Constructive Polynomial Partitioning for Algebraic Curves in $\mathbb{R}^3$ with Applications

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Abstract

In 2015, Guth proved that, for any set of $k$-dimensional varieties in $\mathbb{R}^d$ and for any positive integer $D$, there exists a polynomial of degree at most $D$ whose zero-set divides $\mathbb{R}^d$ into open connected “cells,” so that only a small fraction of the given varieties intersect each cell. Guth’s result generalized an earlier result of Guth and Katz for points.

Guth’s proof relies on a variant of the Borsuk-Ulam theorem, and for $k > 0$, it is unknown how to obtain an explicit representation of such a partitioning polynomial and how to construct it efficiently. In particular, it is unknown how to effectively construct such a polynomial for curves (or even lines) in $\mathbb{R}^3$.

We present an efficient algorithmic construction for this setting. Given a set of $n$ input curves and a positive integer $D$, we efficiently construct a decomposition of space into $O(D^3 \log^3 D)$ open cells, each of which meets $O(n/D^2)$ curves from the input. The construction time is $O(n^2)$. For the case of lines in 3-space we present an improved implementation, whose running time is $O(n^{4/3} \text{polylog } n)$. The constant of proportionality in both time bounds depends on $D$ and the maximum degree of the polynomials defining the input curves.

As an application, we revisit the problem of eliminating depth cycles among non-vertical lines in 3-space, recently studied by Aronov and Sharir (2018), and show an algorithm that cuts $n$ such lines into $O(n^{3/2+\varepsilon})$ pieces that are depth-cycle free, for any $\varepsilon > 0$. The algorithm runs in $O(n^{3/2+\varepsilon})$ time, which is a considerable improvement over the previously known algorithms.

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1 Introduction

Partitioning Polynomials. In [21], Guth developed an efficient space decomposition adapted to a set of varieties in Euclidean space. Specifically, he proved the following:

**Theorem 1.1 (Polynomial Partitioning for Varieties [21]).** Let $\Gamma$ be a set of $k$-dimensional varieties in $\mathbb{R}^d$, each defined by at most $m$ polynomials of degree at most $b$. For each $D \geq 1$, there is a $d$-variate non-zero “partitioning polynomial” $f$ of degree at most $D$, so that $\mathbb{R}^d \setminus Z(f)$ is a union of $O(D^d)$ connected components, each of which intersects at most $C \frac{|\Gamma|}{D}$ varieties from $\Gamma$. Here $C > 0$ is a constant that depends on $b, m, \text{ and } d$.

In particular, when $\Gamma$ is a set $\mathcal{L}$ of algebraic curves in $\mathbb{R}^3$ (defined by polynomials of degree at most $b$), or just lines, Theorem 1.1 guarantees the existence of a polynomial $f$ of degree at most $D$ partitioning $\mathbb{R}^3$ into $O(D^3)$ connected components, each of which intersect $O(|\mathcal{L}|/D^2)$ curves of $\mathcal{L}$. Here and throughout this paper, we will think of $b$ as being fixed as $|\mathcal{L}|$ becomes large, and all implicit constants are allowed to depend on $b$.

Aronov, Miller, and Sharir [9] used Theorem 1.1 to prove that $n$ pairwise disjoint non-vertical triangles in $\mathbb{R}^3$ can be cut into $O(n^{3/2+\varepsilon})$ pieces that form a “depth order,” for any $\varepsilon > 0$ (see below for the definition of depth order and a further discussion). This work extended an earlier result of Aronov and Sharir [10], who proved a bound of $O(n^{3/2} \log \log n)$, for the analogous problem for pairwise disjoint non-vertical lines in $\mathbb{R}^3$. Apart from the $\varepsilon$ loss in the exponent of the triangle bound and polylog $n$ factor in the line bound, these results are optimal.

Theorem 1.1 uses a variant of the Borsuk-Ulam theorem to obtain the partitioning polynomial. However, there is no known effective method to construct such a polynomial. Therefore, despite the recent progress on eliminating depth cycles, there is no matching algorithmic bound for the results established in [9]. The best known result in this direction is the algorithm presented by de Berg [13], which exploits a different technique and achieves a suboptimal bound on the number of pieces. For the case of lines, the work in [10] describes several slow polynomial-time algorithms to compute a depth order, among which is an approximation algorithm by Aronov, de Berg, Gray, and Mumford [8]. Theorem 3.1] that produces a set of cuts whose size is larger than that of the smallest possible by only a polylogarithmic factor. Still, these algorithms do not apply the construction in [10] directly, and the question of whether one can compute an explicit polynomial $f$ with the above properties remains elusive.

Obtaining an explicit decomposition into $O(D^3)$ connected components such that on average each cell meets $O(|\mathcal{L}|/D^2)$ curves of $\mathcal{L}$ is straightforward. Indeed, if $f$ is a degree-$D$ polynomial, then $\mathbb{R}^3 \setminus Z(f)$ is a union of $O(D^3)$ cells (connected components), and every algebraic curve not contained in $Z(f)$ intersects $Z(f)$ in $O(D)$ points, and thus intersects $O(D)$ cells. In particular, on average each cell intersects $O(|\mathcal{L}|/D^2)$ lines. Obtaining a bound of $O(|\mathcal{L}|/D^2)$ in the worst case is much more difficult. In fact, even achieving a more modest bound, say, of the form $O(|\mathcal{L}|/D^{1+\varepsilon})$, for some $\varepsilon > 0$, is already challenging. Using the approach of “$\varepsilon$-cuttings,” one can produce a space decomposition, such that each cell meets roughly $O\left(\frac{|\mathcal{L}|}{D}\right)$ curves of $\mathcal{L}$ (this bound is larger by more than an order of magnitude than our target bound), and such that the total number of curve-cell intersections is close to $O(|\mathcal{L}|D)$, see, e.g., [20,23]. However, we are not aware of an approach based on $\varepsilon$-cuttings where the worst-case bound on the number of curves meeting a cell is $o\left(\frac{|\mathcal{L}|}{D}\right)$.

Theorem 1.1 is an extension of the polynomial partitioning theorem by Guth and Katz [22], based on the polynomial ham-sandwich theorem of Stone and Tukey [28]. Namely, Guth and Katz
showed that, if $\Gamma$ is a finite set of points in $\mathbb{R}^d$ and $D \geq 1$ is an integer parameter, then there is a non-zero polynomial $f$ of degree at most $D$ so that each connected component of $\mathbb{R}^d \setminus Z(f)$ contains $O(|\Gamma|/D^d)$ points of $\Gamma$, with a constant of proportionality depending on $d$. Adapting a definition from [4], let $r = O(D^d)$ be an integer parameter (with an appropriate constant of proportionality). We say in this case that $f$ is an $r$-partitioning polynomial. Agarwal, Matoušek, and Sharir [4] presented an algorithm that efficiently computes such a polynomial $f$.

**Theorem 1.2 (Effective Polynomial Partitioning for Points [4]).** Given a set $P$ of $n$ points in $\mathbb{R}^d$ and an integer parameter $r \leq n$, an $r$-partitioning polynomial $f$ for $P$ of degree $O(r^{1/d})$, with an implicit constant depending on $d$, can be computed in randomized expected time $O(nr + r^3)$.

Although the authors do not state so explicitly, the proof of Theorem 1.2 also applies to multi-sets of points, or, equivalently, sets of points with positive integer weights, where a weight of a point corresponds to the number of times that it appears. This multi-set formulation will be useful for our analysis below.

**Our Result.** We present an efficient algorithm that, given a set of algebraic curves in $\mathbb{R}^3$, partitions $\mathbb{R}^3$ into interior disjoint “cells” (plus a “boundary”) so that only a small fraction of the curves intersect each cell. Informally, we prove a theorem of the following kind:

**Theorem 1.3 (Informal Version).** Let $L$ be a collection of $n$ irreducible algebraic curves in $\mathbb{R}^3$ that satisfy certain general position assumptions. Let $D$ be a positive integer. Then there is a space decomposition of $\mathbb{R}^3$ into $O(D^3 \log^3 D)$ disjoint open cells, plus a boundary, so that each cell intersects $O(n/D^2)$ curves of $L$. The boundary is the union of an algebraic variety of degree $O(D \log D)$ and dimension two, plus an additional semi-algebraic set (with empty interior) that has finite and well-behaved intersection with all but a small number of curves from $L$. Moreover, this space decomposition can be computed in $O(n^2)$ randomized expected time. For the special case where $L$ is a set of lines in $\mathbb{R}^3$, the expected running time improves to $O(n^{4/3} \text{polylog } n)$.

A precise statement of Theorem 1.3 appears in Section 2. The proof of Theorem 1.3 is based on a two-level decomposition. The first level produces a polynomial partitioning for points using Theorem 1.2, and in the second level we apply the method of “$\varepsilon$-cuttings” for efficiently partitioning curve segments in the plane, provided that few pairs of curves intersect. This technique also allows us to efficiently partition curve segments in $\mathbb{R}^3$, provided few pairs of curves intersect when projected to the $xy$-plane. These two ingredients are combined as follows. For each curve in $L$, we consider all points on the curve whose projection onto the $xy$-plane lies on the projection of another curve from $L$; we call such points “points of vertical visibility.” Using Theorem 1.2 we partition $\mathbb{R}^3$ into cells, so that each cell either intersects few curves from $L$, or it contains few points of vertical visibility. Cells of the first kind satisfy the conclusions of our theorem. Cells of the second kind are further decomposed using the $\varepsilon$-cutting machinery mentioned above.

Theorem 1.3 produces a space decomposition with very similar properties to that of Theorem 1.1 for the case $d = 3$, $k = 1$, though our decomposition is weaker by a polylogarithmic factor. However, because our theorem employs a two-level construction, the “boundary” of the cells is not an algebraic variety. Instead, it is the union of an algebraic variety (representing the zero set of an appropriate

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1We note that the polynomial $f$ computed in [4] forms a partition approximating the one shown in [22]. Therefore the constant of proportionality in the degree bound in Theorem 1.2 is slightly worse than that in [22].

2See Section 2.
polynomial obtained at the primary partition) and a semi-algebraic set resulting from the secondary partition.

Armed with Theorem 1.3, we revisit the analysis of Aronov and Sharir [10], in order to algorithmically cut \( n \) pairwise disjoint non-vertical lines in \( \mathbb{R}^3 \) into \( O(n^{3/2+\varepsilon}) \) pieces, such that they form a proper depth order. Whereas the analysis in [10] produces \( O(n^{3/2} \text{polylog } n) \) pieces, which is slightly better than our bound, our procedure (presented in Section 3) can be performed in \( O(n^{3/2+\varepsilon}) \) expected time. This is a considerable improvement over the performance of the algorithms presented in [10] and by de Berg [13], which exploit matrix multiplication.

**An application: Eliminating depth cycles for lines in** \( \mathbb{R}^3 \). With a slight abuse of notation, let \( \mathcal{L} \) be a set of \( n \) pairwise-disjoint lines in \( \mathbb{R}^3 \). We assume that the lines satisfy the following general position assumptions: none of the lines are vertical and their \( xy \)-projections are in general position. Namely, no triple of projected lines meet at a point, and no two projected lines coincide.

For a pair of lines \( \ell, \ell' \in \mathcal{L} \), we say that \( \ell \) passes above \( \ell' \) if a vertical line that meets both \( \ell \) and \( \ell' \) intersects \( \ell \) at a point that has larger \( z \)-coordinate than that of its intersection with \( \ell' \); this line is unique if \( \ell, \ell' \) have non-parallel \( xy \)-projections. We denote this relation as \( \ell' \prec \ell \). This relation is not necessarily transitive, and is likely to form cycles that consist of three or more lines.

Our goal is to efficiently cut the lines in \( \mathcal{L} \) into a finite number of pieces that do not form any cycle under the relation \( \prec \); this is also referred to as depth order. In our setting, these resulting pieces are lines segments, rays, or just lines in 3-space. Aronov and Sharir [10] used Theorem 1.1 for the case \( d = 3, k = 1 \) to obtain a near optimal subquadratic bound on the number of cuts required to create a depth order for lines. Our main result in this direction is the following efficient implementation of their method.

**Theorem 1.4.** Let \( \mathcal{L} \) be a collection of \( n \) pairwise-disjoint lines in \( \mathbb{R}^3 \) in general position. Then, for any prescribed \( \varepsilon > 0 \), we can cut the lines in \( \mathcal{L} \) into \( O(n^{3/2+\varepsilon}) \) pieces, whose depth relation is acyclic. This cutting can be computed in expected \( O(n^{3/2+\varepsilon}) \) time.

The main motivation for eliminating depth cycles comes from hidden surface removal—a technique for rendering a scene in computer graphics [14]. We refer the reader to the earlier work in [3][17], as well as the more recent studies in [9][10][13] for a comprehensive overview, which also include the more intricate problem of eliminating depth cycles among pairwise disjoint triangles in 3-space. In [9], Miller, Sharir, and the first author used Theorem 1.1 for the case \( d = 3, k = 1 \) in order to obtain a near optimal subquadratic bound on the number of pieces required to eliminate depth cycles for triangles in 3-space. In contrast to the case of lines, however, Theorem 1.3 cannot be used directly to efficiently implement their technique. This is due to the fact that the structure resulting from Theorem 1.3 is different than the one resulting from Theorem 1.1. After the publication of the proceeding version of this paper [7], Agarwal and the authors [2] have shown, using a different set of tools, that the polynomial partitioning stated in Theorem 1.1 can be computed in an efficient manner. Integrating this result with the mechanism in [9] eventually yields an efficient algorithm for the setting of triangles. We do not discuss this further here. While the follow-up work [2] provides very general methods for performing partitioning, the current work contains novel techniques that exploit geometry specific to curve arrangements, and we hope these ideas will be incorporated into future problems that involve arrangements of curves.
2 Polynomial Partitioning for Algebraic Curves in 3-Space

In this section we prove Theorem 1.3. Let \( \mathcal{L} \) be a collection of \( n \) irreducible algebraic curves in \( \mathbb{R}^3 \), each defined by polynomials of degree at most \( b \). We will think of \( b \) as being fixed, so all implicit constants may depend on \( b \). In particular, we write \( X(n) = O(Y(n)) \) to mean that there exists a constant \( C \) depending only on \( b \) so that \( X(n) \leq CY(n) \) for all positive integers \( n \). We write \( X(n) = O_t(Y(n)) \) to mean that there exists a constant \( C \) depending only on \( b \) and \( t \) so that \( X(n) \leq CY(n) \) for all positive integers \( n \).

For a set \( X \subset \mathbb{R}^3 \), we denote by \( X^* \) its projection onto the \( xy \)-plane. Let \( \mathcal{L}^* = \{ \gamma^* \mid \gamma \in \mathcal{L} \} \); this is the set of \( xy \)-projections of the curves of \( \mathcal{L} \).

Definition 2.1. Let \( \gamma, \gamma' \) be two distinct irreducible curves in \( \mathbb{R}^3 \). A pair of points \( (p, p') \in \gamma \times \gamma' \), are called points of “vertical visibility” (with respect to \( \gamma \) and \( \gamma' \)) if \( p^* = (p')^* \).

If neither \( \gamma \) nor \( \gamma' \) is a vertical line, and if the projections of \( \gamma \) and \( \gamma' \) have a finite intersection, then \( \gamma \) and \( \gamma' \) have finitely many pairs of points of vertical visibility.

If \( \mathcal{L} \) is a set of irreducible curves, we define \( V(\mathcal{L}) \) to be the multi-set of all points of vertical visibility admitted by the curves in \( \mathcal{L} \). If none of the curves in \( \mathcal{L} \) are vertical lines, and if the \( xy \)-projection of each pair of curves from \( \mathcal{L} \) have finite intersection, then each pair of curves from \( \mathcal{L} \) contribute \( O(1) \) points to \( V(\mathcal{L}) \), and thus \( |V(\mathcal{L})| = O(n^2) \).

Remark. We do not make any further general position assumptions besides requiring that the curves are non-vertical, and the projections of each pair of curves have finite intersection. In particular, two or more curves of \( \mathcal{L} \) may intersect in a single point, and that the \( xy \)-projections of three or more curves may meet in a single point. In such a case where the \( xy \)-projection \( p^* \) of a point \( p \in V(\mathcal{L}) \) meets \( k \) (distinct) \( xy \)-projections of curves from \( \mathcal{L} \), the weight of \( p \) is \( \binom{k}{2} \). This fact is exploited when we apply Theorem 1.2—see below. Note also that a single curve \( \gamma \in \mathcal{L} \) may intersect itself. Moreover, a vertical line may intersect \( \gamma \) at several points, implying that the \( xy \)-projection of \( \gamma \) intersects itself. However, we do not view these points as points of vertical visibility. In fact, this self-intersection might be a continuous set, in which case there is an overlap in the \( xy \)-projection of \( \gamma \). We revisit this scenario in Section 2.2, where we describe how to incorporate that into our analysis.

Hereafter we fix a parameter \( D \geq 1 \). If \( D > C_b n^{1/2} \) (where \( C_b \) is an appropriately chosen constant depending only on \( b \)), then there is a polynomial \( P \) of degree at most \( D \) that vanishes on each curve in \( \mathcal{L} \). This polynomial satisfies the conclusions of Theorem 1.3 and can be computed in \( O(\text{poly}(D)) \) time. Henceforth we will assume that \( D = O(n^{1/2}) \).

Our space decomposition is constructed by two main partitioning steps. In the first, we iteratively partition space by overlaying the zero sets of polynomials of degree \( D \geq 1 \), each of which partitions a subset of \( V(\mathcal{L}) \), so that the overall majority of resulting cells meet only \( O(n/D^2) \) curves of \( \mathcal{L} \) each, and the remaining cells together cover a small fraction of \( V(\mathcal{L}) \) (Lemma 2.2). By applying this process \( O(\log D) \) times, we obtain a trivariate polynomial \( P \) of degree \( O(D \log D) \), which partitions space into \( O((D \log D)^3) \) open cells, each of which either intersects only \( O(n/D^2) \) curves from \( \mathcal{L} \) (we call such a cell acceptable), or contains \( O(n^2/D^4) \) points of \( V(\mathcal{L}) \) (we call such a cell unacceptable). This step is performed in Corollary 2.3.

An unacceptable cell may intersect a large number of curves, but the fact that it contains \( O(n^2/D^4) \) points from \( V(\mathcal{L}) \) allows us to further decompose it into a small number of acceptable subcells. This leads to the second decomposition step, in which we build an \( \varepsilon \)-cutting” within each such cell. This is described in Section 2.2. The construction is based on the random sampling
2.1 The First Decomposition Step: Iteratively Partition Space

For a polynomial \( f : \mathbb{R}^3 \to \mathbb{R} \), define \( Z(f) \) to be the zero set of \( f \), i.e., the set \( \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = 0\} \). We refer to an open connected component of \( \mathbb{R}^3 \setminus Z(f) \) as a cell. We first show the following main property.

**Lemma 2.2.** Let \( \mathcal{L} \) be a collection of \( n \) irreducible algebraic curves in \( \mathbb{R}^3 \), each defined by polynomials of degree at most \( b \) and none of which are vertical lines. Suppose that the projections of each pair of curves from \( \mathcal{L} \) to the \( xy \)-plane have finite intersection. Let \( D \) be a positive integer. For each non-negative integer \( k \geq 0 \), there is a set \( V_k \subseteq V(\mathcal{L}) \) and a polynomial \( F_k \) with the following properties:

1. \( \deg(F_k) \leq kD \).
2. \( |V_k| \leq |V(\mathcal{L})|/2^k \).
3. For each cell \( \Omega \) of \( \mathbb{R}^3 \setminus Z(F_k) \), at least one of the following must hold:
   - \( \Omega \) intersects at most \( C_b n/D^2 \) curves from \( \mathcal{L} \) (where \( C_b \) is an appropriate constant depending on \( b \)), or
   - \( \Omega \cap V(\mathcal{L}) \subset V_k \).

**Proof.** First note that the curves from \( \mathcal{L} \) fully contained in \( Z(F_k) \) can be disregarded, since they do not meet the open cells \( \Omega \) of \( \mathbb{R}^3 \setminus Z(F_k) \), and are therefore irrelevant for the assertions of the lemma.

We prove properties (A)–(C) by induction on \( k \). For \( k = 0 \), the assertions are satisfied by putting \( V_0 = V(\mathcal{L}) \) and \( F_0 = 1 \). For \( k \geq 1 \), let \( V_{k-1} \) be a set of points and let \( F_{k-1} \) be a polynomial satisfying properties (A)–(C) above. Apply Theorem 1.2 to find a partitioning polynomial \( f \) of degree at most \( D \) for the multi-set of points \( V_{k-1} \). Each cell of \( \mathbb{R}^3 \setminus Z(f) \) contains \( O(|V_{k-1}|/D^3) \) points from \( V_{k-1} \).

We call a cell of \( \mathbb{R}^3 \setminus Z(f) \) acceptable if it intersects at most \( C_b n/D^2 \) curves from \( \mathcal{L} \) (we specify the choice of \( C_b \) shortly); otherwise the cell is unacceptable.

Since each curve \( \gamma \in \mathcal{L} \) not contained in \( Z(f) \) intersects \( O(D) \) cells of \( \mathbb{R}^3 \setminus Z(f) \), at most \( O(D^3)/C_b \) cells are unacceptable. This means that \( O(|V_{k-1}|/C_b) \) points from \( V_{k-1} \) are contained in unacceptable cells; we choose \( C_b \) large enough so that this number is at most \( |V_{k-1}|/2^k \).

Define

\[
V_k = \bigcup_{\tau \text{ unacceptable}} \tau \cap V_{k-1},
\]

with the union taken over all unacceptable cells \( \tau \) of \( \mathbb{R}^3 \setminus Z(f) \). To complete the inductive step, we define \( F_k = F_{k-1} \cdot f \). In other words, \( Z(F_k) \) is formed by overlaying \( Z(F_{k-1}) \) and \( Z(f) \). Then

\[
\deg(F_k) \leq \deg(F_{k-1}) + \deg(f) = (k - 1)D + D = kD,
\]

so property (A) is satisfied. We have \( |V_k| \leq |V_{k-1}|/2 \leq |V_0|/2^k \), thus property (B) is satisfied. It remains to verify property (C). Let \( \Omega \) be a cell of \( \mathbb{R}^3 \setminus Z(F_k) \); this cell is contained in the
Then there is a polynomial \( L \) of \( \Omega \) cells. Each cell \( \Omega \) satisfies the following holds: either \( \Omega \) intersects \( L \) or \( \Omega \) is unacceptable. This follows from Harnack’s curve theorem and Bézout’s theorem, see, e.g., [27, Section 2.2].

Let \( \Omega \) be a positive integer. Then there is a polynomial \( P \in \mathbb{R}[x,y,z] \) of degree \( O(D \log D) \) such that \( \mathbb{R}^3 \setminus Z(P) \) is a union of \( O((D \log D)^3) \) cells. For each such cell \( \Omega \), one of the following holds: either \( \Omega \) intersects \( O(n/D^2) \) curves from \( L \), or \( \Omega \) contains \( O(n^2/D^4) \) points of \( V(L) \).

Next, we will remove the requirement that none of the curves in \( L \) are vertical lines.

**Corollary 2.4.** Let \( L \) be a collection of \( n \) irreducible algebraic curves in \( \mathbb{R}^3 \), each defined by polynomials of degree at most \( b \) and none of which are vertical lines. Suppose that the projections of each pair of curves from \( L \) to the \( xy \)-plane have finite intersection. Let \( D \) be a positive integer. Then there is a polynomial \( P \in \mathbb{R}[x,y,z] \) of degree \( O(D \log D) \) such that \( \mathbb{R}^3 \setminus Z(P) \) is a union of \( O((D \log D)^3) \) cells. Each cell \( \Omega \) intersects \( O(n/D^2) \) curves from \( L \). Furthermore, for each such cell \( \Omega \), one of the following holds: either \( \Omega \) intersects \( O(n/D^2) \) curves from \( L \) (such cells are called acceptable), or \( \Omega \) contains \( O(n^2/D^4) \) points of \( V(L) \) (such cells are called unacceptable).

**Proof.** Let \( L^*_1 \subset \mathbb{R}^2 \) be the set of points obtained by projecting the curves in \( L \) to the \( xy \) plane. Apply Theorem 1.2 to find a partitioning polynomial \( P_1 \) of degree at most \( D \) for \( L^*_1 \). Each cell of \( \mathbb{R}^2 \setminus Z(P_1) \) contains \( O(|L^*_1|/D^2) \) points from \( L^*_1 \). We now lift \( Z(P_1) \) in the \( z \)-direction (algebraically, this is a vacuous operation), thereby obtaining a partitioning of \( \mathbb{R}^3 \) into (cylindrical open) cells.

Let \( P_2 \) be the output of Corollary 2.3 applied to \( L \), and define \( P = P_1P_2 \). \( \square \)

### 2.2 The Second Decomposition Step: Random Sampling

In this section we show how to further refine the decomposition obtained in Corollary 2.4 so that all cells are acceptable.

Write \( L = L_1 \cup L_2 \), as in the statement of Corollary 2.4. Fix an unacceptable cell \( \Omega \in \mathbb{R}^3 \setminus Z(P) \). From Corollary 2.3 it follows that \( \Omega \) contains \( O(n^2/D^4) \) points of \( V(L_2) \) (counting multiplicity). Let \( L_\Omega \subset L_2 \) be the subset of curves that meet \( \Omega \). We now intersect each curve \( \gamma \in L_\Omega \) with \( \Omega \). Let \( S_\Omega \) be the collection of the resulting open curve segments lying in \( \Omega \), and let \( S^*_\Omega \) be the set of their projections onto the \( xy \)-plane. Recall that we allow the curves in \( L_2 \) to self-intersect. Moreover, a vertical line might intersect a curve of \( L_2 \) at several points. This implies that the projected curves in \( S^*_\Omega \) may form self-intersections, of which we dispose as follows. We cut each curve in \( S^*_\Omega \) into \( x \)-monotone (open) Jordan arcs, and let \( W_\Omega \) be the resulting set of arcs; we have \( |W_\Omega| = O(|S^*_\Omega|) \). This follows from Harnack’s curve theorem and Bézout’s theorem, see, e.g., [27, Section 2.2].
Let \( r > 0 \) be a real parameter to be fixed shortly. Our goal is to construct a \((1/r)\)-cutting for the arcs in \( W_\Omega \). This is a partition of the plane into simple open cells (in our case these are pseudo-trapezoids determined by the vertical decomposition of the two-dimensional arrangement of some arcs from \( W_\Omega \)), each of which meets at most \(|W_\Omega|/r\) of the arcs in \( W_\Omega \). From [15, Lemma 2.2] it follows that there is a \((1/r)\)-cutting for \( W_\Omega \) consisting of \( O(\tau(r)) \) trapezoidal cells. Here \( \tau(r) \) is the expected number of cells in the vertical decomposition \( A^v(R^*_\Omega) \) of the arrangement \( A(R^*_\Omega) \) of a random subset \( R^*_\Omega \subseteq W_\Omega \), where every arc in \( W_\Omega \) is drawn independently with probability \( p := \frac{cD^2}{n} \) for an appropriate constant \( c > 0 \). We set \( r := p|W_\Omega| \). The expected number of trapezoidal cells in \( A^v(R^*_\Omega) \) is proportional to the expected complexity of the arrangement \( A(R^*_\Omega) \); this is a standard property of planar vertical decompositions [5]. We next show the following claim.

**Claim 2.5.** Letting \( m_\Omega := \mathbb{E}[|R^*_\Omega|] \), the expected complexity of \( A(R^*_\Omega) \) is \( O(m_\Omega + 1) \), where \( \mathbb{E}[\cdot] \) denotes expectation.

**Proof.** Let \( X \) be the set of vertices of \( A(W_\Omega) \). For each vertex \( x \in X \), let \( w(x) \) be the weight of \( x \), that is, the number of pairs of curves from the arrangement \( A(W_\Omega) \) that contain \( x \). Since \( \Omega \) contains \( O(n^2/D^4) \) points of \( V(L) \) (counting multiplicity), we have that \( \sum_{x \in X} w(x) = O(n^2/D^4) \), and thus in particular \(|X| = O(n^2/D^4)\).4

Let \( R^*_\Omega \) be a random subset of \( W_\Omega \), where every arc in \( W_\Omega \) is drawn independently with probability \( p = \frac{cD^2}{n} \). Since \( D = O(n^{1/2}) \), if \( c \) is chosen sufficiently small (depending on \( b \)), then we can ensure that \( p < 1/2 \). For each \( x \in X \), let \( w^*(x) \) be the weight of the vertex \( x \) in this random set. Since \( p < 1 \), we have

\[
\mathbb{E}[w^*(x)] = \sum_{k=2}^{\infty} \binom{k}{2} p^k < \sum_{k=2}^{\infty} \binom{k}{2} p^k = \frac{p^2}{(1 - p)^3}.
\]

By linearity of expectation, the expected combined weight of the vertices in \( A(R^*_\Omega) \) is

\[
\mathbb{E} \left[ \sum_{x \in X} w^*(x) \right] = \sum_{x \in X} \mathbb{E}[w^*(x)] < \sum_{x \in X} \frac{p^2}{(1 - p)^3} = \frac{p^2}{(1 - p)^3}|X|.
\]

Having \( p < 1/2 \) as above, we obtain \(|X| \frac{p^2}{(1 - p)^3} \leq 8p^2 |X| = O(1) \). The claim now follows from the fact that the arrangement complexity of \( A(R^*_\Omega) \) is bounded (up to multiplicative constants) by the number of elements in \( R^*_\Omega \) plus the number of intersections between pairs of curves in the corresponding arrangement. \( \square \)

We next bound the total expected complexity of \( A(R^*_\Omega) \) (and thus the total number of cells in \( A^v(R^*_\Omega) \)), over all unacceptable cells \( \Omega \in \mathbb{R}^3 \setminus Z(P) \). Put

\[
W := \bigcup_{\Omega \text{ unacceptable}} W_\Omega,
\]

3The original formulation in [15] is for canonical triangulations, but in our case, they can be replaced with vertical decompositions.

4The number of vertical visibilities might be considerably smaller, as such a visibility is relevant for a pair of curve segments \( \gamma_1, \gamma_2 \in S_\Omega \) only if both points \( v_1 \in \gamma_1 \) and \( v_2 \in \gamma_2 \) (which lie vertically above the other) are contained in \( \Omega \), but it may happen that only one of \( v_1, v_2 \) is in \( \Omega \).
with the disjoint union taken over all unacceptable cells $\Omega$ of $\mathbb{R}^3 \setminus Z(P)$, and recall that within each such cell $\Omega$ an arc of $W_\Omega$ is selected independently with probability $p = \frac{6D^2}{n}$. Since $\deg(P) = O(D \log D)$, we have $|W| = O(nD \log D + n)$, as we started with $n$ curves and cut them into pieces at the $O(nD \log D)$ points of intersection with $Z(p)$. We also recall that we cut the $xy$-projections of these pieces into $x$-monotone Jordan arcs. Therefore the total expected number of arcs in the samples $R^*_\Omega$, over all unacceptable cells $\Omega$, is $O(pnD \log D) = O(D^3 \log D)$.

Claim 2.5 now implies that

$$\mathbb{E} \left[ \sum_{\Omega} (|R^*_\Omega| + 1) \right] = O(D^3 \log^3 D),$$

with the summation taken over all unacceptable cells $\Omega$ of $\mathbb{R}^3 \setminus Z(P)$. In other words, we have just shown that the expected total number of cells in $A_i(R^*_\Omega)$ over all such cells $\Omega$ is $O(D^3 \log^3 D)$.

We finally describe the actual refinement of the unacceptable cells $\Omega$. Each trapezoidal cell $\Delta^*_\Omega \in A_i(R^*_\Omega)$ is turned into a vertical prism $\sigma_\Omega$ by taking its Cartesian product with the $z$-axis. We now form intersections of $\Omega$ with each prism $\sigma_\Omega$; $\Omega$ is only intersected with prisms arising from the trapezoidal cells of its own decomposition $A_i(R^*_\Omega)$. We refer to these intersections as the (open) second-stage cells and observe that they might not be connected, since $\Omega$ need not be $xy$-monotone. Despite this oddity, our decomposition does have the desired properties.

Indeed, since each second-stage cell $\xi = \Omega \cap \sigma_\Omega$ corresponds to a unique trapezoid $\Delta^*_\Omega$, the overall expected number of second-stage cells is $O(D^3 \log^3 D)$. By the properties of $(1/r)$-cuttings, each trapezoidal cell $\Delta^*_\Omega \in A_i(R^*_\Omega)$ meets $(1/r)$-fraction of the arcs in $W_\Omega$. Since $r = \frac{6D^2}{n} \cdot |W_\Omega|$, each $\Delta^*_\Omega$ meets $O(n/D^2)$ arcs of $W_\Omega$. Therefore $\Delta^*_\Omega$ meets $O(n/D^2)$ curves of $L_2$. So the number of curves from $L_2$ met by $\sigma_\Omega$, and, in particular, by the actual cell $\xi = \Omega \cap \sigma_\Omega$ is $O(n/D^2)$, as claimed. Since $\xi$ is a subset of a cell from Corollary 2.4, we have that $\xi$ intersects $O(n/D^2)$ curves from $L_1$. Thus $\xi$ intersects $O(n/D^2)$ curves from $L$.

Recall that, by Corollary 2.4, the number of the remaining (that is, acceptable) cells in $\mathbb{R}^3 \setminus Z(P)$ is $O(D^3 \log^3 D)$, and each of these cells meets $O(n/D^2)$ curves from $L$. To summarize, in both levels of the decomposition we obtain $O(D^3 \log^3 D)$ cells in total, each meeting $O(n/D^2)$ curves of $L$.

**Wrapping up.** We claim that the cell decomposition described above satisfies the properties stated in Theorem 1.3. We state these properties more formally below.

**Theorem 2.6 (Theorem 1.3 restated).** Let $L$ be a collection of $n$ irreducible algebraic curves in $\mathbb{R}^3$, each defined by polynomials of degree at most $b$. Suppose that the projections of each pair of curves from $L$ to the $xy$-plane have finite intersection. Let $D$ be a positive integer. Then there is a number $N = O(D^3 \log^3 D)$ and a partition $\mathbb{R}^3 = Z \cup \bigcup_{i=1}^N K_i$ (into a boundary and cells) with the following properties.

- Each $K_i$ is an open cell, consisting of a union of connected components of $\mathbb{R}^3 \setminus Z$.
- Each such cell is intersected by $O(n/D^2)$ curves from $L$.
- The interior of $Z$ is empty, and there is a trivariate polynomial $P$ of degree $O(D \log D)$, with $Z(P) \subset Z$.

\[This is potentially an overestimate, since $\Delta^*_\Omega$ may meet several arcs of the same original curve in $L_2$.\]
The curves from $\mathcal{L}$ not contained in $Z(P)$ intersect $Z$ in relatively few points, excluding a subset $\mathcal{L}' \subset \mathcal{L}$ of $O(D^3 \log D)$ curves. Specifically,

$$\sum_{\gamma \in \mathcal{L} \setminus \mathcal{L}' \atop \gamma \notin Z(P)} |\gamma \cap Z| = O(nD \log^3 D). \quad (3)$$

This partition can be computed in $O_D(n^2)$ randomized expected time, where the algorithm outputs for each cell $K_i$ the list of curves from $\mathcal{L}$ that it intersects. For the special case where $\mathcal{L}$ is a set of lines in 3-space in general position the expected running time is improved to $O_D(n^{4/3} \text{polylog } n)$.

The analysis of $(1/r)$-cuttings in [15] guarantees that there exists a choice of the random samples $R^*_\Omega$, such that each of the unacceptable cells has been subdivided into subcells that intersect $O(n/D^2)$ curves from $\mathcal{L}$. If a curve $\gamma \in \mathcal{L}$ is not contained in $Z(P)$, then it intersects $Z(P)$ in $O(D \log D)$ points. Thus the total number of intersections between curves in $\mathcal{L}$ not contained in $Z(P)$ and $Z(P)$ is $O(nD \log D)$. A curve $\gamma$ (not contained in $Z(P)$) intersects a vertical wall of a second-level cell in $O(1)$ points, if $\gamma^*$ does not have any curve segments comprising the sets $W_\Omega$ defined above, which participate in the samples $R^*_\Omega$. Otherwise, $\gamma$ intersects some of the vertical walls of the second-level cells constructed within $\Omega$ in a curve segment. Curves $\gamma$ of the latter kind comprise the set $\mathcal{L}' \subset \mathcal{L}$, and, as argued above, their total expected number is $O(D^3 \log D)$. Thus the total number of intersections between curves from $\mathcal{L} \setminus \mathcal{L}'$ (not contained in $Z(P)$) and cells, resulting either in the first or second stage of the decomposition, is $O((D \log D)^3(n/D^2)) = O(nD \log^3 D)$. This establishes (3).

The implementation details concerning the expected running time of the algorithm, as stated in Theorem 2.6, are given in Section 2.3 below.

Remark. In higher dimensions $d$, the analogue of Theorem 2.6 is a decomposition of $\mathbb{R}^d$ into $O((D \log D)^d)$ cells, each of which intersects $O(n/D^{d-1})$ curves from $\mathcal{L}$. We remark that most of the steps from the proof of Theorem 2.6 extend to higher dimensions. The main difficulty in this extension is handling a curve from $\mathcal{L}$ that projects to a point in the $xy$-plane. Unlike the three-dimensional case, such a curve is not necessarily “vertical” (i.e., parallel to an axis), and therefore the solution from Corollary 2.3 does not apply.

One simple way to resolve this issue is to impose an additional general position assumption on the curves in $\mathcal{L}$. For example, we could require that for each curve $\gamma \in \mathcal{L}$, any fiber of the projection map $\pi: \gamma \to \mathbb{R}^2$ from $\gamma$ to the $xy$ plane has finite cardinality. With this additional assumption, the proof of Theorem 2.6 extends with only minor modifications. Indeed, Lemma 2.2 extends to $\mathbb{R}^d$ by applying it with $k = (2d - 2)|\log_2 D|$, which yields an analogue of Corollary 2.3 where each acceptable cell intersects $O(n/D^{d-1})$ curves from $\mathcal{L}$, and each unacceptable cell contains $O(n^2/D^{2d-2})$ points of vertical visibility. The analysis of the second-level cell decomposition proceeds almost verbatim where we produce a planar vertical decomposition. In this extension the sampling probability $p$ becomes $cD^{n-1}/n$ (we can assume that $D = O(n^{1/(d-1)})$ and thus $p < 1/2$, since otherwise there exists a polynomial of degree $O(D)$ whose zero set contains all of the curves in $\mathcal{L}$), and the planar prisms (once constructed) are lifted to $\mathbb{R}^d$ in all $d - 2$ residual directions.

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\footnote{See Section 2.4}
2.3 Algorithmic Aspects

We now outline how to efficiently implement Theorem 2.6. This involves identifying curves that are non-vertical lines, and constructing $V(L)$ (which can be accomplished in quadratic time by brute-force examination of all pairs of curves), followed by several rounds of invocation of Theorem 1.2 the random sampling step and the construction of vertical decomposition in the plane are standard and are not the bottleneck of the algorithm. A major technicality arises from the fact that we need to keep track of the points of $V(L)$ contained in each cell, to determine which cells are acceptable. This appears to require actually computing the cells and their intersections with the curves of $L$ (see below for further discussion). In what follows, and for the sake of completing this computation, we make the following assumption on the model of computation, which is common in the computational geometry literature (the other common alternative is to assume integer coefficients and express the running time in the number of bit operations, as a function of the size of input and the bit length of the input coefficients):

Assumption 2.7. We assume a model of computation where the roots of a real univariate polynomial of degree $B$ can be constructed in $C = C(B)$ arithmetic operations.

Then, as a main tool, we use a result of Basu, Pollack, and Roy [11] Algorithm 16.6 concerning arrangements of zero sets of polynomials (see also [4] for a similar formulation):

Theorem 2.8 (Basu, Pollack, and Roy [11]). Let $F = \{f_1, \ldots, f_s\}$ be a set of $s$ real $d$-variate polynomials, each of degree at most $D$. Then the arrangement $A(F)$ in $\mathbb{R}^d$ has $O(1)^d (sD)^d$ cells, and it can be computed in time at most $T = s^{d+1}D^{O(d^2)}$. Each cell is described as a semi-algebraic set using at most $T$ polynomials of degree bounded by $D^{O(d^2)}$.

Following the notation of Lemma 2.2 at each step $k > 1$, we are given a subset $V_{k-1} \subseteq V(L)$ and a previously computed partitioning polynomial $F_{k-1}$. We first apply Theorem 1.2 to compute a partitioning polynomial $f$ of degree $D$ for the point set $V_{k-1}$. Then, to determine if a cell $\tau$ in $\mathbb{R}^3 \setminus Z(f)$ is acceptable, we test whether $\tau$ intersects at most $C_b n / D^2$ curves from $L$. By applying Theorem 2.8 to our polynomial $f$, we represent each cell $\tau \in \mathbb{R}^3 \setminus Z(f)$ as a semi-algebraic set (a Boolean formula with polynomial sign tests as atoms), and then test, for each curve $\gamma \in L$, whether $\gamma$ intersects $\tau$. We do this by constructing the intersection points of $\gamma$ with $Z(f)$ (if $\gamma \subset Z(f)$, we exclude $\gamma$ from further consideration), which takes $O_D(1)$ time by Assumption 2.7, and then test points between them for membership in each cell $\tau$, which can be done in time $O_D(1)$ by the Boolean formula just computed (see also [4] for similar considerations). Thus the total running time, over all curves in $L$, is $nO_D(1) + nD^{O(1)} = D(n)$. Next, we compute the new polynomial $F_k = f \cdot F_{k-1}$, and then form the subset $V_k$, by testing for each point of $V_{k-1}$ whether it lies in an unacceptable cell, using the membership test already discussed. Applying Theorem 2.8 and the fact that $|V_{k-1}| = |V(L)| = O(n^2)$, we can complete this task in time $O_D(n^2)$. The process repeats $O(\log D)$ times and takes $O_D(n^2)$ time in total.

We next need to construct a decomposition into vertical prisms within each unacceptable cell $\Omega \in \mathbb{R}^3 \setminus Z(P)$. This involves the computation of $(1/r)$-cuttings within $\Omega$. Recall that we apply the randomized algorithm described in [15] Theorem 2.1, which constructs a $(1/r)$-cutting in an arrangement of $m$ segments in expected time $O(m \log r + A \cdot r/m)$, where $A$ is the total number of intersections among these segments. Applying this to each of the arrangements $A(W_\Omega)$, by substituting $m := |W_\Omega|$, $A := |A(W_\Omega)|$ (where $|A(W_\Omega)|$ is the underlying arrangement complexity), we obtain an expected running time of $O\left(|W_\Omega| \log r + |A(W_\Omega)| \cdot \frac{r}{|W_\Omega|}\right)$. Recall that $r = \frac{cD^2}{n} \cdot |W_\Omega|$, 10
and that $|A(W)| = O(|W| + n^2/D^4)$. Then by applying (2), it is easy to verify that the expected running time, over all unacceptable cells $\Omega$, is $O_D(n)$. We conclude the construction by associating with each second-stage cell $\xi = \Omega \cap \sigma$ the set of curves $L_\xi \subset L$ that it intersects, by testing for each curve $\gamma \in L_\Omega$ whether it also meets the prism $\sigma$. Omitting any further details, we have shown:

**Theorem 2.9.** The decomposition described in Theorem 2.6 can be computed in randomized expected time $O_D(n^2)$.

**Remark.** A major open problem is to improve the running time to subquadratic. The bottleneck is having to explicitly process $V(L)$, or, more generally $V_k$, at each iteration $k$. In the worst case, this set could contain $\Theta(n^2)$ points. The remaining steps of the algorithm can be completed in $O_D(n)$ time. Thus the key to obtaining subquadratic running time lies in having an efficient implicit representation for $V(L)$. We next present such an efficient implementation, based on a range-search mechanism, for the case where $L$ is a set of lines in 3-space.

### 2.4 A Faster Construction for the Case of Lines

In this section, we present an improved implementation of our algorithm for the case of lines in $\mathbb{R}^3$ in general position or, more generally, line segments in 3-space. Our approach is to use a compact representation for the points of vertical visibility instead of storing them explicitly. We note that, in contrast with the algorithm of Theorem 1.3, we are able to track only those pairs of vertically visible points that lie in the same cell of the current decomposition, we refer to them as *unsplit visibility pairs*: pairs in which the two points end up in different cells are not tracked at all, we refer to them as *split visibility pairs*. This, however, does not violate the analysis of Sections 2.1 and 2.2 and the assertions in Theorem 1.3 for the case of lines continue to hold.

We exploit the mechanism of Agarwal [1] to efficiently represent intersections among line segments in general position in the plane. We revisit the algorithm of Agarwal, Matoušek, and Sharir [4] summarized in Theorem 1.2 and modify the procedures that were originally designed to manipulate the input points explicitly, to instead perform the required operations implicitly. A closer inspection of the analysis in [4] shows that we need to support the following two operations: (i) select uniformly a point at random among a collection of points contained in a specific cell $\tau \in \mathbb{R}^3 \setminus Z(f)$, for an appropriate polynomial $f$, and (ii) count the number of points contained in $\tau$.

Fix such a cell $\tau$. As in Section 2.2, let $L_\tau$ be the subset of lines meeting $\tau$. We take the intersection of each line in $L_\tau$ with $\tau$ (using Assumption 2.7), obtain a collection $S_\tau$ of open line segments contained in $\tau$, and consider the set of their projections $S^*_\tau$ to the $xy$-plane. Put $s_\tau := |S^*_\tau|$. Using the algorithm in [1] we construct a compact representation for the pairwise intersecting segments in $S^*_\tau$ in overall $O(s^{4/3}_\tau \log s_\tau)$ time. In particular, this implies that operation (ii) can be completed in the same time bound. Concerning operation (i), the resulting compact representation consists of a union of complete bipartite intersection graphs (each such graph is stored as a pair $(A, B)$ of sets of segments, in which every segment of $A$ intersects every segment of $B$; the pair is stored using $\Theta(|A| + |B|)$ space rather than $\Theta(|A| \times |B|)$; hence the space savings). Once such an implicit representation is available, it is possible to randomly sample a point of intersection in logarithmic time, by first picking the bipartite graph and then randomly

---

7$L$ consists of $n$ non-vertical lines in $\mathbb{R}^3$, with $xy$-projections in general position. Namely, no pair of projected lines coincide and no triple of projected lines meet at a point.
and uniformly picking a segment from \( A \) and a segment of \( B \)—see [1] for more details concerning this construction.

Inspecting the analysis in [1], and integrating it with our efficient implementation of steps (i)–(ii) yields a partitioning polynomial for the points of vertical visibility among \( L \) (or, more generally, \( n \) line segments in \( \mathbb{R}^3 \)) in time \( O_D(n^{4/3} \text{polylog } n) \); refer to [4] for more details. We once again emphasize that with this implementation we can only guarantee to control the number of unsplit visibility pairs inside a cell (that is, both defining lines meet that cell and the two vertically visible points are contained in the cell). The number of split visibility points within a cell can be arbitrarily large. To summarize we have shown:

**Lemma 2.10.** Let \( L \) be a collection of \( n \) line segments in \( \mathbb{R}^3 \) in general position, and let \( V(L) \) be the set of points of their vertical visibilities. Let \( D \) be a positive integer. Then one can compute in expected \( O_D(n^{4/3} \text{polylog } n) \) time a partitioning polynomial \( f \) of degree \( D \), such that each cell in \( \mathbb{R}^3 \setminus Z(f) \) contains \( O(|V(L)|/D^3) \) pairs of unsplit points of vertical visibility from \( V(L) \).

We next describe the modifications to Lemma 2.2 and Corollary 2.3 needed to apply our algorithm. Beginning with the first decomposition step, we observe that Lemma 2.2 continues to hold if instead of considering the entire set \( V_k \), we consider only the subset of unsplit visibility pairs with respect to unacceptable cells, that is, those points of vertical visibility, for which both defining lines intersect the same unacceptable cell generated at step \( k \). With this refinement of \( V_k \) we modify property (C.2) in the statement of Lemma 2.2 accordingly, and inside \( \Omega \) consider only the unsplit visibility pairs of \( V_k \). Then in the assertion of Corollary 2.3 concerning \( \Omega \) we can guarantee that either \( \Omega \) intersects \( O(n/D^2) \) curves from \( L \) or \( \Omega \) contains \( O(n^2/D^4) \) pairs of (unsplit) points of vertical visibility from \( V(L) \).

The implementation of the procedure to compute the partitioning polynomial \( P \) (using the notation of Corollary 2.3) is performed by repeatedly invoking Lemma 2.10 initially on the input lines in \( L \), and at step \( k > 1 \), on the set of the line segments obtained by intersecting the lines of \( L \) with the unacceptable cells from step \( k - 1 \) (this replaces the explicit representation of \( V_{k-1} \)). At each step the number of line segments is only linear in \( n \) and in \( D \) (more specifically, every line is cut into at most \( D + 1 \) segments), and therefore the total running time of computing the partitioning polynomial for \( V_k \), over all \( O(\log D) \) iterations, is \( O_D(n^{4/3} \text{polylog } n) \). In addition, we need to apply some of the operations already discussed above, including the classification of the cells as being either acceptable or unacceptable (recall once again that we use Assumption 2.7); this takes \( O_D(n^{4/3} \text{polylog } n) \) time in total.

The execution of the second decomposition step proceeds verbatim as above, since we consider only the unsplit pairs of vertical visibility in a cell \( \Omega \).

We can again remove the assumption that the lines are non-vertical, by computing a partitioning polynomial \( Q \) for the \( xy \)-projections of the vertical lines using Theorem 1.2, which takes \( O_D(n) \) time in this case. Then we take the product of \( Q \) and \( P \) and continue with the execution of the second decomposition step as just described. We thus conclude:

**Theorem 2.11.** The decomposition described in Theorem 2.6 for the case of \( n \) lines or line segments in \( \mathbb{R}^3 \) whose \( xy \)-projections are in general position can be computed in randomized expected time \( O_D(n^{4/3} \text{polylog } n) \).
3 An Application: Eliminating Depth Cycles among Lines

In [10], Aronov and Sharir obtain a combinatorial bound on the number of cuts needed to eliminate cycles in a collection of lines. The main obstruction to converting this combinatorial bound into an algorithmic procedure is the absence of a constructive version of Theorem 1.1. More specifically, their proof proceeds by partitioning $\mathbb{R}^3$ using a polynomial $f$ of degree $D$ (see more below on the choice of $D$), and then cutting each line not contained in $Z(f)$ at the points where it crosses $Z(f)$ (lines contained in $Z(f)$ need slightly different treatment; we omit the details here; this does not affect the asymptotics of the algorithm runtime or of the number of cuts required). This procedure produces at most $D$ cuts per line. Every line $\ell$ is also cut at $O(D^2)$ additional points, which correspond to locations where $\ell$ passes above a critical point of $f$; more precisely, this is a point $(x_0, y_0, z_0) \in \ell$ such that $f$ and $\partial f/\partial z$ are simultaneously zero at $(x_0, y_0, z_1)$ for some $z_1 < z_0$. This results in a total of $O(D^2 n)$ cuts.

Now, for each open cell $\tau$ of $\mathbb{R}^3 \setminus Z(f)$, Aronov and Sharir [10] collect the set $L_\tau$ of the lines of $\mathcal{L}$ meeting $\tau$, and each set $L_\tau$ is handled recursively, producing a recurrence of the form $C(n) = O(D^3) \cdot C(cn/D^2) + O(D^2 n)$ for the number of cuts $C(n)$ sufficient to eliminate all cycles in a set of $n$ lines, for a suitable absolute constant $c$. The bound of $O(n^{3/2} \log^{O(1)} n)$ is obtained by setting $D$ to $\Theta(n^{1/4})$.

We next sketch how to efficiently implement the steps outlined above. We construct the partitioning in time $O_D(n^{4/3} \text{polylog } n)$, using Theorem 2.11 where we are now forced to choose $D$ a constant; our polynomial $P$ has degree at most $D \log D$, which increases the number of cuts to $O((D \log D)^2 n)$ and the number of cells to $O((D \log D)^3)$. Determining where each line is cut can be done in time $O_D(1)$ as described in Section 2.3. Finding the cuts of the second type along a line can be done by constructing the solution set of the system $\{P = 0, \partial P/\partial z = 0\}$ in the vertical halfplane bounded by the line, in time $O_D(1)$; this follows from properties of resultants [11] Chapter 4] and our Assumption 2.7. Additional work required to process the secondary subdivision involves simply cutting each line meeting a primary subdivision cell $\Omega$ (recall that the sets $L_\Omega$ are constructed by the algorithm of Theorem 2.11 at the points where it crosses the boundary of each prism $\sigma_\Omega$, or equivalently finding the points where the projection of such a line enters and exits each trapezoid of the vertical decomposition $\mathcal{A}_v(R^1_\Omega)$; this computation is already performed in Theorem 2.11. We thus obtain $O(D^3 \log^3 D)$ additional cuts for each line.

A close examination of [10] shows that, even though our partition is not a strictly polynomial one, due to the presence of the secondary subdivision, the correctness argument of Aronov and Sharir [10] carries through here as well. To summarize, the number of cuts made by our algorithm is described by the recurrence

$$C(n) = O(D^3 \log^3 D) \cdot C(cn/D^2) + O_D(n),$$

where $c$ is an absolute constant and $D$ is a constant of our choice. The expected running time on the other hand is governed by the recurrence

$$T(n) = O(D^3 \log^3 D) \cdot T(cn/D^2) + O_D(n^{4/3} \text{polylog } n).$$

They both solve to $O_D(n^{3/2+\varepsilon(D)})$, once we pick a sufficiently large constant $D > 0; \varepsilon = \varepsilon(D) > 0$ depends on $D$ and can be made arbitrarily small by increasing $D$, so we can rewrite the bound as $O(c(n^{3/2+\varepsilon})$. Note that, since $D$ cannot be set to grow with $n$, the number of cuts guaranteed by our algorithm is slightly larger than that guaranteed by the upper bound of [10], namely $O(n^{3/2} \log^{O(1)} n)$. 

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Theorem 3.1. Let $\mathcal{L}$ be a collection of $n$ non-vertical pairwise disjoint lines in $\mathbb{R}^3$ in general position\footnote{See Section 2.4.}, then one can apply $O_\varepsilon(n^{3/2+\varepsilon})$ cuts eliminating all depth cycles among the lines in $\mathcal{L}$. These cuts can be computed in expected time $O_\varepsilon(n^{3/2+\varepsilon})$, for any $\varepsilon > 0$. The number of cuts is near optimal in the worst case.

Remarks. (a) Previous algorithms that solve this problem apply an approximation algorithm of Aronov, de Berg, Gray, and Mumford \footnote{B. Aronov, M. de Berg, C. Gray, and E. Mumford, Cutting cycles of rods in space: Hardness and approximation, In Proc. Nineteenth Annu. ACM-SIAM Sympos. Discr. Alg., 2008, 1241–1248.}, which involves matrix multiplication. The running time is close to $O(n^{4+2\omega})$, where $\omega < 2.373$ is the exponent of matrix multiplication; this was later improved by de Berg \footnote{P. K. Agarwal, J. Matoušek, and M. Sharir, On range searching with semialgebraic sets II. SIAM J. Comput., 42 (2013), 2039–2062.} to $O(n^{3+\omega})$. In spite of the fact our bound $O(n^{3/2+\varepsilon})$ on the number of cuts is slightly inferior with respect to the bound $O(n^{3/2} \log^{O(1)} n)$ in \cite{10}, our algorithm is considerably more efficient.

(b) Note that the algorithm described above works equally well with non-vertical pairwise disjoint algebraic curves of constant degree, with only superficial modifications, mirroring the combinatorial analysis of Aronov and Sharir \cite{10} as well as Sharir and Zahl \cite{27}. The current analysis, however, can only guarantee quadratic running time.

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References

[1] P. K. Agarwal, Partitioning arrangements of lines II: Applications, Discrete Comput. Geom., 5 (1990), 533–573.

[2] P. K. Agarwal, B. Aronov, E. Ezra, and J. Zahl. An efficient algorithm for generalized polynomial partitioning and its applications, Proc. 35th Int. Symp. Comput. Geom., 2019, to appear. Also in arXiv:1812.10269.

[3] P. K. Agarwal, E. F. Grove, T. M. Murali, and J. S. Vitter, Binary space partitions for fat rectangles, SIAM J. Comput. 29(5) (2000), 1422–1448.

[4] P. K. Agarwal, J. Matoušek, and M. Sharir, On range searching with semialgebraic sets II. SIAM J. Comput., 42 (2013), 2039–2062.

[5] P. K. Agarwal and M. Sharir, Arrangements and their applications, In Handbook of Computational Geometry, J. Sack and J. Urrutia, eds., Elsevier, Amsterdam, pp. 973–1027, 2000.

[6] B. Aronov, M. de Berg, C. Gray, and E. Mumford, Cutting cycles of rods in space: Hardness and approximation, In Proc. Nineteenth Annu. ACM-SIAM Sympos. Discr. Alg., 2008, 1241–1248.

[7] B. Aronov, E. Ezra, and J. Zahl. Constructive polynomial partitioning for algebraic curves in $\mathbb{R}^3$ with applications. In Proc. 30th Ann. ACM-SIAM Symp. Discr. Alg, SODA 2019, pp. 2636–2648.
[8] B. Aronov, V. Koltun, and M. Sharir, Cutting triangular cycles of lines in space, *Discrete Comput. Geom.* 33 (2005), 231–247.

[9] B. Aronov, E. Y. Miller, and M. Sharir, Eliminating depth cycles among triangles in three dimensions, In *Proc. 28th Annu. ACM-SIAM Sympos. Discr. Alg.*, 2017, pp. 2476–2494. Also in arXiv:1607.06136v2.

[10] B. Aronov and M. Sharir, Almost tight bounds for eliminating depth cycles in three dimensions, *Discrete Comput. Geom.*, 59 (2018), 725–741. Also in *Proc. 48th ACM Sympos. Theory of Computing*, (2016), pp. 1–8.

[11] S. Basu, R. Pollack, and M. F. Roy, *Algorithms in Real Algebraic Geometry*, 2nd edition, *Springer-Verlag*, Berlin Heidelberg, 2006.

[12] M. de Berg, Linear size binary space partitions for uncluttered scenes. *Algorithmica* 28 (2000) 353–366.

[13] M. de Berg, Removing depth-order cycles among triangles: an efficient algorithm generating triangular fragments, In *Proc. 58th IEEE Annu. Sympos. Found. of Computer Science*, (2017), 272–282.

[14] M. de Berg, O. Cheong, M. van Kreveld, and M. Overmars, *Computational Geometry: Algorithms and Applications*, 3rd Ed., *SpringerVerlag*, Berlin-Heidelberg, 2008.

[15] M. de Berg, O. Schwarzkopf, Cuttings and applications. *Int. J. Comput. Geometry Appl.*, 5(4):343-355 (1995).

[16] B. Chazelle, H. Edelsbrunner, L. Guibas, and M. Sharir, A singly exponential stratification scheme for real semi-algebraic varieties and its applications, *Theoret. Comput. Sci.* 84:77–105, 1991. Also in *Proc. 16th Int. Colloq. on Automata, Languages and Programming*, 1989, pp. 179–193.

[17] B. Chazelle, H. Edelsbrunner, L. J. Guibas, R. Pollack, R. Seidel, M. Sharir, J. Snoeyink, Counting and cutting cycles of lines and rods in space. *Comput. Geom.* 1(6), 305–323 (1992).

[18] B. Chazelle and J. Friedman, A deterministic view of random sampling and its use in geometry, *Combinatorica* 10 (1990), 229–249.

[19] K. L. Clarkson and P. W. Shor, Applications of random sampling in computational geometry, II, *Discrete Comput. Geom.*, 4 (1989), 387–421.

[20] E. Ezra and M. Sharir, Counting and representing intersections among triangles in three dimensions, *Comput. Geom. Theory Appl.* 32 (2005), 196–215.

[21] L. Guth, Polynomial partitioning for a set of varieties, *Math. Proc. Camb. Phil. Soc.*, 159 (2015) 459–469.

[22] L. Guth and N. H. Katz, On the Erdős distinct distance problem in the plane, *Annals Math*. 181 (2015), 155–190. Also in arXiv:1011.4105

[23] V. Koltun and M. Sharir, Curve-sensitive cuttings, *SIAM J. Comput.*, 34(4), 863–878 (2005).
[24] M. S. Paterson, and F. F. Yao, Efficient binary space partitions for hidden-surface removal and solid modeling. *Discrete Comput. Geom.*, 5 (1990), 485–503.

[25] M. S. Paterson and F. F. Yao. Optimal binary space partitions for orthogonal objects. *J. Alg.* 13 (1992), 99–113.

[26] M. Sharir and P. K. Agarwal, *Davenport-Schinzel Sequences and Their Geometric Applications*, Cambridge University Press, New York, 1995.

[27] M. Sharir and J. Zahl, Cutting algebraic curves into pseudo-segments and applications, *J. Combinatorial Theory*, Series A, 150 (2017), 1–35.

[28] A. H. Stone and J. W. Tukey, Generalized sandwich theorems, *Duke Math. J.* 9 (1942), 356–359.

[29] C. D. Tóth, Binary space partitions for axis-aligned fat rectangles. *SIAM J. Comput.*, 38(1) (2008), 429–447.