Classification of Emergent Weyl Spinors in Multi-Fermion Systems

M. A. Zubkov\textsuperscript{a, b, *}

\textsuperscript{a} Physics Department, Ariel University, Ariel, 40700 Israel

\textsuperscript{b} Alikhanov Institute for Theoretical and Experimental Physics, National Research Center Kurchatov Institute, Moscow, 117259 Russia

*e-mail: zubkov@itep.ru

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In the fermionic systems with topologically stable Fermi points, the emergent two-component Weyl fermions appear. We propose the topological classification of these fermions based on the two invariants composed of the two-component Green’s function. We define these invariants using the Wigner–Weyl formalism also in case of essentially inhomogeneous systems. In the case where values of these invariants are minimal (±1), we deal with emergent relativistic symmetry. The emergent gravity appears, and our classification of Weyl fermions gives rise to the classification of vielbein. Transformations between emergent relativistic Weyl fermions of different types correspond to parity conjugation, time reversal, and charge conjugation.

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1. INTRODUCTION

Electrons in solids are described by multicomponent spinors carrying band index. However, at low energies we may describe electrons by effective spinors with the essentially reduced number of components. Only those energy bands are relevant that cross Fermi energy. Emergent spinors in Dirac/Weyl semimetals \cite{1–6} with Fermi points are two-component if the Fermi energy is close to the position of the Fermi points. Because of the repulsion of energy levels, the Fermi points are unstable unless they are protected by topology. Therefore, the effective description in terms of the two-component spinors typically survives in the case where the topological invariants protecting the Fermi points are nonzero \cite{7}. The earlier discussion of these issues may be found in \cite{8–10}. For the review of the topological invariants (in the momentum space) protecting Fermi points and Fermi surfaces see \cite{11}. A well-known example is graphene with emergent Weyl fermions \cite{12, 13}. It is also worth mentioning that the topological invariants are responsible for gapless edge modes of topological insulators \cite{14–16}. Similar phenomena are observed also in superfluids \cite{17, 18}.

The minimum value of the topological invariant $N_3$ (composed of the Green’s functions \cite{11}) responsible for the stability of the Weyl points is ±1. Weyl points with the larger values of $N_3$ can be split into several Weyl points with $N_3 = ±1$. Action of Weyl fermions with $N_3 = ±1$ is relativistic. This results in the appearance of emergent vielbein \cite{11, 19–21}. The emergent Weyl fermions move effectively in a curved spacetime. The emergent gravity fluctuates, but these fluctuations are not described by a theory invariant under the diffeomorphisms.

The nonminimal values of $N_3$ give rise to more exotic types of Weyl fermions, typically with the nonlinear touching points of positive and negative energy branches. An example of such exotic Weyl points is given by the $2 + 1$ D multilayer graphene with the $ABC$ stacking \cite{22}. In addition, such exotic Weyl fermions appear in effective gravitational theories with anisotropic scaling \cite{23–26}, see \cite{27–32}.

It is worth mentioning that the classification of topological insulators \cite{33–35} may be obtained by a certain reduction of the topological classification of Fermi surfaces and emergent spinors incident on them \cite{7}, for details see \cite{11}.

In this work, we are going to extend the mentioned above classification of Weyl points into two directions. First, we notice that in general case of interacting systems the two topological invariants $N_3$ and $N_3^{(3)}$ may be introduced. Both are composed of the Green’s functions, and both of them are reduced to the above mentioned $N_3$ in the absence of interactions. However, they become different in general case. As a result, in the case of the minimum values $N_3, N_3^{(3)} = ±1$ we obtain the four topologically distinct types of Weyl fermions. We call two of them the left- and right-handed particles, and the other two, the left- and right-handed “antiparticles.” The latter types of Weyl points are referred to as the anti-Weyl points; they may appear only in the presence of interactions. This classification extends the conventional one, which considers the two types of relativistic Weyl fermions: left- and right-handed. Such
an extension was noticed, in particular, in [36], though without reference to appropriate topological invariants. Transformations between topologically distinct types of emergent relativistic Weyl fermions are given by various combinations of charge conjugation, time reversal, and parity conjugation. These transformations lead to the specific transformation of vielbein (see [37]).

Another direction for the extension of the classification of emergent Weyl fermions is related to consideration of the inhomogeneous systems following methodology of [38–40]. Namely, the conventional expression for invariant $N_3$ is defined as an integral in the momentum space and is valid for the homogeneous systems only. Using the Wigner–Weyl formalism, we extend the expressions of both $N_3$ and $N_3^{(3)}$ to the inhomogeneous case, when they are given by integrals in the phase space.

### 2. Emergent Weyl Spinors in the Multi-Fermion Systems

Here, we closely follow [41]. We start from the consideration of equilibrium condensed matter system with the $n$-component spinors $\psi$ at zero temperature. Its real time partition function has the form

$$Z = \int D\psi D\overline{\psi} \exp\left(i \int dt \sum_x \overline{\psi}_x(t)(i\partial_t - \hat{H} e^{-i\epsilon})\psi_x(t)\right).$$

The Hamiltonian $\hat{H}$ is a Hermitian matrix depending on the momentum $\hat{p}$. Sum over points of the coordinate space is to be understood as an integral over $d^3x$ for continuous coordinate space. However, we may also consider lattice tight-binding models, in which case we have the sum over the lattice points. In the absence of dependence of $\hat{H}$ on $\hat{p}$, the Hamiltonian $\hat{H}$ depends on $\hat{p}$ only. The factor $e^{-i\epsilon}$ is introduced here in order to point out how the poles in Feynman diagrams are to be treated. $\epsilon$ is assumed to be very small, its appearance may be explained, for example, via equilibrium limit of the Keldysh non-equilibrium path integral in lattice regularization (of time axis). Typically, the factor $e^{-i\epsilon}$ is omitted and is restored only if it is necessary to speak about the ways the singularities in Feynman diagrams are treated.

Several branches of spectrum of $\hat{H}$ repel each other. Therefore, the minimum number $n_{\text{reduced}} = 2$ of branches can cross each other. In addition, this minimum number is fixed by the topology of the momentum space.

Let us consider the position $p_j^{(0)}$ of the crossing of two branches of $\hat{H}$. Transformation via a certain Hermitian matrix $\hat{\Omega}$: $\hat{H}(\hat{p}) = \hat{\Omega} \hat{H}\hat{\Omega}$ diagonalizes the Hamiltonian. Then, the first $2 \times 2$ block $\hat{H}_\text{reduced}$ corresponds to the two crossed branches. The remaining block $\hat{H}_\text{gapped}$ corresponds to the branches of spectrum separated from the crossing point by a gap. We may represent the functional integral as the product of the functional integral over “gapped” modes $\Theta$ and the integral over two reduced fermion components $\Psi$. At low energies (near the Fermi level coinciding with the branch crossing point) the contribution to the physical observables of the reduced fermions dominates over the contribution of the gapped ones because $\Theta$ contributes the physical quantities with the fast oscillating factors. Therefore, we effectively describe the system near the Fermi point in terms of the partition function

$$Z = \int D\Psi D\overline{\Psi} \exp\left(i \int dt \sum_x \overline{\Psi}_x(t)(i\partial_t - \hat{H}_\text{reduced} e^{-i\epsilon})\Psi_x(t)\right).$$

Below, we will omit the subscript “reduced” of $\hat{H}$ for simplicity. The general form of Hermitian matrix $2 \times 2$ brings the partition function to the form

$$Z = \int D\Psi D\overline{\Psi} \exp\left(i \int dt \sum_x \overline{\Psi}_x(t)(i\partial_t - (\hat{m}_L(p)\sigma^k + m(p)) e^{-i\epsilon})\Psi_x(t)\right),$$

where $m_L$ and $m$ are real-valued functions. The nontrivial topology appears when the topological invariant composed of $m_L(p)$ has the nontrivial value

$$N_3 = \frac{-e_{ijk}}{8\pi} \int_\sigma dS^i \hat{m}_L \frac{\partial \hat{m}_L}{\partial p_j} \times \frac{\partial \hat{m}_L}{\partial p_k}, \quad \hat{m}_L = \frac{m_L}{|m_L|}.$$  

In terms of the Matsubara Green’s function

$$G_H(\omega, p) = \frac{1}{i\omega - \hat{H}(p)},$$

the value of $N$ may be written as ($\omega = p_\lambda$)

$$N_3^{(3)} = \frac{1}{3\pi^2} \int_\Sigma \text{Tr}(G_H(\omega, p) dG_H^{-1}(\omega, p)) \wedge dG_H(\omega, p) \wedge dG_H^{-1}(\omega, p)).$$

Here, $\Sigma$ is the three-dimensional hypersurface surrounding the Fermi point in the 4D momentum space (composed of the 3D momentum space and the axis of $\omega = p_\lambda$).

If $N = \pm 1$ in Eq. (4), then near $p = p_j^{(0)}$, the following expansion takes place:

$$m_L(p) = f^j_\alpha (p_j - p_j^{(0)})$$

with $\text{sgn} \text{det} f^j_\alpha = N$, and

$$m(p) = f^0_\alpha (p_j - p_j^{(0)})$$

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with \( j = 1, 2, 3 \). The coefficients \( f'_a \) \((a = 0, 1, 2, 3)\) of expansion may be considered as the source of emergent gravity. As a result, Eq. \( (3) \) acquires the form \((\sigma^0 \equiv 1)\)

\[
Z = \int D\Psi D\overline{\Psi} \exp \left[ i \int dt \sum_x \Psi_x(t)(i\partial, - \hat{H}(\Phi)e^{-i\varepsilon t})\overline{\Psi}_x(t) \right].
\]

The dispersion of quasiparticles near the point \( p^{(0)} \) acquires the form of the Dirac cone. Because of non-zero values of \( f'_0 \), this cone is tilted. Moreover, if \( \|f'_0[\tilde{f}^{-1}]_0\| > 1 \) then the cone becomes over-tilted and we come to the concept of the so-called type II Weyl point [42] (see Remark 2.1. in [41], where this kind of Weyl point was proposed for the first time), when near the Fermi point, the two Fermi pockets appear. The form of the latter is given by solution of equation

\[
\det \sigma^a f'_a(p_j - p_j^{(0)}) = 0, \quad j = 1, 2, 3; \quad a = 0, 1, 2, 3.
\]

The intermediate state between the ordinary (type I) Weyl point and the type II Weyl point corresponds to the case where \( \|f'_0[\tilde{f}^{-1}]_0\| = 1 \). In this case, the Fermi point (pair of Fermi pockets) is reduced actually to the Fermi line.

### 3. TOPOLOGICAL INVARIANTS AND EMERGENT VIELBEIN IN HOMOGENEOUS SYSTEMS

In the presence of interactions, the partition function can be written as

\[
Z = \int D\Psi D\overline{\Psi} D\Phi \exp(iR[\Phi] + i\int dt \sum_x \overline{\Psi}_x(t)(i\partial, - \hat{H}(\Phi)e^{-i\varepsilon t})\Psi_x(t)).
\]

Here, \( \Phi \) is a set of fields that provide interactions and \( R \) is a certain effective action of \( \Phi \). The operator \( \hat{H} \) also depends on \( \Phi \). In the mean field approximation, we substitute the values of \( \Phi \) by their “mean” values. Fluctuations around these mean values will give us the partition function of the form

\[
Z = \int D\Psi D\overline{\Psi} D\Phi \times \exp(iR[\Phi]) \exp \left[ i \int dt \sum_x \overline{\Psi}_x(t)(i\partial, m^0_{\Phi,\omega}\sigma^a) + [\mu m^0_{\Phi,\omega}\sigma^a - (\hat{m}_{\Phi, k}(\hat{p})\sigma^k - m_{\Phi}(\hat{p})e^{-i\varepsilon t})]\Psi_x(t) \right].
\]

Here, it is assumed that \( \Psi \) carries an index enumerating the Weyl points of the system.

Let us consider the Green’s function of the original multi-fermion system

\[
G(t_1 - t_2, x - y) = \frac{-i}{Z} \int D\Psi D\overline{\Psi} D\Phi \exp(iR[\Phi]) + i\int dt \sum_x \overline{\Psi}_x(t)(i\partial, - \hat{H}(\Phi)e^{-i\varepsilon t})\Psi_x(t))
\]

\[
\times \Psi_x(t_1)\overline{\Psi}_x(t_2).
\]

In order to construct the topological invariants responsible for the stability of Fermi points, we may also use the Green’s function of the reduced low energy theory

\[
G(t_1 - t_2, x - y) = \frac{-i}{Z} \int D\Psi D\overline{\Psi} D\Phi \exp(iR[\Phi]) \times \exp(\mu[\Phi]) \exp \left[ i \int dt \sum_x \overline{\Psi}_x(t)(i\partial, m^0_{\Phi,\omega}\sigma^a) + [\mu m^0_{\Phi,\omega}\sigma^a - (\hat{m}_{\Phi, k}(\hat{p})\sigma^k - m_{\Phi}(\hat{p})e^{-i\varepsilon t})]\Psi_x(t) \right]
\]

\[
\times \Psi_x(t_1)\overline{\Psi}_x(t_2).
\]

Then we compose the Fourier transform \( G(P_0, P_1, P_2, P_3) \) of the Green’s function either from Eq. \( (11) \) or from Eq. \( (12) \). We substitute \( P_0 = io \) and \( P_j = p_j, j = 1, 2, 3, \) into \( G(P_0, P_1, P_2, P_3) \).

The first topological invariant is given by the expression

\[
N_3 = \frac{1}{3\pi^2} \int \left[ \text{Tr}[G(\omega, \mathbf{p})dG^{-1}(\omega, \mathbf{p})] \right] dG_H(\omega, \mathbf{p}) \wedge dG_H^{-1}(\omega, \mathbf{p}).
\]

Here, \( \Sigma \) is the three-dimensional hypersurface surrounding the Fermi point in the 4D momentum space (composed of the 3D momentum space and the axis of \( \omega = p_4 \)).

In addition, we define

\[
G_H(\omega, \mathbf{p}) = \frac{1}{i\omega - G(0, \mathbf{p})^{-1}}.
\]

(Again, we may use \( G \) given either by Eq. \( (11) \) or by Eq. \( (12) \).) The second topological invariant is defined as

\[
N_3^{(3)} = \frac{1}{3\pi^2} \int \left[ \text{Tr}[G_H(\omega, \mathbf{p})dG_H^{-1}(\omega, \mathbf{p})] \right] dG_H(\omega, \mathbf{p}) \wedge dG_H^{-1}(\omega, \mathbf{p}).
\]

For the minimum values of \( N_3 \) and \( N_3^{(3)} \) at each Weyl point, we represent

\[
m^0_{\Phi,\omega}(\mathbf{p}) = ee^0_i, \quad \hat{m}_{\Phi, k}(\mathbf{p}) = ee^k_i(p_j - B_j),
\]

\[
m_{\Phi}(\mathbf{p}) = ee^0_i(p_j - B_j), \quad i, j = 1, 2, 3.
\]
Here, $e$ is the determinant of the matrix inverse to the $4$-matrix $e^a_i$ and we assume that the fields $\Phi$ as well as $e^a_i$ are slowly varying. (Again, the index referring to the number of the Fermi point is omitted.) We introduce here notations that match relativistic quantum field theory. The appearance of the field $B_\mu = -\mu$ means that the Fermi energy at the position of the crossing of several branches of the spectrum may differ from zero in the presence of interaction. Thus, $\mu$ is the chemical potential measured from the crossing point.

As a result, the partition function of the model may be rewritten as

$$Z = \int D\Psi D\bar{\Psi} D\xi^a DB_k e^{-S[\xi^a, B, \Psi, \bar{\Psi}]}$$

with

$$S = S_0[e, B] + \frac{1}{2} \left( \int d^4 x \bar{\Psi}_a(t)e^a_i\tilde{\sigma}^\mu\Psi_a(t) + \text{H.c.} \right),$$

where the sum is over $a, j = 0, 1, 2, 3$ while $\sigma^0 \equiv 1$, and $\tilde{D}$ is the covariant derivative that includes the $U(1)$ gauge field $B$:

$$\tilde{D}_0 = \partial_\mu e^{\mu j} - i\mu_0, \quad \tilde{D}_k = \partial_k + iB_k.$$  

$S_0[e, B]$ is the part of the effective action that depends only on $e$ and $B$.

In the absence of an external source of inhomogeneity in the leading order the field $e^a_i$ may be considered as independent of coordinates. The same refers to $B$, which then may be gauged off. In this case, in order to define topological invariants $N_3$ and $N_3^{(3)}$, we use the Green’s function

$$G(\alpha, \beta) = \frac{1}{e(\epsilon^0_i\tilde{\sigma}^\mu\omega - \epsilon^0_i\tilde{\sigma}^\mu\gamma_j)}.$$  

Obviously, $N_3^{(3)}$ and $N_3$ may differ from each other due to the nontrivial components of the emergent vielbein $\epsilon^a_i$. The direct calculation gives

$$N_3 = -\text{sgn det}_{3\times3} \epsilon^a_i.$$  

(Here, $a, j = 0, 1, 2, 3$; i.e., $\epsilon^a_i$ is a $4 \times 4$ matrix.) In the similar way, we get

$$N_3^{(3)} = -\text{sgn det}_{3\times3} \epsilon^a_i,$$

where $a, j = 0, 1, 2, 3$; i.e., $\epsilon^a_i$ is a $3 \times 3$ matrix composed of the space components of the vielbein.

4. $T, C, P$ TRANSFORMATIONS AND $N_3, N_3^{(3)}$

Let us consider the interplay of these topological invariants and $T$, $C$, and $P$ transformations. For the left-handed fermion (the one with $N_3 = N_3^{(3)} = 1$), the transformations are (we use matrix parts of those transformations, without transitions between the left-handed and the right-handed spinors)

$$T : \Psi(t,x) \rightarrow \sigma_2 \Psi^T(-t,x), \quad \Psi(t,x) \rightarrow \Psi^T(-t,x)\sigma_3;$$
$$P : \Psi(t,x) \rightarrow -i\Psi(t,-x), \quad \Psi(t,x) \rightarrow \Psi(t,-x)i;$$
$$C : \Psi(t,x) \rightarrow \sigma_3 \Psi^T(t,x), \quad \Psi(t,x) \rightarrow -\Psi^T(t,x)\sigma_2.$$

At the same time for the right-handed ones (those with $N_3 = N_3^{(3)} = -1$):

$$T : \Psi(t,x) \rightarrow \sigma_2 \Psi^T(-t,x), \quad \Psi(t,x) \rightarrow \Psi^T(-t,x)\sigma_3;$$
$$P : \Psi(t,x) \rightarrow -i\Psi(t,-x), \quad \Psi(t,x) \rightarrow \Psi(t,-x)i;$$
$$C : \Psi(t,x) \rightarrow -\sigma_3 \Psi^T(t,x), \quad \Psi(t,x) \rightarrow -\Psi^T(t,x)\sigma_2.$$

Let us also introduce the additional transformation $R$ (both for the right-handed and for the left-handed fermions):

$$R : \Psi(t,x) \rightarrow -\Psi(t,x), \quad \Psi(t,x) \rightarrow \Psi(t,x).$$

These transformations lead to the same form of the action for the transformed spinor fields, in which the vielbein is transformed as follows:

$$T : e^0_a \rightarrow -e^0_a, \quad e^0_a \rightarrow +e^0_a, \quad e^a_k \rightarrow +e^a_k, \quad e^a_k \rightarrow -e^a_k;$$
$$P : e^0_a \rightarrow e^0_a, \quad e^0_a \rightarrow e^0_a, \quad e^a_k \rightarrow e^a_k, \quad e^a_k \rightarrow -e^a_k;$$
$$C : e^0_a \rightarrow e^0_a, \quad e^0_a \rightarrow -e^0_a, \quad e^a_k \rightarrow +e^a_k, \quad e^a_k \rightarrow -e^a_k;$$

$$a, k = 1, 2, 3.$$  

One can check that the above transformations lead to the transformations of the topological invariants $N_3, N_3^{(3)}$:

$$T : N_3 \rightarrow N_3, \quad N_3^{(3)} \rightarrow -N_3^{(3)};$$
$$C : N_3 \rightarrow -N_3, \quad N_3^{(3)} \rightarrow -N_3^{(3)};$$
$$P : N_3 \rightarrow -N_3, \quad N_3^{(3)} \rightarrow -N_3^{(3)};$$
$$CPT : N_3 \rightarrow N_3, \quad N_3^{(3)} \rightarrow -N_3^{(3)};$$
$$R : N_3 \rightarrow N_3, \quad N_3^{(3)} \rightarrow -N_3^{(3)}.$$  

In the limit of small $e$, the operator $Q = e^a_i\tilde{D}_a e^{-i\epsilon}$ is Hermitian. The same refers to operator $Q$ standing in the fermionic action $\int d^3 x dt \bar{\Psi} Q \Psi$ also in general case. When $Q$ remains in the class of such (almost) Hermitian operators the smooth modifications of the system cannot lead to the change of the values of both $N_3$ and $N_3^{(3)}$. Different values of the topological invari-
nants \( N_3 \) and \( N_3^{(3)} \) determine the topological classification of Weyl fermions. Namely, we define those with \( N_3 = N_3^{(3)} = +1 \) as the left-handed particles. Notice that our definition of \( N_3 \) differs by sign from that of [11]. The fermions with \( N_3 = -N_3^{(3)} = 1 \) may be defined as the left-handed antiparticles. Those with \( N_3 = N_3^{(3)} = -1 \) may be identified with the right-handed particles while \( N_3 = -N_3^{(3)} = -1 \) would correspond to the right-handed antiparticles. This is summarized in Table 1. One can see that parity conjugation changes chirality of particles while time reversal transforms particles to antiparticles. At the same time, the \( RCPT \) transformation (that includes \( R : \Psi \rightarrow -\Psi \)) is identical to unity, which reflects the \( CPT \) invariance of fermionic systems.

According to the definition of the topological invariants \( N_3 \) and \( N_3^{(3)} \), the sum of the values of \( N_3 \) or \( N_3^{(3)} \) over all Fermi points is equal to zero if the momentum space is compact. The proof of this statement is as follows. The sum of \( N_3 \) (\( N_3^{(3)} \)) over the Fermi points is given by Eq. (13) (Eq. (15)) with hypersurface \( \Sigma \) surrounding all Fermi points. If the total momentum space is compact, \( \Sigma \) may be deformed smoothly to a point. The result is, obviously, zero. We deal with compact momentum space if lattice systems are considered with compact Brillouin zone, and, in addition, the axis of imaginary time is discretized. The latter condition is not always fulfilled, and we deal in condensed matter systems without interactions with the Green’s functions of the form of Eq. (14). Then at \( \omega \rightarrow \pm \infty \) the expression standing in the integrals of Eq. (15) tends to zero. As a result, the mentioned topological theorem is back, and the sum of \( N_3^{(3)} \) over all Weyl points is zero. However, in the presence of interactions the dependence of the Green’s function on \( \omega \) may be more complicated. Then the sum over the Weyl points of \( N_3 \) may appear to be nonzero while the sum of \( N_3^{(3)} \) remains vanishing.

In the latter case, the weakened topological theorem allows breakdown of the conventional state with equal number of left- and right-handed fermions in the lattice models. Namely, the situation is possible, when there are two right-handed fermions, but one of them is of the “particle” type, while another one is of the type of “antiparticles.” This situation is considered in the next section.

5. TOY MODEL

Let us consider the toy model, which illustrates unconventional properties of the systems with the anti-Weyl points (i.e., Fermi points corresponding to the antiparticles). In this model, the partition function is given by the formula

\[
Z = \int D\Psi D\bar{\Psi} \times \exp \left( i \int dt \sum_{x} \bar{\Psi}(x,t) \dot{\Psi}(x,t) \right).
\]

The operator \( \hat{Q} \) in the momentum space has the form

\[
\hat{Q}(p_0, p) = p_0 (\alpha \cos \theta + (\alpha - 1) \sin \theta) - (\alpha \sin \theta - (\alpha - 1) (-\cos \theta_0 + \cos \theta_0 - 2)) \sigma^1 e^{-i\epsilon} \sin p_0 \sigma^2 e^{-i\epsilon} + (\sin p_0 + 2 - \cos p_0 - \cos p_0) \sigma^3 e^{-i\epsilon}. \]

Here, \( \alpha \) is a real-valued parameter. In this system, there are two Fermi points \( K^+ \) and \( K^- \):

\[
K^+ : p_1 = 2\arctan(1/\alpha - 1) = 2\theta_0, \quad p_2 = p_3 = 0,
\]

\[
K^- : p_1 = \pi, \quad p_2 = p_3 = 0.
\]

The operator \( \hat{Q} \) may be written in the form

\[
\hat{Q}(p_0, p) = \sqrt{2\alpha^2 - 2\alpha + 1} \left( p_0 \cos \theta + q_0 \sigma^1 e^{-i\epsilon} - q_0 \sigma^2 e^{-i\epsilon} + q_0 \sigma^3 e^{-i\epsilon} \right) / \sqrt{2\alpha^2 - 2\alpha + 1}. \]

Near the Fermi points, we have

\[
K^+ : \frac{\hat{Q}(p_0, p)}{\sqrt{2\alpha^2 - 2\alpha + 1}} = \left( p_0 \cos \phi_0 - q_0 \sigma^1 e^{-i\epsilon} - q_0 \sigma^2 e^{-i\epsilon} + q_0 \sigma^3 e^{-i\epsilon} \right) / \sqrt{2\alpha^2 - 2\alpha + 1}
\]

and

\[
K^- : \frac{\hat{Q}(p_0, p)}{\sqrt{2\alpha^2 - 2\alpha + 1}} = \left( -p_0 \cos \phi_0 + q_0 \sigma^1 e^{-i\epsilon} - q_0 \sigma^2 e^{-i\epsilon} + q_0 \sigma^3 e^{-i\epsilon} \right) / \sqrt{2\alpha^2 - 2\alpha + 1}.
\]

\[
\begin{array}{|c|c|c|}
\hline
\text{Fermion type} & N_3 & N_3^{(3)} \\
\hline
\text{Left-handed Weyl point} & +1 & +1 \\
\text{Right-handed Weyl point} & -1 & -1 \\
\text{Left-handed anti-Weyl point} & +1 & -1 \\
\text{Right-handed anti-Weyl point} & -1 & +1 \\
\hline
\end{array}
\]
Here, \( q = p - K^\pm \). One can see that the topological invariants for those two Weyl points mentioned above at \( 0 < \alpha \leq 1 \) are

\[
K^+ : N_3 = N_3^{(3)} = -1
\]

and

\[
K^- : N_3 = -N_3^{(3)} = -1.
\]

Both Weyl points are right-handed. However, one of them is of the type of particles while the other is of the type of the antiparticles, i.e., the anti-Weyl point. At \( \alpha = 1 \) we deal with the Fermi points separated in the momentum space by \( \pi \). When \( \alpha \) approaches zero the two Weyl points approach each other and merge for \( \alpha = 0 \) at the position of \( K^- \). This is the marginal Weyl point with \( N_3 = -2 \) and \( N_3^{(3)} = 0 \). In its small vicinity, we have

\[
K^- : \hat{\mathcal{O}}(p_0, p) = p_0 q_1 + \frac{1}{2}(q_1^2 + p_0^2)\sigma^2 e^{-i\alpha} - q_2\sigma^2 e^{-i\alpha} - q_1\sigma^2 e^{-i\alpha}.
\]

One can calculate directly the values of \( N_3 \) and \( N_3^{(3)} \) for this Weyl point using machinery developed in Appendix C of [43].

6. INHOMOGENEOUS AND INTERACTING SYSTEMS

In the systems with weak inhomogeneity,\(^1\) we come to the partition function (for \( N_3, N_3^{(3)} = \pm 1 \))

\[
Z = \int D\Psi D\bar{\Psi} De_b DB_\alpha e^{i\mathcal{S}[\Psi, \bar{\Psi}, e_b, B_\alpha]} \tag{23}
\]

with

\[
\mathcal{S} = S_0[e, B] + \frac{1}{2}(\int dt \sum_x \overline{\Psi}_x(t)e_{j}^*\overline{\sigma}^j\hat{D}_j e_{j}\Psi_x(t) + \overline{\Psi}_x(t)\hat{D}^+_j e_{j}^*\overline{\sigma}^j e^{-i\alpha}\Psi_x(t)),
\]

where the sum is over \( a, j = 0, 1, 2, 3 \) while \( \sigma^0 \equiv 1 \). \( \hat{D} \) is the covariant derivative that includes the \( U(1) \) gauge field \( B \), and \( S_0[e, B] \) is the part of the effective action that depends on \( e \) and \( B \) only. Now both fields \( e \) and \( B \) depend on coordinates. We denote here

\[
\hat{D}_j^+ = -\partial e_{j}^* - i\mu, \quad \hat{D}_j^+ = -\partial e_{j}^* + iB_k.
\]

One can see that this is the Hermitian conjugation of \( iD \) except for the factor \( e^{i\alpha} \). In principle, one may include into covariant derivative \( D \) the spin connection. However, its appearance may be taken into account effectively by the renormalization of \( e \) and \( B \). Namely, the spin connection may be represented as \( C_{\alpha\beta}\sigma^\alpha \), where \( C_{\alpha\beta} \) is complex-valued, and \( a, k = 0, 1, 2, 3 \). Then, we obtain

\[
\overline{\Psi} e_b^j C_{\alpha\beta}\sigma^\beta e^{-i\alpha}\Psi + \text{H.c.}
\]

\[
= \overline{\Psi} e_b^j (C_{\beta\alpha} + iC_{\alpha\beta}) e^{-i\alpha}\Psi + \text{H.c.}
\]

\[
= 2\overline{\Psi} (e_b^j \text{Re} C_{\alpha\beta} - e_b^j \text{Im} C_{\alpha\beta} e^{-i\alpha}\Psi + \text{H.c.} \tag{25}
\]

Here, H.c. means the Hermitian conjugation complementary by transformation \( \Psi \to \overline{\Psi} \). One can check that combinations \( e_b^j \text{Re} C_{\alpha\beta} \) and \( e_b^j \text{Im} C_{\alpha\beta} e^{-i\alpha} \) may be absorbed by redefinition of emergent gauge field.

As above one may define the two topological invariants. Let us introduce the Green’s function

\[
\hat{G} = \frac{2}{ee_b^j \overline{\sigma}^j i\hat{D}_j + i\hat{D}_j ee_b^j \overline{\sigma}^j}.
\]

Next, we define its Wigner transformation \( G_{\text{W}}^{(M)}(p, x) = \int d^4 \langle x + r/2|\hat{G}|x - r/2\rangle e^{i(p \cdot r - m)} \).

After Wick rotation, we introduce \( p_0 = i\omega = iP_4 \), and \( P_j = p_j \) for \( j = 1, 2, 3 \); \( x_0 = -iX_4 \), \( X_j = x_j \), and denote the Euclidean Wigner transformation of Green’s function: \( G_{\text{W}}(P, X) = G_{\text{W}}^{(M)}(p, x) \). We also define \( Q_{\text{W}} \) that obeys \( Q_{\text{W}} \ast G_{\text{W}} = 1 \). Here, \( \ast \) denotes the Moyal product

\[
* = \exp \left( i\frac{1}{2} \left( \partial_{X^j} \partial_{\omega} - \partial_{\omega} \partial_{X^j} \right) \right),
\]

The first topological invariant is given by

\[
N_3 = \frac{1}{34\pi^3 |V|} \int \int d^3 X \text{Tr} G_{\text{W}}(P, X) \ast dQ_{\text{W}}(P, X) \tag{26}
\]

\[
\ast \wedge dG_{\text{W}}(P, X) \ast \wedge dQ_{\text{W}}(\omega, P, X) \]

\(|V|\) is the 3-volume of the system. Here, \( \Sigma \) is the three-dimensional hypersurface surrounding the given singularity \( M^{(i)} \) of expression standing inside the integral. This expression resembles the one of [40]. The general procedure for the construction of such invariants was proposed in [38]. In [39], such topological invariant was considered for the interacting quantum Hall effect systems.

In addition, we define

\[
Q_{H,W} = i\omega - Q_{\text{W}}(P, X)|_{\omega = 0}
\]

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and $G_{H,W}$ obeys

$$Q_{H,W} \ast G_{H,W} = 1.$$  

Then the second topological invariant can be defined as

$$N_3^{(3)} = \frac{1}{3\eta} \int \frac{d^3X}{\Omega} \text{Tr}(G_{H,W}(P, X) \ast dQ_{H,W}(P, X) \ast \text{ad}Q_{H,W}(\alpha, P, X)).$$  

Again, $N_3^{(3)}$ and $N_3$ may differ from each other due to the nontrivial components of the emergent vielbein $e^0_y$.

For the weakly dependent on coordinates $e^0_y(X)$ and $B$ (such that $\text{sgn det}_4 e^0_k$ and $\text{sgn det}_3 e^0_k$ do not depend on $X$), we still have

$$N_3 = -\text{sgn det}_4 e^0_k$$

and

$$N_3^{(3)} = -\text{sgn det}_3 e^0_k.$$  

Those two invariants provide topological classification of vielbeins for the inhomogeneous case. In the presence of interactions, we are to use the complete interacting Green’s function $\hat{G}$:

$$\langle \chi | \hat{G} | \psi \rangle = \frac{i}{Z} \int D\Psi D\bar{\Psi} D\bar{e}_k D\bar{B}_k e^{iS[\Phi, \Psi, \bar{\Psi}]}$$

$$\times \Psi(x)\bar{\Psi}(y)$$

instead of the noninteracting one in Eqs. (26) and (27). Moreover, we may use equivalently the Green’s function of the original multi-fermion system

$$\langle \chi_1, t_1 | \hat{G} | \chi_2, t_2 \rangle = \frac{i}{Z} \int D\Psi D\bar{\Psi} D\Phi \text{exp} \left( iR[\Phi] \right)$$

$$+ i \int d\tau \sum \Psi_x(t) \left( \partial \tau - \hat{H}(\Phi) e^{-i\Phi} \right) \bar{\Psi}_x(t) \Psi_x(t_1) \bar{\Psi}_x(t_2)$$

and its Wigner transform.

This will give us the modified expressions for the above topological invariants. $\Sigma$ of these expressions is the hypersurface in the phase space $(P, X)$ that surrounds a singularity of expression standing inside the integral. Positions $\mathcal{M}_i$ of distinct singularities play the role of the Fermi surfaces/Fermi points in case of the inhomogeneous systems. Each $\mathcal{M}_i$ is reduced to a Fermi point for the case of homogeneous system. $N_3$ and $N_3^{(3)}$ defined via integrals of Eqs. (26) and (27) with $\Sigma$ surrounding $\mathcal{M}_i$ are responsible for the topological stability of the given singularities. Only the singular transformation that leads to a change of the values of $N_3$ and $N_3^{(3)}$ may bring the singularity of one type to the singularity of another type. Correspondingly, in the cases $N_3 = \pm |1$ and $N_3^{(3)} = \pm |1$ we speak of the Weyl fermions existing in the presence of emergent gauge field and emergent vielbein.

In the homogeneous case, we may prove that

$$\sum_n \delta(n) = 0$$

and

$$\sum_n \delta(n) N_3^{(3)} = 0$$

if the corresponding Euclidean momentum space is compact. This is always true in lattice-regularized models, when the axis of imaginary time is discretized as well as the coordinate space. In the inhomogeneous case, the number of the left-handed fermions coincides with the number of the right-handed fermions. In condensed matter systems with noncompact axis of time, this topological theorem may be broken partially as it was explained above, and

$$\sum_n \delta(n) N_3^{(3)} \neq 0$$

and

$$\sum_n \delta(n) N_3 = 0.$$  

7. CONCLUSIONS

In this paper, we discuss topological classification of emergent Weyl fermions in multi-fermion systems. In Euclidean spacetime, there would be only one topological invariant responsible for stability of Fermi points. This is $N_3$ of Eqs. (13) or (26). Correspondingly, there exist two types of low energy emergent Weyl spinor fields with minimum values of $N_3$ that describe left- and right-handed particles/antiparticles. The low energy subsystem (of the given multi-fermion system) incident in a vicinity of the given right-handed Weyl point cannot be transformed continuously to that of the left-handed Weyl point.

In the real-time dynamics, the operator $\hat{Q}$ (inverse of the Green’s function) is Hermitian up to an infinitely small correction that points out the way the poles are treated in Feynman diagrams. Now, we can define the two distinct topological invariants composed of $\hat{Q}$ and $\hat{G}$. The second invariant is of Eq. (15) or Eq. (27). In the class of Hermitian operators we cannot transform smoothly those $\hat{Q}$ with different values of $N_3^{(3)}$ to each other.

We come to an unexpected conclusion: in general case in multi-fermion systems with minimum values of $N_3, N_3^{(3)} = \pm 1$ the emergent Weyl fermions appear in four rather than two topologically different classes. We feel this appropriate to divide the Weyl fermions into the left- and the right-handed ones, and also into the fields defining particles and antiparticles, or into the Weyl points and anti-Weyl points. Without interactions, we meet the fields of particles, i.e., Weyl points only. Then the antiparticles are understood as the

$\text{sgn det}_4 e^0_k$ and $\text{sgn det}_3 e^0_k$.  

More precisely, the operator $\hat{Q}$ is Hermitian, but its inverse $\hat{G}$ is considered in space of generalized (rather than ordinary) operator-valued functions, and the mentioned would be infinitely small correction to $\hat{Q}$ actually has the meaning of the proper definition of $\hat{Q}^{-1}$. In space of ordinary operators, the inverse to $\hat{Q}$ does not exist.
holes, i.e., the absence of particles in the set of occupied states.

In the presence of interactions, the sign of emergent $\varepsilon_0^0$ component may become negative, and the Weyl fermion of a marginal type with $N_3 = -N_3^{(3)}$ appears. We feel this instructive to call the field describing the corresponding Weyl fermion the field of the antiparticle type, and the Fermi point itself may be referred to as the anti-Weyl point. This is natural because charge conjugation being applied to the ordinary (left- or right-handed) Weyl fermion (with $N_3 = N_3^{(3)}$) results in the Weyl fermion with $N_3 = -N_3^{(3)}$, i.e., in the Weyl fermion of the type of antiparticle existing in a vicinity of anti-Weyl point.

The appearance of anti-Weyl points with $N_3 = -N_3^{(3)}$ has sense only in the presence of the conventional Weyl fermions with $N_3 = N_3^{(3)}$. Without the latter $R$ transformation $\Psi \rightarrow -\Psi$, $\overline{\Psi} \rightarrow \overline{\Psi}$ brings Weyl fermions to the type of particles with $N_3 = N_3^{(3)}$. At the same time if both types of Weyl fermions coexist, interesting phenomena may occur. For example, a pair of Weyl fermions $(N_3, N_3^{(3)}) = (-1, -1)$ and $(-1, +1)$ may merge giving marginal Weyl point with $(N_3, N_3^{(3)}) = (-2, 0)$. We gave an example of the lattice condensed matter system, in which two right-handed Weyl points exist. One of them is of the type of a Weyl point while another one is of the type of the anti-Weyl point. Changing smoothly parameter $\alpha$ of the system it is possible to bring it to the state, in which the Weyl point and the anti-Weyl point merge giving the marginal Weyl point with $N_3 = -2$ and $N_3^{(3)} = 0$. (The way to calculate the values of $N_3$ and $N_3^{(3)}$ for this Fermi point may be read off from Appendix C of [43].) Notice that if imaginary time axis is discretized as well as coordinate space, the sum over the Weyl points of both $N_3$ and $N_3^{(3)}$ is equal to zero. As a result, in such systems the total number of Weyl points is equal to the total number of anti-Weyl points, while the total number of left-handed Weyl/anti-Weyl points is equal to the total number of right-handed Weyl/anti-Weyl points. This theorem is broken partially in the mentioned toy model, in which the imaginary time axis is not discretized, and the axis of $\alpha$ remains open.

In practice, the anti-Weyl points may appear in solids in the presence of sufficiently strong inter-electron interactions. Therefore, prediction of such materials cannot be given on the basis of density functional theory calculations only. For the purpose of engineering of such materials (in addition to the density functional theory calculation of energy bands), the methods that take into account interactions between Bloch electrons are needed. One may also suppose that existence of anti-Weyl points may be found in fermionic superfluids, possibly, under specific external conditions.

It is worth mentioning that in addition to the topological classification of Fermi points presented here there should exist the complimentary topological classification of the systems, in which the energy bands cross each other at the points in the momentum space, but when those Weyl points form various types of Fermi surfaces rather than the Fermi points. In the case where the values of $N_1$ and $N_3^{(3)}$ are minimal $\pm 1$ such a classification was presented in [44]. In addition to the type I Weyl points there exist the type II resulting in the Fermi surface rather than a single Fermi point, and the type III, in which energy dispersion near the Weyl point becomes degenerate. The type IV combines properties of the type II and III. The extension of this classification to the case of nonminimal values of $N_1$ and $N_3^{(3)}$ is also possible, but it is beyond the scope of this work.

Finally, we would like to notice that the classification presented here may be relevant for the high-energy physics and applications of quantum field theory to cosmology (see [36, 37] and references therein). Then the appearance of the four (rather than two) topologically distinct types of Weyl fermions may be assumed from the very beginning. Such a construction may be relevant for the proper theory of quantum gravity, which should include the strong fluctuations of vielbein giving rise to all four types of the Weyl points [37].

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