Abstract
We review the general systematic formalism describing dynamics of bulk and fluctuations for an arbitrary relativistic hydrodynamic flow carrying baryon charge. Feedback of fluctuations renormalizes hydrodynamic variables including transport coefficients and introduces nonlocal and non-instantaneous terms in constitutive equations. Assembled by the confluent connections and correlation functions, we derive a set of cutoff-independent deterministic equations suitable for numerical implementation. Focusing on the critical modes we show how this formalism matches existing Hydro+ description of fluctuations near the QCD critical point and nontrivially extends it inside and outside of the critical region.

Keywords: hydrodynamic fluctuations, renormalization, long-time tails, QCD critical point, Hydro++

1. Introduction
The quark-gluon plasma created in the heavy-ion collision experiment has been manifested as a nearly perfect fluid, whose dynamic evolution is well described by hydrodynamics as the system is large enough to be treated hydrodynamically. On the other hand, the system is still small enough such that the hydrodynamic fluctuations are not negligible and indeed, it is in an essential ingredient for establishing a theoretical framework to interpret the experimental data from the RHIC Beam Energy Scan (BES) Program aiming to search for the QCD critical point. A systematic treatment for hydrodynamic fluctuations can be dated back to 1950s by Landau and Lifshitz, based on the stochastic Langevin dynamics that requires sampling with the “infinite noise” and thus quite challenging in numerical implementation. In 1970s, Andreev developed a complimentary description [1], again only for nonrelativistic fluid, by performing a renormalization procedure such that all divergences coming from the noises are absorbed into the bare hydrodynamic variables, and the resulting deterministic equations are cutoff independent and therefore simulation friendly. In recent years, this deterministic (also referred to as hydro-kinetic) approach was extended to a relativistic fluid that is boost invariant [2, 3, 4], and was generalized to a neutral fluid with arbitrary background [5], where a general systematic formalism describing dynamics of fluctuations was presented. However, it is not adequate for applying hydrodynamics to the critical region, since the charges play an important role due to critical slowing down. Extending our previous work [5] to charged fluid and applying it to the region near the critical point is the major goal of the work formulated in Ref. [6], the main ideas and key results of which are briefly reviewed in this proceeding.
2. General Formalism

The theory of fluctuating hydrodynamics relies on the separation of scales. In a strongly coupled system such as quark-gluon plasma, the thermal length $1/T \sim \ell_{\text{mic}} \sim 1$ fm is microscopic. Hydrodynamics, however, describes coarse-grained variables defined at the macroscopic scale $b \sim 1/\Lambda$ that is the size of hydrodynamic cells set by the cutoff $\Lambda$. The resulting stochastic variables denoted by the breve accent (e.g., energy density $\tilde{\epsilon}$, charge density $\tilde{n}$ and fluid velocity $\tilde{u}$), could be used to describe the evolution of system at the typical hydrodynamic gradient scale $L \sim 1/k$, around the size of the nucleus (~10 fm) in heavy-ion collisions. The fluctuation scale $y \sim 1/q$ is assumed to satisfy the following hierarchy:

$$\ell_{\text{mic}} \ll b < y \ll L \quad \text{or} \quad T \gg \Lambda > q \gg k.$$  

Nonetheless, there exists a typical fluctuation scale $y$, (conjugate to $q$, $\sim \sqrt{c/\gamma}$) characterizing the transition between equilibrium and non-equilibrium wavenumber modes, resulted from the competition of the evolution rate of background, $\omega \sim c/k$, and the relaxation rate of fluctuations, $\Gamma = \gamma q^2$, where $c$ is the speed of sound and $\gamma$ the relaxation coefficient. Since it is those modes $q \lesssim y$, that play a nontrivial role in the feedback of fluctuations, we would consider $yq$ in the same order as $k$.

The hydrodynamic variables fluctuate between each ensemble member (i.e., event by event in heavy-ion collisions), driven by the noises. Written in terms of the energy-momentum tensor and charge current

$$\tilde{T}^\mu = T^\mu(\tilde{\epsilon}, \tilde{n}, \tilde{u}) + \tilde{S}^\mu, \quad \tilde{J}^\mu = J^\mu(\tilde{\epsilon}, \tilde{n}, \tilde{u}) + \tilde{P}^\mu,$$

$$T^\mu(\epsilon, n, u) = \epsilon u^\mu u^\nu + p \Delta^\mu - 2\eta (\theta^\mu - \frac{1}{3} \Delta^\mu \theta) - \zeta \Delta^\mu \theta, \quad J^\mu(\epsilon, n, u) = nu^\mu - \lambda \partial^\mu \alpha,$$  

where the constitutive relations are well known, they obey the conservation equations

$$\partial_\mu T^\mu = 0, \quad \partial_\mu P^\mu = 0.$$  

Here $\tilde{S}^\mu$ and $\tilde{P}^\mu$ are the random noises whose amplitudes are set by the fluctuation-dissipation theorem and are assumed to be statistically independent and local, i.e.,

$$\langle \tilde{S}^{\mu}(x) \rangle = \langle \tilde{J}^{\mu}(x) \rangle = \langle \tilde{S}^{\mu}(x) \tilde{J}^{\mu}(x') \rangle = 0, \quad \langle \tilde{P}^{\mu}(x) \tilde{P}^{\mu}(x') \rangle \sim \delta^{(4)}(x - x'), \quad \langle \tilde{S}^{\mu}(x) \tilde{S}^{\lambda}(x') \rangle \sim \delta^{(4)}(x - x').$$  

In the stochastic approach of fluctuating hydrodynamics, one can not avoid this ‘infinite noise’ problem due to the Dirac delta functions which make numerical simulation challenging. This problem could be resolved if one switches the gear to the deterministic approach, where the ingredients one deals with are the ensemble averages of stochastic quantities, e.g., $s/n \equiv m \equiv \langle \tilde{m} \rangle$, $p \equiv \langle \tilde{p} \rangle$, $u \equiv \langle \tilde{u} \rangle$ and two-point correlation functions $G_{\lambda\lambda}(x, y) \equiv \langle \phi_\lambda(x) \phi_\lambda(x') \rangle$. Here we choose another set of variables suitable for renormalization procedure and introduce $\phi_{m} \equiv (T_{\lambda\rho\sigma}, \delta \rho/c, \omega \delta \omega_{\mu})$ as the properly rescaled fluctuating variables. $x^\pm = x \pm y/2$ are two space-time variables with the midpoint $x$ and fluctuation separation scale $y$. One can then expand the stochastic quantities around their ensemble averages to higher orders in fluctuations $\phi_{m}$. Truncating the expansion to second order, the averaged energy-momentum tensor and charge current can be separated by the bare part and the fluctuation part, i.e.,

$$\langle \tilde{T}^{\mu}(x) \rangle = T^{\mu}(\epsilon, n, u) + L \left[ G^{\text{mm}}, G^{\text{pp}}, G^{\text{pp}}, G^{\text{mm}}, G^{\text{pp}} \right], \quad \langle \tilde{J}^{\mu}(x) \rangle = J^{\mu}(\epsilon, n, u) + L \left[ G^{\text{mm}}, G^{\text{pp}}, G^{\text{mm}}, G^{\text{pp}} \right].$$

The bare part satisfies the evolution equation for one-point functions that is well known in non-fluctuating hydrodynamics, while the fluctuation part contains the feedback from the two-point functions, read off generically as

$$G_{\lambda\lambda}(x) \equiv G_{\lambda\lambda}(x, y = 0) = \int \frac{d^2 q}{(2\pi)^2} W_{\lambda\lambda}(x, q).$$

1In this proceeding we use consistent notations introduced by Ref. [6] and will not clarify each notation’s meaning if not necessary.

2In this proceeding we introduce $L[\ldots]$ to represent a linear combination of quantities in the square bracket.
i.e., the phase space integration on $q$ of $W_{AB}(x,q)$ being the Wigner transform of $G_{AB}(x,y)$ defined later more comprehensively. The two-point functions will be driven out of thermodynamic equilibrium in the presence of system gradients. It is therefore convenient to decompose the two-point functions as

$$G_{AB}(x) = G_{AB}^{(eq)}(x) + G_{AB}^{(neq)}(x), \quad G_{AB}^{(neq)}(x) = G_{AB}^{(1)}(x) + \bar{G}_{AB}(x),$$

(7)

where $G^{(eq)} \sim \Lambda^3$ coming from the “infinite noise” gives rise to the divergence that can be regularized by the integration cutoff $\Lambda$. Such divergence can be absorbed into the bare ideal constitutive relations by renormalizing relevant hydrodynamic variables. The non-equilibrium part $G^{(neq)}$ consists of two contributions: $G^{(1)} \sim \partial(u, a)\Lambda$ that renormalizes the transport coefficients including viscosities and conductivity being consistent with the first law of thermodynamics, and the remaining non-analytic part $\bar{G}_{AB} \sim \zeta_1 \sim \kappa^{1/2}$ referred to as the long-time tails. Denoting the renormalized quantities by “$\bar{R}$”, Eq. (5) now reads

$$\langle \bar{T}^{\mu\nu}(x) \rangle = T^{\mu\nu}(\epsilon_R, n_R, u_R) + \Lambda \left[ \bar{G}^{\mu\nu} \right], \quad \langle \bar{J}^\mu(x) \rangle = J^\mu(\epsilon_R, n_R, u_R) + \Lambda \left[ \bar{G}^{\mu\nu}, \bar{G}^{\nu\rho} \right],$$

(8)

and the resulting cutoff-independent equations suitable for numerical simulation are (c.f. Eq. (3))

$$\partial_\mu(\bar{T}^{\mu\nu}(x)) = 0, \quad \partial_\mu(\bar{J}^\mu(x)) = 0.$$

(9)

To close the above equations what in demand is the evolution equations describing the dynamics of correlation function $G_{AB}(x,y)$ (or more precisely, the Wigner function $W_{AB}(x,y)$). One may naively expect that we could achieve this by linearize Eq. (3) for $\phi_A$ and substitute which into $u \cdot \partial G(x,y)$. However, dealing with relativity is a bit tricky. In our choice, the concept of “time” is no longer global but frame-dependent, since $G_{AB}(x,y)$ is defined as equal-time correlator subjected to the condition $u \cdot y = 0$, where $y$ represents a three dimensional equal-time surface that is perpendicular to the local velocity $u$ at each space-time point. Thus when we take the derivative of a quantity with respect to $x$, what shall be fixed is $y_a = \epsilon_a(x) y_a$ in each local rest frame, expanded by the triad $\epsilon_a^\mu$. In addition, we are only interested in the “internal” changes unrelated to the boost in the relativistic evolution, thus we define the \textit{confluent} derivatives (connections) and correlation functions that are adjusted by the fluid and denoted by $^\sim$ (e.g., $\partial \to \vec{\nabla}$, $G \to \bar{G}$) [5], assembled by which the evolution (matrix) equation for the two-point confluent Winger functions turns out to be

$$u \cdot \vec{\nabla} W(x,q) = - \left[ i L^{(q)}(x), W \right] + O(k, \gamma q^2) \quad \text{with} \quad W_{AB}(x,q) \equiv \int d^4y \delta(u(x) \cdot y) e^{\sim q y} G_{AB}(x,y).$$

(10)

If we diagonalize this equation in the eigenbasis of the dominant matrix $\Xi^{(q)} \sim c_i q$ and average out the fast (propagating) modes, it will be significantly simplified. In terms of the rescaled Winger function $N_{AB} = \psi_i \psi_i^* W_{AB} \psi_i^2$ where $\psi$ is the eigenbasis transformation matrix, we obtain the decoupled equation for sound modes which is precisely the kinetic equation for phonons:

$$L_L[N_{++}] = 0,$$

(11)

It describes the phonon propagation on top of an arbitrary background, in the speed of sound and driven by the relativistic inertia and Coriolis forces due to acceleration and vorticity of the flow, as well as the “Hubble” force leading to the phonon red shift due to the Hubble-like expansion [5]. The remaining modes mix entropy and velocity fluctuations and schematically satisfy

$$L[N_{mm}] = -2 \Gamma_\alpha \left( N_{mm} - \frac{\mu}{n} \right) + L[N_{m(i)}],$$

$$L[N_{m(i)}] = - (\Gamma_q + \Gamma_\alpha) N_{m(i)} + L[N_{mm}, N_{m(i)}, N_{(i)j}],$$

$$L[N_{(i)j}] = -2 \Gamma_q \left( N_{(i)j} - \frac{T_{ij}}{n} \right) + L[N_{m(i)}, N_{(i)j}].$$

(12a)

(12b)

(12c)

where $L$ is a Liouville-like operator, $\Gamma_\alpha = \gamma_{LL} q^2$, $\Gamma_q = \gamma_{QQ} q^2$, and the subscripts of $N$ denote the $(A, B)$ basis detailed in Ref. [6]. Eq. (9) together with Eqs. (11) and (12) form a closed set of cutoff-independent, deterministic equations describing dynamics of both bulk and fluctuations.
3. Application to the Critical Point

In this section we outline the application of our general formalism to the region near the QCD critical point. The most important feature of the critical region is that the correlation length $\xi$, which is microscopic away from the critical point, becomes macroscopic, i.e., the equilibrium correlation length diverges in the thermodynamic limit. In heavy-ion collisions, it is limited by the size of the system. In order to maintain our scale hierarchy we assume $\xi \ll L$.

There are two modifications (approximations) we shall make on Eqs. (12). First, the specific heat

$$c_p \rightarrow c_p(q) = \frac{c_p}{1 + (q\xi)^2}$$

(13)

is given by the Lorentzian approximation form and as a consequence the equilibrium correlation function $N^{(eq)} = c_p(q)/n$ is no longer local. Second, the relaxation coefficient for diffusive mode is also $q$-dependent,

$$\gamma_\perp \rightarrow \gamma_\perp(q) = \frac{\kappa_0}{c_p} + \frac{T}{6\pi\eta}K(q\xi)$$

(14)

where $\kappa_0$ is the noncritical heat conductivity and $K(x)$ is given by the Kawasaki approximation. Eqs. (12) with the above two modifications are the key equations we proposed for an extended formalism we called Hydro++. Away from the critical point, in the long-wavelength limit $\omega \sim c_s k \ll \Gamma_1 \sim \Gamma_0$, most wavenumber modes equilibrate rapidly and non-fluctuating hydrodynamics is sufficient for describing the system, yet if $\omega$ is large enough to compete with $\Gamma$, our general framework shall be applied, in particular near the critical point. Specifically, due to critical slowing down, we have $\Gamma_1 \sim 1/\xi^2 \ll \Gamma_0 \sim 1/\xi^2$, i.e., different modes may relax with parametrically different rate. Hydro+ applies to the region $\Gamma_1 \ll \omega \ll \Gamma_0$ where hydrodynamics breaks down, and the slowest mode $N_{mm}$ satisfying Eq. (12a) now has to be taken into account [7]. Hydro++, however, nontrivially extends the applicability of Hydro+ further to $\Gamma_1 \ll \Gamma_0 \ll \omega$, i.e., closer to the critical point or towards a smaller system, such that the whole set of equations in Eqs. (12) must be considered. We shall emphasis that Hydro++ is limited by $\omega \ll 1/\xi$, in which case the non-locality at scale $L$ is not negligible. Last but not least, we notice that the bulk viscosity and conductivity are suppressed due to the time-delayed response to gradients. Their frequency-dependent behaviors are also discussed in Ref. [6].

4. Outlook

In this proceeding we reviewed the work presented in Ref. [5, 6], where the state-of-the-art formalism for deterministic fluctuating hydrodynamics, as an integral part of the theoretical framework for BES experiment, is developed. To accomplish this framework, we shall extend this formalism by including higher-point functions that are more sensitive to the critical point, and connect it to the freezeout kinetics and observables. Moreover, it would be interesting to consider its application to the first-order phase transition, providing a more complete picture of fireball evolution in high baryon density region. We defer these to future work.

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