s-Compressible and s-Prime Modules

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Abstract
Let $R$ be a ring with identity and $A$ a left $R$-module. In this article, we introduce new generalizations of compressible and prime modules, namely s-compressible module and s-prime module. An $R$-module $A$ is s-compressible if for any nonzero submodule $B$ of $A$ there exists a small $f$ in $\text{Hom}_R(A, B)$. An $R$-module $A$ is s-prime if for any submodule $B$ of $A$, ann$_R(B)A$ is small in $A$. These concepts and related concepts are studied in as well as many results consist properties and characterizations are obtained.

Keywords: critically s-compressible module, retractable module s-compressible module, s-prime module, small submodule.

1. Introduction
Compressible module was introduced by Zelmanowitz [1] simultaneous with introducing the concept of weakly primitive ring in the way of generalizing the Jacobson density theorem. He also introduced critically compressible module. In[2], the author studied those concepts in details. A left $R$-module is compressible if it can be embedded in any of its nonzero submodule[1]. A compressible module $A$ is critically compressible if it cannot be embedding in any factor $A/B$, where $B$ is a nonzero submodule of $A$. In[1], Zelmanowitz defined a ring to be weakly primitive if it possesses a faithful critically compressible module. In[3]–[6], authors have been extensively studied compressible, critically compressible and prime modules. By using small submodules one direction of generalizations of compressible and prime modules e appeared in [7]–[9]. A small compressible module is defined as a module that can be embedded in its small submodules, as well as small prime module is defined as a
module $A$ in which $\text{ann}_RB=\text{ann}_RA$ for each small submodule $B$ of $A$. Note that a module $A$ is prime, if $\text{ann}_RB=\text{ann}_RA$ for each nonzero submodule $B$ of $A$ [7].

Throughout this work, we use the notion of small submodule. Different generalizations are given. We recall that, an $R$-homomorphism in $\text{Hom}(A, B)$ is said to be small if its kernel is small in $A$[10]. In the new generalization the zero kernel will be replaced by small kernel. An $R$-module $A$ is said to be $s$-compressible if for each nonzero submodule $B$ of $A$ there exists a small element $f$ in $\text{Hom}(A, B)$, that is $\text{ker}f$ is small in $A$. Note that this definition is also appeared in [11] with different abbreviation, sk-compressible.

An $s$-compressible module $A$ is critically $s$-compressible if $\text{Hom}(A, A/ B)$ has no small element for any non-small submodule $B$ of $A$. A module $A$ is $s$-prime if $(\text{ann}_RB) A$ is small in $A$ for any nonzero submodule $B$ of $A$. These concepts are studied, and their relationships among them and with other related concepts are discussed. Some properties and characterizations are obtained. Firstly, it is shown that $s$-compressible with small compressible modules are independent, as well as the $s$-prime and small prime modules are also independent. The class of compressible modules contains both classes of $s$-compressible and small compressible modules. As well as the class of prime modules contains both classes of $s$-prime and small prime modules.

Throughout this article some definitions and notations are given. A module is a left unitary module over a ring $R$ with identity. A submodule $B$ of a module $A$ will be abbreviated by $B \leq A$. A submodule $B$ of a module $A$ is said to be small in $A$ (abbreviated by $B \ll A$) if it is proper and its sum with any other proper submodule of $A$ is again proper, "in other word if $B + C = A$, where $C \leq A$, then $C = A$ [10]. A is said to be hollow if all its proper submodules are small. $\text{Hom}(D, E)$ denotes the set of all $R$-homomorphisms from $D$ into $E$. If $f \in \text{Hom}(D, E)$, then $\text{ker}f = \{d \in D | f(d) = 0\}$, $f$ is a monomorphism if $\text{ker}f = 0$ and it is small if $\text{ker}f \ll D$[10].

If $B \leq A$, then $\text{ann}_RB = \{r \in R | rb = 0\}$ for all $b \in B$ which is called the annihilator of $B$ in $R$ and it is a left ideal of $R$ if $b \in B$, then $\text{ann}_RB = \text{ann}_R\{b\}$. If $B \leq A$, then $[B: R A] = \{ r \in R | rA \subseteq B \}$ is a left ideal of $R$. An $R$-module $A$ is multiplication if for any submodule $B$ of $A$ there exists an ideal $I$ of $R$ such that $B = IA$, in this case $I = [B: RA]$[12]. An $R$-module $A$ is retractable if $\text{Hom}_R(A, B) \neq 0$ for any nonzero submodule $B$ of $A$ [13].

In Section 2 $s$-compressible and critically $s$-compressible modules are introduced and investigated. The notion $s$-compressible is appeared in [11]. It is abbreviated by $sk$-compressible. In this work this notion is studied in details and more results are given. Section 3 devotes to introduce $s$-prime module and study the relationships between the present notions and old related notions.

2. s-Compressible and Critically s-Compressible Modules

Definition (2.1): A nonzero $R$-module $A$ is called $s$-compressible if for any nonzero submodule $B$ of $A$ there exists a small $R$-homomorphism from $A$ into $B$.

Remark (2.2): Any compressible module is $s$-compressible, however the converse is not true.

Remark (2.3): Any simple module is $s$-compressible.

Example (2.4): Consider the $\mathbb{Z}$-module $\mathbb{Z}_n$, if $n = mp^k$ where $p$ is a prime which is not dividing $m$, thus if $s\mathbb{Z}_n$ is a small submodule of $\mathbb{Z}_n$, then $s = pt$ for some $t$.

Note that, in a $R$-module $A$, the submodule $Ra$ is small in $A$ if and only if $a$ belongs to all maximal submodules of $A$ [10].

Now, if $f: \mathbb{Z}_n \rightarrow p^k \mathbb{Z}_n$ is a $\mathbb{Z}$-homomorphism such that $\ker f = s \mathbb{Z}_n$ small in $\mathbb{Z}_n$, then $|\ker f| = n/s$, so that $|\mathbb{Z}_n/\ker f| = s = pt$, while $|p^k \mathbb{Z}_n| = m$, this gives a contradiction with the fact that $\mathbb{Z}_n/\ker f$ is isomorphic to a submodule of $p^k \mathbb{Z}_n$. Therefore, there is no small $\mathbb{Z}$-homomorphism from $\mathbb{Z}_n$ into $p^k \mathbb{Z}_n$, that is, $\mathbb{Z}_n$ is not $s$-compressible if $n = mp^k$ and $p$ is a prime which is not dividing $m$.
On the other hand \( n=p^k \), the \( \mathbb{Z} \)-module \( \mathbb{Z}_n \) is hollow, all its proper submodules are small. It is easy to see that it is s-compressible. Therefore the \( \mathbb{Z} \)-module \( \mathbb{Z}_n \) is s-compressible if and only if \( n=p^k \) where \( p \) is prime.

We note that the two notions small compressible and s-compressible are independent. For example \( \mathbb{Z}_6 \) is small compressible \( \mathbb{Z} \)-module which is not s-compressible, while \( \mathbb{Z}_4 \) is s-compressible that is not small compressible \( \mathbb{Z} \)-module[8]. Both of two \( \mathbb{Z} \)-modules are not compressible. The two classes of small and s-compressible modules contain the class of compressible modules.

Remark 2.5: It is clear that any s-compressible module is retractable. However the converse is not true to see that \( \mathbb{Z}_6 \) as a \( \mathbb{Z} \)-module is retractable but not s-compressible.

Next proposition gives However, a condition can be added to a retractable module to get s-compressible module, see the following.

**Proposition 2.6:** Any hollow retractable module is s-compressible.

**Proof:** Assume that \( A \) is a hollow retractable module, and \( B \) is a nonzero submodule of \( A \), then there exists \( 0 \neq f \in \text{Hom}(A, B) \) such that \( \ker f \) is a proper submodule of \( A \), hence small in \( A \). Therefore \( A \) is s-compressible.

This proposition can be applied to example 2.4 so that \( \mathbb{Z}_{p^k} \) is s-compressible.

We note that the \( \mathbb{Z} \)-module \( \mathbb{Z} \) is s-compressible but not hollow, and this proves that the converse of proposition 2.6 is not true.

**Proposition 2.7:** If \( B \) is a submodule of an s-compressible module \( A \) such that \( J(B)=J(A) \), then \( B \) is s-compressible.

**Proof:** Assume that \( B \) is a submodule of an s-compressible module \( A \) and \( J(B)=J(A) \). If \( K \leq B \), then \( K \leq A \), hence there exists \( f \in \text{Hom}(A, K) \) with \( \ker f \ll A \). Now if \( g=\frac{f}{p} \) then \( g \in \text{Hom}(B, K) \), and \( \ker g \cap \ker f \subseteq B \cap J(A) = B \cap J(B) \leq J(B) \), so that \( \ker g \ll B \). Therefore \( B \) is s-compressible. \( \square \)

**Example 2.8:**

(i) Consider \( A = \mathbb{Q} \oplus \mathbb{Z}_p \), where \( p \) is prime, as a \( \mathbb{Z} \)-module and \( B = \mathbb{Q} \oplus 0 \), then \( B \leq A \) and \( J(B)=J(A)=\mathbb{Q} \oplus 0 \).

(ii) Let \( A = \mathbb{Z} \), as a \( \mathbb{Z} \)-module and \( B = n \mathbb{Z} \), then \( J(B)=J(A)=0 \).

**Corollary 2.9:** If \( J(A)=0 \) and \( A \) is an s-compressible module, then any submodule of \( A \) is s-compressible.

**Proposition 2.10:** If \( A \) is an s-compressible module and \( B \) is a nonzero submodule of \( A \), then \( \text{ann} B \ll A \).

**Proof:** Since \( A \) is an s-compressible, then there exists \( f \in \text{Hom}(A, B) \) with \( \ker f \ll A \). Let \( r \in \text{ann} B \), so that for each \( m \in A \), \( f(mr)=rf(m)=0 \), then \( rm \notin \ker f \ll A \), this implies that \( \text{ann} B \) \( A \subseteq \ker f \ll A \). Therefore \( \text{ann} B \ll A \). \( \square \)

The converse of Proposition 2.10 is not true, for example if \( A \) is a torsion free \( R \)-module, then \( \text{ann} B =0 \) for any non zero submodule \( B \) of \( A \), hence \( \text{ann} B \ll A \ll A \). While there are many torsion free modules not s-compressible, e.g. the \( \mathbb{Z} \)-module \( \mathbb{Q} \).

**Proposition 2.11:** If \( A \) is an \( R \)-module with \( J(A)=0 \), then \( A \) is s-compressible if and only if it is compressible.

**Proof:** The sufficiency is clear. Conversely, \( J(A)=0 \) implies that \( A \) has no nonzero small submodule, so, if \( A \) is s-compressible, there exists \( f \in \text{Hom}(A, B) \) with \( \ker f \) small in \( A \) which implies \( \ker f=0 \) and \( f \) is a monomorphism. \( \square \)

It is well known that a nonzero submodule of a compressible module is compressible. In the following this property will be discussed under certain condition for s-compressibility.

Recall that, an \( R \)-module \( A \) is said to be fully stable, if for each submodule \( B \) of \( A \) and for each \( f \in \text{Hom}(B, A) \), it follows \( f(B) \subseteq B \) [12]. In fact \( A \) is fully stable if and only if \( \text{Hom}(B, A)=\text{End}(B) \) for each submodule \( B \) of \( A \) and more details about fully stable modules can be found in [12]. For completeness a proof will be given.
Lemma 2.12: If $A$ is a fully stable module, $B = B_1 \oplus B_2$ and $K$ are submodules of $A$, then $K \cap B = (K \cap B_1) \oplus (K \cap B_2)$.

Proof: The natural projections of $B$ onto $B_1$ and $B_2$, respectively $\pi_1$, $\pi_2$ are elements of $\text{Hom}(B, B) = \text{End}(B)$, in fact, $\pi_1 \in \text{Hom}(B, B_1)$ and $\pi_2 \in \text{Hom}(B, B_2)$. On the other hand $\pi_1 + \pi_2 = 1_B$, so, $K \cap B = \pi_1(K \cap B) + \pi_2(K \cap B)$. Since $A$ is fully stable, $\pi_i(K \cap B_i) \subseteq K \cap B_i$, $\; (i = 1, 2)$ but $\pi_i(K \cap B) \subseteq B_i$ so $\pi_i(K \cap B) \subseteq K \cap B_i$. Hence $K \cap B \subseteq (K \cap B_1) \oplus (K \cap B_2) \subseteq K \cap B$. □

It is known that any small submodule of a module is contained in its Jacobson radical, while a submodule that contained in the Jacobson radical of the module is small if it is finitely generated [10].

Proposition 2.13: A finitely generated direct summand of a fully stable s-compressible module is s-compressible.

Proof: Assume that $A = A_1 \oplus A_2$ is an s-compressible module and $B$ is a submodule of $A_1$, then $B$ is a submodule of $A$, by assumption there exists $f \in \text{Hom}(A, B)$ with $\ker f \ll A$. Let $g = f|_{A_1}$, then $\ker g = A_1 \cap \ker f$. It is known that $(\text{J}(A) = \text{J}(A_1) \oplus \text{J}(A_2))$ implies $A_1 \cap \ker f \subseteq A_1 \cap (\text{J}(A_1) \oplus \text{J}(A_2) = \text{J}(A_1))$(by full stability) so that $\ker g \subseteq \text{J}(A_1)$ and $\ker f \ll A_1$. Therefore $A_1$ is s-compressible. □

Remark 2.14: The converse of Proposition 2.13 is not true to see that let $\mathbb{Z}_6 = (\bar{2}) \oplus (\bar{3})$ is fully stable [11] and both $(\bar{2})$ and $(\bar{3})$ are s-compressible, however $\mathbb{Z}_6$ is not s-compressible, as we have seen in Example 2.4.

Remark 2.15: It is clear that a homomorphic image of an s-compressible module need not be s-compressible. For instance $\mathbb{Z}$ is an s-compressible Z-module, however $\mathbb{Z}/6\mathbb{Z}$ is not.

Proposition 2.16: If $A_1$ and $A_2$ are two isomorphic modules, then $A_1$ is s-compressible if and only if $A_2$ is s-compressible.

Proof: Assume that $\varphi: A_1 \rightarrow A_2$ is an isomorphism and $A_1$ is s-compressible. Let $B$ be a nonzero submodule of $A_2$. Then $\varphi^{-1}(B)$ is a nonzero submodule of $A_1$, by assumption there exists $\alpha: A_1 \rightarrow \varphi^{-1}(B)$ with $\ker \alpha \ll A_1$. Let $\delta = \alpha \varphi^{-1}$, where $j = \varphi|_{\varphi^{-1}(B)}$, then $\delta \in \text{Hom}(A_2, B)$ and $\ker \delta = \varphi(\ker \alpha) \ll A_2$. Hence $A_2$ is s-compressible. The proof of the other direction is similar. □

Lemma 1.17: If $A$ is a multiplication module and $A = A_1 \oplus A_2$, then $\text{ann}_A A_i = \{ A_j: A \}$, $i \neq j$, $i, j = 1, 2$.

Proof: Let $r \in \text{ann}_A A_1$, then for each $m = m_1 + m_2$, $r(m_1 + m_2) = r m_2 \in A_2$, so that $r \in \{ A_2: A \}$. Conversely. The $r \in \{ A_2: A \}$ implies that for each $m_1 \in A_1$, if $m_2$ is any element of $A_2$, then $m_1 + m_2 \in A$ and $r(m_1 + m_2) \in A_2$, which implies $r m_1 \in A_1 \cap A_2$, hence $r m_1 = 0$ and $r \in \text{ann}_A A_1$. This proves $\text{ann}_A A_1 = \{ A_2: A \}$. By the same manner the other case can be proved.

Proposition 2.18: If $A$ is a multiplication and s-compressible $R$-module then it is indecomposable.

Proof: Assume that $A = A_1 \oplus A_2$, since $A$ is multiplication, we have $A_1 = \{ A_1: A \}$ and $A_2 = \{ A_2: A \}$. By Lemma 1.17, $A_1 = (\text{ann}_R A_2) A$ and $A_2 = (\text{ann}_R A_1) A$, then $A = (\text{ann}_R A_2) A \oplus (\text{ann}_R A_1) A$. But by Proposition 2.3 ($\text{ann}_R A_1) A$ and $(\text{ann}_R A_2) A$ are both small in $A$, which is a contradiction. Therefore $A$ is indecomposable. □

An $R$-module $A$ is said to be duo if any submodule of $A$ is full invariant, that is, for each $f \in \text{End}(A)$ and for each $B \subseteq A$, $f(B) \subseteq B$ [14], and it is said to be torsion free if $rm \neq 0$ whenever $0 \neq r \in R$ and $0 \neq m \in A$, or equivalently $0 \neq m \in A$ and $rm \neq 0$ implies $r \neq 0$.

Next theorems give a characterization of Duo modules, we will start with the following lemma.

Lemma 2.19: "An $R$-module $A$ is duo if and only if for each $f \in \text{End}(A)$ and for each $m \in A$ there exists $r \in R$ such that $f(m) = rm$" [14].
Theorem 2.20: Let $A$ be a duo torsion free $R$-module. Then $A$ is compressible if and only if it is retractable.

Proof: ($\Rightarrow$) It is clear so that it is omitted .

($\Leftarrow$) Assume that $A$ is a duo torsion free $R$-module and retractable, let $0 \neq B \leq A$, then there exists $0 \neq f \in \text{Hom}(A, B)$, it can be considered that $f \in \text{End}(A)$. By Lemma 2.19, for each $m \in A$ there exists $r \in R$ such that $f(m)=rm$. So $\ker f= \{ m \in A \mid rm=0 \text{ for some } r \in R \}$, as $A$ is torsion free and $0 \neq f$, it follows $\ker f=0$, that is $A$ embed in $B$. Therefore $A$ is compressible. □

A compressible module is said to be critically compressible if it cannot be embedded in any of its proper factors [2]. This notion was generalized in [7] using small submodule this way gives that a small compressible module $A$ is called small critically compressible if $A$ cannot be embedded in any proper quotient module $A/B$ with $0 \neq B \ll A$.

Another generalization will be given by using small submodule.

Definition 2.21: An $R$-module $A$ is called critically s-compressible if it is s-compressible and for any not small submodule $B$ of $A$, $\text{Hom}(A, A/B)$ contains no small element.

Remark 1.22: The two classes small critically compressible modules, and critically s-compressible modules are different (see Example 2.23(ii)), and their intersection contains the class of critically compressible modules.

Example 2.23: (i) The $\mathbb{Z}$-module $\mathbb{Z}_n$ is critically s-compressible if and only if $n=p^k$ where $p$ is a prime.

Proof: In Example 2.4, we proved that $\mathbb{Z}_n$ is s-compressible if and only if $n=p^k$ where $p$ is a prime. Since $\mathbb{Z}_{p^k}$ has no proper submodule which is not small, so it is critically s-compressible.

(ii) $\mathbb{Z}_{p^k}$, is not small critically compressible module for $k>1$. While $\mathbb{Z}_n$ is small critically compressible $\mathbb{Z}$-module but not critically s-compressible.

(iii) The $\mathbb{Z}$-module $\mathbb{Z}$, also is critically s-compressible.

(iv) Any critically compressible module is critically s-compressible. But the converse is not true.

(v) Any simple module is critically s-compressible.

By partial endomorphism of a module $A$ it means an element of $\text{Hom}(B, A)$ where $B$ is a submodule of $A$.

Proposition 2.24: If $A$ is a critically s-compressible module, then any nonzero partial endomorphism of $A$ has kernel small in $A$.

Proof: Assume that $A$ is a critically s-compressible module and $0 \neq f \in \text{Hom}(B, A)$, where $B \leq A$, suppose that $\ker f$ is not small in $A$. Then $\text{Im} f \neq 0$ and there exists $0 \neq g \in \text{Hom}(A, \text{Im} f)$ such that $kerg \ll A$ since $A$ is s-compressible. On the other hand $\text{Im} f \cong N/\ker f \leq A/\ker f$, let $h: \text{Im} f \rightarrow B/\ker f$ be an isomorphism and $i: B/\ker f \rightarrow \text{ker} f$ be the inclusion map. Then ihg$\in \text{Hom}(A, A/\ker f)$ and $kerg=\ker g \ll A$. This contradicts the assumption that $A$ is critically s-compressible.

To prove the converse of Proposition 2.24, we need a condition this is given in next proposition.

Proposition 2.25: Let $A$ be an s-compressible module such that for any $L \leq A$ and $K \ll A$, any element of $\text{Hom}(L, A/K)$ has kernel small in $A$. Then $A$ is critically s-compressible.

Proof: Assume that $A$ is an s-compressible module satisfying the above condition. Let $B$ be a submodule of $A$ which is not small and $f \in \text{Hom}(A, A/B)$ and $\ker f \ll A$. Then $A/\ker f \cong L/B$, where $L$ is a submodule of $A$ containing $B$. Let $v: L \rightarrow L/B$ be the natural epimorphism and $q: L/B \rightarrow A/\ker f$ be an isomorphism, then $q \circ v \in \text{Hom}(L, A/\ker f)$ and $\ker q \circ v = B$ which is not small in $A$, a contradiction with the assumed condition. Therefore, the kernel of any element of $\text{Hom}(A, A/B)$ is not small in $A$, that is, $A$ is critically s-compressible. □

3. s-Prime Modules

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Prime modules are defined and investigated in the literatures see [3][4][15]. An R-module A is said to be prime if for any nonzero submodule B of A, \( \text{ann}_R B = \text{ann}_R A \). This notion is generalized in [7] using the concept of small submodules in this way \( \text{ann}_R B = \text{ann}_R A \) for each non-zero small submodule B of A. We also use small submodules to give a different generalization for prime module, its properties, and characterizations as well as relations with s-compressible is also studied.

**Definition 3.1:** A nonzero R-module A is called s-prime if for any nonzero submodule B of A, \( (\text{ann}_R B) A \ll A \).

**Remark 3.2:**
(i) The two notions small prime, and s-prime are independent. For example the \( \mathbb{Z} \)-module \( \mathbb{Z}_4 \) is an s-prime but not small prime (can be easily checked), while the \( \mathbb{Z} \)-module \( \mathbb{Z}_{24} \) is small prime , this is also shown in[7]. However it is not s-prime since \( (\text{ann}_\mathbb{Z}(\langle 8 \rangle )) \mathbb{Z}_{24}=3\mathbb{Z}(\mathbb{Z}_{24}) = \langle 3 \rangle \) not small in \( \mathbb{Z}_{24} \).
(ii) We have seen that any torsion free R -module is s-prime.
(iii) It is clear that any prime module is s-prime. However the converse is not true, for example the \( \mathbb{Z} \)-module \( \mathbb{Z}_8 \) is not prime since \( \text{ann}_\mathbb{Z}(4 \mathbb{Z}_8)=2\mathbb{Z} \), while \( \text{ann}_\mathbb{Z}(\mathbb{Z}_8)=8 \). But \( \mathbb{Z}_8 \) is s-prime \( \mathbb{Z} \)-module (can be easily checked).

**Proposition 3.3:** An s-compressible module is s-prime.

**Proof:** See Proposition 2.10.

It is clear the converse of Proposition 3.3 is not true, the \( \mathbb{Z} \)-module \( \mathbb{Q} \) is s-prime since it is torsionfree (Remark 3.2(ii)) but not s-compressible since \( \text{Hom}(\mathbb{Q}, \mathbb{Z})=0 \).

**Proposition 3.4:** A nonzero R-module A is s-prime if and only if \( (\text{ann}_R Rx) A \ll A \) for each \( 0 \neq x \in A \).

**Proof:** (\( \Rightarrow \)) It is clear.
\( (\Leftarrow) \) Assume that \( (\text{ann}_R Rx) A \ll A \) for each \( 0 \neq x \in A \), and let \( 0 \neq B \subseteq A \), then there exists \( 0 \neq x \in B \) and \( (\text{ann}_R B) \subseteq (\text{ann}_R Rx) \) which implies \( (\text{ann}_R B) A \subseteq(\text{ann}_R Rx) A \ll A \), hence \( (\text{ann}_R B) A \ll A \). Therefore A is s-prime.

**Proposition 3.5:** A nonzero R-module A is s-prime if and only if for each \( 0 \neq B \subseteq A \) and for each ideal I of R, \( IB=0 \) implies \( IA \ll A \).

**Proof:** It is clear that \( IB=0 \) means \( I \subseteq (\text{ann}_R B) \).

**Theorem 3.6:** Let A be a multiplication retractable R-module, then A is s-compressible if and only if it is s-prime.

**Proof:** (\( \Rightarrow \)) See Proposition 3.3.
\( (\Leftarrow) \) Assume that A is a multiplication retractable R-module and \( 0 \neq B \subseteq A \). Then, there exists \( 0 \neq f \in \text{Hom}(A, B) \), since A is retractable, that is \( \text{Im} f \neq 0 \). As A is s-prime, it follows \( (\text{ann}_R \text{Im} f) A \ll A \).

Now, \( \text{ann}_R (\text{Im} f)=\{ r \in R \mid rf=0, \ \forall m \in A \} = \{ r \in R \mid rm \in \ker f, \ \forall m \in A \} = \ker f : A \). Hence \( (\text{ann}_R \text{Im} f) A = [\ker f : A] A = \ker f \), since A is multiplication. Therefore, we have \( \ker f \ll A \), and A is s-compressible.

In [16] author proved that a faithful multiplication R-module is retractable according to this result Theorem 3.6 can be rewritten as following.

**Corollary 3.7:** Let A be a faithful multiplication R-module, then A is s-compressible if and only if it is s-prime.

Recall that a ring R is called left duo if any left ideal is two sided ideal [14].

**Proposition 3.8:** Let R be a left duo ring. A nonzero R-module A is s-prime if and only if for each \( 0 \neq x \in A \) and for each ideal I of R, \( Ix=0 \) implies \( IA \ll A \).

**Proof:** (\( \Rightarrow \)) Assume that \( 0 \neq x \in A \) and \( Ix=0 \), then \( IRx=RIx=0 \) where \( 0 \neq Rx \ll A \) and by assumption \( IA \ll A \).
(⇐) Let $0 \not= B \leq A$ and $IB=0$, if $0 \not= x \in B$, then $Ix=0$, by assumption $IA \ll A$, therefore $A$ is s-prime.

Remark 3.9: The $\mathbb{Z}$-module $\mathbb{Z}_n$ is s-prime if and only if $n= p^k$ where $p$ is a prime number.

Proof: If $n= p^k$, then $\mathbb{Z}_n$ is s-compressible (see Example 2.4), and by Proposition 3.3, it is s-prime. If $n= mk$ with $(m, k)=1$, then $ann_\mathbb{Z}(\overline{m}) = k \mathbb{Z}$, $(k \mathbb{Z}) \mathbb{Z}_n=(\overline{k})$ which is not small in $\mathbb{Z}_n$ since $(\overline{m})+(\overline{k})=\mathbb{Z}_n$.

Proposition 3.10: Let $B$ be a finitely generated submodule of an $R$-module $A$ and $J(B)=J(A)$. If $A$ is s-prime then $B$ is also s-prime.

Proof: Assume that $A$ is s-prime and $K \leq B$, then $K \leq A$ and $(ann_R K) A \ll A$.

Now, $(ann_R K) B \leq (ann_R K) A$ and $(ann_R K) A \ll A$ implies $(ann_R K)A \leq J(A) = J(B)$. Therefore $(ann_R K) B \leq J(B)$, that is, $(ann_R K) B \ll B$. □

It is known that, if $R$ is a commutative ring and $0 \not= x \in A$, where $A$ is an $R$-module, then $ann_x x$ is an ideal of $R$ and $ann_R R x = ann_R x$. The following lemma is needed to get the next result.

Lemma 3.11: Let $R$ be a commutative ring with identity and $A$ a finitely generated faithful multiplication $R$-module. If $I$ is any ideal of $R$, then $I \ll R$ if and only if $IA \ll A$.

Proof: (⇒) Assume that $I \ll R$ and $I A + B = A$ where $B \leq A$. Since $A$ is multiplication, $B=IA$ for some ideal $J$ of $R$, then $IA + JA = A$, hence $(I+ J) A = A$. This implies $I+ J= R$ (see Theorem 3.1,[16]), then $J= R$ (since $I \ll R$).Therefore, $B=IA$, that is $IA \ll A$.

(⇐) Assume that $IA \ll A$ and $I+ J= R$ for some ideal $J$ of $R$, then $IA + JA = RA = A$, and so, $JA = A$ since $IA \ll A$. Again, by (Theorem 3.1,[17]) $J= R$, hence $I \ll R$.

Theorem 3.12: Let $R$ be a commutative ring with identity and $A$ a finitely generated faithful multiplication $R$-module. Then, $A$ is s-prime if and only if $(ann_R x) \ll R$ for each $0 \not= x \in A$.

Proof: See Proposition 3.4 and Lemma 3.11.

Corollary 3.13: Let $R$ be a commutative ring with identity and $A$ a finitely generated faithful multiplication $R$-module, then $A$ is s-compressible.

Proof: Let $A$ be a finitely generated faithful multiplication $R$-module. By (Lemma 4.1,[17]), faithful multiplication modules are torsion free, then $(ann_R x)=0$, then $(ann_R x) \ll R$ for all $0 \not= x \in A$. By Theorem 3.12, $A$ is s-prime. Then by Corollary 3.7, $A$ is s-compressible.

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