MOTIVIC HOMOTOPY OF GROUP SCHEME ACTIONS

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Abstract. To smooth schemes equipped with a smooth affine group scheme action, we associate an equivariant motivic homotopy category. Underlying our construction is the choice of an ‘equivariant Nisnevich topology’ induced by a complete, regular, and bounded cd-structure. We show equivariant K-theory of smooth schemes is represented in the equivariant motivic homotopy category. This is used to characterize equivariantly contractible smooth affine curves and equivariant vector bundles on such curves. Generalizations of the purity and blow-up theorems in motivic homotopy theory are shown for actions of finite cyclic groups.

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1. Introduction

In this paper we develop motivic homotopy theory of smooth affine group scheme actions. We show the main results in the pioneering work of Morel-Voevodsky on motivic homotopy theory [24] generalize to the equivariant setting, e.g., the purity theorem for Thom spaces, the blow-up theorem and representability of $K$-theory.

A major motivation for motivic homotopy theory of group actions is to construct a convenient setting for equivariant cohomology theories on the category of smooth schemes with a group action. For the group of order two, the most important examples are Real algebraic $K$-theory, Real motivic cobordism [16], and a Bredon type theory of equivariant motivic cohomology [13]. We complete these results by representing equivariant $K$-theory of group actions, as introduced by Thomason in the mid 1980’s [30]. By considering actions by the multiplicative group scheme $\mathbb{G}_m$ and its subgroup schemes $\mu_n$ of sheaves of roots of unity, our results point towards an algebro-geometric version of $S^1$-equivariant homotopy theory [11].

As for the trivial group, every equivariant $\mathbb{A}^1$-homotopy equivalence becomes an isomorphism in the equivariant motivic homotopy category. This basic observation implies the same result holds for every equivariant vector bundle, by patching of equivariant $\mathbb{A}^1$-homotopies on a Zariski open covering. Recall that for an algebraic group $G$ over a field $k$, a $G$-equivariant vector bundle $\mathcal{V}$ over a $k$-scheme $X$ with $G$-action is trivial if there exists a $G$-representation $V$ such that $\mathcal{V} = V \times_k X$. For $G$ a finite cyclic group, representability of equivariant $K$-theory allows us to show that every $G$-equivariant vector bundle on an equivariantly $\mathbb{A}^1$-contractible smooth affine curve is trivial. It is an open question (even when $G$ is trivial) whether the same result holds in higher dimensions, starting with surfaces.

We relate the equivariant motivic homotopy category $\mathbf{Ho}^G_{\mathbb{A}^1}(k)$ to other existing settings for homotopy theory. For example, there is a naturally induced adjunction between $\mathbf{Ho}^G_{\mathbb{A}^1}(k)$ and the motivic homotopy category $\mathbf{Ho}_{\mathbb{A}^1}(k)$ corresponding to the trivial group. When the base scheme is the complex numbers, taking complex points furnishes a ‘realization’ functor to the equivariant homotopy category of topological spaces equipped with an action by the complex points of $G$. Under base change the corresponding equivariant motivic homotopy categories are related by standard adjunctions. These and many other functorial properties deserve a thorough treatment using cross-functors in the sense of Voevodsky, following Ayoub’s work [2], which is somehow beyond the scope of this paper.

Our main results can be summarized as follows. We leave precise statements to the main body of the paper.
Theorem 1.1. Let $G$ be a smooth affine group scheme over a field $k$. Denote by $\text{Sm}_k^G$ the category of separated finite type smooth $k$-schemes equipped with a $G$-action. Then we have the following.

1. The equivariant Nisnevich topology on $\text{Sm}_k^G$ is given by a complete, regular and bounded $cd$-structure.

2. The category of simplicial presheaves on $\text{Sm}_k^G$ admits local and motivic model structures in which the local weak equivalence is determined by the equivariant Nisnevich topology and the motivic weak equivalence is governed by equivariant vector bundle projections.

3. The equivariant algebraic $K$-theory of smooth $G$-schemes is represented in the equivariant motivic homotopy category. Motivic weak equivalences between smooth $G$-schemes induce isomorphisms on equivariant $K$-theory.

4. If $k$ is infinite and $G$ is a finite cyclic group, all equivariant vector bundles on an equivariantly contractible smooth affine curve are trivial.

5. If $k$ is algebraically closed and $G$ is a finite cyclic group of prime order, the purity and blow-up theorems for closed immersions of smooth $G$-schemes hold in the equivariant motivic homotopy category.

Herrmann [14, Proposition 3.5.4] proved that equivariant $K$-theory cannot be represented in the equivariant motivic homotopy category for the intermediate Nisnevich topology (see §2.3.1), which follows a construction common in topology by defining weak equivalences via fixed point loci of all subgroup schemes. This approach to equivariant motivic homotopy theory is intuitively very clear and works well in many aspects, but alas does not mesh well with cohomology theories such as equivariant $K$-theory. Our representability theorem is in some sense made possible by the fine differences between equivariant Nisnevich topologies, cf. §3.

Related works: The first version of this paper was written in 2011 and several related papers have appeared during its hiatus period. The subject of motivic homotopy of group actions can be traced back to Deligne’s lecture notes [7] emphasizing the role of quotients by finite group actions in the Rost-Voevodsky proof of the Bloch-Kato conjecture. Hu-Kriz-Ormsby [16] used the equivariant Nisnevich topology to introduce Real algebraic $K$-theory and Real motivic cobordism for the group of order two. Herrmann [14] worked out an unstable and stable equivariant motivic homotopy theory based on fixed points using the intermediate Nisnevich topology, cf. §2.3.1. An alternate construction carried out by Carlsson-Joshua [5] allowing for actions of discrete groups is bootstrapped for solving Carlsson’s conjecture relating $K$-theory of fields to representation theory. A Bredon style motivic cohomology theory related to Real algebraic $K$-theory and equivariant higher Chow groups was introduced by Heller-Voineagu-Østvær [13].

Brief Outline of the paper: We describe the equivariant Nisnevich site on $\text{Sch}_k^G$ in §2 and give a comparison of this site with other known equivariant topologies. A $cd$-structure on the equivariant Nisnevich site and several of its consequences are described in §3.

In §4–5 and 6 we work out the model structures on motivic $G$-spaces based on the equivariant Nisnevich topology and Bousfield localization with respect to
the affine line $\mathbb{A}^1$. A comparison with the motivic homotopy category and certain base change properties are investigated in §4. Proofs of the equivariant purity and blow-up theorems occupy §8 and 9. Finally, we prove representability of equivariant $K$-theory, and derive an algebraic analogue of Segal’s theorem in §10.

Generalizations: One may observe that the equivariant Nisnevich topology can be defined over any noetherian base scheme $S$ using the cd-topology defined by the distinguished squares (3.3). Proposition 3.2 remains valid in this set up. The results of sections 4-7 and 10 also generalize mutatis mutandis.

2. The Nisnevich site for $G$-schemes

Let $k$ be a field and $G$ a smooth affine group scheme over $k$. Recall that the identity component $G^0$ of $G$ is a normal closed subgroup of $G$ which is smooth over $k$. Moreover, the quotient $\overline{G}$ is a finite étale group scheme over $k$. We shall assume throughout this text that $\overline{G}$ is a finite constant group scheme over $k$.

Let $\text{Sch}_k$ denote the category of separated schemes of finite type over $k$ and let $\text{Sm}_k$ denote its full subcategory consisting of smooth schemes over $k$. A scheme in this paper will mean an object of $\text{Sch}_k$ and the scheme $X \times Y$ will mean the fiber product of schemes $X$ and $Y$ over $k$.

Let $\text{Sch}^G_k$ (or $\text{Sm}^G_k$) denote the category of (smooth) separated schemes of finite type over $k$ which are equipped with a $G$-action such that maps between two $G$-schemes are $G$-equivariant and commute with the structure maps to Spec ($k$). In particular, an object of $\text{Sch}^G_k$ is a pair $(X, \mu_X)$ such that $X \in \text{Sch}_k$ and there is an action map $\mu_X : G \times X \to X$ which satisfies the usual axioms of group actions. A morphism $f : (X, \mu_X) \to (Y, \mu_Y)$ is a morphism $f : X \to Y$ in $\text{Sch}_k$ such that $f \circ \mu_X = \mu_Y \circ (\text{id}_G \times f)$. A scheme $X$ can be viewed as a $G$-scheme via the trivial action, in which case we write it as the pair $(X, t_X)$. In this case, $t_X$ is nothing but the projection map $t_X : G \times X \to X$. This yields a full and faithful embedding

\[(2.1) \quad \iota_k : \text{Sch}_k \to \text{Sch}^G_k; \quad X \mapsto (X, t_X)\]

of categories.

2.1. Stabilizer subgroups for $G$-actions. Let $\text{Top}$ denote the category of topological spaces and for a given topological group $G$, let $\text{Top}^G$ denote the category of topological spaces with continuous $G$-actions. There are forgetful functors $|-| : \text{Sch}_k \to \text{Top}$ and $|-| : \text{Sch}^G_k \to \text{Top}^G$. These functors take a scheme $X$ to the underlying Zariski topological space of $X$. Cartesian products in $\text{Top}$ are given the product topology.

Let $G$ be a smooth affine group scheme over $k$ and let $X \in \text{Sch}^G_k$. Given a point $x \in X$, the scheme-theoretic stabilizer of $x$ is the $k(x)$-group scheme $G_x$ defined by the Cartesian square:

\[(2.2) \quad \begin{array}{ccc}
G_x & \to & G \times X \\
\downarrow & & \downarrow \langle \mu_X, \text{id}_X \rangle \\
\text{Spec} (k(x)) & \to & X \times X.
\end{array}\]
The set-theoretic stabilizer of \( x \) is the topological group \( S_x \) defined by the Cartesian square:

\[
\begin{array}{ccc}
S_x & \longrightarrow & |G| \times |X| \\
\downarrow & & \downarrow \left( \mu_{|X|}, \id_{|X|} \right) \\
|\spec (k(x))| & \longrightarrow & |X| \times |X|.
\end{array}
\]

There is a commutative diagram in \( \topological \):

\[
\begin{array}{ccc}
|G_x| & \longrightarrow & |G \times X| \longrightarrow |G| \times |X| \\
\downarrow & & \downarrow \left( \mu_{|X|}, \id_{|X|} \right) \\
|\spec (k(x))| & \Delta_X & |X \times X| \longrightarrow |X| \times |X|.
\end{array}
\]

It is well known that the left square is Cartesian and the horizontal arrows in the right square are surjective. Furthermore, these horizontal arrows are isomorphisms on the sets of closed points if \( k \) is algebraically closed. We conclude:

**Lemma 2.1.** Given a point \( x \in X \), there is a natural morphism of topological groups \( |G_x| \to S_x \to |G| \) such that the second morphism is an inclusion. It is an inclusion of a closed subgroup if \( x \) is a closed point of \( X \).

If \( G \) is a finite constant group scheme over \( k \), we can identify \( G \) with \( |G| \) and there are inclusions of closed subgroups \( G_x \hookrightarrow S_x \hookrightarrow G \). If \( k \) is algebraically closed and \( x \) is a closed point of \( X \), then the map \( |G_x| \to S_x \) is bijective on the sets of closed points.

If \( G \) is a finite constant group scheme over \( k \) one can get the following explicit descriptions of \( G_x \) and \( S_x \). One can check that a \( G \)-action on \( X \) is the same as a homomorphism of groups \( \sigma : G \to \text{Aut}_{\text{Sch}}(X) \) and one has \( S_x = \{ g \in G | g : x := \sigma(g)(x) = x \} \). If \( g \in S_x \), then it acts on \( \spec (k(x)) \) which is just the restriction of \( S_x \)-action on the scheme \( X \). In other words, \( \sigma \) restricts to a homomorphism \( \sigma_x : S_x \to \text{Aut}_{\text{Sch}}(\spec (k(x))) \). Here, \( \text{Sch} \) denotes the category of all schemes over \( k \). One checks using (2.2) that \( G_x = \ker (\sigma_x) \).

**Example 2.2.** Let \( G \) be the cyclic group of order two acting on \( X = \spec (\mathbb{C}) \) in the category \( \text{Sch}_\mathbb{R} \) by complex conjugation. One checks easily that \( G_x \) is trivial while \( S_x = G \), where \( x \) is the unique point of \( X \).

**Example 2.3.** If \( X \in \text{Sch}_k \) and \( g \in S_x \) for some \( x \in X \), observe that \( g \) acts on the local ring \( \mathcal{O}_{X,x} \) by \( k \)-algebra automorphisms. We show by an example that \( G_x \) may act as the identity on \( k(x) \) and differently on \( \mathcal{O}_{X,x} \). Let \( G = \langle \sigma \rangle \) be the cyclic group of order two acting on \( \mathbb{A}_k^1 \) by \( \sigma(x) = -x \). It is clear that \( S_0 = G \) and hence \( G_0 = G \). This action is algebraically described by the \( k \)-algebra automorphism \( \sigma : k[x] \to k[x] \) given by \( \sigma(x) = -x \). It is then clear that \( G \) acts on the local ring \( \mathcal{O}_{\mathbb{A}_k^1,0} \simeq k[x]_{(x)} \) by \( \sigma(x) = -x \). This is the identity map if and only if \( \text{char} (k) = 2 \).
2.2. Equivariant Nisnevich topology. We now define our Nisnevich site for $G$-schemes in terms of equivariant Nisnevich coverings. We shall show that these coverings yield a Grothendieck topology on $\text{Sch}_k^G$. The exact same definitions and results hold for $\text{Sm}_k^G$.

Definition 2.4. Let $X \in \text{Sch}_k^G$. A family of étale morphisms \( \{ Y_i \rightarrow X \}_{i \in I} \) in $\text{Sch}_k^G$ is called a ($G$-equivariant) Nisnevich cover of $X$ if for any point $x \in X$, there is an index $i = i(x) \in I$ and a point $y \in Y_i$ such that

1. $f_i(y) = x$,
2. the induced map of the residue fields $k_x \rightarrow k_y$ is an isomorphism, and
3. the induced map $S_y \rightarrow S_x$ is an isomorphism.

It is immediate from this definition that a $G$-equivariant Nisnevich cover is the same as a Nisnevich cover in the sense of [24] if $G$ is trivial.

Proposition 2.5. The category $\text{Sch}_k^G$ with the $G$-equivariant Nisnevich coverings constitutes a Grothendieck site.

Proof. By the definition it is clear that $G$-equivariant isomorphisms are Nisnevich coverings, and any refinement of a $G$-equivariant Nisnevich cover is also of the same type. We need to check that coverings are preserved under base change. This part is not automatically true by reduction to ordinary (non-equivariant) Nisnevich covers because for some point $x$, a point $y$ in the cover mapping to $x$ may satisfy condition (2) of Definition 2.4, but not condition (3).

We consider the Cartesian diagram in $\text{Sch}_k^G$:

$$
\begin{array}{ccc}
W & \xrightarrow{u'} & Y \\
\downarrow{u} & & \downarrow{u} \\
Z & \xrightarrow{v} & X,
\end{array}
$$

where $u$ is a $G$-equivariant Nisnevich cover. It is clear that $u'$ is étale. Let us now fix a point $z \in Z$ and let $v(z) = x$. Choose a point $y \in Y$ such that $k(x) \xrightarrow{\sim} k(y)$ and $S_y \xrightarrow{\sim} S_x$.

It is easy to check (see [22, Exercise 3.1.7]) that there is a natural homeomorphism of topological spaces

$$
\text{Spec} \left( k(y) \otimes_{k(x)} k(z) \right) \xrightarrow{\sim} \{ w \in W | v'(w) = y, \ u'(w) = z \}.
$$

On the other hand, the isomorphism $k(x) \xrightarrow{\sim} k(y)$ implies that the map $k(z) \rightarrow k(z) \otimes_{k(x)} k(y)$ is an isomorphism. In particular, $k(z) \otimes_{k(x)} k(y)$ is a field and defines a unique point $w = (y, z) \in W$ such that $v'(w) = y$ and $u'(w) = z$. Moreover, the map $k(z) \rightarrow k(w)$ is an isomorphism. We are only left with showing that $S_w \rightarrow S_z$ is an isomorphism. This map is clearly injective. To prove its surjectivity, notice that $g \in S_z$ implies $g \in S_x$ as $v(z) = x$ and $v$ is $G$-equivariant. But then, our assumption implies $g \in S_y$. Since $G$ acts diagonally on $W$ we conclude that $g \in S_w = S_{(y,z)}$. \(\Box\)
Notations: For the rest of this text, we shall abbreviate the term ‘equivariant Nisnevich topology’ by simply calling it the eN-topology. An equivariant Nisnevich cover of a G-scheme will be called an eN-cover. We shall denote the (G-equivariant) Nisnevich Grothendieck site on the category of G-schemes over k by $\text{Sch}^G_{k/\text{Nis}}$, and the corresponding site of smooth G-schemes by $\text{Sm}^G_{k/\text{Nis}}$. We refer to these sites as eN-sites. Throughout the text the following notations will be used.

1. $\text{PSh}^G_{\text{Sch}} :=$ the category of presheaves of sets on $\text{Sch}^G_k$.
2. $\text{Shv}^G_{\text{Sch}} :=$ the category of sheaves of sets on $\text{Sch}^G_{k/\text{Nis}}$.
3. $\text{PSh}^G_{\text{Sm}} :=$ the category of presheaves of sets on $\text{Sm}^G_k$.
4. $\text{Shv}^G_{\text{Sm}} :=$ the category of sheaves of sets on $\text{Sm}^G_{k/\text{Nis}}$.

Suppose $C$ and $D$ are Grothendieck sites. A functor $f^{-1} : C \to D$ is a continuous map of sites if for every sheaf $F$ on $D$, the presheaf $f_*(F) = F \circ f^{-1}$ is a sheaf on $C$. Such a map of sites is written $f : D \to C$. A continuous map of sites $f$ is called a morphism of sites if the left adjoint $f^*$ of $f_*$ commutes with finite limits. Since we shall discuss functors between Grothendieck sites, the following criterion for these notions will be used repeatedly in order to decide about the nature of these functors.

Proposition 2.6. ([24, Remarks 1.1.44, 1.1.45]) Suppose the functor $f^{-1} : C \to D$ commutes with fiber products.

1. Then the map of sites $f : D \to C$ is continuous if and only if $f^{-1}$ preserves coverings.
2. Suppose furthermore the topology on $D$ is sub-canonical. Then $f$ is a morphism of sites if and only if it is continuous.

It follows that there is a continuous map of sites $\tau_G : \text{Sch}^G_{S/\text{Nis}} \to \text{Sm}^G_{S/\text{Nis}}$. One knows, however, that $\tau_G$ is not a morphism of sites (see [24, Example I.1.46]).

2.3. Comparison with other topologies on G-schemes.

2.3.1. The intermediate Nisnevich topology. Suppose G is a finite constant group scheme over k. Replacing set-theoretic stabilizers by scheme-theoretic stabilizers in condition (3) of Definition 2.4 yields a Grothendieck topology on $\text{Sm}^G_k$. This topology is called the H-Nisnevich topology on $\text{Sm}^G_k$ by Herrmann [14] and the intermediate Nisnevich topology by Williams [36]. We let $\text{Sch}^G_{k/\text{Nis}}$ denote the corresponding site, and note below that the intermediate Nisnevich topology on $\text{Sch}^G_k$ is finer than the eN-topology.

Lemma 2.7. ([14, Lemma 2.1.14]) Let $f : Y \to X$ be a G-equivariant morphism of schemes. Let $x \in X$ and suppose that there is a point $y \in Y$ such that $f(y) = x$, $k(x) \xrightarrow{\sim} k(y)$ and $S_y \xrightarrow{\sim} S_x$. Then there is a naturally induced isomorphism $G_y \xrightarrow{\sim} G_x$. 

\textbf{Proof.} Let $g \in G_x$ and consider the commutative diagram

\begin{equation}
\begin{array}{c}
k(x) \xrightarrow{id} k(x) \\
\downarrow \sim \downarrow \\
k(y) \xrightarrow{g_*} k(y).
\end{array}
\end{equation}

Since the vertical arrows are induced by $f$, it follows easily from this diagram that $g_*$ is the identity. That is, $g \in G_y$. \hfill \Box

2.3.2. \textit{The Isovariant Nisnevich topology.} Recall that for $X \in \text{Sch}^G_k$, the isotropy group scheme is a group scheme $G_X$ over $X$ defined by the Cartesian square

\begin{equation}
\begin{array}{c}
G_X \longrightarrow G \times X \\
i_X \downarrow \downarrow (\mu_X, \text{id}_X) \\
X \Delta_X \longrightarrow X \times X.
\end{array}
\end{equation}

A $G$-equivariant étale cover \{\(X_i \to X\)\}_{i \in I} is called isovariant if the induced map of isotropy group schemes is an isomorphism for each $i \in I$. An isovariant étale cover which is also Nisnevich, is called an isovariant Nisnevich cover. The isovariant étale site on smooth schemes was introduced by Thomason \[31\] in order to prove the étale descent for Bott-inverted equivariant $K$-theory with finite coefficients. Its Nisnevich analogue was introduced by Serpe \[27\] in an attempt to prove descent theorems for equivariant algebraic $K$-theory with integral coefficients. (However, most of the results claimed in \[27\] are either false or need amendments.) The intermediate Nisnevich topology is clearly finer than the isovariant Nisnevich topology. Let $\text{Sch}^{G-\text{Iso}}_{k/Nis}$ denote the isovariant Nisnevich site on $\text{Sch}^G_k$.

It is known (see \[14\, \text{Lemma 3.2.8}\]) that the intermediate Nisnevich topology on $\text{Sch}^G_k$ (for $G$ finite) is sub-canonical. It follows that the isovariant Nisnevich topology (being coarser than the intermediate Nisnevich topology) is also sub-canonical (see Corollary \[3.13\] for a more general result). We conclude from Proposition \[2.6\] and Lemma \[2.7\] that for $G$ finite, the identity functor on $\text{Sch}^G_k$ induces morphisms of Grothendieck sites

\begin{equation}
\begin{array}{c}
\text{Sch}^{G-H}_{k/Nis} \overset{\iota^G}{\longrightarrow} \text{Sch}^G_{k/Nis} \\
\downarrow \nu^G \\
\text{Sch}^{G-\text{Iso}}_{k/Nis}.
\end{array}
\end{equation}

The following examples show that the equivariant Nisnevich topology is distinct from the intermediate Nisnevich topology. Moreover, the equivariant Nisnevich and the isovariant Nisnevich topologies are in general not comparable.

\textbf{Example 2.8.} We view the complex numbers $\mathbb{C}$ as an $\mathbb{R}$-algebra and consider the map of $\mathbb{R}$-algebras

\[ f : \mathbb{C} \to \mathbb{C} \times \mathbb{C}; \quad a \mapsto (a, \overline{a}). \]

Let $G = \langle \sigma \rangle$ be the cyclic group of order two acting by complex conjugation on $\mathbb{C}$ and by switching the coordinates on $\mathbb{C} \times \mathbb{C}$. Note that $f$ is a $G$-equivariant...
\[ \mathbb{R} \text{-algebra map and an isovariant Nisnevich covering. Let } Y = \text{Spec} (\mathbb{C} \times \mathbb{C}) \text{ and } X = \text{Spec} (\mathbb{C}). \text{ For the unique point } \eta \in X \text{ we have } S_\eta = G. \text{ On the other hand, the set-theoretic stabilizer of any point in } f_*^{-1}(\eta) \text{ is trivial. Hence } f_* : Y \to X \text{ is not an } eN\text{-cover.} \]

**Example 2.9.** For the inclusion of \( \mathbb{R} \text{-algebras } \mathbb{R} \to \mathbb{C} \) we let \( G = \langle \sigma \rangle \) (as above) act trivially on \( \mathbb{R} \) and by complex conjugation on \( \mathbb{C} \). The inclusion is \( G \)-equivariant étale, but it is neither isovariant nor Nisnevich. However, the map \( \mathbb{R} \to \mathbb{R} \times \mathbb{C} \) is a \( G \)-equivariant Nisnevich (hence an intermediate Nisnevich) cover of \( \text{Spec} (\mathbb{R}) \), which is not isovariant since the first map is not isovariant.

The intermediate Nisnevich topology resembles closely the situation in topology in the sense that \( Y \to X \) is an intermediate Nisnevich cover if and only if the induced maps of fixed point loci \( Y^H \to X^H \) are ordinary Nisnevich covers for all subgroups \( H \subseteq G \). On the other hand, it is also known (see [14, Remark 3.5.5]) that descent and representability of equivariant \( K \)-theory fail in the intermediate and isovariant Nisnevich topologies. This makes the \( eN \)-topology more suitable for studying cohomology theories for schemes with group actions. We have also observed that coverings in the intermediate and isovariant Nisnevich topologies do not necessarily split. It is unlikely that these topologies arise from \( cd \)-structures.

### 3. A \( cd \)-structure on the \( eN \)-topology

Our goal in this section is to show that the \( eN \)-topology can be described in terms of a \( cd \)-structure in the sense of Voevodsky [33]. We shall further show that this \( cd \)-structure is in fact regular, complete, and bounded. Applications of this will appear later in the paper.

#### 3.1. \( eN \)-neighborhoods

Let \( X \in \text{Sch}_k^G \) and let \( i : Z \hookrightarrow X \) be a \( G \)-invariant locally closed subset with the reduced subscheme structure. Let us denote this datum by \((X, Z)\). An \( eN \)-neighborhood of \((X, Z)\) is a commutative square

\[
\begin{array}{ccc}
Z' & \xrightarrow{f'} & U \\
\downarrow & & \downarrow f \\
Z & \xrightarrow{i} & X
\end{array}
\]

in \( \text{Sch}_k^G \) such that \( f \) is étale. We shall denote such a neighborhood by \((U, Z)\). If the square is Cartesian, we shall say that \((U, Z)\) is a distinguished \( eN \)-neighborhood of \((X, Z)\). Notice that in this case, \( Z' \) is automatically reduced.

Given an \( eN \)-neighborhood \( f : (U, Z) \to (X, Z) \) and a \( G \)-invariant locally closed subset \( Y \subseteq X \), we shall write the \( G \)-scheme \( Y \times_X U \) in short as \( Y \cap U \) or \( Y_U \).

#### 3.1.1. \( eN \)-neighborhood refinement

Assume that \( G = \{e = g_0, \ldots, g_n\} \) is a finite constant group scheme over \( k \). Given a Nisnevich neighborhood \((U, Z)\) (not necessarily \( G \)-invariant) and \( g \in G \), the translate of \( U \) by \( g \) is the scheme \( g(U) \) defined
by the Cartesian square

\[
\begin{array}{ccc}
g(U) & \rightarrow & U \\
g(f) & \downarrow & \downarrow f \\
X & \rightarrow_{g^{-1}} & X,
\end{array}
\]

where \(\tau_{g^{-1}} : X \rightarrow X\) is the automorphism of \(X\) defined by \(g^{-1}\) via the \(G\)-action on \(X\). We can iteratively form the fiber product

\[U_G := U \times_X g_1(U) \times_X \cdots \times_X g_n(U),\]

using the maps \(g_i(f) : g_i(U) \rightarrow X\). Since \(Z \hookrightarrow X\) is \(G\)-invariant, it is easy to check that \((U_G, Z)\) is in fact an \(eN\)-neighborhood of \((X, Z)\) and there is a factorization \((U_Z, Z) \rightarrow (U, Z) \rightarrow (X, Z)\). We conclude that every Nisnevich neighborhood of \((X, Z)\) contains, i.e., is dominated by, an \(eN\)-neighborhood.

3.2. \textit{cd}-structure on \(\text{Sch}^G_k\). The notion of \(cd\)-structure on Grothendieck sites was introduced by Voevodsky in \cite{33} in order to streamline the study of homotopy theory of schemes with respect to various topologies. We refer to \cite{33} for the definition of \(cd\)-structure on a category and its various properties.

\textbf{Definition 3.1.} A distinguished \(eN\)-square is a commutative diagram in \(\text{Sch}^G_k\)

\[
\begin{array}{ccc}
B & \rightarrow & Y \\
\downarrow & & \downarrow p \\
A & \rightarrow_j & X,
\end{array}
\]

with \(j\) an open immersion and \((Y, (X \setminus A)_{\text{red}})\) a distinguished \(eN\)-neighborhood of \((X, (X \setminus A)_{\text{red}})\).

The equivariant Nisnevich \(cd\)-structure on \(\text{Sch}^G_k\) is the collection of distinguished \(eN\)-squares \cite{33}

It is straightforward to check that we obtain a \(cd\)-structure on \(\text{Sch}^G_k\) in the sense of \cite{33}, i.e., a commutative diagram isomorphic to a distinguished \(eN\)-square is again a distinguished \(eN\)-square. The equivariant Nisnevich \(cd\)-structure on \(\text{Sm}^G_k\) is defined in the same way using distinguished \(eN\)-squares in the smooth category. Our next result is an equivariant analogue of Voevodsky’s \cite{34} Theorem 2.2]. The proof is obtained by following the steps in the non-equivariant case with suitable modifications at various stages. We refer to \cite{33} § 2] for the definition of a complete, regular, and bounded \(cd\)-structure.

\textbf{Proposition 3.2.} The equivariant Nisnevich \(cd\)-structures on \(\text{Sch}^G_k\) and \(\text{Sm}^G_k\) are complete, regular, and bounded.

\textbf{Proof.} We write a proof for the category \(\text{Sch}^G_k\) as the smooth case is no different. The completeness is a direct consequence of \cite{33} Lemma 2.4] since the distinguished \(eN\)-squares of the form \(3.3\) are closed under pullbacks.
To prove regularity, we observe that given a distinguished $eN$-square (3.3) in $\text{Sch}_G^S$, the derived square

\[
\begin{array}{ccc}
B & \longrightarrow & Y \\
\downarrow_{\Delta_B} & & \downarrow_{\Delta_Y} \\
B \times_A B & \rightarrow & Y \times_X Y
\end{array}
\]

is a distinguished square in $\text{Sch}_k$ by [34, Theorem 2.2] and hence, a distinguished square in $\text{Sch}_G^S$ since all the underlying maps in (3.4) are $G$-equivariant. The regularity condition now follows from [33, Lemma 2.11].

The boundedness condition is not straightforward from the non-equivariant case. First we need to define a density structure on $\text{Sch}_G^S$. For $X \in \text{Sch}_G^S$ and $i \geq 0$, let $D_i(X)$ denote the class of open embeddings $U \rightarrow X$ in $\text{Sch}_G^S$ that define an element of the density structure on $\text{Sch}_S$ [33, Proposition 2.10] under the forgetful functor $\text{Sch}_G^S \rightarrow \text{Sch}_S$. That is, for every $z \in X \setminus U$ there exists a sequence of points $z = x_0, x_1, \ldots, x_i$ in $X$ such that for $0 \leq j < i$, $x_j \neq x_{j+1}$ and $x_j \in \{x_{j+1}\}$. One verifies easily that this defines a density structure on $\text{Sch}_G^S$, and it is locally of finite dimension.

To prove boundedness, it is enough to show that every distinguished $eN$-square is reducing with respect to the above density structure. Consider a distinguished $eN$-square of the form (3.3) and suppose $B_0 \in D_{i-1}(B), A_0 \in D_i(A)$ and $Y_0 \in D_i(Y)$. Applying Lemma 3.3 below to the morphism $\overline{j} \coprod_{\overline{p}}$ we can find $X_0 \in D_i(X)$ such that $j(A_0) \cap p(Y_0) \subseteq X_0$. Replacing $Y$ by $Y_0$, $A$ by $A_0$, $B$ by $B' = A_0 \times Y_0$, $X$ by $X_0$, and applying [34, Lemma 2.5] we are reduced to consider the distinguished $eN$-square

\[
\begin{array}{ccc}
B' & \longrightarrow & Y_0 \\
\downarrow & & \downarrow_{\overline{p}} \\
A_0 & \overrightarrow{j} & X_0.
\end{array}
\]

We now set

\[
B_0' = B' \cap B_0, \quad Z = B' \setminus B_0', \quad Y' = Y_0 \setminus cl_{Y_0}(Z), \quad A' = A_0 \text{ and } X' = j(A_0) \cup p(Y').
\]

In [34, Proposition 2.10] it is noted that

\[
\begin{array}{ccc}
B_0' & \longrightarrow & Y' \\
\downarrow & & \downarrow_{\overline{p}} \\
A_0 & \overrightarrow{j} & X'.
\end{array}
\]

is a distinguished Nisnevich square which satisfies the required properties. To complete the proof we observe that the inclusions in (3.6) are $G$-invariant.

**Lemma 3.3.** Let $f : X \rightarrow Y$ be a morphism in $\text{Sch}_G^S$ and assume that there exists a $G$-invariant dense open subset $U$ in $Y$ such that $f^{-1}(U)$ is dense and
easy to check that $Z$ nonempty intersections (if there are any) of the irreducible components of the closed subscheme (with reduced structure) which is the union of all possible $U$.

Proposition. By [34, Lemma 2.9], there exists $W' \in D_i(Y)$ such that $f^{-1}(W') \subseteq V$. But $W'$ may not be $G$-invariant. However, since $V \subseteq X$ is $G$-invariant (by definition of our density structure), it follows that $f^{-1}(GW') = G(f^{-1}(W')) \subseteq V$. Since the map $\mu_Y : G \times Y \to Y$ is smooth and in particular open, we see that $GW' \subseteq Y$ is a $G$-invariant open subset.

Setting $W = GW'$, it is clear that $W \subseteq Y$ is a $G$-invariant open subset such that $f^{-1}(W) \subseteq V$. Furthermore, as $W' \subseteq W$ and $W' \in D_i(Y)$, we see that $W \in D_i(Y)$. This proves the lemma. □

3.3. $cd$-property of the $eN$-topology. In order to show that the $eN$-topology on $\text{Sch}_k^G$ (and $\text{Sm}_k^G$) is induced by the above $cd$-structure, we need to produce a splitting of $eN$-covers. We do this in the next result. Recall that $G$ is a smooth affine group scheme over $k$ such that $\overline{G} = G/G^0$ is a constant group scheme over $k$. We write $G = \bigcap_{i=0}^r g_i G^0$, where $\{e = g_0, g_1, \ldots, g_r\}$ are points in $G(k)$ which represent the left cosets of $G^0$.

Definition 3.4. A family of morphisms $\{Y_i \xrightarrow{f_i} X\}_{i \in I}$ in $\text{Sch}_k^G$ splits if there is a filtration of $X$ by $G$-invariant closed subschemes

\[
\emptyset = X_{n+1} \subsetneq X_n \subsetneq \cdots \subsetneq X_0 = X,
\]

and for each $0 \leq j \leq n$ there is an $i = i(j) \in I$ such that the map

\[
(X_j \setminus X_{j+1}) \times_X Y_i \to X_j \setminus X_{j+1}
\]

has a $G$-equivariant section. If each $f_i$ is also étale, the family of morphisms is called a split étale cover of $X$.

Proposition 3.5. A family of morphisms $\{Y_i \xrightarrow{f_i} X\}_{i \in I}$ in $\text{Sch}_k^G$ is an $eN$-cover if and only if it is a split étale cover.

Proof. It is clear that a split étale $G$-equivariant family of morphisms is an $eN$-cover. The core of the proof is to show the converse.

Suppose $\{Y_i \xrightarrow{f_i} X\}_{i \in I}$ is a $G$-equivariant Nisnevich cover of $X$. Let $Z \subset X$ be the closed subscheme (with reduced structure) which is the union of all possible nonempty intersections (if there are any) of the irreducible components of $X$. It is easy to check that $Z$ is $G$-invariant. This follows from the fact that every left coset $g_i G$ takes any given irreducible component $X_j$ of $X$ onto some (same or different) irreducible component of $X$ and $g_i G X_j = g_i G X_{j'}$ if and only if $X_j = X_{j'}$. Let $W$ be the $G$-invariant open subscheme of $X$ given by the complement of $Z$ and set $U_i = Y_i \times_X W$. Then $\{U_i \xrightarrow{f_i} W\}$ is an $eN$-cover of $W$. Notice that $W$ is a disjoint union of its irreducible components and each $f_{U_i}$ being étale, it follows that each $U_i$ is also a disjoint union of its irreducible components.

Let $x \in W$ be a generic point of $W$. Then the closure $W_x = \overline{\{x\}}$ in $W$ is an irreducible component of $W$. By our assumption, there is a point $y$ lying in some $U_i$ such that

\[
f_i(y) = x, \ k_x \xrightarrow{\sim} k_y, \text{ and } S_y \xrightarrow{\sim} S_x.
\]
Then the closure \( U_y = \{ y \} \) in \( U_i \) is an irreducible component of \( U_i \). Since \( U_y \to W_x \) is étale and generically an isomorphism, it must be an open immersion. Thus \( f_i \) maps \( U_y \) isomorphically onto an open subset of \( W_x \). We replace \( W_x \) by this open subset \( f_i(U_y) \) and call it our new \( W_x \).

Let \( GU_y \) be the image of the action morphism \( \mu : G \times U_y \to U_i \). Notice that \( \mu \) is a smooth map and hence open. This in particular implies that \( GU_y \) is a \( G \)-invariant open subscheme of \( U_i \) as \( U_y \) is one of the disjoint irreducible components of \( U_i \) and hence open. By the same reason, \( GW_x \) is a \( G \)-invariant open subscheme of \( W \).

Since the identity component \( G^0 \) is connected, it keeps \( U_y \) invariant. In other words, the point \( y \in U_i \) is fixed by \( G^0 \) and hence \( G \) acts on this point via its quotient \( \overline{G} = G/G^0 \). Recall that \( \overline{G} \) is a finite constant group scheme over \( k \).

Since each \( g_j \) takes \( U_y \) onto an irreducible component of \( U_i \) and since \( U_i \) has only finitely many irreducible components which are all disjoint, we see that \( GU_y = U_{i_0} \coprod U_{i_1} \coprod \cdots \coprod U_{i_n} \) is a disjoint union of some irreducible components of \( U_i \) with \( U_{i_0} = U_y \). In particular, for each \( U_{i_j} \), we have \( U_{i_j} = g_jGU_y = g_jU_y \) for some \( g_j \).

Since \( f_i \) maps \( U_y \) isomorphically onto \( W_x \), we conclude from the above that \( f_i \) maps each \( U_{i_j} \) isomorphically onto one and only one \( W_j \) such that \( GW_x = f_i(GU_y) = W_0 \coprod W_1 \coprod \cdots \coprod W_m \) (with \( m \leq n \)) is a disjoint union of open subsets of some irreducible components of \( W \) with \( W_0 = W_x \). The morphism \( f_i \) will map the open subscheme \( GU_y \) isomorphically onto the open subscheme \( GW_x \) if and only if no two components of \( GU_y \) are mapped onto one component of \( GW_x \). This is ensured by using the third condition of the definition of the \( eN \)-covering.

If two distinct components of \( GU_y \) are mapped onto one component of \( GW_x \), we can (using the equivariance of \( f_i \)) apply automorphisms by \( g_j \)'s and assume that one of these components is \( U_y \). In particular, we find that there are some \( j, j' \geq 1 \) such that

\[
(3.10) \quad W_x = f_i(U_y) = f_i(U_{i_j}) = f_i(g_{j'}U_y) = g_{j'}f_i(U_y) = g_{j'}W_x.
\]

But this implies that \( g_{j'} \in S_x \) and \( g_{j'} \notin S_y \). This violates the condition in (3.9) that the set-theoretic stabilizers \( S_y \) and \( S_x \) are isomorphic. We have thus shown that the morphism \( f_i \) has a \( G \)-equivariant splitting over a nonempty \( G \)-invariant open subset \( GW_x \). Letting \( X_1 \) be the complement of this open subset in \( X \), we see that \( X_1 \) is a proper \( G \)-invariant closed subscheme of \( X \) and by restricting our \( eN \)-cover to \( X_1 \), we get such a cover for \( X_1 \). The proof of the proposition is now completed by the Noetherian induction. \( \square \)

**Remark 3.6.** One cannot conclude from (3.10) that \( g_j \) lies in the scheme-theoretic stabilizer of \( x \). We thank Ben Williams for pointing this out soon after the first version of this paper was shared with him in 2011.

**Remark 3.7.** One can easily check that Example 2.8 is also an example of an invariant Nisnevich cover (hence and intermediate Nisnevich cover) which cannot admit an equivariant splitting. This provides a counterexample to [27, Lemma 2.12].

**Remark 3.8.** It is straightforward to see that a split étale cover has the base change property. Using Proposition 3.5 this gives another proof of Proposition 2.3.

**Proposition 3.9.** The \( eN \)-topology on \( \text{Sch}_k^G \) and \( \text{Sm}_k^G \) coincides with the topology induced by the equivariant Nisnevich cd-structure.
Proof. It is easy to see from the definitions that for a distinguished square \((3.1)\), the family \(\{Y \xrightarrow{f_i} X, A \xrightarrow{g_j} X\}\) is an \(eN\)-cover of \(X\). So we only need to prove that any \(eN\)-cover has a refinement which is an equivariant Nisnevich \(cd\)-cover. Let \(\{Y_i \xrightarrow{f_i} X\}_{i \in I}\) be an \(eN\)-cover of \(X\). By Proposition 3.5, we can assume that this is a split étale cover. In particular, there is a finite filtration of \(X\) by the \(G\)-invariant closed subschemes such that the covering map is split in the complementary open subsets. We prove our assertion by induction on the minimal length of this splitting.

If the length of the splitting is zero, then the cover has an equivariant section \(s: X \to Y_i\) for some \(i \in I\). Since each \(f_i\) is étale, \(s\) must be étale too. In particular, this section maps \(X\) isomorphically onto a \(G\)-invariant open subscheme \(X'\) of \(Y_i\). In this case, the square

\[
\begin{array}{ccc}
X' & \longrightarrow & X' \\
\downarrow & & \downarrow \\
X & \longrightarrow & X
\end{array}
\]

is a distinguished \(eN\)-square which refines our cover. To conclude, it suffices now to construct a distinguished \(eN\)-square of the form \((3.3)\) such that the pullback of the covering map \(\{Y_i \xrightarrow{f_i} X\}\) to \(Y\) has a \(G\)-equivariant section and the pullback to \(A\) has an equivariant splitting sequence of length strictly less than \(n\).

Given the splitting sequence of \((3.8)\), we see that \(\{X_n \times_X Y_i \to X_n\}\) is an \(eN\)-cover with a \(G\)-equivariant section \(s: X_n \to X_n \times_X Y_i\) for some \(i\). Let \(X'_n\) be the image of this section. We have seen above that \(X'_n\) is a \(G\)-invariant open subscheme of \(X_n \times_X Y_i\). In particular, its complement \(W_n\) is a \(G\)-invariant closed subscheme. By setting \(A = X \setminus X_n\) and \(Y = (X_n \times_X Y_i) \setminus W_n\), we see that the square defined by \(\{A \xrightarrow{f} X, Y \xrightarrow{g} X\}\) is a distinguished \(eN\)-square. Furthermore, the pullback of this square to \(Y\) has a \(G\)-equivariant section and its pullback to \(A\) is an \(eN\)-cover which has a splitting sequence of length less than \(n\). This completes the proof of the proposition. \(\square\)

Combining Propositions 3.2 and 3.9, we get the following results.

**Theorem 3.10.** The \(eN\)-topology on \(\text{Sch}_k^G\) and \(\text{Sm}_k^G\) is induced by a \(cd\)-structure, which is complete, regular, and bounded.

**Corollary 3.11.** A presheaf \(\mathcal{F}\) of sets on the site \(\text{Sch}_k^G\) (or \(\text{Sm}_k^G\)) is a sheaf if and only if \(\mathcal{F}(\emptyset) = \ast\) and it takes a square of the form \((3.3)\) to a Cartesian square.

**Proof.** This is an immediate consequence of Theorem 3.10 and [33, Lemma 2.9, Proposition 2.15]. \(\square\)

**Corollary 3.12.** For any sheaf \(\mathcal{F}\) of abelian groups on the site \(\text{Sch}_k^G\), one has \(H^i_{eN}(X, \mathcal{F}) = 0\) for \(i > \dim(X)\).

**Proof.** This follows immediately from Proposition 3.2 Theorem 3.10 and [33, Theorem 2.7]. \(\square\)

**Corollary 3.13.** The \(eN\)-topology on \(\text{Sch}_k^G\) is sub-canonical.
Proof. Let \( U \in \text{Sch}^G_k \) and let us consider a square of the form (3.3). By corollary 3.11 it suffices to show that this square is Cartesian after applying the functor \( \text{Hom}_{\text{Sch}^G_k}(-, U) \). So let \( f_1 \in \text{Hom}_{\text{Sch}^G_k}(Y, U) \) and \( f_2 \in \text{Hom}_{\text{Sch}^G_k}(A, U) \) be such that their restriction to \( B \) coincide.

Since the \( eN \)-topology on \( \text{Sch}^G_k \) is known to be sub-canonical for \( G \) trivial, we find a unique \( f \in \text{Hom}_{\text{Sch}_k}(X, U) \) such that \( f \circ p = f_1 \) and \( f \circ j = f_2 \). It remains to show that \( f \) is \( G \)-equivariant. Since the map \( p^{-1}(X \setminus A) \to X \setminus A \) is a \( G \)-equivariant isomorphism, we see that the restrictions of \( f \) to the \( G \)-invariant subsets \( A \) and \( X \setminus A \) are \( G \)-equivariant. It follows that \( f \) is \( G \)-equivariant. \( \Box \)

More applications of Theorem 3.10 will appear \( \S 4.2 \).

3.4. Points in the \( eN \)-topology. Recall that a point \( x \) on a Grothendieck site \( C \) is a functor \( x^* : \text{Shv}(C) \to \text{Sets} \) which commutes with all small colimits and finite limits. Such a functor acquires a right adjoint \( x_* : \text{Sets} \to \text{Shv}(C) \) by Freyd’s adjoint functor theorem. Having enough points is convenient for expressing weak equivalences in the homotopy theory of simplicial presheaves on a site. Below we describe a set of points on the \( eN \)-site of \( G \)-schemes for \( G \) a finite constant group scheme.

Given \( X \in \text{Sch}^G_k \) and \( x \in X \), let \( Gx \) denote the set-theoretic \( G \)-orbit of \( x \). Let \( \mathcal{O}_{X,Gx}^h \) denote the henselization of the semi-local ring \( \mathcal{O}_{X,Gx} \) along the ideal defining the scheme \( Gx \). Set \( X^h_{Gx} = \text{Spec} (\mathcal{O}_{X,Gx}^h) \). One observes that the pair \((X^h_{Gx}, Gx)\) is nothing but the filtering limit of all Nisnevich neighborhoods \((U, Gx)\) of \((X, Gx)\). Since every Nisnevich neighborhood of \((X, Gx)\) contains an \( eN \)-neighborhood (see \( \S 3.1 \)), we see that \( X^h_{Gx} \) is the filtered limit of all \( eN \)-neighborhoods of \((X, Gx)\).

In particular, it acquires a canonical \( Gx \)-preserving compatible \( G \)-action.

Given a pair \( \underline{x} = (X, Gx) \), one gets a functor \( \underline{x} : \text{Sch}^G_k \to \text{Sets} \) by setting \( \underline{x}(U) = \text{Hom}_{\text{Sch}_k}(X^h_{Gx}, U) \). Here, \( \text{Sch}^G_k \) denotes the category of all \( k \)-schemes with \( G \)-action (not necessarily of finite type). The left Kan extension of this gives a functor \( \underline{x} : \text{PSh}^G_{\text{Sch}_k} \to \text{Sets} \) and one checks at once that its restriction to the subcategory \( \text{Shv}^G_{\text{Sch}_k} \) indeed gives a point on \( \text{Sch}^G_k/\text{Nis} \). We shall write this functor on presheaves as \( F \mapsto F(X^h_{Gx}) \).

Proposition 3.14. The collection \( \{\underline{x} \mid X \in \text{Sch}^G_k, x \in X\} \) is a conservative family of points on the site \( \text{Sch}^G_k/\text{Nis} \).

Proof. By \([\Box \text{ Proposition 6.5.a}]\), it is enough to show that if \( U \in \text{Sch}^G_k \) and if \( \{f_i : U_i \to U\}_{i \in I} \) is a family of \( G \)-equivariant maps such that \( \{\underline{x}(U_i) \to \underline{x}(U)\}_{i \in I} \) is surjective for all \( X \in \text{Sch}^G_k \) and all \( x \in X \), then \( \{f_i\} \) is dominated by an \( eN \)-cover of \( U \).

So suppose that \( \{\underline{x}(U_i) \to \underline{x}(U)\}_{i \in I} \) is a surjective family for all pairs \((X, Gx)\). Let \( u \in U \) and let \( v : (U^h_{Gu}, Gu) \to (U, Gu) \) be the resulting \( G \)-equivariant map. By our assumption, we get an \( i \in I \) and a \( G \)-equivariant factorization

\[
\begin{array}{ccc}
U_i & \xrightarrow{w} & U^h_{Gu} \\
& \searrow^{f_i} & \downarrow^v \\
& & U
\end{array}
\]
Notice that \( w \) has to be an isomorphism on \( Gu \) and hence gives a section of \( f_i \) along \( Gu \). Since \( (U'_h, Gu) \) is the filtered limit of \( eN \)-neighborhoods of \( Gu \) and since \( f_i \) is a \( G \)-equivariant finite type morphism, we conclude that there is an \( eN \)-neighborhood \((U'_i, Gu) \) and a \( G \)-equivariant factorization \((U'_i, Gu) \xrightarrow{w} (U_i, Gu) \xrightarrow{f_i} (U, Gu)\). Since \( u \in U \) was chosen as an arbitrary point, we get the desired domination of \( \{ f_i \} \).

\[ \square \]

4. Model structures on simplicial presheaves on \( \text{Sm}^G_{k/Nis} \)

Let \( \mathcal{S} \) denote the category of simplicial sets with internal hom objects \( \mathcal{S}(-, -) \) defined, for example, in [10, I.5]. The category of pointed simplicial sets will be denoted by \( \mathcal{S}_\bullet \). We have the pointed version of internal hom as well. A motivic \( G \)-space is a contravariant functor \( \text{Sm}^G_k \to \mathcal{S} \) and a pointed motivic \( G \)-space is a contravariant functor \( \text{Sm}^G_k \to \mathcal{S}_\bullet \). Due to the finite type condition on \( G \)-schemes, the category \( \text{Sm}^G_k \) is essentially small, i.e., it is locally small with a small set of isomorphism classes of objects. Let \( \mathcal{M}^G_k \) (resp. \( \mathcal{M}^G_{k, \bullet} \)) denote the category of motivic (resp. pointed motivic) \( G \)-spaces. We may identify \( \mathcal{S} \) with the full subcategory of \( \mathcal{M}^G_k \) comprised of constant motivic \( G \)-spaces. The Yoneda lemma yields a fully faithful embedding of \( \text{Sm}^G_k \) into \( \mathcal{M}^G_k \) by sending \( X \in \text{Sm}^G_k \) to the representable motivic \( G \)-space \( h^G_X = \text{Hom}_{\text{Sm}^G_k}(-, X) \) taking values in simplicial sets of dimension zero. Recall from Corollary 3.13 that \( h^G_X \) is a sheaf in the \( eN \)-topology. A pointed motivic \( G \)-space is just a motivic \( G \)-space \( A \) with a map \( pt = h^G_A \to A \). In particular, a pointed \( G \)-scheme \((X, x)\) amounts to a \( G \)-scheme \( X \) together with a \( k \)-rational \( G \)-fixed point \( x \in X \). In the following, we make no notational distinction between \( X \) and \( h^G_X \). For \( X \in \text{Sm}^G_k \), the symbol \( X_+ \) will denote the pointed motivic \( G \)-scheme \( (X \coprod pt, pt) \). We note the following useful fact about \( \mathcal{M}^G_{k, \bullet} \).

**Lemma 4.1.** The category \( \mathcal{M}^G_{k, \bullet} \) is both a closed symmetric monoidal category and a locally finitely presented bicomplete \( \mathcal{S}_\bullet \)-category. In particular, filtered colimits commute with finite limits.

The tensor product in \( \mathcal{M}^G_{k, \bullet} \) is defined by taking pointwise (schemewise) smash product \( (\mathcal{X} \wedge \mathcal{Y})(U) = \mathcal{X}(U) \wedge \mathcal{Y}(U) \). With this definition, \( S^0 = pt \coprod pt = \text{Spec}(k) \coprod \text{Spec}(k) \) is the unit of the product and the limits, colimits are defined pointwise. The functor \( \text{Ev}_U \), evaluating motivic \( G \)-spaces at a fixed \( G \)-scheme \( U \) is strict symmetric monoidal, preserves limits and colimits, and there is an adjunction:

\[ (4.1) \quad \text{Fr}_U : \mathcal{S}_\bullet \rightleftarrows \mathcal{M}^G_{k, \bullet} : \text{Ev}_U. \]

The left adjoint \( \text{Fr}_U \), defined by \( \text{Fr}_U(K) = U_+ \wedge K \), is lax symmetric monoidal for any \( G \)-scheme and strict symmetric monoidal when \( U = pt \). For any \( \mathcal{X} \in \mathcal{M}^G_{k, \bullet} \) and \( K \in \mathcal{S}_\bullet \), we define \( \mathcal{X} \wedge K \) and \( \mathcal{X}^K \) by sending \( U \) to \( \mathcal{X}(U) \wedge K \) and \( \mathcal{S}_\bullet(K, \mathcal{X}(U)) \), respectively.
The $S\_\bullet$-enrichment of motivic $G$-spaces is given degreewise by the pointed simplicial set
\[(4.2) \quad S(\mathcal{X}, \mathcal{Y})_n = \text{Hom}_{\mathcal{M}_G^G}(\mathcal{X} \wedge \Delta[n]_+, \mathcal{Y}).\]

The internal hom in $\mathcal{M}_G^G$ is defined pointwise as $\text{Hom}(\mathcal{X}, \mathcal{Y})(U) = S(\mathcal{X} \wedge U_+, \mathcal{Y})$.

A motivic $G$-space $\mathcal{X}$ is finitely presentable if $\text{Hom}_{\mathcal{M}_G^G}(\mathcal{X}, -)$ commutes with filtered colimits. Using the natural isomorphism $\text{Hom}(U_+ \wedge \mathcal{K}, \mathcal{X}) \cong \mathcal{X}(U \times -)^K$, one deduces that $\mathcal{X}$ is finitely presentable if and only if $S(\mathcal{X}, -)$ commutes with filtered colimits. The pointed finite simplicial sets and the $G$-schemes form the building blocks for $\mathcal{M}_G^G$ in the following sense (see [4, 5.2.2b, 5.2.5]):

**Lemma 4.2.** Every pointed motivic $G$-space is a filtered colimit of finite colimits of pointed motivic $G$-spaces of the form $(U \times \Delta[n]_+)$, where $U \in \text{Sm}_G^G$ and $\Delta[n]$ is the standard $n$-simplex for $n \geq 0$. The motivic $G$-spaces $(U \times \Delta[n]_+)$ are finitely presented. The finitely presented motivic $G$-spaces are closed under retracts, finite colimits and tensor product.

In the above we described the monoidal structure on pointed motivic $G$-spaces. This story works verbatim for motivic $G$-spaces $\mathcal{M}_G^G$ by replacing the smash product with the schemewise defined product $\mathcal{X} \times \mathcal{Y}$.

**4.1. Schemewise model structures.** The goal of this section is to construct model structures on motivic $G$-spaces. We first describe these model structures for the unpointed motivic $G$-spaces and show in the end how these model structures induce such structures on the pointed motivic $G$-spaces. We refer the reader to [15] for standard notions related to model structures. We only recall here that a model structure on $\mathcal{M}_G^G$ is a simplicial model structure if the simplicial structure interacts with cofibrations, fibrations and weak equivalences: If $i : \mathcal{X} \to \mathcal{Y}$ is a cofibration and $p : \mathcal{Z} \to \mathcal{W}$ is a fibration in $\mathcal{M}_G^G$, then the map of simplicial sets
\[S(\mathcal{Y}, \mathcal{Z}) \overset{(i_* \times p_*)}{\to} S(\mathcal{X}, \mathcal{Z}) \times_{S(\mathcal{X}, \mathcal{W})} S(\mathcal{Y}, \mathcal{W})\]
is a Kan fibration, which is a weak equivalence if either $i$ or $p$ is a weak equivalence.

We shall say that a map $f : \mathcal{X} \to \mathcal{Y}$ of motivic $G$-spaces is a schemewise weak equivalence (resp. schemewise fibration) if the map of simplicial sets $\mathcal{X}(X) \to \mathcal{Y}(X)$ is a weak equivalence (resp. Kan fibration) of simplicial sets for every $X \in \text{Sm}_G^G$. Moreover, $f$ is called a projective cofibration if it has the left lifting property with respect to all maps which are schemewise fibrations and weak equivalences. It follows from [15] Theorems 11.6.1, 11.7.3, 13.1.14, Proposition 12.1.5] that $\mathcal{M}_G^G$ acquires the so-called projective model structure:

**Theorem 4.3.** (Projective model structure) The schemewise fibrations and weak equivalence, and projective cofibrations form a combinatorial and simplicial model structure on $\mathcal{M}_G^G$ with respect to the $S\_\bullet$-enrichment in (4.2).

The set of generating cofibrations
\[I^{\text{sch}}_{\text{proj}}(\text{sm}_G^G) = \{U_+ \wedge (\partial \Delta^n \subseteq \Delta^n)_+\}_{n \geq 0, U \in \text{Sm}_G^G}\]
and trivial cofibrations

\[ J_{pro}^{sch}(\text{sm}_k^G) = \{U_+ \land (A_i^n \subset \Delta^n)_+ \}_{n \geq 0, 0 \leq i \leq n, U \in \text{Sm}_k^G} \]

are induced from the corresponding maps in \( S \). The domains and codomains of the maps in these generating sets are finitely presented. The projective model structure is proper. For every \( U \in \text{Sm}_k^G \), the pair \((\text{Fr}_U, \text{Ev}_U)\) forms a Quillen pair.

Let \( \kappa \) be the first cardinal number greater than the cardinality of the set of maps in \( \text{PSh}^G_{\text{Sm}_k} \). If \( \omega \) denotes, as usual, the cardinal of continuum, we define \( \gamma \) as \( \kappa \omega^{\kappa \omega} \). Now let \( I_{\text{inj}}^{sch, \kappa}(\text{sm}_k^G) \) be the set of maps \( \mathcal{X} \to \mathcal{Y} \) such that \( \mathcal{X}(U) \to \mathcal{Y}(U) \) is a cofibration of simplicial sets of cardinality less than \( \kappa \) for every \( U \in \text{Sm}_k^G \). Likewise, we define \( J_{\text{inj}}^{sch, \gamma}(\text{sm}_k^G) \) for schemewise trivial cofibrations of simplicial sets bounded by \( \gamma \). With these definitions, the following holds for the so-called injective model structure on \( \mathcal{M}_k^G \), see \cite{12, 20}.

**Theorem 4.4.** (Injective model structure) There is a cofibrantly generated model structure on \( \mathcal{M}_k^G \) with schemewise cofibrations and weak equivalences, and injective fibrations. The cofibrations and trivial cofibrations are generated by \( I_{\text{inj}}^{sch, \kappa}(\text{sm}_k^G) \) and \( J_{\text{inj}}^{sch, \gamma}(\text{sm}_k^G) \), respectively. The injective model structure is combinatorial, proper and simplicial with the \( S \)-enrichment in \( \cite{12} \).

The third model structure one can consider is an example of a so-called flasque model structure \cite{17}. It is obtained by considering equivariant embeddings of smooth \( G \)-subschemas, generalizing the cognate schemewise model structure in \cite{25} Theorem A.9 for the trivial group. For \( U \in \text{Sm}_k^G \), we consider a finite set of \( G \)-equivariant monomorphisms \( V_i = \{ V_i \to U \}_{i \in I} \). The categorical union \( \bigcup_{i \in I} V_i \) is the coequalizer of the diagram

\[
\coprod_{i,j \in I} V_i \times_U V_j \xrightarrow{i} \coprod_{i \in I} V_i
\]

formed in \( \mathcal{M}_k^G \). We denote by \( i_I \) the induced monomorphism \( \bigcup_{i \in I} V_i \to U \). Note that \( \emptyset \to U \) arises in this way. The push-out product of maps of \( i_I \) and a map between simplicial sets exists in \( \mathcal{M}_k^G \). In particular, we are entitled to form the sets

\[ I_{\text{clo}}^{sch}(\text{sm}_k^G) = \{ i_I \square (\partial \Delta^n \subset \Delta^n)_+ \}_{I,n \geq 0} \]

and

\[ J_{\text{clo}}^{sch}(\text{sm}_k^G) = \{ i_I \square (A_i^n \subset \Delta^n)_+ \}_{I,n \geq 0, 0 \leq i \leq n} \].

A map between motivic \( G \)-spaces is called a closed schemewise fibration if it has the right lifting property with respect to \( J_{\text{clo}}^{sch}(\text{sm}_k^G) \). A closed schemewise cofibration is a map having the left lifting property with respect to every trivial closed schemewise fibration.

**Theorem 4.5.** (Flasque model structure) The schemewise weak equivalences, closed schemewise cofibrations and fibrations form a combinatorial and simplicial model structure on \( \mathcal{M}_k^G \) with respect to the \( S \)-enrichment in \( \cite{12} \). The closed schemewise cofibrations and fibrations are generated by \( I_{\text{clo}}^{sch}(\text{sm}_k^G) \) and \( J_{\text{clo}}^{sch}(\text{sm}_k^G) \), respectively. Moreover, the flasque model structure is cellular and proper.
4.2. Local model structures. Recall from [15, Chapter 3] that if $\Sigma$ is a class of morphisms in a simplicial model structure on $\mathcal{M}_k^G$, then an object $Z$ of $\mathcal{M}_k^G$ is called $\Sigma$-local, if it is fibrant and for every element $f : X \to Y$ in $\Sigma$, the induced map of homotopy function complexes $\mathcal{S}(Y, Z) \to \mathcal{S}(X, Z)$ is a weak equivalence (see [15, Definitions 3.1.4, 17.1.1]). Moreover, a map $f : X \to Y$ in $\mathcal{M}_k^G$ is a $\Sigma$-local equivalence if for every $\Sigma$-local object $Z$, the induced map of homotopy function complexes $\mathcal{S}(Y, Z) \to \mathcal{S}(X, Z)$ is a weak equivalence. Clearly every element of $\Sigma$ is a $\Sigma$-local equivalence.

The left Bousfield localization of $\mathcal{M}_k^G$ with respect to $\Sigma$ is a model category structure $L_\Sigma \mathcal{M}_k^G$ on the underlying category $\mathcal{M}_k^G$ such that

1. weak equivalences coincide with the $\Sigma$-local equivalences of $\mathcal{M}_k^G$,
2. cofibrations coincide with the cofibrations of $\mathcal{M}_k^G$, and
3. fibrations coincide with the maps having the right lifting property with respect to cofibrations that are simultaneously $\Sigma$-local equivalences.

We shall employ the technique of Bousfield localization to define local model structures on $\mathcal{M}_k^G$. One basic idea underlying the local model structures is that the distinguished $eN$-squares inform our definition of locally fibrant motivic $G$-spaces, and hence the accompanying (co)homology theories on $\text{Sm}_k^G$.

Definition 4.6. A motivic $G$-space $X$ is called locally projective fibrant if it is schemewise fibrant and flasque; i.e., $X(\emptyset)$ is contractible and the square

\[
\begin{array}{ccc}
X(X) & \xrightarrow{X(j)} & X(A) \\
\downarrow^{X(p)} & & \downarrow \\
X(Y) & \xrightarrow{} & X(B)
\end{array}
\]

is homotopy Cartesian for every distinguished $eN$-square of the form (3.3).

The locally injective fibrant and locally flasque fibrant motivic $G$-spaces are defined analogously by means of schemewise injective and flasque model structures, respectively.

Let $(-)^{\text{cof}} : \mathcal{M}_k^G \to \mathcal{M}_k^G$ be a cofibrant replacement functor in the schemewise projective model structure.

Definition 4.7. A map $X \to Y$ of motivic $G$-spaces is called a local projective weak equivalence if the induced map

\[
\mathcal{S}(Y^{\text{cof}}, Z) \to \mathcal{S}(X^{\text{cof}}, Z)
\]

is a weak equivalence for every locally projective fibrant motivic $G$-space $Z$. A map is a local projective fibration if it has the right lifting property with respect to projective cofibrations which are simultaneously local projective weak equivalence. The local injective and local flasque weak equivalences and fibrations are defined analogously.

Theorem 4.8. The category $\mathcal{M}_k^G$ acquires local projective, injective and flasque model structures. All of these model structures are combinatorial, proper and simplicial. The identity functors from the local projective model structure to the local flasque and local injective model structures are left Quillen equivalences.
Proof. The schemewise model structures are combinatorial and left proper ones, and hence suitable fodder for Bousfield localizations $L_{\Sigma} M^G_k$, where we define $\Sigma$ by means of distinguished $eN$-squares. In order to identify these Bousfield localizations with the definitions above, we shall make repeated use of the fact that the cofibrations and the fibrant objects determine the weak equivalences in any model structure. The existence of the model structures follows by reconciling the locally fibrant motivic $G$-spaces in the sense of Definition 4.6 with the fibrant objects in the Bousfield localizations determined by $\Sigma$. Once we do these identifications, the claim about the simplicial and the left properness property follows because these properties are preserved under Bousfield localization (see [15, Theorem 4.1.1]).

We start by defining $\Sigma$ in the case of the local projective model structure. For a distinguished $eN$-square $Q$ as in (3.3), let $Q^{hp}$ be the homotopy push-out of $A \leftarrow B \rightarrow Y$ in the schemewise projective model structure. There is a canonical map $Q^{hp} \rightarrow X$ and we set
\begin{equation}
\Sigma^{hp}_{\text{Nis}} = \{Q^{hp} \rightarrow X\}_Q \cup \{* \rightarrow \emptyset\}.
\tag{4.4}
\end{equation}
In the case of the local injective and flasque model structures, we consider the categorical push-out $Q^p$ of $A \leftarrow B \rightarrow Y$ in $M^G_k$. There is a canonical map $Q^p \rightarrow X$ and we set
\begin{equation}
\Sigma^p_{\text{Nis}} = \{Q^p \rightarrow X\}_Q \cup \{* \rightarrow \emptyset\}.
\tag{4.5}
\end{equation}
We claim that the fibrant objects in $L_{\Sigma^{hp}_{\text{Nis}}}^{\text{proj}}$ coincide with the local projective fibrant objects introduced in Definition 4.6. In effect, an object in the localization $L_{\Sigma^{hp}_{\text{Nis}}}^{\text{proj}}$ is fibrant if and only if it is schemewise fibrant and $\Sigma^{hp}_{\text{Nis}}$-local. But by adjunction, this is same as saying that it takes a distinguished $eN$-square to a homotopy Cartesian square and this in turn is same as saying that it is locally projective fibrant. The right properness of the local projective model structure $L_{\Sigma^{hp}_{\text{Nis}}}^{\text{proj}}$ follows from [3, Theorem 1.5]. There is a parallel story for the injective and flasque model structures. In these cases, $B \rightarrow Y$ is a cofibration so that the categorical push-out $Q^p$ is a model for the homotopy push-out. For this reason, it suffices to consider the set $\Sigma^p_{\text{Nis}}$ when constructing the local injective and flasque model structures.

We also observe that the weak equivalences in the local projective, injective and flasque model structures are same. It follows from Lemma 4.9 that a map which is either a local injective fibration or a local flasque fibration, is also a local projective fibration. We conclude from this that the local injective and the local flasque model structures are also right proper.

By [17, Theorems 2.2, 3.7], it follows that the identity functors from the schemewise projective model structure to schemewise flasque and the schemewise injective model structures on $M^G_k$ are left Quillen equivalences. We have just shown that the local projective, local injective and local flasque model structures are obtained by the Bousfield localizations of the corresponding schemewise model structures with respect to the same set. The second part of the theorem now follows from [15, Theorem 3.3.20]. This completes the proof. \qed

Lemma 4.9. Let $\Sigma$ be a set of maps in $M^G_k$. Suppose $f : X \rightarrow Y$ is a fibration in the Bousfield localization $L_{\Sigma} M^G_k$ with respect to the schemewise injective model
structure or the schemewise flasque model structure. Then \( f \) is a fibration in the Bousfield localization with respect to the schemewise projective model structure.

**Proof.** Every schemewise projective cofibration is also a schemewise injective and flasque cofibration, cf. [15, Proposition 11.6.3.], [17, Theorem 3.7]. The result follows now since the weak equivalences in the Bousfield localized model structures on \( L_{\Sigma}M^G_k \) coincide, cf. the proof of Theorem 4.8. □

Combining Theorems 3.10 and 4.8, we get the following explicit description of the weak equivalences in the local projective, injective and flasque model structures on \( M^G_k \) when \( G \) is a finite constant group scheme over \( k \). This description is closest to the description of local weak equivalence of simplicial presheaves in the non-equivariant Nisnevich topology and reflects our usage of the \( eN \)-topology.

**Theorem 4.10.** Assume that \( G \) is a finite constant group scheme over \( k \). A map \( f : \mathcal{X} \to \mathcal{Y} \) in \( M^G_k \) is a weak equivalence in the local projective, injective, and flasque model structures if and only if for all \( X \in Sm^G_k \) and all \( x \in \mathcal{X} \), the map of simplicial sets \( \mathcal{X}(X^h_{Gx}) \to \mathcal{Y}(X^h_{Gf(x)}) \) (see § 3.4) is a weak equivalence.

**Proof.** It follows from Theorems 3.10 and 4.8 and [33, Theorem 3.8] that \( f \) is a weak equivalence in the local projective, injective and flasque model structure if and only if the following hold.

1. The map \( f_* : \pi_0(\mathcal{X}) \to \pi_0(\mathcal{Y}) \) induces isomorphism of the associated sheaves.
2. For all \( X \in Sm^G_k \), all choices of base points \( x \in \mathcal{X}(X)_0 \) and all \( n \geq 1 \), the map \( f_* : \pi_n(\mathcal{X},x) \to \pi_n(\mathcal{Y},f(x)) \) induces an isomorphism of the associated \( eN \)-sheaves on the site \( Sm^G_k \downarrow X \).

But this is same as saying that for all points \( x^* : M^G_k \to S \) of the \( eN \)-site \( Sm^G_{k/Nis} \), the map \( f_* : x^*(\mathcal{X}) \to x^*(\mathcal{Y}) \) is a weak equivalence. Recall here that the \( eN \)-topology on \( Sm^G_k \) is sub-canonical (Corollary 3.13) and hence every point \( x^* : Shv^G_{Sm_k} \to Sets \) (of the site \( Sm^G_{k/Nis} \)) has the left Kan extension to a functor \( x^* : M^G_k \to S \).

Now, it follows from [24, Remark 2.1.3] that \( f : \mathcal{X} \to \mathcal{Y} \) is a local weak equivalence if and only if \( f_* : x^*(\mathcal{X}) \to x^*(\mathcal{Y}) \) is a weak equivalence for all \( x \) lying in a conservative family of points of \( Sm^G_{k/Nis} \). The theorem now follows by applying Proposition 3.14. □

The following result is another consequence of Theorem 3.10. A refined version, see Theorem 5.3, will be used to prove representability of equivariant \( K \)-theory in the equivariant motivic homotopy category.

**Proposition 4.11.** Let \( \mathcal{X} \) be a motivic \( G \)-space and \( \mathcal{X} \to \hat{\mathcal{X}} \) a fibrant replacement in the local projective model structure on \( M^G_k \). Then \( \mathcal{X} \) is flasque if and only if the map \( \mathcal{X} \to \hat{\mathcal{X}} \) is a schemewise weak equivalence. The same result holds for the local injective and local flasque model structures on \( M^G_k \).

**Proof.** First suppose that \( \hat{\mathcal{X}} \) is a fibrant replacement of \( \mathcal{X} \) in the local projective model structure on \( M^G_k \). It follows from Theorem 3.10 and [3] Lemma 4.1] that \( \hat{\mathcal{X}} \) is a flasque presheaf.
If \( \hat{\mathcal{X}} \) is a fibrant replacement of \( \mathcal{X} \) in the local injective or flasque model structure, then it follows from Lemma 4.9 that it is a fibrant replacement of \( \mathcal{X} \) in the local projective model structure too. Hence \( \hat{\mathcal{X}} \) is flasque as shown above.

Suppose now that \( \mathcal{X} \) is flasque. Theorem 3.10 and [33, Lemma 3.5] imply that \( \mathcal{X} \to \hat{\mathcal{X}} \) is a schemewise weak equivalence. The converse implication is trivial. \( \Box \)

5. The equivariant motivic homotopy category \( \mathcal{H}_{\mathbb{A}^1}(k) \)

In this section we construct the unstable homotopy category of motivic \( G \)-spaces. This is done by the following \( \mathbb{A}^1 \)-localization of our local model structures.

5.1. \( \mathbb{A}^1 \)-localization of \( \mathcal{M}_k^G \). Let \( T \) be a site with category of presheaves \( \text{PSh}(T) \). Let \( pt \) denote the terminal object of \( \text{PSh}(T) \). Recall from [24, 2.2.3] that an interval on a site \( T \) is a presheaf \( I \in \text{PSh}(T) \) together with morphisms:

\[
\mu : I \times I \to I; \quad i_0, i_1 : pt \to I
\]

where \( pt \) is the terminal object in \( \text{Psh}(T) \) with the canonical morphism \( p : I \to pt \) such that

\[
\mu(i_0 \times \text{id}_I) = \mu(\text{id}_I \times i_0) = i_0 \circ p
\]

and the morphism \( i_0 \bigcup i_1 : pt \bigcup pt \to I \) is a monomorphism.

In what follows, we let \( I = \mathbb{A}^1_k \) with trivial \( G \)-action and \( pt = \text{Spec} (k) \) such that \( i_0(s) = (s, 0), i_1(s) = (s, 1) \) and \( \mu(a, b) = ab \). It is then immediate that the pair \( (\text{Sm}^G_{k/Nis}, \mathbb{A}^1_k) \) is a site with interval. Since the base field \( k \) is fixed throughout, we shall write \( \mathbb{A}^1 \) for the affine line over \( k \).

**Definition 5.1.** The motivic projective (resp. injective, flasque) model structure on \( \mathcal{M}_k^G \) is the left Bousfield localization of its local projective (resp. injective, flasque) structure with respect to the set of projection maps

\[
\{ X \times \mathbb{A}^1 \xrightarrow{p} X \mid X \in \text{Sm}^G_k \}.
\]

The motivic \( G \)-spaces which are local with respect to this set of maps are called \( \mathbb{A}^1 \)-local. The fibrant objects in the motivic projective (resp. injective, flasque) model structure will be called \( \mathbb{A}^1 \)-fibrant. A weak equivalence in the motivic projective (resp. injective, flasque) model structure will be called a motivic weak equivalence.

In the motivic injective and flasque model structures, the \( \mathbb{A}^1 \)-local objects can be described using the following simpler criterion. We say that a motivic \( G \)-space \( \mathcal{X} \) is \( \mathbb{A}^1 \)-weak invariant if for all \( X \in \text{Sm}^G_k \), the naturally induced map

\[
\mathcal{X}(X) \to \mathcal{X}(X \times \mathbb{A}^1)
\]

is a weak equivalence.

**Lemma 5.2.** Suppose that a motivic \( G \)-space \( \mathcal{X} \) is fibrant in the local injective (or flasque) model structure. Then it is \( \mathbb{A}^1 \)-fibrant if and only if it is \( \mathbb{A}^1 \)-weak invariant.

**Proof.** We first observe that as \( \mathcal{X} \) is already locally fibrant, it is \( \mathbb{A}^1 \)-fibrant if and only if it is \( \mathbb{A}^1 \)-local. The lemma is now a consequence of [15, Definition 3.1.4, Proposition 16.1.3] using the observation that every \( X \in \text{Sm}^G_k \) is cofibrant in the local injective and flasque model structures. \( \Box \)
The motivic weak equivalences in $\mathcal{M}^G_{\mathbb{A}^1}$ are those maps which are $\Sigma_{\text{Nis}}^p$-local (resp. $\Sigma_{\text{Nis}}^p$-local) and $\mathbb{A}^1$-local weak equivalences. The cofibrations coincide with the cofibrations of the underlying local model structures and the fibrations are maps having the right lifting property with respect to cofibrations which are simultaneously motivic weak equivalences.

Theorem 4.8 and [15, Theorem 4.1.1] imply that the motivic projective, injective and flasque model structures on $\mathcal{M}^G_{\mathbb{A}^1}$ are left proper, cellular and simplicial. Moreover, right properness of the motivic model structures follows from [3, Lemma 3.1] and Lemma 4.9.

It follows from [15, Theorem 3.3.20] that the identity functors from the motivic projective to the motivic flasque and injective model structures are left Quillen equivalences. In particular, these model structures have equivalent homotopy categories, which will be denoted by $\text{Ho}^G_{\mathbb{A}^1}(k)$. Given motivic $G$-spaces $\mathcal{X}$ and $\mathcal{Y}$, the set $\text{Hom}_{\text{Ho}^G_{\mathbb{A}^1}(k)}(\mathcal{X}, \mathcal{Y})$ will be denoted by $[\mathcal{X}, \mathcal{Y}]_{\mathbb{A}^1}(k)$.

5.1.1. $\mathbb{A}^1$-flasque sheaves. We shall say that a motivic $G$-space is $\mathbb{A}^1$-flasque if it is flasque and $\mathbb{A}^1$-weak invariant. As another application of Theorem 3.10, we get the following extension of Proposition 4.11 to the motivic model structures. This result is very useful in determining the schemewise weak equivalences of motivic $G$-spaces as demonstrated in our proof of representability of equivariant $K$-theory.

**Theorem 5.3.** A motivic $G$-space $\mathcal{X}$ is $\mathbb{A}^1$-flasque if and only if every fibrant replacement in the motivic injective (resp. flasque) model structure is a schemewise weak equivalence. A map $f : \mathcal{X} \to \mathcal{Y}$ of $\mathbb{A}^1$-flasque motivic spaces is a motivic weak equivalence if and only if it is a schemewise weak equivalence.

*Proof.* The ‘if’ part of the first assertion follows from Proposition 4.11 and Lemma 5.2. To prove the converse, suppose that $\mathcal{X}$ is an $\mathbb{A}^1$-flasque motivic $G$-space and let $f : \mathcal{X} \to \widehat{\mathcal{X}}$ be an $\mathbb{A}^1$-fibrant replacement. By Proposition 4.11 it is enough to show that $f$ is also a locally (i.e., in the local injective or flasque model structure) fibrant replacement.

We factor $f$ as a composition $\mathcal{X} \xrightarrow{g} \mathcal{X}' \xrightarrow{f'} \widehat{\mathcal{X}}$, where $g$ is a local trivial cofibration (in particular, motivic trivial cofibration) and $f'$ is a local fibration. It follows from the 2-out-of-3 axiom that $f'$ is a motivic weak equivalence. We need to show that $f'$ is a local weak equivalence.

Since $\widehat{\mathcal{X}}$ is locally fibrant and $f'$ is a local fibration, it follows that $\mathcal{X}'$ is locally fibrant. In particular, $g$ defines a locally fibrant replacement of $\mathcal{X}$. We conclude from Proposition 4.11 that $g$ is a schemewise weak equivalence. We now apply the $\mathbb{A}^1$-weak invariance of $\mathcal{X}$ and Lemma 5.2 to conclude that $\mathcal{X}'$ is $\mathbb{A}^1$-fibrant. In particular, it is $\Sigma_{\text{Nis}}^p$-local as well as $\mathbb{A}^1$-local. We have thus shown that $f'$ is a motivic weak equivalence of $\mathbb{A}^1$-fibrant motivic $G$-spaces. It follows from the local Whitehead theorem (see [15, Theorem 3.2.12]) that $f'$ is in fact a schemewise weak equivalence. This proves the first part of the theorem.
To prove the second assertion of the theorem for the motivic weak equivalence $f: \mathcal{X} \to \mathcal{Y}$ of $\mathbb{A}^1$-flasque motivic $G$-spaces, we can form a commutative diagram

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\
\downarrow & & \downarrow \\
\hat{\mathcal{X}} & \xrightarrow{\hat{f}} & \hat{\mathcal{Y}}
\end{array}
\]

where the vertical arrows are $\mathbb{A}^1$-fibrant replacements. It follows from the 2-out-of-3 axiom that $\hat{f}$ is a motivic weak equivalence. In this case, $\hat{f}$ is a schemewise weak equivalence by [15, Theorem 3.2.12]. The two vertical arrows are also schemewise weak equivalences by the first assertion of the theorem. It follows that $f$ is a schemewise weak equivalence. \hfill \Box

5.2. Equivariant vector bundles. To justify the construction of the motivic model structure by inverting the trivial line bundle $\mathbb{A}^1$, we show that this in fact makes all equivariant vector bundle projections into motivic weak equivalences.

For maps $f, g: \mathcal{X} \to \mathcal{Y}$ of motivic $G$-spaces, an elementary $\mathbb{A}^1$-homotopy from $f$ to $g$ is a morphism $H: \mathcal{X} \times \mathbb{A}^1 \to \mathcal{Y}$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$. Two maps are called equivariantly $\mathbb{A}^1$-homotopic if they can be connected by a sequence of elementary $\mathbb{A}^1$-homotopies. A map $f: \mathcal{X} \to \mathcal{Y}$ is called a strictly equivariant $\mathbb{A}^1$-homotopy equivalence if there is a morphism $g: \mathcal{Y} \to \mathcal{X}$ such that $f \circ g$ and $g \circ f$ are equivariantly $\mathbb{A}^1$-homotopic to the respective identity maps.

**Proposition 5.4.** Let $X \in \Sm^G_k$ and let $V \xrightarrow{f} X$ be a $G$-equivariant vector bundle. Then the map of associated motivic $G$-spaces is a motivic weak equivalence.

**Proof.** Recall that the $eN$-site has an interval $I$ defined in the beginning of §5.1. Moreover, it is clear that our equivariant motivic homotopy category is obtained precisely by inverting the $I$-local morphisms in the sense of [24 § 2.2.3]. Hence it follows from [24 Lemma 2.3.6] that a strict $\mathbb{A}^1$-homotopy equivalence of motivic $G$-spaces is a motivic weak equivalence. Thus it suffices to show that the map $f: \mathcal{V} \to X$ is a strict $\mathbb{A}^1$-homotopy equivalence.

We can assume that $X$ is $G$-connected in the sense that $G(k)$ acts transitively on the set of connected components of $X$. Suppose $\mathcal{V}$ has rank $n$ and let $\mathcal{U} = \{U_1, \ldots, U_s\}$ be a Zariski open cover (not necessarily $G$-invariant) of $X$ such that each $U_i = \text{Spec}(R_i)$ is affine and $\mathcal{V}_i = f^{-1}(U_i) \to U_i$ is a trivial ordinary bundle given by $\mathcal{V}_i = \text{Spec}(R_i[X^0_i, \ldots, X^n_i])$.

Define the ring map $H: R_i[X^0_i, \ldots, X^n_i] \to R_i[T, X^0_i, \ldots, X^n_i]$ by $X^j_i \mapsto TX^j_i$. It is straightforward to check that since these maps are natural once we fix the $T$-coordinate over $X$, they glue together to give an elementary homotopy

\[
H: \mathcal{V} \times \mathbb{A}^1 \to \mathcal{V}
\]

such that $H \circ i_0 = i_X \circ f$ and $H \circ i_1 = \text{id}_{\mathcal{V}}$, where $i_X: X \to \mathcal{V}$ is the zero-section. Note that $i_0$, $i_1$ and $i_X$ are all $G$-equivariant. Thus, we shall be done if we show that $H$ is $G$-equivariant.

Now $f$ is a $G$-equivariant vector bundle, so that over every point $x \in X$, the fiber of $f$ is a $k_x$-vector space $\mathcal{V}_x$ of rank $n$. Moreover, if $g \in G(k(x))$ is such
that \( gx = x' \), then \( g \) acts on \( \mathcal{V}_x \) by a \( k_x (\simeq k_{x'}) \)-linear isomorphism \( \mathcal{V}_x \rightarrow \mathcal{V}_{x'} \). At the level of the coordinate rings of these fibers, the \( G \)-action and the map \( H \) are described by the diagram

\[
egin{array}{c}
k(x)[x_1, \ldots, x_n] \xrightarrow{\tau_g} k(x')[x'_1, \ldots, x'_n] \\
\downarrow H \downarrow H \\
k(x)[t, x_1, \ldots, x_n] \xrightarrow{\tau_g} k(x')[t, x'_1, \ldots, x'_n],
\end{array}
\]

where \( \tau_g \) is the map on the coordinate rings induced by \( g \in G(k(x)) \). It is straightforward to check that this diagram commutes, which shows that \( H \) is \( G \)-equivariant. \( \square \)

6. The equivariant motivic homotopy category \( \text{Ho}^G_{\mathbb{A}^1, \ast}(k) \)

Recall from §4 the category \( \mathcal{M}^G_k, \ast \) of pointed motivic \( G \)-spaces. Lemma 4.1 shows that \( \mathcal{M}^G_k, \ast \) is closed symmetric monoidal with respect to the smash product and pointed internal homs. There is an adjoint functor pair

\[
\mathcal{M}^G_k \xrightarrow{\text{left adjoint}} \mathcal{M}^G_k \xleftarrow{\text{right adjoint}} \mathcal{M}^G_k,
\]

where the left adjoint adjoins a disjoint base point, \( \mathcal{X} \mapsto \mathcal{X}_\ast = (\mathcal{X} \coprod pt, pt) \) and the right adjoint is the forgetful functor. Since \( \mathcal{M}^G_k, \ast \) is the slice category \( pt \downarrow \mathcal{M}^G_k \), we conclude the existence of the following motivic injective model structure from [15, Theorem 7.6.5].

**Theorem 6.1.** The category \( \mathcal{M}^G_k, \ast \) admits a model structure where a map \( f : (\mathcal{X}, x) \rightarrow (\mathcal{Y}, y) \) is a weak equivalence (resp. cofibration, resp. fibration) if and only if \( f : \mathcal{X} \rightarrow \mathcal{Y} \) is a weak equivalence (resp. cofibration, resp. fibration) in the motivic injective model structure (cf. Definition 5.1) after applying the forgetful functor. This model structure is proper, cellular and simplicial.

The motivic projective model structure on \( \mathcal{M}^G_k, \ast \) is defined by replacing the local injective model structure in Theorem 6.1 by the local projective model structure. Likewise for the motivic flasque model structure. As in the unpointed case, the three model structures are Quillen equivalent and hence have equivalent homotopy categories, which justifies the following definition.

**Definition 6.2.** The equivariant pointed motivic homotopy category \( \text{Ho}^G_{\mathbb{A}^1, \ast}(k) \) is the homotopy category of pointed motivic \( G \)-spaces with respect to either of the motivic model structures. For pointed motivic \( G \)-spaces \( \mathcal{X} \) and \( \mathcal{Y} \), we let \( [\mathcal{X}, \mathcal{Y}]_{G, \mathbb{A}^1} \) denote the set \( \text{Hom}_{\text{Ho}^G_{\mathbb{A}^1, \ast}(k)}(\mathcal{X}, \mathcal{Y}) \). Let \( \text{Ho}^G_{G, \ast}(k) \) denote the homotopy category of pointed motivic \( G \)-spaces with respect to either of the local model structures.

**Proposition 6.3.** The smash product preserves weak equivalences and cofibrations in the motivic injective model structure on \( \mathcal{M}^G_k, \ast \). This induces a symmetric closed monoidal category structure on \( \text{Ho}^G_{\mathbb{A}^1, \ast}(k) \).
Proof. Since the weak equivalences in the motivic projective and injective model structures are same, it follows from [8, Lemma 2.20] that smashing with any pointed motivic $G$-space preserves motivic weak equivalence. Since the cofibrations in the motivic (injective) model structure are monomorphisms, it follows immediately that smash product preserves cofibrations.

The first assertion implies that the smash product defines a structure of symmetric monoidal structure on $\text{Ho}^G_{A^1}(k)$. We need to show that this monoidal structure is closed to complete the proof. Since the motivic projective and injective model structures have equivalent homotopy categories, it suffices to show that the motivic projective model structure on $\mathcal{M}_{k^*}^G$ is monoidal. But this follows from [8, Corollary 2.19].

Recall that the simplicial circle $S^1_s$ is the constant presheaf $\Delta[1]/\partial\Delta[1]$ pointed by the image of $\partial\Delta[1]$. We shall write $(S^1_s)^\wedge n$ as $S^n_s$. Smashing with the simplicial circle gives a functor $\Sigma_s(F, x) = S^1_s \wedge (F, x)$. Let $\Omega^1_s((\mathcal{X}, x))$ be the right adjoint of $S^1_s \wedge (-)$. Proposition 6.3 implies that $(\Sigma_s(-), \Omega^1_s(-))$ is a Quillen pair of endofunctors on $\mathcal{M}_{k^*}^G$. In particular, we get an adjoint pair of endofunctors

(6.1) $\Sigma_s(-) : \text{Ho}^G_{A^1}(k) \longrightarrow \text{Ho}_{A^1}(k) : R\Omega^1_s(-)$.

The functor $R\Omega^1_s((\mathcal{X}, x))$ is given as $\Omega^1_s(Ex((\mathcal{X}, x)))$, where $Ex((\mathcal{X}, x))$ is a cofibrant fibrant replacement of $((\mathcal{X}, x))$ in the motivic model structure.

6.1. Equivariant motivic homotopy groups. We end this section with the definition of equivariant motivic homotopy groups of motivic $G$-spaces and show that these groups coincide with the actual homotopy groups of an $A^1$-fibrant replacement. The results of this section will be used in proving representability of equivariant algebraic $K$-theory in the unstable homotopy category.

Recall from (4.1) that given $X \in \text{Sm}_{k^*}^G$, there is an adjoint pair of functors $(\text{Fr}_X, \text{Ev}_X)$ between $S_*$ and $\mathcal{M}_{k^*}^G$.

Lemma 6.4. The functors $(\text{Fr}_X, \text{Ev}_X)$ form a Quillen pair with respect to the schemewise projective, local projective, and motivic projective model structures on $\mathcal{M}_{k^*}^G$. The same holds for the various localizations of the injective model structure.

Proof. Recall that this adjunction is given by the maps

$$\theta : \text{Hom}_{S_*}(K, S(X_+, \mathcal{X})) \rightarrow \text{Hom}_{\mathcal{M}_{k^*}^G}(K \wedge X_+, \mathcal{X})$$

$$\theta(f)(a \wedge x) = f_x(a)$$

and

$$\phi : \text{Hom}_{\mathcal{M}_{k^*}^G}(K \wedge X_+, \mathcal{X}) \rightarrow \text{Hom}_{S_*}(K, S(X_+, \mathcal{X}))$$

$$\phi(g)(a) = (x \mapsto g(a \wedge x)) .$$

It is straightforward to check that the maps are inverses to each other.

To show that $(\text{Fr}_X, \text{Ev}_X)$ is a Quillen pair, we shall note that $\text{Fr}_X$ preserves cofibrations and trivial cofibrations with respect to all the model structures given in the lemma. First we reduce to the schemewise projective model structure on
This follows because schemewise weak equivalences are the coarsest types of weak equivalences under consideration (see [15, Proposition 3.3.3]), and likewise for the projective cofibrations.

Suppose that \( f : K \to L \) be a cofibration (which is same as a monomorphism) of pointed simplicial sets. If \( f \) is a weak equivalence, then for any pointed simplicial set \( M \), the map \( K \wedge M \to L \wedge M \) is also a weak equivalence. In particular, the map \( K \wedge \mathcal{S}(U, X) \to L \wedge \mathcal{S}(U, X) \) is a weak equivalence for any \( U \in \text{Sm}_G \).

Equivalently, the map \( (K \wedge X)(U) \to (L \wedge X)(U) \) is a weak equivalence for every \( U \in \text{Sm}_G \). But this is same as saying that the map \( K \wedge X \to L \wedge X \) is a schemewise weak equivalence.

We now show that \( K \wedge X \to L \wedge X \) is a projective cofibration. We consider a diagram in \( \mathcal{M}_G \).

\[
\begin{array}{ccc}
K \wedge X_+ & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \psi \\
L \wedge X_+ & \longrightarrow & \mathcal{Y}
\end{array}
\]

where \( \psi \) is a projective trivial fibration. It follows from the definitions of the maps \( \theta \) and \( \psi \) above that the assignments

\[
\text{Hom}_{\mathcal{M}_G} (L \wedge X, \mathcal{X}) \to \text{Hom}_{\mathcal{S}_G} (L, \mathcal{X}(X)) \to \text{Hom}_{\mathcal{M}_G} (L \wedge X, \mathcal{X})
\]

\[
h \mapsto (a \mapsto h(a, \text{id}_X)); \quad h' \mapsto \left( (a \wedge (U \xrightarrow{u} X)) \mapsto h'(a) \circ u \right)
\]

give bijective correspondences of the sets. Thus giving a lifting in (6.2) is equivalent to giving a lifting in the parallel diagram of simplicial sets

\[
\begin{array}{ccc}
K \wedge X_+ & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \psi \\
L \wedge X_+ & \longrightarrow & \mathcal{Y}
\end{array}
\]

Since the fibrations and weak equivalences in the schemewise projective model structure are objectwise, we see from our assumption that the right vertical arrow in (6.3) is a trivial fibration in \( \mathcal{S}_G \). Since \( K \to L \) is assumed to be a cofibration, we get the desired lifting using the model structure on simplicial sets. This completes the proof of the lemma.

**Proposition 6.5.** Let \( (\mathcal{X}, x) \) be a fibrant pointed motivic \( G \)-space in the local injective model structure. Then for any pointed simplicial set \( K \) and any \( X \in \text{Sm}_G \), the Quillen pair \( (\text{Fr}_X, \text{Ev}_X) \) of Lemma 6.4 gives a canonical isomorphism

\[
\text{Hom}_{\text{Ho}^G_{\mathcal{S}_G}(k)} (K \wedge X_+, \mathcal{X}) \xrightarrow{\tilde{\sim}} [K, \mathcal{X}(X)].
\]

If \( \mathcal{X} \) is also \( \mathbb{A}^1 \)-local, then there is a canonical isomorphism

\[
\text{Hom}_{\text{Ho}^G_{\mathcal{S}_G}(k)} (K \wedge X_+, \mathcal{X}) \xrightarrow{\tilde{\sim}} [K, \mathcal{X}(X)].
\]
Proof. Since the functor $K \mapsto K \land X_+$ preserves weak equivalence in all our model structures and since $\mathcal{X}$ is fibrant in the local injective model structure, we conclude from Lemma 6.4 that there are isomorphisms

$$\text{Hom}_{\text{Ho}^G_{\mathcal{A}_1, \bullet}(k)}(K \land X_+, \mathcal{X}) \cong \text{Hom}_{\text{Ho}^G_{\mathcal{A}_1, \bullet}(k)}(L\text{Fr}_X(K), \mathcal{X})$$

$$\cong \text{Hom}_{\mathcal{S}_k}(K, \text{REv}_X(\mathcal{X}))$$

$$\cong \text{Hom}_{\mathcal{S}_k}(K, \text{Ev}_X(\mathcal{X}))$$

$$\cong \text{Hom}_{\mathcal{S}_k}(K, \mathcal{X}(X)).$$

Since $\mathcal{X}$ is fibrant in the local injective model structure, it is schemewise fibrant. In particular, $\mathcal{X}(X)$ is a Kan complex and hence the last term is same as $[K, \mathcal{X}(X)]$.

If $\mathcal{X}$ is also $\mathbb{A}^1$-local, then it is $\mathbb{A}^1$-fibrant by Lemma 5.2. We can now repeat the above argument using Lemma 6.4. □

Corollary 6.6. A map $f : \mathcal{X} \to \mathcal{Y}$ of $\mathbb{A}^1$-fibrant pointed motivic $G$-spaces is a schemewise weak equivalence if and only if the map

$$\text{Hom}_{\text{Ho}^G_{\mathcal{A}_1, \bullet}(k)}(S^i_s \land X_+, \mathcal{X}) \to \text{Hom}_{\text{Ho}^G_{\mathcal{A}_1, \bullet}(k)}(S^i_s \land X_+, \mathcal{Y})$$

is an isomorphism for all $X \in \text{Sm}_k^G$ and all $i \geq 0$.

Proof. We only need to show the ‘if’ part. Since $\mathcal{X}$ and $\mathcal{Y}$ are $\mathbb{A}^1$-fibrant, it follows from Proposition 6.5 that the terms on the left and the right in (6.4) are $\pi_i(\mathcal{X}(X))$ and $\pi_i(\mathcal{Y}(X))$, respectively. This implies that $\mathcal{X} \to \mathcal{Y}$ is a schemewise weak equivalence (and hence motivic weak equivalence). □

6.1.1. Homotopy groups. For a motivic $G$-space $\mathcal{X}$, let $\pi_0^{G, \mathbb{A}^1}(\mathcal{X})$ be the $\epsilon N$-sheaf associated to the presheaf $U \mapsto [U, \mathcal{X}]_{G, \mathbb{A}^1}$ on $\text{Sm}_k^G$. We shall say that $\mathcal{X}$ is equivariantly $\mathbb{A}^1$-connected if $\pi_0^{G, \mathbb{A}^1}(\mathcal{X})$ is constant.

For a pointed motivic $G$-space $(\mathcal{X}, x)$, let $\pi_i^{G, \mathbb{A}^1}(\mathcal{X}, x)$ be the $\epsilon N$-sheaf associated to the presheaf $U \mapsto [S^i_s \land U_+, (\mathcal{X}, x)]_{G, \mathbb{A}^1}$.

It follows from Corollary 6.6 that if $\mathcal{X} \to \mathcal{F}$ is an $\mathbb{A}^1$-fibrant replacement, then $\pi_i^{G, \mathbb{A}^1}(\mathcal{X}, x)$ is same as the sheaf associated to the presheaf of homotopy groups of the simplicial presheaf $\mathcal{F}$. It follows that $\pi_i^{G, \mathbb{A}^1}(\mathcal{X}, x)$ is a sheaf of groups for $i \geq 1$ and a sheaf of abelian groups for $i \geq 2$. Using the functorial fibrant replacements and Corollary 6.6, we obtain the following result.

Proposition 6.7. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of equivariantly $\mathbb{A}^1$-connected motivic $G$-spaces is a motivic weak equivalence if and only if for any choice of base point $x \in \mathcal{X}$, the induced map

$$\pi_i^{G, \mathbb{A}^1}(\mathcal{X}, x) \to \pi_i^{G, \mathbb{A}^1}(\mathcal{Y}, f(x))$$

is an isomorphism for all $i \geq 1$.

7. Comparison with the Nisnevich site and base change

In this section, we study the connection of our $\epsilon N$-site with various other sites associated with group scheme actions.
7.1. **Comparison with the Nisnevich site.** Suppose that \( H \subseteq G \) is a subgroup. We then have the canonical restriction functor \( r^G_H : \text{Sch}^G_k \to \text{Sch}^H_k \). This functor has a left adjoint \( e_H^G : \text{Sch}^H_k \to \text{Sch}^G_k \) given by \( e_H^G(X) = G^H \times X \). In particular, \( r^G_H \) commutes with limits. Thus we get the map of \( eN \)-sites \( \tilde{r}^G_H : \text{Sch}^H_{k/\text{Nis}} \to \text{Sch}^G_{k/\text{Nis}} \) and \( \tilde{e}^G_H : \text{Sch}^G_{k/\text{Nis}} \to \text{Sch}^H_{k/\text{Nis}} \). These are not continuous since they do not in general preserve \( eN \)-covers (see Proposition 2.6). However, if \( H \) is the trivial subgroup scheme, then \( \tilde{r}^G_H \) and \( \tilde{e}^G_H \) preserves covers. Since the underlying topologies are sub-canonical (see Corollary 3.13), Proposition 2.6 implies that \( \tilde{r}^G : \text{Sch}^G_{k/\text{Nis}} \to \text{Sch}_{k/\text{Nis}}^G \) is a morphism of sites. In the smooth setting, we denote the analogous functor by

\[
\text{res} : \text{Sm}_{k/\text{Nis}}^G \to \text{Sm}_{k/\text{Nis}}^G.
\]

**Lemma 7.1.** The pullback functor \( \text{res}^* : \mathcal{M}_k^G \to \mathcal{M}_k \) preserves representable sheaves. It preserves local and motivic weak equivalences.

**Proof.** To show that \( \text{res}^*(X)(U) = \text{Hom}_{\text{Sm}_S}(U, X) \) for \( X \in \text{Sm}_S^G \) and \( U \in \text{Sm}_S \), notice that the term involving the pullback functor is \( \text{colim}_{\{U \to V | V \in \text{Sm}_S^G\}} \text{Hom}_{\text{Sm}_S^G}(V, X) \).

But the colimit is clearly same as the set \( \text{Hom}_{\text{Sm}_S}(U, X) \).

As \( \text{res} \) is a morphism of sites, \( \text{res}^* \) preserves local (with respect to the equivariant and ordinary Nisnevich topologies) weak equivalences by [24, Proposition 2.1.47].

Suppose now that \( \mathcal{X} \) is an \( \mathbb{A}^1 \)-local object of \( \mathcal{M}_k \) and let \( X \times \mathbb{A}^1 \to X \) be the projection map for some \( X \in \text{Sm}_S^G \). Then \( S_{\mathcal{M}_k^G}(X, \mathcal{X}) \) identifies with \( \mathcal{X}(r^G(X)) \) and likewise for \( X \times \mathbb{A}^1 \). It follows that \( \text{res}_*(\mathcal{X}) \) is \( \mathbb{A}^1 \)-local in \( \mathcal{M}_k^G \). Combined with the adjunction it follows that \( \text{res}^* \) preserves motivic weak equivalences. \( \square \)

Using Lemma 7.1 and [24, Proposition 2.3.17], we get the following result.

**Proposition 7.2.** The map \( \text{res} : \text{Sm}_{k,\text{Nis}}^G \to \text{Sm}_{k,\text{Nis}}^G \) is a morphism of sites such that \( \text{res}^* \) preserves local and motivic weak equivalences. Furthermore, there is an adjoint pair of functors

\[
L\text{res}^* : \text{Ho}_{\mathbb{A}^1}(k) \rightleftarrows \text{Ho}_{\mathbb{A}^1}(k) : R\text{res}_*.
\]

We note the following immediate corollary in connection with representability of equivariant \( K \)-theory; see §10.

**Corollary 7.3.** Let \( f : X \to Y \) be a \( G \)-equivariant map of smooth \( G \)-schemes. Suppose that \( f \) is a motivic weak equivalence in \( \mathcal{M}_k^G \). Then the induced map \( K_*(Y) \xrightarrow{f^*} K_*(X) \) is an isomorphism of ordinary \( K \)-theory.

Recall from [24] that there is a full and faithful embedding \( \text{Sm}_k \to \text{Sm}_k^G \), which takes a scheme \( X \) to itself with the trivial \( G \)-action. This functor commutes with fiber product and takes a Nisnevich cover to an \( eN \)-cover. Corollary 3.13 and Proposition 2.6 imply that there is an induced morphism of sites \( \iota : \text{Sm}_{k/\text{Nis}}^G \to \text{Sm}_{k/\text{Nis}} \). Note that \( \iota^* \) is identity and \( \iota_* \) takes any \( G \)-scheme \( X \) to the fixed point subscheme \( X^G \). Recall that \( X^G \) is smooth. We get the following result.
**Proposition 7.4.** The morphism of sites $\iota : \text{Sm}_{k,\text{Nis}}^G \to \text{Sm}_{k,\text{Nis}}$ induces a pair of adjoint functors
\[
\iota^* : \text{Ho}_{\mathbb{A}^1}(k) \rightleftarrows \text{Ho}_{\mathbb{A}^1}(k) : R\iota_*.
\]
The functor $\iota^*$ is a full and faithful embedding of the motivic homotopy category of smooth schemes into the equivariant motivic homotopy category.

7.2. **Change of base field.** Suppose now that $k \hookrightarrow k'$ is an extension of fields and set $G' = G \times \text{Spec}(k')$. Notice that $G'$ is identified with $G$ if the latter is a finite constant group scheme over $k$. The base change functor $f^{-1} : \text{Sm}_k^G \to \text{Sm}_{k'}^G$ is defined by $X \mapsto X \times_{\text{Spec}(k')} \text{Spec}(k)$. It is clear that $f^{-1}$ preserves distinguished $eN$-squares. Thus Corollary 3.14 shows that the site map $f : \text{Sch}_{k'/\text{Nis}}^G \to \text{Sch}_k^G$ is continuous. Since $f^{-1}$ clearly commutes with fiber products, it follows from Corollary 3.13 and Proposition 2.6 that $f$ is a morphism of sites.

**Proposition 7.5.** Given an extension of fields $k \hookrightarrow k'$, the base change functor $f^{-1}$ induces a morphism of sites $f : \text{Sm}_{k'/\text{Nis}}^G \to \text{Sm}_k^G$. This yields an adjoint pair of functors
\[
Lf^* : \text{Ho}_{\mathbb{A}^1}(k) \rightleftarrows \text{Ho}_{\mathbb{A}^1}(k') : Rf_*.
\]

If $k \hookrightarrow k'$ is a finite separable extension, then $f^*$ has a left adjoint $f_# : \mathcal{M}_k^G \to \mathcal{M}_{k'}^G$ which takes any $U \in \text{Sm}_k^G$ to itself, viewed as a $G$-scheme over $k$. This functor preserves motivic weak equivalences and $f^*$ preserves $\mathbb{A}^1$-local motivic $G$-spaces. There is an adjoint pair of functors
\[
Lf_# : \text{Ho}_{\mathbb{A}^1}(k') \rightleftarrows \text{Ho}_{\mathbb{A}^1}(k) : Lf^*.
\]

8. **Local $eN$-linearization of $G$-schemes**

The homotopy purity theorem (see [24, Theorem 3.2.23]) is one of the most important tools in $\mathbb{A}^1$-homotopy theory, e.g., in the construction of Gysin long exact sequences and for Poincaré duality in its most concise form. Our goal in this and the following section is to establish the purity theorem for $G$-schemes when $G$ is a finite cyclic group of prime order. This theorem turns out to have many applications in the equivariant motivic stable homotopy category. As part of proving the purity theorem, we first establish a local equivariant linearization of smooth $G$-schemes in the Zariski topology.

8.1. **$eN$-linearization near a fixed point.** We shall assume throughout this section that $G$ is a finite constant group scheme over $k$ of order prime to the characteristic of $k$. This is mainly to ensure that $G$ is linearly reductive. A (finite) $G$-module will mean a (finite-dimensional) rational representation of $G$. We begin with the following elementary result about $G$-modules.
**Lemma 8.1.** Consider a commutative diagram of $G$-modules

\[
0 \to M_1 \to M \xrightarrow{u} M_2 \to 0
\]

\[
\begin{array}{c c c c}
\downarrow u_1 & \downarrow u & \downarrow u_2 \\
0 & N_1 & \to N & \to N_2 \to 0
\end{array}
\]

in which the rows are exact and the vertical maps are surjective. Assume that $N$ is a finite $G$-module. Then there exists a finite $G$-submodule $M' \subseteq M$ and commutative diagram of finite $G$-modules

\[
0 \to M'_1 \to M' \to M'_2 \to 0
\]

\[
\begin{array}{c c c c}
\downarrow u'_1 & \downarrow u' & \downarrow u'_2 \\
0 & N_1 & \to N & \to N_2 \to 0
\end{array}
\]

with exact rows such that the vertical maps are the restriction of the vertical maps of (8.1) to $G$-submodules. Moreover, they are all isomorphisms.

**Proof.** This is an application of the fact that $G$ is linearly reductive. We give a sketch of the proof. Since $N$ is a finite $G$-module, so are $N_1$ and $N_2$. Hence, we can first find a finite-dimensional $k$-linear subspace $V \subseteq M$ such that $u(V) = N$. We can then find inclusions of linear subspaces $V \subseteq V' \subseteq M$ such that $V'$ is a finite $G$-submodule and $u(V') = N$. Set $L = \ker(V' \to N)$.

Since $G$ is linearly reductive, its representation theory tells us that there is a decomposition $N = N_1 \oplus N_2'$ of finite $G$-modules such that $N_2'$ is mapped isomorphically onto $N_2$. Similarly, there is a direct sum decomposition of finite $G$-modules $V' = L \oplus N'$ such that $N'$ is mapped isomorphically onto $N$ via $u$.

We now set $M' = N'$, $M_2' = \langle u^{-1}(N_2') \cap M' \rangle$ and $M'_1 = \ker(M' \to M_2')$. It is easy to check that we get a diagram as required in (8.2). $\square$

Given a smooth scheme $X$ and a closed point $x \in X$, let $T_x X$ denote the tangent space of $X$ at $x$. Notice that if $X \in \text{Sm}_k^G$ and if $x \in X^G$, then $G$ naturally acts $k(x)$-linearly on $T_x X$. For an affine scheme $X$, its ring of regular functions will be denoted by $k[X]$.

**Lemma 8.2.** Let $X \in \text{Sm}_k^G$ be an affine scheme and let $Z \subseteq X$ be a smooth $G$-invariant closed subscheme. Let $x \in Z$ be a $k$-rational point such that $x \in X^G$. Then there is a $G$-invariant affine neighborhood $U \subseteq X$ of $x$ and a $G$-equivariant étale map $f : U \to T_x X$ such that $f^{-1}(T_x Z) = Z \cap U$.

**Proof.** Let $m_X \subseteq k[X]$ denote the maximal ideal defining the closed point $x$. Since $x \in X^G$, we see that $m_X$ (and all its powers) acquires natural $G$-action coming from the $G$-action on $k[X]$ and the surjection $u : m_X \twoheadrightarrow m_X/m_X^2 = (T_x X)^*$ is an $H$-equivariant $k$-linear map. Let $I \subseteq k[X]$ denote the ideal defining the closed subscheme $Z$. Then $I$ is also $G$-invariant under $G$-action on $k[X]$. Thus we get a...
commutative diagram of $G$-modules and $G$-linear maps:

\[
\begin{array}{cccccc}
0 & \rightarrow & I & \rightarrow & m_X & \rightarrow & m_Z & \rightarrow & 0 \\
0 & \rightarrow & I & \rightarrow & m_X & \rightarrow & m_Z & \rightarrow & 0
\end{array}
\]

in which the rows are exact, the vertical maps are surjective and the bottom row consists of finite $G$-modules.

We can now apply Lemma 8.1 to get a commutative diagram of exact sequences of finite $G$-modules:

\[
\begin{array}{cccccc}
0 & \rightarrow & M_{(X,Z)} & \rightarrow & M_X & \rightarrow & M_Z & \rightarrow & 0 \\
0 & \rightarrow & (N_x Z)^* & \rightarrow & (T_x X)^* & \rightarrow & (T_x Z)^* & \rightarrow & 0
\end{array}
\]

such that the vertical maps are all isomorphisms, Here $N_x Z$ denotes the normal space of $Z \hookrightarrow X$ at $x$. Moreover, the top row is a sequence of $G$-submodules of the top row of (8.3). Notice also that as part of the proof of Lemma 8.1 we have shown that there is a $k$-basis of $M_X$ which maps onto the $k$-bases of $M_Z$ as well as $(T_x X)^*$.

Using these bases, we can now construct a commutative diagram of exact sequences of finite $G$-modules:

\[
\begin{array}{cccccc}
0 & \rightarrow & (N_x Z)^* & \rightarrow & (T_x X)^* & \rightarrow & (T_x Z)^* & \rightarrow & 0 \\
0 & \rightarrow & M_{(X,Z)} & \rightarrow & M_X & \rightarrow & M_Z & \rightarrow & 0
\end{array}
\]

such that the vertical maps are isomorphisms.

The maps $u_X^{-1}$ and $u_Z^{-1}$ induce the corresponding $G$-equivariant maps of the associated symmetric algebras over $k$ (recall that $x \in X(k)$) and composing these maps of symmetric algebras with inclusions $\text{Sym}^*(M_X) \hookrightarrow k[X]$ and $\text{Sym}^*(M_Z) \hookrightarrow k[Z]$, we get a commutative diagram of $G$-equivariant morphisms

\[
\begin{array}{cccc}
\text{Sym}_k^*((T_x X)^*) & \rightarrow & \text{Sym}_k^*((T_x Z)^*) \\
\downarrow & & & \downarrow \\
k[X] & \rightarrow & k[Z].
\end{array}
\]

To check that the kernel of the top row maps onto the ideal $I$ locally at the closed point $x$, we just have to observe from (8.3) that $(N_x Z)^*$ is nothing but $I/(I \cap m_X^2)$ and it maps to the ideal of $Z$ near $x$ via $u_X^{-1}$.

It is easy to check from the local criterion of flatness that $u_X^{-1}$ is flat near $x$. Furthermore, $u_X^{-1}$ clearly induces an isomorphism of the tangent spaces at $m_X$ and $u_X^{-1}(m_X)$. If we set $f$ to be the morphism $f : X \rightarrow T_x X$ defined by $u_X^{-1}$, we see that $f$ is an $G$-equivariant morphism which is étale at $x$ and $f^{-1}(T_x Z) = Z$ near $x$. We conclude that there is an affine neighborhood $U' \subseteq X$ of $x$ such that the
restriction \( f_U \) on \( U' \) is étale and \( f_U^{-1}(T_xZ) = Z \cap U' \). Finally, using the fact that \( x \in X^G \), we set \( U = \bigcap_{g \in G} gU' \) and conclude that \( U \subseteq X \) is a \( G \)-invariant affine neighborhood of \( x \) and there is a \( G \)-equivariant étale map \( f : U \to T_xX \) such that \( f^{-1}(T_xZ) = Z \cap U \). The proof of the lemma is now complete. \( \square \)

**Proposition 8.3.** Let \( G \) be a finite cyclic group of prime order \( p \) which is different from the characteristic of \( k \). Let \( Z \hookrightarrow X \) be a closed immersion in \( \text{Sm}^G_k \) and \( x \in Z \) a \( k \)-rational point. Then, there is a \( G \)-invariant affine neighborhood \( U \) of \( x \), a \( G \)-representation \( V \) with \( G \)-submodule \( Z_V \) and a \( G \)-equivariant étale map \( f : U \to V \) such that \( f^{-1}(Z_V) = Z \cap U \).

**Proof.** Let \( X^G \) denote the closed subscheme of fixed points for the \( G \)-action on \( X \). We first assume that \( x \notin X^G \). Since \( X \setminus X^G \) is \( G \)-invariant, we can assume that \( X^G = \emptyset \). Since \( G \) is a cyclic group of prime order, it acts freely on \( X \). In particular, the quotient map \( \pi : X \to X/G \) is finite étale of degree \( p \). Set \( X' = X/G \) and \( Z' = Z/G \). Then we see that \( p \) is a \((G \text{-equivariant})\) finite étale map with \( \pi^{-1}(Z') = Z \).

Since \((X', Z')\) is a closed immersion of smooth schemes over \( k \), we know that there is an affine neighborhood \( U' \) of \( x' = \pi(x) \) in \( X' \) and an étale map \( f' : U \to \mathbb{A}^d_k \) such that \( f'^{-1}((\mathbb{A}^d_k \times \{0\})) = Z' \) for some \( 1 \leq c \leq d \). Setting \( f = f' \circ \pi \) and \( U = \pi^{-1}(U') \), we conclude that \( U \) is a \( G \)-invariant affine neighborhood of \( x \). Moreover, there is a \( G \)-equivariant étale map \( f : U \to \mathbb{A}^d_k \) (with respect to the trivial action on \( \mathbb{A}^d_k \)) such that \( f^{-1}((\mathbb{A}^d_k \times \{0\})) = Z \cap U \).

We next suppose that \( x \in X^G \). Let \( U' \) be an affine neighborhood of \( x \) in \( X \). Since \( G_x = G \), we see that \( S_x = G \). In particular, \( U = \bigcap_{g \in G} gU' \) is a \( G \)-invariant affine neighborhood of \( x \). We can thus assume that \( X \) is affine. It follows now from Lemma 8.2 that there is a \( G \)-invariant affine neighborhood \( U \) of \( x \) in \( X \) and a \( G \)-equivariant étale map \( f : U \to T_xX \) such that \( f^{-1}(T_xZ) = Z \cap U \). Moreover, as \( p \neq \text{char}(k) \), there is a \( G \)-equivariant decomposition \( T_xX = T_xX \times N_xX \). \( \square \)

**Definition 8.4.** Given a closed immersion \( Z \hookrightarrow X \) in \( \text{Sm}_k^G \), an \( eN \)-linearization of the pair \((X, Z)\) is a pair \((p, q)\) of maps in \( \text{Sm}_k^G \) given by

\[
(X, Z) \xleftarrow{p} (U, Z) \xrightarrow{q} (N_{Z/X}, Z)
\]

such that \( p \) and \( q \) are both distinguished \( eN \)-neighborhoods. We shall say that \((X, Z)\) admits an \( eN \)-linearization if the pair \((p, q)\) as in (8.7) exists.

**Proposition 8.5.** Let \( G \) be a finite cyclic group of prime order \( p \) which is different from the characteristic of \( k \). Let \( Z \hookrightarrow X \) be a closed immersion in \( \text{Sm}_k^G \) and \( x \in Z \) a \( k \)-rational point. Then, there is a \( G \)-invariant affine neighborhood \( U \) of \( x \) such that the pair \((W, w^{-1}(Z \cap U))\) admits an \( eN \)-linearization for any \( G \)-equivariant étale map \( w : W \to U \).

**Proof.** Given any map \( W \to X \), we set \( Z_W = Z \times_X W \). We choose a \( G \)-invariant affine neighborhood \( U \) of \( x \), and a \( G \)-equivariant étale map \( f : U \to V \) as in Proposition 8.3. Let \( f_Z : Z_U \to Z_V \) denote the restriction of \( f \) to \( Z \).

Since \( p \neq \text{char}(k) \), there is a \( G \)-equivariant decomposition \( V = Z_V \times N_{Z/V} \). Let \( j : N_{Z/V} \hookrightarrow V \) be the inclusion map and let \( f' : Z_U \times N_{Z/V} \to V \) be the
$G$-equivariant map $f_Z \times j$. We now consider a commutative diagram in $\text{Sm}_k^G$:

\begin{align}
Z_U &\xrightarrow{i} \tilde{U} \xrightarrow{q} Z_U \times N_{Z/V} \\
U &\xrightarrow{f} V
\end{align}

in which $\tilde{U}$ is defined so that the square is Cartesian and $i'$ is the zero section inclusion. Notice that $\tilde{U}$ is smooth since $(f$ and hence) $q$ is étale.

It is easy to check that $(f' \circ q)^{-1}(Z_V)$ is the $G$-invariant closed subscheme $Z_U \times Z_U$.

Since $Z_U \rightarrow Z_V$ (obtained by the restriction of $f$) is étale by Proposition 8.3, we see that this closed subscheme is a disjoint union of diagonal $\Delta_{Z_U} : Z_U \hookrightarrow Z_U \times Z_U$ and a $G$-invariant closed subscheme $Y$. In particular, $Y$ is a $G$-invariant closed subscheme of $\tilde{U}$. Setting $\tilde{U} = \tilde{U} \setminus Y$, we get $G$-equivariant étale maps $p : \tilde{U} \rightarrow U$ and $q : \tilde{U} \rightarrow V$ and one checks from the construction that $p^{-1}(Z_U) = q^{-1}(Z_V) = \tilde{i}(Z_U)$.

If $w : W \rightarrow U$ is a $G$-equivariant étale map, then we have $N_{Z_W/W} \simeq Z_W \times N_{Z_U/U}$.

Let $w_Z : N_{Z_W/W} \rightarrow N_{Z_V/V} \simeq V$ denote the projection map. This yields an analogous commutative diagram:

\begin{align}
Z_W &\xrightarrow{i} \tilde{W} \xrightarrow{q} N_{Z_W/W} \\
W &\xrightarrow{f \circ w_Z} V
\end{align}

where the lower square is Cartesian. We now repeat the above construction for $f : U \rightarrow V$ verbatim to get $G$-equivariant étale maps $p : \tilde{W} \rightarrow W$ and $q : \tilde{W} \rightarrow N_{Z_W/W}$ and one checks from the construction that $p^{-1}(Z_W) = q^{-1}(Z_W) = \tilde{i}(Z_W)$, where $Z_W \subset N_{Z_W/W}$ is the zero-section.

\section{The equivariant homotopy purity and blow-up theorems}

The \textit{equivariant Thom space} of a $G$-equivariant vector bundle $V \rightarrow X$ in $\text{Sm}_k^G$ is the pointed motivic $G$-space $V/(V \setminus X)$, where $X \hookrightarrow V$ is the zero section. We prove the following purity theorem for normal bundles.

\begin{theorem}
Let $k$ be an algebraically closed field and let $G$ be a finite cyclic group of prime order $p$ which is different from the characteristic of $k$. Let $Z \hookrightarrow X$ be a closed immersion in $\text{Sm}_k^G$. Then there is a canonical isomorphism in $\text{Ho}_k^G \cdot (k)$ of pointed motivic $G$-spaces

$$X/(X \setminus Z) \simeq \text{Th}(N_{Z/X}).$$
\end{theorem}
Proposition 5.4. We conclude that the map

\( \alpha \colon \text{Th}(N_{Z/X}) \rightarrow \text{Th}(N_{Z/X} \times \mathbb{A}^1) \)

is a motivic weak equivalence. On the other hand, the composite map

\( \beta \colon \text{Th}(N_{Z/X}) \rightarrow \text{Th}(N_{Z/X} \times \mathbb{A}^1) \)

is a local weak equivalence. Since \( q \) is a motivic weak equivalence, we conclude that \( \beta \) is a motivic weak equivalence. \( \square \)
9.2. Purity in general. Let \((X, Z)\) be a closed pair in \(\mathbf{Sm}^G_{\text{Sm}}\) as in Theorem 9.1. It follows from Proposition 8.5 that for every \(x \in Z\), there exists a \(G\)-invariant affine neighborhood \(U\) of \(x\) such that the pair \((U, Z \cap U)\) admits an \(eN\)-linearization. Since \(X\) is noetherian, there exists a finite set \(\{U_1, \ldots, U_r\}\) of \(G\)-invariant affine open subsets of \(X\) such that \(X = \bigcup_{i=1}^r U_i\) and each pair \((U_i, Z \cap U_i)\) admits an \(eN\)-linearization.

Set \(U = \prod_{i=1}^r U_i\) and \(Z_U = \prod_{i=1}^r (Z \cap U_i)\). Then \((U, Z_U)\) is a pair of objects in the category \(\mathbf{Shv}^G_{\text{Sm}_{\text{Nis}}}\) and there is a canonical map of sheaves \(U : (U, Z_U) \rightarrow (X, Z)\).

Let \(\mathcal{U}\) (resp. \(\mathcal{Z}_U\)) denote the simplicial sheaf on \(\mathbf{Sm}^G_{\text{Nis}/k}\) whose term at level \(n\) is the \((n+1)\)-fold product \(U \times \cdots \times U\) (resp. \(Z_U \times \cdots \times Z_U\)). This yields a pair of motivic \(G\)-spaces \((\mathcal{U}, \mathcal{Z}_U)\) and a map of pairs of motivic \(G\)-spaces \(f : (\mathcal{U}, \mathcal{Z}_U) \rightarrow (X, Z)\). Setting \(U^X\) to be the \((n+1)\)-fold product \(U \times \cdots \times U\), we see that \(U^X\) is the coproduct of smooth \(G\)-schemes each of which is a fiber product (over \(X\)) of \(n+1\) components of \(U\). Set \(U \setminus \mathcal{Z}_U = u^{-1}(X \setminus Z)\).

Let \(B\) denote the motivic \(G\)-space obtained by applying the \(B(X, Z)\) construction levelwise to the inclusion \(\mathcal{Z}_U \hookrightarrow \mathcal{U}\) (see [24, p. 117]). Observe here that this inclusion is the coproduct of closed embeddings of smooth \(G\)-schemes at each level. Moreover, we have \(B(X \setminus Z, \emptyset) \simeq (X \setminus Z) \times \mathbb{A}^1\) and \(\text{Th}(N_{\emptyset/(X \setminus Z)}) = \text{Spec}(k)\) (as a pointed motivic \(G\)-space). Let \(\text{Th}(N_{\mathcal{Z}_U/U})\) denote the motivic \(G\)-space which is obtained by applying the levelwise Thom space construction for the inclusion \(\mathcal{Z}_U \hookrightarrow N_{\mathcal{Z}_U/U}\). This makes sense because \(\mathcal{Z}_U \hookrightarrow N_{\mathcal{Z}_U/U}\) is the coproduct of 0-section embeddings of equivariant vector bundles over smooth \(G\)-schemes at each level. We obtain a commutative diagram of pointed motivic \(G\)-spaces

\[
\begin{array}{ccc}
U & \xrightarrow{\mathcal{U}} & B \\
\downarrow & & \downarrow \\
\mathcal{Z}_U & \xrightarrow{\mathcal{B}} & \text{Th}(N_{\mathcal{Z}_U/U}) \\
\downarrow & & \downarrow \\
X & \xrightarrow{B(X, Z)} & \text{Th}(N_{Z/X}).
\end{array}
\]

Lemma 9.3. The vertical arrows in (9.4) are local weak equivalences in the \(eN\)-topology.

Proof. It suffices to show that the left vertical arrow is a local weak equivalence as the same argument shows this weak equivalence for the other two vertical arrows. We first claim that the map of sheaves \(U \rightarrow X\) is an epimorphism in the \(eN\)-topology. Using Proposition 8.1 it suffices to show that for any \(Y \in \mathbf{Sch}^G_{\text{Nis}}\) and a point \(y \in Y\), the map \(U(Y^h_{G_Y}) \rightarrow X(Y^h_{G_Y})\) is surjective. So let \(v : Y^h_{G_Y} \rightarrow X\) be a \(G\)-equivariant morphism where \(Y^h_{G_Y}\) is the henselization of a \(G\)-scheme \(Y\) along the \(G\)-orbit \(G_Y\). By our construction of \(U\), there is a component \(U_x\) of the scheme \(U\) such that \(u : (U_x, G_x) \rightarrow (X, G_x)\) is an affine (Zariski) neighborhood of \(G_x\). In particular, the map \((U_x)_{G_x}^h \rightarrow X^h_{G_x}\) is a \(G\)-equivariant isomorphism of semi-local \(G\)-schemes.

Now, the map \(v\) induces a \(G\)-equivariant map \(Y^h_{G_Y} \rightarrow X^h_{G_x}\) which takes \(G_Y\) onto \(G_x\). Since \(u\) is a \(G\)-equivariant isomorphism, we immediately get a \(G\)-equivariant
morphism \( w : Y^h_{G_y} \to (U_x)^h_{G_x} \) such that \( v = u \circ w \). Composing \( w \) with the canonical maps \((U_x)^h_{G_x} \to U_x \leftarrow U\), we get a map \( Y^h_{G_x} \to U \) which factors \( v \). This proves the claim.

Since the \( eN \)-topology on \( \text{Sm}_k^G \) admits a conservative family of points by Proposition 3.14, we can use the above claim and [24, Lemma 2.1.15] to conclude that the map \( U \to X \) is a local weak equivalence. For the same reason, the map \( U \setminus Z_U \to X \setminus Z \) is a local weak equivalence. We conclude from [24, Lemma 2.2.11] that also \( U \setminus Z_U \to X \setminus Z \) is a local weak equivalence. \( \square \)

**Proof of Theorem 9.1:** It suffices to show that the maps \( \alpha_{X,Z} \) and \( \beta_{X,Z} \) are motivic weak equivalences. By Lemma 9.3, this is equivalent to showing that the top horizontal maps in (9.4) are motivic weak equivalences. By [24, Proposition 2.2.14], it suffices to show that the top horizontal maps of simplicial sheaves in (9.4) are motivic weak equivalences at each level \( n \geq 0 \).

The top horizontal maps in (9.4) are isomorphisms for \( X \setminus Z \) and all its \( G \)-invariant open subsets. Thus we are left with showing that the top horizontal maps in (9.4) are motivic weak equivalences for closed pairs of the form \((U_x, Z_{U_x})\) and \((W, \psi^{-1}(Z_{U_x}))\), where \( \psi : W \to U_x \) is a \( G \)-equivariant étale map. By Proposition 8.5, we are reduced to proving the theorem under the assumption that the closed pair \((X, Z)\) admits an \( eN \)-linearization.

So let \((X, Z) \xleftarrow{\phi} (U, Z) \xrightarrow{\eta} (N_{Z/X}, Z)\) be an \( eN \)-linearization of \((X, Z)\) and consider the commutative diagram

\[
\begin{array}{ccc}
U \setminus Z_U & \xrightarrow{B(U, Z_U)} & B(U, Z_U) \setminus (Z_U \times \mathbb{A}^1) \\
\downarrow & & \downarrow \\
X \setminus Z & \xrightarrow{B(X, Z)} & B(X, Z) \setminus (Z \times \mathbb{A}^1) \\
\end{array}
\]

\[
\text{Th}(N_{Z/U}) \quad \text{Th}(N_{Z/X}).
\]

The vertical arrows in (9.5) are local weak equivalences by our definition of local weak equivalence in the \( eN \)-topology. Hence, the top horizontal maps are motivic weak equivalences if and only if so are the bottom horizontal maps.

If we apply this argument for \((N_{Z/X}, Z)\) in place of \((X, Z)\), it follows from the local weak equivalence \( U \setminus Z_U \xrightarrow{\xi} \text{Th}(N_{Z/X}) \) and Lemma 9.2 that the top horizontal maps in (9.5) are motivic weak equivalences. We conclude that the maps \( \alpha_{X,Z} \) and \( \beta_{X,Z} \) are motivic weak equivalences. \( \square \)

Using the same line of proof as for Theorem 9.1 verbatim, we obtain the following result for equivariant blow-ups.

**Theorem 9.4.** Let \( k \) be an algebraically closed field and let \( G \) be a finite cyclic group of prime order \( p \) which is different from the characteristic of \( k \). Let \( Z \hookrightarrow X \) be a closed immersion in \( \text{Sm}_k^G \) with complement \( U = X \setminus Z \). Let \( p : X' \to X \)
denote the blow-up of \( X \) along \( Z \). Then the square
\[
\begin{array}{ccc}
p^{-1}(Z) & \longrightarrow & X'/U \\
\downarrow & & \downarrow \\
Z & \longrightarrow & X/U
\end{array}
\]
is homotopy cocartesian for the motivic (injective) model structure on \( \mathcal{M}_k^G \). In other words, the map \( X'/U \coprod_{p^{-1}(Z)} Z \to X/U \) is a motivic weak equivalence.

10. \( eN \)-descent and unstable representability for equivariant \( K \)-theory

In this section, we establish Nisnevich descent for equivariant \( K \)-theory for \( k \)-schemes (not necessarily smooth). It follows that equivariant \( K \)-theory of smooth schemes is represented by an object in the equivariant motivic homotopy category. As an application, we characterize all equivariantly contractible smooth affine curves with a group action, and moreover all equivariant vector bundles on such curves.

10.1. The equivariant \( K \)-theory presheaf on \( \text{Sm}^G_k \). Quillen’s \( Q \)-construction associates to an exact category \( \mathcal{E} \) with a chosen zero object \( \{0\} \), the category \( Q \mathcal{E} \) whose objects are same as those of \( \mathcal{E} \) but the morphisms between two objects \( M' \) and \( M'' \) are diagrams \( M' \leftarrow N \to M'' \), where the first arrow is an admissible epimorphism and the second arrow is an admissible monomorphism. Taking the classifying space of \( Q \mathcal{E} \), one obtains a simplicial space \( BQ \mathcal{E} \) and the \( K \)-theory space of \( \mathcal{E} \) is defined as
\[
K(\mathcal{E}) = \Omega BQ \mathcal{E}.
\]
Alternate approaches include the \( S_\bullet \)-construction of Waldhausen [35] and the \( G \)-construction of Gillet-Grayson [9]. An advantage of the \( G \)-construction is that the resulting simplicial space \( G \mathcal{E} \) defining \( K \)-theory is homotopy equivalent to \( \Omega BQ \mathcal{E} \).

Let \( G \) be a smooth affine group scheme over \( k \). If \( \mathcal{E}_X \) is the exact category of \( G \)-equivariant vector bundles on any \( X \in \text{Sch}_k^G \), there is a simplicial set \( G \mathcal{E}_X \) homotopy equivalent to \( \Omega BQ \mathcal{E}_X \). With either of these approaches, algebraic \( K \)-theory is only a pseudo-presheaf of simplicial sets (or spectra) on the category of \( G \)-schemes, and not an honest simplicial presheaf. This is remedied by rectification of pseudo-functors as in e.g., [28, Lemma 3.2.6] and the rectification procedure as explained in [19, Chapter 5, page 179], this process yields the following equivariant \( K \)-theory presheaf on \( \text{Sch}_k^G \), and hence on \( \text{Sm}_k^G \) via the full embedding \( \text{Sm}_k^G \hookrightarrow \text{Sch}_k^G \).

**Proposition 10.1.** There is a presheaf of pointed simplicial sets \( K^G \) on \( \text{Sch}_k^G \) with a monoidal structure
\[
K^G \times K^G \to K^G
\]
such that for every $X \in \mathbf{Sch}_k^G$ and $i \geq 0$, there is a canonical isomorphism
\[
\pi_i(K^G(X)) \cong K^G_i(X).
\]

10.2. Nisnevich descent theorems for equivariant $K$-theory. We refer to
Theorem 4.8 for the $eN$-local injective model structures on $\mathbf{Sch}_k^G$ and $\mathbf{Sm}_k^G$. The
descent problem amounts to the following in our setting.

Definition 10.2. A simplicial presheaf $F$ on $\mathbf{Sch}_k^G$ or $\mathbf{Sm}_k^G$ satisfies $eN$-descent
if every fibrant replacement $F \to \text{Ex}F$ in the $eN$-local injective model structure
is an objectwise weak equivalence.

We shall say that a scheme $X \in \mathbf{Sch}_k^G$ is locally $G$-affine if every point in $X$ has a
$G$-invariant affine neighborhood. Observe that all quasi-projective $G$-schemes are
locally $G$-affine if $G$ is finite. Moreover, all locally $G$-affine schemes admit good
quotients for the $G$-action. We denote the category of locally $G$-affine schemes by
$\mathbf{Sch}_k^{G,\text{Aff}}$.

Proposition 10.3. Let $f : Y \to X$ be a $G$-equivariant étale morphism in $\mathbf{Sch}_k^G$
such that one of the following holds.

(1) $X$ and $Y$ are locally $G$-affine and $G$ is finite with $(\text{char}(k), |G|) = 1$.
(2) $X$ and $Y$ are smooth.

Suppose there is a Cartesian square in $\mathbf{Sm}_k^G$
\[
\begin{array}{ccc}
W & \to & Y \\
\downarrow & & \downarrow f \\
U & \to & X \\
\end{array}
\]
where $j$ is an open immersion such that the map $f$ is an isomorphism over the
complement of $U$ (with the reduced structures). Then the diagram of simplicial
sets
\[
\begin{array}{ccc}
K^G(X) & \to & K^G(Y) \\
\downarrow & & \downarrow \\
K^G(U) & \to & K^G(W) \\
\end{array}
\]
is homotopy Cartesian.

Proof. If we are in the case (1) of the theorem, then our assumption implies that
$[Y/G] \to [X/G]$ is a representable morphism of tame Deligne-Mumford stacks
which admit coarse moduli schemes. Hence the theorem follows from [21 Corol-
mary 3.8].

Suppose now that we are in case (2). We then have a commutative diagram of
fibration sequences (see [30]):
\[
\begin{array}{ccc}
G^G(X \setminus U) & \to & K^G(X) \\
\downarrow & & \downarrow \\
G^G(U \setminus W) & \to & K^G(Y) \\
\end{array}
\]
where $G^G(X)$ denotes the $K$-theory of the exact category of the equivariant coherent sheaves on a $G$-scheme $X$. Our hypothesis implies that the left vertical arrow is a weak equivalence. The theorem now follows. □

The following are the main results of this section.

**Theorem 10.4.** Let $G$ be a smooth affine group scheme over $k$. The simplicial presheaf $K^G$ satisfies $eN$-descent on $Sm^G_k$.

If $G$ is finite with $(\text{char}(k), |G|) = 1$, then $K^G$ satisfies $eN$-descent on $Sch^G_{k, 1\text{Aff}}$.

**Proof.** Let $K^G \to \text{Ex}K^G$ be a fibrant replacement in the local injective model structure on the simplicial presheaves on $Sch^G_{k, \mathbf{1}}$ (or $Sm^G_k$). We have to show that this is a schemewise weak equivalence. By Proposition 4.11 we only have to show that $K^G$ is flasque. We remark here that even though this proposition is stated for smooth schemes, it is valid (with same proof) for all $G$-schemes (see [34, Lemma 3.5]). Using Proposition [10.1] and [15, Proposition 13.3.13], it is enough to show that $K^G$ takes a distinguished $eN$-square (3.3) to a homotopy Cartesian square. This follows directly from Proposition [10.3]. □

**Theorem 10.5.** Let $k$ be any field and $G$ a smooth affine group scheme over $k$. For any $X \in Sm^G_k$ and $i \geq 0$, there is a canonical isomorphism

$$K^G_i(X) \xrightarrow{\sim} \text{Hom}_{\text{Ho}^G_{k, \mathbf{1}}(k)} \left(S^i_s \times X_+, K^G \right).$$

**Proof.** Let $K^G \to \text{Ex}K^G$ be a fibrant replacement of $K^G$ in the motivic injective model structure on $\mathcal{M}^G_{k, \mathbf{1}}$. The homotopy invariance property of equivariant $K$-theory for smooth $G$-schemes implies that the motivic $G$-space $K^G$ is $A^1$-weak invariant (see Definition 5.1). Combining this with Proposition 10.3 we deduce that $K^G$ is $A^1$-flasque (see § 5.1.1). We conclude from Theorem 5.3 that the map $K^G \to \text{Ex}K^G$ is a schemewise weak equivalence.

We now apply Proposition 6.5 to get a canonical isomorphism

$$\text{Hom}_{\text{Ho}^G_{k, \mathbf{1}}(k)} \left(S^i_s \times X_+, K^G \right) \xrightarrow{\sim} \text{Hom}_{\text{Ho}^G_{k, \mathbf{1}}(k)} \left(S^i_s \times X_+, \text{Ex}K^G \right) \xrightarrow{\sim} \left[S^i_s, \text{Ex}K^G(X) \right] \xrightarrow{\sim} \pi_i(\text{Ex}K^G(X)) \xrightarrow{\sim} \pi_i(K^G(X)) \xrightarrow{\sim} K^G_i(X).$$

This completes the proof of the theorem. □

### 10.3. Algebraic analogue of Segal’s theorem.

Recall that if $G$ is a topological group, two $G$-equivariant continuous maps $\phi_0, \phi_1 : X \to Y$ between topological $G$-spaces are called $G$-homotopic if there exists a continuous $G$-equivariant map $H : X \times [0, 1] \to Y$ (with trivial $G$-action on $[0, 1]$) such that $H \circ i_j = \phi_j$ for $i_j : \{j\} \to [0, 1], j = 0, 1$.

It was shown by Segal in [26, Proposition 2.3] that $G$-homotopic maps induce the same maps on equivariant topological $K$-theory. As an application of Theorem 10.5 we prove the following algebraic analogue of Segal’s theorem.
Corollary 10.6. Let \( G \) be a smooth affine group scheme over a field \( k \), and \( \phi_0, \phi_1 : X \to Y \) equivariantly \( \mathbb{A}^1 \)-homotopic maps in \( \text{Sm}_k^G \) (see § 5.2). Then \( \phi_0^* = \phi_1^* : K_*^G(Y) \to K_*^G(X) \).

Proof. It is enough to consider the case when \( \phi_0 \) and \( \phi_1 \) are elementary \( \mathbb{A}^1 \)-homotopic. Let \( i_0, i_1 : \text{Spec}(k) \to \mathbb{A}^1 \) be the two inclusions with \( i_0(pt) = 0 \) and \( i_1(pt) = 1 \). It suffices to show that \( i_0^* = i_1^* : K_*^G(X \times \mathbb{A}^1) \to K_*^G(X) \).

Let \( p : X \times \mathbb{A}^1 \to X \) denote the projection map. It follows from Theorem [10.5] that \( p^* : K_*^G(X) \to K_*^G(X \times \mathbb{A}^1) \) is an isomorphism. Hence, it suffices to show that \((p \circ i_0)^* = (p \circ i_1)^* : K_*^G(X) \to K_*^G(X) \). Both these maps equal the identity on \( K_*^G(X) \). \( \square \)

10.4. Equivariantly contractible affine curves. We shall say that a motivic \( G \)-space \( X \) is equivariantly \( \mathbb{A}^1 \)-contractible if the map \( X \to pt \) is a motivic weak equivalence. A \( G \)-equivariant vector bundle \( V \) on \( X \in \text{Sm}_k^G \) is called trivial if there is a \( G \)-representation \( V \) such that \( V = V \times X \).

As an application of Theorem [10.5], we prove the following desired geometric result on equivariant vector bundles.

Theorem 10.7. Let \( k \) be an infinite field and let \( G = \langle \sigma \rangle \) be a finite cyclic group of order prime to the characteristic of \( k \) such that \( \mu_{|G|} \subset k \). Let \( X \) be a smooth affine curve over \( k \) with \( G \)-action. Then \( X \) is equivariantly \( \mathbb{A}^1 \)-contractible if and only if it is isomorphic to an 1-dimensional linear representation of \( G \). In particular, all \( G \)-equivariant vector bundles on \( X \) are trivial if \( X \) is equivariantly \( \mathbb{A}^1 \)-contractible.

Proof. The assertion that a finite-dimensional representation of \( G \) is equivariantly \( \mathbb{A}^1 \)-contractible, follows immediately from Proposition [5.4]. Below we prove the more difficult converse statement.

Suppose that \( X \) is equivariantly \( \mathbb{A}^1 \)-contractible. Since the action of \( G \) on a smooth scheme is linearizable, we can assume that there is smooth projective curve \( \overline{X} \in \text{Sm}_k^G \) and an open immersion \( j : X \hookrightarrow \overline{X} \) in \( \text{Sm}_k^G \). Let \( f : X \to \text{Spec}(k) \) be the structure map.

Claim 1: The curve \( X \) is rational.

Proof of claim 1: Consider the commutative diagram

\[
\begin{array}{ccc}
K_*^G(k) \otimes_{R(G)} \mathbb{Z} & \xrightarrow{f^*_k} & K_i(k) \\
\downarrow f^* & & \downarrow f^* \\
K_*^G(X) \otimes_{R(G)} \mathbb{Z} & \xrightarrow{f^*_X} & K_i(X)
\end{array}
\]

with forgetful horizontal maps from equivariant to ordinary \( K \)-theory. Theorem [10.5] shows the left vertical arrow is an isomorphism for all \( i \geq 0 \). The top horizontal arrow is an isomorphism for all \( i \geq 0 \) by [29] Lemma 5.6. Applying these facts for \( i = 0 \), we see that the composite map

\[
K_0^G(X) \otimes_{R(G)} \mathbb{Z} \to K_0(X) \to \mathbb{Z}
\]
is an isomorphism. On the other hand, the first map is surjective over $\mathbb{Z}[1/|G|]$ by \cite{32} Theorem 1. It follows that $\text{Pic}(X)$ is a torsion group of exponent $|G|$, which happens if and only if $X$ is rational. This proves the claim.

**Claim 2:** $X$ is isomorphic (not necessarily equivariantly) to $\mathbb{A}^1$.

**Proof of claim 2:** Claim 1 implies that $X \simeq \mathbb{P}^1_k$. Inserting $i = 1$ in \cite{10.3} shows the composite map

$$K_1^G(X) \otimes_{\mathbb{Z}[1/|G|]} \mathbb{Z} \xrightarrow{f_1^X} K_1(X) \to \mathcal{O}^\times(X)$$

is just the inclusion $k^\times \hookrightarrow \mathcal{O}^\times(X)$. On the other hand, $f_1^X$ is surjective over $\mathbb{Z}[1/|G|]$ by \cite{32} Theorem 1. It follows that $k^\times[1/|G|] \simeq \mathcal{O}^\times(X)[1/|G|]$, which happens if and only if $X \simeq \mathbb{A}^1$ as an open subscheme of $\mathbb{P}^1_k$.

By the above claims $X$ is the affine line with $G = \langle \sigma \rangle$-action $\sigma(x) = ax + b$ for some fixed $a, b \in k$ with $a^{[G]} = 1$. If $b \neq 0$, then $\sigma$ acts on $\mathbb{A}^1$ without fixed points. This means that the identity map of $\mathbb{A}^1$ gives an element of $[\mathbb{A}^1, \mathbb{A}^1]_{G, \mathbb{A}^1}$ which can not be equivariantly contracted to any fixed point. In particular, $\pi_0^{G,\mathbb{A}^1}(X)$ is not constant and hence $X \to \text{Spec}(k)$ is not a motivic weak equivalence, which contradicts our assumption. We conclude that $b = 0$ and $G$ acts linearly on $\mathbb{A}^1$.

Finally, the claim about the triviality of all $G$-equivariant vector bundles on $X$ follows from the above combined with \cite{6} and \cite{23} Theorem 1. \hfill $\Box$

**Example 10.8.** Theorem \cite{10.7} shows that equivariant $\mathbb{A}^1$-contractibility is a strictly stronger condition than ordinary $\mathbb{A}^1$-contractibility, as one would expect. As an example, let the cyclic group of order two $G = \langle \sigma \rangle$ act on $\mathbb{A}^1$ by $\sigma(x) = 1 - x$. This action is fixed point free and hence not isomorphic to a $G$-representation. Thus $\mathbb{A}^1$ with this action is not equivariantly $\mathbb{A}^1$-contractible.

**Remark 10.9.** One can ask whether the assertion of Theorem \cite{10.7} is true in higher dimensions as well. This seems to be a very difficult question. We do not know the answer even when $G$ is trivial and $X$ is a surface. That is, it is unknown whether an $\mathbb{A}^1$-contractible smooth affine surface is isomorphic to the affine plane. It is known, however, that such surfaces do not admit any non-trivial vector bundle.

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