SLIGHTLY IMPROVED SUM-PRODUCT ESTIMATES IN FIELDS OF PRIME ORDER

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Abstract. Let \( \mathbb{F}_p \) be the field of residue classes modulo a prime number \( p \) and let \( A \) be a nonempty subset of \( \mathbb{F}_p \). In this paper we show that if \( |A| \preceq p^{0.5} \), then
\[
\max\{|A \pm A|, |AA|\} \succeq |A|^{13/12};
\]
if \( |A| \geq p^{0.5} \), then
\[
\max\{|A \pm A|, |AA|\} \succsim \min\{|A|^{13/12}(\frac{|A|}{p^{0.5}})^{1/12}, |A|^{1/11}\}.
\]
These results slightly improve the estimates of Bourgain-Garaev and Shen. Sum-product estimates on different sets are also considered.

1. Introduction

Let \( \mathbb{F}_p \) be the field of residue classes modulo a prime number \( p \) and let \( A, B \) be two nonempty subsets of \( \mathbb{F}_p \). Define the sum set, difference set and product set of \( A \) and \( B \) respectively by
\[
A + B = \{a + b : a \in A, b \in B\},
A - B = \{a - b : a \in A, b \in B\},
AB = \{ab : a \in A, b \in B\}.
\]
From the work of Bourgain, Katz, Tao [4] and Bourgain, Glibichuk, Konyagin [3], it is known that if \( |A| \preceq p^{\delta} \) (see Section 2 for the definitions of \( \preceq, \succeq, \simeq, \succsim \), and \( \sim \)), where \( \delta < 1 \), then one has the sum-product estimate
\[
\max\{|A + A|, |AA|\} \succeq |A|^{1+\epsilon}
\]
for some \( \epsilon = \epsilon(\delta) > 0 \). These kinds of results have found many important applications in various areas of mathematics, and people want to know some quantitative relationships between \( \delta \) and \( \epsilon \) in certain ranges of \( |A| \). For the case \( |A| \succeq p^{0.5} \), the pioneer work was due to Hart, Iosevich and Solymosi [12] via Kloosterman sums. See [6, 8, 18] for further improvements. Note also all lower bounds in [6, 8, 12, 18] are trivial if \( |A| \sim p^{0.5} \).

In the very beginning of the story for the case \( |A| \preceq p^{0.5} \), Garaev [7] proved
\[
\max\{|A + A|, |AA|\} \succeq |A|^{15/14},
\]
which was immediately improved by Katz and Shen [13] with a refinement of the Plünnecke-Ruzsa inequality to
\[
\max\{|A + A|, |AA|\} \succeq |A|^{14/13}.
\]
Later on, Bourgain and Garaev [2] considered difference-product estimates and proved
\begin{equation}
\max\{|A - A|, |AA|\} \geq \frac{|A|^{13/12}}{(\log_2 |A|)^{1/11}},
\end{equation}
which was slightly improved by Shen [15, 16] with elegant covering arguments to
\begin{equation}
\max\{|A + A|, |AA|\} \geq \frac{|A|^{13/12}}{(\log_2 |A|)^{1/3}}.
\end{equation}
With a technique of Chang [5], we can completely drop the logarithmic term in (2).

**Theorem 1.1.** Suppose $A \subset \mathbb{F}_p$ with $|A| \leq p^{0.5}$. Then
\[
\max\{|A + A|, |AA|\} \geq |A|^{13/12}.
\]

Note the Bourgain-Garaev estimate (1) also holds for $p^{0.5} \leq |A| \leq p^{12/23}$ (see [2]). In these ranges and beyond, our next result says that:

**Theorem 1.2.** Suppose $A \subset \mathbb{F}_p$ with $|A| \geq p^{0.5}$. Then
\[
\max\{|A + A|, |AA|\} \geq \min\{|A|^{13/12}(\frac{|A|}{p^{0.5}})^{1/12}, |A|(\frac{p}{|A|})^{1/11}\}.
\]

To compare, if $|A| \leq p^{12/23}$, then $|A|^{13/12} \leq |A|(\frac{p}{|A|})^{1/11}$. Particularly, if $|A| \sim p^{35/68}$, then $\max\{|A + A|, |AA|\} \gtrapprox |A|^{38/35}$. This shows Theorem 1.2 is an improvement of (1).

Similarly, we may consider sum-product estimates on different sets in $\mathbb{F}_p$. Bourgain [1] proved that if $p^{1-\delta} \leq |B| \leq |A| \leq p^\delta$, then for some $\epsilon = \epsilon(\delta) > 0$,
\[
\max\left\{\frac{|A + B|}{|A|}, \frac{|AB|}{|A|}\right\} \gtrapprox p^\epsilon.
\]
Shen [14] quantitatively proved that
(a) if $|B| \leq |A| \leq p^{0.5}$, then
\[
\max\left\{\frac{|A + B|}{|A|}, \frac{|AB|}{|A|}\right\} \gtrapprox (\frac{|B|^{14}}{|A|^{13}})^{1/18};
\]
(b) if $|B| \sim |A| \leq p^{0.5}$, then
\[
\max\{|A + B|, |AB|\} \gtrapprox |A|^{15/14}.
\]
We can also give some improvements.

**Theorem 1.3.** Suppose $A, B \subset \mathbb{F}_p$ with $|B| \leq |A| \leq p^{0.5}$. Then
\[
\max\left\{\frac{|A + B|}{|A|}, \frac{|AB|}{|A|}\right\} \gtrapprox (\frac{|B|^6}{|A|^5})^{1/14}.
\]

**Theorem 1.4.** Suppose $A, B \subset \mathbb{F}_p$ with $|A| \geq |B| \geq p^{0.5}$. Then
\[
\max\left\{\frac{|A + B|}{|A|}, \frac{|AB|}{|A|}\right\} \gtrapprox \min\left\{ (\frac{|B|^7}{|A|^5p^{0.5}})^{1/14}, (\frac{|B|^3p}{|A|^4})^{1/12}\right\}.
\]

**Theorem 1.5.** Suppose $A, B \subset \mathbb{F}_p$ with $|B| \sim |A| \leq p^{0.5}$. Then
\[
\max\{|A + B|, |AB|\} \gtrapprox |A|^{15/14}.
\]
2. Notations and Lemmas

Throughout this paper \( A \) will denote a fixed nonempty set in \( \mathbb{F}_p \). For \( B \), any set, we will denote by \(|B|\) its cardinality. Whenever \( E \) and \( F \) are quantities we will use \( E \preceq F \) or \( F \succeq E \) to mean \( E \leq CF \), where the constant \( C \) is universal (i.e. independent of \( A \) and \( p \)). We will use \( E \gtrsim F \) or \( F \lesssim E \) to mean \( E \leq C(\log |A|)^{\alpha}F \), where the universal constants \( C \) and \( \alpha \) may vary from line to line. Besides, \( E \sim F \) means \( E \preceq F \) and \( F \preceq E \).

For \( Y, Z \subset \mathbb{F}_p \), denote by \( E^+(Y, Z) \) the additive energy between \( Y \) and \( Z \), that is,
\[
E^+(Y, Z) = \sum_{x \in Y} \sum_{y \in Y} |(x + Z) \cap (y + Z)|;
\]
denote by \( E^\times(Y, Z) \) the multiplicative energy between \( Y \) and \( Z \), that is,
\[
E^\times(Y, Z) = \sum_{x \in Y} \sum_{y \in Y} |xZ \cap yZ|.
\]
It is well-known [17] that
\[
E^\circ(Y, Z) \geq \frac{|Y|^2 |Z|^2}{|Y \odot Z|},
\]
where \( \odot \in \{+, \times\} \).

In the following we will give some preliminary lemmas. Lemma 2.1 may be found in [14, 15, 16], while Lemma 2.2 in [11, 13]. Lemma 2.3, following from the work of Glibichuk and Konyagin [9, 10] on additive properties of product sets, was proved in [2, 7, 13, 15]. Since the author have not found a proof of Lemma 2.4 in some popular references, we include a short proof here. Lemma 2.5 is due to Chang [5], whereas we present a slightly different variant.

**Lemma 2.1.** Suppose \( B_1, B_2 \subset \mathbb{F}_p \). Then there exist \( \preceq \min\{\frac{|B_1 + B_2|}{|B_2|}, \frac{|B_1 - B_2|}{|B_2|}\} \) translates of \( B_2 \) such that these copies can cover (in cardinality) 99% of \( B_1 \).

**Lemma 2.2.** Suppose \( B_0, B_1, \ldots, B_k \subset \mathbb{F}_p \). Given any \( \epsilon \in (0, 1) \), there exist a universal constant \( C_{k, \epsilon} \) and an \( X \subset B_0 \) with \(|X| \geq (1 - \epsilon)|B_0|\) such that
\[
|X + B_1 + B_2 + \cdots + B_k| \leq C_{k, \epsilon} \cdot \left( \prod_{i=1}^{k} \frac{|B_i + B_0|}{|B_0|} \right) \cdot |X|.
\]

**Lemma 2.3.** Suppose \( A_1 \subset \mathbb{F}_p \) with \( \frac{A_1 - A_1}{A_1} \neq \mathbb{F}_p \). Then (1) \(|A_1| \leq p^{0.5}\); (2) there exist fixed elements \( a_1, b_1, c_1, d_1 \in A_1 \) (\( a_1 \neq b_1 \)) such that for any \( A' \subset A_1 \) with \(|A'| \geq |A_1|\),
\[
|(b_1 - a_1)A' + (b_1 - a_1)A' + (d_1 - c_1)A'| \geq |A_1|^2;
\]
(3) there exist fixed elements \( a_2, b_2, c_2, d_2 \in A_1 \) (\( a_2 \neq b_2 \)) such that for any \( A'' \subset A_1 \) with \(|A''| \geq |A_1|\),
\[
|(b_2 - a_2)A'' - (b_2 - a_2)A'' + (d_2 - c_2)A''| \geq |A_1|^2.
\]

**Lemma 2.4.** Suppose \( A_1 \subset \mathbb{F}_p \) with \( \frac{A_1 - A_1}{A_1} = \mathbb{F}_p \). Then there exist fixed elements \( a_3, b_3, c_3, d_3 \in A_1 \) (\( a_3 \neq b_3 \)) such that for any \( A''' \subset A_1 \) with \(|A'''| \geq |A_1|\),
\[
|(b_3 - a_3)A''' + (d_3 - c_3)A'''| \geq \min\{|A_1|^2, p\}.
\]

**Lemma 2.5.** There exist fixed elements \( a, b, c \) such that for any \( A \subset \mathbb{F}_p \) with \(|A| \geq |A_1|\),
\[
|(b - a)A + (d - c)A| \geq \min\{|A|^2, p\}.
\]
Proof. There exists $\xi \in \mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\}$ (cf. Formula (5) in [9]) such that

$$E^+(A_1, \xi A_1) \leq |A_1|^2 + \frac{|A_1|^4}{p - 1}.$$  

Since $\frac{A_1 - A_1}{A_1 - A_1} = \mathbb{F}_p$, we can write $\xi = \frac{A_1 - A_1}{A_1 - A_1}$ for some $a_3, b_3, c_3, d_3 \in A_1$. Thus

$$|A'' + \xi A''| \geq \frac{|A''|^4}{E^+(A'', \xi A'')} \geq \frac{|A''|^4}{E^+(A_1, \xi A_1)} \geq \frac{|A_1|^4}{E^+(A_1, \xi A_1)} \geq \min\{|a_1|, p\}.$$  

This proves the lemma. \hfill $\Box$

Lemma 2.5. Suppose $Y, Z \subset \mathbb{F}_p$. Choose a fixed element $y_0 \in Y$ so that

$$\sum_{y \in Y} |y_0 Z \cap yZ| \geq \frac{E^\times(Y, Z)}{|Y|}.$$  

For each $j \leq \lfloor \log_2 |Z| \rfloor$, let $Y_j$ be the set of all $y \in Y$ for which $|y_0 Z \cap yZ| \in N_j$, where $N_1 = \{1, 2\}, N_2 = \{3, 4\}, N_3 = \{5, 6, 7, 8\}, N_4 = \{9, 10, 11, 12, 13, 14, 15, 16\}, \ldots$. Then

$$\max_j 16^j |Y_j|^3 \geq \frac{E^\times(Y, Z)^4}{|Y|^4 |Z|^4}. $$  

Proof. Define $j_s = \max\{j : |Y_j| \in N_s\}$ for each $s \leq \lfloor \log_2 |Z| \rfloor$ (assume $m = 0$). Clearly,

$$\sum_{s: j_s \geq 1} 2^{j_s} 2^s \sim \sum_{j=1}^{\lfloor \log_2 |Z| \rfloor} 2^j |Y_j| \sim \sum_{y \in Y} |y_0 Z \cap yZ|.$$  

Note also

$$\sum_{s: j_s \geq 1} 2^{j_s} 2^s \leq \left( \max_{s: j_s \geq 1} 2^{j_s} 2^{0.75s} \right) \sum_{s=1}^{\lfloor \log_2 |Z| \rfloor} 2^{0.25s} \leq \left( \max_{j \geq 1} 2^j |Y_j|^{0.75} \right) \cdot |Z|^{0.25}. $$  

This proves the lemma. \hfill $\Box$

3. Sum-product estimates on small sets

In this section we prove Theorem 1.1. Suppose $A \subset \mathbb{F}_p$ with $|A| \leq p^{0.5}$. Applying Lemma 2.2 twice with $\epsilon = \sqrt{\frac{1}{2}}$, one can find a subset $Z \subset A$ with $|Z| \geq \left\lfloor \frac{|A|}{2} \right\rfloor$ such that

$$|Z \pm Z \pm Z \pm Z| \leq |Z \pm A \pm A \pm A| \leq \left( \frac{|A + A|}{|A|} \right)^3 |Z| \leq \left| \frac{A + A}{|A|^2} \right|^3.$$  

Choose a fixed element $z_0 \in Z$ so that

$$\sum_{z \in Z} |z_0 Z \cap zZ| \geq \frac{E^\times(Z, Z)}{|Z|}.$$  

For each $j \leq \lfloor \log_2 |Z| \rfloor$, let $Z_j$ be the set of all $z \in Z$ for which $|z_0 Z \cap zZ| \in N_j$ (see Lemma 2.5 for the meaning of $N_j$). Then we can deduce from [15, 16] or mimic the proof of Proposition 5.1 (see also Formula (17) in Section 5) to know that

$$\max_j 16^j |Z_j|^3 \leq |Z \pm Z|^5 \cdot |Z \pm Z \pm Z \pm Z|.$$  


By Lemma 2.5,
\[(5)\quad \max_j 16^j |Z_j|^3 \leq \frac{E^x(Z, Z)^4}{|Z|^9} \leq \frac{|Z|^{11}}{|Z^2|^3} \leq \frac{|A|^{11}}{|AA|^3}.\]
Combining (3), (4) and (5) yields
\[|A + A|^8 |AA|^4 \geq |A|^{13}.\]
This proves Theorem 1.1.

**Remark 3.1.** To establish (1), Bourgain and Garaev [2] actually proved that for any \(A \subset \mathbb{F}_p\) one has
\[E^x(A, A)^4 \leq (|A - A| + \frac{|A|^3}{p}) \cdot |A|^5 \cdot |A - A|^4 \cdot |A + A - A| \cdot (\log |A|)^4.\]
Particularly, if \(|A| \leq p^{0.5}\), then
\[(6)\quad E^x(A, A)^4 \leq |A|^5 \cdot |A - A|^5 \cdot |A + A - A| \cdot (\log |A|)^4.\]
Based on the arguments in [2] and this section one can drop the logarithmic term in (6):
\[(7)\quad E^x(A, A)^4 \leq |A|^5 \cdot |A - A|^5 \cdot |A + A - A|.\]
Besides, two byproducts of the proof of Theorem 1.1 are the estimates (suppose \(|A| \leq p^{0.5}\)):
\[(8)\quad E^x(A, A)^4 \leq |A|^5 \cdot |A + A|^5 \cdot |A + A + A|,\]
\[(9)\quad E^x(A, A)^4 \leq |A|^3 \cdot |A + A|^8.\]

4. **SUM-PRODUCT ESTIMATES ON LARGE SETS**

In this section we give a proof of Theorem 1.2. Suppose \(A \subset \mathbb{F}_p\) with \(|A| \geq p^{0.5}\). Similar to the analysis in Section 3, there exist a subset \(Z \subset A\) with \(|Z| \geq \frac{|A|}{2}\) such that
\[|Z \pm Z \pm Z \pm Z| \geq \frac{|A + A|^3}{|A|^2},\]
and a fixed element \(z_0 \in Z\) so that
\[\sum_{z \in Z} |z_0 z \cap xZ| \geq \frac{|Z|^3}{|Z^2|} \geq \frac{|A|^3}{|AA|}.\]
For each \(j \leq \lceil \log_2 |Z| \rceil\), let \(Z_j\) be the set of all \(z \in Z\) for which \(|z_0 z \cap xZ| \in N_j\). Choose some \(j_0 \leq \lceil \log_2 |Z| \rceil\) so that
\[2^{j_0} |Z_{j_0}| \geq \frac{|A|^3}{|AA|}.\]
There are two cases to consider.

\((\star)\) Suppose \(\frac{Z_{j_0} - Z_{j_0}}{Z_{j_0}} \neq \mathbb{F}_p\). By Lemma 2.3, \(|Z_{j_0}| \leq p^{0.5}\). Similar to (4) one can establish
\[16^{j_0} |Z_{j_0}|^3 \leq |Z \pm Z|^5 \cdot |Z + Z \pm Z|.\]
Consequently,
\[\frac{|A|^{12}}{|AA|^7} \leq 16^{j_0} |Z_{j_0}|^4 \leq |Z \pm Z|^5 \cdot |Z + Z \pm Z| \cdot |Z_{j_0}| \leq \frac{|A + A|^8}{|A|^2} \cdot p^{0.5},\]
which yields
\begin{equation}
|A + A|^8|AA|^4 \approx \frac{|A|^{14}}{p^{0.5}}.
\end{equation}

(\&) Suppose \(Z_{j_0} - Z_{j_0} = \mathbb{F}_p\). If \(|Z_{j_0}| \leq p^{0.5}\), then follow the analysis in (\&\&\&) to obtain (10). Next suppose \(|Z_{j_0}| \geq p^{0.5}\). Similar to the proof of (4) in [15, 16] one can establish
\[p \leq \left(\frac{|Z + Z|}{2^{j_0}}\right)^4 \cdot |Z + Z + Z + Z|.
\]
Consequently,
\[
\frac{|A|^8}{|AA|^4} \leq \frac{|A|^{12}}{|AA|^4 |Z_{j_0}|^4} \approx 16^{j_0} \leq \frac{|Z + Z|^4 |Z + Z + Z + Z|}{p} \leq \frac{|A + A|^7}{p|A|^2},
\]
which yields
\begin{equation}
|A + A|^7|AA|^4 \approx |A|^{10}p.
\end{equation}
Thus Theorem 1.2 follows from (10) and (11).

5. Sum-product estimates on different sets

In this section we prove Theorem 1.3, Theorem 1.4 and Theorem 1.5 together. Suppose \(A, B \subset \mathbb{F}_p\). Choose a fixed element \(a_0 \in A\) so that
\[
\sum_{a \in A} |aB \cap a_0B| \geq \frac{|A||B|^2}{|AB|}.
\]
For each \(j \leq \lfloor \log_2 |B| \rfloor\), let \(A_j\) be the set of all \(a \in A\) for which \(|aB \cap a_0B| \in N_j\). With such preparation and notations we establish the following proposition (the idea of this proposition is due to Chun-Yen Shen [14, 15, 16]).

**Proposition 5.1.** (a) If \(\frac{A_j - A_j}{A_j - A_j} \neq \mathbb{F}_p\), then
\begin{equation}
16^j |A_j|^3 \approx \frac{|A + B|^{10}}{|A|^3 |B|}.
\end{equation}
(b) If \(\frac{A_j - A_j}{A_j - A_j} = \mathbb{F}_p\), then
\begin{equation}
16^j \cdot \min\{|A_j|^2, p\} \leq \frac{|A + B|^8}{|A|^3}.
\end{equation}
(c) No matter what happens, one always has (12) if \(|A_j| \leq p^{0.5}\).

**Proof.** We only prove this proposition for the case \(\frac{A_j - A_j}{A_j - A_j} \neq \mathbb{F}_p\), and the interested reader can similarly deal the case \(\frac{A_j - A_j}{A_j - A_j} = \mathbb{F}_p\) and (c) without difficulty. By Lemma 2.3 (if \(\frac{A_j - A_j}{A_j - A_j} = \mathbb{F}_p\), then apply Lemma 2.4), one can find \(a, b, c, d \in A_j\) (\(a \neq b\)) such that for any \(E \subset A_j\) with \(|E| \geq 0.5|A_j|\),
\begin{equation}
|(b - a)E + (b - a)E + (d - c)E| \geq |A_j|^2.
\end{equation}
By Lemma 2.1, there exist
\[
| - aA_j - aB \cap a_0B | \leq \frac{| aA_j - aB |}{aB \cap a_0B} \leq \frac{1}{2^j}
\]
translates of \( aB \cap a_0B \) such that these copies can cover 99% of \(-aA_j\), there exist
\[
\frac{| bA_j + bB \cap a_0B |}{bB \cap a_0B} \leq \frac{| bA_j + bB |}{2^j} \leq \frac{1}{2^j}
\]
translates of \( bB \cap a_0B \) such that these copies can cover 99% of \( bA_j \). Similar facts hold for \(-cA_j\) and \( dA_j\) with corresponding translates of \( cB \cap a_0B \) and \( dB \cap a_0B \). Hence there exist a subset \( A' \subset A_j \) covering 80% of \( A_j \), and \( \frac{| A+B |}{2^j} \) translates of \( aB \cap a_0B \) such that these copies can totally cover \(-aA'\), \( \frac{| A+B |}{2^j} \) translates of \( bB \cap a_0B \) such that these copies can totally cover \( bA'\), \( \frac{| A+B |}{2^j} \) translates of \( cB \cap a_0B \) such that these copies can totally cover \(-cA'\), \( \frac{| A+B |}{2^j} \) translates of \( dB \cap a_0B \) such that these copies can totally cover \( dA'\). Thus
\[
| - aA' + bA' - cA' + dA' | \leq \left( \frac{| A+B |}{2^j} \right)^4 |a_0B + a_0B + a_0B + a_0B|.
\]
By Lemma 2.2, there exist an \( E \subset A' \) with \( |E| \geq 0.8|A'| \geq 0.64|A_j| \geq 0.5|A_j| \) such that
\[
| (b - a)E + (b - a)A' + (d - c)A' | \leq \frac{| A' + A' |}{| A' |} \cdot | (b - a)A' + (d - c)A' |.
\]
Combining (14), (15) and (16) yields
\[
|A_j|^3 \leq |A + A| \cdot \left( \frac{| A+B |}{2^j} \right)^4 \cdot |B + B + B|.
\]
Thus we can conclude the proof by simply applying the Plünnecke-Ruzsa inequality:
\[
|A + A| \leq \frac{| A + B |^2}{| B |}, \quad |B + B + B| \leq \frac{ | A + B |^4 }{ | A |^3 }.
\]
\[\blacktriangleleft\]

**Proof of Theorem 1.3:** Suppose \( |A| \preceq p^{0.5}, |B| \preceq p^{0.5} \). Choose \( j_0 \leq [ \log_2 |B| ] \) so that
\[
|A| |B|^2 \preceq |AB| \cdot \log_2 |B|.
\]
By Proposition 5.1 (c),
\[
\frac{|A|^4 |B|^8}{|AB|^4 \cdot ( \log_2 |B| )^4} \preceq |A|^4 \cdot 16^{j_0} \leq |A| \cdot |A_j|^3 \cdot 16^{j_0} \preceq \frac{|A+B|^{10}}{|A|^2 |B|},
\]
which yields
\[
|A + B|^{10} |AB|^4 \preceq \frac{|A|^6 |B|^9}{( \log_2 |B| )^4}.
\]
By symmetry,
\[
|B + A|^{10} |BA|^4 \preceq \frac{|B|^6 |A|^9}{( \log_2 |A| )^4}.
\]
This proves Theorem 1.3. \[\blacktriangleleft\]
Proof of Theorem 1.4: Suppose $|A| \geq p^{0.5}, |B| \geq p^{0.5}$. Choose $j_0 \leq \lfloor \log_2 |B| \rfloor$ so that

$$|A_{j_0}|^2 \geq \frac{|A||B|^2}{|AB| \cdot \log |B|}.$$ 

There are two cases to consider.

(♣) Suppose $A_{j_0} - A_{j_0} \neq \mathbb{F}_p$. By Lemma 2.3, $|A_{j_0}| \leq p^{0.5}$. By Proposition 5.1 (c),

$$|A_{j_0}|^3 16^{j_0} \leq |A + B|^{10} \quad |A||B|.$$ 

Consequently,

$$\frac{|A|^4|B|^8}{|AB|^4 \cdot (\log_2 |B|)^4} \leq 16^{j_0} |A_{j_0}|^4 = 16^{j_0} |A_{j_0}|^3 \cdot |A_{j_0}| \leq \frac{|A + B|^{10}}{|A||B|} \cdot p^{0.5},$$

which yields

$$|A + B|^{10} |AB| \geq \frac{|A|^7|B|^9}{p^{0.5} \cdot (\log_2 |B|)^4}.$$ 

By symmetry,

(18) $$|B + A|^{10} |BA| \geq \frac{|B|^7|A|^9}{p^{0.5} \cdot (\log_2 |A|)^4}.$$ 

(♠) Suppose $A_{j_0} - A_{j_0} = \mathbb{F}_p$. If $|A_{j_0}| \leq p^{0.5}$, then follow the analysis in (♣) to obtain (18). Next suppose $|A_{j_0}| \geq p^{0.5}$, then by Proposition 5.1 (b),

$$p 16^{j_0} \leq \frac{|A + B|^8}{|A|^3}.$$ 

Hence

$$\frac{|B|^8}{|AB|^4 \cdot (\log_2 |B|)^4} \leq \frac{|B|^8 \cdot |A|^4}{|AB|^4 \cdot (\log_2 |B|)^4 \cdot |A_{j_0}|^4} \leq 16^{j_0} \leq \frac{|A + B|^8}{p |A|^3},$$

which yields

$$|A + B|^8 |AB| \geq \frac{p |A|^3 |B|^8}{(\log_2 |B|)^4}.$$ 

By symmetry,

(19) $$|B + A|^8 |BA| \geq \frac{p |B|^3 |A|^8}{(\log_2 |A|)^4}.$$ 

Thus Theorem 1.4 follows from (18) and (19). □

Proof of Theorem 1.5: Suppose $|A| \sim |B| \leq p^{0.5}$. By Proposition 5.1 (c),

$$\max_j 16^j |A_j|^3 \leq \frac{|A + B|^{10}}{|A|^4}.$$ 

By Lemma 2.5,

$$\max_j 16^j |A_j|^3 \geq \frac{E^* (A, B)^4}{|A|^5} \geq \frac{|A|^{11}}{|AB|^4}.$$ 

Consequently,

$$|A + B|^{10} |AB| \geq |A|^{15}.$$ 

This proves Theorem 1.5. □
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