An Algorithm for Hinge Vertex Problem on Circular Trapezoid Graphs

HIROTOSHI HONMA¹,a) YOKO NAKAJIMA¹ SHIGERU MASUYAMA²,b)

Received: June 8, 2017, Accepted: August 9, 2017

Abstract: Let \( G = (V,E) \) be a simple connected graph. A vertex \( u \in V \) is called a hinge vertex if there exist two vertices \( x \) and \( y \) in \( G \) whose distance increases when \( u \) is removed. Finding all hinge vertices of a given graph is called the hinge vertex problem. This problem can be applied to improve the stability and robustness of communication network systems. In a number of graph problems, it is known that more efficient sequential or parallel algorithms can be developed by restricting classes of graphs. Circular trapezoid graphs are proper super-classes of trapezoid graphs. In this paper, we propose an \( O(n^3) \) time algorithm for the hinge vertex problem of circular trapezoid graphs.

Keywords: design and analysis of algorithms, hinge vertex problem, intersection graphs, circular trapezoid graphs

1. Introduction

Let \( G = (V,E) \) be a simple undirected graph with a vertex set \( V \) and an edge set \( E \). For a vertex \( u \in V \), we denote the subgraph induced by the vertex set \( V - \{u\} \) as \( G - \{u\} \). The distance \( \delta_G(x,y) \) is defined as the length (i.e., the number of edges) of the shortest path between vertices \( x \) and \( y \) in \( G \). Chang et al. defined \( u \in V \) as a hinge vertex in \( G \) if two vertices \( x, y \in V - \{u\} \) exist, such that \( \delta_{G - \{u\}}(x,y) > \delta_G(x,y) \) \([2]\). Hence, a vertex \( u \in V \) is a hinge vertex if there exist two vertices \( x \) and \( y \) in \( G \) whose distance increases when \( u \) is removed. Note that articulation vertices are a special case of hinge vertices in that the removal of an articulation vertex \( u \) changes the finite distance of some nonadjacent vertices \( x \) and \( y \) to infinity. Finding all hinge vertices of a given graph is called the hinge vertex problem. For a simple graph \( G \) with \( n \) vertices, the hinge vertex problem can be solved in \( O(n^3) \) time by the results in Ref. \([2]\), e.g., Lemma 1 in this study.

The computation of topological properties is a very important research topic, which influences the design and analysis of distributed networks. For example, the overall cost of communication in a network will increase if a computer that corresponds to a hinge vertex stalls. Therefore, identifying the set of hinge vertices in a graph can help detect critical nodes, which can be useful for constructing more stable communication network systems \([6]\).

Numerous studies of hinge vertices on several intersection graphs have been published. For example, Ho et al. \([3]\) presented an \( O(n) \) time algorithm for the hinge vertex problem on permutation graphs. Moreover, Hsu et al. \([8]\) presented an \( O(n) \) time algorithm on interval graphs. The class of trapezoid graphs properly contains both interval graphs and permutation graphs. Honma and Masuyama \([4]\) and Bera \([1]\) presented \( O(n \log n) \) time algorithms for the hinge vertex problem on trapezoid graphs, respectively. Recently, Honma et al. presented an algorithm that runs in \( O(n^2) \) time for identifying the maximum detour hinge vertex on interval graphs \([6]\) and permutation graphs \([7]\).

Lin \([9]\) introduced a circular trapezoid graph (CTG for short), which is a proper superclass of trapezoid graphs and circular-arc graphs. They presented that the maximum weighted independent set can be found in \( O(n^2 \log \log n) \) time on a circular trapezoid graph \([9]\). In this study, we propose an \( O(n^2) \) time algorithm for solving the hinge vertex problem on a CTG.

2. Preliminaries

In this section, we propose some useful data structures and interesting properties on CTGs. We show the circular trapezoid model (CTM for short) before defining the CTG. The model consists of inner and outer circles \( C_1 \) and \( C_2 \) with radius \( r_1 < r_2 \). Each circle is assigned counterclockwise with consecutive integer values \( 1, 2, \ldots, 2n \), where \( n \) is the number of trapezoids. Consider two arcs, \( A_1 \) and \( A_2 \), on \( C_1 \) and \( C_2 \), respectively. Points \( a \) and \( b \) (resp., \( c \) and \( d \)) are the first points encountered when traversing the arc \( A_1 \) (resp., \( A_2 \)) counterclockwise and clockwise, respectively. A trapezoid is the region in circles \( C_1 \) and \( C_2 \) that lies between two non-crossing chords \( ac \) and \( bd \). A trapezoid \( CT_i \) is defined by four corner points \( [a_i, b_i, c_i, d_i] \). Each trapezoid \( CT_i \) is numbered in the ascending order of their corner points \( b_i \)'s, i.e., \( i < j \) if \( b_i < b_j \). The geometric representation described above is called the CTM. Figure 1 (a) illustrates an example of a CTM \( M \) with 12 trapezoids. Table 1 shows the details of \( M \) in Fig. 1 (a).

An undirected graph \( G \) is a CTG if it can be represented by the following CTM \( M \): each vertex of the graph corresponds to a trapezoid, and two vertices are adjacent in \( G \) if and only if their corresponding trapezoids intersect. Figure 1 (b) illustrates the CTG \( G \) corresponding to \( M \) shown in Fig. 1 (a). In this example, \( \delta_G(3,7) = 2 \) and \( \delta_G(5,7) = 4 \), therefore, vertex 5 is a
hinge vertex for 3 and 7. All hinge vertices in G are 5, 7, 9 and 12.

In the following, we introduce an extended circular trapezoid model (ECTM for short) constructed from a CTM. Let n be the number of trapezoids in CTM M. Consider a fictitious line p that connects the points placed between 1 and 2n of C1 and C2. An ECTM EM is obtained by opening a CTM M along a fictitious line p. The ECTM EM consists of two horizontal parallel lines called top and bottom channel, respectively. To avoid confusion, we denote a trapezoid in CTM and ECTM by CTi and Ti, respectively. For each Ti, 1 ≤ i ≤ n, copies Ti,∞ are created by shifting 2n to the right. A procedure for constructing ECTM EM from CTM M in O(n) time is presented in Ref. [5]. Figure 2 illustrates an ECTM EM constructed from the CTM M shown in Fig. 1 (a).

The following properties can be derived in a straightforward manner from the processes of constructing an ECTM [5].

(1) Ti and Ti,∞ in ECTM EM correspond to the vertex i in CTG G.

(2) A vertex i is adjacent to j in G if and only if Ti and Tj, or Tj and Ti,∞ intersect in EM.

3. Useful Lemmas for Hinge Vertex Problem

We introduce some notations that will be used in our algorithm. Let EM be an ECTM constructed from CTM M. We define mt(i), smt(i), mb(i), and smb(i) as follows. Here, the set (including i) of all trapezoids that intersect Ti in EM is denoted by NT[i].

- mt(i) = k such that b_k = max{ b_j | j ∈ N[i]},
- smt(i) = k such that b_k = max{ b_j | j ∈ (N[i] − mt(i) ∪ {i})},
- mb(i) = k such that d_k = max{ d_j | j ∈ N[i]},
- smb(i) = k such that d_k = max{ d_j | j ∈ (N[i] − mb(i) ∪ {i})}.

In the following, we define Yt(i) and Yb(i) as follows.

\[ Yt(i) = \begin{cases} \{ j \mid b_{mt(i)} < a_j < b_{mb(i)}, d_{smb(i)} < c_j \} : mt(i) = mb(i), \\ \{ j \mid b_{mt(i)} < a_j < b_{smb(i)}, d_{mb(i)} < c_j \} : otherwise. \end{cases} \]

\[ Yb(i) = \begin{cases} \{ j \mid d_{smb(i)} < c_j < d_{mb(i)}, b_{smb(i)} < a_j \} : mt(i) = mb(i), \\ \{ j \mid d_{smb(i)} < c_j < d_{mb(i)}, b_{mt(i)} < a_j \} : otherwise. \end{cases} \]

For the example shown in Fig. 2, for the vertex 5, we have mt(5) = 7, smt(5) = 6, mb(5) = 6, smb(5) = 7, Yt(5) = [8, 9], and Yb(5) = ∅. Table 2 shows details of mt(i), smt(i), mb(i), smb(i), Yt(i), and Yb(i) for EM shown in Fig. 2.

We present some lemmas of hinge vertices on CTGs, which are useful for efficiently identifying the hinge vertices. Lemma 1 is proposed by Chang et al. [2] characterizes the hinge vertices of a simple graph.

**Lemma 1** For a simple graph G, a vertex u is a hinge vertex of G if and only if there exist two nonadjacent vertices x, y such that u is the only vertex adjacent to both x and y in G. □

We can easily obtain the following Lemma 2 from Lemma 1.

**Lemma 2** For a CTG G, a vertex u is a hinge vertex of G if and only if there exist two trapezoids CTx and CTy such that CTx and CTy do not intersect, and CTu is the only trapezoid intersecting both CTx and CTy in a CTM M. □

For the example shown in Fig. 1, CT5 and CT8 do not intersect and CT7 is the only trapezoid intersecting both CT5 and CT8 in M. Therefore, vertex 7 is a hinge vertex for 5 and 8 in the corresponding CTG G.

The following Lemma 3 provides the necessary and sufficient condition for hinge vertices in a trapezoid graph presented by Honma and Masuyama [4].

**Lemma 3** A vertex u is a hinge vertex of a trapezoid graph if and only if there exist two vertices x, y satisfying either of the following conditions.

(1) u = mt(x) and y ∈ Yt(x),
(2) u = mb(x) and y ∈ Yb(x). □

The following Lemma 4 provides the necessary and sufficient condition for hinge vertices in a CTG.

**Lemma 4** Let EM be an ECTM constructed from CTM M. A vertex u is a hinge vertex of a CTG G if and only if there exist two vertices x, y satisfying either of the following conditions.

(1) u = mt(x), y ∈ Yt(x), b_{mt(i)} < a_{x,s,n}, and d_{smb(i)} < c_{x,s,n} in ECTM EM.
(2) u = mb(x), y ∈ Yb(x), b_{mb(i)} < a_{x,s,n}, and d_{mb(i)} < c_{x,s,n} in ECTM EM.

(Proof) We only prove this lemma for Condition (1). Condition (2) can be handled in a similar manner.
We first prove the necessity. By Lemma 2, if a vertex \( u \) is a hinge vertex of a CTG, there exist two trapezoids \( CT_s \) and \( CT_y \) such that \( CT_s \) and \( CT_y \) do not intersect, and \( CT_y \) is the only trapezoid intersecting both \( CT_s \) and \( CT_y \) in \( CTM \). From the properties of \( ECTM \), \( T_1 \) and \( T_{ intim } \) correspond to the vertex \( i \) in \( G \), and a vertex \( i \) is adjacent to \( j \) in \( G \) if and only if \( T_1 \) and \( T_j \) or \( T_j \) and \( T_{ intim } \) intersect in \( EM \). Therefore, if a vertex \( u \) is a hinge vertex of a CTG, there exist two trapezoids \( T_1 \) and \( T_y \) such that \( T_1 \) and \( T_y \) do not intersect, and \( T_y \) is the only trapezoid intersecting both \( T_1 \) and \( T_y \), and neither \( T_{ intim } \) nor \( T_{ intim } \) intersect \( T_{ intim } \) in \( ECTM \).

By Lemma 3(1), if \( T_1 \) and \( T_y \) do not intersect, and \( T_u \) is the only trapezoid intersecting both \( T_1 \) and \( T_y \), we have \( u = m(t(x)) \) and \( y = Y(t(x)) \). Moreover, if neither \( T_{ intim } \) nor \( T_{ intim } \) intersect \( T_{ intim } \), then \( b_{ intim } < a_{ intim } \) and \( d_{ intim } < c_{ intim } \). Thus, Condition (1) holds (Fig. 3(a)).

We prove the sufficiency. By Lemma 3(1), if \( u = m(t(x)) \) and \( y = Y(t(x)) \) in \( EM \), then \( T_1 \) and \( T_y \) do not intersect, and \( T_u \) is the only trapezoid intersecting both \( T_1 \) and \( T_y \). Assume that \( b_{ intim } < a_{ intim } \) and \( d_{ intim } < c_{ intim } \), and there exist some trapezoid \( T_2 \) intersecting both \( T_1 \) and \( T_y \). It means that \( b_{ intim } < b_2 \) or \( d_{ intim } < d_2 \), contradicting the definitions of \( m(t(y)) \) and \( m(b(y)) \). Thus, if \( b_{ intim } < a_{ intim } \) and \( d_{ intim } < c_{ intim } \), then neither \( T_{ intim } \) nor \( T_{ intim } \) intersect \( T_{ intim } \) (Fig. 3(a)).

We show how to identify a hinge vertex by applying Condition (1) of Lemma 4. For \( x \) and \( y \) in \( Y(t(x)) \), we check whether \( b_{ intim } < a_{ intim } \) and \( d_{ intim } < c_{ intim } \). For example, for \( x = 5 \) and \( y = 8 \) (\( Y(t(5)) = (8, 9) \)), we have \( b_{ intim } = b_{ intim } = 18 < a_{ intim } = a_{ intim } = 24 \) and \( d_{ intim } = d_{ intim } = 17 < c_{ intim } = c_{ intim } = 23 \). Hence, vertex \( m(t(5)) = 7 \) is a hinge vertex for 5 and 8.

4. Algorithm IHV and its Analysis

In this section, we present Algorithm IHV for identifying all hinge vertices of a CTG \( G \). Algorithm IHV takes a CTM \( M \) as an input. We formally describe Algorithm IHV and analyze its

Table 2 Details of ECTM \( EM \) shown in Fig. 2.

| \( i \) | \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) | \( 5 \) | \( 6 \) | \( 7 \) | \( 8 \) | \( 9 \) | \( 10 \) | \( 11 \) | \( 12 \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( a \) | \(-2\) | \(-3\) | \(-1\) | \(4\) | \(0\) | \(6\) | \(8\) | \(10\) | \(11\) | \(13\) | \(15\) | \(17\) |
| \( b \) | \(1\) | \(2\) | \(3\) | \(5\) | \(7\) | \(9\) | \(12\) | \(14\) | \(16\) | \(18\) | \(19\) | \(20\) |
| \( c \) | \(-5\) | \(-4\) | \(2\) | \(1\) | \(-1\) | \(7\) | \(5\) | \(10\) | \(11\) | \(13\) | \(16\) | \(14\) |
| \( d \) | \(-3\) | \(-2\) | \(3\) | \(4\) | \(6\) | \(9\) | \(8\) | \(12\) | \(15\) | \(17\) | \(18\) | \(24\) |

\[ c_1 \, c_2 \, d_1 \, d_2 \, c_3 \, d_3 \, c_4 \, d_4 \, c_5 \, d_5 \, c_6 \, d_6 \, c_7 \, d_7 \, c_8 \, d_8 \, c_9 \, d_9 \, c_{10} \, d_{10} \, c_{11} \, d_{11} \, c_{12} \, d_{12} \, c_{13} \, d_{13} \, c_{14} \, d_{14} \, c_{15} \, d_{15} \, c_{16} \, d_{16} \, c_{17} \, d_{17} \, c_{18} \, d_{18} \, c_{19} \, d_{19} \, c_{20} \, d_{20} \, c_{21} \, d_{21} \, c_{22} \, d_{22} \, c_{23} \, d_{23} \, c_{24} \, d_{24} \, c_{25} \, d_{25} \, c_{26} \, d_{26} \, c_{27} \, d_{27} \, c_{28} \, d_{28} \, c_{29} \, d_{29} \, c_{30} \, d_{30} \]

\[ a_2 \, a_3 \, a_4 \, a_5 \, a_6 \, a_7 \, a_8 \, a_9 \, a_10 \, a_11 \, a_12 \, a_13 \, a_14 \, a_15 \, a_16 \, a_17 \, a_18 \, a_19 \, a_20 \, a_21 \, a_22 \, a_23 \, a_24 \, a_25 \, a_26 \, a_27 \, a_28 \, a_29 \, a_30 \]

Fig. 2 Extended circular trapezoid model \( EM \).

Fig. 3 Illustration of Lemma 4.
Algorithm 1: Identify Hinge Vertices (IHV)

Input: Each trapezoid’s corner points $a_i, b_i, c_i, d_i$ for $n$ circular trapezoids in a CMT $M$.

Output: A set of all hinge vertices $HV$ in the CTG $G$.

(Step 1) Construct an ECTM $EM$ from $M$;

(Step 2) Compute $mt(i)$ and $mb(i)$ for $1 \leq i \leq 2n$;
Compute $smt(i)$ and $smb(i)$ for $1 \leq i \leq n$;

(Step 3) Compute $Yt(i)$ and $Yh(i)$ for $1 \leq i \leq n$;

(Step 4) Set $HV := \emptyset$;

/* Condition (1) of Lemma 6 */
for $1 \leq i \leq n$ do
for $j \in Yt(i)$ do
if $b_{mt(i)} < a_{mb(i)}$ and $d_{mb(i)} < c_{mt(i)}$ then
$HV := HV \cup \{\text{Normalize}(mt(i))\}$;
end if
end for
end for

/* Condition (2) of Lemma 6 */
for $1 \leq i \leq n$ do
for $j \in Yh(i)$ do
if $b_{smt(i)} < a_{smb(i)}$ and $d_{smb(i)} < c_{smt(i)}$ then
$HV := HV \cup \{\text{Normalize}(mb(i))\}$;
end if
end for
end for

Function Normalize($v$)
if $v > n$ then return $v - n$;
else return $v$;
end function

inherent complexity as follows.

Steps 1 to 3 are preparatory steps for identifying all hinge vertices of $G$. In Step 1, we construct an ECTM $EM$ that can be executed in $O(n)$ time [5]. In Step 2, $mt(i)$, $mb(i)$, $smt(i)$, and $smb(i)$ are computed. This step can be done in $O(n)$ time using prefix computation [1], [4]. Step 3 computes $Yt(i)$ and $Yh(i)$ for $1 \leq i \leq n$. This step runs in $O(n^2)$ time because the size of $\sum_{i=1}^{n} Yt(i)$ is proportional to $n^2$. In Step 4, we find all hinge vertices by applying Lemma 4 that can be executed in $O(n^2)$ time. Thus, we obtain the following theorem.

Theorem 1 Algorithm IHV identifies all hinge vertices of a CTG $G$ in $O(n^2)$ time by taking its CMT $M$ as an input. □

5. Conclusion

In this study, we proposed Algorithm IHV, which operates in $O(n^2)$ time to identify all hinge vertices on a CTG. Identifying all hinge vertices requires an $O(n^3)$ time by a simple method. Therefore, our algorithm outperforms the simple method. Algorithm IHV partly uses the algorithms of Honma et al. [4]. Reducing the complexity of the algorithm and extending the results to other graphs will be addressed in future research.

Acknowledgments This work was partially supported by JSPS KAKENHI Grant Number 25330019, 26330359, and 17K00324.

References
[1] Bera, D., Pal, M. and Pal, T.K.: An efficient algorithm for finding all hinge vertices on trapezoid graphs, Theory of Computing Systems, Vol.36, No.1, pp.17–27 (2003).
[2] Chang, J.M., Hsu, C.C., Wang, Y.L. and Ho, T.Y.: Finding the set of all hinge vertices for strongly chordal graphs in linear time, Inf. Sci., Vol.99, No.3–4, pp.173–182 (1997).
[3] Ho, T.Y., Wang, Y.L. and Yuan, M.T.: A linear time algorithm for finding all hinge vertices of a permutation graph, Inf. Process. Lett., Vol.59, No.2, pp.103–107 (1996).
[4] Honma, H. and Masuyama, S.: A parallel algorithm for finding all hinge vertices of a trapezoid graph, IEICE Trans. Fundamentals, Vol.E85-A, No.5, pp.1031–1040 (2002).
[5] Honma, H., Nakajima, Y., Aoshima, Y. and Masuyama, S.: A Linear-time Algorithm for Constructing a Spanning Tree on Circular Trapezoid Graphs, IEICE Trans. Fundamentals, Vol.E96-A, No.6, pp.1051–1058 (2013).
[6] Honma, H., Nakajima, Y., Igarashi, Y. and Masuyama, S.: Algorithm for Finding Maximum Detour Hinge Vertices of Interval Graphs, IEICE Trans. Fundamentals, Vol.E97-A, No.6, pp.1365–1369 (2014).
[7] Honma, H., Nakajima, Y., Igarashi, Y. and Masuyama, S.: Algorithm for Identifying the Maximum Detour Hinge Vertices of a Permutation Graph, IEICE Trans. Fundamentals, Vol.E98-A, No.6, pp.1161–1167 (2015).
[8] Hsu, F.R., Shan, K., Chao, H.S. and Lee, R.C.: Some optimal parallel algorithms on interval and circular-arc graphs, J. Inf. Sci. Eng., Vol.21, pp.627–642 (2005).
[9] Lin, W.L.: Circular and circle trapezoid graphs, J. Sci. Eng. Tech., Vol.2, No.2, pp.11–17 (2006).

Hiroshi Honma is an Associate Professor at the Department of Creative Engineering, National Institute of Technology, Kushiro College. He received his B.E., M.E., and D.E. degrees in Engineering from Toyohashi University of Technology, in 1990, 1992 and 2009, respectively. His research interest includes computational graph theory and parallel algorithms. He is a member of IEICE and ORSJ.

Yoko Nakajima is a Senior Lecturer at the Department of Creative Engineering, National Institute of Technology, Kushiro College. She received her D.E. degree from Kitami Institute of Technology in 2016. She is working on extraction of future trend information with future reference sentences and its application in future event prediction. She is a member of IPSJ and IEICE.

Shigeru Masuyama is a Professor at the Department of Knowledge-Based Information Engineering, Toyohashi University of Technology. His research interest includes computational graph theory, combinatorial optimization, distributed algorithms and natural language processing. He is a member of IEICE, ORSJ, IPJ, ISCIEJ, and NLPIJ etc.