Rényi and Tsallis entropies: three analytic examples

O Olendski

Department of Applied Physics and Astronomy, University of Sharjah, PO Box 27272, Sharjah, United Arab Emirates

E-mail: oolendski@sharjah.ac.ae

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Abstract
A comparative study of 1D quantum structures which allows analytic expressions for the position and momentum Rényi \( R(\alpha) \) and Tsallis \( T(\alpha) \) entropies, focuses on extracting the most characteristic physical features of these one-parameter functionals. Consideration of the harmonic oscillator reconfirms the special status of the Gaussian distribution: at any parameter \( \alpha \) it converts into the equality both Rényi and Tsallis uncertainty relations removing for the latter an additional requirement \( 1/2 \leq \alpha \leq 1 \) that is a necessary condition for all other geometries. It is shown that the lowest limit of the semi-infinite range of the dimensionless parameter \( \alpha \) where momentum components exist strongly depends on the position potential and/or boundary condition for the position wave function. Asymptotic limits reveal that in either space the entropies \( R(\alpha) \) and \( T(\alpha) \) approach their Shannon counterpart, \( \alpha = 1 \), along different paths. Similarities and differences between the two entropies and their uncertainty relations are exemplified. Some unsolved problems are also indicated.

Keywords: Rényi entropy, Tsallis entropy, quantum information, harmonic oscillator, Robin wall, 1D hydrogen atom

(Some figures may appear in colour only in the online journal)

1. Introduction
To describe how much we know about the location and motion of a nano object, quantum-information theory operates with some functionals of the position \( \rho_\sigma(\mathbf{r}) \) and momentum\(^1 \) \( \rho_\rho(\mathbf{k}) \)

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\(^1\) To keep the results in the most symmetric form, throughout the whole research instead of momentum \( \mathbf{p} \), if not specified otherwise, we will actually operate with the corresponding wave vector \( \mathbf{k} = \mathbf{p}/\hbar \).
densities that are squared amplitudes of the corresponding one-particle wavefunctions \( \Psi_n(\mathbf{r}) \) and \( \Phi_n(\mathbf{k}) \):

\[
\rho_n(\mathbf{r}) = |\Psi_n(\mathbf{r})|^2, \quad (1a)
\]

\[
\gamma_n(\mathbf{k}) = |\Phi_n(\mathbf{k})|^2, \quad (1b)
\]

where a discrete index \( n \) counts all possible bound quantum states. In general \( l \)-dimensional space, \( \Psi_n(\mathbf{r}) \) and \( \Phi_n(\mathbf{k}) \) are related through the Fourier transformation:

\[
\Phi_n(\mathbf{k}) = \frac{1}{(2\pi)^{l/2}} \int_{\mathbb{R}^l} \Psi_n(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}} d\mathbf{r}, \quad (2a)
\]

\[
\Psi_n(\mathbf{r}) = \frac{1}{(2\pi)^{l/2}} \int_{\mathbb{R}^l} \Phi_n(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}} d\mathbf{k}, \quad (2b)
\]

with integrations carried out over the whole available region. They both satisfy orthonormality conditions:

\[
\int_{\mathbb{R}^l} \Psi^*_n(\mathbf{r}) \Psi_m(\mathbf{r}) d\mathbf{r} = \int_{\mathbb{R}^l} \Phi^*_n(\mathbf{k}) \Phi_m(\mathbf{k}) d\mathbf{k} = \delta_{nm}, \quad (3)
\]

where \( \delta_{nm} = \begin{cases} 1, & n = n' \\ 0, & n \neq n' \end{cases} \) is a Kronecker delta, \( n, n' = 1, 2, \ldots \). Position wavefunctions \( \Psi_n(\mathbf{r}) \) and associated eigen energies \( E_n \) are found from the \( l \)-dimensional Schrödinger equation:

\[
-\frac{\hbar^2}{2m_p} \nabla_\mathbf{r}^2 \Psi_n(\mathbf{r}) + V(\mathbf{r}) \Psi_n(\mathbf{r}) = E_n \Psi_n(\mathbf{r}), \quad (4)
\]

with \( m_p \) being the mass of the particle and \( V(\mathbf{r}) \) being an external potential, and \( \nabla_\mathbf{r} = \frac{\partial}{\partial \mathbf{r}} \).

Among these functionals, a very special role is played by Rényi \( R_{\rho_n}(\alpha) \) \([1, 2]\) and Tsallis \( T_{\gamma_n}(\alpha) \) \([3]\) entropies\(^2\) whose expressions in the position (subscript \( \rho \)) and momentum (subscript \( \gamma \)) spaces are:

\[
R_{\rho_n}(\alpha) = \frac{1}{1 - \alpha} \ln \left( \int_{\mathbb{R}^l} \rho_n^\alpha(\mathbf{r}) d\mathbf{r} \right), \quad (5a)
\]

\[
R_{\gamma_n}(\alpha) = \frac{1}{1 - \alpha} \ln \left( \int_{\mathbb{R}^l} \gamma_n^\alpha(\mathbf{k}) d\mathbf{k} \right), \quad (5b)
\]

\[
T_{\rho_n}(\alpha) = \frac{1}{\alpha} - 1 \left( 1 - \int_{\mathbb{R}^l} \rho_n^\alpha(\mathbf{r}) d\mathbf{r} \right), \quad (5c)
\]

\[
T_{\gamma_n}(\alpha) = \frac{1}{\alpha} - 1 \left( 1 - \int_{\mathbb{R}^l} \gamma_n^\alpha(\mathbf{k}) d\mathbf{k} \right). \quad (5d)
\]

These attempt to quantify information with the help of a non-negative parameter, \( 0 < \alpha < \infty \), which can be considered as a factor describing the reaction of the system to its deviation from the equilibrium. Equations (5) show that its small, decreasing to zero values treat the outcomes of the random events more and more in the same manner, regardless of their actual occurrence, and in the extreme case \( \alpha \to 0 \), they all give equal contribution to the entropies. As a result, if the limits of integration in any of equations (5) are infinite or semi-

\(^2\) From a historical point of view, the latter measures should be more correctly called Havrda-Charvát–Daróczy–Tsallis entropies \([4, 5]\).
infinite, the corresponding entropy (provided it exists) does diverge. In the opposite asymptote of the very large parameter, the events with the highest probabilities are the only important donors to the integrals in equations (5) with the relative weight of the low-probability occurrences being negligibly small. Special case $\alpha = 1$ with the help of the l’Hôpital’s rule degenerates for both Rényi $R_{\alpha, \gamma}(\alpha)$ and Tsallis $T_{\alpha, \gamma}(\alpha)$ functionals to the celebrated Shannon entropies [6]:

$$S_\alpha = -\int f_{\alpha}(r) \ln f_{\alpha}(r) \, dr,$$

$$S_\gamma = -\int g_{\gamma}(k) \ln g_{\gamma}(k) \, dk.$$

Dependence $S$ was introduced in 1948 for the random discrete distribution (see equation (14c) below) during the mathematical analysis of communication as a measure of “information, choice and uncertainty” [6]. From a quantum-informational point of view, it is a quantitative descriptor of the lack of our knowledge about the corresponding property of the system: the greater (smaller) the value of this functional is, the less (more) information we have about a nano object. Accordingly, the physical meaning of the Rényi entropy and parameter $\alpha$ can be construed as follows: the equilibrium distribution corresponds to $\alpha = 1$ and any $\alpha \neq 1$ is a deviation from it. Then, the Rényi entropy is a measure of the sensitivity of the system to the deviation from the equilibrium. If parameter $\alpha$ is greater than unity, the corresponding entropy decreases, which means that such a configuration provides more information about the object than its equilibrium counterpart. On the other hand, a distortion in the opposite direction, $\alpha < 1$, increases the entropy with the corresponding decrease of the available information and in the extreme case of $\alpha \to 0$, it reaches its maximal value, which for the infinite or semi-infinite interval leads to a logarithmic divergence, as mentioned above. Within this limit, the integrand in equations (5) is just a flat unit line, which means that, say, for the position entropy $R_{\rho, \gamma}(\alpha)$ we know nothing about particle location in space. The rate of change of the entropy with the Rényi parameter just shows the sensitivity of the system to the degree of non-equilibricity.

Another particular case of these entropies is Onicescu energies [7]:

$$O_{\rho, \gamma} = \int f_{\rho, \gamma}(r) \, dr \equiv e^{-R_{\rho, \gamma}(2)} \equiv 1 - T_{\rho, \gamma}(2),$$

$$O_{\gamma, \gamma} = \int g_{\gamma, \gamma}(k) \, dk \equiv e^{-R_{\gamma, \gamma}(2)} \equiv 1 - T_{\gamma, \gamma}(2),$$

which measure the deviations of the corresponding distribution from the uniformity.

Rényi and Tsallis entropies are related as

$$T = \frac{1}{\alpha - 1} [1 - e^{(1-\alpha)R}],$$

$$R = \frac{1}{1 - \alpha} \ln(1 + (1 - \alpha)T),$$

as directly follows from equations (5). Both are decreasing functions of parameter $\alpha$. One important difference between them lies in the fact that the Rényi entropies are additive (or extensive) whereas the Thallis functionals are non-additive (or non-extensive). Additivity means that if there are two independent events characterized by their probability functions $f(r)$ and $g(r)$, then the following relation holds:
which physically means that the total information obtained by the two independent measurements is exactly the same as the sum of the knowledge received from each separate experiment. The postulate of additivity of the entropy was a cornerstone in Rényi’s search for the measure that generalizes the Shannon one [1]. In turn, the Tsallis entropy is only pseudo-additive:

\[ T_{B}(\alpha) = T_{F}(\alpha) + T_{S}(\alpha) + (1 - \alpha)T_{F}(\alpha)T_{S}(\alpha). \]

Mathematically, this difference is explained by the presence of the logarithm in the expressions for the Rényi entropies. As, for example, equation (10) shows, the Shannon entropy, similar to its Rényi generalization, is an additive functional, too:

\[ S_{B}(\alpha) = S_{F}(\alpha) + S_{S}(\alpha). \]

Further comparative analysis between the Rényi and Tsallis entropies can be found, e.g. in [8–10].

It has to be also noted that the functionals for the continuous distributions from equations (5) were obtained from their discrete counterparts; namely, for the discrete set of all \( N \) possible events with their probabilities:

\[ 0 \leq p_{n} \leq 1, \quad n = 1, 2, \ldots, N, \]

so that

\[ \sum_{n=1}^{N} p_{n} = 1, \]

and one defines the entropies as

\[ R(\alpha) = \frac{1}{1 - \alpha} \ln \left( \sum_{n=1}^{N} p_{n}^{\alpha} \right), \]

\[ T(\alpha) = \frac{1}{\alpha - 1} \left( 1 - \sum_{n=1}^{N} p_{n}^{\alpha} \right), \]

\[ S = -\sum_{n=1}^{N} p_{n} \ln p_{n}, \]

which are, obviously, dimensionless. However, it follows from equations (5a), (5b) and (6), that the Rényi and Shannon entropies for the continuous distributions are measured in units of the logarithm of the length whereas their Tsallis counterparts from equations (5c) and (5d) represent the sum of the dimensionless unity and the quantity measured in units of the distance raised to power \( \ell(1 - \alpha) \) or its inverse. Accordingly, their actual numerical values strongly depend on the units in which the calculations are carried out. Nevertheless, most importantly, the sum of the position and momentum Rényi entropies is a scaling-independent dimensionless quantity. The same is true for the Shannon entropies. In the case of the Tsallis functionals, a dimensional inconsistency of the two factors in equations (5c) and (5d) limits their applications per se suggesting instead the use of the forms \( 1 + (1 - \alpha)T(\alpha) \), as done, for instance, in the associated uncertainty relation, equation (26).

Another important difference between the entropies for the continuous and discrete distributions can be shown in the example of the Shannon functionals [11]; namely, the quantity from equation (14c) under the requirement from equation (12) is always non-negative, but its counterparts from equations (6) can fall below zero if in some region the
corresponding density is greater than unity and its contribution to the integral does outweigh
that from the interval where $\rho(r)$ or $\gamma(k)$ lies between zero and one [12–14]. Historically, Rényi and Tsallis functionals were introduced initially for the discrete distributions in the form from equations (14a) [1] and (14b) [4] first of all for the need of the theory of information only and their physical applications followed (as was the case with the original Tsallis postulate [3]) with a subsequent generalization to the continuous case. In contrast, long before Shannon, the sum from equation (14c) was well known exclusively in physics; namely, the very term ‘entropy’ was introduced into thermodynamics by Clausius in 1865 as a function of state of the thermal system that determines its irreversible scattering of energy. Classical statistical mechanics based on the Boltzmann–Gibbs (BG) approach treats the quantity from equation (14c) (with its right-hand side multiplied by the Boltzmann constant $k_B$) as a measure of disorder with $p_n$ describing the probability of the microstate with the energy $E_n$ to occur and with the total number of microscopic configurations in the system being $N$ whereas the quantum consideration replaces it by von Neumann entropy,

$$S = -k_B \text{Sp}(\hat{\rho} \ln \hat{\rho}),$$

(15)

with $\hat{\rho}$ being the density matrix operator. A pivotal role of the BG entropy can be shown in the example of the canonical distribution when the condition of the maximization of $S$ with the additional requirement of the total internal energy,

$$U = \sum_{n=1}^{N} p_n E_n,$$

(16)

staying unchanged leads to the Gibbs distribution. Tsallis [3] was the first to point out that the entropy from equation (14b) does generalize the BG statistics; namely, its extremization with the help of the Lagrange parameters yields

$$p_n(\alpha; T_t) = \frac{[1 - (\alpha - 1)E_n/(k_B T_t)]^{1/(\alpha - 1)}}{Z(\alpha; T_t)},$$

(17a)

$$Z(\alpha; T_t) = \sum_{n=1}^{N} [1 - (\alpha - 1)E_n/(k_B T_t)]^{1/(\alpha - 1)},$$

(17b)

where $T_t$ is an absolute thermodynamic temperature with subindex ‘t’ distinguishing it from the Tsallis entropy $T(\alpha)$. It can be immediately seen that the limit $\alpha \to 1$ recovers the BG expressions:

$$p_n(\alpha; T_t) = \frac{1}{Z(1; T_t)} e^{-E_n/(k_B T_t)}$$

$$\times \left(1 - \frac{1}{2} \sum_{n'=1}^{N} \left( \frac{E_{n'}}{k_B T_t} \right)^2 - \frac{1}{2} \sum_{n'=1}^{N} \left( \frac{E_{n'}}{k_B T_t} \right)^2 e^{-E_{n'}/(k_B T_t)} \right)\frac{1}{Z(1; T_t)} \left(\alpha - 1\right) + \ldots,$$

(18a)

$$Z(\alpha; T_t) = \sum_{n=1}^{N} e^{-E_n/(k_B T_t)} - \frac{1}{2} (\alpha - 1) \sum_{n'=1}^{N} \left( \frac{E_{n'}}{k_B T_t} \right)^2 e^{-E_{n'}/(k_B T_t)} + \ldots,$$

(18b)

where

$$Z(1; T_t) \equiv Z(T_t) = \sum_{n=1}^{N} e^{-E_n/(k_B T_t)},$$

(19)
is a BG partition function. In this way, Tsallis statistics supplement the BG one by expanding it to \(\alpha \neq 1\). Non-Gaussian distributions from equation (17a) have been predicted theoretically [15] and successfully applied for the explanation of the experiments in miscellaneous branches of physics [16–18], including high-energy collisions [19, 20]. For understanding thermodynamic interpretation of the Rényi entropy [21–24], one needs to substitute the BG distribution \(p_T(1; T_1)\) from equation (18a) at the temperature \(T_1\) into equation (14a) (right-hand side of which, similar to the Shannon entropy, has to be multiplied by \(k_B\)) and represent the Rényi parameter as the ratio of the two temperatures, \(\alpha = T_1/T_2\), arriving in this way at

\[
R \left( \frac{T_1}{T_2} \right) = -\frac{F(T_2) - F(T_1)}{T_2 - T_1},
\]

where \(F(T_i)\) is Helmholtz free energy [25]:

\[
F(T_i) = -k_B T_i \ln Z(T_i).
\]

Thus, the Rényi entropy replaces the derivative of the free energy in the well-known thermodynamic relation between \(S\) and \(F\) [25]:

\[
S(T_i) = -\frac{\partial F}{\partial T_i},
\]

by the ratio of the finite differences at the two temperatures. Drawing a parallel with the velocities in mechanics [26], one can say that the Shannon entropy describes the instantaneous speed of change of the Helmholtz energy with temperature whereas the Rényi functional averages this variation (and, accordingly, the maximal amount of work the system can do) over some interval \(T_2 - T_1\).

For any quantum orbital, the position and momentum components of the entropies are not independent of each other; for example, the Rényi entropies obey the following fundamental inequality [27, 28]:

\[
R_{\rho_\alpha}(\alpha) + R_{\rho_\beta}(\beta) \geq -\frac{l}{2} \left( \frac{1}{1 - \alpha} \ln \frac{\alpha}{\pi} + \frac{1}{1 - \beta} \ln \frac{\beta}{\pi} \right),
\]

with the constraint on the interrelation between the indexes:

\[
\frac{1}{\alpha} + \frac{1}{\beta} = 2.
\]

Within the limit of \(\alpha \to 1\), equation (23) degenerates to the Shannon uncertainty relation [29, 30]:

\[
S_{\rho_\alpha} + S_{\rho_\beta} \geq l(1 + \ln \pi).
\]

Inequality similar to equation (23) exists for the Tsallis entropies too [8]:

\[
\left( \frac{\alpha}{\pi} \right)^{(1/4\alpha)} \left[ 1 + (1 - \alpha) T_{\alpha}(\alpha) \right]^{1/(2\alpha)} \geq \left( \frac{\beta}{\pi} \right)^{(1/4\beta)} \left[ 1 + (1 - \beta) T_{\alpha}(\beta) \right]^{1/(2\beta)},
\]

which is a direct consequence of the Sobolev inequality of the Fourier transform [31]:

\[
\left( \frac{\alpha}{\pi} \right)^{(1/4\alpha)} \left[ \int_{\mathbb{R}^n} \rho^{\alpha}(\mathbf{r}) d\mathbf{r} \right]^{1/(2\alpha)} \geq \left( \frac{\beta}{\pi} \right)^{(1/4\beta)} \left[ \int_{\mathbb{R}^n} \rho^{\beta}(\mathbf{k}) d\mathbf{k} \right]^{1/(2\beta)},
\]

where
in the proof of which, in addition to the requirement from equation (24), an extra restriction,
\[ \frac{1}{2} \leq \alpha \leq 1, \]  
(28)
is imposed [31]. As immediately follows from equations (26) and (27), they are saturated at \( \alpha = \beta = 1 \) when either side turns to a dimensionless \( \pi^{-l/4} \). Observe also that Rényi inequality (23) is obtained by taking the logarithms of its Sobolev counterpart, equation (27), and it is extremely important to point out that as a result of this operation, the condition from equation (28) is removed for the Rényi entropies [28]. This difference between the two uncertainty relations, equations (23) and (26), will be illustrated below in the particular examples.

Rényi and Tsallis entropies find countless applications in science, technology, engineering, medicine and economics, among others. For example, soon after a Hungarian mathematician proposed the functional now bearing his name [1], Rényi measure was successfully applied in coding theory [32]. Its applications in physics include but not are limited to the analysis of the processes of hadronic multiparticle high-energy collisions [33, 34], fractional diffusion processes [35], properties of the XY spin chain [36], flow to the system in thermal equilibrium that is weakly coupled to an arbitrary system out of equilibrium subject to arbitrary time-dependent forces [37], conformal field theory [38, 39], black hole area law [40] and many, many others (see, for example, sources in [38–40]). Recent state-of-the-art experiments directly measured Rényi entropy [41], which opens up new horizons in using quantum entanglement to characterize the dynamics of strongly correlated many-body systems. For our subsequent discussion, one has to note that Rényi, equation (23), Shannon, equation (25), and Tsallis, equation (26), inequalities are more general in establishing a relationship between the two non-commuting operators than the standard Heisenberg uncertainty relation that for the 1D geometry, \( l = 1 \), we will write as

\[ \Delta x \Delta k \geq \frac{1}{2}. \]  
(29)

Here, \( \Delta x \) and \( \Delta k \) are, respectively, the position and wave vector standard deviations:

\[ \Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}, \]  
(30a)

\[ \Delta k = \sqrt{\langle k^2 \rangle - \langle k \rangle^2}, \]  
(30b)

where the associated moments \( \langle x^j \rangle \) and \( \langle k^j \rangle \), \( j = 1, 2, \ldots \), are expressed with the help of the corresponding position \( \rho(x) \) and momentum \( \gamma(k) \) densities:

\[ \langle x^j \rangle = \int_{\mathbb{R}} x^j \rho(x) dx, \]  
(31a)

\[ \langle k^j \rangle = \int_{\mathbb{R}} k^j \gamma(k) dk. \]  
(31b)

To show the non-universality of the Heisenberg relation, one might consider the lowest level of the 1D Neumann well of width \( a \): its energy is zero and its position waveform is just a constant \( \Psi_N^0(x) = a^{-1/2}, -a/2 \leq x \leq a/2 \). Then, its momentum wave function reads

\[ \Psi_N^0(k) = \left( \frac{2}{\pi a} \right)^{1/2} \frac{1}{k} \sin \frac{ka}{2}, -\infty \leq k \leq +\infty. \]  
(32)

From this expression, it can immediately be seen that the second moment \( \langle k^2 \rangle \) and, together with it, dispersion diverge: \( \Delta k = \infty \) [42, 43], which makes the Heisenberg relation
meaningless, since it does not present any new information about position-momentum limitations; infinity is always greater than any finite number. If, instead of defining momentum moments, as is done in equation (31b), one uses alternately the wave vector operator \( \hat{k} = -i\partial/\partial x \) acting upon the space of the position functions:

\[
\langle \hat{k}^j \rangle = \int_{\mathbb{R}} \Psi(x) \hat{k}^j \Psi(x) dx, \tag{31b'}
\]

then it can be seen that for the Neumann ground level the Heisenberg relation is violated since the momentum dispersion turns to zero, \( \Delta k = 0 \) \cite{12}. However, both Rényi \cite{44, 45}, equation (23), and Shannon \cite{12, 43–45}, equation (25), inequalities hold true with, for example, the left-hand side of the latter expression yielding \( 2.6834 \ldots \) \cite{12} whereas \( 1 + \ln \pi = 2.1447 \ldots \).

Returning to the applications of the entropies, let us mention that, besides physics and information theory, Rényi measure was used in the investigation of the spatial distribution of earthquake epicenters \cite{46}, for the analysis of the landscape diversity and integrity \cite{47–50}, the examination of the behavior of the stock markets \cite{51, 52}, characterization of scalp electroencephalogram records corresponding to secondary generalized tonic-clonic epileptic seizures \cite{53}, study of biological signals \cite{54}, exploration and modification of brain activity \cite{55}, in digital image analysis \cite{56}, etc. In turn, Tsallis entropies are widely employed in non-extensive systems, which include the structures and processes characterized by non-ergodicity, long-range correlations and space-time (multi)fractal geometry; see, for example \cite{57} for more details. The literature of both entropies is growing at an impressive rate.

Despite the paramount significance of these two entropies, concrete results on the specific physical quantum systems to date are scarce, which is probably explained by the tremendous difficulties of evaluating (even numerically) the integrals entering them, especially for the momentum components. By means of semi-classical approximation, the general asymptotic formulae were derived for the 1D position Rényi entropy and checked for the infinite potential well \cite{58}. For the same structure the expressions for the entropies \( R_\alpha (\alpha) \) and \( T_\alpha (\alpha) \), which contain double finite sums, were provided for the integer values of parameter \( \alpha \) only \cite{59}. The entropies of the highly excited levels for the 3D hydrogenic system \cite{60} and harmonic oscillator (HO) \cite{61} were scrutinized too and it was shown, in particular, that the ground orbital of the latter structure saturates the entropic relation, equation (23). The results were generalized to any number of dimensions \cite{62, 63}.

In view of these remarks, it is extremely important to investigate the quantum structures where analytic expressions for \( R \) and \( T \) are possible. Below, we consider three such 1D systems. They are

- HO,
- attractive Robin wall, and
- quasi-1D (Q1D) hydrogenic atom.

For the second structure, only one bound state exists whereas the first and last geometries are characterized by an infinite number of localized levels. For the HO, analytic expressions of the entropies can be derived for the ground and first excited orbitals, and for the Q1D hydrogenic atom exact results in terms of the \( \Gamma \)-function \cite{64} are derived for the momentum components of any orbital and for the position part of the ground state. The obtained formulae for the entropies allow a simple analysis of the asymptotic cases. In particular, it is explicitly shown that the position components do exist at any non-negative Rényi parameter whereas their momentum counterparts are valid for the range of index \( \alpha \), which is limited from below by the positive threshold \( \alpha_{TH} > 0 \). This lower bound on the momentum entropies strongly
depends on the position potential and the type of boundary condition imposed on the position waveform. For example, it is equal to one half for the Robin wall, $\alpha_{1}^{\text{RW}} = 1/2$, and to one quarter for the hydrogenic atom, $\alpha_{1}^{\text{Q1D}} = 1/4$. It is shown that the entropic uncertainty relation, equation (23), is always satisfied and is saturated at the limiting values of the parameters from equation (24). Since the results below are easily comprehensible (both mathematically as well as physically) by the students, they can serve as part of a graduate introduction to quantum-information theory. The previous pedagogical approach to studying quantum-information measures [65] addressed Fisher information [66, 67] of the HO, infinitely deep Dirichlet well and Q1D hydrogen atom. Unfortunately, that research suffered a severe blunder in the analysis of the momentum waveform of the latter structure [68]. We shall revisit this issue in more detail in chapter 4. As a result of our analysis, we present some appropriate points that require further consideration.

2. 1D HO

Consider a motion along the whole $x$ axis of the quantum particle in the potential of the form

$$V(x) = \frac{1}{2} m_p \omega^2 x^2.$$  \hfill (33)

Then, eigen energies $E_n$ and associated position eigen functions $\Psi_n(x)$ of the 1D Schrödinger equation take the form [69]

$$E_n = \hbar \omega \left( n + \frac{1}{2} \right),$$  \hfill (34)

$$\Psi_n(x) = \frac{1}{r_n^{1/2}} \frac{1}{\pi^{1/4}} \frac{1}{(2^n n!)^{1/2}} e^{-(x/r_n)^2/2} H_n \left( \frac{x}{r_n} \right),$$  \hfill (35)

where $n = 0, 1, \ldots$. Here, $H_n(x)$ is an $n$th-order Hermite polynomial [64, 70], and $r_n = [\hbar/(m_p \omega)]^{1/2}$. Corresponding momentum waveforms can also be easily obtained from equation (2a) employing, for example, properties of the generating function of the Hermite polynomials [70]:

$$\Phi_n(k) = (-i)^n r_n^{1/2} \frac{1}{\pi^{1/4}} \frac{1}{(2^n n!)^{1/2}} e^{-(r_n k)^2/2} H_n(r_n k).$$  \hfill (36)

An alternative equivalent way of arriving at the same expression is to use a momentum representation of the Schrödinger equation when the coordinate $x$ is an operator, $\hat{x} = \text{id}/\partial k$ [69]. A comparison of equations (35) and (36) reveals that the position $\rho_n(x)$ and momentum $\gamma_n(k)$ densities are related as

$$\gamma_n(k) = r_n^2 \rho_n(r_n^2 k).$$  \hfill (37)

As a result, corresponding Rényi entropies read

$$R_{\alpha}^\rho (\alpha) = \ln r_n + \frac{1}{1 - \alpha} \ln I_\rho (\alpha),$$  \hfill (38a)

$$R_{\alpha}^\gamma (\alpha) = -\ln r_n + \frac{1}{1 - \alpha} \ln I_\gamma (\alpha),$$  \hfill (38b)
It can immediately be seen that the sum of the two entropies from the left-hand side of inequality (23) is a dimensionless scale-independent quantity, as it should be. Accordingly, below in this section, while discussing the quantities $R_{\rho}$ and $R_{\gamma}$, we will use the units in which the length is measured in terms of $r_\omega$, which means that the HO position and momentum Rényi entropies are equal to each other:

$$R_{\rho}(\alpha) \equiv R_{\gamma}(\alpha).$$

Then, if it will not cause any confusion, we will drop the subindex $\rho$ or $\gamma$. Remarkably, for the ground, $n = 0$, and first excited, $n = 1$, states these quantities are evaluated analytically:

$$R_0(\alpha) = \frac{1}{2} \ln \pi - \frac{1}{2} \frac{\ln \alpha}{1 - \alpha},$$

$$R_1(\alpha) = \frac{1}{1 - \alpha} \ln \left( \frac{2^\alpha}{\pi^{1/2} e^{\alpha + 1/2} \Gamma \left( \alpha + \frac{1}{2} \right)} \right).$$

They are plotted in figure 1 together with their $n = 2$ and $n = 3$ counterparts, which can be evaluated only numerically. It can be seen that, as expected, the Rényi entropies are decreasing functions of parameter $\alpha$ and for the higher-lying states they are greater than their lower-lying counterparts. At the vanishing $\alpha$ all the entropies diverge logarithmically:

$$I_\alpha(\alpha) = \frac{1}{(\pi^{1/2} 2^\alpha n!)^\alpha} \int_{-\infty}^{\infty} \left[ e^{-z^2} H_\alpha(z) \right]^\alpha dz.$$
\[ R_0(\alpha) = \frac{1}{2} [\ln \pi - (1 + \alpha) \ln \alpha], \quad \alpha \to 0, \]  
\[ R_1(\alpha) = \frac{1}{2} (\ln \pi - \ln \alpha) - [\gamma + \ln (2\pi^{1/2}) + \ln \alpha] \alpha, \quad \alpha \to 0, \]  
which is explained by the infinite limits of integration of the constant unit function, as mentioned in the Introduction, and at the huge Rényi parameter, \( \alpha \to \infty \), for the two lowest states they are
\[ R_0(\alpha) = \frac{1}{2} \ln \pi + \frac{1}{2} \ln \alpha + \ldots, \]  
\[ R_1(\alpha) = 1 - \ln 2 + \frac{1}{2} \ln \pi + \frac{1}{2} \ln \alpha - 3 \ln 2 + \frac{2}{\alpha} + \ldots, \]
with the leading terms in these asymptotic expansions being \( \frac{1}{2} \ln \pi = 0.572 \ldots \) and \( 1 - \ln 2 + \frac{1}{2} \ln \pi = 0.879 \ldots \), respectively, while in the vicinity of the Shannon case, \( \alpha \to 1 \), they turn to
\[ R_0(\alpha) = \frac{1}{2} (1 + \ln \pi) - \frac{1}{4} (\alpha - 1) + \frac{1}{6} (\alpha - 1)^2 + \ldots, \]  
\[ R_1(\alpha) = \ln 2 + \gamma + \frac{1}{2} \ln \pi - \frac{1}{2} - \frac{\pi^2 - 9}{4} (\alpha - 1) \]  
\[ + \frac{7\zeta(3) - 8}{3} (\alpha - 1)^2 + \ldots \]
Here, \( \gamma \) is Euler’s constant \([64]\):
\[ \gamma = \lim_{n \to \infty} \left( \sum_{i=1}^{n} \frac{1}{i} - \ln n \right) = 0.577 \ldots, \]
and \( \zeta(z) \) is the Riemann zeta function \([64]\):
\[ \zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z}, \]
with \( \zeta(3) = 1.202 \ldots \). Accordingly, the leading terms in equations (44) are 1.072 3\ldots and 1.342 7\ldots. For our subsequent analysis, it is important to point out that the momentum entropies are defined for any positive Rényi parameter \( \alpha \).

To gain deeper insight into the structure of these entropies, it is worthwhile, by using relation (24), to bring inequality (23) to the following form \([28]\):
\[ R_\alpha(\alpha) + R_\beta(\beta) \geq \ln \pi - \frac{1}{2} \left( \frac{\ln \alpha}{1 - \alpha} + \frac{\ln \beta}{1 - \beta} \right). \]
Then, one can see that the Gaussian probability distribution, i.e. ground level, \( n = 0 \), densities with \( \rho_0(x) = \pi^{-1/2} e^{-x^2} \) and \( \gamma_0(k) = \pi^{-1/2} e^{-k^2} \), do saturate the uncertainty relation at any \( \alpha \) and \( \beta \) that obey restriction from equation (24), as was pointed out before \([27]\). Denoting the right-hand-side of inequalities (23) and (47) where, as follows from equation (24), parameter \( \beta \) is a function of \( \alpha \):
\[ \beta = \frac{\alpha}{2\alpha - 1}, \]
by \( f(\alpha) \):

\[
f(\alpha) = \ln \pi - \left[ \ln \alpha - \frac{\alpha - 1/2}{\alpha - 1} \ln(2\alpha - 1) \right].
\]

(49)

it is easy to get its asymptotic limits:

\[
f(\alpha) = \ln 2\pi - [1 + \ln(2\alpha - 1)](2\alpha - 1) + \ldots, \quad \alpha \to \frac{1}{2},
\]

(50a)

\[
f(\alpha) = 1 + \ln \pi - \frac{1}{6}(\alpha - 1)^2 + \frac{1}{3}(\alpha - 1)^3 + \ldots, \quad \alpha \to 1,
\]

(50b)

\[
f(\alpha) = \ln 2\pi + \frac{\ln 2\alpha - 1}{2\alpha} + \ldots, \quad \alpha \to \infty,
\]

(50c)

with \( \ln 2\pi = 1.8378 \ldots \) and numerical value of \( 1 + \ln \pi \) provided in the Introduction. Equation (50b) manifests that the Shannon case, \( \alpha = 1 \), presents a maximum of the function \( f(\alpha) \) and, accordingly, of \( R_\rho(\alpha) + R_\gamma(\beta) \). As the solid curve in figure 2 depicts, it is its global extremum. Observe that equations (50a) and (50c) turn into each other under the transformation,

\[
\alpha \to \beta = \alpha/(2\alpha - 1),
\]

(51)

which is a reflection of the symmetry of the system upon introduction of the symmetrized parameter \( s \), so that \( \alpha = 1/(1 - s) \), \( \beta = 1/(1 + s) \) and \(-1 \leq s \leq +1 \), as was done in [27]. With the help of equation (41b) one can also find the corresponding limits of the first excited

Figure 2. Sum of the position and momentum Rényi entropies \( R_\rho(\alpha) + R_\gamma(\beta) \) with parameter \( \beta \) from equation (48) of the four lowest HO states as functions of parameter \( \alpha \).
state:
\[ R_{\rho_l}(\alpha) + R_{\rho_l}(\beta) = 1 + \ln 4 - 2\left[ \gamma + \ln \pi + \ln \left( \alpha - \frac{1}{2} \right) \right] \left( \alpha - \frac{1}{2} \right) + \ldots, \quad \alpha \to \frac{1}{2}. \]  
(52a)

\[ R_{\rho_l}(\alpha) + R_{\rho_l}(\beta) = 2\gamma - 1 + \ln 4\pi + \left[ \frac{14}{3} \zeta(3) - \frac{\pi^2}{2} - \frac{5}{6} \right] (\alpha - 1)^2 + \ldots, \quad \alpha \to 1, \]  
(52b)

\[ R_{\rho_l}(\alpha) + R_{\rho_l}(\beta) = 1 + \ln 4 + \frac{\ln^4}{\alpha} - \gamma + \ln \alpha + \ldots, \quad \alpha \to \infty. \]  
(52c)

First, note that since \( \frac{14}{3} \zeta(3) - \frac{\pi^2}{2} - \frac{5}{6} \approx -0.1585 \ldots \) is negative, the Shannon case of \( \alpha = 1 \) is again a maximum of the sum of the two Rényi entropies. Next, equations (52a) and (52c), similar to equations (50a) and (50c), transform into each other under the conversion from equation (51). Third, the leading terms in equations (52), namely, \( 1 + \ln 4 = 2.3862 \ldots \) and \( 2\gamma - 1 + \ln 4\pi = 2.6854 \ldots \), are greater than their ground-orbital counterparts from equations (50). This remains true for any positive \( \alpha \). These properties, i.e. the increase of the sum \( R_{\rho_l}(\alpha) + R_{\rho_l}(\beta) \) with the quantum index \( n \) and its global extremum at the Shannon entropy, hold for any excited level, as figure 2 demonstrates where the values for \( n = 2 \) and \( n = 3 \) were computed numerically. Let us also point out that the Rényi inequalities hold true for any \( \alpha \) greater than one half, without additional restriction from equation (28).

Tsallis entropies for the two lowest-lying states read

\[ T_{\rho_l}(\alpha) = \left( \frac{1}{\alpha - 1} \right) \left[ 1 - \frac{1}{\alpha^{1/2}(\pi^{1/2} \omega)^{\alpha - 1}} \right], \]  
(53a)

\[ T_{\rho_l}(\alpha) = \left( \frac{1}{\alpha - 1} \right) \left[ 1 - \frac{1}{\alpha^{1/2} (\omega^{1/2})^{\alpha - 1}} \right], \]  
(53b)

\[ T_{\rho_l}(\alpha) = \left( \frac{1}{\alpha - 1} \right) \left[ 1 - \frac{2}{\pi^{\alpha/2} \omega^{1/2}} \Gamma \left( \frac{\alpha + 1}{2} \right) \right], \]  
(53c)

\[ T_{\rho_l}(\alpha) = \left( \frac{1}{\alpha - 1} \right) \left[ 1 - \frac{2}{\pi^{\alpha/2} \omega^{1/2}} \Gamma \left( \frac{\alpha + 1}{2} \right) \right], \]  
(53d)

where we switched back to the regular dimensional units. First, let us point out that taking the limit \( \alpha \to 1 \) of, for example, equations (53a) and (53b):

\[ T_{\rho_l}(\alpha) = \ln r_\omega + \frac{1}{2}(1 + \ln \pi) - \frac{1}{2} \left[ \ln^2 (\pi^{1/2} \omega) + \ln (\pi^{1/2} \omega) + \frac{3}{4} \right] (\alpha - 1) + \ldots, \]  
(54a)
\[ T_f(\alpha) = -\ln r_\omega + \frac{1}{2}(1 + \ln \pi) \]

\[ -\frac{1}{2} \ln^2 \left( \frac{\pi^{1/2}}{r_\omega} \right) + \ln \left( \frac{\pi^{1/2}}{r_\omega} \right) + \frac{3}{4} (\alpha - 1) + \ldots, \quad (54b) \]

and comparing them with equation (44a) (where, as we recall, the convention \( r_\omega \equiv 1 \) was used), one sees that Rényi and Tsallis entropies approach their Shannon asymptote in different ways. The same can be shown for the first exiting level, equations (53c) and (53d), but since the resulting expressions are bulky, they are not provided here. Next, inserting equations (53a) and (53b) into the entropy inequality (26), one calculates its left- and right-hand sides as \( \pi^{-1/4} r^{1/2}_\omega \) and \( \pi^{-1/4} r^{2/2}_\omega \), respectively, which, due to the requirement from equation (24), means that at any parameter \( \alpha \) the lowest level saturates the Tsallis entropy relation too. So, the Gaussian dependencies not only turn the Tsallis inequality into the identity but, the requirement for its applicability, equation (28), is waived for them. This example again singles them out from all other probability distributions; for example, for the first excited orbital, inequality (26), and accordingly, (27), are

\[ \left( \frac{2}{\alpha} \right)^{1/2} \pi^{-1/4} \Gamma \left( \alpha + \frac{1}{2} \right) r^{1/2}_\omega \geq \left( \frac{2}{\beta} \right)^{1/2} \pi^{-1/4} \Gamma \left( \beta + \frac{1}{2} \right) r^{2/2}_\omega. \quad (55) \]

Equation (48) guarantees that this relation is dimensionally correct. Note that apart from the factor containing \( r_\omega \), both sides of this expression have the same dependence on their parameters \( \alpha \) or \( \beta \). Accordingly, general Tsallis inequality, equation (26), at \( n = 1 \) is saturated at \( \alpha = \beta = 1 \) only when either of its sides becomes a dimensionless \( \pi^{-1/4} = 0.751 \ldots \), as was mentioned in the Introduction. Figure 3 depicts dimensionless parts of inequality (55) (i.e. it is assumed here that \( r_\omega \equiv 1 \)) as functions of parameter \( \alpha \). This manifests that for the first excited HO orbital the Tsallis inequality is satisfied at the interval defined by equation (28) only, in contrast to the Rényi entropies relations, which, as mentioned above, are valid at any \( \alpha \geq 1/2 \). For completeness, let us mention that the position and momentum parts of inequality (55) at \( \alpha = 1/2 \) take the values of \( 2/\pi^{3/4} = 0.847 \ldots \) and \( 2/ (\pi^{1/2})^{1/2} = 0.644 \ldots \) whereas at \( \alpha = \infty \) these magnitudes interchange. Also, in the vicinity of \( \alpha = 1 \), this relation turns to

\[ \frac{1}{\pi^{1/4}} \left[ 1 - \frac{1}{2} \ln^2 (\alpha - 1) \right] \geq \frac{1}{\pi^{1/4}} \left[ 1 + \frac{1}{2} \ln^2 (\alpha - 1) \right], \quad (56) \]

and since the term \( (\gamma - 1 + \ln 2)/2 = 0.135 \ldots \) is positive, the last inequality is satisfied for the Tsallis parameter approaching unity from the left, as expected.

### 3. Attractive Robin wall

By Robin wall we mean a 1D structure that limits the motion of the particle to a half-line, say, \( x > 0 \), and at the confining surface, \( x = 0 \), the following requirement is imposed on the position function \( \Psi(x) \):

\[ \frac{d\Psi(x)}{dx} \bigg|_{x=0} = \frac{1}{\lambda} \Psi(0). \quad (57) \]
From the expression for the current density [69],

\[ j = -\frac{e\hbar}{mp} \mathcal{J}(\psi^* \nabla \psi), \tag{58} \]

it is elementary to check that at any real Robin length \( \Lambda \) no current flows through the point \( x = 0 \):

\[ j|_{x=0} = 0 \quad \text{at} \quad \mathcal{J}(\Lambda) = 0. \tag{59} \]

A remarkable property of this geometry is the fact that at the negative extrapolation parameter \( \Lambda \) it has, in addition to the continuous spectrum with \( E \geq 0 \), a single localized level with the energy \( [13, 71–76] \),

\[ E = -\frac{\hbar^2}{2m_p|\Lambda|^2}, \quad \Lambda < 0, \tag{60} \]

with the corresponding position waveform vanishing at infinity:

\[ \Psi(x) = \left(\frac{2}{|\Lambda|}\right)^{1/2} \exp\left(-\frac{x}{|\Lambda|}\right), \quad x \geq 0, \tag{61} \]

and obeying the normalization condition

\[ \int_0^{\infty} \Psi^2(x)dx = 1. \tag{62} \]
Its momentum counterpart [13],

$$\phi(k) = \left(\frac{|\lambda|}{\pi}\right)^{1/2} \frac{1}{1 + i|\lambda|k},$$

(63)

obeys the normalization too:

$$\int_{-\infty}^{\infty} |\phi(k)|^2 dk = 1.$$  

(64)

Accordingly, the corresponding densities are

$$\rho(x) = \frac{2}{|\lambda|} \exp\left(-\frac{x}{|\lambda|}\right),$$

$$\gamma(k) = \frac{|\lambda|}{\pi} \frac{1}{1 + |\lambda|^2 k^2}.$$  

(65a)

(65b)

Compared to the other two structures considered here, the attractive Robin wall is very peculiar, since for it a standard Heisenberg uncertainty suffers the same shortcomings as the Neumann well discussed in the Introduction, while the Shannon entropy inequality, equation (25), remains true [13]. This is further evidence of the fact that quantum-information entropies present a more general base for defining ‘uncertainty’ than standard deviations $\Delta x$ and $\Delta k$. As shown below, the Rényi inequality that generalizes its Shannon counterpart also holds true for any $\alpha$ saturating at $\alpha = 1/2$. The same is true for the Tsallis inequality, equation (26), too.

The Rényi entropies of this single orbital read

$$R_\alpha^p(\alpha) = \ln|\lambda| - \ln 2 - \frac{\ln \alpha}{1 - \alpha},$$

(66a)

$$R_\alpha^s(\alpha) = -\ln|\lambda| + \ln \pi + \frac{1}{1 - \alpha} \ln \frac{\Gamma\left(\alpha - \frac{1}{2}\right)}{\pi^{1/2} \Gamma(\alpha)}.$$  

(66b)

The first thing to note from these equations is the fact that the momentum Rényi entropy exists (or, rather, has real values) not at any arbitrary non-negative parameter $\alpha$, as was the case for the HO, but only for those indexes $\alpha$ that are greater than the threshold value that in this case is equal to one half:

$$\alpha_{\text{TH}} = \frac{1}{2}.$$  

(67)

Around this point, the entropy diverges logarithmically:

$$R_\alpha^p(\alpha) = -2 \ln \left(\alpha - \frac{1}{2}\right) - \ln|\lambda| - \ln \pi + \ldots, \quad \alpha \to \frac{1}{2},$$

(68)

as does its position counterpart near its own threshold value of zero, as follows from equation (66a). In the opposite limit of the huge Rényi parameter, $\alpha \to \infty$, the entropies are

$$R_\alpha^p(\alpha) = \ln|\lambda| - \ln 2 + \frac{\ln \alpha}{\alpha} + \ldots,$$

(69a)

$$R_\alpha^s(\alpha) = -\ln|\lambda| + \ln \pi + \frac{1}{2} \frac{\ln \alpha}{\alpha} + \ldots,$$

(69b)
and in the neighborhood of unity, $\alpha \to 1$, one has

$$R_\rho(\alpha) = \ln|\Lambda| - \ln 2 + 1 - \frac{1}{2}(\alpha - 1) + \frac{1}{3}(\alpha - 1)^2 + \ldots,$$  

(70a)

$$R_\gamma(\alpha) = -\ln|\Lambda| + \ln \pi + \ln 4 - \frac{\pi^2}{6}(\alpha - 1) + 2\zeta(3)(\alpha - 1)^2 + \ldots,$$  

(70b)

which obviously means that the Rényi entropies just approach their Shannon counterparts [13]. Next, it directly follows from equations (66) that in equations (23) or (47) the sum of the two entropies is, similar to the HO, a scale-independent dimensionless quantity. Accordingly, below in this section, during the discussion of the Rényi entropies the distances will be measured in units of $|\Lambda|$, which is a characteristic length of this system.

Position and momentum Rényi entropies are shown in figure 4 as functions of parameter $\alpha$. As discussed above, the position component is defined on the whole non-negative axis decreasing from the infinitely high values at the infinitely small $\alpha$, passing through zero at $\alpha = 2$, $R_\rho(2) = 0$, and approaching $-\ln 2 = -0.6931 \ldots$ at the parameter tending to infinity. Its momentum counterpart stays positive at any variable that is greater than its threshold monotonically decreasing from unrestrictedly high values at $\alpha = \alpha_{TH}$ to $\ln \pi = 1.1447 \ldots$ at the very large $\alpha$.
Turning to the discussion of the entropic uncertainty relation, we first provide asymptotic limits of the sum from the left-hand side of equations (23) or (47):

\[ R_\rho(\alpha) + R_\gamma(\beta) = \ln 2\pi + 2[\ln(2\pi^{1/2}) - \ln(2\alpha - 1)](2\alpha - 1) + \ldots, \quad \alpha \to \frac{1}{2}, \]

\[ R_\rho(\alpha) + R_\gamma(\beta) = 1 + \ln \pi + \ln 2 + \left(\frac{\pi^2}{6} - \frac{1}{2}\right)(\alpha - 1) \]
\[ + \frac{1}{3} + 2\zeta(3) - \frac{\pi^2}{3}(\alpha - 1)^2 + \ldots, \quad \alpha \to 1, \]

\[ R_\rho(\alpha) + R_\gamma(\beta) = 2\ln\alpha + \ln\frac{8}{\pi} + \frac{\ln(8/\pi) - 1}{\alpha} + 2\ln\alpha \]
\[ + \ldots, \quad \alpha \to \infty. \]

As the dotted line in figure 4 shows, the sum \( R_\rho(\alpha) + R_\gamma(\beta) \) is a monotonically increasing function of parameter \( \alpha \) logarithmically diverging at \( \alpha \) tending to infinity, according to equation (71c). The entropic uncertainty relation, equations (23) or (47), is always satisfied with its saturation taking place at the threshold of the momentum Rényi entropy, as the comparison between equations (50a) and (71a) demonstrates. The behavior near the threshold is shown in enlarged format in the lower inset of the figure. For comparison, the sum \( R_\rho(\beta) + R_\gamma(\alpha) \) is also depicted in figure 4. Its dependence is opposite to the one just described, which is the result of inversion from equation (51); namely, its divergence at \( \alpha_{TH} \) after its monotonic decrease turns at \( \alpha = \infty \) into \( \ln 2\pi \), thus transforming the entropic uncertainty relation into the equality.

Tsallis entropies read

\[ T_\rho(\alpha) = \frac{1}{\alpha - 1}\left[1 - \frac{1}{\alpha} \left(\frac{2}{|A|}\right)^{\alpha - 1}\right], \]

\[ T_\gamma(\alpha) = \frac{1}{\alpha - 1}\left[1 - \frac{|A|^{\alpha - 1} - \Gamma \left(\gamma - \frac{1}{2}\right)}{\Gamma(\alpha)\pi^{\alpha - 1/2}}\right], \]

with their Shannon limits, \( \alpha \to 1 \), being achieved as

\[ T_\rho(\alpha) = \ln|A| - \ln 2 + 1 - \left[-\frac{1}{2}\ln^2\left(\frac{2}{|A|}\right) + \ln\left(\frac{2}{|A|}\right) - 1\right](\alpha - 1) + \ldots, \]

\[ T_\gamma(\alpha) = -\ln|A| + \ln \pi + \ln 4 \]
\[ + \left[-\frac{\pi^2}{6} - 2\ln^2(2) - \frac{1}{2}\ln^2(\pi) - 2\ln(\pi)\ln(2) \right. \]
\[ - \frac{1}{2}\ln^2|A| + \ln(4\pi)|A|\right](\alpha - 1) + \ldots, \]

which again is different to the Rényi asymptotes, equations (70). The corresponding uncertainty relation becomes
which once more is dimensionally correct. Figure 5 shows the dimensionless parts of this inequality. Note that this relation turns into the equality not only at \( \alpha = 1 \), but at the Rényi parameter being one half when its dimensionless parts, \( L^\circ \), degenerate to \( \pi^{-1/2} = 0.5641 \ldots \). Near these points, inequality (74) simplifies to

\[
\frac{1}{(\pi \alpha)^{1/(4\beta)}} \left( \frac{2}{|\Lambda|} \right)^{\frac{\alpha-1}{n}} \geq \frac{|\Lambda|^{\frac{\alpha-1}{2}}}{\pi^{1/2}} \beta^{1/4} \left[ \frac{\Gamma\left(\beta - \frac{1}{2}\right)}{\Gamma(\beta)} \right]^{1/(2\beta)},
\]

which, obviously, are satisfied for the interval from equation (28) only whereas the Rényi relation is valid for the whole region \( \alpha \geq 1/2 \). Note the different lowest powers of the expansion coefficient in inequality (75b).
4. Hydrogen atom

1D potential of the form,

\[ V(x) = \begin{cases} \frac{-\lambda}{x}, & x > 0, \\ \infty, & x \leq 0, \end{cases} \]

(76)

where \( \lambda > 0 \), which enters equation (4), is used, for example, for the analysis of Rydberg atoms irradiated by short half-cycle pulses [77] and description of the electrons trapped above a micrometer-thick film of liquid helium [78]. The energy spectrum coincides with the 3D hydrogen atom [69],

\[ E_n = -\frac{m_p \lambda^2}{2 \hbar^2 n^2}, \quad n = 1, 2, \ldots, \]

(77)

whereas the corresponding waveforms are [68, 79]

\[ \Psi_n(x) = \frac{1}{x_0^{1/2} n^{3/2}} e^{-\tau/n} L_n^{(1)} \left( \frac{2\tau}{n} \right), \]

(78)

with \( L_n^{(1)}(x) \) being a generalized Laguerre polynomial [64]. Here, we introduced a characteristic (Coulomb) length of the atom,

\[ x_0 = \frac{\hbar^2}{m_p \lambda}, \]

(79)

and the line over the symbol means that it is a dimensionless variable:

\[ \bar{x} = \frac{x}{x_0}. \]

(80)

The normalizable wave vector function reads [68, 79]

\[ \Phi_n(k) = (-1)^{n+1} x_0^{1/2} \sqrt{\frac{2n}{\pi}} \frac{(1 - i n k)^n - 1}{(1 + i n k)^{n+1}}, \]

(81)

where the dimensionless wave vector is

\[ \bar{k} = x_0 k. \]

(82)

The corresponding densities are

\[ \gamma_n(k) = \frac{2n \bar{x}}{\pi (1 + (n \bar{k})^2)^{3/2}}. \]

(83)

Some of them are depicted in figure 6, which shows the sharpening of \( \gamma_n(k) \) around \( k = 0 \) with the quantum number \( n \) increasing. Didactically, it is important to note that at infinitely large index \( n \) the distribution from equation (83) turns into the \( \delta \)-function:

\[ \gamma_n(k) \rightarrow \delta(k), \quad n \rightarrow \infty, \]

(84)

which means that the quantum particle can have a zero momentum only. On the other hand, this statement is corroborated by equation (77), which confirms that in this quasi-classical regime the energy, and accordingly, the momentum \( p_n = \sqrt{2m_p E_n} \), are zeros. Simultaneously, as follows from equation (78), the position waveform for index \( n \) grows, gets flatter and approaches zero. Such opposite behavior to the original function and its image, equations (2), is a very general property of Fourier transformation [80].
Observe that the expression for the 1D position waveform, equation (78), is very similar to the angle-independent, $l = 0$, function of the 3D hydrogen-like atom [81]:

$$q_j(q_j r r l)_{nlm} = \frac{1}{x_0^{3/2}} \frac{2}{n^2} \sqrt{(n - l - 1)! (n + l)!} \left(\frac{2\pi}{n}\right)^l e^{-\pi n} \frac{L_{n-l-1}^{2l+1}}{L_{n-l-1}} \left(\frac{2\pi}{n}\right) Y_l^m(\theta, \varphi),$$  

(85)

written in the spherical coordinates $r \equiv (r, \theta, \varphi)$. Here, $n = 1, 2, \ldots$ is a principal quantum number, $l = 0, 1, \ldots, n-1$ is an azimuthal quantum number, $m = -l, -l + 1 \ldots, l - 1, l$ is a magnetic index and $Y_l^m(\theta, \varphi)$ is a standard spherical harmonics. Saha, Talukdar and Chatterjee (STC) [65], inspired by the formal identity of 1D equation (78) and its spherically symmetric, $l = 0, 3D$ counterpart from equation (85), claim that since the 1D and 3D Fourier originals are the same, their Fourier images should be identical too. Accordingly, instead of using a direct Fourier transform,

3 Due to the several different definitions of the Laguerre polynomials, different expressions are used for the function $\Psi_{nl}^{1D}(r, \theta, \varphi)$; for example, Landau and Lifshitz [69] write $L_{n-l-1}^{2l+1}$ instead of our $L_{n-l-1}^{2l+1}$ since their terminology is different from Abramowitz and Stegun [64] and Bateman [82] to which we adhere here (cf. equation (d.13) in [69] versus equations (10.12.5) and (10.12.7) in [82]). Also, Messiah’s definition, equation (B.13) in [81], is slightly different from ours too.

4 Actually, for $l = 0$ there is nothing in the resulting expression for the term corresponding to the factor $\pi$ in equation (78) but, probably, it went unnoticed by STC.
\[ \Phi_n(k) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \Psi_n(x)e^{-ikx}dx, \] (86)

as follows from equation (2a) and was done in [68, 79], they take a general expression of the 3D momentum function derived soon after the birth of quantum mechanics [83]:

\[ \Phi_{nlm}^{3D}(k, \theta_k, \varphi_k) = x_0^{3/2} \sqrt{\frac{2}{\pi} \frac{(n-l-1)!}{(n+l)!}} 2^{2l+1}n^2l! \frac{1}{1 + (n\kappa)^2} \nabla C_{n-l}^{l+1} \left( \frac{(n\kappa)^2 - 1}{(n\kappa)^2 + 1} \right) Y_l^m(\theta_k, \varphi_k), \] (87)

(where \( C_n^l(x) \) is a Gegenbauer polynomial [64, 82]), zero in it the azimuthal quantum number, \( l = 0 \) (and, accordingly, magnetic index, too, \( m = 0 \)) and announce that the ultimate expression containing the Chebyshev polynomial of the second kind \( U_l(x) \) [64, 82] is a correct representation of the momentum wavefunction of the Q1D hydrogen atom. In addition, they claim that since \( r \) in equation (85) with \( l = m = 0 \) changes from zero to infinity, the same holds true in their expression for the momentum component too, \( 0 \leq k < \infty \), thus arbitrarily forbidding motion of the particle to the left. To refute this faulty analysis, let us point out first that even though equations (78) and (85) with \( l = 0 \) look formally the same (let us forget now about the missing multiplier corresponding to \( \tau \)), they describe completely different geometries: for the Q1D case, the particle motion is strictly limited to the positive half-line and for the 3D configuration the length \( r \) is an absolute value of the radius vector \( \mathbf{r} \), which by definition cannot be negative, whereas each Cartesian coordinate:

\[
\begin{align*}
x &= r \sin \theta \cos \varphi, \\
y &= r \sin \theta \sin \varphi, \\
z &= r \cos \theta,
\end{align*}
\]

varies along the whole axis, \( -\infty < x, y, z < +\infty \). Then, for finding its Fourier transform, one uses

\[ \Phi_{nlm}^{3D} = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz \Psi_{nlm}^{3D}e^{-ikx}e^{i(kx+y+\varphi)} \]

\[ = \int_{0}^{+\infty} dr \int_{0}^{\pi} d\theta \int_{0}^{2\pi} d\varphi r^2 \sin \theta \Psi_{nlm}^{3D}(r, \theta, \varphi) \]

\[ \times e^{-ik \left[ \cos \theta \cos \varphi \right]} \cos \theta \sin \theta \sin \varphi, \] (88)

and the last integral is calculated as was done by Podolsky and Pauling [83] leading to equation (87). The first equality was inserted into equation (88) in order to explicitly underline that each Cartesian component is running through the whole axis. Accordingly, \( k \) in equation (87) is the length of the wave vector which is non-negative but each of its Cartesian components, due to isotropy of the problem, can take either a positive or negative axis, \( -\infty < k_x, k_y, k_z < +\infty \). In the same way, in equation (81), which was derived through the integration from equation (86), reference [68], momentum \( k \) runs over the whole of the real values, \( -\infty < k < +\infty \), since the motion to the right \( (k > 0) \) for this geometry does not have any advantages compared with the movement to the left \( (k < 0) \). Not surprisingly, the 1D, equation (86), and 3D, equation (88), integrations yield different results. The flawed nature of the STC momentum wave function can be further exemplified by calculating with its help the momentum Shannon entropy that, as mentioned below, violates the corresponding inequality, equation (25).

Position Rényi and Tsallis entropies can be calculated analytically for the ground state only when the corresponding Laguerre polynomial is just unity, \( L_0^{(j)}(x) \equiv 1 \), whereas their
momentum counterparts can be easily evaluated for any level:

\[
R_\rho(\alpha) = \ln x_0 + \frac{1}{1 - \alpha} \ln \Gamma(2\alpha + 1) - \frac{\Gamma(2\alpha + 1)}{2\alpha^{2\alpha + 1}}.
\]  

(89a)

\[
R_\gamma(\alpha) = -\ln x_0 - \ln n + \frac{1}{1 - \alpha} \ln \left( \frac{2^n}{\pi^{\alpha - 1/2}} \frac{\Gamma \left( \frac{2\alpha - \frac{1}{2}}{2} \right)}{\Gamma(2\alpha)} \right).
\]  

(89b)

\[
T_\rho(\alpha) = \frac{1}{\alpha - 1} \ln \left( \frac{1}{x_0^{\alpha - 1}} \frac{\Gamma(2\alpha + 1)}{2\alpha^{2\alpha + 1}} \right).
\]  

(89c)

\[
T_\gamma(\alpha) = \frac{1}{\alpha - 1} \ln \left( \frac{1}{n^{\alpha - 1}} \frac{\pi^{1/2}(2^n)^{1/2}}{\Gamma(2\alpha)} \right).
\]  

(89d)

Note that momentum components are finite for \( \alpha \geq \alpha_{\text{Q1D}}^{\text{Q1D}} = 1/4 \) only whereas position entropies have their values defined for any non-negative parameter. It is important to stress that the Q1D critical magnitude \( \alpha_{\text{Q1D}}^{\text{Q1D}} \) is different to the one of the Robin wall \( \alpha_{\text{RW}}^{\text{Q1D}} \), equation (67), and HO, which is zero. Thus, the range where the momentum Rényi and Tsallis entropies are defined strongly depends on the corresponding position potential \( V(x) \) and the boundary conditions. Asymptotic limits are

for the Rényi entropies:

\[
R_\rho(\alpha) = \ln x_0 - \ln(2\alpha) - (2\gamma + \ln 2 + 3\ln \alpha)\alpha + \ldots, \quad \alpha \to 0,
\]  

(90a)

\[
R_\rho(\alpha) = \ln x_0 + 2\gamma + \left( 3 - \frac{\pi^2}{3} \right)(\alpha - 1) + \frac{8}{3} \zeta(3) - 3\right)(\alpha - 1)^2 + \ldots, \quad \alpha \to 1,
\]  

(90b)

\[
R_\rho(\alpha) = \ln x_0 + 2 - \ln 4 + \frac{4 - \ln(16\pi) + \ln \alpha}{\alpha} + \ldots, \quad \alpha \to \infty,
\]  

(90c)

\[
R_\gamma(\alpha) = -\ln x_0 - \frac{4}{3} \ln \left( \frac{\pi^{1/4}(2^n)^{1/4}}{\Gamma(2\alpha)} \right) + \ldots, \quad \alpha \to \frac{1}{4} + 0,
\]  

(90d)

\[
R_\gamma(\alpha) = -\ln x_0 + 2 + \ln \frac{8\pi}{n} + \left( 6 - \frac{2}{3} \pi^2 \right)(\alpha - 1) + 16\zeta(3) - \frac{56}{3} \right)(\alpha - 1)^2 + \ldots, \quad \alpha \to 1,
\]  

(90e)

\[
R_\gamma(\alpha) = -\ln x_0 + \ln \frac{\pi}{2n} + \frac{\ln \alpha + \ln \frac{\pi}{2\alpha}}{2\alpha} + \ldots, \quad \alpha \to \infty;
\]  

(90f)

for the Tsallis entropies:

\[
T_\rho(\alpha) = \frac{x_0}{2\alpha} - 1 - \frac{1}{2}(2\gamma + \ln x_0)x_0 - x_0 \ln \alpha + \frac{x_0}{2} + \ldots, \quad \alpha \to 0,
\]  

(91a)
\[ T_{\rho_1}(\alpha) = \ln x_0 + 2\gamma + \left(3 - \frac{\pi^2}{3} - 2\gamma^2 - 2\gamma \ln x_0 - \frac{1}{2} \ln^2 x_0\right)(\alpha - 1) \]
\[ + \left[\frac{1}{6} \ln^3 x_0 + \gamma \ln^2 x_0 + \frac{1}{6}(2\pi^2 + 12\gamma^2 - 18) \ln x_0 \right. \]
\[ + \left. \frac{8}{3} \zeta(3) + \frac{2}{3} \gamma \pi^2 + \frac{4}{3} \gamma^3 - 6\gamma - 3\right](\alpha - 1)^2 + \ldots, \quad \alpha \to 1, \quad (91b) \]
\[ T_{\rho_1}(\alpha) = \frac{1}{\alpha} + \frac{1}{\alpha^2} + \ldots, \quad \alpha \to \infty, \quad (91c) \]
\[ T_{\rho_1}(\alpha) = \frac{2}{3} \left(\frac{2}{\pi n^2}\right)^{1/4} \frac{1}{x_0^{3/2}} \frac{1}{\alpha - \frac{1}{4}} - \frac{4}{3} + \ldots, \quad \alpha \to \frac{1}{4} + 0, \quad (91d) \]

where in equation (91c) it is also assumed that \( x_0 \geq 1 \). Since the expressions for the Tsallis momentum components at \( \alpha \to 1 \) and \( \alpha \to \infty \) are quite unwieldy, they are not shown here. It is observed once again that Rényi and Tsallis functionals approach their Shannon limit in different ways. Note that for the ground state the Shannon entropies are \( S_{\rho_1} = \ln x_0 + 2\gamma \) and \( S_{\rho_0} = -\ln x_0 - 2 + \ln(8\pi) \) with their sum \( S_{\rho_1} + S_{\rho_0} = 2\gamma - 2 + \ln(8\pi) = 2.3786 \ldots \) satisfying, of course, Shannon uncertainty relation, equation (25). In contrast, the STC momentum Shannon entropy with the wave function from [65] that can be calculated only numerically violates this inequality since \( S_{\rho_1}^{STC} = -\ln x_0 + 0.5575 \ldots \) and, accordingly, \( S_{\rho_1} + S_{\rho_0}^{STC} = 1.7119 \ldots \), which is further proof of its faulty nature.

Rényi entropies of the four lowest-energy levels are shown in figure 7 where the distances are measured in units of \( x_0 \) and the position parts for \( n = 2 \) were evaluated numerically. As derived in equations (90a) and (90d), position and momentum components diverge logarithmically when parameter \( \alpha \) approaches their corresponding threshold values that are zero and \( 1/4 \), respectively. Observe that at any Rényi parameter the position part is an increasing function of the quantum index whereas its momentum counterpart decreases as \( -\ln n \), equation (89b), in contrast to the HO where both \( R_{\rho_1} \) and \( R_{\rho_0} \), which are equal to each other, get larger for the larger \( n \), see figure 1.

For the ground orbital, the sum of the two entropies entering Rényi relations (23) or (47) has the following limits:

\[ R_{\rho_1}(\alpha) + R_{\rho_0}(\beta) = \ln 2\pi \]
\[ + \left[\ln 8 - \gamma - 1 - \ln(2\alpha - 1)\right](2\alpha - 1) + \ldots, \quad \alpha \to \frac{1}{2}, \quad (92a) \]
\[ R_{\rho_1}(\alpha) + R_{\rho_0}(\beta) = -2 + 2\gamma + \ln 8\pi + \left(\frac{\pi^2}{3} - 3\right)(\alpha - 1) \]
\[ + \left[\frac{4}{3} \pi^2 + \frac{56}{3} \zeta(3) - \frac{29}{3}\right](\alpha - 1)^2 + \ldots, \quad \alpha \to 1, \quad (92b) \]
\[ R_{\rho_1}(\alpha) + R_{\rho_0}(\beta) = 2 + \ln \frac{\pi}{2} + \frac{2 - \ln 8 - \frac{1}{2} \ln \pi + \frac{1}{2} \ln \alpha}{\alpha}, \quad \alpha \to \infty, \quad (92c) \]

with \(-2 + 2\gamma + \ln 8\pi = 2.3786 \ldots \) and \( 2 + \ln \frac{\pi}{2} = 2.6515 \ldots \). Comparing equation (92a) with its counterpart for the function \( f(\alpha) \), equation (50a), one sees that the limit \( \alpha \to 1/2 \) turns the Rényi uncertainty relation into the identity and, since \( \ln 8 - \gamma - 1 = 0.5022 \ldots \) is...
positive, equation (23) in the vicinity of this asymptote holds true. Figure 8 depicts the sum $R_{\lambda}(\alpha) + R_{\gamma}(\beta)$ for the four lowest-energy states together with the function $f(\alpha)$. As just discussed, for the ground orbital the point $\alpha = 1/2$ saturates the Rényi inequality. Numerical analysis shows the $R_{\lambda}(\alpha) + R_{\gamma}(\beta)$ has a very broad maximum of 2.5273 located at $\alpha \approx 4.55$. Strictly speaking, one can find this extremum by zeroing a derivative with respect to $\alpha$ of the corresponding sum, but the resulting equation has a quite complicated form and its analytic solution is impossible. The figure also exemplifies that the sum of the two entropies is an increasing function of the quantum index, $R_{\lambda}(\alpha) + R_{\gamma}(\beta) > R_{\lambda}(\alpha) + R_{\gamma}(\beta)$. Let us also point out that for the larger $n$ its maximum shifts to the smaller Rényi parameter; for example, for the first excited level the extremum of 2.8876 is achieved at $\alpha \approx 3.53$, and for next two orbitals these numbers are: 3.1370, $\alpha \approx 2.77$, and 3.3277, $\alpha \approx 2.42$, respectively.

Tsallis inequality, equation (26), for the lowest bound level degenerates to

$$x_0^{(1-\alpha)/(2\alpha)} \frac{\Gamma(2\alpha + 1)^{1/(2\alpha)}}{\pi^{1/(4\alpha)} 2^{1/(2\alpha)} \alpha^{1+1/(4\alpha)}} \approx x_0^{(\beta-1)/(2\beta)} \left( \frac{2}{\pi} \right)^{1/2} \left[ \frac{\Gamma \left( 2\beta - \frac{1}{2} \right)}{\Gamma(2\beta)} \right]^{1/(2\beta)}.$$  \hspace{1cm} (93)

As before, the dimension of the left side matches that of the right. Accordingly, in our analysis below we drop the term with $x_0$. In the limiting cases, this relation simplifies to
where $(2/\pi)^{1/2} = 0.7978\ldots$ Note that since $2\gamma + 1 - \ln(2\pi) = 0.3165\ldots$ is smaller than $\ln 2 = 0.6931\ldots$ and since $(1 - 4\gamma + \ln \pi)/4 = -0.04103\ldots$ is negative whereas $[\ln(64\pi) - 5]/4 = 0.07590\ldots$ is positive, equations (94) prove that the Tsallis uncertainty relation holds only inside the interval from equation (28). The behavior of the dimensionless parts from relation (93) is shown in panel (a) of figure 9. It can be seen that, as follows from equation (94a), saturation of the corresponding relation takes place not only at $\alpha = 1$, equation (94b), but at $\alpha = 1/2$ too, as was the case for the Robin wall, chapter 3. However, this property is not a universal characteristic of the Q1D hydrogen atom, since for the higher-lying states the only saturation point is just the right edge of this interval. This is exemplified in figure 9(b), which depicts both sides of the Tsallis inequality (26) for the first two excited levels. In fact, as our numerical results disclose, the difference between the position and momentum parts of equation (26) at $\alpha = 1/2$ increases with the quantum index $n$. 

Figure 8. Sum of the position and momentum Rényi entropies $R_{\rho}(\alpha) + R_{\gamma}(\beta)$ with parameter $\beta$ from equation (48) of the four lowest states of Q1D hydrogen atom as functions of parameter $\alpha$. Function $f(\alpha)$ from equation (49) is shown by the dash-dot-dotted line.
5. Conclusions

One-parameter quantum-information measures find more and more applications in different branches of science and other spheres of human activity. We provided in chapter 1 a (by no means complete) list of the fields where the Rényi $R(\alpha)$ and Tsallis $T(\alpha)$ entropies have been used. There is no doubt that this catalog will expand. To introduce to students the basic concepts of these two functionals, we considered here three 1D quantum systems that allow (fully or at least partially) an analytic calculation of the position and momentum components of these measures; namely, position Schrödinger equation (4) yields solutions $\Psi(x)$ in terms known to the undergraduate students’ functions, such as Hermite or Laguerre polynomials,
and subsequent calculation of their Fourier transforms according to equation (2a) leads to the momentum waveforms $\Phi(k)$, which again are expressed analytically. After this, the mathematical road to the entropies lies through the integration according to equations (5). In this way, during the consideration of the HO, section 2, a special cachet of the Gaussian distribution [84], which corresponds to the lowest-energy state of this geometry, was confirmed: at any dimensionless parameter $\alpha$ it saturates not only the Rényi uncertainty relation, equation (23), but its Tsallis counterpart too, equation (26), without taking into account a restriction from equation (28), which is crucial for any other probability arrangement. The right-hand side of the Rényi inequality (23), which determines the limit of the simultaneous knowledge of the position and momentum, has a maximum at $\alpha = 1$, which physically means, according to our interpretation provided in the Introduction, that just the Shannon entropy among all other $\alpha$ provides less total information about particle location and motion. Higher-energy HO orbitals obey the rule $R_{n+1}(\alpha) > R_n(\alpha)$ and the sum from the uncertainty relation is an increasing function of the quantum index too with its maximum for arbitrary $n$ being achieved in the Shannon case, $\alpha = 1$. Interestingly, for the Q1D hydrogen-like atom the position Rényi entropies, similar to the HO, increase with the quantum numbers move in the opposite direction as $-\ln n$. Importantly, for all three geometries the obtained results simplify in the limiting values of parameter $\alpha$; for example, comparing the entropy behavior near the Shannon case, $\alpha \rightarrow 1$, we have shown that even though at $\alpha = 1$ both Rényi $R$ and Tsallis $T$ functionals reduce to the Shannon counterparts, the trajectories along which this transformation takes place, for all three configurations are different for $R$ and $T$ in either space. Each system exhibits its own behavior of the sum $R_a(\alpha) + R_a(\beta)$ that enters the uncertainty relation from equation (23): for the HO, it has an extremum at $\alpha = 1$ for all levels whereas the $n$-dependent maximum of the hydrogenic orbitals is shifted on the $\alpha$ axis to the left with the energy growing, and for the Robin wall it monotonically and unrestrictedly increases from its saturation value at the Rényi parameter of one half. The next important conclusion lies in the fact that depending on the form of the position potential $V(r)$ in the Schrödinger equation or boundary conditions imposed on the position function $\Phi(r)$, the semi-infinite range of the Rényi or Tsallis parameter where the momentum component exists is different with its lowest edge being zero for the HO, one half for the attractive Robin wall and one quarter for the hydrogen structure. This topic of the influence of the position parameters on the properties of the momentum components needs further development and analysis. Also, it follows from the above consideration that at $\alpha = 1/2$ the lowest orbital of all three structures transforms the Rényi and Tsallis uncertainty relations into the equalities. To check whether or not this is a general feature of any quantum object, we have performed corresponding calculations for the zero-energy state of the 1D Neumann well mentioned in section 1: for it, the left-hand side of equation (23) takes the form

$$R_{\tilde{\alpha}_0}^N(\alpha) + R_{\tilde{\alpha}_0}^N(\beta) = \frac{1}{1 - \beta} \ln \left( \left( \frac{2}{\beta} \right)^{\frac{1}{\beta}} \int_{-\infty}^{\infty} \left[ \left( \frac{\sin \frac{\pi}{2}}{z} \right)^2 \right]^\beta \right),$$

(85a)

and, as our numerical results show, at $\alpha \rightarrow 1/2$ this expression really does converge to $\ln 2\pi$. At the same time, its Tsallis uncertainty relation reads

$$a^{1 - \alpha} \left( \frac{\alpha}{\pi} \right)^{1/\pi} \geq a^{1 - \beta} \left( \frac{2}{\pi} \right)^{1/2} \left( \frac{\beta}{\pi} \right)^{1/2} \int_{-\infty}^{\infty} \left[ \left( \frac{\sin \frac{\pi}{2}}{z} \right)^2 \right]^\beta \frac{dz}{2\pi},$$

(85b)

28
which is indeed satisfied as inequality at any Tsallis parameter smaller than unity, while the asymptote \( \alpha = 1/2 \) turns it into the identity with either side becoming \( (2\pi)^{-1/2} = 0.3989… \). The same property persists for the Dirichlet well too (not shown here). In this way, we arrive at the following.

**Conjecture:**

*Lowest orbital of the \( l \)-dimensional quantum structure in the limit \( \alpha = 1/2 \) transforms Rényi, equation (23), and Tsallis, equation (26), uncertainty relations into the equalities.*

General proof of this statement (if it is true) lies beyond the scope of the present pedagogical article.

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**ORCID iDs**

O Olendski @ https://orcid.org/0000-0001-7891-1793

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