SUBSPACE CODES FROM FERRERS DIAGRAMS

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Abstract. In this paper we give new constructions of Ferrer diagram rank metric codes, which achieve the largest possible dimension. In particular, we prove several cases of a conjecture by T. Etzion and N. Silberstein. We also establish a sharp lower bound on the dimension of linear rank metric anticodes with a given profile. Combining our results with the multilevel construction, we produce examples of subspace codes with the largest known cardinality for the given parameters.

Introduction

Network coding is a branch of information theory concerned with data transmission over noisy networks. A typical scenario where network coding techniques may be applied is multicast, i.e., one source transmitting messages to multiple sinks through a network of intermediate nodes. Network coding provides techniques for optimizing the transmission and storage of data in digital file distribution, streaming television, peer-to-peer networking, and distributed storage, among other applications.

In [11] and [8] it is proved that maximal communication rate in multicasting transmission can be achieved by allowing the intermediate nodes to perform random linear combinations of the inputs that they receive, provided that the cardinality of the ground field is large enough. Motivated by this result, in [10] R. Kötter and F. R. Kschischang define a subspace code of constant dimension \(k\) and length \(n\) over the finite field \(\mathbb{F}_q\) as a collection of \(k\)-dimensional subspaces of \(\mathbb{F}_q^n\). The novel framework developed in [10] also translates errors and erasures correction in network communications into the problem of finding a matrix of a given form, satisfying a rank constraint.

Linear spaces of matrices with rank bounded below by a given \(\delta\) and a fixed Ferrers diagram as shape are introduced in [3], where they are used for constructing subspace codes. Since the cardinality of the subspace codes obtained increases with the dimension of the linear spaces of matrices, it is natural to ask what the maximum possible dimension of such linear spaces is. In [3] T. Etzion and N. Silberstein derive an upper bound in terms of invariants of the Ferrers diagram, and conjecture that the bound is sharp over finite fields. Some examples of Ferrers diagrams that provide evidence for the conjecture may be found in [3] and [16]. In this paper, we establish the conjecture in several cases, including the most relevant case of “large” Ferrer diagrams (see Theorem 23 for a precise statement).

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We also study the natural dual problem of linear spaces of matrices with rank bounded above by \( \delta - 1 \) and a given profile \( \mathcal{P} \) as shape (see Section 3 for the definition of profile). Such spaces also appear in the literature under the name of linear antico-des. In this paper, we determine the largest possible dimension for any profile and over any field, and as a consequence we obtain an upper bound on the dimension of a linear space of matrices with rank bounded below by \( \delta \) and a given profile \( \mathcal{P} \) as shape.

Finally, we discuss how to apply our results to construct subspace codes over \( \mathbb{F}_q \) of the largest known cardinality, for many choices of the parameters and for arbitrary \( q \). Notice that the network coding scheme proposed in [8] (which inspires the definition of subspace code) asymptotically achieves the maximum communication rate only for \( q \) large enough, from which our interest in finding constructions for all values of \( q \). In addition, we show in an example that using constant weight lexicodes in the multilevel construction of [3] may not the best choice, in contrast to what is suggested in previous works.

The paper is organized as follows. In Section 1 and 2 we give some preliminary definitions and results. In Section 3 we prove several relevant cases of the conjecture by T. Etzion and N. Silberstein. Section 4 is concerned with the study of the corresponding linear anticodes, for which we compute the maximum dimension for all possible shapes and over any field. Finally, in Section 5 we show how the results from Section 3 together with the multilevel construction of [3] provide new lower bounds on the size of subspace codes.

New constructions of maximum dimensional Ferrer rank metric codes were obtained independently by A. Wachter-Zeh and T. Etzion in [19]. In particular, their Construction 1 is the same as the construction which appears in the proof of our Theorem 32.

1. Preliminaries

Throughout this paper, we denote by \([k]\) the set \(\{1, \ldots, k\}\). A set of \(k \times m\) matrices over a field is said to be a \(\delta\)-space (resp., a \(\overline{\delta}\)-space) if it is a linear space and every non-zero element of \(V\) has rank at least \(\delta\) (resp., at most \(\overline{\delta}\)). In this paper we study linear spaces of matrices whose shape is a Ferrers diagram in the sense of [3]. For the convenience of the reader, we recall the definitions and results that we will use.

**Definition 1.** Given positive integers \(k\) and \(m\), a **Ferrers diagram** \(\mathcal{F}\) of size \(k \times m\) is a subset of \([k] \times [m]\) with the following properties:

1. If \((i, j) \in \mathcal{F}\) and \(i > 1\), then \((i-1, j) \in \mathcal{F}\),
2. If \((i, j) \in \mathcal{F}\) and \(j < m\), then \((i, j+1) \in \mathcal{F}\).

For any \(1 \leq i \leq k\), the \(i\)-th **row** of \(\mathcal{F}\) is the set of \((i, j) \in \mathcal{F}\) with \(j \in [m]\). Similarly, for any \(1 \leq j \leq m\) the \(j\)-th **column** of \(\mathcal{F}\) is the set of \((i, j) \in \mathcal{F}\) with \(i \in [k]\).

Notice that we do not require \((1, 1) \in \mathcal{F}\) or \((k, m) \in \mathcal{F}\).

**Notation 2.** We often identify a Ferrers diagram \(\mathcal{F}\) with the cardinalities of its rows. Indeed, given positive integers \(m\) and \(k\), there exists a unique Ferrers diagram \(\mathcal{F}\) of size \(k \times m\) such that the \(i\)-th row of \(\mathcal{F}\) has cardinality \(r_i\) for any \(1 \leq i \leq k\). In this case we write \(\mathcal{F} = [r_1, \ldots, r_k]\).

**Remark 3.** Let \(\mathcal{F} = [r_1, \ldots, r_k]\) and \(\mathcal{F}' = [r'_1, \ldots, r'_k]\) be Ferrers diagrams of size \(k \times m\). We have \(\mathcal{F}' \subseteq \mathcal{F}\) if and only if \(r'_i \leq r_i\) for all \(i = 1, \ldots, k\).

Ferrers diagrams may be graphically represented as rows of right-justified dots of decreasing cardinalities. If \(\mathcal{F} = [r_1, \ldots, r_k]\), the first row of the graphical representation of \(\mathcal{F}\) contains \(r_1\) dots, the second row \(r_2\) dots, and so on.
Example 4. Let $F := [6, 3, 2, 2]$ be a Ferrers diagram of size $6 \times 4$. The graphical representation of $F$ is as follows:

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Definition 5. Let $M = (M_{i,j})$ be a $k \times m$ matrix. The support of $M$ is the set of its non-zero entries, i.e., $\text{supp}(M) := \{(i, j) \in [k] \times [m] | M_{i,j} \neq 0\}$. Let $F$ be a Ferrers diagram of size $k \times m$. We say that $M$ has shape $F$ if $\text{supp}(M) \subseteq F$.

Example 6. The two $4 \times 6$ matrices over $F_2$

$$
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
$$

have shape $F := [6, 3, 2, 2]$.

Notation 7. Fix a Ferrer diagram $F$. The set of matrices with entries in a field $F$ which have shape $F$ form a $|F|$-dimensional $F$-vector space, which we denote by $F[F]$. Equivalently,

$$F[F] = \{M \in \text{Mat}_{k \times m}(F) \mid \text{supp}(M) \subseteq F\}.$$ 

A main open problem from [3, Section VI] is the following.

Question 8. Given integers $1 \leq \delta \leq k \leq m$ and a Ferrers diagram $F$ of size $k \times m$, what is the largest possible dimension of a $\delta$-space of $k \times m$ matrices with shape $F$ and entries in a finite field $F$?

Remark 9. Up to a transposition, the assumption $k \leq m$ in Question 8 is not restrictive. In the sequel we always work with Ferrers diagrams of size $k \times m$ with $k \leq m$. This is also the relevant case for network coding applications.

Notice that Question 8 makes sense over any field $F$. We will show in Remark 10 that the answer depends on the choice of $F$. We denote by

$$\text{MaxDim}_F(F, \delta) = \max \{\dim V \mid V \subseteq F[F] \text{ is a } \delta \text{-space}\}$$

the largest possible dimension of a $\delta$-space of $k \times m$ matrices with shape $F$ and entries in $F$.

Notation 10. Given integers $1 \leq \delta \leq k \leq m$, $0 \leq i \leq \delta - 1$, and a Ferrers diagram $F$ of size $k \times m$, we denote by $T_\delta(F, i)$ the cardinality of the set obtained from $F$ by removing the topmost $i$ rows and the rightmost $\delta - i - 1$ columns. Moreover, we set

$$T_\delta(F) := \min_{0 \leq i \leq \delta - 1} T_\delta(F, i).$$

One always has $T_1(F) = |F| = T_1(F, 0)$.

Example 11. Let $F := [6, 3, 2, 2]$. We have $T_4(F, 0) = 3$, $T_4(F, 1) = 1$, $T_4(F, 2) = 2$, $T_4(F, 3) = 2$. Hence $T_4(F) = 1$. Similarly one can check that $T_3(F) = 4$ and $T_2(F) = 7$.

The following lemma collects some properties which will be useful in the sequel. The proof is straightforward.

Lemma 12. Let $F$ be a field, $F$ and $F'$ be Ferrers diagrams. Assume that $F' \subseteq F$. We have:

1. $T_\delta(F) \geq T_\delta(F')$. 

(2) \( \text{MaxDim}_\delta(\mathcal{F}, \mathbb{F}) \geq \text{MaxDim}_\delta(\mathcal{F}', \mathbb{F}) \).

The authors of \cite{3} prove that \( T_\delta(\mathcal{F}) \) is an upper bound for \( \text{MaxDim}_\delta(\mathcal{F}, \mathbb{F}) \) for any \( \delta \). Moreover, they conjecture that the bound is attained when the field \( \mathbb{F} = \mathbb{F}_q \) is finite, for any choice of \( \delta \) and \( \mathcal{F} \). Notice that while in \cite{3} the upper bound is stated only for finite fields, the proof works over an arbitrary field. Here we state the result in the general form.

**Theorem 13** (\cite{3}, Theorem 1). We have

\[
\text{MaxDim}_\delta(\mathcal{F}, \mathbb{F}) \leq T_\delta(\mathcal{F})
\]

for any field \( \mathbb{F} \), any Ferrers diagram \( \mathcal{F} \), and any \( \delta \geq 1 \).

**Conjecture 14** (\cite{3}, Conjecture 1). When \( \mathbb{F} = \mathbb{F}_q \) is a finite field, equality holds in Theorem 13 for any choice of the parameters \( q, \mathcal{F} \) and \( \delta \).

**2. From unrestricted matrices to matrices of prescribed shape**

A well-studied case of Question 8 is when \( \mathcal{F} = [k] \times [m] \). This is the case of *unrestricted matrices*, solved by Delsarte in 1978. The same result was established in the context of network coding by Kötter and Kschischang.

**Theorem 15** (\cite{1}, Theorem 5.4 and Theorem 6.3; \cite{10}, Theorem 14). Let \( 1 \leq \delta \leq k \leq m \) be integers. We have

\[
\text{MaxDim}_\delta([k] \times [m], \mathbb{F}_q) = m(k - \delta + 1)
\]

for any finite field \( \mathbb{F}_q \). In particular, Conjecture 14 holds for \( \mathcal{F} = [k] \times [m] \).

For any choice of the parameters \( \delta, k, \) and \( m \), Delsarte provides in \cite{1} a construction of a \( \delta \)-space of \( k \times m \) matrices of maximum dimension \( m(k - \delta + 1) \) over \( \mathbb{F}_q \). The properties of finite fields play a central role in his argument. It is interesting to notice that the answer to Question 8 (hence the validity of Conjecture 14) depends on the choice of the field \( \mathbb{F} \), while the upper bound \( T_\delta(\mathcal{F}) \) does not. To illustrate this phenomenon, we show that Theorem 15 is false over an algebraically closed field.

**Example 16.** Let \( \mathbb{F} \) be an algebraically closed field. We have \( \text{MaxDim}_\delta(\mathcal{F}, \mathbb{F}) \leq 1 \) for any \( k \geq 1 \) and any Ferrers diagram \( \mathcal{F} \) of size \( k \times k \). By contradiction, assume that \( A \) and \( B \) are linearly independent \( k \times k \) invertible matrices over \( \mathbb{F} \) that span a \( \delta \)-space. Since \( \mathbb{F} \) is algebraically closed, the polynomial \( \det(\lambda B + A) \in \mathbb{F}[\lambda] \) has a root, say \( \lambda \in \mathbb{F} \). As a consequence, \( \lambda A + B \) is not invertible. Since \( A \) and \( B \) span a \( \delta \)-space, we conclude that \( \lambda A + B = 0 \), a contradiction.

Hence we have shown the following.

**Proposition 17.** Let \( \mathbb{F} \) be an algebraically closed field and \( \mathcal{F} = [r_1, \ldots, r_k] \) be a Ferrers diagram of size \( k \times k \), such that \( r_i \geq k - i + 1 \) for \( i = 1, \ldots, k \). Then \( \text{MaxDim}_\delta(\mathcal{F}, \mathbb{F}) = 1 \). In particular, Conjecture 14 and Theorem 15 do not hold over an algebraically closed field.

An easy way to obtain a \( \delta \)-space of \( k \times m \) matrices with a given shape \( \mathcal{F} \) is the following. Take a \( \delta \)-space of unrestricted \( k \times m \) matrices, and select the ones with the appropriate shape \( \mathcal{F} \). The dimension of the \( \delta \)-space obtained can be lower bounded as follows.

**Proposition 18.** Let \( 1 \leq \delta \leq k \leq m \) be integers, and let \( \mathcal{F} \) be a Ferrers diagram of size \( k \times m \). Then for any field \( \mathbb{F} \)

\[
\text{MaxDim}_\delta(\mathcal{F}, \mathbb{F}) \geq \text{MaxDim}_\delta([k] \times [m], \mathbb{F}) - km + |\mathcal{F}|.
\]

In particular, if \( \mathbb{F} = \mathbb{F}_q \) we have

\[
\text{MaxDim}_\delta(\mathcal{F}, \mathbb{F}_q) \geq |\mathcal{F}| - m(\delta - 1).
\]
Proof. Let \( \mathbb{F}[\mathcal{F}] \) be the \( \mathbb{F} \)-vector space of \( k \times m \) matrices with entries in \( \mathbb{F} \) and shape \( \mathcal{F} \). Clearly, \( \dim \mathbb{F}[\mathcal{F}] = |\mathcal{F}| \). Consider a \( \mathcal{F} \)-space \( V \) of \( k \times m \) matrices of dimension MaxDim\( \mathcal{F}([k] \times [m], \mathbb{F}) \). Then

\[
\dim V \cap \mathbb{F}[\mathcal{F}] \geq \text{MaxDim}_{\mathcal{F}}([k] \times [m], \mathbb{F}) + |\mathcal{F}| - km.
\]

If \( \mathbb{F} = \mathbb{F}_q \), the inequality follows from \((1)\) and Theorem \( \Box \).

We then obtain the following easy consequence of Proposition \( \Box \).

**Corollary 19** [3, Theorem 2]. Let \( 1 \leq \delta \leq k \leq m \) be integers, and let \( \mathcal{F} \) be a Ferrers diagram of size \( k \times m \) with \( r_{\delta - 1} = m \). Then Conjecture [14] holds.

**Remark 20.** Corollary \( \Box \) implies that Conjecture \( \Box \) holds for any Ferrers diagram, if \( \delta = 2 \).

**Remark 21.** The dimension of \( V \cap \mathbb{F}[\mathcal{F}] \) depends on the choice of \( V \), where \( V \) is a \( \mathcal{F} \)-space of unrestricted matrices of maximum dimension. Let e.g. \( \mathcal{F} := [3, 2, 1] \). The linear spaces

\[
V_1 := \{ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \}, \quad V_2 := \{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \}
\]

are both \( \mathcal{F} \)-spaces of unrestricted matrices over \( \mathbb{F}_2 \) of maximal dimension 3. However we have

\[
V_1 \cap \mathbb{F}_2[\mathcal{F}] = \{ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \} \quad \text{and} \quad V_2 \cap \mathbb{F}_2[\mathcal{F}] = \{0\}.
\]

**Remark 22.** It is well-known that the rank distribution of a \( \mathcal{F} \)-space of maximum dimension of unrestricted matrices with entries in a finite field is completely determined by \( \delta, k \), and \( m \) (see \( \Box \) or \( \Box \)). This is in general not the case for \( \mathcal{F} \)-spaces of matrices with prescribed shape and maximum dimension. For example, let \( \mathcal{F} := [3, 2, 1] \). The two \( \mathcal{F} \)-spaces of matrices over \( \mathbb{F}_2 \)

\[
W_1 := \{ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \}, \quad W_2 := \{ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \}
\]

have shape \( \mathcal{F} \) and maximum dimension 3. However, they have different rank distributions.

3. Evidence for the conjecture

In this section, we give explicit constructions of \( \mathcal{F} \)-spaces of matrices with prescribed shapes. This allows us to compute the value of MaxDim\( \mathcal{F}([k] \times [m], \mathbb{F}) \) for many choices of \( \mathcal{F} \) and \( \mathbb{F} \). As a consequence, we establish several new cases of Conjecture \( \Box \).

**Theorem 23.** Let \( 2 \leq \delta \leq k \leq m \) be integers, and let \( \mathcal{F} := [r_1, \ldots, r_k] \) be a Ferrers diagram of size \( k \times m \). Assume \( r_{\delta - 1} \geq k \). We have

\[
\text{MaxDim}_{\mathcal{F}}(\mathcal{F}, \mathbb{F}_q) = T_k(\mathcal{F}) = \sum_{i=\delta}^{k} r_i
\]

for any finite field \( \mathbb{F}_q \). In particular, Conjecture \( \Box \) holds.

**Proof.** Define the Ferrers diagram of size \( k \times r_{\delta - 1} \)

\[
\mathcal{F}^\prime := [r_{\delta - 1}, \ldots, r_{\delta - 1}, r_{\delta}, r_{\delta + 1}, \ldots, r_k] \subseteq \mathcal{F}.
\]
Since \( r_{\delta-1} \geq k \), by Corollary 19 there exists a \( \delta \)-space of matrices with shape \( \mathcal{F}' \) and dimension \( |\mathcal{F}'| - r_{\delta-1}(\delta - 1) = \sum_{i=\delta}^{k} r_i \). Hence we have

\[
T_{\delta}(\mathcal{F}') \geq \text{MaxDim}_{\delta}(\mathcal{F}', \mathbb{F}_q) \geq \sum_{i=\delta}^{k} r_i \geq T_{\delta}(\mathcal{F}'),
\]

where the first inequality follows from Theorem 13 and the last from the definition of \( T_{\delta}(\mathcal{F}') \). Hence we have

\[
T_{\delta}(\mathcal{F}') \geq \text{MaxDim}_{\delta}(\mathcal{F}', \mathbb{F}_q) \geq \sum_{i=\delta}^{k} r_i \geq T_{\delta}(\mathcal{F}'),
\]

where the first inequality follows from Theorem 13 and the last from the definition of \( T_{\delta}(\mathcal{F}') \). Therefore

\[
\text{MaxDim}_{\delta}(\mathcal{F}', \mathbb{F}_q) = \sum_{i=\delta}^{k} r_i ,
\]

where the first inequality follows from the definition of \( T_{\delta}(\mathcal{F}) \), the second from Theorem 13, and the third from Lemma 12. Therefore all the inequalities in (2) are equalities.

Remark 24. To construct applicable subspace codes using the multilevel construction, we usually need Ferrers diagrams with \( m \gg k \). In addition, the vector spaces of matrices that contribute the most to the cardinality of the resulting subspace code correspond to Ferrers diagrams of large cardinality. Hence the case treated in Theorem 23 is most relevant in the applications.

For some Ferrers diagrams of size \( k \times k \), the maximum dimension of a \( \delta \)-space of matrices can be lower-bounded as follows.

**Theorem 25.** Let \( k \geq 1 \) be an integer, and let \( \mathcal{F} \) be a Ferrers diagram of size \( k \times k \). Assume that \( k/2 \leq T_k(\mathcal{F}) \leq k - 1 \). We have

\[
\text{MaxDim}_{\delta}(\mathcal{F}, \mathbb{F}_q) \geq \max \left\{ 2T_k(\mathcal{F}) - k + 1, \left\lfloor \frac{k}{2} \right\rfloor \right\}.
\]

In particular, Conjecture 14 holds in the following cases:

- \( \delta = k = m \) even and \( T_k(\mathcal{F}) = k/2 \),
- \( \delta = k = m \) and \( T_k(\mathcal{F}) = k - 1 \).

**Proof.** By definition of \( T_k(\mathcal{F}) \), both the first column and the \( k \)-th row of \( \mathcal{F} \) have cardinality at least \( t := T_k(\mathcal{F}) \). As a consequence, \( \mathcal{F} \) contains the Ferrers diagram \( \mathcal{F}' := [k, \ldots, k, t, \ldots, t] \).

Since \( \mathcal{F}' \subseteq \mathcal{F} \) and \( T_k(\mathcal{F}') = t \), by Lemma 12 it suffices to prove the thesis for \( \mathcal{F}' \). The graphical representation of \( \mathcal{F}' \) is:

\[
\begin{array}{c}
\begin{pmatrix}
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
\end{pmatrix}
\end{array}
\]

Let \( k_1 = \lfloor k/2 \rfloor \) and \( k_2 = \lceil k/2 \rceil \). We have \( t \geq k_2 \) by assumption. By Theorem 15 there exists a \( k_1 \)-space \( V_1 \) (resp., a \( k_2 \)-space \( V_2 \)) of \( k_1 \times k_1 \) (resp., \( k_2 \times k_2 \)) matrices with entries in \( \mathbb{F}_q \) of
dimension $k_1$ (resp., $k_2$). Let $\{M_1, ..., M_{k_1}\}$ be a basis of $V_1$ and let $\{N_1, ..., N_{k_2}\}$ be a basis of $V_2$. The matrices
\[ H_i := \begin{bmatrix} M_i & 0 \\ 0 & N_i \end{bmatrix}, \quad i = 1, ..., k_1 \]
span a $k$-space of matrices with entries in $\mathbb{F}_q$ and shape $\mathcal{F}'$, of dimension $k_1 = \lfloor k/2 \rfloor$. Therefore $\text{MaxDim}_k(\mathcal{F}', \mathbb{F}_q) \geq \lfloor k/2 \rfloor$.

Let us prove that $\text{MaxDim}_k(\mathcal{F}', \mathbb{F}_q) \geq 2t - k + 1$. If $k = t + 1$, then by Corollary \[19\]
$\text{MaxDim}_k(\mathcal{F}', \mathbb{F}_q) \geq |\mathcal{F}'| - k(k - 1) = k - 1 = 2t - k + 1$. If $k \geq t + 2$, let $\{1, \alpha, ..., \alpha^{k-1}\}$ be an $\mathbb{F}_q$-basis of $\mathbb{F}_{q^k} = \mathbb{F}_q(\alpha)$. For $0 \leq i \leq 2t - k$ define the $\mathbb{F}_q$-linear map
\[ f_i : \mathbb{F}_{q^k} \to \mathbb{F}_{q^k} \quad x \mapsto \alpha^i x. \]
Let $W := \text{Span}_{\mathbb{F}_q}\{f_0, \ldots, f_{2t-k}\} \subseteq \text{Hom}_{\mathbb{F}_q}(\mathbb{F}_{q^k}, \mathbb{F}_{q^k})$. Since $t < k$, then $\dim W = 2t - k + 1$, and any $f \in W \setminus \{0\}$ is invertible. Moreover, the matrices associated to the elements of $W$ with respect to the basis $\{\alpha^{k-1}, ..., \alpha, 1\}$ and putting the images in the rows have shape $\mathcal{F}'$. In fact, for $0 \leq i \leq 2t - k$ we have $f_i(\alpha^{k-j}) = \alpha^{k+i-j}$ with $0 \leq k + i - j \leq t - 1$ for $t + 1 \leq j \leq k$. This proves that $\text{MaxDim}_k(\mathcal{F}', \mathbb{F}_q) \geq \dim W = 2t - k + 1$. \hfill $\square$

**Example 26.** Let $q := 5$, $k := 4$ and $\mathcal{F} := [4, 4, 2, 2]$. We apply the first part of the proof of Theorem \[25\] to construct a 2-dimensional $\mathbb{F}$-space of shape $\mathcal{F}$. Let $V = V_1 = V_2$ be the vector space generated over $\mathbb{F}_5$ by
\[ \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 1 \\ 3 & 4 \end{bmatrix}. \]
$V$ is a 2-space, hence the vector space generated by the two matrices
\[ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 3 & 4 \end{bmatrix} \]
is a 2-dimensional 4-space.

**Remark 27.** The lower bound of Theorem \[25\] is not sharp for all choices of the parameters. Let e.g. $k := 5$, $q := 3$ and $\mathcal{F} := [5, 5, 5, 3, 3]$. We have $T_5(\mathcal{F}) = 3$, hence
\[ \max \{2T_5(\mathcal{F}) - 5 + 1, \lfloor 5/2 \rfloor \} = 2. \]
On the other hand, the three matrices over $\mathbb{F}_3$
\[ \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \]
span a 3-dimensional 5-space. Hence $\text{MaxDim}_5(\mathcal{F}, \mathbb{F}_3) = 3$.

The remainder of this section is concerned with Ferrers diagrams with an “upper triangular” profile. We will give a lower-bound on $\text{MaxDim}_k(\mathcal{F}, \mathbb{F})$ in terms of the lengths of the diagonals of $\mathcal{F}$, provided that the field $\mathbb{F}$ is large enough. As a corollary, we compute the maximum possible dimension of $\mathbb{F}$-spaces of upper triangular matrices over sufficiently large fields, establishing Conjecture \[14\] for some families of diagrams. Before proving the next theorem, we recall some elementary results from classical coding theory.
Definition 28. A linear code of length \( n \geq 1 \) and dimension \( k \) over a field \( \mathbb{F} \) is a \( k \)-dimensional vector subspace of \( \mathbb{F}^n \). The weight of a vector in \( \mathbb{F}^n \) is the number of its non-zero components. The minimum distance of a non-zero linear code \( \mathcal{C} \subseteq \mathbb{F}^n \) is the minimum of the weights of the elements of \( \mathcal{C} \setminus \{0\} \).

Lemma 29. Let \( \mathbb{F} \) be a field. For any integers \( 1 \leq \delta \leq n \) there exists a code \( \mathcal{C} \subseteq \mathbb{F}^n \) of minimum distance \( \delta \) and dimension \( n - \delta + 1 \), provided that \( |\mathbb{F}| \geq n - 1 \).

Proof. If \( |\mathbb{F}| = n - 1 \) the result follows from [12] Chapter 11, Theorem 9. Now assume \( |\mathbb{F}| \geq n \). Let \( \alpha_1, \ldots, \alpha_n \in \mathbb{F} \) distinct. Denote by \( \mathbb{F}[x]_{\leq n-\delta} \) the \( \mathbb{F} \)-space of polynomials with coefficients in \( \mathbb{F} \) and degree at most \( n - \delta \). The \( \mathbb{F} \)-linear map \( \varphi : \mathbb{F}[x]_{\leq n-\delta} \to \mathbb{F}^n \) defined by \( \varphi(f) = (f(\alpha_1), \ldots, f(\alpha_n)) \) is injective by the Fundamental Theorem of Algebra. The image of \( \varphi \) is a code with the expected properties. \( \square \)

Definition 30. Let \( \mathcal{F} \) be a Ferrers diagram of size \( k \times m \). The \( r \)-th diagonal of \( \mathcal{F} \) is the set of elements of \( \mathcal{F} \) of the form \((i, j)\) with \( i - j + m = r \). Notice that we enumerate diagonals from right to left. Similarly, if \( M \) is a matrix with shape \( \mathcal{F} \), we define the \( r \)-th \( \mathcal{F} \)-diagonal of \( M \) as the vector with entries \( M_{i,i+m-r} \) such that \( (i, i + m - r) \in \mathcal{F} \).

Example 31. Let \( \mathcal{F} := [4,2,2,1] \). The second diagonal of \( \mathcal{F} \) has cardinality two, the third and the fourth have cardinality three. Consider the matrix \( M \) of shape \( \mathcal{F} \) given by

\[
M := \begin{bmatrix}
a & b & c & d \\
0 & 0 & e & f \\
0 & 0 & g & h \\
0 & 0 & 0 & i \\
\end{bmatrix}.
\]

The second \( \mathcal{F} \)-diagonal of \( M \) is \((c, f)\), the third is \((b, e, h)\), and the fourth is \((a, g, i)\).

A similar construction to the one that we use to prove the next theorem appears in [15]. We thank T. Etzion for bringing this work to our attention.

Theorem 32. Let \( 1 \leq \delta \leq k \leq m \) be integers, and let \( \mathcal{F} \) be a Ferrers diagram of size \( k \times m \). Assume that \( \mathcal{F} \) has \( n \) diagonals \( D_1, \ldots, D_n \) of cardinality at least \( \delta - 1 \). \( D_i \) is the \( \alpha_i \)-th diagonal of \( \mathcal{F} \), for some \( \alpha_1 \leq \ldots \leq \alpha_n \). If \( |\mathbb{F}| \geq \max_{i=1}^n |D_i| - 1 \), then

\[
\text{MaxDim}_\delta(\mathcal{F}, \mathbb{F}) \geq \sum_{i=1}^n (|D_i| - \delta + 1).
\]

Proof. First we notice that the summands corresponding to diagonals of cardinality \( \delta - 1 \) give no contribution to the lower bound. Hence we may assume without loss of generality that \( |D_i| \geq \delta \) for \( i = 1, \ldots, n \). By Lemma 29 for any \( i = 1, \ldots, n \) there exists a code \( \mathcal{C}_i \subseteq \mathbb{F}^{|D_i|} \) of minimum distance \( \delta \) and dimension \( |D_i| - \delta + 1 \). Given vectors \( v_1, \ldots, v_n \) of lengths \( |D_1|, \ldots, |D_n| \) respectively, denote by \( M(v_1, \ldots, v_n, \mathcal{F}) \) the unique \( k \times m \) matrix with the following properties:

1. the shape of \( M(v_1, \ldots, v_n, \mathcal{F}) \) is \( \mathcal{F} \),
2. the vector \( v_i \) is the \( \alpha_i \)-th \( \mathcal{F} \)-diagonal of \( M(v_1, \ldots, v_n, \mathcal{F}) \),
3. all the remaining entries of \( M(v_1, \ldots, v_n, \mathcal{F}) \) are zero.

We claim that the linear space

\[
V := \text{Span}_\mathbb{F} \{ M(v_1, \ldots, v_n, \mathcal{F}) : (v_1, \ldots, v_n) \in C_1 \times \cdots \times C_n \}
\]

is a \( \delta \)-space of \( k \times m \) matrices with shape \( \mathcal{F} \), of dimension \( \sum_{i=1}^n (|D_i| - \delta + 1) \). To compute \( \dim V \), observe that the map \( C_1 \times \cdots \times C_n \to V \) given by \( (v_1, \ldots, v_n) \mapsto M(v_1, \ldots, v_n, \mathcal{F}) \) is an \( \mathbb{F} \)-isomorphism. Since \( \dim(C_i) = |D_i| - \delta + 1 \) for all \( i \), then \( \dim V = \sum_{i=1}^n (|D_i| - \delta + 1) \). It remains
to show that an arbitrary non-zero matrix in \( V \) has rank at least \( \delta \). Fix \( M \in V \setminus \{0\} \), and let \( r \) denote the maximum integer such that the \( r \)-th diagonal of \( M \) is non-zero. By definition of \( V \), we have \( r = \alpha_i \) for some \( i \). Since \( C_i \) has minimum distance \( \delta \), the \( r \)-th diagonal of \( M \) has at least \( \delta \) non-zero entries. By the maximality of \( r \), the entries of \( M \) which lie below such diagonal are all zero. It is easy to see that a matrix \( M \) of this form has rank at least \( \delta \). \( \square \)

**Corollary 33.** Let \( 1 \leq \delta \leq k \leq m \) be integers, and let \( \mathcal{F} = \{r_1, \ldots, r_k\} \) be a Ferrers diagram of size \( k \times m \). Assume \( r_i \geq m - i + 1 \) for \( i = 1, \ldots, \delta - 1 \) and \( r_i \leq m - i + 1 \) for \( i = \delta, \ldots, k \). We have

\[
\text{MaxDim}_\delta(\mathcal{F}, F) = T_\delta(\mathcal{F})
\]

for any field \( F \) such that \( |F| \geq \max_{i=\delta}^{m} |D_i| - 1 \), where \( D_i \) denotes the \( i \)-th diagonal of \( \mathcal{F} \). In particular, Conjecture \([14]\) holds.

**Proof.** Since \( r_i \geq m - i + 1 \) for \( i = 1, \ldots, \delta - 1 \), we have \( |D_\delta|, \ldots, |D_m| \geq \delta - 1 \). By Theorem \([32]\) \( \text{MaxDim}_\delta(\mathcal{F}, F) \geq \sum_{i=\delta}^{m} (|D_i| - \delta + 1) \). By Theorem \([13]\) and the definition of \( T_\delta(\mathcal{F}) \), it suffices to prove that \( T_\delta(\mathcal{F}, \delta - 1) = \sum_{i=\delta}^{m} (|D_i| - \delta + 1) \). Since \( r_i \leq m - i + 1 \) for \( i = \delta, \ldots, k \), and \( r_i \geq m - i + 1 \) for \( i = 1, \ldots, \delta - 1 \), when we remove from \( \mathcal{F} \) the first \( \delta - 1 \) rows we obtain a set of cardinality \( \sum_{i=\delta}^{m} (|D_i| - \delta + 1) \), as claimed. \( \square \)

**Corollary 34.** Let \( 1 \leq \delta \leq k \) be integers. The maximum dimension of a \( \delta \)-space of \( k \times k \) upper (or lower) triangular matrices over any field \( F \) is \( \binom{k-\delta+2}{2} \), provided that \( |F| \geq k - 1 \). In particular, Conjecture \([14]\) holds.

**Proof.** The Ferrers diagram corresponding to upper triangular \( k \times k \) matrices is \( \mathcal{F} := \{k, k-1, \ldots, 1\} \), which satisfies the assumptions of Corollary \([33]\). Hence we only need to check that \( T_\delta(\mathcal{F}) = \binom{k-\delta+2}{2} \). Fix any \( 0 \leq i \leq \delta - 1 \). We have

\[
T_\delta(\mathcal{F}, i) = \sum_{j=k-i+1}^{k} j + \sum_{j=1}^{k-i} j = \sum_{j=1}^{k-\delta+1} j = \binom{k-\delta+2}{2}.
\]

It follows that \( T_\delta(\mathcal{F}) = \binom{k-\delta+2}{2} \). \( \square \)

**Remark 35.** The requirement \( |F| \geq k - 1 \) in the statement of Corollary \([33]\) is not necessary, in general, for the existence of a \( \delta \)-space of \( k \times k \) upper triangular matrices of dimension \( \binom{k-\delta+2}{2} \). For example, the three upper triangular \( 4 \times 4 \) matrices over \( F_2 \)

\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

span a 3-dimensional \( F_2 \)-space.

4. **\( \delta \)-spaces of maximum dimension**

The problem stated in Question \([8]\) has the following natural dual version, in terms of \( \delta \)-spaces of matrices. Notice that \( \delta \)-spaces of matrices are by definition linear rank metric antcodes.

**Question 36.** Given integers \( 1 \leq \delta \leq k \leq m \) and a Ferrers diagram \( \mathcal{F} \) of size \( k \times m \), what is the largest possible dimension of a \( \delta \)-space of \( k \times m \) matrices with shape \( \mathcal{F} \) and entries in a finite field \( F_q \)?
Definition 37. Given positive integers $k$ and $m$, define a profile of size $k \times m$ as a subset $\mathcal{P} \subseteq [k] \times [m]$. For any $1 \leq i \leq k$, the $i$-th row of $\mathcal{P}$ is the set of $(i,j) \in \mathcal{P}$ with $j \in [m]$. Similarly, for any $1 \leq j \leq m$ the $j$-th column of $\mathcal{P}$ is the set of $(i,j) \in \mathcal{P}$ with $i \in [k]$.

A $k \times m$ matrix $M$ has shape $\mathcal{P}$ when $\text{supp}(M) \subseteq \mathcal{P}$.

Notice that Ferrers diagrams are examples of profiles. In this section, using an idea from [14], we answer the following generalization of Question 36.

Question 38. Given integers $1 \leq \delta \leq k \leq m$ and a profile $\mathcal{P}$ of size $k \times m$, what is the largest possible dimension of a $\delta$-space of $k \times m$ matrices with shape $\mathcal{P}$ and entries in an arbitrary field $\mathbb{F}$?

Let $\mathbb{F}$ be a field and $\mathcal{P}$ be a profile of size $k \times m$. We denote by

$$\text{MaxDim}_{\mathcal{P}}(\mathcal{P}, \mathbb{F})$$

the maximum dimension of a $\delta$-space of $k \times m$ matrices with entries in $\mathbb{F}$ and shape $\mathcal{P}$.

Notation 39. Let $1 \leq \delta \leq k \leq m$ be integers, and let $\mathcal{P}$ be a profile of size $k \times m$. Given subsets $I \subseteq [k]$, $J \subseteq [m]$ such that $|I| + |J| = \delta - 1$, we denote by $T_\delta(\mathcal{P}, I, J)$ the cardinality of the set obtained from $\mathcal{P}$ by removing the rows of index $i \in I$ and the columns of index $j \in J$.

Moreover, we set

$$T_\delta(\mathcal{P}) := \min \{ T_\delta(\mathcal{P}, I, J) \mid I \subseteq [k], J \subseteq [m] \text{ and } |I| + |J| = \delta - 1 \}.$$ 

Finally, recall that a line of a matrix is either a row, or a column of the matrix.

Remark 40. When $\mathcal{P} = \mathcal{F}$ is a Ferrers diagram, the definition of $T_\delta(\mathcal{F})$ given in Notation 11 and the definition of $T_\delta(\mathcal{P})$ given in Notation 39 coincide.

Lemma 41. Let $1 \leq \delta \leq k \leq m$ be integers, and let $\mathcal{P}$ be a profile of size $k \times m$. We have

$$\text{MaxDim}_{\mathcal{P}}(\mathcal{P}, \mathbb{F}) \geq |\mathcal{P}| - T_\delta(\mathcal{P})$$

for any field $\mathbb{F}$.

Proof. Choose $I \subseteq [k]$ and $J \subseteq [m]$ such that $|I| + |J| = \delta - 1$ and $T_\delta(\mathcal{P}, I, J) = T_\delta(\mathcal{P})$. Let

$$\mathcal{P}' = \{(i, j) \in \mathcal{P} \mid i \in I \text{ or } j \in J\}.$$ 

Because of the choice of $I$ and $J$, $|\mathcal{P}'| = |\mathcal{P}| - T_\delta(\mathcal{P})$. Denote by $\mathbb{F}[\mathcal{P}']$ the vector space of $k \times m$ matrices over $\mathbb{F}$ with shape $\mathcal{P}'$. We have $\dim_{\mathbb{F}} \mathbb{F}[\mathcal{P}'] = |\mathcal{P}'| = |\mathcal{P}| - T_\delta(\mathcal{P})$. Since the support of any $M \in \mathbb{F}[\mathcal{P}'] \subseteq \mathbb{F}[\mathcal{P}]$ is contained in at most $\delta - 1$ lines, we have $\text{rank}(M) \leq \delta - 1$. Hence $\text{MaxDim}_{\mathcal{P}}(\mathcal{P}, \mathbb{F}) \geq |\mathcal{P}| - T_\delta(\mathcal{P})$, as claimed.

It is now easy to prove the following generalization of Theorem 13.

Theorem 42. Let $1 \leq \delta \leq k \leq m$ be integers, and let $\mathcal{P}$ be a profile of size $k \times m$. For any field $\mathbb{F}$ we have

$$\text{MaxDim}_{\mathcal{P}}(\mathcal{P}, \mathbb{F}) \leq T_\delta(\mathcal{P}).$$

Proof. Let $V$ be a $\delta$-space of matrices with shape $\mathcal{P}$ of dimension $\text{MaxDim}_{\mathcal{P}}(\mathcal{P}, \mathbb{F})$. Similarly, let $W$ be a $\delta - 1$-space of matrices with shape $\mathcal{P}$ of dimension $\text{MaxDim}_{\mathcal{P}}(\mathcal{P}, \mathbb{F})$. Denote by $\mathbb{F}[\mathcal{P}]$ the $|\mathcal{P}|$-dimensional $\mathbb{F}$-vector space of $k \times m$ matrices with shape $\mathcal{P}$ and entries in $\mathbb{F}$. We have $V \cap W = \{0\}$ and $V \oplus W \subseteq \mathbb{F}[\mathcal{P}]$. By Lemma 11 $\dim V \leq |\mathcal{P}| - (|\mathcal{P}| - T_\delta(\mathcal{P}))$. \qed
Remark 43. By Lemma\[\text{[11]}\] Conjecture\[\text{[14]}\] can be restated as follows: Over a finite field $\mathbb{F}_q$ and for any $\delta$, the vector space $\mathbb{F}_q[\mathcal{F}]$ of matrices of fixed shape $\mathcal{F}$ decomposes as

$$\mathbb{F}_q[\mathcal{F}] = V \oplus \overline{V},$$

where $V$ is a $\delta$-space and $\overline{V}$ is a $\delta-1$-space. We stress that this is in general false when the underlying field is not finite (see Proposition\[\text{[17]}\]).

Notation 44. For integers $1 \leq k \leq m$, let $\prec$ denote the lexicographic order on $[k] \times [m]$, i.e., $(i, j) \prec (i', j')$ if and only if either $i < i'$ or $i = i'$ and $j < j'$. For a $k \times m$ matrix $M$ over a field $\mathbb{F}$ we set

$$p(M) := \min\{(i, j) \mid M_{i,j} \neq 0\}.$$

For a set $A$ of $k \times m$ matrices define the 0-1 matrix $M(A)$ over $\mathbb{F}$ as follows:

- $(1)$ $M(A)_{i,j} = 1$ if $(i, j) = p(A)$ for some $A \in \mathcal{A}$,
- $(2)$ $M(A)_{i,j} = 0$ otherwise.

Finally, denote by $\rho(\mathcal{A})$ the minimal cardinality of a set of lines of $M(\mathcal{A})$ which contain all the 1's appearing in $M(\mathcal{A})$.

Lemma 45. (\cite{[14]}, Theorem 1) Let $\mathcal{A}$ be a set of $k \times m$ matrices over a field $\mathbb{F}$. Then $\text{Span}_{\mathbb{F}}(\mathcal{A})$ contains a matrix of rank at least $\rho(\mathcal{A})$.

The following theorem provides an answer to Question\[\text{[36]}\] and Question\[\text{[38]}\]. It is inspired by Theorem 2 of\[\text{[14]}\].

Theorem 46. Let $1 \leq \delta \leq k \leq m$ be integers, and let $\mathcal{P}$ be a profile of size $k \times m$. We have

$$\text{MaxDim}_{\delta-1}(\mathcal{P}, \mathbb{F}) = \left|\mathcal{P}\right| - T_\delta(\mathcal{P})$$

for any field $\mathbb{F}$.

Proof. By Lemma\[\text{[11]}\] it suffices to show that $\text{MaxDim}_{\delta-1}(\mathcal{P}, \mathbb{F}) \leq \left|\mathcal{P}\right| - T_\delta(\mathcal{P})$. Let $V$ be a $\delta-1$-space of $k \times m$ matrices over $\mathbb{F}$ with shape $\mathcal{P}$ of dimension $r := \text{MaxDim}_{\delta-1}(\mathcal{P}, \mathbb{F})$. Choose a basis $\{N_1, \ldots, N_r\}$ of $V$. Let $\varphi$ be the $\mathbb{F}$-isomorphism that sends a $k \times m$ matrix $M$ to the vector of length $km$ whose entries are the entries of $M$ ordered lexicographically. Define $w_i := \varphi(N_i)$ for $i = 1, \ldots, r$. Perform Gaussian elimination on $w_1, \ldots, w_r$ and get vectors $v_1, \ldots, v_r$. Set $M_i := \varphi^{-1}(v_i)$ for $i = 1, \ldots, r$. It is clear that $\mathcal{A} := \{M_1, \ldots, M_r\}$ is a basis of $V$.

Since $p(M_i) \neq p(M_j)$ for $i \neq j$, the support $\mathcal{P}'$ of $M(\mathcal{A})$ has cardinality exactly $r$.

Since $V$ is a $\delta-1$-space, by Lemma\[\text{[15]}\] the support $\mathcal{P}'$ is contained in a set of $i$ rows and $\delta - i - 1$ columns for some $0 \leq i \leq \delta - 1$. Since $\mathcal{P}' \subseteq \mathcal{P}$, we conclude that $|\mathcal{P}'| \leq |\mathcal{P}| - T_\delta(\mathcal{P})$.  

5. Applications and examples

In this section we show how one can apply the results of Section\[\text{[3]}\] to construct large subspace codes with given parameters, for any size of the ground field $\mathbb{F}_q$. In particular we show how to construct the largest known codes for $q \geq 3$ and many choices of the parameters. Moreover, being systematic, the constructions that we propose may be useful for designing efficient decoding algorithms.

For $q = 2, \delta = 2, 3$ and small values of $n$ and $k$, there exists subspace codes which have larger cardinality than the codes we can construct using the results contained in this paper (see e.g.,\[\text{[5]}, \text{[16]}\] and\[\text{[9]}\]). The techniques employed to produce such codes include a computer search, which is not feasible for large values of $q$ and of the other parameters.
Definition 47. Given \( k \)-dimensional vector subspaces \( X, Y \subseteq \mathbb{F}_q^n \), the injection distance between \( X \) and \( Y \) is defined as \( d_I(X, Y) := k - \dim(X \cap Y) \). Denote by \( G_q(k, n) \) the set of \( k \)-dimensional subspaces of \( \mathbb{F}_q^n \). The minimum distance of a subspace code \( C \subseteq G_q(k, n) \) with \( |C| \geq 2 \) is defined as the minimum of all the pairwise distances between distinct elements of \( C \).

Let us briefly recall the multilevel construction for subspace codes proposed by T. Etzion and N. Silberstein in [3].

Notation 48. Let \( X \) be a \( k \)-dimensional subspace of \( \mathbb{F}_q^n \) and let \( M(X) \) be the unique \( k \times n \) matrix in row-reduced echelon form with rowspace \( X \). We associate to \( X \) the binary vector \( v(X) \) of length \( n \) and weight \( k \), which has a 1 in position \( i \) if and only if \( M(X) \) has a pivot in the \( i \)-th column. The vector \( v(X) \) is called the pivot vector associated to \( X \) and \( M(X) \).

Lemma 49 ([3], Lemma 2). Let \( X, Y \in G_q(k, n) \). We have \( d_I(X, Y) \geq \frac{1}{2}d_H(v(X), v(Y)) \), where \( d_H \) denotes the Hamming distance.

Notation 50. Let \( v \) be a binary vector of length \( n \) and weight \( k \), and let \( 1 \leq p_1 < p_2 < \cdots < p_k \leq n \) be the positions of the \( k \) ones of \( v \). The Ferrers diagram associated to \( v \) is the Ferrers diagram \( \mathcal{F}_v = [r_1, \ldots, r_k] \) of size \( k \times (n-k) \) with \( r_i = n-k-p_i+i \) for all \( i = 1, \ldots, k \).

The following result is straightforward. See [3], Section III and IV for examples and details.

Lemma 51. Let \( v \) be a binary vector of length \( n \) and weight \( k \), and let \( 1 \leq p_1 < p_2 < \cdots < p_k \leq n \) be the positions of the \( k \) ones of \( v \). Let \( M \in \mathbb{F}_q[\mathcal{F}_v] \). For \( j = 1, \ldots, n \) define \( n_j := \# \{ 1 \leq i \leq k \mid p_i \leq j \} \). There exists a unique \( k \times n \) matrix \( N \) over \( \mathbb{F}_q \) in row-reduced echelon form having \( v \) as pivot vector and \( N_{i,j} = M_{i,j-n_j} \) for all \( i \in \{1, \ldots, k\} \) and \( j \in \{1, \ldots, n\} \setminus \{p_1, \ldots, p_k\} \).

We denote the matrix \( N \) of Lemma 51 by \( N(v, M) \). The multilevel construction of [3] is summarized in the following result.

Theorem 52 ([3], Theorem 3). Let \( C \) be a binary code of constant weight \( k \), length \( n \) and minimum distance at least \( 2\delta \). For any \( v \in C \) let \( S(v) \subseteq \mathbb{F}_q[\mathcal{F}_v] \) be a \( \delta \)-space. The set

\[
\{\text{rowsp } N(v, M) \mid v \in C, M \in S(v)\} \subseteq G_q(k, n)
\]

is a subspace code of minimum distance at least \( \delta \) and cardinality \( \sum_{v \in C} q^{\dim S(v)} \).

Remark 53. Large subspace codes with \( \delta > 2 \) were obtained in [3] combining the multilevel construction and a computer search, for small values of \( q \). The computer search part is employed to find large spaces of matrices of rank \( \geq \delta \) and given shape. The results of Section 3 of [3] allow us to construct in a systematic way (i.e., without a computer search) linear spaces of matrices with the same parameters as those found via computer search in [3]. In particular, we can construct subspace codes with the same parameters for any \( q \).

Remark 54. In [18], A-L. Trautmann and J. Rosenthal propose the pending dots construction to improve the multilevel construction of [3]. As the multilevel construction, the pending dots construction also depends on the existence of large spaces of matrices with bounded rank and given shape. Using the idea of pending dots, A-L. Trautmann and N. Silberstein construct large subspace codes in \( G_q(k, n) \) of minimum injection distance \( \delta = k - 1 \) (see Section V of [16] and Section III of [17]) for arbitrary values of \( q \). The Ferrers diagrams that they consider for the case \( \delta = k - 1 \) ([16], Lemma 23 and [17], Lemma 18) are special cases of the diagrams studied in Theorem 32.
**Remark 55.** Spread and partial spread codes ([13], [6], [7]) can be obtained through the multilevel construction for a special choice of the pivot vectors. The Ferrers diagrams associated to those pivot vectors are studied in Theorem 15.

We now give some examples of how to combine the results of Section 3 and the multilevel construction to obtain subspace codes with the largest known cardinality for given $k, n, \text{ and } \delta$.

**Example 56.** Let $(n, k, \delta) := (10, 5, 3)$, and let $q$ be any prime power. Consider the binary code $C := \{1111100000, 1100011100, 0011011010, 1000110011, 0010101101, 0101000111\}$.

Observe that $C$ has constant weight 5 and minimum distance 6. Let $v_1, \ldots, v_6$ be the elements of $C$ in the displayed order. It follows from Theorem 23 that:

1. $\text{MaxDim}_\delta(F_{v_1}, F_q) = 15$,
2. $\text{MaxDim}_\delta(F_{v_2}, F_q) = 6$,
3. $\text{MaxDim}_\delta(F_{v_3}, F_q) = 2$.

Notice moreover that $F_{v_4}$ has the following graphical representation.

By Theorem 15 there exist a 2-dimensional 2-space $V$ of $2 \times 2$ matrices over $F_q$ and a 2-dimensional 1-space $W$ of $1 \times 2$ matrices over $F_q$. Let $\{M_1, M_2\}$ and $\{N_1, N_2\}$ be bases for $V$ and $W$, respectively. Then

$$\text{Span}_{F_q} \left\{ \begin{bmatrix} 0 & N_i \\ 0 & 0_{2 \times 2} \end{bmatrix} \begin{bmatrix} 0_{1 \times 2} \\ M_i \end{bmatrix} \mid i = 1, 2 \right\}$$

is a 2-dimensional $\delta$-space of matrices with shape $F_{v_4}$. Since $T_\delta(F_{v_4}) = 2$, it follows that $\text{MaxDim}_\delta(F_{v_4}, F_q) = 2$. Finally, $F_{v_5}$ has $T_\delta(F_{v_5}) = 1$ and contains the Ferrers diagram

Hence by Corollary 34 and Lemma 12 we have $\text{MaxDim}_\delta(F_{v_5}, F_q) = 1$. Using Theorem 52 we obtain a subspace code $C \subseteq G_q(5, 10)$ of minimum distance $\delta = 3$ with

$$|C| = q^{15} + q^6 + 2q^2 + q + 1.$$ 

For $q \geq 3$ this is the subspace code of parameters $(n, k, \delta) = (10, 5, 3)$ with largest known cardinality.

Let us briefly recall the definition of lexicode. The vectors of $F_2^n$ can be lexicographically ordered as follows. Let $v, w \in F_2^n, v \neq w$, and let $i := \min \{ j \mid v_j \neq w_j \}$. We say that $w \prec v$ if $v_i = 1$. Given a binary vector $v \in F_2^n$ of weight $k$, the constant weight lexicode originated by $v$ of minimum distance $2\delta$ is constructed through iterated steps as follows. Start with $C = \{v\}$. List the elements of $F_2^n$ in decreasing lexicographic order. At each step add to $C$ the first vector of the list of weight $k$ and Hamming distance at least $2\delta$ from all the elements of $C$, until there is no such vector left.

According to Theorem 52 the cardinality of a subspace code obtained through the multilevel construction depends on the choice of the binary constant weight code. Since lexicodes are
known to have large cardinality among constant weight binary codes with the same parameters, T. Etzion and N. Silberstein suggest in [3] to use the lexicode originated by the vector
\[
\begin{array}{c}
1 \\
\cdots \\
1 \\
0 \\
\cdots \\
0 \\
k \\
\cdots \\
n-k
\end{array}
\]
in the multilevel construction. However this choice is not always optimal, as we show in the following example.

**Example 57.** Let \( n := 10, \ k := 5, \ \delta = 3 \). Consider the binary constant weight lexicode
\[
C' := \{1111100000, \ 1100011100, \ 1010010011, \ 0101001011, \ 00010101110, \ 0001110101\}.
\]
Let \( w_1, ..., w_6 \) be the elements of \( C' \) in the displayed order. The graphical representations of the \( \mathcal{F}_{w_i} \)'s are as follows.

One can easily check that
\[
T_3(\mathcal{F}_{w_1}) = 15, \ T_3(\mathcal{F}_{w_2}) = 6, \ T_3(\mathcal{F}_{w_3}) = 2, \ T_3(\mathcal{F}_{w_4}) = 1, \ T_3(\mathcal{F}_{w_5}) = 1, \ T_3(\mathcal{F}_{w_6}) = 0.
\]
Therefore, by Theorem 52 and Theorem 13, choosing \( C' \) as the pivot code produces a subspace code of cardinality at most
\[
q^{15} + q^6 + q^2 + 2q + 1.
\]
However, the binary code \( C \) considered in Example 56 produces a subspace code with the same parameters \( n, k, \delta \) and larger cardinality, for all values of \( q \).

Theorem 23 allows us to give a lower bound the cardinality of subspace codes obtained from given pivot vectors through the multilevel construction.

**Theorem 58.** Fix integers \( n, k, \delta \) with \( 2 \leq \delta \leq k \leq n/2 \). Let \( D \subseteq \mathbb{F}_2^n \) be a code of constant weight \( k \) and minimum distance at least \( 2\delta \). For \( v \in D \) let \( p_i(v) \) denote the position of the \( i \)-th one of \( v \). Let
\[
D' := \{v \in D \mid p_{\delta-1}(v) \leq n - 2k + \delta - 1\}
\]
and
\[
D'' = \{v \in D \mid p_i(v) \leq n - k - \delta + 2i - 1, \ i = 1, \ldots, \delta\}.
\]
Then there exists a subspace code \( C' \subseteq \mathcal{G}_q(k, n) \) of injection distance at least \( \delta \) and
\[
|C'| = \sum_{v \in D'} q^{T_3(\mathcal{F}_v)} + \sum_{v \in D'' \setminus D'} q + |D \setminus D''|.
\]
Moreover any subspace code \( C \) obtained from \( D \) through the multilevel construction has
\[
|C| \leq \sum_{v \in D''} q^{T_3(\mathcal{F}_v)} + |D \setminus D''| \leq |C'| + O(q^{(k-\delta+1)(k-1)})
\]
asymptotically in \( q \).
Proof. For \( v \in D \) we denote by \( r_i(v) \) the cardinality of the \( i \)-th row of \( F_v \) for \( i = 1, \ldots, k \). Fix any \( v \in D' \). As in Notation 50 the cardinality of the \( i \)-th row of \( F_v \) is \( r_i(v) = n - k - p_i(v) + i \).

Let \( v \not\in D'' \), then there exists \( i \in \{1, \ldots, \delta\} \) such that \( r_i(v) \leq n - k - (n - k - \delta + 2i) + i = \delta - i \). Therefore,

\[
T_\delta(F_v, i - 1) = \sum_{u=i}^{k} \max\{r_u(v) - (\delta - u + 1), 0\} = 0 = T_\delta(F_v).
\]

This proves that any subspace code \( C \) obtained from \( D \) through the multilevel construction has

\[
|C| \leq \sum_{v \in D'} q^{T_\delta(F_v)} + |D \setminus D''|.
\]

Let now \( v \in D' \). Since \( p_{\delta - 1}(v) \leq n - 2k + \delta - 1 \), we have \( r_{\delta - 1}(v) \geq k \). Combining Theorem 23 and the multilevel construction using the vectors of \( D' \), we construct a code of cardinality \( \sum_{v \in D'} q^{T_\delta(F_v)} \) and minimum distance at least \( \delta \). For any \( v \in D'' \), the associated Ferrers diagram \( F_v \supseteq [\delta, \delta - 1, \ldots, 1, 0, \ldots, 0] \) by the definition of \( D'' \). Hence we have at least one matrix of rank \( \delta \) and shape \( F_v \), namely the \( k \times (n - k) \) matrix containing a top-right justified \( \delta \times \delta \) identity matrix and zeroes everywhere else. Adding the lift of these codewords to the previous code through the multilevel construction, we construct a code \( C' \) with

\[
|C'| = \sum_{v \in D'} q^{T_\delta(F_v)} + \sum_{v \in D'' \setminus D'} q + |D \setminus D''|
\]

as claimed.

It follows from the previous argument that the cardinality of a code \( C \) obtained from \( D \) through the multilevel construction can be increased only by producing larger linear spaces of matrices of rank at least \( \delta \) and support contained in \( F_v \) for \( v \in D'' \setminus D' \). Observe that for any such \( v \) we have \( r_{\delta - 1}(v) \leq k - 1 \), hence \( r_{i}(v) \leq k - 1 \) for \( i = \delta, \ldots, k \). Hence

\[
T_\delta(F_v) \leq \sum_{i=\delta}^{k} r_{i}(v) \leq (k - \delta + 1)(k - 1).
\]

Since \( |D \setminus D''| \) and \( |D'' \setminus D'| \) are constant in \( k \), we have

\[
|C| - |C'| \in \mathcal{O}\left(q^{(k-\delta+1)(k-1)}\right).
\]

Example 59. In Table 59 we give some cardinalities of subspace codes which we find combining the results of Section 3 with the multilevel construction of 23 as shown in Example 56. For \( q \geq 3 \) and the given values of \( k, n \) and \( \delta \), the codes have the largest known size.

\[
\begin{array}{|c|c|c|c|}
\hline
n & k & \delta & \text{size} \\
\hline
10 & 5 & 3 & q^{13} + q^{11} + 2q^2 + q + 1 \\
11 & 5 & 3 & q^{15} + q^9 + q^6 + q^4 + 4q^3 + 3q^2 \\
14 & 4 & 3 & q^{20} + q^{14} + q^{11} + q^9 + q^5 + 2(q^6 + q^5 + q^4) + q^3 + q^2 \\
14 & 5 & 4 & q^{18} + q^{10} + q^3 + 1 \\
15 & 6 & 5 & q^{15} + q^9 + 1 \\
\hline
\end{array}
\]

Table 1. Some large subspace codes in \( G_q(k, n) \) with minimum injection distance at least \( \delta \).
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