ON THE RANDERS METRICS ON TWO-STEP HOMOGENEOUS NILMANIFOLDS OF DIMENSION FIVE

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Abstract. In this paper we study the geometry of simply connected two-step nilpotent Lie groups of dimension five. We give the Levi-Civita connection, curvature tensor, sectional and scalar curvatures of these spaces and show that they have constant negative scalar curvature. Also we show that the only space which admits left invariant Randers metric of Berwald type has three dimensional center. In this case the explicit formula for computing flag curvature is obtained and it is shown that flag curvature and sectional curvature have the same sign.

1. Introduction

A connected Riemannian manifold which admits a transitive nilpotent Lie group $N$ of isometries is called a nilmanifold. E. Wilson showed that for a given homogeneous nilmanifold $M$, there exists a unique nilpotent Lie subgroup $N$ of $I(M)$ acting simply transitively on $M$, and $N$ is normal in $I(M)$ (see [20]). Therefore the Riemannian manifold $M$ will be identified with the Lie group $N$ equipped with a left-invariant Riemannian metric $g$.

Between nilpotent Lie groups the family of two-step nilpotent Lie groups endowed with left invariant Riemannian metrics are studied specially in the recent years (see [5], [6] and [9]). J. Lauret classified all homogeneous nilmanifolds of dimension three and four in [10]. Also simply connected two-step nilpotent Lie groups of dimension five equipped with left invariant Riemannian metrics are classified by S. Homolya and O. Kowalski in [9]. For this reason they classified metric Lie algebras with one, two and three dimensional center. We use their results in this article.

On the other hand studying invariant Finsler metrics on Lie groups and homogeneous spaces developed in these years, (for example see [2, 3, 4, 7, 8, 12, 13, 14, 15, 16, 17, 19]). A. Tóth and Z. Kovács studied the geometry of some special two-Step nilpotent groups with left invariant Finsler metrics in [19]. In the present paper we study the geometry of simply connected two-step nilpotent Lie groups of dimension five endowed with left invariant Riemannian metrics. Then we discuss the existence of left invariant Randers metrics of Berwald type (an interesting Finsler metric which have many applications in physics) on these spaces. Finally we give the explicit formula for computing flag curvature of these metrics and show that the
flag curvature and sectional curvature of the base left invariant Riemannian metric have the same sign.

2. Preliminaries

In this section we give some preliminaries about invariant (Riemannian and Finsler) metrics. A Riemannian metric $g$ on a Lie group $G$ is called left invariant if

$$g(a)(Y, Z) = g(e)(T_a l_{a^{-1}} Y, T_a l_{a^{-1}} Z), \quad \forall a \in G, \forall Y, Z \in T_a G,$$

where $e$ is the unit element of $G$.

For a Lie group $G$ equipped with a left invariant Riemannian metric $g$ the Levi-Civita connection is defined by the following formula:

$$2 < \nabla_U V, W > = < [U, V], W > - < [V, W], U > + < [W, U], V >,$$

for any $U, V, W \in g$, where $g$ is the Lie algebra of $G$ and $<, >$ is the inner product induced by $g$ on $g$.

Now we can generalize this definition to Finsler manifolds. A Finsler metric on a manifold $M$ is a non-negative function $F :TM \rightarrow \mathbb{R}$ with the following properties:

1. $F$ is smooth on the slit tangent bundle $TM^0 := TM \setminus \{0\}$,
2. $F(x, \lambda Y) = \lambda F(x, Y)$ for any $x \in M, Y \in T_x M$ and $\lambda > 0$,
3. the $n \times n$ Hessian matrix $[g_{ij}] = [\frac{1}{2} \frac{\partial^2 F^2}{\partial y_i \partial y_j}]$ is positive definite at every point $(x, Y) \in TM^0$.

A special type of Finsler metrics are Randers metrics which have been introduced by G. Randers [11] in his research on general relativity. Randers metrics are constructed on Riemannian metrics and vector fields (1-forms).

Let $g$ and $X$ be a Riemannian metric and a vector field on a manifold $M$ respectively such that $\|X\| = \sqrt{g(X, X)} < 1$. Then a Randers metric $F$, defined by $g$ and $X$, is a Finsler metric as follows:

$$F(x, Y) = \sqrt{g(x)(Y, Y) + g(x)(X(x), Y)}, \quad \forall x \in M, Y \in T_x M.$$

Similar to the Riemannian case, a Finsler metric $F$ on a Lie group $G$ is called left invariant if

$$F(a, Y) = F(e, T_a l_{a^{-1}} Y), \quad \forall a \in G, Y \in T_a G.$$

A special family of Randers metrics (or in general case Finsler metrics) is the family of Berwaldian Randers metrics. A Randers metric of the form 2.3 is of Berwald type if and only if the vector field $X$ is parallel with respect to the Levi-Civita connection of $g$. In these metrics the Chern connection of the Randers metric $F$ coincide with the Levi-Civita connection of the Riemannian metric $g$.

One of the important quantities which associates with a Finsler manifold is flag curvature
which is a generalization of sectional curvature to Finsler manifolds. Flag curvature is defined as follows.

\[ K(P,Y) = \frac{g_Y(R(U,Y)Y,U)}{g_Y(Y,Y)g_Y(U,U) - g_Y^2(Y,U)}, \]

where \( g_Y(U,V) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} (F^2(Y + sU + tV)) \big|_{s=t=0}, \) \( P = \text{span}\{U,Y\}, \) \( R(U,Y)Y = \nabla_U \nabla_Y Y - \nabla_Y \nabla_U Y - \nabla_{[U,Y]} Y \) and \( \nabla \) is the Chern connection induced by \( F \) (see [1] and [18]).

From now we consider \( N \) is a simply connected two-step nilpotent Lie group of dimension five and \( \mathfrak{n} \) is its Lie algebra.

3. Lie algebras with 1-dimensional center

We can use left invariant Riemannian metrics and left invariant vector fields for constructing left invariant Randers metrics on Lie groups.

Suppose that \( G \) is Lie group, \( g \) is a left invariant Riemannian metric and \( X \) is a left invariant vector field on \( G \) such that \( \sqrt{g(X,X)} < 1 \). Then we can define a left invariant Riemannian metric on \( G \) by using formula (2.3). In fact we have the following proposition on these metrics.

**Proposition 3.1.** There is a one-to-one correspondence between the invariant Randers metrics on the Lie group \( G \) with the underlying Riemannian metric \( g \) and the left invariant vector fields with length \( < 1 \). Therefore the invariant Randers metrics are one-to-one corresponding to the set

\[ V = \{X \in \mathfrak{g}| < X, X > < 1\}. \]

**Proof.** It is suffix to let \( H = \{e\} \) in theorem 2.2 of [3]. \( \square \)

In this section we consider the Lie algebra \( \mathfrak{n} \) has 1−dimensional center. In [9] S. Homolya and O. Kowalski showed that there exist an orthonormal basis \( \{e_1, e_2, e_3, e_4, e_5\} \) of \( \mathfrak{n} \) such that

\[ [e_1, e_2] = \lambda e_5, \quad [e_3, e_4] = \mu e_5, \]

where \( \{e_5\} \) is a basis for the center of \( \mathfrak{n} \), and \( \lambda \geq \mu > 0 \). Also it is considered that the other commutators are zero.

By using the equation (2.2) for the Levi-Civita connection we have:
By using table 3.4 and above equations we have

\[
\begin{align*}
\{ A, B \} & \quad \text{be an orthonormal basis for a two dimensional subspace of } T_eN, \quad \text{where } e \text{ is the unit element of } N, \text{ as follows:} \\
A & = a e_1 + be_2 + ce_3 + de_4 + fe_5 \\
B & = a e_1 - b e_2 - e_3 + d e_4 + f e_5
\end{align*}
\]

(3.5)

By using table 3.4 and above equations we have

\[
R(B, A)A = (b\tilde{a} - a\tilde{b})(\frac{3\lambda^2}{4}(ae_2 - be_1) + \frac{\lambda\mu}{2}(ce_4 - de_3)) \\
+ (d\tilde{c} - c\tilde{d})(\frac{\lambda\mu}{2}(ae_2 - be_1) + \frac{3\mu^2}{4}(ce_4 - de_3)) \\
+ \frac{\lambda^2}{4}\{(f\tilde{b} - b\tilde{f})(fe_2 - be_5) + (f\tilde{a} - a\tilde{f})(fe_1 - ae_5)\} \\
+ \frac{\mu^2}{4}\{(f\tilde{c} - c\tilde{f})(fe_3 - ce_5) + (f\tilde{d} - d\tilde{f})(fe_4 - de_5)\} \\
+ \frac{\lambda\mu}{4}\{(c\tilde{a} - a\tilde{c})(be_4 - de_2) + (d\tilde{a} - a\tilde{d})(ce_2 - be_3) \\
+ (b\tilde{c} - c\tilde{b})(de_1 - ae_4) + (d\tilde{b} - b\tilde{d})(ae_3 - ce_1)\}.
\]

(3.6)
Now a direct computation shows that the sectional curvature $K^R$ can be obtained with the following equation.

$$K^R(A, B) = -\frac{3}{4}(\lambda^2(a\tilde{b} - b\tilde{a})^2 + \mu^2(c\tilde{d} - d\tilde{c})^2)$$

$$+ \frac{\lambda^2}{4}\{(f\tilde{a} - a\tilde{f})^2 + (f\tilde{b} - b\tilde{f})^2\}$$

$$+ \frac{\mu^2}{4}\{(f\tilde{c} - c\tilde{f})^2 + (f\tilde{d} - d\tilde{f})^2\}$$

$$+ \frac{3\lambda\mu}{2}(a\tilde{b} - b\tilde{a})(d\tilde{c} - c\tilde{d})$$

This formula shows that the Riemannian Lie group $N$ admits positive and negative sectional curvature. This fact has been proved by J. A. Wolf in [21] which any non-commutative Lie group admits positive and negative sectional curvatures.

Also for any $p \in N$ the scalar curvature is

$$S(p) = -\frac{\lambda^2 + \mu^2}{2} < 0,$$

which shows that this Riemannian manifold is of constant negative scalar curvature.

**Proposition 3.2.** There is not any left invariant Randers metric of Berwald type on simply connected two-step nilpotent Lie groups of dimension five with 1-dimensional center.

**Proof.** Proposition 3.1 says that for an invariant Randers metric we need a left invariant vector field with length $< 1$. On the other hand we know that a Randers metric is of Berwald type if and only if the vector field is parallel with respect to the Levi-Civita connection (see [1].) Let $Q \in \mathfrak{n}$ be a left invariant vector field on $N$ which is parallel with respect to the Levi-Civita connection. By using table 3.3 and a direct computation we have $Q = 0$. \[\square\]

4. Lie algebras with 2-dimensional center

In this section we study simply connected two-step nilpotent Lie groups of dimension five equipped with left-invariant Riemannian metric and 2-dimensional center. Let $N$ be as above. S. Homolya and O. Kowalski in [9] showed that $\mathfrak{n}$ admits an orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\}$ such that

$$[e_1, e_2] = \lambda e_4, \quad [e_1, e_3] = \mu e_5,$$

where $\{e_4, e_5\}$ is a basis for the center of $\mathfrak{n}$, the other commutators are zero and $\lambda \geq \mu > 0$. 
The equation 2.2 and some computations shows that the Levi-Civita connection of this Riemannian manifold is as the following table.

\[
\begin{array}{c|cccc}
 & e_1 & e_2 & e_3 & e_4 & e_5 \\
\hline 
\nabla e_1 & 0 & \frac{\lambda}{2} e_4 & \frac{\mu}{2} e_5 & -\frac{\lambda}{2} e_2 & -\frac{\mu}{2} e_3 \\
\nabla e_2 & -\frac{\lambda}{2} e_4 & 0 & 0 & \frac{\lambda}{2} e_1 & 0 \\
\nabla e_3 & -\frac{\mu}{2} e_5 & 0 & 0 & 0 & \frac{\mu}{2} e_1 \\
\nabla e_4 & -\frac{\lambda}{2} e_2 & \frac{\lambda}{2} e_1 & 0 & 0 & 0 \\
\nabla e_5 & -\frac{\mu}{2} e_3 & 0 & \frac{\mu}{2} e_1 & 0 & 0 \\
\end{array}
\]

\[(4.2)\]

Now by using the Levi-Civita connection we can compute the curvature tensor as follows.

\[
\begin{array}{c|cccc}
 & e_1 & e_2 & e_3 & e_4 & e_5 \\
\hline 
R(e_1, e_2) & 3\frac{\lambda^2}{4} e_2 & -3\frac{\lambda^2}{4} e_1 & 0 & 0 & 0 \\
R(e_1, e_3) & 3\frac{\mu^2}{4} e_3 & 0 & -3\frac{\mu^2}{4} e_1 & 0 & 0 \\
R(e_1, e_4) & -\frac{\lambda^2}{4} e_4 & 0 & 0 & \frac{\lambda^2}{4} e_1 & 0 \\
R(e_1, e_5) & -\frac{\mu^2}{4} e_5 & 0 & 0 & 0 & \frac{\mu^2}{4} e_1 \\
R(e_2, e_3) & 0 & 0 & 0 & \frac{\lambda \mu}{4} e_5 & -\frac{\lambda \mu}{4} e_4 \\
R(e_2, e_4) & 0 & -\frac{\lambda^2}{4} e_4 & 0 & \frac{\lambda^2}{4} e_2 & 0 \\
R(e_2, e_5) & 0 & 0 & -\frac{\lambda \mu}{4} e_4 & \frac{\lambda \mu}{4} e_3 & 0 \\
R(e_3, e_4) & 0 & -\frac{\lambda \mu}{4} e_5 & 0 & 0 & \frac{\lambda \mu}{4} e_2 \\
R(e_3, e_5) & 0 & 0 & -\frac{\mu^2}{4} e_5 & 0 & \frac{\mu^2}{4} e_3 \\
R(e_4, e_5) & 0 & \frac{\lambda \mu}{4} e_3 & -\frac{\lambda \mu}{4} e_2 & 0 & 0 \\
\end{array}
\]

\[(4.3)\]

For \(A\) and \(B\) similar to the equations 3.5 we have

\[
R(B, A)A = \frac{3\lambda^2}{4} (b\tilde{a} - a\tilde{b})(ae_2 - be_1) + \frac{3\mu^2}{4} (c\tilde{a} - a\tilde{c})(ae_3 - ce_1) \\
+ \frac{\lambda^2}{4} \{(d\tilde{a} - a\tilde{d})(de_1 - ae_4) + (d\tilde{b} - b\tilde{d})(de_2 - be_4)\} \\
+ \frac{\mu^2}{4} \{(f\tilde{a} - a\tilde{f})(fe_1 - ae_5) + (f\tilde{c} - c\tilde{f})(fe_3 - ce_5)\} \\
+ \frac{\lambda \mu}{4} \{(c\tilde{b} - b\tilde{c})(de_5 - fe_4) + (f\tilde{b} - b\tilde{f})(de_3 - ce_4) \\
+ (d\tilde{c} - c\tilde{d})(fe_2 - be_5) + (f\tilde{d} - d\tilde{f})(be_3 - ce_2)\}
\]

\[(4.4)\]
This shows that for the plan \( \{A, B\} \) the sectional curvature is of the following form.

\[
K^R(A, B) = -\frac{3}{4}(\lambda^2(b\tilde{a} - a\tilde{b})^2 + \mu^2(c\tilde{a} - a\tilde{c})^2)
+ \frac{\lambda^2}{4}((d\tilde{a} - a\tilde{d})^2 + (d\tilde{b} - b\tilde{d})^2)
+ \frac{\mu^2}{4}((f\tilde{a} - a\tilde{f})^2 + (f\tilde{c} - c\tilde{f})^2)
+ \frac{\lambda\mu}{2}((f\tilde{b} - b\tilde{f})(d\tilde{c} - c\tilde{d}) + (c\tilde{b} - b\tilde{c})(d\tilde{f} - f\tilde{d}))
\]

(4.5)

The scalar curvature of this Riemannian manifold is

\[
S(p) = -\frac{\lambda^2 + \mu^2}{2} < 0.
\]

(4.6)

The last equation shows that simply connected two-step nilpotent Lie groups of dimension five equipped with left-invariant Riemannian metrics with two dimensional center have constant negative scalar curvature.

Also similar to pervious case we have the following proposition.

**Proposition 4.1.** There is not any left invariant Randers metric of Berwald type on simply connected two-step nilpotent Lie groups of dimension five with 2—dimensional center.

### 5. Lie algebras with 3-dimensional center

Now we study simply connected two-step nilpotent Lie groups of dimension five equipped with left-invariant Riemannian metrics with three dimensional center. This case is different from the pervious cases because in this case the Lie group \( N \) admits a family of invariant Randers metrics of Berwald type.

The Lie algebra structure of this case is as follows.

\( n \) admits an orthonormal basis \( \{e_1, e_2, e_3, e_4, e_5\} \) such that for \( \lambda > 0 \)

\[
[e_1, e_2] = \lambda e_3,
\]

(5.1)

where \( \{e_3, e_4, e_5\} \) is a basis for the center of \( n \) and the other commutators are zero (see [9].).

The Levi-Civita connection of the Riemannian manifold can be obtained by the equation (2.2) as the following table.

|      | \( e_1 \) | \( e_2 \) | \( e_3 \) | \( e_4 \) | \( e_5 \) |
|------|---------|---------|---------|---------|---------|
| \( \nabla_{e_1} \) | 0 | \( \frac{\lambda}{2} e_3 \) | \( -\frac{\lambda}{2} e_2 \) | 0 | 0 |
| \( \nabla_{e_2} \) | \( -\frac{\lambda}{2} e_3 \) | 0 | \( \frac{\lambda}{2} e_1 \) | 0 | 0 |
| \( \nabla_{e_3} \) | \( -\frac{\lambda}{2} e_2 \) | \( \frac{\lambda}{2} e_1 \) | 0 | 0 | 0 |
| \( \nabla_{e_4} \) | 0 | 0 | 0 | 0 | 0 |
| \( \nabla_{e_5} \) | 0 | 0 | 0 | 0 | 0 |

(5.2)
Similar to the above cases the curvature tensor can compute as this table.

|       | $e_1$     | $e_2$     | $e_3$     | $e_4$     | $e_5$     |
|-------|-----------|-----------|-----------|-----------|-----------|
| $R(e_1, e_2)$ | $\frac{3\lambda^2}{4} e_2$ | $-\frac{3\lambda^2}{4} e_1$ | 0         | 0         | 0         |
| $R(e_1, e_3)$ | $-\frac{\lambda^2}{4} e_3$ | 0         | $\frac{\lambda^2}{4} e_1$ | 0         | 0         |
| $R(e_1, e_4)$ | 0         | 0         | 0         | 0         | 0         |
| $R(e_1, e_5)$ | 0         | 0         | 0         | 0         | 0         |
| $R(e_2, e_3)$ | 0         | $-\frac{\lambda^2}{4} e_3$ | $\frac{\lambda^2}{4} e_2$ | 0         | 0         |
| $R(e_2, e_4)$ | 0         | 0         | 0         | 0         | 0         |
| $R(e_2, e_5)$ | 0         | 0         | 0         | 0         | 0         |
| $R(e_3, e_4)$ | 0         | 0         | 0         | 0         | 0         |
| $R(e_3, e_5)$ | 0         | 0         | 0         | 0         | 0         |
| $R(e_4, e_5)$ | 0         | 0         | 0         | 0         | 0         |

(5.3)

Also for orthonormal set $\{A, B\}$ we have

\begin{equation}
R(B, A)A = \frac{\lambda^2}{4} \{3(b\tilde{a} - a\tilde{b})(ae_2 - be_1) + (c\tilde{a} - a\tilde{c})(ce_1 - ae_3)
\end{equation}

\begin{equation}
+ (c\tilde{b} - b\tilde{c})(ce_2 - be_3)\}.
\end{equation}

The last equation shows that the sectional curvature of the Riemannian manifold $N$ for the plan $\text{span}\{A, B\}$ is

\begin{equation}
K^R(A, B) = \frac{\lambda^2}{4} \{(c\tilde{a} - a\tilde{c})^2 + (c\tilde{b} - b\tilde{c})^2 - 3(b\tilde{a} - a\tilde{b})^2\}.
\end{equation}

Also by using the orthonormal basis $\{e_1, \cdots, e_5\}$ for any $p \in N$ we have

\begin{equation}
S(p) = -\frac{\lambda^2}{2} < 0.
\end{equation}

The above equation shows that the Riemannian manifold $N$ is of constant negative scalar curvature.

Now we discuss the left invariant Randers metrics of Berwald type on this manifold.

Let $Q \in \mathfrak{n}$ be a left invariant vector field on $N$ which is parallel with respect to the Levi-Civita connection induced by the left invariant Riemannian metric of $N$. By using table 5.2 and a simple computation we have

\begin{equation}
Q = q_1 e_4 + q_2 e_5.
\end{equation}

On the other hand the length of $Q$ must be less than 1, therefore we have $0 < \sqrt{q_1^2 + q_2^2} < 1$. 

\begin{equation}
(5.4)
\end{equation}
Proposition 5.1. Let $F$ be the Randers metric induced by the Riemannian metric $g$ and the left invariant vector field $Q$ as the formula 2.3. Then the flag curvature of the flag $(P, A)$ in $T_e N$ is given by

$$K(P, A) = \frac{K^R(A, B)}{(1 + q_1 d + q_2 f)^2},$$

where $B$ is any vector in $P$ such that $\{A, B\}$ is an orthonormal basis for $P$.

Proof. The Randers metric is of Berwald type therefore the Levi-Civita connection of $g$ and the Chern connection of $F$ coincide. Hence $F$ and $g$ have the same curvature tensor. Now a direct computation shows that

$$g_A(R(B, A)A, B) = <R(B, A)A, B>(1+<Q, A>)+<Q, B><R(B, A)A, Q + A>$$

$$= K^R(A, B)(1 + q_1 d + q_2 f),$$

$$g_A(A, A) = (1 + <Q, A>)^2 = (1 + q_1 d + q_2 f)^2,$$

$$g_A(B, B) = 1 + <Q, B]^2 + <Q, A> = 1 + (q_1 \tilde{d} + q_2 \tilde{f})^2 + q_1 d + q_2 f,$$

and

$$g_A(A, B) = <Q, B>(1 + <Q, A>) = (q_1 \tilde{d} + q_2 \tilde{f})(1 + q_1 d + q_2 f).$$

Therefore by using formula 2.5 the proof is completed.

The last proposition shows that the Finsler manifold $(N, F)$ admits negative, zero and positive flag curvature in any point.

Acknowledgment

This work was supported by the research grant from Shahrood university of technology.

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