EXAMPLES OF FANO VARIETIES OF INDEX ONE THAT ARE NOT BIRATIONALY RIGID

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1. PUKHLIKOV’S CONJECTURE

The following is a conjecture of Pukhlikov:

Conjecture 1.1. [P4, Conjecture 5.1] Let $V$ be a smooth Fano variety of dimension $\dim V \geq 4$ with Picard group $\text{Pic}(V)$ generated by the canonical class $K_V$. Then $V$ is birationally rigid. If $\dim V \geq 5$, then $V$ is superrigid.

Conjecture 1.1 was proved for a large class of Fano complete intersections [P1], [P3], [P4], [dF], [dFEM]. More examples are given by complete intersections in weighted projective spaces [P2]. Conjecture 1.1 is a generalization of the famous theorem of Iskovskikh and Manin [IM] that states that a smooth quartic threefold is birationally rigid, and therefore not rational.

In this note we give counterexamples (in arbitrarily large dimension) to Conjecture 1.1 using moduli spaces of bundles on curves. I am grateful to Professor János Kollár for telling me about this question and suggesting to look at sections of the theta divisor on moduli spaces of bundles on curves. I thank Jenia Tevelev and Sean Keel for reading this note and for helpful suggestions.

We recall the basic definitions about birational rigidity from [P4]. Let $V$ be a uniruled $\mathbb{Q}$-Gorenstein variety with terminal singularities. For an effective divisor $D \neq 0$ on $V$, one defines the canonical threshold of canonical adjunction $c(D)$ of the divisor $D$ as follows:

$$c(D) = \sup \{ b/a \mid b, a \in \mathbb{Z}_+ \setminus \{0\}, \quad |aD + bK_X| \neq \emptyset\}.$$

(If there are no $a, b \in \mathbb{Z}_+ \setminus \{0\}$ such that $|aD + bK_X| \neq \emptyset$, then we set $c(D) = 0$.)

Note, for an effective divisor $D \neq 0$ on a Fano variety $V$ with $\text{Pic}(V) = \mathbb{Z}K_V$, the canonical threshold $c(D)$ is the number $-m$, if $D = mK_V$ in $\text{Pic}(V)$. Clearly, in this case $c(D) > 0$.

In what follows all varieties are assumed to be $\mathbb{Q}$-factorial with terminal singularities. For simplicity, we work over an algebraically closed field of characteristic zero.

Definition 1.2. [P4, Def. 5.1] A variety $V$ is called birationally rigid, if for any $V'$, any birational map $\phi : V \dasharrow V'$ and any moving linear system $\Sigma'$ on $V'$, there exists a birational self-map $\alpha : V \dasharrow V$ such that if $\Sigma$ is the birational transform of $\Sigma'$ via the composition $\phi \circ \alpha$ (i.e., the linear system induced on $V$ when composing with $\phi \circ \alpha$), one has $c(\Sigma) \leq c(\Sigma')$. The variety $V$ is called birationally superrigid, if one may always take $\alpha = \text{id}$.

The following Proposition is an immediate consequence of the above definitions:

Proposition 1.3. [P4, Prop. 5.1] Let $V$ be a smooth Fano variety with $\text{Pic}(V) = \mathbb{Z}K_V$. If $V$ is birationally rigid, then it is impossible to have a birational map $V \dasharrow V'$, with $V' \to S'$ a morphism with uniruled general fiber and $S'$ a projective variety of dimension $\dim S' \geq 1$.
Proposition 2.1. \( \end{proof}

\[ \text{We start with a general construction. Let } V \to V' \text{ a morphism with uniruled general fiber and } \dim S' \geq 1. \text{ Let } D' \text{ be an effective Cartier divisor on } S' \text{ and let } \Sigma' = \pi^* D'. \text{ By Definition [2]} \text{ there is a birational map } \alpha : V \to V \text{ such that if } \Sigma \text{ is the birational transform of } \Sigma' \text{ via the composition } \phi \circ \alpha, \text{ then } c(\Sigma) \leq c(\Sigma'). \text{ Note that since } V \text{ is a Fano variety with } \text{Pic}(V) = \mathbb{Z} K_V, \text{ one has } c(\Sigma) > 0. \]

We claim that \( c(\Sigma') = 0 \): if there are \( a, b > 0 \) such that \( a \pi^* D' + b K_{V'} \) is effective on \( V' \), then for a general fiber \( F \) of \( \pi \) (choose \( F \) uniruled, in the smooth locus of \( \pi \) and such that it is not contained in the divisor \( a \pi^* D' + b K_{V'} )\), one has that the divisor

\[ (a \pi^* D' + b K_{V'})_{|F} = (b K_{V'})_{|F} = b K_F \]

is effective on \( F \). This is a contradiction, since \( F \) uniruled implies that \( H^0(b K_F) = 0 \) for all \( b > 0 \) \([5]\). \( \Box \)

Note that since the condition of being uniruled is a closed condition \([5]\), one may drop the word “general” from the statement of Proposition [12]. Moreover, since the condition of being uniruled is a birational property, one may as well replace the condition of having a morphism \( V' \to S' \) with having a birational map \( V' \to S' \) with the same properties.

To motivate geometrically Definition [12], we recall the other definition of birational rigidity from \([3]\). Recall that a morphism \( f : V \to S \) is called a Mori fiber space if it has relative Picard number 1, \(-K_V \) is relatively ample for \( f \) and \( \dim S < \dim V \). Note, Fano varieties with Picard number 1 are trivially Mori fiber spaces. A birational map \( \phi : V \to V' \) to another Mori fiber space \( f' : V' \to S' \) is square if there is a birational map \( h : S \to S' \) such that \( h \circ f = f' \circ \phi \) and the map \( \phi \) induces an isomorphism on the general fibers.

**Definition 1.4.** \((3) \text{ Def. 1.3}\) A Mori fiber space \( f : V \to S \) is called birationally rigid if for any birational map \( \phi : V \to V' \) to another Mori fiber space \( f' : V' \to S' \), there is a birational self-map \( \alpha : V \to V \) such that \( \phi \circ \alpha = \text{square} \). If for any \( \phi \) as above it follows that \( \phi \) is square, then \( V \) is called birationally superrigid.

A Mori fiber space that satisfies Definition [12] also satisfies Definition [13] (by the Sarkisov program, based on the Noether-Fano-Iskovskikh inequalities; this is known in dimension 3 and conjectured in higher dimensions). By the Mori program, any uniruled variety is birational to a Mori fiber space; hence, if in addition one assumes the Mori program, the two definitions are equivalent.

It follows from Definition [13] that if \( V \) is a smooth Fano variety with Picard number 1 and birationally rigid, then there is no birational map \( \phi : V \to V' \) with \( f' : V' \to S' \) another Mori fiber space with \( \dim S' > 0 \). From the Mori program for the relative case, one deduces Prop. [13]. In particular, note that if \( V \) is a smooth Fano variety with Picard number 1 and birationally rigid, then for any \( V' \) another Fano variety of Picard number 1, if \( V' \) is birational to \( V \), then \( V \cong V' \).

2. Counterexamples using moduli spaces of bundles on curves

Let \( C \) be a smooth projective curve over \( \mathbb{C} \) of genus \( g \geq 3 \). Fix \( \xi \) to be a degree 1 line bundle on \( C \). Let \( M \) be the moduli space of stable, rank 2 vector bundles on \( C \) with determinant \( \xi \). The moduli space \( M \) is a smooth, projective, variety of dimension \( 3g - 3 \). The Picard group of \( M \) is \( \mathbb{Z} \) \([4]\). Let \( \Theta \) be the ample generator. In fact, \( \Theta \) is very ample \([5]\). Then \( K_M \cong -2 \Theta \) \([5]\).

Let \( N \) be a nonsingular element of the linear system \( |\Theta| \). Let \( \Theta' \in \text{Pic}(N) \) be the restriction of \( \Theta \) to \( N \). The canonical bundle of \( N \) is \( -\Theta' \). Since \( g \geq 3 \), by Lefschetz’s theorem, \( \Theta' \) generates \( \text{Pic}(N) \). Therefore, the variety \( N \) satisfies the conditions in Conjecture [14]. We prove the following:

**Proposition 2.1.** If \( N \) is a general element of the linear system \( |\Theta| \), then \( N \) is a smooth Fano variety with \( \text{Pic}(N) \cong \mathbb{Z} K_N \) that is not birationally rigid.

**Proof.** We start with a general construction. Let \( e \geq 0 \) and let \( L \) be a line bundle of degree \(-e\) on \( C \). Denote by \( V_L \) the space of extensions \( \text{Ext}^1(C^{e-1} \otimes \xi, L) \cong H^1(C, L^2 \otimes \xi^{-1}) \). Then \( V_L \) parametrizes extensions of the form

\[ 0 \to L \to \mathcal{E} \to C^{e-1} \otimes \xi \to 0 \]

(*)

\[ \text{Castravet} \]
Remark 2.5.\[\kappa_{\mathcal{L}} : \mathbb{P}(\mathcal{L}) \setminus Z_{\mathcal{L}} \rightarrow M\]that associates to an extension \(\mathcal{E}\) the isomorphism class of \(\mathcal{E}\). By [Ca, Lemma 2.1] or Lemma A.1\[\kappa_{\mathcal{L}}^*\Theta \cong \mathcal{O}(2e+1)\].

Consider the case when \(e = g - 1\). Then \(\mathbb{P}(\mathcal{L}) \cong \mathbb{P}^{3g-3}\) and \(\kappa_{\mathcal{L}}^*\Theta \cong \mathcal{O}(2g-1)\). The following Claim is a standard fact. For convenience, we include a short proof.

**Claim 2.2.** If \(\mathcal{L}\) is a line bundle of degree \(1 - g\), the morphism \(\kappa_{\mathcal{L}}\) in (2.1) is birational.

**Proof of Claim 2.2.** Note that for any \(\mathcal{E} \in M\), by Riemann-Roch one has

\[\chi(\mathcal{E}^* \otimes \mathcal{L}^{-1} \otimes \xi) = 1.\]

Hence, for any \(\mathcal{E} \in M\) one has:

\[\text{Hom}(\mathcal{E}, \mathcal{L}^{-1} \otimes \xi) \cong H^0(\mathcal{E}^* \otimes \mathcal{L}^{-1} \otimes \xi) \neq 0\]

If for general \(\mathcal{E} \in M\), any non-zero morphism \(\phi : \mathcal{E} \rightarrow \mathcal{L}^{-1} \otimes \xi\) is surjective, then we are done, as \(\phi\) determines uniquely (up to scaling) the extension \([1]\), and by dimension considerations, one must have that

\[h^0(\mathcal{E}^* \otimes \mathcal{L}^{-1} \otimes \xi) = 1,\]

(i.e., the fiber of \(\kappa\) at a general point contains a unique closed point).

We prove that for general \(\mathcal{E}\), a non-zero morphism \(\phi : \mathcal{E} \rightarrow \mathcal{L}^{-1} \otimes \xi\) must be surjective. Note that if \(\phi\) is not surjective, then its image is \(\mathcal{L}^{-1} \otimes \xi(-D)\), for some effective divisor \(D\) of degree \(d > 0\) and there is an exact sequence:

\[0 \rightarrow \mathcal{L}(D) \rightarrow \mathcal{E} \rightarrow \mathcal{L}^{-1} \otimes \xi(-D) \rightarrow 0\]  \(\text{(2.2)}\)

It follows from the stability of \(\mathcal{E}\) that \(d < g\). For each \(0 < d < g\) construct the total space \(\mathbb{P}_d\) of extensions \([2.2]\) by letting \(D\) vary in \(\text{Sym}^d(C)\): the space \(\mathbb{P}_d\) is a projective bundle over \(\text{Sym}^d(C)\) with fiber at \(D\) isomorphic to \(\mathbb{P}(\mathcal{L}(D))\). The dimension is:

\[\dim \mathbb{P}_d = d + \dim \mathbb{P}_d(D) - 1 = d + 2(g - 1 - d) + g - 1 = 3g - 3 - d.\]

The vector bundle which is the middle term of the universal extension over \(\mathbb{P}_d\) induces a rational map \(\varphi : \mathbb{P}_d \dashrightarrow M\). Since \(d > 0\), the map \(\varphi\) is not dominant. Therefore, a general \(\mathcal{E} \in M\) will not sit in an exact sequence \((2.2)\).

From the previous discussion and since \(\Theta\) is very ample, one has the following:

**Claim 2.3.** If \(N\) is a general element of the linear system \(\{\Theta\}\), then \(N\) is birational to an irreducible hypersurface \(X_{2g-1}\) in \(\mathbb{P}^{3g-3}\) of degree \(2g - 1\).

**Proposition 2.1** follows now from Claim 2.3 and Lemma 2.4.

**Lemma 2.4.** For any irreducible hypersurface \(X_d \subset \mathbb{P}^n\) of degree \(d < n\) (possibly singular) there is a rational map \(\rho : X \dashrightarrow S\), \(\dim S > 0\), with uniruled fibers.

**Proof.** Let \(S\) be a general pencil of hyperplanes \(\{H_s\}_{s \in S}\) in \(\mathbb{P}^n\). Then \(X_s = X \cap H_s\) are hypersurfaces of degree \(d\) in \(H_s \cong \mathbb{P}^{n-1}\). A smooth hypersurface of degree \(d < n\) in \(\mathbb{P}^{n-1}\) is Fano; hence, it is rationally connected [KMM]. It follows by deformation theory that any irreducible hypersurface of degree \(d < n\) in \(\mathbb{P}^{n-1}\) is uniruled. Hence, for all \(s \in S\) such that \(X_s\) is irreducible, \(X_s\) is uniruled. Therefore, the induced rational map \(\rho : X \dashrightarrow S\) has uniruled fibers.

**Remark 2.5.** The cohomology group \(H^4(M; \mathcal{Q})\) has two independent generators [N]. Hence, by Lefschetz’s theorem, if \(g \geq 4\), the rank of \(H^4(N; \mathcal{Q})\) is also 2. Since again by Lefschetz’s theorem, the cohomology group \(H^4\) of any smooth complete intersection of dimension \(\geq 5\) is of rank 1, it follows that \(N\) is not a complete intersection.
3. Description of the hypersurface $X_{2g-1} \subset \mathbb{P}^{3g-3}$

Our construction of the map $\kappa_L$ is a variant of the construction of Bertram [B]. The above results about the map $\kappa_L$ (deg $L = 1 - g$) also follow from [B]. We chose to include the above considerations (which are enough for the purpose of this note) because of their simplicity and to avoid referring to the technical results in [B]. However, Bertram’s powerful construction gives a precise description of the hypersurface $X_{2g-1} \subset \mathbb{P}^{3g-3}$. We describe this below.

Let $L$ be a line bundle of degree $1 - g$ on $C$. Then $C$ has a natural embedding $C \subset \mathbb{P}(V_L) \cong \mathbb{P}^{3g-3}$ given by $L^{-2} \otimes \xi \otimes K_C$, since by Serre duality one has:

$$V_L \cong H^1(C, L^2 \otimes \xi^{-1}) \cong H^0(C, L^{-2} \otimes \xi \otimes K_C)^*.$$  

Let $\text{Sec}^k(C)$ be the $(k + 1)$-secant variety of $C$ (i.e., the closure in $\mathbb{P}^{3g-3}$ of the union of all the $k$-planes spanned by $k + 1$ distinct points on $C$).

**Theorem 3.1.** [B Thm. 1] There is a sequence of blow-ups $\pi : \tilde{P}_L \rightarrow \mathbb{P}(V_L)$ with smooth centers (starting with the blow-up of $\mathbb{P}(V_L)$ along $C$) that resolves the rational map $\kappa_L : \mathbb{P}(V_L) \dasharrow M$ into a morphism $\tilde{P}_L \rightarrow M$. There are $g$ exceptional divisors $E_0, E_1, \ldots, E_{g-1}$ and $E_k$ dominates the secant variety $\text{Sec}^k(C)$ for every $k$.

**Theorem 3.2.** [B Thm. 2, Prop. 4.7] There is a natural identification

$$H^i(M, \Theta) \cong H^i(\tilde{P}_L, (2g - 1)H - (2g - 3)E_0 - (2g - 5)E_1 - \ldots - E_{g-2}).$$

(3.1)

It follows from Theorem 3.2 that the proper transform $\tilde{X}$ in $\tilde{P}_L$ of the hypersurface $X_{2g-1} \subset \mathbb{P}^{3g-3}$ of Proposition 2.3 is a general member of the linear system in (3.1). Hence, by Bertini, $\tilde{X}$ is smooth.

**Proposition 3.3.** The singular locus of the hypersurface $X_{2g-1} \subset \mathbb{P}^{3g-3}$ has codimension $\geq g - 2$. Hence, if $g \geq 4$ then $X$ is normal and the canonical bundle $K_X$ is Cartier. Moreover, in this case $X$ has terminal singularities.

**Proof.** Since the proper transform $\tilde{X}$ of $X$ is smooth, it follows that $X$ is smooth outside $\text{Sec}^{g-1}(C) \cap X$. Since a general $X$ does not contain $\text{Sec}^{g-1}(C)$, it follows that the singular locus of $X$ has dimension at most $\dim \text{Sec}^{g-1}(C) - 1 = 2g - 2$. It is well-known that hypersurfaces $X$ whose singular locus has codimension at least 2 are normal and the canonical class $K_X$ is Cartier. Hence, if $g \geq 4$ then $X$ is normal and has the canonical class:

$$K_X = \mathcal{O}_X(1 - g).$$

Consider the resolution $\pi : \tilde{X} \rightarrow X$. The canonical class of $\tilde{P}_L$ is given by:

$$K_{\tilde{P}_L} = -(3g - 2)H + (3g - 5)E_0 + (3g - 7)E_1 + \ldots + (g - 1)E_{g-2}$$

The canonical class of $\tilde{X}$ is given by:

$$K_{\tilde{X}} = (K_{\tilde{P}_L} + \tilde{X})|_{\tilde{X}} = -(g - 1)H + (g - 2)E_0 + (g - 2)E_1 + \ldots + (g - 2)E_{g-2}.$$  

Hence, $X$ has terminal singularities.

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