Periodic orbit bifurcations and scattering time delay fluctuations

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We study fluctuations of the Wigner time delay for open (scattering) systems which exhibit mixed dynamics in the classical limit. It is shown that in the semiclassical limit the time delay fluctuations have a distribution that differs markedly from those which describe fully chaotic (or strongly disordered) systems: their moments have a power law dependence on a semiclassical parameter, with exponents that are rational fractions. These exponents are obtained from bifurcating periodic orbits trapped in the system. They are universal in situations where sufficiently long orbits contribute. We illustrate the influence of bifurcations on the time delay numerically using an open quantum map.

§1. Introduction

In closed systems, for example closed billiards, quantum energy levels and eigenfunctions are related to classical periodic orbits in the semiclassical limit via trace formulae. In the case of the energy level spectrum, the trace formula was first developed by Gutzwiller.\textsuperscript{1) Combining trace formulae with statistical information about the classical dynamics (e.g. ergodicity) underpins the semiclassical theory of quantum fluctuation statistics in closed systems. For example, this approach forms the basis of attempts to understand universality and the connection with random matrix theory in the spectral statistics of fully chaotic systems, and likewise the connection with Poisson statistics in regular systems.

In closed systems that have a mixed (i.e. partly regular and partly chaotic) classical limit, some quantum fluctuation statistics are well described semiclassically by weighted averages over the regular and chaotic components. However others, for example moments of the fluctuations of the spectral counting function around the Weyl mean, or moments of the fluctuations of eigenfunctions about the quantum ergodic limit, are dominated in the semiclassical limit by classical periodic orbits that are close to bifurcation, where their contribution to the trace formulae is enhanced by a power of Planck’s constant that depends on the nature of the bifurcation in question.\textsuperscript{2–5) When long orbits contribute it is assumed that one can average over all of the generic bifurcations. There is then a competition as to which dominates the semiclassical moment asymptotics: essentially the more complicated a bifurcation is the larger is its contribution but the smaller is its range of influence. In the cases studied so far\textsuperscript{3,4) it happens that for any given moment there is a bifurcation which dominates the competition. Different moments are dominated by different
bifurcations. The result is that each moment scales semiclassically as a power of Planck’s constant. These exponents are universal. They take the form of rational fractions whose values are given by simple formulae. This differs markedly from the behaviour one sees in either fully chaotic or fully regular systems.

Open (scattering) systems also exhibit universal quantum fluctuation statistics, for example in the conductance and the time delay, that depend on whether the classical dynamics is regular or chaotic.\(^6\) In chaotic scattering systems the fluctuation statistics coincide with those of random matrix theory. The question then arises as to what happens in open systems in which the classical dynamics is mixed: is there any analogue of the bifurcation-dominated fluctuation statistics found in closed systems? One might initially think not, because the classical trajectories that underlie the semiclassical expression for the \(S\)-matrix are scattering orbits, not periodic orbits. However, it turns out that some quantities related to scattering can be re-expressed directly in terms of the periodic orbits trapped inside the scattering region. For example, if the scattering system is a billiard with holes cut in the perimeter, these quantities can be expressed either in terms of orbits entering and exiting through the holes, or in terms of periodic orbits that never hit the holes. Thus in mixed open systems bifurcating periodic orbits can semiclassically dominate the related fluctuation statistics in exactly the same way as in closed systems, giving rise to new classes of universal scaling exponents that cannot be described within a random matrix model. One example where this was recently shown to be the case is the conductance fluctuations in antidot lattices.\(^7\)

Our purpose here is to point out that the Wigner time delay is another general example. Specifically, we argue that in mixed open systems fluctuations in the time delay may be dominated by classical periodic orbit bifurcations, and that when long orbits contribute the fluctuation moments scale semiclassically with universal exponents whose values, again rational fractions, are related directly to those calculated previously. We illustrate our theory with numerical computations for a class of quantum maps.

\section{The Wigner time delay}

The concept of time delay in quantum scattering was introduced by Eisenbud\(^8\) and Wigner\(^9\) in the context of one-channel spherical wave scattering. Later, Smith\(^10\) extended the notion to the \(M\)-channel case by introducing the lifetime matrix

\[ Q_{ij} = -i\hbar \sum_{c=1}^{M} S_{ic}^\dagger \frac{d}{dE} S_{cj}(E), \]

where \(S\) is the standard scattering matrix and the sum runs over all \(M\) open channels denoted by \(c\). The time delay is defined to be the average of the eigenvalues of \(Q\):

\[ \tau(E) = -\frac{i\hbar}{M} \text{tr} S^\dagger \frac{d}{dE} S = -\frac{i\hbar}{M} \frac{d}{dE} \log \det S. \]

It can be interpreted as the typical time spent by a scattered particle in the interaction region.
The time delay turns out to be very closely related to the density of states in a closed system. This allows one to use the well developed semiclassical apparatus for the density of states of closed systems to unravel features of open ones. The connection was identified (independently) by Friedel and Lifshitz: the average time delay \( \langle \tau(E) \rangle \) is related to the difference between the level density of an interacting Hamiltonian \( H \) with respect to a free or reference Hamiltonian \( H_0 \).

Friedel’s formalism was used by Balian and Bloch to derive a semiclassical expression for the time delay which is closely related to the Gutzwiller trace formula for the density of states. This splits into a smooth part \( \tau(E) \) and a fluctuating part \( \tau^f(E) \) determined by periodic orbits:

\[
\tau(E) = \frac{2\pi\hbar}{M} \rho(E) \approx \frac{2\pi\hbar}{M} \left( \tau(E) + \rho^f(E) \right),
\]

where \( \rho \) is the renormalized density of states. It has a natural interpretation in the context of inside-outside duality.

The smooth term on the right-hand side of (3) can be interpreted as the mean density of scattering resonances. It represents the mean time spent in the scattering system. The fluctuating term is given by

\[
\rho^f(E) \approx \frac{1}{\pi\hbar} \operatorname{Re} \sum_{\gamma} \sum_{m=1}^{\infty} \frac{T_{\gamma}}{\sqrt{\det(M_{\gamma} - I)}} \exp \left( \frac{imS_{\gamma}}{\hbar} - \frac{i\pi}{2} m\mu_{\gamma} \right),
\]

where the sum runs over the periodic orbits \( \gamma \) which are trapped in the repeller and their repetitions. \( T_{\gamma}, S_{\gamma}, M_{\gamma}, \) and \( \mu_{\gamma} \) are respectively period, action, stability matrix and Maslov index of the orbit \( \gamma \).

The semiclassical analysis of the time delay is often based on the relation to the trapped periodic orbits. For example, in fully chaotic open systems, statistical properties of these orbits can be invoked to justify the use of random matrix theory to model the fluctuations. It is worth pointing out that all the emphasis so far has been placed on systems in which the classical limit is fully chaotic. This is not however the typical situation in actual experiments, where real cavities or soft potentials lead to mixed dynamics. Our aim here is to go beyond this idealization by incorporating the contributions from orbit bifurcations. This is important, because the contributions are, in certain regimes, actually the dominant ones.

§3. The time delay for quantum maps

We illustrate the general theory of the last section using quantum maps, since in this case the relation between the time delay and the trapped periodic orbits can be derived straightforwardly.

We start with a map that acts on a phase space corresponding to the unit torus. Its quantization is defined by a unitary time evolution operator \( U \) with finite dimension \( N \). The energy dependence of Hamiltonian systems is simulated by including a phase factor \( \tilde{U} = e^{i\varepsilon U} \) that depends on the quasi-energy \( \varepsilon \). The map is then opened up by removing vertical strips from phase space. These play the role of
holes in the boundary of open billiards. The corresponding scattering matrix $S$ is an $M$-dimensional matrix, where $M < N$ is the total number of position states in the opening. It is the unitary matrix for the transition from the opening onto itself after an arbitrary number of iterations of the internal map $^{19)}$

$$S(\epsilon) = \hat{U}_{OO} + \sum_{n=0}^{\infty} \hat{U}_{OI} \tilde{U}_{II}^{n} \hat{U}_{IO} .$$

(5)

The quantities $\tilde{U}_{kl}$ with $k, l \in \{O, I\}$ are restrictions of the evolution operator to the inside and/or outside. They are defined by $\tilde{U}_{kl} = P_{k} \tilde{U} P_{l}^{T}$, where $P_{O}$ and $P_{I}$ are the projection matrices onto the opening and the interior, respectively. $P_{O}$ and $P_{I}$ have dimensions $M \times N$ and $(N - M) \times N$, respectively, and $P_{O}^{T}$ and $P_{I}^{T}$ are the corresponding transpose matrices. The projection operators satisfy $P_{O}^{T} P_{O} + P_{I}^{T} P_{I} = I_{N}$, $P_{O} P_{O}^{T} = I_{M}$, and $P_{I} P_{I}^{T} = I_{N-M}$ where $I_{L}$ denotes the $L \times L$ unitary matrix.

The sum over $n$ in (5) can be performed and the $S$-matrix written as

$$S(\epsilon) = P_{O} \frac{1}{I_{N} - \hat{U} P_{I}^{T} P_{I}} \hat{U} P_{O}^{T} .$$

(6)

Planck’s constant for quantum torus maps is given by $\hbar = (2\pi N)^{-1}$, and so the time delay has the form

$$\tau(\epsilon) = -\frac{i}{2\pi N M} \frac{d}{d\epsilon} \log |S| .$$

(7)

In order to derive a formula for the time delay in terms of the periodic orbits of the map, it is useful to apply an identity for determinants due to Jacobi. Let $A$ be an $N \times N$ matrix given in terms of the auxiliary block matrices $B, C, D$ and $E$, and $A^{-1}$ its inverse matrix given in terms of $W, X, Y$ and $Z$:

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix} .$$

(8)

$B$ and $W$ are assumed to be square matrices with the same dimension. Jacobi’s determinant identity then states that $\det B = \det Z \det A$.

Let us identify $A$ with the $N \times N$ matrix $[I_{N} - \hat{U} P_{I}^{T} P_{I}]^{-1} \hat{U}$ and the subblock $B$ with $P_{O} A P_{O}^{T} = S$. Then $Z$ follows as $P_{I} A^{-1} P_{I}^{T}$ and the Jacobi identity leads to

$$\det S = \det([I_{N} - \hat{U} P_{I}^{T} P_{I}]^{-1} \hat{U}) \det(P_{I} \hat{U}^{-1} [I_{N} - \hat{U} P_{I}^{T} P_{I}] P_{I}^{T}).$$

(9)

A little elementary linear algebra then gives

$$\det S = \det \hat{U} \frac{\det(\hat{U} I_{II}^{-1} I_{N-M})}{\det(I_{N-M} - \hat{U} I_{II})} .$$

(10)

For the evaluation of the derivative of the logarithm of $\det S$ we need

$$\frac{d}{d\epsilon} \log \det \hat{U} = \frac{d}{d\epsilon} \log e^{iN\epsilon} \det U = iN ,$$

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and

\[
\frac{d}{d\varepsilon} \log \det(I_{N-M} - \tilde{U}_{II}) = -i \sum_{n=1}^{\infty} e^{in\varepsilon} \text{tr} U_{II}^n. \tag{12}
\]

This leads to the final result

\[
\tau(\varepsilon) = \frac{1}{MN} \left( \frac{N}{2\pi} + \frac{1}{\pi} \text{Re} \sum_{n=1}^{\infty} e^{in\varepsilon} \text{tr} U_{II}^n \right), \tag{13}
\]

which is in agreement with the general formula \(3\). The first term on the r.h.s. of \(13\) represents the mean time spent in the scattering region, \(\langle \tau(\varepsilon) \rangle\), while the second term is the fluctuating part of the time delay \(\tau^f(\varepsilon)\). It should be noted that \(\tau^f(\varepsilon)\) is written in terms of \(\tilde{U}_{II}\) rather than \(\tilde{U}\), the evolution operator for the unopened map. The powers of traces of \(U_{II}\) are semiclassically related to the periodic orbits which lie completely in the interior of the open map. If the traces are evaluated in the semiclassical approximation one reproduces \(3\) and \(4\).

§4. Contribution of bifurcations to the time delay

Generic systems have a phase space in which regular islands and regions of chaotic sea coexist. One of the main characteristics of mixed systems is the bifurcation of periodic orbits. Bifurcations are events where different periodic orbits coalesce when parameters of the system are varied. They are important in semiclassical approximations because bifurcating orbits carry a semiclassical weight that is higher than that of the isolated (unstable) periodic orbits and sometimes even that of tori of regular orbits.

Consider the Gutzwiller contribution of a periodic orbit \(\gamma\) and its repetitions \(m\). The amplitude in \(4\) diverges if \(M_{\gamma}^m\) has an eigenvalue one, which happens at a bifurcation. This is because periodic orbits are assumed to be isolated in the derivation of the trace formula.\(^1\) More specifically, the trace formula can be derived by integrating over Poincaré sections perpendicular to periodic orbits, and periodic orbits appear as stationary points of these integrals. Consequently, the usual stationary phase approximation breaks down when stationary point coalesce. The remedy is to perform a uniform asymptotic expansion valid throughout the bifurcation process.\(^20\)–\(^24\) This is obtained by rederiving the trace formula using the appropriate generating function for the Poincaré map \(\Phi(Q', P)\) from \((Q, P)\) to \((Q', P')\) in normal form coordinates. For two-dimensional systems the semiclassical contribution of a bifurcation to the density of states is then given by

\[
\rho^b \propto \frac{1}{\hbar^2} \int dQdP \exp \left( i\Phi(Q, P) / \hbar \right). \tag{14}
\]

As an example we take a saddle node bifurcation, \(\Phi(Q, P) = P^2 + x_1 Q + Q^3\), where \(x_1\) depends on the energy or other system parameters and vanishes at the bifurcation. The stationary points occur for negative \(x_1\) at \((Q, P) = (\pm \sqrt{-x_1}, 0)\). The
contribution \(14\) at the bifurcation is \(\propto \hbar^{-\beta}\), where \(\beta = 7/6\). Away from the bifurcation (i.e. when \(-x_1/\hbar\) is large) one obtains contributions of isolated orbits that are \(\propto \hbar^{-1}\). The two regimes are interpolated by an Airy function.

Besides \(\beta\) there are further exponents that are important for the semiclassical influence of the bifurcation. They describe the size of the parameter intervals over which the bifurcation is semiclassically stronger than isolated periodic orbits. Consider \(\Phi(Q, P) = P^2 + x_1 Q + Q^3\) in the example. We can make the exponent in the integral \(14\) \(\hbar\)-independent by scaling \(Q = \tilde{Q}_1 h^{1/3}, P = \tilde{P}_1 h^{1/2}\) and \(x_1 = \tilde{x}_1 h^{2/3}\). Hence the relevant \(x_1\) interval scales like \(\hbar^{\sigma_1}\), where \(\sigma_1 = 2/3\) in this example. The same analysis can be applied to more complicated bifurcations which have more parameters \(x_i, i = 1, \ldots, K\), in their normal form. \((K\) denotes the codimension of the bifurcation.) One then obtains a characteristic exponent \(\sigma_i\) for every parameter \(x_i\). Because of this finite extension in parameter space, bifurcations of higher codimension have to be taken into account even if only one parameter is varied.

Generic bifurcations of periodic orbits are characterized by the codimension \(K\) and the repetition number \(m\) of the orbit for which the bifurcation occurs. A systematic investigation of the influence of the different bifurcations on moments of the density of states was carried out by Berry, Keating and Schomerus.\(^3\) This has subsequently been extended to determine their influence on the statistics of wavefunctions\(^4\),\(^25\) and, more recently, on the moments of the conductance fluctuations in antidot lattices.\(^7\)

Fluctuations of the Wigner time delay can be characterized by their moments, defined as

\[
\mathcal{M}_{2k} = \left(\frac{M}{2\pi \hbar}\right)^{2k} \left\langle \left\langle \tau^{2k}\right\rangle_{E,X}\right.
\]

(15)

where \(\langle \cdots \rangle_{E,X}\) denotes averaging over energy and over parameter space. In any parameter interval of a system with mixed dynamics infinitely many bifurcations occur, most of them for very long periodic orbits. If \(\hbar\) is small enough, then these bifurcations are important. To determine the influence of a particular bifurcation with codimension \(K\) and repetition number \(m\) on the moments of the time delay, we replace the average in \(15\) by an average over parameters in the normal form. Performing the scaling procedure\(^3\) we can extract the \(\hbar\) dependence for the different bifurcations \(\mathcal{M}_{2k,m,K} \sim \hbar^{-\eta_{k,m,K}}\) where \(\eta_{k,m,K} = 2k \beta_{m,K} - \sum_{i=1}^{K} \sigma_{i,m,K}\). Hence we see that the importance of a bifurcation depends on the quantity \(\eta_{k,m,K}\), which in turn depends on the characteristic exponents \(\beta_{m,K}\) and \(\sigma_{i,m,K}\). The bifurcation that is most important for a particular moment is that for which \(\eta_{k,m,K}\) is largest, and hence one finds that \(\mathcal{M}_{2k} \sim \hbar^{-\eta_k}\) where \(\eta_k = \max_{m,K}(\eta_{k,m,K})\). There is a different winner of this competition between bifurcations for every \(k\). It follows from \(8\) that all the different exponents for the time delay \(15\) coincide with those for the corresponding moments of the density of states.\(^3\),\(^7\)
§5. Numerical results

We now illustrate some of the general ideas described above using a family of perturbed cat maps.26,27) These are maps of the form

\[
\begin{pmatrix}
q' \\
p'
\end{pmatrix} =
\begin{pmatrix}
2 & 1 \\
3 & 2
\end{pmatrix}
\begin{pmatrix}
q \\
p
\end{pmatrix} + \frac{\kappa}{2\pi} \cos(2\pi q) \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mod 1
\] (16)

where \(q\) and \(p\) are coordinates on the unit two-torus, and are taken to be a position and its conjugate momentum. We will concentrate on one particular bifurcation and investigate its influence on the second moment of the time delay. Using (15), (3) and (13) one can write

\[
M_2 \equiv \frac{1}{2\pi} \int_0^{2\pi} (\rho^{fl}(\varepsilon))^2 d\varepsilon \approx \frac{1}{2\pi^2} \sum_n |\text{tr} U_n|^2 e^{-n^2 \Delta^2},
\] (17)

where \(\rho^{fl}\) is the fluctuating part of \(\rho\) convoluted with a normalized gaussian of width \(\Delta\). In our computations \(\Delta\) was taken large enough so that the dominant contributions to \(\rho^{fl}\) come from the \(n = 1\) term in the sum in (17).

For \(\kappa = 0\) the map (16) is uniformly hyperbolic. The perturbed map is guaranteed by Anosov’s theorem to be strongly chaotic for \(\kappa \leq (\sqrt{3} - 1)/\sqrt{5} \approx 0.333\). The period-1 fixed points at

\[
q_j = \frac{1}{2} \left( j - \frac{\kappa}{2\pi} \cos(2\pi q_j) \right)
\] (18)

for integers \(j=0,1\) are then unstable.27) Outside this parameter range bifurcations can occur leading to a mixed phase space. When \(\kappa = \kappa_{\text{bif}} = 5.94338\) the phase space is almost entirely ergodic but a saddle node bifurcation gives rise to a new pair of period-1 orbits.2) For this map, it has been shown27,28) that \(\text{tr} U\) can be expressed in the form

\[
\text{tr} U = \sqrt{(N/i)} \sum_{j=0}^{1} \int_{-\infty}^{\infty} \exp(2\pi i N S_j(q)) dq
\] (19)

where \(S_j(q) = q^2 + \frac{\kappa}{4\pi^2} \sin(2\pi q) - jq\), so that the phase is stationary at the fixed points (18).

After removing strips from phase space one gets the open map. In all the cases illustrated in this section, the ratio \(M/N\) of the dimension of the quantum map to the number of open channels is 0.28. In order to see the contribution of bifurcating points to the moments (17) we have computed \(|\text{tr} U_{ij}|^2\) in two different configurations. First, two strips are located in such a way that they block the unstable fixed points \(q_0 = 0.81425\) and \(q_1 = 0.6857\). Second, the two strips are placed such that they block the bifurcating points \(q_0^+ = 0.4453\) and \(q_1^+ = 0.0548\). These are period-1 fixed points which originate in a saddle node bifurcation at \(\kappa_{\text{bif}}\). The logarithm of the second moment \(M_2\) is plotted in figure 1 as a function of the perturbation parameter \(\kappa\). The two strips block either the bifurcating points (solid black curve and crosses) or the isolated points (dashed red curve and circles). In this and the subsequent two figures
the theoretical curves correspond to asymptotic evaluations of (19), as described in outline in the previous section.

Figure 2 illustrates the contribution of isolated points to $\mathcal{M}_2$ as a function of the dimension $N$. In this case, strips are centred at the bifurcating points so that only the isolated

By contrast, figure 3 illustrates the situation where the strips block the isolated orbits. The contribution of the remaining, bifurcating orbits to $|\text{tr } U|^2$ grows semiclassically like $N^{1/3}$ in this case (i.e. the scaling exponent is $1/3$). For $\kappa$ around $\kappa_{\text{bif}}$, as in figure 1, $\text{tr } U_{II}$ is more intricate and involves Airy functions.

These figures illustrate clearly how the bifurcating trapped periodic orbit dominates the fluctuations of the time delay semiclassically.
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Fig. 3. $\log\mathcal{M}_2$ versus $\log N$ for $\Delta = 1.5$ and $\kappa = \kappa_{\text{crit}}$. Strips centred at the isolated fixed points. Dashed red curve and circles for theory and direct numerical evaluation of the trace using (17) respectively.

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