Global exponential stability of classical solutions to the hydrodynamic model for semiconductors

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Abstract

In this paper, the global well-posedness and stability of classical solutions to the multi-dimensional hydrodynamic model for semiconductors on the framework of Besov space are considered. We weaken the regularity requirement of the initial data, and improve some known results in Sobolev space. The local existence of classical solutions to the Cauchy problem is obtained by the regularized means and compactness argument. Using the high- and low-frequency decomposition method, we prove the global exponential stability of classical solutions (close to equilibrium). Furthermore, it is also shown that the vorticity decays to zero exponentially in the 2D and 3D space. The main analytic tools are the Littlewood-Paley decomposition and Bony’s para-product formula.

Keywords: Hydrodynamic; exponential stability; classical solutions; spectral localization.

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1 Introduction and Main Results

In the modern semiconductors industry, the numerical simulation of device modeling has become very important. With the fast development of miniaturization devices, the traditional drift-diffusion model is no more valid, especially in submicron devices or in the occurrence of high field phenomena. Some kinetic models such as Boltzmann equation which describe the evolution of the distribution function \( f(t, x, v) \) of charged particles in the phase space are more accurate, but they need much computing power in practical application. Therefore, the hydrodynamic model which represents a reasonable compromise between the physical accuracy and the reduction of computational cost has recently received increasing attention in statistical physics and applied mathematics. By applying a moment method and appropriate closure conditions, it can be derived from the semiclassical Boltzmann equation coupled with the electric potential through a Poisson equation (see, e.g., Refs. [12]). In this paper, we are interested in the simplified hydrodynamic model where the energy equation is replaced by a pressure-density relation, which was first analyzed by Degond and Markowich [7]. The main objective is to study the global well-posedness and stability of classical solutions to the (unipolar) hydrodynamic model on the framework of Besov space. After an appropriate scaling, it can be written as

\[
\begin{align*}
\n_t + \nabla \cdot (n\mathbf{u}) &= 0 \\
\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{n} \nabla p(n) &= \nabla \Phi - \frac{\mathbf{u}}{\tau} \\
\Delta \Phi &= n - \bar{n}, & \Phi \to 0 \text{ as } |x| \to +\infty
\end{align*}
\]

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for \((t,x) \in [0, +\infty) \times \mathbb{R}^N, N \geq 2\), where \(n, u = (u^1, u^2, \cdots, u^N)^T\) (\(T\) represents transpose) and \(\Phi\) denote the electron density, the electron velocity and the electrostatic potential respectively. \(\nabla\) is the gradient operator and \(\Delta\) is Laplacian operator. The constant \(\tau\) is the momentum relaxation time of electrons. The pressure \(p\) satisfies the usual \(\gamma\)-law:

\[
p = p(n) = An^\gamma,
\]

where the case \(\gamma > 1\) corresponds to the isentropic gas and \(\gamma = 1\) corresponds to the isothermal gas, \(A\) is a positive constant. The constant \(\bar{n} > 0\) stands for the density of positively charged background ions. The system is supplemented with the initial data

\[
(n, u)(x, 0) = (n_0, u_0)(x), \quad x \in \mathbb{R}^N.
\]

There are many contributions in mathematical analysis for (1.1)-(1.2), like well-posedness of steady state solutions, global existence of classical or entropy weak solutions, large time behavior of classical solutions, relaxation limit problems and so on, we may refer to Refs. [1, 7, 8, 9, 10] and the references therein. For the evolutionary system (1.1)-(1.2), Luo, Natalini and Xin [14] first established the global exponential stability of small classical solutions to the Cauchy problem. Subsequently, there are many well-posedness and stability results for the Cauchy problem and the initial boundary value problem in one dimensional or multi-dimensional space. Physically, it is more important and more interesting to study (1.1)-(1.2) in the multidimensional case. However, up to now, only partial results are available. Hsiao and Wang et al. [10, 18] studied the spherically symmetrical solutions to (1.1)-(1.2) with \(\gamma = 1\), \(\Omega = \{x \in \mathbb{R}^N| 0 < R_1 < |x| < R_2 < +\infty\}\) and \(\gamma > 1\), \(\Omega = \{x \in \mathbb{R}^N| |x| \geq R_1 > 0\}\) respectively. Guo [8] investigated the irrotational Euler-Poisson equation \((N = 3)\) without relaxation and constructed the global smooth irrotational solutions based on the Klein-Gordon effect, which decay to the equilibrium state uniformly as \((1 + t)^\nu(1 < \nu < 3/2)\). Recently, Hsiao, Markowich and Wang [9] dealt with the multidimensional unbounded domain problem \((N = 2, 3)\) without any geometrical assumptions. The main ingredient was to arrive at an \(a\)-\(priori\) estimate in terms of the \emph{classical energy argument}. Later, All [1] discussed the extended thermodynamic model \((N \geq 2)\) and reached the global existence, uniqueness and stability of classical solutions. The key \(a\)-\(priori\) estimate was obtained by the positive definiteness of some Liapunov functions. The above results are established on the framework of Sobolev space \(H^s(\mathbb{R}^N)\) and the regularity index is required to be high \((\ell > 1 + \frac{N}{2}, \ell \in \mathbb{Z})\) when one deals with them by classical analysis methods. To lower the regularity, using the Littlewood-Paley decomposition and Bony’s paraproduct formula, we prove the following well-posedness results for the system (1.1)-(1.3) in the critical nonhomogeneous Besov space \(B_{2,1}^s(\mathbb{R}^N)\) \((\sigma = 1 + \frac{N}{2})\). First of all, we give a local existence and uniqueness theorem of classical solutions to (1.1)-(1.3) away from the vacuum.

**Theorem 1.1.** Let \(\bar{n} > 0\) be a constant reference density. Suppose that \(n_0 - \bar{n}, u_0\) and \(e_0 \in B_{2,1}^s(\mathbb{R}^N)\) with \(n_0 > 0\), then there exist a time \(T_0 > 0\) and a unique solution \((n, u, \nabla \Phi)\) of the system (1.1)-(1.3) such that

\[
(n, u, \nabla \Phi) \in C^1([0, T_0] \times \mathbb{R}^N) \quad \text{with} \quad n > 0 \quad \text{for all} \quad t \in [0, T_0]
\]

and

\[
(n - \bar{n}, u, \nabla \Phi) \in C([0, T_0], B_{2,1}^s(\mathbb{R}^N)) \cap C^1([0, T_0], B_{2,1}^{s-1}(\mathbb{R}^N)),
\]

where \(e_0 := \nabla \Delta^{-1}(n_0 - \bar{n})\).

**Remark 1.1.** The symbol \(\nabla \Delta^{-1}\) means

\[
\nabla \Delta^{-1} f = \int_{\mathbb{R}^N} \nabla_y G(x - y)f(y)dy,
\]

where \(G(x, y)\) is a solution to \(\Delta_x G(x, y) = \delta(x - y)\) with \(x, y \in \mathbb{R}^N\).

**Remark 1.2.** (1) When one estimates the nonlinear pressure term by the spectral localization method, there appear many commutators, which make computation very tedious. Thanks to the ideas in Ref. [10], we introduce a function transform (sound speed) to reduce the system (1.1)-(1.3) to a symmetric hyperbolic system (3.1)-(3.2) where the nonlinear term becomes linear and bi-linear terms. But, under the
transform there is a new function \( h(m) \) in Poisson equation, which is well-defined and smooth on the domain \( \{m | \frac{\gamma - 1}{\gamma} m + \psi > 0 \} \). Theorem 1.1 follows from Proposition 4.1 and Remark 3.1. We extend the local well-posedness result in \( H^3(\mathbb{R}^N) \) (see Ref. \[1\] or \[9\]) to that in Besov space \( B^{s,1}_{2,1}(\mathbb{R}^N) \) in virtue of the regularized means and compactness argument.

(2) Although our local existence result is proved via a symmetric hyperbolic system, there are some especial contents from Poisson equation to be dealt with. Therefore, we can not use the result of Ifimie \[11\] directly.

Under a smallness assumption, we establish the global existence, uniqueness and exponential decay of classical solutions to (1.1)-(1.3).

**Theorem 1.2.** Let \( \bar{n} > 0 \) be a constant reference density. Suppose that \( n_0 - \bar{n}, u_0 \) and \( e_0 \in B^{s,1}_{2,1}(\mathbb{R}^N) \). There exists a positive constant \( \delta_0 \) depending only on \( A, \gamma, \tau \) and \( \bar{n} \) such that if

\[
\|(n_0 - \bar{n}, u_0, e_0)\|_{B^{s,1}_{2,1}(\mathbb{R}^N)} \leq \delta_0,
\]

then there exists a unique global solution \((n, u, \nabla \Phi)\) of the system (1.1)-(1.3) satisfying

\[
(n, u, \nabla \Phi) \in C^1([0, \infty) \times \mathbb{R}^N)
\]

and

\[
(n - \bar{n}, u, \nabla \Phi) \in C([0, \infty), B^{s,1}_{2,1}(\mathbb{R}^N)) \cap C^1([0, \infty), B^{-s,-1}_{2,1}(\mathbb{R}^N)).
\]

Moreover, we have the decay estimate

\[
\|(n - \bar{n}, u, \nabla \Phi)(\cdot, t)\|_{B^{s,1}_{2,1}(\mathbb{R}^N)} + \|(n_t, u_t, \nabla \Phi_t)(\cdot, t)\|_{B^{-s,-1}_{2,1}(\mathbb{R}^N)} \leq C_0 \|(n_0 - \bar{n}, u_0, e_0)\|_{B^{s,1}_{2,1}(\mathbb{R}^N)} \exp(-\mu_0 t), \quad t \geq 0,
\]

where the positive constants \( \mu_0 \) and \( C_0 \) depend only on \( A, \gamma, \tau \) and \( \bar{n} \).

**Remark 1.3.** From the proof of Theorem 1.2, one can see that the decay rate \( \mu_0 \) satisfies \( \mu_0 = K \tau \) if \( 0 < \tau \leq 1 \) and \( \mu_0 = K/\tau \) if \( \tau > 1 \), where \( K \) is a positive constant independent of \( \tau \).

**Remark 1.4.** Theorem 1.2 follows from Proposition 5.1 and Remark 3.1. Proposition 5.1 relies on a crucial a-priori estimate in Proposition 5.2 and the standard bootstrap argument. In the proof of a-priori estimate, we use high- and low-frequency decomposition method. Different from the classical energy argument in Ref. \[9\], our method shows that Poisson equation plays a key role in the low frequency estimates (see (5.3), (5.4) and (5.10)). Such fact leads to the global exponential stability of classical solutions to (3.1)-(3.2).

Based on Theorem 1.2, we can characterize the exponential decay of the vorticity \( \omega = \nabla \times u \) in Besov space \( B^{-s,-1}_{2,1}(\mathbb{R}^N) \).

**Theorem 1.3.** (\( N=2,3 \)) Let \((n, u, \nabla \Phi)\) be the solution in Theorem 1.2. If

\[
\|(n_0 - \bar{n}, u_0, e_0)\|_{L^2(\mathbb{R}^N)} \leq \delta_0',
\]

then the vorticity \( \omega = \nabla \times u \) decays exponentially in \( B^{-s,-1}_{2,1}(\mathbb{R}^N) \)(\( \omega(x, 0) = \nabla \times u_0 \)):

\[
\|\omega(\cdot, t)\|_{B^{-s,-1}_{2,1}(\mathbb{R}^N)} \leq \|\omega(\cdot, 0)\|_{B^{-s,-1}_{2,1}(\mathbb{R}^N)} \exp \left( -\frac{t}{\tau} \right), \quad t \geq 0,
\]

where the positive constant \( \delta_0' = \min \{\delta_0, \frac{1}{2\pi C_5 N \tau} \} \) depends only on \( A, \gamma, \tau \) and \( \bar{n} \) (\( C_5 \) a constant given in (5.22)).

The paper is arranged as follows. In Section 2, we present some definitions and basic facts on the Littlewood-Paley decomposition and Bony’s para-product formula. In Section 3, we reformulate the system (1.1)-(1.3) in order to arrive at the effective a-priori estimates by the spectral localization method. In Section 4, we obtain the local existence and uniqueness of classical solutions to (3.1)-(3.2) with general initial data. In the last section, we deduce a crucial a-priori estimate under a smallness assumption, which is used to achieve the proof of global existence. Furthermore, it is also shown that the vorticity decays to zero exponentially in the 2D and 3D space.

Throughout this paper, the symbol \( C \) denotes a harmless constant and all functional spaces are considered in \( \mathbb{R}^N \), so we may omit the space dependence for simplicity. Moreover, the integration \( \int_{\mathbb{R}^N} f dx \) is labeled as \( \int f \) without any ambiguity.
2 Littlewood-Paley Analysis

In this section, these definitions and basic facts can be found in Darchin’s [6] mini-course.

Let $S$ be the Schwarz class. $(\varphi, \chi)$ is a couple of smooth functions valued in $[0,1]$ such that $\varphi$ is supported in the shell $C(0, \frac{3}{4}, \frac{3}{8}) = \{ \xi \in \mathbb{R}^N | \frac{3}{4} \leq |\xi| \leq \frac{3}{8} \}$ and $\chi$ is supported in the ball $B(0, \frac{4}{3}) = \{ \xi \in \mathbb{R}^N | ||\xi|| \leq \frac{4}{3} \}$ and

$$\chi(\xi) + \sum_{q=0}^{\infty} \varphi(2^{-q}\xi) = 1, \quad q \in \mathbb{Z}, \quad \xi \in \mathbb{R}^N.$$ 

For $f \in S'$(denote the set of temperate distributes which is the dual one of $S$), we can define the nonhomogeneous dyadic blocks as follows:

$$\Delta_{-1} f := \chi(D)f = \hat{h} * f \quad \text{with} \quad \hat{h} = F^{-1} \chi,$$

$$\Delta_q f := \varphi(2^{-q}D)f = 2^{qN} \int h(2^q y)f(x - y)dy \quad \text{with} \quad h = F^{-1} \varphi, \quad \text{if} \quad q \geq 0.$$ 

where $\ast$, $F^{-1}$ represent the convolution operator and the inverse Fourier transform respectively. The nonhomogeneous Littlewood-Paley decomposition is

$$f = \sum_{q \geq -1} \Delta_q f \quad \text{in} \quad S'.$$

Define the low frequency cut-off by

$$S_q f := \sum_{p \leq q - 1} \Delta_p f.$$ 

Of course, $S_0 f = \Delta_{-1} f$. The above Littlewood-Paley decomposition is almost orthogonal in $L^2$.

**Proposition 2.1.** For any $f, g \in S'$, the following properties hold:

$$\Delta_p \Delta_q f \equiv 0 \quad \text{if} \quad |p - q| \geq 2,$$

$$\Delta_q (S_{p-1} f \Delta_p g) \equiv 0 \quad \text{if} \quad |p - q| \geq 5.$$ 

Besov space can be characterized in virtue of the Littlewood-Paley decomposition.

**Definition 2.1.** Let $1 \leq p \leq \infty$ and $s \in \mathbb{R}$. For $1 \leq r < \infty$, the Besov spaces $B^s_{p,r}$ are defined by

$$f \in B^s_{p,r} \iff \left( \sum_{q \geq -1} 2^{qs} \| \Delta_q f \|_{L^p}^r \right)^{\frac{1}{r}} < \infty$$

and $B^s_{p,\infty}$ are defined by

$$f \in B^s_{p,\infty} \iff \sup_{q \geq -1} 2^{qs} \| \Delta_q f \|_{L^p} < \infty.$$ 

**Definition 2.2.** (J.-M. Bony [2]) Let $f, g$ be two temperate distributions. The product $f \cdot g$ has the Bony’s decomposition:

$$f \cdot g = T_f g + T_g f + R(f, g),$$

where $T_f g$ is paraproduct of $g$ by $f$,

$$T_f g = \sum_{p \leq q-2} \Delta_p f \Delta_q g = \sum_q S_{q-1} f \Delta_q v$$

and the remainder $R(f, g)$ is denoted by

$$R(f, g) = \sum_q \Delta_q f \Delta_q v \quad \text{with} \quad \Delta_q := \Delta_{q-1} + \Delta_q + \Delta_{q+1}.$$
Lemma 2.1. (Bernstein) Let $k \in \mathbb{N}$ and $0 < R_1 < R_2$. There exists a constant $C$ depending only on $R_1, R_2$ and $N$ such that for all $1 \leq a < b < \infty$ and $f \in L^a$, we have

$$\text{Supp } \mathcal{F}f \subset B(0, R_1 \lambda) \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^b} \leq C^{k+1} \lambda^{k+N(\frac{1}{b} - \frac{1}{a})} \|f\|_{L^a};$$

$$\text{Supp } \mathcal{F}f \subset C(0, R_1 \lambda, R_2 \lambda) \Rightarrow C^{-k-1} \lambda^k \|f\|_{L^s} \leq \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^s} \leq C^{k+1} \lambda^k \|f\|_{L^a}.$$  

Here, $\mathcal{F}$ represents the Fourier transform.

A result of compactness in Besov space is:

**Proposition 2.2.** Let $1 \leq p, r \leq \infty$, $s \in \mathbb{R}$ and $\varepsilon > 0$. For all $\phi \in C_c^\infty$, the map $f \mapsto \phi f$ is compact from $B^{s+\varepsilon}_{p,r}$ to $B^s_{p,r}$.

Finally, we state a result of continuity for the composition to end this section.

**Proposition 2.3.** Let $1 \leq p, r \leq \infty$, $I$ be open interval of $\mathbb{R}$. Let $s > 0$ and $n$ be the smallest integer such that $n \geq s$. Let $F : I \to \mathbb{R}$ satisfy $F(0) = 0$ and $F' \in W^{n, \infty}(I; \mathbb{R})$. Assume that $v \in B^n_{p,r}$ takes values in $J \subset I$. Then $F(v) \in B^n_{p,r}$ and there exists a constant $C$ depending only on $s, I, J$ and $N$ such that

$$\|F(v)\|_{B^n_{p,r}} \leq C(1 + \|v\|_{L^\infty})^n \|F'\|_{W^{n, \infty}(I)} \|v\|_{B^n_{p,r}}.$$  

## 3 Reformulation of the Original System

In this section, we are going to reformulate (1.1)-(1.3) in order to obtain the effective a-priori estimates by spectral localization method. For the isentropic case ($\gamma > 1$), let the sound speed

$$\psi(n) = \sqrt{p'(n)},$$

and denote the sound speed at a background density $\bar{n}$ by $\bar{\psi} = \psi(\bar{n})$. Similar to that in Ref. [16], we define

$$m = \frac{2}{\gamma - 1} (\psi(n) - \bar{\psi}).$$

Then the system (1.1) can be reduced to the following system for $C^1$ solutions:

$$\begin{cases}
 m_t + \bar{\psi} \text{div} u = -u \cdot \nabla m - \frac{n-1}{2} m \text{div} u, \\
 u_t + \bar{\psi} \nabla m + \frac{n}{\gamma} u = -u \cdot \nabla u - \frac{n-1}{2} m \nabla m + e, \\
 e_t = -\nabla \Delta^{-1} \nabla \{h(m) u + n u\},
\end{cases}$$

(3.1)

where $e := \nabla \Phi$, $h(m) = \{(A\gamma)^{-1} (\frac{n-1}{2} m + \bar{\psi})\}^{\frac{1}{\gamma-1}} - \bar{n}$ is a smooth function on the domain $\{m^{\frac{n-1}{2}} + \psi > 0\}$ satisfying $h(0) = 0$, the non-local term $\nabla \Delta^{-1} \nabla \cdot f$ is the product of Riesz transforms of $f$. The initial data (1.3) becomes into

$$(m, u, e)|_{t=0} = (m_0, u_0, e_0)$$

with

$$m_0 = \frac{2}{\gamma - 1} (\psi(n_0) - \bar{\psi}), \quad e_0 = \nabla \Delta^{-1} (n_0 - \bar{n}).$$

**Remark 3.1.** For any $T > 0$, $(n, u, e) \in C^1([0, T] \times \mathbb{R}^N)$ is a solution of the system (1.1)-(1.2) with $n > 0$, then $(m, u, e) \in C^1([0, T] \times \mathbb{R}^N)$ solves the system (3.1)-(3.2) with $\frac{n-1}{2} m + \bar{\psi} > 0$; Conversely, if $(m, u, e) \in C^1([0, T] \times \mathbb{R}^N)$ solves the system (3.1)-(3.2) with $\frac{n-1}{2} m + \bar{\psi} > 0$, then $(n, u, e) \in C^1([0, T] \times \mathbb{R}^N)$ is a solution of the system (1.1)-(1.2) with $n > 0$, where $n = \psi^{-1}(\frac{n-1}{2} m + \bar{\psi})$.  

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For the isothermal case \((\gamma = 1)\), let \(\tilde{n} = \sqrt{A} (\ln n - \ln \tilde{n})\), \(e = \nabla \Phi\), then the system (1.1) is transformed into the following one for \(C^1\) solutions:

\[
\begin{aligned}
\dot{n} + \sqrt{A} \text{div} u &= -u \cdot \nabla \tilde{n}, \\
\dot{u} + \sqrt{A} \nabla \tilde{n} \frac{\partial}{\partial t} &= -u \cdot \nabla u + e, \\
e_t &= -\nabla \Delta^{-1} \nabla \cdot [h(\tilde{n})u + \tilde{n}u],
\end{aligned}
\]

where \(h(\tilde{n}) = \tilde{n}(\exp(A^{-\frac{2}{\gamma}}n) - 1)\) is a smooth function on the domain \(\{\tilde{n} | -\infty < \tilde{n} < +\infty\}\) satisfying \(h(0) = 0\). The initial data (1.3) turns into

\[
(\tilde{n}, u, e)|_{t=0} = (\sqrt{A}(\ln n_0 - \ln \tilde{n}), u_0, e_0).
\]

**Remark 3.2.** For any \(T > 0\), if \((n, u, e) \in C^1([0, T] \times \mathbb{R}^N)\) is a solution of the system (1.1)-(1.2) with \(n > 0\), then \((\tilde{n}, u, e) \in C^1([0, T] \times \mathbb{R}^N)\) solves the system (3.3)-(3.4); Conversely, if \((\tilde{n}, u, e) \in C^1([0, T] \times \mathbb{R}^N)\) solves the system (3.3)-(3.4), then \((n, u, e) \in C^1([0, T] \times \mathbb{R}^N)\) is a solution of the system (1.1)-(1.2) with \(n > 0\), where \(n = \tilde{n} \exp(A^{-\frac{2}{\gamma}}n)\).

In what follows, we shall only study the system (3.1)-(3.2) and prove the main results in this paper, since (3.3)-(3.4) can be discussed through a similar process.

## 4 Local Existence

In this section, we first give some estimates of commutators in Besov space \(B_{p,1}^s\). Then, using the regularized means and compactness argument, we complete the proof of proposition 4.1.

By using the first and third equation of Eq.(1.1), we get the following formulations under the variable transform immediately.

**Lemma 4.1.**

\[
\text{div} e = h(m), \quad \text{div} u = -\frac{\text{div} e_t + \text{div}(h(m)u)}{\tilde{n}}.
\]

Applying the operator \(\Delta_q\) to (3.1) yields

\[
\begin{aligned}
\partial_t \Delta_q m + (u \cdot \nabla) \Delta_q m &= -\psi \Delta_q \text{div} u + [u, \Delta_q] \cdot \nabla m - \frac{\gamma - 1}{2} \Delta_q (m \text{div} u), \\
\partial_t \Delta_q u + (u \cdot \nabla) \Delta_q u + \frac{\gamma - 1}{2} \Delta_q u &= -\psi \Delta_q (\nabla m) + [u, \Delta_q] \cdot \nabla u \\
&\quad - \frac{\gamma - 1}{2} \Delta_q (m \nabla m) + \Delta_q e,
\end{aligned}
\]

where the commutator \([f, g] = fg - gf\).

Multiplying the first equation of Eq.(4.1) by \(\Delta_q m\), the second one by \(\Delta_q u\) and adding the resulting equations together, then integrating it over \(\mathbb{R}^N\), we obtain

\[
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left(\|\Delta_q m\|^2_{L^2} + \|\Delta_q u\|^2_{L^2}\right) + \frac{1}{\gamma} \|\Delta_q u\|^2_{L^2} \\
= &\frac{1}{2} \int \text{div} u (|\Delta_q m|^2 + |\Delta_q u|^2) + \int ([u, \Delta_q] \cdot \nabla m \Delta_q m + [u, \Delta_q] \cdot \nabla u \Delta_q u) \\
&- \frac{\gamma - 1}{2} \int (\Delta_q (m \text{div} u) \Delta_q m + \Delta_q (m \nabla m) \Delta_q u) + \int \Delta_q e \Delta_q u.
\end{aligned}
\]

The electric field term can be estimated as

\[
\int \Delta_q e \Delta_q u = -\int \Delta_q \Phi \Delta_q \text{div} u.
\]
Proof.

Suppose that \( \varepsilon > 0 \) and \( \int |u| \left( 1 + |\nabla u|^2 \right) dt > 0 \).

Here, we give a lemma to estimate these commutators in (4.2) and (4.4).

\[
\int \{ \Delta_q (m \Delta u) \Delta_q m + \Delta_q (m \nabla m) \Delta_q u \}
\]

\[= - \int \Delta_q m(\nabla m \cdot \Delta_q u) + \int [\Delta_q, m] \nabla m \cdot \Delta_q u + \int [\Delta_q, m] \text{div} u \Delta_q m. \quad (4.4)
\]

Note that the bi-linear spectral localization term, we have

\[
2^n ||[f, \Delta_q]A||_{L^p} \leq C c_q ||f||_{L^{N/p}} ||g||_{L^{N/p}}, \quad s > 0.
\] (4.6)

where the operator \( A = \text{div} \) or \( \nabla \), \( C \) is a harmless constant and \( \{c_q\} \) denotes a sequence such that \( \|c_q\|_{L^\infty} \leq 1 \).

\[\text{Remark 4.1. Similar estimates of commutators have been obtained by Chemin[3], Danchin[4, 5, 6] and Iftimie[11] et al., therefore, we omit the details.}\]

Now, we give a proposition on the local existence and uniqueness of classical solutions to (3.1)-(3.2).

\[\text{Proposition 4.1. Suppose that } (m_0, u_0, \varepsilon_0) \in B^{2}_{2,1} \text{ satisfying } \frac{\gamma - 1}{2} m_0 + \bar{\psi} > 0, \text{ then there exist a time } T_0 > 0 \text{ and a unique solution } (m, u, e) \text{ of (3.1)-(3.2)} \]

\[\text{such that } (m, u, e) \in C^1([0, T_0] \times \mathbb{R}^N) \text{ with } \frac{\gamma - 1}{2} m + \bar{\psi} > 0 \text{ for all } t \in [0, T_0] \text{ and } (m, u, e) \in C([0, T_0], B^{2}_{2,1}) \cap C^1([0, T_0], B^{\ast -1}_{2,1}).\]

\[\text{Proof. (Existence) Let } U_0 = (m_0, u_0, \varepsilon_0)^\top \in B^{2}_{2,1}.\]

\[\text{Claim 1: There exist two positive constants } \eta', \eta'' \text{ such that}\]

\[0 < \eta' \leq \frac{\gamma - 1}{2} m_0 + \bar{\psi} \leq \eta''. \quad (4.7)
\]

In fact, by the imbedding \( B^{\ast -1}_{2,1} \rightarrow C_0(\text{continuous bounded functions which decay to zero at infinity}) \), we know \( m_0 \in C_0 \). For any \( \varepsilon > 0 \), \( \exists M = M(\varepsilon) > 0 \) such that \( |m_0| \leq \varepsilon \) for all \( |x| > M \). We may choose \( \varepsilon = \frac{1}{\frac{\gamma - 1}{2} m_0 + \bar{\psi}} \), then \( 0 < \frac{\bar{\psi}}{2} \leq \frac{\gamma - 1}{2} m_0 + \bar{\psi} \leq \frac{3\bar{\psi}}{2} \). On the other hand, when \( x \in [-M, M] \), we have

\[0 < \frac{\gamma - 1}{2} m_0 + \bar{\psi} \leq \frac{\gamma - 1}{2} m_0 + \bar{\psi} \leq \frac{\gamma - 1}{2} \max_{x \in [-M, M]} m_0 + \bar{\psi}.
\]

So we obtain (4.7) only by choosing

\[\eta' = \max \left\{ \frac{\bar{\psi}}{2}, \frac{\gamma - 1}{2} \min_{x \in [-M, M]} m_0 + \bar{\psi} \right\}, \quad \eta'' = \min \left\{ \frac{3\bar{\psi}}{2}, \frac{\gamma - 1}{2} \max_{x \in [-M, M]} m_0 + \bar{\psi} \right\}.
\]

From (4.7), we can obtain \( 0 < \eta' \leq n_0 \leq \theta'' \) for two positive constants \( \theta', \theta'' \). Note that

\[h(m_0) = \left( (A\gamma)^{-\frac{1}{4}} \left( \frac{\gamma - 1}{2} m_0 + \bar{\psi} \right) \right)^{\frac{2}{\eta'}} - \tilde{n} = n_0 - \tilde{n}
\]
is a smooth function on \( \{m_0 | \gamma - 1 \leq m_0 + \tilde{\psi} > 0\} \), by Proposition 2.3, we have
\[
\|n_0 - \bar{n}\|_{B^2} = \|h(m_0)\|_{B^2} \leq C\|m_0\|_{B^2}.
\]

There exists a sequence \( \{\hat{U}_0^k\} := \{n_0^k, u_0^k, e_0^k\} \) such that \( \{n_0^k - \bar{n}, u_0^k, e_0^k\} \in \mathcal{H}(\ell > \sigma, \ell \in \mathbb{Z}) \) converges to \( (n_0 - \bar{n}, u_0, e_0) \) in \( B^2 \) and satisfies \( 0 < \tilde{\sigma} \leq n_0^k \leq \tilde{\sigma}' \). Furthermore, we also have \( \|U_0^k\|_{B^2} \leq \|U_0\|_{B^2} + 1(\|U_0^k\|_{B^2} \leq (m_0^k, u_0^k, e_0^k)^T) \). We define a sequence \( \{U^k\} := \{(n_0^k, u_0^k, e_0^k)^T\} \), which solves the following equations:
\[
\begin{align*}
&n_0^k + \nabla \cdot (n_0^k u_0^k) = 0 \\
u_0^k + (u_0^k \cdot \nabla) u_0^k + \frac{1}{\tilde{n}_0^k} \nabla p(n_0^k) = e_0^k - \frac{u_0^k}{\tilde{n}_0^k} \\
e_0^k = -\nabla \Delta^{-1} \nabla \cdot (n_0^k u_0^k)
\end{align*}
\]
with the initial data
\[
(n_0^k, u_0^k, e_0^k)\big|_{t=0} = (n_0^k, u_0^k, e_0^k).
\]

It is easy to see (4.8) is a strict hyperbolic symmetric system on \( \hat{G} = \{U^k | n_0^k > 0\} \) in the sense of Friedrichs. Using Kato’s classical result in Ref. [13] or [15] and the \( L^2 \)- boundedness of Riesz transformation, we can get the following local existence result: there exist a time \( T_k \) and a solution \( \hat{U}^k \) to (4.8)-(4.9) such that
\[
\hat{U}^k \in C^1([0, T_k] \times \mathbb{R}^N) \quad \text{with} \quad 0 < \tilde{\sigma}' \leq n_k^0 \leq \tilde{\sigma}'' \quad \text{for all} \ t \in [0, T_k]
\]
\[
(\tilde{\sigma}', \tilde{\sigma}'' \text{ are two positive constants depending on } k)
\]
and
\[
(n_k^0 - \bar{n}, u_k^0, e_k^0) \in C([0, T_k], H^\ell) \cap C^1([0, T_k], H^{\ell-1}).
\]

From Remark 3.1, the sequence \( \{U^k\} = \{(m_0^k, u_0^k, e_0^k)^T\} \) solves the following equations:
\[
\begin{align*}
m_k^0 + \psi \text{div} u_k^0 &= -u_k^0 \cdot \nabla m_k^0 - \frac{1}{2} m_k^0 \text{div} u_k^0 \\
u_k^0 + \psi \nabla m_k^0 + \frac{u_k^0}{n_k} &= -u_k^0 \cdot \nabla u_k^0 - \frac{1}{2} m_k^0 \nabla m_k^0 + e_k^0 \\
e_k^0 &= -\nabla \Delta^{-1} \nabla \cdot (h(m_k^0) u_k^0 + \bar{n} u_k^0)
\end{align*}
\]
with the initial data
\[
(m_k^0, u_k^0, e_k^0)\big|_{t=0} = \left( \frac{2}{\gamma - 1} \left( \psi(n_0^k) - \tilde{\psi} \right), u_0^k, e_0^k \right)
\]
\[
(\psi_n, \psi_m \text{ are two positive constants depending on } k)
\]
satisfying
\[
U^k \in C^1([0, T_k] \times \mathbb{R}^N) \quad \text{with} \quad 0 < \eta_k^0 \leq \frac{\gamma - 1}{2} m_k^0 + \psi \leq \eta_k'' \quad \text{for all} \ t \in [0, T_k]
\]
\[
(\eta_k^0, \eta_k'' \text{ are two positive constants depending on } k)
\]
and
\[
U^k \in C([0, T_k], H^\ell) \cap C^1([0, T_k], H^{\ell-1}).
\]

Let \( [0, T_k^*] \) be the maximal existence interval of above solutions to (4.10)-(4.11). Similar to the discussion in Ref. [13], we have the blow-up criterion:
\[
T_k^* < \infty \iff \limsup_{t \to T_k^*} (\|U^k\|_{L^\infty} + \|\nabla U^k\|_{L^\infty}) = +\infty \quad \text{or}
\]
for any compact subset \( K \subset G, \ U^k(x, t) \) escapes \( K \) as \( t \to T_k^* \),
\[
\text{where } G = \left\{ U^k \big| \frac{\gamma - 1}{2} m_k^0 + \psi > 0 \right\}.
\]

**Claim 2:** For \( t \in [0, \min\{T_k^*, T_0\}) \), it yields \( \|U^k(t)\|_{B^2} \leq 4\lambda_0 \), where \( T_0 = \frac{1}{2c\lambda_0} \), \( \lambda_0 = \|U_0\|_{B^2} + 1 \) and \( \hat{C} \) is a positive constant (independent of \( k \)) given in (4.18).
The proof of Claim 2 relies on the standard continuity method. Let
\[ G = \left\{ T \in [0, \min\{T_k^*, T_0\}) : \sup_{0 \leq t \leq T} E(t) \leq 4\lambda_0 \right\}, \]
where
\[ E(t) = \|U^k(t)\|_{\mathcal{B}_{Z,1}^2}. \]
In fact, we have already known
\[ \|U^k_0\|_{\mathcal{B}_{Z,1}^2} \leq \lambda_0 \leq 4\lambda_0. \]
Then from the continuity of \( E(t) \) on \([0, \min\{T_k^*, T_0\})\), we may see that the set \( G \) is nonempty and relatively close in \([0, \min\{T_k^*, T_0\})\). To show that it is also relatively open in \([0, \min\{T_k^*, T_0\})\), and hence the entire interval \([0, \min\{T_k^*, T_0\})\), it suffices to the weaker bound in (4.12) implies
\[ \|U^k(t)\|_{\mathcal{B}_{Z,1}^2} \leq 2\lambda_0, \quad t \in [0, T] \subset [0, \min\{T_k^*, T_0\}). \]
To do this, we need to make the best use of flow map.

The flow map \( X(t; \bar{t}, x) \) of \( u^k \) starting from \( x \in \mathbb{R}^N \) at time \( \bar{t} \in [0, T] \) can be defined as
\[ \frac{dX}{dt} = u^k(t, X(t; \bar{t}, x)), \quad X(t; \bar{t}, x)|_{t=\bar{t}} = x, \]
then we have
\[ \frac{d}{dt} m^k(t, X(t; \bar{t}, x)) = (\partial_t + u^k \cdot \nabla)m^k(t, X(t; \bar{t}, x)). \]
Together with the first equation of Eq.(4.10), it is easy to get (for any \( \bar{t} \in [0, T] \))
\[ \frac{\gamma - 1}{2} m^k(\bar{t}, x) + \bar{\psi} \]
\[ = \left( \frac{\gamma - 1}{2} m^k(0, X(0; \bar{t}, x)) + \bar{\psi} \right) \exp \left( -\frac{\gamma - 1}{2} \int_{0}^{\bar{t}} \text{div} u^k(\zeta, X(\zeta; \bar{t}, x))d\zeta \right). \]
Thus, there exist two positive constants \( \eta_1, \eta_2 \) (independent of \( k \)) such that
\[ 0 < \eta_1 \leq \frac{\gamma - 1}{2} m^k(\bar{t}, x) + \bar{\psi} \leq \eta_2, \quad (\bar{t}, x) \in [0, T] \times \mathbb{R}^N. \]
Here, we need not consider the effect of relaxation term. Therefore, by (4.2)-(4.4), Hölder’s inequality and Lemma 4.2 (take \( p = 2 \) and \( s = \sigma \)), we get
\[ \frac{d}{dt} \left( 2^{2q\sigma} \|\Delta u^k\|_{L^2}^2 \right) \leq C \left\{ \|\nabla u^k\|_{L^2} 2^{2q\sigma} (\|\Delta u^k\|_{L^2}^2 + \|\Delta u^k\|_{L^2}) + c_q 2^{2q\sigma} \|u^k\|_{\mathcal{B}_{Z,1}^2} \|m^k\|_{\mathcal{B}_{Z,1}^2} \|\Delta m^k\|_{L^2} \right. \]
\[ + c_q 2^{2q\sigma} \|\nabla u^k\|_{L^\infty} \|u^k\|_{\mathcal{B}_{Z,1}^2} \|\Delta u^k\|_{L^2} + 2^{2q\sigma} \|\nabla m^k\|_{L^\infty} \|\Delta u^k\|_{L^2} \|\Delta m^k\|_{L^2} \right. \]
\[ + c_q 2^{2q\sigma} \|\nabla m^k\|_{L^\infty} \|m^k\|_{\mathcal{B}_{Z,1}^2} \|\Delta u^k\|_{L^2} + c_q 2^{2q\sigma} \|m^k\|_{\mathcal{B}_{Z,1}^2} \|u^k\|_{\mathcal{B}_{Z,1}^2} \|\Delta m^k\|_{L^2} \]
\[ + 2^{2q\sigma} \|\Delta (\rho (m^k) u^k)\|_{L^2} \|\Delta (\rho e^k)\|_{L^2}, \quad t \in [0, T]. \]
(4.16)
Dividing (4.16) by \( 2^{2q\sigma} \|u^k\|_{L^2}^2 + \epsilon \) \( (\epsilon > 0 \text{ a small quantity}) \), we have
\[ \frac{d}{dt} \left( 2^{2q\sigma} \|\Delta u^k\|_{L^2}^2 + \epsilon \right) \]
\[ \leq C \left\{ \|\nabla u^k\|_{L^2} 2^{2q\sigma} (\|\Delta u^k\|_{L^2} + \|\Delta u^k\|_{L^2}) + c_q \|u^k\|_{\mathcal{B}_{Z,1}^2} \|m^k\|_{\mathcal{B}_{Z,1}^2} \right\}. \]
\[ +c_q \|\nabla u^k\|_{L^\infty} \|u^k\|_{B^{2}_{2,1}} + \|\nabla m^k\|_{L^\infty} 2^{q \alpha} \|\Delta_q u^k\|_{L^2} + c_q \|\nabla m^k\|_{L^\infty} \times \|m^k\|_{B^{2}_{2,1}} + c_q \|m^k\|_{B^{2}_{2,1}} \|u^k\|_{B^{2}_{2,1}} + 2^{q \alpha} \|\Delta_q (h(m^k)u^k)\|_{L^2} \}, \ t \in [0, T]. \] (4.17)

Integrating (4.17) on the variable \( t \), then taking \( \epsilon \to 0 \) and using Proposition 2.3, we obtain the \textit{a-priori} estimate of \( U^k \):

\[ \|U^k(t)\|_{B^{2}_{2,1}} \leq \ |U_0^k|_{B^{2}_{2,1}} + \tilde{C} \int_0^t \|U^k(\zeta)\|_{B^{2}_{2,1}}^2 d\zeta. \] (4.18)

Furthermore, we have

\[ \sup_{0 \leq \zeta \leq t} \|U^k(\zeta)\|_{B^{2}_{2,1}} \leq \ (1 + \|U_0^k\|_{B^{2}_{2,1}} + 1) + \tilde{C} \int_0^t \sup_{0 \leq \zeta' \leq \zeta} \|U^k(\zeta')\|_{B^{2}_{2,1}}^2 d\zeta', \ t \in [0, T]. \] (4.19)

Set

\[ \lambda_1(t) \equiv (1 + \|U_0^k\|_{B^{2}_{2,1}} + 1) + \tilde{C} \int_0^t \sup_{0 \leq \zeta' \leq \zeta} \|U^k(\zeta')\|_{B^{2}_{2,1}}^2 d\zeta'. \]

Then, we have

\[ \frac{d}{dt} \lambda_1 \leq \tilde{C} \lambda_1^2, \quad \lambda_1(0) = \|U_0^k\|_{B^{2}_{2,1}} + 1, \ t \in [0, T]. \] (4.20)

Let \( \lambda(t) \) solves Riccati equation:

\[ \frac{d}{dt} \lambda = \tilde{C} \lambda^2, \quad \lambda(0) = \|U_0^k\|_{B^{2}_{2,1}} + 1. \] (4.21)

The time \( T_0 = 1/(2\tilde{C}\lambda_0) \) is less than the blow-up time for (4.21). Then by solving the differential inequality (4.20), we have \( \lambda_1(t) \leq \lambda(t) \) for \( t \in [0, T_0] \). Solving (4.21) yields

\[ \lambda_1(t) \leq \|U_0^k\|_{B^{2}_{2,1}} + 1 \quad \lambda(t) \equiv \lambda(t), \ t \in [0, T_0]. \] (4.22)

Therefore, we get

\[ \|U^k(t)\|_{B^{2}_{2,1}} \leq 2\lambda_0, \ t \in [0, T] \subset [0, \min\{T^*_k, T_0\}], \]

which completes the proof of Claim 2.

Furthermore, using Eq.(4.10), we have

\[ \|U^k(t)\|_{B^{2}_{2,1}} \leq \lambda_0', \ t \in [0, \min\{T^*_k, T_0\}], \] (4.23)

where \( \lambda_0' \) is a positive constant only depending on the initial data \( U_0 \). There exist two positive constants \( \check{\eta}, \check{\eta} \) (independent of \( k \)) such that

\[ 0 < \check{\eta} \leq \gamma - \frac{1}{2} m^k(t, x) + \check{\psi} \leq \check{\eta}, \quad (t, x) \in [0, \min\{T^*_k, T_0\}) \times \mathbb{R}^N. \] (4.24)

From Claim 2, (4.23) and (4.24), the blow-up criterion implies \( 0 < T_0 < T^*_k \), moreover, we have \( 0 < T_0 \leq \inf_k T^*_k \).

That is, we find a positive time \( T_0 \) (only depending on the initial data \( U_0 \)) such that the approximative solution sequence \( \{U^k\} \) to (4.10)-(4.11) is uniformly bounded in \( C([0, T_0], B^{2}_{2,1}) \cap C^1([0, T_0], B^{2\sigma-1}_{2,1}) \). Moreover, it weak*-converges (up to a subsequence) to some \( U \) in \( L^\infty([0, T_0], B^{2\sigma-1}_{2,1}) \) in terms of the Banach-Alaoglu Theorem (see Ref. [17] Remark 2 on p.180 in Triebel, 1983). Since \( \{U^k\} \) is also uniformly bounded in \( C([0, T_0], B^{2\sigma-1}_{2,1}) \) (it weak*-converges to \( U \) in \( L^\infty([0, T_0], B^{2\sigma-1}_{2,1}) \)), then \( \{U^k\} \) is uniformly bounded in
Lip([0, T_0], B_{2,1}^{-1}), hence uniformly equicontinuous on [0, T_0] with the norm in B_{2,1}^{-1}. By Proposition 2.2, Ascoli-Arzela theorem and Cantor diagonal process, we deduce that

\[ \phi U^k \to \phi U \text{ in } C([0, T_0], B_{2,1}^{-1}) \text{ as } k \to \infty, \text{ for any } \phi \in C_c^\infty. \]

The properties of strong convergence enable us to pass to the limit in (4.10)-(4.11). Indeed, U is a solution to (3.1)-(3.2). Now, what remains is to check U has the required regularity. First, from \( U \in C([0, T_0], B_{2,1}^{-1}) \) and an interpolation argument, we know \( U \in C([0, T_0], B_{2,1}^{-1}) \) for any \( \sigma' < \sigma \). Furthermore, we have \( S_T U \in C([0, T_0], B_{2,1}^{-1}) \) for any \( q \in \mathbb{N} \). Then, combining with (4.18) (throw off the superscript \( k \)), we derive that \( S_T U \) converges uniformly to \( U \) on \([0, T_0]\) with the norm in \( B_{2,1}^{-1} \). This achieves \( U \in C([0, T_0], B_{2,1}^{-1}) \). And then, using Eq.(3.1), we see that \( U_t \in C([0, T_0], B_{2,1}^{-1}) \), so \( U(t, x) \in C^1([0, T_0] \times \mathbb{R}^N) \). By virtue of the flow map, we get \( 0 < \theta' \leq \frac{2}{1 - m(t, x)} \leq \theta'' \) for \( (t, x) \in [0, T_0] \times \mathbb{R}^N \) according to Claim 1 (\( \theta', \theta'' \) are two positive constants).

(Uniqueness) Let \( \tilde{m} = m_1 - m_2, \tilde{u} = u_1 - u_2, \tilde{e} = e_1 - e_2 \) where \( U_1 = (m_1, u_1, e_1)^\top \) and \( U_2 = (m_2, u_2, e_2)^\top \) are two solutions to the system (3.1)-(3.2) with the same initial data respectively. Then \( \bar{U} = (\tilde{m}, \tilde{u}, \tilde{e})^\top \) satisfies the following equations:

\[
\left\{
\begin{align*}
\dot{\tilde{m}} + \tilde{u} \Delta \tilde{m} &= -u_1 \cdot \nabla \tilde{m} - \tilde{u} \nabla m_1 - \frac{2}{1 - m_1} \nabla \tilde{u} - \frac{2}{1 - m_2} \tilde{m} \nabla \tilde{u}_2, \\
\dot{\tilde{u}} + \tilde{u} \Delta \tilde{u} &= -u_1 \cdot \nabla \tilde{u} - \tilde{u} \nabla u_2 - \frac{2}{1 - m_1} \tilde{m} \nabla \tilde{u}_2 - \frac{2}{1 - m_2} \tilde{m} \nabla \tilde{u}_2 + \dot{\tilde{e}}, \\
\dot{\tilde{e}} &= -\tilde{u} \Delta \tilde{e} - \tilde{e} h((m_1) + \tilde{n}) \tilde{u},
\end{align*}
\right.
\]

where the smooth function \( h(m) := \frac{h(m_1)}{m} \) for \( (m_1) < \sigma \). Then \( \tilde{u} \) is uniformly equicontinuous.

Similar to the derivation of (4.18), from Lemma 4.2 (take \( p = 2 \) and \( s = \sigma - 1 \)), we obtain the following estimate:

\[
\| \tilde{U}(t) \|_{B_{2,1}^{-1}} \leq C \int_0^t \| \tilde{U}(s) \|_{B_{2,1}^{-1}} \left( \| U_1(s) \|_{B_{2,1}} + \| U_2(s) \|_{B_{2,1}} \right) ds, \text{ for } t \in [0, T_0].
\]

By Gronwall’s inequality, we have \( \tilde{U} \equiv 0. \)

\[ \square \]

## 5 Global Existence and Exponential Stability

In this section, we first state a proposition on the global existence and exponential stability of classical solutions to (3.1)-(3.2).

**Proposition 5.1.** Suppose that \( U_0 \in B_{2,1}^{-1} \). There exists a positive constant \( \delta_2 < \min\{\frac{\delta_1}{M_{h,2}}, \frac{\delta_1}{C_1}\} \) depending only on \( A, \gamma, \tau \) and \( \bar{n} \) such that if

\[
\| U(\cdot, 0) \|_{B_{2,1}^{-1}} \leq \delta_2,
\]

then there exists a unique global solution \( U \) to (3.1)-(3.2) satisfying

\[
U \in C([0, \infty), B_{2,1}^{-1}) \cap C^1([0, \infty), B_{2,1}^{-1})
\]

and

\[
\| U(\cdot, t) \|_{B_{2,1}^{-1}} + \| U_1(\cdot, t) \|_{B_{2,1}^{-1}} \leq C_1 \| U(\cdot, 0) \|_{B_{2,1}^{-1}} \exp(-\mu_1 t), \text{ for } t \geq 0,
\]

where \( \delta_1, \mu_1, C_1 \) are three positive constants given by Proposition 5.2 and the positive constant \( C_2 \) is given in (5.19), \( U = (m, u, e)^\top \) and \( U_t = (m_1, u_1, e_1)^\top. \)

The proof of above proposition mainly depends on a crucial a-priori estimate (Proposition 5.2). To do this, we need some lemmas.

**Lemma 5.1.** If \((m, u, e) \in C([0, T], B_{2,1}^{-1}) \cap C^1([0, T], B_{2,1}^{-1})\) is a solution of Eq. (3.1) for any given \( T > 0 \), then

\[
\frac{d}{dt} \left( \| \Delta_q m_t \|_{L^2}^2 + \| \Delta_q u_t \|_{L^2}^2 + \frac{1}{n} \| \Delta_q e_t \|_{L^2}^2 \right) + \frac{2}{\gamma} \| \Delta_q u_t \|_{L^2}^2 \]

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Lemma 5.2. If \((m, u, e) \in C([0, T], B_{2,1}^2) \cap C^1([0, T], B_{2,1}^{-1})\) is a solution of Eq. (3.1) for any given \(T > 0\), then

\[
\Delta_q m_t \leq \frac{\gamma - 1}{2} \left( |\Delta_q m_t|_{L^2} + |\Delta_q \nabla m|_{L^2} \right) + \frac{\gamma - 1}{2} \left( |\Delta_q \nabla u|_{L^2} + |\Delta_q \Delta u|_{L^2} \right) + \frac{\gamma - 1}{2} \left( |\nabla \Delta u|_{L^2} + |\nabla \Delta u|_{L^2} \right) + \frac{\gamma - 1}{2} \left( |\Delta_q \nabla m|_{L^2} + |\Delta_q \nabla \nabla u|_{L^2} \right)
\]

where \(\Delta_q m_t = \Delta_q m_t + \Delta_q \nabla m\) and \(\Delta_q \nabla u = \Delta_q \nabla u + \Delta_q \Delta u\).

Proof. By differentiating the first two equations of Eq. (4.1) with respect to variable \(t\) once, integrating them over \(\mathbb{R}^N\) after multiplying \(\Delta_q m_t, \Delta_q \nabla u\) respectively, similar to the derivation of (4.16), through tedious but straightforward calculations, we can obtain (5.1).

In addition, we give some auxiliary estimates, which are divided into high- and low- frequency cases.

**Lemma 5.2.** If \((m, u, e) \in C([0, T], B_{2,1}^2) \cap C^1([0, T], B_{2,1}^{-1})\) is a solution of Eq. (3.1) for any given \(T > 0\), then

\[
\|\Delta_q m_t\|_{L^2} \leq \left( \frac{\gamma - 1}{2} \left( |\Delta_q m_t|_{L^2} + |\Delta_q \nabla m|_{L^2} \right) + \frac{\gamma - 1}{2} \left( |\Delta_q \nabla u|_{L^2} + |\Delta_q \Delta u|_{L^2} \right) + \frac{\gamma - 1}{2} \left( |\nabla \Delta u|_{L^2} + |\nabla \Delta u|_{L^2} \right) + \frac{\gamma - 1}{2} \left( |\Delta_q \nabla m|_{L^2} + |\Delta_q \nabla \nabla u|_{L^2} \right) \right) + \frac{\gamma - 1}{2} \left( |\Delta_q \nabla m|_{L^2} + |\Delta_q \nabla \nabla u|_{L^2} \right) + \frac{\gamma - 1}{2} \left( |\nabla \Delta u|_{L^2} + |\nabla \Delta u|_{L^2} \right) + \frac{\gamma - 1}{2} \left( |\Delta_q \nabla m|_{L^2} + |\Delta_q \nabla \nabla u|_{L^2} \right)
\]

\[
\|\Delta_q e\|_{L^2} \leq \left( \frac{\gamma - 1}{2} \left( |\Delta_q m_t|_{L^2} + |\Delta_q \nabla m|_{L^2} \right) + \frac{\gamma - 1}{2} \left( |\Delta_q \nabla u|_{L^2} + |\Delta_q \Delta u|_{L^2} \right) + \frac{\gamma - 1}{2} \left( |\nabla \Delta u|_{L^2} + |\nabla \Delta u|_{L^2} \right) + \frac{\gamma - 1}{2} \left( |\Delta_q \nabla m|_{L^2} + |\Delta_q \nabla \nabla u|_{L^2} \right) \right) + \frac{\gamma - 1}{2} \left( |\Delta_q \nabla m|_{L^2} + |\Delta_q \nabla \nabla u|_{L^2} \right) + \frac{\gamma - 1}{2} \left( |\nabla \Delta u|_{L^2} + |\nabla \Delta u|_{L^2} \right) + \frac{\gamma - 1}{2} \left( |\Delta_q \nabla m|_{L^2} + |\Delta_q \nabla \nabla u|_{L^2} \right)
\]

where \(\Delta_q m_t = \Delta_q m_t + \Delta_q \nabla m\) and \(\Delta_q \nabla u = \Delta_q \nabla u + \Delta_q \Delta u\).

Proof. By differentiating the first two equations of Eq. (4.1) with respect to variable \(t\) once, integrating them over \(\mathbb{R}^N\) after multiplying \(\Delta_q m_t, \Delta_q \nabla u\) respectively, similar to the derivation of (4.16), through tedious but straightforward calculations, we can obtain (5.1).

In addition, we give some auxiliary estimates, which are divided into high- and low- frequency cases.
Proof. (1) Using the first equation of Eq.(3.1), we have
\[ m_t = -\left(\tilde{\psi} \text{div} u + u \cdot \nabla m + \frac{\gamma - 1}{2} m \text{div} u\right). \tag{5.8} \]

By applying the operator $$\Delta_q$$ ($$q \geq -1$$) to (5.8), integrating it over $$\mathbb{R}^N$$ after multiplying $$\Delta_q m_t$$, we can arrive at (5.2) with the aid of Hölder’s inequality.

(2) Using the second equation of Eq.(3.1), we get
\[ \tilde{\psi} \nabla m = -\left( u_t + \frac{u}{\tau} + u \cdot \nabla u + \frac{\gamma - 1}{2} m \nabla m - e\right). \tag{5.9} \]

By Lemma 4.1, we have $$\text{div} = h(m)$$. Integration by parts gives
\[
\int \Delta_q e \cdot \nabla \Delta_q m = -\int \Delta_q \text{div} \Delta_q m = -\int \Delta_q h(m) \Delta_q m
= -\int \Delta_q (h(m) - h(0)) \Delta_q m
= -(A\gamma)^{-\frac{1}{2}} \bar{n} \frac{-\bar{m}}{2} \| \Delta_q m \|_{L^2}^2 - \int \Delta_q (\tilde{h}(m) m) \Delta_q m,
\]

where $$\tilde{h}(m) = \int_0^1 h'(\varsigma m) d\varsigma - (A\gamma)^{-\frac{1}{2}} \bar{n} \frac{-\bar{m}}{2}$$ is a smooth function on $$\{m | \frac{-\bar{m}}{2} \varsigma m + \tilde{\psi} > 0, \varsigma \in [0, 1]\}$$ satisfying $$\tilde{h}(0) = 0$$. By applying the operator $$\Delta_q$$ to (5.9), integrating it over $$\mathbb{R}^N$$ after multiplying $$\Delta_q \nabla m$$, we can obtain
\[
\tilde{\psi} \| \Delta_q \nabla m \|_{L^2}^2 + (A\gamma)^{-\frac{1}{2}} \bar{n} \frac{-\bar{m}}{2} \| \Delta_q m \|_{L^2}^2
\leq \left( \frac{1}{\gamma} \| \Delta_q u \|_{L^2} + \| \Delta_q u_t \|_{L^2} + \| u \|_{L^\infty} \| \Delta_q \nabla u \|_{L^2} + \| [u, \Delta_q] \cdot \nabla u \|_{L^2} \right) \| \nabla \Delta_q m \|_{L^2} + \| \Delta_q (\tilde{h}(m) m) \|_{L^2} \| \Delta_q m \|_{L^2}. \tag{5.10} \]

By Lemma 2.1, we know
\[ \| \Delta_q \nabla m \|_{L^2} \approx 2^q \| \Delta_q m \|_{L^2} \quad (q \geq 0), \]
so (5.3) follows. For the low frequency ($$q = -1$$), by (5.10), we get (5.4) immediately.

(3) By applying the operator $$\Delta_q$$ to both sides of $$\text{div} = h(m)$$ ($$q \geq 0$$), integrating it over $$\mathbb{R}^N$$ after multiplying $$\Delta_q \text{div}$$, we reach (5.5) in virtue of Hölder’s inequality. For $$q = -1$$, using the second equation of Eq.(3.1), we can obtain (5.6).

(4) From the $$L^2$$- boundedness of Riesz transform, we can derive (5.7) directly from the last equation of Eq.(3.1).

The crucial a-priori estimate is comprised in the following proposition.

**Proposition 5.2.** There exist three positive constants $$\delta_1, C_1$$ and $$\mu_1$$ depending only on $$A, \gamma, \bar{n}$$ and $$\tau$$ such that for any $$T > 0$$, if
\[
\sup_{0 \leq t \leq T} \left( \| U(\cdot, t) \|_{B_{2,1}^\gamma} + \| U_t(\cdot, t) \|_{B_{2,1}^\gamma} \right) \leq \delta_1, \tag{5.11} \]
then
\[ \| U(\cdot, t) \|_{B_{2,1}^\gamma} + \| U_t(\cdot, t) \|_{B_{2,1}^{\gamma-1}} \leq C_1 \| U(\cdot, 0) \|_{B_{2,1}^\gamma} \exp(-\mu_1 t), \quad t \in [0, T]. \]
Proof. From the a-priori assumption (5.11), we have

\[
\sup_{0 \leq t \leq T} \left( \|U(\cdot, t)\|_{W^{1,\infty}} + \|U_t(\cdot, t)\|_{L^\infty} \right) \leq C\delta_1. \tag{5.12}
\]

To ensure the smoothness of functions \(h(m), \mathcal{H}(m)\) and \(\hat{h}(m)\), we may choose

\[
0 < \delta_1 \leq \frac{\tilde{\psi}}{(\gamma - 1)C},
\]

then

\[
\frac{\gamma - 1}{2} m(t, x) + \tilde{\psi} \geq \frac{\tilde{\psi}}{2} > 0, \quad (t, x) \in [0, T] \times \mathbb{R}^N
\]

and

\[
\frac{\gamma - 1}{2} \varsigma m(t, x) + \tilde{\psi} \geq \frac{\tilde{\psi}}{2} > 0, \quad \varsigma \in [0, 1], \quad (t, x) \in [0, T] \times \mathbb{R}^N.
\]

From (4.2)-(4.4), we set

\[
I_{1,q} := \|\nabla u\|_{L^\infty} (\|\Delta_q m\|_{L^2}^2 + \|\Delta_q u\|_{L^2}^2) + 2\|\Delta_q m\|_{L^2} \|\nabla m\|_{L^2} \|\Delta_q m\|_{L^2} + 2\|\Delta_q u\|_{L^2} \|\nabla m\|_{L^2} \|\Delta_q u\|_{L^2} + (\gamma - 1)\|\nabla m\|_{L^\infty} \|\Delta_q m\|_{L^2} \|\Delta_q u\|_{L^2} + (\gamma - 1)\|\Delta_q \text{div} u\|_{L^2} \|\Delta_q m\|_{L^2} + \frac{2}{\bar{n}} \|\Delta_q (h(m) u)\|_{L^2} \|\Delta_q e\|_{L^2}. \tag{5.13}
\]

From (5.12), (5.13), and (5.14), we have

\[
I_{2,q} := \|\nabla u\|_{L^\infty} (\|\Delta_q m\|_{L^2}^2 + \|\Delta_q u\|_{L^2}^2) + \beta_3 \|\Delta_q \nabla \delta q\|_{L^2} + \frac{1}{\bar{n}} \|\Delta_q e\|_{L^2}
\]

where \(C > 0\) is a harmless constant depending only on \(A, \gamma, \tau\) and \(\bar{n}\).

Proof of Lemma 5.3. Combining (4.2)-(4.4), Lemma 5.1 and 5.2, we have \((q \geq 0)\)

\[
\frac{d}{dt} \left\{ 2^{q} \|\Delta_q m\|_{L^2}^2 + \|\Delta_q u\|_{L^2}^2 + \frac{1}{\bar{n}} \|\Delta_q e\|_{L^2} \right\} + \beta_3 \|\Delta_q \nabla \delta q\|_{L^2} + \beta_3 \|\Delta_q e\|_{L^2}
\]

where \(C > 0\) is a harmless constant depending only on \(A, \gamma, \tau\) and \(\bar{n}\).
respectively. We introduce them in order to eliminate quadratic terms in the right-hand side of (5.15).

where these positive constants $\beta_1, \beta_2, \beta_3$ and $\beta_4$ satisfy

$$\beta_1 = \min \left\{ \frac{\bar{\beta}_\gamma}{2C^2}, \frac{\bar{\beta}_\psi}{C^2} \right\}, \quad \beta_2 = \frac{A\gamma \bar{\psi}}{C^2 n^{1-2\gamma}}, \quad \beta_3 = \frac{1}{\tau C^2 \bar{\psi}^2}, \quad \beta_4 = \frac{1}{n C^2},$$

respectively. We introduce them in order to eliminate quadratic terms in the right-hand side of (5.15). There are no quadratic terms in $I_{1,q}$ and $I_{2,q}$. By Young’s inequality, the first quadratic term can be estimated in this way:

$$\beta_1 \bar{C} \frac{1}{\tau} 2^q \| \Delta_q m \|_{L^2} \| \Delta_q u \|_{L^2} \leq \sqrt{\beta_1 \bar{C} \frac{1}{\tau} 2^q \| \Delta_q u \|_{L^2} \cdot 2^q \sqrt{\bar{\beta}_1 \bar{\psi}} \| \Delta_q m \|_{L^2}}$$

$$\leq \frac{C^2 \bar{C} \frac{1}{\tau} 2^q \| \Delta_q u \|_{L^2}^2 + \frac{1}{4} \bar{\beta}_1 \bar{\psi} 2^{2q} \| \Delta_q m \|_{L^2}^2}{2^q \| \Delta_q u \|_{L^2}^2 + \frac{1}{4} \bar{\beta}_1 \bar{\psi} 2^{2q} \| \Delta_q m \|_{L^2}^2}.$$
\[ + \frac{\beta_3}{2} \| \Delta_\gamma m_e \| L^2_t^2 + \frac{1}{\bar{\tau}} \| \Delta_\gamma u \| L^2_t^2 + \frac{\beta_4}{2n} \| \Delta_\gamma e_i \| L^2_t^2 \]
\[ \leq 2^{2q} I_{1,q} + I_{2,q} + \beta_1 \tilde{C} \tilde{q} \left( \| u \| L^{\infty} \| \Delta_\gamma \nabla u \| L^2_t + \frac{\gamma - 1}{2} \| m \| L^\infty \| \Delta_\gamma \nabla m \| L^2_t \right) \\
+ \| [u, \Delta_\gamma] \| L^2_t + \frac{\gamma - 1}{2} \| m, \Delta_\gamma \| L^2_t + \beta_3 \left( \| u \| L^{\infty} \| \Delta_\gamma \nabla m \| L^2_t + \frac{\gamma - 1}{2} \| m \| L^\infty \right) \\
\times \| \Delta_\gamma \text{div} u \| L^2_t + \| [u, \Delta_\gamma] \| L^2_t + \frac{\gamma - 1}{2} \| m, \Delta_\gamma \| L^2_t \| \Delta_\gamma m_e \| L^2_t \right) \\
+ \frac{\beta_4 \tilde{C}}{n} \| \Delta_\gamma h(m) \| L^2_t \| \Delta_\gamma e_i \| L^2_t. \] (5.16)

Dividing (5.16) by
\[ 2^{2q} \left( \| \Delta_\gamma m_e \| L^2_t^2 + \| \Delta_\gamma u \| L^2_t^2 + \frac{1}{n} \| \Delta_\gamma e_i \| L^2_t^2 \right) + \left( \| \Delta_\gamma m_e \| L^2_t^2 + \| \Delta_\gamma u \| L^2_t^2 + \frac{1}{n} \| \Delta_\gamma e_i \| L^2_t^2 \right)^{\frac{q}{2}} \]
and multiplying (5.16) by the factor $2^{q(\sigma - 1)}$, we get (5.14) immediately with the help of Lemma 4.2, which completes the proof of Lemma 5.3. \[ \square \]

For the case of low frequency ($q = -1$), we also have the following a-priori estimate in a similar way.

**Lemma 5.4.** ($q = -1$) There exists a positive constant $\mu_4$ depending only on $A, \gamma, \tau$ and $\bar{n}$ such that the following estimate holds:

\[ \frac{1}{2^{q-1}} \frac{d}{dt} \left\{ \frac{1}{2} \| \Delta_{-1} m \| L^2_t^2 + \| \Delta_{-1} u \| L^2_t^2 + \frac{1}{n} \| \Delta_{-1} e_i \| L^2_t^2 \right\} + \left( \| \Delta_{-1} m_e \| L^2_t^2 + \| \Delta_{-1} u \| L^2_t^2 + \frac{1}{n} \| \Delta_{-1} e_i \| L^2_t^2 \right)^{\frac{q}{2}} \]
\[ \leq C \left\{ \| U \| W^{1, \infty} + \| U \| L^\infty \right\} \left( \frac{1}{2} \| \Delta_{-1} U \| L^2_t + \| \Delta_{-1} U \| L^2_t \right) + c_{-1} \| U \| B_{2,1} \]
\[ + \| U \| B_{2,1} \| U \| B_{2,1} + \frac{1}{2} \| \Delta_{-1} (h(m) u) \| L^2_t + \frac{1}{2^{q-1}} \| \Delta_{-1} (h(m) u) \| L^2_t \]
\[ + \| \Delta_{-1} (m_e u) \| L^2_t + \| \Delta_{-1} (h(m) u) \| L^2_t + \| \Delta_{-1} (h(m) u) \| L^2_t \right\}, \] (5.17)

where $C > 0$ is a harmless constant depending only on $A, \gamma, \tau$ and $\bar{n}$.

Summing (5.14) on $q \in \mathbb{N} \cup \{0\}$ and adding (5.17) together, according to Proposition 2.3 and a-priori assumption (5.11)-(5.12), we get the following differential inequality:

\[ \frac{d}{dt} Q + \mu_4 \left( \| U \| B_{2,1} + \| U \| B_{2,1} \right) \leq C \delta_1 \left( \| U \| B_{2,1} + \| U \| B_{2,1} \right), \] (5.18)

where

\[ Q := \sum_{q \geq -1} 2^{q(\sigma - 1)} \left\{ 2^{2q} \left( \| \Delta_\gamma m \| L^2_t^2 + \| \Delta_\gamma u \| L^2_t^2 + \frac{1}{n} \| \Delta_\gamma e_i \| L^2_t^2 \right) \right. \]
\[ \left. + \left( \| \Delta_\gamma m_e \| L^2_t^2 + \| \Delta_\gamma u \| L^2_t^2 + \frac{1}{n} \| \Delta_\gamma e_i \| L^2_t^2 \right)^{\frac{q}{2}} \right\} \]
and the constant $\mu_4$ depends only on $A, \gamma, \tau$ and $\bar{n}$. Furthermore, it is easy to show that $Q$ satisfies

\[ C_3 \left( \| U \| B_{2,1} + \| U \| B_{2,1} \right) \leq Q \leq C_4 \left( \| U \| B_{2,1} + \| U \| B_{2,1} \right) \]

for two positive constants $C_3, C_4$. Choosing $\delta_1 = \min \left\{ \frac{\mu_4}{2C_1}, \frac{\mu_4}{(\gamma - 1)c} \right\}$, we complete the proof of Proposition 5.2 with $\mu_1 = \frac{\mu_4}{c}$. \[ \square \]
Proof of Proposition 5.1. From the assumption
\[ \|U(\cdot, 0)\|_{B^2_t} \leq \delta_2, \]
by Proposition 4.1 (local existence), we can determine a time \( T_1 > 0 \) \((T_1 < T_0)\) such that
\[ \|U(\cdot, t)\|_{B^2_t} + \|U(t, t)\|_{B^2_t} \leq C_2 \|U(\cdot, t)\|_{B^2_t} \leq 2C_2 \delta_2 \]
for all \( t \in [0, T_1] \).

Claim: One can choose a positive constant \( \delta_2 \) satisfying \( \delta_2 < \min\{\frac{\delta_1}{2C_2}, \frac{\delta_1}{C_1}\} \) to ensure
\[ \|U(\cdot, t)\|_{B^2_t} + \|U(t, t)\|_{B^2_t} \leq \delta_1, \quad \text{for all } t \in [0, T_0]. \]

Otherwise, we may assume that there exists a time \( T_2 \) \((T_1 < T_2 \leq T_0)\) such that (5.20) is satisfied for all \( t \in [0, T_2) \) and
\[ \|U(\cdot, T_2)\|_{B^2_t} + \|U(t, T_2)\|_{B^2_t} = \delta_1, \]
because (5.20) is satisfied as \( t \in [0, T_1] \) for such a choice of \( \delta_2 \). By Proposition 5.2, for all \( t \in [0, T^k] \) \((T^k \to T_2^-)\),
\[ \|U(\cdot, t)\|_{B^2_t} + \|U(t, t)\|_{B^2_t} \leq C_1 \|U(\cdot, 0)\|_{B^2_t} \exp(-\mu t) \]
In particular,
\[ \|U(\cdot, T^k)\|_{B^2_t} + \|U(t, T^k)\|_{B^2_t} \leq C_1 \|U(\cdot, 0)\|_{B^2_t} \exp(-\mu T^k) \]
\[ \leq C_1 \delta_2 \exp(-\mu T^k) < C_1 \delta_2. \]
By the continuity on \( t \in [0, T_0] \), we get
\[ \|U(\cdot, T_2)\|_{B^2_t} + \|U(t, T_2)\|_{B^2_t} \leq C_1 \delta_2 < \delta_1, \]
which contradicts (5.21). So, (5.20) holds. From Proposition 4.1 and 5.2, using the boot-strap argument, we can prove Proposition 5.1. \( \square \)

From the imbedding property in Besov space \( B^2_t \), \((m, u, e) \in C^1([0, \infty) \times \mathbb{R}^N)\) solves (3.1)-(3.2). The choice of \( \delta_1 \) is sufficient to ensure \( m - \frac{\text{div} u}{2} + \psi > 0 \). According to Remark 3.1, we know \((n, u, e) \in C^1([0, \infty) \times \mathbb{R}^N)\) is a solution of (1.1)-(1.3) with \( n > 0 \). Furthermore, we may attain Theorem 1.2.

Finally, we show the exponential decay of the vorticity.

Proof of Theorem 1.3. When \( N = 2 \) and 3, the curl of the velocity equation in Eq.(1.1) gives
\[ \partial_t \omega + \frac{1}{r} \omega + u \cdot \nabla \omega - \omega \cdot \nabla u = 0 \quad \text{(in particular, } \omega \cdot \nabla u = 0 \text{ when } N = 2). \]
Then, we may get
\[ \frac{1}{2} \frac{d}{dt} \|\Delta \omega\|_{L^2}^2 + \frac{1}{r} \|\Delta \omega\|_{L^2}^2 \leq C_5 (\|\nabla u\|_{L^\infty} \|\Delta \omega\|_{L^2} + \|\omega\|_{L^\infty} \|\Delta \omega\|_{L^2} + c_q \|\nabla u\|_{B^{-1}_2} \|\omega\|_{B^{-1}_2}^2) \|\Delta \omega\|_{L^2}. \] (5.22)
Dividing (5.22) by \( \|\Delta \omega\|_{L^2} \) and summing it on \( q \geq -1 \) \((q \in \mathbb{Z})\) after multiplying the factor \( 2^q \sigma^{-1} \), from Theorem 1.2, we have
\[ \frac{1}{2} \frac{d}{dt} \|\omega(\cdot, t)\|_{B^{-1}_2}^2 + \frac{1}{r} \|\omega(\cdot, t)\|_{B^{-1}_2}^2 \leq C_5 \|u(\cdot, t)\|_{B^{-1}_2} \|\omega(\cdot, t)\|_{B^{-1}_2}^2 \]
\[ \leq C_5 C_0 (\|u(\cdot, t)\|_{B^{-1}_2} \|\omega(\cdot, t)\|_{B^{-1}_2} \|u(\cdot, t)\|_{B^{-1}_2}^2 \]
\[ \leq C_5 C_0 \min \left\{ \frac{\delta_0}{2}, \frac{1}{2C_5 C_0 r} \right\} \|\omega(\cdot, t)\|_{B^{-1}_2}^2 \]
\[ \leq \frac{1}{2r} \|\omega(\cdot, t)\|_{B^{-1}_2}^2. \] (5.23)
Therefore, we obtain the exponential decay of \( \|\omega(\cdot, t)\|_{B^{-1}_2} \). \( \square \)
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