Abstract. We study cumulative scattering effects on wave front propagation in time dependent randomly layered media. It is well known that the wave front has a deterministic characterization in time independent media, aside from a small random shift in the travel time. That is, the pulse shape is predictable, but faded and smeared as described mathematically by a convolution kernel determined by the autocorrelation of the random fluctuations of the wave speed. The main result of this paper is the extension of the pulse stabilization results to time dependent randomly layered media. When the media change slowly, on time scales that are longer than the pulse width and the time it takes the waves to traverse a correlation length, the pulse is not affected by the time fluctuations. In rapidly changing media, where these time scales are similar, both the pulse shape and the random component of the arrival time are affected by the statistics of the time fluctuations of the wave speed. We obtain an integral equation for the wave front, that is more complicated than in time independent media, and cannot be solved analytically, in general. We also give examples of media where the equation simplifies, and the wave front can be analyzed explicitly. We illustrate with these examples how the time fluctuations feed energy into the pulse. Explicitly, we quantify the trade-off between pulse enhancement in dynamic media and pulse fading due to scattering by the random layers.

Key words. randomly layered media, time dependent, pulse stabilization.

1. Introduction. We study wave front propagation in time dependent, randomly layered media. The initial condition corresponds to a plane wave normally incident to the layers, in the range direction, so that the mathematical model for wave propagation is the one dimensional wave equation. Extensions to three dimensional wave fields generated by spatially localized sources are not considered here, but may be done with plane wave decompositions that lead to a family of one dimensional problems, as explained in [8, Chapter14]. The medium fluctuations are around a reference speed \( c_0 \) and are modeled with a random process \( \nu(t,z) \) that is statistically stationary in time \( t \) and homogeneous in range \( z \), that is, the propagation direction. The problem is to describe how a pulse impinging on the layered medium changes due to cumulative scattering effects. The answer depends on the strength of the wave speed fluctuations and the relation between the fundamental length and time scales in the problem: The distance of propagation \( L \), the typical wavelength \( \lambda_0 \), the correlation length \( \ell \) of the layers, the localization length \( L_{loc} \), and the correlation time \( T_\nu \) that quantifies the lifespan of a given spatial realization of \( \nu \). Localization means that the waves cannot penetrate too deep in the random medium, and the incident energy is scattered back. In layered media a pulse is being transformed as it propagates a distance that is on the scale of the localization length as described by the pulse stabilization theory in [1, 4, 5, 15] and in this paper.

There are two high frequency regimes (\( \lambda_0 \ll L \)) that have been studied extensively in the context of wave propagation in time independent randomly layered media (i.e., \( \nu = \nu(z) \)) because they capture wave scattering effects in a canonical way [4, 8]. They assume distances of propagation comparable to the localization length \( (L \sim L_{loc}) \) so that there is significant energy transfer between the forward going and backscattered waves, and the regimes differ by the strength of the fluctuations and the relation between the wavelength and the correlation length [8, Section 5.1.4]. In the

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in the strongly heterogeneous regime the fluctuations are of order one and \( \ell \ll \lambda_o \). Interestingly, the wave front has a similar, nearly deterministic characterization in both regimes, as shown in [1, 4, 5, 15]. The travel time is only slightly affected by the fluctuations, via a small random shift defined by a standard Brownian motion, and the pulse shape is a faded and smeared version of the initial one, described by a deterministic convolution kernel determined by the autocorrelation of \( \nu(z) \). This is called pulse stabilization and was first noted by O’Doherty and Anstey in a geophysical context [12]. A generalization of the pulse stabilization phenomenon persists in media with long range correlations and in multi-scale media [10, 14].

We consider time dependent randomly layered media. The scaling regime is the weakly heterogeneous regime. The waves interact strongly with the medium in this regime, because the wavelength is on the scale of the medium fluctuations, and the behavior of the pulse depends on the full autocorrelation function of the fluctuations. In the strongly heterogeneous regime the waves do not see the layers in detail, because \( \lambda_o \gg \ell \), and the fluctuations take the “effective” form of white noise, independent of the detailed structure of the random process \( \nu \). The main result of the paper is the characterization of the wave front in time dependent media. We show that as long as the correlation time \( T_\nu \) is long in comparison with the time \( T_\ell = \ell/c_o \) of travel over a correlation length, the time changes in the medium have no effect on the pulse. The pulse perceives the medium as if it were time independent. In rapidly changing media, where \( T_\nu \sim T_\ell \), the wave front has a more complicated behavior, since the medium is being transformed during the passage time of the pulse. We derive an integral equation for the wave front. It has a deterministic kernel shape, however with a random dilation, reflecting the random medium modulation during the pulse passage time. This equation is no longer solvable analytically, in general. Nevertheless, there are examples of rapidly changing media where the equation simplifies and the wave front has an explicit and deterministic expression. We illustrate with such examples how the temporal fluctuations feed energy into the wave front, thus enhancing the pulse. That is to say, there is a trade-off between pulse fading due to the spatial fluctuations and pulse enhancement due to the rapid time changes in the medium.

Wave propagation in time dependent randomly layered media has also been considered in [9], for correlation times \( T_\nu \) that are small with respect to the mean scattering time, but larger than \( T_\ell \). The formal stochastic iterative approach in [9] gives closed equations for the statistical moments of the forward and backward going wave fields. However, it relies on the assumption that the wave fields are statistically independent from the medium fluctuations in certain range intervals, which is difficult to justify rigorously.

The paper is organized as follows: We begin in section 2 with the mathematical formulation of the problem and the asymptotic scaling regime. The statement of the pulse stabilization result is in section 3 and its proof is in section 4. We end with a summary in section 5.

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\*The mean scattering time is defined in [9] Equation (1.10)]. It is an order one time scale in our regime.
2. Formulation of the problem. We consider the same one dimensional acoustic wave equation in time changing media as in [9]

\[
\rho \frac{\partial u(t, z)}{\partial t} + \frac{\partial p(t, z)}{\partial z} = F(t, z),
\]

\[
\frac{1}{K(t, z)} \frac{\partial p(t, z)}{\partial t} + \frac{\partial u(t, z)}{\partial z} = 0.
\]

(2.1)

It can be derived from the linearization of the mass and linear momentum conservation equations in fluid dynamics, with \(u\) the displacement of the particles in the medium and \(p\) the time integral of the acoustic pressure. The bulk modulus \(K\) varies with time \(t\) and range \(z\). The density \(\rho\) is assumed constant for simplicity, but the results can be extended to variable \(\rho\). We model the variations of \(K\), and thus of the wave speed \(c = \sqrt{K/\rho}\), with the random process \(\nu(t, z)\)

\[
\frac{1}{c(t, z)} = \begin{cases} 
\left[1 + \nu(t, z)\right]/c_o, & \text{for } z > 0, \\
1/c_o & \text{otherwise},
\end{cases}
\]

(2.2)

and take for simplicity a constant reference wave speed \(c_o\). At any time instant \(t \geq 0\) the fluctuations are confined to the half line \(z > 0\).

The medium is quiescent for \(t < 0\)

\[
p(t, z) = 0, \quad u(t, z) = 0, \quad t < 0,
\]

(2.3)

and the waves are generated by a source located at \(z = 0\), emitting a pulse \(F(t)\)

\[
F(t, z) = \frac{\zeta_o^{1/2} F(t) \delta(z/L)}{2}
\]

(2.4)

for \(L\) the range scale of propagation. Here \(\zeta_o = \rho c_o\) is the reference acoustic impedance, and we normalize [2.4] with it and \(L\) to simplify the formulas below. The source generates a left (backward) going wave that never interacts with the layered medium, and a right (forward) going wave \(A(t, z)\) satisfying the condition

\[
A(t, 0) = F(t).
\]

(2.5)

The problem is to describe the wave front as it travels through the medium. It is defined as the right going wave \(A(t, z)\) observed near the expected travel time \(\tau = z/c_o\) for \(z > 0\), on a time scale comparable to the width of the pulse \(F(t)\).

2.1. Random model of the fluctuations. The random process \(\nu(t, z)\) is a mathematical model of small scale fluctuations of the wave speed. Small scale means that \(c(t, z)\) varies rapidly in \(z\) on the range scale \(L\) of propagation of the waves, and it changes over time intervals that are smaller than the travel time scale \(T = L/c_o\).

The fluctuations are not known in detail, as is typical in applications, and the random process \(\nu(t, z)\) models their uncertainty.

We take \(\nu\) to be a bounded stationary random field, so that the right hand side in [2.2] remains positive. For the analysis we also require that \(\nu\) be twice continuously differentiable with bounded second order partial derivatives and finite dependence ranges in both coordinates. The process \(\nu\) has mean zero and standard deviation \(\sigma\)

\[
\mathbb{E}\{\nu(t, z)\} = 0, \quad \sigma = \sqrt{\mathbb{E}\{\nu^2(t, z)\}},
\]

(2.6)
and it fluctuates on the range scale $\ell$ called the correlation length and on the time scale $T_\nu$ called the correlation time. These are related to the covariance of $\nu$ as

$$
\ell = \frac{1}{\sigma^2} \int_{-\infty}^{\infty} dz' \mathbb{E} \{ \nu(t, z) \nu(t, z + z') \} \tag{2.7}
$$

$$
T_\nu = \frac{1}{\sigma^2} \int_{-\infty}^{\infty} dt' \mathbb{E} \{ \nu(t, z) \nu(t + t', z) \}. \tag{2.8}
$$

2.2. Scaling. We model the pulse $F(t)$ using a real valued function $f(s)$ of dimensionless argument

$$
F(t) = f \left( \frac{t}{T_x} \right), \tag{2.9}
$$

and denote by $T_x$ the time scale that quantifies the pulse duration. We assume that $f$ is continuously differentiable and compactly supported in the interval $[0, S]$, with Fourier transform $\hat{f}(w)$ that peaks at $w = 2\pi$. Since $f$ is real valued, $\hat{f}(-w)$ is defined by the complex conjugate of $\hat{f}(w)$. The Fourier transform of the pulse

$$
\hat{F}(\omega) = \int_{-\infty}^{\infty} dt f \left( \frac{t}{T_x} \right) e^{i\omega t} = T_x \hat{f} \left( \omega T_x \right) \tag{2.10}
$$

has the central frequency $\omega_o = 2\pi/T_x$, which gives the central wavelength $\lambda_o = c_o T_x$.

Let us rewrite the fluctuations in scaled form

$$
\nu(t, z) = \sigma \mu \left( \frac{t}{T_\nu}, \frac{z}{\ell} \right), \tag{2.11}
$$

using the function $\mu$ of dimensionless arguments. Its autocorrelation is given by

$$
\Phi(t', z') = \mathbb{E} \{ \mu(t'' + t', z'' + z') \mu(t'', z'') \}, \tag{2.12}
$$

for dimensionless $t', z' \in \mathbb{R}$. It is integrable and normalized by

$$
\Phi(0, 0) = 1, \quad \int_{-\infty}^{\infty} dt' \Phi(t', 0) = 1, \quad \int_{-\infty}^{\infty} dz' \Phi(0, z') = 1, \tag{2.13}
$$

to be consistent with definitions (2.6)-(2.8).

The reference range scale is $L$, the typical distance of propagation through the medium, and the time is scaled by $T = L/c_o$. We assume that we observe the waves for a maximum time $T \sim T$, where “$\sim$” denotes equal up to a dimensionless order one constant, and use causality to truncate mathematically the fluctuations for $z > L$. This is because until time $T$ the waves are not affected by the medium beyond the range $L \approx c_o T$.

In the scaled coordinates

$$
t' = \frac{t}{T}, \quad z' = \frac{z}{L}, \tag{2.14}
$$

the pulse becomes

$$
F(t) = f \left( \frac{t'}{T_x / T} \right), \tag{2.15}
$$
and the model of the wave speed is

\[
\frac{c_o}{c(t, z)} = \begin{cases} 
1 + \sigma \mu \left( \frac{t'}{T_F}, \frac{z'}{L} \right) / c'_o, & \text{for } z' > 0, \quad z' \leq L' = \frac{T}{L} \sim 1, \\
1/c'_o, & \text{otherwise.} 
\end{cases}
\] (2.16)

In our case, \( c'_o = 1 \) because the reference speed is constant, but in general it would be a dimensionless function of \( z' \). We keep \( c'_o \) in the equations to distinguish between the range and time variables, even though \( c'_o = 1 \).

### 2.3. The asymptotic regime.

We assume from now on that the time, range, and speed are scaled and drop the primes. Our analysis is in an asymptotic regime modeled with the small parameter \( \varepsilon \) defined as

\[
\varepsilon^2 = \frac{T_F}{T} = \frac{\lambda_o}{L} \ll 1. \tag{2.17}
\]

It corresponds to a long distance of propagation compared to the wavelength. The fluctuations of the wave speed are weak, with standard deviation\(^1\)

\[
\sigma = \varepsilon, \tag{2.18}
\]

but they have a significant net scattering effect when the correlation length \( \ell \) is similar to \( \lambda_o \), or the correlation time \( T_\nu \) is similar to the pulse width \( T_F \). For example, it is shown in [8, Chapter 14] that in time independent media with \( \ell \sim \lambda_o \), the localization length [8, Section 5.1.4] \( L_{loc} \) is similar to the distance of propagation \( L \), which is our reference order one length scale:

\[
\frac{L_{loc}}{L} \sim \frac{1}{\sigma^2 L (2\pi/\lambda_o)^2 \ell} = \frac{1}{4\pi^2}. \tag{2.19}
\]

This means that the net scattering is so strong that the waves cannot penetrate much deeper than the range \( L \) in the medium, and are reflected back toward the source.

We model the range and temporal scales of the fluctuations by

\[
\varepsilon^\beta = \frac{\ell}{L}, \quad \varepsilon^\alpha = \frac{T_\nu}{T}, \tag{2.20}
\]

using the dimensionless parameters \( \alpha \) and \( \beta \) in the interval \([0, 2]\), and consider three asymptotic regimes:

- **Regime 1** defined by \( \alpha < 2 \) and \( \beta = 2 \).
- **Regime 2** defined by \( \alpha = 2 \) and \( \beta < 2 \).
- **Regime 3** defined by \( \alpha = \beta = 2 \).

The first regime corresponds to media that change slowly during the duration of the pulse. The second corresponds to rapidly changing media where the lifespan of a spatial realization of the fluctuations is of order the duration of the pulse, but the range scale of the fluctuations is large with respect to the wavelength. The third regime is for media that change rapidly in time and range. We consider these regimes because on one hand they give significant net scattering effects and on the other hand they allow the waves to maintain some residual coherence up to ranges of order \( L \).

\(^1\)The standard deviation does not have to be exactly \( \varepsilon \), but of order \( \varepsilon \), that is \( \sigma = \sigma_o \varepsilon \), for some \( \sigma_o = O(1) \). For simplicity we set \( \sigma_o = 1 \). A similar simplification is made in equation (2.17).
In our scaling the pulse becomes $f(\frac{t}{\varepsilon^2})$ and the model of the wave speed in the random medium is

$$c_\varepsilon(t, z) = \frac{c_0}{1 + \varepsilon \mu \left( \frac{z}{\varepsilon^2}, \frac{t}{\varepsilon^2} \right)},$$

where we use the index $\varepsilon$ to remind us of the dependence on the asymptotic parameter $\varepsilon \ll 1$. The acoustic impedance is given by the product of the wave speed and the constant medium density, so it takes the form

$$\zeta_\varepsilon(t, z) = \frac{\zeta_0}{1 + \varepsilon \mu \left( \frac{z}{\varepsilon^2}, \frac{t}{\varepsilon^2} \right)},$$

(2.2.2)

### 3. Statement of results.

The goal of the paper is to characterize the wave front in the asymptotic limit $\varepsilon \to 0$. To define it we introduce the right and left going waves $A_\varepsilon(t, z)$ and $B_\varepsilon(t, z)$, moving with the local speed $c_\varepsilon(t, z)$, as in [8, Section 8.1]

$$A_\varepsilon(t, z) = \zeta^{-1/2}(t, z)p_\varepsilon(t, z) + \zeta^{1/2}(t, z)u_\varepsilon(t, z),$$

$$B_\varepsilon(t, z) = -\zeta^{-1/2}(t, z)p_\varepsilon(t, z) + \zeta^{1/2}(t, z)u_\varepsilon(t, z).$$

(3.1)

Here $A(Tt, Lz) = A_\varepsilon(t, z)$, with index $\varepsilon$ indicating the dependence on the asymptotic parameter $\varepsilon$. The same notation applies to the displacement $u_\varepsilon(t, z)$, the pressure field $p_\varepsilon(t, z)$ and the left going, backscattered waves $B_\varepsilon(t, z)$. The right going wave satisfies the initial condition at $z = 0$

$$A_\varepsilon(t, 0) = f\left( \frac{t}{\varepsilon^2} \right),$$

(3.2)

and we impose at the order one scaled final range $\bar{L}' = L\varepsilon$ the condition

$$B_\varepsilon(t, \bar{L}') = 0.$$  

(3.3)

It follows from the truncation of the random medium in (2.15), justified by the causality of the wave equation, as explained in section 2.2. It says that up to time $\bar{T}$ we cannot observe waves backscattered beyond the range $L$.

#### 3.1. The wave front and the random travel time.

We define the wave front at range $z$ as the right going wave observed in an $\varepsilon^2$ time window around the reference travel time $\tau = z/c_0$,

$$a_\varepsilon(\tau, s) = A_\varepsilon(t_\varepsilon(c_0\tau, s), c\tau).$$

(3.4)

Here

$$t_\varepsilon(c_0\tau, s) = \varepsilon^2 s + \int_0^{c_0\tau} \frac{dz'}{c_\varepsilon(t_\varepsilon(z', s), s')} = \tau + \varepsilon^2 s + \varepsilon^2 W_\varepsilon(\tau, s),$$

(3.5)

is the travel time along the characteristics of the right going wave, as explained in section 4.1. The characteristics are parameterized by the range $z = c_0\tau$, and start at times $\varepsilon^2 s$ at $z = 0$, with $s$ parameterizing the pulse. Note that $t_\varepsilon(c_0\tau, s)$ fluctuates randomly around the value $\tau + \varepsilon^2 s$, on the scale of the duration of the pulse, as modeled by the process

$$W_\varepsilon(\tau, s) = \frac{1}{\varepsilon} \int_0^\tau du \mu \left( \frac{u}{\varepsilon^2} + \varepsilon^{2-\alpha}(s + W_\varepsilon(u, s)), \frac{c_0u}{\varepsilon^3} \right).$$

(3.6)
We show in appendix B that as \( \varepsilon \to 0 \), \( W_\varepsilon(\tau, s) \) converges in distribution, for fixed \( s \), to the Markov diffusion process

\[
W(\tau) = \theta \tau + D B(\tau),
\]

where \( B(\tau) \) is standard Brownian motion, and the drift and diffusion coefficients \( \theta \) and \( D \) depend on \( \alpha \) and \( \beta \). We have

\[
D = \left[ \int_{-\infty}^\infty dh \Phi(h, c_\alpha h) \right]^{1/2} \quad \text{and} \quad \theta = \int_0^\infty dh \partial_t \Phi(h, c_\alpha h),
\]

in the third asymptotic regime, with \( \alpha = \beta = 2 \), and

\[
D = \left[ \int_{-\infty}^\infty dh \Phi(h, 0) \right]^{1/2} \quad \text{and} \quad \theta = \int_0^\infty dh \partial_t \Phi(h, 0) = -1,
\]

in the second regime, with \( \beta < \alpha = 2 \). In both these regimes the media change rapidly in time and the nonzero drift \( \theta \) means that the waves travel at effective speed that is slightly different than \( c_\alpha \). In the asymptotic regime 1 of slowly changing media, where \( \alpha < \beta = 2 \), we have

\[
D = \left[ \int_{-\infty}^\infty dh \Phi(0, c_\alpha h) \right]^{1/2} \quad \text{and} \quad \theta = 0.
\]

The wave front satisfies the integral equation stated in the next proposition, and proved in section 4.2.

**Lemma 3.1.** The wave front \( a_\varepsilon(\tau, s) \) satisfies to leading order the equation

\[
a_\varepsilon(\tau, s) = f(s) - \frac{c_\alpha^2}{8} \int_0^\tau d\tau' \mu^+_{\alpha,\beta} \left[ \frac{\tau' + \varepsilon^2 (s + W_\varepsilon(\tau', s))}{\varepsilon^\alpha}, \frac{c_\alpha \tau'}{\varepsilon^\beta} \right] \int_{-S_{\alpha,\beta}}^s dy \times
\]

\[
a_\varepsilon(\tau', S_\varepsilon(y; \tau', s)) \mu^-_{\alpha,\beta} \left[ \frac{\tau' + \varepsilon^2 \left( \frac{y + s}{2} + W_\varepsilon(\tau', s) \right)}{\varepsilon^\alpha}, \frac{c_\alpha \tau' + \varepsilon^2 c_\alpha (s - y)/2}{\varepsilon^\beta} \right],
\]

where the interval of integration is defined by the time parameter \( S_{\alpha,\beta} \) which depends on \( \alpha \) and \( \beta \) but not on \( \varepsilon \), and is bounded with high probability. The random processes \( \mu^\pm_{\alpha,\beta} \)

\[
\mu^\pm_{\alpha,\beta}(t, z) = \varepsilon^{2-\beta} \mu_z(t, z) \pm \varepsilon^{2-\alpha} c_\alpha^{-1} \mu_t(t, z),
\]

where \( \mu_t \) and \( \mu_z \) are the partial derivatives in \( z \) and \( t \) of \( \mu \), and the mapping \( S_\varepsilon(y; \tau, s) \) is defined by the equation

\[
S_\varepsilon(y; \tau, s) + W_\varepsilon(\tau, S_\varepsilon(y; \tau, s)) = y + W_\varepsilon(\tau, s),
\]

and in particular \( S_\varepsilon(y = s; \tau, s) = s \). The essential difference in the description of the wave front in the asymptotic regimes 1-3 lies in the dependence of the random travel fluctuations \( W_\varepsilon(\tau, s) \) on the starting point \( s \) of the characteristic. The next lemma states that in the first two regimes \( W_\varepsilon(\tau, s) \) is independent of \( s \) to leading order. This implies that

\[
S_{\alpha,\beta} = o(1) \quad \text{and} \quad S_\varepsilon(y; \tau, s) = y + o(1), \quad \text{for} \ \varepsilon \to 0,
\]
as shown in section 4.2 and we can then simplify the integral equation in Lemma 3.1 to analyze explicitly the wave front and obtain the pulse stabilization results stated in the next section.

Note that in the third regime we may have in general quite different random fluctuations of the travel time along characteristics stemming from far apart points in the support of the pulse, as shown in appendix C. Furthermore, $S_{\alpha,\beta}$ and $S_\varepsilon(y;\tau,s)$ are in general order one random processes, and the wave front description is no longer explicit. There is however an important special case corresponding to a translational medium where we can obtain an explicit characterization and we discuss this below.

The next lemma states that in regimes 1 and 2 the whole wave front experiences the same random travel time correction to leading order. It allows us to get an explicit characterization of the statistically stable pulse shape in these regimes.

**Lemma 3.2.** The random travel time fluctuations $W_\varepsilon(\tau,s)$ do not depend on $s$ to leading order in regimes 1 and 2. We have

\[
\partial_s W_\varepsilon(\tau,s) \to 0 \quad \text{as} \quad \varepsilon \to 0,
\]

in probability, for all $\tau \in (0,\bar{T})$ and $s \in [0,S]$. More explicitly, in regime 1

\[
W_\varepsilon(\tau,s) = \frac{1}{\varepsilon} \int_0^\tau du \mu\left(\frac{u}{\varepsilon^\alpha} + \varepsilon^{2-\alpha}(s + W_\varepsilon(u,s)), \frac{c_\varepsilon u}{\varepsilon^\beta}\right)
\]

\[
= \frac{1}{\varepsilon} \int_0^\tau du \mu\left(\frac{u}{\varepsilon^\alpha}, \frac{c_\varepsilon u}{\varepsilon^\beta}\right) + q_\varepsilon(\tau,s),
\]

with

\[
P\left(\sup_{s \in [0,S], \tau \in [0,\bar{T}]} |q_\varepsilon(\tau,s)| > c\varepsilon^{(2-\alpha-\Delta)/2}\right) < \varepsilon^\Delta,
\]

for $0 < \Delta < 2 - \alpha$, $0 < \varepsilon < \varepsilon_0$ and $C$ and $\varepsilon_0$ independent constants. A similar estimate holds for regime 2, as well.

We refer to appendix C for the proof of (3.14). The more detailed estimate (3.15) is obtained with a standard, but lengthy argument using the submartingale inequality and the Gronwall’s lemma, which we do not include here. This estimate can be extended to regime 2 using the change of variables (B.11) as in appendix B.

**3.2. Pulse stabilization.** The implication of Lemma 3.2 is that the integral equation of the wave front $a_\varepsilon(\tau,s)$ in Lemma 3.1 simplifies in regimes 1-2, with $S_{\alpha,\beta}$ replaced by 0 and $S_\varepsilon(y;\tau,s)$ replaced by $y$. Moreover, $a_\varepsilon(\tau,s)$ converges in probability to a deterministic function $\bar{a}(\tau,s)$, as stated in the next theorem.

**Theorem 3.3.** Assume regime 1 or 2. In the limit $\varepsilon \to 0$, the wave front $a_\varepsilon(\tau,s)$ converges in distribution to the deterministic solution $\bar{a}(\tau,s)$ of the initial value problem

\[
\frac{\partial}{\partial \tau} \bar{a}(\tau,s) = -\frac{c_\varepsilon^2}{8} \int_0^s dy \bar{a}(\tau,y) \psi\left(\frac{s-y}{2}\right), \quad \tau > 0,
\]

\[
\bar{a}(0,s) = f(s),
\]

with integral kernel

\[
\psi(s) = \begin{cases} 
-\partial_s^2 \Phi(0,c_\varepsilon s) & \text{in regime 1}, \\
\frac{c_\varepsilon^2}{8} \partial_s^2 \Phi(s,0) & \text{in regime 2}.
\end{cases}
\]

(3.17)
The limit is given explicitly by

\[
\overline{\alpha}(\tau, s) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hat{f}(\omega) \exp \left[ -i\omega s - \frac{c_0^2}{4} \int_0^\infty du \Psi(u) e^{2i\omega u} \right].
\] (3.18)

The convergence of \(a_\varepsilon\) to \(a\), the solution of (3.16), is derived in section 4.4. To obtain (3.18) note that the right hand side in (3.16) is a convolution in \(y\). Indeed, let us change variables in (3.16) as \(y = s - u\), for \(u \in [0, s]\), and extend the integral over \(u\) to infinity

\[
\frac{\partial}{\partial \tau} \overline{\alpha}(\tau, s) = -\frac{c_0^2}{8} \int_0^\infty du \overline{\alpha}(\tau, s - u) \overline{\Psi}(\frac{u}{2}),
\] (3.19)

because \(\overline{\alpha}(\tau, s - u) = 0\) for \(s - u < 0\), by the compact support of the pulse in \(R^+\).

Taking the Fourier transform with respect to \(s\),

\[
\overline{\alpha}(\tau, s) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hat{\overline{\alpha}}(\tau, \omega) e^{-i\omega s}
\]
we obtain

\[
\frac{\partial}{\partial \tau} \hat{\overline{\alpha}}(\tau, \omega) = -\frac{c_0^2}{4} \hat{\overline{\alpha}}(\tau, \omega) \int_0^\infty du \overline{\Psi}(u) e^{2i\omega u}, \quad \tau > 0,
\]

with initial condition \(\hat{\overline{\alpha}}(0, \omega) = \hat{\overline{f}}(\omega)\). This can be solved explicitly and (3.18) follows by inverting the Fourier transform.

### 3.3. Interpretation of the results.

Theorem 3.3 states that when we observe the right going wave at range \(z\) around the random travel time \(t_\varepsilon(z, s)\), in a time interval equal to the duration of the pulse, the result is deterministic. This phenomenon is called pulse stabilization. Note however that the pulse shape is not the same as the emitted \(f(s)\), like it would be in the homogeneous medium. The pulse is deformed, as modeled by the convolution kernel

\[
K_f(z, s) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega s - \gamma(\omega)z},
\] (3.20)

where we introduced the notation

\[
\gamma(\omega) = \frac{c_0}{4\omega^2} \int_0^\infty du \overline{\Psi}(u) e^{2i\omega u},
\] (3.21)

and factored out the \(\omega^2\) as \(\overline{\Psi}\) depends on the second derivative of the autocorrelation function \(\Phi\), as stated in equation (3.17). This equation shows how the pulse shaping kernel depends on the asymptotic regime (1 or 2).

When we observe the wave at range \(z\) in a time window centered at the reference travel time \(\tau = z/c_0\), the result is no longer deterministic. We state it in the corollary below, which follows from Theorem 3.3 and the limit (3.7) of the random fluctuations of the travel time.

**Corollary 3.4.** As \(\varepsilon \to 0\),

\[
A_\varepsilon(\tau + \varepsilon^2 s, c\tau) \to \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hat{f}(\omega) e^{-i\omega[\theta s - DB(\tau)] - c_0 \tau \omega^2 \gamma(\omega)},
\] (3.22)

where the convergence is in distribution, \(B(\tau)\) is standard Brownian motion and \(\theta\) and \(\mathcal{D}\) are as in (3.7).
3.3.1. Slowly time changing media. In regime 1 we have

\[
\text{real}[\gamma(\omega)] = \frac{c_o}{4\omega^2} \int_0^\infty du \Psi(u) \cos(2\omega u)
\]

\[
= -\frac{1}{4\omega^2} \int_0^\infty dz \partial_z^2 \Phi(0, z) \cos \left( \frac{2\omega}{c_o} z \right)
\]

\[
= \frac{1}{2c_o^2} \int_{-\infty}^{\infty} dz \Phi(0, z) \cos \left( \frac{2\omega}{c_o} z \right)
\]

\[
= \gamma^{(c)}(\omega) \geq 0, \quad (3.23)
\]

where the second equality is from definition (3.17), the third equality is obtained with integration by parts, and the fourth equality is a definition. That \(\gamma^{(c)}\) is non-negative follows by Bochner’s theorem. The imaginary part of \(\gamma(\omega)\) is obtained similarly

\[
\text{imag}[\gamma(\omega)] = \frac{c_o}{4\omega^2} \int_0^\infty du \Psi(u) \sin(2\omega u)
\]

\[
= -\frac{1}{4\omega^2} \int_0^\infty dz \partial_z^2 \Phi(0, z) \sin \left( \frac{2\omega}{c_o} z \right)
\]

\[
= \frac{1}{c_o^2} \int_0^\infty dz \Phi(0, z) \sin \left( \frac{2\omega}{c_o} z \right) - \frac{1}{2\omega c_o}
\]

\[
= \gamma^{(s)}(\omega) - \frac{1}{2\omega c_o}. \quad (3.24)
\]

We recovered the pulse stabilization result [8, Proposition 8.1] in time independent media. The random changes in the medium, on the time scale \(T_\nu = \varepsilon^{\alpha} T\) that is much larger than the pulse width \(T_F\) and the time \(T_\ell\) of traversal of a correlation length, play no role in the leading behavior of the wave front. The result is as if the medium were time independent. The kernel

\[
K(z, s) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega s - \frac{\omega^2}{2}\left[\gamma^{(c)}(\omega) + i\gamma^{(s)}(\omega)\right]} = K_f \left( z, s + \frac{z}{2c_o} \right), \quad (3.26)
\]

is analyzed in detail in [8, 10]. It satisfies the initial value problem

\[
\frac{\partial K(z, s)}{\partial z} = L K(z, s), \quad z > 0,
\]

\[
K(z = 0, s) = \delta(s), \quad (3.27)
\]

with pseudo-differential operator \(L\) describing the pulse deformation caused by scattering in the medium. We can decompose it in two parts, \(L = L^{(c)} + L^{(s)}\), where

\[
\int_{-\infty}^{\infty} L^{(c)} K(z, s) e^{i\omega s} ds = -\omega^2 \gamma^{(c)}(\omega) \int_{-\infty}^{\infty} K(z, s) e^{i\omega s} ds, \quad (3.28)
\]

\[
\int_{-\infty}^{\infty} L^{(s)} K(z, s) e^{i\omega s} ds = -i\omega^2 \gamma^{(s)}(\omega) \int_{-\infty}^{\infty} K(z, s) e^{i\omega s} ds.
\]

The first part \(L^{(c)}\) acts as an effective diffusion operator. It models the frequency-dependent attenuation over range \(z\), at the rate \(\omega^2 \gamma^{(c)}(\omega)\). The second part \(L^{(s)}\) is an effective dispersion operator that preserves energy. The dispersive effect increases with \(z\) at the rate \(\omega^2 \gamma^{(s)}(\omega)\).
When we rewrite (3.28) in the time domain using (3.23)-(3.25)

\[ \mathcal{L}K(z, s) = \left[ \frac{1}{2c_0} \Phi \left( 0, \frac{c_0 s}{2} \right) \mathbf{1}_{[0, \infty)}(s) \right] \ast \left[ \frac{\partial^2 K}{\partial s^2}(z, s) \right] \]

\[ = \frac{1}{2c_0} \int_0^\infty \Phi \left( 0, \frac{c_0 \xi}{2} \right) \frac{\partial^2 K}{\partial s^2}(z, s - \xi) d\xi. \]  

(3.29)

The indicator function \( \mathbf{1}_{[0, \infty)} \) entails that if \( K \) is vanishing for \( s < s_0 \), then \( \mathcal{L}K \) is also vanishing for \( s < s_0 \). That is to say, the operator \( \mathcal{L} \) preserves causality. The operator also satisfies the Kramers-Kronig relations \([6, 11]\), as shown in \([10]\).

Corollary 3.4 gives that

\[ A_\varepsilon \left( \tau + \varepsilon^2 s, c\tau \right) \to f_\tau \left( s - \frac{\tau}{2} - DB(\tau) \right), \]

(3.30)

in distribution, as \( \varepsilon \to 0 \), where

\[ f_\tau(s) = f(s) \ast K(c\tau, s), \]

(3.31)

is the deterministic pulse shape given by the convolution of \( f(s) \) with the kernel \( K(z, s) \). The diffusion coefficient \( D \) in front of the Brownian motion is given in (3.10) and we observe that it satisfies \( D = \sqrt{2c_0 \gamma^{(c)}(0)} \). Note that the result (3.30) is the same as in \([8, Proposition 8.1]\), except for the deterministic delay of \( \tau/2 \). This is because we have a different model of the random wave speed. In \([8, Proposition 8.1]\) the denominator in (2.21) has a square root, and this results in a deterministic correction of the travel time that cancels the delay in (3.30).

### 3.3.2. The wave front in rapidly time changing media.

In regime 2 we have

\[ \text{real}[\gamma(\omega)] = \frac{c_0}{4\omega^2} \int_0^\infty du \Psi(u) \cos(2\omega u) \]

\[ = \frac{1}{4c_0 \omega^2} \int_0^\infty dt \frac{\partial^2}{\partial t^2} \Phi(t, 0) \cos(2\omega t) \]

\[ = -\frac{1}{2c_0} \int_\infty^{\infty} dt \Phi(t, 0) \cos(2\omega t) \]

\[ = \gamma^{(c)}(\omega) \leq 0, \]

(3.32)

where the second equality is from definition (3.17), the third equality is obtained with integration by parts, and the fourth equality is a definition. Note that the real part of \( \gamma \) is now negative. The imaginary part of \( \gamma(\omega) \) is obtained similarly

\[ \text{imag}[\gamma(\omega)] = \frac{c_0}{4\omega^2} \int_0^\infty du \Psi(u) \sin(2\omega u) \]

\[ = \frac{1}{4c_0 \omega^2} \int_0^\infty dt \frac{\partial^2}{\partial t^2} \Phi(t, 0) \sin(2\omega t) \]

\[ = -\frac{1}{c_0} \int_0^\infty dt \Phi(t, 0) \sin(2\omega t) + \frac{1}{2\omega c_0} \]

\[ = \gamma^{(s)}(\omega) + \frac{1}{2\omega c_0}. \]

(3.34)
We define the pulse shaping kernel $K(z, s)$ as before,

$$K(z, s) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega s - \omega^2 [\gamma^{(c)}(\omega) + i\gamma^{(r)}(\omega)]} = K_f \left( z, s - \frac{z}{2c_0} \right), \quad (3.35)$$

and obtain from Corollary 3.4 and definition (3.9) that

$$A_x \left( \tau + \varepsilon^2 s, c\tau \right) \to f_\tau \left( s + \frac{3\tau}{2} - DB(\tau) \right), \quad (3.36)$$

in distribution, as $\varepsilon \to 0$, where $f_\tau(s)$ is the deterministic pulse shape given by the convolution of $f(s)$ with the kernel $K(z, s)$.

Unlike the result in slowly changing media where the pulse fades as it travels through the medium, in rapidly changing media the pulse is enhanced. This is because $\gamma^{(c)}(\omega)$ is negative and energy is not conserved in time dependent media. Thus, the random fluctuations of the wave speed in time and range have opposite effects. We illustrate this with a simple example in section 3.4. Before that, we draw an analogy between the pulse stabilization result in rapidly changing media and the random harmonic oscillator problem studied in [8, Section 7.5].

3.3.3. Analogy to the random harmonic oscillator problem. Let us suppose for a moment that there were no spatial fluctuations, just the temporal ones. We denote these fluctuations by $\tilde{\mu}$ to distinguish them from the time and range fluctuations in (2.21). To help us with the analogy to the random oscillator problem, we change variables

$$z = c_0 T, \quad t = Z/c_0,$$

and obtain a new wave equation for the displacement $u$,

$$\frac{\partial^2 u(T, Z)}{\partial Z^2} - \left[ 1 + \varepsilon \tilde{\mu} \left( \frac{Z/c_0}{\varepsilon^2} \right) \right] \frac{\partial^2 u(T, Z)}{\partial T^2} = 0, \quad Z > 0,$$

for

$$\left[ 1 + \varepsilon \tilde{\mu} \left( \frac{Z/c_0}{\varepsilon^2} \right) \right] = \left[ 1 + \varepsilon \mu \left( \frac{Z/c_0}{\varepsilon^2}, 0 \right) \right]^{-2}.$$

Before, we had a two point boundary value problem for the right and left going waves. The right going wave amplitude was specified in terms of the impinging pulse $f(t/\varepsilon^2)$ by (3.2) at $z = 0$. The left going wave amplitude was set to zero at range $L$, using causality and the finite time $T$ of observation. Now, after the change of variables, we have initial conditions at $Z = 0$, which corresponds to $t = 0$,

$$u(Z = 0, T) = f(T/\varepsilon^2)/(2\sqrt{\zeta_0}),$$

$$\frac{\partial u(Z = 0, T)}{\partial Z} = f'(T/\varepsilon^2)/(2\varepsilon^2 c_0 \sqrt{\zeta_0}).$$

The first initial condition follows from the wave decomposition (3.1) evaluated at $Z = 0$ (i.e., $t = 0$), where the wave is right going with amplitude given by the pulse $f(T/\varepsilon^2)$. The second condition follows from the same decomposition and the relation $\partial_z u(0, T) = -1/(c_0 \zeta_0) \partial_T p(0, T)$ obtained from the first order acoustic system of equations.
Fourier transforming in $\mathcal{T}$, we obtain that

$$\hat{u}_\varepsilon(\omega, Z) = \hat{u} \left( \frac{\omega}{\varepsilon^2}, Z \right)$$

satisfies the Cauchy problem

$$\frac{\partial^2 \hat{u}_\varepsilon(\omega, Z)}{\partial Z^2} + \frac{\omega^2}{\varepsilon^4 c_o^2} \left[ 1 + \varepsilon \hat{\mu} \left( \frac{Z}{c_o} \right) \right] \hat{u}_\varepsilon(\omega, Z) = 0, \quad Z > 0,$$

with initial conditions

$$\hat{u}_\varepsilon(\omega, 0) = \frac{\hat{f}(\omega)}{2\sqrt{c_o}}, \quad \frac{\partial \hat{u}_\varepsilon(\omega, 0)}{\partial Z} = -\frac{i\omega \hat{f}(\omega)}{2c_o \sqrt{c_o}},$$

studied in [8, Section 7.5.3], for random harmonic oscillators. The rate corresponding to exponential growth of the square root of the energy $|\hat{u}_\varepsilon|^2 + |\partial Z \hat{u}_\varepsilon|^2$ with range $Z$ is the Lyapunov exponent defined in [8, Proposition 7.9]. It coincides with the rate $\omega^2|\gamma^{(c)}|$ predicted by our analysis.

### 3.4. Example of a medium in translational motion

The integral equation of the wave front given in Lemma 3.1 is complicated in the asymptotic regime 3, because of the strong dependence of the random travel time fluctuations on the starting point $s$ of the characteristics. We do not have an explicit description of the wave front for an arbitrary model $\mu$ of the fluctuations, but there are models where the problem simplifies. We illustrate here such a model, corresponding to media in uniform translational motion

$$\mu \left( \frac{t}{\varepsilon^2}, \frac{z}{\varepsilon^2} \right) = \eta \left( \frac{z - vt}{\varepsilon^2} \right),$$

where $\eta$ is a mean zero stationary random process with autocorrelation $\mathcal{R}$. The process $\eta$ is as smooth as before and with finite dependence range. The speed $v$ may be positive or negative, but it satisfies the inequality $|v| < c_o$.

The random travel time fluctuations (3.8) are

$$W_\varepsilon(\tau, s) = \frac{1}{\varepsilon} \int_0^{\tau} du \eta \left( \frac{(c_o - v)u}{\varepsilon^2} - vs - vW_\varepsilon(u, s) \right),$$

and changing variables $\zeta = (c_o - v)u - \varepsilon^2 vs - \varepsilon^2 vW_\varepsilon(u, s)$, with $\zeta$ lying in the interval $\zeta \in [O(\varepsilon^2), (c_o - v)\tau + O(\varepsilon^2)]$, we obtain

$$W_\varepsilon(\tau, s) = \frac{1}{\varepsilon(c_o - v)} \int_0^{(c_o - v)\tau} d\zeta \eta \left( \frac{\zeta}{\varepsilon^2} \right) + O(\varepsilon).$$

This is independent of $s$ to leading order, and $W_\varepsilon$ converges in distribution, as $\varepsilon \to 0$, to the diffusion process

$$\mathcal{W}(\tau) = DB(\tau),$$

where $B(\tau)$ is standard Brownian motion and

$$D = \sqrt{\frac{\mathcal{R}(0)}{c_o - v}}.$$

(3.41)
Thus, we can set $S_{\alpha,\beta} = 0$ and $S_{\varepsilon}(y; \tau, s) = y$ in Lemma 3.1 and the random processes (3.11) are

$$
\mu_{\alpha,\beta}(t, z) = \left(1 \mp \frac{v}{c_0}\right) \eta'(z - vt).
$$

The pulse stabilization results in Theorem 3.3 and Corollary 3.4 hold as stated, with

$$
\Psi(s) = \left(\frac{v^2}{c_0^2} - 1\right) \mathcal{R}''((c_o + v)s),
$$

and the parameter $\gamma(\omega)$ in the pulse shaping kernel is

$$
\gamma(\omega) = \frac{(v^2 - c_o^2)}{4c_0^2c_o} \int_0^\infty du \mathcal{R}''((c_o + v)u) e^{2i\omega u}
$$

$$
= i \frac{(v - c_o)}{2\omega c_0(c_o + v)} + \gamma^{(c)}(\omega) + i\gamma^{(s)}(\omega).
$$

Here we used integration by parts and the normalization $\mathcal{R}(0) = 1$, and let

$$
\gamma^{(c)}(\omega) = \frac{(c_o - v)}{2c_o(c_o + v)^2} \hat{\mathcal{R}} \left(\frac{2\omega}{c_o + v}\right)
$$

$$
\gamma^{(s)}(\omega) = \frac{(c_o - v)}{c_o(c_0 + v)^2} \int_0^\infty dz \mathcal{R}(z) \sin \left[2\omega z/(c_o + v)\right].
$$

These results show that as $v \uparrow c_0$ the transformation of the pulse shape becomes small as if the pulse had propagated only through a thin section of the random medium since the right propagating wave component travels with a speed only slightly larger than the medium. However, the random travel time correction (3.41) becomes large since the averaging becomes “less effective”. In the limit $v \downarrow -c_o$ we see that the transformation of the pulse becomes large, since now the left propagating wave component travels with a speed similar to the medium and backscattering of this energy to the right propagating component is relatively delayed.

4. Proof of results. The proof builds on the approach in [8, Section 8.1]. See also [3, 4]. We use random travel time coordinates to derive in section 4.2 a time domain integral equation for the wave front. The coordinates are motivated by the method of characteristics applied to the first order system of partial differential equations satisfied by the right and left going waves, as described in section 4.1. The integral equation for the wave front is stated in Lemma 3.1 and holds in all the three asymptotic regimes. However, it simplifies in regimes 1-2, where the random fluctuations of the travel time are independent of the characteristics in the limit $\varepsilon \to 0$. The simplification is explained in section 4.3 and it leads to the explicit expression of the wave front stated in Theorem 3.3 as shown in section 4.4.

4.1. The wave decomposition and random travel time coordinates. The right and left going waves $A_\varepsilon(t, z)$ and $B_\varepsilon(t, z)$ defined by (3.1) satisfy the following system of first order partial differential equations derived in appendix A.

$$
\frac{1}{c_\varepsilon(t, z)} \frac{\partial A_\varepsilon(t, z)}{\partial t} + \frac{\partial A_\varepsilon(t, z)}{\partial z} = \frac{1}{\varepsilon} [M_\varepsilon(t, z) + N_\varepsilon(t, z)] B_\varepsilon(t, z), \quad (4.1)
$$

$$
\frac{1}{c_\varepsilon(t, z)} \frac{\partial B_\varepsilon(t, z)}{\partial t} - \frac{\partial B_\varepsilon(t, z)}{\partial z} = \frac{1}{\varepsilon} [-M_\varepsilon(t, z) + N_\varepsilon(t, z)] A_\varepsilon(t, z), \quad (4.2)
$$

\[ 14 \]
with initial condition (3.2) for $A_\varepsilon$ and final condition (3.3) for $B_\varepsilon$. Here
\[ M_\varepsilon(t, z) = -\varepsilon^{2-\beta} \mu_z \left( \frac{t}{\varepsilon}, \frac{z}{\varepsilon} \right) \frac{1}{2 \left[ 1 + \varepsilon \mu \left( \frac{t}{\varepsilon}, \frac{z}{\varepsilon} \right) \right]}, \] (4.3)
and
\[ N_\varepsilon(t, z) = -\varepsilon^{2-\alpha} \mu_t \left( \frac{t}{\varepsilon}, \frac{z}{\varepsilon} \right) \frac{1}{2 \varepsilon \left[ 1 + \varepsilon \mu \left( \frac{t}{\varepsilon}, \frac{z}{\varepsilon} \right) \right]^{1/2}}, \] (4.4)
where we use the notation
\[ \mu_z(t, z) = \frac{\partial \mu(t, z)}{\partial z}, \quad \mu_t(t, z) = \frac{\partial \mu(t, z)}{\partial t}. \]

We solve the first order system (4.1)-(4.2) using the method of characteristics. The characteristics of the right going waves are the curves $(t_\varepsilon(z, s), z)$ in the $(t, z)$ plane, where $t_\varepsilon(z, s)$ is the travel time (3.5), and $s$ parameterizes the impinging pulse at $z = 0$, giving the initial condition
\[ A_\varepsilon(t = \varepsilon^2 s, z = 0) = f(s), \quad s \in [0, S]. \] (4.5)
The right going wave observed at the random travel time is obtained by integrating equation (4.1) along the characteristics
\[ A_\varepsilon(t_\varepsilon(z, s), z) = f(s) + \frac{1}{\varepsilon} \int_0^z dz' G_\varepsilon(t_\varepsilon(z', s), z'), \] (4.6)
where
\[ G_\varepsilon(t, z) = [M_\varepsilon(t, z) + N_\varepsilon(t, z)] B_\varepsilon(t, z). \] (4.7)

To determine the left going wave $B_\varepsilon(t_\varepsilon(z', s), z')$, we integrate (4.2) along its characteristic passing through $(t_\varepsilon(z', s), z')$. As illustrated in Figure 4.1, we parameterize this characteristic by $-\eta$, starting from the point of intersection. We have
\[ B_\varepsilon(t_\varepsilon(z', s) + \tilde{t}_\varepsilon(\eta; z', s), z' + \eta) = B_\varepsilon(t_\varepsilon(z', s), z') - \frac{1}{\varepsilon} \int_0^\eta d\eta' Q_\varepsilon(t_\varepsilon(z', s) + \tilde{t}_\varepsilon(\eta'; z', s), z' + \eta'), \] (4.8)
where
\[ Q_\varepsilon(t, z) = [-M_\varepsilon(t, z) + N_\varepsilon(t, z)] A_\varepsilon(t, z), \] (4.9)
and
\[ \tilde{t}_\varepsilon(\eta; z', s) = -\int_0^\eta \frac{d\eta'}{c_\varepsilon \left( t_\varepsilon(z', s) + t_\varepsilon(\eta'; z', s), z' + \eta' \right)}. \] (4.10)

By causality, the left going wave is zero ahead of the front given by the curve $(t_\varepsilon(z, 0), z)$ for $z > 0$. We denote $\eta$ at the front by $\overline{\eta}$. It is defined by
\[ t_\varepsilon(z', s) + \tilde{t}_\varepsilon(\overline{\eta}; z', s) = t_\varepsilon(z' + \overline{\eta}, 0), \]
Fig. 4.1. The characteristics curves \((t_ε(z,s), z)\) of the right going waves start at times \(ε^2 s\) on the initial curve \(z = 0\), with \(s \in [0, S]\) parameterizing the pulse. We draw the characteristics approximately, as straight lines of slope \(1/c_o\), but we recall that \(t_ε(z,s)\) has random fluctuations of order \(ε^2\). The characteristic of the left going wave passing through the point \((t_ε(z',s), z')\) is parameterized by \(-η\), with \(η = 0\) at the point of intersection. The front of the pulse is on the characteristic curve \((t_ε(z,0), z)\), stemming from \(s = 0\) at \(z = 0\).

or equivalently, using (3.5),

\[
\tilde{t}_ε(\eta'; z', s) = t_ε(z' + \eta', 0) - t_ε(z', s) = \frac{\eta}{c_o} - ε^2 \left[ s + W_ε \left( \frac{z'}{c_o}, s \right) - W_ε \left( \frac{z' + \eta}{c_o}, 0 \right) \right].
\]

Now substitute this result in (4.10) and use the model (2.21) of the random wave speed to obtain

\[
\frac{2\eta}{c_o} + ε \int_0^\eta \frac{d\eta'}{c_o} \left( \frac{t_ε(z', s) + \tilde{t}_ε(\eta'; z', s)}{ε^2}, \frac{z' + \eta'}{ε^3} \right) = ε^2 \left[ W_ε \left( \frac{z'}{c_o}, s \right) - W_ε \left( \frac{z' + \eta}{c_o}, 0 \right) \right].
\]

Recalling that \(∂_s W_ε(τ, s) = O(1/ε)\), we conclude that

\[
η = \frac{ε^2 c_o}{2} \left[ s + W_ε \left( \frac{z'}{c_o}, s \right) - W_ε \left( \frac{z'}{c_o}, 0 \right) \right] + O(ε^3).
\]  

Lemma 3.2 states that in regimes 1-2 the random process \(W_ε(τ, s)\) is independent of \(s\) to leading order, so \(η ≈ ε^2 c_o s/2\). In the third regime the process \(W_ε(τ, s)\) depends on \(s\) and \(η\) is random to leading order. In either case \(η\) is guaranteed to be positive, because the characteristic stemming from \(ε^2 s\) at \(z = 0\) cannot intersect the front of the pulse, the characteristic stemming from the origin. Explicitly, we have with \(s > 0\)

\[
t_ε(z', s) > t_ε(z', 0), \quad ∀z' ≥ 0,
\]

\(^1\)The characteristics do not intersect because they are solutions of a flow (ordinary differential equation) problem that is uniquely solvable [7, Section 3.2.5.a].
and therefore
\[ s + W_\varepsilon \left( \frac{z'}{c_o}, s \right) > W_\varepsilon \left( \frac{z'}{c_o}, 0 \right). \]

The left going wave at the point \((t_\varepsilon(\zeta', s), \zeta')\) of intersection with the right going characteristic is
\[ B_\varepsilon(t_\varepsilon(\zeta', s), \zeta') = \frac{1}{\varepsilon} \int_{0}^{\varepsilon} d\eta' Q_\varepsilon(t_\varepsilon(\zeta', s) + \tilde{t}_\varepsilon(\eta'; \zeta', s), \zeta' + \eta'). \]

(4.12)

It follows by setting \(\eta = \eta\) in (4.8), because the left hand side is the left going wave which vanishes at the wave front.

4.2. The integral equation. The integral equation for the wave front is obtained by substituting (4.12) in (4.6). We write it in more convenient form by changing variables in (4.12) as
\[ \eta' = \frac{\varepsilon^2 c_o (s - y)}{2}. \]

(4.13)

That \(y \leq s\) follows from \(\eta' \geq 0\), and in the regimes 1-2 where (4.11) simplifies to \(\eta \approx \varepsilon^2 c_o s/2\) we can take \(S_{\alpha,\beta} = 0\). Otherwise, we have from (4.11) that
\[ W_\varepsilon \left( \frac{z'}{c_o}, 0 \right) - W_\varepsilon \left( \frac{z'}{c_o}, s \right) \leq y \leq s. \]

Moreover, since the pulse is compactly supported in \([0, S]\) and the characteristics cannot cross, we obtain from (3.5) that
\[ S + W_\varepsilon \left( \frac{z'}{c_o}, S \right) > s + W_\varepsilon \left( \frac{z'}{c_o}, s \right) > W_\varepsilon \left( \frac{z'}{c_o}, 0 \right), \quad \forall s \in [0, S]. \]

This leads to the \(s\) independent bound on \(y\)
\[ y \geq W_\varepsilon \left( \frac{z'}{c_o}, 0 \right) - W_\varepsilon \left( \frac{z'}{c_o}, s \right) \]
\[ > W_\varepsilon \left( \frac{z'}{c_o}, 0 \right) - W_\varepsilon \left( \frac{z'}{c_o}, S \right) + s - S \]
\[ > W_\varepsilon \left( \frac{z'}{c_o}, 0 \right) - W_\varepsilon \left( \frac{z'}{c_o}, S \right) - S, \]

which is order one with high probability in view of Lemma 3.2 and Appendix B.

The equation for the left going wave becomes
\[ B_\varepsilon(t_\varepsilon(\zeta', s), \zeta') = \frac{\varepsilon c_o}{2} \int_{-S_{\alpha,\beta}}^{s} d\eta Q_\varepsilon \left[ t_\varepsilon(\zeta', s) + \tilde{t}_\varepsilon \left( \frac{\varepsilon^2 c_o (s - y)}{2}; \zeta' \right), \zeta' \right. \]
\[ \left. + \frac{\varepsilon^2 c_o (s - y)}{2} \right], \]

where we note that
\[ \tilde{t}_\varepsilon \left( \frac{\varepsilon^2 c_o (s - y)}{2}; \zeta', s \right) = - \int_{0}^{\varepsilon^2 c_o (s - y)^3} d\eta' \frac{c_o}{\varepsilon} \left( t_\varepsilon(\zeta', s) + \tilde{t}_\varepsilon(\eta'; \zeta', s), \zeta' + \eta' \right) \]
\[ = - \frac{\varepsilon^2 (s - y)}{2} + O(\varepsilon^3). \]
Then substituting in (4.6) we get
\[ A_\varepsilon \left[ t_\varepsilon(z,s), z \right] = f(s) - \frac{c_o}{2} \int_0^s dz' (M_\varepsilon + N_\varepsilon) (t_\varepsilon(z',s), z') \times \]
\[ \int_{-S_{\alpha,\beta}} dz \: A_\varepsilon \left[ t_\varepsilon(z',s) + i_\varepsilon \left( \frac{\varepsilon^2 c_o (s-y)}{2}; z', s \right), z' + \frac{\varepsilon^2 c_o (s-y)}{2} \right] \times \]
\[ (M_\varepsilon - N_\varepsilon) \left[ t_\varepsilon(z',s) + i_\varepsilon \left( \frac{\varepsilon^2 c_o (s-y)}{2}; z', s \right), z' + \frac{\varepsilon^2 c_o (s-y)}{2} \right]. \]

(4.14)

We show next that \( A_\varepsilon \) is bounded by a constant and then derive the equation for the wave front
\[ a_\varepsilon \left( \frac{z}{c_o}, s \right) := A_\varepsilon \left[ t_\varepsilon(z,s), z \right]. \]

Introducing \( A_\varepsilon \) by
\[ A_\varepsilon (t) = \sup_z |A_\varepsilon (t, z)|, \]
we find, using that the rays in Figure [4.1] are bounded away from the horizontal, that
\[ A_\varepsilon (t) \leq ||f||_{\infty} + C \int_0^t A_\varepsilon (s) ds, \]
for \( C \) a constant and \( \varepsilon < 1 \), where \( || \cdot ||_{\infty} \) denotes the sup norm in \( s \in [0, S] \). The boundedness of \( A_\varepsilon \) and therefore of \( A_\varepsilon \) follows by Gronwall’s lemma.

To close the equation for the wave front \( a_\varepsilon \), we rewrite
\[ t_\varepsilon(z',s) + \tilde{t}_\varepsilon \left( \frac{\varepsilon^2 c_o (s-y)}{2}; z', s \right) = \frac{z'}{c_o} + \frac{\varepsilon^2 (y-s)}{2} + \varepsilon^2 \left[ s + W_\varepsilon \left( \frac{z'}{c_o}, s \right) \right] + O(\varepsilon^3), \]
using the mapping \( S_\varepsilon(y; \tau, s) \) defined implicitly\(^5\) in (3.12). We get
\[ t_\varepsilon(z',s) + \tilde{t}_\varepsilon \left( \frac{\varepsilon^2 c_o (s-y)}{2}; z', s \right) = \frac{z'}{c_o} + \frac{\varepsilon^2 (s-y)}{2} + \varepsilon^2 \left[ y + W_\varepsilon \left( \frac{z'}{c_o}, s \right) \right] + O(\varepsilon^3) \]
\[ = \frac{z'}{c_o} + \frac{\varepsilon^2 (s-y)}{2} + \varepsilon^2 \left[ S_\varepsilon \left( y; \frac{z'}{c_o}, s \right) + W_\varepsilon \left( \frac{z'}{c_o} + \frac{\varepsilon^2 (s-y)}{2}, S_\varepsilon \left( y; \frac{z'}{c_o}, s \right) \right) \right], \]
and therefore
\[ A_\varepsilon \left[ t_\varepsilon(z',s) + \frac{\varepsilon^2 (y-s)}{2}, z' + \frac{\varepsilon^2 c_o (s-y)}{2} \right] = a_\varepsilon \left[ \frac{z'}{c_o} + \frac{\varepsilon^2 (s-y)}{2}, S_\varepsilon \left( y; \frac{z'}{c_o}, s \right) \right]. \]
The integral equation for the wave front becomes
\[ a_\varepsilon (\tau, s) = f(s) - \frac{c_o}{2} \int_0^\tau d\tau' (M_\varepsilon + N_\varepsilon) (t_\varepsilon(c_o \tau', s), c_o \tau') \times \]
\[ \int_{-S_{\alpha,\beta}} d\tau \: a_\varepsilon \left( \tau' + \varepsilon^2 (s-y)/2, S_\varepsilon (y; \tau', s) \right) \times \]
\[ (M_\varepsilon - N_\varepsilon) \left[ t_\varepsilon(c_o \tau', s) + \frac{\varepsilon^2 (y-s)}{2}, c_o \tau' + \frac{\varepsilon^2 c_o (s-y)}{2} \right] + O(\varepsilon). \]

(4.18)

\(^5\)In fact, we use here a slightly modified version of the map \( S_\varepsilon \) which coincides with that in (3.12) to leading order.
It is clear from (4.18) that \( \partial_s a_\varepsilon \) is bounded, so we can neglect the \( \varepsilon^2(s - y)/2 \) term in the argument of \( a_\varepsilon \). The statement of Lemma 3.1 follows from definitions (4.3)-(4.4) of \( M_\varepsilon \) and \( N_\varepsilon \).

**4.3. Simplification in regimes 1-2.** Using Lemma 3.2 in equations (3.12), (4.11), (4.13), and (4.17), we conclude that

\[
S_{\alpha,\beta} = o(1) \quad \text{and} \quad S_\varepsilon(y; \tau, s) = y + o(1),
\]

as \( \varepsilon \to 0 \). It remains to prove that \( \partial_s a_\varepsilon(\tau, s) \) is bounded independent of \( \varepsilon \), in order to simplify the integral equation of the wave front as

\[
a_\varepsilon(\tau, s) = f(s) + \int_0^\tau \int_0^y dy \, \mathcal{G}_\varepsilon(\tau', s, y) a_\varepsilon(\tau', y) + o(1).
\]

Here we introduced the notation

\[
\mathcal{G}_\varepsilon(\tau', s, y) := -\frac{c^2}{8} \mu_{\alpha,\beta} \left[ \frac{\tau' + \varepsilon^2 (s + \mathcal{W}_\varepsilon(\tau', s))}{\varepsilon^\alpha}, \frac{c_\alpha \tau'}{\varepsilon^\beta} \right] \times \mu_{\alpha,\beta}^{\pm} \left[ \frac{\tau' + \varepsilon^2 (\frac{W_\varepsilon(\tau', s)}{\varepsilon} + \mathcal{W}_\varepsilon(\tau', s))}{\varepsilon^\alpha}, \frac{c_\alpha + \varepsilon^2 c_\alpha (s - y)/2}{\varepsilon^\beta} \right],
\]

with \( \mu_{\alpha,\beta}^{\pm} \) defined in (3.11).

To estimate \( \partial_s a_\varepsilon \), we begin by taking the derivative with respect to \( s \) in the equation in Lemma 3.1 rewritten here as

\[
a_\varepsilon(\tau, s) = f(s) + \int_0^\tau \int_0^s dy \, \mathcal{G}_\varepsilon(\tau', s, y) a_\varepsilon(\tau', \mathcal{S}_\varepsilon(y; \tau, s)).
\]

We obtain that

\[
\partial_s a_\varepsilon(\tau, s) = f'(s) + \int_0^\tau \int_0^s dy \, \partial_s \mathcal{G}_\varepsilon(\tau', s, y) a_\varepsilon(\tau', \mathcal{S}_\varepsilon(y; \tau, s)) + \int_0^\tau \int_0^s dy \, \partial_s \mathcal{G}_\varepsilon(\tau', s, y) a_\varepsilon(\tau', \mathcal{S}_\varepsilon(y; \tau, s)) + \]

\[
\int_0^\tau \int_0^s dy \, \partial_s \mathcal{G}_\varepsilon(\tau', s, y) \, \partial_s \mathcal{S}_\varepsilon(y; \tau, s) \, \partial_s a_\varepsilon(\tau', \mathcal{S}_\varepsilon(y; \tau, s)).
\]

It is clear from the definition of \( \mathcal{G}_\varepsilon \) and the assumption that \( \mu \) is twice differentiable with bounded derivatives that both \( \mathcal{G}_\varepsilon \) and \( \partial_s \mathcal{G}_\varepsilon \) are bounded. Moreover, taking the derivative with respect to \( s \) in definition (4.17) of \( \mathcal{S}_\varepsilon \) and using Lemma 3.2, we obtain that \( \partial_s \mathcal{S}_\varepsilon = o(1) \). Thus, using the boundedness of \( a_\varepsilon \) we have

\[
\| \partial_s a_\varepsilon(\tau, \cdot) \|_\infty \leq \left[ C_1 + C_2 \int_0^\tau d\tau' \| \partial_s a_\varepsilon(\tau', \cdot) \|_\infty \right],
\]

with high probability for \( \varepsilon < \varepsilon_0 \), and some constants \( C_1 \) and \( C_2 \). We conclude that \( \partial_s a_\varepsilon \) is bounded.

**4.4. Stochastic averaging.** To show that \( a_\varepsilon \to \bar{\pi} \) as stated in Theorem 3.3 it suffices to show that

\[
\sup_{\tau \in [0, T]} \mathbb{E} \{ \| a_\varepsilon(\tau, \cdot) - \bar{\pi}(\tau, \cdot) \|_\infty \} \to 0, \quad \text{as} \quad \varepsilon \to 0.
\]

The derivation of (4.24) in regime 1 is basically the same as that in [8; Appendix 8.4]. We present here a derivation that is convenient for the case of rapidly changing media of regime 2 under the present parameterization.
4.4.1. The linear integral operators. Let us rewrite the equation (4.20) of the wave front as

\[ a_\epsilon(\tau, s) = f(s) + \int_0^\tau du \left[ L_\epsilon(u)a_\epsilon(u, \cdot) \right](s), \] (4.25)

using the linear operator \( L_\epsilon(u) \) defined on the space \( C_{[0,S]} \) of continuous functions of \( s \in [0,S] \), with values in \( C_{[0,S]} \). The operator is given by

\[ [L_\epsilon(u)\varphi](s) = \int_0^s dy \mathcal{G}_\epsilon(u, s, y)\varphi(y), \quad \forall \varphi \in C_{[0,S]}, \] (4.26)

with \( \mathcal{G}_\epsilon \) defined by (4.21) for \( \tau' = u \), and \( \mu_{\alpha,\beta}^\pm \) obtained from definition (3.11)

\[ \mu_{\alpha,\beta}^\pm(t, z) = \pm c_o^{-1}\mu(t, z) + \epsilon^{2-\beta}\mu_z(t, z). \] (4.27)

Since \( \mu_t \) is bounded, we conclude that \( L_\epsilon(u) \) is Lipschitz continuous (i.e., bounded) with deterministic Lipschitz constant \( C_L \) that is independent of \( \epsilon \) and \( u \),

\[ \|L_\epsilon(u)\varphi\|_\infty \leq C_L \|\varphi\|_\infty, \quad \forall \varphi \in C_{[0,S]}. \] (4.28)

Similarly, we write the equation satisfied by the limit \( a_\) as

\[ a(\tau, s) = f(s) + \int_0^\tau du \left[ \overline{L}(u)a(u, \cdot) \right](s), \] (4.29)

using the bounded linear operator \( \overline{L}(u) : C_{[0,S]} \to C_{[0,S]} \), defined by

\[ \left[ \overline{L}(u)\varphi \right](s) = -\frac{c_o^2}{8} \int_0^s dy \varphi(y) \Psi \left( \frac{s-y}{2} \right), \quad \forall \varphi \in C_{[0,S]}, \] (4.30)

where

\[ \Psi(s) = -c_o^{-2}E \left\{ \mu(t, z)\mu(t+s, z) \right\} = c_o^{-2}\partial_2^s\Phi(s, 0). \] (4.31)

Note that (4.30) is the same as the expectation of (4.26) up to a term of order \( \epsilon^{2-\beta} \).

4.4.2. Averaging. We now show that the limit (4.24) holds. We have

\[ a_\epsilon(\tau, s) - \overline{a}(\tau, s) = g_\epsilon(\tau, s) + \int_0^\tau du \left[ L_\epsilon(u)(a_\epsilon(u, \cdot) - \overline{a}(u, \cdot)) \right](s), \] (4.32)

where

\[ g_\epsilon(\tau, s) = \int_0^\tau du \left[ (L_\epsilon(u) - \overline{L}(u))\overline{a}(u, \cdot) \right](s). \] (4.33)

Recalling that \( L_\epsilon(u) \) is bounded (equation (4.28)), and using the triangle inequality, we get from (4.32) that

\[ \|a_\epsilon(\cdot) - \overline{a}(\cdot)\|_\infty \leq \|g_\epsilon(\cdot, \cdot)\|_\infty + C_L \int_0^\tau du \|a_\epsilon(u, \cdot) - \overline{a}(u, \cdot)\|_\infty. \] (4.34)

Now Gronwall’s inequality gives

\[ \|a_\epsilon(\tau, \cdot) - \overline{a}(\tau, \cdot)\|_\infty \leq e^{\tau C_L} \sup_{0 < t < \tau} \|g_\epsilon(t, \cdot)\|_\infty, \] (4.35)
so it remains to show that $\mathbb{E}[\|g_\varepsilon(\tau, \cdot)\|_\infty] \to 0$, uniformly in $\tau \in [0, T]$.

Let us write $g_\varepsilon$ explicitly, using the definitions (4.26) and (4.30) of the linear operators

$$g_\varepsilon(\tau, s) = \frac{1}{8} \int_0^s dy V_\varepsilon(\tau; s, y), \quad (4.36)$$

in terms of the process

$$V_\varepsilon(\tau; s, y) = \int_0^\tau du \pi(u, y) c_0^2 \Psi \left( \frac{s-y}{2} \right) + \mu_t \left( \frac{u}{\varepsilon^2} + s + W_\varepsilon(u, s), \frac{c_0 u}{\varepsilon^\beta} \right) \times$$

$$\mu_t \left( \frac{u}{\varepsilon^2} + \frac{s+y}{2} + W_\varepsilon(u, s), \frac{c_0 u}{\varepsilon^\beta} + \varepsilon^{2-\beta} c_0 (s-y) \right). \quad (4.37)$$

We can obtain its limit as $\varepsilon \to 0$ using the diffusion approximation theorem in [13] for the joint Markov process

$$X_\varepsilon(\tau) = \left( \begin{array}{c} W_\varepsilon(\tau, s) \\ V_\varepsilon(\tau; s, y) \end{array} \right),$$

where we exclude the fixed parameters $s$ and $y$ in the arguments of the left hand side. We refer to appendix B for details on the theorem. Here we note that $X_\varepsilon(\tau)$ satisfies a system of stochastic differential equations of the form (B.2), with random vector fields

$$F(\tau, x, \sigma) = e_1 \mu(\sigma + s + x_1, c_0 \varepsilon^{2-\beta} \sigma)$$

for $x = x_1 e_1 + x_2 e_2$, and

$$G(\tau, x, \sigma) = e_2 \left[ \pi(\tau, s) \mu_1 (\sigma + s + x_1, c_0 \varepsilon^{2-\beta} \sigma) \mu_t (\sigma + (s + y)/2 + x_1, c_0 \varepsilon^{2-\beta} \sigma) + \pi(\tau, s) c_0^2 \Psi \left( \frac{s-y}{2} \right) + O(\varepsilon^{2-\beta}) \right].$$

The first argument of these fields is the slow variable (the travel time without any scaling factor), and the last argument is the fast variable (the travel time scaled by $\varepsilon^2$). Only $G$ depends on $\tau$, via the continuously differentiable function $\pi(\tau, s)$.

Applying the diffusion approximation theorem we get that $X_\varepsilon(\tau)$ converges in distribution to

$$X(\tau) = \left( \begin{array}{c} W(\tau) \\ 0 \end{array} \right)$$

where $W(\tau)$ is the process defined by equations (3.7). This gives that

$$V_\varepsilon(\tau; s, y) \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

in distribution and therefore also in probability.

Taking expectations in

$$|g_\varepsilon(\tau, s)| = \frac{1}{8} \left| \int_0^s dy V_\varepsilon(\tau; s, y) \right| \leq \frac{1}{8} \int_0^s dy |V_\varepsilon(\tau; s, y)|,$$

and the limit $\varepsilon \to 0$, we get

$$\lim_{\varepsilon \to 0} \mathbb{E}[|g_\varepsilon(\tau, s)|] \leq \lim_{\varepsilon \to 0} \frac{1}{8} \int_0^s dy \mathbb{E}[|V_\varepsilon(\tau; s, y)|] = 0,$$
where the last equality follows from the dominated convergence theorem. This holds pointwise for all \( s \in [0, S] \) and \( \tau \in [0, \bar{T}] \). However, \( \mathcal{V}_\varepsilon(\tau; s, y) \) is bounded and differentiable in \( s \) and \( \tau \), with bounded derivatives uniformly in \( s \in [0, S] \) and \( \tau \in [0, \bar{T}] \). This follows by taking derivatives in definition \( (4.37) \), using the chain rule, Lemma 3.2 and the assumption that \( \mu \) is twice differentiable with bounded derivatives. We conclude that the family of mappings \( \{ f_0 \int_0^s dy \mathcal{V}_\varepsilon \} \) is equicontinuous as \( \varepsilon \to 0 \), and obtain the uniform convergence result

\[
\lim_{\varepsilon \to 0} \sup_{\tau \in [0, \bar{T}]} E[\|g_\varepsilon(\tau, \cdot)\|_\infty] = 0.
\]

5. Summary. We have considered wave propagation in one dimensional random media in the situation when the wave speed varies randomly in both space and time. We derive a mathematical framework involving a random integral equation that can be used to characterize the pulse in this case. The main result is that the pulse stabilization phenomenon known in time independent random media extends to the time variable case. Pulse stabilization means that the amplitude of the wave transmitted from the source to range \( z \) has a deterministic form, when observed in a time window centered at \( \tau + \delta \tau \), and of width comparable to the support of the initial pulse. Here \( \tau \) is the deterministic travel time to range \( z \), and \( \delta \tau \) is a small random correction, on the scale of the support of the initial pulse. It is well known that in time independent media the transmitted pulse is faded and broadened, because scattering in the medium produces delays and also transfers energy to the incoherent waves.

We assume random temporal changes in the medium, modeled by temporal fluctuations of the wave speed. We call the time fluctuations slow, when the life span of a spatial realization of the random wave speed is longer than the time of traversal of a correlation length and the pulse width. The fluctuations are rapid when these time scales are similar. The results show that the slow fluctuations have to leading order no effect on the wave front. The rapid fluctuations affect both the random arrival time correction \( \delta \tau \) and the deterministic pulse deformation. In particular, there is a trade-off between frequency dependent attenuation due to scattering in the medium, and energy gain induced by the temporal fluctuations.

The analysis is based on a stochastic time domain integral equation satisfied by the wave front. The approach is similar to that in [8] Section 8.1 for time independent media, and the pulse stabilization follows from stochastic averaging. Furthermore, the energy gain induced by the temporal fluctuations is explained by drawing an analogy to the random harmonic oscillator problem considered in [8] Section 7.5.

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Appendix A. Derivation of the first order system for the right and left going waves. Let us take the \( z \) derivative in the first equation in (3.1) to obtain

\[
\frac{\partial A_\varepsilon}{\partial z} = \zeta_\varepsilon^{-1/2} \frac{\partial p_\varepsilon}{\partial z} - \frac{p_\varepsilon}{2\zeta_\varepsilon^{3/2}} \frac{\partial \zeta_\varepsilon}{\partial z} + \zeta_\varepsilon^{1/2} \frac{\partial u_\varepsilon}{\partial z} + \frac{u_\varepsilon}{2\zeta_\varepsilon^{1/2}} \frac{\partial \zeta_\varepsilon}{\partial z}.
\]

(A.1)

Since \( p_\varepsilon \) and \( u_\varepsilon \) satisfy (2.1), we have for \( z \neq 0 \)

\[
\frac{\partial A_\varepsilon}{\partial z} = -\rho_\varepsilon^{-1/2} \frac{\partial u_\varepsilon}{\partial t} - \frac{p_\varepsilon}{2\zeta_\varepsilon^{3/2}} \frac{\partial \zeta_\varepsilon}{\partial z} - \zeta_\varepsilon^{1/2} K_\varepsilon^{-1} \frac{\partial p_\varepsilon}{\partial t} + \frac{u_\varepsilon}{2\zeta_\varepsilon^{1/2}} \frac{\partial \zeta_\varepsilon}{\partial z}.
\]

(A.2)
Equations (3.1) give that $p_\varepsilon = \zeta^{1/2}(A_\varepsilon - B_\varepsilon)/2$ and $u_\varepsilon = \zeta^{-1/2}(A_\varepsilon + B_\varepsilon)/2$, and therefore

$$\frac{\partial p_\varepsilon}{\partial t} = \frac{\zeta^{1/2}}{2} \left( \frac{\partial A_\varepsilon}{\partial t} - \frac{\partial B_\varepsilon}{\partial t} \right) + \frac{1}{4\zeta^{1/2}} \frac{\partial \zeta}{\partial t} (A_\varepsilon - B_\varepsilon),$$

$$\frac{\partial u_\varepsilon}{\partial t} = \frac{\zeta^{-1/2}}{2} \left( \frac{\partial A_\varepsilon}{\partial t} + \frac{\partial B_\varepsilon}{\partial t} \right) - \frac{1}{4\zeta^{3/2}} \frac{\partial \zeta}{\partial t} (A_\varepsilon + B_\varepsilon).$$

(A.3)

Substituting in (A.2) and using that $\rho/\zeta_\varepsilon = 1/c_\varepsilon$, we obtain

$$\frac{\partial A_\varepsilon}{\partial z} = -\frac{1}{c_\varepsilon} \frac{\partial A_\varepsilon}{\partial t} + \left( \frac{1}{2\zeta_\varepsilon} \frac{\partial \zeta_\varepsilon}{\partial z} + \frac{1}{2c_\varepsilon \zeta_\varepsilon} \frac{\partial \zeta_\varepsilon}{\partial t} \right) B_\varepsilon.$$  

(A.4)

Equation (4.1) follows with

$$M_\varepsilon(t, z) = \varepsilon^2 \zeta_\varepsilon \frac{\partial \zeta_\varepsilon}{\partial z}, \quad N_\varepsilon(t, z) = \varepsilon^2 c_\varepsilon \zeta_\varepsilon \frac{\partial \zeta_\varepsilon}{\partial t}.$$  

(A.5)

Equation (4.2) follows similarly.

**Appendix B. The random travel time correction.** Let us consider first the regime 3, where the random process (3.6) is

$$W_\varepsilon(\tau, s) = \frac{1}{\varepsilon} \int_0^\tau du \mu \left( \frac{u}{\varepsilon^2} + s + W_\varepsilon(u, s), \frac{c_\varepsilon u}{\varepsilon^2} \right).$$  

(B.1)

To obtain its limit as $\varepsilon \to 0$, we use the theorem in [13] that applies to stochastic differential equations of the form

$$\frac{dX_\varepsilon(\tau)}{d\tau} = \frac{1}{\varepsilon} F(\tau, X_\varepsilon(\tau), \frac{\tau}{\varepsilon^2}) + G(\tau, X_\varepsilon(\tau), \frac{\tau}{\varepsilon^2}), \quad \tau \geq 0,$$  

(B.2)

for $X_\varepsilon(\tau) \in \mathbb{R}^n$, $n \geq 1$, satisfying the initial condition

$$X_\varepsilon(0) = x.$$  

(B.3)

The right hand-side is assumed smooth, with derivatives that grow at most polynomially in $|X_\varepsilon|$ and for any fixed $\tau$, $X$ and $t$,

$$\mathbb{E}\{F(\tau, X, t)\} = 0.$$  

(B.4)

Obviously, $W_\varepsilon$ satisfies a problem like (B.2), with $F = \mu$ and $G = 0$. The initial condition is $W_\varepsilon(0) = 0$.

The theorem in [13] gives that $X_\varepsilon(\tau) \to X_0(\tau)$ in distribution, as $\varepsilon \to 0$, where $X_0(\tau)$ is a Markov diffusion with drift $\theta \in \mathbb{R}^n$ given by

$$\theta(\tau, x) = \lim_{\varepsilon \to 0} \int_0^{\tau+\varepsilon} \frac{d\sigma}{\varepsilon} \left[ \int_0^\sigma \frac{du}{\varepsilon^2} \mathbb{E} \left\{ F \left( \tau, x, \frac{u}{\varepsilon^2} \right) \right\} \cdot \nabla_{x} F \left( \tau, x, \frac{\sigma}{\varepsilon^2} \right) \right] + \mathbb{E} \left\{ G \left( \tau, x, \frac{\sigma}{\varepsilon^2} \right) \right\},$$  

(B.5)

and matrix valued diffusion matrix $D(\tau, x) \in \mathbb{R}^{n \times n}$ with entries

$$D_{ij}(\tau, x) = \lim_{\varepsilon \to 0} \int_0^{\tau+\varepsilon} \frac{d\sigma}{\varepsilon} \int_0^\sigma \frac{du}{\varepsilon^2} \mathbb{E} \left\{ F_i \left( \tau, x, \frac{u}{\varepsilon^2} \right) F_j \left( \tau, x, \frac{\sigma}{\varepsilon^2} \right) \right\}, \quad i, j = 1, \ldots n.$$  

(B.6)
Our process is scalar valued, with
\[
\mathbf{D} = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu \left( \frac{u}{\epsilon^2} + s + \mathbf{x}, \frac{c_0u}{\epsilon^2} \right) \mu \left( \frac{\sigma}{\epsilon^2} + s + \mathbf{x}, \frac{c_0\sigma}{\epsilon^2} \right) \]
\[
= \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} du \Phi \left( \frac{u-\sigma}{\epsilon^2}, \frac{c_0(u-\sigma)}{\epsilon^2} \right)
\]
\[
= \frac{1}{2} \int_{-\infty}^{\infty} dh \Phi (h, c_0h). \quad (B.7)
\]
This is what we call $D^2/2$ in (3.8). The drift coefficient is
\[
\theta = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} du \frac{\partial}{\partial t} \Phi \left( \frac{\sigma-u}{\epsilon^2}, \frac{c_0(\sigma-u)}{\epsilon^2} \right) = \int_{0}^{\infty} dh \frac{\partial}{\partial t} \Phi (h, c_0h), \quad (B.8)
\]
as stated in (3.8).

The results in [13] extend to regime 1 with $\alpha < 2$ and $\beta = 2$. The diffusion coefficient is $D = 2\mathbf{D}$, where
\[
\mathbf{D} = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} du \mu \left( \frac{u}{\epsilon^2} + s + \mathbf{x}, \frac{c_0u}{\epsilon^2} \right) \mu \left( \frac{\sigma}{\epsilon^2} + s + \mathbf{x}, \frac{c_0\sigma}{\epsilon^2} \right)
\]
\[
= \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} du \Phi \left( \frac{u-\sigma}{\epsilon^2}, \frac{c_0(u-\sigma)}{\epsilon^2} \right)
\]
\[
= \frac{1}{2} \int_{-\infty}^{\infty} dh \Phi (0, c_0h), \quad (B.9)
\]
where we neglected the $\epsilon^{2-\alpha}(s+\mathbf{x})$ terms in the first argument of the smooth function $\mu$, because they play no role in the limit $\epsilon \to 0$. The drift coefficient vanishes $\theta = 0$.

Finally, the second regime where $\beta < \alpha = 2$ and
\[
\mathcal{W}_\epsilon (\tau, s) = \frac{1}{\epsilon} \int_{0}^{\tau} du \mu \left( \frac{u}{\epsilon^2} + s + \mathcal{W}_\epsilon (u, s), \frac{c_0u}{\epsilon^2} \right), \quad (B.10)
\]
can be handled with the change of variables
\[
\zeta = u + \epsilon^2 s + \epsilon^2 \mathcal{W}_\epsilon (u, s), \quad (B.11)
\]
so that
\[
d\zeta = \left[ 1 + \epsilon \mu \left( \frac{u}{\epsilon^2} + s + \mathcal{W}_\epsilon (u, s), \frac{c_0u}{\epsilon^2} \right) \right] du. \quad (B.12)
\]
Since $\mu$ is bounded by assumption, we know from the implicit function theorem that we can invert (B.11) to get a unique function $u = u(\zeta)$, and evidently, $u = \zeta + O(\epsilon^2)$. Substituting (B.11) in (B.10), we obtain that
\[
\mathcal{W}_\epsilon (\tau, s) \approx \frac{1}{\epsilon} \int_{0}^{\tau} d\zeta \frac{\mu \left( \frac{\zeta}{\epsilon^2}, \frac{c_0\zeta}{\epsilon^2} - \frac{\epsilon^2-\alpha}{\epsilon^{2-\beta}} c_0 (s + \mathcal{W}_\epsilon (\zeta, s)) \right)}{1 + \epsilon \mu \left( \frac{\zeta}{\epsilon^2}, \frac{c_0\zeta}{\epsilon^2} - \frac{\epsilon^2-\beta}{\epsilon^{2-\beta}} c_0 (s + \mathcal{W}_\epsilon (\zeta, s)) \right)}
\]
and expanding the right hand side in $\varepsilon$,

$$W_{\varepsilon}(\tau, s) \approx \frac{1}{\varepsilon} \int_0^\tau d\zeta \mu \left( \frac{\zeta}{\varepsilon^2}, \frac{c_0 \zeta}{\varepsilon^3} - \varepsilon^{2-\beta} c_0 (s + W_{\varepsilon}(\zeta, s)) \right) - \int_0^\tau d\zeta \mu^2 \left( \frac{\zeta}{\varepsilon^2}, \frac{c_0 \zeta}{\varepsilon^3} - \varepsilon^{2-\beta} c_0 (s + W_{\varepsilon}(\zeta, s)) \right).$$  \hspace{1cm} (B.13)

Here the approximate sign stands for equal up to absolute errors that tend to zero as $\varepsilon \to 0$. The limit of $W_{\varepsilon}(\tau, s)$ to the Markov diffusion process $W(\tau, s)$ follows from this equation as before. The diffusion coefficient is $D = 2D$, where

$$D = \lim_{\varepsilon \to 0} \int_{\tau + \varepsilon}^{\tau + \varepsilon} \frac{d\sigma}{\varepsilon} \int_{\tau}^{\sigma} \frac{du}{\varepsilon^2} \mathbb{E} \left\{ \mu\left( \frac{u}{\varepsilon^2}, \frac{c_0 u}{\varepsilon^3} \right) \mu\left( \frac{\sigma}{\varepsilon^2}, \frac{c_0 \sigma}{\varepsilon^3} \right) \right\} \int_{\tau}^{\sigma} \frac{d\mu}{\varepsilon^2} \Phi \left( \frac{u - \sigma}{\varepsilon^2}, \frac{c_0 (u - \sigma)}{\varepsilon^3} \right)$$

and the drift is

$$\theta = -\lim_{\varepsilon \to 0} \int_{\tau + \varepsilon}^{\tau + \varepsilon} \frac{d\sigma}{\varepsilon} \mathbb{E} \left\{ \mu^2\left( \frac{u}{\varepsilon^2}, \frac{c_0 u}{\varepsilon^3} \right) \right\} = -1,$$  \hspace{1cm} (B.14)

and the drift is

$$\theta = -\lim_{\varepsilon \to 0} \int_{\tau + \varepsilon}^{\tau + \varepsilon} \frac{d\sigma}{\varepsilon} \left\{ \mu^2\left( \frac{u}{\varepsilon^2}, \frac{c_0 u}{\varepsilon^3} \right) \right\} = -1,$$  \hspace{1cm} (B.15)

where we used the normalization (2.13).

**Appendix C. Dependence on the starting point of the characteristic.** We use the diffusion approximation theorem in [13] to study the dependence of $W_{\varepsilon}(\tau, s)$ on $s$. We begin with the case of rapidly changing media in regime 3, and then consider slowly changing media, in regime 1. The analysis in the second regime is similar to that in regime 1, once we make the change of variables (B.11).

In regime 3 the travel time fluctuations are defined by (B.1), and to analyze how they decorrelate in $s$, consider the vector valued process $X_{\varepsilon}(\tau) = \left( W_{\varepsilon}(\tau, s_1), W_{\varepsilon}(\tau, s_2) \right)$ which clearly satisfies an equation of the form (B.2). We obtain that as $\varepsilon \to 0$, $X_{\varepsilon}(\tau)$ converges in distribution to the solution $X(\tau) = \left( X_1(\tau), X_2(\tau) \right)$ of the Itô stochastic differential equations

$$dX_i(\tau) = \theta d\tau + \sum_{j=1}^{2} D_{ij}(X(\tau)) dB_j(\tau), \hspace{1cm} i = 1, 2,$$  \hspace{1cm} (C.1)

where $B_j(\tau)$ are independent standard Brownian motions and $D$ is the square root of $D$, the symmetric diffusion matrix with constant diagonal entries

$$D_{ii} = \frac{1}{2} \int_{-\infty}^{\infty} d\Phi(h, ch), \hspace{1cm} i = 1, 2,$$

and $X$ dependent off-diagonal entries

$$D_{ij}(X) = \frac{1}{2} \int_{-\infty}^{\infty} d\Phi(h + s_1 - s_2 + X_1 - X_2, ch).$$
This shows that the random travel time corrections are correlated along characteristics stemming from nearby points at \( z = 0 \). However, the fluctuations decorrelate for points that are far apart, because of the decay of the autocorrelation \( \Phi \) in its arguments. That is to say, \( D_{ij} \) is small for \( |s_1 - s_2| \gg 1 \), and the diffusion matrix \( D \) is basically diagonal. The two components of the limit process \( X(\tau) \) are driven by two independent Brownian motions, as stated in (C.1).

To show that \( \partial W_{\varepsilon}(\tau, s) \) is order one as \( \varepsilon \to 0 \), we proceed similarly. Consider the vector valued process \( X_{\varepsilon}(\tau) = \left( \frac{W_{\varepsilon}(\tau, s)}{\partial_s W_{\varepsilon}(\tau, s)} \right) \), which satisfies the homogeneous initial condition \( X_{\varepsilon}(0) = 0 \) and equation (B.2), with

\[
F \left( \tau, x, \frac{\tau}{\varepsilon^2} \right) = \left( \begin{array}{c}
\mu \left( \frac{\tau}{\varepsilon^2} + s + x_1, \frac{\tau}{\varepsilon^2} \right) \\
\mu_t \left( \frac{\tau}{\varepsilon^2} + s + x_1, \frac{\tau}{\varepsilon^2} \right) (1 + x_2)
\end{array} \right), \quad x = \left( \begin{array}{c}
x_1 \\
x_2
\end{array} \right),
\]

and \( G = 0 \). The convergence in distribution of \( X_{\varepsilon}(\tau) \) to a vector valued Markov diffusion \( X(\tau) = \left( \begin{array}{c} W(\tau) \\ W^{(1)}(\tau) \end{array} \right) \), follows from the limit theorem in [13], as explained above. The first component of \( X(\tau) \) is the process defined by (3.7)-(3.8).

In the case of slowly changing media in regime 1, the random fluctuations of the travel time are given by

\[
W_{\varepsilon}(\tau, s) = \frac{1}{\varepsilon} \int_0^\tau du \mu \left( \frac{u}{\varepsilon^\alpha} + \varepsilon^{2-\alpha}(s + W_{\varepsilon}(u, s)), \frac{c_\alpha u}{\varepsilon^2} \right)
\]

and the process \( X_{\varepsilon}(\tau) = \left( \frac{W_{\varepsilon}(\tau, s)}{\partial_s W_{\varepsilon}(\tau, s)} \right) \) satisfies the equation (B.2) with

\[
F \left( \tau, x, \frac{\tau}{\varepsilon^2} \right) \approx \mu \left( \frac{\tau}{\varepsilon^2} + \varepsilon^{2-\alpha}(s + x_1), \frac{c \tau}{\varepsilon^2} \right) e_1, \quad x = x_1 e_1 + x_2 e_2,
\]

\( G = 0 \) and initial condition \( X_{\varepsilon}(0) = 0 \). Here the approximation means equal to leading order, as \( \varepsilon \to 0 \). Then, the diffusion approximation theorem in [13] gives that \( X_{\varepsilon}(\tau) \) converges in distribution to \( X(\tau) = \left( \begin{array}{c} W(\tau) \\ 0 \end{array} \right) \), with \( W(\tau) \) defined in (3.7) and (3.10).

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