Euler-Calogero-Moser system from $SU(2)$ Yang-Mills theory

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The relation between $SU(2)$ Yang-Mills mechanics, originated from the 4-dimensional $SU(2)$ Yang-Mills theory under the supposition of spatial homogeneity of the gauge fields, and the Euler-Calogero-Moser model is discussed in the framework of Hamiltonian reduction. Two kinds of reductions of the degrees of freedom are considered: due to the gauge invariance and due to the discrete symmetry. In the former case, it is shown that after elimination of the gauge degrees of freedom from the $SU(2)$ Yang-Mills mechanics the resulting unconstrained system represents the $\text{ID}_3$ Euler-Calogero-Moser model with an external fourth-order potential. Whereas in the latter, the $\text{IA}_6$ Euler-Calogero-Moser model embedded in an external potential is derived whose projection onto the invariant submanifold through the discrete symmetry coincides again with the $SU(2)$ Yang-Mills mechanics. Based on this connection, the equations of motion of the $SU(2)$ Yang-Mills mechanics in the limit of the zero coupling constant are presented in the Lax form.

PACS: 03.20.+i, 11.10.Ef, 11.15.Tk

I. INTRODUCTION

The present note is devoted to the discussion of the correspondence between the dynamics of 3-particles with internal degrees interacting by pairwise $1/r^3$ forces on a line (Euler-Calogero-Moser system \cite{1,2}) and $SU(2)$ Yang-Mills theory with spatially constant gauge fields ($SU(2)$ Yang-Mills mechanics \cite{3} (see also \cite{4,5} and references therein)).

The Euler-Calogero-Moser model is the extension of the famous Calogero-Sutherland-Moser models \cite{6,7} (for its generalizations see \cite{8} and reviews \cite{9,10}) with additional dynamical internal degrees of freedom included. It is interesting that these types of models arises in various areas of theoretical physics like the 2-dimensional Yang-Mills theory \cite{12}, black hole physics \cite{13}, spin chain systems \cite{14}, generalized statistics \cite{15}, higher spin theories \cite{16}, level dynamics for quantum systems \cite{17}, quantum Hall effect \cite{18} and many others. An attractive feature of these generalizations is that it maintains the integrability property of the original Calogero-Sutherland-Moser system. For the general elliptic version of the Euler-Calogero-Moser system, the action-angle type variables have been constructed and the equations of motion have been solved in terms of Riemannian theta-functions \cite{19}, the canonical symplectic form of this model is represented in terms of algebro-geometric data \cite{20} using the general construction of Krichever and Phong \cite{21}.

During the past years a remarkable relation between the Calogero-Moser systems and the exact solutions of 4-dimensional supersymmetric gauge theories has been found \cite{22}. It has been recognized that the so-called Seiberg-Witten spectral curves are identical to the spectral curves of the elliptic $SU(N)$ Calogero-Moser system \cite{23}. Furthermore the generalization of these relations to the $N=2$ supersymmetric gauge theories with general Lie algebras and an adjoint representation of matter hypermultiplet have been derived in \cite{24} (for review of the recent results see, e.g., \cite{25}).
Despite the existence of such a correspondence established on very general grounds, relations between gauge theories and integrable models are far from being understood. In the present note, we would like to point out a simple direct correspondence between the $SU(2)$ Yang-Mills theory and the Euler-Calogero-Moser model. This correspondence follows from the sequence of reductions of degrees of freedom thanks to different kinds of symmetries. At first, supposing the spatial homogeneity of gauge fields, the field theory is reduced to the 9-dimensional degenerate Lagrangian model. Then the pure gauge variables are eliminated by applying the method of Hamiltonian reduction. Finally, rewriting the derived unconstrained matrix model in terms of special coordinates adapted to the action of rigid symmetry, one can arrive at the conventional form of the Euler-Calogero-Moser Hamiltonian. More precisely, we shall demonstrate that the unconstrained $SU(2)$ Yang-Mills mechanics represents the Euler-Calogero-Moser system of type $\text{ID}_3$, i.e., the inverse-square interacting 3-particle system with internal degrees of freedom related to the root system of simple Lie algebra $D_3$ \cite{10,11}, and is embedded in a fourth order external potential written in the superpotential form.

Besides this reduction due to the continuous symmetry of the system, we discuss another possibility of relating the Yang-Mills mechanics to higher order matrix models using the discrete symmetries. We shall explore the method of constructing generalizations of the Calogero-Sutherland-Moser models elaborated recently by A. Polychronakos \cite{26}. This method consists in the usage of the appropriate reduction of the original Calogero model by a subset of its discrete symmetries to an invariant submanifold of the phase space. Representing the Euler-Calogero-Moser system with a special external potential as a $6 \times 6$ symmetric matrix model, we shall show that such a matrix model, after projection onto the invariant submanifold of the phase space using a certain subset of discrete symmetries, is equivalent to the unconstrained $SU(2)$ Yang-Mills mechanics. Finally, we give a Lax pair representation for the equations of motion of the $SU(2)$ Yang-Mills mechanics in the limit of the zero coupling constant.

II. HAMILTONIAN REDUCTION OF THE YANG-MILLS MECHANICS

A. The equivalent unconstrained matrix model

The dynamics of the $SU(2)$ Yang-Mills 1-form $A$ in 4-dimensional Minkowski space-time $M_4$ is governed by the conventional local functional

$$S_{YM} = \frac{1}{2} \int_{M_4} \text{tr} F \wedge \ast F,$$

(2.1)

defined in terms of the curvature 2-form $F = dA + g A \wedge A$, with the coupling constant $g$. After the supposition of the spatial homogeneity of the connection $A$,

$$\mathcal{L}_\partial A = 0,$$

(2.2)

the action (2.1) reduces to the action for a finite dimensional model, the so-called Yang-Mills mechanics (YMM) described by the degenerate matrix Lagrangian

$$L_{YMM} = \frac{1}{2} \text{tr} \left( (D_t A)(D_t A)^T \right) - V(A),$$

(2.3)

The entries of the $3 \times 3$ matrix $A$ are nine spatial components $A_{a i} := A^a_i$ of the connection $A := Y_a e_a dt + A_{ai} e_a dx^i$, where $e_a = \sigma_a / 2i$ with the Pauli matrices $\sigma_a$ and $D_t$ denotes the covariant derivative $(D_t A)_{ai} = A_{ai} + g e_{abc} Y_b A_{ci}$. Due to the spatial homogeneity condition (2.2), all dynamical variables $Y_a$ and $A_{ai}$ are functions of time only. The part of the Lagrangian corresponding to the self-interaction of the gauge fields is gathered in the potential $V(A)$. 

\[ V(A) = \frac{g^2}{4} \left( \text{tr}^2(AA^T) - \text{tr}(AA^T)^2 \right). \quad (2.4) \]

To express the Yang-Mills mechanics in a Hamiltonian form, let us define the phase space endowed with the canonical symplectic structure and spanned by the canonical variables \((Y_a, P_{Y_a})\) and \((A_{ai}, E_{ai})\) where

\[ P_{Y_a} = \frac{\partial L}{\partial \dot{Y}_a} = 0, \quad E_{ai} = \frac{\partial L}{\partial \dot{A}_{ai}} = \dot{A}_{ai} + g \varepsilon_{abc} Y_b A_{ci}. \quad (2.5) \]

According to these definitions of the canonical momenta (2.5), the phase space is restricted by the three primary constraints

\[ P_a Y_a = 0 \quad (2.6) \]

and the evolution of the system is governed by the total Hamiltonian

\[ H_T = H_C + u_a^Y(t) P_{Y_a}, \]

where the canonical Hamiltonian is given by

\[ H_C = \frac{1}{2} \text{tr}(E E^T) + \frac{g^2}{4} \left( \text{tr}^2(AA^T) - \text{tr}(AA^T)^2 \right) + g Y_a \text{tr}(J_a A E^T), \quad (2.7) \]

and the matrix \((J_a)_{bc}\) is defined by \((J_a)_{bc} = -\varepsilon_{abc}\). The conservation of constraints (2.6) in time entails the further condition on the canonical variables

\[ \Phi_a = g \text{tr}(J_a A E^T) = 0, \quad (2.8) \]

that reproduces the homogeneous part of the conventional non-Abelian Gauss law constraints. They are the first class constraints obeying the Poisson brackets algebra

\[ \{\Phi_a, \Phi_b\} = \varepsilon_{abc} \Phi_c. \quad (2.9) \]

In order to project onto the reduced phase space, we use the well-known polar decomposition for an arbitrary \(3 \times 3\) matrix

\[ A_{ai}(\phi, Q) = O_{ak}(\phi) Q_{ki}, \quad (2.10) \]

where \(Q_{ij}\) is a positive definite \(3 \times 3\) symmetric matrix and \(O(\phi_1, \phi_2, \phi_3) = e^{\phi_1 J_3} e^{\phi_2 J_1} e^{\phi_3 J_3}\) is an orthogonal matrix \(O \in SO(3)\). Assuming the nondegenerate character of the matrix \(A_{ai}\), we can treat the polar decomposition as uniquely invertible transformation from the configuration variables \(A_{ai}\) to a new set of six Lagrangian coordinates \(Q_{ij}\) and three coordinates \(\phi_i\). As it follows from further consideration, the variables parameterizing the elements of the \(SO(3)\) group (Euler angles \((\phi_1, \phi_2, \phi_3)\)) are the pure gauge degrees of freedom.

The field strength \(E_{ai}\) in terms of the new canonical variables is

\[ E_{ai} = O_{ak}(\phi) \left[ P_{ki} + \varepsilon_{kil}(\gamma^{-1})_{lj} [\xi^L_j - S_j] \right], \quad (2.11) \]

where \(\xi^L_0\) are three left-invariant vector fields on \(SO(3)\)

\[ \xi^L_0 = \frac{\sin \phi_3}{\sin \phi_2} P_1 + \cos \phi_3 P_2 - \cot \phi_2 \sin \phi_3 P_3, \quad (2.12) \]

\[ \xi^L_2 = \frac{\cos \phi_3}{\sin \phi_2} P_1 - \sin \phi_3 P_2 - \cot \phi_2 \cos \phi_3 P_3, \quad (2.13) \]

\[ \xi^L_3 = P_3. \quad (2.14) \]
Here $S_j = \varepsilon_{jmn}(PQ)_{mn}$ is the spin vector of the gauge field and
\[ \gamma_{ik} = Q_{ik} - \delta_{ik} \text{tr} Q. \] (2.15)

Reformulation of the theory in terms of these variables allows one to easily achieve the Abelianization of the secondary Gauss law constraints. Using the representations (2.10) and (2.11), one can convince oneself that the variables $Q_{ij}$ and $P_{ij}$ make no contribution to the secondary constraints (2.8)
\[ \Phi_a = O_{ab}(\phi) \xi_b^L = 0. \] (2.16)

Hence, assuming nondegenerate character of the matrix
\[ M = \begin{pmatrix} \sin \phi_1, & \cos \phi_1, & -\sin \phi_1 \cot \phi_2 \\ -\cos \phi_1, & \sin \phi_1, & \cos \phi_1 \cot \phi_2 \\ 0, & 0, & 1 \end{pmatrix}, \] (2.17)
we find the set of Abelian constraints equivalent to the Gauss law (2.8)
\[ \tilde{\Phi}_a = P_a = 0. \] (2.18)

After having rewritten the model in this form, we are able to reduce the theory to physical phase space by a straightforward projection onto the constraint shell. The resulting unconstrained Hamiltonian, defined as a projection of the total Hamiltonian onto the constraint shell
\[ H_{YMM} := H_C(Q_{ab}, P_{ab}) \bigg|_{P_a=0, P_0^a=0}, \] (2.19)
can be written in terms of $Q_{ab}$ and $P_{ab}$ as
\[ H_{YMM} = \frac{1}{2} \text{tr} P^2 - \frac{1}{\det^2 \gamma} \text{tr} (\gamma M \gamma)^2 + \frac{g^2}{4} (\text{tr}^2 Q^2 - \text{tr} Q^4), \] (2.20)
where $M_{mn} = (QP - PQ)_{mn}$ denotes the gauge field spin tensor.

### B. Unconstrained model as particle motion on stratified manifold

In the previous section, the unconstrained dynamics of the SU(2) Yang-Mills mechanics was identified with the dynamics of the nondegenerate matrix model (2.20). The configuration space $Q$ of the real symmetric $3 \times 3$ matrices can be endowed with the flat Riemannian metric
\[ ds^2 = \text{Tr} (dQ^2) \] (2.21)
whose group of isometry is formed by orthogonal transformations
\[ Q' = RQR^T. \] (2.22)

Since the unconstrained Hamiltonian system (2.20) is invariant under the action of this rigid group, we are interested in the structure of the orbit space given as a quotient $Q/SO(3)$. The important information on the stratification of the space $Q/SO(3)$ of orbits can be obtained from the so-called isotropy group of points of configuration space which is defined as a subgroup of $SO(3)$ leaving point $x$ invariant $RxR^T = x$. Orbits with the same isotropy group are collected into classes, called by strata. So, as for the case of symmetric matrix, the orbits are uniquely parameterized by the set of ordered eigenvalues of the matrix $Q$, $x_1 \leq x_2 \leq x_3$. One can classify the orbits according to the isotropy groups which are determined by the degeneracies of the matrix eigenvalues:
1. **Principal orbit-type strata**, when all eigenvalues are unequal \( x_1 < x_2 < x_3 \) with the smallest isotropy group \( Z_2 \otimes Z_2 \).

2. **Singular orbit type strata** forming the boundaries of orbit space with

   (a) two coinciding eigenvalues \( x_1 = x_2, x_2 = x_3 \) or \( x_1 = x_3 \), the isotropy group is \( SO(2) \otimes Z_2 \).

   (b) all three eigenvalues are equal \( x_1 = x_2 = x_3 \), here the isotropy group coinciding with the isometry group \( SO(3) \).

In the subsequent sections, we shall demonstrate that the dynamics of the Yang-Mills mechanics, which takes place on the principal orbits is governed by the \( ID_3 \) Euler-Calogero model Hamiltonian with the external potential \( V^{(3)} := g^2/2 \sum_{i<j} x_i^2 x_j^2 \), while for singular orbits the corresponding system is either the \( A_2 \) Calogero model with the external potential \( V^{(2)} := g^2/2(x^4 + 2 x^2 y^2) \) for singular orbits of type (a) or one dimensional system with quartic potential \( V^{(1)} := 3/2g^2 x^4 \) for singular orbits of type (b).

1. **Hamiltonian on principal orbit strata**

   To write down the Hamiltonian describing the motion on the principal orbit strata, we introduce the coordinates along the slices \( x_i \) and along the orbits \( \chi \). Namely, we decompose the nondegenerate symmetric matrix \( Q \) as

   \[
   Q = \mathcal{R}^T (\chi_1, \chi_2, \chi_3) \mathcal{D} \mathcal{R}(\chi_1, \chi_2, \chi_3)
   \]

   with the \( SO(3) \) matrix \( \mathcal{R} \) parameterized by the three Euler angles \( \chi_i := (\chi_1, \chi_2, \chi_3) \) and the diagonal matrix \( \mathcal{D} = \text{diag} \ (x_1, x_2, x_3) \) and consider it as point transformation from the physical coordinates \( Q_{ab} \) and \( P_{ab} \) to \( x_i, p_i \) and \( \chi_i, \pi_i \). The Jacobian of this transformation is the relative volume of orbits

   \[
   J := \left| \det \left( \frac{\partial Q}{\partial x_k}, \frac{\partial Q}{\partial \chi_k} \right) \right| = \prod_{i<k} |x_i - x_k|
   \]

   and is regular for this stratum \( x_1 < x_2 < x_3 \).

   By using the generating function

   \[
   F \left[ x_i, \chi_i; \ P \right] = \text{tr} \ (QP) = \text{tr} \ (\mathcal{R}^T(\chi) \mathcal{D}(x) \mathcal{R}(\chi) P)
   \]

   the canonical conjugate momenta can be found in the form

   \[
   p_i = \frac{\partial F}{\partial x_i} = \text{tr} \ (P \mathcal{R}^T \pi_i \mathcal{R}) \quad \text{and} \quad p_{\chi_i} = \frac{\partial F}{\partial \chi_i} = \text{tr} \left( \mathcal{R}^T \frac{\partial \mathcal{R}}{\partial \chi_i} (PQ - QP) \right),
   \]

   where \( \pi_i \) are the diagonal members of the orthogonal basis for the symmetric \( 3 \times 3 \) matrices \( \alpha_A = (\pi_i, \alpha_i) \ i = 1, 2, 3 \) under the scalar product

   \[
   \text{tr}(\bar{\alpha}_a, \bar{\alpha}_b) = \delta_{ab}, \quad \text{tr}(\alpha_a, \alpha_b) = 2\delta_{ab}, \quad \text{tr}(\bar{\alpha}_a, \alpha_b) = 0.
   \]

   The original physical momenta \( P_{ik} \) can then be expressed in terms of the new canonical variables as

   \[
   P = \mathcal{R}^T \left( \sum_{s=1}^{3} \bar{P}_s \pi_s + \sum_{s=1}^{3} \bar{P}_s \alpha_s \right) \mathcal{R}
   \]

   with \( \bar{P}_s = p_s \).
\[ \mathcal{P}_i = -\frac{1}{2} \frac{\xi^R_i}{x_j - x_k}, \quad \text{(cyclic permutation } i \neq j \neq k) \] (2.28)

and the SO(3) right-invariant Killing vectors

\begin{align*}
\xi^1_R &= p_{\chi_1}, \\
\xi^2_R &= -\sin \chi_1 \cot \chi_2 \ p_{\chi_1} + \cos \chi_1 \ p_{\chi_2} + \frac{\sin \chi_1}{\sin \chi_2} \ p_{\chi_3}, \\
\xi^3_R &= \cos \chi_1 \cot \chi_2 \ p_{\chi_1} + \sin \chi_1 \ p_{\chi_2} - \frac{\cos \chi_1}{\sin \chi_2} \ p_{\chi_3}.
\end{align*}

They satisfy the Poisson bracket algebra

\[ \{\xi^R_a, \xi^R_b\} = \varepsilon_{abc} \xi^R_c. \] (2.32)

Thus, finally, we get the following physical Hamiltonian defined on the unconstrained phase space

\[ H_{YM,M} = \frac{1}{2} \sum_{a=1}^{3} p_{a}^2 + \frac{1}{4} \sum_{a=1}^{3} k_a^2 e^2 + V^{(3)}(x), \] (2.33)

where

\[ k_a^2 = \frac{1}{(x_b + x_c)^2} + \frac{1}{(x_b - x_c)^2}, \quad \text{cyclic } a \neq b \neq c \] (2.34)

and

\[ V^{(3)} = \frac{g^2}{2} \sum_{a < b} x_a^2 x_b^2. \] (2.35)

Note that the potential term in (2.35) has symmetry beyond the cyclic one. This fact allows us to write \( V^{(3)}(x_1, x_2, x_3) \) in the form

\[ V^{(3)}(x_1, x_2, x_3) = \frac{\partial W^{(3)}}{\partial x_a} \frac{\partial W^{(3)}}{\partial x_a}, \quad a = 1, 2, 3 \] (2.36)

with the superpotential \( W^{(3)} = x_1 x_2 x_3 \).

This completes our reduction of the spatially homogeneous Yang-Mills system to the equivalent unconstrained system describing the dynamics of the physical dynamical degrees of freedom. We see that the reduced Hamiltonian \( H_{YM,M} \) on the principal orbit strata is exactly the Hamiltonian of the Euler-Calogero-Moser system of type ID_3, i.e., is of the inverse-square interacting 3-particle system with internal degrees of freedom and related to the root system of the simple Lie algebra D_3 [10,11] embedded in the fourth order external potential (2.36).

C. Singular stratum

Introduction of the additional constraints

\[ x_1 - x_2 = 0 \] (2.37)

or

\[ x_1 - x_2 = 0, \quad x_1 - x_3 = 0 \] (2.38)

defines the invariant two- and one-dimensional strata. One can repeat the above consideration for these singular strata and derive, correspondingly, the following unconstrained Hamiltonians:
1. **Two-dimensional strata**

\[
H_{\text{Sing}}^{(2)} = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{4}(l(l+1)}{(x-y)^2} + \frac{g^2}{2}(x^4 + 2x^2y^2),
\]

(2.39)

where the constant \(l(l+1)\) denotes a value of the square of the particle internal spin.

2. **One-dimensional strata**

\[
H_{\text{Sing}}^{(1)} = \frac{1}{2}p_x^2 + \frac{3}{2}g^2x^4.
\]

(2.40)

### III. EULER-CALOGERO-MOSER SYSTEM AS A FREE MOTION ON SPACE OF SYMMETRIC MATRICES

In order to discuss the relation between the Yang-Mills mechanics and the Euler-Calogero-Moser system, it is useful to represent the latter in the form of a nondegenerate matrix model. Let us consider the Hamiltonian system with the phase space spanned by the \(N \times N\) symmetric matrices \(X\) and \(P\) with the noncanonical symplectic form

\[
\{X_{ab}, P_{cd}\} = \frac{1}{2} (\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}).
\]

(3.1)

The Hamiltonian of the system defined as

\[
H = \frac{1}{2} \text{tr} P^2
\]

(3.2)

describes a free motion in the matrix configuration space. The following statement is fulfilled:

*The Hamiltonian (3.2) rewritten in special coordinates coincides with the Euler-Calogero-Moser Hamiltonian*

\[
H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + \frac{1}{2} \sum_{i \neq j}^{N} \frac{l_{ij}^2}{(x_i - x_j)^2}.
\]

(3.3)

with nonvanishing Poisson brackets for the canonical variables \(x_i, p_j\)

\[
\{x_i, p_j\} = \delta_{ij}, \quad \{l_{ab}, l_{cd}\} = \frac{1}{2} (\delta_{ac} l_{bd} - \delta_{ad} l_{bc} + \delta_{bd} l_{ac} - \delta_{bc} l_{ad}).
\]

(3.5)

To find the adapted set of coordinates in which the Hamiltonian (3.3) coincides with the Euler-Calogero-Moser Hamiltonian (3.3), let us introduce new variables

---

1. This system is the spin generalization of the Calogero-Moser model. Particles are described by their coordinates \(x_i\) and momenta \(p_i\) together with internal degrees of freedom of angular momentum type \(l_{ij} = -l_{ji}\). The analogous model has been introduced in [1] where the internal degrees of freedom satisfy the following Poisson brackets relations

\[
\{l_{ab}, l_{cd}\} = \delta_{bc} l_{ad} - \delta_{ad} l_{cb}.
\]

(3.4)
\[ X = O^{-1}(\theta)Q(q)O(\theta), \quad (3.6) \]

where the orthogonal matrix \( O(q) \) is parameterized by the \( \frac{N(N-1)}{2} \) elements, e.g., the Euler angles \((\theta_1, \cdots, \theta_{\frac{N(N-1)}{2}})\) and \( Q = \text{diag}[q_1, \cdots, q_N] \) denotes a diagonal matrix. This point transformation induces the canonical one which we can obtain using the generating function

\[ F_4 = \left[ P, q_1, \cdots, q_N, \theta_1, \cdots, \theta_{\frac{N(N-1)}{2}} \right] = \text{tr}[X(q, \theta)P]. \quad (3.7) \]

Using the representation

\[ P = O^{-1} \left[ \sum_{a=1}^{N} \bar{\alpha}_a \tilde{P}_a + \sum_{i<j=1}^{N} \alpha_{ij} P_{ij} \right] O, \quad (3.8) \]

where the matrices \((\bar{\alpha}_a, \alpha_{ij})\) form an orthogonal basis in the space of the symmetric \( N \times N \) matrices under the scalar product

\[ \text{tr}(\bar{\alpha}_a \bar{\alpha}_b) = \delta_{ab}, \quad \text{tr}(\alpha_{ij} \alpha_{kl}) = 2\delta_{ik}\delta_{jl}, \quad \text{tr}(\alpha_{a} \alpha_{ij}) = 0, \quad (3.9) \]

one can find that \( \tilde{P}_a = p_a \) and components \( P_{ab} \) are represented via the \( O(N) \) right invariant vectors fields \( l_{ab} \)

\[ P_{ab} = \frac{1}{2} \frac{l_{ab}}{x_a - x_b}. \quad (3.10) \]

From this, it is clear that the Hamiltonian \((3.2)\) coincides with the Euler-Calogero-Moser Hamiltonian \((3.3)\).

IV. YANG-MILLS MECHANICS THROUGH THE DISCRETE REDUCTION OF EULER-CALOGERO-MOSER SYSTEM

In this section, we shall demonstrate how the \( SU(2) \) Yang-Mills mechanics arises from the higher dimensional matrix model after projection onto a certain invariant submanifold determined by the discrete symmetries. Let us consider the classical Hamiltonian system of \( N \) particles on a line with internal degrees of freedom embedded in external field with the potential \( V(x_1, x_2, \ldots, x_N) \) and described by the Hamiltonian

\[ H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + \frac{1}{2} \sum_{i \neq j}^{N} \frac{l_{ij}^2}{(x_i - x_j)^2} + V^{(N)}(x_1, x_2, \ldots, x_N). \quad (4.1) \]

The particles are described by their coordinates \( x_i \) and momenta \( p_i \) together with the internal degrees of freedom of angular momentum type \( l_{ij} = -l_{ji} \). The nonvanishing Poisson brackets are

\[ \{x_i, p_j\} = \delta_{ij}, \quad \{l_{ab}, l_{cd}\} = \delta_{ac}l_{bd} - \delta_{ad}l_{bc} + \delta_{bd}l_{ac} - \delta_{bc}l_{ad}. \quad (4.2) \]
The external potential \( V^{(N)}(x_1, x_2, \ldots, x_N) \) is constructed in terms of the superpotential \( W^{(N)} \)

\[
V^{(N)}(x_1, x_2, \ldots, x_N) = -\frac{1}{4} \sum_{a=1}^{N} \frac{\partial W^{(N)}}{\partial x_a} \frac{\partial W^{(N)}}{\partial x_a},
\]

(4.3)

with \( W^{(N)} \) given as

\[
W^{(N)} = i \sqrt{x_1 x_2 \ldots x_N}.
\]

(4.4)

Below it is useful to treat the internal degrees of freedom entering into the Hamiltonian (4.1) in the Cartesian form

\[
l_{ab} = y_a \pi_b - y_b \pi_a,
\]

(4.5)

where the internal variables \( y_a \) and \( \pi_a \) combine the canonical pairs with the canonical symplectic form. The Hamiltonian (4.1) has the following discrete symmetries [26]:

- Parity \( P \)

\[
\begin{pmatrix} x_i \\ p_i \end{pmatrix} \mapsto \begin{pmatrix} -x_i \\ -p_i \end{pmatrix}, \quad \begin{pmatrix} y_i \\ \pi_i \end{pmatrix} \mapsto \begin{pmatrix} -y_i \\ -\pi_i \end{pmatrix},
\]

(4.6)

- Permutation symmetry \( M \)

\[
\begin{pmatrix} x_i \\ p_i \end{pmatrix} \mapsto \begin{pmatrix} x_{M(i)} \\ p_{M(i)} \end{pmatrix}, \quad \begin{pmatrix} y_i \\ \pi_i \end{pmatrix} \mapsto \begin{pmatrix} y_{M(i)} \\ \pi_{M(i)} \end{pmatrix},
\]

(4.7)

where \( M \) is the element of the permutation group \( S_N \). The manifold of phase space defined as

\[
x_a + x_{N-a+1} = 0, \quad p_a + p_{N-a+1} = 0,
\]

(4.8)

\[
y_a + y_{N-a+1} = 0, \quad \pi_a + \pi_{N-a+1} = 0
\]

(4.9)

is invariant under the action of the symmetry group \( z = D(z) \) where

\[
D = P \times M
\]

(4.10)

and \( M \) is specified as \( M(a) = N - a + 1 \).

In order to project onto the manifold described by constraints (4.8)-(4.9), we use the Dirac method to deal with the second class constraints. Let us introduce the Dirac brackets between the arbitrary functions \( F \) and \( G \) of all variables \( (x_a, p_a, y_a, \pi_a) \) as

\[
\{ F, G \}_D = \{ F, G \} - \{ F, Z_a \} Z_a \{ Z_b, G \}^{-1} Z_b
\]

(4.11)

Writing the superpotential in an invariant form as

\[
W^{(N)} = i \sqrt{\det X},
\]

with the help of a symmetric \( N \times N \) matrix \( X \) whose eigenvalues are \( x_1, x_2, \ldots, x_N \), the external potential reads

\[
V^{(N)}(x_1, x_2, \ldots, x_N) = \det X \text{tr}(X^{-2}).
\]
where $Z_a$ denote all second class constraints $Z_a := (\chi_a, \Pi_a, \bar{\chi}_a, \bar{\Pi}_a)$, $a = 1, \cdots, N$

\begin{align}
\chi_a &= \frac{1}{\sqrt{2}}(x_a + x_{N-a+1}) , \quad \bar{\chi}_a = \frac{1}{\sqrt{2}}(y_a + y_{N-a+1}) , \\
\Pi_a &= \frac{1}{\sqrt{2}}(p_a + p_{N-a+1}) , \quad \bar{\Pi}_a = \frac{1}{\sqrt{2}}(\pi_a + \pi_{N-a+1})
\end{align}

(4.12)

(4.13)

with the canonical algebra

\begin{align}
\{\chi_a, \bar{\chi}_b\} &= \{\Pi_a, \bar{\Pi}_b\} = \{\chi_a, \Pi_b\} = \{\bar{\chi}_a, \Pi_b\} = 0 , \\
\{\chi_a, \Pi_b\} &= \delta_{ab} , \quad \{\bar{\chi}_a, \Pi_b\} = \delta_{ab} .
\end{align}

(4.14)

(4.15)

Thus, the fundamental Dirac brackets are

\[ \{x_a, p_b\}_D = \frac{1}{2} \delta_{ab} , \quad \{y_a, \pi_b\}_D = \frac{1}{2} \delta_{ab} . \]

(4.16)

After the introduction of these new brackets, one can treat all constraints in the strong sense. Letting the constraint functions vanish, the system with Hamiltonian (4.1) reduces to the following one

\[ H_{\text{red}} = \frac{1}{2} \sum_{a=1}^N p_a^2 + \frac{1}{2} \sum_{a \neq b}^N k_{ab}^2 + \frac{g^2}{2} \sum_{a \neq b}^N x_a^2 x_b^2 , \]

(4.17)

where

\[ k_{ab}^2 = \frac{1}{(x_a + x_b)^2} + \frac{1}{(x_a - x_b)^2} \]

(4.18)

Expression (4.17) for $N = 6$ coincides with the Hamiltonian of the $SU(2)$ Yang-Mills mechanics after taking into account that after projection onto the constraint shell (CS) (4.12)-(4.13), the potential (4.3) reduces to the potential of Yang-Mills mechanics

\[ V^{(6)}(x_1, \cdots, x_6)_{\mid \text{CS}} = \frac{1}{2} (x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2) . \]

(4.19)

V. LAX PAIR REPRESENTATION FOR YANG-MILLS MECHANICS IN ZERO COUPLING LIMIT

The conventional perturbative scheme of non-Abelian gauge theories starts with the zero approximation of the free theory. However, the limit of the zero coupling constant is not quite trivial. If the coupling constant in the initial Yang-Mills action vanish, the non-Abelian gauge symmetry reduces to the $U(1) \times U(1) \times U(1)$ symmetry. In this section, we shall discuss this free theory limit for the case of the unconstrained Yang-Mills mechanics. The solution of the corresponding zero coupling limit of the Yang-Mills mechanics in the form of a Lax representation will be given. The relation between (4.1) and (4.17) allows one to construct the Lax pair for the free part of the Hamiltonian (4.17) ($g = 0$) using the known Lax pair for the Euler-Calogero-Moser system (4.1) without an external potential term ($g = 0$).

According to the work of S.Wojciechowski [3], the Lax pair for the system with Hamiltonian

\[ H_{\text{ECM}} = \frac{1}{2} \sum_{a=1}^N p_a^2 + \frac{1}{2} \sum_{a \neq b}^N \frac{l_{ab}^2}{(x_a - x_b)^2} \]

(5.1)
The equations of motion for the free Calogero-Moser model (namely where the matrix $L_{ab} = p_a \delta_{ab} - (1 - \delta_{ab}) \frac{l_{ab}}{x_a - x_b}$, (5.2)

$$A_{ab} = (1 - \delta_{ab}) \frac{l_{ab}}{(x_a - x_b)^2}.$$ (5.3)

and the equations of motion in Lax form are

$$\hat{L} = [A, L],$$ (5.4)

$$\hat{l} = [A, l],$$ (5.5)

where the matrix $(l)_{ab} = l_{ab}$.

The introduction of Dirac brackets allows one to use the Lax pair of higher dimensional Euler-Calogero-Moser model (namely $A_b$) for the construction of Lax pairs $(L_{YMM}, A_{YMM})$ of free Yang-Mills mechanics by performing the projection onto the constraint shell \((1.13, 1.13)\)

$$L_{6 \times 6}^{ECM} |_{CS} = L_{YMM}, \quad A_{6 \times 6}^{ECM} |_{CS} = A_{YMM}. \quad (5.6)$$

Thus, the explicit form of the Lax pair matrices for the free $SU(2)$ Yang-Mills mechanics is given by the following $6 \times 6$ matrices

$$L_{YMM} = \begin{pmatrix}
\frac{l_{12}}{x_1 - x_2} & -\frac{l_{13}}{x_1 - x_3} & -\frac{l_{13}}{x_2 - x_3} & \frac{l_{12}}{x_1 + x_2} & \frac{l_{12}}{x_1 + x_3} & 0 \\
\frac{l_{12}}{x_1 - x_2} & 0 & -\frac{l_{12}}{x_2 - x_3} & 0 & -\frac{l_{12}}{x_1 + x_2} & -\frac{l_{12}}{x_1 + x_3} \\
\frac{l_{12}}{x_1 + x_3} & -\frac{l_{12}}{x_2 + x_3} & 0 & -\frac{l_{12}}{x_1 + x_3} & \frac{l_{12}}{x_1 + x_2} & -\frac{l_{12}}{x_1 + x_3} \\
-x_1 & -x_2 & -x_3 & 0 & 0 & 0 \\
x_1 & x_2 & x_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$ (5.7)

and

$$A_{YMM} = \begin{pmatrix}
\frac{l_{12}}{x_1 - x_2} & \frac{l_{13}}{x_1 - x_3} & \frac{l_{13}}{x_2 - x_3} & \frac{l_{12}}{x_1 + x_2} & \frac{l_{12}}{x_1 + x_3} & 0 \\
\frac{l_{12}}{x_1 - x_2} & 0 & -\frac{l_{12}}{x_2 - x_3} & 0 & -\frac{l_{12}}{x_1 + x_2} & -\frac{l_{12}}{x_1 + x_3} \\
\frac{l_{12}}{x_1 + x_3} & -\frac{l_{12}}{x_2 + x_3} & 0 & -\frac{l_{12}}{x_1 + x_3} & \frac{l_{12}}{x_1 + x_2} & -\frac{l_{12}}{x_1 + x_3} \\
-x_1 & -x_2 & -x_3 & 0 & 0 & 0 \\
x_1 & x_2 & x_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$ (5.8)

The equations of motion for the $SU(2)$ Yang-Mills mechanics in the zero constant coupling limit read in a Lax form as

$$\hat{L}_{YMM} = [A_{YMM}, L_{YMM}],$$ (5.9)

$$\hat{l}_{YMM} = [A_{YMM}, l_{YMM}],$$ (5.10)

where the matrix $l_{YMM}$ is

$$l_{YMM} = \begin{pmatrix}
0 & l_{12} & l_{13} & -l_{13} & -l_{12} & 0 \\
-l_{12} & 0 & l_{23} & -l_{23} & 0 & l_{12} \\
-l_{13} & -l_{23} & 0 & 0 & l_{23} & l_{13} \\
l_{13} & l_{23} & 0 & 0 & -l_{23} & -l_{13} \\
l_{12} & 0 & -l_{23} & l_{23} & 0 & -l_{12} \\
0 & -l_{12} & -l_{13} & l_{13} & l_{12} & 0
\end{pmatrix}. \quad (5.11)$$
We are grateful to V.I. Inozemtsev, M.D. Mateev and H-P. Pavel for discussions. We would like to thank E. Langmann for useful comments on the relations between 2-dimensional Yang-Mills theory on a cylinder and Calogero-Moser systems. B.G. Dimitrov is also acknowledged for reading of the manuscript. The work of A.M.K. was supported in part by the Russian Foundation for Basic Research under grant No. 96-01-00101.

[1] J. Gibbons and T. Hermsen, A generalizations of the Calogero-Moser system, Physica D 11 (1984) 337-348.

[2] S. Wojciechowski, An integrable marriage of the Euler equations with the Calogero-Moser system, Phys. Lett. A 111 (1985) 101-103.

[3] S.G. Matinyan, G.K. Savvidy, and N.G. Ter-Arutyunyan-Savvidy, Classical Yang-Mills mechanics. Non-linear colour oscillations, Sov. Phys. JETP 53 (1981) 421-429.

[4] H.M. Asatryan and G.K. Savvidy, Configuration manifold of Yang-Mills classical mechanics, Phys. Lett. A 99 (1983) 290-292;
M.A. Soloviev, On the geometry of classical mechanics with nonabelian gauge symmetry, Teor. Mat. Fyz. 73 (1987) 3-15;
M.J. Gotay, Reduction of homogeneous Yang-Mills fields, J.Geom. Phys. 6 (1989) 349-365;
B. Dahmen and B. Raabe, Unconstrained SU(2) and SU(3) Yang-Mills classical mechanics, Nucl. Phys. B 384 (1992) 352-380.

[5] S.A. Gogilidze, A.M. Khvedelidze, D.M. Mladenov, and H.-P. Pavel, Hamiltonian reduction of SU(2) Dirac-Yang-Mills mechanics, Phys. Rev. D 57 (1998) 7488-7500.

[6] F. Calogero, Solution of a three-body problem in one dimension, J. Math. Phys. 10 (1969) 2191-2196;
F. Calogero, Ground state of a one-dimensional N-body system, J. Math. Phys. 10 (1969) 2197-2200;
F. Calogero, Solution of the one-dimensional n-body problem with quadratic and/or inversely quadratic pair potentials, J. Math. Phys. 12 (1972) 419-436;
F. Calogero and C. Marchioro, Exact solution of a one-dimensional three-body scattering problem with two-body and/or three-body inverse square potential, J. Math. Phys. 15 (1974) 1425-1430;
F. Calogero, Exactly solvable one-dimensional many body problems, Let. Nuovo Cim. 13 (1975) 411-416.

[7] B. Sutherland, Exact results for a quantum many-body problem in one dimension, Phys. Rev. A 4 (1971) 2019-2021;
B. Sutherland, Exact results for a quantum many-body problem in one dimension. II, Phys. Rev. A 5 (1972) 1372-1376.

[8] J. Moser, Three integrable Hamiltonian systems connected with isospectral deformations, Adv. Math. 16 (1975) 197-220.

[9] A.M. Perelomov, Completely integrable classical systems related to semisimple Lie algebras, Lett. Math. Phys. 1, (1977) 531-540.

[10] M.A. Olshanetsky and A.M. Perelomov, Classical integrable finite-dimensional systems related to Lie algebras, Phys. Rep. 71, (1981) 313-400;
Referencing:

M.A. Olshanetsky and A.M. Perelomov, Quantum integrable systems related to Lie algebras, Phys. Rep. 94, (1983) 313-404.

A.M. Perelomov, *Integrable Systems of Classical Mechanics and Lie algebras*, Vol. I, (Birkhäuser, Boston, 1990).

E. Langmann and G. Semenoff, Gauge theories on a cylinder, Phys. Lett. B 296 (1992) 117-120; A. Gorsky and N. Nekrasov, Hamiltonian systems of Calogero type, and two-dimensional Yang-Mills theory, Nucl. Phys. B 414 (1994) 213-238; E. Langmann, M. Salmhofer, and A. Kovner, Consistent axial-like gauge fixing on hypertori, Mod. Phys. Lett. A 9 (1994) 2913-2926; J.A. Minahan and A.P. Polychronakos, Interacting fermion systems from two-dimensional QCD, Phys. Lett. B 326 (1994) 288-294; A. Niemi and P. Pasanen, On the infrared limit of two-dimensional QCD, Phys. Lett. B 323 (1994) 46-52; J. Blom and E. Langmann, Novel integrable spin-particle models from gauge theories on a cylinder, Phys. Lett. B 429 (1998) 336-342.

G.W. Gibbons and P.K. Townsend, Black holes and Calogero models, hep-th/9812034.

F.D.M. Haldane, Exact Jastrow-Gutzwiller resonating-valence-bond ground state of the spin-$\frac{1}{2}$ antiferromagnetic Heisenberg chain with $1/r^2$ exchange, Phys. Rev. Lett 60 (1988) 635-638; B.S. Shastry, Exact solution of an $S = \frac{1}{2}$ Heisenberg antiferromagnetic chain with long ranged interactions, Phys. Rev. Lett 60 (1988) 639-642; A.P. Polychronakos, Lattice integrable systems of Haldane-Shastry type, Phys. Rev. Lett 70 (1993) 2329-2331; A.P. Polychronakos, Exact spectrum of $SU(N)$ spin chain with inverse-square exchange, Nucl.Phys.B 419 (1994) 553-566.

A.P. Polychronakos, Nonrelativistic bosonization and fractional statistics, Nucl. Phys. B 324 (1989) 597-622; A.P. Polychronakos, Generalized statistics in one dimension, Presented at Les Houches Summer School of Theoretical Physics, Les Houches, France, 7-31 July 1998, hep-th/9902157; L. Brink, T.H. Hansson, S. Konstein, and M.A. Vasiliev, The Calogero model — anyonic representation, fermionic extension and supersymmetry, Nucl. Phys. B 401 (1993) 591-612; S.E. Konstein and M.A. Vasiliev, Supertraces on the algebras of observables of the rational Calogero model with harmonic potential, J. Math. Phys. 37, (1996) 2872-2891; J.M. Leinaas, Generalized statistics and the algebra of observables, hep-th/9611167.

M.A. Vasiliev, More on equations of motion for interacting massless fields of all spins in $(3 + 1)$ dimensions, Phys. Lett. B 285 (1992) 225-234; M.A. Vasiliev, Unfolded representation for relativistic equations in (2+1) anti-de Sitter space, Class. Quant. Grav. 11 (1994) 649-664; M.A. Vasiliev, Higher spin gauge theories in four-dimensions, three-dimensions, and two-dimensions, Int. J. Mod. Phys. D 5 (1996) 763-797, hep-th/9611024.

M. Kuś, F. Haake, D. Zaitsev, and A. Huckleberry, Level dynamics for conservative and quantum systems, J. Phys. A: Math. Gen. 30 (1997) 8635-8651.

N. Kawakami, Novel hierarchy of the $SU(N)$ electron models and edge states of fractional Hall effect, Phys. Rev. Lett. 71 (1993) 275-278; O. Azuma and S. Iso, Explicit relation of quantum Hall effect and Calogero-Sutherland model, Phys. Lett. B 331 (1994) 107-113, hep-th/9312001.

I.M. Krichever, O. Babelon, E. Billey, and M. Tallon, Spin generalization of the Calogero-Moser system.
[20] O. Babelon and M. Talon, The symplectic structure of the spin Calogero-Moser model, Phys.Lett. A 236 (1997) 462-468.

[21] I.M. Krichever and D.H. Phong, On the integrable geometry of soliton equations and the $N = 2$ supersymmetric gauge theories, hep-th/9604199.

[22] N. Seiberg and E. Witten, Electro-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory, Nucl. Phys. B 426 (1994) 19-52, hep-th/9407087.

[23] R. Donagi and E. Witten, Supersymmetric Yang-Mills and integrable systems, Nucl. Phys. 460 (1996) 288-334, hep-th/9510101.

[24] E. D’Hoker and D.H. Phong, Calogero-Moser Lax pairs with spectral parameter for general Lie algebras, Nucl. Phys. B 530 (1998) 537-610, hep-th/9804124.

[25] E. D’Hoker and D.H. Phong, Lectures on supersymmetric Yang-Mills theory and integrable systems, hep-th/991271; based on lectures delivered at "Integrability, the Seiberg-Witten and Whitham equations", Edinburg, September 1998, "Workshop on Gauge Theory and Integrable Models", Kyoto, January 1999, "Supersymmetry and Unified Theory of Elementary Particles", Kyoto, February 1999, hep-th/9903068.

[26] A.P. Polychronakos, Generalized Calogero models through reductions by discrete symmetries, Nucl. Phys. B 543 (1999) 485-498, hep-th/9810211.