Loop analysis of adaptive notch filters

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Abstract: Adaptive notch filters have been the focus of intense research for more than three decades. Low computational requirements and good performance make them attractive for tracking frequency modulated signals. Despite the extensive literature on adaptive notch filters, algorithm extensions and new models for describing the dynamics of the notch frequency continue to be proposed. In this study, the equivalence between adaptive notch filters using a plain gradient (PG) algorithm and frequency lock loops (FLLs) with exponential filtering is established. FLL theory is then used to analyse the noise performance and signal tracking capabilities of adaptive notch filters. A linear model describing the dynamics of the filter adaptation process is derived and the concepts of loop and Doppler bandwidths are introduced. Criteria based on the loop and Doppler bandwidths are suggested to set the PG adaptation step. Finally, algorithm extensions based on the FLL theory are proposed. Theoretical results are supported by Monte Carlo simulations which show the validity of the analysis performed.

1 Introduction

Adaptive notch filters have been studied for more than three decades. More specifically, the infinite impulse response (IIR) adaptive notch filter with constrained poles and zeros suggested by Nehorai [1] has been the focus of intense research for its low computational requirements and good performance. Adaptive notch filters are widely used for removing instantaneous narrow band interference [2] such as unwanted continuous waves and chirp signals [3]. For example, they find application in biomedical signal processing [4] and direct sequence spread spectrum communications [5]. In Global Navigation Satellite Systems (GNSS), the adaptive notch filter has been adopted for interference removal [6, 7] for its ability of attenuating only a narrow part of the spectrum, thus preserving most of the wideband Galileo/GPS signals.

Despite the extensive literature on the subject, algorithm extensions and new models for describing the dynamics of the adaptive notch filter continue to be proposed. For example, Punchalard et al. [8] suggested a modified sign algorithm (SA) to be used in conjunction with the filter structure proposed by Nehorai [1]. In [9], new error criteria were introduced for adapting the notch filter parameters. Loetwassana et al. [10, 11] suggested a modified plain gradient (PG) algorithm to avoid biases in the parameters estimated by a second-order adaptive notch filter. These are just a few examples of recent developments on adaptive notch filters.

The steady-state error of the adaptive notch filter has been mainly studied using the mean-square-error criterion [11, 12] and the linear filter approximation (LFA) approach [13, 14] has been developed for characterising the filter dynamics. In the LFA, two transfer functions, defining two equivalent filters, are introduced and the frequency estimate provided by the adaptive algorithm of the notch filter is described as the linear combination of filtered noise and filtered input signal frequency.

The LFA has been recently extended to generalised adaptive notch filters by Niedzwiecki and Kaczmarek [15] whereas a linear approximation accounting for the presence of a frequency acceleration in the input signal frequency has been considered by Niedzwiecki and Meller [16].

In this paper, a new approach is proposed for the analysis of adaptive notch filters. In particular, the equivalence between adaptive notch filters and standard frequency lock loops (FLLs) [17, 18] is first shown. The equivalence is valid when the PG algorithm used to update the adaptive notch filter is designed to minimise the energy at the output of the moving average (MA) part of the notch filter. The equivalence is then extended to the general case that considers the minimisation of the energy of the notch filter output. In this case, the adaptive notch filter is equivalent to an FLL with exponential filtering [19, 20].

FLL theory is then used to analyse the dynamics of the system. In particular, it is shown that the evolution of the frequency of the filter notch can be described using two equivalent filters as in the LFA approach. The noise transfer function found is equivalent to that obtained in [13, 14]. However, a different parameterisation for the signal component is obtained. The linear approximation obtained using the FLL theory is used to introduce the concepts of loop and Doppler bandwidths which characterise the filter performance in terms of noise rejection and tracking capabilities. Criteria based on the loop and Doppler bandwidths are finally suggested for setting the adaptation step of the algorithm used for updating the notch frequency: in the previous literature, the adaptation step was selected using empirical approaches.

Finally, algorithm extensions based on FLL theory are discussed: the theory developed provides a unified analysis of the adaptive notch filter and its extensions.

Theoretical results are supported by Monte Carlo simulations that show the validity of the analysis developed. Simulations also illustrate the impact of loop and Doppler bandwidths on the design of the adaptive notch filter.

The remaining of this paper is organised as follows: the adaptive notch filter is introduced in Section 2. The equivalence with FLLs is demonstrated in Sections 3 and 4. The linear equivalent model describing the filter dynamics is derived in Section 5, whereas the concepts of loop and Doppler bandwidths are introduced in Section 6. Algorithms extensions are discussed in Section 7 and simulation results are provided in Section 8. Finally, Section 9 concludes the paper.

2 Adaptive notch filter

In this section, the adaptive IIR notch filter is described. The time-invariant version of the filter is at first introduced and the
equations governing the IIR notch filter detailed. Then, the filter zero, \( z_0 \), is allowed to be time-varying leading to the adaptive notch filter.

The notch filter considered in this paper is characterised by the transfer function \([6, 21]\)

\[
H(z) = \frac{1 - z_0 z^{-1}}{1 - k_0 z^{-1}}
\]

where \( z_0 \) is the complex filter zero, which determines the frequency of the filter notch. \( k_0 \) is the pole contraction factor and assumes values in the range \([0, 1)\). This restriction is applied in order to guarantee the stability of the filter. Equation (1) defines the input–output relationship

\[
y[n] - k_0 z_0 y[n - 1] = x[n] - z_0 x[n - 1]
\]

where \( x[n] \) and \( y[n] \) are the samples taken at the input and output of the system, respectively. The filter zero, \( z_0 \), is constrained to lie on the unit circle and thus it can be expressed as

\[
z_0 = \exp[j f_0 T_s]
\]

where \( f_0 \) is the frequency of the filter notch expressed in units of radians/second. In this paper, frequencies will be expressed in units of radians/second and conversion in hertz (Hz) can be obtained through scaling by \( 2\pi \). \( T_s \) is the sampling interval of the signal at the input of the filter.

The output of the MA part of the filter is given by

\[
y_m[n] = x[n] - z_0 x[n - 1] = x[n] - \exp[j f_0 T_s] x[n - 1].
\]

The final output of the filter, \( y[n] \), can be rewritten exploiting the equivalent representation of (1)

\[
H(z) = \frac{1 - z_0 z^{-1}}{1 - k_0 z^{-1}}.
\]

In particular, \( y[n] \) can be expressed as

\[
y[n] = x[n] - k_0 z_0 x[n - 1]
\]

where

\[
x[n] = (1 - k_0) x[n] + k_0 z_0 x[n - 1]
\]

is the output of the auto regressive (AR) part of the filter. Equation (6) has the same functional form of (4) where \( x[n - 1] \) has been replaced by \( x_0[n - 1] \), the output of the AR part of the filter.

Equations (1)–(7) describe the basic structure of a time-invariant IIR notch filter. An adaptive version of such filter is obtained by allowing \( f_0 \) and \( z_0 \) to be time varying. In this case, an adaptation block is used to progressively update \( f_0 \) and \( z_0 \). Since the adaptation block can update \( f_0 \) and \( z_0 \) at each time instant, the dependence on the time index, \( n \), is explicit in the following.

The structure of the adaptive notch filter is detailed in Fig. 1: the adaptation block uses the input samples, \( x[n] \), to estimate the notch frequency, \( f_0[n] \), and to adjust the filter zero, \( z_0[n] \). The adaptation block used to estimate the notch frequency is the main focus of this paper and is analysed in detail in the following sections. In particular, PG algorithms are considered for the estimation of \( f_0[n] \). When the PG approach is adopted, the filter zero, \( z_0[n] \), is adapted using a gradient descent algorithm \([22]\). For example, Borio et al. and Calmettes et al. \([6, 23]\) adapted \( z_0[n] \) by minimising the instantaneous energy of filter output (6)

\[
J(z_0[n]) = |y[n]|^2.
\]

In \([1, 12]\), the energy of the filter output was averaged considering the past \( L \) output samples

\[
J(z_0[n]) = \frac{1}{L} \sum_{m=0}^{L-1} |y[n - m]|^2
\]

whereas Tichavsky and Handel \([13]\) considered the minimisation of the energy filtered using a truncated exponential filter

\[
J(z_0[n]) = \sum_{m=0}^{L-1} \rho^m |y[n - m]|^2
\]

where \( \rho \in (0, 1) \) is the filter forgetting factor. Note that for \( L = 1 \), (9) and (10) degenerate to (8). In the following, the analysis is restricted to PG algorithms minimising (8). Cost function (8) has been selected for its simplicity: more complex cases such as (9) and (10) are left for future work. The minimisation of the energy at the output of the MA part of the filter is considered at first

\[
J(z_0[n]) = |y_m[n]|^2.
\]

### 3 FLL equivalence

When a new sample, \( x[n] \), enters the adaptive notch filter, three operations have to be performed:

(i) Computation of the gradient of the selected cost function. The gradient computation is required for the implementation of PG algorithms.

(ii) Update of \( f_0 \) and determination of its value at the current time instant, \( n \); only at this point, \( f_0[n] \) and \( z_0[n] \) become available.

(iii) Filtering of the input signal using (6) and (7).

![Fig. 1](image-url) Structure of the adaptive notch filter: the adaptation block adjusts the filter zero, \( z_0[n] \), in order to minimise a specific cost function.

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Thus, when the gradient is computed, $f_0[n]$ is not available and the adaptation block can only use a past estimate of the notch frequency, $f_0[n−1]$. For this reason, the output of the MA part of the filter is not directly available for gradient computation and the adaptation block can only evaluate the approximation

$$\tilde{y}_m[n] = x[n] - z_0[n−1]x[n−1]$$

(12)

Thus, the instantaneous energy at the output of the MA part of the notch filter is estimated by the adaptation block using $\tilde{y}_m[n]$ instead of (11). Note that in the steady-state conditions, we have

$$z_0[n] \approx z_0[n−1]$$

and the minimisation of the approximated cost function implies the minimisation of the actual one. Using (12), the instantaneous energy at the output of the MA part of the notch filter can be expressed as

$$|\tilde{y}_m[n]|^2 = |x[n]|^2 + |x[n−1]|^2 - 2\Re[x[n]z_0^*[n−1]x^*[n−1]].$$

(13)

In the absence of noise, the input signal, $x[n]$, can be expressed as

$$x[n] = A[n]\exp\{j\varphi[n]\}$$

(14)

and instantaneous power (13) becomes

$$|\tilde{y}_m[n]|^2 = |x[n]|^2 + |x[n−1]|^2 - 2A[n]A[n−1]\cos\left(f_r[n] - f_0[n−1]\right)T_s$$

(15)

where

$$f_r[n] = \frac{1}{T_s} (\varphi[n] - \varphi[n−1])$$

(16)

is the instantaneous frequency of $x[n]$. PG algorithms update the notch filter frequency according to

$$f_0[n] = f_0[n−1] - \mu g[n]$$

(17)

where $\mu$ is the adaptation step and $g[n]$ is the stochastic gradient given by

$$g[n] = \frac{\partial |\tilde{y}_m[n]|^2}{f_0}$$

$$= -2T_sA[n]A[n−1]\sin\left(f_r[n] - f_0[n−1]\right)T_s$$

$$= -2T_sA[n]z_0^*[n−1]x^*[n−1]$$

(18)

where the gradient has been computed using (15).

Finally, the notch frequency is updated according to

$$f_0[n] = f_0[n−1] + 2\mu T_s\Re\left[x[n]z_0^*[n−1]x^*[n−1]\right].$$

(19)

Equation (19) defines the structure provided in Fig. 2a. The input signal, $x[n]$, is at first multiplied by a delayed and complex conjugated version of itself. This operation is equivalent to a phase differentiation on the input signal. After phase differentiation, the signal is multiplied by the complex conjugate of $z_0$ and the discriminator output is computed. The discriminator output corresponds to the opposite of gradient (18). The discriminator output is then used to update the frequency estimate, $f_0$, and to adapt the notch filter zero.

The diagram in Fig. 2a can be restated as in the bottom part of the figure which shows that the simple adaptive notch filter considered in this section is equivalent to a standard FLL [18, 24–26]. In Fig. 2b, $x[n]$ is at first multiplied by $i^*n−1$, a delayed and complex conjugated version of the complex carrier

$$i[n] = z_0[n]i[n−1] = \prod_{m=0}^{n} z_0[m] = \exp\left(jT_s \sum_{m=0}^{n} f_0[m]\right)$$

(20)

obtaining the signal

$$x_0[n] = x[n]i^*[n−1].$$

(21)

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**Fig. 2** Schematic representation of

a Update (19) used to minimise the instantaneous energy of the output of the MA part of the notch filter

b Adaptive notch filter which minimises the energy at the output of its MA part as standard FLL
If the adaptive notch filter is tracking the signal frequency, \( f[n] \), then \( x_\text{a}[n] \) is a baseband version of the input signal, \( x[n] \). Phase differentiation is then performed and the product
\[
x_\text{a}[n]x_\text{a}^*[n-1] = x[n]x^*[n-1] = x[n]z_\text{a}[n-1]x^*[n-1] = x[n]z_\text{a}[n-1]x^*[n-1] - 2x[n-2]x^*[n-1] (22)
\]
is computed. Signal (22) is equal to the product at the input of the discriminator in Fig. 2a. The discriminator output is then computed and an additional integrator is used in Fig. 2b to evaluate \( \tilde{f}[n] \) and to close the loop.

The analysis provided above shows that the adaptive notch filter using a PG algorithm minimising the energy at the output of the MA part of the filter is equivalent to a standard FLL of the first order. The gradient operator defines the loop discriminator and \( \mu \), the adaptation step, defines the loop filter. This equivalence will be exploited in Section 6 to analyse the tracking properties of the adaptive notch filter and provide a methodology for selecting the adaptation step, \( \mu \).

4 FLL with exponential filtering

When cost function (8) is considered, gradient (18) assumes the following form
\[
g[n] = -2T_s^2[x[n]z_\text{a}[n-1]x^*[n-1]] (23)
\]
where \( x[n-1] \) has been replaced by the output of the AR part of the filter, \( x_\text{a}[n-1] \).

An equivalent FLL model for the adaptive notch filter using a PG algorithm minimising the power at the filter output can be obtained by following the procedure introduced in Section 3. The product in brackets in (23)
\[
P_g = x[n]z_\text{a}[n-1]x^*[n-1] (24)
\]
can be rewritten introducing a complex carrier (20)
\[
P_g = x[n]z_\text{a}[n-1]x^*[n-1] = x[n]z_\text{a}[n-1]x^*[n-1] = x_\text{a}[n-2]x^*[n-1] = x_\text{a}[n]x_\text{a}[n-1] (25)
\]
where the property
\[
T^{-1}[n] = T^n[n] (26)
\]
has been exploited. The term
\[
x_\text{a}[n] = x[n]x^*[n-1] (27)
\]
is a baseband version of \( x[n] \). The recurrence equation governing the generation of \( x_\text{a}[n] \) is directly derived from (7) and it is given by
\[
x_\text{a}[n] = (1 - k_\text{a})x_\text{a}[n] + k_\text{a}z_\text{a}[n-1]x_\text{a}[n-1]. (28)
\]
Assuming that \( z_\text{a} \) is slowly varying with time, the following approximation can be made
\[
z_\text{a}[n]e_{\text{a}}[n-1] \simeq 1. (29)
\]
It is noted that the time variations of \( z_\text{a} \) are governed by (17): since the algorithm operates for small values of the adaptation step, \( \mu \), small variations of \( z_\text{a} \) can be assumed.

Using (29), recurrence (28) becomes
\[
x_\text{a}[n] = (1 - k_\text{a})x_\text{a}[n] + k_\text{a}x_\text{a}[n-1]. (30)
\]
Equations (30) and (25) allow the derivation of the equivalent FLL model for the adaptive notch filter with constrained pole and zero as shown in Fig. 3. The adaptive notch filter is equivalent to an FLL with a pre-filtering stage inserted before phase differentiation. Pre-filtering is implemented by \( h_3[n] \), which is the impulse response of the exponential filter defined by recurrence (30). The transfer function associated to \( h_3[n] \) is
\[
H_3(z) = \frac{1 - k_\text{a}}{1 - k_\text{a}z^{-1}}. (31)
\]
Fig. 3 Equivalent FLL model for the adaptive notch filter with constrained pole and zero

Note that \( H_3(z) \) is a low-pass filter with unit gain at the zero frequency. The bandwidth of the filter is regulated by \( k_\text{a} \), which can also be interpreted as a forgetting factor that leads to sharper transfer functions as it approaches unity. For \( k_\text{a} = 0 \), the standard FLL considered in Section 3 is obtained.

5 Equivalent model

In this section, an equivalent model for the FLL with exponential filtering is derived. The analysis follows the standard approach adopted for the characterisation of tracking loops \[17, 18\] where an equivalent model is at first derived.

For the analysis, signal model (14) is modified by considering the presence of a noise term, \( \eta[n] \), and by assuming that the signal amplitude is slowly varying with time, i.e. \( A[n] \approx A \). In this way, (14) becomes
\[
x[n] = A \exp[j\varphi[n]] + \eta[n] (32)
\]
where \( \eta[n] \) is a white circularly symmetric complex Gaussian random process with independent and identically distributed real and imaginary parts. \( \eta[n] \) has total variance \( \sigma^2 \).
Using (32), \( x_h[n] \) can be expressed as

\[
x_h[n] = x[n]r[n-1] = A \exp[i\Delta \phi[n]] + \eta_h[n]
\]

where \( \Delta \phi[n] \) is a residual phase term. Under lock conditions, i.e. when \( f_i[n] \) is close to the signal instantaneous frequency, \( s_q[n] = A \exp[i\Delta \phi[n]] \) is almost constant. Note that due to the circular symmetry of \( \eta[n], \eta_h[n] \) has the same statistical properties of input noise in (32). Model (33) implies that also \( x_h[n] \) can be decomposed as the sum of a signal and a noise component which are filtered versions of \( s_h[n] \) and \( \eta_h[n] \), respectively,

\[
x_h[n] = s_h[n] + \eta_h[n]. \quad (34)
\]

The discriminator output in Fig. 3 can then be expressed as

\[
D[n] = \Delta \sum_{m=0}^{\infty} h_l[m] s_h[n-m] \quad \text{where} \quad h_l[m] = (1-k_a)h_a^m. \quad (36)
\]

The coefficients, \( h_l[m] \), have been obtained by expanding the geometric series defined by transfer function (31). Using expansion (36), it is possible to compute the signal component at the discriminator output. In particular

\[
2\{x_h[n]s_h'[n-1]\} = 2\sum_{m=0}^{\infty} h_l[m]s_h[n]s'_h[n-m] \]

\[
= A^2 \sum_{m=0}^{\infty} h_l[m] \left\{ \exp\left[i(\Delta \phi[n] - \Delta \phi[n-m])\right] \right\}
\]

\[
= A^2 \sum_{m=0}^{\infty} h_l[m] \sin(\Delta \phi[n] - \Delta \phi[n-m]).
\]

(37)

Assuming that the arguments inside the sine functions in (37) are small, then (37) can be linearised as

\[
2\{x_h[n]s_h'[n-1]\} \approx A^2 \sum_{m=0}^{\infty} h_l[m] (\Delta \phi[n] - \Delta \phi[n-m]).
\]

(38)

Approximation (38) is based on the assumption that the algorithm has been properly initialised and thus that the residual phase term, \( \Delta \phi[n] \), is small for all time instants, \( n \). In this way, differences inside the sine functions in (37) remain small for all values of \( m \). This assumption corresponds to the equilibrium state (ES) hypothesis introduced by Tichavsky and Handel [13], Tichavsky and Nehorai [14] for the derivation of the LFA. The proper initialisation of the adaptation algorithm can be performed using frequency acquisition techniques [18], for example based on the fast Fourier transform algorithm. Alternatively, the properties of the adaptive notch filter can be exploited and unaided frequency acquisition can be performed [18]. When the frequency of the filter notch is far from the frequency of the input signal, the adaptation algorithm progressively sweeps all possible frequencies until frequency lock is achieved. Thus, condition (38) is also valid when data have been processed for a long time and the algorithm is operating in the steady-state conditions. In particular, coefficients, \( h_l[m] \), decay exponentially and only differences obtained for small values of \( m \) influence the summation in (37).

Since small values of \( m \) are considered and assuming that, in steady state, \( \Delta \phi[n] \) is affected only by small variations, differences in (37) remain close to zero, justifying (38). The step size of the adaptation algorithm influences the unaided frequency acquisition process which can be improved using variable step size algorithms [27]. The analysis of variable step size algorithms is outside the scope of this paper.

The difference, \( \Delta \phi[n] - \Delta \phi[n-1-m] \) defines the telescopic sum

\[
\Delta \phi[n] - \Delta \phi[n-1-m] = \sum_{k=0}^{m} (\Delta \phi[n-k] - \Delta \phi[n-1-k])
\]

\[
= T_s \sum_{k=0}^{w} \Delta \phi[n-k]
\]

(39)

where

\[
\Delta \phi[n] = \frac{1}{T_s} (\Delta \phi[n] - \Delta \phi[n-1])
\]

(40)

is the residual frequency error at the instant \( n \). In this way, (38) becomes

\[
\sum_{m=0}^{\infty} h_l[m] \sum_{l=0}^{m} \Delta \phi[n-l]
\]

\[
= T_s A^2 \sum_{m=0}^{\infty} h_l[m] \sum_{l=0}^{m} \Delta \phi[n-l]
\]

\[
= T_s A^2 \sum_{l=0}^{w} g[l] \Delta \phi[n-l].
\]

(41)

Coefficients \( g[l] \) assume the following form

\[
g[l] = (1 - k_a) \sum_{m=0}^{\infty} k_m^a = (1 - k_a) \sum_{m=0}^{\infty} k_m^a = k_m^a = h_l[l]
\]

(42)

and define an exponential filter with a direct current (DC) gain different from 1.

Finally

\[
2\{x_h[n]s_h'[n-1]\} \approx 2T_s A^2 \sum_{m=0}^{\infty} h_l[m] \Delta \phi[n-m]
\]

\[
= G_d \sum_{m=0}^{\infty} h_l[m] \Delta \phi[n-m]
\]

(43)

where

\[
G_d = \frac{2T_s A^2}{1 - k_a}
\]

(44)

is the discriminator gain. From these considerations, it emerges that the signal component at the discriminator output is a scaled and filtered version of the residual frequency error, \( \Delta \phi[n] \).
5.2 Noise component

The noise component at the discriminator output is composed of two terms

\[ C[n] = 2A \{ s_n[n] \eta_n[n] - 1 + \eta_n[n] s_n[n] - 1 \} \]  

(45)

and

\[ N[n] = 2A \{ \eta_n[n] \eta_n[n] - 1 \} \]  

(46)

The first term is due to the interaction between the noise and the signal components, whereas \( N[n] \) is a pure noise term. The two terms are uncorrelated.

\( C[n] \) can be characterised by assuming that, in lock conditions, \( s_n[n] \) is almost constant and

\[ s_n[n] - 1 \approx s_n[n] = A \exp \{ j \phi \} \]  

(47)

Using this approximation, cross-product (45) can be expressed as

\[ C[n] = 2A \exp \{ j \phi \} \eta_n[n] - 1 + \eta_n[n] \exp \{ - j \phi \} \]  

(48)

\[ = 2A \left\{ \eta_n[n] - 1 + \eta_n[n] \right\} \]

where \( \eta_n[n] \) is obtained by rotating \( \eta_n[n] \) by \( -\phi \), the useful signal phase. \( \eta_n[n] \) has the same statistical properties of \( \eta_n[n] \). Superscript \( Q \) is used to denote the imaginary part (quadrature component) of \( \eta_n[n] \). Note that \( \eta_n[n] - 1 \) can be obtained by filtering \( \eta_n[n] \).

From the analysis reported above, it emerges that the cross-product term is obtained by scaling and filtering a zero mean white Gaussian process with variance \( \sigma^2 \). The filter transfer function can be derived from (48) and it is given by

\[ H_d(\zeta) = 1 - \zeta^{-1} \frac{1 - k_a}{1 - k_a \zeta^{-1}} = 1 - \zeta^{-1} \frac{1}{1 - k_a \zeta^{-1}} \]  

(49)

which is the baseband version of (5) and it is a combination of a numerical derivative and an exponential filter.

Noise (46) can be expressed as

\[ N[n] = 2 \left( \eta_n^Q[n] \eta_n^I[n] - 1 \right) \]  

(50)

where the indices ‘\( T \)’ and ‘\( Q \)’ are used to denote real and imaginary parts, respectively. Note that \( \eta_n^Q[n] \) and \( \eta_n^I[n] \) are independent processes.

In Appendix 1, it is shown that \( N[n] \) is a white sequence with variance

\[ \sigma_N^2 = 2 \frac{1 - k_n^2}{1 + k_n^2} \sigma^2 \]  

(51)

It is noted that this component can be neglected for a wide range of \( k_n \) and of input signal-to-noise ratio (SNR) values where the SNR is defined as

\[ \text{SNR} = \frac{A^2}{\sigma^2} \]  

(52)

Theoretical results are supported by Monte Carlo simulations. In particular, it has been verified through simulations that for \( k_n = 0 \), the case of a standard FLL, noise (50) contributes to about 13.7% of the total noise power. When \( k_n = 0.9 \), the contribution of \( N[n] \) is about 1.55%. As the SNR increases the contribution of \( N[n] \) further diminishes. For this reason, only the cross-product term will be considered.

5.3 Equivalent model

The results reported above allow one to derive the equivalent linear model depicted in Fig. 4 that is expressed in the signal phase/frequency domain. In Fig. 4, an equivalent white noise term, \( N_d[n] \) has been introduced. \( N_d[n] \) is zero mean and has variance

\[ \sigma_d^2 = \frac{2T^2 \sigma^2}{(1 - k_n)^2} \]  

(53)
The proof of (53) is provided in Appendix 2. After filtering, \( \text{N}_d[n] \) becomes a coloured noise which models the cross-product term analysed in Section 5.2. The phase term, \( \psi_o \), is the argument of the local carrier, \( \iota[n] \).

\[
\psi_o[n] = \angle \iota[n] = T_s \sum_{m=0}^{n} f_0[m]. \tag{54}
\]

The model in Fig. 4 can be simplified by collecting terms common to the noise and signal branches thus obtaining the final loop depicted in Fig. 5. This model is equivalent to that obtained in [19, 28] for tracking loops with exponential memory discriminators. Thus, the theory developed in [19, 28] can be adopted for the analysis of the adaptive notch filter. When \( k_o = 0 \), the equivalent FLL loop derived in [29] is obtained. From the model in Fig. 5, it is possible to show that the Z-transform of the notch filter frequency, \( f_0[n] \), is given by the linear combination of filtered signal phase and noise

\[
f_0(z) = \frac{\mu H_r(z)}{1 + \mu G_0 \text{N}_d(z) H_s(z)} N_s(z) + \frac{\mu G_0 H_s(z)}{1 + \mu G_0 \text{N}_d(z) H_s(z)} \phi(z) \tag{55}
\]

where

\[
N_s(z) = \frac{T_s z^{-1}}{1 - z^{-1}} \tag{56}
\]

is the transfer function of the numerically controlled oscillator model adopted in Fig. 5. Using (31), the transfer function multiplying the noise term in (8) can be written as

\[
H_s(z) = \frac{\mu(1 - k_o)(1 - z^{-1})}{1 - (1 + k_o - \mu T_s G_{z0} + \mu T_s G_{d0}) z^{-1} + z^{-2}}, \tag{57}
\]

which is equivalent to the noise transfer function obtained in [13, 14] in the LFA. A different transfer function is obtained for the signal component. In (55), the final frequency estimate is expressed as a function of the phase of the input signal: the equivalence with FLLs provides an alternative to the LFA approach for the analysis of notch filters.

From (57), it is possible to note that the term, \( \mu T_s G_{d0} \), has to be dimensionless. Thus, it is convenient to express \( \mu \) as a function of the normalised update step

\[
\hat{\mu} = \mu T_s G_{d0}. \tag{58}
\]

Note that the signal before multiplication by \( \mu \) is in units of radians times \( G_{d0} \). After multiplication by \( \mu = (\hat{\mu}/T_s G_{d0}) \), \( f_0[n] \) is obtained and is in units of radians/second.

The signal phase, \( \phi[n] \), is related to the signal frequency through (16) and thus (55) can be rewritten as a function of \( f_0[n] \)

\[
f_0(z) = H_s(z) N_d(z) + \frac{\mu G_0 H_s(z)}{1 + \mu G_0 N_d(z) H_s(z)} \frac{T_s}{1 - z^{-1}} f_0(z). \tag{59}
\]

Finally, (59) can be restated introducing normalised adaptation step (58)

\[
f_0(z) = H_s(z) N_f(z) + H_s(z) f_0(z) = \frac{\hat{\mu}(1 - k_o)(1 - z^{-1})}{1 - (1 + k_o - \hat{\mu}(1 - k_o)) z^{-1} + k_o z^{-2}} N_f(z) + 1 - (1 + k_o - \hat{\mu}(1 - k_o)) z^{-1} + k_o z^{-2} f_0(z) \tag{60}
\]

where \( N_f(z) \) is the Z-transform of \( N_f[n] \), the normalised frequency noise

\[
N_f[n] = \frac{1}{T_s G_{d0}} N_d[n] \tag{61}
\]

with variance

\[
\sigma_f^2 = \frac{T_s^2}{2 \sigma_f^2} = \frac{1}{2 \text{SNR}}. \tag{62}
\]

Equation (60) will be used in the following section to characterise the tracking properties of the adaptive notch filter.

### 6 Loop bandwidths and adaptation step

The two transfer functions, \( H_s(z) \) and \( H_s(z) \), derived in Section 5 define the Doppler and loop bandwidths, respectively [29]. Under the hypothesis that the term which depends on \( f_0[n] \) in (60) is deterministic, the Doppler bandwidth allows one to determine the variance of the final frequency estimate, \( f_0[n] \). In particular, by neglecting the contribution of \( f_0[n] \), the variance of \( f_0[n] \) can be expressed as

\[
\text{Var}\{f_0[n]\} = \frac{\sigma_f^2}{2 \pi} \int_{-\pi}^{\pi} |H_{sf}(e^{j\omega})|^2 \, d\omega. \tag{63}
\]

The quantity

\[
B_d = \frac{1}{4 \pi T_s} \int_{-\pi}^{\pi} |H_{sf}(e^{j\omega})|^2 \, d\omega \tag{64}
\]

is the Doppler bandwidth expressed in Hz [29]. Using tables in [30] (p. 298), the Doppler bandwidth can be expressed as

\[
B_d = \frac{\hat{\mu}^2(1 - k_o)}{2 T_s(1 + k_o) - \hat{\mu}(1 - k_o)}. \tag{65}
\]

Using (65) and (62), the variance of \( f_0[n] \) becomes

\[
\text{Var}\{f_0[n]\} = 2 B_d T_s \sigma_f^2 = \frac{B_d}{\text{SNR}} \tag{66}
\]

Equation (66) is in agreement with the findings obtained by Tichavsky and Handel [13] using the LFA approach. Moreover, the square root of (66) corresponds to the Doppler tracking jitter derived in [29] for standard FLLs.

The transfer function, \( H_s(z) \), can be used to evaluate the loop bandwidth. Note that in standard tracking loops, the loop bandwidth is defined with respect to a noise transfer function obtained considering a different linear model for the loop
representation [17, 29]. In this case, \( H(z) \) is used and the loop bandwidth characterises the tracking capabilities of the adaptive notch filter in the absence of noise. Noise performance is determined by the Doppler bandwidth introduced above. A discussion about the difference between loop and Doppler bandwidths can be found in [29].

The loop bandwidth is defined as

\[
B_s = \frac{1}{4\pi T_s} \int_{-\infty}^{\infty} \frac{\left| H_f(e^{j\omega}) \right|^2}{\omega^2} \, d\omega
\]

and can be computed in closed form as [30]

\[
B_s = \frac{1}{2T_s} \frac{\bar{\mu}(1 + k_0)}{(1 + k_a) - \bar{\mu}(1 - k_a)}.
\]

It is noted that for \( k_a = 0 \), (68) becomes

\[
B_s = \frac{1}{2T_s} \frac{\bar{\mu}}{2 - \bar{\mu}}
\]

which is the standard expression for the bandwidth of first-order digital tracking loops [31] (Table IV). In this respect, the normalised adaptation step, \( \bar{\mu} \), can be interpreted as the coefficient defining the filter of a first-order digital tracking loop [25, 31].

Thus, \( \bar{\mu} \) can be obtained either by fixing the Doppler bandwidth and inverting (65) or by fixing the loop bandwidth and inverting (68). The Doppler bandwidth allows one to control the variance of the final frequency estimate, \( f_0[n] \), whereas the loop bandwidth allows one to adjust the response of the adaptive loop filter to changes in the input frequency, \( f_0[n] \).

The equivalent model derived in Section 5 implies that the final frequency estimate, \( f_0[n] \), is a linear combination of filtered noise and of the filtered input signal frequency. The signal transfer function, \( H_f(z) \), defines a low-pass filter with bandwidth equal to \( B_s \). The noise filter is the combination of a digital differentiator and a low-pass filter. Thus, \( f_0[n] \) is essentially a low-pass process and, if the condition

\[
\frac{B_s}{T_s} \ll 1
\]

is respected, then \( f_0[n] \) is slowly varying with time. In particular, (70) implies that \( f_0[n] \) has only low frequency components. Condition (70) is usually required for the stability of digital tracking loops [31] and it is necessary in order to have smooth estimates of the input signal frequency and to cope with noise. This fact justifies the assumption of slowly varying processes and (29).

### 7 Algorithm extensions

The PG algorithm can be extended exploiting the equivalence with the FLL. In particular:

- Gradient (23) can be replaced using modified FLL discriminator functions.
- The normalised adaptation step, \( \bar{\mu} \), can be replaced by integrator-based filters which can lead to loops of order higher than 1.

It has been shown that the discriminator output in the FLL equivalent model is obtained as a scaled version of the opposite of gradient (23). In particular, (35) was directly obtained by scaling (23). Alternative PG algorithms can be obtained by replacing \( 2\Delta \) [1] with a different function. Possible candidates are [25]

\[
D[n] = \angle \{ x[n]z_0^2[n-1]v^2[n-1] \} \tag{71a}
\]

Equations (71a)–(71c) are directly derived from FLL discriminators commonly used in the relevant literature [25] and have a gain, \( G_s \), which is independent from the amplitude of the input signal, \( A \). The analysis developed in Section 5 showed that (35) led to gain (44), which is proportional to \( A^2 \). This dependence has to be removed by normalising \( \mu \). However, \( A \) and \( A^2 \) are in general unknown and need to be estimated. The problem of amplitude dependency is well known in the literature and, for example [32] highlighted the need of an automatic gain control to maintain the signal amplitude within a known range and avoid instabilities in the adaptive notch filter. Normalised gradient algorithms were introduced in [19, 33] to remove the amplitude dependency: \( A^2 \) can be determined using a reference signal [33] or employing a signal energy estimator as in normalised least mean squares algorithms [22]. A solution commonly adopted in the FLL literature [25] is the use of amplitude independent discriminators such as (71a)–(71c). Discriminator (71a) leads to the multi-frequency tracker considered in [13] when a single sinusoid is present. A form of SA [8, 32] is obtained using (71c). From this discussion, it emerges that several adaptive algorithms employed for IIR notch filters can be obtained exploiting the FLL theory.

The adaptive notch filter can be further extended by replacing the adaptation step, \( \mu \), by an integrator-based filter with transfer function

\[
F(z) = \sum_{m=0}^{K-1} \mu_m(1 - z^{-1})^m
\]

where \( \mu_m \) are the filter coefficients. For \( K = 1 \) (first-order loops), transfer function (72) reduces to a constant gain that corresponds to the adaptation step, \( \mu \). The integrator gains, \( \{ \mu_m \}_{m=0}^{K-1} \), can be determined using the modified controlled-root formulation [31] for loops with exponential filtering [19].

### 8 Simulation results

In this section, simulation results supporting the theoretical findings described in Sections 5 and 6 are provided. In particular, the adaptive notch filter described in Section 2 has been tested in the presence of signal (32) for different SNR conditions. The tests have been conducted considering a sampling frequency \( f_s = (1/T_s) = 10 \text{ MHz} \).

For the noise performance analysis, a complex sinusoid with constant amplitude and constant frequency was generated at each simulation run. The frequency of the sinusoid was selected randomly and was changed at each simulation run. The sinusoid was corrupted by a white complex Gaussian noise the variance of which was selected according to the SNR considered for the experiment.

Fig. 6a shows the standard deviation of \( f_0[n] \) for a constant Doppler bandwidth, \( B_s = 10 \text{ kHz} \), and for several values of \( k_0 \). In Fig. 6a, the standard deviation has been expressed in Hz in order to provide a more immediate understanding of the results. The curve denoted as ‘theoretical’ has been computed using (66) whereas the other curves have been obtained through Monte Carlo simulations: for each SNR value, signal (32) has been generated and processed using the adaptive notch filter. The frequency estimates, \( f_0[n] \), provided by the PG algorithm have then been used to estimate the frequency standard deviation. For each SNR condition, \( N = 10^7 \) frequency values have been used for estimating the standard deviation. In order to have a constant Doppler bandwidth, the normalised adaptation step, \( \bar{\mu} \), has been obtained by solving (65)

\[
\bar{\mu} = \frac{1}{2B_s T_s} \left[ -B_s T_s + \frac{B_s T_s}{B_s T_s + 8 \left( 1 + k_0 \right) - k_0} \right].
\]

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From Fig. 6a, it emerges that the noise performance of the adaptive notch filter is essentially determined by the Doppler bandwidth and that the frequency standard deviation only marginally depends on \( k_{\alpha} \). For \( k_{\alpha} > 0 \) there is a good agreement between (66) and Monte Carlo simulations. For \( k_{\alpha} = 0 \), simulation and theoretical results start deviating for SNR values lower than 15 dB. This result is to be expected since, for low SNR values and for small contraction factors, the contribution of noise (50) cannot be neglected as in (66).

The impact of the Doppler bandwidth is further analysed in Fig. 6b where \( k_{\alpha} \) has been set to 0.9 and several values of \( B_d \) have been considered. A good agreement between theoretical and simulation results can be observed. For \( B_d = 5 \text{ kHz} \), a small deviation between Monte Carlo and theoretical results can be observed. This is due to the fact that the theory developed in Section 4 assumed that \( f_0[n] \) is slowly varying with time. As the Doppler bandwidth increases also the loop bandwidth increases and condition (70) is no longer respected.

The impact of the loop bandwidth is investigated in Fig. 7 that shows the evolution of the residual frequency error as a function of time and in the absence of noise. The residual frequency error is the difference between the signal frequency, \( f_x[n] \), and \( f_0[n] \). In Fig. 7, simulated curves have been obtained using the adaptive notch filter to process a noiseless complex sinusoid with a constant frequency, \( f_x = 3 \text{ kHz} \), whereas the linear approximations have been computed by processing a frequency step of amplitude \( f_x \) with the filter defined by the signal transfer function, \( H(z) \). The residual error quickly converges to zero and the convergence speed is mainly determined by the loop bandwidth. Although different contraction factors are considered in Fig. 7, the residual frequency errors show similar behaviour. This is due to the fact that, the loop bandwidth has been kept constant during the different experiments. In this case, the normalised adaptation step, \( \overline{\mu} \), has been selected according to
In Fig. 7, a good agreement between simulations and the linear approximation developed in Section 5 is observed. These results support the validity of the theory developed in this paper.

The impact of the loop bandwidth is further analysed in Fig. 8 that shows the convergence behaviour of the residual frequency error for several values of $B_n$ and a constant pole contraction factor, $k_s = 0.9$. Large loop bandwidths improve the convergence speed of the algorithm. Also in this case, a good agreement between simulation and theoretical results can be observed.

Finally, the response of the notch filter adaptation algorithm is considered in Fig. 9 in the presence of noisy sinusoids. In particular, model (60) is additive with respect to the input signal and noise components since non-linear effects in discriminator poles and zeros. Thus, the model derived can also be used to analyse the frequency impulse response which effectively describes the average adaptation algorithm are compared with the noiseless equivalent signal impulse response which effectively describes the average adaptation algorithm are compared with the noiseless equivalent signal impulse response which effectively describes the average behaviour of the algorithm. Also in this case, a good agreement between simulation and theoretical results is obtained.

9 Conclusions

In this paper, the equivalence between adaptive notch filters and FLLs using an exponential filter was established. In particular, the pole contraction factor of the notch filter is equivalent to the forgetting factor of the FLL exponential filter. The equivalence allows the usage of the FLL theory for analysing the dynamics of the adaptive notch filter. In particular, FLL theory was used to derive an equivalent linear model and to introduce the concepts of loop and Doppler bandwidths. These two parameters characterise the performance of the adaptive notch filter in terms of noise rejection and signal tracking capabilities. Two criteria for setting the adaptation step of the PG algorithm adopted by the adaptive notch filter were also provided as a function of the loop and Doppler bandwidths, respectively. Algorithm extensions were also suggested and explained using results from the FLL theory. The theory developed is supported by Monte Carlo simulations and provides valuable insight on the behaviour of the adaptive notch filter and its extensions.

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The two variances in the second line of (75) are equal due to the circular symmetry of $\eta_{\theta}(n)$ which implies that the real and imaginary parts of $\eta_{\theta}(n)$ can be exchanged. Moreover

$$\text{Var} \left[ \eta_{\theta}(n) \eta_{\theta}^*(n-1) \right] = \text{Var} \left[ \eta_{\theta}(n) \eta_{\theta}(n-1) \right]$$

$$= \text{Var} \left[ \eta_{\theta}(n) \sum_{m=0}^{\infty} h[m] \eta_{\theta}(n-m-1) \right]$$

$$= \text{Var} \left[ \eta_{\theta}(n) \left| \sum_{m=0}^{\infty} h[m] \eta_{\theta}(n-m) \right| \right]$$

$$= \sigma_n^2 \sum_{m=0}^{\infty} (1-k_a)^m k_a^{2m} = \frac{\sigma_n^2 - k_a}{1+k_a}$$

(76)

The covariance term in (75) is zero since it involves products with zero mean independent terms.

$$\text{Cov} \left[ \eta_{\theta}(n) \eta_{\theta}(n-1) \right] = \text{E} \left[ \eta_{\theta}(n) \eta_{\theta}(n-1) \right] - \text{E} \left[ \eta_{\theta}(n) \right] \text{E} \left[ \eta_{\theta}(n-1) \right]$$

$$= \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} h[m] h[j] \text{E} \left[ \eta_{\theta}(n-m) \eta_{\theta}(n-j) \right] = 0$$

(77)

From these results, it finally follows

$$\sigma^2_N = 4 \frac{\sigma_n^2 - k_a}{1+k_a} = 2 \sigma_n^2 - \frac{k_a}{1+k_a}$$

(78)

The fact that $\mathcal{N}(n)$ is a white sequence, i.e. it has uncorrelated samples, can be verified by computing the covariance between $\mathcal{N}(n)$ and $\mathcal{M}(n)$ for arbitrary $n$ and $m$. This involves the evaluation of moments of fourth order in $\eta_{\theta}(n)$ and $\eta_{\theta}(m)$. The covariance is zero for the presence of independent and zero mean terms in the resulting products.

12 Appendix 2: Proof of (53)

In Section 5.2, it has been shown that $C(n)$ is obtained by scaling and filtering a zero mean white Gaussian process with variance $\sigma^2$. The filter used to generate $C(n)$ is defined by (49). In the equivalent model depicted in Fig. 4, $N_d(n)$ is processed by a filter with a transfer function

$$\frac{1}{T_s} (1 - z^{-1}) - \frac{k_a}{1-k_a} z^{-1} = \frac{1 - k_a}{T_s} H_d(z)$$

(79)

which is a scaled version of (49). Thus, the noise term entering the loop in Fig. 4 has the same spectral properties of $C(n)$. In order to assign the same power of $C(n)$ to the coloured noise term, the following condition has to be fulfilled

$$\text{Var} \left[ \frac{N_d(n)}{T_s} \frac{1-k_a}{1-k_a} \right] = 4A^2 \frac{\sigma_N^2}{2}$$

(80)

where $4A^2$ is the square of the scaling factor present in (48) which defines $C(n)$. Variance (53) directly follows from (80).