Optimal bilinear control of stochastic nonlinear Schrödinger equations: mass-(sub)critical case

Deng Zhang

Abstract. We study optimal bilinear control problems for stochastic nonlinear Schrödinger equations in both the mass subcritical and critical case. For general initial data of the minimal $L^2$ regularity, we prove the existence and first order Lagrange condition of an open loop control. Furthermore, we obtain uniform estimates of (backward) stochastic solutions in new spaces of type $U^2$ and $V^2$, adapted to evolution operators related to linear Schrödinger equations with lower order perturbations. In particular, we obtain a new temporal regularity of rescaled (backward) stochastic solutions, which is the key ingredient in the proof of tightness of approximating controls induced by Ekeland’s variational principle.

Keywords: Backward stochastic equation, nonlinear Schrödinger equation, optimal control, $U^p$-$V^p$ spaces, Wiener process.

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1Department of Mathematics, Shanghai Jiao Tong University, 200240 Shanghai, China. Email address: dzhang@sjtu.edu.cn
1 Introduction

We are concerned with the controlled stochastic system governed by the nonlinear Schrödinger equation

\[ idX(t, x) = \Delta X(t, x)dt + \lambda |X(t, x)|^{\alpha-1}X(t, x)dt - i\mu(x)X(t, x)dt \]

\[ + V_0(x)X(t, x)dt + \sum_{j=1}^{m} u_j(t)V_j(x)X(t, x)dt \]

\[ + iX(t, x)dW(t, x), \quad t \in (0, T), \quad x \in \mathbb{R}^d, \]

\[ X(0) = X_0 \text{ in } \mathbb{R}^d. \]  

(1.1)

Here, \( \lambda = -1 \) (resp. \( \lambda = 1 \)) corresponds to the defocusing (resp. focusing) case, \( \alpha > 1 \), \( V_j \in L^\infty(\mathbb{R}^d) \) are real valued functions, \( 0 \leq j \leq m \), \( u(t) = (u_1(t), \ldots, u_m(t)) \in \mathbb{R}^m \) represents the control, \( W \) is the colored Wiener process,

\[ W(t, x) = \sum_{j=1}^{N} \mu_je_j(x)\beta_j(t), \quad t \geq 0, \quad x \in \mathbb{R}^d, \]

(1.2)

and

\[ \mu(x) = \frac{1}{2} \sum_{j=1}^{N} |\mu_j|^2 e_j^2(x), \quad x \in \mathbb{R}^d, \quad d \geq 1, \]

where \( \mu_j \) are purely imaginary numbers (i.e. \( \text{Re} \mu_j = 0 \)), \( e_j \) are real-valued functions, and \( \beta_j \) are independent real Brownian motions on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with natural filtration \((\mathcal{F}_t)_{t \geq 0}, 1 \leq j \leq N\). For simplicity, we assume \( N < \infty \), but the arguments in this paper can be extended to the case where \( N = \infty \) under appropriate summable conditions.

In a quantum mechanical interpretation, \( X = X(t, x, \omega), x \in \mathbb{R}^d, t \geq 0, \omega \in \Omega, \) represents the quantum state at time \( t \), while the stochastic perturbation \( iX dW \) represents a stochastic continuous measurement via the pointwise quantum observables \( R_j(X) = \mu_j e_j X \) (see \[ Section 2\]). The functions \( V_j, 0 \leq j \leq m \), describe an external potential. In most situations the control \( u \) represents an external applied force due to the interaction of the quantum system with an electric field or a laser pulse applied to a quantum system.

Here, we are mainly concerned with the conservative case where \( \text{Re} \mu_j = 0, 1 \leq j \leq N \). In this case, \(-i\mu X dt + iX dW\) is indeed the Stratonovitch
differential, and so, via Itô’s formula, one has the pathwise conservation $|X(t)|^2_{L^2} = |X_0|^2_{L^2}, \forall t \geq 0$. Hence, with the normalized initial state $|X_0|_{L^2} = 1$, the quantum system evolves on the unit ball of $L^2$ and verifies the conservation of probability. In particular, in the focusing case $\lambda = 1$, stochastic Schrödinger equations with cubic nonlinearity in dimension two arises in molecular aggregates with thermal fluctuations ([2, 3]). See also [48] for stochastic Schrödinger equations with quartic nonlinearity in dimension one. Actually, in both cases the nonlinearities are critical when the initial data are of only $L^2$ regularity.

We would like also to mention that the non-conservative case (i.e., $\text{Re} \mu_j \neq 0$ for some $1 \leq j \leq N$) plays an important role in the theory of open quantum systems. See [8, 9, 10] and references therein.

For the global well-posedness of (1.1) in the mass-subcritical case (i.e., $1 < \alpha < 1 + \frac{4}{d}$), see [5, 6, 18, 19, 34, 36]. See also [15] for the compact manifold setting, and [12, 13, 14, 20, 21] for the study of martingale solutions.

Optimal control problem and exact controllability of Schrödinger equations have been extensively studied in the deterministic case. See, e.g., [11, 35, 38, 46]. In the stochastic situation, there are many results on optimal control problems of dissipative equations, see, e.g., 30, 31.

However, less results of optimal control problems are known for stochastic Schrödinger equations, which is of dispersive type. One of main difficulties lies in the weak regularization effect of free Schrödinger group $\{e^{-it\Delta}\}$ which, unlike the heat semigroup, cannot raise global Sobolev regularity. Another difficulty arises from the defect of compact embeddings in the probability space. More precisely, even if a space $\mathcal{Y}$ is compactly imbedded into another space $\mathcal{Z}$, one generally does not have the compact imbedding from $L^p(\Omega; \mathcal{Y})$ to $L^p(\Omega; \mathcal{Z}), 1 \leq p \leq \infty$.

For optimal control of stochastic Schrödinger equations with Lipschitz nonlinearity and additive noise, we refer to [39, 40]. For the stochastic nonlinear Schrödinger equation (1.1), the existence of an open-loop optimal control and first-order Lagrange optimality conditions were proved in the recent work [7] for a part of mass subcritical exponents where $2 \leq \alpha < 1 + \frac{4}{d}$ with $1 \leq d \leq 3$. The primary idea of [7] is based on Ekeland’s variational principle ([26, 27]) and the subdifferential in the sense of Rockafellar ([19]), which enable one to obtain geometric characterizations of approximating controls and so the tightness of associated distributions by using appropriate estimates of controlled solutions and related dual backward stochastic solutions.

In the present work, we are mainly interested in the optimal control prob-
lems for (1.1) in the defocusing mass-critical case ($\lambda = -1$, $\alpha = 1 + \frac{4}{d}$, $d \geq 1$).

It should be mentioned that, the global well-posedness and scattering of deterministic nonlinear Schrödinger equations in the defocusing case was one of main conjectures in the field of dispersive equations. The mass-critical case with general initial data has been proved in the recent work by Dodson [22, 23, 24], where one main ingredient is the long-time Strichartz estimate. We would also like to mention [16, 50, 53] for the proof of this conjecture in the defocusing energy-critical case. In the stochastic case, the global well-posedness was proved in [28, 29] for the defocusing mass-critical case with dimension $d = 1$. In the work [55], the global well-posedness of (1.1) has been proved in the defocusing mass-critical case for all dimensions $d \geq 1$. Moreover, the defocusing energy-critical case, scattering as well as Stroock-Varadhan type support theorem have been also studied in [55].

Here we give a unified treatment of optimal bilinear control problems for (1.1) in both the mass subcritical and critical case. In particular, the results apply to the cubic and quartic nonlinear Schrödinger equations in dimensions $d = 2$ and $d = 1$, respectively. Moreover, the results are also applicable to the whole subcritical range of exponents of nonlinearity including the cases $\alpha \in (1, 2)$ with $1 \leq d \leq 3$ and $\alpha \in (1, 1 + \frac{4}{d})$ with $d \geq 4$, which were previously excluded in [7].

Another novelty of the present work is to treat the situation where the initial data of (1.1) are of only $L^2$ regularity, which is actually the minimal regularity required to define the objective functional and is less regular than the $H^1$ regularity assumed in [7]. The key role here is played by the spaces $U^p$ and $V^p$, which were first introduced by Koch and Tataru [41, 42] and have been very effective in the study of dispersive equations. See e.g. [1, 32, 33, 43] and references therein.

We obtain uniform estimates of solutions to (1.1) and dual backward stochastic equation (see (2.13) below) in new spaces of type $U^2$ and $V^2$, adapted to evolution operators related to linear Schrödinger equations with lower order perturbations. These estimates are sharper than those in the Strichartz spaces obtained in [7] and, more importantly, reveal a new temporal regularity of rescaled (backward) stochastic solutions, which is the key ingredient to obtain the tightness of approximating controls induced by Ekeland’s variational principle.

We would also like to mention that the rescaling approach, developed in [5, 6, 7], also plays an important role in constructing appropriate spaces for this new temporal regularity of (backward) stochastic solutions. This,
in spirit, has similarity with the work [4], where a new temporal regularity of stochastic solutions has been obtained for a large class of stochastic dissipative partial differential equations.

Notations. We denote by $| \cdot |_m$ the Euclidean norm in $\mathbb{R}^m$ and by $v \cdot w$ the scalar product of vectors $v, w \in \mathbb{R}^m$. For any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and any multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$, $|\gamma| = \sum_{j=1}^{d} \gamma_j$, $\langle x \rangle = (1 + |x|^2)^{1/2}$, $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$, and $\langle \nabla \rangle = (I - \Delta)^{1/2}$.

For $1 \leq p \leq \infty$, $L^p = L^p(\mathbb{R}^d)$ is the space of $p$-integrable (complex-valued) functions, endowed with the norm $| \cdot |_{L^p}$. In particular, the Hilbert space $L^2(\mathbb{R}^d)$ is endowed with the scalar product $\langle v, w \rangle_2 = \int_{\mathbb{R}^d} v(x)\bar{w}(x)dx$.

As usual, $L^q(0, T; L^p)$ means the space of all $L^q(0, T)$-integrable $L^p$-valued functions, and $C([0, T]; L^p)$ denotes the space of all $L^p$-valued continuous functions on $[0, T]$ with the sup norm in $t$.

For any $2 \leq q < \infty$, we use the $U^p$ and $V^p$ spaces as in [1, 32, 33, 43]. We also use the local smoothing space defined by, for $\alpha, \beta \in \mathbb{R}$, $L^2(I; H^-\beta) = \{u \in \mathcal{D}'; \int_I \int \langle x \rangle^{2\beta} |\langle \nabla \rangle^\alpha u(t, x)|^2dxdt < \infty\}$, where $\mathcal{D}'$ means distributions.

For any two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is the space of linear continuous operators from $\mathcal{X}$ to $\mathcal{Y}$. Throughout this paper, we use $C$ for various constants that may change from line to line.

2 Formulation of main results

To begin with, we recall the definition of a strong solution to equation (1.1).

**Definition 2.1.** Let $X_0 \in L^2$, $0 < T < \infty$. Let $\alpha$ satisfy $1 < \alpha \leq 1 + \frac{4}{d}$, $d \geq 1$. A strong $L^2$-solution to (1.1) on $[0, T]$ is an $L^2$-valued continuous $(\mathcal{F}_t)_{t \geq 0}$-adapted process $X$ such that $|X|^{\alpha - 1}X \in L^1(0, T; H^{-1})$ and $\mathbb{P}$-a.s.

\[
X(t) = X_0 - \int_0^t \left( i\Delta X(s) + \mu X(s) + \lambda i|X(s)|^{\alpha - 1}X(s) + iV_0 X(s) 
+ i \sum_{j=1}^{m} u_j(s)V_j X(s) \right) ds + \int_0^t X(s)dW(s), \ t \in [0, T], \quad (2.1)
\]

as an Itô equation in $H^{-2}$ (resp. $H^{-1}$). Here, the last integral in (2.1) above is taken in the sense of Itô. See, e.g., [17] and [15].
We assume the asymptotically flat condition below as in \([7]\) (see also \([5, 6, 54, 55]\)), mainly for the global well-posedness of (1.1).

\[(H_0)\] For each \(1 \leq j \leq N\) and each \(1 \leq k \leq m\), \(e_j, V_k \in C_b^\infty(\mathbb{R}^d)\) satisfy that for any multi-index \(\gamma \neq 0\),

\[
\lim_{|x| \to \infty} \langle x \rangle^2 |\partial_x^\gamma e_j(x)| + |\partial_x^\gamma V_k(x)| = 0. \tag{2.2}
\]

A pair \((p, q)\) is called a Strichartz pair, if \(\frac{2}{q} = \frac{d(\frac{1}{2} - \frac{1}{p})}{2}, (p, q) \in [2, \infty] \times (2, \infty]\). For any interval \(I \subseteq \mathbb{R}^+\), define the Strichartz spaces by

\[
S^0(I) = \bigcap_{(p,q):\text{Strichartz pair}} L^q(I; L^p), \quad N^0(I) = \bigcup_{(p,q):\text{Strichartz pair}} L^q(I; L^p).
\]

The following hypothesis is assumed for the integrability of solutions to (1.1) in the defocusing mass-critical case.

\[(H0)^* \{e_j\}_{j=1}^N\] are constants with \(d \geq 1\). Or, \(\{e_j\}_{j=1}^N \subseteq L^{\frac{2p}{p-1}}(\mathbb{R}^d)\) for some \(p\) satisfying \(\frac{1}{p} \in (\max\{\frac{1}{2a}, \frac{1}{2} - \frac{1}{2d}, \frac{1}{\alpha}(\frac{1}{2} + \frac{1}{d})\})\) with \(\alpha = 1 + \frac{1}{d}\) and \(1 \leq d \leq 3\).

(Note that, the interval \((\max\{\frac{1}{2a}, \frac{1}{2} - \frac{1}{2d}, \frac{1}{\alpha}(\frac{1}{2} + \frac{1}{d})\})\) is nonempty in dimensions \(1 \leq d \leq 3\). The condition on \(p\) ensures that there exists another Strichartz pair \((\tilde{p}, \tilde{q})\) such that \((\frac{1}{\tilde{p}}, \frac{1}{\tilde{q}}) = (\frac{a}{2}, \frac{a}{q})\), where \(q \in (2, \infty)\) is such that \((p, q)\) is a Strichartz pair, and \(\tilde{p}, \tilde{q}\) denote the conjugate numbers of \(\tilde{p}, \tilde{q}\) respectively. The restriction \(p < \frac{2d}{d-1}\) arises from the integrability (2.7) below.)

Denote by \(U(t, s), t, s \in \mathbb{R}^+\), the evolution operators related to \(-ie^{-W}\Delta(e^{W} \cdot)\), i.e., for any \(v \in L^2\), \(v(t) := U(t, s)v\) solves the linear Schrödinger equation with lower order perturbations below

\[
i\partial_t v(t) = e^{-W}\Delta(e^{W} v(t))(= \Delta v(t) + (b(t) \cdot \nabla + c(t))v(t)), \tag{2.3}
\]

where \(b(t) = 2\nabla W(t), c(t) = \Delta W(t) + \sum_{j=1}^{d}(\partial_j W(t))^2\). It is known (see \([27]\), see also Theorem 3.1 below) that \(\{U(t, s)\}\) are bounded operators in \(L^2\). In particular, in the case where \(\{e_j\}\) are constants, \(-ie^{-W}\Delta(e^{W} \cdot) = -i\Delta\), and so \(U(t, s) = e^{-i(t-s)\Delta}\).

\(^2\)The endpoint case where \(q = 2\) is not considered here.
In order to obtain the temporal regularity of controlled solutions to (1.1), we introduce a new space adapted to the evolution operators as follows

$$U^2(0,T) := \{ v \in \mathcal{D}'((0,T) \times \mathbb{R}^d) : t \mapsto U(0,t)v(t) \in U^2(0,T;L^2) \}$$

endowed with the norm $$\| v \|_{U^2(0,T)} := \| U(0,\cdot)v \|_{U^2(0,T;L^2)}$$, where $$U^2(0,T;L^2)$$ is the space of type $$U^2$$ (see Section 3 below for the precise definition).

The global well-posedness, uniform estimates and the temporal regularity of controlled solutions to (1.1) are summarized below.

**Theorem 2.2.** Consider the mass-subcritical case $$\lambda = \pm 1$$, $$1 < \alpha < 1 + \frac{4}{d}$$, or the defocusing mass-critical case $$\lambda = -1$$, $$\alpha = 1 + \frac{4}{d}$$, $$d \geq 1$$. Assume $$(H_0)$$. For each $$X_0 \in L^2$$, $$u \in U_{ad}$$ (see (2.1) below) and $$0 < T < \infty$$, there exists $$\mathbb{P}$$-a.s. a unique strong $$L^2$$-solution $$X_u$$ to (1.1), satisfying that

$$\sup_{u \in U_{ad}} \| e^{-W} X^u \|_{U^2(0,T) \cap L^2(0,T;H_{-\frac{1}{2}})} < \infty, \quad \mathbb{P} - a.s. \quad (2.4)$$

In particular,

$$\sup_{u \in U_{ad}} \| X^u \|_{S^0(0,T) \cap L^2(0,T;H_{-\frac{1}{2}})} < \infty, \quad \mathbb{P} - a.s., \quad (2.5)$$

and we have $$\mathbb{P}$$-a.s. the temporal regularity

$$\sup_{u \in U_{ad}} \sup_{h \geq 0} h^{-\frac{1}{2}} \left\| U(0,\cdot + h)e^{-W(\cdot + h)} X^u(\cdot + h) - U(0,\cdot)e^{-W} X^u \right\|_{L^2(0,T;L^2)} < \infty. \quad (2.6)$$

Moreover, assume additionally that $$(H_0)^*$$ holds in the defocusing mass-critical case. Then, for any $$1 \leq \rho < \infty$$,

$$\mathbb{E} \sup_{u \in U_{ad}} \| e^{-W} X^u \|_{L^2(0,T;H_{-\frac{1}{2}})}^{\rho} < \infty. \quad (2.7)$$

**Remark 2.3.** The global well-posedness in the mass-subcritical case was proved in [5, 7]. However, the energy method used there is not applicable to the mass-critical case. The proof of global well-posedness in the critical case in Theorem 2.2 benefits from the idea of recent work [55], based on the work of Dodson [22, 23, 24], new rescaling transformations and stability results for nonlinear Schrödinger equations with lower order perturbations.
Remark 2.4. The estimate (2.4) is sharper than (2.5) in the Strichartz space (see also Corollary 3.11 below) and, more importantly, reveals the new temporal regularity (2.6) of controlled solutions.

Remark 2.5. Hypothesis \( (H_0) \) is sufficient to yield the pathwise global well-posedness of (1.1) in both the subcritical and critical case, and it also suffices to get the integrability of solutions in the subcritical case. We also expect it to imply the integrability in the critical case, however, we will not treat this technical problem in the present paper.

Similarly to (2.6), we also have the temporal regularity of \( V_X \) as specified in Theorem 2.6 below, which is crucial to yield the tightness in Section 6 below when the initial data are of the minimal \( L^2 \) regularity.

We shall use another new space adapted to the evolution operators \( \{U(t,s)\} \) as follows

\[ V^2(0,T) := \{ v \in \mathcal{D}'(0,T) : t \mapsto U(0,t)v(t) \in V^2(0,T;L^2) \}, \]

equipped with the norm \( \|v\|_{V^2(0,T)} = \|U(0,\cdot)v\|_{V^2(0,T)} \), where \( V^2(0,T;L^2) \) is the space of type \( V^2 \) (see Section 3 below).

**Theorem 2.6.** Consider the situations of Theorem 2.2. Assume \( (H_0) \). Assume additionally \( (H_0)^* \) in the defocusing mass-critical case. We have for each \( 1 \leq k \leq m \) and for any \( 1 \leq \rho < \infty \),

\[ \mathbb{E} \sup_{u \in U_{ad}} \left\| e^{-W} V_k X u \right\|_{V^2(0,T)}^\rho < \infty. \tag{2.8} \]

In particular, we have the temporal regularity

\[ \mathbb{E} \sup_{u \in U_{ad}} \sup_{h \geq 0} h^{-\frac{\delta}{2}} \left\| U(0,\cdot + h)e^{-W(\cdot + h)} V_k X u(\cdot + h) - U(0,\cdot)e^{-W} V_k X u \right\|_{L^2(0,T;L^2)}^\rho < \infty. \tag{2.9} \]

Now, let us introduce the optimal control problem as in [7].

Let \( L^2_{ad}(0,T;\mathbb{R}^m) \) denote the space of all \( (\mathcal{F}_t)_{t \geq 0} \)-adapted \( \mathbb{R}^m \)-valued processes \( u : [0,T] \rightarrow \mathbb{R}^m \) such that \( u \in L^2((0,T) \times \Omega;\mathbb{R}^m) \). Similarly, let \( L^2_{ad}(0,T;L^2(\Omega;L^2)) \) denote the space of \( L^2 \)-valued \( (\mathcal{F}_t)_{t \geq 0} \)-adapted processes \( v \) such that \( \mathbb{E} \int_0^T |v(t)|_{L^2}^2 dt < \infty. \)
Given \( X_T \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2) \) and \( X \in L^2_{ad}(0, T; L^2(\Omega; L^2)) \), we define the objective functional \( \Phi : L^2_{ad}(0, T; \mathbb{R}^m) \to \mathbb{R} \) by

\[
\Phi(u) = \mathbb{E}|X^u(T) - X_T|^2_{L^2} + \gamma_1 \mathbb{E} \int_0^T |X^u(t) - X(t)|^2_{L^2} dt \\
+ \gamma_2 \mathbb{E} \int_0^T |u(t)|^2_m dt + \gamma_3 \mathbb{E} \int_0^T |u'(t)|^2_m dt.
\]  

(2.10)

Here, the coefficients \( \gamma_1, \gamma_3 \geq 0, \gamma_2 > 0 \), \( | \cdot |_m \) denotes the Euclidean norm in \( \mathbb{R}^m \), and \( u' \) denotes the time derivative of \( u \) if it exists. In most situations, \( X \) is a given trajectory of the uncontrolled system or, in particular, a steady state solution. The control \( u \) belongs to the admissible set \( U_{ad} \) defined by

\[
U_{ad} := \left\{ u \in L^2_{ad}(0, T; \mathbb{R}^m); \ u \in K, \ a.e. \ on \ (0, T) \times \Omega \right\},
\]  

(2.11)

where \( K \) is a compact convex subset of \( \mathbb{R}^m \). Note that, \( \sup_{u \in U_{ad}} \| u \|_{L^\infty(0, T; \mathbb{R}^m)} \leq D_K < \infty \), where \( D_K \) denotes the diameter of \( K \).

The optimal control problem considered in this paper is formulated below

(P) \textbf{Minimize}

\[
\Phi(u) = \mathbb{E} \left( |X(T) - X_T|^2_{L^2} + \gamma_1 \int_0^T |X(t) - X(t)|^2_{L^2} dt \\
+ \gamma_2 \int_0^T |u(t)|^2_m dt + \gamma_3 \int_0^T |u'(t)|^2_m dt \right)
\]  

on all \( (X, u) \in L^2_{ad}(0, T; L^2(\Omega; L^2)) \times U_{ad} \) subject to (1.1).

Heuristically, the objective of control process is to steer the quantum system from an initial state \( X_0 \) to a target state \( X_T \) and also in the neighborhood of a given trajectory \( X \). The last two terms in the cost functional represent the energy cost to obtain the desired objective.

As mentioned in the Introduction, because of the defect of compact imbedding in the stochastic situation, the existence of a solution in Problem (P) does not follow from standard compactness techniques used in deterministic optimization problems. Below we consider the relaxed version of Problem (P) as in [7] which, actually, resembles standard weak solutions to stochastic equations.
**Definition 2.7.** Let $\mathcal{Y} := L^2(\mathbb{R}^d) \times L^2((0,T) \times \mathbb{R}^d) \times C([0,T]; \mathbb{R}^N) \times L^2(0,T; \mathbb{R}^m)$ \times $L^2((0,T) \times \mathbb{R}^d)$ and $(\Omega^*, \mathcal{F}^*, (\mathcal{F}^*_t)_{t \geq 0})$ be a new filtered probability space, carrying $(X_T^*, X^*, \beta^*, u^*, X^*)$ in $\mathcal{Y}$. Define $L^2_{ad^*}(0,T; \mathbb{R}^m)$, $L^2_{ad^*}(0,T; L^2(\Omega; L^2))$, $\mathcal{U}_{ad^*}$ and $\Phi^*(u^*)$ similarly as above on this new filtered probability space.

The system $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*, (\mathcal{F}^*_t)_{t \geq 0}, (X_T^*, X^*, \beta^*, u^*, X^*))$ is said to be admissible, if $X_T^* \in L^2(\Omega, \mathcal{F}_T^*, \mathbb{P}^*; L^2)$, $X^* \in L^2_{ad^*}(0,T; L^2(\Omega; L^2))$, $\beta^* = (\beta_1^*, \ldots, \beta_N^*)$ is an $(\mathcal{F}^*_t)_{t \geq 0}$-adapted $\mathbb{R}^N$-valued Wiener process, the joint distributions of $(X_T^*, X_1^*, \beta^*)$ and $(X_T, X, \beta)$ coincide, $u^* \in \mathcal{U}_{ad^*}$, and $X^*$ is an $L^2$-valued continuous $(\mathcal{F}^*_t)_{t \geq 0}$-adapted process that satisfies equation (1.1) corresponding to $(\beta^*, u^*)$.

The admissible system $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*, (\mathcal{F}^*_t)_{t \geq 0}, (X_T^*, X^*, \beta^*, u^*, X^*))$ is said to be a relaxed solution to the optimal control problem (P), if

$$
\Phi^*(u^*) \leq \inf \{ \Phi(u); u \in \mathcal{U}_{ad^*}, X^u \text{ satisfies } (1.1) \}.
$$

(2.12)

The first result concerning the optimal control problem of (1.1) in the case where $\gamma_3 > 0$ is formulated below.

**Theorem 2.8.** Consider the mass-subcritical or the defocusing mass-critical case. Assume (H0). Then, for each $X_0 \in L^2$, $X_T \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2)$ and $X \in L^2_{ad}(0,T; L^2(\Omega; L^2))$, $0 < T < \infty$, there exists a relaxed solution in the sense of Definition 2.7 to the optimal control problem (P) in the case $\gamma_3 > 0$.

In the case where $\gamma_3 = 0$, we are able to obtain a relaxed solution satisfying (2.12) with “=” replacing “$\leq$”. For this purpose, we introduce the dual linearized backward stochastic equation below

$$
dY = -i \Delta Y dt - \lambda h_1(X^u)Y dt + \lambda ih_2(X^u)\bar{Y} dt + \mu Y dt - iV_0 Y dt - iu \cdot VY dt
\quad + \gamma_1(X^u - \bar{X}) dt - \sum_{k=1}^N \mu_k e_k Z_k dt + \sum_{k=1}^N Z_k d\beta_k(t)
$$

(2.13)

$$
Y(T) = - (X^u(T) - X_T),
$$

where

$$
h_1(X^u) := \frac{\alpha + 1}{2} |X^u|^{\alpha - 1}, \quad h_2(X^u) := \frac{\alpha - 1}{2} |X^u|^{\alpha - 3}(X^u)^2.
$$

(2.14)

The functions $h_j, j = 1, 2$, are the usual complex derivatives of the function $z \mapsto |z|^{\alpha - 1} z$, i.e., $h_1(z) = \partial_z(|z|^{\alpha - 1} z)$ and $h_2(z) = \partial^2_z(|z|^{\alpha - 1} z)$, $z \in \mathbb{C}$.
It should be mentioned that, although equation (2.13) is linear with respect to $Y$, the presence of singular coefficient $h_2(X^u)$ destroys standard Lipschitz property or monotonicity required in the literature of stochastic backward equations (see, e.g., [31, 37, 52, 50]). Hence, the standard energy method, based on the Itô formula of $|Y(t)|^2$, is not applicable to (2.13). Moreover, unlike the heat semigroup, the free Schrödinger group $\{e^{-it\Delta}\}$ is a unitary evolution in $L^2$ and so has no global regularization effect to raise the Sobolev regularity of solutions, which also makes the analysis of (2.13) more difficult. Actually, a great effect of [7] is dedicated to this issue. The key idea is to use the duality arguments to reduce the analysis of (2.13) to that of an associated variational equation (see (5.1) below). See also [30, 44] for duality arguments in analyzing backward stochastic equations arising from control problems.

The global well-posedness and temporal regularity of stochastic backward solutions are formulated below.

**Theorem 2.9.** Consider the mass subcritical or the defocusing mass-critical case. Assume Hypothesis (H0). Assume additionally that $e_j$ are constants, $1 \leq j \leq N$.

Then, given any $X_T \in L^{2+\nu}(\Omega, \mathcal{F}_T; L^2)$ and $X \in L^{2+\nu}(\Omega; L^2(0, T; L^2))$ for some $\nu \in (0, 1)$, there exists a unique $(\mathcal{F}_t)$-adapted solution $(Y^u, Z^u)$ to (2.13) corresponding to $u \in U_{ad}$, satisfying that $Y^u \in C([0, T]; L^2)$, a.s., for any $1 \leq \rho < 2 + \nu$,

$$\sup_{u \in U_{ad}} \mathbb{E}\|e^{-W}Y^u\|^\rho_{Y^2(0, T)} < \infty,$$

and for each $1 \leq k \leq N$,

$$\sup_{u \in U_{ad}} \mathbb{E}\|Z^u\|^\rho_{L^2(0, T; L^2)} < \infty.$$

In particular, for any $1 \leq \rho < 2 + \nu$,

$$\sup_{u \in U_{ad}} \mathbb{E}\|Y^u\|^\rho_{S^0(0, T)} < \infty,$$

and we have the temporal regularity

$$\sup_{u \in U_{ad}} \mathbb{E}\sup_{h \geq 0} h^{-\frac{\rho}{2}} \left\| U(0, \cdot + h)e^{-W(\cdot + h)}Y^u(\cdot + h) - U(0, \cdot)e^{-W}Y^u \right\|^\rho_{L^2(0, T; L^2)} < \infty.$$ 

(2.18)
Remark 2.10. As in Remark 2.4, the estimate (2.15) is sharper than (2.17) in the Strichartz space and yields the temporal regularity (2.18) of backward solutions. The latter fact enables us to obtain the tightness of approximating controls with initial data of minimal $L^2$ regularity. See Section 6 below.

The main result of this paper is formulated below.

**Theorem 2.11.** Consider the mass-subcritical case or the defocusing mass-critical case. Assume Hypothesis (H0). Assume additionally that $e_j$ are constants, $1 \leq j \leq N$.

Then, for each $X_0 \in L^2$, $X_T \in L^{2+\nu}(\Omega; \mathcal{F}_T; L^2)$ and $X \in L^{2+\nu}(\Omega; L^2(0, T; L^2))$ for some $\nu \in (0, 1)$, $0 < T < \infty$, there exists a relaxed solution $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*, (\mathcal{F}^*_t)_{t \geq 0}, X^*_T, X^*, \beta^*, u^*, X^*)$ in the sense of Definition 2.7 to Problem (P) in the case $\gamma_3 = 0$, satisfying

$$\Phi^*(u^*) = \inf\{\Phi(u); u \in U_{ad}, X^u \text{ satisfies } (1.1)\}.$$  

Moreover, we have the explicit characterization of $u^*$ below

$$u^*(t) = PK\left(\frac{1}{\gamma_2} \text{Im} \int_{\mathbb{R}^d} V(x)X^*(t, x)Y^*(t, x)dx\right), \forall t \in [0, T], \mathbb{P}^* - a.s.$$ 

(2.20)

where $PK$ is the projection on $K$, and $Y^*$ is the first component of backward solution $(Y^*, Z^*)$ to (2.13) with $X^*_T, X^*, \beta^*, u^*, X^*$ replacing $X_T, X, \beta, u, X^u$, respectively.

Remark 2.12. Theorem 2.11 is applicable to the whole mass-subcritical case where $1 < \alpha < 1 + \frac{4}{d}$, $d \geq 1$, which includes the ranges $\alpha \in (1, 2)$ with $1 \leq d \leq 3$ and $\alpha \in (1, 1 + \frac{4}{d})$ with $d \geq 4$ that were previously excluded in [7]. More importantly, Theorem 2.11 applies also to the defocusing mass-critical case, which includes the cubic and quartic stochastic nonlinear Schrödinger equations in dimension $d = 2$ and $d = 1$, respectively.

Remark 2.13. The additional condition that $e_j$ are constants is imposed in Theorem 2.11 above mainly for the integrability of variational solutions (see Proposition 5.1 below). This condition can be reduced to that $\mathbb{E}\|X^u\|_{S^\rho([0, T])} + \mathbb{E}\|\psi^u\|_{C([0, T]; L^2)} < \infty$, $\forall 1 \leq \rho < \infty$, where $X^u$ and $\psi^u$ solve (1.1) and (5.1) below respectively. See Remark 5.3 below for detail explanations. As a matter of fact, Hypothesis (H0) suffices to give the pathwise global well-posedness and estimates of solutions to both (1.1) and (5.1), and so it is also expected to yield the integrability of solutions.
Remark 2.14. It is also interesting to study the case where the controls also enter into the diffusion coefficients. In that case, one usually needs to deal with a second adjoint equation (see e.g. [30, 44, 47] and references therein). In the present work, we will not treat this issue.

As a byproduct, we have the following result in the deterministic case.

Corollary 2.15. In the deterministic case (i.e. $\mu_k = 0, 1 \leq k \leq N$), consider the mass-subcritical case or the defocusing mass-critical case as in Theorem 2.11.

Then, for each $X_0 \in L^2$, $X_T \in L^2$ and $\mathbb{X} \in L^2(0, T; L^2)$, $0 < T < \infty$, there exists an optimal control $u$ to Problem (P) in the case $\gamma_3 = 0$, such that

$$\Phi(u) = \inf \{\Phi(v); v \in U_{ad}, X^v \text{ satisfies } (2.1)\}.$$  

Moreover,

$$u(t) = P_K \left( \frac{1}{\gamma_2} \text{Im} \int_{\mathbb{R}^d} V(x)X(t, x)\overline{Y}(t, x)dx \right), \quad \forall t \in [0, T],$$

where $P_K$ is the projection on $K$, and $Y$ is the solution to backward equation (2.13) with $Z_k = 0, 1 \leq k \leq N$.

The remaining part of this paper is organized as follows. In Section 3 we present preliminaries used in this paper, including the Strichartz and local smoothing estimates, $U^p-V^p$ spaces and stability results for nonlinear Schrödinger equations with lower order perturbations. Sections 4 and 5 constitute the technical parts of this paper. In Section 4 we treat the controlled stochastic equation (1.1). We first prove Theorems 2.2 and 2.6 in Subsection 4.1 and then in Subsection 4.2 we prove stability results for controlled solutions and objective functionals, which consequently lead to the proof of Theorem 2.8. In Section 5 we mainly analyze the variational equations and the dual backward stochastic equations. As a consequence, we obtain the directional derivative of objective functional. Section 6 is mainly devoted to the proof of main result Theorem 2.11. For simplicity of the exposition, some technical proofs are postponed to the Appendix.
3 Preliminaries

3.1 Strichartz and local smoothing estimates

We start with the Strichartz and local smoothing estimates related to operators of the form $e^{-\Phi \Delta(e^{\Phi \cdot})}$. In the sequel, we assume that $\text{Re}\Phi = 0$.

**Theorem 3.1.** Let $I = [t_0, T] \subseteq \mathbb{R}^+$. Consider the equation

$$i\partial_t v = e^{-\Phi \Delta(e^{\Phi \cdot})} + f.$$  \hspace{1cm} (3.1)

Here, the function $\Phi = \Phi(t, x)$ is continuous on $t$ for each $x \in \mathbb{R}^d$, $d \geq 1$, $\text{Re}\Phi = 0$, and for each multi-index $\gamma$,

$$\sup_{t \in I} |\partial_x^\gamma \Phi(t, x)| \leq C(\gamma) \sup_{t \in I} g(t) \langle x \rangle^{-2}$$  \hspace{1cm} (3.2)

for some positive and continuous function $g$. Then, for any $v(t_0) \in L^2$ and $f \in N^0(I) + L^2(I; H^{-\frac{1}{2}}_1)$, the solution $v$ to (3.1) satisfies

$$\|v\|_{S^0(I) \cap L^2(I; H^{\frac{1}{2}}_1)} \leq C_T(\|v(t_0)\|_{L^2} + \|f\|_{N^0(I) + L^2(I; H^{-\frac{1}{2}}_1)})$$  \hspace{1cm} (3.3)

Moreover, if $\mathbb{E} \sup_{0 \leq t \leq T}(g(t))^\rho \leq C(T, \rho)$, $\forall 1 \leq \rho < \infty$. then so is $C_T$, i.e.,

$$\mathbb{E}(C_T)^\rho \leq C(\rho, T) < \infty, \ \forall 1 \leq \rho < \infty.$$  \hspace{1cm} (3.4)

**Proof.** The proof of (3.3) is quite similar to that of [55, Theorem 3.3], based on the pseudo-differential calculus. Actually, the asymptotical flatness condition (3.2) ensures that the lower order perturbations arising in the operator $e^{-\Phi \Delta(e^{\Phi \cdot})}$ can be controlled, via the Gårding inequality, by the Poisson bracket $i[\Psi_h, \Delta]$ for some appropriate symbol $h \in S^0$. Concerning the constant $C_T$, similar arguments as in the proof of [54, Theorem 2.6] show that $C_T$ depends polynomially on $\sup_{0 \leq t \leq T} g(t)$, which implies (3.4).

Let $\{V(t, s)\}$ be the evolution operators related to $-ie^{-\Phi \Delta(e^{\Phi \cdot})}$, i.e., for any $v \in L^2$, $v(t) := V(t, s)v$ satisfies the equation

$$i\partial_t v(t) = e^{-\Phi \Delta(e^{\Phi \cdot})} v(t), \ t \in \mathbb{R}^+,$$

with $v(s) = v$. Denote by $V^*(t, s)$ the dual operator of $V(t, s)$, $t, s \in \mathbb{R}^+$. We have the following result concerning the evolution operators.
Proposition 3.2. Let $\Phi$ be as in Theorem 3.1. Then, for any $v \in L^2$, $t \in \mathbb{R}^+$,

\begin{align}
V^*(t,0)v &= V(0,t)v, \quad (3.5) \\
V^*(0,t)v &= V(t,0)v. \quad (3.6)
\end{align}

In particular, for any $v_1, v_2 \in L^2$, $t \in \mathbb{R}^+$,

$$
\langle v_1, v_2 \rangle_2 = \langle V(0,t)e^{-\Phi(t)}v_1, V(0,t)e^{-\Phi(t)}v_2 \rangle_2. \quad (3.7)
$$

The proof is postponed to the Appendix for simplicity.

We end this subsection with Theorem 3.3 below, which relates the stochastic equation (1.1) to a random Schrödinger equation with lower order perturbations. Note that, the local well-posedness of (1.1) in the mass-critical case can be proved by using similar arguments as in [5].

Theorem 3.3. Consider the defocusing mass-critical case, i.e., $\lambda = -1$, $\alpha = 1 + \frac{4}{d}$, $d \geq 1$. Let $X$ be the $L^2$-solution to (1.1) on $[0, \tau^*)$ with $X(0) = X_0 \in L^2$, where $\tau^*$ is the maximal existing time. Given any $(\mathcal{F}_t)$-stopping time $\sigma$ satisfying $0 \leq \sigma < \tau^*$, we define the rescaling transformation

$$
v_\sigma(t) := e^{W_\sigma(t)}X(\sigma + t), \quad t \in [0, \tau], \quad (3.8)
$$

where $W_\sigma(t) := W(\sigma + t) - W(\sigma)$. Set $f(u_\sigma) := V_0 + u_\sigma \cdot V$ with $u_\sigma(t) = u(\sigma + t)$. Then, $v_\sigma$ satisfies $\mathbb{P}$-a.s. the random equation

\begin{align}
&i\partial_t v_\sigma = e^{-W_\sigma(t)}\Delta(e^{W_\sigma}v_\sigma) - F(v_\sigma) + f(u_\sigma)v_\sigma, \quad (3.9) \\
v_\sigma(0) = X(\sigma),
\end{align}

on $[0, \tau^* - \sigma)$ in the space $H^{-2}$.

The proof is similar to that of [55] Theorem 2.17 and is omitted here. Note that, since $\text{Re} \mu_j = 0$, $1 \leq j \leq N$, the term $\hat{\mu}$ defined in [55] (2.22) equals to zero and so does not appear in (3.9).

### 3.2 $U^p$-$V^p$ spaces

In this subsection, we first recall some basic facts of the spaces $U^p$ and $V^p$, following [11, 32, 33, 43]. Then, we enhance the Strichartz estimates in Theorem 3.1 to the spaces of type $U^2$ and $V^2$, adapted to the evolution operators
Theorem 2.2, which will play an important role in the proof of tightness in Section 6 below.

Let \( H \) be a separable Hilbert space over \( \mathbb{C} \), it will be chosen as either \( L^2(\mathbb{R}^d) \) or \( \mathbb{C} \). Let \( \mathcal{Z} \) be the set of finite partitions \( \{ t_k \}_{k=0}^K \) of \( \mathbb{R} \). If \( t_K = \infty \), we use the convention that \( u(t_K) = 0 \) for all functions \( u : \mathbb{R} \mapsto H \). We also use the notation \( \mathcal{X}_I \) for the characteristic function of a set \( I \subseteq \mathbb{R} \).

**Definition 3.4.** Let \( 1 \leq p < \infty \).

(i) A \( U^p \)-atom is a right continuous step function \( u : \mathbb{R} \mapsto H \) of the form

\[
a = \sum_{k=1}^K \mathcal{X}_{(t_{k-1}, t_k)} \phi_{k-1},
\]

for some \( \{ t_k \}_{k=0}^K \in \mathcal{Z} \) and \( \{ \phi_k \}_{k=0}^K \subseteq H \) with \( \sum_{k=0}^{K-1} \| \phi_k \|_H^p = 1 \). The \( U^p \)-space \( U^p(\mathbb{R}; H) \) is defined by

\[
U^p := \left\{ u = \sum_{j=1}^\infty \lambda_j a_j ; a_j \text{ are } U^p \text{ - atoms and } \{ \lambda_j \}_{j \in \mathbb{N}} \in l^1(\mathbb{N}; \mathbb{C}) \right\}
\]

with the norm

\[
\| u \|_{U^p} := \inf \left\{ \sum_{j=1}^\infty |\lambda_j| ; u = \sum_{j=1}^\infty \lambda_j a_j , \lambda_j \in \mathbb{C}, a_j \text{ are } U^p \text{ - atoms} \right\}.
\]

Likewise, \( U^p_c \) denotes the closed subspace in \( U^p \) of all continuous functions \( u : \mathbb{R} \mapsto H \).

(ii) The \( V^p \)-space \( V^p(\mathbb{R}; H) \) is the space of all functions of bounded \( p \)-variation \( v : \mathbb{R} \mapsto H \) satisfying \( v(\infty) = 0 \), endowed with the norm

\[
\| v \|_{V^p(\mathbb{R}; H)} := \sup_{\{ t_k \}_{k=0}^K \in \mathcal{Z}} \left( \sum_{k=1}^K \| v(t_k) - v(t_{k-1}) \|_H^p \right)^{\frac{1}{p}}.
\]

Likewise, let \( V^p_{rc} \) denote the closed subspace of all right continuous functions \( v : \mathbb{R} \mapsto H \) such that \( \lim_{t \to -\infty} v(t) = 0 \).

**Lemma 3.5.** The spaces \( U^p, U^p_c, V^p, V^p_{rc} \) are Banach spaces. For any \( 1 \leq p < q < \infty \), we have the embeddings

\[
U^p \hookrightarrow V^p_{rc} \hookrightarrow U^q \hookrightarrow L^\infty.
\]

(3.10)
Moreover, for $1 < p < \infty$, $V^p \hookrightarrow B^p_{p,\infty}$, i.e., for $v \in V^p$,

$$\sup_{h>0} h^{-\frac{1}{p}} \|v(\cdot + h) - v\|_{L^p(\mathbb{R};H)} \leq C\|v\|_{V^p(\mathbb{R};H)} \tag{3.11}$$

The duality between spaces $U^p$ and $V^p$ are related to a bilinear form $B(u, v) : U^p \times V^p \mapsto B(u, v)$ defined as follows: for $u \in U^p$, $v \in V^p$ and any $t = \{t_k\}_{k=0}^K \in \mathcal{Z}$, $B_t(u, v) := \sum_{k=1}^K \langle u(t_{k-1}), v(t_k) - v(t_{k-1}) \rangle_2$. Then, there exists a unique number $B(u, v)$ such that for all $\varepsilon > 0$, there exists $t \in \mathcal{Z}$ such that for any $t' \supseteq t$, $|B_{t'}(u, v) - B_t(u, v)| < \varepsilon$. (See \cite[Proposition 2.7]{32}.)

**Lemma 3.6.** \cite[Propositions 2.7, 2.10]{32} Let $1 \leq p < \infty$. For $u \in U^p$, $v \in V^p$, $|B(u, v)| \leq \|u\|_{U^p} \|v\|_{V^p}$. Moreover, if in addition $u \in V^1$ and $u$ is absolutely continuous on compact intervals, then,

$$B(u, v) = -\int_\mathbb{R} \langle u'(t), v(t) \rangle_2 \, dt.$$

Given any time interval $I = [a, b] \subseteq \mathbb{R}$, we also consider the restriction space $U^p(I)$ with the norm $\|u\|_{U^p(I)} := \inf\{\|v\|_{U^p(\mathbb{R})}; v|_I = u\}$. Actually, the infimum is attained by $v = X_I u$ (See \cite[Remark A.2]{1}).

**Lemma 3.7.** \cite[Lemmas A.3, A.6]{1} Given any interval $I = [a, b] = \bigcup_{j=1}^\infty I_j$, we have $\|u\|_{U^p(I)} \leq \sum_{j=1}^\infty \|u\|_{U^p(I_j)}$. Moreover, for $u \in U^p_c(I)$, the mapping $t \mapsto \|u\|_{U^p([a, t])}$ is continuous.

We have the following result from \cite[Corollary 2.4]{43} for $v \in C([0, T]; H)$.

**Lemma 3.8.** \cite[Corollary 2.4]{43} Let $1 < p, p' < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, and $v \in V^p(0, T; H) \cap C([0, T]; H)$. We have

$$\|v\|_{V^p(0, T; H)} = \sup\{B(u, v) : u \in C^\infty_0(0, T; H), \|u\|_{U^{p'}(0, T; H)} = 1\}.$$

Similarly to (3.11), we have

**Lemma 3.9.** Let $v \in V^p(0, T; H) \cap C([0, T]; H)$, $1 \leq p < \infty$, $0 < T < \infty$. Then,

$$\|v(\cdot + h) - v\|_{L^p(0, T^*; H)} \leq 2^{\frac{1}{p}}h^{\frac{1}{2}}\|v\|_{V^p(0, T)} \tag{3.12}$$

(See the Appendix for the proof.)

The following estimates relates the $U^p - V^p$ spaces and Strichartz spaces.
Proposition 3.10. Consider the situations in Theorem 3.1 and let $V(t,s)$, $t,s \in \mathbb{R}^+$, be the evolution operators related to $-ie^{-\Phi}\Delta(e^{\Phi})$. Then, for any $I = [t_0, T] \subseteq \mathbb{R}^+$ and any Strichartz pair $(p,q)$,

$$\|u\|_{L^q(I;L^p)} \leq C(T)\|V(t_0,\cdot)u\|_{U^q(I)},$$  

and

$$\left\| \int_{t_0}^T V(t_0,s)f(s)ds \right\|_{V^q(I)} \leq C(T)\|f\|_{L^q(I;L^p)},$$

and

$$\left\| \int_{t_0}^T V(t_0,s)f(s)ds \right\|_{U^2(I)} \leq C(T)(1 + \|f\|_{L^q(I;U^p)+L^2(I;H^1)^{\frac{1}{2}}}).$$

where $C(T)$ is independent of $p,q$ and $C(T) \in L^p(\Omega)$ for any $1 \leq \rho < \infty$.

We postpone the proof to the Appendix. As a consequence, we have

Corollary 3.11. Consider the situations in Theorem 3.1. Then,

$$\|u\|_{S^0(I)} \leq C(T)\|V(t_0,\cdot)u\|_{V^2(I)} \leq C(T)\|V(t_0,\cdot)u\|_{U^2(I)},$$

and

$$\left\| \int_{t_0}^T V(t_0,s)f(s)ds \right\|_{U^2(I)} \leq C(T)\|f\|_{N^0(I)},$$

where $C(T) \in L^p(\Omega), \forall 1 \leq \rho < \infty$.

Proof. The proof follows from Proposition 3.10 and the embedding $V^q \hookrightarrow U^2(I) \hookrightarrow V^2(I) \hookrightarrow U^q(I)$ for $2 < q < \infty$. \(\square\)

3.3 Mass-critical stability results

We first present the recent the global well-posedness results obtained by Dodson [22, 23, 24] in the defocusing mass-critical case.

Theorem 3.12. ([22, 23, 24]) For any $v_0 \in L^2$, there exists a unique global $L^2$-solution $v$ to the equation

$$i\partial_t v = \Delta v - |v|^\frac{4}{n}v,$$

$$v(0) = v_0$$

(3.18)
with $d \geq 1$. Moreover,

$$
\|v\|_{L^{2+\frac{4}{d}}(\mathbb{R} \times \mathbb{R}^d) \cap L^2(\mathbb{R}; H_{-\frac{1}{2}})} \leq B_0(\|v_0\|_{L^2}) < \infty,
$$

(3.19)

where $B_0(\|v_0\|_{L^2})$ depends continuously on $v_0$ in $L^2$.

**Remark 3.13.** The proof of global well-posedness (and also scattering) in [22, 23, 24] is highly nontrivial and is based on the concentration-compact arguments. One key ingredient there is the long-time Strichartz estimate. The bound of $\|v\|_{L^2(\mathbb{R}; H_{-\frac{1}{2}})}$ in (3.19) follows from Strichartz estimates and that of $\|v\|_{L^{2+\frac{4}{d}}(\mathbb{R} \times \mathbb{R}^d)}$, and the continuity of $B_0(\cdot)$ follows from the mass-critical stability result Lemma 3.6 of [51]. We also refer to [16, 50, 53] for the defocusing energy-critical case.

Below we state the mass-critical stability result for Schrödinger equations with lower order perturbations which is crucial in the proof of Theorem 2.2.

**Theorem 3.14.** (Mass-Critical Stability Result). Fix $I = [t_0, T] \subseteq \mathbb{R}^+$. Take the Strichartz pair $(p, q)$, $p = 2 + \frac{4}{d}$ with $d \geq 1$, or $\frac{1}{p} \in (\max\{\frac{1}{2}, \frac{1}{d} - \frac{1}{2d}\}, \frac{1}{\alpha}(\frac{1}{2} + \frac{1}{d}))$ with $\alpha = 1 + \frac{4}{d}$ and $1 \leq d \leq 3$. Let $V(t, s), t, s \in I$, be the evolution operators related to the operator $-ie^{-\Phi}\Delta(e^{\Phi})$, where $\Phi$ is as in Theorem 3.1 with the time function $g$. Let $v$ be the solution to

$$
v(t) = V(t, t_0)v(t_0) + \int_{t_0}^t V(t, s)(i|v|^\frac{4}{d}v + Gv)ds + R(t) \tag{3.20}
$$

for some functions $R$ with $R(t_0) = 0$, $v(t_0) \in L^2$, and $G \in L^\infty(I \times \mathbb{R}^d)$. Let $\tilde{v}$ solve the equation

$$
\tilde{v}(t) = V(t, t_0)\tilde{v}(t_0) + \int_{t_0}^t V(t, s)(i|\tilde{v}|^\frac{4}{d}\tilde{v} + G\tilde{v} + e)ds \tag{3.21}
$$

for some functions $e$ and $\tilde{v}(t_0) \in L^2$. Assume that $\|\tilde{v}\|_{L^\infty(I; L^p)} \leq L$ for some positive constant $L$. Assume also the smallness conditions

$$
\|V(\cdot, t_0)(v(t_0) - \tilde{v}(t_0))\|_{L^\infty(I; L^p)} \leq \varepsilon, \tag{3.22}
$$

$$
\|R\|_{L^\infty(I; L^p)} \leq \varepsilon, \|e\|^\infty_{N^\infty + L^2} \leq \varepsilon \tag{3.23}
$$
for some $0 < \varepsilon \leq \varepsilon_*, \varepsilon_* = \varepsilon_*(C_T, L) > 0$ is a small constant, and $C_T$ is the Strichartz constant in Theorem 3.1. Then,

$$
\|v - \tilde{v} - R\|_{L^q(I;L^p) \cap C(I;L^2)} \leq C_*(C_T, L)\varepsilon, \quad (3.24)
$$

$$
\|v\|_{L^q(I;L^p)} \leq C_*(C_T, L). \quad (3.25)
$$

where $(\varepsilon_*(C_T, L))^{-1}$, $C_*(C_T, L)$ can be taken to be nondecreasing with respect to arguments.

The proof of Theorem 3.14 is quite similar to that of [55, Theorem 4.1] and [28, Proposition 4.6]. Actually, the linear terms $Gv$, $G\tilde{v}$ and the error terms $R$ and $e$ act as perturbations of the nonlinearity. So, the two solutions stay close to each other as long as these perturbations are small enough. For simplicity, the proof is postponed to the Appendix.

4 The controlled equations

4.1 Global well-posedness and integrability

In this subsection, we prove Theorems 2.2 and 2.6. Let us first treat the subcritical case in Theorem 2.2.

Proof of Theorem 2.2. (Mass-subcritical case.) Using $v := e^{-W}X$ we have

$$
i\partial_t v = e^{-W}\Delta(e^Wv) + \lambda F(v) + f(u)v, \quad (4.1)
$$

where $F(v) := |v|^{\alpha-1}v$ and $f(u) := V_0 + u \cdot V$. It has been proved in [7, (3.19)] that for the Strichartz pair $(p, q) = (\alpha + 1, \frac{4(\alpha+1)}{d(\alpha-1)}), d \geq 1$,

$$
\sup_{u \in U_a} \|X^u\|_{L^q(0,T;L^p)} \leq \left(\left[\frac{T}{t}\right] + 1\right)^{\frac{1}{\alpha}} \frac{\alpha}{\alpha - 1} C_T(1 + D_*T)|X_0|_2, \quad (4.2)
$$

where $\theta = 1 - \frac{\alpha-1}{4} > 0$, $t = \alpha^{-\theta}(\alpha-1)\frac{\alpha-1}{\alpha+1}(\|X_0\|_2+1)^{-\frac{\alpha-1}{\alpha+1}}C_T^{-\theta}(1+D_*T)^{-\frac{\alpha-1}{\alpha+1}}$, $D_* = |V_0|_{L^\infty} + D_K\|V\|_{L^\infty(\mathbb{R}^d,\mathbb{R}^m)}$, and $C_T$ is the constant in Theorem 3.1.

Then, using (3.3) we infer that

$$
\mathbb{E} \sup_{u \in U_{ad}} \|X^u\|_{L^q(0,T;L^p)}^\rho \leq C(\rho, T) < \infty, \quad \forall 1 \leq \rho < \infty, \quad (4.3)
$$

which along with Strichartz estimates (3.3) yields (2.3).
In order to obtain (2.4), we reformulate (4.1) as follows

\[ v(t) = U(t, 0)X_0 + (-i) \int_0^t U(t, s)(\lambda F(v) + f(u)v)ds, \]

where \( U(t, s) \), \( t, s \geq 0 \), are the evolution operators related to \(-ie^{-W} \Delta(e^W\cdot)\).

Since \( U(0, t)U(t, s) = U(0, s) \) for all \( 0 \leq s \leq t \), we have

\[ U(0, t)v(t) = X_0 + (-i) \int_0^t U(0, s)(\lambda F(v) + f(u)v)ds. \]

Then, by Corollary 3.11 Hölder’s inequality and \(|v|_{C([0,T]; L^2)} = |X_0|_{L^2}\),

\[ |U(0, \cdot)v|_{L^2} \leq |X_0|_{L^2} + C_T(||F(v)||_{L^q(0,T; L^{q'})} + ||f(u)v||_{L^1(0,T; L^2)}) \]
\[ \leq |X_0|_{L^2} + C_T(T^{\theta ||X^u||_{L^2(0,T; L^p)}} + DsT|X_0|_{L^2}). \]

This along with (4.3) yields (2.7), so (2.4) follows. Using (3.12) we get (2.6).

Therefore, the proof in the mass-subcritical case is complete. □

Next we consider (1.1) in the defocusing mass-critical case. We shall first prove the global well-posedness of (1.1) by using the idea of [55], based on stability result Theorem 3.14 and a series of rescaling transformations. Then, we adapt the arguments in [28] to get the integrability of global solutions.

**Proof of Theorem 2.2** (Mass-critical case.) First, by using Theorem 3.1 and similar arguments as in [5], we have a unique \( L^2 \)-solution \( X^u \) to (1.1) on a maximal existing time interval \([0, \tau^*]\), where \( \tau^*(\leq T) \) is an \( (\mathcal{F}_t) \) stopping time. Moreover, \( \tau^* = T \) a.s. if

\[ ||X^u||_{L^{2+\frac{4}{n}}(0, \tau^*) \times \mathbb{R}^d} < \infty, \quad a.s.. \quad (4.4) \]

Below we prove the global existence and \( L^p(\Omega) \)-integrability of \( X^u \) separately.

**Global existence.** We shall prove the global estimate (4.4) above. For this purpose, we reformulate (3.18) as follows

\[ i\partial_t \tilde{v} = e^{-W} \Delta(e^W \tilde{v}) + F(\tilde{v}) + f(u)\tilde{v} + e, \quad (4.5) \]
\[ \tilde{v}(0) = X_0, \]

with the error term

\[ e = -(b(t) \cdot \nabla + c(t))\tilde{v} - f(u)\tilde{v}, \]

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where \( b(t) \) and \( c(t) \) are as in (2.3).

In order to estimate the error term, we see that

\[
\langle x \rangle \langle \nabla \rangle^{-\frac{1}{2}} (b(t) \cdot \nabla + c(t)) = \Psi_{p(t)} \langle x \rangle^{-1} \langle \nabla \rangle^{\frac{1}{2}},
\]

where \( \Psi_{p(t)} := \langle x \rangle \langle \nabla \rangle^{-\frac{1}{2}} (b(t) \cdot \nabla + c(t)) \langle \nabla \rangle^{-\frac{1}{2}} \langle x \rangle \) is a pseudo-differential operator of order 0 satisfying that for any \( l \geq 1 \), the semi-norms \( |p(t)|_{s_0} \leq C(l) \sup_{0 \leq s \leq t} |\beta(s)| \), where \( |\beta(s)| = \max_{1 \leq j \leq N} |\beta_j(s)| \). (Here we may take \( t \) small enough such that \( |\beta_j(t)| \leq 1 \) and so \( |\beta_j(t)|^2 \leq |\beta_j(t)| \).)

Then, using the \( L^2 \)-boundedness of \( \Psi_{p(t)} \) (see [54, Lemma 2.4]) and Theorem 3.12 we get

\[
\| (b(t) \cdot \nabla + c(t)) \tilde{v} \|_{L^2(0,t;H^1_t)} \leq C_1 \| \tilde{v} \|_{L^2(0,t;H^1_t)} \sup_{0 \leq s \leq t} |\beta(s)| \leq C_1 B_0(|X_0|_{L^2}) \sup_{0 \leq s \leq t} |\beta(s)|. \tag{4.6}
\]

Moreover, since \( |\tilde{v}(t)|_{L^2} = |\tilde{v}(0)|_{L^2} = |X_0|_{L^2} \), we have, if \( D_* \) is as in (4.2),

\[
\| f(u) \tilde{v} \|_{L^1(0,t;L^2)} \leq D_* |X_0|_{L^2} t. \tag{4.7}
\]

Hence, letting \( h(t) := |\beta(t)| + t \), \( D_0 := (C_1 + D_*)(B_0(|X_0|_{L^2}) + |X_0|_{L^2}) \), we obtain

\[
\| e \|_{N^0(0,t) + L^2(0,t;H^1_t)} \leq \| (b(t) \cdot \nabla + c(t)) \tilde{v} \|_{L^2(0,t;H^1_t)} + \| f(u) \tilde{v} \|_{L^1(0,t;L^2)} \leq D_0 \sup_{s \in [0,t]} h(s). \tag{4.8}
\]

Thus, set \( \tau_1 := \inf \{ 0 \leq t \leq \tau^* : D_0 \sup_{s \in [0,t]} h(s) \geq \varepsilon_1(t) \} \cap \tau^* \), where the small constant \( \varepsilon_1(t) := \varepsilon_*(C_t, B_0(|X_0|_{L^2})) \) is as in Theorem 3.14 with \( B_0(|X_0|_{L^2}) \) replacing \( L \). We deduce from (4.8) that

\[
\| e \|_{N^0(0,\tau_1) + L^2(0,\tau_1;H^1_t)} \leq \varepsilon_*(C_t, B_0(|X_0|_{L^2})).
\]

This, via Theorem 3.14 with \( G = -if(u) \) and \( R = 0 \), yields that

\[
\| X^u \|_{L^2 + \frac{1}{2} ((0,\tau_1) \times \mathbb{R}^d)} = \| z \|_{L^2 + \frac{1}{2} ((0,\tau_1) \times \mathbb{R}^d)} \leq C_*(C_{\tau_1}, B_0(|X_0|_{L^2})) =: C_*(C_{\tau_1}).
\]
Next, we define the random times $\tau_{j+1}$ and $\sigma_{j+1}$ inductively as follows: $\sigma_0 := 0$, $\sigma_1 := \tau_1$, and for $j \geq 0$,

$$\sigma_{j+1} := \sigma_j + \tau_{j+1}(\leq \tau^*)$$

where $h_{\sigma_j}(t) := |\beta(\sigma_j + t) - \beta(\sigma_j)| + t$, and $\varepsilon_j(t) := \varepsilon_\star(C_{\sigma_j+t}, B_0(|X_0|_{L^2}))$ is as Theorem 3.14.

We have that $L < \infty$, a.s., by using similar arguments as in the proof of [55, Theorem 2.4] involving the $(\frac{1}{2} - \kappa)$-Hölder continuity of $t \mapsto h_{\sigma_j}(t)$, $0 < \kappa < \frac{1}{2}$.

Then, for each $1 \leq j \leq L - 1$, let $v_{\sigma_j}(t) := e^{-W_{\sigma_j}(t)}X(\sigma_j + t)$ with $W_{\sigma_j}(t) := W(\sigma_j + t) - W(\sigma_j)$ and set $u_{\sigma_j}(t) := u(\sigma_j + t)$, $t \geq 0$. In view of Theorem 3.3, we get

$$i \partial_t v_{\sigma_j} = e^{-W_{\sigma_j}} \Delta(e^{W_{\sigma_j}} v_{\sigma_j}) - F(v_{\sigma_j}) + f(u_{\sigma_j})v_{\sigma_j},$$

$$v_{\sigma_j}(0) = X(\sigma_j).$$

Similarly as above, we compare $v_{\sigma_j}$ with $\tilde{v}_j$ which solves the equation

$$i \partial_t \tilde{v}_j = \Delta \tilde{v}_j - F(\tilde{v}_j)$$

$$\tilde{v}_j(0) = v_{\sigma_j}(0) = X(\sigma_j),$$

or, equivalently,

$$i \partial_t \tilde{v}_j = e^{-W_{\sigma_j}} \Delta(e^{W_{\sigma_j}} \tilde{v}_j) - F(\tilde{v}_j) + f(u_{\sigma_j})\tilde{v}_j + e_j$$

with the error term

$$e_j = -(b_{\sigma_j} \cdot \nabla + c_{\sigma_j})\tilde{v}_j - f(u_{\sigma_j})\tilde{v}_j,$$

and $b_{\sigma_j}(t) := 2\nabla W_{\sigma_j}(t)$, $c_{\sigma_j}(t) := \Delta W_{\sigma_j}(t) + \sum_{j=1}^d (\partial_j W_{\sigma_j}(t))^2$.

Since $|\tilde{v}_j(t)||_{L^2} = |X(\sigma_j)||_{L^2} = |X_0||_{L^2}$, applying Theorem 3.12 we have

$$\|\tilde{v}_j\|_{L^{2+\frac{d}{2}}(\mathbb{R}^d)} \leq B_0(|X(\sigma_j)||_{L^2}) = B_0(|X_0||_{L^2}).$$

Then, estimating as in (4.8), we have

$$\|e_j\|_{N^0(0,\tau_{j+1};H^{\frac{d}{2}}_1)} \leq \|b_{\sigma_j} \cdot \nabla + c_{\sigma_j}\tilde{v}_j\|_{L^2(0,\tau_{j+1};H^{\frac{d}{2}}_1)} + \|f(u_{\sigma_j})\tilde{v}_j\|_{L^1(0,\tau_{j+1};L^2)} \leq D_0 \sup_{t \in [0,\tau_{j+1}]} h_{\sigma_j}(t) \leq \varepsilon_\star(C_{\sigma_{j+1}}, B_0(|X_0||_{L^2})).$$
This, via Theorem 3.14 implies that \( \mathbb{P} \)-a.s. for each \( 1 \leq j \leq L - 1 \),
\[
\|X^u\|_{L^{2+\frac{4}{q}}((\sigma_j,\sigma_{j+1}) \times \mathbb{R}^d)} = \|v_{\sigma_j}\|_{L^{2+\frac{4}{q}}((\sigma_j,\sigma_{j+1}) \times \mathbb{R}^d)} \\
\leq C_*(C_{\sigma_{j+1}}, B_0(|X_0|_{L^2})) =: C_*(C_{\sigma_{j+1}}).
\]

Thus, taking into account \( L < \infty \), a.s., we conclude that
\[
\sup_{u \in \mathcal{U}_d} \|X^u\|_{L^{2+\frac{4}{q}}((0,\sigma_1) \times \mathbb{R}^d)} \leq \sum_{j=0}^{L-1} C_*(C_{\sigma_{j+1}}) \leq LC_*(C_T) < \infty, \ a.s., \ (4.9)
\]
which, in particular, yields (4.4) and so the global existence of \( X^u \) on \([0, T]\).

Therefore, applying Corollary 3.11 to (4.1) and using (4.9) we obtain (2.4), which along with (3.11) and (3.16) implies (2.5) and (2.6).

\( L^p(\Omega) \)-integrability. In the case that \( \{e_j\} \) are constants, equation (4.1) reduces to the deterministic equation
\[
iv \partial_t v = \Delta v + \lambda F(v) + f(u)v, \quad (4.10)
\]
which implies that \( \|v\|_{S^0(0,T)} \in L^\infty(\Omega) \) and so (2.7) follows.

Below we consider the case that \( e_j \in L^{\frac{2d}{d-1}} \) for some \( p \) satisfying the condition in Hypothesis \( (H0)^* \). Take \( q \in (2, \infty) \) such that \( (p, q) \) is a Strichartz pair. Set \( M_1^\epsilon(t) := \sup_{0 \leq \tau < \tau_2 \leq t} |\int_{\tau_1}^{\tau_2} e^{-i(t-s)\Delta} X(s) dW(s)|_{L^2} \), \( M_2^\epsilon(t) := \sup_{0 \leq \tau < \tau_2 \leq t} |\int_{\tau_1}^{\tau_2} e^{-i(t-s)\Delta} X(s) dW(s)|_{L^p} \), \( t \in (0, T) \). Using similar arguments as in the proof of [28, Proposition 2.7] we have that for any \( 1 \leq \rho < \infty \),
\[
\|M_1^\epsilon\|_{L^\rho(\Omega; L^1(0,T))} + \|M_2^\epsilon\|_{L^\rho(\Omega; L^\infty(0,T))} < \infty, \quad (4.11)
\]
(See the Appendix for the proof. Note that, the restrictions \( p < \frac{2d}{d-1} \) and \( 1 \leq d \leq 3 \) are used here.) In particular, (4.11) implies that
\[
\|M_1^\epsilon\|_{L^1(0,T)} + \|M_2^\epsilon\|_{L^\infty(0,T)} < \infty, \ a.s.,
\]

Then, we take a partition \( \{[t'_j, t'_{j+1}]\}_{j=0}^{l'} \) of \([0, T]\) such that \( t'_{j+1} = \inf\{t > t'_j : \|M_1^\epsilon\|_{L^1(t,t)} = \epsilon_*/2\} \land T \), where \( \epsilon_* := \epsilon_*(C_0, B_0(|X_0|_{L^2})) \) is as in Theorem 3.14. \( C_0 \) is the deterministic Strichartz constant related to \( \{e^{-it\Delta}\} \). Note that, \( l' \leq 2\|M_1^\epsilon\|_{L^1(0,T)}/\epsilon_* \).

Similarly, take another partition \( \{[t''_j, t''_{j+1}]\}_{j=0}^{l''} \) of \([0, T]\), such that \( t''_{j+1} = \inf\{t > t''_j : \|M_2^\epsilon\|_{L^\infty(t,t)} = \epsilon_*/2\} \land T \). Then, \( l'' \leq (2\|M_2^\epsilon\|_{L^1(0,T)}/\epsilon_*)^q \).
Moreover, taking $\delta := \varepsilon_\ast / (|D_\ast + |\mu|_{L^\infty}||X_0||_{L^2})$ and $\{t_j\}_{j=0}^{L'+1} = \{t'_j\}_{k=0}^{l'+1} \cup \{j\delta\}_{j=0}^{[T/\delta]}$, we get that for any $1 \leq \rho < \infty$,

$$L' \leq l' + \frac{T}{\delta} \leq \frac{2}{\varepsilon_\ast} ||M_1^\ast||_{L^1(0,T)} + \frac{2}{\varepsilon_\ast} ||M_2^\ast||_{L^q(0,T)} + \frac{1}{\varepsilon_\ast} T|D_\ast + |\mu||_{L^\infty}||X_0||_{L^2} \in L^\rho(\Omega). \quad (4.12)$$

Now, for each $t \in I_j := [t_j, t_{j+1}]$, $0 \leq j \leq L'$, we infer from (1.1) that

$$X(t) = e^{-i(t-t_j)\Delta} X(t_j) + \int_{t_j}^{t} e^{-i(t-s)\Delta}(iF(X) - \mu X - if(u)X)ds + M_j(t), \quad (4.13)$$

where $F(X) = |X|^{\frac{1}{2}}X$, $f(u) = V_0 + u \cdot V$, and $M_j(t) = \int_{t_j}^{t} e^{-i(t-s)\Delta} X(s)dW(s)$, $t \in [0, T]$. In order to apply Theorem 3.14 to obtain the boundedness of $\|X\|_{L^q(I_j; L^p)}$, we compare (4.13) with the equation

$$i\partial_t \tilde{v}_j = \Delta \tilde{v}_j - F(\tilde{v}_j),$$

with $\tilde{v}_j(t_j) = X(t_j)$, or equivalently,

$$\tilde{v}_j(t) = e^{-i(t-t_j)\Delta} X(t_j) + \int_{t_j}^{t} e^{-i(t-s)\Delta}(iF(\tilde{v}_j) - \mu \tilde{v}_j - if(u)\tilde{v}_j + \epsilon)ds$$

with the error term $\epsilon = \mu \tilde{v}_j + if(u)\tilde{v}_j$.

Note that, Theorem 3.12 implies

$$\|\tilde{v}_j\|_{L^q(\mathbb{R}; L^p)} \leq B_0(|\tilde{v}_j|_{L^2}) = B_0(||X(t_j)||_{L^2}) = B_0(||X_0||_{L^2}).$$

Moreover, by the construction of $I_j$,

$$\|M_j\|_{L^q(I_j; L^p)} + \|M_j\|_{L^1(I_j; L^2)} \leq \|M_1^\ast\|_{L^1(I_j)} + \|M_2^\ast\|_{L^q(I_j)} \leq \varepsilon_\ast,$$

and

$$\|\epsilon\|_{N_0(I_j)} \leq \|\mu \tilde{v}_j + if(u)\tilde{v}_j\|_{L^1(I_j; L^2)} \leq (D_\ast + |\mu|_{L^\infty}) \|\tilde{v}_j\|_{C(I_j; L^2)} \leq (D_\ast + |\mu|_{L^\infty})||X_0||_{L^2}\delta \leq \varepsilon_\ast,$$

which implies the conditions (3.22) and (3.23) of Theorem 3.14.
Thus, applying Theorem 3.14 with \( \Phi \equiv 0 \), \( R = M_j \), \( G = -\mu - if(u) \) and \( V(t, s) = e^{-i(t-s)\Delta} \), we obtain

\[
\|X\|_{L^q(I_j;L^p)} \leq C_*(C_0, B_0(|X_0|_{L^2})) \in L^\infty(\Omega).
\]

Summing over \( j \) and using (4.12) we get

\[
\|X\|_{L^q(0,T;L^p)} \leq L'C_*(C_0, B_0(|X_0|_{L^2})) \in L^\rho(\Omega), \quad \forall 1 \leq \rho < \infty.
\]

Then, taking the Strichartz pair \((\tilde{p}, \tilde{q})\) such that \( (\frac{1}{\tilde{p}}, \frac{1}{\tilde{q}}) = (\frac{\alpha}{p}, \frac{\alpha}{q}) \), applying Theorem 3.1 to (4.1), using Hölder’s inequality and (3.17) we finally obtain

\[
\|X\|_{L^2(0,T;H^\frac{1}{2}X)} + \|e^{-W}X\|_{L^p(0,T)}
\]

\[
\leq C(T)|X_0|_{L^2} + C(T)\|X\|^{1+\frac{2}{q}}_{L^q(0,T;L^p)} + C(T)D_*T|X_0|_{L^2} \in L^\rho(\Omega), \quad 1 \leq \rho < \infty.
\]

Therefore, the proof in the defocusing mass-critical case is complete. \( \square \)

Below we prove Theorem 2.6, which is important to derive the tightness in Section 6 below.

**Proof of Theorem 2.6.** First, choose the Strichartz pair \((p, q) = (\alpha + 1, \frac{4(\alpha+1)}{d(\alpha-1)})\) and divide \([0,T]\) into \( \{I_j\}_{j=0}^L := \{[t_j, t_{j+1}]\}_{j=0}^L \) such that \( t_{j+1} = \inf\{t > t_j : \|X\|^{\alpha-1}_{L^q(t_j,t;L^p)} = (2C^*(T))^{-1}\} \land T \), where \( C^*(T) \in L^\rho(\Omega) \) is to be specified in (4.18) below. Then, \( L \leq (2C^*(T))^\frac{1}{\alpha-1}\|X\|_{L^q(0,T;L^p)} \in L^\rho(\Omega) \), \( \forall 1 \leq \rho < \infty \).

Letting \( z_k := e^{-W}V_kX \), we have

\[
i\partial_t z_k = e^{-W}\Delta(e^{-W}z_k) + \lambda|X|^\alpha z_k + f(u)z_k - 2e^{-W}\nabla V_k \cdot \nabla X - e^{-W}\Delta V_k X,
\]

\[
z_k(0) = V_kX_0,
\]

(4.14)

where \( f(u) = V_0 + u \cdot V \). This yields that for every \( 0 \leq j \leq L \),

\[
z_k(t) = U(t, t_j)z_k(t_j) + (-i)\int_{t_j}^t U(t, s) \left[ \lambda|X|^\alpha z_k + f(u)z_k - 2e^{-W}\nabla V_k \cdot \nabla X - e^{-W}\Delta V_k X \right] ds.
\]

Then, using \( U(t_j, t) = U(t_j, s) \) we get

\[
U(t_j, t)z_k(t) = z_k(t_j) + (-i)\int_{t_j}^t \left[ \lambda|X|^\alpha z_k + f(u)z_k - 2e^{-W}\nabla V_k \cdot \nabla X - e^{-W}\Delta V_k X \right] ds.
\]

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Applying Proposition 3.10 we get
\[
\|U(t_j, \cdot)z_k\|_{V^2(I_j)} \leq C(T) \left[ 1 + |z_k(t_j)|_{L^2} + \|X|^{\alpha-1}z_k\|_{L^p(I_j; L^p)} + \|f(u)z_k\|_{L^1(I_j; L^2)} + 2\|e^{-W}\nabla V_k \cdot \nabla X\|_{L^2(I_j; H^\frac{1}{2})} + \|e^{-W}\Delta V_k X\|_{L^1(I_j; L^2)} \right].
\]
(4.15)

Since \(|z_k(t)|_{L^2} \leq \|V_k\|_{L^\infty} \|X_0\|_{L^2}\), we have
\[
|z_k(t_j)|_{L^2} + \|f(u)z_k\|_{L^1(I_j; L^2)} + \|e^{-W}\Delta V_k X\|_{L^1(I_j; L^2)} \leq (\|V_k\|_{L^\infty} + \|V_k\|_{L^\infty} D_\text{ct}} T + \|\Delta V_k\|_{L^\infty} T) \|X_0\|_{L^2}
\]
(4.16)

with \(D_\text{ct} \) as in (4.12). Moreover, for any multi-index \(\gamma\), by \((H0)\),
\[
\sup_{0 \leq t \leq T} |\partial^\gamma_x (e^{-W} \nabla V_k)| \leq C \sup_{0 \leq t \leq T} |\beta(t)|^{\gamma} \langle x \rangle^{-2}.
\]

Then, estimating as in the proof of (4.16) we have for some \(l \geq 1\),
\[
\|e^{-W} \nabla V_k \cdot \nabla X\|_{L^2(I_j; H^\frac{1}{2})} \leq C_0 \sup_{0 \leq t \leq T} |\beta(t)|^l \|X\|_{L^2(I_j; H^\frac{1}{2})}.
\]
(4.17)

Thus, plugging (4.16) and (4.17) into (4.15) and using \(\|X|^{\alpha-1}z_k\|_{L^p(I_j; L^p)} \leq T^\theta \|X\|^{\alpha-1}_{L^p(I_j; L^p)} \|z\|_{L^\theta(I_j; L^\theta)}\), where \(\theta = 1 - \frac{d(\alpha-1)}{4} \geq 0\), we obtain
\[
\|U(t_j, \cdot)z_k\|_{V^2(I_j)} \leq C^*(T)(1 + |X_0|_{L^2} + \|X\|_{L^2(0,T; H^\frac{1}{2})}) + \|X\|^{\alpha-1}_{L^p(I_j; L^p)} \|z\|_{L^\theta(I_j; L^\theta)}
\]
\[
\leq C^*(T)(1 + |X_0|_{L^2} + \|X\|_{L^2(0,T; H^\frac{1}{2})}) + \frac{1}{2} \|z\|_{L^\theta(I_j; L^\theta)}
\]
(4.18)

with \(C^*(T) := C(T)(1 + |V_k|_{L^\infty}(1 + D_\text{ct}} T) + |\Delta V_k|_{L^\infty} T + T^\theta + 2C_0 \sup_{0 \leq t \leq T} |\beta(t)|^l\). It follows that
\[
\|U(t_j, \cdot)z_k\|_{V^2(I_j)} \leq 2C^*(T)(1 + |X_0|_{L^2} + \|X\|_{L^2(0,T; H^\frac{1}{2})}).
\]

Moreover, taking into account \(U(0, \cdot) = U(0, t_j)U(t_j, \cdot), \|U(0, t_j)\|_{L^2(L^2, L^2)} \leq C(T) \in L^\rho(\Omega) \) for any \(1 \leq \rho < \infty\), we have
\[
\|U(0, \cdot)z_k\|_{V^2(I_j)} \leq C(T)\|U(t_j, \cdot)z_k\|_{V^2(I_j)} \leq 2C(T)C^*(T)(1 + |X_0|_{L^2} + \|X\|_{L^2(0,T; H^\frac{1}{2})}).
\]
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Thus, summing over $0 \leq j \leq L$ and using (2.7) we obtain
\[
\|z_k\|_{V^2(0,T)} \leq \sum_{j=0}^{L} \|U(0,\cdot)z_k\|_{V(I_j)} \\
\leq 2LC(T)C^*(T)(1 + \|X_0\|_{L^2} + \|X\|_{L^2(0,T;H^\frac{1}{2}_x)}) \in L^\rho(\Omega)
\]
for any $1 \leq \rho < \infty$, which implies (2.8).

The temporal regularity (2.9) now follows from (2.8) and the embedding (3.12). Therefore, the proof of Theorem 2.6 is complete. \hfill \Box

4.2 Stability with respect to controls

In this Subsection, we prove the continuous dependence of controlled solutions and objective functionals on control and noise.

Consider the sequence $((X_T)_n, X_n, \beta_n, u_n)$, $((X_T, X, \beta, u) \in \mathcal{Y}, n \geq 1$, where $\mathcal{Y}$ is the space as in Definition 2.7, $u_n, u \in U_{ad}$, and $\beta_n, \beta$ are $\mathbb{R}^N$-dimensional Wiener processes. Assume $\mathbb{P}$-a.s.,

\[
\begin{align*}
  u_n &\to u, \text{ in } L^2(0,T;\mathbb{R}^m), \quad \beta_n \to \beta, \text{ in } C([0,T];\mathbb{R}^N), \\
  X_n &\to X, \text{ in } L^2((0,T) \times \mathbb{R}^d), \quad (X_T)_n \to X_T, \text{ in } L^2(\mathbb{R}^d).
\end{align*}
\]

Let $X_n$ (resp. $X$) be the controlled solution to (1.1) corresponding to $(u_n, \beta_n)$ (resp. $(u, \beta)$), $n \geq 1$. We have

Lemma 4.1. Consider the mass-subcritical case or the defocusing mass-critical case. Assume (H0). Then, for each $X_0 \in L^2$, $0 < T < \infty$,

\[
\|X_n - X\|_{S^0(0,T)} \to 0, \quad \text{as } n \to \infty, \text{ a.s.}
\]

Proof. In the subcritical case, (4.21) follows from [7, (3.11)] and Strichartz estimates (3.3). Below we prove the defocusing mass-critical case by using the stability result Theorem 3.14.

Set $W_n(t, x) = \sum_{j=1}^{N} \mu_j \epsilon_j(x) \beta_{j,n}(t)$, $W(t, x) = \sum_{j=1}^{N} \mu_j \epsilon_j(x) \beta_j(t)$, where $\beta_{j,n}$ and $\beta_j$ are the $j$-th component of $\beta_n$ and $\beta$ respectively, $1 \leq j \leq N$, $t \geq 0$, $x \in \mathbb{R}^d$. We may assume $T \geq 1$ without loss of generality.

Using the rescaling transformations $v_n := e^{-W_n}X_n$ we have

\[
\begin{align*}
  i\partial_t v_n &= e^{-W_n} \Delta (e^{W_n}v_n) - F(v_n) + f(u_n)v_n, \\
  v_n(0) &= X_0,
\end{align*}
\]

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where \( f(u_n) := V_0 + u_n \cdot V \). Similarly \( v := e^{-W}X \) solves (4.1) with \( \lambda = -1 \).

We shall compare \( v_n \) and \( v \). For this purpose, we reformulate (4.1)

\[
i \partial_t v = e^{-W_n} \Delta (e^{W_n} v) - F(v) + f(u_n)v + e_n,
\]

where the error term

\[
e_n = ((b(t) - b_n(t)) \cdot \nabla + (c(t) - c_n(t)))v + (f(u) - f(u_n))v
\]

with \( b_n(t) = 2\nabla W_n(t) \), \( c_n(t) = \Delta W_n(t) + \sum_{j=1}^d (\partial_j W_n(t))^2 \), and \( b(t), c(t) \) defined similarly as in (2.3).

Note that, by (H0), \( \mathbb{P} \)-a.s. for any multi-index \( \gamma \),

\[
|\partial_x^n W_n(t, x)| \leq C \sup_{n \geq 1} |\beta_n(t)| \langle x \rangle^{-2} < \infty,
\]

where \( C \) is independent of \( n \). Hence, Theorem 3.1 yields that Strichartz and local smoothing estimates hold for the operator \( e^{-W_n} \Delta (e^{W_n} \cdot) \) and the related Strichartz constants are uniformly bounded for all \( n \).

Moreover, in order to estimate the error term, we see that

\[
\langle x \rangle \langle \nabla \rangle^{-\frac{1}{2}} ((b(t) - b_n(t)) \cdot \nabla + (c(t) - c_n(t))) = \Psi_{p_n(t)} \langle x \rangle^{-1} \langle \nabla \rangle^{\frac{1}{2}},
\]

where \( \Psi_{p_n(t)} := \langle x \rangle \langle \nabla \rangle^{-\frac{1}{2}} ((b(t) - b_n(t)) \cdot \nabla + (c(t) - c_n(t)) \langle \nabla \rangle^{-\frac{1}{2}} \langle x \rangle \) is a pseudo-differential operator of order 0, satisfying \( |p_n(t)|_{S_0} \leq C(l) \| \beta_n \|_{C([0, t]; \mathbb{R}^N)} \| \beta - \beta_n \|_{C([0, t]; \mathbb{R}^N)} \) for any \( l \geq 1 \). Then, estimating as in (4.10) we get

\[
\| ((b(t) - b_n(t)) \cdot \nabla + (c(t) - c_n(t)))v \|_{L^2(0, T; H^{-\frac{1}{2}})} \leq C \| \beta_n \|_{C([0, T]; \mathbb{R}^N)} \| \beta - \beta_n \|_{C([0, T]; \mathbb{R}^N)} \| v \|_{L^2(0, T; H^{-\frac{1}{2}})}.
\]

This along with Hölder’s inequality and (4.19) yields that \( \mathbb{P} \)-a.s. as \( n \to \infty \),

\[
\| e_n \|_{N^0(0, T) + L^2(0, T; H^{-\frac{1}{2}})} \leq \| ((b(t) - b_n(t)) \cdot \nabla + (c(t) - c_n(t)))v \|_{L^2(0, T; H^{-\frac{1}{2}})}
\]

\[
+ \| (f(u) - f(u_n))v \|_{L^1(0, T; L^2)} \leq C \sup_{n \geq 1} \| \beta_n \|_{C([0, T]; \mathbb{R}^N)} \| v \|_{L^2(0, T; H^{-\frac{1}{2}})} \| \beta - \beta_n \|_{C([0, T]; \mathbb{R}^N)}
\]

\[
+ CT^{\frac{1}{2}} \| v \|_{C([0, T]; L^2)} \| u_n - u \|_{L^2(0, T; \mathbb{R}^m)} \to 0.
\]

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Thus, by virtue of Theorem 3.14 with $R = 0$, we obtain
\[
\|v_n - v\|_{L^2 + \frac{4}{3}(0,T) \times R^d} \to 0, \quad \text{as } n \to \infty, \ a.s. \tag{4.25}
\]

Note that, for any $n$ large enough,
\[
\|X_n - X\|_{L^2 + \frac{4}{3}(0,T) \times R^d} \leq \|v_n - v\|_{L^2 + \frac{4}{3}(0,T) \times R^d} + \|e^{-W}X - e^{-W_n}X\|_{L^2 + \frac{4}{3}(0,T) \times R^d} \\
\leq \|v_n - v\|_{L^2 + \frac{4}{3}(0,T) \times R^d} + \|e\|_{L^2((0,T) \times R^d)} \|X\|_{L^2 + \frac{4}{3}(0,T) \times R^d},
\]
where in the last step we used the inequality $|e^x - 1| \leq e|x|$ for $|x| \leq 1.$

Therefore, using (4.19), (4.25) we obtain (4.21) and finish the proof. \(\square\)

**Lemma 4.2.** Assume the conditions of Lemma 4.1 to hold. Assume additionally that \(\sup_n (\mathbb{E}|(X_T)_n|^2 + \mathbb{E}\|X_n\|_{L^2(0,T;L^2)}^2) < \infty\). Then, as $n \to \infty$,

\[
\mathbb{E} \Re \langle X_n(T), (X_T)_n \rangle_2 \to \mathbb{E} \Re \langle X(T), X_T \rangle_2,
\]
\[
\mathbb{E} \int_0^T \Re \langle X_n(t), X_n(t) \rangle_2 dt \to \mathbb{E} \int_0^T \Re \langle X(t), X(t) \rangle_2 dt.
\]

**Proof.** First, we infer from (4.20) and Lemma 4.1 that $\mathbb{P}$-a.s.,
\[
\Re \langle X_n(T), (X_T)_n \rangle_2 \to \Re \langle X(T), X_T \rangle_2.
\]

Moreover, since $|X_n(T)|_{L^2} = |X_0|_{L^2},$
\[
\sup_{n \geq 1} \mathbb{E}|\Re \langle X_n(T), (X_T)_n \rangle_2|^2 \leq |X_0|_{L^2} \sup_{n \geq 1} \mathbb{E}|(X_T)_n|^2_{L^2} < \infty,
\]
which implies the uniform integrability of $\{\Re \langle X_n(T), (X_T)_n \rangle_2\}$. Hence, taking into account (4.28) we obtain (4.26). The proof of (4.27) is similar. \(\square\)

Below we prove the continuous dependence of objective functionals with respect to control and noise when $\gamma_3 = 0$. Similarly to (2.10), define
\[
\Phi_n(u) := \mathbb{E}|X_n(T) - (X_T)_n|^2_{L^2} + \gamma_1 \mathbb{E} \int_0^T |X_n(t) - X_n(t)|^2_{L^2} dt \\
+ \gamma_2 \mathbb{E} \int_0^T |u_n(t)|^2_{m} dt.
\]

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Proposition 4.3. Let $\Phi$ be as in (2.10) with $\gamma_3 = 0$ and $\Phi_n$ be as in (4.29) above. Assume $(H0)$. Assume additionally that $E\{X_T\}_n^2 \to E\{X_T\}_T^2$ and $E\int_0^T |X_n(t)|_2^2 dt \to E\int_0^T |X(t)|_2^2 dt$. Then, for each $X_0 \in L^2$, $0 < T < \infty$,

$$\Phi_n(u_n) - \Phi(u) \to 0, \text{ as } n \to \infty, \text{ a.s..} \quad (4.30)$$

The proof follows by expanding the objective functions $\Phi_n$ and $\Phi$ and then using Lemma 4.2.

We are now ready to prove Theorem 2.8.

Proof of Theorem 2.8. The proof is similar to that of Theorem 2.5 of [7]. Actually, take any approximating controls $\{u_n\} \subseteq U_{ad}$ such that, if $I := \inf \{\Phi(u); u \in U_{ad}, X_u \text{satisfies (1.1)}\}$,

$$I \leq \Phi(u_n) \leq I + \frac{1}{n}, \quad n \geq 1.$$

When $\gamma_3 > 0$, we have additionally the differentiability of $\{u_n\}$, which yields the tightness of associated distributions on $C([0,T];\mathbb{R}^m)$. Hence, one can apply the Skorohod representation theorem and the stability result Proposition 4.3 to obtain a relaxed solution to Problem $(P)$ with $\gamma_3 > 0$. We refer to Section 3 of [7] for more details. \hfill \Box

5 The dual backward stochastic equations

This section is mainly devoted to the proof of Theorem 2.9. We first analyze the variational equation in Subsection 5.1, and then in Subsection 5.2 we prove Theorem 2.9 for the dual backward equation. As a direct application, we obtain the directional derivative of objective functional.

5.1 The variational equations

Consider the equation

$$id\psi = \Delta \psi dt + \lambda f_1(X^u)\psi dt + \lambda f_2(X^u)\bar{\psi} dt - i\mu \psi dt + V_0 \psi dt + u \cdot \nabla \psi dt - i\Psi dt + i\psi dW(t), \quad (5.1)$$

$$\psi(0) = 0.$$
Here, $f_j$ satisfies $\left| f_j(z) \right| \leq C_* |z|^{\alpha - 1}$, $z \in \mathbb{C}$, $j = 1, 2$, $X^u$ is the controlled solution to (1.1) related to $u \in U_{ad}$, and $\Psi$ satisfies $e^{-W(\cdot)}(0,T)$, i.e.,

$$\left\| \int_0^t U(0,s)e^{-W(s)}\Psi(s)ds \right\|_{U^2(0,T)} < \infty.$$ 

We mention that, in the special case where $\Psi = i\tilde{\nu} \cdot VX^u$ with $\tilde{\nu} = v - u$, $u, v \in U_{ad}$, equation (5.1) is indeed the variational equation (see Proposition 5.4 below). Here, we consider more general case, in order to analyze the dual linearized backward stochastic equation (2.13) in Subsection 5.2 later.

**Proposition 5.1.** Consider the mass-subcritical case or the defocusing mass-critical case. Assume (H0). Let $u \in U_{ad}$ and $e^{-W(\cdot)}(0,T)$, $0 < T < \infty$. Then, there exists a unique global $L^2$-solution $\psi_u$ to (5.1), satisfying that

$$\sup_{u \in U_{ad}} \left\| e^{-W(\cdot)}\psi_u \right\|_{L^2(0,T)} \leq C(T)\left\| e^{-W(\cdot)}\Psi \right\|_{N^2(0,T)}, \text{ a.s..}$$

(5.2)

If in addition $\epsilon_j$ are constants, $1 \leq j \leq N$, then $C(T) \in L^\infty(\Omega)$.

**Remark 5.2.** The additional condition that $\{\epsilon_j\}$ are constants can be reduced to the integrability of $\|X^u\|_{S^0(0,T)}$ and $\|\psi^u\|_{C(0,T;L^2)}$ (see Remark 5.3 below) of which the proof is, however, technically unclear due to the singular coefficient $h_2(X^u)$.

**Proof of Proposition 5.1.** We use the idea of [7] and give a unified treatment in the subcritical and critical cases. For convenience, we use $C_T(\geq 1)$ for the constants in Theorem 3.1, Proposition 3.10 and Corollary 3.11.

We first see that $z := e^{-W(\cdot)}\psi_u$ satisfies the equation

$$i\partial_t z = e^{-W(\cdot)}\Delta e^{-W(\cdot)}z + \lambda h(X^u, z) + f(u)z - ie^{-W(\cdot)}\Psi,$$

$$z(0) = 0,$$

where $h(X^u, z) = f_1(X^u)z + f_2(X^u)e^{-2W(\cdot)}$, and $f(u) := V_0 + u \cdot V$. Note that, $\sup_{u \in U_{ad}} \left\| f \right\|_{L^\infty(0,T) \times \mathbb{R}^d} \leq |V_0|_{L^\infty(\mathbb{R}^d)} + D_K \left\| V_0 \right\|_{L^\infty(\mathbb{R}^d;\mathbb{R}^m)} =: D_*$. 

Define the operator $F_1$ on $U^2(0,T)$ by

$$F_1(\phi)(t) := (-i) \int_0^t U(t,s)\left[ \lambda h(X^u, \phi)(s) + f(u(s))\phi(s) - ie^{-W(s)}\Psi(s) \right]ds,$$

where $\phi \in U^2(0,T)$, $0 \leq t \leq T$. 

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Take $(p, q) = (\alpha + 1, \frac{d(\alpha + 1)}{d(\alpha - 1)})$ and set $R_1(t) := 2C_1t^\alpha X^u_{L^p(0, t; L^p)} + D_{\ast}t$, where $\theta = 1 - \frac{d(\alpha - 1)}{4} \in [0, 1)$. Define $\tau_1 := \inf\{t \in [0, T) : C^2_1R_1(t) \geq \frac{1}{4}\} \cap T$, and $\mathcal{Z}_{M_1}^{\tau_1} := \{\phi \in U^2(0, T) : \|\phi\|_{L^2(0, \tau_1)} \leq M_1 := 2C^2_{\tau_1}\|e^{-W}\Psi\|_{L^2(0, \tau_1)}\}$. Note that, $R(t) \rightarrow 0$ as $t \rightarrow 0$, even in the critical case where $\theta = 0$. Moreover, $\tau_1$ is an $\{\mathcal{F}_t\}$-stopping time and $\mathcal{Z}_{M_1}^{\tau_1}$ is a closed ball of Banach space.

Since $U(0, t)U(t, s) = U(0, s)$, we deduce that

$$U(0, t)(F_1(\phi) - F_1(\phi_2)) = (i) \int_0^t U(0, s)[\lambda h(X^u, \phi_1 - \phi_2)(s) + f(u(s))(\phi_1 - \phi_2)(s)] ds.$$  

Then, by Corollary 3.11 and the Hölder inequality, for any $\phi_1, \phi_2 \in \mathcal{Z}_{M_1}^{\tau_1}$,

$$\|F_1(\phi_1) - F_1(\phi_2)\|_{L^2(0, \tau_1)} \leq C_{\tau_1}\|h(X^u, \phi_1 - \phi_2)\|_{L^p(0, \tau_1; L^p)} + C_{T_1}\|f(u)(\phi_1 - \phi_2)\|_{L^1(0, \tau_1; L^2)}$$

$$\leq C_{\tau_1}R_1(\tau_1)(\|\phi_1 - \phi_2\|_{L^p(0, \tau_1; L^p)} + \|\phi_1 - \phi_2\|_{L^2(0, \tau_1; L^2)})$$

$$\leq \frac{1}{2}\|\phi_1 - \phi_2\|_{L^2(0, \tau_1)}.$$  \hspace{1cm} (5.4)

Similarly, we have for any $\phi \in \mathcal{Z}_{M_1}^{\tau_1}$,

$$\|F_1(\phi)\|_{L^2(0, \tau_1)} \leq C^2_{\tau_1}R_1(\tau_1)\|\phi\|_{L^2(0, \tau_1)} + C_{\tau_1}\|e^{-W}\Psi\|_{L^2(0, \tau_1)} \leq M_1.  \hspace{1cm} (5.5)$$

Thus, we infer from (5.4) and (5.5) that $F_1 : \mathcal{Z}_{M_1}^{\tau_1} \rightarrow \mathcal{Z}_{M_1}^{\tau_1}$ is a contraction map, and so there exists a solution $z_1 \in \mathcal{Z}_{M_1}^{\tau_1}$ such that $z_1 = F_1(z_1)$, which implies that $z_1$ solves (5.1) on $[0, \tau_1]$. Moreover, (5.5) yields that

$$\|z_1\|_{L^2(0, \tau_1)} \leq 2C_{\tau_1}\|e^{-W}\Psi\|_{L^2(0, \tau_1)}.$$  

Next we use inductive arguments to extend $z_1$ to the whole time regime $[0, T]$. For this purpose, we set $\sigma_1 := \tau_1$, and for each $j \geq 1$,

$$\tau_{j+1} := \{t \in [0, T - \sigma_j) : \frac{C^2_{\sigma_j + t}R_{j+1}(t)}{C^2_{\sigma_j + t}R_{j+1}}(T - \sigma_j),$$

$$\sigma_{j+1} := \sigma_j + \tau_{j+1}(\leq T), \quad L := \inf\{j \geq 1 : \sigma_j = T\},$$

where $R_{j+1}(t) = 2C_{\sigma_j + t}t^\alpha X^u_{L^p(\sigma_j, \sigma_j + t; L^p)} + D_{\ast}t$.

Suppose that for some $1 \leq j \leq L - 1$ there exists a solution $z_j$ to (5.1) on $[0, \sigma_j]$, satisfying $z_j(\cdot) \equiv z_j(\cdot \land \sigma_j)$, and

$$\|z_j\|_{L^2(0, \sigma_j)} \leq M_j := j(2C_{\sigma_j})^j\|e^{-W}\Psi\|_{L^2(0, \sigma_j)}.  \hspace{1cm} (5.6)$$
We shall construct a solution \( z_{j+1} \) to (5.1) on \([\sigma_j, \sigma_{j+1}]\) below.

For this purpose, we define the operator \( F_j \) on \( U^2(0, T) \) by

\[
F_j(\phi)(t) := U(\sigma_j + t, \sigma_j) z_j(\sigma_j) + (-i) \int_0^t U(\sigma_j + t, \sigma_j + s) \left[ f(u(\sigma_j + s)) \phi(s) + \lambda h(X^n(\sigma_j + s), \phi(s)) - i e^{-W(\sigma_j + s)} \Psi(\sigma_j + s) \right] ds,
\]

where \( \phi \in U^2(0, T) \). Set \( \|\phi\|_{U^2(\sigma_j, \sigma_{j+1})} := \|U(\sigma_j, \sigma_j + \cdot) \phi\|_{U^2(0, \tau_{j+1})} \) and \( Z^{\tau_{j+1}}_{M_{j+1}} := \{ \phi \in U^2(0, T) : \|\phi\|_{U^2(\sigma_j, \sigma_{j+1})} \leq M_{j+1} \} \), where \( M_{j+1} \) is as in (5.6) with \( j + 1 \) replacing \( j \).

Then, estimating as in (5.4) and (5.5), we get for any \( \phi_1, \phi_2, \phi \in Z^{\tau_{j+1}}_{M_{j+1}} \),

\[
\|F_j(\phi_1) - F_j(\phi_2)\|_{U^2(\sigma_j, \sigma_{j+1})} \leq 2 C_{\sigma_{j+1}}^2 R_{j+1}(\tau_{j+1}) \|\phi_1 - \phi_2\|_{U^2(\sigma_j, \sigma_{j+1})} \leq \frac{1}{2} \|\phi_1 - \phi_2\|_{U^2(\sigma_j, \sigma_{j+1})}, \tag{5.7}
\]

and

\[
\|F_j(\phi)\|_{U^2(\sigma_j, \sigma_{n+1})} \leq |z_j(\sigma_j)|_{L^2} + C_{\sigma_{j+1}}^2 R_{j+1}(\tau_{j+1}) \|\phi\|_{U^2(\sigma_j, \sigma_{j+1})} + \left\| \int_0^t U(\sigma_j, \sigma_j + s) e^{-W(\sigma_j + s)} \Psi(\sigma_j + s) ds \right\|_{U^2(0, \tau_{j+1})}. \tag{5.8}
\]

Since \( U(\sigma_j, \sigma_j + s) = U(\sigma_j, 0) U(0, \sigma_j + s) \) and \( \|U(\sigma_j, 0)\|_{L^2 \rightarrow L^2} \leq C_{\sigma_j} \),

\[
\left\| \int_0^t U(\sigma_j, \sigma_j + s) e^{-W(\sigma_j + s)} \Psi(\sigma_j + s) ds \right\|_{U^2(0, \tau_{j+1})} \leq C_{\sigma_j} \left\| \int_0^t U(0, s) e^{-W(s)} \Psi(s) ds \right\|_{U^2(\sigma_j, \sigma_{j+1})}.
\]

Splitting the integral \( \int_0^t = \int_0^{\sigma_j} + \int_{\sigma_j}^{\tau_{j+1}} \) and using \( \|v\|_{U^2(\sigma_j, \sigma_{j+1})} \leq \|v\|_{U^2(0, \sigma_{j+1})} \) for any \( v \in U^2 \), we obtain

\[
\left\| \int_0^t U(\sigma_j, \sigma_j + s) e^{-W(\sigma_j + s)} \Psi(\sigma_j + s) ds \right\|_{U^2(0, \tau_{j+1})} \leq 2 C_{\sigma_j} \|e^{-W} \Psi\|_{L^2(0, \sigma_{j+1})}, \tag{5.9}
\]

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Plugging this into (5.8) and using (5.6) and (3.13) we obtain
\[
\|F_j(\phi)\|_{L^2_t(\sigma_j,\sigma_{j+1})} \\
\leq C_{\sigma_j} \|z_j\|_{L^2_t(0,\sigma_j)} + C_{\sigma_{j+1}}^2 R_{j+1}(\tau_{j+1}) \|\phi\|_{L^2_t(0,\sigma_j)} + 2C_{\sigma_{j+1}} \|e^{-W}\|^1_{L^2(0,\sigma_{j+1})} \\
\leq jC_{\sigma_j} (2C_{\sigma_j}^2 \|e^{-W}\|^1_{L^2(0,\sigma_j)} + \frac{1}{2} M_{j+1} + 2C_{\sigma_{j+1}} \|e^{-W}\|^1_{L^2(0,\sigma_{j+1})}) \\
\leq M_{j+1}.
\]

(5.10)

Hence, we deduce from (5.7) and (5.11) that \(F_j : \mathcal{Z}_{M_{j+1}}^{\tau_{j+1}} \mapsto \mathcal{Z}_{M_{j+1}}^{\tau_{j+1}}\) is a contraction map and so, for some \(\tilde{z}_{j+1} \in \mathcal{Z}_{M_{j+1}}^{\tau_{j+1}}\), \(F_j(\tilde{z}_{j+1}) = \tilde{z}_{j+1}\) on \([0, \tau_{j+1}]\).

Then, letting
\[
z_{j+1}(t) := \begin{cases} z_j(t), & t \in [0, \sigma_j]; \\ \tilde{z}_{j+1}(t - \sigma_j), & t \in [\sigma_j, T], \end{cases}
\]
we obtain that \(z_{j+1}\) is an \(L^2\)-solution to (5.1) on \([0, \sigma_{j+1}]\).

Thus, using inductive arguments we obtain a solution \(z\) to (5.1) on the maximal existing time interval \([0, \sigma_{j+1}]\).

Below we prove that \(L < \infty\), a.s.

For this purpose, we set \(\delta := (16D_\delta C_7^2)^{-1}\) and divide \([0, T]\) into finite subintervals \(\{[t_j', t_{j+1}']_{j=0}^{L^*}\}\), such that \(t_{j+1} = \inf\{t > t_j : C_\delta C_7^2 T^{\theta} \|X^u\|_{L^q(t_j', t; L^p)} \leq \frac{1}{32}\} \land T\). Then, \(l' \leq (32C_\delta C_7^2 T^{\theta})^{\frac{2}{q}} \|X^u\|_{L^q(0, T; L^p)}\).

Hence, letting \(\{t_j\}_{j=0}^{L^*} = \{j\delta : 0 \leq j \leq \frac{T}{\delta} \} \cup \{t_j' : 0 \leq j \leq l' + 1\}\), we obtain a partition \(\{[t_j, t_{j+1}]\}_{j=0}^{L^*}\) of \([0, T]\) satisfying that
\[
L^* \leq \frac{T}{\delta} + l' \leq 16D_\delta C_7^2 T + (32C_\delta C_7^2 T^{\theta})^{\frac{2}{q}} \|X^u\|_{L^q(0, T; L^p)} < \infty, \text{ a.s.}, \quad (5.11)
\]
and
\[
C_7^2 (2C_\delta T^{\theta} \|X^u\|_{L^q(t_j, t_{j+1}; L^p)} + D_\delta(t_{j+1} - t_j)) \leq \frac{1}{8}. \quad (5.12)
\]

We claim that \(L \leq L^* + 1, \text{ a.s.}\). \quad (5.13)

To this end, suppose that \(L > L^* + 1\). Then, for some \(0 \leq k \leq L^*\) and some \(1 \leq j \leq L\), \([t_k, t_{k+1}]\) contains both \(\sigma_j\) and \(\sigma_{j+1}\). If \(0 \leq k < L^*\), we have \(t_k \leq \sigma_j < \sigma_{j+1} \leq t_{k+1} < T\), and so, via the definition of \(\tau_{j+1}\) and (5.12),
\[
\frac{1}{4} = C_{\sigma_{j+1}}^2 R_{j+1}(\tau_{j+1}) \leq C_7^2 (2C_\delta T^{\theta} \|X^u\|_{L^q(t_k, t_{k+1}; L^p)} + D_\delta(t_{k+1} - t_k)) \leq \frac{1}{8},
\]

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yielding a contradiction. If \( k = L^* \), since \( L > L^* + 1 \), we have \( j < L - 1 \). But this yields that \( t_{L^*} \leq \sigma_j < \sigma_{j+1} < t_{L^*+1} = T \), which also leads to a contradiction by reasoning as above. Thus, we prove (5.13), as claimed.

Therefore, \( L < \infty \) and \( \sigma_L = T \), a.s.. So, the solution exists globally. Moreover, we infer from (5.6) that

\[
\| e^{-W} \psi^u \|_{L^2(0,T)} \leq L(2C_T)^2 \| e^{-W} \Psi \|_{L^2(0,T)}, \ a.s.,
\]

which implies (5.2) with \( C(T) = L(2C_T)^2 \).

The uniqueness of solutions to (5.1) can be proved similarly as in [7] by using Strichartz estimates.

In the case \( \sigma_j \) are constants, equation (4.1) reduces to (4.10), which implies that \( \| v \|_{S^0(0,T)} \in L^\infty(\Omega) \) and so is \( \| X^u \|_{S^0(0,T)} \). Moreover, we have \( U(t,s) = e^{-i(t-s)\Delta} \), implying that the related Strichartz constant \( C_T \) is deterministic and is independent of \( T \). Hence, taking into account (5.11), (5.13) and (5.14), we obtain \( C(T) \in L^\infty(\Omega) \).

Therefore, the proof of Proposition 5.1 is complete. \( \square \)

**Remark 5.3.** The \( L^p(\Omega) \)-integrability of \( \| e^{-W} \psi^u \|_{L^2(0,T)} \) can be reduced to that of \( \| X^u \|_{S^0(0,T)} \) and \( \| \psi^u \|_{C([0,T];L^2)} \). Actually, by (5.8) and (5.9),

\[
\| z \|_{L^2(\sigma_j,\sigma_{j+1})} \leq 2|z(\sigma_j)|_{L^2} + 4C_{\sigma_{j+1}} \| e^{-W} \Psi \|_{L^2(0,\sigma_{j+1})}, \ 0 \leq j < L.
\]

Since \( U(0, \cdot) = U(0, \sigma_j)U(\sigma_j, \cdot) \) and \( \| U(0, \sigma_j) \|_{L^2(0,T)} \leq C_{\sigma_j} \), by Lemma 3.7,

\[
\| z \|_{L^2(0,T)} \leq \sum_{j=0}^{L-1} \| U(0, \cdot)z \|_{L^2(\sigma_j,\sigma_{j+1})} \leq \sum_{j=0}^{L-1} C_{\sigma_j} \| z \|_{L^2(0,\sigma_{j+1})}.
\]  

Then, plugging (5.15) into (5.16) and using \( z = e^{-W} \psi^u \) we obtain \( \mathbb{P} \)-a.s.,

\[
\| e^{-W} \psi^u \|_{L^2(0,T)} \leq 4C_T^2 L \left( \| \psi^u \|_{C([0,T];L^2)} + \| e^{-W} \Psi \|_{L^2(0,T)} \right),
\]

which along with (5.11) and (5.13) yields the statement above.

Below we concentrate on the variational equation (5.1) with \( \Psi = i\tilde{u} \cdot VX^u \), \( \tilde{u} = v - u, \ v, u \in \mathcal{U}_{ad}, \) i.e.,

\[
\begin{align*}
\text{id} \varphi &= \Delta \varphi \text{dt} + \lambda h_1(X^u)\varphi \text{dt} + \lambda h_2(X^u)\overline{\varphi} \text{dt} - i\mu \varphi \text{dt} \\
&\quad + V_0 \varphi \text{dt} + u \cdot V \varphi \text{dt} + \tilde{u} \cdot VX^u \text{dt} + i\varphi \text{dW}(t), \\
\varphi(0) &= 0,
\end{align*}
\]

\[ (5.18) \]
where \(h_1(X^u), h_2(X^u)\) are defined in (2.14).

In this case, since \(L^1(0, T; L^2) \hookrightarrow N^0(0, T)\), by (3.17),
\[
\| e^{-W} \Phi \|_{H^2(0, T)} \leq C(T)\| \Phi \| \leq V e^{-W} X^u \|_{L^1(0, T; L^2)} \\
\leq C(T) D_K \| V \|_{L^\infty([0,T],[X_0])} X_0 \|_{L^2} T < \infty, \quad a.s..
\]

We deduce from Proposition 5.1 that there exists a unique global \(L^2\)-solution \(\varphi^{u, \tilde{u}}\) to (5.18), satisfying that
\[
\| e^{-W} \varphi^{u, \tilde{u}} \|_{H^2(0, T)} \leq C(T)\| X_0 \|_{L^2}, \quad a.s.. \quad (5.19)
\]

Proposition 5.4 below shows that \(\varphi^{u, \tilde{u}}\) is indeed the first order approximation of the controlled solution to (1.1).

**Proposition 5.4.** Consider the situations in Proposition 5.1. Assume (H0) and that \(e_j\) are constants, \(1 \leq j \leq N\). Given any \(u, v \in U_{ad}\), let \(\tilde{u} := v - u\) and \(\varphi^{u, \tilde{u}}\) be the solution to (5.18), \(\varepsilon \in (0, 1)\). Set \(u_\varepsilon := u + \varepsilon \tilde{u}\) and let \(X^{u_\varepsilon}\) (resp. \(X^u\)) be the controlled solution to (1.1) related to \(u_\varepsilon\) (resp. \(u\)). Then,

\[
\lim_{\varepsilon \to 0^+} \mathbb{E} \sup_{t \in [0, T]} |\varepsilon^{-1}(X^{u_\varepsilon}(t) - X^u(t)) - \varphi^{u, \tilde{u}}(t)|^2_{L^2} = 0. \quad (5.20)
\]

**Proof.** Set \(\tilde{X}^{u, \tilde{u}}_\varepsilon := \varepsilon^{-1}(X^{u_\varepsilon} - X^u) - \varphi^{u, \tilde{u}}, \tilde{y}^{u, \tilde{u}}_\varepsilon := e^{-W} \tilde{X}^{u, \tilde{u}}_\varepsilon\). We have
\[
\partial_t \tilde{y}^{u, \tilde{u}}_\varepsilon = -i \Delta \tilde{y}^{u, \tilde{u}}_\varepsilon + \lambda i \int_0^1 h(X_{u, r, \varepsilon}, \tilde{y}^{u, \tilde{u}}_\varepsilon) dr - i f(u_\varepsilon) \tilde{y}^{u, \tilde{u}}_\varepsilon + e^{-W} R(\varepsilon, \varphi^{u, \tilde{u}}), \quad (5.21)
\]

where \(f(u_\varepsilon) = V_0 + u_\varepsilon \cdot V, h(X_{u, r, \varepsilon}) = h_1(X_{u, r, \varepsilon}) \tilde{y}^{u, \tilde{u}}_\varepsilon + h_2(X_{u, r, \varepsilon}) e^{-2W} \tilde{y}^{u, \tilde{u}}_\varepsilon, X_{u, r, \varepsilon} := X^u + r(X^{u_\varepsilon} - X^u), r \in (0, 1)\), and \(R(\varepsilon, \varphi^{u, \tilde{u}}) := -i(\lambda R_1(\varepsilon) \varphi^{u, \tilde{u}} + \lambda R_2(\varepsilon) \varphi^{u, \tilde{u}} + \varepsilon \tilde{u} \cdot V \varphi^{u, \tilde{u}})\) with \(R_j(\varepsilon) = \int_0^1 (h_j(X_{u, r, \varepsilon}) - h_j(X^u)) dr, j = 1, 2\).

In order to prove (5.20), it is equivalent to prove that
\[
\lim_{\varepsilon \to 0^+} \mathbb{E} \| \tilde{y}^{u, \tilde{u}}_\varepsilon \|^2_{C([0,T];L^2)} = 0. \quad (5.22)
\]

For this purpose, by virtue of Proposition 5.1 we have
\[
\| \tilde{y}^{u, \tilde{u}}_\varepsilon \|_{H^2(0, T)} \leq C(T)\| e^{-W} R(\varepsilon, \varphi^{u, \tilde{u}}) \|_{H^2(0, T)},
\]

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where \( C(T) \in L^\rho(\Omega), \forall 1 \leq \rho < \infty \). This along with Corollary 3.11 yields

\[
\|y^u_{\varepsilon}\|_{S^0(0,T)} \leq C'(T)\|e^{-W}R(\varepsilon, \varphi^u_{\varepsilon})\|_{N^0(0,T)}
\]

\[
\leq C'(T) \left( \|R_1(\varepsilon)\varphi^u_{\varepsilon}\|_{L^p'(0,T;L^{p'})} + \|R_2(\varepsilon)\varphi^u_{\varepsilon}\|_{L^p'(0,T;L^{p'})} \right)
\]

\[
+ \|\varepsilon u \cdot V \varphi^u_{\varepsilon}\|_{L^1(0,T;L^2)}
\]

where \( C'(T) \in L^\rho(\Omega), (p,q) = (\alpha + 1, \frac{4(\alpha+1)}{d(\alpha-1)}), \) and \( p', q' \) are the conjugate numbers of \( p \) and \( q \), respectively. Thus, by Cauchy’s inequality,

\[
E\|y^u_{\varepsilon}\|^2_{S^0(0,T)} \leq C \left( E\|R_1(\varepsilon)\varphi^u_{\varepsilon}\|^4_{L^p'(0,T;L^{p'})} + E\|R_2(\varepsilon)\varphi^u_{\varepsilon}\|^4_{L^p'(0,T;L^{p'})} \right)
\]

\[
+ E\|\varepsilon u \cdot V \varphi^u_{\varepsilon}\|^4_{L^1(0,T;L^2)} \right)^\frac{1}{2}.
\]

Note that, by Corollary 3.11 and (5.19),

\[
\left( E\|\varepsilon u \cdot V \varphi^u_{\varepsilon}\|^4_{L^1(0,T;L^2)} \right)^\frac{1}{2} \leq C\varepsilon^2 \rightarrow 0.
\]

Thus, it remains to prove that as \( \varepsilon \rightarrow 0 \),

\[
E\|R_1(\varepsilon)\varphi^u_{\varepsilon}\|^4_{L^p'(0,T;L^{p'})} + E\|R_2(\varepsilon)\varphi^u_{\varepsilon}\|^4_{L^p'(0,T;L^{p'})} \rightarrow 0. \quad (5.23)
\]

For this purpose, we take \( R_1(\varepsilon)\varphi^u_{\varepsilon} \) for an example below, the argument for \( R_2(\varepsilon)\varphi^u_{\varepsilon} \) is similar. We shall prove that

\[
E\left\| \int_0^1 (h_1(X_{u,\varepsilon}) - h_1(X_u))\varphi^u_{\varepsilon} dr \right\|^4_{L^p'(0,T;L^{p'})} \rightarrow 0, \text{ as } \varepsilon \rightarrow 0. \quad (5.24)
\]

To this end, we first note that, Lemma 4.11 implies that there exists a null set \( N' \) such that \( \mathbb{P}(N') = 0 \) and for each \( \omega \notin N' \),

\[
\|X^u_{\varepsilon}(\omega) - X^u(\omega)\|_{L^p(0,T;L^p)} \rightarrow 0, \text{ as } \varepsilon \rightarrow 0. \quad (5.25)
\]

Below we omit the argument \( \omega \) for simplicity. This implies that for any subsequence \( \{\varepsilon_n\} \), we can extract a further subsequence (still denoted by \( \{\varepsilon_n\} \)) such that

\[
|X^{u_{\varepsilon_n}}(t) - X^u(t)|_{L^p} \rightarrow 0, \text{ as } n \rightarrow \infty, dt - a.e.. \quad (5.26)
\]
In particular, for $dt$-a.e. $t \in (0, T)$,
\[ X^{u_{\varepsilon_n}}(t) \to X^u(t), \text{ in measure } dx, \]
which yields that for $dt$-a.e. $t \in (0, T)$,
\[ h_1(X_{u,r,\varepsilon_n})(t) \to h_1(X^u)(t), \text{ in measure } dx. \] (5.27)
Moreover, by (5.26), for $dt$-a.e. $t \in (0, T)$, as $n \to \infty$,
\[ |h_1(X_{u,r,\varepsilon_n})(t) - h_1(X^u)(t)|_{L^p}^{a+1} \to 0, \text{ as } n \to \infty. \] (5.28)
We infer from (5.27) and (5.28) that \( \forall r \in (0, 1), dt \text{-a.e. } t \in (0, T), \)
\[ |h_1(X_{u,r,\varepsilon_n})(t) - h_1(X^u)(t)|_{L^p}^{a+1} \to 0, \text{ as } n \to \infty. \] (5.29)
This, via Hölder’s inequality, yields that
\[ \|h_1(X_{u,r,\varepsilon_n})(t) - h_1(X^u)(t))\varphi^{u,\tilde{u}}(t)|_{L^{p'}} \to 0, \text{ as } n \to \infty. \] (5.30)
Next, we claim that \( \forall r \in (0, 1), \{|(h_1(X_{u,r,\varepsilon_n}) - h_1(X^u))\varphi^{u,\tilde{u}}|_{L^{p'}}\} \) is uniformly integrable.
Actually, for any Borel set $A \subseteq (0, T)$, Hölder’s inequality implies that
\[
\int_A \|h_1(X_{u,r,\varepsilon_n}) - h_1(X^u))\varphi^{u,\tilde{u}}|_{L^{p'}} dt \\
\leq C T^q \|\varphi^{u,\tilde{u}}\|_{L^{q}(0,T;L^p)} (\int_A |X^{u_{\varepsilon_n}}|^q_{L^p} dt + \int_A |X^u|^q_{L^p} dt) \frac{(q-1)q'}{q} \\
\leq C(T)\|X^{u_{\varepsilon_n}} - X^u|_{L^{q'}(0,T;L^p')} + C(T)(\int_A |X^u|^p_{L^p} dt) \frac{(q-1)q'}{q},
\]
where $\theta = 1 - \frac{d(a-1)}{4} \in [0, 1)$. Taking into account (5.25) we get
\[
\limsup_{n \to \infty} \int_A \|h_1(X_{u,r,\varepsilon_n}) - h_1(X^u)(t))\varphi^{u,\tilde{u}}|_{L^{p'}} dt \\
\leq C(T)(\int_A |X^u|^p_{L^p} dt) \frac{(q-1)q'}{q} \to 0, \text{ as } |A| \to 0,
\]
which implies the statement, as claimed. Hence, taking into account (5.30), we get that for $\forall r \in (0, 1)$,
\[
(h_1(X_{u,r,\varepsilon_n}) - h_1(X^u))\varphi_{u,\tilde{u}} \to 0, \text{ as } n \to \infty. \quad (5.31)
\]
We also see that
\[
sup_{n \geq 1}\|h_1(X_{u,r,\varepsilon_n})\varphi_{u,\tilde{u}}\|_{L^{p'}(0,T;L^{p'})} \leq C(\alpha)T^\theta(\|X^u\|^{q-1}_{L^{q}(0,T;L^{p})}\|\varphi_{u,\tilde{u}}\|_{L^{q}(0,T;L^{p})}) + sup_{n \geq 1}\|X_{u,n}\|^{\alpha-1}_{L^{q}(0,T;L^{p})}\|\varphi_{u,\tilde{u}}\|_{L^{q}(0,T;L^{p})} < \infty, \quad (5.32)
\]
This, via the bounded convergence theorem, yields that as $n \to \infty$,
\[
\left\| \int_{0}^{1}(h_1(X_{u,r,\varepsilon_n}) - h_1(X^u))\varphi_{u,\tilde{u}} \, dr \right\|_{L^{p'}(0,T;L^{p'})} \leq \int_{0}^{1}\|h_1(X_{u,r,\varepsilon_n}) - h_1(X^u))\varphi_{u,\tilde{u}}\|_{L^{p'}(0,T;L^{p'})} \, dr \to 0 \quad (5.33)
\]
for any subsequence $\{\varepsilon_n\}$. Since $\{\varepsilon_n\}$ is arbitrary, we conclude that $P$-a.s. (5.33) holds with $\varepsilon$ replacing $\varepsilon_n$.

Finally, in view of (2.5), (5.19) and (5.32), we infer that the left-hand side of (5.33) with $\varepsilon$ replacing $\varepsilon_n$ is uniformly integrable, and so we obtain (5.24). The proof for $R_2(\varepsilon)$ is similar.

Therefore, we obtain (5.23) and finish the proof of Proposition 5.4. \hfill \Box

\section{5.2 \textbf{Proof of Theorem 2.9}}

We prove Theorem 2.9 for the stochastic backward equation (2.13).

\textbf{Proof of Theorem 2.9}. We use the duality arguments as in [7] to reduce the analysis of (2.13) to that of the forward equation (5.1).

First, we construct approximating solutions to (2.13). Precisely, we use the truncation $h_{j,n}(X^u) := g(\frac{X^u_n}{n})h_j(X^u)$, $j = 1, 2$, where $g$ is a radial smooth cut-off function such that $g = 1$ on $B_1(\mathbb{R})$, and $g = 0$ on $B_2^c(\mathbb{R})$. Note that $|h_{1,n}(X^u)| + |h_{2,n}(X^u)| \leq \min\{\alpha 2^{\alpha-1}|g|_{L^\infty n^{\alpha-1}}, \alpha|X^u|^{\alpha-1}\}.$

Consider the approximating backward stochastic equation
\[
dY_n = -i\Delta Y_n dt - \lambda i h_{1,n}(X^u)Y_n dt + \lambda i h_{2,n}(X^u)Y_n dt + \mu Y_n dt - iV_0 Y_n dt
\]
\[
- \frac{iu \cdot V Y_n dt + \gamma_1(X^u - \bar{X}_1) dt}{\sum_{k=1}^{N} \mu_k \epsilon_k Z_{k,n} dt + \sum_{k=1}^{N} Z_{k,n} d\beta_k(t)},
\]
\[
Y_n(T) = -(X^u(T) - \bar{X}_T). \quad (5.34)
\]
For $n \geq 1$, there exists a unique $({\mathcal F}_t)$-adapted solution $(Y_n, Z_n)(:= (Y^u_n, Z^u_n)) \in L^2(\Omega; C([0, T]; L^2)) \times (L^2_{ad}(0, T; L^2(\Omega; L^2)))^N$ to (5.34) (see, e.g., [31], [37]).

In order to pass to the limit $n \to \infty$, we prove the uniform estimate blow

$$
\sup_{n \geq 1} \sup_{u \in \mathcal{U}_{ad}} \| e^{-W} Y_n \|_{L^\rho(\Omega; \mathcal{M}^2(0, T))} < \infty, \quad 1 \leq \rho < \rho_\nu := 2 + \nu.
$$

(5.35)

In particular, by (5.13) and $|e^{-W}| = 1$,

$$
\sup_{n \geq 1} \sup_{u \in \mathcal{U}_{ad}} \| Y_n \|_{L^\rho(\Omega; \mathcal{M}^2(0, T))} < \infty, \quad 1 \leq \rho < \rho_\nu.
$$

(5.36)

To this end, we define the functional $\Lambda_n$ on the space $L^\infty(\Omega \times (0, T) \times \mathbb{R}^d)$,

$$
\Lambda_n(\Psi) := \mathbb{E} \Re \langle X^u(T) - X_T, \psi_n(T) \rangle_2 + \gamma_1 \mathbb{E} \int_0^T \Re \langle X^u(t) - X(t), \psi_n(t) \rangle_2 dt,
$$

(5.37)

where $\Psi \in L^\infty(\Omega \times (0, T) \times \mathbb{R}^d)$, $X^u$ is the controlled solution to (1.1), and $\psi_n(\cdot; u, \Psi)$ satisfies (5.11) with $h_j, n(X^u)$ replacing $f_j(X^u), j = 1, 2.$

By Itô’s formula, we have for any $\Psi \in L^\infty(\Omega \times (0, T) \times \mathbb{R}^d),

$$
\Lambda_n(\Psi) = \mathbb{E} \int_0^T \Re \langle \Psi, Y_n \rangle_2 dt.
$$

(5.38)

Moreover, by virtue of Proposition 5.1 we have

$$
\sup_{n \geq 1} \sup_{u \in \mathcal{U}_{ad}} \| e^{-W} \psi_n \|_{L^\rho(0, T)} \leq C(T) \| e^{-W} \Psi \|_{\mathcal{M}^2(0, T)}, \quad a.s.,
$$

(5.39)

where $C(T) \in L^\rho(\Omega), \forall 1 \leq \rho < \infty$, and is independent of $\Psi$. This, via Hölder’s inequality, yields that for any $1 \leq \rho_1 < \rho_2 < \infty,

$$
\sup_{n \geq 1} \sup_{u \in \mathcal{U}_{ad}} \| e^{-W} \psi_n \|_{L^\rho_1(\Omega; L^\rho_2(0, T))} \leq C(\rho_1, \rho_2, T) \| e^{-W} \Psi \|_{L^\rho_1(\Omega; \mathcal{M}^\rho_2(0, T))},
$$

(5.40)

where $C(\rho_1, \rho_2, T)$ is independent of $\Psi$. Then, combining together (2.7), (5.37) and (5.40), we obtain that for any $\varepsilon \in (0, 1),

$$
| \Lambda_n(\Psi) | \leq \gamma_1 \| X^u - X_T \|_{L^\rho(\Omega; L^2(0, T; L^2))} \| \psi_n \|_{L^\rho_1(\Omega; L^\rho_2(0, T; L^2))}
\quad + \| X^u(T) - X_T \|_{L^\rho_1(\Omega; L^2)} \| \psi_n(T) \|_{L^\rho_2(\Omega; L^2)}
\leq C(\rho_\nu, \varepsilon, T) \| e^{-W} \Psi \|_{L^{(\rho_\nu - \varepsilon)'(\Omega; \mathcal{M}^2(0, T))}},
$$

(5.41)
Hence, we conclude from (5.38) and (5.41) that
\[
\sup_{n \geq 1} \sup_{u \in U_{ad}} \left| \mathbb{E} \int_0^T \Re \langle \Psi, Y_n \rangle_2 dt \right| \leq C(\rho, \varepsilon, T) \| e^{-W} \Psi \|_{L^{(\rho, \varepsilon)'}(\Omega; N^2(0, T))},
\]
(5.42)

Now, setting \( Z := \{ v \in L^\infty(\Omega; C_0^\infty(0, T; L^2)) ; \| v \|_{L^{(\rho, \varepsilon)'}(\Omega; U^2(0, T))} \leq 1 \} \) and using Lemmas 3.6 and 3.8 and (3.6) we get
\[
\| e^{-W} Y_n \|_{L^{(\rho, \varepsilon)}(\Omega; U^2(0, T))} = \sup_{v \in Z} |\mathbb{E} ReB(v, U(0, \cdot)e^{-W}Y_n)|
\]
\[
= \sup_{v \in Z} |\mathbb{E} \int_0^T \Re \langle v'(t), U(0, t)e^{-W(t)}Y_n(t) \rangle_2 dt|
\]
\[
= \sup_{v \in Z} |\mathbb{E} \int_0^T \Re \langle e^{W(t)}U(t, 0)v'(t), Y_n(t) \rangle_2 dt|,
\]

Then, by (5.42), the right-hand side above is bounded by
\[
C(\rho, \varepsilon, T) \sup_{v \in Z} \left\| \int_0^T |v'(t)| dt \right\|_{L^{(\rho, \varepsilon)'}(\Omega; U^2(0, T))} = C(\rho, \varepsilon, T) \sup_{v \in Z} \| v \|_{L^{(\rho, \varepsilon)'}(\Omega; U^2(0, T))}
\]
\[
\leq C(\rho, \varepsilon, T),
\]
where \( C(\rho, \varepsilon, T) \) is independent of \( n \) and \( u \in U_{ad} \). This yields (5.39) and so (5.43), as claimed.

Below we choose the Strichartz pair \((p, q) = (\alpha + 1, \frac{4(\alpha + 1)}{d(\alpha - 1)})\) and fix \( 2 \leq \rho < \rho_* \). Estimate (5.36) implies that for some \( \tilde{Y} \in L^\rho(\Omega; S^0(0, T)) \), along a subsequence of \( \{n\} \to \infty \) (still denoted by \( \{n\} \)),
\[
Y_n \overset{w}{\rightharpoonup} \tilde{Y}, \text{ in } L^\rho(\Omega; S^0),
\]
(5.43)
where \( \overset{w}{\rightharpoonup} \) denotes the weak-star convergence.

Moreover, since for each \( j = 1, 2 \), \( h_{j,n}(X^u) \to h_j(X^u) \), \( d\mathbb{P} \otimes dt \otimes dx \)-a.e., and \( \sup_{n \geq 1} |h_{j,n}(X^u)| \leq C|X^u|^{\alpha - 1} \in L^{2\rho'}(\Omega; L^{\frac{2}{\alpha - 1}}(0, T; L^{\frac{2}{\alpha - 1}})) \), by the dominated convergence theorem,
\[
h_{j,n}(X^u) \to h_j(X^u), \text{ in } L^{2\rho'}(\Omega; L^{\frac{2}{\alpha - 1}}(0, T; L^{\frac{2}{\alpha - 1}})), \text{ as } n \to \infty.
\]
(5.44)

Hence, using Hölder’s inequality, (5.43) and (5.44) we obtain
\[
h_{1,n}(X^u)Y_n \overset{\omega}{\rightharpoonup} h_1(X^u)\tilde{Y}, \quad h_{2,n}(X^u)\nabla_n \overset{\omega}{\rightharpoonup} h_2(X^u)\nabla\tilde{Y}, \text{ in } L^{(2\rho)'}(\Omega; L^{\frac{2}{\alpha}}(0, T; L^{\frac{2}{\alpha}})),
\]
(5.45)
where \( \omega \) denotes the weak convergence.

Regarding \( Z_{k,n} \), \( 1 \leq k \leq N \), apply Itô's formula to \( e^{\eta t}|Y_n(t)|^2_{L^2} \) for \( \eta \) large enough and using similar arguments as in the proof of [7, (4.44)], involving the Burkholder-Davis-Gundy inequality, we obtain

\[
\sup_{n \geq 1} \frac{1}{2} \sum_{k=1}^{N} \mathbb{E} \left( \int_0^T e^{\eta s} |Z_{k,n}|_{L^2}^2 ds \right) \leq C(\rho) \mathbb{E} V_{T,n}^\rho + C(\rho, N) e^{\frac{1}{2} \rho \eta T} \sup_{n \geq 1} \mathbb{E} \|Y_n\|_{C([0,T];L^2)},
\]

where

\[
V_{T,n} : = e^{\eta T} |X^u(T)|^2_{L^2} - \mathbb{X}_T^2 + 2 \gamma_1 \int_0^T e^{\eta s} |X^u - \mathbb{X}|^2_{L^2} ds + \alpha \|g\|_{L^\infty} e^{\eta T} T^q \|X^u\|_{L^q(0,T;L^p)}^{q-1} \|Y_n\|_{L^q(0,T;L^p)}^2.
\]

Hence, using (5.36) we get

\[
\sup_{n \geq 1} \sup_{u \in \mathcal{U}_{ad}} \|Z_{k,n}\|_{L^\rho(\Omega;L^2(0,T;L^2))} \leq C < \infty, \quad 1 \leq k \leq N. \tag{5.46}
\]

In particular, there exists \( Z_k^u \in L^\rho(\Omega;L^2(0,T;L^2)) \) such that

\[
Z_{k,n} \xrightarrow{\omega} Z_k^u, \quad \text{in} \ L^2(\Omega;L^2(0,T;L^2)) \tag{5.47}
\]

for a further subsequence if necessary, which implies that

\[
\int_0^T Z_{k,n} d\beta_k(s) \xrightarrow{\omega} \int_0^T Z_k^u d\beta_k(s), \quad \text{in} \ L^2(\Omega;L^2(0,T;L^2)). \tag{5.48}
\]

Now, since equation (5.34) is linear with respect to \( Y_n \), \( Z_{k,n} \), \( 1 \leq k \leq N \), we can use (5.43) - (5.48) to pass to the limit in (5.34). Moreover, using similar arguments as those below (4.44) of [7], we can obtain a continuous version \( Y^u \) of \( \tilde{Y} \), such that \( (Y^u, Z^u) \) solves (2.13) in \( H^{-2} \) for all \( t \in [0,T], \mathbb{P}\)-a.s.. This yields the global existence of solutions to (2.13). The uniqueness of solutions can be proved via the duality (5.38), as in the proof of [7, Proposition 7.2].

Moreover, we obtain (2.15), (2.16) and (2.17) from (5.35), (5.36) and (5.40). Then, in view of (3.12), we also get (2.18).

Therefore, the proof of Theorem 2.9 is complete. \( \square \)

As a consequence of Proposition 5.1, Theorem 2.9 and the duality (5.38) with \( \Psi = i\tilde{u} \cdot VX^u \), we obtain the directional derivative of objective functional \( \Phi \) in the case where \( \gamma_3 = 0 \).
Proposition 5.5. Assume the conditions of Theorem 2.11 to hold. Then, for each $X_0 \in L^2$ and any $u,v \in U_{ad}$, we have
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (\Phi(u + \varepsilon \tilde{u}) - \Phi(u)) = \mathbb{E} \int_0^T \eta(u)(t) \cdot \tilde{u}(t) dt,
\]
(5.49)
where $\tilde{u} = v - u$, and
\[
\eta(u) = 2 \left( \gamma_2 u - \text{Im} \int_{\mathbb{R}^d} V(x)X^u(x)\overline{Y^u(x)} dx \right).
\]
(5.50)
Here, $(Y^u, Z^u)$ is the solution to the dual backward stochastic equation (2.13).

6 Proof of Theorem 2.11

The key idea here is to apply Ekeland’s principle (see [27, Theorem 1] or [26]) to obtain approximating controls, which are minimizers of perturbed objective functionals and so can be characterized explicitly by using the calculus of subdifferential in the sense of Rockafellar ([49]). Then, by virtue of the new temporal regularities (2.9) and (2.18) of controlled and backward solutions respectively, we can obtain the tightness of the associated distributions of approximating controls which, combined with Skorohod’s representation theorem, enables us to obtain a relaxed optimal control for Problem (P).

To be precise, let us first deduce from Proposition 4.3 that, when $\gamma_3 = 0$, $\Phi$ is continuous on the metric space $U_{ad}$ endowed with the distance $d(u, v) = \|u - v\| = (\mathbb{E} \int_0^T |u(t) - v(t)|^2 dt)^{1/2}$. Then, by virtue of Ekeland’s variational principle, for every $n \in \mathbb{N}$, we have $u_n \in U_{ad}$ such that
\[
\Phi(u_n) \leq \Phi(u) + \frac{1}{n} d(u_n, u), \quad \forall u \in U_{ad},
\]
(6.1)
which implies that
\[
u_n = \arg \min \left\{ \Phi(u) + \frac{1}{n} \|u_n - u\|; \ u \in U_{ad} \right\}.
\]
(6.2)
We have the geometric characterization of $\{u_n\}$ as specified below.
Lemma 6.1. Assume the conditions of Theorem 2.11 to hold. Let $X_n$ (resp. $(Y_n, Z_n)$) be the solution to (1.1) (resp. (2.13)), corresponding to $u_n$ above, $n \geq 1$. We have

$$u_n(t) = P_K \left( \frac{1}{\gamma_2} \text{Im} \int V(x)X_n(t,x)\gamma_n(t,x)dx - \frac{1}{2\gamma_2 n} \eta_n(t) \right),$$

(6.3)

where $\eta_n \in \partial(\|u_n - u\|)|_{u=u_n}$ satisfying $\mathbb{E} \int_0^T |\eta_n(t)|^2 dt = 1$, “$\partial$” means the subdifferential, and $P_K$ is the projection operator on $K$.

**Proof.** We reformulate (6.2) as follows

$$u_n = \arg \min \left\{ \tilde{\Phi}(u) + \frac{1}{n} \|u_n - u\|, u \in L^2_{ad}(0,T;\mathbb{R}^m) \right\},$$

where $\tilde{\Phi}(u) = \Phi(u) + I_{\mathcal{U}_{ad}}(u)$, $I_{\mathcal{U}_{ad}}(u) = 0$ if $u \in \mathcal{U}_{ad}$, or $I_{\mathcal{U}_{ad}}(u) = \infty$ otherwise. Then, applying the subdifferential in the sense of Rockafellar (see [49], see also [7]) we obtain

$$0 \in \partial(\tilde{\Phi}(u) + \frac{1}{n} \|u_n - u\||_{u=u_n}).$$

(6.4)

Moreover, by virtue of Theorem 2 of [49] we get

$$\partial(\tilde{\Phi}(u) + \frac{1}{n} \|u_n - u\||_{u=u_n}) \subseteq \partial \tilde{\Phi}(u)|_{u=u_n} + \frac{1}{n} \partial(\|u_n - u\|)|_{u=u_n}$$

$$\subseteq \eta(u_n) + \partial I_{\mathcal{U}_{ad}}(u_n) + \frac{1}{n} \partial(\|u_n - u\||_{u=u_n})$$

$$= \eta(u_n) + \mathcal{N}_{\mathcal{U}_{ad}}(u_n) + \frac{1}{n} \partial(\|u_n - u\||_{u=u_n}).$$

(6.5)

Here, $\eta(u_n)$ is the directional derivative of $\Phi$ at $u_n$ obtained in (5.50), and $\mathcal{N}_{\mathcal{U}_{ad}}(u_n)$ is the normal cone to $\mathcal{U}_{ad}$ at $u_n$, i.e.,

$$\mathcal{N}_{\mathcal{U}_{ad}}(u_n) = \{ v \in L^2_{ad}(0,T;\mathbb{R}^m); \langle v, u_n - \tilde{v} \rangle \geq 0, \ \forall \tilde{v} \in \mathcal{U}_{ad} \},$$

where $\langle , \rangle$ denotes the inner product of $L^2_{ad}(0,T;\mathbb{R}^m)$. We also have the following characterization of $\mathcal{N}_{\mathcal{U}_{ad}}(u_n)$ (see [7] (5.6))

$$\mathcal{N}_{\mathcal{U}_{ad}}(u_n) = \{ v \in L^2_{ad}(0,T;\mathbb{R}^m); v \in \mathcal{N}_K(u_n), \ a.e. \ on \ \Omega \times (0,T) \}.$$
where \( N_K(u_n) \) is the normal cone to \( K \subset \mathbb{R}^m \) at \( u_n \in K \), i.e.,

\[
N_K(u_n) = \{ v \in \mathbb{R}^m; \ v \cdot (u_n - \tilde{v}) \geq 0, \ \forall \tilde{v} \in K \}.
\]

Thus, plugging (6.5) into (6.4) we come to

\[
0 \in \eta(u_n) + N_{ad}(u_n) + \frac{1}{n} \partial(\|u_n - u\|)|_{u=u_n},
\]

which, via (6.6), yields that for some \( \zeta_n \in N_K(u_n) \), \(\eta \in \partial(\|u_n - u\|)|_{u=u_n}\),

\[
\eta(u_n) + \zeta_n + \frac{1}{n} \eta_n = 0.
\]

(6.7)

Taking into account (5.50), we arrive at

\[
u_n(t) + \frac{1}{2\gamma_2} \zeta_n(t) = \frac{1}{\gamma_2} \text{Im} \int_{\mathbb{R}^d} V(x) X_n(t, x) Y_n(t, x) dx - \frac{1}{2\gamma_2 n} \eta_n(t),
\]

(6.8)
a.e. on \((0, T) \times \Omega\), where \(X_n(\cdot := X^{u_n})\) and \((Y_n, Z_n)(\cdot := (Y^{u_n}, Z^{u_n}))\) are the solutions to (1.1) and (2.13) corresponding to \(u_n\), respectively.

Therefore, applying the projection operator \(P_K\) to both sides of (6.8) we obtain (6.3) and finish the proof. □

Below we prove the crucial tightness of distributions of \(\{u_n\}\) on \(L^1(0, T; \mathbb{R}^m)\).

**Lemma 6.2.** Consider the situations in Lemma 6.1. Then, the induced probability measures \(P \circ u_n^{-1}\), \(n \geq 1\), are tight on \(L^1(0, T; \mathbb{R}^m)\).

**Proof.** In view of [7, Lemma A.2], we need to verify that

\[
\lim_{R \to \infty} \limsup_{n \to \infty} \mathbb{P}\left\{ \int_0^T |u_n(t)|_m dt > R \right\} = 0,
\]

(6.9)

and for any \( \varepsilon > 0 \),

\[
\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{P}\left\{ \sup_{0 < h \leq \delta} \int_0^{T-h} |u_n(t + h) - u_n(t)|_m dt > \varepsilon \right\} = 0.
\]

(6.10)

The estimate (6.9) follows immediately from the uniform boundedness of \(\{u_n\}\). In order to prove (6.10), using Markov’s inequality, we only need to prove that for some positive exponent \(b > 0\) and for any \(\delta \in (0, 1)\),

\[
\limsup_{n \to \infty} \mathbb{E} \sup_{0 < h \leq \delta} \int_0^{T-h} |u_n(t + h) - u_n(t)|_m dt \leq C \delta^b.
\]

(6.11)
For this purpose, since $P_K$ is Lipschitz, using (6.3) we have
\[
|u_n(t + h) - u_n(t)|_m \leq \frac{1}{2\gamma_2 n}(|\eta_n(t + h)|_m + |\eta_n(t)|_m) + \frac{1}{\gamma_2} \left| \int VX_n(t + h)\bar{Y}_n(t + h)dx - \int VX_n(t)\bar{Y}_n(t)dx \right|_m. \tag{6.12}
\]

First note that, by Hölder’s inequality, as $n \to \infty$,
\[
\mathbb{E} \sup_{0 \leq h \leq \delta} \int_0^{T-h} \frac{1}{2\gamma_2 n}(|\eta_n(t + h)|_m + |\eta_n(t)|_m)dt \leq \frac{1}{\gamma_2 n} T^{\frac{1}{2}} (\mathbb{E} \int_0^T |\eta_n(t)|^2 m dt)^{\frac{1}{2}} \leq \frac{1}{\gamma_2} T^{\frac{1}{2}} \to 0. \tag{6.13}
\]

Let us treat the second term on the right-hand side of (6.12). For simplicity, set $\tilde{X}_n := U(0, t)e^{-W(t)}(VX_n(t))$, $\tilde{Y}_n := U(0, t)e^{-W(t)}Y_n(t)$. By (3.5),
\[
\left| \int VX_n(t + h)\bar{Y}_n(t + h)dx - \int VX_n(t)\bar{Y}_n(t)dx \right|_m = \left| \langle \tilde{X}_n(t + h), \tilde{Y}_n(t + h) \rangle_2 - \langle \tilde{X}_n(t), \tilde{Y}_n(t) \rangle_2 \right|_m \tag{6.14}
\]
\[
\leq \left| \langle \tilde{X}_n(t + h) - \tilde{X}_n(t), \tilde{Y}_n(t + h) \rangle_2 \right|_m + \left| \langle \tilde{X}_n(t), \tilde{Y}_n(t + h) - \tilde{Y}_n(t) \rangle_2 \right|_m.
\]

Note that, by Cauchy’s inequality,
\[
\int_0^{T-h} \left| \langle \tilde{X}_n(t + h) - \tilde{X}_n(t), \tilde{Y}_n(t + h) \rangle_2 \right|_m dt \leq T^{\frac{1}{2}} \|\tilde{Y}_n\|_{C([0, T]; L^2)} \left( \int_0^{T-h} |\tilde{X}_n(t + h) - \tilde{X}_n(t)|^2_2 dt \right)^{\frac{1}{2}}. \tag{6.15}
\]

Taking into account (3.10) and (3.12) we obtain
\[
\int_0^{T-h} \left| \langle \tilde{X}_n(t + h) - \tilde{X}_n(t), \tilde{Y}_n(t + h) \rangle_2 \right|_m dt \leq CT^{\frac{1}{2}} h^{\frac{1}{2}} \|e^{-W}Y_n\|_{L^2(0, T)} \|e^{-W}VX_n\|_{L^2(0, T)}. \tag{6.16}
\]

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Similarly, estimating as above and using (3.10) and (3.12) we obtain
\[
\left| \sum_{k=0}^{n} \left( X_n(t+k) - X_n(t) \right) \right|_m \leq C T^{3/4} \left( \sum_{k=0}^{n} \left| X_n(t+k) - X_n(t) \right|^2 \right)^{1/2}
\]
Thus, combining (6.14), (6.16) and (6.17) together we obtain
\[
|\bar{X}_n(t) - \bar{Y}_n(t)|_m^2 \leq C T^{3/2} \left( \int_0^T |\bar{Y}_n(t+h) - \bar{Y}_n(t)|_2^2 \, dt \right)^{1/2}
\]
\[
\leq C T^{3/2} h^{1/2} \left( |\bar{Y}_n(t+h)|_{L^2(0,T)}^2 + |\bar{Y}_n(t)|_{L^2(0,T)}^2 \right)
\]
By virtue of uniform estimates (2.7), (2.8) and (2.15), we obtain
\[
\int_0^T \left| \sum_{k=0}^{n} \left( X_n(t+k) - X_n(t) \right) \right|_m \, dt \leq C T^{3/2} \delta^{1/2},
\]
where \( C \) is independent of \( n \). This along with (6.12) and (6.13) implies (6.11) with \( b = 1/2 \), thereby yielding the tightness of \( \{X_n\}_{n \geq 1} \) on \( L^1(0,T; \mathbb{R}^m) \).
Therefore, the proof of Lemma 6.2 is complete. \( \square \)

We are now ready to prove Theorem 2.11.

**Proof of Theorem 2.11.** We use similar arguments as in Section 3 of [7]. Set \( \mathcal{X}_n := (X_T, X, \beta, u_n) \) in the space \( \mathcal{Y} := L^2(\mathbb{R}^d) \times L^2((0,T) \times \mathbb{R}^d) \times C([0,T]; \mathbb{R}^N) \times L^1(0,T; \mathbb{R}^m) \), where \( \beta = (\beta_1, \ldots, \beta_N) \). Lemma 6.2 yields the tightness of the induced probability measures of \( \{X_n\}_{n \geq 1} \) on \( \mathcal{Y} \). This, via Prohorov’s theorem and Skorohod’s representation theorem, implies that there exist a probability space \( (\Omega^*, \mathcal{F}^*, \mathbb{P}^*) \) and sequences \( X_n^* := (X_T^*, X^*, \beta_n^*, u_n^*) \), \( X^* := (X_T^*, X^*, \beta^*, u^*) \) in \( \mathcal{Y} \), \( n \in \mathbb{N} \), such that the joint distributions of \( X_n^* \) and \( X_n \) coincide, \( u_n^* \), \( u^* \in K \), \( d\mathbb{P}^* \times dt \)-a.e., and
\[
X_n^* \to X^*, \text{ in } \mathcal{Y}, \text{ as } n \to \infty, \mathbb{P}^* - a.s. \tag{6.18}
\]
In particular, by the bounded dominated convergence theorem,
\[
u_n^* \to u^*, \text{ in } L^2(0,T; \mathbb{R}^m), \text{ as } n \to \infty, \mathbb{P}^* - a.s. \tag{6.19}
\]
We also note that \(((X^*_T), n, X^*_n; \beta^*_n)\) and \((X^*_T, X^*, \beta^*)\) have the same distributions as \((X^*_T, X, \beta)\).

Then, for each \(n \geq 1\), define \(F_{t,n}^* := \sigma(\mathcal{X}^*_n(s), s \leq t)\). It follows that \(X^*_n(T) \in L^2(\Omega, F_{T,n}^*; \mathbb{P}^*; L^2)\), \(X^*_n \in L^2_{ad^*}(0, T; L^2(\Omega; L^2))\), \(u^*_n \in \mathcal{U}_{ad^*}\), and that \((\beta^*_n(t), F_{t,n}^*)\), \(t \in [0, T]\), is a Wiener process. Moreover, Theorem 2.2 implies a unique strong \(L^2\)-solution \(X^*_n\) to (1.1) related to \((\beta^*_n, u^*_n)\). Hence, we deduce that \((\Omega^*, F^*, \mathbb{P}^*, (F_t^*)_{t \geq 0}, (X^*_T, n, X^*_n, \beta^*_n, u^*_n, X^*_n)\) is an admissible system in the sense of Definition 2.7.

Similarly, let \(F_t^* := \sigma(\mathcal{X}^*(s), s \leq t)\) and \(X^*\) be the unique strong \(L^2\)-solution to (1.1) corresponding to \((\beta^*, u^*)\). Then, \((\Omega^*, F^*, \mathbb{P}^*, (F^*_t)_{t \geq 0}, X^*_T, X^*, \beta^*, u^*, X^*)\) is also an admissible system.

We also note that, since the controlled solution to (1.1) is a measurable map of Wiener processes and controls, the distributions of \(((X^*_T, n, X^*_n, \beta^*_n, u^*_n), X^*_n)\) and \((X^*_T, X^*, \beta, u^*, X^{u^*_n})\) coincide. In particular, if \(\Phi^*_n\) is defined similarly as in (4.29) on the new filtrated probability space, we have \(\Phi^*_n(u^*_n) = \Phi(u_n)\).

Thus, in view of Proposition 4.3 and (6.1), we obtain that, if \(\gamma_3 = 0\),

\[
\Phi^*(u^*) = \lim_{n \to \infty} \Phi^*_n(u^*_n) = \lim_{n \to \infty} \Phi(u_n) = \inf \{ \Phi(u); u \in \mathcal{U}_{ad} \},
\]

which yields (2.19).

Finally, the characterization formula (2.20) of optimal control \(u^*\) can be proved similarly as in the proof of Lemma 6.1.

Therefore, the proof of Theorem 2.11 is complete. \(\square\)

7 Appendix

Proof of Proposition 3.2 First, since \(\text{Re}\Phi = 0, \overline{e^{-\Phi}} = e^\Phi\) and \(V(t, 0)V(0, t) = Id\), we see that (3.7) follows from (3.5).

Below we prove (3.5). It is equivalent to prove it for \(v \in H^2\), since \(H^2\) is dense in \(L^2\). Let \(A(t)(\cdot) := -i e^{-\Phi(t)} \Delta(e^{\Phi(t)} \cdot)\) and \(A^*(t)\) be the dual operator of \(A(t)\), \(t \in \mathbb{R}^+\). Note that, \(A^*(t) = -A(t)\).

On the one hand, since \(\partial_t \langle V(t, 0)v, w\rangle = A(t)V(t, 0)v\) and \((A(t))^* = -A(t)\), we have for any \(w \in H^2\),

\[
\partial_t \langle V^*(t, 0)v, w\rangle_2 = \langle v, \partial_t V(t, 0)w\rangle_2 = \langle u, A(t)V(t, 0)w\rangle_2 = \langle -V^*(t, 0)A(t)v, w\rangle_2,
\]

which implies that

\[
\partial_t V^*(t, 0)v = -V^*(t, 0)A(t)v, \quad V^*(0, 0)v = v. \quad (7.1)
\]
On the other hand, since $V(0, t)V(t, 0) = Id$, we have that
\[
\partial_t V(0, t)v = -V(0, t)A(t)v, \quad V(0, 0)v = v. \tag{7.2}
\]
Thus, we infer from (7.1) and (7.2) that $V^*(t, 0)$ and $V(0, t)$ satisfy the same equation with the same initial data. In view of the uniqueness of solutions (see e.g. \cite{25}), we obtain (3.5).

Regarding (3.6), since $Id = V(t, 0)V(t, 0)$, we deduce from (3.5) that
\[
Id = V^*(t, 0)V^*(0, t) = V(0, t)V^*(0, t).
\]
Then, applying $V(t, 0)$ to both sides above we prove (3.6).

**Proof of Lemma 3.9.** Let $I_j = [jh, (j + 1)h]$, $0 \leq j \leq \lfloor \frac{T}{h} \rfloor - 1 =: L^*$. Since $v \in C([0, T]; H)$, there exists $t_j \in I_j$ such that $\sup_{t \in I_j} \|v(t + h) - v(t)\|_H = \|v(t_j + h) - v(t_j)\|_H$, $0 \leq j \leq L^*$. Then,
\[
\int_{0}^{T-h} \|v(t + h) - v(t)\|^p_H dt \leq h \sum_{j=0}^{L^*} \|v(t_j + h) - v(t_j)\|^p_H.
\]
Moreover, letting $\{t_j^\prime\}_{j=0}^{L^*} = \{t_j + h, t_j\}_{j=0}^{L^*}$ we have that
\[
\sum_{j=0}^{L^*} \|v(t_j + h) - v(t_j)\|^p_H \leq 2^{p+1} \sum_{j=0}^{L-1} \|v(t_j^\prime) - v(t_j\prime)\|^p_H \leq 2^{p+1} \|v\|^p_{V^p(0, T)}.
\]
Thus, combining the estimates above we prove (3.12).

**Proof of Proposition 3.10.** First we prove (3.13). It suffices to prove the case that $V(t_0, \cdot)u$ is a $U^q$-atom of the form $V(t_0, t)u(t) = \sum_{j=1}^{n} u_j \chi_{[t_j, t_{j+1}]}(t)$, $t \in I$, such that $\{t_j\}_{j=1}^{n} \subseteq I$, $\sum_{j=1}^{n} |u_j|_{L^2}^q = 1$. This yields that $u(t) = \sum_{j=1}^{n} V(t, t_0)u_j \chi_{[t_j, t_{j+1})}(t)$, and so
\[
\|u\|_{L^q(I; L^p)} = \sum_{j=1}^{n} \|V(\cdot, t_0)u_i\|_{L^q(t_j, t_{j+1}; L^p)}^q. \tag{7.3}
\]
Using Theorem 3.1 and $\|V(t_j, t_0)\|_{L^q(L^2, L^2)} \leq C(T) \in L^p(\Omega)$ we have
\[
\|V(\cdot, t_0)u_i\|_{L^q(t_j, t_{j+1}; L^p)}^q = \|V(\cdot, t_j)V(t_j, t_0)u_j\|_{L^q(t_j, t_{j+1}; L^p)}^q \leq C_T^q \|V(t_j, t_0)u_j\|_{L^2}^q \leq (C'(T))^q |u_j|_{L^2}^q, \tag{7.4}
\]

where \( C'(T) \) is independent of \((p, q)\) and \( C'(T) \in L^p(\Omega) \) for any \( 1 \leq \rho < \infty \). Plugging (7.4) into (7.3) yields

\[
\|u\|_{L^q(I; L^p)} \leq C'(T) \left( \sum_{j=1}^{n} |u_j|^q L_j \right)^{\frac{1}{q}} \leq C'(T),
\]

which implies (3.13).

In order to prove (3.14), we see that for any partition \( \{t_j\}_{j=0}^m \in \mathcal{Z} \), since

\[
\|V(t_0, t_j)\|_{L^q L^p} \leq C(T) \in L^p(\Omega), \text{ if } \tilde{f} := \mathcal{X}_f,
\]

\[
\left\| \int_{t_{j-1}}^{t_j} V(t_0, s) \tilde{f}(s) ds \right\|_{L^2} \leq C(T) \left( \int_{t_{j-1}}^{t_j} V(t_j, s) \tilde{f}(s) ds \right)_{L^2}
\]

\[
= C(T) \sup_{|z| \leq 1} \left( \int_{t_{j-1}}^{t_j} V(t_{j-1}, s) \tilde{f}(s) ds, z \right)_{L^2}
\]

\[
\leq C(T) \| \tilde{f} \|_{L^q(t_{j-1}, t_j; L^{p'})} \sup_{|z| \leq 1} \| V^*(t_{j-1}, \cdot) \mathcal{X}_I z \|_{L^p(t_{j-1}, t_j; L^p)}.
\]

Estimating as in [54, Lemma 5.3], we have

\[
\| V^*(t_{j-1}, \cdot) \mathcal{X}_I z \|_{L^p(t_{j-1}, t_j; L^p) \cap L^q(t_{j-1}, t_j; H^{\frac{1}{2}})} \leq C(T) |z|_{L^2} \leq C(T),
\]

where \( C(T) \in L^p(\Omega) \) for any \( 1 \leq \rho < \infty \). Then, we get

\[
\sum_{j=1}^{m} \left\| \int_{t_{j-1}}^{t_j} V(t_0, s) \tilde{f}(s) ds \right\|_{L^2}^{q'} \leq (C''(T))^{q'} \sum_{j=1}^{m} \| \tilde{f} \|_{L^p(t_{j-1}, t_j; L^{p'})}^{q'}
\]

\[
= (C''(T))^{q'} \| f \|_{L^p(I; L^{p'})}^{q'},
\]

where \( C''(T) \) is independent of \((p, q)\) and \( C''(T) \in L^p(\Omega), \forall 1 \leq \rho < \infty \). This implies (3.14).

Regarding (3.14), let \( f = f_1 + f_2 \) with \( f_1 \in L^{q'}(I; L^{p'})\), \( f_2 \in L^2(I; H^{1/2})\), and set \( \tilde{f}_j = \mathcal{X}_I f_j, j = 1, 2 \). We can take a finer partition \( \{t_j\}_{j=0}^m \) such that

\[
\| f \|_{L^{q'}(t_{j-1}, t_j; L^{p'})} \leq 1, 1 \leq j \leq m.
\]

Then, estimating as in (7.8), since \( q > 2, q' < 2 \), we have

\[
\sum_{j=1}^{m} \left\| \int_{t_{j-1}}^{t_j} V(t_0, s) \tilde{f}_1(s) ds \right\|_{L^2}^{2} \leq (C''(T))^{2} \sum_{j=1}^{m} \| \tilde{f}_1 \|_{L^{q'}(t_{j-1}, t_j; L^{p'})}^{2}
\]

\[
\leq (C''(T))^{2} \| f_1 \|_{L^{q'}(I; L^{p'})}^{2},
\]

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which implies that
\[
\left\| \int_{t_0}^{t} V(t_0, s) f_1(s) ds \right\|_{L^2(I)} \leq C''(T) \| f_1 \|_{L^{q'}(I; L^{r'})} \leq C''(T)(1 + \| f_1 \|_{L^{r'}(I; L^{q'})}). \tag{7.9}
\]

Moreover, arguing as in the proof of (7.6) and using (7.7) we have
\[
\left\| \sum_{j=1}^{m} \int_{t_{j-1}}^{t_j} V(t_0, s) \tilde{f}_2(s) ds \right\|_{L^2}^2 \leq (C''(T))^2 \| f_2 \|_{L^2(I; H^{-\frac{1}{2}})}^2 .
\]
This yields that
\[
\left\| \int_{t_0}^{t} V(t_0, s) f_2(s) ds \right\|_{L^2(I)} \leq C''(T) \| f_2 \|_{L^2(I; H^{-\frac{1}{2}})} \tag{7.10}
\]

Therefore, combining (7.9) and (7.10) together we prove (3.15). □

In order to prove Theorem 3.14, we first prove the short-time perturbation result below as in [55].

**Proposition 7.1.** *(Mass-Critical Short-time Perturbation).* Consider the situations in Theorem 3.14. Assume also the smallness conditions
\[
|I| + \| \tilde{v} \|_{L^q(I; L^p)} \leq \delta, \tag{7.11}
\]
\[
\| V(\cdot, t_0)(v(t_0) - \tilde{v}(t_0) - R(t_0)) \|_{L^q(I; L^p)} \leq \varepsilon, \| R \|_{L^q(I; L^p) \cap L^1(I; L^2)} \leq \varepsilon, \tag{7.12}
\]
\[
\| e \|_{N^0(I) + L^2(I; H^{-\frac{1}{2}})} \leq \varepsilon \tag{7.13}
\]
for some \(0 < \varepsilon \leq \delta\) where \(\delta = \delta(C_T) > 0\) is a small constant, and \(C_T\) is as in Theorem 3.14. Then, we have
\[
\| v - \tilde{v} - R \|_{L^q(I; L^p) \cap C(I; L^2)} \leq C(C_T)\varepsilon, \tag{7.14}
\]
\[
\| (F(v) - F(\tilde{v})) + G(v - \tilde{v} - R) \|_{N^0(I)} \leq C(C_T)\varepsilon, \tag{7.15}
\]
where \((\delta(C_T))^{-1}, C(C_T)\) can be taken to nondecreasing with respect to \(C_T\).

**Proof.** We mainly prove Proposition 7.1 for the case where \(p\) satisfies that \(\frac{1}{p} \in (\max\{\frac{1}{2a}, \frac{1}{2d} - \frac{1}{2d}\}, \frac{1}{\alpha}(\frac{1}{2} + \frac{1}{d}))\) with \(1 \leq d \leq 3, \alpha = 1 + \frac{4}{d}\). The case \(p = 2 + \frac{4}{d}\) with \(d \geq 1\) can be proved similarly.
As mentioned below Hypothesis \((H0)^*\), there exists another Strichartz pair \((\tilde{p}, \tilde{q})\) such that \(\left(\frac{1}{p}, \frac{1}{q} \right) = \left(\frac{\alpha}{\tilde{p}}, \frac{\alpha}{\tilde{q}} \right)\), where \(q \in (2, \infty)\) is such that \((p, q)\) is a Strichartz pair, and \(\tilde{p}', \tilde{q}'\) are the conjugate numbers of \(\tilde{p}, \tilde{q}\) respectively.

Let \(z := v - \tilde{v} - R, F(\tilde{v}) := |\tilde{v}|^{\frac{4}{d}} \tilde{v}\) and \(F(z + R + \tilde{v})\) be defined similarly. By equations (3.20) and (3.21),
\[
z(t) = V(t, t_0)z(t_0) + \int_{t_0}^t V(t, s)\left(i(F(z + R + \tilde{v}) - F(\tilde{v})) + Gz + GR + e\right) ds.
\]
(7.16)

We set \(S(I) := \|i(F(z + R + \tilde{v}) - F(\tilde{v})) + Gz\|_{N^0(I)}\). By Hölder’s inequality and (7.11)-(7.13),
\[
S(I) \leq \|(F(z + R + \tilde{v}) - F(\tilde{v}))\|_{L^{\tilde{p}'}(0, T; L^p)} + \|Gz\|_{L^1(0, T; L^2)}
\leq C\left(\|\tilde{v}\|_{L^8(I; L^p)}^{\frac{4}{d}} \|z + R\|_{L^8(I; L^p)} + \|z + R\|_{L^8(I; L^p)}^{1 + \frac{4}{d}} + \|Gz\|_{L^1(I; L^2)}\right)
\leq C_1(\delta^{\frac{4}{d}} + \delta \|\tilde{v}\|_{L^8(I; L^p)} + \delta \|z\|_{C(I; L^2)} + \|z\|_{L^8(I; L^p)}^{1 + \frac{4}{d}}),
\]
(7.17)
where \(C_1(\geq 1)\) depends on \(d\) and \(|G|_{L^\infty(I \times \mathbb{R}^d)}\). Moreover, applying Theorem 3.1 to (7.16) and using (7.12) and (7.13) we have
\[
\|\tilde{z}\|_{L^8(I; L^p) \cap C(I; L^2)}
\leq C_T \|V(\cdot, t_0)z(t_0)\|_{L^8(I; L^p)} + S(I) + \|GR\|_{L^1(I; L^2)} + \|e\|_{N^0(I) + L^2(I; H_{\frac{1}{2}})}
\leq C_2 C_T(\delta + S(I)),
\]
(7.18)
where \(C_2\) depends on \(|G|_{L^\infty(I \times \mathbb{R}^d)}\). Then, combining (7.17), (7.18) we get
\[
\|\tilde{z}\|_{L^8(I; L^p) \cap C(I; L^2)} \leq C_1 C_2 C_T\left(2\delta + (\delta^{\frac{4}{d}} + \delta) \|\tilde{v}\|_{L^8(I; L^p)} \cap C(I; L^2) + \|z\|_{L^8(I; L^p)}^{1 + \frac{4}{d}}\right).
\]
Thus, in view of Lemma 6.1 in [7], taking \(\delta = \delta(C_T)\) small enough such that \(C_1 C_2 C_T(\delta^{\frac{4}{d}} + \delta) \leq \frac{1}{2}\) and \(4C_1 C_2 C_T(\delta) \leq (1 - \frac{1}{2}) (2\alpha C_1 C_2 C_T)^{-\frac{1}{\alpha - 1}}\) we obtain (7.14). Moreover, plugging (7.14) into (7.17) we obtain (7.15).

Therefore, the proof is complete. \(\square\)

**Proof of Theorem 3.14.** First fix \(\delta = \delta(C_T)\) as in Proposition 7.1. We divide \([t_0, T]\) into finitely many subintervals \([t_j', t_{j+1}']\), \(0 \leq j \leq l'\), such that \(t_{j+1}' = \inf \{t > t_j'; \|\tilde{v}\|_{L^p(t_j', t; L^p)} = \frac{\delta}{2}\} \wedge T\). Then, \(l' \leq (2L/\delta)^{2 + \frac{4}{d}} < \infty\).
Take a new partition \{I_j \}_{j=0}^l := \{[t_j, t_{j+1}]\}_{j=0}^l of \([0, T]\), such that \{t_j; 0 \leq j \leq l + 1\} = \{t_j'; 0 \leq j \leq l' + 1\} \cup \{t_0 + \frac{j\delta}{2}; 0 \leq j \leq \left\lceil \frac{2(T-t_0)}{\delta} \right\rceil \}. Then,
\[
l \leq (2L/\delta)^{2+\frac{3}{2}} + 2(T-t_0)/\delta, \quad |I_j| + \|\tilde{v}\|_{L^q(I_j; L^p)} \leq \delta, \quad 0 \leq j \leq l.
\]

Let \(C(0) = C(C_T), C(j + 1) = C(0)C_T^2(\sum_{k=0}^j C(k) + |G|_{L^\infty} + 2), 0 \leq j \leq l - 1\), where \(C(C_T)\) and \(C_T\) are the constants in Proposition 7.1 and Theorem 3.1 respectively. Choose \(\varepsilon_\ast = \varepsilon_\ast(C_T, L)\) sufficiently small such that
\[
C_T^2 \left( \sum_{k=0}^l C(k) + |G|_{L^\infty} + 2 \right) \varepsilon_\ast \leq \delta. \tag{7.19}
\]

We claim that (7.14) and (7.15) hold on \(I_j\) with \(C(j)\) replacing \(C(C_T)\) for every \(0 \leq j \leq l\).

To this end, we first see that Proposition 7.1 yields the claim for \(j = 0\). Suppose that the claim is also valid for each \(0 \leq k \leq j < l\). We shall use Proposition 7.1 to show that it also holds on \(I_{j+1}\).

For this purpose, applying Theorem 3.1 to (7.16) again and using (3.23), (7.19) and the inductive assumption we have
\[
\begin{align*}
&\|V(\cdot, t_{j+1})(v(t_{j+1}) - \tilde{v}(t_{j+1}) - R(t_{j+1}))\|_{L^q(I_{j+1}; L^p)} \\
&\leq \|V(\cdot, t_0)(v(t_0) - \tilde{v}(t_0))\|_{L^q(I; L^p)} \\
&\quad + C_T^2 \left( S(t_0, t_{j+1}) + \|GR\|_{L^1(t_0, t_{j+1}; L^2)} + \|e\|_{N^0(t_0, t_{j+1}) + L^2(t_0, t_{j+1}; H_1^{\frac{1}{2}})} \right) \\
&\leq \varepsilon + C_T^2 \left( \sum_{k=0}^j C(k) \varepsilon + |G|_{L^\infty(I \times \mathbb{R}^d)} \varepsilon + \varepsilon \right) \leq \delta.
\end{align*}
\]

Thus, the conditions (7.11)-(7.13) of Proposition 7.1 are satisfied with \(C_T^2 \left( \sum_{k=0}^j C(k) + |G|_{L^\infty} + 2 \right) \varepsilon\) replacing \(\varepsilon\). Applying Proposition 7.1 we prove the claim on \(I_{j+1}\).

Therefore, using inductive arguments we prove the claim on \(I_j\) for every \(0 \leq j \leq l\), thereby proving Theorem 3.14. The proof is complete. \(\square\)

**Proof of (4.11).** The proof is similar to that in [28]. Without loss of generality, we may assume \(\rho \geq q\). By Minkowski’s inequality and Burkholder’s inequality,
\[
\|M^r_* \|_{L^\rho(\Omega; L^1(0, T))} \leq C \left\| M^r_1 \right\|_{L^1(0, T; L^\rho(\Omega))} \leq C \left( \int_0^T \|e^{-i(-\Delta)^{\frac{3}{2}}} X(s) \Phi \|_{HS(\mathbb{R}^N; L^2)}^2 ds \right)^{\frac{1}{2}} \left\| 1_{L^1(0, T)} \right\|,
\tag{7.20}
\]

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where the operator \( \Phi : \mathbb{R}^N \to L^\infty \) is defined by \( \Phi(x) = \sum_{j=1}^N \mu_j e_j x_j \) for \( x = (x_1, \cdots, x_N) \), and \( \| \cdot \|_{HS(\mathbb{R}^N; L^2)} \) denotes the Hilbert-Schmidt norm (see, e.g., [45]). Note that,

\[
\| e^{-i(t-s)\Delta} X(s) \Phi \|_{HS(\mathbb{R}^N; L^2)}^2 \leq C \sum_{j=1}^N |\mu_j|^2 |e_j|_{L^\infty}^2 |X_0|^2_{L^2}.
\]

Plugging this into (7.20) yields immediately that \( \| M_1^* \|_{L^p(\Omega; L^\rho(0,a,b))} < \infty \).

Similarly, again by Minkowski’s inequality and Burkholder’s inequality,

\[
\| M_2^* \|_{L^p(\Omega; L^q(0,T))} \leq \| M_2^* \|_{L^q(0,T; L^p(\Omega))} \leq C \left\| \int_0^T \| e^{-i(t-s)\Delta} X(s) \Phi \|_{\mathcal{R}(\mathbb{R}^N; L^p)}^2 ds \right\|_{L^q(0,T)}, \tag{7.21}
\]

where \( \| \cdot \|_{\mathcal{R}(\mathbb{R}^N; L^p)} \) denotes the \( \gamma \)-radonifying norm (see, e.g., [18, 19, 28]).

Note that, by the dispersive inequality \( \| e^{-i(t-s)\Delta} \|_{L^{p'}(L^p)} \leq C(t-s)^{-\frac{d}{2}(1-\frac{2}{p})} \),

\[
\| e^{-i(t-s)\Delta} X(s) \Phi \|_{\mathcal{R}(\mathbb{R}^N; L^p)} \leq \| \Phi \|_{\mathcal{R}(\mathbb{R}^N; L^{\frac{2p}{p-2}})} \left\| X(s) \right\|_{L^{\frac{2p}{p-2}}(L^{2p})} \| e^{-i(t-s)\Delta} \|_{L^{p'}(L^p)} \leq C(N) |X_0|_{L^2} \left( \sum_{j=1}^N |\mu_j|^2 |e_j|_{L^\infty}^2 \right) \left( t-s \right)^{-\frac{d}{2}(1-\frac{2}{p})}.
\]

Plugging this into (7.21) yields that \( \| M_2^* \|_{L^p(\Omega; L^q(0,T))} < \infty \) if \( p < \frac{2d}{d-1} \), thereby finishing the proof. \( \Box \)

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