K-HOMOLOGICAL FINITENESS AND HYPERBOLIC GROUPS

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Abstract. This paper is concerned with a strong finiteness property for K-homology, which we term uniform summability. We say that a C*-algebra has uniformly summable K-homology if all its K-homology classes can be represented by Fredholm modules which are finitely summable over the same dense subalgebra, and with the same degree of summability. We show that the following C*-algebras have uniformly summable K-homology:

- the C*-crossed product $C(\partial \Gamma) \rtimes \Gamma$ defined by the action of a torsion-free hyperbolic group $\Gamma$ acting on its boundary;
- the reduced group C*-algebra $C^*_r \Gamma$ of a torsion-free hyperbolic group belonging to one of the following classes: uniform lattices in SO($n, 1$) or SU($n, 1$); free groups; or $C'(1/6)$ small-cancellation groups of deficiency 1.

We provide explicit summability degrees, as well as explicit finitely summable representatives for the K-homology classes.

1. Introduction

A Fredholm module over a unital C*-algebra $A$ consists of a representation $\pi: A \to B(H)$ of $A$ on a Hilbert space, and an operator $T \in B(H)$, which in the even case is an essential unitary and in the odd case an essential projection, and which in both cases is required to essentially commute with the representation: $[\pi(a), T] \in K(H)$ for all $a \in A$. The definition of Fredholm module was motivated by elliptic operator theory: if $M$ is a smooth, compact manifold, then any zero-order elliptic pseudodifferential operator $T$ on sections $L^2(E)$ of a bundle over $M$, determines a Fredholm module with the obvious representation $\pi: C(M) \to B(L^2(E))$ of functions by multiplication operators. Since elliptic operators induce maps on K-theory by an index theoretic construction, this led Atiyah, and subsequently Kasparov, to describe the K-homology of a C*-algebra $A$ as equivalence classes of Fredholm modules over $A$.

Connes’ early work on cyclic cohomology, the noncommutative analogue of de Rham theory, and on the noncommutative Chern character, a map from K-homology to cyclic theory, suggested the importance of the finite summability condition on a Fredholm module that $[\pi(a), T] \in L^p(H)$ for dense $a \in A$.

where $L^p(H)$ is the Schatten ideal of $p$-summable compact operators. Connes showed how to associate a canonical cyclic cocycle, the so-called character, to a finitely summable Fredholm module, and how to use the character for computing the index pairing between the K-theory of $A$ and the K-homology class of the Fredholm module. This is just one aspect of a larger landscape, that of quantized calculus [8, Ch.IV], depending on finite summability.

Examples of finitely summable Fredholm modules over C*-algebras are thus of considerable interest in noncommutative geometry, and by this stage there have been numerous constructions of them, but one can state a theorem about their existence in the classical situation, using elliptic operator theory and Poincaré duality. If $M$ is a closed manifold, then $M$ has Poincaré duality with its tangent bundle. This implies that every K-homology class for $M$ is represented by
a pseudodifferential operator of order zero. Classical spectral estimates for pseudodifferential operators imply that the singular values of commutators $[f, T]$, where $f$ is a smooth function and $T$ is pseudodifferential, satisfy the asymptotic law $s_n = O(n^{-1/\dim M})$. Therefore every $K$-homology class for $M$ is represented by a $p$-summable Fredholm module over $C^\infty(M)$ for $p > \dim M$—both the smooth subalgebra and the degree of summability may be taken uniform across all $K$-homology classes. The main focus of the present article is on this strong finiteness phenomenon: that all $K$-homology classes of a $C^*$-algebra can be represented by Fredholm modules which are finitely summable over the same smooth subalgebra, and with the same degree of summability. Our main results give noncommutative examples of $C^*$-algebras having such uniformly summable $K$-homology.

**Theorem A.** Let $\Gamma$ be a regular, torsion-free hyperbolic group, and let $\partial \Gamma$ denote its boundary. Then the crossed-product $C^*$-algebra $C(\partial \Gamma) \rtimes \Gamma$ has uniformly summable $K$-homology.

The summability degree is the Hausdorff dimension of the boundary, more precisely a suitable interpretation thereof, and it is obtained by analytic means from the $\Gamma$-invariant Hölder structure of the boundary. The same structure is responsible for the smooth subalgebra. We are able to find finitely summable representatives for all the $K$-homology classes of $C(\partial \Gamma) \rtimes \Gamma$ by exploiting the Poincaré duality isomorphism of the first author [10].

By restricting the finitely summable Fredholm modules over the crossed-product $C^*$-algebra $C(\partial \Gamma) \rtimes \Gamma$ to the reduced group $C^*$-algebra $C^*_{\text{r}} \Gamma$, and using further ingredients, we deduce the following.

**Theorem B.** Let $\Gamma$ be a finitely generated free group, or a torsion-free cocompact lattice in $\text{SO}(n, 1)$ or $\text{SU}(n, 1)$, or a torsion-free $C^\prime_{1/6}$ small-cancelation group with one more generator than relators. Then the reduced group $C^*$-algebra $C^*_{\text{r}} \Gamma$ has uniformly summable $K$-homology.

Hyperbolicity, in the sense of Gromov, is a coarse notion of negative curvature. A hyperbolic space is a geodesic metric space all of whose geodesic triangles are uniformly thin. A group is said to be hyperbolic if it admits a geometric— that is, isometric, proper, and cocompact—action on a hyperbolic space. Equivalently, and somewhat more intrinsically, a hyperbolic group is a finitely generated group all of whose Cayley graphs are hyperbolic. Fundamental examples of hyperbolic groups are finitely generated free groups, cocompact lattices in $\text{SO}(n, 1)$, $\text{SU}(n, 1)$, or $\text{Sp}(n, 1)$ (‘classical hyperbolic groups’), as well as $C^\prime(1/6)$ small-cancellation groups.

In what follows, hyperbolic groups are assumed to be non-elementary, meaning that we discard the virtually cyclic groups.

The boundary $\partial \Gamma$ of a hyperbolic group $\Gamma$ is a compact Hausdorff space carrying a natural action of $\Gamma$ by homeomorphisms. Our standing assumption that $\Gamma$ is non-elementary translates into $\partial \Gamma$ having uncountably many points. For instance, the boundary of a free group is a Cantor set, and the boundary of a classical hyperbolic group is a sphere. The $\Gamma$-action on $\partial \Gamma$ is topologically amenable [1]. This means that the $C^*$-algebra $C(\partial \Gamma) \rtimes \Gamma$ is nuclear, and that the full and the reduced crossed products coincide [3]. The boundary compactification $\overline{\Gamma} = \Gamma \cup \partial \Gamma$ carries a $\Gamma$-action as well, and it is a coarse compactification—in the sense that a ball of uniform size in $\Gamma$ becomes small in the topology of the compact space $\overline{\Gamma}$ when translated out to the boundary. The action of $\Gamma$ on the boundary $\partial \Gamma$ is minimal and exhibits a north-south dynamics, making the $C^*$-algebra crossed-product $C(\partial \Gamma) \rtimes \Gamma$ purely infinite and simple [2, 29].

If $\Gamma$ is a free group, then $C(\partial \Gamma) \rtimes \Gamma$ is canonically a Cuntz-Krieger algebra with transition matrix simply coding that a generator cannot be followed by its inverse. If $\Gamma$ is a surface group, $\Gamma = \pi_1(M)$ where $M$ is a closed surface of genus at least 2, then $\partial \Gamma$ is the 1-sphere $S^1$ and the groupoid $\Gamma \rtimes \partial \Gamma$ is strongly Morita equivalent to the holonomy groupoid of the canonical foliation of $M \times_\Gamma S^1$ by the projections to $M \times_\Gamma S^1$ of copies of $M$. 


The K-theory of the boundary crossed-products $C(\partial \Gamma) \rtimes \Gamma$ has been investigated in [11]. The result is a Gysin sequence for boundary actions which, in particular, computes the map on K-theory induced by the inclusion $i: C^*_r \Gamma \to C(\partial \Gamma) \rtimes \Gamma$. Its K-homology version, described in the last part of this article, plays an important role herein. Another K-theoretic feature of $C(\partial \Gamma) \rtimes \Gamma$ is that it exhibits Poincaré self-duality: the K-theory and the K-homology of $C(\partial \Gamma) \rtimes \Gamma$ are canonically isomorphic with a parity shift [10]. More precisely, the isomorphism is implemented by a cup-cap product map

$$\Delta \cap: K_* (C(\partial \Gamma) \rtimes \Gamma) \to K^{*+1} (C(\partial \Gamma) \rtimes \Gamma)$$

with a suitable class $\Delta \in K^1 ((C(\partial \Gamma) \rtimes \Gamma) \otimes (C(\partial \Gamma) \rtimes \Gamma))$. For the purposes of this paper, the important feature of Poincaré self-duality is the surjectivity of $\Delta \cap$, which leads to an explicit description of the K-homology of $C(\partial \Gamma) \rtimes \Gamma$ as a function of its K-theory. The proof of Poincaré duality given in [10] requires $\Gamma$ to be torsion-free, and its boundary $\partial \Gamma$ to admit a continuous self-map without any fixed points. This technical condition on the boundary is the regularity assumption in Theorem A. We do not know of any hyperbolic group which fails to be regular.

We now describe our results and our approach in more detail. Let us start with a conceptual clarification of what exactly is the boundary of a hyperbolic group. Having fixed a (non-elementary) hyperbolic group $\Gamma$, by a geometric model for $\Gamma$ we mean a hyperbolic space on which $\Gamma$ acts geometrically. Some groups come with ready-made geometric models, e.g., for a cocompact lattice in $\text{SO}(n,1)$ the $n$-dimensional real hyperbolic space $\mathbb{H}^n$ is such a space. Otherwise, a geometric model can be manufactured as the Cayley graph with respect to a finite generating set - for instance, free groups admit regular trees as geometric models. The important point is that boundaries of geometric models for $\Gamma$ are $\Gamma$-equivariantly homeomorphic. Thus, the boundary of $\Gamma$ should be understood as the boundary of any geometric model for $\Gamma$.

Let $X$ be a geometric model for $\Gamma$, so $\partial X$ is a topological avatar of the boundary of $\Gamma$. There is a natural collection of visual metrics on $\partial X$, all assigning a finite Hausdorff dimension to $\partial X$. The visual dimension of the boundary, denoted visdim $\partial X$, is the infimal Hausdorff dimension over the family of visual metrics. In particular, the visual dimension is at least as large as the topological dimension. The Hausdorff measures defined by visual metrics are comparable, in the sense that they are within constant multiples of each other. This prompts us to define a visual probability measure on $\partial X$ as a Borel probability measure which is comparable with some (equivalently, each) Hausdorff measure defined by a visual metric.

Consider now the crossed product $\mathcal{C}(\partial X) \rtimes \Gamma$, a C*-avatar of $C(\partial \Gamma) \rtimes \Gamma$. Endowing $\partial X$ with a visual probability measure $\mu$, we obtain a faithful representation of $\mathcal{C}(\partial X)$ on $L^2 (\partial X, \mu)$ by multiplication. This induces, in turn, a faithful representation $\lambda_\mu$, the left regular representation with respect to $\mu$, of $C(\partial \Gamma) \rtimes \Gamma$ on $L^2 (\Gamma, L^2 (\partial X, \mu))$. We also let $P_{\ell^2 \Gamma}$ be the projection onto $\ell^2 \Gamma$, regarded as constant functions on $\partial X$.

The basic idea for our construction of Fredholm modules, and the relationship to Poincaré self-duality for $C(\partial \Gamma) \rtimes \Gamma$ begins with the following result.

**Theorem 1.1 (Basic K-cycle).** With the above notations, the pair

$$(\lambda_\mu, P_{\ell^2 \Gamma})$$

is an odd Fredholm module over $C(\partial \Gamma) \rtimes \Gamma$. Moreover, $(\lambda_\mu, P_{\ell^2 \Gamma})$ is $p$-summable for every $p > \max \{2, \text{visdim } \partial X\}$, and it represents the Poincaré dual $\Delta \cap [1] \in K^1 (C(\partial \Gamma) \rtimes \Gamma)$ of the unit class $[1] \in K_0 (C(\partial \Gamma) \rtimes \Gamma)$.

A certain compatibility between the constructions going into Theorem[11] and the Poincaré duality of [10] implies that one can ‘twist’ the basic K-cycle above with projections or unitaries in $C(\partial \Gamma) \rtimes \Gamma$ in a certain way, generalizing Theorem[11] to cover arbitrary K-homology classes — leading to the following essential fact about K-homology classes for $C(\partial \Gamma) \rtimes \Gamma$. 

Theorem 1.2 (Twisted K-cycles). Let $\Gamma$ be regular and torsion-free. Then the following hold.

- Every class in $K^1(C(\partial \Gamma) \rtimes \Gamma)$ is represented by an odd Fredholm module of the form
  \[(\lambda_\mu, P_{t_2 \Gamma} \lambda_\mu^{op}(e) P_{t_2 \Gamma}), \quad e \text{ projection in } C(\partial \Gamma) \rtimes \Gamma.
  \]

  Moreover, the projection $e \in C(\partial \Gamma) \rtimes \Gamma$ can be chosen so that the Fredholm module is $p$-summable for every $p > \max\{2, \text{visdim } \partial X\}$.

- Every class in $K^0(C(\partial \Gamma) \rtimes \Gamma)$ is represented by a balanced even Fredholm module of the form
  \[(\lambda_\mu, P_{t_2 \Gamma} \lambda_\mu^{op}(u) P_{t_2 \Gamma} + (1 - P_{t_2 \Gamma})), \quad u \text{ unitary in } C(\partial \Gamma) \rtimes \Gamma.
  \]

  Moreover, the unitary $u \in C(\partial \Gamma) \rtimes \Gamma$ can be chosen so that the Fredholm module is $p$-summable for every $p > \max\{2, \text{visdim } \partial X\}$.

Here $\lambda_\mu^{op}$ is the right regular representation of $C(\partial X) \rtimes \Gamma$ on $\ell^2(\Gamma, L^2(\partial X, \mu))$, the conjugate of $\lambda_\mu$ by an appropriate self-adjoint unitary $J: C(\partial X) \rtimes \Gamma \to C(\partial X) \rtimes \Gamma$.

While all projections and all unitaries in $C(\partial \Gamma) \rtimes \Gamma$ yield Fredholm modules as above, one needs to restrict to a suitable smooth subalgebra in order to get finite summability. For each $p$, the twisted $K$-cycles are $p$-summable over one and the same dense $*$-subalgebra, the algebraic crossed-product Lip$(\partial X, d) \rtimes_{\text{alg}} \Gamma$ where $d$ is a visual metric on $\partial X$ of Hausdorff dimension at most $p$. We thus obtain Theorem A, in the following more precise form.

Theorem 1.3 (Uniform summability). Let $\Gamma$ be regular and torsion-free. Then the $K$-homology of $C(\partial \Gamma) \rtimes \Gamma$ is uniformly $p$-summable for every $p > \max\{2, \text{visdim } \partial X\}$.

We now turn our attention to the $K$-homology of the reduced $C^*$-algebra $C_r^* \Gamma$. Here issues related to the Baum-Connes conjecture mean that, in general, our methods only yield results about the `$\gamma$-part' of the $K$-homology of $C_r^* \Gamma$. The key tool is the following Gysin sequence, which computes the restriction map $i^*: K^*(C(\partial \Gamma) \rtimes \Gamma) \to \gamma K^*(C_r^* \Gamma)$ on $K$-homology induced by the inclusion $i: C_r^* \Gamma \to C(\partial \Gamma) \rtimes \Gamma$. This sequence is the $K$-homology version of the one in [11], which computes the map $i_*: K_*(C_r^* \Gamma) \to K_*(C(\partial \Gamma) \rtimes \Gamma)$ induced by $i$ on $K$-theory.

Theorem 1.4 (Gysin sequence for $K$-homology). Let $\Gamma$ be torsion-free. Let $\gamma \in KK_0^\Gamma(C, C)$ be the $\gamma$-element for $\Gamma$, and $\gamma K^*(C_r^* \Gamma)$ the corresponding summand of the $K$-homology of $C_r^* \Gamma$. Then there is an exact sequence

\[
0 \to K_1(B \Gamma) \to K^0(C(\partial \Gamma) \rtimes \Gamma) \xrightarrow{i^*} \gamma K^0(C_r^* \Gamma) \xrightarrow{\text{Eul}} K^1(C(\partial \Gamma) \rtimes \Gamma) \to K^1(B \Gamma) \to \gamma K^1(C_r^* \Gamma) \to 0
\]

where Eul is the map $\text{Eul}(a) = \chi(\Gamma) \text{index}(a) \text{[put]} \in K_0(B \Gamma)$, and where $\text{index}$ is the ordinary Fredholm index map $KK^\Gamma(C, C) \to \mathbb{Z}$, $\text{[put]}$ the $K$-homology class of a point.

We note that the torsion assumption could be dropped, at the expense of elaborating the sequence in the way that was done in [11]. The Gysin sequence, combined with Theorem 1.2 yields the following.

Theorem 1.5 (Twisted $K$-cycles over the reduced $C^*$-algebra). Let $\Gamma$ be regular and torsion-free. Then the following hold.

- Every class in the $\gamma$-part $\gamma K^3(C^*_\Gamma)$ of the odd $K$-homology of $C^*_\Gamma$ is represented by an odd Fredholm module of the form
  \[(\lambda, P_{t_2 \Gamma} \lambda_\mu^{op}(e) P_{t_2 \Gamma}), \quad e \text{ projection in } C(\partial \Gamma) \rtimes \Gamma.
  \]
Moreover, the projection $e \in C(\partial \Gamma) \rtimes \Gamma$ can be chosen so that the Fredholm module is $p$-summable over $\mathbb{C} \mathcal{T}$ for every $p > \max\{2, \text{visdim} \partial X\}$.

• If $\chi(\Gamma) = 0$, then, similarly, every class in the $\gamma$-part $\gamma K^0(C^*_\Gamma)$ is represented by a balanced even Fredholm module of the form

$$(\lambda, P_{\mathcal{T} \Gamma}^\gamma(n) P_{\mathcal{T} \Gamma}), \quad u \text{ unitary in } C(\partial \Gamma) \rtimes \Gamma$$

Moreover, the unitary $u \in C(\partial \Gamma) \rtimes \Gamma$ can be chosen so that the Fredholm module is $p$-summable over $\mathbb{C} \mathcal{T}$ for every $p > \max\{2, \text{visdim} \partial X\}$.

• If $\chi(\Gamma) \neq 0$ and if $\gamma_r \in \gamma K^0(C^*_\Gamma)$ is a reduced $\gamma$-element, then every class in the $\gamma$-part $\gamma K^0(C^*_\Gamma)$ is, up to an integral multiple of $\gamma_r$, represented by a balanced even Fredholm module as above.

Here $\lambda$ denotes, as usual, the regular representation of $\Gamma$. A ‘reduced’ $\gamma$-element is roughly the same as a $\gamma$-element (a class in $K^0(C^*_\Gamma)$ which factors in a certain way), but one which is defined over $C^*_\Gamma$ rather than $C^* \Gamma$.

We do not know whether, in general, there exists a reduced $\gamma$-element with a finitely summable representative, for general hyperbolic groups. We also do not know whether, in general, the $\gamma$-element acts as the identity on $K^*(C^*_\Gamma)$. But for the class of a-T-menable groups we do know, thanks to Higson - Kasparov [17], that $\gamma = 1$. Specializing Theorem 1.5 to this class, we obtain:

**Theorem 1.6** (Uniform summability for a-T-menable groups). Assume that $\Gamma$ is regular, torsion-free, and a-T-menable. Then the odd K-homology $K^1(C^*_\Gamma)$ is uniformly $p$-summable over $\mathbb{C} \mathcal{T}$ for every $p > \max\{2, \text{visdim} \partial X\}$. If $\chi(\Gamma) = 0$, then the even K-homology $K^0(C^*_\Gamma)$ is uniformly $p$-summable over $\mathbb{C} \mathcal{T}$ for every $p > \max\{2, \text{visdim} \partial X\}$. If the $\gamma$-element $\gamma_r \in \gamma K^0(C^*_\Gamma)$ can be represented by a $p(\gamma_r)$-summable Fredholm module over $\mathbb{C} \mathcal{T}$, then the even K-homology $K^0(C^*_\Gamma)$ is uniformly $p$-summable over $\mathbb{C} \mathcal{T}$ for every $p > \max\{2, \text{visdim} \partial X, p(\gamma_r)\}$.

A-T-menable hyperbolic groups include finitely generated free groups, cocompact lattices in $SO(n, 1)$ and $SU(n, 1)$, and $C'(1/6)$ small-cancellation groups – the latter by [20]. Applying the previous theorem to each one of these classes, we obtain the following consequences.

**Corollary 1.7.** Let $\Gamma$ be a finitely generated free group. Then the K-homology of $C^*_\Gamma$ is uniformly $p$-summable over $\mathbb{C} \mathcal{T}$ for every $p > 2$.

**Corollary 1.8.** If $\Gamma$ is a torsion-free cocompact lattice in $SO(n, 1)$, then the K-homology of $C^*_\Gamma$ is uniformly $n^+$-summable over $\mathbb{C} \mathcal{T}$ when $n \geq 3$, respectively $p$-summable over $\mathbb{C} \mathcal{T}$ for every $p > 2$, when $n = 2$. If $\Gamma$ is a torsion-free cocompact lattice in $SU(n, 1)$, then the K-homology of $C^*_\Gamma$ is uniformly $(2n)^+$-summable over $\mathbb{C} \mathcal{T}$.

These two corollaries rely on the existence of finitely summable representatives for the $\gamma$-element, due to Julg - Valette [20] in the free group case, Kasparov [24] in the $SO(n, 1)$ case, respectively Julg - Kasparov [19] in the $SU(n, 1)$ case. For small-cancellation groups, the outcome is less satisfactory. We have to apply the vanishing Euler characteristic criterion of Theorem 1.6 as we are lacking information on the finite summability of the $\gamma$-element, and we also do not have an explicit formula for the visual dimension.

**Corollary 1.9.** Let $\Gamma$ be a torsion-free group given by a $C'(1/6)$ presentation $(S | R)$. Then the odd K-homology $K^1(C^*_\Gamma)$ is uniformly summable over $\mathbb{C} \mathcal{T}$, and the same is true for the even K-homology $K^0(C^*_\Gamma)$ provided that $|S| - |R| = 1$.

We note that Corollary 1.8 on real and complex uniform lattices in rank 1, is in stark contrast to the higher rank situation. As shown recently by Puschnigg [35], no non-trivial K-homology class for the reduced C*-algebra of a higher rank lattice can be represented by a Fredholm
module which is finitely summable over the group algebra. This very opposite behaviour with respect to K-homological finiteness is reminiscent of another sharp distinction between rank-1 lattices and higher rank lattices, also involving a notion of finite summability: although every hyperbolic group admits a proper isometric action on an $L^p$-space for large enough $p > 1$ \cite{HI, Mc}, every isometric action of a higher rank lattice on an $L^p$-space, $p > 1$, fixes a point \cite{B}. It should be noted that, despite our strong finiteness results at the level of Fredholm modules (i.e., bounded K-cycles), neither the boundary crossed-product $C(\partial \Gamma) \rtimes \Gamma$ nor the reduced C*-algebra $C_r^* \Gamma$ support any finitely summable spectral triples (i.e., unbounded K-cycles). For $C(\partial \Gamma) \rtimes \Gamma$ this is due to the lack of a trace \cite[Thm.8]{BHI}, whereas for $C_r^* \Gamma$ the reason is the non-amenability of $\Gamma$ \cite[Thm.19]{BHI}, \cite[Thm.1 in IV.9.α]{BHI}. The present work expands, and supersedes, our preprint \cite{EMN}.

2. Preliminaries on K-homology

2.1. Fredholm modules and K-homology. We recall some definitions, while fixing notations along the way. For further details, we refer to Connes \cite[Ch.IV]{Co} and Higson - Roe \cite[Ch.8]{HR}.

Let $A$ be a unital C*-algebra. As usual, $\mathcal{B}(H)$ and $\mathcal{K}(H)$ denote the bounded operators, respectively the ideal of compact operators on a (separable) Hilbert space $H$.

**Definition 2.1** (Atiyah, Kasparov). An odd Fredholm module for $A$ is a pair $(\pi, P)$, where $\pi : A \to \mathcal{B}(H)$ is a representation, $P : H \to H$ is an essential projection in the sense that $P^* - P, P^2 - P \in \mathcal{K}(H)$, and such that $[P, \pi(a)] = P\pi(a) - \pi(a)P \in \mathcal{K}(H)$ for all $a \in A$.

An even Fredholm module for $A$ is a pair $(\pi_\pm, U)$, where $\pi_\pm : A \to \mathcal{B}(H_\pm)$ are representations, $U : H_+ \to H_-$ is an essential unitary in the sense that $U^*U - 1 \in \mathcal{K}(H_+), UU^* - 1 \in \mathcal{K}(H_-)$, and such that $\pi_+(a) - U^*\pi_-(a)U \in \mathcal{K}(H_\pm)$ for all $a \in A$. If $H_+ = H_- = H$ and $\pi_+ = \pi_- =: \pi$, then we say that the even Fredholm module is balanced, and we simply write it $(\pi, U)$.

Fredholm modules are the cycles in Kasparov’s K-homology groups, and for that reason they are also called K-cycles. Here is an outline of the odd case, leading to the odd K-homology group $K^1(A)$. The equivalence relation defined by Kasparov on odd Fredholm modules is generated by unitary equivalence, operator homotopy, and addition of degenerates. Unitary equivalence has the obvious meaning. Two Fredholm modules $(\pi, P_0)$ and $(\pi, P_1)$ are operator homotopic if there is a norm-continuous path of essential projections $(P_t)_{t \in [0, 1]}$ such that $(\pi, P_t)$ is a Fredholm module at all times $t \in [0, 1]$. Thus, the representation $\pi$ is fixed throughout an operator homotopy. A Fredholm module $(\pi, P)$ is degenerate if $P$ is a projection which commutes with the representation $\pi$. Under direct summation of K-homology classes, $K^1(A)$ is an abelian group. Modulo essentially the same equivalence relation as in the odd case, even Fredholm modules up to equivalence are the classes in the even K-homology group $K^0(A)$. Every class in $K^0(A)$ can be represented by a balanced even Fredholm module.

2.2. Finitely summable Fredholm modules. The singular values $\{s_n(T)\}_{n \geq 1}$ of a compact operator $T \in \mathcal{K}(H)$ are the eigenvalues of $|T|$, arranged in non-increasing order and repeated according to their multiplicity. The compactness of $T$ means that $s_n(T) \to 0$. For $p \geq 1$, the Schatten ideals $\mathcal{L}^p(H)$ and $\mathcal{L}^{p+}(H)$ are defined as follows:

$$\mathcal{L}^p(H) = \left\{ T \in \mathcal{K}(H) : \sum s_n(T)^p < \infty \right\}, \quad \mathcal{L}^{p+}(H) = \left\{ T \in \mathcal{K}(H) : s_n(T) = O(n^{-1/p}) \right\}.$$  

(Actually, the definition of $\mathcal{L}^{1+}(H)$ is slightly different, and it will not be used in this paper.) We have $\mathcal{L}^p(H) \subset \mathcal{L}^{p+}(H) \subset \mathcal{L}^q(H)$ for all $q > p$.

The summable Fredholm modules are those which satisfy a restricted version of Definition 2.1, in which the ideals $\mathcal{L}^p(H)$ or $\mathcal{L}^{p+}(H)$ replace the ideal of compact operators $\mathcal{K}(H)$. 

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Definition 2.2 (Connes). An odd Fredholm module $(\pi, P)$ is $p$-summable (over $\mathcal{A}$) if $P^* - P, P^2 - P \in \mathcal{L}^p(\mathcal{H})$ and $|P, \pi(a)| \in \mathcal{L}^p(\mathcal{H})$ for all $a$ in a dense subalgebra $\mathcal{A}$ of $\mathcal{A}$. A balanced even Fredholm module $(\pi, U)$ is $p$-summable (over $\mathcal{A}$) if $UU^* - 1, U^*U - 1 \in \mathcal{L}^p(\mathcal{H})$, and $[U, \pi(a)] \in \mathcal{L}^p(\mathcal{H})$ for all $a$ in a dense subalgebra $\mathcal{A}$ of $\mathcal{A}$.

The notion of $p^+$-summable Fredholm module is defined analogously. We note that $p$-summability implies $p^+$-summability, which in turn implies $q$-summability for all $q > p$.

The property that every K-homology class is representable by a finitely summable Fredholm module could be deemed as K-homological finiteness. Even sharper would be to require that finite summability can be achieved in a uniform way throughout the K-homology classes. Such a uniformity could be imposed on the degree of summability, or on the summability subalgebra, or both. We propose the following definition.

Definition 2.3. The K-homology of a C*-algebra $A$ is uniformly $p$-summable (over $\mathcal{A}$) if there is a dense subalgebra $\mathcal{A}$ of $A$ such that every K-homology class of $A$ can be represented by a Fredholm module which is $p$-summable over $\mathcal{A}$.

There is an obvious variation for $p^+$-summability. The motivating example for this strong notion of K-homological finiteness is the following: the K-homology of the commutative C*-algebra $C(M)$, where $M$ is a smooth closed manifold, is uniformly $(\dim M)^+$-summable over the smooth subalgebra $C^\infty(M)$.

3. The basic K-cycle

Throughout this section, $G$ is a discrete countable group acting by homeomorphisms on a compact metrizable space $X$. To avoid trivialities, we assume that $X$ is not a singleton. We consider the reduced crossed-product $C(X) \rtimes_r G$ associated to the topological dynamics $G \curvearrowright X$.

3.1. Left regular representation, G-expectation and G-deviation. Let $\mu$ be a Borel probability measure on $X$ with full support, meaning that non-empty open subsets have positive measure. The faithful representation of $C(X)$ on $L^2(X, \mu)$ by multiplication induces a faithful representation $\lambda_\mu$ of $C(X) \rtimes_r G$ on $\ell^2(G, L^2(X, \mu))$, the left regular representation with respect to $\mu$. In fact, the C*-algebra $C(X) \rtimes_r G$ can be defined as the norm completion of the algebraic crossed-product $C(X) \rtimes_{al} G$ in the regular representation $\lambda_\mu$. Concretely, $\lambda_\mu$ is given as follows:

$$\lambda_\mu(\phi) \left( \sum \psi_h \delta_h \right) = \sum (h^{-1} \phi) \psi_h \delta_h,$$

where $\phi \in C(X)$, $g \in G$, and $\sum \psi_h \delta_h \in \ell^2(G, L^2(X, \mu))$. The covariance relation $\lambda_\mu(g, \phi) = \lambda_\mu(g)\lambda_\mu(\phi)\lambda_\mu(g^{-1})$ holds.

On the probability space $(X, \mu)$, momentarily devoid of the $G$-action, there are two important numerical characteristics attached to a continuous functions on $X$: the expectation and the standard deviation. Namely, for $\phi \in C(X)$ we put

$$E\phi = \int \phi \, d\mu, \quad \sigma\phi = \sqrt{\mathbb{E}(|\phi|^2) - (E\phi)^2}.$$  

When we bring in the $G$-action, we are led to consider the following dynamical counterparts.

Definition 3.1. The G-expectation and the G-deviation of $\phi \in C(X)$ with respect to $\mu$ are the functions $E\phi : G \to \mathbb{C}$ and $\sigma\phi : G \to [0, \infty)$ given as follows:

$$E\phi(g) = \int g^{-1} \phi \, d\mu = \int \phi \, dg_*\mu, \quad \sigma\phi = \sqrt{\mathbb{E}(|\phi|^2) - (E\phi)^2}.$$
An explicit, and useful, formula for the $G$-deviation is

$$
\sigma(\phi(g)) = \sqrt{\frac{1}{2} \int \int |\phi(gx) - \phi(gy)|^2 \, d\mu(x) \, d\mu(y)}.
$$

As an illustration of the dynamical expectation for a non-trivial group action, consider the case of a group $G \subseteq SU(1, 1)$ acting by linear fractional transformations on the unit circle $S^1 = \{ \zeta \in \mathbb{C} : |\zeta| = 1 \}$. With respect to the normalized Lebesgue measure, $E\phi(g)$ is the value of the Poisson transform of $\phi$ on the unit disk at the point $g(0)$.

**Definition 3.2.** We say that $\mu$ has $C_0$-deviation if $\sigma(\phi) \in C_0(G)$ for all $\phi \in C(X)$, respectively $\ell^p$-deviation if $\sigma(\phi) \in \ell^p(G)$ for all $\phi$ in a dense subalgebra of $C(X)$.

### 3.2. The basic $K$-cycle.

We view $\ell^2G$ as the constant-coefficient subspace of $\ell^2(G, L^2(\mu))$. The corresponding projection $P_{\ell^2G}$ is given by coefficient-wise integration:

$$
P_{\ell^2G} \left( \sum \psi_h \delta_h \right) = \sum \left( \int \psi_h \, d\mu \right) \delta_h.
$$

We are interested in the event that $(\lambda(\mu), P_{\ell^2G})$ is a Fredholm module – or, even better, a summable one – for $(C(X) \rtimes G)$. When this happens, we refer to $(\lambda(\mu), P_{\ell^2G})$ as the basic $K$-cycle associated to $\mu$. The Fredholmness and the summability of $(\lambda(\mu), P_{\ell^2G})$ can be conveniently expressed in terms of the decay of the $G$-deviation. First, we record a general observation regarding Fredholmness and summability in the odd case.

**Lemma 3.3.** Let $A$ be a unital $C^*$-algebra, let $\pi : A \to B(H)$ be a representation, and let $P$ be a projection in $B(H)$. Denote by $s(a) := P\pi(a)P$ the corresponding compression. Then $(\pi, P)$ is a Fredholm module for $A$ if and only if $\sqrt{s(|a|^2) - |s(a)|^2} \in K(H)$ for all $a \in A$. Furthermore, $(\pi, P)$ is $p$-summable over a dense subalgebra $A \subseteq A$ if and only if $\sqrt{s(|a|^2) - |s(a)|^2} \in \ell^p(H)$ for all $a \in A$.

**Proof.** Let $\Pi(a) = (1 - P)\pi(a)P$; this is the lower left corner of the $2$-by-$2$ matrix defined by the decomposition of $\pi$ with respect to $P$. Using the relations

$$
[\pi(a), P] = \Pi(a) - \Pi(a^*)^*, \quad \Pi(a) = (1 - P)\pi(a)P
$$

we see that $(\pi, P)$ is a Fredholm module if and only if $\Pi(a) \in K(H)$ for all $a \in A$, and that $(\pi, P)$ is $p$-summable over $A$ if and only if $\Pi(a) \in \ell^p(H)$ for all $a \in A$. Now

$$
\Pi(a)^*\Pi(a) = P\pi(a^*)P(1 - P)\pi(a)P = P\pi(a^*)P - (P\pi(a)P)(P\pi(a)P) = s(a^*a) - s(a)s(a)
$$

shows that $|\Pi(a)| = \sqrt{s(|a|^2) - |s(a)|^2}$.

**Proposition 3.4.** The pair $(\lambda(\mu), P_{\ell^2G})$ is a Fredholm module if and only if $\mu$ has $C_0$-deviation. If $\mu$ has $\ell^p$-deviation, then $(\lambda(\mu), P_{\ell^2G})$ is a $p$-summable Fredholm module; for $p \geq 2$, the converse holds.

**Proof.** The projection $P_{\ell^2G}$ compresses the space restriction $\lambda|_{C(X)}$ to multiplication by the $G$-expectation on $\ell^2G$:

$$
P_{\ell^2G}\lambda(\mu)(g)P_{\ell^2G} = M(E\phi(g))
$$

for all $\phi \in C(X)$. Hence for $s(\mu)(\phi) := P_{\ell^2G}\lambda(\mu)(\phi)P_{\ell^2G}$ we have

$$
\sqrt{s(\mu)(\phi^2) - |s(\mu)(\phi)|^2} = M(\sigma(\phi)).
$$

By the proof of Lemma 3.3, $\mu$ has $C_0$-deviation if and only if $|\lambda(\mu)(\phi), P_{\ell^2G}]$ is compact for all $\phi \in C(X)$. As $P_{\ell^2G}$ commutes with the group restriction $\lambda|_{G}$, the latter condition is equivalent
to having $[\lambda_\mu, P_{C^2G}]$ compact for all $a \in C(X) \rtimes_{\text{alg}} G$, which is equivalent to $(\lambda\mu, P_{C^2G})$ being Fredholm.

The summable analogue is argued in a similar way. For sufficiency, assume that $\mu$ has $\ell^p$-deviation. Then there is a $G$-invariant, dense $*$-subalgebra $A(X) \subseteq C(X)$ such that $\sigma \phi \in \ell^p G$ for all $\phi \in A(X)$. As above, we deduce that $[\lambda_\mu, P_{C^2G}]$ is a $p$-summable operator for all $a \in A(X) \rtimes_{\text{alg}} G$. Thus $(\lambda\mu, P_{C^2G})$ is $p$-summable.

For the converse, we bring in another expectation, namely the bounded linear map $E : C(X) \rtimes G \to C(X)$ defined by $E(\sum \phi g) = \phi_1$ over $C(X) \rtimes_{\text{alg}} \Gamma$. We claim that

$$\|\Pi(a)\delta_h\|_2 \geq \sigma(E(a))(h)$$

for all $h \in G$ and $a \in C(X) \rtimes G$. Indeed, using along the way the fact that $\Pi(\phi g_2)^* \Pi(\phi g_1)$ is a multiplication operator on $\ell^2 G$, we have:

$$\langle \Pi(a)\delta_h, \Pi(a)\delta_h \rangle = \sum_{g_1, g_2} \langle \Pi(\phi g_2)\delta_{g_1 h}, \Pi(\phi g_2)\delta_{g_2 h} \rangle = \sum_{g_1, g_2} \langle \Pi(\phi g_2)^* \Pi(\phi g_1)\delta_{g_1 h}, \delta_{g_2 h} \rangle$$

$$= \sum_g \langle \Pi(\phi g)^* \Pi(\phi g)\delta_{gh}, \delta_{gh} \rangle = \sum_g \langle M(\sigma \phi g)^2 \delta_{gh}, \delta_{gh} \rangle$$

$$= \sum_g (\sigma \phi g)^2(gh) \geq (\sigma \phi_1)^2(h) = (\sigma(E(a))(h))^2$$

Now assume that $(\lambda\mu, P_{C^2G})$ is a $p$-summable Fredholm module for $C(X) \rtimes G$. Then $\Pi(a)$ is a $p$-summable operator for all $a$ in a dense subalgebra $A$ of $C(X) \rtimes G$. For $p \geq 2$, the $p$-summability of $\Pi(a)$ implies the $p$-summability of $\{\|\Pi(a)\xi\|_2\}_{\xi \in \ell^p}$ for any orthonormal system $(\xi_\ell)_{\ell \in I}$ (Theorem 1.18). In particular $\{\|\Pi(a)\delta_h\|_2\}_{h \in G}$ is $p$-summable, so $\sigma(E(a)) \in \ell^p G$ by (3.3). Thus, we have shown that $\sigma \phi \in \ell^p G$ for all $\phi \in \mathcal{E}(A)$. It follows that $\{\phi \in C(X) : \sigma \phi \in \ell^p G\}$, which is always a subalgebra of $C(X)$, is dense. We conclude that $\mu$ has $\ell^p$-deviation.

At the current level of generality, we cannot address the question whether $(\lambda\mu, P_{C^2G})$, when a Fredholm module, is homologically non-trivial or not. It is, however, clear that it is non-degenerate, given our assumptions that $X$ is not a singleton and $\mu$ is fully-supported.

3.3. $C_0$-deviation and the convergence property. Let $\text{Prob}(X)$ denote the space of Borel probability measures on $X$, and equip $\text{Prob}(X)$ with the weak* convergence induced by $C(X)$: by definition, $\nu_\mu \to \nu$ if $\int \phi d\nu_\mu \to \int \phi d\nu$ for all $\phi \in C(X)$. The space $\text{Prob}(X)$ is compact. In particular, push-forwards of $\mu$ by elements of $G$ must accumulate. We make the following definition.

**Definition 3.5.** The probability measure $\mu$ is said to have the convergence property if the accumulation points of the $G$-orbit $G\mu \subseteq \text{Prob}(X)$ are all point measures.

We think of the convergence property for a probability measure as a measurable analogue of an established notion in topological dynamics, that of a convergence group action. (Hence our choice of terminology.) Let us recall the definition, originally due to Gehring and Martin for the case of group actions on spheres or closed balls, and then subsequently extended by Tukia, Freden, Bowditch to the general case of group actions on compact metrizable spaces. The action of $G$ on $X$ is said to be a convergence action if the following holds: for each sequence $(g_n) \subseteq G$ with $g_n \to \infty$, there is a subsequence $(g_{n_k})$ with attracting and repelling points $x^+, x^- \in X$ in the sense that $g_{n_k} z \to x^+$ uniformly outside neighbourhoods of $x^-$. We note the following simple fact.

**Proposition 3.6.** Let $G \curvearrowright X$ be a convergence action. If points in $X$ are $\mu$-negligible, then $\mu$ has the convergence property.
Proof. Let \((g_n) \ast \mu\) converge in \(\text{Prob}(X)\), where \(g_n \to \infty\) in \(G\). Without loss of generality \((g_n)\) has attracting and repelling points \(x^+, x^- \in X\). We claim that \((g_n) \ast \mu \to \delta_{x^+}\). Indeed, let \(\phi \in C(X)\). Then \(g_n^{-1} \ast \phi\) converges pointwise to the constant function \(\phi(x^+)\) on \(X - \{x^+\}\). As \(x^-\) is \(\mu\)-negligible, we get \(\int \phi \, d(g_n) \ast \mu = \int g_n^{-1} \ast \phi \, d\mu \to \phi(x^+)\) by Lebesgue’s dominated convergence theorem. \(\square\)

The convergence property is relevant for our discussion, in light of the following characterization.

**Proposition 3.7.** The probability measure \(\mu\) has \(C_0\)-deviation if and only if it has the convergence property.

*Proof.* Assume that \(\mu\) has \(C_0\)-deviation, and let \(\nu \in \text{Prob}(X)\) be the limit of a sequence \((g_n) \ast \mu\) with \(g_n \to \infty\) in \(G\). For each \(\phi \in C(X)\) we have, on the one hand, that \(\sigma \phi(g_n)\) converges to 0, and on the other hand that \(\sigma \phi(g_n)\) converges to the standard deviation of \(\phi\) with respect to \(\nu\). Therefore, \(\int |\phi|^2 \, d\nu = \int |\phi| \, d\nu^2\) for all \(\phi \in C(X)\). This continues to hold throughout \(L^2(X, \nu)\), by the density of \(C(X)\) in \(L^2(X, \nu)\) – Borel probability measures on compact metrizable spaces are automatically Radon. Taking characteristic functions of measurable sets, we see that \(\nu\) is \(\{0, 1\}\)-valued. But the only \(\{0, 1\}\)-valued Borel probability measures on \(X\) are the point measures: choosing a compatible metric on \(X\), there exists a sequence of full-measure balls with radius converging to 0, hence a point having full measure.

The converse implication is left to the reader. \(\square\)

In \cite{7} we will show that suitable measures on the boundary of a Gromov hyperbolic group have the convergence property with respect to the boundary action of the group (which is a convergence action.)

### 3.4. Double ergodicity and \(\ell^p\)-deviation.

We address the condition \(p \geq 2\), encountered in Proposition \ref{3.3}. Namely, we show that double ergodicity of \(\mu\) is an obstruction to having \(\ell^p\)-deviation with \(p \leq 2\).

**Proposition 3.8.** If \(\mu \times \mu\) is ergodic for the diagonal action of \(G\) on \(X \times X\), and \(X\) has no isolated points, then a function \(\phi \in C(X)\) with \(\sigma_G \phi \in \ell^2 G\) must be constant. In particular, if \(\mu\) has \(\ell^p\)-deviation then \(p > 2\).

*Proof.* Arguing by contradiction, we assume that \(\phi \in C(X)\) is a non-constant function with the property that \(\sigma \phi \in \ell^2 G\). By \(\ell^1\), we have

\[
\|\sigma \phi\|^2_{\ell^2 G} = \frac{1}{2} \int \sum_{g \in G} |\phi(gx) - \phi(gy)|^2 \, d\mu(x) \, d\mu(y)
\]

Therefore \(S(x, y) = \sum_{g \in G} |\phi(gx) - \phi(gy)|^2\) defines a \(G\)-invariant \(L^2\) map on \(X \times X\). By ergodicity, \(S\) is a.e. constant, say \(S(x, y) = C\) for almost all \((x, y) \in X \times X\).

There exists \(c > 0\) such that the open subset \(V = \{(x, y) : |\phi(x) - \phi(y)| > c\} \subseteq X \times X\) is non-empty. As \(X \times X\) has no isolated points, for each positive integer \(N\) there exist disjoint, non-empty open subsets \(U_1, \ldots, U_N \subseteq V\). Using again the ergodicity assumption, we have that each \(G \cdot U_i = \cup_{g \in G} gU_i\) is either negligible or of full measure. Since non-empty open subsets of \(X \times X\) have positive measure, the latter alternative must occur. It follows that \(\cap_{i=1}^N G \cdot U_i\) has full measure. Let \((x, y)\) in \(\cap_{i=1}^N G \cdot U_i\) with \(S(x, y) = C\). Thus, for each \(i\) we have some \(g_i \in G\) such that \((g_ix, g_iy) \in U_i\). Now the \(g_i\)’s are distinct since the \(U_i\)’s are disjoint, and \(|\phi(g_ix) - \phi(g_iy)| > c\) since \(U_i \subseteq V\), so

\[
C = S(x, y) \geq \sum_{i=1}^N |\phi(g_ix) - \phi(g_iy)|^2 > Nc^2.
\]
As \( N \) is arbitrary, this is a contradiction. \( \square \)

4. Further properties of the basic K-cycle

We now investigate the behaviour of the basic K-cycle under two operations: changing the measure \( \mu \), respectively passing to a finite-index subgroup of \( G \). We keep the notations of the previous section.

4.1. Comparable measures. The pair \( (\lambda_\mu, P_{\ell^2 G}) \) is constructed in reference to the measure \( \mu \), which is not part of the given topological setting. Nevertheless, its relevant features – Fredholmness, degree of summability, K-homology class – are canonical over the measure class of \( \mu \).

The suitable equivalence here is the following: a Borel probability measure \( \mu' \) on \( X \) is said to be comparable to \( \mu \) if \( \mu' \asymp \mu \), in the sense that \( C_1 \mu \leq \mu' \leq C_2 \mu \) for some positive constants \( C_1, C_2 \).

Clearly, comparability is finer than the usual equivalence of measures which, we recall, means that each measure is absolutely continuous with respect to the other. Formula (3.1) shows that comparable measures have comparable \( G \)-deviations, hence the following:

**Proposition 4.1.** Let \( \mu \) and \( \mu' \) be comparable probability measures. Then \( (\lambda_\mu, P_{\ell^2 G}) \) is a Fredholm module if and only if \( (\lambda_{\mu'}, P_{\ell^2 G}) \) is a Fredholm module. For \( p \geq 2 \), \( (\lambda_\mu, P_{\ell^2 G}) \) is \( p \)-summable if and only if \( (\lambda_{\mu'}, P_{\ell^2 G}) \) is \( p \)-summable.

Most importantly, basic K-cycles associated to comparable measures define one and the same homology class:

**Proposition 4.2.** Let \( \mu \) and \( \mu' \) be comparable probability measures having \( C_0 \)-deviation. Then the Fredholm modules \( (\lambda_\mu, P_{\ell^2 G}) \) and \( (\lambda_{\mu'}, P_{\ell^2 G}) \) are K-homologous.

**Proof.** Let \( \rho = d\mu' / d\mu \) be the Radon-Nikodym derivative, so \( \rho \) is essentially bounded from above and from below by the comparability constants of \( \mu \) and \( \mu' \). First, we have a unitary

\[
U : \ell^2(G, L^2(X, \mu)) \to \ell^2(G, L^2(X, \mu)), \quad \sum \psi_h \delta_h \mapsto \sum \sqrt{\rho} \psi_h \delta_h
\]

which intertwines the corresponding regular representations of \( C(X) \rtimes_r G \), that is, \( U \lambda_\mu U^* = \lambda_{\mu'} \). We may therefore exchange \( (\lambda_{\mu'}, P_{\ell^2 G}) \) for \( (\lambda_\mu, U P_{\ell^2 G} U^*) \), where the notation \( P_{\ell^2 G} \) is used in order to emphasize the dependence on \( \mu' \). We now claim that the Fredholm modules \( (\lambda_{\mu}, U P_{\ell^2 G} U^*) \) and \( (\lambda_\mu, P_{\ell^2 G}) \) are operator homotopic. Note that

\[
U P_{\ell^2 G} U^* \left( \sum \psi_h \delta_h \right) = \sum \sqrt{\rho} \left( \int \sqrt{\rho} \psi_h \, d\mu \right) \delta_h,
\]

and that \( \sqrt{\rho} \in L^\infty(X, \mu) \) with \( \| \sqrt{\rho} \|_{L^2(X, \mu)} = 1 \). For \( \eta \in L^\infty(X, \mu) \) satisfying \( \| \eta \|_{L^2(X, \mu)} = 1 \), let \( M(\eta) \) be the corresponding multiplication operator on \( \ell^2(G, L^2(X, \mu)) \). Then

\[
P(\eta) = M(\eta) P_{\ell^2 G} M(\eta), \quad \sum \psi_h \delta_h \mapsto \sum \sqrt{\eta} \left( \int \eta \psi_h \, d\mu \right) \delta_h
\]

is a projection, namely the projection of \( \ell^2(G, L^2(X, \mu)) \) onto \( M(\eta) \ell^2 G \). We have \( |P(\eta), \lambda_\mu| = M(\eta) |P_{\ell^2 G}, \lambda_\mu| M(\eta) \) since \( M(\eta) \) and \( M(\eta) \) commute with \( \lambda_\mu \), so \( (\lambda_\mu, P(\eta)) \) is a Fredholm module. On the other hand, we have \( \| P(\eta_1) - P(\eta_2) \| \leq 2 \| \eta_1 - \eta_2 \|_{L^2(X, \mu)} \); this follows from the fact that

\[
\| \eta_1 \int \eta_1 \psi \, d\mu - \eta_2 \int \eta_2 \psi \, d\mu \|_2 \leq 2 \| \eta_1 - \eta_2 \|_2 \| \psi \|_2
\]

for all \( \psi \in L^2(X, \mu) \). Now let \( \eta(t) = (\cos t) \, 1 + (i \sin t) \sqrt{\rho} \), where \( 0 \leq t \leq \pi/2 \). Then \( \eta(t) \in L^\infty(X, \mu) \), and \( \eta(t) \) describes a continuous path in the unit sphere of \( L^2(X, \mu) \) between the constant function 1 and \( i \sqrt{\rho} \). Consequently, \( P(\eta(t)) \) describes a norm-continuous path between \( P(1) = P_{\ell^2 G} \) and \( P(i \sqrt{\rho}) = P(\sqrt{\rho}) = U P_{\ell^2 G} U^* \). \( \square \)
4.2. Finite-index subgroups. Let \( H \leq G \) be a subgroup. Restriction of representations from \( C(X) \rtimes_r G \) to \( C(X) \rtimes_r H \) takes Fredholm modules for \( C(X) \rtimes_r G \) to Fredholm modules for \( C(X) \rtimes_r H \), and it defines a natural homomorphism of abelian groups

\[
\text{res} : K^1(C(X) \rtimes_r G) \to K^1(C(X) \rtimes_r H).
\]

Assume that \( \mu \) has \( C_0 \)-deviation. One the one hand, restricting \( (\lambda^G_\mu, P_{\ell^2 G}) \) yields a Fredholm module for \( C(X) \rtimes_r H \). On the other hand, we can form the Fredholm module \( (\lambda^H_\mu, P_{\ell^2 H}) \) for \( C(X) \rtimes_r H \). The homological relation between these two Fredholm modules for \( C(X) \rtimes_r H \) is particularly simple in the case when \( H \) has finite index in \( G \).

**Proposition 4.3.** Assume that \( \mu \) has \( C_0 \)-deviation, and that \( \{g_\mu \}_{g \in G} \) forms a family of mutually comparable measures. If \( H \) is a finite-index subgroup of \( G \), then

\[
\text{res} \left[ (\lambda^G_\mu, P_{\ell^2 G}) \right] = [G : H] \left[ (\lambda^H_\mu, P_{\ell^2 H}) \right]
\]

in \( K^1(C(X) \rtimes_r H) \).

**Proof.** Put \( n = [G : H] \), and pick a transversal \( t_1, \ldots, t_n \) for the right \( H \)-cosets. The coset decomposition \( \ell^2(G, L^2(X, \mu)) = \bigoplus^n \ell^2(\text{Ht}_i, L^2(X, \mu)) \) yields

\[
\text{res} \left[ (\lambda^G_\mu, P_{\ell^2 G}) \right] = \bigoplus^n \left[ (\lambda^G_{\mu, t_i}, P_{\ell^2(\text{Ht}_i)}) \right]
\]

in \( K^1(C(X) \rtimes_r H) \), where \( \lambda_{t_i} \) denotes the representation of \( C(X) \rtimes_r H \) on \( \ell^2(\text{Ht}_i, L^2(X, \mu)) \).

Now consider \( (\lambda_{t_i}, P_{\ell^2(\text{Ht}_i)}) \) for \( t \in \{t_1, \ldots, t_n\} \). The unitary

\[
R_{t_i} : \ell^2(H, L^2(X, \mu)) \to \ell^2(\text{Ht}_i, L^2(X, \mu)), \quad \sum \psi_h \delta_h \mapsto \sum \psi_h \delta_{ht_i}
\]

implements an equivalence between \( (\lambda_{t_i}, P_{\ell^2(\text{Ht}_i)}) \) and \( (R^*_{t_i} \lambda_{t_i} R_{t_i}, P_{\ell^2 H}) \). The representation \( R^*_{t_i} \lambda_{t_i} R_{t_i} \) on \( \ell^2(H, L^2(X, \mu)) \) is given by

\[
R^*_{t_i} \lambda_{t_i} R_{t_i}(\phi) \left( \sum \psi_h \delta_h \right) = \sum t^{-1}(h^{-1}, \phi) \psi_h \delta_h, \quad R^*_{t_i} \lambda_{t_i} R_{t_i}(h') \left( \sum \psi_h \delta_h \right) = \sum \psi_{ht_i} \delta_{h'ht_i}
\]

for \( \phi \in C(X) \) and \( h' \in H \). Next, the unitary

\[
V_{t_i} : \ell^2(H, L^2(X, \mu)) \to \ell^2(H, L^2(X, t_* \mu)), \quad \sum \psi_h \delta_h \mapsto \sum (t_* \psi_h) \delta_h
\]

makes \( (R^*_{t_i} \lambda_{t_i} R_{t_i}, P_{\ell^2 H}) \) and \( (\lambda^H_{t_i, \mu}, P_{\ell^2 H}) \) equivalent. On the other hand, the assumption that \( \{g_\mu \}_{g \in G} \) consists of mutually comparable measures implies, in light of Proposition 4.2, that \( (\lambda^H_{t_i, \mu}, P_{\ell^2 H}) \) and \( (\lambda^H_\mu, P_{\ell^2 H}) \) are homologous. Summarizing, we have

\[
[(\text{res} (\lambda^G_\mu, P_{\ell^2 G}))] = \bigoplus^n [(\lambda^H_\mu, P_{\ell^2 H})]
\]

in \( K^1(C(X) \rtimes_r H) \), as desired. \( \square \)

5. Metric-measure structure on the boundary of a hyperbolic space

This section is devoted to the metric-measure structure on the boundary of a hyperbolic space in the sense of Gromov [18], for purposes of establishing that appropriate families of measures on the boundary have the convergence property of Proposition 4.3, and making various fine estimates.

In [5.1] we recall some basic facts on hyperbolic spaces and their boundaries. In [5.2] we focus on the family of visual metrics, and their induced Hausdorff measures, on the boundary of a hyperbolic space. The content of these two subsections is essentially standard [14], [39, Sec.5]. The next subsection, [5.3] on geometric group actions is important for technical and conceptual reasons, where we describe results of Coornaert [9] on Hausdorff measures for visual metrics.
5.1. The boundary of a hyperbolic space. Let \((X, d)\) be a geodesic space which is proper, in the sense that closed balls are compact. The Gromov product of \(x, y \in X\) with respect to \(o \in X\) is defined by the formula
\[
(x, y)_o := \frac{1}{2}(d(o, x) + d(o, y) - d(x, y)).
\]

**Definition 5.1** (Gromov). The space \(X\) is hyperbolic if there exists a constant \(\delta \geq 0\) such that, for all \(x, y, z, o \in X\), we have
\[
(x, y)_o \geq \min \{(x, z)_o, (y, z)_o\} - \delta.
\]

Let \(X\) be hyperbolic, and fix a basepoint \(o \in X\). A sequence \((x_i)\) converges to infinity if \((x_i, x_j)_o \to \infty\) as \(i, j \to \infty\). Two sequences \((x_i), (y_i)\) converging to infinity are asymptotic if \((x_i, y_i)_o \to \infty\) as \(i \to \infty\). The asymptotic relation is an equivalence on sequences converging to infinity. A basepoint change modifies the Gromov product by a uniformly bounded amount, so convergence to infinity and the asymptotic relation are independent of the chosen basepoint \(o \in X\). The boundary of \(X\), denoted \(\partial X\), is the set of asymptotic classes of sequences converging to infinity. A sequence \((x_i) \subseteq X\) converges to \(\xi \in \partial X\) if \((x_i)\) converges to infinity, and the asymptotic class of \((x_i)\) is \(\xi\).

The Gromov product on \(\partial X \times \partial X\) is defined as follows:
\[
(\xi, \xi')_o := \inf \left\{ \liminf (x_i, x'_i)_o : x_i \to \xi, x'_i \to \xi' \right\}
\]
If \(\xi = \xi'\), then \((\xi, \xi')_o = \infty\). If \(\xi \neq \xi'\), then the sequence \((x_i, x'_i)_o\) is bounded whenever \(x_i \to \xi\) and \(x'_i \to \xi'\), hence \((\xi, \xi')_o < \infty\). It turns out that
\[
(\xi, \xi')_o \leq \liminf (x_i, x'_i)_o \leq \limsup (x_i, x'_i)_o \leq (\xi, \xi')_o + 2\delta \quad (x_i \to \xi, x'_i \to \xi').
\]

Similarly, the Gromov product on \(X \times \partial X\) is defined by setting
\[
(x, \xi)_o := \inf \left\{ \liminf (x_i, x_i)_o : x_i \to \xi \right\}.
\]

In this case we have:
\[
(\xi, \xi')_o \leq \liminf (x_i, x_i)_o \leq \limsup (x_i, x_i)_o \leq (x, \xi)_o + \delta \quad (x_i \to \xi)
\]

In particular, (5.1) and (5.2) show that one could take sup instead of inf, or lim sup instead of lim inf, in the definition of the Gromov product on \(\partial X \times \partial X\), respectively \(X \times \partial X\); all these variations would be within \(2\delta\), respectively \(\delta\), of each other.

5.2. Visual metrics. Equipped with a canonical topology defined in terms of the Gromov product, the boundary \(\partial X\) is compact and metrizable (see [14 Ch.7, §2]). But the metric structure on \(\partial X\), which is of great importance in this paper, is a more subtle issue.

**Definition 5.2.** A visual metric on \(\partial X\) is a metric \(d_\epsilon\) satisfying \(d_\epsilon \preceq \exp(-\epsilon(\cdot, \cdot)_o)\) for some \(\epsilon > 0\), called the visual parameter of \(d_\epsilon\).

This definition is independent of the chosen basepoint \(o \in X\), and every visual metric determines the canonical topology on \(\partial X\). If \(d_\epsilon\) is a visual metric, then so is the snowflaked metric \(d_\epsilon^{\alpha}\) for any \(\alpha \in (0, 1]\): consequently, if \(\epsilon\) is a visual parameter then any \(\epsilon' \in (0, \epsilon]\) is a visual parameter as well. In general, there is no natural choice of visual metric on \(\partial X\).

**Fact 5.3** (Scaling). Let \(d_\epsilon\) and \(d_{\epsilon'}\) be two visual metrics. Then:
- \(d_\epsilon\) and \(d_{\epsilon'}\) are Hölder equivalent: \(d_\epsilon^{1/\epsilon} \preceq d_{\epsilon'}^{1/\epsilon'}\);
- the corresponding Hausdorff dimensions are inversely proportional to the visual parameter: \(\epsilon \cdot \text{hdim}(\partial X, d_\epsilon) = \epsilon' \cdot \text{hdim}(\partial X, d_{\epsilon'})\);
- the corresponding Hausdorff measures are comparable: \(\mu_\epsilon \preceq \mu_{\epsilon'}\).
A priori, Hausdorff dimensions and Hausdorff measures corresponding to visual metrics could degenerate to 0 or $\infty$. In other words, a visual metric need not generate a meaningful measure-theoretic structure. As we shall see in §5.3, a geometric group action on $X$ brings a remarkable measure-theoretic regularity to the visual structure of $\partial X$.

Visual metrics do exist, provided that the visual parameter is small with respect to $1/\delta$ where $\delta$ is a constant of hyperbolicity. Furthermore, there is a companion metric-like map on $X \times \partial X$, which is visual in the corresponding way [39, Prop.5.16]:

**Fact 5.4** (Small visual range). Let $\epsilon > 0$ be such that $\epsilon \delta < 1/5$. Then:

- there exists a visual metric $d_\epsilon$ on $\partial X$, having visual parameter $\epsilon$;
- there exists $d_\epsilon : X \times \partial X \to [0, \infty)$ satisfying
  \[
  \frac{1}{2} \exp(-\epsilon(x, \xi)_o) \leq d_\epsilon(x, \xi) \leq \exp(-\epsilon(x, \xi)_o)
  \]
  and
  \[
  |d_\epsilon(x, \xi) - d_\epsilon(x, \xi')| \leq d_\epsilon(\xi, \xi') \leq d_\epsilon(x, \xi) + d_\epsilon(x, \xi')
  \]
  for all $x \in X$ and $\xi, \xi' \in \partial X$.

The small range for visual parameters is by no means optimal, and we have to allow for the possibility that visual metrics may exist for parameters outside of the small range - which is, in fact, what we did by considering the entire ‘cone’ of visual metrics on $\partial X$. Statements about visual metrics on $\partial X$ are sometimes proved by first dealing with visual parameters in the small range, and then extended by scaling (Fact 5.3).

### 5.3. Geometric group actions.

A fundamental feature of hyperbolicity is its invariance under quasi-isometries. This allows for the following perspective on hyperbolic groups.

**Definition 5.5** (Gromov). A group $\Gamma$ is hyperbolic if it acts geometrically, that is to say isometrically, properly and cocompactly, on a hyperbolic space.

We refer to a space carrying a geometric action of a hyperbolic group $\Gamma$ as a geometric model for $\Gamma$. For example, Cayley graphs of $\Gamma$ with respect to various finite generating sets are geometric models for $\Gamma$. There could be, however, more natural geometric models for a given hyperbolic group, e.g., for a surface group of genus at least 2 the natural geometric model is the standard hyperbolic plane. Geometric models for $\Gamma$ have $\Gamma$-equivariantly homeomorphic boundaries, and each one of them is a topological realization of $\partial \Gamma$.

Now let $X$ be a hyperbolic space admitting a geometric action of a (hyperbolic) group $\Gamma$. In what follows, we assume that $\Gamma$ is non-elementary, that is, $\Gamma$ is neither finite, nor virtually infinite cyclic. In terms of the space $X$, the non-elementary hypothesis on $\Gamma$ means that $\partial X$ is infinite as a set.

The action of $\Gamma$ on $X$ extends to the boundary $\partial X$. The boundary action is a convergence action, in the sense of §5.3. We also have $(gx, gx')_o \geq (x, x')_o - d(o, go)$ for all $g \in \Gamma$ and $x, x' \in X$, which implies that

\[
(g\xi, g\xi')_o \geq (\xi, \xi')_o - d(o, go)
\]

for all $g \in \Gamma$ and $\xi, \xi' \in \partial X$. Therefore $\Gamma$ acts by Lipschitz maps on $(\partial X, d_\epsilon)$ for any choice of visual metric $d_\epsilon$ on $\partial X$.

**Definition 5.6.** The exponent, or the volume entropy of $X$ is the finite positive number given by

\[
e_X = \inf \left\{ s > 0 : \sum_{g \in \Gamma} \exp(-sd(o, go)) < \infty \right\} = \limsup_{R \to \infty} \frac{1}{R} \ln \left| \{ g \in \Gamma : d(o, go) \leq R \} \right|.
\]
The two formulas give two interpretations of the exponent, namely critical exponent as well as growth exponent. As the notation $e_X$ already suggests, the definition is independent of the basepoint $o \in X$ and of the group $\Gamma$ acting geometrically on $X$.

The Patterson-Sullivan theory developed by Coornaert in [9] plays a crucial role in understanding the growth of $\Gamma$-orbits in $X$, and the Hausdorff dimensions and measures associated to visual metrics on $\partial X$. The following hold.

**Fact 5.7** (Orbit growth). Let $o$ be a basepoint in $X$. Then $|\{g \in \Gamma : d(o, go) \leq R\}| \approx \exp(e_X R)$.

**Fact 5.8** (Hausdorff measure and dimension). Let $d_\epsilon$ be a visual metric on $\partial X$. Then:

- the Hausdorff dimension $\text{hdim}(\partial X, d_\epsilon)$ equals $e_X/\epsilon$;
- the Hausdorff measure $\mu_\epsilon$ is Ahlfors regular, that is $\mu_\epsilon(B_r) \approx r^{\text{hdim}(\partial X, d_\epsilon)}$ uniformly over all closed balls $B_r$ of radius $0 \leq r \leq \text{diam}(\partial X, d_\epsilon)$.

Both results are due to Coornaert. Fact 5.7 is [9, Thm.7.2]. For sufficiently small visual parameters, Fact 5.8 follows from [9, Prop.7.4, Cor.7.5, Cor.7.6]; using Fact 5.3 it extends to arbitrary visual parameters.

**Definition 5.9.** A visual probability measure on $\partial X$ is a Borel probability measure $\mu$ satisfying $\mu \approx \mu_\epsilon$ for some (equivalently, each) Hausdorff measure $\mu_\epsilon$ defined by a visual metric $d_\epsilon$.

### 6. Visual dimension of the boundary of a hyperbolic space

Let $X$ be a hyperbolic space. We cannot really assign a Hausdorff dimension to the boundary $\partial X$, since there is no canonical choice of metric on $\partial X$. Instead, we work with the following notion of dimension.

**Definition 6.1.** The visual dimension of $\partial X$, denoted $\text{visdim} \partial X$, is the infimal Hausdorff dimension of $(\partial X, d)$ as $d$ runs over the visual metrics on $\partial X$.

One could take this notion a step further, and define a visual dimension for the group $\Gamma$ as the infimal visual dimension of $\partial X$, as $X$ runs over the (isometry classes) of geometric models for $\Gamma$. Our results are, in fact, most conveniently expressed in terms of such a visual dimension for the group.

We illustrate Definition 6.1 on the following important examples.

**Example 6.2** (Regular trees). Let $T$ be a regular tree of degree greater than 2. Topologically, the boundary $\partial T$ is a Cantor set. The Gromov product $(\cdot, \cdot)_\bullet$ on $T$ extends canonically and continuously to $\partial T$, and $\exp(\cdot(\cdot)_\bullet)$ is an ultrametric on $\partial T$. Each $\epsilon > 0$ is a visual parameter, so $\text{visdim} \partial T = 0$.

**Example 6.3** (Rank-1 symmetric spaces). The boundary of $\mathbb{H}^n$, where $n \geq 2$, is the sphere $S^{n-1}$. The usual spherical metric is a visual metric with visual parameter $\epsilon = 1$. We claim that $\epsilon = 1$ is the largest possible parameter. Indeed, the Lipschitz functions with respect to a visual metric are dense in $C(S^{n-1})$. On the other hand, they are the $\epsilon$-Hölder functions with respect to the spherical metric, $\epsilon$ being the visual parameter. Now observe that, on a geodesic metric space, only the constant functions are $\epsilon$-Hölder for $\epsilon > 1$. Thus $\text{visdim} \partial \mathbb{H}^n = n - 1$, the Hausdorff dimension with respect to the spherical metric.

More generally, let us consider the (non-compact) rank-1 symmetric spaces. These are the real, complex, quaternionic, or octonionic hyperbolic spaces $\mathbb{H}^n_K$, where $n \geq 2$ respectively $n = 2$ in the exceptional octonionic case. Put $k = \dim K \in \{1, 2, 4, 8\}$. Topologically, the boundary of $\mathbb{H}^n_K$ is a sphere of dimension $nk - 1$. The standard metric on $\partial \mathbb{H}^n_K$, the so-called Carnot metric, is a visual metric with visual parameter $\epsilon = 1$. As the Carnot metric is geodesic, no parameter greater than 1 is a visual parameter. Therefore $\text{visdim} \partial \mathbb{H}^n_K = \text{hdim} \partial \mathbb{H}^n_K$, the
Hausdorff dimension of \( \partial \mathbb{H}^n_K \) equipped with the Carnot metric, which is explicitly given by the Mitchell- Pansu formula: \( \text{ldim } \partial \mathbb{H}^n_K = \text{topdim } \partial \mathbb{H}^n_K + k - 1 ( = nk + k - 2) \).

Clearly \( \text{visdim } \partial X \geq \text{topdim } \partial X \), as the topological dimension of a metric space is at most the Hausdorff dimension with respect to a compatible metric. In fact, the following chain of inequalities holds:

\[
\text{visdim } \partial X \geq \text{A-confdim } \partial X \geq \text{confdim } \partial X \geq \text{topdim } \partial X.
\]

The conformal dimension of \( \partial X \), denoted \( \text{confdim } \partial X \), is a notion of metric dimension which only depends on the quasi-isometry type of \( X \). It resolves the metric ambiguity at the boundary by taking all possible metrics which are equivalent to a visual metric in a suitable sense. The original definition, due to Pansu, uses quasi-conformal equivalence; more recently, the closely related quasi-Möbius (equivalently, quasi-symmetric) equivalence seems to be favored. Then \( \text{confdim } \partial X \) is defined as the infimal Hausdorff dimension of \( (\partial X, d) \) as \( d \) runs over all metrics which are equivalent to a visual metric. See \[22, \text{Section 14}\].

From a measure-theoretic point of view, the equivalence relation used for defining the conformal dimension is too loose. For hyperbolic spaces admitting geometric group actions, the notion of Ahlfors conformal dimension strikes a compromise by restricting the equivalence relation to Ahlfors regular metrics. Namely, \( \text{A-confdim } \partial X \) is defined as the infimal Hausdorff dimension of \( (\partial X, d) \) as \( d \) runs over all Ahlfors regular metrics on \( \partial X \) which are quasi-Möbius equivalent to a visual metric. The Ahlfors conformal dimension is a key concept for much of the current work on boundaries of hyperbolic spaces from the perspective of analysis on metric spaces \[27\].

If \( X \) admits a geometric group action, then Fact 5.8 shows that the visual dimension measures the range of visual parameters:

\[
\text{visdim } \partial X = \text{volume entropy of } X / \text{vispar } \partial X
\]

where \( \text{vispar } \partial X = \sup\{\epsilon > 0 : \epsilon \text{ is the parameter of a visual metric on } \partial X\} \). The point is now that \( \text{vispar } \partial X \) measures the (coarse) negative curvature of \( X \). The simplest manifestation of this idea is the following fact: if \( X \) is a \( \text{CAT}(\kappa) \) space, where \( \kappa < 0 \), then \( \sqrt{-\kappa} \) is a visual parameter so \( \text{vispar } \partial X \geq \sqrt{-\kappa} \). A coarse version of this fact was investigated by Bonk and Foertsch in \[5\]. They define a notion of asymptotic upper curvature for hyperbolic spaces which is invariant under rough isometries, and which agrees with the metric notion of curvature: if \( X \) is a \( \text{CAT}(\kappa) \) space, then \( X \) has asymptotic upper curvature \( K_u(X) \leq \kappa \). Bonk and Foertsch go on to show that \( \text{vispar } \partial X = \sqrt{-K_u(X)} \) for a hyperbolic space \( X \).

7. The basic K-cycle for boundary actions of hyperbolic groups

In this section, we realize the paradigm of Section 3 in the case of a hyperbolic group acting on its boundary. First, some standing notations for the rest of the paper. The main characters are

- \( \Gamma \): a non-elementary hyperbolic group;
- \( X \): a hyperbolic space on which \( \Gamma \) acts geometrically, i.e., a geometric model for \( \Gamma \);
- \( \mu \): a visual probability measure on \( \partial X \).

In order to be able to do geometric analysis on the boundary, we also fix

- \( d_c \): a visual metric on \( \partial X \),

and we denote

- \( D_c \): the Hausdorff dimension of \( (\partial X, d_c) \).
Lemma 7.1. Let \( o \in X \) be a basepoint. Then there exists \( C' > 0 \) such that, for all \( g \in \Gamma \), we have
\[
\left( \int \int d_r(g\xi, g\xi')^2 \, d\mu(\xi) \, d\mu(\xi') \right)^{1/2} \geq C' \exp(-\epsilon \, d(o, go)).
\]
If \( D_\epsilon > 2 \), then there exists \( C > 0 \) such that, for all \( g \in \Gamma \), we have
\[
\left( \int \int d_r(g\xi, g\xi')^2 \, d\mu(\xi) \, d\mu(\xi') \right)^{1/2} \leq C \exp(-\epsilon \, d(o, go)).
\]

The proof is deferred to the end of the section. The important part is the second estimate; the purpose of the first part is to show that we are getting the correct asymptotics.

Theorem 7.2. \((\lambda_\mu, P_{\epsilon^1})\) is a Fredholm module for \( C(\partial X) \rtimes \Gamma \) which is \( D_\epsilon^+ \)-summable when \( D_\epsilon > 2 \), respectively \( p \)-summable for every \( p > 2 \) when \( D_\epsilon \leq 2 \). The summability holds over the dense \( * \)-subalgebra \( \text{Lip}(\partial X, d_\epsilon) \rtimes_{\text{alg}} \Gamma \), where \( \text{Lip}(\partial X, d_\epsilon) \) is the \( \Gamma \)-invariant algebra of Lipschitz functions on \( \partial X \).

Proof. We first prove the claim in the case when \( D_\epsilon > 2 \). Using formula (3.1), we have
\[
\sigma \phi(g) \leq \|\phi\|_{\text{Lip}} \sqrt{2} \int \int d_r(g\xi, g\xi')^2 \, d\mu(\xi) \, d\mu(\xi')
\]
for all \( \phi \in \text{Lip}(\partial X, d_\epsilon) \). It follows from Lemma 7.1 that there exists \( C > 0 \) such that
\[
\sigma \phi(g) \leq C\|\phi\|_{\text{Lip}} \exp(-\epsilon \, d(o, go))
\]
for all \( \phi \in \text{Lip}(\partial X, d_\epsilon) \) and \( g \in \Gamma \). Let \( T \) denote the multiplication by \( g \mapsto \exp(-\epsilon \, d(o, go)) \), viewed as an operator on \( \ell^2 \Gamma \). We claim that \( T \in L^{D_\epsilon^+}(\ell^2 \Gamma) \). Once we know this, it follows that multiplication by \( \sigma \phi \) is in \( L^{D_\epsilon^+}(\ell^2 \Gamma) \) for all \( \phi \in \text{Lip}(\partial X, D_\epsilon) \), and the proof of Proposition 3.4 shows that \((\lambda_\mu, P_{\epsilon^1})\) is a \( D_\epsilon^+ \)-summable Fredholm module. In order to prove our claim that \( s_n(T) = O(n^{-1/D_\epsilon}) \), we first control a subsequence of singular values for \( T \). Let \( m_k \) denote the size of the “ball” \( \{ g \in \Gamma : d(o, go) \leq k \} \); thus \( m_k \geq \exp(\epsilon \, n \, k) \) by Fact 5.7. We have
\[
s_{m_{k+1}}(T) < \exp(-\epsilon \, k) = \exp(\epsilon \, n \, k)^{-1/D_\epsilon} \leq C_1 m_k^{-1/D_\epsilon}.
\]
For an arbitrary positive integer \( n \), let \( k \) be such that \( m_k + 1 \leq n \leq m_{k+1} + 1 \). Then
\[
s_n(T) \leq s_{m_{k+1}}(T) \leq C_1 m_k^{-1/D_\epsilon} \leq C_1 n^{-1/D_\epsilon} \left( \frac{m_{k+1} + 1}{m_k} \right)^{1/D_\epsilon} \leq C_2 n^{-1/D_\epsilon}
\]
for some constant \( C_2 \) independent of \( n \) and \( k \). The claim is proved for \( D_\epsilon > 2 \).

Now assume that \( D_\epsilon \leq 2 \), and let \( p > 2 \). The idea is to increase the Hausdorff dimension by snowflaking. Namely, let \( \alpha \in (0, 1] \) so that \( D_\epsilon / \alpha \), which is the Hausdorff dimension of \( (\partial X, d_\epsilon^\alpha) \), satisfies \( p > D_\epsilon / \alpha > 2 \). By the previous part of the proof, we know that \((\lambda_\mu, P_{\epsilon^1})\) is \((D_\epsilon / \alpha)^+\)-summable over \( \text{Lip}(\partial X, d_\epsilon^\alpha) \rtimes_{\text{alg}} \Gamma \). As \( \text{Lip}(\partial X, d_\epsilon) \) is contained in \( \text{Lip}(\partial X, d_\epsilon^\alpha) \), we conclude that \((\lambda_\mu, P_{\epsilon^1})\) is \( p \)-summable over \( \text{Lip}(\partial X, d_\epsilon) \rtimes_{\text{alg}} \Gamma \). \( \square \)

A noticeable aspect of Theorem 7.2 is the fact that the summability of the basic K-cycle \((\lambda_\mu, P_{\epsilon^1})\) always occurs above 2. We do not know whether this phenomenon is due to some structural obstruction. There is, however, an obstruction to our method of controlling summability by the decay of the \( \Gamma \)-deviation, and this is the fact that visual probability measures are doubly ergodic \cite{[21]}. Indeed, by Proposition 3.3, the \( \Gamma \)-deviation has to decay faster than \( \ell^2 \) in the presence of double ergodicity.

We may optimize Theorem 7.2 by varying the visual metric. Our notion of visual dimension is in fact tailored for this very purpose.
Corollary 7.3. \((\lambda_\mu, P_{\mathcal{F}T})\) is \(p\)-summable for every \(p > \max\{2, \text{visdim } \partial X\}\). Furthermore, 
\((\lambda_\mu, P_{\mathcal{F}T})\) is \((\text{visdim } \partial X)^+\)-summable provided \text{visdim } \partial X\ is attained and greater than 2.

Example 7.4. Let \(\Gamma\) be a cocompact lattice in \(SO(n,1)\). We let \(\Gamma\) act geometrically on \(\mathbb{H}^n\), and we endow \(\partial \mathbb{H}^n = S^{n-1}\) with the (normalized) spherical measure \(\sigma\). Then the regular representation on \(g\) for all \(g \in \Gamma\) is the complex or quaternionic hyperbolic space \(\mathbb{H}^n\). Endow the boundary \(\partial \mathbb{H}^n = S^{n-1}\), where \(k = 2\) or 4, with the (normalized) spherical measure \(\sigma\) coming from the Carnot metric. Then the regular representation on \(\ell^2(\Gamma, L^2(S^{n-1}, \sigma))\) together with the projection \(P_{\mathcal{F}T}\) define a Fredholm module for \(C(S^{n-1}) \rtimes \Gamma\) which is \((n-1)^+\)-summable when \(n \geq 4\), and \(p\)-summable for every \(p > 2\) when \(n = 2, 3\).

Similarly, let \(\Gamma\) be a cocompact lattice in \(SU(n,1)\) or \(Sp(n,1)\). The natural geometric model for \(\Gamma\) is the complex or quaternionic hyperbolic space \(\mathbb{H}^n\). Endow the boundary \(\partial \mathbb{H}^n = S^{n-1}\), where \(k = 2\) or 4, with the (normalized) spherical measure \(\sigma\) coming from the Carnot metric. Then the regular representation on \(\ell^2(\Gamma, L^2(S^{n-1}, \sigma))\) together with the projection \(P_{\mathcal{F}T}\) define an \((nk + k - 2)^+\)-summable Fredholm module for \(C(S^{n-1}) \rtimes \Gamma\).

Now let us return to Lemma 7.1 whose proof we have postponed.

Proof of Lemma 7.1 The first estimate is straightforward. As in (5.3), we have \((g\xi, g\xi')_o \leq d(o, go) + \xi, \xi' \in \partial X\). Therefore \(d_o(g\xi, g\xi')_o \geq c_1 \exp(-\epsilon d(o, go))\) for all \(\xi, \xi' \in \partial X\) for some \(c_1 > 0\), which implies that

\[
\left( \int \int d_o(g\xi, g\xi')^2 \, d\mu(\xi) \, d\mu(\xi') \right)^{1/2} \geq c_2 \exp(-\epsilon d(o, go)).
\]

The second, conditional estimate is more involved. First, we assume that the visual parameter \(\epsilon\) is in the small visual range and that \(d_o\) is a visual metric enjoying the properties listed in Fact 5.3. We let \(\alpha > 0\), and we show the following: if \(d_o > 2\alpha\), then there exists \(C > 0\) such that for all \(g \in \Gamma\) we have

\[
\left( \int \int d_o(g\xi, g\xi')^{2\alpha} \, d\mu(\xi) \, d\mu(\xi') \right)^{1/2} \leq C \exp(-\alpha d(o, go)).
\]

Let \(\xi, \xi' \in \partial X\). Observe that \((gx, gx')_o + (g^{-1}o, x)_o \geq d(o, go)\) whenever \(x, x' \in X\); indeed, this is just a complicated rewriting of the triangle inequality \(d(o, x) + d(o, x') \geq d(x, x')\). Letting \(x \to \xi, x \to \xi'\) and using (5.1) and (5.2), we obtain that

\[
(g\xi, g\xi')_o + (g^{-1}o, \xi)_o + (g^{-1}o, \xi')_o \geq d(o, go) - 4\delta.
\]

Thus there is \(C_1 > 0\) such that

\[
d_o(g\xi, g\xi') \leq C_1 \exp(-\epsilon d(o, go)) \, d_o(g^{-1}o, \xi)^{-1} d_o(g^{-1}o, \xi')^{-1}
\]

for all \(g \in \Gamma\) and \(\xi, \xi' \in \partial X\). It follows that

\[
\left( \int \int d_o(g\xi, g\xi')^{2\alpha} \, d\mu(\xi) \, d\mu(\xi') \right)^{1/2} \leq C_1 \exp(-\alpha d(o, go)) \int d_o(g^{-1}o, \xi)^{-2\alpha} \, d\mu(\xi)
\]

and our next goal is to show that the integral on the left hand side is bounded above independently of \(g \in \Gamma\). As shown at this point, the technical advantage of using \(\epsilon\) in the small visual range becomes apparent: the function \(d_o(g^{-1}o, \cdot)\) is Lipschitz, in particular measurable, on \(\partial X\). For each positive integer \(k\) we put

\[
\Delta_k = \{ \xi \in \partial X : \exp(-ck) \leq d_o(g^{-1}o, \xi) \leq \exp(-\epsilon(k-1))\}.
\]

(Although we will not need this fact, we remark that \(d_o(g^{-1}o, \cdot)\) is bounded below by a constant multiple of \(\exp(-\epsilon d(o, go))\), so \(\Delta_k\) is in fact empty for \(k \gg d(o, go)\).) From the hyperbolic inequality \(\xi, \xi' \in \partial X\), we deduce that there is \(C_2 \geq 0\) such that

\[
d_o(\xi, \xi') \leq C_2 \max\{d_o(\xi, \xi), d_o(\xi, \xi')\}
\]
for all $\xi, \xi' \in \partial X$. This inequality implies that $\text{diam}(\Delta_k) \leq e^\epsilon C_2 \exp(-ek)$. It now follows from Fact 5.3 that there exists $C_3 > 0$, independent of $k$, such that

$$
\mu(\Delta_k) \leq C_3 \left( \exp(-ek) \right)^{e_X/\epsilon} = C_3 \exp(-e_X k).
$$

Using this diameter bound, we immediately get the desired integral estimate:

$$
\int d_\epsilon(g^{-1}o, \xi)^{-2\alpha} \, d\mu(\xi) \leq \sum_{k \geq 1} \int d_\epsilon(g^{-1}o, \xi)^{-2\alpha} \, d\mu(\xi) \leq \sum_{k \geq 1} \exp(2\alpha e k) \mu(\Delta_k)
\leq C_3 \sum_{k \geq 1} \left( (2\alpha e - e_X) k \right)
$$

and the latter series converges since $D_\epsilon = e_X/\epsilon > 2\alpha$ by assumption.

Now let $\epsilon$ be an arbitrary visual parameter. Pick $\epsilon_0$ in the small visual range, and let $d_{\epsilon_0}$ be a corresponding visual metric. By Fact 5.3 we have

$$
\left( \int \int d_\epsilon(g\xi, g\xi')^2 \, d\mu(\xi) \, d\mu(\xi') \right)^{1/2} \leq \left( \int \int d_{\epsilon_0}(g\xi, g\xi')^{2/\epsilon_0} \, d\mu(\xi) \, d\mu(\xi') \right)^{1/2}.
$$

According to the lemma’s hypothesis, $\text{hdim}(\partial X, d_{\epsilon_0}) = (\epsilon/\epsilon_0) \text{hdim}(\partial X, d_\epsilon)$ is greater than $2\epsilon/\epsilon_0$. Hence the previous part of the proof shows that the right-hand side is bounded above by a constant multiple of $\exp(-(\epsilon/\epsilon_0)\epsilon_0 d(o, go)) = \exp(-\epsilon d(o, go))$. \hfill \Box

For the sake of conciseness, we adopt the following for the rest of the paper.

**Notation.** We write $D_\epsilon^{-\text{\emph{sumnable}}}$ to mean

\[
\begin{cases}
D_\epsilon^+\text{-sumnable} & \text{when } D_\epsilon > 2, \\
\text{p-sumnable for every } p > 2 & \text{when } D_\epsilon \leq 2.
\end{cases}
\]

8. THE BOUNDARY EXTENSION

All the basic K-cycles coming from visual probability measures on $\partial X$ are K-homologous, by Proposition 14.2. The purpose of this section is to describe the $K^1$-class of a basic K-cycle as the class of a certain canonical extension of $C(\partial X) \rtimes \Gamma$ by the compact operators on $\ell^2 \Gamma$. This extension encodes the compactification of $\Gamma$ obtained by attaching the boundary $\partial X$.

Let $\bar{\Gamma} = \Gamma \cup \partial X$ be the compactification of $\Gamma$ by the boundary of the geometric model $X$. By definition, $g \rightarrow \omega \in \partial X$ in $\bar{\Gamma}$ if $go \rightarrow \omega$ in $\bar{X}$ for some (equivalently, each) base point $o \in X$. From the exact sequence of $\Gamma$-C*-algebras $0 \rightarrow C_0(\Gamma) \rightarrow C(\bar{\Gamma}) \rightarrow C(\partial X) \rightarrow 0$ we obtain an exact sequence of $\Gamma$-C*-crossed products by $\Gamma$:

\[
0 \rightarrow C_0(\Gamma) \rtimes \Gamma \rightarrow C(\bar{\Gamma}) \rtimes \Gamma \rightarrow C(\partial X) \rtimes \Gamma \rightarrow 0
\]

Each C*-algebra in (8.1) is nuclear; in particular, the full and the reduced crossed products agree. The faithful representation of $C(\Gamma)$ on $\ell^2 \Gamma$ by multiplication induces a faithful representation $\pi : C(\bar{\Gamma}) \rtimes \Gamma \rightarrow \mathcal{B}(\ell^2 \Gamma)$, which restricts to the standard isomorphism between the ideal term $C_0(\Gamma) \rtimes \Gamma$ and the compact operators $\mathcal{K}(\ell^2 \Gamma)$. Thus (8.1) defines a class in the Brown - Douglas - Filmore group $\text{Ext}(C(\partial \Gamma) \rtimes \Gamma)$. The nuclearity of $C(\partial \Gamma) \rtimes \Gamma$ implies that $\text{Ext}(C(\partial \Gamma) \rtimes \Gamma)$ and $K_1(C(\partial \Gamma) \rtimes \Gamma)$ are isomorphic. The map $\text{Ext} \rightarrow K^1$ is given by the Stinespring construction, which dilates a completely positive map to an odd Fredholm module.

The compactification of $\Gamma$, hence the exact sequence (8.1) as well, is defined in reference to the chosen geometric model $X$. However, and this is an important point, the boundaries of two geometric models for $\Gamma$ are $\Gamma$-equivariantly homeomorphic. It follows that, up to isomorphism of extensions, the exact sequence (8.1) is independent of the chosen geometric model $X$. 
Definition 8.1. The boundary extension class \([\partial \Gamma] \in \text{Ext}(C(\partial \Gamma) \times \Gamma)\) is the class defined by the extension \([8.1]\).

We will show that the Fredholm module \((\lambda, P_{\partial \Gamma})\) represents \([\partial \Gamma]\). The initial ingredient is the following lemma, which should be compared with Proposition \([5.6]\) and its proof.

Lemma 8.2. If \(g \to \omega \in \partial \Gamma\) in \(\overline{\Gamma}\), then \(g, \mu \to \delta_\omega\) in \(\text{Prob}(\partial \Gamma)\).

Proof. Fix \(\phi \in C(\partial \Gamma)\). We have
\[
\left| \int \phi \, dg \, \mu - \phi(\omega) \right| = \left| \int \phi(g \xi) - \phi(\omega) \, d\mu(\xi) \right| \leq \int |\phi(g \xi) - \phi(\omega)| \, d\mu(\xi).
\]
and we show that the right-hand integral converges to 0 as \(g \to \omega\) in \(\overline{\Gamma}\). Let \(t > 0\), and let \(d_\epsilon\) be a visual metric on \(\partial \Gamma\) with parameter \(\epsilon\) in the small visual range so that Fact \([5.1]\) applies. The uniform continuity of \(\phi\) provides us with some \(R > 0\) such that \(|\phi(\xi) - \phi(\xi')| < t\) whenever \(d_\epsilon(\xi, \xi') < R\). The set
\[
Z(g) = \{ \xi \in \partial \Gamma : d_\epsilon(g \xi, \omega) \geq R \}
\]
is measurable, since \(\xi \mapsto d_\epsilon(g \xi, \omega)\) is continuous. We write:
\[
\int \left| \phi(g \xi) - \phi(\omega) \right| \, d\mu(\xi) = \int_{\partial \Gamma \setminus Z(g)} \left| \phi(g \xi) - \phi(\omega) \right| \, d\mu(\xi) + \int_{Z(g)} \left| \phi(g \xi) - \phi(\omega) \right| \, d\mu(\xi)
\]
\[
\leq t + 2\|\phi\|_\infty \mu(Z(g))
\]
It suffices to show that \(\mu(Z(g)) \to 0\) as \(g \to \omega\). Let \(o \in X\) be a basepoint. One easily checks that \((gx, w)_o + (g^{-1}o, x)_o \geq (go, w)_o\) for \(x, w \in X\). Letting \(x \to \xi\) and \(w \to \omega\), and using \([5.1]\) and \([5.2]\), we get \((g_\xi, \omega)_o + (g^{-1}o, \xi)_o \geq (go, \omega)_o - 3\delta\). It follows that there is \(C_1 > 0\) such that
\[
d_\epsilon(g_\xi, \omega) \leq C_1 \, d_\epsilon(go, \omega)
\]
for all \(g \in \Gamma\) and \(\xi, \omega \in \partial \Gamma\). Now by hyperbolicity there exists \(C_2 > 0\) such that \(d_\epsilon(\xi, \xi') \leq C_2 \max \{ d_\epsilon(g^{-1}o, \xi), d_\epsilon(g^{-1}o, \xi') \}\) for all \(\xi, \xi' \in Z(g)\). In turn, both \(d_\epsilon(g^{-1}o, \xi)\) and \(d_\epsilon(g^{-1}o, \xi')\) are at most \(C_1 R^{-1} \, d_\epsilon(go, \omega)\) by the inequality above. Thus \(\text{diam}(Z(g)) \leq C_3 d_\epsilon(go, \omega)\). By Ahlfors regularity (Fact \([5.8]\)) \(\mu(Z(g)) \leq C_4\, d_\epsilon(go, \omega)^{e_\Gamma/e_\Gamma} \). Now if \(g \to \omega\) then \(d_\epsilon(go, \omega) \to 0\), so \(\mu(Z(g)) \to 0\) as desired. \(\square\)

In terms of the \(\Gamma\)-expectation, Lemma \([8.2]\) can be interpreted as follows: if \(g \to \omega \in \partial \Gamma\) in \(\overline{\Gamma}\), then \(E\phi(g) \to \phi(\omega)\) for all \(\phi \in C(\partial \Gamma)\). This means that we may extend continuous maps on the boundary \(\partial \Gamma\) to continuous maps on the compactification \(\overline{\Gamma}\) by gluing a map to its \(\Gamma\)-expectation. Namely, for \(\phi \in C(\partial \Gamma)\) we define \(\overline{\Gamma}\phi \in C(\overline{\Gamma})\) by
\[
\overline{\Gamma}\phi = \begin{cases} \phi & \text{on } \partial \Gamma \setminus Z(g), \\ E\phi & \text{on } \Gamma. \end{cases}
\]
We view \(\overline{\Gamma} : C(\partial \Gamma) \to C(\overline{\Gamma})\) as a \(\Gamma\)-equivariant, completely positive section for the quotient map \(C(\overline{\Gamma}) \to C(\partial \Gamma)\) given by restriction. We immediately obtain a completely positive section for the quotient map \(C(\overline{\Gamma}) \times \Gamma \to C(\partial \Gamma) \times \Gamma\).

Theorem 8.3. Define
\[
s_\mu : C(\partial \Gamma) \times_{\text{alg}} \Gamma \to C(\overline{\Gamma}) \times \Gamma, \quad s_\mu \left( \sum_g \phi_g g \right) = \sum \left( \overline{\Gamma}\phi_g \right) g.
\]
Then \(s_\mu\) extends to a completely positive section for the quotient \(C(\overline{\Gamma}) \times \Gamma \to C(\partial \Gamma) \times \Gamma\). Consequently, \((\lambda_\mu, P_{\partial \Gamma})\) is a Fredholm module representing the boundary extension class \([\partial \Gamma]\).
Proof. Recall that \( \pi : C(\bar{\Gamma}) \rtimes \Gamma \to B(\ell^2 \Gamma) \) is the representation induced by the multiplication representation of \( C(\bar{\Gamma}) \) on \( \ell^2 \Gamma \). We claim that

\[
\pi s_\mu(a) = P_{\ell^2 \Gamma} \lambda_\mu(a) P_{\ell^2 \Gamma}
\]

for all \( a \in C(\partial X) \rtimes_{\text{alg}} \Gamma \). Indeed, for \( \phi \in C(\partial X) \) and \( g \in \Gamma \) we have

\[
\pi s_\mu(\phi g) = \pi(\mathcal{E} \phi) \pi(g) = (P_{\ell^2 \Gamma} \lambda_\mu(\phi) P_{\ell^2 \Gamma})(P_{\ell^2 \Gamma} \lambda_\mu(g) P_{\ell^2 \Gamma}) = P_{\ell^2 \Gamma} \lambda_\mu(\phi g) P_{\ell^2 \Gamma}
\]

by using \((3.2)\), and the fact that \( \lambda_\mu \) isometric, \((8.3)\) implies that \( C \) is a representation of \( \lambda_\mu \) in our setting. Since \( \Gamma \) acts by Lipschitz maps on \( \partial X \), \((8.3)\) implies that \( \lambda_\mu |_\Gamma \) commutes with \( P_{\ell^2 \Gamma} \). Since \( \pi \) is faithful, and therefore isometric, \((8.3)\) implies that \( s_\mu \) extends by continuity to a completely positive section for the quotient \( C(\bar{\Gamma}) \rtimes \Gamma \to C(\partial \Gamma) \rtimes \Gamma \). The Stinespring dilation \( \pi s_\mu = P_{\ell^2 \Gamma} \lambda_\mu P_{\ell^2 \Gamma} \) precisely means that \( (\lambda_\mu, P_{\ell^2 \Gamma}) \) represents \([\partial \Gamma]\).

Theorem \((8.3)\) and Proposition \((4.3)\) imply the following.

**Proposition 8.4.** Let \( \Lambda \) be a finite-index subgroup of \( \Gamma \). Then the restriction homomorphism \( \text{res} : K^1(C(\partial \Gamma) \rtimes \Gamma) \to K^1(C(\partial \Lambda) \rtimes \Lambda) \) sends \([\partial \Gamma]\) to \([\Gamma : \Lambda] : [\partial \Lambda]\).

Indeed, the comparability condition required in the statement of Proposition \((4.3)\) is satisfied in our setting. Since \( \Gamma \) acts by Lipschitz maps on \( \partial X \) for any choice of visual metric \( d_e \), the corresponding Hausdorff measure satisfies \( g_* \mu_e \cong \mu_e \) for all \( g \in \Gamma \); the same will then hold for any visual probability measure on \( \partial X \).

9. Poincaré duality and twisted K-cycles

9.1. Poincaré duality. Poincaré duality for \( C(\partial \Gamma) \rtimes \Gamma \), proved in \([10]\), plays an essential role in this paper. The proof from \([10]\), though most likely not Poincaré duality itself, needs the following mild symmetry condition on the boundary.

**Definition 9.1.** A hyperbolic group is **regular** if its boundary admits a continuous self-map without fixed-points.

Regularity in the above sense is satisfied whenever the boundary is a topological sphere, a Cantor set or a Menger compactum. We are not aware of any example where regularity fails. The ‘topologically rigid’ hyperbolic groups of Kapovich and Kleiner \([25]\) come close, though, for their boundaries admit no self-homeomorphisms without fixed points (but regularity does not require a homeomorphism, merely a map).

Poincaré duality is defined by a cap-cap procedure explained in \([10]\). A C*-algebra is **Poincaré self-dual** in this sense if there are two ‘fundamental classes’,

\[
\Delta \in KK_1(A \otimes A, \mathbb{C}), \quad \hat{\Delta} \in KK_1(\mathbb{C}, A \otimes A)
\]

satisfying certain equations which we do not specify here (the zig-zag equations of the theory of adjoint functors.) Given \( \Delta \) as above, a ‘Poincaré duality’ map is defined by

\[
\Delta \cap : K_n(A) \to K^{n+1}(A), \quad \Delta \cap x = (x \otimes \mathbb{C}, \mathbb{1}_A) \otimes_{A \otimes A} \Delta,
\]

or in other words, by composing in KK, the morphisms \( x \otimes \mathbb{1}_A \in KK_n(A, A \otimes A) \) with \( \Delta \in KK_1(A \otimes A, \mathbb{C}) \), to get a morphism in \( KK_{n+1}(A, \mathbb{C}) = K^{n+1}(A) \).

For \( C(\partial \Gamma) \rtimes \Gamma \), such fundamental classes \( \Delta \) and \( \hat{\Delta} \) are constructed in \([10]\), and, using the Baum-Connes machinery, it is shown that \( \Delta \cap \) is an isomorphism when \( \Gamma \) is regular and torsion-free. The inverse isomorphism comes from \( \hat{\Delta} \). The class \( \Delta \) is defined as an **extension**, i.e. as a cycle for the Brown - Douglas - Filmore group \( \text{Ext}(C(\partial \Gamma) \rtimes \Gamma \otimes \partial \Gamma, \mathbb{C}) \), whose Busby invariant is as follows. First let

\[
\tau : C(\partial \Gamma) \rtimes \Gamma \to B(\ell^2 \Gamma)/K(\ell^2 \Gamma)
\]
be the Busby invariant of the extension $[\mathbb{S}, \mathbb{T}]$. Thus $\tau$ is the integrated form of the covariant pair associated to the regular representation $\lambda: \Gamma \rightarrow B(\ell^2 \Gamma)$ followed by the quotient map $B \rightarrow B/\mathcal{K}$, and the map $\phi \mapsto M(\hat{\phi})$ followed by the quotient map, where $\hat{\phi}$ is an extension of $\phi \in C(\partial \Gamma)$ to a continuous function on $\mathbb{T}$. Let

$$\tau^{op}: C(\partial \Gamma) \times \Gamma \rightarrow B(\ell^2 \Gamma)/K(\ell^2 \Gamma), \quad \tau^{op}(a) := J\tau(a)J$$

where $J$ is the symmetry

$$J: \ell^2 \Gamma \rightarrow \ell^2 \Gamma, \quad J(\delta_g) := \delta_{g^{-1}}.$$ 

Note that $s^{op}: C(\partial \Gamma) \times \Gamma \rightarrow JC(\partial \Gamma) \times \Gamma J \subset B(\ell^2 \Gamma)$, defined by $s^{op}(a) := Js(a)J$ is a completely positive splitting of $\tau^{op}$.

As proved in [10], $\tau$ and $\tau^{op}$ commute so they combine to give a unital $*$-homomorphism $C(\partial \Gamma) \times \Gamma \times C(\partial \Gamma) \times \Gamma \rightarrow B(\ell^2 \Gamma)/K(\ell^2 \Gamma)$, $a \otimes b \mapsto \tau(a)\tau^{op}(b)$. The class $\Delta$ is by definition the pre-image of this extension under the canonical map

$$\text{KK}_1(C(\partial \Gamma) \times \Gamma \otimes C(\partial \Gamma) \times \Gamma, \mathbb{C}) \rightarrow \text{Ext}(C(\partial \Gamma) \times \Gamma \otimes C(\partial \Gamma) \times \Gamma, \mathbb{C}),$$

which is an isomorphism since $C(\partial \Gamma) \times \Gamma \otimes C(\partial \Gamma) \times \Gamma$ is nuclear. However, a cycle in $\text{KK}_1$ representing $\Delta$ is difficult to describe because we do not know of a concrete completely positive splitting of the extension defining $\Delta$.

Nevertheless, we show that the ideas of the previous sections can be used to compute the Poincaré duality map $\Delta^\wedge$ in explicit terms.

**Lemma 9.2.** Let $e \in C(\partial \Gamma) \times \Gamma$ be a projection. Then the Poincaré dual $\Delta \cap [e]$ of the class $[e] \in \text{K}_0(C(\partial \Gamma) \times \Gamma)$ is the class in $K^1(C(\partial \Gamma) \times \Gamma) = \text{KK}_1(C(\partial \Gamma) \times \Gamma, \mathbb{C}) \cong \text{Ext}(C(\partial \Gamma) \times \Gamma, \mathbb{C})$ of the extension with Busby invariant

$$\tau_e: C(\partial \Gamma) \times \Gamma \rightarrow B(\ell^2 \Gamma)/K(\ell^2 \Gamma), \quad \tau_e(a) := \tau^{op}(e)\tau(a)\tau^{op}(e).$$

Let $u \in C(\partial \Gamma) \times \Gamma$ be a unitary, and let

$$\bar{u}: C_0(\mathbb{R}) \subset C(S^1) \rightarrow C(\partial \Gamma) \times \Gamma$$

denote the composition of the usual inclusion of $C_0(\mathbb{R}) \cong C_0(S^1 - \{1\})$ into $C(S^1)$, functional calculus for $u$. Then the Poincaré dual $\Delta \cap [u]$ of the class $[u] \in \text{K}_1(C(\partial \Gamma) \times \Gamma)$ is the class in $\text{KK}_1(C_0(\mathbb{R}) \otimes C(\partial \Gamma) \times \Gamma, \mathbb{C})$ of the extension of $C_0(\mathbb{R}) \otimes C(\partial \Gamma) \times \Gamma$ by $K(\ell^2 \Gamma)$ whose Busby invariant is

$$\tau_u(f \otimes a) = (\tau^{op} \circ \bar{u})(f)\tau(a).$$

**Proof.** Both assertions follow from the functoriality of the Kasparov and Brown - Douglas - Filmore theories. To prove 1, let $e$ be a projection in $C(\partial \Gamma) \times \Gamma$, then $[e] = e_\ast([1])$ where $e_\ast: \text{K}_\ast(\mathbb{C}) \cong \mathbb{Z} \rightarrow \text{K}_\ast(C(\partial \Gamma) \times \Gamma)$ is the $*$-homomorphism induced by $e$, and the element

$$[e] \otimes 1_{C(\partial \Gamma) \times \Gamma} \in \text{KK}_0(C(\partial \Gamma) \times \Gamma, C(\partial \Gamma) \times \Gamma \otimes C(\partial \Gamma) \times \Gamma)$$

is represented by the $*$-homomorphism $e \otimes 1_{C(\partial \Gamma) \times \Gamma}$, and composing with $\Delta$ amounts to composing the Busby invariant for $\Delta$ and the $*$-homomorphism $e \otimes 1$. By the definitions and the fact that $\tau^{op}(e)$ is a projection in the Calkin algebra commuting with $\tau(a)$ for all $a$, this yields $\tau_e$. The second assertion is proved similarly. 

Note that the boundary extension class $[\partial \Gamma] \in K^1(C(\partial \Gamma) \times \Gamma)$ is the Poincaré dual of the unit class $[1_{C(\partial \Gamma) \times \Gamma}] \in \text{K}_0(C(\partial \Gamma) \times \Gamma)$. On the other hand, the K-theory Gysin sequence of [11] shows that the order of $[1_{C(\partial \Gamma) \times \Gamma}]$ in $\text{K}_0(C(\partial \Gamma) \times \Gamma)$ is determined by the Euler characteristic of $\Gamma$. Consequently:
Proposition 9.3 (Emerson, Emerson - Meyer). Assume that \( \Gamma \) is regular and torsion-free. Then \( [\partial_\Gamma] \neq 0 \) in \( K^1(C(\partial \Gamma) \times \Gamma) \) if and only if \( \chi(\Gamma) = \pm 1 \). Furthermore, \( [\partial_\Gamma] \) has infinite order in \( K^1(C(\partial \Gamma) \times \Gamma) \) if and only if \( \chi(\Gamma) = 0 \).

We may pass from torsion-free to virtually torsion-free groups, and establish a version of Proposition 9.3 for this much larger class, by using Proposition 9.4. The rational Euler characteristic of a virtually torsion-free group \( \Gamma \) is defined by the formula \( \chi(\Gamma) = \chi(\Lambda)/[\Gamma : \Lambda] \) where \( \Lambda \) is any torsion-free subgroup of finite index.

Corollary 9.4. Assume that \( \Gamma \) is regular and virtually torsion-free. If \( \chi(\Gamma) \notin 1/\mathbb{Z} \) then \( [\partial_\Gamma] \neq 0 \) in \( K^1(C(\partial \Gamma) \times \Gamma) \). If \( \chi(\Gamma) = 0 \) then \( [\partial_\Gamma] \) has infinite order in \( K^1(C(\partial \Gamma) \times \Gamma) \).

The assumption of virtual torsion-freeness is a very mild one. A long-standing open problem asks whether all hyperbolic groups are virtually torsion-free.

9.2. Twisted K-cycles. We now show how to compute \( \Delta \cap \) in terms of certain canonical Fredholm modules obtained by ‘twisting’ the basic K-cycle \( (\lambda_\mu, P_{2\Gamma}) \).

The right regular representation \( \lambda_\mu^{op} \) of \( C(\partial X) \times \Gamma \) on \( \ell^2(\Gamma, L^2(\partial X, \mu)) \) is given as follows:

\[
\lambda_\mu^{op}(\phi) \left( \sum \psi_h \delta_h \right) = \left( \sum (h, \phi) \psi_h \delta_h \right), \quad \lambda_\mu^{op}(g) \left( \sum \psi_h \delta_h \right) = \left( \sum \psi_h \delta_{hg^{-1}} \right)
\]

for \( \phi \in C(\partial X) \), \( g \in \Gamma \). Note the covariance relation \( \lambda_\mu^{op}(g, \phi) = \lambda_\mu^{op}(g) \lambda_\mu^{op}(\phi) \lambda_\mu^{op}(g^{-1}) \).

The right and the left regular representations do not commute, but do satisfy:

\[
[\lambda_\mu(\phi) , \lambda_\mu^{op}(\phi')] = 0, \quad [\lambda_\mu(g) , \lambda_\mu^{op}(g')] = 0
\]

for all \( \phi, \phi' \in C(X) \) and \( g, g' \in \Gamma \).

The symmetry \( J \) on \( \ell^2 \Gamma \) has an obvious extension \( J \otimes id \) to \( \ell^2(\Gamma, L^2(\partial X, \mu)) \), and, using the same notation for this extension, we have

\[
\lambda_\mu^{op} = J \lambda_\mu J.
\]

Since \( s \) splits \( \tau \), the image in the Calkin algebra of \( P_{2\Gamma} \lambda_\mu(a) P_{2\Gamma} = \tau(a) \) is \( \tau(a) \). Also, \( P_{2\Gamma} \lambda_\mu^{op}(a) P_{2\Gamma} = s^{op}(a) \), and its image in the Calkin algebra is \( \tau^{op}(a) \). Hence

\[
P_{2\Gamma} \lambda_\mu(a) P_{2\Gamma} = \tau(a) \mod \mathcal{K}, \quad \text{and} \quad P_{2\Gamma} \lambda_\mu^{op}(a) P_{2\Gamma} = \tau^{op}(a) \mod \mathcal{K}.
\]

Theorem 9.5. Let \( \Gamma \) be regular and torsion-free. Then the following hold.

- Every class in \( K^1(C(\partial \Gamma) \times \Gamma) \) is represented by an odd Fredholm module of the form
  \[
  (\lambda_{\mu}, P_{2\Gamma} \lambda_\mu^{op}(e) P_{2\Gamma}), \quad \ell \text{ projection in } C(\partial \Gamma) \times \Gamma.
  \]

- Every class in \( K^0(C(\partial \Gamma) \times \Gamma) \) is represented by a balanced even Fredholm module of the form
  \[
  (\lambda_{\mu}, P_{2\Gamma} \lambda_\mu^{op}(u) P_{2\Gamma} + (1 - P_{2\Gamma})), \quad u \text{ unitary in } C(\partial \Gamma) \times \Gamma.
  \]

Proof. Let \( Q_e := P_{2\Gamma} \lambda_\mu^{op}(e) P_{2\Gamma} \). By the discussion preceding the Theorem, \( Q_e = \tau^{op}(e) \mod \text{compact operators.} \) Hence \( Q_e \) is a self-adjoint, essential projection. Furthermore, the commutator \( [\lambda_{\mu}(a) , Q_e] \) is compact for all \( a \in C(\partial \Gamma) \times \Gamma \). For mod compacts we have

\[
\lambda_{\mu}(a) Q_e = \lambda_{\mu}(a) P_{2\Gamma} \lambda_\mu^{op}(e) P_{2\Gamma} = P_{2\Gamma} \lambda_{\mu}(a) P_{2\Gamma} \lambda_\mu^{op}(e) P_{2\Gamma} = \tau(a) \tau^{op}(e) = \tau^{op}(e) \tau(a) = Q_e \lambda_{\mu}(a).
\]

This shows that \( (\lambda_{\mu}, P_{2\Gamma} \lambda_\mu^{op}(e) P_{2\Gamma}) \) is a Fredholm module. The map \( KK_1 \) to \( \text{Ext} \) sends its class to the extension with Busby invariant \( a \mapsto Q_e \lambda_{\mu}(a) Q_e \mod K \) and this equals \( \tau_e(a) := \tau^{op}(e) \tau(a) \tau^{op}(e) \). Hence \( (\lambda_{\mu}, P_{2\Gamma} \lambda_\mu^{op}(e) P_{2\Gamma}) \) represents \( [\tau_e(a)] \), which equals \( \Delta \cap [e] \) by Lemma 9.2.

The second assertion is proved by combining the same observations with Bott Periodicity, see Lemma 2 of [10].
10. Summability of the Twisted K-cycles

In this section we prove the following theorem, of which Theorem 15.3 is an immediate corollary.

**Theorem 10.1.** Let $\Gamma$ be regular and torsion-free. Then $C(\partial X) \times \Gamma$ has uniformly $D^\alpha_\epsilon$-summable K-homology over $\text{Lip}(\partial X, d_\epsilon) \times_{\text{alg}} \Gamma$.

To illustrate the theorem, we point out the following explicit applications.

**Corollary 10.2.** If $\Gamma$ is a finitely generated free group, then the K-homology of $C(\partial \Gamma) \times \Gamma$ is uniformly $p$-summable for every $p > 2$. If $\Gamma$ is a torsion-free cocompact lattice in $\text{SO}(n, 1)$, then the K-homology of $C(S^{n-1}) \times \Gamma$ is uniformly $(n-1)^+$-summable when $n \geq 4$, respectively uniformly $p$-summable for every $p > 2$ when $n = 2, 3$. If $\Gamma$ is a torsion-free cocompact lattice in $\text{SU}(n, 1)$, then the K-homology of $C(S^{2n-1}) \times \Gamma$ is uniformly $(2n)^+$-summable. If $\Gamma$ is a torsion-free cocompact lattice in $\text{Sp}(n, 1)$, then the K-homology of $C(S^{4n-1}) \times \Gamma$ is uniformly $(4n+2)^+$-summable.

To prove Theorem 10.1 we start with an integral estimate.

**Lemma 10.3.** Let $o \in X$ be a basepoint, and assume that $D_\epsilon > 2$. Then there exists $C > 0$ such that, for all $g, h \in \Gamma$, we have

\[
\int d_\epsilon(g\xi, h\xi) \, d\mu(\xi) \leq C \exp(-\epsilon (go, ho)_o).
\]

**Proof.** The proof is similar to that of Lemma 7.4. First, we assume that the parameter $\epsilon$ is in the small visual range and we let $\alpha > 0$. We claim the following: if $D_\epsilon > 2\alpha$, then there exists $C > 0$ such that for all $g, h \in \Gamma$ we have

\[
(10.1) \quad \int d_\epsilon(g\xi, h\xi)^\alpha \, d\mu(\xi) \leq C \exp(-\alpha \epsilon (go, ho)_o).
\]

Pick $\xi \in \partial X$. Observe that $(gx, hx)_o + (g^{-1}o, x)_o + (h^{-1}o, x)_o \geq (go, ho)_o$ for $x \in X$; indeed, this amounts to $d(gx, hx) - d(go, ho) \leq 2d(o, x)$. Letting $x \to \xi$ and using (5.1) and (5.2), we get

\[
(g\xi, h\xi)_o + (g^{-1}o, \xi)_o + (h^{-1}o, \xi)_o \geq (go, ho)_o - 4\delta.
\]

Hence, there is $C_1 \geq 0$ such that, for all $\xi \in \partial X$, we have

\[
(10.2) \quad d_\epsilon(g\xi, h\xi) \leq C_1 \exp(-\epsilon (go, ho)_o) \, d_\epsilon(g^{-1}o, \xi)^{-1} \, d_\epsilon(h^{-1}o, \xi)^{-1}.
\]

Recall from the proof of Lemma 7.4 that

\[
\int d_\epsilon(g^{-1}o, \xi)^{-2\alpha} \, d\mu(\xi) \leq C_2
\]

independently of $g \in \Gamma$. By the Cauchy-Schwartz inequality, it follows that

\[
(10.3) \quad \int d_\epsilon(g^{-1}o, \xi)^{-\alpha} \, d_\epsilon(h^{-1}o, \xi)^{-\alpha} \, d\mu(\xi) \leq C_2
\]

independently of $g, h \in \Gamma$. Now (10.2) and (10.3) yield (10.1).

The remainder of the proof goes just like the last step in the proof of Lemma 7.4. Let $\epsilon$ be an arbitrary visual parameter, let $\epsilon_0$ be in the small visual range, and let $d_{\epsilon_0}$ be a visual metric as in Fact 5.3. We have

\[
\int d_\epsilon(g\xi, h\xi) \, d\mu(\xi) \asymp \int d_{\epsilon_0}(g\xi, h\xi)^{\epsilon/\epsilon_0} \, d\mu(\xi)
\]

by Fact 5.3. As $\text{hdim}(\partial X, d_{\epsilon_0}) > 2\epsilon/\epsilon_0$, the previous part says that a constant multiple of $\exp(-\epsilon/\epsilon_0)\epsilon_0 (go, ho)_o = \exp(-\epsilon (go, ho)_o)$ is an upper bound for the right hand side. \(\square\)
Lemma 10.4. For all $a, b \in \text{Lip}(\partial X, d_e) \rtimes_{\text{alg}} \Gamma$, the commutator $[\lambda_\mu(a), P_{E_{\Gamma}} \lambda_\mu^o(b) P_{E_{\Gamma}}]$ is $D_e^\alpha$-summable.

Proof. Assume that $D_e > 2$. The case when $D_e \leq 2$ is deduced as in the proof of Theorem 7.2 by snowflaking.

We recall from Theorem 7.2 that $\lambda_\mu(a)$ commutes mod $\mathcal{L}^{D_\alpha^+}$ with $P_{E_{\Gamma}}$. It follows that

$$[\lambda_\mu(a), P_{E_{\Gamma}} \lambda_\mu^o(b) P_{E_{\Gamma}}] = [P_{E_{\Gamma}} \lambda_\mu(a) P_{E_{\Gamma}}, P_{E_{\Gamma}} \lambda_\mu^o(b) P_{E_{\Gamma}}] \mod \mathcal{L}^{D_\alpha^+}$$

and the right-hand commutator is, with our notations, $[s_\mu(a), s_\mu^o(b)]$. Clearly, the property that $[s_\mu(a), s_\mu^o(b)] \in \mathcal{L}^{D_\alpha^+}$ is additive in $a$ and $b$. Now $s_\mu(aa') = s_\mu(a)s_\mu(a')$ mod $\mathcal{L}^{D_\alpha^+}$ for $a, a' \in \text{Lip}(\partial X, d_e) \rtimes_{\text{alg}} \Gamma$, hence $s_\mu^o = Js_\mu J$ is multiplicative mod $\mathcal{L}^{D_\alpha^+}$ on $\text{Lip}(\partial X, d_e) \rtimes_{\text{alg}} \Gamma$ as well. Therefore, the property that $[s_\mu(a), s_\mu^o(b)] \in \mathcal{L}^{D_\alpha^+}$ is also multiplicative in $a$ and $b$. We thus see that it suffices to treat the case when $a$ and $b$ are either Lipschitz functions or group elements.

For all $g, g' \in \Gamma$ and $\phi, \phi' \in \text{Lip}(\partial X, d_e)$ we have

$$[s(g), s^o(g')] = 0, \quad [s(\phi), s^o(\phi')] = 0, \quad [s(g), s^o(\phi)] = -J[s(\phi), s^o(g)]J.$$ 

It therefore suffices to analyze the summability of the commutator $[s(\phi), s^o(g)]$, or, more conveniently, the summability of $s^o(g^{-1})[s(\phi), s^o(g)]$ which is readily verified to be multiplication by $h \mapsto E\phi(hg^{-1}) - E\phi(h)$ on $\ell^2\Gamma$.

For $h_1, h_2 \in \Gamma$ and $\phi, \phi' \in \text{Lip}(\partial X, d_e)$ we have

$$|E\phi(h_1) - E\phi(h_2)| \leq \int |\phi(h_1 \xi) - \phi(h_2 \xi)| \, d\mu(\xi) \leq \|\phi\|_{\text{Lip}} \int d_\Gamma(h_1 \xi, h_2 \xi) \, d\mu(\xi)$$

so, by Lemma 10.3, there exists a constant $C > 0$ such that

$$|E\phi(h_1) - E\phi(h_2)| \leq C\|\phi\|_{\text{Lip}} \exp(-\varepsilon(h_1 o, h_2 o)_o).$$

Put $h_1 = hg^{-1}$ and $h_2 = h$. As $(hg^{-1} o, h_1 o) = (g^{-1} o, h_1 o) \geq d(o, h o) - d(o, go)$, we obtain

$$|E\phi(hg^{-1}) - E\phi(h)| \leq C\|\phi\|_{\text{Lip}} \exp(\varepsilon d(o, go)) \exp(-\varepsilon d(o, ho)).$$

Finally, we recall from the proof of Theorem 7.2 that multiplication by $h \mapsto \exp(-\varepsilon d(o, ho))$ is in $\mathcal{L}^{D_\alpha^+}(\ell^2\Gamma)$.

Theorem 10.5. There is a smooth subalgebra $A \subseteq C(\partial X) \rtimes \Gamma$, containing $\text{Lip}(\partial X, d_e) \rtimes_{\text{alg}} \Gamma$, such that the odd, respectively even, Fredholm modules

$$(\lambda_\mu, P_{E_{\Gamma}} \lambda_\mu^o(e) P_{E_{\Gamma}}), \quad e \text{ projection in } A$$

$$(\lambda_\mu, P_{E_{\Gamma}} \lambda_\mu^o(u) P_{E_{\Gamma}} + (1 - P_{E_{\Gamma}})), \quad u \text{ unitary in } A$$

are $D_e^\alpha$-summable Fredholm modules over $\text{Lip}(\partial X, d_e) \rtimes_{\text{alg}} \Gamma$.

We recall that a subalgebra $A$ of a $C^*$-algebra $\mathcal{A}$ is said to be smooth if it is dense and stable under holomorphic calculus. Then the projections of $A$ are dense in the projections of $\mathcal{A}$, and the unitaries of $A$ are dense in the unitaries of $\mathcal{A}$. In particular, the K-theory classes of $\mathcal{A}$ can be represented by projections, respectively unitaries, from $\mathcal{A}$. In light of this fact, Theorem 10.1 follows by combining Theorem 10.5 and Theorem 9.5.
Then $A_b$ is stable under holomorphic calculus in $C(\partial X) \rtimes \Gamma$, therefore the same holds true for $A$, the intersection of the family of subalgebras $\{A_b : b \in \text{Lip}(\partial X, d_\ast) \bowtie_{\text{alg}} \Gamma\}$.

Next, we prove the summability claim. Note first that
\[ [\lambda_\mu(a), P_{\ell^1}] \in L^{D^+,+}, \quad \text{for all } a \in A. \]
Then also $[\lambda_\mu^0(a), P_{\ell^1}] \in L^{D^+,+}$ for all $a \in A$. It follows that $P_{\ell^1} \lambda_\mu^0(e) P_{\ell^1}$ is a projection mod $L^{D^+,+}$ whenever $e \in A$ is a projection, and that $P_{\ell^1} \lambda_\mu^0(u) P_{\ell^1} + (1 - P_{\ell^1})$ is a unitary mod $L^{D^+,+}$ whenever $u \in A$ is a unitary.

By definition, if $a \in A$ then the commutator $[\lambda_\mu(a), P_{\ell^1} \lambda_\mu^0(b) P_{\ell^1}]$ is $D^+_\ast$-summable for every $b \in \text{Lip}(\partial X, d_\ast) \bowtie_{\text{alg}} \Gamma$. As already hinted in the proof of Lemma 10.4, the summability of the above commutators is in fact symmetric in $a$ and $b$. Indeed, using the fact that $[\lambda_\mu(a), P_{\ell^1}] \in L^{D^+,+}$ for all $a \in A$, we may write
\[
[\lambda_\mu(a), P_{\ell^1} \lambda_\mu^0(b) P_{\ell^1}] = [P_{\ell^1} \lambda_\mu(a) P_{\ell^1}, P_{\ell^1} \lambda_\mu^0(b) P_{\ell^1}] \mod L^{D^+,+},
\]
\[
[\lambda_\mu(b), P_{\ell^1} \lambda_\mu^0(a) P_{\ell^1}] = [P_{\ell^1} \lambda_\mu(b) P_{\ell^1}, P_{\ell^1} \lambda_\mu^0(a) P_{\ell^1}] \mod L^{D^+,+}.
\]
Now observe that the right-hand side commutators are conjugate by the symmetry $J$. This shows that, if $b \in \text{Lip}(\partial X, d_\ast) \bowtie_{\text{alg}} \Gamma$, then the commutator $[\lambda_\mu(b), P_{\ell^1} \lambda_\mu^0(a) P_{\ell^1}]$ is $D^+_\ast$-summable for every $a \in A$. We conclude that the indicated Fredholm modules are $D^+_\ast$-summable over $\text{Lip}(\partial X, d_\ast) \bowtie_{\text{alg}} \Gamma$. \hfill \qed

11. The K-homology Gysin sequence for boundary actions

$\Gamma$ remains a Gromov hyperbolic group; in this section, we attack the problem of proving that the K-homology of the reduced C*-algebra of $\Gamma$ has uniformly summable K-homology. This involves some tools from KK-theory. We start by summarizing the basic facts we will need about ‘$\gamma$-elements’ and the Dirac dual-Dirac method.

11.1. Descent, $\gamma$-elements. For any discrete group (or more generally locally compact group) $\Gamma$, ‘descent,’ in Kasparov theory, refers to a natural map
\[ j : KK(T)(A, B) \to KK_\ast(A \rtimes \Gamma, B \rtimes \Gamma) \]
which extends to equivariant KK-cycles (and homotopies) the process of integrating a $\Gamma$-equivariant $\ast$-homomorphism $A \to B$ to an ordinary $\ast$-homomorphism $A \rtimes \Gamma \to B \rtimes \Gamma$. Either the maximal or the reduced crossed-product can be used; thus there is also a ‘reduced’ descent map
\[ j_r : KK(T)(A, B) \to KK_\ast(A \rtimes_r \Gamma, B \rtimes_r \Gamma) \]
in which the reduced is used.

Descent $j$ (respectively reduced descent $j_r$) makes the abelian group $KK(A \rtimes \Gamma, B \rtimes \Gamma)$ (respectively $KK(A \rtimes_r \Gamma, B \rtimes_r \Gamma)$) a left module over the ring $KK(T)(\mathbb{C}, \mathbb{C})$, and likewise a right module, using the structure of $KK(T)(A, B)$ as a module over $KK(T)(\mathbb{C}, \mathbb{C})$, for any $\Gamma$-C*-algebras $A, B$.

The $\gamma$-element is defined contingent on the existence of a proper $\Gamma$-C*-algebra $P$ and classes $\eta \in KK(T)(\mathbb{C}, P)$ and $D \in KK(T)(\mathbb{C}, P)$ such that $D \otimes_{\mathbb{C}} \eta = 1_P \in KK(T)(\mathbb{C}, \mathbb{C})$. If such classes exist, $\gamma$ is defined to be the (idempotent) $\eta \otimes_P D \in KK(T)(\mathbb{C}, \mathbb{C})$. This determines $\gamma$ uniquely, but its existence is not clear in general. The issue is the existence of $\eta$, called the dual-Dirac morphism: it can be shown (see [32] and [12]) that for any $\Gamma$, there exist proper $P$ and a morphism $D \in KK(T)(P, \mathbb{C})$ (the Dirac morphism) such that existence of $\eta$ is equivalent to a coarse geometric condition on the group, namely, that the ‘coarse co-assembly map’ for $\Gamma$ is an isomorphism (the coarse co-assembly map is described in [12]). The coarse co-assembly map is, however, an isomorphism for all hyperbolic groups, and more generally, for groups which
uniformly embed in a Hilbert space, so all such groups have $\gamma$-elements. The first explicit construction of them in the case of hyperbolic groups is due to Kasparov and Skandalis [25].

It is not true that $\gamma = 1 \in KK_0^\Gamma (\mathbb{C}, \mathbb{C})$ for general hyperbolic groups, 1 being the class $1 := [\varepsilon] \in KK_0^\Gamma (\mathbb{C}, \mathbb{C})$ of the trivial representation $\varepsilon : C^\ast \Gamma \to \mathbb{C}$. An argument of Skandalis [38] even gives examples where $j_r(\gamma) \neq 1_{C_r^\ast \Gamma} \in KK_0 (C_r^\ast \Gamma, C_r^\ast \Gamma)$.

For cocompact lattices in $SO(n, 1)$ or $SU(n, 1)$, or free groups, $\gamma = 1$ is true due to results of Kasparov [24], and Kasparov and Julg [19]. These groups are also known to be a-T-menable, so $\gamma = 1$ follows from the Higson - Kasparov theorem (see [17]) as well.

For our purposes, we are mostly concerned about whether $\gamma$ acts as the identity on various $KK$-groups, especially $K^\ast (C_r^\ast \Gamma)$. When $\Gamma$ is hyperbolic, recent work of Lafforgue and others [28, 33] shows that $\gamma$ does act as the identity on $K$-theory of $C_r^\ast \Gamma$, but nothing seems to be known at present about the case of $K$-homology.

The point of the $\gamma$-part, is that it is the ‘topologically accessible’ part of the $K$-homology, in the sense of the following seminal theorem of Kasparov.

**Lemma 11.1 (Kasparov).** Let $\Gamma$ be a discrete group and $\mathcal{E} \Gamma$ its classifying space for proper actions. Then the canonical inflation map of [25]

$$p_\mathcal{E}^\Gamma : KK_\Gamma^\ast (A, B) \to RKK_\Gamma^\ast (\mathcal{E} \Gamma; A, B)$$

is an isomorphism from the $\gamma$-part of $KK_\Gamma^\ast (A, B)$ onto its target.

Here $RKK_\Gamma^\ast (\mathcal{E} \Gamma; A, B)$ is the $\Gamma$-equivariant representable $K$-theory of $\mathcal{E} \Gamma$. If $A = B = \mathbb{C}$ it is the ordinary $K$-theory of $C_0 (\partial \Gamma) \rtimes \Gamma$, and if $\Gamma$ is torsion-free and $\mathcal{E} \Gamma$ is a $co$-$compact$ model for the classifying space, it is $K^\ast (\Gamma \backslash \mathcal{E} \Gamma)$.

Finally, we remind the reader that since a Gromov hyperbolic group acts amenably on its boundary, $\gamma$ acts as the identity on $KK_\Gamma^\ast (C(\partial \Gamma) \otimes A, B)$ for any $A, B$. (The Dirac dual-Dirac method gives a $KK_\Gamma^\ast$-equivalence between $C(\partial \Gamma)$ and a proper $\Gamma$-$C^\ast$-algebra, while $\gamma$ acts as the identity on any $KK_\Gamma^\ast (P, B)$-group when $P$ is proper, by properties of $\gamma$ – see [32].)

Equivalently, such rings are isomorphic to their topological counter-parts:

**Corollary 11.2.** For any $\Gamma$-$C^\ast$-algebras $A, B$,

$$KK_\Gamma^\ast (C(\partial \Gamma) \otimes A, B) \cong RKK_\Gamma^\ast (\mathcal{E} \Gamma; C(\partial \Gamma) \otimes A, B)$$

$$KK_\Gamma^\ast (C(\mathcal{T}) \otimes A, B) \cong RKK_\Gamma^\ast (\mathcal{E} \Gamma; C(\mathcal{T}) \otimes A, B)$$

by the inflation map $p_\mathcal{E}^\Gamma$.

11.2. **The $\gamma$-element regarded as a K-homology class for $C_r^\ast \Gamma$.** Let $\lambda : C^\ast \Gamma \to C_r^\ast \Gamma$ be the projection from the maximal group $C^\ast$-algebra to the reduced group $C^\ast$-algebra, and let us call any element $\gamma_\tau \in K_0 (C_r^\ast \Gamma)$ such that

$$\lambda^\ast (\gamma_\tau) = \gamma \in K_0 (C^\ast \Gamma) \cong KK_0^\Gamma (\mathbb{C}, \mathbb{C})$$

a **reduced $\gamma$-element** for $\Gamma$.

**Proposition 11.3.** The map $\lambda^\ast : K^\ast (C_r^\ast \Gamma) \to K^\ast (C^\ast \Gamma) \cong KK_0^\Gamma (\mathbb{C}, \mathbb{C})$ induces an isomorphism between the $\gamma$-parts of these two rings. In particular, if $\Gamma$ has a $\gamma$-element, then it has a reduced $\gamma$-element.

The (or any such) element $\gamma_\tau$ as in the Proposition will play a role in the ‘Gysin sequence’ developed in the next subsection.

**Proof.** We produce a map $\gamma K^\ast (C^\ast \Gamma) \to \gamma K^\ast (C_r^\ast \Gamma)$ inverting $\lambda^\ast$ as follows. We first recall that the standard identification $KK_\Gamma^\ast (A, \mathbb{C}) \to KK_\Gamma (A \rtimes \Gamma, \mathbb{C})$, coming from the fact that the groups have the same cycles when $\Gamma$ is discrete, can be expressed in terms the ‘descent’ construction
In particular, taking \( \gamma \in KK_0(C, C) \), its image under the isomorphism with \( KK_0(C^\ast \Gamma, C) = K^0(C^\ast \Gamma) \) is \( j(\gamma) \otimes \Gamma \[e\] \). With these formalities aside, we next factor the \( \gamma \)-element, or rather, its image in \( KK_0(C^\ast \Gamma, C) \) as follows. Let \( \eta \in KK(C^\ast \Gamma, P) \) be the dual-Dirac morphism, and let \( \Delta \in KK(P, C) \) be the Dirac morphism for \( \Gamma \). Then \( j(\eta) \otimes_{P^\ast \Gamma} j(D) \otimes_{\Gamma} \[e\] \) factors the image of \( \gamma \) in \( KK_0(C^\ast \Gamma, C) \). This is because \( \gamma = \eta \otimes_P D \), and the naturality of the descent map.

More generally, any \( a \in \gamma KK^*_\Gamma(C, C) \), interpreted as an element of \( K^*(C^\ast \Gamma) \), can be thus factored as

\[
j(a) \otimes_{\Gamma} \[e\] = j(a \otimes_C \gamma) \otimes_{\Gamma} \[e\] = j(a) \otimes_{\Gamma} \eta \otimes_P D \otimes_{\Gamma} \[e\] = j(a) \otimes_{\Gamma} j(\eta) \otimes_{P^\ast \Gamma} j(D) \otimes_{\Gamma} \[e\]
\]

where the first equality is due to \( \gamma \otimes_C a = a \) for \( a \) in the \( \gamma \)-part, the third by the naturality of the descent map.

Now to obtain an element \( a' \) such that \( \lambda^*(a') = a \), consider the element

\[
b' = j_r(a) \otimes_{C^\ast \Gamma} j_r(\eta) \in KK_\ast(C^\ast \Gamma, P \rtimes_r \Gamma)
\]

defined using the reduced descent map. Now \( P \) being proper implies \( P \rtimes_r \Gamma \cong P \rtimes \Gamma \). Applying this isomorphism to \( b' \) gives a class \( b \in KK_\ast(C^\ast \Gamma, P \rtimes \Gamma) \). Then the required element \( a' \) such that \( \lambda^*(a') = a \) is

\[
a' = b \otimes_{P^\ast \Gamma} j(D) \otimes_{P^\ast \Gamma} \[e\].
\]

\[\Box\]

**Remark 11.4.** In particular, Kasparov’s Theorem (Lemma 11.1) can be alternately phrased in terms of the K-homology of the **reduced** \( C^\ast \Gamma \)-algebra: the \( \gamma \)-part of the K-homology of \( C^\ast \Gamma \) is isomorphic to the topological group ring \( RKK^\Gamma(\mathcal{E} \Gamma; C, C) \) (by the composition of \( \lambda^* \) and the inflation map.)

### 11.3. The Gysin sequence

Let \( \Gamma \) be a hyperbolic group, \( \partial \Gamma \) its Gromov boundary, etc. Let \( i : C \to C(\partial \Gamma) \) be the natural inclusion of \( C \) as constant functions on \( \partial \Gamma \), defining a morphism \( [i] \in KK^0(C, C(\partial \Gamma)) \) and then, by reduced descent, a morphism \( [i] := j_r([i]) \in KK_0(C^\ast \Gamma, C(\partial \Gamma) \rtimes \Gamma) \), which is nothing but the Kasparov morphism determined by the \( C^\ast \)-algebra monomorphism \( i : C^\ast \Gamma \to C(\partial \Gamma) \rtimes \Gamma \) of the reduced \( C^\ast \)-algebra in the reduced crossed-product.

Then composition with \( [i] \) induces a map \( i^* : K^*(C(\partial \Gamma) \rtimes \Gamma) \to K^*(C^\ast \Gamma) \). The aim of this section is to compute this map.

**Lemma 11.5.** The range of \( i^* : K^*(C(\partial \Gamma) \rtimes \Gamma) \to K^*(C^\ast \Gamma) \) is contained in the \( \gamma \)-part of \( K^*(C^\ast \Gamma) \).

**Proof.** Since \( \gamma \) acts as the identity on \( K^*(C(\partial \Gamma) \rtimes \Gamma) \) and \( i^* \) is a \( KK^\Gamma(C, C) \)-module map, for any \( x \in KK^1(C(\partial \Gamma), C) \) it holds that \( i^*(x) = i^*(\gamma x) = \gamma i^*(x) \in \gamma KK^\Gamma(C, C) \). \[\Box\]

Let \( X \) be a Rips complex for \( \Gamma \) which models \( \mathcal{E} \Gamma \) (see [31]). Let

- \( r : C(X) \to C(\partial X) \cong C(\partial \Gamma) \) the \( \Gamma \)-equivariant map of restriction to the boundary.
- \( u : C \to C(X) \) be the inclusion as constant functions

Both maps are \( \Gamma \)-equivariant.

**Lemma 11.6.** The map

\[
u^* : KK^\Gamma(C(X), C) \to KK^\Gamma(C, C)
\]
on $\text{KK}^\Gamma$-theory induced by composition with $u \in \text{KK}^\Gamma_0(\mathbb{C}, C(X))$ is an isomorphism onto the $\gamma$-part of $\text{KK}^\Gamma(\mathbb{C}, \mathbb{C})$. Moreover, the composition

$$
\text{KK}^\Gamma(C(\partial \Gamma), \mathbb{C}) \xrightarrow{r^*} \text{KK}^\Gamma_*(C(\overline{\Gamma}), \mathbb{C}) \xrightarrow{u^*} \gamma\text{KK}^\Gamma(\mathbb{C}, \mathbb{C})
$$

equals $i^*_r$.

**Proof.** Recalling that $X = \mathcal{E}\Gamma$, Lemma 11.6 says that the inflation map

$$p_X^* : \text{KK}^\Gamma_*(A, B) \to \text{RKK}^\Gamma(X; A, B)$$

is an isomorphism from the $\gamma$-part of $\text{KK}^\Gamma_*(A, B)$ to $\text{RKK}^\Gamma(X; A, B)$. On the other hand, $\overline{X}$ is $H$-equivariantly contractible for every finite subgroup $H$ of $\Gamma$. In other worlds, $C(\overline{X})$ is $H$-equivariantly homotopy equivalent to $\mathbb{C}$ for every finite $H \subset \Gamma$, equivalently, $u : \mathbb{C} \to C(\overline{\Gamma})$ regarded as an element of $\text{KK}^H_0(\mathbb{C}, C(\overline{X}))$ is invertible for every such $H$. Hence by [32], $p_X^*(u)$ is invertible. The last statement is a routine verification left to the reader. □

Now the $\Gamma$-exact sequence

$$0 \to C_0(X) \to C(\overline{X}) \to C(\partial X) \cong C(\partial \Gamma) \to 0$$

of $\Gamma$-$C^*$-algebras induces a long exact sequence

$$
\cdots \to \text{KK}^\Gamma_0(C_0(X), \mathbb{C}) \xrightarrow{\varphi^*} \text{KK}^\Gamma_0(C(\overline{X}), \mathbb{C}) \xrightarrow{\tau^*} \text{KK}^\Gamma_0(C(\partial X), \mathbb{C}) \xrightarrow{\Delta^*} \text{KK}^\Gamma_0(C_0(X), \mathbb{C}) \to \cdots
$$

in $\Gamma$-equivariant $K$-homology, where $\varphi^*$ is the map on $K$-homology induced by the equivariant $*$-homomorphism $C_0(X) \to C(\overline{X})$, and $r^*$ is induced by the restriction homomorphism $r : C(\overline{X}) \to C(\partial X)$.

By Lemma 11.3 we can replace the middle term $\text{KK}^\Gamma_0(C(\overline{X}), \mathbb{C})$ by $\gamma\text{KK}^\Gamma(\mathbb{C}, \mathbb{C})$. With this replacement, the map $r^*$ is replaced by $i^*_r$. Exactly as in [11] one computes that the map $\varphi^*$ becomes the restriction to $\gamma\text{KK}^\Gamma_0(\mathbb{C}, \mathbb{C})$ of the map $\text{KK}^\Gamma_0(\mathbb{C}, \mathbb{C}) \to \text{KK}^\Gamma_0(C_0(X), \mathbb{C})$ of external product with the Euler class defined to be

$$\text{Eul}_\Gamma := (p_X^*)^{-1}(\Delta_X) \in \text{KK}^\Gamma_0(C_0(X), \mathbb{C})$$

where $\Delta_X \in \text{RKK}^\Gamma_0(X; C_0(X), \mathbb{C})$ is the morphism induced by the diagonal embedding $X \to X \times X$. (See §3 of [11].)

Now if $\Gamma$ is any discrete group with co-compact $\mathcal{E}\Gamma$, then $\text{KK}^\Gamma_0(C_0(X), \mathbb{C}) \cong K_0(\Gamma \setminus X) \cong K_0(B\Gamma)$, and under this identification, the Euler class for $\Gamma$ is just the ordinary Euler characteristic of $\Gamma$ (an integer, equal to the Euler characteristic of $B\Gamma$) multiplied by the class of a point in $K$-homology (see [11]). So we can insert this into the sequence (11.2).

**Theorem 11.7 (Gysin sequence for $K$-homology).** Let $\Gamma$ be a torsion-free hyperbolic group. Then there is an exact sequence

$$0 \to K_1(B\Gamma) \to K^0(\mathcal{C}(\partial \Gamma) \rtimes \Gamma) \xrightarrow{i^*} \gamma\text{KK}^\Gamma_0(\mathbb{C}, \mathbb{C}) \xrightarrow{\text{Eul}_\Gamma} K_0(B\Gamma)$$

$$\to K^1(\mathcal{C}(\partial \Gamma) \rtimes \Gamma) \xrightarrow{i^*} \gamma\text{KK}^\Gamma_0(\mathbb{C}, \mathbb{C}) \to 0$$

where $i^* : K^*(\mathcal{C}(\partial \Gamma) \rtimes \Gamma) \to K^*(\mathcal{C}(\partial \Gamma) \rtimes \Gamma)$ is the map induced by the inclusion $i : \mathcal{C}(\partial \Gamma) \rtimes \Gamma \to \mathcal{C}(\partial \Gamma) \rtimes \Gamma$, and where $\text{Eul}$ is the map $\text{Eul}(a) = \chi(\Gamma) \text{index}(a) [\text{put}] \in K_0(B\Gamma)$, with index the ordinary Fredholm index map $\text{KK}^\Gamma(\mathbb{C}, \mathbb{C}) \to \mathbb{Z}$, and $[\text{put}]$ is the class in $K$-homology of a point in $B\Gamma$. 
Corollary 11.8. The restriction homomorphism $i^*: K^*(C(\partial \Gamma) \times \Gamma) \to \gamma K^*(C^*_r \Gamma)$ is a surjection in dimension $*=1$, and a surjection in both dimensions if $\chi(\Gamma) = 0$. When $\chi(\Gamma) \neq 0$, let $\gamma_r \in \gamma K^0(C^*_r \Gamma)$ be a reduced $\gamma$-element. Then for each $a \in \gamma K^0(C^*_r \Gamma)$ there exists $b \in K^0(C(\partial \Gamma) \times \Gamma)$ such that

$$a = \text{index}(a) \gamma_r + i^*(b).$$

Proof. The statement regarding $*=1$ and the one when the Euler characteristic is zero are both obvious from the Gysin sequence. For the second statement, let $a \in \gamma K^0(C^*_r \Gamma)$, then since $\text{index}(\gamma_r) = 1$, $a - \text{index}(a) \gamma_r$ has index zero and hence is in the kernel of the map $\text{Eul}$. Hence it is in the range of $i^*$, by the Gysin sequence. Thus $a = \text{index}(a) \gamma_r + i^*(b)$ for $b \in \text{ran}(i^*)$ as claimed. \hfill \Box

The results of this section show that, up to a cyclic summand, the $K$-homology of the reduced $C^*$-algebra of $\Gamma$ comes entirely from restricting $\Gamma$-equivariant $K$-homology classes from the boundary.

12. Uniformly summable $K$-cycles over the reduced group $C^*$-algebra

Let $\Gamma$ be regular and torsion-free. Recall that every class in $K^1(C(\partial X) \times \Gamma)$ can be represented by an odd Fredholm module of the form

$$(\lambda_{\mu}, P_{\ell^2 \Gamma} \lambda_{\mu}^{\text{op}}(e) P_{\ell^2 \Gamma})$$

for some projection $e \in C(\partial X) \times \Gamma$, and every class in $K^1(C(\partial X) \times \Gamma)$ can be represented by a balanced even Fredholm module of the form

$$(\lambda_{\mu}, P_{\ell^2 \Gamma} \lambda_{\mu}^{\text{op}}(u) P_{\ell^2 \Gamma} + (1 - P_{\ell^2 \Gamma}))$$

for some unitary $u \in C(\partial X) \times \Gamma$. At the level of cycles, the map $i^*$ on $K$-homology induced by $i: C^*_r \Gamma \to C(\partial \Gamma) \times \Gamma$ merely restricts the representation of $C(\partial \Gamma) \times \Gamma$ to the subalgebra $C^*_r \Gamma$. Thus we restrict the representation $\lambda_{\mu}$ to $C^*_r \Gamma$. Then, as each $\lambda_{\mu}(g)$ commutes with $P_{\ell^2 \Gamma}$, we can remove the degenerate summand $(1 - P_{\ell^2 \Gamma}) \cdot \ell^2(\Gamma, L^2(\partial X, \mu))$. Note that the restriction of $\lambda_{\mu}$ to the remaining summand $P_{\ell^2 \Gamma} \cdot \ell^2(\Gamma, L^2(\partial X, \mu)) = \ell^2 \Gamma$ is the regular representation $\lambda$. Thus, over $C^*_r \Gamma$ the above Fredholm modules take the form

$$\Phi(a) := (\lambda, P_{\ell^2 \Gamma} \lambda_{\mu}^{\text{op}}(a) P_{\ell^2 \Gamma})$$

where $a$ is a projection or a unitary in $C(\partial X) \times \Gamma$. If $a$ is a projection or a unitary in $\mathcal{A}$, where $\mathcal{A}$ as in Theorem 10.7, then $\Phi(a)$ is $D^\gamma$-summable over the group algebra $\mathbb{C} \Gamma$.

Lemma 12.1. Assume that $\Gamma$ is regular and torsion-free, and let $\mathcal{A}$ be the smooth subalgebra of Theorem 10.7. Then every class in the image of the restriction map $i^*: K^*(C(\partial X) \times \Gamma) \to K^*(C^*_r \Gamma)$ is represented by a Fredholm module of the form $[\Phi(a)]$ for some projection, respectively unitary $a \in \mathcal{A}$. In particular, every class in $i^* K^1(C^*_r \Gamma)$ has a representative which is $D^\gamma$-summable over $\mathbb{C} \Gamma$.

Combining Lemma 12.1 and Corollary 11.8 we obtain:

Theorem 12.2. Assume that $\Gamma$ is regular and torsion-free, and let $\mathcal{A}$ be the smooth subalgebra of Theorem 10.7. Then the following hold.

- Every class in $\gamma K^1(C^*_r \Gamma)$ is of the form $[\Phi(e)]$ for some projection $e \in \mathcal{A}$. In particular, every class in $\gamma K^1(C^*_r \Gamma)$ is represented by a Fredholm module which is $D^\gamma$-summable over $\mathbb{C} \Gamma$. 

• If $\chi(\Gamma) = 0$, then every class in $\gamma K^0(C^*_r \Gamma)$ is of the form $[\Phi(u)]$ for some unitary $u \in \mathcal{A}$. In particular, every class in $\gamma K^0(C^*_r \Gamma)$ is represented by a Fredholm module which is $D^*_r$-summable over $C\Gamma$.

If $\chi(\Gamma) \neq 0$, and $\gamma_r$ is a reduced $\gamma$-element, then every class in $\gamma K^0(C^*_r \Gamma)$ is of the form $k\gamma_r + [\Phi(u)]$ for some integer $k$ and some unitary $u \in \mathcal{A}$. In particular, if $\gamma_r$ is represented by a Fredholm module which is $p(\gamma_r)$-summable over $C\Gamma$, then every class in $\gamma K^0(C^*_r \Gamma)$ is represented by a Fredholm module which is $\max\{p(\gamma_r), D^*_r\}$-summable over $C\Gamma$.

We now specialize Theorem 12.2 to four families of a-T-menable groups; note that $\gamma = 1$ for a-T-menable groups by [17], so $\gamma K^*(C^*_r \Gamma) = K^*(C^*_r \Gamma)$.

12.1. Free groups. Let $\Gamma$ be a finitely generated free group of rank at least 2. Given any $p > 2$, every class in $i^*K^*(C^*_r \Gamma)$ has a $p$-summable representative over $C\Gamma$. On the other hand, the Julg - Valette model for the $\gamma$-element [20] is $1$-summable over $C\Gamma$, hence the same holds true for the reduced $\gamma$-element $\gamma_r$. We conclude that $C^*_r \Gamma$ has uniformly $p$-summable K-homology over $C\Gamma$ for every $p > 2$.

12.2. Real uniform lattices. Let $\Gamma$ be a torsion-free uniform lattice in $SO(n, 1)$. Then classes in $i^*K^*(C^*_r \Gamma)$ are $(n-1)^+$-summable over $C\Gamma$ when $n \geq 4$, respectively $p$-summable over $C\Gamma$ for every $p > 2$, when $n = 2, 3$.

If $n$ is odd, then $\chi(\Gamma) = 0$ so $i^*K^*(C^*_r \Gamma)$ covers in fact all the K-homology of $C^*_r \Gamma$. On the other hand, but still in this odd case, Kasparov shows in [24] that the $\gamma$-element for $SO(n, 1)$ is represented by a Fredholm module in which the operator $F$ is the phase of a degree 1 elliptic operator on the sphere $S^{n-1}$. Moreover, the unitary action of the group $\Gamma$ commutes with $F$ modulo pseudodifferential operators of order $-1$ because the action is conformal (and so the operators $F$ and $gFg^{-1}$ have the same symbol). Hence the commutators $[g, F]$ have singular values satisfying $s_k \approx k^{-(n-1)}$, that is, Kasparov’s Fredholm module is $(n-1)^+$-summable over $C\Gamma$.

If $n$ is even, we may set $\gamma_r$ to be the pull-back of the $\gamma$-element in $KK^0_{SO(n+1)}(\mathbb{C}, \mathbb{C})$ to an element of $KK^0_{SO}(\mathbb{C}, \mathbb{C})$ under the inclusion $\Gamma \subset SO(n, 1) \subset SO(n+1, 1)$ of $\Gamma$ as a closed subgroup of $SO(n+1, 1)$. The resulting class clearly satisfies the conditions of Proposition 11.3 and, by the previous paragraph, it is represented by an $n^+$-summable Fredholm module.

We conclude in the end that the K-homology of $C^*_r \Gamma$ is uniformly $n^+$-summable over $C\Gamma$ when $n \geq 3$, respectively $p$-summable over $C\Gamma$ for every $p > 2$, when $n = 2$.

12.3. Complex uniform lattices. Let $\Gamma$ be a torsion-free uniform lattice in $SU(n, 1)$. Then classes in $i^*K^*(C^*_r \Gamma)$ are $(2n)^+$-summable over $C\Gamma$. A model for the $\gamma$-element of $SU(n, 1)$ has been given by Julg and Kasparov in [19]. In this case, the method involves construction of an appropriate hypoelliptic operator on the contact manifold $S^{2n-1}$ (the contact structure is $SU(n, 1)$-invariant.) Inspection of the article [19] reveals that the relevant commutators $[g, F]$ are pseudodifferential operators in the class $\Psi^{-1}(S^{2n-1})$, and it is well-known that the singular values in this case satisfy $s_k \approx k^{-(2n)}$. We conclude that the K-homology of $C^*_r \Gamma$ is uniformly $(2n)^+$-summable over $C\Gamma$.

12.4. Small-cancellation groups. Let $\Gamma$ be a torsion-free group given by a finite presentation $(S \mid R)$ satisfying the $C'(1/6)$ small-cancellation condition. As a geometric model for $\Gamma$ we take the Cayley graph with respect to $S$, denoted $\Gamma(S)$. Combining Fact 5.4 and Fact 5.8 we get the coarse estimate $\text{visdim} \partial \Gamma(S) \leq 5\delta \log \left(2|S| - 1\right)$ where $\delta$ is the hyperbolicity constant of $\Gamma(S)$. In a $C'(1/6)$ situation, it is possible to give a combinatorial estimate for $\delta$, namely, $\delta \leq 3 \max\{|r| : r \in R\}$ by [14] Appendix, Thm.36. We thus get the explicit, though far from
optimal, estimate
\[ \text{visdim } \partial \Gamma(S) \leq 15 \log(2|S| - 1) \max\{|r| : r \in \mathcal{R}\} =: \kappa(S|R). \]

As \( \Gamma \) is torsion-free, the 2-complex defined by the presentation \( \langle S \mid \mathcal{R} \rangle \) is aspherical. Hence \( \chi(\Gamma) = 1 - |S| + |\mathcal{R}| \), and \( \Gamma \) has cohomological dimension at most 2. If \( \text{cd } \Gamma = 1 \) then \( \Gamma \) is a free group, a case we have already discussed. So let us assume that \( \text{cd } \Gamma = 2 \), in which case \( \partial \Gamma(S) \) has topological dimension 1. We then have the following fact, due to Misha Kapovich (personal communication):

**Lemma 12.3.** If \( \Gamma \) is a hyperbolic group with 1-dimensional boundary, then \( \Gamma \) is regular in the sense of Definition 9.1.

**Proof.** A result of Bonk and Kleiner \([6]\) says that the boundary of a hyperbolic group contains a quasi-circle provided that the group is not virtually free. In particular, \( \partial \Gamma \) contains a topological circle. The proof is completed by the following general claim: if \( Z \) is a \( d \)-dimensional compact space containing a \( d \)-dimensional topological sphere \( S^d \), then \( Z \) admits a continuous self-map without fixed points.

To prove the claim, recall the following alternative definition of topological dimension: a compact space \( X \) has dimension at most \( n \) if and only if every continuous map \( X_0 \to S^n \), defined on a compact subset \( X_0 \subseteq X \), can be continuously extended to the entire \( X \). Applying this fact to the space \( Z \) and the identity map \( S^d \to S^d \), we obtain a retraction \( \rho : Z \to S^d \). The composition \( \tau \rho \), where \( \tau : S^d \to S^d \) is the antipodal involution, is clearly fixed-point free. \( \square \)

Thus \( \Gamma \) meets the conditions of Theorem 1.6. It follows that the odd K-homology \( \text{K}^1(\text{C}^*_r \Gamma) \) is uniformly \( p \)-summable over \( \text{C}^*_r \Gamma \) for every \( p > \kappa(S|R) \), and that the same is true for the even K-homology \( \text{K}^0(\text{C}^*_r \Gamma) \) provided that \( \chi(\Gamma) = 0 \), i.e., \( |S| - |\mathcal{R}| = 1 \).

13. Coda

**Problems and related work.** We start by recording two problems that we have already mentioned in the introduction.

*Problem.* Let \( \Gamma \) be a hyperbolic group. Does there exist a finitely summable representative for a reduced \( \gamma \)-element \( \gamma_r \in \text{K}^0(\text{C}^*_r \Gamma) \)?

*Problem.* Let \( \Gamma \) be a hyperbolic group. Does \( \gamma \) act as the identity on the K-homology of \( \text{C}^*_r \Gamma \)?

We believe the answer to the first problem to be positive, since the case \( \chi(\Gamma) = 0 \) appears as a by-product of our results, and since there is no reason to expect the Euler characteristic to be actually relevant to the problem. The second problem may well have a negative answer. The examples of Skandalis, where K-amenability fails for \( \text{C}^*_r \Gamma \), have \( j_r(\gamma) \neq 1 \in \text{KK}(\text{C}^*_r \Gamma, \text{C}^*_r \Gamma) \).

One of our main results, Theorem B of the introduction, suggests the following.

*Problem.* Let \( \Gamma \) be a hyperbolic group. Is the K-homology of \( \text{C}^*_r \Gamma \) uniformly summable over \( \text{C}^*_r \Gamma \)?

Of course if the first problem has a positive answer, we obtain at least a solution to this third problem for the \( \gamma \)-part of the K-homology of \( \text{C}^*_r \Gamma \). Conjecturally, fundamental groups of negatively curved compact manifolds fit this class (i.e., have finitely summable models for the \( \gamma \)-element) but we have not attempted to resolve this here.

**Other results.** Lott \([30]\) employs elliptic operator methods to construct K-homology cycles, some of which are finitely summable, for the crossed-product C*-algebra arising from the action of a subgroup of \( \text{SO}(n, 1) \) on its limit set. Lott’s constructions are not obviously related to ours.
In Rave’s thesis [36] it is proved that AF C*-algebras have, in the terminology of this paper, uniformly summable K-homology. (Note that [36] also contains a detailed account of the fact, already mentioned in this introduction, that the commutative C*-algebra $C^*(M)$ of a closed manifold $M$ has $(\dim M)^{r}$-summable K-homology over $C^\infty(M)$.)

Very recently, Goffeng and Mesland [15] have addressed the issue of uniform summability for K-homology in the case of Cuntz-Krieger C*-algebras. This is a family which is analogous, in many ways, to the boundary C*-crossed products considered herein. In [15] it is shown that the odd K-homology of a Cuntz-Krieger C*-algebra is uniformly summable.

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