Functional Forms for the Squeeze and the
Time-Displacement Operators

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ABSTRACT

Using Baker-Campbell-Hausdorff relations, the squeeze and harmonic-oscillator time-displacement operators are given in the form

$$\exp[\delta I] \exp[\alpha(x^2)] \exp[\beta(x\partial)] \exp[\gamma(\partial)^2],$$

where $$\alpha, \beta, \gamma,$$ and $$\delta$$ are explicitly determined. Applications are discussed.
1 Introduction

In group theory and quantum optics, the use of Baker-Campbell-Hausdorff relations, to write unitary operators in more useful (often normal-ordered) forms, is a well-known technique. For instance, the displacement operator,

\[ D(\alpha) = \exp[\alpha a^\dagger - \alpha^* a] , \quad \alpha = \alpha_1 + i\alpha_2 \equiv (x_0 + ip_0)/\sqrt{2} , \]  

and the squeeze operator,

\[ S(z) = \exp \left[ \frac{1}{2} za^\dagger a^\dagger - \frac{1}{2} z^* aa \right] , \quad z = re^{i\phi} = z_1 + iz_2 , \]  

can be written as

\[ D(\alpha) = \exp \left[ -\frac{1}{2} |\alpha|^2 \right] \exp[\alpha a^\dagger] \exp[-\alpha^* a] \]  

and

\[ S(z) = \exp \left[ \frac{1}{2} e^{i\theta} (\tanh r)a^\dagger a^\dagger \right] \left( \frac{1}{\cosh r} \right)^{\frac{1}{2}+a^\dagger a} \exp \left[ -\frac{1}{2} e^{-i\theta} (\tanh r)aa \right] \]  

\[ = \exp \left[ \frac{1}{2} e^{i\theta} (\tanh r)a^\dagger a^\dagger \right] (\cosh r)^{-1/2} \sum_{n=0}^{\infty} \frac{(\sec r - 1)^n}{n!} (a^\dagger)^n (a)^n \times \exp \left[ -\frac{1}{2} e^{-i\theta} (\tanh r)aa \right] . \]  

However, such transformations are not as well-known in \( x - p \) space, in terms of the variables

\[ x = \frac{1}{\sqrt{2}} (a + a^\dagger) , \quad \partial = ip = \frac{1}{\sqrt{2}} (a - a^\dagger) , \]  

\[ [a, a^\dagger] = 1 , \quad [x, \partial] = -1 . \]  

Granted, the displacement operator is easily expressed as

\[ D(\alpha) = \exp[-ix_0p_0/2] \exp[ip_0x] \exp[-x_0\partial] . \]  

but the squeeze operator

\[ S(z) = \exp[-z_1(x\partial + 1/2) + iz_2(x^2 + \partial^2)/2] . \]
is not. If it were, it could be easily applied to any wave function, using the operator properties

\[ \exp[c \partial] h(x) = h(x + c) \]  \hspace{1cm} (10) \\
\[ \exp[\tau(x \partial)] h(x) = h(xe^\tau) \]  \hspace{1cm} (11) \\
\[ \exp[c(\partial^2)] h(x) = \frac{1}{\sqrt{4\pi c}} \int_{-\infty}^{\infty} \exp \left[ -\frac{(y-x)^2}{4c} \right] h(y) dy . \]  \hspace{1cm} (12)

In Section 2 we describe the BCH method to obtain such transformations. In Section 3 we explicitly apply it to the squeeze operator, and in Section 4 we do the same for the unitary, harmonic-oscillator, time-displacement operator

\[ T = \exp[-i(a^\dagger a + 1/2)] = \exp[-i(x^2 + \partial^2)/2] . \]  \hspace{1cm} (13)

In Section 5 we give examples and indicate further work.

## 2 The Method

There are a number of early papers that deal with this subject, but a complete description, that we follow, was given by Wei and Norman [3]. It has been used widely. (See, e.g., Refs. [4, 5].) Consider a unitary operator \( U(t) \) in terms of the exponentiations of \( I, x^2, (x \partial), \) and \((\partial)^2\) times a parameter, \( t \):

\[ U(t) = \exp[t\{b_1 I + b_2 x^2 + b_3 (x \partial) + b_4 (\partial^2)\}] , \]  \hspace{1cm} (14) 

where the \( b_i \) are not functions of \( x \) or \( \partial \). Set this equal to the following ordered product:

\[ U_1(t) = \exp[\delta I] \exp[i\alpha x^2] \exp[\beta (x \partial)] \exp[i\gamma (\partial)^2] , \]  \hspace{1cm} (15)

where \( \alpha(t) \) (not to be confused with the coherent state label \( \alpha \)), \( \beta(t) \), \( \gamma(t) \), and \( \delta(t) \) are the functions to be determined. (Note that even in the case where \( U \) contains only \( I, x^2, \) and \((\partial)^2\) terms, one still needs a \((x \partial)\) term on the right since this operator is needed to close the algebra.)
Take the time derivative of both sides of the equality $U = U_1$, and then multiply on the right by $U^\dagger$ and $U_1^\dagger$, respectively. This yields

$$b_1 I + b_2 x^2 + b_3 (x \partial) + b_4 (\partial^2)$$

$$= [e^{i \alpha x^2} + \beta \exp[i \alpha x^2] (x \partial) \exp[-i \alpha x^2]$$

$$+ \gamma \exp[i \alpha x^2] \exp[\beta (x \partial)] (\partial^2) \exp[-\beta (x \partial)] \exp[-i \alpha x^2] + \dot{\delta}] , \quad (16)$$

where “dot” signifies $\frac{d}{dt}$. The factor $[e^{2\delta} e^{-\beta}]$ is unity by $UU^\dagger = 1$. Now the main line operators on the right-hand side of the equation should be commuted to the right.

First do this with

$$X = \exp[i \alpha x^2] (x \partial) \exp[-i \alpha x^2] . \quad (17)$$

Using

$$e^A Be^{-A} = B + [A, B] + [A, [A, B]]/2 + \ldots , \quad (18)$$

and

$$[x^2, x \partial] = -2x^2 , \quad [x^2, [x^2, x \partial]] = 0 , \quad (19)$$

one has

$$X = (x \partial) - i2\alpha x^2 . \quad (20)$$

Similarly one has

$$Y = \exp[\beta (x \partial)] (\partial^2) \exp[-\beta (x \partial)]$$

$$= \partial^2 \sum_{n=0}^{\infty} \frac{(-2\beta)^{2n}}{n!} = \partial^2 e^{-2\beta} . \quad (21)$$

This leaves to calculate

$$Z = \exp[i \alpha x^2] \partial^2 \exp[-i \alpha x^2]$$

$$= -i2\alpha - i4\alpha (x \partial) - 4\alpha^2 (x^2) + \partial^2 , \quad (22)$$

where the last line comes from direct differentiation.

Putting Eqs. (20), (21), and (22) into Eq. (16), one has an equation with coefficients multiplying $I, x^2, (x \partial)$, and $\partial^2$. Because these are independent variables, that
means we can write the coefficients multiplying each of these variables as a separate equation:

\[
\begin{align*}
b_1 &= \dot{\gamma} e^{-2\beta}(2\alpha) + \dot{\delta} , \\
b_2 &= i\dot{\alpha} - i2\alpha \dot{\beta} + i\gamma e^{-2\beta}(-4\alpha^2) , \\
b_3 &= \dot{\beta} + \dot{\gamma} e^{-2\beta}(4\alpha) , \\
b_4 &= i\dot{\gamma} e^{-2\beta} .
\end{align*}
\] (23)

(24)

(25)

(26)

These are four first-order differential equations in four unknowns. They should be solved subject to the boundary conditions that the \( b_i(0) = 0 \). Then set \( t = 1 \) and one has the unitary operator in product form. (For the time-displacement case, one lets \( t \) simply remain \( t \), the time.)

### 3 The Squeeze Operator

For the squeeze operator one has, from Eq. (9),

\[
\begin{align*}
b_1 &= -\frac{z_1}{2} , \\
b_2 &= i\frac{z_2}{2} , \\
b_3 &= -z_1 , \\
b_4 &= i\frac{z_2}{2} .
\end{align*}
\] (27)

Put this into Eqs. (23)-(26). With linear combinations of Eqs. (24) to (26) one finds

\[
\dot{\alpha} = -2z_2\alpha^2 - 2z_1\alpha + z_2/2 .
\] (28)

The solution that goes to zero at \( t = 0 \) is

\[
\alpha(t) = \frac{z_2^2 \sinh rt}{2r S(t)} ,
\] (29)

where

\[
S(t) = \cosh rt + \frac{z_1}{r} \sinh rt .
\] (30)

Putting this result into a linear combination of Eqs. (24) and (26),

\[
\dot{\beta} = -z_1 - 2z_2\alpha .
\] (31)
or
\[ \beta = -\ln S(t) . \]  

(32)

Now Eq. (26) gives us
\[ \dot{\gamma} = \frac{z_2}{2} e^{2\beta}, \]  

(33)

or
\[ \gamma = \frac{z^2}{2r} \sinh rt \frac{S(t)}{S} = \alpha . \]  

(34)

Finally, Eq. (23) gives us
\[ \dot{\delta} = -\frac{z_1}{2} - 2\alpha \dot{\gamma} e^{2\beta}, \]  

(35)

or
\[ \delta = -\frac{1}{2} \ln S(t) = \frac{\beta}{2} . \]  

(36)

Therefore, setting \( t = 1 \), we obtain the squeeze operator,
\[ S = S^{-1/2} \exp \left[ \frac{i z_2 \sinh r}{2r} \frac{S}{S} (x^2) \right] \exp[-(\ln S)(x\partial)] \exp \left[ \frac{i z_2 \sinh r}{2r} \frac{S}{S} (\partial^2) \right] , \]  

(37)

where
\[ S = \cosh r + \frac{z_1}{r} \sinh r = e^r \cos^2 \frac{\phi}{2} + e^{-r} \sin^2 \frac{\phi}{2} . \]  

(38)

4 The Time-Displacement Operator

The harmonic-oscillator time-evolution operator can now be calculated in the same way, but with
\[ b_1 = 0, \quad b_2 = -\frac{i}{2} , \quad b_3 = 0 , \quad b_4 = +\frac{i}{2} . \]  

(39)

Then the solutions are found in the same way, this yielding, respectively,
\[ \dot{\alpha} = -2\alpha^2 - 1/2 , \quad \alpha(t) = -\frac{\tan t}{2} , \]  

(40)

\[ \alpha \dot{\beta} = 1/2 + \dot{\alpha} , \quad \beta(t) = -\ln(\cos t) , \]  

(41)

\[ \dot{\gamma} = e^{2\beta}/2 , \quad \gamma(t) = [\tan t]/2 , \]  

(42)

\[ \dot{\delta} = -\alpha , \quad \delta(t) = -[\ln(\cos t)]/2 . \]  

(43)
This means that the harmonic-oscillator time-displacement operator is

\[ T = [\cos t]^{-1/2} \exp \left[ -\frac{i}{2} \tan t(x^2) \right] \exp \left[ - (\ln \cos t)(x \partial) \right] \exp \left[ \frac{i}{2} \tan t(\partial^2) \right]. \]  

(44)

This result can be viewed as complementary to others [6] in the study of time-evolution.

5 Discussion

It is enlightening to look at specific examples.

Using Eqs. (8) and (38) on the harmonic-oscillator ground state,

\[ \psi_0 = \pi^{-1/4} \exp[-x^2/2], \]  

(45)

one finds

\[
\psi_{ss} = D(\alpha)S(z)\psi_0 \\
= \frac{1}{\pi^{1/4}} \exp[-ix_0p_0/2] \exp \left[ -(x - x_0)^2 \left( \frac{1}{2S^2(1 + 2\kappa)} - i\kappa \right) + ip_0x \right], \]

(46)

where

\[ \kappa = \frac{z_2 \sinh r}{2rs}. \]  

(47)

This is the most general squeezed state. Setting \( z \) to be real and positive yields the most commonly studied example:

\[ \psi_{ss} = [\pi^{1/2}s]^{-1/2} \exp \left[ -\frac{(x - x_0)^2}{2s^2} - ip_0x \right], \]  

(48)

where

\[ s = e^r. \]  

(49)

As a test, the time-evolution operator can be applied to the coherent states, which are Eq. (48) with \( s = 1 \). Then one finds

\[
T\psi_{cs} = \frac{e^{-it/2}}{\pi^{1/4}} \exp \left[ - \frac{1}{2} \left\{ x - (x_0 \cos t + p_0 \sin t) \right\}^2 \right] \exp[ip(x_0 \cos t - x_0 \sin t)] \\
\exp \left[ -\frac{i}{2} (x_0 \cos t + p_0 \sin t)(p_0 \cos t - x_0 \sin t) \right]. 
\]

(50)
Of course, a simpler calculation is possible by replacing $\alpha$ with $\alpha e^{-it}$ in the series defining the coherent states as an infinite sum of number states.

But the time-evolution operator can be applied to more complicated systems, for instance, the even and odd states of the harmonic oscillator. There, starting from the real and $p_0 = 0$ squeezed states one can calculate

$$
\psi_{s\pm}(x,t) = T\psi_{s\pm}(x) .
$$

(51)

One finds a closed form expression for $\psi_{s\pm}$:

$$
\psi_{s\pm}(x,t) = \left[ \frac{s}{2\pi^{1/2}(1 \pm e^{-x_0^2 \sin^2 t})} \right]^{1/2} \frac{s^2 \cos t - i \sin t}{s^4 \cos^2 t + \sin^2 t} \left\{ \exp \left[ -\frac{(x - x_0 \cos t)^2}{2} \left( \frac{s^2 - i \tan t}{s^4 \cos^2 t + \sin^2 t} \right) - \frac{i}{2} (\tan t) x^2 \right] \pm \exp \left[ -\frac{(x + x_0 \cos t)^2}{2} \left( \frac{s^2 - i \tan t}{s^4 \cos^2 t + \sin^2 t} \right) - \frac{i}{2} (\tan t) x^2 \right] \right\} .
$$

(52)

(Observe that the terms $\exp[-i(t\tan t)x^2/2]$ are necessary to cancel the singularities of the terms $\exp[ix^2\tan t/(2\sin^2 t)]$ when $t$ is an odd multiple of $\pi/2$.) This then yields an analytic description of the probability densities as a function of $x,t$:

$$
\rho_{s\pm}(x,t) = \frac{\exp[-(x^2 + x_0^2 \cos^2 t)/d^2]}{\pi^{1/2}d[1 \pm d \exp[-x_0^2/s^2]]} \left\{ \cosh \left( \frac{2xx_0(\cos t)}{d^2} \right) \pm \cos \left( \frac{2x_0 \sin t}{d^2s^2} \right) \right\} ,
$$

where

$$
d^2 = s^2 \cos^2 t + \sin^2 t/s^2 .
$$

(53)

(54)

Finally, it is amusing to apply the simple time-evolution operator for a particle in a box,

$$
T_0 = \exp[+i(\partial)^2/2] ,
$$

(55)

to a number eigenstate of a particle in a box. One finds

$$
T_0 \sin(\pi nx) = \sin(\pi nx) \exp[-i\pi^2 n^2/2] ,
$$

(56)
the correct time-evolution of a number state. Here the operator is repeating the calculation for a continuous set of boxes along the real axis.

These techniques can be applied elsewhere, such as in obtaining the time-evolution operator for a system with different potentials. For example, the time-dependent system \[8, 9\]

\[ V(x, t) = g^{(2)}(t)x^2 + g^{(1)}(t)x + g^{(0)}(t) \]  

(57)
can be studied.

**Acknowledgements**

I thank D. R. Truax for many useful conversations, and S. l. Braunstein for helpful comments. This work was supported by the U.S. Department of Energy and the Alexander von Humboldt Stiftung.
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