On $\lambda'$-sets

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Abstract

A set $X \subseteq 2^\omega$ is a $\lambda'$-set iff for every countable set $Y \subseteq 2^\omega$ there exists a $G_\delta$ set $G$ such that $(X \cup Y) \cap G = Y$. In this paper we prove two forcing results about $\lambda'$-sets. First we show that it is consistent that every $\lambda'$-set is a $\gamma$-set. Secondly we show that is independent whether or not every $(\dagger)$-$\lambda'$-set is a $\lambda'$-set.

1 $\lambda'$-sets and $\gamma$-sets

A set $X \subseteq 2^\omega$ is a $\lambda'$-set iff for all countable $A \subseteq 2^\omega$ there exists a $G_\delta$ set $G$ such that

$$(X \cup A) \cap G = A$$

An $\omega$-cover of $X$ is a countable set of open sets such that every finite subset of $X$ is contained an element of the cover. A $\gamma$-cover of $X$ is a countable sequence of open subsets of $X$ such that every element of $X$ is in all but countably many elements of the sequence.

Define. $X$ is a $\gamma$-set iff any $\omega$-cover of $X$ contains a $\gamma$-cover of $X$.

In this section we answer a question of Gary Gruenhage who asked if there is always a $\lambda'$-set which is not a $\gamma$-set. We answer this in the negative.

It is well known (see Gerlitz and Nagy [4]) that MA($\sigma$-centered) implies that every set of reals of cardinality less than the continuum is a $\gamma$-set. The standard model for MA($\sigma$-centered) (see Kunen and Tall [7]) is obtained as follows:

Suppose that $M$ is a countable standard model of ZFC+CH and we iterate $\sigma$-centered forcings of size $\omega_1$ in $M$ with a finite support iteration of length $\omega_2$. In the final model $M_{\omega_2}$, we have that MA($\sigma$-centered) is true and the continuum is $\omega_2$.

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Theorem 1.1 In the standard model for MA(\(\sigma\)-centered) every \(\lambda'\) set has cardinality \(\leq \omega_1\), and (it follows from MA(\(\sigma\)-centered)) every set of size \(\omega_1\) is a \(\gamma\)-set. Hence, in this model, every \(\lambda'\)-set is a \(\gamma\)-set.

Proof
We will use the following Lemma in our proof.

Lemma 1.2 Suppose that \(P\) is a \(\sigma\)-centered forcing such that
\[|\exists \tau \in 2^\omega|\]
Then there exists a countable set \(A \subseteq 2^\omega\) in the ground model such that for every \(p \in P\) and open set \(U \supseteq A\) coded in the ground model there exists \(q \leq p\) such that \(q|\exists \tau \in U\).

Proof
To prove the Lemma we will use the following Claim.

Claim. Suppose \(\Sigma \subseteq P\) is a centered subset. Then there exists \(x \in 2^\omega\) such that for every \(p \in \Sigma\) and for every \(n < \omega\) there exists \(q \leq p\) such that
\[p|\exists x \mid n = \tau \mid n.\]

pf: Otherwise by the compactness of \(2^\omega\) there exists a finite set
\[\{p_m : m < N\} \subseteq \Sigma\] and \(\{s_m : m < N\} \subseteq 2^{<\omega}\)
such that \(\{[s_m] : m < N\}\) covers \(2^\omega\) and for each \(m < N\) we have that
\[p_m|\exists \tau \notin [s_m].\]
But this is a contradiction since there exists some \(p \in P\) below all of the \(p_m\).
This proves the Claim.

Let \(P = \bigcup_{n<\omega} \Sigma_n\) be a sequence of centered sets. Then for each \(n\) there exists \(x_n \in 2^\omega\) such that for every \(p \in \Sigma_n\) and for every \(m \in \omega\) there exists \(q \leq p\) such that
\[q|\exists x_n \mid m = \tau \mid m.\]
Now let \(A = \{x_n : n < \omega\}\). This proves the Lemma.

QED

Suppose \(X \subseteq 2^\omega\) is a \(\lambda'\)-set in \(M_{\omega_2}\). For each \(\alpha \leq \omega_2\) define
\[X_\alpha = X \cap M_\alpha\]
By a standard Lowenheim-Skolem argument we can find \(\alpha < \omega_2\) such that
1. $X_\alpha \in M_\alpha$ and

2. for every countable $A \subseteq 2^\omega$ which is in $M_\alpha$ there exists a $G_\delta$-set $G$ coded in $M_\alpha$ such that

$$(X_\omega \cup A) \cap G = A$$

We claim that $X = X_\omega = X_\alpha$ and hence has cardinality $\leq \omega_1$. Suppose that $\tau$ is any term for an element of $2^\omega$ in $M_\omega$. Since $\tau$ is added at some latter stage $\beta$ with $\alpha \leq \beta < \omega_2$ and the iteration of $\sigma$-centered forcings of length $< \omega_2$ is $\sigma$-centered, it follows that $\tau$ is added by a $\sigma$-centered forcing over $M_\alpha$. Let $A \subseteq 2^\omega$ be the countable set given by the Lemma. By the Lemma it follows that $\tau$ must be an element of any $G_\delta$ set coded in $M_\alpha$ which contains $A$. Using item (2) above we see that $\tau$ must be in $A$ if it is in $X_\omega$. Therefore $X_\omega \setminus X_\alpha = \emptyset$. QED

Remark. This argument is similar to the proof that there are no $\lambda'$-sets of size $\omega_2$ in Laver’s model, see Miller [10].

Remark. A set of reals $X$ is a $\lambda$-set iff every countable subset of $X$ is a relative $G_\delta$. In ZFC we must always have a $\lambda$-set which is not a $\gamma$-set. To see this let

$$X = \{f_\alpha \in \omega^\omega : \alpha < b\}$$

be well-ordered by eventual dominance and unbounded. Then Rothberger [13] (or see Miller [7]) showed that $X$ is a $\lambda$-set. However $X$ is not a $\gamma$-set as is witnessed by the sequences of $\omega$-covers

$$U_m = \{U_n^m : n \in \omega\} \text{ where } U_n^m = \{f \in \omega^\omega : f(m) < n\}.$$ 

In fact the set $X$ is a $\lambda'$-set with respect to $\omega^\omega$. This follows from the following lemma.

**Lemma 1.3 (Rothberger)** Suppose $Z_\beta = \{f_\alpha : \alpha < \beta\} \subseteq \omega^\omega$ is well-ordered by eventual dominance, and $A \subseteq \omega^\omega$ is countable and for every $g \in A$ there exists $\alpha < \beta$ such that $\exists n g(n) < f_\alpha(n)$. Then there exists a $G_\delta$ set $G$ with

$$G \cap (Z_\beta \cup A) = A$$
Proof
This is proved by induction on $\beta$. and assume the lemma is true for all $\delta < \beta$.
If $\beta$ is a successor ordinal, then the induction is trivial.
Case 1. $\beta$ is a limit ordinal of uncountable cofinality.
Find $\delta_0 < \beta$ so that for each $g \in A \exists \infty n \ g(n) < f_{\delta_0}(n)$. Then by induction there exists a $G_\delta$ set $G$ with
\[ G \cap (Z_{\delta_0} \cup A) = A \]
Let $H = \{g \in \omega : \exists \infty n \ g(n) < f_{\delta_0}(n)\}$ Then $H$ is a $G_\delta$ set containing $A$ and missing $Z_{\beta} \setminus Z_{\delta}$ and so
\[ (G \cap H) \cap (Z_{\beta} \cup A) = A \]
Case 2. $\beta$ is a limit ordinal of countable cofinality.
Let $\beta_n$ be an increasing $\omega$-sequence with limit $\beta$ and let
\[ A_n = \{g \in A : \exists \infty m \ g(m) < f_{\beta_n}(m)\} \]
By inductive assumption there exists $G_\delta$ sets $G_n$ so that
\[ G_n \cap (Z_{\beta_n} \cup A_n) = A_n \]
Define
\[ G_n^* = G_n \cup \{g \in \omega^\omega : \exists \infty m \ f_{\beta_n}(m) \leq g(m)\} \]
Note that $G_n^*$ is a $G_\delta$ set which contains $A$ but still
\[ G_n^* \cap (Z_{\beta_n} \cup A_n) = A_n \]
Define $G = \cap_{n<\omega} G_n^*$. Then $G$ is a $G_\delta$-set with
\[ G \cap (Z_{\beta} \cup A) = A \]
QED

Remark. A Hausdorff gap is an example of a $\lambda'$ set of cardinality $\omega_1$. $\gamma$-sets have strong measure zero and Laver [3] proved that it consistent that every strong measure zero set is countable.

Suppose there exists $X, Y \subseteq 2^\omega$ such that $|X| = |Y|$ and $X$ is a $\lambda'$-set and $Y$ is not a $\gamma$-set. Then there exists $Z$ which is a $\lambda'$-set and not a $\gamma$-set. To see this let $X = \{x_\alpha : \alpha < \kappa\}$ and $Y = \{y_\alpha : \alpha < \kappa\}$. Put
$Z = \{(x_\alpha, y_\alpha) : \alpha < \kappa\}$. The first $\kappa$ for which $\text{MA}(\sigma\text{-centered})$ fails is $p$ (Bell [4]) and $p$ is also the size of the smallest non $\gamma$-set. Hence any model where every $\lambda'$-set is $\gamma$-set and $\mathfrak{c} \leq \omega_2$ must satisfy $\text{MA}(\sigma\text{-centered})$ and $\mathfrak{c} = \omega_2$.

Remark. Gruenhage and Szeptychi [6] were interested in obtaining a set of reals $X \subseteq 2^\omega$ which is $\gamma$-set and not a $\lambda'$-set because of the following two topological games.

Let $X$ be a topological space and $x \in X$.

Game: $G_{\mathcal{O}, \mathcal{P}}(X, x)$: On round $n$ player $\mathcal{O}$ chooses an open neighborhood $U_n$ of $x$ and player $\mathcal{P}$ chooses a point $p_n \in U_n \setminus \{x\}$. Player $\mathcal{O}$ wins iff the sequence $p_n$ converges to $x$.

Game: $G_{\mathcal{O}, \mathcal{P}}^f(X, x)$: The same except we allow player $\mathcal{P}$ to choose a finite set of points $P_n \subseteq U_n \setminus \{x\}$ on his move and $\mathcal{O}$ wins iff $\bigcup_{n<\omega} P_n$ converges to $x$.

It is not hard to check that player $\mathcal{O}$ has a winning strategy in $G_{\mathcal{O}, \mathcal{P}}(X, x)$ iff player $\mathcal{O}$ has a winning strategy in $G_{\mathcal{O}, \mathcal{P}}^f(X, x)$. Also if player $\mathcal{P}$ has a winning strategy in $G_{\mathcal{O}, \mathcal{P}}(X, x)$, then it is a winning strategy in $G_{\mathcal{O}, \mathcal{P}}^f(X, x)$.

Given $X \subseteq 2^\omega$ consider the topology on $2^{<\omega} \cup \infty$ generated by

1. $\{\sigma\}$ for each $\sigma \in 2^{<\omega}$ and
2. $\{\infty\} \cup (2^{<\omega} \cup \{x \upharpoonright n : n < \omega\})$ for each $x \in X$.

Let $X_F$ denote this countable topological space.

Gruenhage [4], Nyikos [13], Sharma [14], and Gruenhage and Szeptycki [6] can be combined to show that:

$X$ is not a $\gamma$-set iff player $\mathcal{P}$ has a winning strategy in $G_{\mathcal{O}, \mathcal{P}}^f(X_F, \infty)$.

If $X$ is a $\lambda'$-set, then $\mathcal{P}$ has no winning strategy in $G_{\mathcal{O}, \mathcal{P}}(X_F, \infty)$.

Hence, if there is a set $X$ which is a $\lambda'$-set and not a $\gamma$-set, then $\mathcal{P}$ has a winning strategy in $G_{\mathcal{O}, \mathcal{P}}^f(X_F, \infty)$ but not in $G_{\mathcal{O}, \mathcal{P}}(X_F, \infty)$.

Dow [2] results imply that in Laver’s model [8]:

$X$ is a $\lambda'$-set iff $\mathcal{P}$ has no winning strategy in $G_{\mathcal{O}, \mathcal{P}}(X_F, \infty)$. 

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But, it also consistent that they are not the same. In Galvin and Miller \cite{3} it is shown that assuming MA(\(\sigma\)-centered) there is a \(\gamma\)-set \(X\) which is concentrated on a countable subset of itself. Hence \(P\) has no winning strategy in \(G^f_{O,P}(X_F, \infty)\) hence none in \(G_{O,P}(X_F, \infty)\), but \(X\) is not a \(\lambda'\)-set.

**Question 1.4** Is it consistent with ZFC that for every \(X \subseteq 2^\omega\) that \(P\) has no winning strategy in \(G_{O,P}(X_F, \infty)\) iff \(P\) has no winning strategy in \(G^f_{O,P}(X_F, \infty)\)?

To better see the connection with \(\gamma\)-sets consider the following game:

**Game:** \(G^\gamma_{F,C}(X)\): Two players \(F\) finite and \(C\) clopen alternate plays as follows. On round \(n\) player \(F\) plays a finite set \(F_n \subseteq X\) and player \(C\) responds with a clopen set \(C_n\) in \(2^\omega\) with \(F_n \subseteq C_n\). Player \(F\) wins iff \(\langle C_n : n < \omega \rangle\) is a \(\gamma\)-cover of \(X\), ie. for all \(x \in X\) for all but finitely many \(n\) we have \(x \in C_n\).

This game is exactly the same as \(G^f_{O,P}(X_F, \infty)\). A neighborhood basis for \(\infty\) in \(X_F\) consists of sets of the form \(2^{<\omega} \setminus \{x \upharpoonright n : x \in F, n < \omega\}\) for \(F \subseteq X\) finite. So we can regard \(O\) as player \(F\) playing a finite subset of \(X\). Instead of \(P\) playing a finite set \(P_n \subseteq 2^{<\omega}\) just regard him as \(C\) playing the clopen set

\[
C_n = 2^\omega \setminus \bigcup \{[s] : s \in P_n\}.
\]

**Theorem 1.5** (Gruenhage, Szeptycki, Nyikos) For \(X \subseteq 2^\omega\) the following are equivalent:

1. \(X\) is not a \(\gamma\)-set
2. \(C\) has a winning strategy in \(G^\gamma_{F,C}(X)\).

**Proof**
Suppose \(X\) is is not a \(\gamma\)-set and let \(U\) be an \(\omega\)-cover with no \(\gamma\)-subcover. Without loss of generality we may assume the elements of \(U\) are clopen. Given any \(F_n\) let \(C\) choose \(C_n \in U\) with \(F_n \subseteq C_n\). Then since \(\langle C_n : n < \omega \rangle\) is not a \(\gamma\)-cover, \(C\) wins.

For the other direction suppose Player \(C\) has a winning strategy \(\tau\) in \(G^\gamma_{F,C}(X)\). Construct \(\langle F_s, C_s : s \in \omega^{<\omega}\rangle\) so that
1. for each $s \in \omega^<\omega$ the set $\mathcal{U}_s = \{C_{sn} : n < \omega\}$ is an $\omega$-cover of $X$ and

2. for each $s \in \omega^<\omega$ and the set $C_s$ is the response of player $C$ using the strategy $\tau$ against the play $F_s[1], F_s[2], \ldots, F_s$.

To do this just let

$$\mathcal{U}_s = \{C : \exists F C = \tau(F_s[1], F_s[2], \ldots, F_s)\}$$

This is countable since there are only countably many clopen sets and by the rules of the game it must be an $\omega$-cover. For each element of $\mathcal{U}_s$ choose a witness $F$.

Suppose for contradiction that $X$ is a $\gamma$-set. It is well known (Gerlits and Nagy [4]) that for a $\gamma$ set $X$ that given a sequence of $\omega$-covers, we may choose one element of each to get a $\gamma$-cover. This is denoted $X \in S_1(\Omega, \Gamma)$. Hence we may choose $C_{sn_s}$ for each $s \in \omega^<\omega$ such that every $x \in X$ is in all but finitely many $C_{sn_s}$. But now just look at the branch

$$m_0, m_1, m_2, \ldots \text{ where } m_0 = n_{i()}, \ldots, m_{k+1} = n_{(m_0, m_1, m_2, \ldots, m_k)}$$

But

$$F_{(m_0)}, C_{(m_0)}, \ldots, F_{(m_0, m_1, \ldots, m_k)}, C_{(m_0, m_1, \ldots, m_k)}, \ldots$$

is a play using the strategy $\tau$ with yields a $\gamma$ cover. This is a contradiction. QED

2. $(\dagger)$-\(\lambda\)'-set

In this section we answer Problem 2.12 from Nowik and Weiss [11] which asks basically whether it is true that every $(\dagger)$-\(\lambda\)'-set is a $\lambda$-set.

Definition. For any $f \in \omega^\omega$

$$G_f = \{a \in [\omega]^{<\omega} \subseteq 2^\omega : \forall n \exists m > n \ a_n < f(n)\} \quad a = \{a_0 < a_1 < \cdots\}$$

Definition. A set $X \subseteq 2^\omega$ is a $(\dagger)$-\(\lambda\)'-set iff for every $f \in \omega^\omega$ we have $X \cap G_f$ is a $\lambda$-set.

**Theorem 2.1** Suppose that the continuum hypothesis is true or even just $b = \delta$. Then there exists a $(\dagger)$-\(\lambda\)'-set which is not a $\lambda$-set.
**Theorem 2.2** In the Cohen real model (Cohen’s original model for not CH) every $(\uparrow)$-$\lambda'$-set is a $\lambda'$-set.

Proof of Theorem 2.1

Assume CH. Let $\{f_\alpha \in \omega^\omega : \alpha < \omega_1\}$ be a scale. That is, for $\alpha < \beta$ we have that $f_\alpha <^* f_\beta$ and for all $g \in \omega^\omega$ there exists $\alpha < \omega_1$ such that $g <^* f_\alpha$. We may also assume that the $f_\alpha$ are strictly increasing. Let $X \subseteq [\omega]^\omega$ be the set of ranges of the elements of the scale. Then for any $g \in \omega^\omega$ we have that $G_g \cap X$ is countable and hence a $\lambda'$-set. On the other hand $X$ is not a $\lambda'$-set because of the countable set $[\omega]<\omega$. If $U \subseteq P(\omega)$ is an open set containing $[\omega]<\omega$, then $P(\omega) \setminus U$ is a compact subset of $[\omega]^\omega$. If we identify $\omega^\omega$ with $[\omega]^\omega$ this means that there exists $f \in \omega^\omega$ such that for all $g \in K$ we have $\forall n \ g(n) < f(n)$. It follows that for all but countably many $\alpha$ we have that the range($f_\alpha$) $\in U$.

The proof using $b = d$ is similar. Start with a scale indexed by $b$ and note that any set $Y \subseteq P(\omega)$ of size less than $b$ is a $\lambda'$-set (this is due to Rothberger, see the proof of Lemma 2.4).

QED

Proof of Theorem 2.2

Assume that $M$ is a countable transitive standard model of ZFC+CH.

For any $\alpha \leq \omega^M_2$ let $\mathbb{P}_\alpha$ be the finite partial functions from $\alpha$ into 2. We claim that for any $G$ a $\mathbb{P}_{\omega_2}$-generic filter over $M$ that in the model $M[G]$ every $(\uparrow)$-$\lambda'$-set is a $\lambda'$-set.

**Lemma 2.3** Suppose $N$ is a countable standard model of ZFC+CH, $\mathbb{P}$ is a countable poset in $N$, and

$N \models X \subseteq \omega^\omega$ is unbounded in $\leq^*$

Then for any $G$ which is $\mathbb{P}$-generic over $N$ we have that

$N[G] \models X$ is unbounded in $\leq^*$

Proof

Let $\{g_\alpha : \alpha < \omega_1^N\}$ be a scale in $N$. Working in $N$ choose $f_\alpha \in X$ so that

$\exists^\infty n \ f_\alpha(n) > g_\alpha(n)$
Note that for every \( g \in \omega^\omega \cap N \) there exists \( \alpha < \omega_1 \) such that
\[
\forall \beta > \alpha \ \exists^\infty n \ f_\beta(n) > g(n).
\]

Suppose for contradiction that for some \( g \in N[G] \cap \omega^\omega \) and all \( \alpha < \omega_1 \) we have that \( g \geq^* f_\alpha \). Then for some \( \Sigma \in [\omega_1]^{\omega_1} \) and \( n < \omega \) we have that
\[
\forall m > n \ \forall \alpha \in \Sigma \ f_\alpha(m) \leq g(m)
\]
Let \( q \in G \) force this fact. Now since \( \mathbb{P} \) is a countable poset, there exists some \( p \in G \) with \( p \leq q \) such that
\[
\Gamma = \{ \alpha < \omega_1 : p \models \alpha \in \hat{\Sigma} \}
\]
is uncountable (and by definability of forcing it is in \( N \)). But note that \( \{ f_\alpha : \alpha \in \Gamma \} \) is unbounded and so for some \( m > n \) the set \( \{ f_\alpha(m) : \alpha \in \Gamma \} \) is unbounded in \( \omega \).

Let \( r \leq p \) decide \( g(m) \), i.e., for some \( k < \omega \) suppose
\[
r \models \hat{g}(m) = k.
\]
Choose \( \alpha \in \Gamma \) such that \( f_\alpha(m) > k \), then \( r \) forces a contradiction and the Lemma is proved.
QED

**Lemma 2.4** Suppose \( N \) is a countable standard model of \( \text{ZFC}+\text{CH} \), \( \mathbb{P} \) is a countable poset in \( N \), and
\[
N \models Y \subseteq 2^\omega \text{ is not a } \lambda' \text{ set}
\]
Then for \( G \mathbb{P} \)-generic over \( N \) we have that
\[
N[G] \models Y \text{ is not a } \lambda' \text{ set}
\]
Proof
Let \( D \subseteq 2^\omega \) be countable in \( N \) and witness that \( Y \) is not a \( \lambda' \)-set, i.e. there is no \( G_\delta \) set \( \bigcap_n U_n \) coded in \( N \) with
\[
\bigcap_n U_n \cap (Y \cup D) = D
\]
Working in $N$ let $D = \{ x_n : n < \omega \}$ and let $Z = Y \setminus D$ and for each $z \in Z$ define $f_z \in \omega^\omega$ such that $f_z(n)$ is the least $m$ such that $x_n \upharpoonright m \neq z \upharpoonright m$. Now the family $X = \{ f_z : z \in Z \}$ must be unbounded in $\leq^*$ in $N$. Suppose not, then there exists $g \in \omega^\omega \cap N$ which eventually dominates each element of $X$. It follows that if we let

$$U_n = \bigcup_{m<n} [x_m \upharpoonright n] \cup \bigcup_{m\geq n} [x_m \upharpoonright g(m)]$$

then

$$\bigcap_{n<\omega} U_n \cap (Y \cup D) = D$$

which is a contradiction.

It follows from Lemma 2.3 that $X$ is unbounded in $N[G]$. I claim that $D$ cannot be $G_\delta$ in $Y \cup D$ in the model $N[G]$. Suppose not, and let $\bigcap_{n<\omega} U_n$ be a $G_\delta$ in $N[G]$ such that

$$\bigcap_{n<\omega} U_n \cap (Y \cup D) = D$$

For each $n$ let $g_n \in \omega^\omega$ be such that for every $m$ we have that

$$[x_m \upharpoonright g_n(m)] \subseteq U_n.$$

Now for any $z \in Z$ there exist a $n$ such that $z \notin U_n$. But this means that $f_z(m) \leq g_n(m)$ for every $m$ since otherwise

$$x_m \upharpoonright g_n(m) = z \upharpoonright g_n(m)$$

and then $z \in U_n$. This proves the Lemma.

QED

Now we prove Theorem 2.2. Suppose that $X \subseteq 2^\omega$ is in $M[G]$ where $G$ is $\mathbb{P}_{\omega_2}$-generic over $M$ and

$$M[G] \models X \text{ is not a } \lambda'\text{-set}$$

By Lowenheim-Skolem arguments there exists $\alpha < \omega_2$ such that

$$X_\alpha = \text{def } X \cap M[G_\alpha], \ X_\alpha \in M[G_\alpha], \text{ and } M[G_\alpha] \models X_\alpha \text{ is not a } \lambda'\text{-set} \tag{10}$$
Since being a $\lambda'$-set only depends on codes for $G_\delta$-sets and reals are added by countable suborders of $\mathbb{P}_{(\alpha,\omega_2)}$ it follows from Lemma 2.4 that

$$M[G] \models X_\alpha \text{ is not a } \lambda'-\text{set}$$

But if $f \in \omega^\omega \in M[G]$ is $\omega^{<\omega}$-generic over $M[G_\alpha]$ then $X_\alpha \subseteq G_f$. It follows that

$$M[G] \models X \text{ is not } (\dagger)-\lambda'-\text{set}$$

as was to be proved.

QED

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