SOME IMPROVEMENTS OF JORDAN-STEČKIN AND BECKER-STARK INEQUALITIES

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The aim of this article is to propose some improvements of the Jordan-Stečkin and Becker-Stark inequalities discussed in L. Debnath, C. Mortici, L. Zhu: Refinements of Jordan-Stečkin and Becker-Stark inequalities, Results Math. 67(1-2)(2015), 207–215.

1. INTRODUCTION

L. Debnath, C. Mortici and L. Zhu discuss in [1] Jordan’s inequality:

\[
\frac{\sin x}{x} \geq \frac{2}{\pi}, \quad x \in (0, \pi/2]
\]

and its improvements

\[
\frac{2}{\pi} + \frac{1}{\pi^3} (\pi^2 - 4x^2) \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{\pi - 2}{\pi^3} (\pi^2 - 4x^2), \quad x \in (0, \pi/2],
\]

and

\[
\frac{2}{\pi} + \frac{1}{2\pi^3} (\pi^4 - 16x^4) \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{\pi - 2}{\pi^5} (\pi^4 - 16x^4), \quad x \in (0, \pi/2].
\]

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They conclude that the equalities in (2) and (3) hold if and only if \( x = \pi/2 \). In the case where \( x \to 0^+ \), we have equalities on the right-hand side of (2) and (3), and strict inequalities on the left-hand side of (2) and (3).

In [1] (Theorem 1, Theorem 2), the left-hand side of (2) and (3) near zero was improved.

The following inequality:

\[
\tan x \geq \frac{4}{\pi} \cdot \frac{x}{\pi - 2x}, \quad x \in [0, \pi/2).
\]

well known as Stečkin's inequality, was also analysed in [1].

As noted in [1], this inequality becomes an equality for \( x = 0 \), and

\[
\lim_{x \to (\pi/2)^-} \left( \tan x - \frac{4}{\pi} \cdot \frac{x}{\pi - 2x} \right) = \frac{2}{\pi}.
\]

Some improvements of (4), in the left neighbourhood of \( \pi/2 \), were presented in [1] (Theorem 3, Theorem 4).

M. Becker and L. E. Stark present in [2] the inequality

\[
\frac{8}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{\pi^2}{\pi^2 - 4x^2}, \quad 0 < x < \frac{\pi}{2}.
\]

Certain double inequalities of the Becker-Stark type were proposed in [1] (Theorem 5, Theorem 6).

In this paper, we generalise and improve the inequalities stated in Theorem 1, Theorem 2, Theorem 3, Theorem 4, Theorem 5 and Theorem 6 from [1]. They are cited below for readers' convenience.

**Statement 1** ([1], Theorem 1) For every \( x \in (0, \pi/2) \), it holds that

\[
\frac{2}{\pi} + \frac{1}{\pi^2} (\pi^2 - 4x^2) + \left( 1 - \frac{3}{\pi} \right) - \left( \frac{1}{6} - \frac{4}{\pi^3} \right) x^2 < \frac{\sin x}{x} < \frac{2}{\pi} + \frac{1}{\pi^3} (\pi^4 - 16x^4) + \left( 1 - \frac{5}{2\pi} \right) - \frac{1}{6} x^2 + \frac{8}{\pi^5} + \frac{1}{120} x^4.
\]

**Statement 2** ([1], Theorem 2) For every \( x \in (0, \pi/2) \), it holds that

\[
\frac{2}{\pi} + \frac{1}{2\pi^5} (\pi^4 - 16x^4) + \left( 1 - \frac{5}{2\pi} \right) - \frac{1}{6} x^2 < \frac{\sin x}{x} < \frac{2}{\pi} + \frac{1}{\pi^3} (\pi^4 - 16x^4) + \left( 1 - \frac{5}{2\pi} \right) - \frac{1}{6} x^2 + \left( \frac{8}{\pi^5} + \frac{1}{120} \right) x^4.
\]
Statement 3 ([1], Theorem 3) For every \( x \in (0, \pi/2) \), it holds that
\[
\frac{2}{\pi} - \frac{1}{2} \left( \frac{\pi}{2} - x \right) < \tan x - \frac{4}{\pi} \cdot \frac{x}{\pi - 2x} < \frac{2}{\pi} - \frac{1}{3} \left( \frac{\pi}{2} - x \right).
\]  

Statement 4 ([1], Theorem 4) For every \( x \in (0, 1) \), it holds that
\[
\left( 1 - \frac{4}{\pi^2} \right) x - \frac{8}{\pi^3} x^2 < \tan x - \frac{4}{\pi} \cdot \frac{x}{\pi - 2x} < \left( 1 - \frac{4}{\pi^2} \right) x.
\]  

Statement 5 ([1], Theorem 5) For every \( x \in (0.373, \pi/2) \) on the left-hand side and every \( x \in (0.301, \pi/2) \) on the right-hand side, the following inequalities hold true:
\[
8 + a(x) \left( \frac{\pi}{2} - x \right) + \left( \frac{16}{\pi^2} - \frac{8}{3} \right) \left( \frac{\pi}{2} - x \right)^2 < \tan x < 8 + b(x) \left( \frac{\pi}{2} - x \right) + \left( \frac{32}{\pi^3} - \frac{8}{3\pi} \right) \left( \frac{\pi}{2} - x \right)^3,
\]  

where
\[
a(x) = \frac{8}{\pi} \left( \frac{\pi}{2} - x \right) + \left( \frac{16}{\pi^2} - \frac{8}{3} \right) \left( \frac{\pi}{2} - x \right)^2
\]  

and
\[
b(x) = a(x) + \left( \frac{32}{\pi^3} - \frac{8}{3\pi} \right) \left( \frac{\pi}{2} - x \right)^3.
\]  

Statement 6 ([1], Theorem 6) For every real number \( x \in (0, 1.371) \), the following inequality holds true:
\[
\tan x < \frac{\pi^2 - (4 - \frac{4}{\pi^2}) x^2 - (\frac{4}{3} - \frac{2}{\pi} a(x)) x^4}{\pi^2 - 4x^2}.
\]  

2. PRELIMINARIES

Let \( T_n^{\varphi,a}(x) \) be the TAYLOR polynomial of the order \( n \in N \), associated to the function \( \varphi(x) \) at the point \( x = a \). \( T_n^{\varphi,a}(x) \) and \( T_n^{\varphi,a}(x) \) represent the TAYLOR polynomial of the order \( n \in N \), associated to the function \( \varphi(x) \) at the point \( x = a \), in the case \( T_n^{\varphi,a}(x) > \varphi(x) \), respectively \( T_n^{\varphi,a}(x) \leq \varphi(x) \), for every \( x \in (a, b) \). We call \( T_n^{\varphi,a}(x) \) and \( T_n^{\varphi,a}(x) \) an upward and a downward approximation of \( \varphi \) on \( (a, b) \), respectively.

As discussed in [4], for the sine function the following inequalities hold:
\[
T_{13}^{\sin,0}(x) < T_{7}^{\sin,0}(x) < T_{11}^{\sin,0}(x) < T_{15}^{\sin,0}(x) < \ldots < \sin x < \ldots < T_{13}^{\sin,0}(x) < T_{9}^{\sin,0}(x) < T_{5}^{\sin,0}(x) < T_{1}^{\sin,0}(x),
\]
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for $x \in (0, \sqrt{20}) = (0, 4.472...)$.

We have the following Taylor series of $\text{sinc}$ $x$:

$$\text{sinc} \ x = \frac{\sin \ x}{x} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k+1)!}$$

for $x \neq 0$.

According to [6], for $x \in (0, \pi/2)$ we have the following series representations:

$$\tan \ x = \sum_{k=1}^{\infty} 2^{2k} \frac{(2^{2k} - 1)}{(2k)!} |B_{2k}| x^{2k-1}$$

and

$$\cot \ x = \frac{1}{x - \sum_{k=1}^{\infty} 2^{2k} |B_{2k}| \frac{1}{(2k)!} x^{2k-1}$$

where $B_i$ $(i \in N)$ are Bernoulli’s numbers.

Suppose that $f(x)$ is a real function on $(a, b)$, and that $n$ is a positive integer such that $f^{(k)}(a+), f^{(k)}(b-), (k \in 0, 1, 2, \ldots, n-1)$ exist. Let us denote:

$$T_n^{f,b,a}(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(b-)}{k!} (x-b)^k + \frac{1}{(a-b)^n} \left( f(a+) - \sum_{k=0}^{n-1} \frac{(a-b)^k f^{(k)}(b-)}{k!} \right) (x-b)^n$$

and

$$T_n^{f,a,b}(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a+)}{k!} (x-a)^k + \frac{1}{(b-a)^n} \left( f(b-) - \sum_{k=0}^{n-1} \frac{(b-a)^k f^{(k)}(a+)}{k!} \right) (x-a)^n$$

S. Wu and L. Debnath proved the following theorem in [7]:

**Theorem WD** Suppose that $f(x)$ is a real function on $(a, b)$, and that $n$ is a positive integer such that $f^{(k)}(a+), f^{(k)}(b-), (k \in 0, 1, 2, \ldots, n)$ exist.

(i) Supposing that $(-1)^n f^{(n)}(x)$ is increasing on $(a, b)$, then for all $x \in (a, b)$ the following inequality holds:

$$T_n^{f,b,a}(x) < f(x) < T_n^{f,a,b}(x)$$

Furthermore, if $(-1)^n f^{(n)}(x)$ is decreasing on $(a, b)$, then the reverse inequality holds.
(ii) Supposing that \( f^{(n)}(x) \) is increasing on \((a, b)\), then for all \( x \in (a, b) \) the following inequality holds:

\[
T_n^{f,a,b}(x) > f(x) > T_n^{f,a}(x).
\]  

Furthermore, if \( f^{(n)}(x) \) is decreasing on \((a, b)\), then the reverse inequality holds.

Some interesting applications of the previous theorem can be found in \([5, 19, 20, 32]\).

3. MAIN RESULTS

3.1 IMPROVEMENTS OF INEQUALITIES IN STATEMENT 1

According to (12), we can approximate the sinc \( x \) function as follows:

\[
T_n^{\text{sinc},0}(x) < T_{2n}^{\text{sinc},0}(x) < T_{4n}^{\text{sinc},0}(x) < \ldots < \text{sinc} x < \ldots < T_{12}^{\text{sinc},0}(x) < T_8^{\text{sinc},0}(x) < T_4^{\text{sinc},0}(x) < T_0^{\text{sinc},0}(x),
\]

for \( x \in (0, \pi/2) \subset (0, \sqrt{20}) \).

Based on approximation (18), we have the following theorem

**Theorem 1** For every \( x \in (0, \pi/2) \) we have:

\[
T_n^{\text{sinc},0}(x) = \frac{2}{\pi} + \frac{1}{\pi^3} \left( \pi^2 - 4x^2 \right) + \left( 1 - \frac{3}{\pi} \right) - \left( \frac{1}{6} - \frac{4}{\pi^3} \right) x^2 \leq \leq T_{4k}^{\text{sinc},0}(x) < \text{sinc} x < T_{4k^2}^{\text{sinc},0}(x) \leq \frac{2}{\pi} + \frac{1}{\pi^3} \left( \pi^2 - 4x^2 \right) + \left( 1 - \frac{3}{\pi} \right) - \left( \frac{1}{6} - \frac{4}{\pi^3} \right) x^2 + \frac{1}{120} x^4 = T_4^{\text{sinc},0}(x) < T_0^{\text{sinc},0}(x),
\]

for \( k, k_2 \in \mathbb{N} \).

**Remark 1** It is obvious that Statement 1 is a special case of Theorem 1.
3.2 IMPROVEMENTS OF INEQUALITIES IN STATEMENT 2

Consider the following polynomials in inequality (7) from Statement 2:

\[ Q_4(x) = \frac{2}{\pi} + \frac{1}{2\pi^5} (\pi^4 - 16x^4) + \left(1 - \frac{5}{2\pi}\right) - \frac{1}{6}x^2 = -\frac{8x^4}{\pi^5} - \frac{x^2}{6} + 1 \]

and

\[ R_4(x) = \frac{2}{\pi} + \frac{\pi - 2}{\pi^5} (\pi^4 - 16x^4) + \left(1 - \frac{5}{2\pi}\right) - \frac{1}{6}x^2 + \left(\frac{8}{\pi^5} + \frac{1}{120}\right)x^4 \]

\[ = \left(-\frac{16}{\pi^4} + \frac{40}{\pi^5} + \frac{1}{120}\right)x^4 - \frac{x^2}{6} - \frac{5}{2\pi} + 2. \]

We have the following theorem:

**Theorem 2** For every \( x \in (0, \pi/2) \) we have:

\[ (20) \quad Q_4(x) < T_{6k_1-2}^{\text{sinc},0}(x) < \text{sinc} x < T_{4k_2}^{\text{sinc},0}(x) < R_4(x), \]

for \( k_1, k_2 \in \mathbb{N} \).

**Proof** In order to prove (20), it is sufficient to prove that for every \( x \in (0, \pi/2) \) the inequalities \( Q_4(x) < T_{6k_1-2}^{\text{sinc},0}(x) \) and \( T_{4k_2}^{\text{sinc},0}(x) < R_4(x) \) are true.

According to (13) we have:

\[ T_{4k_2}^{\text{sinc},0}(x) = 1 - \frac{x^2}{6} + \frac{x^4}{120} \]

\[ T_{4k_2}^{\text{sinc},0}(x) = 1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040}. \]

It is obvious that

\[ T_{6k_1-2}^{\text{sinc},0}(x) - Q_4(x) > \left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040}\right) - \left(-\frac{8x^4}{\pi^5} - \frac{x^2}{6} + 1\right) = \]

\[ = \left(\frac{1}{120} + \frac{8}{\pi^5}\right)x^4 - \frac{x^6}{5040} > 0 \]

and

\[ R_4(x) - T_{6k_1-2}^{\text{sinc},0}(x) > \left(-\frac{16}{\pi^4} + \frac{40}{\pi^5} + \frac{1}{120}\right)x^4 - \frac{x^2}{6} - \frac{5}{2\pi} + 2 \]

\[ - \left(1 - \frac{x^2}{6} + \frac{x^4}{120}\right) = \left(-\frac{16}{\pi^4} + \frac{40}{\pi^5}\right)x^4 - \frac{5}{2\pi} + 1 > 0 \]

hold for \( x \in (0, \pi/2) \).

**Remark 2** Statement 2 is a special case of Theorem 2.
3.3 IMPROVEMENTS OF INEQUALITIES IN STATEMENT 3

In a monograph [3], D. S. Mitrinović discussed about Stečkin’s inequality:

$$\tan x > \frac{4}{\pi} \cdot \frac{x}{\pi - 2x},$$

for $x \in (0, \pi/2)$. Let us denote:

$$f(x) = \tan x - \frac{4x}{\pi (\pi - 2x)},$$

for $x \in (0, \pi/2)$ and let us notice:

$$\lim_{x \to \pi/2^-} f(x) = \frac{\pi}{2}.$$

In [1], inequalities (8) are proposed as adequate approximations of the function $f(x)$ in the left neighbourhood of the point $x = \pi/2$.

By replacing $x$ with $\pi/2 - t$ in the function $f(x)$, we obtain:

$$g(t) = f\left(\frac{\pi}{2} - t\right) = \cot t - \frac{1}{2} + \frac{2}{\pi},$$

for $t \in (0, \pi/2)$. According to (15), we have that

$$\cot t < \frac{\pi}{2} - T_n\cot,0(t) = \frac{1}{l} - \sum_{k=1}^{n} \frac{2^{2k} |B_{2k}|}{(2k)!} t^{2k-1}$$

for $t \in (0, \pi/2]$ and $n \in \mathbb{N}$. Further, we have the following:

$$g(t) < \frac{\pi}{2} - T_n\cot,0(t) - \frac{1}{l} + \frac{2}{\pi} \tag{22}$$

and according to Theorem WD

$$\cot t > T_n\cot,0,\pi/2(t) = T_{n-1}\cot,0(t) + \left(\frac{\pi}{2}\right)^n \left(\frac{\pi}{2} - T_{n-1}\cot,0\left(\frac{\pi}{2}\right)\right)^n, \tag{23}$$

for $t \in (0, \pi/2]$ and $n \in \mathbb{N}$. According to (22) and (23), we have:

$$g(t) > \frac{\pi}{2} - T_n\cot,0,\pi/2(t) - \frac{1}{l} + \frac{2}{\pi} \tag{24}$$

for $t \in (0, \pi/2]$. Let us denote:

$$F_n^g(t) = T_n\cot,0(t) - \frac{1}{l} + \frac{2}{\pi}$$

and

$$E_n^g(t) = T_n\cot,0,\pi/2(t) - \frac{1}{l} + \frac{2}{\pi}.$$

Returning replacement $t = \pi/2 - x$ in (22) and (24), we have the following theorem:
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**Theorem 3** For \( x \in (0, \pi/2) \) and \( n \in \mathbb{N} \), we have:

\[
F_n^g\left(\frac{\pi}{2} - x\right) < f(x) < F_n^g\left(\frac{\pi}{2} - x\right)
\]

**Corollary 1** We have the following improvements for inequality (8) given in Statement 3.

1. For \( n = 1 \) and for \( x \in (0, \pi/2) \), we have:

\[
Q_1(x) < F_1^g\left(\frac{\pi}{2} - x\right) = \frac{2}{\pi} - \frac{4}{\pi^2} \left(\frac{\pi}{2} - x\right) < f(x) < \frac{2}{\pi} - \frac{1}{3} \left(\frac{\pi}{2} - x\right) = F_1^g\left(\frac{\pi}{2} - x\right) = R_1(x).
\]

2. For \( n = 3 \) and for \( x \in (0, \pi/2) \), we have:

\[
Q_1(x) < F_1^g\left(\frac{\pi}{2} - x\right) < F_3^g\left(\frac{\pi}{2} - x\right) = \frac{2}{\pi} - \frac{1}{3} \left(\frac{\pi}{2} - x\right) - \left(\frac{2}{\pi}\right)^3 \left(\frac{2}{\pi} - \frac{\pi}{6}\right)\left(\frac{\pi}{2} - x\right)^3
\]

\[
< f(x) < \frac{2}{\pi} - \frac{1}{3} \left(\frac{\pi}{2} - x\right) - \left(\frac{2}{\pi}\right)^3 \left(\frac{2}{\pi} - \frac{\pi}{6}\right)\left(\frac{\pi}{2} - x\right)^3 = F_3^g\left(\frac{\pi}{2} - x\right) < F_1^g\left(\frac{\pi}{2} - x\right) = R_1(x).
\]

### 3.4 Improvements of Inequalities in Statement 4

For the function \( f(x) \) defined in (21), and according to the Taylor series of the \( \tan x \) function in (14) and the binomial expansion of \( \frac{1}{1-\left(\frac{2}{\pi}\right)x} \) over the interval \( (0, \pi/2) \), we have:

\[
f(x) = \tan x - \frac{4}{\pi} \frac{x}{\pi - 2x}
\]

\[
= \sum_{i=1}^{\infty} \frac{2^i (2^i - 1) |B_{2i}| x^{2i-1}}{(2i)!} - \frac{4}{\pi^2} \frac{x}{1 - \left(\frac{2}{\pi}\right)x}
\]

\[
= \sum_{i=1}^{\infty} \frac{2^i (2^i - 1) |B_{2i}| x^{2i-1}}{(2i)!} - \sum_{j=1}^{\infty} \frac{2^{j+1}}{j+1} x^j
\]

\[
= \sum_{k=1}^{\infty} (-1)^{k-1} \alpha_k x^k,
\]

where

\[
\alpha_k = \begin{cases} 
\frac{2^{k+1}}{\pi^{k+1}} : & k = 2\ell \\
\frac{2^{2k+1} (2^{2k+1} - 1) |B_{2k+1}|}{(k+1)!} - \frac{2^{k+1}}{\pi^{k+1}} : & k = 2\ell - 1
\end{cases}
\]
for \( \ell \in N \). It is not hard to check that:

\[
\alpha_k > 0, \quad \lim_{k \to \infty} \alpha_k = 0 \quad \text{and} \quad (\alpha_k)_{\downarrow},
\]

for \( k \in N \). Finally, based on (26) and (27) and based on Leibnitz theorem, we have the following theorem:

**Theorem 4** For every \( x \in (0, 1) \) and \( \ell \in N \), the following holds:

\[
T_{2\ell}^f(x) < f(x) < T_{2\ell-1}^f(x).
\]

**Remark 3** Inequality (28) for \( \ell = 1 \) represents inequality (9) from Statement 4.

3.5 Improvements of Inequalities in Statement 5

Consider the following function:

\[
\phi(x) = (\pi^2 - 4x^2) \tan x,
\]

for \( x \in (0, \pi/2) \).

By replacing \( x \) with \( \pi/2 - t \) in the function \( \phi(x) \), we obtain:

\[
\psi(t) = \phi\left(\frac{\pi}{2} - t\right) = \frac{8t(\pi - t) \cot t}{\pi - 2t}
\]

for \( t \in (0, \pi/2) \). The improvement or inequalities from (10) are given with the following theorem:

**Theorem 5** For every \( x \in (0, \pi/2) \), the following holds:

\[
T_4^{\psi, 0}\left(\frac{\pi}{2} - x\right) = 8 + \frac{8}{\pi} \left(\frac{\pi}{2} - x\right) + \left(\frac{16}{\pi^2} - \frac{8}{3}\right) \left(\frac{\pi}{2} - x\right)^2 + \left(\frac{32}{\pi^3} - \frac{8}{27}\right) \left(\frac{\pi}{2} - x\right)^3
\]

\[
+ \left(\frac{64}{\pi^4} - \frac{16}{3\pi^2} - \frac{8}{45}\right) \left(\frac{\pi}{2} - x\right)^4 < \psi(x) \quad < \quad \psi\left(\frac{\pi}{2} - x\right) = 8 + \frac{8}{\pi} \left(\frac{\pi}{2} - x\right) + \left(\frac{16}{\pi^2} - \frac{8}{3}\right) \left(\frac{\pi}{2} - x\right)^2 + \left(\frac{32}{\pi^3} - \frac{8}{27}\right) \left(\frac{\pi}{2} - x\right)^3
\]

\[
+ \left(\frac{64}{\pi^4} - \frac{16}{3\pi^2} - \frac{8}{45}\right) \left(\frac{\pi}{2} - x\right)^4 + \left(\frac{128}{\pi^5} - \frac{32}{3\pi^3} - \frac{8}{45\pi}\right) \left(\frac{\pi}{2} - x\right)^5.
\]

One proof of this statement is based on equivalent mixed trigonometric polynomial inequalities:

\[
f(x) = (\pi^2 - 4x^2) \sin x - x T_4^{\psi, 0}\left(\frac{\pi}{2} - x\right) \cos x > 0
\]
and
\[ g(x) = (\pi^2 - 4x^2) \sin x - x T_5^{\psi,0} \left( \frac{\pi}{2} - x \right) \cos x < 0, \]
for \( x \in (0, \pi/2) \). References [15, 16] show that the problem of proving mixed trigonometric polynomial inequalities is a decidable problem. In these two references are presented appropriate algorithms that follow mentioned inequalities. Some interesting applications of the algorithmic approach in proving mixed trigonometric inequalities can be found in [21, 31]; see also [17, 18]. G. Bercu in [33, 34] presented some interesting approximations of trigonometric functions using Padé approximant.

**Remark 4** It is obvious that Statement 5 is a consequence of Theorem 5.

Further, let us observe the array \( (\alpha_k)_{k \in \mathbb{N}} \) defined by:
\[ \alpha_1 = 1, \quad \alpha_{2j} = 0, \quad \alpha_{2j+1} = -\frac{2^{2j} |B_{2j}|}{(2j)!} \]
for \( j \in \mathbb{N} \). Then based on [6], we have the following series representations:
\[
\psi(t) = \frac{8}{\pi} t (\pi - t) \cdot \frac{1}{1 - \left(\frac{2t}{\pi}\right)^2} \cot t \\
= \frac{8}{\pi} t (\pi - t) \left( \sum_{i=0}^{\infty} \left(\frac{2^i t}{\pi}\right)^i \right) \left( \sum_{j=0}^{\infty} \alpha_{2j+1} t^{2j-1} \right)
\]
for \( t \in (0, \pi/2) \). Let \( r_2(m) \) be the remainder after division of the natural number \( m \) by 2. We are posing the following conjecture:

**Conjecture 1**

1. For the function \( \psi(t) \) on \( t \in (0, \pi/2) \), the following equality holds:
\[
\psi(t) = \sum_{m=0}^{\infty} \left( \frac{8}{\pi} \frac{\alpha_{m+1-r_2(m)}}{\pi^{r_2(m)}} + \sum_{i=1}^{[m/2]} \frac{2^{i+2+r_2(m)} \alpha_{m+1-2i-r_2(m)}}{\pi^{2i+r_2(m)}} \right) t^m.
\]
2. For the function \( \psi(t) \) on \( t \in (0, \pi/2) \) and \( \ell \in \mathbb{N} \), the following inequalities are true:
\[
T_{2\ell}^{\psi,0}(t) < \psi(t) < T_{2\ell+1}^{\psi,0}(t) \quad (t \in (0, \pi/2) \land \ell \in \mathbb{N}).
\]

### 3.6 Improvements of Inequality in Statement 6

Let us denote the following function:
\[ f(x) = (\pi^2 - 4x^2) \tan x \frac{x}{x} \]
for \( x \in (0, \pi/2) \).
According to [6] and (14), we have:

\[ f(x) = \sum_{k=1}^{\infty} C_k x^{2k-2} \tag{31} \]

where

\[ C_k = \frac{\pi^2 \cdot 2^{2k} (2^{2k} - 1) |B_{2k}|}{(2k)!} - \frac{4 \cdot 2^{2k-2} (2^{2k-2} - 1) |B_{2k-2}|}{(2k-2)!}, \tag{32} \]

and \( x \in (0, c) \) and \( 0 < c < \pi/2 \). It is not hard to check \( C_k < 0 \) for \( k \in \mathbb{N} \).

Finally, based on Theorem WD we get the following theorem:

**Theorem 6** For every \( x \in (0, c) \), where \( 0 < c < \pi/2 \), the following inequalities hold:

\[
T_{m_1}^{f,0,c}(x) = \sum_{k=1}^{m_1-1} C_k x^{2k-2} + \left( \frac{1}{c} \right)^{2m_1-2} \left( f(c) - \left( \sum_{k=1}^{m_1-1} C_k c^{2k-2} \right) \right) x^{2m_1-2} < f(x) < \sum_{k=1}^{m_2} C_k x^{2k-2} = T_{m_2}^{f,0,c}(x),
\]

for \( m_1, m_2 \in \mathbb{N} \).

**Remark 5** It is obvious that Statement 6 is a consequence of Theorem 6 for \( m_2 = 3 \).

The approximations discussed in this paper can be of great significance for potential application of analytic inequalities in engineering. Some specific inequalities of a similar type are considered in [11, 12, 13].

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