CHARACTERIZING WEAK SOLUTIONS FOR VECTOR OPTIMIZATION PROBLEMS

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Abstract. This paper provides characterizations of the weak solutions of optimization problems where a given vector function \( F \), from a decision space \( X \) to an objective space \( Y \), is “minimized” on the set of elements \( x \in C \) (where \( C \subset X \) is a given nonempty constraint set), satisfying \( G(x) \leq S 0 \), where \( G \) is another given vector function from \( X \) to a constraint space \( Z \) with positive cone \( S \). The three spaces \( X, Y, \) and \( Z \) are locally convex Hausdorff topological vector spaces, with \( Y \) and \( Z \) partially ordered by two convex cones \( K \) and \( S \), respectively, and enlarged with a greatest and a smallest element. In order to get suitable versions of the Farkas lemma allowing to obtain optimality conditions expressed in terms of the data, the triplet \( (F, G, C) \), we use non-asymptotic representations of the \( K \)-\epigraph of the conjugate function of \( F + I_A \), where \( I_A \) denotes the indicator function of the feasible set \( A \), that is, the function associating the zero vector of \( Y \) to any element of \( A \) and the greatest element of \( Y \) to any element of \( X \setminus A \).

Key words. Vector optimization, weak minimal solutions, qualification conditions, Farkas-type results for vector functions, strong duality

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1. Introduction. This paper analyzes vector optimization problems of the form

\[
(VOP) \quad \text{WMin} \{ F(x) : x \in C, \ G(x) \leq S 0 \},
\]

where \( \text{WMin} \) stands for the task consisting of determining the weakly minimal elements (concept defined in Section 2) of some subset of the objective space \( Y \), which is equipped with a partial ordering \( \leq_K \) induced by a convex cone \( K \), the constraint set \( C \) is a given subset of the decision space \( X \), \( \leq_S \) denotes the partial ordering induced in the constraint space \( Z \) by a convex cone \( S \), \( 0_Z \) is the zero vector in \( Z \), \( F: X \to Y \cup \{ +\infty_Y \} \) and \( G: X \to Z \cup \{ +\infty_Z \} \), with \( +\infty_Y \) and \( +\infty_Z \) denoting ”greatest elements” aggregated to \( Y \) and \( Z \), respectively. We assume that \( X, Y, Z \) are locally convex Hausdorff topological vector spaces. The main purpose of this paper is to give conditions for a given \( \pi \in A \) to satisfy \( F(\pi) \in \text{WMin} F(A) \), where \( A := \{ x \in C, \ G(x) \leq S 0 \} \) is the feasible set of \( (VOP) \). Moreover, we are particularly interested in conditions which are expressed in terms of the data, that is, in conditions only involving mathematical objects related to the triplet \( (F, G, C) \).

Different concepts of solutions in vector optimization have been proposed, each one having its own set of advantages and disadvantages. For instance, regarding multiobjective optimization (when \( Y = \mathbb{R}^m \) and \( K = \mathbb{R}^m_+ \)), it is usually admitted that weakly efficient solutions, efficient solutions, and super efficient solutions are preferable from the computational, practical, and stability perspectives, respectively (see, e.g., [1], [2], [4], [13], [14], [16], and references therein). On the other hand, weak orders allow to apply the elegant conjugate duality machinery (\( \mathbb{R} \)). So, computability
A.

The feasible set abstract versions of the corresponding results in [9, Sections 4 and 5], where from the optimality conditions. The results in Sections 4 and 5 can be seen as non-asymptotic representations of epi (f + iA) under some optimality conditions. A strong duality theorem for (VOP) is obtained through the introduction of qualification conditions on (f, G, C) allowing to represent epi (f + iA) in terms of (f, G, C) (see, e.g., \[3, \text{Theorem 8.2}\]).

In the same vein, in this paper we reformulate (VOP) as an unconstrained vector optimization problem with problem as an unconstrained scalar one with objective function \( f + i_A \), where \( i_A \) denotes the indicator function of the feasible set \( A \), i.e., \( i_A (x) = 0 \) if \( x \in A \) and \( i_A (x) = +\infty \) if \( x \in X \setminus A \). Optimality conditions involving \( A \) (as the classical one that the null functional is a subgradient of \( f + i_A \) at \( \pi \)) are considered too abstract for practical purposes as a manageable description of \( A \) is seldom available. The epigraph of the Fenchel-Moreau conjugate of \( f + i_A \), denoted by epi \((f + i_A)^*\), plays an important role in order to obtain checkable optimality conditions for (SOP), that is, conditions which are expressed in terms of the data, the triple \((f, G, C)\). This is usually done through the introduction of qualification conditions on \((f, G, C)\) (called constraint qualifications when they only involve \( G \) and \( C \)) allowing to represent epi \((f + i_A)^*\) in terms of \((f, G, C)\) (see, e.g., \[3, \text{Theorem 8.2}\]).

In the same vein, in this paper we reformulate (VOP) as an unconstrained vector optimization problem with problem as an unconstrained scalar one with objective function \( f + I_A \), where \( I_A \) denotes the indicator function of the feasible set \( A \), i.e., \( I_A (x) = 0 \) if \( x \in A \) and \( I_A (x) = +\infty \) if \( x \in X \setminus A \). After the introductory Section 2, Section 3 provides different representations of the \( K \)-epigraph of the conjugate of \( f + I_A \), say epi \(_K (F + I_A)^*\) (concept to be introduced in Section 2) in terms of \((f, G, C)\). These representations are called asymptotic when they involve a limiting process (typically, in the form of closure of some set depending on \( F, G, \) and \( C \)) and non-asymptotic otherwise. The main results in this paper are Theorems 3.7 and 3.14 which are alternative extensions of \[3, \text{Theorem 8.2}\] to the vector framework providing non-asymptotic representations of epi \(_K (F + I_A)^*\) under the same qualification conditions (Theorems 3.15 and 3.16). We provide in Section 4 two kinds of Farkas-type results oriented to (VOP): those which characterize the inclusion of \( A \) in a second subset \( B \) of \( X \) depending on \( F \) under certain set of assumptions \( P \) are said to be Farkas lemmas while those establishing the equivalence of \( P \) with some characterization of the inclusion \( A \subset B \) are called characterizations of Farkas lemma. Similarly, the final Section 5 provides optimality conditions establishing characterizations of the weakly minimal solutions to (VOP) under \( P \) and characterizations of optimality conditions asserting the equivalence of \( P \) with some optimality conditions. A strong duality theorem for (VOP) is obtained from the optimality conditions. The results in Sections 4 and 5 can be seen as non-asymptotic versions of the corresponding results in \[9, \text{Sections 4 and 5}\], where \( P \) involves the feasible set \( A \).

2. Preliminaries. Throughout the paper \( X, Y, Z \) are three given locally convex Hausdorff topological vector spaces with topological dual spaces denoted by \( X^*, Y^*, Z^* \), respectively. The only topology we consider on dual spaces is the weak*-topology.

For a set \( U \subset X \), we denote by cl \( U \), conv \( U \), cl conv \( U \), lin \( U \), int \( U \), ri \( U \), and sqri \( U \) the closure, the convex hull, the closed convex hull, the linear hull, the interior, the relative interior, and the strong quasi-relative interior of \( U \), respectively. Note that
cl conv $U = \text{cl}(\text{conv} U)$. The null vector in $X$ is denoted by $0_X$ and the dimension of a linear subspace $U$ of $X$ by $\dim U$. Given two subsets $A$ and $B$ of a topological space, one says that $A$ is closed regarding $B$ if $B \cap \text{cl} A = B \cap A$ (see, e.g., [3, Section 9]).

We assume that $K$ is a given closed, pointed, convex cone in $Y$ with nonempty interior, i.e., $\text{int} K \neq \emptyset$. A weak ordering in $Y$, "\(<_K\)" is defined as follows: for $y_1, y_2 \in Y$,

$$y_1 <_K y_2 \iff y_1 - y_2 \in - \text{int} K,$$

or equivalently, $y_1 \not<_K y_2$ if and only if $y_1 - y_2 \notin - \text{int} K$.

We enlarge $Y$ by attaching a greatest element $+\infty_Y$ and a smallest element $-\infty_Y$ with respect to $<_K$, which do not belong to $Y$, and we denote $Y^\bullet := Y \cup \{-\infty_Y, +\infty_Y\}$. By convention, $-\infty_Y <_K y <_K +\infty_Y$ for any $y \in Y$. We also assume by convention that

$$-(+\infty_Y) = -\infty_Y, \quad -(\infty_Y) = +\infty_Y;$$

$$(+\infty_Y) + y = y + (+\infty_Y) = +\infty_Y, \quad \forall y \in Y \cup \{+\infty_Y\},$$

$$(\infty_Y) + y = y + (-\infty_Y) = -\infty_Y, \quad \forall y \in Y \cup \{-\infty_Y\}.$$  \hspace{1cm} (2.1)

The sums $(-\infty_Y) + (+\infty_Y)$ and $(+\infty_Y) + (-\infty_Y)$ are not considered in this paper.

Notice that in the space $Y$ the cone $K$ also generates another order $\leq_K$ defined as, for $y_1, y_2 \in Y$,

$$y_1 \leq_K y_2 \text{ if and only if } y_2 \in y_1 + K.$$  

It is obvious that the order $\leq_K$ also can be extended to $Y^\bullet$ with the convention that $-\infty_Y \leq_K y \leq_K +\infty_Y$ for any $y \in Y$ together with the others in (2.1).

We now recall the following basic definitions regarding the subsets of $Y^\bullet$ (see, e.g., [3, 4, Definition 7.4.1], [12], [14], [15], [18], etc.):

**Definition 2.1.** Consider a set $M$ such that $\emptyset \neq M \subset Y^\bullet$.

1. An element $v \in Y^\bullet$ is said to be a weakly infimal element of $M$ if for all $v \in M$ we have $v \not<_K v$ and if for any $\bar{v} \in Y^\bullet$ such that $\bar{v} <_K v$ there exists some $v \in M$ satisfying $v <_K \bar{v}$. The set of all weakly infimal elements of $M$ is denoted by $\text{WInf} M$ and is called the weak infimum of $M$.

2. An element $\bar{v} \in Y^\bullet$ is said to be a weakly supremal element of $M$ if for all $v \in M$ we have $v \not<_K \bar{v}$ and if for any $\bar{v} \in Y^\bullet$ such that $\bar{v} <_K v$ there exists some $\bar{v} \in M$ satisfying $\bar{v} <_K v$. The set of all weakly supremal elements of $M$ is denoted by $\text{WSup} M$ and is called the weak supremum of $M$.

3. The weak minimum of $M$ is the set $\text{WMin} M = M \cap \text{WInf} M$ and its elements are the weakly minimal elements of $M$.

4. The weak maximum of $M$ is the set $\text{WMax} M = M \cap \text{WSup} M$ and its elements are the weakly maximal elements of $M$.

5. An element $\overline{v} \in M$ is called a strongly maximal element of $M$ if it holds $v \leq_K \overline{v}$ for all $v \in M$. The set of all strongly maximal elements of $M$ is denoted by $\text{SMax} M$.

Observe that, if $M \subset Y$, then $\overline{v} \in \text{SMax} M$ if and only if $M \subset \overline{v} - K$. Thus, if $M \subset Y$ then

$$\text{SMax} M = \{v \in M : M \subset \overline{v} - K\}. \hspace{1cm} (2.2)$$

Moreover, in this case, if $K$ is a pointed cone and $\text{SMax} M \neq \emptyset$ then $\text{SMax} M$ is a singleton, i.e., the strongly maximum element of the set $M$ in this case, if exists, will be unique. In such a case, we write $\overline{v} = \text{SMax} M$ instead of $\text{SMax} M = \{\overline{v}\}$. 

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The next elementary properties will be used in the sequel:

- According to the definition above,
  \[ +\infty_Y \in \text{Sup} \implies \text{Sup} = \{ +\infty_Y \} \]
  \[ \forall \hat{v} \in Y, \exists v \in M : \hat{v} <_K v. \] (2.3)

- If \( \emptyset \neq M \subset Y \) and \( \text{Sup} M \neq \{ +\infty_Y \} \), by [9] Proposition 2.7(i), one has
  \[ \text{Sup} M = \text{cl}(M - \text{int } K) \setminus (M - \text{int } K). \] (2.4)

- If \( M \neq \emptyset \) and \( +\infty_Y \notin M \neq \{ -\infty_Y \} \), then, by [9] Proposition 2.7(ii), one gets
  \[ \text{Sup} M = M \setminus (M - \text{int } K). \] (2.5)

- If \( \emptyset \neq M \subset Y \) and \( \text{Sup} M \neq \{ +\infty_Y \} \), from [18] Proposition 2.4, one gets
  \[ \text{Sup} M - \text{int } K = M - \text{int } K. \] (2.6)

We shall also use the next lemma.

**Lemma 2.2.** Given \( \emptyset \neq M \subset Y^* \), the following statements hold true:

(i) If \( +\infty_Y \notin M \) and \( M \cap \text{int } K = \emptyset \), then \( \text{Sup} M \neq \{ +\infty_Y \} \).

(ii) If there exists \( v_0 \in M \cap \text{int } K \) such that \( \lambda v_0 \in M \) for all \( \lambda > 0 \), then \( \text{Sup} M = \{ +\infty_Y \} \).

(iii) If \( M \subset -K \) and \( 0_Y \in M \) then \( \text{Sup} M = \text{Sup}(-K) \).

**Proof.** (i) Assume that \( M \cap \text{int } K = \emptyset \). Then, \( 0_Y \not\prec_K v \) for all \( v \in M \) and it follows from (2.3) that \( \text{Sup} M \neq \{ +\infty_Y \} \).

(ii) Assume that there is \( v_0 \in M \cap \text{int } K \) such that \( \lambda v_0 \in M \) for all \( \lambda > 0 \). If \( \text{Sup} M \neq \{ +\infty_Y \} \), then by (2.3), there exists \( \hat{v} \in Y \) such that \( \hat{v} \not\prec_K v \) for any \( v \in M \). We get

\[
\hat{v} \not\prec_K \lambda v_0, \forall \lambda > 0 \implies \lambda v_0 - \hat{v} \not\in \text{int } K, \forall \lambda > 0
\]

\[
\implies v_0 - \frac{1}{\lambda} \hat{v} \not\in \text{int } K, \forall \lambda > 0.
\] (2.7)

On the other hand, because of the continuity of the map \( t \mapsto v_0 - t\hat{v} \) at 0 and \( v_0 \in \text{int } K \), there exists \( \epsilon > 0 \) such that

\[
i \in ]-\epsilon, \epsilon[ \implies v_0 - t\hat{v} \in \text{int } K,
\]

which contradicts (2.7) and the proof is complete.

(iii) Assume that \( M \subset -K \) and \( 0_Y \in M \). Then \( M - \text{int } K = -\text{int } K \). Indeed, \( M - \text{int } K \subset -K - \text{int } K = -\text{int } K \). Since \( 0_Y \in M \), we also have \( -\text{int } K \subset M - \text{int } K \). On the other hand, because \( K \) is a pointed cone, \( M \subset -K \) yields \( M \cap \text{int } K = \emptyset \). So we get from (i) that \( \text{Sup} M \neq \{ +\infty \} \). According to (2.4),

\[ \text{Sup} M = \text{cl}((M - \text{int } K) \setminus (M - \text{int } K) = \text{cl}(-\text{int } K) \setminus (-\text{int } K) = \text{Sup}(-K) \]

and we are done. \( \Box \)

**Lemma 2.3.** Assume \( \emptyset \neq M \subset Y \), \( \text{Sup} M \subset Y \), and there exists \( v_0 \in Y \setminus (-K) \) such that \( \lambda v_0 \in M \) for all \( \lambda > 0 \). Then \( \text{Max}(\text{Sup} M) = \emptyset \).
Proof. Let us suppose by contradiction that \( \text{SM}ax(\text{WSup}M) \neq \emptyset \) and \( \bar{v} = \text{SM}ax(\text{WSup}M) \). Since \( \text{WSup}M \subset Y \), one has \( \text{WSup}M \subset \bar{v} - K \) (see (2.2)). It follows from (2.4) and (2.6) that

\[
M \subset \text{cl}(M - \text{int}K) = [\text{cl}(M - \text{int}K) \setminus (M - \text{int}K)] \cup (M - \text{int}K) = \text{WSup}M \cup (\text{WSup}M - \text{int}K) \subset \text{WSup}M - K,
\]

and consequently, \( M \subset \bar{v} - K = \bar{v} - K \). Thus, from the assumption that \( \lambda v_0 \in M \) for all \( \lambda > 0 \), one has

\[
\lambda v_0 \in \bar{v} - K, \, \forall \lambda > 0 \implies v_0 - \frac{1}{\lambda} \bar{v} \in -K, \, \forall \lambda > 0. \quad (2.8)
\]

On other hand, because \( v_0 \in Y \setminus (-K) \), the set \( -K \) is closed, and the map \( t \mapsto v_0 - t\bar{v} \) is continuous at \( t = 0 \), we can find \( \epsilon > 0 \) such that

\[
t \in [\epsilon, 0] \implies v_0 - t\bar{v} \in Y \setminus (-K),
\]

which contradicts (2.8) and the proof is complete. \( \square \)

We denote by \( \mathcal{L}(X, Y) \) the space of linear continuous mappings from \( X \) to \( Y \), and by \( 0_L \in \mathcal{L}(X, Y) \) the zero mapping defined by \( 0_L(x) = 0_Y \) for all \( x \in X \). Obviously, \( \mathcal{L}(X, Y) = X^* \) whenever \( Y = \mathbb{R} \). We consider \( \mathcal{L}(X, Y) \) equipped with the so-called weak topology, that is, the one defined by the pointwise convergence. In other words, given a net \( (L_i)_{i \in I} \subset \mathcal{L}(X, Y) \) and \( L \in \mathcal{L}(X, Y) \), \( L_i \to L \) means that \( L_i(x) \to L(x) \) in \( Y \) for all \( x \in X \).

Given a vector-valued mapping \( F : X \to Y^* \), the domain of \( F \) is defined by

\[
\text{dom} F := \{ x \in X : F(x) \neq +\infty_Y \}
\]

and \( F \) is proper when \( \text{dom} F \neq \emptyset \) and \( -\infty_Y \notin F(X) \). The \( K \)-epigraph of \( F \), denoted by \( \text{epi}_K F \), is defined by

\[
\text{epi}_K F := \{(x, y) \in X \times Y : y \in F(x) + K\}.
\]

We say that \( F \) is \( K \)-convex (\( K \)-epi closed) if \( \text{epi}_K F \) is a convex set (a closed set in the product space, respectively). If \( F \) is \( K \)-convex, it is evident that \( \text{dom} F \) is a convex set in \( X \).

**Definition 2.4.** The set-valued map \( F^* : \mathcal{L}(X, Y) \rightrightarrows Y^* \) defined by

\[
F^*(L) := \text{WSup} \{ L(x) - F(x) : x \in X \} = \text{WSup} \{ L - F)(X) \},
\]

is called the conjugate map of \( F \). The domain and the (strong) ”max-domain” of \( F^* \) are defined as

\[
\text{dom} F^* := \{ L \in \mathcal{L}(X, Y) : F^*(L) \neq +\infty_Y \},
\]

and

\[
\text{dom}_M F^* := \{ L \in \mathcal{L}(X, Y) : F^*(L) \subset Y \text{ and } \text{SM}ax F^*(L) \neq \emptyset \},
\]

respectively, while the \( K \)-epigraph of \( F^* \) is

\[
\text{epi}_K F^* := \{(L, y) \in \mathcal{L}(X, Y) \times Y : y \in F^*(L) + K\}.
\]
Let $S$ be a nonempty closed and convex cone in $Z$ and $\leq_s$ be the ordering on $Z$ induced by the cone $S$, i.e.,

$$z_1 \leq_s z_2 \text{ if and only if } z_2 - z_1 \in S.$$ (2.9)

We also enlarge $Z$ by attaching a greatest element $+\infty_Z$ and a smallest element $-\infty_Z$ (with respect to $\leq_s$) which do not belong to $Z$, and define $Z^* := Z \cup \{-\infty_Z, +\infty_Z\}$. In $Z^*$ we adopt the same conventions as in (2.1).

For $T \in \mathcal{L}(Z, Y)$ and $G : X \to Z \cup \{+\infty_Z\}$, we define the composite function $T \circ G : X \to Y^*$ as follows:

$$(T \circ G)(x) := \begin{cases} T(G(x)), & \text{if } G(x) \in Z, \\ +\infty_Y, & \text{if } G(x) = +\infty_Z. \end{cases}$$

Recall that $S$ is a nonempty closed and convex cone in $Z$. Let us set

$$\mathcal{L}_+(S, K) := \{T \in \mathcal{L}(Z, Y) : T(S) \subset K\}$$

and

$$\mathcal{L}_+^w(S, K) := \{T \in \mathcal{L}(Z, Y) : T(S) \cap (-\text{int } K) = \emptyset\}.$$ (2.10)

It is clear that $\mathcal{L}_+(S, K) \subset \mathcal{L}_+^w(S, K)$. Indeed, for any $T \in \mathcal{L}_+(S, K)$, one has $T(S) \subset K$. So, $T(S) \cap (-\text{int } K) = \emptyset$ (as $K$ is pointed cone) and hence, $T \in \mathcal{L}_+^w(S, K)$. However, the inclusion $\mathcal{L}_+(S, K) \subset \mathcal{L}_+^w(S, K)$ is generally strict (see Example 2.6 below).

It is worth noticing that, when $Y = \mathbb{R}$ and $K = \mathbb{R}_+$, the conjugate, the domain and the $K$-epigraph of $f : X \to \mathbb{R} \cup \{+\infty\}$ are nothing else than the ordinary conjugate, the domain, and the epigraph of the scalar function $f$, i.e.,

$$f^*(x^*) := \sup_{x \in X} \langle x^*, x \rangle - f(x), \forall x^* \in X^*,$$

$$\text{dom } f := \{x \in X : f(x) \neq +\infty\},$$

and

$$\text{epi } f := \{(x, r) \in X \times \mathbb{R} : x \in \text{dom } f, f(x) \leq r\},$$

respectively. Moreover, since

$$T(S) \cap (-\text{int } \mathbb{R}_+) = \emptyset \iff T(S) \subset \mathbb{R}_+,$$

we have

$$\mathcal{L}_+^w(S, \mathbb{R}_+) = \mathcal{L}_+(S, \mathbb{R}_+) = S^+ := \{z^* \in Z^* : \langle z^*, s \rangle \geq 0 \text{ for all } s \in S\},$$

in other words, the (positive) dual cone $S^+$ of $S$ in the sense of convex analysis.

In order to obtain a suitable interpretation of $\mathcal{L}_+^w(S, K)$ we must extend the concept of indicator function from scalar to vector functions: the indicator map $I_D : X \to Y^*$ of a set $D \subset X$ is defined by

$$I_D(x) = \begin{cases} 0_Y, & \text{if } x \in D, \\ +\infty_Y, & \text{otherwise}. \end{cases}$$
In the case $Y = \mathbb{R}$, $I_D$ is the usual indicator function $i_D$.

**Lemma 2.5.** One has

$$\mathcal{L}^w_+(S, K) = \text{dom } I^*_S \text{ and } \mathcal{L}_+(S, K) = \text{dom}_M I^*_S.$$ 

**Proof.** Taking an arbitrary $T \in \mathcal{L}(Z, Y)$, one has

$$I^*_S(T) = \text{WSup}\{T(z) : z \in -S\} = \text{WSup} T(-S),$$

and so, $T \in \text{dom } I^*_S$ if and only if $\text{WSup } T(\mathcal{-S}) \neq \{+\infty_Y\}$. Two cases are possible.

If $T \in \mathcal{L}^w_+(S, K)$, $T(\mathcal{S}) \cap (\mathcal{-int} K) = \emptyset$, and consequently, $T(-\mathcal{S}) \cap \mathcal{int} K = \emptyset$. So, it follows from Lemma 2.4(i) that $\text{WSup } T(-\mathcal{S}) \neq \{+\infty_Y\}$.

If $T \in \mathcal{L}(X, Y) \setminus \mathcal{L}^w_+(S, K)$, there exists $v_0 \in T(\mathcal{S}) \cap (\mathcal{-int} K)$. Then, $-v_0 \in \text{int } K$ and $\lambda(-v_0) \in T(-\mathcal{S})$ for all $\lambda > 0$ because $S$ is a cone. So, by Lemma 2.4(ii), $\text{WSup } T(-\mathcal{S}) = \{+\infty_Y\}$. Consequently, $\text{dom } I^*_S = \mathcal{L}^w_+(S, K)$.

• Take an arbitrary $T \in \mathcal{L}_+(S, K)$. Then one has $T(\mathcal{S}) \subset K$, or equivalently, $T(-\mathcal{S}) \subset -K$. It is clear that $0_Y = T(0_Z) \in T(-\mathcal{S})$. According to Lemma 2.4(iii), $I^*_S(T) = \text{WSup } T(-\mathcal{S}) = \text{WSup} (-K)$. So, $\text{SMax } I^*_S(T) = \{0_Y\} \neq \emptyset$, and consequently, $T \in \text{dom}_M I^*_S$.

Now take an arbitrary $T \in \mathcal{L}(Z, Y) \setminus \mathcal{L}_+(S, K)$. One has $T(\mathcal{S}) \not\subset K$, or equivalently, there exists $s_0 \in -\mathcal{S}$ such that $T(s_0) \notin -K$. Thus, applying Lemma 2.4 with $M = T(-\mathcal{S})$ and $v_0 = T(s_0)$, we get that, if $\text{WSup } T(-\mathcal{S}) \subset Y$, then

$$\text{SMax } \{\text{WSup } T(-\mathcal{S})\} = \text{SMax } \{I^*_S(T)\} = \emptyset.$$

So, $T \notin \text{dom}_M I^*_S$ and we are done. □

We shall use the following simple example for illustrative purposes throughout the paper.

**Example 2.6.** Take $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}_{+}^2$, $S = \mathbb{R}_{+}$, $F : \mathbb{R} \longmapsto \mathbb{R}^2$ the null mapping, and $G : \mathbb{R} \longmapsto \mathbb{R}$ such that $G(x) = -x$ for all $x \in \mathbb{R}$. Then $\mathcal{L}(Z, Y) = \mathbb{R}^2$, $\mathcal{L}_+(S, K) = \mathbb{R}_+^2$, and $\mathcal{L}_0^+(S, K) = \{ (t_1, t_2) \in \mathbb{R}^2 : t_1 \geq 0 \vee t_2 \geq 0 \}$. Moreover, given $(\alpha, \beta) \in \mathbb{R}^2$, $F^*(\alpha, \beta) = \text{WSup } \{\mathbb{R} (\alpha, \beta)\} = \{ (-\mathbb{R}_+^2) \times \{0\} \} \cup \{ \{0\} \times (-\mathbb{R}_+) \} = \{ (+\infty\mathbb{R}^2) \}$ if $\alpha \beta > 0$, or otherwise.

Thus, $\text{epi}_K F^* = \bigcup_{i=1}^{4} N_i$, where

\begin{align*}
N_1 & = \{ (0, 0, y_1, y_2) : y_1 \geq 0 \vee y_2 \geq 0 \}, \\
N_2 & = \{ (\alpha, \beta, y_1, y_2) : \alpha \beta < 0 \wedge y_2 \geq \frac{L}{G}y_1 \}, \\
N_3 & = \{ (\alpha, 0, y_1, y_2) : \alpha \neq 0 \wedge y_2 \geq 0 \}, \\
N_4 & = \{ (0, \beta, y_1, y_2) : \beta \neq 0 \wedge y_1 \geq 0 \}.
\end{align*}

Observe that, given $(\alpha, \beta, y_1, y_2) \in \text{cl } N_2$, we have

\begin{align*}
& \alpha \beta < 0 \quad \Rightarrow \quad (\alpha, \beta, y_1, y_2) \in N_2, \\
& \alpha = 0 = \beta \quad \Rightarrow \quad (\alpha, \beta, y_1, y_2) \in N_1, \\
& \alpha < 0 = \beta \quad \Rightarrow \quad (\alpha, \beta, y_1, y_2) \in N_3, \\
& \alpha = 0 > \beta \quad \Rightarrow \quad (\alpha, \beta, y_1, y_2) \in N_4.
\end{align*}
so that $\text{cl} \cap N_2 \subset \text{epi}_K F^*$. Thus,

$$
\text{cl} \cap epi K F^* \subset \bigcup_{i=1}^{4} \text{cl} \cap N_i \subset \cap N_1 \cup \cap epi K F^* \cup (N_3 \cup N_1) \cup (N_4 \cup N_1) = \text{epi}_K F^*,
$$

showing that $\text{epi}_K F^*$ is closed. However, $\text{epi}_K F^*$ is not convex as its image by the projection mapping $(\alpha, \beta, y_1, y_2) \mapsto (\alpha, \beta)$ is the domain of $F^*$, $\text{dom} F^* = \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha \beta \leq 0\}$, which is obviously non-convex.

3. Representing $\text{epi}_K (F + I_A)^*$. Let $X, Y, Z, F$ and $G$ be as in Section 2. Assume further that $F$ and $G$ are proper mappings, $K$ is a closed, pointed, convex cone in $Y$ with nonempty interior, and $S$ is a convex cone in $Z$. Moreover, $C$ is a nonempty convex subset of $X$ and $A := C \cap G^{-1}(-S)$.

The following lemmas are useful for the representation of $\text{epi}_K (F + I_A)^*$ in this section.

**Lemma 3.1.** Let $L: X \to Y^*$. It holds

$$(L, y) \in \text{epi}_K F^* \iff y - L(x) + F(x) \notin -\text{int} K, \forall x \in X.$$  

**Proof.** It is a direct consequence of [9, Theorem 3.1] with $f = F$, $g \equiv 0_Z$, and $C = X$.

The main results in this section are two extensions of the following characterization of the epigraph of $(f + i_A)^*$ for a given scalar function $f$ (recall that $L_+(S, \mathbb{R}_+)$ and $L_+^\infty(S, \mathbb{R}_+)$ are alternative generalizations of the dual cone $S^*$ to the vector setting).

**Lemma 3.2.** [3, Theorem 8.2] Let $C$ be a nonempty closed convex subset of $X$, $f: X \to \mathbb{R} \cup \{-\infty\}$ be a proper lower semicontinuous (lsc) convex function, and $G: X \to Z^*$ be a proper $S$-convex and $S$-epi closed mapping. Assume that $A \cap \text{dom} f \neq \emptyset$. Then,

$$
\text{epi}(f + i_A)^* = \text{cl} \left[ \bigcup_{z^* \in S^+} \text{epi}(f + i_C + z^* \circ G)^* \right].
$$

**Lemma 3.3.** Let $F: X \to Y^*$ be a proper $K$-convex mapping and $C$ be a convex subset of $X$ such that $C \cap \text{dom} F \neq \emptyset$. Then $F(C \cap \text{dom} F) + \text{int} K$ is a convex subset of $Y$.

**Proof.** Let arbitrary $y_1, y_2 \in F(C \cap \text{dom} f) + \text{int} K$ and $\lambda \in [0, 1]$, we will prove that $\lambda y_1 + (1 - \lambda)y_2 \in F(C \cap \text{dom} f) + \text{int} K$.

Since $y_1, y_2 \in F(C \cap \text{dom} f) + \text{int} K$, there exists $x_1, x_2 \in C \cap \text{dom} F$ such that $y_1 \in F(x_1) + \text{int} K$, $y_2 \in F(x_2) + \text{int} K$ and consequently,

$$
\lambda y_1 + (1 - \lambda)y_2 = \lambda F(x_1) + (1 - \lambda)F(x_2) + \text{int} K. \quad (3.1)
$$

Now, because $(x_1, F(x_1)), (x_2, F(x_2)) \in \text{epi}_K F$ and $F$ is a $K$-convex mapping, one has $\lambda(x_1, F(x_1)) + (1 - \lambda)(x_2, F(x_2)) \in \text{epi}_K F$, which means

$$
\lambda F(x_1) + (1 - \lambda)F(x_2) \in F(\lambda x_1 + (1 - \lambda)x_2) + K. \quad (3.2)
$$

It follows from (3.1), (3.2) and the equality

$$
K + \text{int} K = \text{int} K \quad (3.3)
$$
(see e.g. [9 (7)]) that
\[ \lambda y_1 + (1 - \lambda)y_2 \in F(\lambda x_1 + (1 - \lambda)x_2) + \text{int } K, \]
and we are done (note that \( \lambda x_1 + (1 - \lambda)x_2 \in C \cap \text{dom } F \)) since \( C \cap \text{dom } f \) is convex.

The next lemma is proved in [6] Lemma 1.3] under the assumption that \( Y \) is a normed space.

**Lemma 3.4.** Let \( M \subset Y \) be a nonempty open convex set and let \( \bar{y} \in Y \) with \( \bar{y} \notin M \). Then there exists \( y^* \in Y^* \) such that
\[ y^*(u) < y^*(\bar{y}), \quad \forall u \in M. \]

**Proof.** It is a straightforward consequence of Theorem 3.4 in [17].

**Lemma 3.5.** Let \( \bar{y} \in Y, y^* \in Y^* \) and \( \emptyset \neq M \subset Y \), and assume that
\[ y^*(u) < y^*(\bar{y}), \quad \forall u \in M - \text{int } K. \quad (3.4) \]

Then, the following statements hold:
(i) \( y^*(v) \leq y^*(\bar{y}), \quad \forall v \in M; \)
(ii) \( y^*(k) > 0 \) for all \( k \in \text{int } K \) and, consequently, \( y^* \in K^+ \).

**Proof.** (i) Take \( k_0 \in \text{int } K \). Then, for any \( v \in M \), it follows from (3.4) that
\[ y^*(v - \lambda k_0) < y^*(\bar{y}), \quad \forall \lambda > 0, \]
and by letting \( \lambda \to 0 \), we get \( y^*(v) \leq y^*(\bar{y}) \).

(ii) Take arbitrarily \( k \in \text{int } K \). We firstly show that there exists \( \lambda > 0 \) such that \( \bar{y} - \lambda k \in M - \text{int } K \). Indeed, take \( m_0 \in M \) and \( k_0 \in K \). Because of the continuity of the mapping \( t \mapsto (m_0 - k_0 - \bar{y})t + k \) at \( t = 0 \), there is a \( \epsilon > 0 \) such that
\[ (m_0 - k_0 - \bar{y})t + k \in \text{int } K. \]
Taking \( \lambda = \frac{\epsilon}{2} \), we obtain \( m_0 - k_0 - \bar{y} + \lambda k \in \lambda \text{int } K \), and consequently, applying (3.3),
\[ \bar{y} - \lambda k \in m_0 - k_0 - \lambda \text{int } K \subset M - \text{int } K \]
\[ = M - \text{int } K. \]
It now follows from (3.4) that \( y^*(\bar{y} - \lambda k) < y^*(\bar{y}) \), which yields \( y^*(k) > 0 \). Since \( K = \text{cl}(\text{int } K) \), \( y^*(k) \geq 0 \) for all \( k \in K \) which means that \( y^* \in K^+ \) and the proof is complete.

**Lemma 3.6.** If \( F : X \rightarrow Y \cup \{ +\infty \} \) is a proper mapping, then \( \text{epi}_K F^* \) is a closed subset of \( \mathcal{L}(X, Y) \times Y \).

**Proof.** Let \( \{(L_i, y_i)\}_{i \in I} \subset \text{epi}_K F^* \) be a net such that \((L_i, y_i) \to (L, y)\). We will show that \((L, y) \in \text{epi}_K F^*\). Let us suppose the contrary, that is \((L, y) \notin \text{epi}_K F^*\). Then, by Lemma [3.1] there exists \( \bar{x} \in \text{dom } F \) such that
\[ y - L(\bar{x}) + F(\bar{x}) \in - \text{int } K. \]
As \( y_i - L_i(\bar{x}) + F(\bar{x}) \to y - L(\bar{x}) + F(\bar{x}) \), there is a \( i_0 \in I \) such that for all \( i \in I \), \( i \acleq i_0 \), where \( \acleq \) is the net order,
\[ y_i - L_i(\bar{x}) + F(\bar{x}) \in - \text{int } K, \]
which again by Lemma 3.1 yields \((L, y) \notin epi_K F^*\) for all \(i > i_0\), and this is a contradiction. 

**Theorem 3.7** (1st asymptotic representation of \(epi_K (F + I_A)^*\)). Let \(C\) be a nonempty closed convex subset of \(X\), \(F: X \to Y \cup \{+\infty\}\) be a proper \(K\)-convex mapping such that \(y^* \circ F\) is lsc for all \(y^* \in Y^*\), and \(G: X \to Z \cup \{+\infty\}\) be a proper \(S\)-convex and \(S\)-epi closed mapping. Assume that \(A \cap \text{dom } F \neq \emptyset\). Then

\[
epi_K (F + I_A)^* = \overline{\text{cl}} \left[ \bigcup_{T \in \mathcal{L}^+(S, K)} epi_K (F + I_C + T \circ G)^* \right]. \tag{3.5}
\]

**Proof.**  

- According to [9, Lemma 4.1],

\[
epi_K (F + I_A)^* \supset \bigcup_{T \in \mathcal{L}^+(S, K)} epi_K (F + I_C + T \circ G)^* ,
\]

which together with Lemma 3.6 yields

\[
epi_K (F + I_A)^* \supset \overline{\text{cl}} \left[ \bigcup_{T \in \mathcal{L}^+(S, K)} epi_K (F + I_C + T \circ G)^* \right]. \tag{3.6}
\]

- To prove (3.6), it is sufficient to show that the converse inclusion in (3.6) also holds. For this, take arbitrarily \((L, y) \in epi_K (F + I_A)^*\) and we will show that

\[
(L, y) \in \overline{\text{cl}} \left[ \bigcup_{T \in \mathcal{L}^+(S, K)} epi_K (F + I_C + T \circ G)^* \right]. \tag{3.7}
\]

Observe that if \((L, y) \in epi_K (F + I_A)^*\) then, by Lemma 3.1

\[
y \notin L(x) - F(x) - \text{int } K, \quad \forall x \in A \cap \text{dom } F,
\]

or equivalently, \(y \notin (L - F)(A \cap \text{dom } F) - \text{int } K\).

- Now, since \(G\) is \(S\)-convex, \(G^{-1}(-S)\) is a convex set, and hence, \(A = C \cap G^{-1}(-S)\) is convex, too. Moreover, \(F - L\) is a \(K\)-convex mapping (as \(F\) is \(K\)-convex), and we get from Lemma 3.3 that \((F - L)(A \cap \text{dom } F) + \text{int } K\) is convex, or equivalently, \((L - F)(A \cap \text{dom } F) - \text{int } K\) is convex.

On the one hand, as \(y \notin (L - F)(A \cap \text{dom } F) - \text{int } K\), Lemma 3.4 ensures the existence of \(y^* \in Y^*\) satisfying

\[
y^*(u) < y^*(y), \quad \forall u \in (L - F)(A \cap \text{dom } F) - \text{int } K.
\]

It then follows from Lemma 3.5 that

\[
y^* \circ (L - F)(x) \leq y^*(y), \quad \forall x \in A \cap \text{dom } F, \tag{3.8}
\]

\[
y^* \in K^+ \quad \text{and} \quad y^*(k) > 0 \quad \forall k \in \text{int } K. \tag{3.9}
\]

- On the other hand, since \(y^* \circ F\) is a proper convex lsc function, applying Lemma 3.2 to the scalar function \(y^* \circ F\), one gets

\[
epi(y^* \circ F + i_A)^* = \overline{\text{cl}} \left[ \bigcup_{z^* \in S^+} epi(y^* \circ F + i_C + z^* \circ G)^* \right]. \tag{3.10}
\]
Note that (3.8) is equivalent to \( y^*(y) \geq (y^* \circ F + i_A)^*(y^* \circ L) \) or, equivalently, 
\( (y^* \circ L, y^*(y)) \in \text{epi}(y^* \circ F + i_A)^* \). Hence, by (3.10), there exist nets \( \{z^*\}_{i \in I} \subset S^+, \{x_i^*\}_{i \in I} \subset X^* \) and \( \{r_i\}_{i \in I} \subset \mathbb{R} \) such that \( x_i^* \to y^* \circ L, r_i \to y^* (y) \) and
\[
(x_i^*, r_i) \in \text{epi}(y^* \circ F + i_C + z^*_i \circ G)^*, \quad \forall i \in I.
\] (3.11)

Take an arbitrary \( k_0 \in \text{int} K \). Then \( y^*(k_0) > 0 \) (see (3.9)).

Now for each \( i \in I \), set 
\[
y_i := y + \frac{r_i - y^*(y)}{y^*(k_0)} k_0 ,
\]
and define the mapping \( L_i : X \to Y \) by 
\[
L_i(x) := L(x) + \frac{x_i^*(x) - (y^* \circ L)(x)}{y^*(k_0)} k_0, \quad \forall x \in X.
\]

It is easy to check that
\[
y^*(y_i) = r_i, \quad L_i \in \mathcal{L}(X, Y), \quad y^* \circ L_i = x_i^*, \quad \forall i \in I \quad \text{and} \quad (y_i, L_i) \to (y, L).
\] (3.12)

- We now claim that
\[
(L_i, y_i) \in \bigcup_{T \in \mathcal{L}_+(S, K)} \text{epi} K(F + I_C + T \circ G)^*, \quad \forall i \in I.
\]

Indeed, for each \( i \in I \), combining (3.11) and (3.12) we get
\[
y^*(y_i) \geq (y^* \circ F + i_C + z_i^* \circ G)^*(y^* \circ L_i),
\]
or equivalently,
\[
y^*(y_i) \geq (y^* \circ L_i)(x) - (y^* \circ F)(x) - (z_i^* \circ G)(x), \quad \forall x \in C \cap \text{dom} F.
\] (3.13)

For each \( i \in I \), define \( T_i : Z \to Y \) by
\[
T_i(z) := \frac{z_i^*(z)}{y^*(k_0)} k_0, \quad \forall z \in Z.
\]

Then \( T_i \in \mathcal{L}(Z, Y) \). Moreover, if \( z \in S \), then \( z_i^*(z) \geq 0 \) (as \( z_i^* \in S^+ \)) and so, \( T_i(z) \in K \) (as \( k_0 \in \text{int} K \) and \( y^*(k_0) > 0 \)). Consequently, \( T_i \in \mathcal{L}_+(S, K) \).

Since \( y^* \circ T_i = z_i^* \), with the help of the mappings \( T_i \in \mathcal{L}_+(S, K), \quad i \in I \), (3.13) can be rewritten as
\[
y^*(y_i) \geq (y^* \circ L_i)(x) - (y^* \circ F)(x) - (y^* \circ T_i \circ G)(x), \quad \forall x \in C \cap \text{dom} F,
\]
or equivalently,
\[
y^*(L_i(x) - F(x) - (T_i \circ G)(x) - y_i) \leq 0, \quad \forall x \in C \cap \text{dom} F.
\]
The last inequality, together with (3.9), implies that
\[
y_i \notin L_i(x) - F(x) - (T_i \circ G)(x) - \text{int} K, \quad \forall x \in C \cap \text{dom} F,
\]
which, together with Lemma 3.1 yields \( (L_i, y_i) \in \text{epi} K(F + I_C + T_i \circ G)^* \).
Finally, taking (3.12) into account, (3.7) follows and we are done. □

We now show that the closure in the right-hand side of (3.5) in Theorem 3.7 can be removed if certain qualification condition holds. To do this we need the lemma below on scalar functions.

**Lemma 3.8.** Let $C$ be a nonempty convex subset of $X$, $f: X \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function, and $G: X \to Z \cup \{+\infty\}$ be a proper $S$-convex mapping. Let $D := G(C \cap \text{dom} f \cap \text{dom} G) + S$. Assume that $A \cap \text{dom} f \neq \emptyset$ and one of the following conditions is fulfilled:

(i) There exists $\bar{z} \in C \cap \text{dom} f$ such that $G(\bar{z}) \in -\text{int} S$;
(ii) $X, Z$ are Fréchet spaces, $C$ is closed, $f$ is lsc, $G$ is $S$-epi closed and $0_Z \in \text{sri} D$;
(iii) $\dim \text{lin} D < +\infty$ and $0_Z \in \text{ri} D$.

Then

$$\text{epi}(f + i_A) = \bigcup_{z^* \in S^+} \text{epi}(f + i_C + z^* \circ G).$$

**Proof.** Take an arbitrary $x^* \in X^*$. Applying [3, Theorem 3.4], with $f - x^*$ playing the role of primal objective function, we get the existence of $\bar{z}^* \in S^+$ satisfying

$$\inf_{x \in C} [f(x) - x^*(x)] = \max_{z^* \in S^+} \inf_{x \in C} [f(x) - x^*(x) + (z^* \circ G)(x)]$$

$$= \inf_{x \in C} [f(x) - x^*(x) + (\bar{z}^* \circ G)(x)],$$

which is equivalent to

$$(f + i_A)^*(x^*) = (f + i_C + \bar{z}^* \circ G)^*(x^*),$$

for some $\bar{z}^* \in S^+$, showing that

$$\text{epi}(f + i_A) = \bigcup_{z^* \in S^+} \text{epi}(f + i_C + z^* \circ G)^*$$

and we are done. □

**Theorem 3.9** (1st non-asymptotic representation of $\text{epi}_K(F + I_A)^*$). Let $C$ be a nonempty convex subset of $X$, $F: X \to Y \cup \{+\infty\}$ be a proper $K$-convex mapping, and $G: X \to Z \cup \{+\infty\}$ be a proper $S$-convex mapping. Consider the set $E := G(C \cap \text{dom} F \cap \text{dom} G) + S$. Assume that $A \cap \text{dom} F \neq \emptyset$ and at least one of the following qualification conditions holds:

(c1) There exists $\bar{x} \in C \cap \text{dom} F$ such that $G(\bar{x}) \in -\text{int} S$;
(c2) $X, Z$ are Fréchet spaces, $C$ is closed, $y^* \circ F$ is lsc for all $y^* \in Y^*$, $G$ is $S$-epi closed and $0_Z \in \text{sri} D$;
(c3) $\dim \text{lin} E < +\infty$ and $0_Z \in \text{ri} E$.

Then

$$\text{epi}_K(F + I_A) = \bigcup_{T \in \mathcal{L}_+(S, K)} \text{epi}_K(F + I_C + T \circ G)^*.$$

**Proof.** The proof goes in parallel to the one of Theorem 3.7, using Lemma 3.8 instead of Lemma 3.2. For an easy reading, the main ideas will be repeated below.
• By [9] Lemma 4.1, it is sufficient to show that
\[
\text{epi}_K(F + I_A)^* \subset \bigcup_{T \in \mathcal{L}_+(S, K)} \text{epi}_K(F + I_C + T \circ G)^*.
\]  
(3.14)

• Take an arbitrary \((L, y) \in \text{epi}_K(F + I_A)^*\). Then, by the same argument as the one in the proof of Theorem 3.4 using Lemmas 3.1, 3.3, 3.4, and 3.5, there exists \(y^* \in Y^*\) such that (3.8) and (3.9) hold.

Observe also that (3.8) is equivalent to \(y^*(y) \geq (y^* \circ F + i_A)^*(y^* \circ L)\), which accounts for
\[
(y^* \circ L, y^*(y)) \in \text{epi}(y^* \circ F + i_A)^*.
\]  
(3.15)

• Because \(y^* \in K^+\) and \(F\) is a \(K\)-convex mapping, \(y^* \circ F\) is a convex function. If one of the qualification conditions \((c_1), (c_2), (c_3)\) holds then, by Lemma 3.8 one has
\[
\text{epi}(y^* \circ F + i_A)^* = \bigcup_{z^* \in S^+} \text{epi}(y^* \circ F + i_C + z^* \circ G)^*.
\]  
(3.16)

This and (3.15) ensure the existence of \(z^* \in S^+\) satisfying \((y^* \circ L, y^*(y)) \in \text{epi}(y^* \circ F + i_C + z^* \circ G)^*\), which means that
\[
y^*(y) \geq (y^* \circ L)(x) - (y^* \circ F)(x) - (z^* \circ G)(x), \quad \forall x \in C \cap \text{dom} F.
\]  
(3.17)

• Now, pick \(k_0 \in \text{int} K\) and consider the linear mapping \(T: Z \rightarrow Y\) such that
\[
T(z) := \frac{z^*(z)}{y^*(k_0)}, \forall z \in Z.
\]

Then \(T \in \mathcal{L}_+(S, K)\) and \(y^* \circ T \in Z^*\). Hence, (3.17) can be rewritten as
\[
y^*(y) \geq (y^* \circ L)(x) - (y^* \circ F)(x) - (y^* \circ T \circ G)(x), \quad \forall x \in C \cap \text{dom} F,
\]
or equivalently,
\[
y^*(L(x) - F(x) - (T \circ G)(x) - y) \leq 0, \quad \forall x \in C \cap \text{dom} F.
\]

So, by (3.9),
\[
L(x) - F(x) - (T \circ G)(x) - y \notin \text{int} K, \quad \forall x \in C \cap \text{dom} F,
\]
which in turn yields, by Lemma 3.1, \((L, y) \in \text{epi}_K(F + I_C + T \circ G)^*\). Hence, (3.14) has been proved and the proof is complete. \(\Box\)

**Example 3.10.** Let \(X, Y, Z, F, \text{ and } G\) be as in Example 2.6. Let \(C = \mathbb{R}\). Due to the extreme simplicity of \(A = C \cap G^{-1}(-S) = \mathbb{R}_+\) in this case, \(\text{epi}_K(F + I_A)^*\) can be calculated directly. In fact, since \((F + I_A)^* (\alpha, \beta) = \text{WSup}([R_+(\alpha, \beta)]\), one gets

\[
(F + I_A)^* (\alpha, \beta) = \begin{cases} 
\{+\infty_{\mathbb{R}_+}\}, & \text{if } \alpha > 0 \text{ and } \beta > 0, \\
\{(-\mathbb{R}_+) \times \{0\} \} \cup \{(0) \times (-\mathbb{R}_+)\}, & \text{if } \alpha \leq 0 \text{ and } \beta \leq 0, \\
\mathbb{R}_+ (\alpha, \beta), & \text{if } \alpha \beta = 0 \text{ and } \alpha + \beta > 0, \\
\mathbb{R}_+ (\alpha, \beta) \cup \{(-\mathbb{R}_+) \times \{0\}\}, & \text{if } \alpha > 0 \text{ and } \beta < 0, \\
\mathbb{R}_+ (\alpha, \beta) \cup \{(0) \times (-\mathbb{R}_+)\}, & \text{if } \alpha < 0 \text{ and } \beta > 0.
\end{cases}
\]

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Thus, $\text{epi}_K(F + I_A)^* = \bigcup_{i=1}^{5} P_i$, where

- $P_1 = \{(\alpha, \beta, y_1, y_2) : \alpha \leq 0 \land \beta \leq 0 \land (y_1 \geq 0 \lor y_2 \geq 0)\}$,
- $P_2 = \{(0, \beta, y_1, y_2) : \beta > 0 \land y_1 \geq 0\}$,
- $P_3 = \{(\alpha, 0, y_1, y_2) : \alpha > 0 \land y_2 \geq 0\}$,
- $P_4 = \{(\alpha, \beta, y_1, y_2) : \alpha > 0 \land \beta < 0 \land y_2 \geq \min \{0, \frac{\beta}{\alpha} y_1\}\}$,
- $P_5 = \{(\alpha, \beta, y_1, y_2) : \alpha < 0 \land \beta > 0 \land y_1 \geq \min \{0, \frac{\beta}{\alpha} y_2\}\}$.

Since $\text{dom}(F + I_A)^* = \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha \leq 0 \lor \beta \leq 0\}$ is not convex, $\text{epi}_K(F + I_A)^*$ cannot be convex while its closedness follows from Lemma 5.6 applied to the proper vector function $F + I_A = I_A$.

According to Theorem 3.9, as the interior type condition (c1) is satisfied by any positive number, we can also express

$$\text{epi}_K(F + I_A)^* = \bigcup_{(t_1, t_2) \in \mathbb{R}^2_+} \text{epi}_K((t_1, t_2) \circ G)^*,$$

where

$$((t_1, t_2) \circ G)^* (\alpha, \beta) = \text{WSup} \{\mathbb{R} (\alpha + t_1, \beta + t_2)\} = F^* (\alpha + t_1, \beta + t_2).$$

So, $\text{epi}_K((t_1, t_2) \circ G)^* = \bigcup_{i=1}^{4} Q_i (t_1, t_2)$, with

- $Q_1 (t_1, t_2) = \{(-t_1, -t_2, y_1, y_2) : y_1 \geq 0 \lor y_2 \geq 0\}$,
- $Q_2 (t_1, t_2) = \{(\alpha, \beta, y_1, y_2) : (\alpha + t_1) (\beta + t_2) < 0 \land y_2 \geq \left(\frac{\beta + t_2}{\alpha + t_1}\right) y_1\}$,
- $Q_3 (t_1, t_2) = \{(\alpha, -t_2, y_1, y_2) : \alpha \neq -t_1 \land y_2 \geq 0\}$,
- $Q_4 (t_1, t_2) = \{(-t_1, \beta, y_1, y_2) : \beta \neq -t_2 \land y_1 \geq 0\}$.

From Theorem 3.9 and the inclusion $\mathcal{L}_+(S, K) \subset \mathcal{L}_+^w(S, K)$, one has

$$\text{epi}_K(F + I_A)^* \subset \bigcup_{T \in \mathcal{L}_+^w(S, K)} \text{epi}_K(F + IC + T \circ G)^*.$$

Next we show that this inclusion might be strict under the assumptions of Theorem 3.9. Indeed,

$$(1, 0, 0, -1) \in Q_1 (-1, 0) \setminus \left(\bigcup_{i=1}^{5} P_i\right) \subset \left[\bigcup_{T \in \mathcal{L}_+^w(S, K)} \text{epi}_K(F + IC + T \circ G)^*\right] \setminus \text{epi}_K(F + I_A)^*.$$

The rest of this section is devoted to derive representations of $\text{epi}_K(F + I_A)^*$ where the set

$$\mathcal{L}_+^w(S, K) = \{T \in \mathcal{L}(X, Y) : T(S) \cap (- \text{int } K) = \emptyset\}$$

replaces $\mathcal{L}_+(S, K)$ as index set at the right-hand side union of sets.
LEMMA 3.11. One has

\[ \text{epi}_K(F + I_A)^* \supset \bigcap_{v \in I_{-S}^*(T)} \left[ \text{epi}_K(F + I_C + T \circ G)^* + (0_L, v) \right], \quad \forall T \in \mathcal{L}_+^u(S, K). \quad (3.18) \]

**Proof.** Take arbitrarily \( T \in \mathcal{L}_+^u(S, K) \) and

\[ (L, y) \in \bigcap_{v \in I_{-S}^*(T)} \left[ \text{epi}_K(F + I_C + T \circ G)^* + (0_L, v) \right]. \]

Then,

\[ (L, y - v) \in \text{epi}_K(F + I_C + T \circ G)^*, \quad \forall v \in I_{-S}^*(T), \]

and, by Lemma 3.11 and (2.10), the last inclusion is equivalent to

\[ y - v - L(x) + F(x) + (T \circ G)(x) \notin - \text{int } K, \quad \forall x \in C, \forall v \in I_{-S}^*(T) \]

\[ \Leftrightarrow y - L(x) + F(x) + (T \circ G)(x) \notin I_{-S}^*(T) - \text{int } K, \quad \forall x \in C \]

\[ \Leftrightarrow y - L(x) + F(x) + (T \circ G)(x) \notin \text{WSup } T(-S) - \text{int } K, \quad \forall x \in C \]

\[ \Leftrightarrow y - L(x) + F(x) + (T \circ G)(x) \notin T(-S) - \text{int } K, \quad \forall x \in C \]

\[ \Leftrightarrow y - L(x) + F(x) \notin u - (T \circ G)(x) - \text{int } K, \quad \forall u \in T(-S), \forall x \in C. \quad (3.19) \]

Now, for any \( x \in A \), taking \( u = (T \circ G)(x) \) in (3.19) (note that \( x \in A \) yields \( G(x) \in -S \)), we get \( y - L(x) + F(x) \notin - \text{int } K \). Hence, again by Lemma 3.11, we obtain \((L, y) \in \text{epi}_K(F + I_A)^* \) and (3.18) follows. □

**LEMMA 3.12.** If \( T \in \mathcal{L}_+^u(S, K) \) then

\[ \text{epi}_K(F + I_C + T \circ G)^* = \bigcap_{v \in I_{-S}^*(T)} \left[ \text{epi}_K(F + I_C + T \circ G)^* + (0_L, v) \right]. \]

**Proof.** Assume that \( T \in \mathcal{L}_+^u(S, K) \). One has \( T(-S) \subset -K \) and \( 0_Y \in T(-S) \) (as \( 0_Y = T(0_X) \)). So, by Definition (2.4) and (2.10), \( 0_Y \in \text{WSup } T(-S) = I_{-S}^*(T) \). Hence, \( \text{epi}_K(F + I_C + T \circ G)^* \) is a member of the collection in the right-hand side of (3.20), and we obtain

\[ \bigcap_{v \in I_{-S}^*(T)} \left[ \text{epi}_K(F + I_C + T \circ G)^* + (0_L, v) \right] \subset \text{epi}_K(F + I_C + T \circ G)^*. \]

Conversely, take an arbitrary \((L, y) \in \text{epi}_K(F + I_C + T \circ G)^* \). We will prove that \((L, y) \in \bigcap_{v \in I_{-S}^*(T)} \left[ \text{epi}_K(F + I_C + T \circ G)^* + (0_L, v) \right] \), or equivalently,

\[ (L, y) \in \text{epi}_K(F + I_C + T \circ G)^* + (0_L, v), \quad \forall v \in I_{-S}^*(T). \]

Since \( T(-S) \subset -K \), it follows from (2.3) that \( \text{WSup } T(-S) \subset \text{cl } [T(-S) - \text{int } K] \subset -K \), and consequently \( I_{-S}^*(T) \subset -K \).

Since \((L, y) \in \text{epi}_K(F + I_C + T \circ G)^* \), one has \( y \in (F + I_C + T \circ G)^*(L) + K \) for any \( v \in I_{-S}^*(T) \), as \( I_{-S}^*(T) \subset -K \), we get

\[ y - v \in (F + I_C + T \circ G)^*(L) + K + K = (F + I_C + T \circ G)^*(L) + K, \]
which accounts for \((L, y) \in \text{epi}_K(F + IC + T \circ G)^*(L) + (0_L, v)\) and we are done. □

**Proposition 3.13.** One has

\[
\text{epi}_K(F + IA)^* \supset \bigcup_{T \in \mathcal{L}^+(S, K)} \left[ \bigcap_{v \in I^*_S(T)} \left[ \text{epi}_K(F + IC + T \circ G)^* + (0_L, v) \right] \right]
\]

\[
\supset \bigcup_{T \in \mathcal{L}^+(S, K)} \text{epi}_K(F + IC + T \circ G)^*.
\]

**Proof.** The first inclusion follows from Lemma 3.11 while the second one follows from the fact that \(\mathcal{L}^+(S, K) \subset \mathcal{L}^*_u(S, K)\) and Lemma 3.12. □

**Theorem 3.14** (2nd asymptotic representation of \(\text{epi}_K(F + IA)^*\)). Assume all the assumptions of Theorem 3.7 hold. Then,

\[
\text{epi}_K(F + IA)^* = \text{cl} \left\{ \bigcup_{T \in \mathcal{L}^+(S, K)} \left[ \bigcap_{v \in I^*_S(T)} \left[ \text{epi}_K(F + IC + T \circ G)^* + (0_L, v) \right] \right] \right\}.
\]

**Proof.** It follows from Lemma 3.10 and Proposition 3.13 that

\[
\text{epi}_K(F + IA)^* \supset \text{cl} \left\{ \bigcup_{T \in \mathcal{L}^+(S, K)} \left[ \bigcap_{v \in I^*_S(T)} \left[ \text{epi}_K(F + IC + T \circ G)^* + (0_L, v) \right] \right] \right\}.
\]

and the conclusion now follows from this double inclusion and Theorem 3.7. □

**Theorem 3.15** (2nd non-asymptotic representation of \(\text{epi}_K(F + IA)^*\)). Assume all the assumptions of Theorem 3.9. Then

\[
\text{epi}_K(F + IA)^* = \bigcup_{T \in \mathcal{L}^+(S, K)} \left[ \bigcap_{v \in I^*_S(T)} \left[ \text{epi}_K(F + IC + T \circ G)^* + (0_L, v) \right] \right].
\]

**Proof.** By Proposition 3.13,

\[
\text{epi}_K(F + IA)^* \supset \bigcup_{T \in \mathcal{L}^+(S, K)} \left[ \bigcap_{v \in I^*_S(T)} \left[ \text{epi}_K(F + IC + T \circ G)^* + (0_L, v) \right] \right]
\]

\[
\supset \bigcup_{T \in \mathcal{L}^+(S, K)} \text{epi}_K(F + IC + T \circ G)^*.
\]

The conclusion now follows from this double inclusion and Theorem 3.9. □

4. **Farkas-type results for vector-valued functions.** Let \(X, Y, Z, F, G, C,\) and \(A\) be as in Section 3. We also assume that \(A \cap \text{dom} F \neq \emptyset.\)

This section provides stable reverse Farkas-type results in the sense of [9] for the constraint system of (VOP):

\[
\{ x \in C, \ G(x) \in -S \} \equiv \{ x \in C, \ G(x) \leq_S 0_Z \}.
\]
We first recall a general result which will be useful in the sequel.

**Lemma 4.1.** [12] Theorem 4.5] The following statements are equivalent:

(a) $\text{epi}_K(F + I_A) = \bigcup_{T \in \mathcal{L}(S,K)} \text{epi}_K(F + I_C + T \circ G)^*$;

(b) For any $y \in Y$ and any $L \in \mathcal{L}(X,Y)$, the following assertions are equivalent:

\[(b_1) \ G(x) \in -S, \ x \in C \implies F(x) - L(x) + y \notin -\text{int } K; \]
\[(b_2) \ \exists T \in \mathcal{L}(S,K) \text{ such that } F(x) + (T \circ G)(x) - L(x) + y \notin -\text{int } K \forall x \in C. \]

**Theorem 4.2** (1st characterization of Farkas lemma). Let $C$ be a nonempty closed convex subset of $X$, $F : X \to Y \cup \{+\infty\}$ be a proper $K$-convex mapping satisfying $y^* \circ F$ is lsc for all $y^* \in Y^*$, and $G : X \to Z \cup \{+\infty\}$ be a proper $S$-convex and $S$-epi closed mapping. Then, the following statements are equivalent:

\[ (a') \text{ The set} \]
\[
\bigcup_{T \in \mathcal{L}(S,K)} \text{epi}_K(F + I_C + T \circ G)^*
\]

is closed in $\mathcal{L}(X,Y) \times Y$.

(b) $\forall (L, y) \in \mathcal{L}(X,Y) \times Y$, $(b_1) \iff (b_2)$.

**Proof.** It follows from Theorem 3.7 that

\[ \text{epi}_K(F + I_A)^* = \text{cl} \left[ \bigcup_{T \in \mathcal{L}(S,K)} \text{epi}_K(F + I_C + T \circ G)^* \right]. \]

Hence, $(a')$ is equivalent to $\text{epi}_K(F + I_A)^* = \bigcup_{T \in \mathcal{L}(S,K)} \text{epi}_K(F + I_C + T \circ G)^*$ and the conclusion follows from Lemma 4.1.

**Theorem 4.3** (1st Farkas lemma). Let $C$ be a nonempty convex subset of $X$, $F : X \to Y \cup \{+\infty\}$ be a proper $K$-convex mapping, and $G : X \to Z \cup \{+\infty\}$ be a proper $S$-convex mapping. Assume that $A \cap \text{dom } F \neq \emptyset$ and that one of the conditions $(c_1)$, $(c_2)$ and $(c_3)$ holds. Then for all $(L, y) \in \mathcal{L}(X,Y) \times Y$, one has $(b_1) \iff (b_2)$.

**Proof.** We get from Theorem 3.9 that

\[ \text{epi}_K(F + I_A)^* = \bigcup_{T \in \mathcal{L}(S,K)} \text{epi}_K(F + I_C + T \circ G)^* \]

and hence, the conclusion also follows from Lemma 4.1.

**Theorem 4.4** (2nd characterization of Farkas lemma). Assume all the assumptions of Theorem 4.4. Then the following statements are equivalent:

(c) $\text{epi}_K(F + I_A)^* = \bigcup_{T \in \mathcal{L}^+(S,K)} \left[ \bigcap_{v \in \mathcal{F}^-_T(S)} \left[ \text{epi}_K(F + I_C + T \circ G)^* + (0_L, v) \right] \right]$.

(d) For any $y \in Y$ and any $L \in \mathcal{L}(X,Y)$, the following assertions are equivalent:

\[(b_1) \ G(x) \in -S, \ x \in C \implies F(x) - L(x) + y \notin -\text{int } K; \]
\[(b_2) \ \exists T \in \mathcal{L}^+(S,K) : F(x) + (T \circ G)(x) - L(x) + y \notin T(-S) - \text{int } K \forall x \in C. \]
Proof. [(c) $\implies$ (d)] Take an arbitrary $(L, y) \in \mathcal{L}(X, Y) \times Y$. On the one hand, by Lemma 3.3, one has

$$(L, y) \in \text{epi}_K(F + I_A)^*$$

$$\iff y - L(x) + F(x) + I_A(x) \notin - \text{int } K, \forall x \in X$$

$$\iff y - L(x) + F(x) \notin - \text{int } K, \forall x \in A.$$ 

On the other hand, by an argument similar to that of (3.19),

$$(L, y) \in \bigcup_{T \in \mathcal{L}^u(S, K)} \left( \bigcap_{v \in I^*_{-S}(T)} \left[ \text{epi}_K(F + I_C + T \circ G)^* + (0_L, v) \right] \right)$$

$$\iff \exists T \in \mathcal{L}^u(S, K) : (L, y) \in \bigcap_{v \in I^*_{-S}(T)} \left[ \text{epi}_K(F + I_C + T \circ G)^* + (0_L, v) \right]$$

$$\iff \exists T \in \mathcal{L}^u(S, K) : (L, y - v) \in \text{epi}_K(F + I_C + T \circ G)^*, \forall v \in I^*_{-S}(T)$$

$$\iff \exists T \in \mathcal{L}^u(S, K) : y - v - L(x) + F(x) + I_C(x) + (T \circ G)(x) \notin - \text{int } K,$n \forall x \in X, \forall v \in I^*_{-S}(T),$$

$$\iff \exists T \in \mathcal{L}^u(S, K) : y - L(x) + F(x) + (T \circ G)(x) \notin - \text{int } K, \forall x \in C,$n$$

$$\iff \exists T \in \mathcal{L}^u(S, K) : y - L(x) + F(x) + (T \circ G)(x) \notin \text{WSup } T(-S) - \text{int } K, \forall x \in C$$

$$\iff \exists T \in \mathcal{L}^u(S, K) : y - L(x) + F(x) + (T \circ G)(x) \notin T(-S) - \text{int } K, \forall x \in C.$$ 

Then, the implication (c) $\implies$ (d) follows.

[(d) $\implies$ (c)] Thanks to Lemma 3.11 we only need to prove the inclusion "⊂" in (c). In fact, if $(L, y) \in \text{epi}_K(F + I_A)^*$, then $y - L(x) + F(x) \notin - \text{int } K$, for all $x \in A$, and by the equivalence $(b_1) \iff (b_3)$, there exists $T \in \mathcal{L}^u(S, K)$ such that

$$F(x) + (T \circ G)(x) - L(x) + y \notin T(-S) - \text{int } K, \forall x \in C.$$ 

(4.1)

Since $T(-S) - \text{int } K = \text{WSup } T(-S) - \text{int } K = I^*_{-S}(T) - \text{int } K$ (see (2.6)), it turns out that (4.1) is equivalent to

$$y - v - L(x) + F(x) + I_C(x) + (T \circ G)(x) \notin - \text{int } K, \forall x \in X, \forall v \in I^*_{-S}(T),$$

which yields $(L, y) \in \text{epi}_K(F + I_C + T \circ G)^* + (0_L, v)$ for all $v \in I^*_{-S}(T)$, and the aimed inclusion follows. \qed

Theorem 4.5 (3rd characterization of Farkas lemma). Assume all the assumptions of Theorem 4.3. Then the following statements are equivalent:

(c') The set

$$\bigcup_{T \in \mathcal{L}^u(S, K)} \left( \bigcap_{v \in I^*_{-S}(T)} \left[ \text{epi}_K(F + I_C + T \circ G)^* + (0_L, v) \right] \right)$$

is closed in $\mathcal{L}(X, Y) \times Y$;

(d) $\forall (L, y) \in \mathcal{L}(X, Y) \times Y$, $(b_1) \iff (b_3)$. 

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Proof. It follows from Theorem 3.14 that
\[
\operatorname{epi} K(F + I_A)^* = \operatorname{cl} \left\{ \bigcup_{T \in \mathcal{L}_+^+(S, K)} \left[ \bigcap_{v \in I^*_+(T)} \operatorname{epi} K(F + I_C + T \circ G)^* + (0_L, v) \right] \right\}.
\]
Hence, (c’) is equivalent to
\[
\operatorname{epi} K(F + I_A)^* = \bigcup_{T \in \mathcal{L}_+^+(S, K)} \left[ \bigcap_{v \in I^*_+(T)} \operatorname{epi} K(F + I_C + T \circ G)^* + (0_L, v) \right],
\]
and the conclusion follows from Theorem 4.4.

Theorem 4.6 (2nd Farkas lemma). Assume all the assumptions of Theorem 4.3. Then, for all \((L, y) \in \mathcal{L}(X, Y) \times Y\), one has \((b_1) \iff (b_3)\).

Proof. Under the assumptions of this theorem, it follows from Theorem 3.15 that
\[
\operatorname{epi} K(F + I_A)^* = \bigcup_{T \in \mathcal{L}_+^+(S, K)} \left[ \bigcap_{v \in I^*_+(T)} \operatorname{epi} K(F + I_C + T \circ G)^* + (0_L, v) \right].
\]
The conclusion now comes from Theorem 4.4.

It should be mentioned that the Farkas-type results of the forms \([(b_1) \iff (b_2)]\) or \([(b_1) \iff (b_3)]\) in Theorems 4.2-4.6, when specified to the case where \(Y = \mathbb{R}\), following the way as in [9], either cover or extend many Farkas-type results and their stable forms in the literature, such as [5], [7], [10], [11], etc.

5. Optimality conditions for vector optimization problems. Let \(X, Y, Z, F, G, C,\) and \(A\) be as in Section 4. In this section we provide optimality conditions for the feasible and non-trivial vector optimization problem
\[
(VOP) \quad \text{WMin} \{ F(x) : x \in C, G(x) \in -S \}.
\]
Using the Farkas-type results established in the last section, we get the following optimality conditions for (VOP).

Theorem 5.1 (1st characterization of optimality conditions). Let \(\bar{x} \in A \cap \operatorname{dom} F\). Assume all the assumptions of Theorem 4.2 Then the following statements are equivalent:

(e) The set
\[
\bigcup_{T \in \mathcal{L}_+^+(S, K)} \operatorname{epi} K(F + I_C + T \circ G)^*
\]
is closed regarding \((0_L, -F(\bar{x}))\);

(f) \(\bar{x}\) is a weak solution of (VOP) if and only if there exists \(T \in \mathcal{L}_+^+(S, K)\) such that
\[
-F(\bar{x}) \in (F + I_C + T \circ G)^*(0_L) + K;
\]

(g) \(\bar{x}\) is a weak solution of (VOP) if and only if there exists \(T \in \mathcal{L}_+^+(S, K)\) such that
\[
F(x) + (T \circ G)(x) - F(\bar{x}) \notin \operatorname{int} K, \forall x \in C.
\]
Proof. Under the current assumptions, we get from Theorem 5.3 in [9] that

$$
\text{epi}_K(F + I_A)^* = \text{cl} \left[ \bigcup_{T \in \mathcal{L}(S,K)} \text{epi}_K(F + I_C + T \circ G)^* \right].
$$

Hence, (e) is equivalent to

$$
\text{epi}_K(F + I_A)^* \cap \{(0_L, -F(\bar{x}))\} = \left( \bigcup_{T \in \mathcal{L}(S,K)} \text{epi}_K(F + I_C + T \circ G)^* \right) \cap \{(0_L, -F(\bar{x}))\}
$$

and the conclusion follows from Theorem 5.3 in [9]. □

Theorem 5.2 (1st optimality conditions). Let $\bar{x} \in A \cap \text{dom} F$. Assume all the assumptions of Theorem 4.3. Then (f) and (g) hold.

Proof. Under the assumptions of this theorem, we get from Theorem 5.2 and Lemma 5.6 that $\bigcup_{T \in \mathcal{L}(S,K)} \text{epi}_K(F + I_C + T \circ G)^*$ is a closed subset of $\mathcal{L}(X, Y) \times Y$.

So, (e) holds and the conclusion comes from Theorem 5.1. □

Theorem 5.3 (2nd characterization of optimality conditions). Let $\bar{x} \in A \cap \text{dom} F$. Assume all the assumptions of Theorem 4.2. Then the following statements are equivalent:

(h) The set

$$
\bigcup_{T \in \mathcal{L}(S,K)} \left[ \bigcap_{v \in I^*_S(T)} \left[ \text{epi}_K(F + I_C + T \circ G)^* + (0_L, v) \right] \right]
$$

is closed regarding $(0_L, -F(\bar{x}))$;

(i) $\bar{x}$ is a weak solution of (VOP) if and only if there exists $T \in \mathcal{L}(S, K)$ such that

$$
-F(\bar{x}) - I_{-S}^*(T) \subset (F + I_C + T \circ G)^*(0_L) + K;
$$

(j) $\bar{x}$ is a weak solution of (VOP) if and only if there exists $T \in \mathcal{L}(S, K)$ such that

$$
F(x) + (T \circ G)(x) - F(\bar{x}) \notin T(-S) - \text{int } K, \forall x \in C.
$$

Proof. Arguing as in the proof of Theorem 4.4 with $(0_L, -F(\bar{x}))$ instead of $(L, y)$, we get

$$
(0_L, -F(\bar{x})) \in \bigcup_{T \in \mathcal{L}(S, K)} \left[ \bigcap_{v \in I^*_S(T)} \text{epi}_K(F + I_C + T \circ G)^* + (0_L, v) \right]
$$

$$
\iff \exists T \in \mathcal{L}(S, K) : (0_L, -F(\bar{x}) - v) \in \text{epi}_K(F + I_C + T \circ G)^*, \forall v \in I^*_S(T)
$$

$$
\iff \exists T \in \mathcal{L}(S, K) : -F(\bar{x}) - v \in (F + I_C + T \circ G)^*(0_L) + K, \forall v \in I^*_S(T)
$$

$$
\iff \exists T \in \mathcal{L}(S, K) : -F(\bar{x}) - I_{-S}^*(T) \subset (F + I_C + T \circ G)^*(0_L) + K,
$$

and also

$$
(0_L, -F(\bar{x})) \in \bigcup_{T \in \mathcal{L}(S, K)} \left[ \bigcap_{v \in I^*_S(T)} \text{epi}_K(F + I_C + T \circ G)^* + (0_L, v) \right]
$$

$$
\iff \exists T \in \mathcal{L}(S, K) : F(x) + (T \circ G)(x) - F(\bar{x}) \notin T(-S) - \text{int } K, \forall x \in C.
$$
Next, under the assumptions of this theorem, we get from Theorem 3.14 that
\[ \text{epi}_K(F + I_A)^* = \text{cl} \left\{ \bigcup_{T \in \mathcal{L}^+(S,K)} \left[ \bigcap_{v \in I^-_S(T)} \text{epi}_K(F + I_C + T \circ G)^* + (0_{\mathcal{L}}, v) \right] \right\}. \]

Hence, \((h)\) is equivalent to
\[ \text{epi}_K(F + I_A)^* \cap \{(0_{\mathcal{L}}, -F(\bar{x}))\} = \left\{ \bigcup_{T \in \mathcal{L}^+(S,K)} \left[ \bigcap_{v \in I^-_S(T)} \text{epi}_K(F + I_C + T \circ G)^* + (0_{\mathcal{L}}, v) \right] \right\} \cap \{(0_{\mathcal{L}}, -F(\bar{x}))\}. \] (5.3)

Now, take an arbitrary \(\bar{x} \in A \cap \text{dom} F\). Let us recall that \(\bar{x}\) is a weak solution of \((\text{VOP})\) if and only if \((0_{\mathcal{L}}, -F(\bar{x})) \in \text{epi}_K(F + I_A)^*\) (see [9, Proposition 5.1]). Hence, we get \([h] \iff (i)\) from (5.3) and (5.1), and \([h] \iff (j)\) from (5.3) and (5.2). \(\square\)

**Theorem 5.4 (2nd optimality conditions)**. Let \(\bar{x} \in A \cap \text{dom} F\). Assume all the assumptions of Theorem 5.3. Then \((i)\) and \((j)\) hold.

**Proof.** Under the current assumptions, we get from Theorem 3.15 and Lemma 3.6 that
\[ \bigcup_{T \in \mathcal{L}^+(S,K)} \left[ \bigcap_{v \in I^-_S(T)} \text{epi}_K(F + I_C + T \circ G)^* + (0_{\mathcal{L}}, v) \right] \]
is a closed subset of \(\mathcal{L}(X,Y) \times Y\). So, \((h)\) holds and the conclusion comes from Theorem 5.3. \(\square\)

We now revisit Example 3.10 paying attention to statements \((f)\) and \((i)\), whose common right-hand side set is
\[ (F + I_C + (t_1, t_2) \circ G)^* (0_{\mathcal{L}}) + \mathbb{R}^2_+ = \begin{cases} (\mathbb{R} \times \mathbb{R}_+) \cup (\mathbb{R}_+ \times \mathbb{R}), & \text{if } t_1 = t_2 = 0, \\ \{+\infty\}, & \text{if } t_1 t_2 > 0, \\ \mathbb{R} (t_1, t_2) + \mathbb{R}^2_+, & \text{else}. \end{cases} \]

It can be easily realized that the elements of \(\mathcal{L}^+(S,K)\) and \(\mathcal{L}^+_+(S,K)\) satisfying the optimality conditions in \((f)\) and \((i)\) are those of
\[ \{(t_1, t_2) \in \mathbb{R}^2_+ : t_1 t_2 \leq 0\} = [\mathbb{R}_+ \times \{0\}] \cup \{0\} \times \mathbb{R}_+ \]
and
\[ \{(t_1, t_2) \in (\mathbb{R} \times \mathbb{R}_+) \cup (\mathbb{R}_+ \times \mathbb{R}) : t_1 t_2 \leq 0\} = [- (\mathbb{R}_+) \times \mathbb{R}_+] \cup (\mathbb{R}_+ \times \mathbb{R}_+) \cup (\mathbb{R}_+ \times - (\mathbb{R}_+) \}, \]
respectively. Since both sets are nonempty, we get the trivial conclusion (as \(F \equiv 0_{\mathcal{L}}\)) that any feasible solution \(\mathcal{F}\) is weakly minimal.

It is worth mentioning that the Farkas-type results in Section 4 and the optimality conditions in this section can be used to derive duality results for \((\text{VOP})\). For instance, the corollary proved below extends the strong duality result [4, Theorem 4.2.7], which was established under the assumption that \((c_1)\) holds.
In [3] p. 138], the dual problem of (VOP) with respect to weakly efficient solutions is defined as:

\[(DVOP) \quad \text{WMax} \{ y : (T, y) \in B \}, \]

where the dual feasible set \(B\) is the set of pairs \((T, y) \in \mathcal{L}(S, K) \times Y\) such that there is no \(x \in C \cap \text{dom} G\) such that \((F + T \circ G)(x) <_K y\). Equivalently,

\[
B = \{(T, y) \in \mathcal{L}(S, K) \times Y : y \notin (F + T \circ G)(C \cap \text{dom} G) + \text{int} K\}.
\]

**Corollary 5.5** (Strong duality for the pair (VOP) - (DVOP)). Let \(F : X \to Y \cup \{+\infty\}\) be a proper \(K\)-convex mapping, \(C\) be a nonempty convex subset of \(X\), and \(G : X \to Z \cup \{+\infty\}\) be a proper \(S\)-convex mapping. Assume that \(A \cap \text{dom} F \neq \emptyset\) and one of the conditions \((c_1), (c_2),\) or \((c_3)\) is fulfilled. If \(\bar{x} \in A\) is a solution of (VOP), then there exists a solution \((T, \bar{y})\) of (DVOP) such that \(F(\bar{x}) = \bar{y}\).

**Proof.** Denote \(M := \{y : (T, y) \in B\}\) and assume that \(\bar{x}\) is a solution of (VOP). It follows from Theorem 5.2 that \((g)\) holds, i.e., there exists \(T \in \mathcal{L}(S, K)\) such that \((\bar{T}, F(\bar{x})) \in B\). Hence, \(F(\bar{x}) \in M\).

Now, take an arbitrary \(y \in M\). By the definition of the set \(M\),

\[
\exists T_0 \in \mathcal{L}(S, K) : y \notin (F + T_0 \circ G)(C \cap \text{dom} G) + \text{int} K. \tag{5.4}
\]

Since \(\bar{x} \in A\), \(G(\bar{x}) \in -S\) and so \(-T_0 \circ G(\bar{x}) \in K\). On the other hand, one gets from (5.4) that \([y - F(\bar{x})] + [-T_0 \circ G](\bar{x})] \notin \text{int} K\), and hence, \(y - F(\bar{x}) \notin \text{int} K\) as \(\text{int} K + K = \text{int} K\). Since this holds for any \(y \in M\), one gets \(F(\bar{x}) \notin M - \text{int} K\).

We have just shown that \(F(\bar{x}) \in M \setminus (M - \text{int} K)\), which yields that \(M \neq \emptyset\) and \(\text{WSup} M \neq \{+\infty\}\) (see 2.3). It now follows from (2.4) that \(F(\bar{x}) \in \text{WMax} M\). So \((\bar{T}, F(\bar{x}))\) is a solution of (DVOP) and the proof is complete. \(\square\)

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