PRINCIPAL MINORS OF GAUSSIAN ORTHOGONAL ENSEMBLE

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ABSTRACT. In this paper, we study the extremal process of the maxima of all the largest eigenvalues of principal minors of the classical Gaussian orthogonal ensemble (GOE). We prove that the fluctuation of the maxima is given by the Gumbel distribution in the limit. We also derive the limiting joint distribution of the maxima and the corresponding eigenvector, which implies that these two random variables are asymptotically independent.

1. Introduction

Motivated by high-dimensional statistics and signal processing, the authors in [4] derived the growth order of the extremal process of the maxima of all the largest eigenvalues of the principal minors of the classical random matrices of GOE and Wishart matrices, where the results have applications for the construction of the compressed sensing matrices as in [8]. In this paper, we further study the fluctuation of such maxima for the GOE case. Our main result is that the fluctuation is given by the Gumbel distribution with some Poisson structure involved in the limit, and we also derive the limiting joint distribution of such maxima and its corresponding eigenvector which indicates that these two random variables are asymptotically independent.

The Gaussian orthogonal ensemble (GOE) is the Gaussian measure defined on the space of real symmetric matrices, i.e., \( G = (g_{ij})_{1 \leq i,j \leq n} \) is a symmetric matrix whose upper triangular entries are independent real Gaussian variables with the following distribution

\[
g_{ij} \overset{d}{=} \begin{cases} 
N_R(0, 2) & \text{if } i = j; \\
N_R(0, 1) & \text{if } i < j.
\end{cases}
\]

Let \( \lambda_1(G) > \lambda_2(G) > \cdots > \lambda_n(G) \) be eigenvalues of GOE, then the distribution of these eigenvalues is invariant under the orthogonal group action and the joint density is

\[
\frac{1}{Z_n} \prod_{k=1}^{n} e^{-\frac{1}{4} \lambda_k^2} \prod_{i<j} (\lambda_i - \lambda_j),
\]

where

\[
Z_n = 2^{n(n+1)/4} (2\pi)^{n/2} (n!)^{-1} \prod_{j=1}^{n} \frac{\Gamma(1 + j/2)}{\Gamma(3/2)}
\]

is the partition function. And the limit of the empirical measure of these eigenvalues is given by the classical semicircle law [1].
Let’s first introduce some notations in order to present our main results. Given symmetric matrices \( G = (g_{ij})_{1 \leq i \leq j \leq n} \) sampled from GOE, for \( \alpha \subset \{1, \ldots, n\} \) with cardinality \(|\alpha| = m \in \mathbb{Z}^+\), we denote \( G_\alpha = (g_{ij})_{i,j \in \alpha} \) as the principal minors of \( G \) of size \( m \times m \), then \( G_\alpha \) is also symmetric. Let \( \lambda_1(G_\alpha) > \lambda_2(G_\alpha) > \cdots > \lambda_m(G_\alpha) \) be the ordered eigenvalues of \( G_\alpha \). Now we define the extremal process of the maxima of all the largest eigenvalues of the principal minors as

\[
T_{m,n} = \max_{\alpha \subset \{1, \ldots, n\}, |\alpha| = m} \lambda_1(G_\alpha).
\]

In [4], the authors studied the asymptotic properties of \( T_{m,n} \) and proved that under the assumption that \( m \) fixed or \( m \to +\infty \) with \( m = o \left( \frac{\ln n}{\ln \ln n} \right) \), it holds

\[
T_{m,n} - 2\sqrt{m \ln n} \to 0 \quad \text{in probability as } n \to +\infty.
\]

In this paper, we further derive the fluctuation of \( T_{m,n} \) when \( m \) is fixed as \( n \) tends to infinity. Our first result is the following

**Theorem 1.** For GOE, we have the following convergence in distribution

\[
T_{m,n}^2 - 4m \ln n - 2(m - 2) \ln \ln n \xrightarrow{d} Y
\]

as \( n \to +\infty \) for fixed \( m \), where the random variable \( Y \) has the Gumbel distribution

\[
F_Y(y) = \exp(-ce^{-y/4}), \quad y \in \mathbb{R}.
\]

Here, the constant

\[
c_m = \frac{(2m)^{(m-2)/2}K_m}{(m - 1)!2^{3/2}\Gamma(1 + m/2)},
\]

where \( K_m = \mu(S_m) \) is the probability of the event

\[
S_m := \left\{ x \in S^{m-1}: \sum_{j \in \beta} x_j^2 \leq \sqrt{k/m}, \ \forall \beta \subset \{1, \cdots, m\} \text{ with } \forall 1 \leq |\beta| = k < m \right\}
\]

under the uniform distribution \( \mu \) on the unit sphere \( S^{m-1} \). In particular,

\[
c_1 = \frac{1}{2\sqrt{\pi}}, \quad c_2 = -\frac{1}{2\sqrt{2}} + \frac{\sqrt{2}}{\pi} \arcsin \left( \frac{1}{2} \right)^{1/4}.
\]

The following classical result in the theory of extremal processes is regarded as a special case of our main result. Let \( a_1, \ldots, a_n \) be i.i.d. Gaussian random variables \( N_\mathbb{R}(0, 2) \), for the extremal process

\[
M_n := \max\{a_1, \ldots, a_n\},
\]

let

\[
a_n = 2\sqrt{\ln n}
\]

and

\[
b_n = 2\sqrt{\ln n} - \frac{\ln \ln n + \ln(4\pi)}{2\sqrt{\ln n}}.
\]

Then for any \( y \in \mathbb{R} \), the following classical result holds (Theorem 1.5.3 in [6])

\[
\lim_{n \to +\infty} \mathbb{P}[a_n (M_n - b_n) \leq y] = e^{-e^{-y/2}}.
\]
One can check that Theorem 1 for $T_{1,n}^2$ when $m = 1$ is equivalent to this classical result for $M_n$. In this sense, our result is fundamental which is a natural generalization of such classical result for the extremal process of the scalar-valued random variables to the matrix-valued random variables with correlations.

The constant $K_m$ appearing in the extremal index $c_m$ represents the strength of the correlation between different principal minors. For a comprehensive overview of the extremal theory of stochastic processes, we refer to [6, 7].

Our proof for Theorem 1 also implies the joint distribution of the maxima of the largest eigenvalues of principal minors and its corresponding eigenvector. To be more precise, given $n$, let $v^* \in S^{m-1}$ be the unit eigenvector corresponding to the largest eigenvalue of the principal minor that attains the maxima $T_{m,n}$. By symmetry, $-v^*$ is also the corresponding eigenvector. Now we have the following limit for the joint distribution of $(T_{m,n}, v^*)$.

**Theorem 2.** Given any $y \in \mathbb{R}$ and symmetric Borel set $Q \subset S^{m-1}$ such that $-Q = Q$, let $y_m^2 = 4m \ln n + 2(m-2) \ln \ln n + y$, then the joint distribution satisfies

$$
\mathbb{P}(T_{m,n} > y_m, v^* \in Q) \to (1 - F_Y(y))\nu(Q)
$$

as $n \to +\infty$, which implies that $T_{m,n}$ and $v^*$ are asymptotically independent. Here, $\nu$ is the uniform distribution on $S_m$, i.e.,

$$
\nu(Q) = \mu(Q \cap S_m)/\mu(S_m).
$$

The study of principal minors of random matrices holds significant importance and finds applications in various fields such as statistics, machine learning, deep learning, information theory, and more. In the following, we only list two of them.

In compressed sensing, one has to recover an input vector $f$ from the corrupted measurements $y = Af + e$, where $A$ is a coding matrix and $e$ is an arbitrary and unknown vector of errors. The famous result by Candès-Tao [8] is that if the coding matrix $A$ satisfies the restricted isometry property (Definition 1.1 in [8]), then the input $f$ is the unique solution to some $\ell_1$-minimization problem provided that the support $S$ (the number of nonzero entries) of errors $e$ is not too large. Therefore, one of the major goals in compressed sensing is to construct the coding matrix $A$ that satisfies the restricted isometry property. In Section 3 of [8], Candès-Tao proved that the Gaussian random matrices $A$ can satisfy such property with overwhelming probability, and they can derive the estimate about the support $S$ via the probabilistic estimate on the maxima of the largest eigenvalues of principal minors of Wishart matrices $A^T A$. A simple proof based on the concentration measure theory is present in [3].

In statistics, the problem of finding the largest eigenvalue over principal minors (of some fixed order) of the sample covariance matrices is also called *sparse principal component analysis*. Gamarnik and Li [4, Section 6] asked about the distribution of the largest eigenvalue as well as the algorithmic complexity of this problem, i.e., whether there exists efficient algorithms to find the principal minor which attains the (near-)optimal value for the largest eigenvalue. Consider the Wishart version of this problem, i.e., let $X$ be an $n \times p$ matrices with independent $\mathcal{N}_R(0, 1)$ distributed entries and take $R_{m,n,p}$ to be the maximal eigenvalue of all $m$-th order principal minors of $X^T X$. It is proved in [4] that

$$
\frac{R_{m,n,p} - n}{\sqrt{n}} - \sqrt{2m \log p} \to 0
$$
in probability as $n \to \infty$, under conditions requiring $\log p = o(n^{1/2})$. It seems that our method in the current paper can also be applied to study the Wishart case and it’s expected that some Gumbel fluctuation will be observed as well. We will postpone this to further investigations.

Notation. In this paper, $c$, $C$ and $C'$ stand for positive constants, but their values may change from line to line. For simplicity, the notation $a_n \sim b_n$ means $\lim_{n \to +\infty} a_n/b_n = 1$.

2. Proof of Theorem 1

In this section, we will prove Theorem 1 by assuming some technical lemmas where the proofs of these lemmas are postponed to §3.

The proof of Theorem 1 is based on Lemma 1 with the proof given in [2] by the Stein-Chen method. It provides a criteria to prove the convergence of the total number of occurrences of the point process to the Poisson distribution, and thus it provides a method to derive the distribution for some extremal processes.

Lemma 1. Let $I$ be an index set, and for $\alpha \in I$, let $X_\alpha$ be a Bernoulli random variable with $p_\alpha = P(X_\alpha = 1) = 1 - P(X_\alpha = 0)$. For each $\alpha \in I$, let $N_\alpha$ be a subset of $I$ with $\alpha \in N_\alpha$, that is, $\alpha \in N_\alpha \subset I$. Let

$$S = \sum_{\alpha \in I} X_\alpha, \quad \lambda = \mathbb{E}S = \sum_{\alpha \in I} p_\alpha \in (0, +\infty),$$

let $Z$ be the Poisson random variable with intensity $\mathbb{E}Z = \lambda$, then it holds that

$$\|\mathcal{L}(S) - \mathcal{L}(Z)\| \leq 2(b_1 + b_2 + b_3),$$

and the probability of no occurrence has the estimate

$$|\mathbb{P}(S = 0) - e^{-\lambda}|$$

$$= |\mathbb{P}(X_\alpha = 0, \forall \alpha \in I) - e^{-\lambda}|$$

$$\leq \min(1, \lambda^{-1})(b_1 + b_2 + b_3).$$

Here, $\|\mathcal{L}(S) - \mathcal{L}(Z)\|$ is the total variation distance between the distributions $S$ and $Z$, and

$$b_1 = \sum_{\alpha \in I} \sum_{\beta \in N_\alpha} p_\alpha p_\beta,$$

$$b_2 = \sum_{\alpha \in I} \sum_{\alpha \neq \beta \in N_\alpha} \mathbb{E}[X_\alpha X_\beta],$$

$$b_3 = \sum_{\alpha \in I} \mathbb{E}\mathbb{E}[X_\alpha | \sigma(X_\beta, \beta \notin N_\alpha)] - p_\alpha,$$

and $\sigma(X_\beta, \beta \notin N_\alpha)$ is the $\sigma$-algebra generated by $\{X_\beta, \beta \notin N_\alpha\}$. In particular, if $X_\alpha$ is independent of $\{X_\beta, \beta \notin N_\alpha\}$ for each $\alpha$, then $b_3 = 0$.

One may think of $N_\alpha$ as a ‘neighborhood of dependence’ for $\alpha$ such that $X_\alpha$ is independent or nearly independent of all of $X_\beta$ for $\beta \notin N_\alpha$. And Lemma 1 indicates that when $b_1$, $b_2$ and $b_3$ are all small enough, then $S$ which is the total number of occurrences tends to a Poisson distribution.
2.1. **Proof of Theorem 1.** Given any $\ell \times \ell$ symmetric matrix $S$, we rearrange the eigenvalues of $S$ in descending order $\lambda_1(S) \geq \cdots \geq \lambda_\ell(S)$, and we denote

$$|S|^2 := \text{Tr } S^2$$

and

$$\lambda_1^*(S) := [\lambda_1(S)^2 + |S|^2]/2]^{1/2}.$$  

For fixed $m \geq 2$, we define the index set

$$I_m = \{\alpha \subset \{1, \cdots, n\}, |\alpha| = m\}$$

and the neighborhood set

$$N_\alpha = \{\beta \in I_m : \alpha \cap \beta \neq \emptyset\} \text{ for } \alpha \in I_m.$$  

Throughout the article, for any fixed real number $y$, we define

$$y_k^2 := 4k \ln n + 2k \ln \ln n, \ y_k > 0, \ 1 \leq k < m$$

and

$$y_m^2 := 4m \ln n + 2(m - 2) \ln \ln n + y, \ y_m > 0.$$  

For symmetric matrices $G = (g_{ij})_{1 \leq i, j \leq n}$ sampled from GOE, for $\alpha \in I_m$ where $|\alpha| = m$, we denote $G_\alpha = (g_{ij})_{i, j \in \alpha}$ as the principal minor of size $m \times m$ and we define the event

$$A_\alpha = \{\lambda_1(G_\alpha) > y_m; \ \lambda_1^*(G_\beta) \leq y_k, \ \forall 1 \leq k < m, \ \beta \subset \alpha, |\beta| = k\}.$$  

Recall the definition of $T_{m,n}$ in [11] we first have

$$0 \leq \mathbb{P}(\cap_{\alpha \in I_m} A_\alpha^c) - \mathbb{P}(T_{m,n} \leq y_m) \leq \mathbb{P}(\lambda_1^*(G_\beta) > y_k \text{ for some } \beta \in I_k, \ 1 \leq k < m)$$

$$\leq \sum_{k=1}^{m-1} \sum_{\beta \in I_k} \mathbb{P}(\lambda_1^*(G_\beta) > y_k)$$

$$= \sum_{k=1}^{m-1} \binom{n}{k} \mathbb{P}(\lambda_1^*(G_{\{1, \cdots, k\}}) > y_k).$$

Now we need the following lemma and we postpone its proof to the next section.

**Lemma 2.** For fixed $k \geq 1$, there exists a constant $C > 0$ (depending on $k$) so that for all $x > 1$,

$$\mathbb{P}(|G_{\{1, \cdots, k\}}|^2 > x^2) \leq C x^{k(k+1)/2-2} e^{-x^2/4},$$

$$\mathbb{P}(\lambda_1(G_{\{1, \cdots, k\}}) > x) \leq C x^{k-2} e^{-x^2/4},$$

$$\mathbb{P}(\lambda_1^*(G_{\{1, \cdots, k\}}) > x) \leq C x^{k-2} e^{-x^2/4}.$$  

Using (12) we have

$$\sum_{k=1}^{m-1} \binom{n}{k} \mathbb{P}(\lambda_1^*(G_{\{1, \cdots, k\}}) > y_k) \leq C \sum_{k=1}^{m-1} n^k y_k^{k-2} e^{-y_k^2/4} \leq C/\ln n.$$  

Combining this with (9) we get

$$0 \leq \mathbb{P}(\cap_{\alpha \in I_m} A_\alpha^c) - \mathbb{P}(T_{m,n} \leq y_m) \leq C/\ln n,$$
and thus we have
\[
\lim_{n \to +\infty} \mathbb{P}(T_{m,n} \leq y_m) = \lim_{n \to +\infty} \mathbb{P}(\cap_{\alpha \in I_m} A_\alpha^c).
\]
Therefore, it’s enough to derive the limit of \(\mathbb{P}(\cap_{\alpha \in I_m} A_\alpha^c)\) to prove Theorem \([\text{I}]\) By Lemma \([\text{I}]\), we have
\[
|\mathbb{P}(\cap_{\alpha \in I_m} A_\alpha^c) - e^{-t_n}| \leq b_{n,1} + b_{n,2},
\]
where
\[
t_n = \binom{n}{m} \mathbb{P}(A_{(1, \ldots, m)}),
\]
\[
b_{n,1} \leq \binom{n}{m} \left( \binom{n}{m} - \binom{n-m}{m} \right) \mathbb{P}(A_{(1, \ldots, m)})^2,
\]
\[
b_{n,2} \leq \sum_{k=1}^{m-1} \binom{n}{k} \binom{n-k}{m-k} \mathbb{P}(A_{(1, \ldots, k, \ldots, m)} \cap A_{(1, \ldots, k, m+1, \ldots, 2m-k)}).
\]
Using \([\text{I}]\), we have
\[
\mathbb{P}(A_{(1, \ldots, m)}) \leq \mathbb{P}(\lambda_1(G_{(1, \ldots, m)}) > y_m) \leq Cy_m^{-2}e^{-y_m^2/4} \leq Cn^{-m}.
\]
And thus we have
\[
b_{n,1} \leq Cn^{2m-1} \mathbb{P}(A_{(1, \ldots, m)})^2 \leq C'n^{-1},
\]
which tends to 0 in the limit.

It remains to find the limit of \(t_n\) and show that \(b_{n,2}\) tends to 0 in order to complete the proof of Theorem \([\text{I}]\).

Let \(\alpha = \{1, \ldots, m\}, \gamma = \{m-k+1, \ldots, m\}, \zeta = \{m-k+1, \ldots, 2m-k\}\), then \(\alpha \cap \zeta = \gamma, |\alpha| = |\gamma| = m\) and \(|\gamma| = k\). By rearranging the indices, we have
\[
\mathbb{P}(A_{(1, \ldots, k, \ldots, m)} \cap A_{(1, \ldots, k, m+1, \ldots, 2m-k)})
\]
\[
= \mathbb{P}(A_{(1, \ldots, m)} \cap A_{(m-k+1, \ldots, 2m-k)})
\]
\[
\leq \mathbb{P}(\lambda_1(G_\alpha) > y_m, \lambda_1(G_\zeta) > y_m, \lambda_1^*(G_{\alpha \gamma}) \leq y_{m-k}, \lambda_1^*(G_{\zeta \gamma}) \leq y_k)
\]
\[
= \mathbb{E}[\mathbb{P}(\lambda_1(G_\alpha) > y_m, \lambda_1(G_\zeta) > y_m, \lambda_1^*(G_{\alpha \gamma}) \leq y_{m-k}, \lambda_1^*(G_{\zeta \gamma}) \leq y_{m-k}| G_\gamma)
\]
\[
\times 1[\lambda_1^*(G_\gamma) \leq y_k)]
\]
\[
= \mathbb{E}[\mathbb{P}(\lambda_1(G_\alpha) > y_m, \lambda_1^*(G_{\alpha \gamma}) \leq y_{m-k}| G_\gamma)^2 1[\lambda_1^*(G_\gamma) \leq y_k]].
\]
The following lemma will imply that \(b_{n,2}\) tends to 0 as \(n \to +\infty\).

Lemma 3. For \(\alpha \in I_m, \gamma \subset \alpha, |\gamma| = k, 1 \leq k < m, \beta = \alpha \setminus \gamma, x > 1, \delta, \delta' \in (0, 1), \) then there are some constants \(C\) and \(C'\) (depending on \(m, \delta\) and \(\delta'\)) such that
\[
\mathbb{P}(\lambda_1(G_\alpha) > x| G_\beta, G_\gamma) 1[\lambda_1^*(G_\beta) \leq (1-\delta)x, \lambda_1^*(G_\gamma) \leq (1-\delta')x]
\]
\[
\leq Cx^{-1}(x/(\lambda_1^*(G_\beta) + 1) + x/ (\lambda_1^*(G_\gamma) + 1))^{k(m-k)-1}e^{-(x-\lambda_1^*(G_\beta))(x-\lambda_1^*(G_\gamma))/2}
\]
and
\[
\mathbb{P}(\lambda_1(G_\alpha) > x, \lambda_1^*(G_\beta) \leq (1-\delta)x| G_\gamma) 1[\lambda_1^*(G_\gamma) \leq (1-\delta')x]
\]
\[
\leq C'x^{m-k-2}(x/(\lambda_1^*(G_\gamma) + 1))^{k(m-k)-1}e^{((\lambda_1^*(G_\gamma))^2-x^2)/4}.\]
By assuming Lemma 3, we have

\[
\mathbb{E}[\mathbb{P}(\lambda_1(G_\alpha) > y_m, \lambda_\gamma^*(G_\alpha) \leq y_m \mid |G_\gamma| \geq y_k)] \\
\leq C \mathbb{E}[y_m^{2m-2k-4}(y_m/(\lambda_1^*(G_\gamma) + 1))^{2k(m-k)-2}e^{((\lambda_1^*(G_\gamma))^2-y_m^2)/2} \mathbb{1}_{\{\lambda_1^*(G_\gamma) \leq y_k\}}]
\]

(19)  \quad = C \mathbb{E}[f(\lambda_1^*(G_\gamma)) \mathbb{1}_{\{\lambda_1^*(G_\gamma) \leq y_k\}}]

where we define

(20)  \quad f(t) = y_m^{2m-2k-4}(y_m/(t+1))^{2k(m-k)-2}e^{(t^2-y_m^2)/2}, \ t \geq 0.

Integration by parts, we have

(21)

\[
\mathbb{E}(f(\lambda_1^*(G_\gamma)) \mathbb{1}_{\lambda_1^*(G_\gamma) \leq y_k}) = \int_0^{y_k} f'(t)\mathbb{P}(\lambda_1^*(G_\gamma) > t)dt \\
- f(y_k)\mathbb{P}(\lambda_1^*(G_\gamma) > y_k) + f(0).
\]

Note that

(22)  \quad f'(t) = -(2k(m-k)-2)(t+1)^{-1}f(t) + tf(t) \leq tf(t).

Hence, using (12) for \( t > 1 \) and (22), we have

(23)

\[
\int_0^{y_k} f'(t)\mathbb{P}(\lambda_1^*(G_\gamma) > t)dt \\
\leq \int_0^1 tf(t)dt + \int_1^{y_k} f'(t)\mathbb{P}(\lambda_1^*(G_\gamma) > t)dt \\
\leq \max_{0 \leq t \leq 1} f(t) + C \int_0^{y_k} ty_m^{2m-2k-4}(y_m/(t+1))^{2k(m-k)-2}e^{(t^2-y_m^2)/2}k-2e^{-t^2/4}dt.
\]

Therefore, by (21) we further have

(24)

\[
\mathbb{E}(f(\lambda_1^*(G_\gamma)) \mathbb{1}_{\lambda_1^*(G_\gamma) \leq y_k}) \leq 2 \max_{0 \leq t \leq 1} f(t) \\
+ C \int_1^{y_k} ty_m^{2m-2k-4}(y_m/(t+1))^{2k(m-k)-2}e^{(t^2-y_m^2)/2}k-2e^{-t^2/4}dt.
\]

We now separate this integration into \( 1 < t < y_k/2 \) and \( y_k/2 < t < y_k \). For \( 1 < t < y_k/2 \) the integrand is bounded by

\[
Cy_m^{2m-6k-2k(m-k)-1}y_m^2e^{16-y_m^2/2},
\]

where we used the fact that \( y_k \leq y_m \) for \( n \) large enough. For \( y_k/2 < t < y_k \), we can bound the integrand by

\[
Cy_m^{2m-6k-2k/4}e^{2-y_m^2/2},
\]
where we used the fact that \( y_m/y_k \leq 2\sqrt{m/k} \) as \( n \) large enough. Therefore, as \( n \) large enough, we have

\[
\int_1^{y_k} f'(t)p(\lambda_+^*(G, \beta) > t)dt \\
\leq Cy_m^{2m-k-6}e^{-y_m^2/2n} \left( \int_1^{y_k/2} y_m^{2k(m-k)-1}e^{y_m^2/16}dt + \int_{y_k/2}^{y_k} te^{t^2/4}dt \right) \\
\leq Cy_m^{2m-k-6}e^{-y_m^2/2} \left( y_m^{2k(m-k)}e^{y_m^2/16} + e^{y_m^2/4} \right) \\
\leq Cy_m^{2m-k-6}e^{y_m^2/16-y_m^2/2},
\]

where in the last inequality we used the fact that \( y_m^{2k(m-k)}e^{-3y_m^2/16} \) can be bounded from above uniformly for all \( n \).

The definition of \( f(t) \) in (20) and the fact that \( y_m/y_k \sim \sqrt{m/k} \) imply

\[
\max_{0 \leq t \leq 1} f(t) \leq y_m^{2m-2k-4}y_m^{2k(m-k)-2}e^{(1-y_m^2)/2} \leq Cy_m^{2m-k-6}e^{y_m^2/4-y_m^2/2}
\]
as \( n \) large enough. It follows from (19), (24), (25) and (26) that

\[
\mathbb{E}[p(\lambda_1(G) > y_m, \lambda_1^*(G, \gamma) \leq y_m-k|G,y)] \leq C y_m^{2m-k-6}e^{y_m^2/4-y_m^2/2}
\]
as \( n \) large enough. Therefore, we have

\[
b_{n,2} \leq \sum_{k=1}^{m-1} \frac{m}{k} \left( \frac{n-k}{m-k} \right) \left( \frac{n-m}{m} \right) P(A_{1,\ldots,k,\ldots,m} \cap A_{1,\ldots,k,m+1,\ldots,2m-k})
\leq \sum_{k=1}^{m-1} n^{2m-k}P(A_{1,\ldots,m} \cap A_{1,\ldots,k,m+1,\ldots,2m-k})
\leq C \sum_{k=1}^{m-1} n^{2m-k}y_m^{2m-k-6}e^{y_m^2/4-y_m^2/2}
\leq C \sum_{k=1}^{m-1} y_m^{2m-k-6}(\ln n)^{k/2-m+2} \leq C(\ln n)^{-1},
\]

which tends to 0 as \( n \to +\infty \).

The following lemma gives the limit of \( t_n \).

**Lemma 4.** We have the limit

\[
\lim_{n \to +\infty} t_n = c_m e^{-y/4},
\]

where

\[
c_m = \frac{(2m)^{(m-2)/2}K_m}{(m-1)^{3/2}\Gamma(1+m/2)},
\]

where \( K_m \) is the constant defined in Theorem 4.

By assuming Lemma 4 by (16) together with the facts that \( b_{n,1} \to 0 \) and \( b_{n,2} \to 0 \), we can conclude that

\[
\lim_{n \to +\infty} P(\cap_{a \in A} A_{n,a}) = \exp(-c_m e^{-y/4}).
\]
By (15), this further implies
\[ \lim_{n \to +\infty} P(T_{m,n} \leq y_m) = \exp(-c_m e^{-y/4}), \]
which proves Theorem 1.

3. Proofs of lemmas

In this section, we will prove Lemma 2, Lemma 3 and Lemma 4, and thus we
complete the proof of Theorem 1.

3.1. Proof of Lemma 2

Proof. We simply have the following estimates. For \( s \in \mathbb{R} \), there are some constants \( C, C' > 0 \) depending on \( s \) such that for all \( x > 1 \) we have

\[ \int_x^{+\infty} r^s \exp(-r) dr \leq C x^s \exp(-x) \]

and

\[ \int_x^{+\infty} r^s \exp(-r^2/2) dr \leq C' x^{s-1} \exp(-x^2/2). \]

Now let \( g_1, \ldots, g_\ell \) be \( \ell \) independent \( \mathcal{N}(0, 1) \) random variables, then for all \( t \geq 1 \), we have

\[ \mathbb{P}(\sum_{i=1}^{\ell} g_i^2 \geq t) \leq C t^{\ell/2-1} \exp(-t/2), \]

where \( C > 0 \) only depends on \( \ell \). The proof of (32) follows if we combine the estimate (30) and the fact that the probability density of the chi-squared distribution \( \chi^2(\ell) := \sum_{i=1}^{\ell} g_i^2 \) with \( \ell \) degrees of freedom is given by

\[ \frac{1}{2^{\ell/2} \Gamma(\ell/2)} x^{\ell/2-1} e^{-x/2}. \]

Lemma 2 holds obviously when \( k = 1 \) (note that \( g_{ii} \overset{d}{=} \mathcal{N}(0, 2) \)), and thus in the followings we consider the case when \( k \geq 2 \). By the definition of the principal minor, \( G_{1,\ldots,k} = (g_{ij})_{1 \leq i,j \leq k} \) is also sampled from GOE. Now let \( \tilde{g}_{ij} = g_{ij} \) if \( j \neq i \) and \( g_{ij}/\sqrt{2} \) otherwise, then \( \tilde{g}_{ij}, 1 \leq i \leq j \leq k \) are i.i.d. \( \mathcal{N}(0, 1) \) random variables.

To prove (10), we note that by definition,

\[ |G_{1,\ldots,k}|^2 = \text{Tr}(G_{1,\ldots,k}^2) = \sum_{i,j=1}^{k} g_{ij}^2 = 2 \sum_{1 \leq i \leq j \leq k} \tilde{g}_{ij}^2, \]

therefore, by (30) and (32) we have

\[ \mathbb{P}(|G_{1,\ldots,k}|^2 > x^2) = \mathbb{P}\left( \sum_{1 \leq i \leq j \leq k} \tilde{g}_{ij}^2 > x^2/2 \right) \leq C x^{k(k+1)/2-2} e^{-x^2/4}. \]

This proves (10). To prove (11), by formula (1), the joint density of eigenvalues \( \lambda_1 \geq \cdots \geq \lambda_k \) of \( G_{1,\ldots,k} \) is

\[ J_k(\lambda_1, \ldots, \lambda_k) = \frac{1}{Z_k} \prod_{1 \leq i < j \leq k} (\lambda_i - \lambda_j) \exp\left( \frac{\sum_{j=1}^{k} \lambda_j^2}{4} \right). \]
Note that if \( \lambda_1 > x > 1 \), we have \( \lambda_1 - \lambda_j \leq (\lambda_1 + 1)(|\lambda_j| + 1) \leq 2\lambda_1(|\lambda_j| + 1) \), and thus we have

\[
\prod_{1 \leq i < j \leq k} (\lambda_i - \lambda_j) \leq C\lambda_i^{k-1} \prod_{2 \leq i \leq k} (|\lambda_i| + 1) \prod_{2 \leq i < j \leq k} (\lambda_i - \lambda_j),
\]

which further implies

\[
\int_{\lambda_1 > x} J_k(\lambda_1, \ldots, \lambda_k)d\lambda_1 \cdots d\lambda_k \leq C \int_{x}^{\infty} \lambda_1^{k-1} \exp(-\lambda_1^2/4)d\lambda_1 \\
\times \int_{\infty > \lambda_2 > \cdots > \lambda_k > x} \prod_{2 \leq i \leq k} (|\lambda_i| + 1) \prod_{2 \leq i < j \leq k} (\lambda_i - \lambda_j) \exp \left( -\frac{\sum_{i=2}^{k} \lambda_i^2}{4} \right) d\lambda_2 \cdots d\lambda_k,
\]

which is bounded from above by \( C' x^{k-2} \exp(-x^2/4) \) by (34), thereby proving (11).

Now we prove (12). By definition of \( \lambda_1^*(G_1, \ldots, k) \), we have

\[
P(\lambda_1^*(G_1, \ldots, k) > x) \\
= P \left( \sum_{i=2}^{k} \lambda_i^2 + 2\lambda_1^2 > 2x^2 \right) \\
\leq \sum_{y=0}^{\lfloor \sqrt{2x} + 1 \rfloor} P \left( 2\lambda_i^2 > (2x^2 - (y + 1)^2) \right) \leq 1 + \sum_{i=2}^{k} \lambda_i^2 \leq (y + 1)^2 \)

\[
= P \left( \sum_{i=2}^{k} \lambda_i^2 > 2x^2 \right)
\]

\[
:= I_1 + I_2.
\]

By the fact \( \lambda_1 - \lambda_j \leq (|\lambda_1| + 1)(|\lambda_j| + 1) \) for all \( 2 \leq j \leq k \), we first have

\[
\prod_{1 \leq i < j \leq k} (\lambda_i - \lambda_j) \leq C(|\lambda_1| + 1)^{k-1} \prod_{2 \leq i \leq k} (|\lambda_i| + 1) \prod_{2 \leq i < j \leq k} (\lambda_i - \lambda_j).
\]

Then by the inequality of arithmetic and geometric means, for \( \sum_{i=2}^{k} \lambda_i^2 \geq 1 \), we have

\[
\prod_{2 \leq i \leq k} (|\lambda_i| + 1) \prod_{2 \leq i < j \leq k} (\lambda_i - \lambda_j) \leq \prod_{2 \leq i \leq k} (|\lambda_i| + 1) \prod_{2 \leq i < j \leq k} (\lambda_i + |\lambda_j|)
\]

\[
\leq \left( \frac{1 + \sqrt{k-1} \sum_{i=2}^{k} \lambda_i^2}{k/2} \right)^{k-1} \leq C \left( \frac{\sum_{i=2}^{k} \lambda_i^2}{k} \right)^{k(k-1)/4}.
\]

Therefore, for \( x > 1 \), combining (34) and (35), we can bound \( I_2 \) as follows

\[
\int_{\sum_{i=2}^{k} \lambda_i^2 > 2x^2} J_k(\lambda_1, \ldots, \lambda_k)d\lambda_1 \cdots d\lambda_k \leq C \int_{0}^{\infty} (|\lambda_1| + 1)^{k-1} \exp(-\lambda_1^2/4)d\lambda_1 \\
\times \int_{\sum_{i=2}^{k} \lambda_i^2 > 2x^2} \left( \sum_{i=2}^{k} \lambda_i^2 \right)^{k(k-1)/4} \exp \left( -\frac{\sum_{i=2}^{k} \lambda_i^2}{4} \right) d\lambda_2 \cdots d\lambda_k.
\]
The first integral is bounded. Using the polar coordinate to the second integral, and by (31) we have the bound

\[
I_2 \leq C \int_{r > \sqrt{2}x} r^{k(k-1)/2} \exp(-r^2/4) r^{k-2} dr \leq C x^{k(k+1)/2 - 3} \exp(-x^2/2),
\]
which can be further bounded by \(C' x^{k-2} \exp(-x^2/4)\) for \(x > 1\) by choosing \(C'\) large enough.

Now we estimate \(I_1\) for \(x > 1\). For the case \(x^2 - (y + 1)^2/2 > 1\), by (34) and (43), we have

\[
P \left( 2\lambda_1^2 > (2x^2 - (y + 1)^2)_+, \ y^2 \leq \sum_{i=2}^k \lambda_i^2 \leq (y + 1)^2 \right)
\]

\[
\leq C \int_{2\lambda_1^2 > x^2 - (y + 1)^2/2 > 1} |\lambda_1|^{k-1} \exp(-\lambda_1^2/4) d\lambda_1
\]

\[
\times \int_{\lambda_2 > \cdots > \lambda_k > (y + 1)^2/2 > \sum_{i=2}^k \lambda_i^2 \geq y^2} \prod_{2 \leq i \leq k} (|\lambda_i| + 1) \prod_{2 \leq i < j \leq k} (\lambda_i - \lambda_j) \exp \left( - \sum_{i=2}^k \lambda_i^2/4 \right) d\lambda_2 \cdots d\lambda_k
\]

\[
\leq C x^{k-2} \exp(-x^2 - (y + 1)^2/2)/4)
\]

\[
\times \int_{R^{k-1}: (y + 1)^2/2 > \sum_{i=2}^k \lambda_i^2 \geq y^2} \prod_{2 \leq i \leq k} (|\lambda_i| + 1) \prod_{2 \leq i < j \leq k} (|\lambda_i| + |\lambda_j|) \exp \left( - \sum_{i=2}^k \lambda_i^2/4 \right) d\lambda_2 \cdots d\lambda_k.
\]

We denote the last integral as

\[
A_k(y) := \int_{R^{k-1}: (y + 1)^2/2 > \sum_{i=2}^k \lambda_i^2 \geq y^2} \prod_{2 \leq i \leq k} (|\lambda_i| + 1) \prod_{2 \leq i < j \leq k} (|\lambda_i| + |\lambda_j|) \exp \left( - \sum_{i=2}^k \lambda_i^2/4 \right) d\lambda_2 \cdots d\lambda_k.
\]

For the case \(x^2 - (y + 1)^2/2 \leq 1\), by (34) and the arguments as above, we simply have

\[
P \left( 2\lambda_1^2 > (2x^2 - (y + 1)^2)_+, \ y^2 \leq \sum_{i=2}^k \lambda_i^2 \leq (y + 1)^2 \right)
\]

\[
\leq C \left( \int_{R} (|\lambda_1| + 1)^{k-1} \exp(-\lambda_1^2/4) d\lambda_1 \right) A_k(y)
\]

\[
\leq C A_k(y)
\]

\[
\leq C x^{k-2} \exp(-(x^2 - (y + 1)^2/2)/4) A_k(y)
\]

for \(x > 1\). Therefore, in both cases, by the polar coordinate, we further have the upper bound,

\[
P \left( 2\lambda_1^2 > (2x^2 - (y + 1)^2)_+, \ y^2 \leq \sum_{i=2}^k \lambda_i^2 \leq (y + 1)^2 \right)
\]

\[
\leq C x^{k-2} e^{-x^2/4} \int_{y + 1 > r > y} (1 + r)^{k-1} r^{k-1}(k-2)/2 r^{k-2} \exp((r + 1)^2/8 - r^2/4) dr.
\]
By taking the summation, $I_1$ can be bounded from above by

$$I_1 \leq Cx^{k-2} \exp(-x^2/4) \int_0^{[\sqrt{2x}] + 1} (1 + r)^{k-1} (k-1) \exp((r + 1)^2/8 - r^2/4) dr$$

$$\leq Cx^{k-2} \exp(-x^2/4) \int_0^{+\infty} \cdots = C' x^{k-2} \exp(-x^2/4).$$

This will complete the proof of (12) by the estimates of $I_1$ and $I_2$. $\square$

### 3.2. Proof of Lemma 3

**Proof.** We first prove (17). We claim that (17) is equivalent to the following statement: for any $\delta > 0$, $x > 1$, there exists a constant $C$ depending on $\delta$, such that

$$\mathbb{P}(\lambda_1(G_\alpha) > x|G_\beta, G_\gamma) 1_{\{\lambda_1(G_\beta) \leq (1-\delta)x, \lambda_1(G_\gamma) \leq (1-\delta)x\}}$$

$$\leq C x^{-1} (x/(\lambda_1^2(G_\beta) + 1) + x/(\lambda_1^2(G_\gamma) + 1))^{k(m-k)-1} e^{-((x-\lambda_1^2(G_\beta))((x-\lambda_1^2(G_\gamma))/2)$$

The implication that $17 \Rightarrow 37$ is trivial. We now show that $37$ implies $17$. For any $\delta, \delta' \in (0,1)$, we define $\delta = \min\{\delta, \delta'\}$. $37$ implies that there exists a constant $C(\tilde{\delta})$ such that

$$\mathbb{P}(\lambda_1(G_\alpha) > x|G_\beta, G_\gamma) 1_{\{\lambda_1^2(G_\beta) \leq (1-\tilde{\delta})x, \lambda_1^2(G_\gamma) \leq (1-\tilde{\delta})x\}}$$

$$\leq C x^{-1} (x/(\lambda_1^2(G_\beta) + 1) + x/(\lambda_1^2(G_\gamma) + 1))^{k(m-k)-1} e^{-((x-\lambda_1^2(G_\beta))((x-\lambda_1^2(G_\gamma))/2)$$

implies $17$ since

$$\left\{\lambda_1^2(G_\beta) \leq (1-\delta)x, \lambda_1^2(G_\gamma) \leq (1-\delta)x\right\} \subset \left\{\lambda_1^2(G_\beta) \leq (1-\tilde{\delta})x, \lambda_1^2(G_\gamma) \leq (1-\tilde{\delta})x\right\}.$$

This completes the proof of the equivalence between $17$ and $37$. We now prove $37$. Without loss of generality, we may assume

$$\alpha = \{1, \ldots, m\}, \gamma = \{1, \ldots, k\}, \beta = \{k+1, \ldots, m\}.$$

Since $G_\beta$ and $G_\gamma$ are both symmetric matrices sampled from GOE (independently), we can find orthogonal matrices $U$ and $U'$ such that

$$UG_\beta U^t = X, \quad U'G_\gamma U'^t = Z,$$

where the diagonal matrices

$$X = \begin{pmatrix} \lambda_1(G_\beta) \\ \lambda_2(G_\beta) \\ \ddots \\ \lambda_\ell(G_\beta) \end{pmatrix}, \quad Z = \begin{pmatrix} \lambda_1(G_\gamma) \\ \lambda_2(G_\gamma) \\ \ddots \\ \lambda_\ell(G_\gamma) \end{pmatrix},$$

where $\ell := m - k$.

It follows that

$$\begin{pmatrix} U \\ U' \end{pmatrix} G_\alpha \begin{pmatrix} U^t \\ U'^t \end{pmatrix} = \begin{pmatrix} X & V \\ V^t & Z \end{pmatrix},$$

where $V$ is an $\ell \times k$ matrix with i.i.d. $N_\mathbb{R}(0,1)$ entries.

Given any $1 \times m$ vector $v$, we decompose it as

$$v = (p, q), \quad p = (p_1, \ldots, p_\ell), \quad q = (q_1, \ldots, q_k),$$
then we have

\[
\begin{bmatrix} X & V & i \\ i V & Z \end{bmatrix} v' = \sum_{i=1}^{\ell} \lambda_i(G_\beta)p_i^2 + \sum_{j=1}^{k} \lambda_j(G_\gamma)q_j^2 + 2 \sum_{i=1}^{\ell} \sum_{j=1}^{k} p_i v_{ij}q_j.
\]

For simplicity, we use \( P^* \) to denote the conditional probability (conditional on \( G_\beta \) and \( G_\gamma \)). By Rayleigh quotient, for \( x > 1 \), we have

\[
P^*(\lambda_1(G_\alpha) > x) \leq \max\{\lambda_1(G_\beta), \lambda_1(G_\gamma)\}.
\]

We define the event

\[
\Omega := \left\{ \lambda_1^*(G_\beta) \leq (1 - \delta)x, \lambda_1^*(G_\gamma) \leq (1 - \delta)x \right\}.
\]

For the rest of the proof, all the arguments are restricted on the event \( \Omega \) and \( x > 1 \).

On \( \Omega \), by the definition \( \lambda^*(A) \) for any symmetric matrix \( A \) in \( \mathbb{F} \), we simply have

\[
x + \sqrt{2}(1 - \delta)x \leq x - \lambda_i(G_\beta) \geq x - \lambda_1^*(G_\beta) \geq \delta x > 0 \quad \text{and} \quad x + \sqrt{2}(1 - \delta)x \leq x - \lambda_j(G_\gamma) \geq x - \lambda_1^*(G_\gamma) \geq \delta x > 0.
\]

Therefore, on \( \Omega \) it holds that

\[(39) \quad z_{ij} := (x - \lambda_i(G_\beta))(x - \lambda_j(G_\gamma)) \in [\delta^2x^2, (\sqrt{2} - \sqrt{2}\delta + 1)^2x^2].\]

We denote

\[
\bar{p}_i = \sqrt{x - \lambda_i(G_\beta)}p_i, \quad \bar{q}_j = \sqrt{x - \lambda_j(G_\gamma)}q_j, \quad \bar{v}_{ij} = \frac{v_{ij}}{\sqrt{z_{ij}}}.
\]

Using Cauchy-Schwartz inequality we have

\[
2 \sum_{i=1}^{\ell} \sum_{j=1}^{k} \bar{p}_i \bar{v}_{ij} \bar{q}_j \leq 2 \left( \sum_{i=1}^{\ell} \bar{p}_i^2 \right) \left( \sum_{i=1}^{\ell} \sum_{j=1}^{k} \bar{v}_{ij} \bar{q}_j \right)^2 \leq 2 \left( \sum_{i=1}^{\ell} \bar{p}_i^2 \right) \left( \sum_{j=1}^{k} \bar{q}_j^2 \right) \left( \sum_{i=1}^{\ell} \sum_{j=1}^{k} \bar{v}_{ij}^2 \right) \leq \sqrt{\sum_{i=1}^{\ell} \sum_{j=1}^{k} \bar{v}_{ij}^2} \left( \sum_{i=1}^{\ell} \bar{p}_i^2 + \sum_{j=1}^{k} \bar{q}_j^2 \right).
\]
Therefore, we further have the probabilistic estimate of $\lambda_1(G_\alpha)$ as

$$P^*(\lambda_1(G_\alpha) > x)$$

$$= P^* \left( \exists(\bar{p}, \bar{q}) \neq 0, 2 \sum_{i=1}^{\ell} \sum_{j=1}^{k} \bar{p}_i \bar{q}_j \geq \sum_{i=1}^{\ell} \bar{p}_i^2 + \sum_{j=1}^{k} \bar{q}_j^2 \right)$$

$$\leq P^* \left( \sum_{i=1}^{\ell} \sum_{j=1}^{k} \bar{v}_{ij}^2 \geq 1 \right).$$

If $k = \ell = 1$, on the event $\Omega$, by (31) for Gaussian random variable $v_{11}$ we have

$$P^*(\bar{v}_{11}^2 \geq 1) = P^*(v_{11}^2 > z_{11}) \leq \frac{C}{x} \exp(-z_{11}/2)$$

$$\leq \frac{C}{x} \exp(-(x - \lambda_1(G_\beta))(x - \lambda_1(G_\gamma))/2)$$

$$\leq \frac{C}{x} \exp(-(x - \lambda_1^*(G_\beta))(x - \lambda_1^*(G_\gamma))/2),$$

where we have used the fact that $z_{11} \geq \delta^2 x^2$. This will imply (37) in the case $m = 2, k = \ell = 1$. Now we consider the case $k \geq 2$ or $\ell \geq 2$. We define

$$\kappa = \begin{cases} 
\min \left\{ \frac{\lambda_1(G_\alpha) - \lambda_2(G_\beta)}{x - \lambda_2(G_\beta)}, \frac{\lambda_1(G_\gamma) - \lambda_2(G_\beta)}{x - \lambda_2(G_\beta)} \right\} & \text{if } \ell, k \geq 2; \\
\frac{\lambda_1(G_\alpha) - \lambda_2(G_\beta)}{x - \lambda_2(G_\beta)} & \text{if } k = 1, \ell \geq 2; \\
\frac{\lambda_1(G_\gamma) - \lambda_2(G_\beta)}{x - \lambda_2(G_\beta)} & \text{if } \ell = 1, k \geq 2.
\end{cases}$$

On the event $\Omega$, one can show that

$$0 \leq \kappa \leq \frac{(1 - \delta)x - \lambda_2(G_\beta)}{x - \lambda_2(G_\beta)} \leq \frac{(1 - \delta)x - (-\sqrt{2}(1 - \delta)x)}{x - (-\sqrt{2}(1 - \delta)x)} = \frac{(1 + \sqrt{2})(1 - \delta)}{1 + \sqrt{2}(1 - \delta)} := c_\delta < 1.$$

Note that

$$\sum_{i=1}^{\ell} \sum_{j=1}^{k} \bar{v}_{ij}^2 = \sum_{i=1}^{\ell} \sum_{j=1}^{k} \frac{v_{ij}^2}{(x - \lambda_i(G_\beta))(x - \lambda_j(G_\gamma))}$$

$$= \frac{1}{(x - \lambda_1(G_\beta))(x - \lambda_1(G_\gamma))} \left( v_{11}^2 + \sum_{(i,j) \neq (1,1)} \frac{(x - \lambda_1(G_\beta))(x - \lambda_1(G_\gamma))}{(x - \lambda_i(G_\beta))(x - \lambda_j(G_\gamma))} v_{ij}^2 \right)$$

$$\leq \frac{1}{(x - \lambda_1(G_\beta))(x - \lambda_1(G_\gamma))} \left( v_{11}^2 + (1 - \kappa) \sum_{(i,j) \neq (1,1)} v_{ij}^2 \right),$$

where we used the fact that

$$\frac{(x - \lambda_1(G_\beta))(x - \lambda_1(G_\gamma))}{(x - \lambda_1(G_\beta))(x - \lambda_1(G_\gamma))} \leq \max \left\{ \frac{x - \lambda_1(G_\beta)}{x - \lambda_2(G_\beta)}, \frac{x - \lambda_1(G_\gamma)}{x - \lambda_2(G_\gamma)} \right\}$$

$$= 1 - \min \left\{ \frac{\lambda_1(G_\beta) - \lambda_2(G_\beta)}{x - \lambda_2(G_\beta)}, \frac{\lambda_1(G_\gamma) - \lambda_2(G_\gamma)}{x - \lambda_2(G_\gamma)} \right\}.$$
for $(i, j) \neq (1, 1)$. Hence, by (32) we can conclude the following bounds on the event $\Omega$

$$
\mathbb{P}^*\left(\lambda_1(G_\alpha) > x \right) \\
\leq \mathbb{P}^*\left( \sum_{i=1}^{\ell} \sum_{j=1}^{k} \tilde{v}_{ij}^2 \geq 1 \right) \\
\leq \mathbb{P}^*\left( \sum_{i=1}^{\ell} \sum_{j=1}^{k} \left(1 - \kappa \right) v_{ij}^2 \geq z_{11} \right) \\
\leq \mathbb{P}^*\left( \sum_{i=1}^{\ell} \sum_{j=1}^{k} v_{ij}^2 \geq z_{11} \right) \\
\leq C x^{k\ell/2 - 1} \exp(-z_{11}/2) \\
\leq C x^{k\ell/2 - 2} \exp(-(x - \lambda_1(G_\beta))(x - \lambda_1(G_\gamma))/2),
$$

where in the last step we used (37), and $C$ is a constant depending on $\delta$ and $m$.

We now consider the case when one of the following two conditions holds.

Condition 1: $\lambda_1'(G_\beta) \leq 1$ or $\lambda_1'(G_\gamma) \leq 1$. Under this condition, since $\lambda_1(G_\beta) \leq \lambda_1'(G_\beta)$ and $\lambda_1(G_\gamma) \leq \lambda_1'(G_\gamma)$, we have

$$
\mathbb{P}^*\left( \sum_{i=1}^{\ell} \sum_{j=1}^{k} \tilde{v}_{ij}^2 \geq 1 \right) \leq \exp(\lambda_1'(G_\beta) - \lambda_1(G_\beta)) \leq \exp(\lambda_1'(G_\gamma) - \lambda_1(G_\gamma)),
$$

which further yields the bound (31) by the assumption $\lambda_1'(G_\beta) \leq 1$ or $\lambda_1'(G_\gamma) \leq 1$.

Condition 2: max\{\{|\lambda_i(G_\beta)|, 2 \leq i \leq \ell\} > |\lambda_1(G_\beta)|/2 or max\{|\lambda_i(G_\gamma)|, 2 \leq i \leq k\} > |\lambda_1(G_\gamma)|/2. If condition 1 fails (i.e., $\lambda_1'(G_\beta) > 1$ and $\lambda_1'(G_\gamma) > 1$) and condition 2 holds (say, max\{\{|\lambda_i(G_\beta)|, 2 \leq i \leq \ell\} > |\lambda_1(G_\beta)|/2), then we have

$$
\lambda_1'(G_\beta) - \lambda_1(G_\beta) = \frac{2 \sum_{i=2}^{\ell} \lambda_i^2(G_\beta)}{\lambda_1^2(G_\beta) + \lambda_1(G_\beta)} \geq C \left( \frac{\lambda_1'(G_\beta)}{4 \lambda_1'(G_\beta)} \right)^2 = C' \lambda_1'(G_\beta).
$$

In this case, on $\Omega$ we have

$$
(x - \lambda_1'(G_\beta))(x - \lambda_1'(G_\gamma)) \leq (x - \lambda_1(G_\beta))(x - \lambda_1(G_\gamma)) - (\lambda_1'(G_\beta) - \lambda_1'(G_\gamma)) \delta x \\
\leq (x - \lambda_1(G_\beta))(x - \lambda_1(G_\gamma)) - C' \lambda_1'(G_\beta) \delta x \\
\leq (x - \lambda_1(G_\beta))(x - \lambda_1(G_\gamma)) - C' \delta x.
$$

Using (42), we have

$$
\mathbb{P}^*(\lambda_1(G_\alpha) > x) \leq C x^{k\ell/2 - 2} \exp(-\left( (x - \lambda_1'(G_\beta))(x - \lambda_1'(G_\gamma)) / 2 \right)) \exp(-C' \delta x).
$$

This can be further bounded above from above by

$$
C x^{-1} \left( \frac{x}{1 + \lambda_1'(G_\beta)} + \frac{x}{1 + \lambda_1'(G_\gamma)} \right)^{k\ell - 1} \exp(-(x - \lambda_1'(G_\beta))(x - \lambda_1'(G_\gamma))/2),
$$

this is because on $\Omega$ it holds

$$
\exp(-C' \delta x) \leq C \left( \frac{1}{1 + (1 - \delta)x} \right)^{k\ell - 1} \leq C \left( \frac{1}{1 + \lambda_1'(G_\beta)} + \frac{1}{1 + \lambda_1'(G_\gamma)} \right)^{k\ell - 1}
$$
for $x > 1$ by choosing $C$ large enough. Note that the constant $C > 0$ only depends on $m$ and $\delta$, and does not depend on (the conditional) $\lambda_1^*(G_\beta)$ and $\lambda_1^*(G_\gamma)$.

As a summary, we have verified \((44)\) when either of the two conditions is satisfied. Now we assume that both conditions fail so that $\lambda_1^*(G_\beta) > 1$, $\lambda_1^*(G_\gamma) > 1$, $\max(|\lambda_i(G_\beta)|, 2 \leq i \leq \ell) \leq |\lambda_1(G_\beta)|/2$ and $\max(|\lambda_i(G_\gamma)|, 2 \leq i \leq k) \leq |\lambda_1(G_\gamma)|/2$. Then there exists a constant $c > 0$ such that

$$\lambda_1(G_\beta) - \lambda_2(G_\beta) \geq c\lambda_1^*(G_\beta) \geq c \quad \text{and} \quad \lambda_1(G_\gamma) - \lambda_2(G_\gamma) \geq c\lambda_1^*(G_\gamma) \geq c.$$ 

Therefore, by definition of $\kappa$, \((41)\) and the assumptions $\lambda_1^*(G_\beta) > 1$, $\lambda_1^*(G_\gamma) > 1$, we have

\begin{equation}
1 > c\delta \geq \kappa \geq c \min\left\{ \frac{\lambda_1^*(G_\beta)}{x - \lambda_2(G_\beta)}, \frac{\lambda_1^*(G_\gamma)}{x - \lambda_2(G_\gamma)} \right\} = \frac{c}{2} \min\left\{ \frac{\lambda_1^*(G_\beta) + 1}{x - \lambda_2(G_\beta)}, \frac{\lambda_1^*(G_\gamma) + 1}{x - \lambda_2(G_\gamma)} \right\}. 
\end{equation}

By the fact that $0 < \delta x \leq x - \lambda_2(G_\beta), x - \lambda_2(G_\gamma) \leq x + \sqrt{2}(1 - \delta)x$ on $\Omega$, we further have

\begin{equation}
c\delta \geq \kappa \geq C\delta \min\left\{ \frac{\lambda_1^*(G_\beta) + 1}{x}, \frac{\lambda_1^*(G_\gamma) + 1}{x} \right\}, 
\end{equation}

which implies

\begin{equation}
\frac{1}{\kappa} \leq Cx \left( \frac{1}{\lambda_1^*(G_\beta) + 1} + \frac{1}{\lambda_1^*(G_\gamma) + 1} \right)
\end{equation}

for some constant $C > 0$ that depends on $m$ and $\delta$. Note that the probability density function $p(t)$ for $v_{11}^2$ is $\exp(-t/2)/\sqrt{2\pi t}$, recall \((22)\), we have

$$\mathbb{P}^* \left( \sum_{i=1}^{t} \sum_{j=1}^{k} \bar{v}_{ij}^2 \geq 1 \right) \leq \mathbb{P}^* \left( v_{11}^2 + (1 - \kappa) \sum_{(i,j) \neq (1,1)} v_{ij}^2 > z_{11} \right)$$

$$= \int_{0}^{z_{11}} \frac{\exp(-(z_{11} - t)/2)}{\sqrt{2\pi(z_{11} - t)}} \mathbb{P}^* \left( \left( 1 - \kappa \right) \sum_{(i,j) \neq (1,1)} v_{ij}^2 > t \right) dt + \mathbb{P}^* (v_{11}^2 > z_{11})$$

$$\leq C \exp(-z_{11}/2) \left( \frac{1}{\sqrt{z_{11}}} + \int_{0}^{z_{11}} \frac{\exp(t/2)}{\sqrt{z_{11} - t}} \mathbb{P}^* \left( \left( 1 - \kappa \right) \sum_{(i,j) \neq (1,1)} v_{ij}^2 > t \right) dt \right),$$

where we have used $\mathbb{P}^*(v_{11}^2 > z_{11}) \leq \frac{C_{\kappa} v_{11}^{\kappa-1/2} - 1}{\sqrt{z_{11}}}$ by \((32)\). By \((32)\) again, we have

$$\mathbb{P}^* \left( \left( 1 - \kappa \right) \sum_{(i,j) \neq (1,1)} v_{ij}^2 > t \right) \leq C \left( \frac{t}{1 - \kappa} \right)^{(k\ell - 1)/2 - 1} \exp \left( - \frac{t}{2(1 - \kappa)} \right)$$

$$\leq C t^{(k\ell - 1)/2 - 1} \exp \left( - \frac{t}{2(1 - \kappa)} \right),$$
where in the last inequality we used the fact that $1 - \kappa \geq 1 - c_\delta > 0$ by (31). It follows that

$$P^* \left( \sum_{i=1}^{\ell} \sum_{j=1}^{k} \bar{v}_{ij}^2 \geq 1 \right)$$

$$\leq C \exp(-z_{11}/2) \left( \frac{1}{\sqrt{z_{11}}} + \int_{0}^{z_{11}} \frac{t^{(k\ell-3)/2}}{\sqrt{z_{11} - t}} \exp \left( \frac{t}{2} - \frac{t}{2(1 - \kappa)} \right) dt \right)$$

$$\leq C' \exp(-z_{11}/2) \left( \frac{1}{\sqrt{z_{11}}} + \frac{1}{\sqrt{z_{11}}} \int_{0}^{z_{11}/2} t^{(k\ell-3)/2} \exp \left( -\frac{kt}{2(1 - \kappa)} \right) dt \right. \left. + \frac{(k\ell-3)/2}{\sqrt{z_{11}}} \exp \left( -\frac{\kappa z_{11}}{4(1 - \kappa)} \right) \right) \int_{z_{11}/2}^{z_{11}} \frac{1}{\sqrt{z_{11} - t}} dt \right)$$

$$\leq C' \exp(-z_{11}/2) \left( 1 + \left( \frac{1}{\kappa} \right)^{(k\ell-1)/2} \int_{0}^{\infty} s^{(k\ell-3)/2} \exp(-s) ds \right. \left. + \frac{(k\ell-1)/2}{\sqrt{z_{11}}} \exp \left( -\frac{\kappa z_{11}}{4} \right) \right).$$

Note that the integration of $s^{(k\ell-3)/2} \exp(-s)$ converges since $k\ell \geq 2$, and thus the second term can be bounded from above by $c(1/\kappa)^{(k\ell-1)/2}$. For the third term, the global maxima of the function $x^{(k\ell-1)/2} e^{-\kappa x/4}$ is obtained at the point $x = c'/\kappa$, here $c'$ depends on $k, \ell$, and thus the third term can be bounded from above by $C(1/\kappa)^{(k\ell-1)/2}$, where $C$ depends on $\delta$ and $m$. Therefore, on $\Omega$ we have

$$P^* \left( \sum_{i=1}^{\ell} \sum_{j=1}^{k} \bar{v}_{ij}^2 \geq 1 \right)$$

$$\leq C' \exp(-z_{11}/2) \left( 1 + c(1/\kappa)^{(k\ell-1)/2} + C(1/\kappa)^{(k\ell-1)/2} \right)$$

$$\leq C' \exp(-z_{11}/2) \left( (1/\kappa)^{(k\ell-1)/2} + c(1/\kappa)^{(k\ell-1)/2} + C(1/\kappa)^{(k\ell-1)/2} \right) \text{ [since } 1 < 1/\kappa\text{]}$$

$$= C' \exp(-z_{11}/2) \left( 1/\kappa \right)^{(k\ell-1)/2}$$

$$\leq C' \exp(-z_{11}/2) \left( 1/\kappa \right)^{k\ell-1}$$

$$\leq C \exp(-z_{11}/2) \left( \frac{x}{\lambda_1^*(G_\beta) + 1} + \frac{x}{\lambda_1^*(G_\gamma) + 1} \right)^{k\ell-1} \exp(-\left( x - \lambda_1^*(G_\beta) \right) (x - \lambda_1^*(G_\gamma) / 2)) \text{ [by (39), (40)]}.$$}

This proves (37), and thus we complete the proof of (17).

We now prove (18) using (12) and (17). By convexity of the function $s \to s^{(k(m-k)-1)/2}$ for $s > 0$, we have

$$x^{-1} \left( x / (\lambda_1^*(G_\beta) + 1) + x / (\lambda_1^*(G_\gamma) + 1) \right)^{k(m-k)-1} e^{-\left( x - \lambda_1^*(G_\beta) \right) (x - \lambda_1^*(G_\gamma) / 2)}$$

$$\leq C x^{-1} \left( x / (\lambda_1^*(G_\beta) + 1) \right)^{k(m-k)-1} e^{-\left( x - \lambda_1^*(G_\beta) \right) (x - \lambda_1^*(G_\gamma) / 2)}$$

$$+ C x^{-1} \left( x / (\lambda_1^*(G_\gamma) + 1) \right)^{k(m-k)-1} e^{-\left( x - \lambda_1^*(G_\beta) \right) (x - \lambda_1^*(G_\gamma) / 2)}$$

$$:= I_3 + I_4.$$
Let $\bar{E}$ be the conditional expectation with respect to $G_\gamma$ and $\ell := m - k$ as before, and we define
\[
f(t) = x^{-1}(x/(t + 1))^{\ell t - 1} \exp(-(x - t)(x - \lambda_1^*(G_\gamma))/2)
\]
and
\[
h(t) = x^{-1}(x/(\lambda_1^*(G_\gamma) + 1))^{\ell t - 1} \exp(-(x - t)(x - \lambda_1^*(G_\gamma))/2).
\]
We define the event
\[
\Omega' = \{\lambda_1^*(G_\gamma) \leq (1 - \delta')x\}.
\]
Using (17) and (47), we have
\[
\mathbb{E}(\frac{\lambda_1(G_\alpha)}{x}, \lambda_1^*(G_\beta) \leq (1 - \delta)x|G_\gamma) \mathbb{1}_{\{\lambda_1^*(G_\gamma) \leq (1 - \delta')x\}}
\]
\[
\mathbb{E}(E \{\lambda_1(G_\alpha) > x, \lambda_1^*(G_\beta) < (1 - \delta)x\} \mathbb{1}_{\{\lambda_1^*(G_\gamma) \leq (1 - \delta')x\}})
\]
\[
\mathbb{E}(E \{\lambda_1(G_\alpha) > x\} \mathbb{1}_{\{\lambda_1^*(G_\beta) < (1 - \delta)x\}} \mathbb{1}_{\{\lambda_1^*(G_\gamma) \leq (1 - \delta')x\}})
\]
\[
\leq C \mathbb{E}(x^{-1}(x/(\lambda_1^*(G_\beta) + 1) + x/(\lambda_1^*(G_\gamma) + 1))^\ell t - 1)
\]
\[
\exp(-(x - \lambda_1^*(G_\beta))(x - \lambda_1^*(G_\gamma))/2) \mathbb{1}_{\{\lambda_1^*(G_\beta) \leq (1 - \delta)x\}} \mathbb{1}_{\{\lambda_1^*(G_\gamma) \leq (1 - \delta)\}}
\]
\[
\leq C \mathbb{E}(I_3 \mathbb{1}_{\{\lambda_1^*(G_\beta) \leq (1 - \delta)x\}}) \mathbb{I}_{\Omega'} + C \mathbb{E}(I_4 \mathbb{1}_{\{\lambda_1^*(G_\beta) \leq (1 - \delta)x\}}) \mathbb{I}_{\Omega'}
\]
\[
\leq C \left( \mathbb{E}(f(\lambda_1^*(G_\beta)) \mathbb{1}_{\{\lambda_1^*(G_\beta) < (1 - \delta)x\}} \mathbb{I}_{\Omega'} + \mathbb{E}(h(\lambda_1^*(G_\beta)) \mathbb{1}_{\{\lambda_1^*(G_\beta) < (1 - \delta)x\}}) \mathbb{I}_{\Omega'} \right).
\]
Note that $G_\beta$ and $G_\gamma$ are independent, therefore, $\lambda_1^*(G_\beta)$ and $\lambda_1^*(G_\gamma)$ are independent as well. Hence for any $t > 0$, $\mathbb{P}(\lambda_1^*(G_\beta) > t)$ is equal to unconditional probability $\mathbb{P}(\lambda_1^*(G_\beta) > t)$. Then using integration by parts, we have
\[
\mathbb{E}(f(\lambda_1^*(G_\beta)) \mathbb{1}_{\{\lambda_1^*(G_\beta) < (1 - \delta)x\}} = \int_0^{(1 - \delta)x} f'(t) \mathbb{P}(\lambda_1^*(G_\beta) > t) dt
\]
\[
- f((1 - \delta)x) \mathbb{P}(\lambda_1^*(G_\beta) > (1 - \delta)x) + f(0).
\]
On the event $\Omega' = \{\lambda_1^*(G_\gamma) \leq (1 - \delta')x\}$, one has
\[
f'(t) \leq \frac{1}{2} x/(t + 1)^{\ell t - 1} \exp(-(x - t)(x - \lambda_1^*(G_\gamma))/2).
\]
Therefore, we have
\[
\int_0^1 f'(t) \mathbb{P}(\lambda_1^*(G_\beta) > t) dt \leq \frac{1}{2} x^{\ell t - 1} \exp(-(x - 1)(x - \lambda_1^*(G_\gamma))/2)
\]
\[
= \frac{1}{2} x^{\ell t - 1} e^{1/4} \exp((\lambda_1^*(G_\gamma)^2 - x^2)/4) \exp(-((x - \lambda_1^*(G_\gamma))^2 - 1/4)/4).
\]
On the event $\Omega'$, we have $x - \lambda_1^*(G_\gamma) \geq \delta'x$. Therefore, for $C$ large enough, one has
\[
\exp(-((x - \lambda_1^*(G_\gamma))^2 - 1/4)/4) \leq C \exp(-\delta'^2 x^2/8).
\]
It further follows from (50) and (51) that for $x > 1$, there exists $C$ large enough such that
\[
\int_0^1 f'(t) \mathbb{P}(\lambda_1^*(G_\beta) > t) dt \leq C x^{\ell - 2} \exp(((\lambda_1^*(G_\gamma))^2 - x^2)/4).
\]
If \((1 - \delta)x \leq 1\), then we trivially have
\[
\int_0^{(1 - \delta)x} f'(t) \mathbb{P}(\lambda_1^*(G_\beta) > t) dt \leq \int_0^1 \cdots \leq Cx^{k-2} \exp((\lambda_1^*(G_\gamma))^2 - x^2)/4).
\]
Now we consider the case when \((1 - \delta)x \geq 1\). Recall (24) in Lemma 2 we have
\[
\mathbb{P}(\lambda_1^*(G_\beta) > t) \leq Ct^{k-2} \exp(-t^2/4), \quad t \geq 1,
\]
this implies that
\[
\int_1^{(1 - \delta)x} f'(t) \mathbb{P}(\lambda_1^*(G_\beta) > t) dt \leq Cx^{k-2} \exp((\lambda_1^*(G_\gamma))^2 - x^2)/4)
\]
\times \int_1^{(1 - \delta)x} (x/(t + 1))^{k-1} \exp(-(t - (x - \lambda_1^*(G_\gamma))^2/4) dt,
\]
where we have used the identity
\[
-(x - t)(x - \lambda_1^*(G_\gamma))/2 - t^2/4
= - (t - (x - \lambda_1^*(G_\gamma))^2) + \frac{(x - \lambda_1^*(G_\gamma))^2}{4} - \frac{x(x - \lambda_1^*(G_\gamma))}{2}
= - \frac{(t - (x - \lambda_1^*(G_\gamma))^2}{4} + \frac{(x - \lambda_1^*(G_\gamma))^2}{4} - x^2/2.
\]
On \(\Omega'\), we have,
\[
\int_1^{(1 - \delta)x} (x/(t + 1))^{k-1} \exp(-(t - (x - \lambda_1^*(G_\gamma))^2/4) dt
\]
\leq \int_0^{\min\{\delta'x/2, (1 - \delta)x\}} x^{k-1} \exp(-t^2/16) dt
\]
\]
\leq \int_0^{\min\{\delta'x/2, (1 - \delta)x\}} \left(\frac{2}{1 - \delta}\right) x^{k-1} \exp(-t^2/16) dt
\leq Cx^{k-2} \exp(-\delta'^2x^2/16) + C \leq C'.
\]
Combining (22), (55) and (53), for \((1 - \delta)x > 1\), we have
\[
\int_0^{(1 - \delta)x} f'(t) \mathbb{P}(\lambda_1^*(G_\beta) > t) dt = \int_0^1 \cdots \int_1^{(1 - \delta)x} \cdots \leq Cx^{k-2} \exp((\lambda_1^*(G_\gamma))^2 - x^2)/4).
\]
By (53), the above estimate is actually true for all \((1 - \delta)x > 1\) on \(\Omega'\). We also have
\[
f(0) = x^{k-2} \exp(-x(x - \lambda_1^*(G_\gamma))/2) \leq Cx^{k-2} \exp((\lambda_1^*(G_\gamma))^2 - x^2)/4)
\]
for \(C\) large enough, this is because on \(\Omega'\) we have
\[
-x(x - \lambda_1^*(G_\gamma))/2 - ((\lambda_1^*(G_\gamma))^2 - x^2)/4 = -(x - \lambda_1^*(G_\gamma))^2/4 \leq -\delta'^2x^2.
\]
Combining (49), (57) and (55), on \(\Omega'\) we get
\[
\mathbb{E}(f(\lambda_1^*(G_\beta))1_{\{\lambda_1^*(G_\beta) < (1 - \delta)x\}}) \leq C'x^{k-2} \exp((\lambda_1^*(G_\gamma))^2 - x^2)/4).
\]
On \(\Omega'\), for \((1 - \delta)x > 1\), it holds that \(x/(\lambda_1^*(G_\gamma) + 1) \geq x/(1 - \delta')x + 1 \geq 1/(2 - \delta')\), therefore, we further have the upper bound,
\[
\mathbb{E}(f(\lambda_1^*(G_\beta))1_{\{\lambda_1^*(G_\beta) < (1 - \delta)x\}}) \leq C'x^{k-2}(x/(\lambda_1^*(G_\gamma) + 1))^{k-1} \exp((\lambda_1^*(G_\gamma))^2 - x^2)/4).
\]
We can similarly control the conditional expectation of \( h(\lambda_1^\gamma(G_\beta))1_{\{\lambda_1^\gamma(G_\beta) \leq (1-\delta)x\}} \).

Analogously to (49), we have

\[
\mathbb{E}(h(\lambda_1^\gamma(G_\beta))1_{\{\lambda_1^\gamma(G_\beta) < (1-\delta)x\}}) = \int_0^{(1-\delta)x} h'(t)P(\lambda_1^\gamma(G_\beta) > t)dt - h((1-\delta)x)P(\lambda_1^\gamma(G_\beta) > (1-\delta)x) + h(0).
\]

On the event \( \Omega' = \{\lambda_1^\gamma(G_\gamma) \leq (1-\delta')x\} \), one has

\[
h'(t) \leq \frac{1}{2}(x/(\lambda_1^\gamma(G_\gamma) + 1))^{kt-1} \exp(-(x-t)(x-\lambda_1^\gamma(G_\gamma))/2).
\]

We basically repeat the proofs of (57) and (58) to get

\[
\mathbb{E}(h(\lambda_1^\gamma(G_\beta))1_{\{\lambda_1^\gamma(G_\beta) < (1-\delta)x\}}) \leq Cx^{t-2}(x/(\lambda_1^\gamma(G_\gamma) + 1))^{kt-1} \exp(((\lambda_1^\gamma(G_\gamma))^2 - x^2)/4).
\]

Then (53) follows from (58), (60) and (62) (recall that \( \ell = m-k \)).

3.3. Proof of Lemma 4

Proof. By the definition of \( A_\alpha \) in (8) with \( \alpha = \{1, 2, \ldots, m\} \), we have

\[
P(A_{1,2,\ldots,m}) = \mathbb{P}(\lambda_1(G_\alpha) > y_m; \lambda_1^\gamma(G_\beta) \leq y_k, \forall 1 \leq k < m, \beta \subset \alpha, |\beta| = k) = \mathbb{P}(\lambda_1(G_\alpha) > y_m) \mathbb{P}(\lambda_1^\gamma(G_\beta) \leq y_k, \forall 1 \leq k < m, \beta \subset \alpha, |\beta| = k|\lambda_1(G_\alpha) > y_m).
\]

Lemma 4 follows from the following two limits,

\[
\lim_{n \to +\infty} \mathbb{P}(\lambda_1(G_\alpha) > y_m)/[m2^{-(1+m)/2}/\Gamma(1+m/2) \cdot y_m^{-2}e^{-y_m^2/4}] = 1
\]

and

\[
\lim_{n \to +\infty} \mathbb{P}(\lambda_1^\gamma(G_\beta) \leq y_k, \forall 1 \leq k < m, \beta \subset \alpha, |\beta| = k|\lambda_1(G_\alpha) > y_m) = K_m,
\]

where \( K_m \) is the constant defined in Theorem 1. Indeed, (63), (64), (65) and the definition of \( y_m \) imply that

\[
\lim_{n \to +\infty} t_n = \lim_{n \to +\infty} \left( \frac{n}{m} \right) P(A_{1,2,\ldots,m}) = \lim_{n \to +\infty} \frac{n^m}{m!} \left( m2^{-(1+m)/2}/\Gamma(1+m/2) \cdot y_m^{-2}e^{-y_m^2/4} \right) K_m
\]

\[
= \lim_{n \to +\infty} \frac{y_m^{-2}(-\ln n)^{-2} K_m e^{-y_m^4}}{(m-1)!2^{m+1} \Gamma(1+m/2)}
\]

\[
= \frac{(2m)^{m-2}/2 K_m}{(m-1)!2^{3/2} \Gamma(1+m/2)} e^{-y_m^4},
\]

as desired.
We first prove (64). By formula (1) for $G_\alpha$ where $|\alpha| = m$, we have
\[
P(\lambda_1(G_\alpha) > y_m) = \frac{1}{Z_m} \int_{\lambda_1 > \cdots > \lambda_m: \lambda_1 > y_m} e^{-\frac{m}{4} \sum_{i=1}^{m} \lambda_i^2 / 4} \prod_{1 \leq i < j \leq m} |\lambda_i - \lambda_j| d\lambda_1 \cdots d\lambda_m
\]
\[
= \frac{1}{Z_m} \int_{\lambda_1 > y_m} e^{-\lambda_1^2 / 4} g(\lambda_1) d\lambda_1,
\]
where we denote
\[
g(\lambda_1) = \int_{\lambda_1 > \cdots > \lambda_m} e^{-\frac{m}{4} \sum_{i=2}^{m} \lambda_i^2 / 4} \prod_{1 \leq i < j \leq m} |\lambda_i - \lambda_j| d\lambda_2 \cdots d\lambda_m.
\]

We claim that
\[
\lim_{\lambda_1 \to +\infty} g(\lambda_1)/(\lambda_1^{m-1} Z_{m-1}) = 1.
\]

Dividing into two cases $\lambda_m > -\sqrt{\lambda_1}$ and $\lambda_m < -\sqrt{\lambda_1}$, $g(\lambda_1)$ is bounded from above by
\[
\int_{\lambda_2 > \cdots > \lambda_m > -\sqrt{\lambda_1}} (\lambda_1 + \sqrt{\lambda_1})^{m-1} \prod_{2 \leq i < j \leq m} (\lambda_i - \lambda_j) e^{-\sum_{i=2}^{m} \lambda_i^2 / 4} d\lambda_2 \cdots d\lambda_m
\]
\[
+ \int_{\lambda_2 > \cdots > \lambda_m < -\sqrt{\lambda_1}} \prod_{1 \leq i < j \leq m} (\lambda_i - \lambda_j) e^{-\sum_{i=2}^{m} \lambda_i^2 / 4} d\lambda_2 \cdots d\lambda_m
\]
\[
:= I_5 + I_6.
\]

Note that $I_5$ is further bounded from above by $(\lambda_1 + \sqrt{\lambda_1})^{m-1} Z_{m-1}$. For $I_6$, we note that
\[
\prod_{1 \leq i < j \leq m} (\lambda_i - \lambda_j) \leq \prod_{1 \leq i < j \leq m} (|\lambda_i| + 1)(|\lambda_j| + 1) \leq \prod_{i=1}^{m} (|\lambda_i| + 1)^{m-1}.
\]

Therefore, we can bound $I_6$ from above by
\[
(|\lambda_1| + 1)^{m-1} \int_{\lambda_m < -\sqrt{\lambda_1}} (|\lambda_m| + 1)^{m-1} \exp(-\lambda_m^2 / 4) d\lambda_m
\]
\[
\times \left( \int_{\mathbb{R}} (|\lambda_2| + 1)^{m-1} \exp(-\lambda_2^2 / 4) d\lambda_2 \right)^{m-2}.
\]

For $\lambda_1$ large enough, (69) can be further bounded from above by (using (51))
\[
C \lambda_1^{m-1} \sqrt{\lambda_1}^{m-2} \exp(-\lambda_1 / 4) \leq C'(\lambda_1^{m-2}.
\]

Combining the estimates for $I_5$ and $I_6$ we get
\[
g(\lambda_1) \leq Z_{m-1}(\lambda_1 + \sqrt{\lambda_1})^{m-1} + C \lambda_1^{m-2}
\]
for $\lambda_1$ large enough. It follows that
\[
\limsup_{\lambda_1 \to +\infty} \frac{g(\lambda_1)}{\lambda_1^{m-1}} \leq Z_{m-1}.
\]
For the lower bound, for all \( \lambda_1 > 1 \) we have
\[
g(\lambda_1) \geq (\lambda_1 - \sqrt{\lambda_1})^{m-1} \int_{\sqrt{\lambda_1} \geq \lambda_2 > \cdots > \lambda_m} e^{-\frac{\alpha}{2} \sum_{i=2}^{m} \lambda_i^2 / 2} \prod_{2 \leq i < j \leq m} |\lambda_i - \lambda_j| d\lambda_2 \cdots d\lambda_m.
\]

It follows that
\[
\liminf_{\lambda_1 \to +\infty} \frac{g(\lambda_1)}{\lambda_1^{m-1}} \geq \liminf_{\lambda_1 \to +\infty} \int_{\sqrt{\lambda_1} \geq \lambda_2 > \cdots > \lambda_m} e^{-\frac{\alpha}{2} \sum_{i=2}^{m} \lambda_i^2 / 2} \prod_{2 \leq i < j \leq m} |\lambda_i - \lambda_j| d\lambda_2 \cdots d\lambda_m = Z_{m-1}.
\]

Then (72) follows from the upper and lower bounds.

Therefore, as \( n \to +\infty \), i.e., \( y_m \to +\infty \), we have
\[
\int_{\lambda_1 > y_m} e^{-\lambda_1^2 / 4} g(\lambda_1) d\lambda_1 \sim Z_{m-1} \int_{\lambda_1 > y_m} e^{-\lambda_1^2 / 4 \lambda_1^{m-1}} d\lambda_1.
\]

Standard results for the upper incomplete Gamma function imply
\[
\int_{\lambda_1 > y_m} e^{-\lambda_1^2 / 4 \lambda_1^{m-1}} d\lambda_1 = \int_{s > y_m / 4} e^{-s^{m-1} / 2} ds \sim 2^{m-1} e^{-y_m^2 / 4 (y_m / 4)^{m-2} / 2} = 2 e^{-y_m^2 / 4 y_m^{m-2}}.
\]

Combining (66), (70) and (71) we get
\[
\mathbb{P}(\lambda_1(G_a) > y_m) \sim \frac{Z_{m-1}}{Z_m} \int_{\lambda_1 > y_m} e^{-\lambda_1^2 / 4 \lambda_1^{m-1}} d\lambda_1
\]
\[
\sim \frac{(Z_{m-1} / Z_m)2y_m^{m-2}e^{-y_m^2 / 4}}{m(2\pi)^{-1/2} \Gamma(3/2) / \Gamma(1+m/2) \cdot 2^{1-m/2}y_m^{-2} \cdot 2^{-1+m/2} \Gamma(1+m/2)} \cdot y_m^{m-2} e^{-y_m^2 / 4}.
\]

This completes the proof of (64).

Now we prove (65). We define the following two auxiliary events:
\[
B_\alpha := \left\{ y_m < \lambda_1(G_a) < y_m + 1, \lambda_2(G_a) < \sqrt{\log \log n}, \lambda_m(G_a) > -\sqrt{\log \log n} \right\}
\]
and
\[
D_\alpha := \left\{ y_m < \lambda_1(G_a) < y_m + 1 \right\}.
\]

Then it holds that
\[
B_\alpha \subset D_\alpha \subset \left\{ \lambda_1(G_a) > y_m \right\}
\]
and we first show that
\[
\lim_{n \to +\infty} \mathbb{P}(B_\alpha | \lambda_1(G_a) > y_m) = 1.
\]

Note that (75) follows if we can prove the following two limits,
\[
\lim_{n \to +\infty} \mathbb{P}(D_\alpha | \lambda_1(G_a) > y_m) = 1
\]
and
\[
\lim_{n \to +\infty} \mathbb{P}(B_\alpha | D_\alpha) = 1.
\]
By (75), we find that
\[
\lim_{n \to +\infty} \frac{\mathbb{P}(\lambda_1(G_n) > y_m + 1)}{\mathbb{P}(\lambda_1(G_n) > y_m)} = 0,
\]
which implies (74). We now prove (75) by finding a sequence of numbers \( \epsilon_n \to 0 \) such that for \( n \) large enough it holds that
\[
\frac{\mathbb{P}(\lambda_2(G_n) > \sqrt{\log \log n} \text{ or } \lambda_m(G_n) < -\sqrt{\log \log n}|D_\alpha)}{\mathbb{P}(\lambda_2(G_n) < 1, \lambda_m(G_n) > 0|D_\alpha)} \leq \epsilon_n.
\]
Indeed, (76) implies that
\[
\mathbb{P}(\lambda_2(G_n) > \sqrt{\log \log n} \text{ or } \lambda_m(G_n) < -\sqrt{\log \log n}|D_\alpha) \leq \frac{\epsilon_n}{\epsilon_n + 1},
\]
which converges to 0 as \( n \to +\infty \), and thus this yields (75). To prove (76), we divide the set \( \{ \lambda_2(G_n) > \sqrt{\log \log n} \text{ or } \lambda_m(G_n) < -\sqrt{\log \log n} \} \) as the union of two disjoint subsets \( S_1 \cup S_2 \) as follows. On the subset \( S_1 := \{ \lambda_m(G_n) \leq -\sqrt{\log \log n} \} \), if we condition on \( D_\alpha \), we have
\[
\prod_{1 \leq i < j \leq m} (\lambda_i - \lambda_j)
\leq C(\lambda_1 + |\lambda_m|)^{m-1}(|\lambda_2| + |\lambda_m|)^{(m-1)(m-2)/2}
\leq C\lambda_1^{m-1}|\lambda_m|^{m-1}(|\lambda_2| + 1)^{(m-1)(m-2)/2}|\lambda_m|^{(m-1)(m-2)/2}
= C\lambda_1^{m-1}|\lambda_m|^{(m-1)m/2}(|\lambda_2| + 1)^{(m-1)(m-2)/2}.
\]
On the subset \( S_2 := \{ \lambda_2(G_n) > \sqrt{\log \log n}, \lambda_m(G_n) > -\sqrt{\log \log n} \} \), if we condition on \( D_\alpha \) where \( \lambda_1(G_n) \sim 2\sqrt{m\log n} \), then we easily have the upper bound
\[
\prod_{1 \leq i < j \leq m} (\lambda_i - \lambda_j) \leq C\lambda_1^{m-1}\lambda_2^{(m-1)(m-2)/2}.
\]
It follows that the left hand side of (76) can be bounded from above by the summation of
\[
C\int_{y_m}^{y_m + 1} \lambda_1^{m-1}e^{-\lambda_2^2/4}d\lambda_1 \int_{\lambda_m < -\sqrt{\log \log n}} |\lambda_m|^{(m-1)m/2}(|\lambda_2| + 1)^{(m-1)(m-2)/2}e^{-\sum_{i=2}^{m} \lambda_i^2/4}d\lambda_2 \cdots d\lambda_m
\]
and
\[
C\int_{y_m}^{y_m + 1} \lambda_1^{m-1}e^{-\lambda_2^2/4}d\lambda_1 \int_{\lambda_m > \sqrt{\log \log n}} |\lambda_m|^{(m-1)m/2}e^{-\sum_{i=2}^{m} \lambda_i^2/4}d\lambda_2 \cdots d\lambda_m
\]
The above summation can be further bounded from above by
\[
C\left( \int_{\lambda_m < -\sqrt{\log \log n}} |\lambda_m|^{(m-1)m/2} \exp(-\lambda_m^2/4)d\lambda_m
+ \int_{\lambda_m > \sqrt{\log \log n}} \lambda_2^{(m-1)(m-2)/2} \exp(-\lambda_2^2/4)d\lambda_2 \right)
\]
as \( n \) large enough. Clearly it holds that \( \epsilon_n \to 0 \) since \( \sqrt{\log \log n} \to +\infty \). This completes the proof of (75) and thus the proof of (73).
Now we are ready to prove (65). We define the event
\[ H_\alpha = \left\{ \lambda_1^*(G_\beta) \leq y_k, \quad \forall \ 1 \leq k < m, \ \beta \subset \alpha, \ |\beta| = k \right\}. \]
Using (73), we see that (65) is equivalent to
\[ \lim_{n \to +\infty} \mathbb{P}(H_\alpha | B_\alpha) = K_m. \]
To prove (79) we need to use the fact that
\[ G_\alpha \overset{d}{=} U^T \text{diag}(\lambda_1, \ldots, \lambda_m) U \]
where $|\alpha| = m$ and
\[ U \overset{d}{=} \mathcal{U}(O(m)) \]
is sampled from the uniform measure on the orthogonal group $O(m)$ which is independent of $(\lambda_1, \ldots, \lambda_m)$ and
\[ (\lambda_1, \ldots, \lambda_m) \overset{d}{=} (\lambda_1(G_\alpha), \ldots, \lambda_m(G_\alpha)). \]
Given any
\[ X := U^T \text{diag}(\lambda_1, \ldots, \lambda_m) U, \ U \in O(m), \ \lambda_1 \geq \cdots \geq \lambda_m, \]
by definition it holds
\[ \lambda_1(X) = \lambda_1, \ \lambda_1^*(X)^2 = \lambda_1^2 + \sum_{k=2}^m \lambda_k^2 / 2. \]
Let $\lambda_1^-(X) := \sqrt{\sum_{k=2}^m \lambda_k^2} = \sqrt{|X|^2 - \lambda_1^2(X)}$, $X_1 := U^T \text{diag}(\lambda_1, 0, \ldots, 0) U$, $X_2 := X - X_1$, then we have $|X_2| = \sqrt{\text{Tr}(X_2^2)} = \lambda_1^-(X)$. For $\beta \subset \alpha = \{1, \ldots, m\}$ we have
\[ |\lambda_1^*(X_\beta) - \lambda_1^*((X_1)_{\beta})| \leq |(X_2)_{\beta}| \leq |X_2| = \lambda_1^-(X). \]
Here, by the Lipschitz continuity of eigenvalues for symmetric matrices (see Corollary A.6 in [1]), we can further derive the fact that
\[ |\lambda_1^*(A) - \lambda_1^*(B)| \leq |A - B|. \]
Since $(X_1)_{\beta}$ is a matrix of rank at most 1, we have
\[ \lambda_1^*((X_1)_{\beta}) = (\text{Tr}((X_1)_{\beta}^2))^{1/2} = |\lambda_1| \sum_{k \in \beta} u_k^2, \]
here $u = (u_1, \ldots, u_m)$ is the first row of $U$. Note that $u$ has the uniform distribution on the unit sphere $S^{m-1}$. By (80) and (81), we have
\[ |\lambda_1^*(X_\beta) - |\lambda_1| \sum_{k \in \beta} u_k^2| \leq \lambda_1^-(X). \]
Now we replace $X$ by $G_\alpha$. On the event $B_\alpha$ we have
\[ \lambda_1^-(G_\alpha) \leq \sqrt{m \log \log n} \quad \text{and} \quad y_m < \lambda_1 < y_m + 1, \]
which together with (82) imply
\[ y_m \sum_{k \in \beta} u_k^2 - \sqrt{m \log \log n} \leq \lambda_1^*(G_\beta) \leq (y_m + 1) \sum_{k \in \beta} u_k^2 + \sqrt{m \log \log n} \]
for all $\beta \subset \alpha$. 
As in §(86) we define the random variable \( \hat{\alpha} \). By (9) and (13), we have
\[
\lim_{n \to +\infty} \mathbb{P}(H_{\alpha}|B_{\alpha}) = \mathbb{P}\left( \sum_{k \in \beta} u_{k}^{2} \leq \frac{y_{k}}{y_{m}+1}, \forall \beta \subset \alpha, |\beta| = k \right),
\]
which is exactly the definition of \( K_{m} \). This proves (79), and hence (85). Therefore, we complete the proof of Lemma 4. \( \square \)

4. Proof of Theorem 2

Now we prove Theorem 2. As before, for \( G = (g_{ij})_{1 \leq i \leq j \leq n} \) sampled from GOE, we denote \( G_{\alpha} = (g_{ij})_{i,j \in \alpha} \) as the principal minor of size \( m \times m \) for \( \alpha \subset \{1, \ldots, n\} \) with \( |\alpha| = m \) and \( T_{m} \alpha \) is the collection of all such \( \alpha \). Let \( v_{1}(G_{\alpha}) \) be the eigenvector of the largest eigenvalue of \( G_{\alpha} \) and \( v^{*} \in S^{m-1} \) be the eigenvector of the largest eigenvalue of the principal sub-matrix that attains the maximal eigenvalue \( T_{m,n} \), i.e., we have
\[
\alpha^{*} := \arg \max_{\alpha \in T_{m}} \lambda_{1}(G_{\alpha})
\]
and
\[
v^{*} = v_{1}(G_{\alpha^{*}}).
\]
As in §(83) we recall the definition of the event
\[
H_{\alpha} := \{ \lambda_{1}^{*}(G_{\beta}) \leq y_{k}, \forall 1 \leq k < m, \beta \subset \alpha, |\beta| = k \}.
\]
We define the random variable \( \hat{\alpha} \) as follows. If the event \( \bigcup_{\alpha \in T_{m}} H_{\alpha} \) holds, then we set
\[
\hat{\alpha} := \arg \max_{\alpha \in T_{m}, H_{\alpha} \text{ holds}} \lambda_{1}(G_{\alpha}).
\]
Otherwise, we set \( \hat{\alpha} \) to be \( \{1, \ldots, m\} \). We now set
\[
\hat{v} := v_{1}(G_{\hat{\alpha}}).
\]
In other words, \( \hat{v} \) is the eigenvector of the largest eigenvalue of the principal sub-matrix \( G_{\alpha} \) that achieves the maximal eigenvalue under the constraint that \( H_{\alpha} \) is true. By (8) and (12), we have
\[
\lim_{n \to +\infty} \mathbb{P}(\cap_{\alpha \in T_{m}} H_{\alpha}) = 1.
\]
On the event \( \cap_{\alpha \in T_{m}} H_{\alpha} \) we clearly have \( \hat{\alpha} = \alpha^{*} \) and \( \hat{v} = v^{*} \) since the constraint for \( \hat{\alpha} \) doesn’t have any effect. Recall the definition of \( A_{\alpha} \) in §(8) where \( A_{\alpha} = H_{\alpha} \cap \{ \lambda_{1}(G_{\alpha}) > y_{m} \} \), on \( \cap_{\alpha \in T_{m}} H_{\alpha} \), the two events \( \{ T_{m,n} \geq y_{m} \} \) and \( \cup_{\alpha \in T_{m}} A_{\alpha} \) coincide. In other words, for symmetric \( Q \), we have
\[
(\cap_{\alpha \in T_{m}} H_{\alpha}) \cap \{ T_{G,m,n} \geq y_{m}, v^{*} \in Q \} = (\cap_{\alpha \in T_{m}} H_{\alpha}) \cap \{ \cup_{\alpha \in T_{m}} A_{\alpha}, \hat{v} \in Q \}.
\]
It follows that
\[
|\mathbb{P}(T_{m,n} \geq y_m, v^* \in Q) - \mathbb{P}(\cup_{\alpha \in I_m} A_\alpha, \hat{v} \in Q)|
\]
\[
= |\mathbb{P}((\cap_{\alpha \in I_m} H_\alpha)^c \cap \{T_{m,n} \geq y_m, v^* \in Q\}) - \mathbb{P}((\cap_{\alpha \in I_m} H_\alpha)^c \cap (\cup_{\alpha \in I_m} A_\alpha, \hat{v} \in Q))|
\]
\[
\leq 2\mathbb{P}((\cap_{\alpha \in I_m} H_\alpha)^c),
\]
which converges to 0 as \( n \to +\infty \). Hence, (5) is equivalent to the limit
\[
\mathbb{P}(\cup_{\alpha \in I_m} A_\alpha, \hat{v} \in Q) \to (1 - F_Y(y))^\nu(Q).
\]
All of the rest is to prove this convergence. Now we define four quantities
\[
k_1(\alpha) = \mathbb{P}(A_\alpha, v_1(G_\alpha) \in Q; \forall \alpha' \neq \alpha, A_{\alpha'} \text{ fails or } \lambda_1(G_{\alpha'}) < \lambda_1(G_\alpha)),
\]
\[
k_2(\alpha) = \mathbb{P}(A_\alpha, v_1(G_\alpha) \in Q; \forall \alpha' \cap \emptyset = \emptyset, A_{\alpha'} \text{ fails or } \lambda_1(G_{\alpha'}) < \lambda_1(G_\alpha)),
\]
\[
k_3(\alpha) = \mathbb{P}(A_\alpha; \forall \alpha' \neq \alpha, A_{\alpha'} \text{ fails or } \lambda_1(G_{\alpha'}) < \lambda_1(G_\alpha)),
\]
\[
k_4(\alpha) = \mathbb{P}(A_\alpha; \forall \alpha' \cap \emptyset = \emptyset, A_{\alpha'} \text{ fails or } \lambda_1(G_{\alpha'}) < \lambda_1(G_\alpha)).
\]
We note that
\[
\mathbb{P}(\cup_{\alpha \in I_m} A_\alpha, \hat{v} \in Q)
\]
\[
= \sum_{\alpha \in I_m} \mathbb{P}(A_\alpha, v_1(G_\alpha) \in Q; \forall \alpha' \neq \alpha, A_{\alpha'} \text{ fails or } \lambda_1(G_{\alpha'}) < \lambda_1(G_\alpha))
\]
\[
= \sum_{\alpha \in I_m} k_1(\alpha).
\]
In fact \( k_1(\alpha), \ldots, k_4(\alpha) \) don’t depend on the specific choice of \( \alpha \). Recall the definition of \( b_{n,2} \) and (28), we have
\[
\lim_{n \to +\infty} \sum_{\alpha, \alpha' \in I_m; \alpha' \neq \emptyset} \mathbb{P}(A_\alpha \cap A_{\alpha'}) = \lim_{n \to +\infty} b_{n,2} = 0.
\]
By the definition of \( k_1(\alpha) \) and \( k_2(\alpha) \) together with the union bound we have
\[
\sum_{\alpha \in I_m} |k_1(\alpha) - k_2(\alpha)| \leq \sum_{\alpha \in I_m} \mathbb{P}(A_\alpha \cap (\cup_{\alpha' \cap \emptyset = \emptyset} A_{\alpha'})) \leq \sum_{\alpha, \alpha' \in I_m; \alpha' \neq \emptyset} \mathbb{P}(A_\alpha \cap A_{\alpha'}).
\]
Combining (92), (93) and (94) we see that
\[
\lim_{n \to +\infty} \left| \mathbb{P}(\cup_{\alpha \in I_m} A_\alpha, \hat{v} \in Q) - \sum_{\alpha \in I_m} k_2(\alpha) \right| = 0.
\]
We can similarly show that
\[
\lim_{n \to +\infty} \left| \mathbb{P}(\cup_{\alpha \in I_m} A_\alpha) - \sum_{\alpha \in I_m} k_4(\alpha) \right| \leq \lim_{n \to +\infty} \sum_{\alpha \in I_m} |k_3(\alpha) - k_4(\alpha)| = 0.
\]
Recall that we have already shown
\[
\mathbb{P}(\cup_{\alpha \in I_m} A_\alpha) \to 1 - F_Y(y)
\]
in the proof of Theorem 1 as \( n \to +\infty \). Hence, combining (96) and (97) we have
\[
\lim_{n \to +\infty} \sum_{\alpha \in I_m} k_4(\alpha) = 1 - F_Y(y).
\]
The advantage of introducing $k_2(\alpha)$ is that, we have the conditional probability

$$P(v_1(G_\alpha) \in Q|A_\alpha; \forall \alpha' \cap \alpha = \emptyset, A_{\alpha'} \text{ fails or } \lambda_1(G_{\alpha'}) < \lambda_1(G_\alpha))$$

(99)\[=P(v_1(G_\alpha) \in Q|A_\alpha),\]

since $G_\alpha$ is independent of $\{G_{\alpha'} : \alpha' \cap \alpha = \emptyset\}$. Consequently, we have

$$k_2(\alpha) = P(A_\alpha; \forall \alpha' \cap \alpha = \emptyset, A_{\alpha'} \text{ fails or } \lambda_1(G_{\alpha'}) < \lambda_1(G_\alpha))$$

(100)\[\times P(v_1(G_\alpha) \in Q|A_\alpha; \forall \alpha' \cap \alpha = \emptyset, A_{\alpha'} \text{ fails or } \lambda_1(G_{\alpha'}) < \lambda_1(G_\alpha))\]

$$=P(v_1(G_\alpha) \in Q|A_\alpha)k_4(\alpha).$$

Clearly the term $P(v_1(G_\alpha) \in Q|A_\alpha)$ is the same for all $\alpha \in I_m$. Let $\alpha_1 = \{1, \ldots, m\}$. Then we have

$$\sum_{\alpha \in I_m} k_2(\alpha) = \sum_{\alpha \in I_m} P(v_1(G_\alpha) \in Q|A_\alpha)k_4(\alpha) = P(v_1(G_{\alpha_1}) \in Q|A_{\alpha_1}) \sum_{\alpha \in I_m} k_4(\alpha).$$

(101)

Let $u$ be sampled from the uniform distribution on the unit sphere $S^{m-1}$. Inspecting the proof of (95) in §3.3, especially (73) and (84), we have

$$\lim_{n \to +\infty} P(H_\alpha, v_1(G_\alpha) \in Q|\lambda_1(G_\alpha) > y_m) = P(u \in S_m \cap Q),$$

(102)

where $S_m$ has been defined in (4).

By the fact $A_\alpha = H_\alpha \cap \{\lambda_1(G_\alpha) > y_m\}$, we have

$$\lim_{n \to +\infty} P(v_1(G_{\alpha_1}) \in Q|A_{\alpha_1})$$

$$= \lim_{n \to +\infty} \frac{P(H_\alpha, v_1(G_\alpha) \in Q|\lambda_1(G_\alpha) > y_m)}{P(H_\alpha|\lambda_1(G_\alpha) > y_m)}$$

$$= \frac{P(u \in S_m \cap Q)}{P(u \in S_m)}$$

$$= \nu(Q),$$

(103)

where $\nu$ is the uniform distribution on the set $S_m$.

Combining (95), (98), (101) and (103) we get

$$\lim_{n \to +\infty} P(\cup_{\alpha \in I_m} A_\alpha, \tilde{v} \in Q)$$

$$= \lim_{n \to +\infty} \sum_{\alpha \in I_m} k_2(\alpha)$$

(104)\[= \lim_{n \to +\infty} P(v_1(G_{\alpha_1}) \in Q|A_{\alpha_1}) \sum_{\alpha \in I_m} k_4(\alpha)

= \nu(Q)(1 - F_Y(y)).

This proves (90), and thus the proof of Theorem 2.

REFERENCES

[1] G. W. Anderson, A. Guionnet and O. Zeitouni, An introduction to random matrices. Cambridge Studies in Advanced Mathematics, 118. Cambridge University Press, Cambridge, 2010.

[2] Arratia R., Goldstein L., Gordon L. Two moments suffice for Poisson approximations: The Chen-Stein method. Ann. Probab. 17(1989):9-25.

[3] Baraniuk, R., Davenport, M., DeVore, R., and Wakin, M. (2008). A simple proof of the restricted isometry property for random matrices. Constructive Approximation, 28(3):253-263.
[4] Cai T.-T., Jiang T., Li X. Asymptotic Analysis for Extreme Eigenvalues of Principal Minors of Random Matrices, Ann. Appl. Probab. 31 (6) 2953 - 2990, December 2021.
[5] Gamarnik, D., and Li, Q. (2018). Finding a large submatrix of a Gaussian Random matrix. The Annals of Statistics, 46(6A), 2511-2561.
[6] Leadbetter, M.R., Lindgren, G., Rootzen, H.: Extremes and related properties of random sequences and processes. Springer Series in Statistics, Springer (1983).
[7] Leadbetter, M. R. and Rootzen, H. (1988). Extremal theory for stochastic processes. Ann. Probab., 431-478.
[8] Tao, Terence; Candès, Emmanuel J. (2006), Decoding by linear programming, IEEE transactions on information theory 51 (12), 4203-4215.

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