ELLIPSE EQUATIONS IN DIVergENCE FORM WITH DRIFTS IN L²

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Abstract. We consider the Dirichlet problem for second-order linear elliptic equations in divergence form

\[- \text{div}(A \nabla u) + b \cdot \nabla u + \lambda u = f + \text{div} F \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial \Omega,
\]

in bounded Lipschitz domain \(\Omega\) in \(\mathbb{R}^2\), where \(A : \mathbb{R}^2 \rightarrow \mathbb{R}^2\), \(b : \Omega \rightarrow \mathbb{R}^2\), and \(\lambda \geq 0\) are given. If \(2 < p < \infty\) and \(A\) has a small mean oscillation in small balls, \(\Omega\) has small Lipschitz constant, and \(\text{div } A, b \in L^2(\Omega; \mathbb{R}^2)\), then we prove existence and uniqueness of weak solutions in \(W^{1,p}_0(\Omega)\) of the problem. Similar result also holds for the dual problem.

1. Introduction

This paper is devoted to complementing known results on \(W^{1,p}\)-estimates for second-order linear elliptic equations with singular drifts terms. Let \(\Omega\) be a bounded Lipschitz domain in \(\mathbb{R}^n, n \geq 2\). For a fixed constant \(\lambda \geq 0\) and a given vector field \(b = (b^1, b^2, \ldots, b^n) : \Omega \rightarrow \mathbb{R}^n\), we consider the following Dirichlet problems of linear elliptic equations of second-order:

\[
\begin{align*}
- \text{div}(A \nabla u) + b \cdot \nabla u + \lambda u &= f + \text{div} F \quad \text{in } \Omega, \\
ue &= 0 \quad \text{on } \partial \Omega. 
\end{align*}
\]
\((D)\)

and

\[
\begin{align*}
- \text{div}(A^T \nabla v) - \text{div}(v b) + \lambda v &= g + \text{div} G \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \partial \Omega, 
\end{align*}
\]
\((D')\)

Here \(A = (a^{ij}) : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}\) denotes an \(n \times n\) real matrix-valued measurable function which is uniformly elliptic, that is, there exists \(0 < \delta < 1\) such that

\[
\delta |\xi|^2 \leq \sum_{i,j=1}^n a^{ij}(x) \xi_i \xi_j \quad \text{and} \quad \max_{1 \leq i,j \leq n} |a^{ij}(x)| \leq \delta^{-1} \quad \text{for all } x, \xi \in \mathbb{R}^n. 
\] (1.1)

\(W^{1,p}\)-estimates for the problems \((D)\) and \((D')\) were established by several authors under various assumptions on the leading coefficients \(a^{ij}\) and the domains \(\Omega\) when \(b = 0\) or more generally \(b \in L^\infty(\Omega; \mathbb{R}^n)\); see \(\cite{2, 4, 6, 8-10, 16, 22}\) and references therein. Also, see the recent survey paper of Dong \(\cite{7}\). In particular, Dong-Kim \(\cite{10}\) proved \(W^{1,p}\)-estimates for the problems \((D)\) and \((D')\) when the leading coefficients satisfy small mean oscillations in small balls and \(b \in L^\infty(\Omega; \mathbb{R}^n)\) on a bounded Lipschitz domain with small Lipschitz constant, see Assumptions \(\cite{2, 22}\) and \(\cite{2, 3}\) for
precise statements. One may ask whether we can obtain $W^{1,p}$-estimates for the problems (D) and (D') with unbounded drifts.

Suppose that $b \in L^q(\Omega; \mathbb{R}^n)$, where $n \leq q < \infty$ if $n \geq 3$ and $2 < q < \infty$ if $n = 2$. In Ladyzhenskaya-Ural’tseva [21, Chapter 3], they considered the following Dirichlet problem

$$-\text{div}(A \nabla u + ub) + c \cdot \nabla u + du = f + \text{div} F \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega$$

where $A$ satisfies (1.1), $b, c \in L^q(\Omega; \mathbb{R}^n)$, $d \in L^{n/2}(\Omega)$, and $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^n$. It was shown that under some restricted condition on the zeroth order term $d$, for every $f \in L^{2n/(n+2)}(\Omega)$ and $F \in L^2(\Omega; \mathbb{R}^n)$, there exists a unique weak solution $u \in W^{1,2}_0(\Omega)$ of the problem. Here $n = n$ if $n \geq 3$ and $n = 2 + \varepsilon$ if $n = 2$. Stampacchia [28] also considered a similar problem, but the result is similar to that of Ladyzhenskaya-Ural’tseva.

To the best knowledge of the author, Trudinger [29] first proved that given $\lambda \geq 0$, $f \in L^2(\Omega)$, and $F \in L^2(\Omega; \mathbb{R}^n)$, there exists a unique weak solution $u$ in $W^{1,2}_0(\Omega)$ for the problem (D). The key tools to prove the theorem are the weak maximum principle and the Fredholm alternative theorem. Later, Droniou [11] gave another proof by showing $W^{1,2}$-estimates for the problem (D') and duality method. This result was extended by Kim-Kim [19] who proved $W^{1,p}$-estimates for the problem (D) with $\lambda = 0$ when $A$ is the identity matrix, $q' < p < \infty$, and $\Omega$ is a bounded $C^1$-domain. Later, Kang-Kim [17] proved $W^{1,p}$-estimates for the problem (D) when $q' < p < \infty$, $A$ has small mean oscillation, and $\Omega$ is a bounded Lipschitz domain with small Lipschitz constant. Similar results also hold for the problem (D'). We also mention that there are some recent results on the problems (D) and (D') when the drift $b$ is in weak $L^p$-space; see Moscariello [27], Kim-Tsai [21], and the recent result of the author [23].

The purpose of this paper is to complement $W^{1,p}$-results on elliptic equations with the drift $b \in L^2(\Omega; \mathbb{R}^2)$, which were not considered in [17, 19]. We remark that our result is new even if $A$ is the identity matrix. The motivation for writing this paper is the recent paper due to Krylov [24] who proved $W^{2,p}$-result for second-order elliptic equations of non-divergence form with the drift $b \in L^p(\Omega; \mathbb{R}^n)$, $n \geq 2$. More precisely, if $1 < p < n$, $\Omega$ is a bounded $C^{1,1}$-domain, $A$ has small mean oscillation (see Assumption 2.2), $b \in L^p(\Omega; \mathbb{R}^n)$, and $\lambda \geq 0$, then there exists a unique $u \in W^{1,p}_0(\Omega) \cap W^{2,p}(\Omega)$ satisfying

$$\sum_{i,j=1}^n a^{ij}D_{ij}u + b \cdot \nabla u - \lambda u = f \quad \text{in } \Omega.$$

Related to our result, this is the first result on solvability of elliptic equations with $b \in L^2(\Omega; \mathbb{R}^2)$. Motivated by this result, one may consider $W^{1,p}$-results for the problems (D) and (D') when $b \in L^2(\Omega; \mathbb{R}^2)$.

In this paper, it will be shown in Theorem 2.5 that if $2 < p < \infty$, $A$ has a small mean oscillation in small balls, $\text{div} A, b \in L^2(\Omega; \mathbb{R}^2)$, and $\Omega$ has small Lipschitz constant, then for every $\lambda \geq 0$, $f \in L^p(\Omega)$, and $F \in L^p(\Omega; \mathbb{R}^n)$, there exists a unique weak solution $u \in W^{1,p}_0(\Omega)$ of the problem (D). By duality, we have a similar result for the problem (D'), see Section 2 for the precise statements and the definition of $\text{div} A \in L^2(\Omega; \mathbb{R}^2)$.

Our method to prove Theorem 2.5 is to use a functional analytic argument as in [17, 18, 20]. A key tool is the following $c$-inequality (Proposition 3.1) inspired by
Gerhardt [15]: suppose that $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^2$ and $2 < p < \infty$. Then for each $\varepsilon > 0$, there exists a constant $C_\varepsilon = C_\varepsilon(\varepsilon, p, b, \Omega) > 0$ such that
\[
\|b \cdot \nabla u\|_{L^{1,p}(\Omega)} \leq \varepsilon \|
abla u\|_{L^p(\Omega)} + C_\varepsilon \|u\|_{L^p(\Omega)}
\] (1.2)
for all $u \in W^{1,p}(\Omega)$. Using this estimate, we prove that if $2 < p < \infty$, $A$ has small mean oscillations on small balls and $\Omega$ has small Lipschitz constant, then for sufficiently large $\lambda_1$, the following holds for $\lambda \geq \lambda_1$: if $f \in L^p(\Omega)$, and $F \in L^p(\Omega; \mathbb{R}^2)$, then there exists a unique weak solution $u \in W^{1,p}_0(\Omega)$ of the problem $[D]$. Similar results also hold for the problem $[D^1]$. This result induces an operator $L_p + \lambda_1 I_p : W^{-1,p}(\Omega) \to W^{1,p}_0(\Omega)$, whose inverse can be regarded as a compact operator on $L^p(\Omega)$. Hence by the Fredholm alternative theorem (see [3, Theorem 6.6] e.g.), it suffices to prove the uniqueness of weak solutions in $W^{1,p}_0(\Omega)$ for the problem $[D]$, see Section 5 for the definition of $L_p + \lambda_1 I_p$ and the reduction. To show the uniqueness of weak solutions in $W^{1,p}_0(\Omega)$ of the problem $[D]$, we use an Alexandrov type maximum principle, which was recently proved by Krylov [23, Corollary 3.1], see Theorem 4.2. To use this theorem in our setting, we assume in addition that $\text{div} A \in L^2(\Omega; \mathbb{R}^2)$ to convert an elliptic equation in divergence form into an equation in non-divergence from. It seems to be open whether we can remove the additional assumption $\text{div} A \in L^2(\Omega; \mathbb{R}^2)$.

The organization of this paper is as follows. We introduce some notations and state the main theorem in the next section. In Section 3, we prove $W^{1,p}$-results for the problems $[D]$ and $[D^1]$ for sufficiently large $\lambda$. Next, we prove the uniqueness of weak solutions of the problem $[D]$ in Section 4. Proof of the main theorem is presented in Section 5. For the reader’s convenience, we give all necessary details that can be found in [15, 18, 20].

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2. Notation and Main result

In this section, we introduce several notations used in this article. Also, we give the main theorem of this paper. We use “: =” to denote a definition. As usual, $\mathbb{R}^n$ stands the standard Euclidean space of $n$-points and $| \cdot |$ is the standard Euclidean norm on $\mathbb{R}^n$. For $r > 0$ and $x \in \mathbb{R}^n$, we write $B_r(x) := \{ y \in \mathbb{R}^n : |x - y| < r \}$. We also write $B_r := B_r(0)$. For $x \in \mathbb{R}^n$, we write $x = (x', x_n)$ where $x' \in \mathbb{R}^{n-1}$ and $B_r(x') := \{ y' \in \mathbb{R}^{n-1} : |x' - y'| < r \}$. For $1 \leq j, k \leq n$, we denote
\[
D_{kj}u = \frac{\partial u}{\partial x_j}; \quad D_{kj}u = D_j D_k u = u_{x_k x_j}.
\]
We also use the notation $\nabla u := (D_1 u, \ldots, D_n u)$ for the gradient of $u$.

We denote by $X'$ the dual space of a Banach space $X$. The dual pairing of $X$ and $X'$ is denoted by $\langle \cdot, \cdot \rangle_{X', X}$ or simply $\langle \cdot, \cdot \rangle$. For $k \in \mathbb{N} \cup \{\infty\}$, let $C^k_b(\Omega)$ be the space of all functions in $C^k(\mathbb{R}^n)$ with compact supports in $\Omega$ and let $C^k(\overline{\Omega})$ the space of the restrictions to $\overline{\Omega}$ of all functions in $C^k(\mathbb{R}^n)$. For $k \in \mathbb{N}$ and $1 \leq p < \infty$, $L^p(\Omega)$ and $W^{k,p}(\Omega)$ denote the standard $L^p$-space on $\Omega$ with Lebesgue measure and
the Sobolev spaces on $\Omega$, respectively. We define $W^{1,p}_0(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$. For $1 < p < \infty$, we write $p' := p/(p-1)$ the conjugate exponent to $p$. For such $p$, we define $W^{-1,p}_0(\Omega) := (W^{1,p}_0(\Omega))'$. For $1 \leq p < \infty$, by $L^p(\Omega; \mathbb{R}^m)$, we denote the set of all $\mathbb{R}^m$-valued measurable functions $u = (u^1, \ldots, u^m)$ on $\Omega$ satisfying

$$
\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u(x)|^p \, dx\right)^{1/p} < \infty.
$$

Similarly, $C_c^\infty(\Omega; \mathbb{R}^m)$ denotes the set of all $\mathbb{R}^m$-valued measurable functions $\Phi = (\phi^1, \ldots, \phi^m)$ on $\Omega$ satisfying $\phi^i \in C_c^\infty(\Omega)$ for all $1 \leq i \leq m$.

For open sets $U$ and $V$, we write $V \subseteq U$ if $\overline{V}$ is compact and $\overline{V} \subset U$. For $1 \leq p < \infty$ and $k \in \mathbb{N} \cup \{0\}$, we write $W^{k,p}_0(\Omega)$ if $u : \Omega \to \mathbb{R}$ satisfy $u \in W^{k,p}(\Omega')$ for any $\Omega' \subseteq \Omega$. Similarly, a vector field $u : \Omega \to \mathbb{R}^m$ is in $L^p_{\text{loc}}(\Omega; \mathbb{R}^m)$ if $u \in L^p(\Omega'; \mathbb{R}^m)$ for any $\Omega' \subseteq \Omega$. For a measurable function $f$ on $E \subseteq \mathbb{R}^n$, we write

$$(f)_E = \frac{1}{|E|} \int_E f \, dx = \int_E f \, dx,$$

where $|E|$ denotes the $n$-dimensional Lebesgue measure of $E$. Finally, by $C = C(p_1, \ldots, p_k)$, we denote a generic positive constant depending only on the parameters $p_1, \ldots, p_k$.

We define weak solutions of the problem $(D)$ and $(D')$ as follows.

**Definition 2.1.** Let $\lambda \geq 0$, $1 < p < \infty$, and $b : \Omega \to \mathbb{R}^n$ be a given measurable vector field.

1. Given $f \in L^p(\Omega)$ and $F \in L^p(\Omega; \mathbb{R}^n)$, we say that $u \in W^{1,p}_0(\Omega)$ is a weak solution of $(D)$ if $b \cdot \nabla u \in L^1_{\text{loc}}(\Omega)$ and

$$
\int_\Omega A \nabla u \cdot \nabla \phi \, dx + \int_\Omega (b \cdot \nabla u) \phi \, dx + \lambda \int_\Omega u \phi \, dx = \int_\Omega f \phi \, dx - \int_\Omega F \cdot \nabla \phi \, dx \quad (2.1)
$$

for all $\phi \in C_c^\infty(\Omega)$.

2. Given $g \in L^p(\Omega)$ and $G \in L^p(\Omega; \mathbb{R}^n)$, we say that $v \in W^{1,p}_0(\Omega)$ is a weak solution of $(D')$ if $vb \in L^1_{\text{loc}}(\Omega; \mathbb{R}^n)$ and

$$
\int_\Omega (A^T \nabla v + vb) \cdot \nabla \psi \, dx + \lambda \int_\Omega v \psi \, dx = \int_\Omega g \psi \, dx - \int_\Omega G \cdot \nabla \psi \, dx \quad (2.2)
$$

for all $\psi \in C_c^\infty(\Omega)$.

Let $b \in L^n(\Omega; \mathbb{R}^n)$. If $n' \leq p < \infty$, then by Hölder’s inequality, we have $b \cdot \nabla v \in L^1(\Omega)$ for any $v \in W^{1,p}(\Omega)$. If $1 < p \leq n$, then it follows from Hölder’s inequality and Sobolev’s embedding theorem that

$$
\|vb\|_{L^1(\Omega)} \leq \|b\|_{L^n(\Omega)} \|v\|_{L^{n'}(\Omega)} \leq C\|b\|_{L^n(\Omega)} \|v\|_{W^{1,p}(\Omega)}
$$

for all $v \in W^{1,p}(\Omega)$. Hence if we have an unbounded drift $b \in L^n(\Omega; \mathbb{R}^n)$, then the range of $p$ is limited to ensure the well-definedness of weak solutions in $W^{1,p}_0(\Omega)$ for problems $(D)$ and $(D')$, respectively.

We impose the following regularity assumption on the leading coefficients:

**Assumption 2.2** $(\gamma)$. There exists a constant $R_0 \in (0, 1]$ such that

$$
\max_{1 \leq r, s \leq n} \int_{B_r(x)} |a^{ij}(y) - (a^{ij})_{B_r(x)}| \, dy \leq \gamma
$$
for any $x \in \mathbb{R}^n$ and $0 < r \leq R_0$.

Note that if $A$ is in VMO (see e.g. [22]), then Assumption 2.2 ($\gamma$) is satisfied for any $\gamma > 0$.

Next, we impose the following regularity assumption on the boundary of the domain $\Omega$:

**Assumption 2.3 ($\theta$).** There is a constant $R_0 \in (0,1]$ such that for any $x_0 \in \partial \Omega$,
there exists a Lipschitz function $\sigma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that
$$\Omega \cap B_{R_0}(x_0) = \{ x \in B_{R}(x_0) : x_n > \sigma(x') \}$$
and
$$\sup_{x',y' \in B_{R_0}(x_0), x' \neq y'} \frac{|\sigma(x') - \sigma(y')|}{|x' - y'|} \leq \theta$$
in some coordinate system.

It is easy to check that every $C^1$-domain satisfies Assumption 2.3 ($\theta$) for any $\theta > 0$.

To state our main theorem, we introduce the definition of weak $L^2$-divergence for a bounded measurable matrix-valued function.

**Definition 2.4.** A bounded measurable matrix-valued function $A : \mathbb{R}^2 \to \mathbb{R}^{2 \times 2}$ has weak $L^2$-divergence in $\Omega$ if there exists a vector field $c$ in $L^2(\Omega; \mathbb{R}^2)$ such that
$$\int_{\Omega} \sum_{i,j=1}^{2} a^{ij} D_i \phi_j \, dx = - \int_{\Omega} c \cdot \Phi \, dx$$
for all $\Phi = (\phi^1, \phi^2) \in C^0_c(\Omega; \mathbb{R}^2)$. In this case, we write $c = \text{div} A$ and $\text{div} A \in L^2(\Omega; \mathbb{R}^2)$.

Now we state the main theorem of this paper.

**Theorem 2.5.** Let $\Omega$ be a bounded domain in $\mathbb{R}^2$, $2 < p < \infty$, and $\lambda \geq 0$. Suppose that $A$ satisfies (1.1), $\text{div} A \in L^2(\Omega; \mathbb{R}^2)$, and $b \in L^2(\Omega; \mathbb{R}^2)$. Then there exist constants $\gamma = \gamma(\delta, p, \|\text{div} A\|_{L^2(\Omega)}$, $\|b\|_{L^2(\Omega)}$) and $\theta = \theta(\delta, p) > 0$ such that under Assumptions 2.2 ($\gamma$) and 2.3 ($\theta$), the following hold:

(i) For every $f \in L^p(\Omega)$ and $F \in L^p(\Omega; \mathbb{R}^2)$, there exists a unique weak solution $u \in W^{1,p}_0(\Omega)$ of (1.1). Moreover we have
$$\|\nabla u\|_{L^p(\Omega)} + \lambda^{1/2} \|u\|_{L^p(\Omega)} \leq C \left[ \min (1, \lambda^{-1}) \|f\|_{L^p(\Omega)} + \|F\|_{L^p(\Omega)} \right]$$
for some constant $C$ independent of $u$, $f$, $F$, and $\lambda$.

(ii) For every $g \in L^{p'}(\Omega)$ and $G \in L^{p'}(\Omega; \mathbb{R}^2)$, there exists a unique weak solution $v \in W^{1,p'}_0(\Omega)$ of (1.1). Moreover we have
$$\|\nabla v\|_{L^{p'}(\Omega)} + \lambda^{1/2} \|v\|_{L^{p'}(\Omega)} \leq C \left[ \min (1, \lambda^{-1}) \|g\|_{L^{p'}(\Omega)} + \|G\|_{L^{p'}(\Omega)} \right]$$
for some constant $C$ independent of $v$, $g$, $G$, and $\lambda$.

**Remark.** (i) Theorem 2.5 complements the result of Kim-Kim [19] when $a^{ij} = \delta^{ij}$ and $b \in L^2(\Omega; \mathbb{R}^2)$, where $\delta^{ij}$ is the Kronecker delta.
(ii) The range $2 < p < \infty$ in Theorem 2.5 is optimal. Define
$$a^{ij} = \delta^{ij}, \quad u(x) = \ln \ln |x|, \quad \text{and} \quad b(x) = - \frac{x}{|x|^2 \ln |x|}.$$
Then $b \in L^2(B_{1/\epsilon}; \mathbb{R}^2)$ and $u \in W^{1,2}_0(B_{1/\epsilon})$ is a nontrivial weak solution satisfying the problem (D) with $\lambda = 0$, $f = 0$, and $F = 0$. Hence the uniqueness of weak solutions in $W^{1,2}_0(\Omega)$ of the problem (D) fails in general. More related examples can be found in Filonov [13] and Filonov-Shilkin [14].

(iii) In Kang-Kim [17, Theorem 2.5], they obtained $W^{1,p}$-estimates for the problems (D) and $(D')$ when $a^{ij}$ has small mean oscillations in small balls and $b \in L^q(\Omega; \mathbb{R}^2)$ with $q > 2$. It seems to be open whether we can remove additional assumption $\text{div} \, A \in L^2(\Omega; \mathbb{R}^2)$ when $b \in L^2(\Omega; \mathbb{R}^2)$.

3. SOLVABILITY OF THE PROBLEMS (D) AND (D') FOR LARGE $\lambda$

In this section, we obtain $W^{1,p}$-estimates for the problems (D) and $(D')$ for sufficiently large $\lambda$.

We first show basic estimates for the drift terms and the $\varepsilon$-inequalities inspired by Gerhardt [11], which play crucial roles in the proof of the main theorem of this paper.

Proposition 3.1. Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and $n' < p < \infty$. Suppose that $b \in L^p(\Omega; \mathbb{R}^n)$. Then there exists a constant $C = C(n,p,\Omega) > 0$ such that

$$
\int_\Omega |(v b) \cdot \nabla u| \, dx \leq C \|b\|_{L^p(\Omega)} \|u\|_{W^{1,p}(\Omega)} \|v\|_{W^{1,p'}(\Omega)}
$$

(3.1)

for all $u \in W^{1,p}(\Omega)$ and $v \in W^{1,p'}(\Omega)$.

For each $\varepsilon > 0$, there exists a constant $C_\varepsilon = C_\varepsilon(\varepsilon, n, p, b, \Omega) > 0$ such that

$$
b \cdot \nabla u \in W^{-1,p}(\Omega) \quad \text{and} \quad \|b \cdot \nabla u\|_{W^{-1,p}(\Omega)} \leq \varepsilon \|u\|_{W^{1,p}(\Omega)} + C_\varepsilon \|u\|_{L^p(\Omega)}
$$

(3.2)

for all $u \in W^{1,p}(\Omega)$. Similarly, for each $\varepsilon > 0$, there exists a constant $C_\varepsilon' = C_\varepsilon'(\varepsilon, n, p, b, \Omega) > 0$ such that

$$
\text{div}(v b) \in W^{-1,p'}(\Omega) \quad \text{and} \quad \|\text{div}(v b)\|_{W^{-1,p'}(\Omega)} \leq \varepsilon \|v\|_{W^{1,p'}(\Omega)} + C_\varepsilon' \|v\|_{L^{p'}(\Omega)}
$$

(3.3)

for all $v \in W^{1,p'}(\Omega)$.

Proof. By Hölder’s inequality and Sobolev’s embedding theorem, we have

$$
\int_\Omega |(v b) \cdot \nabla u| \, dx \leq \|b\|_{L^p(\Omega)} \|\nabla u\|_{L^p(\Omega)} \|v\|_{L^{p'}(\Omega)} 
\leq C(n,p,\Omega) \|b\|_{L^p(\Omega)} \|u\|_{W^{1,p}(\Omega)} \|v\|_{W^{1,p'}(\Omega)}
$$

for all $u \in W^{1,p}(\Omega)$ and $v \in W^{1,p'}(\Omega)$. This proves (3.1).

Note that $b \cdot \nabla u \in W^{-1,p}(\Omega)$ by the estimate (3.1). To show $\varepsilon$-inequality (3.2), note that integration by part shows that the identity

$$
\int_\Omega (b \cdot \nabla u) v dx = - \int_\Omega (b \cdot \nabla v) u dx - \int_\Omega (\text{div} \, b) uv dx
$$

(3.4)

holds for any $b \in C_c^\infty(\bar{\Omega}; \mathbb{R}^n)$, $u \in C^\infty(\Omega)$, and $v \in C^\infty(\Omega)$. Since $C^\infty(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$ (see [12, Theorem 4.3]) and the estimate (3.1) holds, a standard density argument shows that identity (3.4) holds for any $b \in C_c^\infty(\Omega; \mathbb{R}^n)$, $u \in W^{1,p}(\Omega)$, and $v \in C^\infty(\Omega)$.

Let $\varepsilon > 0$ be given. Since $C_c^\infty(\Omega; \mathbb{R}^n)$ is dense in $L^n(\Omega; \mathbb{R}^n)$, there exists $b_\varepsilon \in C_c^\infty(\Omega; \mathbb{R}^n)$ such that

$$
\|b_\varepsilon - b\|_{L^p(\Omega)} < \varepsilon / C,
$$
where $C$ is the same constant in (3.1). Fix $v \in C_c^\infty(\Omega)$. Then by (3.3), we have
\[
\int_\Omega (b \cdot \nabla u)v \, dx = \int_\Omega [(b - b_\varepsilon) \cdot \nabla u]v \, dx + \int_\Omega (b_\varepsilon \cdot \nabla u)v \, dx
\]
\[
= \int_\Omega [(b - b_\varepsilon) \cdot \nabla u]v \, dx - \int_\Omega (b_\varepsilon \cdot \nabla v)u \, dx - \int_\Omega (\text{div } b_\varepsilon)uv \, dx.
\]
By (3.1) and Hölder’s inequality, we get
\[
\int_\Omega (b \cdot \nabla u)v \, dx \leq C\|b - b_\varepsilon\|_{L^\infty(\Omega)}\|u\|_{W^{1,p}(\Omega)}\|v\|_{W^{1,p'}(\Omega)}
\]
\[
+ \|b_\varepsilon\|_{L^\infty(\Omega)}\|u\|_{L^p(\Omega)}\|v\|_{L^{p'}(\Omega)} + \|\text{div } b_\varepsilon\|_{L^\infty(\Omega)}\|u\|_{L^p(\Omega)}\|v\|_{L^{p'}(\Omega)}
\]
\[
\leq (\varepsilon\|u\|_{W^{1,p}(\Omega)} + C\varepsilon\|u\|_{L^p(\Omega)}) \|v\|_{W^{1,p'}(\Omega)},
\]
where $C_\varepsilon = \|b_\varepsilon\|_{L^\infty(\Omega)} + \|\text{div } b_\varepsilon\|_{L^\infty(\Omega)}$. Since $v \in C_c^\infty(\Omega)$ was arbitrary chosen, this implies that
\[
\|b \cdot \nabla u\|_{W^{-1,p'}(\Omega)} \leq \varepsilon\|u\|_{W^{1,p}(\Omega)} + C\varepsilon\|u\|_{L^p(\Omega)},
\]
which proves (3.2).

To show (3.3), suppose that $u \in C_c^\infty(\Omega)$ and $v \in W^{1,p'}(\Omega)$. Since
\[
\int_\Omega (v b) \cdot \nabla u \, dx = \int_\Omega [v(b - b_\varepsilon)] \cdot \nabla u \, dx + \int_\Omega (v b_\varepsilon) \cdot \nabla u \, dx,
\]
it follows from (3.1) and Hölder’s inequality that
\[
\int_\Omega (v b) \cdot \nabla u \, dx \leq (\varepsilon\|v\|_{W^{1,p'}(\Omega)} + \|b_\varepsilon\|_{L^\infty(\Omega)}\|v\|_{L^{p'}(\Omega)})\|u\|_{W^{1,p}(\Omega)}. \tag{3.5}
\]
Since
\[
\langle \text{div } (v b), u \rangle = -\int_\Omega (v b) \cdot \nabla u \, dx
\]
for all $u \in C_c^\infty(\Omega)$, it follows from (3.5) that $\text{div } (v b) \in W^{-1,p'}(\Omega)$ and the estimate (3.3) holds. This completes the proof of Proposition 3.1 \hfill \square

We use the following special case of Dong-Kim [10, Theorem 7].

**Theorem 3.2.** Let $1 < p < \infty$ and $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. Then there exist $\gamma = \gamma(n,p,\delta) > 0$, $\theta = \theta(n,p,\delta) > 0$, and $\lambda_0 = \lambda_0(n,p,\delta,R_0,\Omega) \geq 1$ such that under Assumptions 3.2 (\(\gamma\)) and 3.3 (\(\theta\)), the following holds for any $\lambda \geq \lambda_0$: for any $f \in L^p(\Omega;\mathbb{R}^n)$ and $F \in L^p(\Omega;\mathbb{R}^n)$, there exists a unique weak solution $u \in W^{1,p}(\Omega)$ such that
\[
- \text{div}(A \nabla u) + \lambda u = f + \text{div } F \quad \text{in } \Omega.
\]
Moreover we have
\[
\lambda^{1/2}\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)} \leq C \left(\lambda^{-1/2}\|f\|_{L^p(\Omega)} + \|F\|_{L^p(\Omega)}\right)
\]
for some constant $C = C(n,p,\delta,R_0,\Omega) > 0$.

**Remark.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and $1 < p < \infty$. If $f_0 \in W^{-1,p}(\Omega)$, then there exists $F_0 \in L^p(\Omega;\mathbb{R}^n)$ such that
\[
\text{div } F_0 = f_0 \quad \text{in } \Omega \quad \text{and} \quad \|F_0\|_{L^p(\Omega)} \leq C(n,p,\Omega)\|f_0\|_{W^{-1,p}(\Omega)},
\]
(see e.g. [21], Lemma 3.9). By Theorem 3.2 there exist $\gamma = \gamma(n,p,\delta) > 0$, $\theta = \theta(n,p,\delta,R_0) > 0$, and $\lambda_0 = \lambda_0(n,p,\delta,R_0) \geq 1$ such that under Assumptions
we have for \( \lambda \geq \lambda_0 \) and for each \( f \in L^p(\Omega) \), there exists a unique weak solution \( u \in W^{1,p}_0(\Omega) \) such that

\[
\int_{\Omega} (A \nabla u) \cdot \nabla \phi \, dx + \lambda \int_{\Omega} u \phi \, dx = \int_{\Omega} f \phi \, dx - \int_{\Omega} F_0 \cdot \nabla \phi \, dx
\]

for all \( \phi \in C_c^\infty(\Omega) \). Moreover we have

\[
\lambda^{1/2} \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)} \leq C (\lambda^{-1/2} \|f\|_{L^p(\Omega)} + \|F_0\|_{L^p(\Omega)})
\]

(3.6)

for some constant \( C = C(n, p, \delta, R_0, \Omega) > 0 \).

Now we present the main theorem of this section. From now on, we mainly focus on the case \( b \in L^2(\Omega; \mathbb{R}^2) \) since other cases \( b \in L^q(\Omega; \mathbb{R}^n) \) are already considered in Kang-Kim \( 14 \) when \( n \leq q < \infty \) if \( n \geq 3 \) and \( 2 < q < \infty \) if \( n = 2 \).

**Theorem 3.3.** Let \( 2 < p < \infty \) and \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \). Suppose that \( A \) satisfies \( 11 \) and \( b \in L^2(\Omega; \mathbb{R}^2) \). Then there exist \( \gamma = \gamma(p, \delta) > 0 \), \( \theta = \theta(p, \delta) > 0 \), and \( \lambda_1 = \lambda_1(p, \delta, R_0, \Omega, b) \geq 1 \) such that under Assumptions \( 22 \)(\( \gamma \)) and \( 23 \)(\( \theta \)), the following results hold for any \( \lambda \geq \lambda_1 \):

(i) If \( f \in L^p(\Omega) \) and \( F \in L^p(\Omega; \mathbb{R}^2) \), then there exists a unique weak solution \( u \in W^{1,p}_0(\Omega) \) of \[[2] \]. Moreover

\[
\|\nabla u\|_{L^p(\Omega)} + \lambda^{1/2} \|u\|_{L^p(\Omega)} \leq C \left[ \lambda^{-1/2} \|f\|_{L^p(\Omega)} + \|F\|_{L^p(\Omega)} \right]
\]

for some constant \( C = C(p, \delta, R_0, \Omega) > 0 \).

(ii) If \( g \in L^{p'}(\Omega) \) and \( G \in L^{p'}(\Omega; \mathbb{R}^2) \), then there exists a unique weak solution \( v \in W^{1,p'}_0(\Omega) \) of \[[3] \]. Moreover

\[
\|\nabla v\|_{L^{p'}(\Omega)} + \lambda^{1/2} \|v\|_{L^{p'}(\Omega)} \leq C \left[ \lambda^{-1/2} \|g\|_{L^{p'}(\Omega)} + \|G\|_{L^{p'}(\Omega)} \right]
\]

for some constant \( C = C(p, \delta, R_0, \Omega) > 0 \).

**Proof.** Let \( f \in L^p(\Omega) \) and \( F \in L^p(\Omega; \mathbb{R}^2) \). Then \( \text{div} \, F \in W^{-1,p}(\Omega) \). For each \( u \in W^{1,p}_0(\Omega) \), it follows from Proposition \( 31 \) that \( b \cdot \nabla u \in W^{-1,p}(\Omega) \). Moreover, for each \( \varepsilon > 0 \), there exists a constant \( C_\varepsilon = C(\varepsilon, p, b, \Omega) > 0 \) such that

\[
\|b \cdot \nabla u\|_{W^{-1,p}(\Omega)} \leq \varepsilon \|\nabla u\|_{L^p(\Omega)} + C_\varepsilon \|u\|_{L^p(\Omega)}
\]

(3.7)

for all \( u \in W^{1,p}_0(\Omega) \). By the remark of Theorem \( 32 \) there exist \( \gamma = \gamma(p, \delta) > 0 \), \( \theta = \theta(p, \delta) > 0 \), and \( \lambda_0 = \lambda_0(p, \delta, R_0, \Omega) \geq 1 \) such that under Assumptions \( 22 \)(\( \gamma \)) and \( 23 \)(\( \theta \)), for \( \lambda \geq \lambda_0 \), there exists a unique \( \pi = \pi(u) \in W^{1,p}_0(\Omega) \) satisfying

\[
\int_{\Omega} A \nabla \pi \cdot \nabla \phi \, dx + \lambda \int_{\Omega} \pi \phi \, dx = \langle \text{div} \, F - b \cdot \nabla u, \phi \rangle + \int_{\Omega} f \phi \, dx
\]

for all \( \phi \in C_c^\infty(\Omega) \). Moreover, we have

\[
\lambda^{1/2} \|\pi\|_{L^p(\Omega)} + \|\nabla \pi\|_{L^p(\Omega)} \leq C_0 \left( \lambda^{-1/2} \|f\|_{L^p(\Omega)} + \|\text{div} \, F - b \cdot \nabla u\|_{W^{-1,p}(\Omega)} \right)
\]

for some constant \( C_0 = C_0(p, \delta, R_0, \Omega) > 0 \). Choose \( \varepsilon > 0 \) so that \( \varepsilon C_0 = 1/2 \). Then we have

\[
\lambda^{1/2} \|\pi\|_{L^p(\Omega)} + \|\nabla \pi\|_{L^p(\Omega)} \leq C_0 \lambda^{-1/2} \|f\|_{L^p(\Omega)} + C_0 \|F\|_{L^p(\Omega)}
\]

\[
+ \frac{1}{2} \left( \|\nabla u\|_{L^p(\Omega)} + C_\varepsilon \|u\|_{L^p(\Omega)} \right),
\]
where \( C = C_*(p, b, \rho, R_0, \Omega) > 0 \). Moreover, we have
\[
\lambda^{1/2}||T(u_1) - T(u_2)||_{L^p(\Omega)} + ||\nabla(T(u_1) - T(u_2))||_{L^p(\Omega)} \\
\leq \frac{1}{2} (||\nabla(u_1 - u_2)||_{L^p(\Omega)} + C_* ||u_1 - u_2||_{L^p(\Omega)})
\]
for \( u_1, u_2 \in W^{1,p}_0(\Omega) \). Let \( \lambda_1 := \max(C^2_*, \lambda_0) + 1 \). Then for \( \lambda \geq \lambda_1 \), \( T \) is a contraction on \( W^{1,p}_0(\Omega) \) which is a Banach space equipped with the equivalent norm \( ||\nabla||_{L^p(\Omega)} + |||L^p(\Omega) \rangle \). Hence by the Banach fixed point theorem, there exists a unique \( u \in W^{1,p}_0(\Omega) \) such that \( T(u) = u \), that is, \( u \) is a weak solution of the problem (4.1). Moreover, \( u \) satisfies
\[
\lambda^{1/2}||u||_{L^p(\Omega)} + ||u||_{L^p(\Omega)} \leq 2C_0 \left( \lambda^{-1/2}||f||_{L^p(\Omega)} + ||F||_{L^p(\Omega)} \right).
\]
This completes the proof of (i). Following the exactly same argument, one can also prove (ii) whose proof is omitted. This completes the proof of Theorem 4.3. \( \square \)

4. Uniqueness of weak solutions for the problem (1.1)

This section is devoted to a proof of the uniqueness part of Theorem 2.5 (i). Below is the main theorem of this section.

**Theorem 4.1.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \), \( 2 < p < \infty \). Suppose that \( A \) satisfies (1.1), \( \text{div} A, b \in L^2(\Omega; \mathbb{R}^2) \), and \( \lambda \geq 0 \). Then there exists a constant \( \gamma = \gamma(p, \rho, \delta, ||\text{div} A||_{L^2(\Omega)}, ||b||_{L^\infty(\Omega)}) \) such that under Assumption 2.2 (\( \gamma \)), if \( u \in W^{1,p}_0(\Omega) \) satisfies
\[
\int_{\Omega} A \nabla u \cdot \nabla \phi + (b \cdot \nabla u + \lambda u) \phi = 0 \quad (4.1)
\]
for all \( \phi \in C_0^\infty(\Omega) \), then \( u \) is identically zero in \( \Omega \).

To prove Theorem 4.1 we use recent results due to Krylov [23, 24]. To state these results, for \( 0 < \delta < 1 \), let \( S_\delta \) be the set of \( n \times n \) real symmetric matrices which are measurable and whose eigenvalues are in \([\delta, \delta^{-1}]\) and \( b : \Omega \rightarrow \mathbb{R}^n \) be a vector field. Write
\[
Lu = \sum_{i,j=1}^n a^{ij} D_{ij} u - b \cdot \nabla u.
\]
The following theorem can be found in [23, Corollary 3.1], which generalizes the classical theorem due to Alexandrov [1].

**Theorem 4.2.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) and \( c \) be a nonnegative measurable function on \( \Omega \). Suppose that \( 0 < \delta < 1 \) and \( A \) is a \( S_\delta \)-valued function on \( \mathbb{R}^n \) and \( b \in L^n(\Omega; \mathbb{R}^n) \). Then there exists a number \( n/2 < n_0 < n \) depending on \( n \), \( \delta \), and \( ||b||_{L^n(\Omega)} \) such that if \( n_0 \leq p < \infty \), then there exists a constant \( C \) depending on \( n \), \( p \), \( \delta \), \( ||b||_{L^n(\Omega)} \), and the diameter of \( \Omega \) such that
\[
u(x) \leq C ||(Lu - cu) - ||_{L^p(\Omega)} + \sup_{\partial \Omega} u_+ \quad \text{in} \ \Omega \quad (4.2)
\]
for all \( u \in W^{2,p}_{loc}(\Omega) \cap C(\overline{\Omega}) \).

The following theorem is a special case of [24, Theorem 4.2].
Theorem 4.3. Let $\Omega$ be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 2$, $1 < p < n$, and $\lambda \geq 0$. Assume that $A$ satisfies $[1.1]$ and $b \in L^n(\Omega; \mathbb{R}^n)$. Then there exists $\gamma = \gamma(n, p, \delta) > 0$ such that under Assumption $[2.2](\gamma)$, for every $g \in L^p(\Omega)$, there exists a unique strong solution $u \in W^{1,p}_0(\Omega) \cap W^{2,p}(\Omega)$ satisfying

$$- \sum_{i,j=1}^n a^{ij} D_{ij} u + b \cdot \nabla u + \lambda u = g \quad \text{in } \Omega.$$ 

Now we are ready to prove the main theorem of this section.

Proof of Theorem 4.1. By Theorem 4.2, there exists a number $1 < n_0 < 2$ depending on $\delta$, $\|\text{div } A\|_{L^2(\Omega)}$, and $\|b\|_{L^2(\Omega)}$ such that for $n_0 < q < 2$, there exists a constant $C$ depending on $q$, $\delta$, $\|\text{div } A\|_{L^2(\Omega)}$, $\|b\|_{L^2(\Omega)}$, and the diameter of $\Omega$ such that

$$u(x) \leq C\|\{Lu - \lambda u\}_-\|_{L^q(\Omega)} + \sup_{\partial \Omega} u_+ \quad \text{in } \Omega$$

for all $u \in W^{2,q}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$, where

$$Lu = \sum_{i,j=1}^2 a^{ij} D_{ij} u - (b \cdot \text{div } A) \cdot \nabla u.$$ 

Define $q = 1/2\left(\text{max}\{2p/(p+2), n_0\} + 2\right)$. Then $\max\{n_0, \frac{2p}{p+2}\} < q < 2$. Since $u \in W^{1,p}_0(\Omega)$ and $p > 2$, it follows from the Sobolev embedding theorem that $u \in C(\overline{\Omega})$. Hence it suffices to show that $u \in W^{1,p}_{\text{loc}}(\Omega)$. To show this, we fix a bounded smooth subdomain $\Omega' \subseteq \Omega$ and let $\zeta \in C^\infty_c(\Omega')$. Since $u$ satisfies $[4.1]$, we have

$$\int_\Omega A \nabla u \cdot \nabla (\zeta \phi) \, dx + \int_\Omega (b \cdot \nabla u + \lambda u)(\zeta \phi) \, dx = 0 \quad (4.3)$$

for all $\phi \in C^\infty_c(\Omega)$. By $\text{div } A \in L^2(\Omega; \mathbb{R}^2)$ and $u \in W^{1,p}_0(\Omega)$, $p > 2$, it follows that

$$- \int_\Omega \text{div } A \cdot (u \phi \nabla \zeta) \, dx = \sum_{i,j=1}^2 \int_\Omega a^{ij} D_{ij} u (\zeta \phi) \, dx \quad (4.4)$$

and

$$\int_\Omega A \nabla (u \zeta) \cdot \nabla \phi + (b \cdot \nabla (u \zeta) + \lambda_1 (\zeta u)) \phi \, dx = - \int_\Omega g \phi \, dx, \quad (4.5)$$

where

$$g = u \text{div } A \cdot \nabla \zeta + \sum_{i,j=1}^n (a^{ij} D_{ij} \zeta) u + \nabla u \cdot A \nabla \zeta + A \nabla u \cdot \nabla \zeta - u b \cdot \nabla \zeta + (\lambda - \lambda_1)(\zeta u).$$

Note that $g \in L^2(\Omega)$ since $u \in W^{1,p}_0(\Omega)$, $p > 2$, and $\text{div } A, b \in L^2(\Omega; \mathbb{R}^2)$. 


By Theorem 3.3 there exist $γ_0 = γ_0(p, δ) > 0$ and $λ_1 = λ_1(p, δ, b, Ω') ≥ 1$ such that under Assumption 2.2, for $λ ≥ λ_1$, $f ∈ L^p(Ω')$, and $F ∈ L^p(Ω'; R^2)$, there exists a unique weak solution $v ∈ W^{1,p'}(Ω)$ such that

$$-\text{div}(A∇v) + b · ∇v + λv = f + \text{div} F \quad \text{in} \ Ω', \quad v = 0 \quad \text{on} \ ∂Ω'.$$

For such $λ_1$ and $q = \frac{1}{2} (\max\{2p/(p + 2), n_0\} + 2)$, it follows from Theorem 4.3 that there exists $γ_1 = γ_1(p, n_0, δ) > 0$ such that under Assumption 2.2 ($γ_1$), there exists a unique $w ∈ W^{1,q}_0(Ω') \cap W^{2,q}(Ω')$ satisfying

$$-\sum_{i,j=1}^{2} a^{ij} D_{ij} w + (b - \text{div} A) · ∇w + λ_1 w = -g \quad \text{in} \ Ω'.$$

Set $γ = \min\{γ_0, γ_1\}$ and choose $A$ so that $A$ satisfies (1.1), $\text{div} A ∈ L^2(Ω; R^2)$, and Assumption 2.2 ($γ$). Since $w ∈ W^{1,q}_0(Ω') \cap W^{2,q}(Ω')$ and $\text{div} A ∈ L^2(Ω; R^2)$, it follows from the Sobolev embedding theorem that $w ∈ W^{1,q}_0(Ω')$ and

$$\int_{Ω'} \text{div} A · (φ∇w) \, dx = -\sum_{i,j=1}^{2} \int_{Ω'} a^{ij} D_i(φD_j w) \, dx$$

$$= -\sum_{i,j=1}^{2} \int_{Ω'} (a^{ij} D_{ij} w)φ \, dx - \int_{Ω'} A∇w · ∇φ \, dx$$

for all $φ ∈ C_0^∞(Ω')$. Hence $w$ satisfies

$$\int_{Ω'} A∇w · ∇φ + (b · ∇w + λ_1 w)φ \, dx = -\int_{Ω'} gφ \, dx$$

for all $φ ∈ C_0^∞(Ω')$. Since $2p/(p + 2) < q < 2$ and $Ω'$ is bounded, it follows that $W^{1,q}_0(Ω') ⊂ W^{1,p}_0(Ω')$. Hence by (4.5) and Theorem 3.3 we have $w = ζu$ in $Ω'$, which implies that $ζu ∈ W^{2,q}(Ω')$ for any $ζ ∈ C_0^∞(Ω')$. Since there exists a sequence of smooth bounded subdomains $\{Ω_k\}$ satisfying $Ω_k ⊂ Ω_{k+1} ⊂ Ω$ and $\bigcup_k Ω_k = Ω$ (see e.g. Proposition 8.2.1), we conclude that $u ∈ W^{2,q}_0(Ω)$. Since $\text{div} A ∈ L^2(Ω; R^2)$, we have

$$\sum_{i,j=1}^{2} \int_{Ω} a^{ij} D_i [(D_j u)φ] \, dx = -\int_{Ω} φ \text{div} A · ∇u \, dx$$

for all $φ ∈ C_c^∞(Ω)$. Hence $u$ satisfies

$$-\sum_{i,j=1}^{2} a^{ij} D_{ij} u + (b - \text{div} A) · ∇u + λu = 0 \quad \text{in} \ Ω.$$

Therefore by Theorem 1.2 we conclude that $u$ is identically zero in $Ω$. This completes the proof of Theorem 4.3.

5. PROOF OF THEOREM 2.5

This section is devoted to a proof of Theorem 2.5. For the case of sufficiently large $λ$, this was done by Theorem 3.3. To treat the case of small $λ$, we use a functional analytic argument as in Kang-Kim [17, 19] and Kim-Kwon [20] to deduce that it
suffices to prove the uniqueness part of (i). To explain this, let $2 < p < \infty$ be fixed. Then
\[
(u, v) \mapsto \langle \mathcal{L}_p u, v \rangle = \int_{\Omega} (A \nabla u) \cdot \nabla v \, dx + \int_{\Omega} (b \cdot \nabla u) v \, dx
\]
and
\[
(u, v) \mapsto \langle \mathcal{L}_p^* v, u \rangle = \int_{\Omega} (A^T \nabla v) \cdot \nabla u \, dx + \int_{\Omega} (v b) \cdot \nabla u \, dx = \langle \mathcal{L}_p u, v \rangle
\]
define bounded linear operators
\[
\mathcal{L}_p : W^{1,p}_0(\Omega) \to W^{-1,p}(\Omega) \quad \text{and} \quad \mathcal{L}_p^* : W^{1,p'}_0(\Omega) \to W^{-1,p'}(\Omega).
\]
Let $\mathcal{I}_p : L^p(\Omega) \to (L^p(\Omega))'$ be the isomorphism defined by
\[
\langle \mathcal{I}_p u, v \rangle = \int_{\Omega} u v \, dx \quad \text{for all } (u, v) \in L^p(\Omega) \times L^{p'}(\Omega).
\]
We also define $\text{div}_p : L^p(\Omega; \mathbb{R}^2) \to W^{-1,p}(\Omega)$ by
\[
\langle \text{div}_p F, \varphi \rangle = -\int_{\Omega} F \cdot \nabla \varphi \, dx \quad \text{for all } \varphi \in W^{1,p'}(\Omega).
\]
Then it is easy to show that $u \in W^{1,p}_0(\Omega)$ is a weak solution of the problem (12) if and only if $(\mathcal{L}_p + \lambda I_p) u = \mathcal{I}_p f + \text{div}_p F$, while $v \in W^{1,p'}_0(\Omega)$ is a weak solution of (13) if and only if $(\mathcal{L}_p^* + \lambda I_{p'}) v = \mathcal{I}_p^* g + \text{div}_{p'} G$.

By Theorem 3.3 there exist $\gamma = \gamma(p, \delta) > 0$, $\theta = \theta(p, \delta) > 0$, and $\lambda_1 \geq 1$ such that under Assumptions 2.2 \((\gamma)\) and 2.3 \((\theta)\), the operators $\mathcal{L}_p + \lambda_1 I_p$ and $\mathcal{L}_p^* + \lambda_1 I_{p'}$ are invertible. Then for $0 \leq \lambda \neq \lambda_1$, the linear operators
\[
K_{p,\lambda} = (\lambda_1 - \lambda)(\mathcal{L}_p + \lambda_1 I_p)^{-1} \circ \mathcal{I}_p : L^p(\Omega) \to L^p(\Omega)
\]
and
\[
K_{p',\lambda}^* = (\lambda_1 - \lambda)(\mathcal{L}_p^* + \lambda_1 I_{p'})^{-1} \circ \mathcal{I}_{p'} : L^{p'}(\Omega) \to L^{p'}(\Omega)
\]
are bounded and even compact by Rellich-Kondrachov’s embedding theorem. Since
\[
\int_{\Omega} (K_{p,\lambda} u) v \, dx = \int_{\Omega} (K_{p',\lambda}^* v) u \, dx
\]
for all $(u, v) \in L^p(\Omega) \times L^p(\Omega)$, it follows that
\[
K_{p',\lambda} = I_{p'}^{-1} \circ K_{p,\lambda} \circ I_p,
\]
where $K_{p,\lambda} : (L^p(\Omega))^\prime \to (L^p(\Omega))^\prime$ is the adjoint operator of $K_{p,\lambda}$. Note also that
\[
\ker(Id - K_{p,\lambda}) = \ker(\mathcal{L}_p + \lambda I_p) \quad \text{and} \quad \ker(Id - K_{p',\lambda}^*) = \ker(\mathcal{L}_p^* + \lambda I_{p'}).
\]
By (5.1) and the Fredholm alternative theorem (see e.g. 
\[\text{Im}(Id - K_{p,\lambda}) = \left\{ u \in L^p(\Omega) : \int_{\Omega} uv \, dx = 0 \quad \text{for all } v \in \ker(\mathcal{L}_p^* + \lambda I_{p'}) \right\}, \]
(5.2)
\[\text{Im}(Id - K_{p',\lambda}^*) = \left\{ v \in L^{p'}(\Omega) : \int_{\Omega} vu \, dx = 0 \quad \text{for all } u \in \ker(\mathcal{L}_p + \lambda I_p) \right\}, \]
(5.3)
and
\[
\dim \ker(\mathcal{L}_p + \lambda I_p) = \dim \ker(\mathcal{L}_p^* + \lambda I_{p'}) < \infty
\]
(5.4)
for all $\lambda \geq 0$.

Now we are ready to prove Theorem 2.5.
Proof of Theorem 2.5. Choose \((\gamma_0, \theta_0, \lambda_1)\) from Theorem 3.3 and choose \(\gamma_1\) from Theorem 4.1 and let \(\gamma = \min\{\gamma_0, \gamma_1\}\). Under Assumptions 2.2 \((\gamma)\) and 2.3 \((\theta)\), Theorem 2.5 holds for all \(\lambda \geq \lambda_1\) by Theorem 3.3. Suppose that \(0 \leq \lambda < \lambda_1\). By (5.4) and Theorem 4.1, we have

\[
\ker(L_p + \lambda I_p) = \ker(L_{p'} + \lambda I_{p'}) = \{0\} \quad (5.5)
\]

for any \(\lambda \geq 0\). Hence by (5.2) and (5.3), we have

\[
\text{Im}(Id - K_{p,\lambda}) = \text{Im}(Id - K_{p',\lambda}) = L^p(\Omega). \quad (5.6)
\]

Given \(f \in L^p(\Omega)\) and \(F \in L^p(\Omega; \mathbb{R}^2)\), let

\[
w = (L_p + \lambda I_p)^{-1}(I_p f + \text{div}_p F).
\]

By (5.6), there exists \(u \in L^p(\Omega)\) such that

\[
u - K_{p,\lambda} u = w.
\]

Since \((L_p + \lambda I_p)^{-1}\) maps \(W^{-1,p}(\Omega)\) to \(W^{1,p}_0(\Omega)\), it follows from the definition of \(K_{p,\lambda}\) that \(w, K_{p,\lambda} u \in W^{1,p}_0(\Omega)\). Hence

\[
u \in W^{1,p}_0(\Omega) \quad \text{and} \quad (L_p + \lambda I_p) u = I_p f + \text{div}_p F.
\]

This proves that \(u\) is a weak solution of the problem \((D)\). Similarly, we can also prove the existence of weak solution of the problem \((D')\). By (5.5), we also have uniqueness of weak solution of the problem \((D')\). It remains to show the desired estimate.

Let \(f \in L^p(\Omega)\) and \(F \in L^p(\Omega; \mathbb{R}^2)\). Define

\[
\langle \ell, \phi \rangle = \int_{\Omega} f \phi \, dx - \int_{\Omega} F \cdot \nabla \phi \, dx \quad \text{for all } \phi \in \mathcal{C}_c^\infty(\Omega).
\]

Then

\[
\ell \in W^{-1,p}(\Omega) \quad \text{and} \quad \|\ell\|_{W^{-1,p}(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|F\|_{L^p(\Omega)}.
\]

Since the operator \(L_p + \lambda I_p\) is bijective, it follows that there exists \(u_\lambda \in W^{1,p}_0(\Omega)\) such that

\[
(L_p + \lambda I_p) u_\lambda = \ell.
\]

Moreover, it follows from the bounded inverse theorem (see e.g. [3, Corollary 2.7]) that there exists a constant \(C_\lambda > 0\) independent of \(u_\lambda\) and \(\ell\) such that

\[
\|u_\lambda\|_{W^{1,p}(\Omega)} \leq C_\lambda \|\ell\|_{W^{-1,p}(\Omega)} \leq C_\lambda (\|f\|_{L^p(\Omega)} + \|F\|_{L^p(\Omega)}).
\]

In fact, we can choose a constant independent of \(\lambda\). Indeed, for \(0 \leq \lambda \leq \lambda_1\), let

\[
E_\lambda = \left\{ \mu \in [0, \lambda_1] : C_\lambda |\lambda - \mu| < \frac{1}{2} \right\}.
\]

For \(\mu \in E_\lambda\), since the operator \(L_p + \lambda I_p\) is bijective, there exists \(u_\mu \in W^{1,p}_0(\Omega)\) such that

\[
(L_p + \lambda I_p) u_\mu = I_p f + \text{div}_p F + (\lambda - \mu) I_p u_\mu.
\]

Moreover, we have

\[
\|u_\mu\|_{W^{1,p}(\Omega)} \leq 2C_\lambda (\|f\|_{L^p(\Omega)} + \|F\|_{L^p(\Omega)})
\]

for all \(\mu \in E_\lambda\). Hence by the compactness of \([0, \lambda_1]\), there is a constant \(C > 0\) independent of \(u_\lambda, f, F,\) and \(\lambda\) such that

\[
\|u_\lambda\|_{W^{1,p}(\Omega)} \leq 2C (\|f\|_{L^p(\Omega)} + \|F\|_{L^p(\Omega)}).
\]
By the Poincaré inequality, we conclude that
\[
\|\nabla u_\lambda\|_{L^p(\Omega)} + \lambda^{1/2} \|u_\lambda\|_{L^p(\Omega)} \leq C(\|f\|_{L^p(\Omega)} + \|F\|_{L^p(\Omega)})
\]
for all \(\lambda \in [0, \lambda_1]\). This proves the desired estimate for a solution of the problem \(\mathcal{D}\). Following exactly the same argument, one can derive a similar estimate for a weak solution of the problem \(\mathcal{D}'\). This completes the proof of Theorem 2.5. \(\square\)

REFERENCES
1. A. D. Aleksandrov, Uniqueness conditions and bounds for the solution of the Dirichlet problem, Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. 18 (1963), no. 3, 5–29. MR 0164135
2. P. Auscher and M. Qafsaoui, Observations on \(W^{1,p}\) estimates for divergence elliptic equations with VMO coefficients, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 5 (2002), no. 2, 487–509. MR 1911202
3. H. Brezis, Functional analysis, Sobolev spaces and partial differential equations, Universitext, Springer, New York, 2011. MR 2759829
4. S.-S. Byun, Elliptic equations with BMO coefficients in Lipschitz domains, Trans. Amer. Math. Soc. 357 (2005), no. 3, 1025–1046. MR 2110431
5. D. Daners, Domain perturbation for linear and semi-linear boundary value problems, Handbook of differential equations: stationary partial differential equations. Vol. VI, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2008, pp. 1–81. MR 2569323
6. G. Di Fazio, \(L^p\) estimates for divergence form elliptic equations with discontinuous coefficients, Boll. Un. Mat. Ital. A (7) 10 (1996), no. 2, 409–420. MR 1405255
7. H. Dong, Recent progress in the \(L_p\) theory for elliptic and parabolic equations with discontinuous coefficients, Anal. Theory Appl. 36 (2020), no. 2, 161–199. MR 4156495
8. H. Dong and D. Kim, Elliptic equations in divergence form with partially BMO coefficients, Arch. Ration. Mech. Anal. 196 (2010), no. 1, 25–70. MR 2601069
9. ________, Higher order elliptic and parabolic systems with variably partially BMO coefficients in regular and irregular domains, J. Funct. Anal. 261 (2011), no. 11, 3279–3327. MR 2835999
10. ________, On the \(L_p\)-solvability of higher order parabolic and elliptic systems with BMO coefficients, Arch. Ration. Mech. Anal. 199 (2011), no. 3, 889–941. MR 2771670
11. J. Droniou, Non-coercive linear elliptic problems, Potential Anal. 17 (2002), no. 2, 181–203. MR 1908676
12. L. C. Evans and R. F. Gariepy, Measure theory and fine properties of functions, revised ed., Textbooks in Mathematics, CRC Press, Boca Raton, FL, 2015. MR 3409135
13. N. Filonov, On the regularity of solutions to the equation \(-\Delta u + b \cdot \nabla u = 0\), Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 410 (2013), no. Kraevye Zadachi Matematicheskoi Fiziki i Smezhnye Voprosy Teorii Funktsiĭ. 43, 168–186, 189. MR 3048265
14. N. Filonov and T. Shilkin, *On some properties of weak solutions to elliptic equations with divergence-free drifts*, Mathematical analysis in fluid mechanics—selected recent results, Contemp. Math., vol. 710, Amer. Math. Soc., Providence, RI, 2018, pp. 105–120. MR 3818670
15. C. Gerhardt, *Stationary solutions to the Navier-Stokes equations in dimension four*, Math. Z. 165 (1979), no. 2, 193–197. MR 520820
16. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition. MR 1814364
17. B. Kang and H. Kim, *$W^{1,p}$-estimates for elliptic equations with lower order terms*, Commun. Pure Appl. Anal. 16 (2017), no. 3, 799–821. MR 3623550
18. , *On $L^p$-resolvent estimates for second-order elliptic equations in divergence form*, Potential Anal. 50 (2019), no. 1, 107–133. MR 3900848
19. H. Kim and Y.-H. Kim, *On weak solutions of elliptic equations with singular drifts*, SIAM J. Math. Anal. 47 (2015), no. 2, 1271–1290. MR 3328143
20. H. Kim and H. Kwon, *Dirichlet and Neumann problems for elliptic equations with singular drifts on Lipschitz domains*, [arXiv:1811.12619](https://arxiv.org/abs/1811.12619)
21. H. Kim and T.-P. Tsai, *Existence, uniqueness, and regularity results for elliptic equations with drift terms in critical weak spaces*, SIAM J. Math. Anal. 52 (2020), no. 2, 1146–1191. MR 4075335
22. N. V. Krylov, *Parabolic and elliptic equations with VMO coefficients*, Comm. Partial Differential Equations 32 (2007), no. 1-3, 453–475. MR 2304157
23. , *On stochastic equations with drift in $L_d^p$*, 2020.
24. , *Elliptic equations with VMO $A, B \in L_d$, and $C \in L_{d/2}^d$*, Trans. Amer. Math. Soc. 374 (2021), no. 4, 2805–2822. MR 4223034
25. H. Kwon, *Existence and uniqueness of weak solution in $W^{1,2+\varepsilon}$ for elliptic equations with drifts in weak-$L^n$ spaces*, J. Math. Anal. Appl. 500 (2021), no. 1, 125165. MR 4235253
26. O. A. Ladyzhenskaya and N. N. Ural’tseva, *Linear and quasilinear elliptic equations*, Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis, Academic Press, New York-London, 1968. MR 0244627
27. G. Moscariello, *Existence and uniqueness for elliptic equations with lower-order terms*, Adv. Calc. Var. 4 (2011), no. 4, 421–444. MR 2844512
28. G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier (Grenoble) 15 (1965), no. fasc. 1, 189–258. MR 192177
29. N. S. Trudinger, *Linear elliptic operators with measurable coefficients*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) 27 (1973), 265–308. MR 369884