Quantifying quantum coherence based on the generalized $\alpha - z$–relative Rényi entropy

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We present a family of coherence quantifiers based on the generalized $\alpha - z$–relative Rényi entropy. These quantifiers satisfy all the standard criteria for well-defined measures of coherence, and include some existing coherence measures as special cases.

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I. INTRODUCTION

Coherence, being at the heart of interference phenomena, plays a central role in quantum physics as it enables applications that are impossible within classical mechanics or ray optics. Coherence is also a vital physical resource with various applications in biology [1–3], thermodynamical systems [4, 5], transport theory [6, 7] and nanoscale physics [8]. Recent developments in our understanding of quantum coherence [9–14] and nonclassical correlation have come from the burgeoning field of quantum information science. One important pillar of the field is the study on quantification of coherence.

In Ref. [15] the authors established a rigorous framework (BCP framework) for quantifying coherence. The BCP framework consists of the following postulates that any quantifier of coherence $C$ should fulfill:

(C1) Faithfulness: $C(\rho) \geq 0$, with equality if and only if $\rho$ is incoherent.

(C2) Monotonicity: $C(\Phi_I(\rho)) \leq C(\rho)$, for any incoherent operation $\Phi_I$.

(C3) Convexity: $C$ is a convex function of the state, i.e.,

$$\sum_n p_n C(\rho_n) \geq C(\sum_n p_n \rho_n),$$

where $p_n \geq 0$, $\sum_n p_n = 1$.

(C4) Strong monotonicity: $C$ does not increase on average under selective incoherent operations, i.e,

$$C(\rho) \geq \sum_n p_n C(\varrho_n),$$

with probabilities $p_n = tr(K_n \rho K_n^\dagger)$, post measurement states $\varrho_n = \frac{K_n \rho K_n^\dagger}{p_n}$, and incoherent operators $K_n$.

The authors of Ref. [16] provided a simple and interesting condition to replace (C3) and (C4) with the additivity of coherence for block-diagonal states,

$$C(\rho_1 \oplus \rho_2) = C(\rho_1) + C(\rho_2),$$

for any $p \in [0,1]$, $\rho_i \in \varepsilon(\mathcal{H}_i)$, $i = 1, 2$, and $p \rho_1 \oplus (1 - p) \rho_2 \in \varepsilon(\mathcal{H}_1 \oplus \mathcal{H}_2)$, where $\varepsilon(\mathcal{H})$ denotes the set of density matrices on the Hilbert space $\mathcal{H}$.

For a given $d$-dimensional Hilbert space $\mathcal{H}$, let us fix an orthonormal basis $\{|i\rangle\}_{i=1}^d$. We call all density matrices that are diagonal in this basis incoherent and label this set of quantum states by $\mathcal{I} \subset \mathcal{H}$. All density operators $\delta \in \mathcal{I}$ are of the form:

$$\delta = \sum_i p_i |i\rangle\langle i|,$$

where $p_i \geq 0$ and $\sum_i p_i = 1$. Otherwise the states are coherent. Let $\Lambda$ be a completely positive trace preserving (CPTP) map:

$$\Lambda(\rho) = \sum_i K_n \rho K_n^\dagger,$$
where \( \{K_n\} \) is a set of Kraus operators satisfying \( \sum_n K_n K_n^\dagger = I_d \), with \( I_d \) the identity operator. If \( K_n^\dagger I K_n \in \mathcal{I} \) for all \( n \), we call \( \{K_n\} \) a set of incoherent Kraus operators, and the corresponding operation \( \Lambda \) an incoherent operational one.

II. THE FUNCTION \( f_{\alpha,z}(\rho, \sigma) \)

Quantifying coherence is a key task in both quantum mechanical theory and practical applications. In Ref. \([17\), \([18\)](\text{raw_text})\) the following function has been presented,

\[
f_{\alpha,z}(\rho, \sigma) = \text{Tr}(\sigma^{\frac{1}{1-z}} \rho^{\frac{z}{1-z}}),
\]

(2)

for arbitrary two density matrices \( \rho \) and \( \sigma \). Here, \( \alpha, z \in \mathbb{R} \). To study the limit when \( \alpha \to 1 \) and \( z \to 0 \), the authors in Ref. \([18\)](\text{raw_text}) parameterized \( z \) in terms of \( \alpha \) as \( z = r(\alpha - 1) \), where \( r \) is a non-zero finite real number, and considered the limit when \( \alpha \to 1 \): \( \lim_{\alpha \to 1} f_{\alpha,r(\alpha-1)}(\rho, \sigma) = \rho \). For fixed \( \alpha \neq 1 \), \( z \to 0 \) is exactly related to the anti-Lie-Trotter problem \([19\)](\text{raw_text})..

For a finite dimensional Hilbert space \( \mathcal{H} \), the set of linear operators is denoted by \( \mathcal{L}(\mathcal{H}) \). The adjoint of \( X \in \mathcal{L}(\mathcal{H}) \) is denoted by \( X^\dagger \). For \( X \in \mathcal{L}(\mathcal{H}) \) and real \( p \neq 0 \), \( \|X\|_p \) is defined by \([20\)](\text{raw_text})

\[
\|X\|_p = (\text{tr}|X|^p)^{\frac{1}{p}},
\]

where \( |X| = \sqrt{X^\dagger X} \). Here, for a self-adjoint operator \( X \), \( X^{-1} \) means the inverse restricted to \( \text{supp}(X) \), so \( X^{-1}X = XX^{-1} \) equals to the orthogonal projection on \( \text{supp}(X) \).

The Hölder’s inequality belongs to a richer family of inequalities. For every \( p_1, ..., p_k, r > 0 \) with \( \frac{1}{r} = \frac{1}{p_1} + ... + \frac{1}{p_k} \) one has \([21\)](\text{raw_text})

\[
\|X_1 ... X_k\|_r \leq \|X_1\|_{p_1} ... \|X_k\|_{p_k}.
\]

(3)

From this inequality and the fact that \( \|X^{-1}\|_{-p} = \|X\|_p^{-1} \), the following reverse Hölder’s inequality is derived. Let \( r > 0 \) and \( p_1, ..., p_k \) be such that \( \frac{1}{r} = \frac{1}{p_1} + ... + \frac{1}{p_k} \) and that exactly one of \( p_i \)'s is positive and the rests are negative \([20\)](\text{raw_text})

\[
\|X_1 ... X_k\|_r \geq \|X_1\|_{p_1} ... \|X_k\|_{p_k}.
\]

(4)

Moreover, equalities holds in \([3\)](\text{raw_text}) and \([4\)](\text{raw_text}) if and only if \( |X_i|^{p_i}, i = 1, 2, ..., k, \) are proportional.

**Lemma 1** For states \( \rho \) and \( \sigma \),

(1) If \( 0 < \alpha < 1 \) and \( z > 0 \), we have

\[
f_{\alpha,z}(\rho, \sigma) \leq 1;
\]

(2) If \( \alpha > 1 \) and \( z > 0 \), we have

\[
f_{\alpha,z}(\rho, \sigma) \geq 1.
\]

(3) \( f_{\alpha,z}(\rho, \sigma) = 1 \) if and only if \( \rho = \sigma \), for \( \alpha \in (0,1) \cup (1, +\infty) \) and \( z > 0 \).

**[Proof]** Let \( r = z, p_1 = \frac{2z}{1-\alpha}, p_2 = \frac{z}{\alpha}, X_1 = \sigma^{\frac{1}{p_1}}, X_2 = \rho^{\frac{1}{p_2}} \). When \( \alpha \in (0,1) \) and \( z > 0 \), we have

\[
f_{\alpha,z}(\rho, \sigma) = \text{tr}(X_1 X_2 X_1)^z = \text{tr}(\{\text{supp}(X_1)\}^r)
\]

\[
= (\|X_1 X_2 X_1\|_r)^r
\]

\[
\leq (\|X_1\|_{p_1} \|X_2\|_{p_2} \|X_1\|_{p_1})^r
\]

\[
= 1,
\]

where the second equality is due to \( X_i^\dagger = X_i \) for \( i = 1, 2 \). From \([3\)](\text{raw_text}) we obtain the first inequality.

When \( \alpha > 1 \) and \( z > 0 \), we have

\[
f_{\alpha,z}(\rho, \sigma) = (\|X_1 X_2 X_1\|_r)^r
\]

\[
\geq (\|X_1\|_{p_1} \|X_2\|_{p_2} \|X_1\|_{p_1})^r
\]

\[
= 1,
\]

(5)
where the first inequality is due to \( \text{[1]} \).

In the above proof of inequalities \( \text{[5]} \) and \( \text{[10]} \), \( ||X_1X_2X_1||_2 = ||X_1||_p||X_2||_p||X_1||_p \) if and only if \( |X_1|^p_1 \) and \( |X_2|^p_2 \) are proportional, i.e., there is a number \( k \) which satisfies \( \sigma = kp \). Since \( \text{tr}(\rho) = \text{tr}(\sigma) = 1 \), then we obtain \( k = 1 \). \( \square \)

Let \( P(\mathcal{H}) \) be the set of positive semidefinite operators on \( \mathcal{H} \). For non-normalized states \( \rho, \sigma \in P(\mathcal{H}) \) with \( \text{supp } \rho \subseteq \text{supp } \sigma \), it has been defined in Ref. \( \text{[18]} \),

\[
D_{\alpha,z}(\rho||\sigma) := \frac{1}{\alpha - 1} \log \frac{f_{\alpha,z}(\rho, \sigma)}{\text{tr}\rho}.
\] \( \text{(7)} \)

For any states \( \rho, \sigma \) such that \( \text{supp } \rho \subseteq \text{supp } \sigma \), and for any CPTP map \( \Lambda \): \( D_{\alpha,z}(\Lambda(\rho)||\Lambda(\sigma)) \leq D_{\alpha,z}(\rho||\sigma) \) holds in each of the following cases \( \text{[13]} \):

- \( \alpha \in (0, 1] \) and \( z \geq \max\{\alpha, 1 - \alpha\} \);
- \( \alpha \in [1, 2] \) and \( z = 1 \);
- \( \alpha \in [1, 2] \) and \( z = \frac{2}{\alpha} \);
- \( \alpha \geq 1 \) and \( z = \alpha \).

For two states \( \rho \) and \( \sigma \), one has \( f_{\alpha,z}(\rho, \sigma) = e^{(\alpha - 1)D_{\alpha,z}(\rho||\sigma)} \). Hence \( f_{\alpha,z}(\rho, \sigma) \) has the following properties:

**Lemma 2** For any quantum states \( \rho \) and \( \sigma \), such that \( \text{supp } \rho \subseteq \text{supp } \sigma \), and for any CPTP map \( \Lambda \), we have

- If \( \alpha \in (0, 1] \) and \( z \geq \max\{\alpha, 1 - \alpha\} \), then
  \[
f_{\alpha,z}(\Lambda(\rho), \Lambda(\sigma)) \geq f_{\alpha,z}(\rho, \sigma);
  \]

- If \( \alpha \in [1, 2] \) and \( z \in \{1, \frac{2}{\alpha}\} \); or \( \alpha \geq 1 \) and \( z = \alpha \), then
  \[
f_{\alpha,z}(\Lambda(\rho), \Lambda(\sigma)) \leq f_{\alpha,z}(\rho, \sigma).
  \]

### III. COHERENCE QUANTIFICATION

The coherence \( C(\rho) \) in Ref. \( \text{[21]} \) can be expressed as

\[
C(\rho) = 1 - \left[ \max_{\sigma \in I} f_{\frac{2}{\alpha},1}(\rho, \sigma) \right]^2.
\] \( \text{(8)} \)

In Ref. \( \text{[22]} \) a bona fide measure of quantum coherence \( C(\rho) \) has been presented by utilizing the Hellinger distance:

\[
D_H(\rho, \sigma) = \text{Tr}(\sqrt{\rho} - \sqrt{\sigma})^2;
\]

\[
C(\rho) = \min_{\sigma \in I} D_H(\rho, \sigma)
\]

\[
= 2 \left[ 1 - \max_{\sigma \in I} f_{\frac{2}{\alpha},1}(\rho, \sigma) \right],
\] \( \text{(9)} \)

which is the coherence \( C_2(\epsilon|\rho) \) of Theorem 3 in Ref. \( \text{[23]} \).

In Ref. \( \text{[23]} \) the coherence has been quantified based on the Tsallis relative \( \alpha \) entropy,

\[
D_{\alpha}(\rho||\sigma) = \frac{1}{\alpha - 1} (f_{\alpha,1}(\rho, \sigma) - 1).
\] \( \text{(10)} \)

But it was shown that it violates the strong monotonicity, even though it can unambiguously distinguish the coherent state from the incoherent ones with the monotonicity. In Ref. \( \text{[24]} \) a family of coherence quantifiers has been presented, which are closely related to the Tsallis relative \( \alpha \) entropy:

\[
C'_\alpha(\rho) = \min_{\sigma \in I} \frac{1}{\alpha - 1} \left( f_{\alpha,1}(\rho, \sigma) - 1 \right),
\] \( \text{(11)} \)

where \( \alpha \in (0, 2] \).

In the following we define a generalized \( \alpha - z \)-relative R\'enyi entropy:

\[
D_{\alpha,z}(\rho, \sigma) = \frac{f_{\alpha,z}(\rho, \sigma) - 1}{\alpha - 1}.
\] \( \text{(12)} \)

It is worthwhile noting that several coherence measures like relative entropy \( \text{[12]} \), geometric coherence \( \text{[25]} \), the sandwiched R\'enyi relative entropy \( \text{[26]} \) and max-relative entropy \( \text{[9]} \) are related to the generalized \( \alpha - z \)-relative R\'enyi entropy.

Based on the relation \( f_{\alpha,z}(\rho, \sigma) \) and \( D_{\alpha,z}(\rho, \sigma) \), and Lemma \( \text{[2]} \) we have
Corollary 1 For any quantum states $\rho$ and $\sigma$ for which $\text{supp} \rho \subseteq \text{supp} \sigma$, and for any CPTP map $\Lambda$: $D_{\alpha,z}(\Lambda(\rho), \Lambda(\sigma)) \leq D_{\alpha,z}(\rho, \sigma)$ holds in each of the following case:
- $\alpha \in (0, 1]$ and $z \geq \max\{\alpha, 1 - \alpha\}$;
- $\alpha \in [1, 2]$ and $z = 1$;
- $\alpha \in [1, 2]$ and $z = \frac{4}{3}$;
- $\alpha > 1$ and $z = \alpha$.

With the above properties, based on the generalized $\alpha - z$-relative Rényi entropy we define the quantity: $C_{\alpha,z}(\rho) = \min_{\sigma \in \mathcal{I}} D_{\alpha,z}(\rho, \sigma)$. The following statement takes place.

Theorem 1 The quantum coherence $C_{\alpha,z}(\rho)$ of a state $\rho$ given by

$$C_{\alpha,z}(\rho) = \min_{\sigma \in \mathcal{I}} D_{\alpha,z}(\rho, \sigma)$$

is a well-defined measure of coherence for the following case:
- $\alpha \in (0, 1]$ and $z \geq \max\{\alpha, 1 - \alpha\}$;
- $\alpha \in [1, 2]$ and $z = 1$;
- $\alpha \in [1, 2]$ and $z = \frac{4}{3}$;
- $\alpha > 1$ and $z = \alpha$.

[Proof] Because of (2), (12) and (13), we have

$$C_{\alpha,z}(\rho) = \begin{cases} \frac{1 - \max_{\sigma \in \mathcal{I}} f_{z,\alpha}^1(\rho, \sigma)}{1 - \alpha}, & 0 < \alpha < 1, \\ \min_{\sigma \in \mathcal{I}} f_{z,\alpha}^1(\rho, \sigma)^{-1}, & \alpha > 1. \end{cases}$$

From Lemma 1, we have $C_{\alpha,z}(\rho) \geq 0$, and $C_{\alpha,z}(\rho) = 0$ if and only if $\rho = \sigma$. Let $\sigma$ be the optimal incoherent state such that $C_{\alpha,z}(\rho) = D_{\alpha,z}(\rho, \sigma)$. Taking into account Corollary 1 we have that $C_{\alpha,z}(\rho)$ does not increase under any incoherent operations.

Next we prove that $C_{\alpha,z}(\rho)$ satisfies Eq. (11). Suppose $\rho$ is block-diagonal in the reference basis $\{|j\}_j=1^d$, $\rho = p_1 \rho_1 \oplus p_2 \rho_2$ with $p_1 \geq 0, p_2 \geq 0, p_1 + p_2 = 1$, $\rho_1$ and $\rho_2$ are density operators. Let $\sigma = q_1 \sigma_1 \oplus q_2 \sigma_2$ with $q_1 \geq 0, q_2 \geq 0$, $q_1 + q_2 = 1$, and $\sigma_1, \sigma_2$ are diagonal states similar to $\rho_1, \rho_2$, respectively.

Denote $\Delta$ either max or min. Set $t_i = \Delta_{\rho_i, \sigma_i} \text{tr}(\sigma^{1/\alpha} \rho_i^{1/\alpha} \sigma^{1/\alpha} \rho_i^{1/\alpha})^z$, $i = 1, 2$. We have

$$\Delta_{\rho_i, \sigma_i} \text{tr}(\sigma^{1/\alpha} \rho_i^{1/\alpha} \sigma^{1/\alpha} \rho_i^{1/\alpha})^z = \Delta_{q_1, q_2} (q_1^{1-\alpha} p_1^{\alpha} t_1 + q_2^{1-\alpha} p_2^{\alpha} t_2).$$

Due to the Hölder inequality with $0 < \alpha < 1$, we have

$$q_1^{1-\alpha} p_1^{\alpha} t_1 + q_2^{1-\alpha} p_2^{\alpha} t_2 \leq \left( \sum_{i=1,2} p_i^{1/\alpha} \right)^{\alpha},$$

where the equality holds if and only $q_1 = l p_1^{1/\alpha}$ and $q_2 = l p_2^{1/\alpha}$ with $l = \left[ p_1^{1/\alpha} + p_2^{1/\alpha} \right]^{-1}$, i.e.,

$$\max_{q_1, q_2} (q_1^{1-\alpha} p_1^{\alpha} t_1 + q_2^{1-\alpha} p_2^{\alpha} t_2) = \left( \sum_{i=1,2} p_i^{1/\alpha} \right)^{\alpha}. \quad (15)$$

Similarly, for the inequality with $\alpha > 1$, we have

$$q_1^{1-\alpha} p_1^{\alpha} t_1 + q_2^{1-\alpha} p_2^{\alpha} t_2 \geq \left( \sum_{i=1,2} p_i^{1/\alpha} \right)^{\alpha}.$$

When $q_1 = l p_1^{1/\alpha}$ and $q_2 = l p_2^{1/\alpha}$, we obtain

$$\min_{q_1, q_2} (q_1^{1-\alpha} p_1^{\alpha} t_1 + q_2^{1-\alpha} p_2^{\alpha} t_2) = \left( \sum_{i=1,2} p_i^{1/\alpha} \right)^{\alpha}. \quad (16)$$
Combining (14), (15) and (16), we have
\[
\Delta_{\sigma} = \frac{1}{\alpha} f_{\alpha, z}(\rho, \sigma) = p_{1} \Delta_{\sigma} = \frac{1}{\alpha} f_{\alpha, z}(\rho_{1}, \sigma_{1}) + p_{2} \Delta_{\sigma} = \frac{1}{\alpha} f_{\alpha, z}(\rho_{2}, \sigma_{2}).
\]
Thus, \( C_{\alpha, z} \) satisfies additivity of coherence for block-diagonal states: \( C_{\alpha, z}(p_{1} \rho_{1} \oplus p_{1} \rho_{1}) = p_{1} C_{\alpha, z}(\rho_{1}) + p_{2} C_{\alpha, z}(\rho_{2}). \) \( \Box \)

Corollary 2
For any incoherent state \( \sigma \) and \( \sigma = \rho \), \( C_{\alpha, z}(\rho) \) actually defines a family of coherence measures which includes several typical coherence measures.

IV. THE PROPERTIES OF \( C_{\alpha, z}(\rho) \)

From Theorem 1, \( C_{\alpha, z}(\rho) \) is a well-defined measure of coherence for \( \alpha \in (0, 1) \cup (1, 2] \),
\[
C_{\alpha, z} = \min_{\sigma \in I} \left[ f_{\alpha, z}(\rho, \sigma) - 1 \over \alpha - 1 \right],
\]
where \( f(\rho, \sigma) = tr(\rho^{\alpha} \sigma^{1-\alpha}) \), since for any pair of square matrices \( A \) and \( B \), the eigenvalues of \( AB \) and \( BA \) are the same. For any incoherent state \( \sigma = \sum_{k=1}^{d} \delta_{kk} |k\rangle \langle k| \), we have
\[
tr(\sigma^{1-\alpha} \rho^{\alpha}) = \sum_{k=1}^{d} \delta_{kk}^{\alpha} |k\rangle \langle k|^{\alpha} = Q \sum_{k=1}^{d} \frac{|k\rangle \langle k|}{Q} \delta_{kk}^{\alpha-1},
\]
where \( Q = \left( \sum_{k=1}^{d} \langle k| \rho^{\alpha} |k\rangle^{\frac{1}{\alpha}} \right)^{\alpha} \). Denote
\[
\varepsilon(\alpha) = \begin{cases} 0, & 0 < \alpha < 1, \\ 1, & 1 < \alpha. \end{cases}
\]

According to the Hölder inequality and the converse Hölder inequality, we have
\[
\varepsilon(\alpha) \sum_{k=1}^{d} \frac{|k\rangle \langle k|}{Q} \delta_{kk}^{\alpha-1} \geq \varepsilon(\alpha) \left( \sum_{k=1}^{d} \delta_{kk} \right)^{\frac{1}{\alpha}} \left( \sum_{k=1}^{d} \frac{|k\rangle \langle k|}{Q} \right)^{\frac{\alpha}{\alpha}} = \varepsilon(\alpha),
\]
where the equality is attained when \( \delta_{kk}^{\alpha-1} = \frac{|k\rangle \langle k|}{Q} \). Then one finds the following conclusion.

Corollary 2 For \( \alpha \in (0, 1) \cup (1, 2] \),
\[
C_{\alpha, 1}(\rho) = \frac{\sum_{k=1}^{d} \langle k| \rho^{\alpha} |k\rangle^{\frac{1}{\alpha}} - 1}{\alpha - 1}.
\]
And the maximal coherence can be achieved by the maximally coherent states.
That the maximal coherence can be achieved by the maximally coherent states for $C_{\alpha,1}(\rho)$, with $\alpha \in (0, 1) \cup (1, 2]$, can be seen in the following. Based on the eigen-decomposition of a $d$-dimensional state $\rho = \sum_{j=1}^{d} \lambda_j |\varphi_j\rangle \langle \varphi_j|$, with $\lambda_j$ and $|\varphi_j\rangle$ representing the eigenvalue and eigenvectors, we have:

$$
\varepsilon(\alpha) \sum_{k=1}^{d} \langle k | \rho^\alpha | k \rangle^{\frac{1}{\alpha}} = \varepsilon(\alpha) \sum_{k=1}^{d} \left( \frac{\lambda_j^\alpha}{\langle \varphi_j | \langle \varphi_j | \varphi_j \rangle^{2}} \right)^{\frac{1}{\alpha}} \\
\leq \varepsilon(\alpha)^d \sum_{k=1}^{d} \lambda_j^\alpha \langle \varphi_j | \langle \varphi_j | \varphi_j \rangle^{2}} \right)^{\frac{1}{\alpha}} \\
= \varepsilon(\alpha)^d \sum_{k=1}^{d} \lambda_j^\alpha \right)^{\frac{1}{\alpha}}.
$$

where the first inequality is due to

$$\sum_{k=1}^{n} \lambda_k x_k^p \leq \left( \sum_{k=1}^{n} \lambda_k \right)^{1-p} \left( \sum_{k=1}^{n} \lambda_k x_k^p \right)^{p}, 0 < p \leq 1, \quad \sum_{k=1}^{n} \lambda_k \right)^{1-p} \left( \sum_{k=1}^{n} \lambda_k x_k^p \right)^{p}, p > 1,$$

with $x_k = \sum_{j=1}^{d} \lambda_j^\alpha \langle \varphi_j | \langle \varphi_j | \varphi_j \rangle^{2}} \geq 0, \lambda_k = 1 \ (k = 1, 2, ..., n)$ and $p = \frac{1}{\alpha}$. Then one can easily find that the upper bound of the coherence can be attained by the maximally coherent states $\rho_d = |\varphi\rangle \langle \varphi|$ with $|\varphi\rangle = \frac{1}{\sqrt{d}} \sum_{j=1}^{d} e^{i\phi_j} |j\rangle$, $C_{\alpha,1}(\rho_d) = \frac{d \varepsilon(z_1)}{\alpha-1}$. □

**Theorem 2** For $\alpha \in (0, 1), \beta \in (1, 2], \gamma > 1, \max\{\alpha, 1-\alpha\} \leq \alpha_1 \leq 1, \alpha_2 \geq 1$, we have

$$C_{\alpha,2}(\rho) \leq C_{\alpha,1}(\rho) \leq C_{\alpha,2}(\rho); \quad C_{\beta,1}(\rho) \leq C_{\beta,2}(\rho);$$

And

$$C_{\gamma,\gamma}(\rho) \leq \sum_{k=1}^{d} \langle k | \rho^\gamma | k \rangle^{\frac{1}{\gamma}} \leq \frac{1}{\gamma-1}.
$$

[Proof] Set

$$\varepsilon(z_i) = \begin{cases} -1, & 0 \leq z_i \leq 1, \\ 1, & z_i > 1, \end{cases}$$

where $i = 1, 2$. According to the Araki-Lieb-Thirring inequality, for matrixes $A, B \geq 0$, $q \geq 0$ and for $0 \leq r \leq 1$, the following inequality holds $[28]$.

$$tr(A^r B^r A^r)^q \leq tr(ABA)^{rq}.
$$

While for $r \geq 1$, the inequality is reversed $[28]$,

$$tr(A^r B^r A^r)^q \geq tr(ABA)^{rq}.
$$

From $[21]$ and $[22]$, we have

$$\varepsilon(z_i)f_{\alpha,z_i}(\rho, \sigma) = \varepsilon(z_i)tr(\frac{1}{\rho} \sigma)^{\frac{1}{\alpha}} \rho \sigma \frac{1}{\rho} \sigma^{\frac{1}{\alpha}})^{z_i} \\
\leq \varepsilon(z_i)tr(\frac{1}{\rho} \sigma^{\frac{1}{\alpha}} \sigma \frac{1}{\rho}) \\
= \varepsilon(z_i)tr(\rho \sigma^{1-\alpha}) \\
= \varepsilon(z_i)f_{\alpha,1}(\rho, \sigma).$$
Combining (13) and \( \alpha \in (0, 1) \), we have \( C_{\alpha,z_1}(\rho) \leq C_{\alpha,1}(\rho) \leq C_{\alpha,z_2}(\rho) \). (19) can be obtained in a similar way.

Since \( \gamma > 1 \), we have \( f_{\gamma,\gamma}(\rho, \sigma) \leq tr(\rho \sigma^{1-\gamma}) \). Similar to the proof of (17), \( \min_{\sigma \in I} tr^{\frac{1}{\gamma}}(\rho^{1-\gamma}) = \sum_{k=1}^{d} (k|\rho^\gamma|k)^{\frac{1}{\gamma}} \), we obtain (20). \( \square \)

**Example 1:** Let us consider a single-qubit pure state,

\[ \rho = \frac{1}{2}(I_2 + \sum_i c_i \sigma_i), \]

where \( \sum_i c_i^2 = 1 \), \( I_2 \) is the \( 2 \times 2 \) identity matrix and \( \sigma_i \) (\( i = 1, 2, 3 \)) are Pauli matrices. By Ref. [17], one has

\[ \max_{\sigma \in I} tr^2(\sqrt{\sigma} \sqrt{\rho}) = \frac{1}{2}(1 + |c_3|), \]

and

\[ \max_{\sigma \in I} tr^2(\sqrt{\rho} \sqrt{\sigma}) = \frac{1}{2}(1 + c_3^2). \]

For the single-qubit pure state \( \rho \), one has

\[ \rho^{\frac{1}{2}} = \rho = \left( \frac{1 + c_3}{2} \frac{1 - c_3}{2} \frac{i c_3}{2} \frac{1 - i c_3}{2} \right). \]  

(23)

Since \( tr(\sigma^{\frac{1}{2}} \rho^{\frac{1}{2}} \sigma^{\frac{1}{2}})^2 = tr(\sigma^{\frac{1}{2}} \rho^{\frac{1}{2}})^2 \), we now compute \( \max_{\sigma \in I} \left[ tr(\sigma^{\frac{1}{2}} \rho^{\frac{1}{2}})^2 \right]^2 \). Suppose that \( \sigma = \sum_i p_i |i\rangle \langle i| \) with \( p_1 + p_2 = 1 \) and \( 0 \leq p_1, p_2 \leq 1 \). We have

\[ \sqrt{tr(\sigma^{\frac{1}{2}} \rho^{\frac{1}{2}})^2} = \frac{1 + c_3}{2} p_1^{\frac{1}{2}} + \frac{1 - c_3}{2} p_2^{\frac{1}{2}} \]

\[ \leq \left[ \left( \frac{1 + c_3}{2} \right)^{\frac{1}{2}} + \left( \frac{1 - c_3}{2} \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}, \]

by using the H"older inequality and that the equality holds if and only \( p_1 = c(\frac{1 + c_3}{2})^{\frac{1}{2}} \) and \( p_2 = c(\frac{1 - c_3}{2})^{\frac{1}{2}} \) with \( c = \left[ (\frac{1 + c_3}{2})^{\frac{1}{2}} + (\frac{1 - c_3}{2})^{\frac{1}{2}} \right]^{-1} \). Therefore we have

\[ \max_{\sigma \in I} \left[ tr(\sigma^{\frac{1}{2}} \rho^{\frac{1}{2}})^2 \right]^2 = \left[ \left( \frac{1 + c_3}{2} \right)^{\frac{1}{2}} + \left( \frac{1 - c_3}{2} \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}. \]

Due to (13), we obtain

\[ C_{\frac{1}{2},z}(\rho) = 2 \left[ 1 - \max_{\sigma \in I} tr^2(\sigma^{\frac{1}{2}} \rho^{\frac{1}{2}} \sigma^{\frac{1}{2}})^2 \right], \]

then we have

\[ C_{\frac{1}{2},z}(\rho) = 1 - |c_3|, \]

\[ C_{\frac{1}{2},1}(\rho) = 1 - c_3^2 \]

and

\[ C_{\frac{1}{2},2}(\rho) = 2 - 2 \left[ \left( \frac{1 + c_3}{2} \right)^{\frac{1}{2}} + \left( \frac{1 - c_3}{2} \right)^{\frac{1}{2}} \right]^3. \]

It is obvious that \( C_{\frac{1}{2},z}(\rho) \leq C_{\frac{1}{2},1}(\rho) \leq C_{\frac{1}{2},2}(\rho) \), see Fig. 1.
FIG. 1: The red dotted line is the vale of $C_{\frac{1}{2},2}(\rho)$; The blue solid line is the vale of $C_{\frac{1}{2},1}(\rho)$; The dashed line is the vale of $C_{\frac{1}{2},\frac{1}{2}}(\rho)$.

V. CONCLUSION

In summary, we have proposed four classes of coherence $C_{\alpha,z}(\rho)$ measures based on the generalized $\alpha - z$-relative Rényi entropy. It has been proven that these coherence measures satisfy all the required criteria for a satisfactory coherence measure. Moreover, we have obtained the analytical formulas for special quantifiers with $z = 1$ and also studied relations among the four classes of coherence $C_{\alpha,z}(\rho)$.

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