Surfaces containing two circles through each point and decomposition of quaternionic matrices

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We find all surfaces in $\mathbb{R}^3$ such that through each point of the surface one can draw two circular arcs fully contained in the surface. Due to the natural statement and obvious architectural motivation, this is a problem which must be solved by mathematicians. It turned out to be hard and remained open in spite of many partial advances starting from Darboux’s works in the 19th century (see also [1]–[3]).

**Theorem 1.** If through each point of an analytic surface in $\mathbb{R}^3$ one can draw two transversal circular arcs fully contained in the surface (and depending analytically on the point), then some composition of inversions takes this surface to a subset of one of the following sets:

- (E) the set $\{p + q: p \in \alpha, q \in \beta\}$, where $\alpha$ and $\beta$ are two circles in $\mathbb{R}^3$;
- (C) the set $\{2[p \times q]/|p + q|^2: p \in \alpha, q \in \beta, p + q \neq 0\}$, where $\alpha$ and $\beta$ are two circles on the unit sphere $S^2$ and $[p \times q]$ is the cross product of the vectors $p$ and $q$;
- (D) the set $\{(x, y, z): Q(x, y, z, x^2 + y^2 + z^2) = 0\}$, where $Q \in \mathbb{R}[x, y, z]$ has degree 2 or 1.

Figure 1. Euclidean (E) and Clifford (C) translation surfaces and a Darboux cyclide (D).

Let us give a general plan for proving the theorem and prove one particular lemma. We solve the more general problem of finding all such surfaces in $S^4$ instead of $\mathbb{R}^3$. Using Schicho’s parametrization of surfaces containing two conic sections through each point [5], we reduce the problem to solving the equation $X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 = X_6^2$ in polynomials $X_1, \ldots, X_6 \in \mathbb{R}[u, v]$ of degree at most 2 in each of the variables $u$ and $v$. Such ‘Pythagorean 6-tuples’ of polynomials define a surface $X_1(u, v): \ldots: X_6(u, v)$ in $S^4$ containing two (possibly degenerate) circles $u = \text{const}$ and $v = \text{const}$ through each point.

Our approach uses decomposition of quaternionic matrices. The above equation holds if and only if the matrix

\[
\begin{pmatrix}
X_6-X_5 & X_1+iX_2+jX_3+kX_4 \\
X_1-iX_2-jX_3-kX_4 & X_6+X_5
\end{pmatrix}
\]

is degenerate, that is, has linearly dependent rows (multiplication by scalars, that is, elements of the ring $\mathbb{H}[u, v]$, is performed from the left). A $2 \times 2$ matrix $M$ splits if it is a Kronecker product of two vectors, that is, $M_{ij} = A_iB_j$ for $1 \leq i, j \leq 2$ and some $A_1, A_2, B_1, B_2 \in \mathbb{H}[u, v]$. Each

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splitting matrix must be degenerate, and over a unique commutative factorization domain the converse is also true. Several results measuring the relation between the two notions over $\mathbb{H}[u,v]$ lead to the solution of our equation. We prove just one of them now.

**Lemma 2.** Each degenerate $2 \times 2$ matrix $M$ with entries in $\mathbb{H}[u,v]$ of degree at most 1 in $v$ splits.

**Proof.** Consider the following 2 cases.

Case 1: the matrix entries are independent of $v$. In this case we prove the lemma by induction on the minimal degree of the entries, using division with remainders in $\mathbb{H}[u]$ [4]. The induction base is the case when one of the entries vanishes, say, $M_{22} = 0$. Since the matrix is degenerate, we have $M_{21} = 0$ or $M_{12} = 0$. Then the matrix splits as $(1, 0)(M_{11}, M_{12})$ or $(M_{11}, M_{21})(1, 0)$, respectively. To perform the induction step, assume that each entry is non-zero. Without loss of generality let $M_{22}$ be the entry with minimal degree. By a result in [4], §2, one can divide $M_{21}$ by $M_{22}$ from the left with a remainder: $M_{21} = M_{22}X + M'_{21}$ for some $X, M'_{21} \in \mathbb{H}[u]$ such that $\deg M'_{21} < \deg M_{22}$. Subtract the second column of the matrix $M$, right-multiplied by $X$, from the first column. The resulting matrix is also degenerate and has an entry $M'_{21}$ with degree smaller than the degree of $M_{22}$. By the inductive hypothesis, the resulting matrix splits. Thus the initial matrix $M$ splits.

Case 2: some entries of $M$ depend non-trivially on $v$. In this case we prove the lemma by induction on the number of such entries. The induction base is the case when there are only one or two such entries on one (for instance, the main) diagonal and one of them is $M_{11}$. By assumption $M_{12}$ and $M_{21}$ do not depend on $v$. Since $M$ is degenerate, it follows that then $M_{22} = 0$. Just as in the induction base of Case 1 we get that $M$ splits.

Let $M_{ij}(u,v) = M^{(1)}_{ij}(u)v + M^{(0)}_{ij}(u)$. The induction step is performed using another induction, on the minimal degree of $M^{(1)}_{ij}(u)$ over all $i, j$ such that $M^{(1)}_{ij}(u) \neq 0$. Assume without loss of generality that $M^{(2)}_{22}$ is non-zero and has minimal degree. One of the polynomials $M^{(1)}_{21}$ and $M^{(1)}_{12}$ must also be non-zero. Let $M^{(1)}_{21}$ be non-zero (otherwise consider the conjugate matrix). We divide $M^{(1)}_{21}$ by $M^{(1)}_{22}$ from the left with a remainder: $M^{(1)}_{21} = M^{(1)}_{22}X + M'_{21}$, where $X, M'_{21} \in \mathbb{H}[u]$ and $\deg M'_{21} < \deg M^{(1)}_{22}$. Subtract the second column of the matrix $M$, right-multiplied by $X$, from the first column. The resulting matrix is also degenerate and has either fewer non-zero elements $M^{(1)}_{ij}$ or a smaller minimal degree of them (depending on whether $M'_{21}$ vanishes or not). The zero entries of the matrix $M^{(1)}$, if any, remain zero: if $M^{(1)}_{11} = 0$, then also $M^{(1)}_{12} = 0$, because $M^{(1)}_{21} \neq 0$ and the matrix $M$ is degenerate. By induction $M$ splits.

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