A Duality for Yang-Mills Moduli Spaces on Noncommutative Manifolds

Hiroshi TAKAI
Department of Mathematics,
Tokyo Metropolitan University

Abstract

Studied are the moduli spaces of Yang-Mills connections on finitely generated projective modules associated with noncommutative flows. It is actually shown that they are homeomorphic to those on dual modules associated with dual noncommutative flows. As a corollary, the result is also affirmative to the case of noncommutative multiflows. As an important application, computed are the moduli spaces of the instanton bundles over noncommutative Euclidean 4-spaces with respect to the canonical action of space translations without using the ADHM-construction.
§1. Introduction  
Among miscellaneous topics in super string theory or M-theory, one of their most important problems is concerned with the compactification of fields, which means that either 10 or 11 dimensional field theory would be reduced to 4 dimensional one by compactifying either 6 or 7 dimensional space time respectively. For instance, an 11 dimensional M-theory has a circle compactification to deduce a IIA-type super string theory, which describes a nonchiral field theory of closed strings due to BFSS ([9]). Moreover, this theory has also one more circle compactification to deduce a IIB-type superstring theory, which describes a chiral field theory of closed strings via the so-called T-transformations ([10]). Recently, Connes, Douglas and Schwarz [2] have shown that the field theory to such a 2-torus compactification cited above has a complete solution by taking the moduli spaces of Yang-Mills connections of appropriate modules for the gauge action of the 2-torus on either commutative or noncommutative 2-torus. Actually, Connes and Rieffel [3] have proved that the latter Yang-Mills moduli space is homeomorphic to the 2-torus. From this point of view, the problem of finding the Yang-Mills moduli space for a given smooth noncommutative dynamical system is a quite important one to determine the unified 4 dimensional field theory having the unique compactification.

In this paper, we present a certain duality of Yang-Mills moduli spaces for noncommutative flows. More precisely, we show that the Yang-Mills moduli spaces for smooth noncommutative flows are homeomorphic to those for the associated dual flows. This could be interpreted as no physical data is changed under dimension reduction of space time. The method itself is also applicable to noncommutative multi flows in principle. As an important application, we determine topologically the moduli spaces of the instanton bundles over noncommutative Euclidean 4-spaces with respect the canonical action of the space translations without using the ADHM-construction (cf:[5],[7],[8]).

§2. Noncommutative Yang-Mills Theory  
In this section, we review the noncommutative Yang-Mills theory due to Connes-Rieffel[3]. Let \((A, G, \alpha)\) be a C*-dynamical system, \(A^\infty\) the set of all smooth elements of \(A\) under \(\alpha\) and \(\alpha^\infty\) the restriction of \(\alpha\) to \(A^\infty\) where \(G\) is a connected Lie group. Then the system \((A^\infty, G, \alpha^\infty)\) becomes a noncommutative smooth dynamical system. In what follows, we only treat such a dynamical system, so that we notationally write it by \((A, G, \alpha)\). Let \(\delta\) be the differentiation map of \(\alpha\). Then it is a Lie homomorphism from the Lie algebra \(\mathfrak{G}\) of \(G\) to the Lie
algebra $\text{Der}(A)$ of all $^*$-derivations of $A$. Let $\Xi$ be a finitely generated projective right $A$-module. Then it has a Hermitian structure $<\cdot | \cdot>_A$ with the property that

$$
<\xi | \eta>^*_A = <\eta | \xi>_A, \quad <\xi | \eta a> = <\xi | \eta>_A a
$$

$(\xi, \eta \in \Xi, a \in A)$. Now we can define a noncommutative version of connections on vector bundles over manifolds in the following fashion: Let $\nabla$ be a linear map from $\Xi$ to $\Xi \otimes G^*$. Then it is called a connection of $\Xi$ if it satisfies

$$
\nabla_X(\xi a) = \nabla_X(\xi) a + \xi \delta_X(a)
$$

$(\xi \in \Xi, a \in A, X \in G)$. Moreover, a connection $\nabla$ is said to be compatible with respect to $<\cdot | \cdot>_A$ (or compatible) if it satisfies

$$
\delta_X(<\xi | \eta>) = <\nabla_X(\xi) | \eta> + <\xi | \nabla_X(\eta)>
$$

$(\xi \in \Xi, a \in A, X \in G)$. We denote by $CC(\Xi)$ the set of all compatible connections of $\Xi$. Then it is nonempty because it contains the so-called Grassmann connection $\nabla^0$, which is defined as follows: By assumption, $\Xi = P(A^n)$ for some $n \geq 1$ and a projection $P \in M_n(M(A))$ where $M(A)$ is the multiplier algebra of $A$. Then $\nabla^0 = P[\delta^n]$ becomes a compatible connection of $\Xi$, where $\delta^n$ is the differentiation map of the action $\alpha^n = \alpha \otimes id_n$ on $M_n(A)$. Now for any $\nabla \in CC(\Xi)$, there exists an element $\Omega_X \in \text{End}_A(\Xi)$ such that

$$
\nabla_X = \nabla^0_X + \Omega_X
$$

$(X \in G)$, where $E = \text{End}_A(\Xi)$ is the set of all $A$-endomorphisms of $\Xi$. Since $\nabla$ and $\nabla^0$ are compatible, then $\Omega_X$ ($X \in G$) are all skew-adjoint. Given a $\nabla \in CC(\Xi)$, there exists a skew adjoint $E$-valued 2-form $\Theta_\nabla$ of $G$ such as

$$
\Theta_\nabla(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}
$$

$(X, Y \in G)$. It is called the curvature of $\nabla$ associated with $(A, G, \alpha)$. Then $\nabla$ is said to be flat if there exists a 2-form $\omega$ of $G$ such that

$$
\Theta_\nabla(X,Y) = \omega(X,Y) \ Id_E
$$

$(X, Y \in G)$. We now assume an existence of a continuous $\alpha$-invariant faithful trace $\tau$ on $A$ as a noncommutative version of integrability of manifolds. Then there also exists a continuous faithful trace $\tilde{\tau}$ on $E$ such that

$$
\tilde{\tau}(<\xi | \eta>_E) = \tau(<\eta | \xi>_A)$$
\((\xi, \eta \in \Xi)\), where

\[
< \xi \mid \eta >_E (\zeta) = \xi < \eta \mid \zeta >_A
\]

\((\xi, \eta, \zeta \in \Xi)\). In fact, it is well defined because of the assumption of \(\Xi\). Using \(\tilde{\tau}\), we define a noncommutative version of the Yang-Mills functional on manifolds as follows:

\[
YM(\nabla) = -\tilde{\tau}(\{\Theta_\nabla\}^2)
\]

where

\[
\{\Theta_\nabla\}^2 = \sum_{i<j} \Theta_\nabla(X_i \wedge X_j)^2 \in E
\]

for an orthonormal basis \(\{X_i\}_i\) of \(\mathcal{G}\) with respect to the Killing form. Since \(\Theta_\nabla(X_i \wedge X_j)\) are all skew adjoint, then \(\{\Theta_\nabla\}^2\) has negative spectra only. Therefore \(YM(\nabla) \geq 0\) for all \(\nabla \in CC(\Xi)\). Moreover, it is independent of the choice of a hermitian structure \(<\cdot \mid \cdot>_A\) on \(\Xi\). Now let \(U(E)\) be the set of all unitaries of \(E\). It is called the gauge group of \(\Xi\). For any \(u \in U(E)\), we define the gauge transformation \(\gamma_u\) on \(CC(\Xi)\) by

\[
(\gamma_u(\nabla))_X(\xi) = u(\nabla_X)(u^*\xi)
\]

\((u \in U(E), X \in \mathcal{G}, \xi \in \Xi)\). Then \(\gamma\) calls the gauge action of \(U(E)\) on \(CC(\Xi)\). The Yang-Mills functional \(YM\) is \(\gamma\)-invariant, namely

\[
YM(\gamma_u(\nabla)) = YM(\nabla)
\]

\((u \in U(E), \nabla \in CC(\Xi))\). We then consider the first variational problem of \(YM\), namely find a \(\nabla \in CC(\Xi)\) such that

\[
\frac{d}{dt}(YM(\nabla_t)) \bigg|_{t=0} = 0
\]

for any smooth path \(\nabla_t \in CC(\Xi)\) \((|t| < \epsilon)\) with \(\nabla_0 = \nabla\), which is called a Yang-Mills connection of \(\Xi\) with respect to the system \((A, G, \alpha, \tau)\). Let \(MC(\Xi)\) be the set of all Yang-Mills connections of \(\Xi\) with respect to \((A, G, \alpha, \tau)\). Then the orbit space \(\mathcal{M}^{(A,G,\alpha,\tau)}(\Xi)\) of \(MC(\Xi)\) by the gauge action \(\gamma\) of \(U(E)\) is called the moduli space of the Yang-Mills connections of \(\Xi\) with respect to the system \((A, G, \alpha, \tau)\). We then state the following theorem due to Connes-Rieffel[3] which is quite powerful to construct a Yang-Mills connection:

**Theorem 2.1([3])** Let \((A, G, \alpha)\) be a \(C^\infty\)-dynamical system and \(\tau\) be a faithful \(\alpha\)-invariant continuous trace on \(A\). Let \(\Xi\) be a
finitely generated projective right $A$-module. If $G$ is an abelian connected Lie group, then $\nabla \in MC(\Xi)$ if and only if it is flat for any $\nabla \in CC(\Xi)$.

§3. Dual Yang-Mills Moduli spaces  In this section, we only take Frechet flows (or multi-flows) as a special case of $C^\infty$-dynamical systems. According to Elliott-Natsume-Nest[4], let $(A, R, \alpha)$ be a Frechet *-flow in the sense that

1. $(A, \{\| \cdot \|_n\}_{n \geq 1})$ is a Frechet *-algebra (which is dense in a $C^*$-algebra),
2. $t \mapsto \alpha_t(a)$ is $C^\infty$-class with respect to $\| \cdot \|_n$ ($n \geq 1$),
3. For any $m, k \geq 1$, there exist $n, j \geq 1$ and $C > 0$ such that

$$\left\| \frac{d^k}{dt^k} \alpha_t(a) \right\|_m < C(1 + t^2)^{j/2} \|a\|_n \quad (a \in A, \ t \in \mathbb{R})$$

In what follows, we state Frechet *-flows by $F^*$-flows. Typical are the following three examples as $F^*$-flows:

Examples 3.1 Let $S(\mathbb{R})$ be the abelian $F^*$-algebra of all complex valued rapidly decreasing smooth functions on $\mathbb{R}$ and $\lambda$ the shift action of $\mathbb{R}$ on $S(\mathbb{R})$. Then the triplet $(S(\mathbb{R}), \mathbb{R}, \lambda)$ is a $F^*$-flow.

Example 3.2 Let $K^\infty(\mathbb{R})$ be the $F^*$-algebra consisting of all compact operators on $L^2(\mathbb{R})$ with their integral kernels in $S(\mathbb{R}^2)$, and $Ad(\lambda)$ the adjoint action of $\mathbb{R}$ on $K^\infty(\mathbb{R})$. Then the triplet $(K^\infty(\mathbb{R}), \mathbb{R}, Ad(\lambda))$ is a $F^*$-flow.

Example 3.3 Let $\mathbb{R}^2_\theta$ be the $F^*$-algebra $S(\mathbb{R}^2)$ with Moyal product $*_\theta$ ($\theta \in \mathbb{R}$), and $\tilde{\theta}$ the dual action of the canonical action $\theta$ on $S(\mathbb{R})$. Then the triplet $(\mathbb{R}^2_\theta, \mathbb{R}, \tilde{\theta})$ is a $F^*$-flow.

Now let $(A, R, \alpha)$ be a $F^*$-flow with a continuous $\alpha$-invariant faithful trace $\tau$, and let $S(\mathbb{R}, A)$ be the $F^*$-algebra consisting of all $A$-valued rapidly decreasing smooth functions on $\mathbb{R}$ with its seminorms $\| \cdot \|_{m,n}$ given by

$$\|x\|_{m,n} = \sup_{t \in \mathbb{R}} (1 + t^2)^{m/2} \left\| \frac{d^n}{dt^n} x(t) \right\|_m \quad (x \in S(\mathbb{R}, A)).$$

Moreover it has the following product and involution:

1. $(x*_\alpha y)(t) = \int_{\mathbb{R}} x(s) \alpha_s(y(t-s)) ds$, (2) $x^*(t) = \alpha_t(x(-t))$
\((x, y \in S(\mathbb{R}, A))\). Then we call \(S(\mathbb{R}, A)\) the \(F^*\)-crossed product of \(A\) by the action \(\alpha\) of \(\mathbb{R}\), which is written by \(A \rtimes_\alpha \mathbb{R}\). In fact, the three examples cited above are isomorphic to \(\mathbb{C} \rtimes_\iota \mathbb{R}, S(\mathbb{R}) \rtimes_\lambda \mathbb{R}\) and \(S(\mathbb{R}) \rtimes_\theta \mathbb{R}\) respectively. Then we define two actions \(\widehat{\alpha}, \tilde{\alpha}\) of \(\mathbb{R}\) on \(A \rtimes_\alpha \mathbb{R}\) given by

\[
\widehat{\alpha}_s(x)(t) = e^{2\pi i st} x(t), \quad \tilde{\alpha}_s(x)(t) = \alpha_s(x(t)), \quad (i = \sqrt{-1})
\]

\((x \in A \rtimes_\alpha \mathbb{R}, s, t \in \mathbb{R})\). The triplets \((A \rtimes_\alpha \mathbb{R}, \mathbb{R}, \widehat{\alpha})\) and \((A \rtimes_\alpha \mathbb{R}, \mathbb{R}, \tilde{\alpha})\) become \(F^*\)-flows. The former is called to be the dual \(F^*\)-flow of \((A, \mathbb{R}, \alpha)\). Then the same duality holds as in the case of \(C^*\)-crossed products in the following:

**Theorem 3.4** ([4]) Given a \(F^*\)-flow \((A, \mathbb{R}, \alpha)\), its double dual \(F^*\)-flow \((A \rtimes_\alpha \mathbb{R} \rtimes_\hat{\alpha} \mathbb{R}, \mathbb{R}, \hat{\alpha})\) is isomorphic to the \(F^*\)-flow \((A \otimes \mathcal{K}^\infty(\mathbb{R}), \mathbb{R}, \alpha \otimes Ad(\lambda))\).

In fact, the equivariant isomorphism \(\Psi^0_\alpha : A \rtimes_\alpha \mathbb{R} \rtimes_\hat{\alpha} \mathbb{R} \mapsto A \otimes \mathcal{K}^\infty(\mathbb{R})\) is given by

\[
\Psi^0_\alpha(x)(t, s) = \int_\mathbb{R} e^{2\pi i rs} \alpha_{-t}(x(t - s, r)) \, dr
\]

\((x \in A \rtimes_\alpha \mathbb{R} \rtimes_\hat{\alpha} \mathbb{R}, t, s \in \mathbb{R})\). Then the inverse isomorphism \((\Psi^0_\alpha)^{-1}\) of \(\Psi^0_\alpha\) is given by

\[
(\Psi^0_\alpha)^{-1}(x)(t, s) = \int_\mathbb{R} e^{2\pi i (t-r)s} \alpha_r(x(r, r - t)) \, dr
\]

\((x \in A \otimes \mathcal{K}^\infty(\mathbb{R}), t, s \in \mathbb{R})\). Now let \((A \rtimes_\alpha \mathbb{R}, \mathbb{R}, \widehat{\alpha})\) be the dual \(F^*\)-flow of \((A, \mathbb{R}, \alpha)\). If there exists a continuous faithful \(\alpha\)-invariant trace \(\tau\) on \(A\), then so does it for the \(F^*\)-flow \((\hat{A}, \mathbb{R}, \hat{\alpha})\) given by

\[
\hat{\tau}(x) = \tau(x(0)) \quad (x \in \hat{A})
\]

where \(\hat{A} = A \rtimes_\alpha \mathbb{R}\). Then \(\hat{\tau}\) is called the dual trace of \(\tau\). Then we consider the Yang-Mills moduli spaces for such dual systems. Namely, let \(\Xi\) be a finitely generated projective right \(A\)-module and \(\hat{\Xi} = \Xi \otimes_A \hat{A}\). Then it becomes a finitely generated projective right \(\hat{A}\)-module. Indeed, the action of \(\hat{A}\) on \(\hat{\Xi}\) is given by

\[
(\xi x)(t) = \int_\mathbb{R} \xi(s) \alpha_{s}(x(t - s)) \, ds
\]

\((\xi \in \hat{\Xi}, x \in \hat{A})\). On the other hand, the action \(\tilde{\alpha}\) is implemented by an unitary multiplier flow on \(\hat{A}\), namely there exists a strictly
continuous unitary flow $\tilde{u}$ of the multiplier algebra $\widehat{A}$ of $\widehat{A}$ such that $\tilde{\alpha}_t = Ad(\tilde{u}_t)$ on $\widehat{A}$. Then it follows that $\tilde{\tau}$ is $\tilde{\alpha}$-invariant. Since the action $\tilde{\alpha}$ commutes with $\tilde{\alpha}$, we can define the action $\tilde{\alpha}$ of $\hat{A}$ on $\hat{A}$ by $\tilde{\alpha} \circ \tilde{\alpha}$ which makes $\tilde{\tau}$ invariant. Then we propose another Yang-Mills moduli space $\mathcal{M}(\hat{A}, \mathbb{R}, \hat{A})$ of $\hat{A}$ with respect to the $F^*$-flow $(\hat{A}, \mathbb{R}, \hat{A})$ and the dual trace $\tilde{\tau}$, which is called the dual Yang-Mills moduli space of $\mathcal{M}(\hat{A}, \mathbb{R}, \hat{A})$.

§4. Main result In this section, we prove the following theorem, which means physically that in quantum field theory, all physical data are invariant under dimension reduction:

Theorem 4.1 [Duality] Let $(\hat{A}, \mathbb{R}, \alpha)$ be a $F^*$-flow with a continuous $\alpha$-invariant faithful trace $\tau$ and let $\Xi$ be a finitely generated projective right $\hat{A}$-module. Then there exist a $F^*$-flow $(\hat{A}, \mathbb{R}, \hat{\tau})$ with a dual trace $\hat{\tau}$ of $\tau$ and a finitely generated projective right $\hat{A}$-module $\hat{\Xi}$ whose Yang-Mills moduli space $\mathcal{M}(\hat{A}, \mathbb{R}, \hat{\tau})(\hat{\Xi})$ is homeomorphic to $\mathcal{M}(\hat{A}, \mathbb{R}, \hat{\alpha})(\hat{\Xi})$.

Applying Theorems 3.4 and 4.1, we have the following corollary:

Corollary 4.2 [Dimension Reduction] Let $(\hat{A}, \mathbb{R}, \hat{\alpha})$ be the $F^*$-flow cited in Theorem 4.1 and $\beta$ a smooth flow on $\hat{A}$ commuting with $\hat{\alpha}$. Suppose there exists a continuous faithful $\beta$-invariant trace $\tau$, then given a finitely generated projective right $\hat{A}$-module $\Xi$, there exists a $F^*$-flow $(A, \mathbb{R}, \beta_A)$ with a continuous faithful $\beta_A$-invariant trace $\tau_A$ and a finitely generated projective $A$-module $\Xi_A$ such that $\mathcal{M}(\hat{A}, \mathbb{R}, \beta, \tau)(\Xi)$ is homeomorphic to $\mathcal{M}(A, \mathbb{R}, \beta_A, \tau_A)(\Xi_A)$.

Proof of Theorem 4.1: By the assumption of $\Xi$, there exist a natural number $n$ and a projection $P \in M_n(\mathcal{M}(A))$ such that $\Xi = P(A^n)$, where $\mathcal{M}(A)$ is the multiplier algebra of $A$. Let us take a Hermitian structure $< \cdot | \cdot >_A$ on $\Xi$ by

$$< \xi | \eta >_A = \sum_{j=1}^n \xi_j^* \eta_j$$

($\xi, \eta \in \Xi$). Then if $\nabla^0$ is the Grassmann connection of $\Xi$, then it belongs to $CC(\Xi)$. Moreover it follows from Theorem 2.1 that $\nabla^0 \in MC(\Xi)$. Now for any $\nabla \in MC(\Xi)$ and $X \in \text{Lie}(\mathbb{R})$, there exists a skew adjoint element $\Omega_X \in E$ such that

$$\nabla_X = \nabla_X^0 + \Omega_X.$$
As \( \Xi = P(A^n) \), it follows that \( \widehat{\Xi} = \widehat{P}(\widehat{\Lambda}^n) \) where 
\[
\widehat{P} = P \otimes I_A \in M_n(M(\Lambda)) .
\]
Then we know that 
\[
\text{End}_\Lambda(\widehat{\Xi}) = \widehat{P} M_n(\widehat{\Lambda}) \widehat{P} ,
\]
which is denoted by \( \widehat{E} \). From now on, we want to define a mapping from \( M(A,R,\alpha,\tau)(\Xi) \) into \( M(\widehat{A},R,\pi,\bar{\tau})(\widehat{\Xi}) \) in the following way: Since \( E = \text{End}_A(\Xi) \) is no longer \( \alpha^n \)-invariant in general, it follows using the same idea in Connes[1] that there exists a \( F^* \)-flow \((M_n(\Lambda, \alpha^n R, \pi^n ), (M_n(\Lambda, \pi^n R, \bar{\tau}))) \) such that 
\[
(1) \quad \beta_t(Pa) = P \beta_t(a) \quad (a \in M_n(A), \ t \in \mathbb{R}),
\]
\[
(2) \quad (M_n(\Lambda, \alpha^n R, \pi^n )) \text{ is outer equivalent to } (M_n(A, \beta R, \bar{\beta})).
\]
By [1], let \( \iota_u \) be the equivariant isomorphism from \((M_n(\Lambda, \alpha^n R, \pi^n ), \iota_u) \) onto \((M_n(A, \beta R, \bar{\beta}), \iota_u) \) such that 
\[
\iota_u \circ \alpha^n \circ \iota_u^{-1} = \bar{\beta} .
\]
Then we have the following lemma which would be applied later:

**Lemma 4.3([1])** The next two statements holds:

1. There is an equivariant isomorphism \( \iota_u \) from 
\[
(M_n(A, \alpha^n, \alpha^n R, \pi^n )) \text{ onto } (M_n(A, \beta R, \bar{\beta})).
\]
2. There exists a unitary multiplier \( W \) of \( M_n(A) \otimes K(\mathbb{R}) \) such that 
\[
\text{Ad}(W) \circ \Psi^0_\beta \circ \iota_u = \Psi^0_{\alpha^n},
\]
where \( \Psi^0_{\cdot} \) are the equivariant isomorphisms as in Theorem 3.4 associated with \( \cdot \), and 
\[
\widetilde{\iota_u}(a)(s) = \iota_u\{a(s)\}
\]
for all 
\[
a \in \mathcal{S}((\mathbb{R}, M_n(A) \times_{\alpha^n} R) .
\]
Since \( M_n(\Xi) = (P \otimes I_n)([M_n(A)]^n) \), it is a finitely generated projective \( M_n(A) \)-module. Let \( d\bar{\beta} \) be the infinitesimal generator of \( \bar{\beta} \). Now for any \( \nabla \in MC(M_n(\Lambda, \beta R, \pi^n ))(M_n(\Xi)) \), there exists a skew adjoint element \( \Omega \in E_n = \text{End}_{M_n(A)}(M_n(\Xi)) \) such that 
\[
\nabla = (P \otimes I_n)d\beta^n + \Omega .
\]
Let $\hat{E}_n = \text{End}_{M_n(A)}(M_n(\Xi))$ be the set of all $E_n$-valued rapidly decreasing smooth functions on $\mathbb{R}$. Then it becomes a $\mathbb{F}^*$-algebra with respect to the $\beta^n$-twisted convolution product. By definition, we see that

$$\hat{E}_n = \text{End}_{M_n(A)}(M_n(\Xi)),$$

where

$$\widehat{M_n(A)} = M_n(A) \ltimes \mathbb{R} , \quad \widehat{M_n(\Xi)} = M_n(\Xi) \otimes_{M_n(A)} M_n(\widehat{A}).$$

Let us define the element $\hat{\Omega} \in \widehat{E}_n$ by

$$\hat{\Omega}(\xi)(t) = \Omega(\xi(t))$$

$(\xi \in \widehat{M_n(\Xi)}, t \in \mathbb{R})$ In fact, we check that

$$\hat{\Omega}(\xi \otimes f)(t) = \Omega(\xi(t))$$

$(\xi \in \widehat{M_n(\Xi)}, f \in M_n(\widehat{A})).$ As $\beta^n$ is used as the restriction of the natural extension of $\beta$ of $M_n^2(A)$ to $E_n$, then it follows from the definition that

$$\hat{E}_n = E_n \ltimes \beta^n \mathbb{R}.$$

Then we obtain that

$$\hat{\Omega} \in E_n \ltimes \beta^n \mathbb{R}.$$

We then have the following lemma:

**Lemma 4.4** \( \hat{\Omega} \in \widehat{E}_n \) is skew adjoint.

**Proof.** Since $\Omega \in E_n$ is skew adjoint, we compute that

$$\langle \hat{\Omega}(\xi \otimes f) | \eta \otimes g \rangle_{M_n(A)}$$

$$= \sum_{j=1}^{n} \hat{\Omega}(\xi \otimes f)_j^*(\eta \otimes g)_j$$

where $\hat{\Omega}(\xi \otimes f)_j, (\eta \otimes g)_j \in \widehat{M_n(\widehat{A})}$. Then it is easy to check that

$$\hat{\Omega}(\xi \otimes f)_j = \Omega(\xi)_j \otimes f$$
, using which we deduce that

\[
\langle \hat{\Omega}(\xi \otimes f) \mid \eta \otimes g \rangle_{\overline{M_n(\mathcal{A})}}(t)
= \sum_{j=1}^{n} \int_{\mathbb{R}} \beta_s(\Omega(\xi)_{j}^* \eta_j) \overline{f(-s)}g(t - s) \, ds
= \int_{\mathbb{R}} \beta_s \{ \langle \Omega(\xi) \mid \eta \rangle_{\overline{M_n(\mathcal{A})}} \} \overline{f(-s)}g(t - s) \, ds
\]

As \( \Omega \) is skew adjoint, it follows that

\[
\langle \Omega(\xi) \mid \eta \rangle_{\overline{M_n(\mathcal{A})}} = -\langle \xi \mid \Omega(\eta) \rangle_{\overline{M_n(\mathcal{A})}}.
\]

Then we obtain that

\[
\langle \hat{\Omega}(\xi \otimes f) \mid \eta \otimes g \rangle_{\overline{M_n(\mathcal{A})}}(t) = -\langle \xi \otimes f \mid \hat{\Omega}(\eta \otimes g) \rangle_{\overline{M_n(\mathcal{A})}}(t)
\]

(\( \xi, \eta \in M_n(\Xi), \ f, g \in S(\mathbb{R}) \)). This implies the conclusion. Q.E.D.

Let \( d\hat{\beta}^n \) and \( d\tilde{\beta}^n \) be the infinitesimal generators of the dual action \( \hat{\beta}^n \) and the canonical extension \( \tilde{\beta}^n \) of \( \beta^n \) to \( \hat{\mathcal{E}}_n \) respectively. Since \( \hat{\beta}^n \) commutes with \( \tilde{\beta}^n \), then \( d\hat{\beta}^n + d\tilde{\beta}^n \) is the infinitesimal generator of \( \overline{\beta^n} \). Then we have the following lemma:

**Lemma 4.5**

\[
(\hat{P} \otimes I_n)(d\overline{\beta^n}) + \hat{\Omega} \in MC(\overline{M_n(\mathcal{A})}, \mathbb{R}, \overline{\beta^n})(\overline{M_n(\Xi)}),
\]

Proof. Since \( \hat{P} \otimes I_n)(d\overline{\beta^n}) \) is the Grassmann connection of \( \overline{M_n(\Xi)} \) with respect to the action \( \overline{\beta} \), Theorem 2.1 implies that it belongs to

\[
MC(\overline{M_n(\mathcal{A})}, \mathbb{R}, \overline{\beta^n})(\overline{M_n(\Xi)}).
\]

As \( \hat{\Omega} \in \overline{\mathcal{E}}_n \) is skew adjoint by Lemma 4.4, the conclusion follows from Theorem 2.1. Q.E.D.

By the Lemma 4.5, we then define a mapping

\[
\Phi_\beta : \mathcal{M}(\overline{M_n(\mathcal{A})}, \mathbb{R}, \overline{\beta^n})(\overline{M_n(\Xi)}) \longrightarrow \mathcal{M}(\overline{M_n(\mathcal{A})}, \mathbb{R}, \overline{\beta^n})(\overline{M_n(\Xi)}),
\]

by the following fashion

\[
\Phi_\beta([\nabla]_{U(\mathcal{E}_n)}) = [(\hat{P} \otimes I_n)(d\overline{\beta^n}) + \hat{\Omega}]_{U(\overline{\mathcal{E}}_n)},
\]
where $[\nabla]_{\{*\}}$ means the equivalence class of $\nabla$ under the gauge action of $\{*\}$.

We then check the following lemma:

**Lemma 4.6** $\Phi_{\beta}$ is well defined.

**Proof.** Let $\Omega, \Omega_1$ be two skew-adjoint elements in $E_n$, and suppose $u(\nabla_\beta^0 + \Omega)u^* = \nabla_\beta^0 + \Omega_1$ for some unitary $u \in E_n$, then

$$\Omega_1 = u\nabla_\beta^0 u^* - \nabla_\beta^0 + u\Omega u^*$$

We have to show that $\nabla_\beta^0 + \hat{\Omega}$ is equal to $\nabla_\beta^0 + \hat{\Omega}_1$ up to the gauge automorphisms of $U(E_n)$. Now we compute that

$$\left(\nabla_\beta^0 + \hat{\Omega}_1\right)(\xi)(t) = \left(\hat{P} \otimes I_n\right)(\delta)(\xi)(t) + (u\nabla_\beta^0 u^* - \nabla_\beta^0 + u\Omega u^*)\{\xi(t)\}$$

$$= 2\pi i t \xi(t) + u\hat{\Omega}u^*(\xi)(t) + (u\nabla_\beta^0 u^* - \nabla_\beta^0)\{\xi(t)\}$$

$(\xi \in \hat{M}_n(\Xi))$ where $\hat{u}(\xi)(t) = u\{\xi(t)\}$. Since we see that

$$(u\nabla_\beta^0 u^* - \nabla_\beta^0)\{\xi(t)\} = (u\nabla_\beta^0 u^* - \nabla_\beta^0)(\xi)(t),$$

and $\hat{u}\nabla_\beta^0 u^*(\xi)(t) = 2\pi i t \xi(t)$, then we obtain that

$$\nabla_\beta^0 + \nabla_\beta^0 + \hat{\Omega}_1)(\xi)(t) = \gamma_\Xi(\nabla_\beta^0 + \nabla_\beta^0 + \hat{\Omega})(\xi)(t)$$

$(\xi \in \hat{M}_n(\Xi))$. As we know that

$$\nabla_\beta^0 + \nabla_\beta^0 = \nabla_\beta^0$$

, then the conclusion follows. Q.E.D.

By definition, $\tilde{\beta}$ is a weakly inner action of $\hat{M}_n(A)$ implemented by a unitary multiplier flow $\mu$ of $\hat{M}_n(A)$ faithfully acting on $L^2(\mathbb{R}, H_{\tau^n})$ for the Hilbert space $L^2(M_n(A), \tau^n)$. Actually, as $\tilde{\beta}_t(a)(s) = \beta_t(a(s))$ for all $a \in \hat{M}_n(A)$, $s, t \in \mathbb{R}$, then $\mu_t(a)(s) = a(s - t)$ for all $a \in \mathcal{S}(\mathbb{R}, M_n(A))$, $s, t \in \mathbb{R}$ Then we have the following lemma:
Lemma 4.7 The $F^\ast$-system $(\hat{M}_n(A), \mathbb{R}, \hat{\beta})$ is inner conjugate to the $F^\ast$-system $(\overline{M}_n(A), \mathbb{R}, \overline{\beta})$. Then it implies that the $F^\ast$-system 

$(\overline{M}_n(A) \rtimes_{\overline{\beta}} \mathbb{R}, \mathbb{R}, \overline{\beta})$ is isomorphic to the system $(\hat{M}_n(A), \mathbb{R}, \hat{\beta})$

via the map:

$$\Lambda(x)(t) = \mu_{-1}x(t)$$

for all $x \in \hat{M}_n(A) \rtimes_{\hat{\beta}} \mathbb{R}$, where 

$$\hat{M}_n(A) = \hat{M}_n(A) \rtimes_{\hat{\beta}} \mathbb{R}.$$ 

By Lemmas 4.5 and 4.7, we deduce the following lemma:

Lemma 4.8 Let $\Lambda_\beta : M(\hat{M}_n(A) \rtimes_{\hat{\beta}} \mathbb{R}, \mathbb{R}, \hat{\beta}) \to M_n(\hat{\beta}(\hat{M}_n(A)))$ defined by

$$\Lambda_\beta([\nabla]_{U(\hat{M}_n(A) \rtimes_{\hat{\beta}} \mathbb{R})}) = [(\Lambda^n \circ \nabla \circ (\Lambda^n)^{-1}]_{U(\hat{M}_n(A))}$$

where $\nu$ is the unitary multiplier flow of $M_n(A)$ implementing $\hat{\beta}$ on $\hat{M}_n(A)$, and $\Lambda^n$ is the isomorphism from $\text{End}_{\hat{M}_n(A) \rtimes_{\hat{\beta}} \mathbb{R}}(\hat{M}_n(A))$ onto $\text{End}_{\hat{M}_n(A) \rtimes_{\hat{\beta}} \mathbb{R}}(\hat{M}_n(A))$ induced by $\Lambda$. Then it implies that $\Lambda_\beta$ is a homeomorphism.

Proof. By the definition of $\Lambda$, we check that

$$\Lambda \circ \hat{\beta} \circ \Lambda^{-1} = \hat{\beta}, \quad \Lambda \circ \hat{\beta} \circ \Lambda^{-1} = \hat{\beta}.$$ 

By the same reason as for $\hat{\beta}$, there exists a unitary multiplier flow $\nu$ of $\hat{M}_n(A)$ such that $\hat{\beta} = \text{Ad}(\nu)$ on $\hat{M}_n(A)$. The rest is easily seen. Q.E.D.

Let $\Psi_0^{\beta}$ be the isomorphism from the $F^\ast$-system $(\hat{M}_n(A), \mathbb{R}, \hat{\beta})$ onto the $F^\ast$-system $(M_n(A) \otimes \mathcal{K}^\infty(\mathbb{R}), \mathbb{R}, \beta \otimes \text{Ad}(\lambda))$ defined by

$$\Psi_0^{\beta}(x)(t, s) = \int_\mathbb{R} e^{2\pi i s r} \beta_{-1}(x(t - s, r)) \, dr,$$ 

where $\beta$ is the unitary multiplier flow implementing $\hat{\beta}$ on $\hat{M}_n(A)$. The rest is easily seen. Q.E.D.
and
\[
(\Psi_0^\beta)^{-1}(x)(t, s) = \int_{\mathbb{R}} e^{2\pi i (t-r)s} \beta_r(x(r, r - t)) \, dr
\]
\[(x \in M_n(A) \otimes K^\infty(\mathbb{R}), t, s \in \mathbb{R}).\] By definition, we compute that
\[
\Psi_0^\beta \circ \text{Ad}(\nu_p) \circ (\Psi_0^\beta)^{-1}(x)(t, s)
\]
\[= \int_{\mathbb{R}} e^{2\pi i (rs + p(t-s))} \beta_{-1}((\Psi_0^\beta)^{-1}(x)(t, s, r)) \, dr,
\]
which is equal to
\[
\int \int e^{2\pi i (p(t-s) + (t-r')r')} \beta_{-t}(x(r', r' - t + s)) \, dr' \, dr.
\]
\[(x \in M_n(A) \otimes K^\infty(\mathbb{R}), t, s \in \mathbb{R}).\] Therefore it follows that
\[
\Psi_0^\beta \circ \text{Ad}(\nu_p) \circ (\Psi_0^\beta)^{-1}(x)(t, s) = e^{2\pi ip(t-s)} x(t, s),
\]
\[(x \in M_n(A) \otimes K^\infty(\mathbb{R}), p, t, s \in \mathbb{R}),\] which implies that there exists a unitary multiplier flow \(\nu_\beta\) of \(K^\infty(\mathbb{R})\) with the property that
\[
\Psi_0^\beta \circ \text{Ad}(\nu_p) \circ (\Psi_0^\beta)^{-1} = \text{Ad}(I \otimes (\nu_p)_p) \quad (p \in \mathbb{R})
\]
on \(M_n(A) \otimes K^\infty(\mathbb{R}).\) Then it turns out that
\[
\Psi_0^\beta \circ (\hat{\beta} \circ \text{Ad}(\nu)) \circ (\Psi_0^\beta)^{-1} = \beta \otimes \text{Ad}(\lambda \circ \nu_\beta).
\]
Let \(\Psi_\beta\) be the map: \(\mathcal{M}(\widehat{M_n(A), \mathbb{R}, \hat{\beta} \otimes \text{Ad}(\nu), \hat{\gamma}}) \rightarrow (M_n(\Xi) \otimes K^\infty(\mathbb{R}))\)
induced by the equivariant isomorphism \(\Psi_0^\beta\). Then we also show the following lemma by the same way as Lemma 4.8:

**Lemma 4.9** \(\Psi_\beta\) is a homeomorphism induced by the equivariant isomorphism \(\Psi_0^\beta\).

Let us now consider the following map:
\[
\Pi_\beta : \mathcal{M}(\widehat{M_n(A) \otimes K^\infty(\mathbb{R}), \mathbb{R}, \beta \otimes \text{Ad}(\lambda \nu_\beta), \tau^n \otimes \text{Tr}}) \rightarrow \mathcal{M}(\widehat{M_n(\Xi) \otimes K^\infty(\mathbb{R}), \mathbb{R}, \beta, \tau^n})
\]
defined by the natural one induced from the map \(\Pi : \)
\[
M_n(A) \otimes K^\infty(\mathbb{R}) \rightarrow M_n(A)
\]
given by $\Pi : x \mapsto (I \otimes e)x(I \otimes e)$, where $e$ is a rank one projection of $\mathcal{K}^\infty(\mathbb{R})$, $Tr$ the canonical trace of $\mathcal{K}^\infty(\mathbb{R})$. Now let $\nabla \in \text{MC}(M_n(\Xi) \otimes \mathcal{K}^\infty(\mathbb{R}))$ and put

$$\nabla_e(\xi) = (I_n \otimes e)\nabla(\xi \otimes e)$$

($\xi \in M_n(\Xi)$), where $I_n$ is the identity of $E_n$. Then $\nabla_e$ is well defined and independent of the choice of $e$ up to the gauge equivalence because of the existence of a unitary multiplier of $\mathcal{K}^\infty(\mathbb{R})$ which sends $e$ to another rank one projection. Then we see that given any $u \in U(E_n \otimes \mathcal{K}^\infty(\mathbb{R}))$,

$$(\gamma_u(\nabla))_e = \gamma_{u_e}(\nabla_e),$$

where $u_e = (I_n \otimes e)u(I_n \otimes e) \in U(E_n)$. Here we define a map $\Pi_\beta$ by

$$\Pi_\beta([\nabla]_{U(E_n \otimes \mathcal{K}^\infty(\mathbb{R}))}) = [\nabla_e]_{U(E_n)} = [\Pi^n \circ \nabla \circ (\Pi^n)^{-1}]_{U(E_n)}.$$

Then it is well defined and independent of the choice of $e$. Moreover, we have the following lemma:

Lemma 4.10

$$\Pi_\beta : \mathcal{M}(M_n(A) \otimes \mathcal{K}^\infty(\mathbb{R}), \mathbb{R}, \beta \otimes \text{Ad}(\lambda \circ \nu_\beta)) \rightarrow \mathcal{M}(M_n(\Xi) \otimes \mathcal{K}^\infty(\mathbb{R}))$$

is a homeomorphism.

Proof. Let us define the mapping $\Pi^{-1}_\beta$ by

$$\Pi^{-1}_\beta([\nabla]) = [\nabla \otimes I_{\mathcal{K}^\infty(\mathbb{R})}].$$

Then it is easily seen that both $\Pi^{-1}_\beta \circ \Pi_\beta$ and $\Pi_\beta \circ \Pi^{-1}_\beta$ are identities. Moreover if $[\nabla'] \rightarrow [\nabla]$ with respect to $< \cdot \mid \cdot >_{M_n(A)}$, then it follows from the definition that there exists a unitary net $\{u_\iota\}$ (by choosing a subnet) of $E_n$ such that $\gamma_{u_\iota}(\nabla') \rightarrow \nabla$, which implies that $[(\nabla')_e] \rightarrow [\nabla_e]$, so that $\Pi_\beta$ is continuous and so is also $\Pi^{-1}_\beta$ by the same way. Q.E.D.

Let $\nabla^0$ be the Grassmann connection of $\cdot$. Then we easily check the following lemma:

Lemma 4.11

$$\Psi^0_{\beta^n} \circ \nabla^0_{\beta \circ \text{Ad}(\nu)} \circ (\Psi^0_{\beta^n})^{-1} = \nabla^0_{\beta \circ \text{Ad}(\lambda \circ \nu_\beta)}.$$  

Proof. It follows from Lemma 4.9 that

$$\Psi^0_{\beta} \circ (\tilde{\gamma} \circ \text{Ad}(\nu) \circ (\Psi^0_{\beta})^{-1} = \beta \otimes \text{Ad}(\lambda \circ \nu_\beta).$$
Since
\[ \widehat{M_n}(\Xi) = S(\mathbb{R}^2, M_n(\Xi)), \]
and
\[ \nabla^0_\beta = \widehat{P} d \widehat{\beta^n}, \]
where
\[ \widehat{P} = P \otimes I_n \otimes I_{S(\mathbb{R}^2)}, \]
and \( d \widehat{\beta^n} \) is the infinitesimal generator of \( \widehat{\beta^n} \), then this implies the conclusion. Q.E.D.

Moreover, we need the following lemma which is directly shown:

**Lemma 4.12** There exists a
\[ U \in U(\text{End}_{M_n(A) \otimes K^\infty(\mathbb{R})}(M_n(\Xi) \otimes S(\mathbb{R}^2))) \]
such that
\[ \Psi^0_{\beta^n} \circ \Lambda^n \circ \widehat{\Omega} \circ (\Psi^0_{\beta^n} \circ \Lambda^n)^{-1} = \gamma_U(\Omega \otimes \text{Id}) \]
Actually, \( U \) is defined as \( U(\xi)(s, t) = u^n_{s, t} \xi(s, t) \) for all \( \xi \in M_n(\Xi) \otimes S(\mathbb{R}^2) \), where \( u^n \) is the unitary flow to \( E_n \) induced by \( \beta^n \).

**Proof.** We know by definition that
\[ \Psi^0_{\beta^n}(\xi)(s, t) = \int_{\mathbb{R}} e^{2\pi i rt} u^n_{-s} \xi(s - t, r) \, dr \]
(\( \xi \in \widehat{M_n}(\Xi) \)), and
\[ (\Psi^0_{\beta^n})^{-1}(\xi)(s, t) = \int_{\mathbb{R}} e^{2\pi i (s-r)t} u^n_{r} \xi(r, r - s) \, dr \]
(\( \xi \in M_n(\Xi) \otimes S(\mathbb{R}^2) \)). Then we compute that
\[ \Psi^0_{\beta^n} \circ \Lambda^n \circ \widehat{\Omega}(\xi)(s, t) = \int e^{2\pi i t} u^n_{-s}(\Lambda^n \circ \widehat{\Omega}(\xi)(s - t, r)) \, dr \]
(\( \xi \in \widehat{M_n}(\Xi) \)). Then as we know that
\[ \Lambda^n \circ \widehat{\Omega}(\xi)(s - t, r) = \nu^n_{s, t} \Omega(\xi(r)) \}
Therefore we have that
\[ \Psi^0_{\beta n} \circ \Lambda^n \circ \tilde{\Omega}(\xi)(s, t) \]
\[ = \int e^{2\pi ir} u_{s-r} \Omega \{(\xi)(s - t + r, r)\} \, dr \]

\((\xi \in \hat{M}_n(\Xi))\). Replacing \(\xi\) by \((\Psi^0_{\beta n} \circ \Lambda^n)^{-1}(\xi)\), we obtain that
\[ (\Psi^0_{\beta n} \circ \Lambda^n)^{-1}(\xi)(s - t + r, r) \]
\[ = \int_{\mathbb{R}} e^{2\pi i(s-t-r)r} u_{r'}(r', r' - s + t) \, dr' \]

\((\xi \in \hat{M}_n(\Xi) \otimes \mathcal{S}(\mathbb{R}^2))\). Combining the argument discussed above, we deduce that
\[ (\Psi^0_{\beta n} \circ \Lambda^n) \circ \tilde{\Omega} \circ (\Psi^0_{\beta n} \circ \Lambda^n)^{-1}(\xi)(s, t) \]
\[ = \int \int e^{2\pi i(s-r)r} u_{r'} \Omega \{u_{r'}(r', r' - s + t)\} \, dr' \, dr' \]

which is equal to
\[ u_{s-r} \Omega \{u_{r}(s, t)\} = \gamma_U(\Omega \otimes Id)(\xi)(s, t) \]
where \(U(\xi)(s, t) = u_{s-r} \xi (s, t)\). This implies the conclusion. Q.E.D.

We next show the following lemma which seems to be essential to prove our main theorem:

**Lemma 4.13**
\[ \Pi_\beta \circ \Psi_\beta \circ \Lambda_\beta \circ \Phi_\beta \circ \Phi_\beta = Id \]
on \(\mathcal{M}(\hat{M}_n(A, \mathbb{R}, \beta, \tau^n)(\hat{M}_n(\Xi)))\), where \(\Psi_\beta\) is the homeomorphism on the moduli space induced from the isomorphism \(\Psi^0_{\beta n}\) given in Lemma 4.9.

**Proof.** Let \(\nabla = \nabla^0_{\beta n} + \Omega \in \mathcal{M}(\hat{M}_n(A, \mathbb{R}, \beta, \tau^n))(\hat{M}_n(\Xi))\). Then we know by Lemmas 4.11 and 4.12 that there exist a
\[ U \in U(\text{End}_{\hat{M}_n(A) \otimes K^\infty(\mathbb{R})}(\hat{M}_n(\Xi) \otimes \mathcal{S}(\mathbb{R}^2))) \]
such that
\[ (\Psi^0_{\beta n} \circ \Lambda^n) \circ \nabla^0_{\beta} \circ (\Psi^0_{\beta n} \circ \Lambda^n)^{-1} = \nabla^0_{\beta \otimes \text{Ad}(\Lambda \nu_{\beta})}, \]
and
\[ (\Psi^0_{\beta n} \circ \Lambda^n) \circ \tilde{\Omega} \circ (\Psi^0_{\beta n} \circ \Lambda^n)^{-1} = \gamma_U(\Omega \otimes Id) \].
By Lemmas 4.11 and 4.12, we obtain that
\[ \Psi_\beta \circ \Lambda_\beta \circ \Phi_\beta \circ \Phi_\beta ([\nabla]_{U(E_n)}) = \left( [\nabla^0_{\beta_n} \circ \text{Ad}(\lambda \circ \nu)] + \gamma U(\Omega \otimes Id) \right)_{U(E_n) \otimes K^\infty(\mathbb{R})} \]
then we see that
\[ \left( \nabla^0_{\beta_n} \circ \text{Ad}(\lambda \circ \nu) \right)_e = \nabla^0_{\beta_n} \otimes e , \]
where \( e \) is a rank one projection of \( K^\infty(\mathbb{R}) \). In fact, we check that
\[ (P \otimes I_n \otimes e)d ((\beta^n) \otimes \text{Ad}(\lambda \circ \nu))^{n}(\xi \otimes e) = (P \otimes I_n \otimes e)\{d ((\beta^n) \otimes I) + d(I \otimes \text{Ad}(\lambda \circ \nu)^n)\}(\xi \otimes e) . \]
(\( \xi \in M_n(\Xi) \)). Then it follows that
\[ e\text{Ad}(\lambda_t \circ (\nu)^n)(e) = e. \]
for all \( t \in \mathbb{R} \). Therefore, it follows that
\[ (P \otimes I_n \otimes e)d (I \otimes \text{Ad}(\lambda \circ \nu)^n)(\xi \otimes e) = 0 \]
(\( \xi \in M_n(\Xi) \)). On the other hand, we know that
\[ \gamma U(\Omega \otimes Id)_e = \gamma U_e(\Omega) \otimes e , \]
where \( U_e \) belongs to \( U(E_n) \) with \( U_e \otimes e = (I \otimes e)U(I \otimes e) \). By the definition of \( U \), \( U_e \) commutes with \( \beta^n \). Consequently, it follows that
\[ [\nabla^0_{\beta_n} \otimes e + \gamma U_e(\Omega) \otimes e]_{U(E_n) \otimes e} = [\nabla^0_{\beta_n} + \Omega]_{U(E_n)}, \]
which deduce the conclusion. Q.E.D.

Applying Lemma 4.13 to the system \((\widehat{M_n(A)}, \mathbb{R}, \overline{\beta})\), we obtain the following corollary:

Corollary 4.14
\[ \Pi_{\overline{\beta}} \circ \Psi_{\overline{\beta}} \circ \Lambda_{\overline{\beta}} \circ \Phi_{\overline{\beta}} \circ \Phi_{\overline{\beta}} = \text{Id} \]
holds on
\[ \mathcal{M}^{(\widehat{M_n(A)}, \mathbb{R}, \overline{\beta}, \overline{\tau^n})}(\widehat{M_n(\Xi)}) , \]
where the map \( \Pi_{\overline{\beta}} \) is a homeomorphism from
\[ \mathcal{M}(\widehat{M}_n(A) \otimes \mathcal{K}^\infty(\mathbb{R}), \mathcal{R} \otimes \text{Ad}(\lambda \otimes \tau_n), \tau_n \otimes \text{Tr}) (\widehat{M}_n(\Xi) \otimes \mathcal{K}^\infty(\mathbb{R})) \]

onto

\[ \mathcal{M}(\widehat{M}_n(A), \mathcal{R}, \tau_n) (\widehat{M}_n(\Xi)) \]
induced by the mapping:

\[ \widehat{M}_n(A) \otimes \mathcal{K}^\infty(\mathbb{R}) \hookrightarrow \widehat{M}_n(A) \]
given by \( x \mapsto (I \otimes e)x(I \otimes e) \).

As we have seen in Lemma 4.8, 

\[ \Lambda \circ \widetilde{\beta} \circ \Lambda^{-1} = \tilde{\beta} \circ \text{Ad}(\nu) \]

Then the relation between \( \Phi_{\tilde{\beta} \circ \text{Ad}(\nu)} \) and \( \Phi_{\beta} \) is as follows:

Lemma 4.15

\[ \Phi_{\tilde{\beta} \circ \text{Ad}(\nu)} = \Lambda_{\beta} \circ (\Pi_{\beta} \circ \Psi_{\beta} \circ \Lambda_{\beta})^{-1} \circ \Phi_{\beta} \circ \Pi_{\beta} \circ \Psi_{\beta} \]

holds on

\[ \mathcal{M}(\widehat{M}_n(A), \mathcal{R}, \tilde{\beta} \circ \text{Ad}(\nu), \tau_n) (\widehat{M}_n(\Xi)) \].

Proof. By Lemmas 4.8 \sim 4.13 and Corollary 4.14, we have that

\[ (\Pi_{\beta} \circ \Psi_{\beta} \circ \Lambda_{\beta})^{-1} = \Phi_{\beta} \circ \Phi_{\beta} \].

By the equality written just above this Lemma, we see that

\[ \Lambda_{\beta} \circ \Phi_{\beta} \circ (\Lambda_{\beta})^{-1} = \Phi_{\beta} \circ \Phi_{\beta} \].

By Lemma 4.13 and Corollary 4.14, \( \Phi_{\beta} \) is bijective. By Lemma 4.12, we have that

\[ \Phi_{\beta}^{-1} = \Phi_{\beta} \circ \Pi_{\beta} \circ \Psi_{\beta} , \]

which implies the conclusion. Q.E.D.

Summing up the above argument, we obtain the following lemma:

Lemma 4.16 \( \Phi_{\beta} \) and \( \Phi_{\overline{\beta}} \) are homeomorphisms.

Proof. By Lemmas 4.8 \sim 4.15, \( \Phi_{\beta} \) and \( \Phi_{\overline{\beta}} \) are bijective and bicontinuous, which completes the proof. Q.E.D.

We then define a map \((\Phi_{\alpha^n})^{-1}\):

\[ \mathcal{M}(M_n(A) \times \alpha^n, \mathcal{R}, \alpha^n, \tau_n) (\widehat{M}_n(\Xi)) \]
\[ \mapsto \mathcal{M}^{(M_n(A), \mathbb{R}, \alpha^n, \tau^n)}(M_n(\Xi)) \]
defined by
\[ \Pi_{\alpha^n} \circ \Psi_{\alpha^n} \circ (\tilde{t})^{-1} \circ \Lambda_{\beta} \circ \Phi_{\beta} \circ t , \]
where \( t \) is the extended map on \( \mathcal{M}^{(M_n(A), \mathbb{R}, \alpha^n, \tau^n)}(M_n(\Xi)) \) induced by \( t_u \) in Lemma 4.3, and so is \( \tilde{t} \) on \( \mathcal{M}^{(M_n(A), \mathbb{R}, \alpha^n, \tau^n)}(M_n(\Xi)) \) induced by \( \tilde{t}_u \), because the map \( t_u \) intertwines \( \alpha^n \) and \( \beta \), and the map \( \tilde{t}_u \) intertwines \( \alpha^n \) and \( \tilde{\beta} \).

**Lemma 4.17** \( (\Phi_{\alpha^n})^{-1} \) is a homeomorphism.

**Proof.** By Lemma 4.3 (2), there exists a unitary multiplier \( W \) of \( M_n(A) \otimes K_\infty(\mathbb{R}) \) such that
\[ \text{Ad}(W) \circ \Psi_0^0 \circ \tilde{t}_u = \Psi_{\alpha^n}^0 , \]
which implies that \( \Psi_0^0 \circ \tilde{t}_u = \Psi_{\alpha^n}^0 \). Moreover, \( t \) and \( \tilde{t} \) are homeomorphisms. Then it follows from Lemma 4.16 that \( (\Phi_{\alpha^n})^{-1} \) is a homeomorphism. Q.E.D.

By Lemmas 4.17, we deduce the following corollary:

**Corollary 4.18** \( \Phi_{\alpha^n} \) is a homeomorphism:
\[ \mathcal{M}^{(M_n(A), \mathbb{R}, \alpha^n, \tau^n)}(M_n(\Xi)) \mapsto \mathcal{M}^{(M_n(A) \times \mathbb{R}, \mathbb{R}, \alpha^n, \tau^n)}(M_n(\Xi)) . \]

Finally, using \( \Phi_{\alpha^n} \), we define the map \( \Phi_{\alpha} : \mathcal{M}^{(A, \mathbb{R}, \alpha, \tau)}(\Xi) \mapsto \mathcal{M}^{(A, \mathbb{R}, \alpha, \tau)}(\hat{\Xi}) \) by
\[ \Phi_{\alpha} = \hat{\Pi}_n \circ \Phi_{\alpha^n} \circ (\Pi_n)^{-1} , \]
where \( \Pi_n \) is a homeomorphism:
\[ \mathcal{M}^{(A \otimes M_n(C), \mathbb{R}, \alpha \otimes \text{Ad}(\lambda \circ \nu_\alpha), \tau \otimes Tr_n)}(\Xi \otimes M_n(C)) \mapsto \mathcal{M}^{(A, \mathbb{R}, \alpha, \tau)}(\Xi) . \]

Finally, we show the following main lemma:

**Lemma 4.19** \( \Phi_{\alpha} \) is a homeomorphism.

**Proof.** In Lemma 4.10, replacing \( (M_n(A) \otimes K_\infty(\mathbb{R}), \beta \otimes \text{Ad}(\lambda \circ \nu_\beta)) \) and \( M_n(\Xi) \otimes K_\infty(\mathbb{R}) \) by \( A \otimes M_n((C)) \), \( \alpha^n \) and \( \Xi \otimes M_n(C) \) respectively, we deduce that both \( \hat{\Pi}_n \) and \( (\Pi_n)^{-1} \) are homeomorphisms. Then it implies the conclusion by Corollary 4.18. Q.E.D.
Summing up all the arguments discussed above, we obtain the main result of Theorem 4.1.

In what follows, we compute the moduli spaces of some concrete examples by means of Theorem 4.1:

Example 4.20
\[ \mathcal{M}(^\infty K(\mathbb{R}), \mathbb{R}, Ad(\lambda), Tr) (^\infty K(\mathbb{R})) \approx \mathcal{M}(S(\mathbb{R}), \mathbb{R}, \lambda) (S(\mathbb{R})) \approx \mathcal{M}(^C \mathbb{R}, \text{Id}, 1) (\mathbb{C}) \approx \mathbb{R}, \]
where \( \approx \) means a symbol of homeomorphism.

Examples 4.21  Given a \( \theta \in \mathbb{R} \), let us take the Moyal product \( \star_\theta \) on \( S(\mathbb{R}^2) \). Then \( (S(\mathbb{R}^2), \star_\theta) \) becomes a \( F^* \)-algebra, which is denoted by \( \mathbb{R}^2 \). Since \( \mathbb{R}^2 \) is isomorphic to \( S(\mathbb{R}) \times _\theta \mathbb{R} \), then it follows from Theorem 4.1 that
\[ \mathcal{M}(^2 \mathbb{R}, \mathbb{R}, \tau_\theta) (\mathbb{R}^2) \approx \mathbb{R}, \]
where \( \tau_\theta \) is the canonical trace of \( \mathbb{R}^2 \).

Even though changing \( F^* \)-flows into \( F^* \)-multiflows, the same result as Theorem 4.1 is obtained by applying it repeatedly. Actually, we now take a \( F^* \)-dynamical system \( (A, \mathbb{R}^2, \alpha) \) with a faithful \( \alpha \)-invariant trace \( \tau \) on \( A \). Let \( \alpha^i = \alpha_{(i,0)} \) if \( i = 1 \), \( = \alpha_{(0,i)} \) if \( i = 2 \). By definition, \( \alpha^1 \) commutes with \( \alpha^2 \) and \( \alpha = \alpha^1 \circ \alpha^2 \). Let us take \( \tilde{A}^1 = A \times _{\alpha^1} \mathbb{R} \). Then we know that \( \tilde{A} = \tilde{A}^1 \times _{\alpha^2} \mathbb{R} \). We now apply Theorem 4.1 to the \( F^* \)-dynamical system \( (A, \alpha^1, \mathbb{R}), \Xi \) and \( \tau \). Then it follows that
\[ \mathcal{M}(A, \mathbb{R}, \alpha^1, \tau) (\Xi) \approx \mathcal{M}(\tilde{A}^1, \mathbb{R}, \tilde{\alpha}^1, \tilde{\tau}^1) (\tilde{\Xi}^1), \]
where \( \tilde{\tau}^1 \) is the dual trace of \( \tau \) on \( \tilde{A}^1 \) and \( \tilde{\Xi}^1 \). Since \( \alpha^1 \) commutes with \( \alpha^2 \), it follows from definition that \( \tilde{\alpha}^1 \) commutes with \( \tilde{\alpha}^2 \). Then by the same way as the proof of Theorem 4.1 again, we obtain that
\[ \mathcal{M}(A, \mathbb{R}, \alpha^2, \tau) (\Xi) \approx \mathcal{M}(\tilde{A}^2, \mathbb{R}^2, \tilde{\alpha}^2, \tilde{\tau}^2) (\tilde{\Xi}^2), \]
In fact, the method used in the main proof of Theorem 4.1 takes place only on the \( F^* \)-dynamical systems \( (A, \mathbb{R}^2, \alpha) \) and \( (\tilde{A}^1, \mathbb{R}^2, \tilde{\alpha}^1, \tilde{\alpha}^2) \), so that there is no affection from the action \( \alpha^2 \). Actually, the moduli map from the former one to the latter one is defined by
\[ [d\alpha^n + \Omega_\alpha] \longrightarrow [d(\tilde{\alpha}^1 \circ \tilde{\alpha}^2)^n + \tilde{\Omega}_\alpha]. \]
Then the next step is to be shown that
\[ M(\hat{A}, R^n, \alpha, \tau, \hat{\tau}) (\Xi) \approx M(\hat{A}, R^n, \alpha, \tau, \hat{\tau}) (\hat{\Xi}) , \]
where \( \hat{\Xi} = \Xi \otimes_A R \) and \( \hat{\tau} \) is the dual trace of \( \tau \) on \( A \times_A R^2 \).
Under the isomorphism \( \Phi \) from \( \hat{A} \times_A R^2 \) onto \( A \times_A R^2 \), we then check that the action \( \alpha^2 \circ \hat{\alpha} \) is equivalent to \( \alpha \). Actually the former action is transformed into \( \alpha^2 \circ \hat{\alpha} \) of \( R \times_A R^2 \) on \( A \times_A R^2 \) by taking \( Ad(\Phi) \), which is nothing but the latter action. We then obtain the following theorem:

**Theorem 4.22** Let \((A, R^n, \alpha) (n \geq 2)\) be a \( F^* \)-multiflow with a faithful continuous \( \alpha \)-invariant trace \( \tau \), and \( \Xi \) a finitely generated projective \( A \)-module. Then there exist a \( F^* \)-multiflow \((\hat{A}, R^n, \overline{\alpha})\) with a dual trace \( \hat{\tau} \), and a dual \( \hat{A} \)-module \( \hat{\Xi} \) such that
\[ M(A, R^n, \alpha, \tau, \hat{\tau}) (\Xi) \approx M(\hat{A}, R^n, \overline{\alpha}, \hat{\tau}) (\hat{\Xi}) . \]
where \( \hat{\Xi} = \Xi \otimes_A \hat{A} \), and \( \hat{\tau} \) is the dual trace of \( \tau \) on \( \hat{A} \).

**Remark.** In multiflow cases, the curvatures of Yang-Mills connections are non-zero 2-tensors in general although they always vanish in single flow cases.

Using the Theorem 4.22, the similar statement as Corollary 4.2 is obtained in the following:

**Corollary 4.23** Let \((A, R^n, \alpha) \) be a \( F^* \)-multiflow with a faithful continuous \( \alpha \)-invariant trace \( \tau \), and \( \Xi \) a finitely generated projective \( A \)-module. Then there exist a \( F^* \)-multiflow \((\hat{A}, R^n, \overline{\alpha})\) with the dual trace \( \hat{\tau} \). Suppose \((\hat{A}, R^n, \beta)\) is another \( F^* \)-multiflow such that
\[ \hat{\tau} \circ \beta = \overline{\alpha} \circ \beta \]
then given a finitely generated projective \( \hat{A} \)-module \( \hat{\Xi} \), there exist a \( F^* \)-multiflow \((A, R^n, \beta_A)\), a finitely generated projective \( A \)-module \( \Xi_A \) and a faithful \( \beta_A \)-invariant trace \( \tau_A \) of \( A \) such that
\[ M(\hat{A}, R^n, \beta, \hat{\tau}) (\Xi) \approx M(A, R^n, \beta_A, \tau_A) (\Xi_A) . \]

**§ 5. Application** In this section, we apply Theorem 4.22 and Corollary 4.23 to compute the moduli space in the case of the instanton bundles on the noncommutative Euclidean 4-space with respect to the canonical space translations without using the ADHM.
construction. Actually, it appears as a Higgs branch of the theory of D0-branes bound to D4-branes by the expectation value of the B-field as well as a regularized version of the target space of supersymmetric quantum mechanics arising in the light cone description of (2,0) superconformal theories in six dimensions, although its algebraic structure has already been established in the example 10.1 of [12] (cf. [7], [8]):

Let $\mathbb{R}^4_\theta$ be the noncommutative $\mathbb{R}^4$ for an antisymmetric 4x4 matrix $\theta = (\theta_{i,j})$, in other words, $\mathbb{R}^4_\theta$ is the $\mathbb{F}^*$-algebra generated by 4-selfadjoint elements $\{x_i\}_{i=1}^4$ with the property that

$$[x_i, x_j] = \theta_{i,j}$$

($i, j = 1, \cdots, 4$). In other words,

$$\mathbb{R}^4_\theta = \{ \int_{\mathbb{R}^4} f(t_1, t_2, t_3, t_4) x_1^{2\pi i t_1} x_2^{2\pi i t_2} x_3^{2\pi i t_3} x_4^{2\pi i t_4} dt_1 dt_2 dt_3 dt_4 \mid f \in S(\mathbb{R}^4) \}$$

as a $\mathbb{F}^*$-algebra, where $S(\mathbb{R}^4)$ is the set of all rapidly decreasing complex valued functions on $\mathbb{R}^4$. Let $x^i = \theta^{i,j} x_j$ ($i, j = 1, \cdots, 4$) where $(\theta^{i,j})$ is the inverse matrix of $(\theta_{i,j})$. Then the $\mathbb{F}^*$-algebra $\mathbb{R}^4_\theta$ depends essentially on one positive real number denoted by the same symbol $\theta$, which satisfy the following relation:

$$[z^*_i, z_i] = \theta, \ [z_i, z_j] = [z^*_i, z_j] = 0 \ (i, j = 0, 1, i \neq j)$$

where $z_0 = x^1 + \sqrt{-1} x^2, z_1 = x^3 + \sqrt{-1} x^4$ and $z^*_i$ are the conjugate operators of $z_i$. Let us consider the canonical action $\alpha$ of $\mathbb{R}^4$ on $\mathbb{R}^4_\theta$ defined by

$$\alpha_{t_i}(x_i) = x_i + t_i$$

($t_i \in \mathbb{R}, i = 1, \cdots, 4$). Then it is easily seen that the triplet $(\mathbb{R}^4_\theta, \mathbb{R}^4, \alpha)$ is a $\mathbb{F}^*$-dynamical system, and we easily see that

$$\alpha_{w_i}(z_i) = z_i + w_i$$

($w_i \in \mathbb{C}, i = 0, 1$). By (2), $\mathbb{R}^4_\theta$ is nothing but the $\mathbb{F}^*$-tensor product $A_0 \otimes A_1$ where $A_i$ are the $\mathbb{F}^*$-algebras generated by $z_i$ ($i = 0, 1$). We now check the algebraic structure of $A_i$. By (2), it follows from [8](cf. [11]) that there exist two Fock spaces $H_i$ such that
$z_t(\xi_n^i) = \sqrt{(n+1)\theta} \xi_{n+1}^i$, $z_{it}(\xi_n^i) = \sqrt{n\theta} \xi_{n-1}^i$,

where $\{\xi_n^i\}$ are complete orthonormal systems of $H_i$ with respect to the following inner product:

$$< f \mid g > = \sum (n+1)\theta f(n)\overline{g(n)}$$

for two $\mathbb{C}$-valued functions $f, g$ on $\mathbb{N}$ such that

$$\sum (n+1)\theta |f(n)|^2 < \infty, \quad \sum (n+1)\theta |g(n)|^2 < \infty.$$  

for $i = 0, 1$. We may assume that the $A_i$ act on $H_i$ irreducibly. Then it also follows from [11] that the $F^*$-algebras $A_i$ are isomorphic to the $F^*$-algebras $K^\infty(H_i)$ defined by

$$K^\infty(H_i) = \{ T \in K(H_i) \mid \{ \lambda_k \} \in S(\mathbb{N}) \}$$

where $\{\lambda_k\}$ are all eigen values of $T$ and $S(\mathbb{N})$ are the set of all sequences $\{c_n\}$ of $\mathbb{C}$ with $\sup_{n\geq 1} (1+|n|)^k|c_n| < \infty$ for all $k \geq 0$. Therefore, the $F^*$-algebra $\mathbb{R}_4^\alpha$ is isomorphic to $K^\infty(H_0 \otimes H_1)$. We then have the following proposition:

**Proposition 1 (cf:[12]).** If $\theta \neq 0$, then $\mathbb{R}_\theta^4$ is isomorphic to $K^\infty(L^2(\mathbb{C}^2))$ as a $F^*$-algebra.

By the above Proposition, $K^\infty(L^2(\mathbb{C}^2))$ is the $F^*$-crossed product $S(\mathbb{C}^2) \rtimes_\tau \mathbb{C}^2$ of $S(\mathbb{C}^2)$ by the shift action $\tau$ of $\mathbb{C}^2$. We then consider the action $\alpha$ defined before. By (4), it follows from [R] that $\alpha$ plays a role of the dual action of $\tau$. Then the $F^*$-crossed product $\mathbb{R}^4_\alpha$ of $\mathbb{R}_\theta^4$ by the action $\alpha$ of $\mathbb{R}^4$ is isomorphic to the $F^*$-crossed product $K^\infty(L^2(\mathbb{C}^2)) \rtimes_\tilde{\tau} \mathbb{C}^2$, where $\tilde{\tau}$ is the dual action of $\tau$. Then it is isomorphic to $S(\mathbb{C}^2) \otimes K^\infty(L^2(\mathbb{C}^2))$ as a $F^*$-algebra.

We now consider a finitely generated projective right $\mathbb{R}^4_\alpha$-module $\Xi$. Then there exist an integer $n \geq 1$ and a projection $P \in M_n(\mathcal{M}(\mathbb{R}^4_\alpha))$ such that $\Xi = P((\mathbb{R}_\alpha^4)^n)$. where $\mathcal{M}(\mathbb{R}^4_\alpha)$ is the $F^*$-algebra consisting of all bounded linear operators $T$ on $L^2(\mathbb{C}^2)$ whose kernel functions $T(\cdot, \cdot)$ are $\mathbb{C}$-valued bounded $C^\infty$-functions of $\mathbb{C}^2 \times \mathbb{C}^2$. Let us take the canonical faithful trace $\text{Tr}$ on $\mathbb{R}^4_\alpha$ because of Proposition 1. Then we consider the moduli space:

$$\mathcal{M}(K^\infty(L^2(\mathbb{C}^2)), \mathbb{C}^2, \alpha, \text{Tr}) (\Xi)$$

of $\Xi$ for $(K^\infty(L^2(\mathbb{C}^2)), \mathbb{C}^2, \alpha, \text{Tr})$.

We want to describe $P$ cited above as a precise fashion. Actually, we know that

$$K^\infty(L^2(\mathbb{C}^2)) \cong S(\mathbb{C}^2) \rtimes_\lambda \mathbb{C}^2$$
where \(\cong\) means isomorphism as a \(F^*\)-algebra. \(\lambda\) is the shift action of \(\mathbb{C}^2\) on \(S(\mathbb{C}^2)\). Then it follows that

\[
M_n(\mathcal{K}^\infty(L^2(\mathbb{C}^2))) \cong M_n(S(\mathbb{C}^2)) \rtimes_{\lambda^n} \mathbb{C}^2
\]

where

\[
\lambda^n_w(f)(w') = f(w' - w)
\]

for \(f \in M_n(S(\mathbb{C}^2)), w, w' \in \mathbb{C}^2\). Let \(\lambda^n\) be the action of \(\mathbb{C}^2\) on \(M_n(\mathcal{K}^\infty(L^2(\mathbb{C}^2)))\) associated with \(\lambda^n\) satisfying Theorem 2. It follows from Proposition 3 that

\[
\mathcal{M}^{(\mathbb{R}^4,\mathbb{R}^4,\alpha,T\gamma)}(\Xi) \approx \mathcal{M}^{(\mathcal{K}^\infty(L^2(\mathbb{C}^2)),\mathbb{C}^2,\alpha,T\gamma)}(\Xi_1)
\]

\[
\approx \mathcal{M}^{(S(\mathbb{C}^2) \rtimes_{\lambda} \mathbb{C}^2,\tilde{\lambda},\int_{\mathbb{C}^2} dz)}(\Xi_2),
\]

where

\[
\Xi_1 = P_1(\mathcal{K}^\infty(L^2(\mathbb{C}^2))^n), \quad \Xi_2 = P_2((S(\mathbb{C}^2) \rtimes_{\lambda} \mathbb{C}^2))^n)
\]

for the two projections \(P_j\) \((j = 1, 2)\) with the property that

\[
P_1 \in M_n(\mathcal{M}(\mathcal{K}^\infty(L^2(\mathbb{C}^2)))) \quad P_2 \in M_n(\mathcal{M}(S(\mathbb{C}^2)) \rtimes_{\lambda} \mathbb{C}^2)
\]

corresponding to \(\Xi\), where \(\mathcal{M}(S(\mathbb{C}^2))\) is the \(F^*\)-algebra consisting of all \(\mathbb{C}\)-valued bounded \(C^\infty\)-functions on \(\mathbb{C}^2\) and \(\lambda\) is the shift action of \(\mathbb{C}^2\) on \(\mathcal{M}(S(\mathbb{C}^2))\). By its definition, we know that

\[
\tilde{\lambda}_w = \hat{\lambda}_w \circ \tilde{\lambda}_w, \quad (w \in \mathbb{C}^2)
\]

where \(\tilde{\lambda}\) is the dual action of \(\lambda\) and

\[
\tilde{\lambda}_w(x)(w') = \lambda_w(x(w'))
\]

for all \(x \in S(\mathbb{C}^2) \rtimes_{\lambda} \mathbb{C}^2\) and \(w, w' \in \mathbb{C}^2\). Hence \(\tilde{\lambda}\) commutes with \(\lambda\), which implies by Theorem 2 that there exist a \(F^*\)-dynamical system \((S(\mathbb{C}^2), \mathbb{C}^2, \hat{\lambda}_{S(\mathbb{C}^2)}, \int_{\mathbb{C}^2} dz)\) and a finitely generated projective right \(A\)-module \((\Xi_2)_{S(\mathbb{C}^2)}\) such that

\[
\mathcal{M}^{(S(\mathbb{C}^2) \rtimes_{\lambda} \mathbb{C}^2,\mathbb{C}^2,\hat{\lambda},\int_{\mathbb{C}^2} dz)}(\Xi_2) \approx \mathcal{M}^{(S(\mathbb{C}^2),\mathbb{C}^2,\hat{\lambda}_{S(\mathbb{C}^2)},\int_{\mathbb{C}^2} dz)}((\Xi_2)_{S(\mathbb{C}^2)}).
\]

We know that there exist an integer \(m \geq 1\) and a projection \(Q \in M_m(\mathcal{M}(S(\mathbb{C}^2)))\) such that

\[
(\Xi_2)_{S(\mathbb{C}^2)} = Q(S(\mathbb{C}^2)^m).
\]
Moreover, it follows from the definition that the action $\tilde{\lambda}_{\mathbb{S}(\mathbb{C}^2)}$ is nothing but $\lambda$. We now determine the moduli space

$$\mathcal{M}^{(S(\mathbb{C}^2),\mathbb{C}^2,\lambda,f_{\mathbb{C}^2}dz)}(Q(S(\mathbb{C}^2)^m))$$

in what follows: Since $Q \in M_m(\mathcal{M}(S(\mathbb{C}^2)))$ and

$$M_m(S(\mathbb{C}^2)) \cong S(\mathbb{C}^2, M_m(\mathbb{C})),$$

then it also follows from Theorem 2 that there exist a finitely generated projective right $\mathbb{C}$-module $Q(S(\mathbb{C}^2)^m)$ such that

$$\mathcal{M}^{(S(\mathbb{C}^2),\mathbb{C}^2,\tilde{\lambda}_{\mathbb{S}(\mathbb{C}^2)}, f_{\mathbb{C}^2}dz)}(Q(S(\mathbb{C}^2)^m)) \cong \mathcal{M}^{(\mathbb{C},\mathbb{C}^2,\lambda,1)}(Q(S(\mathbb{C}^2)^m)).$$

Since $Q(S(\mathbb{C}^2)^m)$ is a finitely generated projective right $\mathbb{C}$-module, then its construction tells us that there exists a projection $R \in M_m(\mathbb{C})$ such that

$$Q(S(\mathbb{C}^2)^m) = R(\mathbb{C}^m).$$

Summing up the argument discussed above, we deduce that

$$\mathcal{M}^{(\mathbb{R}^4,\mathbb{R}^4,\mathfrak{r},\alpha,Tr)}(\Xi) \approx \mathcal{M}^{(\mathbb{C},\mathbb{C}^2,\mathfrak{r},1)}(R(\mathbb{C}^m)).$$

By the definition of the moduli space, we deduce that

$$\mathcal{M}^{(\mathbb{C},\mathbb{C}^2,\mathfrak{r},1)}(R(\mathbb{C}^m)) \cong \text{End}_\mathbb{C}(R(\mathbb{C}^m))_{sk}/U(\text{End}_\mathbb{C}(R(\mathbb{C}^m))),$$

where

$$\text{End}_\mathbb{C}(R(\mathbb{C}^m))_{sk} \text{ and } U(\text{End}_\mathbb{C}(R(\mathbb{C}^m))).$$

are the set of all skew adjoint and unitary elements in

$$\text{End}_\mathbb{C}(R(\mathbb{C}^m))$$

respectively. Since $\text{End}_\mathbb{C}(R(\mathbb{C}^m)) = M_k(\mathbb{C})$ for some natural number $k$ ($m \geq k$), it follows by using diagonalization that

$$\text{End}_\mathbb{C}(R(\mathbb{C}^m))_{sk}/U(\text{End}_\mathbb{C}(R(\mathbb{C}^m))) \approx \mathbb{R}^k,$$

which implies the following theorem:

**Theorem 2.** Let $\mathbb{R}^4_\theta$ be the deformation quantization of $\mathbb{R}^4$ with respect to a skew symmetric matrix $\theta$ and take the $F^*$-dynamical system $(\mathbb{R}^4_\theta, \mathbb{R}^4, \alpha)$ with a canonical faithful $\alpha$-invariant trace $Tr$ of
$\mathbb{R}^4_\theta$, where $\alpha$ is the translation action of $\mathbb{R}^4$ on $\mathbb{R}^4_\theta$. Suppose $\Xi$ is a finitely generated projective right $\mathbb{R}^4_\theta$-module, then there exists a natural number $k$ such that

$$\mathcal{M}^{(\mathbb{R}^4_\theta, \mathbb{R}^4, \alpha, \text{Tr})} (\Xi) \approx \mathbb{R}^k.$$

Remark. The above theorem only states the topological data of the moduli spaces of Yang-Mills connections. We would study their both differential and holomorphic structures in a forthcoming paper (cf:[7]).
Acknowledgement

I would like to express my sincere gratitude to Professor T. Natsume for his careful reading and many pieces of advice to my manuscript, to Dr. S. Satomi for his valuable comments, and to Ms. J. Takai for her constant encouragement.

References

[1] A. Connes: An Analogue of the Thom Isomorphism for Crossed Products of a C*-Algebra by an Action of \( \mathbb{R} \), Adv. Math. 39, (1981), 31-55.

[2] A. Connes, M.R. Douglas and A. Schwarz: Noncommutative Geometry and Matrix Theory: Compactification on Tori, JHEP, 9802 (1998) 003.

[3] A. Connes and M.A. Rieffel: Yang-Mills for noncommutative two tori, Contemp. Math. Oper. Alg. Math. Phys. 62, A.M.S. (1987), 237-266.

[4] G.A. Elliott, T. Natsume and R. Nest: Cyclic cohomology for one parameter smooth crossed products, Acta Math., 160 (1988), 285-305.

[5] K. Furuuchi: Instantons on Noncommutative \( \mathbb{R}^4 \) and Projection Operators. [arXiv:hep-th/9912047]

[6] H. Kawai: Constructive Formulation of String Theory, Math. Sci., 4 (2002), 41-48.

[7] H. Nakajima: Resolutions of Moduli Spaces of Ideal Instantons on \( \mathbb{R}^4 \). World Scientific. 129-136 (1994).

[8] N. Nekrasov and A. Schwarz: Instantons on Noncommutative \( \mathbb{R}^4 \) and (2, 0) Superconformal Six Dimensional Theory. Commun. Math. Phy. 198, 689-703 (1998).

[9] Y. Ohkawa, Matrix Models of M-Theory, Math. Sci., 4 (2002), 35-40.
[10] N.Ohta, Duality of Superstring Theory and M-Theory, Math.Sci.,4(2002),16-22.

[11] C.R.Putnam: Commutation Properties of Hilbert Space Operators and Related Topics. Springer-Verlag (1967).

[12] M.A.Rieffel: Deformation Quantization for Actions of \( \mathbb{R}^d \). Memoires AMS.506(1993).

[13] H.Takai: Yang-Mills Theory for Noncommutative Flows. arXiv:math-ph/0403026

[14] H.Takai: Yang-Mills Theory for Noncommutative Flows, Addendum. arXiv:math-ph/0407038