Towards a solution of pure Yang-Mills theory in 3 + 1 dimensions

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Abstract

We discuss an analytic approach towards the solution of pure Yang-Mills theory in 3 + 1 dimensional spacetime. The approach is based on the use of local gauge invariant variables in the Schrödinger representation and the large \(N\) planar limit. In particular, within this approach we point out unexpected parallels between pure Yang-Mills theory in 2 + 1 and 3 + 1 dimensions. The most important parallel shows up in the analysis of the ground state wave-functional especially in view of the numerical similarity of the existing large \(N\) lattice simulations of the spectra of 2 + 1 and 3 + 1 Yang Mills theories.

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1 Introduction

Recently, new analytical results pertaining to the spectrum of 2 + 1 dimensional Yang-Mills theory (YM\(_{2+1}\)), that are in excellent agreement with the lattice data in the planar limit [1], have been derived in [2]. These analytic results regarding the mass gap, string tension and the glueball spectrum resulted from a determination of the ground state wave-functional in the planar limit [3]. The analytic calculations in YM\(_{2+1}\) were based, as in [4], on the remarkable work of Karabali and Nair [5].

Because in 2 + 1 dimensions space is two dimensional, one can work in a complex basis in the Hamiltonian picture. One might wonder whether a similar success can be achieved in the more realistic case of a 3 + 1 dimensional Yang-Mills theory (YM\(_{3+1}\)). As we will explain in this letter, a formalism does exist which closely parallels that of KN. The ingredients of this formalism were introduced long ago by I. Bars [6] in his work on local gauge invariant ‘corner variables’. (Other related work can be found in [7].)

In this rather programmatic letter, we outline and extend the general formalism, and explain the physical interpretation and use of corner variables both in YM\(_{2+1}\) and YM\(_{3+1}\). In particular, in the context of YM\(_{2+1}\), the corner variables are appropriate to a real coordinate basis as opposed to the complex basis of KN. Also, as pointed out by Bars in [6], the corner variables can be also used in a covariant Lagrangian picture, which is an obvious additional attractive feature of the formalism. In the Hamiltonian picture the use of corner variables also reveals unexpected parallels between pure Yang-Mills theory in 2 + 1 and 3 + 1 dimensions. We concentrate here on the discussion of the vacuum wave-functional in the Hamiltonian picture, especially in light of the large N lattice simulations.

The themes outlined in this paper are explored in more detail in a companion paper [8] and in [9]. The organization of this letter is as follows: in Section 2 we discuss the formalism of corner variables in view of our recent work on pure Yang Mills theory in 2+1 dimensions [2]. Then we turn to the more dynamically relevant issues in Section 3. Section 4 is devoted to the general analysis of the structure of the vacuum wave-functional. We conclude the paper with a short outline of some of our current work in progress.

2 The Formalism of Corner Variables

The key insight we start from is that the KN variables [5] in the 2 + 1 dimensional setting are related to line integrals in from infinity to a point \(x\). The analogous variables in 3 + 1 were introduced by Bars and we call them \(M_i(x)\) where \(M_i(x)\) are unitary matrices.\(^1\) They satisfy the defining equation

\[
A_i = -\partial_i M_i M_i^{-1} \quad \text{(no sum on } i) \tag{1}
\]

In other words

\[
M_i(x) = Pe^{xp[-\int_{-\infty}^{x} A]} \quad \tag{2}
\]

\(^1\)We use notation similar to that of KN and warn the reader of notation difference compared to Bars.
where the integral is a straight spatial contour for fixed \( x^j, i \neq j \). The interesting thing about these variables is that the natural lattice formulation parallels the continuum formulation. The above formulation is appropriate to the Hamiltonian formalism, if \( j \) is a spatial index. The translation to 2+1 KN variables is then just

\[
M_z = M, \quad M_{\bar{z}} = M^{-1}
\]

(3)

Note that if we had used a real coordinate basis, then we would have had a pair of (unrelated) unitary matrices \( M_1, M_2 \). Next, define the corner variables

\[
H_{ij} = M_i^{-1}M_j
\]

(4)

Note that \( H_{jj} = 1 \) and \( H_{ji} = H_{ij}^{-1} \) – this just means traversing the corner in the opposite direction is precisely the inverse group element. The \( H_{ij} \) are unitary in a real coordinate basis. There is also a constraint (here written for 3 + 1 — it is trivial in 2 + 1)

\[
H_{ij}H_{jk}H_{ki} = 1
\]

(5)

Another description is in terms of a “semi-complex” coordinate basis \( \{u, z, \bar{z}\} \); then one may parameterize as

\[
H_{uz} = H, \quad H_{\bar{z}u} = H^\dagger, \quad H_{\bar{z}z} = H^\dagger H.
\]

(6)

For example, one could use the notation \( M_z = M, M_{\bar{z}} = M^{-1}, M_u^\dagger M_u = 1 \), with \( H = M_u^\dagger M \). The constraint takes the form \( H_{\bar{z}u}H_{uz}H_{\bar{z}z} = 1 \). In other words, there is in \( D = 3 \) in the semi-complex coordinate basis a complex \( H \)-field (compared to a Hermitian field in \( D = 2 \)); thus there are twice as many degrees of freedom, as expected. Returning to the general notation, gauge transformations act as

\[
M_j \mapsto gM_j
\]

(7)

so the \( H_{ij} \) are gauge invariant.

The formalism of Bars also contains the notion of (or a generalization of) holomorphic invariance (which is familiar from the 2+1 KN setting). Indeed, note that one may introduce ‘currents’

\[
J_{ij} = (\partial_j H_{ij})H_{ij}^{-1}
\]

(8)
(in 2+1, \( J \sim J_{\bar{z}z} \) and \( J^\dagger \sim -J_{z\bar{z}} \)). The extra symmetry acts as

\[ M_i \mapsto M_i h^{-1}_i(x^j), \quad j \neq i \]  

(9)

(this apparently, as far as we can tell, was not pointed out in [6].) The condition \( j \neq i \) on the function \( h_i \) is the analogue of holomorphy. This leaves the gauge fields invariant, and one finds

\[ H_{ij} \mapsto h_i H_{ij} h^{-1}_j \]  

(10)

In the semi-complex basis, we would have \( M_u \rightarrow M_u h^{-1}_u(z, \bar{z}) \), \( M_z \rightarrow M_z h^1(u, \bar{z}) \) and so \( H \rightarrow h_u(z, \bar{z}) H h^1(u, \bar{z}) \), \( H^\dagger \rightarrow h(u, z) H^\dagger h^{-1}_u(z, \bar{z}) \). The \( J_{ij} \) transform as connections

\[ J_{ij} \mapsto h_i J_{ij} h^{-1}_i + \partial_j h_i h^{-1}_i \]  

(11)

In the real coordinate basis, it appears that there are six currents that are apparently distinct. However, there is a ‘reality’ condition on their derivatives of the form

\[ \partial_i J_{ij} = -H_{ij}(\partial_j J_{ji}) H^{-1}_{ij} \]  

(12)

(in \( D = 2 \), there is a similar relation which reads \( \bar{\partial} J = H(\partial J^\dagger) H^{-1} \).) By defining \( \bar{J}_{ij} = -H_{ij} J_{ji} H^{-1}_{ij} \), we may rewrite this as

\[ \partial_i J_{ij} = \partial_j \bar{J}_{ij} - [J_{ij}, \bar{J}_{ij}] \]  

(13)

and so there are covariant derivatives \( D_{ij} = \partial_j - J_{ij} \). These currents are related to the magnetic field \( F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j] \) by

\[ \partial_i J_{ij} = -M^{-1}_i F_{ij} M_i. \]  

(14)

As a short form then, we will denote

\[ B_{i-1} = \partial_i J_{i,i+1} \quad (\text{cyclic}). \]  

(15)

These fields transform homogeneously under the ‘holomorphic’ symmetry

\[ B_{i-1} \mapsto h_i B_{i-1} h^{-1}_i \]  

(16)

Note that there are only really two of these fields, because, solving the constraint [5], there are only two independent \( H_{ij} \)'s, say \( H_{12} \) and \( H_{13} \).

Note that the covariant derivatives can be written in terms of the usual derivative and \( M_i \) as

\[ \nabla_i = \partial_i + A_i = M_i \partial_i M^{-1}_i. \]  

(17)

Finally we note some notations and conventions used in this paper. The connection is expanded in terms of anti-hermitean generators \( T_a \), \( A_i = A^a_i T_a \) satisfying the algebra \([T_a, T_b] = f_{abc} T_c \). Also, we denote by \( tr \) the trace in the fundamental representation, so that \( tr(1) = N \). This trace is normalized as \( -2 tr(T_a T_b) = \delta_{ab} \). In the adjoint representation the generators are given by \((T^a)^{bc} = -f_{abc} \), where indices are raised or lowered with the metric \( \delta_{ab} \) and the trace in the adjoint is denoted by \( tr_{ad} \). A group element \( M \) is represented
in the adjoint by \( M_{ab} = -2 \text{tr}(T_a M T_b M^{-1}) \), clearly we have \((M^{-1})_{ab} = M_{ba}\) and also \( M T_b M^{-1} = T_a M^{a b} \).

We will need a variety of Green’s functions. We denote by \( G_i(x,y) \) the inverse of \( \partial_i \), such that 

\[
\partial_i G_i(x,y) = 2 \delta^{(D)}(x,y) \quad \text{(no sum)}.
\]

Since \( \partial_i \) admits zero modes, its inverse is not uniquely defined and we will work with the explicit choice \( G_{i}(x, y) = \theta(x_1) \theta(x_2) \delta(x_3) \) and \( G_i(x, y) \equiv G_i(x - y) \) (which is not antisymmetric). This choice is not arbitrary, it is the unique choice consistent with the definition of the variable \( M_i \) as an ordered exponential.

Indeed we can write

\[
M_i(x) = \sum_{n} (-1)^n \int dy (G_i A_i)^n(x, y),
\]

where \((G_i A_i)^2(x, y) = \int dz G_i(x, z) A_i(z) G_i(z, y) A_i(y)\).

\section{Further Technical Details}

The Yang-Mills action is taken to be \( S_{YM} = \frac{1}{2g^2} \int \text{tr}(F_{\mu \nu} F^{\mu \nu}) \) and the Hamiltonian in the Yang-Mills variables is

\[
\mathcal{H} = \sum_{i,a} \left\{ -\frac{g^2}{2} \left( \frac{\delta}{\delta A_i^a} \right)^2 + \frac{1}{2g^2} (F_i^a)^2 \right\}
\]

with \( F_i^a = \frac{1}{2} \epsilon_{ijk} F_{jk}^a \) and should be supplemented by the gauss law constraint \( \nabla_i \frac{\delta}{\delta A_i} = 0 \). The wave-functionals in the Schrödinger representation of pure Yang-Mills are gauge invariant functionals of \( A_i \) and the scalar product is given by

\[
||\Psi||^2 = \int_{A/\mathcal{G}} D\mu(A) \bar{\Psi}[A] \Psi[A],
\]

where the integral is over the space of gauge connections modulo gauge transformations, \( D\mu(A) = \frac{DA}{\text{Vol}(\mathcal{G})} \). From the previous section we know that we can equivalently describe gauge invariant wave functionals as ‘holomorphic’ invariant wave-functionals of \( H_{ij} \) or \( J_{ij} \). This change of variables can be explicitly performed both at the level of the measure and the Hamiltonian \( \mathcal{H} \); we recall here some of these results.

Since \( A_i = -\partial_i M_i M_i^{-1}, \delta A_i = -(\nabla_i \delta M_i) M_i^{-1} \), the change of variables involves a determinant

\[
\Gamma^{\delta} = \det \left( \frac{\delta A_i}{\delta M_i M_i^{-1}} \right) = \det(\nabla_1 \nabla_2 \nabla_3).
\]

The variational derivative of the action is found to be trivial

\[
\frac{\delta \Gamma}{\delta A_i^a(x)} = tr_{ad} \left[ (\nabla_i)^{-1}(x, x) T_a \right] = 1.
\]

The scalar product written in term of the Bars variables is simply

\[
||\Psi||^2 = \int DH_{12} DH_{13} \bar{\Psi}[H] \Psi[H] = \int \prod_{i \neq j} DH_{ij} \delta(H_{12} H_{13} H_{23}) \bar{\Psi}[H] \Psi[H]
\]
where $DH$ denotes the product over left-right Haar measures on $SU(N)$. In the last equality we inserted a delta function on the group to emphasize the symmetric form of the measure under permutation of indices. The interpretation of this delta function constraint should be clear: it is the integrated version of the Bianchi identity $\epsilon^{ijk} \nabla_i F_{jk} = 0$ expressed in terms of gauge invariant observables. One important feature of this scalar product is the fact that the identity functional $\Psi[H] = 1$, which can be can be viewed as a limit of the identity cylindrical functional, is a normalisable wave-functional.

The potential term of the Yang-Mills Hamiltonian is easily expressed in terms of the Bars variables

$$ V \equiv \frac{1}{2} \sum_{i,a} (F^a_i)^2 = \frac{1}{2} \sum_{i,a} \int (\partial_i J^a_{i,i+1})^2(x)dx. \quad (24) $$

and the kinetic term reads

$$ T = -\frac{1}{2} \sum_{a,i} \int \delta A^p_a(x) \frac{\delta}{\delta A^q_a(x)} = \frac{1}{2} \sum_{a,i} \int dydz \ P^i_a(y) \Theta^{ab}_i(y,z) P^j_b(z) \quad (25) $$

where $P^i_a$ are right derivatives on the group

$$ [P^i_a(x), M_j(y)] \equiv \delta^i_j \ M_i(y) T_a \ \delta^{(D)}(x,y) \quad (26) $$

and we have introduced the kernel

$$ \Theta^{ab}_i(y,z) = \delta^{ab} (G_i G_i)(y,z) = \delta^{ab} \int dG_i(y,x)G_i(x,z). \quad (27) $$

It is important to note that the form (25) for the kinetic term is valid only when $T$ acts on holomorphic invariant states. This Hamiltonian can be checked to be hermitian [8].

Two technical points are in order here. First, in the KN formalism, a non-trivial Jacobian appeared and is given by the exponent of a gauged WZW action. This is an artifact of the complex polarization adopted in the formalism, and it is not a universal feature. In the real basis that we have discussed here, such a Jacobian does not arise. Indeed, consider the computation of such a Jacobian determinant, which one can write conveniently as a fermionic determinant. Depending on the choice of a chiral (in the complex KN-like polarization) or non-chiral basis (in the real Bars-like polarization) for the auxiliary adjoint fermions, in the case of $2 + 1$ Hamiltonian formulation, one gets the usual anomalous term represented by the WZW action, or one gets a unit Jacobian, respectively. Thus one also has a trivial unit Jacobian in $3 + 1$ dimensions if one works with Bars’ corner variables.

Second, there is a delicate issue of holomorphically invariant regularization procedure (which extends the techniques used by [8]). And indeed, in order to make sense of the computation of the determinant or the hermiticity property of the kinetic term, one needs a regularisation scheme which preserves all the symmetries. Such a regularisation scheme is fully described in [8] and is absolutely crucial for actual computations.

### 3.1 Semi-complex coordinates

One of the most important observations about the Bars’ formalism is that the kinetic terms have the form $T = \int (Gpq)^2$ (where $p$ and $q$ denote the canonical generalized momentum and coordinate, $[q,p] = i$) with a regularized Green’s function $G$. 

The significance of this will be appreciated by considering a toy example with a similar Hamiltonian. Consider a free 1d rotator. The kinetic term is \( T = -\delta^2 \theta / \delta \theta^2 \). Here, \( \theta \) is a periodic variable and the spectrum is discrete, with eigenvalues \( n^2 \), and with periodic eigenfunctions \( e^{in\theta} \). Now if we change variables \( \rho = e^{i\theta} \), the kinetic term becomes homogeneous

\[
T = \rho \frac{\delta}{\delta \rho} \frac{\delta}{\delta \rho}
\]

where obviously \( \rho \) and its conjugate momentum \( \frac{\delta}{\delta \rho} \) have the canonical commutation relations. So this has a \((pq)^2\) form, as required. Now we can rewrite this using the canonical commutator as

\[
T = \rho \frac{\delta}{\delta \rho} + \rho^2 \left( \frac{\delta}{\delta \rho} \right)^2
\]

which looks like the usual kinetic term for pure Yang-Mills theory in terms of complex corner variables as in \([5]\). We look at the action of this kinetic term on \( \rho^n \), \( n = 1, 2, 3... \) which correspond to the eigenstates \( e^{in\theta} \). The first term gives a homegeneity factor (the Euler factor) of \( n \), which is of course not the correct eigenvalue! To get the correct eigenvalue \( n^2 \) we need also the second term in \( T^2 \).

The upshot here is that we see a close parallel between this simple toy model and the formulation of pure Yang-Mills theory in terms of corner variables.

In view of this toy example we discuss the complex versus real corner variables and the form of the corresponding Hamiltonian. The similarity between the Karabali-Nair Hamiltonian written in terms of complex corner variables and the collective field theory \([10]\) has been noted before \([5, 11, 2]\). Yet the Hamiltonian written in terms of real variables apparently does not have this form. In particular, it seems that the homogeneous piece of the Hamiltonian (as illustrated by the above toy example) is missing. One would have such a term however, if we write the theory in the semi-complex basis in \( 3 + 1 \) that we discussed in Section 2. Such a term acts homogeneously on suitable functionals and thus acts as an effective dimension counting operator. In what follows we will further comment on this issue when we discuss the constituent picture of glueballs.

More explicitly in terms of the semi-complex coordinates from section 2 \((u, z, \bar{z})\), the gauge invariant information is encoded in a complex \( H \equiv H_{uvz} \). In this case the variational derivative of the determinant \( \frac{\delta \Gamma}{\delta A^\dagger_{(x)} (x)} \) can be computed to give \([8]\)

\[
\delta \Gamma = -\frac{2N\mu}{\pi^{3/2}} \int \text{tr}[(H^\dagger H)^{-1}\delta(H H^\dagger)\partial((H^\dagger H)^{-1}\partial H^\dagger H)]
\]

This expression integrates to a 3 dimensional generalization of the WZW action

\[
\Gamma = -\frac{2N\mu}{\pi^{3/2}} \left[ \frac{1}{2} \int du d^2z \text{tr}(\partial(H^\dagger H)\partial(H^\dagger H)^{-1}) + \frac{i}{12} \int du \int_{B_u} \text{tr}(H^\dagger H)^{-1}d(H^\dagger H) \right]^3
\]

where \( B_u \) denotes a 3-ball bounding the plane \( u = \text{const} \).

In this form the parallel with the KN formalism \([5]\) becomes very striking. The 2 + 1 dimensional results developed in \([3]\) can be thus viewed as a natural reduction of the 3 + 1

\[\text{Note that in the quantum field theory setting the factor of } n^2 \text{ is misleading as it can be seen in the detailed analysis of the 2 + 1 dimensional Yang-Mills theory } [3]. \text{ Further discussion is in Section 4.}\]
dimensional formulation in the semi-complex basis. In particular, the Hamiltonian in the semi-complex basis is \[ \mathcal{H} \sim \frac{g^2 N \mu}{2 \pi^{3/2}} \left\{ \int du d^2 z J_{z\bar{z}} \delta \frac{\delta}{\delta J_{z\bar{z}}} + \int du \int d^2 w_1 d^2 w_2 \left( \frac{\delta}{\delta J_{z\bar{z}}(w_1)} \frac{\delta}{\delta J_{z\bar{z}}(w_2)} + \ldots \right) \right\} \] (32)

where \( \Omega \) is essentially fixed by the two-point function of the WZW model as in the 2 + 1 dimensional context [3].

4 Vacuum wave-functional: general discussion

In this section, we make some general comments about the vacuum wave-functional in 3 + 1 dimensional Yang Mills theory in view of the already discussed parallel between pure gauge theories in 2+1 and 3+1 dimensions. First, on general grounds one knows that the vacuum wave-functional \( \Psi_0 \) is gauge invariant, parity even (in the absence of \( \theta \) terms) and most importantly should be a strictly positive functional. This can be written as

\[ \Psi_0 = e^P \] (33)

where \( P \) is a functional of the ‘position’ variables. Second, we also know that the vacuum wave-functional can be formally written as a path integral on half space-time as

\[ \Psi_0[\tilde{A}] = \int_{A(t=0)=\tilde{A}} DA \ e^{\int_0^\infty dt \int d^3 x \ tr(F_{\mu\nu} F^{\mu\nu})(t,x)} \] (34)

where we integrate over all fields satisfying the boundary condition \( A(0, x) = \tilde{A}(x) \) and the integral is computed in the \( A_0 = 0 \) gauge. This implies that \( P \) is a gauge invariant sum over connected Feynman diagrams and in the large \( N \) limit is a non-local functional involving only a single trace over gauge indices.

A physical mass scale \( m \) must emerge in this theory. In fact, in order to write a wave-functional that involves arbitrary number of derivatives acting on fields, a scale must be introduced by hand. Presumably then, physical mass scales would be determined self-consistently. Furthermore, if the 3 + 1 theory follows the 2 + 1 theory, we can suppose that in the IR, the mass scale enforces locality of \( P \) in this limit. Ultimately, this question can only be answered for sure by computing physical quantities (such as the Wilson loop expectation value). Given this assumption, we can formally develop the non-local functional \( P \) in terms of local operators. By the previous argument, this expression involves only local, single trace gauge invariant functionals. Such operators are naturally labeled by two integers \( n, k \) which count the number of laplacians and powers of magnetic field insertions respectively. We can write this expansion, in parallel with 2 + 1, symbolically as

\[ P = \sum_{k,n} \int tr \left[ \left( \frac{\Delta}{m^2} \right)^n \left( \frac{F}{m^2} \right)^k \right] \] (35)

We will advocate, as in 2 + 1, that it is useful to take \( P \) to have a certain quadratic form. In particular, when written in terms of the gauge invariant variables, we will consider as an
\[ \Psi_0 = \exp \left( \frac{1}{g^2 m} \int \text{tr } B_i K(L/m^2)B_i + \ldots \right). \]  

(36)

where \( K \) is a kernel constructed out of ‘holomorphic’ covariant derivatives scaled by \( m \), but does not contain the curvature \( B_i \). The kernel \( K \) should be determined (although we do not do so here) consistent with gauge, ‘holomorphic’ and spacetime symmetries. Most importantly, it should be determined such that the wave-functional is normalizable. In 2+1, such a kernel was found, and it seems reasonable to suppose that this could be repeated in 3+1, given the similarity of the formalism.

We note, independent of these remarks, that \( P \) should have a specific asymptotic form in the far UV — the theory is free in that limit, and thus the vacuum wave-functional should be appropriate to free gluons. In this limit, we should find

\[ P_{UV} = \frac{1}{g^2} \int_k \text{tr } B_i(-k) \frac{1}{\sqrt{k^2}} B_i(k). \]  

(37)

This is determined essentially by dimensional analysis and is consistent with the transversality of gluons. We note that the mass scale \( m \) introduced to define the kernel, disappears in this limit. It is a straightforward exercise to demonstrate that the Schrödinger formalism does indeed reproduce this perturbative result.

The UV behavior of the wave-functional is required but does not address the question of normalizability. Also, to be consistent with confinement, (in the sense of no phase transition occurring between UV and IR) the kernel \( K \) should behave, thought of as a function of momentum, in a smooth way, and as we have discussed, take a certain simple form in the far IR. In particular, we are supposing here that this form is

\[ P_{IR} = \frac{1}{2g^2 m} \int_k \text{tr } B_i(-k)B_i(k). \]  

(38)

Another way to motivate this form would be to compute how the Hamiltonian acts on this operator, and build the Schrödinger equation. Using the ‘holomorphic’ regularization with a regulator scale \( \mu \) and in the context of real variables, one finds

\[ g^2 T : \int B_i B_i = 2M \int B_i B_i \]  

(39)

where

\[ 2M = \frac{g^2 N}{2(2\pi)^{3/2}} \mu. \]  

(40)

This is reminiscent of the behavior in 2+1, and is consistent with the kinetic energy acting to ‘count’ derivatives. Perhaps the simplest way to understand this result is to resort to the semi-complex basis in which the intuition gained in 2+1 dimensions becomes useful. In that basis, there is a homogeneous part in the kinetic term (as in our toy example) which acts as a dimension operator.

\[ \mathcal{H} \sim M \int J_{zz} \frac{\delta}{\delta J_{zz}} + \ldots \]  

(41)
where \( M \) is an effective mass scale. This is a first order operator. The rest of the kinetic term, a second order operator, acts to properly normal order the operators in the regularized calculation as well as provide the necessary invariant counterterms so that the final result is indeed holomorphic invariant.

Note though that the difference between 2 + 1 and 3 + 1 is in the power-counting. In 2 + 1, \( M \) is naturally given by the coupling constant due to its dimensionality. In 3 + 1 the coupling constant is dimensionless, and \( M \) is generated dynamically from the regularization of \( \mathcal{T} \), and appears for obvious dimensional reasons! It becomes clear though that this result is regulator dependent, and what happens in the continuum limit is not clear.

The result (39) is consistent with (38), relating \( M \) to \( m \). The IR vacuum wave-functional provides a probability measure \( \Psi_0^* \Psi_0 \) equivalent to the partition function of the Euclidean three-dimensional Yang-Mills theory with an effective Yang-Mills coupling \( g_{3D}^2 \equiv mg^2 \). One could use this to compute the expectation value of a large Wilson loop and deduce the area law behavior. Presumably this would mean that the square root of the string tension scales with \( m \). This remark ties the scale introduced into the vacuum wave-functional to this physical parameter.

Let us remark further on the generalized Gaussian form of the vacuum wave-functional. In particular, the neglect of higher order terms in the exponent of the wave-functional needs to be justified. We want to be perfectly clear that the validity of the generalized Gaussian form for the vacuum wave-functional cannot be established by appealing only to standard large \( N \) simplifications. The large \( N \) limit only selects single trace expressions in the ansatz for the wave-functional. Thus we also do not expect that this wave-functional is exact; if this were true, \( B_i \) would indeed represent the right variables appropriate to the large \( N \) limit. Nevertheless, as in the 2 + 1 dimensional case [2], these local gauge invariant variables are the correct constituent degrees of freedom, even though they do not appear as physical asymptotic states! The physical states (an infinite number of glueballs, or non-local gauge invariant variables, such as Wilson loops) can be built out of these local degrees of freedom, so that all expected predictions based on the large \( N \) counting (factorization, suppression of vertices etc) are fulfilled.

The neglect of the higher order terms in \( B_i \) suggests that there is a second expansion parameter at work, involving a new length scale. One such candidate scale is the size of glueballs. The approximation employed here amounts to considering only ‘free’ glueballs, which are point-like and non-interacting. This is consistent with the large \( N \) picture where one expects that all glueball interactions are suppressed by powers of \( 1/N \). This wave-functional has the form of a “generalized coherent state” appropriate to large \( N \) [10]. One way to intuitively motivate the Gaussian form of the wave-functional is to think of \( B_i \) as the relevant local probes of real physical states, but not as actual asymptotic states! Then apart from the rank of the gauge group \( N \), there should exist another expansion parameter, which is related to the size of the glueballs. The quadratic term in the wave functional should be then interpreted as the leading term in the expansion in the inverse of that effective glueball size. This would be very reminiscent of the \( \alpha' \) expansion in string theory.

As we mentioned above the confinement should be implied by the normalizability of the wave-functional, as in the 2 + 1 dimensional counterpart [2]. Note that the constituent \( B_i \) should not appear as an asymptotic state. Also, we further expect the glueball constituents
to be “seeds” for constituent quarks once the fermionic degrees of freedom are included.

Furthermore we also want to reiterate that the “QCD scale” as given by \( m \) should be self-consistently determined: on one side \( m \) is needed in the wave-functional for dimensional reasons, and thus it sets the mass scale for the spectrum, and on the other hand, the square root of the string tension is given in terms of \( m \) up to a multiplicative numerical constant. Note that the cut-off \( \mu \) appears both in the expression for the gap and the string tension, and should consistently cancel in the ratio (assuming they both persist in the continuum). Also, in the limit of small coupling, the ratio of the string tension and the scale \( m \) (or equivalently the cut-off \( \mu \)) should only depend on the dimensionless coupling \( g^2 N \), and thus given the fact that the wave-functional describes a free theory in the UV, is then corrected logarithmically, as implied by asymptotic freedom.

Finally, one of the amazing features of large \( N \) lattice simulations [12] is that the actual numerical values for the ratio of masses and the square root of the string tension are of the same order of magnitude, both in \( 2 + 1 \) and \( 3 + 1 \) dimensions. The actual numbers are very close, up to 15\% to 20\%. We think that this is evidence that the large \( N \) Yang Mills theories in \( 2 + 1 \) and \( 3 + 1 \) theories are “close” in the sense of the respective continuum limits. We think that this is not a coincidence in view of the formalism we have discussed.

5 Conclusion

In this short programmatic paper we have discussed an analytic approach towards the solution of pure Yang-Mills theory in \( 3 + 1 \) dimensional spacetime. Our approach is based on the use of local gauge invariant corner variables in the Schrödinger representation and the large \( N \) limit. In particular, within this approach one finds unexpected parallels between pure Yang-Mills theory in \( 2 + 1 \) and \( 3 + 1 \) dimensions.

There are obviously many important questions to be addressed. Perhaps the most important is to determine a definite form of the kernel in the quadratic wave-functional in parallel with the successful discussion of the \( 2 + 1 \) dimensional situation [2] which did lead to the computation of the large \( N \) glueball spectra/Regge trajectories. The large \( N \) lattice results are already available [12].

There are of course many other obvious questions, both in the \( 2 + 1 \) and \( 3 + 1 \) dimensional contexts: the inclusion of fermions and the computation of the meson (and baryon) masses in the large \( N \) limit; the development of the covariant approach; the elucidation of the role of topology; the proof of confinement and many others. We plan to address some of these questions in [9].

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