1. Introduction

We work with compact Riemann surfaces, and simply refer to them as “curves”, although in some instances we allow for singular points. We propose a generalization of the $\sigma$-function for $(n, s)$ curves, which was developed recently, based on Klein’s and Baker’s theories [3]; we parallel constructions given in Mumford’s Tata lectures on theta II [57], namely algebraic expressions for abelian functions, based on nineteenth-century work; by the use of specific abelian variables and the introduction of a generalized Weierstrass $\wp$-function, defined for hyperelliptic curves [57] (IIIa. Section 10), Mumford expresses the transcendental function theory on the Jacobian in terms of meromorphic functions on the curve; applications are given to integrable dynamical systems.

1.1. The $\sigma$-function. We say that a pointed algebraic curve is of W-type $(n_1, \ldots, n_s)$ $(n_1 < n_2 < \ldots < n_s)$ if its Weierstrass semigroup at the point has minimal set of generators $\{n_1, \ldots, n_s\}$. The Kleinian $\sigma$-function (cf. [3, 4, 33, 71]), originally devised for hyperelliptic curves, which have W-type $(2, 2g+1)$, is closely related to Mumford’s $\wp$. Baker showed in essence that $\sigma$ satisfies the KdV hierarchy and KP equation (as they became known in the 1970s). He introduced a bilinear equation, which was newly proposed in the 1960s as Hirota’s bilinear operator [5].
Baker’s work was recently revisited and expanded by several authors, among whom co-authors Buchstaber, Enolskii, and Leykin [6, 7, 10, 11]; Grant [30]; Matsutani [48]; Onishi [61]. In subsequent work [17], Eilbeck, Enolskii and Leykin gave a construction of the Kleinian \( \sigma \)-function of [10] generalized to curves of W-type \((n, s)\), where \(n\) and \(s\) are any two relatively prime positive integers; we call them \((n, s)\) curves for short, as in the literature (these are the same as the \(C_{rs}\) curves of [50, 52]); they did so by defining the fundamental differential of the second kind over every \((n, s)\) curve. We call their construction “EEL construction”. Using the EEL construction, properties such as addition formulae and the order of vanishing of the \(\sigma\)-function of \((n, s)\) curves are obtained in [18, 19, 50]. Nakayashiki [60] investigated properties of the function also by the use of differentials on symmetric products of the curve. Further Eilbeck, Enolski and Gibbons [16] and Nakayashiki [59] exhibited connections between \(\sigma\)-function and Sato’s \(\tau\)-function.

In our program, we extend the study to affine curves in higher-dimensional space. The construction was given by the first- and second-named authors in [49] for a curve of W-type \((3, 4, 5)\). As in the \((n, s)\) case, we henceforth omit “W-type” for brevity. Here, we generalize the \(\sigma\)-function to a \((3, 7, 8)\) curve with affine model in 3-dimensional space, and to a \((6, 13, 14, 15, 16)\) curve which covers it. The motivation of this generalization comes from the the Monstrous Moonshine and the Weierstrass semigroup problem, described below in subsection 1.2. Following the EEL construction, we define the fundamental differential of the second kind, which gives a generalized Legendre relation; this in turn gives us the \(\sigma\)-function and we obtain the Jacobi inversion formulae following previous work [50]. We note that our Kleinian \(\sigma\)-function is again a generalization of Weierstrass’ elliptic \(\sigma\).

Korotkin with Shramchenko defined a sigma function for any compact Riemann surface [42], which so far has not been explicitly related with meromorphic functions on the curve; on the other hand, Ayano investigated sigma functions for space curves of a special type [2], called telescopic curves, but those do not include a \((3, 7, 8)\) or a \((6, 13, 14, 15, 16)\) curve.

1.2. **Monstrous Moonshine.** Our motivation for focusing on the curves of genus 4, \((3, 7, 8)\) and genus 12, \((6, 13, 14, 15, 16)\), is due to a suggestion by John McKay. The curve \((6, 13, 14, 15, 16)\) is associated with a Jacobi variety whose real dimension is 24 and has the Weierstrass gap sequence \(\{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 17, 23\}\), similar to the Norton numbers \(\{1, 2, 3, 4, 5, 7, 8, 9, 11, 17, 19, 23\}\). The Norton numbers, as stated in [26, 53, 54, 60], e.g., are the coefficients of replicable functions in a Norton basis; in turn, replicable functions are related to the Monstrous Moonshine. In VI.3 of [28], the **Prize Question** reads, for \(\hat{A}\) and the (twisted) \(\hat{A}\)-genus (since we are quoting verbatim, we refer to [28] for technical definitions): \textit{Does there exist a 24-dimensional, compact, orientable, differentiable manifold \(X\) with } \(p_1(X) = 0, w_2(X) = 0, \hat{A}(X) = 1\) \textit{and } \(\hat{A}(X, T_C) = 0?\) For such a manifold, suitable twisted \(\hat{A}\) genera would equal dimensions of irreducible representations of the Monster; and if the Monster acted on \(X\) by diffeomorphisms, \textit{one would possess a key to construct a great many representations of the Monster.} The Witten genus is expressed in terms of the Weierstrass \(\sigma\) function [28].

Of course, although a Jacobian is compact, orientable, differentiable and admits a spin structure [29] (equivalently, its \(w_2\) is zero), \(p_1\) is non-zero, being the genus of the curve. Therefore,
though our Jacobian has real dimension 24, we do not propose a link with the above problem, but we give some remarks in Section 7 on relevant geometric properties from a viewpoint of the construction of the generalized \( \sigma \) function associated with a \((6, 13, 14, 15, 16)\) curve.

With such motivation, we sought a \((6, 13, 14, 15, 16)\) curve. However, as recalled in Section 2 it is a non-trivial problem whether there exists a pointed curve with given numerical semigroup at the point. This article is based upon J. Komeda’s proof that there exists a non-singular curve with Weierstrass gap sequence \(\{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 17, 23\}\) at a given point (see Proposition 2.1); its Weierstrass semigroup is indeed \(H_{12}\), the sub-semigroup of the non-negative integers \(\mathbb{N}_0\) generated by \(M_{12} := \{6, 13, 14, 15, 16\}\). The genus-12 curve constructed by Komeda covers a (genus-4) \((3, 7, 8)\) curve; we begin with the construction and analysis of its \(\sigma\)-function; relationships between the algebraic functions of the curves exhibit similar properties to those of the Monstrous Moonshine.

Let \(H\) denote a subsemigroup of the additive group of non-negative integers such that only finitely many positive integers, called the gaps of \(H\), are missing from \(H\). Let \(B_H\) denote the subring of the polynomial ring \(k[t]\) generated by the monomials \(t^h, h \in H\). Then \(\text{Spec} \ B_H\) is a monomial curve; i.e. an irreducible affine curve with \(\mathbb{G}_m\)-action, where \(\mathbb{G}_m\) denotes the multiplicative group of \(k\) (\(\mathbb{C}\) in our case). There is an induced \(\mathbb{G}_m\)-action on \(T^1 := T^1(B_H)\), the Zariski tangent space of \(\text{Spec} \ B_H\) [62], p. 4, and, decomposing \(T^1\) into eigenspaces, one can write \(T^1 = \sum_{\nu} T^1(\nu)\), where \(\nu\) ranges from \(-\infty\) to \(+\infty\). One says that \(B_H\) is negatively graded if \(T^1(\nu) = 0\) for all positive \(\nu\).

In his thesis [62], Pinkham showed that \(H\) occurs as a Weierstrass semigroup if and only if the corresponding monomial curve, \(\text{Spec} \ k[t^h : h \in H]\), can be smoothed negatively. Pinkham used this to show that if \(H\) is a complete intersection (in particular, if \(H\) is generated by two elements), then \(H\) occurs as a Weierstrass semigroup. This also holds for almost complete intersection semigroups generated by three elements and was generalized to almost complete intersection semigroups generated by four elements. Rim and Vitulli [66] give a complete characterization of such negatively graded semigroup rings. Moreover, they extend Pinkham’s result by showing that every negatively graded semigroup ring can be smoothed. By the work of Pinkham, this implies that if \(H\) is any negatively graded semigroup, then there exist a smooth projective curve \(X\) and a point \(x \in X\) such that the gaps of \(H\) are the Weierstrass gaps at \(x\).

In Section 2 we give a brief review of Weierstrass semigroups. In Sections 3 and 4 we proceed as follows: we define monomial curves with given Weierstrass semigroups, with motivation from the Norton numbers; when the semigroup falls outside the verifiably smoothable case, we give Komeda’s proof that one smooth curve with such Weierstrass semigroup exists; Pinkham’s calculation of the expected dimension yields a positive number, therefore we can conclude that the monomial curve is smoothable. In Section 5 we use the local coordinates given by the monomial presentation of the curve, to manufacture a local section of certain meromorphic differentials, and we construct an abelian function on the curve, the \(\sigma\)-function, by integrating those differentials. In [49], the original idea was implemented for the semigroup \((3, 4, 5)\), including the \(\sigma\)-function construction and natural extensions of the ones previously developed for \((n, s)\) curves. The \(\sigma\)-function provides a stratification of the Jacobian in Section
Weierstrass. The question was revived in the 1980s, viewed as the question of deformations of the Monster, starting with the above-mentioned numerical observation connecting the Norton numbers and the differentials we constructed \([A]\).

\section{Weierstrass Semigroups}

For reference, we give a brief overview of the recent study of the “numerical semigroups”, namely those (additive) sub-semigroups of the non-negative integers \(\mathbb{N}_0\) whose complement is finite. We then give a new result due to Komeda concerned with the analogy with the Norton basis; this was already presented in \([49]\).

We use standard notation, e.g. \(h^i\) for the dimension of the cohomology group \(H^i\) of a sheaf over the curve. Also, a sheaf \(\mathcal{O}_X(D)\) where \(D\) is a divisor on \(X\) may be denoted by \(D\) for short, as well as \(h^i(X, \mathcal{O}_X(D))\) by \(h^i(D)\).

For a numerical semigroup \(H\) generated by a set \(M\), the number of elements of \(L(H) := \mathbb{N}_0 \setminus H\), the “gap sequence” of \(H\), is called “genus” if it is finite, and denoted by \(g(H)\). We focus on the numerical semigroup \(H_4\) generated by \(M_4 := \{3, 7, 8\}\), i.e., \(H_4 = \langle 3, 7, 8 \rangle\) as well as the problem of the numerical semigroup \(H_{12}\) generated by \(M_{12}\), i.e., \(H_{12} = \langle M_{12}\rangle\) for the reasons explained in subsection \([1, 2]\).

The genus of \(H_4, H_{12}\) (respectively) is \(g = 4, 12\), and

\[
L(H_4) = \{1, 2, 4, 5\}, \quad L(H_{12}) = \{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 17, 23\}.
\]

For a gap sequence \(L := \{\ell_0 < \ell_1 < \cdots < \ell_{g-1}\}\) of genus \(g\), let \(M(L)\) be the minimal set of generators for the semigroup consisting of the complement of \(L\) (it is easy to show that a minimal set of generators must be unique) and

\[
\alpha(L) := \{\alpha_0(L), \alpha_1(L), \ldots, \alpha_{g-1}(L)\},
\]

where \(\alpha_i(L) := \ell_i - i - 1\). When an \(\alpha_i\) is repeated \(j \geq 1\) times we write \(\alpha_i^j(L)\) in \(\alpha(L)\). We set \(\text{wt}(L) = \sum_{i=0}^{g-1} \alpha_i(L)\), and refer to it as the weight of \(L\). We let \(\alpha_{\min}(L)\) be the smallest positive integer of \(M(L)\). We call a semigroup \(H\) an \(\alpha_{\min}(L)\)-semigroup, so that \(H_4\) is a 3-semigroup and \(H_{12}\) is a 6-semigroup. Moreover,

\[
\alpha(L(M_4)) = \{0^2, 1^2\} \quad \text{and} \quad \alpha(L(M_{12})) = \{0^5, 1^5, 6, 11\}.
\]

The 6-semigroups with 4 generators were studied by Komeda in \([38]\).

For a curve \(X\) of genus \(g\) and a point \(P \in X\), the semigroup

\[
H(X, P) := \{n \in \mathbb{N}_0 \mid \text{there exists } f \in k(X) \text{ such that } (f)_{\infty} = nP\}
\]

is called the Weierstrass semigroup of the point \(P\). If \(L(H(X, P))\) differs from the set \(\{1, 2, \cdots, g\}\), we say that \(P\) is a Weierstrass point of \(X\).

A numerical semigroup \(H\) is said to be Weierstrass if there exists a pointed curve \((X, P)\) such that \(H = H(X, P)\). Hurwitz posed the problem whether any numerical semigroup \(H\) is Weierstrass. The question was revived in the 1980s, viewed as the question of deformations of
a reduced complex curve singularity \((X_0, \infty)\). In \([8, 9]\), a counterexample was given, for the semigroup \(H_B\) generated by 13, 14, 15, 16, 17, 18, 20, 22 and 23, whose genus is 16: this can be seen as follows. Assume that \(H_B = H(X, P)\) for a pointed curve \((X, P)\), then \(X\) would have a holomorphic differential vanishing to order \(\ell - 1\) at \(P\) for any \(\ell \in L(H_B)\). By letting \(L_2(H)\) be the set of all sums of two elements of \(L(H)\) for a semigroup \(H\), \(X\) would have a holomorphic quadratic differential vanishing to order \(m - 2\) at \(P\) for any \(m \in L_2(H_B)\), which contradicts the fact that \(#L_2(H_B) = 46 > 45 = (3 \times 16) - 3 = h^0(X, \mathcal{K}_X^{\otimes 2})\). Note that we write \(\mathcal{K}_X^{\otimes 2}\) for the sheaf of regular quadratic differentials on \(X\).

We mainly refer to the work of Komeda and co-authors for further information \([34]-[41]\). D. Eisenbud with J. Harris \([22]\), C. Maclachlan and I. Morrison jointly with H. Pinkham are other authors who wrote on this issue. We sketch a list of sufficient properties for \(H\) to be Weierstrass:

(1) semigroup whose cardinality of the generators is less than 4,
(2) semigroup whose genus is less than 9,
(3) semigroup which is primitive, i.e., twice the smallest positive integer in \(H(L) > 1\) the largest integer in \(L\), of genus 9,
(4) semigroup whose \(a_{\min}(L) = 2, 3, 4, 5,\)
(5) semigroup which is primitive and \(\text{wt}(L) \leq g - 1,\)
(6) semigroup whose \(\alpha(L) = (g^r - 2, m, n)\) for genus \(g\) and \(\text{wt}(L) = g,\)
(7) semigroup whose \(\alpha(L) = (g^r, m')\) for genus \(g\), and
(8) other many special cases of \(n\)-semigroups generated by 4 elements.

Due to (1) or (2), \(H_4\) is Weierstrass.

As it is stated in \([49]\), we have the following crucial proposition due to Komeda:

**Proposition 2.1.** The numerical semigroup \(\langle 6, 13, 14, 15, 16 \rangle\) is Weierstrass.

*Proof.\* We consider a pointed curve \((X, P)\) with \(H(X, P) = \langle 3, 7, 8 \rangle\), which has genus 4, and \(\mathcal{K}_X\) a canonical divisor on \(X\). Then,

\[
2 = h^0(4P) = 4 + 1 - 4 + h^0(\mathcal{K}_X - 4P) = 1 + h^0(\mathcal{K}_X - 4P);
\]

this implies that \(\mathcal{K}_X - 4P \sim P_1 + P_2\) for some points \(P_1\) and \(P_2 \in X\). Moreover, since

\[
2 = h^0(5P) = 5 + 1 - 4 + h^0(\mathcal{K}_X - 5P) = 2 + h^0(\mathcal{K}_X - 5P),
\]

then \(h^0(\mathcal{K}_X - 5P) = 0\), and \(P_i \neq P\) for \(i = 1, 2\). This implies that \(\mathcal{K}_X \sim 4P + P_1 + P_2\) with \(P_i \neq P\) for \(i = 1, 2\). By setting \(D := 7P - P_1 - P_2\), we have \(\deg(2D - P) = 9 = 2 \times 4 + 1\), and this implies that the complete linear system \([2D - P]\) is very ample, hence base-point free. Therefore, \(D \sim P + \sum Q_i\) (= a reduced divisor). Let \(\mathcal{L}\) be the invertible sheaf \(\mathcal{O}_X(-D)\) on \(X\) and \(\phi\) an isomorphism \(\mathcal{L}^{\otimes 2} \cong \mathcal{O}_X(-P - Q_1 - \cdots - Q_9) \subset \mathcal{O}_X\). The vector bundle \(\mathcal{O}_X \oplus \mathcal{L}\) has an \(\mathcal{O}_X\)-algebra structure through \(\phi\). The canonical \(2 : 1\) morphism \(\pi : \tilde{X} = \text{Spec} (\mathcal{O}_X \oplus \mathcal{L}) \rightarrow X\) has branch locus \(\{P, Q_1, \ldots, Q_9\}\). For \(\tilde{P}\) the ramification point of \(\pi\) over \(P\), we see that \(H(\tilde{X}, \tilde{P}) = \langle 6, 13, 14, 15, 16 \rangle\) using the following formula:

\[
h^0(2n\tilde{P}) = h^0(nP) + h^0(nP - D)
\]
for any non-negative integer \( n \). We have \( h^0(12\tilde{P}) = h^0(6P) + h^0(6P - 7P + P_1 + P_2) = 3 + h^0(P_1 + P_2 - P) = 3 \) because of \( P_i \neq P \) for \( i = 1, 2 \). Moreover, we have \( h^0(14\tilde{P}) = h^0(7P) + h^0(P_1 + P_2) = 4 + 1 = 5 \), which implies that \( 13, 14 \in H(\tilde{X}, \tilde{P}) \). The equalities
\[
h^0(16\tilde{P}) = h^0(8P) + h^0(P + P_1 + P_2) = 5 + 3 + 1 - 4 + h^0(K_X - P - P_1 - P_2) = 5 + h^0(3P) = 7
\]
guarantee that \( 15, 16 \in H(\tilde{X}, \tilde{P}) \). We have
\[
h^0(18\tilde{P}) = h^0(9P) + h^0(2P + P_1 + P_2) = 6 + 4 + 1 - 4 + h^0(K_X - 2P - P_1 - P_2) = 7 + h^0(2P) = 8,
\]
which implies that \( 17 \notin H(\tilde{X}, \tilde{P}) \) and \( 18 \in H(\tilde{X}, \tilde{P}) \). Moreover, we get
\[
h^0(20\tilde{P}) = h^0(10P) + h^0(3P + P_1 + P_2) = 7 + 5 + 1 - 4 + h^0(K_X - 3P - P_1 - P_2) = 9 + h^0(P) = 10.
\]
Hence, we have \( 19, 20 \in H(\tilde{X}, \tilde{P}) \). We obtain
\[
h^0(22\tilde{P}) = h^0(11P) + h^0(4P + P_1 + P_2) = 8 + 6 + 1 - 4 + h^0(K_X - 4P - P_1 - P_2) = 11 + 1 = 12.
\]
Hence, we have \( 21, 22 \in H(\tilde{X}, \tilde{P}) \). Finally, we get
\[
h^0(24\tilde{P}) = h^0(12P) + h^0(5P + P_1 + P_2) = 9 + 7 + 1 - 4 = 13.
\]
Hence, we have \( 23 \notin H(\tilde{X}, \tilde{P}) \) and \( 24 \in H(\tilde{X}, \tilde{P}) \). We conclude that \( H(\tilde{X}, \tilde{P}) = \langle 6, 13, 14, 15, 16 \rangle \).

\[\square\]

3. Semigroup \( H_4 = \langle 3, 7, 8 \rangle \)

In our construction of \( \sigma \)-functions, the tool is a description of the affine ring of the curve minus \( \infty \), the Weierstrass point in question. We begin with the semigroup \( H_4 = \langle 3, 7, 8 \rangle \) because to treat our motivating example \( H_{12} \), the existence proof in Proposition 2.1 uses an explicit double cover of a curve with semigroup \( H_4 \).

3.1. The curve as a monomial curve. A monomial curve is an irreducible affine curve with \( \mathbb{G}_m \)-action, where \( \mathbb{G}_m \) is the multiplicative group of the complex numbers. To construct our curve, we consider the semigroup ring of \( H_4 \). We follow Pinkham’s strategy [62] both in the \( H_4 \) and \( H_{12} \) cases, namely, we start with an irreducible curve singularity with \( \mathbb{G}_m \) action, described parametrically by its semigroup ring. We deform the matrix of the relations. This deformation space \( \tilde{U} \) “classifies the set of pairs consisting of a smooth and proper algebraic curve \( X \) together with a point \( P \in X \) with given semigroup \( H(X, P) \). If \( \tilde{U} \) is non void, then we have constructed directly a compactification of a moduli space for curves with points of semi group \( H \)” ([62], Introduction). We have a result that ensures that \( \tilde{U} \) is nonempty in both cases. In the case of \( H_4 \), Pinkham again [62] Section 14) (Sec. 14) proves nonemptiness for semigroups with two generators and conjectures it for three, proved later in [66].

**Proposition 3.1.** For the \( k \)-algebra homomorphism,
\[
\varphi_4 : k[Z] := k[Z_3, Z_7, Z_8] \to k[t^a]_{a \in M_4}, \quad Z_a \mapsto t^a,
\]
the kernel of \( \varphi_4 \) is generated by \( f_b^{(Z)} = 0 \) (\( b = 14, 15, 16 \)) where
\[
(3.1) \quad f_{14}^{(Z)} = Z_7^2 - Z_3^2 Z_8, \quad f_{15}^{(Z)} = Z_7 Z_8 - Z_3^5, \quad f_{16}^{(Z)} = Z_8^2 - Z_3^3 Z_7.
\]
Proof: This follows from a result of Herzog’s [27], who showed that for any W-semigroup, the number of generators of the kernel is two or three. □

Here we note that these relations are given by the $2 \times 2$ minors of

$$
\begin{vmatrix}
Z_3^2 & Z_7 & Z_8 \\
Z_7 & Z_8 & Z_3^3
\end{vmatrix}.
$$

The monomial ring $B_{H_4} := k[t^a]_{a \in M_4}$ is given by

$$
B_{H_4} \simeq k[Z_3, Z_7, Z_8]/\ker \varphi.
$$

We say that $Z_a$ has weight $a$.

To indicate the action of $\mathbb{G}_m$, defined as $Z_a \mapsto g^a Z_a$ for $g \in \mathbb{G}_m$, and the induced action on the monomial ring $B_{H_4}$, we use the notation $\mathbb{G}_m^{(4)}$: the superscript (4) is a convenience since we are going to relate several curves and different actions.

To construct a smooth curve $X_4$ with $H(X_4, \infty) = H_4$, we consider two plane affine curves $X_6$ and $X_7$, both with a unique point at infinity which we always denote by $\infty$, corresponding to the equations:

$$
f_{6,21}(x, y_7) := y_7^3 - k_7(x), \quad k_7(x) := k_3(x)k_2(x)^2,
$$

and

$$
f_{7,24}(x, y_8) := y_8^3 - k_8(x), \quad k_8(x) := k_3(x)^2k_2(x),
$$

where for distinct $b_a \in \mathbb{C}$ ($a = 1, 2, \ldots, 5$),

$$
k_3(x) := (x - b_1)(x - b_2)(x - b_3) = x^3 + \lambda_1^{(3)}x^2 + \lambda_2^{(3)}x + \lambda_3^{(3)},
$$

$$
k_2(x) := (x - b_4)(x - b_5) = x^2 + \lambda_2^{(2)}x + \lambda_2^{(2)}.
$$

Both curves have singular points in their affine parts; the singular points of the affine part of $X_6$ are $(b_a, 0)$, $a = 4, 5$, whereas those of $X_7$ are $(b_a, 0)$, $a = 1, 2, 3$. We consider the rings $R_6 := \mathbb{C}[x, y_7]/(f_{6,21}(x, y_7))$ and $R_7 := \mathbb{C}[x, y_8]/(f_{7,24}(x, y_8))$ associated with $X_6$ and $X_7$ respectively. The genera of the semigroups of $X_6$ and $X_7$ at $\infty$ are 6 = $g(\langle 3, 7 \rangle)$ and $7 = g(\langle 3, 8 \rangle)$ respectively.

Due to the normalization theorem [32, p.5, p.68](p.5,p.68) (Theorem B.1), every complete, reduced, irreducible algebraic curve admits a normalization which is unique up to isomorphism. Although the functions $y_7$ and $y_8$ introduced are defined on different curves, the following identities are consistent because the zeroes of $f_{6,21}(x, y_7)$ and $f_{7,24}(x, y_8)$ are related,

$$
y_7y_8 = k_2(x)k_3(x), \quad y_8 = \frac{y_7^2}{(x - b_4)(x - b_5)}, \quad y_7 = \frac{y_8^2}{(x - b_1)(x - b_2)(x - b_3)}.
$$

Consistency is achieved by implementing the natural action of a primitive third root of unity $\zeta_3$ on $y_7$ and $y_8$ in $R_6$ and $R_7$: on the first relation, we make the choice $a = 0$ out of the possible $0 \leq a \leq 2$ for $y_7y_8 \mapsto \zeta_3^a k_2(x)k_3(x)$.

As normalization of these singular curves, we consider the commutative ring,

$$
R_4 := \mathbb{C}[x, y_7, y_8]/(f_{14}, f_{15}, f_{16}),
$$
and the curve Spec $R_4$. Here we define $f_{14}, f_{15}, f_{16} \in \mathbb{C}[x, y_7, y_8]$ by

$$f_{14} = y_7^2 - y_8 k_2(x), \quad f_{15} = y_7 y_8 - k_2(x) k_3(x), \quad f_{16} = y_8^2 - y_7 k_3(x),$$

which are also viewed as the $2 \times 2$ minors of

$$
\begin{pmatrix}
  k_2(x) & y_7 & y_8 \\
  y_7 & y_8 & k_3(x)
\end{pmatrix},
$$

as a deformation of \eqref{3.2}. Note that when $x$ is infinite, so are $y_7$ and $y_8$. $\mathbb{G}_m^{(4)}$ acts on $R_4$ by sending $x$ and $y_a$ to $g^{-3}x, g^{-a}y_a$ for $g \in \mathbb{G}_m^{(4)}$ and $a = 7, 8$, since $-3$ and $-a$ are the weights (meaning the valuation of the point, negative of the order of pole) of $x$ and $y_a$ at $\infty$.

Due to Nagata’s Jacobian criterion in Th. 30.10 of \cite[Theorem 30.10]{Ref}, Spec $R_4$ is non-singular except at $\infty$:

**Proposition 3.2.** For every $(x, y_7, y_8)$ where $f_{14}, f_{15}$ and $f_{16}$ simultaneously vanish,

$$\text{rank } U_4 = 2,$$

where

$$U_4 := \begin{pmatrix}
  \frac{\partial f_{14}}{\partial x} & \frac{\partial f_{14}}{\partial y_7} & \frac{\partial f_{14}}{\partial y_8} \\
  \frac{\partial f_{15}}{\partial x} & \frac{\partial f_{15}}{\partial y_7} & \frac{\partial f_{15}}{\partial y_8} \\
  \frac{\partial f_{16}}{\partial x} & \frac{\partial f_{16}}{\partial y_7} & \frac{\partial f_{16}}{\partial y_8}
\end{pmatrix}.$$

**Proof.** The matrix $U_4$ is equal to

$$U_4 = \begin{pmatrix}
y_8 k_2'(x) & y_7 & -k_2(x) \\
y_8 & y_8 & -k_2(x) \\
y_7 k_3'(x) & 2y_7 & y_7
\end{pmatrix}.$$

When $y_7$ vanishes, and in consequence $y_8$ vanishes, $x$ must be one of $\{b_a\}_{a=1, \ldots, 5}$. For $x = b_1$,

$$U_4 = \begin{pmatrix}
0 & y_8 y_7 & -k_2(b_1) \\
y_8 & y_8 & -k_2(b_1) \\
y_7 & y_7 & -k_2(b_1)
\end{pmatrix}.$$

Here $k_2(b_1)$ and $(b_1 - b_2)(b_1 - b_3)$ don’t vanish so the rank is 2. Similarly for $x = b_a a = 2, \ldots, 5$.

On the other hand, when $y_7$ and $y_8$ are nonzero,

$$
\begin{pmatrix}
1 & -1 & 1 \\
1 & -1 & 1 \\
1 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
y_8^2 y_7 y_8 \\
y_8^2 \\
y_8^2
\end{pmatrix}
\begin{pmatrix}
1 & -1 & 1 \\
1 & -1 & 1 \\
1 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
y_8^3 k_2'(x) & y_7 y_8^2 & -y_8^3 k_2(x) \\
y_8^3 k_2'(x) & y_7 y_8^2 & -y_8^3 k_2(x) \\
y_8^3 k_3'(x) & -y_8^3 k_3(x) & 2y_7 y_8^2
\end{pmatrix}
$$

shows that the rank is 2. □

Lastly, the smooth curve $X_4$ associated to the quotient field of the ring $R_4$ is obtained by completing the affine piece Spec $R_4$ by one smooth point at infinity, which we still denote by $\infty$; this can be seen by introducing a local parameter at $\infty$ as in Sec.2 of \cite[Section 2]{Ref} (see Appendix B). There is a cyclic action on $X_4$ by

$$\hat{\zeta}_3(x, y_7, y_8) = (x, \zeta_3 y_7, \zeta_3^2 y_8).$$
Proposition 3.3. There is a natural homomorphism $R_a \to R_4$ for each $(a = 6, 7)$.

Proof. The natural injection $\varphi : \mathbb{C}[x, y_7] \to \mathbb{C}[x, y_7, y_8]$ is a ring homomorphism, because the relation $\varphi^{-1}(f_{14}, f_{15}, f_{16}) = (f_{0,21})$ holds. Similarly we have the result for $a = 7$. \hfill \Box

In conclusion, the above shows that $X_4$ is the normalization of $X_6$ and $X_7$.

3.2. The Weierstrass gaps and holomorphic one forms. The Weierstrass gap sequences at $\infty$ are given by the following table. Indeed, we have $H(X_4, \infty) = H_4$. In view of the construction of the curve, functions that have the order of poles in the complement of the gaps can be given by monomials in $R_4$. We denote by $\phi_i^{(g)}$ the sequence of such monic monomials in $R_6$ for $g = 6, 7$ and we add the notational device $H^0_I$ in the subscript to signify functions, as in the group $H^0(X_4, \infty, \mathcal{O}_{X_4})$: $\phi_{R_4}^{(4)}$ for $R_4$, e.g., $\phi_{R_4}^{(4)} = 1, \phi_{R_4}^{(4)} = x, \phi_{R_4}^{(4)} = x^2, \phi_{R_4}^{(4)} = y_7, \phi_{R_4}^{(4)} = y_8, \phi_{R_4}^{(4)} = x^3$, e.g.

|       | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|
| $X_6$ |   | - | - | - | - | - | $x^2$ | $y_7$ | - | $x^3$ | $y_7$ | - | $x^4$ | $y_7$ | $x^5$ | $y_7$ | $x^3$ |
| $X_7$ |   | - | - | - | - | - | $x^2$ | $y_8$ | - | $x^3$ | $y_8$ | - | $x^4$ | $y_8$ | - | $x^5$ | $y_8$ |
| $X_4$ |   | - | - | - | - | - | $x^2$ | $y_7$ | $y_8$ | - | $x^3$ | $y_7$ | $y_8$ | - | $x^4$ | $y_7$ | $y_8^2$ |

We define the weight $N^{(g)}(n)$ and $N^{(4)}_{H_0}(n)$ by

$$N^{(4)}_{H_0}(n) = -\text{wt}(\phi_{H_0}^{(4)}), \quad N^{(g)}(n) = -\text{wt}(\phi_{G}^{(g)})$$

where $\text{wt}()$ is the negative of the order of pole at $\infty$. These are consistent with the weights corresponding to the action $\mathbb{G}_m^{(4)}$, whereas $R_6$ ($g = 6, 7$) provides the Weierstrass gap sequences associated with the semigroups $(3, 7)$ and $(3, 8)$.

The differentials of the first kind of the singular curves of $X_6$ and $X_7$ are given by

$$\nu_i^{i,6} := \frac{\phi_{i-1}^{(7)}dx}{3y_7^2}, \quad (i = 1, 2, 3, 4, 5, 6), \quad \nu_i^{i,7} := \frac{\phi_{i-1}^{(8)}dx}{3y_8^2}, \quad (i = 1, 2, 3, 4, 5, 6, 7).$$

We introduce the monomial $\phi_{R_4}^{(4)}$ of $R_4$ in the following Proposition, where we add the notational device $H^1_I$ in the subscript to indicate differentials, as in the group $H^1(X_4, \infty, \mathcal{O}_{X_4})$. The properties of these monomials will be investigated and used in Proposition 3.5, Lemma 6.3, Proposition 6.4, and by Serre duality.

Proposition 3.4. The holomorphic one forms over $X_4$ can be expressed by

$$\nu_i^{i,4} := \frac{\phi_{R_4}^{(4)}dx}{3y_7y_8}, \quad (i = 1, 2, 3, 4),$$
where the monomial $\phi_{H^i}^{(4)} \in R_4$ is defined by

$$\phi_{H^0}^{(4)} := y_7, \quad \phi_{H^1}^{(4)} := y_8, \quad \phi_{H^2}^{(4)} := xy_7, \quad \phi_{H^3}^{(4)} := xys.$$

Further we define $\phi_{H^4}^{(4)} := x^2 y_7, \phi_{H^5}^{(4)} := x^2 y_8$, and for $j > 5$, and let $\phi_{H^j}^{(4)}$ be defined as $\phi_{H^j}^{(4)} := \phi_{H^{j+5}}^{(4)}$. We also define the weight $N_{H^i}^{(4)}(n)$ by

$$N_{H^i}^{(4)}(n) = -\text{wt}(\phi_{H^i}^{(4)}),$$

namely the order of pole of $\phi_{H^i}^{(4)}$ at $\infty$, which is consistent with the action coming from $H^4_\infty$.

**Remark 3.5.** We make a remark on the nature of the canonical divisor $K_4$ of $X_4$. By writing coordinates $(x, y_7, y_8)$ for a point on $X_4$, the divisors of our basis of one-forms are:

$$\begin{align*}
(\nu_1^{I^4}) &= B_4 + B_5 + 4\infty, \\
(\nu_2^{I^4}) &= B_1 + B_2 + B_3 + 3\infty, \\
(\nu_3^{I^4}) &= \sum_{a=0}^{2} (0, \zeta_3^a \sqrt[3]{k_7(0)}, \zeta_3^{2a} \sqrt[3]{k_8(0)}) + B_4 + B_5 + \infty, \\
(\nu_4^{I^4}) &= \sum_{a=0}^{2} (0, \zeta_3^a \sqrt[3]{k_7(0)}, \zeta_3^{2a} \sqrt[3]{k_8(0)}) + B_1 + B_2 + B_3,
\end{align*}$$

where $B_a := (b_a, 0, 0)$ ($a = 1, 2, 3, 4, 5$). Looking at the divisor of $(\nu_1^{I^4})$ we see that $B_4, B_5$ and $\infty$ correspond to $P_1$ and $P_2$ and $P$ in Proposition 2.1. We should note that we have the following linear equivalences which play an important role in subsection 6.1:

$$\begin{align*}
K_{X_4} &\sim 2(3\infty - (B_4 + B_5 - 2\infty)) \\
&\sim 2(3\infty - (B_1 + B_2 + B_3 - 3\infty)) \\
&\sim 6\infty - (B_1 + B_2 + B_3 + B_4 + B_5 - 5\infty)
\end{align*}$$

deduced by using

$$\begin{align*}
B_1 + B_2 + B_3 + B_4 + B_5 - 5\infty &\sim -(B_4 + B_5 - 2\infty) \\
&\sim 2(B_4 + B_5 - 2\infty) \\
&\sim -(B_1 + B_2 + B_3 - 3\infty) \\
&\sim 2(B_1 + B_2 + B_3 - 3\infty).
\end{align*}$$

In fact, any positive canonical divisor must include points in $X_4 \setminus \infty$. This is contrast to the fact that for every $(n, s)$ curve, there is a canonical divisor given by $(2g - 2)\infty$, cf. [51]; recall (letting $n = r$ as in this reference) that $g = (r - 1)(s - 1)/2$. This is the reason we used the monomials $\phi_{H^i}^{(4)}$ besides the $\phi_{H^0}^{(4)}$’s.

The following proposition is self-evident:
Proposition 3.6. For $n > 0$, \( \sum_{i=0}^{n} a_i \frac{x^i dx}{y \gamma y^8} \) belongs to \( H^0(X_4 \setminus \infty, \Omega_{X_4}) \), for any constants \( a_i \), in particular to \( H^0(X_4, \Omega_{X_4}) \) for \( n \leq 4 \), where \( \Omega_{X_4} \) is the sheaf of holomorphic one-forms.

Lemma 3.7. For a non-negative integer \( n < 5 \), if the one form,
\[
\sum_{i=0}^{n} a_i \frac{x^i dx}{k_2(x)k_3(x)},
\]
is holomorphic over \( X_4 \), then each \( a_i \) vanishes.

Proof. For \( n < 5 \), every term in \( \sum_{i=0}^{n} a_i \frac{x^i dx}{y \gamma y^8} \) has singularities of different valences at points in \( X_4 \setminus \infty \).

By letting
\[
\Lambda_i^{(g)} = N^{(g)}(g) - N^{(g)}(i - 1) - g + i - 1, \quad \Lambda_i^{(4)} = N_H^{(4)}(4) - N^{(4)}(i - 1) - 5 + i,
\]
the related Young diagrams \( Y_4 = (\Lambda_1^{(4)}, \Lambda_2^{(4)}, \Lambda_3^{(4)}, \Lambda_4^{(4)}) \), \( Y_6 = (\Lambda_1^{(6)}, \Lambda_2^{(6)}, \cdots, \Lambda_6^{(6)}) \) and \( Y_7 = (\Lambda_1^{(7)}, \Lambda_2^{(7)}, \cdots, \Lambda_7^{(7)}) \) are given by

Since the Young diagram \( Y_4 \) is not symmetric, the semigroup \( H_4 \) is also not symmetric, where a numerical semigroup is symmetric if and only if \( 2g - 1 \) occurs in the gap sequence \([64]\). We should note that the elements of a minimal set of generators, \( \alpha_{g-i}(L) \) in (2.1), correspond to \( \Lambda_i^{(g)} \) for each \( X_g \) (\( g = 4, 6, 7 \)).

Also self-evident is the following:

Lemma 3.8. By using the same letters in source and target under the homomorphism of Proposition [3,3] (as well as identifying a coset by a standard representative), the holomorphic one forms \( \nu_i^{(4)} \) over \( X_4 \) satisfy:
\[
\nu_i^{(4)} = \tilde{\nu}_{i+3}^{(4)} = \tilde{\nu}_{i+4}^{(4)} \quad (i = 1, 2, 3, 4),
\]
where
\[
\tilde{\nu}_i^{(4)} := \nu_i^{(4)}, \quad (i = 4, 6), \quad \tilde{\nu}_3^{(4)} := \nu_3^{(4)} + \lambda_1^{(2)} \nu_2^{(4)} + \lambda_2^{(2)} \nu_1^{(4)}, \quad \tilde{\nu}_5^{(4)} := \nu_5^{(4)} + \lambda_1^{(2)} \nu_4^{(4)} + \lambda_2^{(2)} \nu_2^{(4)},
\]
\[
\tilde{\nu}_4^{(7)} := \nu_4^{(7)}, \quad (i = 4, 6), \quad \tilde{\nu}_5^{(7)} := \nu_5^{(7)} + \lambda_1^{(3)} \nu_3^{(7)} + \lambda_2^{(3)} \nu_2^{(7)} + \lambda_3^{(3)} \nu_1^{(7)}, \quad \tilde{\nu}_7^{(7)} := \nu_7^{(7)} + \lambda_1^{(3)} \nu_5^{(7)} + \lambda_2^{(3)} \nu_4^{(7)} + \lambda_3^{(3)} \nu_2^{(7)}.\]
As is standard practice, we choose a basis \( \alpha_i^{(4)} \), \( \beta_j^{(4)} \) \((1 \leq i, j \leq 4)\) of \( H_1(X, \mathbb{Z}) \) such that the intersection numbers are \( \alpha_i^{(4)} \cdot \alpha_j^{(4)} = \beta_i^{(4)} \cdot \beta_j^{(4)} = 0 \) and \( \alpha_i^{(4)} \cdot \beta_j^{(4)} = \delta_{ij} \), and we denote the period matrices by

\[
\left[ \omega^{(4)\nu} \omega^{(4)\mu} \right] = \frac{1}{2} \left[ \int_{\alpha_i^{(4)}} \nu^{(4)}_j \int_{\beta_i^{(4)}} \nu^{(4)}_j \right]_{i, j = 1, 2, \ldots, 4}.
\]

Let \( \Lambda_4 \) be a lattice generated by \( \omega^{(4)\nu} \) and \( \omega^{(4)\mu} \).

For a point \( P \in X_4 \), we define the Abel map,

\[
\hat{u}_o(P) := \int_{\infty}^{P} \nu^{(4)} \in \mathbb{C}^4,
\]

and for \( k \) points \( P_1, P_2, \ldots, P_k \in X_4 \), also a shifted Abel map

\[
\hat{u}(P_1, \ldots, P_k) := \hat{u}_o(P_1, \ldots, P_k) + \hat{u}_o(B_4, B_5),
\]

where \( \hat{u}_o(P_1, \ldots, P_k) := \sum_{i=1}^{k} \hat{u}_o(P_i) \). As shown in (6.2), the shift is devised so that the \((-1)\)-operation on a complex vector \( \hat{u}(P_1, \ldots, P_k) \) is consistent with equivalence of divisors on the curve. The “\( u \)” are not really functions of ordered \( k \)-tuples of points, since one needs to choose a path of integration, but it is conventional (and harmless) to use this abbreviated notation.

Then we obtain the Jacobian \( J_4 \) by

\[
\kappa : \mathbb{C}^4 \to J_4 = \mathbb{C}^4 / \Lambda_4,
\]

and the subvariety \( \mathcal{W}^{(4)k} \) by

\[
\mathcal{W}^{(4)k} := \kappa \hat{u}(S^k X_4).
\]

Hereafter we use the convention that for \( P_a \in X_4 \), \( P_a \) is expressed by \( (x_a, y_{7, a}, y_{8, a}) \) or \( (x_{P_a}, y_{7, P_a}, y_{8, P_a}) \). By letting \( u := \hat{u}(P_1, \ldots, P_4) \), we have

\[
\begin{pmatrix}
\frac{\partial}{\partial u_1} \\
\frac{\partial}{\partial u_2} \\
\frac{\partial}{\partial u_3} \\
\frac{\partial}{\partial u_4}
\end{pmatrix}
\Psi_4^{(4)}
\begin{pmatrix}
3 y_{7,1} y_{8,1} \partial / \partial x_1 \\
3 y_{7,2} y_{8,2} \partial / \partial x_2 \\
3 y_{7,3} y_{8,3} \partial / \partial x_3 \\
3 y_{7,4} y_{8,4} \partial / \partial x_4
\end{pmatrix},
\]

where

\[
\Psi_4^{(4)} := 
\begin{pmatrix}
\phi_{H_1}^{(4)}(P_1) & \phi_{H_1}^{(4)}(P_1) & \phi_{H_2}^{(4)}(P_1) & \phi_{H_3}^{(4)}(P_1) \\
\phi_{H_1}^{(4)}(P_2) & \phi_{H_1}^{(4)}(P_2) & \phi_{H_2}^{(4)}(P_2) & \phi_{H_3}^{(4)}(P_2) \\
\phi_{H_1}^{(4)}(P_3) & \phi_{H_1}^{(4)}(P_3) & \phi_{H_2}^{(4)}(P_3) & \phi_{H_3}^{(4)}(P_3) \\
\phi_{H_1}^{(4)}(P_4) & \phi_{H_1}^{(4)}(P_4) & \phi_{H_2}^{(4)}(P_4) & \phi_{H_3}^{(4)}(P_4)
\end{pmatrix}.
\]
In other words,
\[
\sum_{i=1}^{4} \epsilon_i \partial_{\epsilon_i} = \left| \Psi_4^{(4)} \right|^{-1} \begin{vmatrix}
\phi_{H^0_1}(P_1) & \phi_{H^1_1}(P_1) & \phi_{H^2_1}(P_1) & \phi_{H^3_1}(P_1) & 3y_7,1y_8,1\partial/\partial x_1 \\
\phi_{H^0_2}(P_2) & \phi_{H^1_2}(P_2) & \phi_{H^2_2}(P_2) & \phi_{H^3_2}(P_2) & 3y_7,2y_8,2\partial/\partial x_2 \\
\phi_{H^0_3}(P_3) & \phi_{H^1_3}(P_3) & \phi_{H^2_3}(P_3) & \phi_{H^3_3}(P_3) & 3y_7,3y_8,3\partial/\partial x_3 \\
\phi_{H^0_4}(P_4) & \phi_{H^1_4}(P_4) & \phi_{H^2_4}(P_4) & \phi_{H^3_4}(P_4) & 3y_7,4y_8,4\partial/\partial x_4 \\
\epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 & 0
\end{vmatrix}.
\]

As these relations hold for the image of the Abel map, we also have similar relations for a stratum \(\kappa\mu(S^k X_4)\) when \(k < 4\) using the submatrices of \(\Psi_4^{(4)}\) as in [52]. Further we have a natural relation:
\[
\sum_{i,j=1}^{4} \epsilon_{i,j}^{(4)} \phi_{H^{i,j-1}}^{(4)}(P_1) \phi_{H^{i,j}}^{(4)}(P_2) \frac{\partial^2}{\partial \epsilon_i \partial \epsilon_j} = 9y_7,1y_8,1y_7,2y_8,2 \frac{\partial^2}{\partial x_1 \partial x_2}.
\]

### 3.3. Differentials of the second and the third kinds.
Following the EEL-construction [17, 7] for \((n, s)\) curves, we produce an algebraic representation of a differential form which, up to a tensor of holomorphic one-forms, is equal to the fundamental normalized differential of the second kind in Cor.2.6 of [25, Corollary 2.6]. We extend the EEL-construction to the space curve \(X_4\).

**Definition 3.9.** A two-form \(\Omega^{(4)}(P_1, P_2)\) on \(X_4 \times X_4\) is called a fundamental differential of the second kind if it is symmetric,
\[
\Omega^{(4)}(P_1, P_2) = \Omega^{(4)}(P_2, P_1),
\]
has its only pole (of second order) along the diagonal of \(X_4 \times X_4\), and in the vicinity of each point \((P_1, P_2)\) is expanded in power series as
\[
\Omega^{(4)}(P_1, P_2) = \left( \frac{1}{(t_{P_1} - t_{P_2})^2} + d_{\geq 1}(1) \right) dt_{P_1} \otimes dt_{P_2} \quad (\text{as } P_1 \to P_2),
\]
where \(t_P\) is a local coordinate at the point \(P \in X_4\).

**Proposition 3.10.** Let \(\Sigma^{(4)}(P_1, P_2)\) be the following form,
\[
\Sigma^{(4)}(P, Q) := \frac{y_7,P y_8, P + y_7, P y_8, Q + y_7, Q y_8, P}{(x_P - x_Q)3y_7,P y_8, P} dx_P.
\]
Then \(\Sigma^{(4)}(P, Q)\) has the properties
1. \(\Sigma^{(4)}(P, Q)\) as a function of \(P\) is singular at \(Q = (x_Q, y_7,Q, y_8,Q)\) and \(\infty\), and is not singular at \(\zeta_3(Q) = (x_Q, \zeta_3 y_7,Q, \zeta_3^2 y_8,Q)\).
2. \(\Sigma^{(4)}(P, Q)\) as a function of \(Q\) is singular at \(P\) and \(\infty\).

**Proof.** Direct computation. □

The following holds for non-singular \(X_4\):
Proposition 3.11. There exist differentials $\nu_{j}^{I:4} = \nu_{j}^{I:4}(x, y)$ ($j = 1, 2, \ldots, 4$) of the second kind such that they have a simple pole at $\infty$ and satisfy the relation,

$$d_Q \Sigma^{(4)}(P, Q) - d_P \Sigma^{(4)}(Q, P)$$

$$= \sum_{i=1}^{4} \left( \nu_{i}^{I:4}(Q) \otimes \nu_{i}^{I:4}(P) - \nu_{i}^{I:4}(P) \otimes \nu_{i}^{I:4}(Q) \right),$$

where

$$d_Q \Sigma^{(4)}(P, Q) := dx_P \otimes dx_Q \frac{\partial}{\partial x_Q} \frac{y_{7, P} y_{8, P} + y_{7, P} y_{8, Q} + y_{7, Q} y_{8, P}}{(x_P - x_Q)^3 y_{7, P} y_{8, P}} dx_P.$$ 

The set of differentials $\{\nu_{1}^{I:4}, \nu_{2}^{I:4}, \nu_{3}^{I:4}, \nu_{4}^{I:4}\}$ is determined modulo the linear space spanned by $\langle \nu_{j}^{I:4} \rangle_{j=1, \ldots, 4}$ and it has representatives

$$\nu_{4}^{I:4} = \frac{-x^2 dx}{3 y_8},$$

$$\nu_{3}^{I:4} = \frac{-(2x^2 + \lambda^2(2) x) dx}{3 y_7},$$

$$\nu_{2}^{I:4} = \frac{-\left(4x^3 + (3\lambda^3(1) + 2\lambda^2(2))x^2 + (2\lambda^2(3) + \lambda^2(2)\lambda^2(3))x + \lambda^2(3)\right) dx}{3 y_8},$$

$$\nu_{1}^{I:4} = \frac{-\left(5x^3 + (3\lambda^3(1) + 4\lambda^2(2))x^2 + (\lambda^2(3) + 2\lambda^2(2)\lambda^3(3) + 3\lambda^2(2))x + \lambda^2(2)\lambda^3(3)\right) dx}{3 y_7}.$$

We will henceforth use this basis $\nu_{i}^{I:4}$.

Proof.

$$\frac{\partial}{\partial x_Q} \frac{y_{7, P} y_{8, P} + y_{7, P} y_{8, Q} + y_{7, Q} y_{8, P}}{(x_P - x_Q)^3 y_{7, P} y_{8, P}} dx_P$$

$$= \frac{1}{(x_P - x_Q)^9 y_{7, P} y_{8, P} y_{7, Q} y_{8, Q}} \left[ \frac{3(y_{7, P} y_{8, P} + y_{7, P} y_{8, Q} + y_{7, Q} y_{8, P}) y_{7, Q} y_{8, Q}}{(x_P - x_Q)^2} \right. + \left. \left( y_{7, P} \frac{y_{7, Q} y_{8, Q}}{y_{7, Q}} (2k_{3, Q} k_{3, Q} k_{2, Q} + k_{3, Q}^2 k_{2, Q}^2) - y_{8, P} \frac{y_{8, Q}}{y_{7, Q}} (2k_{3, Q} k_{2, Q} k_{2, Q} + k_{3, Q} k_{2, Q}^2) \right) \right].$$

Here we set $k_{a, P} = k_{a}(x_P)$ and $k_{a, P}' = dk_{a}(x_P)/dx_P$. The result follows from the equalities:

$$\frac{\partial}{\partial x_Q} \frac{y_{7, P} y_{8, P} + y_{7, P} y_{8, Q} + y_{7, Q} y_{8, P}}{(x_P - x_Q)^3 y_{7, P} y_{8, P}} = \frac{\partial}{\partial x_P} \frac{y_{7, Q} y_{8, Q} + y_{7, Q} y_{8, P} + y_{7, P} y_{8, Q}}{(x_Q - x_P)^3 y_{7, Q} y_{8, Q}}.$$
where monomial representatives for elements of Proposition 3.11, gives the result. □

Lemma 3.13. We have

\[
\lim_{P \to \infty} \frac{F^{(4)}(P_1, P_2)}{\phi^{(4)}_{H^3}(P_1)(x_1 - x_2)^2} = \phi^{(4)}_{H^4}(P_2) = x_P y_T P_2.
\]

Proof. A calculation, using the formula for \( F^{(4)} \) from [3, 12] and the expression for \( A_4 \) in the proof of Proposition 3.11 gives the result. □

Hereafter for \( P = (x, y_T, y_8) \), we expand \( h_i(P) = 3y_T y_8 \nu^{H^4}_i(P)/dx \), \( 1 \leq i \leq g \), in the monomial representatives for elements of \( R_4 \) to avoid carrying holomorphic one-forms.
For later convenience we introduce the notation:

\[
\Omega^{(4)}_{Q_1, Q_2} := \int_{P_2}^{P_1} \int_{Q_2}^{Q_1} \Omega^{(4)}(P, Q)
\]

(3.17)

\[= \int_{P_2}^{P_1} (\Sigma^{(4)}(P, Q_1) - \Sigma^{(4)}(P, Q_2)) + \sum_{i=1}^{4} \int_{P_2}^{P_1} \nu_1^{I_4}(P) \int_{Q_2}^{Q_1} \nu_1^{II_4}(P).\]

4. Semi-group \(H = \langle 6, 13, 14, 15, 16 \rangle\)

In this section, we investigate a non-singular curve whose Weierstrass semigroup at one point \(\infty\) is the numerical semigroup \(H_{12}\) generated by \(M_{12} = \{6, 13, 14, 15, 16\}\). Again we follow Pinkham’s program, deform the local equations of a smooth affine curve with a given monomial ring at \(\infty\), and use Komeda’s result to ensure that there exist a global curve in the deformation space.

4.1. The curve as a monomial curve. The proof of the following Proposition is contributed by Komeda.

**Proposition 4.1.** Let \(B_{H_{12}}\) be the monomial ring which is given by \(k[t^a]_{a \in M_{12}}\) for the numerical semigroup \(H_{12}\). For the \(k\)-algebra homomorphism,

\[\varphi_{12} : k[Z] := k[Z_6, Z_{13}, Z_{14}, Z_{15}, Z_{16}] \to k[t^a]_{a \in M_{12}}, Z_a \mapsto t^a,\]

where \(Z_a\) has weight \(a = 6, 13, 14, 15, 16\), the kernel of \(\varphi_{12}\) is generated by the following relations \(f_{12}^{(Z)}(b = 1, \ldots, 9),\)

\[
\begin{align*}
    f_{12, 1}^{(Z)} &= Z_{13}^2 - Z_6^2 Z_{14}, & f_{12, 2}^{(Z)} &= Z_{13} Z_{14} - Z_6^2 Z_{15}, & f_{12, 3}^{(Z)} &= Z_{14}^2 - Z_{13} Z_{15}, \\
    f_{12, 4}^{(Z)} &= Z_{14}^2 - Z_6^2 Z_{16}, & f_{12, 5}^{(Z)} &= Z_{13} Z_{16} - Z_{14} Z_{15}, & f_{12, 6}^{(Z)} &= Z_{15}^2 - Z_{16}^2, \\
    f_{12, 7}^{(Z)} &= Z_{14} Z_{16} - Z_6^2, & f_{12, 8}^{(Z)} &= Z_{15} Z_{16} - Z_6^2 Z_{13}, & f_{12, 9}^{(Z)} &= Z_{16}^2 - Z_6^2 Z_{14}.
\end{align*}
\]

**Proof.** We set \(a_1 = 6, a_2 = 13, a_3 = 14, a_4 = 15\) and \(a_5 = 16\). Let \(\varphi_{12} : k[Z_6, Z_{13}, Z_{14}, Z_{15}, Z_{16}] \to k[H_{12}] = k[t^a]_{a \in H_{12}}\) be the \(k\)-algebra homomorphism which sends \(Z_i\) to \(t^{a_i}\) for each \(i\). Then the ideal \(I = \ker \varphi\) is generated by

\[
\begin{align*}
    f_{12, 1}^{(Z)} &= Z_{13}^2 - Z_6^2 Z_{14}, & f_{12, 2}^{(Z)} &= Z_{13} Z_{14} - Z_6^2 Z_{15}, & f_{12, 3}^{(Z)} &= Z_{14}^2 - Z_{13} Z_{15}, \\
    f_{12, 4}^{(Z)} &= Z_{14}^2 - Z_6^2 Z_{16}, & f_{12, 5}^{(Z)} &= Z_{13} Z_{16} - Z_{14} Z_{15}, & f_{12, 6}^{(Z)} &= Z_{15}^2 - Z_{16}^2, \\
    f_{12, 7}^{(Z)} &= Z_{14} Z_{16} - Z_6^2, & f_{12, 8}^{(Z)} &= Z_{15} Z_{16} - Z_6^2 Z_{13}, & f_{12, 9}^{(Z)} &= Z_{16}^2 - Z_6^2 Z_{14}.
\end{align*}
\]

We set \(\alpha_i = \min \{ \alpha \in \mathbb{N}_0 \mid \alpha > 0, \alpha a_i \in \langle a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_5 \rangle \} \). Then it is easy to check that \(\alpha_1 = 5, \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 2\) and

\[5a_1 = 2a_2, 2a_2 = 2a_1 + a_3, 2a_4 = 2a_6 + a_5, 2a_4 = a_3 + a_5, 2a_5 = a_1 + 2a_2.\]
We look for all identities of type:

$$\beta_i a_i + \beta_j a_j = \gamma_k a_k + \gamma_\ell a_\ell + \gamma_m a_m$$

with $0 < \beta_i < \alpha_i$, $0 < \beta_j < \alpha_j$, $0 < \gamma_k < \alpha_k$, $0 < \gamma_\ell < \alpha_\ell$ and $0 \leq \gamma_m < \alpha_m$. Then we get the following four relations:

$$a_2 + a_3 = 2a_1 + a_4, \quad a_2 + a_4 = 2a_1 + a_5, \quad a_2 + a_5 = a_3 + a_4, \quad a_4 + a_5 = 3a_1 + a_3.$$

It is easy to check that the polynomials given in the statement belong to $I$. Let $J$ be the ideal generated by the above nine polynomials. To prove $I = J$ it suffices to show the following: If

$$f = \prod_i Z_{a_i}^{\beta_i} - \prod_i Z_{a_i}^{\gamma_i} \in \ker \varphi \text{ with } \beta_i \gamma_i = 0, \text{ every } i,$$

then $f \in J$.

**Case 1.** Let $Z_{a_i}^{\beta_i} - Z_{a_i}^{\gamma_i} Z_{a_k}^{\gamma_k} Z_{a_\ell}^{\gamma_\ell} Z_{a_m}^{\gamma_m} \in I$ with $\beta_i \geq \alpha_i$, $\gamma_j > 0$, $\gamma_k \geq 0$, $\gamma_\ell \geq 0$ and $\gamma_m \geq 0$.

Let $i = 6$. We have

$$f = Z_6^{\beta_1-5}(-f_{12,6}) + Z_6^{\beta_2-5}Z_{15}^2 - Z_{a_j}^{\gamma_j} M \equiv Z_6^{\beta_1-5}Z_{15} - Z_{a_j}^{\gamma_j} M \mod J,$$

where $M$ is some monomial. Hence, if $j = 4$, then we may decrease the degree of $f$. If $j = 3$ or $5$, using $f_{12,6} - f_{12,7}$ we may decrease the degree of $f$. We may assume that $f = Z_6^{\beta_1} - Z_1^{\gamma_2}$ with $\gamma_2 \geq \alpha_2$. Hence, we have

$$f = Z_6^{\beta_1} - Z_{13}^{\gamma_2-2} f_{12,1} - Z_{13}^{\gamma_2-2} Z_6^2 Z_{14} \equiv Z_6^{\beta_1} - Z_{13}^{\gamma_2-2} Z_6^2 Z_{14} \mod J,$$

which implies that we may decrease the degree of $f$.

Let $i = 2$. We have

$$Z_{13}^{\beta_2} = Z_{13}^{\beta_2-2} f_{12,1} + Z_{13}^{\beta_2-2} Z_6^2 Z_{14}.$$

Hence, if $j = 1$ or $3$, then we may decrease the degree of $f$. Let $f = Z_{13}^{\beta_2} - Z_{15}^{\gamma_4} Z_{16}^{\gamma_5}$. Using $f_{12,8} \in J$ we may assume that $\gamma_4 \geq 2$ or $\gamma_5 \geq 2$. If $\gamma_5 \geq 2$, by $f_{12,9} \in J$ we may decrease the degree of $f$. Let $\gamma_4 \geq 2$. Then

$$f = Z_{13}^{\beta_2-2} f_{12,1} + Z_{13}^{\beta_2-2} Z_6^2 Z_{14} - Z_{15}^{\gamma_4-2}(f_{12,6} - f_{12,7}) Z_{16} = Z_{15}^{\gamma_4-2} Z_{14} Z_{16}^{\gamma_5+1}.$$

Hence, we may decrease the degree of $f$.

Let $i = 4$. Since we obtain

$$Z_{14}^{\beta_3} = Z_{14}^{\beta_3-2} f_{12,4} + Z_{14}^2 Z_{14}^{\beta_3-2} Z_{16} = Z_{14}^{\beta_3-2} f_{12,3} + Z_{13} Z_{14}^{\beta_3-2} Z_{15},$$

which implies that we may decrease the degree of $f$.

Let $i = 4$. In view of

$$Z_{15}^{\beta_4} = Z_{15}^{\beta_4-2}(f_{12,6} - f_{12,7}) + Z_{14} Z_{15}^{\beta_4-2} Z_{16} = Z_{15}^{\beta_4-2} f_{12,6} + Z_6^2 Z_{15}^{\beta_4-2},$$

we may assume that $f = Z_{15}^{\beta_4} - Z_{13}^{\gamma_2}$. Hence, we get

$$f = Z_{15}^{\beta_4-2} f_{12,6} - Z_{13}^{\gamma_2-2} f_{12,1} + Z_6 Z_{15}^{\beta_4-2} - Z_6^2 Z_{13}^{\gamma_2-2} Z_{14}.$$

We may decrease the degree of $f$. 
Let $i = 16$. Since we have
\[ Z_1^{\beta_5} = Z_1^{\beta_5 - 2} f_{12,9}^{(Z)} + Z_6 Z_{13}^{\gamma_5} Z_{16}^{\beta_5 - 2}, \]
we may assume that $f = Z_1^{\beta_5} - Z_1^{\gamma_5} Z_{16}^{5}$. In view of $f_{12,5} \in J$ we may assume that $\gamma_{14} \geq 2$ or $\gamma_4 \geq 2$. By $f_{12,4}$ and $f_{12,6} - f_{12,7} \in J$ we may decrease the degree of $f$.

*Case 2.* Let $f = Z_{a_1}^{\beta_5} Z_{a_j}^{\gamma_k} Z_{a_m}^{\gamma_m} \in I$ with $\beta_i > 0$, $\beta_j > 0$, $\gamma_k > 0$, $\gamma_\ell > 0$ and $\gamma_m \geq 0$. By the proof of Case 1 we may assume that $0 < \beta_i < \alpha_i$, $0 < \beta_j < \alpha_j$, $0 < \gamma_k < \alpha_k$, $0 < \gamma_\ell < \alpha_\ell$ and $0 \leq \gamma_m < \alpha_m$. In this case $f$ is one of the four polynomials $f_{12,2}$, $f_{12,4} - f_{12,3}$, $f_{12,5}$ and $f_{12,8}$.

We label $G_{m_{12}}^{(12)}$ the action of $G_m$ on $B_{H_{12}} \simeq k[Z]/\ker \varphi_{12}$ so that for $g \in G_{m_{12}}^{(12)}$ $Z_a \mapsto g^a Z_a$ and each $a$ agrees with the weight.

The relations listed in Proposition 4.1 are given by the $2 \times 2$ minors of
\[
\begin{vmatrix}
Z_6^2 & Z_{14} & Z_{16} \\
Z_{14} & Z_{16} & Z_6^2
\end{vmatrix},
\begin{vmatrix}
Z_6^2 & Z_{13} & Z_{14} & Z_{15} \\
Z_{13} & Z_{14} & Z_{15} & Z_{16}
\end{vmatrix},
\]
although $Z_{13} Z_{15} - Z_6^2 Z_{16}$ is not listed, and $f_{12,8}$ is not one of the minors they are compatible: for example, the given minor follows by combining $f_{12,8}$ with $f_{12,1}$ and $f_{12,9}$. We note that the first matrix is the same as \( \text{(3.2)} \). Thus, we reprise the notation used in Section 3.

We construct a non-singular curve $X_{12}$ by giving an affine patch, an ideal in the ring $\mathbb{C}[x, y_{13}, y_{14}, y_{15}, y_{16}]$. For any complex numbers $b_0$ and $b_1$ distinct from the previous $b$'s, we let
\[
\hat{k}_2(x) := (x - b_6)(x - b_7) = x^2 + \hat{k}_2(x), \quad \hat{k}_5(x) := \hat{k}_2(x) k_3(x),
\]
\[
k_{13}(x) := k_3(x) k_2(x)^2 k_2(x)^3, \quad k_{14}(x) := k_7(x)^2 = k_3(x)^2 k_2(x)^4,
\]
\[
k_{15}(x) := k_5(x)^3, \quad k_{16}(x) := k_8(x)^2 = k_3(x)^4 k_2(x)^2.
\]

Let the prime ideal $P$ in $\mathbb{C}[x, y_{13}, y_{14}, y_{15}, y_{16}]$ be defined by
\[
P := (f_{12,1}, f_{12,2}, f_{12,3}, f_{12,4}, f_{12,5}, f_{12,6}, f_{12,7}, f_{12,8}, f_{12,9}),
\]
where
\[
f_{12,1} := y_{13}^2 - \hat{k}_2(x) y_{14}, \quad f_{12,2} := y_{13} y_{14} - k_2(x) y_{15}, \quad f_{12,3} := \hat{k}_2(x) y_{14}^2 - y_{13} y_{15} k_2(x),
\]
\[
f_{12,4} := y_{14}^2 - k_2(x) y_{16}, \quad f_{12,5} := y_{13} y_{16} - y_{14} y_{15}, \quad f_{12,6} := y_{15}^2 - \hat{k}_2(x) k_3(x),
\]
\[
f_{12,7} := y_{14} y_{16} - k_2(x) k_3(x), \quad f_{12,8} := y_{15} y_{16} - k_3(x) y_{15}, \quad f_{12,9} := y_{16}^2 - k_3(x) y_{14},
\]
which are the $2 \times 2$ minors of
\[
\begin{vmatrix}
k_2(x) & y_{14} & y_{16} \\
y_{14} & y_{16} & k_3(x)
\end{vmatrix}, \quad \begin{vmatrix}
\hat{k}_2(x) & y_{13} & y_{14} \\
y_{13} & y_{14} & \hat{k}_2(x)
\end{vmatrix}, \quad \begin{vmatrix}
\hat{k}_2(x) & y_{13} & y_{15} \\
y_{13} & y_{15} & \hat{k}_2(x)
\end{vmatrix}.
\]

\[\]
again, the minor $y_{13}y_{15} - \hat{k}_2(x)y_{16}$ is not in the list of $f_{i,j}$ and $f_{12,8}$ is not a minor, but they are compatible—the minor follows by combining $f_{12,8}$ with $f_{12,1}$ and $f_{12,9}$. Here the first matrix is the same as (3.4) and the latter is related to the double covering of $X_4$. In other words, we essentially identify $y_a$ and $y_{2a}$ for $a = 7, 8$. There is only one point at infinity in this affine model because, as is clear from the equations, $x$ approaches $\infty$ if and only if each $y_b$ ($b = 13, 14, 15, 16$) approaches $\infty$. We define the $G_m^{(12)}$ action on $x$ and $y_a$ by $g_m^{-6}x$ and $g_m^{-a}y_a$ ($a = 13, 14, 15, 16$) near $\infty \in X_{12}$.

Corresponding to Proposition 4.1 we will consider a commutative ring,

$$R_{12} = \mathbb{C}[x, y_{13}, y_{14}, y_{15}, y_{16}]/\mathcal{P}.$$  

Proposition 4.2. There is a ring homomorphism $R_4 \rightarrow R_{12}$.

Following Nagata’s Jacobian criterion in Th.30.10 of [47, Theorem 30.10], we show that Spec$R_{12}$ is non-singular.

Proposition 4.3. For every $(x, y_{13}, y_{14}, y_{15}, y_{16})$ which is a zero of every $(f_{12,a})_{a=1,\ldots,9}$, we have

$$\text{rank } \mathcal{U}_{12} = 4, \quad \mathcal{U}_{12} := \left( \frac{\partial}{\partial x} f_{12,a} \quad \frac{\partial}{\partial y_{13}} f_{12,a} \quad \frac{\partial}{\partial y_{14}} f_{12,a} \quad \frac{\partial}{\partial y_{15}} f_{12,a} \quad \frac{\partial}{\partial y_{16}} f_{12,a} \right)_{a=1,\ldots,9}.$$  

Proof. $\mathcal{U}_{12}$ is

$$\begin{pmatrix}
-\hat{k}'_2 y_{14} & 2y_{13} & -\hat{k}_2 \\
-\hat{k}'_2 y_{15} & y_{14} & y_{13} & -k_2 \\
\hat{k}'_2 y_{14} - y_{13}y_{15}k'_2 & -y_{15}k_2 & 2\hat{k}_2 y_{14} & -y_{13}k_2 \\
-k'_2 y_{16} & y_{16} & -y_{15} & y_{13} & 2y_{15} \\
-(\hat{k}_2 k_3)' & y_{16} & y_{14} & 2y_{15} \\
-k'_2 y_{13} & -k_3 & y_{16} & y_{15} & 2y_{16} \\
-k'_2 y_{14} & -k_3 & y_{16} & 2y_{16} & 2y_{16}
\end{pmatrix}.$$  

When $x = b_1$ as a zero of $k_3(x)$, every $y_a$ vanishes and thus

$$\begin{pmatrix}
\hat{k}_2(b_1) & -k_2(b_1) \\
-k_2(b_1) & 2y_{15} \\
-k_2(b_1)(b_1 - b_2)(b_1 - b_3) & 2y_{15}
\end{pmatrix}.$$
is evidently a matrix of rank 4. When \(x = b_4\) as a zero of \(k_2(x)\), \(y_{13}, y_{14}\) and \(y_{16}\) vanish and \(U_{12}\) is equal to

\[
\begin{pmatrix}
-(b_4 - b_5) y_{15} & -k_2(b_4) \\
y_{15} & -y_{15} \\
-(k_2 k_3)'(b_1) & -k_3(b_4) \\
-(b_4 - b_5) k_3(b_4) & 2y_{15}
\end{pmatrix}
\]

whose rank is obviously 4. When \(x = b_6\) as a zero of \(\hat{k}_2(x)\), \(y_{13}\) and \(y_{15}\) vanish and thus the matrix is

\[
\begin{pmatrix}
-(b_6 - b_7) y_{14} & y_{14} & -k_2(b_6) \\
(b_6 - b_7) y_{14}^2 & 2y_{14} & k_2(b_6) \\
-k_2(b_6)' y_{16} & y_{16} & y_{14} \\
-(b_6 - b_7) k_3(b_6) & -k_3 & y_{16} \\
-(k_2 k_3)'(b_6) & -k_3(b_6) & 2y_{16}
\end{pmatrix}
\]

Since \(\frac{y_{14}}{k_3(b_6)} = \frac{y_{16}}{y_{14}} = \frac{k_3(b_6)}{y_{16}} = \frac{y_{14}}{k_2(b_6)}\), the rank is 4. When \(x\) differs from \(b_1, b_2, \ldots, b_7\), we compute

\[
\begin{pmatrix}
1 & 1 & -y_{15}/y_{16} & -y_{14}/y_{16} \\
y_{14} & -y_{13} & 1 & 1 \\
-k_3/y_{14} & y_{15}/y_{14} & 1 & -\hat{k}_2/y_{13} \\
-k_3/y_{15} & -y_{16}/y_{13} & 1 & -y_{14}/y_{13}
\end{pmatrix}
\]

\(U_{12}\),
which is equal to a matrix of rank 4,
\[
\begin{pmatrix}
-k'_2 y_{14} & 2 y_{13} & -k'_2 \\
-k'_2 y_{16} & 2 y_{14} & k_2 \\
-y_{16} & -y_{15} & -y_{14} & y_{13} \\
-k'_3 y_{13} & -k_3 & y_{16} & y_{15}
\end{pmatrix}.
\]

\[\square\]

There is a Riemann surface, which we denote \(X_{12}\), obtained from the affine smooth curve given by \(\text{Spec}(\mathbb{C}[x, y_{13}, y_{14}, y_{15}, y_{16}])\) by adding one point \(\infty\) (see [3]).

In the local ring of the place \(\infty\), we express \(x\) and \(y\)'s as
\[
y_a = \frac{1}{t_{12+a}}(1 + \cdots), \quad (a = 1, 2, 3, 4), \quad x = \frac{1}{t^6},
\]
using a local parameter \(t\) at \(\infty\).

For later use, we introduce the following ring,
\[
\hat{R}_{12} := \mathbb{C}[x, w_3, w_2, \hat{w}_2]/(w_3^6 - k_3(x), w_2^6 - k_2(x), \hat{w}_2^6 - \hat{k}_2(x)),
\]
and then we have a natural ring homomorphism \(i_{12} : R_{12} \to \hat{R}_{12}\). Since in \(\hat{R}_{12}\), we have
\[
w_3^6 = k_3(x), \quad w_2^6 = k_2(x), \quad \hat{w}_2^6 = \hat{k}_2(x),
\]
in \(i_{12}(R_{12})\), the following holds
\[
y_{13} = w_3 w_2^2 \hat{w}_2^3, \quad y_{14} = w_3^2 w_2^4, \quad y_{15} = w_3^3 \hat{w}_2^3, \quad y_{16} = w_3^4 w_2^2.
\]
This gives a projection
\[
\text{Spec } \hat{R}_{12} \to \text{Spec } R_{12}.
\]

The curve \(X_{12}\) has a cyclic action of a primitive sixth root of unity \(\zeta_6\):
\[
\hat{\zeta}_6(x, y_{13}, y_{14}, y_{15}, y_{16}) = (x, \zeta_6 y_{13}, \zeta_6^2 y_{14}, \zeta_6^3 y_{15}, \zeta_6^4 y_{16}).
\]
Using the above expression of \(w\)'s, it is obvious that the prime ideal \(P\) is stable for the cyclic action.
4.2. **The Weierstrass gaps and holomorphic one forms.** The Weierstrass gap sequence of $R_{12}$ at $\infty$ is given by the following table: in particular, the semigroup $H(X_{12}, \infty)$ equals $H_{12}$.

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | - | - | - | - | - | - | - | x | - | - | - | - | - | y | 13 | y | 14 | y | 15 | - | x | 3 | - | x | y | 13 | y | 14 | y | 15 |

We introduce a notation for the monomials in $R_{12}$ whose orders of pole at $\infty$ correspond to the non-gaps:

\[
\phi_0^{(12)} = 1, \quad \phi_1^{(12)} = x, \quad \phi_2^{(12)} = x^2, \quad \phi_{a+2}^{(12)} = y_a, \quad \phi_7^{(12)} = x^3, \quad \phi_{a+7}^{(12)} = xy_a, \\
\phi_{6i-12}^{(12)} = x^i, \quad \phi_{6i-12+a}^{(12)} = x^{i-1}y_a, \quad \phi_{6i-7}^{(12)} = x^{i-3}y_{13}y_{16}, \quad (a = 1, 2, 3, 4, i = 4, 5, 6, \ldots).
\]

We define the weight $N^{(12)}(n)$ by

\[
N^{(12)}(n) = -\text{wt}(\phi_n^{(12)}),
\]

where $\text{wt}(\cdot)$ is the negative of the order of pole at $\infty$, consistent with the $G_m^{(12)}$ action.

The related Young diagram $\mathcal{Y}_{12}$ of $M_{12}$ is given by

\[
\Lambda_i = N^{(12)}(g) - N^{(12)}(i - 1) - 13 + i,
\]

which is equal to $\alpha_{12-i}(L(H_{12}))$ as defined in (2.1).

When $b$'s are distinct, a basis of the holomorphic one-forms is given by $\nu_{i}^{(12)}$, $i = 1, 2, \ldots, 12,$

\[
\nu_{i}^{(12)} := \nu_{i}^{f;12} := \frac{\phi_{i-1}^{(12)}dx}{6y_{13}y_{16}} = \frac{\phi_{i-1}^{(12)}dx}{6y_{14}y_{15}},
\]
or

\[
\begin{align*}
\nu_{1,12} & = \frac{dx}{6y_{13}y_{16}}, & \nu_{2,12} & = \frac{xdx}{6y_{13}y_{16}}, & \nu_{3,12} & = \frac{x^2dx}{6y_{13}y_{16}}, & \nu_{4,12} & = \frac{dx}{6y_{16}}, \\
\nu_{5,12} & = \frac{dx}{6y_{15}}, & \nu_{6,12} & = \frac{xdx}{6y_{15}}, & \nu_{7,12} & = \frac{xdx}{6y_{14}}, & \nu_{8,12} & = \frac{dx}{6y_{13}}, \\
\nu_{9,12} & = \frac{xdx}{6y_{16}}, & \nu_{10,12} & = \frac{xdx}{6y_{15}}, & \nu_{11,12} & = \frac{xdx}{6y_{14}}, & \nu_{12,12} & = \frac{dx}{6y_{13}},
\end{align*}
\]

(4.7)

and the corresponding Abel map for a point \( P \in X_{12} \) is defined:

\[
\hat{u}(P) \equiv \hat{u}_o(P) := \int_{\infty}^{P} \nu_{i,12} \in \mathbb{C}^{12},
\]

for \( k \) points \( P_1, P_2, \ldots, P_k \in X_{12} \), the Abel map \( \hat{u}(P_1, \ldots, P_k) \equiv \hat{u}_o(P_1, \ldots, P_k) \) is defined by \( \sum_{i=1}^{k} \hat{u}(P_i) \).

We also choose a basis \( \alpha_i^{(12)}, \beta_j^{(12)} \ (1 \leq i, j \leq 12) \) of \( H_1(X_{12}, \mathbb{Z}) \) such that the intersection numbers are \( \alpha_i^{(12)} \cdot \alpha_j^{(12)} = \beta_i^{(12)} \cdot \beta_j^{(12)} = 0 \) and \( \alpha_i^{(12)} \cdot \beta_j^{(12)} = \delta_{ij} \). Let the period matrices be denoted by

\[
\begin{align*}
[\omega^{(12)}, \omega^{(12)}] & := \frac{1}{2} \left[ \int_{\alpha_i^{(12)}} \nu_{j,12} \int_{\beta_j^{(12)}} \nu_{i,12} \right]_{i,j=1,2,\ldots,12}.
\end{align*}
\]

Then we define the Jacobian \( J_{12} \) by

\[
\kappa : \mathbb{C}^{12} \to J_{12} = \mathbb{C}^{12}/\Lambda_{12},
\]

where \( \Lambda_{12} \) is generated by \( \omega^{(12)}_\alpha \) and \( \omega^{(12)}_\nu \).

The Abel map of for \( (P_1, \ldots, P_k) \in S^k X_{12} \) to \( \mathbb{C}^{12} \) is also expressed by

\[
t_{23-N^{(12)}(12-i)}^{[k]} := u_i^{[k]} := \hat{u}(P_1, \ldots, P_k) \in \mathbb{C}^{12}.
\]

Further we define the subvariety \( \mathcal{W}^{(12)k} \) by

\[
\mathcal{W}^{(12)k} := \hat{u}(S^k X_{12})/\Lambda_{12}.
\]

By letting

\[
\begin{align*}
t_{23-N^{(12)}(12-i)} & := u_i := t_{23-N^{(12)}(12-i)}^{[12]} \equiv u_{i}^{[12]},
\end{align*}
\]

the weights are given by

\[
\{\text{wt}(t_i)\} = \{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 17, 23\}.
\]

Here we use the convention that for \( P_a \in X_{12}, \) \( P_a \) is expressed by \( (x_{a,1}, y_{13,a}, y_{14,a}, y_{15,a}, y_{16,a}) \)
or \( (x_{b,a}, y_{13,a}, y_{14,a}, y_{15,a}, y_{16,a}) \) and \( w_{b,a} \) is also \( w_b \) associated with \( P_a \).
By letting \( u := \hat{u}(P_1, \ldots, P_{12}) \), we have
\[
\begin{pmatrix}
\frac{\partial}{\partial u_1} \\
\frac{\partial}{\partial u_2} \\
\vdots \\
\frac{\partial}{\partial u_{11}} \\
\frac{\partial}{\partial u_{12}}
\end{pmatrix}
= \Psi_{12}^{(12)} \begin{pmatrix}
6y_{13,1}y_{14,1}\frac{\partial}{\partial x_1} \\
6y_{13,2}y_{14,2}\frac{\partial}{\partial x_2} \\
\vdots \\
6y_{13,11}y_{14,11}\frac{\partial}{\partial x_{11}} \\
6y_{13,12}y_{14,12}\frac{\partial}{\partial x_{12}}
\end{pmatrix},
\]
where
\[
\Psi_{12}^{(12)} := \begin{pmatrix}
\phi_0^{(12)}(P_1) & \phi_1^{(12)}(P_1) & \phi_2^{(12)}(P_1) & \cdots & \phi_{11}^{(12)}(P_1) \\
\phi_0^{(12)}(P_2) & \phi_1^{(12)}(P_2) & \phi_2^{(12)}(P_2) & \cdots & \phi_{11}^{(12)}(P_2) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\phi_0^{(12)}(P_{11}) & \phi_1^{(12)}(P_{11}) & \phi_2^{(12)}(P_{11}) & \cdots & \phi_{11}^{(12)}(P_{11}) \\
\phi_0^{(12)}(P_{12}) & \phi_1^{(12)}(P_{12}) & \phi_2^{(12)}(P_{12}) & \cdots & \phi_{11}^{(12)}(P_{12})
\end{pmatrix}.
\]

In particular, for any 12-tuple of numbers \((\epsilon_i)_{i=1,\ldots,12}\), we have
\[
\sum_{i=1}^{12} \epsilon_i \frac{\partial}{\partial u_i} = |\Psi_{12}^{(12)}|^{-1} \begin{vmatrix}
\phi_0^{(12)}(P_1) & \phi_1^{(12)}(P_1) & \phi_2^{(12)}(P_1) & \cdots & \phi_{11}^{(12)}(P_1) & 3y_{13,1}y_{16,1}\frac{\partial}{\partial x_1} \\
\phi_0^{(12)}(P_2) & \phi_1^{(12)}(P_2) & \phi_2^{(12)}(P_2) & \cdots & \phi_{11}^{(12)}(P_2) & 3y_{13,2}y_{16,2}\frac{\partial}{\partial x_2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_0^{(12)}(P_{11}) & \phi_1^{(12)}(P_{11}) & \phi_2^{(12)}(P_{11}) & \cdots & \phi_{11}^{(12)}(P_{11}) & 3y_{13,3}y_{16,3}\frac{\partial}{\partial x_3} \\
\phi_0^{(12)}(P_{12}) & \phi_1^{(12)}(P_{12}) & \phi_2^{(12)}(P_{12}) & \cdots & \phi_{11}^{(12)}(P_{12}) & 3y_{13,4}y_{16,4}\frac{\partial}{\partial x_4} \\
\epsilon_1 & \epsilon_2 & \epsilon_3 & \cdots & \epsilon_{12}
\end{vmatrix}.
\]

We express the change of variables:
\[
\sum_{i,j=1}^{12} \phi_{i-1}^{(12)}(P_1) \phi_{j-1}^{(12)}(P_2) \frac{\partial^2}{\partial u_i(P_1) \partial u_j(P_2)} = 9y_{13,1}y_{16,1}y_{13,2}y_{16,2} \frac{\partial^2}{\partial x_1 \partial x_2}.
\]

4.3. **Differentials of the second and the third kinds.** We also give an algebraic representation of the fundamental normalized differential of the second kind \(\Omega^{(12)}(P_1, P_2)\) on \(X_{12} \times X_{12}\), whose defining properties are the same as those given for \(\Omega^{(4)}(P_1, P_2)\) on \(X_4 \times X_4\).

**Proposition 4.4.** Let \(\Sigma^{(12)}(P_1, P_2)\) be the following form,
\[
\Sigma^{(12)}(P_1, P_2) := \frac{y_{13,1}y_{16,1} + y_{13,2}y_{16,1} + y_{13,1}y_{16,2} + y_{14,2}y_{15,1} + y_{14,1}y_{15,2} + y_{14,2}y_{15,2}}{6(x_1 - x_2)y_{13,1}y_{16,1}} dx_1.
\]

Then \(\Sigma^{(12)}(P, Q)\) has the following properties:

1. \(\Sigma^{(12)}(P, Q)\) as a function of \(P\) is singular at \(Q = (x_Q, y_{13,3}, y_{14,4}, y_{15,5}, y_{16,6})\) and \(\infty\) and is not singular at \(\hat{\zeta}_6(Q) = (x_Q, \zeta_6 y_{13,3}, \zeta_6^2 y_{14,4}, \zeta_6^3 y_{15,5}, \zeta_6^4 y_{16,6})\).

2. \(\Sigma^{(12)}(P, Q)\) as a function of \(Q\) is singular at \(P\) and \(\infty\).
\textbf{Proposition 4.5.} There exist differentials \( \nu_j^{I;12} = \nu_j^{I;12}(x, y) \) \((j = 1, 2, \cdots, 12)\) of the second kind such that they have their only pole at \( \infty \) and satisfy the relation,
\begin{equation}
\frac{d_Q \Sigma^{(12)}(P, Q) - d_P \Sigma^{(12)}(Q, P)}{d_Q \Sigma^{(12)}(P, Q) := dx_P \otimes dx_Q \frac{\partial}{\partial x_Q} \Sigma^{(12)}}.
\end{equation}

The set of differentials \( \{ \nu_1^{I;12}, \nu_2^{I;12}, \cdots, \nu_{12}^{I;12}\} \) is determined modulo the \( \mathbb{C} \)-linear space spanned by \( \langle \nu_j^{I;12}\rangle_{j=1,\cdots,12} \).

\textbf{Proposition 4.6.} The differentials of the second kind are given by
\begin{equation*}
\nu_1^{I;12} = \left( -23x^5 + (19\lambda_1^{(2)} + 18\lambda_1^{(3)} + 20\lambda_1^{(2)})x^4 + (14\lambda_1^{(3)}\lambda_1^{(2)} + 15\lambda_1^{(3)}\lambda_1^{(2)} + 16\lambda_1^{(2)}\lambda_1^{(2)} + 13\lambda_2^{(3)} + 15\lambda_2^{(2)} + 17\lambda_2^{(2)}x^3 + (10\lambda_2^{(3)}\lambda_2^{(2)} + 9\lambda_2^{(3)}\lambda_2^{(2)} + 10\lambda_2^{(3)}\lambda_1^{(2)} + 12\lambda_2^{(2)}\lambda_1^{(2)} + 12\lambda_2^{(3)}\lambda_2^{(2)} + 13\lambda_2^{(2)}\lambda_2^{(2)} + 8\lambda_3^{(3)} + 11\lambda_3^{(2)}\lambda_1^{(2)}\lambda_1^{(2)} + x^2 + (4\lambda_3^{(3)}\lambda_1^{(2)} + 5\lambda_3^{(2)}\lambda_2^{(2)} + 5\lambda_3^{(2)}\lambda_1^{(2)} + 7\lambda_2^{(3)}\lambda_2^{(2)} + 9\lambda_2^{(2)}\lambda_2^{(2)} + 6\lambda_3^{(3)}\lambda_2^{(2)} + \lambda_1^{(2)} + 8\lambda_3^{(3)}\lambda_2^{(2)} + 7\lambda_1^{(3)}\lambda_2^{(2)} + \lambda_1^{(2)}\lambda_1^{(2)} + 3\lambda_1^{(3)}\lambda_1^{(2)}\lambda_2^{(2)} + 4\lambda_1^{(3)}\lambda_2^{(2)}\lambda_2^{(2)} + 2\lambda_2^{(3)}\lambda_2^{(2)}\lambda_1^{(2)} + \lambda_3^{(3)}\lambda_1^{(2)}\lambda_1^{(2)} + 2\lambda_3^{(3)}\lambda_2^{(2)}\lambda_2^{(2)} \right) dx/6y_{13},
\end{equation*}
\[ \nu_2^{II,12} = - \left( 17x^4 + (13\lambda_1^{(2)} + 12\lambda_1^{(3)} + 14\hat{\lambda}_1^{(2)})x^3 \
+ (10\lambda_1^{(2)} \hat{\lambda}_1^{(2)} + 9\lambda_1^{(3)} \hat{\lambda}_1^{(2)} + 8\lambda_1^{(3)} \lambda_1^{(2)} + 11\hat{\lambda}_2^{(2)} + 9\lambda_2^{(2)} + 7\lambda_2^{(3)})x^2 \
+ (7\lambda_1^{(2)} \lambda_2^{(2)} + 6\lambda_1^{(3)} \lambda_2^{(2)} + 4\lambda_1^{(3)} \lambda_2^{(2)} + 6\lambda_2^{(2)} \lambda_1^{(2)} + 4\lambda_2^{(3)} \lambda_1^{(2)} + 3\lambda_2^{(3)} \lambda_2^{(2)} \
+ 2\lambda_3^{(3)} + 5\lambda_1^{(3)} \lambda_1^{(2)} \hat{\lambda}_1^{(2)})x \
+ \lambda_1^{(3)} \lambda_2^{(2)} \lambda_1^{(2)} + 2\lambda_1^{(3)} \lambda_1^{(2)} \hat{\lambda}_1^{(2)} + 3\lambda_2^{(2)} \lambda_1^{(2)} + \lambda_2^{(3)} \lambda_1^{(2)} \right) dx/6y_{13}, \]

\[ \nu_3^{II,12} = \left( 11x^3 + (6\lambda_1^{(3)} + 7\lambda_1^{(2)} + 8\hat{\lambda}_1^{(2)})x^2 + (4\lambda_1^{(2)} \hat{\lambda}_1^{(2)} + 3\lambda_1^{(3)} \hat{\lambda}_1^{(2)} + 2\lambda_2^{(3)} \lambda_1^{(2)} \
+ \lambda_2^{(3)} + 3\lambda_2^{(2)} + 5\hat{\lambda}_2^{(2)})x - 3\lambda_2^{(3)} \lambda_1^{(2)} + \lambda_2^{(2)} \hat{\lambda}_1^{(2)} \right) dx/6y_{13}, \]

\[ \nu_4^{II,12} = - \left( (8x^3 + (4\lambda_1^{(2)} + 6\lambda_1^{(3)})x^2 + (2\lambda_1^{(3)} \lambda_1^{(2)} + 4\lambda_2^{(3)})x) + 2\lambda_3^{(3)} \right) dx/6y_{14}, \]

\[ \nu_5^{II,12} = - \left( 9x^3 + (6\lambda_1^{(2)} + 6\lambda_1^{(3)})x^2 + (3\lambda_1^{(3)} \hat{\lambda}_1^{(2)} + 3\hat{\lambda}_2^{(2)} + 3\lambda_2^{(3)})x \right) dx/6y_{15}, \]

\[ \nu_6^{II,12} = - \left( 10x^3 + (8\lambda_1^{(2)} + 6\lambda_1^{(3)})x^2 + (4\lambda_1^{(3)} \lambda_1^{(2)} + 2\lambda_2^{(3)} + 6\lambda_2^{(2)})x + 2\lambda_1^{(3)} \lambda_2^{(2)} \right) dx/6y_{16}, \]

\[ \nu_7^{II,12} = - \left( 7x^5 + (6\lambda_1^{(3)} + 5\lambda_1^{(2)} + 4\hat{\lambda}_1^{(2)})x^4 \
+ (3\lambda_2^{(2)} + 3\lambda_1^{(3)} \hat{\lambda}_1^{(2)} - \hat{\lambda}_2^{(2)} + 5\lambda_2^{(3)} + 2\lambda_1^{(2)} \hat{\lambda}_1^{(2)} + 4\lambda_1^{(3)} \lambda_1^{(2)})x^3 \
+ (\lambda_1^{(3)} \lambda_1^{(2)} \hat{\lambda}_1^{(2)} + 2\lambda_2^{(3)} \hat{\lambda}_1^{(2)} + 3\lambda_1^{(3)} \lambda_1^{(2)} + 2\lambda_1^{(3)} \lambda_2^{(2)} + 4\lambda_3^{(3)})x^2 \
+ (-2\lambda_3^{(3)} \lambda_1^{(2)} - \lambda_2^{(3)} \lambda_2^{(2)} - \lambda_3^{(3)} \hat{\lambda}_1^{(2)}) \right) dx/6y_{13}y_{16}, \]

\[ \nu_8^{II,12} = - \left( 5x^2 + (2\lambda_1^{(2)} + \lambda_1^{(3)})x \right) / 6y_{13}, \]

\[ \nu_9^{II,12} = -2x^2 dx/6y_{14}, \quad \nu_{10}^{II,12} = -3x^2 dx/6y_{15}, \quad \nu_{11}^{II,12} = -(2\lambda_1^{(2)} x + 4x^2) dx/6y_{16}, \]

\[ \nu_{12}^{II,12} = -x^4 / 6y_{13}y_{16}. \]

**Proof.** By letting the numerator in (4.12) and (4.13) be denoted by

\[ A(P_1, P_2) := y_{13}y_{16,1} + y_{13}y_{16,1} + y_{13}y_{16,2} + y_{14}y_{15,1} + y_{14}y_{15,2} + y_{14}y_{15,2} + y_{14}y_{15,2} \]

\[ = w_{3,1}w_{2,1}w_{2,2}w_{3,3}w_{3,1}w_{2,1} + w_{3,2}w_{2,2}w_{2,2}w_{3,3}w_{3,1}w_{2,1} + w_{3,1}w_{2,1}w_{2,3}w_{3,2}w_{3,1}w_{2,1} \]

\[ + w_{3,1}w_{2,1}w_{2,3}w_{3,2}w_{3,1}w_{2,1} + w_{3,1}w_{2,1}w_{2,3}w_{3,2}w_{3,1}w_{2,1} + w_{3,1}w_{2,1}w_{2,3}w_{3,2}w_{3,1}w_{2,1} \]

\[ = w_{3,1}w_{2,1}w_{2,3}w_{3,2}w_{3,1}w_{2,1} + w_{3,1}w_{2,1}w_{2,3}w_{3,2}w_{3,1}w_{2,1} + w_{3,1}w_{2,1}w_{2,3}w_{3,2}w_{3,1}w_{2,1} \]

\[ + w_{3,2}w_{2,2}w_{3,1}w_{2,2}w_{3,1}w_{2,2} + w_{3,1}w_{2,1}w_{2,3}w_{3,2}w_{3,1}w_{2,1} + w_{3,1}w_{2,1}w_{2,3}w_{3,2}w_{3,1}w_{2,1} \]

Proof. By letting the numerator in (4.12) and (4.13) be denoted by

\[ A(P_1, P_2) := y_{13}y_{16,1} + y_{13}y_{16,1} + y_{13}y_{16,1} + y_{13}y_{16,1} + y_{14}y_{15,1} + y_{14}y_{15,2} + y_{14}y_{15,2} + y_{14}y_{15,2} \]

\[ = w_{3,1}w_{2,1}w_{2,2}w_{3,3}w_{3,1}w_{2,1} + w_{3,2}w_{2,2}w_{2,2}w_{3,3}w_{3,1}w_{2,1} + w_{3,1}w_{2,1}w_{2,3}w_{3,2}w_{3,1}w_{2,1} \]

\[ + w_{3,1}w_{2,1}w_{2,3}w_{3,2}w_{3,1}w_{2,1} + w_{3,1}w_{2,1}w_{2,3}w_{3,2}w_{3,1}w_{2,1} + w_{3,1}w_{2,1}w_{2,3}w_{3,2}w_{3,1}w_{2,1} \]

\[ = w_{3,1}w_{2,1}w_{2,3}w_{3,2}w_{3,1}w_{2,1} + w_{3,1}w_{2,1}w_{2,3}w_{3,2}w_{3,1}w_{2,1} + w_{3,1}w_{2,1}w_{2,3}w_{3,2}w_{3,1}w_{2,1} \]

\[ + w_{3,1}w_{2,1}w_{2,3}w_{3,2}w_{3,1}w_{2,1} + w_{3,1}w_{2,1}w_{2,3}w_{3,2}w_{3,1}w_{2,1} + w_{3,1}w_{2,1}w_{2,3}w_{3,2}w_{3,1}w_{2,1}. \]
we have the equality,
\[
\frac{\partial \Sigma^{(12)}(P_1, P_2)}{\partial x_2} - \frac{\partial \Sigma^{(12)}(P_2, P_1)}{\partial x_1} = \frac{(y_{13,2}y_{16,2}A(P_1, P_2) - y_{13,1}y_{16,1}A(P_2, P_1))/(x_1 - x_2)}{(x_1 - x_2)y_{13,1}y_{16,1}y_{13,2}y_{16,2}} + \frac{y_{13,1}y_{16,1} \frac{\partial A(P_1, P_2)}{\partial x_2} + y_{13,2}y_{16,2} \frac{\partial A(P_2, P_1)}{\partial x_1}}{(x_1 - x_2)y_{13,1}y_{16,1}y_{13,2}y_{16,2}}.
\]
\[\tag{4.16}\]

We must evaluate the numerator of (4.16); we denote it by \(B(P_1, P_2)\). The former terms are written as
\[
y_{13,2}y_{16,2}A(P_1, P_2) - y_{13,1}y_{16,1}A(P_2, P_1)
= w_{3,2}^6w_{2,2}^6\hat{w}_{2,2}w_{3,1}w_{2,1}^4 + w_{3,1}w_{2,1}^2w_{3,2}^9w_{2,2}^6\hat{w}_{2,2}^3 + w_{3,2}w_{2,2}^2w_{3,1}\hat{w}_{2,2}^3
+ w_{3,1}w_{2,1}^2w_{3,2}^8w_{2,2}^6 + w_{3,2}w_{2,2}^2w_{3,1}\hat{w}_{2,2}^3 - (P_1 \leftrightarrow P_2)
= y_{16,1}(k_{3,2}\hat{w}_{2,2} + y_{13,1}y_{15,2}k_{3,2}\hat{w}_{2,2} + y_{13,2}y_{15,1}k_{3,2}\hat{w}_{2,2} + y_{14,1}y_{14,2}k_{3,2}\hat{w}_{2,2} + y_{16,2}k_{3,2}k_{2,2}\hat{w}_{2,2} - k_{3,1}k_{2,1}\hat{w}_{2,2})
+ (y_{13,1}y_{15,2} + y_{13,2}y_{15,1})(k_{3,2}\hat{w}_{2,2} - k_{3,1}k_{2,1}) + y_{14,1}y_{14,2}(k_{3,2}\hat{w}_{2,2} - k_{3,1}k_{2,1}).
\]

Here \(k_{a,b} := k_a(x_b)\) and \(\hat{k}_{2,b} := \hat{k}_2(x_b)\) for \(a = 2, 3\) and \(b = 1, 2, 2\) and \(k' := dk/dx\). We reinforce that these are expressions in the affine coordinates \(x's\) and \(y's\), in other words the \(w's\) in these expressions, which are algebraic over the ring \(R_{12}\), appear only within algebraic combinations which belong to \(R_{12}\).

The differential terms of (4.16) are given as follows:
\[
6y_{13,2}y_{16,2} \frac{\partial A(P_2, P_1)}{\partial x_2} = w_{3,2}^5w_{2,2}^4\hat{w}_{2,2}^2 + \frac{y_{16,1}}{w_{3,2}^5} \left( \frac{k_{3,2}w_{2,2}^2w_{3,2}^9\hat{w}_{2,2}w_{3,1}}{w_{2,2}^2} + \frac{2}{w_{3,2}^2} \frac{k_{2,2}w_{3,2}^9\hat{w}_{2,2}}{w_{2,2}^2} + \frac{3}{w_{3,2}^3} \frac{w_{3,2}w_{2,2}^2w_{3,1}w_{2,1}^4}{w_{2,2}^2} \right)
+ y_{13,1} \left( \frac{4}{w_{3,2}^5} k_{3,2}w_{2,2}^2w_{3,2}^9\hat{w}_{2,2}w_{3,1} + \frac{4}{w_{3,2}^2} k_{3,2}w_{2,2}^2w_{3,2}^9\hat{w}_{2,2} + \frac{4}{w_{3,2}^2} k_{3,2}w_{2,2}^2w_{3,2}^9\hat{w}_{2,2} \right)
+ y_{14,1} \left( \frac{3}{w_{3,2}^5} k_{3,2}w_{2,2}^2w_{3,2}^9\hat{w}_{2,2}w_{3,1} + \frac{3}{w_{3,2}^2} k_{3,2}w_{2,2}^2w_{3,2}^9\hat{w}_{2,2} + \frac{3}{w_{3,2}^2} k_{3,2}w_{2,2}^2w_{3,2}^9\hat{w}_{2,2} \right)
\]
\[
= y_{16,1} \left( \frac{k_{3,2}w_{2,2}^2w_{3,2}^9\hat{w}_{2,2}}{w_{2,2}^2} + 2k_{3,2}k_{3,2}^2w_{2,2}^2w_{3,2}^9\hat{w}_{2,2} + 3k_{3,2}k_{2,2}w_{2,2}^2w_{3,2}^9\hat{w}_{2,2} \right)
+ y_{13,1}y_{15,2} \left( \frac{4}{w_{3,2}^2} k_{3,2}w_{2,2}^2w_{3,2}^9\hat{w}_{2,2}w_{3,1} + \frac{4}{w_{3,2}^2} k_{3,2}w_{2,2}^2w_{3,2}^9\hat{w}_{2,2} + \frac{4}{w_{3,2}^2} k_{3,2}w_{2,2}^2w_{3,2}^9\hat{w}_{2,2} \right)
+ y_{14,1}y_{14,2} \left( \frac{3}{w_{3,2}^2} k_{3,2}w_{2,2}^2w_{3,2}^9\hat{w}_{2,2}w_{3,1} + \frac{3}{w_{3,2}^2} k_{3,2}w_{2,2}^2w_{3,2}^9\hat{w}_{2,2} + \frac{3}{w_{3,2}^2} k_{3,2}w_{2,2}^2w_{3,2}^9\hat{w}_{2,2} \right).
\]
\[
6y_{13,2}y_{16,2} \frac{\partial A(P_2, P_1)}{\partial x_2} + 6y_{13,1}y_{16,1} \frac{\partial A(P_1, P_2)}{\partial x_1} \\
= y_{16,1} \left( k'_{3,2} k_{2,2} \hat{k}_{2,2} + 2k_{3,2} k'_{2,2} \hat{k}_{2,2} + 3k_{3,2} k_{2,2} \hat{k}'_{2,2} + 5k'_{3,1} k_{2,1} \hat{k}_{2,1} + 4k_{3,1} k'_{2,1} \hat{k}_{2,1} + 3k_{3,1} k_{2,1} \hat{k}'_{2,1} \right) \\
+ y_{13,1}y_{15,2} \left( 4k'_{3,2} k_{2,2} + 2k_{3,2} k'_{2,2} + 2k'_{3,1} k_{2,1} + 4k_{3,1} k'_{2,1} \right) \\
+ y_{15,1}y_{13,2} \left( 2k'_{3,2} k_{2,2} + 4k_{3,2} k'_{2,2} + 4k'_{3,1} k_{2,1} + 2k_{3,1} k'_{2,1} \right) \\
+ y_{14,1}y_{14,2} \left( 3k'_{3,2} \hat{k}_{2,2} + 3k_{3,2} \hat{k}'_{2,2} + 3k_{3,1} \hat{k}_{2,1} + 3k_{3,1} \hat{k}'_{2,1} \right) \\
+ y_{16,2} \left( 5k'_{3,2} k_{2,2} \hat{k}_{2,2} + 4k_{3,2} k'_{2,2} \hat{k}_{2,2} + 3k_{3,2} k_{2,2} \hat{k}'_{2,2} + k'_{3,1} k_{2,1} \hat{k}_{2,1} + 2k_{3,1} k'_{2,1} \hat{k}_{2,1} + 3k_{3,1} k_{2,1} \hat{k}'_{2,1} \right).
\]

The numerator \( B(P_1, P_2) \) in (4.16) is

\[
B(P_1, P_2) = y_{16,1}B_{16}(P_1, P_2) + y_{13,1}y_{15,2}B_{13,15}(P_1, P_2) + y_{14,1}y_{14,2}B_{14}(P_1, P_2) \\
+ y_{13,2}y_{15,1}B_{13,15}(P_2, P_1) + y_{16,2}B_{16}(P_2, P_1),
\]

where \( B_{16}, B_{13,14} \) and \( B_{15} \) are rational functions of \( x_1 \) and \( x_2 \), we evaluate:

\[
B_{16}(P_1, P_2) = \frac{(k_{3,2} k_{2,2} \hat{k}_{2,2} - k_{3,1} k_{2,1} \hat{k}_{2,1})}{x_1 - x_2} + k'_{3,2} k_{2,2} \hat{k}_{2,2} + 2k_{3,2} k'_{2,2} \hat{k}_{2,2} + 3k_{3,2} k_{2,2} \hat{k}'_{2,2} \\
+ 5k'_{3,1} k_{2,1} \hat{k}_{2,1} + 4k_{3,1} k'_{2,1} \hat{k}_{2,1} + 3k_{3,1} k_{2,1} \hat{k}'_{2,1},
\]

\[
B_{13,15}(P_1, P_2) = \frac{k_{3,2} k_{2,2} - k_{3,1} k_{2,1}}{x_1 - x_2} + 2k'_{3,2} k_{2,2} + 4k_{3,2} k'_{2,2} - 4k'_{3,1} k_{2,1} - 2k_{3,1} k'_{2,1},
\]

and

\[
B_{14}(P_1, P_2) = \frac{k_{3,2} \hat{k}_{2,2} - k_{3,1} \hat{k}_{2,1}}{x_1 - x_2} + 3k'_{3,2} \hat{k}_{2,2} + 3k_{3,2} \hat{k}'_{2,2} - 3k'_{3,1} \hat{k}_{2,1} - 3k_{3,1} \hat{k}'_{2,1}.
\]

We are concerned with \( B(P_1, P_2) / (x_1 - x_2) \) and thus compute

\[
\frac{B_{16}(P_1, P_2)}{(x_1 - x_2)} = -7x_2^5 + \left( -6 \lambda_1^{(3)} - x_1 - 5 \lambda_1^{(2)} - 4 \lambda_1^{(2)} \right) x_2^4 \\
+ \left( 5x_2^2 + (2 \lambda_1^{(2)} + \lambda_1^{(2)}) x_1 - 3 \lambda_1^{(2)} - 3 \lambda_1^{(2)} \lambda_1^{(2)} - \lambda_2^{(2)} - 5 \lambda_2^{(2)} - 2 \lambda_1^{(2)} \lambda_1^{(2)} - 4 \lambda_1^{(3)} \lambda_1^{(2)} \right) x_2^3 \\
+ \left( 11x_1^3 + (6 \lambda_1^{(3)} + 7 \lambda_1^{(2)}) x_1 + 8 \lambda_1^{(2)} \right) x_2^2 + \left( 4 \lambda_1^{(2)} \lambda_1^{(2)} + 3 \lambda_1^{(3)} \lambda_1^{(2)} + 2 \lambda_1^{(3)} \lambda_1^{(2)} \right) \\
\lambda_1^{(2)} + 3 \lambda_2^{(2)} + 5 \lambda_2^{(2)} \right) x_1 - \lambda_1^{(3)} \lambda_1^{(2)} - 2 \lambda_2^{(2)} \lambda_1^{(2)} - 3 \lambda_2^{(3)} \lambda_1^{(2)} - 2 \lambda_1^{(3)} \lambda_1^{(2)} + \lambda_1^{(2)} \lambda_2^{(2)} - 4 \lambda_1^{(3)} \lambda_1^{(2)}} x_2^2
By these results we obtain the $\nu^{12}$'s in the statement.

By these results we obtain the $\nu^{12}$'s in the statement.

Corollary 4.7. (1) The one-form

$$\Pi^{(12)}_{P_2}(P) := \Sigma^{(12)}(P, P_1)dx - \Sigma^{(12)}(P, P_2)dx$$

is a differential of the third kind, whose only (first-order) poles are $P = P_1$ and $P = P_2$ with residues $+1$ and $-1$ respectively.
(2) The fundamental differential of the second kind $\Omega^{(12)}(P_1, P_2)$ is given by

$$
\Omega^{(12)}(P_1, P_2) = d_{P_2} \Sigma^{(12)}(P_1, P_2) + \sum_{i=1}^{12} \nu_i^{I:12}(P_1) \otimes \nu_i^{II:12}(P_2)
$$

(4.17)

$$
= \frac{F^{(12)}(P_1, P_2)dx_1 \otimes dx_2}{(x_1 - x_2)^26^2y_{13,P_1}y_{16,P_1}y_{13,P_2}y_{16,P_2}},
$$

where $F^{(4)}$ is an element of $R_{12} \otimes R_{12}$.

The following result is easily obtained by observing the coefficient of the largest degree term with respect to $P_1$,

**Lemma 4.8.** We have

$$
\lim_{P_1 \to \infty} \frac{F^{(12)}(P_1, P_2)}{\phi^{(12)}_1(P_1)(x_1 - x_2)^2} = \phi^{(12)}_{12}(P_2) = x_{P_2}^4.
$$

(4.18)

For later convenience we introduce the notation:

$$
\Omega^{(12)}_{P_1,P_2, Q_1,Q_2} := \int_{P_1}^{P_2} \int_{Q_1}^{Q_2} \Omega^{(12)}(P, Q)
$$

(4.19)

$$
= \int_{P_1}^{P_2} (\Sigma^{(12)}(P, Q_1) - \Sigma^{(12)}(P, Q_2)) + \sum_{i=1}^{12} \int_{P_1}^{P_2} \nu_i^{I:12}(P) \int_{Q_2}^{Q_1} \nu_i^{II:12}(P).
$$

5. **The Sigma Function for (3, 7, 8) and (6, 13, 14, 15, 16) Curves**

5.1. **Legendre relation.** In this section, $g$ will be 4 or 12. In the monomial notation, as there were two conventions for $X_4$ and $X_{12}$, $\phi^{(g)}$ will be $\phi^{(4)}_H$ for $g = 4$ and $\phi^{(12)}$ for $g = 12$.

We write the periods:

$$
[\eta^{(g)}', \eta^{(g)''}] := \frac{1}{2} \left[ \int_{\alpha_i^{(g)}} \nu_j^{I:2:g} \int_{\beta_j^{(g)}} \nu_j^{II:2:g} \right]_{i,j=1,2,\ldots,g}.
$$

(5.1)

Let $\tau_{Q_1,Q_2}^{(g)}$ be the normalized differential of the third kind that has residues $+1, -1$ at $Q_1, Q_2$, is regular everywhere else, and is normalized. $\int_{\alpha_i} \tau_{P,Q}^{(g)} = 0$ for every $i$ as in III.3.5 of [24, III.3.5]. The following Lemma corresponding to Corollary 2.6 (ii) in [25] holds:

**Lemma 5.1.**

$$
\Omega^{(g)}_{P_1,P_2, Q_1,Q_2} = \int_{P_1}^{P_2} \tau_{Q_1,Q_2}^{(g)} + \sum_{i,j=1}^{g} \gamma_{ij}^{(g)} \int_{P_1}^{P_2} \nu_i^{I:2:g} \int_{Q_2}^{Q_1} \nu_j^{II:2:g},
$$

where $\gamma^{(g)} = \omega^{(g)'}\eta^{(g)''}$. 

Proof. \( \partial_{P_1} \Omega^{(g)}_{P_1, P_2} - \tau^{(g)}_{Q_1, Q_2} \) must be expressed by the linear combination of the holomorphic one forms. Noting \( \int_{P_1}^{P_2} \tau^{(g)}_{Q_1, Q_2} = \int_{Q_1}^{Q_2} \tau^{(g)}_{P_1, P_2} \) in III.3.5 of [24, III.3.5], \( \gamma_{ij}^{(g)} \) is symmetric. Due to (3.17) and (4.19),

\[
(\Sigma(P_1, Q_1) - \Sigma(P_1, Q_2)) + \sum_{i=1}^{g} \nu_i^{I,g}(P_1) \int_{Q_2}^{Q_1} \nu_i^{II,g}(P) = \tau^{(g)}_{Q_1, Q_2}(P_1) + \sum_{i,j=1}^{g} \gamma_{ij}^{(g)} \nu_i^{I,g}(P_1) \int_{Q_2}^{Q_1} \nu_j^{I,g}.
\]

By choosing an appropriate path \( \Gamma \) from \( Q_1 \) to \( Q_2 \), homotopic to \( \alpha_k^{(g)} \), we have \((\eta^{(g)'} \nu^{I,g})^t = \gamma^{(g)} \cdot (\omega^{(g)'} \nu^{I,g})^t \).

The following Proposition provides \textit{generalized Legendre relation} [3, 10, 11], which determines a symplectic structure in the Jacobian [63, Ch.3,4](Ch.3,4).

\begin{proposition}
\textbf{(generalized Legendre relation)}
\textit{The matrix},

\begin{equation}
M := \begin{bmatrix}
2\omega^{(g)'} & 2\omega^{(g)''} \\
2\eta^{(g)'} & 2\eta^{(g)''}
\end{bmatrix},
\end{equation}

\textit{satisfies}

\begin{equation}
M \begin{bmatrix}
1 \\
-1
\end{bmatrix} = 2\pi \sqrt{-1} \begin{bmatrix}
1 \\
-1
\end{bmatrix}.
\end{equation}
\end{proposition}

\textit{Proof.} By comparing Lemma 5.1 and (3.15) in Corollary 3.12 and (4.17) in Corollary 4.7, we choose appropriately \((2g)^2\) paths and take the integrals along these paths. For example, for

\[
\int_{P_2}^{P_1} (\Sigma(P, Q_1) - \Sigma(P, Q_2)) + \sum_{i=1}^{g} \int_{P_2}^{P_1} \nu_i^{I,g}(P) \int_{Q_2}^{Q_1} \nu_i^{II,g}(P)
\]

the contour of \( Q_1 \) to \( Q_2 \) along \( \alpha_k \) gives

\[
\sum_{i=1}^{g} \int_{P_2}^{P_1} \nu_i^{I,g}(P) \eta_{k_i}^{(g)''}(P) = \sum_{i,j=1}^{g} \gamma_{ij}^{(g)} \int_{P_2}^{P_1} \nu_i^{I,g} \omega_{ kj}^{(g)''},
\]

and further contour integration provides \((t^{\omega^{(g)''} \eta^{(g)''}})^t = (t^{\omega^{(g)''} \eta^{(g)''}})^t \). The contour of \( Q_1 \) to \( Q_2 \) along \( \beta_k^{(g)} \) gives

\[
\sum_{i=1}^{g} \int_{P_2}^{P_1} \nu_i^{I,g}(P) \eta_{k_i}^{(g)''}(P) = 2\pi \sqrt{-1} \int_{P_2}^{P_1} (\omega^{(g)''})^{-1} \nu^{I,g}_k + \sum_{i,j=1}^{g} \gamma_{ij}^{(g)} \int_{P_2}^{P_1} \nu_i^{I,g} \omega_{kj}^{(g)''},
\]

which, in view of the contour integral, turns out to be

\[
(t^{\omega^{(g)''} \eta^{(g)''}})^t = 2\pi \sqrt{-1} T + (t^{\omega^{(g)''} \gamma^{(g)} \omega^{(g)''}}),
\]

\[
(t^{\omega^{(g)''} \eta^{(g)''}})^t = 2\pi \sqrt{-1} T + (t^{\omega^{(g)''} \gamma^{(g)} \omega^{(g)''}}).
\]
Here we use $\int_{\beta_k} \tau_{Q_1,Q_2}^{(g)} = 2\pi \sqrt{-1} \int_{Q_2} \omega^{(g)\nu-1} \nu^{I:g} k$ in III.3.5 of [24 III.3.5].

5.2. $\sigma$ function. By the Riemann relations [25], it is known that $\text{Im} (\omega^{(g)\nu-1} \omega^{(g)\nu})$ is positive definite.

We let

\begin{equation}
\delta := \left[ \begin{array}{c} \delta'' \\ \delta' \end{array} \right] \in \left( \frac{1}{2\mathbb{Z}} \right)^{2g}
\end{equation}

be the theta characteristic which is equal to the Riemann constant $\xi_R$ with respect to the base point $\infty$ and the period matrix $[2\omega^{(g)\nu} 2\omega^{(g)\nu}]$. This will be given explicitly in terms of divisors by Proposition [6.7] and Corollary [6.9].

We define an entire function of (a column-vector) $u = (u_1, u_2, \ldots, u_g) \in \mathbb{C}^g$,

\begin{equation}
\sigma^{(g)}(u) = \sigma^{(g)}(u; M) = \sigma^{(g)}(u_1, u_2, \ldots, u_g; M)
= c \exp(-\frac{1}{2} t u \eta^{(g)\nu} \omega^{(g)\nu-1} u) \vartheta[\delta] \left( \frac{1}{2} \omega^{(g)\nu-1} u; \omega^{(g)\nu-1} \omega^{(g)\nu} \right)
= c \exp(-\frac{1}{2} t u \eta^{(g)\nu} \omega^{(g)\nu-1} u)
\times \sum_{n \in \mathbb{Z}^g} \exp \left[ \pi \sqrt{-1} \left\{ \left( n + \delta'' \right) \omega^{(g)\nu-1} \omega^{(g)\nu} \left( n + \delta'' \right) + t (n + \delta'') \left( \omega^{(g)\nu-1} u + 2\delta' \right) \right\} \right],
\end{equation}

where $c$ is a certain constant, in fact a rational function of the $b$'s.

For a given $u \in \mathbb{C}^g$, we introduce $u'$ and $u''$ in $\mathbb{R}^g$ so that

$$u = 2\omega^{(g)\nu} u' + 2\omega^{(g)\nu} u''.$$

Proposition 5.3. For $u, v \in \mathbb{C}^g$, and $\ell = 2\omega^{(g)\nu} \ell' + 2\omega^{(g)\nu} \ell'' \in \Lambda$, we define

$$L(u, v) := 2 t u \eta^{(g)\nu} \ell', \quad \chi(\ell) := \exp \left[ \pi \sqrt{-1} \left( 2 t \ell' \delta'' - t \ell'' \delta' + t \ell' \ell'' \right) \right] \left( \in \{ 1, -1 \} \right).$$

The following holds

\begin{equation}
\sigma^{(g)}(u + \ell) = \sigma^{(g)}(u) \exp(L(u + \frac{1}{2} \ell, \ell)) \chi(\ell).
\end{equation}

Proof. Direct computations using the definition of $\sigma^{(g)}$ give the result as in Chapter VI of [43].

Remark 5.4. Following a formula proven for the rational/polynomial case by Buchstaber, Leykin and Enolskii [7], in [58] Nakayashiki showed that the leading term in the Taylor expansion of the $\sigma$ function associated with $(r, s)$ curve is expressed by a Schur function by normalizing the constant factor $c$. We also expect that

$$\sigma^{(g)}(u) = S_\Lambda(T) |_{T_{\lambda_i+g-1} \equiv u_i} + \sum \alpha a_\alpha u^\alpha,$$
where \( a_\alpha \in \mathbb{Q}[b_1, \ldots, b_k] \) for \( \ell = 5 \) for \( X_4 \) and \( \ell = 7 \) for \( X_{12} \), \( \alpha = \alpha_1 \cdots \alpha_g \) and \( u^\alpha = u_1^{\alpha_1} \cdots u_g^{\alpha_g} \). Here for a Young diagram \( \Lambda \), \( S_\Lambda \) and \( s_\Lambda \) are the Schur functions defined by

\[
S_\Lambda(T) = s_\Lambda(t), \quad T_k := \frac{1}{k} \sum_{i=1}^{g} t_i^k.
\]

Let us simply write \( \mathcal{W}^k \) instead of \( \mathcal{W}^{(g)k} \). The vanishing locus of \( \sigma^{(g)} \) is:

\[
\Theta^{g-1} = (\mathcal{W}^{g-1} \cup [-1]\mathcal{W}^{g-1}) = \mathcal{W}^{g-1}.
\]

The last equality is due to Proposition 5.3, which shows that \( \sigma^{(g)} \) is an even or odd function under the action of \([-1] \); the reason for introducing \( \mathcal{W}^{g-1} \cup [-1]\mathcal{W}^{g-1} \) is that the analogous loci when \( g - 1 \) is replaced by \( k \) play an important role and \( \mathcal{W}^k \) is not \([-1]-\)invariant in general.

5.3. The Riemann fundamental relation. Using (6.2) to detect the divisors corresponding to the minus-sign operation on \( \mathcal{J}_4 \), we review a relation which we call the Riemann fundamental relation ([65], §95 in [3]):

**Proposition 5.5.** For \((P, Q, P_1, P'_1) \in X^2 \times (S^g(X) \setminus S_1^g(X)) \times (S^g(X) \setminus S_1^g(X))\),

\[
\exp \left( \sum_{i,j=1}^{g} \Omega^{(g) P_i Q_j} \right) = \frac{\sigma^{(g)}(\hat{u}_o(P) - u)\sigma^{(g)}(\hat{u}_o(Q) - v)}{\sigma^{(g)}(\hat{u}_o(Q) - u)\sigma^{(g)}(\hat{u}_o(P) - v)}
\]

\[
= \frac{\sigma^{(g)}(\hat{u}_o(P) - \hat{u}(P_1, \ldots, P_g))\sigma^{(g)}(\hat{u}_o(Q) - \hat{u}(P'_1, \ldots, P'_g))}{\sigma^{(g)}(\hat{u}_o(Q) - \hat{u}(P_1, \ldots, P_g))\sigma^{(g)}(\hat{u}_o(P) - \hat{u}(P'_1, \ldots, P'_g))}.
\]

**Proof.** The right-hand side can be expressed as

\[
\exp(\text{bilinear term in } u's) \theta(\omega^{(g)r-1}(\hat{u}_o(P) - u) + \xi_R)\theta(\omega^{(g)r-1}(\hat{u}_o(Q) - v) + \xi_R),
\]

where \( \xi_R \) is the Riemann constant. By Riemann’s theorem for theta functions in [25] (p. 23), the above becomes

\[
\exp(\text{bilinear term in } u's) \exp \left( \sum_{j=1}^{g} \int_{P_j}^{P} \tau^{(g)}_{P_j P'_j} \right).
\]

The exponential part of the bilinear term is given by

\[
(u - v)^{\gamma^{(g)}}(\hat{u}_o(P) - \hat{u}_o(Q)) = \sum_{i,j,k} \gamma^{(g)}_{i,j,k} \int_{P_i}^{P_k} \nu_{i,j}^{k} \int_{Q}^{P} \nu_{j}^{P},
\]

which equals (Lemma (5.1))

\[
\sum_{i=1}^{g} \Omega^{(g) P_i Q_j} = \sum_{i=1}^{g} \int_{Q}^{P} \tau^{(g)}_{P_i P'_j}.
\]
The above integrals depend upon the paths we choose, but (5.6) shows that such dependence cancels. Thus the right-hand side coincides with the left-hand side.

Proposition 5.6. For \((P, P_1, \ldots, P_g) \in X \times S^g(X) \setminus S_1^g(X)\) and \(u := \hat{u}(P_1, \ldots, P_g)\), the equality,

\[
\sum_{i,j=1}^{g} \varphi_{ij}^{(g)}(\hat{u}_a(P) - u) \phi_{i-1}^{(g)}(P) \phi_{j-1}^{(g)}(P_a) = \frac{F^{(g)}(P, P_a)}{(x - x_a)^2},
\]

holds for every \(a = 1, 2, \ldots, g\), where we set

\[
\varphi_{ij}^{(g)}(u) := -\frac{\sigma_i^{(g)}(u) \sigma_j^{(g)}(u) - \sigma^{(g)}(u) \sigma_{ij}^{(g)}}{\sigma^{(g)}(u)^2} \equiv -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma^{(g)}(u).
\]

Here for \(g = 4\) case, \(\phi_{i}^{(4)}\) is interpreted as \(\phi_{H_4,ij}^{(4)}\).

Proof. For the case \(X_4\), using the relation (3.9) and taking logarithm of both sides in the Riemann fundamental relation and differentiating along \(P_1 = P\) and \(P_2 = P_a\), we obtain the claim. Similarly we have the result for the case \(X_{12}\) using (4.11).

\[\square\]

6. JACOBI INVERSION FORMULAE FOR \((3, 7, 8)\) AND \((6, 13, 14, 15, 16)\) CURVES

In this section, we will consider a Jacobi-inversion type formula associated with \(X_4\) and \(X_{12}\).

6.1. The \(\mu_n^{(g)}\) functions over \((3, 7, 8)\) and \((6, 13, 14, 15, 16)\) curves. In this subsection, \(g\) will be 4 or 12, and \(\phi^{(g)}\) accordingly, \(\phi_{H_4}^{(4)}, \phi_{H_4}^{(4)}\), or \(\phi^{(12)}\). (We refer to \(\psi^{(k)}\) instead of \(\psi^{(g)^{(k)}}\).) In [50], we introduced meromorphic functions on the curve, reviewed here in Definition 6.1, which generalize the polynomial \(U\) in Mumford’s \((U, V, W)\) parameterization of a hyperelliptic Jacobian (which he attributes to Jacobi) [57].

For the definition of the function \(\mu\)’s, we introduce the Frobenius-Stickelberger (FS) matrix associated with \(X_4\) and \(X_{12}\). As in (3.8) and (4.10), we define the \(\ell\)-reduced Frobenius-Stickelberger (FS) matrix by:

\[
\psi_n^{(g;\ell)}(P_1, P_2, \ldots, P_n) := \begin{pmatrix}
\phi_0^{(g)}(P_1) & \phi_1^{(g)}(P_1) & \phi_2^{(g)}(P_1) & \cdots & \phi_\ell^{(g)}(P_1) & \cdots & \phi_n^{(g)}(P_1) \\
\phi_0^{(g)}(P_2) & \phi_1^{(g)}(P_2) & \phi_2^{(g)}(P_2) & \cdots & \phi_\ell^{(g)}(P_2) & \cdots & \phi_n^{(g)}(P_2) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\phi_0^{(g)}(P_n) & \phi_1^{(g)}(P_n) & \phi_2^{(g)}(P_n) & \cdots & \phi_\ell^{(g)}(P_n) & \cdots & \phi_n^{(g)}(P_n)
\end{pmatrix},
\]

and \(\psi_n^{(g;\ell)}(P_1, P_2, \ldots, P_n) := \det \psi_n^{(g;\ell)}(P_1, P_2, \ldots, P_n)\) (a check on top of a letter signifies deletion). It is also convenient to introduce the simpler notation:

\[
\phi_n^{(g)}(P_1, \ldots, P_n) := \det(\psi_n^{(g;\ell)}(P_1, \ldots, P_n)), \quad \psi_n^{(g)}(P_1, \ldots, P_n) := \psi_n^{(g;\ell)}(P_1, \ldots, P_n),
\]

\[\square\]
for the un-bordered matrix. We call this matrix Frobenius-Stickelberger (FS) matrix and its determinant Frobenius-Stickelberger (FS) determinant. These become singular for some tuples in \((X_g \setminus \infty)^n\).

**Definition 6.1.** For \(P, P_1, \ldots, P_n \in (X_g \setminus \infty) \times SS^n(X_g \setminus \infty)\), we define \(\mu^{(g)}_n(P)\) by

\[
\mu^{(g)}_n(P) := \mu^{(g)}_n(P; P_1, \ldots, P_n) := \lim_{P'_i \to P_i} \frac{1}{\psi^{(g)}_{n+1}(P'_1, \ldots, P'_n, P)},
\]

where the \(P'_i\) are generic, the limit is taken (irrespective of the order) for each \(i\); and \(\mu^{(g)}_{n,k}(P_1, \ldots, P_n)\) by

\[
\mu^{(g)}_{n,k}(P) = \phi^{(g)}_{n,k}(P) + \sum_{k=0}^{n-1} (-1)^{n-k} \mu^{(g)}_{n,k}(P_1, \ldots, P_n) \phi^{(g)}_{k}(P),
\]

with the convention \(\mu^{(g)}_{n,0}(P_1, \ldots, P_n) \equiv 1\). Let \(\mu^{(4)}_{H_a,n}\) be \(\mu^{(4)}_n\) for \(\phi^{(4)}_{H_a,n}\) for \(a = 0, 1\).

Meromorphic functions, viewed as divisors on the curve, allow us to express the addition structure of \(\text{Pic}X_g\) in terms of FS-matrices. For \(n\) points \((P_i)_{i=1,\ldots,n} \in X_g \setminus \infty\), we find an element of \(R\) associated with any point \(P = (x, y)\) in \((X_g \setminus \infty)\), \(\alpha_n(P) := \alpha_n(P; P_1, \ldots, P_n) = \sum_{i=0}^n a_i \phi_i(P)\), \(a_i \in \mathbb{C}\) and \(\alpha_n = 1\), which has a zero at each point \(P_i\) (with multiplicity, if the \(P_i\) are repeated) and has smallest possible order of pole at \(\infty\) with this property; \(\alpha_n\) can be identified with \(\mu^{(4)}_{H_a,n}\) for \(X_4\) and \(\mu^{(12)}\) for \(X_{12}\).

For \(\alpha_n\) for \(X_g\) \((g = 4, 12)\) and \(N^{(g)}(n)\) \((N^{(4)}_H(n), N^{(12)}(n))\), we have the following lemma:

**Lemma 6.2.** Let \(n\) be a positive integer. For \((P_i)_{i=1,\ldots,n} \in SS^n(X \setminus \infty)\), the function \(\alpha_n\) over \(X_g\) induces the map (which we call by the same name):

\[
\alpha_n : SS^n(X_g \setminus \infty) \to SS^{N^{(g)}(n)-n}(X_g),
\]

i.e., to \((P_i)_{i=1,\ldots,n} \in SS^n(X_g \setminus \infty)\) there corresponds an element \((Q_i)_{i=1,\ldots,N^{(g)}(n)-n} \in SS^{N^{(g)}(n)-n}(X_g)\), such that

\[
\sum_{i=1}^n P_i - n\infty \sim - \sum_{i=1}^{N^{(g)}(n)-n} Q_i + (N^{(g)}(n) - n)\infty.
\]

On the other hand, \(\frac{\mu^{(4)}_{H_{a,n}} dx}{3y^7 y_8}\) does not have a singularity over \(X \setminus \infty\) whereas \(\frac{\mu^{(4)}_{H_{a,n}} dx}{3y^7 y_8}\) does. The divisor of \(\mu^{(4)}_{H_{1,n}}(P)\) contains \(\sum_{i=1}^5 B_i - 5\infty \sim 2(B_4 + B_5) - 4\infty\). For \(\alpha_{H^1,n} := \mu^{(4)}_{H^1,n}\) for \(X_4\) and \(N^{(4)}_{H^1}(n)\), we have the following lemma:

**Lemma 6.3.** Let \(n\) be a positive integer. For \((P_i)_{i=1,\ldots,n} \in SS^n(X_4 \setminus \infty)\), the function \(\alpha_{H^1,n}\) over \(X_4\) induces the map (which we call by the same name):

\[
\alpha_{H^1,n} : SS^n(X_4 \setminus \infty) \to SS^{N^{(4)}_{H^1}(n)-n-5}(X_4),
\]
The third relation, 
\[
\hat{u}(P_1) = \hat{u}(Q_1''), \\
-\hat{u}(P_1', P_2') = -\hat{u}(Q_1'', Q_2'', Q_3), \\
\hat{u}(P_1', P_2', P_3') = \hat{u}(Q_1, Q_2, Q_3), \\
-\hat{u}(P_1, P_2, P_3) = -\hat{u}_o(Q_1, Q_2, Q_3) + 2\hat{u}_o(B_4, B_5).
\]
is quite important since it shows (see \cite{[56]} p.166) that Serre duality on $X_4$ is given as $\iota_3 : W^3 \to W^3$ by

$$P_1 + P_2 + P_3 + B_4 + B_5 - 5\infty \sim - (Q_1 + Q_2 + Q_3 + B_4 + B_5 - 5\infty).$$

**Proposition 6.6.** For $X_4$ we have the relation

$$W^3 \subset \hat{u}_o(S^3(X_4)).$$

**Proof.** Noting $(y^7/y_8) = B_4 + B_5 + \infty - B_1 - B_2 - B_3$, there are $P'_a \in X_4$ ($a = 1, 2, 3$) such that

$$P_1 + P_2 + P_3 + B_4 + B_5 - 5\infty \sim - (Q_1 + Q_2 + Q_3 + B_4 + B_5 - 5\infty)$$

$$\sim - (Q_1 + Q_2 + Q_3 + B_1 + B_2 + B_3 - 6\infty)$$

$$\sim P_1' + P_2' + P_3' - 3\infty.$$

In the last relation, we consider the divisor of $\mu_4^{(4)}(P; Q_1, Q_2, Q_3, B_1, B_2, B_3)$ noting $N_{H_{y}}^{(4)} = 9$.

Let image($\iota_3$) be denoted by $[-1]W^g$, especially $\iota_g : W^g \to [-1]W^g$. Noting for $n \geq g$, $\iota_g \circ \iota_n$ gives the Abel sum

$$W^y \xrightarrow{\iota_n} W^y \xrightarrow{\iota_g} W^y, \quad (\hat{u}(P_1, \ldots, P_n) \equiv \hat{u}(Q_1, \ldots, Q_g) \mod \Lambda).$$

In particular, the addition law on the Jacobian is given by $\iota_g \circ \iota_2 g$

$$W^{2g} \xrightarrow{\iota_2 g} W^g \xrightarrow{\iota_g} W^g, \quad (\hat{u}(P_1, \ldots, P_3, P'_1, \ldots, P'_3) \equiv \hat{u}(Q_1, \ldots, Q_g) \mod \Lambda).$$

**Proposition 6.7.** There are points $P_{R,1}, P_{R,2}$ and $P_{R,3}$ of $X_4 \setminus \infty$ such that $2\hat{u}(P_{R,1}, P_{R,2}, P_{R,3}) = 0$ and the Riemann constant associated with $X_4$ is given by

$$\xi_R = \hat{u}(P_{R,1}, P_{R,2}, P_{R,3}).$$

**Proof.** Let us consider the points $P_{0,1}, P_{0,2}$ and $P_{0,3}$ of $X_4 \setminus \infty$ satisfying

$$(P_{0,1} + P_{0,2} + P_{0,3}) + (B_4 + B_5) - 10\infty \sim 0,$$

and

$$(P_{0,1} + P_{0,2} + P_{0,3}) + (B_4 + B_5) - 5\infty \not\sim 0.$$

From Lemma[6.8] the condition means that every permutation $(P_1, P_2, P_3)$ of $(P_{0,1}, P_{0,2}, P_{0,3})$ is a non-trivial solution of $\beta(P_1, P_2, P_3) = 0$ for

$$\beta(P_1, P_2, P_3) = \begin{vmatrix} y_7(P_1) & y_8(P_1) & (xy_7)(P_1) & (xy_8)(P_1) \\ \frac{dy_7}{dx}(P_1) & \frac{dy_8}{dx}(P_1) & \frac{d(xy_7)}{dx}(P_1) & \frac{d(xy_8)}{dx}(P_1) \\ y_7(P_2) & y_8(P_2) & (xy_7)(P_2) & (xy_8)(P_2) \\ y_7(P_3) & y_8(P_3) & (xy_7)(P_3) & (xy_8)(P_3) \end{vmatrix}.$$

The non-triviality of the solution means

1) $(P_{0,1} + P_{0,2} + P_{0,3}) - 3\infty \sim 0$ does not hold,

2) $P_{0,a}$ is not equal to $B_b, \ a = 1, 2, 3, \ b = 1, \ldots, 5.$
By employing the notation \([A, B] := A_7B_8 - A_8B_7\) in this proof, we have the relation,

\[
\beta(P_1, P_2, P_3) = [y(P_1), \frac{dy}{dx}(P_1)] [y(P_2), y(P_3)] (x(P_1) - x(P_2)) (x(P_1) - x(P_3)) + [y(P_1), y(P_2)] [y(P_1), y(P_3)] (x(P_2) - x(P_3)).
\]

Further the relations \(\beta(P_1, P_2, P_3) = 0, \beta(P_2, P_3, P_1) = 0\) and \(\beta(P_3, P_1, P_2) = 0\) mean

\[
\begin{align*}
[y(P_1), \frac{dy}{dx}(P_1)] [y(P_2), y(P_3)] &\quad [y(P_1), y(P_2)] [y(P_1), y(P_3)] = 0, \quad \text{for } x \neq 5, \\
[y(P_2), \frac{dy}{dx}(P_2)] [y(P_3), y(P_1)] &\quad [y(P_2), y(P_1)] [y(P_2), y(P_3)] = 0, \\
[y(P_3), \frac{dy}{dx}(P_3)] [y(P_1), y(P_2)] &\quad [y(P_3), y(P_1)] [y(P_3), y(P_2)] = 0.
\end{align*}
\]

By arranging them, these relations imply that \(\hat{\beta}(P_1, P_2) = 0, \hat{\beta}(P_2, P_3) = 0,\) and \(\hat{\beta}(P_3, P_1) = 0,\) where

\[
\hat{\beta}(P_1, P_2) := \left( \frac{[y(P_1), y(P_2)]}{x(P_1) - x(P_2)} \right)^2 - [y(P_1), \frac{dy}{dx}(P_1)] [y(P_2), \frac{dy}{dx}(P_2)].
\]

It should be noted that \(\hat{\beta}(P_1, P_2) = 0\) is a relation in \(X_4^2\) rather than \(X_4^3\). Then it is obvious that \(\hat{\beta}(P_1, P_2) = \hat{\beta}(P_2, P_1)\), and \(\hat{\beta}(B_a, P_2) = 0\) for \(a = 1, \ldots, 5\). By expanding the first term, we have

\[
\hat{\beta}(P_1, P_2) = \left( \sum_{\ell=1}^4 \frac{1}{\ell!} [y(P_2), \frac{d^\ell y}{dx^\ell}(P_2)] (x(P_1) - x(P_2))^{\ell-1} \right)
\]

\[
\times \left( \sum_{\ell=1}^4 \frac{1}{\ell!} [y(P_1), \frac{d^\ell y}{dx^\ell}(P_1)] (x(P_1) - x(P_2))^{\ell-1} \right)
\]

\[
- [y(P_1), \frac{dy}{dx}(P_1)] [y(P_2), \frac{dy}{dx}(P_2)].
\]

Noting the order of \(\hat{\beta}\) with respect to \((x(P_1) - x(P_2))\), \(\hat{\beta}\) obviously has a single zero at \((x(P_1) - x(P_2))\). However we are not concerned with the case \(x(P_1) = x(P_2)\) due to (6.4). Thus, we consider

\[
\hat{\beta}(P_1, P_2) := \frac{\hat{\beta}(P_1, P_2)}{x(P_1) - x(P_2)},
\]

which does not vanish at \(x(P_1) = x(P_2)\). Noting that \(dy_a/dx\) belongs to \(\frac{1}{y_a^b} \mathbb{C}[x]\) and \(\hat{\beta}(B_b, P_2) = 0\) for \(b = 1, \ldots, 5\) (\(B_b\) is a zero of \(y_a\)), we find that \(\hat{\beta}(P_1, P_2)\) is a meromorphic function belonging to \(H^0((X_4 \setminus \infty)^2, \mathcal{O}_{X_4^2})\). Thus \(\hat{\beta}(P_1, P_2) = 0\) defines a hypersurface \(S\) in \((X_4 \setminus \infty)^2\), namely a curve. By letting \(p_a : X_4^2 \to X_4\) be the natural projections \((a = 1, 2)\), there is a continuous surjective map \(\varpi_a : S \to p_a((X_4 \setminus \infty)^2)\), which therefore is finite-to-one.
Finally, if we consider inside \((X_4 \setminus \infty)^3\) the three hypersurfaces \(S_{12}, S_{23}\) and \(S_{31}\), defined by \(\tilde{\beta}(P_1, P_2) = 0, \tilde{\beta}(P_2, P_3) = 0\) and \(\tilde{\beta}(P_3, P_1) = 0\), we can say that they have non-empty intersection by Bézout’s theorem, since the equations can be extended to the compactification. If one of the \(P_a\) is \(B_a (a = 1, 2, 3, b = 1, \ldots, 5)\), it is obvious that they don’t satisfy \((6.3)\). Hence any common zero of \(\tilde{\beta}\)’s must be a non-trivial solution.

In other words, we find non-trivial points in \(X_4^3\) satisfying \(\beta = 0\)’s and we let them be \(\{(P_{0,1}, P_{0,2}, P_{0,3})\} \subset (X_4 \setminus \infty)^3\).

Hence letting one of \((P_{0,1}, P_{0,2}, P_{0,3})\) be \((P_{R,1}, P_{R,2}, P_{R,3})\), we have

\[
\left( \frac{\mu^{(4)}_{H,3}(P; P_{R,1}, P_{R,2}, P_{R,3}) dx}{3y_s^2} \right) \sim 2(P_{R,1} + P_{R,2} + P_{R,3}) + 2(B_4 + B_5) - 4\infty
\]

by noting that the function \(\gamma\) in Lemma 6.8 is essentially the same as \(\Psi^{(4)}_4(P_1, \ldots, P_4)\). Due to Theorem 11 in [Lew],

\[
\hat{u}_o(2(P_{R,1} + P_{R,2} + P_{R,3}) + 2(B_4 + B_5)) = -2\xi_R,
\]

where \(\xi_R\) is the Riemann constant with base point \(\infty\).

**Lemma 6.8.** For a point \((P_{0,1}, P_{0,2}, P_{0,3}) \in (X_4 \setminus \infty)^3\) such that every permutation \((P_1, P_2, P_3)\) of \((P_{0,1}, P_{0,2}, P_{0,3})\) satisfies \(\beta(P_1, P_2, P_3) = 0\),

\[
2(P_{0,1} + P_{0,2} + P_{0,3}) + 2(B_4 + B_5) - 10\infty \sim 0.
\]

**Proof.** Let us consider the divisor of

\[
\gamma(P; P_1, P_2, P_3) = \begin{vmatrix}
y_7(P) & y_8(P) & (xy_7)(P) & (xy_8)(P) 
y_7(P_1) & y_8(P_1) & (xy_7)(P_1) & (xy_8)(P_1) 
y_7(P_2) & y_8(P_2) & (xy_7)(P_2) & (xy_8)(P_2) 
y_7(P_3) & y_8(P_3) & (xy_7)(P_3) & (xy_8)(P_3)
\end{vmatrix}.
\]

It is trivial that its divisor contains \(P_1 + P_2 + P_2\) and \((B_1 + B_2 + B_3 + B_4 + B_5)\). However there might be other three points as its zeros. By considering a local parameter \(t_c := x - x_c\) around \(P_c\), we have

\[
(y_7(P), y_8(P), x(P)y_7(P), x(P)y_8(P))
\]

\[
= (y_7(P_c) + \frac{d}{dx}y_7(P_c)t_c + \frac{1}{2!} \frac{d^2}{dx^2}y_7(P_c)t_c^2 + \cdots, 
\]

\[
\ldots, x(8y(P_c) + \frac{d}{dx}(xy_8(P_c))t_c + \frac{1}{2!} \frac{d^2}{dx^2}(xy_8(P_c))t_c^2 + \cdots).
\]
In other words, we have
\[
\gamma(P; P_1, P_2, P_3) = \begin{vmatrix}
    y_r(P_c) + \frac{d y_r(P_c)}{d x} t_c & y_s(P_c) + \frac{d y_s(P_c)}{d x} t_c \\
    y_r(P_1) & y_s(P_1) \\
    y_r(P_2) & y_s(P_2) \\
    y_r(P_3) & y_s(P_3)
\end{vmatrix} + (\text{terms with } t_c^\ell (\ell > 1)).
\]

As an example, in the \( P = P_1 \) case, \( \gamma(P; P_1, P_2, P_3) \) vanishes to second order because of the condition \( \beta(P_1, P_2, P_3) = 0 \). Hence the divisor of \( \gamma(P; P_1, P_2, P_3) \) must be
\[
2(P_1 + P_2 + P_3) + (B_1 + B_2 + B_3 + B_4 + B_5) - 11 \infty \sim 0,
\]
and the divisor of \( \frac{\partial \gamma(P; P_1, P_2, P_3)}{\partial y(P)} \) is given by
\[
2(P_1 + P_2 + P_3) + 2(B_4 + B_5) - 10 \infty \sim 0.
\]

Due to Proposition 6.7 and Theorem 1.1 in [25], we have the following corollary:

**Corollary 6.9.** There is a theta characteristic,
\[
\delta := \begin{bmatrix}
    \delta'' \\
    \delta'
\end{bmatrix} \in \left(\frac{1}{2}\mathbb{Z}\right)^{2g},
\]
which is equal to the Riemann constant \( \xi_R \) associated with \( X_g \) \( g = 4, 12 \) with respect to the base point \( \infty \) and the corresponding period matrix. In other words, for every \( (P_1, P_2, \ldots, P_{g-1}) \in S^{g-1} X_g \),
\[
\theta(\hat{u}(P_1, \ldots, P_{g-1}) + \xi_R) = 0.
\]

**Proof.** The \( g = 12 \) case is trivial, whereas the \( g = 4 \) case is due to Proposition 6.6 and Proposition 6.7. □ 

6.2. **Jacobi inversion formulae over \( \Theta^k \).** For \( X = X_4, X_{12} \), we also introduce
\[
S^m(X) := \{ D \in S^n(X) \mid \dim|D| \geq m \},
\]
where \( |D| \) is the complete linear system \( \hat{u}^{-1}(\hat{u}(D)) \) in IV.1 of [1], IV.1.

**Theorem 6.10. (Jacobi inversion formula)**

1. For \( (P, P_1, \ldots, P_4) \in X_4 \times S^4(X_4) \setminus S^4_1(X_4) \), we have
   - (a) \( \mu^{(4)}_{H^4,4}(P; P_1, \ldots, P_4) = \phi^{(4)}_{H^4,4}(P) - \sum_{j=1}^4 \phi^{(4)}_{H^4,j}(\hat{u}(P_1, \ldots, P_4)) \phi^{(4)}_{H^4,j-1}(P) \).
   - (b) \( \phi^{(4)}_{3-k,4}(\hat{u}(P_1, \ldots, P_4)) = (-1)^{3-k} \mu^{(4)}_{H^4,4}(P_1, \ldots, P_4), \ (k = 0, \ldots, 3) \).
2. For \( (P, P_1, \ldots, P_{12}) \in X_{12} \times S^{12}(X_{12}) \setminus S^{12}_1(X_{12}) \), we have
\[
(a) \; \mu_{12}^{(12)}(P, P_1, \ldots, P_{12}) = \phi_{12}^{(12)}(P) - \sum_{j=1}^{12} \varphi_{12j}^{(12)}(\hat{u}(P_1, \ldots, P_{12})) \phi_{j-1}^{(12)}(P).
\]
\[
(b) \; \varphi_{12,k+1}^{(12)}(\hat{u}(P_1, \ldots, P_{12})) = (-1)^{12-k-1} \mu_{12,k}^{(12)}(P_1, \ldots, P_{12}), \quad (k = 0, \ldots, 12).
\]

**Proof.** Same as in Prop.4.6 of [50], Prop. 4.6. \qed

We introduce
\[
\Theta^k := \mathcal{W}^k \cup [-1] \mathcal{W}^k.
\]
For \( X_4 \) case, noting Corollary 6.9, we have considered the locus \( \mathcal{W}^k \cup [-1] \mathcal{W}^k \) \( k = 0, 1, 2 \) of the theta divisor \( \mathcal{W}^3 \), rather than \( \hat{u}_o(S^k X_4) \cup [-1] \hat{u}_o(S^k X_4) \).

We have the following theorem, in which both sides are obtained by taking a limit by \( P_{k+1} \to \infty \) as mentioned in Th.5.1 of [50]. Some of the left-hand sides in the following theorem are given by zero over zero but they are well-defined in the limit.

**Theorem 6.11.** The following relations for \( \mu_{4i}^{(4)} \) to \( X_4 \), and for \( \mu_{12}^{(12)} \) to \( X_{12} \) hold.

1. **\( \Theta^g \) case:** for \( (P_1, \ldots, P_g) \in S^g(X_g) \setminus S_1^g(X_g) \) and \( u = \hat{u}(P_1, \ldots, P_g) \in \kappa^{-1}(\Theta^g) \),
   \[
   \frac{\sigma_i^{(g)}(u) \sigma_g(u) - \sigma_{g,i}^{(g)}(u) \sigma(g)(u)}{\sigma(g)^2(u)} = (-1)^{g+1-i} \mu_{g,i-1}^{(g)}(P_1, \ldots, P_g), \quad \text{for } 0 < i \leq g.
   \]

2. **\( \Theta^{g-1} \) case:** for \( (P_1, \ldots, P_{g-1}) \in S^{g-1}(X_g) \setminus S_1^{g-1}(X_g) \) and \( u = \pm \hat{u}(P_1, \ldots, P_{g-1}) \in \kappa^{-1}(\Theta^{g-1}) \),
   \[
   \frac{\sigma_i^{(g)}(u)}{\sigma_g^{(g)}(u)} = \begin{cases} (-1)^{g-i} \mu_{g-1,i-1}^{(g)}(P_1, \ldots, P_{g-1}) & \text{for } 0 < i < g, \\ 1 & \text{for } i = g. \end{cases}
   \]

3. **\( \Theta^k \) case:** for \( (P_1, \ldots, P_k) \in S^k(X_g) \setminus S_1^k(X_g) \) and \( u = \pm \hat{u}(P_1, \ldots, P_k) \in \kappa^{-1}(\Theta^k) \), \( (k = 1, 2, \ldots, g - 2) \),
   \[
   \frac{\sigma_i^{(g)}(u)}{\sigma_{k+1}^{(g)}(u)} = \begin{cases} (-1)^{k-i+1} \mu_{k,i-1}^{(g)}(P_1, \ldots, P_k) & \text{for } 0 < i \leq k, \\ 1 & \text{for } i = k + 1, \\ 0 & \text{for } k + 1 < i \leq g. \end{cases}
   \]

**Proof.** Essentially the same as in Th.5.1 of [50], Th. 5.1. \qed

We should note that for the \( k = 1 \) case, we have
\[
\frac{\sigma_1^{(12)}(u)}{\sigma_2^{(12)}(u)} = -x, \quad \text{for } (x, y_{13}, y_{14}, y_{15}, y_{16}) \in X_{12}.
\]
7. Configurations of Curves and Monstrous Moonshine

We highlight some relationships between the curves $X_4$ and $X_{12}$, and an additional smooth curve $\tilde{X}_4$ of genus 4, with affine plane model of type $(3, 4)$, and their algebraic functions; the degrees of these functions and of the differentials on the curves are related to numbers that appear in the Monstrous Moonshine. The idea for relating the curves is to factor out a (local) group action; this will be achieved by retaining only some of the functions in the affine rings of the curves, and manufacturing the relations they satisfy.

7.1. Covering spaces of $(3, 7, 8)$, and $(6, 13, 14, 15, 16)$ curves. We will relate to $X_{12}$ certain covers and subcovers, with the goal of presenting observations on symmetries, acting on algebraic or transcendental functions, that appear to be related to the Monster.

First we should note that by sending $y_7$ and $y_8$ to $y_{14}$ and $y_{16}$, the relations $f_{12,4}$, $f_{12,7}$ and $f_{12,9}$ for $X_{12}$ map to $f_{14}$, $f_{15}$, and $f_{16}$ for $X_4$, respectively. Hence $X_{12}$ has a natural projection to $X_4$. More precisely, $X_{12}$ is a double covering of $X_4$ as mentioned in the proof of Proposition 2.1.

The Jacobian $J^{(4)}$ has 120 odd and 136 even $\theta$ characteristics for a total of $2^8 = 256$. Coble in Ch. 5 of [13, Ch. 5] identified several group actions on the characteristics and Vakil [70] introduced an $E_8$ action. This sporadic group plays a role in the representation of the Monster [14]. Through the double covering $X_{12}$ over $X_4$, the basis of the tangent space of $J^{(4)}$ is naturally embedded in those of $J^{(12)}$, $\iota : T J^{(4)} \rightarrow T J^{(12)}$,

\[
\frac{1}{2} \iota(\nu_{1}^{J^{(4)}}) = \nu_{4}^{J^{(12)}}, \quad \frac{1}{2} \iota(\nu_{2}^{J^{(4)}}) = \nu_{6}^{J^{(12)}}, \quad \frac{1}{2} \iota(\nu_{3}^{J^{(4)}}) = \nu_{9}^{J^{(12)}}, \quad \frac{1}{2} \iota(\nu_{4}^{J^{(4)}}) = \nu_{11}^{J^{(12)}}.
\]

We have also have morphisms $X_4 \rightarrow X_6$ and $X_4 \rightarrow X_7$. Similarly $X_{12}$ admits projections, from $X_{12}$ to $X_4$, $X_6$, $X_7$ and the following two further curves $X_2$ and $X_{30}$:

1. a hyperelliptic curve $X_2$ of genus two given by $f_{12,9} = y_{15}^2 - \hat{k}_2(x)k_3(x)$, and
2. a $(6, 13)$ singular curve $X_{30}$ of genus $30 = g((6, 13))$ given by $y_{13}^6 - \hat{k}_2(x)^3k_2(x)^2k_3(x)$.

As we showed in (4.6), we also have the natural projections from the affine curve $\hat{C}_{12}$ to the affine curve $C_{12} \subset X_{12}$, where $\hat{C}_{12} := \text{Spec} \hat{R}_{12}$ and $C_{12} := \text{Spec} R_{12}$. Since $R_{12}$ is decomposed as $\mathbb{C}[x]$-algebra;

\[
\hat{R}_{12} = \mathbb{C}[x, w_3]/(w_3^6 - k_3(x)) \otimes_{\mathbb{C}[x]} \mathbb{C}[x, w_2]/(w_2^6 - k_2(x)) \otimes_{\mathbb{C}[x]} \mathbb{C}[x, \hat{w}_2]/(\hat{w}_2^6 - \hat{k}_2(x)),
\]

$\hat{C}_{12}$ is isomorphic as a fiber product of $C_1 \times_C C_0 \times_C \hat{C}_0$, where $C := \text{Spec} \mathbb{C}[x]$, $C_1 := \text{Spec} \mathbb{C}[x, w_3]/(w_3^6 - k_3(x))$, $C_0 := \text{Spec} \mathbb{C}[x, w_2]/(w_2^6 - k_2(x))$ and $\hat{C}_0 := \text{Spec} \mathbb{C}[x, \hat{w}_2]/(\hat{w}_2^6 - \hat{k}_2(x))$.

Additionally, $\hat{C}_{12}$ has a projection to $\hat{C}_4 := \text{Spec} \hat{R}_4$, $(\hat{C}_{12} \rightarrow \hat{C}_4)$ where

\[
\hat{R}_4 := \mathbb{C}[x, \hat{y}_4]/(\hat{y}_4^6 - k_3(x)k_2(x)).
\]
by identifying \(w_3w_2\) with \(\hat{y}_4\) as in \([45,44]\), i.e., \(\hat{y}_4 := w_3w_2\). This curve \(\hat{C}_4\) is birational to the sextic affine curve \(\hat{C}'_4\) for \(w^3 = \prod_{i=0}^{5} (z - c_i)\) via the change of variables:

\[
z := \frac{1}{x - \hat{c}_0} + c_0, \quad w := \frac{1}{3 - \prod_{i=1}^{5} (b_i - c_0)} (z - c_0)^2, \quad c_i := \frac{1}{b_i - \hat{c}_0} + c_0
\]

giving local coordinates at \(\infty\) for the Riemann surface \(\hat{X}_4 = \hat{C}_4 \cup \hat{C}'_4\). The Jacobian \(J^{(4)}\) of \(\hat{X}_4\) also has 256 theta characteristics, the order of \((\mathbb{Z}/2\mathbb{Z})^8\), which Coble used in \([13]\) (cf. p. 250). It is known that the reflection group of \(E_8\) is related to an algebraic curve of genus 4 and degree 6 via a Del Pezzo surface and these theta characteristics \([45,72]\). The surface \(y^3 = x^5 + q^2\) has a Du Val singularity and is also connected with \(E_8\) and McKay correspondence \([21, 20, 68, 69]\).

7.2. **Numbers 10 and 19 in \(L(6, 13, 14, 15, 16)\) and Norton Numbers.** Our first remark is that the complementary sequence of the sub-semigroup \(H_{12}\) of the non-negative integers \(\mathbb{N}_0\) generated by \(M_{12} := \{6, 13, 14, 15, 16\}\), namely the Weierstrass semigroup of \(X_{12}\) at \(\infty\), is

\[
L(H_{12}) := \{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 17, 23\}
\]

while the Norton sequence is given by \([26, 53, 54, 60]\) (see Appendix A),

\[
\{1, 2, 3, 4, 5, 7, 8, 9, 11, 17, 19, 23\}
\]

The sequences differ only by the elements 10 and 19. Using the covering \(\hat{C}_{12}\) of \(C_{12}\), we notice that the order of pole of the denominator of the holomorphic one form \((4.7)\) of \(X_{12}\) decomposes as 10 + 19:

\[
\text{wt} \left( \frac{1}{y_{13}y_{16}} \right) = \text{wt} \left( \frac{1}{y_{14}y_{15}} \right) = \text{wt} \left( \frac{1}{w_3^2w_2^3} \right) = 10 + 19,
\]

because \(\text{wt}(w_3^2w_2^3) = 10\) and \(\text{wt}(w_3^2w_2^3w_3^2) = 19\) in \(H^0(\hat{C}_{12}, \mathcal{O}_{\hat{C}_{12}})\). This points to the significance of the configurations of curves we consider.

7.3. **A similarity between \(\sigma^{(12)}\) and replicable functions.** For a replicable function (see Appendix A)

\[
f(q) = \frac{1}{q} + \sum_{i=1}^{\infty} c_i q^i,
\]

the coefficients generated by a finite number of \(h_i\)'s,

\[
c_i \in \mathbb{Q}[h_1, h_2, h_3, h_4, h_5, h_7, h_8, h_9, h_{11}, h_{17}, h_{19}, h_{23}],
\]

plays an essential role in the moonshine phenomena as in \([26, 53, 54]\). The Grunsky coefficients \(h_{m,n}\) of \(f(q)\) are defined by \([31]\)

\[
\sum_{m,n} h_{m,n} p^m q^n := \log \left( \frac{pq}{p-q} \right),
\]

and have the property,

\[
h_{m,n} \in \mathbb{Q}[h_1, h_2, h_3, h_4, h_5, h_7, h_8, h_9, h_{11}, h_{17}, h_{19}, h_{23}].
\]
Further it is known that the SL(2,\mathbb{Z}) action on replicable functions can be expressed in terms of Faber polynomials \cite{54}. Since the \(\tau\)-function solutions to the dKP hierarchy is also given by the Faber polynomials \cite{12}, we hope that the sigma function can be expressed in terms of theta, tau, sigma functions \cite{16,59,55}.

**Remark 7.1.**

1. As mentioned in Remark \cite{5,4}, the \(\sigma^{(12)}\) function over \(\mathbb{C}^{12}\) is an entire function and is also expected to be expressed by the Schur function. If the moduli parameters \((b_1, \ldots, b_7)\) are polynomials of \(p\) and \(q\), it is expected that
   \[
   \sigma^{(12)}(u) - S_{y_{12}}(T)|_{t_i=t_i} = \sum_{m,n} \hat{h}_{m,n} q^m p^n,
   \]
   where
   \[
   \hat{h}_{m,n} \in \mathbb{Q}[t_1, t_2, t_3, t_4, t_5, t_7, t_9, t_{10}, t_{11}, t_{17}, t_{23}].
   \]
2. \(\sigma^{(12)}\) is a generalization of Weierstrass sigma function whereas the Weierstrass sigma function plays an important role in \cite{28}; if we consider a suitable degeneration of the curve \(X_{12}\) associated with elliptic curves, it is expected that \(\sigma^{(12)}\) might be written in terms of Weierstrass’ elliptic \(\sigma\).
3. By the Riemann-Kempf theorem, using \cite{4,9}, a suitable derivative of \(\sigma, \sigma_{t_1, \ldots, t_k}\), is a function of \(\{t_1, t_2, \ldots, t_{23-N^{(12)(13-k)}}\}\) over an open subset of the subvariety \(W_k\) \cite{52}. The other \(t\)’s of the hierarchy are functions of \(t_1, t_2, \ldots, t_{23-N^{(12)(13-k)}}\) \cite{52}. This is analogous to the fact that for a suitable replicable function, every \(c_i\) belongs to a subring \(\mathbb{Q}[h_1, \ldots, h_k]\) of \(\mathbb{Q}[h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9, h_{11}, h_{17}, h_{19}, h_{23}]\), and a subset of \(h\)’s are functions of the other \(h\)’s, e.g., for the case of the \(j\)-function, \(c_i \in \mathbb{Q}[h_1, h_2, h_3, h_5]\) \cite{46}.
4. (For an \((r, s)\) curve), algebraic solutions to the dKP equation exist \cite{51}: it should be possible to generalize them to curves covered locally by affine patches such as \(X_4\) and \(X_{12}\). As an illustration, let us consider the case that \(k_2 = k_3\): Then \(\phi^{(12)}_{11} \phi^{(12)}_9 = (\phi^{(12)}_{10})^2\) and for \(v := y_{14}/y_{15} = \frac{d}{dt_2} \frac{\sigma^{(12)}}{\sigma^{(12)}_2} / \frac{d}{dt_3} \frac{\sigma^{(12)}}{\sigma^{(12)}_2}\) over \(\tilde{u}(X_{12}) \subset \mathbb{C}^{12}\), we have a \(X_{12}\) curve solution of the dKP equation,
   \[
   \frac{\partial}{\partial t_3} \frac{\partial}{\partial t_1} v + \frac{\partial}{\partial t_1} \left( v \frac{\partial}{\partial t_1} v \right) - 2 \frac{\partial}{\partial t_2} \frac{\partial}{\partial t_2} v = 0.
   \]
   The proof is the same as in \cite{51}. It is expected that we might have the dKP hierarchy for \((t_1, t_2, t_3, t_4, t_5, t_7, t_8, t_9, t_{10}, t_{11}, t_{17}, t_{23})\). It is noted that J. McKay, and A. Sebbar considered the relation between replicable functions and \(\tau\)-functions of the dKP hierarchy \cite{54}.
APPENDIX A. NORTON CONDITION

Let $A$ be a $\mathbb{Q}$ ring, with filtration associated to multiplication, $A = \bigcup_{i} A_i$ and $A_j A_i \subset A_{i+j}$. Let us consider $q = e^{2 \pi \sqrt{-1} \tau}$ for $z \in H_+ := \{\tau \in \mathbb{C} \mid \Re \tau \geq 0\}$ and a function

$$f(q) = q^{-1} + h_1 q + h_2 q^2 + \cdots$$

where $h_j \in A_j$. We also write $f(\tau) = f(q)$.

The Grunsky coefficient $h_{m,n} \in A_{m+n-1}$ of $f(q)$ is defined by \[31\]

$$\sum_{m,n} h_{m,n} p^m q^n := \log \left( \frac{p q f(q) - f(p)}{p - q} \right)$$

and the Faber polynomial $F_{f,n}(f)$ \[67\] \[23\] is defined by

$$F_{f,n}(f(q)) = \frac{1}{p^n} + n \sum_{m>0} h_{m,n} p^m,$$

where $h_{1,m} = h_m$. From the definition of Grunsky coefficients, we have this property:

**Lemma A.1.**

$$h_{r,s} = h_{r+s-1} + \frac{1}{r+s} \sum_{m=1}^{r-1} \sum_{n=1}^{s-1} (n+m) h_{r-s-m-n-1} h_{m,n}.$$

These appear in the dispersionless KP hierarchy \[12\].

The replicable functions are generalizations of the elliptic modular function $j(\tau)$, which is characterized by its expansion and the following property under the action of Hecke operators $T_n$ for every $n \geq 1$,

$$n T_n(j(\tau)) = \sum_{a,d, n, 0 \leq b < d} j \left( \frac{a \tau + b}{d} \right) = F_{j,n}(j(\tau)).$$

This gives an $SL_2(\mathbb{Z})$ action on $j$.

The replicable functions were characterized by Norton, as having certain properties under the action of the Hecke operators, as members of the finite family of functions \{ $f^{(a)}$ \} given by

$$\sum_{a,d, n, 0 \leq b < d} f^{(a)} \left( \frac{a \tau + b}{d} \right) = F_{f,n}(f(\tau)).$$

Norton considered characterized the coefficients $h_{m,1}$ due to such action.

**Definition A.2.** If whenever $nm = rs$ and $(n,m) = (r,s)$, we have the identity $h_{n,m} = h_{r,s}$, we say that $H$ is replicable. The condition is called Norton condition.

**Example A.3.**

1. $h_6 = h_{3,2} = h_4 + h_1 h_2$,
2. $h_{12} = h_{3,4} = h_6 + h_2^2 h_2 + 2h_2 h_3 + h_1 h_4$,
3. $h_{10} = h_{5,2} = h_6 + h_1 h_4 + h_2 h_3$,
4. $h_{14} = h_{7,2} = h_8 + h_1 h_6 + h_2 h_5 + h_3 h_4$, and
5. $h_{15} = h_{3,5} = h_7 + 2h_2 h_4 + h_3^2 + h_3 h_1 + h_1^2 h_3 + h_1 h_2^2$. 
The Norton condition is not consistent with the grading of $A$ unless we modify the grading, e.g., $h_{-1} = 1$ with the weight $-1$.

Norton proved that [15],

**Theorem A.4.** For a replicable function $f$ with Fourier coefficients $h_n$, every $h_n$ belongs to $\mathbb{Q}[H]$, where $H := \{h_i \mid i \in \Phi\}$ and $\Phi := \{1, 2, 3, 4, 5, 7, 8, 9, 11, 17, 19, 23\}$.

The set of the numbers $\ell = 1, 2, 3, 4, 5, 7, 8, 9, 11, 17, 19, 23$ is weakly symmetric except for the pair 4 and 10 or the pair 10 and 19 as shown in Table A.1.

Table A.1: Weakly symmetric property of Norton numbers.

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|---|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| - | - | 1 | 2 | 3 | 4 | 5 | - | 7 | 8 | 9 | - | 11 | - | - | - | - | 17 | - | 19 | - | - | - | - | 23 |
| 23 | - | - | - | 19 | - | 17 | - | 11 | 9 | 8 | 7 | 5 | 4 | 3 | 2 | 1 | - | - | - | - | - | 23 |
| 23 | 22 | 21 | 20 | 19 | 18 | 17 | 16 | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |

Lastly, let’s note that the complement of $\{1, 2, 3, 4, 5, 7, 8, 9, 11, 17, 19, 23\}$ is not a numerical semigroup because, for example, 10 and 13 belong to the complement but $23 = 10 + 13$ does not.

**APPENDIX B. COVERING OF CURVES**

As the correspondence between an algebraic variety and a commutative ring, we have the well-known normalization theorem [32, p.5, p.68]:

**Theorem B.1.** For any irreducible algebraic curve $X \subset P^2 \mathbb{C}$, there exists a compact Riemann surface $\tilde{X}$ and a holomorphic mapping $s: \tilde{X} \rightarrow P^2 \mathbb{C}$ such that $s(\tilde{X}) = X$ and $s$ is injective on the inverse image of the set of smooth points of $X$. Further the Riemann surface is unique up to its isomorphism; if there are two Riemann surfaces $\tilde{X}$ and $\tilde{X}'$ given by normalizations of $X$, there is a biholomorphic from $\tilde{X}$ to $\tilde{X}'$.

Using this theorem, we have the following results:

**Proposition B.2.** Without loss of generality, we assume that every $b_a$ does not vanish. By letting $y_7 := (y_7/x^3)$, $y_8 := (y_8/x^3)$ and $x := 1/x$,

$$
\begin{align*}
\mathcal{U}_4 &:= \begin{pmatrix}
\frac{\partial f_{14}}{\partial y_7} & \frac{\partial f_{14}}{\partial y_8} & \frac{\partial f_{14}}{\partial x} \\
\frac{\partial f_{15}}{\partial y_7} & \frac{\partial f_{15}}{\partial y_8} & \frac{\partial f_{15}}{\partial x} \\
\frac{\partial f_{16}}{\partial y_7} & \frac{\partial f_{16}}{\partial y_8} & \frac{\partial f_{16}}{\partial x}
\end{pmatrix}
\end{align*}
$$

where $k_2(x) = (1 - b_1 x)(1 - b_2 x)(1 - b_3 x)$ and $k_3(x) = (1 - b_1 x)(1 - b_5 x)$,
whose rank is 2 in a neighborhood of \((x, y_x, y_y) = (0, 0, 0)\).

**Proof.** Since \(U_4\) is
\[
U_4 = \left[ \begin{array}{ccc} y_x(xk_3(x))' & 2y_7 & -xk_2(x) \\ (xk_2(x)k_3(x))' & y_8 & y_7 \\ y_2k_2(x) & -k_3(x) & 2y_4 \end{array} \right],
\]
it is obvious that its rank is 2 at \((x, y_x, y_y) = (0, 0, 0)\). For the case that \((x, y_x, y_y) \neq (0, 0, 0)\), since the structure is the same as that of \(U_4\), its rank is also 2.

We can define
\[R_4 := \mathbb{C}[x, y_7, y_8]/(f_{14}^{(b,z)}, f_{15}^{(b,z)}, f_{16}^{(b,z)}),\]
and \(\text{Spec } R_4\). The curve \(X_4\) defined by affine patches are \(\text{Spec } R_4\) and \(\text{Spec } R_4\) is non-singular.

**B.1.1. Homogeneous ring of \(X_4\).** By partially following the arguments on the space curve in Pinkham [62], in this subsection, we mention relations among \(\text{Spec } R_4, X_4, \) a homogeneous ring, its Proj, the numerical semigroup \(H_4\), and the monomial curve related to \(B_{H_4}\).

The coefficient ring is given by \( \mathcal{C}^{(4)} := \mathbb{C}[[b_1, b_2, b_3, b_4]] \) and its two projections are introduced by
\[\varphi_b^{(4)} : \mathcal{C}^{(4)} \to \mathbb{C} = \mathcal{C}^{(4)}/(b_1 - b, b_2 - b, b_3 - b, b_4 - b),\]
and
\[\varphi_0^{(4)} : \mathcal{C}^{(4)} \to \mathbb{C} = \mathcal{C}^{(4)}/(b_1, b_2, b_3, b_4).\]

We are now considering
\[R_4^{(b,z)} := \mathcal{C}^{(4)}[x, y_7, y_8, z]/(f_{14}^{(b,z)}, f_{15}^{(b,z)}, f_{16}^{(b,z)}),\]
where for \(k_3^{(b,z)} := (x - b_1 z^3)(x - b_2 z^3)(x - b_3 z^3)\) and \(k_2^{(b,z)} := (x - b_4 z^3)(x - b_5 z^3)\),
\[f_{14}^{(b,z)}(x, y_7, y_8, z) := y_7^2 - y_8 k_2^{(b,z)}, \quad f_{15}^{(b,z)}(x, y_7, y_8, z) := y_7 y_8 - k_2^{(b,z)} k_3^{(b,z)}(x),\]
\[f_{16}^{(b,z)}(x, y_7, y_8, z) := y_8^2 - y_7 k_3^{(b,z)}(x).\]
Then we have \(\varphi_b^{(4)}(f_{a}^{(b)}(x, y_7, y_8, 1)) = f_a(x, y_7, y_8)\) and \(\varphi_0^{(4)}(f_{a}^{(b)}) = f_a^{(z)}\) for \(a = 14, 15, 16\).

By letting \(R_4^{(b,z)} := R_4^{(b,z)}/(z - 1)\), we have natural homomorphisms,
\[
\begin{align*}
\text{Spec } B_{H_4} & \longrightarrow \text{Spec } R_4^{(b,z)} & \longrightarrow \text{Spec } R_4 \\
\downarrow & & \downarrow \\
\text{Spec } \mathbb{C} & \longrightarrow \text{Spec } \mathcal{C}^{(4)} & \longrightarrow \text{Spec } \mathbb{C}
\end{align*}
\]
This shows that \(\text{Spec } R_4\) and \(\text{Spec } B_{H_4}\) are fibers over the moduli space of pointed curves with given Weierstrass semigroup \(H_4\). Note that \(R_4^{(b,z)}\) is related to the moduli space \(\mathcal{M}_4\) of \((3, 7, 8)\) curves [62 Ch.IV](Ch.IV).
We deal with the projectivization by defining $R^{(z)}_4 := \varphi^{(4)}_b R^{(b,z)}_4$. Here $z^3$ has the same weight as $x$. The weighted homogeneous ideal in $R^{(z)}_4$ under $\mathbb{C}^*_m$-action provides $\text{Proj} R^{(z)}_4$. We have

$$R^{(z)}_4/(z-1) \approx R_4, \quad R^{(z)}_4/(z) \approx B_{H_4}.$$  

Thus we have the unique point $\infty$ as a ramified point by letting $z = 0$, which recovers (3.1). More precisely, $Z_3$, $Z_7$ and $Z_8$ correspond to $z^3/x$, $z^7/y_7$, and $z^8/y_8$ whereas $x$, $y_7$ and $y_8$ correspond to $z^3/x$, $y_7z^2/x^3$, and $y_8z/x^3$; for example, the relation $f^{(b,z)}_{14}(x, y_7, y_8, z)$ becomes

$$f^{(b,z)}_{14}(x, y_7, y_8, z) = \frac{y_7^2 y_8 x^2}{z^4} \left( \left( \frac{z^8}{y_8} \right) \left( \frac{z^3}{x} \right)^2 - \left( \frac{z^7}{y_7} \right)^2 \left( 1 - b_4 \frac{z^3}{x} \right) \left( 1 - b_5 \frac{z^3}{x} \right) \right),$$

around $z = 0$. Hence we can identify $\text{Proj} R^{(z)}_4$ with $X_4$ and the unique infinity point $\infty$ with the point $z = 0$, where $z$ can be regarded as element of $\mathbb{C}^*_m$.

The following relations hold, for functions $w_i$ which are algebraic over the ring $R_4$, and parametrizations of $y_7$ and $y_8$:

\begin{equation}
(\text{B.2}) \quad y_7 = w_3 w_2^2, \quad y_8 = w_3^2 w_2,
\end{equation}

where

\begin{equation}
(\text{B.3}) \quad w_2^3 = k_2, \quad w_3^3 = k_3.
\end{equation}

Since $\hat{R}_4 := \mathbb{C}[x, w_2, w_3]/(w_3^2 - k_2(x), w_3^3 - k_3(x)) = \mathbb{C}[x, w_2]/(w_3^2 - k_2(x)) \otimes_{\mathbb{C}[x]} \mathbb{C}[x, w_3]/(w_3^3 - k_3(x))$, its spectrum has a fiber structure.

### B.2. Riemann surface $X_{12}$ and affine curves.

We consider a different affine model for $X_{12}$ than the one in Section 4. As for the $(3, 7, 8)$ curve, we can parametrize this curve with two smooth charts; we construct the one that contains $\infty$:

**Proposition B.3.** Without loss of generality, we assume that every $b_a$ does not vanish. Let us define $y_{2a} = y_{2a}/x^3$ for $a = 13, 14, 15, 16$.

\begin{align*}
& \hat{f}_{12,1} := y_{13}^2 - x \hat{k}_2(x) y_{14}; \quad \hat{f}_{12,2} := y_{13} y_{14} - x \hat{k}_2(x) y_{15}; \quad \hat{f}_{12,3} := \hat{k}_2(x) y_{14}^2 - y_{13} y_{15} \hat{k}_3(x); \\
& \hat{f}_{12,4} := y_{14}^2 - x \hat{k}_2(x) y_{16}; \quad \hat{f}_{12,5} := y_{13} y_{16} - y_{14} y_{15}; \quad \hat{f}_{12,6} := y_{15}^2 - x \hat{k}_2(x) k_3(x); \\
& \hat{f}_{12,7} := y_{14} y_{16} - x \hat{k}_2(x) k_3(x); \quad \hat{f}_{12,8} := y_{15} y_{16} - k_3(x) y_{13}; \quad \hat{f}_{12,9} := y_{16}^2 - k_3(x) y_{14};
\end{align*}

where $\hat{k}_2(x) = (1 - b_3 x)(1 - b_7 x)$. For every $(x, y_{13}, y_{14}, y_{15}, y_{16})$ which is zero of every $(\hat{f}_{12,a})_{a=1,\ldots,9}$, we have

$$\text{rank } U_{12} = 4, \quad U_{12} := \left( \begin{array}{cccc}
\partial/\partial x \hat{f}_{12,a} & \partial/\partial y_{23} \hat{f}_{12,a} & \partial/\partial y_{14} \hat{f}_{12,a} & \partial/\partial y_{215} \hat{f}_{12,a} \\
\partial/\partial y_{23} \hat{f}_{12,a} & \partial/\partial y_{14} \hat{f}_{12,a} & \partial/\partial y_{215} \hat{f}_{12,a} & \partial/\partial y_{216} \hat{f}_{12,a} \\
\end{array} \right)_{a=1,\ldots,9}. \right.$$
around \((x, y_{13}, y_{14}, y_{15}, y_{16}) = (0, 0, 0, 0, 0)\).

Proof. \(U_{12}\) is

\[
\begin{pmatrix}
-\bigl(\frac{x}{z_1}\bigr)'y_{14} & 2y_{13} & -\dot{k}_2 \\
-\bigl(\frac{x}{z_1}\bigr)'y_{15} & y_{14} & y_{13} & -x\dot{k}_2 \\
\bigl(\frac{x}{z_1}\bigr)'y_{14}^2 - y_{13}^2 & -y_{15}xk_2 & 2x\dot{k}_2y_{14} & -y_{13}x\dot{k}_2 \\
-\bigl(\frac{x}{z_1}\bigr)'y_{16} & y_{16} & -y_{15} & -y_{14} & y_{13} & \dot{k}_2 \\
-\bigl(\frac{x}{z_1}\bigr)'y_{15} & -\dot{k}_3 & y_{16} & y_{14} & y_{13} \\
-\bigl(\frac{x}{z_1}\bigr)'y_{14} & -\dot{k}_3 & -\dot{k}_3 & y_{16} & y_{15} & 2y_{16}
\end{pmatrix}.
\]

At \((x, y_{13}, y_{14}, y_{15}, y_{16}) = (0, 0, 0, 0, 0)\), it is obvious that its rank is 4. For the case that \((x, y_{13}, y_{14}, y_{15}, y_{16}) \neq (0, 0, 0, 0, 0)\), since the structure is the same as that of \(U_{12}\), its rank is also 4.

The scheme related to \(\text{Spec } R_{12}\) and \(\text{Spec } R_{12}\) is denoted by \(X_{12}\). It has a unique \(\infty\) point as \(1/x = 0\).

B.2.1. Homogeneous ring of \(X_{12}\). As mentioned in Section 4, we partially follow the arguments on the space curve by Pinkham [62] and describe \(X_{12}\) more precisely. The coefficient ring is given by

\[
\mathcal{C}^{(12)} := \mathbb{C}[b_1, b_2, b_3, b_4, b_5, b_6, b_7]
\]

and its two projections are introduced by

\[
\varphi^{(12)}_b : \mathcal{C}^{(12)} \to \mathbb{C} = \mathcal{C}^{(12)}/(b_1 - b_1, b_2 - b_2, b_3 - b_3, b_4 - b_4, b_5 - b_5, b_6 - b_6, b_7 - b_7),
\]

and

\[
\varphi^{(12)}_0 : \mathcal{C}^{(12)} \to \mathbb{C} = \mathcal{C}^{(12)}/(b_1, b_2, b_3, b_4, b_5, b_6, b_7).
\]

We are now considering

\[
R^{(b, z)}_{12} := \mathcal{C}^{(12)}[x, y_{13}, y_{14}, y_{15}, y_{16}, z]/\mathcal{P}^{(b, z)},
\]

where for \(k^{(b, z)}_{1} := (x - \hat{b}_1z^6)(x - \hat{b}_2z^6)(x - \hat{b}_3z^6), k^{(b, z)}_{2} := (x - \hat{b}_4z^6)(x - \hat{b}_5z^6), k^{(b, z)}_{3} := (x - \hat{b}_6z^6)(x - \hat{b}_7z^6), \mathcal{P}^{(b, z)} := (f^{(b)}_{12, 1}, f^{(b)}_{12, 2}, f^{(b)}_{12, 3}, f^{(b)}_{12, 4}, f^{(b)}_{12, 5}, f^{(b)}_{12, 6}, f^{(b)}_{12, 7}, f^{(b)}_{12, 8}, f^{(b)}_{12, 9}),
\]
This shows that numerical semigroup which we propose to relate to our work on the Moonshine: his work on replicable functions led him to observations and conjectures (cf. [54]). Then we have

\[ f_{12.1} := y_{13} - k_2(z)y_{14}, \quad f_{12.2} := y_{13}y_{14} - k_2(z)y_{15}, \quad f_{12.3} := k_2^2(x)y_{14} - y_{13}y_{15}k_2(z)(x) \]
\[ f_{12.4} := y_{14} - k_2(z)y_{16}, \quad f_{12.5} := y_{13}y_{16} - y_{14}y_{15}, \quad f_{12.6} := y_{15}^2 - k_2(z)(x)k_3(z)(x), \]
\[ f_{12.7} := y_{14}y_{16} - k_2(z)(x)k_3(z)(x), \quad f_{12.8} := y_{15}y_{16} - k_3(z)(x)y_{13}, \quad f_{12.9} := y_{16} - k_3(z)(x)y_{14}. \]

Then we have \( \phi_b^{(12)}(f_{12.a}(x, y_{13}, y_{14}, y_{15}, y_{16}, 1)) = f_{12.a} \), and essentially \( \phi_0^{(12)}(f_{12.a}(x, y_{13}, y_{14}, y_{15}, y_{16}, 1)) = f_{12.a}^{(Z)} \) for \( a = 1, 2, \ldots, 9 \).

By letting \( R^{(z)}_{12} := R^{(b,z)}_{12}/(\hat{z} - 1) \), we have natural homomorphisms,

\[
\begin{array}{ccc}
\text{Spec } B_{H_{12}} & \longrightarrow & \text{Spec } R^{(b,z)}_{12} \leftarrow \text{Spec } R_{12} \\
\downarrow & & \downarrow \\
\text{Spec } \mathbb{C} & \longrightarrow & \text{Spec } \mathbb{C}^{(12)} \leftarrow \text{Spec } \mathbb{C}
\end{array}
\]

This shows that \( \text{Spec } R_{12} \) and \( \text{Spec } B_{H_{12}} \) are fibers over the moduli space corresponding to the numerical semigroup \( H_{12} \). We note that \( R^{(b)}_{12} \) is related to the moduli space \( M_{12} \) of \( (6, 13, 14, 15, 16) \) curves.

We deal with projective space by defining \( R^{(z)}_{12} := \phi_b^{(12)}R^{(b,z)}_{12} \). Here \( \hat{z}^6 \) has the same weight as \( x \). Every homogeneous ideal in \( R^{(z)}_{12} \) under the \( \mathbb{G}_m^{(12)} \)-action provides \( \text{Proj } R^{(z)}_{12} \), which is identified with \( X_{12} \). We have

\[ R^{(z)}_{12}/(\hat{z} - 1) \approx R_{12}, \quad R^{(z)}_{12}/(\hat{z}) \approx B_{H_{12}}. \]

Thus we have the unique point \( \infty \) as a ramified point by letting \( \hat{z} = 0 \), which recovers (4.1), similarly to (3.1); \( Z_6 \), and \( Z_a \) (\( a = 13, 14, 15, 16 \)) correspond to \( \hat{z}^6/x \) and \( \hat{z}^a/y_a \) whereas \( x \) and \( y_a \) (\( a = 13, 14, 15, 16 \)) correspond to \( \hat{z}^6/x \) and \( \hat{z}^{18-a}y_a/x^3 \).

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