Group Partitions via Commutativity and Related Topics

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Abstract

Let $G$ be a nonabelian group, $A \subseteq G$ an abelian subgroup and $n \geq 2$ an integer. We say that $G$ has an $n$-abelian partition with respect to $A$, if there exists a partition of $G$ into $A$ and $n$ disjoint commuting subsets $A_1, A_2, \ldots, A_n$ of $G$, such that $|A_i| > 1$ for each $i = 1, 2, \ldots, n$. We first classify all nonabelian groups, up to isomorphism, which have an $n$-abelian partition for $n = 2, 3$. Then, we provide some formulas concerning the number of spanning trees of commuting graphs associated with certain finite groups. Finally, we point out some ways to finding the number of spanning trees of the commuting graphs of some specific groups.

Keywords: $n$-abelian partition, AC-group, projective special linear group, commuting graph, spanning tree, Laplacian matrix.

1 Introduction and Motivation

Let $\Gamma$ be a simple graph and $m, n$ two non-negative integers. We say that $\Gamma$ is $(m, n)$-partitionable if its vertex set can be partitioned into $m$ independent sets $I_1, \ldots, I_m$ and $n$ cliques $C_1, \ldots, C_n$; that is

$$V_{\Gamma} = I_1 \uplus I_2 \uplus \cdots \uplus I_m \uplus C_1 \uplus C_2 \uplus \cdots \uplus C_n.$$  

Such a partition of $V_{\Gamma}$ is called a $(m, n)$-partition of $\Gamma$ (see [5]). We shall note some special cases: $(1, 1)$-partitionable graphs are called split graphs (see [8]), $(1, 0)$-partitionable graphs are called edgeless graphs, $(0, 1)$-partitionable graphs are called complete graphs. In particular, in the case when $m = 0$ or $n = 0$, we essentially split $\Gamma$ into $n$ cliques, $V_{\Gamma} = C_1 \uplus C_2 \uplus \cdots \uplus C_n$, or $m$ independent sets, $V_{\Gamma} = I_1 \uplus I_2 \uplus \cdots \uplus I_m$, respectively.

We now focus our attention on a graph associated with a finite group- the so-called commuting graph. Let $G$ be a finite group and $X$ a nonempty subset of $G$. The commuting graph $\mathcal{C}(X)$, has $X$ as its vertex set with two distinct elements of $X$ joined by an edge when they commute in $G$. Commuting graphs have been

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investigated by many authors in various contexts, see for instance \cite{6, 7, 11}. Clearly, \( \mathcal{C}(G) \) is \((0, n)\)-partitionable if and only if \( G \) can be partitioned into \( n \) commuting subsets. This suggests the following definition.

**Definition 1.1** Let \( G \) be a nonabelian group, \( A \subseteq G \) an abelian subgroup and \( n \geq 2 \) an integer. We say that \( G \) has an \( n \)-abelian partition with respect to \( A \), if there exists a partition of \( G \) into \( A \) and \( n \) disjoint commuting subsets \( A_1, A_2, \ldots, A_n \) of \( G \), \( G = A \uplus A_1 \uplus A_2 \uplus \cdots \uplus A_n \), such that \( |A_i| > 1 \) for each \( i = 1, 2, \ldots, n \).

Note that, the condition \( n \geq 2 \) in Definition 1.1 is needed, because if \( n = 1 \), we have \( G = A \uplus A_1 \), and so \( G = \langle A_1 \rangle \). Since \( A_1 \) is a commuting set, this would imply \( G \) is abelian, which is not the case. On the other hand, the structure of groups \( G \) which have an \( n \)-abelian partition for \( n = 2, 3 \), is obtained (Theorems 3.1 and 3.2).

**Remark.** Let \( Z = Z(G) \). If \( |Z| \geq 2 \), \( |G : Z| = 1+n \geq 4 \) and \( T = \{x_0, x_1, \ldots, x_n\} \) is a transversal for \( Z \) in \( G \), where \( x_0 \in Z \), then, as the cosets of the centre are abelian subsets of \( G \), we have the following \( n \)-abelian partition for \( G \) with respect to \( Z \):

\[
G = Z \uplus Zx_1 \uplus Zx_2 \uplus \cdots \uplus Zx_n.
\]

However, there are centerless groups \( G \) for which there is no \( n \)-abelian partition with respect to an abelian subgroup, for every \( n \). For example, consider the symmetric group \( S_3 \) on 3 letters.

Clearly, when \( 1 \in X \), \( \mathcal{C}(X) \) is connected, so one can talk about the number of spanning trees (or tree-number) of this graph, which is denoted by \( \kappa(X) \). Moreover, it is easy to see that \( X \) is a commuting subset of \( G \) (i.e., all pairs of its elements commute) if and only if \( \mathcal{C}(X) \) is a complete graph. Thus, if \( X \) is a commuting subset of \( G \), then by Cayley’s formula we obtain \( \kappa(X) = |X||X|-2 \).

In \cite{11}, in particular, it was proved that, for every Frobenius group \( G \) with core \( F \) and complement \( C \), \( \kappa(G) = \kappa(F) \kappa(C)|F| \). Moreover, it was shown in \cite{11} that the alternating group \( A_5 \) can be characterized by \( \kappa(A_5) \) in the class of nonsolvable groups. Here, we are interested in the problem of finding the \( \kappa(G) \) for some specific groups \( G \) (Theorems \ref{thm4.2}-\ref{thm4.4} Corollaries \ref{cor4.5} \ref{cor4.6} and Table \ref{table1}).

The outline of the paper is as follows. In Section 2, we provide a number of basic results related to the tree-numbers. In Section 3, the structure of groups \( G \) which have a \( n \)-abelian partition for \( n \in \{2, 3\} \), is obtained. Finally, in Section 4, some explicit formulas for the tree-numbers of commuting graphs associated with certain groups are obtained.

All notation and terminology for groups and graphs are standard. In addition, the spectrum \( \omega(G) \) of a finite group \( G \) is the set of its element orders. It is closed under divisibility relation and so determined uniquely through the set \( \mu(G) \) of those elements in \( \omega(G) \) that are maximal under the divisibility relation. Following S. M. Belcastro and G. J. Sherman \cite{2} we denote by \( \#\text{Cent}(G) \) the number of distinct centralizers in \( G \). We shall say that a group \( G \) is \( n \)-centralizer if \( \#\text{Cent}(G) = n \).
2 Auxiliary Results

Let $G$ be a nonabelian group and $A \subset G$ an abelian subgroup. If $G$ has an $n$-abelian partition with respect to $A$, $G = A \cup A_1 \cup \cdots \cup A_n$, then we have
\[
\mathcal{C}(A) = K_{|A|}, \quad \mathcal{C}(A_i \cup \{1\}) = K_{|A_i|+1}, \quad i = 1, 2, \ldots, n.
\]
Therefore, by Corollary 2.7 in [11], we get:
\[
\kappa(G) \geq |A|^{|A|-2} \prod_{i=1}^{n} (|A_i| + 1)^{|A_i|-1}.
\]
In particular, if the center $Z(G)$ of the group $G$ is nontrivial, and
\[
G = Z(G) \cup Z(G)x_1 \cup Z(G)x_2 \cup \cdots \cup Z(G)x_n,
\]
is an $n$-abelian partition for $G$ (a coset decomposition of $G$), then
\[
\kappa(G) \geq |Z(G)|^{2|Z(G)|-2} (|Z(G)| + 1)^{|Z(G)|-1}.
\]

A noncommuting set of a group $G$ (i.e., an independent set in commuting graph $C(G)$) has the property that no two of its elements commute under the group operation. We denote by $nc(G)$ the maximum cardinality of any noncommuting set of $G$ (the independence number of $C(G)$). Denote by $k(G)$ the number of distinct conjugacy classes of $G$. If $G$ has an $n$-abelian partition, then the pigeon-hole principle gives $nc(G) \leq n + 1$. Thus, by Corollary 2.2 (a) in [3], we obtain
\[
|G| \leq nc(G) \cdot k(G) \leq (n + 1)k(G),
\]
which immediately implies that
\[
n \geq \left\lfloor \frac{|G|}{k(G)} \right\rfloor - 1.
\]
Therefore, we have found a lower bound for $n$, when $k(G)$ is known.

Some Examples. Let $G = L_2(q)$, where $q \geq 4$ is a power of 2. We know that $|G| = q(q^2 - 1)$ and $k(L_2(q)) = q + 1$. Thus, if $G$ has an $n$-abelian partition, then by Eq. (1), we get $n \geq q^2 - q - 1$. In particular, since $A_5 \cong L_2(4)$, if $A_5$ has an $n$-abelian partition, then $n \geq 11$. In fact, $A_5$ has a 20-abelian partition, as follows:
\[
A_5^\# = A \cup A_1^\# \cup A_2^\# \cup \cdots \cup A_20^\#,
\]
where $A_i^\# = A_i \setminus \{1\}$, for every $i$, and
\[
A, A_1, \ldots, A_5 \text{ are Sylow 5-subgroups of order 5},
A_6, A_7, \ldots, A_{15} \text{ are Sylow 3-subgroups of order 3},
A_{16}, A_{17}, \ldots, A_{20} \text{ are Sylow 2-subgroups of order 4}.
\]
Similarly, if $G_1 = \text{GL}(2, q)$ and $G_2 = \text{GL}(3, q)$, $q$ a prime power, then we have

$$|G_1| = (q^2 - q)(q^2 - 1)$$

and

$$k(G_1) = q^2 - 1,$$

while

$$|G_2| = (q^3 - 1)(q^3 - q)(q^3 - q^2)$$

and

$$k(G_2) = q^3 - q.$$

Again, if $G_i$ has an $n_i$-abelian partition, for $i = 1, 2$, by Eq. (1), we obtain

$$n_1 \geq q(q - 1) - 1$$

and

$$n_2 \geq q^2(q^3 - 1)(q - 1) - 1.$$
Lemma 2.2 Let $\Gamma$ be any graph on $n$ vertices with Laplacian spectrum $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. If $m$ is an integer, then the following product

$$(\mu_1 + m)(\mu_2 + m) \cdots (\mu_{n-1} + m),$$

is also an integer which is divisible by $m$.

Proof. Consider the characteristic polynomial of the Laplacian matrix $L = L_\Gamma$:

$$\sigma(\Gamma; \mu) = \det(\mu I - L) = \mu^n + c_1 \mu^{n-1} + \cdots + c_{n-1} \mu + c_n.$$  

First, we observe that the coefficients $c_i$ are integers \cite[Theorem 7.5]{4}, and in particular, $c_n = 0$. This forces $\sigma(\Gamma; -m)$ is an integer, which is divisible by $m$. Moreover, we have

$$\sigma(\Gamma; \mu) = (\mu - \mu_1)(\mu - \mu_2) \cdots (\mu - \mu_n),$$

and since $\mu_n = 0$, we obtain

$$\sigma(\Gamma; -m) = (-1)^m (\mu_1 + m)(\mu_2 + m) \cdots (\mu_{n-1} + m).$$

The result now follows. \qed

A universal vertex is a vertex of a graph that is adjacent to all other vertices of the graph.

Lemma 2.3 Let a graph $\Gamma$ with $n$ vertices contain $m < n$ universal vertices. Then $\kappa(\Gamma)$ is divisible by $n^m - 1$.

Proof. Let $W$ be the set of universal vertices. Clearly, $\Gamma = K_m \vee (\Gamma - W)$. Since the Laplacian for the complete graph $K_m$ has eigenvalue 0 with multiplicity 1 and eigenvalue $m$ with multiplicity $m - 1$, it follows by Lemma 2.1 that $L_\Gamma$ has eigenvalue $n$ with multiplicity at least $m$. The result is now immediate from Lemma 2.2 and Eq. (2). \qed

It is easy to see that for a group $G$, its center consists of the universal vertices of $C(G)$. Therefore, if $Z(G)$ is of order $m$, then we have

$$C(G) = K_m \vee \Delta(G),$$

where $\Delta(G) = C(G \setminus Z(G))$. The following corollary is now immediate from Lemma 2.3

Corollary 2.4 Let $G$ be a nonabelian group of order $n$ with the center of order $m$. Then $\kappa(G)$ is divisible by $n^{m-1}$.

We add further information about the commuting graph $C(G)$. Let $G$ be a nonabelian group with $|G| = n$ and $|Z(G)| = m$. Then $\Delta = \Delta(G)$ is a graph on $\nu = n - m > 1$ vertices. For the characteristic polynomial of the Laplacian matrix $L_\Delta$, we shall write

$$\sigma(\Delta; \mu) = \det(\mu I - L_\Delta) = \mu^\nu + c_1 \mu^{\nu-1} + \cdots + c_{\nu-1} \mu.$$  

(note that $c_{\nu} = 0$.)
If \( \text{Spec}(L_\Delta) = (\mu_1, \mu_2, \ldots, \mu_{\nu-1}, 0) \), then we may write
\[
\sigma(\Delta; \mu) = \mu(\mu - \mu_1)(\mu - \mu_2) \cdots (\mu - \mu_{\nu-1}).
\]
Moreover, by Lemma 2.1, the eigenvalues of Laplacian matrix \( L_{C(G)} \) are:
\[
\begin{align*}
&n, n, \ldots, n, \\
&m - 1, &m + \mu_1, &m + \mu_2, \ldots, &m + \mu_{\nu-1}, 0.
\end{align*}
\]
It follows immediately using Eq. (2) that
\[
\kappa(G) = n^{m-1}(\mu_1 + m)(\mu_2 + m) \cdots (\mu_{\nu-1} + m),
\]
or equivalently
\[
\kappa(G) = n^{m-1}(-1)^\nu \sigma(\Delta; -m)/m.
\]
As \( \sigma(\Delta; -m)/m \) is an integer, so \( n^{m-1} \) divides \( \kappa(G) \).

As pointed out in the Introduction, if a group \( G \) has the nontrivial center \( Z = Z(G) \) and \( |G : Z| = 1 + n \geq 4 \), then the coset decomposition
\[
G = \bigcup_{i=0}^{n} Z x_i,
\]
is an \( n \)-abelian partition of \( G \). Let \( |G| = n \) and \( |Z| = m > 1 \). Put \( t = n/m \), and
\[
C_0(G) = K_m \vee (t - 1) K_m.
\]
Clearly, \( C_0(G) \) is a subgraph of \( C(G) \), and so \( \kappa(G) \geq \kappa(C_0(G)) \).

**Corollary 2.5** With the above notation, we have
\[
\kappa(G) \geq n^{m-1} m n^{m-1} \frac{2^{t-1}(t-1)(m-1)}{t-2}.
\]

**Proof.** By Lemma 2.1, the eigenvalues of Laplacian matrix \( L_{C_0(G)} \) are:
\[
\begin{align*}
&n, n, \ldots, n, \\
&m - 1, &2m, &2m, \ldots, &2m, &m, \ldots, &m, 0.
\end{align*}
\]
Using Eq. (2) and simple computations, we obtain
\[
\kappa(G) \geq \kappa(C_0(G)) = n^{m-1} m n^{m-1} \frac{2^{t-1}(t-1)(m-1)}{t-2},
\]
and the result follows. \( \square \)

**Lemma 2.6** [13, Lemma 4.1] Let \( \{g_1, \ldots, g_m\} \) be a largest noncommuting subset of \( G \). Then \( \cap_{i=1}^{m} C_G(g_i) \) is an abelian subgroup of \( G \).

**Proof.** Assume the contrary and choose \( a, b \in \cap_{i=1}^{m} C_G(g_i) \) such that \( ab \neq ba \). Then it is easy to see that \( \{a, bg_1, bg_2, \ldots, bg_m\} \) is a noncommuting subset of \( G \), a contradiction. \( \square \)
3 Groups having an $n$-abelian partition

Given a finite group $G$, we denote by $\text{cs}(G)$ the set of conjugacy class sizes of $G$. Itô proved that \cite{10} Theorem 1] if $\text{cs}(G) = \{1, m\}$, then $G$ is a direct product of a Sylow $p$-group of $G$ with an abelian group. In particular, then $m$ is a power of $p$.

Theorem 3.1 The following conditions on a nonabelian group $G$ are equivalent:

1. $G$ has a 2-abelian partition with respect to an abelian subgroup $A$.
2. $G = P \times Q$, where $P \in \text{Syl}_2(G)$ with $P/Z(P) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $Q$ is abelian, and $A = \langle Z(G), t \rangle$, where $t$ is an involution outside of $Z(G)$.

Proof. $(1) \Rightarrow (2)$ Suppose that $G$ is a nonabelian group, which has a 2-abelian partition $G = A \cup A_1 \cup A_2$. First of all, we notice that every noncommuting set of $G$ can have at most three elements. Now fix a noncentral element $x$ of $G$. Since $C_G(x) < G$, we can choose $y \in G$, such that $x$ and $y$ do not commute. It is well known that a group cannot be written as the union of two proper subgroups, thus $C_G(x) \cup C_G(y) < G$, and so we can choose $z$ in $G$, such that $B = \{x, y, z\}$ is a noncommuting set of $G$. Now, we have

$$G = C_x \cup C_y \cup C_z,$$

where $C_x = C_G(x)$, $C_y = C_G(y)$ and $C_z = C_G(z)$. Put $K = C_x \cap C_y \cap C_z$, which is an abelian subgroup of $G$, by Lemma 2.6. Indeed, by a result of Scorza \cite{14}, we have

(a) $[G : C_x] = [G : C_y] = [G : C_z] = 2$,
(b) $K = C_x \cap C_y = C_x \cap C_z = C_y \cap C_z$, and
(c) $K$ is a normal subgroup of $G$ and the factor group $G/K$ is isomorphic to the Klein Four Group.

Thus $|x^G| = 2$, and since $x \in G \setminus Z(G)$ was arbitrary, it follows that $\text{cs}(G) = \{1, 2\}$. By Itô’s result \cite{10} Theorem 1], $G = P \times Q$, where $P \in \text{Syl}_2(G)$ is a nonabelian and $Q \leq Z(G)$.

On the other hand, $B$ is a maximal noncommuting set of $G$, which forces $C_t \setminus K$ to be a commuting set of $G$ for each $t \in B$, and so the centralizer $C_t$ is abelian, because $C_t = \langle C_t \setminus K \rangle$. This implies that $K = Z(G)$, and so

$$\frac{P}{Z(P)} \cong \frac{P \times Q}{Z(P) \times Q} = \frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2,$$

and the proof is complete.

$(2) \Rightarrow (1)$ Let $\{t_1, t_2, t_3, t_4\}$ be a transversal for $Z(G)$ in $G$, with $t_1 \in Z(G)$. Then, $G$ is a disjoint union:

$$G = Z(G) \cup Z(G)t_2 \cup Z(G)t_3 \cup Z(G)t_4.$$
Put $A = Z(G) \cup Z(G)t_2$, $A_1 = Z(G)t_3$ and $A_2 = Z(G)t_4$. Then $A$ is an abelian group (since $t_2^3 \in Z(G)$), $A_1$ and $A_2$ are commuting sets, and $G = A \cup A_1 \cup A_2$ is a 2-abelian partition of $G$. \hfill \Box$

**Theorem 3.2** The following conditions on a nonabelian group $G$ are equivalent:

(1) $G$ has a 3-abelian partition with respect to an abelian subgroup $A$.

(2) $|Z(G)| \geq 2$ and $G/Z(G)$ is isomorphic to one of the following groups:

\[
\mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_3 \times \mathbb{Z}_3, \quad S_3.
\]

In the first case, $A = Z(G)$, and in two other cases $A = \langle Z(G), x \rangle$, where $x$ is an element of order 3 outside of $Z(G)$.

**Proof.** (1) $\Rightarrow$ (2) Suppose that $G$ is a nonabelian group, which has a 3-abelian partition $G = A \cup A_1 \cup A_2 \cup A_3$. First of all, we notice that $nc(G) = 3$ or 4. It now follows from Lemma 2.4 in [1] that $G$ is either 3-centralizer or 4-centralizer, respectively. Therefore, by Theorems 2 and 4 in [2], we conclude that $G$ modulo its center is isomorphic to one of the groups: $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, or $S_3$, as required. Finally, since $G$ is a nonabelian group with at least 7 elements, $|Z(G)| \geq 2$.

(2) $\Rightarrow$ (1) Let $Z = Z(G)$. We treat separately the different cases:

(a) $G/Z \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. In this case, there are noncentral elements $x_1$, $x_2$, and $x_3$ of $G$ such that $G = Z \cup Zx_1 \cup Zx_2 \cup Zx_3$, which is a 3-abelian partition of $G$.

(b) $G/Z(G) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. In this case, we have

\[
G/Z \cong \langle Zx, Zy \mid x^3, y^3, [x, y] \in Z \rangle,
\]

which implies that

\[
G = Z \cup Zx \cup Zx^2 \cup Zy \cup Zy^2 \cup Zxy \cup Zx^2y \cup Zxy^2 \cup Zx^2y.
\]

We put

\[
A := Z \cup Zx \cup Zx^2 = \langle Z, x \rangle,
\]

\[
A_1 := Zy \cup Zy^2 = \langle Z, y \rangle \setminus Z,
\]

\[
A_2 := Zxy \cup Zx^2y^2 = \langle Z, xy \rangle \setminus Z,
\]

\[
A_3 := Zxy^2 \cup Zx^2y = \langle Z, xy^2 \rangle \setminus Z.
\]

Then $G = A \cup A_1 \cup A_2 \cup A_3$ is a 3-abelian partition of $G$.

(c) $G/Z \cong S_3$. In this case, we have $G/Z \cong \langle Zx, Zy \mid x^3, y^2, (xy)^2 \in Z \rangle$, which implies that

\[
G = Z \cup Zx \cup Zx^2 \cup Zy \cup Zyx \cup Zyx^2.
\]

Put $A := Z \cup Zx \cup Zx^2 = \langle Z, x \rangle$. Then, $G = A \cup Zy \cup Zyx \cup Zyx^2$ is a 3-abelian partition of $G$.

The proof is complete. \hfill \Box
4 Computing some explicit formulas for $\kappa(G)$

We here consider the problem of finding the tree-number of commuting graphs associated with certain finite groups. Our first result concerns Frobenius groups.

**Theorem 4.1** Let $H \subset G$ be a subgroup of nonabelian group $G$ such that the commuting graph $C(G \setminus H)$ is empty. Then, $H$ is abelian of odd order and $G$ is a Frobenius group with kernel $H$ and complement $\mathbb{Z}_2$. In particular, $\kappa(G) = |H|^{[H]-2}$.

**Proof.** Let $H = G \setminus H$. Clearly, $H$ is nonempty and $G = H \cup H$. If $A < G$ is an abelian subgroup, then either $A \subseteq H$ or $|A| = 2$. In order to prove this, note that $A \cap H$ can have at most one element, so if $A$ is not contained in $H$, then the subgroup $A \cap H$ has order $|A| - 1$. Since this must divide $|A|$, we conclude that $|A| = 2$.

Let $\bar{h}$ be an arbitrary element in $H$. If $1 \neq g \in C_G(\bar{h})$, then $\langle g, \bar{h} \rangle$ is abelian and is not contained in $H$, so it has order 2, and thus $g = \bar{h}$ and $\bar{h}$ has order 2 and $C_G(\bar{h}) = \langle \bar{h} \rangle$. Now $\bar{h}$ is contained in some Sylow 2-subgroup $S$ of $G$. Since $Z(S)$ is nontrivial, $Z(S) \leq C_G(\bar{h}) = \langle \bar{h} \rangle$, so $\bar{h}$ is central in $S$, and thus $S \leq C_G(\bar{h}) = \langle \bar{h} \rangle$ has order 2. Then $|G| = 2n$, where $n$ is odd. It follows that $G$ has a normal subgroup $N$ of order $n$. Every element of $H$ has order 2 so $N \leq H$, and thus $N = H$, and $H$ has order $n$. Finally, since $\bar{h}$ centralizes no nonidentity element of $H$, the result follows.

For the last statement, we note that

$$C(G) = K_1 \cup (K_{h-1} \oplus K_h^c),$$

where $h = |H|$. We apply Lemma 2.4 to $\Gamma_1 = K_1$ and $\Gamma_2 = K_{h-1} \oplus K_h^c$, and use Eq. (2). □

In what follows, we shall give an explicit formula for $\kappa(L_2(2^n))$. Let $G = L_2(q)$, where $q = 2^n \geq 4$. Before we start, we need some well known facts about the simple groups $G$, which are proven in [9]:

1. $|G| = q(q^2 - 1)$ and $\mu(G) = \{2, q - 1, q + 1\}$.

2. Let $P$ be a Sylow 2-subgroup of $G$. Then $P$ is an elementary abelian 2-group of order $q$, which is a TI-subgroup, and $|N_G(P)| = q(q - 1)$.

3. Let $A \subset G$ be a cyclic subgroup of order $q - 1$. Then $A$ is a TI-subgroup and the normalizer $N_G(A)$ is a dihedral group of order $2(q - 1)$.

4. Let $B \subset G$ be a cyclic subgroup of order $q + 1$. Then $B$ is a TI-subgroup and the normalizer $N_G(B)$ is a dihedral group of order $2(q + 1)$.

We recall that a subgroup $H \subseteq G$ is a **TI-subgroup** (trivial intersection subgroup) if for every $g \in G$, either $H^g = H$ or $H \cap H^g = \{1\}$.
Applying Cayley’s formula to commuting graphs associated with these abelian subgroups yields

\[ \kappa(L_2(q)) = q^{(q-2)(q+1)}(q-1)^{(q-3)q(q+1)/2}(q+1)^{(q-1)^2q/2}. \]

In particular, when \( q = 4 \), we have \( L_2(4) \cong L_2(5) \cong \mathbb{A}_5 \) and so

\[ \kappa(L_2(5)) = \kappa(\mathbb{A}_5) = 2^{20} \cdot 3^{10} \cdot 5^{18}. \]

**Proof.** Let \( G = L_2(q), \ q = 2^n \geq 4 \). As already mentioned, \( G \) contains abelian subgroups \( P, A, B \), of orders \( q, q - 1, q + 1 \), respectively, every two distinct conjugates of them intersect trivially and every element of \( G \) is a conjugate of an element in \( P \cup A \cup B \). Let

\[ G = N_P u_1 \cup \cdots \cup N_P u_r = N_A v_1 \cup \cdots \cup N_A v_s = N_B w_1 \cup \cdots \cup N_B w_t, \]

be coset decompositions of \( G \) by \( N_P = N_G(P), N_A = N_G(A) \) and \( N_B = N_G(B) \), where \( r = [G : N_P] = q + 1, s = [G : N_A] = q(q + 1)/2 \) and \( t = [G : N_B] = (q - 1)q/2 \). Then, we have

\[ G = P^{u_1} \cup \cdots \cup P^{u_r} \cup A^{v_1} \cup \cdots \cup A^{v_s} \cup B^{w_1} \cup \cdots \cup B^{w_t}. \]

Note that if \( x \) is a nonidentity element of \( P^{u_i} \) (resp. \( A^{v_j}, B^{w_k} \)), the centralizer \( C_G(x) \) coincides with \( P^{u_i} \) (resp. \( A^{v_j}, B^{w_k} \)) for \( 1 \leq i \leq r \) (resp. \( 1 \leq j \leq s, 1 \leq k \leq t \)). This shows that

\[ C(G) = K_1 \vee (rK_{q-1} \oplus sK_{q-2} \oplus tK_q). \]

Applying Cayley’s formula to commuting graphs associated with these abelian subgroups yields

\[ \kappa(G) = \kappa(P)^r \cdot \kappa(A)^s \cdot \kappa(B)^t = q^{(q-2)r} \cdot (q - 1)^{(q-3)s} \cdot (q + 1)^{(q-1)t}, \]

and the result follows. \( \square \)

Remark. We know that \( L_2(2) \cong S_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \) is a Frobenius group of order 6. It is routine to check that \( \kappa(L_2(2)) = 3 \) (see also [11, Lemma 2.9]).

In what follows, we will concentrate on nonabelian groups \( G \) in which the centralizer of every noncentral element of \( G \) is abelian. Such groups are called AC-groups. The smallest nonabelian AC-group is \( S_3 \). As a matter of fact, there are many infinite families of AC-groups, such as:

- **Dihedral groups** \( D_{2k} \ (k \geq 3) \), defined by
  \[ D_{2k} = \langle x, y \mid x^k = y^2 = 1, yxy^{-1} = x^{-1} \rangle. \]

- **Semidihedral groups** \( SD_{2k} \ (k \geq 4) \), defined by
  \[ SD_{2k} = \langle x, y \mid x^{2k-1} = y^2 = 1, yxy^{-1} = x^{-1+2^{k-2}} \rangle. \]
• Generalized quaternion groups $Q_{4k}$ ($k \geq 2$), defined by
  \[Q_{4k} = \langle x, y \mid x^{2k} = 1, y^2 = x^k, yxy^{-1} = x^{-1}\rangle.\]

• Simple groups $L_2(2^k)$ ($k \geq 2$), and general linear groups $\text{GL}(2, q)$, $q = p^k > 2$, $p$ a prime.

We now return to the general case. Let $G$ be a nonabelian AC-group of order $n$ with center $Z = Z(G)$ of order $m$. Then, by Eq. (2), we have
  \[C(G) = K_m \vee \Delta(G),\]
where $\Delta(G) = C(G \setminus Z)$. It is easy to see that $\Lambda := \{C_G(x) \mid x \in G \setminus Z\}$ is the set of all maximal abelian subgroups of $G$. Let $t := |\Lambda|$ and
  \[\Lambda = \{C_G(x_1), C_G(x_2), \ldots, C_G(x_t)\}.\]
Put $C_i := C_G(x_i) \setminus Z$ and $m_i = |C_i|$, for $i = 1, 2, \ldots, t$. Then, we get
  \[\Delta(G) = \bigoplus_{i=1}^{t} C_i = \bigoplus_{i=1}^{t} K_{m_i}.\]
It follows by Lemma 2.4 (1) that the eigenvalues of Laplacian matrix $L_{\Delta(G)}$ are:
  \[0, m_1, m_1, \ldots, m_1, 0, m_2, m_2, \ldots, m_2, \ldots, 0, m_t, m_t, \ldots, m_t.\]
Therefore, using Lemma 2.4 (2), the eigenvalues of Laplacian matrix $L_{\Omega(G)}$ are:
  \[n, n, \ldots, n, m_1 + m_1, m_1, m_1, \ldots, m_1 + m_1, m_2 + m_2, \ldots, m_2, \ldots, m_t + m_t, m_t, m_t, \ldots, m_t, 0,\]
and using Eq. (2), we get the following theorem (see also [11, Corollary 2.5.]):

**Theorem 4.3** Let $G$ be a finite nonabelian AC-group of order $n$ with center of order $m$. Let $C_G(x_1), C_G(x_2), \ldots, C_G(x_t)$ be all distinct centralizers of noncentral elements of $G$ and $m_i = |C_G(x_i) \setminus Z(G)|$, for $i = 1, 2, \ldots, t$. Then, there holds
  \[\kappa(G) = n^{m-1}t^{t-1} \prod_{i=1}^{t} (m_i + m)^{m_i-1}.\]
In particular, if $G$ is a centerless AC-group, then we have
  \[\kappa(G) = \prod_{i=1}^{t} (m_i + 1)^{m_i-1} = \prod_{i=1}^{t} |C_G(x_i)|^{|C_G(x_i)|-2}.\]

Theorem 4.3 together with some rather technical computations (see proofs of Corollary 3.7 and Propositions 4.1–4.3 in [7]) yields some special results which are summarized in Table 1.
Table 1. $\kappa(G)$ for some special AC-groups $G$.

| $G$  | $n$  | $m$  | $t$  | $m_i$   | $\kappa(G)$ |
|------|------|------|------|---------|-------------|
| $D_{2k}$ | $k$ odd | $2k$ | $1$  | $k+1$ | $k-1,1,\ldots,1$ | $k^{k-2}$ |
| $D_{2k}$ | $k$ even | $2k$ | $2$  | $\frac{k}{2}+1$ | $k-2,2,\ldots,2$ | $2^{\frac{k+2}{2}}k^{k-2}$ |
| $Q_{4k}$ | $k \geq 2$ | $4k$ | $2$  | $k+1$ | $2k-2,2,\ldots,2$ | $2^{5k-1}k^{2k-2}$ |
| $SD_{2k}$ | $k \geq 4$ | $2k$ | $2k-1+1$ | $2k-1-2,2,\ldots,2$ | $2(2k-2)(2k+1)+4$ |
| $P$ | $p$ prime | $p^3$ | $p$  | $p+1$ | $p^2-p,\ldots,p^2-p$ | $p^2p^3-5$ |

Similarly, if $G$ is one of the almost simple groups $L_2(2^k)$ or $GL(2,q)$, where $k \geq 2$ and $q = p^f > 2$ ($p$ is a prime), then using technical computations similar to those in the proofs of Propositions 4.4 and 4.5 in [7], we obtain:

$$G = L_2(2^k): \quad n = 2k(2^k-1), \quad m = 1, \quad t = 2^{4k-2} + 2^k + 1, \quad \text{and} \quad m_i:$$

$$\frac{2^k-1, \ldots, 2^k-1, 2^k-2, \ldots, 2^k-2, 2^k, \ldots, 2^k}{2^k+1, 2^k+1, 2^k+1}.$$

$$G = GL(2,q): \quad n = (q^2-1)(q^2-2), \quad m = q-1, \quad t = q^2+q+1, \quad \text{and} \quad m_i:$$

$$\frac{q^2-3q+2, \ldots, q^2-3q+2, q^2-2q+1, \ldots, q^2-2q+1}{q+1}.$$

Therefore, by Theorem 4.3 we get

$$\kappa(L_2(2^k)) = 2^{(2^k-2)(2^k+1)}(2^k-1)^{2k-1}(2^k-3)(2^k+1)(2^k+1)^{2k-1}(2^k-1)^2,$$

(compare this with Theorem 4.2), and

$$\kappa(GL(2,q)) = q^{q^3-q^2-q+2} - 2q^{q^3-2q^2-4}q + (q+1)^{q(q+1)}.$$  

In the next result, we deal with an AC-group $G$ for which $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

**Theorem 4.4** Let $G$ be a group of order $n$ with nontrivial center $Z(G)$ of order $m$ such that $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where $p$ is a prime. Then, there holds

$$\kappa(G) = p^{n+m-p-3}m^{n-2}.$$

**Proof.** First, we claim that $G$ is an AC-group. To show this, suppose $x$ is a noncentral element of $G$. Then $Z(G) \leq C_G(x) < G$, and since $|G : Z(G)| = p^2$, we conclude that $|C_G(x) : Z(G)| = p$. This shows that $C_G(x) = \langle x, Z(G) \rangle$, which is an abelian group, and so $G$ is an AC-group, as claimed.

Since $G/Z(G)$ is an elementary abelian $p$-group of order $p^2$, it has exactly $p+1$ distinct subgroups of order $p$, say $\langle Z(G)x_1 \rangle, \langle Z(G)x_2 \rangle, \ldots, \langle Z(G)x_{p+1} \rangle$. We claim that $C_G(x_1), C_G(x_2), \ldots, C_G(x_{p+1})$ are all distinct centralizers of noncentral elements of $G$. Suppose $y$ is an arbitrary noncentral element of $G$. Then
Let \( G \) be a group having a 2-abelian partition and let \( Z(G) \) be its center of order \( m \). Then
\[
\kappa(G) = 2^{5m-5}m^{4m-2}.
\]

**Proof.** Using Theorem 3.1, \( G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \). Now, it follows by Theorem 4.4 that \( \kappa(G) = 2^{5m-5}m^{4m-2} \), as required. □

**Corollary 4.6** Let \( G \) be a group having a 3-abelian partition and let \( Z(G) \) be its center of order \( m \). The following conditions hold:

(a) \( G/Z(G) \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \) and \( \kappa(G) = 3^{10m-6}m^{9m-2} \).

(b) \( G/Z(G) \cong \mathbb{S}_3 \) and \( \kappa(G) = 2^{4m-4}3^{3m-2}m^{6m-1} \).

**Proof.** Using Theorem 3.2, we treat separately the different cases:

(1) \( G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p \) where \( p \in \{2, 3\} \). In this case, it follows from Theorem 4.4 that
\[
\kappa(G) = p^{m(p^2+1)-p-3}m^{mp^2-2},
\]
whence also (a) and (b) hold.

(2) \( G/Z(G) \cong \mathbb{S}_3 \). Using Theorem 5 in [2], \( G \) has only five distinct centralizers. In fact, in the notation of Theorem 3.2 the distinct centralizers of \( G \), written as unions of right cosets of \( Z = Z(G) \), are
\[
C_1 = Z \cup Zx \cup Zx^2 = \langle Z, x \rangle,
C_2 = Z \cup Zy = \langle Z, y \rangle,
C_3 = Z \cup Zyx = \langle Z, xy \rangle,
C_4 = Z \cup Zyx^2 = \langle Z, xy^2 \rangle.
\]

This shows that \( G \) is an AC-group, and thus using Theorem 4.3 we obtain \( \kappa(G) = 2^{4m-4}3^{3m-2}m^{6m-1} \), and (c) follows.

The proof is complete. □
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