An energy formula for fully nonlinear degenerate parabolic equations in one spatial dimension

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Abstract

Energy (or Lyapunov) functions are used in order to prove stability of equilibria, or to indicate a gradient-like structure of a dynamical system. Matano constructed a Lyapunov function for quasilinear non-degenerate parabolic equations with gradient dependency. We modify Matano’s method to construct an energy formula for fully nonlinear degenerate parabolic equations. In particular, we provide a new energy formula for the porous medium equation.

Keywords: fully nonlinear degenerate parabolic equations; Lyapunov function.

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1 Main results

We consider the scalar fully nonlinear partial differential equation

\[ f(x, u, u_x, u_{xx}, u_t) = 0, \]

for \( x \in (0, 1) \) and \( t > 0 \). Here indices abbreviate partial derivatives. We assume that \( f \in C^2 \) satisfies the following degenerate parabolic conditions

\[ f_q \cdot f_r \leq 0, \quad f_r \neq 0, \quad \text{and} \quad f_q(x, u, p, 0, 0) \neq 0, \]

for every argument \( (x, u, p, q, r) := (x, u, u_x, u_{xx}, u_t) \in [0, 1] \times \mathbb{R}^4 \). Conditions (1.2) imply that the diffusion coefficient \( f_q \) may vanish. Without loss of generality, we consider \( f_r < 0 \) and thereby \( f_q \geq 0 \). Indeed, if \( f_r > 0 \), then \( f_{\hat{r}} < 0 \) for \( \hat{r} := -r \).

We consider (1.1) with two types of separated boundary conditions at \( x = \iota \in \{0, 1\} \). For each boundary point \( x = \iota \), separately, we either assume homogeneous Dirichlet boundary conditions or nonlinear boundary conditions of Robin type, respectively

\begin{align*}
(1.3a) \quad & u = 0, \\
(1.3b) \quad & u_x = b'(u).
\end{align*}

We assume \( b' \in C^1 \). Neumann boundary conditions occur if \( b'(u) = 0 \). See [1] and [22] for abstract settings involving nonlinear boundary conditions of the type (1.3b).

Equation (1.1) includes classical examples, such as evolution involving the \( p \)-laplacian diffusion, the porous medium equation or the mean curvature flow for graphs. In case of gradient dependent nonlinear forcing, these classical quasilinear degenerate diffusion did not have any apparent variational structure, which we are now able to display. It is the scope of this paper to provide a unifying variational formulation to several degenerate fully nonlinear parabolic equations in one spatial dimension given by (1.1).

Below we construct a Lyapunov function

\[ E := \int_0^1 L(x, u, u_x) \, dx \quad \text{such that} \quad \frac{dE}{dt} < 0 \]

along nonequilibrium solutions \( u = u(t, x) \) of (1.1). Therefore the time dependent energy \( t \mapsto E(u(t, .)) \) decreases strictly, except at equilibria, i.e. \( u_t \equiv 0 \).
To construct a Lyapunov function $E$ as in (1.4), we rewrite the fully nonlinear parabolic equation (1.1) suitably, following the spirit of [19]. Then we continue with a modification of Matano’s original idea for non-degenerate quasilinear equations in order to incorporate degeneracies.

Indeed, we split the equation (1.1) in order to emphasize the degenerate diffusion,

\[(1.5)\]

\[F(x, u, u_x, u_{xx}, u_t) = f_q(x, u, p, 0, 0)u_{xx},\]

where $F(x, u, p, q, r) := -f(x, u, p, q, r) + f_q(x, u, p, 0, 0)u_{xx}$. Here the degenerate parabolicity conditions (1.2) become

\[(1.6)\]

\[F_r(x, u, p, q, r) > 0, \quad \text{and} \quad f_q(x, u, p, 0, 0) \geq F_q(x, u, p, q, r) \neq f_q(x, u, p, q, r),\]

for every argument $(x, u, p, q, r) := (x, u, u_x, u_{xx}, u_t)$.

Next, we split $F$ into two parts: one which is independent of $u_{xx}$ and $u_t$, whereas another that depends on them. First we distinguish a term $F^0$ related to reaction, when $u_{xx} = u_t = 0$. Second we describe the time evolution term $F^1$, the only term that depends on $u_t$. Specifically, we define

\[(1.7)\]

\[
F^0(x, u, p) := F(x, u, p, 0, 0), \\
F^1(x, u, p, q, r) := F(x, u, p, q, r) - F^0(x, u, p),
\]

where $F^0 \in C^2$ and $F^1 \in C^1$, since $f \in C^2$.

The parabolic equation (1.5) can be rewritten as

\[(1.8)\]

\[F^1(x, u, p, q, r) = f_q(x, u, p, 0, 0)u_{xx} - F^0(x, u, p).\]

The degenerate parabolicity conditions (1.2) incarnated in (1.6) imply

\[(1.9)\]

\[F_r^1 > 0 \quad \text{and} \quad f_q(x, u, p, 0, 0) \geq F_q^1(x, u, p, q, r) \neq f_q(x, u, p, q, r),\]

for every argument $(x, u, p, q, r) := (x, u, u_x, u_{xx}, u_t)$.

Before we present the main result that constructs a Lyapunov function, we suppose that along the characteristic equation given by

\[(1.10)\]

\[
\begin{align*}
\dot{x} &= f_q(x, u, p, 0, 0), \\
\dot{u} &= f_q(x, u, p, 0, 0)p, \\
\dot{p} &= F^0(x, u, p),
\end{align*}
\]

there is a solution $g$ of the following equation:

\[(1.11)\]

\[
\dot{g} = -F^0_p(x, u, p) - f_{qx}(x, u, p, 0, 0) - f_{qu}(x, u, p, 0, 0)p.
\]

Note that the characteristic equation (1.10) for degenerate PDEs is different than the one obtained by Matano in the non-degenerate case. Nevertheless these equations can be transformed into each other by a suitable ‘time’ rescaling that absorbs $1/f_q(x, u, p, 0, 0) < \infty$ in case of non-degenerate equations. Note that global existence of the characteristic equations (1.10)-(1.11) might fail, in general.
Theorem 1.1. Assume $f \in C^2$ satisfies (1.2). Suppose the characteristic equations (1.10)–(1.11) have global solutions.

Then there exists a Lagrange function $L = L(x, u, p)$ on bounded sets of $(u, p) \in \mathbb{R}^2$ such that

$$(1.12) \quad E := \int_0^1 L(x, u, u_x) \, dx$$

is a Lyapunov function (1.4) for the equation (1.1). More precisely, bounded solutions $u(t, x)$ of (1.1) satisfy

$$(1.13) \quad \frac{dE}{dt} = -\int_0^1 \exp(g(x, u, u_x))F^1(x, u, u_x, u_t, u_{xx}, u_{tt}) \, dx$$

where $g(x, u, u_x)$ solves (1.11); $F^1 \cdot u_t \geq 0$, and it is zero if, and only if, $u_t \equiv 0$.

For non-degenerate quasilinear equations, $f(x, u, p, q, r) = -r + a(x, u, p)q + h(x, u, p)$, where $a > 0$, a Lyapunov function $E$ was constructed, independently, by Zelenyak [32] and Matano [25]. See also [10] for concise expositions of Matano’s method. This method was extended to fully nonlinear non-degenerate parabolic equations, when $f_q \cdot f_r < 0$, in [19]. An analogous method for Jacobi systems, a spatially discrete variant, was developed in [11]. For an adaptation to diffusion with singular coefficients see [17].

We emphasize that the procedure to construct the energy function in (1.12) is formal. Once the Lagrange function $L$ is obtained, one needs to verify various properties needed for a well-defined Lyapunov function, such as integrability, bounds, regularity, etc.

Thus, the properties and applicability of each energy function have to be dealt with on a case-by-case basis. In Section 3, we provide some examples. Even if the Lyapunov function is only well-defined for sufficiently regular initial data, we may still be able to obtain dynamical information on invariant subspaces of regular enough initial data. For a deeper regularity analysis, see [15, 3, 4, 5, 27] and references therein.

We comment on limitations, modifications and possible applications of our result.

The hypothesis $f_r \neq 0$ in (1.2) excludes time-evolution type degeneracies such as in Trudinger’s equation, $(u^\alpha)_t = u_{xx}$, for $\alpha > 0$; see [28]. However, time-evolution degeneracies can be transformed into singular equations which we may be able to construct an energy; see Problem 3.6 in [31]. The condition $f_q(x, u, p, 0, 0) \neq 0$ in (1.2) prevents degeneracies of the same order of $u_{xx}$, such as the dual porous medium equation, $u_t = |u_{xx}|^{m-1}u_{xx}$, for $m > 1$, see [31]; and the double-phase equation, given by $u_t = (|u_{xx}|^p + a(x)|u_{xx}|^q)u_{xx}$ with $q \geq p \geq 2$, see [6].

Note that our method can potentially treat singular diffusion, i.e., when $f_q(x, u, p, 0, 0)$ may be unbounded. An example is $u_t = (u^n u_x/|u_x|)_x$ for $m \geq 0$, which is called the total variation flow for $m = 0$, or the heat equation in transparent media for $m = 1$. See [12] and references therein. However, there are two delicate issues to obtain a Lyapunov function. First, the characteristic equations (1.10) may not have global solutions. Second, the Ansatz for $L_{pp}$ still to be defined in (2.4) might not be twice integrable (in $p$) in order to obtain a well-defined formula for $L$ in (2.13).
An alternative splitting of the fully nonlinear equation was pursued in [19], different than (1.8), yielding an energy that decays according to

\[ \frac{dE}{dt} = -\int_0^1 L_{pp} \tilde{F}^1 u_t^2 \, dx \]

for some \( \tilde{F}^1 > 0 \). Instead of the decay in (1.13), one may also be able to obtain a Lyapunov function that decays according to (1.14), which extracts the \( L^2 \)-gradient flow with weight \( L_{pp} \tilde{F}^1 > 0 \). However, we believe that these different splittings do not change the Lyapunov function itself, only the aesthetics of the abstract formulae.

A semiflow treatment of fully nonlinear degenerate equations of the type (1.1) on an appropriate phase space \( X \supseteq C^1([0,1]) \) has been lacking in its full generality, akin to the one provided by [22] for non-degenerate parabolic equations. We expect that under additional growth conditions on \( f \), similar to the non-degenerate case in [21, Proposition 3.5] and [29, Chapter 6, Sec. 5], imply that solutions of (1.1) are bounded, global and generate a dissipative semiflow. In particular, this would guarantee the global existence of the characteristics (1.10)–(1.11) after an appropriate cut-off of \( f \) outside a sufficiently large set of \( X \), and thus the existence of a Lyapunov function \( E \) in such a bounded set. In more general settings, including solutions which blow-up in finite time, boundedness of \( E \) from below might of course fail. In fact, applications to fully nonlinear blow-up may require delicate analysis of the characteristic equation (2.11) beyond such crude cut-off.

In addition, it would be desirable to extract dynamical information on the long time behaviour of solutions of (1.1). Indeed, under certain conditions on \( f \) that also guarantee asymptotic compactness of the semiflow, there should be a global attractor \( \mathcal{A} \subseteq X \) as in the non-degenerate case in [13] or [16, Theorem 2.2]. For particular cases of degenerate type, see [7, 9]. Thus, as a consequence of the Lyapunov function (1.12), bounded trajectories should converge to (sets of) equilibria, according to the LaSalle invariance principle; see [14, Section 4.3] for the non-degenerate case. Note that the complete description of \( \omega \)-limit sets is a delicate issue: for the porous medium equation, the \( \omega \)-limit sets consist of a singleton in case of hyperbolic (and therefore isolated) equilibria, see [2]; whereas equations with \( p \)-laplacian diffusion admit continua of equilibria, see [26]. See also [8]. Finally, the connection problem for the equations (1.1)–(1.2) that describes which equilibria are connected by means of a heteroclinic orbit remains open, see [18] and references therein for the non-denegerate case.

The remainder of the paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, we compute several significant examples of Lyapunov functions.
2 Proof

We recall the equation (1.8),

\[ f_q(x, u, p, 0, 0)u_{xx} = F^0(x, u, p) + F^1(x, u, p, q, r), \]

with degenerate parabolicity conditions \( F^1_r > 0 \) and \( f_q \geq F^1_q \). See (1.9).

Differentiating the definition (1.12) of the Lyapunov function \( E \) with respect to time \( t \) along classical solutions \( u(t, x) \) of (1.1), we obtain

\[ \frac{dE}{dt} = \int_0^1 (L_u u_t + L_p u_{xt}) \, dx. \]

Here we used that \( u_{xt} = p_t \). The Lagrange function \( L \) depends on \( (x, u, p) = (x, u, u_x) \), only. It remains to determine \( L \) such that \( \frac{dE}{dt} < 0 \), except at equilibria. Integrating the term \( L_p u_{xt} \) in (2.2) by parts, and carrying out the differentiation of \( L_p \) with respect to \( x \), we obtain

\[ \frac{dE}{dt} = L_p u_t \bigg|_{x=1}^{x=0} + \int_0^1 \left( L_u - \frac{d}{dx} L_p \right) u_t \, dx \]

\[ = L_p u_t \bigg|_{x=1}^{x=0} + \int_0^1 (L_u - L_{px} - L_{pu} u_x - L_{pp} u_{xx}) u_t \, dx. \]

At this point, Matano would plug in the non-degenerate PDE in \( u_{xx} \). However, this can not be performed for degenerate equations, since we can not isolate \( u_{xx} \) in equation (2.1), as \( f_q \) may be zero. In order to remedy this, we modify Matano’s original Ansatz, \( L_{pp} = \exp(g(x, u, p)) \), which would yield a first order PDE to be solved for \( g(x, u, p) \). Instead, we consider a different Ansatz

\[ L_{pp} := f_q(x, u, p, 0, 0) \exp(g(x, u, p)), \]

for some function \( g(x, u, p) \) to be found. Note (2.4) is not identically zero, since \( f_q(x, u, p, 0, 0) \neq 0 \) due to (1.2). Thus

\[ \frac{dE}{dt} = L_p u_t \bigg|_{0}^{1} + \int_0^1 (L_u - L_{px} - L_{pu} u_x - \exp(g) f_q u_{xx}) u_t \, dx. \]

We then substitute the PDE (1.1) recast in (2.1), to obtain

\[ \frac{dE}{dt} = L_p u_t \bigg|_{0}^{1} + \int_0^1 (L_u - L_{px} - L_{pu} u_x - \exp(g) F^0) u_t \, dx - \int_0^1 \exp(g) F^1 u_t \, dx. \]

We seek to construct the Lagrange function \( L \) such that the boundary terms vanish, the parenthesis in the first integral (2.6) also vanishes, and satisfies the Ansatz (2.4) for some function \( g(x, u, p) \). This yields a Lyapunov function such that

\[ \frac{dE}{dt} = - \int_0^1 \exp(g) F^1 u_t \, dx. \]
Note $F^1 u_t \geq 0$, due the parabolicity condition $F^1_r > 0$. Next, we guarantee that there exists a function $g(x, u, p)$ such that

$$\tag{2.8} L_u - L_{px} - pL_{pu} - \exp(g)F^0 = 0,$$

for all $(x, u, p) \in [0, 1] \times \mathbb{R}^2$, and also $L_p u_t = 0$ on the boundaries $x = 0, 1$. In this part, note that $(u, p) \in \mathbb{R}^2$ are real variables rather than solutions $u, u_x$ of PDEs depending on $(t, x)$.

Differentiating (2.8) with respect to $p$, the terms $L_{pu}$ cancel, yielding

$$\tag{2.9} L_{ppx} + pL_{ppu} + \exp(g)g_p F^0 = -\exp(g)F^0_p.$$

Rewriting (2.9) in terms of $g$, according to (2.4), amounts to the first order PDE,

$$\tag{2.10} f_q g_x + pf_q g_u + F^0 g_p = -F^0_p - f qx - pf qu.$$

The method of characteristics can solve (2.10): along solutions of the auxiliary ODEs

$$\tag{2.11} \begin{align*}
\dot{x} &= \frac{dx}{d\tau} = f_q(x, u, p, 0, 0), \\
\dot{u} &= \frac{du}{d\tau} = f_q(x, u, p, 0) p, \\
\dot{p} &= \frac{dp}{d\tau} = F^0(x, u, p),
\end{align*}$$

the function $g$ must satisfy

$$\tag{2.12} \dot{g} = \frac{dg}{d\tau} = -F^0_p(x, u, p) - f qx(x, u, p, 0, 0) - f qu(x, u, p, 0, 0) p,$$

with the initial condition $g(0, u_0, p_0)$, where $(u_0, p_0) := (u(0, 0), u_x(0, 0))$. Our differentiability assumptions on $f$ imply $g \in C^0$, at least.

Without further assumptions on the nonlinearity $f$ in (1.1), solutions to (2.11) may not exist on the whole required interval $x \in [0, 1]$. For this reason, we have assumed the global existence of solutions for the characteristic equations.

After this construction we now have to reverse gear and ascend from a function $g$ satisfying (2.10) to a Lagrange function $L$ satisfying (2.8). The general solution $L$ of $L_{pp} = f_q \exp(g)$ can be obtained by integrating it twice with respect to $p$:

$$\tag{2.13} L(x, u, p) := \int_0^p \int_0^{p_1} f_q(x, u, p_2) \exp(g(x, u, p_2)) \, dp_2 \, dp_1 + L^0(x, u) + L^1(x, u)p.$$

This solves (2.9). To ensure that $L$ is also a solution of (2.8), we have to determine the integration “constants” $L^0$ and $L^1$, appropriately.

Recall that (2.9) was obtained through differentiation of (2.8) with respect to $p$. Conversely, the left-hand side of (2.8) is therefore independent of $p$. Hence (2.8) is satisfied
for all $p$ if it holds for some fixed value $p = p_*$. At $p = p_*$, the construction of $L$ yields $L_p = L^1_x$, $L_{px} = L^1_x$ and $L_u = L^0_u$. Insertion in (2.8) at $p = p_*$ yields

(2.14) $L^0_u = L^1_x + p_* \cdot L^1_u \cdot F^0 \cdot \exp(g)$.

Integrating with respect to $u$, we obtain

(2.15) $L^0(x, u) := \int^u \left[ L^1_x(x, u_1) + p_* \cdot L^1_u(x, u_1) + \exp(g(x, u_1, p_*)) \cdot F^0(x, u_1, p_*) \right] \, du_1,$

where we neglect an irrelevant additive constant $L^{00}(x)$ for $E$.

To complete the proof it only remains to show that $L_p u_t$ vanishes at the boundaries $x = 0, 1$. At any boundary of Dirichlet type (1.3a) this is trivial because $r = u_t = 0$. Thus we let $L^1 \equiv 0$.

In the case of a nonlinear Robin boundary condition (1.3b) at only one boundary, either $x = 0$ or $x = 1$, we have to choose $L$ such that $L_p(t, u, b'(u)) = 0$. By our construction (2.13) of $L$, this is equivalent to

(2.16) $L^1(t, u) := -\int_0^{b'(u)} \exp(g(t, u, p)) \, dp,$

and we may choose $L^1$ to be independent of $x$.

For nonlinear Robin boundary conditions (1.3b) at both boundaries, $x = 0$ and $x = 1$, we define $L^1(t, u)$ as in (2.16). Linear interpolation $L^1(x, u) := (1 - x) L^1(0, u) + x L^1(1, u)$ then provides $L^1 \in C^1$ such that $L_p(t, u, b'(u)) = 0$.

3 Examples

We explicitly compute examples of energies using the method in the previous section. Since our construction is new even for the case of degenerate quasilinear diffusion with gradient dependent reaction, we focus on well-known examples of quasilinear diffusion with the simplest polynomial forcing term, i.e., $u^{n_x}$.

For the sake of simplicity, we consider Dirichlet boundary conditions throughout the examples, which yields $L^1 \equiv 0$.

3.1 Gradient-degenerate quasilinear diffusion with nonlinear forcing

Consider the equation

(3.1) $u_t = a(u_x)u_{xx} + h(u),$

with $a(u_x) \geq 0$. In the abstract setting in the previous section, we have that $f_q = a(p)$, $F^0 = -h(u)$ and $F^1 = u_t$. 

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Thus, the characteristic equations (2.11) are given by

\begin{align}
\dot{x} &= a(p), \\
\dot{u} &= a(p) p, \\
\dot{p} &= -h(u),
\end{align}

(3.2)

and \( g \) evolves according to (2.12), i.e.,

\begin{equation}
\dot{g} = 0.
\end{equation}

(3.3)

Note that \( g(x(\tau), u(\tau), p(\tau)) \) is a constant function, and therefore we obtain the trivial solution \( g \equiv 0 \) for the initial condition \( g_0 := g(0, u(0), p(0)) = g(0, u_0, p_0) = 0 \). According to the construction in the previous section, the equation (2.14) yields \( L_{pp} = a(p) \).

Note that \( L^1 = 0 \) due to Dirichlet boundary conditions, and \( L^0 = -\int_0^u h(u_1) \, du_1 \) due to the equation (2.15) for \( p_\ast = p_0 \). Hence (2.13) implies that

\begin{equation}
E = \int_0^1 \left( \int_0^{P_1} a(p_2) \, dp_2 \, dp_1 - \int_0^u h(u_1) \, du_1 \right) \, dx,
\end{equation}

(3.4)

which decays according to

\begin{equation}
\frac{dE}{dt} = -\int_0^1 u_t^2 \, dx.
\end{equation}

(3.5)

In particular, the \( p \)-Laplacian equation occurs when \( a(u_x) = (\rho - 1)|u_x|^{\rho-2} u_x \) and \( h \equiv 0 \), and thus we recover its well-known energy \( E = \int_0^1 |u_x|^{\rho}/\rho \, dx \). Also, the mean curvature flow for one dimensional entire graphs occurs when \( a(u_x) = (1+u_x^2)^{-3/2} \) and \( h \equiv 0 \), and thereby we also recover the energy \( E = \int_0^1 \sqrt{1+u_x^2} \, dx \). This energy accounts for the perimeter of the curve, which decreases under evolution of mean curvature according to (3.5). Note that the mean curvature flow only degenerates at infinity, i.e., when \( |u_x| \to \infty \), and thus we expect that an appropriate compactification of the semiflow will be described by a degenerate equation at infinity, see [20].

### 3.2 \( \rho \)-Laplacian diffusion and polynomial gradient reaction

Consider the equation

\begin{equation}
\frac{\partial u}{\partial t} = \partial^{\rho} u + u_t^n = (\rho - 1)|u_x|^{\rho-2} u_{xx} + u_x^n,
\end{equation}

(3.6)

where \( \rho \geq 2 \) and \( n \geq 0 \). In terms of the formulation in the previous section, we have that \( f_q = (\rho - 1)|u_x|^{\rho-2}, F^0 = -u_x^n \), and \( F^1 = u_t \).

Hence the characteristic equations (2.11) are given by

\begin{align}
\dot{x} &= (\rho - 1) |p|^{\rho-2}, \\
\dot{u} &= (\rho - 1) |p|^{\rho-1}, \\
\dot{p} &= -p^n,
\end{align}

(3.7)

\(^{1}\)In the literature, this operator is called the \( p \)-Laplacian. However, in our notation \( p := u_x \) and thus we replace the parameter \( p \) by \( \rho \) in the degenerate diffusion operator, i.e., \( \partial^{\rho} u := (|u_x|^{\rho-2} u_x)_x \).
and (2.12) becomes

\[ \dot{g} = np^{n-1}. \]

We can solve these equations explicitly,

\[ p(\tau) = \begin{cases} p_0 e^{-\tau} & \text{for } n = 1 \\ \frac{p_0}{(1+(n-1)p_0^{n-1}\tau)^{\frac{1}{n-1}}} & \text{for } n > 1, \end{cases} \]

and

\[ g(\tau) = \begin{cases} \tau + g_0 & \text{for } n = 1 \\ \frac{n}{n-1} \log\left(1 + (n - 1)p_0^{n-1}\tau\right) + g_0 & \text{for } n > 1. \end{cases} \]

Consequently,

\[ g(p) = n \log\left(\frac{p_0}{p}\right) + g_0 \]

Thus equations (2.4), (2.15) with \( p_* = p_0 \), and the Dirichlet boundary conditions imply that

\[ L_{pp} = (\rho - 1) \exp(g_0)|p_0|^n |p|^{\rho-n-2}, \quad L^0 = -\exp(g_0)|p_0|^n u, \quad \text{and} \quad L^1 = 0. \]

Hence the Lagrangian \( L \) can be obtained according to (2.13), yielding the following energy formula, up to a multiplicative constant \( \exp(g_0)|p_0|^n \):

\[ E = \int_0^1 \left(\frac{(\rho - 1)}{(\rho - n)(\rho - n - 1)}|u_x|^{\rho-n} - u\right) \, dx, \]

which decays according to

\[ \frac{dE}{dt} = -\int_0^1 \frac{u_x^2}{|u_x|^n} \, dx. \]

Since we have not proved any further regularity of the energy \( E \), its derivative is also formal. For equilibria, \( u_t \equiv 0 \), the energy \( E \) in (3.13) is constant, and its derivative \( dE/dt \) given by (3.14) either vanishes or it attains the value \(-\infty\) in case the integrand in (3.14) is not integrable, which means the derivative is not well defined. Similarly for time dependent solutions: either (3.14) is integrable yielding negative bounded values, or (3.14) is not integrable and thereby ill-defined. Thus, equilibria may be critical points of a non-differentiable energy. However, one can still obtain dynamical information for continuous Lyapunov functions, see [14, Chapter 4].

Note that the energy in this example remains true for \( n < 0 \), even though the reaction term in the vector field is singular when \( u_x = 0 \). In this case, the decay rate of the energy in (3.14) is bounded along bounded solutions of (3.6). Thus, our methods can be applied in certain cases of quenching phenomena, whenever the hypothesis (1.2) holds true and one can solve the characteristic equations. See [30] for an example of quenching in a fully nonlinear equation.
3.3 Mean curvature flow diffusion and polynomial gradient reaction

Consider the equation

\begin{equation}
(3.15) \quad u_t = \left( \frac{u_x}{\sqrt{1 + u_x^2}} \right)_x + u_n = \frac{u_{xx}}{(1 + u_x^2)^{3/2}} + u_n.
\end{equation}

In this case, we have that \( f_q = (1 + u_x^2)^{-3/2} \), \( F^0 = -u_n \) and \( F^1 = u_t \). Hence the characteristic equations \((2.11)\) are given by

\begin{equation}
(3.16) \quad \dot{x} = \frac{1}{(1 + p^2)^{3/2}}, \quad \dot{u} = \frac{p}{(1 + p^2)^{3/2}}, \quad \dot{p} = -p^n,
\end{equation}

and \((2.12)\) is given by

\begin{equation}
(3.17) \quad \dot{g} = np^{n-1}.
\end{equation}

Since the relevant equations coincide with \((3.7)\), we obtain the same solutions in \((3.9)\) and \((3.10)\). Hence the equations \((2.4), (2.15)\) with \( p_* = p_0 \), and Dirichlet boundary conditions yield

\begin{equation}
(3.18) \quad L_{pp} = \exp(g_0) |p_0|^n \frac{1}{|p|^n (1 + p^2)^{3/2}} \quad L^0 = -\exp(g_0) |p_0|^n u \quad \text{and} \quad L^1 = 0.
\end{equation}

Thus the energy \((2.13)\), up to a multiplicative constant \( \exp(g_0) |p_0|^n \), is given by

\begin{equation}
(3.19) \quad E = \int_0^1 \left( 2 \tilde{F}_1 \left( \frac{3}{2}, 1 - \frac{n}{2}, \frac{n-2}{2}; -u_x^2 \right) k - 2 \tilde{F}_1 \left( \frac{3}{2}, 1 - \frac{n}{2}, 2 - \frac{2}{2}; -u_x^2 \right) k_* - u \right) dx,
\end{equation}

where \( 2 \tilde{F}_1(a_1, a_2; b_1; z) \) is the regularized generalized hypergeometric function,

\begin{equation}
(3.20) \quad k := -\frac{\Gamma \left( \frac{3-n}{2} \right)}{(n-1)}, \quad \text{and} \quad k_* := -\frac{\Gamma \left( 1 - \frac{n}{2} \right) \Gamma \left( \frac{3-n}{2} \right)}{(n-1) \Gamma \left( \frac{1-n}{2} \right)},
\end{equation}

with \( \Gamma(\cdot) \) denoting the Gamma function. Thus the energy \((3.19)\) decays according to

\begin{equation}
(3.21) \quad \frac{dE}{dt} = -\int_0^1 \frac{u_x^2}{|u_x|^n} dx.
\end{equation}

In particular, the energy \((3.19)\) for \( n = 2 \) has a simpler formulation:

\begin{equation}
(3.22) \quad E = \int_0^1 \left( \coth^{-1} \left( \sqrt{1 + u_x^2} \right) - 2 \sqrt{1 + u_x^2} - u \right) dx.
\end{equation}
3.4 Inverse Mean Curvature Flow for Certain Graphs

Consider the equation

\[
(3.23) \quad u_t = \frac{1 + u_x^2}{1 - \left(1 - \frac{u_x^2}{1 + u_x^2}\right) u_{xx}}.
\]

This equation has been considered in higher dimensions in [23, Section 3], and we construct a different monotone quantity in comparison to [24].

In this case, we have that

\[
(3.24) \quad f_q = \frac{(1 + u_x^2)^2}{(1 + u_x^2 - u_{xx})^2}, \quad F^0 = -(1 + u_x^2), \quad F^1 = u_t + \frac{u_{xx}^2}{u_{xx} - (1 + u_x^2)}
\]

Note that this equation is not degenerate, since \( f_q(x, u, p, 0, 0) = 1 \). However, the method in [19] is not applicable, because simply isolating \( u_{xx} \) in (3.23) yields an ill-defined vector field for \( u_t = 0 \). Thus we proceed to show that our current method is also applicable for non-degenerate equations and overcomes certain problems in [19].

The characteristic equation (2.11) is given by

\[
\dot{x} = 1, \quad \dot{u} = p, \quad \dot{p} = -(1 + p^2),
\]

and (2.12) is given by

\[
\dot{g} = 2p.
\]

We can solve these equations explicitly. We obtain

\[
(3.27) \quad p(\tau) = -\tan(\tau + \arctan(-p_0))
\]

and

\[
(3.28) \quad g(\tau) = g_0 + 2 \log \left( \frac{\cos(\tau + \arctan(-p_0))}{\cos(\arctan(-p_0))} \right).
\]

Consequently,

\[
(3.29) \quad g(p) = g_0 + 2 \log \left( \frac{\cos(\arctan(-p))}{\cos(\arctan(-p_0))} \right) = g_0 + \log \left( \frac{1 + p_0^2}{1 + p^2} \right),
\]

Thus equations (2.4), (2.15) with \( p_* = p_0 \), and the Dirichlet boundary imply that

\[
(3.30) \quad L_{pp} = \exp(g_0) \frac{1 + p_0^2}{1 + p^2} \quad L^0 = -\exp(g_0)(1 + p_0^2)u \quad \text{and} \quad L^1 = 0.
\]
Thus the energy (2.13), up to a multiplicative constant \(\exp(g_0)(1 + p_0^2)\), is given by

\[
E = \int_0^1 \left( u_x \arctan(u_x) - \log(1 + u_x^2) - u \right) dx,
\]

Thus the energies (3.31) decay according to

\[
\frac{dE}{dt} = -\int_0^1 \frac{(2 + u_x^2)u_x^2}{(1 + u_x^2)^3} u_t^2 dx,
\]

since \(F^1\) given by (3.24) can be rewritten as \(F^1 = \frac{(2+u_x^2)u_x^2}{(1+u_x^2)^2} u_t\) by substituting (3.23).

### 3.5 Porous Medium equation

Consider the porous medium equation (PME) for \(m \geq 1\),

\[
(3.33) \quad u_t = (u^m)_{xx} = mu^{m-1}u_{xx} + m(m-1)u^{m-2}u_x^2.
\]

Note this is a degenerate parabolic equation for non-negative solutions \(u \geq 0\), only.\(^2\)

In the previous setting, \(f_q = mu^{m-1}\), \(F^0 = -m(m-1)u^{m-2}u_x^2\) and \(F^1 = u_t\). Therefore the characteristic equations (2.11) are given by

\[
\begin{align*}
\dot{x} &= mu^{m-1}, \\
\dot{u} &= mu^{m-1}p, \\
\dot{p} &= -m(m-1)u^{m-2}p^2,
\end{align*}
\]

and (2.12) reduces to

\[
\begin{align*}
\dot{g} &= m(m-1)u^{m-2}p.
\end{align*}
\]

If \(u_0 = 0\), then \(g(x, u, p) \equiv g_0\), which in turn implies that \(L_{pp} = m \exp(g_0)u^{m-1}\).

For \(u_0 \neq 0\), we introduce the variable \(\tilde{\tau}\) such that \(d\tilde{\tau}/d\tau = mu^{m-2}\), with notation \((\cdot)' = d(\cdot)/d\tilde{\tau}\), the characteristic equations become

\[
\begin{align*}
x' &= u, \\
\dot{u} &= up, \\
p' &= -(m-1)p^2,
\end{align*}
\]

and

\[
\begin{align*}
g' &= (m-1)p.
\end{align*}
\]

We can solve this explicitly, which yields

\[
(3.38) \quad p(\tilde{\tau}) = \frac{1}{1/p_0 + (m-1)\tilde{\tau}},
\]

\(^2\)The diffusion given by \((|u|^{m-1}u)_{xx} = m|u|^{m-1}u_{xx} + m(m-1)|u|^{m-3}uu_x^2\) is a natural extension that takes sign-changing solutions into account, which is thereby called the **signed PME** in [31]. For the sake of simplicity, we proceed with the usual PME in the main text.
and
\[(3.39) \quad g(\tilde{\tau}) = g_0 + \log \left((m - 1)p_0\tilde{\tau} + 1\right).\]
Hence
\[(3.40) \quad g(p) = g_0 + \log \left(\frac{p_0}{p}\right).\]
Thus equations (2.4), (2.15) with \(p_* = 0\), and the Dirichlet boundary imply that
\[(3.41) \quad L_{pp} = m \exp(g_0) \left| \frac{p_0}{p} \right| u^{m-1}, \quad L^0 = 0, \quad \text{and} \quad L^1 = 0.\]
Hence the Lagrangian \(L\) can be obtained according to (2.13), yielding the following energy formula, up to a multiplicative constant \(|p_0| \exp(g_0)\),
\[(3.42) \quad E = \int_0^1 mu^{m-1}|u_x| \left(\log |u_x| - 1\right) dx,
\]
which decays according to
\[(3.43) \quad \frac{dE}{dt} = -\int_0^1 \frac{u_x^2}{|u_x|} dx.\]
Note that (3.42) is different from the usual energy given by
\[(3.44) \quad \tilde{E} = \int_0^1 \left| \frac{u}{m + 1} \right|^m dx \quad \text{such that} \quad \frac{d\tilde{E}}{dt} = -\int_0^1 \left[\left(|u|^m\right)_x\right]^2 dx.\]
The energy \(E\) decays with respect to the \(L^2\)-norm of \(u\) with weight \(1/|u_x|\), whereas \(\tilde{E}\) decays with respect to the \(L^2\)-norm of \(|u|^m\)\(\_x\). Therefore, the new energy \(E\) in (3.42) decays except at equilibria and it may be more suitable to infer dynamical properties of the porous medium equation, such as the results in [2] and [31].

For general filtration equations, \(u_t = [a(u)]_{xx}\) with \(a(u) \geq 0\), we can recover the well-known energy formula in (3.44) by modifying the method in the previous Section. Indeed, the Ansatz that \(L \equiv L(u)\) and \(L_u = a(u)\) yields
\[(3.45) \quad \tilde{E} = \int_0^1 \left( \int_0^u a(u_1) du_1 \right) dx \quad \text{such that} \quad \frac{d\tilde{E}}{dt} = -\int_0^1 \left(a_u(u)u_x\right)^2 dx.\]
Recall that in the process of integrating the Ansatz, we obtain a constant of integration, \(L^1(x, p)\), that should be added to the above energy (3.45). This constant can be considered to be zero, due to Dirichlet boundary conditions; whereas it should be treated similarly as in the proof in case of Neumann or Robin boundary conditions. For \(a(u) = |u|^m\) with \(m > 0\), we recover (3.44); for \(a(u) = \exp(-1/u)\), we recover the superslow diffusion equation.

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