Non-uniqueness of cosmic ray two-fluid equations at shocks and possible remedies

Siddhartha Gupta\textsuperscript{a,b}, Prateek Sharma\textsuperscript{a,c}, Andrea Mignone\textsuperscript{d}

\textsuperscript{a}Department of Physics and Joint Astronomy Programme, Indian Institute of Science, Bangalore, India 560012
\textsuperscript{b}Baman Research Institute, Sadashiva Nagar, Bangalore 560080, India
\textsuperscript{c}MPI für Astrophysik, Karl-Schwarzschild str 1, D-85741 Garching, Germany
\textsuperscript{d}Dipartimento di Fisica, Università di Torino, via P. Giuria 1, I-10125 Torino, Italy

Abstract

Cosmic rays (CRs) are frequently modeled as an additional fluid in hydrodynamic (HD) and magnetohydrodynamic (MHD) simulations of astrophysical flows. The standard CR two-fluid model is described in terms of three conservation laws (expressing conservation of mass, momentum and total energy) and one additional equation (for the CR pressure) that cannot be cast in a satisfactory conservative form. The presence of non-conservative terms with spatial derivatives in the model equations prevents a unique weak solution behind a shock. We investigate a number of methods for the numerical solution of the two-fluid equations and find that, in the presence of shock waves, the results depend on the choice of the numerical methods (spatial reconstruction, time stepping, and the CFL number) and the adopted discretization. Nevertheless, all methods converge to a unique result only if the energy partition between the thermal and non-thermal fluids at the shock is prescribed a priori. This highlights the closure problem of the two-fluid equations at shocks. We suggest a robust method where the solutions are insensitive to the numerical method. Comparison with the currently used methods, critical test problems, and future directions are discussed.

Keywords: shock waves – cosmic rays – hydrodynamics – methods: numerical

1. Introduction

Macroscopic extension of the cosmic ray (CR) transport equation in the form of the two-fluid CR-HD/MHD equations is very useful to study the effects of non-thermal processes in astrophysical systems (Skilling 27; Drury & Voelk 7). It provides important insights about the dynamical effects of CRs, which are difficult and expensive to capture from the CR transport equation (see Drury 6 for a review). In many astrophysical systems (e.g., the Milky Way disk), CR energy density is comparable to the thermal/magnetic energy density, and their pressure can be as important as other components (e.g., thermal and magnetic pressures).

A fluid description of CRs is justified because the Larmor radius of energy-dominating CRs ($r_L \sim 10^{-3} E_{10GeV}/B_{\mu G}$ parsec) is much smaller than the scales of interest and they are expected to be confined along the direction of magnetic fields by self-generated magnetic fluctuations at this scale (Kulsrud & Pearce 16). Therefore, the two-fluid model is applicable in a variety of astrophysical systems, ranging from a star-forming cloud to clusters of galaxies.

Many astrophysical phenomena are studied with the two-fluid model of CRs. CRs are an attractive agent for feedback heating because their energy loss time scale is much longer than the cooling time of the thermal gas. CRs retain energy in the cloud for a long time and provide extra pressure leading to a moderate star formation rate. For a similar reason, CRs help to launch galactic winds (e.g., Salem & Bryan 23; Wiener et al. 32). The studies with the two-fluid models also found that CRs can reduce the temperature of the circumgalactic medium and explain the observed absorption lines of various elements (Simpson et al. 26; Girichidis et al. 8; Butsky & Quinn 3). CR heating can be an efficient heating mechanism in galaxy clusters (e.g., Guo & Oh 9; Jacob & Pfrommer 12) and can affect buoyancy instabilities in the intracluster medium (e.g., Sharma et al. 25). These important conclusions are based on the numerical simulations of the two-fluid model and therefore it is necessary to closely examine the properties of the these equations and their numerical implementation.

Similar two-fluid equations are also used in other contexts. For example, the two-fluid MHD equations (with pressures along and perpendicular to field lines and equations governing their evolution, and the generalization to include electrons and ions separately) are used to study pressure anisotropy in astrophysical and fusion plasmas (e.g., Sharma et al. 24; Jardin et al. 13). Both CR-HD/MHD and MHD equations with anisotropic pressure have two internal energy (equivalently pressure) equations but they do not have a separate density/velocity equation for the second fluid. In the CR two-fluid model, the equation that determines the evolution of CR energy density contains term(s) on the right hand side (RHS) with spatial derivative (also known as a source or coupling term) and this makes the weak solution across the shock non-unique and dependent on numerics. Note that there are other two-fluid systems that do not have the spatial derivatives in the source terms and for these the solution across a shock is unique (e.g., Balsara et al. 1).
The non-uniqueness problem that we discuss in this paper applies to both the two-fluid hydro and MHD equations but we exclusively consider the former to focus on this problem and its possible solution in a simpler setting. Generalization to MHD is straightforward.

The two-fluid CR-HD model contains three conservation laws (for mass, momentum, and total energy) plus one additional equation, which governs the evolution of CR energy density. Since the CR particle density is usually negligible compared to the thermal particle density, the evolution of the CR particle density is not considered. However, a CR particle carries significant kinetic energy, which makes the energy density contribution of CRs (integrated over entire spectrum) comparable to that of the thermal plasma. The CR energy equation contains a coupling term (\(p_{\text{cr}} \nabla \cdot \mathbf{v}\) on the RHS in Eq. 4) representing interactions between CRs and the thermal plasma. In absence of further assumptions, this coupling term cannot be written in a flux-conservative form. Although this term does not represent a major problem in smooth flows, it poses serious numerical difficulties at shocks as the weak solution to the underlying hyperbolic system is not unique.

The numerical discretization of these non-conservative terms in the two-fluid CR-HD/MHD system can be done in different ways. However, these implementations may not produce an identical solution. This is a crucial point, which was highlighted by Kudoh & Hanawa [15]. They suggested that if the CR energy density is estimated from the advection of a passive scalar, namely \(\chi = p_{\text{cr}} \nabla \cdot \mathbf{v}\) (where \(\gamma_{\text{cr}}\) is adiabatic index for CR fluid, \(p_{\text{cr}}\) is CR pressure, \(\rho\) is gas density), then the results are consistent with the underlying mathematical formalism. This is equivalent to postulating that the CR entropy \(p_{\text{cr}} / \rho\) is conserved across a shock. The advantage of this method is that the two-fluid equations apparently become conservative, which makes numerical application of Godunov-type shock-capturing schemes straightforward. However, this formalism generates spurious waves (see figures 6 – 10 in Kudoh & Hanawa [15]). Moreover, the assumption of a constant CR entropy across a shock is inconsistent with the idea that most CRs are accelerated at shocks. This motivates us to explore alternative strategies.

In this paper, we discuss several numerical discretizations of the two-fluid CR-HD equations by implementing the equations in the PLUTO code (Mignone et al. 18). We find that for most of the common methods, the numerical solutions depend on the choice of spatial reconstruction, time stepping, and even the CFL number. We suggest a method which gives robust numerical results. A physically faithful two-fluid implementation of CRs must be calibrated with the results from kinetic (e.g., Particle-In-Cell) simulations.

We organize this paper as follows. After presenting the basic equations in section 2, we discuss different methods in section 3. Section 4 presents results of various test problems. Section 5 discusses some prescriptions for partitioning the energy downstream of shocks into gas thermal and CR energies and broader implications of our work. Our main results are summarized in section 6.

### 2. Governing Equations

The two-fluid CR-HD/MHD equations are obtained from the Fokker-Planck CR transport equation (Skilling [28]), which is given by

\[
\frac{\partial f}{\partial t} + (\mathbf{v} + \mathbf{v}_{\text{st}}) \cdot \nabla f = \frac{p}{\rho^2} \nabla \cdot (\mathbf{v} + \mathbf{v}_{\text{st}}) + \nabla \cdot \left[ D_e \hat{b} \left( \hat{b} \cdot \nabla f \right) \right] + \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial f}{\partial \rho} \right) \tag{1}
\]

Here \(f(x, \mathcal{P}, t)\) is the CR distribution function assumed to be isotropic in momentum space, \(\mathcal{P}\) is the CR momentum, \(D_e\) and \(D_p\) are the diffusion coefficients in spatial and momentum space, \(\mathbf{v}\) is the velocity of thermal plasma. The term \(\mathbf{v}_{\text{st}} = (v_{\text{st}}^x, v_{\text{st}}^y, v_{\text{st}}^z)\) represents the bulk velocity of the scattering centers of CR particles w.r.t. the background plasma, known as the streaming velocity, which is along the direction of the magnetic field (\(\hat{b}\)) but down the gradient of CR pressure (for a brief discussion see Appendix A in Pfrommer et al. [22]). The first term on the RHS represents the CR convection term, the second term is spatial diffusion while the third term represents CR diffusion in the momentum space (Skilling [28]).

Drury & Voelk [7] suggested that Eq. (1) can be simplified if one rewrites it in terms of macroscopic variables. This is done by taking the energy moment of Eq. (1), which yields

\[
\frac{\partial e_{\text{cr}}}{\partial t} + (\mathbf{v} + \mathbf{v}_{\text{st}}) \cdot \nabla e_{\text{cr}} = - (e_{\text{cr}} - p_{\text{cr}}) \nabla \cdot (\mathbf{v} + \mathbf{v}_{\text{st}}) + \nabla \cdot \left[ \kappa_{\text{cr}} \hat{b} \left( \hat{b} \cdot \nabla e_{\text{cr}} \right) \right] + \Gamma_m \tag{2}
\]

Here \(\kappa_{\text{cr}}\) is the CR diffusion coefficient integrated over the CR distribution function (see equation 7 in Drury & Voelk [7]). CR streaming and anisotropic diffusion cannot be captured in hydrodynamics, therefore, we take \(|\mathbf{v}|/|\mathbf{v}_{\text{st}}| \gg 1\) and \(\kappa_{\text{cr}} \hat{b} \left( \hat{b} \cdot \nabla e_{\text{cr}} \right) \approx \kappa_{\text{cr}} \nabla e_{\text{cr}}\). It is also assumed that CR diffusion in momentum space is negligible, so that \(\Gamma_m \rightarrow 0\). This leads to the two-fluid CR-HD equations described as follows.

The two-fluid CR-HD equations can be written as

\[
\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F} = \mathbf{S}, \tag{3}
\]

where \(\mathbf{U} = (\rho, \rho \mathbf{v}, e_t, e_{\text{cr}})^T\) is the conservative variable array, while the flux tensor and the source term are given, respectively, by

\[
\begin{align*}
\mathbf{F} = & \begin{bmatrix} 
\rho \mathbf{v} \\
\rho \mathbf{v} \mathbf{v} + p_t \\
(e_t + p_t) \mathbf{v} - \kappa_{\text{cr}} \nabla e_{\text{cr}} \\
e_{\text{cr}} \mathbf{v} - \kappa_{\text{cr}} \nabla e_{\text{cr}} 
\end{bmatrix}^T, \\
\mathbf{S} = & \begin{bmatrix} 
0 \\
0 \\
-p_{\text{cr}} \nabla \cdot \mathbf{v} \\
-p_{\text{cr}} \nabla \cdot \mathbf{v}
\end{bmatrix}
\end{align*}
\tag{4}
\]

where

\[
e_t = e_g + e_{\text{cr}} = \left( \frac{1}{2} \rho \mathbf{|v|}^2 + \frac{p_g}{\gamma_g - 1} \right) + \frac{p_{\text{cr}}}{\gamma_{\text{cr}} - 1} \tag{5}
\]
Figure 1: Wave diagram of two-fluid (gas + CR) hydro system. Three straight lines originating from (0,0) represent eigenvectors, where the eigenvalues are labeled by \( \lambda_p \).

\[
p_i = p_g + p_{cr}
\]

are, respectively, the total energy density (the sum of kinetic energy density, gas thermal energy density, and CR energy density \( e_{cr} \)) and the total pressure (the sum of gas pressure \( p_g \) and CR pressure \( p_{cr} \)). The adiabatic constants of the thermal and CR fluids are defined as \( \gamma_g = 1 + p_g/e_{th} \) (where thermal energy density \( e_{th} = e_g - \rho |\mathbf{v}|^2/2 \)) and \( \gamma_{cr} = 1 + p_{cr}/e_{cr} \), and their values are taken to be 5/3 and 4/3 respectively. Note that the last element of \( \mathbf{U} (e_{cr}) \) does not obey a conservative equation. CRs can lose/gain energy due to the term \(-p_{cr}\mathbf{\nabla} \cdot \mathbf{v}\) representing coupling between thermal and CR fluids. This term, involving a derivative, gives a non-zero and non-unique contribution across a shock which depends on numerical implementation. Later we explore various implementations of this term and the associated numerical challenges.

In order to understand the characteristic structure of the two-fluid equations, we neglect the CR diffusion term (i.e., the term \( k_{cr}\mathbf{v}_{cr} \mathbf{\nabla} e_{cr} \) of Eq. 4)\(^1\) and focus only on the hyperbolic structure of the equations. Considering 1D Cartesian coordinate with three velocity components, the primitive form of Eq. (3) then becomes

\[
\frac{\partial \mathbf{V}}{\partial t} + A_p(\mathbf{V}) \frac{\partial \mathbf{V}}{\partial x} = 0 \, , \tag{7}
\]

where

\[
\mathbf{V} = \begin{bmatrix}
\rho \\
v_x \\v_y \\v_z \\p_g \\p_{cr}
\end{bmatrix}, \quad A_p = \begin{bmatrix}
\rho & 0 & 0 & 0 & 0 & 0 \\
0 & v_x & 0 & 0 & \frac{1}{\gamma_g} & \frac{1}{\gamma_g} \\
0 & 0 & v_y & 0 & 0 & 0 \\
0 & 0 & 0 & v_z & 0 & 0 \\
0 & 0 & 0 & 0 & \rho a_{g}^2 & 0 \\
0 & 0 & 0 & 0 & 0 & \rho a_{cr}^2
\end{bmatrix} . \tag{8}
\]

Here \( a_{g} = (\gamma_g p_g/\rho)^{\frac{1}{2}} \) and \( a_{cr} = (\gamma_{cr} p_{cr}/\rho)^{\frac{1}{2}} \). Defining the effective sound speed as \( a = (a_{g}^2 + a_{cr}^2)^{\frac{1}{2}} \), the eigenvalues of the characteristic matrix \( A_p \) are found to be (also see Fig. 1)

\[
\lambda_p = v_x - a, \ v_x, \ v_y, \ v_z, \ v_x + a. \tag{9}
\]

Right eigenvectors of the characteristic matrix \( A_p \) can be written as,

\[
R_p = [r_p^{mu}] = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 1 \\
\frac{-a_{cr}^2}{p_{cr}} & 0 & 0 & 0 & 0 & \frac{a_{cr}^2}{p_{cr}} \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
a_{cr}^2 & 0 & 0 & 0 & -1 & a_{cr}^2 \\
a_{cr}^2 & 0 & 0 & 1 & 0 & a_{cr}^2
\end{bmatrix}. \tag{10}
\]

In the absence of CR fluid, the 5\(^{th}\) column and the 6\(^{th}\) row of the matrix \( R_p \) are absent, i.e., the right-going eigenvalue is \( \lambda = v_x + a_g \) and the corresponding eigenvector \( r_5 = [1, a_g/\rho, 0, 0, a_{cr}^2]^T \). This implies that the ratios between the fluctuations in density, velocity, and pressure scale as \( 1 : a_g/\rho : 0 : 0 : a_{cr}^2 \). Therefore, any change in the x-velocity in a right going sound wave causes a simultaneous change in the density and the pressure, which increase/decrease without any ambiguity. However, the presence of CRs can introduce ambiguity in the fluctuations, which is illustrated as follows.

As shown in Eq. (10), for the right-going sound waves in the two-fluid CR equations the ratios between the fluctuations in density, velocity, thermal pressure, and CR pressure scale as \( 1 : a_g/\rho : 0 : 0 : a_{cr}^2 : a_{cr}^2 \), i.e., the fluctuation of the x-velocity can be satisfied for multiple values of \( a_g \) and \( a_{cr} \) [because \( a = (a_{g}^2 + a_{cr}^2)^{\frac{1}{2}} \)]. This ambiguity affects the fluctuations of thermal and CR pressures. Moreover, the 5\(^{th}\) column of the right eigenvector shows that at the contact discontinuity (represented by \( r_2 \) to \( r_5 \)), the ratio between the fluctuations of the thermal pressure and CR pressure is \( -1 \). This means that, although the total pressure \( (p_i, \text{see Eq. 6}) \) on both side of the contact discontinuity is constant, the thermal and CR pressures may not remain the same across a contact discontinuity. The fluctuation in thermal pressure across the contact discontinuity is compensated by a gain/loss of the CR pressure or vice versa. This subtlety in the characteristic structure of the two fluid CR equations leads to additional numerical issues which are addressed in the next sections.

3. Numerical framework

In this section, we explore a variety of different discretization strategies for the numerical solution of the two-fluid equations. All methods differ essentially in the representation of the coupling term (the non-conservative term in Eq. 4). Let \( \mathcal{L}_g() \) and \( \mathcal{L}_c() \) be the discrete operators corresponding to the evolution of the homogeneous part of the equations and to the source term alone, respectively. The update step can then be achieved via operator splitting,

\[
\mathcal{U} = \mathcal{L}_g(\mathcal{U}^g) \quad \mathcal{U}^{g+1} = \mathcal{L}_c(\mathcal{U}) , \tag{11}
\]

\(^1\)Diffusion term is usually separately implemented using the standard parabolic schemes (e.g., Vaidya et al. [31]).
Figure 2: Various methods for implementing the non-conservative term of CR-HD equations. All methods differ w.r.t. to the evolution of gas/total energy and CR energy equations. The mass and momentum conservation equations are identical in all the methods. Depending on the implementation of fluid energy densities, these methods are classified into three different options. In Option 1: \( E_g + E_{cr} \), the energy density of CRs \( \varepsilon_{cr} = p_{cr}/(\gamma_{cr} - 1) \) and gas \( \varepsilon_g = \rho|v|^2/2 + p_g/(\gamma_g - 1) \) are separately evolved. In this case, the coupling term can either be \( p_{cr} \nabla \cdot v \) (pdv method) or \( v \cdot \nabla p_{cr} \) (vdp method). In Option 2: \( E_t + E_{cr} \), the total energy density, i.e., \( \varepsilon_g + \varepsilon_{cr} \) is used instead of \( \varepsilon_g \) in order to update the energy density of the system. In this case, the coupling term can either be \( p_{cr} \nabla \cdot v \) or \( v \cdot \nabla p_{cr} \), similar to Option 1. However, in this Option 2, the coupling term does not appear in the total energy equation. In Option 3: \( E_t + S_{cr} \), CR energy density is updated from the advection of a passive scalar and coupling terms are absent. In Option 1 and 2, depending on the implementation of the coupling terms, they can be divided into two sub methods: (i) operator-splitting (OpSplit) and (ii) Unsplit. Naming of different methods is given in the right side of this diagram.

or in a fully unsplit fashion:

\[
U^{n+1} = (L_b + L_s)(U^n).
\]

We label these two approaches as OpSplit (Eq. 11) and Unsplit (Eq. 12), respectively. Notice that the operator split method presented here is only first-order accurate in time, but it can be made formally second order accurate using Strang (or alternate) splitting (Strang [29]).

The rationale for choosing \( L_b \) and \( L_s \) depends on the implementation of fluid energy densities and for this reason we classify them into three different options: Option 1 - “\( E_g + E_{cr} \)”, 2 - “\( E_t + E_{cr} \)”, and 3 - “\( E_t + S_{cr} \)”. The differences between various methods are briefly illustrated in Fig. 2 and nomenclature of different indices is shown in Fig. 3. Note that since the implementation of mass and momentum equations remain the same in all of the methods, we focus solely on the energy equations in what follows.

3.1. Option 1: \( E_g + E_{cr} \)

Here the energies of thermal and CR fluids, coupled by the source term, are evolved separately; i.e., we solve the following equations

\[
\frac{\partial \varepsilon_g}{\partial t} = -\nabla \cdot \left([\varepsilon_g + p_t]v\right) + p_{cr} \nabla \cdot v, \tag{13}
\]

\[
\frac{\partial \varepsilon_{cr}}{\partial t} = -\nabla \cdot (\varepsilon_{cr}v) - p_{cr} \nabla \cdot v. \tag{14}
\]

By suitably rewriting the RHS of Eqs. (13) and (14), the coupling term may be implemented either as \( p_{cr} \nabla \cdot v \) (pdv method) or \( v \cdot \nabla p_{cr} \) (vdp method). Although the former choice (pdv method) is found to be numerically robust and more common, we discuss both implementations separately in the following sections. Note that total energy conservation is ensured by adopting the same discretization for the source term in Eq. (13) and (14).

A. pdv method

Figure 3: Nomenclature of the used indices. Black circles denote grid points. Dashed vertical lines represent cell interfaces. \( V^- \) represents primitive variables w.r.t. the 0th grid; i.e., for piecewise constant reconstruction (the 1st order scheme), both \( V^- \) and \( V^+ \) are identical. The notation \( i - \frac{1}{2}, i + \frac{1}{2} \) ... represents the cell-interface (in the code, they are labeled by \( ..., i - 1, i \)).
In this method, the CR energy flux is defined as the CR energy density times the advection velocity of the thermal fluid (Eq. 14). Since the source term is $p_{cr} \nabla \cdot v$, we refer to this method as the pdv method.

**A1. OpSplit-pdv method:** In many studies, the $p_{cr} \nabla \cdot v$ term has been implemented using an operator splitting method (e.g., Pfrommer et al. 21; Sharma et al. 25; Salem & Bryan 23; Gupta et al. 10). Following Eq. (11), $e_{cr}$ is first evolved along with the other hydrodynamic variables to yield $e_{cr}^n$. This value is then used to calculate $p_{cr} \nabla \cdot v$ at the next step. For a first-order time-stepping scheme this amounts to

$$e_{cr,i}^n = e_{cr,i}^{n-1} - \Delta t \left( f_{i+\frac{1}{2}}^n - f_{i-\frac{1}{2}}^n \right),$$

and

$$e_{cr,i}^{n+1} = e_{cr,i}^n - \Delta t p_{cr,i}^{n} \left( v_{i+\frac{1}{2}}^{n+1} - v_{i-\frac{1}{2}}^{n+1} \right).$$

where $f_{i+\frac{1}{2}}^n$ is the HLL flux at the cell interface (see Chapter 10 in Toro 30) while the interface velocity (see e.g., Fig. 2) is defined as

$$v_{i+\frac{1}{2}}^n = \frac{1}{2}(v_{i}^n + v_{i+1}^n).$$

In other words, the quantity $\nabla \cdot v$ is estimated using a cell centered method. The gas energy equation (13) is updated using a similar discretization as Eqs. (15) and (16).

Eq. (15) is readily generalized to second-order temporal accuracy by using a Runge-Kutta method already available in the code while direct extension to curvilinear geometries is thoroughly described in the work by Mignone [17].

**A2. Unsplit-pdv method:** In the fully unsplit scheme (Eq. 12), the $p_{cr} \nabla \cdot v$ term is directly added to the RHS of Eqs. (13) and (14). This yields, for the CR energy equation, the following update:

$$e_{cr,i}^{n+1} = e_{cr,i}^n - \Delta t \left( f_{i+\frac{1}{2}}^n - f_{i-\frac{1}{2}}^n \right)$$

$$-\Delta t p_{cr,i}^{n} \left( v_{i+\frac{1}{2}}^{n+1} - v_{i-\frac{1}{2}}^{n+1} \right).$$

Although several options are possible for choosing $p_{cr,i}^{n}$ and $v_{i+\frac{1}{2}}^{n+1}$ (see e.g., Appendix B), one can take advantage of the Riemann solver to estimate $p_{cr,i}^{n}$ and $v_{i+\frac{1}{2}}^{n+1}$. We find that the most robust choice for the cell interface velocity is

$$v_{i+\frac{1}{2}}^{n+1} = \frac{\mathcal{U}(\rho_{[i,i+\frac{1}{2}]})}{\mathcal{U}(\rho_{[i,i+\frac{1}{2}]})},$$

where $\mathcal{U}(\rho)$ and $\mathcal{U}(\rho)$ are the momentum- and density- state variables obtained from the HLL Riemann solver (see chapter 10 in Toro [30]). Such a choice is also used by Pfrommer et al. [22].

The CR pressure in Eq. (18) can also be chosen in various ways. We find that the most robust selection is

$$p_{cr,i}^{n} = \frac{\gamma_{cr} - 1}{2} \left( e_{cr,i-\frac{1}{2}}^{n} + e_{cr,i+\frac{1}{2}}^{n} \right),$$

where $e_{cr,i-\frac{1}{2}}^{n}$ is the state-variable obtained from the Riemann solver. We label this method as “Eg+Ecr (Unsplit-pdv)”.

**B. vdp method**

Alternatively, the coupling term can be implemented in $\nabla \cdot v p_{cr}$ form. The evolution of CR energy density is obtained by rewriting the CR energy equation (Eq. 14) as

$$\frac{\partial e_{cr}}{\partial t} = -\nabla \cdot \left[ (e_{cr} + p_{cr}) \nabla v \right] - \nabla \cdot \nabla p_{cr},$$

which, at the discrete level, becomes similar to Eq. (15) and (18) by exchanging $v$ and $p_{cr}$.

Although, Eqs. (14) and (21) are mathematically equivalent, the numerical discretisations are not the same. Since the coupling term is $\nabla \cdot v p_{cr}$, we call this method as vdp method. As before, this coupling term can be included via an operator splitting method (OpSplit-vdp) or an unsplit method (Unsplit-vdp).

One may now choose a cell-centered discretization for $v$, and the arithmetic average between the left and right interface values for $p_{cr,i+\frac{1}{2}}$. However, as it will be shown in section 4, this choice leads to results that depend on the type and order of the spatial reconstruction algorithm. This effect is considerably less pronounced with the pdv method.

3.2. Option 2: Ets+Ecr

In this case, we replace Eq. (13) with the total energy equation

$$\frac{\partial e_{t}}{\partial t} = -\nabla \cdot \left[ (e_{t} + p_{t}) \nabla v \right],$$

where $e_{t}$ and $p_{t}$ are given by Eqs. (5) and (6).

The total energy density now directly obeys a conservative equation. However, the CR energy equation still contains the coupling term, which can be implemented using the pdv or vdp method as described earlier. After extensive numerical testing, we have found that the numerical results are consistent for various reconstruction schemes only with the pdv method in a fully unsplit fashion, and $v$ and $p_{cr}$ are chosen as given in Eqs. (19) and (20). We thus label this method as “Ets+Ecr (Unsplit-pdv)”. It is worth mentioning that both “Ets+Ecr (Unsplit-pdv)” and “Eg+Ecr (Unsplit-pdv)” give an identical result when Eqs. (19) and (20) are used.

3.3. Option 3: Ets+Scr

Kudoh & Hanawa [15] suggested that the difficulties related to the presence of the coupling term may be avoided if the evolution of CR energy density is obtained from the advection of a passive scalar. They redefined the CR pressure as $p_{cr} \equiv p_{cr}^{\star}$ and
used a passive scalar: \( \chi = \rho c_r / \rho \equiv \rho^{1/\gamma_{cr}} / \rho \) to update the CR energy equation. Under this formalism, Eq. (14) or (21) reads

\[
\frac{\partial}{\partial t} (\chi \rho) + \nabla \cdot (\chi \rho \mathbf{v}) = 0 .
\] (23)

The remaining equations maintain the same form as for Option 2. The advantage of this method is that the two-fluid equations evidently become conservative, which makes the application of Godunov-type formalism straightforward.

Since Eq. (23) is a tracer equation, the CR entropy \( S_{cr} \equiv h_{cr} / \rho^{3/2} \) does not experience a jump across a shock front\(^3\). However, we argue that CRs, like the thermal plasma, should not be adiabatic across a shock. Indeed, CRs are accelerated with non-negligible efficiencies across strong shocks, but this possibility is not allowed by the strict isentropic evolution imposed by the above scalar equation. We show later in section 5.1 that the choice of isentropic evolution of CRs across shocks is one of the several equation-of-states that one can impose. The results from this method are labeled by “Et+Scr”.

4. Test problems & results

In this section we compare the previously presented numerical methods for the solution of selected one- and multi-dimensional benchmarks. The initial conditions are listed in Table 1. Computations are performed using either a 1\(^{st}\)-order scheme (Euler time stepping with flat reconstruction), a 2\(^{nd}\)-order scheme (RK2 time stepping with linear reconstruction) or a 3\(^{rd}\)-order scheme (RK3 time stepping with WENO reconstruction). The fiducial CFL number is set to \( C_a = 0.6 \) unless otherwise stated.

4.1. Shock tube A

The initial condition for this test consists of two constant states separated by a discontinuity placed at \( x = 0 \). Left and right states are defined as \((\rho, v_x, p, c_r, h) = (1, 0, 2, 1)\) and \((\rho, v_x, p, c_r, h) = (0.2, 0, 0.02, 0.1)\) respectively. In all cases we employ 1000 equally spaced zones. The set-up for this problem is identical to Kudoh & Hanawa [15].

Fig. 4 shows various profiles at \( t = 0 \) (black dashed curves) and \( t = 0.1 \) (circles) where different colours correspond to the four solution methods discussed in section 3. The solutions have been obtained with the 1\(^{st}\)-order scheme and represent the best possible solution from our numerical experiments. The comparison between different curves shows that the solutions are not identical. Blue and green curves - representing the \( Eg+Ecr\) (OpSplit-pdv) and \( Et+Ecr\) (Unsplit-pdv) methods respectively - look similar although, as we shall see in Fig. 5, the solution obtained with the OpSplit method depends on the details of spatial reconstruction and time-stepping.

We wish to spend some time discussing the top rightmost panel of Fig. 4, which shows the CR entropy profile \((p_{cr} / \rho^{3/2})\). In the two-fluid CR-HD model, it is sometimes assumed that CRs are adiabatically compressed in the post-shock region. This assumption also helps to obtain an analytical solution of the shock tube problem (Pfrommer et al. 21). In numerical simulations this assumption is not automatically fulfilled. Some recent studies have highlighted this point (see the footnote 10 in Jiang & Oh 14; also see Kudoh & Hanawa 15). In fact, we find that CRs are adiabatic only for the \( Et+Scr \) method which is constructed to do so in the first place. In general, the post-shock solution depends on the method one uses.

Now we come to a problem that is even more serious than the post-shock solutions not matching across different methods. Namely that the solution (for all methods except unsplit-pdv and \( Et+Scr \)) depends on numerical details such as reconstruction and the CFL number. While the \( Et+Scr \) method is expected to yield robust results at shocks, there are problems e.g., spurious contact waves as we show in section 4.2.

Fig. 5 shows the zoomed-in CR entropy profile from the shock tube problem. Four vertically aligned panels display results from four different methods. In each panel, black, green and blue curves correspond to solutions obtained using 1\(^{st}\), 2\(^{nd}\) and 3\(^{rd}\)-order schemes. The top panels show that the results are not unique. Most importantly, the solutions obtained using the method \( Et+Ecr\) (Unsplit-vdp) and \( Eg+Ecr\) (OpSplit-pdv) depend on spatial reconstruction. This can be seen by comparing different curves in the 1\(^{st}\) and 2\(^{nd}\) panels in the top row. Other than 3\(^{rd}\) and 4\(^{th}\) columns [i.e., other than \( Et+Scr \) and \( Et+Ecr\) (Unsplit-pdv) methods], the numerical solution depends on spatial reconstruction.

Conversely, the results in the bottom panels of Fig. 5 show very similar profiles and have been obtained by fixing the CR pressure behind the shock, as explained in the following. First, we employ a shock detection algorithm (c.f., section 5.2) to identify shocked zones. Then we redistribute the shocked zone’s thermal and CR energies using a parameter defined as

\[
\epsilon_{\text{shock}} = \frac{\epsilon_{cr}}{\epsilon_{cr} + \epsilon_{th}} = \frac{w_{cr}(\gamma_{cr} - 1)}{[\gamma_{cr} - 1 + w_{cr}(\gamma_{cr} - \gamma_{cr})]}
\] (24)

where

\[
w_{cr} = \frac{p_{cr}}{(p_{cr} + p_{cr})} .
\] (25)

The redistribution of energy among CRs and the thermal fluid does not change the total energy (in particular \( \epsilon_{cr} + \epsilon_{th} \)) of the cell. Thus, we are able to obtain identical solutions irrespective of the numerical method. This prescription is tantamount to a sub-grid injection model for CRs at shocks (see e.g., Caprioli & Spitkovsky 4, Mignone et al. 20). The choice of \( w_{cr} \) or a similar parameter provides a closure to the two-fluid equations at shocks (and other dissipative structures in general). In section 5.1, we discuss some other possible equation-of-states (EqS) which can be used.
Fig. 6 shows the robustness of the ‘Et+Ecr (Unsplit-pdv)’ method for different spatial reconstructions and time stepping. As a reference solution, we have performed a simulation using an extremely high grid resolution. The snapshot of various profiles at \( t = 0.1 \) are shown. Four different solid curves, i.e., black/grey (1st order), green (2nd order), and blue (3rd order) curves show the solutions with different spatial reconstruction. Since the profiles are identical, our implementation of the ‘Et+Ecr (Unsplit-pdv)’ method is numerically robust.

Lessons from this section are: (i) the solution of the two-fluid equations in the post-shock region depends on the choice of numerical methods, (ii) all methods give identical result only when the CR pressure behind the shock is imposed ‘by hand’ (e.g., using a similar equation as shown in Eq. 25), (iii) the results from our method (i.e., “Et+Ecr (Unsplit-pdv)” using Eqs. 19 and 20) do not seem to depend on the numerics (such as spatial reconstruction and time-stepping).

4.2. Pressure balance

This is an important test problem designed to check the evolution of a pressure balance mode. At \( t = 0 \), the left and right states are defined as \((\rho, v_x, p_g, p_{cr})_L = (1, 0, 2, 1)\) and \((\rho, v_x, p_g, p_{cr})_R = (0.2, 0.02, 0.1)\). The sub-plot in some panels displays the zoomed-in view of the post-shock region and it shows that different methods do not give a unique solution.
(A) Without fixing $p_{cr}$ behind the shock

(B) With fixing $p_{cr}$ behind the shock

Figure 5: The zoomed-in view of CR entropy profile ($p_{cr}/\rho^{4/3}$) from the shock tube A (Table 1). The three colours display results from the 1st− (black), 2nd− (green), and 3rd− (blue) order numerical schemes. Four vertically aligned panels stand for the different methods. For each of these methods, we show the results using the best possible combination of $p_{cr}$ and $v$ from our experiments, except for the 3rd column where the coupling term is not present. The results in the top panels are not identical and depend on spatial reconstruction in 1st and 2nd panels. However, in 3rd and 4th panels, the results are numerically robust. In bottom panels, we fix the CR pressure of the shocked-zones by using a parameter, $w_{cr} = p_{cr}/(p_{cr} + p_{g})$ (setting it to 0.5 for the illustration purpose). Figure shows that if the energy exchange between thermal and CR fluid at the shock is fixed by using a parameter like $w_{cr}$ then all methods give an identical solution.

Figure 6: Snapshot of various profiles from the two-fluid shock tube A (see Table 1) obtained using the $Et+Ecr$ (Unsplit-pdv) method. Black dashed curves show the profiles at $t = 0$ and circles at $t = 0.1$. Different colours represent results of different numerical schemes where ‘n’ denotes the number of grid points. All runs are performed using the CFL number 0.6 (the results do not seem to depend on CFL numbers; see Appendix B) and without fixing $p_{cr}$ across the shock. The figure shows that the results do not depend on numerics, i.e., higher order solvers reproduce results identical to a high-resolution 1st order solvers.
Figure 7: Advection of pressure balance test problem (problem 2 in Table 1). Black dashed lines display initial conditions, and circles represent various profiles after an advection time across the box ($t = 1$). In all the methods shown by different colours, spatial reconstruction is linear, time stepping is RK2, and CFL number is 0.6. At $t = 0$, the total pressure in the left and right states (i.e., $x \leq 0.5$ and $x > 0.5$) is equal, i.e., various profiles are not expected to show difference after one period (i.e., at $t = 1$). However, here we see that only blue and green curves reproduce the expected result. Red and magenta curves representing “Unsplit- vdp” and “Et+Scr” methods do not give the correct solution. The vdp method is not preferred over pdv because although velocity is continuous across $x = 0$, CR pressure is not and therefore taking its derivative produces spurious disturbances at the contact discontinuity.

(\(\rho, v_x, p_L, p_C\)) = (1, 1, 0.9, 0.1). We use the 2nd-order scheme and impose periodic boundary conditions. Since, the total pressure is the same across the interface ($p_L + p_C = 1$) the profiles should not change in time.

The snapshots of fluid quantities at $t = 1$ (i.e., after one advection time across the domain) are shown in Fig. 7. Blue (OpSplit-pdv) and green (Unsplit-pdv) curves do not show spurious oscillations. However, red (Unsplit- vdp) and magenta (Et+Scr for both HLL and HLLC Riemann solvers) curves fail to maintain the pressure balance mode.

Although the Et+Scr method yielded a numerically robust result in the previous shock tube problem, it does not produce an equally robust solution in this case. Kudoh & Hanawa [15] showed that the amplitude of the spurious waves in Fig. 7 (for magenta curves) can be reduced by increasing the grid resolution or including additional numerical diffusion. However, both options are computationally expensive and rely on problem-dependent fine-tuning which makes the choice of resolution hardly conclusive.

From the next section onwards, we discuss the results from our best performing method, “Et + Ecr (Unsplit-pdv)”.

4.3. Shock tube B

The initial condition for this test (problem 3 in Table 1) leads to the formation of a strong shock/rarefaction wave, which allows us to assess the sensitivity of the results on the left- and right-going wave speed estimates. Such estimates are typically required to calculate both the Riemann flux and the cell interface state of the conservative variables in HLL or HLLC-type (e.g., HLLC and HLLD) solvers. The exact wave speeds cannot be derived from first principles.

In the literature, various choices are available for the left- and right-going wave speeds. (see e.g., Chapter 10 in Toro [30]). The simplest choice is the direct wave speed estimates suggested by Davis [5], yielding

\begin{align}
S_{R,i+1/2} &= \text{max}(v_x^i + a_L^i, v_x^{i+1} + a_R^{i+1}) \\
S_{L,i+1/2} &= \text{min}(v_x^i - a_L^i, v_x^{i+1} - a_R^{i+1}).
\end{align}

Here we have used the notation introduced in Fig. 3. For some test problems, the present implementation (i.e., “Et+ Ecr (Unsplit-pdv)”; see section 3) shows an unexpected behaviour near the opening of the Riemann fan (c.f., shown by red curves in Fig. 8). We solve this issue by increasing the spatial width of the Riemann fan as follows.
We redefine the right- and left- moving wave speeds as

$$\begin{align*}
\hat{S}_{R,i+1/2} &= \max(v_i^+ + \phi a_i^+, v_{i+1} + \phi a_{i+1}^-) \\
\hat{S}_{L,i+1/2} &= \min(v_i^- - \phi a_i^-, v_{i+1}^- - \phi a_{i+1}^-),
\end{align*}$$

respectively, and choose $\phi = 1.1$. Note that, when $\phi = 1$ then Eq. (27) reduces to Eq. (26).

This technique basically increases the width of the mixed state, which becomes $2\phi a$ at $t + dt$, instead of $2a$ (here $a = (a_h^2 + a_c^2)^{1/2}$ is the sound speed of composite fluid). Increasing the width of the mixed state is justified by the fact that during the solution of the Riemann problem we do not include the coupling term which changes the effective signal speed (see e.g., Appendix A). Although in Unsplit-pdv method we update the CR energy equation (Eq. 18) in a single step, the required sound wave speed may lie between $[a_h^2 + a_c^2]^{1/2}$ and $[a_h^2 + (\gamma_f/\gamma_c)a_c^2]^{1/2}$. Therefore, our choice of $\phi = 1.1$ is meaningful. Note that, the factor $\phi$ introduced here is obtained from our numerical experiments, with no rigorous justification. In Fig. 8 we explicitly show the effect of the factor $\phi$.

Fig. 8 shows the snapshot of pressure profiles at $t = 10^{-4}$ (problem 3 in Table 1). Red and green curves show numerical solution for $\phi = 1$ and $\phi = 1.1$ respectively. Figure shows that the solutions are smooth only for $\phi = 1.1$ (green curves), implying the necessity of a different wave speed estimate in order to obtain a robust solution.

4.4. Sedov-Taylor Blast Wave

We have extended our implementation of CR-HD equations to multi-dimensions and different geometries in the PLUTO code. We perform the standard blast wave problem in 3D Cartesian and 1D spherical geometries. The numerical set-up is discussed below.

At $t = 0$, we create high pressure in a small region by setting $p_{inj} = (\gamma_f - 1)E/\Delta V$ where $\Delta V = 4/3\pi r_{inj}^3$ is the small volume, $r_{inj} = 0.01$ (in code units), and $E = 10^{51}$ erg. In the rest of the computational domain, the density, velocity, and pressures are set to 1, 0, and 60 respectively. The unit of density, velocity, and length are $m_H\text{ cm}^{-3} (1.67 \times 10^{-24} \text{ g cm}^{-3})$, $10^5 \text{ cm s}^{-1}$ and $3.086 \times 10^{18} \text{ cm}$ (1 parsec) respectively. For 1D spherical geometry, we set the inner radial boundary at $r = 0.001$ to reflective, and for 3D Cartesian, we set all boundaries to outflow. For the 3D run, we set 200 equally spaced zones along $x, y,$ and $z$ directions, i.e., $N_x N_y N_z = 100^3$ (but we also perform a low resolution run using $N_x N_y N_z = 10^3$ to check convergence), and for the 1D run we use 200 equally spaced zones. Simulations are performed using a CFL number $C_x = 0.2$ for both geometries. Since we do not inject mass, the Sedov-Taylor phase starts right at the beginning of the shock evolution. The initial injection is purely thermal, but we inject CRs at the shock as follows.

First, we identify shocked zones using a shock detection algorithm (see section 5.2) and then inject CRs in the shocked zones using a parameter $w_{cr}$ (Eq. 25) as done previously in 1D. We set $w_{cr} = 0.5$, i.e., equipartition between thermal and CR pressures in the post-shock zones. The results from 3D (Cartesian) and 1D (spherical) runs are shown in Fig. 9. In the top panels we show the snapshots of density, thermal and CR pressures in the $z = 0$ plane at $t = 2 \times 10^{-6}$ (code unit) obtained from our 3D run. The second panel (top row) shows the zoomed-in view of the shocked zones where CR energy is injected. In the bottom panels we compare the results between the 3D case and the 1D runs, showing good agreement. We, therefore, conclude that our implementation (i.e., $Et+Cr$ (Unsplit-pdv) method) is well suited for multi-dimensional calculations.

5. Discussion

In previous sections we have shown that the numerical solution of the two-fluid equations depends on the choice of discretization of the coupling term and the results become identical only when the post-shock CR pressure is set using a parameter $w_{cr}$ (Fig. 5). In section 5.1 we discuss various closure models for the post-shock thermal and CR pressures. In a physically motivated two-fluid model, CRs must be injected at shocks and...
the identification of shocked zones becomes critical. We discuss this in section 5.2. In section 5.3 we discuss broader implications of our work.

5.1. Closure problem at shocks and possible solutions

In the shock rest frame, the mass, momentum, and energy conservation equations for the two-fluid CR-HD model are

\[
\rho_1 v_1 = \rho_2 v_2 ,
\]

\[
\rho_1 v_1^2 + p_{g,1} + p_{cr,1} = \rho_2 v_2^2 + p_{g,2} + p_{cr,2} ,
\]

\[
\left\{ \frac{1}{2} \rho_1 v_1^2 + \frac{\gamma_g}{\gamma_g - 1} p_{g,1} + \frac{\gamma_{cr}}{\gamma_{cr} - 1} p_{cr,1} \right\} v_1 = \left\{ \frac{1}{2} \rho_2 v_2^2 + \frac{\gamma_g}{\gamma_g - 1} p_{g,2} + \frac{\gamma_{cr}}{\gamma_{cr} - 1} p_{cr,2} \right\} v_2,
\]

respectively, where the subscripts 1 and 2 represent the upstream and downstream fluid variables. These equations show that there are three conservation laws (for mass, momentum, and total energy) and four unknowns (\(\rho, v, p_{g}, p_{cr}\)). This constitutes a closure problem illustrated as follows.

We wish to find the shock jump conditions such as the compression ratio, \(R = \rho_2/\rho_1 = v_1/v_2\), in terms of the upstream gas and CR Mach numbers which are defined as

\[
M_{g,1} = \frac{v_1}{(\gamma_g p_{g,1}/\rho_1)^{1/2}} \quad \text{and} \quad M_{cr,1} = \frac{v_1}{(\gamma_{cr} p_{cr,1}/\rho_1)^{1/2}}
\]

respectively. The effective upstream Mach number of the composite fluid can be defined as \(M_1 = (M_{g,1}^{-2} + M_{cr,1}^{-2})^{-1/2}\). Note that, for a given set of upstream parameters: \(M_1(M_{g,1}, M_{cr,1})\), \(p_{g,1}\), and \(p_{cr,1}\), we have to find \(R\), \(p_{g,2}\), and \(p_{cr,2}\). For simplicity, we normalize the gas and CR pressures relative to the upstream gas pressure and denote them by \(P_{g,1} = p_{g,1}/p_{g,1}\) and \(P_{cr,i} = p_{cr,i}/p_{g,1}\) respectively, where \(i \in 1, 2\). Therefore, we have

\[
M_1^2 + \frac{1 + P_{cr,1}}{\gamma_g + \gamma_{cr} P_{cr,1}} = \frac{M_{g,1}^2}{R} + \frac{P_{g,2} + P_{cr,2}}{\gamma_g + \gamma_{cr} P_{cr,2}}
\]

\[
R \left\{ \frac{M_{g,1}^2}{2} + \frac{1}{\gamma_g + \gamma_{cr} P_{cr,1}} \left( \frac{\gamma_g}{\gamma_g - 1} + \frac{\gamma_{cr}}{\gamma_{cr} - 1} P_{cr,1} \right) \right\} = \left\{ \frac{M_{g,1}^2}{2 R} + \frac{1}{\gamma_g + \gamma_{cr} P_{cr,1}} \left( \frac{\gamma_g}{\gamma_g - 1} P_{g,2} + \frac{\gamma_{cr}}{\gamma_{cr} - 1} P_{cr,2} \right) \right\}
\]

where the normalized upstream CR pressure, \(P_{cr,1}\) is written as

\[
P_{cr,1} = \frac{p_{cr,1}}{p_{g,1}} = \left( \frac{\gamma_{cr}}{\gamma_g} \right) \left( \frac{M_{g,1}}{M_{cr,1}} \right)^2.
\]

Eqs (32) and (33) contain three unknowns: \(P_{g,2}\), \(P_{cr,2}\), and \(R\). To get a unique solution, we need an additional equation and this is the closure problem. We can fix the downstream CR pressure (\(P_{cr,2}\)) in several ways. Here we discuss three possible equation-of-states (EoSs) in detail.
5.1.1. $w_{cr}$-EoS

In this case, the downstream CR pressure is set to a fraction $w_{cr}$ of the downstream total pressure, i.e.,

$$ w_{cr} = \frac{p_{cr,2}}{p_{g,2} + p_{cr,2}} = \frac{p_{cr,2}}{p_{g,2} + p_{cr,2}}, \tag{35} $$

where $0 \leq w_{cr} \leq 1$. Using Eq. (35) we substitute $p_{cr,2}$ in Eqs. (32) and (33). After combining them, we obtain a quadratic equation whose solutions can be written as $R_s$ (for details see Appendix C.1). The solution $R_s$ represents compression ratio $\geq 1$ (for a shock) and its values as a function of upstream Mach numbers for four different $w_{cr}$ are shown in Fig 10. The figure indicates that the compression ratio increases with $w_{cr}$. It can also be shown analytically:

$$ \lim_{M_1 \to \infty} R_s = \frac{\gamma_g + 1}{\gamma_g - 1} + 2 w_{cr} \left( \frac{\gamma_g}{\gamma_c - 1} - \frac{\gamma_g}{\gamma_c} \right). \tag{36} $$

Taking $\gamma_g = 5/3$ and $\gamma_c = 4/3$, we obtain $R_s = 4 + 3w_{cr}$, which shows that the compression ratio approaches 7 when $w_{cr} \to 1$.

5.1.2. $\epsilon_{cr}$-EoS

For this EoS, the downstream CR enthalpy flux is set as a fraction $\epsilon_{cr}$ of the upstream total energy flux, i.e.,

$$ \epsilon_{cr} = \frac{\gamma_g}{\gamma_c - 1} p_{cr,2} v_2 \left( \frac{1}{\gamma_c \rho_1 v_1^2} + \frac{\gamma_g}{\gamma_c - 1} p_{g,1} + \frac{\gamma_c}{\gamma_c - 1} p_{cr,1} \right) \tag{37} $$

where $0 \leq \epsilon_{cr} \leq 1$ ensures energy conservation. Substituting Eq. (37) in Eqs. (32) and (33) again gives a quadratic equation (see section Appendix C.2). We find that the qualitative nature of the solutions remains the same as shown in Fig. 10, i.e., the compression ratio increases with $\epsilon_{cr}$. In the limit $M_1 \to \infty$, $w_{cr}$ can be written as a function of $\epsilon_{cr}$ (for details see Appendix C.3), as

$$ w_{cr,s} = \frac{4 \left\{ -3(1 - \epsilon_{cr}) \pm (9 + 6\epsilon_{cr})^{1/2} \right\}}{3(8 - 3\epsilon_{cr})} \tag{38} $$

Note that, the constrain $0 \leq \epsilon_{cr} \leq 1$ implies that $w_{cr,s}$ is the physical solution. The value of $w_{cr}$ for various upstream Mach numbers and $\epsilon_{cr}$ are shown in Fig. 11.

5.1.3. Adiabatic-EoS

In this case, the CR entropy across a shock is assumed to be constant (this corresponds to the 'Et+Scr' method discussed in section 3.3), i.e., the downstream CR pressure can be defined as

$$ p_{cr,2} = p_{cr,1} \left( \frac{\rho_2}{\rho_1} \right)^{\gamma_c} \to p_{cr,2} = p_{cr,1} R^{\gamma_c} \tag{39} $$

This EoS is sometimes used to find an analytical solution of a shock tube problem (see e.g., Pfrommer et al. [21]; Kudoh & Hanawa [15]). However, as discussed previously, this is only one of the possible EoSs. To find the compression ratio, we substitute $p_{cr,2}$ in Eq. (32) and obtain $p_{cr,2}$ as a function of $R$, $p_{cr,1}$, and $M_1$. Replacing this $p_{cr,2}$ in Eq. (33) we obtain

$$ R^{\gamma_c+1} A - R^2 B + RC + D = 0, \tag{40} $$

where

$$ A = \left( \frac{\gamma_c}{\gamma_c - 1} - \frac{\gamma_g}{\gamma_g - 1} \right) p_{cr,1} \tag{41} $$

$$ B = \left( \frac{\gamma_g}{\gamma_c} + \gamma_c p_{cr,1} \right) \left( \frac{M_1^2}{2} + \frac{\gamma_g}{\gamma_g - 1} + \frac{\gamma_c}{\gamma_c - 1} p_{cr,1} \right) \tag{42} $$

$$ C = \frac{\gamma_g}{\gamma_g - 1} \left( M_1^2 (\gamma_g + \gamma_c p_{cr,1}) + 1 + p_{cr,1} \right), \tag{43} $$

$$ D = M_1^2 (\gamma_g + \gamma_c p_{cr,1}) \left( \frac{1}{2} - \frac{\gamma_g}{\gamma_g - 1} \right). \tag{44} $$
Figure 11: Dependence of $w_{cr}$ (Eq. 35) on the upstream Mach numbers and $\epsilon_{cr}$ (Eq. 37). Each symbol in left and middle panels represents solution of $w_{cr}$ for a given $\epsilon_{cr}$, where the colours represent the upstream CR ($M_{cr,1}$) and gas ($M_{g,1}$) Mach numbers respectively. These two panels show that for a high Mach number flow the parameter $w_{cr}$ mainly depends on $\epsilon_{cr}$. The right panel shows the dependence of $w_{cr}$ on $\epsilon_{cr}$. The black solid curve shows the analytic solution (Eq. 38).

Figure 12: Solution for an adiabatic EoS. [Left] Compression ratio ($R$) as a function of upstream gas ($M_{g,1}$) and CR ($M_{cr,1}$) Mach numbers, which shows that the compression ratio $\leq 4$. [Right] Dependences of $w_{cr}$ on the upstream Mach numbers. It shows that if the upstream CR pressure is very small compared to the upstream ram pressure (i.e., when $M_{cr,1} \gg 1$) then the post-shock CR pressure is almost negligible compared to gas pressure. For a large upstream CR pressure, the value of $w_{cr} \rightarrow 1$, i.e., the downstream is mostly dominated by CR pressure. However, even in this regime, the compression ratio $\leq 4$, which can be seen by comparing the left and right panels. These results are easy to understand intuitively because CRs are adiabatic and the density jump is bounded by 4.

5.2. Shock detection

Shock is defined as a surface where a sharp transition between upstream and downstream occurs. However, in numerical simulations, a shock is often broadened over several grid zones due to non-negligible numerical viscosity at the shock. Therefore it is important to discuss how one can detect shocked zones and implement different EoSs introduced in the previous section.
For shock detection, we use the following three conditions:

1. Compressibility: $\nabla \cdot \mathbf{v} < 0$.
2. Pressure jump: $(\nabla p_e \cdot \Delta x) / \rho_{min} \geq \delta_{\text{threshold}}$, and
3. Bypassing spurious waves at contact discontinuity which may fulfil conditions (1) and (2): $\nabla T \cdot \nabla \rho > 0$,

where $T$ is the fluid temperature (see e.g. section 3.1.1 in Pfrommer et al. [22]; also see Appendix B in Mignone et al. [19]). We find that the choice $\delta_{\text{threshold}} = 0.5 – 1$ works quite well for the test problems discussed in this work. An example is already shown in the top panel of Fig. 9. The shock zones are spread over 3 – 4 grid zones and for each grid zone, $p_e$ and $p_{cr}$ will not be the same. While using a parameter $w_{cr}$ to fix the post-shock CR pressure, we notice that the post-shock conditions mainly depend on the total pressure of the shocked-zone nearest to the downstream.

5.3. Broader implications

A physically faithful value of $w_{cr}$ (or an equivalent parameter; e.g., see section 5.1.2) depends on upstream parameters (e.g., Mach number and magnetic field orientation w.r.t. shock normal) and it needs to be prescribed using Particle-In-Cell simulation (see e.g., Caprioli & Spitkovsky 4). In MHD, the injection of CRs will also depend on the nature of the shock (e.g., fast versus slow) and detailed calibration with kinetic plasma simulations will be needed. This will be addressed in a future work.

Analytic work on two-fluid shocks (Drury & Voelk 7, Becker & Kazanas 2) have considered the impact of CR diffusion on the shock structure. A non-negligible CR diffusion implies that the CR pressure (unlike the gas pressure, in general) is continuous across the shock transition. Two possible shocks are possible in this case: a discontinuous, gas-mediated sub-shock with CRs diffusing ahead of the shock; and a CR-dominated smooth “shock” (across which the gas properties vary smoothly) if the gas Mach number is large, $\geq 12$ (see Fig. 2 in Becker & Kazanas 2; see the right panels in Fig. 5 of Gupta et al. 10 for an astrophysical realization in a wind-driven shock). In latter, most of the upstream kinetic energy flux goes into CR acceleration rather than heating the thermal plasma, and the astrophysical implications of this are enormous (see e.g., Gupta et al. 11). However, the physical existence of such shocks is yet to be established by kinetic plasma simulations. Nonetheless, we expect our method to capture these two types of shocks in presence of CR diffusion. A detailed investigation is left for future.

In this paper we focus on the closure problem in two-fluid equations at shocks but there is a similar problem at other dissipative structures (that can arise in even highly subsonic flows) such as reconnection sheets and turbulent eddies at small scales. One is faced with an analogous problem of how to partition the dissipated energy among the thermal plasma and CRs (or electrons and ions for an electron-ion two fluid system). This motivates subgrid models based on kinetic simulations of such dissipative regions. There is no such ambiguity for the standard hydrodynamic equations because the energy lost from the fields/flows (say due to numerical averaging in simulations without explicit dissipation) appears as thermal energy in conservative evolution. Since most CRs are believed to be accelerated at shocks, we can start with subgrid models of CR injection at shocks before bothering about other dissipative structures.

6. Summary

The two-fluid CR-HD model suffers from a closure problem. It contains three conservation laws and one additional equation (for the CR pressure) which cannot be cast in a satisfactory conservative form and causes difficulties in its numerical implementation. A unique shock jump condition is possible only if one makes an additional assumption. The steady-state shock structures can be predicted by assuming a suitable downstream CR pressure/energy (as discussed in section 5.1). There is a degeneracy between the gas and CR pressures (see last paragraph of section 2) because of which the solutions may depend on numerics. Without fixing the fraction of upstream energy transformed into CR energy and simply relying on the numerical discretization of the non-conservative exchange term involving derivatives $p_e \nabla \cdot \mathbf{v}$ (or equivalently, $\mathbf{v} \cdot \nabla p_e$), make the solutions of the two-fluid equations across shocks depend on the details of numerics (e.g., spatial reconstruction, time-stepping, even the CFL number). In this work, we have investigated numerical implementation of the two-fluid CR-HD equations. Our findings are summarized as follows:

- Numerical solution of the two-fluid equations depends on implementation of the coupling term ($p_e \nabla \cdot \mathbf{v}$ or $\mathbf{v} \cdot \nabla p_e$). We show that the different discretizations do not show an identical solution (Figs. 4 and 5). This is because of the non-negligible and non-unique contribution of the source term involving a derivative at the shock.

- We suggest a method (“Et+Ecr (Unsplit-pdv)”, section 3) for which the solutions are robust to the choice of spatial reconstruction, time-stepping, and the CFL number (see e.g., Fig. 6). In order to ensure that the characteristic speed remains within the fastest signal propagation speed, this method demands a slightly higher signal speed than the standard estimate of the two-fluid sound speed (Fig. 8).

- We show that all methods give an identical solution only when the CR pressure across the shock is fixed by an imposed equation of state. This can be done, for example, by specifying CR pressure in the shocked zones (as done in Fig. 5; also see section 5.1). A physically realistic implementation of the CR pressure across a shock is possible by calibrating with kinetic simulations using different upstream parameters.

In summary, this work highlights the critical aspects of the two-fluid CR-HD equations. Although here we have not discussed the CR-MHD system, these problems are also present there. The implementation of CR-MHD will be discussed in a future work.
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Appendix A. Eigenvalues without the coupling term

To obtain the HLL flux at left- and right-interface of the computational cell, we need to provide an estimate of the signal speed to the solver. Since in the operator splitting method the solver does not have any information of the $p_{cr} \nabla \cdot \mathbf{v}$ (or $\mathbf{v} \cdot \nabla p_{cr}$) coupling term, the signal speed can be different from the actual speed of the complete system. This is illustrated as follows.

Consider that the source term in Eq. (3), $p_{cr} \nabla \cdot \mathbf{v}$, is absent. In 1D Cartesian geometry, the Jacobian matrix without the coupling and diffusion terms is found to be

$$
\begin{pmatrix}
0 & 1 & 0 \\
\frac{1}{2}(\gamma_g - 3)v^2 & (3 - \gamma_g)v & (\gamma_g - 1)(\gamma_{cr} - \gamma_g) \\
\frac{1}{2}(\gamma_g - 3)v^2 - v(\frac{\alpha_1^2}{\gamma_{cr} - 1} + \frac{\alpha_1^2}{\gamma_{cr} + 1}) & \frac{1}{2}(3 - 2\gamma_g)v^2 + (\frac{\alpha_1^2}{\gamma_{cr} - 1} + \frac{\alpha_1^2}{\gamma_{cr} + 1}) & \gamma_g v \\
-\frac{\alpha_1^2}{\gamma_{cr} - 1} & 0 & v
\end{pmatrix}
$$

(EA.1)

Eigenvalues of this matrix are $\lambda = v - a_{eff}, v, v + a_{eff}$ where $a_{eff} = (\alpha_1^2 + (\gamma_g/\gamma_{cr})\alpha_0^2)^{1/2}$. The methods involving the pdv source term give the same set of eigenvalues. As $\gamma_g > \gamma_{cr}$ we note that the effective propagation speed $a_{eff}$ is slightly larger than the actual sound speed of the composite fluid, which is $(\alpha_0^2 + \alpha_1^2)^{1/2}$. Without using the effective sound speed as an estimate of the maximum signal propagation speed, one may find spurious oscillations in the solution. Such an example has been discussed in section 4.3. The effective sound speed $(a_{eff})$ with vdp splitting is also different from $(\alpha_0^2 + \alpha_1^2)^{1/2}$, and we advice caution even in this case for the signal speed estimate.

Appendix B. Implementation of the coupling term

Other than for the Et+Scr method, the coupling term has to be implemented in the two-fluid equations. In order to implement this term, we have to choose $p_{cr}$ and $v$ at the cell center/interf ace, which can be chosen in various ways. We find that all of these choices do not produce a unique/consistent result. To explicitly show this, we run the shock tube A (problem 1 in Table 1) with seven possible combinations of $p_{cr}$ and $v$ in Eq. (18). These choices are given below.

Possibility 1:

$$
v_{i\to i+\frac{1}{2}} = \frac{1}{2}(v_{i-1}^{n} + v_{i}^{n} + v_{i+1}^{n} + v_{i+2}^{n}) \quad ; \quad v_{i\to i+1} = \frac{1}{2}(v_{i}^{n} + v_{i+1}^{n} + v_{i+2}^{n} + v_{i+3}^{n}) \quad ; \quad p_{cr,i} = p_{cr,i+\frac{1}{2}} \quad \text{when} \quad v_{i}^{n} \geq 0 \\
v_{i\to \frac{1}{2}-} = \text{same} ; \quad v_{i\to i+\frac{1}{2}} = \text{same} ; \quad p_{cr,i} = p_{cr,i+\frac{1}{2}} ; \quad v_{i\to \frac{1}{2}+} = 0 \quad \text{if} \quad v_{i}^{n} < 0
$$

Possibility 2:

$$
v_{i \to i+\frac{1}{2}} = \frac{1}{2}(v_{i-1}^{n} + v_{i}^{n} + v_{i+1}^{n} + v_{i+2}^{n}) \quad ; \quad v_{i \to i+1} = \frac{1}{2}(v_{i}^{n} + v_{i+1}^{n} + v_{i+2}^{n} + v_{i+3}^{n}) ; \quad p_{cr,i} = \frac{1}{2}(p_{cr,i-1} + p_{cr,i+1} + p_{cr,i+2} + p_{cr,i+3}) \quad ; \quad v_{i}^{n} \geq 0 \\
v_{i \to i+\frac{1}{2} +} = \text{same} ; \quad v_{i \to i+\frac{1}{2} -} = \text{same} ; \quad p_{cr,i} = \frac{1}{2}(p_{cr,i-1} + p_{cr,i+1} + p_{cr,i+2} + p_{cr,i+3}) ; \quad v_{i}^{n} < 0
$$

Possibility 3:

$$
v_{i \to i+\frac{1}{2}} = \frac{1}{2}(v_{i-1}^{n} + v_{i}^{n} + v_{i+1}^{n}) \quad ; \quad v_{i \to i+1} = \frac{1}{2}(v_{i}^{n} + v_{i+1}^{n} + v_{i+2}^{n}) ; \quad p_{cr,i} = \frac{1}{2}(p_{cr,i-1} + p_{cr,i+1}) ; \quad v_{i}^{n} \geq 0 \\
v_{i \to i+\frac{1}{2} +} = \text{same} ; \quad v_{i \to i+\frac{1}{2} -} = \text{same} ; \quad p_{cr,i} = \frac{1}{2}(p_{cr,i-1} + p_{cr,i+1}) ; \quad v_{i}^{n} < 0
$$

Possibility 4:

$$
v_{i \to i+\frac{1}{2}} = \frac{1}{2}(v_{i-1}^{n} + v_{i}^{n} + v_{i+1}^{n}) \quad ; \quad v_{i \to i+1} = \frac{1}{2}(v_{i}^{n} + v_{i+1}^{n} + v_{i+2}^{n}) ; \quad p_{cr,i} = \frac{1}{2}(p_{cr,i-1} + p_{cr,i+1}) ; \quad v_{i}^{n} \geq 0 \\
v_{i \to i+\frac{1}{2} +} = \text{same} ; \quad v_{i \to i+\frac{1}{2} -} = \text{same} ; \quad p_{cr,i} = \frac{1}{2}(p_{cr,i-1} + p_{cr,i+1}) ; \quad v_{i}^{n} < 0
$$

Possibility 5:

$$
v_{i \to i+\frac{1}{2}} = \frac{1}{2}(v_{i-1}^{n} + v_{i}^{n} + v_{i+1}^{n}) \quad ; \quad v_{i \to i+1} = \frac{1}{2}(v_{i}^{n} + v_{i+1}^{n} + v_{i+2}^{n}) ; \quad p_{cr,i} = \frac{1}{2}(p_{cr,i-1} + p_{cr,i+1}) ; \quad v_{i}^{n} \geq 0 \\
v_{i \to i+\frac{1}{2} +} = \text{same} ; \quad v_{i \to i+\frac{1}{2} -} = \text{same} ; \quad p_{cr,i} = \frac{1}{2}(p_{cr,i-1} + p_{cr,i+1}) ; \quad v_{i}^{n} < 0
$$

Possibility 6:

$$
v_{i \to i+\frac{1}{2}} = \frac{q_{i\to i+\frac{1}{2}}^{n+1} - q_{i\to i-\frac{1}{2}}^{n+1}}{q_{i\to i-\frac{1}{2}}^{n+1}} ; \quad v_{i \to i+1} = \frac{q_{i+\frac{1}{2}\to i+1}^{n+1} - q_{i+\frac{1}{2}\to i+\frac{1}{2}}^{n+1}}{q_{i+\frac{1}{2}\to i+\frac{1}{2}}^{n+1}} ; \quad p_{cr,i} = \frac{1}{2}(p_{cr,i-1} + p_{cr,i+1}) ; \quad v_{i}^{n} \geq 0 \\
v_{i \to i+\frac{1}{2} +} = \text{same} ; \quad v_{i \to i+\frac{1}{2} -} = \text{same} ; \quad p_{cr,i} = \frac{1}{2}(p_{cr,i-1} + p_{cr,i+1}) ; \quad v_{i}^{n} < 0
$$

Possibility 7:

$$
v_{i \to i+\frac{1}{2}} = \frac{q_{i\to i+\frac{1}{2}}^{n+1} - q_{i\to i-\frac{1}{2}}^{n+1}}{q_{i\to i-\frac{1}{2}}^{n+1}} ; \quad v_{i \to i+1} = \frac{q_{i+\frac{1}{2}\to i+1}^{n+1} - q_{i+\frac{1}{2}\to i+\frac{1}{2}}^{n+1}}{q_{i+\frac{1}{2}\to i+\frac{1}{2}}^{n+1}} ; \quad p_{cr,i} = \frac{1}{2}(p_{cr,i-1} + p_{cr,i+1}) ; \quad v_{i}^{n} \geq 0 \\
v_{i \to i+\frac{1}{2} +} = \text{same} ; \quad v_{i \to i+\frac{1}{2} -} = \text{same} ; \quad p_{cr,i} = \frac{1}{2}(p_{cr,i-1} + p_{cr,i+1}) ; \quad v_{i}^{n} < 0
$$
This work

Figure B.13: Two-fluid shock tube A (problem 1 in Table 1). The dashed black curves show profiles at \( t = 0 \) and the solid curves show profiles at \( t = 0.1 \). For all runs, the used method is “Unsplit-pdv”, however, the choice of \( v \) and \( p_{cr} \) are different (see the Possibilities 1 – 7 described in Appendix B). The solutions are obtained using a 1st order numerical scheme, where \( N_x = 5000 \) and \( C_a = 0.6 \). The solid black curve shows the results from our implementation, i.e., \( v \) and \( p_{cr} \) are chosen as given by Eqs. (19) and (20) respectively. The figures show that the solution in the post-shock region depends on the choice of the numerical method. In Fig. B.14, we show that the solution depends on the CFL number for most of the methods.

Figure B.14: Same as Fig. B.13 with a different CFL number \( C_a = 0.1 \). The colour code is the same as in Fig. B.13. A comparison between the same colour curves in Figs. B.13 and B.14 shows that for most choices of \( p_{cr} \) and \( v \), which are required to implement the coupling term, the solution depends on the CFL number. For our preferred method, shown with solid black lines, solutions are robust to the choice of \( p_{cr} \) and \( v \) given above. This experiment shows that our choice of \( v \) and \( p_{cr} \) (i.e., Eqs. 19 and 20) provides a robust result (black solid curves), i.e., solutions are robust to the choice of spatial reconstructions, time stepping, (as shown in Fig. 6) and the CFL number (compare black solid curves in Figs. B.13 and B.14).

Note that the possible choices are not limited to the above seven combinations. Figs. B.13 and B.14 show the solutions for two different CFL numbers 0.6 and 0.1 respectively. Each colour in these figures represents different possible combinations of \( p_{cr} \) and \( v \) given above. This experiment shows that our choice of \( v \) and \( p_{cr} \) (i.e., Eqs. 19 and 20) provides a robust result (black solid curves), i.e., solutions are robust to the choice of spatial reconstructions, time stepping, (as shown in Fig. 6) and the CFL number (compare black solid curves in Figs. B.13 and B.14).
Appendix C. Details of various EoSs

Here we present the detailed mathematical formulae related to sections 5.1.1 and 5.1.2.

Appendix C.1. \( w_{cr}\)-EoS

The quadratic equation for the compression factor in section 5.1.1 is

\[
R^2 A - RB + C = 0,
\]

where

\[
A = \left\{ \frac{\gamma_g + \gamma_{cr}P_{cr,1}}{2} + \frac{\gamma_g}{\gamma_g - 1} + \frac{\gamma_{cr}M_1}{\gamma_{cr} - 1} \right\},
\]

\[
B = \left\{ \frac{\gamma_g}{\gamma_g - 1} (1 - w_{cr}) + \frac{\gamma_{cr}}{\gamma_{cr} - 1} w_{cr} \right\} \left\{ (\gamma_g + \gamma_{cr}P_{cr,1})M_1^2 + (1 + P_{cr,1}) \right\},
\]

\[
C = \left\{ \frac{\gamma_g}{\gamma_g - 1} (1 - w_{cr}) + \frac{\gamma_{cr}}{\gamma_{cr} - 1} w_{cr} + \frac{1}{2} \right\} \left\{ (\gamma_g + \gamma_{cr}P_{cr,1})M_1^2 \right\}.
\]

Here \( w_{cr} \) is a parameter specified in the downstream, and \( P_{cr,1} \) (Eq. 34) and \( M_1(M_{cr,1}, M_{cr,1}) \) are upstream parameters. The solutions of Eq. (C.1) is given by

\[
R_g = \frac{B \pm (B^2 - 4AC)^{1/2}}{2A},
\]

Since \( R_g \leq 1 \), \( R_g \) represents the physical solution.

Appendix C.2. \( \epsilon_{cr}\)-EoS

To obtain a solution for the compression factor in section 5.1.2, we normalize the gas and CR pressures of Eq. (37) w.r.t. the upstream gas pressure (as done previously) and obtain

\[
P_{cr,2} = \epsilon_{cr} \left\{ \frac{\gamma_g\gamma_{cr} - 1}{\gamma_g\gamma_{cr}} \right\} \frac{\gamma_{cr}}{\gamma_g} \frac{1}{\gamma_{cr} P_{cr,1}} \left\{ \frac{\gamma_g}{\gamma_g - 1} + \frac{\gamma_{cr}}{\gamma_{cr} - 1} P_{cr,1} \right\}.
\]

Using Eq. (C.6), we substitute \( P_{cr,2} \) from Eq. (32) and (33) to again get a quadratic form

\[
R^2 A - RB + C = 0
\]

where

\[
A = (\gamma_g + \gamma_{cr}P_{cr,1})M_1^2 \left\{ \frac{\gamma_g - 1}{\gamma_g\gamma_{cr}} - \frac{\gamma_{cr} - 1}{\gamma_g\gamma_{cr}} \right\} + 1 + \epsilon_{cr} \left\{ \frac{\gamma_g - 1}{\gamma_g\gamma_{cr}} - \frac{\gamma_{cr} - 1}{\gamma_g\gamma_{cr}} \right\} + P_{cr,1} \left\{ (1 - \epsilon_{cr}) \left\{ \frac{\gamma_g - 1}{\gamma_g\gamma_{cr}} - \frac{\gamma_{cr} - 1}{\gamma_g\gamma_{cr}} \right\} \right\}
\]

\[
B = (\gamma_g + \gamma_{cr}P_{cr,1})M_1^2 + (1 + P_{cr,1})
\]

\[
C = (\gamma_g + \gamma_{cr}P_{cr,1})M_1^2 \left\{ \frac{\gamma_g + 1}{2\gamma_g} \right\}
\]

Therefore, the solutions can be obtained using Eq. (C.5).

Appendix C.3. Relation between \( w_{cr} \) and \( \epsilon_{cr} \)

The relation between \( w_{cr}\)-EoS and \( \epsilon_{cr}\)-EoS is obtained as follows. From Eq. (35), we note that the value of \( w_{cr} \) depends on the downstream CR and gas pressures. The CR pressure \( (P_{cr,2}) \) is already provided in Eq. (C.6). The remaining quantity, \( P_{cr,2} \), can be obtained using momentum conservation equation, Eq. (32), which gives

\[
P_{cr,2} = \left\{ (\gamma_g + \gamma_{cr}P_{cr,1})M_1^2 \left\{ \frac{1}{\gamma_g} - \frac{1}{\gamma_{cr}} \right\} + 1 + P_{cr,1} \right\} - P_{cr,2}.
\]

It is worth mentioning that the only constrain on the above equation is \( P_{cr,2} > 0 \), implying that the downstream CR pressure \( P_{cr,2} < \left\{ (\gamma_g + \gamma_{cr}P_{cr,1})M_1^2 \left\{ \frac{1}{\gamma_g} - \frac{1}{\gamma_{cr}} \right\} + 1 + P_{cr,1} \right\} \). From the values of \( P_{cr,2} \) and \( P_{cr,2} \), one can obtain \( w_{cr} \) as a function of \( R, M_1, P_{cr,1}, \) and \( \epsilon_{cr} \) where \( R(M_1, P_{cr,1}, \epsilon_{cr}) \) can be calculated from the solution of Eq. (C.7). Therefore, the parameter \( w_{cr} \) becomes a function of \( M_{g,1}, M_{cr,1}, \) and \( \epsilon_{cr} \). In the limit \( M_1 \to \infty \), we find

\[
w_{cr} = \frac{\epsilon_{cr}^2}{8} \frac{R_s(w_{cr})}{1 - \frac{1}{R_s(w_{cr})}} \to w_{cr,\epsilon} = \frac{4\left(-3(1 - \epsilon_{cr}) \pm (9 + 6\epsilon_{cr})^{1/2}\right)}{3(8 - 3\epsilon_{cr})},
\]

where we have taken \( R_s \) from Eq. (36). Note that, the constrain \( 0 \leq w_{cr} \leq 1 \) implies that \( w_{cr,\epsilon} \) is the physical solution (shown by the black curve in the rightmost panel of Fig. 11).