Controllability of Networked Sampled-Data Systems

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Abstract—The controllability of networked sampled-data systems with zero-order holders on the control and transmission channels is explored, where single- and multirate sampling patterns are considered, respectively. The effects of sampling on the controllability of networked systems are analyzed, with some necessary and/or sufficient controllability conditions derived. Different from the sampling control of single systems, the pathological sampling of node systems could be eliminated by an appropriate design of network structure and inner couplings. Whereas the pathological sampling of single nodes will cause the entire system with a singular topology matrix to lose controllability. Moreover, any periodic sampling will not affect the controllability of networked systems with specific node dynamics. All the results indicate that whether a networked system is under pathological sampling or not is jointly determined by mutually coupled factors.

Index Terms—Multirate sampling, network controllability, networked systems, pathological sampling, sampled-data systems.

I. INTRODUCTION

CONTROLLABILITY, as a prerequisite of effective control of system performance and behavior [1], has been extensively investigated during the past decades, with various rank conditions and graphic properties developed [2], [3], [4], [5], [6], [7]. Recently, the upsurge of network science has promoted the expansion of research on controllability. The maximum matching theorem was applied to ensure structural controllability by driver node selection [8] and the relationship between the number of driver nodes and network structure properties was investigated [9], [10]. The notion of target control was proposed in [10], and the control algorithm was optimized through preferential matching [11]. Multiple descriptions of node importance were presented [12], [13] to analyze the robustness of network controllability [14]. By the output Gramian of infinite lattice systems, Klickstein and Sorrentino [15] found that the control energy is exponentially related to the maximum distance between driver and target nodes. So far, the controllability of complex networks has been applied to real-world control problems in biology, medicine, and other fields [16], [17].

Due to the development of computer control platforms and problems such as limited bandwidth, signal instability, and transmission delay, the control and transmission signals of real systems are usually sampled data. Therefore, how to maintain the controllability of a system during sampling is a problem worth of attention. It was pointed out in [18] that the controllability of linear time-invariant (LTI) systems may not be preserved during the periodic sampling. In [2], the sampling that destroys controllability was defined as “pathological sampling,” and a nonpathological sampling condition related to eigenvalues of the state matrix was provided. The nonpathological sampling of switched linear systems was studied in [19]. The influence of sampling on controllability indices was analyzed in [20]. In [21], a method of adding steps of nonequidistant sampling based on periodic sampling was proposed to maintain the controllability of LTI systems, and is further applied to time-varying systems [22]. The multirate sampling pattern, where control channels have nonidentical sampling periods was studied in [23] with a sufficient controllability condition given.

The research works above are either for single network control systems with no sampling on transmission channels between nodes, or under the assumption that each node system in the network is 1-D. However, node states in real-world systems are often high-dimensional and coupled with each other through multiple transmission channels. In [24], some controllability conditions for general networked systems were presented based on the transfer function matrix. An easier-to-verify necessary and sufficient controllability criterion was derived in [25] by matrix similarity transformation. In [26], it was clarified that the coupling of network structure and node dynamics jointly determines the controllability of a networked system. A controllability decomposition approach for networked systems was proposed in [27] to investigate each node system when the
network is not completely controllable. The controllability of heterogeneous networked systems was investigated in [28]. The controllability conditions for networked continuous-time LTI (CLTI) systems, their sampling effects have not attracted much attention. Some results showed that the controllability of a special type of networked systems, multiagent systems (MASs), can be decoupled into two independent parts related to single nodes and network topology, respectively [31], [32], [33]. Some research on the sampling controllability of MASs provided necessary and sufficient conditions under synchronous and asynchronous sampling protocols [34], [35]. In [36], it was found that MASs can always maintain controllability after sampling and Lu et al. [37] discovered that topology switching can ensure the controllability of MASs with uncontrollable subsystems. However, to date, little work has been devoted to the controllability of more general networked sampled-data systems that cannot be decoupled.

This article studies the state controllability of networked sampled-data systems. Based on the above analysis, we find that most existing research on the controllability of networked sampled-data systems did not consider multirate cases and general topologies, and only control sampling was taken into account, without sampling on transmission channels. The contributions of this article are fourfold.

1) In the proposed model, sampling on both control and transmission channels is considered, including single- and multirate sampling patterns. More general networked systems with directed, weighted topology, and multidimensional node dynamics are studied in this article.

2) Necessary and/or sufficient controllability conditions are developed (mainly Theorems 1, 2, Corollaries 1, 2) for networked sampled-data systems, which have much lower computational complexity than classic criteria.

3) The influence of sampling on the networked system’s controllability is depicted, which is coupled with network topology, inner couplings, external inputs, and node dynamics. Especially, systems with 1-D or self-loop-state node dynamics can retain controllability during any periodic sampling (Corollaries 10, 12).

4) The influence of network structure on sampling controllability is analyzed. The loss of controllability caused by the pathological sampling of single node systems can be eliminated by an appropriate design of network topology and inner couplings (Remark 6) unless the topology matrix is singular (e.g., tree, chain, star) (Corollary 3).

The rest of this article is organized as follows. The model formulation and preliminaries are introduced in Section II. In Section III, some controllability conditions are developed for general networked sampled-data systems. Networked systems with special topologies and dynamics are studied in Sections IV and V, respectively. Section VI preliminarily inspects the controllability of networked multirate sampled-data systems. Finally, Section VII concludes this article.

II. PRELIMINARIES AND MODEL FORMULATION

In this section, some useful preliminaries, and the model of networked systems with a general structure and periodic sampling pattern are introduced.

A. Notations and Definitions

Let $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{N}$ denote the fields of real, complex, and natural numbers, respectively. Denote by $I_n$ the identity matrix of size $n \times n$. By $e_i$ the row vector with all zero entries except that the $i$th element is 1. By diag$\{a_1, a_2, \ldots, a_n\}$ the $n \times n$ matrix with diagonal elements $a_1, a_2, \ldots, a_n$, and by diagblock$\{A_1, A_2, \ldots, A_n\}$ the block matrix with matrices $A_1, A_2, \ldots, A_n$ on the diagonal.

Moreover, denote the set of all eigenvalues of matrix $A \in \mathbb{C}^{n \times n}$ by $\sigma(A) = \{\lambda_1, \ldots, \lambda_r\}$, $1 \leq r \leq n$, and by $M(\lambda_i|A)$ the eigenspace of $A$ with respect to $\lambda_i$. $D(A)$ denotes the sum of the geometric multiplicity of all eigenvalues of matrix $A$. Let $\mathcal{R}(\cdot)$ and $\mathcal{N}^T(\cdot)$ denote the column space and left null space, respectively. Denote by $\dim(\cdot)$ the dimension of space. The complex linear span of row vectors $v_1, v_2, \ldots, v_n$ is denoted by $\text{span}\{v_1, v_2, \ldots, v_n\} = \{\sum_{i=1}^{n} c_i v_i | c_i \in \mathbb{C}\}$, which is the set of all their complex linear combinations. Let $A \otimes B$ denote the Kronecker product of matrices $A$ and $B$, and $V_1 \oplus V_2$ the direct sum of space $V_1$ and $V_2$. It is assumed that the dimensions of matrices are compatible with algebraic operations if they are not specified.

Definition 1 (see [38]): A row vector, $v^m$ is called the $m$th-order generalized left eigenvector of matrix $A \in \mathbb{C}^{n \times n}$ corresponding to $\lambda \in \sigma(A)$ if

$$v^m(A - \lambda I_n)^m = 0 \quad \text{and} \quad v^m(A - \lambda I_n)^{m-1} \neq 0$$

then $v^1, v^2, \ldots, v^m$ form a left Jordan chain of $A$ about $\lambda$, the vector with the maximum value of $\alpha$ is the length of this Jordan chain.

Definition 2 (see [25]): Let $E \in \mathbb{C}^{n \times n}$, $H \in \mathbb{C}^{n \times n}$, $\theta \in \sigma(E)$. For row vectors $\xi_1, \xi_2, \ldots, \xi_\gamma$ satisfy

$$\xi_1(\theta I_n - E) = 0, \quad \text{and} \quad \xi_1(\theta I_n - E) = \xi_{j-1} e^{H}, \quad j = 2, \ldots, \gamma$$

then $\xi_1, \xi_2, \ldots, \xi_\gamma$ form a generalized left Jordan chain of $E$ about $H$ corresponding to $\theta$, where $\xi_1$ is the top vector and the maximum value of $\gamma$ is the length of this generalized Jordan chain.

B. Control Sampling of Single Systems

Describe a CLTI system by $\dot{x}(t) = Ax(t) + Bu(t)$, where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^p$ represent the state and input, respectively, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$. By periodic sampling on control channels, a corresponding sampled-data system can be obtained and described by $x((k + 1)h) = e^{Ah}x(kh) + \int_0^h e^{A\tau} dBu(kh)$, where $h$ denotes the sampling period, $k \in \mathbb{N}$. 

According to the Popov–Belevitch–Hautus (PBH) rank condition [1], a CLTI system $(A, B)$ is controllable if and only if $\forall s \in \mathbb{C}$, rank$\{|sI_n - A, B|\} = n$. Due to the difference between the reachable subspace and controllable subspace of
discrete-time LTI (DLTI) systems [39], the sampled-data system is controllable if but not only if \( \forall s \in \mathbb{C}, \text{rank}([sI_n - e^{Ah}, \int_0^h e^{A\tau}B]) = n \).

An uncontrollable CLTI system cannot gain controllability by sampling. However, a controllable CLTI system may become uncontrollable after sampling due to the improper selection of \( h \). The sampling that destroys controllability is called pathological sampling. In [2], a definition of (non-) pathological sampling is provided.

**Definition 3 (see [2]):** The control sampling with period \( h \) is nonpathological about \( A \) if \( \forall \lambda_i, \lambda_j \in \sigma(A), i, j \in \{1, \ldots, r\} \) satisfy

\[
\lambda_i - \lambda_j \neq \frac{2k\pi}{h}, \quad k = \pm 1, \pm 2, \ldots,
\]

otherwise \( h \) is pathological about \( A \). Here, “i” refers to the imaginary unit.

**C. General Model of Networked Sampled-Data Systems**

Consider a general directed and weighted network consisting of identical node systems:

\[
\begin{align*}
\dot{x}_i(t) &= Ax_i(t) + \sum_{j=1}^N w_{ij}y_j(t) + \delta_iBu_i(t) \quad (i = 1, 2, \ldots, N) \\
y_i(t) &=Cx_i(t)
\end{align*}
\]

where \( x_i \in \mathbb{R}^n \), \( y_i \in \mathbb{R}^m \), and \( u_i \in \mathbb{R}^p \) denote the state, output, and external control input of node \( i \), respectively. \( A \in \mathbb{R}^{n \times n} \) is the state matrix describing the node dynamics. \( B \in \mathbb{R}^{n \times p} \) is the input matrix, \( \delta_i = 1 \) if node \( i \) is under control, otherwise \( \delta_i = 0 \). \( C \in \mathbb{R}^{m \times n} \) is the output matrix, and \( H \in \mathbb{R}^{m \times m} \) describes the inner couplings among the nodes. \( w_{ij} \neq 0 \) if there is an edge from node \( j \) to node \( i \), otherwise \( w_{ij} = 0 \). Specially, \( w_{ii} \neq 0 \) if there is a self-ring of node \( i \) in the network, otherwise \( w_{ii} = 0 \).

Let \( W = [w_{ij}] \in \mathbb{R}^{N \times N} \) and \( \Delta = \text{diag}\{\delta_1, \ldots, \delta_N\} \in \mathbb{R}^{N \times N} \) represent the transmission channels and control channels of the networked system, respectively. Let \( X(t) = [x_1^T(t), \ldots, x_N^T(t)]^T \) be the total state of the networked system, and \( U(t) = [u_1^T(t), \ldots, u_N^T(t)]^T \) be the total external control input. Then, the networked system can be rewritten in a compact form

\[
\dot{X}(t) = \Phi X(t) + \Psi U(t)
\]

where

\[
\Phi = I_N \otimes A + W \otimes HC, \quad \Psi = \Delta \otimes B.
\]

Then consider the associated sampled-data case. Assume that the sampling is performed on all control channels and transmission channels simultaneously by samplers with zero-order hold. Let \( h \) be the sampling period, \( k \in \mathbb{N} \). As a result, the corresponding sampled-data system of (2), (3) can be represented as follows:

\[
X((k + 1)h) = \Phi_sX(kh) + \Psi_sU(kh)
\]

with

\[
\Phi_s = I_N \otimes e^{Ah} + W \otimes \mathcal{H}(h), \quad \Psi_s = \Delta \otimes \mathcal{B}(h)
\]

By the PBH rank condition, one has \( \text{rank}([sI_A - \Phi_s, \Psi_s]) = 2 \) when \( s = 23.14 \). However, for the corresponding CLTI system \( (\Phi, \Psi) \), \( \text{rank}([sI_A - \Phi, \Psi]) = 4 \) no matter what value \( s \) takes.

It can be seen that the controllability of the networked system has been lost during the sampling. However, the above verification may result in a curse of dimensionality when the network scale is large, so lower dimensional criteria are needed. Compared with the single LTI system, there are more coupled factors playing a role in the controllability of networked systems, such as topologies and inner couplings, because the sampling is performed on not only control channels, but transmission channels. Therefore, the controllability analysis for control sampling of single systems cannot be directly applied to networked sampled-data systems, either.

**III. MAIN RESULTS**

In this section, necessary and/or sufficient controllability conditions are derived for the networked sampled-data system (4), (5). At first, a general condition is given in Theorem 1, followed by other conditions for systems with special structures. Examples are provided to illustrate the verification.
First of all, Lemmas 1–2 are given as preliminaries for the controllability rank conditions later on.

**Lemma 1:** Suppose that \( \sigma(W) = \{\lambda_1, \lambda_2, \ldots, \lambda_r\} \), and \( \sigma(E_i) = \{\theta_1^i, \theta_2^i, \ldots, \theta_p^i\} \), where
\[
E_i = e^{Ah} + \lambda_i \mathcal{H}(h)
\]
\( i = 1, \ldots, r \). Then, \( \sigma(\Phi_s) = \{\theta_1^1, \theta_1^2, \ldots, \theta_p^s\} \), where \( \xi_{ij} \) is top vector, and \( \gamma_{ij} \) is the length, \( i = 1, 2, \ldots, r \), \( j = 1, 2, \ldots, p_i \).

**Lemma 2:** The eigenspace of \( \Phi_s \) about \( \theta_1^i \) is \( M(\theta_1^i|\Phi_s) = V(\theta_1^i) \), where \( V(\theta_1^i) = \text{span}\{v_{11}^i, v_{12}^i, \ldots, v_{1r}^i\} \), \( v_{ij}^i = v_i^1 \otimes \xi_{ij}^1 \), \( v_{ij}^i = v_i^2 \otimes \xi_{ij}^2 + v_i^l \otimes \xi_{ij}^l \), \( v_{ij}^i = v_i^{l-1} \otimes \xi_{ij}^{l-1} \), \( \xi_{ij}^{l-1} \), \( \beta_{ij} = \min(\alpha_{ij}, \gamma_{ij}) \), \( i = 1, 2, \ldots, r \) and \( j = 1, 2, \ldots, p_i \).

**Remark 1:** Lemmas 1–2 are the sampled-data version of the results of [25, Th. 1]. In order to verify the networked sampled-data system based on the PBH rank condition, it is necessary to calculate all the eigenvalues and the whole eigenspace of the state matrix \( \Phi_s \). Lemmas 1–2 provide a decomposition method to obtain the eigenvalues by lower dimensional subspaces (represented by \( E_i \)), and to calculate the eigenspace by direct representation of Kronecker products of vectors with lower dimensions (i.e., \( v_i^k \) and \( \xi_{ij}^k \)). In this way, the curse of dimensionality can be avoided because there is no need to perform matrix operations directly on high dimensional \( \Phi_s \).

Based on the above analysis, a sufficient controllability criterion for the networked sampled-data system (4), (5) is given as follows.

**Theorem 1:** The networked sampled-data system (4), (5) is controllable if \( \forall \eta \in M(\theta_1^i|\Phi_s) \) and \( \eta \neq 0, \eta(\Delta \otimes B(h)) \neq 0 \), for every \( i = 1, \ldots, r \) and \( j = 1, \ldots, p_i \), where \( M(\theta_1^i|\Phi_s) \) can be calculated by Lemma 2.

**Proof:** Based on the PBH rank condition, a DLTI system is controllable if the input matrix left multiplied by any eigenvector of the state matrix is nonzero. By Lemmas 1–2, \( M(\theta_1^i|\Phi_s) \) with \( i = 1, \ldots, r \) and \( j = 1, \ldots, p_i \) is the whole eigenspace of the state matrix \( \Phi_s \), thus, Theorem 1 holds.

Theorem 1 reveals that the controllability of a networked sampled data system can be inferred from lower dimensional operations by matrices and vectors related to some subsystems. The following example demonstrates the specific verification process by Theorem 1.

**Example 1:** Consider a networked sampled-data system consisting of three identical node systems in a chain structure, where
\[
w_{21} = w_{32} = 1, \delta_1 = 1, h = 0.1, \text{ and } B = I_2,
\]
\[
A = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad H = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}, \quad e^{Ah} = \begin{bmatrix}
1.1052 & 0 \\
0.1105 & 1.1052
\end{bmatrix}, \quad \mathcal{H}(h) = \begin{bmatrix}
0.1052 & 0 \\
0.0053 & 0.1052
\end{bmatrix}.
\]

It can be computed that the associated generalized left eigenvectors of \( W \) with respect to \( \lambda_1 = 0 \) are \( v_1^1 = [1, 0, 0], v_1^2 = [0, 1, 0] \), and \( v_1^3 = [0, 0, 1] \). Then
\[
E_1 = e^{Ah} + \lambda_1 H(h) = \begin{bmatrix}
1.1052 & 0 \\
0.1105 & 1.1052
\end{bmatrix}.
\]

The eigenvalue of \( E_1 \) is \( \theta_1^1 = 1.1052 \), with the generalized left Jordan chain of \( E_1 \) about \( \mathcal{H}(h) \) corresponding to \( \theta_1^1 \) being \( \xi_{11}^1 = [1, 0], \xi_{12}^1 = [0, -0.9520]. \) Then, \( \eta_{11}^1 = v_1^1 \otimes \xi_{11}^1 = [1, 0, 0, 0, 0, 0] \) and \( \eta_{11}^2 = v_1^2 \otimes \xi_{12}^1 = [0, -0.952, 1, 0, 0, 0] \)

One can verify that for any \( a_1, a_2 \in \mathbb{C} \) and \( [a_1, a_2] \neq 0 \),
\[
(a_1 \eta_{11}^1 + a_2 \eta_{11}^2)(\Delta \otimes B(h)) = \begin{bmatrix}
0.1052 & 0 & \cdots & 0 \\
0.0053 & 0.1052 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0 
\end{bmatrix} 
eq 0.
\]

Therefore, this networked sampled-data system is controllable.

**Remark 2:** Different from [25, Th. 2], Theorem 1 is a sufficient but not necessary controllability condition. Though the reachable subspace and controllable subspace of the CLTI system are always equivalent to each other, this does not hold for the sampled-data system when \( \Phi_s \) is singular (according to the first Note on [39, p. 102]). Since the PBH rank condition is derived based on the image space of the controllability matrix (which is equal to the reachable subspace but strictly included in the controllable subspace), it is sufficient but not necessary. In this case, the controllable subspace can still be \( \mathbb{R}^n \) even if the condition of Theorem 1 is not satisfied, for it is derived from the PBH rank condition. Also, if \( \Phi_s \) is nonsingular, which means \( E_i \) has no zero eigenvalues for every \( i = 1, 2, \ldots, r \), Theorem 1 turns to be necessary and sufficient (by the last Note on [39, P. 101]).

The following is a counterexample showing that the condition in Theorem 1 is sufficient but not necessary.

**Example 2:** Consider a networked sampled-data system consisting of three identical nodes in a chain structure with self-rings, where \( w_{11} = w_{22} = w_{33} = -1, w_{12} = w_{23} = 1, \delta_1 = 1, h = 0.1 \), and
\[
A = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
1 \\
0
\end{bmatrix}, \quad H = \begin{bmatrix}
10 & 0 \\
0 & 10
\end{bmatrix}.
\]

According to the definition of controllability of DLTI systems, \( (\Phi_s, \Psi_s) \) is controllable if and only if \( \forall X(0) \in \mathbb{R}^p \), there exist
By simple calculation, it is easy to find that $\Phi_s$ is a nilpotent matrix. Therefore, if one takes $U(\cdot) = 0$ and $k \geq 6$, (7) holds, which indicates that the system is controllable. However, by Theorem 1, $\sigma(W) = \{-1\}$, and $E_1 = e^{Ah} - B(h)$. It can be calculated that $\sigma(E_1) = \{0\}$. By Definitions 1–2, one has $v^1_1 = [0, 0, 1]$, $v^2_1 = [0, 1, 0]$, $\xi^1_{11} = [0, 1]$, $\xi^2_{11} = [-20, \ast]$. Therefore, the eigenvectors of $\Phi_s$ corresponding to the eigenvalue 0 are $\eta^1_{11} = v^1_1 \otimes \xi^1_{11} = [0, 0, 0, 0, 0, 1]$, and $\eta^2_{11} = v^1_1 \otimes \xi^2_{11} + v^2_1 \otimes \xi^1_{11} = [0, 0, 0, 1, -20, \ast]$. It is obvious that $\eta^1_{11}(\Delta \otimes B(h)) = 0$ and $\eta^2_{11}(\Delta \otimes B(h)) = 0$. If the condition in Theorem 1 is necessary, it means that the system is completely uncontrollable, which leads to a contradiction.

Remark 3: In [25], it was revealed that the controllability of an LTI system is affected by network topology, node dynamics, external control inputs, and inner couplings. Theorem 1 further develops the condition for the corresponding sampled-data system, where the subsystem $(A + \lambda_i H, B)$ in [25] is replaced by $(e^{Ah} + \lambda_i H, B(h))$. Therefore, it can be explicitly found that the sampling period $h$ also has effects on the controllability of the networked sampled-data system and is coupled with other network-related factors.

When the network topology matrix is diagonalizable, an easily-verified condition is given below.

Theorem 2: Assume that $W$ is diagonalizable, the networked sampled-data system (4), (5) is controllable if the following conditions hold simultaneously:

1) $(W, \Delta)$ is controllable;
2) $(E_i, B(h))$ is controllable for every $i = 1, 2, \ldots, N$;
3) If $\theta \in \mathbb{C}$ is a common eigenvalue of $E_{k_1}, E_{k_2}, \ldots, E_{k_q}$, $1 < q \leq N$, then $(v_{k_1} \otimes \xi_{k_1} + v_{k_2} \otimes \xi_{k_2} + \cdots + v_{k_q} \otimes \xi_{k_q})(\Delta \otimes B(h)) \neq 0$ for all $\xi_j \in M(\theta E_j)$, with $j = k_1, \ldots, k_q$, and $[\xi_{k_1}, \ldots, \xi_{k_q}] \neq 0$.

Proof: According to the PBH rank condition for DLTI systems, system (4), (5) is controllable if $[sI_{Nn} - \Phi_s, \Psi_s]$ is of full row rank for all $s \in \mathbb{C}$. Since all the Jordan blocks of $W$ are 1-D, denote the $N$ linearly independent eigenvectors of $W$ by $v_1, v_2, \ldots, v_N$, and

$$T = [v_1, \ldots, v_N]^T, \quad T\Delta = [d_1, \ldots, d_N]^T$$

where $v_i \in \mathbb{C}^{1 \times N}$, $d_i \in \mathbb{C}^{1 \times N}, i = 1, \ldots, N$. Then, make a similarity transformation:

$$[sI_{Nn} - \Phi_s, \Psi_s]$$

$$= [sI_{Nn} - I_N \otimes e^{Ah} - W \otimes H(h), \Delta \otimes B(h)]$$

$$= (T^{-1} \otimes I_n)[sI_{Nn} - \mathcal{F}, (T\Delta) \otimes B(h)] [T \otimes I_n 0 \quad 0 \quad I_{Np}]$$

$$= (T^{-1} \otimes I_n)[sI_{Nn} - \mathcal{F}, \mathcal{G}] [T \otimes I_n 0 \quad 0 \quad I_{Np}]$$

where

$$[sI_{Nn} - \mathcal{F}, \mathcal{G}] = \begin{bmatrix} sI_n - E_1 & \cdots & G_1 \\ & \ddots & \vdots \\ & & sI_n - E_N & G_N \end{bmatrix}$$

$$G_i = d_i \otimes B(h), \quad i = 1, 2, \ldots, N.$$ 

Since $T$ is invertible, the rank of $[sI_{Nn} - \Phi_s, \Psi_s]$ equals to the rank of $[sI_{Nn} - \mathcal{F}, \mathcal{G}]$. Then the proof can be completed by showing: If $(\mathcal{F}, \mathcal{G})$ is not controllable, it will lead to the contradiction of one of the conditions in Theorem 2.

Firstly, consider the case that $(E_k, G_k)$ is not controllable for some $k \in \{1, 2, \ldots, N\}$. If $d_k = 0$, $v_k[\lambda_k I - W, \Delta] = [0, d_k] = 0$, which implies that $(W, \Delta)$ is uncontrollable and condition (1) does not hold. Then, assume that $d_k \neq 0$. If rank([$sI_n - E_k$, $d_k \otimes B(h)$]) = rank([$sI_n - E_k$, $B(h)$]) < $n$, condition (2) does not hold.

Finally, discuss the last possibility that $(E_i, G_i)$ is controllable for every $i = 1, 2, \ldots, N$, but there exists some $\theta$ being a common eigenvalue of $E_{k_1}, \ldots, E_{k_q}$, $1 < q \leq N$, making $[\theta I_{Nn} - \mathcal{F}, \mathcal{G}]$ not of full row rank. Then, there exists some $\xi = [\xi_{k_1}, \ldots, \xi_{k_q}], \xi_j \in M(\theta E_j), j = k_1, \ldots, k_q$, satisfying

$$[\theta I_{Nn} - E_{k_1} \otimes d_{k_1} \otimes B(h)] = 0, \quad (\xi_{k_1} \otimes d_{k_1} \otimes B(h)) + \cdots + \xi_{k_q} (d_{k_q} \otimes B(h))] = 0.$$ 

Thus, condition 3) is contradicted.

Remark 4: To verify the state controllability of the networked sampled-data system by Theorem 1, the computational complexity is no more than $O(N^4 + n^4 N + G^3 n^3)$, which is much lower than $O(N^4 n^4)$ of the Kalman criterion or the PBH rank condition (consistent with the result in [25, Remark 1]). As for Theorem 2, it is more intuitive that the computational complexity of conditions 1–3) is no more than $O(N^4 + n^4 N + G^3 n^3)$, respectively. Since each eigenvalue only needs to be checked once by condition 2) or condition 3), the actual computational complexity can be even lower. In addition, the steps such as the eigenspace decomposition of state matrices and the verification of subsystems can be performed in parallel in practical applications. Therefore, the proposed criteria can reduce the computational burden.

Note that Theorem 1 can also verify the networked sampled-data system with a diagonalizable topology matrix. However, in any case, it requires calculating the whole eigenspace of $Nn$-dimension, and multiplying eigenvectors with the input matrix $\Delta \otimes B(h) \in \mathbb{R}^{Nn \times Np}$. Therefore, the operations are still high-dimensional in the verification process. By Theorem 2, $Nn$-dimensional vector operations are involved only when there are common eigenvalues [corresponding to condition 3)]. For noncommon eigenvalues, the controllability can be checked just by conditions 1) and 2), which only involve vector operations.

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of N-dimensional and n-dimensional. The following example illustrates the utilization of Theorem 2.

**Example 3:** Consider a networked sampled-data system consisting of three identical nodes, where A, B, Δ, h are the same as that in Example 1, so are \( e^{Ah} \) and B(h), and \( w_{12} = w_{21} = w_{23} = w_{12} = 1, HC = I_2 \). It is easy to compute that \( \sigma(W) = \{0, 1.4142, -1.4142\} \), then

\[
E_1 = e^{Ah} = \begin{bmatrix} 1.1052 & 0 \\ 0.1105 & 1.1052 \end{bmatrix}
\]

\[
E_2 = e^{Ah} + 1.4142 \mathcal{H}(h) = \begin{bmatrix} 1.2539 & 0 \\ 0.1181 & 1.2539 \end{bmatrix}
\]

\[
E_3 = e^{Ah} - 1.4142 \mathcal{H}(h) = \begin{bmatrix} 0.9564 & 0 \\ 0.1030 & 0.9564 \end{bmatrix}
\]

Since W is diagonalizable, the controllability of the networked sampled-data system can be tested by Theorem 2.

Obviously, \( (W, \Delta) \) is controllable. Next, inspect the controllability of \((E_1, B(h))\), \((E_2, B(h))\), and \((E_3, B(h))\). By simple calculation, it shows that \( \forall s \in \mathbb{C} \),

\[
\text{rank}([sI_N - E_1, B(h)]) = \begin{cases} 2 & \text{if } \theta_k E_1 - E_k, d_k \otimes B(h) \text{ not of full row rank when } s = \theta_k \end{cases}
\]

\[
\text{rank}([sI_N - E_2, B(h)]) = \begin{cases} 2 & \text{if } \theta_k E_2 - E_k, d_k \otimes B(h) \text{ not of full row rank when } s = \theta_k \end{cases}
\]

\[
\text{rank}([sI_N - E_3, B(h)]) = \begin{cases} 2 & \text{if } \theta_k E_3 - E_k, d_k \otimes B(h) \text{ not of full row rank when } s = \theta_k \end{cases}
\]

Therefore, one has \( \theta_k E_k - E_k, d_k \otimes B(h) \text{ not of full row rank when } s = \theta_k \), hence \( \theta_k E_1, E_2, E_3, \text{ not of full row rank} \), and \( \theta_k E_k, d_k \otimes B(h) \text{ not of full row rank when } s = \theta_k \).

Remark 5: For the networked system with an undirected (or bidirectional) topological structure, its \( W \) is a real symmetric matrix and thus can be diagonalized. Therefore, the conditions in Theorem 2 can be applied to the undirected (or bidirectional) networked sampled-data systems. In addition, many other topological structures can also be represented by diagonalizable W, such as cycles discussed in the following section.

The system is controllable according to Theorem 2.

**Corollary 1:** Assume that \( W \) is diagonalizable and \( E_j \) is nonsingular for every \( i = 1, \ldots, N \). The networked sampled-data system (4), (5) is controllable if and only if conditions 1), 2), and 3) in Theorem 2 hold simultaneously.

**Proof:** The sufficiency part of the proof has been shown as the proof of Theorem 2.

**Necessity:** If \( \Phi_N \) is nonsingular, system (4), (5) is controllable only if \([sI_N - W, \mathcal{G}] \) is of full row rank.

If \( (W, \Delta) \) is uncontrollable, there exists some \( \lambda_k \in \sigma(W), k \in \{1, \ldots, N\} \) and its corresponding eigenvector \( v_k \), satisfying \( v_k \Delta = d_k = 0 \). When the geometric multiplicity of \( \lambda_k \) is 1, let \( \theta_k \in \sigma(E_k), E_k = e^{Ah} + \lambda_k \mathcal{H}(h) \), then rank(\( \lambda_k I_n - E_k, d_k \otimes B(h) \)) = rank(\( \lambda_k I_n - E_k \)) < m, making \( [sI_N - W, \mathcal{G}] \) is of full row rank when \( s = \theta_k \). When the geometric multiplicity of \( \lambda_k \) is \( q > 1 \), it has \( q \) linearly independent eigenvectors \( v_{k_1}, v_{k_2}, \ldots, v_{k_q} \), where \( k_i \in \{1, \ldots, N\} \), \( i = 1, \ldots, q \). Assume that there exists some \( v_k = a T_k, a = [a_1, a_2, \ldots, a_q] \in \mathbb{C}^{1 \times q}, a \neq 0, T_k = [v_{k_1}^\top, v_{k_2}^\top, \ldots, v_{k_q}^\top]^\top, D_k = [d_{k_1}, \ldots, d_{k_q}]^\top \), such that \( v_k \Delta = a T_k \Delta = a D_k = 0 \). It is obvious that \( E_{k_1} = \ldots = E_{k_q} = e^{Ah} + \lambda_k \mathcal{H}(h) \). Let \( \theta_k \in \sigma(E_{k_1}) \), \( \xi_{k_i} \in M(\theta_k E_{k_i}), \) denote \( \xi = a \otimes \xi_{k_i} \), one has

\[
\begin{bmatrix}
\theta_k I_N - E_{k_1} & d_{k_1} \otimes B(h) \\
\vdots & \vdots \\
\theta_k I_N - E_{k_q} & d_{k_q} \otimes B(h)
\end{bmatrix} = [0, (a D_k) \otimes \xi_{k_i} B(h)]]
\]

which also makes \([sI_N - W, \mathcal{G}] \) not of full row rank when \( s = \theta_k \).

If \( (E_k, B(h)) \) is uncontrollable for some \( k \in \{1, \ldots, N\} \), then \( \theta_k E_k - E_k, d_k \otimes B(h) \) and furthermore \( \theta_k I_N - W, \mathcal{G} \) are not of full row rank.

Finally, consider the case that \( \theta \) is a common eigenvalue of \( E_{k_1}, \ldots, E_{k_q} \), \( 1 < q \leq N \) and \( k_1, \ldots, k_q \in \{1, \ldots, r\} \). Assume that \( \exists \xi_{k_1} \in M(\theta E_{k_1}), j = k_1, \ldots, k_q \), such that \( (v_{k_1} \otimes \xi_{k_1} + v_{k_2} \otimes \xi_{k_2} + \ldots + v_{k_q} \otimes \xi_{k_q})(\Delta \otimes B(h)) = 0 \). Let \( \eta = [\eta_1, \ldots, \eta_N] \), where \( \eta_j \in \mathbb{C}^{1 \times n}, \eta_j = \xi_{j_1} \) if \( j \in \{k_1, \ldots, k_q\} \), otherwise \( \eta_j = 0 \). It follows that

\[
\begin{bmatrix}
\theta I_N - W, \mathcal{G} \\
\vdots \\
\theta I_N - W, \mathcal{G}
\end{bmatrix} = [0, (v_{k_1} \otimes \xi_{k_1} + \ldots + v_{k_q} \otimes \xi_{k_q})(\Delta \otimes B(h)))] = 0.
\]

Therefore, \([sI_N - W, \mathcal{G}] \) is not of full row rank.

The following example intuitively shows the efficiency of Corollary 1 to identify the uncontrollability of the networked sampled-data system.

**Example 4:** Reconsider the system in Example 3, but let \( HC = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \).

Then one has

\[
E_1 = e^{Ah} = \begin{bmatrix} 1.1052 & 0 \\ 0.1105 & 1.1052 \end{bmatrix}
\]

\[
E_2 = e^{Ah} + 1.4142 \mathcal{H}(h) = \begin{bmatrix} 1.2539 & 0 \\ 0.1181 & 1.2539 \end{bmatrix}
\]

\[
E_3 = e^{Ah} - 1.4142 \mathcal{H}(h) = \begin{bmatrix} 0.9564 & 0 \\ 0.1030 & 0.9564 \end{bmatrix}
\]

Since W is diagonalizable and 0 is not an eigenvalue of \( E_1, E_2, \) or \( E_3 \), the controllability of the networked sampled-data system can be tested by Corollary 1.

Note that 1.052 is a common eigenvalue of \( E_1, E_2, \) and \( E_3 \), and \( \xi_1 = [1, 0], \xi_2 = [0.7939, 0], \xi_3 = [0.6922, 1], v_1 = [-1, 0, 1], v_2 = [1, 1.4142, 1], v_3 = [-1, -1.4142, 1] \). Then, take \( \eta = v_1 \otimes a_1 \xi_1 + v_2 \otimes a_2 \xi_2 + v_3 \otimes a_3 \xi_3 \). When \( a_1 = 0.7939, \)
where \( a_2 = 1, a_3 = 0, \eta(\Delta \otimes B(h)) = 0 \). According to condition (3) of Corollary 1, the system is uncontrollable.

Some necessary conditions are listed as follows.

**Corollary 2:** Assume that \( 0 \notin \sigma(E_i) \) for every \( i = 1, \ldots, r \). The networked sampled-data system (4), (5) is controllable only if \((W, \Delta)\) and \((E_i, B(h))\), \( i = 1, 2, \ldots, r \) are all controllable.

**Proof:** Corollary 2 is induced from the necessity part of the proof of Corollary 1.

**Corollary 3:** Assume that \( W \) is singular, and \( 0 \notin \sigma(E_i) \) for every \( i = 1, \ldots, r \). The networked sampled-data system (4), (5) is controllable only if \((e^{Ah}, B(h))\) is controllable.

**Proof:** If \( W \) is singular, there exists some \( D \) has to be controllable to ensure the controllability of \((e^{Ah}, B(h))\). Since \( E_k = e^{Ah} \), the controllability of \((e^{Ah}, B(h))\) is necessary.

Corollary 3 can be used to effectively identify the uncontrollability of the whole networked sampled-data system by its node dynamics independently, which is demonstrated by the following example. In this case, the uncontrollability of the whole networked system can be diagnosed even if the information about the precise network topology and inner couplings is unknown. Note that Corollary 3 is not a sufficient condition, since the controllability of the system (4), (5) is determined by multiple coupled factors. Even if \((e^{Ah}, B(h))\) is controllable, \( W \) may have another nonzero eigenvalue \( \lambda_i \), such that \( \theta_i^2 I_1 - E_i, B(h) \) is not of full row rank for some \( \theta_i^2 \in \sigma(E_i), i \in \{1, \ldots, r\} \).

**Example 5:** Consider a networked system consisting of three connected identical nodes with a chain structure, where \( w_{21} \) and \( w_{32} \) are nonzero.

\[
A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]

and the sampling period is \( h = 0.1 \). It is obvious that the eigenvalue of \( W \) is 0 and \( E_1 \) is nonsingular.

\[
E_1 = e^{Ah} = \begin{bmatrix} 1.1052 & 0 \\ 0.1105 & 1.1052 \end{bmatrix}, \quad B(h) = \begin{bmatrix} 0 & 0 \\ 0 & 0.1052 \end{bmatrix}
\]

Then, it can be calculated that

\[
\text{rank}([sI_2 - e^{Ah}, B(h)]) = \text{rank}\left(\begin{bmatrix} s - 1.1052 & 0 \\ -0.1105 & s - 1.1052 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0.1052 \end{bmatrix}\right) = 1 < 2
\]

when \( s = 1.1052 \). By Corollary 3, the networked sampled-data system is uncontrollable.

**Remark 6:** The pathological sampling of \((A, B)\) will not inevitably result in the loss of controllability of the whole networked system, which will be shown in Example 6. However, Corollary 3 reveals that for the networked sampled-data system with singular \( W \) and nonsingular \( \Phi_s \), the pathological sampling of single node system \((A, B)\) cannot be eliminated in the network. Especially, if \( W \) only has zero eigenvalues (e.g., the tree structure, including chains and stars), the pathological sampling of \((A, B)\) will always result in the uncontrollability of the whole system.

**Example 6:** Reconsider the networked sampled-data system in Fig. 1, but extend the topology to three nodes in a cycle structure, where \( \delta_1 = 1 \), \( w_{13} = w_{21} = w_{32} = 1 \), and

\[
e^{Ah} = \begin{bmatrix} -23.1407 & 0 \\ 0 & -23.1407 \end{bmatrix}, \quad B(h) = \begin{bmatrix} -12.0703 & 0 \\ 0 & -12.0703 \end{bmatrix}.
\]

The sampling is pathological about \( A \), i.e., the controllability of the single node system is lost during the control sampling because

\[
\text{rank}([sI_2 - e^{Ah}, B(h)]) = \text{rank}\left(\begin{bmatrix} s + 23.1407 & 0 \\ 0 & s + 23.1407 \end{bmatrix}\begin{bmatrix} -12.0703 & 0 \\ 0 & -12.0703 \end{bmatrix}\right) = 1
\]

when \( s = -23.1407 \). However, one can find that the whole networked sampled-data system is controllable by verifying Corollary 1 as follows.

The eigenvalues of \( W \) are \( \lambda_1 = 1, \lambda_2 = -0.5 - 0.866i, \lambda_3 = -0.5 + 0.866i \). Then

\[
H(h) = \begin{bmatrix} -12.0703 & 12.0703 \\ -12.0703 & -12.0703 \end{bmatrix}
\]

\[
E_1 = e^{Ah} + \lambda_1 H(h) = \begin{bmatrix} -35.2110 & 12.0703 \\ -12.0703 & -35.2110 \end{bmatrix}
\]

\[
E_2 = e^{Ah} + \lambda_2 H(h)
\]

\[
e^{Ah} + \lambda_3 H(h) = \begin{bmatrix} -17.11 - 10.45i & -6.035 - 10.45i \\ 6.035 + 10.45i & -17.11 - 10.45i \end{bmatrix}
\]

It is easy to check that \((W, \Delta)\) is controllable, and \(\text{rank}([sI_2 - E_1, B(h)])\), \(\text{rank}([sI_2 - E_2, B(h)])\), and \(\text{rank}([sI_2 - E_3, B(h)])\) are all equal to 2 for \( \forall s \in \mathbb{C} \), and there are no common eigenvalues between \( E_1, E_2, \) and \( E_3 \).

**Corollary 4:** Assume that \( 0 \notin \sigma(E_i) \) for every \( i = 1, \ldots, r \). If \((W, \Delta)\) is uncontrollable, and \( \lambda_{k_1}, \ldots, \lambda_{k_s} \) are uncontrollable modes, corresponding to eigenvectors \( v_{k_1}, \ldots, v_{k_s} \), respectively, \( k_1, \ldots, k_q \in \{1, \ldots, r\} \). Then, the dimension of controllable subspace of system (4), (5) is no more than \( N_f - \sum_{j=1}^q \text{rank}(\Phi_{s_j}) \).

**Proof:** For the state matrix \( \Phi_s \) is nonsingular, the controllable subspace of \((\Phi_s, \Psi_s)\) is the column space of \( \mathcal{C} = [\Psi_s, \Phi_s \Psi_s, \ldots, \Phi_s^{N_f-1} \Psi_s] \), i.e., \( R(\mathcal{C}) \). According to Lemma 2, for \( j = 1, \ldots, q \), assume that \( v_{k_j} \in \Delta = 0 \), and \( E_{k_j} \) has \( \mathcal{D}(E_{k_j}) \) independent eigenvectors, namely, \( \xi_{k_1}^j, \ldots, \xi_{k_l}^j, \), \( \mathcal{E}_{k_j} \) is an independent eigenvector of \( \Phi_s \), and \( \xi_{k_j}^j, \Phi_s \Psi_s = 0 \).
for every $l = 1, \ldots, D(E_{k})$. Therefore, $\eta_{k,l}^1 \in N^T(\mathcal{C})$ for every $l = 1, \ldots, D(E_{k})$. Consider all these eigenvectors of uncontrollable modes of $W$, i.e., $v_{k,1}, \ldots, v_{k}^1$, one has

$$\sum_{j=1}^{q} D(E_{k_j}) \leq \dim(N^T(\mathcal{C})) = N - \dim(R(\mathcal{C})).$$

Then, it comes to the result that the dimension of the controllable subspace of the system (4), (5) is no more than $N - \sum_{j=1}^{q} D(E_{k_j})$.

IV. SAMPLED-DATA SYSTEMS WITH SPECIAL TOPOLOGIES

In this section, the topology of directed trees and cycles are considered, with some easy-verified controllability conditions developed.

A. Trees

Consider a networked sampled-data system, where the topology is a directed tree. For example, a three networked sampled-data system with six nodes is shown in Fig. 2(a), where nodes 1, 2, 5 are under control. The topology matrix can be written in lower triangular form with all the elements on the diagonal being zero. Then, it is obvious that all the eigenvalues of the topology matrix are zero. Denote

$$\sigma(\Phi_s) = \sigma(e^{Ah}) = \{\sigma_1, \ldots, \sigma_q\}, \quad 1 \leq q \leq n.$$ 

It is easy to find that $0 \notin \sigma(\Phi_s)$, for $e^{Ah}$ is always nonsingular with arbitrary $A$ and $h$. Therefore, the controllability conditions derived from the PBH rank condition become necessary and sufficient for systems with tree structures.

Note that a chain network is a tree network with only one leaf, and a star network is also a tree network, composed of one root node and multiple leaf nodes. Examples of networked sampled-data systems with chain and star topology are shown in Fig. 2(b) and (c), respectively. The controllability of networked sampled-data systems with these two types of topologies is analyzed as follows.

1) Chains: Consider a networked sampled-data system with a directed chain topology, where the topology matrix is in the form of

$$W_{\text{chain}} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ w_{2,1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & w_{N,N-1} & 0 \end{bmatrix}$$

where $w_{i,i-1}$ is the weight of edge $\{i-1, i\}$, and $w_{i,i-1} \neq 0$ for $i = 2, \ldots, N$. It is straightforward that the network topology is controllable if the first node is under control, i.e., $(W_{\text{chain}}, \Delta)$ is controllable if $\Delta = \Delta_1$,

$$\Delta_1 = \text{diag}(1, 0, \ldots, 0).$$

Recall the notions of the Jordan chain and the generalized Jordan chain. Let $v_{1,1}^1, v_{1,2}^1, \ldots, v_{1,N}^1$ be a Jordan chain of $W_{\text{chain}}$ about 0, and $\xi_1^1, \ldots, \xi_\gamma^1$ be a generalized Jordan chain of $e^{Ah}$ about $H(h)$ corresponding to $\sigma_i$, $i = 1, \ldots, q$. Then, an easier-to-verify controllability condition can be obtained for networked sampled-data systems with chain topology $(W_{\text{chain}}, \Delta_1)$.

**Corollary 5:** The networked sampled-data system with chain topology $(W_{\text{chain}}, \Delta_1)$ is controllable if and only if $n_{B(h)} \neq 0$, for $\forall \eta \in V_i$ and $\gamma \neq 0$, where $V_i = \text{span}\{\xi_1^1, \ldots, \xi_\gamma^1\}$, $\beta_i = \min\{N, \gamma_i\}$, $i = 1, \ldots, q$. Specially, if $\sigma_i = \gamma_i = 1$, $V_i = \text{span}\{\xi_1^1, \ldots, \xi_\gamma^1\}$.

**Proof:** It can be calculated that the left Jordan chain of $W_{\text{chain}}$ with respect to 0 is $v_{1,1}^1 = e_1$, $v_{1,1}^2 = (1/w_{2,1}) e_2$, $\ldots$, $v_{1,1}^N = (1/\prod_{k=2}^{N} w_{k,k-1}) e_N$. The generalized Jordan chain of $e^{Ah}$ about $H(h)$ corresponding to $\sigma_i$ is $\xi_1^i, \ldots, \xi_\gamma^i$, $i = 1, \ldots, q$. Similar to the proof of Theorem 1, the eigenvectors of $\Phi_s$, about $\sigma_i$ are $\eta_i^1 = v_{1,1}^1 \otimes \xi_1^i, \ldots, \eta_i^{\beta_i} = v_{1,1}^1 \otimes \xi_\beta_i^i + \cdots + v_{1,1}^1 \otimes \xi_1^i$, where $\beta_i = \min\{N, \gamma_i\}$. According to the PBH rank condition, the networked sampled-data system is controllable if and only if $\forall i \in \{1, \ldots, q\}$, $\forall \eta \in \text{span}\{\eta_i^1, \ldots, \eta_i^{\beta_i}\}$ and $\eta \neq 0$, $\eta(\Delta_1 \otimes B(h)) \neq 0$. Since $\Delta_1 = \text{diag}(1, 0, \ldots, 0)$, the above condition is equivalent to that $\forall \eta \in \text{span}\{\eta_i^1, \ldots, \eta_i^{\beta_i}\}$ and $\eta \neq 0, n_{B(h)} \neq 0$. Specially, if the geometric multiplicity of some eigenvalue of $e^{Ah}$ is greater than 1, the test needs to consider all the general Jordan chains corresponding to this eigenvalue.

2) Stars: Consider a networked sampled-data system with a directed star topology, where the topology matrix is in the form of

$$W_{\text{star}} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ w_{2,1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ w_{N,1} & \cdots & 0 & 0 \end{bmatrix}$$

where $w_{i,1}$ denotes the weight of edge $\{i, 1\}$, and $w_{i,1} \neq 0$ for $i = 2, \ldots, N$. It can be simply found that the network topology of a star network is controllable if the root and at least $N - 2$
leaf nodes are under control. Without loss of generality, let node 2 be not under control, i.e., \((W_{\text{star}}, \Delta)\) is controllable if \(\Delta = \Delta_2\).

\[
\Delta_2 = \text{diag}\{1, 0, 1, \ldots, 1\}.
\]

Then another easier-to-verify controllability condition can be obtained for networked sampled-data systems with star topology \((W_{\text{star}}, \Delta_2)\).

**Corollary 6:** The networked sampled-data system with star topology \((W_{\text{star}}, \Delta_2)\) is controllable if and only if \(\eta B(h) \neq 0\), for \(\forall \eta \in V_i\) and \(\eta \neq 0\), where \(V_i = \text{span}\{\xi_1^i, \ldots, \xi_{\beta_i}^i\}\), \(\beta_i = \min\{2, \gamma_i\} \), \(i = 1, \ldots, q\). Specially, if \(\sigma_i = \ldots = \sigma_{i_1}, l > 1\), \(V_i = \text{span}\{\xi_1^i, \ldots, \xi_{\beta_i}^i\}\).

**Proof:** The generalized left eigenvectors of \(W_{\text{star}}\) with respect to 0 are \(v_1^i = e_1, v_2^i = (1/w_{2,1})e_2, v_3^i = v_2^i - (w_{3,1}/w_{2,1})e_2, \ldots, v_{\beta_i - 1}^i = v_{\beta_i - 2}^i - (w_{\beta_i,1}/w_{\beta_i - 1,1})e_2\). For \(i = 1, \ldots, q\), let the generalized left Jordan chain of \(e^{Ah}\) about \(H(h)\) corresponding to \(\sigma_i\) be \(\xi_1^i, \ldots, \xi_{\beta_i}^i\).

**Necessity:** If \(\gamma_i = 1\), the eigenvectors of \(\Phi_a\) about \(\sigma_i\) are \(\eta_1^i = v_1^i \otimes \xi_1^i, \eta_2^i = v_2^i \otimes \xi_1^i, \ldots, \eta_{\beta_i}^i = v_{\beta_i}^i \otimes \xi_1^i\). According to the PBH rank condition, the networked sampled-data system is controllable only if \(\forall \eta \in \text{span}\{\eta_1^i, \ldots, \eta_{\beta_i}^i\}\) and \(\eta \neq 0, \eta (\Delta_2 \otimes B(h)) \neq 0\). Since \(\Delta_2 = \text{diag}\{1, 0, 1, \ldots, 1\}\), the above condition is equivalent to \(\xi_1^i \otimes B(h) \neq 0\). Otherwise, if \(\gamma_i > 1\), the eigenvectors of \(\Phi_a\) about \(\sigma_i\) are \(\eta_1^i = v_1^i \otimes \xi_1^i, \ldots, \eta_{\beta_i - 1}^i = v_{\beta_i - 2}^i \otimes \xi_1^i, \eta_{\beta_i}^i = v_{\beta_i}^i \otimes \xi_1^i\). Then the networked sampled-data system is controllable only if \(\forall \eta \in \text{span}\{\eta_1^i, \ldots, \eta_{\beta_i}^i\}\) and \(\eta \neq 0, \eta (\Delta_2 \otimes B(h)) \neq 0\). Since \(\Delta_2 = \text{diag}\{1, 0, 1, \ldots, 1\}\), the above condition is equal to \(\forall \eta \in \text{span}\{\xi_1^i, \xi_2^i\}\) and \(\eta \neq 0, \eta B(h) \neq 0\). 

**Sufficiency:** The networked sampled-data system with star structure is controllable if any eigenvector \(\eta_1^i\) of \(\Phi_a\) with respect to \(\sigma_i, i = 1, \ldots, q\) satisfies \(\eta_1^i (\Delta_2 \otimes B(h)) \neq 0\). If \(\gamma_i = 1, j = 1, \ldots, N - 1\); If \(\gamma_i > 1, j = 1, \ldots, N\). The remained proof is similar to the necessity part. 

Specially, if the geometric multiplicity of some eigenvalue of \(e^{Ah}\) is greater than 1, the test needs to consider all the general Jordan chains corresponding to this eigenvalue. In summary, the result in Corollary 6 holds.

**Corollary 7:** Assume that \(\gamma_i = 1\) for every \(i = 1, \ldots, q\), and \(h\) is nonpathological about \(A\). The networked sampled-data system with star topology \((W_{\text{star}}, \Delta_2)\) is controllable if and only if \((A, B)\) is controllable.

**Proof:** Corollary 7 can be proved based on the proof of Corollary 6 and the nonpathological sampling condition of single systems.

### B. Cycles

Consider a networked sampled-data system with a directed cycle topology, where the topology matrix is in the form of

\[
W_{\text{cycle}} = \begin{bmatrix}
0 & 0 & \cdots & w_{1,N} \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & w_{N,N-1} & 0
\end{bmatrix}
\]

The networked sampled-data system with cycle topology \((W_{\text{cycle}}, \Delta)\) is controllable if \(\Delta = \Delta_2\).

\[
\Delta_2 = \text{diag}\{1, 0, 1, \ldots, 1\}.
\]

with \(w_{1,N} \neq 0\) and \(w_{i,i-1} \neq 0\) for \(i = 2, \ldots, N\). It can be learned that one external input added to an arbitrary node is enough for the controllability of the network topology. Without loss of generality, assume that the external input is added to the first node, i.e., \((W_{\text{cycle}}, \Delta)\) is controllable if \(\Delta = \Delta_1\). For instance, a cycle networked sampled-data system with five nodes is shown in Fig. 3, where only node 1 is under control.

The characteristic polynomial of \(W_{\text{cycle}}\) is

\[
|\lambda I_N - W_{\text{cycle}}| = \lambda^N - w_{1,1}N \prod_{i=2}^{N} w_{i,i-1}.
\]

Denote \(w_{1,1}N \prod_{i=2}^{N} w_{i,i-1}\) by \(\bar{w}\). Since \(\bar{w}\) is a real number, the solutions of \(|\lambda I_N - W_{\text{cycle}}| = 0\) are

\[
\lambda_i = \begin{cases}
\sqrt{-\bar{w}}e^{i\theta_i}, & \theta_i = \frac{2i\pi}{N}, \text{ if } \bar{w} > 0 \\
\sqrt{-\bar{w}}e^{i\theta_i}, & \theta_i = \frac{(2i-1)\pi}{N}, \text{ if } \bar{w} < 0
\end{cases}, \quad i = 1, \ldots, N.
\]

Therefore, \(N\) eigenvalues of \(W_{\text{cycle}}\) are different from each other, i.e., all the Jordan blocks of \(W_{\text{cycle}}\) are 1-D.

**Corollary 8:** The networked sampled-data system with cycle topology \((W_{\text{cycle}}, \Delta_1)\) is controllable if conditions 2) and 3) in Theorem 2 hold simultaneously.

**Proof:** Since \(W_{\text{cycle}}\) is diagonalizable, Corollary 8 can be derived from the result in Corollary 1.

### V. NETWORKED SAMPLED-DATA SYSTEMS WITH SPECIAL DYNAMICS

In this section, two types of special node dynamics are considered: 1-D dynamics and self-loop-state dynamics.

#### A. One-Dimensional Dynamics

Consider a networked system with 1-D dynamics, whose state matrix and input matrix are reduced to scalars denoted by \(a \neq 0\) and \(b\), respectively. Apparently, \(HC\) is also a scalar, denoted by \(c\) and \(a, b, c \in \mathbb{R}\). The model of networked systems with 1-D node dynamics is a special variant of the networked system introduced in Section II-C. In the continuous-time system (2),

\[
\Phi = aI_N + c W, \quad \Psi = b\Delta
\]
and in the related sampled-data system (4),
\[ \Phi_s = e^{ah}I_N + \frac{c}{a}(e^{ah} - 1)W, \quad \Psi_s = \frac{b}{a}(e^{ah} - 1)\Delta. \quad (10) \]

A controllability condition for the system (4), (10) can be derived based on Theorem 2 as follows.

**Corollary 9:** The networked sampled-data system with 1-D node dynamics (4), (10) is controllable if \( W, \Delta \) is controllable and \( b \neq 0, c \neq 0 \).

**Proof:** Similar to the derivation of Theorem 2, system (4), (10) is controllable if
\[
\begin{align*}
[sI_N - e^{ah}I_N - \frac{c}{a}(e^{ah} - 1)W, & \quad \frac{b}{a}(e^{ah} - 1)\Delta] \\
= T^{-1} \left[ \left(s - e^{ah}\right)I_N - \frac{c}{a}(e^{ah} - 1)J, bT\Delta \right] \\
= T^{-1} \begin{bmatrix} T & 0 \\
0 & \frac{1}{a}(e^{ah} - 1)I_N \end{bmatrix} A \\
= T^{-1}(sI_N - F, G) \begin{bmatrix} T & 0 \\
0 & \frac{1}{a}(e^{ah} - 1)I_N \end{bmatrix}
\end{align*}
\]
is of full row rank, where \( J \) is the Jordan normal form of \( W \), and \( T = \left[(v_1^1)^T, \ldots, (v_1^r)^T, (v_2^2)^T, \ldots, (v_r^r)^T, \ldots, (v_{10}^{10})^T \right]^T \)
satisfying \( W = T^{-1}JT \). \( F = \text{diagblock}\{F_1, \ldots, F_r\}, \quad G = [G_1^1, \ldots, G_r^r]^T \), and for \( i = 1, 2, \ldots, r \),
\[
F_i = \begin{bmatrix} E_i & \frac{c}{a}(e^{ah} - 1) & \cdots & \frac{c}{a}(e^{ah} - 1) \\
\cdots & \ddots & \ddots & \cdots \\
E_i & \cdots & \frac{c}{a}(e^{ah} - 1) & \cdots \\
\end{bmatrix} (a_i \times a_i)
\]
\[
G_i = \begin{bmatrix} bu_i^{a_i}\Delta \\
\frac{c}{a}(e^{ah} - 1) + \lambda_i \frac{c}{a}(e^{ah} - 1) \end{bmatrix} \in C.
\]

Then prove by contradiction. Assume that system (4), (10) is uncontrollable. Since \( E_1, \ldots, E_r \) are scalars, \( \sigma(\Phi_s) = \{E_1, \ldots, E_r\} \). Consider \( E_{k}, k \in \{1, \ldots, r\} \). If \( [E_kI_N - F, G] \) is not of full row rank, there exists some nonzero \( \eta = [\eta_1, \ldots, \eta_r], \) such that \( \eta G = 0 \), where \( \eta = a_i v_i^1 \in C^{1 \times a_i} \), \( a_i \) is an arbitrary nonzero complex number if \( E_i = E_{k_i} \), otherwise \( a_i = 0, i = 1, \ldots, r \). If \( a, b, c \neq 0, \) \( \forall i, j \in \{1, \ldots, r\}, i \neq j \), \( E_i = E_j \) if and only if \( \lambda_i = \lambda_j \). Therefore, \( \eta G = 0 \) is equivalent to \( \sum_{i=1}^{r} a_i v_i^1 \Delta = 0 \), indicating that \( (W, \Delta) \) is uncontrollable.

**Remark 7:** Note that \( c \neq 0 \) in Corollary 9 is not necessary. Assume that \( c = 0 \), let \( \Delta = I_N \), system (4), (10) is still controllable. It shows that when the transmission channels are cut off, external inputs must be added to each node to maintain the controllability of the overall system. This also reveals the enhanced effects of interactions between nodes on the controllability of networked sampled-data systems.

Note that \( e^{ah} \) is always a nonzero scalar. It seems that no effects of sampling are reflected in the controllability of the system (4), (10). In fact, when the node dynamics are 1-D, the periodic sampling does not influence the controllability of the networked system, which is shown in Corollary 10. Therefore, the controllability of the networked sampled-data system can be directly inferred from its original continuous-time system.

**Corollary 10:** The networked sampled-data system with 1-D node dynamics (4), (10) is controllable if its original continuous-time system (2), (9) is controllable.

**Proof:** Proof by contradiction. If system (4), (10) is uncontrollable, there exists some \( \xi \in C^{1 \times N} \) and \( s_1 \in C \), such that \( (s_1 - e^{ah})\xi = \frac{a}{\xi}(e^{ah} - 1)\xi W + \frac{b}{a}(e^{ah} - 1)b\Delta = 0 \) simultaneously. System (2), (9) is uncontrollable if there exists some \( \xi \in C^{1 \times N} \) and \( s_0 \in C \), satisfying \( s_0\xi = \xi(aI_N + cW) \) and \( bc\Delta = 0 \) simultaneously. The latter condition is met by setting \( s_0 = a(1 - \frac{s_1 - e^{ah}}{e^{ah} - 1}) \).

**Corollary 11:** If \( 0 \neq \sigma(\Phi_s) \), the networked sampled-data system with 1-D node dynamics (4), (10) is controllable if and only if its original continuous-time system (2), (9) is controllable.

**Proof:** The sufficiency part has been proved in the proof of Corollary 10. Necessity: Since \( \sigma(\Phi_s) = \{E_1, E_2, \ldots, E_r\}, 0 \neq \sigma(\Phi_s) \) is equivalent to: \( \forall i \in \{1, \ldots, r\}, c\lambda_i \neq \frac{a e^{ah}}{1 - e^{ah}} \). In this case, if \( (s_1 - e^{ah})I_N + \xi \left(\frac{a}{\xi}(e^{ah} - 1)W + \frac{b}{a}(e^{ah} - 1)b\Delta\right) \) is not of full row rank, system (4), (10) is uncontrollable. Assume that the related continuous system (2), (9) is uncontrollable, then there exists some \( \xi \in C^{1 \times N} \) and \( s_0 \in C \) such that \( s_0\xi = \xi(aI_N + cW) \) and \( bc\Delta = 0 \) simultaneously. Let \( s_1 = 1 - \frac{a}{\xi}(1 - e^{ah}) \), it follows that \( \xi(s_1 - e^{ah})I_N + \xi \left(\frac{a}{\xi}(e^{ah} - 1)W + \frac{b}{a}(e^{ah} - 1)b\Delta\right) = \frac{a e^{ah} - 1}{a}(s_0\xi - a\xi - \xi cW), 0 = 0 \), which leads to the uncontrollability of system (4), (10).

**B. Self-Loop-State Dynamics**

Consider a networked sampled-data system with identical multidimensional self-loop-state node dynamics, i.e., \( A = I_n \). In such a networked system, there are no interactions between the internal states of each node. The dynamics will become
\[
\Phi = I_N + W \otimes HC, \quad \Psi = \Delta \otimes B. \quad (11)
\]
for the continuous-time system (2), and for the corresponding sampled-data system (4)
\[
\Phi_s = e^{ah}I_N + (e^{ah} - 1)(W \otimes HC) \]
\[
\Psi_s = (e^{ah} - 1)(\Delta \otimes B). \quad (12)
\]

**Corollary 12:** The networked sampled-data system with self-loop-state node dynamics (4), (12) is controllable if its original continuous-time system (2), (11) is controllable.

**Proof:** Similar to the proof of Corollary 10, assume that the sampled-data system (4), (12) is uncontrollable. Then there exists some \( \xi \in C^{1 \times N} \) and \( s_1 \in C \) such that \( (e^{ah} - 1)(W \otimes HC) = (s_1 - e^{ah})\xi \) and \( (e^{ah} - 1)(\Delta \otimes B) = 0 \) simultaneously. Let \( s_0 = \frac{a e^{ah}}{1 - e^{ah}} + 1 \), it follows that \( (W \otimes HC) = (s_0 - 1)\xi \) and \( (\Delta \otimes B) = 0 \) at the same time, which indicates that the corresponding continuous-time system (2), (11) is also uncontrollable.

**Corollary 13:** If \( 0 \neq \sigma(\Phi_s) \), the networked sampled-data system with self-loop-state node dynamics (4), (12) is controllable
if and only if its original continuous-time system (2), (11) is controllable.

Proof: The sufficiency part has been proved in the proof of Corollary 12. Necessity: Suppose that the continuous-time system (2), (11) is uncontrollable, then there exist some $\xi \in \mathbb{C}^{1 \times N_n}$ and $s_0 \in \mathbb{C}$ satisfying $\xi (W \otimes C) = (s_0 - 1)\xi$ and $\xi (\Delta \otimes B) = 0$ simultaneously. Let $s_1 = (e^{h} - 1)s_0 + 1$, it follows that $(s_1 - e^{h})\xi - (e^{h} - 1)\xi (W \otimes C) = (e^{h} - 1)((s_0 - 1)\xi - \xi (W \otimes C)) = 0$ and $(e^{h} - 1)\xi (\Delta \otimes B) = 0$ hold simultaneously. Therefore, the corresponding sampled-data system (4), (12) is also uncontrollable.

VI. NETWORKED MULTIRATE SAMPLED-DATA SYSTEMS

The sampling period on control channels may be different from that on transmission channels in the networked sampled-data system. According to the multiple relationships between the two sampling periods, the networked multirate sampled-data systems are divided into transmission multirate sampled-data (TMS) systems and control multirate sampled-data (CMS) systems. The controllability of these two categories of multirate sampled-data systems is discussed in this section.

A. Networked TMS Systems

Here, a networked TMS system means that all nodes are sampled periodically at the same time on both transmission channels and control channels, but the control sampling period is an integer multiple of the transmission sampling period. As illustrated in Fig. 4(a), in the TMS pattern, $h$ is the transmission sampling period and $lh$ is the control sampling period, with $l$ being a positive integer.

The dynamics of node $i$ in the networked TMS system are represented as

$$
\begin{align*}
\dot{x}_i(t) &= A x_i(t) + \sum_{j=1}^{N} w_{ij} y_j((k + l)r) + \delta_i B u_i(klh) \\
y_i((k + l)r) &= C x_i((k + l)r)
\end{align*}
$$

where $t \in \{ (k + l)r, (k + l + 1)r \}, r = 0, 1, \ldots, l - 1$ and $k \in \mathbb{N}$. The associated compact form can be written as

$$
X((k + l)r + 1)h) = \Phi_s X((k + l)r)h + \Psi_s U(klh)
$$

where $\Phi_s$ and $\Psi_s$ are the same as that in (5). Furthermore, system (14) can be transformed into a DLTI system with sampling period $lh$:

$$
X((k + 1)lh) = \tilde{\Phi}_s X((k + 1)lh) + \tilde{\Psi}_s U((k + 1)lh)
$$

with

$$
\tilde{\Phi}_s = \Phi_s^l, \tilde{\Psi}_s = (\Phi_s^{-1})^l \Psi_s.
$$

By Lemmas 1 and 2, the eigenvalues and the corresponding eigenspace of $\Phi_s$ can be obtained. Then, the eigenvalues and eigenspace of $\tilde{\Phi}_s$ can also be calculated by Lemma 3.

Lemma 3: Assume that $\sigma(\Phi) = \{ \theta_1, \ldots, \theta_p \}$, then $\sigma(\tilde{\Phi}_s) = \{ \theta_1^l, \ldots, \theta_p^l \}$. Moreover, $M(\theta_i^l) \subset M(\theta_j^r)\Phi_s$ for $i = 1, 2, \ldots, p, j = 1, 2, \ldots, p$. Specially, if $\theta_i^r$ is $\theta_i^r$ is $\theta_j^r \cup \theta_j^r$ $\cup \theta_j^r$ and $\tilde{\Phi}_s = \Phi_s, \sigma(\tilde{\Phi}_s)$ can be easily obtained according to the spectral mapping theorem. In addition, $\eta \in M(\theta_i^l)\Phi_s, \eta(\Phi_s) = (\theta_i^l)\eta$, and therefore $\eta \in M(\theta_j^r)\Phi_s$. Specially, assume that $(\theta_i^r)^l = (\theta_i^r)^l = (\theta_i^r)^l \Phi_s$ and consider $\eta \in M(\theta_i^l)\Phi_s, \eta(\Phi_s) = (\theta_i^l)\eta, i.e., \eta \in M(\theta_j^r)\Phi_s$. Thus $\tilde{\Phi}_s = (\Phi_s^l, \Phi_s^l) \subset M(\theta_j^r)\Phi_s$

Based on Lemma 3, the controllability of the networked TMS system can be verified by Corollary 14.

Corollary 14: Assume that $\Phi_s$ is nonsingular. The Networked TMS system (15), (16) is controllable only if the following conditions hold simultaneously:

1) $\forall \theta_i^r \in \sigma(\Phi_s), \sum_{c=0}^{1} (\theta_i^c)^c \neq 0$, for every $i = 1, \ldots, r$ and $j = 1, \ldots, p$;

2) $\forall \eta \in M(\theta_i^r)\Phi_s$ and $\eta \neq 0, \eta(\Delta \otimes B) \neq 0$, for every $i = 1, \ldots, r$ and $j = 1, \ldots, p$.

Proof: Corollary 14 can be proved based on Lemma 3. According to the PBH rank condition for DLTI systems, system (15), (16) is controllable only if $\forall \theta_i^r \in \sigma(\Phi_s), \sum_{c=0}^{1} (\theta_i^c)^c \neq 0$, for every $i = 1, \ldots, r$ and $j = 1, \ldots, p$, and $\eta \neq 0, \eta(\Delta \otimes B) \neq 0$, for every $i = 1, \ldots, p$. In addition, consider the case that $(\theta_i^r)^l = (\theta_i^r)^l = (\theta_i^r)^l \neq \theta$, where $q > 1, i.e., \eta(\Phi_s \Phi_s) = \eta(\Phi_s \Phi_s)$ and $\sum_{c=0}^{1} (\theta_i^c)^c \neq 0$ according to condition (1), $a_k \sum_{c=0}^{1} (\theta_i^c)^c$ can be an arbitrary complex number for $k = 1, \ldots, q$. Consequently, the condition $\sum_{c=0}^{1} (\theta_i^c)^c = 0$ holds for all $i = 1, \ldots, p$. Therefore, the proof of Corollary 14 is complete.

B. Networked CMS Systems

Here, a networked CMS system means that all nodes are sampled periodically at the same time on both transmission
channels and control channels, but the transmission sampling period is an integer multiple of the control sampling period. As shown in Fig. 4(b), in the CMS pattern, \( h \) is the control sampling period and \( lh \) is the transmission sampling period, with \( l \) being a positive integer.

The dynamics of the node in the networked CMS system are described as

\[
\begin{align*}
\dot{x}_i(t) &= Ax_i(t) + \sum_{j=1}^{N} w_{ij} H y_j(klh) + \delta_i B u_i((kl+r)h) \\
y_i(klh) &= C x_i(klh)
\end{align*}
\]

(17)

where \( t \in [(kl+r)h, (kl+r+1)h) \), \( r = 0, 1, \ldots, l-1 \) and \( k \in \mathbb{N} \). The corresponding compact form can be written as

\[
X((kl+r+1)h) = \Lambda X((kl+r)h) + \Gamma X(klh) + \Psi_s U((kl+r)h)
\]

(18)

where \( \Psi_s \) is the same as that in (5), with

\[
\Lambda = I_N \otimes e^{Ah}, \Gamma = W \otimes \mathcal{H}(lh).
\]

(19)

Furthermore, denote \( \hat{U}(kh) = [U^T(klh), U^T((kl+1)h), \ldots, U^T((kl+l-1)h)]^T \), then system (18), (19) can be transformed into a DLTI system with a sampling \( lh \):

\[
X((k+1)h) = \hat{\Phi}_s X(klh) + \hat{\Psi}_s \hat{U}(kh)
\]

(20)

with

\[
\hat{\Phi}_s = I_N \otimes e^{Ah} + W \otimes \mathcal{H}(lh)
\]

\[
\hat{\Psi}_s = \left[ \Delta \otimes e^{Ah}(lh) B(h), \ldots, \Delta \otimes e^{Ah} B((kl)h), \Delta \otimes B(h) \right].
\]

(21)

Based on Theorem 1, a sufficient condition for the controllability of the networked CMS system can be derived. Denote \( \hat{E}_1 = e^{Ah} + \lambda_1 \mathcal{H}(lh) \), \( \sigma(\hat{E}_1) = \{\hat{\theta}_1^*, \ldots, \hat{\theta}_r^*\}, r = 1, 2, \ldots, r \).

**Corollary 15:** The networked CMS system (20), (21) is controllable if \( \forall \eta \in M(\hat{\theta}_1^*, \ldots, \hat{\theta}_r^*) \) and \( \eta \neq 0 \), \( \eta \Delta \otimes e^{Ah(l-1)h} B(h), \ldots, \Delta \otimes e^{Ah} B((kl)h), \Delta \otimes B(h) \neq 0 \), for every \( i = 1, \ldots, r \) and \( j = 1, \ldots, p_i \).

**Proof:** The expression of \( \hat{\Phi}_s \) is in the same form as \( \Phi_s \), except that the sampling period of \( \hat{\Phi}_s \) is \( lh \) times that of \( \Phi_s \). Therefore, Corollary 15 can be proved based on Theorem 1.

The above corollaries show that the controllability conditions of networked multirate sampled-data systems can be simplified based on the single-rate system (4), (5). As a result, verifying a networked multirate sampled-data system by Corollaries 14 or 15 greatly reduces the computational complexity than using the PBH rank condition directly. However, it has not been proved that the controllability of system (4), (5) can ensure that system (15), (16) or system (20), (21) is controllable, and vice versa. But the above analysis has shed light on the increasing controllability of networked sampled-data systems by adjusting the multiple relationships between the sampling periods on control channels and transmission channels.

**VII. CONCLUSION**

The controllability of networked sampled-data systems is investigated. The sampling is periodic on the transmission and control channels, with single- and multirate patterns considered, respectively. Necessary or sufficient controllability conditions are developed, indicating that the controllability of networked sampled-data systems is jointly determined by external inputs, network topology, inner couplings, node dynamics, and the sampling period. Results show that the pathological sampling of single-node systems can be eliminated by an appropriate design of network topology and inner couplings. However, when the topology matrix is singular, the pathological sampling of single-node systems will inevitably lead to the loss of controllability of the whole system. For systems with 1-D or self-loop-state node dynamics, the sampling will not affect the controllability of the networked systems. In further studies, we will consider more general systems with heterogeneous node dynamics. In addition, more complex sampling patterns will be investigated. More complex and deeper network structures will be studied, and the nonpathological sampling conditions of networked systems will be further explored.

**REFERENCES**

[1] L. Xiang, F. Chen, W. Ren, and G. Chen, “Advances in network controllability,” IEEE Circuits Syst. Mag., vol. 19, no. 2, pp. 8–32, Second Quarter 2019.

[2] T. Chen and B. A. Francis, Optimal Sampled-Data Control Systems. London, U.K.: Springer, 1995.

[3] R. E. Kalman, “Canonical structure of linear dynamical systems,” Proc. Nat. Acad. Sci. USA, vol. 48, no. 4, pp. 596–600, 1962.

[4] E. Davison and S. Wang, “New results on the controllability and observability of general composite systems,” IEEE Trans. Autom. Control, vol. 20, no. 1, pp. 123–128, Feb. 1975.

[5] E. G. Gilbert, “Controllability and observability in multivariable control systems,” SIAM J. Control, vol. 1, no. 2, pp. 128–151, 1963.

[6] H. Kobayashi, H. Hanafusa, and T. Yoshikawa, “Controllability under decentralized information structure,” IEEE Trans. Autom. Control, vol. 23, no. 2, pp. 182–188, Apr. 1978.

[7] C.-T. Lin, “Structural controllability,” IEEE Trans. Autom. Control, vol. 19, no. 3, pp. 201–208, Jun. 1974.

[8] Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabási, “Controllability of complex networks,” Nature, vol. 473, no. 7346, pp. 167–173, 2011.

[9] G. Menichetti, L. Dall’Asta, and G. Bianconi, “Network controllability is determined by the density of low in-degree and out-degree nodes,” Phys. Rev. Lett., vol. 113, no. 7, 2014, Art. no. 078701.

[10] M. Pósfai, Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabási, “Effect of correlations on network controllability,” Sci. Rep., vol. 3, no. 1, pp. 1–7, 2013.

[11] X. Zhang, H. Wang, and T. Lv, “Efficient target control of complex networks based on preferential matching,” PLoS One, vol. 12, no. 4, 2017, Art. no. e0175375.

[12] Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabási, “Control centrality and hierarchical structure in complex networks,” PLoS One, vol. 7, no. 9, 2012, Art. no. e44459.

[13] T. Jia, Y.-Y. Liu, E. Csóka, M. Pósfai, J.-J. Slotine, and A.-L. Barabási, “Emergence of bimodality in controlling complex networks,” Nat. Commun., vol. 4, no. 1, pp. 1–6, 2013.

[14] C.-L. Pa, W.-J. Pei, and A. Michaels, “Robustness analysis of network controllability,” Physica A, vol. 391, no. 18, pp. 4420–4425, 2012.

[15] I. Klickstein and F. Sorrentino, “The controllability gramian of lattice graphs,” Automatica, vol. 114, 2020, Art. no. 108833.

[16] G. Yan et al., “Network control principles predict neuron function in Caenorhabditis elegans connectome,” Nature, vol. 550, no. 7677, pp. 519–523, 2017.

[17] W.-F. Guo, S.-W. Zhang, Q.-Q. Shi, C.-M. Zhang, T. Zeng, and L. Chen, “A novel algorithm for finding optimal driver nodes to target control complex networks and its applications for drug targets identification,” BMC Genomic., vol. 19, no. 1, pp. 67–79, 2018.

[18] Y. Bar-Ness and G. Langholz, “Preservation of controllability under sampling,” Int. J. Control, vol. 22, no. 1, pp. 39–47, 1975.
[19] M. Babaali and M. Egerstedt, “Nonpathological sampling of switched linear systems,” IEEE Trans. Autom. Control, vol. 50, no. 12, pp. 2102–2105, Dec. 2005.

[20] T. Hagiwara and M. Araki, “Controllability indices of sampled-data systems,” Int. J. Syst. Sci., vol. 19, no. 12, pp. 2449–2457, 1988.

[21] G. Kreisselmeier, “On sampling without loss of observability/controllability,” IEEE Trans. Autom. Control, vol. 44, no. 5, pp. 1021–1025, May 1999.

[22] G. Guo, “Systems with nonequidistant sampling: Controllable? Observable? Stable?,” Asian J. Control, vol. 7, no. 4, pp. 455–461, 2005.

[23] M. M. S. Pasand and M. Montazeri, “Controllability and stabilizability of multi-rate sampled data systems,” Syst. Control Lett., vol. 113, pp. 27–30, 2018.

[24] T. Zhou, “On the controllability and observability of networked dynamic systems,” Automatica, vol. 52, pp. 63–75, 2015.

[25] Y. Hao, Z. Duan, G. Chen, and F. Wu, “New controllability conditions for networked, identical LTI systems,” IEEE Trans. Autom. Control, vol. 64, no. 10, pp. 4223–4228, Oct. 2019.

[26] L. Wang, G. Chen, X. Wang, and W. K. Tang, “Controllability of networked MIMO systems,” Automatica, vol. 69, pp. 405–409, 2016.

[27] F. L. Iadicicce, F. Sorrentino, and F. Garofalo, “On node controllability and observability in complex dynamical networks,” IEEE Control Syst. Lett., vol. 3, no. 4, pp. 847–852, Oct. 2019.

[28] L. Xiang, P. Wang, F. Chen, and G. Chen, “Controllability of directed networked MIMO systems with heterogeneous dynamics,” IEEE Trans. Control Netw. Syst., vol. 7, no. 2, pp. 807–817, Jun. 2020.

[29] J.-N. Wu, X. Li, and G. Chen, “Controllability of deep-coupling dynamical networks,” IEEE Trans. Circuits Syst. I. Reg. Papers, vol. 67, no. 12, pp. 5211–5222, Dec. 2020.

[30] Y. Hao, Q. Wang, Z. Duan, and G. Chen, “Controllability of kroenecker product networks,” Automatica, vol. 110, 2019, Art. no. 108597.

[31] Z. Ji, H. Lin, and H. Yu, “Protocols design and uncontrollable topologies construction for multi-agent networks,” IEEE Trans. Autom. Control, vol. 60, no. 3, pp. 781–786, Mar. 2015.

[32] W. Ni, X. Wang, and C. Xiong, “Consensus controllability, observability and robust design for leader-following linear multi-agent systems,” Automatica, vol. 49, no. 7, pp. 2199–2205, 2013.

[33] L. Xiang, J. J. Zhu, F. Chen, and G. Chen, “Controllability of weighted and directed networks with nonidentical node dynamics,” Math. Probl. Eng., vol. 2013, 2013, Art. no. 405034.

[34] Y. Lou and Y. Hong, “Controllability analysis of multi-agent systems with directed and weighted interconnection,” Int. J. Control, vol. 85, no. 10, pp. 1486–1496, 2012.

[35] B. Zhao and Y. Guan, “Data-sampling controllability of multi-agent systems,” IMA J. Math. Control Inform., vol. 37, no. 3, pp. 794–813, 2020.

[36] Z. Ji, T. Chen, and H. Yu, “Controllability of sampled-data multi-agent systems,” in Proc. 33rd Chin. Control Conf., 2014, pp. 1534–1539.

[37] Z. Lu, Z. Ji, and Z. Zhang, “Sampled-data based structural controllability of multi-agent systems with switching topology,” J. Franklin Inst., vol. 357, no. 15, pp. 10886–10899, 2020.

[38] S. Roman, S. Axler, and F. Gehring, Advanced Linear Algebra, vol. 3. New York, NY, USA: Springer, 2005.

[39] J. P. Hespanha, Linear Systems Theory. Princeton, NJ, USA: Princeton Univ. Press, 2018.