Classical decays in decoherent quantum maps

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We study the asymptotic long-time behavior of open quantum maps and relate the decays to the eigenvalues of a coarse-grained superoperator. In specific ranges of coarse graining, and for chaotic maps, these decay rates are given by the Ruelle-Pollicott resonances of the classical map.

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The study of the emergence of classical features in the quantum behavior of hamiltonian systems has centered on two areas: on the one hand, the statistical fluctuation properties of the quantum spectrum were the first quantities to reveal universality classes related to classical - chaotic or integrable - behavior. On the other hand, the study of time dependent quantities has also provided quantum signatures of the classical world. The growth of Von Neuman entropy in decohering quantum systems, by the Ruelle-Pollicott resonances. We show in this letter we report the emergence and the relevance in quantum dephasing maps of yet other quantities of a classical nature - the Ruelle-Pollicott resonances - and show that they rule the asymptotic decay of time dependent quantities.

In a simple model for decoherence in quantum maps was introduced based on the Kraus representation of superoperators, with a straightforward interpretation as quantum coarse-graining in phase space. The model was employed to study the entropy evolution and to confirm the prediction that for certain regimes the rate of growth is independent of the coarse graining and is given by the Lyapunov exponent of the classical motion. This is a short time regime, of the order of the Ehrenfest time, after which relaxation towards the uniform density occurs. In this letter we study this asymptotic relaxation regime for quantum maps and show that it is ruled by the eigenvalues of a similar coarse-graining superoperator. These eigenvalues can be effectively calculated and they determine -in certain well defined ranges- classical decay rates independent of the coarse graining. These decay rates are given, in the semiclassical limit of chaotic systems, by the Ruelle-Pollicott resonances. We show in particular that, beyond the Ehrenfest time, these resonances determine the asymptotic behavior of the linear entropy and of the Loschmidt echo.

The unitary evolution of the density matrix for a quantum system is given by $\rho_{n+1} = U \rho_n U^\dagger$ where $U$ is a unitary evolution operator that quantizes a classical map. For simplicity we consider area preserving maps on a phase space with periodic boundary conditions, i.e. a torus (normalized to unit area). In this case the Hilbert space has a finite dimension $N$ related to $\hbar$ by $2\pi\hbar N = 1$.

$$\rho_{n+1} = L \rho_n$$

which is still unitary, of dimension $N^2 \times N^2$. The eigenvalues of $L$ are $e^{i(E_i - E_j)}$, where $E_i$ are the Floquet eigenvalues of the map.

The quantum coarse graining is implemented by a Kraus superoperator

$$D_i = \sum_{p,q=0}^{N-1} c_i(p,q) \hat{T}(p,q) \otimes \hat{T}^\dagger(p,q),$$

where $c_i(p,q)$ is a smooth positive periodic function on the torus to be defined below. In order for $D_i$ to be trace preserving the condition $\sum_{p,q} c_i(p,q) = 1$ must be satisfied. Inasmuch as $c_i(p,q)$ is a narrow gaussian-like function, $D_i$ implements an incoherent sum of slightly displaced density matrices over a phase space region of order $\epsilon$, this being a measure of the amount of coarse graining. To avoid an overall drift we further assume that $c_i(p,q)$ is an even function of its arguments. The spectral properties of $D_i$ are easily calculated. $D_i$ is hermitian and using the fact that $\hat{T}(p,q)\hat{T}(\lambda,\mu)\hat{T}^\dagger(p,q) = \hat{T}(\lambda,\mu)e^{i\frac{2\pi}{N}(\lambda q - \mu p)}$ its spectral properties are given by

$$D_i \hat{T}(\lambda,\mu) = \hat{c}_i(\lambda,\mu) \hat{T}(\lambda,\mu),$$

where $\hat{c}_i(\lambda,\mu) = \sum_{p,q} c_i(p,q)e^{i\frac{2\pi}{N}(\lambda \lambda - \mu p)}$ is the DFT of $c_i(p,q)$. It is irrelevant whether we specify the diffusion...
by \( c_\epsilon(p, q) \) or \( \hat{c}_\epsilon(\lambda, \mu) \). We found more convenient to define the latter as

\[
\hat{c}_\epsilon(\lambda, \mu) = e^{-\frac{\epsilon}{2}(\frac{\pi}{N})^2(\sin^2[\pi \lambda / N] + \sin^2[\pi \mu / N])},
\]

(5)

whose Fourier transform is a smooth periodic gaussian-like coarse graining kernel, of width \( \epsilon / 2 \pi \). The coarse-grained dynamics of the quantum map will be given by:

\[
\rho_{n+1} = D_\epsilon L \rho_n = L_\epsilon \rho_n.
\]

(6)

For any finite value of \( \epsilon \), \( L_\epsilon \) is a convex sum of unitary matrices and is therefore a contracting map whose spectrum is contained in the unit circle, and has no degenerate eigenvalue equal to 1, corresponding to the uniform eigenvector \( \hat{T}(0, 0) \), and \((N^2 - 1)\) eigenvalues of modulus smaller than 1. Therefore, the time evolution of the density matrix can be decomposed as a sum of exponentially decaying modes plus a constant. In particular, the asymptotic decay towards the uniform density will be given by the eigenvalue closer to the unit circle. Although this is strictly true for any value of \( N \) and \( \epsilon \geq 0 \), the behavior of the eigenvalues in the limits \( N \to \infty \) and \( \epsilon \to 0 \) is strongly dependent on the classical features—chaotic or regular—of the map in question.

Before studying these limits we review the equivalent procedure for classical maps: densities in phase space evolve with the Frobenius-Perron (F-P) operator \( L \) arising from the Liouville equation. For a classical map, the general form of \( L \) is \( L(x, y) = \delta[y - f(x)] \) where \( x = (q, p), y = (q', p') \). As it stands, the operator \( L \) is unitary on \( L^2 \), the space of square integrable functions on phase space. When it is convoluted with a narrow coarse graining operator \( D_\epsilon \), \( \epsilon \) being the coarse graining parameter, we obtain

\[
L_\epsilon = D_\epsilon \circ L.
\]

(7)

\( D_\epsilon \) has the role of damping the high frequency components of phase space distributions in \( L^2 \) thus producing an effective truncation of \( L \) and drastically changing its spectrum. In fact now \( L_\epsilon \) is compact and its spectrum consists of isolated resonances contained in the unit circle [10, 11]. The behavior of these resonances as \( \epsilon \to 0 \) is radically different for chaotic or regular maps. If the map is ergodic and mixing, \( L_\epsilon \) has an isolated non-degenerate eigenvalue equal to 1, corresponding to the uniform density, and isolated eigenvalues with finite multiplicity, and modulus strictly smaller than 1. In the singular limit \( \epsilon \to 0 \), these eigenvalues are the Ruelle resonances [10, 11] and they determine the exponential decay of correlation functions in phase space [12, 13]. An equivalent procedure to uncover these resonances consists in introducing a basis of functions ordered by resolution in phase space, then constructing \( L \) in this basis and subsequently truncating to a finite dimension \( M \), resulting in a matrix \( L^{(M)} \) whose largest eigenvalues stay fixed as \( M \to \infty \) (see [12, 13, 14, 15, 16] for a review and [17] for a quantum application).

It was proved recently, under quite broad conditions [11] (Theorem 1), that for smooth maps on the torus and for any fixed \( \epsilon \) the spectrum of the coarse grained quantum propagator \( L_\epsilon \) converges to that of \( L_\epsilon \) in the semiclassical limit \( N \to \infty \). This in turn implies that at finite \( \epsilon \) the quantum decay rates are given by the corresponding classical eigenvalues. We study below the issue of letting \( \epsilon \to 0 \).

We first compare these spectra for the perturbed Arnold cat map (and its quantization [15])

\[
p' = p + q - 2\pi k \sin[2\pi q], \quad q' = q + p' + 2\pi k \sin[2\pi p'] \tag{8}
\]

The classical Arnold cat map is ergodic and mixing, has uniform hyperbolicity and its Lyapunov coefficient is \( \chi = (1/2)(3 + \sqrt{5}) \). The perturbation is added to avoid the the non-generic behavior of the quantized version and \( \chi \) is not changed substantially by it. On the contrary, the Ruelle resonances are very sensitive to the perturbation. In

Fig. 1 we compare the eigenvalues of \( L_\epsilon \) and \( L_\epsilon \). The main difficulty in both calculations is the large dimensions of the matrices involved. For the classical calculation we took the coarse graining operator \( D_\epsilon \) to be a periodic

![FIG. 1: Comparison of the leading eigenvalues of the quantum \( L \) and classical \( L_\epsilon \) coarse-grained propagators for the perturbed Arnold cat map, with different coarse graining and perturbation (a) \( k = 0.01, \epsilon = 0.4 \); (b) \( k = 0.02, \epsilon = 0.314 \). (• ≡ classical, grid of 50 × 50 ; ⊕ ≡ quantum, \( N = 150 \).) 

![Graph showing the comparison of the leading eigenvalues of the quantum and classical propagators for the perturbed Arnold cat map.](image_url)
gaussian of width $\epsilon/(2\pi)$. The spectrum of $L_\epsilon$ was obtained by projecting eq. (3) on a grid of $50 \times 50$ sites and diagonalizing. The procedure is stable for the leading eigenvalues as long as $\epsilon$ is much larger than the grid size 10. The spectrum of $L_\epsilon$ was obtained by a variational method similar to that proposed in 12. A basis of right and left trial functions, which are smooth on the unstable and stable manifold respectively, are used to construct a subspace that contains the leading eigenspace. The advantage of this method is that the resulting generalized eigenvalue problem has a small dimension as the basis contains information about the dynamics. Details about the computation of the spectrum of $L_\epsilon$, with this method will be reported in a future article 20. Good agreement for the leading eigenvalues of both spectra was found in all the cases we computed, provided $N$ was large enough, in agreement with 11. In order to obtain the Ruelle resonances we now have to take the limit $\epsilon \to 0$. Clearly this limit cannot be taken at constant $N$ because for finite matrices unitarity is eventually recovered and all eigenvalues go back to the unit circle.

In Fig. 2 we study this process as a function of $N$ and $\epsilon$ for the leading eigenvalue $\lambda_1$. We observe that for each $N$ there is a range of values of $\epsilon$ for which the eigenvalue is independent of $\epsilon$ and for which the limit $\epsilon \to 0$ can be extrapolated safely, thus defining a property of the classical map, independent of $N$ and $\epsilon$. A similar situation occurs for the other eigenvalues, but the safe range becomes smaller as the distance to the unit circle increases. We conclude therefore that the leading portion of the spectrum of $L_\epsilon$ for large $N$ is of a purely classical nature and provides the rates of asymptotic decay of quantum time dependent quantities. For chaotic maps these classical rates are further identified with the spectrum of Ruelle resonances. 21

To test this prediction consider first the evolution of the linear entropy $S_n = -\ln |\rho_0^n|$ . It was shown in 21 that for short times the entropy growth is linear with a slope independent of $\epsilon$ and determined by the Lyapunov exponent of the map. This is shown in the inset of Fig. 3 where $S_n$ is plotted as a function of $n$ for $N = 450$, $\epsilon = 0.05$ and various values of the perturbation. As expected for an ergodic map, for longer times the state relaxes to the uniform distribution and $S$ converges to $S_\infty = -\ln(1/N)$. To uncover the rate of relaxation we subtract the uniform density from the initial state and obtain, using the long time decomposition of $\rho_0$, into decaying modes, $S_n = -2\ln|\lambda_1|n$. Again a linear growth is obtained but this time with a slope dependent on the leading Ruelle resonance. This feature is clearly seen in Fig. 3 where we plot $S_n$ for different values of $k$.

FIG. 2: The leading eigenvalue $|\lambda_1|$ as a function of $\epsilon$, for different values of $N$. There is a range, growing with $N$ where the eigenvalue is independent of $\epsilon$ so that it can be extrapolated to $\epsilon \to 0$. $k = 0.01$.

A very similar pattern emerges for the decay of the so called Loschmidt echo, or fidelity 4. This is defined as $M_n = \text{tr}(\rho_n^\dagger \rho_n^0)$ where $\rho_n$ and $\rho_n^0$ are states evolved with slightly different maps from an initial state $\rho_0$. This quantity, originally considered by Peres 22 as a measure of sensitivity to perturbations, has acquired renewed importance as a measure of environment independent decay of spin echoes 4 and of the accuracy of implementation of quantum algorithms. It is usually considered only in the case of strictly unitary evolution where it equals $M_n(0) = \text{tr}[L_n^\dagger \rho_0 L_\epsilon^n \rho_0]$. For (slightly) open systems it is natural to replace this quantity by $M_n(\epsilon) = \text{tr}[L_\epsilon^\dagger \rho_0 L_\epsilon^n \rho_0]$ and then consider the limit $\epsilon \to 0$ to extract features that are independent of the coarse-graining. The same considerations regarding the non-commutativity of the $N$ and $\epsilon$ limits apply here and in the “safe” regime of $N$ and $\epsilon$ the asymptotic decay of $M_n$ is again ruled by the Ruelle resonance. In the
classical case, a similar result for the echo was obtained recently [22]. Our discussion shows that the quantum decay must follow the classical one. In fact the echo is closely related to the linear entropy discussed above. Schwartz inequality $\text{tr}(\rho_n\rho_n') \leq \sqrt{\text{tr}(\rho_n^2)\text{tr}(\rho_n'^2)}$ leads to $\ln M_n \leq -(1/2)(S_n + S_n')$. Again the uniform density is subtracted from the initial state to reveal the exponential relaxation regime. In the asymptotic regime the inequality becomes an equality and we find that beyond the Ehrenfest time the decay proceeds as

$$\ln M_n = -(1/2)(\ln |\lambda_1| + \ln |\lambda_1'|)n.$$  

(9)

This is illustrated in Fig. 4, where it is clearly seen that, after the initial Lyapunov regime the echo decays with the average of the two linear entropies.

Our spectral analysis shows that for any observable whose quantum evolution depends on the operator $L$, and this is not restricted to maps only, the best procedure to extract its classical behavior is to replace $L$ by $L_\epsilon$ and let $\epsilon \to 0$ and $N \to \infty$ (meaning $\hbar \to 0$) in such a way as to remain in the “safe” region where unitarity is not recovered. The convergence of the spectra of $L$ and $L_\epsilon$ is proven under very specific conditions in [12] and demonstrated here in numerical experiments, is expected to be much more general, and in practice provides a very powerful means of extracting classical behavior from quantum maps [23]. Whether different models of decoherence, based on physical mechanisms in small quantum devices, lead to the same results remains to be explored but, at least from the numerical point of view, are also subject to this spectral analysis and can reveal other modes of decay not related to classical properties.

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FIG. 4: Decay of the Loschmidt echo beyond the Ehrenfest time. Two values of $k,(+ \equiv k = 0.005, + \equiv k = 0.017$ perturbed by $\Delta k = 0.002$ are plotted.$N = 450, \epsilon = 0.1$. The full line shows the echo while the dotted lines give the perturbed and unperturbed entropies $-S_n$ and $-S_n'$.

[1] O. Bohigas, Random Matrix Theories in Chaotic Dynamics. Proceedings of the Les Houches Summer School, Session LI, edited by M.-J. Giannoni, A. Voros and J. Zinn-Justin (North Holland, Amsterdam,1991).
[2] W. H. Zurek and J. P. Paz, Phys. Rev. Lett 72, 2508 (1994).
[3] D. Montelova, J. P. Paz. Phys. Rev. Lett 85, 3373 (2000).
[4] R. A. Jalabert and H. M. Pastawski. Phys. Rev. Lett. 86, 2490 (2001); Ph. Jaquod, P. G. Silverstrov and C. W. J. Beenakker. Phys. Rev. E 64 055203(R) (2001); F. M. Cucchietti, H. M. Pastawski and D. A. Wisnacki. Phys. Rev. E 65 045206 (2002); G. Benenti and G. Casati. Phys. Rev. E 65, 066205 (2002).
[5] D. Ruelle. Phys. Rev. Lett. 56, 405 (1986)
[6] D. Ruelle. J. Stat. Phys 44, 281 (1986)
[7] P. Bianucci, J. P. Paz and M. Saraceno. Phys. Rev. E 65, 046226 (2002).
[8] K. Kraus States, Effects and Operations, Springer-Verlag, Berlin, 1983.
[9] J. Schwinger, Proc. Nat. Acad, Sci. 46 (1960), 570, 893.
[10] M. Blank, G. Keller and C. Liverani. Nonlinearity 15, 1905-1973 (2002).
[11] S. Nonnenmacher Spectral Properties of Noisy classical and Quantum propagators nlin.CD/03011014
[12] S. Fishman, Wave Functions, Wigner Functions and Green Functions of Chaotic Systems and M. R. Zirnbauer, Pair Correlations of Quantum Chaotic Maps from Supersymmetry in “Supersymmetry and Trace Formulae Chaos and Disorder”, NATO ASI Series, edited by I. V. Lerner, J. P. Keating and D. E. Khmelnitskii. (Kluwer Academic/Plenum Publishers, New York, 1999).
[13] J. Weber, F. Haake, P. Seba. Phys. Rev. Lett. 85, 3560-3563 (2000).
[14] H. H. Hasegawa and W. C. Saphir. Phys. Rev. A 46, 7401 (1992).
[15] M. Khodas, S. Fishman and O. Agam. Phys. Rev. E. 62, 4760(2000)
[16] S. Fishman and S. Rahav. nlin.CD/0204608
[17] C. Manderfeld, J. Weber and F. Haake. J. Phys. A 34, 9893 (2001).
[18] J. H. Hannay and M. V. Berry, Physica 1D,267-290 (1980).
[19] G. Blum and O. Agam. Phys. Rev. E 62, 1977 (2000).
[20] I. García-Mata and M. Saraceno. In preparation.
[21] The situation is quite different for regular maps, because the classical spectrum now consists of many eigenvalues converging to the unit circle as $\epsilon \to 0$. Thus the decay is not exponential but power-law.
[22] A. Peres Phys. Rev A 30, 1610 (1984)
[23] G. Benenti, G. Casati, G. Vebve nlin.CD/0208003
[24] D. Braun. CHAOS, 9,730 (1999). D. Braun. Physica D, 131, 265 (1999).
[25] Lately work on the relation between classical and quantum trace formulas for maps has come to our attention [24].