Discriminantal Groups and Zariski Pairs of Sextic Curves

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Abstract
A series of Zariski pairs and four Zariski triplets were found by using lattice theory of K3 surfaces. There is a Zariski triplet of which one member is a deformation of another.

1 Introduction
In [13] Zariski showed that there are two irreducible sextic curves $C_1, C_2$ with six cusps and the fundamental groups of $\mathbb{P}^2 \setminus C_1$ and $\mathbb{P}^2 \setminus C_2$ are not isomorphic. Such pairs are called Zariski pairs. The precise definition of Zariski pair differs from paper to paper. Here we adopt the following definition: Two plane curves $C_1, C_2$ of the same degree form a Zariski pairs if $C_1, C_2$ have the same combinatorial data (cf. [2]) and $(\mathbb{P}^2, C_1)$ and $(\mathbb{P}^2, C_2)$ are not homeomorphic. The Zariski triplet and $k$-plet are defined similarly (cf. [4]). A brief account of the history of Zariski pairs can be found in [4]. It is remarkable that the degrees of all known Zariski pairs are at least six.

Let $C$ be a reduced sextic curve with simple singularities only and let $X$ be the K3 surface obtained from the double cover branched over $C$. Let $N_C$ be the orthogonal complement in $H^2(X, \mathbb{Z})$ of the sublattice generated by all irreducible components of the inverse image of $C$ in $X$. Shimada shows in [9] that $N_C$ is a topological invariant of the pair $(\mathbb{P}^2, C)$. When $C$ is maximizing, i.e., the Milnor number of $C$ is 19, $N_C$ is the transcendental lattice of the K3 surface $X$. Let $\gamma_X$ be the discriminantal form of the Picard lattice of $X$. For some special maximizing sextics there are two non-isomorphic positive definite lattices of rank two whose discriminantal forms are isomorphic to $-\gamma_X$. By Shimada’s theorem they are Zariski pairs, called arithmetic Zariski pairs. Shimada was able to enumerate all such pairs ([8, 9]).

For any reduced sextic with simple singularities, not necessarily maximizing, let $M$ be the primitive hull of the sublattice generated by all irreducible components of

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the inverse image of $C$ in $X$. By Shimada’s theorem and Nikulin’s lattice theory, the discriminant group $A$ of $M$ is a topological invariant of $(\mathbb{P}^2, C)$, which is weaker than $N_C$. In this paper we use this invariant to obtain a series of Zariski pairs and four Zariski triplets of reduced sextics. Among them the most interesting one is a Zariski triplet of three conics with $3A_5 + 3A_1$, of which one member of the triplet is the deformation of another member (Theorem 5.2). To our knowledge this is the first such example.

One significant difference between our Zariski pairs and Shimada’s arithmetic Zariski pairs is that in our examples although two members of a pair have the same combinatorial data but for one member there is a plane curve of low degree whose intersection number with the sextic at every point is even. This geometric property is not shared by arithmetic Zariski pairs of Milnor number 19, since the Picard groups of both members of such a pair are isomorphic.

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After the finishing of this paper, the authors were kindly informed by Professor Shimada that he obtained similar results, (cf. [7]).

2 Discriminantal group of a sextic curve with simple singularities

Let $C$ be a reduced sextic curve with simple singularities only. Let $p : Y \to \mathbb{P}^2$ be the double cover branched over $C$ and let $\mu : X \to Y$ be the minimal resolution of singularities of $Y$. Then $X$ is a K3 surface and $H^2(X, \mathbb{Z})$ is a unimodular lattice of signature $(3, 19)$. Let Pic$(X)$ denote the Picard lattice of $X$. It is a primitive sublattice of $H^2(X, \mathbb{Z})$. Let $G$ be the sublattice of Pic$(X)$ generated by all irreducible components of the pull-back of $C$ in $X$ and let $\tilde{G}$ be the primitive hull of $G$ in $H^2(X, \mathbb{Z})$. We define the discriminant group of $C$ to be the finite group $\tilde{G}^\vee / \tilde{G}$, where $\tilde{G}^\vee$ is the dual lattice of $\tilde{G}$.

**Lemma 2.1.** Let $C$ and $C_1$ be two reduced sextic curves with simple singularities only. If $(\mathbb{P}^2, C)$ is homeomorphic to $(\mathbb{P}^2, C_1)$, then the discriminantal groups of $C$ and $C_1$ are isomorphic.

Proof. Let $G$ be the sublattice generated by all irreducible components of $(p \mu)^{-1}(C)$ and let $\tilde{G}$ be the primitive hull of $G$ in $H^2(X, \mathbb{Z})$. According to a theorem of Shimada ([9]) the orthogonal complement $G^\perp$ of $G$ in $H^2(X, \mathbb{Z})$ is a topological invariant of the pair $(\mathbb{P}^2, C)$. By Nikulin’s lattice theory ([6]) the discriminantal group of $G^\perp$ is isomorphic to that of $\tilde{G}$. Hence the lemma holds. $\Box$

Denote the unimodular even lattice of signature $(3, 19)$ by $\Lambda$, called the K3 lattice. Recall that for any lattice $L$, an overlattice $M$ of $L$ is a sublattice of the dual lattice $L^\vee$ such that $M \supset L$ and $|M/L| < \infty$. 

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Theorem 2.2 (Urabe [10, 11]). Let $G = \sum_k a_k A_k + \sum_i d_i D_i + \sum_m e_m E_m$ be a finite Dynkin graph. Let $L(G)$ denote the negative definite lattice of $G$. Let $\mathbb{Z}\lambda$ be a lattice of rank one generated by $\lambda$ with $\lambda^2 = 2$. Then there is a reduced sextic curve in $\mathbb{P}^2$ whose singularities correspond to $G$ if and only if there is an overlattice $M$ of $\mathbb{Z}\lambda \oplus L(G)$ such that there is a primitive embedding of $M$ into the K3 lattice $\Lambda$ such that

i) if $u \in M$, $u\lambda = 0$, $u^2 = -2$, then $u \in L(G)$;

ii) there is no $u \in M$ with $u\lambda = 1$ and $u^2 = 0$.

The main tool in Urabe’s proof of the theorem is the surjectivity of the period map for K3 surfaces. The sextic curve in the theorem can be so chosen that the overlattice $M$ is exactly the Picard group of the corresponding K3 surface and the pull-back of a line on the plane belongs to the divisor class $\lambda$.

3 Classical Zariski pair

In this section we give a lattice-theoretic interpretation of Zariski’s classical example mentioned at the beginning of the paper.

Let $M$ be the lattice of $A_2$. Let $L = \mathbb{Z}\lambda \oplus M^6$. Then $L$ has a primitive embedding into the K3 lattice. By Theorem 2.2 there is a sextic curve $C_1$ such that the Picard group of the corresponding double sextic is isomorphic to $L$.

Denote the 12 generators of the root lattice of $6A_2$ by $e_i (1 \leq i \leq 12)$ such that $e_i e_{i+1} = 1$ for $i = 1, 3, 5, 7, 9, 11$. Let

$$u = \sum_{i=1}^{6} \frac{e_{2i-1} + 2e_{2i}}{3}.$$ 

Then $uv \in \mathbb{Z}$ for every $v \in L$ and $u^2 \in 2\mathbb{Z}$. Hence the subgroup $L'$ of $L^\vee$ generated by $u$ and $L$ is an overlattice of $L$. By Nikulin’s embedding criterion there is a primitive embedding of $L'$ into $\Lambda$. It is easy to check that the two additional conditions of Theorem 2.2 are also satisfied. Thus there is a sextic curve $C_2$ such that the Picard group of the corresponding double sextic is isomorphic to $L'$.

Both $C_1$ and $C_2$ are irreducible sextic curves with six $A_2$ cusps as their only singularities. However, their discriminantal groups are $L^\vee / L$ and $L'^\vee / L'$ respectively, which are not isomorphic. Hence $\{C_1, C_2\}$ is a Zariski pair.

Next we show that the six cusps of $C_2$ are located on a conic.

Let $p : Y \to \mathbb{P}^2$ be the double cover of $\mathbb{P}^2$ branched over $C_2$ and let $\mu : X \to Y$ be the minimal resolution of singularities of $Y$. Identify $\text{Pic}(X)$ with $L'$. The map $p\mu$ is
determined by the linear system \(|\lambda|\). Let

\[
D = \lambda - \sum_{i=1}^{6} \frac{e_{2i-1} + 2e_{2i}}{3}.
\]

Then \(D \in \text{Pic}(X)\). Since \(D^2 = -2\), the Riemann-Roch theorem implies that either \(h^0(X, D) > 0\) or \(h^0(X, -D) > 0\). Since \(\lambda D = 2 > 0\), we have \(h^0(X, D) > 0\). Hence we may assume that \(D\) is an effective divisor. Choose an irreducible component \(D_1\) of \(D\) such that \(\lambda D_1 > 0\).

Suppose that \(\lambda D_1 = 1\). Then the divisor \(D_1\) would be in the class \(\lambda/2 + \sum_{i=1}^{12} k_i e_i\), in which \(k_i \in \mathbb{Q}\). However, the latter is not in \(\text{Pic}(X)\). This leads to a contradiction. Hence \(\lambda D_1 = 2\).

Let \(D = D_1 + E\). Then \(\lambda F = 0\) for each irreducible component \(F\) of \(E\). Thus \(F\) is contracted to a point in \(\mathbb{P}^2\). This means that \(E = \sum_{j=1}^{12} k_j e_j\) where \(k_j\) is a nonnegative integer. For any \(1 \leq i \leq 6\), if \(k_{2i-1} > 0\) or \(k_{2i} > 0\) then

\[
0 \leq k_{2i} = D(k_{2i-1}e_{2i-1} + k_{2i}e_{2i})
= D_1(k_{2i-1}e_{2i-1} + k_{2i}e_{2i}) + (k_{2i-1}e_{2i-1} + k_{2i}e_{2i})^2
< D_1(k_{2i-1}e_{2i-1} + k_{2i}e_{2i})
\]

implies that either \(D_1 e_{2i-1} > 0\) or \(D_1 e_{2i} > 0\). Hence \(pp(D_1)\) is a conic passing through all six cusps of \(C_2\).

4 Sextics of Milnor number 19 with simple singularities

Shimada finds all arithmetic Zariski pairs for sextics with simple singularities of Milnor number 19 ([8]). In this section we present a few more Zariski pairs of sextics of Milnor number 19 with different discriminant groups, all reducible.

Example 1 \(E_6 + A_{11} + 2A_1\)

Let \(L\) denote the negative definite lattice of the Dynkin graph \(E_6 + A_{11} + 2A_1\). The 19 generators of \(L\) are labeled according to Figure [1]. Let \(V = \mathbb{Q} \otimes \mathbb{Z}(\mathbb{Z} \lambda \oplus L)\), in which \(\lambda^2 = 2\). Let

\[
u = \frac{\sum_{i=1}^{11} i e_{i+6}}{2} + \frac{e_{18}}{2} + \frac{e_{19}}{2} \in V.
\]

It can be verified that \(u^2 \in 2\mathbb{Z}\) and \(uw \in \mathbb{Z}\) for any \(w \in \mathbb{Z} \lambda \oplus L\). Let \(M_1\) be the lattice generated by \(\mathbb{Z} \lambda \oplus L\) and \(u\). Then \(M_1\) is an overlattice of \(\mathbb{Z} \lambda \oplus L\). Using Nikulin’s criterion for lattice embedding ([6] 1.12.2), one verifies that there is a primitive embedding from \(M_1\) into the K3 lattice \(\Lambda\). Moreover, it is not hard to check that \(M_1\) satisfies
the two additional conditions in Theorem 2.2. It follows that there is a reduced sextic curve $C_1$ with $E_6, A_{11}, A_1, A_1$ as its singularities. Although we can use the algorithm in [12] to determine the irreducible decomposition of $C_1$, the following lemma uses an elementary argument to serve the same purpose.

**Lemma 4.1.** Let $C$ be a reduced sextic curve with $E_6, A_{11}, A_1, A_1$ as its only singularities. Then $C = B + D$ where $B$ and $D$ are irreducible curves of degree 2 and degree 4 respectively satisfying the following conditions:

1) $D$ has an $E_6$ singularity;
2) $B \cap D = \{p, q_1, q_2\}$, in which $p$ is an $A_{11}$ point of $C$ and $B, D$ meet at $q_1$ and $q_2$ transversally.

**Proof.** Let $d$ be the maximal degree of all irreducible components of $C$. Since $E_6$ is locally irreducible, $d$ is at least 4. Since the arithmetic genus of an irreducible sextic curve is 10, there is no irreducible sextic curve with $E_6, A_{11}, A_1, A_1$ as its singularities. Hence $d \leq 5$. It is obvious that there are two irreducible components passing through the $A_{11}$ point and the intersection number of these two components at $A_{11}$ point is 6. The only possibility is that $d = 4$ and the other component is a conic. The rest is clear.

Let $v = \frac{3e_1 + 2e_2 + 4e_3 + 6e_4 + 5e_5 + 4e_6}{3} + \frac{\sum_{i=1}^{11} ie_{i+6}}{6} + \frac{e_{18}}{2} + \frac{e_{19}}{2} \in V$.

Then $v^2 \in 2\mathbb{Z}$ and $vw \in \mathbb{Z}$ for any $w \in \mathbb{Z} \lambda \oplus L$. Let $M_2$ be the lattice generated by $\mathbb{Z} \lambda \oplus L$ and $v$. Then $M_2$ is an overlattice of $\mathbb{Z} \lambda \oplus L$. Note that $3v \equiv u \pmod{L}$.

Using the same method as before, we assert that $M_2$ satisfies the conditions in Theorem 2.2. Let $C_2$ be the sextic determined by $M_2$. By Lemma 4.1 $C_2$ has the same configuration as $C_1$. The discriminantal groups of $C_1$ and $C_2$ are $M_1^\vee / M_1$ and $M_2^\vee / M_2$ respectively. They are finite groups of different size. Hence they are not isomorphic. This shows that $\{C_1, C_2\}$ is a Zariski pair.

Next we show that $C_1$ and $C_2$ are distinguished by the existence of a special conic on $\mathbb{P}^2$.

**Lemma 4.2.** Let $C$ be either $C_1$ or $C_2$. Let $p_1, p_2, p_3, p_4$ be its singularities of types $E_6, A_{11}, A_1, A_1$ respectively. Then the following statements hold:

1) There is a conic $Q$ on $\mathbb{P}^2$ such that $(C_2, Q)_{p_1} = (C_2, Q)_{p_2} = 4, (C_2, Q)_{p_3} = (C_2, Q)_{p_4} = 2$;
2) There is no conic $Q$ on $\mathbb{P}^2$ such that $(C_1, Q)_{p_1} = (C_1, Q)_{p_2} = 4, (C_1, Q)_{p_3} = (C_1, Q)_{p_4} = 2$. 

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Proof. Let $p : Y \to \mathbb{P}^2$ be the double cover branched over $C$ and let $\mu : X \to Y$ be the minimal resolution of singularities of $Y$. There are nineteen $-2$ curves on $X$ arising from the singularities of $C$. They are still denoted by $e_1, \ldots, e_{19}$ by abuse of notation.

Case 1) $C = C_2$:

The map $p\mu$ is determined by the linear system $|\lambda|$. Let

$$D = \lambda - \frac{3e_1 + 2e_2 + 6e_4 + 5e_5 + 4e_6}{3} - \sum_{i=1}^{11} i e_{i+6} + e_{17} - \frac{e_{18}}{2} - \frac{e_{19}}{2}.$$  

It follows from $D \equiv -v \pmod{\mathbb{Z}\lambda \oplus L}$ that $D \in \text{Pic}(X)$. Since $D^2 = -2$, the Riemann-Roch theorem shows that either $h^0(X, D) > 0$ or $h^0(X, -D) > 0$. Since $\lambda D = 2 > 0$, we have $h^0(X, D) > 0$. Hence we may assume that $D$ is an effective divisor. Choose an irreducible component $D_1$ of $D$ such that $\lambda D_1 > 0$.

Suppose that $\lambda D_1 = 1$. Then image of $D_1$ in $L^*/L$ would be $\lambda/2 + \sum_{i=1}^{19} n_i e_i$ for some $n_i (1 \leq i \leq 19)$, which is not in the lattice $M_2$. This leads to a contradiction. Hence $\lambda D_1 = 2$.

Let $D = D_1 + E$. Then $\lambda F = 0$ for each irreducible component $F$ of $E$. Thus $F$ is contracted to a point in $\mathbb{P}^2$. This means that $E$ consists of exceptional curves. Since $De_i > 0$ for $i = 6, 16, 18, 19$, the image $Q$ of $D$ in $\mathbb{P}^2$ passes through all singularities of $C_2$. Since there is no line with this property, it must be a conic.

Following the process of the canonical resolution of a double cover, it is easy to see that $(C_2, Q)_{p_1} = (C_2, Q)_{p_2} = 4$.

Case 2) $C = C_1$:

Suppose that there is such a conic $Q$ for $C_1$. Since $Q$ has even intersection number with $C_1$ at each point of $Q \cap C_1$, it splits into two components $\tilde{Q}_1$ and $\tilde{Q}_2$ in $X$. Since $\lambda \tilde{Q}_i > 0$ for $i = 1, 2$ and $\lambda (\tilde{Q}_1 + \tilde{Q}_2) = 2$, we have $\lambda \tilde{Q}_1 = \lambda \tilde{Q}_2 = 1$. Since $(Q, C_1)_{p_2} = 4$, one of $\tilde{Q}_1$ and $\tilde{Q}_2$ meets $e_{16}$ transversally. We may assume that $\tilde{Q}_1 e_{16} = 1$. Since
respectively.

The intersection numbers of are labeled according to Figure 2. Let 

\[ \lambda = \sum_{i=1}^{19} m_i e_i \]

for some \( n_i \) and \( m_i \). Obviously there is no such element in \( M_1 \).

The equations are as follows:

\[ C_1 : (3x_0^3 x_2 + 3x_0^2 x_1^2 - 3x_0 x_1^3 + 2x_1^4)(3x_0 x_2 - x_2^3 + 3x_1 x_2 + 3x_1^2) = 0, \]

for which \( E_6, A_{11}, A_1, A_1 \) are located at \((0 : 0 : 1), (1 : 0 : 0), (1 : 3/2 + \sqrt{-3} : 33/4 + 15 \sqrt{-3}/4), (1 : 3/2 - \sqrt{-3} : 33/4 - 15 \sqrt{-3}/4)\) respectively.

\[ C_2 : (x_1 x_2 + 3x_0 x_1 + 8x_0^2 + 3x_0 x_2)(3x_1^3 x_2 + x_0 x_1^3 - 2x_1^2 x_2^3 + 3x_0 x_1^2 + x_0 x_2 + x_1 x_2) = 0, \]

for which \( E_6, A_{11}, A_1, A_1 \) are located at \((1 : 0 : 0), (1 : -2 : -2), (0 : 1 : 0), (0 : 0 : 1)\) respectively.

For \( C_2 \) the conic \( Q : x_0 x_1 + x_0 x_2 + x_1 x_2 = 0 \) passes through all singularities. The intersection numbers of \( Q \) with the quartic at \( E_6 \) and \( A_{11} \) are equal to 4.

**Example 2** \( A_{11} + A_5 + 3A_1 \)

Let \( L \) denote the lattice of the Dynkin graph \( A_{11} + A_5 + 3A_1 \). The 19 generators of \( L \) are labeled according to Figure 2. Let \( V = \mathbb{Q} \otimes \mathbb{Z} \lambda \otimes L \). Let

\[ u_1 = \frac{\sum_{i=1}^{11} i e_i}{2} + \frac{e_{18}}{2} + \frac{e_{19}}{2}, \]

\[ u_2 = \frac{\lambda}{2} + \frac{\sum_{i=1}^{11} i e_{i+11}}{2} + \frac{e_{18}}{2} + \frac{e_{19}}{2}. \]

Let \( M_1 \) be the lattice generated by \( u_1, u_2 \) and \( \mathbb{Z} \lambda \otimes L \). Then \( M_1 \) is an overlattice of \( \mathbb{Z} \lambda \otimes L \) satisfying the conditions in Theorem 2.2. The configuration of \( C_1 \) can be described as
follows. $C_1$ consists of three components $Q_1, Q_2, Q_3$ of degrees $1, 2, 3$ respectively. $Q_3$
has a node. $Q_1 \cap Q_3$ is an $A_{11}, Q_1 \cap Q_3$ is an $A_3$ point and $Q_1$ intersects $Q_2$ at two
distinct points.
Let
$$v = \frac{\sum_{i=1}^{11} i e_i}{6} + \frac{\sum_{i=1}^{5} i e_{i+11}}{3} + \frac{e_{18}}{2} + \frac{e_{19}}{2}.$$ 
Let $M_2$ be the lattice generated by $u_1, u_2, v$ and $\mathbb{Z} \lambda \oplus L$. Then $M_2$ is an overlattice of $\mathbb{Z} \lambda \oplus L$
satisfying the conditions in Theorem [2,2]. The corresponding sextic curves $C_2$ has the
same configuration as $C_1$. However their discriminantal groups are not isomorphic,
since their orders are different. It follows that $\{C_1, C_2\}$ is a Zariski pair.
The equations are as follows:

$C_1 : (2x_0 - 5x_1 + 3x_2)(16x_0x_2 - 16x_1^2 - 4x_1x_2 + 3x_2^2)(4x_0^2x_2 - 4x_0x_1x_2 + x_1^2 + x_2^2) = 0,$
in which the $A_{11}, A_5, A_1, A_1, A_1$ points are located at $(1 : 0 : 0), (1 : 1 : 1), (0 : 0 : 1), (1 : \frac{24}{43} + \frac{4\sqrt{3}}{43} : \frac{28}{43} + \frac{20\sqrt{3}}{129}), (1 : \frac{-24}{43} - \frac{4\sqrt{3}}{43} : \frac{28}{43} - \frac{20\sqrt{3}}{129})$ respectively.

$C_2 : (x_2 - x_0)(x_0^2x_2 - x_0x_1^2 + x_1^2x_2)(x_0x_2 - x_1^2 + x_2^2) = 0,$
in which the $A_{11}, A_5, A_1, A_1, A_1$ points are located at $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (1 : \sqrt{2} : 1), (1 : -\sqrt{2} : 1)$ respectively.

For $C_2$ the line $L : x_2 = 0$ passes $A_{11}, A_5$ and is tangent to the conic at $A_{11}$. This
property is not shared by $C_1$.

**Example 3** $A_{17} + 2A_1$

Let $L$ denote the lattice of the Dynkin graph $A_{17} + 2A_1$. The 19 generators of $L$ are
labeled according to Figure 3. Let $V = \mathbb{Q} \otimes_2 (\mathbb{Z} \lambda \oplus L)$. Let
$$u = \frac{\lambda}{2} + \frac{\sum_{i=1}^{17} i e_i}{2}.$$ 
Let $M_1$ be the lattice generated by $u$ and $\mathbb{Z} \lambda \oplus L$. Then $M_1$ is an overlattice of $\mathbb{Z} \lambda \oplus L$
satisfying the conditions in Theorem [2,2]. Let $C_1$ be the corresponding sextic.
Let
$$v = \frac{\lambda}{2} + \frac{\sum_{i=1}^{17} i e_i}{6}.$$ 
Let $M_2$ be the lattice generated by $v$ and $\mathbb{Z} \lambda \oplus L$. Then $M_2$ is an overlattice of $\mathbb{Z} \lambda \oplus L$
satisfying the conditions in Theorem [2,2]. Let $C_2$ be the corresponding sextic.

Both curves $C_1$ and $C_2$ have two nodal cubics as irreducible components and these
two components meet at $A_{17}$. They form a Zariski pair.
Figure 3: Dynkin graph of $A_{17} + 2A_1$

Denote the $A_{17}$ point by $p$. Let $L$ be the common tangent line of the two cubics at $p$. Then $(L, C_1)_p = 4$ and $(L, C_2)_p = 6$.

The equations are as follows:

$$C_1 : (x_0^2 x_2 - x_0 x_1^2 + x_1^3 + x_0 x_1 x_2 + 7 x_1^2 x_2)(29 + 9 \sqrt{-3})x_0 x_2 - (29 + 9 \sqrt{-3})x_0 x_1^2$$

$$- (484 - 18 \sqrt{-3})x_0 x_1 x_2 + 542 x_1^3 + (3078 - 54 \sqrt{-3})x_0 x_2^2 - (3901 - 135 \sqrt{-3})x_1^2 x_2$$

$$+ (13851 - 243 \sqrt{-3})x_2^3 + (1539 - 27 \sqrt{-3})x_1 x_2^2] = 0,$$

in which the $A_{17}, A_1, A_1$ are located at $(1 : 0 : 0), (0 : 0 : 1)$ and $(1 : 1 : 13 + 3 \sqrt{-3} : 3 : 1170 : 1)$ respectively.

$$C_2 : (x_0^2 x_2 + x_1^3 + x_0 x_1 x_2 + x_1^2 x_2 - \frac{x_1^3}{16})(x_0^2 x_2 + x_1^3 + x_0 x_1 x_2 + x_1^2 x_2) = 0,$$

in which the $A_{17}, A_1, A_1$ are located at $(1 : 0 : 0), (1 : -2 : 4), (0 : 0 : 1)$ respectively.

**Example 4** $A_{15} + A_3 + A_1$

Let $L$ denote the lattice of the Dynkin graph $A_{15} + A_3 + A_1$. The 19 generators of $L$ are labeled according to Figure 4. Let $V = \mathbb{Q} \otimes (\mathbb{Z} \lambda \oplus L)$. Let

$$u = \frac{\lambda}{2} + \sum_{i=1}^{15} ie_i + \frac{e_{16} + 2e_{17} + 3e_{18}}{2} + \frac{e_{19}}{2}.$$

Let $M_1$ be the lattice generated by $u$ and $\mathbb{Z} \lambda \oplus L$. Then $M_1$ is an overlattice of $\mathbb{Z} \lambda \oplus L$ satisfying the conditions in Theorem 2.2. Let $C_1$ be the corresponding sextic.

Let

$$v = \frac{\lambda}{2} + \sum_{i=1}^{15} ie_i + \frac{e_{16} + 2e_{17} + 3e_{18}}{4}.$$

Let $M_2$ be the lattice generated by $u, v$ and $\mathbb{Z} \lambda \oplus L$. Then $M_2$ is an overlattice of $\mathbb{Z} \lambda \oplus L$ satisfying the conditions in Theorem 2.2. Let $C_2$ be the corresponding sextic.
Both curves $C_1$ and $C_2$ have two irreducible components of degree 4 and 2 respectively. The quartic component contains an $A_3$ and an $A_1$. The two components meet at $A_{15}$. $\{C_1, C_2\}$ is a Zariski pair.

The equations are as follows:

$$C_1 : (2x_0x_2 + 2x_1^2 - 2x_2 + \sqrt{-2}x_1x_2)(4x_0^2x_2 + 4x_0^2x_1^2 + 4x_0^2x_2^2 - 2\sqrt{-2}x_0x_1x_2 - 4\sqrt{-2}x_0x_3^2 + 34x_0x_1^2x_2 + 12\sqrt{-2}x_0x_1x_3^2 + 22x_1^4 + 15\sqrt{-2}x_1x_2 - 18x_1^2x_2^2) = 0,$$

in which $A_{15}, A_3$ and $A_1$ are located at $(1 : 0 : 0), (0 : 0 : 1)$ and $(1 : -\frac{\sqrt{2}}{3} : \frac{2}{9})$ respectively. The line $x_1 = 0$ passes $A_{15}$ and $A_3$ but its intersection number with $C_1$ at $A_{15}$ is equal to 2.

$$C_2 : (x_0x_2 + x_1^2 - \frac{x_2^2}{2})(x_0^3x_2 + x_0^2x_1^2 + 2x_0x_1^2x_2 + x_1^3x_2^2 + \frac{3x_0^2x_2^2}{2}) = 0,$$

in which $A_{15}, A_3$ and $A_1$ are located at $(1 : 0 : 0), (0 : 1 : 0)$ and $(0 : 0 : 1)$ respectively. The intersection numbers of the line $x_2 = 0$ with $C_2$ at $A_{15}$ and $A_3$ are 4 and 2 respectively.

**Example 5** $2A_9 + A_1$

Let $L$ denote the lattice of the Dynkin graph $2A_9 + A_1$. The generators of $L$ are labeled according to Figure 5. Let $V = \mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z} \lambda \oplus L)$. Let

$$u = \frac{\lambda}{2} + \frac{\sum_{i=1}^{9} i e_i}{2}.$$

Let $M_1$ be the lattice generated by $u$ and $\mathbb{Z} \lambda \oplus L$. Then $M_1$ is an overlattice of $\mathbb{Z} \lambda \oplus L$ satisfying the conditions in Theorem 2.2. Let $C_1$ be the corresponding sextic. Let

$$v = \frac{\lambda}{2} + \frac{\sum_{i=1}^{9} i e_i}{5} + \frac{\sum_{i=1}^{9} i e_{i+9}}{10}.$$
Figure 5: Dynkin graph of $2A_9 + A_1$

Let $M_2$ be the lattice generated by $u, v$ and $\mathbb{Z}\lambda \oplus L$. Then $M_2$ is an overlattice of $\mathbb{Z}\lambda \oplus L$ satisfying the conditions in Theorem 2.2. Let $C_2$ be the corresponding sextic.

Both curves $C_1$ and $C_2$ have two irreducible components of degree 5 and 1 respectively. The quintic component contains an $A_9$ and an $A_1$. The two components meet at the second $A_9$. Let $L$ be the line connecting the two $A_9$ points. Let $p$ be the $A_9$ of the quintic component. Then $(L, C_1)_p = 2$ and $(L, C_2)_p = 4$.

The equations of $C_1$ and $C_2$ are as follows:

$C_1 : x_0[x_0^3x_2^2 + 2x_0^2x_1^2x_2 + x_0x_1^4 + (25/2 - 5\sqrt{5}/2)x_1^5 + (27 - 5\sqrt{5})x_0x_1^3x_2$

$+(-22 + 6\sqrt{5})x_0^2x_2^3 + (29/2 - 5\sqrt{5}/2)x_0^2x_1^3x_2 + (-21 + 6\sqrt{5})x_0^2x_1^2x_2^2$

$+(-19/2 + 7\sqrt{5}/2)x_0x_1x_2^3 + (-9 + 4\sqrt{5})x_0x_2^4] = 0,$

in which the quint component has an $A_9$ at $(1 : 0 : 0)$ and an $A_1$ at $(1 : -39 - 87\sqrt{5}/5 : 648 + 1449\sqrt{5}/5)$, and the two components meet at another $A_9$ point $(0 : 1 : 0)$.

$C_2 : x_1(x_0^2x_1^3 + 2x_0x_1^2x_2^2 + x_1^4x_2 + \frac{3x_0^3x_1^2}{4} + x_0^2x_1^2x_2 + \frac{9x_0^4x_1}{64}$

$+\frac{3x_0^3x_1x_2}{8} + x_0^2x_1x_2^2 + x_0x_1x_2^3 + \frac{125x_0^5}{27648}) = 0,$

in which $A_1$ is at $(1 : -5/48 : -1/4)$ and the two $A_9$ points are at $(0 : 1 : 0)$ and $(0 : 0 : 1)$ respectively.

In summary, we obtain the following result.

**Theorem 4.3.** There are five Zariski pairs of sextic curves of Milnor number 19, whose combinations of singularities are

$$E_6 + A_{11} + 2A_1, A_{17} + 2A_1, A_{15} + A_3 + A_1, A_{11} + A_5 + 3A_1, 2A_0 + A_1.$$
5 Sextic Zariski pairs with Milnor number less than 19

Many Zariski pairs of sextics with lower Milnor numbers can be obtained using the same method. We choose several special ones to analyze in details. The others can be obtained by the same method and are listed in the tables at the end of this paper.

5.1 Zariski triplets

Four Zariski triplets were found among reduced sextics with simple singularities.

5.1.1 Zariski triplet of three conics

Namba and Tsuchihashi constructed a Zariski pair of octive curves which consists of four conics in [5]. Here we give a Zariski triplet consisting of three conics.

Let $L$ be the root lattice of $3A_5 + 3A_1$. Let $N = \mathbb{Z} \lambda \oplus L$. Denote the 18 generators of $L$ by $e_i (1 \leq i \leq 18)$ in a natural way such that $e_i e_{i+1} = 1$ for $1 \leq i \leq 4, 6 \leq i \leq 9$ and $11 \leq i \leq 14$. Let

$$u_1 = \left( \sum_{i=1}^{5} i(e_{i+5} + e_{i+10}) + e_{17} + e_{18} \right) / 2,$$

$$v_1 = \left( \sum_{i=1}^{5} i(e_i + e_{i+10}) + e_{16} + e_{18} \right) / 2,$$

$$u_2 = \sum_{i=1}^{5} i(e_i + e_{i+5}) / 6 + \sum_{i=1}^{5} ie_{i+10} / 3 + (e_{17} + e_{18}) / 2,$$

$$v_2 = \left( \sum_{i=1}^{5} i(e_{i+5} + e_{i+10}) + e_{16} + e_{18} \right) / 2$$

and

$$w = \lambda / 2 + \sum_{i=1}^{5} i(e_i + e_{i+5} + e_{i+10}) / 2.$$

Let $A_i$ be the sublattice of $N^\vee$ generated by $u_i, v_i$ and $N$ for $i = 1, 2$. Let $A_3$ be the sublattice of $N^\vee$ generated by $u_1, v_1$ and $w$. Then $A_1, A_2, A_3$ are overlattices of $N$ satisfying the conditions in Theorem 2.22. Let $E_1, E_2, E_3$ be their corresponding reduced sextic curves. The configurations of these two curves turn out to be the same: three conics. Since the orders of $A_1^\vee / A_1, A_2^\vee / A_2$ and $A_3^\vee / A_3$ are different, $\{E_1, E_2, E_3\}$ is a Zariski triplet.

Let us call a sextic curve corresponding to the overlattice $A_i$ a sextic of type $i$ for $i = 1, 2, 3$. 


Theorem 5.1. There is a Zariski triplet \( \{E_1, E_2, E_3\} \) of sextics of three conics where \( E_i \) is of type \( i \). They are distinguished by the following conditions:

1) For \( E_2 \) there is a conic on \( \mathbb{P}^2 \) passing through the three \( A_5 \) points such that the intersection number of this conic with \( E_2 \) at each \( A_5 \) point is 4. This property is not shared by \( E_1 \) and \( E_3 \).

2) For \( E_3 \) there is a nodal cubic with the node at one \( A_5 \) point such that the intersection number of this nodal cubic with \( E_3 \) at each \( A_5 \) point is equal to 6. This property is not shared by \( E_1 \) and \( E_2 \).

Sketch of the proof. Let \( X_i \) denote the \( K3 \) surface obtained by the double cover branched over \( E_i \) for \( i = 1, 2, 3 \).

Let \( D = \lambda - \frac{e_1 + 2e_2 + 3e_3 + 4e_4 + 2e_5}{3} - \frac{e_6 + 2e_7 + 3e_8 + 4e_9 + 2e_{10}}{3} - \frac{2e_11 + 4e_12 + 3e_13 + 2e_14 + e_15}{3} \),

Then \( D \in \text{Pic}(X_2) \) and \( D^2 = -2, D\lambda = 2 \). Using the same argument as before, we proved that the image of one irreducible component of a member of \( |D| \) under the double cover map is a conic with the desired property.

Let

\[
C = \frac{3\lambda/2 - \frac{e_1 + 2e_2 + 3e_3 + 2e_4 + e_5}{2} - \frac{e_6 + 2e_7 + 3e_8 + 2e_9 + e_{10}}{2} - \frac{e_11 + 2e_12 + 3e_13 + 4e_14 + 3e_15}{2}}{3},
\]

Then \( C \in \text{Pic}(X_3) \) and \( C^2 = -2, C\lambda = 3 \). For the same reason as before the image of one irreducible component of a member of \( |C| \) in \( \mathbb{P}^2 \) is a nodal cubic with the desired property. □

In the remaining part of this subsection we calculate in details the explicit equations of all such sextics.

Let \((x_0 : x_1 : x_2)\) be the homogeneous coordinates of \( \mathbb{P}^2 \). Let \( Y \) be a sextic curve consisting of three conics \( C_1, C_2, C_3 \) satisfying the conditions in Theorem 5.1. After a suitable linear change of coordinates, we may assume that the three \( A_5 \) points are located at

\((1 : 0 : 0) \in C_2 \cap C_3, (0 : 1 : 0) \in C_1 \cap C_3, (0 : 0 : 1) \in C_1 \cap C_2\),

and the tangent lines of \( C_1 \) at \((0 : 1 : 0)\) and \((0 : 0 : 1)\) are given by the equations \( x_2 - x_0 = 0 \) and \( x_1 - x_0 = 0 \) respectively. The equations of the conics are written as
\[ C_1 : x_1 x_2 - x_0 x_1 - x_0 x_2 + \lambda x_0^2 = 0, \]
\[ C_2 : x_1 x_2 - ax_0 x_1 - x_0 x_2 + bx_1^2 = 0, \]
\[ C_3 : x_1 x_2 - x_0 x_1 - \frac{1}{a} x_0 x_2 + cx_1^2 = 0 \]

where \( \lambda, a, b, c \) are parameters to be determined. With generic values for \( \lambda, a, b, c \) the sextic has at least three \( A_3 \) singularities at \((1 : 0 : 0), (0 : 1 : 0) \) and \((0 : 0 : 1) \) already. The requirement of \( A_5 \) poses three conditions on the parameters, which are determined as follows.

1) \( A_5 \) at \((1 : 0 : 0)\):
Under the standard affine coordinates \( x = x_1/x_0, y = x_2/x_0 \) the equations of \( C_2 \) and \( C_3 \) are

\[ C_2 : xy - ax - y + bx^2 = 0, \]
\[ C_3 : xy - x - \frac{1}{a} y + cy^2 = 0. \]

After the change of coordinates \( y = y' - ax \) these two equations become

\[ y' + (a - b)x^2 - xy' = 0 \tag{1} \]

and

\[ y' + a^2(1 - ac)x^2 - a(1 - 2ac)xy' + cy'^2 = 0. \tag{2} \]

In order that the intersection number of \( C_2 \) and \( C_3 \) at \((1 : 0 : 0)\) is greater than two, the coefficients of the term \( x^2 \) in \((1) \) and \((2) \) should be equal. This gives the following relation

\[ b = a - a^2 + a^3 c. \tag{3} \]

2) \( A_5 \) at \((0 : 1 : 0)\):
Under the affine coordinates \( w = x_0/x_1, y = x_2/x_1 \) the equations of \( C_1 \) and \( C_3 \) are

\[ C_1 : y - w - wy + \lambda w^2 = 0, \]
\[ C_3 : y - w - \frac{1}{a} wy + cy^2. \]

After the change of coordinates \( y = y' + w \) these two equations become

\[ y' + (\lambda - 1)w^2 - wy' = 0 \tag{4} \]
and

\[ y' + (c - \frac{1}{a})w^2 + \cdots = 0. \tag{5} \]

The coefficients of the term \( w^2 \) in (4) and (5) should be equal. This gives the following relation

\[ c = \frac{1}{a} + \lambda - 1. \tag{6} \]

3) \( A_5 \) at \((0 : 0 : 1)\):
Under the affine coordinates \( w = x_0/x_2, x = x_1/x_2 \) the equations of \( C_1 \) and \( C_3 \) are

\[
C_1 : x - xw - w + \lambda w^2 = 0,
\]
\[
C_2 : x - axw - w + bx^2 = 0.
\]

The requirement of an \( A_5 \) at \((0 : 0 : 1)\) gives the following condition

\[ b = a + \lambda - 1. \tag{7} \]

The three conditions (3), (6) and (7) yield

\[
(\lambda - 1)(a^3 - 1) = 0.
\]

Since \( \lambda \neq 1 \) (otherwise the conic \( C_1 \) would become the union of two lines), \( a \) is equal to one of \( 1, \zeta, \zeta^2 \) with \( \zeta = e^{2\pi i/3} \). Thus we obtain three sets of solutions for the parameters \( a, b, c \):

- \( a = 1, \quad b = \lambda, \quad c = \lambda, \)
- \( a = \zeta, \quad b = \lambda + \zeta - 1, \quad c = \lambda + \zeta^2 - 1, \)
- \( a = \zeta^2, \quad b = \lambda + \zeta^2 - 1, \quad c = \lambda + \zeta - 1, \)

Hence there are three families of sextic curves satisfying our conditions defined by

\[
(x_1x_2 - x_0x_1 - x_0x_2 + \lambda x_0^2)(x_1x_2 - x_0x_1 - x_0x_2 + \lambda x_1^2) = 0,
\tag{8}
\]

\[
(x_1x_2 - x_0x_1 - x_0x_2 + \lambda x_2^2) = 0,
\]

\[
(x_1x_2 - x_0x_1 - x_0x_2 + \lambda x_0^2)(x_1x_2 - \zeta x_0x_1 - x_0x_2 + (\lambda + \zeta - 1)x_1^2) = 0,
\tag{9}
\]

\[
(x_1x_2 - x_0x_1 - \zeta^2 x_0x_2 + (\lambda + \zeta^2 - 1)x_2^2) = 0,
\]

Hence there are three families of sextic curves satisfying our conditions defined by
and

\[(x_1x_2 - x_0x_1 - x_0x_2 + \lambda x_0^2)(x_1x_2 - \zeta x_0x_2 + (\lambda + \zeta - 1)x_2^2) = 0\]  \hspace{1cm} (10)

respectively. The equation (9) becomes (10) if the variables \(x_1\) and \(x_2\) are exchanged. Hence they are essentially the same.

Let \(X_\lambda\) and \(Y_\lambda\) denote the sextic curves defined by (8) and (9) respectively. When \(\lambda \neq 0, 1\), the sextic \(X_\lambda\) is composed of three conics with \(3A_5 + 3A_1\) as singularities. The three \(A_1\) points are located at

\[\left(1 : \frac{2}{\lambda + 1} : \frac{2}{\lambda + 1} \right), \left(1 : -1 : \frac{\lambda + 1}{2} \right), \left(1 : \frac{\lambda + 1}{2} : -1 \right)\].

When \(\lambda = -1\), \(X_\lambda\) is special in the sense that every two \(A_5\) points are collinear with an \(A_1\) point.

When \(\lambda \neq 0, \frac{3}{2}, 2, \frac{3 \pm \sqrt{-3}}{2}, \frac{3 \pm \sqrt{-3}}{4}, \frac{1 \pm \sqrt{-3}}{2}\) the sextic \(Y_\lambda\) has the desired configuration. The three \(A_1\) points of \(Y_\lambda\) are located at

\[\left(1 : \frac{2\lambda}{2\lambda - 3 + \sqrt{-3}} : \frac{\lambda(2\lambda - 1 + \sqrt{-3})}{4\lambda - 3 + \sqrt{-3}} \right), \left(1 : \frac{\lambda(2\lambda - 1 - \sqrt{-3})}{4\lambda - 3 - \sqrt{-3}} : \frac{2\lambda}{2\lambda - 3 - \sqrt{-3}} \right)\]

and

\[\left(1 : \frac{(1 - \sqrt{-3})(2\lambda - 3)}{(\lambda - 2)(2\lambda - 3 + \sqrt{-3})} : \frac{(1 + \sqrt{-3})(2\lambda - 3)}{(\lambda - 2)(2\lambda - 3 - \sqrt{-3})} \right)\].

**Theorem 5.2.** Let \(X_\lambda\) and \(Y_\lambda\) denote the sextic curves defined by (8) and (9) respectively. Then

\[X_\lambda \begin{cases} 
\text{is of type 2,} & \text{if } \lambda \neq 0, 1 \\
\text{degenerates,} & \text{if } \lambda = 0, 1 
\end{cases}\]

and

\[Y_\lambda \begin{cases} 
\text{degenerates,} & \text{if } \lambda = 0, \frac{3}{2}, 2, \frac{3 \pm \sqrt{-3}}{2}, \frac{3 \pm \sqrt{-3}}{4}, \frac{1 \pm \sqrt{-3}}{2} \\
\text{is of type 3,} & \text{if } \lambda = 3, \pm \sqrt{-3}, \\
\text{is of type 1,} & \text{otherwise.} 
\end{cases}\]

In particular, every sextic of type 3 with this configuration is a deformation of the ones of type 1.
Proof.
The values of \( \lambda \) for which \( X_4 \) or \( Y_4 \) degenerates have been discussed. Here we only consider the non-degenerate values of \( \lambda \).

Let \( Q \) be the conic defined by \( x_1x_2 - x_0x_1 - x_0x_2 = 0 \). It is obvious that for every \( X_4 \) or \( Y_4 \) the conic \( Q \) is the unique one passing three \( A_5 \) points and whose intersection numbers with the sextic at both \((0 : 1 : 0)\) and \((0 : 0 : 1)\) are greater than 2. However, the intersection number of \( Q \) with the sextic at \((1 : 0 : 0)\) is greater than 2 if and only if the sextic is some \( X_4 \). Hence \( X_4 \) is of type 2 and \( Y_4 \) is of type 1 or 3.

Let \( N \) be a nodal cubic characterizing some \( Y_4 \) as a sextic of type 3. There are three choices \((1 : 0 : 0)\), \((0 : 1 : 0)\), \((0 : 0 : 1)\) for the location of the node. They will give three values of \( \lambda \). First assume that the node of \( N \) is at \((1 : 0 : 0)\).

Since the tangent lines of \( N \) at \((0 : 1 : 0)\) and \((0 : 0 : 1)\) are the same as those of \( C_2 \), the equation of \( N \) takes the form

\[
x^2 - x^2y + ry^2 - rx^2 - sxy = 0
\]

under the affine coordinates \( x = x_1/x_0, y = x_2/x_0 \). The tangent cone of \( N \) at \( x = 0, y = 0 \) is \( x^2 + ry^2 + sxy \) and the tangent line of \( Y_4 \) is \( y + \zeta x \). In order that \( N \) and \( Y_4 \) have higher contact at \( x = 0, y = 0 \) it is necessary that

\[
1 - \zeta s + r\zeta^2 = 0. \tag{11}
\]

The other two conditions of \( N \) at \((0 : 1 : 0)\) and \((0 : 0 : 1)\) are computed. They yields the following two relations:

\[
r - s = \lambda - 1, \tag{12}
\]

\[
r(\lambda - 1) - 1 + s = 0. \tag{13}
\]

The solution of the simultaneous equations (11), (12), and (13) is \( r = 1, s = -1, \lambda = 3 \). Hence \( Y_3 \) has type 3.

If the node is chosen to be at \((0 : 1 : 0)\) or \((0 : 0 : 1)\) then the similar computation shows that \( \lambda = \pm \sqrt{-3} \). Therefore \( Y_3, Y_\sqrt{-3}, Y_-\sqrt{-3} \) are all possible \( Y_4 \) of type 3. \( \square \)

5.1.2 2\( A_7 + A_3 \)

Let \( L \) be the root lattice of the Dynkin graph \( 2A_7 + A_3 \). Let \( N = \mathbb{Z}\lambda \oplus L \). Denote the 17 generators of \( L \) by \( e_i (1 \leq i \leq 17) \) in a natural way such that \( e_ie_{i+1} = 1 \) for \( 1 \leq i \leq 6, 8 \leq i \leq 13 \) and \( 15 \leq i \leq 16 \). Let

\[
u_1 = \sum_{i=1}^{7} i(e_i + e_{i+1})/2,
\]

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be three elements in $N^\nu$. It is easy to verify that $u_i^2 \in 2\mathbb{Z}$ for $i = 1, 2, 3$. Let $M_i$ be the sublattice of $N^\nu$ generated by $u_i$ and $N$ for $i = 1, 2, 3$. Note that $N \subset M_1 \subset M_2 \subset M_3$. It can be verified that all these three overlattices of $N$ satisfy the conditions in Theorem [2.2]. Therefore there are three reduced sextic curves $C_1, C_2, C_3$ whose discriminantal groups are $M_1^\nu/M_1, M_2^\nu/M_2, M_3^\nu/M_3$ respectively. By the algorithm in [12] we determine that all these three curves have the same configuration. That is the union of an irreducible quartic curve with an $A_3$ singularity and a conic such that the conic meets the quartic at two points to form $2A_7$. Thus we have the following theorem.

**Theorem 5.3.** The collection of three sextic curves $\{C_1, C_2, C_3\}$ forms a Zariski triplet.

These three curves are distinguished by the following two conditions:
1) The three singularities on $C_3$ are collinear while those on $C_1$ or $C_2$ are not.
2) For $C_1$ and $C_2$ let $Q$ be the conic passing through the three singularities and infinitely near points at both $A_7$. Then $Q$ passes the infinitely near point of $A_3$ for $C_2$ but does not pass the one for $C_1$.

The equations are given as following:

$$C_1 : \begin{bmatrix} x_0^2 + ax_0x_1 + (\sqrt{2} - a)x_0x_2 + x_1x_2 \end{bmatrix}[2(\frac{\sqrt{2}}{2}a - 1)^2x_0^2(x_2 - x_1)^2 + c_{2,2}x_1^2x_2^2$$

$$+c_{3,0}x_0x_1^3 + c_{2,1}x_0x_1^2x_2 + c_{1,2}x_0x_1x_2^2 - c_{0,3}x_0x_2^3 + c_{3,1}x_1x_2 + c_{1,3}x_1x_2^3] = 0$$

where

- $c_{2,2} = 2 \sqrt{2}ab - 2a^2b - b + 2 \sqrt{2}a - 3$,
- $c_{3,0} = a^3b - 2 \sqrt{2}a^2 + 2a$,
- $c_{2,1} = 3 \sqrt{2}a^2b - 3a^3b - 2ab + 4 \sqrt{2}a^2 - 8a + 2 \sqrt{2}$,
- $c_{1,2} = 3a^3b - 6 \sqrt{2}a^2b + 8ab - 2 \sqrt{2}a^2 - 2 \sqrt{2}b + 6a - 2 \sqrt{2}$,
- $c_{0,3} = 3 \sqrt{2}a^2b - a^3b - 6ab + 2 \sqrt{2}b$,
- $c_{1,3} = a^2b - 2 \sqrt{2}ab + 2b$,
- $c_{3,1} = a^2b - 2 \sqrt{2}a + 2$. 

\[ \sum_{i=1}^{7} i(e_i + e_{i+7})/4 + (e_{15} + 2e_{16} + 3e_{17})/2, \]

$$u_3 = \lambda/2 + \sum_{i=1}^{7} i(e_i + e_{i+7})/8 + (e_{15} + 2e_{16} + 3e_{17})/4$$
in which \( a \) and \( b \) are generic parameters.

The \( A_7, A_7, A_3 \) are located at \((0 : 1 : 0), (0 : 0 : 1) \) and \((1 : 0 : 0) \) respectively.

\[ C_2 : (x_0^3 + ax_0x_1 - ax_0x_2 + x_1x_2)[(a^2x_0^2(x_2 - x_1))^2 + (1 - b - 2a^2b)x_1x_2^2 + a^3bx_0x_1^3 + a(2 - 2b - 3a^2b)(x_0x_1x_2 - x_0x_1x_2) - a^3bx_0x_2^3 + a^2bx_1x_2^2 + a^2bx_1x_3^3] = 0 \]

for generic parameters \( a \) and \( b \).

The \( A_7, A_7, A_3 \) are located at \((0 : 1 : 0), (0 : 0 : 1) \) and \((1 : 0 : 0) \) respectively. The conic \( ax_0x_1 - ax_0x_2 + x_1x_2 \) is tangent to the quartic component at \((1 : 0 : 0) \) and to the conic at both \((0 : 1 : 0) \) and \((0 : 0 : 1) \).

\[ C_3 : (x_0x_1 + x_2^3)(x_0^3x_1 + x_0^2x_2^2 + b^2x_0x_1x_2 - 2x_0^2x_1^2 + bx_0x_1^3 + bx_0x_2^3 + \left(\frac{b^2}{4} - 2\right) x_0x_1x_2 - bx_0x_1^2x_2 + x_0x_1^3 + ax_0^4 - bx_1x_2^3 + x_1^3x_2) = 0 \]

for generic parameters \( a \) and \( b \).

The three collinear \( A_7, A_7, A_3 \) points are located \((1 : 0 : 0), (0 : 1 : 0), (1 : 0 : 1) \) respectively.

5.1.3 2\( A_7 + A_3 + A_1 \)

Let \( L \) be the root lattice of the Dynkin graph \( 2A_7 + A_3 + A_1 \). Let \( N = \mathbb{Z}l \oplus L \). Denote the 18 generators of \( L \) by \( e_i (1 \leq i \leq 18) \) in a natural way such that \( e_ie_{i+1} = 1 \) for \( 1 \leq i \leq 6, 8 \leq i \leq 13 \) and \( 15 \leq i \leq 16 \). Let

\[ u_1 = \sum_{i=1}^{7} i(e_i + e_{i+7})/2, \]

\[ u_2 = \sum_{i=1}^{7} i(e_i + e_{i+7})/4 + (e_{15} + 2e_{16} + 3e_{17})/2, \]

\[ u_3 = \sum_{i=1}^{7} i(e_i + 3e_{i+7})/8 + (e_{15} + 2e_{16} + 3e_{17})/4 + e_{18}/2 \]

and

\[ v = l/2 + \sum_{i=1}^{7} i(e_{i+7})/2 + e_{18}/2 \]
be four elements in $N'$. It is easy to verify that $u_i^2 \in 2\mathbb{Z}$ for $i = 1, 2, 3$ and $v^2 \in 2\mathbb{Z}$. Let $M_i$ be the sublattice of $N'$ generated by $u_i, v$ and $N$ for $i = 1, 2, 3$. It can be verified that all these three overlattices of $N$ satisfy the conditions in Theorem 2.2. Therefore there are three reduced sextic curves $C'_1, C'_2, C'_3$ whose discriminantal groups are $M'_1/M_1, M'_2/M_2, M'_3/M_3$ respectively. By the algorithm in [12] we determine that all these three curves have the same configuration: the union of an irreducible quartic curve with an $A_3$ singularity and two lines such that the each line meets the quartic with intersection number 4. Thus we have the following

**Theorem 5.4.** The collection of three sextic curves $\{C'_1, C'_2, C'_3\}$ forms a Zariski triplet.

These three curves are distinguished by the following two conditions:

1) The three singularities on $C'_3$ are collinear while those on $C'_1$ or $C'_2$ are not.

2) For $C'_1$ and $C'_2$ let $Q$ be the conic passing through the $A_7, A_7, A_3$ and infinitely near points at both $A_7$. Then $Q$ passes the infinitely near point of $A_3$ for $C'_2$ but does not pass the one for $C'_1$.

The equations of the curves are given as following:

$$C'_1 : (x_1 - x_0)(x_2 - x_0)(\sqrt{-1}x_0^2x_1^2 + 2x_0^2x_1x_2 - \sqrt{-1}x_0^2x_2^2 - (2\lambda - 2\sqrt{-1})x_0x_2^3$$

$$-(2\lambda + 2 - 2\sqrt{-1})x_0x_1x_2^2 - (2\lambda + 2)x_0^2x_1x_2 - 2\lambda x_0x_1^3$$

$$+(2\lambda - 2\sqrt{-1})x_1x_2^3 + (2\lambda + 2 - \sqrt{-1})x_1^2x_2^2 + 2\lambda x_1^3x_2] = 0$$

for a generic parameter $\lambda$.

The $A_7, A_7, A_3$ and $A_1$ points are located at $(0 : 1 : 0), (0 : 0 : 1), (1 : 0 : 0)$ and $(1 : 1 : 1)$ respectively.

$$C'_2 : (x_1 - x_0)(x_2 - x_0)(x_0^2x_1^2 + 2x_0^2x_1x_2 + x_0^2x_2^2 - 2\lambda x_0x_1^3 - (2\lambda + 2)x_0x_1^2x_2$$

$$-(2\lambda + 2)x_0x_1x_2^2 - 2\lambda x_0x_2^3 + 2\lambda x_1^3x_2 + (2\lambda + 1)x_1^2x_2^2 + 2\lambda x_1^3x_2] = 0$$

for a generic parameter $\lambda$.

The $A_7, A_7, A_3, A_1$ points are located at $(0 : 1 : 0), (0 : 0 : 1), (1 : 0 : 0)$ and $(1 : 1 : 1)$ respectively. The conic $x_1x_2 - x_0x_1 - x_0x_2$ has intersection number 4 with $C'_2$ at $A_7$ and $A_3$ points.

$$C'_3 : x_1x_2(x_1^3x_2 - 2x_1^2x_2^2 + x_1^3x_2^2 + \lambda x_0^4) = 0$$

for a generic parameter $\lambda$.

The $A_7, A_7, A_3, A_1$ points are located at $(0 : 1 : 0), (0 : 0 : 1), (1 : 1 : 1)$ and $(1 : 0 : 0)$ respectively. The line $x_0 = 0$ passes through the two $A_7$ points and the $A_3$ point.
Let $L$ be the root lattice of $A_{15} + A_3$. Let $N = \mathbb{Z} \lambda \oplus L$. Denote the 18 generators of $L$ by $e_i (1 \leq i \leq 18)$ in a natural way such that $e_i e_{i+1} = 1$ for $1 \leq i \leq 14$ and $16 \leq i \leq 17$. Let

$$u_1 = \frac{1}{2} \sum_{i=1}^{15} ie_i,$$

$$u_2 = \frac{1}{4} \sum_{i=1}^{15} ie_i + \frac{e_{16} + 2e_{17} + 3e_{18}}{2},$$

$$u_3 = \frac{\lambda}{2} + \frac{1}{8} \sum_{i=1}^{15} ie_i + \frac{e_{16} + 2e_{17} + 3e_{18}}{4}$$

be three elements in $N^\vee$. It is easy to verify that $u_i^2 \in 2\mathbb{Z}$ for $i = 1, 2, 3$. Let $M_i$ be the sublattice of $N^\vee$ generated by $u_i$ and $N$ for $i = 1, 2, 3$. It can be verified that all these three overlattices of $N$ satisfy the conditions in Theorem 2.2. Therefore there are three reduced sextic curves $D_1, D_2, D_3$ whose discriminantal groups are $M_1^\vee / M_1, M_2^\vee / M_2$ and $M_3^\vee / M_3$ respectively. By the algorithm in [12] we determine that all these three curves have the same configuration. That is the union of an irreducible quartic curve with an $A_3$ singularity and a conic such that the conic meets the quartic at an $A_{15}$ point. Thus we have the following theorem

**Theorem 5.5.** The collection of three sextic curves $\{D_1, D_2, D_3\}$ forms a Zariski triplet.

These three curves are distinguished by the following two conditions:

1) The line passing $A_{15}$ and $A_3$ is tangent to the quartic component for $D_3$, but not for $D_1$ and $D_2$.

2) For $D_1$ and $D_2$, let $Q$ be the conic passing through the two singularities together with their infinitely near points such that the intersection number of the conic component and $Q$ at $A_{15}$ is at least 3. Then $Q \cap D_2 = \{A_{15}, A_3\}$ while $Q \cap D_1$ contains four distinct points.

The equations of the curves are:

$$D_1 : (2x_0x_2 + 2x_1^2 + x_2^2)[(3\lambda + 2)x_1^4 - 4x_0^2x_2 + \lambda x_0^2x_2^2 - 4\lambda x_0^2x_1x_2 - 4x_0^2x_1^2 - 2\lambda x_0x_1x_2^2 + (4\lambda + 4)x_0x_1^2x_2 - 4\lambda x_0x_1^3 + \lambda x_1^2x_2^2] = 0$$

for a generic parameter $\lambda$ and the $A_{15}, A_3$ are located at $(1:0:0)$ and $(0:0:1)$ respectively.
\[ D_2 : (x_0x_2 - x_1^2 + \lambda x_2^2)(x_0^3x_2 - x_0^2x_1^2 - 5\lambda x_1^4 + 10\lambda x_0x_2^2 - 4\lambda x_0^2x_1^2) = 0 \]
for a generic parameter \( \lambda \) and the \( A_{15}, A_3 \) are located at (1 : 0 : 0) and (0 : 0 : 1) respectively. the conic \( x_0x_2 - x_1^2 \) intersects the sextic at \( A_{15} \) and \( A_3 \) only.

\[ D_3 : (x_0x_2 + x_1^2 - x_1x_2 + x_2^2)(x_0^3x_2 + x_0^2x_2^2 + \lambda x_2^4 - x_0^2x_1x_2 + x_0^2x_2^2) = 0, \]
and the \( A_{15}, A_3 \) are located at (1 : 0 : 0) and (0 : 1 : 0) respectively. The line \( x_2 = 0 \) passes \( A_{15}, A_3 \) and is tangent to the conic component at (1 : 0 : 0).

### 5.2 More than one Zariski pairs for the same combination of singularities

Our search shows several occasions where more than one Zariski pairs pop up for the same combination of singularities. Here is a typical example.

**Theorem 5.6.** There are four Zariski pairs of reduced sextic curves with \( 3A_5 + 2A_1 \) as their singularities. Their configurations are as follows:

1) a quintic plus a line;
2) a quartic plus a conic;
3) two cubics;
4) a cubic plus a conic plus a line.

**Proof.** Let \( L \) be the root lattice of the Dynkin graph \( 3A_5 + 2A_1 \). Let \( N = \mathbb{Z}\lambda \oplus L \). Denote the 17 generators of \( L \) by \( e_i \) (1 \( \leq i \leq 17 \)) in a natural way such that \( e_ie_{i+1} = 1 \) for 1 \( \leq i \leq 4, 6 \leq i \leq 9 \) and 11 \( \leq i \leq 14 \). Let

\[ u_1 = (\lambda + \sum_{i=1}^{5} ie_i + e_{16} + e_{17})/2, \]
\[ u_2 = \lambda/2 + \sum_{i=1}^{5} ie_i/6 + \sum_{i=1}^{5} i(e_{i+5} + e_{i+10})/3 + (e_{16} + e_{17})/2, \]
\[ v_1 = \sum_{i=1}^{5} i(e_i + e_{i+5})/2 + (e_{16} + e_{17})/2, \]
\[ v_2 = \sum_{i=1}^{5} i(e_i + e_{i+5})/6 + \sum_{i=1}^{5} ie_{i+10}/3 + (e_{16} + e_{17})/2, \]
\[ w_1 = (\lambda + \sum_{i=1}^{5} i(e_i + e_{i+5} + e_{i+10}))/2. \]
Let $A_i$ be the sublattice of $N^\vee$ generated by $u_i$ and $N$ for $i = 1, 2$. Let $B_i$ be the sublattice of $N^\vee$ generated by $v_i$ and $N$ for $i = 1, 2$. Let $D_i$ be the sublattice of $N^\vee$ generated by $w_i$ and $N$ for $i = 1, 2$. Let $G_i$ be the sublattice of $N^\vee$ generated by $s_i, t_i$ and $N$ for $i = 1, 2$.

It can be verified that all these eight overlattices of $N$ satisfy the conditions in Theorem 2.2. Therefore there are eight reduced sextic curves $C_1, C_2, E_1, E_2, P_1, P_2, Q_1, Q_2$ whose discriminantal groups are

$$A_1^\vee/A_1, A_2^\vee/A_2, B_1^\vee/B_1, B_2^\vee/B_2, D_1^\vee/D_1, D_2^\vee/D_2, G_1^\vee/G_1, G_2^\vee/G_2$$

respectively.

By using the algorithm in [12] we determine the configurations of these curves are:

1) a quintic plus a line for $C_1$ and $C_2$;
2) a quartic plus a conic for $E_1$ and $E_2$;
3) two cubics for $P_1$ and $P_2$;
4) a cubic plus a conic plus a line for $Q_1$ and $Q_2$.

It follows that the pairs $\{C_1, C_2\}, \{E_1, E_2\}, \{P_1, P_2\}$ and $\{Q_1, Q_2\}$ are Zariski pairs. □

### 6 Other Zariski pairs with different discriminantal groups

Other Zariski pairs can be obtained using the method in the main text. They are listed below. The notations in the tables are briefly explained as follows.

The configuration of a sextic is described by the incidence table with its entries to be the local components of a simple singularites according to the convention in Figure 6.
For example,
\[
\begin{array}{c}
A_{17} \\
A_1
\end{array}
\begin{array}{c}
I \\
I,II
\end{array}
\]
means that the sextic has a nodal cubic and a smooth cubic as its irreducible components and they intersect at an \(A_{17}\) point.

The overlattices are denoted by the generators over the root lattice. The trivial lattice, i.e. the root lattice, is denote by “-”. A generator of the overlattice is represented by its image in \(L^\vee/L\) where \(L\) is the root lattice. As long as there is no singularity of type \(D_{2n}\), the discriminant group of every singularity is a cyclic group. Moreover, \((\mathbb{Z}\lambda)^\vee/\mathbb{Z}\lambda \cong \mathbb{Z}/2\mathbb{Z}\) is also cyclic. Hence a generator can be represented by a sequence of numerals of which the first corresponds to its component in \((\mathbb{Z}\lambda)^\vee/\mathbb{Z}\lambda\).

Take the example of \(D_7 + A_{11}\): the element 026 stands for \(2u + 6v\) where \(u\) and \(v\) are generators of \(L(D_7)^\vee/L(D_7)\) and \(L(A_{11})^\vee/L(A_{11})\) respectively.
The last column in the table is a special curve distinguishing two members of the Zariski pair. It is given by the degree of the curve followed by a sequence of intersection numbers with the sextic curve at singularities.
### Table 1. Milnor number 18

| singularities | configuration | overlattices | special curve |
|---------------|---------------|--------------|---------------|
| 3E_6          | irreducible   | -.0111       | 2,(E_6,4),(E_6,4),(E_6,4) |
| 2E_6 + A_5 + A_1 | irreducible   | -.01120      | 2,(E_6,4),(E_6,4),(A_5,4) |
| E_6 + A_11 + A_1 | irreducible   | -.0140       | 2,(E_6,4),(A_11,8) |
| E_6 + A_3 + A_2 + 2A_1 | irreducible | -.013100     | 2,(E_6,4),(A_3,6),(A_2,2) |
| E_6 + 2A_5 + 2A_1 | E_6 A_5 A_5 A_1 A_1 | 100311, 111211 | 2,(E_6,4),(A_5,5),(A_1,6) |
| E_6 + 2A_5 + 2A_1 | E_6 A_5 A_5 A_1 A_1 | 003311, 011111 | 2,(E_6,4),(A_5,2),(A_1,2) |
| D_7 + A_11     | D_7 A_11      | 026,013      | 2,(D_7,6),(A_11,6) |
| D_7 + A_7 + A_3 + A_1 | D_7 A_7 A_3 A_1 | 02420,10401 | 2,(D_7,6),(A_7,4),(A_3,2) |
| A_17 + A_1     | A_17 A_1      | -.060        | 2,(A_17,12) |
| A_17 + A_1     | A_17 A_1      | 190,130      | 1,(A_17,6) |
| A_14 + A_3 + 2A_1 | A_14 A_3 A_1 A_1 | -0.5100     | 2,(A_14,10),(A_2,2) |
| A_11 + A_5 + 2A_1 | A_11 A_5 A_2 A_1 A_1 | 10311, 14111 | 2,(A_11,8),(A_5,4) |
| A_11 + A_5 + 2A_1 | A_11 A_5 A_1 A_1 | 10311, 14111 | 2,(A_11,8),(A_5,4) |
| A_11 + A_5 + 2A_1 | A_11 A_5 A_2 A_1 A_1 | 10311, 14111 | 2,(A_11,8),(A_5,4) |
| A_11 + A_5 + 2A_1 | A_11 A_2 A_1 A_1 | 10311, 14111 | 1,(A_11,4),(A_5,2) |
| A_11 + A_5 + 2A_1 | A_11 A_3 A_1 A_1 | {060011,10311} | 2,(A_11,4),(A_5,4), (A_1,2), (A_1,2) |
| A_11 + A_5 + 2A_1 | A_11 A_2 A_1 A_1 | {02211,10311} | 2,(A_11,4),(A_2,2), (A_2,2), (A_1,2), (A_1,2) |
| A_11 + 2A_3 + 3A_1 | A_11 A_2 A_1 A_1 | 0600911, 0211011 | 2,(A_11,8),(A_2,2),(A_2,2) |
| A_11 + 2A_3 + 3A_1 | A_11 A_3 A_1 A_1 | 1600111, 1211111 | 2,(A_11,8),(A_2,2),(A_2,2) |
| 2A_6           | irreducible    | -.024        | 2,(A_6,4),(A_6,8) |
Table 1 (cont.)

| singularities | configuration | overlattices | special curve |
|---------------|---------------|--------------|---------------|
| 2A9           | A9 A9         | 105,121      | 1, (A9, 4), (A9, 2) |
| A9 + 2A4 + A1 | irreducible   | .02220       | 2, (A9, 4), (A9, 4), (A1, 4) |
| A9 + 2A4 + A1 | A9 A4 A4 A1  | 15000, 11110 | 1, (A9, 2), (A4, 2), (A1, 2) |
| 2A3 + 2A1     | irreducible   | .03300       | 2, (A3, 6), (A3, 6) |
| A8 + A5 + A2 + 3A1 | A8 A5 A5 A2 A1 A1 | 1030011, 1311011 | 2, (A8, 6), (A5, 4), (A2, 2) |
| 2A7 + A3 + A1 | A7 A7 A3 A1  | 01311, 12221 | 2, (A7, 6), (A7, 2), (A3, 2), (A1, 2) |
| 3A5 + 3A1     | A5 A5 A5 A5 A1 A1 | 0033011,1003101 | 2, (A5, 4), (A5, 2), (A5, 2) |
| 3A5 + 3A1     | 3 3 A5 A5 A5 A1 A1 | 0033011,1300011 | 2, (A5, 4), (A5, 2), (A5, 2) |

Table 2. Milnor number 17

| singularities | configuration | overlattices | special curve |
|---------------|---------------|--------------|---------------|
| 2E6 + A5      | irreducible   | -.0112       | 2, (E6, 4), (E6, 4), (A5, 4) |
| 2E6 + 2A2 + A1| irreducible   | -.01110      | 2, (E6, 4), (E6, 4), (A2, 2) |
| E6 + A11      | irreducible   | .014         | 2, (E6, 4), (A11, 8) |
| E6 + A8 + A2 + A1 | irreducible | -.01310 | 2, (E6, 4), (A8, 6), (A2, 2) |
| E6 + 2A5 + A1 | irreducible   | -.01220      | 2, (E6, 4), (A5, 4), (A5, 4) |
| E6 + A3 + 2A2 + 2A1 | irreducible | -.012100 | 2, (E6, 4), (A3, 2), (A2, 2) |
| E6 + A5 + 2A2 + 2A1 | E6 A5 A5 A2 A1 A1 | 1030011,1111111 | 2, (E6, 4), (A5, 4), (A2, 2) |
| D7 + A3 + A1  | D7 A3 A3     | 0242,0121    | 2, (D7, 6), (A3, 4), (A1, 2) |
| D7 + 3A3 + A1 | D7 A3 A3 A3 A1 | 022220,100221 | 2, (D7, 6), (A3, 2) |
| 2D3 + A7      | D3 D3 A7     | 0224,0112    | 2, (D3, 4), (D3, 4), (A7, 4) |
| singularities         | configuration | overlattices               | special curve |
|----------------------|---------------|-----------------------------|---------------|
| 2D₃ + 2A₃ + A₁       | D₂, D₂, A₃, A₃, A₁ | {022220,102021, 011110,102021} | 2, (A₁, 2), (A₁, 2) |
| D₅ + A₁₁ + A₁       | D₅, A₁₁, A₁ | 0260,0131                  | 2, (D₅, 4), (A₁₁, 6), (A₁, 2) |
| D₅ + A₇ + A₃ + 2A₁   | D₅, A₇, A₃, A₁, A₁ | {024200,104001, 012110,104001} | 2, (A₁, 2), (A₁, 2) |
| A₁₇                  | irreducible   | -.06                        | 2, (A₁₇, 12) |
| A₁₇                  | 3 I           | 19,13                       | 1, (A₁₇, 6) |
| A₁₅ + 2A₁            | A₁₅, A₁, A₁ | 0800,0411                  | 2, (A₁₅, 8), (A₂, 2), (A₁, 2) |
| A₁₄ + A₂ + A₁        | irreducible   | -.0510                      | 2, (A₁₄, 10), (A₂, 2) |
| A₁₁ + A₅ + A₁       | irreducible   | -.0420                      | 2, (A₁₁, 8), (A₅, 4) |
| A₁₁ + A₅ + A₁       | 3 I I I, I, II | 1630,1210                  | 1, (A₁₁, 4), (A₅, 2) |
| A₁₁ + 2A₁            | A₁₁, A₃, A₃ | 0620,0312                  | 2, (A₁₁, 6), (A₃, 4), (A₁, 2) |
| A₁₁ + 2A₁ + 2A₁      | irreducible   | -.041100                    | 2, (A₁₁, 8), (A₂, 2), (A₂, 2) |
| A₁₁ + 2A₁ + 2A₁      | A₁₁, A₂, A₂, A₁, A₁ | 060011,021111           | 2, (A₁₁, 4), (A₂, 2), (A₂, 2) |
| A₉ + 2A₄             | irreducible   | -.0222                      | 2, (A₉, 4), (A₄, 4), (A₄, 4) |
| A₉ + 2A₄             | 5 I I I, I    | 1500,1111                  | 1, (A₉, 2), (A₄, 2), (A₄, 2) |
| 2A₄ + A₁             | irreducible   | -.0330                      | 2, (A₄, 6), (A₆, 6) |
| A₈ + A₅ + A₂ + 2A₁   | irreducible   | -.032100                    | 2, (A₈, 6), (A₅, 4), (A₂, 2) |
| A₈ + A₅ + A₂ + 2A₁   | A₈, A₅, A₂, A₁, A₁ | 103011,131111         | 2, (A₈, 6), (A₅, 4), (A₂, 2) |
| A₈ + 3A₂ + 3A₁       | irreducible   | -.0311000                   | 2, (A₈, 6), (A₂, 2), (A₂, 2) |
| 2A₇ + 3A₁            | A₇, A₇, A₁, A₁ | {044000,104001, 022110,104001} | 2, (A₇, 4), (A₇, 4) |

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Table 2 (cont.)

| singularities | configuration | overlattices | special curve |
|---------------|---------------|--------------|---------------|
| $A_7 + 3A_3 + A_1$ | $A_7$ $A_3$ $A_3$ $A_3$ $A_1$ | {040220,1002221} | 2. $(A_7,4), (A_3,4)$ |
| | | {021120,102201} | $(A_1,2), (A_1,2)$ |
| | | 10300011,112111011 | 2. $(A_3,2), (A_5,4), (A_2,2)$ |
| $2A_5 + 2A_2 + 3A_1$ | $A_5$ $A_5$ $A_2$ $A_2$ $A_1$ $A_1$ | 03300011,01111011 | 2. $(A_5,2), (A_5,2), (A_2,2)$ |
| | | 13300111,11111111 | $(A_2,2), (A_2,2)$ |
| | | {03300011,10300101} | 2. $(A_2,2), (A_5,2), (A_2,2)$ |
| | | {01111011,10300101} | $(A_2,2), (A_5,2), (A_2,2)$ |
| $4A_4 + A_1$ | irreducible | -011220 | 2. $(A_4,4), (A_5,4)$ |
| | | | $(A_4,2), (A_1,2)$ |

Table 3. Milnor number 16

| singularities | configuration | overlattices | special curve |
|---------------|---------------|--------------|---------------|
| $2E_6 + 2A_2$ | irreducible | -0111 | 2. $(E_6,4), (E_6,4)$ |
| | | | $(A_2,2), (A_2,2)$ |
| $E_6 + A_3 + A_2$ | irreducible | -0131 | 2. $(E_6,4), (A_6,6), (A_2,2)$ |
| $E_6 + 2A_2$ | irreducible | -0122 | 2. $(E_6,4), (A_2,4), (A_2,4)$ |
| $E_6 + A_3 + 2A_2 + A_1$ | irreducible | -012110 | 2. $(E_6,4), (A_5,4)$ |
| | | | $(A_2,2), (A_2,2)$ |
| $E_6 + 4A_2 + 2A_1$ | irreducible | -0111100 | 2. $(E_6,4), (A_2,2), (A_2,2)$ |
| | | | $(A_2,2), (A_2,2)$ |
| $D_5 + 3A_3$ | $D_5$ $A_3$ $A_3$ $A_3$ | 022220,011111 | 2. $(D_5,6), (A_5,2)$ |
| | | | $(A_2,2), (A_2,2)$ |
| $2D_5 + 2A_3$ | $D_5$ $D_5$ $A_3$ $A_3$ | 022220,011111 | 2. $(D_5,4), (D_5,4)$ |
| | | | $(A_3,2), (A_3,2)$ |
| $D_5 + A_3 + A_3 + A_1$ | $D_5$ $A_7$ $A_3$ $A_1$ | 02420,01211 | 2. $(D_5,4), (A_3,4)$ |
| | | | $(A_3,2), (A_3,2)$ |
| $D_5 + 3A_3 + 2A_1$ | $D_5$ $A_3$ $A_3$ $A_3$ $A_1$ $A_1$ | {0222200,1002201} | 2. $(D_5,4), (A_3,2), (A_3,2)$ |
| | | | {0111101,1002201} | $(A_1,2), (A_1,2)$ |

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| singularities | configuration | overlattices | special curve |
|---------------|---------------|--------------|---------------|
| $A_{14} + A_3$ | irreducible | -0.051 | 2, $(A_{14}, 10), (A_3, 2)$ |
| $A_{11} + A_5$ | irreducible | -0.042 | 2, $(A_{11}, 8), (A_5, 4)$ |
| $A_{11} + A_3$ | $A_1$ $A_3$ $A_3$ $A_5$ | 3 I I | 1, $(A_{11}, 4), (A_3, 2)$ |
| $A_{11} + A_3 + 2A_1$ | $A_1$ $A_3$ $A_3$ $A_1$ $A_1$ | 4 I I I, I, I, I, I | 2, $(A_{11}, 6), (A_3, 2), (A_1, 2)$ |
| $A_{11} + 2A_3 + A_1$ | irreducible | -0.04110 | 2, $(A_{11}, 8), (A_3, 2), (A_1, 2)$ |
| $2A_5$ | irreducible | -0.033 | 2, $(A_5, 6), (A_3, 6)$ |
| $A_5 + A_5 + A_2 + A_1$ | irreducible | -0.03210 | 2, $(A_5, 6), (A_3, 4), (A_2, 2)$ |
| $A_5 + 3A_2 + 2A_1$ | irreducible | -0.031100 | 2, $(A_5, 6), (A_3, 2), (A_2, 2), (A_1, 2)$ |
| $2A_7 + 2A_1$ | $A_7$ $A_7$ $A_1$ $A_1$ $A_1$ | 4 I I I, I, I, I | 0.04400, 0.02211 |
| $A_7 + 3A_3$ | $A_7$ $A_3$ $A_3$ $A_3$ | 4 I I, I, I, I | 0.04022, 0.02112 |
| $A_7 + 2A_3 + 3A_1$ | $A_7$ $A_3$ $A_3$ $A_1$ $A_1$ $A_1$ | 4 I I I, I, I, I | (0.042200, 0.102200) |
| $3A_3 + A_1$ | irreducible | -0.02220 | 2, $(A_3, 4), (A_3, 4), (A_1, 2)$ |
| $3A_3 + A_1$ | $A_3$ $A_3$ $A_3$ $A_3$ | 3 I I I I, II | 1, $(A_3, 2), (A_3, 2), (A_1, 2)$ |
| $2A_5 + 2A_2 + 2A_1$ | irreducible | -0.022100 | 2, $(A_5, 6), (A_3, 4), (A_2, 2)$ |
| $2A_5 + 2A_2 + 2A_1$ | $A_5$ $A_5$ $A_2$ $A_2$ $A_2$ $A_2$ | 5 I, I, I, I, I, I | 0.103001, 0.112111 |
| $2A_5 + 2A_2 + 2A_1$ | $A_5$ $A_2$ $A_2$ $A_2$ $A_2$ $A_2$ | 4 I I I I I I | 0.033001, 0.011111 |
| $A_5 + 4A_2 + 3A_1$ | irreducible | -0.02111000 | 2, $(A_5, 2), (A_2, 2), (A_1, 2)$ |
| $A_5 + 4A_2 + 3A_1$ | $A_5$ $A_2$ $A_2$ $A_2$ $A_2$ $A_2$ | 5 I I I I I I | 0.1300001, 0.111111011 |
| $4A_3$ | irreducible | -0.02211 | 2, $(A_5, 4), (A_3, 4), (A_2, 2),$ |
| $5A_3 + A_1$ | $A_3$ $A_3$ $A_3$ $A_3$ $A_3$ | 4 I I I I I I | (0.002220, 0.000221) |

Table 3 (cont.)
Table 4. Milnor number 15 or less

| singularities | configuration | overlattices | special curve                  |
|---------------|---------------|--------------|--------------------------------|
| $E_6 + A_3 + 2A_2$ | irreducible   | -0.1211      | 2, $(E_6, 4), (A_5, 4), (A_2, 2)$ |
| $E_6 + 4A_2 + A_1$ | irreducible   | -0.011110    | 2, $(E_6, 4), (A_2, 2), (A_2, 2)$ |
| $D_5 + 3A_3 + A_1$ |                | 022220,011111 | 2, $(D_5, 4), (A_3, 2), (A_3, 2)$ |
| $A_{11} + 2A_2$   | irreducible   | -0.0411      | 2, $(A_{11}, 8), (A_2, 2), (A_2, 2)$ |
| $A_8 + A_3 + A_2$ | irreducible   | -0.0321      | 2, $(A_8, 6), (A_5, 4), (A_2, 2)$ |
| $A_8 + 3A_2 + A_1$ | irreducible   | -0.03110     | 2, $(A_8, 6), (A_2, 2), (A_2, 2)$ |
| $A_7 + 2A_3 + 2A_1$ |              | 042200,021111 | 2, $(A_7, 4), (A_5, 2), (A_3, 2)$ |
| $3A_4$         | irreducible   | -0.0222      | 2, $(A_4, 4), (A_5, 4), (A_5, 4)$ |
| $3A_5$         |                | 1333,1111    | 1, $(A_5, 2), (A_5, 2), (A_5, 2)$ |
| $2A_3 + 2A_2 + A_1$ | irreducible   | -0.022110    | 2, $(A_5, 4), (A_5, 4), (A_2, 2), (A_2, 2)$ |
| $A_5 + 4A_2 + 2A_1$ | irreducible   | -0.021110    | 2, $(A_5, 4), (A_2, 2), (A_2, 2)$ |
| $A_5 + 4A_2 + 2A_1$ |                | 13000011,1111111 | 2, $(A_5, 4), (A_2, 2), (A_2, 2)$ |
| singularities | configuration | overlattices | special curve |
|--------------|--------------|-------------|---------------|
| $5A_3$       | $A_3$ $A_3$ $A_3$ $A_3$ $A_3$ | 002222,0111112 | 2, $(A_3, 4), (A_3, 2), (A_3, 2), (A_3, 2)$ |
| $4A_3 + 3A_1$ | $A_3$ $A_3$ $A_3$ $A_1$ $A_1$ | {02222000, 10022001} | 2, $(A_3, 2), (A_3, 2), (A_3, 2), (A_3, 2)$ |
| $6A_2 + 3A_1$ | irreducible | -011111000 | 2, $(A_2, 2), (A_2, 2), (A_2, 2), (A_2, 2)$ |
| $E_6 + 4A_2$ | irreducible | -011111 | 2, $(E_6, 4), (A_2, 2), (A_2, 2)$ |
| $A_8 + 3A_2$ | irreducible | -03111 | 2, $(A_8, 6), (A_2, 2), (A_2, 2)$ |
| $2A_3 + 2A_2$ | irreducible | -02211 | 2, $(A_3, 4), (A_2, 2), (A_2, 2)$ |
| $A_5 + 4A_2 + A_1$ | irreducible | -0211110 | 2, $(A_5, 4), (A_2, 2), (A_2, 2)$ |
| $4A_3 + 2A_1$ | $A_3$ $A_3$ $A_3$ $A_1$ $A_1$ | 0222200, 0111111 | 2, $(A_3, 2), (A_3, 2), (A_3, 2), (A_3, 2)$ |
| $6A_2 + 2A_1$ | irreducible | -011111000 | 2, $(A_2, 2), (A_2, 2), (A_2, 2), (A_2, 2), (A_2, 2)$ |
| $A_5 + 4A_2$ | irreducible | -021111 | 2, $(A_5, 4), (A_2, 2), (A_2, 2), (A_2, 2), (A_2, 2)$ |
| $6A_2 + A_1$ | irreducible | -011111000 | 2, $(A_2, 2), (A_2, 2), (A_2, 2)$ |
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