Stochastic growth equations on growing domains

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Abstract. The dynamics of linear stochastic growth equations on growing substrates is studied. The substrate is assumed to grow in time following the power law $t^\gamma$, where the growth index $\gamma$ is an arbitrary positive number. Two different regimes are clearly identified: for small $\gamma$ the interface becomes correlated, and the dynamics is dominated by diffusion; for large $\gamma$ the interface stays uncorrelated, and the dynamics is dominated by dilution. In this second regime, for short time intervals and spatial scales the critical exponents corresponding to the non-growing substrate situation are recovered. For long time differences or large spatial scales the situation is different. Large spatial scales show the uncorrelated character of the growing interface. Long time intervals are studied by means of the auto-correlation and persistence exponents. It becomes apparent that dilution is the mechanism by which correlations are propagated in this second case.

Keywords: correlation functions (theory), driven diffusive systems (theory), persistence (theory), kinetic roughening (theory)

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1. Introduction

Fluctuating interfaces have been the object of study of many different works over the last few decades. Together with the interest in possible technological applications that their understanding may bring, for instance in the thin film industry, there is a genuine theoretical interest in unveiling their dynamical properties. This is so to the extent that the Kardar–Parisi–Zhang (KPZ) equation [1], one of the most influential models for surface growth, is being currently considered as a prototypical model of nonequilibrium dynamics.

Usually, stochastic equations modeling surface growth have been studied in static domains. On the other hand, some kinds of growing interfaces, for instance radial ones, present a domain size that grows over time. The direct study of radial interfaces is complicated by nonlinear effects, including the possibility of instabilities affecting the radial symmetry [2]–[4]. This suggests studying first the dynamics of linear stochastic growth equations on growing domains. Once the effect of substrate growth on the interface dynamics is understood, one could move to the more complicated case of radial growth.

Of course, considering linear growth equations has a limited applicability to real physical systems, as important nonlinearities are being neglected. Still, the detailed analysis of linear stochastic growth equations has revealed important physical properties of rough surfaces [5]. Additionally, the study of linear equations has served as a basis for the approach to the more complicated nonlinear ones [6]. In this sense, we expect that the results presented here might provide useful insights into the dynamics of nonlinear equations on growing domains.

Stochastic growth equations for radial interfaces have been previously considered in the literature [2]–[4], [7,8]. In [8], radial interfaces with a domain growing linearly in time are studied, and several dynamical quantities are calculated and compared to the classical values. Under the approximations made in this work, the only genuine radial effect that is being considered is domain growth, while the possible nonlinear effects are disregarded.

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This allows a perfect comparison among the results presented there and the ones that we will introduce here. We will also examine how decorrelation might appear in the growing interface and what is the resulting large scale structure [4], how the classical values of the critical exponents are recovered [9], and in what precise limits this occurs [10].

The goal of this work is to put previous studies considering domains growing linearly in time in a broader context. This will be achieved by proposing an arbitrary power law growth model for the domain. We will focus on rough interfaces, i.e., models for which the growth exponent \( \beta \) is strictly positive. The paper is organized as follows. In section 2 we describe the phenomenology of domain growth. In section 3 we focus on the nonequilibrium dynamics of the Edwards–Wilkinson (EW) equation, and in section 4 we extend these results to the general linear Langevin dynamics. In section 5 the temporal correlations of the fluctuating interface are studied, and in section 6, its persistence properties. In section 7 the connection to radial growth is investigated, and the conclusions of this work are drawn in section 8.

2. Growing domains

In order to study the dynamics of stochastic growth equations on growing domains we begin considering the EW equation [11], which reads

\[
\partial_t h = D \nabla^2 h + F + \xi(y, t),
\]

where \( \xi(y, t) \) is a zero-mean Gaussian white noise whose correlation is

\[
\langle \xi(y, t) \xi(y', t') \rangle = \epsilon \delta(y - y') \delta(t - t'),
\]

\( D \) is the diffusion constant, \( F \) the constant deposition rate and \( \epsilon \) the noise intensity, all these parameters being positive. To derive the EW equation on a growing domain we will follow the theory introduced in [12], focused on reaction–diffusion dynamics on uniformly growing domains. We start by considering the conservation law in integral form:

\[
\frac{d}{dt} \int_{S_t} h(y, t) \, dy = \int_{S_t} [\nabla \cdot j + F(y, t)] \, dy,
\]

where \( S_t \) is the uniformly growing domain, \( j = -D \nabla h \) is the current generated by diffusion, and \( F(y, t) = F + \xi(y, t) \) is the EW growth mechanism. By applying the Reynolds transport theorem we find

\[
\frac{d}{dt} \int_{S_t} h(y, t) \, dy = \int_{S_t} [\partial_t h + \nabla \cdot (vh)] \, dy,
\]

where \( v(y, t) \) denotes the flow velocity generated by the growing domain. Valid as it is for any domain, the integral conservation law may be expressed in the local form

\[
\partial_t h + \nabla \cdot (vh) = D \nabla^2 h + F(y, t).
\]

In this equation we readily identify two new terms, the advection one \( v \cdot \nabla h \), and the dilution one \( h \nabla \cdot v \). For every \( y \in S_t \), that has evolved from \( y_0 \in S_{t_0} \), we find \( v(y, t) = \partial y/\partial t \). Let us now concentrate on one-dimensional substrates and then move to higher dimensionalities. In this case uniform growth translates into \( y = g(t)y_0 \), where
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\( g(t) \) is a temporal function such that \( g(t_0) = 1 \). This yields \( v = \frac{y \dot{g}}{g} \), and thus

\[
\partial_t h + \frac{\dot{g}}{g} (y \partial_y h + h) = D \partial_y^2 h + F + \xi(y, t). \tag{6}
\]

For a one-dimensional substrate \((0, L(t))\), with \( L(t) = g(t)L_0 \), we change the spatial coordinate \( x = yL_0/L(t) \), where \( L_0 = L(t_0) \), in order to map the problem into the interval \((0, L_0)\). This transformation counterbalances advection, and so the resulting equation reads

\[
\frac{\partial h}{\partial t} = \left( \frac{L_0}{L(t)} \right)^2 D \frac{\partial^2 h}{\partial x^2} - \frac{\dot{g}}{g} h + F + \sqrt{\frac{L_0}{L(t)}} \xi(x, t), \tag{7}
\]

where we have used the fact that the noise is delta correlated. The dilution term has become \( h \nabla \cdot \mathbf{v} = -\frac{\dot{g}}{g} h \). It has been disregarded in reaction–diffusion systems due to its irrelevance in this context [12], but we will keep it here, where it will show its measurable effects on the dynamics. Indeed, dilution has a transparent physical meaning: as the substrate grows the deposited material becomes distributed in a larger \((d\text{-dimensional})\) area. This matter redistribution causes in turn the propagation of correlations, additionally to diffusion, resulting in a different dynamical scenario as the following sections will show. Now we assume that the growth function adopts the power law form \( g(t) = \left(\frac{t}{t_0}\right)^\gamma \), where the growth index \( \gamma \geq 0 \), to find

\[
\frac{\partial h}{\partial t} = \left( \frac{t_0}{t} \right)^{2\gamma} D \frac{\partial^2 h}{\partial x^2} - \frac{\gamma}{t} h + F + \left( \frac{t_0}{t} \right)^{\gamma/2} \xi(x, t). \tag{8}
\]

The growth index \( \gamma \) is a new degree of freedom of this problem; it cannot be deduced from the other model parameters, and has to be measured directly from the physical system under study. Our next step is to assume no flux boundary conditions \( \partial_x h(0, t) = \partial_x h(L_0, t) = 0 \), both due to their physical relevance and because they break translation invariance. It will be interesting to see how translation invariance is recovered as a consequence of decorrelation for rapidly growing domains. Let us decompose the solution in the basis formed by the eigenfunctions of the Laplacian on the domain under consideration:

\[
h(x, t) = \sum_{n=0}^{\infty} h_n(t) \cos\left( \frac{n\pi x}{L_0} \right), \tag{9}
\]

to reduce equation (8) to a stochastic differential equation for the different modes

\[
\frac{dh_n}{dt} = -\frac{n^2 \pi^2}{L_0^2} \left( \frac{t_0}{t} \right)^{2\gamma} D h_n - \frac{\gamma}{t} h_n + \left( \frac{t_0}{t} \right)^{\gamma/2} \xi_n(t), \tag{10}
\]

if \( n \neq 0 \), and

\[
\frac{dh_0}{dt} = -\frac{\gamma}{t} h_0 + F + \left( \frac{t_0}{t} \right)^{\gamma/2} \xi_0(t). \tag{11}
\]
In these equations $\xi_n(t)$ is a Gaussian random variable with zero mean and correlation given by

$$\langle \xi_m(t)\xi_n(t') \rangle = \frac{2\epsilon}{L_0} \delta_{mn} \delta(t - t'), \quad \text{if } n, m \neq 0,$$

$$\langle \xi_0(t)\xi_n(t') \rangle = 0, \quad \text{if } n \neq 0,$$

$$\langle \xi_0(t)\xi_0(t') \rangle = \frac{\epsilon}{L_0} \delta(t - t').$$

(12)

(13)

(14)

3. Edwards–Wilkinson dynamics

We can straightforwardly derive the equation of motion for $\langle h_0 \rangle$:

$$\frac{d}{dt} \langle h_0 \rangle = -\frac{\gamma}{t} \langle h_0 \rangle + F,$$

whose long time solution reads

$$\langle h_0 \rangle = \frac{F}{\gamma + 1} t.$$  

(15)

(16)

For the second moment we find

$$\frac{d}{dt} \langle h_0^2 \rangle = -\frac{2\gamma}{t} \langle h_0^2 \rangle + 2F \langle h_0 \rangle + \frac{\epsilon t_0^\gamma}{L_0 t^{\gamma}},$$

and the corresponding long time solution

$$\langle h_0^2 \rangle = \frac{F^2}{(\gamma + 1)^2 t^2} + \frac{et}{(\gamma + 1) L_0 t^{\gamma}},$$

(17)

(18)

where the second summand in the right-hand side of this equation will be explicitly suppressed due to its subleading character, but it will be taken into account implicitly later on in order to construct Dirac delta functions out of infinite series. For the other modes we find

$$\frac{d}{dt} \langle h_n \rangle = -\left( D \frac{n^2 \pi^2 t_0^{2\gamma}}{L_0^2 t^{2\gamma}} + \frac{\gamma}{t} \right) \langle h_n \rangle,$$

(19)

and we integrate it to find, in the long time limit, $\langle h_n \rangle \to 0$. The correlation obeys the equation

$$\frac{d}{dt} \langle h_m h_n \rangle = -\left[ D \left( \frac{m^2 + n^2 \pi^2 t_0^{2\gamma}}{L_0^2 t^{2\gamma}} + \frac{2\gamma}{t} \right) + \frac{2\epsilon t_0^\gamma}{L_0 t^{\gamma}} \delta_{mn} \right] \langle h_m h_n \rangle,$$

(20)

whose solution is readily computable in terms of Misra functions [13]; however its concrete form is complicated and not particularly illuminating, so we will not reproduce it here. The long time asymptotics of this solution depends on the value of $\gamma$. For $\gamma < 1/2$ the

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1 The Misra function is $\varphi_n(x) = \int_1^\infty s^n e^{-sx} \, ds$. 

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interface dynamics is dominated by diffusion, and one finds
\[ \langle h_m h_n \rangle = \frac{2c\delta_{mn}L_0}{D(m^2 + n^2)\pi^2} \left( \frac{t}{t_0} \right)^\gamma. \]

(21)

If \( \gamma > 1/2 \), then the interface dynamics is dominated by dilution, and the asymptotic solution reads
\[ \langle h_m h_n \rangle = \frac{2c\delta_{mn}}{(1 + \gamma)L_0} \left( \frac{t}{t_0} \right)^{1-\gamma}. \]

(22)

Now we can reconstruct the first moments of the solution in coordinate space. For the long time mean value we have
\[ \langle h(x, t) \rangle = \frac{F}{\gamma + 1} t, \]

(23)

and for the long time correlation when \( x \neq x' \)
\[ \langle h(x, t)h(x', t) \rangle = \frac{F^2}{(\gamma + 1)^2} t^2 + \frac{\epsilon L_0}{D\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \left( \frac{n\pi x}{L_0} \right) \cos \left( \frac{n\pi x'}{L_0} \right), \]

(24)

if \( \gamma < 1/2 \), and
\[ \langle h(x, t)h(x', t) \rangle = \frac{F^2}{(\gamma + 1)^2} t^2 + \frac{2c\epsilon t_0}{(\gamma + 1)L_0} \left( \frac{t}{t_0} \right)^{1-\gamma} \sum_{n=1}^{\infty} \cos \left( \frac{n\pi x}{L_0} \right) \cos \left( \frac{n\pi x'}{L_0} \right), \]

(25)

if \( \gamma > 1/2 \) and asymptotically for long times, where we have used the decomposition of the Dirac delta function in the same basis as in equation (9) and we have implicitly taken into account the subleading term in equation (18). Changing back to the original Lagrangian coordinates, \( y = xL(t)/L_0 \), we find
\[ \langle h(y, t)h(y', t) \rangle = \frac{F^2}{(\gamma + 1)^2} t^2 + \frac{c\epsilon}{\gamma + 1} \delta(y - y'); \]

(26)

the solution reduces to random deposition for long times. We thus see that the surface stays uncorrelated if \( \gamma > 1/2 \), while it becomes correlated when \( \gamma < 1/2 \). We can further analyze the correlated phase summing equation (24) to find
\[ \langle h(x, t)h(x', t) \rangle = \frac{F^2}{(\gamma + 1)^2} t^2 + \frac{c\epsilon}{12DL_0} \left( \frac{t}{t_0} \right)^\gamma \left[ 2L_0^2 - 6L_0 \max(x, x') + 3(x^2 + x'^2) \right], \]

(27)

and in \( y \) coordinates
\[ \langle h(y, t)h(y', t) \rangle = \frac{F^2}{(\gamma + 1)^2} t^2 + \frac{c\epsilon}{12DL(t)} \left[ 2L(t)^2 - 6L(t) \max(y, y') + 3(y^2 + y'^2) \right]. \]

(28)

The one-point correlation function reads
\[ \langle h(x, t)^2 \rangle = \frac{F^2}{(\gamma + 1)^2} t^2 + \frac{c\epsilon}{12DL_0} \left( \frac{t}{t_0} \right)^\gamma \left[ 2L_0^2 - 6L_0x + 6x^2 \right], \]

(29)
correlation function is not spatially homogeneous due to the no flux boundary conditions in agreement with the long time classical EW equation. Note that the one-point to find a coordinate independent result we define the surface width $W_{(\text{periodic boundary conditions or unbounded domains preserve the spatial homogeneity);}$ $\text{otherwise}$ Maclaurin formula representation of the series. This result is valid as long as $0 \approx \sqrt{\Delta t}$ in the long time limit, where we have used the asymptotic expansion of the Euler–Maclaurin formula to $\text{summing this expression,}$ revealing that the interface becomes globally correlated only if diffusion is large enough, or alternatively if the initial system size and growth rate are small enough; otherwise, the interface only becomes partially correlated.

It is clear that equation (22) is valid if the system is observed from spatial distances $|x - x'| \gg (t/t_0)^{1/2-\gamma}$. This might constitute a good approximation for the two-point correlation function, but it definitely breaks down when considering the one-point correlation. In this case we have to consider again the solution of equation (20), but this time in the range $n, m > (t/t_0)^{\gamma-1/2}$. In this scale the interface is again dominated by diffusion and dilution may be disregarded, and we obtain the same long time solution as in equation (21). The asymptotic behavior is obtained by summing this expression, but including in the sum only the modes that contribute to the short scale behavior $n, m > (t/t_0)^{\gamma-1/2}$. The result is

$$\langle h(x, t)^2 \rangle - \langle h(x, t) \rangle^2 = \frac{\epsilon L_0}{D \pi^2} \left( \frac{t}{t_0} \right)^\gamma \sum_{n > (t/t_0)^{\gamma-1/2}} \frac{1}{n^2} \cos^2 \left( \frac{n \pi x}{L_0} \right) \approx \frac{\epsilon L_0}{2 \pi^2 D} \left( \frac{t}{t_0} \right)^{1/2},$$

in the long time limit, where we have used the asymptotic expansion of the Euler–Maclaurin formula representation of the series. This result is valid as long as $0 \neq x \neq L_0$; otherwise

$$\langle h(0, t)^2 \rangle - \langle h(0, t) \rangle^2 = \langle h(L_0, t)^2 \rangle - \langle h(L_0, t) \rangle^2 \approx \frac{\epsilon L_0}{\pi^2 D} \left( \frac{t}{t_0} \right)^{1/2}.$$
Note that the behavior is now perfectly homogeneous, due to the uncorrelated character of the interface in this case. The only exceptions are the boundary points \( x = 0, L_0 \), because they are affected by growth only along one direction, and as a consequence their auto-correlation is twice the value of the auto-correlation of any other point not located at the boundary. These results hold independently of the reference frame, be it Lagrangian or Eulerian, precisely because the interface is uncorrelated. This allows us to calculate the height difference correlation function

\[
\langle [h(y, t) - h(y', t)]^2 \rangle = \frac{\epsilon L_0}{2\pi^2 D} \left( \frac{t}{t_0} \right)^{1/2},
\]

when \( y \neq y', 0 \neq y \neq L(t) \) and \( 0 \neq y' \neq L(t) \). This shows the agreement with the short time classical EW equation. Note that the dependence on the system size (both initial and time dependent) is the same as in the static domain case, allowing the definition of the roughness exponent, which takes on its classical value.

4. General linear Langevin equation

We now move to a more general situation in which we consider an arbitrary diffusion operator of order \( \zeta \) and an arbitrary spatial dimension \( d \); see equation (5). From now on the \( d \)-dimensional coordinates will be denoted as \( x \rightarrow x \) and \( y \rightarrow y \) for simplicity. In this case, we can proceed in exactly the same way as in the one-dimensional situation to find, instead of equation (8), the equation

\[
\partial_t h = D \left( \frac{t_0}{t} \right)^{\zeta \gamma} |\nabla|^\zeta h - \frac{d^\gamma}{t} h + F + \left( \frac{t_0}{t} \right)^{d\gamma/2} \xi(x, t),
\]

where the fractional operator \( |\nabla|^\zeta \) acts in Fourier space as

\[
(|\nabla|^\zeta h)_n = -\frac{|n|^\zeta \pi^\zeta}{L_0^\zeta} h_n.
\]
where $(\cdot)_n$ denotes the corresponding Fourier transformed quantity, and $\mathbf{n} = (n_1, \ldots, n_d)$. However, the solution of a boundary value problem for an arbitrary fractional power of the Laplacian is not obtained so straightforwardly. For some values of the parameter $\zeta$ (such as $\zeta \in (1, 2)$) this operator describes Lévy flight dispersal, for which even the definition of a boundary presents difficulties [14]. To overcome this pitfall we will decompose the solution using the same basis as in the EW equation case (see equation (9)) in the case of a $d$-dimensional cubic box. This harmonic decomposition implies the no flux boundary conditions not only for $\zeta = 2$, but also for all positive even integer values of this exponent. Indeed, for these values of $\zeta$ it is easy to check that expanding the solution in the $d$-dimensional analog of (9) is equivalent to prescribe the vanishing of all derivatives of odd order smaller than $\zeta$ of the solution at the boundary. This allows us to propose a plausible definition of solution to the no flux initial–boundary value problem (40) as the solution to this differential equation which is expressed in terms of the $d$-dimensional version of expression (9). This way the solution to the initial–boundary value problem is defined in the context of evolution semigroup theory as an initial value problem in a prescribed functional domain [15], which in this case is generated by the selected eigenfunctions of the Laplacian, as in equation (9).

In the present case the superuniversal threshold above which the interface becomes uncorrelated turns out to be $\gamma = 1/\zeta$. For $\gamma$ greater than this value (and for large spatial scales in the case of the second moment), the two first moments of the function height are given by

$$
\langle h(y, t) \rangle = \frac{F}{d\gamma + 1} t, \quad (42)
$$

$$
\langle h(y, t)h(y', t) \rangle = \frac{F^2}{(d\gamma + 1)^2} t^2 + \frac{\epsilon t}{d\gamma + 1} \delta(y - y'). \quad (43)
$$

Let us now take a look at the one-point correlation function. We can proceed in the same way as in section 3 to find

$$
\langle h(x, t)^2 \rangle - \langle h(x, t) \rangle^2 \sim L_0^{-(d-\zeta)} \left( \frac{t}{t_0} \right)^{(\zeta-d)\gamma} \sum_{\mathbf{n} > (t/t_0)^{\gamma-1/\zeta}} \frac{1}{|\mathbf{n}|^\zeta} \sim L_0^{(\zeta-d)} \left( \frac{t}{t_0} \right)^{1-d/\zeta}, \quad (44)
$$

by means of the asymptotic expansion of the Euler–Maclaurin formula for the series in the last step, and where

$$
\sum_{\mathbf{n}} = \sum_{n_1} \cdots \sum_{n_d}. \quad (45)
$$

In equation (44) we have assumed $x = 0$ for ease of analysis, as we know that, in the limit of uncorrelated interface, all points are equivalent up to a numerical prefactor present in the boundary points$^2$; the rough interface inequality $\zeta > d$ was assumed too. When this inequality is reversed, $\zeta \leq d$, then the series does not converge. The situation is the same in the case of a non-growing domain, where for $\zeta \leq d$ the interface is flat or at most logarithmically rough, but this series is divergent. Note that the one-point correlation

$^2$ We conjecture that this factor is $2^b$, where $b$ is the number of faces of the hypercubic substrate that the point in question belongs to; for the origin, or any other corner, $b = d$. 

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equation (44) grows as $t^{2\beta}$, where $\beta$ is the corresponding growth exponent of the model without domain growth, and as $L_0^{2\alpha}$ (and also as $L(t)^{2\alpha}$) for the corresponding classical roughness exponent $\alpha$. These exponents arise as a consequence of the local behavior of the model, which is analogous to the static unbounded domain one; see below.

The behavior of the one-point correlation function also helps in establishing some dynamical properties of the interface when $\gamma < 1/\zeta$:

$$\langle h(y, t)^2 \rangle - \langle h(y, t) \rangle^2 \sim L_0^{\zeta-d} \left( \frac{t}{t_0} \right)^{(\zeta-d)\gamma}. \quad (46)$$

As in previous cases, we note that there is no saturation, unlike in the non-growing domain situation, for any $\gamma > 0$. One can define the roughness exponent $\alpha$ from the second-moment dependence on the system size for long times. In the case of a growing domain one would have in principle two possible choices: the initial system size $L_0$ and the time dependent size $L(t)$. It turns out that these yield the same value $\alpha = (\zeta - d)/2$, which allows an unambiguous definition of this exponent. It is worthy of note that this exponent is exactly the same as in the regime with $\gamma = 0$, with the difference that this latter case is characterized by the saturation of the fluctuations. If we allow the definition of the long time growth exponent $\beta_{\infty}$ as the power law dependence of the second moment on the temporal variable for long times, we find $\beta_{\infty} = \gamma(\zeta - d)/2$. To calculate an effective dynamic exponent $z_{\text{eff}}$ we note that the correlation length $\Lambda$ travels as $\Lambda(t) \approx \left( \frac{D t_0^\zeta}{L_0^\gamma} \right)^{1/\zeta}$, and so the correlated interface fraction at time $t$ is

$$\frac{\Lambda(t)}{L(t)} \approx \left( \frac{D t_0^\zeta}{L_0^\gamma} \right)^{1/\zeta}, \quad (47)$$

which leaves us with

$$z_{\text{eff}} = \frac{\zeta}{1 - \zeta \gamma}, \quad (48)$$

if $\gamma < 1/\zeta$ and infinite if $\gamma > 1/\zeta$. One sees $z_{\text{eff}}(\gamma = 0) = \zeta$ and we recover the classical case, and $\lim_{\gamma \to 1/\zeta} z_{\text{eff}}(\gamma) = \infty$, showing the limit in which the interface becomes uncorrelated.

In the marginal situation characterized by $\gamma = 1/\zeta$, diffusion and dilution balance each other and so the resulting dynamics is given by the concrete values of the equation parameters. As we have seen, the effective dynamic exponent becomes divergent, and the resulting fraction of correlated interface is

$$\frac{\Lambda(t)}{L(t)} \approx \left( \frac{D t_0^\zeta}{L_0^\gamma} \right)^{1/\zeta}, \quad (49)$$

which shows that for large diffusion and small initial system size and growth rate the interface becomes globally correlated. Alternatively, for small diffusion and large initial system size and growth rate the interface becomes only partially correlated.

As we have seen, the roughness exponent can be defined as $\alpha = (\zeta - d)/2$ independently of the value of $\gamma$, as in the static domain case. For a supercritical $\gamma$, the long time growth exponent reads $\beta_{\infty} = (1 - d/\zeta)/2$, equaling the growth exponent in the classical case. For any $\gamma \neq 1/(2\zeta)$ the inequality $\alpha \neq \beta_{\infty} z_{\text{eff}}$ holds, while the equality $\alpha = \beta_{\infty} \zeta$ is true only if $\gamma \geq 1/\zeta$. On the other hand, we have the alternative relation $\alpha = \beta_{\infty} z_{\infty}$, for $z_{\infty} = \max\{1/\gamma, \zeta\}$. Indeed, there is a close connection between

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1/\gamma and the dynamic exponent, i.e., this quantity describes quantitatively the speed at which correlations propagate along the interface. This relation is further explored in the following sections.

One can clarify things further by calculating the two-point correlation function for those bulk points that lie closer than the correlation length \(|x - x'| \ll (Dt)^{1/\zeta - \gamma}\). For \(d = 1\) we obtain

\[
\langle h(x,t)h(x',t) \rangle - \langle h(x,t) \rangle^2 = \left[ |x - x'| \left( \frac{t}{t_0} \right)^\gamma \right]^{-1/\zeta} \mathcal{F} \left[ |x - x'| \left( \frac{t}{t_0} \right)^{-1/\zeta} \right],
\]

or alternatively

\[
\langle h(y,t)h(y',t) \rangle - \langle h(y,t) \rangle^2 = |y - y'|^{-1} \mathcal{F} \left[ |y - y'| \left( \frac{t}{t_0} \right)^{-1/\zeta} \right],
\]

where the scaling function reads

\[
\mathcal{F}(u) = \frac{\epsilon L_0^{\xi-1}}{4\pi^\xi D} u^{1-\xi} \int_1^\infty s^{-\xi} \cos \left( \frac{u\pi s}{L_0} \right) ds,
\]

whose integral can be considered as a trigonometric variant of the Misra function \([13]\) (see footnote 1). These formulae have been found in the long time limit after adiabatic elimination of highly oscillatory functions (which results from a direct application of the Riemann–Lebesgue lemma). Result (44) together with (51) allows us to recover the classical critical exponents \(\alpha = (\zeta - 1)/2, \beta = 1/2 - 1/(2\zeta)\), and \(z = \zeta\), when we consider Lagrangian coordinates \(y\) and distances shorter than the correlation length. Note that equations (50) and (51) only depend on \(|x - x'|\) and \(|y - y'|\) respectively and thus they are translationally invariant: this is a consequence of decorrelation, which makes the interface bulk behave like it does in the case of an unbounded domain for short spatial scales. In this limit, boundary conditions do not affect the dynamics of bulk points, but boundary points show a different prefactor, as in the strictly local situation; see (35) and (44).

5. Temporal correlations

In order to calculate the temporal correlations we need to consider the EW dynamics equation (8) in the short time limit, where the growth exponent \(\beta\) becomes apparent. The homogeneous solution of its Fourier transformed representation equation (10) is

\[
h_n(t) = \left( \frac{t}{t_0} \right)^{-\gamma} \exp \left[ - \frac{n^2\pi^2 D t_0^2 \tau^{1-2\gamma} - t_0^\gamma}{L_0^\gamma} \right] h_n(t_0) \equiv G_n(t)h_n(t_0),
\]

which yields the following complete solution when the initial condition vanishes:

\[
h_n(t) = G_n(t) \int_{t_0}^t G_n^{-1}(\tau) \left( \frac{t_0}{\tau} \right)^{\gamma/2} \xi_n(\tau) d\tau.
\]

The one-point two-times correlation function then reads

\[
\langle h_n(t)h_n(t') \rangle = \frac{2\epsilon}{L_0} G_n(t)G_n(t') \int_{t_0}^{\min(t,t')} G_n^{-2}(\tau) \left( \frac{t_0}{\tau} \right)^\gamma d\tau,
\]

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and after inverting Fourier we arrive at the real space expression

$$\langle h(x,t)h(x,t') \rangle = \sum_{n=0}^{\infty} \langle h_n(t)h_n(t') \rangle \cos^2 \left( \frac{n\pi x}{L_0} \right).$$

The propagator $G_n(t)$ suggests the scaling variable $v_n \sim nt^{1/2-\gamma}$ in Fourier space, which corresponds to the real space scaling variable $u \sim xt^{-1/2}$, as can be read directly from the last equation. This again suggests the definition of the effective dynamical exponent $z_{\text{eff}} = 2/(1 - 2\gamma)$. If we express the correlation equation (55) for $t = t'$ in terms of the scaling variable $v_n$ (and we refer to it as $C(v_n)$ multiplied by a suitable power of $t$) and we introduce the ‘differential’ $1 \equiv \Delta n \sim t^{-1/2} \Delta v$, we can cast the last expression in the integral form

$$\langle h(x,t)^2 \rangle - \langle h(x,t) \rangle^2 = t^{1/2} \int_{v_1}^{\infty} C(v) \cos^2 \left( \frac{v_n \pi u}{L_0} \right) dv_n,$n

where the series converges as a Riemann sum to the above integral when

$$Dt \ll (L_0^2 + Dt_0)^{\frac{1}{2\gamma}} t_0^{\frac{2}{2\gamma}},$$

or equivalently $t \ll t_c \sim L_0^{\frac{z_{\text{eff}}}{\gamma}}$, for $t_c$ being the time it takes the correlations reaching the substrate boundaries, assuming that the substrate initial size is very large. If $\gamma < 1/2$, the whole substrate becomes correlated, yielding a finite $t_c$; for $\gamma > 1/2$ the convergence of the Riemann sum to the integral is assured for all times, corresponding to the physical fact that the substrate never becomes correlated. In front of the integral we find the factor $t^{1/2}$, compatible with the growth exponent $\beta = 1/4$, and the integral can be shown to be absolutely convergent due to the Gaussian dependence of $G_n(t)$ on $n$.

The general situation in which we deal with a $d$-dimensional substrate and the diffusion is mediated by an operator of order $\zeta$ can be constructed following the same steps. In this case the propagator reads

$$G_n(t) = \left( \frac{t}{t_0} \right)^{-d\gamma} \exp \left[ -\frac{n^z \pi^z d t_0^{z-1-\gamma} - t_0}{L_0^z} \right],$$

suggesting that the scaling variables are $v_n \sim nt^{1/2-\gamma}$ and $u \sim xt^{-1/2}$, and a definition for the effective dynamical exponent $z_{\text{eff}} = \zeta/(1 - \gamma \zeta)$. We find again convergence of the Riemann sum to an integral for any time if $\gamma > 1/\zeta$, and for short times

$$Dt \ll (L_0^2 + Dt_0)^{\frac{\zeta\gamma}{t_0^{\gamma}}},$$

if $\gamma < 1/\zeta$, in agreement with the expression for the correlation time $t_c \sim L_0^{z_{\text{eff}}}$. This integral is again absolutely convergent as it decays superexponentially for large values of the scaling variable $v_n$, leading to the result

$$\langle h(x,t)^2 \rangle - \langle h(x,t) \rangle^2 \sim t^{1-d/\zeta},$$

in agreement with the classical growth exponent $\beta = 1/2 - d/(2\zeta)$.
We are now in a position to calculate the temporal auto-correlation

\[ A(t, t') \equiv \frac{\langle h(x, t)h(x, t') \rangle_0}{\langle h(x, t)^2 \rangle_0^{1/2} \langle h(x, t')^2 \rangle_0^{1/2}} \sim \left( \frac{\min\{t, t'\}}{\max\{t, t'\}} \right)^{\lambda}, \tag{61} \]

where \( \lambda \) is the auto-correlation exponent and \( \langle \cdot \rangle_0 \) denotes the average with the zeroth-mode contribution suppressed, as in (60). The remaining ingredient is the correlation \( \langle h(x, t)h(x, t') \rangle_0 \). Going back to equation (56) we see that the Fourier space scaling variable now reads

\[ v_n = \left[ \frac{t^{1-\gamma \zeta} + (t')^{1-\gamma \zeta} - 2x^{1-\gamma \zeta}}{1 - \gamma \zeta} \right]^{1/\zeta} n. \tag{62} \]

If \( \gamma < 1/\zeta \) the term \( \max\{t, t'\}^{1-\gamma \zeta} \) is dominant and the factor in front of the convergent Riemann sum reads

\[ \max\{t, t'\}^{-d/\zeta} \min\{t, t'\}, \tag{63} \]

after the time integration has been performed and in the limit \( \max\{t, t'\} \gg \min\{t, t'\} \). In this same limit, but when \( \gamma > 1/\zeta \), the term \( \min\{t, t'\}^{1-\gamma \zeta} \) becomes dominant and the prefactor reads

\[ \max\{t, t'\}^{-d\gamma} \min\{t, t'\}^{1-d/\zeta + d\gamma}. \tag{64} \]

The resulting temporal correlation adopts the form indicated in the right-hand side of (61), where

\[ \lambda = \begin{cases} \beta + d/\zeta & \text{if } \gamma < 1/\zeta, \\ \beta + \gamma d & \text{if } \gamma > 1/\zeta, \end{cases} \tag{65} \]

or alternatively

\[ \lambda = \beta + \frac{d}{z_\lambda}, \tag{66} \]

where \( \beta = 1/2 - d/(2\zeta) \) and the new dynamical exponent is defined as

\[ z_\lambda = \min\{\zeta, 1/\gamma\}. \tag{67} \]

This last form is the natural generalization of the corresponding one in [16], and tells us that correlations are propagated either by diffusion or dilution: the dominant mechanism is chosen in each regime. We can extract more information about the correlation function, such as its decay properties

\[ \langle h(x, t)h(x, t') \rangle_0 \sim \max\{t, t'\}^{-d/\zeta}, \quad \text{when } \max\{t, t'\} \to \infty, \tag{68} \]

signaling that it decays to zero for long times. Also, the short time behavior of the auto-correlation function is

\[ A(t, t') \approx 1 - R \left( 1 - \frac{\min\{t, t'\}}{\max\{t, t'\}} \right)^{1-d/\zeta}, \quad \text{when } \max\{t, t'\} \approx \min\{t, t'\}, \tag{69} \]

homogeneously in \( \gamma \), where \( R = R(\zeta, d, \gamma) \) is a universal function of its arguments. This behavior is compatible with the one found in the \( \gamma = 0 \) case [17]. It indicates that the short time properties of the auto-correlation are independent of the substrate growth velocity, but the long time behavior is influenced by the mechanism by which correlations are propagated, be it diffusion or dilution.
6. Persistence

The persistence of a stochastic process denotes its tendency to continue in its current state. When considering the dynamics of a fluctuating interface, one refers to the persistence probability \( P_+(t_1, t_2) \) (\( P_-(t_1, t_2) \)) as the pointwise probability that the interface remains above (below) its profile at \( t_1 \) up to time \( t_2 > t_1 \) [16, 17]. Herein, as in [8], we concentrate on the case in which the initial profile is flat, and we suppress the contribution coming from the zeroth mode as in section 5. For the stochastic growth equations under consideration the symmetry \( h_n \rightarrow -h_n \) for all Fourier modes \( n > 0 \) holds, implying the equality \( P_+ = P_- \equiv P \). For long times \( t_2 \gg t_1 \) we have the power law behavior [16, 17]

\[
P(t_1, t_2) \sim (t_1/t_2)^\theta,
\]

(70)

defining the persistence exponent \( \theta \). It was previously calculated in the limit \( \zeta \rightarrow \infty \) when \( \gamma = 0 \) [17]

\[
\theta \approx \frac{1}{2} + \frac{2\sqrt{2} - 1}{2} \frac{d}{\zeta},
\]

(71)

up to higher order terms, and in this same limit when \( d = 1 \) and \( \gamma = 1 \) [8]

\[
\theta \approx \frac{1}{2} - \frac{1}{2\zeta},
\]

(72)

up to higher order terms. The goal of this section is to calculate the persistence exponent \( \theta \) in the limit \( \zeta \rightarrow \infty \) for a finite, but otherwise arbitrary, value of \( \gamma \). In order to proceed with the calculation, we need to consider again the normalized auto-correlation function, i.e., the left-hand side of (61). This time we will not focus on the limit \( \max\{t, t'\} \gg \min\{t, t'\} \); instead we will consider an arbitrary relation between \( t \) and \( t' \). In this case we have\(^3\)

\[
\langle h(x, t)h(x, t') \rangle_0 \sim \max\{t, t'\}^{-d\gamma} \min\{t, t'\} \{\max\{t, t'\}^{1-\gamma z} - \min\{t, t'\}^{1-\gamma z}\}^{-d/z} \\
\times \left[ 2 \min\{t, t'\} \max\{t, t'\}^{\gamma z} - \min\{t, t'\} \right]^{-d/z} \\
\times 2F_1 \left[ \gamma d - 1 \frac{d}{\gamma z - 1}; 1 + \frac{\gamma d - 1}{\gamma z - 1}; 2 \min\{t, t'\} \max\{t, t'\}^{\gamma z} \right] / t(t')^{\gamma z + t^\gamma t'},
\]

(73)

where \( 2F_1(x_1, x_2; x_3; x_4) \) is the Gauss hypergeometric function [18]\(^4\). In order to derive the persistence exponent we consider the auto-correlation function in logarithmic time \( T = \ln(t) \)

\[
A(T, T') \equiv \frac{\langle h(x, e^T)h(x, e^{T'}) \rangle_0}{\langle h(x, e^{T})^2 \rangle_0^{1/2} \langle h(x, e^{T'})^2 \rangle_0^{1/2}} \sim e^{-(\beta + d\gamma)|T - T'|},
\]

when \( |T - T'| \rightarrow \infty \). (74)

\(^3\) In this expression we have suppressed all constant prefactors, including divergent terms of the form \( 1/(1 - d\gamma) \). Therefore the limit \( \gamma \rightarrow 1/d \) must be taken with caution, and one cannot rely on this simplified formula. This includes as a particular case the correlation calculated in [8]. The rest of the results in this section do not depend on this technicality.

\(^4\) The Gauss hypergeometric function is \( 2F_1(x_1, x_2; x_3; x_4) = \sum_{n=0}^{\infty} (x_1)_n (x_2)_n / (x_3)_n x_4^n / n! \), where \((x_1)_n = x_1(x_1 + 1)(x_1 + 2) \cdots (x_1 + n - 1)\) is the rising factorial.
Note that this is the correlation function for the normalized (to unit variance) function height, which becomes stationary in the logarithmic temporal variable. For short times we have

\[ A(T, T') = 1 + \mathcal{O}(|T - T'|^{2\beta}) , \quad \text{when } |T - T'| \to 0, \]

homogeneously in \( \gamma \). The first-order term is a power \( 2\beta = 1 - d/\zeta < 1 \), classifying the process as a Slepian non-smooth one \[19\]. This fact, together with the asymptotics \( A(T, T') \sim e^{-(1/2+d\gamma)|T-T'|} \) when \( \zeta \to \infty \), means that we can calculate the persistence exponent \( \theta \) perturbatively about \( \theta = 1/2 + d\gamma \) for large \( \zeta \) (and finite \( d \) and \( \gamma \)) in the following fashion \[8,17\]:

\[ \theta \approx \left( \frac{1}{2} + d\gamma \right) \left[ \frac{1}{\pi} \int_0^\infty \{ A(\tau) - e^{-(1/2+d\gamma)\tau} \} \left\{ 1 - e^{-(1+2d\gamma)\tau} \right\}^{-3/2} d\tau \right], \tag{76} \]

for \( \tau = |T - T'| \), yielding, in the limit \( \zeta \to \infty \), the result

\[ \theta \approx \frac{1}{2} + d\gamma - \frac{d}{2\zeta}, \tag{77} \]

up to higher order terms. This last result is reminiscent of the one obtained in \[8\], but it is corrected by the effect of dilution, as we shall discuss in section 7. We can see that, in the limit considered, the interface is less persistent than in the case of a static domain. This is so because in the uncorrelated phase dilution acts as a relaxation mechanism on the strictly local scale with a higher efficiency than diffusion. Note that in the limit \( \gamma \to 0 \) one does not recover the static domain result \[17\]. This is so because in this calculation we have assumed \( \gamma > 1/\zeta \), and so the vanishing \( \gamma \) limit implicitly implies a faster vanishing \( 1/\zeta \) limit. As a conclusion we find that when \( \gamma \to 0 \) the persistence exponent \( \theta \to 1/2 \).

7. Connection to radial growth

As mentioned in section 1, one of the caracteristics of radial growth is its growing domain interface. Herein, we will use the results derived in previous sections to put former derivations in radial geometry \[4,8\] in a broader context. In this case as well dilution plays an important role on the interface dynamics. As radial interfaces grow in time, the interfacial matter becomes diluted among the new deposited matter and the correlations transported simultaneously. Note that the physical origin of dilution here is the same as in section 2, and thus it is not related to the surface curvature.

The one-dimensional radial counterpart of the general linear Langevin equation (40) could be defined as \[8\]

\[ \frac{\partial r}{\partial t} = \gamma F t^{\gamma-1} + \frac{1}{r^\zeta} |\nabla \phi|^\zeta r + \frac{1}{\sqrt{r}} \eta(\theta, t), \tag{78} \]

for the field \( r(\theta, t) \), where reparameterization invariance \[7\] has been taken into account, but dilution has been disregarded. Its analysis yields, for \( \gamma > 1/\zeta \) and performing a van Kampen system size expansion about the homogeneously growing state, the long time large angular scale correlation function \[4,21\]

\[ \langle r(\theta, t)r(\theta', t') \rangle_0 \sim t^{1-\gamma}\delta(\theta - \theta') \sim t\delta(s - s'), \tag{79} \]

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for $\gamma < 1$, 
\[ \langle r(\theta, t)r(\theta', t) \rangle_0 \sim \ln(t) \delta(\theta - \theta') \sim t \ln(t) \delta(s - s'), \]  

(80)

for $\gamma = 1$, and 
\[ \langle r(\theta, t)r(\theta', t) \rangle_0 \sim t^{1-\gamma} \delta(\theta - \theta') \sim t^\gamma \delta(s - s'), \]  

(81)

for $\gamma > 1$, where $s - s' \sim t^\gamma(\theta - \theta')$ is the arc-length scale. For $\gamma < 1$ we recover the random deposition correlation, while for $\gamma \geq 1$ we found, in the arc-length variable, an average rapid roughening version of it, this is,
\[ \int_{s(\theta=0)}^{s(\theta=2\pi)} \langle r(s, t)r(s', t) \rangle_0 \, ds \sim \begin{cases} t & \text{if } \gamma < 1, \\ t \ln(t) & \text{if } \gamma = 1, \\ t^\gamma & \text{if } \gamma > 1. \end{cases} \]  

(82)

This result emerges when the dilution term is not taken into account. When we contemplate the effect of dilution, as in the previous sections, we find a pure random deposition correlation in Lagrangian coordinates
\[ \langle r(s, t)r(s', t) \rangle_0 \sim t \delta(s - s'), \]  

(83)

homogeneously in $\gamma$ (provided $\gamma > 1/\zeta$), as in (43). Correspondingly, the prefactor of the Dirac delta is $t^{1-\gamma}$ in the Eulerian setting. One can see that the large scale results derived for radial interfaces [4,21] are identical to the ones found here for homogeneously growing domains, once dilution is introduced. These results can be straightforwardly generalized to an arbitrary dimension $d$. In this case one needs $d$ angles to parameterize the interface in the Eulerian setting, which will lead to $d$ different arc-lengths in Lagrangian coordinates and to a $d$-dimensional Dirac delta specifying the spatial properties of the uncorrelated interface. As a consequence one finds
\[ \int_{S_d} \langle r(s, t)r(s', t) \rangle_0 \, ds \sim \begin{cases} t & \text{if } \gamma < 1/d, \\ t \ln(t) & \text{if } \gamma = 1/d, \\ t^{\gamma d} & \text{if } \gamma > 1/d, \end{cases} \]  

(84)

when dilution is not considered; $s = (s_1, \ldots, s_d)$ is the set of all arc-lengths and the integral extends to the whole domain. If we contemplate dilution, then the resulting correlation in the large spatial scale and for long times is pure $d$-dimensional random deposition. The physical reason for this enhanced stochasticity when dilution is not present is the following. Dilution distributes the fluctuations along the interface at the growth rate, keeping the total amount of noise constant. In its absence, the interface is composed of unconnected sites, and new ones are added in the process of domain growth. They act as independent sources of noise, and so they contribute to augmenting the overall fluctuations.

The study of radial growth performed in [8] on the strictly local scale does not contemplate the effect of dilution either. For the sake of completeness, and to facilitate comparisons, we have derived the previous section’s results deliberately disregarding the effect of dilution. As expected, the long time short spatial scale behavior is reminiscent of the unbounded substrate situation, as happened in (50) and (51). In contrast, the large spatial scale dynamics shows a different phenomenology characterized by a particular random deposition effective behavior. Equivalently, the long time interval
(max\{t, t'\} \gg \min\{t, t'\}) asymptotics is affected by substrate growth, and the absence of dilution modifies the corresponding results. If \( \gamma < 1/\zeta \), the correlation function decays to zero at infinity as a power law
\[
\langle h(x, t)h(x, t') \rangle_0 \sim (\max\{t, t'\})^{d\gamma - d/\zeta}, \quad \text{when} \ \max\{t, t'\} \to \infty,
\]
while for \( \gamma > 1/\zeta \) this correlation function approaches a non-zero value. This fact is related to the behavior of the Fourier modes (59), which in the absence of dilution decay to zero for \( \gamma < 1/\zeta \) in the long time limit, while for \( \gamma > 1/\zeta \) they approach a non-zero value asymptotically in time, analogously to the particular case analyzed in [8].

The temporal auto-correlation function, in the absence of dilution, is
\[
\frac{\langle h(x, t)h(x, t') \rangle_0}{\langle h(x, t)^2 \rangle_0^{1/2} \langle h(x, t')^2 \rangle_0^{1/2}} \sim \left( \frac{\min\{t, t'\}}{\max\{t, t'\}} \right)^\lambda,
\]
where
\[
\lambda = \begin{cases} 
\frac{1}{2} + \frac{d}{2\zeta} - d\gamma & \text{if} \ \gamma < 1/\zeta, \\
1 - \frac{d}{2\zeta} & \text{if} \ \gamma > 1/\zeta,
\end{cases}
\]
or alternatively
\[
\lambda = \beta + \frac{d}{z_{\text{eff}}},
\]
where \( \beta = 1/2 - d/(2\zeta) \) and
\[
z_{\text{eff}} = \begin{cases} 
\zeta/(1 - \gamma\zeta) & \text{if} \ \gamma < 1/\zeta, \\
\infty & \text{if} \ \gamma > 1/\zeta.
\end{cases}
\]
From these formulae one can clearly read that when dilution is suppressed there is no mechanism for correlation propagation and thus the system behaves as an effective particular random deposition model in this limit. Similar information can be obtained from the persistence exponent, obtained this time perturbatively about \( \theta \approx 1/2 \)
\[
\theta \approx \frac{1}{2} - \frac{d}{2\zeta}, \quad \zeta \to \infty,
\]
generalizing the previous result [8]. This, together with numerical simulations suggesting \( \theta \approx 1/2 \) almost homogeneously in \( \zeta \) [8], reinforces the idea of an effective particular random deposition behavior. However, for \( \zeta \) large enough \( \theta < 1/2 \), and in consequence the process is more persistent than random deposition. This fact admits a transparent physical explanation. For a static domain \( \theta \) decreases for increasing \( \beta \) [16]: the reason is that in this case the exponent \( \beta \) contains information on the relaxation properties of the interface (through its dependence on the dynamic exponent \( z \)). In this case relaxation is mediated by diffusion, which connects the different interface points and thus pushes the interface towards its mean value, diminishing the persistence of the fluctuations. For a growing domain, dilution acts as the relaxation mechanism when diffusion becomes inoperative (in the uncorrelated phase). Suppressing dilution, there is no relaxation mechanism left, following from the exponent \( \beta \) only containing information about the
strictly local fluctuational properties of the interface. Indeed, the interface variance grows as $t^{2\beta}$ (see section 4), and so for smaller $\beta$ we have weaker fluctuations intensity, implying a longer first-passage time. This implies in turn a smaller value for the persistence exponent $\theta$, as persistence is nothing but a first-passage problem [17]. This explains how $\theta$ may increase for increasing $\beta$, although understanding the whole numerical sequence of values for the persistent exponent in [8] would require a deeper analysis.

These last results for the temporal auto-correlation and persistence show that in the uncorrelated phase, dilution is the only thing responsible for correlations propagation. Interestingly, the simulations performed in [8] for the Eden model [22] show a temporal auto-correlation function fully compatible with the one described here for an uncorrelated interface in the absence of dilution. The two persistence exponents, for above and below the mean fluctuations (which are different for the less symmetric Eden interface), are greater than $\theta = 1/2$, but are notably smaller than the static domain ones [8]. This result is surprising because for a radial cluster grown according to the Eden rules, to which new cells are added at random positions on its interface, dilution is expected to occur. It would be very interesting to unveil the mechanism counterbalancing dilution in this case. It might be a consequence of the particular way in which an Eden cluster grows, or perhaps due to some possible nonlinear effect acting on the interface as a consequence of geometry.

8. Conclusions

In this work the dynamics of linear stochastic growth equations whose domain size grows in time as power law $t^\gamma$ has been studied. The growth index possesses one critical value $\gamma = 1/\zeta$, for $\zeta$ being the order of the diffusion operator. If $\gamma < 1/\zeta$ the interface correlations are propagated by means of diffusion at a faster speed than domain growth, resulting in a fully correlated interface. The time it takes correlations to travel the whole interface depends on the initial substrate size $t_c \sim L_0^{z_{\text{eff}}}$ for $z_{\text{eff}} = \zeta/(1 - \gamma \zeta)$, or alternatively $t_c \sim L(t_c)^\zeta$. The roughness exponent $\alpha$ can be defined from the strictly local properties of the interface uniformly in the growth index. This value of $\alpha$ is exactly the same one as is obtained in the static domain case. For any $\gamma > 0$ saturation never occurs, and the interface width continues to grow for all times with a long time growth exponent $\beta_\infty = \alpha \gamma$ when $\gamma < 1/\zeta$; note that in this regime $\beta_\infty < \beta$ so there is partial saturation of the fluctuations. The relation $\beta_\infty = \alpha \gamma$ also implies that correlations travel like $t^{\gamma}$ in the long time regime, after they have spread globally on the interface. Prior to that they propagate as $t^{1/\zeta}$, because diffusion is a faster mechanism for information transfer. Once they have reached the interface limits, this transfer speed is limited by the slower process of domain growth, resulting in the stated exponent relation. This phenomenology is independent of whether we contemplate dilution or not: it is a strictly direct consequence of domain growth, not of dilution (although dilution carries information at the same velocity).

The regime in which $\gamma > 1/\zeta$ is characterized by a loss of correlation along the interface. This translates into a delta correlated spatial correlation for long times and large spatial scales. The correlations for short spatial scales and time intervals are reminiscent of the ones found in the case of a static unbounded domain, revealing that diffusion is acting at this level. On the other hand, large spatial scales and long time intervals both reveal
that the dominant mechanism for correlations propagation is the dilution effect created by domain growth. By means of dilution, correlations travel at the speed at which the domain grows, so a global correlation of the interface becomes impossible. The situation is further clarified by the calculation of the auto-correlation and persistence exponents. The auto-correlation exponent \( \lambda = \beta + d/\zeta \) for all \( \gamma < 1/\zeta \) (including \( \gamma = 0 \) shows that one site interacts with itself at former times by means of the growth process (indicated by the first summand \( \beta \)) and with neighboring sites by means of diffusion (indicated by the second summand \( d/\zeta \)), which is dominant in this regime. If \( \gamma > 1/\zeta \), then the auto-correlation exponent reads \( \lambda = \beta + d\gamma \). This illustrates how, for fast domain growth, dilution replaces diffusion and becomes responsible for the interaction with the neighboring sites. A similar conclusion is reached by analyzing the persistence of the surface fluctuations. While the auto-correlation exponent yields information about the long time interval asymptotics, the persistence exponent carries complementary information obtained from averaging over all possible time interval lengths. It reads \( \theta = 1/2 + d\gamma - d/(2\zeta) \) when \( \zeta \to \infty \) and \( \gamma > 1/\zeta \), and so it is greater than in the static domain situation, which implies that the interface is less persistent. A reduced persistence is associated with a stronger tendency to go back to the mean, which is mediated by a stronger coupling with the neighboring points through dilution, more efficient than diffusion in the limit considered. Additionally, the persistence exponent increases with the growth exponent: a higher \( \beta \) corresponds to stronger fluctuations and thus to a shorter first-passage time. This is in contrast to what happens in the \( \gamma = 0 \) situation, where the persistence exponent decreases for increasing \( \beta \) [16]. The reason for this is that in this case \( \beta \) contains information about the strength of the coupling among the interface sites. Smaller \( \beta \) implies a stronger coupling and a correspondingly less persistent interface.

The crossover situation \( \gamma = 1/\zeta \) is characterized by a nonuniversal behavior. There is a strong dependence on the parameter values that enter in competition to yield the resulting system dynamics. In the generic case \( \gamma \neq 1/\zeta \), a number of the results derived in this paper can be connected to the Family–Vicsek ansatz [23] by means of the simple substitution \( L \to L(t) \) taking into account the temporal evolution of the system size directly in this ansatz. From the results of this paper, one would expect this to be so in the regime \( \gamma < 1/\zeta \), or for \( \gamma > 1/\zeta \) provided short spatiotemporal distances are under consideration. When \( \gamma > 1/\zeta \), and for long time intervals and/or large spatial scales one would perhaps expect a different result. Indeed, in this regime correlations are propagated by means of dilution, which becomes more effective than diffusion. However, the surface dynamics is still well described by the Family–Vicsek ansatz. To see this, consider for instance the unbounded domain situation uniformly growing with a growth index \( \gamma \). In this case the Family–Vicsek ansatz tells us that the two-point correlation is \( C(|x - x'|, t) = t^{2\beta} F(|x - x'|/t^{1/\zeta - \gamma}) \) in Eulerian coordinates \( x \) and \( 2\beta = 1 - d/\zeta \) for the linear stochastic growth equations considered here. For long times and \( \gamma > 1/\zeta \) we find \( C(|x - x'|, t) = t^{1-d\gamma} t^{d\gamma-d/\zeta} F(|x - x'|/t^{1/\zeta - \gamma}) \to t^{1-d\gamma} \delta(x - x') \) when \( t \to \infty \), i.e., we recover the standard random deposition correlation. This result fully agrees with the correlations derived from the stochastic growth equations which take into account dilution. On the other hand, the correlations shown at the beginning of section 7 were obtained suppressing the dilution term and cannot be derived from the Family–Vicsek ansatz. So we see that this ansatz implicitly takes into account dilution. This miracle occurs because the Family–Vicsek ansatz neglects the memory with respect to the initial condition at \( t_0 \).
However, this memory effect is present in the stochastic growth equations with no dilution, leading to a different result [20]. Introducing dilution asymptotically erases the memory with respect to the initial condition, which implies in turn the coincidence of the results from the stochastic growth equations and from the Family–Vicsek ansatz. Of course, in this reasoning we have assumed the rough interface inequality $\zeta > d$; otherwise the appearance of nonuniversal anomalous dimensions is indeed possible as we have shown for $\zeta = d$ in [21].

One of the characteristics of radial growth is the growing domain size. We have isolated this effect, which will hopefully allow a better understanding of the dynamics of radial interfaces. In order to facilitate the comparison with previous work we have deliberately neglected the dilution term in the dynamics, although this term arises naturally in the scenario considered. Our results compared favorably to and extended those of [8]. The resulting analysis in the fast growth regime showed an interface without relaxation mechanisms, and a corresponding temporal auto-correlation function decaying to a non-zero value in the long time limit. Once dilution is suppressed, and because diffusion is inoperative in the large scale for $\gamma > 1/\zeta$, there is no remaining coupling among the surface sites, as revealed by the auto-correlation exponent $\lambda = \beta$ signaling that one site only interacts with itself at different temporal points. The information obtained from the persistence exponent is the same. Note that the no dilution assumption allows growth characteristics beyond what one would expect from the static domain results. For instance, an average rapid roughening effect (not real rapid roughening which would imply pointwise estimates out of reach in this context) is possible for $\gamma \geq 1/d$ (see equation (84)), because fluctuations are no longer being distributed along the growing domain. Values of the persistence exponent $\theta < 1/2$ strictly smaller than the random deposition one are possible for large values of $\zeta$. Both become impossible if the domain is non-growing or if we contemplate dilution. Interestingly, simulations performed with the Eden model (for which $\gamma = 1$) show that it behaves as if dilution were not present [8]. Basically, this means that an Eden cluster grows as a random deposition process, with all its interface points being completely decorrelated (even at the local scale for long time intervals), but with weaker surface fluctuations as given by a smaller value of $\beta (=1/3$ for this particular model). The only dissenting factor is the numerically measured value of the persistence exponents, greater than the expected $\theta \approx 1/2$ [8]. Anyway, these values are consistent with the rest of results, as they are considerably smaller than the static domain ones [16], implying a more persistent interface. They may be the consequence of a KPZ nonlinearity (expected from the $\beta = 1/3$ exponent) acting on the interface, as it propagates correlations linearly in time [4], and so it can compete with the linearly growing Eden interface.

When studying a radial interface, it seems necessary to take into account both reparameterization invariance [7] and dilution [12]. The Eden model seems to be, in principle, not an exception, as its interface grows by the addition of new cells randomly placed on the cluster surface. However, numerical results pointed to the fact that dilution is not operative in this case [8]. It would be very interesting to understand what mechanism is counterbalancing dilution in this model. Some candidates are the presumed KPZ nonlinearity present in the Eden surface dynamics, which is able to propagate correlations linearly in time (exactly the velocity at which dilution would operate at the linearly growing Eden interface), or the nonlinearities implied by reparameterization invariance.
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