Cosmological Evolution of Einstein-Aether Models with Power-law-like Potential

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ABSTRACT

The so-called Einstein-Aether theory is General Relativity coupled (at second derivative order) to a dynamical unit time-like vector field (the aether). It is a Lorentz-violating theory, and gained much attention in the recent years. In the present work, we study the cosmological evolution of Einstein-Aether models with power-law-like potential, by using the method of dynamical system. In the case without matter, there are two attractors which correspond to an inflationary universe in the early epoch, or a de Sitter universe in the late time. In the case with matter but there is no interaction between dark energy and matter, there are only two de Sitter attractors, and no scaling attractor exists. So, it is difficult to alleviate the cosmological coincidence problem. Therefore, we then allow the interaction between dark energy and matter. In this case, several scaling attractors can exist under some complicated conditions, and hence the cosmological coincidence problem could be alleviated.

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I. INTRODUCTION

The so-called Einstein-Aether (Æ) theory [1–6] is General Relativity (GR) coupled (at second derivative order) to a dynamical unit time-like vector field (the aether). In this theory, the local structure of spacetime is described by the metric tensor \( g_{ab} \) (as in GR) together with the aether vector \( u_a \). The aether defines a preferred rest frame at each point of spacetime, but preserves rotational symmetry in that frame. The Æ theory is a Lorentz-violating theory, while it preserves locality, general covariance and the successes of GR [1–6].

The Æ theory gained much attention in the recent years. For instance, some observational and experimental constraints on the Æ theory have been considered [7–9, 36] (see also [1]). It has been claimed in [3] that if the aether is restricted in the action to be hypersurface orthogonal, the Æ theory is identical to the IR limit of the healthy extension of Horava-Lifshitz gravity [10]. Black holes, neutron stars and binary pulsars in the Æ theory have been extensively considered in e.g. [11–15, 36]. Also, supersymmetric aether has been discussed in e.g. [16, 17]. In [18], magnetic fields in the Æ theory have been studied. In [19], Godel type metrics in the Æ theory were considered. The Æ theory as a quantum effective field theory was considered in [21]. A positive energy theorem for the Æ theory has been shown in [22]. Of course, it is not surprising to find that the Æ theory has been extensively considered in cosmology. For example, some exact solutions of the inflationary Æ cosmological models have been found in [5]. In [23], it has been considered as an alternative to dark matter or dark energy. Also, the cosmological perturbations in the Æ theory have been studied in [24, 25]. The stability of Æ cosmological models has been discussed in [8]. The inflaton coupled with an expanding aether was considered in [2]. The cosmic microwave background polarization in the Æ theory has been investigated in e.g. [26]. The observational constraints on the Æ models as an alternative to dark energy were obtained in e.g. [27]. The cosmological constraints on modified Chaplygin gas in the Æ theory has also been found in [28]. The anisotropic Æ cosmological models were discussed in [29]. The Lorentz-violating inflation in the Æ theory has been studied in [30]. In fact, there are many other interesting works on the Æ cosmology in the literature, and we cannot mention all of them here.

In a Friedmann-Robertson-Walker (FRW) universe, the aether field will be aligned with the cosmic frame, and is related to the expansion of the universe [2, 5, 6], namely

\[ \nabla_a u_b = \frac{\theta}{3} (g_{ab} - u_a u_b) , \]

where the expansion \( \theta = 3H \) in the FRW cosmology, and \( H \equiv \dot{a}/a \) is the Hubble parameter; \( a \) is the scale factor; a dot denotes the derivative with respect to the cosmic time \( t \). If there exists a self-interacting scalar field \( \phi \) (which might play the role of inflaton or dark energy) in the universe, it will couple to the expanding aether. The modified stress tensor is given by [2, 5, 6]

\[ T_{ab} = \nabla_a \phi \nabla_b \phi - \left( \frac{1}{2} 
abla_c \phi \nabla^c \phi - V + \theta V_{\theta} \right) g_{ab} + \dot{V}_{\theta} (u_a u_b - g_{ab}) , \]

where the potential \( V = V(\theta, \phi) \) is now a function of both \( \theta \) and \( \phi \). Notice that \( V_{\theta} \equiv \partial V / \partial \theta \). So, the corresponding energy density \( \rho_{\phi} \) and pressure \( p_{\phi} \) of the homogeneous scalar field read [2, 5, 6]

\[ \rho_{\phi} = \frac{1}{2} \dot{\phi}^2 + V - \theta V_{\theta} , \]

\[ p_{\phi} = \frac{1}{2} \dot{\phi}^2 - V + \theta V_{\theta} + \dot{V}_{\theta} . \]

Using Eqs. (3) and (4), it is easy to see that the energy conservation equation of the scalar field,

\[ \dot{\rho}_{\phi} + \theta (\rho_{\phi} + p_{\phi}) = 0 , \]

is actually equivalent to the equation of motion

\[ \ddot{\phi} + \theta \dot{\phi} + V_{\phi} = 0 , \]

where \( V_{\phi} \equiv \partial V / \partial \phi \). Interestingly, in the Æ cosmology, Eq. (5) still has the same form as in GR.
In [5], Barrow proposed an exponential-like potential

\[ V(\theta, \phi) = V_0 e^{-\lambda \phi} + \sum_{r=0}^n \mu_r \theta^r e^{(r-2)\lambda \phi/2}, \]  

(7)

and found some exact solutions for the $\mathcal{E}$ models with this potential. Barrow [5] noted that this choice of potential subsumes the simple cases with $V(\theta, \phi) = f(\phi)\theta^2$ of [30] and $V(\theta, \phi) = f(\theta^2)$ of [23], but cannot include the choice $V(\theta, \phi) = \frac{1}{2}m_\phi^2 \phi^2 + \mu \phi \theta$ considered in [2]. Then, Sandin et al. [6] studied the cosmological evolution of the $\mathcal{E}$ models with a special exponential-like potential,

\[ V(\theta, \phi) = V_0 e^{-\lambda \phi} + \mu_1 \sqrt{V_0} \theta e^{-\lambda \phi/2} + \mu_2 \theta^2, \]

(8)

and found some interesting results.

In the present work, we would like to suggest a power-law-like potential for the $\mathcal{E}$ models, namely

\[ V(\theta, \phi) = V_0 \theta^n + \sum_{r=0}^m \mu_r \theta^r \phi^{n(2-r)/2}, \]

(9)

where $V_0$, $n$ and $\{\mu_r\}$ are all constants. There are two fine motivations to consider this type of potential. First, as is well known, the power-law potential is important and hence has been extensively used in the literature since it can arise from various fundamental theories. For instance, a power-law potential might be motivated from the dilatation symmetry [20]. Although we cannot directly derive the power-law-like potential for the $\mathcal{E}$ models, it is still reasonable to consider such a potential phenomenologically. Second, the power-law-like potential given in Eq. (9) subsumes the simple cases with not only $V(\theta, \phi) = f(\phi)\theta^2$ of [30] and $V(\theta, \phi) = f(\theta^2)$ of [23], but also $V(\theta, \phi) = \frac{1}{2}m_\phi^2 \phi^2 + \mu \phi \theta$ of [2], in contrast to the exponential-like potential given in Eq. (7). Therefore, it is of interest to study the $\mathcal{E}$ models with this type of power-law-like potential.

Here, we focus on the cosmological evolution of the $\mathcal{E}$ models with the power-law-like potential proposed in Eq. (9). However, the potential is too complicated when $r$ is large. On the other hand, if we only consider the term $\theta f(\phi)$ (namely $r = 1$), it will be canceled in $V - \theta V_\phi$, and then the energy density $\rho_\phi$ and pressure $p_\phi$ in Eqs. (3) and (4) become trivial in this case. Therefore, in this work we consider the case with $r$ up to 2, namely

\[ V(\theta, \phi) = V_0 \theta^n + \mu_1 \theta \phi^{n/2} + \mu_2 \theta^2. \]

(10)

Note that we consider a flat FRW universe and set $8\pi G = \hbar = c = 1$ throughout this work. We use the method of dynamical system [31] (see also e.g. [32–35]) to investigate the cosmological evolution of the $\mathcal{E}$ models with the power-law-like potential given in Eq. (10). There might be some scaling attractors in the dynamical system, and both the fractional densities of dark energy and matter are non-vanishing constants over there. The universe will eventually enter these scaling attractors regardless of the initial conditions, and hence it is not so surprising that we are living in an epoch in which the densities of dark energy and matter are comparable. Therefore, the cosmological coincidence problem could be alleviated without fine-tunings.

II. THE UNIVERSE WITHOUT MATTER

At first, we consider the cosmological evolution of the universe without matter, namely the total energy density is dominated by $\rho_\phi$. This case corresponds to the early universe dominated by inflaton, or the far future universe dominated by dark energy. The Lagrangians for the aether and scalar field read [2]

\[ \mathcal{L}_\text{ac} = -\frac{M_p^2}{2} \left[ R + K_{cd}^{ab} \nabla_a u^c \nabla_b u^d + \lambda (u^a u_a - 1) \right], \]  
\[ \mathcal{L}_\phi = \frac{1}{2} \nabla_a \phi \nabla^a \phi - V(\theta, \phi), \]

where $M_p$ is the reduced Planck mass, and

\[ K_{cd}^{ab} = c_1 g^{ab} g_{cd} + c_2 \delta_c^a \delta^b_d + c_3 \delta^a_c \delta_d^b + c_4 u^a u^b g_{cd}, \]
in which \( c_i \) are all dimensionless free parameters. After some algebra (see e.g. [2, 5, 6]), one can find the stress tensor in Eq. (2), and then the corresponding energy density \( \rho_\phi \) and pressure \( p_\phi \) of the homogeneous scalar field in Eqs. (3) and (4). Considering a flat FRW universe and using Eqs. (3), (4), the corresponding Friedmann equation and Raychaudhuri equation are given by [2, 5, 6]

\[
\frac{1}{3} \frac{\dot{\theta}^2}{\theta^2} = \rho_\phi = \frac{1}{2} \dot{\phi}^2 + V - \theta V_\theta, \tag{11}
\]

\[
\frac{1}{3} \frac{\dot{\theta}}{\theta} = -\frac{1}{2} (\rho_\phi + p_\phi) = -\frac{1}{2} \left( \dot{\phi}^2 + V_\theta \right). \tag{12}
\]

Following e.g. [31–35], we introduce three dimensionless variables

\[
x_1 \equiv \sqrt{\frac{3}{2(1 + 3\mu_2)}} \frac{\dot{\phi}}{\theta}, \quad x_2 \equiv \sqrt{\frac{3V_0}{1 + 3\mu_1}} \frac{\dot{\phi}^{n/2}}{\theta}, \quad x_3 \equiv \phi^{-1}. \tag{13}
\]

Then, the Friedmann equation (11) can be recast as

\[
1 = x_1^2 + x_2^2. \tag{14}
\]

From Eqs. (11), (12) and (3), (4), we have

\[
s \equiv -\frac{\ddot{\theta}}{\dot{\theta}^2} = x_1 \left( 1 + \frac{1}{3} \nu_1 x_2 x_3 \right). \tag{15}
\]

Note that for convenience we introduce two new constants

\[
\nu_1 \equiv \frac{3\nu_1}{2\sqrt{2V_0}}, \quad \nu_2 \equiv \sqrt{\frac{3(1 + 3\mu_2)}{2}}. \tag{16}
\]

By the help of Eqs. (11), (12) and (3), (4), the evolution equation (10) can be recast as a dynamical system, namely

\[
x_1' = 3(s - 1) x_1 - (\nu_1 + \nu_2 x_2) x_2 x_3, \tag{17}
\]
\[
x_2' = (3s + \nu_2 x_1 x_3) x_2, \tag{18}
\]
\[
x_3' = -2\nu_1 x_1 x_3^2, \tag{19}
\]

where \( s \) is given in Eq. (15), and a prime denotes the derivative with respect to the so-called e-folding time \( N \equiv \ln a \). On the other hand, using Eqs. (11) and (12), the equation-of-state parameter (EoS) is given by

\[
w_\phi \equiv \frac{p_\phi}{\rho_\phi} = 2s - 1. \tag{20}
\]

Of course, by definition, the deceleration parameter reads

\[
q \equiv -\frac{\ddot{a}}{aH^2} = -1 - 3 \frac{\ddot{\theta}}{\dot{\theta}^2} = 3s - 1. \tag{21}
\]

We can obtain the critical points \((\bar{x}_1, \bar{x}_2, \bar{x}_3)\) of the above autonomous system by imposing the conditions \( x_1' = x_2' = x_3' = 0 \) and the Friedmann constraint (14), i.e., \( x_1^2 + x_2^2 = 1 \). On the other hand, by definitions, \( \bar{x}_1, \bar{x}_2 \) and \( \bar{x}_3 \) should be real. There are four critical points, and we present them in Table I. Points (D.1p) and (D.1m) correspond to a decelerated universe while the scalar field mimics a stiff fluid. Thus, they are not desirable in fact. Points (D.2p) and (D.2m) correspond to a de Sitter universe while the scalar field mimics a cosmological constant. They are suitable to describe the inflationary universe in the early epoch, or the accelerated universe in the late time.

We study the stability of these critical points, by substituting the linear perturbations \( x_1 \rightarrow \bar{x}_1 + \delta x_1, \)

\( x_2 \rightarrow \bar{x}_2 + \delta x_2, \)

\( x_3 \rightarrow \bar{x}_3 + \delta x_3 \) about the critical point \((\bar{x}_1, \bar{x}_2, \bar{x}_3)\) into the autonomous system (17),
Because of the Friedmann constraint \((14)\), there are only two independent evolution equations, namely

\[
\delta x_1' = 3 \left[ (\delta - 1) \delta x_1 + \bar{x}_3 \delta x_3 - \nu_1 \epsilon \sqrt{1 - \bar{x}_1^2} \right] \left( \delta x_3 - \frac{\bar{x}_3 \delta x_1}{1 - \bar{x}_1^2} \right) - \nu_2 \left[ (1 - \bar{x}_1^2) \delta x_3 - 2 \bar{x}_1 \bar{x}_3 \delta x_1 \right],
\]

\[
\delta x_3' = -2 \nu_2 \left( \bar{x}_3^2 \delta x_1 + 2 \bar{x}_1 \bar{x}_3 \delta x_3 \right),
\]

where \(\epsilon\) is the sign of \(\bar{x}_2\), and

\[
\bar{s} = \bar{x}_1 \left( \bar{x}_1 + \frac{x_3}{3} \nu_1 \epsilon \sqrt{1 - \bar{x}_1^2} \right),
\]

\[
\delta \bar{s} = 2 \bar{x}_1 \delta x_1 + \frac{1}{3} \nu_1 \epsilon \sqrt{1 - \bar{x}_1^2} \left[ (1 - \bar{x}_1^2) \bar{x}_3 \delta x_1 + \bar{x}_1 \delta x_3 \right].
\]

The two eigenvalues of the coefficient matrix of Eqs. (22) and (23) determine the stability of the critical point. The eigenvalues of both Points (D.1p) and (D.1m) are \(\{6, 0\}\), and hence they are unstable. On the other hand, the eigenvalues of both Points (D.2p) and (D.2m) are \(\{-3, 0\}\), and hence they are stable. So, in the case without matter, there are only two attractors (D.2p) and (D.2m), which correspond to an inflationary universe in the early epoch, or a de Sitter universe in the late time.

### III. ADDING MATTER

Here, we consider the universe containing also matter, which is described by a perfect fluid with barotropic EoS, namely

\[
p_m = w_m \rho_m = (\gamma - 1) \rho_m,
\]

where the barotropic index \(\gamma\) is a constant, and \(0 < \gamma < 2\). In particular, \(\gamma = 1\) and \(4/3\) correspond to pressureless matter and radiation, respectively. In this case, the corresponding Friedmann equation and Raychaudhuri equation become

\[
\frac{1}{3} \theta^2 = \rho_\phi + p_m = \frac{1}{2} \dot{\phi}^2 + V - \theta \dot{\theta} + \rho_m,
\]

\[
\frac{1}{3} \dot{\theta} = -\frac{1}{2} \left( \rho_\phi + p_\phi + \rho_m + p_m \right) = -\frac{1}{2} \left( \dot{\phi}^2 + \dot{V} + \gamma \rho_m \right).
\]

The energy conservation equation of matter is given by

\[
\dot{\rho}_m + \gamma \theta \rho_m = 0.
\]
In addition to $x_1, x_2, x_3$ defined in Eq. (13), we introduce another dimensionless variable

$$x_4 \equiv \sqrt{\frac{3}{1 + 3\mu_2}} \cdot \sqrt{\rho_m / \theta}. \quad (30)$$

Now, the Friedmann equation (27) can be recast as

$$1 = x_1^2 + x_2^2 + x_4^2. \quad (31)$$

From Eqs. (27), (28) and (3), (4), we find that

$$s = -\frac{\dot{\theta}}{\theta^2} = x_1^2 + \frac{1}{3} \nu_1 x_1 x_2 x_3 + \frac{\gamma}{2} x_4^2. \quad (32)$$

By the help of Eqs. (27), (28) and (3), (4), the evolution equations (6) and (29) can be recast as dynamical system, namely

$$x_1' = 3 (s - 1) x_1 - \left(\nu_1 + n\nu_2 x_2 x_3\right) x_2 x_3, \quad (33)$$
$$x_2' = \left(3 s + n\nu_2 x_1 x_3\right) x_2, \quad (34)$$
$$x_3' = -2\nu_2 x_1 x_3^2, \quad (35)$$
$$x_4' = 2x_4 \left(s - \frac{\gamma}{2}\right), \quad (36)$$

where $s$ is given in Eq. (32). On the other hand, using Eqs. (3), (31) and (16), it is easy to find that the fractional energy densities $\Omega_i \equiv 3\rho_i / \theta^2$ of the scalar field and matter are given by

$$\Omega_\phi = \left(1 + 3\mu_2\right) \left(x_1^2 + x_2^2\right) - 3\mu_2 = 1 - \frac{2}{3} \nu_2^2 x_4^2, \quad \Omega_m = \frac{2}{3} \nu_2^2 x_4^2, \quad (37)$$

and they satisfy $\Omega_\phi + \Omega_m = 1$. Note that due to the $\mu_2\theta^2$ term in the potential $V(\theta, \phi)$ (see Eq. (10)), one cannot naively write $\Omega_\phi = x_1^2 + x_2^2$ and $\Omega_m = x_4^2$. Using Eqs. (27) and (28), the total EoS reads

$$w_{\text{tot}} \equiv p_{\text{tot}}/\rho_{\text{tot}} = \Omega_\phi w_\phi + \Omega_m w_m, \quad (38)$$

Noting that $w_\phi \equiv p_\phi/\rho_\phi = w_{\text{tot}} - \Omega_m (\gamma - 1) / \Omega_\phi = \gamma - 1 + \frac{3}{2} (2s - \gamma) (s - \gamma - 2)/3 + 2\nu_2^2 x_4^2. \quad (39)$

By definition, the deceleration parameter $q$ is the same in Eq. (21), but in which $s$ has been changed to the one in Eq. (32). We can obtain the critical points $(x_1, x_2, x_3, x_4)$ of the above autonomous system by imposing the conditions $x_1' = x_2' = x_3' = x_4' = 0$ and the Friedmann constraint (31), i.e., $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$. On the other hand, by definitions, $x_1, x_2, x_3, x_4$ should be real, and $x_4 \geq 0$. There are five critical points, and we present them in Table 1. Points (M.1p) and (M.1m) correspond to a decelerated universe while the scalar field mimics a stiff fluid. Points (M.2p) and (M.2m) correspond to a de Sitter universe while the scalar field mimics a cosmological constant. Note that these four solutions all correspond to a universe dominated by the scalar field (dark energy). Point (M.3) is a scaling solution if $\nu_2^2 \neq 0$ or 3/2. Since its $q = 3\gamma/2 - 1$, the universe is decelerated ($q > 0$) if $\gamma > 2/3$, and is accelerated ($q < 0$) if $\gamma < 2/3$. So, in the case of $\gamma = 1$ (pressureless matter) and $\gamma = 4/3$ (radiation), the universe cannot be accelerated.

To study the stability of these critical points, we substitute the linear perturbations $x_1 \rightarrow x_1 + \delta x_1, x_2 \rightarrow x_2 + \delta x_2, x_3 \rightarrow x_3 + \delta x_3, x_4 \rightarrow x_4 + \delta x_4$ about the critical point $(x_1, x_2, x_3, x_4)$ into the autonomous system (33) and linearize them. Because of the Friedmann constraint (31), there are only three independent evolution equations, namely

$$\delta x_1' = 3 [(s - 1) \delta x_1 + x_2 \delta s] - \nu_1 \sqrt{1 - x_1^2 - x_2^2} \left[\delta x_3 - \frac{x_3 (x_1 \delta x_1 + x_4 \delta x_4)}{1 - x_1^2 - x_2^2}\right], \quad (40)$$
$$\delta x_2' = -\nu_2 \left[\left(1 - x_1^2 - x_2^2\right) \delta x_3 - 2x_3 (x_1 \delta x_1 + x_4 \delta x_4)\right], \quad (41)$$
$$\delta x_4' = 3 (x_4 \delta s + s \delta x_4), \quad (42)$$
where $\epsilon$ is the sign of $\bar{x}_2$, and

$$s = \bar{x}_1^2 + \frac{1}{3} \nu_1 \bar{x}_1 \bar{x}_3 \epsilon \sqrt{1 - \bar{x}_1^2 - \bar{x}_3^2 + \frac{1}{2} \bar{x}_4^2}, \quad \text{(43)}$$

$$\delta s = 2 \bar{x}_1 \delta x_1 + \gamma \bar{x}_3 \delta x_4 + \frac{1}{3} \nu_1 \epsilon \sqrt{1 - \bar{x}_1^2 - \bar{x}_4^2} \left[ \bar{x}_1 \delta x_3 + \bar{x}_3 \delta x_1 - \bar{x}_1 \bar{x}_3 (\bar{x}_1 \delta x_1 + \bar{x}_3 \delta x_4) \right]. \quad \text{(44)}$$

The three eigenvalues of the coefficient matrix of Eqs. (40)–(42) determine the stability of the critical point. The eigenvalues of both Points (M.1p) and (M.1m) are $\{6, 3, 0\}$, and hence they are unstable. On the other hand, the eigenvalues of both Points (M.2p) and (M.2m) are $\{-3, 0, 0\}$, and hence they are stable. The eigenvalues of Point (M.3) is $\{9 \gamma / 2, 3(\gamma - 2) / 2, 0\}$, and hence it is unstable, since $\gamma$ is positive. So, for the case with matter, there are only two attractors (M.2p) and (M.2m), which correspond to an inflationary universe in the early epoch, or a de Sitter universe in the late time. Unfortunately, there is no scaling attractor in this case, since Point (M.3) is unstable. However, the scaling attractor is necessary to alleviate the cosmological coincidence problem. So, we should try to find a way out.

| Label | Critical Point $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$ | $\Omega_\phi$ | $\Omega_m$ | $w_{\text{tot}}$ | $w_\phi$ | $q$ |
|-------|----------------------------------------------------|----------------|-----------|----------------|---------|-----|
| M.1p  | 1, 0, 0, 0                                         | 1              | 0         | 1              | 1       | 2   |
| M.1m  | -1, 0, 0, 0                                        | 1              | 0         | 1              | 1       | 2   |
| M.2p  | 0, 1, 0, 0                                         | 1              | 0         | -1             | -1      | -1  |
| M.2m  | 0, -1, 0, 0                                        | 1              | 0         | -1             | -1      | -1  |
| M.3   | 0, 0, any, 1                                       | $1 - \frac{2}{3} \nu_2^2$ | $\frac{2}{3} \nu_2^2$ | $\gamma - 1$ | $\gamma - 1$ | $\frac{3}{2} \gamma - 1$ |

TABLE II: Critical points for the autonomous system (33)–(36) and their corresponding physical quantities.

**IV. ALLOWING THE INTERACTION BETWEEN DARK ENERGY AND MATTER**

As mentioned in the previous section, we should try to find a suitable way to alleviate the cosmological coincidence problem. In the literature, the common way is allowing the interaction between dark energy and matter. If dark energy can decay into matter (vice versa), these two components might achieve a balance and then their fractional energy densities become constant at some scaling attractors (if any). This is the key to alleviate the cosmological coincidence problem. Therefore, we allow the interaction between dark energy and matter in this section.

We assume that dark energy and matter interact through a coupling term $Q$, according to

$$\dot{\rho}_\phi + \theta (\rho_\phi + p_\phi) = -Q, \quad \text{(45)}$$

$$\dot{\rho}_m + \theta (\rho_m + p_m) = Q, \quad \text{(46)}$$

which preserves the total energy conservation equation $\dot{\rho}_{\text{tot}} + \theta (\rho_{\text{tot}} + p_{\text{tot}}) = 0$. Due to the interaction $Q$ in Eq. (45), the equation of motion should be modified accordingly, i.e.,

$$\ddot{\phi} + \theta \dot{\phi} + V_\phi = -\frac{Q}{\phi}. \quad \text{(47)}$$

By the help of Eqs. (27), (28) and (3), (4), the evolution equations (45) and (46) can be recast as a...
dynamical system, namely

\[ x'_1 = 3(s - 1)x_1 - (\nu_1 + n\nu_2x_2)x_2x_3 - Q_1, \]
\[ x'_2 = (3s + n\nu_2x_1x_3)x_2, \]
\[ x'_3 = -2\nu_2x_1x_2^2, \]
\[ x'_4 = 3x_4\left(s - \frac{\gamma}{2}\right) + Q_2, \]

where \( s \) is given in Eq. (52), and

\[ Q_1 \equiv \sqrt{\frac{3}{2(1 + 3\mu_2)}} \cdot \frac{3Q}{\theta^2\phi}, \quad Q_2 \equiv \frac{3x_4Q}{2\theta\rho_m}. \]  

Eqs. (48)–(51) could be an autonomous system if the interaction term \( Q \) is chosen to be suitable forms. In the present work, we will consider four most familiar interaction terms extensively considered in the literature (see e.g. [31–35]), namely, Case (I) \( Q = \eta_1\rho_m\dot{\phi} \), Case (II) \( Q = \eta_2\rho_m \), Case (III) \( Q = \eta_3\rho_m \), and Case (IV) \( Q = \eta_4\rho_{tot} \), where \( \eta_i \) are all constants. Once the interaction term \( Q \) is specified, we can obtain the critical points \((x_1, x_2, x_3, x_4)\) of the above autonomous system by imposing the conditions \( x'_1 = x'_2 = x'_3 = x'_4 = 0 \) and the Friedmann constraint (31), i.e., \( x'_1 + x'_2 + x'_3 = 1 \). On the other hand, by definitions, \( x_1, x_2, x_3, x_4 \) should be real, and \( x_4 \geq 0 \). The physical quantities of the critical points, namely \( \Omega_\phi, \Omega_m, \omega_{tot}, w_\phi \), and \( q \), are given in Eqs. (57), (58), (59), (60), and in which \( s \) is given in Eq. (52).

Once the critical points are available, their stability should be investigated. To this end, we substitute the linear perturbations \( x_1 \to x_1 + \delta x_1, x_2 \to x_2 + \delta x_2, x_3 \to x_3 + \delta x_3, x_4 \to x_4 + \delta x_4 \) about the critical point \((x_1, x_2, x_3, x_4)\) into the autonomous system (48)–(51) and linearize them. Because of the Friedmann constraint (51), there are only three independent evolution equations, namely

\[ \delta x'_1 = 3[(s - 1)\delta x_1 + x_1\delta s] - \nu_1\epsilon\sqrt{1 - \bar{x}_1^2 - \bar{x}_2^2}\left[\delta x_3 - \frac{x_1\delta x_1 + x_4\delta x_4}{1 - \bar{x}_1^2 - \bar{x}_2^2}\right] - \nu_2\left[(1 - \bar{x}_1^2 - \bar{x}_2^2)\delta x_3 - 2x_3(x_1\delta x_1 + x_4\delta x_4)\right] - \delta Q_1, \]
\[ \delta x'_3 = -2\nu_2\left(x_2^2\delta x_1 + 2x_1x_3\delta x_3\right), \]
\[ \delta x'_4 = 3(x_4\delta s + s\delta x_4) + \delta Q_2, \]

where \( \epsilon \) is the sign of \( x_2 \), and \( \delta Q_1, \delta Q_2 \) are the linear perturbations coming from \( Q_1, Q_2 \), respectively. Note that \( s \) and \( \delta s \) are given in Eqs. (53) and (43). The three eigenvalues of the coefficient matrix of Eqs. (53)–(55) determine the stability of the critical point.

A. Case (I) \( Q = \eta_1\rho_m\dot{\phi} \)

At first, we consider the case with \( Q = \eta_1\rho_m\dot{\phi} \). The corresponding \( Q_1 = \eta_1\nu_2x_2^2 \) and \( Q_2 = \eta_1\nu_2x_1x_4 \).

In this case, there are five critical points, and we present them in Table (1). Points (I.1p) and (I.1m) correspond to a de Sitter universe while the scalar field mimics a stiff fluid. Points (I.2p) and (I.2m) correspond to a de Sitter universe while the scalar field mimics a cosmological constant. Note that these four solutions all correspond to a universe dominated by the scalar field (dark energy). Point (I.3) is a scaling solution, and its physical quantities read

\[ \Omega_\phi = 1 - \frac{2}{3}v_2^2 + \frac{8\eta_1^2v_2^4}{27(\gamma - 2)^2}, \quad \Omega_m = \frac{2}{3}v_2^2 - \frac{8\eta_1^2v_2^4}{27(\gamma - 2)^2}, \]
\[ w_{tot} = \gamma - 1 - \frac{4\eta_1^2v_2^2}{9(\gamma - 2)}, \quad w_\phi = \gamma - 1 + \frac{12(\gamma - 2)\eta_1^2v_2^2}{(18v_2^2 - 27)(\gamma - 2)^2 - 8\eta_1^2v_2^4}, \]
\[ q = \frac{3}{2}\gamma - 1 + \frac{2\eta_1^2v_2^2}{3(2 - \gamma)}. \]
Since $0 < \gamma < 2$, it is worth noting that at Point (I.3) the universe is decelerated ($q > 0$) when $\gamma > 2/3$. Unfortunately, the universe cannot be accelerated if $\gamma < 0$. Nevertheless, we would like to say more. Since $0 < \gamma < 2$, at least one of its eigenvalues is positive (when $\sigma_1$ is a real number), or the real parts of both last two eigenvalues are positive (when $\sigma_1$ is an imaginary number). Therefore, Point (I.3) is certainly unstable for $\gamma > 1/2$. So, if $\gamma = 1$ (pressureless matter) and $4/3$ (radiation), Point (I.3) cannot be an attractor. It can be a scaling attractor only when $\gamma < 1/2$ (necessary but not sufficient condition), in this case the EoS of matter $w_m = \gamma - 1 < -1/2$, which is dark energy in fact.

The second choice of interaction is $Q = \eta_2 \theta \rho_m$. The corresponding $Q_1 = \frac{2}{3} \eta_2 x_1^2 x_4^{-1}$ and $Q_2 = \frac{3}{3} \eta_2 x_4$. In this case, there are four critical points, and we present them in Table III. Points (II.1p) and (II.1m) correspond to a decelerated universe while the scalar field mimics a stiff fluid. Note that these two solutions both correspond to a universe dominated by the scalar field (dark energy). Points (II.2p) and (II.2m) are scaling solutions, and their physical quantities read

$$
\Omega_\phi = 1 - \frac{2}{3} \eta_2 \left(1 - \frac{\eta_2}{\gamma - 2}\right), \quad \Omega_m = \frac{2}{3} \eta_2 \left(1 - \frac{\eta_2}{\gamma - 2}\right),
$$

$$
w_{\text{tot}} = \gamma - 1 - \eta_2, \quad w_\phi = \gamma - 1 + \frac{3(\gamma - 2) \eta_2}{6 - 3\gamma + 2(\gamma - 2 - \eta_2) \eta_2}, \quad q = \frac{1}{2} \left(3\gamma - 2 - 3\eta_2\right). \tag{59}
$$
Since $0 < \gamma < 2$, both Points (II.2p) and (II.2m) can exist under the conditions $\gamma - 2 \leq \eta_2 \leq 0$ and $0 \leq \Omega_m \leq 1$. From Eq. (59), the condition to accelerate the universe ($q < 0$) is $\gamma - \eta_2 < 2/3$. On the other hand, if $\gamma > 2/3$, the universe is certainly decelerated ($q > 0$) because $\eta_2 \leq 0$ is required by the existence of Points (II.2p) and (II.2m). Unfortunately, the universe cannot be accelerated if $\gamma = 1$ (pressureless matter) and 4/3 (radiation).

We then consider the stability of these critical points. The corresponding $\delta Q_1 = \frac{3}{2} \eta_2 \delta x_4 \bar{x}_1^{-1}(2 \delta x_4 - x_4 \bar{x}_1^{-1} \delta x_1)$ and $\delta Q_2 = \frac{1}{2} \eta_2 \delta x_4$. The eigenvalues of both Points (II.1p) and (II.1m) are $\{6, 0, 3(\eta_2 + 2)/2\}$, and hence they are unstable. The eigenvalues of both Points (II.2p) and (II.2m) are

\[
\left\{0, -3 + \frac{15}{4} \gamma - 3\eta_2 - \sigma_2, -3 + \frac{15}{4} \gamma - 3\eta_2 + \sigma_2 \right\},
\]

where

\[
\sigma_2 = \frac{3}{4} \left[ \frac{(\gamma + 4)^2 - 16\gamma \eta_2}{\gamma - 2} \right]^{1/2}.
\]

Points (II.2p) and (II.2m) can exist and are stable under some conditions, which are too complicated to be presented here. Nevertheless, we would like to say more. Since $\gamma - 2 \leq \eta_2 \leq 0$ is required by the existence of Points (II.2p) and (II.2m), $\sigma_2$ is a positive real number. On the other hand, if $\gamma > 4/5$, at least one of the last two eigenvalues is positive. Therefore, Points (II.2p) and (II.2m) is certainly unstable for $\gamma > 4/5$. So, if $\gamma = 1$ (pressureless matter) and 4/3 (radiation), Points (II.2p) and (II.2m) cannot be attractors. They can be scaling attractors only when $\gamma < 4/5$ (necessary but not sufficient condition). As mentioned above, $\gamma < 2/3$ and $\eta_2 < 2/3$ is required to accelerate the universe at Points (II.2p) and (II.2m).

In this case, the EoS of matter $w_m = \gamma - 1 < -1/3$, which is dark energy in fact.

| Label | Critical Point $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$ | $\Omega_0$ | $\Omega_m$ | $w_{\text{tot}}$ | $w_\phi$ | $q$ |
|-------|-------------------------------------------------------------|---------|---------|----------------|--------|-----|
| II.1p | $1, 0, 0, 0$                                               | 1       | 0       | 1              | 1      | 2   |
| II.1m | $-1, 0, 0, 0$                                              | 1       | 0       | 1              | 1      | 2   |
| II.2p | $\sqrt{\frac{\eta_2}{\gamma - 2}}, 0, 0, \sqrt{1 - \frac{\eta_2}{\gamma - 2}}$ | Eq. (59) |         |                |        |     |
| II.2m | $-\sqrt{\frac{\eta_2}{\gamma - 2}}, 0, 0, \sqrt{1 - \frac{\eta_2}{\gamma - 2}}$ | Eq. (59) |         |                |        |     |

TABLE IV: Critical points for the autonomous system (18) – (21) and their corresponding physical quantities, in the Case (II) $Q = \eta_2 \theta \rho_m$.

C. Case (III) $Q = \eta_3 \theta \rho_\phi$

The third choice of interaction is $Q = \eta_3 \theta \rho_\phi$. The corresponding $Q_1 = \frac{9}{4} \eta_3 \nu^2 \left(1 - \frac{2}{3} \nu^2 \bar{x}_4^2\right) x_1^{-1}$ and $Q_2 = \frac{2}{3} \eta_3 \nu^2 x_4^{-1} - \frac{3}{2} \eta_3 x_4$. In this case, there are eight critical points. The first four critical points can exist only when $\eta_3 = 0$, i.e., there is no interaction between dark energy and matter. So, they reduce to the cases considered in Sec. III and here we do not consider them any more. We present the other four critical points in Table V. Note that we have introduced a new constant

\[
\sigma_3 \equiv \frac{\sqrt{(\gamma - 2 + \eta_3)\nu^2 - 6(\gamma - 2)\eta_3}}{(\gamma - 2)\nu^2}.
\]
Critical point \(\bar{\sigma} \) certainly decelerated \((q > 0)\), then we consider the stability of these critical points. In this case, we find the corresponding 

\[
\Omega_\phi = 1 - \frac{\nu^2}{3} \left( 1 - \sigma_3 + \frac{\eta_3}{\gamma - 2} \right), \quad \Omega_m = \frac{\nu^2}{3} \left( 1 - \sigma_3 + \frac{\eta_3}{\gamma - 2} \right), \\
w_{tot} = \frac{1}{2} \left[ \gamma + \eta_3 + (2 - \gamma)\sigma_3 \right], \quad w_\phi = \gamma - 1 - \frac{3(\gamma - 2) [(\gamma - 2)(1 + \sigma_3) - \eta_3]}{2(\gamma - 2)(3 - \nu^2 + \nu_3^2 \sigma_3) - 2\eta_3 \nu^2}, \\
q = \frac{1}{4} \left[ 2 + 3\gamma + 3\eta_3 - 3\sigma_3(\gamma - 2) \right].
\]

Since \(0 < \gamma < 2\), they can exist under the conditions \(2 - \gamma \geq \eta_3 - \sigma_3(\gamma - 2) \geq \gamma - 2\) and \(0 \leq \Omega_m \leq 1\), as well as \(\sigma_3\) is real. Therefore, \(q \geq \frac{[2 + 3\gamma + 3(\gamma - 2)]}{4} = (\gamma - 2)/2\). So, if \(\gamma > 2/3\), the universe is certainly decelerated \((q > 0)\). Unfortunately, the universe cannot be accelerated if \(\gamma = 1\) (pressureless matter) and 4/3 (radiation). Points (III.2p) and (III.2m) are also scaling solutions, and their physical quantities read

\[
\Omega_\phi = 1 - \frac{\nu^2}{3} \left( 1 + \sigma_3 + \frac{\eta_3}{\gamma - 2} \right), \quad \Omega_m = \frac{\nu^2}{3} \left( 1 + \sigma_3 + \frac{\eta_3}{\gamma - 2} \right), \\
w_{tot} = \frac{1}{2} \left[ \gamma + \eta_3 + (\gamma - 2)\sigma_3 \right], \quad w_\phi = \gamma - 1 - \frac{3(\gamma - 2) [(\gamma - 2)(1 - \sigma_3) - \eta_3]}{2(\gamma - 2)(3 - \nu^2 - \nu_3^2 \sigma_3) - 2\eta_3 \nu^2}, \\
q = \frac{1}{4} \left[ 2 + 3\gamma + 3\eta_3 + 3\sigma_3(\gamma - 2) \right].
\]

Since \(0 < \gamma < 2\), they can exist under the conditions \(2 - \gamma \geq \eta_3 + \sigma_3(\gamma - 2) \geq \gamma - 2\) and \(0 \leq \Omega_m \leq 1\), as well as \(\sigma_3\) is real. Therefore, \(q \geq \frac{[2 + 3\gamma + 3(\gamma - 2)]}{4} = (\gamma - 2)/2\). So, if \(\gamma > 2/3\), the universe is certainly decelerated \((q > 0)\). Unfortunately, the universe cannot be accelerated if \(\gamma = 1\) (pressureless matter) and 4/3 (radiation).

Then, we consider the stability of these critical points. In this case, we find the corresponding \(\delta Q_1 = -\frac{\eta_3 \nu_3^2}{2} \left[ (1 - \frac{4\nu_3^2 x_3^2}{\nu_3^2 x_3^2 - \sigma_3}) x_3^{-2} \delta x_1 + \frac{4\nu_3^2 x_3 x_1^{-1}}{\nu_3^2 x_3^2} \delta x_4 \right]\) and \(\delta Q_2 = -\frac{\eta_3 \delta x_4}{2} \left( 1 + \frac{1}{2} \nu_3^2 x_3^{-2} \right)\). The eigenvalues of both Points (III.1p) and (III.1m) are

\[
\left\{ 0, \frac{3}{4} \left[ 2 + 2\gamma + \eta_3 - 3(\gamma - 2)\sigma_3 - \sqrt{\sigma_{31}} \right], \frac{3}{4} \left[ 2 + 2\gamma + \eta_3 - 3(\gamma - 2)\sigma_3 + \sqrt{\sigma_{31}} \right] \right\},
\]

where \(\sigma_{31}\) is a function of \(\gamma, \eta_3\) and \(\nu_2\), which is too complicated to present here. The eigenvalues of both

| Label | Critical Point \((\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)\) |
|-------|--------------------------------------------------|
| III.1p | \(\sqrt{\frac{1}{2} \left( 1 - \frac{\eta_3}{\gamma - 2} + \sigma_3 \right)}, 0, 0, \sqrt{\frac{1}{2} \left( 1 + \frac{\eta_3}{\gamma - 2} - \sigma_3 \right)}\) |
| III.1m | \(-\sqrt{\frac{1}{2} \left( 1 - \frac{\eta_3}{\gamma - 2} + \sigma_3 \right)}, 0, 0, \sqrt{\frac{1}{2} \left( 1 + \frac{\eta_3}{\gamma - 2} - \sigma_3 \right)}\) |
| III.2p | \(\sqrt{\frac{1}{2} \left( 1 - \frac{\eta_3}{\gamma - 2} - \sigma_3 \right)}, 0, 0, \sqrt{\frac{1}{2} \left( 1 + \frac{\eta_3}{\gamma - 2} + \sigma_3 \right)}\) |
| III.2m | \(-\sqrt{\frac{1}{2} \left( 1 - \frac{\eta_3}{\gamma - 2} - \sigma_3 \right)}, 0, 0, \sqrt{\frac{1}{2} \left( 1 + \frac{\eta_3}{\gamma - 2} + \sigma_3 \right)}\) |

TABLE V: Critical points for the autonomous system \([19]-[31]\), in the Case (III) \(Q = \eta_3 \theta \rho_0\).
TABLE VI: Critical points for the autonomous system (48)—(51), in the Case (IV) $Q = \eta_4 \theta_{\text{tot}}$.

Points (III.2p) and (III.2m) read
\[
\left\{ 0, \frac{3}{4} \left( 2 + 2 \gamma + \eta_3 + 3(\gamma - 2)\sigma_3 - \sqrt{\sigma_3^2} \right), \frac{3}{4} \left( 2 + 2 \gamma + \eta_3 + 3(\gamma - 2)\sigma_3 + \sqrt{\sigma_3^2} \right) \right\},
\]
where $\sigma_{32}$ is a function of $\gamma$, $\eta_3$ and $\nu_2$, which is too complicated to present here. All these four critical points can exist and are stable under some conditions, which are very complicated and we do not present them here. Nevertheless, we would like to say more. As mentioned above, if $\gamma > 2/3$, the universe is certainly decelerated ($q > 0$) at all these four critical points. Unfortunately, the universe cannot be accelerated if $\gamma = 1$ (pressureless matter) and $4/3$ (radiation), even when these four critical points can exist and are scaling attractors.

D. Case (IV) $Q = \eta_4 \theta_{\text{tot}}$

The fourth choice of interaction is $Q = \eta_4 \theta_{\text{tot}}$. In this case, the corresponding $Q_1 = \frac{3}{4} \nu_2^2 \eta_4 x_4^{-1}$ and $Q_2 = \frac{3}{4} \nu_2^2 \eta_4 x_4^{-1}$. There are four critical points, and we present them in Table VI. Note that we have introduced a new constant
\[
\sigma_4 \equiv \frac{\sqrt{(\gamma - 2)}[\gamma - 2)]\nu_2^2 - 6\eta_4}{(\gamma - 2)\nu_2}.
\]
Points (IV.1p) and (IV.1m) are scaling solutions, and their physical quantities read
\[
\Omega_\phi = 1 - \frac{\nu_2^2}{3}(1 - \sigma_4), \quad \Omega_m = \frac{\nu_2^2}{3}(1 - \sigma_4),
\]
\[
w_{\text{tot}} = \frac{1}{2}[\gamma + (2 - \gamma)\sigma_4], \quad w_\phi = \gamma - 1 + \frac{3(\gamma - 2)(1 + \sigma_4)}{2[\nu_2^2(1 - \sigma_4) - 3]},
\]
\[q = \frac{1}{4}[2 + 3\gamma + 3\sigma_4(2 - \gamma)].
\]
Points (IV.2p) and (IV.2m) are also scaling solutions, and their physical quantities read
\[
\Omega_\phi = 1 - \frac{\nu_2^2}{3}(1 + \sigma_4), \quad \Omega_m = \frac{\nu_2^2}{3}(1 + \sigma_4),
\]
\[
w_{\text{tot}} = \frac{1}{2}[\gamma + (\gamma - 2)\sigma_4], \quad w_\phi = \gamma - 1 + \frac{3(\gamma - 2)(1 - \sigma_4)}{2[\nu_2^2(1 + \sigma_4) - 3]},
\]
\[q = \frac{1}{4}[2 + 3\gamma + 3\sigma_4(\gamma - 2)].
\]
Since $0 < \gamma < 2$, all these four critical points can exist under the condition $\eta_4 \geq 0$ and $0 \leq \Omega_m \leq 1$. Thus, $-1 \leq \sigma_4 \leq 0$. From Eq. (58), if $\gamma > 2/3$, the universe is certainly decelerated ($q > 0$) at Points (IV.1p) and (IV.1m). Unfortunately, the universe cannot be accelerated if $\gamma = 1$ (pressureless matter) and $4/3$ (radiation). From Eq. (60), the universe is always decelerated ($q > 0$) at Points (IV.2p) and (IV.2m), since $0 < \gamma < 2$ and $\sigma_4 \leq 0$.

We then consider the stability of these critical points. The corresponding $\delta Q_1 = -\frac{9}{4}v_2^2\eta_4\dot{x}_1^2\delta x_1$ and $\delta Q_2 = -\frac{9}{4}v_2^2\eta_4\delta x_4$. The eigenvalues of both Points (IV.1p) and (IV.1m) are
\[
\left\{0, \frac{3}{4} \left[2 + 2\gamma - 3(\gamma - 2)\sigma_4 + \frac{\sqrt{\sigma_4^2}}{\eta_4(\gamma - 2)^2\nu^2_2}\right], \frac{3}{4} \left[2 + 2\gamma - 3(\gamma - 2)\sigma_4 - \frac{\sqrt{\sigma_4^2}}{\eta_4(\gamma - 2)^2\nu^2_2}\right]\right\},
\]
where $\sigma_4$ is a function of $\gamma$, $\eta_4$, and $\nu_2$, which is too complicated to present here. Points (IV.1p) and (IV.1m) can exist and are stable under some conditions, which are very complicated and we do not present them here. Nevertheless, we would like to say more. Since $0 < \gamma < 2$, $\eta_4 \geq 0$ and $-1 \leq \sigma_4 \leq 0$ if Points (IV.1p) and (IV.1m) can exist, we find that $2 + 2\gamma - 3(\gamma - 2)\sigma_4 \geq 5\gamma - 4$. So, if $\gamma > 4/5$, at least one of the last two eigenvalues is positive (when $\sigma_4$ is a real number), or the real parts of both last two eigenvalues are positive (when $\sigma_4$ is an imaginary number). Therefore, Points (IV.1p) and (IV.1m) are certainly unstable for $\gamma > 4/5$. So, if $\gamma = 1$ (pressureless matter) and $4/3$ (radiation), Points (IV.1p) and (IV.1m) cannot be attractors. They can be scaling attractors only when $\gamma < 4/5$ (necessary but not sufficient condition). On the other hand, the eigenvalues of both Points (IV.2p) and (IV.2m) are
\[
\left\{0, \frac{3}{4} \left[2 + 2\gamma + 3(\gamma - 2)\sigma_4 + \frac{\sqrt{\sigma_4^2}}{\eta_4(\gamma - 2)^2\nu^2_2}\right], \frac{3}{4} \left[2 + 2\gamma + 3(\gamma - 2)\sigma_4 - \frac{\sqrt{\sigma_4^2}}{\eta_4(\gamma - 2)^2\nu^2_2}\right]\right\},
\]
where $\sigma_4$ is a function of $\gamma$, $\eta_4$, and $\nu_2$, which is too complicated to present here. Since $0 < \gamma < 2$, $\eta_4 \geq 0$ and $-1 \leq \sigma_4 \leq 0$ if Points (IV.2p) and (IV.2m) can exist, we find that $2 + 2\gamma + 3(\gamma - 2)\sigma_4 > 0$ always. Thus, at least one of the last two eigenvalues is positive (when $\sigma_4$ is a real number), or the real parts of both last two eigenvalues are positive (when $\sigma_4$ is an imaginary number). Therefore, Points (IV.2p) and (IV.2m) are always unstable if they can exist.

V. CONCLUDING REMARKS

In this work, we studied the cosmological evolution of the $\mathcal{E}$ models with the power-law-like potential given in Eq. (10). In the case without matter, there are two attractors which correspond to an inflationary universe in the early epoch, or a de Sitter universe in the late time. In the case with matter but there is no interaction between dark energy and matter, there are only two de Sitter attractors, and no scaling attractor exists. So, it is difficult to alleviate the cosmological coincidence problem. Therefore, we then allow the interaction between dark energy and matter. In this case, several scaling attractors can exist under some complicated conditions, and hence the cosmological coincidence problem could be alleviated. However, their stability and/or the condition to accelerate the universe usually require $\gamma < 2/3$, $4/5$ or $1/2$ (depending on the particular form of interaction $Q$). So, it is invalid for the normal matter, namely $\gamma = 1$ (pressureless matter) and $4/3$ (radiation).

Some remarks are in order. First, in this work we only considered the power-law-like potential given in Eq. (10), namely $r \leq 2$. In fact, it might have new interesting results if we use the power-law-like potential in Eq. (9) with higher $r$. Second, we can even use a more general power-law-like potential,
\[
V(\theta, \phi) = V_0 \phi^n + \sum_{\alpha, \beta} \mu_{\alpha \beta} \theta^\alpha \phi^\beta.
\]
Third, in addition to the interaction terms considered here, there are many exotic interaction terms in the literature, such as $Q \propto \theta (a \rho_m + \beta \rho_\phi)$, $Q \propto \rho_m \rho_\phi$, and $Q \propto g (a \rho + \beta \rho_\phi)$, where $g$ is the deceleration parameter and $\rho$ can be $\rho_m$, $\rho_\phi$ or $\rho_{tot}$. Fourth, we can also consider a non-flat FRW universe with $k \neq 0$. All the above generalizations might have novel results, and deserve further investigation. Finally, it is of interest to study the phase map of the autonomous system more carefully. For instance, one could try to discuss the separatrices (if any) and the perturbations around them (we thank the anonymous referee for bringing this point to our attention).
for pointing out this issue). However, because our main goal of the present work is to find whether the scaling attractors exist to alleviate the cosmological coincidence problem, this type of discussions seems to be beyond our scope, and the relevant discussions might greatly extend the length of this paper. We consider that it is better to study this issue in the future works.

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