An Algorithm to Decompose Permutation Representations of Finite Groups:
A Polynomial Algebra Approach

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Abstract
We describe an algorithm for splitting a permutation representation of a finite group into irreducible components. The algorithm is based on the fact that the components of the invariant inner product in invariant subspaces are operators of projection into these subspaces. An important element of the algorithm is the calculation of Gröbner bases of polynomial ideals. A preliminary implementation of the algorithm splits representations up to dimensions of several thousand. Some examples of computations are given in Appendix A.

1 Introduction
In general, the problem of splitting a module over an associative algebra into irreducible submodules is quite nontrivial. An overview of the algorithmic aspects of this problem can be found in [1]. We consider here a particular, but important from our point of view [2], case of the problem.

Let $G$ be a permutation group on the set $\Omega \cong \{1, \ldots, N\}$. We will denote the action of $g \in G$ on $i \in \Omega$ by $i^g$. To avoid inessential technical complications, we assume that $G$ acts transitively on $\Omega$.

A representation of $G$ in an $N$-dimensional vector space over a field $F$ by the matrices $P(g)$ with the entries $P(g)_{ij} = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker delta, will be called a permutation representation. Our approach uses the concept of inner product, so we have to exclude vector spaces over finite fields — there are no reasonable inner products for such spaces. Thus, we assume that the permutation representation space is a Hilbert space $\mathcal{H}_N$. From a constructive point of view it is sufficient to assume that the base field $F$ is a minimal splitting field of the group $G$. Such field is a subfield of an $m$-th cyclotomic field, where $m$ is a certain divisor of the exponent of the group $G$. The field $F$, being an abelian extension of the field $Q$, is a constructive dense subfield of $R$ or $C$.

An orbit of $G$ on the Cartesian square $\Omega \times \Omega$ is called an orbital [3]. The number of orbitals, called the rank of $G$, will be denoted by $R$. Among the orbitals of a transitive
group there is one diagonal orbital, \( \Delta_1 = \{(i,i) \mid i \in \Omega \} \), which will always be fixed as the first element in the list of orbitals: \( \{ \Delta_1, \ldots, \Delta_R \} \). For transitive \( G \) there is a natural one-to-one correspondence between the orbitals of \( G \) and the orbits of a point stabilizer \( G_i \):
\[
\Delta \leftrightarrow \Sigma_i = \{j \in \Omega \mid (i,j) \in \Delta\}.
\]
The \( G_i \)-orbits are called suborbits and their cardinalities will be called the suborbit lengths. Note that \(|\Delta| = N|\Sigma_i|\).

The invariance condition for a bilinear form \( A \) in the Hilbert space \( \mathcal{H}_N \) can be written as the system of equations \( A = P(g)AP(g^{-1}), \ g \in G \). It is easy to verify that in terms of the entries the equations of this system have the form \((A)_{ij} = (A)_{ig}g_{jg}\). Thus, the basis of all invariant bilinear forms is in one-to-one correspondence with the set of orbitals: with each orbital \( \Delta_r \in \{ \Delta_1, \ldots, \Delta_R \} \) is associated a basis matrix \( A_r \) of the size \( N \times N \) with the entries
\[
(A_r)_{ij} = \begin{cases} 1, & \text{if } (i,j) \in \Delta_r, \\ 0, & \text{if } (i,j) \notin \Delta_r. \end{cases}
\]
In particular, for the diagonal orbital we have \( A_1 = 1_N \). The matrices
\[
A_1, A_2, \ldots, A_R
\]
form a basis of the centralizer algebra of the representation \( P \). The multiplication table for this basis has the form
\[
A_pA_q = \sum_{r=1}^{R} C_{pq}^r A_r,
\]
where \( C_{pq}^r \) are non-negative integers. The commutativity of the centralizer algebra indicates that the permutation representation \( P \) is multiplicity-free.

### 2 Algorithm

Let \( T \) be a transformation (we can assume that \( T \) is a unitary matrix) that splits the permutation representation \( P \) into \( M \) irreducible components:
\[
T^{-1}P(g)T = 1 \oplus U_{d_2}(g) \oplus \cdots \oplus U_{d_m}(g) \oplus \cdots \oplus U_{d_M}(g),
\]
where \( U_{d_m} \) is a \( d_m \)-dimensional irreducible subrepresentation, \( \oplus \) denotes the direct sum of matrices, i.e., \( A \oplus B = \text{diag}(A,B) \).

The identity matrix \( 1_N \) is the standard inner product in any orthonormal basis. In the splitting basis we have the following decomposition of the standard inner product
\[
1_N = 1_{d_1=1} \oplus \cdots \oplus 1_{d_m} \oplus \cdots \oplus 1_{d_M}.
\]
The inverse image of this decomposition in the original permutation basis has the form
\[
1_N = B_1 + \cdots + B_m + \cdots + B_M,
\]
where \( B_m \) is defined by the relation
\[
T^{-1}B_mT = 0_{1+d_2+\cdots+d_{m-1}} \oplus 1_{d_m} \oplus 0_{d_{m+1}+\cdots+d_M}.
\]
The main idea of the algorithm is based on the fact that $B_m$ is a projector, i.e., $B_m^2 = B_m$. Thus, all $B_m$’s can be obtained as solutions of the equation

$$X^2 - X = 0$$ (4)

for the generic invariant form

$$X = x_1 A_1 + \cdots + x_R A_R.$$ 

Using the multiplication table (2), we can write the left-hand side of (4) as a set of polynomials

$$E(x_1, \ldots, x_R) = \{ E_1(x_1, \ldots, x_R), \ldots, E_R(x_1, \ldots, x_R) \}$$ (5)

and equation (4) takes the form

$$E(x_1, \ldots, x_R) = 0.$$ (6)

(This is an abbreviation for the set of equations $\{ E_1 = 0, \ldots, E_R = 0 \}$.) Each polynomial in (5) has the structure $E_r(x_1, \ldots, x_R) = Q_r(x_1, \ldots, x_R) - x_r$, where $Q_r(x_1, \ldots, x_R)$ is a homogeneous polynomial of degree 2 in the indeterminates $x_1, \ldots, x_R$.

In the basis (1) the projector $B_m$ can be represented as

$$B_m = b_{m,1} A_1 + b_{m,2} A_2 + \cdots + b_{m,R} A_R,$$

where the vector

$$B_m = [b_{m,1}, \ldots, b_{m,R}]$$

is a solution of equation (6). Since only $A_1$ has nonzero diagonal elements, we have

$$\text{tr} B_m = b_{m,1} N.$$ 

On the other hand, relation (3) shows that

$$\text{tr} B_m = d_m.$$ 

This allows us to fix the coefficient $b_{m,1}$:

$$b_{m,1} = d_m / N.$$ 

Thus the only relevant values of $x_1$ in (6) are $d/N$ for some $d$’s from the interval $[1..N-1]$. Any relevant natural number $d$ is either irreducible dimension or sum of such dimensions. Using the orthogonality condition for the projectors,

$$B_m B_{m'} = 0 \text{ if } m \neq m',$$

we can exclude the consideration of dimensions that are sums of irreducible ones.

The main part of our algorithm is a loop over the possible dimensions $d$. The loop starts with $d = 1$ and ends when the sum of irreducible dimensions becomes equal to $N$. The current $d$ is processed as follows:

1. Substitute $d$ into the set of polynomials: $E(x_1, x_2, \ldots, x_R) \rightarrow E(d/N, x_2, \ldots, x_R)$.
2. Compute the Gröbner basis (Gb) of the polynomial system $E(d/N, x_2, \ldots, x_R)$.
3. If $\text{Gb} = [1]$ then the system of equations $E(d/N, x_2, \ldots, x_R) = 0$ is inconsistent. Go to the next dimension: $d \rightarrow d + 1$.

4. Otherwise compute the Hilbert dimension ($H_d$) of the Gröbner basis $\text{Gb}$.

5. If $H_d = 0$ then all irreducible components of the dimension $d$ are multiplicity-free. If there are $k$ different $d$-dimensional irreducible components then for the system of polynomial equations $\text{Gb} = 0$ we obtain the following set of solutions:

$$\{B_{m+1} = [d/N, b_{m+1,2}, \ldots, b_{m+1,R}], \ldots, B_{m+k} = [d/N, b_{m+k,2}, \ldots, b_{m+k,R}]\}. \quad (7)$$

Here $m$ is the number of irreducible components constructed before. Apply the procedure $\text{ProcessSingleSolution}$ (described below) to all solutions in (7).

6. If $H_d > 0$ then we encounter an irreducible component with a multiplicity $k > 1$. The corresponding component of the centralizer algebra has the form $A \otimes 1_d$, where $A$ is an arbitrary $k \times k$ matrix, and $\otimes$ denotes the Kronecker product. The idempotency condition $(A \otimes 1_d)^2 = A \otimes 1_d$ implies

$$A^2 - A = 0. \quad (8)$$

The complete family of solutions\(^1\) of (8) is a manifold of dimension $\left\lfloor k^2/2 \right\rfloor = H_d$. Application of Gröbner basis technique for polynomial systems with parameters involves the cumbersome procedure of partitioning the parameter space. To avoid these difficulties, we construct $k$ mutually orthogonal particular solutions of the form (7) and apply to each of them the procedure $\text{ProcessSingleSolution}$.

The input of procedure $\text{ProcessSingleSolution}$ is the current irreducible projector $B_m$. The procedure performs the following.

1. Calculate the orthogonality condition $B_m X = 0$. This is a system of linear equations

$$O_m(x_1, \ldots, x_R) = 0$$

for the indeterminates $x_1, \ldots, x_R$ with the numerical coefficients $b_{m,1}, \ldots, b_{m,R}$.

2. Add the orthogonality relations to the idempotency polynomial set (5)

$$E(x_1, \ldots, x_R) \rightarrow E(x_1, \ldots, x_R) \cup O_m(x_1, \ldots, x_R).$$

This is done in order to exclude from further consideration the subspace of the projector $B_m$.

3. Add $B_m$ to the list of irreducible projectors $\{B_1, \ldots, B_{m-1}\} \rightarrow \{B_1, \ldots, B_{m-1}, B_m\}$.

\(^1\)The computation of the solutions is always algorithmically realizable, since the problem involves only polynomial equations with abelian Galois groups.

\(^2\)It is well known that any solution of equation (8) can be represented as $A = Q^{-1}(I_r \oplus 0_{k-r})Q$, where $Q$ is an arbitrary invertible $k \times k$ matrix, and $0 \leq r \leq k$. 4

3 Implementation

Our approach involves some widely used methods of polynomial computer algebra. Therefore it is reasonable, at least for the preliminary experience, to take advantage of computer algebra systems with the developed tools for working with polynomials.

Our current implementation is a program written in C. The input data for the program is a set of permutations \( S = \{s_1, \ldots, s_K\} \) of degree \( N \) that generates the group \( G \).

The program

1. computes the basis of the centralizer algebra (1) and its multiplication table (2),
2. constructs the system of quadratic polynomials (5) — the left hand sides of the idempotency condition \( X^2 - X = 0 \),
3. constructs the bilinear system \( BX \) corresponding to the orthogonality condition \( BX = 0 \) — the indeterminates \( B = [b_1, \ldots, b_R] \) of this system are replaced by specific numerical values in the procedure \text{ProcessSingleSolution},
4. generates the code for processing the above constructed polynomial data by the computer algebra system \text{Maple}.

Conclusion

The algorithm described here is based on the use of methods of polynomial algebra, which are considered algorithmically difficult. However, our approach leads to a small number of low-degree polynomials. As can be seen in Appendix A even a straightforward implementation of the approach can cope with rather large tasks. There are the obvious ways to improve performance: (1) to write in C Gröbner bases algorithms specialized for the problem under consideration, (2) to use a more efficient system of computer algebra, for example, \text{Magma}.

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References

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[3] Cameron P. J. \textit{Permutation Groups}. Cambridge University Press, 1999.

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A Examples of computation: Mathieu group $M_{22}$ and its cover $3M_{22}$

Main properties of the group. The Mathieu group $M_{22}$ is one of the 26 sporadic simple groups. $|M_{22}| = 443520 = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$. $\text{Exp}(M_{22}) = 9240 = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11$. Schur multiplier: $C_{12}$. $\text{Out}(M_{22}) = C_2$.

Comments. Generators of representations are taken from the ATLAS [4].

The results presented below assume the following ordering for the centralizer algebra basis matrices

\[
\begin{align*}
A_1 &= 1, & A_2, \ldots, A_k, & A_{k+1}, A_{k+2} &= A^{T}_{k+1}, \ldots, A_{R-1}, A_R &= A^{T}_{R-1}.
\end{align*}
\]

The matrices within the first sublist are ordered by the rule: $A < B$ if $i_A < i_B$, where $i_A = \min(i \mid (A)_{i1} = 1)$ (and similarly for $i_B$). The same rule is applied to the first elements of the pairs of asymmetric matrices.

Representations are denoted by their dimensions in bold (possibly with some signs added to distinguish different representations of the same dimension). Permutation representations are underlined. Multiple subrepresentations are underbraced in decompositions.

All timing data were obtained on a PC with 3.30GHz Intel Core i3 2120 CPU.

A.1 462-dimensional representations of $M_{22}$

There are three 462-dimensional permutation representations of $M_{22}$, which are distinguished by the letters $a$, $b$, $c$ in the ATLAS [4].

A.1.1 Representation 462a

Rank: 5. Suborbit lengths: 1, 240, 120, 96, 5.

\[
462a \cong 1 \oplus 21 \oplus 55 \oplus 154 \oplus 231
\]

\[
\begin{align*}
B_1 &= \frac{1}{462} \sum_{k=1}^{5} A_k \\
B_{21} &= \frac{1}{22} \left( A_1 + \frac{1}{12} A_2 + \frac{1}{12} A_3 - \frac{3}{8} A_4 + A_5 \right) \\
B_{55} &= \frac{5}{42} \left( A_1 - \frac{1}{20} A_2 - \frac{1}{20} A_3 + \frac{1}{8} A_4 + A_5 \right) \\
B_{154} &= \frac{1}{3} \left( A_1 - \frac{1}{20} A_2 + \frac{1}{10} A_3 - \frac{1}{5} A_5 \right) \\
B_{231} &= \frac{1}{2} \left( A_1 + \frac{1}{30} A_2 - \frac{1}{15} A_3 - \frac{1}{5} A_5 \right)
\end{align*}
\]

Time C: 11 sec. Time Maple: 2 sec.
A.1.2 Representation 462b

Rank: 8. Suborbit lengths: 1, 320, 60, 20, 20, 1, 20, 20.

\[ 462b \cong 1 \oplus (21 \oplus 21) \oplus 55 \oplus 154 \oplus 210 \]

\[ B_1 = \frac{1}{462} \sum_{k=1}^{8} A_k \]

\[ B_{21} = \frac{1}{22} \left\{ A_1 - \frac{1}{19} A_2 - \frac{1}{19} A_3 + \frac{3}{19} \left( 3 - \sqrt{11} \right) A_4 + \frac{3}{19} \left( 3 + \sqrt{11} \right) A_5 + \frac{1}{19} A_6 \right\} \]

\[ B_{21}' = \frac{1}{22} \left\{ A_1 - \frac{9}{190} A_2 - \frac{9}{190} A_3 + \frac{1}{380} \left( 181 + 60\sqrt{11} \right) A_4 + \frac{1}{380} \left( 181 - 60\sqrt{11} \right) A_5 - \frac{1}{19} A_6 - \frac{1}{20} A_7 - \frac{1}{20} A_8 \right\} \]

\[ B_{55} = \frac{5}{42} \left( A_1 - \frac{1}{20} A_2 + \frac{3}{10} A_3 - \frac{1}{20} A_4 - \frac{1}{20} A_5 + A_6 - \frac{1}{20} A_7 - \frac{1}{20} A_8 \right) \]

\[ B_{154} = \frac{1}{3} \left( A_1 + \frac{1}{40} A_2 - \frac{1}{10} A_3 - \frac{1}{20} A_4 - \frac{1}{20} A_5 + A_6 - \frac{1}{20} A_7 - \frac{1}{20} A_8 \right) \]

\[ B_{210} = \frac{5}{11} \left( A_1 - \frac{1}{20} A_4 - \frac{1}{20} A_5 - \frac{1}{20} A_6 + \frac{1}{20} A_7 + \frac{1}{20} A_8 \right) \]

Time C: 15 sec. Time Maple: 4 sec.

A.1.3 Representation 462c

Rank: 8. Suborbit lengths: 1, 160, 96, 20, 20, 5, 80, 80.

\[ 462c \cong 1 \oplus (21 \oplus 21) \oplus 55 \oplus 154 \oplus 210 \]

\[ B_1 = \frac{1}{462} \sum_{k=1}^{8} A_k \]

\[ B_{21} = \frac{1}{22} \left\{ A_1 + \frac{1}{124} \left( 7 - 3\sqrt{33} \right) A_2 - \frac{1}{248} \left( 53 - 5\sqrt{33} \right) A_3 + \frac{1}{124} \left( 57 + 11\sqrt{33} \right) A_4 - \frac{1}{124} \left( 7 - 3\sqrt{33} \right) A_5 + \frac{1}{31} \left( 15 - 2\sqrt{33} \right) A_6 \right\} \]

\[ B_{21}' = \frac{1}{22} \left\{ A_1 + \frac{1}{620} \left( 58 + 15\sqrt{33} \right) A_2 - \frac{5}{248} \left( 8 + \sqrt{33} \right) A_3 + \frac{1}{620} \left( 304 - 55\sqrt{33} \right) A_4 - \frac{3}{124} \left( 8 + \sqrt{33} \right) A_5 + \frac{1}{155} \left( 49 + 10\sqrt{33} \right) A_6 - \frac{1}{20} A_7 - \frac{1}{20} A_8 \right\} \]

\[ B_{55} = \frac{5}{42} \left( A_1 - \frac{1}{20} A_2 + \frac{1}{8} A_3 - \frac{1}{20} A_4 - \frac{1}{20} A_5 + A_6 - \frac{1}{20} A_7 - \frac{1}{20} A_8 \right) \]

\[ B_{154} = \frac{1}{3} \left( A_1 + \frac{1}{40} A_2 - \frac{1}{20} A_3 + \frac{1}{4} A_5 - \frac{1}{5} A_6 - \frac{1}{20} A_7 - \frac{1}{20} A_8 \right) \]

\[ B_{210} = \frac{5}{11} \left( A_1 - \frac{1}{40} A_2 - \frac{1}{20} A_4 - \frac{1}{20} A_5 - \frac{3}{20} A_6 + \frac{1}{20} A_7 + \frac{1}{20} A_8 \right) \]

Time C: 7 sec. Time Maple: 2 sec.
A.2 672-dimensional representation of $M_{22}$

Rank: 6. Suborbit lengths: 1, 165, 55, 330, 55, 66.

$$672 \cong 1 \oplus 21 \oplus 55 \oplus 154 \oplus 210 \oplus 231$$

\[
B_1 = \frac{1}{672} \sum_{k=1}^{6} A_k
\]

\[
B_{21} = \frac{1}{32} \left( A_1 + \frac{3}{11} A_2 + \frac{3}{11} A_3 - \frac{1}{11} A_4 - \frac{5}{11} A_5 - \frac{1}{11} A_6 \right)
\]

\[
B_{55} = \frac{55}{672} \left( A_1 - \frac{1}{55} A_2 + \frac{13}{55} A_3 - \frac{1}{55} A_4 + \frac{13}{55} A_5 - \frac{3}{11} A_6 \right)
\]

\[
B_{154} = \frac{11}{48} \left( A_1 + \frac{3}{55} A_2 - \frac{3}{55} A_3 - \frac{3}{55} A_4 + \frac{7}{55} A_5 + \frac{1}{11} A_6 \right)
\]

\[
B_{210} = \frac{5}{16} \left( A_1 + \frac{1}{165} A_2 - \frac{7}{55} A_3 + \frac{7}{165} A_4 - \frac{3}{55} A_5 - \frac{1}{11} A_6 \right)
\]

\[
B_{231} = \frac{11}{32} \left( A_1 - \frac{1}{15} A_2 + \frac{1}{11} A_3 + \frac{1}{165} A_4 - \frac{3}{55} A_5 + \frac{1}{11} A_6 \right)
\]

Time C: 43 sec. Time Maple: 2 sec.

A.3 770-dimensional representation of $M_{22}$

Rank: 9. Suborbit lengths: 1, 96, 144, 72, 144, 9, 16, 144, 144.

$$770 \cong 1 \oplus 21 \oplus (55 + 55) \oplus 99 \oplus 154 \oplus 385$$

\[
B_1 = \frac{1}{770} \sum_{k=1}^{9} A_k
\]

\[
B_{21} = \frac{3}{110} \left( A_1 + \frac{1}{12} A_2 + \frac{1}{12} A_3 + \frac{1}{12} A_4 - \frac{3}{8} A_5 + A_6 - \frac{3}{8} A_7 + \frac{1}{12} A_8 + \frac{1}{12} A_9 \right)
\]

\[
B_{55} = \frac{1}{14} \left\{ A_1 + \frac{1}{4} \left( 1 - i \frac{3}{\sqrt{7}} \right) A_2 - \frac{1}{4} \left( 1 - i \frac{1}{\sqrt{7}} \right) A_3 - \frac{1}{4} \left( 1 + i \frac{1}{\sqrt{7}} \right) A_4 + \frac{1}{8} \left( 1 + i \frac{1}{\sqrt{7}} \right) A_5 + \left( 1 + i \frac{2}{\sqrt{7}} \right) A_6 + \frac{1}{8} \left( 1 + i \frac{9}{\sqrt{7}} \right) A_7 \right\}
\]

\[
B_{55}' = \frac{1}{14} \left\{ A_1 - \frac{1}{12} \left( 2 - i \frac{9}{\sqrt{7}} \right) A_2 + \frac{1}{12} \left( 2 - i \frac{3}{\sqrt{7}} \right) A_3 + \frac{1}{36} \left( 16 + i \frac{9}{\sqrt{7}} \right) A_4 + \frac{1}{72} \left( 4 - i \frac{9}{\sqrt{7}} \right) A_5 - \frac{1}{9} \left( 1 + i \frac{18}{\sqrt{7}} \right) A_6 - \frac{1}{8} \left( 4 + i \frac{9}{\sqrt{7}} \right) A_7 - \frac{5}{36} A_8 - \frac{5}{36} A_9 \right\}
\]

\[
B_{99} = \frac{9}{70} \left( A_1 - \frac{1}{24} A_2 - \frac{1}{9} A_3 + \frac{1}{6} A_4 - \frac{1}{24} A_5 - \frac{1}{9} A_6 + \frac{3}{8} A_7 + \frac{1}{36} A_8 + \frac{1}{36} A_9 \right)
\]

\[
B_{154} = \frac{1}{5} \left( A_1 + \frac{1}{12} A_2 + \frac{1}{12} A_3 - \frac{1}{18} A_4 - \frac{1}{36} A_5 - \frac{1}{9} A_6 + \frac{1}{4} A_7 - \frac{1}{18} A_8 - \frac{1}{18} A_9 \right)
\]

\[
B_{385} = \frac{1}{2} \left( A_1 - \frac{1}{24} A_2 - \frac{1}{18} A_4 + \frac{1}{72} A_5 - \frac{1}{9} A_6 - \frac{1}{8} A_7 + \frac{1}{36} A_8 + \frac{1}{36} A_9 \right)
\]

Time C: 38 sec. Time Maple: 5 sec.
A.4 990-dimensional representation of $3.M_{22}$

Rank: 13. Suborbit lengths: 1, 168, 336, 42, 7, 168, 168, 42, 42, 7, 7, 1, 1.

\[
990 \cong 1 \oplus 21_\alpha \oplus 21_\beta \oplus 21_\gamma \oplus 55_\alpha \oplus 99_\alpha \oplus 99_\beta \oplus 99_\gamma \\
\oplus 105^+_\alpha \oplus 105^+_\beta \oplus 105^-_\alpha \oplus 105^-_\beta \oplus 154
\]

\[
B_1 = \frac{1}{990} \sum_{k=1}^{13} A_k
\]

\[
B_{21_\alpha} = \frac{7}{330} \left( A_1 - \frac{5}{28} A_2 + \frac{3}{14} A_3 + \frac{3}{14} A_4 - \frac{4}{7} A_5 - \frac{5}{28} A_6 - \frac{5}{28} A_7 + \frac{3}{14} A_8 + \frac{3}{14} A_9 \\
- \frac{4}{7} A_{10} - \frac{4}{7} A_{11} + A_{12} + A_{13} \right)
\]

\[
B_{21_\beta} = \frac{7}{330} \left( A_1 + \frac{1}{7} A_2 + \frac{3}{7} A_4 + \frac{5}{7} A_5 - \frac{1}{14} \left( 1 - i \sqrt{3} \right) A_6 - \frac{1}{14} \left( 1 + i \sqrt{3} \right) A_7 \\
- \frac{3}{14} \left( 1 - i \sqrt{3} \right) A_8 - \frac{3}{14} \left( 1 + i \sqrt{3} \right) A_9 - \frac{5}{14} \left( 1 - i \sqrt{3} \right) A_{10} \\
- \frac{5}{14} \left( 1 + i \sqrt{3} \right) A_{11} - \frac{1}{2} \left( 1 + i \sqrt{3} \right) A_{12} - \frac{1}{2} \left( 1 - i \sqrt{3} \right) A_{13} \right)
\]

\[
B_{21_\gamma} = \overline{B_{21_\beta}}
\]

\[
B_{55} = \frac{1}{18} \left( A_1 - \frac{1}{28} A_2 - \frac{1}{14} A_3 + \frac{3}{14} A_4 - \frac{4}{7} A_5 - \frac{1}{28} A_6 - \frac{1}{28} A_7 + \frac{3}{14} A_8 + \frac{3}{14} A_9 \\
+ \frac{4}{7} A_{10} + \frac{4}{7} A_{11} + A_{12} + A_{13} \right)
\]

\[
B_{99_\alpha} = \frac{1}{10} \left( A_1 + \frac{1}{12} A_2 - \frac{1}{6} A_4 - \frac{1}{24} \left( 1 - i \sqrt{3} \right) A_6 - \frac{1}{24} \left( 1 + i \sqrt{3} \right) A_7 \\
+ \frac{1}{12} \left( 1 - i \sqrt{3} \right) A_8 + \frac{1}{12} \left( 1 + i \sqrt{3} \right) A_9 - \frac{1}{2} \left( 1 + i \sqrt{3} \right) A_{12} \\
- \frac{1}{2} \left( 1 - i \sqrt{3} \right) A_{13} \right)
\]

\[
B_{99_\beta} = \overline{B_{99_\alpha}}
\]

\[
B_{99_\gamma} = \frac{1}{10} \left( A_1 + \frac{1}{21} A_2 - \frac{1}{14} A_3 + \frac{1}{21} A_4 - \frac{3}{7} A_5 + \frac{1}{21} A_6 + \frac{1}{21} A_7 + \frac{1}{21} A_8 + \frac{1}{21} A_9 \\
- \frac{3}{7} A_{10} - \frac{3}{7} A_{11} + A_{12} + A_{13} \right)
\]

\[
B_{105^+_\alpha} = \frac{7}{66} \left( A_1 - \frac{56}{1} \left( 3 + \frac{\sqrt{33}}{3} \right) A_2 + \frac{1}{28} \left( 1 + \frac{\sqrt{33}}{3} \right) A_4 - \frac{1}{14} \left( 1 \mp \sqrt{33} \right) A_5 \\
+ \frac{1}{112} \left[ 3 + \frac{\sqrt{33}}{3} + i \left( \sqrt{11} \mp 3 \sqrt{3} \right) \right] A_6 + \frac{1}{112} \left[ 3 + \frac{\sqrt{33}}{3} - i \left( \sqrt{11} \mp 3 \sqrt{3} \right) \right] A_7 \\
- \frac{1}{56} \left[ 1 \mp \sqrt{33} \mp i \left( \sqrt{11} \mp 3 \sqrt{3} \right) \right] A_8 - \frac{1}{56} \left[ 1 \mp \sqrt{33} + i \left( \sqrt{11} \mp 3 \sqrt{3} \right) \right] A_9 \right)
\]
\[
\begin{align*}
+ \frac{1}{28} \left[1 \mp \sqrt{3} - i \left(3\sqrt{11} \mp \sqrt{3}\right)\right] A_{10} + \frac{1}{28} \left[1 \mp \sqrt{3} + i \left(3\sqrt{11} \mp \sqrt{3}\right)\right] A_{11} \\
- \frac{1}{2} \left(1 \mp i\sqrt{3}\right) A_{12} - \frac{1}{2} \left(1 \pm i\sqrt{3}\right) A_{13}
\end{align*}
\]

\[B_{105\beta}^\pm = B_{105\beta}^\pm\]

\[B_{154} = \frac{7}{45} \left(A_1 + \frac{1}{28} A_3 - \frac{1}{7} A_4 + \frac{1}{7} A_5 - \frac{1}{7} A_8 - \frac{1}{7} A_9 + \frac{1}{7} A_{10} + \frac{1}{7} A_{11} + A_{12} + A_{13}\right)\]

Time C: 2 min 7 sec. Time Maple: 37 sec.

A.5 2016-dimensional representation of \(3.M_22\)

Rank: 16. Suborbit lengths: \(1, 55, 165, 330, 165, 66, 66, 66, 330, 330, 165, 165, 55, 55, 1, 1\).

\[B_1 = \frac{1}{2016} \sum_{k=1}^{16} A_k\]

\[B_{21\alpha} = \frac{1}{96} \left(A_1 - \frac{5}{11} A_2 + \frac{3}{11} A_3 - \frac{1}{11} A_4 + \frac{3}{11} A_5 - \frac{1}{11} A_6 - \frac{1}{11} A_7 - \frac{1}{11} A_8 - \frac{1}{11} A_9 - \frac{1}{11} A_{10} + \frac{3}{11} A_{11} + \frac{3}{11} A_{12} - \frac{5}{11} A_{13} - \frac{5}{11} A_{14} + A_{15} + A_{16}\right)\]

\[B_{21\beta} = \frac{1}{96} \left(A_1 + \frac{4}{11} A_2 + \frac{3}{11} A_3 - \frac{1}{11} A_4 + \frac{4}{11} A_6 + \frac{2}{11} \left(1 + i\sqrt{3}\right) A_7 + \frac{2}{11} \left(1 - i\sqrt{3}\right) A_8 + \frac{1}{22} \left(1 + i\sqrt{3}\right) A_9 + \frac{1}{22} \left(1 - i\sqrt{3}\right) A_{10} - \frac{3}{22} \left(1 + i\sqrt{3}\right) A_{11} - \frac{3}{22} \left(1 - i\sqrt{3}\right) A_{12} - \frac{2}{11} \left(1 + i\sqrt{3}\right) A_{13} - \frac{2}{11} \left(1 - i\sqrt{3}\right) A_{14} - \frac{1}{2} \left(1 + i\sqrt{3}\right) A_{15} - \frac{1}{2} \left(1 - i\sqrt{3}\right) A_{16}\right)\]

\[B_{21\gamma} = \overline{B_{21\alpha}}\]

\[B_{55} = \frac{55}{2016} \left(A_1 + \frac{13}{55} A_2 - \frac{1}{55} A_3 - \frac{1}{55} A_4 + \frac{13}{55} A_5 - \frac{1}{55} A_6 - \frac{3}{11} A_7 - \frac{3}{11} A_8 - \frac{1}{55} A_9 - \frac{1}{55} A_{10} - \frac{1}{55} A_{11} - \frac{1}{55} A_{12} + \frac{13}{55} A_{13} + \frac{13}{55} A_{14} + A_{15} + A_{16}\right)\]

\[B_{105\beta} = \frac{5}{96} \left(A_1 - \frac{2}{55} \left(1 \pm \sqrt{33}\right) A_2 + \frac{1}{55} \left(4 \pm \frac{\sqrt{33}}{3}\right) A_3 + \frac{1}{110} \left(1 \mp \frac{\sqrt{33}}{3}\right) A_4 + \frac{1}{22} \left(3 \mp \frac{\sqrt{33}}{3}\right) A_6 - \frac{1}{44} \left[3 \mp \frac{\sqrt{33}}{3} - i \left(\sqrt{11} 
\mp 3\sqrt{3}\right)\right] A_7 - \frac{1}{44} \left[3 \mp \frac{\sqrt{33}}{3} + i \left(\sqrt{11} 
\mp 3\sqrt{3}\right)\right] A_8\right)
\[-\frac{1}{220} \left[ 1 + \frac{\sqrt{33}}{3} - i \left( \sqrt{11} \mp \sqrt{3} \right) \right] A_9 - \frac{1}{220} \left[ 1 + \frac{\sqrt{33}}{3} + i \left( \sqrt{11} \pm \sqrt{3} \right) \right] A_{10} \]
\[-\frac{1}{110} \left[ 4 + \frac{\sqrt{33}}{3} + i \left( \sqrt{11} \pm 4\sqrt{3} \right) \right] A_{11} - \frac{1}{110} \left[ 4 + \frac{\sqrt{33}}{3} - i \left( \sqrt{11} \pm 4\sqrt{3} \right) \right] A_{12} \]
\[+ \frac{1}{55} \left[ 1 \pm \sqrt{33} + i \left( 3\sqrt{11} \pm \sqrt{3} \right) \right] A_{13} + \frac{1}{55} \left[ 1 \pm \sqrt{33} - i \left( 3\sqrt{11} \pm \sqrt{3} \right) \right] A_{14} \]
\[-\frac{1}{2} \left( 1 + i\sqrt{3} \right) A_{15} - \frac{1}{2} \left( 1 - i\sqrt{3} \right) A_{16} \right\} \]

\[B_{105_\beta} = B_{105_\delta} \]

\[B_{154} = \frac{11}{144} \left( A_1 + \frac{7}{55} A_2 + \frac{3}{55} A_3 - \frac{3}{55} A_4 - \frac{1}{11} A_5 + \frac{1}{11} A_6 + \frac{1}{11} A_7 + \frac{1}{11} A_8 \]
\[-\frac{3}{55} A_9 - \frac{3}{55} A_{10} + \frac{3}{55} A_{11} + \frac{3}{55} A_{12} + \frac{7}{55} A_{13} + \frac{7}{55} A_{14} + A_{15} + A_{16} \right) \]

\[B_{210_\alpha} = \frac{5}{48} \left( A_1 - \frac{3}{55} A_2 + \frac{1}{165} A_4 - \frac{7}{55} A_5 - \frac{1}{11} A_6 - \frac{1}{11} A_7 - \frac{1}{11} A_8 \]
\[+ \frac{7}{165} A_9 - \frac{7}{165} A_{10} + \frac{1}{165} A_{11} + \frac{1}{165} A_{12} - \frac{3}{55} A_{13} - \frac{3}{55} A_{14} + A_{15} + A_{16} \right) \]

\[B_{210_\beta} = \frac{5}{48} \left( A_1 - \frac{1}{15} A_3 - \frac{1}{15} A_4 + \frac{1}{30} \left( 1 + i\sqrt{3} \right) A_9 + \frac{1}{30} \left( 1 - i\sqrt{3} \right) A_{10} \]
\[+ \frac{1}{30} \left( 1 + i\sqrt{3} \right) A_{11} + \frac{1}{30} \left( 1 - i\sqrt{3} \right) A_{12} \]
\[-\frac{1}{2} \left( 1 + i\sqrt{3} \right) A_{15} - \frac{1}{2} \left( 1 - i\sqrt{3} \right) A_{16} \right) \]

\[B_{210_\gamma} = B_{210_\delta} \]

\[B_{231_\alpha} = \frac{11}{96} \left( A_1 - \frac{3}{55} A_2 - \frac{1}{15} A_3 + \frac{1}{165} A_4 + \frac{1}{11} A_5 + \frac{1}{11} A_6 + \frac{1}{11} A_7 + \frac{1}{11} A_8 \]
\[+ \frac{1}{165} A_9 + \frac{1}{165} A_{10} - \frac{1}{15} A_{11} - \frac{1}{15} A_{12} - \frac{3}{55} A_{13} - \frac{3}{55} A_{14} + A_{15} + A_{16} \right) \]

\[B_{231_\beta} = \frac{11}{96} \left( A_1 - \frac{1}{33} A_3 + \frac{2}{33} A_4 - \frac{1}{11} A_6 + \frac{1}{22} \left( 1 + i\sqrt{3} \right) A_7 + \frac{1}{22} \left( 1 - i\sqrt{3} \right) A_8 \]
\[-\frac{1}{33} \left( 1 + i\sqrt{3} \right) A_9 - \frac{1}{33} \left( 1 - i\sqrt{3} \right) A_{10} \]
\[+ \frac{1}{66} \left( 1 + i\sqrt{3} \right) A_{11} + \frac{1}{66} \left( 1 - i\sqrt{3} \right) A_{12} \]
\[-\frac{1}{2} \left( 1 + i\sqrt{3} \right) A_{15} - \frac{1}{2} \left( 1 - i\sqrt{3} \right) A_{16} \right) \]

\[B_{231_\gamma} = B_{231_\delta} \]

Time C: 21 min 34 sec. Time Maple: 1 h 28 min 17 sec.