Itô calculus for Cramér-Lundberg model

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Abstract

In insurance mathematics, specifically in risk theory, mainly functional analytic techniques are used. In this paper, we give an alternative approach to deriving some well-known, basic, but important results on the classical collective risk model. Applying techniques based on Itô’s calculus, we derive an integro-differential equation for the Gerber-Shiu function, under the Cramér-Lundberg model.

Keywords risk theory, ruin probability, Itô calculus

Research Activity Group Mathematical Finance

1. Introduction

In this paper, we consider a classical model of an insurer’s asset liability, Cramér-Lundberg model, and derive a simple and classical result, on the expected discounted penalty function, so-called Gerber-Shiu function, using Itô’s calculus. Although, in this paper we treat only this simple compound Poisson case, our method is applicable to more general cases and, to the best of the authors’ knowledge, not commonly known in the actuarial mathematics or stochastic calculus communities. The aim of the present paper is to provide the readers with a quick understanding of the method.

The Cramér-Lundberg model was first introduced as a compound Poisson process with drift. The ruin probability was defined as the first time the process becomes negative, and its derivation attracted and inspired a multitude of mathematical methods. An analytical approach, in which the ruin probability of ruin is the solution of some well-crafted integral differential equations, with boundary conditions, received a lot of impetus in the actuarial literature. The derivation of these integro-differential equations employed various techniques, from empirical probabilistic approaches, martingales and generators, to renewal arguments and fixed point random equations, see [1–5], and the references therein. Moreover, when the classical model has been enriched to capture more details of the reality and therefore, more elaborated models where developed and new methods stemmed from it. For instance, in a financial environment setting, [6–8] relies on stochastic calculus.

Here we introduce a stochastic analytic technique, which provides a unified approach for models with jumps. We explore the method in detail for the classical Cramér-Lundberg model. Moreover, as long as we can employ Itô’s formula (for general semimartingales) to characterize discrete-time martingales, exemplified by (7), we can apply the same recipe in the larger setting of jump-diffusion models. See [9] for full details.

2. Setting and the result

Let \((U_t)_{t \geq 0}\) be a stochastic process given by

\[
U_t = u + ct - \sum_{i=1}^{\infty} X_i 1_{\{t \geq \sum_{j=1}^{i-1} T_j\}}, \quad t \geq 0, \tag{1}
\]

where \(c > 0\) and

(1) \(T_j, j = 0, 1, \ldots,\) holding time between \(j - 1\) th jump and \(j\) th jump, are independent, identically exponentially distributed positive random variables;

\[
P(0 \leq T_i \leq t) = 1 - e^{-\lambda t} \quad t \geq 0, i = 0, 1, \ldots.
\]

(2) We assume that \((X_i)_{i \geq 1}\) and \((T_i)_{i \geq 1}\) are independent.

(3) \(X_1, X_2, \ldots,\) the size of claims that the insurance company is liable at the random times \((T_j)_{j \geq 0}\) at which the claims are paid, are independent, identically distributed (i.i.d.) positive random variables, the common distribution function of which has the continuous density \(f_X\) on \([0, \infty),\) that is,

\[
P(X_i \leq x) = \int_0^x f_X(y) dy, \quad \forall i.
\]

Remark 1 The jump-part of the process (1) can be rewritten as

\[
\sum_{i=1}^{N_t} X_i =: R_t, \tag{2}
\]

where

\[
N_t := \sum_{i=1}^{\infty} 1_{\{t \geq \sum_{j=1}^{i} T_j\}}.
\]
which is a Poisson process with intensity $\lambda$. That is to say, (2) is a compound Poisson process.

Let

$$T^* := \inf\{t > 0 : U_t < 0\}.$$  

Our target in this paper is to give a description of the **expected discounted penalty function**, referred to as the Gerber–Shiu function in actuarial mathematics.

For a discount factor $\delta \geq 0$ and a positive bounded continuous penalty function $w$, define

$$h(u; \delta) := E[1_{[T^*, t)}] e^{-\delta T^*} w(U_{T^*}) | U_0 = u].$$  

for $u > 0$. The Gerber–Shiu function $h$ coincides with various functions appearing in risk theory, for particular choices of $\delta$ and $w$ in (3). Here a few examples of Gerber–Shiu functions are given.

(1) When $w \equiv 1$, we have

$$h(u; \delta) = E[1_{[T^*, t)}] e^{-\delta T^*} | U_0 = u],$$

which is the Laplace transform of the time of ruin.

(2) When $w \equiv 1$ and $\delta = 0$, we have

$$h(u; 0) = E[1_{[T^*, t)}] | U_0 = u],$$

which is nothing but the probability of ruin.

For more details on the Gerber–Shiu function, we refer to [10].

The following is a well-established fact.

**Theorem 2** (see e.g. [10]) The Gerber–Shiu function (3) is the unique bounded solution of the integro-differential equation

$$-ch' (u) + (\lambda + \delta) h (u) = \lambda \int_0^\infty h(u-y) f_X(y) dy, \tag{4}$$

for $u > 0$, with the boundary conditions

$$\begin{align*}
\lim_{s \to -\infty} e^{-\delta u} h(u) &= 0, \\
h(u) &= w(u), \quad u < 0. \tag{5}
\end{align*}$$

**Proof** As we mentioned, we will give a proof based on stochastic calculus. We start from recalling Itô’s formula for the compound Poisson process with drift (1). Let us define a random measure $N_R$ on $\mathbb{R}^+ \times \mathbb{R}$ by

$$N_R([0, t] \times \mathcal{A}) = \sum_{i=1}^{N_t} \delta_{X_i}(\mathcal{A}), \quad t \geq 0$$

where $\delta_x$ is the Dirac measure at $x$ in $\mathbb{R}$. Then,

$$R_t = \int_0^t \int_{\mathbb{R}} x \, dN_R(ds, dx),$$

and for any function $g$ and $0 \leq s \leq t$

$$g(R_t) - g(R_s) = \sum_{j=0}^{N_t-1-N_s} (g(R_{N_s+j-1} + X_{N_t-j}) - g(R_{N_t-1-j}))$$

$$= \sum_{j=0}^{N_t-1-N_s} \int_{N_t-j-1}^{N_t-1} (g(R_{u-} + x) - g(R_{u-})) \, dN_R(du, dx).$$

Similarly, for $g \in C^1_c(\mathbb{R})$, $0 \leq s \leq t$,

$$e^{-\delta t} g(U_t) - e^{-\delta s} g(U_s)$$

$$= \sum_{j=0}^{N_t-2-N_s} \left( e^{-\delta (N_t-j)} g(U_{N_s+j-1} - X_{N_t-j}) - e^{-\delta (N_s-j-1)} g(U_{N_s+j-1}) \right)$$

$$+ \int_s^t (g(U_{u-} - x) - g(U_{u-})) \, dN_R(du, dx)$$

$$= \sum_{j=0}^{N_t-2-N_s} \int_{N_t-j-1}^{N_t-1} e^{-\delta u} \left( cg'(U_u) - \delta g(U_u) \right) \, du$$

$$+ \int_s^t (g(U_{u-} - x) - g(U_{u-})) \, dN_R(du, dx).$$

Here, the compensator is easily obtained as

$$\hat{N}_R(du, dx) = \lambda f_X(x) \, du \, dx.$$

We then notice that $\{e^{-\delta T_k} g(U_{T_k})\}_{k=0,1,...}$ is a martingale with respect to

$$\mathcal{F}_k := \sigma(\{T_i, X_i\}_{i=1}^k)$$

if and only if

$$E \left[ \int_{T_{k-1}}^{T_k} e^{-\delta u} \left( cg'(U_u) - \delta g(U_u) \right) \, du \right]$$

$$= E \left[ \int_0^{T_1} e^{-\delta u} \left( cg'(U_u) - \delta g(U_u) \right) \, du \right]$$
for any $U_0 = u > 0$, where we have used the i.i.d. property of $(T_i, X_i)$ for $i$.

By the independence of $T_1$ and $X_1$, the second quantity in (6) can be calculated as

$$\lambda E \left[ \int_0^\infty e^{-\lambda t} \int_0^t e^{-\delta v} \left[ (cg'(u + cv) - \delta g(u + cv)) + \int_R (g(u + cv - x) - g(u + cv)) \lambda f_X(x) dx \right] dv dt \right]$$

$$= \lambda E \left[ \int_0^\infty e^{-\lambda t} \int_0^t e^{-\delta v} \left[ (cg'(u + cv) - (\delta + \lambda)g(u + cv)) + \int_R (g(u + cv - x)\lambda f_X(x) dx \right] dv dt \right]$$

$$= 0.$$

Thus, we have confirmed that

$$\{e^{-\delta T_k} h(U_{T_k})\}_{k=0,1...}$$

is a martingale if and only if $h$ is a solution to (4).

Next, suppose that $\{e^{-\delta T_k} h(U_{T_k})\}_{k \in Z_+}$ is a discrete martingale. Then, by the optional sampling theorem, $\{e^{-\delta (T_k \wedge T^\ast)} h(U_{T_k \wedge T^\ast})\}_{k \in Z_+}$ is also a discrete martingale. Therefore,

$$h(u) = E[e^{-\delta (T_k \wedge T^\ast)} h(U_{T_k \wedge T^\ast}) 1_{\{T^\ast < T_k\}} | U_0 = u] + E[e^{-\delta (T_k \wedge T^\ast)} h(U_{T_k \wedge T^\ast}) 1_{\{T^\ast \geq T_k\}} | U_0 = u]$$

$$= E[e^{-\delta (T_k \wedge T^\ast)} w(U_{T_k \wedge T^\ast}) 1_{\{T^\ast < T_k\}} | U_0 = u] + E[e^{-\delta (T_k \wedge T^\ast)} h(U_{T_k \wedge T^\ast}) 1_{\{T^\ast \geq T_k\}} | U_0 = u]$$

where the second equality is due to the condition that

$$h(u) = w(u) \quad u < 0.$$

By letting $k \to \infty$, we obtain that $h$ is the Gerber-Siu function, thanks to the condition (5).

It remains to prove the existence and the uniqueness of the solution to (4) with the boundary condition (5). This is again a classical result but we give its brief description. Let

$$G_k(u) := \kappa - \frac{\lambda}{c} \int_0^u e^{-\frac{\lambda}{c} y} \int_y^\infty w(v - y)f_X(y)dy dv,$$

$$u \geq 0,$$

for $\kappa \in R$ and

$$F(u) := -\frac{\lambda}{c} \int_0^u e^{-\frac{\lambda}{c} y} f_X(y) dy, \quad u \geq 0.$$

By the assumptions on $f_X$ and $w$, we know that $G_k$ and $F$ are in $C_b \cap C^1$.

We shall consider the following Volterra equation (in g):

$$g(u) = G_k(u) + \int_0^u F(u - v) g(v) dv.$$

(8)

At the moment, let us assume temporarily the existence and the uniqueness of the solution in $C_b([0, \infty)) \cap C^1([0, \infty))$ to the equation (8) are established. Then, by differentiating both sides of (8),

$$g'(u) = G_k'(u) + \int_0^u F'(u - v) g(v) dv$$

$$= -\frac{\lambda}{c} e^{-\frac{\lambda}{c} u} \int_0^\infty w(u - v)f_X(v) dv$$

$$- \frac{\lambda}{c} \int_0^u e^{-\frac{\lambda}{c} (u - v)} f_X(u - v) g(v) dv.$$

Then, we see that $e^{(\lambda - \delta)/\lambda} g(t)$ satisfies (4), noticing the right-hand-side of (4) can be calculated as

$$\lambda \int_0^u w(u - v)f_X(v) dv + \lambda \int_0^u f_X(u - v)h(v) dv$$

since $f_X(x) = 0$ for $x < 0$ and since $h(x) = w(x)$ for $x < 0$. Note also that $e^{(\lambda - \delta)/\lambda} g(t)$ is the solution to (8) with the boundary condition (5) is implied by those with (8), which is a well-established fact.

Now we turn to the construction of the solution to the Volterra equation (8). Let $F^1 \equiv F$ and

$$F^n(u) := \int_0^u F^{n-1}(u - v)F(v) dv$$

for $n \geq 2$, that is, $n$-times convolution: $F^n = F^{*n}$. Since

$$\|F\|_\infty := \sup_u |f_X(u)| = \frac{\lambda}{c} e^{-\frac{\lambda}{c} X} < \infty,$$

we have that

$$\|F^{*n}(u)\| \leq \|F\|_\infty \int_{u \geq u_1 \geq \cdots \geq u_n \geq 0} du_1 du_2 \cdots du_n$$

(9)

Therefore,

$$\sum_n F^n * G_k(u) \leq \exp(\|F\|_\infty u)$$

and thus $\sum_n F^n * G_k(u)$ converges uniformly on every compact interval, defining a continuous function on $[0, \infty)$. Let

$$g(u) := G_k(u) + \sum_{n=1}^\infty F^n * G_k(u).$$

Then,

$$F * g = F * G_k(u) + F * \sum_{n=1}^\infty F^n * G_k(u),$$

(10)

and since the convergence is uniform, we can exchange the order in the last term of (10) to find that it is a solution of (8). Since $G_k$ is differentiable, so are $F^n * G_k$. Again we can use the estimate (9) to see that $g$ is differentiable.

Note that when $G_k \equiv 0$, $g \equiv 0$ is the unique solution.
that is locally integrable since (9) implies
\[
|g(u)| \leq \sup_{x \leq u} |g(x)||F|^{n} \frac{u^n}{n!},
\]
for any \(n\). This fact also implies the uniqueness of the continuous solution of (8) for generic \(G_{\kappa}\).
\[(QED)\]

3. Concluding Remark

We have employed a stochastic calculus approach to derive the integro-differential equation of the Gerber-Shiu function, in the classical Cramér-Lundberg model. As we have seen, the idea is simple. The Itô’s formula characterises martingales. The risk process is a martingale with respect to the discrete-time filtration under which we can retrieve the Markov property. Thanks to the Markov property, we can derive an integro-differential equation. In [9], the method is applied to more general cases where the risk process is a renewal jump-diffusion risk process.

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