Ensuring Network Connectedness in Optimal Transmission Switching

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Abstract—Network connectedness is indispensable for the normal operation of transmission networks. However, there still remains a lack of efficient, necessary and sufficient constraints that can be directly added to optimization models of optimal transmission switching (OTS) to ensure network connectedness. To fill this gap, this letter proposes a necessary and sufficient set of linear connectedness constraints, by leveraging the equivalence between network connectedness and feasibility of DC power flow of an auxiliary power network. With these constraints, connectedness of the optimal topology can always be ensured. Furthermore, exploiting the fact that not all lines are switchable in OTS, we also develop a reduction scheme for the proposed connectedness constraints, seeking for improvement of computational efficiency. Finally, numerical studies with a DC OTS model demonstrate the effectiveness of the proposed constraints. The computational burden caused by the connectedness constraints is moderate and can be relieved by using the reduce version.

Index Terms—optimal transmission switching, network connectedness, network connectivity

I. INTRODUCTION

OPTIMAL transmission switching (OTS) is the problem to find an optimal generation dispatch and transmission network topology to minimize the dispatch cost [1]. Due to decreasing generation-side dispatchability with growing penetration of variable renewable energy, OTS for leveraging grid-side flexibility is expected to be more widely and actively engaged in future network operations [2].

Network connectedness should be ensured for system normal operations; however, this is not fully considered in most formulations of OTS. Solving optimization models of OTS without connectedness constraints may produce a connected topology in particular cases, but is very likely to fail in most cases. To the best of our knowledge, network connectedness has only been addressed in [3], where a necessary but not sufficient condition is employed, [4], where OTS is solved sub-optimally by sensitivity analysis such that network connectedness can be easily ensured when selecting candidate lines, and [5] which designs a branching strategy to preserve network connectedness during heuristics. Despite these efforts, there still remains a lack of efficient, necessary and sufficient constraints that can be directly added to optimization models of OTS to ensure network connectedness.

Inspired by our recent work where the network radiality is imposed partially by power flow equations [6], this letter proposes a sufficient and necessary set of linear constraints to ensure network connectedness in OTS problems. Seeing that not all lines are switchable, a reduced version of these constraints seeking for improvement of computational efficiency is also developed. Finally, the effectiveness and computational efficiency of the proposed approach are numerically demonstrated.

II. METHOD

A. Notations and the Basic Idea

The transmission network is represented as an undirected connected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{v_i\}_{i=1}^{N_v}$ and $\mathcal{E} = \{e_i \in \mathcal{V} \times \mathcal{V}\}_{i=1}^{N_e}$ corresponding to all buses and all branches, respectively. Let $z \in \mathbb{R}^{N_e}$ be the vector of binary variables to represent status of branches, where $z_i = 1$ if branch $e_i$ is switched on and $z_i = 0$ otherwise. We use $\mathcal{G}_z$ to denote the edge-induced subgraph of $\mathcal{G}$ by edges $\{e_i | z_i = 1\}$, which corresponds to the transmission network after line switching assigned by $z$. Here an edge-induced subgraph is a subset of the edges of graph $\mathcal{G}$ together with all of the nodes that are their endpoints. With each edge of $\mathcal{G}$ assigned an arbitrary orientation, denote by $\mathcal{E}_\theta$ and $\mathcal{E}_{\theta,z}$ the oriented incidence matrices of $\mathcal{G}$ and $\mathcal{G}_z$, respectively. Denote by $L_{\theta,z}$ the Laplacian matrix of graph $\mathcal{G}_z$.

Connectedness of graph $\mathcal{G}_z$ should be ensured in OTS problems. The basic idea of the following proposed approach to achieve this is to construct an auxiliary power network with the same topology as $\mathcal{G}_z$ and existence of its DC power flow is guaranteed iff $\mathcal{G}_z$ is connected. In this way, determination of connectedness of $\mathcal{G}_z$ can be converted to that of existence of DC power flow of the auxiliary power network, and the latter condition can be much more easily embedded into optimization models of OTS.

B. Connectedness Constraints

We first introduce an auxiliary power network with the same topology as $\mathcal{G}_z$, unit branch reactance, phase angle denoted as $\theta \in \mathbb{R}^{N_e}$, and real power injection given by $c \in \mathbb{R}^{N_v}$. Suppose that $\mathcal{G}$ has $N_s$ connected node-induced subgraphs and their node sets are collected by $\{\mathcal{V}_i\}_{i=1}^{N_s}$. Here a node-induce subgraph refers to an arbitrary nonempty subset of the nodes of graph $\mathcal{G}$ together with all of the edges whose endpoints are both in this subset. In such manner, $\{\mathcal{V}_i\}_{i=1}^{N_s}$ includes node
sets of connected components of all possible $G_z$. Without loss of generality, it is assumed that $V_1 = \mathcal{V}$ is the node set of the node-induced subgraph equal to $G_z$. Let $J \in \mathbb{R}^{N_x \times N_x}$ be the constant matrix satisfying $\forall i \in \{i' | i'_{v_i}=1\}$, $J_{i,j} = 1$ if $j \in V_j$, and $J_{i,j} = 0$ otherwise. Fig. 1 illustrates set $\{V_i | V_i_{N_x}\}$ and the corresponding matrix $J$ for a 4-node graph. With the matrix $J$, a uniquely-balanced $c$ is defined by Definition 1.

**Definition 1 (Unique balance).** Let $b = Jc \in \mathbb{R}^{N_x}$. Then $c$ is multiply-balanced if $\forall i \in \{i' | V_i_{N_x}\}$, $b_i = 0$ and $\forall i \in \{i' | V_i_{N_x}\}$, $b_i \neq 0$, with $1 \leq N_m \leq N_x$. If $N_m = 1$, $c$ is uniquely-balanced.

**Lemma 1.** Given any uniquely-balanced $c$, solutions of the DC power flow equation of the auxiliary power network, i.e., $L_{G_z} \vartheta = c$, exist if graph $G_z$ is connected.

**Proof.** Sufficiency. Suppose graph $G_z$ is connected. A necessary and sufficient condition for any solution(s) of $L_{G_z} \vartheta = c$ to exist is that $Wc = c$ where $W = L_{G_z}L_{G_z}^{-1}$ with $L_{G_z}$ being the Moore-Penrose pseudoinverse of $G_z$.

Since $G_z$ is connected, by [7, Lemma 3], $W$ is given as $W_{ij} = -\frac{\tau}{\chi N_{x}}$ for $i \neq j$ and $W_{ij} = \sum_{j=1,j \neq i}^{N_{x}} W_{ij} = \frac{N_{x}-1}{\chi N_{x}}$ for $i = j$. Recall that $c$ is uniquely-balanced, thus we have $\forall \{i | V_i_{N_x}\}$, $Wc = c$. Hence, $Wc = -\frac{\tau}{\chi N_{x}}$ and $Wc = c$. Thus solutions of $L_{G_z} \vartheta = c$ exist.

Necessity. Suppose that graph $G_z$ is disconnected and with $N_d$ connected components of size $n_k$, $\forall k \in \{k | k_{V_k}\}$. By rearranging nodes of each connected component together, we have $L_{G_z} = \text{diag}(L_{G_z}^{k_1} | \cdots | L_{G_z}^{k_{N_d}})$ with $L_{G_z}^{k}$ being the Laplacian matrix of the $k$-th connected component of $G_z$. Let $\vartheta^k$ and $c^k$ be subvectors of $\vartheta$ and $c$ corresponding to the $k$-th component, respectively; and $W^k = L_{G_z}^{k+1}L_{G_z}^{k+1}$. Consider existence of solutions of $L_{G_z}^{k} \vartheta^k = c^k$. Following the proof of sufficiency, but since $1_{N_{d}}^{T} c^{k} \neq 0$ by its unique balance, it is trivial that $W^k c^k \neq c^k$. Thus $\forall k \in \{k | V_k_{N_x}\}$, solutions of $L_{G_z}^{k} \vartheta^k = c^k$ do not exist and the same for $L_{G_z} \vartheta = c$.

**Theorem 1.** Graph $G_z$ is connected iff the following set of constraints is feasible:

\[
\begin{align*}
E_{G_z}^T \vartheta - \rho + M(1_{N_x} - z) & \geq 0 \\
E_{G_z}^T \vartheta - \rho - M(1_{N_x} - z) & \leq 0 \\
- Mz & \leq \rho \leq Mz \hspace{1cm} (1c) \\
E_{G_z} \rho & = c \hspace{1cm} (1d)
\end{align*}
\]

where $\rho \in \mathbb{R}^{N_x}$ are auxiliary variables, $M$ is a sufficiently large positive number, and $c$ is any uniquely-balanced one.

**Proof.** We first prove that $L_{G_z} \vartheta = c$ and (1) are equivalent.

First of all, we have $L_{G_z} \vartheta = c \Leftrightarrow E_{G_z}^T E_{G_z} \vartheta = c \Leftrightarrow \{E_{G_z}^T \vartheta = \rho, E_{G_z} \rho = c\} \Leftrightarrow z \in (E_{G_z}^T \vartheta) = \rho \hspace{1cm} (2a)$

$E_{G_z}(z \circ \rho) = c$. \hspace{1cm} (2b)

Regarding the left-hand side of (2a) as bilinear terms of $z$ and $E_{G_z}^T \vartheta$, (1a)-(1c) are just the McCormick envelopes of (2a), which are exact since $z \in \mathbb{B}^{N_x}$. For bilinear terms $z \circ \rho$ in (2b), we have $z \circ \rho = \rho$ deriving from their exact McCormick envelopes with upper and lower bounds of $\rho$ given by (1c). Thus, (2b) and (1d) are equivalent if (1c) holds. Then we conclude equivalence between $L_{G_z} \vartheta = c$ and (1), which together with Lemma 1 gives Theorem 1.

**Remark 1.** (1) are similar to power flow constraints together with branch flow limit constraints used in most optimization models for DC OTS, but where $c$ is to be optimized and $M$ in (1c) is replaced by terms related to branch flow limits.

According to Theorem 1, by adding constraints (1) with auxiliary variables $\vartheta \in \mathbb{R}^{N_x}$ and $\rho \in \mathbb{R}^{N_x}$ to optimization models of OTS, network connectedness will be guaranteed. Notably, constraints (1) are all linear and without new binary variables being introduced. Thus constraints (1) will not change the type of original optimization problems. For the setting of $c$ required to be uniquely-balanced, a simple way is letting any one $c_i$ be $1 - N_n$ and the others be 1.

**Remark 2.** The proposed connectedness constraints can be easily extended to more general cases where power grids are also allowed to operate as certain multiple sub-networks. In this case, we only need to replace the uniquely-balanced $c$ by a multiply-balanced $c$ with $\{V_i | V_i_{N_x}\}$ corresponding to all bus sets of allowable sub-networks.

**C. Reduction of the Connectedness Constraints**

Seeing that some branches, collected by $\mathcal{E}_u$, are unswitchable in OTS, i.e., partial entries of $z$ are fixed to 1, constraints (1) can be reduced to improve computational efficiency. The reduced connectedness constraints are obtained by steps:

S.1: Find the connected components of graph $G_u(\mathcal{V}, \mathcal{E}_u)$, denoted by $\mathcal{N} = \{G_u(k) | \mathcal{V}_k, \mathcal{E}_k\}_{k=1}^{N_c}$, and the set of edges connecting different components in $\mathcal{N}$, denoted by $\mathcal{E}_i$;

S.2: Construct graphs $\{G_u(\mathcal{V}_k, \mathcal{E}_k)\}_{k=1}^{N_c}$ where $\mathcal{V}_k \subseteq \mathcal{V}_k$ containing nodes being involved in $\mathcal{E}_i$, and $\mathcal{E}_k$ is arbitrarily assigned as long as $G_u(k)$ is connected.

S.3: Construct graph $\mathcal{G}'(\mathcal{V}', \mathcal{E})'$ with $\mathcal{V}' = \bigcup_{k=1}^{N_c} \mathcal{V}_k$ and $\mathcal{E}' = \mathcal{E}_i \cup \bigcup_{k=1}^{N_c} \mathcal{E}_k$;

S.4: Introduce vector $z' \in \mathbb{B}^{E'}$ to represent status of branches in $\mathcal{E}'$, whose entries equal to 1 for $\bigcup_{k=1}^{N_c} \mathcal{E}_k$ and equal to corresponding entries in $z$ for $\mathcal{E}_i$;
The reduced version of (1) is obtained by replacing \(E_G, \varrho, \rho, 1_N, \) and \(c\) in (1) by their counterparts for graph \(G'\), and \(z\) by \(z'\).

Here \(S.1-S.3\) are illustrated in Fig. 2 with an 8-bus system.

**Remark 3.** The above reduction of the connectedness constraints is based on the fact that connectedness of \(G_s\) and \(G_s'\), is equivalent, where \(G_s'\) denotes the edge-induced subgraph of \(G'\) by edges \(e_i|z'_i = 1\).

### III. Case Study

The proposed approach to ensure network connectedness is demonstrated using the DC OTS model presented in [1] on two test systems: IEEE 30-bus system and 300-bus system. We use M1 to M4 referring to the original OTS model without connectedness constraints, that with the necessary connectedness constraints, \(E_G^\text{abs} z \geq 1_N\), used in [3], with (1), and with reduced (1), respectively. For each system, \([\alpha N]\) lines are assumed to be switchable with \(\alpha \in \{0.3, 0.4, 0.5, 0.6, 0.7\}\), and for each value of \(\alpha\), we randomly generate 50 different configurations of switchable lines. The total number of lines to be switched off is bounded by \([0.05 N]\) and \([0.15 N]\). Gurobi 9.0 is used to solve optimization models with default solver parameters. All computations are performed on a Linux 64-Bit PC with an Intel(R) Core(TM) i5-6500 CPU@3.20GHz and 16GB RAM. All data and source code can be found in [8].

#### Table I

| \(\alpha\) | IEEE 30-bus system | IEEE 300-bus system |
|-----------|-------------------|-------------------|
|           | M1                | M2                | M3 | M4 | M1 | M2 | M3 | M4 |
| 0.3 78%   | 98%              | 100%             | 100%| 0% | 0% | 100%| 100%| 100%|
| 0.4 70%   | 98%              | 100%             | 100%| 0% | 0% | 100%| 100%| 100%|
| 0.5 64%   | 98%              | 100%             | 100%| 0% | 0% | 100%| 100%| 100%|
| 0.6 70%   | 100               | 100%             | 100%| 0% | 0% | 100%| 100%| 100%|
| 0.7 84%   | 98%              | 100%             | 100%| 0% | 0% | 100%| 100%| 100%|

Note: The rate of connected optimal topologies for each value of \(\alpha\) and OTS model is calculated by dividing the number of connected optimal topologies by the number of configurations of switchable lines (i.e., 50).

Fig. 3. Average solution time for M1 to M4 with varying \(\alpha\). Each optimization model is solved 50 times, then the average solution time is calculated by averaging the solution time of corresponding 50 different line configurations.

Table I compares rates of connected optimal topologies obtained by different OTS models with varying \(\alpha\). It can be found that for the IEEE 30-bus system, connectedness of the optimal topology is probable to be ensured without connectedness constraints, and can almost always be ensured with only the necessary constraints. However, for the IEEE 300-bus system, all obtained optimal topologies are unconnected for M1 and M2. In contrast, with our proposed connectedness constraints or the reduce version, connectedness of the optimal topology is always ensured for both test systems. It shows the necessity of including connectedness constraints into OTS problems, especially for large systems.

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