Three Variants of a Discrete Median Problem Including a Train Network

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Abstract

In this paper, we treat a problem for locating facilities within a train network. Concretely, given the number of facilities, a train network, and a distribution of facility users in an area, the problem objective is to minimize the sum of journey times to the nearest facilities for all users, where each facility is located on a station. The problem is referred to as a discrete median problem including a train network. We first introduce three models for accessing a facility. We then formulate the problem under each model, the first of which reduces to a p-median problem, whilst the remaining two are formulated as two geographical optimization problems. For each problem, we also implement a simple solution method using a standard local search, and show some computational results.

Key words: facility location, weighted Voronoi diagram, discrete median problem, geographical optimization problem, train network

1 Introduction

A facility location problem seeks appropriate locations for facilities within a spatial context. The class of facility location problems has many areas of application. It can be very useful in urban planning when determining the locations for schools, emergency services and hospitals, factories and warehouses, postboxes, etc. As a result, to date, a large amount of research on facility location has been carried out (see e.g. Refs. [2, 10]). Moreover, a bibliography of literature published up to 1984 exists in Ref. [3], and a review of facility location can be found in Ref. [4] from the viewpoint of microeconomic location planning.

A class of minisum problems exists within the class of facility location problems. In a minisum problem, the objective is to minimize the sum of transportation costs, which are assumed to be proportional to the distance travelled. Since the publication of Ref. [7], a minisum problem has often been referred to as a median problem. A median problem formulated using the ordinary Voronoi diagram was presented in Ref. [8], where locations for facilities are assumed to be continuous: all facilities can be located at any point within a spatial context. This continuous median problem aims to minimize the sum of transportation costs to the nearest facilities, when the number of facilities and the distribution of facility users are given. In the simple descent method used to solve this problem numerically, the ordinary Voronoi diagram is iteratively computed. The computational experimentation showed that this problem can be solved within a practicable time. Since the appearance of this seminal paper (i.e. Ref. [8]), a number of variants formulated through Voronoi diagrams have been presented and solved (see e.g. Refs. [11, 12, 14]).

In Ref. [8], it is assumed that only one mode of transport is available to users accessing a facility, for example, each user may access a facility only on foot, only by bicycle, only by car, etc. However, the assumption of the use of one mode of transport could largely be viewed as inappropriate for built-up areas that include a train network, as each user may make part of the journey on foot and part of the journey by train. Therefore, this paper considers two modes of transport: transport on foot and by train. Here, walking distance, walking time, etc. could be viewed as possible transportation costs of travelling on foot, whilst travel time, train fare paid, etc. could be seen as possible transportation costs of travelling by train. In order to address both the transportation costs of travelling on foot and by train, we focus only on time cost. It should be noted that in this paper the sum of walking time and travel time by train is referred to as journey time.
A developed train network in a built-up area could be assumed to be a connected graph because it consists of vertices and edges which correspond to stations and sections of railway line between adjacent stations, respectively. Each vertex has the location information of the corresponding station, and each edge has the travel time by train for the corresponding section of railway line. As facilities that exist near stations offer the greatest convenience for any user travelling by train, for simplicity, each facility is assumed to be located on a station. This is quite realistic, especially if we imagine, for example, that each facility is located in a large station building. Given the number of facilities, the train network, and the user distribution in an area, our objective is to minimize the sum of journey times to the nearest facilities for all users: a discrete median problem including a train network, with the contribution of this paper being to formulate such a problem under several proposed models for accessing a facility.

The discrete median problem including a train network presented here introduces three models that offer three natural ways to access a facility, each incorporating the use of two modes of transport, and each having its own major benefit. The first model assumes that each user initially goes to the nearest station on foot, and then travels to a facility which is reachable from this station by train in the shortest time possible. Note that, in this model, once a user gets on a train, they can not get off the train until they reach a facility, which is called the no getting off constraint. This model could be viewed as reasonable for many users since the walking time is minimized. The second model requires each user to access a facility such that the sum of walking time and travel time by train is minimized, where each user first goes to a station on foot, and then travels to a facility by train. This model minimizes the journey time to a facility for each user under the no getting off constraint, and so is quite realistic, too. Such a model would suit any user who wishes to access a facility as quickly as possible but who wants a simple journey in order to achieve this. Finally, the third model stipulates that each user must access a facility such that the journey time to a facility is minimized, where the user can travel on foot or by train as many times as needed in order to minimize this journey time. This model is also practical since the journey time to a facility for each user is minimized with no travel constraints being imposed. This model would be of greatest value to anyone wishing to access a facility as quickly as possible, regardless of the complexity of the journey. Note that, from the assumptions of these three models, one feature is that, for each user, the journey time under the first model is never less than that under the second model, whilst the journey time under the second model is never less than that under the third model.

In formulating the discrete median problem including a train network, the problem under the first model is formulated as a p-median problem. Under the second and third models, the problem is formulated as two geographical optimization problems, where the weighted Voronoi diagram is applied to their formulation. These two problems differ, however, in that the second problem is formulated using a train network whereas the third problem is formulated using a train network together with direct routes on foot between all stations.

Following this introduction, the paper is organized as follows. Section 2 reviews the well-known continuous median problem and its solution method presented in Refs. [8, 11], in which there is no train network. In Section 3, we detail the three models for accessing a facility using two modes of transport, and formulate the discrete median problem including a train network under each model. Section 4 describes a simple solution method using a standard local search for the problem. In Section 5, we show some computational results from the viewpoint of the average journey time to a facility. Finally, Section 6 summarizes our findings and offers possible areas of future work in a conclusion.

2 Simple Review of the Continuous Median Problem

In this section, we focus on a special case of the continuous median problem presented in Ref. [8]. We make a simple review of the problem that minimizes the sum of walking times to the nearest facilities, when the number of facilities and the user distribution are given within an area on the xy-plane. In the computational experimentation of Section 5, this particular problem is compared with the discrete median problem including a train network.

2.1 Problem Description

The number of facilities is denoted by $k$. The $k$ facilities are labelled as $p_1, p_2, \ldots, p_k$, with each facility being accompanied by its Cartesian coordinates. The Euclidean distance between two points $p$ and $p_i$ is denoted by $\|p - p_i\|$. The catchment area of each facility $p_i$ is denoted by the ordinary Voronoi region $V(p_i)$, which is defined as follows:

$$V(p_i) = \bigcap_{j=1,\ldots,k, j \neq i} \{ p \in \mathbb{R}^2 | \|p - p_i\| \leq \|p - p_j\| \}.$$
The user distribution is represented by the user density function $f(p)$ for any point $p$.

All users in each ordinary Voronoi region $V(p_i)$ go to the facility $p_i$. Moreover, the walking distance is assumed to be the Euclidean distance. From these assumptions, for all users in a given area, the sum of walking distances to the nearest facilities, which is denoted by $T$, is written as

$$T(p_1, p_2, \ldots, p_k) = \sum_{i=1}^{k} \int_{V(p_i)} \|p - p_i\| f(p) dx dy. \quad (1)$$

The expression $T/c$ is the sum of walking times to the nearest facilities, where the notation $c$ is the walking speed and is assumed to be a constant. Therefore, $T$ can be considered to be the objective function for minimizing the sum of walking times to the nearest facilities.

### 2.2 Solution Method Using a Steepest Descent Method

Since the function $T$ in Eq. (1) is non-linear and non-convex (see Ref. [11]), a numerical method is used. The steepest descent method that computes a local minimum of the function $T$ exists in Ref. [11]. Denoting the coordinates of each facility $p_i$ by $(x_i, y_i)$, the formulae on the partial differential of the function $T$ are written as follows (see Ref. [11]): for each $i = 1, 2, \ldots, k$,

$$\frac{\partial T}{\partial x_i} = \int_{V(p_i)} \frac{(x_i - x)f(x, y)}{\sqrt{(x_i - x)^2 + (y_i - y)^2}} dx dy,$$

$$\frac{\partial T}{\partial y_i} = \int_{V(p_i)} \frac{(y_i - y)f(x, y)}{\sqrt{(x_i - x)^2 + (y_i - y)^2}} dx dy.$$

Using these equations, the steepest descent method for the continuous median problem is written algorithmically as follows.

**Steepest descent method**

**Step 1.** Initialization: Determine an initial solution \( \{(x_1^{(0)}, y_1^{(0)}), (x_2^{(0)}, y_2^{(0)}), \ldots, (x_k^{(0)}, y_k^{(0)})\} \) of $k$ facilities, and $l := 0$.

**Step 2.** Judgment of stopping: Determine whether this algorithm stops.

**Step 3.** Repetition: Assuming that a positive value $\alpha_l$ is a step size at the solution $(x_i^{(l)}, y_i^{(l)})$ for each $i = 1, 2, \ldots, k$,

$$x_i^{(l+1)} := x_i^{(l)} - \alpha_l \cdot \frac{\partial T}{\partial x_i} \bigg|_{x_i := x_i^{(l)}, y_i := y_i^{(l)}},$$

$$y_i^{(l+1)} := y_i^{(l)} - \alpha_l \cdot \frac{\partial T}{\partial y_i} \bigg|_{x_i := x_i^{(l)}, y_i := y_i^{(l)}}$$

for each $i = 1, 2, \ldots, k$, $l := l + 1$, and go to Step 2.

The initial solution of facilities determined in Step 1 has an influence on the solution quality obtained by the steepest descent method. Therefore, in the computational experimentation of Section 5, a large number of initial solutions of facilities are randomly generated. Then, Steps 2 and 3 are applied for each initial solution, and from among the obtained solutions the best solution is output.

### 3 A Discrete Median Problem Including a Train Network

Considering a train network and a continuous distribution $f$ of facility users in an area, we present a problem for locating $k$ facilities, where each facility is to be located on a station. A facility user can travel on foot and by train when accessing a facility. The walking speed $c$ is assumed to be a constant. A train network is assumed to be a connected graph which is embedded in a plane, and is denoted by $G = (S, E)$, where $S$ and $E$ are the sets of stations and sections of railway line between adjacent stations, respectively. The number of stations is denoted by $n$ and the number of facilities by $k$, where $k \leq n$. The $n$ stations are labelled as $s_1, s_2, \ldots, s_n$, with each station being accompanied by its Cartesian coordinates. The Cartesian coordinates for each station are assumed to fall within the given area. Each section $e \in E$ of railway line has a travel time $t(e)$ by train. Since we identify a facility with the station on which the facility is located, the set of $k$ facilities is denoted by $S_k \subset S$. When locating $k$ facilities, we would like to minimize the sum of journey times to the nearest facilities for all users in the given area on the $xy$-plane. This problem is referred to as a discrete median problem including a train network.

Three variants of the problem are put forward in this section. Concretely, we present three models for accessing a facility, and formulate this problem under each model. Model 1 minimizes the walking time for each user under the no getting off constraint. Model 2 minimizes the journey time (i.e. the sum of walking time and travel time by train) to a facility for each user under the no getting off constraint. Model 3 minimizes the journey time to a facility for each user with no travel constraints being imposed. For example, Model 3 allows a journey in which a user gets off a train at a station, then goes to another station on foot, and then gets on a train at that station. In all three models, as the goal of each journey is to reach a facility, as soon as a user reaches any facility along the journey, the journey stops.
3.1 Problem Formulation under Model 1

Model 1 is explained as follows: each user first walks to the nearest station from the viewpoint of the Euclidean distance, and then travels to a facility which is reachable from this station by train in the shortest time possible.

Under Model 1, we formulate the discrete median problem including a train network. We first focus on walking time. For all users on a point \( p \in V(s_i) \), where \( V(s_i) \) is the ordinary Voronoi region of station \( s_i \), the sum of walking times from \( p \) to the nearest station \( s_i \) is written as \( \|p - s_i\|/c \cdot f(p) \). Therefore, for all users in \( V(s_i) \), the sum of walking times to the nearest station \( s_i \), which is denoted by \( T(s_i) \), is written as

\[
\iint_{V(s_i)} \|p - s_i\|/c \cdot f(p) dxdy. \tag{2}
\]

Next, we focus on travel time by train. Let \( t(s_i, s_j) \) be the shortest travel time by train between two stations \( s_i \) and \( s_j \). For each station \( s_i \), the shortest travel time by train from \( s_i \) to the nearest facility is denoted by \( w_i \):

\[
w_i = \min_{s_j \in S} t(s_i, s_j).
\]

Moreover, denoting the number of users in the ordinary Voronoi region \( V(s_i) \) of each station \( s_i \) by \( P(s_i) \), \( P(s_i) = \iint_{V(s_i)} f(p) dxdy \) holds. Therefore, for all users whose nearest station is \( s_i \), the sum of travel times by train from \( s_i \) to the nearest facility is given by

\[
w_i P(s_i). \tag{3}
\]

Using Eqs. (2) and (3), for all users in a given area, the sum of journey times to the nearest facilities can be written as

\[
\sum_{i=1}^{n} (T(s_i) + w_i P(s_i)), \tag{4}
\]

which is the objective function of the problem being considered. For each station \( s_i \), \( T(s_i) \) in Eq. (4) can be regarded as a constant, since this value is independent of locations for facilities. Therefore, the objective function for minimizing the sum of journey times to the nearest facilities for all users is written more simply as

\[
\sum_{i=1}^{n} w_i P(s_i). \tag{5}
\]

We note that the problem is equivalent to the p-median problem found in Ref. [5]. Model 1 stipulates that each user must first walk to the nearest station, meaning that, regardless of locations for facilities, the station which each user first accesses is predetermined and, as a result, for each station \( s_i \), the number of users in the ordinary Voronoi region of \( s_i \) is given. Thus, the problem that minimizes Eq. (5) is a p-median problem that minimizes the sum of travel times to the nearest facilities, under the assumption that a demand exists for each vertex in a given graph.

3.2 Problem Formulation under Model 2

Model 2 is explained as follows: each user accesses a facility that minimizes the journey time under the no getting off constraint. Since Model 2 is the model obtained by removing the constraint that each user first walks to the nearest station, Model 2 is a relaxation of Model 1. Therefore, a feature of Models 1 and 2 is that, for each user, the journey time under Model 2 is never greater than that under Model 1.

The sum of walking time from a point \( p \) to a station \( s_i \) and travel time by train from that station \( s_i \) to the nearest facility is given by the weighted time

\[
t_w(p, s_i) = \frac{\|p - s_i\|}{c} + w_i. \tag{6}
\]

Then, the catchment area with weighted time of a station \( s_i \) over a station \( s_j \) (\( j \neq i \)) is written as

\[
\text{Cat}(s_i, s_j) = \{ p \in \mathbb{R}^2 | t_w(p, s_i) \leq t_w(p, s_j) \} = \{ p \in \mathbb{R}^2 | \|p - s_i\| - \|p - s_j\| \leq c(w_j - w_i) \},
\]

where the second equality follows from Eq. (6).

The shape of the catchment area \( \text{Cat}(s_i, s_j) \) of \( s_i \) over \( s_j \) varies according to the parameter values \( \|s_i - s_j\| \) and \( c|w_j - w_i| \). First, if \( \|s_i - s_j\| < c|w_j - w_i| \), then \( \text{Cat}(s_i, s_j) \) covers the whole plane or disappears completely. Second, if \( \|s_i - s_j\| = c|w_j - w_i| \), then \( \text{Cat}(s_i, s_j) \) covers either one of two possible areas: the whole plane except for the half line extending from \( s_j \) that follows the same direction as an imaginary line drawn from \( s_i \) to \( s_j \); or the half line extending from \( s_i \) that follows the same direction as an imaginary line drawn from \( s_j \) to \( s_i \). Third, if \( \|s_i - s_j\| > c|w_j - w_i| \), then the boundary of \( \text{Cat}(s_i, s_j) \) is given by \( \{ p \in \mathbb{R}^2 | \|p - s_i\| - \|p - s_j\| = c(w_j - w_i) \} \). This boundary is the locus of a point \( p \) such that the difference between the Euclidean distance from \( p \) to \( s_i \) and the Euclidean distance from \( p \) to \( s_j \) is constant. This locus is a part of the hyperbolic curve with foci \( s_i \) and \( s_j \).

The catchment area of each station \( s_i \) with weighted time is denoted by the weighted Voronoi region \( V_w(s_i) \), which is written as follows:

\[
V_w(s_i) = \bigcap_{s_j \in S \setminus \{s_i\}} \text{Cat}(s_i, s_j).
\]

Then, the sum of journey times to the nearest facilities for all users can be written as

\[
\sum_{s_i \in S} \iint_{V_w(s_i)} t_w(p, s_i) f(p) dxdy, \tag{7}
\]

which is the objective function of the problem under Model 2. This problem is formulated using the weighted
Voronoi regions for all stations. Therefore, this problem can be seen as a geographical optimization problem, which is a term first used in Ref. [8]. The catchment area of each facility \( s_i \in S_k \) is given by \( \bigcup_{s_j, n(s_j)=s_i} V_w(s_j) \), where \( n(s_j) \) is a facility which is reachable from station \( s_j \) by train in the shortest time possible. This is the area in which the users who access the facility \( s_i \) are distributed.

Example 1 We show an example of weighted Voronoi regions. There are three stations \( s_1, s_2, \) and \( s_3 \), and two sections of railway line \( e_1 = \{s_1, s_2\} \) and \( e_2 = \{s_1, s_3\} \), where \( s_1, s_2, \) and \( s_3 \) are located at \((0, 0)\), \((10, 0)\), and \((0, 10)\), respectively; \( t(e_1) = 2 \) and \( t(e_2) = 3 \); and \( s_2 \) and \( s_3 \) are facilities. Let \( c = 1 \).

In this case, \( w_1 = 2, w_2 = 0, \) and \( w_3 = 0 \). The weighted Voronoi region \( V_w(s_i) \) of each station \( s_i \) is shown in Fig. 1. The boundaries between \( V_w(s_1) \) and \( V_w(s_2) \), \( V_w(s_1) \) and \( V_w(s_3) \), and \( V_w(s_2) \) and \( V_w(s_3) \) are formed from parts of the boundaries satisfying

\[
\sqrt{x^2 + y^2} + 2 = \sqrt{(x - 10)^2 + y^2}, \quad \sqrt{x^2 + y^2} + 2 = \sqrt{x^2 + (y - 10)^2}, \quad \text{and} \quad y = x,
\]

respectively. Moreover, the catchment areas of facilities \( s_2 \) and \( s_3 \) are \( V_w(s_1) \cup V_w(s_2) \) and \( V_w(s_3) \), respectively.

3.3 Problem Formulation under Model 3

Model 3 is explained as follows: each user accesses a facility that minimizes the journey time to a facility under the assumption that the user can travel on foot or by train as many times as needed. Model 3 is the model obtained by removing the no getting off constraint from Model 2. As Model 3 is a relaxation of Model 2, a feature of these two models is that the journey time for each user under Model 3 is never greater than that under Model 2. For example, when a user accesses a facility \( s_k \) from a station \( s_j \), Model 3 considers not only the journey in which the user travels to \( s_k \) by train, but also the journey in which the user gets off a train at an earlier station \( s_j \), and then travels the remaining part of the journey to \( s_k \) on foot; see Fig. 2. Note that the latter journey is not possible under Model 2.

In order to formulate the discrete median problem including a train network under Model 3, we introduce the following network: for every pair of stations in a train network \( G \), an edge is spanned between those two stations, and then the spanned edge is assigned a value for the walking time corresponding to the Euclidean distance between those two stations. This network, denoted by \( G' \), can be seen as a network that comprises the train network \( G \) together with a direct route on foot between every pair of stations. Using \( G' \), we can formulate the problem under Model 3 in the same way as the problem is formulated under Model 2.

4 Solution Method Using a Local Search

To solve the three variants of the discrete median problem including a train network described in Section 3, we apply a standard local search, which is written algorithmically as follows.

Local search

Step 1. Generate an initial solution \( S_k \) of \( k \) facilities.

Step 2. Within the neighborhood \( N(S_k) \) of \( S_k \), search for a solution \( S_k' \) of \( k \) facilities whose objective function value is better than the objective function value for \( S_k \). If such a solution \( S_k' \) is found, let \( S_k := S_k' \) and return to Step 2. Otherwise output the current solution \( S_k \) and stop.

The initial solution generated in Step 1 has an influence on the solution quality obtained by the local search. Therefore, in the computational experimentation of Section 5, a large number of initial solutions are randomly generated. Then, Step 2 is applied for each initial solution, and from among the obtained solutions the best solution is output.

In Step 1, an initial solution \( S_k \) is randomly generated. In Step 2, a better solution is searched for within
Figure 2: Examples of two possible journeys from a station $s_i$ to a facility $s_k$ under Model 3, where (i) shows a journey entirely by train and (ii) shows a journey by train to $s_j$ and then on foot from $s_j$ to $s_k$. The neighborhood of $S_k$. This neighborhood $N(S_k)$ of $S_k$ is defined as follows: in the given train network $G$, $N(S_k)$ consists of the sets of $k$ facilities obtained by permitting the location of facility $s_i$ to be changed to any station adjacent to this facility $s_i$ for each facility $s_i \in S_k$. Therefore, the search space of $N(S_k)$ is written as $\sum_{s_i \in S_k} \deg(s_i)$, being equal to $O(k \max_{s_i \in S} \deg(s_i))$, where $\deg(s_i)$ is the degree of vertex $s_i$ in the train network $G$.

The search within $N(S_k)$ checks the stations adjacent to $s_i$ for each facility $s_i \in S_k$. The search of these stations adjacent to $s_i$ is explained as follows: if there is a station adjacent to $s_i$ that improves the objective function value, let the facility move to the station adjacent to $s_i$ that maximally improves the objective function value; otherwise, the facility remains at $s_i$. After searching within $N(S_k)$, if a better solution is found, the neighborhood of the updated solution is similarly searched; otherwise, the current solution $S_i$ is output. The obtained solution is called locally optimal.

For the problem under Model 1, before applying the local search, the number of users $P(s_i)$ for each station $s_i \in S$ and the shortest travel time by train between every pair of stations are computed in advance. When $P(s_i)$ for each $s_i \in S$ and the shortest travel times are given, the shortest travel times $w_1, \ldots, w_n$ by train to the nearest facilities for all stations $s_1, \ldots, s_n$ in Step 2 can be computed in time complexity $O(nk)$. Moreover, in the search within the neighborhood of Step 2, the number of computations of the objective function value is $O(k \max_{s_i \in S} \deg(s_i))$. As a result, the time complexity for each repetition of Step 2 is $O(nk^2 \max_{s_i \in S} \deg(s_i))$.

For the problem under Model 2, before applying the local search, the shortest travel time by train between every pair of stations is computed in advance. When those shortest travel times are given, the shortest travel times $w_1, \ldots, w_n$ by train to the nearest facilities for all stations $s_1, \ldots, s_n$ in Step 2 can be computed in time complexity $O(nk)$.

For the problem under Model 3, before applying the local search, the network $G'$ is constructed, comprising the train network $G$ and a direct route on foot between every pair of stations in $G$. Then the shortest travel time between every pair of stations in $G'$ is computed. After this preprocessing, the local search is applied in the same way as under Model 2.

5 Computational Experimentation

Using the library functions in LEDA (see Ref. [9]), we created computer programs for the solution method using the steepest descent method in Section 2.2 and the solution method using the local search in Section 4. These solution methods were implemented on a PC (CPU: Intel Core i7 4 GHz, RAM: 32 GB, OS: Windows 10 Professional). Population data was chosen related to the area centered around Shinjuku City in Tokyo. This was simply because the university where the authors are employed is located in this area. To address the actual user density function, we made use of the Grid Square Statistics (see Ref. [6]) compiled by government institutions including the Statistics Bureau of the Ministry of Internal Affairs and Communications. We used the area that corresponds to the Secondary Area Partition mesh codes 5339-34, 35, 36, 44, 45, 46, 54, 55, and 56. Since, for the user density function, we used Quarter Grid Squares, the length/width of each square being approximately 250 m, 14,400 (120 × 120) Quarter Grid Squares exist in this area. We used the raster data of population distribution.
generated from the results of the 2005 census. From this data, the total population in this area was calculated at 10,610,671.

In the computational experimentation for the discrete median problem in Section 3, we used the hypothetical train network based on JR, ODACUY, KEIO, KEIKYU, KEISEI, SEIBU, TOKYO, and TOBU train lines in the aforementioned area, comprising 300 stations. For each section \( e \in E \) of railway line, the travel time \( t(e) \) by train was set to the value of the Euclidean distance between two adjacent stations corresponding to \( e \) divided by train speed. The train speed was set to 800 m/min by referring to the train timetable of the JR Yamanote Line, which is a railway loop line with a circle route fully contained within the area of Tokyo being used for population distribution data.

Throughout the computational experimentation, the walking speed \( c \) was set to 80 m/min based on Ref. [13]. As the number of different initial solutions for each solution method increases, the quality of the local optimal solution increases but with increased calculation time. Therefore, in order to finish the calculation in a practicable time, the number of initial solutions randomly generated was set to 100 in each solution method implemented. Moreover, the step of improvement for each solution method was repeated until we obtained a local optimal solution.

First, the computational results on the average journey times for the three models presented are shown in Fig. 3, where the horizontal axis represents the number of facilities \( k \). The average journey time to a facility was obtained by dividing the sum of journey times to the nearest facilities by the total population. From Fig. 3, when \( k \) is small, the average journey times for Models 2 and 3 are less than those for Model 1. However, as \( k \) increases, the difference in the average journey times between the three models decreases. It can be considered that as \( k \) increases, so too does the number of users whose nearest stations have facilities. When \( k = 300 \), the average journey times for the three models become the same since every station has a facility, and so no train travel is required.

Next, the average CPU times to execute local searches for the three models presented are shown in Fig. 4. From Fig. 4, compared with the local searches for Models 2 and 3, the local search for Model 1 is very fast since each user always goes to the nearest station, irrespective of the locations of facilities. The results on average CPU times of the local searches for Models 2 and 3 show a similar tendency for all \( k \) investigated. Concretely, as \( k \) increases, the average CPU time also increases. However, when \( k \) passes a certain threshold, the average CPU time decreases. This is because the search space in the local search is small when \( k \) is large. From Fig. 4, for the train network including 300 stations, we can confirm that the local searches for the three models run in a practicable time.

Finally, letting Model 0 be the known model for the continuous median problem in Section 2, we compare the computational results on the average journey times for the model considering a train network (i.e. Model 3) and the model not considering a train network (i.e. Model 0). In the computational experimentation under Model 0, we assumed the following: the steepest descent method stops in Step 2 if the value of the objective function Eq. (1) is not improved when compared with the preceding objective function value or if the number of iterations \( l \) exceeds 200,000. Moreover, the step size \( \alpha_l \) in Step 3 was set to 0.00001 for any \( l \). The computational results are shown in Fig. 5. From Fig. 5, when \( k \) is small, the av-
Average journey times for Model 3 are less than those for Model 0. This can be considered the effect of the train network. However, as $k$ increases, the difference in the average journey times between Models 0 and 3 decreases, and when $k = 125$, the average journey time for Model 3 exceeds that for Model 0. With subsequent increases in $k$, the difference between the average journey times for Models 0 and 3 increases. Therefore, we can confirm that when $k$ passes a certain threshold, the model in which facilities can be located at any point within the given area (i.e. Model 0) has an advantage over Model 3 in terms of average journey time.

6 Conclusions

We presented three variants of a discrete median problem including a train network. This problem introduced three reasonable models to access a facility, each incorporating the use of two modes of transport. Then, the problem under each model was formulated, the first of which reduced to a p-median problem. The other two problems were formulated as two geographical optimization problems, where the weighted Voronoi diagram was applied to their formulation. We also implemented a solution method based on a local search for the problem, and show some computational results.

Each of these three models has its own relative benefits that depend on many factors including the nature of a facility, how often it is accessed, and who is accessing it. The discrete median problem in this paper can be applied not only to a train network but also other mass transportation systems such as bus routes, subway networks, etc. Therefore, when locating facilities in an area with a developed transportation system, the problem presented is very useful.

A number of other possible variants of the problem presented are worthy of future consideration. For example, by relaxing the assumption that each facility is located on a station, we could assume that each facility is able to be located anywhere on a given plane. In addition, train waiting time, transfer of trains, time of day, etc. could also be considered. Such variants of the problem could yield even more realistic models for addressing average journey time to a facility. Moreover, further investigation of the features of the three models presented based on empirical observations from, for example, differing facility positions, train network structures, and population distributions is another potential area of future research.

References

[1] Y. Asami: A Note on the Derivation of the First and Second Derivatives of Objective Functions in Geographical Optimization Problems, Journal of the Faculty of Engineering, the University of Tokyo (B), 41 (1) (1991), 1-13.

[2] M. S. Daskin: Network and Discrete Location: Models, Algorithms, and Applications, Second Edition (Wiley, 2013).

[3] W. Domschke and A. Drexl: Location and Layout Planning: An International Bibliography (Springer-Verlag, 1985).

[4] W. Domschke and G. Krispin: Location and Layout Planning: A Survey, OR Spectrum, 19 (3) (1997), 181-194.

[5] M. R. Garey and D. S. Johnson: Computers and Intractability: A Guide to the Theory of NP-Completeness (W. H. Freeman and Company, San Francisco, 1979).

[6] Grid Square Statistics HP: http://www.stat.go.jp/english/data/mesh/index.htm

[7] S. L. Hakimi: Optimum Locations of Switching Centers and the Absolute Centers and Medians of a Graph, Operations Research, 12 (3) (1964), 450-459.

[8] M. Iri, K. Murata, and T. Ohya: A Fast Voronoi-Diagram Algorithm with Applications to Geographical Optimization Problems, Lecture Notes in Control and Information Sciences, 59 (1984), 273-288.
[9] LEDA HP: http://www.algorithmic-solutions.com/index.php/products/leda-for-c

[10] P. B. Mirchandani and R. L. Francis (eds.): Discrete Location Theory (Wiley-Interscience, 1990).

[11] A. Okabe, B. Boots, K. Sugihara, and S. N. Chiu: Spatial Tessellations: Concepts and Applications of Voronoi Diagrams, Second Edition, Chapter 9 (Wiley, 2000).

[12] A. Okabe and A. Suzuki: Mathematics for Optimal Location (Asakura Shoten, 1992) (published in Japanese only).

[13] Regulations for describing the locations of real estate in Japan: http://www.rftc.jp/webkanri/kanri/wp-content/uploads/2019/02/h_sekoukisoku.pdf (in Japanese only)

[14] A. Suzuki and A. Okabe: Using Voronoi diagrams, in Facility Location: A Survey of Applications and Methods, pp. 103-118 (Springer, New York, 1995).