INTERMEDIATE JUSTIFICATION LOGICS: UNIFIED COMPLETENESS RESULTS

NICHOLAS PISCHKE

Abstract. We introduce abstract intermediate justification logics by extending arbitrary intermediate propositional logics with a subset of specific axioms of (classical) justification logic. We study these intermediate justification logics semantically out of various perspectives by combining the well-known semantical access points to intermediate logics through algebraic and Kripke-frame based models with the usual semantic machinery used by Mkrtchyevs, Fittings or Lehmanns and Studers models for classical justification logics. We prove unified completeness theorems for all intermediate justification logics and their corresponding semantics using a respective propositional completeness theorem of the underlying intermediate logic. We consider especially the particular instances of intuitionistic, classical and Gödel justification logics because of their previous presence in the literature.

1. Introduction

Justification logics originated in the 90’s from the studies of Artemov (see [1, 2]) regarding the provability interpretation of the modal logic $S4$ (as initiated by Gödel [19]) and the connected problem of formalizing the Brouwer-Heyting-Kolmogorov interpretation of intuitionistic propositional logic. From there, the prototype justification logic (the logic of proofs) was substantially generalized and the resulting family of justification logics gained importance in the context of general (explicit) epistemic reasoning (see the survey [3]) with two recent textbooks on the subject [4, 24]. The original semantics for the logic of proofs was its intended arithmetical interpretation in Peano arithmetic but since then, various semantics have been proposed which apply not only to the logic of proofs but to the whole family of justification logics. Notable instances important in this paper are the syntactic models of Mkrtchyev [30] as well as the possible-world models of Fitting [12, 14] and the recent subset semantics of Lehmann and Studer [26]. These other semantical access points have been instrumental not only in demonstrating the strength of justification logics in modeling general epistemic scenarios and classifying the ontology of justification terms and formulae in classical justification logics but also in inner-logical investigations for properties like decidability (see e.g. [22, 30, 23]).

Besides the classical justification logics, there is a growing literature on non-classical justification logics including various streams originating from the formalization of explicit but vague knowledge. There, in particular importance for this paper is the work on many-valued justification logics (see [16, 17, 34, 35]) and on intuitionistic justification logics (see [24, 27, 28]). In fact, the Gödel justification logics from [16, 34, 35] also relate to the latter, with Gödel logic, as the base logic, being one of the prime examples of an intermediate logic, originating from Dummetts work [10] (in turn influenced by Gödels remarks on intuitionistic logic [18]).

We give a unified semantical theory of the above examples of intuitionistic, Gödel as well as classical justification logics and beyond by introducing abstract intermediate justification logics (that is intermediate propositional logics over the justification language extended with a collection of designated justification axioms) and classifying them semantically. Starting at the two characteristic semantical access points for the underlying intermediate logics of algebraic semantics based on Heyting algebras and of the semantics of Kripke (that is intuitionistic Kripke frames) based on partial orders, we extended these algebraic and order theoretic approaches to intermediate logics by the usual (appropriately adapted) semantic machinery for treating justification modalities from the classical models of Mkrtchyev [30], Fitting [12, 14] as well as Lehmann and Studer [26]. Here, the algebraic approach extends the three classes of classical Mkrtchyev, Fitting and subset models by allowing the models to take values not only in $\{0,1\}$ (or $[0,1]$ as in the case of Gödel justification logics) but in arbitrary Heyting algebras. The approach via intuitionistic Kripke frames extends the previous considerations for semantics of intuitionistic justification logics by new model classes as well as a wider range of applicable logics.

All these considerations culminate in general unified completeness theorems based on a semantical characterization of the underlying intermediate logic. As the class of intermediate justification logics contains especially the well-known cases of intuitionistic, Gödel and classical justification logics, these completeness results moreover contain all the completeness theorems based on Mkrtchyev, Fitting and subset models for classical justification

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logics as well as the previous completeness theorems for the Gödel justification logics with respect to \([0, 1]\)-valued Mkrtychev and Fitting models and the frame-based completeness theorems for intuitionistic justification logics as special cases.

## 2. Intermediate justification logics

### 2.1. Syntax and proof calculi.

We consider the propositional language

$$L_0 : \phi := \bot | p | (\phi \land \phi) | (\phi \lor \phi) | (\phi \rightarrow \phi)$$

where \(p \in \text{Var} := \{p_i \mid i \in \mathbb{N}\}\). We introduce negation as the abbreviation \(\neg \phi := \phi \rightarrow \bot\). We also define

$$\bigwedge_{i=1}^{n} \phi_i := \phi_1 \land \cdots \land \phi_n$$

for some \(\phi_1, \ldots, \phi_n \in L_0\). The same applies to \(\lor\). In order to define intermediate logics and later intermediate justification logics, we need to briefly review some notions regarding propositional substitutions.

A substitution in \(L_0\) is a function \(\sigma : \text{Var} \rightarrow L_0\). This function \(\sigma\) naturally extends to \(L_0\) by commuting with the connectives \(\land, \lor, \rightarrow\) and \(\bot\) and we write \(\sigma(\phi)\) also for the image of this extended function.

Using this definition of substitutions, we can now give the following definition of an intermediate justification logic.

**Definition 2.1.** A intermediate logic (over \(L_0\)) is a set \(I \subseteq L_0\) which satisfies:

1. the schemes (A1) - (A9) are contained in \(I\);
2. \(I\) is closed under modus ponens, that is \(\phi \rightarrow \psi, \phi \in I\) implies \(\psi \in I\);
3. \(I\) is closed under substitution in \(L_0\).

Here, the schemes (A1) - (A9) are given by:

- **(A1):** \(\phi \rightarrow (\psi \rightarrow \phi)\);
- **(A2):** \((\phi \rightarrow (\chi \rightarrow \psi)) \rightarrow ((\phi \rightarrow \chi) \rightarrow (\phi \rightarrow \psi))\);
- **(A3):** \((\phi \land \psi) \rightarrow \phi\);
- **(A4):** \((\phi \land \psi) \rightarrow \psi\);
- **(A5):** \(\phi \rightarrow (\psi \rightarrow (\phi \land \psi))\);
- **(A6):** \(\phi \rightarrow (\phi \lor \psi)\);
- **(A7):** \(\psi \rightarrow (\phi \lor \psi)\);
- **(A8):** \((\phi \lor \psi) \rightarrow ((\chi \rightarrow \psi) \rightarrow ((\phi \lor \chi) \rightarrow \psi))\);
- **(A9):** \(\bot \rightarrow \phi\).

We define the smallest intermediate propositional logic, that is the logic given by the axiom schemes (A1) - (A9) in \(L_0\) and closed under modus ponens, by \(\text{ITP}\). Given a set of formulæ \(\Gamma \subseteq L_0\), we write

$$\Gamma \vdash \chi \text{ if } \exists \gamma_1, \ldots, \gamma_n \in \Gamma \left( \bigwedge_{i=1}^{n} \gamma_i \rightarrow \phi \in I \right).$$

On the side of justification logics, we consider the following set of justification terms

$$Jt : t ::= x | c | [t + t] | [t \cdot t] | !t$$

where \(x \in V := \{x_i \mid i \in \mathbb{N}\}\) and \(c \in C := \{c_i \mid i \in \mathbb{N}\}\) and the resulting multi-modal language

$$L_J : \phi ::= \bot | p | (\phi \land \phi) | (\phi \lor \phi) | (\phi \rightarrow \phi) | t : \phi$$

where \(p \in \text{Var}\) and \(t \in Jt\). Naturally, the same abbreviations as for \(L_0\) also apply here. Given a set \(\Gamma, \Delta \subseteq L_J\), we write \(\Gamma \oplus \Delta\) for the smallest set containing \(\Gamma \cup \Delta\) which is closed under modus ponens.

In order to formulate intermediate justification logics, we consider especially substitutions in \(L_J\). These are again functions \(\sigma : \text{Var} \rightarrow L_J\) which extend uniquely to \(L_J\) to commuting with \(\land, \lor, \rightarrow, \bot\) and the justification modalities \(t :\). We again write \(\sigma(\phi)\) for the image of a formula \(\phi \in L_J\) under this extension. By \(\overline{\Gamma}\), we denote the closure of \(\Gamma\) under substitutions in \(L_J\).

**Definition 2.2.** Let \(I\) be an intermediate propositional logic. Given the axiom schemes

- **(J):** \(t : (\phi \rightarrow \psi) \rightarrow (s : \phi \rightarrow [t \cdot s] : \psi),\)
- **(+):** \(t : \phi \rightarrow [t + s] : \phi, t : \phi \rightarrow [s + t] : \phi,\)
- **(F):** \(t : \phi \rightarrow \phi,\)
- **(I):** \(t : \phi \rightarrow !t : t : \phi,\)

we consider the following justification logics based on \(I:\)

1. \(IJ_0 := I \oplus (J) \oplus (+);\)
specification for Lemma 2.5. Let with the schemes we consider explicitly in this paper are specific instances of intermediate propositional logics and of the resulting intermediate justification logics which are conservative over their corresponding intermediate logic and lemma and internalization. We omit this here as well (see [3, 4, 25] for these concepts in the classical case).

axiomatically appropriate constant specifications as in the classical case and obtain analogues for the lifting definitions and results regarding constant specifications more clean to state. Note however, that all results can be appropriately adapted to constant specifications over axiomatic bases.

Note also, that naturally Constant specifications can be used to augment proof systems and increase the amount of justified formulae pure convenience as not introducing the notion of axiomatic systems for intermediate logics makes the following

As before in the propositional case, given \( \Gamma \cup \{ \phi \} \subseteq \mathcal{L}_J \), we write

\[
\Gamma \vdash_{\mathcal{I}_J \mathcal{L}_0} \phi \text{ iff } \exists \gamma_1, \ldots, \gamma_n \in \Gamma \left( \bigwedge_{i=1}^{n} \gamma_i \rightarrow \phi \in \mathcal{I}_J \mathcal{L}_0 \right).
\]

Specific instances of intermediate propositional logics and of the resulting intermediate justification logics which we consider explicitly in this paper are

\[
\mathcal{G} := \mathcal{IPC} \oplus (\mathcal{LIN}), \quad \mathcal{CPC} := \mathcal{IPC} \oplus (\mathcal{LEM}),
\]

with the schemes

(\mathcal{LIN}):
(\phi \rightarrow \psi) \lor (\psi \rightarrow \phi),

(\mathcal{LEM}):
\phi \lor \neg \phi,

over \( \mathcal{L}_0 \).

**Definition 2.3.** Let \( \mathcal{I} \) be an intermediate propositional logic and \( \mathcal{I}_J \mathcal{L}_0 \in \{ \mathcal{I}_J 0, \mathcal{I}_J \mathcal{T}_0, \mathcal{I}_J 4_0, \mathcal{I}_J \mathcal{T} 4_0 \} \). A constant specification for \( \mathcal{I}_J \mathcal{L}_0 \) is a set \( CS \) of formulae from \( \mathcal{L}_J \) of the form

\[
c_{i_0} : \cdots : c_{i_k} : \phi
\]

where \( n \geq 1 \), \( c_{i_k} \in CS \) for all \( k \) and \( \phi \) is an axiom instance of \( \mathcal{I}_J \mathcal{L}_0 \), that is \( \phi \in \mathcal{T} \) or \( \phi \) is an instance of the justification axiom schemes (\( J \), (\( + \)), (\( F \)), (\( I \)) (depending on \( \mathcal{I}_J \mathcal{L}_0 \)).

Constant specifications can be used to augment proof systems and increase the amount of justified formulae which they can prove.

**Definition 2.4.** Let \( \mathcal{I} \) be an intermediate propositional logic and \( \mathcal{I}_J \mathcal{L}_0 \in \{ \mathcal{I}_J 0, \mathcal{I}_J \mathcal{T}_0, \mathcal{I}_J 4_0, \mathcal{I}_J \mathcal{T} 4_0 \} \) and let \( CS \) be a constant specification for \( \mathcal{I}_J \mathcal{L}_0 \). We write \( \Gamma \vdash_{\mathcal{I}_J \mathcal{L}_0 CS} \phi \) for \( \Gamma \cup CS \vdash_{\mathcal{I}_J \mathcal{L}_0} \phi \) with \( \Gamma \cup \{ \phi \} \subseteq \mathcal{L}_J \).

Note, that the above definition of a constant specification is different from the usual one in the literature. Normally, one works with a specific set of axiom schemes for the propositional base of the justification logic in question and allows \( \phi \) in

\[
c_{i_0} : \cdots : c_{i_k} : \phi
\]

only to be an instance of these axioms where here, we allow \( \phi \) to be an arbitrary theorem of \( \mathcal{T} \). This is out of pure convenience as not introducing the notion of axiomatic systems for intermediate logics makes the following definitions and results regarding constant specifications more clean to state. Note however, that all results can be appropriately adapted to constant specifications over axiomatic bases.

If one follows this line of defining axiomatic bases of intermediate logics however, one can similarly define axiomatically appropriate constant specifications as in the classical case and obtain analogues for the lifting lemma and internalization. We omit this here as well (see [3, 4, 25] for these concepts in the classical case).

As a straightforward application of classical techniques (see e.g. [25]), one can show directly that all the intermediate justification logics are conservative over their corresponding intermediate logic and \( \mathcal{L}_0 \).

**Lemma 2.5.** Let \( \mathcal{I} \) be an intermediate logic and let \( \mathcal{I}_J \mathcal{L}_0 \in \{ \mathcal{I}_J 0, \mathcal{I}_J \mathcal{T}_0, \mathcal{I}_J 4_0, \mathcal{I}_J \mathcal{T} 4_0 \} \). Let \( CS \) be a constant specification for \( \mathcal{I}_J \mathcal{L}_0 \). For any \( \phi \in \mathcal{L}_0 \): \( \vdash_{\mathcal{I}_J \mathcal{L}_0 CS} \phi \) iff \( \vdash_{\mathcal{I}} \phi \).

The proof is a natural generalization from the classical case, see e.g. [25] for a version for \( \mathcal{CPC} \mathcal{JT} 4 \). Further, we want to mention the deduction theorem for \( \mathcal{I}_J \mathcal{L}_CS \).

**Lemma 2.6.** Let \( \mathcal{I} \) be an intermediate logic and let \( \mathcal{I}_J \mathcal{L}_0 \in \{ \mathcal{I}_J 0, \mathcal{I}_J \mathcal{T}_0, \mathcal{I}_J 4_0, \mathcal{I}_J \mathcal{T} 4_0 \} \). Further, let \( CS \) be a constant specification for \( \mathcal{I}_J \mathcal{L}_0 \). For any \( \Gamma \cup \{ \phi, \psi \} \subseteq \mathcal{L}_J \):

\[
\Gamma \cup \{ \phi \} \vdash_{\mathcal{I}_J \mathcal{L}_0 CS} \psi \text{ iff } \Gamma \vdash_{\mathcal{I}_J \mathcal{L}_0 CS} \phi \rightarrow \psi.
\]

2.2. Extended propositional languages. In later sections, it will be convenient to consider intermediate logics over different sets of variables. For this, we consider the language

\[
\mathcal{L}_0(X) : \phi ::= \perp \mid x \mid (\phi \land \phi) \mid (\phi \lor \phi) \mid (\phi \rightarrow \phi)
\]

where \( X \) is a countably infinite set of variables. The same notational abbreviations as before also apply here. Note also, that naturally \( \mathcal{L}_0(\text{Var}) = \mathcal{L}_0 \). A particular choice different from \( \text{Var} \) for \( X \) in the following will be the set

\[
\text{Var}^* := \text{Var} \cup \{ \phi_t \mid \phi \in \mathcal{L}_j, t \in J_t \}.
\]
Here, we write $L_0^*: = L_0(Var^*)$.

For the following definition, note that any bijection $t: Var \to X$ can be naturally extended to a bijection $t: L_0 \to L_0(X)$ through recursion on $L_0$ by commuting with $\land, \lor, \to$ and $\bot$. Also, such a bijection $t: Var \to X$ always exists as both $X$ and $Var$ are countably infinite.

**Definition 2.7.** Let $I$ be an intermediate logic and let $t: Var \to X$ be some (any) bijection extended to $t: L_0 \to L_0(X)$. We write $I(X): = t[I]$.

Note, that here also $I(Var) = I$.

**Remark 2.8.** In the above definition, it is indeed not important which bijection $f: Var \to X$ is fixed as $I$ is closed under substitutions. Further, naturally $I(X)$ is closed under modus ponens and under substitutions of variables in $X$ by formulae in $L_0(X)$.

Given $I(X)$ and $\Gamma \cup \{\phi\} \subseteq L_0(X)$, we write $\Gamma \vdash_{I(X)} \phi$ if as before

$$\exists \gamma_1, \ldots, \gamma_n \in \Gamma \left( \bigwedge_{i=1}^n \gamma_i \to \phi \in I(X) \right).$$

In the following, we will also write $I^*$ for the particular case of $I(Var^*)$.

### 3. Algebraic semantics for intermediate justification logics

We move on to the first main line of semantics for intermediate justification logics studied here, extending the model-theoretic approaches of Mkrtychev, Fitting as well as Lehmann and Studer to take values in arbitrary Heyting algebras. The models which we introduce, as well as the techniques used later to prove corresponding completeness theorems, are similar to those from [33] where completeness theorems of the particular case of Gödel justification logics with respect to models over the particular Heyting algebra $[0, 1]_G$ (see the last section) were considered.

#### 3.1. Heyting algebras and propositional semantics

We give some preliminaries on Heyting algebras and their relevant notions as a primer for the later definitions.

**Definition 3.1.** A Heyting algebra is structure $A = \langle A, \land^A, \lor^A, \to^A, 0^A, 1^A \rangle$ such that $\langle A, \land^A, \lor^A, 0^A, 1^A \rangle$ is a bounded lattice with largest element $1^A$ and smallest element $0^A$ and $\to^A$ is a binary operation with

1. $x \to^A x = 1^A$,
2. $x \land^A (x \to^A y) = x \land^A y$,
3. $y \land^A (x \to^A y) = y$,
4. $x \to^A (y \land^A z) = (x \to^A y) \land^A (x \to^A z)$,

where we write $a \leq^A b$ for $a \land^A b = a$.

Note, that this order $\leq^A$ on $A$ is always a partial order. Given a Heyting algebra $A$, we write $\neg^A x := x \to^A 0^A$. We call a Heyting algebra $A$ linear if $x \leq^A y$ or $y \leq^A x$ for all $x, y \in A$. $A$ is called a Boolean algebra, if $x \to^A y = \neg^A x \lor^A y$ for all $x, y \in A$.

We collect some facts about Heyting algebras which are of use later.

**Lemma 3.2.** Let $A = \langle A, \land^A, \lor^A, \to^A, 0^A, 1^A \rangle$ be a Heyting algebra. Then, for all $x, y, z, w \in A$:

1. $x \land^A y \leq^A z$ iff $x \leq^A y \to^A z$;
2. $x \leq^A y$ iff $x \to^A y = 1^A$;
3. $1 = \to^A x$;
4. if $x \leq^A y$, then $y \land^A z \leq^A x \to^A z$;
5. $(x \to^A y) \land^A (z \to^A w) \leq^A (x \land^A z) \to^A (y \land^A w)$.

These properties are quite immediate from the definition of Heyting algebras. For a modern reference on basic properties of Heyting algebras, see e.g. [33].

Another particular property of Heyting algebras important in this note is that of completeness.

**Definition 3.3.** A Heyting algebra $A$ is complete if every set $X \subseteq A$ has a join and a meet with respect to $\leq^A$, that is for every $X \subseteq A$ there are $s_X, i_X \in A$ such that:

- $\forall x \in X : x \leq^A s_X$ and if $x \leq^A s$ for all $x \in X$, then $s_X \leq^A s$;
- $\forall x \in X : i_X \leq^A x$ and if $i \leq^A x$ for all $x \in X$, then $i \leq^A i_X$. 
We denote these (unique) joins and meets, $s_X$ and $i_X$, by $\lor X$ and $\land X$, respectively. Given a class of Heyting algebras $C$, we write $C_{fin}$ for the subclass of all finite algebras and $C_{com}$ for the subclass of all complete Heyting algebras in $C$. Naturally, every finite Heyting algebra is complete.

Given an (extended) propositional language $L_0(X)$, we can give an algebraic interpretation using various classes of particular Heyting algebras.

**Definition 3.4.** Let $A$ be a Heyting algebra. A propositional evaluation of $L_0(X)$ is a function $f : L_0(X) \rightarrow A$ which satisfies the following equations:

1. $f(\perp) = 0^A$;
2. $f(\phi \land \psi) = f(\phi)^A \land f(\psi)$;
3. $f(\phi \lor \psi) = f(\phi)^A \lor f(\psi)$;
4. $f(\phi \rightarrow \psi) = f(\phi)^A \rightarrow f(\psi)$.

We denote the set of all $A$-valued propositional evaluations of $L_0(X)$ by $\text{Ev}(A; L_0(X))$.

**Definition 3.5.** Let $C$ be a class of Heyting algebras. $\Gamma \cup \{\phi\} \subseteq L_0(X)$. We write $\Gamma \vDash_C \phi$ if $\forall A \in C : \forall f \in \text{Ev}(A; L_0(X)) : f[\Gamma] \subseteq \{1^A\}$ implies $f(\phi) = 1^A$.

If in particular $C = \{A\}$, we write $\vDash_A$ for the corresponding relation.

**Definition 3.6.** Let $\mathcal{I}$ be an intermediate logic and let $X$ be a set of variables. We say that $\mathcal{I}(X)$ is (strongly) complete with respect to a class $C$ of Heyting algebras if for any $\Gamma \cup \{\phi\} \subseteq L_0(X)$: $\Gamma \vdash_{\mathcal{I}(X)} \phi$ iff $\Gamma \vDash_C \phi$.

Although not particularly important for the rest of the paper, every intermediate logic actually has at least one class of Heyting algebras with respect to which it is strongly complete (namely its variety). We collect this in the following fact.

**Fact 3.7.** For every intermediate logic $\mathcal{I}$ and any set of variables $X$, there is a class of Heyting algebras $C$ such that $\mathcal{I}(X)$ is strongly complete with respect to $C$.

For a modern reference of the proof, see again e.g. [33]. Correspondingly, we introduce the following notation. We write $C \in \text{Alg}(\mathcal{I}(X))$, $C \in \text{Alg}_{com}(\mathcal{I}(X))$ or $C \in \text{Alg}_{fin}(\mathcal{I}(X))$ if $C$ is a class of Heyting algebras, of complete Heyting algebras or of finite Heyting algebras with respect to which $\mathcal{I}(X)$ is strongly complete. Note, that here $C \in \text{Alg}(\mathcal{I}(X))$ iff $C \in \text{Alg}(\mathcal{I}(Y))$

for arbitrary sets of variables $X, Y$ and similarly for $\text{Alg}_{com}(\mathcal{I}(X))$ and $\text{Alg}_{fin}(\mathcal{I}(X))$.

### 3.2. Algebraic Mkrtychev models.

The first class of semantics which we consider are algebraic Mkrtychev models. The classical Mkrtychev models were introduced in [30], originally for the logic of proofs, and mark the first non-provability semantics. The generalization of the Mkrtychev models to the other classical justification logics $\mathcal{J}_0, \mathcal{J}_T, \mathcal{J}_40$ is due to Kuznets [22]. In some contexts, especially [4, 25], these models are also called basic models. The following algebraic models also generalize the work on $[0,1]$-valued Mkrtychev models in [18, 34] for the Gödel justification logics.

**Definition 3.8** (Algebraic Mkrtychev model). Let $A$ be a Heyting algebra. An ($A$-valued) algebraic Mkrtychev model is a structure $\mathfrak{M} = (A, \mathcal{V})$ such that $\mathcal{V} : \mathcal{L}_J \rightarrow A$ fulfills

1. $\mathcal{V}(\perp) = 0^A$;
2. $\mathcal{V}(\phi \land \psi) = \mathcal{V}(\phi)^A \land \mathcal{V}(\psi)$;
3. $\mathcal{V}(\phi \lor \psi) = \mathcal{V}(\phi)^A \lor \mathcal{V}(\psi)$;
4. $\mathcal{V}(\phi \rightarrow \psi) = \mathcal{V}(\phi)^A \rightarrow \mathcal{V}(\psi)$,

for all $\phi, \psi \in \mathcal{L}_J$ and such that it satisfies

(i) $\mathcal{V}(t : (\phi \rightarrow \psi)) \land \mathcal{V}(s : \phi) \leq \mathcal{V}([t \cdot s] : \psi)$,
(ii) $\mathcal{V}(t : \phi) \lor \mathcal{V}(s : \phi) \leq \mathcal{V}([t + s] : \phi)$,

for all $t, s \in Jt$ and $\phi, \psi \in \mathcal{L}_J$.

We write $\mathfrak{M} \models \phi$ if $\mathcal{V}(\phi) = 1^A$ and $\mathfrak{M} \models \Gamma$ if $\mathfrak{M} \models \gamma$ for all $\gamma \in \Gamma$ where $\Gamma \subseteq \mathcal{L}_J$.

**Definition 3.9.** Let $\mathfrak{M} = (A, \mathcal{V})$ be an $A$-valued algebraic Mkrtychev model. We call $\mathfrak{M}$

1. factive if $\mathcal{V}(t : \phi) \leq \mathcal{V}(\phi)$, and
2. introspective if $\mathcal{V}(t : \phi) \leq \mathcal{V}([t : t] : \phi)$.

**Definition 3.10.** Let $C$ be a class of Heyting algebras. Then:

1. $\text{CAMJ}$ denotes the class of all $A$-valued Mkrtychev models, for all $A \in C$;
Definition 3.11. Let $A$ be a Heyting algebra and let $\mathfrak{M} = (A, V)$ be an algebraic Mkrtychev model. Further, let $CS$ be a constant specification (for some proof calculus). We say that $\mathfrak{M}$ respects $CS$ if $V(c : \phi) = 1^A$ for all $c : \phi \in CS$.

If $C$ is a class of algebraic Mkrtychev models, then we denote the subclass of all models from $C$ respecting a constant specification $CS$ by $C_{CS}$.

Definition 3.12. Let $C$ be a class of algebraic Mkrtychev models and let $\Gamma \cup \{\phi\} \subseteq L_J$. We write:

1. $\Gamma \models_C \phi$ if $\forall \mathfrak{M} = (A, V) \in C \left( {\bigwedge}^A \{V(\gamma) \mid \gamma \in \Gamma \} \leq^A V(\phi) \right)$;
2. $\Gamma \models_{\mathfrak{C}} \phi$ if $\forall \mathfrak{M} = (A, V) \in C \left( \mathfrak{M} \models \Gamma \Rightarrow \mathfrak{M} \models \phi \right)$.

Lemma 3.13. Let $I$ be an intermediate logic, $IJ L_0 \in \{IJJ_0, IJJT_0, IJJT_4_0, IJJT_4\}$, let $CS$ be a constant specification logic, and let $C \subseteq \text{Alg}(I)$. For any $\Gamma \cup \{\phi\} \subseteq L_J$:

$$\Gamma \vdash_{IJ L CS} \phi \text{ implies } \Gamma \models_{\text{CAMJL}_{CS}} \phi.$$

Proof. We only show that $\vdash_{IJ L_0} \phi$ implies $\models_{\text{CAMJL}} \phi$. This already suffices for the strong completeness statement above by the following argument using the deduction theorem for the respective logics and compactness of the provability relations:

$$\Gamma \vdash_{IJ L_0} \phi \implies \exists \mathfrak{M}_0 \subseteq \Gamma \cup CS \text{ finite } \left( \Gamma_0 \vdash_{IJ L_0} \phi \right)$$

implies $\exists \mathfrak{M}_0 \subseteq \Gamma \cup CS$ finite $\left( \vdash_{\text{CAMJL}} \left( {\bigwedge}^A \{V(\gamma) \mid \gamma \in \Gamma_0 \} \leq^A V(\phi) \right) \right)$.

If now $\phi \in T$, then by definition there is a substitution $\sigma : \mathcal{V}ar \to L_J$ and a formula $\psi \in \mathcal{I}$ such that $\phi = \sigma(\psi)$. Let $A \in C$ and $\mathfrak{M} = \langle A, V \rangle$ be a CAMJ-model. Then, we may define

$$f : \chi \mapsto V(\sigma(\chi))$$

for $\chi \in \mathcal{L}_0$. By definition of $\mathfrak{M}$ and properties of $\sigma$, we have that $f$ is a well-defined evaluation on $A$. By the choice of $C$, we have that $\psi \in \mathcal{I}$ implies $f(\psi) = 1^A$ and thus $V(\sigma(\psi)) = V(\phi) = 1$. As $\mathfrak{M}$ was arbitrary, we have $\models_{\text{CAMJ}} \phi$. □

3.3. Algebraic Fitting models. The second algebraic semantics which we consider is based on algebraic Fitting models, derived from the fundamental possible-world semantics of Fitting [12, 13] which combined the earlier work of Mkrtychev on syntactic evaluations with the usual semantics of non-explicit modal logics based on modal Kripke models. As a generalization, we allow the accessibility, evidence and evaluation functions to take values in Heyting algebras. We have to restrict to complete Heyting algebras however, as we want certain algebraic equations to be satisfied which involve infima and suprema. The algebraic Fitting models presented here again generalize the previously introduced many-valued Fitting models from [16, 31] from the context of the Gödel justification logics.

Definition 3.14. Let $A$ be a complete Heyting algebra. An ($A$-valued) algebraic Fitting model is a structure $\mathfrak{M} = \langle A, W, R, E, V \rangle$ with
such that it fulfills the conditions

1. \( V(w, \bot) = 0^A \),
2. \( V(w, \phi \land \psi) = V(w, \phi) \land^A V(w, \psi) \),
3. \( V(w, \phi \lor \psi) = V(w, \phi) \lor^A V(w, \psi) \),
4. \( V(w, \phi \rightarrow \psi) = V(w, \phi) \rightarrow^A V(w, \psi) \),
5. \( V(w, t : \phi) = E_w(t, \phi) \land^A \bigwedge \{ R(w, v) \rightarrow^A V(v, \phi) \mid v \in W \} \),

for all \( w \in W \) and such that it satisfies

(i) \( E_w(t, \phi \rightarrow \psi) \land^A E_w(s, \phi) \leq^A E_w(t \cdot s, \psi) \),
(ii) \( E_w(t, \phi) \lor^A E_w(s, \phi) \leq^A E_w(t + s, \phi) \),

for all \( w \in W \), all \( t, s \in Jt \) and all \( \phi, \psi \in \mathcal{L}_J \).

We write \( (M, w) \models \phi \) for \( V(w, \phi) = 1^A \) and \( (M, w) \models \gamma \) if \( (M, w) \models \gamma \) for all \( \gamma \in \Gamma \).

**Definition 3.15.** Let \( \mathfrak{M} = (A, W, R, E, V) \) be an \( A \)-valued Fitting model. We call \( \mathfrak{M} \)

(i) reflexive if \( \forall w \in W (R(w, w) = 1^A) \),
(ii) transitive if \( \forall w, v, u \in W (R(w, v) \land^A R(v, u) \leq^A R(w, u)) \),
(iii) monotone if \( \forall w, v \in W \forall t \in Jt, \phi \in \mathcal{L}_J (E_w(t, \phi) \land^A R(w, v) \leq^A E_v(t, \phi)) \),
(iv) introspective if it is transitive, monotone and satisfies

\[ E_w(t, \phi) \leq^A E_w(t, \phi) \]

for all \( w \in W \) and all \( t \in Jt, \phi \in \mathcal{L}_J \),
(v) accessibility-crisp if \( \forall w, v \in W (R(w, v) \in \{0^A, 1^A\}) \).

**Definition 3.16.** Let \( C \) be a class of complete Heyting algebras. Then:

1. \( \text{CAFJ} \) denotes the class of all \( A \)-valued Fitting models, for all \( A \in C \);
2. \( \text{CAFJT} \) denotes the class of all reflexive \( \text{CAFJ} \)-models;
3. \( \text{CAFJ4} \) denotes the class of all introspective \( \text{CAFJ} \)-models;
4. \( \text{CFJT4} \) denotes the class of all \( \text{CAFJ4} \)-models which are reflexive.

By \( C^\circ \), we denote the class of all accessibility-crisp models in \( C \) for some class \( C \) of algebraic Fitting models.

**Definition 3.17.** Let \( A \) be a complete Heyting algebra and let \( \mathfrak{M} = (A, W, R, E, V) \) be a \( A \)-valued algebraic Fitting model. We say that \( \mathfrak{M} \) respects a constant specification \( CS \) (for some proof system) if

\[ V(w, c : \phi) = 1^A \]

for all \( w \in W \) and all \( c : \phi \in CS \).

Given a class \( C \) of algebraic Fitting model, we denote the subclass of all algebraic Fitting models in \( C \) respecting a constant specification \( CS \) (for some proof system) by \( C_{CS} \).

**Definition 3.18.** Let \( C \) be a class of algebraic Fitting models and \( \Gamma \cup \{ \phi \} \subseteq \mathcal{L}_J \). We write:

1. \( \Gamma \models c \phi \) if \( \forall \mathfrak{M} = (A, W, R, E, V) \in C \forall w \in W \left( \bigwedge \{ V(w, \gamma) \mid \gamma \in \Gamma \} \leq^A V(w, \phi) \right) \);
2. \( \Gamma \models c^L \phi \) if \( \forall \mathfrak{M} = (A, W, R, E, V) \in C \forall w \in W \left( (\mathfrak{M}, w) \models \Gamma \Rightarrow (\mathfrak{M}, w) \models \phi \right) \).

**Lemma 3.19.** Let \( I \) be an intermediate logic and \( I \mathcal{J} \mathcal{L}_0 \in \{ I \mathcal{J}_0, I \mathcal{J} \mathcal{T}_0, I \mathcal{J} \mathcal{T}_40, I \mathcal{J} \mathcal{T}_40 \} \). Let \( CS \) be a constant specification for \( I \mathcal{J} \mathcal{L}_0 \) and let \( C \in \text{Alg}_{\text{com}}(I) \). For any \( \Gamma \cup \{ \phi \} \subseteq \mathcal{L}_J \):

\[ \Gamma \vdash_{I \mathcal{J} \mathcal{L}_0} \phi \text{ implies } \Gamma \models_{\text{CAFJ}_{CS}} \phi \]

**Proof.** As before, we only show \( \Gamma \vdash_{I \mathcal{J} \mathcal{L}_0} \phi \) implies \( \models_{\text{CAFJ}_{CS}} \phi \). The argument from the proof of Lemma 3.13 about how to obtain strong soundness can be straightforwardly adapted to the case of algebraic Fitting models.

To see that \( \Gamma \vdash_{I \mathcal{J} \mathcal{L}_0} \phi \) implies \( \models_{\text{CAFJ}} \phi \), note that it is again enough to show the claim for \( \phi \in \mathcal{T} \) or \( \phi \) being a justification axiom (depending on \( I \mathcal{L}_J \)).

If \( \phi \in \mathcal{T} \), then by the choice of \( C \) one may repeat the argument from the proof of Lemma 3.13 locally for every \( V(w, \cdot) \) with \( w \in W \) to obtain \( \models_{\text{CAFJ}} \phi \).

As the algebraic Fitting models are slightly more complex in their evaluation of the justification modalities, we actually show the validity of the justification axiom schemes in their respective model classes. For this, let \( \mathfrak{M} = (A, W, R, E, V) \in \text{CAFJ} \) and let \( w \in W \).
(1) Consider the axiom scheme (J). Then, we have
\[
\bigwedge^A \{ \mathcal{R}(w, v) \rightarrow^A \mathcal{V}(v, \phi \rightarrow \psi) \mid v \in \mathcal{W} \} \land^A \bigwedge^A \{ \mathcal{R}(w, v) \rightarrow^A \mathcal{V}(v, \phi) \mid v \in \mathcal{W} \} \\
\leq^A \bigwedge^A \{ \mathcal{R}(w, v) \rightarrow^A \mathcal{V}(v, \psi) \mid v \in \mathcal{W} \}
\]

Further we have \( \mathcal{E}_w(t, \phi \rightarrow \psi) \land^A \mathcal{E}_w(s, \phi) \leq^A \mathcal{E}_w(t \cdot s, \psi) \) by condition (i) of Definition 3.14. Thus:
\[
\mathcal{V}(w, t : (\phi \rightarrow \psi)) \land^A \mathcal{V}(w, s : \phi) = \mathcal{E}_w(t, \phi \rightarrow \psi) \land^A \bigwedge^A \{ \mathcal{R}(w, v) \rightarrow^A \mathcal{V}(v, \phi \rightarrow \psi) \mid v \in \mathcal{W} \} \land^A \mathcal{E}_w(s, \phi) \land^A \bigwedge^A \{ \mathcal{R}(w, v) \rightarrow^A \mathcal{V}(v, \phi) \mid v \in \mathcal{W} \} \\
\leq^A \mathcal{E}_w(t \cdot s, \phi) \land^A \bigwedge^A \{ \mathcal{R}(w, v) \rightarrow^A \mathcal{V}(v, \psi) \mid v \in \mathcal{W} \} \\
= \mathcal{V}(w, [t :: s] : \psi).
\]

The claim follows from the above as by residuation, we have \( \mathcal{V}(w, t : (\phi \rightarrow \psi)) \leq^A \mathcal{V}(w, s : \phi) \rightarrow^A \mathcal{V}(w, [t :: s] : \psi) \).

(2) Consider the axiom scheme (+). We only show \( \mathcal{V}(w, t : \phi) \leq^A \mathcal{V}(w, [t + s] : \phi) \). The other part follows similarly. By condition (ii) of Definition 3.14, we have
\[
\mathcal{E}_w(t, \phi) \leq^A \mathcal{E}_w(t, \phi) \lor^A \mathcal{E}_w(s, \phi) \leq^A \mathcal{E}_w(t + s, \phi)
\]
and thus
\[
\mathcal{V}(w, t : \phi) = \mathcal{E}_w(t, \phi) \land^A \bigwedge^A \{ \mathcal{R}(w, v) \rightarrow^A \mathcal{V}(v, \phi) \mid v \in \mathcal{W} \} \\
\leq^A \mathcal{E}_w(t + s, \phi) \land^A \bigwedge^A \{ \mathcal{R}(w, v) \rightarrow^A \mathcal{V}(v, \phi) \mid v \in \mathcal{W} \} \\
= \mathcal{V}(w, [t + s] : \phi).
\]

(3) Consider the axiom scheme (F) and assume that \( \mathfrak{M} \) is reflexive. We have \( \mathcal{R}(w, w) = 1^A \) and thus:
\[
\mathcal{V}(w, t : \phi) = \mathcal{E}_w(t, \phi) \land^A \bigwedge^A \{ \mathcal{R}(w, v) \rightarrow^A \mathcal{V}(v, \phi) \mid v \in \mathcal{W} \} \\
\leq^A \bigwedge^A \{ \mathcal{R}(w, v) \rightarrow^A \mathcal{V}(v, \phi) \mid v \in \mathcal{W} \} \\
\leq^A \mathcal{R}(w, w) \rightarrow^A \mathcal{V}(w, \phi) \\
= \mathcal{V}(w, \phi).
\]

(4) Consider the axiom scheme (I). Assume that \( \mathfrak{M} \) is introspective. By the transitivity of \( \mathcal{R} \), we have at first
\[
\bigwedge^A \{ \mathcal{R}(w, u) \rightarrow^A \mathcal{V}(u, \phi) \mid u \in \mathcal{W} \} \leq^A \mathcal{R}(w, v) \rightarrow^A \bigwedge^A \{ \mathcal{R}(v, u) \rightarrow^A \mathcal{V}(u, \phi) \mid u \in \mathcal{W} \}.
\]

To see this, note that we have
\[
\mathcal{R}(w, v) \land^A \bigwedge^A \{ \mathcal{R}(w, u) \rightarrow^A \mathcal{V}(u, \phi) \mid u \in \mathcal{W} \} \leq^A \mathcal{R}(w, v) \land^A (\mathcal{R}(w, u) \rightarrow^A \mathcal{V}(u, \phi)) \\
\leq^A \mathcal{R}(w, v) \land^A (\mathcal{R}(w, v) \land^A (\mathcal{R}(w, u) \rightarrow^A \mathcal{V}(u, \phi))) \\
= \mathcal{R}(w, v) \land^A (\mathcal{R}(w, v) \rightarrow^A (\mathcal{R}(v, u) \rightarrow^A \mathcal{V}(u, \phi))) \\
\leq^A \mathcal{R}(w, u) \rightarrow^A \mathcal{V}(u, \phi)
\]
for all \( u \in \mathcal{W} \). By taking the infimum over \( u \), we have
\[
\mathcal{R}(w, v) \land^A \bigwedge^A \{ \mathcal{R}(w, u) \rightarrow^A \mathcal{V}(u, \phi) \mid u \in \mathcal{W} \} \leq^A \bigwedge^A \{ \mathcal{R}(v, u) \rightarrow^A \mathcal{V}(u, \phi) \mid u \in \mathcal{W} \}.
\]

Further, we have by monotonicity that
\[
\mathcal{E}_w(t, \phi) \leq^A \mathcal{R}(w, v) \rightarrow^A \mathcal{E}_w(t, \phi).
\]
Therefore, we have
\[
\nu(w, t : \phi) = \mathcal{E}_w(t, \phi) \land^A \bigwedge^A \{ \mathcal{R}(w, u) \rightarrow^A \nu(u, \phi) \mid u \in \mathcal{W} \}
\]
\[
\leq^A (\mathcal{R}(w, v) \rightarrow^A \mathcal{E}_w(t, \phi)) \land^A \left( \mathcal{R}(w, v) \rightarrow^A \bigwedge^A \{ \mathcal{R}(v, u) \rightarrow^A \nu(u, \phi) \mid u \in \mathcal{W} \} \right)
\]
\[
\leq^A \mathcal{R}(w, v) \rightarrow^A \left( \mathcal{E}_w(t, \phi) \land^A \bigwedge^A \{ \mathcal{R}(v, u) \rightarrow^A \nu(u, \phi) \mid u \in \mathcal{W} \} \right)
\]
\[
= \mathcal{R}(w, v) \rightarrow^A \nu(v, t : \phi).
\]

By taking the infimum, we have
\[
\mathcal{E}_w(t, \phi) \land^A \bigwedge^A \{ \mathcal{R}(w, u) \rightarrow^A \nu(u, \phi) \mid u \in \mathcal{W} \} \leq^A \bigwedge^A \{ \mathcal{R}(w, v) \rightarrow^A \nu(v, t : \phi) \mid v \in \mathcal{W} \}.
\]
Thus, we have for any \( v \in \mathcal{W} \) by introspectivity:
\[
\nu(w, t : \phi) \leq^A \mathcal{E}_w(t, t : \phi) \land^A \bigwedge^A \{ \mathcal{R}(w, v) \rightarrow^A \nu(v, t : \phi) \mid v \in \mathcal{W} \}
\]
\[
= \mathcal{V}(w, t : \phi).
\]

3.4. Algebraic subset models. The last algebraic semantics which we consider is based on algebraic generalizations of the subset models for classical justification logic by Lehmann and Studer [26]. Similar as with the previous algebraic Fitting models, we allow all involved functions to take arbitrary values in Heyting algebras, again restricting ourselves to complete Heyting algebras to be able to formulate certain regularity conditions.

Definition 3.20 (Algebraic subset model). Let \( A \) be a complete Heyting algebra with domain \( A \). An \((A\text{-valued})\) algebraic subset model is a structure \( \mathfrak{M} = (A, \mathcal{W}, \mathcal{W}_0, \mathcal{E}, \nu) \) with

- \( \mathcal{W} \neq \emptyset \),
- \( \mathcal{W}_0 \subseteq \mathcal{W}, \mathcal{W}_0 \neq \emptyset \),
- \( \mathcal{E} : Jt \times \mathcal{W} \times \mathcal{W} \rightarrow A \),
- \( \nu : \mathcal{W} \times \mathcal{L}_J \rightarrow A \),

such that for all \( w \in \mathcal{W}_0 \), \( \nu \) fulfills the conditions

1. \( \nu(w, \bot) = 0^A \),
2. \( \nu(w, \phi \land \psi) = \nu(w, \phi) \land^A \nu(w, \psi) \),
3. \( \nu(w, \phi \lor \psi) = \nu(w, \phi) \lor^A \nu(w, \psi) \),
4. \( \nu(w, \phi \rightarrow \psi) = \nu(w, \phi) \rightarrow^A \nu(w, \psi) \),
5. \( \nu(w, t : \phi) = \bigwedge^A \{ \mathcal{E}_t(w, v) \rightarrow^A \nu(v, \phi) \mid v \in \mathcal{W} \} \),

and that it is regular, that is for all \( w \in \mathcal{W}_0 \):

- (i) \( \mathcal{E}_{s+t}(w, v) \leq^A \mathcal{E}_s(w, v) \land^A \mathcal{E}_t(w, v) \) for all \( v \in \mathcal{W} \);
- (ii) for all \( v \in \mathcal{W} \):
  \[
  \mathcal{E}_{s+t}(w, v) \leq^A \bigwedge^A \{ \mathfrak{M}^w_{s+t}(\psi) \rightarrow^A \nu(v, \psi) \mid \psi \in \mathcal{L}_J \}
  \]
  with
  \[
  \mathfrak{M}^w_{s+t}(\psi) := \bigvee^A \{ \nu(w, s : (\phi \rightarrow \psi)) \land \nu(w, t : \phi) \mid \phi \in \mathcal{L}_J \}.
  \]

We write \( (\mathfrak{M}, w) \models \phi \) for \( \nu(w, \phi) = 1^A \) and \( (\mathfrak{M}, w) \models \Gamma \) for \( \nu(w, \gamma) = 1^A \) for all \( \gamma \in \Gamma \).

The function \( \mathcal{E} \) is actually a straightforward \( A \)-valued generalization of the \( \mathcal{E} \)-function from [26] as it is in fact nothing more than a different representation of the function
\[
\mathcal{E} : Jt \times \mathcal{W} \rightarrow A^\mathcal{W}
\]
which maps terms and worlds to \( A \text{-valued subsets of } \mathcal{W} \).

Definition 3.21. Let \( \mathfrak{M} = (A, \mathcal{W}, \mathcal{W}_0, \mathcal{E}, \nu) \) be an \( A \)-valued subset model. We call \( \mathfrak{M} \)

- (i) reflexive if \( \forall w \in \mathcal{W}_0 \forall t \in Jt (\mathcal{E}_t(w, w) = 1^A) \),
- (ii) introspective if \( \forall w \in \mathcal{W}_0 \forall v \in \mathcal{W} \forall t \in Jt (\mathcal{E}_{t}(w, v) \leq^A \bigwedge^A \{ \nu(w, t : \phi) \rightarrow^A \nu(v, t : \phi) \mid \phi \in \mathcal{L}_J \}) \),
- (iii) accessibility-crisp if \( \forall t \in Jt \forall w, v \in \mathcal{W}_0 (\mathcal{E}_t(w, v) \in \{ 0^A, 1^A \}) \).

Definition 3.22. Let \( C \) be a class of complete Heyting algebras. Then:
(1) $\text{CASJ}$ denotes the class of all $A$-valued subset models, for all $A \in C$;
(2) $\text{CASJT}$ denotes the class of all $A$-valued reflexive subset models, for all $A \in C$;
(3) $\text{CASJ4}$ denotes the class of all $A$-valued introspective subset models, for all $A \in C$;
(4) $\text{CASJT4}$ denotes the class of all $A$-valued reflexive and introspective subset models, for all $A \in C$.

Given a class $C$ of algebraic subset models, we denote the class of all accessibility-crisp models in $C$ by $C^\ast$.

**Definition 3.23.** Let $A$ be a complete Heyting algebra and let $\mathfrak{M} = (A, W, W_0, E, V)$ be a $A$-valued algebraic subset model. Further, let $CS$ be a constant specification (for some proof calculus). We say that $\mathfrak{M}$ respects $CS$ if $V(w, c : \phi) = 1^A$ for all $c : \phi \in CS$ and all $w \in W_0$.

Given a class $C$ of algebraic subset models, we write $C_{CS}$ for the class of all models from $C$ which respect $CS$. As before, there are two natural consequence relations to consider here.

**Definition 3.24.** Let $\Gamma \cup \{\phi\} \subseteq L_j$ and $C$ be a class of algebraic subset models. We write

1. $\Gamma \models_C \phi$ if $\forall \mathfrak{M} = (A, W, W_0, E, V) \in C \forall w \in W_0 \left(\bigwedge \{V(w, \gamma) \mid \gamma \in \Gamma\} \subseteq A \ V(w, \phi)\right)$;
2. $\Gamma \equiv_C \phi$ if $\forall \mathfrak{M} = (A, W, W_0, E, V) \in C \forall w \in W_0 \left(\langle \mathfrak{M}, w \rangle \models \Rightarrow (\mathfrak{M}, w) \models \phi\right)$.

We write $\Gamma \models_{A, I} \phi$ or $\Gamma \equiv_{A, I} \phi$ for $\Gamma \models_{\{A\}, I} \phi$ or $\Gamma \equiv_{\{A\}, I} \phi$, respectively.

**Lemma 3.25.** Let $\mathcal{I}$ be an intermediate logic and $\mathcal{I}_j$ be a constant specification for $\mathcal{I}_j$ and let $C \in \text{Alg}_{\text{com}}(\mathcal{I})$. For any $\Gamma \cup \{\phi\} \subseteq L_j$, we have:

$\Gamma \vdash_{\mathcal{I}_j, C_{CS}} \phi$ implies $\Gamma \models_{A, I} \phi$.

**Proof.** By the same reasoning as in Lemmas 8.13 and 9.13 we only show $\vdash_{\mathcal{I}_j, L_j} \phi$ implies $\models_{\text{CASJ}} \phi$. Similarly, it suffices to show the claim for $\phi \in \mathcal{T}$ as well as the justifications axioms (based on $\mathcal{I}, \mathcal{L}_j$).

We may repeat the argument from the previous soundness proofs that $\phi \in \mathcal{T}$ implies $\models_{\text{CASJ}} \phi$ by constructing a similar propositional evaluation $f$ locally for every $V(w, \cdot)$ over every $w \in W_0$.

We thus only show the validity of (1) $(J)$, (2) $(\phi)$, (3) $(F)$ and (4) $(4)$ in their respective model classes. For this, let $\mathfrak{M} = (A, W, W_0, E, V)$ be a $\text{CASJ}_{CS}$-model and let $w \in W_0$.

1. We show

$$\forall \mathfrak{M} = (A, W, W_0, E, V) \in C \forall w \in W_0 \left(\bigwedge \{V(w, \gamma) \mid \gamma \in \Gamma\} \subseteq A \ V(w, \phi)\right)$$

For this, let $v \in W$. We then have

$$\forall \mathfrak{M} = (A, W, W_0, E, V) \in C \forall w \in W_0 \left(\bigwedge \{V(w, t : (\phi \rightarrow \psi)) \land A \ V(w, s : \phi) \subseteq A \ V(w, t : \phi)ight)$$

through condition (ii) of Definition 3.20. Therefore, we have

$$\forall \mathfrak{M} = (A, W, W_0, E, V) \in C \forall w \in W \forall v \in W \forall s \in W \left[\bigwedge \{V(w, t : (\phi \rightarrow \psi)) \land A \ V(w, s : \phi) \subseteq A \ V(w, t \cdot s : \psi)\right]$$

as $v$ was arbitrary, which is (1).

2. Let $v \in W$. We have

$$\forall \mathfrak{M} = (A, W, W_0, E, V) \in C \forall w \in W \forall u \in W \left(\bigwedge \{V(w, t : u) \rightarrow A \ V(u, \phi) \mid u \in W\right)$$

through condition (i) in Definition 3.20. As $v$ was arbitrary, we have

$$\forall \mathfrak{M} = (A, W, W_0, E, V) \in C \forall w \in W \forall t \in W \left(\bigwedge \{V(w, t : \phi) \subseteq A \ V(w, t \cdot s : \phi)\right.$$}

One shows similarly that $\forall \mathfrak{M} = (A, W, W_0, E, V) \in C \forall w \in W \left(\bigwedge \{V(w, t + s : \phi) \right.$

3. $\mathfrak{M}$ is reflexive by assumption. Therefore, we have $E_t(w, w) = 1^A$ as $w \in W_0$ and thus

$$\forall \mathfrak{M} = (A, W, W_0, E, V) \in C \forall w \in W \forall v \in W \left(\bigwedge \{V(w, t : \phi) \subseteq A \ V(w, t \cdot s : \phi)\right.$$}
Thus, we have
\[ \mathcal{V}(w, t : \phi) \leq^A \mathcal{E}_\mathcal{V}(w, v) \rightarrow^A \mathcal{V}(v, t : \phi) \]
for any \( v \in \mathcal{W} \) by the introspectivity and we have therefore:
\[ \mathcal{V}(w, t : \phi) \leq^A \bigwedge \{ \mathcal{E}_\mathcal{V}(w, v) \rightarrow^A \mathcal{V}(v, t : \phi) \mid v \in \mathcal{W} \} = \mathcal{V}(v, t : \phi). \]
\( \square \)

4. Completeness for algebraic semantics

To approach completeness, we translate the language \( \mathcal{L}_J \) to \( \mathcal{L}_0^* \) by introducing the translation
\[ \star : \mathcal{L}_J \rightarrow \mathcal{L}_0^* \]
using recursion on \( \mathcal{L}_J \) with the following clauses:
\begin{itemize}
  \item \( \bot^* := \bot; \)
  \item \( p^* := p; \)
  \item \( (\phi \circ v)^* := \phi^* \circ \psi^* \) with \( o \in \{ \land, \lor, \rightarrow \}; \)
  \item \( (t : \phi)^* := \phi_t. \)
\end{itemize}

Using the above translation, we can convert formulæ containing justification modalities into formulæ of \( \mathcal{L}_0^* \) and use semantic results for the intermediate logic in question over \( \mathcal{L}_0^* \) to derive results for the corresponding intermediate justification logic. This approach, especially in the context of algebra-valued modal logics, goes back to Caicedo and Rodriguez work [6] (see also [39]) and was previously also applied in the context of many-valued justification logics (see [34]).

For this, the following lemma provides a way to interpret modal systems in extended propositional systems. For this, given a proof calculus \( \mathcal{S} \) over a language \( \mathcal{L} \), we write \( \text{Th}_S := \{ \phi \in \mathcal{L} \mid \models_S \phi \} \).

**Lemma 4.1.** Let \( \mathcal{I} \) be an intermediate logic and \( \mathcal{I}\mathcal{J}_0 \in \{ \mathcal{I}\mathcal{J}_0, \mathcal{I}\mathcal{J}_T_0, \mathcal{I}\mathcal{J}_A_0, \mathcal{I}\mathcal{J}_T_4 \} \) and \( \text{CS} \) be a constant specification for \( \mathcal{I}\mathcal{J}_0 \). For any \( \Gamma \cup \{ \phi \} \subseteq \mathcal{L}_J \):

\[ \Gamma \vdash_{\mathcal{I}\mathcal{J}_\mathcal{L}_\text{CS}} \phi \iff \Gamma^* \cup (\text{Th}_{\mathcal{I}\mathcal{J}_\mathcal{L}_\text{CS}})^* \vdash_{\mathcal{I}_\star} \phi^*. \]

**Proof.** We prove both directions separately. In any way, recall that \( \star \) is a bijection between \( \mathcal{L}_J \) and \( \mathcal{L}_0^* \).

For the direction from left to right, notice that it suffices to show \( \Gamma^* \cup (\text{Th}_{\mathcal{I}\mathcal{J}_\mathcal{L}_\text{CS}})^* \vdash_{\mathcal{I}_\star} \phi^* \) for
\begin{itemize}
  \item (i) \( \phi \in \Gamma \), or
  \item (ii) \( \phi \in \text{CS} \), or
  \item (iii) \( \phi \in \mathcal{I}\mathcal{J}_0 \),
\end{itemize}
and that it is preserved under modus ponens. The latter is obvious by definition of \( \star \). For (i) of the former, we have \( \phi^* \in \Gamma^* \) and thus \( \Gamma^* \cup (\text{Th}_{\mathcal{I}\mathcal{J}_\mathcal{L}_\text{CS}})^* \vdash_{\mathcal{I}_\star} \phi^* \). For (ii) and (iii), we have \( \vdash_{\mathcal{I}\mathcal{J}_\mathcal{L}_\text{CS}} \phi \), and thus \( \phi^* \in (\text{Th}_{\mathcal{I}\mathcal{J}_\mathcal{L}_\text{CS}})^* \).

This gives again \( \Gamma^* \cup (\text{Th}_{\mathcal{I}\mathcal{J}_\mathcal{L}_\text{CS}})^* \vdash_{\mathcal{I}_\star} \phi^* \).

For the direction from right to left, note that also here it suffices to show \( \Gamma \vdash_{\mathcal{I}\mathcal{J}_\mathcal{L}_\text{CS}} \phi \) for
\begin{itemize}
  \item (a) \( \phi \in \Gamma \), or
  \item (b) \( \phi \in (\text{Th}_{\mathcal{I}\mathcal{J}_\mathcal{L}_\text{CS}})^* \), or
  \item (c) \( \phi \in \mathcal{I}^* \),
\end{itemize}
and that also here, it is preserved under modus ponens. The latter is again immediate. For (a) of the former, we have \( \phi \in \Gamma \) which gives \( \vdash_{\mathcal{I}\mathcal{J}_\mathcal{L}_\text{CS}} \phi \) directly. For (b), we have \( \vdash_{\mathcal{I}\mathcal{J}_\mathcal{L}_\text{CS}} \phi \) by definition. For (c), we have \( \phi \in \mathcal{I}^* \). To see this, note that by the definition of \( \mathcal{I}^* = \mathcal{I}(\mathcal{L}_0^*) \), we have \( \phi^* = \sigma(\psi) \) for some \( \psi \in \mathcal{I} \) and some bijection \( t : \text{Var} \rightarrow \text{Var}^* \). Now, the function
\[ \sigma_t : p \mapsto \begin{cases} q & \text{if } t(p) = q \\ s : \phi & \text{if } t(p) = \phi_s \end{cases} \]
is a substitution from \( \text{Var} \) to \( \mathcal{L}_J \) and we have
\[ \phi = \sigma_t(\psi). \]
Thus, we have \( \phi \in \mathcal{I} \) and thus \( \phi \in \mathcal{I}\mathcal{J}_\mathcal{L}_0 \), i.e. \( \vdash_{\mathcal{I}\mathcal{J}_\mathcal{L}_\text{CS}} \phi \). \( \square \)

The rest of this section is devoted countermodel constructions, converting algebraic evaluations of \( \mathcal{L}_0^* \) into corresponding algebraic Mkrtchyan, Fitting or subset models and deriving corresponding completeness results for the intermediate justification logics from this.
4.1. Completeness w.r.t. algebraic Mkrtychev models.

**Definition 4.2.** Let $I$ be an intermediate logic and let $I\mathcal{J}\mathcal{L}_0 \in \{I\mathcal{J}_0, I\mathcal{J}T_0, I\mathcal{J}_4, I\mathcal{J}T_4\}$ where $CS$ is a constant specification for $I\mathcal{J}\mathcal{L}_0$. Let $A$ be a Heyting algebra and $v \in \text{Ev}(A; L^*_0)$. The canonical algebraic Mkrtychev model w.r.t. $A$ and $v$ is the structure $\mathfrak{M}^{c,M}_{A,v}(I\mathcal{J}\mathcal{L}_{CS}) = \langle A, V^c \rangle$ defined by:

$$V^c(\phi) := v(\phi^c).$$

**Lemma 4.3.** For any Heyting algebra $A$, any $v \in \text{Ev}(A; L^*_0)$ with $v[(Th_{I\mathcal{J}\mathcal{L}_{CS}})^*] \subseteq \{1^A\}$ and any choice of $I\mathcal{J}\mathcal{L}_{CS}$, $\mathfrak{M}^{c,M}_{A,v}(I\mathcal{J}\mathcal{L}_{CS})$ is a well-defined $A$-valued algebraic Fitting model. Further:

(a) if $(F)$ is an axiom scheme of $I\mathcal{J}\mathcal{L}_{CS}$, then $\mathfrak{M}^{c,M}_{A,v}(I\mathcal{J}\mathcal{L}_{CS})$ is factive;
(b) if $(I)$ is an axiom scheme of $I\mathcal{J}\mathcal{L}_{CS}$, then $\mathfrak{M}^{c,M}_{A,v}(I\mathcal{J}\mathcal{L}_{CS})$ is introspective.

**Proof.** As $v \in \text{Ev}(A; L^*_0)$, we have items (1) - (4) from Definition 3.3. Then, as additionally $v[(Th_{I\mathcal{J}\mathcal{L}_{CS}})^*] \subseteq \{1^A\}$, we have

$$V^c(t : (\phi \rightarrow \psi)) \wedge^A V^c(s : \phi) = v((\phi \rightarrow \psi)_t) \wedge^A v(\phi_s)$$

and

$$V^c(t : \phi) \vee^A V^c(s : \phi) = v(\phi_t) \vee^A v(\phi_s)$$

regarding items (i) and (ii) of Definition 3.3. Now, regarding (a), if $(F)$ is an axiom scheme of $I\mathcal{J}\mathcal{L}_{CS}$, we naturally have

$$V^c(t : \phi) = v(\phi_t) \leq v(\phi^*) = V^c(\phi).$$

As for (b), if $(I)$ is an axiom scheme of $I\mathcal{J}\mathcal{L}_{CS}$, we have

$$V^c(t : \phi) = v(\phi_t) \leq^A v((t : \phi)_t) = V^c(t : \phi).$$

$\square$

**Theorem 4.4.** Let $I$ be an intermediate logic and let $I\mathcal{J}\mathcal{L}_0 \in \{I\mathcal{J}_0, I\mathcal{J}T_0, I\mathcal{J}_4, I\mathcal{J}T_4\}$ where $CS$ is a constant specification for $I\mathcal{J}\mathcal{L}_0$. Further, let $C \in \text{Alg}(I)$.

For any $\Gamma \cup \{\phi\} \subseteq L_J$, the following are equivalent:

1. $\Gamma \vdash_{I\mathcal{J}\mathcal{L}_{CS}} \phi$;
2. $\Gamma \vdash_{\text{CAMJL}_{CS}} \phi$;
3. $\Gamma \vdash_{\text{CAMJL}_{CS}} \phi$.

**Proof.** (1) implies (2) comes from Lemma 3.13 and (2) implies (3) is natural. For (3) implies (1), suppose $\Gamma \not\vdash_{I\mathcal{J}\mathcal{L}_{CS}} \phi$. Then, by Lemma 4.1, we have

$$\Gamma^* \cup (Th_{I\mathcal{J}\mathcal{L}_{CS}})^* \not\vdash_{I^*} \phi^*$$

which gives that there exists a $A \in C$ and $v \in \text{Ev}(A; L^*_0)$ such that

$$v[\Gamma^*] \subseteq \{1^A\}, v[(Th_{I\mathcal{J}\mathcal{L}_{CS}})^*] \subseteq \{1^A\} \text{ and } v(\phi) < 1^A$$

by assumption on $C$. By Lemma 4.3, we have that $\mathfrak{M}^{c,M}_{A,v}(I\mathcal{J}\mathcal{L}_{CS})$ is a well-defined CAMJL-model and by definition, it follows that:

$$\mathfrak{M} \models \Gamma \text{ and } \mathfrak{M} \not\models \phi.$$

Also, $\mathfrak{M}^{c,M}_{A,v}(I\mathcal{J}\mathcal{L}_{CS})$ respects $CS$. As we have $\vdash_{I\mathcal{J}\mathcal{L}_{CS}} c : \phi$ for $c : \phi \in CS$, we have $\phi_c \in (Th_{I\mathcal{J}\mathcal{L}_{CS}})^*$ and thus $v(\phi_c) = 1^A$. By definition, we have

$$V^c(c : \phi) = v(\phi_c) = 1^A.$$

Thus, we have $\Gamma \not\vdash_{\text{CAMJL}_{CS}} \phi$. $\square$
4.2. Completeness w.r.t. algebraic Fitting models.

**Definition 4.5.** Let $\mathcal{I}$ be an intermediate propositional logic and let $\mathcal{IJL}_0 \in \{\mathcal{IJ}_0, \mathcal{IJT}_0, \mathcal{IJL}_0, \mathcal{IJT}_A, \mathcal{IJF}_0\}$ where $CS$ is a constant specification for $\mathcal{IJL}_0$. Let $A$ be a complete Heyting algebra. The canonical algebraic Fitting model w.r.t. $A$ is the structure $\mathfrak{M}_A^F(\mathcal{IJL}_{CS}) = (A, W^e, W_0^e, E^e, V^e)$ defined as follows:

- $W^e := \{ v \in \text{Ev}(A; L_0^e) \mid v[(\text{Th}_{\mathcal{IJL}_{CS}})^*] \subseteq \{1^A\} \}$;
- $R^e(v, w) := \begin{cases} 1^A & \text{if } v \in J t \forall \phi \in L_J (v(\phi_t) \leq^A w(\phi^*)); \\ 0^A & \text{otherwise}; \end{cases}$
- $E^e_v(t, \phi) := v(\phi_t)$;
- $V^e(v, \phi) := v(\phi^*)$.

**Lemma 4.6.** For any complete Heyting algebra $A$ and any choice of $\mathcal{IJL}_{CS}$, $\mathfrak{M}_A^F(\mathcal{IJL}_{CS})$ is a well-defined $A$-valued algebraic Fitting model. Further:

(a) if $(F)$ is an axiom scheme of $\mathcal{IJL}_{CS}$, then $\mathfrak{M}_A^F(\mathcal{IJL}_{CS})$ is reflexive;

(b) if $(I)$ is an axiom scheme of $\mathcal{IJL}_{CS}$, then $\mathfrak{M}_A^F(\mathcal{IJL}_{CS})$ is introspective.

**Proof.** Condition (1) - (4) from Definition 3.1 follow immediately for any $v \in W^e$ as $v \in \text{Ev}(A; L_0^e)$ and by the definition of $\ast$. For item (5), we have

$$v(\phi_t) \leq^A w(\phi^*)$$

for any $w \in W^e$ with $R^e(v, w) = 1^A$. Thus, we have

$$v(\phi_t) \leq^A \bigwedge^A \{ w(\phi^*) \mid w \in W^e, R^e(v, w) = 1^A \} = \bigwedge^A \{ R^e(v, w) \rightarrow^A w(\phi^*) \mid w \in W^e \}.$$Therefore

$$E^e_v(t, \phi) \wedge^A \bigwedge^A \{ R^e(v, w) \rightarrow^A w(\phi^*) \mid w \in W^e \} = v(\phi_t).$$

For item (i), note that

$$E^e_v(t, \phi \rightarrow \psi) \wedge^A E^e_v(s, \phi) = v((\phi \rightarrow \psi)_t) \wedge^A v(\phi_s) \leq^A v(\psi_{[t, s]}) = E^e_v(t \cdot s, \psi)$$

where the inequality follows using the axiom scheme $(J)$ as $v[(\text{Th}_{\mathcal{IJL}_{CS}})^*] \subseteq \{1^A\}$ and $v \in \text{Ev}(A; L_0^e)$.

For item (ii), note that

$$E^e_v(t, \phi) = v(\phi_t) \leq^A v(\phi_{t+1}) = E^e_v(t + s, \phi)$$

and similarly for $s$ through the axiom scheme $(+)$ as again $v[(\text{Th}_{\mathcal{IJL}_{CS}})^*] \subseteq \{1^A\}$ and $v \in \text{Ev}(A; L_0^e)$. Thus, we have

$$E^e_v(t, \phi) \vee^A E^e_v(s, \phi) \leq^A E^e_v(t + s, \phi).$$

On to item (a), if $(F)$ is an axiom scheme of $\mathcal{IJL}_{CS}$, then we naturally have

$$v(\phi_t) \leq^A v(\phi^*)$$

for any $\phi \in L_J$ and any $t \in J t$. Thus, especially we have $R^e(v, v) = 1^A$ by definition and thus $R^e$ is reflexive.

For item (b), note at first that by the axiom scheme $(I)$, we have

$$E^e_v(t, \phi) = v(\phi_t) \leq^A v((t : \phi)_u) = E^e_v(\varphi, t : \phi)$$

for any $\phi \in L_J$ and any $t \in J t$ by definition of the canonical model. Further, we have that $R^e$ is transitive. For this, let $R^e(v, w) = R^e(w, u) = 1^A$. Then, we have for any $\phi \in L_J$ and any $t \in J t$:

$$v(\phi_t) \leq^A v((t : \phi)_u) \leq^A w(\phi_t) \leq^A u(\phi^*)$$

and thus $R^e(v, u) = 1^A$. For the property of monotonicity, suppose $R^e(v, w) = 1^A$. Then, we have

$$E^e_v(t, \phi) = v(\phi_t) \leq^A v((t : \phi)_u) \leq^A w(\phi_t) = E^e_w(t, \phi)$$

which is monotonicity. 

**Theorem 4.7.** Let $\mathcal{I}$ be an intermediate logic and let $\mathcal{IJL}_0 \in \{\mathcal{IJ}_0, \mathcal{IJT}_0, \mathcal{IJL}_0, \mathcal{IJT}_A, \mathcal{IJF}_0\}$ where $CS$ is a constant specification for $\mathcal{IJL}_0$. Further, let $C \in \text{Alg}_{\text{com}}(\mathcal{I})$.

For any $\Gamma \cup \{\phi\} \subseteq L_J$, the following are equivalent:

1. $\Gamma \vdash_{\mathcal{IJL}_{CS}} \phi$;
2. $\Gamma \models_{\mathcal{CAFL}_{CS}} \phi$;
3. $\Gamma \models_{\mathcal{CAFL}_{CS}} \phi$;
4. $\Gamma \models_{\mathcal{CAFL}_{CS}} \phi$. 

Thus, suppose $\Gamma \vdash_{IJL_{CS}} \phi$. Then, by Lemma 3.21 we have that
$$\Gamma^* \cup (Th_{IJL_{CS}})^* \vdash_{\phi^*} \phi^*.$$ 
By the assumption on $C$, we have that there exists a $A \in C$ and a $v \in Ev(A; L^C_0)$ such that
$$v[\Gamma^*] \subseteq \{1^A\}, v[(Th_{IJL_{CS}})^*] \subseteq \{1^A\} \text{ but } v(\phi^*) \neq 1^A.$$ 
Thus, by definition of $M_A^{c,F}(IJL_{CS})$, we have $v \in W^c$. Further, by definition we have
$$V^c(v, \gamma) = 1^A \text{ for all } \gamma \in \Gamma$$ 
but $V^c(v, \phi) \neq 1^A$. Using Lemma 4.6 $M_A^{c,F}(IJL_{CS})$ is again a well-defined CAFJL-model and as before, it respects $CS$. Thus, we have
$$\Gamma \not\vdash_{L^c,F_{CS}} 1 \text{ as } M_A^{c,F}(IJL_{CS}) \text{ is accessibility-crisp.} \tag*{□}$$

4.3. Completeness w.r.t. algebraic subset models.

**Definition 4.8.** Let $I$ be an intermediate propositional logic and let $IJL_0 \in \{IJL_0, IJL_{0}^{0}, IJL_{40}, IJL_{40}^{0}\}$ where $CS$ is a constant specification for $IJL_0$. Let $A$ be a complete Heyting algebra. The **canonical algebraic subset model** w.r.t. $A$ is the structure $M_A^{c,S}(IJL_{CS}) = \langle A, W^c, W^c_0, E^c, V^c \rangle$ defined as follows:
- $W^c := A^J$;
- $W^c_0 := \{v \in Ev(A; L^C_0) \mid v[(Th_{IJL_{CS}})^*] \subseteq \{1^A\}\};$
- $E^c_t(v, w) := \begin{cases} 1^A \text{ if } \forall \phi \in L_J \left(v(\phi) \leq^A w(\phi^*)\right); \\ 0^A \text{ otherwise}; \\ V^c(v, \phi) := v(\phi^*). \end{cases}$

**Lemma 4.9.** For any complete Heyting algebra $A$ and any choice of $IJL_{CS}$, $M_A^{c,S}(IJL_{CS})$ is well-defined $A$-valued algebraic subset model. Further:

(a) if $(F)$ is an axiom scheme of $IJL_{CS}$, then $M_A^{c,S}(IJL_{CS})$ is reflexive;
(b) if $(I)$ is an axiom scheme of $IJL_{CS}$, then $M_A^{c,S}(IJL_{CS})$ is introspective.

**Proof.** To show that $M_A^{c}(IJL_{CS})$ is well-defined, we have to verify the conditions (1) - (5) and (i), (ii) from Definition 3.20. For this, let $v \in W^c_0$. We only show (5) from the former, as (1) - (4) follows naturally from $v \in Ev(A; L^C_0)$.

For (5), we show the equality in two steps. At first, note that
$$\bigwedge^A \{E^c_t(v, w) \rightarrow^A V^c(w, \phi) \mid w \in W^c\} = \bigwedge^A \{E^c_t(v, w) \rightarrow^A w(\phi^*) \mid w \in W^c\} = \bigwedge^A \{w(\phi^*) \mid w \in W^c, E^c_t(v, w) = 1^A\}.$$ 
Now, by definition we have
$$V^c(v, t : \phi) = v(\phi_t) \leq^A w(\phi^*)$$
for any $w \in W^c$. Thus, we naturally have
$$V^c(v, t : \phi) \leq^A \bigwedge^A \{w(\phi^*) \mid w \in W^c, E^c_t(v, w) = 1^A\}.$$ 
For the other direction, consider $v_t : L_J \rightarrow A, \psi^* \mapsto \psi_t$. Then, we have that $v_t \in W^c$ and further
$$v(\psi_t) \leq^A v_t(\phi^*)$$
by definition. Thus $E^c_t(v, v_t) = 1^A$ and therefore
$$\bigwedge^A \{w(\phi^*) \mid w \in W^c, E^c_t(v, w) = 1^A\} \leq^A v_t(\phi^*) = v(\phi_t).$$ 
Let further $w \in W^c$.

(i) Suppose $E^c_t(v, w) = 1^A$. Then, we have (as $v \in Ev(A; L^C_0)$ and $v[(Th_{IJL_{CS}})^*] \subseteq \{1^A\}$)
$$v(\phi) \leq^A v(\phi_{t^+}) \leq^A w(\phi^*)$$
through axiom scheme $(+)$ for any $\phi \in L_J$ and similarly for $v(\phi)$. Thus, we have $E^c_t(v, w) = E^c_t(v, w) = 1^A.$
(ii) Suppose $\mathcal{E}^C_{t,s}(v, w) = 1^A$. We write $(\mathcal{M}^S)^i_{t,s}$ as a shorthand for $(\mathcal{M}^S_{\mathcal{A}}(\mathcal{I}, \mathcal{J}CS))_{t,s}^i$. Then to show

$$(\mathcal{M}^S)^i_{t,s}(\psi) \leq^A w(\psi^*)$$

for every $\psi \in \mathcal{L}_J$, it suffices to show (for an arbitrary $\psi \in \mathcal{L}_J$):

(1)

$$(\mathcal{M}^S)^{\varepsilon}_{t,s}(\psi) \leq^A v(\psi_{t,s}).$$

(\dagger) however follows from

$$(\mathcal{M}^S)^{\varepsilon}_{t,s}(\psi) = \bigvee^A \{\nu^c(v, t : (\psi \rightarrow \psi)) \wedge^A \nu^c(v, s : \phi) \mid \phi \in \mathcal{L}_J\}
\quad = \bigvee^A \{v((\phi \rightarrow \psi)_{t}) \wedge^A v(\phi_{s}) \mid \phi \in \mathcal{L}_J\}
\quad \leq^A v(\psi_{t,s}).$$

It remains to show items (a) and (b).

(a) Assume that $(F)$ is an axiom scheme of $\mathcal{I}, \mathcal{J}L_{CS}$. Let $v \in W_0^c$ and let $t \in Jt$. We then have naturally that

$$v(\phi^* \rightarrow \psi^*) \wedge^A v(\phi^*) \leq^A v(\psi^*)$$

for any $\phi, \psi \in \mathcal{L}_J$ as $v \in \text{Ev}(A; L_0^c)$. Now, using that $(F)$ is an axiom scheme of $\mathcal{I}, \mathcal{J}L_{CS}$, we have for any $\phi \in \mathcal{L}_J$ that

$$v(\phi_t \rightarrow \phi^*) = 1^A$$

through $v[(Th_{\mathcal{I}, \mathcal{J}L_{CS}})^*] \subseteq \{1^A\}$ and thus

$$v(\phi_t) \leq^A v(\phi^*)$$

as $v \in \text{Ev}(A; L_0^c)$ again. This gives $\mathcal{E}_t(v, v) = 1^A$.

(b) Assume that $(I)$ is an axiom scheme of $\mathcal{I}, \mathcal{J}L_{CS}$ and let again $v \in W_0^c$, $w \in W^c$ and $t \in Jt$. Assume $\mathcal{E}_t(v, w) = 1^A$ and let $\phi \in \mathcal{L}_J$ be arbitrary. We have, as

$$v(\phi_t) \leq^A v((t : \phi)_{t})$$

through $v \in W_0^c$, that

$$V(v, t : \phi) = v(\phi_t) \quad \leq^A v((t : \phi)_{t}) \quad \leq^A w(\phi_t) = V(w, t : \phi)$$

where the last inequality following follows from $\mathcal{E}_t(v, w) = 1^A$.

Theorem 4.10. Let $\mathcal{I}$ be an intermediate logic and let $\mathcal{I}, \mathcal{J}L_0 \in \{\mathcal{I}, \mathcal{J}0, \mathcal{I}, \mathcal{J}T_0, \mathcal{I}, \mathcal{J}A_0, \mathcal{I}, \mathcal{J}T_0\}$ where $CS$ is a constant specification for $\mathcal{I}, \mathcal{J}L_0$. Let further $C \in \text{Alg}_{com}(\mathcal{I})$

For any $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$, the following are equivalent:

1. $\Gamma \vdash_{\mathcal{I}, \mathcal{J}L_{CS}} \phi$;
2. $\Gamma \models_{\text{CASJL}_{CS}} \phi$;
3. $\Gamma \vdash_{\text{CASJL}_{CS}} \phi$;
4. $\Gamma \vdash_{\text{CASJL}_{CS}} \phi$.

Proof. (1) implies (2) comes from Lemma 3.26. (2) implies (3) and (3) implies (4) is obvious. So, assume that $\Gamma \vdash_{\mathcal{I}, \mathcal{J}L_{CS}} \phi$. Through Lemma 3.11 we first have

$$\Gamma^* \cup (Th_{\mathcal{I}, \mathcal{J}L_{CS}})^* \nvdash_{\mathcal{I}} \phi^*$$

By the choice of C, there is a complete Heyting algebra $A \in C$ with an evaluation $v \in \text{Ev}(A; L_0^c)$ such that

(i)

$$v[\Gamma^*] \cup v[(Th_{\mathcal{I}, \mathcal{J}L_{CS}})^*] \subseteq \{1^A\} \text{ but } v(\phi^*) \neq 1^A.$$

By Lemma 4.9 we have that $\mathcal{M}^S_{A}(\mathcal{I}, \mathcal{J}L_{CS}) \in \text{CASJL}$ through (i). Also $\mathcal{M}^S_{A}(\mathcal{I}, \mathcal{J}L_{CS})$ naturally respects $CS$ and is accessibility-crisp. Further, through (i) and the definition of $\mathcal{M}^S_{A}(\mathcal{I}, \mathcal{J}L_{CS})$, we have

$$\nu^c(v, \gamma) = 1^A \text{ for all } \gamma \in \Gamma \text{ but } \nu^c(v, \phi) \neq 1^A$$

and thus again per definition $\Gamma \nvdash_{\text{CASJL}_{CS}} \phi$. 

$\square$
5. Frame semantics for intermediate justification logics

As a second semantic approach, we extend not Heyting algebras but intuitionistic Kripke frames for intermediate logics with the semantic machinery of the models of Mkrtychev, Fitting or of Lehmann and Studer.

This extends the work on intuitionistic Mkrtychev and Fitting models (under different terminology) from Marti and Studer in [27] to wider classes of logics. The intuitionistic subset models based on Kripke frames introduced later are completely new in the literature.

5.1. Kripke frames and propositional semantics. We review some concepts from Kripke frames for propositional intermediate logics (see e.g. [15, 31]). For this, we need some terminology from the context of the theory of partial orders first.

**Definition 5.1.** We call a partial order, that is a structure \( \langle F, \leq \rangle \) such that \( \leq \) is a binary relation on the non-empty set \( F \) which satisfies the conditions

1. \( x \leq x \) (reflexivity),
2. \( x \leq y \) and \( y \leq x \) implies \( x = y \) (antisymmetry),
3. \( x \leq y \) and \( y \leq z \) implies \( x \leq z \) (transitivity),

for all \( x, y, z \in F \), a Kripke frame. A set \( X \subseteq F \) is called a cone (or upset), if

\[
\forall x \in X \forall y \in F \ (x \leq y \Rightarrow y \in X).
\]

We denote the smallest cone containing a set \( X \) of a partial order \( \langle F, \leq \rangle \) by \( \uparrow X \). A cone \( X \) is called principal if \( X = \uparrow \{x\} \) for some element \( x \). It is straightforward that

\[
\uparrow X = \bigcup_{x \in X} \uparrow \{x\}.
\]

A Kripke frame \( \mathfrak{G} = \langle G, \leq' \rangle \) is an (induced) subframe of a Kripke frame \( \mathfrak{F} = \langle F, \leq \rangle \), if \( G \subseteq F \) and \( \leq' = \leq \cap (G \times G) \). In this case, we also write \( \mathfrak{G} = \mathfrak{F} | G \). A Kripke frame is called principal if its domain is principal.

**Definition 5.2.** Let \( \mathfrak{F} = \langle F, \leq \rangle \) be a Kripke frame. A \( (\mathcal{L}_0(X),-) \)Kripke model based on \( \mathfrak{F} \) is a structure \( \mathfrak{M} = \langle \mathfrak{F}, \models' \rangle \) with \( \models' \subseteq F \times X \) which satisfies

\[
x \leq y \text{ and } x \models' p \Rightarrow y \models' p
\]

for all \( p \in X \).

A Kripke model \( \mathfrak{M} = \langle \mathfrak{F}, \models' \rangle \) is called an (induced) submodel of a Kripke model \( \mathfrak{M} = \langle \mathfrak{F}, \models \rangle \) if \( \mathfrak{G} \) is an induced subframe of \( \mathfrak{F} \) and for all \( p \in X \):

\[
\{x \in G \mid x \models' p\} = \{x \in F \mid x \models p\} \cap G.
\]

We write \( \mathfrak{M} = \mathfrak{M} | G \) in this case.

Given a Kripke model \( \mathfrak{M} = \langle \mathfrak{F}, \models \rangle \), we introduce the satisfaction relation \( \models \) for formulae from \( \mathcal{L}_0(X) \) as follows. Given a \( x \in F \), we define recursively:

- \( (\mathfrak{M}, x) \not\models \bot \);
- \( (\mathfrak{M}, x) \models p \) if \( x \models p \);
- \( (\mathfrak{M}, x) \models \phi \land \psi \) if \( (\mathfrak{M}, x) \models \phi \) and \( (\mathfrak{M}, x) \models \psi \);
- \( (\mathfrak{M}, x) \models \phi \lor \psi \) if \( (\mathfrak{M}, x) \models \phi \) or \( (\mathfrak{M}, x) \models \psi \);
- \( (\mathfrak{M}, x) \models \phi \rightarrow \psi \) if \( \forall y \in F \ (x \leq y \Rightarrow (\mathfrak{M}, y) \not\models \phi \) or \( (\mathfrak{M}, y) \models \psi \).

We write \( \mathfrak{M} \models \phi \) if \( (\mathfrak{M}, x) \models \phi \) for any \( x \in F \), \( (\mathfrak{M}, x) \models \Gamma \) if \( (\mathfrak{M}, x) \models \gamma \) for all \( \gamma \in \Gamma \) and \( \mathfrak{M} \models \Gamma \) if \( (\mathfrak{M}, x) \models \Gamma \) for all \( x \in F \).

A fundamental property of Kripke models is that the monotonicity of propositional variables extends to all formulae. More precisely, we have the following:

**Lemma 5.3.** Let \( \mathfrak{M} = \langle \mathfrak{F}, \models \rangle \) be a \( \mathcal{L}_0(X) \) Kripke model. Then, for all \( \phi \in \mathcal{L}_0(X) \) and all \( x, y \in F \):

\[
x \leq y \text{ and } (\mathfrak{M}, x) \models \phi \Rightarrow (\mathfrak{M}, y) \models \phi.
\]

The proof is an easy induction on the structure of \( \mathcal{L}_0(X) \). Given a class of Kripke frames \( C \), we write \( \mathcal{M}_0(\mathcal{C}; \mathcal{L}_0(X)) \) for the class of all Kripke models over \( \mathcal{L}_0(X) \) with underlying Kripke frames from \( \mathcal{C} \). Given a single frame \( \mathfrak{F} \), we also write \( \mathcal{M}_0(\mathfrak{F}; \mathcal{L}_0(X)) \) for \( \mathcal{M}_0(\mathfrak{F}; \mathcal{L}_0(X)) \).

Using these definitions, there are now two definitions of consequence to consider.
Lemma 5.6. Let $\mathcal{C} \subseteq \mathcal{L}_0(\mathcal{X})$ and $\mathcal{C}$ be a class of Kripke models. Then, we write:

1. $\Gamma \vdash \phi$ if $\forall \mathcal{M} \in \mathcal{C} \cap \mathcal{D}(\mathcal{M}) \left( (\mathcal{M}, x) \models \Gamma \Rightarrow (\mathcal{M}, x) \models \phi \right)$;
2. $\Gamma \vdash \phi$ if $\forall \mathcal{M} \in \mathcal{C} \left( (\mathcal{M}, x) \models \Gamma \Rightarrow \mathcal{M} \models \phi \right)$.

Further, if $\mathcal{C}$ is now a class of Kripke frames, we write:

3. $\Gamma \models \phi$ if $\Gamma \models \mathcal{M}_0((\mathcal{C} \cap \mathcal{L}_0(\mathcal{X}))) \phi$;
4. $\Gamma \vdash \phi$ if $\Gamma \vdash \mathcal{M}_0((\mathcal{C} \cap \mathcal{L}_0(\mathcal{X}))) \phi$.

Definition 5.5. Let $\mathcal{I}$ be an intermediate logic, $X$ a countably infinite set of variables and $\mathcal{C}$ be a class of Kripke frames.

1. We say that $\mathcal{I}(X)$ is strongly complete w.r.t. $\mathcal{C}$ if $\Gamma \models \mathcal{I}(X) \phi$ iff $\Gamma \models \mathcal{C} \phi$.
2. We say that $\mathcal{I}(X)$ is strongly globally complete w.r.t. $\mathcal{C}$ if $\Gamma \models \mathcal{I}(X) \phi$ iff $\Gamma \models \mathcal{C} \phi$.

Given a class of Kripke frames $\mathcal{C}$, we write $\mathcal{C} \in \mathcal{KFr}(\mathcal{I})$ or $\mathcal{C} \in \mathcal{KFr}^\partial(\mathcal{I})$ if $\mathcal{I}$ is strongly (locally) complete or strongly globally complete w.r.t. $\mathcal{C}$, respectively. We also write $\mathcal{C} \in \mathcal{KFr}(\mathcal{I}) \cap \mathcal{KFr}^\partial(\mathcal{I})$ for $\mathcal{C} \in \mathcal{KFr}(\mathcal{I})$ and $\mathcal{C} \in \mathcal{KFr}^\partial(\mathcal{I})$.

The global version will later prove to be important in the completeness considerations. Two things shall be noted in this context. First, it is well known that there are Kripke incomplete intermediate logics, that is intermediate logics where there is no class of Kripke frames for which the logic is (even weakly) complete. This is well-known to be connected with the corresponding problem of Kripke incomplete modal logics and the first such logic was constructed in [37]. All following considerations involving propositional completeness w.r.t. classes of Kripke frames thus implicitly assume that such a class exists.

Further, if an intermediate logic is characterized by a class of Kripke frames locally, there is a simple extended class of frames which characterizes the logic globally. More precisely, we have the following:

Lemma 5.6. Let $\mathcal{C}$ be a class of Kripke frames and let $\overline{\mathcal{C}}$ be the closure of $\mathcal{C}$ under principal subframes. Let $\Gamma \cup \{ \phi \} \subseteq \mathcal{L}_0(\mathcal{X})$. Then, we have:

1. $\Gamma \vdash \phi$ if $\forall \mathcal{M} \in \mathcal{C} \cap \mathcal{D}(\mathcal{M}) \left( (\mathcal{M}, x) \models \Gamma \Rightarrow (\mathcal{M}, x) \models \phi \right)$;
2. $\Gamma \vdash \phi$ if $\forall \mathcal{M} \in \mathcal{C} \left( (\mathcal{M}, x) \models \Gamma \Rightarrow \mathcal{M} \models \phi \right)$.

Proof. For (1), we have naturally the direction from right to left. For the converse, note that for all $\phi \in \mathcal{L}_0$, all $\mathcal{M}$ over frames from $\mathcal{C}$ and all $x \in \mathcal{D}(\mathcal{M})$, we have

$$(\mathcal{M}, x) \models \phi \text{ if } (\mathcal{M} \upharpoonright \{ x \}, x) \models \phi.$$ 

Thus, the claim follows from the fact that for every $\mathcal{G} \in \overline{\mathcal{C}}$, we have $\mathcal{G} \in \mathcal{C}$ or $\mathcal{G} = \mathcal{G} \upharpoonright \{ x \}$ for some $\mathcal{G} \in \mathcal{C}$ and some $x \in \mathcal{D}(\mathcal{G})$.

For (2), we naturally have the direction from left to right. For the converse, consider $\Gamma \models \mathcal{G} \phi$, that is

$$\forall \mathcal{M} \in \mathcal{M}_0((\mathcal{G}; \mathcal{L}_0)) \left( (\mathcal{M}, x) \models \Gamma \Rightarrow \mathcal{M} \models \phi \right).$$

Let $\mathcal{G} \in \overline{\mathcal{C}}$ and $\mathcal{M} \in \mathcal{M}_0((\mathcal{G}; \mathcal{L}_0))$ as well as $x \in F$ and suppose $(\mathcal{M}, x) \models \Gamma$. Consider

$$\mathcal{M}' := \mathcal{M} \upharpoonright \{ x \}.$$ 

Then, we have $\mathcal{M}' \models \Gamma$ by Lemma 5.3 and as $\mathcal{M}' \in \mathcal{M}_0((\mathcal{G}; \mathcal{L}_0))$ as $\overline{\mathcal{C}}$ is closed under principal subframes. We thus have $\mathcal{M}' \models \phi$ by $\Gamma \models \mathcal{G} \phi$, i.e. especially $(\mathcal{M}, x) \models \phi$. Thus, we have $\Gamma \models \mathcal{G} \phi$. \qed

5.2. Intuitionistic Mkrtychev models. We continue our semantical investigations into intermediate justification logics by extending the approach of Mkrtchyevs syntactic models by intuitionistic Kripke frames. These intuitionistic Mkrtychev models are akin to the previously considered models from [27] for $\mathcal{IPCJ} T \mathcal{4}$ (under the name of intuitionistic basic models).

Definition 5.7. Let $\mathcal{G} = (F, \leq)$ be a Kripke frame. An intuitionistic Mkrtychev model based on $\mathcal{G}$ is a structure $\mathcal{M} = (\mathcal{G}, \mathcal{E}, \models)$ such that $\models \subseteq F \times Var$ and $\mathcal{E} : JT \times F \to 2^{\mathcal{G} \times 1}$ satisfy

1. $x \leq y$ and $x \vdash p$ implies $y \vdash p$ for all $p \in Var$,
2. $x \leq y$ and $\phi \in \mathcal{E}(x)$ implies $\phi \in \mathcal{E}(y)$ for all $\phi \in \mathcal{L}_J$ and all $t \in JT$;

for all $x, y \in F$ as well as

(i) $\mathcal{E}(x) \supseteq \mathcal{E}(y)$ for all $x \in F$ and all $t, s \in JT$;

(ii) $\mathcal{E}(x) \supseteq \mathcal{E}(y)$ for all $x \in F$ and all $t, s \in JT$, where

$$\Gamma \vdash \Delta := \{ \phi \in \mathcal{L}_J \mid \psi \vdash \phi \in \Gamma, \psi \in \Delta \text{ for some } \psi \in \mathcal{L}_J \}$$

for $\Gamma, \Delta \subseteq \mathcal{L}_J$. 
Given an intuitionistic Mkrtychev model $\mathcal{M}$ over a Kripke frame $\mathfrak{g} = (F, \leq)$, we also write $D(\mathcal{M}) := F$ and call $F$ the domain of $\mathcal{M}$. Note that we use $F$ to denote the domain of a model but also $(F)$ to denote the axiom scheme of factivity for the intermediate justification logics.

Over an intuitionistic Mkrtychev model $\mathcal{M} = \langle \mathfrak{g}, \mathcal{E}, \vdash \rangle$, we introduce the following local satisfaction relation by recursion:

- $(\mathfrak{g}, x) \not\models \bot$;
- $(\mathfrak{g}, x) \models p$ if $x \models p$;
- $(\mathfrak{g}, x) \models \phi \land \psi$ if $(\mathfrak{g}, x) \models \phi$ and $(\mathfrak{g}, x) \models \psi$;
- $(\mathfrak{g}, x) \models \phi \lor \psi$ if $\forall y \in F (x \leq y \Rightarrow (\mathfrak{g}, x) \not\models \phi$ or $(\mathfrak{g}, x) \models \psi)$;
- $(\mathfrak{g}, x) \models t : \phi$ if $\phi \in \mathcal{E}(x)$.

We write $(\mathfrak{g}, x) \models \phi$ if $(\mathfrak{g}, x) \models \phi$ for all $\gamma \in \Gamma$. Further, we have the following immediate lemma.

**Lemma 5.8.** Let $\mathfrak{g} = (F, \leq)$ be a Kripke frame and let $\mathcal{M}$ be a intuitionistic Mkrtychev model over $\mathfrak{g}$. For any $\phi \in \mathcal{L}_J$ and all $x, y \in F$:

$x \leq y$ and $(\mathfrak{g}, x) \models \phi$ implies $(\mathfrak{g}, y) \models \phi$.

**Definition 5.9.** Let $\mathfrak{g} = (F, \leq)$ be a Kripke frame and $\mathcal{M} = \langle \mathfrak{g}, \mathcal{E}, \vdash \rangle$ be an intuitionistic Mkrtychev model. We call $\mathcal{M}$

1. *factive* if $\phi \in \mathcal{E}(x)$ implies $(\mathfrak{g}, x) \models \phi$, and
2. *introspective* if $t : \mathcal{E}(x) \subseteq \mathcal{E}(x)$ where $t : \Gamma = \{ t : \gamma \mid \gamma \in \Gamma \}$.

**Definition 5.10.** Let $C$ be a class of Kripke frames. Then, we write:

1. $\text{CKMJ}$ for the class of all intuitionistic Mkrtychev models over frames from $C$;
2. $\text{CKMJT}$ for the class of all factive intuitionistic Mkrtychev models over frames from $C$;
3. $\text{CKM4}$ for the class of all introspective intuitionistic Mkrtychev models over frames from $C$;
4. $\text{CKMJT4}$ for the class of all factive and introspective intuitionistic Mkrtychev models over $C$.

**Definition 5.11.** Let $\mathcal{M} = \langle \mathfrak{g}, \mathcal{E}, \vdash \rangle$ be an intuitionistic Mkrtychev model and let $CS$ be a constant specification (for some proof calculus). We say that $\mathcal{M}$ respects $CS$ if for all $x \in F$ and all $c : \phi \in CS : \phi \in \mathcal{E}(x)$.

Given a class $C$ of intuitionistic Mkrtychev models, we denote the class of all intuitionistic Mkrtychev models respecting a constant specification $CS$ by $C_{CS}$.

**Definition 5.12.** Let $C$ be a class of intuitionistic Mkrtychev models. We write $\Gamma \models_C \phi$ if for all $\mathcal{M} \in C$ and all $x \in D(\mathcal{M})$:

$(\mathfrak{g}, x) \models \phi$ implies $(\mathfrak{g}, x) \models \phi$.

**Lemma 5.13.** Let $I$ be an intermediate logic and $IJL_0 \in \{ IJ_0, IJT_0, IJT_4, IJT_4_0 \}$. Let $CS$ be a constant specification for $IJL_0$ and let $C$ be a finite logic for $I$. For any $\Gamma \cup \{ \phi \} \subseteq \mathcal{L}_J$:

$\Gamma \vdash_{IJL_{CS}} \phi$ implies $\Gamma \models_{CKMJ_{CS}} \phi$.

**Proof.** By an argument similar to the one of Lemma 5.13 we may reduce strong to weak soundness:

$\Gamma \vdash_{IJL_0} \phi$ implies $\exists \mathfrak{g} \leq \Gamma \cup CS$ finite $\left( \Gamma \vdash_{IJL_0} \bigwedge \Gamma_0 \rightarrow \phi \right)$

implies $\exists \mathfrak{g} \leq \Gamma \cup CS$ finite $\left( \models_{CKMJ} \bigwedge \Gamma_0 \rightarrow \phi \right)$

implies $\exists \mathfrak{g} \leq \Gamma \cup CS$ finite $\forall \mathcal{M} \in CKMJ \forall x \in D(\mathcal{M}) \forall y \geq x ((\mathfrak{g}, y) \models \Gamma_0 \Rightarrow (\mathfrak{g}, y) \models \phi)$

implies $\exists \mathfrak{g} \leq \Gamma \cup CS$ finite $\forall \mathcal{M} \in CKMJ \forall x \in D(\mathcal{M}) ((\mathfrak{g}, x) \models \Gamma_0 \Rightarrow (\mathfrak{g}, x) \models \phi)$

implies $\forall \mathcal{M} \in CKMJ \forall x \in D(\mathcal{M}) ((\mathfrak{g}, x) \models \Gamma \cup CS \Rightarrow (\mathfrak{g}, x) \models \phi)$

implies $\forall \mathcal{M} \in CKMJ_{CS} \forall x \in D(\mathcal{M}) ((\mathfrak{g}, x) \models \Gamma \Rightarrow (\mathfrak{g}, x) \models \phi)$.

Thus, we only show that $\Gamma \vdash_{IJL_0} \phi$ implies $\models_{CKMJ} \phi$. As before, by definition of $IJL_0$, it suffices to show $\models_{CKMJ} \phi$ for $\phi \in \mathcal{E}$ or $\phi$ being an instance of the justification axioms (depending on $IJL_0$). For both, let $\mathcal{M} = \langle \mathfrak{g}, \mathcal{E}, \vdash \rangle \in CKMJ$ as well as $x \in D(\mathcal{M})$.

If $\phi \in \mathcal{E}$, then there is a substitution $\sigma : Var \rightarrow \mathcal{L}_J$ such that $\phi = \sigma(\psi)$ for some $\psi \in \mathcal{E}$. By the choice of $C$, we have that $(\mathfrak{g}, y) \models \psi$ for any $\mathcal{M} = \langle \mathfrak{g}, \vdash \rangle$ and any $y \in D(\mathfrak{g})$. Define a particular $\vdash'$ by

$y \vdash' p$ if $(\mathfrak{g}, y) \models \sigma(p)$

for any $p \in Var$ and any $y \in F$ and define $\mathfrak{M} = \langle \mathfrak{g}, \vdash' \rangle$. Then, it is straightforward to see that $(\mathfrak{M}, y) \models \sigma(\chi)$ iff $(\mathfrak{g}, y) \models \chi$ for any $\chi \in \mathcal{L}_0$ and thus especially, we have $(\mathfrak{M}, x) \models \phi$. This gives $\models_{CKMJ} \phi$. 


If \( \phi \) is an instance of \((J)\) or \((+)\), then the conditions (i) and (ii) of Definition 5.17 respectively, give the validity of \( \phi \) immediately.

Similarly if \( \phi \) is an instance of \((F)\) and \( \mathfrak{M} \) is factive or \( \phi \) is an instance of \((I)\) and \( \mathfrak{M} \) is introspective, the respective validity of \( \phi \) follows immediately by the definition of factive or introspective intuitionistic Mkrtychev models, that is (1) or (2) of Definition 5.19.

\[ \square \]

### 5.3. Intuitionistic Fitting models

We continue with intuitionistic Fitting models, combining various streams of semantics in non-classical modal logics by extending the approach using intuitionistic modal Kripke models of [22] for intuitionistic modal logics by the machinery of evidence functions for explicit modalities in the sense of semantics in non-classical modal logics by extending the approach using intuitionistic modal Kripke models. In any way, the models which we introduce are akin to a model class from [27] for \( \text{IPCJT}4 \) (which are called intuitionistic modal models there).

**Definition 5.14.** Let \( \mathfrak{F} = (F, \leq) \) be a Kripke frame. An **intuitionistic Fitting model based on** \( \mathfrak{F} \) is a structure \( \mathfrak{M} = \langle \mathfrak{F}, \mathcal{R}, \mathcal{E}, \models \rangle \) such that \( \models \subseteq F \times Var \), \( \mathcal{R} \subseteq F \times F \) and \( \mathcal{E} : \text{Jt} \times F \rightarrow 2^{\mathcal{E}_{t}} \) satisfy

1. \( x \leq y \text{ and } x \models y \models p \text{ for all } p \in \text{Var}, \)
2. \( x \leq y \text{ and } \phi \in \mathcal{E}_{t}(x) \text{ imply } \phi \in \mathcal{E}_{t}(y) \text{ for all } \phi \in \mathcal{L}_{t} \text{ and all } t \in \text{Jt}, \)
3. \( x \leq y \text{ implies } \mathcal{R}[y] \subseteq \mathcal{R}[x]. \)

for all \( x, y \in F \) as well as

1. \( \mathcal{E}_{t}(x) \sqcap \mathcal{E}_{s}(x) \subseteq \mathcal{E}_{[t,s]}(x) \text{ for all } x \in F \text{ and all } t, s \in \text{Jt}, \)
2. \( \mathcal{E}_{t}(x) \cup \mathcal{E}_{s}(x) \subseteq \mathcal{E}_{[t,s]}(x) \text{ for all } x \in F \text{ and all } t, s \in \text{Jt}. \)

Over an intuitionistic Fitting model \( \mathfrak{M} = \langle \mathfrak{F}, \mathcal{R}, \mathcal{E}, \models \rangle \), we introduce the following local satisfaction relation by recursion:

- \((\mathfrak{M}, x) \not\models \bot;\)
- \((\mathfrak{M}, x) \models p \text{ if } x \models p;\)
- \((\mathfrak{M}, x) \models \phi \land \psi \text{ if } (\mathfrak{M}, x) \models \phi \text{ and } (\mathfrak{M}, x) \models \psi;\)
- \((\mathfrak{M}, x) \models \phi \lor \psi \text{ if } (\mathfrak{M}, x) \not\models \phi \text{ or } (\mathfrak{M}, x) \models \psi;\)
- \((\mathfrak{M}, x) \models t : \phi \text{ if } \phi \in \mathcal{E}_{t}(x) \text{ and } \forall y \in \mathcal{R}[x] (\mathfrak{M}, y) \models \phi.\)

We write \((\mathfrak{M}, x) \models \Gamma \text{ if } (\mathfrak{M}, x) \models \gamma \text{ for all } \gamma \in \Gamma. \) Also, given an intuitionistic Fitting model \( \mathfrak{M} \) over a Kripke frame \( \mathfrak{F} = (F, \leq, V) \), we write again \( D(\mathfrak{M}) = F. \)

**Lemma 5.15.** Let \( \mathfrak{M} \) be an intuitionistic Fitting model over a Kripke frame \( \mathfrak{F} = (F, \leq) \). For any \( \phi \in \mathcal{L}_{t} \) and all \( x, y \in F: \)

\[
\begin{align*}
\text{if } x \leq y \text{ and } (\mathfrak{M}, x) \models \phi \text{ imply } (\mathfrak{M}, y) \models \phi.
\end{align*}
\]

**Definition 5.16.** Let \( \mathfrak{F} = (F, \leq) \) be a Kripke frame and \( \mathfrak{M} = \langle \mathfrak{F}, \mathcal{R}, \mathcal{E}, \models \rangle \) be an intuitionistic Fitting model. We call \( \mathfrak{M} \)

1. **reflexive** if \( \mathcal{R} \) is reflexive,
2. **transitive** if \( \mathcal{R} \) is transitive,
3. **monotone** if \( \mathcal{E}_{t}(x) \subseteq \mathcal{E}_{t}(y) \) for \( y \in \mathcal{R}[x] ,\)
4. **introspective** if it is transitive, monotone and \( t : \phi \in \mathcal{E}_{t}(x) \) for all \( t \in \text{Jt} \) and \( x \in F.\)

**Definition 5.17.** Let \( \mathcal{C} \) be a class of intuitionistic Fitting models. We write \( \Gamma \models \mathcal{C} \phi \) if for all \( \mathfrak{M} \in \mathcal{C} \) and all \( x \in D(\mathfrak{M}) ; (\mathfrak{M}, x) \models \Gamma \) implies \((\mathfrak{M}, x) \models \phi.\)

**Definition 5.18.** Let \( \mathcal{C} \) be a class of Kripke frames. Then, we write:

1. \( \text{CKFJ} \) for the class of all intuitionistic Fitting models over frames from \( \mathcal{C};\)
2. \( \text{CKFJT} \) for the class of all reflexive intuitionistic Fitting models over frames from \( \mathcal{C};\)
3. \( \text{CKFJ4} \) for the class of all introspective intuitionistic Fitting models over frames from \( \mathcal{C};\)
4. \( \text{CKFJT4} \) for the class of all reflexive and introspective intuitionistic Fitting models over frames from \( \mathcal{C}.\)

**Lemma 5.19.** Let \( \mathcal{I} \) be an intermediate logic and \( \mathcal{I} \mathcal{J} \mathcal{L}_{0} \in \{\mathcal{I} \mathcal{J} \mathcal{L}_{0}, \mathcal{I} \mathcal{J} \mathcal{L}_{0}, \mathcal{I} \mathcal{J} \mathcal{L}_{0}, \mathcal{I} \mathcal{J} \mathcal{L}_{0}\}. \) Let \( \mathcal{C} \mathcal{S} \) be a constant specification for \( \mathcal{I} \mathcal{J} \mathcal{L}_{0} \) and let \( \mathcal{C} \in \mathcal{KFr}(\mathcal{I}). \) Then, for any \( \Gamma \cup \{\phi\} \subseteq \mathcal{L}_{1} : \)

\[
\Gamma \models_{\mathcal{I} \mathcal{J} \mathcal{L}_{0}} \phi \text{ implies } \Gamma \models_{\text{CKFJ} \mathcal{C} \mathcal{S}} \phi.
\]

**Proof.** As in Lemma 5.13 we may restrict ourselves to weak soundness only. Here, it again suffices to only verify \((\mathfrak{M}, x) \models \phi \) for any \( \phi \in \mathcal{I} \) or \( \phi \) being an instance of a justification axiom (depending on \( \mathcal{I} \mathcal{J} \mathcal{L}_{0} \)) as well as any \( \mathfrak{M} \in \text{CKFJ} \mathcal{C} \mathcal{S} \) and any \( x \in D(\mathfrak{M}). \)
The case for \( \phi \in \mathcal{F} \) can be handled similarly as in Lemma 5.13. We thus only show the validity of (1) \((J)\), (2) \((+)\) as well as (3) \((F)\) and (4) \((I)\) in their respective model classes. For this, let \( \mathfrak{M} = (\mathfrak{F}, \mathcal{R}, \mathcal{E}, \models) \) be an intuitionistic Fitting model with \( \mathfrak{F} \in \mathcal{C} \).

1. We show

\[
(\mathfrak{M}, x) \models t : (\phi \rightarrow \psi) \text{ impl. } (\mathfrak{M}, x) \models (s : \phi \rightarrow [t : s] : \psi)
\]

for any \( \phi, \psi \in \mathcal{E}_t \), any \( t, s \in Jt \) and any \( x \in F \). For this, suppose \( (\mathfrak{M}, x) \models t : (\phi \rightarrow \psi) \), that is by definition \( \phi \rightarrow \psi \in \mathcal{E}_t(x) \) as well as

\[
\forall y \in \mathcal{R}[x] \, (\mathfrak{M}, y) \models \phi \rightarrow \psi.
\]

Let \( y \geq x \) and suppose \( (\mathfrak{M}, y) \models s : \phi \), that is \( \phi \in \mathcal{E}_s(y) \) and

\[
\forall z \in \mathcal{R}[y] \, (\mathfrak{M}, z) \models \phi.
\]

By condition (2) of Definition 5.14, we have that \( \phi \rightarrow \psi \in \mathcal{E}_t(y) \). By condition (i), we have thus that \( \psi \in \mathcal{E}_{[t : s]}(y) \). Now, let \( z \in \mathcal{R}[y] \). As above, we have \( (\mathfrak{M}, z) \models \phi \) and by condition (3) of Definition 5.14 we have that \( z \in \mathcal{R}[x] \) and thus \( (\mathfrak{M}, z) \models \phi \rightarrow \psi \). Thus, we have especially \( (\mathfrak{M}, z) \models \psi \).

Therefore, we have \( \forall z \in \mathcal{R}[y] \, (\mathfrak{M}, z) \models \psi \) and in combination with \( \psi \in \mathcal{E}_{[t, s]}(y) \), we have \( (\mathfrak{M}, y) \models \psi \) as \( y \) was arbitrary, we have \( (\mathfrak{M}, x) \models s : \phi \rightarrow [t : s] : \psi \).

As \( x \in F \) was arbitrary, we have \( (\mathfrak{M}, x) \models t : (\phi \rightarrow \psi) \rightarrow (s : \phi \rightarrow [t : s] : \psi) \).

2. Let \( x \in F \) be arbitrary. Suppose \( (\mathfrak{M}, x) \models t : \phi \), that is \( \phi \in \mathcal{E}_t(x) \) and \( \forall y \in \mathcal{R}[x] \, (\mathfrak{M}, y) \models \phi \).

The former gives \( \phi \in \mathcal{E}_{[t : s]}(x) \) by condition (ii) of Definition 5.14 and this combined with the latter gives

\[
(\mathfrak{M}, x) \models [t : s] : \phi.
\]

As \( x \in F \) was arbitrary, we have \( (\mathfrak{M}, x) \models t : \phi \rightarrow [t : s] : \phi \). One shows \( (\mathfrak{M}, x) \models s : \phi \rightarrow [t + s] : \phi \) in a similar way.

3. Suppose \( \mathfrak{M} \) is reflexive and let \( x \in F \). Suppose

\[
(\mathfrak{M}, x) \models t : \phi
\]

that is especially we have \( \forall y \in \mathcal{R}[x] \, (\mathfrak{M}, y) \models \phi \). As \( \mathcal{R} \) is reflexive, we have \( x \in \mathcal{R}[x] \) and thus \( (\mathfrak{M}, x) \models \phi \). As \( x \) was arbitrary, we have

\[
(\mathfrak{M}, x) \models t : \phi \rightarrow \phi
\]

4. Let \( \mathfrak{M} \) be introspective and let \( x \in F \). Suppose that \( (\mathfrak{M}, x) \models t : \phi \), that is \( \phi \in \mathcal{E}_t(x) \) and \( \forall y \in \mathcal{R}[x] \, (\mathfrak{M}, y) \models \phi \).

The former gives at first \( s : \phi \in \mathcal{E}_t(x) \) by introspection. Now, let \( y \in \mathcal{R}[x] \) be arbitrary. By the monotonicity aspect of introspection, we have \( \phi \in \mathcal{E}_t(y) \) as \( \phi \in \mathcal{E}_t(x) \). Now, let \( z \in \mathcal{R}[y] \). By transitivity of \( \mathcal{R} \), we have \( z \in \mathcal{R}[x] \) and thus \( (\mathfrak{M}, z) \models \phi \) by assumption. Summarized, we have

\[
(\mathfrak{M}, y) \models t : \phi \text{ for all } y \in \mathcal{R}[x] \text{ and this combined with } t : \phi \in \mathcal{E}_t(x) \text{ gives } (\mathfrak{M}, x) \models t : \phi.
\]

As \( x \) was arbitrary, we have \( (\mathfrak{M}, x) \models t : \phi \rightarrow \phi = t : \phi \).

\[ \square \]

5.4. Intuitionistic subset models. The last semantics which we introduce, based on intuitionistic Kripke frames, extends the considerations of Lehmann and Studer from [26] about their subset models to these intermediate cases. This semantics seems to have not appeared in the literature before.

Definition 5.20. Let \( \mathfrak{F} = (\mathcal{F}_0, \leq) \) be a Kripke frame. An intuitionistic subset model over \( \mathfrak{F} \) is a structure \( \mathfrak{M} = (\mathfrak{F}, \mathcal{E}, \mathcal{L}, \models) \) with \( \mathcal{F} \supseteq \mathcal{F}_0 \) and \( \mathcal{E} : Jt \rightarrow 2^{\mathcal{F} \times \mathcal{F}} \) and \( \models \subseteq \mathcal{F} \times \mathcal{L} \) and which satisfies

1. \( x \leq y \) and \( x \models p \) imply \( y \models p \) for all \( p \in \text{Var} \),

2. \( x \leq y \) implies \( \mathcal{E}_t[y] \subseteq \mathcal{E}_t[x] \) for all \( t \in Jt \),

for all \( x, y \in \mathcal{F}_0 \) as well as

(i) \( x \not\models \bot \),

(ii) \( x \models \phi \land \psi \text{ iff } x \models \phi \text{ and } x \models \psi \),

(iii) \( x \models \phi \lor \psi \text{ iff } x \models \phi \text{ or } x \models \psi \),

(iv) \( x \models \phi \rightarrow \psi \text{ iff } \forall y \geq x : y \not\models \phi \text{ or } y \models \psi \),

(v) \( x \models t : \phi \text{ iff } \forall y \in \mathcal{E}_t[x] : y \not\models \phi \),

for any \( x \in \mathcal{F}_0 \) and such that it satisfies:
Let $\mathfrak{M}$ be a class of intuitionistic subset models over frames from $\mathcal{F}$.

We write
\[ D \mid \mathfrak{M} \mid \mathfrak{N} \] for the subclass of all reflexive intuitionistic subset models over frames from $\mathcal{F}$.

Further, we write $\mathfrak{M} \mid \mathfrak{M} \mid \mathfrak{N}$ for the class of all reflexive intuitionistic subset models over frames from $\mathcal{F}$.

Definition 5.24. Let $\mathfrak{C}$ be a class of Kripke frames.

Then, we write:

1. $\mathfrak{C}$ for the class of all intuitionistic subset models over frames from $\mathcal{C}$;
2. $\mathfrak{C} \mathfrak{J}$ for the class of all reflexive intuitionistic subset models over frames from $\mathcal{C}$;
3. $\mathfrak{C} \mathfrak{S}$ for the class of all introspective intuitionistic subset models over frames from $\mathcal{C}$;
4. $\mathfrak{C} \mathfrak{S} \mathfrak{J}$ for the class of all reflexive and introspective intuitionistic subset models over frames from $\mathcal{C}$.

Definition 5.25. Let $\mathfrak{N} = \langle F_0, \leq \rangle$ be a Kripke frame and $\mathfrak{M} = \langle \mathfrak{N}, \mathfrak{E}, \mathfrak{C}, \mid \rangle$ be an intuitionistic subset model over $\mathfrak{N}$. Let $\mathfrak{C}$ be a constant specification (for some proof system). We say that $\mathfrak{M}$ respects $\mathfrak{C}$ if
\[ x \mid \mathfrak{C} \phi \] for all $\phi \in \mathfrak{C}$ and all $x \in F_0$.

Given a class $\mathfrak{C}$ of intuitionistic subset models, we write $\mathfrak{C} \mathfrak{S}$ for the subclass of all models respecting a constant specification $\mathfrak{C}$.

Lemma 5.26. Let $\mathcal{I}$ be an intermediate logic and $\mathcal{I} \mathcal{J} \mathcal{L}_0 \in \{ \mathcal{I} \mathcal{J} \mathcal{J}_0, \mathcal{I} \mathcal{J} \mathcal{T}_0, \mathcal{I} \mathcal{J} \mathcal{A}_4, \mathcal{I} \mathcal{J} \mathcal{T}_4 \}$. Let $\mathcal{C}$ be a constant specification for $\mathcal{I} \mathcal{J} \mathcal{L}_0$. Let $\mathfrak{C} \in \mathcal{K} \mathfrak{F}(\mathcal{I})$. For any $\Gamma \in \{ \mathfrak{C} \}$ and $\mathfrak{N} \in \mathfrak{M}$:
\[ \Gamma \mid \mathfrak{N} \mathfrak{C} \phi \] implies $\Gamma \mid \mathfrak{C} \mathfrak{S} \mathfrak{N} \mathfrak{C} \phi$.

Proof. Reasoning as in Lemma 5.13 and 5.19, we restrict the argument and only show the validity of $(\mathcal{J})$, $(\mathcal{S})$, $(\mathcal{F})$ and $(\mathcal{I})$ in their respective model classes. For this, let $\mathfrak{M} = \langle \mathfrak{N}, \mathfrak{E}, \mathfrak{C}, \mid \rangle$ be an intuitionistic subset model over a Kripke frame $\mathfrak{N}$ and let $x \in F_0$.

1. For $(\mathcal{J})$, suppose $x \mid \mathfrak{M} t : (\phi \rightarrow \psi)$. Now, we want to show $x \mid \mathfrak{M} s : \phi \rightarrow [t \cdot s] : \psi$. For this, let $y \geq x$. Note that then $y \in F_0$ as $x$ is only a relation on $F_0$.

Suppose $y \mid \mathfrak{M} s : \phi$, that is
\[ \forall z \in \mathcal{E}_1[y] \ z \mid \mathfrak{M} \phi. \]

Further, $x \mid \mathfrak{M} t : (\phi \rightarrow \psi)$ implies $y \mid \mathfrak{M} t : (\phi \rightarrow \psi)$ by Lemma 5.21, that is
\[ \forall z \in \mathcal{E}_1[y] \ z \mid \mathfrak{M} \phi \rightarrow \psi. \]

Let $z \in \mathcal{E}_{t,s}[y]$. Then, by property (b) of Definition 5.20, we have
\[ z \in \{ w \in F \mid \forall \chi \in (\mathfrak{M})^y_x w \mid \mathfrak{M} \chi \} \]
and thus it suffices to show $\psi \in (\mathfrak{M})^y_x z \mid \mathfrak{M} \psi$.

2. Suppose $x \mid \mathfrak{M} t : \phi$. That is, we have
\[ \forall y \in \mathcal{E}_t[x] \ y \mid \mathfrak{M} \phi. \]

Then, by condition (a) of Definition 5.20, we have
\[ \forall y \in \mathcal{E}_{t+s}[x] \subseteq \mathcal{E}_t[x] \ y \mid \mathfrak{M} \phi \]
which is $x \mid \mathfrak{M} [t + s] : \phi$. As $x$ was arbitrary, we have $x \mid \mathfrak{M} t : \phi \rightarrow [t + s] : \phi$ for any $x \in F_0$. Similarly, one shows $x \mid \mathfrak{M} s : \phi \rightarrow [t + s] : \phi$ for any $x \in F_0$. 

(3) Let \( \mathcal{M} \) be reflexive with \( x \vdash t : \phi \). Then, we have
\[
\forall y \in \mathcal{E}_t[x] \ y \vdash \phi,
\]
that is as \( \mathcal{M} \) is reflexive \( x \in \mathcal{E}_t[x] \) and thus \( x \vdash \phi \). As \( x \) was arbitrary, we have \( x \vdash t : \phi \rightarrow \phi \) for any \( x \in F_0 \).

(4) Let \( \mathcal{M} \) be introspective \( x \vdash t : \phi \). Then, we have
\[
\forall y \in \mathcal{E}_t[x] \ y \vdash t : \phi
\]
by definition of introspectivity but this is exactly \( x \vdash t : \phi \). Again we have \( x \vdash t : \phi \rightarrow t : \phi \) for any \( x \in F_0 \) as \( x \) was arbitrary.

6. Completeness for frame semantics

In this section, we prove the corresponding completeness theorems for the intermediate justification logics together with their previously introduced semantics based on Mkrtychev, Fitting or subset models over intuitionistic Kripke frames. The permissible classes of frames for the completeness theorems derive, similarly as the permissible classes of Heyting algebras from the completeness theorems for the algebraic models, from the underlying intermediate logic where we especially rely on the global completeness statement introduced earlier.

6.1. Completeness w.r.t. intuitionistic Mkrtychev models.

Definition 6.1. Let \( \mathfrak{F} = \langle F, \leq \rangle \) be a Kripke frame and let \( \mathfrak{N} \in \mathcal{M}_0(\mathfrak{F}; L^*_0) \). We define the canonical intuitionistic Mkrtychev model over \( \mathfrak{N} \) as the structure \( \mathcal{M}^c_{\mathfrak{N}} = (\mathfrak{F}, \mathcal{E}^c, \| \cdot \| ^c) \) by setting:

1. \( x \vdash^c p \iff x \vdash^* p \);
2. \( \mathcal{E}^c_t(x) := \{ \phi \in \mathcal{L}_J \mid x \vdash^* \phi \} \).

Lemma 6.2. Let \( \mathfrak{F} = \langle F, \leq \rangle \) be a Kripke frame, let \( \mathfrak{N} \in \mathcal{M}_0(\mathfrak{F}; L^*_0) \) and let \( \mathcal{M}^c_{\mathfrak{N}} = (\mathfrak{F}, \mathcal{E}^c, \| \cdot \| ^c) \) as above. Then for all \( \phi \in \mathcal{L}_J \) and all \( x \in F \):
\[
(\mathcal{M}^c_{\mathfrak{N}}, x) \models \phi \iff (\mathfrak{N}, x) \models \phi^*.
\]

Proof. We prove it by induction on \( \phi \). The claim is immediate for \( p \in \text{Var} \) by definition. Suppose the claim is true for \( \phi, \psi \). Then, we have at first
\[
(\mathcal{M}^c_{\mathfrak{N}}, x) \models \phi \land \psi \iff (\mathcal{M}^c_{\mathfrak{N}}, x) \models \phi \text{ and } (\mathcal{M}^c_{\mathfrak{N}}, x) \models \psi
\]
and similarly for \( \lor \). For \( \rightarrow \), we have
\[
(\mathcal{M}^c_{\mathfrak{N}}, x) \models \phi \rightarrow \psi \iff \forall y \geq x : (\mathcal{M}^c_{\mathfrak{N}}, y) \models \phi \text{ implies } (\mathcal{M}^c_{\mathfrak{N}}, y) \models \psi
\]
Lastly, we have
\[
(\mathcal{M}^c_{\mathfrak{N}}, x) \models t : \phi \iff \phi \in \mathcal{E}^c_t(x)
\]
and similarly for \( \phi_t \).

Lemma 6.3. Let \( \mathfrak{F} = \langle F, \leq \rangle \) be a Kripke frame and let \( \mathfrak{N} \in \mathcal{M}_0(\mathfrak{F}; L^*_0) \) such that additionally \( \mathfrak{N} \models (\text{Th}_{\mathcal{I}_J L_{cs}})^* \). Then \( \mathcal{M}^c_{\mathfrak{N}} \) is a well-defined intuitionistic Mkrtychev model. Further:

(a) if \( (F) \) is an axiom scheme of \( \mathcal{I}_J L_0 \), then \( \mathcal{M}^c_{\mathfrak{N}} \) is factive;
(b) if \( (I) \) is an axiom scheme of \( \mathcal{I}_J L_0 \), then \( \mathcal{M}^c_{\mathfrak{N}} \) is introspective.

Proof. We first show properties (1) and (2) of Definition 5.7. For this, let \( x, y \in F \) with \( x \leq y \). For (1), we have
\[
\vdash x \vdash^c p \Rightarrow x \vdash^* p \Rightarrow y \vdash^* p \Rightarrow y \vdash^c p
\]
and for (2), we have
\[
\phi \in \mathcal{E}_t(x) \Rightarrow x \vdash^* \phi_t \Rightarrow y \vdash^* \phi_t \Rightarrow \phi \in \mathcal{E}_t(y).
\]
Both follow from Lemma 6.3 applied to \( \mathfrak{N} \).
For properties (i) and (ii), let $x \in F$ and $t, s \in Jt$. Then, at first for (i), let $\phi \in \mathcal{E}_t^F(x) \supseteq \mathcal{E}_t^c(x)$, that is by definition $\exists \psi \in \mathcal{L}_J$:

$$\psi \to \phi \in \mathcal{E}_t^c(x) \text{ and } \psi \in \mathcal{E}_t^c(x).$$

Untangling the definition of $\mathcal{E}^c$, we have

$$x \vdash^* (\psi \to \phi)_t \text{ and } x \vdash^* \psi_s.$$ 

As we have $\mathcal{R} \models (Th_{\mathcal{I}J\mathcal{L}_{CS}})^\ast$, we have by the $\ast$-translation of the axiom scheme (J) that $\forall y \geq x$:

$$y \vdash^* (\psi \to \phi)_t \text{ and } y \vdash^* \psi_s \implies y \vdash^* \phi_{[t,s]}$$

and thus especially, as $x \geq x$, we have $x \vdash^* \phi_{[t,s]}$ and thus $\phi \in \mathcal{E}_t^c(x)$. 

For (ii), let $\phi \in \mathcal{E}_t^c(x) \cup \mathcal{E}_t^c(y)$. Then, we have $x \vdash^* \phi \ast \phi_{[t,s]}$ or $x \vdash^* \phi_s$. By the $\ast$-translation of the axiom scheme (+) and $\mathcal{R} \models (Th_{\mathcal{I}J\mathcal{L}_{CS}})^\ast$, we have in either case as before $x \vdash^* \phi_{[t,s]}$.

Now, for (a), if (F) is an axiom scheme of $\mathcal{I}J\mathcal{L}_0$, then we have by $\mathcal{R} \models (Th_{\mathcal{I}J\mathcal{L}_{CS}})^\ast$ again, that $x \vdash^* \phi_t$ implies ($\mathcal{R}, x) \models \phi^\ast$. By the definition of $\mathcal{E}^c$ and Lemma 6.2, we have thus $\phi \in \mathcal{E}_t^c(x)$ implies ($\mathcal{R}_0^M, x) \models \phi$ and thus $\mathcal{M}_{01}^{c, M}$ is factive.

For (b), if (I) is an axiom scheme of $\mathcal{I}J\mathcal{L}_0$, then we have

$$x \vdash^* \phi_t \text{ implies } x \vdash^* (t : \phi)_t$$

and by definition that is

$$\phi \in \mathcal{E}_t^c(x) \text{ implies } t : \phi \in \mathcal{E}_t^c(x)$$

which is $t : \mathcal{E}_t^c(x) \subseteq \mathcal{E}_t^c(x)$ and thus $\mathcal{M}_{01}^{c, M}$ is introspective. \hfill $\Box$

**Theorem 6.4.** Let $\mathcal{I}$ be an intermediate logic, $\mathcal{I}J\mathcal{L}_0 \in \{\mathcal{IJ}_0, \mathcal{IJ}_T, \mathcal{IJ}_A_0, \mathcal{IJ}_A_4, \mathcal{IJ}_A_4_1\}$ and let $CS$ be a constant specification for $\mathcal{I}J\mathcal{L}_0$. Let $C \in \text{KFr}(\mathcal{I}) \cap \text{KFr}(\mathcal{I})$. For any $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$, we have:

$$\Gamma \vdash_{\mathcal{I}J\mathcal{L}_{CS}} \phi \iff \Gamma \models_{\text{CKMJL}_{CS}} \phi.$$ 

**Proof.** The direction from left to right follows from Lemma 5.13. For the converse, suppose $\Gamma \not\vdash_{\mathcal{I}J\mathcal{L}_{CS}} \phi$. By Lemma 4.1, we have

$$\Gamma^* \cup (Th_{\mathcal{I}J\mathcal{L}_{CS}})^\ast \not\vdash_{\mathcal{I}J} \phi^*$$

and by assumption on the global strong completeness of $\mathcal{I}$ w.r.t. $\mathcal{C}$, there is a $\mathcal{R} = (\mathcal{I}, \vdash^\ast)$ in $\mathcal{M}_0(\mathcal{C}, \mathcal{L}_0^\ast)$ such that

$$\mathcal{R} \models \Gamma^* \cup (Th_{\mathcal{I}J\mathcal{L}_{CS}})^\ast \text{ but } \mathcal{R} \not\models \phi^*.$$ 

By Lemma 6.3, we have $\mathcal{M}_{01}^{c, M} \in \text{CKMJL}$ for the corresponding canonical intuitionistic Mktychew model. By Lemma 6.2, we have

$$\mathcal{M}_{01}^{c, M} \models CS$$

and thus $\mathcal{M}_{01}^{c, M} \in \text{CKMJL}_{CS}$ as well as

$$\mathcal{M}_{01}^{c, M} \models \Gamma \text{ but } \mathcal{M}_{01}^{c, M} \not\models \phi.$$ 

Thus, we have $\Gamma \not\models_{\text{CKMJL}_{CS}} \phi$. \hfill $\Box$

### 6.2. Completeness w.r.t. intuitionistic Fitting models.

**Definition 6.5.** Let $\mathcal{I} = (F, \leq)$ be a Kripke frame and let $\mathcal{R} \in \mathcal{M}_0(\mathcal{I}; \mathcal{L}_0^\ast)$. We define the canonical intuitionistic Fitting model over $\mathcal{R}$ as the structure $\mathcal{M}_{01}^{c, F} = (\mathcal{I}, \mathcal{R}^c, \mathcal{E}^c, \vdash^c)$ by setting:

1. $x \vdash^c p$ iff $x \vdash^* p$;
2. $\mathcal{E}_t^c(x) := \{ \phi \in \mathcal{L}_J \mid x \vdash^* \phi_t \}$;
3. $(x, y) \in \mathcal{R}^c$ iff $\forall t \in Jt \phi \in \mathcal{L}_J \ (x \vdash^* \phi_t \Rightarrow (\mathcal{R}, x) \models \phi^\ast)$.

**Lemma 6.6.** Let $\mathcal{I} = (F, \leq)$ be a Kripke frame, let $\mathcal{R} \in \mathcal{M}_0(\mathcal{I}; \mathcal{L}_0^\ast)$ and define $\mathcal{M}_{01}^{c, F}$ as above. For any $\phi \in \mathcal{L}_J$ and all $x \in F$:

$$(\mathcal{M}_{01}^{c, F}, x) \models \phi \iff (\mathcal{R}, x) \models \phi^\ast.$$
Proof. The claim is again proved by induction on the structure of the formula. We only consider the modal case. Suppose the claim holds for all \( x \in F \) and some \( \phi \in \mathcal{L}_I \).

At first, suppose \( (\mathfrak{N}, x) \models \phi_t \), i.e. \( x \models^* \phi_t \). Then, naturally \( \phi \in \mathcal{E}_I^c(x) \) by definition. Let further \( y \in \mathcal{R}^c[x] \). Then, as \( x \models^* \phi_t \), we have \( (\mathfrak{N}, y) \models \phi^* \) by definition and thus \( (\mathfrak{M}_N^c, x) \models \phi \) by induction hypothesis. Thus, we have

\[
\phi \in \mathcal{E}_I^c(x) \quad \text{and} \quad \forall y \in \mathcal{R}^c[x] \quad (\mathfrak{M}_N^c, x) \models \phi
\]

and thus \( (\mathfrak{M}_N^c, x) \models t : \phi \).

Conversely, suppose \( (\mathfrak{N}, x) \not\models \phi_t \), that is \( x \not\models^* \phi_t \). Thus, by definition \( \phi \not\in \mathcal{E}_I^c(x) \) and thus \( (\mathfrak{M}_N^c, x) \not\models t : \phi \).

immediately by definition.

\( \square \)

Lemma 6.7. Let \( \mathfrak{F} = (F, \leq) \) be a Kripke frame and let \( \mathfrak{N} \in \mathcal{M}_0(\mathfrak{F}; \mathcal{L}_0^c) \) such that additionally \( \mathfrak{N} \models (Th_{IJ\mathcal{L}_{CS}})^* \). Then \( \mathfrak{M}_N^c \) is a well-defined intuitionistic Fitting model. Further:

(a) if \( (F) \) is an axiom scheme of \( \mathcal{I}\mathcal{J}\mathcal{L}_0 \), then \( \mathfrak{M}_N^c \) is reflexive;

(b) if \( (I) \) is an axiom scheme of \( \mathcal{I}\mathcal{J}\mathcal{L}_0 \), then \( \mathfrak{M}_N^c \) is introspective.

Proof. For properties (1) - (3) of Definition 5.11, let \( x, y \in F \) with \( x \leq y \). For (1) and (2), we have as before

\[
x \models^c p \Rightarrow x \models^* p \Rightarrow y \models^* p \Rightarrow y \models^c p
\]

and

\[
\phi \in \mathcal{E}_I(x) \Rightarrow x \models^* \phi_t \Rightarrow y \models^* \phi_t \Rightarrow \phi \in \mathcal{E}_I(y)
\]

by Lemma 5.3 for \( \mathfrak{N} \). For (3), let \( z \in \mathcal{R}^c[y] \), that is we have

\[
\forall t \in \mathcal{J} t \models \phi \in \mathcal{L}_I \quad (y \models^* \phi_t \Rightarrow (\mathfrak{N}, z) \models \phi^*).
\]

Then, for any \( t \in \mathcal{J}t \) and any \( \phi \in \mathcal{L}_I \) we have, if \( x \models^* \phi_t \) that \( y \models^* \phi_t \) by Lemma 5.3 and thus by the above \( (\mathfrak{N}, x) \models \phi^* \). Thus \( z \in \mathcal{R}^c[y] \) and thus \( \mathcal{R}^c[y] \subseteq \mathcal{R}^c[x] \).

We have properties (1) and (2) of Definition 5.11 in the same way as in the proof of Lemma 6.3. For (3), let \( z \in \mathcal{R}^c[y] \) for \( x \leq y \), that is

\[
\forall t \in \mathcal{J} t \models \phi \in \mathcal{L}_I \quad (y \models^* \phi_t \Rightarrow (\mathfrak{N}, z) \models \phi^*).
\]

If \( x \models^* \phi_t \), then \( y \models^* \phi_t \) by Lemma 5.3 and thus by the above \( (\mathfrak{N}, x) \models \phi^* \). Thus \( z \in \mathcal{R}^c[y] \).

For property (i), let \( \phi \in \mathcal{E}_I^c(x) \supseteq \mathcal{E}_I^c(x) \), i.e.

\[
\exists \psi \in \mathcal{L}_I (\psi \rightarrow \phi \in \mathcal{E}_I^c(x) \land \psi \in \mathcal{E}^c(x)).
\]

Thus, by definition, we have

\[
x \models^* (\psi \rightarrow \phi)_t \land x \models^* \psi_a
\]

and thus, as \( \mathfrak{N} \models (Th_{IJ\mathcal{L}_{CS}})^* \), we have

\[
x \models^* \phi_{[t,a]},
\]

that is \( \phi \in \mathcal{E}_I^c(x) \).

For property (ii), note that \( x \models^* \phi_t \) implies \( x \models^* \phi_{[t,a]} \) again by \( \mathfrak{N} \models (Th_{IJ\mathcal{L}_{CS}})^* \) and thus \( \phi \in \mathcal{E}_I^c(x) \) implies \( \phi \in \mathcal{E}_I^c(x) \) and similarly for \( \phi \in \mathcal{E}^c(x) \).

Suppose that \( (F) \) is an axiom scheme of \( \mathcal{I}\mathcal{J}\mathcal{L}_0 \). Then, we have

\[
\forall t \in \mathcal{J} t \models \phi \in \mathcal{L}_I \quad (x \models^* \phi_t \Rightarrow (\mathfrak{N}, x) \models \phi^*)
\]

as \( \mathfrak{N} \models (Th_{IJ\mathcal{L}_{CS}})^* \) and this is exactly \( (x, x) \in \mathcal{R}^c \).

Suppose that \( (I) \) is an axiom scheme of \( \mathcal{I}\mathcal{J}\mathcal{L}_0 \). As in the case of intuitionistic Mkrtchyan models, one shows

\[
t : \mathcal{E}_I^c(x) \subseteq \mathcal{E}_I^c(x).
\]

For the transitivity of \( \mathcal{R}^c \), let \( (x, y), (y, z) \in \mathcal{R}^c \). that is, we have

\[
\forall t \in \mathcal{J} t \models \phi \in \mathcal{L}_I \quad (x \models^* \phi_t \Rightarrow (\mathfrak{N}, y) \models \phi^*)
\]

as well as

\[
\forall t \in \mathcal{J} t \models \phi \in \mathcal{L}_I \quad (y \models^* \phi_t \Rightarrow (\mathfrak{N}, z) \models \phi^*).
\]

As \( (I) \) is an axiom scheme and \( \mathfrak{N} \models (Th_{IJ\mathcal{L}_{CS}})^* \), we have

\[
w \models^* \phi_t \Rightarrow w \models^* (t : \phi)_t
\]
for any \( w \in F \). Thus, especially we have

\[ x \models^* \phi_t \Rightarrow x \vdash^* (t : \phi)_t \Rightarrow y \models^* \phi_t \Rightarrow (\mathfrak{I}, z) \models \phi^* \]

using Lemma 6.6. Thus, by definition we have \((x, z) \in R^c\).

For the monotonicity, let \( y \in R^c[x] \) and let \( \phi \in \mathcal{E}_1(x) \). The former gives

\[ \forall t \in JT_0 \varphi \in L_J, (x \models^* \phi_t \Rightarrow (\mathfrak{I}, y) \models \phi^*) \]

and the latter gives \( x \models^* \phi_t \). As \( \mathfrak{N} \models (Th_{IJLCS})^* \), we have especially \( x \models^* (t : \phi)_t \). By the above, we have \((\mathfrak{I}, y) \models \phi_t\), that is \( y \models^* \phi_t \) and thus \( \phi \in \mathcal{E}_1(y) \). Thus, \( \mathfrak{M}^{c,F}_{\mathfrak{N}} \) is monotone and thus \( \mathfrak{M}^{c,F}_{\mathfrak{N}} \) is introspective.

**Theorem 6.8.** Let \( \Gamma \) be an intermediate logic, \( \mathcal{I}, \mathcal{J}, \mathcal{L}_0 \in \{ \mathcal{I}, \mathcal{J}, \mathcal{J}, \mathcal{T}, \mathcal{J}, \mathcal{T}, \mathcal{J}, \mathcal{T}, \mathcal{J}, \mathcal{T} \} \) and let \( CS \) be a constant specification for \( \mathcal{I}, \mathcal{J}, \mathcal{L}_0 \). Let \( C \in KFr(I) \cap KFr(J) \). For any \( \Gamma \cup \{ \phi \} \subseteq \mathcal{L}_J \), we have:

\[ \Gamma \vdash_{\mathcal{I}, \mathcal{J}, \mathcal{L}_CS} \phi \Leftrightarrow \Gamma \models_{\mathfrak{C}, \mathfrak{E}LCS} \phi. \]

**Proof.** The direction from left to right now comes from Lemma 6.7. For the converse, we again use Lemma 6.1 and the assumptions on \( C \) to obtain a \( \mathfrak{N} = (\mathfrak{G}, \models^*) \in \mathcal{M}_0(C; \mathcal{L}_0^c) \) with

\[ \mathfrak{N} \models \Gamma^* \cup (Th_{IJLCS})^* \quad \text{but} \quad \mathfrak{N} \not\models \phi^*. \]

Using Lemmas 6.7 and 6.6, we obtain as before that \( \mathfrak{M}^{c,F}_{\mathfrak{N}} \in \mathfrak{C}, \mathfrak{E}LCS \) as well as

\[ \mathfrak{M}^{c,F}_{\mathfrak{N}} \models \Gamma \quad \text{but} \quad \mathfrak{M}^{c,F}_{\mathfrak{N}} \not\models \phi, \]

that is \( \Gamma \not\models_{\mathfrak{C}, \mathfrak{E}LCS} \phi \).

\[ \square \]

### 6.3. Completeness w.r.t. intuitionistic subset models.

**Definition 6.9.** Let \( \tilde{\mathfrak{G}} = (\mathcal{F}_0, \subseteq) \) be a Kripke frame and let \( \mathfrak{N} \in \mathcal{M}_0(\tilde{\mathfrak{G}}; \mathcal{L}_0^c) \). We define the **canonical intuitionistic subset model** over \( \mathfrak{N} \) as the structure \( \mathfrak{M}^{c,S}_{\mathfrak{N}} = (\tilde{\mathfrak{G}}, F^c, E^c, \models^c) \) by setting:

1. \( F^c = F_0 \cup \bigcup_{x \in F_0} \{ x \mid t \in JT \} \);
2. \( (x, y) \in E^c \) if \( \forall \phi \in \mathcal{L}_J (x \models^c \phi_t \Rightarrow y \models^c \phi) \) for all \( x, y \in F^c \);
3. \( \models^c \phi \) if \( (\mathfrak{N}, x) \models \phi^* \).

**Lemma 6.10.** Let \( \mathfrak{G} = (\mathcal{F}_0, \subseteq) \) be a Kripke frame, let \( \mathfrak{N} \in \mathcal{M}_0(\tilde{\mathfrak{G}}; \mathcal{L}_0^c) \) and define \( \mathfrak{M}^{c,S}_{\mathfrak{N}} \) as above. For any \( \phi \in \mathcal{L}_J \) and any \( x \in F_0 \):

\[ (\mathfrak{M}^{c,S}_{\mathfrak{N}}, x) \models \phi \iff (\mathfrak{N}, x) \models \phi^*. \]

**Proof.** This is immediate by definition as we have \( (\mathfrak{M}^{c,S}_{\mathfrak{N}}, x) \models \phi \iff x \models^c \phi \iff (\mathfrak{N}, x) \models \phi^* \), given a \( x \in F_0 \).

The simplicity of the above lemma is in contrast to the truth lemmas for the previous canonical models over Kripke frames. In the context of intuitionistic subset models, the relation \( \models^c \) completely encodes the truth values of formulae to be able to cope with "irregular" worlds. This comes with the expense of conditions of well-definedness for \( \models^c \) and thus, the previous complexity of showing an equivalence like the one of the above lemma is shifted into the following result.

**Lemma 6.11.** Let \( \tilde{\mathfrak{G}} = (\mathcal{F}_0, \subseteq) \) be a Kripke frame and let \( \mathfrak{N} \in \mathcal{M}_0(\tilde{\mathfrak{G}}; \mathcal{L}_0^c) \) such that additionally \( \mathfrak{N} \models (Th_{IJLCS})^* \). Then \( \mathfrak{M}^{c,S}_{\mathfrak{N}} \) is a well-defined intuitionistic subset model. Further:

1. if \( (F) \) is an axiom scheme of \( \mathcal{I}, \mathcal{J}, \mathcal{L}_0 \), then \( \mathfrak{M}^{c,S}_{\mathfrak{N}} \) is reflexive;
2. if \( (I) \) is an axiom scheme of \( \mathcal{I}, \mathcal{J}, \mathcal{L}_0 \), then \( \mathfrak{M}^{c,S}_{\mathfrak{N}} \) is introspective.

**Proof.** We begin with properties (i) - (v) from Definition 5.20. For this, let \( x \in F_0 \). The properties (i) - (iii) are immediate by using the respective properties of \( \models^c \) and the fact that \( \star \) commutes with \( \bot, \land, \lor \).

For (iv), note that we have

\[ x \models^c \phi \rightarrow \psi \iff (\mathfrak{N}, x) \models \phi^* \rightarrow \psi^* \]

\[ \text{if } \forall y \ge x ( (\mathfrak{N}, y) \models \phi^* \text{ or } (\mathfrak{N}, y) \models \psi^*) \]

\[ \text{if } \forall y \ge x (y \models^c \phi \text{ or } y \models^c \psi) \]

where it is instrumental that \( \subseteq \) is a relation on \( F_0 \) only.

For (v), we have for one by definition that

\[ \forall y \in \mathcal{E}_1(x) \forall \phi \in \mathcal{L}_J ((\mathfrak{N}, x) \models \phi_t \Rightarrow (\mathfrak{N}, y) \models \phi^*) \]
that is we have

\[ x \equiv^c t : \phi \Rightarrow x \equiv^* \phi_t \]
\[ \Rightarrow \forall y \in E^c_t[x] ((\mathfrak{M}, y) \models \phi^*) \]
\[ \Leftrightarrow \forall y \in E^c_t[x] (y \equiv^c \phi) . \]

For another, we have \( x_t \in E_t[x] \) as we have \( x_t \equiv^c \phi \) iff \( x \equiv^* \phi_t \) by definition. Thus, if \( x \equiv^c t : \phi \), then \( x \equiv^* \phi_t \) and thus \( x_t \equiv^c \phi \). Therefore

\[ x \equiv^c t : \phi \Rightarrow \exists y \in E^c_t[x] (y \equiv^c \phi) . \]

Concluding, we have \( x \equiv^c t : \phi \) iff \( \forall y \in E^c_t[x] (y \equiv^c \phi) \).

Regarding properties (1) and (2) of Definition 5.20, let \( x \leq y \) for \( x, y \in F_0 \). Property (1) follows as in the proof of Lemma 6.9 by (3).(a) of Definition 6.9. For property (2), let \( z \in E^c_t[y] \). Thus, we have

\[ \forall \phi \in \mathcal{L}_J (y \equiv^c t : \phi \Rightarrow z \equiv^c \phi) \]

and if \( x \equiv^c t : \phi \), then as \( x, y \in F_0 \), we have \( x \equiv^* \phi_t \) and thus \( y \equiv^* \phi_t \) which is \( y \equiv^c t : \phi \). By the above, we have \( y \equiv^c \phi \) and thus \( z \in E^c_t[x] \).

Now, on to properties (a), (b) of Definition 5.20. For (a), let \( y \in E^c_t[x] \), that is we have

\[ \forall \phi \in \mathcal{L}_J (x \equiv^* \phi_{[t+s]} \Rightarrow (\mathfrak{M}, y) \models \phi^*) \]

by definition. Now, by assumption as \( \mathfrak{M} \models (\text{Th}_{\mathcal{I}_JL_{CS}})^* \) we have \( x \equiv^* \phi_t \) implies \( x \equiv^* \phi_{[t+s]} \) and \( x \equiv^* \phi_s \) implies \( x \equiv^* \phi_{[t+s]} \). Therefore, we have

\[ \forall \phi \in \mathcal{L}_J (x \equiv^* \phi_t \Rightarrow x \equiv^* \phi_{[t+s]} \Rightarrow (\mathfrak{M}, y) \models \phi^*) \]

and

\[ \forall \phi \in \mathcal{L}_J (x \equiv^* \phi_s \Rightarrow x \equiv^* \phi_{[t+s]} \Rightarrow (\mathfrak{M}, y) \models \phi^*) \]

which is \( y \in E^c_t[x] \cap E^c_s[x] \).

For (b), let \( y \in E^c_t[x] \), that is

\[ (i) \quad \forall \phi \in \mathcal{L}_J (x \equiv^* \phi_{[t+s]} \Rightarrow (\mathfrak{M}, y) \models \phi^*). \]

Let \( \phi \in (\mathfrak{M}^{c,S}_{\mathfrak{M}})_{E^c_s} \), that is there is a \( \psi \in \mathcal{L}_J \) such that

\[ \forall z \in F^c (z \in E^c_t[x] \Rightarrow z \equiv^c \psi \rightarrow \phi \text{ and } z \in E^c_s[x] \Rightarrow z \equiv^c \psi). \]

By property (v), we have that \( x \equiv^c t : (\psi \rightarrow \phi) \) and \( x \equiv^c s : \psi \), i.e. by definition as \( x \in F_0 \):

\[ (\mathfrak{M}, x) \models (\psi \rightarrow \phi), \text{ and } (\mathfrak{M}, x) \models \psi_s \]

and by \( \mathfrak{M} \models (\text{Th}_{\mathcal{I}_JL_{CS}})^* \) and axiom (J), we have

\[ (\mathfrak{M}, x) \models \phi_{[t+s]} . \]

Thus, by (i), we have \( (\mathfrak{M}, y) \models \phi^* \) and thus by definition \( y \equiv^c \phi \).

Assume second to last that (F) is an axiom scheme of \( \mathcal{I}_JL_0 \). Then, we have

\[ x \equiv^* \phi_t \Rightarrow (\mathfrak{M}, x) \models \phi^* \]

for all \( x \in F_0 \) and all \( \phi \in \mathcal{L}_J \), \( t \in \mathcal{J}t \) as \( \mathfrak{M} \models (\text{Th}_{\mathcal{I}_JL_{CS}})^* \) and thus, by definition we have \( x \in E^c_t[x] \) for all \( t \in \mathcal{J}t \).

Assume last that (I) is an axiom scheme of \( \mathcal{I}_JL_0 \). Let \( y \in E^c_t[x] \), that is

\[ (i) \quad \forall \phi \in \mathcal{L}_J (x \equiv^* \phi_t \Rightarrow (\mathfrak{M}, y) \models \phi^*). \]

Let \( \phi \in \mathcal{L}_J \) and assume \( x \equiv^c t : \phi \), that is \( x \equiv^* \phi_t \). Then, as \( \mathfrak{M} \models (\text{Th}_{\mathcal{I}_JL_{CS}})^* \), we have \( x \equiv^* (t : \phi)_t \). By (i) we have \( (\mathfrak{M}, y) \models (t : \phi)^* \), that is \( (\mathfrak{M}, y) \models \phi_t \) and thus by definition \( y \equiv^c t : \phi \).

**Theorem 6.12.** Let \( \mathcal{I} \) be an intermediate logic, \( \mathcal{I}_JL_0 \in \{ \mathcal{I}_J0, \mathcal{I}_J\mathcal{J}0, \mathcal{I}_J\mathcal{J}40, \mathcal{I}_\mathcal{J}40 \} \) and \( CS \) be a constant specification for \( \mathcal{I}_JL_0 \). Let \( C \in \text{KFr}(\mathcal{I}) \cap \text{KFr}^2(\mathcal{I}) \). Then, for any \( \Gamma \cup \{ \phi \} \subseteq \mathcal{L}_J \), we have:

\[ \Gamma \vdash_{\mathcal{I}_JL_{CS}} \phi \text{ iff } \Gamma \models_{\text{CS} \cup_{\mathcal{L}_J} \phi} . \]
Proof. Lemma 6.26 gives the direction from left to right. As before, Lemma 6.1 as well as the assumptions on C give a \( \mathcal{L}_d^* \)-model \( \mathfrak{N} = (\mathfrak{A}, \mathfrak{B}^+) \in \mathcal{M}_0(\mathcal{L}_d^*) \) with
\[
\mathfrak{N} \models (Th_{\mathcal{I}, \mathcal{J}, \mathcal{L}_d^*})^*, \mathfrak{N} \models \Gamma^* \text{ but } \mathfrak{N} \not\models \phi^*.
\]
The first part gives that \( \mathfrak{M}_{\mathfrak{N}}^S \) is a well-defined CKSJL-model by Lemma 6.11 and Lemma 6.10 gives
\[
\mathfrak{M}_{\mathfrak{N}}^c \models \Gamma \text{ but } \mathfrak{M}_{\mathfrak{N}}^c \not\models \phi.
\]
as well as \( \mathfrak{M}_{\mathfrak{N}}^c \models CS \), i.e. \( \mathfrak{M}_{\mathfrak{N}}^c \in \text{CKSJL}_{CS} \). Therefore, by definition \( \Gamma \not\models \text{CKSJL}_{CS} \phi \). \( \square \)

7. Conclusion

The completeness theorems proved in this paper show that behind any previous completeness result in the literature stands a unified completeness theorem lifting classes of algebras or classes of Kripke frames complete for some intermediate logic to a complete model class for the corresponding justification logic. Key to this is of course the strong completeness assumption of the underlying propositional logic and it remains open whether there are similar liftings of weak propositional completeness to weak completeness on the justification side.

We want to acknowledge that the algebraic results can be generalized in an immediate way. E.g. with an algebraic Fitting model \( \mathfrak{M} = (\mathfrak{A}, \mathfrak{W}, \mathfrak{R}, \mathfrak{E}, \mathfrak{V}) \) over a complete Heyting algebra \( \mathfrak{A} \), \( \mathfrak{A} \) can be generalized to not be complete but only \( \text{card}(\mathfrak{W})^+ \)-complete, similarly as in the case of the Kripke-models taking values in Heyting algebras for intuitionistic modal logics from Ono [32]. This of course also applies to the algebraic subset models.

This study of the general class of intermediate justification logics was initiated, originally, to study extensions of the Gödel-McKinsey-Tarski translation and modal companions (see e.g. [19] [29] and also [7] [11]) to the language of justification logics and to relate the intermediate justification logics with hybrid justification logics (in the sense of [5] [13]). A relationship between hybrid justification logics and intermediate justification logics could prove to be a similarly fruitful connection as between intermediate logics and modal logics over \( \mathcal{S}_4 \), interlinking the classical hybrid justification logics with intermediate justification logics and vice versa e.g. in terms of decidability.

Also, the work on structural proof calculi for intermediate logics from e.g. [9] may be used to prove a (not completely, but extensive) unified realization theorem between modal and justification logics over an (arbitrary) intermediate logic.

We end this section by a review of some old results reobtained as corollaries of the here proved completeness results obtainable through this paper.

7.1. The special cases of \( \text{IPC}, \mathcal{G} \) and \( \text{CPC} \). As special instances of the completeness results, we in particular have the following algebraic and frame-based completeness theorems for \( \text{IPC}, \mathcal{G} \) and \( \text{CPC} \). For this, consider first the following collection of algebraic completeness results:

We use \( \mathfrak{H} \) to denote the class of all Heyting algebras. An important instance of a linear Heyting algebra is the standard Gödel algebra \( \langle 0, 1 \rangle_G \) given by
\[
\langle 0, 1 \rangle_G := ([0, 1], \min, \max, \Rightarrow, 0, 1)
\]
where
\[
x \Rightarrow y := \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}
\]
for \( x, y \in [0, 1] \). The natural choice of a Boolean algebra is the algebra
\[
\{0, 1\}_B := \langle \{0, 1\}, \min, \max, \Rightarrow, 0, 1 \rangle
\]
with the above function \( \Rightarrow \) restricted to \( \{0, 1\} \).

**Theorem 7.1.** We have the following algebraic completeness results:

1. \( \text{IPC} \) is strongly complete with respect to \( \mathfrak{H}_{\text{fin}} \);
2. \( \mathcal{G} \) is strongly complete with respect to \( \langle 0, 1 \rangle_G \);
3. \( \text{CPC} \) is strongly complete with respect to \( \{0, 1\}_B \).

All items are folklore by now. For example, item (2) was proven by Dummett in [10]. Based on Theorems 4.4, 4.7 and 4.10 we obtain the following particular corollaries.
Corollary 7.2. Let $\text{IPCJ}_L \in \{\text{IPCJ}_0, \text{IPCJ}_T, \text{IPCJ}_4, \text{IPCJ}_T 4\}$ where $CS$ is a constant specification for $\text{IPCJ}_L$. For any $\Gamma \cup \{\phi\} \subseteq L_J$, the following are equivalent:

1. $\Gamma \vdash_{\text{IPCJ}_L CS} \phi$;
2. $\Gamma \models_{\text{HJ}_n \text{AMJL}_{CS}} \phi$;
3. $\Gamma \models_{\text{HJ}_n \text{AFJL}_{CS}} \phi$;
4. $\Gamma \models_{\text{HJ}_n \text{ASJL}_{CS}} \phi$.

Corollary 7.3. Let $\mathcal{G}_L \in \{\mathcal{G}_J, \mathcal{G}_JT, \mathcal{G}_J 4, \mathcal{G}_JT 4\}$ where $CS$ is a constant specification for $\mathcal{G}_L$. For any $\Gamma \cup \{\phi\} \subseteq L_J$, the following are equivalent:

1. $\Gamma \vdash_{\mathcal{G}_L CS} \phi$;
2. $\Gamma \models_{\mathcal{G}_J 0 \text{AMJL}_{CS}} \phi$;
3. $\Gamma \models_{\mathcal{G}_J 0 \text{AFJL}_{CS}} \phi$;
4. $\Gamma \models_{\mathcal{G}_J 0 \text{ASJL}_{CS}} \phi$.

Corollary 7.4. Let $\text{CPCJ}_L \in \{\text{CPCJ}_0, \text{CPCJ}_T, \text{CPCJ}_4, \text{CPCJ}_T 4\}$ where $CS$ is a constant specification for $\text{CPCJ}_L$. For any $\Gamma \cup \{\phi\} \subseteq L_J$, the following are equivalent:

1. $\Gamma \vdash_{\text{CPCJ}_L CS} \phi$;
2. $\Gamma \models_{\{0,1\}_n 0 \text{AMJL}_{CS}} \phi$;
3. $\Gamma \models_{\{0,1\}_n 0 \text{AFJL}_{CS}} \phi$;
4. $\Gamma \models_{\{0,1\}_n 0 \text{ASJL}_{CS}} \phi$.

These theorems already contain several well-known results from the literature on semantics for justification logics. At first, the equivalence between (1) and (5), (6) and (7) in Corollary 7.3 are among the completeness results previously obtained for Gödel justification logic in [34], where the authors considered models for $GJ_L$ and $GJT_L$ as well as Lehmann and Studer [26], respectively. Further, the equivalence between (1) and (5), (6) in Corollary 7.4 are the known completeness theorems of Mkrtychev [30], Fitting [12, 14] as well as Lehmann and Studer [26], respectively. These theorems already contain several well-known results from the literature on semantics for justification logics.

The previous work on semantics of intuitionistic justification logic in the sense of the present paper is mainly [27], where the authors considered models for $\text{IPCJ}_T 4$ based on extending Kripke-frames for intuitionistic propositional logic by semantic machinery for justification logics from Mkrtychev and Fitting models. The above corollary [22] gives a different semantic approach to $\text{IPCJ}_T 4$ and considers various other intuitionistic justification logics not present in the literature before.

We reobtain these results of [27] through the completeness theorems proved here regarding Mkrtychev, Fitting and subset models over Kripke frames for intermediate logics. However, considering the wider class of intermediate logics, we also obtain the following alternative semantic characterizations of $\text{IPCJ}_L, \mathcal{G}_L$ and $\text{CPCJ}_L$ using the semantic characterizations of their base logics based on Kripke frames. Let IF be the class of all intuitionistic Kripke frames and let $\text{LIF}, \text{SIF}$ be the class of all linear intuitionistic Kripke frames and of all single-point intuitionistic Kripke frames, respectively. As a well-known result, we have:

Theorem 7.5. For any $\Gamma \cup \{\phi\} \subseteq L_J$:

1. $\Gamma \vdash_{\text{IPC}} \phi$ iff $\Gamma \models_{\text{IF}} \phi$;
2. $\Gamma \vdash_{\mathcal{G}} \phi$ iff $\Gamma \models_{\text{LIF}} \phi$;
3. $\Gamma \vdash_{\text{CPC}} \phi$ iff $\Gamma \models_{\text{SIF}} \phi$.

Item (1) goes back to Kripke’s work [21]. Item (2) and (3) are not that easily traceable ((3) is quite immediate), but for more modern references see e.g. [8]. Combining this with the fact that IF as well as LIF and SIF are closed under principal subframes, we have by Lemma 5.6.

Corollary 7.6 (of Theorem 7.5). For any $\Gamma \cup \{\phi\} \subseteq L_J$:

1. $\Gamma \vdash_{\text{IPC}} \phi$ iff $\Gamma \models_{\text{IF}} \phi$;
2. $\Gamma \vdash_{\mathcal{G}} \phi$ iff $\Gamma \models_{\text{LIF}} \phi$;
3. $\Gamma \vdash_{\text{CPC}} \phi$ iff $\Gamma \models_{\text{SIF}} \phi$.

Thus, the completeness theorems based on Kripke frames apply and we obtain the following completeness theorems.

Corollary 7.7. Let $\text{IPCJ}_L \in \{\text{IPCJ}_0, \text{IPCJ}_T, \text{IPCJ}_4, \text{IPCJ}_T 4\}$ where $CS$ is a constant specification for $\text{IPCJ}_L$. For any $\Gamma \cup \{\phi\} \subseteq L_J$, the following are equivalent:

1. $\Gamma \vdash_{\text{IPCJ}_L CS} \phi$;
Corollary 7.8. Let $GJL_0 \in \{GJL_0, GJL_0, GJL_0, GJL_0, GJL_0\}$ where $CS$ is a constant specification for $GJL_0$. For any $\Gamma \cup \{\phi\} \subseteq L_J$, the following are equivalent:

1. $\Gamma \vdash GJL_{CS} \phi$;
2. $\Gamma \vdash LJKML_{CS} \phi$;
3. $\Gamma \vdash LIFKML_{CS} \phi$;
4. $\Gamma \vdash LSKML_{CS} \phi$.

Corollary 7.9. Let $CPCJL_0 \in \{CPCJL_0, CPCJL_0, CPCJL_0, CPCJL_0, CPCJL_0\}$ where $CS$ is a constant specification for $CPCJL_0$. For any $\Gamma \cup \{\phi\} \subseteq L_J$, the following are equivalent:

1. $\Gamma \vdash CPCJL_{CS} \phi$;
2. $\Gamma \vdash SIFKML_{CS} \phi$;
3. $\Gamma \vdash SIFKML_{CS} \phi$;
4. $\Gamma \vdash SIFKML_{CS} \phi$.

In particular, the equivalences between (1), (2) and (3) in Corollary 7.7, for $0 = \vdash JL$ and let $\phi \in \mathbb{N}$. Then, we define the $n$-valued Gödel logic by

$G_n := G \oplus (BC)n$.

The notation for the axiom scheme comes from its use in intermediate logics of bounded cardinality. The usual semantics for $G_n$ is given by a characteristic matrix through the Heyting algebra

$V_{G_n}(\phi) := (V^{(n)}(\phi), \min, \max, \Rightarrow, 0, 1)$

with

$V^{(n)} := \left\{ 1 - \frac{1}{k} | 1 \leq k \leq n - 1 \right\} \cup \{1\}$

and the operation $\Rightarrow$ as before, restricted to $V^{(n)}$. Indeed, we then have the following completeness theorem.

Theorem 7.10. $G$ is strongly complete with respect to $V^{(n)}_G$.

For the above theorem and more background on the finite valued Gödel logics, we refer to [1, 30]. Using Theorems [4.3, 4.7] and [4.10] we obtain the following corollary:

Corollary 7.11. Let $G_nJL_0 \in \{G_nJL_0, G_nJL_0, G_nJL_0, G_nJL_0, G_nJL_0\}$ and let $CS$ be a constant specification for $G_nJL_0$. For any $\Gamma \cup \{\phi\} \subseteq L_J$, the following are equivalent:

1. $\Gamma \vdash G_nJL_{CS} \phi$;
2. $\Gamma \vdash V^{(n)}_G AMJL_{CS} \phi$;
3. $\Gamma \vdash V^{(n)}_G AFJL_{CS} \phi$;
4. $\Gamma \vdash V^{(n)}_G ASJL_{CS} \phi$.

7.2. Some less-well-known intermediate justification logics.

7.2.1. Finite-valued Gödel logics. Prominent strengthenings of the infinite valued Gödel logic $G$ (or Gödel-Dummet logic) are the finite valued Gödel logics $G_n$. These actually predate $G$ in the sense that this sequence of intermediate logics is the one used by Gödel in [18] for his investigations about intuitionistic logic. $G$ was later defined by Dummett in [10]. Axiomatized, we can give the following description of $G_n$. Consider the axiom scheme

$(BC)_n: \forall n \in \mathbb{N} \forall j < k (p_j \Rightarrow p_k)$

for any $n \in \mathbb{N}$. Then, we define the $n$-valued Gödel logic by

$G_n := G \oplus (BC)_n$.

A classical result of Gabbai [15] (see also Smorynski’s [38]) is the completeness result in terms of special Kripke frames. For this consider the following definitions:

(WLEM): $\neg \phi \lor \neg \phi$

and the corresponding logic of the weak law of the excluded middle, also known as Jankov’s logic (introduced in [20]), given by

$KC := ITP \oplus (WLEM)$.
Definition 7.12. A Kripke frame $\langle F, \leq \rangle$ has topwidth $k$, if there are $k$ maximal nodes $x_1, \ldots, x_k$ such that for every $y \in F$, there is a $i \in \{1, \ldots, k\}$ such that $y \leq x_i$.

Let $\mathbf{TIF}_k$ be the class of all intuitionistic Kripke frames with topwidth $k$. Then, one obtains the following semantical characterization.

Theorem 7.13 (Gabbai [15]). $KC$ is strongly complete w.r.t. $\mathbf{TIF}_1$.

Considering that the model class in question is closed under principal subframes, we have the following corollary based on Lemma 6.4.

Corollary 7.14. $KC$ is strongly globally complete w.r.t. $\mathbf{TIF}_1$.

This results in the following completeness theorem for justification logics based on Jankov’s logic as a corollary of Theorems 6.4 [18] and 6.12.

Corollary 7.15. Let $\mathcal{K}CJL_0 \in \{\mathcal{K}CJ_0, \mathcal{K}CJ_4_0, \mathcal{K}CJ_4_0, \mathcal{K}CJ_4_0\}$ where $CS$ is a constant specification for $\mathcal{K}CJL_0$. For any $\Gamma \cup \{\phi\} \subseteq L_J$, the following are equivalent:

1. $\Gamma \vdash_{\mathcal{K}CJL_{CS}} \phi$;
2. $\Gamma \vdash_{\mathbf{TIF}_1, \mathcal{K}MJ_{CS}} \phi$;
3. $\Gamma \vdash_{\mathbf{TIF}_1, \mathcal{K}FJ_{CS}} \phi$;
4. $\Gamma \vdash_{\mathbf{TIF}_1, \mathcal{K}SJ_{CS}} \phi$.

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Hoch-Weiseler Str. 46, Butzbach, 35510, Hesse, Germany
E-mail address: pischkenicholas@gmail.com