Abundance of Ground States with Positive Parity

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We investigate analytically and numerically a random-matrix model for $m$ fermions occupying $\ell_1$ single-particle states with positive parity and $\ell_2$ single-particle states with negative parity and interacting through random two-body forces that conserve parity. The single-particle states are completely degenerate and carry no further quantum numbers. We compare spectra of many-body states with positive and with negative parity. We show that in the dilute limit defined by $m, \ell_{1,2} \to \infty, m/\ell_{1,2} \to 0$, ground states with positive and with negative parity occur with equal probability. Differences in the ground-state probabilities are, thus, a finite-size effect and are mainly due to different dimensions of the Hilbert spaces of either parity.

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I. MOTIVATION AND AIM

Johnson et al. \cite{1} observed that in the two-body random ensemble (TBRE) of the nuclear shell model, ground states with spin zero occur much more frequently than corresponds to their statistical weight. That observation caused considerable theoretical activity (see the reviews \cite{2} and \cite{3}). A similar preponderance for states with positive parity was found in Ref. \cite{4}. We wish to explore the reason for that preponderance. We focus attention on parity (rather than spin) because that quantum number is analytically more easily accessible. We use a model with spinless fermions that interact via random two-body forces. The degenerate single-particle states carry no orbital angular momentum quantum number but have either positive or negative parity. The model is a modified version of EGOE(2), the embedded two-body ensemble of Gaussian random matrices \cite{5}. We investigate the model by using both, an analytical approach and numerical simulations. The analytical approach evaluates traces of powers of the Hamiltonian up to very high order and uses results of Refs. \cite{6} and \cite{7} to estimate the position of the ground state. The numerical simulations involve diagonalization of matrices drawn at random from the ensemble and can be done only for Hamiltonian matrices of sufficiently small dimension whereas the analytical approach is suited also for large-dimensional matrices.

To motivate our focus on traces of the Hamiltonian, we recall in Section II how the ground-state energy was estimated in Refs. \cite{6} and \cite{7}. That method is used and compared with numerical simulations in Section V. Prior to that, we define our model in Section III. The first and second moments of the Hamiltonian are calculated for both parities in Section IV. After presenting our numerical results, we investigate our model in the limit of large matrix dimension $N$ in Section VI. We show that for $N \to \infty$, both the first and the second moments of the Hamiltonian have the same values for either parity. Combining that fact with the well-known result \cite{2} that the shape of the average spectrum is asymptotically ($N \to \infty$) Gaussian, we conclude that ground states of either parity are equally likely. In Section VII we show that the strong correlations found asymptotically for the first and second moments extend to higher (but not to all) moments. We discuss the implications of that result for correlations between the spectral fluctuation properties of positive- and negative-parity states and show that the result reinforces our conclusions. We conclude with a summary and discussion.

II. SIMPLE ESTIMATE FOR THE GROUND-STATE ENERGY

To estimate the ground-state energy, we use with proper modifications the method introduced for states with spin in Ref. \cite{6} and improved in Ref. \cite{7}. Let $H$ denote the Hamiltonian of the system, $P_{\pm}$ the projectors onto states with positive and negative parity, respectively, and $E_{\text{ground}}(\pm)$ the energies of the lowest state with positive or negative parity, respectively. We estimate $E_{\text{ground}}(\pm)$ by writing

$$E_{\text{ground}}(\pm) = n\text{Trace}(HP_{\pm}) - r_{\pm}\sigma_{\pm}. \tag{1}$$

The symbol $n\text{Trace}$ stands for the normalized trace (the actual trace divided by the dimension $N_{\pm}$ of Hilbert space), and the width $\sigma$ is defined as

$$\sigma_{\pm}^2 = n\text{Trace}(H^2P_{\pm}). \tag{2}$$
In Ref. [1] the analogue of Eq. (1) was used without the first term on the right-hand side. That term was added in Ref. [7]. It represents the fluctuations of the centroid of the spectrum. Inclusion of that term improves the agreement with numerical simulations: the fluctuations of the parameter \( r \) are reduced. Equation (1) has a simple interpretation: shell-model spectra have nearly Gaussian shape \([5]\) and are, thus, essentially characterized by the centroid and the width. In the case of spin, the stochastic fluctuations of \( r \) were found to be small, so that \( r \) can be considered a constant. In Ref. [7], an explicit expression for \( r \) was obtained by fitting the results of numerical calculations. It reads

\[
r = \sqrt{0.99 \ln N + 0.36}.
\]

We actually prefer to determine \( r_{\pm} \) by a fit to numerical data. In Section IV we compare the result with Eq. (3). We also compare the numerically determined probability of finding a ground state of given parity with predictions derived from Eqs. (1) and (2).

### III. MODEL

We consider a system of \( m \) spinless fermions distributed over a set of degenerate single-particle states. There are \( \ell_1 \) states of positive parity and \( \ell_2 \) states of negative parity, with associated creation and destruction operators \( a_{1\mu}^\dagger, a_{1\mu} (\mu = 1, 2, \ldots, \ell_1) \) and \( a_{2\rho}^\dagger, a_{2\rho} (\rho = 1, 2, \ldots, \ell_2) \), respectively. The single-particle states carry no further quantum numbers. The many-body states of the system have positive (negative) parity if the number \( m_1 \) of fermions in negative-parity states is even (odd, respectively). The total numbers \( N_+ \) and \( N_- \) of positive- and negative-parity states are

\[
N_+ = \sum_{m_1, m_2} \delta_{m_1 + m_2, m} \delta_{m_2, \text{even}} \binom{\ell_1}{m_1} \binom{\ell_2}{m_2},
\]

\[
N_- = \sum_{m_1, m_2} \delta_{m_1 + m_2, m} \delta_{m_2, \text{odd}} \binom{\ell_1}{m_1} \binom{\ell_2}{m_2}.
\]

The Hamiltonian \( H \) is a sum of two-body interactions that conserve parity,

\[
H = \frac{1}{4} \sum_{\mu\nu\rho\sigma} V_{\mu\nu;\rho\sigma}^{(1)} a_{1\mu}^\dagger a_{1\nu}^\dagger a_{1\sigma} a_{1\rho} + \frac{1}{4} \sum_{\mu\nu\rho\sigma} V_{\mu\nu;\rho\sigma}^{(2)} a_{2\mu}^\dagger a_{2\nu}^\dagger a_{2\sigma} a_{2\rho}
\]

\[
+ \frac{1}{4} \sum_{\mu\nu\rho\sigma} X_{\mu\nu;\rho\sigma}^{(1)} (a_{1\mu}^\dagger a_{1\nu}^\dagger a_{2\sigma} a_{2\rho} + a_{2\rho}^\dagger a_{2\sigma} a_{1\nu}^\dagger a_{1\mu})
\]

\[
+ \sum_{\mu\nu\rho\sigma} X_{\mu\nu;\rho\sigma}^{(2)} a_{1\mu}^\dagger a_{2\rho}^\dagger a_{2\sigma} a_{1\nu}.
\]

The ranges of the summation indices depend in an obvious way on the creation operators and matrix elements on which they appear. The two-body matrix elements obey the symmetry relations

\[
V_{\mu\nu;\rho\sigma}^{(1)} = V_{\rho\sigma;\mu\nu}^{(1)} = -V_{\nu\mu;\rho\sigma}^{(1)} = (V_{\mu\nu;\rho\sigma}^{(1)})^*,
\]

\[
V_{\mu\nu;\rho\sigma}^{(2)} = V_{\rho\sigma;\mu\nu}^{(2)} = -V_{\nu\mu;\rho\sigma}^{(2)} = (V_{\mu\nu;\rho\sigma}^{(2)})^*,
\]

\[
X_{\mu\nu;\rho\sigma}^{(1)} = -X_{\nu\mu;\rho\sigma}^{(1)} = -X_{\mu\nu;\rho\sigma}^{(1)} = (X_{\mu\nu;\rho\sigma}^{(1)})^*,
\]

\[
X_{\mu\nu;\rho\sigma}^{(2)} = (X_{\mu\nu;\rho\sigma}^{(2)})^*.
\]

An ensemble of Hamiltonians is obtained when we consider the matrix elements in Eq. (5) as Gaussian-distributed random variables. We assume that the \( V_{\mu\nu;\rho\sigma}^{(1)} \) are not correlated with the \( V_{\mu\nu';\rho\sigma'}^{(2)} \) and likewise for the pairs \( V_{\mu\nu;\rho\sigma}^{(1)}, X_{\mu\nu;\rho\sigma}^{(k)} \) for \( i, k = 1, 2 \) and \( \mu, \nu, \rho, \sigma \). All matrix elements have zero mean values. For the variances, we define pairs of indices \( \alpha, \beta \) by writing \( \alpha = \{\mu\nu\} \) and likewise for \( \beta \), and have for \( i = 1, 2 \)

\[
V_{\alpha;\beta}^{(1)} V_{\alpha';\beta'}^{(1)} = \delta_{\alpha\alpha'} \delta_{\beta\beta'} + \delta_{\alpha\beta'} \delta_{\beta\alpha'},
\]

\[
X_{\alpha;\beta}^{(1)} X_{\alpha';\beta'}^{(1)} = \delta_{\alpha\alpha'} \delta_{\beta\beta'}.
\]
The bar denotes the average over the ensemble, and $\delta_{\alpha\beta}$ stands for $(\delta_{\mu\nu}/\delta_{\nu\nu'} - \delta_{\mu\nu'}/\delta_{\nu\nu})$, etc. The matrix elements $X^{(2)}$ do not possess any symmetry properties and obey

$$X^{(2)}_{\mu\nu;\rho\sigma} X^{(2)}_{\mu'\nu';\rho'\sigma'} = v^2 \delta_{\mu\mu'} \delta_{\nu\nu'} \delta_{\rho\rho'} \delta_{\sigma\sigma'} .$$

Without loss of generality, we put $v^2 = 1$ in the sequel.

IV. CALCULATION OF nTrace($H$) AND OF nTrace($H^2$).

These two traces are needed for the evaluation of Eqs. (1) and (2). The only non-vanishing contributions to the two traces arise from terms in $H$ and in $H^2$ which leave the number of fermions in every single-particle state unchanged. These terms are found by using Wick contractions of the creation and annihilation operators in the expressions for $H$ and $H^2$. We indicate the omission of all other terms by an arrow. For $H$ we obtain

$$H \to \frac{1}{2} \sum_{\mu\nu} V^{(1)}_{\mu\nu;\mu\nu} n_{1\mu} n_{1\nu} + \frac{1}{2} \sum_{\mu\nu} V^{(2)}_{\mu\nu;\mu\nu} n_{2\mu} n_{2\nu} + \sum_{\mu\nu} X^{(2)}_{\mu\nu;\mu\nu} n_{1\mu} n_{2\nu} .$$

Here $n_{i\mu}$ is the number operator for state $(i\mu)$ with $i = 1, 2$.

The diagonal element of $n_{1\mu} n_{1\nu}$ taken between one of the states with $m_1$ fermions in positive-parity single-particle states and $m_2$ fermions in negative-parity single-particle states vanishes unless both states $(1\mu)$ and $(1\nu)$ are occupied, in which case the matrix element equals unity. There are altogether $(\ell_1-2)(\ell_2-2)$ such states. We consider separately the normalized traces over the positive-parity and the negative-parity many-body states. We recall that $P_\pm$ are the projection operators onto the many-body states with positive and negative parity, respectively. We obtain

$$\text{nTrace}(H P_+ ) = \frac{1}{2N_+} \sum_{\mu\nu} V^{(1)}_{\mu\nu;\mu\nu} \sum_{m_1 m_2} \delta_{m_1+m_2, m} \delta_{m_2, \text{even}} \left( \ell_1 - 2 \right) \left( \ell_2 \right) \left( m_1 - 2 \right) \left( m_2 \right)$$

$$+ \frac{1}{2N_+} \sum_{\mu\nu} V^{(2)}_{\mu\nu;\mu\nu} \sum_{m_1 m_2} \delta_{m_1+m_2, m} \delta_{m_2, \text{even}} \left( \ell_1 \right) \left( m_1 - 2 \right) \left( m_2 - 2 \right)$$

$$+ \frac{1}{N_+} \sum_{\mu\nu} X^{(2)}_{\mu\nu;\mu\nu} \sum_{m_1 m_2} \delta_{m_1+m_2, m} \delta_{m_2, \text{even}} \left( \ell_1 - 1 \right) \left( m_1 - 1 \right) \left( m_2 - 1 \right) ,$$

$$\text{nTrace}(H P_- ) = \frac{1}{2N_-} \sum_{\mu\nu} V^{(1)}_{\mu\nu;\mu\nu} \sum_{m_1 m_2} \delta_{m_1+m_2, m} \delta_{m_2, \text{odd}} \left( \ell_1 - 2 \right) \left( \ell_2 \right) \left( m_1 - 2 \right) \left( m_2 \right)$$

$$+ \frac{1}{2N_-} \sum_{\mu\nu} V^{(2)}_{\mu\nu;\mu\nu} \sum_{m_1 m_2} \delta_{m_1+m_2, m} \delta_{m_2, \text{odd}} \left( \ell_1 \right) \left( m_1 - 2 \right) \left( m_2 - 2 \right)$$

$$+ \frac{1}{N_-} \sum_{\mu\nu} X^{(2)}_{\mu\nu;\mu\nu} \sum_{m_1 m_2} \delta_{m_1+m_2, m} \delta_{m_2, \text{odd}} \left( \ell_1 - 1 \right) \left( m_1 - 1 \right) \left( m_2 - 1 \right) .$$

Both traces are seen to depend on the same three uncorrelated random variables,

$$z_1 = \sum_{\mu\nu} V^{(1)}_{\mu\nu;\mu\nu} ; \quad z_2 = \sum_{\mu\nu} V^{(2)}_{\mu\nu;\mu\nu} ; \quad z_3 = \sum_{\mu\nu} X^{(2)}_{\mu\nu;\mu\nu} .$$

As sums of uncorrelated random variables with equal Gaussian distributions, $z_1$, $z_2$, and $z_3$ have Gaussian distributions with mean values zero and second moments $[\ell_1(\ell_1-1)/4]$, $[\ell_2(\ell_2-1)/4]$, and $\ell_1\ell_2$, respectively. Thus, the distribution of the traces in Eq. (10) is completely known.

The pattern that emerges in Eq. (10) will be seen to apply quite generally to traces of arbitrary powers of $H$: the traces are sums of products. The first factor in each product depends only on the random variables and is the same for both parities. The second factor differs for states of positive and states of negative parity but is independent of the random variables. That general pattern will be decisive for our understanding of the preponderance of ground states with positive parity.

We turn to Trace $(H^2)$. The following terms yield non-zero contributions: the square of the first term on the right-hand side of Eq. (3), the square of the second term, the product of the first and the second term, the square of the third term, and the square of the fourth term. We consider these terms in turn.
In the square of the first term, there appear the two matrix elements $V^{(1)}$ with their associated creation and annihilation operators. Wick contraction is possible in three different ways: (i) We contract the two creation and the two annihilation operators associated with the same matrix element. That is the same procedure as used in formula (9) and yields a total of 4 contraction patterns. (ii) We contract one of the two creation operators associated with the first matrix element with an annihilation operator associated with the same matrix element, and the other with an annihilation operator associated with the second matrix element. That yields a total of 16 contraction patterns. (iii) We contract the two creation operators associated with the first matrix element with the two annihilation operators associated with the second matrix element. That yields a total of 4 contraction patterns. It is straightforward to check that because of the fermionic anticommutation rules and the symmetry properties of the different contraction patterns in each of the three groups yield identical results. For the square of the second term on the right-hand side of Eq. (5), these considerations apply likewise. For the product of the first and second term, only the contraction patterns used in formula (9) are possible. In the square of the third term on the right-hand side of Eq. (5), only the product of the two terms in round brackets gives a non-vanishing contribution, with obvious contraction patterns. In the square of the fourth term, there are the same three possibilities as in the square of the first term. Altogether this yields

$$H^2 \rightarrow \left( \sum_{i=1}^{2} \frac{1}{2} V^{(i)}_{\alpha \beta; \alpha \beta} n_{i\alpha} n_{i\beta} \right)^2 + \sum_{i=1}^{2} \sum_{\alpha \beta} V^{(i)}_{\alpha \beta; \alpha \beta} V^{(i)}_{\alpha \beta^*; \alpha \beta^*} n_{i\alpha} n_{i\beta} (1 - n_{i\alpha'} n_{i\beta'})$$

$$+ \sum_{i=1}^{2} \left( \frac{1}{2} \sum_{\alpha \beta} (V^{(i)}_{\alpha \beta; \alpha \beta})^2 n_{i\alpha} n_{i\beta} + \sum_{\alpha \beta} (V^{(i)}_{\alpha \beta; \alpha \beta})^2 n_{i\alpha} n_{i\beta} (1 - n_{i\alpha'}) \right)$$

$$+ \sum_{\alpha \beta} \left( X^{(1)}_{\alpha \beta; \alpha \beta'}^2 \left( n_{i\alpha} n_{1\beta} (1 - n_{2\alpha'}) (1 - n_{2\beta'}) \right) \right)$$

$$+ \left( \sum_{\mu \rho} X^{(2)}_{\mu \rho; \mu \rho} n_{1\mu} n_{2\rho} \right)^2$$

$$+ \sum_{\mu \rho} X^{(2)}_{\mu \rho; \mu \rho} X^{(2)}_{\mu \rho; \mu \rho} n_{1\mu} (1 - n_{1\nu}) n_{2\rho} n_{2\sigma}$$

$$+ \sum_{\mu \rho} X^{(2)}_{\mu \rho; \mu \rho} X^{(2)}_{\mu \rho; \mu \rho} n_{1\mu} n_{1\nu} n_{2\rho} (1 - n_{2\sigma})$$

$$+ \sum_{\mu \rho} X^{(2)}_{\mu \rho; \mu \rho} X^{(2)}_{\mu \rho; \mu \rho} n_{1\mu} (1 - n_{1\nu}) n_{2\rho} (1 - n_{2\sigma}) .$$

Before working out the trace of this expression, it is useful to rearrange it in such a way that in all summations no two summation indices take the same values. This yields

$$H^2 \rightarrow \sum_{i=1}^{2} \left( 2 \sum_{\alpha \beta} (V^{(i)}_{\alpha \beta; \alpha \beta})^2 n_{i\alpha} n_{i\beta} \right.$$}

$$+ \sum_{\alpha \beta} (V^{(i)}_{\alpha \beta; \alpha \beta})^2 n_{i\alpha} n_{i\beta} (1 - n_{i\alpha'} n_{i\beta'})$$

$$+ \sum_{\alpha \beta} (1 - \delta_{\beta \beta'}) V^{(i)}_{\alpha \beta; \alpha \beta} V^{(i)}_{\alpha \beta^*; \alpha \beta^*} n_{i\alpha} n_{i\beta} n_{i\beta'}$$

$$+ \frac{1}{4} \sum_{\alpha \beta} (V^{(i)}_{\alpha \beta; \alpha \beta})^2 n_{i\alpha} n_{i\beta} (1 - n_{i\alpha'}) (1 - n_{i\beta'}) \right).$$
\[
\text{nTrace}(H^2 P_+) = \sum_{m_1 m_2} \delta_{m_1+m_2, m} \delta_{m, \text{even}} \\
\times \left\{ \frac{2}{N_+} \sum_{a} (V_{a}^{(1)})^2 \left( \frac{\ell_1 - 2}{m_1 - 2} \right) \left( \frac{\ell_2}{m_2} \right) \right. \\
+ \frac{2}{N_+} \sum_{a} (V_{a}^{(2)})^2 \left( \frac{\ell_1}{m_1} \right) \left( \frac{\ell_2 - 2}{m_2 - 2} \right) \\
+ \frac{2}{N_+} \sum_{a} (1 - \delta_{a \beta'})(V_{a}^{(1)})^2 \left( \frac{\ell_1 - 3}{m_1 - 2} \right) \left( \frac{\ell_2}{m_2} \right) \\
+ \frac{2}{N_+} \sum_{a} (1 - \delta_{a \beta'})(V_{a}^{(2)})^2 \left( \frac{\ell_1}{m_1} \right) \left( \frac{\ell_2 - 3}{m_2 - 2} \right) \\
+ \frac{2}{N_+} \sum_{a} (1 - \delta_{a \beta'})(V_{a}^{(1)}) (V_{a}^{(2)}) \right.
\]

In calculating the trace, we observe that the number of nonequal summation indices in the terms on the right-hand side of Eq. (13) determines the weight factors. The result is
\[
\times \left( \frac{\ell_1 - 3}{m_1 - 3} \right) \left( \frac{\ell_2}{m_2} \right) + \frac{2}{N_4} \sum_{\alpha \beta \gamma} (1 - \delta_{\beta \gamma}) V_{\alpha \beta; \alpha \gamma}^{(2)} V_{\alpha \beta; \alpha \gamma}^{(2)} \times \left( \frac{\ell_1}{m_1} \right) \left( \frac{\ell_2 - 3}{m_2 - 3} \right) + \frac{1}{4N_4} \sum_{\alpha \beta \alpha' \beta'} (1 - \delta_{\beta \beta'}) (1 - \delta_{\beta \beta'}) (1 - \delta_{\beta \beta'}) (1 - \delta_{\beta \beta'}) \times (V_{\alpha \beta; \alpha \gamma}^{(1)})^2 \left( \frac{\ell_1 - 4}{m_1 - 2} \right) \left( \frac{\ell_2}{m_2} \right) + \frac{1}{4N_4} \sum_{\alpha \beta \alpha' \beta'} (1 - \delta_{\beta \beta'}) (1 - \delta_{\beta \beta'}) (1 - \delta_{\beta \beta'}) (1 - \delta_{\beta \beta'}) \times (V_{\alpha \beta; \alpha \gamma}^{(2)})^2 \left( \frac{\ell_1}{m_1} \right) \left( \frac{\ell_2 - 4}{m_2 - 2} \right) + \frac{1}{4N_4} \sum_{\alpha \beta \alpha' \beta'} (1 - \delta_{\beta \beta'}) (1 - \delta_{\beta \beta'}) (1 - \delta_{\beta \beta'}) (1 - \delta_{\beta \beta'}) \times V_{\alpha \beta; \alpha \gamma}^{(1)} V_{\alpha \beta; \alpha \gamma}^{(1)} \left( \frac{\ell_1 - 4}{m_1 - 4} \right) \left( \frac{\ell_2}{m_2} \right) + \frac{1}{4N_4} \sum_{\alpha \beta \alpha' \beta'} (1 - \delta_{\beta \beta'}) (1 - \delta_{\beta \beta'}) (1 - \delta_{\beta \beta'}) (1 - \delta_{\beta \beta'}) \times V_{\alpha \beta; \alpha \gamma}^{(2)} V_{\alpha \beta; \alpha \gamma}^{(2)} \left( \frac{\ell_1}{m_1} \right) \left( \frac{\ell_2 - 4}{m_2 - 4} \right) + \frac{1}{N_4} \sum_{\alpha \beta \alpha' \beta'} (1 - \delta_{\beta \beta'}) (1 - \delta_{\beta \beta'}) \times V_{\alpha \beta; \alpha \gamma}^{(1)} V_{\alpha \beta; \alpha \gamma}^{(1)} \left( \frac{\ell_1 - 4}{m_1 - 4} \right) \left( \frac{\ell_2}{m_2} \right) + \frac{1}{N_4} \sum_{\alpha \beta \alpha' \beta'} (1 - \delta_{\beta \beta'}) (1 - \delta_{\beta \beta'}) \times (V_{\alpha \beta; \alpha \gamma}^{(2)})^2 \left( \frac{\ell_1}{m_1} \right) \left( \frac{\ell_2 - 4}{m_2 - 4} \right) + \frac{1}{2N_4} \sum_{\alpha \beta \alpha' \beta'} V_{\alpha \beta; \alpha \gamma}^{(1)} V_{\alpha \beta; \alpha \gamma}^{(1)} \left( \frac{\ell_1 - 2}{m_1 - 2} \right) \left( \frac{\ell_2 - 2}{m_2 - 2} \right) + \frac{1}{4N_4} \sum_{\alpha \beta \alpha' \beta'} (X_{\alpha \beta; \alpha \gamma}^{(1)})^2 \left( \frac{\ell_1 - 2}{m_1 - 2} \right) \left( \frac{\ell_2 - 2}{m_2 - 2} \right) + \frac{1}{N_4} \sum_{\mu \rho} (X_{\mu \rho; \rho}^{(2)})^2 \left( \frac{\ell_1 - 1}{m_1 - 1} \right) \left( \frac{\ell_2 - 1}{m_2 - 1} \right) + \frac{1}{N_4} \sum_{\mu \rho' \mu} (1 - \delta_{\mu \rho'}) X_{\mu \rho; \rho}^{(2)} X_{\mu \rho; \rho}^{(2)} \left( \frac{\ell_1 - 2}{m_1 - 2} \right) \left( \frac{\ell_2 - 1}{m_2 - 1} \right) + \frac{1}{N_4} \sum_{\mu \rho' \rho} (1 - \delta_{\rho' \rho}) X_{\mu \rho; \rho}^{(2)} X_{\mu \rho; \rho}^{(2)} \left( \frac{\ell_1 - 1}{m_1 - 1} \right) \left( \frac{\ell_2 - 2}{m_2 - 2} \right) + \frac{1}{N_4} \sum_{\mu \rho' \rho} (1 - \delta_{\mu \rho'}) (1 - \delta_{\rho' \rho}) X_{\mu \rho; \rho}^{(2)} X_{\mu \rho; \rho}^{(2)}
\begin{align}
&\times \binom{\ell_1 - 2}{m_1 - 2} \binom{\ell_2 - 2}{m_2 - 2} \\
&+ \frac{1}{N_+} \sum_{\mu\nu\rho\sigma} (1 - \delta_{\mu\nu}) X_{\mu\nu;\rho\sigma}^{(2)} X_{\mu\nu;\rho\sigma}^{(2)} \binom{\ell_1 - 2}{m_1 - 1} \binom{\ell_2 - 2}{m_2 - 1} \\
&+ \frac{1}{N_+} \sum_{\mu\nu\rho\sigma} (1 - \delta_{\mu\nu})(1 - \delta_{\rho\sigma}) X_{\mu\nu;\rho\sigma}^{(2)} X_{\mu\nu;\rho\sigma}^{(2)} \binom{\ell_1 - 1}{m_1 - 1} \binom{\ell_2 - 2}{m_2 - 1} \\
&+ \frac{1}{N_+} \sum_{\mu\nu\rho\sigma} (1 - \delta_{\mu\nu})(1 - \delta_{\rho\sigma}) X_{\mu\nu;\rho\sigma}^{(2)} X_{\mu\nu;\rho\sigma}^{(2)} \binom{\ell_1 - 2}{m_1 - 1} \binom{\ell_2 - 2}{m_2 - 1} \\
&\times \frac{1}{N_+} \sum_{\mu\nu\rho\sigma} (1 - \delta_{\mu\nu})(1 - \delta_{\rho\sigma}) X_{\mu\nu;\rho\sigma}^{(2)} X_{\mu\nu;\rho\sigma}^{(2)} \binom{\ell_1 - 1}{m_1 - 1} \binom{\ell_2 - 2}{m_2 - 1} \left\{ \right. \\
\end{align}

For $n \text{Trace}(H^2 P_{\pm})$ we find exactly the same expression except that the second Kronecker delta in the first line on the right-hand side of Eq. (14) is replaced by $\delta_{m_2, \text{odd}}$, and that $N_+$ is replaced everywhere by $N_-$. As in the case of $\text{Trace}H$, the trace of $H^2$ is a sum of terms each of which is the product of two factors. One factor depends only on the random variables and is the same for both parities. The distribution of these factors can be worked out. That is not done here. Some of the factors are correlated with each other. The other factor is a weight factor that is a sum over products of binomial factors. It does not depend on the random variables and is not obviously the same for the two parities. Our result would not apply in the case of states with spin where the linear or bilinear forms containing the random variables will depend on the total spin. Assuming that Eqs. (1) to (3) hold, we conclude that a preponderance of ground states with even parity - if it exists - can have only one of two causes: it may be due to differences between the non-statistical weight factors, or to differences in the scale factors $r_+$ and $r_-$. (We recall that according to Eq. (3), the latter depend on the matrix dimensions $N_\pm$.)

V. NUMERICAL RESULTS

For a test of Eq. (1), we perform numerical simulations. To this purpose, we consider several systems that differ in the parameters $\ell_1$, $\ell_2$, and $m$. For each set of parameters, we set up the matrix corresponding to the Hamiltonian (5) in a space of Slater determinants. The Gaussian-distributed two-body matrix elements are computed by a pseudo-random number generator, and the ground-state energies $E_{\text{ground}}(\pm)$ are obtained from a numerical diagonalization of the Hamiltonian matrix. For the largest-dimensional matrices, we employ the ARPACK package [9] in the diagonalization. In addition to the ground-state energy we also compute the normalized traces $n \text{Trace}(H^k P_{\pm})$ for $k = 1, 2$. Our ensemble consists of 100 random Hamiltonians for each set of parameters $\ell_1$, $\ell_2$, and $m$, and we record the ground-state energies $E_{\text{ground}}(\pm)$ and the first two moments $n \text{Trace}(H^k P_{\pm})$ (with $k = 1, 2$) of the parity-projected Hamiltonian for each member of the ensemble. We employ Eq. (1) and determine the scale factors $r_{\pm}$ that relate the ground-state energy to the first and second moment of the Hamiltonian by fit. An example is shown for the set of parameters $m = \ell_1 = \ell_2 = 9$ in Fig. 1. The results obtained for the scale factors (with their rms variances) are shown in Table I. The table also shows the probability $p_\pm$ that the ground state has positive parity. Inspection of Table I shows that the parity of the ground state is very sensitive to $r_{\pm}$. A small difference in the scale factors $r_{\pm}$ is more strongly correlated with the parity of the ground state than a small difference in the numbers $N_\pm$ of many-body basis states.

Once the scale factors are determined, we can test how well the right-hand side of Eq. (1) can be used to determine the parity of the ground state. Our results show that the application of Eq. (1) with an energy-independent scale factor does not yield reliable predictions. Indeed, Fig. 1 suggests that a linear relation $r_{\pm} (E_{\text{ground}}(\pm)) = a_{\pm} + b_{\pm} E_{\text{ground}}(\pm)$ should describe the data more accurately. Again, we determine the coefficients $a_{\pm}$ and $b_{\pm}$ by fit, and then employ
FIG. 1: (Color online) Scale factors $r_{\pm}$ of Eq. (1) for a system of $m = 9$ fermions on $\ell_1 = 9$ single-particle orbitals with positive parity and $\ell_2 = 9$ orbitals with negative parity as a function of the ground-state energies $E_{\text{ground}}(\pm)$.

The right-hand side of Eq. (1) with the energy-dependent scale factor to determine the parity of the ground state as

$$\text{sign} \left( \frac{n \text{Tr} HP_- - a_- \sigma_-}{1 + b_- \sigma_-} - \frac{n \text{Tr} HP_+ - a_+ \sigma_+}{1 + b_+ \sigma_+} \right). \quad (15)$$

Though this estimate is not correct for each individual member of the ensemble, it yields reasonably reliable predictions for the estimated probability $p_+(\text{est})$ of finding a ground-state with positive parity. Our results for this probability are shown in the last column of Table I.

TABLE I: Results of numerical simulations. Here, $m$, $\ell_1$, and $\ell_2$ denote the number of fermions and the number of single-particle levels with positive and negative parity, respectively. $N_\pm$ is the number of many-body states with the indicated parity, and $r_{\pm}$ denote the scale factors. $p_+$ denotes the probability that the ground state has positive parity, while $p_+(\text{est})$ is the probability that the estimated ground state has positive parity (based on Eq. (1) with a scale factor that is a polynomial of degree one in the energy).

| $\ell_1$ | $\ell_2$ | $m$ | $N_+$ | $N_-$ | $r_+$ | $r_-$ | $p_+$ | $p_+(\text{est})$ |
|--------|--------|-----|------|------|------|------|------|---------------|
| 6      | 6      | 452 | 472  | 2.39 ± 0.12 | 2.43 ± 0.12 | 0.18 | 0.00 |
| 7      | 7      | 1001| 1001 | 2.42 ± 0.08 | 2.42 ± 0.08 | 0.47 | 0.49 |
| 7      | 6      | 1484| 1519 | 2.51 ± 0.08 | 2.55 ± 0.08 | 0.20 | 0.03 |
| 7      | 7      | 1716| 1716 | 2.61 ± 0.08 | 2.60 ± 0.08 | 0.48 | 0.56 |
| 9      | 9      | 4284| 4284 | 2.47 ± 0.06 | 2.47 ± 0.06 | 0.55 | 0.54 |
| 10     | 8      | 4312| 4256 | 2.48 ± 0.06 | 2.46 ± 0.05 | 0.84 | 1.00 |
| 9      | 7      | 6435| 6435 | 2.77 ± 0.07 | 2.76 ± 0.08 | 0.54 | 0.58 |
| 8      | 8      | 6470| 6400 | 2.78 ± 0.07 | 2.74 ± 0.06 | 0.83 | 1.00 |
| 10     | 6      | 6300| 6490 | 2.73 ± 0.08 | 2.77 ± 0.09 | 0.18 | 0.00 |
| 9      | 9      | 24310| 24310| 2.90 ± 0.08 | 2.90 ± 0.07 | 0.52 | 0.57 |
| 8      | 10     | 24240| 24380| 2.87 ± 0.07 | 2.91 ± 0.07 | 0.20 | 0.00 |
| 7      | 11     | 24310| 24310| 2.88 ± 0.07 | 2.89 ± 0.07 | 0.50 | 0.27 |

VI. DILUTE LIMIT

In canonical random-matrix theory, attention is usually focused on the limit of large matrix dimension. We follow suit by considering our model in the “dilute limit” defined by $\ell_{1,2} \rightarrow \infty$ and $m/\ell_{1,2} \rightarrow 0$. In practice, we compute the leading order of expressions of interest under the strong conditions $1 \ll m \ll \ell_{1,2}$. We show that the weight factors appearing in the traces of $H^k$ with $k = 1, 2$ for positive and for negative parity become asymptotically equal. That statement holds not only for $\ell_1 = \ell_2$ but also for $\ell_1 \neq \ell_2$. 
Equations (10) and (14) show that for the positive parity states, all weight factors have the general form

$$\sum_{m_1, m_2} \delta_{m_1 + m_2, m} \delta_{m_2, \text{even}} \frac{1}{N_+} \left( \frac{\ell_1 - \alpha_1}{m_1 - \beta_1} \right) \left( \frac{\ell_2 - \alpha_2}{m_2 - \beta_2} \right), \tag{16}$$

with $\alpha_1, \alpha_2, \beta_1, \beta_2$ small positive integers. We evaluate the sums in Eq. (10) and the corresponding sums defining $N_+$ with the help of Stirling’s formula, $n! \approx \exp(n \ln n - n)$. With $\mu$ integer, we write $m_2 = 2\mu, m_1 = m - 2\mu$ and have for the numerator of Eq. (16) (all terms except for $(N_+)^{-1}$)

$$\sum_{\mu} \exp \left\{ (\ell_1 - \alpha_1) \ln(\ell_1 - \alpha_1) - (m - 2\mu - \beta_1) \ln(m - 2\mu - \beta_1) \right. - (\ell_1 - \alpha_1 - m + 2\mu + \beta_1) \ln(\ell_1 - \alpha_1 - m + 2\mu + \beta_1) \\
+ (\ell_2 - \alpha_2) \ln(\ell_2 - \alpha_2) - (2\mu + \beta_2) \ln(2\mu + \beta_2) \\
- (\ell_2 - \alpha_2 - 2\mu + \beta_2) \ln(\ell_2 - \alpha_2 - 2\mu + \beta_2) \right\}. \tag{17}$$

We write the sum as an integral over $\mu$. The integrand assumes its maximum value at

$$\mu_+^{(0)} = \frac{1}{2} \frac{(m - \beta_1)(\ell_2 - \alpha_2) + \beta_2(\ell_1 - \alpha_1)}{\ell_1 - \alpha_1 + \ell_2 - \alpha_2}. \tag{18}$$

With $\delta \mu = \mu - \mu_0$, expansion around the maximum yields the negative-definite quadratic form

$$-\frac{2(\delta \mu)^2}{m - 2\mu_+^{(0)} - \beta_1} -\frac{2(\delta \mu)^2}{\ell_1 - \alpha_1 - m + 2\mu_+^{(0)} + \beta_1} - \frac{2(\delta \mu)^2}{2\mu_+^{(0)} - \beta_2} \tag{19}$$

$$-\ell_2 - \alpha_2 - 2\mu_+^{(0)} + \beta_2 = \frac{1}{2} (\delta \mu)^2 \tau^2.$$

Here the last equation defines the width $\tau$. For $\ell_1 \gg 1, \ell_2 \gg 1, m \gg 1$ we have $\mu_0 \gg 1$. For the dilute limit, we neglect terms of higher order, and the resulting integral is Gaussian. We extend the integration from $-\infty$ to $+\infty$. The numerator of expression (16) becomes

$$\sqrt{2\pi \tau} \exp \left\{ (\ell_1 - \alpha_1) \ln(\ell_1 - \alpha_1) + (\ell_2 - \alpha_2) \ln(\ell_2 - \alpha_2) \right\} \times \exp \left\{ - (m - 2\mu_+^{(0)} - \beta_1) \ln(m - 2\mu_+^{(0)} - \beta_1) \right\} \times \exp \left\{ - (2\mu_+^{(0)} - \beta_2) \ln(2\mu_+^{(0)} - \beta_2) \right\} \times \exp \left\{ - (\ell_1 - \alpha_1 - m + 2\mu_+^{(0)} + \beta_1) \ln(\ell_1 - \alpha_1 - m + 2\mu_+^{(0)} + \beta_1) \right\} \times \exp \left\{ - (\ell_2 - \alpha_2 - 2\mu_+^{(0)} + \beta_2) \ln(\ell_2 - \alpha_2 - 2\mu_+^{(0)} + \beta_2) \right\}. \tag{20}$$

Using the same approximations to calculate $N_+$, we obtain a result of the form (20) but with $\alpha_1, \alpha_2, \beta_1, \beta_2$ everywhere (including the definitions of $\tau$ and $\mu_+^{(0)}$) replaced by zero.

We turn to the negative-parity states. For these states, the word “even” in Eq. (10) is replaced by “odd” and $N_+$ by $N_-$. The calculation is completely analogous except for the replacements $\beta_1 \rightarrow \beta_1 + 1$ and $\beta_2 \rightarrow \beta_2 - 1$. For the maximum of the integrand, that implies $2\mu_+^{(0)} = 2\mu_+^{(0)} - 1$. As a consequence, the terms $2\mu_+^{(0)} + \beta_1$ and $2\mu_+^{(0)} - \beta_2$ have the same values for states with positive and with negative parity. This in turn implies that the widths $\tau$ and the terms in the exponential in expression (20) have the same values for states with positive and with negative parity. It follows that in our approximation every weight factor for states with positive parity has the same value as the corresponding weight factor for states with negative parity. This result is valid beyond the Gaussian approximation used in obtaining Eq. (20). Indeed, the fundamental form (17) depends on $\mu$ only through the invariant combinations $\mu + \beta_1$ and $\mu - \beta_2$. Modifications can arise only in cases where the limits of integration (which depend on $\alpha_1, \alpha_2, \beta_1,$ and $\beta_2$) play a role, i.e., for small values of $\ell_1, \ell_2,$ or $m$. 


We have shown that in the dilute limit and for every realization of our random-matrix model, both the first and the second moments of \( H \) coincide in leading order for states with positive and for states with negative parity. The same is true of the matrix dimensions \( N_+ \) and \( N_- \). Thus for every realization, our Eqs. (1) to (3) predict equal values for the ground-state energies for both parities. How reliable is that prediction? We recall that in the dilute limit, the average spectrum of the embedded random two-body ensemble (EGOE(2)) is Gaussian \([5]\). The proof given in Ref. \([5]\) applies likewise to our model. We expect, therefore, that in the dilute limit and to a very high degree of approximation the spectrum of the embedded random two-body ensemble (EGOE(2)) is Gaussian \([5]\). The proof given in Ref. \([5]\) applies to our model. We expect, therefore, that in the dilute limit and to a very high degree of approximation the spectrum of any given realization of the ensemble has also Gaussian shape. (For a single realization, the shape of the spectrum is defined by taking local averages over a number \( n \ll N_\pm \) of neighboring levels.) That expectation rests on the plausible assumption that our random-matrix model is ergodic, at least in the dilute limit, and implies that for every realization, our Eqs. (1) to (3) become even better approximations as the matrix dimension increases. We conclude that a general term in the Wick-contracted form of \( H^k \), characterized by the four integers \( k_1, k_2, k_3, k_4 \) as constrained above, has the form

\[
\sum_{\alpha_1, \alpha_2, \ldots, \alpha_{k_1}} \sum_{\beta_1, \beta_2, \ldots, \beta_{k_2}} \sum_{\gamma_1, \gamma_2, \ldots, \gamma_{k_3}} \sum_{\delta_1, \delta_2, \ldots, \delta_{k_4}} 
\left\{ \prod_{r=1}^{k_1} n_{1\alpha_r} \prod_{s=1}^{k_2} (1 - n_{1\beta_s}) \prod_{t=1}^{k_3} n_{2\gamma_t} \prod_{u=1}^{k_4} (1 - n_{2\delta_u}) \right\}.
\]

The sums in this expression are jointly constrained by the condition that no two summation indices are equal. The form of the function \( f \) depends upon the value of \( k \). \( f \) is a monomial of order \( k \) in the matrix elements \( V^1, V^2, X^1, X^2 \). These carry the summation indices. The Wick-contraction of \( H^k \) yields a sum of terms of the form \((21)\). For the calculation of \( \text{nTrace}[H^k \mathcal{P}_\pm] \), we observe that the expression

\[
\Pi_{\pm}(k_1, k_2, k_3, k_4) = \text{nTrace}\left\{ \prod_{r=1}^{k_1} n_{1\alpha_r} \prod_{s=1}^{k_2} (1 - n_{1\beta_s}) \prod_{t=1}^{k_3} n_{2\gamma_t} \prod_{u=1}^{k_4} (1 - n_{2\delta_u}) \mathcal{P}_\pm \right\}
\]

VII. SPECTRAL FLUCTUATIONS

Given the coincidence of both the first and second moments of \( H \) for states of either parity in the dilute limit, we ask: does that coincidence extend to all higher moments so that the local spectral fluctuation properties of both ensembles are completely locked? We approach the answer by studying higher moments of \( H \).

We consider \( \text{nTrace}(H^k \mathcal{P}_\pm) \) for \( k \) integer and \( k \geq 3 \). These traces are now shown to have the same structure as the first and second moments of \( H \): each trace is a sum of terms each of which is the product of a monomial (or polynomial) of order \( k \) in the two-body matrix elements (the same for the projectors \( \mathcal{P}_+ \) and \( \mathcal{P}_- \)) and a weight factor that does not depend on the random variables but may have a different value for positive and negative parity.

We proceed as in Section [V] but are interested only in the general form of the result. The operator \( H^k \) is a monomial of order \( k \) in the matrix elements \( V^{1,1}, V^{1,2}, X^{1,1}, X^{1,2} \). Each matrix element carries four indices. Thus, in \( H^k \) there occur \( 4k \) independent summations over single-particle level indices. Non-vanishing contributions to the trace of \( H^k \) arise only from Wick-contracted terms. Each pairwise Wick contraction of a creation and an annihilation operator in \( H^k \) produces a factor of the form \( n_{1\alpha}, (1 - n_{1\alpha}), n_{2\beta}, \) or \( (1 - n_{2\beta}) \), as the case may be. At the same time, two summation indices become equal. After all Wick contractions are done, \( H^k \) contains at most \( 2k \) independent summations over level indices. (That number may be smaller than \( 2k \) since two or more of the resulting factors \( n_{1\alpha}, (1 - n_{1\alpha}), n_{2\beta}, \) or \( (1 - n_{2\beta}) \) may carry the same index.) By using the identity \( n^2 = n \) for the number operator, the Wick-contracted \( H^k \) can be written in such a way that the summation indices on all such factors are different. For \( k = 2 \), that was done in Eq. (13). We consider a single term resulting from this procedure and denote by \( k_1, k_2, k_3, k_4 \) the powers of the four types of factors (in the same sequence as listed above) in that term. The maximum power with which all factors jointly can appear, is \( 2k \) so that \( k_1 + k_2 + k_3 + k_4 \leq 2k \). Clearly we must also have \( k_1 + k_2 \leq \ell_1 \) and \( k_3 + k_4 \leq \ell_2 \). We conclude that a general term in the Wick-contracted form of \( H^k \), characterized by the four integers \( k_1, k_2, k_3, k_4 \) as constrained above, has the form

\[
\sum_{\alpha_1, \alpha_2, \ldots, \alpha_{k_1}} \sum_{\beta_1, \beta_2, \ldots, \beta_{k_2}} \sum_{\gamma_1, \gamma_2, \ldots, \gamma_{k_3}} \sum_{\delta_1, \delta_2, \ldots, \delta_{k_4}} 
\left\{ \prod_{r=1}^{k_1} n_{1\alpha_r} \prod_{s=1}^{k_2} (1 - n_{1\beta_s}) \prod_{t=1}^{k_3} n_{2\gamma_t} \prod_{u=1}^{k_4} (1 - n_{2\delta_u}) \right\}.
\]

The sums in this expression are jointly constrained by the condition that no two summation indices are equal. The form of the function \( f \) depends upon the value of \( k \). \( f \) is a monomial of order \( k \) in the matrix elements \( V^i, V^j, X^i, X^j \). These carry the summation indices. The Wick-contraction of \( H^k \) yields a sum of terms of the form \((21)\). For the calculation of \( \text{nTrace}[H^k \mathcal{P}_\pm] \), we observe that the expression

\[
\Pi_{\pm}(k_1, k_2, k_3, k_4) = \text{nTrace}\left\{ \prod_{r=1}^{k_1} n_{1\alpha_r} \prod_{s=1}^{k_2} (1 - n_{1\beta_s}) \prod_{t=1}^{k_3} n_{2\gamma_t} \prod_{u=1}^{k_4} (1 - n_{2\delta_u}) \mathcal{P}_\pm \right\}
\]
the same values for states with positive and with negative parity. Here the asymptotic regime is characterized by the relations 

\[ \ell \approx m \]

accommodate the relation 

\[ \ell \not\approx m \]

not depend on assuming any symmetry such as parity. Indeed, it is known \[8\] that the moments of 

\[ H \]

the moments of 

\[ k \]

always be equal. Consider, for instance, the case where the ground states of either parity carry equal probabilities. That conclusion is relevant for the local spectral fluctuations of both ensembles. Indeed, it is known \[8\] that the moments of 

\[ k \]

properties of both ensembles are uncorrelated in the dilute limit, even though the moments of 

\[ H \]

the moments of 

\[ k \]

grow approximately like 

\[ \approx 1 \]

in the sense of Eq. (23), the two Hamiltonians remain totally correlated. This is a remarkable result in its own right. Indeed, with increasing values of 

\[ k \]

expression (21) are given by 

\[ \sum_{\alpha_1,..,\alpha_4} \sum_{\beta_1,..,\beta_4} \sum_{\gamma_1,..,\gamma_4} \sum_{\delta_1,..,\delta_4} f_{\alpha_1,..,\alpha_4;\beta_1,..,\beta_4;\gamma_1,..,\gamma_4;\delta_1,..,\delta_4} \times \Pi_\pm(k_1, k_2, k_3, k_4) . \]  

(23)

Expression (23) shows that the results derived in Section \[14\] for \( n \text{Trace}[H P_{\pm}] \) and for \( n \text{Trace}[H^2 P_{\pm}] \) hold for arbitrary powers \( k \) of \( H \): Each trace \( n \text{Trace}[H^k P_{\pm}] \) is a sum of terms; every term in the sum is the product of two factors. The first factor contains the random variables and is the same for the states with positive and with negative parity. The second factor, a weight factor, may depend on parity. We have, thus, shown that the Hamiltonians for states with positive and with negative parity are very highly correlated.

We turn to the weight factors appearing in Eq. (23) and show that these are also asymptotically equal. Our statement applies up to a maximum value of \( k \) which we determine approximately. The weight factors \( \Pi_\pm(k_1, k_2, k_3, k_4) \) are explicitly given by

\[ \Pi_+(k_1, k_2, k_3, k_4) = \frac{1}{N_+} \sum_{m_1, m_2} \delta_{m_1 + m_2, m} \delta_{m_2, \text{even}} \left( \frac{\ell_1 - k_2}{m_1 - k_1} \right) \left( \frac{\ell_2 - k_4}{m_2 - k_3} \right) \]  

(24)

and

\[ \Pi_-(k_1, k_2, k_3, k_4) = \frac{1}{N_-} \sum_{m_1, m_2} \delta_{m_1 + m_2, m} \delta_{m_2, \text{odd}} \left( \frac{\ell_1 - k_2}{m_1 - k_1} \right) \left( \frac{\ell_2 - k_4}{m_2 - k_3} \right) . \]  

(25)

In the summations over \( m_1, m_2 \), we obviously must have \( m_1 \geq k_1 \) and \( m_2 \geq k_3 \). Since \( m_1 \) and \( m_2 \) are both bounded by \( m \), that condition in fact limits \( k_1 \) and \( k_3 \). It is obvious that for large values of \( k \), the two weight factors cannot always be equal. Consider, for instance, the case \( k_1 = 0, k_3 = m \). Then we have \( m_1 = 1 \) and \( m_2 = m \). That implies \( \Pi_+(k_1, k_2, k_3, k_4) = 0 \), \( \Pi_-(k_1, k_2, k_3, k_4) \neq 0 \) if \( m \) is odd and \( \Pi_+(k_1, k_2, k_3, k_4) = 0 \), \( \Pi_-(k_1, k_2, k_3, k_4) \neq 0 \) if \( m \) is even. To avoid such cases, we must have \( k < m \). Even then \( \Pi_+ \) and \( \Pi_- \) may differ. This happens when the bounds on the summation indices in Eqs. (24) and (25) are relevant. We avoid these cases by choosing \( k < m \). We recall that the asymptotic regime is characterized by the relations \( \ell_1 < m < \ell_1, \ell_2 \). We thus require that \( m \) is sufficiently large to accommodate the relation \( k < m \) and yet allows \( k \) to assume values large compared to one. With these assumptions, the arguments used above for \( k = 1, 2 \) show that \( \Pi_+ = \Pi_- \).

We have shown that in the asymptotic regime and for all \( k \) with \( k \leq k_0 \), the moments \( \text{Trace}(H^k P_{\pm}) \) pairwise have the same values for states with positive and with negative parity. Here \( k_0 \) obeys \( 1 < k_0 < m \). That conclusion does not depend on assuming any symmetry such as parity. We have also shown that for \( k > k_0 \), the moments differ. As \( k \) increases, the bounds on the summations over products of binomial factors become ever more important. As a consequence, the differences between moments for states with positive and with negative parity increase with \( k \). That statement is relevant for the local spectral fluctuation properties of both ensembles. Indeed, it is known \[8\] that such fluctuation properties depend on the very highest moments of \( H \): In the limit of infinite matrix dimension, there exists a clear separation between the overall shape of the spectrum (defined by averaging over an energy interval large compared to the average level spacing \( d \)), and the local spectral fluctuations (defined on a scale of order \( d \)). Since the moments of \( H \) for states of positive and negative parity differ for \( k > k_0 \), we conclude that the local fluctuation properties of both ensembles are uncorrelated in the dilute limit, even though the moments of \( H \) for both parities coincide up to \( k \approx k_0 \). This excludes the possibility mentioned in Section \[14\] that the local spectral fluctuation properties of the two ensembles are locked in such a way that the positive-parity ground state fluctuates more often towards smaller energies than does its opposite number or vice versa and completes the proof that in the dilute limit, ground states of either parity carry equal probabilities.

**VIII. SUMMARY AND DISCUSSION**

We have shown that in the dilute limit, ground states of either parity carry equal probabilities. That conclusion is based on the following facts. (i) The spectra are asymptotically Gaussian, and Eqs. (1) to (3) become asymptotically
strictly valid. (ii) The first and second moments of $H$ and the dimensions of the Hamiltonian matrices become asymptotically equal for either parity so that Eqs. (1) to (3) predict equal probabilities for either parity. (iii) The local spectral fluctuation properties of the two spectra are asymptotically uncorrelated because very high moments have different values. That fact excludes a locking of these fluctuations.

Deviations from equal ground-state probabilities are, thus, finite-size effects. For values of the parameters $m, \ell_1, \ell_2$ that are sufficiently small for numerical simulations, we have indeed found such deviations. They occur whenever the dimensions $N_+$ and $N_-$ differ. Conversely, for $N_+ = N_-$ we have not found significant deviations from equal probabilities. The small fluctuations found for $r_\pm$ in the fits to the data show that Eqs. (1) and (2) are approximately valid: They do predict correctly which parity has the higher probability to furnish the ground state. The values of the predicted probabilities are semi-quantitatively correct.

Calculations using the two–body random ensemble (TBRE) reported in Ref. [10] displayed correlations between spectra carrying different quantum numbers. One may argue that these results contradict our present findings. This is not the case: Calculations using the TBRE are necessarily restricted to small matrix dimensions while our argument for independence of spectral fluctuation properties of states of positive and negative parity applies only in the dilute limit, i.e. for infinite matrix dimension.

Our results may have interesting implications for the statistical theory of nuclear reactions. There an open question is this: are $S$-matrix elements carrying different quantum numbers like total spin uncorrelated? That assumption is always used in the theory and is consistent with the observed symmetry of compound-nucleus cross sections about 90 degrees in the center-of-mass system. Still, the assumption is not obviously valid for a realistic random-matrix model of nuclear reactions. Normally the statistical theory of nuclear reactions uses the Gaussian orthogonal ensemble (GOE). It would be more realistic to use instead the TBRE. The TBRE differs from the GOE in that it employs a shell-model in which the two-body matrix elements are the random variables. (For a review of the TBRE, see Ref. [11].) But then it is the same set of random variables that govern scattering matrix elements carrying different quantum numbers; just as in the model considered above the same random two-body matrix elements govern the Hamiltonians for states of different parity. To approach the question, we observe that for orthogonally invariant ensembles, universality holds also for elements of the scattering matrix carrying identical quantum numbers [12]. That statement implies that correlations between such elements depend only on local spectral fluctuation properties. This conclusion is supported by the explicit calculation in Ref. [13] of the correlation function of a pair of $S$-matrix elements: Aside from the strength of the coupling to the open channels, the correlation depends solely on the value of the local mean level density. If we assume that these statements carry over to the TBRE, and if we further assume that in the TBRE just as in the model studied above, the local spectral fluctuation properties of spectra carrying different quantum numbers are uncorrelated in the limit of large matrix dimension, we are led to the conclusion that $S$-matrix elements carrying different quantum numbers are, likewise, uncorrelated. The limit of a large matrix dimension is appropriate because the resonances relevant in the statistical theory correspond to states above the ground state.

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