KP line solitons and Tamari lattices

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Abstract

The Kadomtsev–Petviashvili (KP) II equation possesses a class of line soliton solutions which can be qualitatively described via a tropical approximation as a chain of rooted binary trees, except at ‘critical’ events where a transition to a different rooted binary tree takes place. We prove that these correspond to maximal chains in Tamari lattices (which are poset structures on associahedra). We further derive results that allow us to compute details of the evolution, including the critical events. Moreover, we present some insights into the structure of the more general line soliton solutions. All this yields a characterization of possible evolutions of line soliton patterns on a shallow fluid surface (provided that the KP-II approximation applies).

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The Kadomtsev–Petviashvili (KP) II equation possesses exact solutions consisting of an arbitrary number of line solitons [1–5]. More comprehensive studies of the structure of the rather complex networks emerging in this way have been undertaken quite recently [6–18] (also see the review [19] and the references cited therein). Whereas in these works a classification in terms of the asymptotic behavior at large negative and positive times, and large (positive or negative) values of the coordinate transverse to the main propagation direction, has been addressed, in the present work we proceed toward an understanding of the full evolution.

It is rather difficult to generate specific line soliton patterns in a laboratory (but see [18, 19] for recent progress). In order to test the validity of the KP approximation, there is at least the possibility of generating such networks by chance and then to observe their evolution qualitatively, i.e. as a (time-ordered) sequence of certain patterns. For a subclass
The tree on the left side represents the Tamari lattice $\mathcal{T}_1$ which consists of a single node. On the right side, a corresponding representation of $\mathcal{T}_2$ is shown.

The Tamari lattice $\mathcal{T}_n$ can be defined as a partially ordered set (poset) in which the elements consist of different ways of grouping a sequence of $n + 1$ objects into pairs using parentheses (binary bracketing). The partial order is imposed by allowing only a rightward application of the associativity law: $(ab)c \rightarrow a(bc)$. $\mathcal{T}_1$ has a single element, $(ab)$, which can also be represented as the rooted binary tree on the left side in figure 1. $\mathcal{T}_2$ is given by $(ab)c \rightarrow a(bc)$, which corresponds to the two rooted binary trees on the right of figure 1.

For a sequence of four objects $abcd$, the five possible groupings are $((ab)c)d$, $(a(bc))d$, $a((bc)d)$, $a(b(cd))$, and $(ab)(cd)$. The Tamari lattice $\mathcal{T}_3$ then consists of the two chains $((ab)c)d \rightarrow (a(bc))d \rightarrow a((bc)d) \rightarrow a(b(cd))$ and $((ab)c)d \rightarrow (ab)(cd) \rightarrow a(b(cd))$, and thus forms a pentagon.

In section 2, we specify the class of KP line soliton solutions, which is the central object of this work, and demonstrate their rooted tree structure. In section 3, we make further steps toward a classification of such solutions as evolutions of rooted trees. This somewhat pedagogical approach is supplemented by general results derived in appendix A. Section 4 presents some insights concerning the understanding of general line soliton solutions. Section 5 draws some conclusions and briefly summarizes further results, elaborated in additional appendices.

2. Rooted tree structure of the simplest class of KP line soliton solutions

Writing the variable $u$ of the KP equation as

$$u = 2 \log(\tau)_{xx},$$

with a function $\tau$, the KP equation

$$(-4u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0$$

(where e.g. $u_t = \partial u/\partial x$) is transformed into the Hirota bilinear form

$$4(\tau_{xt} - \tau_x \tau_t) - 3(\tau_{xy} \tau_y - \tau_x \tau_{xy}) - \tau_x \tau_{xxx} + 4\tau_x \tau_{xxx} - 3\tau_{xx}^2 = 0.$$  

The simplest class of line soliton solutions is then given by

$$\tau = \sum_{k=1}^{M+1} e^{\theta_k}, \quad \theta_k = p_k x + p_k^2 y + c_k,$$  

\[\text{(2.1)}\]
Figure 2. A contour plot of a line soliton solution with $M = 2$ at a fixed time. This is an example of a ‘Miles resonance’ [37]. The thin lines are boundary lines between two phase regions that are dominated by another phase, so that they are not visible in a line soliton plot. This solution keeps its form for all times (while moving from the right to the left) and can be represented by the trivial Tamari lattice $T_1$. The right figure shows a plot of $\log(\tau)$ at the same time. It confirms the tropical description of the line soliton solution as the boundary between three planes determined by the tropical function $\max\{\theta_1, \theta_2, \theta_3\}$.

provided we make the replacement

$$c_k \mapsto p_1 t + c_k. \quad (2.2)$$

$p_k, c_k$ are real constants and $M \in \mathbb{N}$. The absorption of the time variable $t$ into the parameter $c_k$ (via the inverse of the above redefinition) is very helpful at the moment. Without restriction of generality we can assume that $p_1 < \cdots < p_{M+1}$. The $xy$-plane is divided into regions dominated by one of the phases (also see [8]). Let us call the region where $\theta_i > \theta_k$, for all $k \neq i$, the $\theta_i$-region. There we have

$$\log(\tau) = \theta_i + \log \left( 1 + \sum_{j=1}^{M+1} e^{-\theta_j} \right) \simeq \theta_i,$$

where the approximation is valid sufficiently far away from the boundary. Hence

$$\log(\tau) \simeq \max\{\theta_1, \ldots, \theta_{M+1}\},$$

where the right-hand side can be regarded as a tropical version of $\log(\tau)$ (also see appendix D).

Away from the boundary of a dominating phase region, $\max\{\theta_1, \ldots, \theta_{M+1}\}$ is linear in $x$, hence $u$ vanishes. A line soliton branch thus corresponds to a boundary line between two dominating phase regions. This is the picture that underlies our approach toward a classification of KP line soliton solutions. For $M = 1$ we have a single line soliton. Figure 2 shows the case $M = 2$.

For $i \neq j$, we have

$$\theta_i - \theta_j = (p_i - p_j)[x - x_{ij}(y)],$$

where

$$x_{ij}(y) = -(p_i + p_j)y - c_{ij} = x_{ji}(y), \quad c_{ij} = \frac{c_i - c_j}{p_i - p_j} = c_{ji}.$$  

Hence $\theta_i = \theta_j$ determines the boundary line $x = x_{ij}(y)$ between the region where $\theta_i$ dominates $\theta_j$ and the region where $\theta_j$ dominates $\theta_i$. Such a line cannot be parallel to the $x$-axis. Consequently it divides the plane into a left and a right part.

**Proposition 2.1.** For $p_i < p_j$ we have

$$\theta_i \succeq \theta_j \quad \text{for} \quad x \preceq x_{ij}(y, t),$$

$$\theta_i \preceq \theta_j \quad \text{for} \quad x \succeq x_{ij}(y, t).$$
i.e. $\theta_i$ dominates $\theta_j$ on the left side of the line $x = x_{ij}(y,t)$, and vice versa on the right side.

A particular consequence is that for all $i = 1, \ldots, M+1$, the $\theta_i$-region is convex, and thus in particular connected. For $M > 1$ we have the identity

$$x_{ij}(y) - x_{ik}(y) = (p_k - p_j)(y - y_{ijk}) \quad \text{with} \quad y_{ijk} = -c_{ijk},$$

where

$$c_{ijk} = \frac{c_{ij} - c_{ik}}{p_j - p_k} = \frac{c_i}{(p_i - p_j)(p_i - p_k)} + \frac{c_j}{(p_j - p_k)(p_j - p_i)} + \frac{c_k}{(p_k - p_i)(p_k - p_j)} = \frac{c_i}{(p_i - p_j)(p_i - p_k)} + \text{cyclic permutations}$$

is totally symmetric (i.e. invariant under arbitrary permutations of $i, j, k$). It follows that the boundary lines $(x_{ij}(y), y)$ and $(x_{ik}(y), y)$ meet at the point

$$P_{ijk} = (x_{ijk}, y_{ijk}),$$

where

$$x_{ijk} = x_{ij}(y_{ijk}) = \frac{c_i(p_j^2 - p_k^2) + c_j(p_k^2 - p_i^2) + c_k(p_i^2 - p_j^2)}{(p_i - p_j)(p_j - p_k)(p_k - p_i)}.$$

It further follows that the line $(x_{jk}(y), y)$ also passes through $P_{ijk}$. At the ‘critical point’ $P_{ijk}$ we have $\theta_i = \theta_j = \theta_k$ (also see figure 2).

**Proposition 2.2.** Let $p_i < p_j < p_k$. Then

$$x_{ij}(y) < x_{ik}(y) < x_{jk}(y) \quad \text{for} \quad y < y_{ijk}$$

$$x_{ij}(y) > x_{ik}(y) > x_{jk}(y) \quad \text{for} \quad y > y_{ijk}.$$

**Proof.** This is an immediate consequence of (2.3). \qed

A (part of a) boundary line $x = x_{ij}(y)$ is called non-visible if it lies in a region where $\theta_i$ (or $\theta_j$) is dominated by another phase. Otherwise it is called visible. Correspondingly, the critical points can be classified as visible or non-visible. Visibility of $P_{ijk}$ means that at this point the $\theta_i$, $\theta_j$, and $\theta_k$-region meet.

**Proposition 2.3.** Let $p_i < p_j < p_k$. Then the half-lines \{ $x = x_{ij}(y) \mid y > y_{ijk}$ \}, \{ $x = x_{ik}(y) \mid y > y_{ijk}$ \} and \{ $x = x_{jk}(y) \mid y < y_{ijk}$ \} are non-visible.

**Proof.** The following identity is easily verified:

$$\theta_k - \theta_i = (p_k - p_i)[x - x_{ij}(y) + (p_k - p_j)(y - y_{ijk})].$$

Hence, along $x = x_{ij}(y), y > y_{ijk}$, $\theta_k$ dominates $\theta_i$, so that this half-line is non-visible. The same identity written in the form

$$\theta_i - \theta_k = -(p_k - p_i)[x - x_{jk}(y) - (p_j - p_i)(y - y_{ijk})],$$

respectively

$$\theta_j - \theta_i = (p_j - p_i)[x - x_{ik}(y) - (p_k - p_j)(y - y_{ijk})],$$

implies the non-visibility of the other two half-lines. \qed

As a consequence of the last proposition, if $P_{ijk}$ is visible, then only the half-lines \{ $x = x_{ij}(y) \mid y < y_{ijk}$ \}, \{ $x = x_{ik}(y) \mid y < y_{ijk}$ \} and \{ $x = x_{jk}(y) \mid y > y_{ijk}$ \} are visible in a neighborhood of $P_{ijk}$ (figure 2 shows the case $M = 2$). Figure 3 summarizes the process connected with the passage of $y$ through a critical value, corresponding to a visible critical
Figure 3. The left figure expresses what happens when $y$ passes a critical value $y_{ijk}$ corresponding to a visible critical point $P_{ijk}$ with $p_i < p_j < p_k$. The nodes coincide if $y = y_{ijk}$. For $y < y_{ijk}$ (left side of the left figure), $x_{ij}$ and $x_{jkl}$ are visible but not $x_{ikl}$, and their order $x_{ij} < x_{jkl}$ is expressed by the direction of the edges. For $y > y_{ijk}$ (vertical chain in the left figure), $x_{ikl}$ is visible, but not the other two. The right figure shows the tetrahedron poset obtained in this way for $M = 3$ with $p_1 < p_2 < p_3 < p_4$. With each of its faces a critical value of $y$ is associated (and a corresponding ‘higher order’ arrow), as in the left figure. Hence, e.g. for $y < \min\{y_{ijk}\}$ we have the chain $x_{12} < x_{23} < x_{34}$.

This gives a rule to construct a poset for each $M > 1$. The nodes are the phases and an edge is directed from $\theta_i$ to $\theta_j$ if $i < j$, assuming that $p_1 < \cdots < p_{M+1}$. We assign $x_{ij}$ to the corresponding edge. For $M = 2$ this yields a poset structure on a triangle, and more generally on the complete graph on $M + 1$ nodes, which can be viewed as an $M$-simplex.

**Proposition 2.4.** Let $p_1 < p_2 < \cdots < p_{M+1}$.

(1) For $y > \max\{y_{ijk}\}$, only the half-line $x = x_{1,M+1}(y)$ is visible.

(2) For $y < \min\{y_{ijk}\}$, all the half-lines $x = x_{m,M+1}(y)$, $m = 1, \ldots, M$, are visible, and no other.

**Proof.** The following is a special case of the identity already used in the proof of proposition 2.3:

$$\theta_1 - \theta_n = (p_1 - p_n)[x - x_{1,M+1}(y) - (p_{M+1} - p_n)(y - y_{1,n,M+1})].$$

This implies $\theta_1 > \theta_n$ along $x = x_{1,M+1}(y)$, $y > \max\{y_{ijk}\}$, for $n = 2, \ldots, M$. Hence $x = x_{1,M+1}(y)$ is visible for large enough $y$. According to proposition 2.3, all other lines are non-visible for large enough $y$. This proves (1). We also have

$$\theta_m - \theta_n = (p_m - p_n)[x - x_{m,M+1}(y) - (p_{M+1} - p_n)(y - y_{m,m+1,n})].$$

Along $x = x_{m,M+1}(y)$, $y < \min\{y_{ijk}\}$, it implies $\theta_m > \theta_n$ for all $n \neq m, m + 1$. As a consequence, this line is visible for large enough negative $y$, and this holds for $m = 1, \ldots, M$. Again, proposition 2.3 forbids other lines to be visible for large enough negative $y$, and this proves (2).

The last result (also see [13, 19]) implies the following asymptotic structure of a line soliton graph (from the restricted class considered in this section), see figure 4. For large enough $y$ there is only a single half-line. For large enough negative $y$ one observes $M$ lines. In

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7 The alert reader will notice that figure 3 uses a notation of higher category theory. Indeed, the structures appearing in this work provide corresponding examples.
Figure 4. The asymptotic structure of a line soliton graph with $p_1 < p_2 < \cdots < p_{M+1}$. The order of the dominating phase regions is a consequence of $x_{m,m+1} < x_{m+1,m+2}$, $m = 1, \ldots, M - 1$, for $y$ smaller than all of its critical values, according to proposition 2.2.

3. Time evolution of line soliton patterns

3.1. The first step

Let us reintroduce the time variable $t$ (which we hid away in the preceding section) via the replacement (2.2). Then we have

$$x_{ij}(y, t) = -(p_i + p_j)y - (p_i^2 + p_i p_j + p_j^2)t - c_{ij},$$

$$y_{ijk}(t) = -(p_i + p_j + p_k)t - c_{ijk},$$

$$x_{ijk}(t) = x_{ij}(y_{ijk}(t), t),$$

where $c_{ij}$ and $c_{ijk}$ are given by the previous formulas. The critical points $P_{ijkl}$ now depend on $t$; hence, they constitute ‘critical lines’ in $\mathbb{R}^3$ (with coordinates $x, y, t$). For $M > 2$ we have the identity

$$y_{ijk}(t) - y_{ijl}(t) = (p_l - p_k)(t - t_{ijkl}),$$

with $t_{ijkl} = -c_{ijkl}$, (3.1)

where

$$c_{ijkl} = c_{ijk} - c_{ijl} = \frac{c_i}{p_k - p_l} - \frac{c_j}{p_l - p_k} + \frac{c_k}{p_i - p_l} - \frac{c_l}{p_i - p_k},$$

is totally symmetric in the indices $i, j, k, l$. It follows that two critical points $P_{ijk}(t)$ and $P_{ijl}(t)$ coincide (only) at the time given by $t_{ijkl}$. Furthermore, at this value of time it turns out that $P_{ikl}$ and $P_{jkl}$ also coincide with this point. Hence we actually have a coincidence of (at least) four critical points. At the ‘critical event’

$$(P_{ijkl}, t_{ijkl}) \in \mathbb{R}^3 \quad \text{with} \quad P_{ijkl} := P_{ijl}(t_{ijkl}),$$

where the four critical lines intersect, we have $\theta_i = \theta_j = \theta_k = \theta_l$. For $M = 3$, there is only a single critical time, namely $t_{1234}$, and $P_{1234}$ is visible at $t = t_{1234}$, i.e. a meeting point of line soliton branches (see figure 5). For $M > 3$, there are $(M+1)$ critical times, and the situation is more involved.

8 This is not true for more general line soliton solutions (also see section 4), outside the class considered here.
Figure 5. Evolution of a line soliton structure with $M = 3$. These are snapshots at times $t < t_{1234}$ (left), $t = t_{1234}$ (middle) and $t > t_{1234}$ (right). Again, thin lines are boundary lines between two phase regions that are dominated by another phase and hence not visible in a line soliton plot. Disregarding the degenerate configuration at $t = t_{1234}$, this evolution can obviously be represented by the single chain of which the Tamari lattice $\mathbb{T}_2$ consists (see figure 1).

Proposition 3.1. Let $p_i < p_j < p_k < p_l$. Then,

$$y_{ij}(t) < y_{ij}(t) < y_{ik}(t) < y_{ik}(t) \quad \text{for } t < t_{ijkl},$$
$$y_{ij}(t) > y_{ij}(t) > y_{ik}(t) > y_{ik}(t) \quad \text{for } t > t_{ijkl}.$$ 

Proof. This is an immediate consequence of identity (3.1). □

Proposition 3.2. Let $p_i < p_j < p_k < p_l$. Then,

(1) $P_{ij}(t)$ and $P_{jkl}(t)$ are non-visible for $t < t_{ijkl}$,
(2) $P_{ijk}(t)$ and $P_{ikl}(t)$ are non-visible for $t > t_{ijkl}$.

Proof. An identity used in the proofs of some propositions in section 2 generalizes via (2.2) to

$$\theta_i - \theta_k = (p_l - p_k)(p_k - p_j)(p_j - p_i)(t - t_{ijkl}).$$

Evaluating this at $P_{ij}(t)$, we obtain

$$\theta_i - \theta_k = (p_l - p_k)(p_k - p_j)(p_j - p_i)(t - t_{ijkl}),$$

which is negative if $t < t_{ijkl}$; hence, $P_{ij}(t)$ is then non-visible. A similar argument applies in the other cases. □

Proposition 3.3.

(1) For $t < \min\{t_{ijkl}\}$ only the critical points $P_{1,m,m+1}(t)$, $m = 2, \ldots, M$, are visible,
(2) For $t > \max\{t_{ijkl}\}$ only the critical points $P_{m-1,m,M+1}(t)$, $m = 2, \ldots, M$, are visible.

Proof. At $P_{1,m,m+1}(t)$ we have

$$\theta_1 - \theta_n = -(p_n - p_1)(p_n - p_m)(p_m - p_{m+1})(t - t_{1,m,m+1,n}),$$

which, for $t$ smaller than all of its critical values, is positive for all $n$ different from $1, m, m + 1$. Proposition 3.2, part 1, shows that all other critical points are non-visible.

At $P_{m-1,m,M+1}(t)$ we have

$$\theta_{M+1} - \theta_0 = (p_{M+1} - p_n)(p_n - p_m)(p_m - p_{m-1})(t - t_{n,m-1,m,M+1}).$$
Figure 6. The structure of the solution for $t < \min\{t_{ijkl}\}$ (left tree) and $t > \max\{t_{ijkl}\}$ (right tree). For fixed $M$ these two trees form the maximal and the minimal element, respectively, of a Tamari lattice.

Figure 7. The evolution through a critical time $t_{ijkl}$ with the visible critical point $P_{ijkl}$, where $p_1 < p_j < p_k < p_l$. It amounts to a right-rotation (see e.g. [38]) applied to the first binary tree. This expresses a central feature of Tamari lattices, the rightward application of the associativity law mentioned in the introduction, in the language of binary trees (also see [23, 24, 26, 32, 33, 39]). It has to be considered as a local process, i.e. a binary tree displayed in this figure typically appears as a substructure of a bigger binary tree.

which, for $t$ greater than all of its critical values, is positive for all $n$ different from $m - 1, M, M + 1$. Proposition 3.2, part 2, shows that all other critical points are non-visible. \hfill \square

Collecting our results, for $t$ smaller than all of its critical values, the line soliton pattern can be represented by the left graph in figure 6 (note that $y_{1,m,m+1} < y_{1,m+1,m+2}, m = 2, \ldots, M - 1$, according to proposition 3.1), and for $t$ greater than all of its critical values by the right graph (since then $y_{m,m+1,M+1} > y_{m+1,m+2,M+1}, m = 1, \ldots, M - 2$).

Together with proposition 3.2, the next proposition describes what happens when time passes a critical value $t_{ijkl}$ with a visible critical point $P_{ijkl}$, also see figure 7. In particular it follows that, disregarding the ‘degenerate’ cases at a critical time, for $M > 1$ the graphs have the structure of a rooted binary tree.

**Proposition 3.4.** Let $p_1 < p_j < p_k < p_l$. If $P_{ijkl}$ is visible at $t = t_{ijkl}$ and not a meeting point of more than four phases, then

1. $P_{ijk}(t)$ and $P_{ikl}(t)$ are visible for $t < t_{ijkl}$,
2. $P_{ijl}(t)$ and $P_{jkl}(t)$ are visible for $t > t_{ijkl}$.

Here $t$ is assumed to be close enough to $t_{ijkl}$ so that no other critical time with a visible critical point is in between.

**Proof.** For $t$ close enough to $t_{ijkl}$, a dominating phase in the vicinity of $P_{ijkl}$ can only be one of the four phases $\theta_i, \theta_j, \theta_k, \theta_l$, as a consequence of the assumptions. As in the proof of proposition 3.2, at $P_{ijl}(t)$ we have

$$\theta_i - \theta_k = (p_k - p_i)(p_k - p_j)(p_l - p_k)(t - t_{ijkl}),$$

which is positive if $t > t_{ijkl}$. This excludes $\theta_k$ as a dominating phase. Since $\theta_i = \theta_j = \theta_l$ at $P_{ijl}(t)$, this critical point is visible. Clearly, $P_{ijl}(t)$ remains visible unless $t$ takes another
critical value with a visible critical point. A similar argument applies to the other critical points.

Let us recall that, disregarding critical time values, any line soliton solution from the class defined in section 2 determines a time-ordered sequence of rooted binary trees (with the same number of leaves). Proposition 3.4 states that the rule according to which the transition from a binary tree to the next takes place is precisely the characteristic property of a Tamari lattice (also see figure 7). This leads to the following conclusion.

**Theorem 3.5.** Each line soliton solution with τ of the form (2.1), $M > 1$, and without coincidences of critical times defines a sequence of rooted binary trees which is a maximal chain in a Tamari lattice.

Up to $M = 5$ we will show explicitly how every maximal chain in $T_{M-1}$ is realized by line soliton solutions. Propositions 3.1, 3.2 and 3.4 have generalizations which are elaborated in appendix A and which will be important in the following. In particular, $x_{ij}, y_{ijk}, t_{ijkl}$ are special cases of (A.5).

Based on results of section 2 (in particular propositions 2.2 and 2.3), a simple recipe to construct soliton binary trees can be formulated. A line soliton binary tree at a fixed time is indeed easily constructed from the sequence of ordered coordinates $y_{ijk}$ of the visible critical points $P_{ijk}$ via

$$x_{ik} \xrightarrow{y_{ijk}} (x_{ij}, x_{jk}),$$

(3.2)

to be applied in the top to bottom direction (assuming $p_i < p_j < p_k$). Here we understand momentarily $x_{ij}$ to represent only the visible part of the line between (then dominating) phase regions $\theta_i$ and $\theta_j$. See figure 8 and also appendix C for further consequences.

The transition to another binary tree at the critical time $t_{ijkl}$, i.e. the ‘rotation’ shown in figure 7, can be expressed as

$$(y_{ikl}, y_{ijk}) \xrightarrow{t_{ijkl}} (y_{ijl}, y_{jkl}),$$

(3.3)

assuming $p_i < p_j < p_k < p_l$. Here, $(y_{ikl}, y_{ijk})$ is a pair of neighbors in the decreasingly ordered sequence of critical y-values that determines a rooted binary tree associated with a line soliton solution at some event. In order to apply this map, it may be necessary to first apply a permutation (see example 3.10 below and also appendix C). The initial rooted binary tree, corresponding to a line soliton solution at large negative values of $t$ (cf proposition 3.3), is determined by the sequence $(y_{1,M,M+1}, y_{1,M-1,M}, \ldots, y_{123})$. If we know the order of all

9 This restriction ensures that at a critical time only a single ‘rotation’ takes place. At a coincidence at least two rotations are applied simultaneously and that means a direct transition in the Tamari lattice to a more remote neighbor on a chain.
critical times $t_{ijkl}$ that correspond to visible events, then (3.3) generates a description of the line soliton evolution as a chain of rooted binary trees.

**Remark 3.6.** In section 2, we met a family of posets associated with simplexes, where the (directed) edges correspond to the critical values of $x$. There is a new family of posets where the nodes are given by the maximal chains in the corresponding poset of the first family. The (directed) edges are associated with the critical values of $y$, which are ordered increasingly from top to bottom along a chain. Now we note that the process determined by propositions 3.1, 3.2 and 3.4, hence (3.3), can be expressed as the graph in figure 9.

For $M = 3$, figure 9 already displays the whole poset, which is thus a tetragon. The top node is given by the chain $x_{12} < x_{23} < x_{34}$, the left and right nodes by $x_{13} < x_{34}$ and $x_{12} < x_{24}$, respectively, and the bottom node by $x_{14}$. These data can be read off from the tetrahedron poset in figure 3. For $M = 4$, we obtain the cube poset in figure 10. The top node is given by the longest maximal chain in the $M = 4$ simplex poset of the first family, which is $x_{12} < x_{23} < x_{34} < x_{45}$. Using the rule expressed by the left graph of figure 3, the nodes in
the next row are \(x_{13} < x_{34} < x_{45}, x_{12} < x_{24} < x_{45}\) and \(x_{12} < x_{23} < x_{35}\), respectively\(^{10}\). In the next lower row we have \(x_{14} < x_{45}, x_{13} < x_{35}\) and \(x_{12} < x_{23} < x_{35}\). The bottom node is given by \(x_{15}\).

For \(M > 4\) we obtain a hypercube.

For \(t < \min\{t_{ijkl}\}\) we read off from the cube in figure 10 the chain \(y_{123} < y_{134} < y_{145}\), which is the initial (rooted binary tree) configuration. If the first critical time is \(t_{1234}\), then a transition to the tree determined by \(y_{234} < y_{124} < y_{145}\) takes place, and for the further time development the only possibility is via the critical time \(t_{1245}\) to \(y_{234} < y_{245} < y_{125}\), and afterward via \(t_{2345}\) to \(y_{345} < y_{235} < y_{125}\), which is the configuration for \(t > \max\{t_{ijkl}\}\). If the first critical time is \(t_{1345}\), then we have a transition to \(y_{123} < y_{345} < y_{135}\). As a rooted binary tree, this is equivalent to the tree given by \(y_{345} < y_{123} < y_{135}\), a transition encoded by the top face in figure 10. For the latter tree, the only possible further transition is via \(t_{1235}\) to the unique final configuration for \(t > \max\{t_{ijkl}\}\). All this results in the Tamari lattice \(T_3\) shown in figure 13 below. We will take a somewhat different route to it in order to be able to determine conditions under which the left or the right chain is realized, corresponding to which of the critical time values \(t_{1234}\) and \(t_{1345}\) is the smaller one.

### 3.2. The second step

To further classify the possible line soliton evolutions with \(M > 3\), we have to look at the cases where some of the critical times are equal. This corresponds to particular choices of the constants \(c_k\). In order to analyze this, it turns out to be convenient to redefine the latter via

\[
c_k \mapsto p_k^4 t^{(4)} + c_k \quad k = 1, \ldots, M + 1,
\]

with a new parameter \(t^{(4)}\). If \(t^{(4)}\) is identified with the next to \(t\) evolution variable of the KP hierarchy, then the function \(u\) (see section 2) also solves the second KP hierarchy equation. It should not be a big surprise that the hierarchy structure plays a simplifying role in the classification problem of line soliton solutions. Let us introduce the complete homogeneous symmetric polynomials

\[
h_m(p_1, \ldots, p_n) = \sum_{a_1 + \cdots + a_n = m} p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n},
\]

where \(a_k \in \mathbb{N} \cup \{0\}\). Then we have (also see (A.5))

\[
x_{ij}(y, t, t^{(4)}) = -h_1(p_i, p_j) y - h_2(p_i, p_j) t - h_3(p_i, p_j) t^{(4)} - c_{ij},
\]

\[
x_{ijk}(t, t^{(4)}) = x_{ij}(y_{ijk}(t, t^{(4)}), t, t^{(4)}),
\]

\[
y_{ijk}(t, t^{(4)}) = -h_1(p_i, p_j, p_k) t - h_2(p_i, p_j, p_k) t^{(4)} - c_{ijk},
\]

\[
t_{ijkl}(t^{(4)}) = -h_1(p_i, p_j, p_k, p_l) t^{(4)} - c_{ijkl},
\]

with \(c_{ij}, c_{ijk}\) and \(c_{ijkl}\) given in terms of \(c_k\) by the previous formulas. We note that now a critical point \(P_{ijkl}\) depends on \(t\) and \(t^{(4)}\); hence, it forms a surface in \(\mathbb{R}^4\). The critical point \(P_{ijkl}\) depends on \(t^{(4)}\); hence, it forms a line in \(\mathbb{R}^4\), which is the intersection of the surfaces corresponding to \(P_{ijk}, P_{ikl}, P_{ijl}, P_{jkl}\).

We find (also see proposition A.3)

\[
t_{ijkl}(t^{(4)}) - t_{ijklm}(t^{(4)}) = (p_m - p_i) (t^{(4)} - t^{(4)}_{ijklm}) \quad \text{with} \quad t^{(4)}_{ijklm} = -c_{ijklm},
\]

\(^{10}\) The combinatorics is simply described as follows. Assign the sequence 12345 to the top node (which stands for the list of all phases \(\theta_i, i = 1, \ldots, 5\)). The next neighbor nodes are obtained by deleting the second, third and forth number, respectively. Hence we obtain 1345, 1245 and 1235. Each of them has two next lower neighbors, obtained by deleting one of the two numbers in the middle. For example, 1345 is connected with 145 and 135. Finally, from these we obtain 15 to represent the bottom node.
where
\[ c_{ijklm} = \frac{c_{ijkl} - c_{ikjm}}{p_l - p_m} = \frac{c_l}{(p_l - p_i)(p_l - p_j)(p_l - p_k)(p_l - p_m)} + \text{cyclic permutations}. \]

If two critical times sharing three indices are equal, i.e. \( t_{ijkl} = t_{ikjm} \), then it follows that (at least) five critical times are equal: \( t_{ijkl} = t_{ijkm} = t_{ijkl} = t_{iklm} = t_{ijkl} \). This occurs when \( t^{(4)} = t^{(4)}_{ijklm} \). At the critical event
\[ (P_{ijklm}, t_{ijkl}(t^{(4)}_{ijklm})), t^{(4)}_{ijklm} \in \mathbb{R}^4, \]
with the projection point
\[ P_{ijklm} := P_{ijkl}(t^{(4)}_{ijklm}) \]
in the \( xy \)-plane, we thus have \( \theta_l = \theta_j = \theta_k = \theta_i = \theta_m \). The following is a direct consequence of (3.4) (also see proposition A.4).

**Proposition 3.7.** If \( p_i < p_j < p_k < p_l < p_m \) we have
\[ t_{ijkl}(t^{(4)}_i) < t_{ikjm}(t^{(4)}_i) < t_{ijklm}(t^{(4)}_i) < t_{iklm}(t^{(4)}_i) < t_{ijklm}(t^{(4)}_i) \]
\[ t_{ijkl}(t^{(4)}_i) < t_{ijklm}(t^{(4)}_i) \]
for \( t^{(4)} = t^{(4)}_{ijklm} \).

The next results are special cases of proposition A.7 and proposition A.8 in appendix A.

**Proposition 3.8.** Let \( p_i < p_j < p_k < p_l < p_m \). Then
1. \( P_{ijkl}(t^{(4)}_i) \) and \( P_{ijkm}(t^{(4)}_i) \) are non-visible for \( t^{(4)} < t^{(4)}_{ijklm} \) and
2. \( P_{ijkl}(t^{(4)}_i) \), \( P_{ijkm}(t^{(4)}_i) \) and \( P_{ijklm}(t^{(4)}_i) \) are non-visible for \( t^{(4)} > t^{(4)}_{ijklm} \).

**Proposition 3.9.** Let \( p_i < p_j < p_k < p_l < p_m \) and suppose \( P_{ijklm} \) is visible at \( t = t_{ijkl} \), \( t^{(4)} = t^{(4)}_{ijklm} \) and not a meeting point of more than five phases. The following holds for values of \( t^{(4)} \) that are close enough to \( t^{(4)}_{ijklm} \), so that no other critical value of \( t^{(4)} \) with a visible projection point is in between.
1. \( P_{ijkl}(t^{(4)}_{ijklm}) \), \( P_{ijkm}(t^{(4)}_{ijklm}) \) and \( P_{ijklm}(t^{(4)}_{ijklm}) \) are visible, at the respective critical time, if \( t^{(4)} < t^{(4)}_{ijklm} \),\(^{11}\)
2. \( P_{ijklm}(t^{(4)}_{ijklm}) \) and \( P_{ijklm}(t^{(4)}_{ijklm}) \) are visible, at the respective critical time, if \( t^{(4)} > t^{(4)}_{ijklm} \).

Figure 11 expresses the subgraph structure determined by propositions 3.7, 3.8 and 3.9 as a process.

**Example 3.10.** Let \( M = 4 \). For any fixed \( t^{(4)} \), we have five critical times \( t_{1234}(t^{(4)}) \), \( t_{1235}(t^{(4)}) \), \( t_{1245}(t^{(4)}) \), \( t_{1345}(t^{(4)}) \), \( t_{2345}(t^{(4)}) \). The corresponding critical events have projection points \( P_{2345}(t^{(4)}) \), \( P_{1345}(t^{(4)}) \), \( P_{2345}(t^{(4)}) \), \( P_{1345}(t^{(4)}) \), \( P_{2345}(t^{(4)}) \), \( P_{1345}(t^{(4)}) \), at which four phases meet. All these critical events coincide for \( t^{(4)} = t^{(4)}_{12345} \). At the associated projection point \( P_{12345} \) all the five phases meet, and it is therefore visible at \( t = t_{12345} \) and \( t^{(4)} = t^{(4)}_{12345} \). A description of the evolution of the line soliton pattern thus has to distinguish the cases \( t^{(4)} < t^{(4)}_{12345} \) and \( t^{(4)} > t^{(4)}_{12345} \).

1. \( t^{(4)} < t^{(4)}_{12345} \). From propositions 3.7, 3.8 and 3.9, we obtain all ‘visible’ critical times and they satisfy \( t_{1234} < t_{1245} < t_{2345} \). Via (3.3) this yields
\[ (y_{1245}, y_{1345}, y_{1235}) \xrightarrow{t_{1234}} (y_{1345}, y_{1245}, y_{2345}) \xrightarrow{t_{1245}} (y_{125}, y_{245}, y_{2345}) \xrightarrow{t_{2345}} (y_{125}, y_{235}, y_{2345}), \]
which translates into the first sequence of rooted binary trees in figure 12.

\(^{11}\) For example, for \( t^{(4)} < t^{(4)}_{ijklm} \), \( P_{ijkl}(t^{(4)}_{ijkl}) \) is visible at \( t = t_{ijkl}(t^{(4)}_{ijkl}) \).
The process connected with the passage of $t^{(4)}$ through a critical value $t_{ijklm}^{(4)}$ with the visible critical point $P_{ijklm}$ where $p_i < p_j < p_k < p_l < p_m$. Here $ijkl$ stands for $t_{ijkl}$ and $ijklm$ for $t_{ijklm}^{(4)}$. The left chain corresponds to $t^{(4)} < t_{ijklm}^{(4)}$, and the right to $t^{(4)} > t_{ijklm}^{(4)}$. Also see figure 13 for a special case.

![Figure 11](image1.png)

Figure 11. The process connected with the passage of $t^{(4)}$ through a critical value $t_{ijklm}^{(4)}$ with the visible critical point $P_{ijklm}$ where $p_i < p_j < p_k < p_l < p_m$. Here $ijkl$ stands for $t_{ijkl}$ and $ijklm$ for $t_{ijklm}^{(4)}$. The left chain corresponds to $t^{(4)} < t_{ijklm}^{(4)}$, and the right to $t^{(4)} > t_{ijklm}^{(4)}$. Also see figure 13 for a special case.

![Figure 12](image2.png)

Figure 12. Evolution of line soliton patterns with $M = 4$ and $t^{(4)} < t_{12345}^{(4)}$ (first chain), respectively $t^{(4)} > t_{12345}^{(4)}$ (second chain). These are the two maximal chains in the Tamari lattice $T_3$, which forms a pentagon (see figure 13). Instead of assigning $t$-intervals to the trees, it is convenient to assign the corresponding critical values $t_{ijkl}$ to the arrows, i.e. the edges of the Tamari lattice (as in figures 11 and 13).

(2) $t^{(4)} > t_{12345}^{(4)}$. Then, the ‘visible’ critical times satisfy $t_{1345} < t_{1235}$. This leads to

\[(y_{145}, y_{134}, y_{123}) \xrightarrow{t_{1345}} (y_{135}, y_{345}, y_{123}) \xrightarrow{\text{permutation}} (y_{135}, y_{123}, y_{345}) \xrightarrow{t_{1235}} (y_{125}, y_{235}, y_{345}),\]

which translates into the second chain in figure 12. The tree in the middle allows for the two possibilities $y_{123} < y_{345}$ and $y_{345} < y_{123}$ (in accordance with proposition 3.1). A permutation is necessary in order to be able to apply (3.3) with the second critical time to the respective pair of neighbors. This makes sense if we regard the two possibilities as equivalent (and this has been done in figure 12). Resolving the ‘fine structure’, by determining the event where $y_{123} = y_{345}$, they can be distinguished in a setting of trees with levels [30], see appendix B.

The two sequences of rooted binary trees obtained for $t^{(4)} < t_{12345}^{(4)}$, respectively $t^{(4)} > t_{12345}^{(4)}$, are the two maximal chains in the Tamari lattice $T_3$ (see figure 13).

3.3. The third step

For $M > 4$ we redefine the constants $c_k$ once more,

\[c_k \mapsto p_k t^{(5)} + c_k \quad k = 1, \ldots, M + 1,\]
Figure 13. Representation of the Tamari lattice $T_3$ by line soliton graphs (which is a special case of figure 11; also see remark 3.6). The left chain is realized if $t^{(4)} < t^{(4)}_{12345}$, and the right chain if $t^{(4)} > t^{(4)}_{12345}$ (also see figure 12). At $t = t^{(4)}_{1234}$, a direct transition takes place from the uppermost to the lowermost tree.

with a new parameter $t^{(5)}$. Then we have

$$x_{ij}(y, t, t^{(4)}, t^{(5)}) = -h_1(p_i, p_j)y - h_2(p_i, p_j)t - h_3(p_i, p_j)t^{(4)} - h_4(p_i, p_j)t^{(5)} - c_{ij},$$

$$x_{ijk}(t^{(4)}, t^{(5)}) = x_{ij}(y_{ijk}(t, t^{(4)}, t^{(5)}), t, t^{(4)}, t^{(5)}),$$

$$y_{ijk}(t, t^{(4)}, t^{(5)}) = -h_1(p_i, p_j, p_k)t - h_2(p_i, p_j, p_k)t^{(4)} - h_3(p_i, p_j, p_k)t^{(5)} - c_{ijk},$$

$$t^{(4)}_{ijkl}(t^{(5)}) = h_1(p_i, p_j, p_k, p_l)t^{(4)} - h_2(p_i, p_j, p_k, p_l)t^{(5)} - c_{ijkl},$$

$$t^{(4)}_{ijklm}(t^{(5)}) = -h_1(p_i, p_j, p_k, p_l, p_m)t^{(5)} - c_{ijklm},$$

with $c_{ij}$, $c_{ijk}$, $c_{ijkl}$ and $c_{ijklm}$ as defined previously (also see (A.4)). Coincidences of critical values of $t^{(4)}$ can only occur at the following critical values of $t^{(5)}$:

$$t^{(5)}_{ijklmn} = -c_{ijklmn},$$

where

$$c_{ijklmn} = c_{ijklm} - c_{ijkl} =\frac{c_i}{(p_i - p_j)(p_i - p_k)(p_l - p_m)(p_l - p_n) + \text{cyclic permutations}}.$$}

This follows from the identity (also see proposition A.3)

$$t^{(4)}_{ijklmn} - t^{(4)}_{ijkl} = (p_{n} - p_{m})(t^{(5)} - t^{(5)}_{ijklmn}).$$

Furthermore, at $t^{(5)} = t^{(5)}_{ijklmn}$ we have $t^{(4)}_{ijklmn} = t^{(4)}_{ijkl} = t^{(4)}_{ijklmn} = t^{(4)}_{ijklmn} = t^{(4)}_{ijklmn} = t^{(4)}_{ijklmn}$. At this critical event (now a point in $\mathbb{R}^5$ with coordinates $x, y, t, t^{(4)}, t^{(5)}$) having the projection

$$P_{ijklmn} := P_{ijklmn}(t^{(5)}_{ijklmn})$$

in the $xy$-plane, we have $\theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 = \theta_6$. The following is a consequence of (3.5) (also see proposition A.4).

**Proposition 3.11.** If $p_i < p_j < p_k < p_l < p_m < p_n$, then

$$t^{(4)}_{ijklmn} < t^{(4)}_{ijkl} < t^{(4)}_{ijklmn} < t^{(4)}_{ijklmn} < t^{(4)}_{ijklmn} < t^{(4)}_{ijklmn} < t^{(4)}_{ijklmn}$$

for $t^{(5)} < t^{(5)}_{ijklmn}$.
The next two propositions are special cases of proposition A.7 and proposition A.8, respectively.

**Proposition 3.12.** Let \( p_1 < p_j < p_k < p_l < p_m < p_n \). Then,

1. \( P_{ijkl}(t^{(5)}), P_{ijklm}(t^{(5)}) \) and \( P_{ijklmn}(t^{(5)}) \) are non-visible for \( t^{(5)} < t^{(5)}_{ijklmn} \), and
2. \( P_{ijkl}(t^{(5)}), P_{ijklm}(t^{(5)}) \) and \( P_{ijklmn}(t^{(5)}) \) are non-visible for \( t^{(5)} > t^{(5)}_{ijklmn} \).

**Proposition 3.13.** Let \( p_1 < p_j < p_k < p_l < p_m < p_n \) and suppose that \( P_{ijklmn} \) is visible at \( t = t_{ijkl} \), \( t^{(4)} = t^{(4)}_{ijklmn} \), \( t^{(5)} = t^{(5)}_{ijklmn} \) and not a meeting point of more than six phases. The following holds for values of \( t^{(5)} \) that are close enough to \( t^{(5)}_{ijklmn} \) so that no other critical value of \( t^{(5)} \) with a visible projection point is in between.

1. \( P_{ijkl}(t^{(5)}), P_{ijklm}(t^{(5)}) \) and \( P_{ijklmn}(t^{(5)}) \) are visible, at the respective critical values of \( t \) and \( t^{(4)} \), if \( t^{(5)} < t^{(5)}_{ijklmn} \).
2. \( P_{ijkl}(t^{(5)}), P_{ijklm}(t^{(5)}) \) and \( P_{ijklmn}(t^{(5)}) \) are visible, at the respective critical values of \( t \) and \( t^{(4)} \), if \( t^{(5)} > t^{(5)}_{ijklmn} \).

Figure 14 expresses a consequence of propositions 3.11, 3.12 and 3.13 as a process.

**Example 3.14.** For \( M = 5 \) there is only a single critical value of \( t^{(5)} \), namely \( t^{(5)}_{123456} \), and \( P_{123456} \) is visible at \( t = t_{123456} \), \( t^{(4)} = t^{(4)}_{123456} \) and \( t^{(5)} = t^{(5)}_{123456} \), as a meeting point of all six phases. There are six critical values of \( t^{(4)} \), namely \( t^{(4)}_{123456}, t^{(4)}_{123465}, t^{(4)}_{123546}, t^{(4)}_{125436}, t^{(4)}_{142356} \).

For \( t^{(5)} < t^{(5)}_{123456} \), according to proposition 3.11 we have to distinguish the cases where (1) \( t^{(4)} < t^{(4)}_{123456} \), (2) \( t^{(4)}_{123456} < t^{(4)} < t^{(4)}_{123465} \), (3) \( t^{(4)}_{123456} < t^{(4)} < t^{(4)}_{123546} \) and (4) \( t^{(4)}_{123456} < t^{(4)} \). In case (1) we obtain from proposition 3.7 the inequalities (a) \( t_{1234} < t_{1235} < t_{1245} < t_{1345} < t_{1346} < t_{1246} < t_{1346} < t_{1236} < t_{1234} \), (b) \( t_{1234} < t_{1235} < t_{1245} < t_{1345} < t_{1346} < t_{1246} < t_{1346} < t_{1236} \), (c) \( t_{1235} < t_{1234} < t_{1236} < t_{1245} < t_{1345} < t_{1346} < t_{1246} < t_{1346} \), (d) \( t_{1236} < t_{1234} < t_{1235} < t_{1245} < t_{1345} < t_{1346} < t_{1246} < t_{1346} \), (e) \( t_{1234} < t_{1235} < t_{1245} < t_{1345} < t_{1346} < t_{1246} < t_{1346} < t_{1236} \), and (f) \( t_{1234} < t_{1235} < t_{1245} < t_{1345} < t_{1346} < t_{1246} < t_{1346} \). According to proposition 3.8, the critical points appearing at times \( t_{1234}, t_{1235}, t_{1245}, t_{1345}, t_{1346}, t_{1246} \) are non-visible. Their elimination leads to (a) \( t_{1234} < t_{1235} < t_{1245} < t_{1345} \), (b) \( t_{1234} < t_{1235} < t_{1245} < t_{1345} \), (d) \( t_{1245} < t_{1236} \), (e) \( t_{1345} < t_{1346} < t_{1346} \), and the union determines the second poset in...
Figure 15. Intermediate step in the derivation of the possible line soliton evolutions for $M = 5$ and $t^{(4)} < \min\{t^{(4)}\}_{ijklm}$ (pair of posets on the left), respectively $t^{(4)} > \max\{t^{(4)}\}_{ijklm}$ (pair of posets on the right). In both cases the first diagram is obtained as the union of all sequences of ordered critical times. The second diagram then results by dropping those critical times for which the critical event (or rather its projection in the $xy$-plane) is non-visible (and removing redundant edges). A four-digit number stands for the corresponding critical time.

Figure 16. The two possible evolutions for $M = 5$ and $t^{(4)} < \min\{t^{(4)}\}_{ijklm}$. They correspond to chains in the Tamari lattice $T_4$.

Figure 17. The evolution for $M = 5$ and $t^{(4)} > \max\{t^{(4)}\}_{ijklm}$, another chain in the Tamari lattice $T_4$.

Since proposition A.9 does not identify any of the remaining critical times as ‘non-visible’ (note that $t^{(4)}_{12345}, t^{(4)}_{12356}$ and $t^{(4)}_{13456}$ correspond to visible events according to proposition 3.13), we can refer to proposition A.10 in order to conclude that they all correspond to visible events only.

We are not aware of a general argument why the union of sequences of ordered critical times, as in one of the cases (1)–(4) of example 3.14 (also see figure 15), are posets. At least this turns out to be the case for $M \leq 6$. 

12 We are not aware of a general argument why the union of sequences of ordered critical times, as in one of the cases (1)–(4) of example 3.14 (also see figure 15), are posets. At least this turns out to be the case for $M \leq 6$. 

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Figure 18. Classes of possible evolutions for $M = 5$ and $t^{(5)} < t^{(5)}_{123456}$, corresponding to intervals of $i^{(4)}$ and transitions at critical values of $t^{(4)}$. This corresponds to the left chain in figure 14.

Figure 19. Classes of possible evolutions for $M = 5$ and $t^{(5)} > t^{(5)}_{123456}$. This corresponds to the right chain in figure 14. As indicated by equally encircled parts, the deformation of a class into the next, as $i^{(4)}$ passes through a critical value, proceeds according to the left to right half pentagon structure of $T_3$.

Depending on the order of the critical time values $t_{1236}$ and $t_{2345}$, the evolution follows one of the two sequences of rooted binary trees in figure 16, easily elaborated with the help of (3.3).

In case (4) we have the inequalities (a') $t_{1234} > t_{1235} > t_{1245} > t_{1345} > t_{2345}$, (b') $t_{1234} > t_{1236} > t_{1246} > t_{1346} > t_{2346}$, (c') $t_{1235} > t_{1236} > t_{1246} > t_{1346} > t_{1356} > t_{2356}$, (d') $t_{1245} > t_{1246} > t_{1256} > t_{1456} > t_{2456}$, (e') $t_{1345} > t_{1346} > t_{1356} > t_{1456} > t_{3456}$ and (f')...
Figure 20. The left graph shows the representation of the Tamari lattice $T_4$ in terms of rooted binary trees which represent line soliton patterns. The digraph on the right describes $T_4$ without overlapping edges and makes evident that it consists of six pentagons and three tetragons. (The 2-faces of Tamari lattices are pentagons or tetragons, see e.g. [40].) Here the numbers assigned to the pentagons encode the critical times associated with the edges. For an edge between pentagon $i$ and pentagon $j$ we form the complement of $ij$ in 123456, which then determines the associated critical time. Opposite edges of a tetragon have to be identified for this counting, so that e.g. $t_{1456}$ is assigned to the left uppermost arrow. Another familiar representation is as a poset structure on the associahedron [41–43] in three dimensions.

Figure 21. The evolution of a line soliton pattern of type 1 in table 1. Here we have $t_i = p_1 x + p_2 y + p_3 t + p_4 t^2 + p_5 t^3 + p_6 t^4 + c_i$ with the choice $p_1 = -2$, $p_2 = -3/2$, $p_3 = -1$, $p_4 = 1/2$, $p_5 = 5/4$, $p_6 = 2$ and $c_1 = 10$, $c_2 = c_3 = c_4 = c_5 = 0$, $c_6 = -10$. Furthermore, we set $\mu = -2$ and $\lambda = -1$. The line soliton plots are taken at $t = -10, -7, -3.6, 0, 4, 10, 20$. 

For $t_{2345} > t_{2346} > t_{2356} > t_{2456} > t_{3456}$. Now we have to eliminate $t_{1234}$, $t_{1235}$, $t_{1245}$, $t_{1246}$, $t_{1256}$, $t_{1345}$, $t_{1356}$, $t_{2345}$, $t_{2346}$, $t_{2356}$, $t_{2456}$, $t_{3456}$ (see figure 15). The resulting sequence of rooted binary trees is shown in figure 17. Figure 18 shows the resulting classes of chains in cases (1)–(4). For $t^{(5)} > t^{(123456)}$, the corresponding (classes of) chains are displayed in figure 19. In order to realize a certain chain in $T_4$, the critical times appearing along it all have to be smaller than...
the critical times of neighboring branches. Solving the inequalities arising in this way, one obtains the conditions in table 1 (also see tables B1 and B2 in appendix B). An example of a line soliton solution of type 1 in the table is displayed in figure 21.

The further steps needed to treat the cases $M > 5$ should now be obvious and we refer to appendix A for corresponding general results.

4. On the general class of KP line soliton solutions

In the preceding sections (and appendix A) we restricted our considerations to a special class of line soliton solutions and achieved a complete description (in the tropical approximation) of their evolution. In this section we argue that more general solutions can actually be understood fairly well as superimpositions of solutions from the special class, with rather simple modifications. The somewhat qualitative picture laid out in this section still has to be elaborated in more detail, however.

The general class of line soliton solutions of the KP-II equation (and more generally its hierarchy) in the Hirota form is well known to be given by

$$\tau = f_1 \wedge f_2 \wedge \cdots \wedge f_n,$$

where

$$f_i = \sum_{j=1}^{M+1} e_{ij} e_j, \quad e_j = e^{\theta_j}, \quad \theta_j = \sum_{r=1}^{M} p_{jr}^{(r)} + c_j,$$

and the exterior product on the space of functions generated by the exponential functions $e_j$, $j = 1, \ldots, M + 1$, is defined by

$$e_{i_1} \wedge \cdots \wedge e_{i_n} = \Delta(p_{i_1}, \ldots, p_{i_n})e_{i_1} \cdots e_{i_n},$$

with the Vandermonde determinant (A.3). Hence,

$$\tau = \sum_{1 \leq i_1 < \cdots < i_n \leq M+1} A_{i_1 \cdots i_n} e_{i_1} \cdots e_{i_n} \quad \text{where} \quad A_{i_1 \cdots i_n} = e_{i_1} \cdots e_{i_n} \Delta(p_{i_1}, \ldots, p_{i_n}).$$
Example 4.1. A subclass of the above class of solutions is given by
\[
\tau = (e_1 + e_2) \wedge (e_2 + e_3) \wedge \cdots \wedge (e_M + e_{M+1}) \\
= \sum_{i=1}^{M+1} e_i \wedge e_{i+1} \wedge \cdots \wedge e_{M+1} \\
= \Delta (p_1, \ldots, p_{M+1}) e^{\theta_1 + \cdots + \theta_{M+1}} \sum_{i=1}^{M+1} a_i e^{-\theta_i},
\]
where a hat indicates an omission, and \(a_i = 1/[(p_i - p_1) \cdots (p_i - p_{i-1})(p_{i+1} - p_i) \cdots (p_{M+1} - p_i)]\). Assuming \(p_1 < p_2 < \cdots < p_{M+1}\), \(a_i\) is positive; hence, it can be absorbed into the constant \(c_i\). Moreover, the factor in front of the sum drops out in the expression \(u = 2 \log(\tau)_{xx}\) for the KP soliton solution, so that an equivalent \(\tau\)-function is given by
\[
\tilde{\tau} = \sum_{i=1}^{M+1} e^{-\theta_i}.
\]
Via \(p_i \mapsto -p_i\) and \(c_i \mapsto -c_i\) (and with a renumbering of the \(p\)'s), this is the class of solutions treated in the main part of this work, up to the reflection \(t^{(2r)} \mapsto -t^{(2r)}\), \(r = 1, 2, \ldots\), which includes \(y \mapsto -y\). The corresponding rooted binary trees are hence given by those of our simple class, but drawn upside down.

Remark 4.2. Since the above expression for \(\tau\) determines a KP solution, this also holds for
\[
\tau' = e^{-\theta_1 - \cdots - \theta_{M+1}} \tau = \sum_{1 \leq i_1 < \cdots < i_n \leq M+1} A_{i_1 \cdots i_n} e^{-\theta_{i_1} - \cdots - \theta_{i_n}},
\]
where \([k_1, \ldots, k_n] = [1, \ldots, M + 1] \setminus [i_1, \ldots, i_n]\). Since the reflection \(t^{(r)} \mapsto -t^{(r)}\), \(r = 1, 2, \ldots\), is a symmetry of the KP equation, and since \(c_i \mapsto -c_i\) preserves the above class of solutions, we also conclude that
\[
\tau_\star = \sum_{1 \leq i_1 < \cdots < i_n \leq M+1} A_{i_1 \cdots i_n} e^{\theta_{i_1} + \cdots + \theta_{i_n}} = \sum_{1 \leq i_1 < \cdots < i_n \leq M+1} A_{i_1 \cdots i_n} e_{k_1} \cdots e_{k_n}
\]
is a solution, which we call the dual of \(\tau\).

Let us order the constants \(p_i\) such that \(p_1 < \cdots < p_{M+1}\) and let us assume that no pair of the functions \(f_i\), \(i = 1, \ldots, n\), has an \(e_j\) in common. The cases excluded by this assumption can be recovered by taking a limit where pairs of neighboring \(p\)'s coincide. By absorbing the modulus of a nonvanishing constant \(\epsilon_{ij}\) via a redefinition of the constant \(c_j\), without restriction of generality we can assume that
\[
\epsilon_{ij} \in \{0, \pm 1\}.
\]
By demanding that the coefficients \(A_{i_1 \cdots i_n}\) are all non-negative, and at least one of them different from zero, we ensure that \(\tau\) is positive and the KP solution is then regular. Then we obtain
\[
\tau = \sum_{1 \leq i_1 < \cdots < i_n \leq M+1} |\epsilon_{i_1} \cdots \epsilon_{i_n}| \, e^{\theta_{i_1} - \cdots - \theta_{i_n}},
\]
where
\[
\theta_{i_1 \cdots i_n} = \theta_{i_1} + \cdots + \theta_{i_n} + \log \Delta(p_{i_1}, \ldots, p_{i_n}).
\]
In particular,
\[
\theta_{ij} = \theta_i + \theta_j + \log(p_j - p_i).
For fixed values of the parameters \( t^{(3)} = t, t^{(4)}, \ldots \), the \( xy \)-plane is divided into regions where one of the phases \( \theta_{ij} = \frac{1}{\epsilon} \log(\tau) \) dominates all others. The line soliton segments are given by the visible boundaries of these regions. The tropical approximation now reads

\[
\log(\tau) \simeq \max[\theta_{ij} | 1 \leq i_1 < \ldots < i_n \leq M + 1, \epsilon_{i_1}, \ldots, \epsilon_{i_n} \neq 0].
\]

In principle one can approach a classification of solutions in a similar way as done for the special class in the main part of this work. In the following we set \( t^{(n)} = 0 \) for \( n > 3 \) (more precisely, we absorb these variables into the constants \( c_i \)). Assuming \( p_i < p_j, p_k < p_l \) and \( p_i + p_j \neq p_k + p_l \), we have

\[
\theta_{ij} - \theta_{kl} = (p_i + p_j - p_k - p_l)(x - x_{ij,kl}(y, t)),
\]

where

\[
x_{ij,kl}(y, t) = \frac{1}{p_i + p_j - p_k - p_l} \left( -\left(p_i^2 + p_j^2 - p_k^2 - p_l^2\right)y - \left(p_i^3 + p_j^3 - p_k^3 - p_l^3\right)t \right.
\]

\[
\left. - c_i - c_j + c_k + c_l + \log \left( \frac{p_i - p_k}{p_j - p_l} \right) \right).
\]

The boundary between the regions associated with the two phases \( \theta_{ij} \) and \( \theta_{kl} \) is therefore given by \( x = x_{ij,kl}(y, t) \). In particular, we find that

\[
x_{ik,jk} = x_{ij} + \frac{1}{p_j - p_i} \log \left( \frac{p_k - p_i}{p_k - p_j} \right),
\]

with

\[
x_{ij}(y, t) = -(p_i + p_j)y - \left(p_i^2 + p_j^2 + p_l^3 \right)t - c_{ij}
\]

( an expression that already appeared in section 3.1 ). This in turn implies

\[
x_{il,ji} = x_{ik,jk} = -\frac{1}{p_j - p_i} \ell(p_i, p_j, p_k, p_l),
\]

where

\[
\ell(p_i, p_j, p_k, p_l) = \log \left( \frac{(p_k - p_i)(p_l - p_j)}{(p_l - p_i)(p_k - p_j)} \right)
\]

is the logarithm of the cross ratio of the constants \( p_i, p_j, p_k, p_l \). Hence the boundary lines \( x = x_{ik,jk} \) and \( x = x_{il,ji}, k \neq l \), are always parallel with a constant (i.e. \( y \)- and \( t \)-independent) separation on the \( x \)-axis. We note that these ‘shifts’ also do not depend on the parameters \( c_i \) (hence also not on \( t^{(n)}, n > 3 \)). In particular, they coincide with the asymptotic phase shifts (difference of phase values for \( x \to \pm \infty \)) given in [19].

Furthermore, the boundary lines \( x_{ij,kl}, x_{kl,mn} \) meet at the point with the \( y \)-coordinate

\[
\gamma_{ij,kl,mn} = \frac{\left(p_i^3 + p_j^3 - p_k^3 - p_l^3\right)}{\left(p_i + p_j - p_k - p_l\right)} - \frac{\left(p_k^3 + p_l^3 - p_m^3 - p_n^3\right)}{\left(p_k + p_l - p_m - p_n\right)} \frac{1}{p_i + p_j - p_k - p_l} \times \frac{1}{p_k + p_l - p_m - p_n}
\]

\[
= \frac{1}{p_i + p_j - p_k - p_l} \left( p_i^3 + p_j^3 - p_k^3 - p_l^3 \right) t + c_i - c_j + c_k - c_l + \log \left( \frac{p_j - p_k}{p_l - p_i} \right)
\]

\[
- \frac{1}{p_k + p_l - p_m - p_n} \left( p_k^3 + p_l^3 - p_m^3 - p_n^3 \right) t + c_k + c_l - c_m - c_n + \log \left( \frac{p_l - p_k}{p_n - p_m} \right).
\]
provided that the inverses exist. Moreover, we have the identities

\[
\begin{align*}
\theta_{ij} - \theta_{ii} &= (p_i + p_j - p_m - p_n)(x - x_{ij,kl}) + \frac{1}{p_i + p_j - p_k - p_l} \\
&\times \left( (p_i^2 + p_j^2)(p_m + p_n - p_k - p_l) + (p_k^2 + p_l^2)(p_i + p_j - p_m - p_n) \\
&+ (p_m^2 + p_n^2)(p_k + p_l - p_i - p_j)(y - y_{ij,kl,nn}),
\right) 
\end{align*}
\]

(4.4)

and

\[
\begin{align*}
\theta_{ij} - \theta_{kl} &= (p_i + p_j - p_k - p_l)(x - x_{ij,ik}) \\
&\times (p_i - p_l)(p_i - p_j - p_k + p_l)(y - y_{ij,ik,ij}) + \epsilon(p_i, p_l, p_j).
\end{align*}
\]

(4.5)

They do not explicitly depend on \( t \), nor on the constants \( c_i \) and the log \( \Delta \) terms. The further analysis turns out to be quite involved, though. A fair qualitative understanding can be reached without a deeper analysis, however, as outlined in the following.

According to our assumptions, \( \Omega_i = \{ \theta_{ij} | \epsilon_{ij} \neq 0, j = 1, \ldots, M + 1, i = 1, \ldots, n \} \), are disjoint sets. If we can neglect the effect of all the terms \( \log \Delta(p_i, \ldots, p_n) \), then the tropical approximation is given by

\[
\log(\tau) \simeq \sum_{i=1}^{n} \max(\Omega_i),
\]

which unveils the line soliton configuration as a superimposition of the line soliton configurations corresponding to the constituents \( f_i, i = 1, \ldots, n \).

Superimposing two line soliton configurations, due to the locality of the KP equation there can only be an interaction between them at points where a branch of one of them crosses a branch of the other. This is locally an interaction between two line solitons, where now we should switch on the log \( \Delta \) term. We shall see in the next example what this brings about.

For \( M = 3 \), i.e. four phases, the regularity condition only allows the two 2-forms

\[
\tau_O = (e_1 + e_2) \land (e_3 + e_4) \quad \text{and} \quad \tau_P = (e_1 - e_4) \land (e_2 + e_3),
\]

which belong to classes called ‘O-type’ and ‘P-type’ by some authors (see e.g. [15, 19]).

We will consider the O-type solution in detail in example 4.3. The analysis of the P-type solution is very much the same. In addition to the 2-form solutions, further regular solutions for \( M = 3 \) are given by \( \tau = e_1 + e_2 + e_3 + e_4 \), belonging to our special class, and its dual \( \tau_\ast = e_2 \land e_3 \land e_4 + e_1 \land e_3 \land e_4 + e_1 \land e_2 \land e_4 + e_1 \land e_2 \land e_3 \) (cf example 4.1). Further regular solutions are obtained from solutions with \( M > 3 \) by taking limits where pairs of neighboring \( p \)'s coincide, see example 4.6 below.

Example 4.3. Assuming \( p_1 < p_2 < p_3 < p_4 \), we have

\[
\begin{align*}
\tau_O &= (p_3 - p_4)e_3 + (p_4 - p_1)e_1e_4 + (p_3 - p_2)e_2e_3 + (p_4 - p_2)e_2e_4 \\
&= \epsilon^{03} + \epsilon^{04} + \epsilon^{23} + \epsilon^{24}.
\end{align*}
\]

Our tropical approximation is given by

\[
\log(\tau) \simeq \max\{\theta_{13}, \theta_{14}, \theta_{23}, \theta_{24}\}.
\]

If the constants \( \log(p_j - p_i) \) are negligible, then \( \log(\tau) \simeq \max\{\theta_1, \theta_2\} + \max\{\theta_3, \theta_4\} \) and a plot of \( \log(\tau)_{x,i} \) is simply the result of superimposing the plots of \( \log(e_1 + e_2)_{x,i} \) and \( \log(e_3 + e_4)_{x,i} \),

13 Whereas in section 2 we only used the tropical binary operation \( a \oplus b = \max\{a, b\} \), here also the complementary one shows up, i.e. \( a \odot b = a + b \).
14 In the case of the solution \( \tau_P \) below, we have \( f_1 = e_1 - e_4 \), which leads to a singular solution. However, we note that this strong approximation does not depend on the sign of the coefficients \( \epsilon_{ij} \). As a consequence, in this approximation \( f_1 \) gets replaced by \( e_1 + e_4 \), which determines a line soliton.
hence displaying two crossing lines, corresponding to $\theta_1 = \theta_2$, respectively $\theta_3 = \theta_4$ (see figure 22). In general, however, the constants $\log(p_j - p_i)$ are not negligible, of course, and the situation is more complicated (see the right plot in figure 22). Equation (4.1) together with (4.3) yields

\[ \theta_{13} = \theta_{14} \iff x = x_{34}(y, t) + \frac{1}{p_4 - p_3} \log \left( \frac{p_3 - p_1}{p_4 - p_1} \right) =: x_{13,14}(y, t), \]

\[ \theta_{23} = \theta_{24} \iff x = x_{34}(y, t) + \frac{1}{p_4 - p_3} \log \left( \frac{p_3 - p_2}{p_4 - p_2} \right) =: x_{23,24}(y, t), \]

\[ \theta_{13} = \theta_{23} \iff x = x_{12}(y, t) + \frac{1}{p_2 - p_1} \log \left( \frac{p_3 - p_1}{p_3 - p_2} \right) =: x_{13,23}(y, t), \]

\[ \theta_{14} = \theta_{24} \iff x = x_{12}(y, t) + \frac{1}{p_2 - p_1} \log \left( \frac{p_4 - p_1}{p_4 - p_2} \right) =: x_{14,24}(y, t). \]

Moreover, according to (4.1) and (4.2) we have

\[ \theta_{13} = \theta_{24} \iff x = -\frac{1}{p_1 + p_3 - p_2 - p_4} \left( p_1^2 + p_3^2 - p_2^2 - p_4^2 \right) y + \left( p_1^3 + p_3^3 - p_2^3 - p_4^3 \right) \log \left( \frac{p_4 - p_2}{p_3 - p_1} \right) =: x_{13,24}(y, t), \]

\[ \theta_{14} = \theta_{23} \iff x = -\frac{1}{p_1 + p_4 - p_2 - p_3} \left( p_1^2 + p_4^2 - p_2^2 - p_3^2 \right) y + \left( p_1^3 + p_4^3 - p_2^3 - p_3^3 \right) \log \left( \frac{p_4 - p_2}{p_4 - p_1} \right) =: x_{14,23}(y, t). \]

The boundary $\theta_{14} = \theta_{23}$ cannot be expressed in the form $x = x_{14,23}(y, t)$ if $p_1 + p_4 - p_2 - p_3$ vanishes, it is then parallel to the $x$-axis. The other boundaries can always be expressed in this form (as a consequence of $p_1 < p_2 < p_3 < p_4$). The two boundary lines given by $\theta_{13} = \theta_{14}$ and $\theta_{23} = \theta_{24}$, respectively, and also those given by $\theta_{13} = \theta_{23}$ and $\theta_{14} = \theta_{24}$, respectively, are always parallel, with a constant separation on the $x$-axis given by

\[ x_{13,14}(y, t) - x_{23,24}(y, t) = \frac{1}{p_4 - p_3} \xi(p_1, p_2, p_3, p_4), \]

\[ x_{14,24}(y, t) - x_{13,23}(y, t) = -\frac{1}{p_2 - p_1} \xi(p_1, p_2, p_3, p_4). \]

The point in which the boundary lines $x = x_{13,14}(y, t)$ and $x = x_{14,24}(y, t)$ intersect (at time $t$), and thus the three phases $\theta_{13}, \theta_{14}, \theta_{24}$ meet, has the $y$-coordinate

\[ y_{13,14,24} = -\frac{1}{p_1 + p_2 - p_3 - p_4} \left( p_1^2 + p_1 p_2 + p_2^2 - p_3 p_4 - p_4^2 \right) t + c_1 + c_3 \]

\[ + \frac{1}{p_3 - p_1} \log \left( \frac{p_3 - p_1}{p_4 - p_1} \right) + \frac{1}{p_2 - p_1} \log \left( \frac{p_4 - p_2}{p_4 - p_1} \right). \]

Similarly, the intersection point of the lines $x = x_{23,24}(y, t)$ and $x = x_{13,23}(y, t)$, where the three phases $\theta_{13}, \theta_{23}, \theta_{24}$ meet, has the $y$-coordinate
we find we end up with a Miles resonance in this limit.

As a consequence, (for fixed \( t \)) the half-lines \( \{ x = x_{13,23}(y, t) \mid y > y_{13,23,24} \} \) and \( \{ x = x_{23,24}(y, t) \mid y > y_{13,23,24} \} \) are non-visible. Furthermore, we find

\[
\begin{align*}
\theta_{13} - \theta_{24} &= -(p_4 - p_3 + p_2 - p_1)(x - x_{13,23}) - (p_4 - p_3)(p_3 + p_4 - p_1 - p_2)(y - y_{13,23,24}), \\
\theta_{23} - \theta_{13} &= (p_2 - p_1)(x - x_{23,24}) - (p_2 - p_1)(p_3 + p_4 - p_1 - p_2)(y - y_{13,23,24}), \\
\theta_{13} - \theta_{23} &= -(p_2 - p_1)(x - x_{13,23}) + (p_2 - p_1)(p_3 + p_4 - p_1 - p_2)(y - y_{13,23,24}).
\end{align*}
\]

As a consequence, (for fixed \( t \)) the half-lines \( \{ x = x_{13,23}(y, t) \mid y > y_{13,23,24} \} \) and \( \{ x = x_{23,24}(y, t) \mid y > y_{13,23,24} \} \) are non-visible. Furthermore, we find

\[
\begin{align*}
\theta_{13} - \theta_{24} &= -(p_4 - p_3 + p_2 - p_1)(x - x_{13,24}) + (p_2 - p_1)(p_3 + p_4 - p_1 - p_2)(y - y_{13,14,24}), \\
\theta_{14} - \theta_{13} &= (p_4 - p_3)(x - x_{14,24}) + (p_4 - p_3)(p_3 + p_4 - p_1 - p_2)(y - y_{13,14,24}), \\
\theta_{13} - \theta_{14} &= -(p_4 - p_3)(x - x_{13,24}) - (p_2 - p_1)(p_3 + p_4 - p_1 - p_2)(y - y_{13,14,24}).
\end{align*}
\]

Figure 22. The left phase region plot shows \( \max[\theta_1, \theta_2] + \max[\theta_3, \theta_4] \) as a function of \( x \) (horizontal axis) and \( y \), the right one shows the full tropical approximation \( \max[\theta_{13}, \theta_{14}, \theta_{23}, \theta_{24}] \) at \( t = 0 \) (and all higher \( t^{\cdot \cdot} \) also set to zero). Here we chose the solution of example 4.3 with \( p_1 = -1, p_2 = -1/2, p_3 = 1/4, p_4 = 5/4 \) and \( c_1 = c_4 = 0, c_2 = -c_3 = -10.\)
This shows that the half-lines \( \{ x = x_{13,14}(y,t) \mid y < y_{13,14,24} \} \), \( \{ x = x_{14,24}(y,t) \mid y < y_{13,14,24} \} \) and \( \{ x_{13,24}(y,t) \mid y > y_{13,14,24} \} \) are non-visible. Moreover, one can show that the whole line given by \( x = x_{14,23} \) is non-visible. All this is compatible with the right plot in figure 22. We know that the complementary half-lines are visible in the approximation where we neglect the phase shift terms \( \log \Delta \) (and the two triple-phase coincidences merge). Since (4.4) does not explicitly depend on these terms, we can conclude that they remain visible when switching the phase shifts on. We can confirm this by further explicit computations. For example, at the three-phase coincidence with the \( y \)-coordinate \( y_{13,23,24} \), (4.5) implies \( \theta_{23} - \theta_{14} = \ell(p_2, p_1, p_4, p_3) > 0 \); hence, this point is visible.

**Example 4.4.** In the case of \( \tau_P \), the tropical approximation is \( \log(\tau_P) \simeq \max(\theta_{12}, \theta_{13}, \theta_{24}, \theta_{34}) \). We can proceed as in example 4.3. The line determined by \( \theta_{12} = \theta_{34} \) can always be solved for \( x \) (as a consequence of \( p_1 < p_2 < p_3 < p_4 \)) and turns out to be non-visible (also see figure 23). The slope of the line given by \( \theta_{13} = \theta_{24} \) is \( -(p_1 - p_2 + p_3 - p_4)(p_1^3 - p_2^3 + p_3^3 - p_4^3) \). Furthermore, we obtain

\[
\begin{align*}
x_{12,13} - x_{24,34} &= \frac{1}{p_3 - p_2} \ell(p_2, p_3, p_1, p_4), \\
x_{13,34} - x_{12,24} &= -\frac{1}{p_4 - p_1} \ell(p_2, p_3, p_1, p_4).
\end{align*}
\]

In contrast to the case treated in example 4.3, we need an additional condition, namely \( p_1 + p_4 \neq p_2 + p_3 \), in order to ensure the existence of (then visible) three-phase coincidences, here with the \( y \)-coordinate \( y_{12,13,24} \), respectively \( y_{13,24,34} \). Their distance along the \( y \)-axis is

\[
y_{13,24,34} - y_{12,13,24} = -\frac{p_1 - p_2 + p_3 - p_4}{(p_3 - p_2)(p_4 - p_1)(p_1 + p_4 - p_2 - p_3)} \ell(p_2, p_3, p_1, p_4)
\]

The excluded case where \( p_1 + p_4 = p_2 + p_3 \) is further considered in example 4.5.

**Example 4.5.** Here we consider \( \tau_P \) with \( p_1 + p_4 = p_2 + p_3 \). Writing

\[
p_1 = \frac{1}{2}(q - a - b), \quad p_2 = \frac{1}{2}(q - a), \quad p_3 = \frac{1}{2}(q + a), \quad p_4 = \frac{1}{2}(q + a + b),
\]

with real constants \( a, b > 0 \) and \( q \), we find
\[ x_{12,13} = -qy - \frac{1}{4}(a^2 + 3q^2)t + \frac{1}{a}(c_2 - c_3 - \log(1 + 2a/b)), \]
\[ x_{12,24} = -qy - \frac{1}{4}[(a + b)^2 + 3q^2]t + \frac{1}{a+b}(c_1 - c_4 - \log(1 + 2a/b)), \]
\[ x_{12,34} = -qy - \frac{1}{4}(a^2 + ab + b^2 + 3q^2)t + \frac{1}{2a+b}(c_1 + c_2 - c_3 - c_4), \]
\[ x_{13,24} = -qy - \frac{1}{4}(3a^2 + 3ab + b^2 + 3q^2)t + \frac{1}{b}(c_1 - c_2 + c_3 - c_4), \]
\[ x_{13,34} = -qy - \frac{1}{4}(a^2 + 3q^2)t + \frac{1}{a+b}(c_1 - c_4 + \log(1 + 2a/b)), \]
\[ x_{24,34} = -qy - \frac{1}{4}(a^2 + 3q^2)t + \frac{1}{a}(c_2 - c_3 + \log(1 + 2a/b)). \]

Hence all these lines are parallel with the slope \(-1/q\). The two boundary lines \(x = x_{12,13}\) and \(x = x_{24,34}\) move with the same speed, and the same holds for \(x = x_{12,24}\) and \(x = x_{13,34}\). We note that \(x_{12,13} < x_{24,34}\) and \(x_{12,24} < x_{13,34}\). Furthermore, we find the following coincidence events:

\[ x_{13,24} = x_{13,34} = x_{24,34} \quad t = t_0 - \Delta t =: t_- \]
\[ x_{12,34} = x_{13,24} \quad \text{at} \quad t = t_0 \]
\[ x_{12,13} = x_{12,24} = x_{13,24} \quad t = t_0 + \Delta t =: t_+, \]

where

\[ t_0 = \frac{4ac_1 - c_4 - (a + b)(c_2 - c_3)}{ab(a+b)(2a+b)}, \quad \Delta t = \frac{4\log(1 + 2a/b)}{a(a+b)(2a+b)} > 0. \]

Since \(\theta_{12} - \theta_{13} = \frac{1}{2}ab(a+b)(t - t_0) - \log(1 + 2a/b)\) and \(\theta_{12} - \theta_{24} = -\frac{1}{2}ab(a+b)(t - t_0) - \log(1 + 2a/b)\) on \(x = x_{12,34}\), we conclude that this line is never visible. Hence also the event at \(t_0\) is non-visible. Along \(x = x_{12,13}\) we find \(\theta_{12} - \theta_{24} = -(q - 2p_1)(q - p_1 - p_2)(p_2 - p_1)(t - t_+ + 2\log[(q - p_2 - p_1)/(p_2 - p_1)]\), which are both positive for \(t < t_+\), and the first expression is negative for \(t > t_+\). Hence, \(x = x_{12,13}\) is visible for \(t < t_+\) and non-visible for \(t > t_+\). In the same way we find that \(x = x_{13,34}\) is visible for \(t < t_-\) and non-visible for \(t > t_-\), \(x = x_{12,24}\) is visible for \(t > t_-\) and non-visible for \(t < t_-\), \(x = x_{24,34}\) is visible for \(t > t_+\) and non-visible for \(t < t_+\), and \(x = x_{13,24}\) is visible for \(t < t_+\) and non-visible otherwise. There are no further visible lines. Hence, for \(t < t_-\) and \(t > t_+\) there are two visible boundary lines corresponding to two parallel line solitons\(^\dagger\) (oblique to the \(x\)-axis). But for \(t_- < t < t_+\) there are three parallel visible boundary lines; also see figure 24. This means that for \(t < t_-\) and \(t > t_+\) only three of the four phases are visible, and all four are visible only for \(t_- < t < t_+\).

The tropical description provides us with an interpretation of the soliton interaction process. For \(t < t_-\) (left of the three region plots in figure 24) the lines \(x_{12,13} \simeq x_{23}\) and \(x_{13,34} \simeq x_{14}\) represent two line solitons moving from right to left, where the latter is faster than the first. At \(t_-\), the faster soliton sends off a virtual line soliton (corresponding to \(x_{13,24}\)) and thereby mutates to \(x_{24,34} \simeq x_{23}\), which is a new manifestation of the slower line soliton. At \(t_+\) the original slower soliton swallows the virtual one and mutates to \(x_{12,24} \simeq x_{14}\), which is a new manifestation of the original faster soliton. A generalization of this solution, now with \(n\) parallel line solitons\(^\ddagger\), is given by

\[ \tau = (e_1 - (-1)^nn e_{2n}) \land (e_2 - (-1)^{n-1} e_{2n-1}) \land \cdots \land (e_{n-1} - e_{n+1}) \land (e_n + e_{n+1}). \]

\(^\dagger\) The constants \(a\) and \(b\) determine the amplitudes of these line solitons, see appendix D.

\(^\ddagger\) Actually, this case can be reduced to a discussion of the KdV equation, in the tropical approximation.
where \( p_1 < p_2 < \cdots < p_{2n} \) and \( p_1 + p_{2n} = p_2 + p_{2n-1} = \cdots = p_{n-1} + p_{n+2} = p_n + p_{n+1} \). Moreover, by taking the wedge product of two such functions, we can generate grid-like structures. For example, let

\[
\tau = (e^{-p_6} + e_{-1}) \wedge (e^{-5} - e_{-2}) \wedge (e^{-4} + e_{-3}) \wedge (e_1 + e_6) \wedge (e_2 - e_5) \wedge (e_3 + e_4),
\]

where \( p_{-6} < p_{-5} < \cdots < p_{-1} < p_1 < \cdots < p_6, p_{-6} + p_{-1} = p_{-5} + p_{-2} = p_{-4} + p_{-3} \) and \( p_1 + p_6 = p_2 + p_5 = p_3 + p_4 \). Figure 25 shows a plot of such a solution.

The above results suggest that the line soliton solution is generically obtained as a superimposition of the constituents (i.e. the factors in the wedge product, modulo conversion of negative to positive signs) and in addition with the creation of new line segments of constant length and slope due to the \( \log \Delta \) phase shift terms, as in example 4.3. Typically these new line segments will not be visible in a line soliton plot, with the exception of the extremal cases considered next.

We explore what happens when two neighboring constants, say \( p_i, p_{i+1} \), in the sequence of \( p \)'s approach each other. Writing \( p_{i+1} - p_i = e^{-\alpha_i} \) and \( b_i = e^{\alpha_{i+1} - \alpha_i} \), we have \( e_{i+1} \simeq b_i e_i \) for large positive \( \alpha_i \). If \( i, i+1 \in \{k_1, \ldots, k_n\} \), we find

\[
\log \Delta(p_{k_1}, \ldots, p_{k_n}) \simeq \log \left( \frac{\partial}{\partial p_{i+1}} \Delta(p_{k_1}, \ldots, p_{k_n}) \right)_{p_{i+1} = p_i} - \alpha_i.
\]
As a consequence, the region dominated by $\theta_{k_1,\ldots,i,i+1,\ldots,k_n}$ disappears in the limit $\alpha_i \to \infty$. If $i + 1 \in \{k_1, \ldots, k_n\}$, but $i \not\in \{k_1, \ldots, k_n\}$, then

$$\theta_{k_1,\ldots,i,i+1,\ldots,k_n} = \theta_{k_1,\ldots,i,i+1,\ldots,k_n} + \log \Delta(p_{k_1},\ldots,p_{i+1},\ldots,p_{k_n})$$

$$\simeq \theta_{k_1,\ldots,i,i+1,\ldots,k_n} + \log b_i + \log \Delta(p_{k_1},\ldots,p_{i+1},\ldots,p_{k_n})$$

$$= \theta_{k_1,\ldots,i,i+1,\ldots,k_n} + \log b_i =: \tilde{\theta}_{k_1,\ldots,i,i+1,\ldots,k_n}.$$ 

Hence the $\theta_{k_1,\ldots,i,i+1,\ldots,k_n}$-region passes into a $\tilde{\theta}_{k_1,\ldots,i,i+1,\ldots,k_n}$-region. Boundary lines between regions that do not carry an index $i + 1$ remain unchanged.

We conclude that, as $p_i+1 \to p_i$, each region with a dominating phase of the form $\theta_{k_1,\ldots,i,i+1,\ldots,k_n}$ is shifted away, and the phase regions to its left and to its right meet, a corresponding boundary line is created.

**Example 4.6.** We consider regular 2-form solutions with five phases (i.e. $M = 4$), $p_1 < p_2 < p_3 < p_4 < p_5$, and limits where two neighboring constants coincide.

1. $\tau = (e_1 + e_2) \land (e_3 + e_4 + e_5)$. Setting $p_3 = p_2$, we have $e_3 = a e_2$ with a constant $a > 0$. Recalling that a constant overall factor of $\tau$ does not change the respective KP soliton solution, after a redefinition of $c_4$ and $c_5$, and a renumbering, we obtain $\tau' = (e_1 + e_2) \land (e_3 + e_4 + e_5)$. Figure 26 shows an example for what happens as $p_3 \to p_2$.

The phase region associated with this pair is shifted away to infinity in this limit.

2. $\tau = (e_1 + e_2 - e_3) \land (e_3 + e_4)$. Setting $p_3 = p_4$, after a redefinition of $c_1$ and $c_2$ we end up with $\tau' = (e_1 + e_2 - e_3) \land (e_3 + e_4)$.

3. $\tau = (e_1 - e_3) \land (e_2 + e_3 + e_4)$. Setting $p_5 = p_4$, after a redefinition of $c_1$ we obtain $\tau' = (e_1 + e_4) \land (e_2 + e_3 + e_4)$.

4. $\tau = (e_1 - e_4 - e_5) \land (e_2 + e_3)$. Setting $p_4 = p_3$, redefining $c_1$ and $c_5$, and finally renaming $p_3$ to $p_4$, we find $\tau' = (e_1 - e_3 - e_2) \land (e_2 + e_3)$.

Starting with a regular six-phase solution, via two limits we obtain a four-phase solution:

5. $\tau = (e_1 + e_2 - e_3) \land (e_3 + e_4 + e_5)$. We set $p_6 = p_5$ and $p_3 = p_2$ to obtain $(e_1 + e_2 - a e_3) \land (e_2 + e_4 + e_5)$. We can achieve $a = 1$ with a redefinition of $c_1$ and $c_2$, or $b = 1$ with a redefinition of $c_4$ and $c_5$, but not both simultaneously. After a renaming we obtain $\tau' = (e_1 + e_2 - e_3) \land (a e_2 + e_3 + e_4)$, $a > 0$. Figure 27 shows a structure appearing in the tropical approximation that is not present in the full solution.
Figure 27. The left plot shows the tropical approximation of a solution of type (5) in example 4.6 at a fixed time in a region of size comparable with the width of a line soliton. The other plots show the full solution in the same region as a contour plot and a three-dimensional plot over the xy-plane.

But there are parameter values where the two bounded regions in the left plot in figure 27 indeed become visible (cf figure 4 in [10]).

Together with \( \tau_O \) and \( \tau_P \), we have seven types of four-phase 2-form solutions (cf the seven cases of \((2,2)\)-solutions in [19]). Modulo redefinitions of the constants \( c_i \) in (2) is the dual of that in (1), and also (3) and (4) are related in this way. \( \tau_O, \tau_P \) and \( \tau' \) in (5) are self-dual.

5. Summary of further results and conclusions

For the simplest class of KP-II line soliton solutions, we have shown that the time evolution can be described as a time-ordered sequence of rooted binary trees and that this constitutes a maximal chain in a Tamari lattice.

Moreover, we derived general results (in particular in appendix A) that allow us to compute the data corresponding to transition events (where a rooted binary tree evolves into another). The fact that the soliton solutions extend to solutions of the KP hierarchy plays a crucial role in the derivation of these results.

Tamari lattices are related to quite a number of mathematical structures and our work adds to it by establishing a bridge to an integrable PDE, the KP equation (and moreover its hierarchy)\(^{17}\). The latter is well known for other deep connections with various areas of mathematics.

The family of Tamari lattices is actually not the only family of posets (or lattices) showing up in the line soliton classification problem. We already met in section 2 a family where the nodes are the phases \( \theta_i \) and the edges correspond to critical values of \( x \). The underlying polytopes are a triangle \((M = 2)\), a tetrahedron \((M = 3)\), and their higher-dimensional analogs \((M > 3)\). Another family appeared in section 3.1. Its nodes consist of chains of critical \( x \)-values and the edges correspond to critical \( y \)-values. The underlying polytopes are a tetragon \((M = 3)\), a cube \((M = 4)\), and hypercubes for \( M > 4 \).

According to figures 18 and 19 (also see figure 14) there is a new lattice of the *hexagon* form. Its six nodes are given by the six classes built from the nine maximal chains of \( T_4 \) (see figures 18 and 19, and also appendix C), and its edges correspond to the six critical values of \( t(t^6) \). This lattice is an analog of the pentagon Tamari lattice \( T_3 \) and belongs to a new family. Its next member is obtained for \( M = 6 \). It has 25 nodes, which consist of classes of maximal chains in \( T_5 \) (see appendix C), and its (directed) edges are again determined by the critical values of \( t(t^4) \), see figure 28.

\(^{17}\) Also see [44] for a relation between integrable PDEs and polytopes.
Moreover, we expect a hierarchy of families of lattices. We already mentioned the two families associated with the critical values of $x$ and $y$. The Tamari lattices correspond to the critical values of $t = t^{(3)}$; the next family is associated with the critical values of $t^{(4)}$. More generally, there is a family associated with the critical values of $t^{(n)}$, $n \in \mathbb{N}$. Comparison with the algebra of oriented simplexes formulated in [45] (also see [46]) in terms of higher-dimensional categories shows striking relations which should be further elaborated. An exploration of more general classes of line solitons might exhibit relations with other posets (or lattices) and polytopes.

In this work we solved the classification problem for the simplest class of KP-II line soliton solutions, corresponding to rooted trees. Our classification rests upon the exploration of events where phases coincide. At such an event the tree that describes the line soliton configuration changes its form. A finer description is obtained by also taking events into account at which a transition between two trees with levels (associated with the same rooted binary tree) takes place. Our exposition made contact with such a refinement at various places. A nice example is the ‘missing face’ in the cube poset in figure 10. We elaborated this refinement in appendix B and explained in appendix C how it lifts the Tamari lattices (or associahedra) to permutohedra.

At first sight the classification for the simple class of line soliton solutions appears to be only a small step toward the classification of the whole set of line soliton solutions, which exhibit a much more complicated behavior. But this is not quite so, as outlined in section 4. Any line soliton solution can be written as a (suitably defined) exterior product of $\tau$-functions from the simple class. Generically such a product corresponds to superimposing the soliton graphs associated with the constituents. Since the interaction is local, there can only be a change in a neighborhood of a point where a soliton branch of one constituent meets a branch.

Figure 28. The left figure shows a new lattice. Its nodes are classes of maximal chains in $T_{5}$ and the edges correspond to critical values of $t^{(3)}$. A two-digit number stands for its complement in 1234567. The right figure represents this lattice as a polyhedron. It has 25 nodes, 39 edges and 16 faces, and it consists of seven hexagons and nine tetragons.
of another. At such a point a new line soliton segment (due to a phase shift) is created
and its length does not depend on time and not on the constants \( c_i \), but only on the values
of the constants \( p_i \). For generic parameter values, this effect is hardly visible. It becomes
significant, however, in cases where some of the (a priori assumed to be different) constants
\( p_i \) coincide. These are the more complicated cases which should still be explored in more
detail.

We expect that our tropical approximations of KP line soliton solutions have a place in
the tropical (totally positive) Grassmannian [47, 48]. For other approaches to the KP line
soliton classification problem we refer in particular to the review [19] and the references
therein.

Finally, we would like to stress that the tropical approximation allows one to zoom into
the interaction structure of solitons and enriches it with an underlying quantum particle-like
picture (see example 4.5). We expect that this tropical approach will also be useful in the case
of other (in particular soliton) equations.

Appendix A. Some general results

A.1. Preparations

The phases \( \theta_k \) appearing in the expression for the function \( \tau \) have the form

\[
\theta_k = \sum_{r=1}^{n-1} p_k^r t^{(r)} + c_k,
\]

where \( p_k, c_k \in \mathbb{R} \) and \( t^{(r)} \), \( r = 1, \ldots, n - 1 \), are real variables. The constants \( p_i \) are assumed
to be pairwise different. In previous sections we wrote \( t^{(1)} = x, t^{(2)} = y, t^{(3)} = t \). In order to
find the values of \( t^{(r)} \) for which \( \theta_k = \theta_{k_1} = \cdots = \theta_{k_n} = -t^{(0)} \), we have to solve the linear system

\[
p_k^{n-1} t^{(n-1)} + p_k^{n-2} t^{(n-2)} + \cdots + p_k t^{(1)} + t^{(0)} = -c_k, \quad i = 1, \ldots, n,
\]

which is done with the help of Cramer’s rule. In particular, for \( t^{(n-1)} \) we obtain the solution

\[
t^{(n-1)}_{k_1 \ldots k_n} = -c_{k_1 \ldots k_n}, \quad (A.1)
\]

where

\[
c_{k_1 \ldots k_n} = \frac{\kappa(p_{k_1}, \ldots, p_{k_n})}{\Delta(p_{k_1}, \ldots, p_{k_n})} \quad (A.2)
\]

with the Vandermonde determinant

\[
\Delta(p_{k_1}, \ldots, p_{k_n}) = \begin{vmatrix}
p_{k_1} & \cdots & p_{k_1}^{n-1} 
p_{k_2} & \cdots & p_{k_2}^{n-1} 
\vdots & \ddots & \vdots 
p_{k_n} & \cdots & p_{k_n}^{n-1}
\end{vmatrix} = \prod_{1 \leq i < j \leq n} (p_{k_i} - p_{k_j}) \quad (A.3)
\]

and

\[
\kappa(p_{k_1}, \ldots, p_{k_n}) = \begin{vmatrix}
p_{k_1} & \cdots & p_{k_1}^{n-2} & c_{k_1} 
p_{k_2} & \cdots & p_{k_2}^{n-2} & c_{k_2} 
\vdots & \ddots & \vdots & \vdots 
p_{k_n} & \cdots & p_{k_n}^{n-2} & c_{k_n}
\end{vmatrix}
\]
\[(−1)^{n−1} \sum_{i=1}^{n} (−1)^{i−1} c_ki = (−1)^{n−i} \sum_{i=1}^{n} (−1)^{i−1} c_ik \]

\[= \sum_{i=1}^{n} (−1)^{n−i} c_h \Delta(p_{k1}, \ldots, p_{kn}) \]

Here a hat indicates an omission. Now (A.2) implies

\[c_{k1...kn} = \sum_{i=1}^{n} \left( \frac{c_ki}{p_{ki} - p_{ki}} \right) \]

Proposition A.1.

\[c_{i...kn} = \frac{c_{i...kn} - c_{i...kn}}{p_{ki} - p_{ki}} \quad (i \neq j) \]

Proof. Since \(c_{i...kn}\) is totally symmetric, it suffices to prove the formula for \(i = 1\) and \(j = n + 1\). Using (A.4) we have

\[c_{k2...kn} - c_{k1...kn} = \left(\frac{c_{k1...kn}}{(p_{k2} - p_{k1}) \cdots (p_{kn} - p_{k1})} - \frac{c_{k1...kn}}{(p_{k1} - p_{k1}) \cdots (p_{kn} - p_{k1})}\right)\]

\[+ \sum_{r=2}^{n} \left(\frac{1}{(p_{k2} - p_{k1})} - \frac{1}{(p_{k1} - p_{k1})}\right) \cdot \left(\frac{c_{k1...kn}}{(p_{k1} - p_{r}) \cdots (p_{kn} - p_{r})} \cdots (p_{k1} - p_{k1})\right)\]

hence,

\[\frac{c_{k2...kn} - c_{k1...kn}}{p_{kn} - p_{ki}} = \left(\frac{c_{k1...kn}}{(p_{k2} - p_{k1}) \cdots (p_{kn} - p_{k1})} + \frac{c_{k1...kn}}{(p_{k1} - p_{k1}) \cdots (p_{kn} - p_{k1})}\right)\]

\[+ \sum_{r=2}^{n} \left(\frac{1}{(p_{k2} - p_{k1})} - \frac{1}{(p_{k1} - p_{k1})}\right) \cdot \left(\frac{c_{k1...kn}}{(p_{k1} - p_{r}) \cdots (p_{kn} - p_{r})} \cdots (p_{k1} - p_{k1})\right) = c_{k1...kn} \]

\[\square\]

Proposition A.2. The substitution \(c_k \mapsto c_k + \delta_k^{i(r)}\), with a variable \(i(r)\), has the following effect:

\[c_{k1...kn} \mapsto c_{k1...kn} + \delta_{r-n+1}(p_{k1}, \ldots, p_{kn}) \]

where \(h_{m, m} = 1, 2, \ldots\), are the complete symmetric polynomials, and \(h_{m, m} = 0\) if \(m < 0\), \(h_0 = 1\).

Proof. By linearity of the determinant, the substitution affects \(\kappa(p_{k1}, \ldots, p_{kn})\) as follows:

\[\kappa(p_{k1}, \ldots, p_{kn}) \mapsto \kappa(p_{kr}, \ldots, p_{kn}) + \left| \begin{array}{cccc} 1 & p_{k1} & \cdots & p_{k1}^{r-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & p_{kn} & \cdots & p_{kn}^{r-2} \\ p_{kr} & p_{kr}^{r-2} & \cdots & p_{kr}^{r-2} \end{array} \right| \]

The latter determinant equals \(\delta_{r-n+1}(p_{k1}, \ldots, p_{kn}) \Delta(p_{k1}, \ldots, p_{kn})\) (see e.g. [49]). Now the assertion follows from the expression (A.2) for \(c_{k1...kn}\). \(\square\)
A redefinition $c_k \mapsto c_k + \sum_{r=1}^{N} p'_k t^{(r)}$ changes the expression for the phases to

$$\theta_k = \sum_{r=1}^{N} p'_k t^{(r)} + c_k,$$

and, by application of the last proposition to (A.1), the critical value for the $n$-phase coincidence \( \theta_{k_1} = \theta_{k_2} = \cdots = \theta_{k_n} \) now reads

$$t^{(n-1)}_{k_1 \ldots k_n} = - \sum_{r=1}^{N+1-n} h_r(p_{k_1}, \ldots, p_{k_n}) t^{(n+r-1)} - c_{k_1 \ldots k_n}, \quad \text{(A.5)}$$

where $n = 2, 3, \ldots, N + 1$. In particular, $t^{(N)}_{k_1 \ldots k_{N+1}} = -c_{k_1 \ldots k_{N+1}}$. Equation (A.5) also makes sense for $n = 1$, where $\theta_i = -t_i^{(0)}$. The following proposition presents identities that have the same form irrespective of the value of $N$, i.e. their form is not affected by the redefinitions expressed in the last proposition. Of course, the ingredients (A.5) do depend on $N$.

**Proposition A.3.** For $n = 1, \ldots, N + 1$ we have

$$t^{(n-1)}_{k_1 \ldots k_{N+1}} - t^{(n-1)}_{k_{N+1} \ldots k_1} = (p_{k_1} - p_{k_N}) (t^{(n)} - t^{(n)}_{k_1 \ldots k_N}) \quad i, j = 1, \ldots, n + 1. \quad \text{(A.6)}$$

**Proof.** Using (A.5) for fixed $N$, and proposition A.1, we obtain

$$t^{(n-1)}_{k_1 \ldots k_{N+1}} - t^{(n-1)}_{k_{N+1} \ldots k_1} = (p_{k_1} - p_{k_N}) c_{k_1 \ldots k_{N+1}}$$

$$+ \sum_{r=1}^{N+1-n} \left[ h_r(p_{k_1}, \ldots, \widehat{p_{k_r}}, \ldots, p_{k_{N+1}}) - h_r(p_{k_1}, \ldots, \widehat{p_{k_r}}, \ldots, p_{k_{N+1}}) \right] t^{(n+r-1)}.$$

Eliminating $c_{k_1 \ldots k_{N+1}}$, with the help of (A.5) (with $n$ replaced by $n + 1$), we obtain

$$t^{(n-1)}_{k_1 \ldots k_{N+1}} - t^{(n-1)}_{k_{N+1} \ldots k_1} = \left[ h_1(p_{k_1}, \ldots, \widehat{p_{k_1}}, \ldots, p_{k_{N+1}}) - h_1(p_{k_1}, \ldots, \widehat{p_{k_1}}, \ldots, p_{k_{N+1}}) \right] t^{(n)}$$

$$- (p_{k_1} - p_{k_N}) t^{(n)}_{k_1 \ldots k_{N+1}}$$

$$+ \sum_{r=1}^{N+1-n} \left[ h_r(p_{k_1}, \ldots, \widehat{p_{k_r}}, \ldots, p_{k_{N+1}}) - h_r(p_{k_1}, \ldots, \widehat{p_{k_r}}, \ldots, p_{k_{N+1}}) \right] t^{(n+r-1)}.$$

But the last sum vanishes as a consequence of the identities

$$h_r(p_{k_1}, \ldots, \widehat{p_{k_r}}, \ldots, p_{k_{N+1}}) - h_r(p_{k_1}, \ldots, \widehat{p_{k_r}}, \ldots, p_{k_{N+1}}) = (p_{k_1} - p_{k_N}) h_{r-1}(p_{k_1}, \ldots, p_{k_{N+1}}).$$

□

A.2. Main results

For fixed $M$, we have $M + 1$ phases

$$\theta_i = \sum_{r=1}^{M} p'_i t^{(r)} + c_i \quad i = 1, \ldots, M + 1.$$

In the following we regard the variables $t^{(r)}$, $r = 1, \ldots, M$, as Cartesian coordinates on $\mathbb{R}^M$. The region in $\mathbb{R}^M$ where $\theta_i$ dominates is given by

$$U_i = \{ t \in \mathbb{R}^M \mid \max\{\theta_1(t), \ldots, \theta_{M+1}(t)\} = \theta_i(t) \}.$$

18 Despite our notation, $t^{(n-1)}_{k_1 \ldots k_n}$ depends on the choice of $N$, of course. Note that it is a function of $t^{(n)}$, $\ldots$, $t^{(N)}$.

19 A proof of these identities is obtained via the substitution $c_k \mapsto p'_k$ (cf proposition A.2) in the formula in proposition A.1.
Associated with any set \( \{ k_1, \ldots, k_{n+1} \} \subset \{ 1, \ldots, M + 1 \} \), \( n > 0 \), there is a critical plane,

\[
\mathcal{P}_{k_1, \ldots, k_{n+1}} = \{ t \in \mathbb{R}^M | \theta_{k_i} (t) = \cdots = \theta_{k_{n+1}} (t) \},
\]

which is an affine plane of dimension \( M - n \). Since the \( p_i \) are pairwise different, no pair of hyperplanes \( \mathcal{P}_{ij}, 1 \leq i < j \leq M + 1 \), can be parallel. In particular, they cannot coincide and thus \( U_i \neq \emptyset, i = 1, \ldots, M + 1 \). We also note that \( \bigcup_{1 \leq i \leq M+1} U_i = \mathbb{R}^M \). Some obvious relations are

\[
\mathcal{P}_{k_1, \ldots, k_{n+1}} \subset \mathcal{P}_{k_1, \ldots, k_{n+1}, m} \quad \text{for} \quad m < n
\]

and

\[
\mathcal{P}_{k_1, \ldots, k_{n+1}} = \mathcal{P}_{k_1, \ldots, k_{n+1}} \cap \mathcal{P}_{k_1, \ldots, k_{n+1}, r} \quad \text{for} \quad r \neq s,
\]

where a hat again indicates an omission, hence also

\[
\mathcal{P}_{k_1, \ldots, k_{n+1}} = \bigcap_{r=1}^{n+1} \mathcal{P}_{k_1, \ldots, k_{n+1}}.
\]

We can use \( t^{(n+1)}, \ldots, t^{(M)} \) as coordinates on \( \mathcal{P}_{k_1, \ldots, k_{n+1}} \), since on this subset of \( \mathbb{R}^M \) the remaining coordinates are fixed as solutions of the system \( \theta_{k_1} = \cdots = \theta_{k_{n+1}} = -t^{(0)} \), i.e.

\[
\sum_{r=0}^{n} p'_{k_j} t^{(r)} = -c_j - \sum_{r=n+1}^{M} p'_{k_j} t^{(r)} \quad j = 1, \ldots, n + 1.
\]

We solve this system for \( t^{(r)}, r = 1, \ldots, n \), and denote the solutions as \( t^{(r)}_{k_1, \ldots, k_{n+1}}(t^{(n+1)}, \ldots, t^{(M)}) \), \( r = 1, \ldots, n \). They depend linearly on the parameters \( c_i \). For the highest we already found

\[
t^{(n)}_{k_1, \ldots, k_{n+1}}(t^{(n+1)}, \ldots, t^{(M)}) = -\sum_{r=1}^{M-n} h_i(p_{k_1}, \ldots, p_{k_{n+1}}) t^{(n+r)} - c_{k_1, \ldots, k_{n+1}},
\]

which is totally symmetric in the lower indices. These are called critical values of \( t^{(n)} \). The values of \( t^{(r)}_{k_1, \ldots, k_{n+1}}(t^{(n+1)}, \ldots, t^{(M)}) \), \( r = 1, \ldots, n - 1 \), are then determined iteratively as functions of \( t^{(n+1)}, \ldots, t^{(M)} \). Hence the points of \( \mathcal{P}_{k_1, \ldots, k_{n+1}} \) are given by

\[
t_{k_1, \ldots, k_{n+1}}(t^{(n+1)}, \ldots, t^{(M)}) := t^{(1)}_{k_1, \ldots, k_{n+1}}, \ldots, t^{(n)}_{k_1, \ldots, k_{n+1}}, t^{(n+1)}_{k_1, \ldots, k_{n+1}}, \ldots, t^{(M)}_{k_1, \ldots, k_{n+1}},
\]

where we suppressed the arguments of \( t^{(r)}_{k_1, \ldots, k_{n+1}} \).

**Proposition A.4.** Let \( \{ k_1, \ldots, k_{n+1} \} \subset \{ 1, \ldots, M + 1 \} \) and \( p_k < p_j \). Then we have

\[
t^{(n-1)}_{k_1, \ldots, k_{n+1}} \leq t^{(n-1)}_{k_1, \ldots, k_{n+1}} \quad \text{for} \quad t^{(n)} \leq t^{(n)}_{k_1, \ldots, k_{n+1}}.
\]

**Proof.** This is an immediate consequence of (A.6). \( \square \)

**Proposition A.5.** Let \( \{ k_1, \ldots, k_{n+1} \} \subset \{ 1, \ldots, M + 1 \} \). Then,

\[
\theta_{k_1} - \theta_{k_{n+1}} = -\sum_{r=1}^{n} \left( \prod_{j=1}^{r} (p_{k_{n+1}} - p_{k_j}) \right) (t^{(r)} - t^{(r)}_{k_1, \ldots, k_{n+1}}).
\]

**Proof.** Equation (A.6) with \( n = 1 \) reads

\[
\theta_{k_1} - \theta_{k_2} = -(p_{k_2} - p_{k_1})(x - x_{k_1, k_2}),
\]

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which is the above formula for \( n = 1 \). Assuming that the assertion holds for \( n \), we can apply it with \( \{k_1, \ldots, k_n, k_{n+1}\} \) to obtain

\[
\theta_{k_1} - \theta_{k_{n+1}} = -\sum_{r=1}^{n-1} \left( \prod_{j=1}^{r} (p_{k_{r+1}} - p_{k_j}) \right) \left( t^{(r)} - t^{(r)}_{k_1 \ldots k_{r+1}} \right)
\]

\[
- \left( \prod_{j=1}^{n} (p_{k_{n+1}} - p_{k_j}) \right) \left( t^{(n)} - t^{(n)}_{k_1 \ldots k_{n+1}} \right)
\]

\[
= - \sum_{r=1}^{n} \left( \prod_{j=1}^{r} (p_{k_{n+1}} - p_{k_j}) \right) \left( t^{(r)} - t^{(r)}_{k_1 \ldots k_{n+1}} \right)
\]

\[
- \left( \prod_{j=1}^{n} (p_{k_{n+1}} - p_{k_j}) \right) \left( t^{(n)} - t^{(n)}_{k_1 \ldots k_{n+1}} \right)
\]

Using (A.6) in the last factor, this becomes the asserted formula for \( n + 1 \), which thus completes the induction step. \( \square \)

**Corollary A.6.** Let \( \{k_1, \ldots, k_{n+1}\} \subset \{1, \ldots, M + 1\} \). On \( P_{k_1 \ldots k_{n+1}} \), we have

\[
\theta_{k_1} - \theta_{k_{n+1}} = -(p_{k_{n+1}} - p_{k_1})(p_{k_{n+1}} - p_{k_2}) \cdots (p_{k_{n+1}} - p_{k_n})(t^{(n)} - t^{(n)}_{k_1 \ldots k_{n+1}}).
\]

(A.8)

A point \( t_0 \in P_{k_1 \ldots k_{n+1}} \) will be called non-visible if there is an \( m \notin \{k_1, \ldots, k_{n+1}\} \) such that \( \theta_{k_m}(t_0) > \theta_{k_1}(t_0) \) \(^{20}\) Otherwise it will be called visible. In previous sections we considered the projection into the \( xy \)-plane, \( P_{k_1 \ldots k_{n+1}}(t^{(n+1)}, \ldots, t^{(M)}) \), of a point \( t_{k_1 \ldots k_{n+1}}(t^{(n+1)}, \ldots, t^{(M)}) \in \mathcal{T}_{k_1 \ldots k_{n+1}} \). Our previous notion of visibility of \( P_{k_1 \ldots k_{n+1}}(t^{(n+1)}, \ldots, t^{(M)}) \), which means ordinary visibility in a plot of \( \max \{\theta_1, \ldots, \theta_M\} \), is in fact equivalent to the visibility of the latter point in \( R^M \). In the following, \( [n/2] \) denotes the smallest integer greater than or equal to \( n/2 \), and \( [n/2] \) the largest integer smaller than or equal to \( n/2 \).

**Proposition A.7.** For \( n = 1, 2, \ldots \), let \( \{k_1, \ldots, k_{n+1}\} \subset \{1, \ldots, M + 1\} \), \( p_{k_1} < p_{k_2} < \cdots < p_{k_{n+1}} \) and \( t^{(n+1)}_0, \ldots, t^{(M)}_0 \in R \). The following half-lines are non-visible:

1. \( \{t^{(n+1)}_{k_1 \ldots k_{n+1}}, \ldots, t^{(M)}_{k_1 \ldots k_{n+1}} \} \cap P_{k_1 \ldots k_{n+1}}(t^{(n+1)}_0, \ldots, t^{(M)}_0) \subset P_{k_1 \ldots k_{n+1}}(t^{(n+1)}_0, \ldots, t^{(M)}_0) \), \( r = 0, \ldots, [n/2] - 1 \);
2. \( \{t^{(n)}_{k_1 \ldots k_{n+1}}, \ldots, t^{(M)}_{k_1 \ldots k_{n+1}} \} \cap P_{k_1 \ldots k_{n+1}}(t^{(n+1)}_0, \ldots, t^{(M)}_0) \subset P_{k_1 \ldots k_{n+1}}(t^{(n+1)}_0, \ldots, t^{(M)}_0) \), \( r = 0, \ldots, [n/2] \).

Here \( t^{(n)}_{k_1 \ldots k_{n+1}} \) stands for \( t^{(n)}_{k_1 \ldots k_{n+1}}(t^{(n+1)}_0, \ldots, t^{(M)}_0) \).

**Proof.** On \( P_{k_1 \ldots k_{n+1}} \), (A.8) can be written in the form

\[
\theta_{k_1} - \theta_{k_{n+1}} = -\prod_{j=1}^{m} (p_{k_{n+1}} - p_{k_j}) \left( t^{(n)} - t^{(n)}_{k_1 \ldots k_{n+1}} \right)
\]

\( m = 2, \ldots, n+1 \).

We actually consider this equation on \( P_{k_1 \ldots k_{n+1}} \cap E \), where \( E \) is the plane in \( R^M \) determined by fixing the values of \( t^{(n+1)}_0, \ldots, t^{(M)}_0 \) to \( t^{(n+1)}_0, \ldots, t^{(M)}_0 \). As a consequence of our assumption \( p_{k_1} < \cdots < p_{k_{n+1}} \), for \( m = n+1 \) the above expression is negative if \( t^{(n)} > t^{(n)}_{k_1 \ldots k_{n+1}} \); hence, \( P_{k_1 \ldots k_{n+1}} \cap E \) is then non-visible. For \( m = n \) the expression is negative if \( t^{(n)} < t^{(n)}_{k_1 \ldots k_{n+1}} \); hence, \( P_{k_1 \ldots k_{n+1}} \cap E \) is then non-visible. For \( m = n-1 \), the expression is negative if \( t^{(n)} > t^{(n)}_{k_1 \ldots k_{n+1}} \); hence, \( P_{k_1 \ldots k_{n+1}} \cap E \) is then non-visible. This argument can be continued as long as \( m > 1 \).

\(^{20}\) In this case \( m \) can be chosen such that \( \theta_{k_m} \) is a dominating phase at \( t_0 \).
On the remaining critical plane \( P_{k_2,...,k_{n+1}} \), which appears in case (1) for odd \( n \) and in case (2) for even \( n \), we can write the above equation as

\[
\theta_{k_{n+1}} - \theta_{k_2} = (-1)^{n+1} \left( \prod_{j=2,...,n+1} (p_{k_j} - p_{k_2}) \right) (t^{(n)} - t_{k_2}^{(n)}).
\]

This is negative if either \( n \) is odd and \( t^{(n)} < t_{k_2}^{(n)} \), or if \( n \) is even and \( t^{(n)} > t_{k_2}^{(n)} \). As a consequence, \( P_{k_2,...,k_{n+1}} \cap E \) is then non-visible.

We note that the sets of critical planes in part 1 and part 2 of proposition A.7 are complementary. In the following we call a critical point \( t_0 \in P_{k_1,...,k_{n+1}} \) generic if it is not also a higher order critical point, i.e. if \( t_0 \notin P_{k_2,...,k_{n+1}} \) with any \( k_{n+2} \notin \{ k_1,...,k_{n+1} \} \).

**Proposition A.8.** Let \( \{ k_1,\ldots,k_{n+1} \} \subset \{ 1,\ldots,M+1 \} \) and \( p_{k_1} < p_{k_2} < \cdots < p_{k_{n+1}} \). Let \( \alpha, \beta \) be such that in the open interval \((\alpha, t^{(n)}_{k_1,...,k_{n+1}})\), respectively \((-\infty, t^{(n)}_{k_1,...,k_{n+1}})\), there is no critical value of \( t^{(n)} \) corresponding to a visible critical point. If \( t_{k_1,...,k_{n+1}}^{(n)} \) is generic and visible, then the following line segments are visible:

1. \( \{ t_{k_1,...,k_{n+1}}^{(n+1)}, \ldots, t_{k_1,...,k_{n+1}}^{(M)} \} \cap (\alpha, t^{(n)}_{k_1,...,k_{n+1}}) \)
2. \( \{ t_{k_1,...,k_{n+1}}^{(n)}, \ldots, t_{k_1,...,k_{n+1}}^{(M)} \} \cap (-\infty, t^{(n)}_{k_1,...,k_{n+1}}) \)

Here we set \( t_{k_1,...,k_{n+1}}^{(0)} = t_{k_1,...,k_{n+1}}^{(0+)} \).

**Proof.** In the following we use \( E \) as defined in the proof of proposition A.7. Since we assume \( t_0 := t_{k_1,...,k_{n+1}}^{(0+)} \) to be visible, at \( t_0 \) the phases \( \theta_{k_1}, \ldots, \theta_{k_{n+1}} \) coincide and dominate. Since \( t_0 \) is assumed to be generic, there is a neighborhood of \( t_0 \) in \( E \) which is covered by the polyhedral cones \( U_{k_i} \cap E, j = 1,\ldots,n+1 \). Since each line \( P_{k_i,...,k_{n+1}} \cap E, r \in \{ 1,\ldots,n+1 \} \), contains \( t_0 \), it follows that its visible part \( P_{k_i,...,k_{n+1}} \cap E \cap U_{k_i} \cap \cdots \cap U_{k_{n+1}} \cap U_{k_1} \cap \cdots \cap U_{k_{n+1}} \) extends in the direction complementary to that in proposition A.7, either indefinitely or up to a point of \( E \) where it meets \( U_0 \) with some \( m \notin \{ k_1,\ldots,k_{n+1} \} \). We only need to consider the latter case further. Then \( \{ t_1 \} := U_0 \cap \cdots \cap U_{k_1} \cap U_{k_2} \cap \cdots \cap U_{k_{n+1}} \cap E \) is a visible point in \( P_{k_1,...,k_{n+1}} \cap E \). Since the line between \( t_0 \) and \( t_1 \) is visible, it cannot be a part of the non-visible half-line determined by proposition A.7 applied to \( t_1 \).

The following proposition shows that the existence of a visible critical point requires the existence of a visible critical point one level higher.

**Proposition A.9.** Let \( n \leq M \) and \( t_0^{(0)} \in \mathbb{R}, r = n,\ldots,M \). If all points \( t_{k_1,...,k_{n+1}}^{(r)} \), where \( \{ k_1,\ldots,k_{n+1} \} \supset \{ l_1,\ldots,l_n \} \), are non-visible, then the line \( \{ t_{l_1,...,l_n}^{(r)}, t_0^{(r+1)}, \ldots, t_0^{(M)} \} \) is non-visible.

**Proof.** Let \( E \) again denote the set \( \{ t \in \mathbb{R}^M | t^{(r+1)} = t^{(0+)}_0, \ldots, t^{(M)} = t^{(M)}_0 \} \) and let \( t_0 \in P_{k_1,...,k_{n+1}} \cap E \) be visible. Since \( n \leq M \), a critical point exists one level higher, given by \( \{ t_1 \} = P_{k_1,...,k_{n+1}} \cap E \) with some \( k_1,\ldots,k_{n+1} \) such that \( \{ l_1,\ldots,l_n \} \subset \{ k_1,\ldots,k_{n+1} \} \). If this point is visible, the proposition holds. If \( t_1 \) is non-visible, then \( t_1 \in U_{k_1} \cap E \) with some \( m \notin \{ k_1,\ldots,k_{n+1} \} \). Clearly, it cannot coincide with \( t_0 \). By continuity, on the line segment between \( t_0 \) and \( t_1 \), a visible critical point then exists, which is \( t_{l_1,...,l_n}^{(r+1)}, \ldots, t_{l_1,...,l_n}^{(M)} \). This proves that if \( t_{l_1,...,l_n}^{(r)} = t_0^{(r+1)}, \ldots, t_0^{(M)} \) is visible, then there is a visible critical point
Without restriction of generality we can choose
\[ p_1 \prec p_2 \prec \cdots \prec P_{M+1}. \]
There is only a single critical value \( t(M) \). As a meeting point of all phases, \( t(M) \) is visible. Proposition A.4 shows that for \( 1 \leq i < j \leq M+1, \)
\[ t(M-1)_{i,j,M+1} \leq t(M-1)_{i,j,M+1} \quad \text{for} \quad t(M) \leq t(M)_{k_{i,k_{i+1}}}. \]
According to proposition A.7 and proposition A.8, only the half-lines
\[ \{ t_{1,\ldots,M-2r,\ldots,M+1} | t(M) > t(M)_{k_{i,k_{i+1}}} \} \subset R_{1,\ldots,M-2r,\ldots,M+1} \quad r = 0, \ldots, [(M+1)/2] - 1, \]
\[ \{ t_{1,\ldots,M-2r+1,\ldots,M+1} | t(M) < t(M)_{k_{i,k_{i+1}}} \} \subset R_{1,\ldots,M-2r+1,\ldots,M+1} \quad r = 0, \ldots, [(M+1)/2]. \]
are visible. We note that the two sets of lines are complementary, and each of them is visible in exactly one of the two half-spaces (corresponding to \( t(M) > t_{k_{i,k_{i+1}}}, \) respectively \( t(M) < t_{k_{i,k_{i+1}}}. \)). Proceeding in this way, we find that each critical 2-plane \( P_{k_{i,k_{i+1}}} \) is visible in some region of \( \mathbb{R}^M \), and so forth. Since \( P_{k_{i,k_{i+1}}} \) is contained in all critical planes, they all contain visible points.

The next result is particularly helpful.

**Proposition A.10.** All non-visible critical points are obtained by application of proposition A.7 only to visible critical points, and by proposition A.9.\(^{21}\)

**Proof.** Let \( t_0 = t_0_{[0,0]}(0^{(0)}, \ldots, 0^{(M)} \} \) be non-visible. If there is no visible critical point of the form \( t_{k_{i,k_{i+1}}}^{(0)}, \ldots, 0^{(M)} \), with \( \{0, \ldots, 0\} \subset \{k_{i}, \ldots, k_{i+1}\} \), then the non-visibility of \( t_0 \) is a consequence of proposition A.9. If there is a visible critical point of the above form, then there is also a visible critical point \( t_1 \) such that no other visible critical point exists on the line segment joining \( t_0 \) and \( t_1 \). Since \( t_0 \) cannot lie on the visible side of \( t_1 \) as determined by proposition A.8, it lies on the non-visible side of \( t_1 \) as determined by proposition A.7. Hence the non-visibility of \( t_0 \) is a consequence of proposition A.7.

The chains of rooted binary trees describing line soliton solutions can be constructed from the knowledge of the ‘visible’ critical values \( t_{k_{i,k_{i+1}}}^{(n)}, n = 1, \ldots, M, \) and their order (determined top down via proposition A.4). For \( t^{(n+1)} \) from the interval between two of its critical values (formally including \( \pm \infty \)), the corresponding visible critical values of \( t^{(n)} \) are obtained from all critical values simply by deleting all those that are non-visible by an application of the rules of proposition A.4, proposition A.7 and proposition A.9 (where the latter may not be necessary).

Of course, one can establish further useful results about the visibility or non-visibility of critical points. The following is an example.

**Proposition A.11.**

1. Let \( 0 \leq r < s \leq [(n+1)/2] - 1 \) and \( t_0^{(n+1)} < t_{k_{i',k_{i'+1}}}^{(n+1)}(0^{(n+2)}, \ldots, 0^{(M)}). \) Then the whole line \( \{ t_{k_{i',k_{i'+1}}}^{(n+1)}, 0^{(n+2)}, \ldots, 0^{(M)} \} | t^{(n)} \in \mathbb{R} \} \) is non-visible.
2. Let \( 0 \leq r < s \leq [(n+1)/2] \) and \( t_0^{(n+1)} > t_{k_{i',k_{i'+1}}}^{(n+1)}(0^{(n+2)}, \ldots, 0^{(M)}). \) Then the whole line \( \{ t_{k_{i',k_{i'+1}}}^{(n+1)}, 0^{(n+2)}, \ldots, 0^{(M)} \} | t^{(n)} \in \mathbb{R} \} \) is non-visible.

\(^{21}\) We conjecture that the reference to proposition A.9 can be dropped, provided the application of proposition A.7 is extended to all critical points.
Proof. We only prove (1). If \( t_0^{(n+1)} < t_{1345}^{(n+2)}(t_0^{(n+1)}, \ldots, t_0^{(M)}) \), then proposition A.4 implies that

\[
t_{1345}^{(n+2)}(t_0^{(n+1)}, \ldots, t_0^{(M)}) < t_{1345}^{(n+1)}(t_0^{(n+1)}, \ldots, t_0^{(M)}),
\]

where \( t_0^{(n+1)} < t_{1345}^{(n+2)}(t_0^{(n+1)}, \ldots, t_0^{(M)}) \) is non-visible by the application of proposition A.7. But, again as a consequence of proposition A.7, it is also non-visible for \( t_0^{(n)} > t_{1345}^{(n+2)}(t_0^{(n+1)}, \ldots, t_0^{(M)}) \).

□

Example A.12. Let \( M = 5 \) and \( t^{(5)} < t_{123456}^{(5)} \). Applying proposition A.11 (part 1) with \( n = 4 \), we find that the events associated with the critical times \( t_{1246}, t_{2346}, t_{2456} \) are non-visible. Since these are all possible critical times at which \( y_{246} \) can coincide with other critical \( y \)-values, and since there is no corresponding node in the initial rooted binary tree, it cannot appear during any line soliton evolution (with \( t^{(5)} < t_{123456}^{(5)} \)). The non-visibility of \( y_{246} \) also follows by an application of proposition A.9. As a consequence, the left tree in figure A1 does not appear in figure 18.

If \( t^{(5)} > t_{123456}^{(5)} \), proposition A.11 (part 2) with \( n = 4 \) shows that \( t_{1356} < t_{2345} < t_{1345} \) are non-visible. This in turn implies that \( y_{135} \) can never show up, which excludes the right tree in figure A1, which indeed does not appear in figure 19.

Appendix B. A finer classification in terms of trees with levels

A finer description of line soliton evolutions can be achieved by using the refinement of (rooted) binary trees to ‘trees with levels’ [30]. Figure B1 shows such a refinement of the second chain in figure 12, including also the degenerate trees at \( t = t_{1345} \) and \( t = t_{1235} \) which are not binary. Here we took into account that a time \( t_0 \) exists at which the two subtrees appearing between \( t_{1345} \) and \( t_{1235} \) have the same height, i.e. the same \( y \)-value. The third and the fifth tree are the two trees with levels associated with the fourth tree in figure B1.

Setting \( t^{(5)} \) and higher variables to zero, the condition \( y_{ijk} = y_{lmn} \) determines the ‘critical’ time

\[
t_{ijk;lmn} = \frac{1}{p_i + p_j + p_k - p_l - p_m - p_n} (c_{ijk} - c_{lmn} + [h_2(p_i, p_j, p_k) - h_2(p_l, p_m, p_n)] t^{(4)}),
\]

provided that \( p_i + p_j + p_k \neq p_l + p_m + p_n \).
Example B.1. Let $M = 4$. In the case considered in figure B1, we have $t_0 = t_{123;345}$ and

$$t_0 - t_{1345} = \frac{(p_4 - p_2)(p_5 - p_2)}{p_4 - p_1 + p_5 - p_2},$$

$$t_{1235} - t_0 = \frac{(p_4 - p_1)(p_4 - p_2)}{p_4 - p_1 + p_5 - p_2}.$$

Assuming $p_1 < \cdots < p_5$, these expressions are both positive since the chain is only realized if $t^{(4)} > t^{(4)}_{12345}$ (right chain in figure 13). Furthermore,

$$x_{345}(t_0) - x_{123}(t_0) = \frac{(p_4 - p_1)(p_4 - p_2)(p_5 - p_1)(p_5 - p_2)}{p_4 - p_1 + p_5 - p_2} (t^{(4)} - t^{(4)}_{12345}) > 0.$$

After introduction of $t^{(5)}$, the analogous condition $t_{ijkl} = t_{mnr}$ determines the following critical value of $t^{(4)}$:

$$t^{(4)}_{ijkl;mnr} = \frac{1}{p_i + p_j + p_k + p_l - p_m - p_n - p_r - p_s} \times (c_{ijkl} - c_{mnr} + h_2(p_i, p_j, p_k, p_l) - h_2(p_m, p_n, p_r, p_s)) t^{(5)},$$

provided that $p_i + p_j + p_k + p_l \neq p_m + p_n + p_r + p_s$.\footnote{See figure B2 for an example.}

A further useful formula is

$$t_{ijkl} - t_{mnr} = \frac{(p_m + p_n + p_r + p_s - p_i - p_j - p_k - p_l)(t^{(4)} - t^{(4)}_{ijkl;mnr})}{t^{(4)}_{ijkl;mnr}}.$$\footnote{This determines the relative order of $t_{ijkl}$ and $t_{mnr}$.}

Depending on the order of the $p$'s, and whether $t^{(4)} > t^{(4)}_{ijkl;mnr}$ or $t^{(4)} < t^{(4)}_{ijkl;mnr}$, this determines the relative order of $t^{(4)}_{ijkl}$ and $t_{mnr}$.

Example B.2. Let $M = 5$. The additional critical values of $t^{(4)}$ can be used to refine figures 18 and 19. For $t^{(5)} < t^{(5)}_{12345}$, they have to satisfy the inequalities $t^{(4)}_{1256;345} < t^{(4)}_{12345} < t^{(4)}_{12356} < t^{(4)}_{1236;345}$. Indeed, we find

$$t^{(4)}_{1236;345} - t^{(4)}_{12356} = \frac{(p_4 - p_1)(p_4 - p_2)}{p_4 + p_5 - p_1 - p_2} (t^{(5)} - t^{(5)}_{12345}) > 0,$$

$$t^{(4)}_{12345} - t^{(4)}_{12356} = \frac{(p_4 - p_1)(p_4 - p_2)}{p_4 + p_5 - p_1 - p_2} (t^{(5)} - t^{(5)}_{12345}) > 0,$$

so that $t^{(4)}_{1236;345}$ always exists. This is not so for $t^{(4)}_{1256;345}$. Firstly, it is only defined if $p_1 + p_6 \neq p_3 + p_4$. Secondly,

$$t^{(4)}_{12356} - t^{(4)}_{1256;345} = \frac{(p_6 - p_3)(p_6 - p_4)}{p_1 + p_6 - p_3 - p_4} (t^{(5)} - t^{(5)}_{12345})$$\footnote{It should now be obvious how this extends to a formula for corresponding critical values of $t^{(n)}$, $n > 2$.}
Figure B3. A representation of the Tamari lattice $T_A$ and the conditions on $t^{(4)}$ under which the respective chains are realized by line soliton solutions. Here e.g. $>t^{(4)}_{12346}$ stands for $t^{(4)}_{1234} > t^{(4)}_{12456}$. A number $ijkl$ assigned to an edge represents a critical transition time $t_{ijkl}$. On a dashed line, $t^{(4)}$ is equal to a critical value, and this corresponds to a direct transition, skipping a next neighbor on a maximal chain. For any of the additional critical values that take care of trees with levels, this is a transition in a tetragon (whereas for an ordinary critical value it takes place in a pentagon).

is positive only if $p_1 + p_6 < p_3 + p_4$ holds.

For $t^{(5)} > t^{(4)}_{12345}$, the inequalities $t^{(4)}_{1256;2345} < t^{(4)}_{12346} < t^{(4)}_{1234;1456} < t^{(4)}_{123456} > 0$, and $t^{(4)}_{1234;1456} > 0$, so that $t^{(4)}_{1234;1456}$ always exists. But $t^{(4)}_{1256;2345}$ only shows up if

$$t^{(4)}_{1256;2345} = \frac{(p_5 - p_4)(p_4 - p_1)}{p_1 + p_6 - p_3 - p_4}$$

is positive, which requires $p_1 + p_6 > p_3 + p_4$. The possible orders of the critical $t^{(4)}$-values are summarized in table B1.

The additional critical values of $t^{(4)}$ moreover allow us to express the conditions in table 1 under which a line soliton solution corresponds to one of the maximal chains in $T_A$ in
terms of inequalities involving only $t^{(4)}$ and its critical values. Here we use some of the above and similar expressions for the differences of critical values of $t^{(4)}$. The results are collected in table B2.

Appendix C. A symbolic representation of trees with levels, and a relation between permutahedra and Tamari lattices

This appendix presents some results that should also be of interest beyond the line soliton classification problem. We should stress, however, that not all statements are accompanied by a rigorous proof.

C.1. A poset structure for permutahedra

Let us assign to each node of a rooted binary tree a separate level and number the levels from top to bottom. The node on level $l$ will then be represented by the natural number $n_l$ if it lies on the $n_l$th edge, where the edges are consecutively numbered from left to right along the level. The highest node (root node) thus always corresponds to $n_1 = 1$. In this way any rooted binary tree with levels [30] (also see appendix B) and with $r$ (internal) nodes is uniquely represented by a sequence of natural numbers $n_1, n_2, \ldots, n_r$ with $n_i \leq i$, $i = 1, \ldots, r$, and any such sequence defines a rooted binary tree with levels. Hence we have a bijection between the set of rooted binary trees with levels and with $r$ nodes, and the set

$$\mathcal{S}_r = \{ n = (n_1, n_2, \ldots, n_r)| n_i \in \mathbb{N}, n_i \leq i, i = 1, \ldots, r \}.$$
This set has \( r! \) elements. For example, the chain consisting of the first, third, fifth and last tree in figure B1 corresponds to the chain \((1, 1, 1) \rightarrow (1, 2, 1) \rightarrow (1, 1, 2) \rightarrow (1, 2, 3)\). The left tree in figure B2 corresponds to \((1, 1, 2, 2)\), and the third to \((1, 2, 2, 3)\).

On \( \mathcal{S}_r \), we define an action of the permutation group \( \mathcal{S}_r \) as follows. Let \( \sigma : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N} \) be given by

\[
\sigma(m, n) = \begin{cases} 
(n, m + 1) & \text{if } m \geq n, \\
(n - 1, m) & \text{if } m < n
\end{cases}
\]

Clearly, \( \sigma \) is involutory: \( \sigma^2 = \text{id} \). For \( s = 1, \ldots, r - 1 \), let \( \sigma_s : \mathcal{S}_r \rightarrow \mathcal{S}_r \) be the map given by application of \( \sigma \) to the \( s \)th pair, counted from right to left, in the sequence of natural numbers defining an element of \( \mathcal{S}_r \), i.e.

\[
\sigma_s(n_1, \ldots, n_r) = \begin{cases} 
(n_1, \ldots, n_{r-s-1}, n_{r-s+1}, n_{r-s} + 1, n_{r-s+2}, \ldots, n_r) & \text{if } n_{r-s} \geq n_{r-s+1}, \\
(n_1, \ldots, n_{r-s+1} - 1, n_{r-s}, n_{r-s+2}, \ldots, n_r) & \text{if } n_{r-s} < n_{r-s+1}.
\end{cases}
\]

Then we have the relations

\[
\sigma_s^2 = \text{id}, \quad \sigma_s \sigma_{s+1} \sigma_s = \sigma_{s+1} \sigma_s \sigma_{s+1}, \quad \sigma_{s+1} \sigma_s = \sigma_s \sigma_{s+1} \quad \text{if } |s - s'| > 1,
\]

and we have an action of the symmetric group \( \mathcal{S}_r \) on \( \mathcal{S}_r \).

Let \( \sigma^H \) be the restriction of \( \sigma \) to \( H = \{(n_1, n_2) \in \mathbb{N} \times \mathbb{N} | n_1 \geq n_2 \} \). Defining

\[
n \prec \ n' \quad \text{if} \quad n = \sigma^H(n') \quad \text{for some} \ s,
\]

we obtain in an obvious way a partial order \( \preceq \) on \( \mathcal{S}_r \). Then \((1, \ldots, 1)\) is minimal and \((1, 2, \ldots, r)\) is maximal with respect to this partial order. This results in a poset underlying the permutahedron of order \( r \) \([50, 51]\)\(^{23}\).

For what follows it is convenient to split the operation \( \sigma^H \) into two operations \( a \) and \( b \), according to a split of \( H \) into its diagonal part and the rest. Hence, for \( m, n \in \mathbb{N} \) we have

\[
a(n, n) = (n, n + 1), \quad b(m, n) = (n, m + 1) \quad \forall m > n.
\]

As a consequence of their origin, the operations \( a_s \) and \( b_s \) satisfy the braid relation

\[
a_s a_{s+1} a_s = a_{s+1} b_s a_{s+1}.
\]

(C.1)

This is only defined on a subsequence of the form \( n, n, n \) (with the last \( n \) at position \( s \), counted from the right). For \( r = 3 \), (C.1) applied to the minimal element \((1, 1, 1)\) generates the whole poset underlying the permutahedron of order 3, see figure C1. We observe that it collapses to the Tamari lattice \( \mathbb{T}_3 \) if we identify \((1, 2, 1)\) and \((1, 1, 3)\), which are related by \( b_1 \) and which are trees with levels having the same underlying rooted binary tree.

In addition, we have the identities

\[
b_s b_{s+1} a_s = a_{s+1} b_s b_{s+1}, \quad a_s b_{s+1} b_s = b_{s+1} a_s b_{s+1}, \quad b_s b_{s+1} b_s = b_{s+1} b_s b_{s+1}
\]

(C.2)

(23) Also see [30] for a way to associate a permutation with each tree with levels, and hence with any sequence in \( \mathcal{S}_r \), for some \( r \in \mathbb{N} \).

(24) The brackets are only used to display the structure of these expressions more clearly.

Proposition C.1. A special maximal chain in the permutahedron poset \((\mathcal{S}_r, \preceq)\) is obtained by application of

\[
a_1 (a_2 a_1) (a_3 a_2 a_1) \cdots (a_{r-2} \cdots a_1) (a_{r-1} \cdots a_1)
\]
to the minimal element 11...1 (with r times 1). Its length is $\frac{1}{2}(r - 1)r$.

**Proof.** Stepwise application of $a_{r-1} \ldots a_1$ yields $(1, \ldots, 1) \xrightarrow{a_1} (1, \ldots, 1, 2) \xrightarrow{a_2} (1, 2, \ldots, 2)$. Application of the next subsequence leads to $(1, 2, \ldots, 2) \xrightarrow{a_{r-1}} (1, 2, \ldots, 2, 3) \xrightarrow{a_2} \ldots \xrightarrow{a_{r-1}} (1, 2, 3, \ldots, 3)$. Continuing in this way, we finally obtain the maximal element $(1, 2, \ldots, r)$. The total number of $a$'s in the sequence is $\sum_{a=1}^{r-1} n = (r - 1)r/2$.

□

**Remark C.2.** The application of an $a$ or $b$ to an element $n \in S_r$ raises the weight $|n| = n_1 + \ldots + n_r$ by 1. In order to get from $(1, \ldots, 1)$, which has weight $r$, to $(1, 2, \ldots, r)$ with weight $r(r+1)/2$, we need $r(r+1)/2 - r = (r - 1)/2$ operations of type $a$ or $b$. This shows that all chains in the permutohedron have the same length, namely $r(r - 1)/2$.

Figure C2 shows the permutohedron of order 4 (i.e. $r = 4$), supplied with the poset structure introduced above. The 16 maximal chains are generated via application of the above braid relations to the sequence $a_1a_2a_3a_4a_1$ that determines a maximal chain according to proposition C.1:

$\{a_1a_2a_3a_4a_1, a_1a_2a_3a_4a_1, a_1a_2a_3a_4a_1, a_1a_2a_3a_4a_1, a_1a_2a_3a_4a_1, a_1a_2a_3a_4a_1, a_1a_2a_3a_4a_1, a_1a_2a_3a_4a_1\}$,

$\{a_1a_2a_3a_4a_1, a_1a_2a_3a_4a_1, a_1a_2a_3a_4a_1, a_1a_2a_3a_4a_1\}$,

$\{a_1a_2a_3a_4a_1, a_1a_2a_3a_4a_1, a_1a_2a_3a_4a_1, a_1a_2a_3a_4a_1\}$,

$\{a_1a_2a_3a_4a_1, a_1a_2a_3a_4a_1, a_1a_2a_3a_4a_1, a_1a_2a_3a_4a_1\}$.

Here we grouped those chains together that are related by a braid relation which only involves $b$'s. We shall see that also this permutohedron can be collapsed to the corresponding Tamari lattice $T_4$.

C.2. From permutohedra to Tamari lattices

As explained in figure C3, the operation $a_i$ corresponds to a right rotation in a rooted binary tree, which is the characteristic property of a Tamari lattice.

An application of $b_i$ does not change the respective underlying rooted binary tree, but only exchanges the associated rooted binary trees with levels, see figure C4.

Identifying those rooted binary trees with levels that correspond to the same rooted binary tree (without levels), we can use as representative the sequence for which we also have

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25 If a sequence of $a$'s and $b$'s maps the minimal element $(1, \ldots, 1)$ to the maximal element $(1, 2, \ldots, r)$ of $S_r$, this remains true for any sequence obtained from it via application of the braid rules. Hence, every sequence obtained in this way again generates a maximal chain in $S_r$. 

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A poset structure for the permutohedron of order 4. Here e.g. 1211 stands for $(1, 2, 1, 1) \in \mathfrak{S}_4$.

The operation $a_{i \rightarrow i}$ amounts to a right rotation taking place between the two levels $i$ and $i + 1$. For the left tree we have $n_i = n_{i+1} = n$, and for the right tree $n_i = n$ and $n_{i+1} = n + 1$.

The operation $b_{i \rightarrow i}$ exchanges the nodes of two consecutive levels. For the left tree we have $n_i = n + k$, $k \in \mathbb{N}$, and $n_{i+1} = n$, and for the right tree $n_i = n$ and $n_{i+1} = n + k + 1$.

$n_i \leq n_{i+1}$ (also see [52]). This defines a bijection between the set of rooted binary trees with $r$ nodes and

\[ \mathcal{Q}_r = \{(n_1, n_2, \ldots, n_r)|n_i \in \mathbb{N}, n_i \leq i \text{ and } n_i \leq n_{i+1} \forall i\}. \]

The number of elements of this set is the Catalan number $C_r = \frac{1}{r+1} \binom{2r}{r}$ (see exercise 19 in [52]). The above partial order on $\mathfrak{S}_r$ induces a partial order on $\mathcal{Q}_r$, and in this way a permutohedron collapses to the corresponding Tamari lattice (or associahedron, also see [53]).
Remark C.3. In section 3.1 we described the nodes of a rooted binary tree, describing a line soliton solution at some event, as coincidences of three phases. Ordering the nodes from top to bottom and from left to right, this assigns a sequence \((i_1, j_1, k_1), \ldots, (i_r, j_r, k_r)\) of ordered triples of natural numbers, \(i_m < j_m < k_m\), to the tree. Then the sequence \(i_1, i_2, \ldots, i_r\) of the first indices is precisely the sequence of natural numbers in \(\mathcal{Q}_r\), that characterizes the tree in the way described above. This correspondence does not extend to trees with levels.

By definition, the operation \(a_s\) preserves \(\mathcal{S}_r\) (hence operates on trees with levels), but it does not preserve \(\mathcal{Q}_r\). We can correct this by application of operations \(b_s\) (which are not defined on \(\mathcal{Q}_r\)). Indeed, one can show that for any sequence \(n ∈ \mathcal{S}_r \setminus \mathcal{Q}_r\), there is a finite combination of \(b_s\)'s that transforms it into a sequence in \(\mathcal{Q}_r\).

In describing Tamari lattices, hence disregarding the refinement to trees with levels, we have to regard two sequences of \(a_s\)'s and \(b_s\)'s as equivalent if they only differ by an application of any of the rules (C.2), and those in (C.3) involving \(b_s\)'s. The restriction of the permutohedron poset to \(\mathcal{Q}_r\), selects those sequences in which any application of some \(a_s\) that leads out of \(\mathcal{Q}_r\) is immediately corrected by \(b_s\)'s. Hence these are sequences where all \(b_s\)'s are commuted as far as possible to the right, using the braid rules that involve \(b_s\), with the exception of (C.1). For the permutohedron of order 4, the 16 maximal chains given in appendix C.1 reduce to 9 maximal chains, which (applied to \((1, 1, 1, 1)\)) generate the maximal chains of the Tamari lattice \(\mathbb{T}_4\) (cf table 1).

Stepwise application of the special sequence of \(a_s\)'s in proposition C.1 to the minimal element \((1, \ldots, 1)\) actually generates a sequence of elements in \(\mathcal{Q}_r\) (see the proof of the proposition). Since the application of \(a\) encodes the characteristic property of a Tamari lattice, this determines a maximal chain in a Tamari lattice. Its length is \((r − 1)r/2\), and this is known to be the greatest length of a chain in \(\mathbb{T}_r\) [54].

Proposition C.4. A shortest maximal chain in the Tamari lattice \(\mathbb{T}_r\) is obtained by the application of

\[ a_{r−1}(b_{r−2})a_{r−1}(b_{r−3}b_{r−2})a_{r−1}(b_{r−4}b_{r−3}b_{r−2})a_{r−1} \cdots a_{r−1}(b_1 \cdots b_{r−2})a_{r−1} \]

to the minimal element \((1, \ldots, 1)\) of \(\mathcal{Q}_r\), \(\preceq\).

Proof. Application of \(b_1 \cdots b_{r−2}a_{r−1}\) yields \((1, \ldots, 1) \xrightarrow{a_{r−1}} (1, 2, 1, \ldots, 1) \xrightarrow{b_{r−2}} (1, 1, 3, 1, \ldots, 1) \xrightarrow{b_{r−3}} (1, 1, 1, 4, 1, \ldots, 1) \xrightarrow{b_{r−4}} \cdots \xrightarrow{b_{r−1}} (1, \ldots, 1, r). The next subsequence \(b_2 \cdots b_{r−2}a_{r−1}\) maps \((1, \ldots, 1, r)\) to \((1, \ldots, 1, r−1, r)\). Continuing in this way, we finally obtain \((1, 2, \ldots, r−1, r)\), the final node of \(\mathbb{T}_r\). Hence the chain is maximal. The total number of \(a_s\)'s is \(r−1\), which is known to be the shortest length of a maximal Tamari chain [54].

Two sequences of \(a_s\)'s and \(b_s\)'s are said to belong to the same class if they differ only by an application of \(a_s a_s' = a_s a_s\) for \(|s−s'| > 1\). In particular, for \(n > 3\), this rule creates further longest maximal chains from those in proposition C.1 and proposition C.4. The 'pentagon rule' (C.1) changes a sequence (and hence a Tamari chain) in a more drastic way (since it changes the number of \(a_s\)’s).

\(\mathbb{T}_3\) consists of two chains, each of which is a class: \(a_1 a_2 a_1\) and \(a_2 b_1 a_2\). For \(\mathbb{T}_4\) there are six classes: \((1) a_1 a_2 a_3 a_4 a_2 a_1\) and \(a_1 a_2 a_1 a_3 a_2 a_1\), \((2) a_1 a_2 a_3 a_2 b_1 a_2\), \((3) a_2 b_1 a_2 a_3 a_2 a_1\), \((4) a_1 a_3 b_2 a_3 b_1 a_2\) and \(a_2 a_3 b_1 a_2 b_1 a_2\), \((5) a_3 a_2 a_3 b_1 b_2 a_3\) and \((6) a_3 b_2 a_3 b_1 b_2 a_3\). For \(\mathbb{T}_5\) there are 25 classes and 94 chains, see table C1 and figure 28.
Maslov dequantization and ultra-discretization

After the rescaling $t^{(n)} \rightarrow t^{(n)}/\hbar$ and $c_i \rightarrow c_i/\hbar$, with a constant $\hbar$, the class of solutions studied in sections 2, 3 and appendix A is given by

$$\tau = \sum_{i=1}^{M+1} e^{\theta_i/\hbar}, \quad \theta_i = \sum_{n=1}^{M} p_n^{(n)} t^{(n)} + c_i.$$

Then we have

$$\lim_{\hbar \to 0} \hbar \log \tau = \lim_{\hbar \to 0} \hbar \log \left( \sum_{i=1}^{M+1} e^{\theta_i/\hbar} \right) = \max\{\theta_1, \ldots, \theta_{M+1}\},$$

applying a formula familiar in the context of tropical mathematics\(^{26}\), and regarding $\theta_i$ as $\hbar$-independent. The result confirms our basic approximation formula in section 2. So far we were only interested in the (evolution of the) form of line solitons as contours in the $xy$-plane. But it is also of interest to find a good approximation for the amplitude $u$ of the KP solution e.g. at the meeting points of line soliton branches, hence at the coincidence points of phases

\(^{26}\) This formula underlies what is called ‘Maslov dequantization’ [55]. A related method is ‘ultra-discretization’ [56].
in the tropical approximation. From
\[ \phi = \hbar (\log \tau)_x = \frac{1}{\tau} \sum_{i=1}^{M+1} p_i e^{\theta_i/\hbar} = \frac{p_k + \sum_{i=1, i \neq k}^{M+1} p_ie^{-(\theta_k - \theta_i)/\hbar}}{1 + \sum_{i=1, i \neq k}^{M+1} e^{-(\theta_k - \theta_i)/\hbar}}, \quad k = 1, \ldots, M + 1, \]
we obtain \( \lim_{\hbar \to 0} \phi = p_k \) in the \( \theta_k \)-region, away from coincidences of phases. At a visible coincidence \( \theta_k = \cdots = \theta_{k_n} \), which is generic in the sense that it is not a coincidence of more than \( m \) phases, we find \( \phi = \frac{1}{m} \sum_{i=1}^{m} p_k \). Furthermore,
\[ u = 2h^2 (\log \tau)_{xx} = 2h \phi_x = \frac{2}{\tau^2} \sum_{i=1}^{M+1} p_i^2 e^{\theta_i/\hbar} = \frac{2}{\tau^2} \left( \sum_{i=1}^{M+1} p_i e^{\theta_i/\hbar} \right)^2, \]
which implies \( \lim_{\hbar \to 0} u = \frac{1}{2} (p_k - p_i)^2 \) at a visible generic coincidence \( \theta_k = \theta_i \).

For an asymptotic soliton branch given by \( \theta_m = \theta_{m+1} \) for large negative values of \( y \), for \( \hbar = 1 \) the above formula implies \( u \sim \frac{1}{2} (p_m - p_n)^2 \) as \( y \to -\infty \), which thus coincides with the tropical value. A corresponding relation also holds for the remaining asymptotic soliton branch, given by \( \theta_1 = \theta_{M+1} \), as \( y \to +\infty \).

More generally, we find
\[ \lim_{\hbar \to 0} u = \frac{2}{m^2} \sum_{1 \leq i < j \leq m} (p_k - p_k)^2 \quad \text{at a visible generic coincidence} \ \theta_k = \cdots = \theta_{k_n}. \]

At a highest coincidence, i.e. \( \theta_1 = \cdots = \theta_{M+1} \), the tropical value is precisely the exact value (i.e. the corresponding value of \( u \) for \( \hbar = 1 \)). This is not so at a (generic) visible lower coincidence. But it is clear from the above formula for \( u \) that the corrections involve (only) exponentials of negative phase differences. Hence, the tropical values yield a perfect approximation unless those phase differences become extremely small (which means that we are close to a higher order coincidence).

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