Topological Phase Transition driven by Infinitesimal Instability: Majorana Fermions in Non-Hermitian Spintronics

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Quantum phase transitions are intriguing and fundamental cooperative phenomena in physics. Analyzing a superconducting nanowire with spin-dependent non-Hermitian hopping, we discover a topological quantum phase transition driven by infinitesimal cascade instability. The anomalous phase transition is complementary to the universal non-Bloch wave behavior of non-Hermitian systems. We show that an infinite small magnetic field drastically suppresses the non-Hermitian skin effect, deriving a topological phase with Majorana boundary states. Furthermore, by identifying the bulk topological invariant, we establish the non-Hermitian bulk-boundary correspondence that does not have a Hermitian counterpart. We also discuss an experimental realization of the system by using the spin-current injection to a quantum wire.

Recently, non-Hermitian Hamiltonians have attracted much interest in various fields such as open systems, correlated and disordered systems, quantum critical phenomena, and quantum and classical photonics. Among them, topological properties of such Hamiltonians have been extensively investigated both in gapped and gapless phases, and a lot of essential differences from the Hermitian cases have been pointed out. For instance owing to the complex nature of the energy spectrum, there are several distinct definitions of energy gaps, which amplify the possibility of topological phases.

Although the non-Hermitian physics under the periodic boundary condition (PBC) can be investigated by using mathematical tools developed in the Hermitian physics, it is not easy to treat them under the open boundary condition (OBC) because of the non-Bloch wave behavior. For instance, the phase diagrams under the OBC are different from those under the PBC in several non-Hermitian models, which obscures the conventional bulk-boundary correspondence. Thus far, the non-Hermitian bulk-boundary correspondence has not been established except for several attempts.

In this Letter, we construct and analyze a simple non-Hermitian lattice model that describes a one-dimensional s-wave superconductor with spin-dependent asymmetric hopping. Although this model is topologically nontrivial under the PBC, it is difficult to treat them under the open boundary condition because of the non-Bloch wave behavior. For instance, the phase diagrams under the OBC are different from those under the PBC in several non-Hermitian models, which obscures the conventional bulk-boundary correspondence. Thus far, the non-Hermitian bulk-boundary correspondence has not been established except for several attempts.

In this Letter, we construct and analyze a simple non-Hermitian lattice model that describes a one-dimensional s-wave superconductor with spin-dependent asymmetric hopping. Although this model is topologically nontrivial under the PBC, the SU(2) imaginary gauge transformation reveals that the system under the OBC does not show any topological boundary modes. Interestingly, however, we find that this mismatching is drastically remedied by an infinitesimal transverse magnetic field in the thermodynamic limit. Performing the numerical diagonalization with a small magnetic field, we find the missing Majorana boundary modes, which are protected by the bulk topological invariant. This finding establishes the presence of the non-Hermitian bulk-boundary correspondence that has no analog in the Hermitian physics.

Finally we also discuss an experimental realization by using the spin-current injection to a quantum wire. Periodic boundary condition.—In order to grasp a rough idea, we first analyze the infinite lattice system under the PBC, where the momentum-space picture is useful. For Hermitian systems, the Majorana fermions are known to appear on boundaries of a spinless p-wave superconductor, though it is not experimentally relevant thus far. Instead of the direct realization, several schemes have been proposed to effectively create the spinless Cooper pairing. For example, in a quantum wire with the Rashba spin-orbit interaction and the Zeeman magnetic field, the spin degree of freedom is frozen due to the spin-momentum locking. In this paper, we use the spin-momentum locked dissipation to realize a similar spinless situation.
Let us consider the following Hamiltonian:

\[ H = H_N + H_\Delta, \]
\[ H_N = \sum_{k, \sigma_z = \pm} \left[ -2t \cos k - \frac{i\Gamma}{2}(1 + \sin k\sigma_z) \right] a_{k, \sigma_z}^\dagger a_{k, \sigma_z}, \]
\[ H_\Delta = \sum_k \left[ \Delta a_{k, \uparrow}^\dagger a_{-k, \downarrow} + \text{H.c.} \right], \]

where \((a^\dagger, a)\) are spin-1/2 fermionic (electron) creation and annihilation operators, and real parameters \(t, \Gamma\) and \(\Delta\) describe the kinetic energy, dissipation (loss for particles and gain for holes), and an s-wave gap function, respectively. The spin- and momentum-dependent dissipation is a non-Hermitian variant of spin-orbit interaction. For \(\Gamma > \Delta\), only the left-(right)-going electrons with up-(down)-spin can participate in the Cooper pairing (Fig. 1), and thus we obtain an effective spinless superconductor. Diagonalizing the Hamiltonian, we have

\[ H = \sum_{k, \alpha = \pm} E_{k, \alpha} \bar{\alpha}_{k, \alpha} \alpha_{k, \alpha}, \]

where \((\bar{\alpha}, \alpha)\) are creation and annihilation operators of the Bogoliubov quasi-particles, and \(E_{k, \pm} = \sqrt{\left[ -2t \cos k - \frac{i\Gamma}{2}(1 \pm \sin k) \right]^2 + \Delta^2}\) are their energy dispersion [Fig. 2(b)]. Note that \(\bar{\alpha}\) is no longer the Hermitian conjugate of \(\alpha\) under the non-Hermiticity, while the conventional anti-commutation relation \(\{\alpha_{k, \alpha}, \bar{\alpha}_{k', \alpha'}\} = \delta_{kk'} \delta_{\alpha \alpha'}\) holds, and \(\alpha\) annihilates the BCS vacuum \(|0\rangle\) (see Supplemental Material (SM) or Ref. [82]).

In the real-space picture, Eq. (1) can be written as a simple lattice model

\[ H = \sum_{i, \sigma_z = \pm} \left[ -t_{\sigma_z} a_{i+1, \sigma_z}^\dagger a_{i, \sigma_z} - t_{(-\sigma_z)} a_{i+1, \sigma_z}^\dagger a_{i, \sigma_z} - i(t_+ - t_-) a_{i, \sigma_z}^\dagger a_{i+1, \sigma_z} + \sum_i \left[ \Delta a_{i, \uparrow}^\dagger a_{i, \downarrow} + \text{H.c.} \right] \right], \]

where \(i\) is the site index, and \(t_\pm = t \pm \Gamma/4\). Note that the normal part of the Hamiltonian includes the non-Hermitian asymmetric hopping terms whose asymmetry depends on the z-component spin. These hopping terms are regarded as those of a stacked Hatano-Nelson model \(1\) with up and down spins.

**Open boundary condition.**—Thus far, we have introduced the non-Hermitian spin-orbit interaction for the purpose of the realization of Majorana boundary states. This proposal is based on the momentum-space picture, which corresponds to the PBC in real space. In the presence of non-Hermiticity, however, extensive sensitivity of the energy spectrum to boundary conditions obscures the naive bulk-boundary correspondence.

Let us impose the OBC on the Hamiltonian \(3\). To consider the eigenvalues of Eq. (3), we perform the SU(2) imaginary gauge transformation, which generalizes the imaginary gauge transformation \(1\) used in the analysis of the Hatano-Nelson model:

\[ a_{i, \uparrow} = \left( \sqrt{\frac{t_+}{t_-}} \right)^i b_{i, \uparrow}, a_{i, \downarrow} = \left( \sqrt{\frac{t_+}{t_-}} \right)^{-i} b_{i, \downarrow}, \]
\[ a_{i, \downarrow} = \left( \sqrt{\frac{t_+}{t_-}} \right)^{-i} b_{i, \downarrow}, a_{i, \uparrow} = \left( \sqrt{\frac{t_+}{t_-}} \right)^i b_{i, \uparrow}, \]

where \((b^\dagger, b)\) are creation and annihilation operators of the new basis. Under this transformation, the spin-dependent asymmetric hopping terms are mapped to the spin-independent symmetric ones:

\[ -\sqrt{t_+ t_-} b_{i+1, \sigma_z}^\dagger b_{i, \sigma_z} + \text{H.c.}, \]

while the other terms are invariant. Thus, the Hamiltonian \(3\) is mapped to a conventional s-wave superconductor, apart from a constant dissipation term. Although this transformation changes the eigenfunctions drastically, it does not change the eigenvalues because it is a similarity transformation. Thus, the Hamiltonian \(3\) has the same energy spectrum as that of the mapped s-wave superconductor. This implies that the lattice model \(3\) is topologically trivial under the OBC and has no Majorana boundary modes. The energy spectrum with a constant dissipation does not depend on the boundary condition in the thermodynamic limit as in the case of the Hermitian physics, and it is calculated as

\[ E_{k, \alpha} = \sqrt{\left[ -2t \cos k - i(t_+ - t_-) \right]^2 + \Delta^2}, \]

which does not depend on the band index \(a = \pm \). The obtained spectrum is drastically different from Eq. \(2\) [Fig. 2(a)].

At first sight, the above consideration seems to ruin the scenario of Majorana modes by the non-Hermitian spin-orbit interaction. Interestingly, however, an infinitesimal perturbation resurrects this scenario as shown below.

**Phase transition driven by infinitesimal instability.**—In the case of the OBC, asymmetry of the hopping induces the accumulation of eigenstates near boundaries. The
Roughly speaking, this perturbation cannot be ignored if the order-of-limits changes the physics:
\[
\lim_{\delta h \to 0} \lim_{L \to \infty} \neq \lim_{L \to \infty} \lim_{\delta h \to 0}. \tag{10}
\]

Similar high sensitivity of eigenvalues to the perturbation is also discussed in mathematics [34].

The terms in Eq. (9) grow exponentially near boundaries, getting rid of the accumulated states of the skin effect. This implies that the OBC bulk spectrum would be close to the PBC one [Eq. (2)] in the presence of the perturbation. In the following, we perform the numerical diagonalization to confirm this expectation.

**Numerical diagonalization.**—We rewrite the Hamiltonian (3) with the small perturbation (7) in the Nambu representation:
\[
H + H_{ex} = \frac{1}{2} \sum_{i,j} \Psi_i \mathcal{H}_{i,j}^{\text{BdG}} \Psi_j, \tag{11}
\]
where \( \Psi_i = (a_i^{\dagger}, a_{i+1}^\dagger, a_{i+1}, a_i) \) is the Nambu spinor, and \( \mathcal{H}_{i,j}^{\text{BdG}} \) is the Bogoliubov-de Gennes (BdG) Hamiltonian matrix (explicit form in SM). Using the Nambu representation, we numerically calculate the energy spectra of the finite lattice system \( (L = 100) \) for various transverse magnetic fields and plot them in Fig. 2 (c). The model parameters are \( t_+ = 1, t_- = 0.7 \) and \( \Delta = 0.2 \).
Let us consider the BdG Bloch Hamiltonian constructed from Eq. (11)

$$H_{k}^{\text{BdG}} = \left[ -2t \cos k - \frac{i}{2} \Gamma \right] \hat{\sigma}_z - \frac{i}{2} \Gamma \sin k \hat{\sigma}_z - \Delta \hat{\sigma}_y \hat{\tau}_y, \quad (15)$$

where $\hat{\sigma}$s and $\hat{\tau}$s are the Pauli matrices in the spin and particle-hole space, respectively. This Hamiltonian belongs to class D in the Altland-Zirnbauer classification, and supports the particle-hole symmetry

$$\hat{\tau}_x (H_{k}^{\text{BdG}})^T \hat{\tau}_x = -H_{-k}^{\text{BdG}}. \quad (16)$$

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Note that the transpose in the charge conjugation is not equivalent to the complex conjugation for the non-Hermitian case [87].

The bulk band is not gapped in a usual sense since the bulk spectrum in Fig. 1(b) is totally connected in the complex energy spectrum. Therefore, the conventional class D topological invariant or its non-Hermitian variant is no longer well-defined; our obtained topological phase originates essentially from non-Hermiticity. Hence we use another topological invariant intrinsic to non-Hermitian systems. We propose the following $\mathbb{Z}_2$ invariant to characterize the present topological phase:

$$(-1)^\nu = \frac{\text{Pf}(\tau_x H_{k=\pi})}{\text{Pf}(\tau_x H_{k=0})} \exp \left( -\frac{1}{2} \int_0^\pi \text{d} k \text{Tr}[H_{k}^{-1} \partial_k H_{k}] \right), \quad (17)$$

where $k = 0, \pi$ are the time-reversal-invariant points, and the superscript “BdG” is omitted. The topological invariant is well-defined unless $\det H_k = 0$ (i.e. $|\Gamma| = |\Delta|$). The competition between $\Delta$ and $\Gamma$ determines the topological phase; the $\mathbb{Z}_2$ index is trivial for $|\Gamma| < |\Delta|$ and nontrivial for $|\Gamma| > |\Delta|$ (see SM). Thus, the present strong $\Gamma$ case, where the boundary modes exist, corresponds to the nontrivial phase, while the weak $\Gamma$ case, where the boundary modes are absent (see SM), corresponds to the trivial phase.

**Spintronic application.**—We finally discuss an experimental realization of the Hamiltonian (3). The nontrivial task is to implement the spin-dependent asymmetric hopping terms, or equivalently, the non-Hermitian spin-orbit interaction. Although the full implementation of the $\sin k \sigma_z$ term seems to be difficult, we may introduce the essentially the same effect near the Fermi level, where the superconducting pairing occurs. In order to introduce the spin-momentum locked effect near the Fermi level, we propose to a pure spin current injection to a quantum wire. Under this non-equilibrium circumstance, modes with spin current $\sigma_z \partial E/\partial k$ opposed to the injected spin current would have a shorter life time by scatterings. For sufficiently large imbalance of dissipation $\Gamma$, we obtain the situation in Fig. 3. Another promising platform is ultra-cold atom systems. The possible realization of

In the absence of the magnetic field, the result is well approximated by Eq. (6). For $\delta h \gtrsim 10^{-8}$, the energy spectrum differs from Eq. (6), which is consistent with the value $(t_-/t_+)^{L/2} \simeq 2 \times 10^{-8}$ in Eq. (9). For $\delta h \gtrsim 10^{-4}$, we find two superposition states of Majorana fermions localized on two boundaries of the lattice system (Fig. 3), while the bulk spectrum surrounds the origin of the complex plane and has the similar shape as the spectrum under the PBC. As we expected, the small perturbation changes the spectrum into the topological one and induces Majorana fermions.

The Majorana fermions satisfy the non-Hermitian Majorana condition (see SM):

$$\overline{\gamma}_i = \gamma_i, \quad (12)$$

where $i = 1, 2$ denote the boundaries on which the Majorana fermions localize. The effective theory of the two edges (1 and 2) are given by

$$H_{\text{Boundary}} = \frac{i \epsilon}{2} \gamma_1 \gamma_2 = \epsilon (\gamma_1 \gamma_2 - 1), \quad (13)$$

where $\epsilon$ is the complex finite-size coupling, and $(\alpha, \alpha)$ are the fermion operators constructed from the Majorana fermions:

$$\gamma_1 = \alpha + \alpha, \quad \gamma_2 = \alpha - \alpha. \quad (14)$$

In the present numerical calculation, the fermion energy $\epsilon$ takes the imaginary number ($\epsilon = -4 \times 10^{-5} i$).

**Non-Hermitian topological phase.**—In the presence of the non-Hermitian skin effect, the non-Bloch wave functions in the OBC are necessary to define the topological number. In our case with small magnetic field, however, the numerical calculation indicates that the non-Bloch wave function reduces to the conventional Bloch one in the thermodynamic limit [52]. In the following, we identify the topological invariant that protects the Majorana zero mode, by using the conventional Bloch wave functions in the PBC.

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the asymmetric hopping term has been theoretically proposed in Ref. \[44\], which would be generalized to our model with the spin degrees of freedom.

In summary, we have constructed and analyzed a simple non-Hermitian lattice model of an s-wave superconductor that realizes a novel topological phase. The topological phase transition is driven by an infinitesimal external magnetic field. We have also discussed an experimental realization in spintronics. The present model provides the first concrete example of the non-Hermitian topological phase that does not have Hermitian counterparts.

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Supplemental Material for
“Topological Phase Transition driven by Infinitesimal Instability: Majorana Fermions
in Non-Hermitian Spintronics”

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EXPLICIT FORM OF THE REAL-SPACE BDG HAMILTONIAN

For convenience, we here write down the explicit form of the Bogoliubov-de Gennes Hamiltonian in real space. The
matrix elements are given by

\[
H_{i+1,i}^{\text{BdG}} = \begin{pmatrix}
-t_+ & 0 & 0 & 0 \\
0 & t_- & 0 & 0 \\
0 & 0 & -t_+ & 0 \\
0 & 0 & 0 & t_-
\end{pmatrix}, \quad
H_{i,i+1}^{\text{BdG}} = \begin{pmatrix}
-t_- & 0 & 0 & 0 \\
0 & -t_+ & 0 & 0 \\
0 & 0 & t_- & 0 \\
0 & 0 & 0 & t_+
\end{pmatrix},
\]

(1)

where \( t_{\pm} = t \pm \frac{\Gamma}{4} \).

CREATION AND ANNIHILATION OPERATORS OF EIGENSTATES IN NON-HERMITIAN SYSTEMS

We here discuss how the creation and annihilation operators of eigenstates are defined in non-Hermitian systems. We consider the general quadratic non-Hermitian Hamiltonian

\[
H = \sum_{i,j} a_i^\dagger H_{i,j} a_j,
\]

(2)

where \( H \) is a non-Hermitian Hamiltonian matrix, and \( (a, a^\dagger) \) are creation and annihilation operators that satisfy the bosonic or fermionic commutation relations. Suppose that \( H \) is diagonalizable. In such a case, physical eigenstates are characterized by the right eigenstates of \( H \). Let us define the following two matrices by using the right and left eigenstates:

\[
R := (|u_1\rangle, |u_2\rangle, \cdots), \quad L := (|u_1\rangle\rangle, |u_2\rangle\rangle, \cdots),
\]

(3)

where the right and left eigenstates of \( H \) are defined as

\[
H|u_n\rangle = E_n|u_n\rangle, \quad H^\dagger|u_n\rangle = E_n^*|u_n\rangle.
\]

(4)

By using these matrices, the biorthonormal condition \( \langle m|n\rangle = \delta_{mn} \) and the completeness condition \( \sum_n \langle n|n\rangle = \sum_n |n\rangle\langle n| = 1 \) can be summarized in the following simple form:

\[
R^\dagger L = L^\dagger R = RL^\dagger = LR^\dagger = 1.
\]

(5)

Using these relations, \( H \) can be expressed as

\[
H = REL^\dagger = RER^{-1},
\]

(6)

where \( E = \text{diag}(\cdots, E_n, \cdots) \). Thus, the Hamiltonian (2) can be rewritten as

\[
H = \sum_n (\sum_i a_i^\dagger R_{i,n}) E_n (\sum_j R_{n,j}^{-1} a_j) =: \sum_n E_n \bar{\alpha}_n \alpha_n.
\]

(7)
This is the definition of the creation and annihilation in the new basis. Although $\bar{\alpha}$ is no longer the Hermitian conjugate of $\alpha$, $(\alpha, \bar{\alpha})$ behave as the creation and annihilation operators that satisfy the bosonic or fermionic commutation relation:

$$[\alpha_n, \alpha^{\dagger}_{n'}]_{\pm} = R^{-1}_{n,j}R_{i,n'}[a_j, a_i^\dagger]_{\pm} = [R^{-1}R]_{n,n'} = \delta_{nn'},$$

$$[\alpha_n, \alpha_{n'}]_{\pm} = [\bar{\alpha}_n, \bar{\alpha}_{n'}]_{\pm} = 0,$$

where $[,]_{\pm}$ denotes the bosonic and fermionic commutation relation. The new vacuum and eigenstates are defined as

$$\alpha_n|0\rangle = 0,$$

$$|n\rangle = \alpha_n|0\rangle.$$

(8)

Z$_2$ TOPOLOGICAL INARIANT OF CLASS D POINT-GAPPED PHASE IN ONE DIMENSION

We here construct a Z$_2$ topological invariant in one dimension protected by the particle-hole symmetry

$$\tau_x H_k^T \tau_x = -H_{-k}.$$

(10)

As we noted in the main text, the charge conjugation is defined by using not the complex conjugation but the transpose, and they are inequivalent in non-Hermitian systems. The point-gap topological classification of the non-Hermitian Hamiltonian is mapped to the topological classification of the corresponding Hermitian Hamiltonian [1]:

$$\tilde{H}_k = \left( \begin{array}{cc} 0 & H_k \\ H_k^T & 0 \end{array} \right).$$

(11)

The particle-hole symmetry in the mapped Hamiltonian can be written as the conventional antiunitary operation:

$$\tau_x \tilde{H}_k^T \tau_x = -\tilde{H}_{-k}.$$

(12)

In addition, the Hamiltonian has a chiral symmetry:

$$\Sigma_z \left( \begin{array}{cc} 0 & H_k \\ H_k^T & 0 \end{array} \right) \Sigma_z = - \left( \begin{array}{cc} 0 & H_k \\ H_k^T & 0 \end{array} \right)$$

with $\Sigma_z := \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right).$

(13)

Owing to the additional chiral symmetry, the symmetry class is shifted, and the mapped Hamiltonian turns out to be a class DIII Hermitian matrix. In the Hermitian topological classification, the class DIII topological phases in one dimension are characterized by a Z$_2$ topological invariant. Actually, we can construct the Z$_2$ topological invariant by making use of the off-diagonal basis in Eq. (11):

$$(-1)^{\nu} = \frac{\text{Pf}(\tau_x H_{k=x})}{\text{Pf}(\tau_x H_0)} \exp \left( -\frac{1}{2} \int_0^{\pi} dk \text{Tr}[H_k^{-1}\partial_k H_k] \right),$$

(14)
where \( k = 0, \pi \) denote the time-reversal invariant points in momentum space. We have used the fact that \( \tau_x \mathcal{H}(0) \) and \( \tau_x \mathcal{H}(\pi) \) are antisymmetric matrices due to Eq. (10), and thus the Pfaffian can be naturally defined for them. Note that the topological invariant (14) is written in terms of the original non-Hermitian Hamiltonian \( \mathcal{H} \). This formula enables us to compute the topological invariant of the model used in the main text:

\[
\mathcal{H}_k = \left[ -2t \cos k - \frac{i \Gamma}{2} \right] \hat{\sigma}_z - \frac{i \Gamma}{2} \sin k \hat{\sigma}_y - \Delta \hat{\sigma}_x.
\]  

(15)

Although the analytical expression of the integrand is complicated due to the inverse of the Hamiltonian, we find that the topological invariant for the gapped region \((E \neq 0)\) is given by

\[
(-1)^\nu = \text{sgn} \left| \Delta \right| - \left| \Gamma \right|.
\]

(16)

Note that the topological phase transition occurs at \( |\Delta| = |\Gamma| \), where the point gap of the complex energy band structure of Eq. (15) is closed.

To check the bulk-boundary correspondence, we perform the numerical diagonalization in real space \((L = 400, \text{OBC})\) for several \( \Gamma \)s (Fig. 1). Owing to the slight change of the bulk spectrum that comes from the finite size effect and small but nonzero magnetic field, the exact correspondence between the finite real-space calculation and the momentum-space one does not hold. In fact, the phase transition occurs around \( \Gamma = 0.22 \), which differs from \( \Delta (= 0.2) \). Besides this slight difference, we find that the topological phase transition is clearly accompanied by the near-zero boundary modes.

**MAJORANA CONDITION**

We here discuss the Majorana zero mode in non-Hermitian systems. Suppose that the Hamiltonian matrix \( \mathcal{H} \) has the particle-hole symmetry \( C = \tau_x \):

\[
\tau_x \mathcal{H}^T \tau_x = -\mathcal{H}.
\]

(17)

For convenience, we rewrite the theory in the Majorana basis:

\[
a' := Ua \quad \text{with} \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \quad a'^\dagger = a'.
\]

(18)

In this basis, the particle-hole symmetry \( C \) is equal to unity:

\[
a'^\dagger \mathcal{H} a = a'^\dagger [U \mathcal{H} U^{-1}] a' =: a'^\dagger \mathcal{H}' a',
\]

\[
-\mathcal{H} = \tau_x \mathcal{H}^T \tau_x = \tau_x U^T [(U^T)^{-1} \mathcal{H}^T U^T] (U^T)^{-1} \tau_x
\]

\[
\Leftrightarrow -\mathcal{H}' = [U \tau_x U^T] \mathcal{H}' [(U^T)^{-1} \tau_x U^{-1}]
\]

\[
= \mathcal{H}'^T,
\]

(19)

where we have used the explicit form of \( U \) and \( \tau_x \) in the last line. In the following, we use the Majorana basis and omit \(^'.\)

In the Majorana basis, the following equation holds:

\[
\mathcal{H}' |u_n\rangle = E_n^* |u_n\rangle
\]

\[
\Leftrightarrow \mathcal{H}'^T |u_n\rangle^* = E_n^* |u_n\rangle^*.
\]

\[
\Leftrightarrow \mathcal{H} |u_n\rangle^* = -E_n^* |u_n\rangle^*.
\]

(20)

We have used the definition of the left eigenfunction (8) in the first line and \( \mathcal{H}^T = -\mathcal{H} \) in the last line. Thus, the particle-hole symmetry ensures the existence of the presence of eigenfunction with \(-E_n\) for each eigenfunction with \(E_n\) except for the zero mode. If the number of zero mode is one, then

\[
|u_0\rangle = |u_0\rangle^*.
\]

(21)

By using Eq. (11), we can write the creation and annihilation operators for the zero mode as

\[
\gamma = |u_0\rangle_i a_i^\dagger, \quad \bar{\tau} = |u_0\rangle_i a_i.
\]

(22)
Using the property of the Majorana basis $a_i^\dagger = a_i$ and Eq. (21), we obtain the Majorana condition:

$$\gamma = \gamma.$$  \hspace{1cm} (23)

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[1] K. Kawabata, K. Shiozaki, M. Ueda, and M. Sato, arXiv: 1812.09133.