A family of r-points 1-block implicit methods with optimized region of stability for stiff initial value problems in ordinary differential equations

IM Esuabana, SE Ekoro, FA Adie and UA Abasiekwere

DOI: https://doi.org/10.22271/maths.2022.v7.i4a.842

Abstract
In this paper, we proposed a family of r-points 1-block implicit methods with optimized region of stability. This family of methods is derived with Mathematical 10.4 software and the stability is investigated using boundary locus techniques. The block methods are consistence, zero stable, and A-stable and satisfy other stability requirements which finds them suitable for stiff problems in ODEs. Numerical experiments are presented and results are compared with other block methods and exact solutions of some stiff ordinary differential equations. The methods have been found to show competitiveness with other numerical methods.

Keywords: Block method, stiffness, initial value problems, ordinary differential equation, collocation and interpolation, A-stability, boundary locus, linear multistep methods

Introduction
Differential equations arise in many areas of science and technology, specifically whenever a deterministic relation involving some continuously varying quantities (modeled by functions) and the rate of change in space or time (expressed as derivatives) is known. Such areas include Hamilton’s equations in classical mechanics, Newton’s second law in dynamics; Radioactive decay in nuclear complex analysis; the heat equation in thermodynamics; Verhulst equation in biological population growth, Lotka volterra equations in biological population dynamics; exogenous growth model etc.

The history of differential equations is traced from calculus, which was independently invented by English physicist Isaac Newton in 1676 and a German mathematician Gottfried Leibniz in 1693. Other mathematicians like Jacob Bernoulli and Leonard Euler also made remarkable contributions to the solutions of differential equations which also arise as a result of some physical phenomena in every day’s life.

Most complex differential equations cannot be solved analytically, therefore approximation to the solution becomes imperative. This can be made possible by the application of suitable numerical methods. Researchers like [6, 3, 1, 13, 10, 11, 14, 20], etc have proposed different numerical methods for solving IVPs in ODEs but some of these methods are not self-starting methods and required single step methods to generate other starting values which is computationally cumbersome and prone to enormous error since the starting method and the method are not of the same order. Our interest in this work is to formulate a family of block methods with optimized region of stability suitable for stiff and non-stiff Initial Value problems in Ordinary Differential Equations of the form

\[ y' = f(x), y(x_0) = y_0, x \in [a, b] \]  

The proposed methods can also perform well for system of first order ODEs. Numerical block methods preserve the advantage of being self-starting methods and have minimal errors, it also has lesser time of computation since it solve simultaneously at all the grid points unlike some other numerical methods [11, 5, 15, 1, 7, 18, 19] etc. have worked on some numerical block methods for solution of (1) and have presented results that are competitive.

"22"
Derivation of the Methods

Our interest in this section is to construct the proposed Linear Multistep Methods that generate the block method. We consider polynomial interpolant of the form:

\[ p(x) = \sum_{j=0}^{2r} a_j x^j \] which also approximates the solution of (1) as

\[ y(x) = \sum_{j=0}^{2r} a_j x^j, \text{ ie. } p(x) \approx y(x) \]

and

\[ y'(x) = \sum_{j=1}^{2r} j a_j x^{j-1}. \]

Collocating and interpolating at the different grid points generates the block in the form of

\[ A^0 Y_m = A^1 Y_{m-1} + \frac{h}{s} B^1 F_{m-1} + \frac{h}{d} B^0 F_m \] (2)

where \( A^1, B^0, B^1 \) are \( r \times r \) matrices, \( A^0 \) a unit diagonal matrix, \( h \) is the fixed step size, \( d \) and \( s \) are stability parameters of the methods and \( r = 2, 3, \ldots \).

\( Y_m \) and \( F_m \) are defined as \( Y_m = (y_{m+1}, y_{m+2}, \ldots, y_{m+r})^T \)

\( F_m = (f_{m+1}, f_{m+2}, \ldots, f_{m+r})^T \).

For \( r=2 \) we have

\[ Y_m = \begin{pmatrix} y_{m+1} \\ y_{m+2} \end{pmatrix}, \quad F_m = \begin{pmatrix} f_{m+1} \\ f_{m+2} \end{pmatrix}, \quad Y_{m-1} = \begin{pmatrix} y_{m-1} \\ y_{m} \end{pmatrix}, \quad F_{m-1} = \begin{pmatrix} f_{m-1} \\ f_{m} \end{pmatrix} \]

And (2) above becomes

\[ A^0 Y_m = A^1 Y_{m-1} + \frac{h}{s} B^1 F_{m-1} + \frac{h}{d} B^0 F_m \]

\[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_{m+1} \\ y_{m+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_{m-1} \\ y_{m} \end{bmatrix} + \frac{h}{s} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} f_{m+1} \\ f_{m+2} \end{bmatrix} + \frac{h}{d} \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \begin{bmatrix} f_{m-1} \\ f_{m} \end{bmatrix} \]

Note that \( A^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A^1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B^1 = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, B^0 = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}. \)

Expand (3) we obtain
\[ y_{m+1} = y_m + \frac{h}{s} a_1 f_{m-1} + \frac{h}{s} a_2 f_m + \frac{h}{d} c_1 f_{m+1} + \frac{h}{d} c_2 f_{m+2} \]
\[ y_{m+2} = y_m + \frac{h}{s} a_3 f_{m-1} + \frac{h}{s} a_4 f_m + \frac{h}{d} c_3 f_{m+1} + \frac{h}{d} c_4 f_{m+2} \]

Obtaining the unknown parameters in (4) with Mathematica (10.4) we obtain
\[ a_1 = -\frac{1}{24}, \quad a_2 = \frac{13}{24}, \quad a_3 = 0, \quad a_4 = \frac{1}{3}, \]
\[ B^1 = \begin{bmatrix} -\frac{1}{24} & \frac{13}{24} \\ \frac{1}{3} \end{bmatrix}, \]
\[ c_1 = \frac{13}{24}, \quad c_2 = -\frac{1}{24}, \quad c_3 = \frac{4}{3}, \quad c_4 = \frac{1}{3}. \]
\[ B^0 = \begin{bmatrix} \frac{13}{24} & -\frac{1}{24} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \]

We have
\[ y_{m+1} = y_m + \frac{h}{24s} f_{m-1} + \frac{13h}{24s} f_m + \frac{13h}{24d} f_{m+1} - \frac{h}{24d} f_{m+2} \]
\[ y_{m+2} = y_m + \frac{h}{24s} f_{m-1} + \frac{4h}{3d} f_{m+1} + \frac{h}{3d} f_{m+2} \]

for \( r = 3 \) we have
\[ Y_m = (y_{m+1}, y_{m+2}, y_{m+3})^T, \quad F_m = (f_{m+1}, f_{m+2}, f_{m+3})^T, \quad Y_m-1 = (y_{m-2}, y_{m-1}, y_{m})^T, \quad F_{m-1} = (f_{m-2}, f_{m-1}, f_{m})^T \]

And (2) above becomes
\[ A^0 Y_m = A^1 Y_{m-1} + \frac{h}{s} B^1 F_{m-1} + \frac{h}{d} B^0 F_m \]

\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{m+1} \\ y_{m+2} \\ y_{m+3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{m-2} \\ y_{m-1} \\ y_{m} \end{bmatrix} + \frac{h}{s} \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \begin{bmatrix} f_{m-2} \\ f_{m-1} \\ f_{m} \end{bmatrix} + \frac{h}{d} \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix} \begin{bmatrix} f_{m+1} \\ f_{m+2} \\ f_{m+3} \end{bmatrix}
\]

Note \( A^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A^1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B^1 = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}, \quad B^0 = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix} \]

Expand (6) we obtain

"24"
\[ y_{n+1} = y_n + \frac{h}{s} a_1 f_{n+2} + \frac{h}{s} a_2 f_{n+1} + \frac{h}{s} a_3 f_{n} + \frac{h}{d} c_1 f_{n+2} + \frac{h}{d} c_2 f_{n+1} + \frac{h}{d} c_3 f_{n} \]
\[ y_{n+2} = y_n + \frac{h}{s} a_1 f_{n+2} + \frac{h}{s} a_2 f_{n+1} + \frac{h}{s} a_3 f_{n} + \frac{h}{d} c_1 f_{n+2} + \frac{h}{d} c_2 f_{n+1} + \frac{h}{d} c_3 f_{n} \]
\[ y_{n+3} = y_n + \frac{h}{s} a_1 f_{n+2} + \frac{h}{s} a_2 f_{n+1} + \frac{h}{s} a_3 f_{n} + \frac{h}{d} c_1 f_{n+2} + \frac{h}{d} c_2 f_{n+1} + \frac{h}{d} c_3 f_{n} \]

Obtaining the unknown parameters in (7) with Mathematica (10.4) we have

\[ a_1 = \frac{11}{1440}, \quad a_2 = -\frac{31}{480}, \quad a_3 = \frac{401}{720}, \quad a_4 = 0, \quad a_5 = \frac{1}{90}, \quad a_6 = \frac{17}{45}, \quad a_7 = \frac{3}{160}, \quad a_8 = -\frac{21}{160}, \quad a_9 = \frac{57}{80} \]

\[ B^1 = \begin{bmatrix} 11 & -31 & 401 \\ 1440 & 480 & 720 \\ 0 & -1 & 17 \\ 3 & 90 & 45 \\ 160 & -160 & 80 \end{bmatrix} \]

\[ c_1 = \frac{401}{720}, \quad c_2 = -\frac{31}{480}, \quad c_3 = \frac{11}{1440}, \quad c_4 = \frac{19}{15}, \quad c_5 = \frac{17}{45}, \quad c_6 = -\frac{17}{45}, \quad c_7 = \frac{59}{80}, \quad c_8 = \frac{219}{160}, \quad c_9 = \frac{51}{160} \]

\[ B^0 = \begin{bmatrix} 401 & -31 & 11 \\ 720 & 480 & 1440 \\ 19 & 17 & -17 \\ 15 & 45 & 45 \\ 59 & -219 & 51 \\ 80 & 160 & 160 \end{bmatrix} \]

we have

\[ y_{n+1} = y_n + \frac{h}{1440} f_{n+2} - \frac{31 h}{480} f_{n+1} + \frac{401 h}{720} f_n + \frac{401 h}{720} f_{n+1} - \frac{31 h}{480} f_{n+2} + \frac{11 h}{1440} f_{n+3} \]
\[ y_{n+2} = y_n - \frac{h}{90} f_{n-1} + \frac{17 h}{45} f_{n} + \frac{19 h}{15} c_{4} f_{n+1} + \frac{17 h}{45} f_{n+2} - \frac{17 h}{45} f_{n+3} \]
\[ y_{n+3} = y_n + \frac{3 h}{160} f_{n-2} - \frac{2 h}{160} f_{n-1} + \frac{57 h}{80} f_n + \frac{59 h}{80} f_{n+1} + \frac{219 h}{160} f_{n+2} + \frac{51 h}{160} f_{n+3} \]

when \( r = 4 \) we have

\[ Y_m = \begin{bmatrix} y_{m+1} \\ y_{m+2} \\ y_{m+3} \\ y_{m+4} \end{bmatrix}, \quad F_m = \begin{bmatrix} f_{m+1} \\ f_{m+2} \\ f_{m+3} \\ f_{m+4} \end{bmatrix}, \quad Y_{m-1} = \begin{bmatrix} y_{m-3} \\ y_{m-2} \\ y_{m-1} \\ y_m \end{bmatrix}, \quad F_{m-1} = \begin{bmatrix} f_{m-3} \\ f_{m-2} \\ f_{m-1} \\ f_m \end{bmatrix} \]

\[ A^0 Y_m = A^1 Y_{m-1} + \frac{h}{s} B^1 F_{m-1} + \frac{h}{d} B^0 F_m \]

And (2) above becomes
Expanding (9) we have

\[
y_{m+1} = y_m + \frac{h}{s} a_1 f_{m-1} + \frac{h}{s} a_2 f_{m-2} + \frac{h}{s} a_5 f_{m-2} + \frac{h}{s} a_8 f_{m-2} + \frac{h}{d} c_1 f_{m+1} + \frac{h}{d} c_2 f_{m+2} + \frac{h}{d} c_3 f_{m+3} + \frac{h}{d} c_4 f_{m+4}
\]

\[
y_{m+2} = y_m + \frac{h}{s} a_5 f_{m-2} + \frac{h}{s} a_6 f_{m-3} + \frac{h}{s} a_7 f_{m-3} + \frac{h}{s} a_8 f_{m-3} + \frac{h}{d} c_1 f_{m+1} + \frac{h}{d} c_2 f_{m+2} + \frac{h}{d} c_3 f_{m+3} + \frac{h}{d} c_4 f_{m+4}
\]

\[
y_{m+3} = y_m + \frac{h}{s} a_9 f_{m-3} + \frac{h}{s} a_{10} f_{m-2} + \frac{h}{s} a_{11} f_{m-2} + \frac{h}{s} a_{12} f_{m-2} + \frac{h}{d} c_9 f_{m+1} + \frac{h}{d} c_{10} f_{m+2} + \frac{h}{d} c_{11} f_{m+3} + \frac{h}{d} c_{12} f_{m+4}
\]

\[
y_{m+4} = y_m + \frac{h}{s} a_{13} f_{m-3} + \frac{h}{s} a_{14} f_{m-2} + \frac{h}{s} a_{15} f_{m-2} + \frac{h}{s} a_{16} f_{m-2} + \frac{h}{d} c_{13} f_{m+1} + \frac{h}{d} c_{14} f_{m+2} + \frac{h}{d} c_{15} f_{m+3} + \frac{h}{d} c_{16} f_{m+4}
\]
Obtaining the unknown parameters in (10) with Mathematical (10.4) we have

\[ B^1 = \begin{bmatrix}
191 & 1879 & -353 & 68323 \\
120960 & 120960 & -4480 & 120960 \\
0 & 1 & 2 & 167 \\
13 & 117 & 513 & 2777 \\
4480 & 4480 & 4480 & 4480 \\
8 & 64 & 8 & 106 \\
945 & 945 & 35 & 945
\end{bmatrix}, \quad B^0 = \begin{bmatrix}
68323 & 353 & 1879 & 191 \\
120960 & 4480 & 120960 & 120960 \\
1172 & 167 & 2 & 1 \\
3897 & 1107 & 337 & 9 \\
4480 & 896 & 896 & 896 \\
1784 & 8 & 1448 & 278 \\
945 & 35 & 945 & 945
\end{bmatrix}. \]

For \( r = 5 \) we have

\[ Y_m = \begin{pmatrix} y_{m+1}, y_{m+2}, y_{m+3}, y_{m+4}, y_{m+5} \end{pmatrix}^T, \]

\[ F_m = \begin{pmatrix} f_{m+1}, f_{m+2}, f_{m+3}, f_{m+4}, f_{m+5} \end{pmatrix}^T, \]

\[ Y_{m-1} = \begin{pmatrix} y_{m-4}, y_{m-3}, y_{m-2}, y_{m-1}, y_m \end{pmatrix}^T, \]

\[ F_{m-1} = \begin{pmatrix} f_{m-4}, f_{m-3}, f_{m-2}, f_{m-1}, f_m \end{pmatrix}^T. \]

And (2) above becomes

\[ A^0 Y_m = A^1 Y_{m-1} + \frac{h}{s} B^1 F_{m-1} + \frac{h}{d} B^0 F_m. \]
Note

\[
A^0 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad A^1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad B^1 = \begin{bmatrix}
a_1 & a_2 & a_3 & a_4 & a_5 \\
a_6 & a_7 & a_8 & a_9 & a_{10} \\
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\
a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25}
\end{bmatrix}, \quad B^0 = \begin{bmatrix}
c_1 & c_2 & c_3 & c_4 & c_5 \\
c_6 & c_7 & c_8 & c_9 & c_{10} \\
c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\
c_{16} & c_{17} & c_{18} & c_{19} & c_{20} \\
c_{21} & c_{22} & c_{23} & c_{24} & c_{25}
\end{bmatrix}
\]

the expansion of (11) gives

\[
\begin{align*}
y_{m+1} &= y_m + \frac{h}{s} a_1 f_{m-4} + \frac{h}{s} a_2 f_{m-3} + \frac{h}{s} a_3 f_{m-2} + \frac{h}{s} a_4 f_{m-1} + \frac{h}{d} a_5 f_m + \frac{h}{d} c_1 f_{m+1} + \frac{h}{d} c_2 f_{m+2} + \\
&\quad \frac{h}{d} c_3 f_{m+3} + \frac{h}{d} c_4 f_{m+4} + \frac{h}{d} c_5 f_{m+5} \\
y_{m+5} &= y_m + \frac{h}{s} a_{21} f_{m-4} + \frac{h}{s} a_{22} f_{m-3} + \frac{h}{s} a_{23} f_{m-2} + \frac{h}{s} a_{24} f_{m-1} + \frac{h}{s} a_{25} f_m + \frac{h}{d} c_{21} f_{m+1} + \frac{h}{d} c_{22} f_{m+2} + \\
&\quad \frac{h}{d} c_{23} f_{m+3} + \frac{h}{d} c_{24} f_{m+4} + \frac{h}{d} c_{25} f_{m+5} \\
y_{m-2} &= y_m + \frac{h}{s} a_6 f_{m-4} + \frac{h}{s} a_7 f_{m-3} + \frac{h}{s} a_8 f_{m-2} + \frac{h}{s} a_9 f_{m-1} + \frac{h}{s} a_{10} f_m + \frac{h}{d} c_6 f_{m-1} + \frac{h}{d} c_7 f_{m-2} + \\
&\quad \frac{h}{d} c_8 f_{m-3} + \frac{h}{d} c_9 f_{m-4} + \frac{h}{d} c_{10} f_{m-5} \\
y_{m+2} &= y_m + \frac{h}{s} a_{16} f_{m-4} + \frac{h}{s} a_{17} f_{m-3} + \frac{h}{s} a_{18} f_{m-2} + \frac{h}{s} a_{19} f_{m-1} + \frac{h}{s} a_{20} f_m + \frac{h}{d} c_{16} f_{m+1} + \frac{h}{d} c_{17} f_{m+2} + \\
&\quad \frac{h}{d} c_{18} f_{m+3} + \frac{h}{d} c_{19} f_{m+4} + \frac{h}{d} c_{20} f_{m+5}
\end{align*}
\]

To obtain unknown parameters in (12) with Mathematical (10.4) we have

\[
B^1 = \begin{bmatrix}
2497 & -28939 & 581 & -40111 & 2067169 \\
7257600 & 7257600 & 25920 & 453600 & 3628800 \\
0 & 23 & 56700 & 12850 & 26321 \\
7 & 77 & 351 & 11200 & 44800 \\
12800 & 12800 & 572 & 11200 & 2764 \\
13 & 26 & 11975 & 116775 & 298505 \\
14175 & 2835 & 14175 & 14175 & 14175 \\
25 & 20675 & 11975 & 167775 & 145152 \\
3584 & 290304 & 36288 & 18144 & 145152
\end{bmatrix}, \quad B^0 = \begin{bmatrix}
-2067169 & 40111 & 581 & -28939 & 2497 \\
3628800 & 453600 & 25920 & 7257600 & 7257600 \\
13903 & 23189 & 56700 & 12850 & 113400 \\
11340 & 6609 & 4559 & 351 & 113400 \\
44800 & 5600 & 11200 & 17920 & 89600 \\
20908 & 1628 & 19192 & 4909 & 94 \\
14175 & 2835 & 14175 & 14175 & 14175 \\
131575 & 50425 & 7225 & 478525 & 81385 \\
145152 & 18144 & 36288 & 290304 & 290304
\end{bmatrix}
\]
Consistency of the Method: The LMM method can be proved to be consistent if and only if the following conditions are satisfied [9].

\[ \sum_{i=0}^{k} \alpha_i = 0, \sum_{i=0}^{k} \beta_i = 0 \]

Definition (1.1): Block method is consistent if it has order at least one [4].

Zero Stability: Block method is said to be zero stable if the roots \( r \) of the characteristics polynomial \( \rho(r) \) defined by 

\[ \rho(r) = \det \begin{bmatrix} rA^0 - A^1 \end{bmatrix} \]

satisfies \( |r| \leq 1 \) and every root with \( |r_0| = 1 \) has multiplicity not exceeding one in the limit as \( h \to 0 \).

Therefore, the block method is zero stable.

Theorem (1.1): The necessary conditions for LMM to converge are; it must be consistent and zero stable [8]. Since the method satisfies these conditions, therefore it is convergent.

Stability of the method: Consider the block method below (2) above

\[ A^0 Y_m = A^1 Y_{m-1} + \frac{h}{s} B^1 F_{m-1} + \frac{h}{d} B^0 F_m \]

\[ y'_m = \lambda y_m \ & \ y'_{m-1} = \lambda y_{m-1} \]  

Applying the test equation (13) on (2) we have

\[ A^0 Y_m = A^1 Y_{m-1} + \frac{h}{s} \lambda B^1 y_{m-1} + \frac{h}{d} \lambda B^0 y_m \]

\[ A^0 Y_m - \frac{h}{d} \lambda B^0 y_m = A^1 Y_{m-1} + \frac{h}{s} \lambda B^1 y_{m-1} \]

\[ y_m = \frac{Y_{m-1}(A^1 + \frac{h}{s} \lambda B^1)}{(A^0 - \frac{h}{s} \lambda B^0)} \]

\[ y_m = \left( A^0 - \frac{h}{d} \lambda B^0 \right)^{-1} \left( A^1 + \frac{h}{s} \lambda B^1 \right) Y_{m-1} \]  

let \( z = h\lambda \)

\[ y_m = \left( A^0 - \frac{z}{d} B^0 \right)^{-1} \left( A^1 + \frac{z}{s} B^1 \right) Y_{m-1} \]
Where \( (A^0 - \frac{z}{d} B^0)^{-1} (A^1 + \frac{z}{s} B^1) \) is called amplification matrix of the method?

**Order and Error Constant**

**Definition (1.2):** The order of LMM and its associated linear operator given by

\[
L[y(x); h] = \sum_{j=0}^{k} [\alpha_j y(x + jh) - h\beta_j y'(x + jh)]
\]

Is defined as a unique integer \( p \) such that \( C_p = 0, q = 0(1)p and C_{p+1} \neq 0 \)

Where the \( C_q \) are constant defined as

\[

c_0 = \alpha_0 + \alpha_1 + \cdots + \alpha_k \\
c_1 = \alpha_1 + 2\alpha_2 + \cdots + k\alpha_k - (\beta_1 + \cdots + \beta_k) \\
c_q = \frac{1}{q!} (\alpha_1 + 2q\alpha_2 + \cdots + kq\alpha_k) - \frac{1}{(q - 1)!} (\beta_1 + 2^{q-1}\beta_2 + \cdots + k^{q-1}\beta_k)
\]

\( q = 2, 3, \ldots, k \) \(^7\).

The associated linear operator (2) is

\[
L[y(x); h] = A^0 Y_m - A^1 Y_{m-1} - \frac{h}{s} B^1 F_{m-1} - \frac{h}{d} B^0 F_m
\]

Expanding (16) we obtain

\[
L[y(x); h] = c_0 y(x_n) + c_1 h y'(x_n) + \cdots + c_p h^p y^p(x_n) + \cdots
\]

The methods (2) is said to be of order \( p \) if \( c_0 = c_1 = \cdots = c_p = 0 \) and \( c_{p+1} \neq 0 \) where \( c_{p+1} \) is the error constant of the block method. Therefore the order \( p \) and the associated error constant for each \( r \)-point is given as follows.

For \( r = 2 \) is of order \( p = 4 \) with error constant \[
\begin{bmatrix}
11 \\
720 \\
-1 \\
90
\end{bmatrix}^T
\]

For \( r = 3 \) is of order \( p = 6 \) with error constant, \[
\begin{bmatrix}
191 \\
60480 \\
1 \\
756 \\
-29 \\
2240
\end{bmatrix}^T
\]

For \( r = 4 \) is of order \( p = 8 \) with error constant, \[
\begin{bmatrix}
2497 \\
3628800 \\
-23 \\
113400 \\
81 \\
44800 \\
-107 \\
14175
\end{bmatrix}^T
\]

For \( r = 5 \) is of order \( p = 10 \) with error constant, \[
\begin{bmatrix}
14797 \\
95800320 \\
263 \\
7484400 \\
-127 \\
394240 \\
367 \\
467775 \\
-114985 \\
19160064
\end{bmatrix}^T
\]

**Definition (1.3):** A numerical method is said to be A-Stable if the stability region contain the entire left hand side of the complex plane \(^4\).

One of the basic requirements for every numerical method to solve stiff IVPs in ODEs is that the method must be A-stable. Applying the boundary locus techniques on the amplification matrix of (2) we have
Fig 1: A-stable method for $r=2$, $d=-1$ and $s=-1$.

Fig 2: A-stable method for $r=3$ at $d=-1$ and $s=-6.5$
Implementation of the Method

This section considers numerical experiments of the proposed block methods for solution of system of first order IVPs in ODEs.

Problem 1:
\[ y' = \frac{y(1 - y)}{2(y - 1)}, \quad y(0) = \frac{5}{6}, \quad 0 \leq x \leq 1 \]

Exact Solution
\[ y(x) = \frac{1}{2} + \frac{1}{4} - \frac{x}{36} e^{-x} \]

Problem 2:
\[ y' = -\frac{y^3}{2}, \quad y(0) = 1, \quad 0 \leq x \leq 4 \]

Exact Solution
\[ y(x) = \frac{1}{\sqrt{1 + x}} \]

Problem 3:
\[ y' = y, \quad y(0) = 1, \quad h = 0.1 \]

Exact Solution
\[ y(x) = e^x \]

Problem 4: Consider the system of equations
\[ X'(t) = 998X(t) + 1998Y(t), X(0) = 1, \]
\[ Y'(t) = -999X(t) - 1999Y(t), Y(0) = 0, \]

The exact solutions are
\[ X(t) = 2e^{-t} - e^{-1000t} \]
\[ Y(t) = e^{-1000t} - e^{-t}. \]

Table 1: Numerical solution for problem 1

| X     | Theoretical Solution | 3PBEBDF ORDER P=6 [17] | Proposed Method ORDER P=6 | Error Between Exact Solution and 3PBEBDF [17] | Error Between Exact Solution and Proposed Method of order p=6 |
|-------|---------------------|-----------------------|--------------------------|---------------------------------------------|-------------------------------------------------------------|
| 0.0   | 1.8533333           | 1.8333333             | 1.8333333                | 0                                           | 0                                                           |
| 0.1   | 1.8526020           | 1.8527450             | 1.8526350                | 0.0001430                                  | 0.0000420                                                   |
| 0.2   | 1.8691712           | 1.8690573             | 1.8691573                | 0.0001139                                  | 0.000128                                                    |
| 0.3   | 1.8835474           | 1.8829767             | 1.8829767                | 0.0005707                                  | 0.0002505                                                   |
| 0.4   | 1.8961060           | 1.8960859             | 1.8960959                | 0.0000201                                  | 0.0000101                                                   |
| 0.5   | 0.9071359           | 0.9065019             | 0.9068239                | 0.0004120                                  | 0.00003120                                                  |
| 0.6   | 0.9168647           | 0.9155497             | 0.9167617                | 0.0013150                                  | 0.0001030                                                   |
| 0.7   | 0.9254760           | 0.9253058             | 0.9254158                | 0.0001702                                  | 0.0000602                                                   |
| 0.8   | 0.9331203           | 0.9321562             | 0.9328783                | 0.0009641                                  | 0.0002420                                                   |
| 0.9   | 0.9399227           | 0.9381680             | 0.9391680                | 0.0017547                                  | 0.0007547                                                   |
| 1.0   | 0.9459884           | 0.9439650             | 0.9449762                | 0.0020234                                  | 0.0010122                                                   |

Table 2: Numerical solution for problem 2

| X     | Theoretical Solution | 3PBEBDF ORDER P=6 [17] | Proposed Method ORDER P=6 | Absolute Error Between Exact Solution and 3PBEBDF [17] | Error Between Exact Solution and Proposed Method of order p=6 |
|-------|---------------------|-----------------------|--------------------------|----------------------------------------------------------|-------------------------------------------------------------|
| 0.0   | 1.0000000           | 1.0000000             | 1.0000000                | 0                                                       | 0                                                           |
| 0.1   | 0.9534626           | 0.9531365             | 0.9533494                | 0.0003261                                  | 0.0001132                                                   |
| 0.2   | 0.9128709           | 0.9131317             | 0.9128106                | 0.0002608                                  | 0.0000603                                                   |
| 0.3   | 0.8770580           | 0.8783667             | 0.8767493                | 0.0013087                                  | 0.0003087                                                   |
| 0.4   | 0.8451543           | 0.8452111             | 0.8451295                | 0.0005068                                  | 0.000248                                                    |
| 0.5   | 0.8164966           | 0.8180200             | 0.8161924                | 0.0015044                                  | 0.0003042                                                   |
1. Aijileye GA, Owgumu OD. Hybrid Block Method Algorithms for Solution of First Order Initial Value problems in Ordinary Differential Equations. Journal of Applied & Computational Mathematics. 2018;7(2):2-3.
2. Akinsola VO, Aregebesola YA. The Numerical Solutions of System of Stiff Ordinary Differential Equations using the semi Implicit extrapolation. Journal of Advances in Scientific Research & Application, 2014, 56-66.
3. Esuabana IM, Ekoro SE. Hybrid Linear Multistep Methods with Nested Hybrid Predictors for solving linear and Non linear Initial Value Problems in Ordinary Differential Equations. International Institute for Science, Technology and Education. 2017:77-88.
4. Fatunla SO. Numerical Method for Initial Value Problems in Ordinary Differential Equation. New York Academic press Inc, 1988.
5. Fatunla SO. A Class of Block Method for second order initial value problems. International Journal of Computer Mathematics, 1995;55:119-133.
6. Babangida B, Musa H, Ibrahim LK. A New Numerical Method for Solving Stiff Initial Value problems. 2016;3:1-5.
7. Chu MT, Hamilton H. Parallel Solution of Ordinary Differential Equations by Multi block Methods. Journal of Society for Industrial and Applied Mathematics, 1987.
8. Lambert JD. Computational Methods in ordinary differential equations. New York: John Wiley and Sons, 1973.
9. Lambert JD. Numerical Methods for ordinary differential equations. John Wiley and Sons New York 1991.
10. Ekoro SE, Ikhole MN, Esuabana IM. Implicit Second Derivative Hybrid Linear Multistep Method with Nested Predictors for Ordinary Differential Equations. American Scientific Research Journal for Engineering, Technology and Sciences (ASRJETS). 2018;42(1):297-308.
11. Ajei II, Utalor K, Onumanyi P. A Family of High Order One-Block Methods for the Solution of Stiff Initial Value Problems. Journal of Advances in Mathematics and Computer Science, 2019.
12. Ajei O, Ikhole MN, Onumanyi P. A family of L(α)-stable Block Methods for Stiff Ordinary Differential Equations. American Journal of Computational and Applied Mathematics. 2014;4(1):24-31.
13. Musa H, Bala H. 3-Point Diagonally Implicit Super Class of Block Backward Differentiation Formula for Solving Stiff Initial Value Problems. Dutse Journal of Pure and Applied Sciences. 2019;5(1b):1-10.
14. Ismail F. Explicit and Implicit 3-Point Block Methods for Solving Special Second Order Ordinary Differential Equations Directly. International Journal of Applied Mathematics. 2009;5:239-254.
15. Lee. Two and Three Point Block Methods for Solving First Order Ordinary Differential Equation in Parallel. M.Sc Thesis Universiti Putra Malaysia, 2000.
16. Alvarez J, Rojo J, An improve Class of Generalized Runge Kutta Methods for Stiff Problems. Journal of Apply Mathematics & Computation. 2002;130:537-560.
17. Musa A, Suleiman MB, Senu N. Fully Implicit 3-points Block Extended Backward Differentiation Formula for Stiff Initial Value Problems. Journal of Applied Mathematics and Science. 2012;6:4211-4228.
18. Esuabana IM, Ekoro SE. Derivation and Implementation of new Family of Second Derivative Hybrid Linear Multistep Methods for Stiff Ordinary Differential Equations, Global Journal of Mathematics. 2018;12(2):821-828.
19. Esuabana IM, Ekoro SE, Ojo BO, Abasiekwere UA. Adam’s block with first and second derivative future points for initial value problems in ordinary differential equations, J. Math. Comput. Sci. 2021;11(2):1470-1485.
20. Esuabana IM, Ekoro SE, Ikhile MNO. Implicit Second Derivative Hybrid Linear Multistep Method with Nested Predictors for Ordinary Differential Equations. American Scientific Research Journal for Engineering Technology and Sciences. 2018;42(1):297-308.