BLOW-UP FOR REALIZING HOMOTOPY CLASSES IN THE
THREE-BODY PROBLEM

ABSTRACT. This expository note describes McGehee blow-up \cite{13} in its role as one of the main tools in my recent proof with Rick Moeckel \cite{21} that every free homotopy class for the planar three-body problem can be realized by a periodic solution. The main novelty is my use of energy-balance to motivate the transformation of McGehee. Another novelty is an explicit description of the blown-up reduced phase space for the planar N-body problem, N ≥ 3 as a complex vector bundle over [0, ∞) × \mathbb{CP}^{N-2} where r ∈ [0, ∞) measures the size of a labelled planar N-gon and [s] ∈ \mathbb{CP}^{N-2} describes its shape.

1. INTRODUCTION

Deleting collisions endows the configuration space of the planar three-body problem with non-trivial topology. Modulo rotations, this space is homotopic to a two-sphere minus three points and so has a large set of free homotopy classes of loops. We call these classes the “relative free homotopy classes. We say that a solution is “relatively periodic” if it is periodic modulo rotation, or equivalently, if it is periodic in some rotating frame.

Theorem 1 (\((RM)^2\) \cite{21}). For equal or near-equal masses, and angular momenta J sufficiently small but nonzero, every relative free homotopy class for the planar three-body problem is realized by a relatively periodic orbit for the Newtonian planar three-body problem having energy −1 and angular momentum J.

Remark. “Relative free homotopy classes” are encoded by “reduced syzygy sequences” : periodic lists of 1’s 2’s and 3’s such as 123232... where the symbol i indicates that the three bodies have become instantaneously collinear with mass i in between j and k with i,j,k a permutation of the symbols 1,2,3. See for example (\cite{21}) for details.

History of the theorem. A well-known theorem in Riemannian geometry asserts that on a compact Riemannian manifold every free homotopy class of loops is realized by a periodic geodesic. Inspired by this basic geometric fact Wu-yi Hsiang, in 1996, posed the question: “Is every free homotopy class realized by a (relatively) periodic solution to the planar Newtonian three-body equation?” For 17 years I tried to use variational methods to prove that the answer is “yes” when the total angular momentum J is zero. Finally, at the urging of Carles Simó, in October of 2014, I gave up on variational methods and tested the waters of dynamical methods. Almost as soon as I gave up I realized that Rick Moeckel had come within epsilon of proving theorem 1 in the 1980s [ (\cite{17}, \cite{16}, and \cite{19})].

My purpose in this note is to give an exposition of one of the principal tools in our proof, McGehee blow-up \cite{13}, and a sense of how we use this tool to prove the theorem. The main novelty is the use of energy-balance to motivate the mysterious
transformation of McGehee. Another novelty is an explicit description of the blown-up reduced phase space for the planar N-body problem, $N \geq 3$. For further reading on McGehee blow-up we recommend Moeckel [15], pages 222-225 and Chenciner [5].

2. Background: Equations and Solutions.

2.1. The Equations. The classical three-body problem demands that we solve the system of non-linear ODEs:

$$
\begin{align*}
    m_1 \ddot{q}_1 &= F_{21} + F_{31} \\
    m_2 \ddot{q}_2 &= F_{12} + F_{32} \\
    m_3 \ddot{q}_3 &= F_{23} + F_{13}.
\end{align*}
$$

where

$$
F_{ab} = G m_a m_b \frac{q_a - q_b}{r_{ab}^3}
$$

is the force exerted by mass $m_a$ on mass $m_b$ and $r_{ab} = |q_a - q_b|$, $q_a \in \mathbb{R}^d$, and $m_a, G > 0$.

Here $a, b = 1, 2, 3$ label the bodies. The dimension $d$ for us will be 2. (The standard value is $d = 3$.) The $m_a$ represent the values of point masses whose instantaneous positions are $q_a(t)$. The double dots indicate two time derivatives: $\ddot{q} = \frac{d^2q}{dt^2}$. The constant $G$ is Newton’s gravitational constant and is physically needed to make dimensions match up. Being mathematicians, we can and do set $G = 1$.

2.2. The Solutions of Euler and Lagrange. The only solutions to the three-body problem for which we have explicit formulae were found by Euler [10] and Lagrange [12] in the last half of the 18th century. See figures 1, 2. Their solutions are central to our story.

For Lagrange’s solution, place the three masses at the vertices of an equilateral triangle and drop them: let them go from rest. They shrink homothetically shrink towards their common center of mass, remaining equilateral at each instant. The solution ends in finite time in triple collision. This motion forms half of Lagrange’s triple collision solution. To obtain the other half of Lagrange’s solution use time-reversal invariance to continue this solution backwards in time. In the full solution the three masses explode out of triple collision, reach a maximum size at the instant at which we dropped the three masses, and then shrink back to triple collision, staying equilateral throughout. A surprise is that the Lagrange solution works regardless of the mass ratios $m_1 : m_2 : m_3$.

For Euler’s solutions, place the masses on the line in a certain order: $q_k < q_j < q_i$ so as to form a special ratio $q_k - q_j : q_j - q_i$. (This special ratio depends on the mass ratios and also the choice of mass $m_j$ on the middle and is the root of a fifth degree polynomial whose coefficients depend on the masses.) Again drop them. They stay on the line as they evolve and again the similarity class of the (degenerate) triangle stays constant: this ratio of side lengths stays constant. (In case the two masses at the ends are equal then the special ratio is 1 : 1: place $m_j$ at the midpoint of $m_i$ and $m_k$.)

The solutions just described are part of a family of explicit solutions discovered by Euler and Lagrange. For every one of the solutions in these families the similarity class formed by the three masses stays constant in time during the evolution. Each
mass moves on its own Keplerian conic with the center of mass of the triple as focus, the solutions described above being the special case of degenerate (colinear) ellipses. We derive these families analytically in section 4.3.1 below.

Figure 1. A Lagrange Solution.

All together these solutions form five families. The corresponding shapes are called “central configurations”. The Lagrange solutions count as two, one shape for each orientation of a labelled equilateral triangle. The Euler solutions count as three, one for each choice of mass in the middle.

For almost all (Newtonian) time the solutions of theorem 1 are very close to one of the three Euler solutions. The Lagrange solutions act as bridges between various Eulers.
3. Shape sphere. Blow-up and reduction, first pass.

A basic aid to understanding the planar three-body problem is the shape sphere, a two-sphere whose points represent oriented similarity classes of triangles. At each instant of time three bodies form the vertices of a triangle. Call two triangles “oriented similar” if one can be brought to the other by a composition of translations, rotations, and scalings. The resulting space of equivalence classes forms the shape sphere. See figure 3. This sphere has 8 marked points, the 5 central configurations just described $L_+^+$, $L_-, E_1, E_2, E_3$ and 3 binary collision points labelled $B_{12}, B_{23}, B_{31}$ . The sphere’s equator represents the space of collinear triangles. The 3 binary collision points, and 3 Euler central configurations lie on this equator, interleaved so as to be alternating.

The earliest occuring picture of the shape sphere in the context of celestial mechanics with which I am familiar is [18] . You will find a detailed exposition of the shape sphere and its relation to the three-body problem in [22].

We now summarize how the shape sphere arises out of the three-body problem. The configuration space for the 3 body problem, with collisions allowed, is $\mathbb{C}^3$ with $q = (q_1, q_2, q_3) \in \mathbb{C}^3$ representing the 3 vertices of the triangle - the positions of the 3 bodies. We have identified $\mathbb{C}$ with $\mathbb{R}^2$ in the standard way: $x + iy \in \mathbb{C}$ corresponds $(x, y) \in \mathbb{R}^2$. A standard trick from Freshman physics allows us to restrict the problem to the center-of-mass zero subspace:

$$\mathbb{E}_{cm} = \{q \in \mathbb{C}^2 : m_1 q_1 + m_2 q_2 + m_3 q_3 = 0\} \cong \mathbb{C}^2 \subset \mathbb{C}^3.$$

(See the beginning of section 5 below.) In $\mathbb{E}_{cm}$ the binary collision locus become three complex lines which intersect at the origin 0. The origin represents triple collision. The masses endow $\mathbb{C}^3$ with a canonical metric called the “mass-metric”
(eq. (4)) and relative to that metric the distance from triple collision is given by $r$ where
\[ r^2 = m_1|q_1|^2 + m_2|q_2|^2 + m_3|q_3|^2. \]

(See eq. (8).) Take the sphere
\[ \{ r = 1 \} := S^3 \subset \mathbb{C}^2 \cong E_{cm}. \]

Because the three-body equations are invariant under rotations they descend to ODEs on the quotient of $\mathbb{C}^2 = E_{cm}$ by the group $S^3$ of rotations. This quotient space is topologically an $\mathbb{R}^3$. To understand this quotient note that the rotation action leaves $r$ unchanged but moves points on $S^3$ around according to $(Z_1, Z_2) \mapsto (uZ_1, uZ_2)$, $u \in S^1 \subset \mathbb{C}$. (Here $Z_1, Z_2$ are any complex linear coordinates for $E_{cm}$.)

This is the circle action used to form the Hopf fibration:
\[ \text{Hopf} : S^3 \to S^3/S^1 = S^2 = \text{shape sphere}. \]

Points of the quotient $\mathbb{R}^3$ represent oriented congruence classes of triangles: planar triangles modulo translation and rotation, but not scaling. Express $\mathbb{R}^3$ in spherical coordinates $(r, s), s \in S^2$. Then the origin $r = 0$ corresponds to triple collision. A point $s$ on the sphere represents a ray $rs, r \geq 0$ of triangles all having the same shape. The collision locus $C = \{ r_{12} = 0 \text{ or } r_{23} = 0 \text{ or } r_{31} = 0 \}$ is then represented by the three rays corresponding to the three binary collision points $B_{12}, B_{23}, B_{31} \in S^2$.

Newton's equations break down at triple collision $r = 0$. McGehee blow-up is a change of variables (equations (12)) which converts Newton’s equations to a system of ODEs which is well-defined when $r = 0$. The locus $r = 0$ in the new variables is called “the collision manifold” and forms a bundle over the shape sphere. The blown-up system of ODEs has exactly 10 fixed points, all on the collision manifold, with a pair of fixed points lying over each of the five central configurations. For
a chosen central configuration, one element of the pair corresponds to the homogeneous arc incoming to triple collision, as in our original description of the Lagrange solution, while the other element of the pair corresponds to the initial segment of that solution which explodes out from triple collision.

The 10 fixed points on the collision manifold have stable and unstable manifolds, parts of which stick out of the collision manifold, and which intersect in complicated ways, as per the Smale Horseshoe and heteroclinic tangles. See figure 5. Moeckel investigated these manifolds and their relations in seminal works [15], [16], [20], [18], [17], and [19] where he proved existence of “topological heteroclinic tangles” between them.

One finds the following abstract graph

\[ 
\begin{array}{c}
L_+ \\
E_1 \quad E_2 \quad E_3 \\
L_-
\end{array}
\]

in several of these papers of Moeckel (17, p. 53, Theorem 1′, and 19) whose Figure 2 becomes our graph after deleting the vertices labelled with Bs (for binary collision) as well edges incident to them). Moeckel’s theorem in 17, based on the intersections between stable and unstable manifolds of the 10 fixed points, asserts that all paths in this graph are “realized” by solutions to the three-body problem provided the angular momentum, energy and masses are as per theorem 1. Embed this graph in the shape sphere as indicated by figure 4. Call the embedded graph the “concrete connection graph”.

The dynamical relevance of the concrete connection graph has to do with the Isosceles three-body problem. When two of the masses are equal, say \( m_1 = m_2 \), then the isosceles triangles \( r_{13} = r_{23} \) form an invariant submanifold of the three-body problem whose dynamics is called the “Isosceles three-body problem”. These Isosceles triangles form a great circle passing through both Lagrange points, the binary point \( B_{23} \), and the Euler point \( E_1 \). If all three masses are equal we have three Isosceles subproblems represented by three great circles on the shape sphere. Take one-half of each great circle, namely that half whose endpoints are the two Lagrange points and which contains the Euler point. In this way we form the concrete connection graph in which the edges are parts of the Isosceles great circles.

Observe that the shape sphere minus the three binary collision points retracts onto the concrete connection graph. Theorem 1 follows immediately from this observation and Moeckel’s theorem refered to above, once we know that the realizing solutions of Moeckel’s theorem, projected onto the shape sphere, stay \( C^0 \)-close to corresponding edges in the concrete connection graph. For a few more details see the final section of this article.

4. Set-up. Blow-up

It is no more work to perform the blow-up for the N body problem in d-dimensional Euclidean space, rather than our special case of the three-body problem.
in the plane. The d-dimensional N-body equations are:

\[ m_a \ddot{q}_a = \sum_{b \neq a} F_{ba} \quad , q_a \in \mathbb{R}^d \]

with the forces \( F_{ba} \) as above.

4.1. Metric Reformulation. Let

\[ \mathcal{E} = (\mathbb{R}^d)^N \]

declare the N-body configuration space. Write points of \( \mathcal{E} \) as \( q = (q_1, \ldots, q_N) \) and think of the points as the N-gons in d-space. The masses endow \( \mathcal{E} \) with an inner product, called the mass inner product:

\[ \langle q, v \rangle = \sum m_a q_a \cdot v_a \]

so that the standard kinetic energy is given by

\[ K = \frac{1}{2} \langle \dot{q}, \dot{q} \rangle. \]

Let \( \nabla \) be the gradient associated to this metric: \( df_q(v) = \langle \nabla f(q), v \rangle \), so that \( (\nabla f)_a = \frac{1}{m_a} \frac{\partial f}{\partial q_a} \). Then the N-body equations take the simple form

\[ \ddot{q} = \nabla U(q) \]
where \( U \) is the negative of the standard potential \( V \):

\[
U = -V = \sum_{a < b} \frac{m_a m_b}{r_{ab}},
\]

the sum being over all distinct pairs \( a, b \). As is well known, the total energy is conserved (i.e. constant) along solutions

\[
H = K - U = K + V.
\]

We use

\[
r = \sqrt{(q,q)}
\]

to measure the size of our configuration \( q = (q_1, \ldots, q_N) \). Lagrange proved that

\[
r^2 = \sum_{a < b} m_a m_b r_{ab}^2 / \Sigma m_a
\]

provided we are in center of mass coordinates: \( \Sigma m_a q_a = 0 \). Then

\[
r = 0 \Leftrightarrow \text{total collision: all masses coincide}
\]

while

\[
U = \infty \Leftrightarrow \text{some collision: some pair of masses coincide}.
\]
**Exercise 1.** Use the metric reformulation of Newton’s equations eq (3), the fact that $U$ is homogeneous of degree $-1$ and Euler’s identity for homogeneous functions to derive the “virial identity”, also known as the Lagrange-Jacobi identity: $d^2(r^2)/dt^2 = 4H + 2U$. Also show that $2H + U = H + K$.

4.2. McGehee transformation via Energy Balance. The key property of the potential energy $-U$, as far as McGehee’s transformation is concerned, is that it is homogeneous of degree $-1$: $U(\lambda q) = \lambda^{-1}U(q)$, or $q \mapsto \lambda q = \Rightarrow U \mapsto \lambda^{-1}U$.

Our guiding principle in deriving the McGehee blown-up equations is to require that the kinetic energy $K$ scale the same as potential energy, so that the total energy balance under scaling and thus has a scaling law. We call this principle “energy balance”. Then $K \mapsto \lambda^{-1}K$ and since $K$ is quadratic in velocities $v$ velocities must scale by $v = \dot{q} \mapsto \lambda^{-1/2}v$.

How must time scale? Since $dq \mapsto \lambda dq$ and $v = dq/dt$, we see that for a power law scaling $dt \mapsto \lambda^a dt$ to yield $v \mapsto \lambda^{-1/2}v$ we must have $a = 3/2$. Summarizing, our space-time scaling law must be

$$q \mapsto \lambda q, dt \mapsto \lambda^{3/2} dt,$$

which induces the desired scalings $v \mapsto \lambda^{-1/2}v; (U, K, H) \mapsto \lambda^{-1}(U, K, H)$

**Exercise 2.** Show that $q(t)$ solves (6) if and only also $q(\lambda t) := \lambda q(\lambda^{-3/2} t)$ solves (6). Explain how the exponent $-3/2$ in this transformation-of-paths formula arises from the +3/2 in the time part of the scaling law of eq (9).

McGehee’s genius was to rewrite Newton’s equations, as much as is possible, in scale invariant terms. We cannot completely get rid of scale, but we can encode scale in the single size variable $r = \sqrt{\langle q, q \rangle}$ introduced earlier and through which we remove scale from the remaining variables:

1. \( q = rs \)
2. \( v = r^{-1/2} y \)
3. \( dt = r^{3/2} d\tau \)

These relations define the *McGehee transformation* \((q, v; t) \mapsto (r, s, y; \tau)\). Observe that \( s \) lies on the unit sphere \( r = 1 \) in the configuration space, \( s \in S = S^{dN-1} = \{r = 1\} \subset \mathbb{E} \) so that \( (r, s) \) are spherical coordinates on \( \mathbb{E} \). We sometimes refer to \( s \) as the shape of the configuration \( q \).

**Exercise 3.** Write \( ' \) for \( d/d\tau = r^{3/2} d/dt \). Show that McGehee’s transformation transforms Newton’s equations (6) to the equations

$$r' = rv$$
$$s' = y - \nu s$$
$$y' = \nabla U(s) + \frac{1}{2} by$$

where \( \nu = \langle s, y \rangle \). These equations are the McGehee blown-up equations.
In the last equation $\nabla U(s) \in E$ is the same gradient as in Newton’s equation (9), only that restricted to points $s$ of the sphere $\{r = 1\}$. The blown-up equations are analytic and extend analytically to the total collision manifold $r = 0$. For $N > 2$ the equations still have singularities due to partial collisions eg $r_{12} = 0$, at which $\nabla U(s)$ still blows up.

**Definition 1.** The “extended collision manifold” is the locus $r = 0$ for the blown-up phase space $[0, \infty) \times S \times \mathbb{R}^{dN}$ of McGehee.

The first of the three blown-up ODEs asserts that the extended collision manifold is an invariant submanifold. On the extended collision manifold the flow is non-trivial , as a glance at the last two equations shows. Away from the extended collision manifold, the blown-up equations are equivalent to Newton’s equations. What have we gained by adding this collision manifold?

**4.3. Equilibria!** The first thing one learns in a class in dynamical systems is to look for equilibria. But Newton’s equations have no equilibria! N stars cannot just sit there, still, in space. Adding the extended collision manifold thru blow-up introduces equilibria. When $N = 3$ these equilibria correspond to the solutions of Euler and Lagrange described above. In the general case the equilibria correspond to “central configurations”. (Proposition 1 below.)

*Finding the equilibria.* From the first of the blow-up equations (13) we see that at an equilibrium must lie on the extended collision manifold $r = 0$ (consistent with wha we just said about “stars cannot just sit there”). Plugging the second equilibrium equation $0 = \nu s$ into the third equation of the blown-up equations (13) yields the “shape equation”

$$\nabla U(s) = -\frac{1}{2} \nu^2 s.$$  

Taking the inner product of both sides of the shape equation with $s$ and using Euler’s identity for homogeneous functions yields

$$U(s) = \frac{1}{2} \nu^2$$

or $\nu = \pm \sqrt{2U(s)}$. Now the gradient of the function $r^2$ at a point $s \in S$ is $2s$ so that we can rewrite the shape equation as

$$\nabla U(s) = c\nabla(r^2), c = -\frac{1}{4} \nu^2 = -\frac{1}{2} U(s).$$

Think of $c$ as a Lagrange multiplier. We have proved that the shape $s$ of an equilibrium configuration must be a “central configuration” where:

**Definition 2.** A central configuration is a shape $s \in S$ which is a critical point of $U$ restricted to the sphere $r = 1$.

Conversely, for each central configuration shape $s_{cc} \in S$ we obtain an equilibrium point $(r, s, y) = (0, s_{cc}, y)$ by setting $y = \nu s_{cc}$. with $\nu = \pm \sqrt{2U(s_{cc})}$. We have established

**Proposition 1.** Equilibria of the blown-up equation are in 2:1 correspondence with central configurations. This correspondence associates to a given central configuration $s_{cc}$ the two equilibria $(r, s, y) = (0, s_{cc}, \nu s_{cc})$, with $\nu = \pm \sqrt{2U(s_{cc})} s_{cc}$.
There is another way to arrive at central configurations in keeping with our original discussion of the Euler and Lagrange solutions. Make the ansatz:

\[ q(t) = \lambda(t)s \]

where \( \lambda(t) \) is a time dependent scalar and \( s \in S \) is constant.

**Exercise 4.** Show that the ansatz (14) satisfies Newton’s equations if and only if \( s \) is a central configuration and \( \lambda(t) \) satisfies the “Kepler problem”:

\[ \ddot{\lambda} = -\mu\lambda/|\lambda|^3 \quad \mu = \frac{1}{2}\nu^2 = U(s) \]

All solutions of the one-dimensional Kepler problem end in collision: \( \lambda = 0 \). This value of \( \lambda \) in the ansatz corresponds to total collision. The ansatz (14) with real scalar \( \lambda(t) \) yields the Lagrange and Euler solutions to the three body equations which we first described above by “dropping” bodies.

4.3.1. The Euler and Lagrange family. Planar problems. Assume we are in the planar case so that \( d = 2 \). Identify \( \mathbb{R}^2 \) with \( \mathbb{C} \): \( (x,y) \rightarrow x + iy \) so that \( E = \mathbb{C}^N \) and so that complex scalar multiplication of \( s = (s_1, \ldots, s_N) \in S \subset \mathbb{C}^N \) by \( \lambda \in \mathbb{C} \) corresponds to scaling the \( N \)-gon \( s \) by the factor \( |\lambda| \) while rotating it by \( \text{Arg}(\lambda) \).

**Exercise 5.** Show that \( \nabla U(\lambda q) = \frac{\lambda}{|\lambda|^2} \nabla U(q) \)

**Exercise 6.** Use exercise 5 to show that exercise 4 also holds in the case of the planar \( N \)-body problem, with now \( \lambda(t) \in \mathbb{C} \) a complex scalar.

The solutions of exercise 6 are motions in which all \( N \) bodies move “homographically” : meaning by scaling and rotating. Since \( \lambda(t) \) parameterizes a conic, each body moves along a homographic conic \( \lambda(t)s_a, a = 1, 2, \ldots, N \). We now have, for each planar central configuration, a family of solutions parameterized by the complex solutions to eq. (15), and varying from total collision solutions when \( \lambda(t) \in \mathbb{R} \) to circular motions \( \lambda(t) = e^{i\omega t} \in S^1 \subset \mathbb{C} \). For fixed energy \( h \) we can think of the parameter of the family as the angular momentum \( J \) discussed below, with \( J = 0 \) being total collision and the maximum or minimum value of \( J \) being the circular motion.

4.3.2. An open problem. The potential \( U \) is invariant under rotations and translations. Consequently, the central configurations as we defined them are not isolated, but come in families.

Is the set of central configurations, modulo rotations and translations, a finite set? This problem is attributed to Chazy [6]. See Albouy-Cabral [2] and for perspective and a recent survey.

What is known. Some History. \( N = 3 \): Euler and Lagrange had established the complete list of central configurations as described here. \( N = 4 \): Albouy [1] classified the central configurations in the case of 4 equal masses two centuries two decades and a few years after Euler and Lagrange. One of his main achievements was to show that in the equal mass case the 4-body central configurations all have a reflectional symmetry. Eleven years after Albouy’s work Hampton and Moeckel [11] proved that the central configurations are finite (less than 1856 XXX in number!) \( N = 5 \): In 2012 Albouy and Kaloshin [3] proved that for \( N = 5 \) and away from an algebraic surface in the parameter space \( \mathbb{RP}^4 \) of mass ratios, the number of central configurations is finite. In 1999 Roberts [23] constructed examples for \( N = 5 \) with
one of the five masses negative in which the set of central configurations is infinite, underlining the subtlety of the problem.

4.4. Linear and angular momentum. Besides energy, the only known constants of the motion for the general N-body problem are the components of the linear momentum

\[ P = \sum m_a v_a \]

and the angular momentum

\[ J(q, v) = \sum m_a q_a \wedge v_a \]

These momenta are intimately connected to the fact that the group G of rigid motions acts by symmetries of Newton’s equations.

Exercise 7. v ∈ E is orthogonal to the G orbit thru q ∈ E if and only if P(v) = 0 and J(q, v) = 0

4.5. Center of mass frame. A well-know argument using Galilean symmetry and found in essentially any introductory physics text allows us to suppose that all our solutions satisfy \( P = 0 \) and \( \sum m_a q_a = 0 \)

In this case we say that we are in “center of mass frame” and we set

\[ E_{cm} = \{ q \in E : \Sigma m_a q_a = 0 \} \cong \mathbb{R}^{d(N-1)} \]

The infinitesimal generators of the translation action are the vectors \( q = (c, c, \ldots, c) = c \mathbf{1}^r, c \in \mathbb{R}^d \). Now \( E_{cm} \) is precisely the orthogonal complement to the subspace of vectors of the form \( c \mathbf{1} \). This space of vectors corresponds to the generators of the translation group, or alternatively to the space of all total collision configurations. It follows that \( E_{cm} \) realizes the quotient of \( E \) by translations and that in \( E_{cm} \) only one point represents total collision: the origin.

We can go to center of mass frame before or after blow-up, the result is the same, namely the system of ODEs (13) (with poles on the partial collision locus) but restricted to the the subset of variables

\[ (r, s, y) = [0, \infty) \times S_{cm} \times E_{cm} \]

where

\[ S_{cm} = \{ q \in E_{cm} : \langle s, s \rangle = 1 \} \cong S^{d(N-1)-1} \].

4.6. Energy-momentum level sets and the Standard Collision Manifold. Because energy and angular momentum are invariant as we flow according to Newton, by fixing their values \( h \) and \( J_0 \) we obtain invariant submanifolds of phase space:

\[ M^{int}(h) = \{ H = h, r > 0 \} \]

and

\[ M^{int}(h, J_0) = \{ H = h, r > 0 \} \]

Energy and angular momentum are not defined at \( r = 0 \) so we have excluded \( r = 0 \). Set

\[ M(h) = \text{Closure}(M^{int}(h)), \quad M(h, J_0) = \text{Closure}(M^{int}(h, J_0)) \]

the closure being within within the blown-up phase space. We will need to understand the boundaries of these spaces, which is their intersection with the extended
collision manifold \( r = 0 \); in other words we must understand how these invariant submanifolds approach the extended collision manifold \( \{ r = 0 \} \) as \( r \to 0 \).

The following notation will be useful in this endeavor.

**Definition 3. [Notation]** For \( F = F(q, v) \) a homogeneous function on \( \mathbb{R} \times \mathbb{R} \) write \( \tilde{F} \) for the scale-invariant version of \( F \) achieved by multiplying \( F \) by \( r^{-\alpha} \) where \( \alpha \) is the degree of homogeneity of \( F \) with respect to our weighted scaling. Thus: \( F(q, v) = r^\alpha \tilde{F}(s, y) \).

According to “energy balance” both the potential energy, kinetic energy, and total energy are homogeneous of degree \(-1\). Thus

\[
\tilde{U}(s) = rU(q)
\]

where \( \tilde{U} \) is homogeneous of degree 0 and can be viewed as a function on the sphere \( S_{cm} \). And

\[
\tilde{H}(s, y) = \frac{1}{2} \langle y, y \rangle - U(s) = \tilde{K}(y) - \tilde{U}(s)
\]

and \( \tilde{K}, \tilde{U} \) are homogeneous of degree 0. The angular momentum is homogeneous of degree \( 1/2 \) so that

\[
J = r^{1/2} \tilde{J}(s, y)
\]

where \( \tilde{J} \) is scale invariant and equals \( \sum m_a s_a \wedge y_a \).

If follows immediately from eq (19) that

\[
\partial(M(h)) = \{ \tilde{H} = 0, r = 0 \}.
\]

while using in addition eq (20) we see that

\[
\partial(M(h, 0)) = \{ \tilde{H} = 0, \tilde{J} = 0, r = 0 \}.
\]

These are basic important submanifolds so we give them separate names.

**Definition 4.** The full collision manifold is \( M_0 = \{ \tilde{H} = 0, r = 0 \} \).

**Definition 5.** The “standard collision manifold” is the locus

\[
C := \{ r = \tilde{H} = \tilde{J} = 0 \}.
\]

Thus the extended collision manifold contains the full collision manifold \( M_0 \) which in turn contains the standard collision manifold \( C \). The equilibria all lie on \( C \). Another reason for the importance of the standard collision manifold \( C \) is a theorem of Sundman.

**Theorem 2.** (Sundman) If \( r \to 0 \) along an honest solution, then \( J = 0 \) for that solution and hence that solution tends to \( C \) as \( r \to 0 \). Moreover, the solution tends to the subset of equilibria within \( C \).

Here we are using the hopefully obvious

**Definition 6.** An “honest solution” to the blown-up equations is a solution such that \( r > 0 \).
The honest solutions are just the reparameterizations of solutions to our original Newton’s equations according to the blown-up time.

**Remark.** The standard collision manifold \( C \) is the space most authors refer to when they speak of the “collision manifold” for the \( N \)-body problem. Chenciner (see also [7]) argues that the standard collision manifold is the dilation quotient of the \( N \)-body phase space.

**Exercise 8.** Use eq (13) to show that
\[
\frac{d}{d\tau} \tilde{H} = \nu \tilde{H},
\]
\[
\frac{d}{d\tau} \tilde{J} = -\frac{1}{2} \nu \tilde{J}
\]
hold everywhere on the blown-up phase space.

It follows from this exercise that \( \{ \tilde{H} = 0 \} \) and \( \{ \tilde{J} = 0 \} \) are invariant manifolds, as are \( M_0 \) and \( C \).

5. **Quotient by Rotations.**

Newton’s equations and their McGehee blow-ups (eq 13) are invariant under the group \( G \) of rigid motions and so descend to ODEs on the quotient space of their phase spaces by \( G \). Working on this quotient instead of the original helps our intuition enormously, especially in the case \( N = 3 \) and \( d = 2 \). We describe the quotient and some aspects of the quotient flow.

The group \( G \) of rigid motions is the product of two subgroups, the translation group and the rotation group. We have already formed the quotient of phase space by translations when we went to center-of-mass frame, i.e. by restricting to \( s, y \in \mathbb{E}_{cm} \). To form the remaining quotient by rotations it is much cleaner to restrict to the planar case \( d = 2 \). Henceforth we assume that we are working with the planar \( N \)-body problem, \( d = 2 \).

We identify \( \mathbb{R}^2 \) with \( \mathbb{C} \) as before. Thus \( \mathbb{E} \cong \mathbb{C}^N \) and \( \mathbb{E}_{cm} \cong \mathbb{C}^{N-1} \). Represent rotations as unit complex scalars \( u \in S^1 \subset \mathbb{C} \) acting on \((q,v) \in \mathbb{E}_{cm} \times \mathbb{E}_{cm} \) by \((q,v) \mapsto (uq,uv)\) and on McGehee coordinates by \((r,s,y) \mapsto (r,us,uy)\).

**Definition 7.** The blown up reduced phase space in the planar case is the quotient of the blown-up center of mass phase space \([0, \infty) \times S_{cm} \times \mathbb{E}_{cm} \cong [0, \infty) \times S^{2N-3} \times \mathbb{C}^{N-1} \) by the group of rotations. Upon deleting the collision locus \( C \) we denote this quotient by
\[
\mathcal{P}_N = ([0, \infty) \times (S^{2N-3} \setminus C) \times \mathbb{C}^{N-1}) / S^1.
\]

Momentarily forget the velocities \( v \) or \( y \), and the deletion of the collision locus \( C \) in trying to understand the quotient. The circle action sends a blow-up configuration \((r,s) \) to \((r,us), s \in S_{cm} \). So we need to understand the quotient of the sphere \( S_{cm} = S^{2N-3} \) by this action of \( S^1 \). It is well known that this quotient \( S_{cm} / S^1 \) is isomorphic to the complex projective space \( \mathbb{CP}^{N-2} := \mathbb{P}(\mathbb{E}_{cm}) \) with the projection map \( S_{cm} \rightarrow S_{cm} / S^1 \) being the Hopf fibration. Hence the quotient of the \((r,s) \) by \( S^1 \) yields \([0, \infty) \times \mathbb{CP}^{N-2} \).

To better understand the meaning of points of \( \mathbb{CP}^{N-2} \), work with \( q \in \mathbb{E}_{cm} \) instead of \( q = s \in S_{cm} \), insisting only that \( q \neq 0 \) and now allowing the scalar \( u \) to vary over the larger group \( \mathbb{C}^* \supset S^1 \) of all nonzero complex numbers. The resulting quotient is well-known to be \((\mathbb{C}^{N-1} \setminus \{0\}) / \mathbb{C}^* = \mathbb{CP}^{N-2} \). The action of \( u \in \mathbb{C}^* \) on
$q \in \mathbb{C}^{N-1} \setminus \{0\}$ is precisely the action of rotating and scaling the (centered) N-gon $q$.

**Definition 8.** The projective space $\mathbb{CP}^{N-2}$ just constructed is called shape space. Its points represent oriented similarity classes of planar N-gons.

We have realized the configuration part of the quotient after blow-up as $[0, \infty) \times \mathbb{CP}^{N-2}$ where $\mathbb{CP}^{N-2}$ is the shape space. When $N = 3$ the shape space is the shape sphere described above.

**Collision locus.**

The condition that a configuration $q = (q_1, \ldots, q_N)$ represent a collision is that $q_a = q_b$ for some $a \neq b, 1 \leq a, b \leq N$. This condition is complex linear when viewed in homogeneous coordinates $[q_1, q_1, \ldots, q_N]$ and so defines a complex hyperplane, a $\mathbb{CP}^{N-3} \subset \mathbb{CP}^{N-2}$. There are are $\binom{N}{2}$ pairs $(a,b)$ and so we have to delete $\binom{N}{2}$ hyperplanes from our shape space. The union of these hyperplanes, viewed projectively, is the collision locus:

$$\mathcal{C} = \{[q] = [q_1, q_2, \ldots, q_N] \in \mathbb{CP}^{N-2} : q_a = q_b \text{ some } a \neq b\}.$$  

We use the same symbol for the collision locus before or after quotient.

**Accounting for velocities.** In the last few paragraphs above we dropped the velocity $y$. The quotient map $(r, s) \mapsto (r, [s])$ from $[0, \infty) \times S_{cm} \rightarrow [0, \infty) \times \mathbb{CP}^{N-2}$ expresses $[0, \infty) \times S_{cm}$ as a principal $S^1$ bundle over $[0, \infty) \times \mathbb{CP}^{N-2}$.

Now include the velocity $y$. The quotient procedure with $y$ included is precisely the procedure used to construct an associated vector bundle to a principal bundle. Realizing this, we see that the quotient $P_N$ is a complex vector bundle over $[0, \infty) \times (\mathbb{CP}^{N-2} \setminus \mathcal{C})$ whose rank is $N - 1$ — the fiber being coordinatized by $y \in E_{cm}$. What is this vector bundle?

**Proposition 2.**

$$P_N = [0, \infty) \times T(\mathbb{CP}^{N-2} \setminus \mathcal{C}) \times \mathbb{R}^2$$

as a vector bundle over $[0, \infty) \times (\mathbb{CP}^{N-2} \setminus \mathcal{C})$. The final $\mathbb{R}^2$ factor is coordinatized by $(\nu, J)$ where $\nu = (s, y)$ represents the time rate of change of size and where $J = \langle is, y \rangle$ is also equal to $r^{-1/2}J$ off of $r = 0$ where $J$ is the usual total angular momentum of the system. The fiber variable tangent to shape space $\mathbb{CP}^{N-2}$ represents “shape” velocity.

In the case of $N = 3$ we have $\mathbb{CP}^{N-2} = \mathbb{CP}^1 = S^2$, the shape sphere previously discussed in section 3. Then

$$P_3 = [0, \infty) \times T(S^2 \setminus \mathcal{C}) \times \mathbb{R}^2 = [0, \infty) \times \mathbb{R} \times (S^2 \setminus \mathcal{C}) \times \mathbb{R}^3$$

where

$$\mathcal{C} = \{B_{12}, B_{23}, B_{31}\}$$

is the set of three binary collision points.

5.0.1. **Velocity (Saari) decomposition.** Passing thru a configuration $q \in E_{cm}$, we have two group-defined curves: the scalings $\lambda q, \lambda \in \mathbb{R}$ of $q$ and the rotations $uq, u \in S^1$ of $q$. The tangent spaces to these curves are orthogonal, and together with the orthogonal complement of their span they define a geometric splitting of $T_q E_{cm} = E_{cm}$

$$T_q E_{cm} = (\text{scale}) + (\text{rotation}) + (\text{horizontal})$$

(21)

$$= \mathbb{R} q \oplus i\mathbb{R} q \oplus \{v : J(q, v) = 0, \nu(q, v) = 0\}$$

(22)
Definition 9. The horizontal space at \( q \) is the orthogonal complement (rel. the mass metric) of the sum of first two subspaces \( \mathbb{R} q \) and \( i \mathbb{R} q \), i.e it is the orthogonal complement to the \( C \)-span of \( q \).

Refer to exercise 7 and the definition of \( \nu \) to see why the horizontal space at \( q \) is, as described above, the zero locus of \( J(q,v) \) and \( \nu(q,v) \).

Unit vectors spanning the scale and rotation spaces are \( s \) and \( is \). Consequently, if we take a \( v \in T_q \mathbb{E}_{cm} \) and decompose it accordingly we get

\[
(23) \quad v = \langle s, v \rangle s + \langle is, v \rangle is + v_{\text{hor}}
\]

and the scale invariant version:

\[
(24) \quad y = \nu s + ˜Jis + y_{\text{hor}}; \quad \nu = \langle s, y \rangle, \quad ˜J = \langle is, y \rangle
\]

where the subscript “hor” on \( v \) and \( y \) denote their orthogonal projections onto the horizontal subspace.

Remark D. Saari pointed out the importance to celestial mechanics of the horizontal-vertical splitting of eq (23) and hence this splitting is often called the “Saari decomposition”.

5.0.2. Proof of proposition 2. The decomposition (eq (24)) of \( y \) is \( S^1 \)-equivariant. The coefficients of the first two terms \( \nu \) and \( ˜J = \langle is, y \rangle \) are \( S^1 \)-invariant functions and so are well defined functions on the quotient \( \mathcal{P} \). The horizontal term \( y_{\text{hor}} \), as \( y \) varies at fixed \( s \), sweeps out the horizontal subspace at \( s \) and these subspaces, as \( s \) varies, forms the horizontal distribution associated to a connection on the principal \( S^1 \)-bundle \( S_{cm} \rightarrow \mathbb{C} \mathbb{P}^{N-2} \). It is a basic fact about principal \( G \)-bundles with connection that the union of the horizontal spaces for the connection forms a \( G \)-equivariant vector bundle over the total space, and the quotient of this vector bundle by \( G \) is canonically isomorphic to the tangent space to the base space. Writing \([s,y]\) to denote the \( S^1 \)-equivalence class of the pair \((s,y)\) we see that the set of all \([s,y_{\text{hor}}]\)’s forms \( T \mathbb{C} \mathbb{P}^{N-2} \). Now \( s \), together with \((y_{\text{hor}}, \nu, ˜J)\) determine \( y \) uniquely. It follows that the map \([s,y] \mapsto ([s,y_{\text{hor}}],(\nu, ˜J))\) is a vector bundle isomorphism between the vector bundles \((S_{cm} \times \mathbb{E}_{cm})/S^1 \) and \( T \mathbb{C} \mathbb{P}^{N-2} \times \mathbb{R}^2 \) over \( \mathbb{C} \mathbb{P}^{N-2} \). The radial scaling coordinate \( r \) “goes along for the ride” without change.

QED

Because the decompositions of equations (23, 24) are orthogonal and the second decomposition is scale invariant it follows that total kinetic energy decomposes as

\[
(25) \quad K(q,v) = \frac{1}{2} \frac{\nu^2}{r} + \frac{1}{2} \frac{J^2}{r^2} + \frac{K_{\text{shape}}([s,y_{\text{hor}}])}{r}
\]

The final term \( K_{\text{shape}} \) is formed by computing the squared length of the horizontal factor \( y_{\text{hor}} \) and is canonically identified with the kinetic energy of the standard (Fubini-Study) metric on the shape space \( \mathbb{C} \mathbb{P}^{N-2} \).

Remark. The kinetic energy decomposition (25) shows that for \( J \neq 0 \) the manifolds \( M^{\text{int}}(H_0, J) \) is already closed in \( \mathcal{P} \) so that

\[
(26) \quad M(h,J) = M^{\text{int}}(h,J)
\]
Indeed, the energy equation \( rh = \tilde{H} \) shows that \( \dot{U} \geq \frac{1}{2} J^2/r + O(r) \) holds on \( M^{\text{int}}(h, J) \) which shows that if for a sequence \( p_i \in M^{\text{int}}(H_0, J) \) we have that \( r(p_i) \to 0 \) then \( U(s_i) \to \infty \) so that the shape \( s_i \) of these points \( p_i \) are converging to the collision locus \( C \subset \mathbb{CP}^{N-2} \) on the shape space. But we deleted \( C \) in forming \( \mathcal{P}_N \).

5.1. Euler-Lagrange family in reduced coordinates. We follow Moeckel and look into what a planar central configuration family of section 4.3.1 such as the Euler or Lagrange family looks like in the coordinates of \( \mathcal{P}_N \).

Let \( s_{cc} \) be a planar central configuration and \([s_{cc}] \in \mathbb{CP}^{N-2} \) the corresponding point in shape space. During the evolution of the associated family, this shape does not change. Only the size \( r \) and angle \( \theta \) of of the configuration changes. This size and angle change is specified by \( \lambda = \lambda(t) = re^{i\theta} \) where \( \lambda(t) \) solves the Kepler problem as per exercise 6. Since the shape does not change, the shape velocity \( y_{hor} \) is identically zero along each of these solutions and so \( K_{sh} = 0 \). Thus along such a solution

\[
\dot{K} = \frac{1}{2} \dot{r}^2 + \frac{1}{2} \dot{J}^2 = \frac{1}{2} \nu^2 + \frac{1}{2} J^2/r
\]

(see eq. (25)) and the only variables which change are \((r, \nu, J)\) among the full set of variables \((r, [s, y_{hor}], (\nu, J))\) of \( \mathcal{P}_N = [0, \infty) \times T(\mathbb{CP}^{N-2} \setminus C) \times \mathbb{R}^2 \) But \( J = r^{1/2}J \) and \( J \) is constant along solutions so the change of \( r \) and choice of \( J \) determines the change of \( J \). So we can think of the only variables being \( \nu, r \).

Fix the energy \( h \). We can then view the central configuration family as a one-parameter family of curves in the \((\nu, r)\) plane, the parameter being the angular momentum \( J \). Indeed the energy equation reads:

\[
rh = \frac{1}{2} \nu^2 + \frac{1}{2} J^2/r - U(s_{cc}).
\]

and since \( U(s_{cc}) \) is constant, this defines a one-parameter family of curves. We plot these curves in the \((\nu, r)\) plane for various values of the angular momentum \( J \) below in figure 5.1

Observe the rest point cycle in this picture: the closed curve passing thru the two equilibria. This curve is the union of two solution curves, a top arch which is an honest solution, and a bottom return curve. The top arch is the ejection-collision orbit first described when we described Lagrange’s solution: it explode out of total collision along the shape \( s_{cc} \) achieves a maximum size and shrink back to triple collision. It connects the rest point \( s \in C \) having shape \( s_{cc} \) and \( \nu = \sqrt{2U(s_{cc})} > 0 \) the rest point \( s_a \) having shape \( s_{cc} \) and \( \nu = -\sqrt{2U(s_{cc})} < 0 \). This top arch lies on \( M(h, 0) \). The bottom ‘return road’ lies on the full collision manifold \( \{r = 0, \tilde{H} = 0\} \) and yields a return route from \( s_a \) to \( s \). This rest point cycle is the limit of the family of the periodic central configuration solutions with \( J \neq 0 \) as \( J \to 0 \).

Notational Convenience. We have just used the symbol \( M(h, 0) \subset \mathcal{P}_N \) for what used to be a submanifold of the phase space before quotient. We will continue to use the same notation for any \( G \) invariant submanifold or function on phase space before or after the quotient procedure. Thus we have:

\[
C, M_0, M(h), M^{int}(h), M(h, J_0), \text{ etc.} \subset \mathcal{P}_N.
\]
Figure 6. A central configuration family in $\nu, r$ coordinates. The arch and ‘floor’ $r = 0$ comprise the rest cycle

6. A GRADIENT (LIKE) FLOW!

The dominant aspect of the flow on the full collision manifold $M_0$ is that $-\nu$ acts like a Liapanov function.

**Exercise 9.** Use equations (13) to derive the identity

$$\nu' = \tilde{K} - \frac{1}{2} \nu^2 + \tilde{H}$$

(See for example Moeckel [18, eq. (1.6).])
Exercise 10. Use the “Saari decomposition” of kinetic energy (eq (25)) to show that
\[
\tilde{K} - \frac{1}{2} \nu'^2 = K_{sh} + \frac{1}{2} \tilde{J}^2.
\]
Conclude, using the previous exercise, that
\[
\nu' = K_{sh} + \frac{1}{2} \tilde{J}^2 \geq 0 \text{ on } M_0 = \{ r = 0, \tilde{H} = 0 \}.
\]

You have proved much of Proposition 3. \(\nu' \geq 0\) everywhere on the full collision manifold \(M_0\). Moreover \(\nu\) is constant along a solution lying in \(M_0\) if and only if that solution is one of the equilibria.

Remark. A flow is called “gradient-like” if it admits a continuous function \(f\) which is strictly monotone decreasing along all solution curves except equilibria. (See Robinson [24] p. 357.) The proposition thus asserts that the blown-up flow is gradient-like on the full collision manifold \(M_0\) relative to the function \(f = -\nu\).

Proof of proposition 3. In the exercise you proved that \(\nu'\) is positive everywhere except at the points where it is zero. We must then show that any solution which lies on the locus \(\nu' = 0\) is an equilibrium. We see that \(\nu' = 0\) if and only if \(K_{sh} = 0 = \tilde{J}\). Now \(d\tilde{J}/d\tau = -\frac{1}{2} \nu \tilde{J}\). (This holds both on and off the collision manifold.) It follows that any solution starting on \(\tilde{J} = 0\) remains on the locus \(\tilde{J} = 0\). \(K_{sh}(s, y) = 0\) if and only if \(y_{hor} = 0\) in which case, both the \(y_{hor}\) and the \(is\) term (from \(\tilde{J} = (is, y)\)) in the decomposition of \(y\) are zero so that \(y = \lambda s\) with \(\lambda \in \mathbb{R}\). Take inner products with \(s\) to get \(\lambda = \nu\). Now assume we have a solution curve \((s(\tau), y(\tau))\) lying on the locus \(\tilde{J} = 0, K_{sh} = 0\). Differentiating the equation \(y(\tau) = \nu(\tau)s(\tau)\) using the blow-up equations we see that \(y' = \nu' s + \nu s'\). But \(\nu' = 0\) by assumption and \(s' = y - \nu s = 0\) by the blow-up equations, so \(y' = 0\) along the solution: our curve is an equilibrium.

QED

Pause for a moment. Reflect how different flow on the full collision locus is from a Hamiltonian flow on an energy level set.

6.1. Moeckel’s manifold with corner into a manifold with a T. In [15], at the beginning of section 2, Moeckel constructs a certain manifold with corners in preparation for perturbing the heteroclinic tangles lying on \(M(h, 0)\) into the realms of \(M(h, \epsilon)\). (He denotes his manifold with a corner by \(M_{0+}\) and later simply \(M\).) Dynamics on this manifold-with-corners is essential to our proof of theorem 1. I had a hard time making sense of this manifold. I rederive what Moeckel did in a slightly different way. I get a “manifold with a T” instead of Moeckel’s manifold with a corner. A “T” is made out of two corners, or “L”s (one reflected relative to the other) joined along their vertical edge. One of these corners is Moeckel’s manifold with a corner and the other is a reflection of it. The vertex, or corner itself, is our good friend \(C\) the standard collision manifold. (Figure 6.1.)

Recall that \(M(h)\) is a hypersurface in \(P_N\), and as such is a manifold with boundary, whose boundary is our friend full collision manifold \(M_0 = \{ r = 0, \tilde{H} = 0 \}\).

Definition 10. \(\hat{M}(h) = M(h, 0) \cup M_0 \subset \hat{M}(h)\).
\(\hat{M}(h)\) is a codimension 1 subvariety of the smooth manifold with boundary \(M(h)\). It is the zero locus of the function \(r\tilde{J}\) restricted to \(M(h)\) and as such has two algebraic components: \(r = 0\) which is our full collision manifold \(M_0\), and \(J = 0\) which forms \(M(h,0)\). The singular locus of \(\hat{M}(h)\) is the intersection \(C = \{r = 0, \tilde{J} = 0\}\) of these two components. All the rest point cycles described above associated to the central configurations lie on this \(\hat{M}(h)\). \(\hat{M}(h)\) is comprised of two “manifolds with corners”, namely \(\{r\tilde{J} = 0, \tilde{J} \geq 0\}\) and \(\{r\tilde{J} = 0, \tilde{J} \leq 0\}\). The first of these is Moeckel’s.

\(\hat{M}(h)\) is to be viewed as the limit as \(J \to 0\) of the manifolds \(M(h,J)\).

**Figure 7.** \(\hat{M}(h)\) inside \(M(h)\) is the zero level set of \(r\tilde{J}\).

**Proposition 4.** For \(S \subset \mathbb{R}\) a subset of the line of angular momentum values, set \(M^{\text{int}}(h,S) = \bigcup_{J \in S} M^{\text{int}}(h,J)\). Then \(\hat{M}(h) = \cap_{\epsilon > 0} M^{\text{int}}(h,(\epsilon,\epsilon))\).
The proof of the proposition follows in a routine way from our expressions for energy, from \( rh = \tilde{H}, \ J = r^{-1/2}J \) and the kinetic energy decomposition of eq 25.

It is useful to recall, eq (26) that the \( M(h,J) = M^{int}(h,J) \) are closed for \( J \neq 0 \).

As an alternative to the description of the proposition, we can either let \( J \to 0 \) from above or below. Set

\[
\hat{M}_+(h) = \lim_{J \to 0^+} M(h,J)
\]

and

\[
\hat{M}_-(h) = \lim_{J \to 0^-} M(h,J).
\]

Then one can show without difficulty that

\[
\hat{M}(h) = M_+(h) \cup M_-(h),
\]

with \( M_+(h) = \{ p \in \hat{M}(h), \tilde{J} \geq 0 \} \) and \( M_-(h) = \{ p \in \hat{M}(h), \tilde{J} \leq 0 \} \) being the two manifolds with corners described earlier, Moeckel’s manifold with a corner being \( \hat{M}_+ \).

What is a manifold with a ‘\( T \’ \)? Suppose we have two real-valued functions \( x, y \) on an \( n \)-dimensional manifold \( Q \) such that 0 is a regular value for both functions and \( (0,0) \) is a regular value of the map \( (x,y) : Q \to \mathbb{R}^2 \). Then the locus \( \{ xy = 0, y \geq 0 \} \) is a manifold with a \( T \). Its singular locus is \( \{ x = y = 0 \} \). A manifold with a \( T \) is locally diffeomorphic to the product of the “upside down \( T \)” \( xy = 0, y \geq 0 \) in the \( xy \) plane, by an \( \mathbb{R}^{n-2} \). See figure 6.1.

6.2. **Finishing up the proof of theorem 1.** The idea of Moeckel is that hyperbolic structures persist on perturbation, and that the various stable-unstable connections between Euler and Lagrange central configuration points on on \( \hat{M}(h) \) are sufficiently “hyperbolic” that they persist into \( M(h, \epsilon) \) for \( \epsilon \neq 0 \) small. Nonzero angular momentum is needed to get orbits connecting from \( R’ \)'s to \( R^* \) in finite time since the rest cycle of figure 5.1 takes infinite blown-up time. Moeckel cannot carry out the “perturbation of hyperbolic” idea literally because he cannot establish the needed hyperbolicity or transversality. Instead, following an earlier idea of Easton, he replaces hyperbolicity by a weaker notion of “topologically transverse” between collections of “windows” transverse to the flow. This notion is sufficiently flexible and stable to allow Moeckel to perturb the various formal connections to get actual orbits realizing walks in the abstract graph introduced in section 3. By following the details of his proof, three decades later, we were able to verify that his realizing solutions when projected onto the shape space do indeed stay \( C^0 \)-close to the concrete connection graph as described in section 3.

The hypothesis of equal or near equal masses is needed to insure that (some of) the eigenvalues for the linearization at the Euler equilibria are complex. This complexity implies a “spiralling” of the Lagrange stable/unstable manifolds around the Euler unstable/stable manifolds and is needed to insure that all connections in the abstract connection graph are realized.

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