{0,0}(R)$, the energy of the first non-remnant kelvonic excitation, appears to become larger than the typical inter-level spacing with increasing $R$ [highlighted in the inset of Fig. 2(a)]. Additionally, the presence of the NG mode (especially notable in the hydrodynamic DOS) is not inconsistent with the Bogoliubov spectrum $\omega_B(k)$, which accounts only for density excitations. (Although excitations of the Bogoliubov spectrum entail phase excitations, the NG mode is an excitation of the phase exclusively.)

C. Angular momentum-resolved densities of states

1. AM-DOS

Figure 3 shows plots of the AM-DOS for selected values of $(\epsilon, w)$ and $R$ and angular momenta $m = 0, \pm 1, \pm 2$. Notable features of the hydrodynamic DOS are reproduced here: the NG mode, at $m = 0$, and the LCLS, at $m = -1$, tower over all other low-lying states.

There is a visible growth of the hydrodynamic weight (the height of the peaks in each $D_B^{(H)}$) with energy that ties in with the known breakdown of the hydrodynamic approximation beyond low energies [33]; it signals a departure from a collective sound-wave (phononic) picture of excitations, wherein the spectrum is $\sim k$, to a single-particle (atomic) one, with the spectrum $\sim k^2$. The transition from the phononic to the atomic picture is accompanied by a decrease in the magnitude of the hole-component $v_\lambda$ relative to the particle-component $u_\lambda$ of the excitation [33]. Thus, in this sense, the hydrodynamic weight $c^{(H)}$, Eq. (25), is a measure of the particle-hole imbalance of a bosonic state. We may understand this observation by noting that the hydrodynamic spatial density $\varrho_\lambda^{(H)}$, Eq. (23), amounts to an interference pattern between the amplitudes of the particle and hole components $u_\lambda$ and $v_\lambda$; this interference is essentially destructive, since $\Phi_0 v_\lambda$ has only a phase $e^{i\pi}$ relative to $\Phi_0 u_\lambda$. Thus, the discrepancy in the magnitudes of $u_\lambda$ and $v_\lambda$ determines the intensity of the interference pattern $\varrho_\lambda^{(H)}$ and then $c^{(H)}$, being its integral, functions as a global measure of the particle-hole imbalance of the mode $X_\lambda$.

We note, moreover, that the presence of the vortex is known to lift the angular momentum-degenerate excitations of an otherwise homogeneous BEC [45], a feature that we highlight in Fig. 3.

Contrarily to the DOS, the AM-DOS of a finite system are sparse (compare Figs. 2 and 3), that is, states are separable within each angular momentum sector. It follows that the spatial dimension of the AM-LDOS can be represented faithfully in terms of individual spatial densities $\varrho_{m,n}^{(O)}$, Eqs. (21) and (23), alone. These are displayed in Fig. 4. In particular, the Bogoliubov density of the LCLS is shown to be orders-of-magnitude larger at
FIG. 2. Plots of the (a) symplectic and (b) hydrodynamic DOS (solid), compared with the plots of Eqs. (27) and (28) (dashed), for system sizes $R = 50$ (red), $R = 125$ (orange) and $R = 200$ (blue) (appearing in ascending order in the plots), and pinning $(\epsilon, w) = (0.1, 1)$. Distinct scales in the vertical axes show that $D^{(S)}$ bounds $D^{(H)}$. Insets zoom into the low-energy features in the boxed regions of the respective plots: (a) the highlighted energies are the energy of the first density (non-kelvonic) excitation $\omega_0$, for the two system sizes $R = 125$ and $R = 200$; (b) peaks belonging to the NG mode and LCLS are labelled; the energy $\omega_{-1,0}$ is the energy of the LCLS at the selected configuration of the pinning potential. The notation of the energy levels follows Eq. (19).

the core than other low-lying, core-localized states.

This happens because $\varrho^{(S)}$ in Eq. (21) is sign-indeterminate, so that a bosonic state can be locally particle-like (hole-like) in regions of positive (negative) sign; accordingly, the state is locally characterized by an accumulation (depletion) of atoms proportional to its magnitude. Complementarily to the hydrodynamic weight $c^{(H)}$, which acts as a global measure, the symplectic spatial density $\varrho^{(S)}$ acts as a local measure of particle-hole character of a state. The LCLS is, therefore, largely more particle-like ($|u_{\text{LCLS}}| \gg |v_{\text{LCLS}}|$) than most other low-lying states [4], whence its hydrodynamic weight derives exceptional magnitude. The one exception is the NG mode, which is likewise particle-like but delocalized: its symplectic AM-LDOS is similar in magnitude to that of the first non-kelvonic excitation, for instance, but the hydrodynamic AM-DOS of the latter is vanishing—in fact, the hydrodynamic weight of the first is negligible while that of the NG surpasses the LCLS (see Fig. 3). As energy increases, the magnitudes of the symplectic and hydrodynamic densities become generically comparable, as modes become progressively more particle-like. These features are patent in Fig. 4.

D. Features of low-lying states

1. Effect of the pinning potential

The following results show how the LCLS is strongly dependent on the pinning potential, while other modes
have only a negligible (and indirect) dependence. We begin by noting that the pinning energy competes against the dominant centrifugal barrier \((m + \nu \sigma_3)^2/r^2\) at the vortex core \(r \lesssim 1\) in Eq. (18). A mode \((m,n)\) is, thus, insensitive to the pinning as long as it cannot penetrate the centrifugal barrier: there may be a non-negligible dependence on the pinning potential only in case \(\omega_{m,n} \geq (m \pm \nu)^2\). This observation ensures that we will find a negligible effect for all low-energy modes except at angular momenta \(m = \pm 1\); we have found the numerical evidence to support this observation, and so we focus our discussion on the case \(m = -1\), for definiteness. In Fig. 5, we depict the eigenvalues and hydrodynamic weights \(c^{(H)}\), Eq. (25), (shown as the size of plot markers) at low energies as functions of the pinning parameters. The strong dependence of the LCLS on the pinning potential is visible and, furthermore, we see that the \(c^{(H)}\) cross over at avoided level crossings, suggesting that the LCLS enters the energy region of (non-kelvonic) density excitations. In order to clarify these features, we obtained the minimal model described next.

We consider the reduced Bogoliubov Eq. (19) with the Hamiltonian rewritten as \(\mathcal{H} = \mathcal{H}^{(0)} + \Delta \mathcal{H}\), where

\[
\Delta \mathcal{H} = \sigma_3 V_p + \sigma_3 \delta_{\text{BEC}},
\]

with

\[
\delta_{\text{BEC}} = \begin{pmatrix}
-\delta\mu + 2\delta\phi^2 & \delta\phi^2 \\
\delta\phi^2 & -\delta\mu + 2\delta\phi^2
\end{pmatrix}
\]
yields the algebraic equation

\[ \omega_n = \omega_{n,0} + \Delta_{00} \left( \frac{\omega_0}{\omega_1} + \Delta_{10} \right) \]

where \( \Delta_{ij} = \Delta_{ji} = \langle X_i^{(0)} \Delta H X_j^{(0)} \rangle \), with the eigenvalues \( \omega_{n,0} \).

The first level avoidance involves the \( m = -1 \) modes \( X_i^{(0)} \), for \( n = 0, 1 \), solutions of \( H^{(0)} X_i^{(0)} = \omega_{n,0} X_i^{(0)} \).

Thus, we expand the eigenstates of \( (19) \) in this subspace, i.e.,

\[ X(r) = a_0 X_i^{(0)}(r) + a_1 X_j^{(0)}(r). \]  (32)

Taking the bilinear product \( \langle X_i^{(0)}, H X \rangle \), for \( i = 0, 1 \), yields the algebraic equation

\[
\begin{pmatrix}
\omega_0^{(0)} + \Delta_{00} & \Delta_{01} \\
\Delta_{10} & \omega_1^{(0)} + \Delta_{11}
\end{pmatrix}
\begin{pmatrix}
a_0 \\
a_1
\end{pmatrix}
= \omega
\begin{pmatrix}
a_0 \\
a_1
\end{pmatrix},
\]

where \( \Delta_{ij} = \Delta_{ji} = \langle X_i^{(0)}, \Delta H X_j^{(0)} \rangle \), with the eigenvalues \( \omega_{n,0} \).

FIG. 5. Plots of eigenvalues \( \omega_{m,n} \) of the modes \( n = 0, 1, 2 \) of angular momentum \( m = -1 \) as a function of the maximum optical potential \( \epsilon \), for \( R = 200 \) and beam waist (a) \( w = 0.2 \), (b) \( w = 0.4 \), (c) \( w = 0.6 \), (d) \( w = 0.8 \), (e) \( w = 1.0 \) and (f) \( w = 1.2 \); the size of the plot markers is proportional to the hydrodynamic weight, Eq. (25), of each state, for each value of \( \epsilon \) (scaled logarithmically for comparison). Black dotted lines show the results of the minimal model described in the text.

FIG. 6. Plots of overlaps \( |a_{n,0}|^2 = |\langle X_i^{(0)}, X_n \rangle|^2 \) of modes \( n = 0, 1, 2 \) of angular momentum \( m = -1 \) with the \( n = 0 \) mode of the pinning-less system, as a function of the maximum optical potential \( \epsilon \) for system size \( R = 200 \) and beam waist \( w = 0.4 \) [same as Fig. 5(b)]. Black dotted lines show the results of the minimal model described in the text. The curve for \( n = 3 \) is not included, though it is visible that \( a_{0,3} \) becomes non-negligible within this range.

\[
\omega_{n=01} = \tilde{\omega} + \Delta_+ \mp \sqrt{\Omega - \Delta_-^2 + |\Delta_{01}|^2},
\]

(33)

where \( \tilde{\omega} = (\omega_1^{(0)} + \omega_0^{(0)})/2 \), the half-gap \( \Omega = (\omega_1^{(0)} - \omega_0^{(0)})/2 \) and \( \Delta_\pm = (\Delta_{00} \pm \Delta_{11})/2 \); the minus- (plus-) signed branch in Eq. (33) is the \( n = 0 \) (\( n = 1 \)) solution. The comparisons of Eqs. (33) with the numerical results in Fig. 5 reveal the qualitative agreement of the minimal model; the quantitative inaccuracy results simply from the truncated subspace in Eq. (32) and is of no bearing to the following analysis.

Further comparing Eq. (33) with numerics in Fig. 5, we observe \( \Delta_{00} \) to be much larger than \( \Delta_{01} \) and \( \Delta_{11} \); indeed, we found \( \Delta_{01} \approx 10^{-2} \Delta_{00} \) while \( 10^{-2} < \Delta_{11}/\Delta_{01} \approx 1 \) across the sampled values of \( (\epsilon, w) \). To clarify these disparities, we consider the quantity

\[
\eta_{ij} = \frac{\langle X_i^{(0)}, \sigma_i \delta_{\text{BEC}} X_j^{(0)} \rangle}{\langle X_i^{(0)}, \sigma_i V_p X_j^{(0)} \rangle},
\]

that is, the ratio of the contributions to the \( \Delta_{ij} \), Eq. (30): the term \( V_p \) is the potential energy in the pinning potential; the term \( \delta_{\text{BEC}} \), Eq. (31), is the potential energy in the deformation of the BEC caused by the pinning potential. (Note that both terms are effects of the application of the pinning potential on the system, but that \( V_p \) is the direct effect while \( \delta_{\text{BEC}} \) is an indirect effect on its modes, in particular.) We found that \( \eta_{11} > 0 \) with \( 0.1 < \eta_{11} < 20 \) and, for \( i = 0, 1, \eta_{0i} < 0 \) (negative due to the deformation term) with \( 10^{-2} < |\eta_{0i}| < 0.5 \), increasing with \( w \) in all cases—that is, the term \( V_p \) dominates over \( \delta_{\text{BEC}} \) for both \( \Delta_{00} \) and \( \Delta_{01} \) and vice-versa for \( \Delta_{11} \).
This shows, since $\Delta_{00} \gg \Delta_{01} \gg \Delta_{11}$, that the LCLS is strongly dependent on the pinning potential directly, due to its exceptionally large amplitude at the vortex core; other modes are, at most, negligibly dependent on the pinning potential and indirectly so via the deformation of the BEC, as shown by $\Delta_{11}$.

We analyze the crossover of the $c^{(H)}$ by considering the probability of observing the pinning potential-free LCLS (i.e., the state $X_0^{(0)}$) given a state $X_n$, that is, the overlap $|a_{0,n}|^2 = |\langle X_0^{(0)} | X_n \rangle|^2$, plotted in Fig. 6. This figure of merit differentiates the states that possess a large, particle-like density at the vortex core for each configuration of the pinning potential. In particular, we can conclude that the LCLS (i.e., the mode with quantum numbers $m = -1$ and $n = 0$) eventually loses the characteristic hydrodynamic weight resulting from the large, core-localized density, as this becomes, due to the pinning potential, a feature of higher energy states. Hence the negligible magnitude of the $g_L^{(0)}$ seen in Fig. 1(c.2) relative to Fig. 1(b.2). Moreover, we notice that the crossover $a_{0,0}$ with $a_{0,1}$ is relatively steep while $a_{0,1}$ with $a_{0,2}$ is markedly smoother and can be seen to take place at a value $|a_{0,2}|^2 < 0.5$. This suggests, that, as the intensity of the pinning potential increases (and with it the energy of a density at the vortex core), the large hydrodynamic weight characteristic of the LCLS (for a weak pinning potential) becomes spread out across a number of modes instead of concentrated in a single one, and, so, there will be a number of modes with increased hydrodynamic weights instead of a single dominant one. Thus, we can think of the large hydrodynamic weight, initially concentrated in the LCLS, as becoming diluted under a sufficiently intense pinning potential. This effect is shown in terms of

![FIG. 7. Hydrodynamic AM-DOS of low-energy modes of angular momentum $m = -1$ as a function of the maximum optical potential $\epsilon$ (right-vertical axis) for system size $R = 200$ and beam waist $w = 0.4$ [same as Fig. 5(b)]. Solid lines highlight modes that have a noticeably increased hydrodynamic weight $c^{(H)}$, for each value of $\epsilon$, and colours indicate the mode having the largest hydrodynamic weight with red ($n = 0$), blue ($n = 1$), orange ($n = 2$) and violet ($n = 3$). The fact that the energy of the LCLS is positive in the absence of a blue ($n = 0$), having the largest hydrodynamic weight characteristic of the LCLS (for a weak pinning potential) and, with it the energy of a density at the vortex core), the large hydrodynamic weight characteristic of the LCLS (for a weak pinning potential) becomes spread out across a number of modes instead of

![FIG. 8. Log-log plots of (a) $\omega_{0,0}$, the energy of the first non-kevonic excitation without a pinning potential, and (b) $\Delta\omega$, the mean inter-level spacing in energy between states above $\omega_{0,0}$, as functions of $R$; light blue circles are the numerical data and the darker blue lines are the linear fits performed in the scaling region used to obtain the exponents $\alpha = 0.505 \pm 0.002$ for $\omega_{0,0}$ and $\beta = 1.05 \pm 0.03$ for $\Delta\omega$; the latter we obtained from a fixed sample including the first 250 states above $\omega_{0,0}$ and, in the fitting, we considered the standard error of the mean. Data breaks away from the scaling region at very large $R$ principally due to numerical errors, as local features become too small for at the defined numerical precision. Gray lines in each panel are the corresponding quantity obtained from the homogeneous system used to derive Eqs. (27) and (28), whose energy levels are given explicitly in terms of the homogeneous spectrum by $\omega_{n,m} = \omega_B(k_{m,n})$, $k_{m,n} = j_{m,n+1}/R$ where $j_{m,n+1}$ is the $(n + 1)$-th zero of the Bessel function $J_m$; deviations from the $R^{-1}$ scaling are one order of magnitude below significant digits and covered by error margins.]}
of the hydrodynamic AM-DOS in Fig. 7.

2. Scaling and R-dependence

We carried out scaling analyses concerning the discrepancy between the gap (that is, the energy of the lowest non-kelvonic excitation $\omega_{0,0}$) and the mean inter-level spacing $\Delta \omega$, as apparent in the inset of Fig. 2(a), thus, obtaining information on the appropriate low-energy, continuum description of density excitations; results are shown in Fig. 8. Indeed, we find distinct scalings between the energy of the first density excitation and the mean inter-level spacing, indicative of a gap that scales non-trivially with $R$. For comparison, we show the prediction for a homogeneous BEC (given by the Bogoliubov spectrum $\omega_B$) to scale as $R^{-1}$; this gap is a trivial finite-size effect, as it scales equally to the mean inter-level spacing and approaches a massless spectrum, that is, linearly with $R^{-1}$. Notice that the deviation in the exponent of the mean inter-level spacing from that of the homogeneous spectrum, though small, is not accounted for by the error margin.

Finally, it shows in Fig. (6) that the energy level of the LCLS does not cross zero at the presented system size, that is, that it does not represent an energetic instability. Indeed, we have found that this mode stabilizes spontaneously for a system size $R \gtrsim 73$, in qualitative agreement with Ref. [4]. (Eventual quantitative discrepancies are attributed to the fact that a quasi-2D BEC produced by a box potential is considered here.)

IV. DISCUSSION AND CONCLUSIONS

We have employed two classes of density of states and local density of states to trace three key features of the low-lying excitations of a quantum vortex in a quasi-2D BEC. In particular, we have introduced and computed the hydrodynamic density of states, which properly accounts for impurity-BEC interactions. The first key feature is the exceptionally large hydrodynamic weight of the lowest core-localized state (LCLS), that is, the remnant kelvon mode surviving the dimensionality reduction imposed by the trap. The magnitude of its hydrodynamic weight is a consequence of its pronounced particle-like character at the vortex core, rendering it sensitive to a pinning potential; this, in turn, explains the mechanism behind the energetic stabilization of vortices by pinning. In the presence of impurities, however, the pinning potential takes on a novel role: by tuning the energy of the LCLS, it might provide a switching mechanism between states of different angular momenta of an impurity bound in the vortex core. As a matter of fact, the present findings are not limited to the single-vortex configuration considered in this work—we can expect local properties of the excitations studied here to hold in different physical setups, namely, in the vicinity of vortices in the Abrikosov lattice of a quasi-2D BEC, where each vortex in the array is bound to possess a qualitatively similar remnant kelvon mode [46].

The second key feature is the non-trivial scaling of the spectrum of excitations. The scaling of the gap implies that the low-lying excitations of a large (but finite) BEC hosting a vortex are accounted for by a continuum of momenta with a gap $O(R^{-1/2})$, as suggested by our results, in stark contrast with the typical Bogoliubov spectrum $\omega_B(k) = \sqrt{k^2 + \mu}$, which is linear and gapless, that is, its finite-size gap scales equally to the inter-level spacing as $O(R^{-1})$. The resulting continuum of excitations of the vortex-BEC must be non-trivial as well: the deviation of the scaling exponent of the mean inter-level spacing indicates a non-analytic dependence in momentum; because the condition of a vanishing group velocity at the gap cannot be fulfilled by a linear dispersion, such a deviation is, in fact, necessary. Physically, we attribute these results to the non-locality of the vortex: its profile decays as $\sim 1/r^2$, resulting in the $\log(R)$-divergent energy of the BEC and, notably, in a logarithmically-modified dispersion of its kelvons (in a three-dimensional BEC) [1]. To our knowledge, however, the theory to support the long-wavelength behavior of the in-plane (non-kelvonic) density excitations is yet to be established, and it is one that may have important consequences to the quantum treatment of vortex dynamics in a quasi-2D BEC [47].

The third key feature is that the Nambu-Goldstone mode of a BEC can have a non-negligible effect on an interacting impurity, as suggested by the extraordinary magnitude of its hydrodynamic weight; because it is a consequence of the $U(1)$ spontaneous symmetry-breaking, this becomes a general feature of the BEC phase. We can expect it to play a part in the phenomenology of degenerate fermion-BEC mixtures as well as heterogeneous BEC mixtures, namely, by facilitating non-trivial dynamics between phase fluctuations of each BEC [48, 49]. Regarding the polaron physics of the impurity-BEC system, moreover, we have shown that a proper account of the Nambu-Goldstone mode is crucial, as impurities can become exceptionally sensitive to the phase fluctuations of a BEC.

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Appendix A: Decay width of an impurity bounded in a quantum vortex

We motivate the introduction of the hydrodynamic densities of states by obtaining the decay width between states of an impurity bounded in a quantum vortex.

We consider the BEC described by Hamiltonian (1) to be in the presence of atoms of a distinct species, described by a Hamiltonian

$$\hat{H}_{\text{imp}} = \int d^2r \hat{\Psi}^\dagger(r) \left[ -\frac{\hbar^2}{2M} \nabla^2 + g_{12} \hat{\Phi}^\dagger(r) \hat{\Phi}(r) \right] \hat{\Psi}(r),$$

that is, a field of dilute (i.e., non-interacting) atoms of mass $M_2$ interacting with BEC atoms by a contact potential of strength $g_{12} > 0$; we identify the atoms of mass $M_2$ as impurities with respect to the BEC, since they are assumed to be dilute and of a distinct species; the total Hamiltonian will be $\hat{H} = \hat{H}_{\text{BEC}} + \hat{H}_{\text{imp}}$.

Substituting the BEC fields for (2) in (A1) yields, to leading and sub-leading orders,

$$\hat{H}_{\text{imp}} = \frac{\chi}{n_0 \xi^2} \int d^2r \hat{\Psi}^\dagger h_{\text{imp}} \hat{\Psi} + \frac{\chi \gamma^2}{(n_0 \xi^2)^{3/2}} \int d^2r \hat{\Psi}^\dagger \left( \Phi_{\nu} \hat{\Phi}^\dagger + \Phi_{\nu}^\dagger \hat{\Phi} \right) \hat{\Psi} + \ldots,$$

(A2)

scaled to the natural units $n_0 g$ and $\xi$, according to the prescriptions in the main text, having introduced a Schrödinger operator $h_{\text{imp}} = -\nabla^2 + \gamma^2 |\Phi_{\nu}|^2$ and parameters $\chi = M/M_2$ and $\gamma^2 = M_2 g_{12}/(M g)$; performing a further scaling $\hat{H}_{\text{imp}} \rightarrow -\frac{\chi}{n_0 \xi^2} \hat{H}_{\text{imp}}$, the total Hamiltonian can be written as

$$\hat{H} = \hat{F}_0 + \frac{1}{n_0 \xi^2} \left( \hat{H}_B + \chi \hat{H}_{\text{imp}} \right).$$

Bosonic field operators $\hat{\phi}$ and $\hat{\phi}^\dagger$ are expanded as in Eq (8), a basis of eigenfunctions of $\hat{H}_B$, resulting

$$\Phi_{\nu} \hat{\phi}(r) + \text{h.c.} = \sum_{m,n} \left( \zeta_{m,n}(r) \delta_{m,n} + \overline{\zeta_{m,n}(r)} \delta_{n,m} \right),$$

$$\zeta_{m,n}(r) = \overline{\Phi_{\nu}(r)} u_{m,n}(r) + \Phi_{\nu}(r) v_{m,n}(r),$$

where the identification $\lambda = (m, n)$ is made. Impurity field operators $\hat{\Psi}$ and $\hat{\Psi}^\dagger$ can be expanded in a basis of eigenfunctions of $h_{\text{imp}}$: the vortex profile of the BEC density $|\Phi_{\nu}|^2$ — along with the condition that $g_{12} > 0$ — essentially guarantees the existence of bound states of the impurities localized at the vortex core [26]; these are also eigenstates of angular momentum in the plane, since the density $|\Phi_{\nu}|^2$ is cylindrically symmetric.

For the purpose of this derivation, we consider a two-level truncated basis comprised of the lowest-energy states of angular momenta $\ell \neq 0$ and $\ell' = 0$, i.e., we let $\hat{\Psi}(r) = \Psi_0(r) \hat{a}_0 + \Psi_\ell(r) \hat{a}_\ell$, yielding the effective Hamiltonian

$$\hat{H}_{\text{eff}} = \hat{H}_B + \Delta \left( \hat{a}_\ell^\dagger \hat{a}_\ell - \hat{a}_\ell^\dagger \hat{a}_0 \right) + \sum_{m,n} \left( g^{(n)}_{\ell,0} \hat{b}_{m,n} + g^{(n)}_{\ell,0} \hat{b}^\dagger_{m,n} \right) \hat{a}_\ell^\dagger \hat{a}_0 + \text{h.c.},$$

with $2\Delta$ the energy gap where $g^{(n)}_{\ell,0} = \delta_{\ell,m} g^{(n)}_{\ell}$ and $g^{(n)}_{\ell,0} = \delta_{-\ell,m} g^{(n)}_{\ell}$,

$$g^{(n)}_{\ell} = \frac{\chi \gamma^2}{\sqrt{n_0 \xi^2}} \int d^2r \Phi_{\ell,n} \Psi_0;$$

the selection rules above are made apparent from the fact that $L_{m,n} = m \zeta_{m,n}$ and $L \Psi_\ell = \ell \Psi_\ell$, with $L = -i \partial/\partial \varphi$ the two-dimensional angular momentum operator. An effective model for the dynamics of the impurity coupled to the bosonic bath follows from (A3) by projecting onto a single-particle subspace of the impurity in the rotating-wave approximation and considering only the bosonic modes of angular momentum $m = \ell$.

$$\hat{H}_{\text{eff}} = \sum_n \omega_{\ell,n} \hat{b}_{\ell,n}^\dagger \hat{b}_{\ell,n} + \Delta \sigma \epsilon_n + \sum_n \left( e^{-i(\omega_{\ell,n} - \Delta)} g^{(n)}_{\ell} \hat{b}_{\ell,n} \sigma_+ + \text{h.c.} \right),$$

with $\sigma_+$ the raising operator of impurity levels. A standard approach to this problem is to employ the Wigner-Weisskopf approximation [50], which predicts a decay width

$$\Gamma_{\ell \rightarrow 0} = \pi \sum_n |g^{(n)}_{\ell}|^2 \delta(\Delta - \omega_{\ell,n}).$$

Then, considering that the impurity is localized in the vortex core of a large BEC, the $|g^{(n)}_{\ell}|^2$ can be approximated to rewrite the width as

$$\Gamma_{\ell \rightarrow 0} \approx \pi \frac{\chi \gamma^2}{n_0 \xi^2} \int d^2r D^{(H)}_\ell(r; \Delta) \left| \Psi_\ell(r) \right|^2 \left| \Psi_0(r) \right|^2,$$

where $D^{(H)}_\ell(r; \Delta) = \sum_n \left| \zeta_{\ell,n}(r) \right|^2 \delta(\Delta - \omega_{\ell,n})$ is a hydrodynamic AM-LDOS, Eq. (22).

Appendix B: Computation of the radial BEC profile

A box potential, such as considered in this work, endows the BEC wave function with a nearly-homogeneous profile, with the exception of an exponentially-fast depletion of the wave function at the border [51]. On the one hand, as the size of the BEC increases, it becomes numerically non-trivial to compute its full profile, seeing as the depletion becomes steeper and more abrupt at the scale of the system, as illustrated in Fig. 1(a); on the other hand, provided that any other non-homogeneous
feature is sufficiently localized within the bulk, that is, away from the border of the BEC, the computation of its wave function and the computation of the wave function near the border can be asymptotically separated. We employ this observation to compute the solution of Eq. (29) by introducing an ansatz of the form

\[ \phi_{\nu}(r) = \phi_{\Sigma}(r) \phi_{\partial \Sigma}(r), \]

where \( \phi_{\Sigma} \) is the wave function in the bulk surface of the quasi-2D BEC and \( \phi_{\partial \Sigma} \) the wave function near the border.

The bulk wave function \( \phi_{\Sigma} \) is required to satisfy Eq. (29) in absence of \( V_{\nu} \)—that is, it is not required to satisfy the boundary condition \( \phi_{\nu}(R) = 0 \), but rather \( \phi_{\Sigma}(r \to \infty) = \sqrt{\mu} \) by the asymptotic separation argument. The solution can be obtained using imaginary-time evolution on a transformed radial coordinate \( \theta = 2 \arctan(r) \), with \( \theta \in [0, \pi) \); we did not update the chemical potential at each step of the evolution \[52\] but rather sampled the parametric \( \mu \)-dependence of the numerical solutions. Further, because \( V_{\nu} \) is exponentially localized at the origin, the asymptotic expansion \( \phi_{\Sigma}(r) = \sqrt{\mu} \left( 1 - \frac{\nu^2}{2 \mu r^2} \right) + \ldots \) holds for large \( r \) \[37\].

We now solve for \( \phi_{\partial \Sigma} \) using an asymptotic approximation. We begin by writing \( \phi_{\partial \Sigma}(r) = f(x) \), for \( i = 0, 1, 2 \), with the coordinate \( x = \sqrt{\mu/2(R-r)} \); the resulting equation reads

\[ \begin{align*}
-\frac{1}{2} f'' + \left( \frac{1}{\sqrt{2 \mu R}} + \frac{x}{\mu R^2} \right) f' + \\
+ \frac{\nu^2}{\mu R^2} f - \left[ 1 - \left( 1 - \frac{\nu^2}{\mu R^2} \right)^2 \right] f = O(R^{-3}),
\end{align*} \]

\[ \text{(B1)} \]

where we keep terms up to second order in \( R^{-1} \); likewise, we expand \( f \) asymptotically to order \( R^{-2} \):

\[ f(x) = f_0(x) + R^{-1} f_1(x) + R^{-2} f_2(x) + \ldots. \]

\[ \text{(B2)} \]

Considering \( x \in [0, \infty) \) by the asymptotic separation argument, we require the boundary condition \( f(0) = 0 \) and that \( f(x) \) be bounded as \( x \to \infty \). To 0th order, the equation is \(- \frac{1}{2} f''_0 + \left( \frac{1}{\sqrt{2 \mu R}} + \frac{x}{\mu R^2} \right) f'_0 + \frac{\nu^2}{\mu R^2} f_0 = 0 \), with solution \( f_0(x) = \tanh(x) \), in agreement with Ref. \[51\]. The 1st and 2nd-order equations are not worthwhile displaying here; we mention only that we solved for the \( f_i \) analytically with the aid of symbolic computation software; results are plotted in Fig. 9.

This method was validated against fully imaginary time-evolved solutions for small system sizes (\( R < 70 \)). Because \( R^2 \) amounts to a dimensionless coupling strength \[4\], this approximation constitutes a strong-coupling approximation. Moreover, further terms in the \( R \gg 1 \) asymptotic expansion of \( \phi_{\Sigma} \) can be obtained to compute Eq. (B2) to arbitrary order in \( R^{-1} \).

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