THE TRANSVERSE DENSITY BUNDLE AND MODULAR CLASSES OF LIE GROUPOIDS

MARIUS CRAINIC AND JOÃO NUNO MESTRE

Abstract. In this note we revisit the notions of transverse density bundle and of modular classes of Lie algebroids and Lie groupoids; in particular, we point out that one should use the transverse density bundle $D^A_{tr}$ instead of $Q_A$, which is the representation that is commonly used when talking about modular classes. One of the reasons for this is that, as we will see, $Q_A$ is not really an object associated to the stack presented by a Lie groupoid (in general, it is not a representation of the groupoid!).

1. Introduction

We revisit the transverse density bundle $D^A_{tr}$, a representation that is canonically associated to any Lie groupoid [2], and its relation to the notions of modular classes of Lie groupoids and Lie algebroids. These are characteristic classes of certain 1-dimensional representations [1, 3, 6], that we recall and discuss in this note.

The definition of the modular class of a Lie algebroid always comes with the slogan, inspired by various examples, that it is “the obstruction to the existence of a transverse measure”. Here we would like to point out that the transverse density bundle $D^A_{tr}$ and our discussions make this slogan precise. In particular, we point out that the canonical representation $Q_A$ [3] which is commonly used in the context of modular classes of Lie algebroids should actually be replaced by $D^A_{tr}$, especially when passing from Lie algebroids to Lie groupoids.

Notation and conventions: Throughout the paper $G$ denotes a Lie groupoid over $M$, with source and target maps denoted by $s$ and $t$, respectively. For background material on Lie groupoids, Lie algebroids, and their representations we refer to [8]. We only consider Lie groupoids having all $s$-fibers of the same dimension or, equivalently, whose algebroid $A$ has constant rank. Actually, all vector bundles in this paper are assumed to be of constant rank.

2. Transverse density bundles

2.1. Volume, orientation and density bundles. First of all, note that any group homomorphism $\delta : GL_r \to \mathbb{R}^*$ allows us to associate to any $r$-dimensional vector space a canonical 1-dimensional vector space a canonical 1-dimensional vector space $L_\delta(V) := \{\xi : \text{Fr}(V) \to \mathbb{R} : \xi(e \cdot A) = \delta(A)\xi(e) \text{ for all } e \in \text{Fr}(V), A \in GL_r\}$, where $\text{Fr}(V) = \text{Isom}(\mathbb{R}^r, V)$ is the space of frames on $V$, endowed with the standard (right) action of $GL_r$. The cases that are of interest for us are:

- $\delta = \det$, for which we obtain the top exterior power $\Lambda^{\text{top}} V^*$.
- $\delta = \text{sign} \circ \det$, for which we obtain the orientation space $o_V$ of $V$.
- $\delta = |\det|$ which defines the space $D_V$ of densities of $V$. More generally, for $l \in \mathbb{Z}$, $\delta = |\det|^l$ defines the space $D^l_V$ of $l$-densities of $V$.

It is clear that there is a canonical isomorphism:

$$D_V \otimes o_V \cong \Lambda^{\text{top}} V^*, \ldots$$
When $V = L$ is 1-dimensional, $\mathcal{D}_L$ is also denoted by $|L|$; so, in general,

$$\mathcal{D}_V = |\Lambda^{\text{top}}V^*|,$$

which fits well with the fact that for any $\omega \in \Lambda^{\text{top}}V^*$, $|\omega|$ makes sense as an element of $\mathcal{D}_V$. For 1-dimensional vector spaces $W_1$ and $W_2$, one has a canonical isomorphism $|W_1| \otimes |W_2| \cong |W_1 \otimes W_2|$ (in particular $|W^*| \cong |W|^{\ast}$). From the properties of $\Lambda^{\text{top}}V^*$ (or by similar arguments), one obtains canonical isomorphisms:

1. $\mathcal{D}_V^* \cong \mathcal{D}_{V^*}$ for any vector space $V$.
2. For any short exact sequence of vector spaces $0 \to V \to U \to W \to 0$ (e.g. for $U = V \oplus W$), one has an induced isomorphism $\mathcal{D}_U \cong \mathcal{D}_V \otimes \mathcal{D}_W$.

Since the previous discussion is canonical (free of choices), it can be applied (fiber-wise) to vector bundles over a manifold $M$ so that, for any such vector bundle $E$, one can talk about the associated line bundles over $M$

$$\mathcal{D}_E, \Lambda^{\text{top}}E^*, \sigma_E$$

and the previous isomorphisms continue to hold at this level. However, at this stage, only $\mathcal{D}_E$ is trivializable, and even that is in a non-canonical way.

**Definition 2.1.** A density on a manifold $M$ is any section of the density bundle $\mathcal{D}_{TM}$.

The main point about densities on manifolds is that they can be integrated in a canonical fashion, so that associated to any compactly supported positive density $\rho$ on $M$, one obtains a Radon measure $\mu_\rho$, defined by

$$\mu_\rho : C^\infty_c (M) \to \mathbb{R}, \quad \mu_\rho(f) = \int_M f \cdot \rho.$$

2.2. The transverse volume, orientation and density bundles.

**Definition 2.2.** For a Lie algebroid $A$ over $M$, the the transverse density bundle of $A$ is the vector bundle over $M$ defined by:

$$\mathcal{D}_A^r := \mathcal{D}_A \otimes \mathcal{D}_{TM}.$$

Similarly one can define the transverse volume and orientation bundles

$$\mathcal{V}_A^r := \mathcal{V}_A \otimes \mathcal{V}_{TM} = \Lambda^{\text{top}}A \otimes \Lambda^{\text{top}}T^*M, \quad \sigma_A^r := \sigma_A \otimes \sigma_{TM}$$

and the usual relations between these bundles continue to hold in this setting; e.g.:

$$\mathcal{D}_A^r = |\Lambda^{\text{top}}A \otimes \Lambda^{\text{top}}T^*M| = |\mathcal{V}_A^r|.$$

One of the main properties of these bundles is that they are representations of $A$ and, even better, of $\mathcal{G}$, whenever $\mathcal{G}$ is a Lie groupoid with algebroid $A$; hence they do deserve the name of “transverse” vector bundles. We describe the canonical action of $\mathcal{G}$ on the transverse density bundle $\mathcal{D}_A^r$; for the other two the description is identical. We have to associate to any arrow $g : x \to y$ of $\mathcal{G}$ a linear transformation $g_* : \mathcal{D}_{A,x}^r \to \mathcal{D}_{A,y}^r$.

The differential of $s$ and the right translations induce a short exact sequence

$$0 \to A_y \to T_y \mathcal{G} \xrightarrow{d_s} T_x M \to 0,$$

which, in turn (cf. item 2 in Subsection 2.1), induces an isomorphism:

$$\mathcal{D}(T_y \mathcal{G}) \cong \mathcal{D}(A_y) \otimes \mathcal{D}(T_x M).$$
Using the similar isomorphism at $g^{-1}$ and the fact that the differential of the inversion map gives an isomorphism $T_g \mathcal{G} \cong T_{g^{-1}} \mathcal{G}$, we find an isomorphism
\[ \mathcal{D}(A_y) \otimes \mathcal{D}(T_x M) \cong \mathcal{D}(A_x) \otimes \mathcal{D}(T_y M). \]
and therefore an isomorphism:
\[ \mathcal{D}(A_x^*) \otimes \mathcal{D}(T_x M) \cong \mathcal{D}(A_y^*) \otimes \mathcal{D}(T_y M), \]
and this defines the action $g_*$ we were looking for (it is straightforward to check that this defines indeed an action).

**Definition 2.3.** A transverse density for the Lie groupoid $\mathcal{G}$ is any $\mathcal{G}$-invariant section of the transverse density bundle $\mathcal{D}^*_A$.

**Remark 2.4.** Recall that the canonical integration of densities lets us associate to a compactly supported positive density $\rho$ on a manifold $M$ a Radon measure $\mu_\rho$. Similarly, a positive transverse density for a groupoid $\mathcal{G}$ gives rise to what is called a transverse measure for $\mathcal{G}$. Such measures were studied in [2], and represent measures on the differentiable stack presented by $\mathcal{G}$.

### 3. The modular class(es) revisited

Throughout this section $\mathcal{G}$ is a Lie groupoid over $M$ and $A$ is its Lie algebroid. We will be using the transverse density bundle $\mathcal{D}^*_A$, volume bundle $\mathcal{V}^*_A$ and orientation bundle $\mathfrak{o}^*_A$, viewed as representations of $\mathcal{G}$ as explained in Section 2.2. Let us mention, right away, the relation between these bundles. As vector bundles over $M$, we know (see Section 2.2) that there are canonical vector bundle isomorphism between
- $\mathcal{D}^{\text{tr}}_A$ and $\mathcal{V}^{\text{tr}}_A \otimes \mathfrak{o}^{\text{tr}}_A$.
- $\mathcal{V}^{\text{tr}}_A$ and $\mathcal{D}^{\text{tr}}_A \otimes \mathfrak{o}^{\text{tr}}_A$.
- $\mathfrak{o}^{\text{tr}}_A \otimes \mathfrak{o}^{\text{tr}}_A$ and the trivial line bundle.
- $\mathfrak{o}^{\text{tr}}_A$ and $(\mathfrak{o}^{\text{tr}}_A)^*$.

**Lemma 3.1.** All these canonical isomorphisms are isomorphisms of representations of $\mathcal{G}$ (where the trivial line bundle is endowed with the trivial action).

**Proof.** Given the way that the action of $\mathcal{G}$ was defined (Section 2.2), the direct check can be rather lengthy and painful. Here is a more conceptual approach. The main remark is that these actions can be defined in general, whenever we have a functor $F$ which associates to a vector space $V$ a 1-dimensional vector space $F(V)$ and to a (linear) isomorphism $f : V \rightarrow W$ an isomorphism $F(f) : F(V) \rightarrow F(W)$ such that:

1. $F$ commutes with the duality functor $D$, i.e., $F \circ D$ and $D \circ F$ are isomorphic through a natural transformation $\eta : F \circ D \rightarrow D \circ F$.
2. for any exact sequence $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ there is an induced isomorphism between $F(V)$ and $F(U) \otimes F(W)$, natural in the obvious sense.

Let’s call such $F$’s “good functors”. The construction from Section 2.2 shows that for any good functor $F$,
\[ F^{\text{tr}}_A := F(A^*) \otimes F(TM) \]
is a representation of $\mathcal{G}$. Given two good functors $F$ and $F'$, an isomorphism $\eta : F \rightarrow F'$ will be called good if it is compatible with the natural transformations from 1. and 2. above. It is clear that, for any such $\eta$, there is an induced map $\eta^*\mathfrak{o}$ that is an isomorphism between $F^{\text{tr}}_A$ and $F'^{\text{tr}}_A$, as representations of $\mathcal{G}$. It should also be clear that, for any two good functors $F$ and $F'$, so is their tensor product. We see that we are left with proving that certain isomorphisms involving the functors $\mathcal{D}$, $\mathcal{V}$ and $\mathfrak{o}$ (e.g. $\mathcal{D} \cong \mathcal{V} \otimes \mathfrak{o}$) are good in the previous sense; and that is straightforward. \qed
3.1. The modular class of $G$. Let us concentrate on the question of whether $G$ admits a strictly positive transverse density (these are the “measures” from the slogan at the start of the section, or “geometric measures” in the terminology of [2]). Start with any strictly positive section $\sigma$ of $D^\ast_{tr}A$. Then any other such section is of type $e^f \sigma$ for some $f \in C^\infty(M)$; moreover $e^f \sigma$ is invariant if and only if
\[
e^f(y)\sigma(y) = e^f(x)g(\sigma(x))
\]
for all $g : x \to y$ an arrow of $G$. Considering $c_\sigma := \ln \left( \frac{\sigma(y)}{g(\sigma(x))} \right)$, one has $c_\sigma \in C^\infty(G)$ and one checks right away that it is a 1-cocycle, i.e.,
\[
c_\sigma(gh) = c_\sigma(g) + c_\sigma(h)
\]
for all $g$ and $h$ composable. The condition on $f$ that we were considering reads:
\[
f(x) - f(y) = c_\sigma(g)
\]
for all $g : x \to y$, i.e., $c_\sigma = \delta(f)$ in the differentiable cohomology complex of $G$, $(C^\bullet_{diff}(G), \delta)$. Furthermore, an easy check shows that the class $[c_\sigma] \in H^1_{diff}(G)$ does not depend on the choice of $\sigma$. Therefore it gives rise to a canonical class
\[
mod(G) \in H^1_{diff}(G),
\]
called the modular class of the Lie groupoid $G$. By construction:

**Lemma 3.2.** $G$ admits a strictly positive transverse density iff $mod(G) = 0$.

The result makes precise the expectations of [3, Section 7] for the meaning of the modular class in the absence of superorientability. With this, the existence of transverse densities and measures for proper Lie groupoids of [2] is just about the vanishing of differentiable cohomology of proper groupoids (Proposition 1 in [1]).

The construction of $mod(G)$ can be seen as a very particular case of the construction from [1] of characteristic classes of representations of $G$, classes that live in the odd differentiable cohomology of $G$.

Here we are interested only in the 1-dimensional representations $L$, with corresponding class denoted
\[
\theta_G(L) \in H^1_{diff}(G).
\]
For a direct description, similar to that of $mod(G)$, we first assume that $L$ is trivializable as a vector bundle and we choose a nowhere vanishing section $\sigma$. Then, for $g : x \to y$, we can write
\[
g \cdot \sigma(x) = \tilde{c}_\sigma(g)\sigma(x) \quad (\tilde{c}_\sigma(g) \in \mathbb{R}^*)
\]
and this defines a function
\[
(3.1) \quad \tilde{c}_\sigma : G \to \mathbb{R}^*
\]
that is a groupoid homomorphism. The cocycle of interest is
\[
(3.2) \quad c_\sigma = \ln(\tilde{c}_\sigma) : G \to \mathbb{R};
\]
its cohomology class does not depend on the choice of $\sigma$ and defines $\theta_G(L)$. It is clear that for two such representations $L_1$ and $L_2$ (trivializable as vector bundles),
\[
(3.3) \quad \theta_G(L_1 \otimes L_2) = \theta_G(L_1) + \theta_G(L_2).
\]
This indicates how to proceed for a general $L$: consider the representation $L \otimes L$ which is (noncanonically) trivializable and define:
\[
(3.4) \quad \theta_G(L) := \frac{1}{2}\theta_G(L \otimes L).
\]
The multiplicativity formula for $\theta_G$ remains valid for all $L_1$ and $L_2$. By construction:

**Lemma 3.3.** One has $\text{mod}(G) = \theta_G(D^*_A)$.

**Remark 3.4** (a warning). It is not true (even if $L$ is trivializable as a vector bundle!) that $\theta_G(L)$ is the obstruction to $L$ being isomorphic to the trivial representation. Lemma 3.2 holds because the transverse density bundle is more than trivializable: one can also talk about positivity of sections of $D^*_A$ and $D^*_A$ is trivializable as an oriented representation of $G$.

The tendency in existing literature, at least for the infinitesimal version of the modular class (see below), is to use simpler representations instead of $D^*_A$. Here we would like to clarify the role of the transverse volume bundle $V^*_A$: can one use it to define $\text{mod}(G)$? In short, the answer is: yes, but one should not do it because it would give rise to the wrong expectations (because of the previous warning!). We summarise this into the following:

**Proposition 3.5.** For any Lie groupoid $G$, $\text{mod}(G) = \theta_G(V^*_A)$. However, it is not true that that $\text{mod}(G) = 0$ happens if and only if $G$ admits a transverse volume form (i.e., a nowhere vanishing $G$-invariant section of $V^*_A$).

Counterexamples for the last part are provided already by manifolds $M$, viewed as groupoids with unit arrows only. Indeed, in this case the associated transverse (density, volume) bundles are the usual bundles of $M$; hence the modular class is zero even if $M$ is not orientable. For the first part of the proposition, using the multiplicativity $\theta_A$ of $\theta_G$ and the canonical isomorphisms discussed at the beginning of the section, we have to show that

$$\theta_G(o^*_A) = 0.$$  

In turn, this follows by applying again the multiplicativity of $\theta_G$ and the canonical isomorphism between $o^*_A \otimes o^*_A$ and the trivial representation.

### 3.2. The modular class of $A$.

The construction of the modular class of a Lie algebroid $A$, introduced by Evens, Lu and Weinstein [3], is based on the geometry of a certain 1-dimensional representation $Q_A$ of the Lie algebroid $A$: $\text{mod}(A)$ is the characteristic class of $Q_A$. Let us first recall the construction of the characteristic class $\theta_A(L) \in H^1(A)$ associated to any 1-dimensional representation $L$ of $A$ (the infinitesimal version of the construction of the classes $\theta_G(L)$ of groupoid representations). First one uses the analogue of (3.4) to reduce the construction to the case when $L$ is trivializable as a vector bundle; then, for such $L$, one chooses a nowhere vanishing section $\sigma$ and one writes the infinitesimal action $\nabla$ of $A$ on $L$ as

$$\nabla_{\alpha}(\sigma) = c_{\sigma}(\alpha) \cdot \sigma,$$

therefore defining $c_{\sigma}$ as an element $c_{\sigma}(L) \in \Omega^1(A)$. Similar to the previous discussion, the flatness of $\nabla$ implies that $c_{\sigma}(L)$ is a closed $A$-form and its cohomology class does not depend on the choice of $\sigma$; therefore it defines a class, called the characteristic class of $L$, and denoted

$$\theta_A(L) \in H^1(A).$$

Note that the situation is simpler than at the level of $G$: for $L$ trivializable as a vector bundle, $\theta_A(L) = 0$ if and only if $L$ is isomorphic to the trivial representation of $A$ (compare with the warning from Remark 3.4).

Inspired by the previous subsection, we define:

**Definition 3.6.** The modular class of a Lie algebroid $A$, denoted $\text{mod}(A)$, is the characteristic class of $D^*_A$.
When $A$ is the Lie algebroid of a Lie groupoid $\mathcal{G}$, since $\mathcal{D}_A^{tr}$ is a representation of $\mathcal{G}$, we deduce (cf. Theorem 7 in [1]):

**Proposition 3.7.** For any Lie groupoid $\mathcal{G}$, the Van Est map in degree 1,

$$V:E:H_{\text{diff}}^1(\mathcal{G}) \to H^1(A)$$

sends $\text{mod}(\mathcal{G})$ to $\text{mod}(A)$. In particular, if $A$ is integrable by a unimodular Lie groupoid (e.g. by a proper Lie groupoid), then its modular class vanishes.

In particular, since the Van Est map in degree 1 is injective if $\mathcal{G}$ is s-connected (see e.g. Theorem 4 in [1]) we deduce:

**Corollary 3.8.** If $\mathcal{G}$ is an s-connected Lie group with Lie algebroid $A$, then $\text{mod}(A)$ is the obstruction to the existence of a strictly positive transverse density of $\mathcal{G}$.

3.3. **(Not) $Q_A$.** The modular class of a Lie algebroid $A$ can be defined as the characteristic class of various 1-dimensional representations of $A$. We have used $\mathcal{D}_A^{tr}$, but the common choice in the literature (starting with [3]) is the line bundle

$$Q_A = \Lambda^{\text{top}}A \otimes |\Lambda^{\text{top}}T^*M|.$$  

The infinitesimal action of $A$ on $Q_A$ is explained in [3]; equivalently, one writes

$$(3.6) \quad Q_A = \mathcal{D}_A^{tr} \otimes \sigma_A$$

in which both terms are representations of $A$: $\mathcal{D}_A^{tr}$ was already discussed, while $\sigma_A$ is a representation of $A$ since it is a flat vector bundle over $M$.

**Lemma 3.9.** The representations $Q_A$, $\mathcal{D}_A^{tr}$ and $\mathcal{V}_A^{tr}$ of $A$ have the same characteristic class (namely $\text{mod}(A)$).

*Proof.* Using $\mathcal{D}_A^{tr} \cong \mathcal{V}_A^{tr} \otimes \sigma_A^{tr}$, and the multiplicativity of $\theta_A$, it suffices to show that $\theta_A(\sigma_A^{tr}) = 0$ and similarly for $\sigma_A$. Using again multiplicativity, it suffices to show that $\theta_A(\sigma_A^{tr} \otimes o_A^{tr}) = 0$ - which is true because $\sigma_A^{tr} \otimes o_A^{tr}$ is isomorphic to the trivial representation (Lemma 3.1); and similarly for $\sigma_A$ just that, this time, $\sigma_A \otimes o_A$ is isomorphic to the trivial line bundle already as a flat vector bundle. \(\square\)

Despite the previous lemma, because of Proposition 3.5 and the discussion around it, using $\mathcal{V}_A^{tr}$ to define $\text{mod}(A)$, although correct, may give rise to the wrong expectations. However, using $Q_A$ to define $\text{mod}(A)$ is even more unfortunate, for even more fundamental reasons: in general, $Q_A$ is not a representation of the groupoid $\mathcal{G}$! Indeed, using (3.6), the condition that $Q_A$ can be made into a representation of $\mathcal{G}$ is equivalent to the same condition for $\sigma_A$. But the orientation bundles $\sigma_A$ are the typical examples of algebroid representations that do not come from groupoid ones. That is clear already in the case of the pair groupoid of a manifold $M$, whose representations are automatically trivial as vector bundles, but for which $\sigma_A = o_T M$ is not trivializable if $M$ is not orientable.

Note also that the fact that $\mathcal{D}_A^{tr}$, unlike $Q_A$, is a representation of $\mathcal{G}$, was absolutely essential for obtaining Proposition 3.7 and Corollary 3.8.

3.4. **Transverse orientability and the first Stiefel-Whitney class of $\mathcal{G}$.** It is interesting to look back at the construction of the characteristic class $\theta_{\mathcal{G}}(L)$ of a 1-dimensional representation $L$ of $\mathcal{G}$. The reason for the warning mentioned in Remark 3.4 comes from the fact that, when passing from $c_\mathcal{G}$ to $c_\mathcal{G}$ (in 3.2), one loses information related to orientability.

Following the exposition in [7], we return to the discussion around (3.2); in particular, we assume that $L$ is a 1-dimensional representation of $\mathcal{G}$ that is trivializable as a vector bundle, and $\sigma$ is a nowhere vanishing section of $L$. Then one can either: 
\begin{itemize}
  \item consider the entire \( \bar{c}_\sigma : \mathcal{G} \to \mathbb{R}^* \)
  \item consider only the part of \( \bar{c}_\sigma \) that is not contained in \( c_\sigma \), i.e., \( \text{sign} \circ \bar{c}_\sigma : \mathcal{G} \to \mathbb{Z}_2 \),
\end{itemize}

where we identify \( \mathbb{Z}_2 \) with \( \{ -1, 1 \} \subset \mathbb{R}^* \).

Both of them are (differentiable) cocycles on \( \mathcal{G} \) with coefficients in a (abelian Lie) group. Such cocycles give rise to classes in the cohomology groups \( H^1_{\text{diff}}(\mathcal{G}, \mathbb{R}^*) \) and \( H^1_{\text{diff}}(\mathcal{G}, \mathbb{Z}_2) \), which are abelian groups but no longer vector spaces. As before, the resulting cohomology classes are independent of \( \sigma \); we denote them by

\[
\tilde{\theta}_\mathcal{G}(L) \in H^1_{\text{diff}}(\mathcal{G}, \mathbb{R}^*), \quad w(L) \in H^1_{\text{diff}}(\mathcal{G}, \mathbb{Z}_2),
\]

and we will call them the **extended characteristic class** of \( \mathcal{L} \), and the **Stiefel-Whitney class** of \( \mathcal{L} \), respectively. All these classes can be put together using the decomposition of the group \( \mathbb{R}^* \) as

\[
\mathbb{R}^* \cong \mathbb{R} \times \mathbb{Z}_2, \quad \lambda \mapsto (\ln(|\lambda|)), \text{sign}(\lambda))
\]

and the induced isomorphism

\[
H^1_{\text{diff}}(\mathcal{G}, \mathbb{R}^*) \cong H^1_{\text{diff}}(\mathcal{G}) \times H^1_{\text{diff}}(\mathcal{G}, \mathbb{Z}_2).
\]

With this, the extended class of \( \mathcal{L} \) is

\[
\tilde{\theta}_\mathcal{G}(L) = (\theta_\mathcal{G}(L), w(L)).
\]

Note that, by construction, \( \tilde{\theta}_\mathcal{G}(L) \) is trivial (equal to the identity of the group) if and only if \( \mathcal{L} \) is isomorphic to the trivial representation, and \( w(L) \) is trivial if and only if \( \mathcal{L} \) is \( \mathcal{G} \)-orientable, i.e., if \( \mathcal{L} \) admits an orientation with the property that the action of \( \mathcal{G} \) is orientation-preserving. A warning however: the classes \( \tilde{\theta}_\mathcal{G}(L) \) and \( w(L) \) have been defined so far only when \( \mathcal{L} \) is trivializable as a vector bundle; moreover, these constructions cannot be extended to general \( \mathcal{L} \)'s while preserving their main properties. Hence, when it comes to the canonical representations of \( \mathcal{G} \), one can apply them only to \( \mathcal{D}_\mathcal{M}^1 \), for which one obtains

\[
w(\mathcal{D}_\mathcal{M}^1) = 1, \quad \tilde{\theta}_\mathcal{G}(\mathcal{D}_\mathcal{M}^1) = (\mod(\mathcal{G}), 1).
\]

For the previous discussion we assumed that \( \mathcal{L} \) was trivializable as a vector bundle. To handle general \( \mathcal{L} \)'s one can use covers \( \mathcal{U} \) of \( \mathcal{M} \) by open subsets over which \( \mathcal{L} \) is trivializable. Such an open cover induces a groupoid \( \mathcal{G}_\mathcal{U} \) over the disjoint union of the open subsets in \( \mathcal{U} \), obtained by pulling-back \( \mathcal{G} \) along the canonical map from the disjoint union into \( \mathcal{M} \). The pull-back \( \mathcal{L}_\mathcal{U} \) of \( \mathcal{L} \) is a representation of \( \mathcal{G}_\mathcal{U} \) and, by the choice of \( \mathcal{U} \), one has well-defined classes

\[
\tilde{\theta}(\mathcal{L}_\mathcal{U}) = (\theta(\mathcal{L}_\mathcal{U}), w(\mathcal{L}_\mathcal{U})) \in H^1_{\text{diff}}(\mathcal{G}_\mathcal{U}, \mathbb{R}^*) \cong H^1_{\text{diff}}(\mathcal{G}_\mathcal{U}) \times H^1_{\text{diff}}(\mathcal{G}_\mathcal{U}, \mathbb{Z}_2).
\]

Note that, since \( \mathcal{G}_\mathcal{U} \) is Morita equivalent to \( \mathcal{G} \) (see Example 5.10 in [S]), when passing from \( \mathcal{L} \) to \( \mathcal{L}_\mathcal{U} \) (as representations) one does not lose any information. To obtain a class that is independent of the covers one proceeds as usual and one passes to the filtered colimit (with respect to the refinement of covers) and defines

\[
\tilde{\mathcal{H}}^1_{\text{diff}}(\mathcal{G}, \mathbb{R}^*) \cong \lim_{\rightarrow \mathcal{U}} H^1_{\text{diff}}(\mathcal{G}_\mathcal{U}, \mathbb{R}^*),
\]

and similarly \( \tilde{\mathcal{H}}^1_{\text{diff}}(\mathcal{G}, \mathbb{Z}_2) \). The Morita invariance of differentiable cohomology with coefficients in \( \mathbb{R} \) implies that the restriction to open subsets induces an isomorphism

\[
H^1_{\text{diff}}(\mathcal{G}) \cong H^1_{\text{diff}}(\mathcal{G}_\mathcal{U})
\]

(that sends \( \theta_\mathcal{G}(L) \) to \( \theta_{\mathcal{G}_\mathcal{U}}(L_\mathcal{U}) \)); hence there are induced cohomology classes

\[
\theta_\mathcal{G}(L) \in \tilde{\mathcal{H}}^1_{\text{diff}}(\mathcal{G}, \mathbb{R}^*), \quad w(L) \in \tilde{\mathcal{H}}^1_{\text{diff}}(\mathcal{G}, \mathbb{Z}_2)
\]
and canonical isomorphism of groups

\[ \tilde{H}^1_{\text{diff}}(\mathcal{G}, \mathbb{R}^*) \cong H^1_{\text{diff}}(\mathcal{G}) \times \tilde{H}^1_{\text{diff}}(\mathcal{G}, \mathbb{Z}_2) \]

such that:

- \( \tilde{\theta}_\mathcal{G}(L) = (\theta_\mathcal{G}(L), w(L)) \).
- \( \tilde{\theta}_\mathcal{G}(L) \) is trivial if and only if \( L \) is isomorphic to the trivial representation.

Actually, \( \tilde{\theta}_\mathcal{G} \) gives an isomorphism between the group \( \text{Rep}^1(\mathcal{G}) \) of isomorphism classes of 1-dimensional representations (with the tensor product) with the Čech-type cohomology with coefficients in \( \mathbb{R}^* \):

\[ (3.7) \quad \tilde{\theta}_\mathcal{G} : \text{Rep}^1(\mathcal{G}) \sim \tilde{H}^1_{\text{diff}}(\mathcal{G}, \mathbb{R}^*). \]

Denoting \( \mathbb{R}^* = GL_1(\mathbb{R}) \) by \( H \), this is a particular case of the interpretation of \( \mathcal{G} \)-equivariant principal \( H \)-bundles in terms of transition functions (see for example [5]), valid for any Lie group \( H \), interpretation that is itself at the heart of Haefliger’s work on the transverse geometry of foliations [4]. While \( w \) is trivial on \( \mathcal{D}^H_A \), it gives rise to interesting information when applied to the transverse volume bundle or, equivalently, to the orientation one.

**Definition 3.10.** The **transverse first Stiefel-Whitney class** of \( \mathcal{G} \) is:

\[ w_1^{\text{tr}}(\mathcal{G}) := w(\nu_0^\mathcal{G}) = w(a_0^\mathcal{G}) \in \tilde{H}^1_{\text{diff}}(\mathcal{G}, \mathbb{Z}_2). \]

As a consequence of the previous discussion we state here the following:

**Corollary 3.11.** One has:

1. \( \mathcal{G} \) is transversely orientable iff \( w_1^{\text{tr}}(\mathcal{G}) = 1 \).
2. \( \mathcal{G} \) admits transverse volume forms iff \( \text{mod}(\mathcal{G}) = 0 \) and \( w_1^{\text{tr}}(\mathcal{G}) = 1 \).

**Acknowledgements**

This research was supported by the NWO Vici Grant no. 639.033.312. The second author was supported also by the FCT grant SFRH/BD/71257/2010 under the POPH/FSE programmes. We would also like to acknowledge various discussions with Rui Loja Fernandes, Ioan Mărcuț and David Martínez Torres.

**References**

[1] M. Crainic, *Differentiable and algebroid cohomology, van Est isomorphisms, and characteristic classes*, Comment. Math. Helv. **78** (2013), no. 4, 681–721.
[2] M. Crainic and J. N. Mestre, *Measures on differentiable stacks*, J. Noncommut. Geom **13** (2019), no. 4, 1235–1294.
[3] S. Evens, J.-H. Lu, and A. Weinstein, *Transverse measures, the modular class and a cohomology pairing for Lie algebroids*, Quart. J. Math. Oxford Ser. (2) **50** (1999), no. 200, 417–436.
[4] A. Haefliger, *Homotopy and integrability*, Manifolds–Amsterdam 1970 (Proc. Nuffic Summer School), Lecture Notes in Mathematics, Vol. 197, Springer, Berlin, 1971, pp. 133–143.
[5] D. Husemoller, *Fibre bundles*, third ed., Graduate Texts in Mathematics, vol. 20, Springer-Verlag, New York, 1994.
[6] R. A. Mehta, *Modular classes of Lie groupoid representations up to homotopy*, SIGMA Symmetry Integrability Geom. Methods Appl. **11** (2015).
[7] J. N. Mestre, *Differentiable stacks: stratifications, measures and deformations*, Ph.D. thesis, Utrecht University, 2016.
[8] I. Moerdijk and J. Mrčun, *Introduction to foliations and Lie groupoids*, Cambridge Studies in Advanced Mathematics, vol. 91, Cambridge University Press, Cambridge, 2003.

**Mathematical Institute, Utrecht University, The Netherlands**

**E-mail address:** m.crainic@uu.nl

**Centre for Mathematics, University of Porto, Portugal**

**E-mail address:** jnmestre@gmail.com