KATO-PONCE INEQUALITIES ON WEIGHTED AND VARIABLE
LEBESGUE SPACES

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Abstract. We prove fractional Leibniz rules and related commutator estimates in
the settings of weighted and variable Lebesgue spaces. Our main tools are uniform
weighted estimates for sequences of square-function-type operators and a bilinear
extrapolation theorem. We also give applications of the extrapolation theorem to
the boundedness on variable Lebesgue spaces of certain bilinear multiplier operators
and singular integrals.

1. Introduction and main results

For \( s \geq 0 \), the inhomogeneous and homogeneous \( s \)-th differentiation operators \( J^s \)
and \( D^s \), respectively, are defined via the Fourier transform as
\[
\hat{J}^s(f)(\xi) = (1 + |\xi|^2)^{\frac{s}{2}} \hat{f}(\xi) \quad \text{and} \quad \hat{D}^s(f)(\xi) = |\xi|^s \hat{f}(\xi).
\]
The following inequalities, known as Kato-Ponce inequalities or fractional Leibniz
rules, hold for such operators and for \( f, g \in S(\mathbb{R}^n) \):
\[
\begin{align*}
\|D^s(fg)\|_{L^r} &\lesssim \left( \|D^s f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|D^s g\|_{L^{q_2}} \right), \\
\|J^s(fg)\|_{L^r} &\lesssim \left( \|J^s f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|J^s g\|_{L^{q_2}} \right),
\end{align*}
\]
where \( 1 < p_1, p_2, q_1, q_2 \leq \infty, \frac{1}{r} < \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}, s > \max\{0, n(\frac{1}{r} - 1)\} \)
or \( s \) is a non-negative even integer, and the implicit constants depend only on the
parameters involved. The cases \( 1 < r < \infty, 1 < q_1, p_2 \leq \infty, 1 < p_1, q_2 < \infty \)
and \( s > 0 \) for such estimates have been known for a long time and go back to the
pioneering work in \([5, 27, 28]\) for the study of the Euler, Navier-Stokes and Korteweg-de
Vries equations. Very recently in \([21]\) a different approach was used to extend \((1.1)\)
and \((1.2)\) to the range \( \frac{1}{s} < r \leq 1 \) with \( s > \max\{0, n(\frac{1}{r} - 1)\} \) or \( s \) a non-negative even

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were introduced in [19]. In [27], an important tool in the study of the Cauchy problem for the Euler and Navier-Stokes equations in the setting of $L^p$-based Sobolev spaces was the following commutator estimate closely related to (1.2):

$$
\| J^s(fg) - f J^s(g) \|_{L^p} \lesssim \| J^s f \|_{L^p} \| g \|_{L^\infty} + \| \nabla f \|_{L^\infty} \| J^{s-1} g \|_{L^p},
$$

where $1 < p < \infty$ and $s \geq 0$. For other commutator estimates of the sort, see [4] and the references it contains.

The goal of this paper is to prove Kato-Ponce inequalities in the settings of weighted Lebesgue spaces and variable Lebesgue spaces. The techniques employed in the context of weighted Lebesgue spaces also let us obtain fractional Leibniz rules in weighted Lorentz spaces and Morrey spaces. To state our two main results, let $A$ be a factor and $B$ denote the Muckenhoupt class of weights. For brevity we will use the notation $A \lesssim B$ to mean that $A \leq cB$, where $c$ is a constant that may only depend on some of the parameters and weights used but not on the functions involved. For the notation used in the statement of Theorem 1.1, we refer the reader to Section 2.

**Theorem 1.1.** Let $1 < p, q < \infty$ and $\frac{1}{r} < \frac{1}{r} < \infty$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. If $v \in A_p$, $w \in A_q$, and $s$ is a non-negative even integer, then for all $f, g \in S(\mathbb{R}^n)$,

$$
\| D^s(fg) \|_{L^r(v\tilde{p} w^{-\frac{1}{r}})} \lesssim \| D^s f \|_{L^p(v)} \| g \|_{L^q(w)} + \| f \|_{L^p(v)} \| D^s g \|_{L^q(w)};
$$

$$
\| J^s(fg) \|_{L^r(v\tilde{p} w^{-\frac{1}{r}})} \lesssim \| J^s f \|_{L^p(v)} \| g \|_{L^q(w)} + \| f \|_{L^p(v)} \| J^s g \|_{L^q(w)};
$$

and

$$
\| D^s(fg) - f D^s(g) \|_{L^r(v\tilde{p} w^{-\frac{1}{r}})} \lesssim \| D^s f \|_{L^p(v)} \| g \|_{L^q(w)} + \| \nabla f \|_{L^p(v)} \| D^{s-1} g \|_{L^q(w)};
$$

$$
\| J^s(fg) - f J^s(g) \|_{L^r(v\tilde{p} w^{-\frac{1}{r}})} \lesssim \| J^s f \|_{L^p(v)} \| g \|_{L^q(w)} + \| \nabla f \|_{L^p(v)} \| J^{s-1} g \|_{L^q(w)}.
$$

The implicit constants depend on $p, q, s, [v]_{A_p}$ and $[w]_{A_q}$.

The “factored” bilinear weights in Theorem 1.1 were introduced in [22] and further studied in [20]. In [31] a more general case of bilinear $A_p$ weights was introduced. We do not know if our result can be proved for these weights. However, the factored weights are sufficient to apply extrapolation and prove our second main result. For the notation used in the statement of Theorem 1.2, we refer the reader to Section 3.
Theorem 1.2. Let $p(\cdot), q(\cdot), r(\cdot) \in \mathcal{P}_0$ be exponent functions such that $\frac{1}{r(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)}$. Suppose further that there exist $1 < p < p_-$ and $1 < q < q_-$ such that \((\frac{p(\cdot)}{p})', (\frac{q(\cdot)}{q})' \in \mathcal{B}\). If $s > \max\{0, n(\frac{1}{r} - 1)\}$, where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, or $s$ is a non-negative even integer, then for all $f, g \in \mathcal{S}(\mathbb{R}^n)$,

\begin{align}
\|D^s (fg)\|_{L^r(\cdot)} &\lesssim \|D^s f\|_{L^p(\cdot)} \|g\|_{L^q(\cdot)} + \|f\|_{L^p(\cdot)} \|D^s g\|_{L^q(\cdot)}; \\
\|J^s (fg)\|_{L^r(\cdot)} &\lesssim \|J^s f\|_{L^p(\cdot)} \|g\|_{L^q(\cdot)} + \|f\|_{L^p(\cdot)} \|J^s g\|_{L^q(\cdot)}; \\
\|D^s (fg) - fD^s(g)\|_{L^r(\cdot)} &\lesssim \|D^s f\|_{L^p(\cdot)} \|g\|_{L^q(\cdot)} + \|\nabla f\|_{L^p(\cdot)} \|D^{s-1} g\|_{L^q(\cdot)}; \\
\|J^s (fg) - fJ^s(g)\|_{L^r(\cdot)} &\lesssim \|J^s f\|_{L^p(\cdot)} \|g\|_{L^q(\cdot)} + \|\nabla f\|_{L^p(\cdot)} \|J^{s-1} g\|_{L^q(\cdot)}.
\end{align}

Remark 1.1. Implicit in the proof of Theorem 1.1 is the fact that, in the case $w = v$, it is possible to have different pairs $p, q$ for each term on the righthand side of (1.3), (1.4), (1.5) and (1.6). Also, the proof of Theorem 1.2 allows for different pairs of exponents $p(\cdot), q(\cdot)$ for each term on the righthand side of the inequalities for variable Lebesgue spaces.

Remark 1.2. As we will explain in Section 3, the hypotheses of Theorem 1.2 imply that $p(\cdot)$ and $q(\cdot)$ are both bounded, and this restriction is intrinsic to our proof. However, in the scale of variable Lebesgue spaces it is possible to have unbounded exponents or even exponents that are equal to infinity on sets of positive measure. We conjecture that there is a version of Theorem 1.2 that allows unbounded exponents and that includes the endpoint inequality in [4] as a special case.

Our proofs of (1.3) and (1.4) in Theorem 1.1 exploit the ideas used in [21] to prove their unweighted counterparts (1.1) and (1.2). This approach requires us to use weighted estimates for a certain family of square-functions; moreover we need to have good control on their norms. The central step in this argument is Theorem 2.1 where we obtain uniform estimates for such operators. We note that Theorem 2.1 improves the estimate gotten in [21, Corollary 1]; their proof relies on a weak (1, 1) estimate and interpolation. This is enough for the proof of the unweighted estimates (1.1) and (1.2), but not for their weighted analogs of Theorem 1.1. We instead prove the necessary weighted strong type estimates directly. A novel feature of our approach is that we avoid using the maximal operator to estimate convolution operators and instead use an argument based on averaging operators that was developed in a different context in [11]. We do not know if the weak type estimates in weighted or variable Lebesgue spaces that correspond to the weak-type estimates proved in [21] hold.

The proof of the weighted commutator estimates (1.5) and (1.6) in Theorem 1.1 relies on a decomposition of the operators $D^s (fg)$ and $J^s (fg)$ given in [27] for the case of $J^s$ as well as weighted estimates for certain bilinear operators from [20, 22].
The variable Lebesgue space estimates in Theorem 1.2 are a consequence of Theorem 1.1 and a bilinear extrapolation theorem (Theorem 3.1) that allows us to obtain bilinear estimates in variable Lebesgue spaces from bilinear estimates in weighted Lebesgue spaces. This result generalizes both the extrapolation theorem for variable Lebesgue spaces in [9] (see also [8]) and the bilinear extrapolation theorem in [20]. Our result is interesting in its own right as it lets us easily prove a number of other bilinear estimates in the variable Lebesgue space setting. To illustrate this, we use Theorem 3.1 to prove estimates in variable Lebesgue spaces for several types of bilinear operators, including bilinear Calderón-Zygmund operators, bilinear multiplier operators with symbols that have limited smoothness, and certain rough bilinear singular integrals.

The remainder of this paper is organized as follows. In Section 2 we present definitions and basic results about weights and weighted norm inequalities, and then state and prove Theorem 2.1. In Section 3 we give the necessary definitions and background information on variable Lebesgue spaces, and prove Theorem 3.1. In Section 4 we state and prove the applications of Theorem 3.1 to other bilinear operators. In Section 5 we prove Theorems 1.1 and 1.2. Finally, in Section 6 we present Kato-Ponce inequalities in the settings of weighted Lorentz spaces and Morrey spaces and explain how they follow from tools developed for the proof of Theorem 1.1.

Throughout this paper, all notation will be standard or will be defined as needed. We will let \( \mathbb{Z} \) be the set of integers, \( \mathbb{N} \) the set of natural numbers, and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Unless explicitly indicated otherwise, all function spaces that appear will consist of complex-valued functions defined on \( \mathbb{R}^n \).

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2. Weighted square function estimates

The main result in this section is Theorem 2.1, which gives a uniform weighted estimate for certain families of square-functions. This estimate is in turn used in the proofs of our main results. We divide this section into three parts. In the first we give some definitions and state some known results. In the second we state and prove Theorem 2.1. The proof appears relatively short, but it depends on several technical propositions which we prove in the final part.

Preliminary results. By a weight \( w \) we mean a non-negative, locally integrable function defined on \( \mathbb{R}^n \). Given \( 0 < p < \infty \), let \( L^p(w) \) denote the class of complex-valued, measurable functions \( f \) defined on \( \mathbb{R}^n \) such that

\[
\|f\|_{L^p(w)} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx \right)^{\frac{1}{p}} < \infty.
\]
For brevity, when $w = 1$ we write $L^p$ and $\|f\|_{L^p}$ instead of $L^p(w)$ and $\|f\|_{L^p(w)}$.

Given a locally integrable function $f$ on $\mathbb{R}^n$ and a set $E \subset \mathbb{R}^n$ of positive measure, define

$$\int_E f \, dy = \frac{1}{|E|} \int_E f(y) \, dy.$$  

The Hardy-Littlewood maximal operator $\mathcal{M}$ is defined as follows: for each $x \in \mathbb{R}^n$, let

$$\mathcal{M}(f)(x) = \sup_B \int_B |f(y)| \, dy \chi_B(x),$$

where the supremum is taken over all Euclidean balls in $\mathbb{R}^n$.

Given $1 < p < \infty$, the Muckenhoupt class $A_p$ consists of all weights $w$ such that

$$[w]_p = \sup_Q \left( \int_Q w(y) \, dy \right) \left( \int_Q w^{-\frac{1}{p-1}} \right)^{p-1} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. The class $A_1$ consists of all weights $w$ such that $\mathcal{M}(w)(x) \lesssim w(x)$ for almost every $x \in \mathbb{R}^n$; we define

$$[w]_{A_1} = \sup_Q \left( \int_Q w(y) \, dy \right) \left\|w^{-\frac{1}{p}}\right\|_{L^\infty(Q)},$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. Note that $A_1 \subset A_p$ for $1 < p < \infty$ and $[w]_{A_p} \leq [w]_{A_1}$.

It is well known that for $1 < p < \infty$, the Muckenhoupt condition characterizes the weights $w$ such that $\mathcal{M}$ is bounded on $L^p(w)$ (e.g., see [15]). Below, we will need the vector-valued version of this result, also referred to as the weighted Fefferman-Stein inequality. For a proof, see [1, 10].

**Lemma 2.A.** Given $1 < p, q < \infty$ and $w \in A_p$, for all sequences $\{f_k\}$ of locally integrable functions defined on $\mathbb{R}^n$,

$$\left\| \left( \sum_{k \in \mathbb{Z}} |\mathcal{M}(f_k)|^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} \lesssim \left\| \left( \sum_{k \in \mathbb{Z}} |f_k|^q \right)^{\frac{1}{q}} \right\|_{L^p(w)},$$

where the implicit constant depends on $p, q$, and $[w]_{A_p}$.

The next result we recall is a weighted Littlewood-Paley estimate. To state it, let $\widehat{f}$ denote the Fourier transform of a tempered distribution on $\mathbb{R}^n$: more precisely, for all $\xi \in \mathbb{R}^n$ and $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} \, dx.$$
Lemma 2.B. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\text{supp}(\hat{\varphi}) \subset \{ \xi \in \mathbb{R}^n : c_1 < |\xi| < c_2 \}$ for some $0 < c_1 < c_2 < \infty$. Set $\varphi_k(x) = 2^{kn}\varphi(2^kx)$ and suppose further that for some constant $c_\varphi > 0$ and for all $\xi \neq 0$,

$$
\sum_{k \in \mathbb{Z}} |\hat{\varphi}(2^{-k}\xi)|^2 = c_\varphi.
$$

Then for every $1 < p < \infty$, $w \in A_p$, and $f \in L^p(w)$,

$$(2.2) \quad \left\| \left( \sum_{k \in \mathbb{Z}} |\varphi_k * f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)} \lesssim \|f\|_{L^p(w)},$$

where the implicit constants depend on $\varphi$ and $[w]_{A_p}$ but do not depend on $f$.

For a proof, see [37, Proposition 1.9] and the comment that follows; see also [29].

Statement and proof of Theorem 2.1. In Lemma 2.B the implicit constant in (2.2) depends on the function $\varphi$; in particular, if we replace $\varphi$ by a translation $\varphi(\cdot + z)$, $z \in \mathbb{R}^n$, then we do not know a priori whether the constants will depend on $z$. Our main result in this section shows that in this case they do not.

Theorem 2.1. Let $\Psi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\text{supp}(\hat{\Psi}) \subset \{ \xi \in \mathbb{R}^n : c_1 < |\xi| < c_2 \}$ for some $0 < c_1 < c_2 < \infty$. Given a sequence $\bar{\varepsilon} = \{z_{k,m}\}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} \subset \mathbb{R}^n$, define $\Psi^\varepsilon_{k,m}(x) = 2^{kn}\Psi(2^k(x + z_{k,m}))$ for $x \in \mathbb{R}^n$, $m \in \mathbb{Z}^n$ and $k \in \mathbb{Z}$. Then for every $1 < p < \infty$, $w \in A_p$, and $f \in L^p(w)$,

$$
\left\| \left( \sum_{k \in \mathbb{Z}} |\Psi^\varepsilon_{k,m} * f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)} \lesssim \|f\|_{L^p(w)},
$$

where the implicit constants depend on $\Psi$ and $[w]_{A_p}$ but are independent of $m$ and $\varepsilon$.

Remark 2.1. As we noted in the Introduction, Theorem 2.1 improves the corresponding unweighted estimate from [21, Corollary 1]. There, the authors showed that for the particular sequence $z_{k,m} = 2^{-k}m$, $k \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$, we have that

$$
\left\| \left( \sum_{k \in \mathbb{Z}} |\Psi^\varepsilon_{k,m} * f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \log(1 + |m|) \|f\|_{L^p}.
$$

They proved this inequality by means of an unweighted, weak (1, 1) inequality and then interpolating with the unweighted $L^2$ estimate. They prove the weak (1,1) estimate by showing that the operator satisfies a vector-valued Hörmander condition; such a condition is not sufficient for proving weighted norm inequalities: see [33]. Our proof of Theorem 2.1 makes use of Rubio de Francia extrapolation, and so does not yield a weighted endpoint estimate. We do not know if such an estimate holds, either with a constant independent of $m$ or with a constant of order $\log(1 + |m|)$. 
Remark 2.2. We refer the reader to Section 6 for versions of Theorem 2.1 in the settings of weighted Lorentz spaces and Morrey spaces.

The proof of Theorem 2.1 requires two propositions which we state here; their proofs are given in the final part of this section.

Proposition 2.2. Let \( \varphi \in S(\mathbb{R}^n) \) be such that \( \text{supp} (\hat{\varphi}) \subset \{ \xi \in \mathbb{R}^n : c_1 < |\xi| < c_2 \} \) for some \( 0 < c_1 < c_2 < \infty \) and \( \sum_{k \in \mathbb{Z}} |\hat{\varphi}(2^{-k}\xi)|^2 = c_\varphi \) for some \( c_\varphi > 0 \) and for all \( \xi \neq 0 \). Set \( \varphi_k(x) = 2^{kn}\varphi(2^kx) \) for \( k \in \mathbb{Z} \). If \( 1 < p < \infty \) and \( w \in A_p \), then

\[
\left\| \sum_{k \in \mathbb{Z}} \varphi_k * f \right\|_{L^p(w)} \lesssim \left( \sum_{k \in \mathbb{Z}} |f_k|^2 \right)^{1/2}_{L^p(w)}
\]

for all sequences \( \{f_k\}_{k \in \mathbb{Z}} \subset L^p(w) \) such that the righthand side is finite. The implicit constant depends only on \( \varphi \) and \( [w]_{A_p} \).

Proposition 2.3. Given \( \Psi \in S(\mathbb{R}^n) \) and a sequence \( \tilde{z} = \{z_{k,m}\}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} \subset \mathbb{R}^n \), define \( \Psi_{k,m}^\varepsilon(x) = 2^{kn}\Psi(2^k(x + z_{k,m})) \) for \( x \in \mathbb{R}^n, m \in \mathbb{Z}^n \) and \( k \in \mathbb{Z} \). If \( 1 < p < \infty \) and \( w \in A_p \), then for all \( f \in L^p(w) \),

\[
\sup_{k,m \in \mathbb{Z}} \left\| \Psi_{k,m}^\varepsilon * f \right\|_{L^p(w)} \lesssim [w]_{A_p}^{1/2} \|f\|_{L^p(w)},
\]

where the implicit constant depends on \( \Psi \) and is independent of \( \tilde{z} \) and \( w \).

Proof of Theorem 2.1. Our proof is inspired by the argument in [16], but it is simpler because of the structure of our operator. By Rubio de Francia extrapolation (see [10]), it is enough to prove the desired inequality when \( p = 2 \).

Set \( T_{m,\varepsilon}^\varphi(f) = \sum_{k \in \mathbb{Z}} \varepsilon_k \Psi_{k,m}^\varphi * f \), where \( m \in \mathbb{Z}^n \) and \( \varepsilon = \{\varepsilon_k\}_{k \in \mathbb{Z}} \) with \( \varepsilon_k = \pm 1 \) for each \( k \in \mathbb{Z} \). Without loss of generality we may assume \( c_1 = \frac{1}{2} \) and \( c_2 = 2 \). Fix \( \varphi \in S(\mathbb{R}^n) \) such that \( \hat{\varphi} \equiv 1 \) on \( \{ \xi \in \mathbb{R}^n : \frac{1}{2} < |\xi| < 2 \} \), \( \hat{\varphi} \) is supported on an annulus, and \( \sum_{k \in \mathbb{Z}} |\hat{\varphi}(2^{-k}\xi)|^2 = c_\varphi \) for all \( \xi \neq 0 \) for some constant \( c_\varphi \). Define \( \varphi_k(x) = 2^{kn}\varphi(2^kx) \). Then \( \varepsilon_k \Psi_{k,m}^\varphi * f = \varphi_k * \varepsilon_k \Psi_{k,m}^\varphi * f \) pointwise for all \( k \in \mathbb{Z}, m \in \mathbb{Z}^n \), and \( f \in S' (\mathbb{R}^n) \). By Propositions 2.2 and 2.3, and by Lemma 2.B, it follows that for \( w \in A_2, f \in L^2(w) \), and all \( \tilde{z} \) and \( \varepsilon \),

\[
\left\| T_{m,\varepsilon}^\varphi(f) \right\|_{L^2(w)}^2 = \left\| \sum_{k \in \mathbb{Z}} \varphi_k * \varepsilon_k \Psi_{k,m}^\varphi * \varphi_k * f \right\|_{L^2(w)}^2 \lesssim \left\| \sum_{k \in \mathbb{Z}} \left\| \Psi_{k,m}^\varphi * \varphi_k * f \right\|_{L^2(w)}^2 \right\|_{L^2(w)} \lesssim \left\| \varphi_k * f \right\|_{L^2(w)}^2 \lesssim \|f\|_{L^2(w)}^2;
\]
the implicit constants depend only on \( \varphi, \Psi, \) and \([w]_{A_2}\). In other words, we have shown that \( T_{m,\varepsilon}^z \) is bounded on \( L^2(w) \) for every \( w \in A_2 \) with the operator norm controlled by a constant independent of \( m, \varepsilon \) and \( \bar{z} \). Clearly, the same argument shows that the operators \( T_{m,\varepsilon}^{z,+} \) and \( T_{m,\varepsilon}^{z,-} \) defined as \( T_{m,\varepsilon}^z \) but with \( k \in \mathbb{N}_0 \) and \( -k \in \mathbb{N} \), respectively, instead of \( k \in \mathbb{Z} \), are bounded on \( L^2(w) \) with constants independent of the same quantities.

We can now argue as in [15, p. 177]. Let \( \{r_k\}_{k \in \mathbb{N}_0} \) be the system of Rademacher functions. That is, define \( r_0(t) = -1 \) for \( 0 \leq t < \frac{1}{2} \), \( r_0(t) = 1 \) for \( \frac{1}{2} \leq t < 1 \), and extend it as a periodic function on \( \mathbb{R} \). Then define \( r_k(t) = r_0(2^k t) \) for \( 0 \leq t < 1 \). Recall that \( \{r_k\}_{k \in \mathbb{N}_0} \) is an orthonormal system in \( L^2([0, 1]) \) and we have that for all \( k \in \ell^2 \),

\[
\left\| \sum_{k=0}^{\infty} a_k r_k \right\|_{L^2([0, 1])} = \left( \sum_{k=0}^{\infty} |a_k|^2 \right)^{\frac{1}{2}}.
\]

Therefore, if we set \( \varepsilon_t = \{r_k(t)\}_{k \in \mathbb{N}_0} \) for \( 0 \leq t < 1 \), then we have that

\[
\sum_{k \in \mathbb{Z}} |\Psi_{k,m}^{z} * f(x)|^2 = \int_0^1 \left| \sum_{k=0}^{\infty} r_k(t) \Psi_{k,m}^{z} * f(x) \right|^2 dt + \int_{-\infty}^{1} \left| \sum_{k=-\infty}^{-1} r_{-k}(t) \Psi_{k,m}^{z} * f(x) \right|^2 dt
\]

\[
= \int_0^1 |T_{m,\varepsilon_t}^{z,+} f(x)|^2 dt + \int_{-\infty}^{1} |T_{m,\varepsilon_t}^{z,-} f(x)|^2 dt.
\]

If we fix \( w \in A_2 \) and compute the \( L^2(w) \) norm, then we get

\[
\left\| \left( \sum_{k \in \mathbb{Z}} |\Psi_{k,m}^{z} * f(x)|^2 \right)^{\frac{1}{2}} \right\|_{L^2(w)}^2 = \int_0^1 \left\| T_{m,\varepsilon_t}^{z,+} f \right\|_{L^2(w)}^2 dt + \int_{-\infty}^{1} \left\| T_{m,\varepsilon_t}^{z,-} f \right\|_{L^2(w)}^2 dt \lesssim \|f\|_{L^2(w)}^2,
\]

with the implicit constant independent of \( m, \bar{z} \) and \( f \).

**Proof of Propositions 2.2 and 2.3.**

**Proof of Proposition 2.2.** Fix \( w, \varphi \) and \( \{f_k\}_{k \in \mathbb{Z}} \) as in the hypotheses. As before, we can assume without loss of generality that in defining the support of \( \tilde{\varphi} \), \( c_1 = \frac{1}{2} \) and \( c_2 = 2 \). For all \( N \in \mathbb{N}_0 \), define

\[
F_N = \sum_{|\ell| \leq N} \varphi_{\ell} * f_{\ell}.
\]

Then for all \( N_2 > N_1 \) and all \( k \in \mathbb{Z} \) we have that

\[
\varphi_k * (F_{N_2} - F_{N_1}) = \sum_{N_1 < |\ell| \leq N_2} \varphi_k * \varphi_{\ell} * f_{\ell}.
\]
Comparing supports on the Fourier transform side, it follows that

\[
\varphi_k \ast (F_{N_2} - F_{N_1}) = \begin{cases} 
0 & \text{if } |k| \leq N_1 - 1 \text{ or } |k| \geq N_2 + 2, \\
\sum_{\ell=-1}^{l} \varphi_k \ast \varphi_{k-\ell} \ast f_{k-\ell} & \text{if } N_1 + 2 \leq |k| \leq N_2 - 1, \\
\sum_{\ell=-1}^{0} \varphi_k \ast \varphi_{k-\ell} \ast f_{k-\ell} & \text{if } |k| = N_1 + 1, \\
\sum_{\ell=0}^{l} \varphi_k \ast \varphi_{k-\ell} \ast f_{k-\ell} & \text{if } |k| = N_2, \\
\varphi_k \ast \varphi_{k+1} \ast f_{k+1} & \text{if } |k| = N_1, \\
\varphi_k \ast \varphi_{k-1} \ast f_{k-1} & \text{if } |k| = N_2 + 1.
\end{cases}
\]

Recall that for all \( k \in \mathbb{Z} \), \(|\varphi_k \ast g| \lesssim \mathcal{M}(g)\) with a constant that depends only on \( \varphi \). (See [15].) Since \( \mathcal{M} \) is bounded on \( L^p(w) \), it follows that \( F_N \in L^p(w) \) for all \( N \). Moreover, by Lemma 2.B, this maximal operator inequality and Lemma 2.A, we get that

\[
\|F_{N_1} - F_{N_2}\|_{L^p(w)} \lesssim \sum_{\ell=-1}^{1} \left\| \left( \sum_{N_1 \leq |k| \leq N_2+1} |\varphi_k \ast \varphi_{k-\ell} \ast f_{k-\ell}|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)} \lesssim \sum_{\ell=-1}^{1} \left\| \left( \sum_{N_1 \leq |k| \leq N_2+1} |\mathcal{M}(\mathcal{M}(f_{k-\ell}))|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)} \lesssim \left\| \left( \sum_{N_1-1 \leq |k| \leq N_2+2} |f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)}.
\]

The last term in the above chain of inequalities converges to 0 as \( N_1, N_2 \) tend to infinity; therefore, we have that \( \{F_N\}_{N \in \mathbb{N}_0} \) is a Cauchy sequence in \( L^p(w) \). Hence, \( \sum_{k \in \mathbb{Z}} \varphi_k \ast f_k \) converges in \( L^p(w) \). Moreover, the same argument as before also shows that

\[
\left\| \sum_{|k| \leq N} \varphi_k \ast f_k \right\|_{L^p(w)} \lesssim \left\| \left( \sum_{|k| \leq N+2} |f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)}.
\]

with the implicit constant independent of the sequence \( \{f_k\}_{k \in \mathbb{Z}} \). If we let \( N \) tend to infinity we get the desired inequality. \( \square \)

If we did not care about the size of the constant, we could easily prove a version of Proposition 2.3 using the pointwise inequality \( |\Psi_{k,m}^z \ast f(x)| \leq C \mathcal{M}(f)(x) \). However, the constant may depend on \( \bar{z} \) and blow up with \( k \) and \( m \). Moreover, the uniform pointwise estimate \( |\Psi_{k,m}^z \ast f(x)| \leq C \Psi \mathcal{M}(f)(x - z_{k,m}) \) only gives the desired result if \( w \) is constant since otherwise \( L^p(w) \) is not translation invariant.

Therefore, to prove the desired uniform estimate in weighted spaces we will use an argument developed in [11] for matrix weights. The proof requires two lemmas. The first was proved in [11]; for the convenience of the reader we give the proof for the scalar case here.
Lemma 2.4. If $1 < p < \infty$ and $w \in A_p$, then for every ball $B$ and $f \in L^p(w)$,

$$\| |B|^{-1} (\chi_B * f)\|_{L^p(w)} \lesssim [w]_{A_p}^{1/p} \|f\|_{L^p(w)},$$

with the implicit constant independent of $w$.

Proof. Let $p$, $w$ and $f$ be as in the hypotheses. It is enough to prove this estimate for cubes with sides parallel to the coordinate axes instead of balls: given any ball $B$, if $Q$ is the smallest cube that contains $B$, then $|B|^{-1} (\chi_B * f) \leq C_n |Q|^{-1} (\chi_Q * f)$.

Fix a cube $Q$, denote its side length by $\ell(Q)$ and define the cubes $Q_m = Q + \ell(Q)m$ for $m \in \mathbb{Z}^n$. Then $\{Q_m\}_{m \in \mathbb{Z}^n}$ is a pairwise disjoint partition of $\mathbb{R}^n$. Further, we can divide the family $\{3Q_m\}_{m \in \mathbb{Z}^n}$ into $3^n$ families $Q_J$, $J \in \{1, \ldots, 3^n\}$, of pairwise disjoint cubes ($3Q_m$ denotes the cube with the same center as $Q_m$ and side length $3\ell(Q_m)$). Then for all $x \in Q_m,$

$$|Q|^{-1} |(\chi_Q * f)(x)| \leq |Q|^{-1} \int_{\mathbb{R}^n} |f(y)\chi_Q(x - y)| dy \leq 3^n \int_{3Q_m} |f(y)| dy.$$

Hence,

$$\int_{\mathbb{R}^n} \| |Q|^{-1} (\chi_Q * f)(x)\|^p w(x) dx$$

$$= \sum_{m \in \mathbb{Z}^n} \int_{Q_m} \| |Q|^{-1} (\chi_Q * f)(x)\|^p w(x) dx$$

$$\lesssim \sum_{m \in \mathbb{Z}^n} \int_{Q_m} \left( \int_{3Q_m} |f(y)| dy \right)^p w(x) dx$$

$$\lesssim \sum_{J=1}^{3^n} \sum_{3Q_m \in Q_J} \int_{3Q_m} w(x) dx \left( \int_{3Q_m} |f(y)| w(y)^{1/p} w(y)^{-1/p} dy \right)^p.$$  

If we apply Hölder’s inequality, and use the facts that $w \in A_p$ and the cubes in $Q_J$ are pairwise disjoint, we get

$$\lesssim \sum_{J=1}^{3^n} \sum_{3Q_m \in Q_J} \left( \int_{3Q_m} |f(x)|^p w(x) dx \right) \left( \int_{3Q_m} w(x) dx \right) \left( \int_{3Q_m} w(x)^{-1/p} dx \right)^p$$

$$\lesssim [w]_{A_p} \|f\|^p_{L^p(w)}.$$

The implicit constants are independent of $w$ and $f$, and the proof is complete.  □

The next lemma is also from [11] where it was given implicitly and without proof.
Lemma 2.5. Let $\Phi \in L^1(\mathbb{R}^n)$ be non-negative and radially decreasing. Define $\tau_z(\Phi)(x) = \Phi(x + z)$ for $z \in \mathbb{R}^n$. If $1 < p < \infty$ and $w \in A_p$, then for all $f \in L^p(w)$,

$$\|\tau_z(\Phi) * f\|_{L^p(w)} \lesssim \|\Phi\|_{L^1} [w]_{A_p}^{\frac{1}{p}} \|f\|_{L^p(w)},$$

with the implicit constant independent of $z$, $w$ and $\Phi$.

Proof. Assume first that $\Phi \in L^1(\mathbb{R}^n)$ is of the form

$$\Phi(x) = \sum_{k=1}^{\infty} a_k |B_k|^{-1} \chi_{B_k}(x),$$

where $a_k \geq 0$ and $B_k$ is a ball centered at the origin for each $k \in \mathbb{N}$. Note that $\|\Phi\|_{L^1} = \sum_{k=1}^{\infty} a_k$. Fix $z \in \mathbb{R}^n$; then we have

$$\tau_z(\Phi) = \sum_{k=1}^{\infty} a_k |\tilde{B_k}|^{-1} \chi_{\tilde{B_k}},$$

where $\tilde{B_k} = -z + B_k$ for $k \in \mathbb{N}$. With $p$, $w$ and $f$ as in the hypotheses, by Lemma 2.4 we have that

$$\|\tau_z(\Phi) * f\|_{L^p(w)} \lesssim \sum_{k=1}^{\infty} a_k \|\tilde{B_k}|^{-1} (\chi_{\tilde{B_k}} * f)\|_{L^p(w)}$$

$$\lesssim \sum_{k=1}^{\infty} a_k \|\tilde{B_k}|^{-1} (\chi_{\tilde{B_k}} * f)\|_{L^p(w)} = [w]_{A_p}^{\frac{1}{p}} \|\Phi\|_{L^1} \|f\|_{L^p(w)},$$

where the implicit constant is independent of $z$, $w$, $\Phi$ and $f$.

To complete the proof, note that an arbitrary function as in the hypotheses can be approximated from below by a sequence of functions of the form treated above. The desired inequality then follows by the Monotone Convergence Theorem. \(\square\)

Proof of Proposition 2.3. Since $\Psi \in \mathcal{S}(\mathbb{R}^n)$, there exists a non-negative radially decreasing function $\Phi \in L^1(\mathbb{R}^n)$ such that $|\Psi(x)| \leq \Phi(x)$ for all $x \in \mathbb{R}^n$. Let $\Phi_{k,m}^x(x) = 2^{nk}\Phi(2^k(x + z_{k,m}))$. Fix $1 < p < \infty$ and $w \in A_p$; then by Lemma 2.5 we have that

$$\|\Psi_{k,m}^x * f\|_{L^p(w)} \leq \|\Phi_{k,m}^x * f\|_{L^p(w)}$$

$$\lesssim \|2^{nk}\Phi(2^k.)\|_{L^1} [w]_{A_p}^{\frac{1}{p}} \|f\|_{L^p(w)} = [w]_{A_p}^{\frac{1}{p}} \|\Phi\|_{L^1} \|f\|_{L^p(w)}.$$

\(\square\)
3. Bilinear extrapolation on variable Lebesgue spaces

The main result in this section is Theorem 3.1, a bilinear extrapolation result that allows to deduce bilinear estimates in variable Lebesgue spaces from bilinear estimates in weighted Lebesgue spaces. This result is key for our proof of Theorem 1.2. Extrapolation is an important technique for proving norm inequalities in variable Lebesgue spaces: we refer the reader to [8, 9] for further details in the linear case.

We divide this section into two parts. In the first we give some basic definitions and results about variable Lebesgue spaces. In the second we state and prove Theorem 3.1. In Section 4 below, in order to illustrate the broader utility of bilinear extrapolation, we give applications of Theorem 3.1 to prove estimates for a variety of bilinear operators.

Definitions and preliminary results. For complete information on variable Lebesgue spaces and for proofs of the results stated here, see [8, 13]

Define the collection \( \mathcal{P} \) of exponent functions to be the set of measurable functions \( p(\cdot) : \mathbb{R}^n \to [1, \infty] \). Similarly, let \( \mathcal{P}_0 \) denote the set of measurable functions \( p(\cdot) : \mathbb{R}^n \to (0, \infty] \). Given an exponent \( p(\cdot) \), we let

\[
p_+ = \text{ess sup}_{x \in \mathbb{R}^n} p(x), \quad p_- = \text{ess inf}_{x \in \mathbb{R}^n} p(x).
\]

Given \( p(\cdot) \in \mathcal{P}_0 \), we define the modular

\[
\rho_{p(\cdot)}(f) = \int_{\mathbb{R}^n \setminus \mathbb{R}^n_\infty} |f(x)|^{p(x)} \, dx + \|f\|_{L^\infty(\mathbb{R}^n_\infty)},
\]

where \( \mathbb{R}^n_\infty = \{ x \in \mathbb{R}^n : p(x) = \infty \} \). The variable Lebesgue space \( L^{p(\cdot)} \) consists of all measurable functions \( f \) defined on \( \mathbb{R}^n \) that satisfy \( \rho_{p(\cdot)}(f/\lambda) < \infty \) for some \( \lambda > 0. \) This is a quasi-Banach space (Banach space if \( p(\cdot) \in \mathcal{P}_0 \)) with the quasi-norm (norm if \( p(\cdot) \in \mathcal{P} \)) given by

\[
\|f\|_{L^{p(\cdot)}} = \inf \{ \lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leq 1 \}.
\]

The variable Lebesgue spaces generalize the classical Lebesgue spaces: if \( 0 < p_0 \leq \infty \) and \( p(\cdot) \equiv p_0 \), then \( L^{p(\cdot)} = L^{p_0} \) with equality of norms.

The following lemmas are basic properties of the norm. The first relates the norm and the modular, the second generalizes Hölder’s inequality, and the third gives an equivalent expression for the norm.

**Lemma 3.C.** Given \( p(\cdot) \in \mathcal{P}_0 \), \( \|f\|_{L^{p(\cdot)}} \leq 1 \) if and only if \( \rho_{p(\cdot)}(f) \leq 1 \).

**Lemma 3.D.** Given \( p(\cdot), q(\cdot), r(\cdot) \in \mathcal{P}_0 \), suppose

\[
\frac{1}{r(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)}.
\]
Then for all $f \in L^p(·)$ and $g \in L^q(·)$,
\[ \|fg\|_{L^r(·)} \lesssim \|f\|_{L^p(·)} \|g\|_{L^q(·)}. \]
The implicit constant depends only on $p(·)$ and $q(·)$.

**Remark 3.1.** In [8] both are proved assuming that the exponents are in $\mathcal{P}$; however, the proofs can be easily adapted to this more general setting.

**Lemma 3.E.** If $p(·) \in \mathcal{P}$, then for all $f \in L^p(·)$,
\[ \|f\|_{L^p(·)} \approx \sup_{\mathbb{R}^n} \int f g \, dx, \]
where the supremum is taken over all $g \in L^{p'(·)}$, $\|g\|_{L^{p'(·)}} = 1$, with $p'(·) \in \mathcal{P}$ defined pointwise by
\[ \frac{1}{p(·)} + \frac{1}{p'(·)} = 1. \]
The implicit constants depend only on $p(·)$.

Central to our result is the boundedness of the maximal operator $\mathcal{M}$ on $L^p(·)$. Let $\mathcal{B}$ be the family of all $p(·) \in \mathcal{P}$ such that for all $f \in L^p(·)$,
\[ \|\mathcal{M}f\|_{L^p(·)} \lesssim \|f\|_{L^p(·)}. \]
The norm of $\mathcal{M}$ as a bounded operator on $L^p(·)$ will be denoted by $\|\mathcal{M}\|_{p(·)}$. A necessary condition for $p(·) \in \mathcal{B}$ is that $p_+ > 1$. As a consequence, our hypothesis in Theorems 1.2 and 3.1 that $\left(\frac{p(·)}{p'}\right)'$, $\left(\frac{q(·)}{q'}\right)' \in \mathcal{B}$ immediately implies that $p_+, q_+ < \infty$.

A sufficient condition for $p(·) \in \mathcal{B}$ is that $p(·)$ is log-H"older continuous locally: there exists $C_0 > 0$ such that
\[ |p(x) - p(y)| \leq \frac{C_0}{-\log(|x - y|)}, \quad |x - y| < \frac{1}{2}; \]
and log-H"older continuous at infinity: there exists $p_\infty$, $C_\infty > 0$ such that
\[ |p(x) - p_\infty| \leq \frac{C_\infty}{\log(e + |x|)}. \]

While not strictly necessary to prove Theorem 3.1, we want to note the following result, which in practice makes it easier to apply. Given $p(·) \in \mathcal{P}$ with $1 < p_- \leq p_+ < \infty$, the following statements are equivalent:

1. $p(·) \in \mathcal{B}$,
2. $p'(·) \in \mathcal{B}$,
3. $p(·)/q \in \mathcal{B}$ for some $1 < q < p_-$,
4. $(p(·)/q)' \in \mathcal{B}$ for some $1 < q < p_-$. 
Statement and proof of bilinear extrapolation.

**Theorem 3.1.** Let $\mathcal{F}$ be a family of ordered triples of non-negative, measurable functions defined on $\mathbb{R}^n$. Suppose there are indices $0 < p, q, r < \infty$ satisfying $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, such that for every $v, w \in A_1$,

\begin{equation}
\left( \int_{\mathbb{R}^n} h(x)^r v(x) \tau w(x) \tau dx \right)^{\frac{1}{r}} \lesssim \left( \int_{\mathbb{R}^n} f(x)^p v(x) dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^n} g(x)^q w(x) dx \right)^{\frac{1}{q}}
\end{equation}

for all $(h, f, g) \in \mathcal{F}$ such that the lefthand side is finite and where the implicit constant depends only on $p$, $q$, $\|w\|_{A_1}$ and $\|v\|_{A_1}$. Let $p(\cdot), q(\cdot), r(\cdot) \in P_0(\mathbb{R}^n)$ be such that

\begin{equation}
\frac{1}{r(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)}, \quad 0 < p < p_-, \quad 0 < q < q_-, \quad \text{and} \quad \left( \frac{p(\cdot)}{p} \right)' \left( \frac{q(\cdot)}{q} \right)' \in \mathcal{B}.
\end{equation}

Then

\begin{equation}
\|h\|_{L^{r(\cdot)}} \lesssim \|f\|_{L^{p(\cdot)}} \|g\|_{L^{q(\cdot)}}
\end{equation}

for all $(h, f, g) \in \mathcal{F}$ such that $h \in L^{r(\cdot)}$. The implicit constant only depends on $p(\cdot)$ and $q(\cdot)$.

**Proof.** Set $\bar{r}(\cdot) = \frac{r(\cdot)}{r}$, $\bar{p}(\cdot) = \frac{p(\cdot)}{p}$ and $\bar{q}(\cdot) = \frac{q(\cdot)}{q}$. Then by our hypotheses, $1 < \bar{p}_- \leq \bar{p}_+ < \infty$, and the same is true for $\bar{r}(\cdot)$ and $\bar{q}(\cdot)$.

We first define two Rubio de Francia iteration algorithms: given a non-negative function $\tau$,

\begin{align*}
\mathcal{R}_1 \tau(x) &= \sum_{k=0}^{\infty} \frac{\mathcal{M}^k \tau(x)}{2^k \|\mathcal{M}\|^k_{\bar{p}(\cdot)}}, \\
\mathcal{R}_2 \tau(x) &= \sum_{k=0}^{\infty} \frac{\mathcal{M}^k \tau(x)}{2^k \|\mathcal{M}\|^k_{\bar{q}(\cdot)}}.
\end{align*}

Then $\tau(x) \leq \mathcal{R}_1 \tau(x)$ and, since $\mathcal{M}$ is bounded on $L^{\bar{p}(\cdot)}$, we have that $\|\mathcal{R}_1 \tau\|_{\bar{p}(\cdot)} \leq 2\|\tau\|_{\bar{p}(\cdot)}$ and $\mathcal{R}_1 \tau \in A_1$ with $[\mathcal{R}_1 \tau]_{A_1} \leq 2\|\mathcal{M}\|_{\bar{p}(\cdot)}$. (See [8, Theorem 5.24] for details.) The same is true for $\mathcal{R}_2 \tau$ with $\bar{p}(\cdot)$ replaced by $\bar{q}(\cdot)$ everywhere.

Now fix a triple $(h, f, g) \in \mathcal{F}$ such that $\|h\|_{L^{r(\cdot)}} < \infty$. Then by Lemma 3.E,

\begin{equation}
\|h\|_{L^{r(\cdot)}}^r = \|h^r\|_{L^{r(\cdot)}} \approx \sup \int_{\mathbb{R}^n} h(x)^r \tau(x) dx,
\end{equation}

where the supremum is taken over all non-negative functions $\tau \in L^{\bar{r}(\cdot)}$ with $\|\tau\|_{L^{\bar{r}(\cdot)}} = 1$. Therefore, it is enough to prove that for all such $\tau$,

\begin{equation}
\int_{\mathbb{R}^n} h(x)^r \tau(x) dx \lesssim \|f\|_{L^{\bar{p}(\cdot)}} \|g\|_{L^{\bar{q}(\cdot)}}^r,
\end{equation}

where the implicit constant is independent of $\tau$. 

Define functions $\theta_1(\cdot), \theta_2(\cdot)$ by

$$
\theta_1(\cdot) = \frac{r \bar{p}'(\cdot)}{pp'(\cdot)}, \quad \theta_2(\cdot) = \frac{r \bar{q}'(\cdot)}{qq'(\cdot)}.
$$

Then for all $x$,

$$
\theta_1(x) + \theta_2(x) = \bar{r}'(x) \left( \frac{r p(x) - p}{p} + \frac{r q(x) - q}{q} \right) = \bar{r}'(x) \left( \frac{r}{p} + \frac{r}{q} - \frac{r}{r(x)} \right) = 1.
$$

Hence, by the properties of the iteration algorithms,

$$
\int_{\mathbb{R}^n} h(x)^r \tau(x) \, dx = \int_{\mathbb{R}^n} h(x)^r \tau(x)^{\theta_1(x)} \tau(x)^{\theta_2(x)} \, dx \leq \int_{\mathbb{R}^n} h(x)^r \mathcal{R}_1(\tau (\bar{p}'(\cdot))(x) \mathcal{R}_2(\tau (\bar{q}'(\cdot))(x) \tau \, dx.
$$

We claim that the righthand side of this inequality is finite. To see this, first note that by the computation above for $\theta_1 + \theta_2$, we have that

$$
1 = \frac{1}{\bar{r}'(\cdot)} = \frac{1}{\bar{r}'(\cdot)} + \frac{r}{p \bar{p}'(\cdot)} + \frac{r}{q \bar{q}'(\cdot)}.
$$

Therefore, by Lemma 3.3,

$$
\int_{\mathbb{R}^n} h(x)^r \mathcal{R}_1(\tau (\bar{p}'(\cdot))(x) \mathcal{R}_2(\tau (\bar{q}'(\cdot))(x) \tau \, dx
$$

$$
\lesssim \|h\|_{L^r(\cdot)} \left( \|\mathcal{R}_1(\tau (\bar{p}'(\cdot))(\bar{p})\|_{L^\bar{p}'(\cdot)} \|\mathcal{R}_2(\tau (\bar{q}'(\cdot))(\bar{q})\|_{L^{\bar{q}'(\cdot)}} \right)
$$

$$
\lesssim \|h\|_{L^r(\cdot)} \left( \|\mathcal{R}_1(\tau (\bar{p}'(\cdot))(\bar{p})\|_{L^{\bar{p}'(\cdot)}} \|\mathcal{R}_2(\tau (\bar{q}'(\cdot))(\bar{q})\|_{L^{\bar{q}'(\cdot)}} \right)
$$

Since $\|\tau\|_{L^r(\cdot)} = 1$, by Lemma 3.3 we have that

$$
1 \geq \int_{\mathbb{R}^n} \tau(x)^{\bar{r}'(\cdot)} \, dx = \int_{\mathbb{R}^n} \left( \tau(x) (\bar{r}'(\cdot)) \right) \bar{r}'(x) \, dx,
$$

which again by Lemma 3.3, since $(\bar{p}')_+ < \infty$, implies that $\|\tau (\bar{p}'(\cdot))\|_{L^{\bar{r}'(\cdot)}} \leq 1$. Similarly, we have that $\|\tau (\bar{q}'(\cdot))\|_{L^{\bar{r}'(\cdot)}} \leq 1$. Therefore, since $\|h\|_{L^r(\cdot)} < \infty$, it follows that the righthand side is finite.

Given this, and given that $\mathcal{R}_1(\tau (\bar{p}'(\cdot)), \mathcal{R}_2(\tau (\bar{q}'(\cdot)) \in A_1$ with $A_1$ characteristics independent of $\tau$, we can apply our hypothesis to get

$$
\int_{\mathbb{R}^n} h(x)^r \mathcal{R}_1(\tau (\bar{p}'(\cdot))(x) \mathcal{R}_2(\tau (\bar{q}'(\cdot))(x) \tau \, dx
$$
\[ \left( \int_{\mathbb{R}^n} f(x)^p R_1(\tau_{\mathcal{P}(\cdot)})^q(x) \, dx \right)^\frac{1}{q} \left( \int_{\mathbb{R}^n} g(x)^q R_2(\tau_{\mathcal{P}(\cdot)})^p(x) \, dx \right)^\frac{1}{p}. \]

To estimate the first integral on the right-hand side we apply Lemma 3.3:
\[ \int_{\mathbb{R}^n} f(x)^p R_1(\tau_{\mathcal{P}(\cdot)})^q(x) \, dx \]
\[ \lesssim \|f\|^p_{L^p(\cdot)} \|R_1(\tau_{\mathcal{P}(\cdot)})\|_{L^q(\cdot)} \lesssim \|f\|^p_{L^p(\cdot)} \|\tau_{\mathcal{P}(\cdot)}\|^q_{L^q(\cdot)} \lesssim \|f\|^p_{L^p(\cdot)}. \]

In exactly the same way we have that the second integral is bounded by \( \|g\|^q_{L^q(\cdot)} \). If we combine all of the above estimates, we get the desired inequality and this completes the proof. \qed

4. Further applications of Theorem 3.1

In this section we show that Theorem 3.1 implies boundedness properties in variable Lebesgue spaces for a variety of bilinear operators.

In order to apply Theorem 3.1 to a bilinear operator \( T \), the corresponding family \( \mathcal{F} \) will consist of triples of the form \( (|T(f,g)|, |f|, |g|) \), where \( f, g \) are chosen from the domain of \( T \) or some appropriate (dense) subset of the domain. To insure \textit{a priori} the assumption \( \|T(f,g)\|_{L^r(\cdot)} < \infty \) holds, it will suffice to replace \( |T(f,g)| \) in each triple by \( \min(|T(f,g)|, N) \chi_{B(0,N)} \) and then apply the the monotone convergence theorem for variable Lebesgue spaces (see [8, Theorem 2.59]).

**Bilinear Calderón-Zygmund operators in variable Lebesgue spaces.** Let \( K(x,y,z) \) be a complex-valued, locally integrable function on \( \mathbb{R}^{3n} \setminus \Delta \), where \( \Delta = \{(x,x,x) : x \in \mathbb{R}^n \} \). \( K \) is a Calderón-Zygmund kernel if there exist \( A > 0 \) and \( \delta > 0 \) such that for all \((x,y,z) \in \mathbb{R}^{3n} \setminus \Delta,\)

\[ |K(x,y,z)| \leq \frac{A}{(|x-y|+|x-z|+|y-z|)^{2n}} \]

and

\[ |K(x,y,z) - K(\tilde{x},y,z)| \leq \frac{A |x-\tilde{x}|^\delta}{(|x-y|+|x-z|+|y-z|)^{2n+\delta}} \]

whenever \(|x-\tilde{x}| \leq \frac{1}{2} \max(|x-z|,|x-y|)\). We also assume that the two analogous difference estimates with respect to the variables \( y \) and \( z \) hold. An operator \( T \), continuous from \( \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \) into \( \mathcal{S}'(\mathbb{R}^n) \), is a bilinear Calderón-Zygmund operator if it satisfies two conditions:

1. there exists a bilinear Calderón-Zygmund kernel \( K \) such that

\[ T(f,g)(x) = \int_{\mathbb{R}^{2n}} K(x,y,z)f(y)g(z) \, dy \, dz \]
for all \( f, g \in C_c^\infty(\mathbb{R}^n) \) and all \( x \notin \text{supp}(f) \cap \text{supp}(g) \);

(2) there exist \( 1 \leq p, q, r < \infty \) such that \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \) and \( T \) can be extended to a bounded operator from \( L^p \times L^q \) into \( L^r \).

Bilinear Calderón-Zygmund operators enjoy boundedness properties in various function spaces. We refer the reader to [23, 31] and the references they contain for more information on bilinear Calderón-Zygmund theory. To apply Theorem 3.1 we need a weighted norm inequality from [20, Corollary 8.2] (see also [22]). This is not the best result known, but it enough for our purposes; we refer the reader to [31, Corollary 3.9] for further results in the weighted setting.

**Theorem 4.F.** Let \( T \) be a bilinear Calderón-Zygmund operator. Given \( 1 < p, q < \infty \), \( \frac{1}{2} < r < \infty \), \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \), suppose \( v \in A_p \) and \( w \in A_q \). Then for all \( f \in L^p(v) \), \( g \in L^q(w) \),

\[
\|T(f, g)\|_{L^r(v\bar{\nabla}^{T, T})} \lesssim \|f\|_{L^p(v)} \|g\|_{L^q(w)},
\]

where the implicit constant depends only on \( p, q, \) \([v]_{A_p} \), \([w]_{A_q} \) and the size of the constants for the kernel of \( T \).

As an immediate consequence of Theorem 4.F and Theorem 3.1 we get the boundedness of bilinear Calderón-Zygmund operators in variable Lebesgue spaces.

**Corollary 4.1.** Let \( p(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \) satisfy \( \frac{1}{r(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)}, p_ - > 1, q_- > 1, \) \( \frac{1}{r(\cdot)} > \frac{1}{2} \). Suppose further that there exist \( 1 < p < p_- \) and \( 1 < q < q_- \) such that \( \left( \frac{p(\cdot)}{r(\cdot)} \right) \left( \frac{q(\cdot)}{r(\cdot)} \right) \in \mathcal{P} \). If \( T \) is a bilinear Calderón-Zygmund operator, then for all \( f \in L^{p(\cdot)} \), \( g \in L^{q(\cdot)} \),

\[
\|T(f, g)\|_{L^{r(\cdot)}} \lesssim \|f\|_{L^{p(\cdot)}} \|g\|_{L^{q(\cdot)}},
\]

**Remark 4.1.** The boundedness of bilinear Calderón-Zygmund operators in variable Lebesgue spaces was first proved in [24, Corollary 2.1] and [32, Theorem 3.1]. We improve upon both of these results: the former requires the additional hypothesis \( \frac{r(\cdot)}{r} \in \mathcal{P} \) for some \( 0 < r < r_- \) while the latter assumes \( r(\cdot) \in \mathcal{P} \). In both cases, the proofs use linear extrapolation in the scale of variable Lebesgue spaces.

**Bilinear multiplier operators in variable Lebesgue space.** Examples of bilinear Calderón-Zygmund operators are Coifman-Meyer multiplier operators (see [7, 23]). Such operators are of the form

\[
T_\sigma(f, g)(x) = \int_{\mathbb{R}^{2n}} \sigma(\xi, \eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} \, d\xi d\eta, \quad x \in \mathbb{R}^n,
\]

where \( \sigma \) is a complex-valued, smooth function defined for \( \xi, \eta \in \mathbb{R}^n \), called a (bilinear) symbol or multiplier, that satisfies

\[
\left| \partial_\xi^\beta \partial_\eta^\gamma \sigma(\xi, \eta) \right| \lesssim (|\xi| + |\eta|)^{-|\beta + \gamma|} \quad \xi, \eta \in \mathbb{R}^n, (\xi, \eta) \neq (0, 0),
\]
for all multi-indices $\beta, \gamma \in \mathbb{N}_0^n$ with $|\beta + \gamma| \leq 2n + 1$. In particular, these operators satisfy the weighted estimates of Theorem 4.1 and the variable Lebesgue space estimates in Corollary 4.1.

If (4.2) is only satisfied for $|\beta + \gamma| \leq L$, where $L < 2n + 1$, then $T_\sigma$ may fail to be a bilinear Calderón-Zygmund operator. However, weighted estimates like those given in Theorem 4.1 do hold true for some bilinear multiplier operators with rougher symbols whose regularity is measured in terms of Sobolev norms. Consequently, such operators are also bounded in variable Lebesgue spaces.

We state such results precisely. Given $s, t \in \mathbb{R}$, the Sobolev space $H^{(s,t)}(\mathbb{R}^{2n})$ consists of all $F \in \mathcal{S}'(\mathbb{R}^{2n})$ such that

$$
\|F\|_{H^{(s,t)}} = \left( \int_{\mathbb{R}^{2n}} (1 + |\tau_1|^2)^s(1 + |\tau_2|^2)^t |\hat{F}(\tau_1, \tau_2)|^2 \, d\tau_1 d\tau_2 \right)^{\frac{1}{2}} < \infty.
$$

It follows that $H^s(\mathbb{R}^{2n}) \subset H^{(s,0)}(\mathbb{R}^{2n})$ for $s \geq 0$, where $H^s(\mathbb{R}^{2n})$ is the Sobolev space consisting of $F \in L^2(\mathbb{R}^{2n})$ such that

$$
\left( \int_{\mathbb{R}^{2n}} (1 + |\tau_1|^2)^s |\hat{F}(\tau_1, \tau_2)|^2 \, d\tau_1 d\tau_2 \right)^{\frac{1}{2}} < \infty.
$$

Fix $\Psi \in \mathcal{S}(\mathbb{R}^{2n})$ such that $\text{supp}(\Psi) \subset \{(\xi, \eta) \in \mathbb{R}^{2n} : \frac{1}{2} \leq |(\xi, \eta)| \leq 2\}$ and $\sum_{k \in \mathbb{Z}} \Psi(2^{-k}\xi, 2^{-k}\eta) = 1$ for $((\xi, \eta)) \neq (0,0)$. Given a complex-valued, bounded function $\sigma$ defined on $\mathbb{R}^{2n}$, set $\sigma_k(\xi, \eta) = \sigma(2^k\xi, 2^k\eta)\Psi(\xi, \eta)$ for $\xi, \eta \in \mathbb{R}^n$ and $k \in \mathbb{Z}$. The following result from [17, Theorem 6.2] is a weighted version of a Hörmander type theorem for bilinear Fourier multipliers.

**Theorem 4.6.** Let $1 < p, q < \infty$, $\frac{1}{2} < r < \infty$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and $\frac{n}{2} < s, t \leq n$. Assume $p > \frac{n}{s}$, $q > \frac{n}{t}$, $v \in A^\infty_{\frac{n}{m}}$, $w \in A^\infty_{\frac{n}{m}}$ and $\sigma(\xi, \eta)$ is a complex-valued, bounded function defined for $\xi, \eta \in \mathbb{R}^n$ such that $\sup_{k \in \mathbb{Z}} \|\sigma_k\|_{H^{(s,t)}} < \infty$. Then for all $f \in L^p(v)$, $g \in L^q(w)$,

$$
\|T_\sigma(f, g)\|_{L_r(\mathbb{R}^{2n})} \lesssim \|f\|_{L^p(v)} \|g\|_{L^q(w)},
$$

where the implicit constant depends only on $p$, $q$, $[v]_{A^\infty_{\frac{n}{m}}}$, $[w]_{A^\infty_{\frac{n}{m}}}$ and $\sigma$.

As an immediate consequence of Theorem 4.6 and Theorem 3.1 we get the boundedness of bilinear multipliers in variable Lebesgue spaces.

**Corollary 4.2.** Given $\frac{3}{2} < s, t \leq n$, let $p(\cdot), q(\cdot), r(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ satisfy $\frac{1}{q(\cdot)} = \frac{1}{r(\cdot)} + \frac{1}{q(\cdot)}$, $p_+ > \frac{n}{s}$, $q_+ > \frac{n}{t}$, $r_+ > \frac{1}{2}$ and assume there exist $\frac{n}{s} < p < p_-$ and $\frac{n}{t} < q < q_-$ such that $(\frac{p(\cdot)}{p})'$, $(\frac{q(\cdot)}{q})' \in \mathcal{B}$. If $\sigma(\xi, \eta)$ is a complex-valued, bounded function defined for $\xi, \eta \in \mathbb{R}^n$ such that $\sup_{k \in \mathbb{Z}} \|\sigma_k\|_{H^{(s,t)}} < \infty$, then for all $f \in L^{p(\cdot)}$, $g \in L^{q(\cdot)}$,

$$
\|T_\sigma(f, g)\|_{L_r(\mathbb{R}^{2n})} \lesssim \|f\|_{L^{p(\cdot)}} \|g\|_{L^{q(\cdot)}}.
$$
A slightly different version of Corollary 4.2 for the smaller class of symbols $\sigma$ satisfying $\sup_{k \in \mathbb{Z}} \|\sigma_k\|_{H^s(\mathbb{R}^n)} < \infty$ with $n < s \leq 2n$ was proved in [36]. The proof again used linear extrapolation and required the additional hypothesis that $\frac{p}{r} \in \mathcal{B}$.

**Remark 4.2.** Weighted estimates like those in the hypothesis of Theorem 3.1, and the corresponding estimates on variable Lebesgue spaces, may be obtained for certain bilinear pseudodifferential operators $T_\sigma$, where $\sigma = \sigma(x, \xi, \eta)$ for $x, \xi, \eta \in \mathbb{R}^n$. We refer the reader to [35] and the references it contains for further details on these operators.

**Rough bilinear singular integrals in variable Lebesgue spaces.** Let $\Omega \in L^\infty(\mathbb{S}^{2n-1})$ be such that $\int_{\mathbb{S}^{2n-1}} \Omega d\sigma = 0$ and define the bilinear singular integral operator associated with $\Omega$ by

\[ T_\Omega(f, g)(x) = \text{p.v.} \int_{\mathbb{R}^{2n}} K(x - y, x - z)f(y)g(z) dydz, \]

where $f, g \in S(\mathbb{R}^n), x \in \mathbb{R}^n$ and $K(y, z) = \frac{\Omega((y, z)/(y, z))}{|y, z|^{2n}}$. These operators were introduced by Coifman and Meyer [6]; for their history see [18] and the references it contains. In this latter paper it was proved that $T_\Omega$ is bounded from $L^p \times L^q$ into $L^r$ for $1 < p, q < \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. In fact, we can readily adapt their proof to show that $T_\Omega$ satisfies a weighted version of this result when $1 \leq r < \infty$.

**Theorem 4.3.** Let $\Omega \in L^\infty(\mathbb{S}^{2n-1})$ be such that $\int_{\mathbb{S}^{2n-1}} \Omega d\sigma = 0$. If $1 < p, q < \infty$, $1 \leq r < \infty$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $v \in A_p$ and $w \in A_q$, then for all $f \in L^p(v)$, $g \in L^q(w)$,

\[ \|T_\Omega(f, g)\|_{L^r(v^{1/r}w^{1/r})} \lesssim \|\Omega\|_{L^\infty} \|f\|_{L^p(v)} \|g\|_{L^q(w)}, \]

where the implicit constant depends only on $p, q, [v]_{A_p}, [w]_{A_q}$ and $\Omega$.

**Proof.** Let $p, q, r$ and $v, w$ be as in the hypotheses. In [18] they showed that $T_\Omega = \sum_{j \in \mathbb{Z}} T_j$, where each $T_j$ is a Calderón-Zygmund operator. Moreover, they proved [18, Proposition 3 and Lemma 11] that there exists $\delta > 0$ such that for all $f \in L^p$, $g \in L^q$,

\[ \|T_j(f, g)\|_{L^r} \lesssim \|\Omega\|_{L^\infty} 2^{-|j|\delta} \|f\|_{L^p} \|g\|_{L^q}. \]

To get a similar weighted estimate, we adapt an interpolation argument from [16] to the bilinear setting. In [18, Lemma 10] they prove that for any $0 < \varepsilon < 1$, the kernel constant of $T_j$ is controlled by $C_{\varepsilon, n} \|\Omega\|_{L^\infty} 2^{|j|\varepsilon}$. It follows from the proof of Theorem 4.F that the constant in the weighted norm inequality depends linearly on the kernel constant; hence, this gives us that

\[ \|T_j(f, g)\|_{L^r(v^{1/r}w^{1/r})} \lesssim C_{\varepsilon, n} \|\Omega\|_{L^\infty} 2^{|j|\varepsilon} \|f\|_{L^p(v)} \|g\|_{L^q(w)}. \]
for all \( f \in L^p(v) \), \( g \in L^q(w) \) and \( j \in \mathbb{Z} \). By the sharp reverse H"older inequality \[26\], there exists \( 0 < \theta < 1 \) depending on \([v]_{A_p}\), such that \( v^\frac{1}{\theta} \in A_p \) and \([v^\frac{1}{\theta}]_{A_p} \leq 2[v]_{A_p} \); the same is true for \( w^\frac{1}{\theta} \in A_q \). Therefore, the above argument in fact implies that
\[
\|T_j(f,g)\|_{L^r(\mathbb{R}^n; L^q(w^T))} \lesssim C_{\epsilon,n} \|\Omega\|_{L^\infty} 2^{j|\epsilon|} \|f\|_{L^p(v^T)} \|g\|_{L^q(w^T)}
\]
for all \( f \in L^p(v^T) \), \( g \in L^q(w^T) \) and \( j \in \mathbb{Z} \), where the implicit constant only depends on \( p \), \( q \), \([v]_{A_p}\) and \([w]_{A_q}\).

Since we assumed that \( 1 \leq r < \infty \), we can use complex interpolation with a change of measure (\[2\], Theorem 5.5.3) and bilinear complex interpolation (\[2\], Theorem 4.4.1) to get that (4.4) and (4.5) together imply
\[
\|T_j(f,g)\|_{L^r(\mathbb{R}^n; L^q(w^T))} \lesssim C_{\epsilon,n} \|\Omega\|_{L^\infty} 2^{-|j|(1-\theta)\delta - \theta \epsilon} \|f\|_{L^p(v^T)} \|g\|_{L^q(w^T)}
\]
for all \( f \in L^p(v^\frac{1}{\theta})^\prime \), \( g \in L^q(w^\frac{1}{\theta})^\prime \) and \( j \in \mathbb{Z} \). If we choose \( \epsilon \) small enough that \( \rho = (1 - \theta)\delta - \theta \epsilon > 0 \), we then obtain that for all \( f \in L^p(v), g \in L^q(w) \),
\[
\|T_\Omega(f,g)\|_{L^r(v^\frac{1}{\theta} w^\frac{1}{\theta})} \lesssim \|\Omega\|_{L^\infty} \sum_{j \in \mathbb{Z}} 2^{-jp} \|f\|_{L^p(v)} \|g\|_{L^q(w)}.
\]
Since this series converges, we get the desired result. \(\square\)

Theorems 3.1 and 4.3 together immediately imply that \( T_\Omega \) is bounded in the variable Lebesgue spaces.

**Corollary 4.4.** Let \( p(\cdot), q(\cdot), r(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) satisfy \( \frac{1}{r(\cdot)} = \frac{1}{q(\cdot)} + \frac{1}{p(\cdot)} \), \( p_- > 1 \), \( q_- > 1 \), \( r_- > 1 \), and assume there exist \( 1 < p < p_- \) and \( 1 < q < q_- \) such that \( \frac{1}{p} + \frac{1}{q} \leq 1 \) and \( \left( \frac{p(\cdot)}{p(\cdot)} \right)', \left( \frac{q(\cdot)}{q(\cdot)} \right)' \in \mathcal{B} \). If \( \Omega \in L^\infty(S^{2n-1}) \) with \( \int_{S^{2n-1}} \Omega d\sigma = 0 \) then for all \( f \in L^{p(\cdot)}, g \in L^{q(\cdot)} \),
\[
\|T_\Omega(f,g)\|_{L^{r(\cdot)}} \lesssim \|f\|_{L^{p(\cdot)}} \|g\|_{L^{q(\cdot)}}.
\]

**Remark 4.3.** We conjecture that Theorem 4.3 is true when \( 1/2 < r < 1 \), which would in turn imply that Corollary 4.4 holds when \( 1/2 < r_- < 1 \). In the proof of the theorem, the argument holds when \( r < 1 \) up through the proof of (4.5). It is only in applying complex interpolation that we need the additional hypothesis that \( r > 1 \).

5. **Proof of main results**

In this section we prove Theorems 1.1 and 1.2.

**Proof of Theorem 1.1.** Our proof follows the broad outline of the proof of \[21, Theorem 1\], and we refer the reader there for some details. Let \( p, q, r, s \) and \( v, w \) be as in the hypotheses. Fix a function \( \varphi \in \mathcal{S}(\mathbb{R}^n) \) such that \( \text{supp}(\varphi) \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq 2 \} \).
and $\varphi \equiv 1$ in $\{ \xi \in \mathbb{R}^n : |\xi| \leq 1 \}$. Define $\psi$ by $\psi(\xi) = \varphi(\xi) - \varphi(2\xi)$ for $\xi \in \mathbb{R}^n$. Then for all $k \in \mathbb{Z}$,

$$\text{supp}(\psi(2^{-k} \cdot)) \subset \{ \xi \in \mathbb{R}^n : 2^{k-1} \leq |\xi| \leq 2^{k+1} \},$$

and for $\xi \neq 0$ and $M \in \mathbb{Z}$,

$$\sum_{k \in \mathbb{Z}} \psi(2^{-k} \xi) = 1, \quad \sum_{k \leq M} \psi(2^{-k} \xi) = \varphi(2^{-M} \xi).$$

We first prove the estimate (1.3). Given $f, g \in \mathcal{S}(\mathbb{R}^n)$, decompose $D^s(fg)(x)$ as

$$D^s(fg)(x) = \int_{\mathbb{R}^{2n}} |\xi + \eta|^s \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta,$$

$$= \int_{\mathbb{R}^{2n}} |\xi + \eta|^s \left( \sum_{k \in \mathbb{Z}} \psi(2^{-k} \xi) \hat{f}(\xi) \right) \left( \sum_{\ell \in \mathbb{Z}} \psi(2^{-\ell} \eta) \hat{g}(\eta) \right) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta,$$

$$= T_{1,s}(D^s f, g)(x) + T_{2,s}(f, D^s g)(x) + T_{3,s}(f, D^s g)(x),$$

where

$$T_{1,s}(f, g) = \int_{\mathbb{R}^{2n}} \Phi_1(\xi, \eta) \frac{|\xi + \eta|^s}{|\xi|^s} \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta,$$

$$T_{2,s}(f, g) = \int_{\mathbb{R}^{2n}} \Phi_2(\xi, \eta) \frac{|\xi + \eta|^s}{|\eta|^s} \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta,$$

$$T_{3,s}(f, g) = \int_{\mathbb{R}^{2n}} \Phi_3(\xi, \eta) \frac{|\xi + \eta|^s}{|\eta|^s} \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta,$$

and

$$\Phi_1(\xi, \eta) = \sum_{k \in \mathbb{Z}} \psi(2^{-k} \xi) \varphi(2^{-(k-5)} \eta),$$

$$\Phi_2(\xi, \eta) = \sum_{k \in \mathbb{Z}} \varphi(2^{-(k-5)} \xi) \psi(2^{-k} \eta),$$

$$\Phi_3(\xi, \eta) = \sum_{k \in \mathbb{Z}} \sum_{\ell = -4}^{4} \psi(2^{-k} \xi) \psi(2^{-(k+\ell)} \eta).$$

To complete the proof it will suffice to prove that for all $f, g \in \mathcal{S}(\mathbb{R}^n)$,

\begin{align*}
(5.1) \quad & \|T_{1,s}(f, g)\|_{L^r(v^{p/\tilde{p}}w^{q/\tilde{q}})} \lesssim \|f\|_{L^p(v)} \|g\|_{L^q(w)}, \\
(5.2) \quad & \|T_{2,s}(f, g)\|_{L^r(v^{p/\tilde{p}}w^{q/\tilde{q}})} \lesssim \|f\|_{L^p(v)} \|g\|_{L^q(w)}, \\
(5.3) \quad & \|T_{3,s}(f, g)\|_{L^r(v^{p/\tilde{p}}w^{q/\tilde{q}})} \lesssim \|f\|_{L^p(v)} \|g\|_{L^q(w)},
\end{align*}
The first two inequalities are straightforward. Since $\Phi_1$ is supported in $\{(\xi, \eta) \in \mathbb{R}^{2n} : |\eta| \leq \frac{1}{8} |\xi|\}$ and $\Phi_2$ is supported in $\{(\xi, \eta) \in \mathbb{R}^{2n} : |\xi| \leq \frac{1}{8} |\eta|\}$, it follows that $T_{1,s}$ and $T_{2,s}$ are bilinear Coifman-Meyer multiplier operators; therefore, by Theorem 4.F we have that (5.1) and (5.2) hold.

If $s$ is a non-negative even integer or $s$ is sufficiently large, then $T_{3,s}$ is also a Coifman-Meyer multiplier operator and so (5.3) holds in these cases. Otherwise, $T_{3,s}$ may fail to be a Coifman-Meyer multiplier operator (see [21, Remark 1, p. 1139]). In general, however, as shown in [21, p. 1137], $T_{3,s}$ can be written as a finite sum of terms (one for each value of $\ell$) of the form

$$\sum_{m \in \mathbb{Z}^n} c_{s,m} T_m(f, g),$$

with $|c_{s,m}| \lesssim (1 + |m|)^{-(n+s)}$ and

$$T_m(f, g)(x) = \sum_{k \in \mathbb{Z}} (\Psi_{1,k,m}^1 * f)(x) (\Psi_{2,k,m}^2 * g)(x),$$

where for $j = 1, 2$, $k \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$, $\Psi_{j,k,m}^j(x) = 2^{kn} \Psi_j(2^k x + m)$ for some smooth function $\Psi_j$ such that $\hat{\Psi}_j$ is supported in an annulus. By the Cauchy-Schwarz inequality,

$$\left| \sum_{m \in \mathbb{Z}^n} c_{s,m} T_m(f, g)(x) \right| \leq \sum_{m \in \mathbb{Z}^n} (1 + |m|)^{-(n+s)} \left( \sum_{k \in \mathbb{Z}} |\Psi_{1,k,m}^1 * f(x)|^2 \right)^{1/2} \left( \sum_{k \in \mathbb{Z}} |\Psi_{2,k,m}^2 * g(x)|^2 \right)^{1/2}.$$ 

Define the square function $S_{m}^j$, $j = 1, 2$, by

$$S_{m}^j(f)(x) = \left( \sum_{k \in \mathbb{Z}} |\Psi_{j,k,m}^j * f(x)|^2 \right)^{1/2}.$$ 

By Theorem 2.1 the operators $S_{m}^j$ satisfy weighted estimates that are uniform in $m$; therefore, with $r_* = \min(r, 1)$, we get that for all $f, g \in \mathcal{S}(\mathbb{R}^n)$,

$$\left\| \sum_{m \in \mathbb{Z}^n} c_{s,m} T_m(f, g) \right\|_{L^r(v^{1/r} w^{1/r})}^{r_*} \lesssim \sum_{m \in \mathbb{Z}^n} (1 + |m|)^{-(n+s)r_*} \left\| S_{m}^1 f \right\|_{L^p(v)}^{r_*} \left\| S_{m}^2 g \right\|_{L^q(w)}^{r_*} \lesssim \sum_{m \in \mathbb{Z}^n} (1 + |m|)^{-(n+s)r_*} \left\| f \right\|_{L^p(v)}^{r_*} \left\| g \right\|_{L^q(w)}^{r_*}. $$
By assumption, \( s > \max\{0, n\left(\frac{1}{s} - 1\right)\} \); hence, \( (n + s)r_s > n \) and so the series in \( m \) converges. This proves inequality (5.3) and so completes the proof of (1.3).

The proof of (1.4) is similar. We can decompose \( J^s(f, g) \) as

\[
J^s(f, g)(x) = \tilde{T}_{1,s}(J^s f, g) + \tilde{T}_{2,s}(f, J^s g) + \tilde{T}_{3,s}(f, J^s g);
\]

the operators \( \tilde{T}_{1,s} \) and \( \tilde{T}_{2,s} \) are defined like \( T_{1,s} \) and \( T_{2,s} \) with \( \frac{|x + y|^s}{|x|} \) and \( \frac{|x + y|^s}{|y|} \) replaced by \( \frac{(1 + |x + y|^2)^{\frac{s}{2}}}{(1 + |x|^2)^{\frac{s}{2}}} \) and \( \frac{(1 + |x + y|^2)^{\frac{s}{2}}}{(1 + |y|^2)^{\frac{s}{2}}} \) respectively. Again, both \( \tilde{T}_{1,s} \) and \( \tilde{T}_{2,s} \) are bilinear Coifman-Meyer multiplier operators; therefore, weighted estimates corresponding to (5.1) and (5.2) hold for \( \tilde{T}_{j,s} \) for \( j = 1, 2, \ldots \), respectively.

To estimate \( \tilde{T}_{3,s} \), we use the fact (see [21, pp. 1148-1149]) that it can be written as

\[
\tilde{T}_{3,s}(f, g)(x) = \tilde{T}_{3,s}^1(f, g)(x) + \tilde{T}_{3,s}^2(f, g)(x), \quad x \in \mathbb{R}^n,
\]

where

\[
|\tilde{T}_{3,s}^1(f, g)(x)| \leq \sum_{m, \ell \in \mathbb{Z}^n} b_m^s \tilde{b}_{\ell}^s \sum_{k \geq 0} |(\Psi_{k,m} * f)(x)(\Psi_{k,m+\ell} * g)(x)|
\]

and

\[
|\tilde{T}_{3,s}^2(f, g)(x)| \leq \sum_{m, \ell \in \mathbb{Z}^n} a_m^s \tilde{a}_{\ell}^s \sum_{k \geq 0} |(\Psi_{k,m} * f)(\Psi_{k,m+\ell} * g)|,
\]

where \( \Psi_{k,m}(x) = 2^{kn}\Psi(2^kx + m) \) and \( \tilde{\Psi}_{k,m}(x) = 2^{kn}\Psi(2^k(x - m)) \) with \( \Psi \in \mathcal{S}(\mathbb{R}^n) \) and \( \tilde{\Psi} \) supported in an annulus. The constants satisfy

\[
|b_m^s|, |\tilde{b}_{\ell}^s|, |a_m^s|, |\tilde{a}_{\ell}^s| \lesssim |m|^{-n-s}.
\]

We can now argue as we did for \( T_{3,s} \) above and use Theorem 2.1 to get the corresponding estimate (5.3) for \( \tilde{T}_{3,s} \).

We next prove (1.5). The decompositions given below are inspired by those used in the proof of [27, Lemma X1] corresponding to \( J^s \). We have

\[
D^s(f g)(x) - f D^s(g)(x) = \int_{\mathbb{R}^{2n}} (|\xi + \eta|^s - |\eta|^s) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} \, d\xi d\eta
\]

\[
= Q_{1,s}(f, g)(x) + Q_{2,s}(f, g)(x) + Q_{3,s}(f, g)(x),
\]

where

\[
Q_{1,s}(f, g) = \int_{\mathbb{R}^{2n}} \Phi_1(\xi, \eta) (|\xi + \eta|^s - |\eta|^s) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} \, d\xi d\eta;
\]

\[
Q_{2,s}(f, g) = \int_{\mathbb{R}^{2n}} \Phi_2(\xi, \eta) (|\xi + \eta|^s - |\eta|^s) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} \, d\xi d\eta;
\]

\[
Q_{3,s}(f, g) = \int_{\mathbb{R}^{2n}} \Phi_3(\xi, \eta) (|\xi + \eta|^s - |\eta|^s) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} \, d\xi d\eta.
\]
\[ Q_{3,s}(f, g) = \int_{\mathbb{R}^{2n}} \Phi_3(\xi, \eta) (|\xi + \eta|^s - |\eta|^s) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi ix(\xi + \eta)} \, d\xi d\eta. \]

The operator \( Q_{1,s} \) can in turn be decomposed as
\[ Q_{1,s}(f, g)(x) = Q_{1,s}^1(D^s f, g)(x) - Q_{1,s}^2(f, g)(x), \]
where \( Q_{1,s}^1 \) is the same as \( T_{1,s} \) (and hence satisfies (5.1)) and
\[ Q_{1,s}^2(f, g)(x) = \int_{\mathbb{R}^{2n}} \Phi_1(\xi, \eta) |\eta|^s \hat{f}(\xi) \hat{g}(\eta) e^{2\pi ix(\xi + \eta)} \, d\xi d\eta. \]

Moreover,
\[ \Phi_1(\xi, \eta) |\eta|^s \hat{f}(\xi) \hat{g}(\eta) = \Phi_1(\xi, \eta) |2\pi \xi|^{-2} |2\pi \xi|^2 \hat{f}(\xi) |\eta|^{-1} |\eta|^2 \hat{g}(\eta) = \frac{1}{2\pi} \Phi_1(\xi, \eta) |\xi|^{-2} \sum_{j=1}^{n} \xi_j \hat{f}(\xi) \sum_{k=1}^{n} \eta_k G_k D^{s-1} \hat{g}(\eta), \]
where \( \hat{G_k} h(\eta) = \frac{m}{|\eta|^s} \hat{h}(\eta) \) is a constant multiple of the Riesz transform \( R_k \). Therefore, we have that
\[ Q_{1,s}^2(f, g)(x) = \sum_{j,k=1}^{n} Q_{1,s}^{2,j,k}(\partial_j f, G_k D^{s-1} g), \]
where \( Q_{1,s}^{2,j,k} \) is a bilinear multiplier operator with symbol \( \frac{1}{2\pi} \xi_j \eta_k |\xi|^{-2} \Phi_1(\xi, \eta) \). Such a symbol is a Coifman-Meyer multiplier since \( \Phi_1 \) is supported in the set \( \{ (\xi, \eta) \in \mathbb{R}^{2n} : |\eta| \leq \frac{1}{\xi} |\xi| \} \); therefore, we can apply Theorem 4.F to \( Q_{1,s}^{2,j,k} \) to get that for all \( f, g \in \mathcal{S}(\mathbb{R}^n) \),
\[ \|Q_{1,s}^2(f, g)\|_{L^r(\mathbb{R}^n)} \lesssim \|\nabla f\|_{L^p(\mathbb{R}^n)} \|D^{s-1} g\|_{L^q(\mathbb{R}^n)} \]
where we have used that \( G_k \) is a bounded operator from \( L^q(w) \) into \( L^q(w) \), since the Riesz transforms are (see [15]). If we combine these estimates, we see that for all \( f, g \in \mathcal{S}(\mathbb{R}^n) \),
\[ (5.4) \quad \|Q_{1,s}(f, g)\|_{L^r(\mathbb{R}^n)} \lesssim \|D^s f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)} + \|\nabla f\|_{L^p(\mathbb{R}^n)} \|D^{s-1} g\|_{L^q(\mathbb{R}^n)}. \]

The symbol of the operator \( Q_{2,s} \) is given by
\[ \Phi_2(\xi, \eta)(|\xi + \eta|^s - |\eta|^s) = \Phi_2(\xi, \eta) |\eta|^s \left( \frac{|\xi + \eta|^s}{|\eta|^s} - 1 \right) \]
\[ = \Phi_2(\xi, \eta) |\eta|^s \left[ \left( 1 + \frac{|\xi|^2 + 2\xi \cdot \eta}{|\eta|^2} \right)^{s/2} - 1 \right] \]
\[ = \Phi_2(\xi, \eta) |\eta|^s \sum_{j=1}^{\infty} \left( \frac{s/2}{j} \right) \left( \frac{|\xi|^2 + 2\xi \cdot \eta}{|\eta|^2} \right)^j, \]
where \((s/2)_j = s/(s/2 - 1)(s/2 - 2)\cdots(s/2 - j + 1)\) and the series converges absolutely and uniformly on the support of \(\Phi_2\). Indeed, the support of \(\Phi_2\) is contained in \(\{\|\xi\| \leq \frac{1}{8} \|\eta\|\}\) and for \((\xi, \eta)\) in this set we have \(\xi^2 + 2\xi \cdot \eta < 1\). (Recall that the radius of convergence of the binomial series is 1 for exponents that are not in \(\mathbb{N}_0\); otherwise the sum is finite.) Now, with \(c_{j,s} = (s/2)_j\),

\[
c_{j,s}\Phi_2(\xi, \eta) |\xi|^s \left(\frac{|\xi|^2 + 2\xi \cdot \eta}{|\eta|^2}\right)^j \hat{f}(\xi) \hat{g}(\eta)
\]

setting \(\sigma_{j,\nu}(\xi, \eta) = \Phi_2(\xi, \eta) \left(\frac{|\xi|^2 + 2\xi \cdot \eta}{|\eta|^2}\right)^{j-1}(\xi_\nu + 2\eta_\nu)\), we have that

\[
Q_{2,s}(f, g)(x) = \frac{1}{2\pi} \sum_{j=1}^{\infty} \sum_{\nu=1}^{n} c_{j,s} T_{\sigma_{j,\nu}}(\partial^\nu f, D^{s-1}g)(x) \quad \forall f, g \in \mathcal{S}(\mathbb{R}^n), x \in \mathbb{R}^n,
\]

where \(T_{\sigma_{j,\nu}}\) is the bilinear multiplier operator with symbol \(\sigma_{j,\nu}\). By Lemma 5.1 (whose statement and proof we defer to the end of this section), \(\sigma_{j,\nu}\) is a Coifman-Meyer multiplier for each \(j \in \mathbb{N}\) and \(\nu = 1, \ldots, n\). Further, \(\sum_{j=1}^{\infty} |c_{j,s}| \|T_{\sigma_{j,\nu}}\|_{p,q,v,w} < \infty\), where \(\|T_{\sigma_{j,\nu}}\|_{p,q,v,w}\) is the norm of \(T_{\sigma_{j,\nu}}\) as a bounded operator from \(L^p(v) \times L^q(w)\) into \(L^{r\sqrt{vw}}(\mathbb{R}^n)\). This implies that for \(f, g \in \mathcal{S}(\mathbb{R}^n)\),

\[
(5.5) \quad \|Q_{2,s}(f, g)\|_{L^{r\sqrt{vw}}(\mathbb{R}^n)} \lesssim \|\nabla f\|_{L^p(v)} \|D^{s-1}g\|_{L^q(w)}.
\]

Finally, the operator \(Q_{3,s}\) can be written as

\[
Q_{3,s}(f, g)(x) = Q_{3,s}^1(D^s f, g)(x) - Q_{3,s}^2(D^s f, g)(x),
\]

where

\[
Q_{3,s}^1(f, g)(x) = \int_{\mathbb{R}^n} \Phi_3(\xi, \eta) \frac{|\xi + \eta|^s}{|\xi|^s} \hat{f}(\xi) \hat{g}(\eta) e^{2\pi ix \cdot (\xi + \eta)} d\xi d\eta,
\]

\[
Q_{3,s}^2(f, g)(x) = \int_{\mathbb{R}^n} \Phi_3(\xi, \eta) \frac{|\eta|^s}{|\xi|^s} \hat{f}(\xi) \hat{g}(\eta) e^{2\pi ix \cdot (\xi + \eta)} d\xi d\eta.
\]

The operator \(Q_{3,s}^1\) is the same as the operator \(T_{3,s}\) but with the roles of \(f\) and \(g\) interchanged. The operator \(Q_{3,s}^2\) is a Coifman-Meyer multiplier operator since \(\Phi_3\) is supported in a region where \(|\xi| \sim |\eta|\). Therefore, for \(j = 1, 2\), \(Q_{3,s}^j(f, g)\) satisfies the weighted estimates in the conclusion of Theorem 4.F, and so we have that for all \(f, g \in \mathcal{S}(\mathbb{R}^n)\),

\[
(5.6) \quad \|Q_{3,s}(f, g)\|_{L^{r\sqrt{vw}}(\mathbb{R}^n)} \lesssim \|D^s f\|_{L^p(v)} \|g\|_{L^q(w)}.
\]
The estimate (1.5) now follows from (5.4), (5.5) and (5.6).

Finally to prove (1.6) we describe the changes we need to make in the preceding argument (see [27, Lemma X1]). In this case, the operators $Q_{j,s}$ are replaced by the operators $\tilde{Q}_{j,s}$, $j = 1, 2, 3$, which in their symbols have $(1 + |\xi + \eta|^2)^{s/2} - (1 + |\eta|^2)^{s/2}$ instead of $|\xi + \eta|^s - |\eta|^s$. The operator $\tilde{Q}_{1,s}$ can be decomposed as

$$\tilde{Q}_{1,s}(f, g)(x) = \tilde{Q}_{1,s}^1(J^s f, g)(x) - \tilde{Q}_{1,s}^2(f, g)(x),$$

where $\tilde{Q}_{1,s}^1$ has symbol $\Phi_1(\xi, \eta)\frac{(1 + |\xi + \eta|^2)^{s/2} - 1}{(1 + |\xi|^2)^{s/2}}$, which is a Coifman-Meyer multiplier, and $\tilde{Q}_{1,s}^2$ has symbol $\Phi_1(\xi, \eta)((1 + |\eta|^2)^{s/2} - 1)$. We then get

$$\tilde{Q}_{1,s}^2(f, g)(x) = \sum_{j,k=1}^n \tilde{Q}_{1,s}^{2,j,k}(\partial_j f, \tilde{G}_k J^{s-1} g),$$

where $\tilde{Q}_{1,s}^{2,j,k}$ is a bilinear multiplier operator with symbol $\frac{1}{2\pi}\xi_j \eta_k |\xi|^{-2} \Phi_1(\xi, \eta)$ (i.e. $\tilde{Q}_{1,s}^{2,j,k} = Q_{1,s}^{2,j,k}$) and $\tilde{G}_k$ is the linear multiplier operator given by

$$\tilde{G}_k h(\eta) = \eta_k (1 + |\eta|^2)^{1/2}((1 + |\eta|^2)^{s/2} - 1) \hat{h}(\eta).$$

By the weighted version of the Hörmander-Mihlin theorem (see [30]), $\tilde{G}_k$ is a bounded operator in $L^q(w)$. As a consequence of these arguments we get (5.4) with $Q_{1,s}$ and $D$ replaced with $\tilde{Q}_{1,s}$ and $J$, respectively.

For the operator $\tilde{Q}_{2,s}$, we have that its symbol satisfies

$$\Phi_2(\xi, \eta)((1 + |\xi + \eta|^2)^{s/2} - (1 + |\eta|^2)^{s/2})$$

$$= \Phi_2(\xi, \eta)(1 + |\eta|^2)^{s/2} \sum_{j=1}^\infty \left(\frac{s/2}{j}\right) \left(\frac{|\xi|^2 + 2\xi \cdot \eta}{1 + |\eta|^2}\right)^j,$$

with the series converging uniformly and absolutely on the support of $\Phi_2$, since $\left|\frac{|\xi|^2 + 2\xi \cdot \eta}{1 + |\eta|^2}\right| \leq \frac{17}{64} < 1$ if $(\xi, \eta) \in \text{supp}(\Phi_2)$. Hence, we have that for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\tilde{Q}_{2,s}(f, g)(x) = \frac{1}{2\pi} \sum_{j=1}^\infty \sum_{\nu=1}^n \left(\frac{s/2}{j}\right) T_{\tilde{\sigma}_{j,\nu}}(\partial_{\nu} f, J^{s-1} g)(x),$$

where

$$\tilde{\sigma}_{j,\nu}(\xi, \eta) = \Phi_2(\xi, \eta)\frac{(|\xi|^2 + 2\xi \cdot \eta)^{j-1}}{(1 + |\eta|^2)^{j-\frac{s}{2}}}(\xi_\nu + 2\eta_\nu).$$

The counterpart of (5.5) for $\tilde{Q}_{2,s}$ (with $D$ replaced by $J$) now follows from Lemma 5.1.
Finally, the operator $\tilde{Q}_{3,s}$ is treated in the same way as $Q_{3,s}$. The corresponding operator $\tilde{Q}_{3,s}^1$ is analogous to $\tilde{T}_{3,s}$, while $\tilde{Q}_{3,s}^2$ has symbol $\tilde{\Phi}_3(\xi,\eta) \left( \frac{1+|\eta|^2}{1+|\xi|^2} \right)^{\frac{3}{2}}$, which is a Coifman-Meyer multiplier. Inequality (5.6) with $Q_{3,s}$ and $D$ replaced by $\tilde{Q}_{3,s}$ and $J$, respectively, follows as before.

Proof of Theorem 1.2. The desired inequalities all follow at once from bilinear extrapolation and the weighted estimates derived in the proof of Theorem 1.1. To prove (1.7) and (1.8), it suffices to note that the weighted inequalities proved for the operators $T_{j,s}$, $j = 1, 2, 3$—i.e., (5.1), (5.2) and (5.3)—satisfy the hypotheses of Theorem 3.1, and so we get the corresponding variable Lebesgue space estimates. The same is true for the operators $\tilde{T}_{j,s}$, $j = 1, 2, 3$.

To prove (1.9) and (1.10), we again note that the weighted norm inequalities for $Q_{j,s}$, $j = 1, 2, 3$—i.e., (5.4), (5.5) and (5.6)—again satisfy the hypotheses of Theorem 3.1, and so we get the corresponding variable Lebesgue space estimates. The same is true for the operators $\tilde{Q}_{j,s}$, $j = 1, 2, 3$.

We conclude this section by stating and proving the lemma used in the proof of Theorem 1.1.

Lemma 5.1. For $j \in \mathbb{N}$ and $\nu = 1, \ldots, n$, let $\sigma_{j,\nu}$ and $\tilde{\sigma}_{j,\nu}$ be as in the proof of Theorem 1.1, let $p, q, r, v, w$ be as in the hypotheses of Theorem 1.1, and fix $s > 0$. Then $T_{\sigma_{j,\nu}}$ is a Coifman-Meyer multiplier operator and $\sum_{j=1}^{\infty} \left| \frac{1}{2} \left( \frac{1}{j} \right) \right| ||T_{\sigma_{j,\nu}}||_{p,q,v,w} < \infty$, where $||T_{\sigma_{j,\nu}}||_{p,q,v,w}$ is the norm of $T_{\sigma_{j,\nu}}$ as a bounded operator from $L^p(v) \times L^q(w)$ into $L^{r}(\tilde{v}^{\frac{1}{r}} \tilde{w}^{\frac{1}{w}})$. The analogous result also holds for $\tilde{\sigma}_{j,\nu}$.

Proof. We will prove Lemma 5.1 using Theorem 4.10 and the fact that for Coifman-Meyer multipliers, the constant in the weighted norm inequality depends linearly on the constant in (4.2). Recall that $\sigma_{j,\nu} = \frac{(|\xi|^2 + 2\xi \cdot \eta \cdot 2\eta \cdot \eta)}{|\xi|^2 + 2\xi \cdot \eta \cdot 2\eta \cdot \eta}$ and $\phi_{2}(\xi, \eta)$, where $\phi_{2}$ is supported in the set $\{ (\xi, \eta) \in \mathbb{R}^{2n} : |\xi| \leq \frac{1}{2} |\eta| \}$ and $\xi_{\nu}, \eta_{\nu}$ denote the $\nu$-th coordinates of $\xi$ and $\eta$, respectively. Set $\tau_{j,\nu}(\xi, \eta) = (\xi_{\nu} + 2\eta_{\nu})^{2\nu-1}$ and note that $\tau_{j,\nu}$ is homogeneous of degree 0 in its domain. For $\beta, \gamma \in \mathbb{N}_{0}$, we then have that for all $\xi, \eta \neq 0$,

$$\partial_{\xi}^{\beta} \partial_{\eta}^{\gamma} \tau_{j,\nu}(\xi, \eta) = |(\xi, \eta)|^{1+|\beta+\gamma|} (\tilde{\Phi}_{3,\nu})^{(\xi, \eta)};$$

therefore, for all $(\xi, \eta) \in \text{supp}(\phi_{2}) \setminus \{(0,0)\}$,

$$\left| \partial_{\xi}^{\beta} \partial_{\eta}^{\gamma} \tau_{j,\nu}(\xi, \eta) \right| \lesssim \sup_{(a,b) \in \text{supp}(\phi_{2}) \setminus \{(0,0)\}} \left| (\partial_{a}^{\beta} \partial_{b}^{\gamma} \tau_{j,\nu})^{(a,b)} \right| |(\xi, \eta)|^{-1+|\beta+\gamma|}.$$
Note that $(\partial_a^\beta \partial_b^\gamma \tau_{j,\nu})(\frac{(a,b)}{|(a,b)|})$ is bounded on the set $\text{supp}(\Phi_2) \setminus \{(0,0)\}$ since $\frac{8}{\sqrt{65}} \leq \frac{|(a,b)|}{|a,b|} \leq 1$ for $(a,b) \in \text{supp}(\Phi_2) \setminus \{(0,0)\}$. By the product rule for derivatives, the above inequality combined with the fact that $\Phi_2$ is a Coifman-Meyer multiplier implies that $\sigma_{j,\nu}$ is also a Coifman-Meyer multiplier. Moreover, for all $(\xi, \eta) \neq (0,0)$,

$$\left| \partial_\xi^\beta \partial_\eta^\gamma \sigma_{j,\nu}(\xi, \eta) \right| \lesssim \sup_{(a,b) \in \text{supp}(\Phi_2) \setminus \{(0,0)\}} \left| (\partial_a^\beta \partial_b^\gamma \tau_{j,\nu})(\frac{(a,b)}{|(a,b)|}) \right| |(\xi, \eta)|^{-|\beta+\gamma|},$$

where the implicit constant is independent of $j$. By Theorem 4.7 we have that

$$\|T_{\sigma_{j,\nu}}\|_{p, q, v, w} \lesssim \sup_{|\beta+\gamma| \leq 2n+1} \sup_{(\xi, \eta) \neq (0,0)} \left| (\partial_\xi^\beta \partial_\eta^\gamma \sigma_{j,\nu})(\xi, \eta) \right| |(\xi, \eta)|^{-|\beta+\gamma|},$$

where the implicit constant is again independent of $j$. Together, these two inequalities imply that for all $j$,

$$\|T_{\sigma_{j,\nu}}\|_{p, q, v, w} \lesssim \sup_{|\beta+\gamma| \leq 2n+1} \sup_{(\xi, \eta) \in S} \left| (\partial_\xi^\beta \partial_\eta^\gamma \tau_{j,\nu})(\xi, \eta) \right|,$$

where $S = \mathbb{S}^{2n-1} \cap \{(\xi, \eta) : |\xi| \leq \frac{1}{4} |\eta| \}$. Set $F(\xi, \eta) = \frac{\xi + 2\pi n}{|\eta|}$ and $G(\xi, \eta) = \frac{|\xi|^2 + 2\pi n}{|\eta|^2}$, then $\tau_{j,\nu}(\xi, \eta) = F(\xi, \eta)(G(\xi, \eta))^{-1}$. By induction we have that if $j > 2n+2$ and $|\beta + \gamma| \leq 2n+1$, then

$$(5.7) \quad \partial_\xi^\beta \partial_\eta^\gamma \tau_{j,\nu}(\xi, \eta) = \partial_\xi^\beta \partial_\eta^\gamma F(\xi, \eta)(G(\xi, \eta))^{j-1} + \sum_{\beta_1 + \beta_2 = \beta \atop \gamma_1 + \gamma_2 = \gamma \atop (\beta_2, \gamma_2) \neq (0,0)} \partial_\xi^\beta_1 \partial_\eta^\gamma_1 F(\xi, \eta) \prod_{\ell=2}^{[\beta_2 + \gamma_2] + 1} \sum_{\kappa=1}^{\ell-1} (j - \kappa)(G(\xi, \eta))^{j-\ell} H_{\beta_2, \gamma_2}(\xi, \eta),$$

where $H_{\beta_2, \gamma_2}$ is a sum of products of derivatives of $G$ with each product having $\beta_2$ derivatives with respect to $\xi$ and $\gamma_2$ derivatives with respect to $\eta$. The derivatives of $F$ and $G$ are bounded on $S$ and $|G(\xi, \eta)| \leq \frac{17}{64}$ for $(\xi, \eta) \in S$. Therefore, for all $j > 2n+2$ we have that

$$\sup_{|\beta+\gamma| \leq 2n+1} \sup_{(\xi, \eta) \in S} \left| \partial_\xi^\beta \partial_\eta^\gamma \tau_{j,\nu}(\xi, \eta) \right| \lesssim \left( \frac{17}{64} \right)^{j-1} + \sum_{\ell=2}^{2n+2} (j-1) \cdots (j-\ell+1) \left( \frac{17}{64} \right)^{j-\ell} \lesssim \left( \frac{17}{64} \right)^{j-1} + \left( \frac{17}{64} \right)^{j} \sum_{\ell=2}^{2n+2} \left( \frac{17}{64} \right)^{\ell-1} \lesssim \left( \frac{17}{64} \right)^{j} + \left( \frac{17}{64} \right)^{j}.$$
Thus,
\[
\sum_{j=2n+3}^{\infty} \left| \left( s/2 \right) \right| \left| \left| T_{\sigma_{j,\nu}} \right| \right|_{p, q, v, w} \lesssim \sum_{j=2n+3}^{\infty} \left| \left( s/2 \right) \right| \left| \left( \frac{17}{64} \right) \right|^{j} + \left( \frac{17}{32} \right)^{j} < \infty,
\]
which implies the desired result for \( \sigma_{j,s} \).

To prove the analogous result for \( \tilde{\sigma}_{j,\nu} \), recall that
\[
\tilde{\sigma}_{j,\nu}(\xi, \eta) = \Phi_{2}(\xi, \eta) \frac{(|\xi|^{2} + 2\xi \cdot \eta)^{j-1}}{(1 + |\eta|^{2})^{j-\frac{1}{2}}}(\xi_{\nu} + 2\eta_{\nu}) = \Phi_{2}(\xi, \eta)\tilde{T}_{j,\nu}(\xi, \eta).
\]
The product rule and the fact that \( \Phi_{2} \) is a Coifman-Meyer multiplier imply that
\[
\left| \partial_{\xi}^{\beta} \partial_{\eta}^{\gamma} \tilde{\sigma}_{j,\nu}(\xi, \eta) \right| \lesssim \sup_{(a, b) \in \text{supp}(\Phi_{2}) \setminus \{(0, 0)\}} \left| \left| (a, b) \right| \right|^{\beta + \gamma} \left| \partial_{\nu}^{\beta} \partial_{b}^{\gamma} \tilde{T}_{j,\nu}(a, b) \right| \left| (\xi, \eta) \right|^{-|\beta + \gamma|}
\]
for all \((\xi, \eta) \neq (0, 0)\) and all \( j \). This computation is similar to the estimate for \( \tau_{j,\nu} \) above, which only uses homogeneity to rescale the derivative. The boundedness of the supremum is a consequence of the argument below. Thus for all \( j \) we have that
\[
\left| \left| T_{\tilde{\sigma}_{j,\nu}} \right| \right|_{p, q, v, w} \lesssim \sup_{|\beta + \gamma| \leq 2n+1} \sup_{(\xi, \eta) \neq (0, 0)} \left| \partial_{\xi}^{\beta} \partial_{\eta}^{\gamma} \tilde{\sigma}_{j,\nu}(\xi, \eta) \right| \left| (\xi, \eta) \right|^{\beta + \gamma} \lesssim \sup_{(\xi, \eta) \in \text{supp}(\Phi_{2}) \setminus \{(0, 0)\}} \left| \left| (\xi, \eta) \right| \right|^{\beta + \gamma} \left| \partial_{\nu}^{\beta} \partial_{b}^{\gamma} \tilde{T}_{j,\nu}(\xi, \eta) \right| \left| (\xi, \eta) \right|^{-|\beta + \gamma|}.
\]

Define \( \tilde{F}(\xi, \eta) = \frac{\xi_{\nu} + 2\eta_{\nu}}{(1 + |\eta|^{2})^{\frac{3}{2}}} \) and \( \tilde{G}(\xi, \eta) = \frac{|\xi|^{2} + 2\xi \cdot \eta}{1 + |\eta|^{2}} \), then we can prove a formula analogous to (5.7) for \( \partial_{\xi}^{\beta} \partial_{\eta}^{\gamma} \tilde{T}_{j,\nu}(\xi, \eta) \) with \( j > 2n + 2 \) and \( |\beta + \gamma| \leq 2n + 1 \). It follows that the functions \( |(\xi, \eta)|^{1 + |\gamma|} \partial_{\xi}^{\beta} \partial_{\eta}^{\gamma} \tilde{F}(\xi, \eta) \) and \( |(\xi, \eta)|^{\beta + \gamma} \tilde{H}_{\beta,\gamma}(\xi, \eta) \) are bounded on \( \text{supp}(\Phi_{2}) \setminus \{(0, 0)\} \) for any multi-indices and that \( \left| \tilde{G}(\xi, \eta) \right| \leq \frac{17}{64} \) for \((\xi, \eta) \in \text{supp}(\Phi_{2}) \setminus \{(0, 0)\} \); therefore, we can argue as before to obtain the desired result for \( \tilde{\sigma}_{j,\nu} \).

6. Kato-Ponce inequalities in Lorentz spaces and Morrey spaces

The tools used to prove Theorem 1.1 also lead to fractional Leibniz rules in the settings of weighted Lorentz spaces (more generally, weighted rearrangement invariant quasi-Banach function spaces) and Morrey spaces. We state these results and briefly describe their proofs.
6.1. **Kato-Ponce inequalities in weighted Lorentz spaces.** Given $0 < p < \infty$, $0 < q \leq \infty$ and a weight $w$ defined on $\mathbb{R}^n$, we denote by $L^{p,q}(w)$ the weighted Lorentz space consisting of complex-valued, measurable functions $f$ defined on $\mathbb{R}^n$ such that

$$
\|f\|_{L^{p,q}(w)} = \left( \int_0^{\infty} (t^{\frac{1}{p}} f_w(t))^{q} \frac{dt}{t} \right)^{\frac{1}{q}} < \infty,
$$

where $f_w(t) = \inf\{\lambda \geq 0 : w_f(\lambda) \leq t\}$ with $w_f(\lambda) = w(\{x \in \mathbb{R}^n : |f(x)| > \lambda\})$; the obvious changes apply if $q = \infty$. It follows that $L^{p,p}(w) = L^p(w)$ for $0 < p < \infty$.

**Theorem 6.1.** Let $1 < p, q < \infty$ and $\frac{1}{r} < r < \infty$ be such that $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$ and $0 < a, b, c \leq \infty$ be such that $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$. If $w \in A_{\min(p,q)}$, and $s > \max\{0, n(\frac{1}{r} - 1)\}$ or $s$ is a non-negative even integer, then for all $f, g \in \mathcal{S}(\mathbb{R}^n)$,

(6.1) \[ \|D^s(fg)\|_{L^{r,a}(w)} \lesssim \|D^s f\|_{L^{p,b}(w)} \|g\|_{L^{q,c}(w)} + \|f\|_{L^{p,b}(w)} \|D^s g\|_{L^{r,c}(w)}, \]

(6.2) \[ \|J^s(fg)\|_{L^{r,a}(w)} \lesssim \|J^s f\|_{L^{p,b}(w)} \|g\|_{L^{q,c}(w)} + \|f\|_{L^{p,b}(w)} \|J^s g\|_{L^{r,c}(w)}. \]

The implicit constants depend on $p, q, s$ and $[w]_{A_{\min(p,q)}}$. Moreover, different pairs of $p, q$ and $b, c$ can be chosen for each term on the right hand sides of (6.1) and (6.2).

The proof of Theorem 6.1 will follow from a version of Theorem 2.1 for weighted Lorentz spaces and the boundedness of bilinear Calderón-Zygmund operators in weighted Lorentz spaces.

**Theorem 6.2.** Let $\Psi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\text{supp}(\hat{\Psi}) \subset \{\xi \in \mathbb{R}^n : c_1 < |\xi| < c_2\}$ for some $0 < c_1 < c_2 < \infty$. Given a sequence $z = \{z_k, m\}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} \subset \mathbb{R}_n^9$, define $\Psi_{z_k, m}(x) = 2^{kn} \Psi(2^k(x + z_k, m))$ for $x \in \mathbb{R}^n$, $m \in \mathbb{Z}^n$ and $k \in \mathbb{Z}$. Then for every $1 < p < \infty$, $1 < a < \infty$, $w \in A_p$, and $f \in L^p(w) \cap L^{p,a}(w),$

$$
\left\| \left( \sum_{k \in \mathbb{Z}} |\Psi_{z_k, m} * f|^2 \right)^{\frac{1}{2}} \right\|_{L^{p,a}(w)} \lesssim \|f\|_{L^{p,a}(w)},
$$

where the implicit constants depend on $\Psi$ and $[w]_{A_p}$ but are independent of $m$ and $z$.

**Proof.** This is due to Theorem 2.1 along with [10, Thm 4.10 and comments on p. 70], applied to the family of pairs $\left( \left( \sum_{k \in \mathbb{Z}} |\Psi_{z_k, m} * f|^2 \right)^{\frac{1}{2}}, f \right)$, $f \in L^p(w) \cap L^{p,a}(w)$. $\square$

**Theorem 6.10** (See Corollary 6.11 and comments on page 311 in [12]). Let $1 < p, q < \infty$ and $\frac{1}{r} < r < \infty$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and $0 < a, b, c \leq \infty$ be such that $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$. If $T$ is a bilinear Calderón-Zygmund operator and $w \in A_{\min(p,q)}$, then $T$ is bounded from $L^{p,b}(w) \times L^{q,c}(w)$ into $L^{r,a}(w).$
Proof of Theorem 6.1. The proof of (6.1) and (6.2) now proceeds as in the case of (1.3) and (1.4). Indeed, for \( j = 1, 2, 3 \), let \( T_{j,s} \) and \( \tilde{T}_{j,s} \) be as in the proof of Theorem 1.1. Theorem 6.H gives the needed estimates for the operators \( T_{j,s} \) and \( \tilde{T}_{j,s} \) for \( j = 1, 2 \). The desired control for the operators \( T_{3,s} \) and \( \tilde{T}_{3,s} \) follows the same ideas as those used for (1.3) and (1.4). Recall that \( \| \cdot \|_{L_p^a(w)} \) is comparable to a subadditive quantity for \( 0 < t \leq 1 \) satisfying \( t \leq a \) and \( t < p \) (see [25, p. 258, (2.2)])

and weighted Lorentz spaces satisfy a Hölder-type inequality (see [25, Thm 4.5]); then apply Theorem 6.2. □

Remark 6.1. Theorem 4.10 in [10] and Corollary 6.11 in [12] are also true for weighted versions of rearrangement invariant Banach and quasi-Banach function spaces, respectively, satisfying certain conditions. An argument like the one above for Lorentz spaces leads to some variants of (6.1) and (6.2) in this more general context, and in particular to Orlicz space estimates.

6.2. Kato-Ponce inequalities in Morrey spaces. Given \( 0 \leq \kappa \leq n \) and \( 0 < p < \infty \), the Morrey space \( L^{p,\kappa} \) is defined as the class of functions \( f \in L^p_{\text{loc}}(\mathbb{R}^n) \) such that

\[
\| f \|_{L^{p,\kappa}} = \sup_B \left( \frac{1}{|B|^{1-\kappa/n}} \int_B |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty,
\]

where the supremum is taken over all balls \( B \subset \mathbb{R}^n \). It is a Banach space, which coincides with \( L^p \) for \( \kappa = n \) and with \( L^\infty \) for \( \kappa = 0 \).

Theorem 6.3. Let \( 1 < p, q < \infty \) and \( \frac{1}{2} < r < \infty \) be such that \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \) and \( 0 < \kappa \leq n \). If \( s > \max \{0, n(\frac{1}{r} - 1)\} \) or \( s \) is a non-negative even integer, then for all \( f, g \in \mathcal{S}(\mathbb{R}^n) \),

\[
\begin{align*}
\| D^s (fg) \|_{L^{r,\kappa}} & \lesssim \| D^s f \|_{L^{p,\kappa}} \| g \|_{L^{q,\kappa}} + \| f \|_{L^{p,\kappa}} \| D^s g \|_{L^{q,\kappa}}, \\
\| J^s (fg) \|_{L^{r,\kappa}} & \lesssim \| J^s f \|_{L^{p,\kappa}} \| g \|_{L^{q,\kappa}} + \| f \|_{L^{p,\kappa}} \| J^s g \|_{L^{q,\kappa}}.
\end{align*}
\]

Different triplets of \( p, q, \kappa \) can be chosen on the righthand side of (6.3) and (6.4).

As above, the proof of Theorem 6.3 follows from a version of Theorem 2.1 for Morrey spaces and the boundedness of bilinear Calderón-Zygmund operators in Morrey spaces. We finish this section with the statements of such results.

Theorem 6.4. Let \( \Psi \in \mathcal{S}(\mathbb{R}^n) \) be such that \( \text{supp}(\tilde{\Psi}) \subset \{ \xi \in \mathbb{R}^n : c_1 < |\xi| < c_2 \} \) for some \( 0 < c_1 < c_2 < \infty \). Given a sequence \( \tilde{z} = \{ z_{k,m} \} \subset \mathbb{R}^n \), define \( \Psi_{k,m}(x) = 2^{kn} \Psi(2^k(x + z_{k,m})) \) for \( x \in \mathbb{R}^n \), \( m \in \mathbb{Z}^n \) and \( k \in \mathbb{Z} \). If \( 0 < \kappa \leq n \), then for all \( f \in \mathcal{S}(\mathbb{R}^n) \),

\[
\left\| \left( \sum_{k \in \mathbb{Z}} |\Psi_{k,m} * f|^2 \right)^{\frac{1}{2}} \right\|_{L^{p,\kappa}} \lesssim \left\| f \right\|_{L^{p,\kappa}},
\]
where the implicit constants are independent of \( m \) and \( \bar{z} \).

Theorem 6.4 is a direct consequence of Theorem 2.1 and the following result.

**Proposition 6.1** (see Theorem 6.7 in [14]). Let \( \mathcal{F} \) be a family of ordered pairs of non-negative measurable functions defined on \( \mathbb{R}^n \) and \( 1 \leq p < \infty \). Suppose that for every \( w \in A_1 \),

\[
\int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx \lesssim \int_{\mathbb{R}^n} |g(x)|^p w(x) \, dx
\]

for all \( (f, g) \in \mathcal{F} \), where the implicit constants depend only on \( p \) and \([w]_{A_1} \). If \( 0 < \kappa \leq n \), then

\[
\|f\|_{L^{p, \kappa}} \lesssim \|g\|_{L^{p, \kappa}}
\]

for all \( (f, g) \in \mathcal{F} \).

**Theorem 6.1J** (Particular case of Theorem 1 in [38]). Let \( 1 < p, q < \infty \) and \( \frac{1}{r} < \frac{1}{p} + \frac{1}{q} \) and \( 0 < \kappa \leq n \). If \( T \) is a bilinear Calderón-Zygmund operator, then \( T \) is bounded from \( L^{p, \kappa} \times L^{q, \kappa} \) into \( L^{r, \kappa} \).

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