The integration problem for complex Lie algebroids

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Abstract

A complex Lie algebroid is a complex vector bundle over a smooth (real) manifold $M$ with a bracket on sections and an anchor to the complexified tangent bundle of $M$ which satisfy the usual Lie algebroid axioms. A proposal is made here to integrate analytic complex Lie algebroids by using analytic continuation to a complexification of $M$ and integration to a holomorphic groupoid. A collection of diverse examples reveal that the holomorphic stacks presented by these groupoids tend to coincide with known objects associated to structures in complex geometry. This suggests that the object integrating a complex Lie algebroid should be a holomorphic stack.

1 Introduction

It is a pleasure to dedicate this paper to Professor Hideki Omori. His work over many years, introducing ILH manifolds [30], Weyl manifolds [32], and blurred Lie groups [31] has broadened the notion of what constitutes a “space.” The problem of “integrating” complex vector fields on real manifolds seems to lead to yet another kind of space, which is investigated in this paper.

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Recall that a Lie algebroid over a smooth manifold $M$ is a real vector bundle $E$ over $M$ with a Lie algebra structure (over $\mathbb{R}$) on its sections and a bundle map $\rho$ (called the anchor) from $E$ to the tangent bundle $TM$, satisfying the Leibniz rule

$$[a, fb] = f[a, b] + (\rho(a)f)b$$

for sections $a$ and $b$ and smooth functions $f: M \to \mathbb{R}$. Sections of a Lie algebroid may be thought of as “virtual” vector fields, which are mapped to ordinary vector fields by the anchor.

There is an analogous definition for complex manifolds, in which $E$ is a holomorphic vector bundle over $M$, and the Lie algebra structure is defined on the sheaf of local sections. Such objects are called complex Lie algebroids by Chemla [6], but they will be called in this paper holomorphic Lie algebroids to distinguish them from the “hybrid” objects defined in [5] as follows.

**Definition 1.1** A complex Lie algebroid (CLA) over a smooth (real) manifold $M$ is a complex vector bundle $E$ over $M$ with a Lie algebra structure (over $\mathbb{C}$) on its space $E$ of sections and a bundle map $\rho$ (called the anchor) from $E$ to the complexified tangent bundle $T_{\mathbb{C}}M$, satisfying the Leibniz rule

$$[a, fb] = f[a, b] + (\rho(a)f)b$$

for sections $a$ and $b$ in $E$ and smooth functions $f: M \to \mathbb{C}$.

The unmodified term “Lie algebroid” will always mean “real Lie algebroid.”

Every Lie algebroid may be realized as the bundle whose sections are the left invariant vector fields on a local Lie groupoid $\Gamma$. The integration problem of determining when $\Gamma$ can be taken to be a global groupoid was completely solved in [8], but, for a complex Lie algebroid, it is not even clear what the corresponding local object should be. The main purpose of the present paper is to propose a candidate for this object.

Any CLA $E$ whose anchor is injective may be identified with the involutive subbundle $\rho(E) \subseteq T_{\mathbb{C}}M$. Such subbundles have been studied extensively under the name of “involutive structures” or “formally integrable structures,” for instance by Treves [39]. An important issue in these studies has been to establish the existence (or nonexistence) of “enough integrals,” i.e. smooth functions which are annihilated by all the sections of $E$. In
the general $C^\infty$ case, the question is very subtle and leads to deep problems and results in linear PDE theory. When $E$ is analytic, though, one can sometimes proceed in a fairly straightforward way by complexifying $M$ and extending $E$ by holomorphic continuation to an involutive holomorphic tangent subbundle of the complexification, where it defines a holomorphic foliation. The leaf space of this foliation is then a complex manifold whose holomorphic functions restrict to $M$ to give integrals of $E$. (A succinct example of this may be found at the end of [35]; see Section 3.2 below.)

The leaf space described above may be thought of as the “integration” of the involutive subbundle $E$; this suggests a similar approach to analytic CLAs whose anchors may not be injective. Any analytic CLA $E$ over $M$ may be holomorphically continued to a holomorphic Lie algebroid $E'$ over a complexification $M_C$; $E'$ may then be integrated to a (possibly local) holomorphic groupoid $G$. Since $G$ will generally have nontrivial isotropy, one must take this into account by considering not just the orbit space of $G$, but the “holomorphic stack” associated to $G$.

Some intuition behind the complexification approach to integration comes from the following picture in the real case. If $G$ is a Lie group, there is a long tradition of thinking of its Lie algebra elements as tiny arrows pointing from the identity of $G$ to “infinitesimally nearby” elements. If $G$ is now a Lie groupoid over a manifold $M$, $M$ may be identified with the identity elements of $G$, and an element $a$ of the Lie algebroid $E$ of $G$ may be thought of as an arrow from the base $x \in M$ of $a$ to a groupoid element with its source at $x$ and its target at an infinitesimally nearby $y \in G$. The tangent vector $\rho(a)$ is then viewed as a tiny arrow in $M$ pointing from $x$ to $y$.

Now suppose that $E$ is a complex Lie algebroid over $M$. Then $\rho(a)$ is a complex tangent vector. To visualize it, one may still think of the tail of the tiny arrow as being at $a$, but the imaginary part of the vector will force the head to lie somewhere “out there” in a complex manifold $M_C$ containing $M$ as a totally real submanifold. To invert (and compose) such groupoid elements requires that their sources as well as targets be allowed to lie in this complexification $M_C$. Thus, the integration should be a groupoid over the complexification.

What exactly is this complexification? Haefliger [18], Shrutick [37], and Whitney and Bruhat [44] all showed that every analytic manifold $M$ may be embedded as an analytic, totally real submanifold of a complex manifold $M_C$. Any two such complexifications are canonically isomorphic near $M$.

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1In this paper, “analytic” will always mean “real analytic”, and “holomorphic” will be used for “complex analytic.”
Consequently, the identity map extends uniquely near $M$ to an antiholo-
morphic involution of $M_C$ ("complex conjugation") having $M$ as its fixed
point set. Finally, Grauert [16] showed that the complexification may be
taken to have a pseudoconvex boundary and therefore be a Stein manifold.$M_C$ is then called a Grauert tube.

Of course, constructing the complexification requires that the Lie alge-
broid have a real analytic structure. For the underlying smooth manifold
$M$, such a structure exists and is unique up to isomorphism [43], though the
isomorphism between two such structures is far from canonical. Extending
the analyticity to $E$ is an issue which must be dealt with in each example.

In fact, examples are at the heart of this paper. Except for some brief
final remarks, the many observations and questions about CLAs which arise
naturally by extension from the real theory and from complex geometry will
be left for future work. Concepts such as cohomology, connections, modular
classes, Kähler structure, and quantization are discussed by Block [3] and
Cannas and the author [5] and in work in progress with Eric Leichtnam and
Xiang Tang [24].

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2 Complexifications of real Lie algebroids

A complex Lie algebroid over a point is just a Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. It seems
natural to take as integration of $\mathfrak{g}$ a holomorphic Lie group $G$ with this Lie
algebra. In particular, if $\mathfrak{g}$ is the complexification of a real Lie algebra $\mathfrak{g}_R$,
then $G$ is a complexification of a real Lie group $G_R$.

Next, given any real Lie algebroid $E_R$, its complexification $E$ becomes a
complex Lie algebroid when the bracket and anchor are extended by complex
(bi)linearity. If $E_R$ is integrated to a (possibly local) Lie groupoid $G_R$, then
a natural candidate for $G$ would be a complexification of $G_R$. For this
complexification to exist, $G_R$ must admit an analytic structure, and, when
this structure does exist, it is rarely unique (though it may be unique up to
isomorphism).
2.1 Zero Lie algebroids

Let $E_\mathbb{R}$ be the zero Lie algebroid over $M$. An analytic structure on $E_\mathbb{R}$ is just an analytic structure on $M$, which exists but is unique only up to noncanonical isomorphism. Now the unique source-connected Lie groupoid integrating $E_\mathbb{R}$ is the manifold $M$ itself, which always admits a complexification $M_C$. This complex manifold is far from unique, but its germ along $M$ is unique up to natural (holomorphic) isomorphism, given the analytic structure on $M$. One could say that the choice of analytic structure on $M$ is part of the integration of this zero complex Lie algebroid.

This example suggests that the object integrating $M$ should be the germ along $M$ of a complexification of $M$. Getting rid of all the choices, including that of the analytic structure, requires that the complexification $M_C$ be shrunk even further, to a formal neighborhood of $M$ in $M_C$. Both of these possibilities will be considered in many of the examples which follow.

**Remark 2.1** One could define the germ as an object for which the underlying topological space is $M$, but with a structure sheaf given by germs along $M$ of holomorphic functions on $M_C$. But these are just the analytic functions on $M$. For the formal neighborhood, the structure sheaf becomes simply the infinite jets of smooth complex-valued functions.

2.2 Tangent bundles

Let $E = T_C M$ be the full complexified tangent bundle. Once again, an analytic structure on $E_\mathbb{R} = TM$ is tantamount to an analytic structure on $M$, which leads to many complexifications $M_C$, as above. A source-connected Lie groupoid integrating $TM$ is the pair groupoid $M \times M$, while the source-simply connected groupoid is the fundamental groupoid $\pi(M)$. The pair groupoid $M_C \times M_C$ is then a complexification of $M \times M$ and may be taken as an integration of the complex Lie algebroid $T_C M$. On the other hand, $\pi(M)$ could be complexified to $\pi(M_C)$; however, the result is sensitive, even after restriction to $M$, to the choice of $M_C$. If $M_C$ is taken to be a small neighborhood of $M$, the restriction to $M$ is just $\pi(M)$.

2.2.1 Interlude: The integration as a stack

Some of the dependence on the choice of $M_C$ disappears when two groupoids are declared to be “the same” when they are Morita equivalent. The groupoid is then seen as a presentation of a differential stack (see Behrend [2] and
Tseng and Zhu [40]) or, more precisely, a holomorphic stack. Since a transitive groupoid is equivalent to any of its isotropy groups, the stack represented by a pair groupoid $M \times M$ is just a point. The only difference between this and $M_C \times M_C$ is that the latter represents a “holomorphic point.” Depending on the choice of groupoid, this point as a stack might carry isotropy equal to the fundamental group of $M$ or even of one of its complexifications.

2.3 Action groupoids

Any (right) action of a Lie algebra $\mathfrak{k}$ on $M$ induces an action, or transformation, groupoid structure on the trivial vector bundle $E_R = M \times \mathfrak{k}$. The complexified bundle $E = M \times \mathfrak{k}_C$ becomes a complex Lie algebroid whose anchor maps the constant sections of $E$ to a finite dimensional Lie algebra of complex vector fields on $M$.

When the $\mathfrak{k}$ action comes from a (left) action of a Lie group $K$, $E_R$ integrates to the transformation groupoid $K \times M$; in fact, Dazord [9] showed that $E_R$ is always integrable to a global groupoid $G$ which encodes the (possibly local) integration of the $\mathfrak{k}$ action.

Passing from $E_R$ to $E$ complicates issues significantly. First, complexifying $G$ requires an analytic structure on it, which amounts to an analytic structure on $M$ for which the $\mathfrak{k}$ action is analytic. But this can fail to exist even when when $\mathfrak{k} = \mathbb{R}$, in other words, when the action is simply given by a vector field. For instance, if the vector field vanishes to infinite order at a point $p$ of $M$, but not on a neighborhood of $p$, it can never be made analytic, so complexification of the action groupoid $G$ and hence integration of $E$ become impossible except on the formal level.

In addition, it is conceivable that some smooth action groupoids may be made analytic in essentially different ways, even though, according to Kutzschebauch [21], this cannot happen for proper actions by groups with finitely many connected components. Perhaps there is a smooth actions which admits several quite different complexifications.

When $M$ and the $\mathfrak{k}$ action are analytic, the vector fields generating the action extend to holomorphic vector fields on a complexification $M_C$, leading to a holomorphic Lie algebroid structure on $M_C \times k_C$. This integrates to a holomorphic Lie groupoid $G$, the “local transformation groupoid” of the complexified $K_C$ action.

Note the slightly different strategy here—the Lie algebroid is first extended to the complexification and then integrated, rather than the other way around. This strategy will be used extensively below.
Example 2.2 Let \( \mathfrak{k} = \mathbb{R} \) act on \( M = \mathbb{R} \) via the vector field \( x \frac{\partial}{\partial x} \). When \( \mathfrak{k} \) is considered as the Lie algebra of the multiplicative group \( \mathbb{R}^+ \), the resulting action groupoid is \( \mathbb{R}^+ \times \mathbb{R} \), with the first component acting on the second by multiplication. The orbits of this groupoid are the two open half lines and the origin.

A natural complexification of \( \mathbb{R}^+ \times \mathbb{R} \) is the action groupoid \( \mathbb{C}^+ \times \mathbb{C} \), whose orbits are the origin in \( \mathbb{C} \) and its complement \( \mathbb{C}^\times \). When this groupoid is restricted to the original manifold \( \mathbb{R} \), the two half lines now belong to the same orbit, even if the complexification \( \mathbb{C} \) is replaced by a small neighborhood of the real axis. (In this case, the complexified groupoid would no longer be an action groupoid, but it would still have just the two orbits.) As a stack, the complexified groupoid represents a space with two points, one of which is an ordinary holomorphic point. The second point is in the closure of the first and has isotropy group \( \mathbb{C}^\times \).

After restriction of the groupoid to the germ of \( \mathbb{C} \) around \( M \), or to the formal neighborhood, the notion of “orbit” is harder to pin down, since the groupoid does not directly define an equivalence relation.

A somewhat different result is obtained if the algebroid is first extended and then integrated. The extended complex Lie algebroid is \( \mathbb{C} \times \mathbb{C} \); for its natural integration, the group is the simply connected cover \( \mathbb{C} \) of \( \mathbb{C}^\times \). The action groupoid is now \( \mathbb{C} \times \mathbb{C} \) with the action \( w \cdot z = e^{w} z \), for which the orbits are the same as before, but the isotropy group of nonzero \( z \) (including real \( z \)) is now \( 2\pi i \mathbb{Z} \).

Remark 2.3 A similar but slightly more complicated example is given by the vector field on the phase plane \( M = \mathbb{R}^2 \) which describes a classical mechanical system near a local maximum of the potential function. The complexification of the action groupoid \( \mathbb{R} \times \mathbb{R}^2 \) includes groupoid elements connecting states on opposite sides of the potential maximum which cannot be connected by real classical trajectories. These groupoid elements are not without physical interest, though, since they may be interpreted as representing quantum tunneling.

2.4 Foliations

An analytic foliation \( E_\mathbb{R} \subset TM \) extends to a holomorphic foliation of \( M_\mathbb{C} \), and, if \( M_\mathbb{C} \) is small enough, the leaf stack of the latter is just a straightforward complexification of the (analytic) leaf stack of the former. In particular, if the former is a manifold, so is the latter.
But there are many foliations which admit no compatible analytic structure. Take for example the Reeb \[33\] foliation (or for that matter, according to Haefliger \[18\], any foliation) on \(S^3\). The leaf space of the Reeb foliation consists of two circles and a special point whose only open neighborhood is the entire space. The isotropy group of the holonomy groupoid is trivial for the leaves on the circles and \(\mathbb{Z}^2\) for the special leaf.

To complexify the Lie algebroid by complexifying the foliation groupoid, one might look instead at the equivalent groupoid given by restriction to a cross section to the leaves. This cross section can be taken as a copy of \(\mathbb{R}\) on which \(\mathbb{Z}^2\) acts, fixing the origin, with one of the two generators acting by 1-sided contractions on the left half line and the other by contractions on the right. Complexifying the action of the generators gives maps on \(\mathbb{C}\) which have essential singularities at the origin, and there seems to be no way to make a holomorphic stack out of this data.

### 3 Involutive structures

A complex Lie algebroid \(E\) over \(M\) with injective anchor may be identified with the image of its anchor, which is an involutive subbundle of \(T_{\mathbb{C}}M\). Following Treves \[39\], these subbundles will be called here involutive structures. An analytic structure on \(E\) is just an analytic structure on \(M\) for which \(E\) admits local bases of analytic complex vector fields.

Let \(E\) be an analytic subbundle of \(T_{\mathbb{C}}M\), then, and \(M_{\mathbb{C}}\) a complexification of \(M\). Identifying \(T_{\mathbb{C}}M\) with the restriction to \(M\) of \(TM_{\mathbb{C}}\), one may extend the local bases of analytic sections of \(E\) to local holomorphic sections of \(TM_{\mathbb{C}}\). For \(M_{\mathbb{C}}\) sufficiently small, local bases again determine a holomorphic subbundle \(E'\) of \(TM_{\mathbb{C}}\). Holomorphic continuation of identities implies that \(E'\) is itself involutive; by the holomorphic Frobenius theorem, it determines a holomorphic foliation of \(M_{\mathbb{C}}\). The holonomy groupoid of this foliation determines a holomorphic stack which may be considered as the integration of the complex Lie algebroid \(E\).

The rest of this section is devoted to examples of involutive structures viewed as CLAs.

#### 3.1 Complex structures

An almost complex structure on \(M\) is an endomorphism \(J : TM \to TM\) such that \(-J^2\) is the identity. \(T_{\mathbb{C}}M\) is the direct sum of the \(-i\) and \(+i\) eigenspaces of the complexified operator \(J_{\mathbb{C}}\). These conjugate complex subbundles, denoted by \(T^{0,1}_J M\) and \(T^{1,0}_J M\) respectively, are involutive if and
only if $J$ is integrable in the sense that the Nijenhuis tensor $N_J$ vanishes. The eigenspace $T^{0,1}_J M$ is then a CLA which, like $J$ itself, is called a complex structure. It is a standard fact that every subbundle $E \subset T_C M$ such that $E \oplus \overline{E} = T_C M$ is $T^{0,1}_J M$ for some almost complex structure $J$.

Theorems of Eckmann-Frölicher [11] and Ehresmann [12] (analytic case)\(^2\) and Newlander-Nirenberg [29] (smooth case) tell us that any complex structure on $M$ is locally isomorphic to the standard one on $\mathbb{R}^{2n} = \mathbb{C}^n$; i.e., it gives a reduction of the atlas of smooth charts on $M$ to a subatlas with holomorphic transition functions, making $M$ into a complex manifold. Let us pretend for a moment, though, that we do not know those theorems and look directly at the integration of an analytic complex structure as a holomorphic stack. (The result of this exercise will turn out to be the original 1951 proof!)

According to the discussion above, complexification gives a foliation $E'$ of a suitably small $M_C$ whose leaves, by the condition $E \oplus \overline{E} = T_C M$, have tangent spaces along $M$ which are complementary to the real subbundle $TM$. As a result, shrinking $M_C$ again can insure that each leaf is a holomorphic ball intersecting $M$ exactly once, transversely, so that the leaf space of this foliation may be identified with $M$. This leaf space being a complex manifold, $M$ itself inherits the structure of a complex manifold. Holomorphic local coordinates on $M$ result from sliding open sets in $M$ along the foliation $E'$ to identify them with open sets in holomorphic transversals, e.g. leaves of the holomorphic foliation $\overline{E}$ which extends $E$.

The holomorphic stack in this case may be identified with $M$ as a complex manifold, presented by the holonomy groupoid of the foliation $E'$. An alternate presentation is the etale groupoid obtained by restricting the holonomy groupoid to the union of enough transversals to cover $M$ under projection along $E'$. The latter groupoid is just the equivalence relation associated to a covering of $M$ by holomorphic charts.

When $E$ is given simply as a smooth complex structure, the only recourse is to invoke the Newlander-Nirenberg theorem. This has the consequence that $M$ has an analytic structure in which $E$ is analytic, so the previous situation is obtained.

**Remark 3.1** The analytic structure on $M$ which makes a complex structure $E$ analytic is unique, since it must be the one attached to the holomorphic structure determined by $E$. The situation is therefore different from that for the complex Lie algebroid $T_C M$ and the zero Lie algebroid, whose integration

\(^2\)The cited authors also attribute the result to de Rham.
depends on an arbitrary choice of analytic structure compatible with the
given smooth structure.

3.2 CR structures

A step beyond the complex structures within the class of involutive systems
are the general CR structures. These are subbundles $E$ of $T_C M$ such that
$E$ and $\bar{E}$ intersect only in the zero section, but $E \oplus \bar{E}$ is not necessarily all
of $T_C M$.\footnote{Some authors use the term “CR structure” only when $E \oplus \bar{E}$ is of codimension one in $T_C M$.}

Any “generic” real submanifold $M$ in a complex manifold $X$ inherits a
CR structure, namely the intersection $G_{M,X} = T_C M \cap T^{0,1} J_X$. To be precise,
the submanifold is called generic when $G_{M,X}$ has constant dimension; note
that real hypersurfaces are always generic in this sense. $G_{M,X} \oplus \overline{G_{M,X}}$ is the
complexification of the maximal complex subbundle $F_{M,X}$ of $TM$. A natural
geometric problem, which has led to fundamental developments in linear
PDE theory, is whether a given CR manifold can be realized either locally
or globally as a submanifold in some complex manifold, and in particular in
$\mathbb{C}^n$. For analytic CR structures, the integration method of this paper solves
this problem. What follows below essentially reproduces an argument of
Andreotti and Fredricks \cite{AndreottiFredricks1957}, or more precisely, that in the review by Rossi
\cite{Rossi1985} of that paper.

Let $E'$ be the integrable holomorphic subbundle of $T M_C$ which extends
$E$. The corresponding foliation will be called the CR foliation. If $M$
has (real) dimension $2n + r$ and $E$ has complex dimension $n$, then $M_C$
has complex dimension $2n + r$, and the leaves of the CR foliation have complex
dimension $n$; each of them meets $M$ in a point, with no common tangent
vectors (since $E$ contains no real vectors). It follows that $M_C$ can be chosen
so that the leaves are simply connected; the stack defined by the foliation
groupoid is then simply a complex manifold $N$ of complex dimension $n + r$
containing $M$ as a real hypersurface of real codimension $r$. When $r = 0,$
$N = M$, and $M$ is a complex manifold; when $n = 0$ (zero Lie algebroid),
$N = M_C$. (Andreotti and Fredricks \cite{AndreottiFredricks1957} call $N$ a complexification of $M$ for
any $n$; thus, the complexification of a complex manifold is the manifold
itself.)
3.3 The Mizohata structure

The next example shows that the natural map from $M$ to a stack which integrates a complex Lie algebroid $E \to M$ may not be injective.

As in Example I.10.1 of Treves [39], the Mizohata structure over $M = \mathbb{R}^2$ is defined to be the involutive system $E$ spanned by the complex vector field

$$i \partial / \partial t - t \partial / \partial x.$$  

It is a complex structure except along the $x$-axis, where it is the complexification of the real subspace spanned by $\partial / \partial t$. The holomorphic continuation of $E$ over $\mathbb{C}^2$ is spanned by the same vector field in which $(x, t)$ are taken as complex variables, and the leaves of the corresponding foliation $E'$ are the levels of the invariant function $\zeta = x - it^2/2$. These levels, which can be described as graphs $x = it^2/2 + \zeta$ with the parameter $t$ running through $\mathbb{C}$, are contractible, so the stack defined by the foliation groupoid is isomorphic to $\mathbb{C}$ with $\zeta$ as its complex coordinate. The natural map from $M$ to this stack folds $\mathbb{R}^2$ along the $x$-axis, and the image is the (closed) lower half plane.

The situation becomes more complicated rather than simpler if the complexification is shrunk to a neighborhood of $\mathbb{R}^2$ in $\mathbb{C}^2$, for instance that defined by the bounds $|\Im t| < \epsilon$ and $|\Im z| < \epsilon$ on the imaginary parts. In this case, some of the level manifolds of $\zeta$ split into two components, so that the corresponding part of the leaf space (the complement of a strip near the origin in the lower half plane) bifurcates into two branches. The common closure of these branches is a family of leaves depending on one (real) parameter, so we can describe the integration of the Mizohata structure (or the “complexification”, in the language used in CR geometry) as the non-Hausdorff complex manifold which is the union of an open strip along the real axis in the complex $\zeta$-plane with two copies of the rest of the lower half plane. The map from $M$ to this stack now separates points except those in a strip along the $x$ axis, which is folded as before.

Integrals of the involutive structure on $M$ must be even in $t$ near the $x$ axis; since they are holomorphic away from the $x$ axis, they must be even everywhere. In this case, there are integrals of $E$ which are not the pullback of holomorphic functions on the stack. (See Example III.2.1 in Treves [39].)

It is not clear what kind of geometric object is obtained in the limit as the complexification shrinks down to $M$, or for the formal complexification.

\[4\]There is no bifurcation in the upper half plane.
A test problem for any global theory of integration is to describe the integration of involutive structures on smooth surfaces which have singularities along a collection of simple closed curves and which are complex structures elsewhere.

3.4 Eastwood-Graham and LeBrun-Mason structures

In the next example, due to Eastwood and Graham [10], the map from $M$ to the stack integrating a complex Lie algebroid has nondiscrete fibres.

Consider $\mathbb{C}^2$ with coordinates $z = x + iy$ and $w = s + it$ and the involutive structure spanned by $\partial / \partial x + i \partial / \partial y$ and $\partial / \partial t - (x + iy) \partial / \partial s$, or, in complex notation, $\partial / \partial z$ and $\partial / \partial t - z \partial / \partial s$. When $y \neq 0$, this is a complex structure, while when $y = 0$, it contains the real subspace spanned by $\partial / \partial t - x \partial / \partial s$. The integrals for this structure are generated by $z = x + iy$ and $\zeta = s + zt$.

On the complexification $\mathbb{C}^2_C = \mathbb{C}^4$, $x$, $y$, $s$, and $t$ may have complex values, and then the map $(z, \zeta) : \mathbb{C}^2_C \to \mathbb{C}^2$ is a submersion whose fibres are the leaves of the extended foliation; thus, the leaf space (and hence the stack which integrates the structure) may be identified with the complex $(z, \zeta)$ plane.

What is singular here is the map $\phi$ from the original $\mathbb{C}^2 = \mathbb{R}^4$ to this stack. When the variables $(x, y, s, t)$ are real, $\phi$ is a local diffeomorphism, except on the hypersurface $y = 0$, where each of the orbits of the vector field $\partial / \partial t - x \partial / \partial s$ is mapped to a constant. The image of this hypersurface is the subset of the $(z, \zeta)$ plane on which the variables are both real, and, as is clearly described by Eastwood and Graham [11], the map $\phi$ realizes the (real) blow-up of $\mathbb{R}^2$ in $\mathbb{C}^2$.

A similar involutive structure was constructed by Lebrun and Mason [23] on the projectivized complexified tangent bundle of a surface with affine connection; the singular curves in their example are the geodesics.

4 Boundary Lie algebroids

This section exhibits CLAs which are neither involutive systems nor the complexification of real Lie algebroids. The example is taken from work in progress by Leichtnam, Tang, and the author [24] on Kähler geometry and deformation quantization in the setting of CLAs. The description of the integration of these CLAs is not complete.

Let $X$ be a complex manifold of (complex) dimension $n + 1$ with boundary $M$, and let $\mathcal{E}_{M,X}$ be the space of complex vector fields on $X$ whose values along $M$ lie in the induced CR structure $G_{M,X}$. $\mathcal{E}_{M,X}$ is a module over
$C^\infty(X)$ and is closed under bracket. The following lemma shows that that it may be identified with the space of sections of a complex Lie algebroid $\mathcal{E}_{M,X}$.

**Lemma 4.1** $\mathcal{E}_{M,X}$ is a locally free $C^\infty(X)$-module.

**Proof.** Away from the boundary, $\mathcal{E}_{M,X}$ is the space of sections of $T_C M$, hence locally free. Near a boundary point, choose a defining function $\psi$, i.e. a function which vanishes on the boundary and has no critical points there. Next, choose a local basis $\mathfrak{v}_1, \ldots, \mathfrak{v}_n$ of $G_{M,X}$ and extend it to a linearly independent set of sections of $T^{0,1}X$, still denoted by $\mathfrak{v}_j$, defined in an open subset of $X$, to be shrunk as necessary. Let $\nu_j$ be the complex conjugate of $\mathfrak{v}_j$. These vectors all annihilate $\psi$ on $M$; there is no obstruction to having them annihilate $\psi$ everywhere. Next, choose a local section $\mathfrak{v}_0$ of $T^{0,1}X$ such that $\mathfrak{v}_0 \cdot \psi = 1$, and let $\nu_0$ be its conjugate. This gives a local basis $(\nu, \nu')$ for the complex vector fields. Such a vector field belongs to $\mathcal{E}_{M,X}$ if and only if, when it is expanded with respect to this basis, the coefficients of $\nu_0$ and all the $\nu_j$ vanish along $M$. Since this means that all these coefficients are divisible by $\psi$ with smooth quotient, setting $u'_0 = \psi \mathfrak{v}_0$, $u'_j = \mathfrak{v}_j$ for $j = 1, \ldots, n$, and $u_j = \psi \mathfrak{v}_j$ for $j = 0, \ldots, n$ produces a local basis $(u, u')$ for $\mathcal{E}_{M,X}$.

$\square$

To integrate the boundary Lie algebroid $E_{M,X}$, assuming analyticity as usual, one may begin by extending $X$ slightly beyond $M$, so that $M$ becomes an embedded hypersurface. In the complexification $X_C$, $M$ extends to a submanifold $M_C$ of complex codimension one. The CR structure on $M$ extends (see Section 3.2) to the tangent bundle $E'$ of the CR foliation on $M_C$. The holomorphic continuation of $E_{M,X}$ is then the holomorphic Lie algebroid whose local sections are the vector fields on $X_C$ whose restrictions to $M_C$ have their values in $E'$.

What is the groupoid of this Lie algebroid over $X_C$? Over the complement of $M_C$, the Lie algebroid is the tangent bundle, so the groupoid could be taken to be the pair groupoid. Since $M_C$ has complex codimension one, though, its complement generally has a nontrivial fundamental group, and the fundamental groupoid or one of its nontrivial quotients might be appropriate as well. The choice depends in part on compatibility with the choice made on $M_C$ itself.

Over $M_C$, the image of the anchor of the extended Lie algebroid is the tangent bundle $E'$ to the CR foliation, but now, unlike in the pure CR situation, there is nontrivial isotropy. To describe this isotropy, note that, at
each point $x$ of $M_C$, there is a flag $E'_x \subset T_xM_C \subset T_xX_C$. The isotropy algebra may be identified with the endomorphisms of the normal space $T_xX_C/E'_x$ which vanish on $T_xM_C$. Given two points $x$ and $y$ in $M_C$, there are morphisms in the integrating groupoid from $x$ to $y$ if and only if $x$ and $y$ lie in the same leaf of the CR foliation. Each such morphism is then a linear map $T_xX_C/E'_x \to T_yX_C/E'_y$ whose restriction $T_xM_C/E'_x \to T_yM_C/E'_y$ coincides with the linearized holonomy map along any path in the leaf. (Assume that the complexification is small enough so that the leaves are simply connected.) In particular, when $x = y$, the isotropy group consists of the automorphisms of $T_xX_C/E'_x$ which fix $T_xM_C/E'_x$. (Compare the author’s discussion in Section 6 of [42], where the Lie algebroid and its integrating groupoid are studied for the vector fields tangent to the boundary of a real manifold, as well as the treatment by Mazzeo [25] of vector fields tangent to the fibres of a submersion on the boundary. Finally, a slightly different, class of vector fields on a manifold with fibred boundary is used by Mazzeo and Melrose [26].)

When $x$ lies on the real hypersurface $M$, the space above admits an explicit description in terms of the CR geometry. Over $M$, $TX_C$ restricts to $T_CX$, $TM_C$ is just $T_CM$, and $E'$ is the CR structure $G_{M,X} = T_CM \cap T^{0,1}X$. Thus, the isotropy of the integrating groupoid consists of the automorphisms of $T_CX/T_CM \cap T^{0,1}_C$ which fix its codimension one subspace $T_CM/T_CM \cap T^{0,1}_C$. These automorphisms act on the complexified normal bundle $T_CX/T_CM$, and those which act trivially on the normal bundle are “shears” which may be identified with the additive group of linear maps from that normal bundle to $T_CM/T_CM \cap T^{0,1}_C$. The choice of a defining function trivializes the normal bundle, so the isotropy is an extension of the automorphism (or “dilation”) group of the normal bundle by the abelian group $T_CM/T_CM \cap T^{0,1}_C$.

The preceding description of the integrating groupoid is not complete, since it lacks an explanation of how the piece over the interior and the piece over the boundary fit together. In particular, if one were to use the fundamental groupoid on the interior, as described above, it may be necessary to use a covering of the automorphisms of the line bundle on the boundary.

## 5 Generalized complex structures

In the rapidly developing subject of **generalized geometry**, originated by Hitchin [19], the tangent bundle $TM$ of a manifold with its Lie algebroid structure is replaced by the **generalized tangent bundle** $\mathcal{T}M$, which is
the direct sum $TM \oplus T^*M$ equipped with the Courant algebroid structure consisting of the bracket

$$[(\xi_1, \theta_1), (\xi_2, \theta_2)] = \left( [\xi_1, \xi_2], \mathcal{L}_{\xi_1} \theta_2 - \mathcal{L}_{\xi_2} \theta_1 - \frac{1}{2} d(i_{\xi_1} \theta_2 - i_{\xi_2} \theta_1) \right),$$

the anchor $\mathcal{T}M \to TM$ which projects to the first summand, and the symmetric bilinear form

$$\langle (\xi_1, \theta_1), (\xi_2, \theta_2) \rangle = \frac{1}{2}(i\xi_1 \theta_2 + i\xi_2 \theta_1).$$

Like the tangent bundle, $\mathcal{T}M$ may be complexified to the “complex Courant algebroid” $\mathcal{T}_CM$. It is not a complex Lie algebroid, but it contains many CLAs, in particular the complex Dirac structures, i.e. the (complex) subbundles $E$ which are maximal isotropic for the symmetric form and whose sections are closed under the bracket. For instance, if $A \subseteq \mathcal{T}_CM$ is an involutive system and $A^\perp \subseteq T^*_CM$ is its annihilator, then $A \oplus A^\perp$ is a complex Dirac structure.

Of special interest among the complex Dirac structures are those for which $E \oplus E = \mathcal{T}M$. These are called generalized complex structures and are the $-i$ eigenspaces of (the complexifications of) integrable almost complex structures $J : \mathcal{T}M \to \mathcal{T}M$; the integrability condition here is that the Nijenhuis torsion is zero, the usual bracket of vector fields in the definition of the torsion being replaced by the Courant bracket.

In particular given a complex structure $J : TM \to TM$, with associated CLA $T_J^{0,1}M$, the direct sum with its annihilator is the generalized complex structure $T_J^{0,1}M = T_J^{0,1}M \oplus T_J^{1,0*}M$. The image of the anchor is the involutive system $T_J^{0,1}M$, but $T_J^{0,1}M$ itself is not an involutive system, since the kernel of its anchor is the nontrivial bundle $T_J^{1,0*}M$. Also, $T_J^{0,1}M$ is not isomorphic to the complexification of a real Lie algebroid, since the image of its anchor is not invariant under complex conjugation.

Another kind of example arises from symplectic structures on $M$, viewed as bundle maps $\omega : TM \to T^*M$. Here, the generalized complex structure $E_\omega$ is defined to be the graph of the complex 2-form $i\omega$. This time, the anchor is bijective, so, as a Lie algebroid, $E_\omega$ is isomorphic to $\mathcal{T}_CM$.

What is the integration, in the sense of this paper, of a generalized complex structure? First, let $J$ be a complex structure on $M$, $\mathcal{T}_J^{0,1}M = T_J^{0,1}M \oplus T_J^{1,0*}M$ the corresponding generalized structure. Complexifying $M$ and $J$ as in Section 3.1 gives a foliation on $M_C$. The groupoid which integrates the holomorphic continuation of $\mathcal{T}_J^{0,1}M$ is the semidirect product
groupoid obtained from the action of the holonomy groupoid of the foliation (via the “Bott connection”) on its conormal bundle. (This is just the holomorphic version of a construction by Bursztyn, Crainic, Zhu, and the author [4].) This action groupoid is equivalent to the holomorphic leaf space $M$ carrying the cotangent bundle $T^{1,0}_{J} M$ of additive groups as its isotropy. The corresponding stack is a the bundle over $M$ whose fibres are the “universal classifying stacks” of the cotangent spaces.

Next let $\omega$ be a symplectic structure on $M$. Since the generalized complex structure $E_{\omega}$ is isomorphic to $T_{C}M$, its integration must be that of $T_{C}M$, i.e. the holomorphic point, perhaps carrying the fundamental group of $M$ as isotropy. To see what has become of $\omega$, it is best to look again at (real and complex) Dirac structures.

As a subbundle of $TM$, a Dirac structure $E$ carries a natural skewsymmetric bilinear form, the restriction of

$$B(\xi_1, \theta_1), (\xi_2, \theta_2)) = (1/2)(i\xi_1 \theta_2 - i\xi_2 \theta_1).$$

It is shown in [4] that this form gives rise to a multiplicative closed 2-form on a groupoid integrating $E$, producing a presymplectic groupoid. Applying this construction to the holomorphic extension of any complex Dirac structure $E$ shows that its integration as a CLA is a holomorphic symplectic groupoid over $M_{C}$. In particular, for $E_{\omega}$ or any other complex Poisson structure, it is a holomorphic symplectic groupoid. For $E_{J}$, or any other direct sum of an involutive structure with its annihilator, the restriction of $B$ is zero, and hence so is the presymplectic structure on the integrating groupoid.

### 6 Further topics and questions

A notion of integration for complex Lie algebroids has been proposed in this paper. There are many interesting questions about other extensions of Lie algebroid theory to the complex case, including the relation between these extensions and the integration construction proposed here. Some examples conclude this paper.

#### 6.1 Integrability

Does the integrability criterion of Crainic and Fernandes [8] apply in the holomorphic case? What are the conditions on an analytic CLA which determine whether its holomorphic continuation is integrable? What can one do in the nonanalytic case?
6.2 Cohomology

A “van Est” theorem of Crainic [7] describes the relation between the cohomology of a Lie algebroid and that of its integrating groupoids. The definition of cohomology extends in a straightforward to CLAs (for instance, it gives the Dolbeault cohomology in the case of a complex structure). Is there a van Est theorem in this case, too?

6.3 Bisections

One consequence of the integration of a Lie algebroid \( E \) is that the submanifolds of an integrating groupoid which are sections for the source and target maps form a group whose Lie algebra in some formal sense is the space of sections of \( E \). Is there a similar construction for the case of a complex Lie algebroid? Some hints might come from the constructions by Neretin [27] and Segal [36] (also see Yuriev [45]) of a semigroup which in some sense integrates the complexified Lie algebra of vector fields on a circle. Conversely, a general construction for CLAs could provide complexifications for the diffeomorphism groups of other manifolds.

6.4 Quantization

Once a Lie algebroid \( E \) has been integrated, the groupoid algebra of an integrating groupoid may be considered, following Landsman and Ramazan [22], as a deformation quantization of the Poisson structure on the dual bundle \( E^* \), or as a completion of Rinehart’s universal enveloping algebra of \( E \). Is there a corresponding application for the integration of a CLA?

On the other hand, given a complex Poisson structure \( \Pi \) on \( M \), it defines a CLA structure on the complexified cotangent bundle. Integration of this structure should give a holomorphic symplectic groupoid which should be somehow related to the deformation quantization of \( (M, \pi) \). On the formal level (without integration), it is possible [24] to extend the methods of Karabegov [20] and Nest and Tsygan [28] to construct deformation quantizations of certain boundary Lie algebroids as in Section 4 above.

6.5 Connections and representations

If \( E \) is a CLA over \( M \) and \( V \) is a complex vector bundle \( V \), an \( E \)-connection on \( V \) is a map \( a \mapsto \nabla_a \) from the sections of \( E \) to the \( \mathbb{C} \)-endomorphisms of the sections of \( V \) which satisfies the conditions \( \nabla_{fa} u = f \nabla_a u \) and \( \nabla_a gu =
\[ g\nabla_a u + (\rho(a)g)u. \] The connection is flat and is also called a representation of \( E \) on \( V \) if the map \( a \mapsto \nabla_a \) is a Lie algebra homomorphism.

For instance, if \( E \) is a complex structure, a representation of \( E \) on \( V \) is a holomorphic structure on \( V \). More generally, representations of CR structures correspond to CR vector bundles, as in the work of Webster \[41\]. After complex extension, an analytic representation of an analytic CR structure becomes a flat connection along the leaves of the CR foliation, which leads to a holomorphic vector bundle on the complexification.

If \( E \) is the generalized complex structure associated to a complex structure, a representation is a holomorphic structure together with a holomorphic action of the holomorphic cotangent bundle by bundle endomorphisms of the representation space.

### 6.6 The modular class

The modular class of a Lie algebroid, introduced by Evans, Lu, and the author \[14\] is the obstruction to the existence of an “invariant measure.” Its definition extends directly to the case of CLAs. For a complex structure, the modular class is the obstruction to the existence of a Calabi-Yau structure.

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