ERROR BOUNDS FOR MODEL REDUCTION OF FEEDBACK-CONTROLLED LINEAR STOCHASTIC DYNAMICS ON HILBERT SPACES

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ABSTRACT. We analyze structure-preserving model order reduction methods for Ornstein-Uhlenbeck processes and linear S(P)DEs with multiplicative noise based on balanced truncation. For the first time, we include in this study the analysis of non-zero initial conditions. We moreover allow for feedback-controlled dynamics for solving stochastic optimal control problems with reduced-order models and prove novel error bounds for a class of linear quadratic regulator problems. We provide numerical evidence for the bounds and discuss the application of our approach to enhanced sampling methods from non-equilibrium statistical mechanics.

1. INTRODUCTION

In this paper, we consider optimal control problems for Ornstein-Uhlenbeck processes and linear stochastic systems with multiplicative noise in a Hilbert space setting. These (abstract) equations describe stochastic partial differential equations (SPDEs) or high dimensional stochastic differential equations (SDEs) representing spatial discretizations of underlying SPDEs. Since optimal control problems in large (or infinite) dimensions often require high computational effort, thereby rendering practical applications infeasible, we resort to model order reduction (MOR) techniques. Here, the key idea is to identify low-order approximations capturing the dynamics of the originally large-scale systems such that, subsequently, the optimal control problem can be solved in the dimensionally reduced setting in which the complexity is lower or in which algorithms can be applied that would not be feasible in the original framework.

Even though MOR of linear and bilinear control systems is often justified by the incentive to reduce the computational burden associated with solving optimal control problems, they are usually not designed for approximating feedback-control problems. Most of the standard techniques like Gramian-based (balanced) MOR [CG86, GA04], proper orthogonal decomposition [KV08, Ro05], or interpolation-based MOR [SDS21, GAB08] are open-loop methods. Therefore, most of the error analysis focuses on worst-case error bounds (if any) for a certain class of (e.g. square-integrable) admissible controls. The specifics of the control or the cost functional are not taken into...
account for the identification of the relevant subspace or the bounds on the approximation error, which explains that the typical error bounds, such as the Hankel norm or $L^\infty$-error bounds of balanced truncation, are rather conservative when applied to feedback control, i.e. closed-loop systems. Notable exceptions are linear quadratic Gaussian control (LQG) balancing methods [JS83] and some of their more recent variants (e.g. [Cu03, BMS21]) that are based on balancing a pair of control and filter Riccati equations associated with an underlying linear quadratic regulator problem; other approaches include stochastic (Feynman-Kac or backward SDE) representations of the associated Hamilton-Jacobi-Bellman (HJB) equation [HLPZ14, KNH18].

In this paper, we follow an alternative route to LQG balancing or dynamic programming using stochastic representations of HJB equations and instead consider Gramian-based MOR with the goal to tighten the available open-loop error bounds. The motivation for this strategy is that the computational cost associated with solving Lyapunov equations for the Gramians rather than algebraic Riccati equations or HJB equations (e.g. using monotone finite difference schemes [BS91] or deep learning [NR21]) is greatly reduced. We should mention that for Ornstein-Uhlenbeck type systems with quadratic cost functionals, it is possible to reduce the corresponding HJB equations to Riccati equations, which further reduces the computational overhead of grid-based discretization schemes for HJB equations. Nevertheless, Lyapunov equations in infinite dimensions are relatively well-behaved, which cannot be said for the corresponding operator Riccati equations (e.g. see [OC05]), thereby further motivating our study.

To fix ideas, let $(M_t)_{t \geq 0}$ be a square-integrable mean zero Lévy process and let $(\mathcal{F}_t)_{t \geq 0}$ be its induced filtration. For control functions $u \in L^2_{ad}(\Omega \times (0,T))$ with values in $\mathbb{R}^m$, we study the differential equations

$$\begin{align*}
\frac{dZ_{t}^{\text{ou}}}{dt} &= AZ_{t}^{\text{ou}} \ dt + Bu_{t} \ dt + K \ dM_{t}, \quad Z_{t_0}^{\text{ou}} = \xi \quad (1.1a) \\
\frac{dZ_{t}^{\text{lin}}}{dt} &= AZ_{t}^{\text{lin}} \ dt + Bu_{t} \ dt + NZ_{t}^{\text{lin}} \ dM_{t}, \quad Z_{t_0}^{\text{lin}} = \xi \quad (1.1b)
\end{align*}$$

for $t \in (t_0, T)$ on some separable Hilbert space $X$. We will mostly consider the case $t_0 = 0$, and throughout the paper we use the labels “ou” for Ornstein-Uhlenbeck processes and “lin” for linear systems with multiplicative noise that are sometimes also referred to as “bilinear” in the literature (e.g. [BD11]). In (1.1a) the process is allowed to take values in $\mathbb{R}^d$, whereas in equation (1.1b) the Lévy process is assumed to be scalar\(^1\). The precise assumptions we impose on the OU process (1.1a) are stated in Section 3 and for equation 2 in Section 4. Most of the notation will be explained in Section 1.4. In the equations above $B : \mathbb{R}^m \rightarrow X$, is the linear input operator.

\(^1\)This assumption is only to simplify the notation in this article and an adaptation to multiple noise terms $\sum_{i=1}^{l} N_i Z_{t}^{\text{lin}} \ dM_{t}^i$ is straightforward.
In general, we are interested in outputs $CZ_t$, where $C : X \to \mathcal{H}$ is the linear output operator.

The study of controlled Ornstein-Uhlenbeck processes (1.1a) is of great practical relevance and has various applications such as interest rates models [V77] or pair trading in mathematical finance [ES19], and Langevin equations in physics [K07]. Such processes are also considered to model random perturbations of linear deterministic systems [HNS21]. Linear stochastic differential equations with multiplicative noise generalizes a dissipative geometric Brownian motion and has multiple applications in mathematical finance, where, most prominently, such equations describe stock prices in the Black-Scholes model [H09]. Examples involving SPDEs include stochastic variants of the linearized Navier-Stokes equations [DFV14], stochastic polymer models [MHKZ89], or the Kushner-Stratonovich equations of nonlinear filtering [B65].

1.1. Optimal control. For the optimal control problem associated to the equations in (1.1), we consider quadratic cost functions on a time horizon $T \in (0, \infty)$, given by

$$
J^\text{ou}_{\text{LQR}}(CZ^\text{ou}, u, T) := \frac{1}{T} \left( \|CZ^\text{ou}\|^2_{L^2(\Omega_T)} + \langle u, Ru \rangle_{L^2(\Omega_T)} \right)
$$

$$
J^\text{lin}_r(CZ^\text{lin}, u, T) := \|CZ^\text{lin}\|^2_{L^2_r L^r_\omega(\Omega_T)} + \langle u, Ru \rangle_{L^2(\Omega_T)} \text{ with } r \in (1, 2].
$$

(For the definition of the corresponding norms and scalar products, see Section 1.4 below.) We consider only functionals of quadratic type, as they allow us to use an explicit representation of the optimal feedback control using LGQ theory, which is necessary to obtain our error bounds. Let us remark that the functionals in (1.2) are defined slightly differently compared to some of the control applications appearing in the literature in order to acknowledge the fact that a stable OU dynamics with uniformly bounded second moment is not decaying, in contrast to a (mean-square) stable dynamics with multiplicative noise. For a fixed simulation time $T$, the regularization by $1/T$ of the first control functional can be omitted, however, it becomes necessary for an infinite simulation time. To be precise, in case of $T = \infty$ we define for the Ornstein-Uhlenbeck process

$$
J^\text{ou}_{\text{LQR}}(CZ^\text{ou}, u, \infty) := \limsup_{T \to \infty} J^\text{ou}_{\text{LQR}}(CZ^\text{ou}, u, T).
$$

In (1.2), $R$ is a (strictly) symmetric positive-definite matrix such that all eigenvalues of $R$ are strictly positive. For the ease of notation, we suppress the explicit dependence of the cost in case of a finite time horizon, $T < \infty$, on the initial data $(t_0, \xi)$. In the case $T = \infty$ and under some suitable ergodicity (i.e. stability and complete controllability) assumptions, the optimal cost after taking the infimum over the controls $u$ can be shown to be independent of the initial conditions [ABG12].
1.2. Model order reduction. As mentioned above, MOR shall be applied in order to lower the complexity of the problem discussed in Section 1.1. We mainly focus on a Gramian-based approach called balanced truncation (BT). Gramians can be interpreted as algebraic structures that are constructed to identify less relevant directions in state equations such as (1.1a) and (1.1b) as well as redundant information in the quantity of interest $CZ_t$. Simultaneous diagonalization of these Gramians then allows to easily detect and truncate unimportant states in order to find an accurate reduced system.

It turns out that MOR of control systems is intimately related with MOR of non-zero initial conditions. Therefore a few remarks on the specifics of Gramian-based BT in connection with non-zero initial conditions are in order.

1.2.1. Deterministic systems. BT is very popular in the context of linear and bilinear deterministic control systems, since it features computable error bounds and preserves many structural properties of the dynamics, such as stability or passivity (e.g. [SVR08]). Nevertheless, considered as an approximation tool for the Hankel operator that is underlying the system under consideration, it heavily relies on $L^2$-isometries and the fact that inputs and outputs are square-integrable functions on the positive reals [G84]. With few exceptions (see [BGM17, HRA11, DHQ19]), most of the available error bounds consider the dynamics under zero (or: homogeneous) initial conditions. This is somewhat surprising as, for example, the system-theoretic concepts of finite-time controllability and reachability make assertions about bounded measurable control inputs only and do not assume the initial condition to be zero (see, e.g. [C85, Sec. 4]). It is possible to think of the initial conditions as an extra control input, however, the control input associated with the initial condition is a Dirac delta function, and as a consequence it is neither bounded nor square-integrable; the approach thus requires an appropriate regularization that then leads to Hankel norm error bounds that depend on the particular regularization chosen (see e.g. [HRA11]).

1.2.2. Stochastic setting. In this article, we follow a different route and extend the notion of the Hankel operator to account for the non-zero initial conditions by an appropriate shifting of the underlying reachability and observability Gramians. The details will be given below in Section 2. In doing so, we study balanced MOR methods for (1.1) under non-zero initial states. Reduced order models, based on BT, for (uncontrolled) Ornstein-Uhlenbeck processes ($Z^{\text{ou}}_t$) have been considered in [FR18]; controlled processes ($Z^{\text{lin}}_t$) have been extensively studied within the standard stochastic BT framework and we refer the reader to [BH19, BR15, BD11] and references therein for a general overview. To our knowledge, non-zero initial conditions for equations like (1.1b) have in general not been considered in the BT MOR framework so far.
For stochastic control problems, for which the optimal policies are known to be Markovian feedback controls, the dependence of the controlled dynamics on the initial conditions is crucial [FS06, Sec. III.7]. In contrast to the deterministic case, dynamic programming, i.e. (approximately) solving HJB equations, or stochastic optimization methods are the methods of choice to compute optimal controls, and these methods rely on a careful treatment of the initial data. For example, the solution to the HJB equation, the value function, is a function of the initial conditions, and the optimal control can often be expressed in terms of the derivatives of the value function. As we will detail below, we include the initial states in the MOR process by projecting them on an $L^2$-subspace that is spanned by the admissible initial states, which guarantees that we can treat control and initial data on the same footing.

1.2.3. Differences between (1.1a) and (1.1b). The treatment of OU processes and systems with multiplicative noise seems analogous and follows a similar guiding principle in our work, but has fundamental differences. While we consider the same noise processes ($M_t$) for both equations, the assumptions on the considered dynamics are different. The assumption on the OU-type dynamics requires a strictly dissipative linear part, whereas we require a slightly stronger stability condition for the stochastic dynamics for systems with multiplicative noise. In case of OU processes, we work directly with the underlying semigroup, whereas for systems with multiplicative noise, it is the stochastic flow generated by the uncontrolled part, that takes on the fundamental position. What prevents us from putting the two dynamics (1.1a) and (1.1b) under the same umbrella are the different mathematical structures of the two equations, which force us to use different estimates. Specifically, we end up controlling different norms of the solution, even though we enforce the same square integrability condition on the controls. This is unavoidable, and it is owed to the fact that for the OU process with additive noise large randomness will induce a large norm, whereas for systems with multiplicative noise, the effect of the noise on the norm of the solution depends by the magnitude of the process itself.

1.3. Outline. The rest of the article is organized as follows: Before presenting BT in a nutshell in Section 2, Section 1.4 briefly introduces the basic notation for this article. The OU semigroup and the corresponding model reduction error bound are discussed in Section 3, whereas linear S(P)DEs with multiplicative noise are the subject of Section 4. The OU and S(P)DE error bounds are then revisited from the perspective of optimal control theory in Section 5, where we focus on linear quadratic regulator (LQR) problems. Finally, in Section 6, we illustrate the theoretical findings from Sections 3–5 with suitable numerical examples.

1.4. Further notation. The space of bounded linear operators between Banach spaces $X,Y$ is denoted by $\mathcal{L}(X,Y)$ and just by $\mathcal{L}(X)$ if $X = Y$. The operator norm of a
bounded operator $T \in \mathcal{L}(X, Y)$ is written as $\|T\|$. The trace-class and Hilbert-Schmidt operators between Hilbert spaces $X, Y$ are denoted by $\text{TC}(X, Y)$ and $\text{HS}(X, Y)$, respectively. In particular, we recall that for a linear operator $T \in \text{TC}(X, Y)$, where $X$ and $Y$ are now separable Hilbert spaces, the trace norm is given as

$$\|T\|_{\text{TC}} = \sup \left\{ \sum_{n \in \mathbb{N}} |\langle f_n, Te_n \rangle_Y| : (e_n)_{n \in \mathbb{N}} \text{ ONB of } X \text{ and } (f_n)_{n \in \mathbb{N}} \text{ ONB of } Y \right\}. \quad (1.3)$$

The Hilbert-Schmidt norm is given by

$$\|T\|_{\text{HS}} := \sqrt{\sum_{n, m \in \mathbb{N}} |\langle f_m, Te_n \rangle_Y|^2} \quad (1.4)$$

where $(e_n)_{n \in \mathbb{N}}$ is any ONB of $X$ and $(f_n)_{n \in \mathbb{N}}$ any ONB of $Y$.

We say that $g = O(f)$ if there is a $C > 0$ such that $\|g\| \leq C \|f\|$. The domain of unbounded operators $A$ is denoted by $D(A)$.

We write $\Delta(\Xi)$ to denote the difference of the quantity $\Xi$ for two systems, i.e., $\Delta(\Xi) = \Xi_{\text{System 1}} - \Xi_{\text{System 2}}$. We denote the expectation of a random variable $Y$ by $\mathbb{E}(Y)$ where we throughout the article assume to work on some fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If we want to address an operator $L$ for both OU processes and linear systems with multiplicative noise, we write $L^{\text{ou}}|_{\text{lin}}$.

We write $\Omega_T := \Omega \times (0, T)$ and define, for a Banach space $Y$, the norm associated with the space $L^2(\Omega_T, Y)$

$$\|f\|_{L^2(\Omega_T, Y)} := \sqrt{\mathbb{E} \int_{(0, T)} \|f(t)\|_Y^2 \, dt}. \quad (1.5)$$

When writing $L^p$ spaces, we most often omit the domain and sometimes also the image space to shorten the notation.

We also define the norm on iterated $L^pL^q$ spaces by

$$\|f\|_{L^p_xL^q_y} := \|x \mapsto \|y \mapsto f(x, y)\|_{L^q_y}\|_{L^p_x}, \quad (1.6)$$

where the $L^q$ norm is taken over the second argument, $y$, followed by the $L^p$ norm integration over the first argument, $x$.

We use the subscript $\text{ad}$ for $L^p$ spaces to denote stochastic processes in $L^p$ that are adapted to a canonical filtration.

The convolution of two functions is denoted by

$$(f * g)(x) = \int_{\mathbb{R}} f(x - y)g(y) \, dy.$$

We write $\mathbb{1}_X$ for the indicator function on some measurable set $X$, i.e. $\mathbb{1}_X(x) = 1$ if $x \in X$ and 0 otherwise.
If a sequence \((x_n)\) converges with respect to the weak topology of a Banach space to some element \(x\) of that space, we write \(x_n \rightharpoonup x\).

To include subspaces of relevant initial states in the MOR process, we define for an orthonormal family \(\phi_i \in L^2(\Omega, X)\), the map \(B_{in} : \mathbb{R}^k \to X\) by \(B_{in}v := \sum_{i=1}^{k} \langle v, \hat{e}_i \rangle \mathbb{R}^k \phi_i\). Here, \(\text{span}\{\phi_i; i \in \{1,..,k\}\}\) is the space of admissible initial states. In other words, we define an operator \(B_{in}\) such that \(B_{in}B_{in}^*\) is a projection onto the subspace of admissible initial states.

### 2. Balanced truncation in a nutshell

In this article, we study MOR methods for equations (1.1). To fix ideas, let us for now assume that the underlying Hilbert space \(X\) is finite-dimensional. In the first step of the MOR process, positive semidefinite observability and reachability Gramians \(O_{ou}|lin\) and \(P_{ou}|lin\) are computed from Lyapunov equations, using an auxiliary operator \(S := B_{in}B_{in}^* + KE(M_1M_1^*)K^*\). We note that Gramians are the key ingredient of balancing-related MOR methods like BT, since from their eigenspaces, dominant subspaces of the underlying system can be extracted.

For Ornstein-Uhlenbeck processes, for which we consider two types of reachability Gramians \(P_{ou}\) and \(P_{ou}|lin\), the Lyapunov equations (Prop. 3.3) take the form

\[
A^*\mathcal{O}_{ou} + \mathcal{O}_{ou}A + C^*C = 0,
A\mathcal{P}_{ou} + \mathcal{P}_{ou}A^* + BB^* + KE(M_1M_1^*)K^* = 0 \quad \text{(2.1)}
A(\mathcal{P}_{ou} - S) + (\mathcal{P}_{ou} - S)A^* + BB^* = 0.
\]

- The first reachability Gramian \(\mathcal{P}_{ou}\) is employed to obtain an error bound on the supremum norm with initial state 0 (Theorem 1), which is the basis for a bound with general initial states relying on the same type of Gramian (Corollary 3.4).
- The second reachability Gramian \(\mathcal{P}_{ou}|lin\) depends also on the chosen initial states and allows us to obtain an \(L^2\) error bound (Theorem 2).

For linear systems with multiplicative noise they satisfy (see Prop. 4.2 below)

\[
A(\mathcal{O}_{lin} - B_{in}B_{in}^*) + (\mathcal{O}_{lin} - B_{in}B_{in}^*)A^* + N(\mathcal{P}_{lin} - B_{in}B_{in}^*)N^*
+ BB^* + B_{in}B_{in}^* = 0 \quad \text{and} \quad \text{(2.2)}
A^*\mathcal{O}_{lin} + \mathcal{O}_{lin}A + N^*\mathcal{O}_{lin}N + C^*C = 0.
\]

Since both \(\mathcal{O}_{ou}|lin\) and \(\mathcal{P}_{ou}|lin\) are positive semidefinite, they can be decomposed as \(\mathcal{O}_{ou}|lin = W*W\) and \(\mathcal{P}_{ou}|lin = RR^*\). Let \(\mathcal{O}_{ou}|lin\) and \(\mathcal{P}_{ou}|lin\) have for simplicity full rank, the balanced representation is obtained by first performing a singular value decomposition \(WR = V\Sigma U^*\), to identify a dominant subspace for the dynamics of the system, where \(V, U\) are unitary and \(\Sigma\) is diagonal. The diagonal entries of \(\Sigma\) are called
Hankel singular values of the system. Then, we conjugate the system by operators
\[ T := \Sigma^{-1/2} V^* W \] and \( T^{-1} := RU \Sigma^{-1/2} \) such that
\[
\begin{align*}
A'_b |_{\text{lin}} &:= T A^\text{ou} |_{\text{lin}} T^{-1}, \quad B'_b |_{\text{lin}} := T B^\text{ou} |_{\text{lin}}, \quad C'_b |_{\text{lin}} := C^\text{ou} |_{\text{lin}} T^{-1}, \quad \text{and} \\
N'_b |_{\text{lin}} &:= T N |_{\text{lin}} T^{-1}, \quad K'_b := T K^\text{ou}. 
\end{align*}
\] (2.3)

The state space transformation in (2.3) can be viewed as a procedure in which the Gramians \( O^\text{ou} |_{\text{lin}} \) and \( P^\text{ou} |_{\text{lin}} \) are simultaneously diagonalized. This is done because the Hankel singular values characterize the importance of associated state components as shown in other stochastic settings for additive noise [FR18] and multiplicative noise [R18]. To obtain a reduced system by BT, the operator \( \Sigma \) is now approximated. This approximation is obtained by discarding the smallest singular values of \( \Sigma \) and only capturing the large ones since the corresponding state variables mainly determine the dynamics. Error bounds in this article are commonly expressed in terms of the difference of Hankel operators for the full and the reduced system. This difference of Hankel operators we denote by \( \Delta(H) \). The Hankel operator is one possible decomposition \( WR \) of the Gramians above. The precise definition of the Hankel operator is stated in Definitions 3.2, for OU processes, and 4.1, for linear systems with multiplicative noise, respectively. However, to evaluate the trace norm difference it is not necessary to analyze the Hankel operator directly: To evaluate the singular values of \( \Delta(H) \), and thus the trace norm of \( \Delta(H) \), we introduce an error system
\[
\begin{align*}
\hat{A}^\text{ou} |_{\text{lin}} &:= \begin{pmatrix} A & 0 \\ 0 & \hat{A} \end{pmatrix}, \quad \hat{B}^\text{ou} |_{\text{lin}} := \begin{pmatrix} B \\ B \end{pmatrix}, \quad \hat{B}_\text{in} |_{\text{lin}} := \begin{pmatrix} B^\text{in} \\ B^\text{in} \end{pmatrix}, \quad \hat{C}^\text{ou} |_{\text{lin}} := \begin{pmatrix} C & -\hat{C} \end{pmatrix}, \\
\hat{N} |_{\text{lin}} &:= \begin{pmatrix} N & 0 \\ 0 & \hat{N} \end{pmatrix}, \quad \hat{K}_\text{in} |_{\text{lin}} := \begin{pmatrix} \hat{K}_\text{in} \\ \hat{K}_\text{in} \end{pmatrix}, \quad \text{and state variable } \hat{Z}_t |_{\text{lin}} := \begin{pmatrix} Z_t \\ \hat{Z}_t \end{pmatrix},
\end{align*}
\] (2.4)

where operators/states without tilde belong to System 1, as in (1.1), and with tilde to some System 2. This second system could be any other system with the same structure such as the reduced system, e.g., resulting from applying BT. Certainly, the output of the error system is the error between the outputs of both systems. Then one can define Gramians \( \hat{O} = \hat{W}^* \hat{W} \) and \( \hat{P} = \hat{R} \hat{P}^* \) of this error system (2.4) that satisfy Lyapunov equations (2.1) or (2.2) for the error system, i.e.
\[
\begin{align*}
\hat{A}^* \hat{O} + \hat{O}^* \hat{A} + \hat{C}^* \hat{C} &= 0, \\
\hat{A} (\hat{P}^* - \hat{S}) + (\hat{P} - \hat{S}) \hat{A}^* + \hat{B} \hat{B}^* &= 0
\end{align*}
\] (2.5)

where \( \hat{S} := \hat{B}_\text{in} \hat{B}_\text{in}^* + \hat{K} \hat{E} (M_1 M_1^*) \hat{K}^* \) and analogously for linear systems with multiplicative noise. We can then perform a singular value decomposition \( \hat{W} \hat{R} = \hat{V} \Lambda \hat{U}^* \) with diagonal operator \( \Lambda \) that contains all singular values of the error system (2.4) on
its diagonal [RS14, Theorem 5.1]. It is then easy to check that
\[ \| \Delta(H) \|_{TC} = \sum_{\lambda \in \Lambda} \lambda = \sum_{\mu \in \sqrt{\sigma(\hat{O}\hat{P})}} \mu. \]
This property follows as any decomposition \( \hat{W}\hat{R} \) is equivalent to the Hankel operator \( \hat{H} \) associated with system (2.4):

More precisely, there exist unitary mappings [RS14, Prop. 6.1] \( U : \text{ran}(\hat{W}\hat{R}) \rightarrow \text{ran}(\hat{H}) \) and \( V : \ker^+(\hat{W}\hat{R}) \rightarrow \ker^+(\hat{H}) \) such that
\[ \Delta(H)|_{\ker^+(\hat{H})} = \hat{H}|_{\ker^+(\hat{H})} = U\left(\hat{W}\hat{R}\right)|_{\ker^+(\hat{W}\hat{R})}V^*|_{\ker^+(\hat{H})}. \]
Notice that when BT is used, \( \| \Delta(H) \|_{TC} \) is expected to be small, since a reduced system is constructed that is supposed to keep the large Hankel singular values of the original system such that \( \sum_{\lambda \in \Lambda} \lambda \) has small summands in most of the cases.

We summarize the preceding discussion of the Hankel operator error bounds:

- The trace class norm of the Hankel operator difference is computable by solving in addition the Lyapunov equations for the error system consisting of the original and the reduced system (2.4).
- The error bound does not require the user to compute the Hankel operator directly.
- As a word of caution: The Hankel operators do not have any obvious energy interpretation. In particular, the difference of Hankel operators in trace norm is not the same as the sum of truncated Hankel singular values in the MOR process.

### 3. Ornstein-Uhlenbeck processes

Let \( X \) be a Hilbert space, \( A \) be the generator of a \( C_0 \)-semigroup \((T_t)_{t \geq 0}\) on \( X \), as well as \( K : \mathbb{R}^d \rightarrow X \) and \( B : \mathbb{R}^m \rightarrow X \) both linear and continuous maps. For the OU processes (1.1a), we define the mild solution \((Z_t^{ou})_{t \geq 0}\) with initial state \( \xi \in L^2(\Omega, \mathcal{F}_0, X) \) with output given by the variation of constant formula
\[
Y_t = CZ_t^{ou} = CT_t\xi + \int_0^t CT_{t-s}K \ dM_s + \int_0^t CT_{t-s}Bu_s \ ds. \tag{3.1}
\]
In particular, if \( X \) is finite-dimensional or more general, if \((T_t)\) is uniformly continuous, then the semigroup is just given by \( T_t := e^{tA} \).

For OU processes we make the following stability assumption:

**Assumption 1 (OU processes).** *We assume that \( A \) is the generator of an exponentially stable semigroup \((T_t)_{t \geq 0}\) such that for some \( \omega > 0 \) and \( \nu \geq 1 : \| T_t \| \leq \nu e^{-\omega t} \). Moreover,
we assume that \((M_t)_{t \geq 0}\) is a square-integrable mean zero Lévy process taking values in \(\mathbb{R}^d\).

In the theory of balanced truncation, it is common to introduce two types of Gramians, an observability Gramian and a reachability Gramian. Here, we introduce for Ornstein-Uhlenbeck processes two possible types of such reachability Gramians, from which one of them is also taking into account non-zero initial conditions. Their slightly different definitions are mainly motivated by our two methods of obtaining error bounds that we introduce in this article.

**Definition 3.1 (OU Gramians).** For the controlled OU process, we define the observability Gramian for \(x, y \in \mathcal{X}\) by

\[
\langle x, \mathcal{O}_{\text{ou}} y \rangle_{\mathcal{X}} := \int_0^{\infty} \langle C T_s x, C T_s y \rangle_{\mathcal{H}} \, ds \tag{3.2}
\]

and two types of reachability Gramians for \(x, y \in \mathcal{X}\) by

\[
\langle x, \mathcal{P}_{\text{ou}} y \rangle_{\mathcal{X}} := \int_0^{\infty} \langle x, T_t (K E (M_1 M_1^*) K^* + BB^*) T_t^* y \rangle_{\mathcal{X}} \, dt, \quad \text{and} \tag{3.3}
\]

\[
\langle x, \mathcal{P}_{\text{ou}} y \rangle_{\mathcal{X}} := \int_0^{\infty} \langle x, T_t BB^* T_t^* y \rangle_{\mathcal{X}} \, dt + \langle x, (B_{in} B_{in}^* + K E (M_1 M_1^*) K^*) y \rangle_{\mathcal{X}}.
\]

If \(\mathcal{X}\) is finite-dimensional, then the definition of the Gramians reduces in case of the observability Gramian to

\[
\mathcal{O}_{\text{ou}} = \int_0^{\infty} T_s^* C^* C T_s \, ds \tag{3.4}
\]

and for the reachability Gramians to

\[
\mathcal{P}_{\text{ou}} = \int_0^{\infty} T_t (K E (M_1 M_1^*) K^* + BB^*) T_t^* \, dt, \quad \text{and} \tag{3.5}
\]

\[
\mathcal{P}_{\text{ou}} = \int_0^{\infty} T_t BB^* T_t^* \, dt + B_{in} B_{in}^* + K E (M_1 M_1^*) K^*.
\]

The weak formulation for infinite-dimensional spaces \(\mathcal{X}\) is needed in general, as \(t \mapsto T_t\) is not necessarily measurable but \(t \mapsto T_t x\) for any fixed \(x \in \mathcal{X}\) is.

**Definition 3.2 (OU Hankel operator).** The OU Hankel operator is the operator

\[
H_{\text{ou}} := W_{\text{ou}} R_{\text{ou}} \in \mathcal{L}(L^2((0, \infty), \mathbb{R}^m) \oplus \mathbb{R}^d \oplus \mathbb{R}^k, L^2((0, \infty), \mathcal{H})).
\]

Here, we assume that the controls take values in \(\mathbb{R}^m\), the space of admissible initial states is \(k\)-dimensional, and the noise process takes values in \(\mathbb{R}^d\).

The observability map \(W_{\text{ou}} \in \mathcal{L}(\mathcal{X}, L^2((0, \infty), \mathcal{H}))\) is defined as

\[
W_{\text{ou}} x := C T_t x \text{ such that } \mathcal{O}_{\text{ou}} = W_{\text{ou}}^* W_{\text{ou}},
\]

where \(W_{\text{ou}}\) is a Hilbert-Schmidt operator if \(\mathcal{H}\) is finite-dimensional.
The reachability map $R_{\text{ou}} \in \text{HS}(L^2((0, \infty), \mathbb{R}^m) \oplus \mathbb{R}^d \oplus \mathbb{R}^k, X)$ is defined as

$$R_{\text{ou}}(f, v, u) := \int_0^\infty T_s B f_s \, ds + K \sqrt{\mathbb{E}(M_1 M_1^*)} v + B_{\text{in}} u$$

such that $\mathcal{P}_{\text{ou}} = R_{\text{ou}} R_{\text{ou}}^*$. \hfill (3.6)

The Gramians (3.2) and (3.3) satisfy the following Lyapunov equations:

**Proposition 3.3** (Lyapunov equations). The observability Gramian satisfies for all $x_2, y_2 \in D(A)$

$$\langle Ax_2, \mathcal{P}_{\text{ou}} y_2 \rangle_X + \langle x_2, \mathcal{P}_{\text{ou}} A y_2 \rangle_X + \langle x_2, C^* C y_2 \rangle_X = 0$$

and the reachability Gramians satisfies, for all $x_1, y_1 \in D(A^*)$, with $S := B_{\text{in}} B_{\text{in}}^* + K \mathbb{E}(M_1 M_1^*) K^*$,

$$\langle x_1, \mathcal{P}_{\text{ou}} A^* y_1 \rangle_X + \langle A^* x_1, \mathcal{P}_{\text{ou}} y_1 \rangle_X + \langle x_1, (B B^* + K \mathbb{E}(M_1 M_1^*) K^*) y_1 \rangle_X = 0 \quad \text{and} \quad \langle x_1, (\mathcal{P}_{\text{ou}} - S) A^* y_1 \rangle_X + \langle A^* x_1, (\mathcal{P}_{\text{ou}} - S) y_1 \rangle_X + \langle x_1, B B^* y_1 \rangle_X = 0.$$

If $A$ is bounded, the equations reduce to

$$A^* \mathcal{P}_{\text{ou}} + \mathcal{P}_{\text{ou}} A + C^* C = 0$$

and

$$\mathcal{P}_{\text{ou}} A^* + A \mathcal{P}_{\text{ou}} + B B^* + K \mathbb{E}(M_1 M_1^*) K^* = 0 \quad \text{and} \quad (\mathcal{P}_{\text{ou}} - S) A^* + A (\mathcal{P}_{\text{ou}} - S) + B B^* = 0.$$

**Proof.** The Lyapunov equations follow immediately from the Lyapunov equations for linear deterministic systems [ORW13]:

This is immediate for the observability Gramian, since it coincides with the observability Gramian for linear systems.

For the reachability Gramian it suffices to observe that $\mathcal{P}_{\text{ou}}$ and $\mathcal{P}_{\text{ou}} - S$ are of the form of a linear reachability Gramian. \hfill \Square

### 3.1. Error bounds.

We start by stating a direct bound for two OU processes as in (1.1a) with $(C, A, K, B)$ and $(\tilde{C}, \tilde{A}, \tilde{K}, \tilde{B})$, respectively, both having zero initial conditions. To this end, let $(T_t)$ and $(\tilde{T}_t)$ be the semigroups generated by $A$ and $\tilde{A}$. To state the error bound, we introduce for $i \in \{1, 2\}$ the auxiliary Gramians defined
in terms of $B_1 = B$ and $B_2 = K \sqrt{\mathbb{E}(M_1 M_1^*)}$ by

$$AP_t + P_tA^* = -B_tB_t^* , \quad P_t := \int_0^\infty C T_s B_t B_t^* T_s^* C^* ds,$$

$$AP_{t,g} + P_{t,g}A^* = -B_{t}B_{t}^* , \quad P_{t,g} := \int_0^\infty C T_s B_t B_t^* T_s^* C^* ds,$$

$$\tilde{A}P_t + \tilde{P}_tA^* = -\tilde{B}_t\tilde{B}_t^* , \quad \tilde{P}_t := \int_0^\infty \tilde{C} T_s B_t B_t^* T_s^* \tilde{C}^* ds,$$

and observe that the sums $\mathcal{P}^{ou} = P_1 + P_2$ and $\tilde{\mathcal{P}}^{ou} = \tilde{P}_1 + \tilde{P}_2$ coincide with the reachability Gramian for $B_{\text{in}} = 0$. Moreover, we write $\mathcal{P}^{ou} = P_{1,g} + P_{2,g}$. We then have the following error bound for the outputs of two Ornstein-Uhlenbeck processes starting from zero with possibly two different controls.

**Theorem 1** (Error bound from zero). For control functions $u, \tilde{u} \in L^2_{ad}(\Omega_T, \mathbb{R}^n)$ and initial conditions $Y_0 = \tilde{Y}_0 = 0$, it follows that the difference between the outputs of two OU processes satisfies

$$\sup_{t \in [0,T]} \sqrt{\mathbb{E} \left[ ||Y_t - \tilde{Y}_t||^2 \right]} \leq \sqrt{2} (1 \lor ||u||_{L^2(\Omega_T)}) \left( \text{tr} \left( C \mathcal{P}^{ou} C^* - 2C \mathcal{P}^{ou} \tilde{C}^* \right) + \text{tr}(\tilde{C} \mathcal{P}^{ou} \tilde{C}^*) \right)^{1/2} + (\text{tr}(\tilde{C} \tilde{P}_1 \tilde{C}^*))^{1/2} ||u - \tilde{u}||_{L^2(\Omega_T)}.$$  

**Proof.** The explicit outputs of controlled OU processes are according to (3.1) given by

$$Y_t = C \int_0^t T_{t-s} Bu_s \ ds + C \int_0^t T_{t-s} K \ dM_s \quad \text{and} \quad \tilde{Y}_t = \tilde{C} \int_0^t \tilde{T}_{t-s} B \tilde{u}_s \ ds + \tilde{C} \int_0^t \tilde{T}_{t-s} \tilde{K} \ dM_s.$$  

We insert the representations for $Y_t$ and $\tilde{Y}_t$ from (3.9) and obtain for (3.8)

$$\left( \mathbb{E} ||Y_t - \tilde{Y}_t||^2 \right)^{1/2} = \left( \mathbb{E} ||(I_1(u) - \tilde{I}_1(u)) + (\tilde{I}_1(u) - \tilde{I}_1(\tilde{u})) + (I_2 - \tilde{I}_2)||^2 \right)^{1/2} \leq \left( \mathbb{E} ||I_1(u) - \tilde{I}_1(u)||^2 \right)^{1/2} + \left( \mathbb{E} ||I_2 - \tilde{I}_2||^2 \right)^{1/2} + \left( \mathbb{E} ||\tilde{I}_1(u) - \tilde{I}_1(\tilde{u})||^2 \right)^{1/2} \right)^{1/2}$$  

From [FR18, (31)] we know that

$$\mathbb{E}[||I_2 - \tilde{I}_2||^2] \leq \text{tr} \left[ CP_2 C^* - 2CP_{2,g} \tilde{C}^* + \tilde{C} P_2 \tilde{C}^* \right].$$  

(3.11)
for all $t \in [0, T]$. We can estimate the first term in (3.10) using that
\[
\mathbb{E}[\|I_1(u) - \tilde{I}_1(u)\|^2] = \mathbb{E} \left[ \left\| \int_0^t \left( CT_{t-s}Bu_s - \tilde{C}\tilde{T}_{t-s}\tilde{B}u_s \right) \, ds \right\|^2 \right]
\leq \mathbb{E} \left[ \left( \int_0^t \left\| CT_{t-s}B - \tilde{C}\tilde{T}_{t-s}\tilde{B} \right\|_{HS} \|u_s\| \, ds \right)^2 \right]
\leq \mathbb{E} \left[ \int_0^t \left\| CT_{t-s}B - \tilde{C}\tilde{T}_{t-s}\tilde{B} \right\|_{HS}^2 \, ds \int_0^t \|u_s\|^2 \, ds \right]
= \int_0^t \left\| CT_{t-s}B - \tilde{C}\tilde{T}_{t-s}\tilde{B} \right\|_{HS}^2 \, ds \mathbb{E} \left[ \int_0^t \|u_s\|^2 \, ds \right]
\leq \operatorname{tr} \left[ CP_1C^* - 2CP_{1\tilde{g}}\tilde{C}^* + \tilde{C}\tilde{P}_1\tilde{C}^* \right] \|u\|^2_{L^2(\Omega_T)},
\] (3.12)
where we used Cauchy-Schwarz and took the limit $t \to \infty$ in the first integral and $t \to T$ in the second one. Furthermore, we find for the remaining term in (3.10) that
\[
\left( \mathbb{E}[\|\tilde{I}_1(u) - \tilde{I}_1(\tilde{u})\|^2] \right)^{\frac{1}{2}} \leq \left( \mathbb{E} \left[ \left( \int_0^t \left\| \tilde{C}e^{\tilde{A}(t-s)}\tilde{B}_1 \right\|_{HS} \|u_s - \tilde{u}_s\| \, ds \right)^2 \right] \right)^{\frac{1}{2}}
\leq \left( \int_0^t \left\| \tilde{C}e^{\tilde{A}(t-s)}\tilde{B}_1 \right\|_{HS}^2 \, ds \right)^{\frac{1}{2}} \left( \mathbb{E} \int_0^t \|u_s - \tilde{u}_s\|^2 \, ds \right)^{\frac{1}{2}}
\leq \left( \operatorname{tr}(\tilde{C}\tilde{P}_1\tilde{C}^*) \right)^{\frac{1}{2}} \|u - \tilde{u}\|_{L^2(\Omega_T)}.
\] (3.13)
In order to get (3.8), we estimate
\[
\left( \mathbb{E}[\|I_1(u) - \tilde{I}_1(u)\|^2] \right)^{\frac{1}{2}} + \left( \mathbb{E}[\|I_2 - \tilde{I}_2\|^2] \right)^{\frac{1}{2}} \leq \sqrt{2} \sqrt{\mathbb{E}[\|I_1(u) - \tilde{I}_1(u)\|^2] + \mathbb{E}[\|I_2 - \tilde{I}_2\|^2]}
\] (3.13)
applying $a + b \leq \sqrt{2}\sqrt{a^2 + b^2}$ for $a, b \in \mathbb{R}^+$. We insert (3.11) and (3.12) into (3.13) and enlarge the resulting expression trough 1, $\|u\|^2_{L^2(\Omega_T)} \leq (1 \lor \|u\|^2_{L^2(\Omega_T)})$. The bound (3.8) now follows, by the linearity of the trace. 

A different control $\tilde{u}$ in the reduced order model appears for example if model reduction is applied in the context of optimal control. Solving a control problem in the reduced system then leads to a different control strategy compared to the full model. However, we see from the bound in Theorem 1 that the expression depending on the difference between $u$ and $\tilde{u}$ is scaled by a term depending on $\tilde{P}_1$, an operator that cannot be expected to be small. Hence, one can only guarantee a good approximation if $u$ and $\tilde{u}$ are not too different. Notice that the bound in Theorem 1 is a generalization of the result in [FR18], where $B = 0$ was considered. Moreover, if the second system is a reduced model based on BT, then $\operatorname{tr} \left( C\mathcal{P}_1\mathcal{C}^* - 2C\mathcal{P}_1\tilde{C}^* + \tilde{C}\tilde{P}_1\tilde{C}^* \right)$ in Theorem 1 can be expressed in terms of a weighted sum of truncated Hankel singular values of
the system with zero initial data, which can be shown following the steps of [FR18]. Therefore, the error of BT is low if we choose the reduced system dimension such that the truncated Hankel singular values are small.

We now state an error bound in case the initial condition is not zero.

**Corollary 3.4 (Error bound non-zero initial states).** Let $u, u \in L^2_{ad}(\Omega_T, \mathbb{R}^n)$, $Y$ be the output of (1.1a) with $Z_0^0 = \xi = B_{in}v$ and $\tilde{Y}$ be the output of the reduced system with $\tilde{Z}_0^0 = 0$. We define

$$
\tilde{Y}_t(0) = \tilde{C}(0)\tilde{T}_t(0)\tilde{B}_{in}v + \tilde{Y}_t,
$$

where $(\tilde{T}_t(0))_{t \geq 0}$ is a $C_0$-semigroup generated by some operator $\tilde{A}(0)$ and $\tilde{B}_{in}, \tilde{C}(0)$ are additional input and output operators, respectively. Then, we have

$$
\|Y - \tilde{Y}(0)\|_{L^2(\Omega_T)} \leq \sqrt{2T} (1 + \|u\|_{L^2(\Omega_T)}) \left( \text{tr} \left( C\mathcal{P}_{ou}C^* - 2C\mathcal{P}_{g}\tilde{C}^* + \tilde{C}\mathcal{P}_{ou}\tilde{C}^* \right) \right)^{1/2}
$$

$$
+ \|v\|_{L^2(\Omega)} \left( \text{tr} \left( CP_0C^* - 2CP_{0,g}\tilde{C}^* + \tilde{C}\tilde{P}_0\tilde{C}^* \right) \right)^{1/2}
$$

$$
+ \sqrt{T} \left( \text{tr}(\tilde{C}\tilde{P}_1\tilde{C}^*) \right)^{1/2} \|u - \tilde{u}\|_{L^2(\Omega_T)},
$$

where $P_0, P_{0,g}$ and $\tilde{P}_0$ satisfy

$$
AP_0 + P_0A^* = -B_{in}B_{in}^*, \ AP_{0,g} + P_{0,g}\tilde{A}(0)^* = -B_{in}\tilde{B}_{in}^*, \ \tilde{A}(0)\tilde{P}_0 + \tilde{P}_0\tilde{A}(0)^* = -\tilde{B}_{in}\tilde{B}_{in}^*.
$$

**Proof.** We use the triangle inequality to obtain

$$
\|Y - \tilde{Y}(0)\|_{L^2(\Omega_T)} \leq \left( \mathbb{E} \int_0^T \| (Y_t - CT_tB_{in}v) - \tilde{Y}_t \|^2 dt \right)^{1/2}
$$

$$
+ \left( \mathbb{E} \int_0^T \| CT_tB_{in}v - \tilde{C}(0)\tilde{T}_t(0)\tilde{B}_{in}v \|^2 dt \right)^{1/2}.
$$

Since the function $Y_t - CT_tB_{in}v$, $t \in [0, T]$, is the output to (1.1a) with zero initial state, Theorem 1 yields

$$
\left( \mathbb{E} \int_0^T \| (Y_t - CT_tB_{in}v) - \tilde{Y}_t \|^2 dt \right)^{1/2} \leq \sqrt{T} \left( \text{tr}(\tilde{C}\tilde{P}_1\tilde{C}^*) \right)^{1/2} \|u - \tilde{u}\|_{L^2(\Omega_T)}
$$

$$
+ \sqrt{2T} (1 + \|u\|_{L^2(\Omega_T)}) \left( \text{tr} \left( C\mathcal{P}_{ou}C^* - 2C\mathcal{P}_{g}\tilde{C}^* + \tilde{C}\mathcal{P}_{ou}\tilde{C}^* \right) \right)^{1/2}.
$$

Moreover, as in previous estimates, we find

$$
\mathbb{E} \int_0^T \| CT_tB_{in}v - \tilde{C}(0)\tilde{T}_t(0)\tilde{B}_{in}v \|^2 dt \leq \int_0^T \| CT_tB_{in}v - \tilde{C}(0)\tilde{T}_t(0)\tilde{B}_{in}v \|^2_{HS} dt \mathbb{E} \|v\|^2
$$

$$
\leq \mathbb{E} \|v\|^2 \text{tr} \left( CP_0C^* - 2CP_{0,g}\tilde{C}^* + \tilde{C}\tilde{P}_0\tilde{C}^* \right)
$$

concluding the proof. \qed
**Remark 1.** The choice of $\tilde{Y}^{(0)}$ in (3.14) is motivated by the fact that (1.1a) can be decomposed into a homogeneous and inhomogeneous part. Its output can then be written as $Y_t = C\mathcal{H}_t^{ou} + C\mathcal{T}_t^{ou}$, where

\begin{align*}
    d\mathcal{H}_t^{ou} &= A\mathcal{H}_t^{ou} \, dt, \quad \mathcal{H}_0^{ou} = \xi = B_{in}v, \quad (3.15) \\
    d\mathcal{T}_t^{ou} &= A\mathcal{T}_t^{ou} \, dt + Bu_t \, dt + K \, dM_t, \quad \mathcal{T}_0^{ou} = 0. \quad (3.16)
\end{align*}

As in [BGM17], BT based on the Gramian $P_0$ can be applied to (3.15) in order to get a reduced system with matrices $(\tilde{A}^{(0)}, \tilde{B}_{in}, \tilde{C}^{(0)})$. BT is used a second time but now based on $\mathcal{P}^{ou}$ to find a reduced system to (3.16). The reduced order matrices in this case are $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{K})$. The sum of both reduced order outputs is then a suitable candidate for the choice of $\tilde{Y}^{(0)}$. In the context of BT, it was also shown in [BGM17] that the error term $\text{tr} \left( CP_0 C^* - 2C P_{0, g} \tilde{C}^{(0)*} + \tilde{C}^{(0)*} \tilde{P} \tilde{C}^{(0)*} \right)$ is a function of the truncated Hankel singular values based on $P_0$. Consequently, BT applied to (3.15) and (3.16) yields a small error if one truncates the respective small Hankel singular values only.

We now state another error bound that takes into account the initial states and bounds the norms appearing in the control functional (1.2). In contrast to the previous approach in Remark 1, the second ansatz does not rely on a splitting of the system. It is an all in one reduction procedure which invokes the Hankel operator that relies on the reachability Gramian $\mathcal{P}^{ou}$. However, the error will be bounded by the truncated singular values of the error system (2.4) instead of the truncated Hankel singular values of the large-scale system. First, we need the following lemma, where we employ $\Delta$ introduced in Subsection 1.4.

**Lemma 3.5.** Let $\mathcal{H} \simeq \mathbb{R}^n$ be a finite-dimensional space, then for two systems with the same Lévy noise profile, satisfying Assumption 1, the difference of their Hankel operators $\Delta(H^{ou})$ satisfies

\begin{align*}
    \frac{1}{\sqrt{T}} \left\| \Delta \left( \int_0^t CT_{t-s}K \, dM_s \right) \right\|_{L^2(\Omega_T; HS(\mathbb{R}^m, \mathbb{R}^n))} &\leq \| \Delta(H^{ou})\|_{HS}, \\
    \| \Delta(CTB_{in})\|_{L^2(0, \infty); HS(\mathbb{R}^k, \mathbb{R}^n))} &\leq \| \Delta(H^{ou})\|_{HS}, \text{ and} \\
    \| \Delta(CTB)\|_{L^1((0, \infty); HS(\mathbb{R}^m, \mathbb{R}^n))} &\leq 2 \| \Delta(H^{ou})\|_{TC}. \quad (3.17)
\end{align*}
Proof. To obtain the first bound in (3.17), consider the process
\( X_t := \int_0^t C_t K \, dM_s \)
such that by Ito’s isometry
\[
\frac{1}{T} \| \Delta(X) \|^2_{L^2(\Omega_T)} = \frac{1}{T} \int_0^T \mathbb{E} \| \Delta(X_t) \|^2 \, dt
\]
Ito’s iso. \( \overset{t \rightarrow s}{\Rightarrow} \)
\[
\frac{1}{T} \int_0^T \int_0^t \| \Delta(C_{t-s} K) \sqrt{\mathbb{E}(M_1 M_1^*)} \|^2_{HS} \, ds \, dt
\]
(3.18)
where in (1) we extended the integration range from 0 to \( T \) and in (2) we used that
the integrand is independent of \( t \).

We now derive a lower bound on the Hilbert-Schmidt norm of the Hankel operator. Recall that the Hilbert-Schmidt norm of an operator is defined in (1.4).

Thus, using any ONB \( (e_i)_{i \in \mathbb{N}} \) of \( L^2((0, \infty), \mathbb{R}^n) \) and \( (f_j)_{j \in \{1, \ldots, d\}} \) of \( \mathbb{R}^d \), we have the
lower bound on the Hilbert-Schmidt norm, since we do not take a complete basis of
the input space of the Hankel operator, yields the first estimate in (3.17)
\[
\| \Delta(H^{ou}) \|^2_{HS} \overset{(1.4)}{\geq} \sum_{j=1}^d \sum_{i=1}^\infty \left| \langle \Delta(H^{ou}) (0, f_j, 0), e_i \rangle_{L^2} \right|^2
\]
Def.3.2
\[
\overset{\text{Def.3.2}}{=} \sum_{j=1}^d \sum_{i=1}^\infty \left| \left\langle \Delta \left( CT \cdot K \sqrt{\mathbb{E}(M_1 M_1^*)} \right) f_j, e_i \right\rangle_{L^2} \right|^2
\]
(3.19)
\[
\overset{(1.4)}{=} \int_0^\infty \left\| \Delta(CT_s K \sqrt{\mathbb{E}(M_1 M_1^*)}) \right\|^2_{HS} \, ds \overset{\text{3.18}}{\geq} \frac{1}{T} \| \Delta(X) \|^2_{L^2(\Omega_T)}.
\]
The second bound in (3.17) follows straight from the definition of the Hilbert-Schmidt norm by taking an orthonormal basis \( (e_i)_{i \in \mathbb{N}} \) of \( L^2((0, \infty), \mathcal{H}) \) and \( (f_i)_{i \in \{1, \ldots, k\}} \) an orthonormal system of \( \mathbb{R}^k \). Then, it follows that
\[
\| \Delta(H) \|^2_{HS} \geq \sum_{i=1}^\infty \sum_{j=1}^k \left| \langle e_i, \Delta(H)(0, 0, f_j) \rangle_{L^2} \right|^2
\]
(3.20)
\[
= \| \Delta(CTB_{in}) \|^2_{L^2((0, \infty), \mathbb{HS}(\mathbb{R}^k, \mathbb{R}^n))}.
\]
The last bound in (3.17) follows from linear BT theory [CGP88, Theorem 2.1]. \( \square \)
From the preceding estimates we can now obtain the following error bound on the global dynamics.

**Theorem 2 (OU Error bound).** Consider two OU-processes with the same control function $u \in L^2(\Omega_T, \mathbb{R}^m)$, see (1.5), driven by the same Lévy processes, but (possibly different) initial conditions $\xi := \sum_{i=1}^{k} \langle v, \hat{e}_i \rangle \tilde{R}_k \hat{\phi}_i$, and $\tilde{\xi} := \sum_{i=1}^{k} \langle v, \hat{e}_i \rangle \tilde{R}_k \hat{\xi}_i$. Here, $(\phi_i)$ is the $L^2(\Omega, F_0, X)$-orthonormal system of $B_{in}$. The difference between the outputs of two such processes satisfies

$$
\frac{\| \Delta (C Z_{ou}) \|_{L^2(\Omega_T)}}{\sqrt{T}} \leq \| \Delta (H_{ou}) \|_{TC} \left( 1 + \frac{\| \xi \|_{L^2(\Omega)} + 2 \| u \|_{L^2(\Omega_T)}}{\sqrt{T}} \right). \quad (3.21)
$$

**Proof.** We have for $v \in \mathbb{R}^k$ by orthonormality of $(\phi_i)$ that $\| v \| = \| \xi \|_{L^2(\Omega)}$ and define $X_t := \int_0^t C T_{t-s} K \, dM_s$. By Young's inequality, which implies that for $f(s) := \| \Delta (1_{[0,\infty)} C T_s B) \|$ and $g(s) := 1_{[0,T]} \| u_s \|$ we have

$$
\| f * g \|_{L^2(0,T)} \leq \| f \|_{L^1(0,T)} \| g \|_{L^2(0,T)},
$$

and Lemma 3.5, it follows that

$$
\| \Delta (C Z_{ou}) \|_{L^2(\Omega_T)} \overset{(3.1)}{\leq} \left( \| \Delta (X) \|_{L^2(\Omega_T)} + \| \Delta (C T B_{in}) (v) \|_{L^2(\Omega_T)} + \| \| \Delta (1_{[0,\infty)} C T B) \| \| 1_{[0,T]} \| u \| \|_{L^2(\Omega_T)} \right)
$$

Young's ineq.

$$
\overset{\text{Lemma 3.5}}{\leq} \left( \| \Delta (X) \|_{L^2(\Omega_T)} + \| \Delta (C T B_{in}) \|_{L^1(0,\infty)} \| \xi \|_{L^2(\Omega)} + \| \Delta (C T B) \|_{L^1(0,\infty)} \| u \|_{L^2(\Omega_T)} \right)
$$

$$
\overset{\text{Young's ineq.}}{\leq} \| \Delta (H_{ou}) \|_{TC} \left( \sqrt{T} + \| \xi \|_{L^2(\Omega)} + 2 \| u \|_{L^2(\Omega_T)} \right).
$$

\hfill \square

We can see that the bound in Theorem 2 depends on $\| \Delta (H_{ou}) \|_{TC}$, which is the sum of singular values of the error system. By construction of BT, the associated reduced system keeps the larger Hankel singular values of the original system such that $\| \Delta (H_{ou}) \|_{TC}$ and hence the error is expected to be small whenever the second system is a reduced model by BT with appropriate reduced order dimension.
4. Linear systems with multiplicative noise

In this section, a bound for the output error between two S(P)DEs of the form (1.1b) is proved. It is based on the singular values of the associated error system and therefore requires the study of suitable Gramians. This bound provides an a posteriori criterion for the approximation error, e.g., in the context of model order reduction. The solution to the linear S(P)DE is given as the sum of the homogeneous process

\[ dH_t^{\text{lin}} = AH_t^{\text{lin}} dt + NH_t^{\text{lin}} dM_t, \text{ such that } \]
\[ H_0^{\text{lin}} = \xi \]  

and the solution to the inhomogeneous problem starting from zero

\[ dI_t^{\text{lin}} = AI_t^{\text{lin}} dt + NI_t^{\text{lin}} dM_t + Bu_t dt, \text{ such that } \]
\[ I_0^{\text{lin}} = 0. \]  

The solution to the homogeneous equation (4.1), started at time \( s \) from state \( \xi \), defines a flow \( H_t^{\text{lin}} =: \Phi_{t,s}^{\text{lin}} \xi \). If the initial time is \( s = 0 \), we just write \( \Phi_t^{\text{lin}} := \Phi^{\text{lin}}(t,0) \). We now introduce a stability criterion for linear systems with multiplicative noise which is necessary to ensure dissipative dynamics.

**Assumption 2** (Linear systems with multiplicative noise). *We make the assumption that \( \Phi_{t,s}^{\text{lin}} \) is exponentially stable in mean square sense, i.e. there are \( \gamma, c > 0 \) such that for all \( \xi \in L^2(\Omega, \mathcal{F}, X) \) and \( t \geq s \)

\[ E (\| (\Phi_{t,s}^{\text{lin}})^{-1} \|_2^2) \leq \gamma e^{-c(t-s)}E \| \xi \|^2. \]  

Moreover, we assume that \( (M_t)_{t \geq 0} \) is a square-integrable scalar-valued mean zero Lévy process.

We use the following representation of the homogeneous solution with flow \( H_t^{\text{lin}} =: \Phi_{t,s}^{\text{lin}} \xi \) such that

\[ CZ_t^{\text{lin}} := C H_t^{\text{lin}} + C T_t^{\text{lin}} = C \Phi_t^{\text{lin}} \xi + \int_0^t C \Phi_{t,s}^{\text{lin}} Bu_s \, ds. \]  

This expression coincides with the output of the mild solution as discussed in [BH19, (5.4)ff.]. The observability and reachability Gramian for linear systems with multiplicative noise are for \( x, y \in X \) defined as

\[ \langle x, \mathcal{O}^{\text{lin}} y \rangle_X = E \int_0^\infty \langle C \Phi_s^{\text{lin}} x, C \Phi_s^{\text{lin}} y \rangle_{\mathcal{H}} \, ds \]
\[ \langle x, \mathcal{R}^{\text{lin}} y \rangle_X = E \int_0^\infty \langle x, (\Phi_s^{\text{lin}} B)(\Phi_s^{\text{lin}} B)^* y \rangle_X \, ds + \langle x, B_{\text{in}} B_{\text{in}}^* y \rangle_X. \]  

(4.5)
To decompose the Gramians as
\[ O^{\text{lin}} = W^{\text{lin}} W^{\text{lin}}^* \quad \text{and} \quad P^{\text{lin}} = R^{\text{lin}} R^{\text{lin}}^*, \] (4.6)
we introduce observability \( W^{\text{lin}} \in \mathcal{L}(X, L^2(\Omega_\infty, \mathcal{H})) \) and reachability maps \( R^{\text{lin}} \in \text{HS}(L^2(\Omega_\infty, \mathbb{R}^m) \oplus \mathbb{R}^k, X) \) defined as
\[
(W^{\text{lin}} x)_t := C \Phi^{\text{lin}}_t x \quad \text{and} \quad R^{\text{lin}}(f, u) := \mathbb{E} \int_0^\infty \Phi^{\text{lin}}_s B f_s \, ds + B_{\text{in}} u. \] (4.7)

A straightforward computation shows that the above operators indeed satisfy (4.6). The main theoretical tool for our study is the Hankel operator which we shall introduce next.

Definition 4.1 (Hankel operator). The Hankel operator for the linear system with multiplicative noise is the Hilbert-Schmidt operator defined as
\[ H^{\text{lin}} := W^{\text{lin}} R^{\text{lin}} \in \text{HS}(L^2(\Omega_\infty, \mathbb{R}^m) \oplus \mathbb{R}^k, \mathcal{H}) \]
and is trace-class if \( \mathcal{H} \) is finite-dimensional.

The above Hilbert-Schmidt and trace-class properties follow from the same arguments as in [BH19, Sec. 5.2]. Adding the operator \( B_{\text{in}} \) to \( R^{\text{lin}} \) does not affect these properties as \( B_{\text{in}} \) is a finite rank operator.

The Gramians (4.5) satisfy Lyapunov equations given in the following proposition. This fact is very useful for the practical computation of these Gramians since such equations can be solved even in very high-dimensional settings.

Proposition 4.2 (Lyapunov equations). The stochastic Gramians for the system with multiplicative noise satisfy the following Lyapunov equations for all \( x_1, y_1 \in D(A^*) \) and \( x_2, y_2 \in D(A) \)
\[
\langle x_1, B B^* y_1 \rangle_X + \langle A^* x_1, (\mathcal{O}^{\text{lin}} - B_{\text{in}} B_{\text{lin}}^*) y_1 \rangle_X + \langle x_1, (\mathcal{O}^{\text{lin}} - B_{\text{in}} B_{\text{lin}}^*) A^* y_1 \rangle_X \\
+ \langle N^* x_1, (\mathcal{O}^{\text{lin}} - B_{\text{in}} B_{\text{lin}}^*) N^* y_1 \rangle_X \mathbb{E}(M_1^2) = 0 \quad \text{and} \\
\langle x_2, C^* C y_2 \rangle_X + \langle A x_2, \mathcal{O}^{\text{lin}} y_2 \rangle_X + \langle x_2, \mathcal{O}^{\text{lin}} A y_2 \rangle_X + \langle N x_2, \mathcal{O}^{\text{lin}} N y_2 \rangle_X \mathbb{E}(M_1^2) = 0.
\]

Proof. It suffices to observe that the observability Gramian and \( \mathcal{O}^{\text{lin}} - B_{\text{in}} B_{\text{lin}}^* \) coincide with the observability and reachability Gramian in [BH19]. The Lyapunov equations are then stated in [BH19, Lemma 5.6]. \( \square \)

Our next Lemma provides some auxiliary results that are relevant for the final error estimate of the difference of the stochastic dynamics in terms of the Hankel operator.
Lemma 4.3. Let $\mathcal{H}$ be a finite-dimensional space, we consider two linear multiplicative systems with the same or two i.i.d. square-integrable mean zero Lévy processes $(M_t)_{t \geq 0}$ each, then the difference of Hankel operators $\Delta(H^\text{lin})$ satisfies
\[
\|\Delta(C\Phi^\text{lin}B_m)\|_{L^2(\Omega_\infty, HS(\mathbb{R}^k, \mathbb{R}^n))} \leq \|\Delta(H^\text{lin})\|_{\text{HS}} \text{ and}
\|\Delta(C\Phi^\text{lin}B)\|_{L_1^2L_2^\infty(\Omega_\infty, HS(\mathbb{R}^m, \mathbb{R}^n))} \leq 2\|\Delta(H^\text{lin})\|_{\text{TC}}.
\] (4.8)

Proof. The first bound in (4.8) follows straight from the definition of the Hilbert-Schmidt norm, i.e. let $(f_j)_{j \in \{1, \ldots, k\}}$ be an orthonormal basis of $\mathbb{R}^k$ and $(e_i)_{i \in \mathbb{N}}$ an orthonormal basis of $L^2(\Omega(0, \infty), \mathcal{H})$. This implies that
\[
\|\Delta(H^\text{lin})\|^2_{\text{HS}} \geq \sum_{j=1}^{k} \sum_{i=1}^{\infty} |\langle e_i, \Delta(H^\text{lin})(0, f_j) \rangle|_{L^2}^2 = \|\Delta(C\Phi^\text{lin}B_m)\|^2_{L^2(\Omega_\infty, HS(\mathbb{R}^k, \mathbb{R}^n))}.
\]
The second bound has been derived in [BH19, Theorem 3, (5.11)] under the assumption that the noise profiles are independent. In the case of the same noise profile, the same proof as for [BH19, Theorem 3] applies. This is because the flow of the coupled system $\hat{Z}_t = (Z_t, \tilde{Z}_t)$ is a Markov process, which is the key property used in [BH19, (5.12)].

The Markov property of $\hat{Z}_t$ follows, since $\hat{Z}_t$ is a solution to the S(P)DE
\[
d\hat{Z}_t^{\text{lin}} = \hat{A}^{\text{lin}}\hat{Z}_t^{\text{lin}} dt + \hat{N}^{\text{lin}}\hat{Z}_t^{\text{lin}} dM_t + \hat{B}^{\text{lin}}u_t dt,
\] (4.9)
where we used the notation introduced in (2.4). The solution to this system satisfies the Markov property [PZ07, Sec.9.6].

We are now ready to state our main error bound.

Theorem (Error bound). Consider two linear systems with multiplicative noise. For initial conditions $\xi := \sum_{i=1}^{k} \langle v, \xi_i \rangle_{\mathbb{R}^k} \xi_i$ with $L^2(\Omega, \mathcal{F}_0, X)$-orthonormal system $(\xi_i)$, and $\tilde{\xi} := \sum_{i=1}^{k} \langle v, \tilde{\xi}_i \rangle_{\mathbb{R}^k}\tilde{\xi}_i$, it follows that for two Lévy processes $(M_t)_{t \geq 0}$, which we assume to be either the same or independent, each one of them driving the dynamics of a linear system with multiplicative noise, we have for control functions $u \in L^2_{\text{ad}}(\Omega_\infty, \mathbb{R}^m)$ that
\[
\|\Delta(CZ^{\text{lin}})\|_{L_1^2L_2^\infty(\Omega_\infty)} \leq \|\Delta(H^\text{lin})\|_{\text{TC}} \left(\|\xi\|_{L^2(\Omega)} + 2\|u\|_{L^2(\Omega_\infty)}\right)
\] (4.10)
and for control functions $u \in L_1^2L_\infty^\omega(\Omega_\infty, \mathbb{R}^n)$ we have
\[
\|\Delta(CZ^{\text{lin}})\|_{L_2^2(\Omega_\infty)} \leq \|\Delta(H^\text{lin})\|_{\text{TC}} \left(\|\xi\|_{L^2(\Omega)} + 2\|u\|_{L_1^2L_\infty^\omega(\Omega_\infty)}\right)
\] (4.11)

Proof. From (4.4) we find that
\[
\|\Delta(CZ^{\text{lin}})\|_{L_1^2L_2^\infty(\Omega_\infty)} \leq \|\Delta(C\mathcal{H}^{\text{lin}})\|_{L_1^2L_2^\infty(\Omega_\infty)} + \|\Delta(C\mathcal{T}^{\text{lin}})\|_{L_1^2L_2^\infty(\Omega_\infty)} \text{ and}
\|\Delta(CZ^{\text{lin}})\|_{L_2^2(\Omega_\infty)} \leq \|\Delta(C\mathcal{H}^{\text{lin}})\|_{L_2^2(\Omega_\infty)} + \|\Delta(C\mathcal{T}^{\text{lin}})\|_{L_2^2(\Omega_\infty)}.
\] (4.12)
For the first terms on the right-hand side of (4.12) we have using

- the Cauchy-Schwarz inequality in (1),
- the explicit expression for the homogeneous solution in (2), and
- the first estimate of (4.8) in (3)

that

\[ \| \Delta (C \tilde{H}^\text{lin}) \|_{L^2_t L^2_{\omega}(\Omega_\infty)} \leq \| \Delta (C \tilde{H}^\text{lin}) \|_{L^2(\Omega_\infty)} \]

\[ \leq \| \Delta (C \Phi^\text{lin} B_m) \|_{L^2(\Omega_\infty, \mathcal{HS}(\mathbb{R}^k, \mathbb{R}^n))} \| \xi \|_{L^2(\Omega)} \]  

(4.13)

\[ \leq \| \Delta (H^\text{lin}) \|_{\mathcal{HS}} \| \xi \|_{L^2(\Omega)} . \]

To estimate the second terms on the right-hand side of (4.12) we require some additional estimates on the inhomogeneous flow (4.2)

\[ \| \Delta (C T^\text{lin}) \|^2_{L^2_t L^2_{\omega}(\Omega_\infty)} \leq \int_0^\infty \left( \mathbb{E} \int_0^t \| \Delta (C \Phi_{t,s} B) \| \| u_s \| \, ds \right)^2 \, dt \]

\[ \leq \int_0^\infty \left( \int_0^t \sqrt{\mathbb{E}(\| \Delta (C \Phi_{t,s} B) \|^2) \mathbb{E}(\| u_s \|^2) \, ds} \right)^2 \, dt \]

\[ \leq \int_0^\infty \left( \int_0^t \sqrt{\mathbb{E}(\| \Delta (C \Phi_{t-s} B) \|^2_{\mathcal{HS}}) \mathbb{E}(\| u_s \|^2) \, ds} \right)^2 \, dt \]

\[ \leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \mathbb{I}_{[0, \infty)}(t-s) \sqrt{\mathbb{E}(\| \Delta (C \Phi_{t-s} B) \|^2_{\mathcal{HS}}) \mathbb{I}_{[0, \infty)}(s) \mathbb{E}(\| u_s \|^2) \, ds} \right)^2 \, dt . \]

In (1) we applied Hölder’s inequality in the expectation value and in (2) we use the Markov property, cf. [BH19, (5.15)]. In (3) we just rewrote the expression using indicator functions to make the convolutional structure more apparent. If we then introduce auxiliary functions \( f(s) := \mathbb{I}_{[0, \infty)}(s) \sqrt{\mathbb{E}(\| \Delta (C \Phi_s B) \|^2_{\mathcal{HS}})} \) and \( g(s) := \mathbb{I}_{[0, \infty)}(s) \sqrt{\mathbb{E}(\| u_s \|^2)} \), we can interpret the above estimate as a convolution estimate

\[ \| \Delta (C T^\text{lin}) \|_{L^2_t L^2_{\omega}(\Omega_\infty)} \leq \| f * g \|_{L^2} . \]

If we then apply Young’s convolution inequality we find

\[ \| f * g \|_{L^2} \leq \| f \|_{L^1} \| g \|_{L^2} . \]

Using that \( \| f \|_{L^1} = \| \Delta (C \Phi^\text{lin} B) \|_{L^1_t L^2_{\omega}(\Omega_\infty, \mathcal{HS}(\mathbb{R}^m, \mathbb{R}^n))} \) and \( \| g \|_{L^2} = \| u \|_{L^2(\Omega_\infty)} \) and combining this with the second inequality in (4.8) yields

\[ \| \Delta (C T^\text{lin}) \|_{L^2_t L^2_{\omega}(\Omega_\infty)} \leq 2 \| \Delta (C \Phi^\text{lin} B) \|_{L^1_t L^2_{\omega}(\Omega_\infty, \mathcal{HS}(\mathbb{R}^m, \mathbb{R}^n))} \| u \|_{L^2(\Omega_\infty)} \]

(4.15)
Analogously, we find using Minkowski’s integral inequality in (1) and analogous arguments as presented in estimates (4.14) and (4.15) to obtain (2) and (3) respectively, and using the second estimate in (4.8) to get (4) that

\[
\| \Delta (CT_{\text{lin}}) \|^2_{L^2(\Omega, t)} = \int_0^\infty \mathbb{E} \left( \int_0^t \| \Delta (C\Phi_{t,s}B) \| \| u_s \| \, ds \right)^2 \, dt
\]

\[
\leq \int_0^\infty \left( \int_0^t \sqrt{\mathbb{E}(\| \Delta (C\Phi_{t,s}B) \|^2) \| u_s \|_{L^\infty(\Omega)} \, ds} \right)^2 \, dt
\]

\[
\leq \int_0^\infty \left( \int_0^t (t-s) \sqrt{\mathbb{E}(\| \Delta (C\Phi_{t-s}B) \|^2) \| u_s \|_{L^\infty(\Omega)} \, ds} \right)^2 \, dt
\]

\[
\leq \| \Delta (C\Phi_{t,s}B) \|^2_{L^2(\Omega, \mathbb{E}(\Xi_{t-s}))} \| u \|^2_{L^2(\Omega, \mathbb{E}(\Xi_{t-s}))}
\]

\[
\leq 4 \| \Delta (H_{\text{lin}}) \|^2_{L^2(\Omega, \mathbb{E}(\Xi_{t-s}))} \| u \|^2_{L^2(\Omega, \mathbb{E}(\Xi_{t-s}))}.
\]

Inserting bounds (4.13), (4.15), (4.16) into (4.12) then yields the claim. \(\square\)

We observe that the bounds of Theorem 3 depend on \(\| \Delta (H_{\text{lin}}) \|_{T_C} \), which indicates once more that a reduced order model by BT will lead to a small error also in the case of multiplicative noise. We can (formally) improve our previous convergence result using interpolation to \(q \in (1, 2)\). The convex case \(q = 2\) will be analyzed separately in Section 5.1 for Wiener noise.

**Corollary 4.4.** Consider two linear systems with multiplicative noise profile that we assume to be either i.i.d. or the same for both systems. For initial conditions \(\xi = \sum_{i=1}^k (v, \hat{\xi}_i)_{\mathbb{R}^k} \Xi_i\) with \(L^2(\Omega, \mathcal{F}_0, X)\) orthonormal system \((\xi_i)\), and \(\hat{\xi} = \sum_{i=1}^k (v, \hat{\xi}_i)_{\mathbb{R}^k} \hat{\xi}_i\). Let \(q \in (1, 2)\) then the following estimate holds

\[
\| \Delta (CZ_{\text{lin}}) \|_{L^2(\Omega, t)} \leq \| \Delta (CZ_{\text{lin}}) \|_{L^2(\Omega, t)}^{2(1-1-1)} \| \Delta (CZ_{\text{lin}}) \|_{L^2(\Omega, t)}^{2(1-1-1)}
\]

Moreover, we have that for any \(T \in [0, \infty)\) that for \(u \in L^2_{\text{ad}}(\Omega_T)\) and \(\gamma, c\) as in (4.3)

\[
\| CZ_{\text{lin}} \|_{L^2(\Omega, t)} \leq \gamma \| C \| \left( \frac{\| \xi \|_{L^2(\Omega)}}{\sqrt{2c}} + \frac{\| B \|}{c} \| u \|_{L^2(\Omega, t)} \right).
\]

It follows that for two Lévy processes \((M_t)_{t \geq 0}\) that we assume either to be independent or the same, that drive the dynamics of a linear system with multiplicative noise, we have for control functions \(u \in L^2_{\text{ad}}(\Omega_T)\) that

\[
\| \Delta (CZ_{\text{lin}}) \|_{L^2(\Omega, t)} \leq \left( \| \Delta (H_{\text{lin}}) \|_{T_C} \left( \| \xi \|_{L^2(\Omega)} + 2 \| u \|_{L^2(\Omega, t)} \right) \right)^{2(1-1-1)} \times
\]

\[
\times \left( \gamma \| C \| \left( \frac{\| \xi \|_{L^2(\Omega)}}{\sqrt{2c}} + \frac{\| B \|}{c} \| u \|_{L^2(\Omega, t)} \right) \right)^{2(1-1-1)}.
\]
Proof. The result follows from applying Hölder’s inequality twice: After applying Hölder’s inequality in the expectation with parameters \( p = (2 - q)^{-1} \) and \( \tilde{p} = (q - 1)^{-1} \) for \( q \) as in the statement, we obtain

\[
\mathbb{E} \left( \| \Delta (CZ_t^{\text{lin}}) \|^q \right) \leq \mathbb{E} \left( \| \Delta (CZ_t^{\text{lin}}) \|^{2-q} \| \Delta (CZ_t^{\text{lin}}) \|^{2(q-1)} \right) \\
\leq \left( \mathbb{E} \left( \| \Delta (CZ_t^{\text{lin}}) \| \right) \right)^{2-q} \left( \mathbb{E} \left( \| \Delta (CZ_t^{\text{lin}}) \|^2 \right) \right)^{q-1}.
\]

(4.19)

We thus conclude that after applying Hölder’s inequality with \( p = (2q - 1)^{-1} \) and \( \tilde{p} = (2 - 2q)^{-1} \) in time that

\[
\| \Delta (CZ_t^{\text{lin}}) \|_{L^2_t L^2_s(\Omega_T)}^2 = \int_0^T \mathbb{E} \left( \| \Delta (CZ_t^{\text{lin}}) \|^q \right)^{2/q} \ dt \\
\leq \int_0^T \left( \mathbb{E} \left( \| \Delta (CZ_t^{\text{lin}}) \| \right) \right)^{4q-2} \left( \mathbb{E} \left( \| \Delta (CZ_t^{\text{lin}}) \|^2 \right) \right)^{2(1-q^{-1})} \ dt \\
\leq \left( \int_0^T \mathbb{E} \left( \| \Delta (CZ_t^{\text{lin}}) \| \right)^{2} \ dt \right)^{2q-1} \left( \int_0^T \mathbb{E} \left( \| \Delta (CZ_t^{\text{lin}}) \|^2 \right) \ dt \right)^{2(1-q^{-1})} \\
= \| \Delta (CZ_t^{\text{lin}}) \|_{L^2_t L^2_s(\Omega_T)}^{2(2q^{-1}-1)} \| \Delta (CZ_t^{\text{lin}}) \|_{L^2(\Omega_T)}^{4(1-q^{-1})}.
\]

(4.20)

It therefore suffices to verify the \( L^2(\Omega_T) \)-boundedness of the process \( CZ_t \), which is the second term in the last line of (4.20), since the first term has been estimated in Theorem 3.

We then have from (4.4)

\[
\| \Delta (CZ_t^{\text{lin}}) \|_{L^2(\Omega_T)} \leq \| \Delta (C^{\text{lin}} H) \|_{L^2(\Omega_T)} + \| \Delta (C^{\text{lin}} Z) \|_{L^2(\Omega_T)}.
\]

(4.21)

The first term on the right-hand side, we can easily estimate as in (4.13)

\[
\| \Delta (C^{\text{lin}} H) \|_{L^2(\Omega_T)} \leq \| \Delta (H^{\text{lin}}) \|_{\text{HS}} \| \xi \|_{L^2(\Omega)}.
\]

(4.22)

Thus, it suffices to bound for \( u \in L^2(\Omega_{\infty}) \) the second term on the right-hand side of (4.21). This can be done by looking at

\[
I_t^{\text{lin}} = \int_t^T T_{t-s} N T_{t-s}^{\text{lin}} \ dM_s + \int_0^t T_{t-s} B u_s \ ds.
\]

Recall that the solution is given

\[
CZ_t^{\text{lin}} := C^{\text{lin}} H_t + CT_t^{\text{lin}} = C^{\text{lin}} \Phi_t^{\text{lin}} \xi + \int_0^t C^{\text{lin}} \Phi_{t,s}^{\text{lin}} Bu_s \ ds.
\]

Using that the flow is exponentially stable, we find uniformly for all \( T > 0 \)

\[
\| C^{\text{lin}} \Phi_t^{\text{lin}} \xi \|_{L^2(\Omega)} \leq \gamma \| C \| \| \xi \|_{L^2(\Omega)} \sqrt{\int_0^T e^{-2ct} \ dt} \leq \gamma \frac{\| C \| \| \xi \|_{L^2(\Omega)}}{\sqrt{2c}}
\]
and similarly for \(X_t := \int_0^t C \Phi_{t,s}^\text{lin} B u_s \, ds\) using exponential stability of the flow and Minkowski’s integral inequality in (1) and Young’s convolution inequality in (2)

\[
\|X\|_{L^2(\Omega_T)} \leq \gamma \|C\| \|B\| \sqrt{\|\mathbb{E}u\|^2} \int_0^t \|C\| \|B\| e^{-c(t-s)} \|\mathbb{E}u\|^2 \, ds \, dt
\]

(4.23)

Thus, we have altogether that

\[
\|CZ\|_{L^2(\Omega_T)} \leq \gamma \|C\| \left( \frac{\|\xi\|_{L^2(\Omega)}}{\sqrt{2c}} + \frac{\|B\|}{c} \|u\|_{L^2(\Omega_T)} \right).
\]

The final inequality in the statement of the Corollary then follows from the above estimates together with Theorem 3.

\[\square\]

5. Optimal control theory

As discussed in the introduction of this paper, optimal control of large-scale SDEs (1.1) given cost functionals (1.2) is generally very expensive or even infeasible. Therefore, MOR is used to approximate these high-dimensional equations in order to subsequently solve the optimal control problem in the surrogate model. We denote the output of the full and the reduced system by \(Y(u) = CZ(u)\) and \(\tilde{Y}(u) = \tilde{C}\tilde{Z}(u)\), respectively, and write the dependence on the control \(u\) explicitly. Obtaining an optimal control \(\tilde{u}_*\) in the reduced system, it is known by Theorems 2 and 3 that \(Y(\tilde{u}_*) \approx \tilde{Y}(\tilde{u}_*)\) if we apply BT in order to ensure \(\|\Delta(H_{\text{lin}})|_{\text{lin}}\|_{TC}\) to be small. However, we are more interested in the performance of the reduced optimal control in the original system. This means that we measure the distances between \(Y(u_*)\) and \(\tilde{Y}(\tilde{u}_*)\) as well as \(u_*\) and \(\tilde{u}_*\) in terms of the cost functionals. Here, \(u_*\) represents the optimal control in the original model. Therefore, we establish the following proposition. We start by showing that the abstract optimal control problems for the two stochastic equations (1.1) with control functionals (1.2) are well-posed. Moreover, we state explicit bounds on the cost functionals for the optimal control error under MOR.

**Proposition 5.1.** The optimal control problem (OCP) for stochastic systems (1.1) with associated energy functionals \(J_i\) as in (1.2), is well-posed and there exists a minimizer \(u \in L^2(\Omega_T)\) to the OCP. Let us now consider two systems, with outputs \(CZ\) and \(\tilde{C}\tilde{Z}\) satisfying the conditions of Theorems 2 and 3 respectively, and consider two minimizers, of the two energy functionals systems given by

\[
u_* = \arg \min_u J(CZ(u), u, T) \quad \text{and} \quad \tilde{u}_* = \arg \min_u J(\tilde{C}\tilde{Z}(u), u, T).
\]

\[\text{(5.1)}\]

\[\text{We just write } J \text{ to denote any of the functionals in (1.2)}\]
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In the case of Ornstein-Uhlenbeck processes we have

\[
\left| \sqrt{J_{LQR}^{ou}(CZ_{ad}^{ou}(u_*), u_*, T)} - \sqrt{J_{LQR}^{ou}(\tilde{C}\tilde{Z}_{ad}^{ou}(\tilde{u}_*), \tilde{u}_*, T)} \right| \leq \| \Delta(H^{ou}) \|_{TC} \times (1 + T^{-1/2} \left( \| \xi \|_{L^2(\Omega)} + 2 \max \{ \| u_* \|_{L^2(\Omega_T)}, \| \tilde{u}_* \|_{L^2(\Omega_T)} \} \right))
\]

and for linear systems with multiplicative noise and \( r \in [1, 2] \) there is \( C_T > 0 \) such that

\[
\left| \sqrt{J_r^{lin}(CZ_{ad}^{lin}(u_*), u_*, T)} - \sqrt{J_r^{lin}(\tilde{C}\tilde{Z}_{ad}^{lin}(\tilde{u}_*), \tilde{u}_*, T)} \right| \leq \left( \| \Delta(H^{lin}) \|_{TC} \left( \| \xi \|_{L^2(\Omega)} + 2 \max \{ \| u_* \|_{L^2(\Omega_T)}, \| \tilde{u}_* \|_{L^2(\Omega_T)} \} \right) \right)^{2r-1-1} \left( \gamma \| C \| \left( \frac{\| \xi \|_{L^2(\Omega)}}{\sqrt{2c}} + \frac{\| B \|}{c} \max \{ \| u_* \|_{L^2_{ad}(\Omega_T)}, \| \tilde{u}_* \|_{L^2_{ad}(\Omega_T)} \} \right) \right)^{2(1-r^{-1})}.
\]

Remark 2. The result of Proposition 5.1 is an indicator that the optimal control \( \tilde{u}_* \) obtained from a reduced system is of good quality if we choose the surrogate model such that \( \| \Delta(H_{ad}^{lin}) \|_{TC} \) is small. This can be ensured if a reduced system by BT with appropriate reduced order dimension is chosen.

Proof of Proposition 5.1. We restrict ourselves, for the proof of the existence of minimizers, to systems (1.1b), as controlled OU processes (1.1a) can be studied in a similar way.

Since the control functional is bounded from below, we can find a minimizing sequence of \( u_n \in L^2_{ad}(\Omega_T) \) defining processes \( Z_{ad}^{lin}(u_*), u_*, T) \) such that

\[
\lim_{n \to \infty} J_r^{lin}(CZ_{ad}^{lin}(u_*), u_*, T) = \inf_{u_n \in L^2_{ad}(\Omega_T)} J_r^{lin}(CZ_{ad}^{lin}(u_*), u_*, T)
\]

so that the \( u_n \) satisfy

\[
Z_{ad}^{lin}(t) = T_t \xi + \int_0^t T_{t-s} N Z_{ad}^{lin}(s) \, dM_s + \int_0^t T_{t-s} Bu_n(s) \, ds.
\]

Since the \( L^2_{ad}(\Omega_T) \) norm of the elements \( (u_n) \) is bounded, it follows from (4.17) that \( (Z_{ad}^{lin}) \) is uniformly bounded in \( L^2_{ad}(\Omega_T) \).

Weak compactness implies the existence of weak limits in \( L^2_{ad}(\Omega_T) \) for subsequences, that we denote just as the original sequences, \( Z_{ad}^{lin} \to Z_{ad}^{lin} \in L^2_{ad}(\Omega_T) \) and \( \lim_{n \to \infty} u_n = u \in L^2_{ad}(\Omega_T) \).

Recall that by Ito’s isometry and \( \| T_{t-s} \| \leq \nu e^{-\omega(t-s)} \) in (1), and Young’s inequality (2) with \( f(s) := \mathbb{1}_{[0, \infty)}(s) \nu^2 e^{-2\omega s} \) and \( g(s) := \mathbb{1}_{[0, T]}(s) \mathbb{E} \| \mathcal{S} \|^2 \) we have

\[
\| f \ast g \|_{L^1(\mathbb{R})} \leq \| f \|_{L^1(\mathbb{R})} \| g \|_{L^1(\mathbb{R})} = \nu^2 / (2\omega) \| \mathcal{S} \|^2_{L^2(\Omega_T)}.
\]
there exists a linear continuous operator
\[ I : L^2_{ad}(Ω_T) → L^2_{ad}(Ω_T), \quad I(\mathcal{S})_t = \int_0^t T_{t-s}N\mathcal{S}_s \, dM(s) \]
\[ \|I(\mathcal{S})\|_{L^2(Ω_T)} \leq \sqrt{\mathbb{E}(M_1^2)}\|N\|\sqrt{\|f*g\|_{L^2(\mathbb{R})}} \leq \frac{\sqrt{\mathbb{E}(M_1^2)}\|N\|}{\sqrt{2\omega}} \|\mathcal{S}\|_{L^2(Ω_T)}. \quad (5.5) \]

Similarly, there is a continuous linear operator
\[ D : L^2_{ad}(Ω_T) → L^2_{ad}(Ω_T), \quad D(u)_t = \int_0^t T_{t-s}Bu_s \, ds \]
\[ \|D(u)\|_{L^2(Ω_T)} \leq \frac{\|B\|\mu}{\omega}\|u\|_{L^2(Ω_T)}. \quad (5.6) \]

Thus, by weak convergence \( Z_n^{lin} \rightarrow Z^{lin} \) in \( L^2_{ad}(Ω_T) \), we can take any functional \( f \in L^2_{ad}(Ω_T)^* \). Then \( f \circ I \in L^2_{ad}(Ω_T)^* \) and thus the following weak limit exists
\[ I(Z_n^{lin}) \rightarrow I(Z^{lin}) \text{ in } L^2_{ad}(Ω_T). \]

Furthermore, we have the following weak limits in \( L^2_{ad}(Ω_T) \)
\[ \int_0^t T_{t-s}NZ_n^{lin}(s) \, dM_s \rightharpoonup \int_0^t T_{t-s}NZ^{lin}_s \, dM_s \text{ and} \]
\[ \int_0^t T_{t-s}Bu_n(s) \, ds \rightharpoonup \int_0^t T_{t-s}Bu_s \, ds \quad (5.7) \]
such that the process \( Z^{lin} \) satisfies with optimal control \( u \)
\[ Z_t^{lin} = T_t\xi + \int_0^t T_{t-s}NZ^{lin}_s \, dM_s + \int_0^t T_{t-s}Bu_s \, ds. \]

Finally, to see that this solution actually minimizes the optimal control functional, we use that by weak convergence and lower semicontinuity of the norm
\[ \|CZ(u)\|_{L^2_{ad}(Ω_T)}^2 + \langle u, Ru \rangle_{L^2(Ω_T)} \leq \inf_{u_*, u \in L^2_{ad}(Ω_T)} J_r^{lin}(CZ^{lin}(u_*, u_*, T)) \]
\[ \leq \lim_{n→∞} J_r^{lin}(CZ(u_n), u_n, T) \quad (5.8) \]

which means that by the assumption on the sequence \( u_n \), the control function \( u \) is a minimizer.

We now write \( Z(u) \) or \( \tilde{Z}(u) \) where \( u \) is a control in order to emphasize which control is used. We then observe that from the inverse triangle inequality, we have for Ornstein-Uhlenbeck processes using (1.2)
\[ \sqrt{J_{LQR}^{fu}(\tilde{C}Z^{on}(u_*), u_*, T)} - T^{-1/2}\|\Delta (CZ^{on}(u_*))\|_{L^2(Ω_T)} \leq \sqrt{J_{LQR}^{fu}(CZ^{on}(u_*), u_*, T)} \]
\[ \sqrt{J_{LQR}^{fu}(CZ^{on}(\tilde{u}_*), \tilde{u}_*, T)} - T^{-1/2}\|\Delta (CZ^{on}(\tilde{u}_*))\|_{L^2(Ω_T)} \leq \sqrt{J_{LQR}^{fu}(\tilde{C}Z^{on}(\tilde{u}_*), \tilde{u}_*, T)} \quad (5.9) \]
and for systems with multiplicative noise

\[
\sqrt{J_{\text{LQR}}^\text{lin}(\tilde{C}Z^\text{lin}(u_*), u_*, T)} - \|\Delta (CZ^\text{lin}(u_*))\|_{L^2(\Omega_T)} \leq \sqrt{J_{\text{LQR}}^\text{lin}(CZ^\text{lin}(u_*), u_*, T)} (5.10)
\]

\[
\sqrt{J_{\text{LQR}}^\text{lin}(CZ^\text{lin}(\tilde{u}_*, \tilde{u}_*, T)} - \|\Delta (CZ^\text{lin}(\tilde{u}_*))\|_{L^2(\Omega_T)} \leq \sqrt{J_{\text{LQR}}^\text{lin}(\tilde{C}Z^\text{lin}(\tilde{u}_*), \tilde{u}_*, T)}.
\]

Since \(u_*\) and \(\tilde{u}_*\) are minimizers of the respective functional, we have

\[
J_{\text{LQR}}^\text{lin}(\tilde{C}Z^\text{lin}(\tilde{u}_*), \tilde{u}_*, T) \leq J_{\text{LQR}}^\text{lin}(CZ^\text{lin}(u_*), u_*, T) \leq J_{\text{LQR}}^\text{lin}(\tilde{C}Z^\text{lin}(\tilde{u}_*), \tilde{u}_*, T). (5.11)
\]

Both estimates imply immediately that

\[
\left| \sqrt{J_{\text{LQR}}^\text{lin}(\tilde{C}Z^\text{lin}(\tilde{u}_*), \tilde{u}_*, T)} - \sqrt{J_{\text{LQR}}^\text{lin}(CZ^\text{lin}(u_*), u_*, T)} \right|
\leq T^{-1/2} \max \{ \|\Delta (CZ^\text{lin}(\tilde{u}_*))\|_{L^2(\Omega_T)}, \|\Delta (CZ^\text{lin}(u_*))\|_{L^2(\Omega_T)} \} \quad \text{and}
\]

\[
\left| \sqrt{J_{\text{LQR}}^\text{lin}(\tilde{C}Z^\text{lin}(\tilde{u}_*), \tilde{u}_*, T)} - \sqrt{J_{\text{LQR}}^\text{lin}(CZ^\text{lin}(u_*), u_* T)} \right|
\leq \max \{ \|\Delta (CZ^\text{lin}(\tilde{u}_*))\|_{L^2(\Omega_T)}, \|\Delta (CZ^\text{lin}(u_*))\|_{L^2(\Omega_T)} \} . (5.12)
\]

The bounds then follow from the conditions stated in Theorems 2 and 3 respectively.

\[\square\]

5.1. Infinite time Linear Quadratic Regulator. In the previous subsection we showed that the energy functionals (1.2) with optimal control are well-approximated by the reduced order models, cf. Proposition 5.1.

In this subsection we go one step further and focus on the control itself and discuss techniques to approximate the optimal control using a reduced order model with a focus on infinite time horizons.

5.1.1. Ornstein-Uhlenbeck processes. Before discussing further the links between MOR and optimal control theory, we state in the next Proposition an approximation result on the optimal control \(u\) to a high-dimensional Ornstein-Uhlenbeck process with Gaussian noise (1.1a) and error control.

**Proposition 5.2.** Let \(X\) be finite-dimensional and let \((Z^\text{ou}_t)_{t \geq 0}\) be a controlled Ornstein-Uhlenbeck process (1.1a) satisfying Assumption 1 with standard Wiener noise \((W_t)_{t \geq 0}\) such that the pair \((A, C)\) is observable. The solution to the OCP with \(T = \infty\) and \(R > 0\) in (1.2) is given by the fixed-point equation\(^3\)

\[
u^*_P(t) = -R^{-1}B^*PZ^\text{ou}_t, (5.13)\]

\(^3\)as \(Z^\text{ou}_t\) itself depends on \(u\)
where $P$ is the unique positive-definite solution to the Riccati equation
\[ A^*P + PA + C^*C - PBR^{-1}B^*P = 0. \] (5.14)

For a sequence $P_k \geq 0$ of unique solutions to standard Lyapunov equations
\[ A_k^*P_k + P_kA_k + C^*C + L_k^*RL_k = 0 \] (5.15)
where $L_k := R^{-1}B^*P_{k-1}$ for $k \geq 1$ and $A_k := A - BL_k$ for $k \geq 0$ with $L_0 := 0$,
matrices $P_k$ then converge quadratically and monotonically, in the sense of operators, to $P$. The control functions
\[ u_{P_k}(t) = -R^{-1}B^*P_kZ_t^{ou}, \]
satisfy for $\|P - P_k\|$ sufficiently small, uniformly in the final time parameter $T$,
\[ T^{-1/2}\|u_P - u_{P_k}\|_{L^2(\Omega)} = O(\|P_k - P_{k-1}\|^2). \]

Proof. Substituting (5.13) into (1.1a) yields an Ornstein-Uhlenbeck process
\[ dZ_t^{ou} = (A - BR^{-1}B^*)Z_t^{ou} \, dt + K \, dW_t. \] (5.16)
The operator $A_P := A - BR^{-1}B^*P$ is the generator of an exponentially stable semigroup $\|T_P(t)\| \leq \nu e^{-\omega t}$ [Z75, Theorem 1] for some $\nu, \omega > 0$.

Here, $P$ is the unique positive solution to the Riccati equation such that for all $x, y \in D(A)$
\[ \langle Ax, Py \rangle_X + \langle Px, Ay \rangle_X + \langle (C^*C - PBR^{-1}B^*P)x, y \rangle_X = 0. \]

By Newton’s method, one can approximate $P$ by a sequence $P_k \geq 0$, where $P_k$ solve Lyapunov equations (5.15), for Hurwitz matrices $A_k$ [Kl68, Proof 1)] with quadratic convergence rate [Kl68, (13)] to the solution of the Riccati equation, namely
\[ \|P - P_k\| \leq c\|P_k - P_{k-1}\|^2. \] (5.17)

Standard results from semigroup theory imply that $A_k$ is also a generator with semigroup satisfying [EN00, 1.3, Chap. 3]
\[ \|T_k(t)\| \leq \nu e^{(-\omega+\nu)\|BR^{-1}B^*\|\|P-P_k\|}t. \] (5.18)

For an approximation $P_k$ of $P$ we find using (3.1), (5.13), and (5.16)
\[
\begin{align*}
\|(u_P - u_{P_k})(t)\|_{L^2(\Omega)} & \leq \|R^{-1}B^*\| \|P - P_k\| \|Z_t^{ou}\|_{L^2(\Omega)} + \|R^{-1}B^*P_k\| \|Z_t^{ou} - Z_{k-1}^{ou}\|_{L^2(\Omega)} \\
& \leq \|R^{-1}B^*\| \left( \|P - P_k\| \left( \left\| \int_0^t T_P(t-s)K \, dW_s \right\|_{L^2(\Omega)} + \nu e^{-\omega t} \|\xi\|_{L^2(\Omega)} \right) \\
& + \|P_k\| \left( \left\| \int_0^t (T_P - T_k)(t-s)K \, dW_s \right\|_{L^2(\Omega)} + \|(T_P - T_k)(t)\|_{L^2(\Omega)} \right) \right).
\end{align*}
\]
We then use that by the product rule of differentiation
\[
(T_P - T_k)(t)\xi = -\int_0^t \frac{d}{ds} (T_P(t - s) T_k(s)) \xi \, ds \\
= -\int_0^t T_P(t - s) BR^{-1} B^* (P_k - P) T_k(s) \xi \, ds
\] (5.19)
such that due to (5.18) and (5.19)
\[
\| (T_P - T_k)(t)\xi \|_{L^2(\Omega)} \leq \nu^2 \| BR^{-1} B^* \| \| P - P_k \| \times \int_0^t e^{-\omega(t-s)} e^{-\nu \| BR^{-1} B^* \| \| P - P_k \| s} \| \xi \|_{L^2(\Omega)} \, ds \\
= \nu^2 e^{-\omega t} \| BR^{-1} B^* \| \| P - P_k \| \int_0^t e^{\nu \| BR^{-1} B^* \| \| P - P_k \| s} \| \xi \|_{L^2(\Omega)} \, ds \\
= \nu e^{-\omega t} \left( e^{\nu \| BR^{-1} B^* \| \| P - P_k \| t} - 1 \right) \| \xi \|_{L^2(\Omega)}.
\]
Rearranging and estimating further using Ito’s isometry and the integral identity
\[
\int_0^\infty \left( e^{-at} (e^{ct} - 1) \right)^2 \, dt = \frac{c^2}{4a^3 - 6a^2 c + 2ac^2}, \quad \text{for } \text{Re}(a) > \text{Re}(c), \text{Re}(a) > 0,
\]
we obtain by setting \( \alpha := \nu \| BR^{-1} B^* \| \)
\[
\| (u_P - u_{P_k})(t) \|_{L^2(\Omega)} \leq \nu \| R^{-1} \| \| B \| \left( \| P - P_k \| \left( \| \xi \|_{L^2(\Omega)} + \frac{\| K \|_{HS}}{\sqrt{2\omega}} \right) \right) \\
+ \frac{\alpha \| K \|_{HS} \| P - P_k \| }{\sqrt{4\omega^3 - 6\omega^2 \alpha \| P - P_k \| + 2\omega \alpha^2 \| P - P_k \|^2}} + e^{-\omega t} \left( e^{\alpha \| P - P_k \| t} - 1 \right) \| \xi \|)
\] (5.21)
By taking the \( L^2 \) norm and regularizing the expression by dividing it by \( \sqrt{T} \), we then finally obtain, using \( T^{-1/2} \| 1 \|_{L^2(\Omega_T)} = 1 \) and (5.20) in the last term, the following estimate
\[
T^{-1/2} \| u_P - u_{P_k} \|_{L^2(\Omega_T)} \leq \nu \| R^{-1} \| \| B \| \| P - P_k \| \times \left( \| \xi \|_{L^2(\Omega)} \frac{\| K \|_{HS}}{\sqrt{2\omega}} \right) \\
+ \frac{\alpha \left( \| K \|_{HS} + T^{-1/2} \| \xi \| \right) }{\sqrt{4\omega^3 - 6\omega^2 \alpha \| P - P_k \| + 2\omega \alpha^2 \| P - P_k \|^2}}.
\] (5.22)
Thus, by approximating the solution to the Riccati equation using the scheme outlined in Proposition 5.2, the optimal feedback law (5.13) is approximated by the output
of a new (uncontrolled) linear system
\[
\begin{aligned}
dZ_{t}^{\text{ou}} &= \bar{A}Z_{t}^{\text{ou}} \, dt + K \, dW_{t}, \\
Z_{0}^{\text{ou}} &= \xi, \text{ and} \\
u_{P_{k}}(t) &= \bar{C}Z_{t}^{\text{ou}}
\end{aligned}
\]  \tag{5.23}

with operators
\[
\bar{C} = -R^{-1}B^{*}P_{k}, \quad \bar{A} = A - BR^{-1}B^{*}P_{k}, \quad \text{and } \text{ran}(B_{\text{in}}) \ni \xi.
\]

If we now define a reduced model to (5.23), e.g., by balancing the system (which is (1.1a) with \((A, B)\) replaced by \((\bar{A}, 0)\) and output operator \(\bar{C}\)), we can use Theorem 2 to control the error between the outputs. This allows us to approximate the optimal control of the full high-dimensional system by the output of a reduced system of (5.23).

The method outlined in this section allows us to approximate the (unique) optimal control of the full system using an auxiliary reduced order model. This is a stronger result than the approximation of energy functionals in Proposition 5.1. In general, the approximation of the optimal control may not be possible, since the optimal control may not be unique and may not be given as the output of a linear system, again.

5.1.2. Linear systems with multiplicative noise. We now turn to the infinite time OCP for finite-dimensional linear systems with multiplicative standard Wiener noise \((W_{t})\) (1.1b) and optimal control functionals (1.2) with optimal control
\[
\begin{aligned}
u_{*} &= \arg\min_{u \in L^{2}(\Omega_{T})} J_{\text{LQR}}^{\text{lin}}(CZ_{t}^{\text{lin}}, u, \infty).
\end{aligned}
\]  \tag{5.24}

Let \(P\) then be the solution to the augmented Riccati equation \([RZ00, (5)]\)
\[
\begin{aligned}
A^{*}P + PA + N^{*}PN - PBR^{-1}B^{T}P + C^{*}C &= 0.
\end{aligned}
\]

The optimal control to (5.24) is then given by the fixed-point equation \((Z_{t}^{\text{lin}} \text{ also depends on } u_{*})\)
\[
\begin{aligned}
u_{*}(t) &= -R^{-1}B^{*}PZ_{t}^{\text{lin}}.
\end{aligned}
\]  \tag{5.25}

Thus, by replacing \(u_{*}\) in the above expression by (5.25), we find that \(u_{*}\) is the output of
\[
\begin{aligned}
dZ_{t}^{\text{lin}} &= \bar{A}Z_{t}^{\text{lin}} \, dt + NZ_{t}^{\text{lin}} \, dW_{t} \\
Z_{0}^{\text{lin}} &= \xi, \text{ and} \\
u_{*}(t) &= \bar{C}Z_{t}^{\text{lin}}
\end{aligned}
\]  \tag{5.26}

with operators
\[
\bar{C} = -R^{-1}B^{*}P, \quad \bar{A} = A - BR^{-1}B^{*}P, \quad \text{and } \text{ran}(B_{\text{in}}) \ni \xi.
\]

Reducing (5.26) leads to an approximation for time optimal control that is based on solving a low-dimensional system.
6. Numerical Examples

6.1. Controlled Ornstein-Uhlenbeck. For an illustration of the above bounds we consider an Ornstein-Uhlenbeck process with control $u_t = \sin(t) \mathbf{1} \in \mathbb{R}^d$ governed by

$$dZ_t = AZ_t \, dt + B_1 u_t \, dt + B_2 \, dW_t,$$
$$Y_t = CZ_t, \quad Z_0 = z_0,$$

(6.1)

with $Z_t, W_t \in \mathbb{R}^d, A, B_1, B_2 \in \mathbb{R}^{d \times d}, C \in \mathbb{R}^{m \times d}, d, m = 50$, where we choose the corresponding matrices such that the dynamics is most pronounced in the first $r = 5$ dimensions, namely

$$A, B_1, B_2, C = \text{diag}(-1, \ldots, -1, -0.01, \ldots, -0.01) + (\alpha_{ij}),$$

(6.2)

with random noise $\alpha_{ij} \sim \mathcal{N}(0, 10^{-6})$ i.i.d. being different for each variable. We either choose $z_0^* = (0, \ldots, 0)$ or $z_0^* = (1, \ldots, 1, 0, \ldots, 0)$ as an initial value, take $B_{\text{lin}} = z_0$ and compare the bounds obtained in Theorems 1 and 2 and Corollary 3.4 with a simulation of the full and the reduced dynamics using BT.

Figure 1. Error bounds and simulations of BT of Ornstein-Uhlenbeck systems. The simulation is the numerically simulated error of the norm specified in the respective Theorem/Corollary.
In the top panel of Figure 1 we show the error bounds as well as the Hankel singular values and simulation results with varying dimension $r$ of the reduced model when starting in $z_0^* = (0, \ldots, 0)$. The simulation results are obtained with a simple Euler-Maruyama discretization with step-size 0.01. We see that both bounds are rather conservative, the supremum bound on the left hand side seems to be a bit tighter than the $L^2$ bounds (also naturally due to the $\sqrt{T}$ scaling of the latter) and we in particular realize that the bound from Corollary 3.4 seems to be tighter than the one from Theorem 2. The bottom panel shows the same approach, however, now choosing $z_0^* = (1, \ldots, 1, 0, \ldots, 0)$. Here, we do not have a supremum bound anymore, but realize that the two $L^2$ bounds hold and that model reduction works well. For computing all the Gramians we use the formulas (2.1). The code can be found at github.com/lorenzrichter/balanced-truncation.

6.2. Chain of oscillators. The one-dimensional chain of oscillators is a non-equilibrium statistical mechanics model that describes heat transport through a chain of $N$ particles coupled at each end to heat reservoirs at different temperatures with friction parameter $\gamma$ at the first and last particle. It was first introduced for the rigorous derivation of Fourier’s law, or a rigorous proof of its breakdown: this is well described in [BLR00]. We consider $N$ particles and denote by $q_i$ the location of each particle with respect to their equilibrium position and by $p_i$ its momentum. The Hamilton function $H : \mathbb{R}^{2N} \to \mathbb{R}$ of the system is given by

$$H(q, p) = \frac{\langle p, M^{-1}p \rangle}{2} + V_{\eta, \zeta}(q),$$

$$V_{\eta, \zeta}(q) = \sum_{i=1}^N \eta_i q_i^2 + \sum_{i=1}^{N-1} \xi_i (q_i - q_{i+1})^2$$

with mass matrix $M := m \text{id}_{2N \times 2N}$ and coupling strengths $\eta_i, \xi_i > 0$. The above form of the potential describes particles that are fixed by a quadratic pinning potential $U_{\text{pin}, i}(q) = \eta_i q_i^2$ and interact with their nearest neighbors through a quadratic interaction potential $U_{\text{int}, i}(q_i - q_j) = \xi_i (q_i - q_j)^2$ for $j = i + 1$ and $i \in \{1, \ldots, N\}$. The 1st and $N$th particle are each coupled to a heat bath at inverse temperatures $\beta_1$ and $\beta_N$, respectively. We also assume these two particles $I = \{1, N\}$ to be subject to friction. The dynamics of the system is described by the Langevin dynamics

$$\text{d}q_t = M^{-1}p_t \, \text{d}t$$

$$\text{d}p_t = (-S q_t - \Gamma p_t + \sigma u_t) \, \text{d}t + \sigma \, \text{d}W_t$$

where $u_t \in \mathbb{R}^N$ is an external control and $(W_t)$ an $\mathbb{R}^N$-valued standard Wiener process. Expressing the system using phase-space coordinates $Z_t := (q_t^*, p_t^*)^*$ we see that the entire system is described by the Ornstein-Uhlenbeck process

$$\text{d}Z_t = (AZ_t + Bu_t) \, \text{d}t + K \, \text{d}W_t$$

(6.4)
with
\[ A = \begin{pmatrix} 0 & M^{-1} \\ -S & -\Gamma \end{pmatrix}, \quad K = B = \begin{pmatrix} 0 & 0 \\ 0 & 0 & \sigma \end{pmatrix}, \]
with fluctuation-dissipation relation
\[ \sigma = \text{diag} \left( \frac{\sqrt{2m\gamma}}{\sqrt{\beta_1}}, 0, \ldots, 0, \frac{\sqrt{2m\gamma}}{\sqrt{\beta_N}} \right), \quad \Gamma = \text{diag}(\gamma, 0, \ldots, 0, \gamma). \tag{6.5} \]

Here, we changed the notation so that \( u_t \in \mathbb{R}^{2N} \) is an external control and \((W_t)\) an \( \mathbb{R}^{2N} \)-valued standard Wiener process.

The operator \( S \) is the Jacobi (tridiagonal) matrix for \( f = (f_1, \ldots, f_N) \in \mathbb{R}^N \), defined as
\[ (Sf)_n = -\xi_n f_{n+1} - \xi_{n-1} f_{n-1} + (\eta_n + (2 - \delta_{n\in I})\xi_n)f_n \]
where \( f_0 = f_{N+1} := 0 \). The matrix \( A \) is Hurwitz if all parameters of the model are strictly positive.

The invariant distribution to the uncontrolled process (6.4) is given by \([LLR67]\)
\[ \mu_{\Sigma_\beta}(q, p) := (2\pi)^{-N/2} \det(\Sigma_\beta^{-1/2}) \exp \left(-\frac{1}{2} \langle (q, p), \Sigma_\beta^{-1}(q, p) \rangle \right), \tag{6.6} \]
where the covariance matrix \( \Sigma_\beta \) is the solution to the Lyapunov equation \([LLR67, (2.8)]\)
\[ A\Sigma_\beta + \Sigma_\beta A^* + KK^* = 0. \tag{6.7} \]

6.3. Friction and spectral gap. If in the chain of oscillators one chooses the friction according to (6.5), then the spectral gap of \( A \) closes necessarily as \( N \to \infty \). This is apparent by studying
\[ \sum_{\lambda \in \sigma(A)} \lambda = \text{tr}(A) = \text{tr}(-\Gamma) = -2\gamma. \]

Since we have \( 2N \) (counting multiplicity) eigenvalues with negative real parts, we conclude that the one with largest real part decays to zero at least with rate \( |\text{Re}(\lambda_S)| = \mathcal{O}(N^{-1}) \).

The situation changes once we apply a constant non-zero friction \( \gamma := \gamma_1 = \gamma_2 > 0 \) such that \( \Gamma := \text{diag}(\gamma, \ldots, \gamma) \) to all the particles. In this case, we find for the determinant using the block-determinant formula
\[ \det \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} = \det(Q_{22}Q_{11} - Q_{21}Q_{12}) \text{ if } Q_{11}Q_{12} = Q_{12}Q_{11} \]
the decomposition
\[ \det(A - \lambda I) = \det(\lambda^2 I + \lambda \Gamma + SM^{-1}) = 0. \]

This equation is equivalent to solving \( \lambda^2 + \gamma \lambda + \mu = 0 \) where \( \mu \in \sigma(SM^{-1}) \). By explicitly solving the quadratic equation, one can see that this equation has only solutions with strictly negative real part if \( SM^{-1} \) has a uniform – in the number of
particles – spectral gap. A comprehensive discussion of the spectral gap for this model can be found in [M19, BM19].

For our numerical simulations we do not want the closing of the spectral gap to inflict the simulations. We therefore consider a mild constant friction parameter $\gamma_2$ and a larger friction parameter $\gamma_1$ at the terminal ends of the chain. To be precise, we choose a simulation time $T = 10$, $N = 75$ oscillators and $\gamma_1 = 10, \gamma_2 = 0.25, m = \xi_n = \eta_n = \beta_1 = \beta_N = 1$. Figure 2 shows the BT bound from Theorem 1 on the left hand side and the $L^2$ bound from 2 in the middle subplot along with the simulated errors, again with varying reduced dimension $r$ on the x-axis. The plot on the right hand side shows the Hankel singular values. We can see that indeed one can reduce the dimension of the system significantly with only getting a small error and we note that the $L^2$ error bound seems to saturate for large $r$, which might be due to numerical issues.

![Figure 2. Error analysis of the chain of oscillators when applying BT.](image)

6.4. **Stochastic optimal control.** We now study the set of reachable distributions $\mathcal{N}(0, \Sigma)$ for a controlled OU process (1.1a). To be precise, we are looking for a feedback law of minimal energy

$$ J_{\text{LQR}}^{\text{ou}}(0, u, \infty) := \lim_{T \to \infty} \frac{\|u\|_{L^2(\Omega_T)}^2}{T} \tag{6.8} $$

to maintain an invariant state $\mu_\Sigma$ for some given $\Sigma > 0$, namely

$$ \mu_\Sigma(q, p) := (2\pi)^{-N} \det(\Sigma^{-1/2}) \exp\left( -\frac{1}{2} \langle (q, p), \Sigma^{-1}(q, p) \rangle \right). \tag{6.9} $$

According to [CGP16, Theorem 4] this invariant state can be attained with a control $u_t^* = -K^*\Pi Z_t$, where $\Pi$ is (any) symmetric matrix that satisfies

$$ (A - BB^*\Pi)\Sigma + \Sigma(A - BB^*\Pi)^* + KK^* = 0. \tag{6.10} $$
In our next Proposition we show that, from the invariant distribution for the chain of oscillators associated with some boundary temperatures $\beta = (\beta_1, \beta_N)$, we can reach the invariant state associated with any other boundary temperature $\beta' = (\beta'_1, \beta'_N)$.

**Proposition 6.1.** There exists a control that steers the chain of oscillators (6.4), with physical temperature $\beta = (\beta_1, \beta_N)$, to the invariant distribution $\mathcal{N}(0, \Sigma_{\beta'})$ with temperatures $\beta' = (\beta'_1, \beta'_N)$. If $\beta_1 = \beta_n$ and $\beta'_1 = \beta'_N$ then the invariant state has covariance matrix

$$\Sigma_{\beta'} = \beta'^{-1}_1 \begin{pmatrix} S^{-1} & 0 \\ 0 & M \end{pmatrix}$$

(6.11)

and a solution $\Pi$ to (6.10) reads

$$\Pi = \text{diag} \left( 0, \frac{(\beta'_1 - \beta_1)}{2} M^{-1} \right).$$

(6.12)

**Proof.** A sufficient condition [CGP16, Theorem 4] to be able to reach a state $\mathcal{N}(0, \Sigma_{\beta'})$ is that $\text{im}(B) \subset \text{im}(K)$ and $\Sigma$ solves the Lyapunov equation

$$\Sigma_{\beta'} A^* + A \Sigma_{\beta'} + KK^* + BX^* + XB^* = 0$$

for some $X$. We thus define diagonal matrices $X_\delta$ for $\delta_1, \delta_N \in \mathbb{R}$ by

$$X_\delta := \text{diag}(0, \ldots, 0, \delta_1, 0, \ldots, 0, \delta_N).$$

(6.13)

It is then obvious that for a suitable choice of $\delta$ and any other temperature $\beta' = (\beta'_1, \beta'_N)$ at the terminal ends of the chain we have due to (6.7)

$$A \Sigma_{\beta'} + \Sigma_{\beta'} A^* + K_{\beta'} K_{\beta'}^* = 0$$

such that by choosing $\delta$ such that

$$A K_{\beta} K_{\beta}^* + B_{\beta} X_{\delta}^* + X_{\delta} B_{\beta}^* = K_{\beta'} K_{\beta'}^*$$

where we used the subscript $\beta$ to emphasize the temperature profile used in the respective matrix. This implies that the uncontrolled chain of oscillators (6.4) with equilibrium state (6.6) and temperature $\beta$ can be steered into the equilibrium state (6.6) for any other temperature $\beta'$.

The form of the covariance matrix (6.11) can be directly verified by inserting it into (6.7).

To verify (6.12), we use the fluctuation-dissipation relation $\sigma \sigma^* = \frac{2}{\beta_1} M \Gamma$ and write the symmetric matrix $\Pi$ as a block matrix

$$\Pi = \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{pmatrix},$$
we then get
\[
\left( \begin{array}{cc}
0 & M^{-1} \\
-S & -\Gamma
\end{array} \right) - \left( \begin{array}{cc}
0 & 0 \\
0 & \sigma \sigma^*
\end{array} \right) \left( \begin{array}{cc}
\Pi_{11} & \Pi_{12} \\
\Pi_{21} & \Pi_{22}
\end{array} \right) \left( \begin{array}{cc}
S^{-1} & 0 \\
0 & M
\end{array} \right)
\]
\[
+ \left( \begin{array}{cc}
S^{-1} & 0 \\
0 & M
\end{array} \right) \left( \begin{array}{cc}
0 & -S \\
M^{-1} & -\Gamma
\end{array} \right) - \left( \begin{array}{cc}
\Pi^*_{11} & \Pi^*_{21} \\
\Pi^*_{12} & \Pi^*_{22}
\end{array} \right) \left( \begin{array}{cc}
0 & 0 \\
0 & \sigma \sigma^*
\end{array} \right)
\right)
\]
\[
= -\beta'_1 \left( \begin{array}{cc}
0 & 0 \\
0 & \sigma \sigma^*
\end{array} \right).
\]
which reduces to
\[
\left( \begin{array}{cc}
0 & M^{-1} \\
-S - \sigma \sigma^* \Pi_{21} & -\Gamma - \sigma \sigma^* \Pi_{22}
\end{array} \right) \left( \begin{array}{cc}
S^{-1} & 0 \\
0 & M
\end{array} \right)
\]
\[
+ \left( \begin{array}{cc}
S^{-1} & 0 \\
0 & M
\end{array} \right) \left( \begin{array}{cc}
0 & -S - \Pi^*_{21} \sigma \sigma^* \\
M^{-1} & -\Gamma - \Pi^*_{22} \sigma \sigma^*
\end{array} \right)
\]
\[
= -\beta'_1 \left( \begin{array}{cc}
0 & 0 \\
0 & \sigma \sigma^*
\end{array} \right).
\]
From the block (12) we get
\[
M^{-1} M - S^{-1} S - S^{-1} \Pi^*_{21} \sigma \sigma^* = 0 \quad \text{such that we can choose } \Pi_{21} = 0.
\]
From the block (22) we get
\[
\beta^{-1}_1 \left( -\Gamma M - \sigma \sigma^* \Pi_{22} M - M \Gamma - M \Pi_{22} \sigma \sigma^* \right) = -\sigma \sigma^*.
\]
By symmetry, \( \Pi_{12} = 0 \). One can check that
\[
\Pi_{22} = \left( \frac{\beta'_1 - \beta_1}{2} \right) M^{-1}.
\]
At last, we may then choose \( \Pi_{11} = 0 \) since this matrix does not enter in the Lyapunov equation.

To see that our choice of \( \Pi \) is admissible it remains to verify that \( A - KK^* \Pi \) is Hurwitz. This however follows immediately since \( A \) is Hurwitz and \( -KK^* \Pi \) is diagonal with non-positive entries.

\[
\square
\]

**Remark 3.** If one wants to solve (6.10) for a general covariance matrix \( \Sigma \), vectorization can be used to get
\[
\text{vec}(A \Sigma + \Sigma A^* + KK^*) = \text{vec}(BB^* \Pi \Sigma + \Sigma \Pi^* BB^*)
\]
\[
= (\Sigma \otimes BB^*) \text{vec}(\Pi) + (BB^* \otimes \Sigma) \text{vec}(\Pi^*)
\]
\[
= (\Sigma \otimes BB^* + BB^* \otimes \Sigma) \text{vec}(\Pi),
\]
since we assume $\Pi$ to be symmetric. Note that $\Pi$ is admissible only if the rank condition
\[
\begin{pmatrix}
A\Sigma + \Sigma A^* + KK^* & B \\
B^* & 0
\end{pmatrix}
= \begin{pmatrix}
0 & B \\
B^* & 0
\end{pmatrix}
\]
holds and $A - BB^*\Pi$ is Hurwitz (see [CGP16]).

6.5. Optimal control meets balanced truncation. We now discuss how to use BT to steer subsystems into a designated steady state.

We again consider the high-dimensional Ornstein-Uhlenbeck process (6.4), for which we have discussed in Subsection 6.4 the convergence of
\[
\Sigma_t = \mathbb{E}(Z_tZ_t^*)
\]
to a designated covariance matrix $\Sigma > 0$, under certain conditions.

Now, we want to study the case where we only want to find a control that maintains a certain covariance matrix $\Sigma_{rr} > 0$ for an $r \ll d$-dimensional projection of our original system. In this case, the above method does not apply immediately.

To be precise, we are interested in reaching the sub-covariance matrix $\Sigma_{rr}$ as the limiting covariance matrix of
\[
Q\Sigma_t Q^* = Q\mathbb{E}(Z_tZ_t^*)Q = \mathbb{E}((QZ_t)(QZ_t)^*),
\]
where $Q$ is a suitable projection matrix.

We can now first reduce the model to $r$ dimensions (recall that $r$ is the rank of $Q$) using BT with observability matrix $C = Q$ and then apply the method described in Subsection (6.4) to the reduced system $(\tilde{C}, \tilde{A}, \tilde{K}, \tilde{B})$ by using that
\[
Q\Sigma_t Q^* \approx \mathbb{E}
\left(
\tilde{C}\tilde{Z}_t \tilde{C}\tilde{Z}_t^*
\right).
\]

More precisely, it follows that
\[
\left\|\mathbb{E}((CZ_t)(CZ_t)^*) - \mathbb{E}((\tilde{C}\tilde{Z}_t)(\tilde{C}\tilde{Z}_t)^*)\right\|
\leq \left\|\mathbb{E}((CZ_t - \tilde{C}\tilde{Z}_t)(CZ_t)^*)\right\|
+ \left\|\mathbb{E}((\tilde{C}\tilde{Z}_t)((CZ_t)^* - (\tilde{C}\tilde{Z}_t)^*))\right\|
\leq \|CZ_t - \tilde{C}\tilde{Z}_t\|_{L^2(\Omega)} \left(\|CZ_t\|_{L^2(\Omega)} + \|\tilde{C}\tilde{Z}_t\|_{L^2(\Omega)}\right).
\]

(6.14)

Thus, the covariance matrix $\mathbb{E}((\tilde{Z}_t\tilde{Z}_t^*)$ that the reduced process $\tilde{Z}_t$ is supposed to maintain is the normal distribution (6.9) with (formal inverse) $\Sigma^{-1} = \tilde{C}^*\Sigma_{rr}^{-1}\tilde{C}$. If $\Sigma^{-1}$ has full rank, and thus $\Sigma^{-1}$ is the inverse of an actual matrix $\Sigma$, then this auxiliary distribution for the reduced system can be used to compute an optimal control, as described in Section 6.4, for the full system.
We illustrate the above ideas in the following example.

**Example 1** (Target distribution of outmost oscillators). Let us say we want to prescribe the covariance matrix of the subsystem containing only the leftmost and rightmost oscillators and accordingly choose $Q \in \mathbb{R}^{4 \times d}$, $d = 2N$, with $Q_{11} = 1, Q_{2,N} = 1, Q_{3,N+1} = 1, Q_{4,2N} = 1$, to retain position and momentum variables, and choose all other $Q_{ij} = 0$. We can then employ BT to obtain a reduced system associated with the original system

$$dZ_t = (AZ_t + Bu_t) \, dt + K_\beta \, dW_t$$
$$Y_t = QZ_t.$$  \hspace{1cm} (6.15)

The reduced system is of lower dimension $r$ with $r \ll d$,

$$d\tilde{Z}_t = (\tilde{A}\tilde{Z}_t + \tilde{B}u_t) \, dt + \tilde{K}_\beta \, dW_t$$
$$\tilde{Y}_t = \tilde{Q}\tilde{Z}_t.$$  \hspace{1cm} (6.16)

To run a numerical simulation we choose the sub-covariance to be

$$\Sigma_{kk} = S_{kk} + S^*_{kk}, \quad S_{kk} = \text{diag}(3, \ldots , 3) + (|a_{ij}|), \quad a_{ij} \sim \mathcal{N}(0,1)$$  \hspace{1cm} (6.17)

and compute the optimal control as described above. We have realized that it is important to actually check the speed of convergence as [CGP16] does not say anything about the time needed to be “close” to the stationary distribution. This can for instance be done by looking at the smallest real part of the eigenvalues of the matrix $A - BB^*\Sigma$. To evaluate the closeness to our desired target distribution, we compare the empirical covariance $\hat{\Sigma}_{rr,t}$ to the desired covariance $\Sigma_{rr}$ by means of the scaled Frobenius norm $\frac{1}{d}\|\hat{\Sigma}_{rr,t} - \Sigma_{rr}\|_F$. Figure 3 displays this measure as a function of time by simulating $k$ different realizations of the reduced controlled process up to $T = 30$. We see that we indeed get very close to the desired target, in particular if we choose $k$ large enough. The time discretization of the Euler-Maruyama scheme that we use for discretization seems to be small enough in all trials.
Figure 3. Convergence of the reduced chain of oscillator system to the desired target distribution.

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