Relativistic Calculation of Structure Functions $b_{1,2}(x)$ of the Deuteron.

A.Yu. Umnikov

Department of Physics, University of Perugia, and INFN, Sezione di Perugia, via A. Pascoli, Perugia, I-06100, Italy.

Abstract

The structure functions $b_{1,2}^{D}(x)$ of the deuteron are studied within covariant approach. It is shown that usual nonrelativistic convolution model result in incorrect behavior of this structure functions at small $x$ and violates the exact sum rules. Realistic calculations are carried out with the Bethe-Salpeter amplitude of the deuteron and compared with the nonrelativistic results.

1 Introduction

The study of the deep inelastic lepton scattering with polarized targets and beams provides refined information about the quark content of hadrons. These days much attention is attracted to the nucleon’s spin-dependent Structure Functions (SF), $g_{1,2}^{N}$ (see e.g. review [1] and references therein). The SFs, $g_{1,2}$, is the simplest example of the spin-dependent SFs, which exist for all targets with a nonzero spin, starting $s = 1/2$. At the same time, hadrons with a spin higher than $1/2$ have additional spin-dependent SFs [2].

The spin-dependent SFs $b_{1,2}(x)$ of spin-1 hadrons have been studied on a few occasions, including the vector mesons and deuteron [3, 4, 5, 6]. For mesons some qualitative estimates have been done, but “a real understanding of $b_{1}(x)$ at the quark level is not yet available” [5]. The only exact sum rules for mesons have been proposed by Efremov and Teryaev [3]:

$$\int_{0}^{1} b_{1}(x) dx = 0. \quad (1)$$

$$\int_{0}^{1} b_{2}(x) dx = 0. \quad (2)$$

It would be unrealistic to suggest that these sum rules will be experimentally verified for the mesons any time soon. However, they are independent of the target, i.e. it is supposed to be valid for the spin-1 nuclei as well. The deuteron is the most probable candidate for its SFs $b_{1,2}$ to be measured. Indeed, a number of deep inelastic experiments on the polarized deuterons are run or being prepared at the world’s best facilities, such as SLAC,
CERN, CEBAF and DESY. These experiments are usually aiming to extract a neutron SF \( g_1 \), however, in principle, the same data can be used to obtain SFs \( b_{1,2} \). The problem is caused only by the yet insufficient accuracy of measurements. Realistic calculations with the deuteron wave functions in the Bonn and Paris potentials have been done in ref. [6, 7]. The preliminary results within the Bethe-Salpeter formalism have also been presented [7]. It was shown that relativistic calculations differ from nonrelativistic ones, but the reasons have not been discussed. The sum rules (1)-(2) have not been analyzed in these papers but the displayed shape of the SFs makes questionable the validity of the sum rules.

This letter presents a study of the deuteron SFs \( b_{1,2}^D(x) \) within a covariant approach, based on the relativistic convolution formalism for the deep inelastic scattering [8, 9, 10, 11] and the Bethe-Salpeter formalism for the deuteron bound state [12, 13, 14]. The issue of the sum rules is specially addressed. It is shown that the nonrelativistic convolution model is not relevant to calculate SFs \( b_{1,2}^D(x) \) at small \( x \), because this model inevitably breaks at least one of the sum rules (1) or (2). At the same time, the relativistic calculations based on the Bethe-Salpeter amplitude for the deuteron are consistent with both of the sum rules. The SFs, \( b_{1,2}^D(x) \), are calculated within both relativistic and nonrelativistic approaches.

2 Structure functions in the relativistic impulse approximation

The nucleon contribution to the deep inelastic scattering of electrons off the deuteron is defined by the triangle diagram (see Fig. 1) [8, 9, 10, 11]:

\[
\langle \hat{W} \rangle_M = i \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left\{ \bar{\Psi}_M(p_0, p) \hat{W}(q, p_1) \Psi_M(p_0, p)(\hat{p}_2 - m) \right\},
\]

(3)

where \( p_1,2 = P_D/2\pm p = (M_D/2\pm p_0, \pm p) \), \( P_D \) is the deuteron momentum, \( M_D \) is the deuteron mass, \( p = (p_0, p) \) is the relative momentum of nucleons, \( \hat{W} \) is an operator appropriate for the process (see e.g. and references therein [11, 15, 16]) and \( \Psi_M(p_0, p) \) is the Bethe-Salpeter amplitude for the deuteron with \( M \) being the deuteron’s total momentum projection. The original convolution model for deep inelastic scattering is reproduced by choosing operator \( \hat{W} \) in the form:

\[
\hat{W} = \frac{\hat{q}}{2pq} W_{\mu\nu}^N(q, p_1),
\]

(4)

where \( m \) is the nucleon mass, \( q = (\nu, 0, 0, -\sqrt{\nu^2 + Q^2}) \) is the momentum transfer, \( Q^2 = -q^2 \) and \( W_{\mu\nu}(p, q)^N \) is the hadron tensor of nucleon:

\[
W_{\mu\nu}^N(q, p) = \left(-g_{\mu\nu} + \frac{q_{\mu}q_{\nu}}{q^2}\right) F_1^N(x, Q^2) + \left(p_{\mu} - q_{\mu}\right) \left(p_{\nu} - q_{\nu}\right) \frac{F_2^N(x, Q^2)}{pq},
\]

(5)
where \( x = Q^2/(2pq) \) and \( F_{1,2}^N \) are the nucleon SFs. The small effects of the off-mass-shell deformation of the nucleon tensor [15, 16, 17] are not considered in the present letter, because these effects do not affect sum rules (1)-(2) and do not noticeably change the absolute values of the SFs. That is why SFs \( F_{1,2}^N \) in (\ref{eq:5}) do not depend on \( p^2 \), but \( q^2 \) and \( pq \).

One can extract scalar SFs from the hadron of the deuteron. For instance, for \( F_2 \) one gets:

\[
F_2^D(x_N, Q^2, M) = \frac{i}{2M_D} \int \frac{d^4p}{(2\pi)^4} F_2^N \left( \frac{x_N m}{p_{10} + p_{13}}, Q^2 \right) \textrm{Tr} \left\{ \bar{\Psi}_M(p_0, \mathbf{p})(\gamma_0 + \gamma_3)\Psi_M(p_0, \mathbf{p})(\hat{p}_2 - m) \right\}, \tag{6}
\]

where \( x_N = Q^2/(2m\nu) \) is the Bjorken scaling variable, i.e. this is \( x \) for the on-mass-shell nucleon at rest, \( p_{10} \) and \( p_{13} \) are the time and 3-rd components of the 1-st nucleon momentum. Formula (\ref{eq:6}) has not been averaged over the projection of the deuteron total momentum, \( M \), since in the present form it gives an understanding what the SF \( b_2^D \) is. Indeed, eq. (\ref{eq:6}) gives two independent \( \text{“SFs”} \), with \( M = \pm 1 \) and \( M = 0 \), which are related to the usual spin-independent SF, \( F_2^D \), and a new SF, \( b_2^D \):

\[
F_2^D(x_N, Q^2) = \frac{1}{3} \sum_{M=0,\pm 1} F_2^D(x_N, Q^2, M), \tag{7}
\]

\[
b_2(x_N, Q^2) = F_2^D(x, Q^2, M = +1) - F_2^D(x, Q^2, M = 0), \tag{8}
\]

\[
F_2^D(x_N, Q^2, M = +1) = F_2^D(x_N, Q^2, M = -1). \tag{9}
\]

Of course, instead of (\ref{eq:7}) and (\ref{eq:8}) any other linearly independent combination of the functions \( F_2^D(x, Q^2, M) \) can be chosen.

Note, SFs \( F_2^D(x, Q^2, M) \) are independent of the lepton polarization, therefore, both SFs, \( F_2^D \) and \( b_2^D \), can be measured in experiments with an unpolarized lepton beam and polarized deuteron target. In view of eq. (\ref{eq:9}), only one of the SFs \( F_2^D(x, Q^2, M) \) is needed, in addition to the spin-independent \( F_2^D(x, Q^2) \), in order to obtain \( b_2(x, Q^2) \). The other SF, \( b_1^D \), is related to the deuteron SF \( F_1^D \), the same way as \( b_2^D \) is related to \( F_2^D \), viz. via eqs. (\ref{eq:7}), and \( b_2^D = 2xb_1^D \).

### 3 Singularity of the triangle diagram and sum rules

It has been previously shown [8, 14, 15] how a singular structure of the triangle graph (Fig. 1) rules the behavior of the spin-independent SF \( F_2^D \). In particular, it has been found that the relativistic impulse approximation satisfies the unitarity and provides the correct kinematical region of the variable \( x_N \). However, both these properties of the exact covariant amplitude are broken in practical calculations, when nonrelativistic wave functions of the deuteron are used. In this case one can refer to the argument that such deviations are small, and are not important for phenomenology. At the same time, a realistic Bethe-Salpeter

\[^3\text{Note that the “native” deuteron variable is } x_D = (m/M_D)x_N, \text{ however } x_N \text{ is used more often.}\]
amplitude of the deuteron serves ideally for a consistent phenomenological application of the covariant theory of the processes on the bound nucleons.

In order to calculate SFs, (6)-(8) and analyze the sum rules, the singularities of the triangle diagram should be explicitly taken into account. To do that, eq. (6) is rewritten as:

\[ F_D^2(x_N, Q^2, M) = \frac{i}{2M_D^2} \left( \frac{x_N m}{p_{10} + p_{13}} \right) F_N^2 \left( \frac{x_N m}{M_D - \omega + p_3}, Q^2 \right) \frac{1}{(p_1^2 - m^2 + i\epsilon)(p_2^2 - m^2 + i\epsilon)} \Theta \left\{ \phi_M(p_0, p)(\hat{p}_1 + m)(\gamma_0 + \gamma_3)(\hat{p}_1 + m)\phi_M(p_0, p)(\hat{p}_2 + m) \right\}, \]

where \( \phi_M(p_0, p) = (\hat{p}_1 - m)\Psi_M(p_0, p)(\hat{p}_2 - m) \) is the Bethe-Salpeter vertex functions of the deuteron.

Analysis of singularities in the complex \( p_{2+} \)-plane allows for one analytical integration in (10) \[10\]. Being translated into variables which are used in the present paper, this integration is equivalent to picking the residue in the second nucleon pole, \( p_{20} = \omega = \sqrt{m^2 + p^2} \) or \( p_0 = M_D/2 - \omega \), in the complex \( p_0 \)-plane when both of the following conditions are satisfied:

\[ 0 < \omega - p_3 < M_D. \]

The contribution of the region of \( p \) beyond (11), into integral (10), is zero, i.e. different poles cancel each other. Note, that in the required pole \( p_{10} = M_D - \omega \). Calculating residue in (10), one gets:

\[ F_D^2(x_N, Q^2, M) = \frac{1}{2M_D^2} \int \frac{d^3p}{(2\pi)^3} F_N^2 \left( \frac{x_N m}{M_D - \omega + p_3}, Q^2 \right) \frac{1}{2\omega M_D^2(M_D - 2\omega)^2} \Theta(M_D - \omega + p_3) \Theta \left\{ \phi_M(p_0, p)(\hat{p}_1 + m)(\gamma_0 + \gamma_3)(\hat{p}_1 + m)\phi_M(p_0, p)(\hat{p}_2 + m) \right\}_{p_0 = M_D/2 - \omega}, \]

where the \( \Theta \)-function guaranties the right of conditions (11), the left condition is always satisfied.

It is useful to rewrite (12) in the convolution form:

\[ F_D^2(x_N, Q^2, M) = \int_0^{M_D/m} dy F_N^2 \left( \frac{x_N}{y}, Q^2 \right) f_M^{N/D}(y), \]

where “the effective distribution” of nucleons in the deuteron is defined by

\[ f_M^{N/D}(y) = \frac{1}{2M_D} \int \frac{d^3p}{(2\pi)^3} \delta \left( y - \frac{M_D - \omega + p_3}{m} \right) \Theta(y) \frac{1}{2\omega M_D^2(M_D - 2\omega)^2} \Theta \left\{ \phi_M(p_0, p)(\hat{p}_1 + m)(\gamma_0 + \gamma_3)(\hat{p}_1 + m)\phi_M(p_0, p)(\hat{p}_2 + m) \right\}_{p_0 = M_D/2 - \omega}. \]
Two sum rules can be written down for the effective distribution $f_{M}^{N/D}(y)$:

\[
\int_{0}^{M_{D}/m} f_{M}^{N/D}(y)dy = \langle D | \hat{Q} | D \rangle_{M} = 1, \tag{15}
\]
\[
\int_{0}^{M_{D}/m} yf_{M}^{N/D}(y)dy = \langle D | (\Theta_{N})_{\mu}^{\mu} | D \rangle_{M} = 1 - \delta_{N}, \tag{16}
\]

where $\hat{Q} \propto \bar{\psi}(x)\gamma_{0}\psi(x)$ is the vector charge and $(\Theta_{N})_{\mu}^{\mu} \propto i\bar{\psi}(x)\gamma_{\mu}\partial^{\mu}\psi(x)$ is the trace of energy-momentum tensor. Eq. (15) presents the vector charge conservation generalized for the deuteron states with different $M$. In spite of such clear physical interpretation, some time ago it was a subject of some controversy [8, 9, 10]. Indeed, the derivation of sum rule (15) contains some subtle points and equivalence between it and the expression for the charge

\[
\langle D | \hat{Q} | D \rangle_{M} = \frac{1}{2M_{D}} \int \frac{d^{3}p}{(2\pi)^{3}} \frac{1}{3} \sum_{M} \text{Tr} \left\{ \bar{\Psi}_{M}(p_{0},p)\gamma_{0}\Psi_{M}(p_{0},p)(\hat{p}_{2} - m) \right\} \tag{17}
\]

is a non-trivial fact, particularly, because of the presence of the $\Theta$-function in eq. (14). This $\Theta$-function provide a correct kinematics in variable $x_{N}$ but cuts out a part of the integration domain in $d^{3}p$. This cutting of the integration interval in the polar angle $\theta$ leads to non-zero contribution of the matrix element containing $\gamma_{3}$, which is proportional to $\cos\theta$. The sum rule (16) for the first moment of $f_{M}^{N/D}$ is of a different nature, it presents the nucleon contribution into the total momentum of the deuteron [18, 19] and $\delta_{N}$ is a part of the total momentum carried by the non-nucleon component (mesons). An important property of sum rules (15) and (16) is that their r.h.s. do not depend upon the deuteron spin orientation.

The SFs $F_{1,2}^{D}$ and $b_{1,2}^{D}$ are now calculated as follows:

\[
\left\{ \begin{array}{l}
F_{1}^{D}(x_{N},Q^{2}) \\
b_{1}^{D}(x_{N},Q^{2})
\end{array} \right\} = \int_{0}^{M_{D}/m} \frac{dy}{y} \left\{ \begin{array}{l}
f_{M}^{N/D}(y) \\
\Delta f_{M}^{N/D}(y)
\end{array} \right\} F_{1}^{N} \left( \frac{x_{N}}{y},Q^{2} \right), \tag{18}
\]
\[
\left\{ \begin{array}{l}
F_{2}^{D}(x_{N},Q^{2}) \\
b_{2}^{D}(x_{N},Q^{2})
\end{array} \right\} = \int_{0}^{M_{D}/m} dy \left\{ \begin{array}{l}
f_{M}^{N/D}(y) \\
\Delta f_{M}^{N/D}(y)
\end{array} \right\} F_{2}^{N} \left( \frac{x_{N}}{y},Q^{2} \right), \tag{19}
\]

where distributions $f_{M}^{N/D}$ and $\Delta f_{M}^{N/D}$ are given by

\[
f_{M}^{N/D}(y) = \frac{1}{3} \sum_{M} f_{M}^{N/D}(y), \tag{20}
\]
\[
\Delta f_{M}^{N/D}(y) = f_{1}^{N/D}(y) - f_{0}^{N/D}(y). \tag{21}
\]

The sum rules for $f_{M}^{N/D}(y)$ and $\Delta f_{M}^{N/D}(y)$ follow from sum rules for $f_{M}^{N/D}(y)$ and definitions (20)-(21):

\[
\int_{0}^{M_{D}/m} f_{M}^{N/D}(y)dy = \frac{1}{3} \sum_{M} \langle D | \hat{Q} | D \rangle_{M} = 1, \tag{22}
\]
\[
\int_0^{M_D/m} y f^{N/D}_D(y) dy = \frac{1}{3} \sum_M \langle D| (\Theta_N)^\mu_\mu | D \rangle_M = 1 - \delta_N, \quad (23)
\]

\[
\int_0^{M_D/m} \Delta f^{N/D}_D(y) dy = \langle D| \hat{Q}| D \rangle_{M=1} - \langle D| \hat{Q}| D \rangle_{M=0} = 0, \quad (24)
\]

\[
\int_0^{M_D/m} y \Delta f^{N/D}_D(y) dy = \langle D| (\Theta_N)^\mu_\mu | D \rangle_{M=1} - \langle D| (\Theta_N)^\mu_\mu | D \rangle_{M=0} = 0. \quad (25)
\]

Sum rules for the deuteron SFs \(b_1^D\) and \(b_2^D\) are the immediate result of combining eqs. (24)-(25) and (18)-(19):

\[
\int_0^{M_D/m} dx_D b_1^D(x_D) = 0, \quad \int_0^{M_D/m} dx_D b_2^D(x_D) = 0, \quad (26)
\]

i.e. in a full agreement with the sum rules (1) and (4).

An explicit expression for the distribution function \(f^{N/D}_M(y)\) (and therefore of \(f^{N/D}_N(y)\) and \(\Delta f^{N/D}_N(y)\)) in terms of the components of the Bethe-Salpeter amplitude (eq. (14)) is quite cumbersome and it will be presented elsewhere.

### 4 Nonrelativistic formulae

The nonrelativistic expressions for \(f^{N/D}_M(y)\) and \(\Delta f^{N/D}_M(y)\) can be obtained by using an analogy of the charge densities calculated within the Bethe-Salpeter formalism and corresponding densities calculated with wave functions, i.e.:

\[
\frac{1}{2M_D} \int_{-\infty}^{+\infty} \frac{dp_0}{2\pi} \text{Tr} \left\{ \bar{\Psi}_M(p_0, \mathbf{p}) \gamma_0 \Psi_M(p_0, \mathbf{p}) (\hat{p}_2 - m) \right\} \sim \Psi_M^\dagger(\mathbf{p}) \Psi_M(\mathbf{p}), \quad (27)
\]

\[
\frac{1}{2M_D} \int_{-\infty}^{+\infty} \frac{dp_0}{2\pi} \text{Tr} \left\{ \bar{\Psi}_M(p_0, \mathbf{p}) \gamma_3 \Psi_M(p_0, \mathbf{p}) (\hat{p}_2 - m) \right\} \sim \frac{p_3}{m} \Psi_M^\dagger(\mathbf{p}) \Psi_M(\mathbf{p}), \quad (28)
\]

where \(\Psi_M(\mathbf{p})\) is a nonrelativistic wave function of the deuteron (do not confuse with the Bethe-Salpeter amplitude, \(\Psi_M(p_0, \mathbf{p})\))! The well-known result for the spin-independent distribution is immediately reproduced (see e.g. [11, 13, 18]):

\[
f_{n.r.}^{N/D}(y) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \delta \left( y - \frac{M_D - w + p_3}{m} \right) \Theta(y) \\
\quad \quad \quad \quad \quad \quad \quad \quad (1 + \frac{p_3}{m}) \frac{1}{3} \sum_M \Psi_M^\dagger(\mathbf{p}) \Psi_M(\mathbf{p}) \\
\quad \quad \quad \quad \quad \quad \quad \quad = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \delta \left( y - \frac{M_D - w + p_3}{m} \right) \Theta(y)
\]
\[ (1 + \frac{|p| \cos \theta}{m}) \left\{ u^2(|p|) + w^2(|p|) \right\}, \] (29)

where \( u \) and \( w \) are the \( S \)– and \( D \)-wave components of the deuteron wave function, respectively. The presence of the \( \Theta \)-function in the r.h.s. of eq. (29) slightly violates the sum rule (22). However, this usually phenomenologically is not noticeable, since the only region of large momenta, \( |p| > 0.7 \text{ GeV} \), is affected by the \( \Theta \)-function and it does not contribute much to the norm of the deuteron wave function.

For distribution \( \Delta f_{N/D}^{N/D}(y) \), one gets:

\[
\Delta f_{n.r.}^{N/D}(y) = \int \frac{d^3 p}{(2\pi)^3} \delta \left( y - \frac{M_D - w + p_3}{m} \right) \Theta(y) \\
(1 + \frac{p_{13}}{m}) \left\{ \Psi_1^\dagger(p)\Psi_1(p) - \Psi_0^\dagger(p)\Psi_0(p) \right\} \\
= \int \frac{d^3 p}{(2\pi)^3} \delta \left( y - \frac{M_D - w + p_3}{m} \right) \Theta(y) \\
(1 + \frac{|p| \cos \theta}{m})P_2(\cos \theta) \frac{3}{2} w(|p|) \left\{ 2\sqrt{2}u(|p|) + w(|p|) \right\}, \] (30)

where \( P_2 \) is the Legendre polynomial.

Again, the sum rule (24) is broken by the presence of the \( \Theta \)-function in (30). Neglecting it, one gets

\[
\int_0^1 dx_D b_1^D(x_D) \propto \int_0^{M_D/m} \Delta f_{n.r.}^{N/D}(y) dy \\
\propto \int_{-1}^1 d(\cos \theta)(1 + \frac{|p| \cos \theta}{m})P_2(\cos \theta) = 0, \] (31)

where the orthogonality property of the Legendre polynomials is used.

A deviation from zero, caused by the \( \Theta \)-functions is not large compared to 1, but \textit{anything} is large compared to 0! One can \textit{artificially} adjust formula (30) to fulfill this sum rule. For instance, \textit{small} corrections to the normalization of the both terms with \( M = 1 \) and \( M = 0 \) can be made to satisfy the sum rule in form (24). However, the situation with the second sum rule, (2) and (26), is more difficult and can not be fixed by any simple adjustments of the normalizations. Similar to eq. (31), it can be written (neglecting the \( \Theta \)-function!):

\[
\int_0^1 dx_D b_2^D(x_D) \propto \int_0^{M_D/m} y \Delta f_{n.r.}^{N/D}(y) dy \\
\propto \int_{-1}^1 d(\cos \theta)(M_D - w + \frac{|p| \cos \theta}{m})P_2(\cos \theta) \neq 0. \] (32)
Thus, there is no reason for this sum rule to be satisfied with the non relativistic distribution function (30).

In conclusion to this section, the nonrelativistic formulae, in principle, violates the sum rules for the SFs $b_{1,2}^D$. However, one can still hope that it will be a small effect, one not noticeable in practice. The illustrative numerical calculations are given in the next Section, this helps one to understand a quantitative side of the effects.

5 Numerical calculations

In this Section results of the numerical calculations are presented. The SFs of the deuteron $b_{1,2}^D(x)$ are calculated within both the relativistic and nonrelativistic approaches.

The relativistic calculations are based on the formulae (14), (15)-(21) and utilizes the realistic Bethe-Salpeter amplitude for the deuteron calculated in ref. [14]. This amplitude is essentially just a different presentation of the amplitude obtained by Zuilhof and Tjon [12]. An important feature of the calculations is a numerical “inverse Wick rotation” of the amplitude. This has been done by expanding a non-singular part of the matrix elements (14) into a series in $p_0/m$, up to the fourth order. Singularities in the nucleon propagators have been treated exactly. Note, the numerical approximation made can potentially cause a violation of the exact sum rules.

The nonrelativistic calculations, eq. (30), uses the realistic wave function of the deuteron in the Bonn potential [13]. Another ingredient of the calculations, the nucleon SFs $F_{1,2}^N(x, Q^2)$, is taken from ref. [20] at $Q^2 = 10 \text{ GeV}^2$. The results are neither very sensitive to the particular choice of the parametrization for the nucleon SFs nor to the $Q^2$-dependence of them.

The distribution functions $\Delta f_{N,D}^{N/D}(y)$ are calculated and the results are presented in Fig. 2. The relativistic (solid line) and nonrelativistic (dotted line) give very similar behavior of the distribution function. Indeed, it is difficult to distinguish between them, not speaking about making definite conclusions. The third line in the Fig. 2 is given for an illustration, and presents $y\Delta f_{N,D}^{N/D}(y)$ for the relativistic calculations. The calculation of the sum rules is more representative. To understand the scale of effects, which are discussed below, it is customary to define auxiliary quantities:

$$\int_0^{M_D/m} \text{Abs} \left( \Delta f_{N/D}^{N/D}(y) \right) dy \approx \int_0^{M_D/m} \text{Abs} \left( y\Delta f_{N,D}^{N/D}(y) \right) dy \approx 0.14. \quad (33)$$

The Bethe-Salpeter and nonrelativistic Bonn calculations give the same result in (33), with accuracy of $\sim 5\%$. Thus, the effective distribution functions, $\Delta f_{N,D}^{N/D}$, are an order of magnitude smaller than the usual spin-independent distributions $f_{N,D}^{N/D}$ normalized on 1. This is not a very important circumstance, but it works against accuracy in the numerical calculations, since $\Delta f_{N,D}^{N/D}$ is a difference of two functions normalized on 1 ($M = 1$ and $M = 0$). Numerically, the sum rule (31) is satisfied both in relativistic and nonrelativistic calculations with good accuracy, despite the approximate numerical “inverse Wick rotation” and the discussion after eq. (30). The correspondent integrals are $\sim 5 \cdot 10^{-4}$ and $\sim 3 \cdot 10^{-5}$ and
they should be compared to the estimate (33). The sum rule (13) may be used to improve distributions $\Delta f^{N/D}$ by making integrals for $f^{N/D}_1$ and $f^{N/D}_0$ exactly the same. However, this does not lead to a significant variation of results for SFs, except $x \to 0$ for $b^D_1(x)$.

The behavior of the $b^D_1(x)$ at $x \to 0$ deserves to be considered more closely, especially for numerical calculations, since the nucleon function $F^N_1(x)$ can be divergent at small $x$. Unfortunately it is impossible to estimate $b^D_1(0)$ exactly for the realistic SFs $F^N_1$. However, a contribution of singularity can be evaluated. Indeed, let us assume a singular behavior as $F^N_1 \sim C/x$, then eq. (18) leads for small $x$ to

$$b^D_1(x \to 0) \sim \frac{C}{x} \int_x^{M_D/m} \Delta f^{N/D}(y)dy = \frac{C}{x} \left\{ \int_0^{M_D/m} - \int_0^x \right\} \Delta f^{N/D}(y)dy$$

$$\sim \frac{C \cdot Z}{x} - C \Delta f^{N/D}(0),$$

where $Z = 0$ in exact relativistic formula, but it can be a small number in numerical calculations or in the nonrelativistic formalism. Thus, the limit of the deuteron SF $b^D_1(x)$ as $x \to 0$ is a constant, but one has to exercise great care in performing the numerical computations, since any error leads to a divergent behavior at small $x$. In this context, an adjustment of norms of the two terms in formulae (21) and (30) has a meaning of subtraction of the numerical error from $b^D_1$ at small $x$.

The situation with the second sum rule (2) is quite different. Numerically it is broken more significantly than the previous one. Correspondent integrals are $\sim 1 \cdot 10^{-3}$ and $\sim 3 \cdot 10^{-3}$ for relativistic and nonrelativistic calculations respectively, i.e. about 0.7% and 2% compare to (33). Therefore, numerical approximations slightly damage the relativistic formula. It is attributed to the numerical rotation to the Minkowski space. An adjustment of the normalization, as it has been discussed, slightly improves the accuracy (to 0.5%). On the contrary, the result for the nonrelativistic approach is stable with respect to any adjustments, since it is defined by the formulae (32).

The SFs $b^D_1$ and $b^D_2$ are calculated within two approaches as well. The results are shown in Fig. 3 a) and b). The behavior of the functions in Fig. 3 a) suggests the validity of the sum rule (1). At the same time, the nonrelativistic calculation for $b^D_2$ in Fig. 3 b) (dotted line) obviously does not satisfy sum rule (2). The main difference of the relativistic and nonrelativistic calculations is at small $x$, where these approaches give different signs for the SFs. To illustrate the effect of the presence of the $\Theta$-function under integral in nonrelativistic formula (30), the calculations have been done as well with a restricted interval of integration over $|p|$. The condition $|p| < 0.7$ GeV corresponds to the “softer” deuteron wave function, but makes sum rule (32) exact. Correspondent SFs are shown in Fig. 3 a) and b) (dashed line). The result of this “experiment” is that the effect of $\Theta$-function is not quantitatively significant. It also does not affect the principle conclusion about the second sum rule (2), but makes the defect slightly smaller. This is understandable, since the sum rule breaking terms in (32) is $\propto |p| \cos \theta$. 

9
6 Summary

The structure functions $b_{1,2}^D(x)$ of the deuteron have been studied within the covariant approach based on the Bethe-Salpeter formalism for the deuteron. It is shown that the non-relativistic convolution model results in an incorrect behavior of these structures at small $x$ and violates the exact sum rules. The importance of accurate relativistic calculations for $b_{1,2}^D(x)$ is demonstrated.

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References

[1] M. Anselmino, A.V. Efremov and E. Leader, Phys. Rep. 261 (1995) 1.
[2] R.L. Jaffe and A. Manohar, Nucl. Phys. B321 (1989) 343.
[3] A.V. Efremov and O.V. Teryaev, Sov. J. Nucl. Phys. 36 (1982) 557.
[4] L.I. Frankfurt and M.S. Strickman, Phys. Rep. 76 (1981) 216; Nucl. Phys. A405 (1983) 557.
[5] P. Hoodbhoy, R.L. Jaffe and A. Manohar, Nucl. Phys. B312 (1989) 571.
[6] L.P. Kaptari and A.Yu. Umnikov, Z. Phys. A341 (1992) 353.
[7] A.Yu. Umnikov, L.P. Kaptari, K. Yu. Kazakov and F. Khanna, Alberta-Thy-29-94; e-print archives hep-ph 9410241.
[8] L.I. Frankfurt and M.S. Strickman, Phys. Lett. B64 (1976) 433; B65 (1976) 51; B76 (1978) 333.
[9] P.V. Landshoff and J.C. Polkingorne, Phys. Rev. D18 (1978) 153.
[10] D. Kusno and M.J. Moravcsik, Phys. Rev. D20 (1979) 2734.
[11] P.J. Mulders, A.W. Schreiber and H. Meyer, Nucl. Phys. A549 (1992) 498.
[12] M.J. Zuilhof and J.A. Tjon, Phys. Rev. C22 (1980) 2369.
[13] A.Yu. Umnikov and F. Khanna, Phys. Rev. C49 (1994) 2311.
[14] A.Yu. Umnikov, L.P. Kaptari, K.Yu. Kazakov and F. Khanna, Phys. Lett. B334 (1994) 163.
[15] W. Melnitchouk, A.W. Schreiber and A.W. Thomas, Phys. Rev. D49 (1994) 1183.
[16] S. Kulagin, W. Melnitchouk, G. Piller and W. Weise, Phys. Rev. C52 (1995) 932.
[17] A.Yu. Umnikov, L.P. Kaptari, and F. Khanna, Phys. Rev. C53 (1996) 351.
[18] B.L. Birbrair, E.M. Levin and A.G. Shuvaev, Nucl. Phys. A496 (1989) 704.
[19] R. Machleid, K. Holinde and Ch. Elster, Phys. Rep. 149 (1987) 1.
[20] L.P. Kaptari and A.Yu. Umnikov, Phys. Lett. B259 (1991) 155.
Figure captions

Figure 1: Triangle diagram

Figure 2: $\Delta f^{N/D}(y)$. Curves: the BS - solid, the Bonn - dotted, the BS multiplied on extra factor $y$ - dashed.

Figure 3: $b^{D}_{1,2}(x)$. Curves: the BS - solid, the Bonn - dotted, the Bonn with cut - dashed.
Fig. A. Umnikov, Relativistic calculation...
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