Optimal majority threshold in a stochastic environment

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Received: date / Accepted: date

Abstract Within the model of social dynamics determined by collective decisions in a stochastic environment (the ViSE model), we consider the case of a homogeneous society consisting of classically rational economic agents. We obtain analytical expressions for the optimal majority threshold as a function of the parameters of the environment, assuming that the proposals are generated by means of a continuous distribution. The cases of several specific distributions are considered in more detail.

Keywords ViSE model · social dynamics · voting · stochastic environment · pit of losses

1 Introduction

In Borzenko et al (2006), the ViSE (Voting in a Stochastic Environment) model has been proposed. Its simplest version describes a society that consists of n classically rational economic agents, who are boundedly rational egoists (hereafter, egoists). Each of them maximizes their individual utility in every act of choice, which turns out to be the most profitable noncooperative strategy. Various cooperative and egoistic strategies within the ViSE model have been studied in Borzenko et al (2006), Chebotarev (2006), Chebotarev et al (2009), and Malyshev and Chebotarev (2017), altruistic strategies in Chebotarev et al (2018b).

Each participant (agent) is characterized by the current value of capital (which can also be interpreted as the value of individual utility). A proposal [of the environment] is a vector of proposed capital increments of the participants. A similar model with randomly generated proposals appeared in Compte and Jehiel (2017). The society can accept or reject every proposal by voting. Each
agent votes for those and only those proposals that increase their individual capital. A proposal is accepted and implemented (i.e., the participants’ capitals get the proposed increments) if and only if the proportion of the society supporting this proposal is greater than $\alpha \in [0, 1]$ (the strict relative voting threshold). Otherwise, all capitals remain unchanged. This voting procedure is called “$\alpha$-majority” (cf. Nitzan and Paroush (1982, 1984), Felsenthal and Machover (2001) and O’Boyle (2009)).

The voting threshold $\alpha$ will also be called the majority threshold and, more precisely, the acceptance threshold, since $\alpha < 0.5$ is allowed.

The concept of proposal enables one to model potential changes that are beneficial for some agents and disadvantageous for the others. As a result of the implementation of such a proposal, the capitals of some agents increase, while the capitals of the others decrease.

The proposals are stochastically generated by the environment and put to a general vote over and over again. The subject of the study is the dynamics of the participants’ capitals as a result of this process. A similar dynamic model has been considered in Mirkin (1979), Subsection 1.3 of Chapter 2.

Further, dynamic voting models have been studied in the theory of legislative bargaining (see Duggan and Kalandrakis (2012)), where stochastic generation of proposals has been assumed in some cases (Penn (2009), Dziuda and Loeper (2014, 2016)). On other connections between the ViSE model and various comparable models, we refer to Chebotarev et al (2018a). In accordance with the basic ViSE model, the capital increments that form the proposals are realizations of independent identically distributed random variables. In this paper, we present a general result applicable to any distribution that has a mean and focus on three families of distributions: continuous uniform distributions, normal distributions (cf. Chebotarev et al (2018a)), and symmetrized Pareto distributions (see Chebotarev et al (2018b)).

Each distribution is characterized by its mathematical expectation, $\mu$ and standard deviation, $\sigma$. The ratio $\sigma/\mu$ is called the coefficient of variation of a random variable. The inverse coefficient of variation $\rho = \mu/\sigma$, which we call the adjusted or normalized mean of the environment, measures the relative favorability of the environment. If $\rho > 0$, then the opportunities provided by the environment are favorable on average; if $\rho < 0$, then the environment is unfavorable.

In the present paper, we study:

- the optimal acceptance threshold for a general continuous distribution (Subsection 2.1), i.e., the threshold that maximizes the total capital of the society (this generalizes Theorem 1 in Chebotarev et al (2018a));
- dependence of the optimal acceptance threshold on the model parameters for several specific distributions (Subsections 2.2, 2.4).
2 Optimal majority threshold

2.1 A general result

To familiarize with the problem that the optimal majority threshold solves, let us look at the dependence of the one-step mean capital increment of an agent on the adjusted mean of the environment $\rho$ (Chebotarev et al. (2018a)).

Let $\zeta = (\zeta_1, \ldots, \zeta_n)$ denote a random proposal on some step. Its component $\zeta_i$ is the proposed capital increment of agent $i$. The components $\zeta_1, \ldots, \zeta_n$ are independent identically distributed random variables. $\zeta$ will denote a similar scalar variable without reference to a specific agent. Similarly, let $\eta = (\eta_1, \ldots, \eta_n)$ be the random vector of actual increments of the agents on the same step. If $\zeta$ is adopted, then $\eta = \zeta$; otherwise $\eta = (0, \ldots, 0)$. Consequently,

$$
\eta = \zeta I(\zeta, \alpha n),
$$

where

$$
I(\zeta, \alpha n) = \begin{cases} 
1, & \text{if the number of positive components of } \zeta \text{ is greater than } \alpha n; \\
0, & \text{otherwise.} 
\end{cases}
$$

Let $\eta$ be a random variable similar to every $\eta_i$, but having no reference to a specific agent. We are interested in the one-step mean capital increment of an agent, i.e. $M(\eta)$, where $M(\cdot)$ is the mathematical expectation.

For 21 participants and $\alpha = 0.5$, the dependence of $M(\eta)$ on $\rho = \mu/\sigma$ is presented in Fig. 1, where proposals are generated by the normal distribution.

Fig. 1 shows that for $\rho \in (-0.85, -0.266)$, the mean capital increment is an appreciable negative value, i.e., proposals approved by the majority are, on average, unprofitable and impoverishing for the society. This part of the curve is called a “pit of losses.” For $\rho < -0.85$, the negative mean increment is very close to zero, since the proposals are extremely rarely accepted.

For each specific environment, there is an optimal acceptance threshold $\alpha_0$ that provides the highest possible one-step mean capital increment $M(\eta)$ of an agent.

The optimal acceptance threshold for the normal distribution as a function of the environment parameters has been studied in Chebotarev et al. (2018a). This threshold turns out to be independent of the size of the society $n$.

Voting with the optimal acceptance thresholds always yields positive expected capital increments and so it is devoid of “pits of losses.”

The following theorem provides a general expression for the optimal voting threshold, which holds for any distribution that has a mathematical expectation.

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1 See Nitzan and Paroush (1982) and Azrieli and Kim (2014) on other approaches to optimizing the majority threshold and Rae (1969) and Sekiguchi and Ohtsuki (2015) for a discussion of the case of multiple voting in this context.
Theorem 1 In a society consisting of egoists, the optimal voting threshold is

\[ \alpha_0 = \left(1 + \frac{M^+}{M^-}\right)^{-1}, \tag{2} \]

where \( M^- = |M(\zeta | \zeta \leq 0)|, M^+ = M(\zeta | \zeta > 0), \) and random variable \( \zeta \) determines the capital increment of any agent in a random proposal.

In terms of the value \( R = \frac{M^+}{M^-} \), which we call the \textit{win/loss magnitude ratio}, equation (2) takes the form

\[ \alpha_0 = (1 + R)^{-1}. \]

Proof According to the proof of Lemma 1 in \cite{Chebotarev2006}, we have

\[ M(\eta) = \sum_{x=[n_t]+1}^{n} M(\eta | n^+ = x) b(x | n), \tag{3} \]

where \( \eta \) is the actual one-step capital increment of an agent determined by \cite{1}, \( n^+ \) is the number of positive components in a proposal, \( n_t = an \) is the \textit{absolute voting threshold} (a proposal is accepted if and only if \( n^+ > n_t \)).
\([n_t]\) is the integer part of \([n_t]\). \(b(x \mid n) = P\{n^+ = x\} = \binom{n}{x} p^x q^{n-x}\), \(p\) is the probability that a proposal component is positive, and \(q = 1 - p\).

Let us ascertain how \(M(\eta)\) changes as \(\alpha\) increases. When a continuously increasing \(n_t = \alpha n\) becomes integer, we remove one term from the sum (3). Now we take into account that:

(i) \(b(x \mid n)\) is always positive;

(ii) when \(M(\eta \mid n^+ = [n_t] + 1)\) changes its sign from minus to plus with the increase of \(n_t = \alpha n\), i.e., when the sum (3) collects exactly all the positive terms, we get the maximum \(M(\eta)\), i.e., the maximum one-step mean capital increment. In turn, this determines an optimal absolute voting threshold \(n_0 = n_t\) and the corresponding optimal relative voting threshold \(\alpha_0\) such that \(\alpha_0 n = n_0\).

If a threshold \(\alpha\) is optimal and \([\alpha_1 n] = [\alpha n]\), then \(\alpha_1\) is also an optimal threshold. As follows from the above considerations, all the optimal majority thresholds \(\alpha_0\) can be found from the system of inequalities

\[
\begin{align*}
M(\eta \mid n^+ = [\alpha_0 n]) &< 0 \\
M(\eta \mid n^+ = [\alpha_0 n] + 1) &> 0.
\end{align*}
\]

Observe that for any integer \(x\) in the segment \([n_t] + 1, n - 1\], it holds that

\[
M(\eta \mid n^+ = x) = \frac{\frac{\eta}{x} \cdot p^x q^{n-x}}{x^x n^x}, \quad \frac{\eta}{n} = \frac{\eta}{n}.
\]

and

\[
P\{\eta \leq 0 \mid n^+ = x\} = 1 - \frac{x}{n}.
\]

Substituting (5), (6) into (4) and using the fact that by the independence of the components of the proposal, \(M(\eta \mid n^+ = x, \eta > 0) = M(\eta \mid \eta > 0) = M(\zeta \mid \zeta > 0) = M^+\) and similarly \(|M(\eta \mid n^+ = x, \eta \leq 0)| = |M(\eta \mid n^+ > n_t, \eta \leq 0)| = |M(\zeta \mid \zeta \leq 0)| = M^-\), we obtain

\[
\begin{align*}
M^+ \frac{[\alpha_0 n]}{n} - M^- \left(1 - \frac{[\alpha_0 n]}{n}\right) &< 0 \\
M^+ \frac{[\alpha_0 n] + 1}{n} - M^- \left(1 - \frac{[\alpha_0 n] + 1}{n}\right) &> 0.
\end{align*}
\]

Expressing \(\alpha_0\) from this we get one of the optimal majority thresholds:

\[\text{In the case of } x = n, \text{ there is only the first term of the following equation.}\]
\[ \alpha_0 = \left(1 + \frac{M^+}{M^-}\right)^{-1}. \]

Finally, observe that the left-hand side of the first inequality of (8) with \( \alpha \) substituted for \( \alpha_0 \) grows in \( \alpha \in [0; 1] \) because \( M^+ \geq 0 \) and \( M^- \geq 0 \). Therefore \( M(\eta \mid n^+ = x) \) indeed changes its sign from minus to plus as \( x \) grows. \( \square \)

Let \( \bar{\alpha}_0 \) be the center of the half-interval of optimal majority thresholds for fixed \( n, \sigma, \) and \( \mu \). Then this half-interval is \( [\bar{\alpha}_0 - \frac{1}{2n}, \bar{\alpha}_0 + \frac{1}{2n}] \) and its extreme points have denominator \( n \). Figures 2, 3, and 4 show the dependence of \( \bar{\alpha}_0 \) on \( \rho = \mu/\sigma \) for different distributions that characterize the stochastic environment by determining generation of proposals.

As one can observe for different distributions, outside the segment \( \rho \in [-0.7, 0.7] \), if a majority threshold is close to the optimal one and the number of participants is appreciable, then the proposals are almost always accepted (to the right of the segment) or almost always rejected (to the left of this segment). Therefore, in these cases, the issue of determining the accurate optimal threshold loses its practical value.

2.2 Proposals generated by continuous uniform distributions

Let \(-a < 0\) and \(b > 0\) be the minimum and maximum values of a continuous uniform distribution, respectively.

**Corollary 1** The optimal majority threshold in the case of proposals generated by the continuous uniform distribution on the segment \([-a, b]\) with \(-a < 0\) and \(b > 0\) is

\[ \alpha_0 = \left(1 + \frac{b}{a}\right)^{-1}. \] (9)

Indeed, in this case, \( M^- = \frac{a}{2}, M^+ = \frac{b}{2} \), and \( R = \frac{b}{a} \), hence, (2) provides (9).

If \( b \) approaches 0 from above, then \( \alpha_0 \) approaches 1 from below, and the optimal voting procedure is unanimity. Indeed, profitable proposed capital increments become much smaller in absolute value than disadvantageous ones, therefore, each participant should be able to reject a proposal.

As \(-a\) approaches 0 from below, disadvantageous proposed capital increments become much smaller in absolute value than profitable ones. Therefore, a coalition consisting of any single voter should be able to accept a proposal. In accordance with this, the optimal relative threshold \( \alpha_0 \) decreases to 0.
Corollary 2  In terms of the adjusted mean of the environment $\rho = \mu/\sigma$, it holds that

$$\alpha_0 = \begin{cases} 1, & \rho \leq -\sqrt{3} \\ \frac{1}{2} \left(1 - \frac{\rho}{\sqrt{3}} \right), & -\sqrt{3} < \rho < \sqrt{3} \\ 0, & \rho \geq \sqrt{3}. \end{cases}$$

This follows from (9) and the expressions $\mu = -\frac{a+b}{2}$ and $\sigma = \frac{b-a}{\sqrt{3}}$. It is worth mentioning that the dependence of $\alpha_0$ on $\rho$ is linear, as distinct from (9).

Figure 2 illustrates the dependence of the center of the half-interval of optimal majority thresholds versus $\rho = \mu/\sigma$ for continuous uniform distributions in the segment $\rho \in [-0.5, 0.5]$.

2.3 Proposals generated by normal distributions

For normal distributions, the following corollary holds.
Corollary 3  The optimal majority threshold in the case of proposals generated by the normal distribution with parameters \( \mu \) and \( \sigma \) is

\[
\alpha_0 = F(\rho) \left(1 - \frac{\rho F(-\rho)}{f(\rho)}\right),
\]

where \( \rho = \mu/\sigma \), while \( F(\cdot) \) and \( f(\cdot) \) are the standard normal cumulative distribution function and density, respectively.

Corollary 3 follows from Theorem 1 and the facts that \( M^- = -\sigma \left( \rho - \frac{f(\rho)}{F(\rho)} \right) \), and \( M^+ = \sigma \left( \rho + \frac{f(\rho)}{F(\rho)} \right) \), which can be easily found by integration. Note that Corollary 3 strengthens the first statement of Theorem 1 in Chebotarev et al (2018a).

Figure 3 illustrates the dependence of the center of the half-interval of optimal majority thresholds versus \( \rho = \mu/\sigma \) for normal distributions in the segment \( \rho \in [-2.5, 2.5] \) (to show nonlinearity).

We refer to Chebotarev et al (2018a) for some additional properties (e.g., the rate of change of the optimal voting threshold as a function of \( \rho \)).

2.4 Proposals generated by symmetrized Pareto distributions

Pareto distributions are widely used for modeling social, linguistic, geophysical, financial, and some other types of data. The Pareto distribution with
The center $\bar{a}_0$ of the half-interval of optimal majority thresholds (a “ladder”) for $n=131$ and the optimal majority threshold $[11]$ as functions of $\rho$ for symmetrized Pareto distributions with $k=8$.

Positive parameters $k$ and $a$ can be defined by means of the function $P(\xi > x) = \left(\frac{x}{a}\right)^k$, where $\xi \in [a, \infty)$ is a random variable.

The ViSE model normally involves distributions that enable both positive and negative values. Consider the symmetrized Pareto distributions (see Chebotarev et al. (2018b) for more details). For its construction, the density function $f(x) = \frac{k}{2} \left(\frac{x}{a}\right)^k$ of the Pareto distribution is divided by 2 and combined with its reflection w.r.t. the line $x = a$.

The density of the resulting distribution with mode $\mu$ is

$$f(x) = \frac{k}{2a} \left(\left|\frac{x-\mu}{a}\right| + 1\right)^{-(k+1)}.$$

For symmetrized Pareto distributions with $k > 2$, the following result holds true.

**Corollary 4** The optimal majority threshold in the case of proposals generated by the symmetrized Pareto distribution with parameters $\mu$, $\sigma$, and $k > 2$ is

$$\alpha_0 = \begin{cases} \frac{C+p}{kp+\sigma} \left(1 - \frac{1}{2} \left(\frac{C}{C+p}\right)^k\right), & \mu > 0, \\ 1 + \frac{C-\rho}{kp+\sigma} \left(1 - \frac{1}{2} \left(\frac{C}{C-\rho}\right)^k\right), & \mu \leq 0, \end{cases} \tag{11}$$
where \( \rho = \frac{\mu}{\sigma} \), and \( C = \sqrt{\frac{(k-1)(k-2)}{2}} = \frac{a}{\sigma} \).

Corollary 4 follows from Theorem 1 and the facts (their proof is given below) that:

\[
M^- = \sigma \left( \frac{C + \rho}{k-1} \right), \quad M^+ = \frac{\sigma}{1 - \frac{1}{2} \left( \frac{C + \rho}{\sigma} \right)^k} \left( \rho + \left( \frac{C + \rho}{2(k-1)} \right)^k \right) \quad \text{whenever} \quad \mu > 0;
\]

\[
M^- = -\frac{\sigma}{1 - \frac{1}{2} \left( \frac{C - \rho}{\sigma} \right)^k} \left( \rho - \left( \frac{C - \rho}{2(k-1)} \right)^k \right), \quad M^+ = \sigma \left( \frac{C - \rho}{k-1} \right) \quad \text{whenever} \quad \mu \leq 0.
\]

The “ladder” and the optimal majority threshold curve for symmetrized Pareto distributions are fundamentally different from the corresponding curves for the normal and continuous uniform distributions. Namely, we have \( \alpha_0 = 0.5 \) in a rather wide neighborhood of \( \rho = 0 \) for odd \( n \)’s and an increase of \( \alpha_0 \) when \( \rho \) becomes positive for even \( n \)’s.

Correspondingly, the curve \( \alpha_0(\rho) \) has an abnormal part in a vicinity of zero, where the optimal threshold increases with the adjusted mean. As a result, the optimal relative threshold \( \alpha_0(\rho) \) has two extremes. This is caused by the peculiarities of the symmetrized Pareto distribution. An increase of \( \rho \) through zero decreases \( M^+ \) and increases \( M^- \). By virtue of Eq. (2), this causes an increase of \( \alpha_0 \).

Figure 4 illustrates the dependence of the center of the half-interval of optimal voting thresholds versus \( \rho = \mu/\sigma \) for symmetrized Pareto distributions with \( k = 8 \).

Proof of Corollary 4 Let \( F(\cdot) \) and \( f(\cdot) \) be the cumulative Pareto distribution function and the Pareto density, respectively; \( \rho = \mu/\sigma \), and \( C\sigma = \sigma \sqrt{\frac{(k-1)(k-2)}{2}} = a \).

Let \( \mu > 0 \). Then

\[
M^- = \frac{1}{F(0)} \int_{-\infty}^{0} \frac{k}{2C\sigma} \left( \frac{-x + C\sigma + \rho\sigma}{C\sigma} \right)^{-k+1} \frac{dx}{x} = \frac{k}{2C\sigma} \left( \frac{(C\sigma)^k(-x + C\sigma + \rho\sigma)^{-k+1}}{(k-1)k} \right) \bigg|_{-\infty}^{0} = \sigma \left( \frac{C + \rho}{k-1} \right);
\]
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\[ M^+ = \frac{1}{1 - F(0)} \left[ \int_0^\mu \frac{k}{2C\sigma} \left( \frac{-x + C\sigma + \rho\sigma}{C\sigma} \right)^{-k(k+1)} dx + \int_\mu^\infty \frac{k}{2C\sigma} \left( \frac{x + C\sigma - \rho\sigma}{C\sigma} \right)^{-k(k+1)} dx \right] \]

\[ = \frac{1}{1 - \frac{1}{2} \left( \frac{C}{\rho + \rho} \right)^k} \left( \frac{(C\sigma)^{k+1}(-x + C\sigma + \rho\sigma)^{-k(k-1)}}{(k-1)k} \right) \bigg|_0^{\mu\sigma} \]

\[ - \frac{1}{1 - \frac{1}{2} \left( \frac{C}{\rho + \rho} \right)^k} \left( \frac{(C\sigma)^{k+1}(x + C\sigma - \rho\sigma)^{-k(k-1)}}{(k-1)k} \right) \bigg|_{\rho\sigma}^{\infty} \]

\[ = \frac{\sigma}{1 - \frac{1}{2} \left( \frac{C}{\rho + \rho} \right)^k} \left( \rho + \left( \frac{C}{\rho + \rho} \right)^k \frac{C + \rho}{2(k-1)} \right). \]

Similarly, \( M^- = -\frac{\sigma}{1 - \frac{1}{2} \left( \frac{C}{\rho + \rho} \right)^k} \left( \rho - \left( \frac{C}{\rho + \rho} \right)^k \frac{C - \rho}{2(k-1)} \right) \) and \( M^+ = \sigma \frac{C - \rho}{k-1} \) whenever \( \mu \leq 0 \).

\[ \square \]

3 Conclusion

In this paper, we obtained a general expression for the optimal voting threshold (i.e., the threshold that maximizes the total capital of the society) as a function of the parameters of the stochastic proposal generator in the assumptions of the ViSE model. This expression was given a more specific form for several types of distributions.

Estimation of the optimal majority threshold seems to be a solvable problem in real situations. If the ViSE model is at least approximately adequate and one can estimate \( \rho = \mu/\sigma \) using experiments, then it is possible to obtain such an estimation by means of the formulas provided in this study.

We found that for some distributions of proposals, the plausible at first glance conclusion that it is beneficial to increase the voting threshold when the environment becomes less favorable is not generally true. A more in-depth study of this issue should be the subject of future research.

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