An $L_2$-quotient algorithm for finitely presented groups on arbitrarily many generators

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Abstract

Abstract. We generalize the Plesken-Fabiański $L_2$-quotient algorithm for finitely presented groups on two or three generators to allow an arbitrary number of generators. The main difficulty lies in a constructive description of the invariant ring of $GL(2, K)$ on $m$ copies of $SL(2, K)$ by simultaneous conjugation. By giving this description, we generalize and simplify some of the known results in invariant theory. An implementation of the algorithm is available in the computer algebra system Magma.

Keywords. Finitely presented groups; quotient algorithm; varieties of representations; invariant theory

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1 Introduction

The Plesken-Fabiański $L_2$-quotient algorithm \cite{PF09} takes as input a finitely presented group $G$ on two generators and computes all quotients of $G$ which are isomorphic to $PSL(2, q)$ or $PGL(2, q)$. The algorithm finds all possible prime powers $q$, and also deals with the case when there are infinitely many. This was adapted by Fabiański \cite{Fab09} to allow finitely presented groups on three generators. In particular, the algorithm can decide whether $G$ has infinitely many quotients isomorphic to $PSL(2, q)$ or $PGL(2, q)$, so in some cases it can be used to prove that a finitely presented group is infinite. This has been applied for example in \cite{CHN11}. In this paper, we generalize the algorithm to allow finitely presented groups on an arbitrary number of generators.

The method of Fabiański and Plesken uses the character of representations $F_2 \to SL(2, K)$, where $F_2$ is the free group of rank 2 and $K$ is an arbitrary field. The character is fully determined by the traces of the images of the two generators of $F_2$ and their product. This observation goes as far back as to Vogt \cite{Vog89} and Fricke and Klein \cite{FK65}. Horowitz \cite{Hor72} gives a rigorous proof of this fact, and generalizes it to representations $F_m \to SL(2, K)$ for an arbitrary $m$, by proving that a character is fully determined by $2^m - 1$ traces. While the traces for $m = 2$ are algebraically independent (that is, for every choice of traces for the images of the two generators and their product, there always exists a representation with these traces), this is no longer true for $m > 2$. The problem is thus to describe all relations between the traces, or equivalently, to give a presentation for the invariant ring $K[SL(2, K)^m]^{GL(2, K)}$, where $GL(2, K)$ acts on $m$ copies of $SL(2, K)$ by conjugation. Furthermore, we need this description to be independent of the characteristic of the field $K$. This problem has a long history. Procesi \cite{Pro76} proves that the invariant ring $K[(K^{n \times n})^m]^{GL(n, K)}$ is finitely generated if $K$ has characteristic zero, and Donkin \cite{Don92} generalizes this to arbitrary fields $K$. However, their results are non-constructive. Procesi \cite{Pro84} gives an implicit description of the invariant ring $\mathbb{C}[(\mathbb{C}^{2 \times 2})^m]^{GL(2, \mathbb{C})}$, and Drensky \cite{Dre93} gives an explicit description, however, their results are not valid for fields of characteristic 2. Magnus \cite{Mag80} uses Horowitz’s results to to give a description of the quotient ring of the invariant ring.

We will use the approach of Horowitz and Magnus to get a partial description of the invariant ring. The methods are constructive and the arguments are shorter than the original arguments; at the same time we get more precise results, needed for the algorithm. This theory is developed in Section 2.

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Sections 3 are adaptations of [PF09], where we have to generalize results on characters and traces to work for arbitrarily many generators. Up until the end of Section 4 all results assume that representations restricted to the subgroup generated by the first two generators is absolutely irreducible. The results in Section 5 show how the general case can be reduced to this special case. In Section 6 a new test to recognize epimorphisms onto A4, S4, and A5 is developed, since the test described in [PF09] is inefficient for more than two generators. Section 7 describes the proper notation and theory to deal with an infinite number of L2-quotients. The algorithm is given in Section 8 with several examples in Section 9.

## 2 Fricke characters

Throughout the paper, $K$ is an arbitrary field and $m \geq 2$ an integer, unless specified otherwise. In this section, we adopt the following notation.

**Notation 2.1.** Given matrices $A_1, \ldots, A_m \in \text{SL}(2, K)$ and a list $i_1, \ldots, i_k \in \{\pm 1, \ldots, \pm m\}$, we set $t_{i_1, \ldots, i_k} := \text{tr}(A_{i_1} \cdots A_{i_k})$, where $A_{-i} := A_i^{-1}$ for $i \in \{1, \ldots, m\}$. If $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, m\}$ with $i_1 < i_2 < \cdots < i_k$, then $t_I := t_{i_1, \ldots, i_k}$.

Let $A_1, A_2, A_3 \in \text{SL}(2, K)$. The traces satisfy the following basic identities.

\begin{align*}
t_{1,1,2} &= t_1 t_{1,2} - t_2, & (1) \\
t_{-1,2} &= t_1 t_2 - t_{1,2}, & (2) \\
t_{1,2,1,3} &= t_1 t_{2,1,3} + t_{2,3} - t_{2,2,3}, & (3) \\
t_{1,3,2} &= -t_{1,2,3} + t_{1,2,3} + t_{2,1,3} + t_{3,1,2} - t_{1,2,3}. & (4)
\end{align*}

The first two identities are easy consequences of the Cayley-Hamilton Theorem, and the others are easy consequences of the first two (for 3 consider $\text{tr}((A_1 A_2)^2 (A_2^{-1} A_3))$; for 4 consider $\text{tr}(A_1^{-1} (A_2^{-1} A_3)) = \text{tr}((A_2 A_1)^{-1} A_3)$).

We first prove that all traces of words in the $A_i$ are consequences of the $t_I$ with $\emptyset \neq I \subseteq \{1, \ldots, m\}$. This was already observed by Vogt [Vog89] and later by Fricke and Klein [FK65]. The first rigorous proof of this fact was given by Horowitz [Hor72], and a shorter proof by Fabiańska and Plesken [PF09].

Let $F_m$ be the free group of rank $m$, generated by $g_1, \ldots, g_m$.

**Theorem 2.2 ([Hor72] Theorem 3.1, [PF09] Lemma 2.1).** Let $X_m := \{x_I \mid \emptyset \neq I \subseteq \{1, \ldots, m\}\}$ be a set of indeterminates over $K$. For every $w \in F_m$ there exists a polynomial $\tau(w) \in K[X_m]$, such that for every field $K$ and every $m$-tuple $A = (A_1, \ldots, A_m) \in \text{SL}(2, K)^m$,

$$
\text{tr}(w(A_1, \ldots, A_m)) = \varepsilon_A(\tau(w))
$$

where $\varepsilon_A : K[X_m] \to K$ is the evaluation map which sends $x_I$ to $t_I$.

Since the proof in [Hor72] is lengthy, and the result in [PF09] is not as general, we present a short proof here in its entirety. The basic idea is that of [PF09] Lemma 2.1.

**Proof.** We assume that $w$ is freely and cyclically reduced and proceed by induction on the length of $w$. If $w$ is conjugate to $g_i^{-1} w'$ for some $w' \in F_n$ of length $|w| \geq 1$, set $\tau(w) = \tau(g_i w') - \tau(g_i) \tau(w')$. Thus we may assume that all exponents of $w$ are positive. If $w$ is conjugate to $g_i w g_j w''$ for some $w', w'' \in F_m$ with $|w'| + |w''| = |w| - 2$, set $\tau(w) = \tau(g_i w') \tau(g_j w'') + \tau(w') \tau(w'') - \tau(w') \tau(w'')$. We are left to deal with the case where $w$ is of the form $w = g_{i_1} \cdots g_{i_k}$ where the $i_j$ are pairwise distinct. We may assume $i_1 < i_j$ for all $j \in \{2, \ldots, k\}$. The case $i_1 = \cdots = i_k$ is the induction basis, so there is nothing to do. Otherwise, let $j$ be the smallest index with $i_j > i_{j+1}$. Set $w_1 := g_{i_1} \cdots g_{i_{j-1}}$, $w_2 := g_{i_j}$, and $w_3 := g_{i_{j+1}} \cdots g_{i_k}$, so $w = w_1 w_2 w_3$. By equation 4 we may set $\tau(w) := -\tau(w_1 w_3 w_2) + \tau(w_1) \tau(w_2 w_3) + \tau(w_2) \tau(w_1 w_3) + \tau(w_3) \tau(w_1 w_2) - \tau(w_1) \tau(w_2) \tau(w_3)$. Either $w_1 w_3 w_2$ is of the desired form, or we repeat this process. This terminates after finitely many steps. \qed
We call \( \tau(w) \) the trace polynomial of \( w \). If \( n > 2 \), then \( \tau(w) \) is not unique. For example, define the Fricke polynomial

\[
\phi(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}) := x_{12}^2 + (x_1x_2x_3 - x_1x_{23} - x_2x_{13} - x_3x_{12})x_{123} \\
+ x_1^2 + x_2^2 + x_3^2 + x_{12}^2 + x_{13}^2 + x_{23}^2 - x_1x_2x_{12} - x_1x_3x_{13} - x_2x_3x_{23} + x_{12}x_{13}x_{23} - 4.
\]

Then \( \epsilon_A(\phi) = 0 \) for every choice of \( A \). Proofs appear for example in [Hor72, Section 2] and [Mag80, Lemma 2.2]. We will see below that \( \phi \) is simply a determinant condition (see Proposition 2.4 and Corollary 2.5).

A lot of effort has been put into describing all polynomial relations between the traces. More precisely, let

\[
I_m := \{ f \in \mathbb{Z}[X_m] \mid \epsilon_A(f) = 0 \text{ for all } A_1, \ldots, A_n \in \text{SL}(2, \mathbb{C}) \}
\]

and \( \Phi_m := \mathbb{Z}[X_m]/I_m \), the ring of Fricke characters. It is easy to see that \( \epsilon_A(f) = 0 \) for all \( A \in \text{SL}(2, K)^m \), so the role of \( \mathbb{C} \) is not special. Horowitz [Hor72] Theorem 4.3] proves \( I_3 = (\phi) \), and Whittmore [Whi73, Theorem 1] proves that \( I_m \) is not principal if \( m \geq 4 \). Magnus [Mag80, Theorem 2.1] shows that \( \Phi_m \) can be embedded into a finitely generated extension field of \( \mathbb{Q} \) of transcendence degree \( 3m - 3 \). Note that \( \Phi_m \otimes \mathbb{C} \) is isomorphic to the invariant ring \( \mathbb{C}[\text{SL}(2, \mathbb{C})]^m_{GL(2, \mathbb{C})} \). Procesi [Pro84] gives a description of the invariant ring \( \mathbb{C}(\mathbb{C}^{2\times 2})^m_{GL(2, \mathbb{C})} \), and an explicit presentation of the invariant ring with generators and relations is given by Drensky [Dre03, Theorem 2.3]. However, these results are not valid for fields of characteristic 2, and hence cannot be applied to describe \( \Phi_m \).

Our first aim is to partially describe \( \Phi_m \); we give a presentation of a localisation of \( \Phi_m \), which will be enough for our algorithmic applications. By doing that, we will also find new and shorter proofs of some of the results mentioned above.

We will use the following basic result.

**Proposition 2.3** ([Mac69, Theorem 2], [Mag80, Equation (2.7)], [BH95, Proposition 4.1], [PF09, Proposition 3.1]). Let \( A = (A_1, A_2) \in \text{SL}(2, K)^2 \). Then \( (A_1, A_2) \) is absolutely irreducible if and only if \( (t_1, t_2, t_{12}) \) is a zero of

\[
\rho := x_1^2 + x_2^2 + x_{12}^2 - x_1x_2x_{12} - 4.
\]

This is based on the fact that \( (A_1, A_2) \) is absolutely irreducible if and only if \( (I_2, A_1, A_2, A_1A_2) \) is a \( K \)-basis of \( K^{2\times 2} \) (see for example [PF09]), a result which we will also use several times.

The main result in this section shows that two matrices \( A_1, A_2 \) uniquely determine an arbitrary matrix by the specification of four traces; it also shows that the Fricke polynomial is really a determinant condition. The basic idea of the proof has already been used by Brumfiel and Hilden [BH95, Proposition B.4].

**Proposition 2.4.** Let \( A_1, A_2 \in \text{SL}(2, K) \) such that \( (A_1, A_2) \) is absolutely irreducible, and let \( i \geq 3 \). Given \( T_i, T_{1i}, T_{2i}, T_{12i} \in K \), there exists a unique \( A_1 \in K^{2\times 2} \) such that \( t_i = T_i \) for all \( i \in \{0, 1, 2, 3\} \). Moreover, \( \det(A_1) = 1 \) if and only if \( \phi(t_1, \ldots, t_{12}) = 0 \).

More precisely, let

\[
\lambda_i^0 := (x_1^2 + x_2^2 + x_{12}^2 - x_1x_2x_{12} - 2)x_i - x_1x_{1i} - x_2x_{2i} + (x_1x_2 - x_{12})x_{12i}, \\
\lambda_i^1 := -x_1x_i - x_2x_{12i} + x_{12}x_{2i} + 2x_1i, \\
\lambda_i^2 := -x_2x_i - x_1x_{12i} + x_{12}x_{1i} + 2x_2i, \\
\lambda_i^{12} := -x_1x_{2i} - x_{12}x_{1i} - x_2x_{2i} + x_1x_{2i} + 2x_{12i};
\]

set \( A_i := \lambda_i^1(t_1, t_2, T_i, t_{1i}, t_{12i}, T_{1i}, T_{2i}, T_{12i}) \) for \( i \in \{0, 1, 2, 3\} \). Then

\[
A_i = \frac{1}{\rho(t_1, t_2, t_{12})}(A_0I_2 + A_1A_1 + A_2A_2 + A_{12}A_1A_2).
\]

**Proof.** Since \( (A_1, A_2) \) is absolutely irreducible, \( (I_2, A_1, A_2, A_1A_2) \) is a \( K \)-basis of \( K^{2\times 2} \). Thus if \( A_i \) exists as in the statement, then \( A_i = \mu_0I_2 + \mu_1A_1 + \mu_2A_2 + \mu_{12}A_1A_2 \) for some \( \mu_i \in K \). Multiplying the equation...
from the left by the matrices $I_2, A_1, A_2, A_3$ and taking traces shows that the $\mu_i$ are the unique solution of
\[
\begin{pmatrix}
2 & t_1 & t_2 & t_{12} \\
t_1 & t_1^2 - 2 & t_{12} & t_{1}t_{12} - t_2 \\
t_2 & t_{12} & t_2^2 - 2 & t_{2}t_{12} - t_1 \\
t_{12} & t_{1}t_{12} - t_2 & t_{2}t_{12} - t_1 & t_{12}^2 - 2
\end{pmatrix}
\begin{pmatrix}
\mu_0 \\
\mu_1 \\
\mu_2 \\
\mu_{12}
\end{pmatrix}
= \begin{pmatrix}
T_1 \\
T_{11} \\
T_2 \\
T_{121}
\end{pmatrix},
\]
which is given by $\mu_i = \Lambda_i/\rho(t_1, t_2, t_{12})$. This proves the uniqueness and existence of $A_i$. It remains to show the determinant condition. We use the idea of [PF09, Proposition 3.1]. Let $\alpha$ be a root of $X^2 - t_1X + 1$; by enlarging $K$ if necessary, we may assume $\alpha \in K$. Let $v_1 \in K^{2 \times 1}$ be an eigenvector of $A_1$ with eigenvalue $\alpha$. Set $v_2 := A_2v_1$, and let $M \in \operatorname{GL}(2, K)$ be the matrix with columns $v_1$ and $v_2$. Set $B_j := M^{-1}A_jM$ for $j \in \{1, 2, 1\}$. Then
\[
B_1 = \begin{pmatrix}
\alpha & t_2(\alpha - t_1) + t_{12} \\
0 & t_1 - \alpha
\end{pmatrix}
\quad \text{and} \quad
B_2 = \begin{pmatrix}
0 & -1 \\
1 & t_2
\end{pmatrix}.
\]
Since $B_1 = 1/\rho(t_1, t_2, t_{12})(\Lambda_0I_2 + \Lambda_1B_1 + \Lambda_2B_2 + \Lambda_{12}B_1B_2)$,
\[
\det(A_i) = \det(B_i) = \frac{\phi(t_1, \ldots, t_{12}) + \rho(t_1, t_2, t_{12})}{\rho(t_1, t_2, t_{12})},
\]
which concludes the proof.

**Corollary 2.5.** Let $A_1, A_2, A_3 \in \operatorname{SL}(2, K)$. The $t_I$ satisfy the Fricke relation, that is, $\phi(t_1, \ldots, t_{123}) = 0$.

**Proof.** We may assume without loss of generality that $K$ is algebraically closed. By Proposition 2.4 the statement is true for the Zariski-open subset $U = \{(A_1, A_2, A_3) \in \operatorname{SL}(2, K)^3 \mid \rho(\operatorname{tr}(A_1), \operatorname{tr}(A_2), \operatorname{tr}(A_1A_2)) \neq 0\}$, so by continuity, it is true for all elements in $\operatorname{SL}(2, K)^3$.

The following is a generalization of [Mag80, Theorem 2.2] and [PF09, Proposition 3.1]. Set $T_m := \{(i) \mid 1 \leq i \leq m\} \cup \{(i, j) \mid 1 \leq i, j \leq m\} \cup \{(1, 2, k) \mid 3 \leq k \leq m\}.

**Corollary 2.6.** Let $T_I \in K$ for $I \in T_m$ such that $\rho(T_1, T_2, T_{12}) \neq 0$ and $\rho(T_1, T_2, T_3, T_{123}) = 0$ for all $3 \leq k \leq m$.

Let $L$ be the splitting field of $X^2 - T_1X + 1 \in K[X]$. There exists $A = (A_1, \ldots, A_m) \in \operatorname{SL}(2, L)^m$ such that $t_I = T_I$ for all $I \in T_m$, and $A$ is unique up to conjugation by $\operatorname{GL}(2, L)$.

There exists $A \in \operatorname{SL}(2, K)^m$ such that $t_I = T_I$ for all $I \in T_m$ if and only if $\rho(T_1, T_2, T_{12}) = N_{L/K}(z)$ for some $z \in L$. In this case, $A$ is unique up to conjugation by $\operatorname{GL}(2, K)$.

**Proof.** By [PF09, Proposition 3.1], there exists $A' := (A_1, A_2) \in \operatorname{SL}(2, L)^2$ with $t_1 = T_1$, $t_2 = T_2$, and $t_{12} = T_{12}$; furthermore, $A'$ is unique up to $L$-equivalence, and $A'$ exists in $\operatorname{SL}(2, K)^2$ if and only if $\rho(T_1, T_2, T_{12})$ is a norm, in which case $A'$ is unique up to $K$-equivalence. By Proposition 2.4 the choice of $A'$ and the $T_I$ uniquely determines the matrices $A_3, \ldots, A_m$.

This result implies that the traces $t_j$ with $J \not\subset I_m$ can be expressed in the traces $t_I$ with $I \in T_m$ if $\rho(t_1, t_2, t_{12}) \neq 0$. The next result gives the precise formulae.

**Proposition 2.7.** Let $A_1, \ldots, A_n \in \operatorname{SL}(2, K)$. Let $3 \leq i < n$ and $\emptyset \not\subset \{i, 1, \ldots, n\}$. The tuple $t = (t_j \mid \emptyset \not\subset I \subset \{1, \ldots, n\})$ is a zero of the polynomials
\[
\begin{align*}
x_{i1}\rho - (\lambda_0x_j + \lambda_1x_{1j} + \lambda_2x_{2j} + \lambda_{12}x_{12j}), \\
x_{i2}\rho - (\lambda_0x_{1j} + \lambda_1(x_1x_{1j} - x_j) + \lambda_2x_{2j} + \lambda_{12}(x_1x_{12j} - x_2j)), \\
x_{2i}\rho - (\lambda_0x_{2j} + \lambda_1(-x_{12j} + x_2j + x_2x_{1j} + x_jx_{12j} - x_1x_2x_{1j} + \lambda_2(x_2x_{2j} - x_j) + \lambda_{12}(x_{12j} - x_1x_{1j})), \\
x_{12i}\rho - (\lambda_0x_{12j} + \lambda_1(x_1x_{12j} - x_2j + x_{2j}) + \lambda_2(x_2x_{12j} - x_{1j}) + \lambda_{12}(x_{12j} - x_{1j})).
\end{align*}
\]
Proof. It is enough to prove the statement if $\rho \neq 0$ (see proof of Corollary 2.6). By Proposition 2.4, we see $A_i = A_0 A_2 + A_1 A_1 + A_2 A_2 + A_1 A_2$. Multiplying from the right by $A_j$ and from the left by $I_2, A_1, A_2, A_1 A_2$ and taking traces yields the result. \hfill \square

For a ring $R$ and $r \in R$, let $R_r$ denote the localisation of $R$ at the set $\{1, r, r^2, \ldots\}$. While we do not have an explicit description of the ring $\Phi_m$ of Frick characters, we have one for a localisation of $\Phi_m$.

**Corollary 2.8.** Let
\[
\Phi'_m := (\mathbb{Z}[x_I \mid I \in I_m]/\langle \phi_{123}, \ldots, \phi_{12m} \rangle)_\rho,
\]
where $\phi_{12i} := \phi(x_1, x_2, x_i, x_{12}, x_{11}, x_{2i}, x_{12i})$ for $3 \leq i \leq m$ and $\rho := \rho(x_1, x_2, x_{12})$. The ring homomorphism $\Phi'_m \to (\Phi_m)_\rho$ defined by $x_I \mapsto x_I$ is an isomorphism.

Proof. Define a ring homomorphism $\alpha : \mathbb{Z}[x_I \mid I \in I_m] \to (\Phi_m)_\rho$ by mapping $x_I$ to $x_I$. By Proposition 2.7, $\alpha$ is surjective, and by Proposition 2.3, $\alpha$ factors over $\Phi'_m$. But Proposition 2.4 also shows that the induced map is injective (see also Corollary 2.6). \hfill \square

**Corollary 2.9 (Mang58 Theorem 2.1]).** The quotient field of $\Phi'_m$ is isomorphic to a $(m-2)$-fold quadratic extension of a rational function field of transcendence degree $3m - 3$ over $\mathbb{Q}$.

### 3 Trace tuples

The ultimate goal is to get a bijection between prime ideals of $\Phi'_m$ and equivalence classes of representations $F_m \to \text{SL}(2, K)$, where $K$ ranges over all fields.

**Definition 3.1.** A tuple $t = (t_I \mid I \in I_m) \subseteq K^{2m}$ is a trace tuple if
\[
\rho(t_1, t_2, t_{12}) \neq 0 \quad \text{and} \quad \phi(t_1, t_2, t_i, t_{1i}, t_{2i}, t_{2i}) = 0 \quad \text{for all} \ 3 \leq i \leq m.
\]

If $A \in \text{SL}(2, K)^m$ such that $(A_1, A_2)$ is absolutely irreducible, then the traces $t_I = \text{tr}(A_{i1} \cdots A_{ik})$ for $I = \{i_1 < \cdots < i_k\} \subseteq I_m$ form a trace tuple. We call this the trace tuple of $A$, and $A$ a realization of $t$. The tuple $(t_I \mid \emptyset \neq J \subseteq \{1, \ldots, m\})$ is the full trace tuple of $A$.

**Definition 3.2.** Let $\Gamma$ be a group generated by $\gamma_1, \ldots, \gamma_m$. Let
\[
\mathcal{R}(\Gamma, K) := \{ \Delta : \Gamma \to \text{SL}(2, K) \mid \Delta|_{\langle \gamma_1, \gamma_2 \rangle} \text{ is absolutely irreducible} \}.
\]

**Remark 3.3.** The set $\mathcal{R}(\Gamma, K)$ is in bijection to the set of matrices $A \in \text{SL}(2, K)^m$ such that $(A_1, A_2)$ is absolutely irreducible, so we may talk about trace tuples of $\Delta$ and regard representations as realizations of trace tuples.

We will first prove the results for finite fields and then generalize to arbitrary fields.

#### 3.1 Finite fields

**Definition 3.4.** Let $t, t' \in \mathbb{F}_q^{2m}$ be trace tuples. Let $L$ and $L'$ be the subfields of $\mathbb{F}_q$ generated by $t$ and $t'$, respectively. We say that $t$ and $t'$ are equivalent if there exists an isomorphism $\alpha : L \to L'$ such that $\alpha(t_I) = t'_I$ for all $I \in I_m$.

**Remark 3.5.** By Corollary 2.6, every trace tuple $t \in \mathbb{F}_q^{2m}$ has a realization $A \in \text{SL}(2, \mathbb{F}_q)^m$.

Let $t \in \mathbb{F}_q^{2m}$ be a trace tuple. Define a ring homomorphism $\alpha_t : \Phi'_m \to \mathbb{F}_q$ by $\alpha_t(x_I) := t_I$ for $I \in I_m$. Then $P_t := \ker(\alpha_t)$ is a maximal ideal of $\Phi'_m$.

Conversely, let $P \in \text{MaxSpec}(\Phi'_m)$, where $\text{MaxSpec}(\Phi'_m)$ denotes the set of maximal ideals of $\Phi'_m$. Let $F_q = \Phi'_m/P$, and set $(t_P)_I := x_I + P \in F_q$ for $I \in I_m$. Then $t_P := ((t_P)_I \mid I \in I_m) \in \mathbb{F}_q^{2m}$ is a trace tuple.

**Theorem 3.6.** The maps $P \mapsto t_P$ and $t \mapsto P_t$ induce mutually inverse bijections between $\text{MaxSpec}(\Phi'_m)$ and the set of equivalence classes of trace tuples over finite fields.
Proof. Let $P \in \text{MaxSpec}(\Phi_m')$. Since $\alpha_t(x_I) = x_I + P$ by definition, we see $P = P_t$. Now let $t \in \mathbb{F}^m_q$ be a trace tuple; we may assume that $\mathbb{F}^m_q$ is generated by $t$. Then $\Phi_m'/P_t$ is a field with $q$ elements. Define a homomorphism $\mathbb{F}^m_q \to \Phi_m'/P_t$ by $t_I \mapsto x_I + P_t$. By definition of $P_t$ this is well-defined and it is clearly surjective, hence an isomorphism; it maps $t$ to $t_P$, so $t$ is equivalent to $t_P$. \hfill $\square$

If $q|q'$, then we can embed $\mathcal{R}(F_m, \mathbb{F}_q)$ into $\mathcal{R}(F_m, \mathbb{F}_{q'})$, and we can embed $\mathcal{R}(F_m, \mathbb{F}_q) / \Gamma L(2, q)$ into $\mathcal{R}(F_m, \mathbb{F}_q') / \Gamma L(2, q')$ (where $\Gamma L(2, q)$ acts on $\mathcal{R}(F_m, \mathbb{F}_q)$ by composition).

**Corollary 3.7.** There is a bijection between $\text{MaxSpec}(\Phi_m')$ and $\bigcup_q \mathcal{R}(F_m, \mathbb{F}_q) / \Gamma L(2, q)$, where $q$ ranges over all prime powers.

**Proof.** This follows by Theorem 3.6 and Corollary 2.6. \hfill $\square$

### 3.2 Arbitrary fields

**Definition 3.8.** Let $K$ and $K'$ be fields. Let $t \in K^{I_m}$ and $t' \in (K')^{I_m}$ be trace tuples, and let $S$ and $S'$ be the rings generated by $t$ and $t'$, respectively. We say that $t$ and $t'$ are equivalent if there exists a ring isomorphism $\alpha: S \to S'$ such that $\alpha(t_I) = t'_I$ for all $I \in I_m$.

**Remark 3.9.** By Corollary 2.6, every trace tuple $t \in K^{I_m}$ has a realization, but in general we must allow field extensions. That is, there exist matrices $A \in \text{SL}(2, L)^m$ with $t_I = \text{tr}(A_{i_1} \cdots A_{i_k})$ for all $I = \{i_1 < \cdots < i_k\} \in I_m$, where $L$ is either $K$ or a quadratic extension of $K$.

Let $t \in K^{I_m}$ be a trace tuple. Define a ring homomorphism $\alpha_t: \Phi_m' \to K$ by $\alpha_t(x_I) := t_I$ for $I \in I_m$. Then $P_t := \ker(\alpha_t)$ is a prime ideal of $\Phi_m'$.

Conversely, let $P \in \text{Spec}(\Phi_m')$, where $\text{Spec}(\Phi_m')$ denotes the set of prime ideals of $\Phi_m'$. Let $K$ be the quotient field of $\Phi_m'/P$; set $(t_P)_I := x_I + P \in K$ for $I \in I_m$. Then $t_P := ((t_P)_I | I \in I_m) \in K^{I_m}$ is a trace tuple.

**Theorem 3.10.** The maps $P \mapsto t_P$ and $t \mapsto P_t$ induce mutually inverse bijections between $\text{Spec}(\Phi_m')$ and the set of equivalence classes of trace tuples.

### 4 Actions

**Definition 4.1.** Let $\Sigma_m := \{\pm 1\}^m$, the group of sign changes. Let $\Delta \in \mathcal{R}(F_m, K)$, and let $\chi: F_m \to \mathbb{F}_q^*: w \mapsto \text{tr}(\Delta(w))$ be the character of $\Delta$. Let $t \in K^{I_m}$ be a trace tuple.

1. Let $\sigma \in \Sigma_m$. Define

$$\sigma \Delta: F_m \to \text{SL}(2, K): w \mapsto w(\sigma)\Delta(w);$$

$$\sigma \chi: F_m \to K: w \mapsto w(\sigma)\chi(w);$$

and

$$\sigma t_I := \left(\prod_{\sigma I} \sigma_I\right)t_I.$$

This defines actions of $\Sigma_m$ on representations, characters, and trace tuples.

2. Let $\sigma \in \Sigma_m$. Define a ring automorphism on $\Phi_m'$ by mapping $x_I$ to $\left(\prod_{\sigma I} \sigma_I\right)x_I$. This defines an action of $\Sigma_m$ on $\Phi_m'$ by automorphisms, and hence an action on the set of ideals of $\Phi_m'$.

3. Let $\alpha \in \text{Gal}(K)$. Define

$$\alpha \Delta: F_m \to \text{SL}(2, K): w \mapsto \alpha(\Delta(w));$$

$$\alpha \chi: F_m \to K: w \mapsto \alpha(\chi(w));$$

and

$$\alpha t_I := \alpha(t_I).$$

This defines actions of $\text{Gal}(K)$ on representations, characters, and trace tuples.
Remark 4.2. The actions are compatible with the various bijections. More precisely, let \( \Delta \in \mathcal{R}(F_m, K) \), let \( t \in K^{2m} \) be a trace tuple, and let \( P \in \text{Spec}(\Phi_m') \). Denote by \( \chi_\Delta \) the character of \( \Delta \) and by \( t_\Delta \) the trace tuple of \( \Delta \). Then

\[
\chi(\sigma \Delta) = \sigma(\chi_\Delta), \quad t_\sigma(\Delta) = \sigma(t_\Delta), \quad P_\sigma(\Delta) = \sigma(P_\Delta), \quad \text{and} \quad t_\sigma(P) = \sigma(t_P)
\]

for all \( \sigma \in \Sigma_m \), and

\[
\chi(\alpha \Delta) = \alpha(\chi_\Delta) \quad \text{and} \quad t_\alpha(\Delta) = \alpha(t_\Delta)
\]

for all \( \alpha \in \text{Gal}(K) \).

5 Projective representations and finitely presented groups

Definition 5.1. Let \( \Gamma \) be a group generated by \( \gamma_1, \ldots, \gamma_m \). Set

\[
\mathcal{P}(\Gamma, K) := \{ \delta: \Gamma \to \text{PSL}(2, K) \mid \delta(\gamma_1, \gamma_2) \text{ is absolutely irreducible} \}.
\]

Theorem 5.2. There is a bijection between \( \text{MaxSpec}(\Phi_m')/\Sigma_m \) and \( \bigcup_q \mathcal{P}(F_m, \mathbb{F}_q)/\text{PTL}(2, q) \), where \( q \) ranges over all prime powers.

Proof. This follows from Corollary 3.7, since two representations \( \Delta, \Delta': F_m \to \text{SL}(2, q) \) induce the same projective representation if and only if \( \Delta' = \sigma \Delta \) for some \( \sigma \in \Sigma_m \).

Definition 5.3. Let \( G = \langle g_1, \ldots, g_m \mid w_1, \ldots, w_r \rangle \) be a finitely presented group. For \( s \in \{\pm 1\}^r \) define

\[
I_s(G) := \{ \tau(w_i b) - s_i \tau(b) \mid 1 \leq i \leq r, \ b \in \{1, g_1, g_2, g_1 g_2\} \} \subseteq \Phi_m',
\]

the trace presentation ideal of \( G \) with respect to the sign system \( s \). (We regard the \( \tau(w) \) as elements of \( \Phi_m' \) via the isomorphism of Corollary 2.8.) Set \( I(G) := \bigcap_{s \in \{\pm 1\}^r} I_s(G) \), the full trace presentation ideal of \( G \).

The following result is a reformulation of [PF09, Proposition 3.3].

Proposition 5.4. Let \( G \) be a finitely presented group. Let \( \Delta \in \mathcal{R}(F_m, \text{SL}(2, K)) \) with trace tuple \( t \in K^{2m} \) and prime ideal \( P = P_1 \in \text{Spec}(\Phi_m') \). The following are equivalent:

1. The representation \( \Delta \) induces a projective presentation \( \delta: G \to \text{PSL}(2, K) \).
2. The trace tuple \( t \) is a zero of \( I(G) \).
3. The prime ideal \( P \) contains \( I(G) \).

Proof. The equivalence of (2) and (3) is immediate. We prove the equivalence of (1) and (2). Let \( A_s := \Delta(g_i) \). Then \( \Delta \) induces a projective representation of \( G \) if and only if \( w_i(A_1, \ldots, A_m) = s_i I_2 \) for some \( s = (s_1, \ldots, s_r) \in \{\pm 1\}^r \). Since the trace bilinear form is non-degenerate, this is equivalent to \( \text{tr}(w_i(A_1, \ldots, A_m)B) - s_i \text{tr}(B) = 0 \), where \( B \) runs through a basis of \( K^{2 \times 2} \). Since \( \langle A_1, A_2 \rangle \) is absolutely irreducible, we can choose the basis \( (I_2, A_1, A_2, A_1 A_2) \).

Corollary 5.5. There is a bijection between the maximal elements of \( \text{V}(I(G))/\Sigma_m \), where \( \text{V}(I(G)) = \{ P \in \text{Spec}(\Phi_m') \mid I(G) \subseteq P \} \) and \( \bigcup_q \mathcal{P}(G, \mathbb{F}_q)/\text{PTL}(2, q) \), where \( q \) ranges over all prime powers.

6 Subgroups

Corollary 5.5 describes a bijection between classes of maximal ideals and classes of absolutely irreducible projective representations. In this section, we establish criteria to decide whether a maximal ideal is mapped to a surjective projective representation.

According to Dickson’s classification (see for example [Suz82, Section 3.6]), an absolutely irreducible subgroup \( U \subseteq \text{PSL}(2, q) \) is


- isomorphic to $A_4$, $S_4$, or $A_5$, or
- a dihedral group, or
- isomorphic to $\text{PGL}(2, q')$ for some $q' | r$ if $q = r^2$ is a square, or
- isomorphic to $\text{PSL}(2, q')$ for some $q' | q$.

For a finite group $H$ let $J(H) := \bigcap_{Q} I(G)$, where $G$ ranges over all presentations of $G$ on $m$ generators.

**Proposition 6.1.** Let $H$ be a finite group. Set $J'(H) := (J(H)) : \left( \bigcap_{Q} J(Q) \right) \subseteq \Phi'_m$, where $Q$ ranges over all proper quotients of $H$.

Let $\Delta \in \mathcal{R}(F_m, \text{SL}(2, K))$ with trace tuple $t \in K^{2m}$ and prime ideal $P = P_t \in \text{Spec}(\Phi'_m)$. The following are equivalent:

1. The representation $\Delta$ induces a projective presentation $\delta$ such that $\text{im}(\delta) \cong H$.
2. The trace tuple $t$ is a zero of $J'(H)$.
3. The prime ideal $P$ contains $J'(H)$.

**Proof.** It suffices to prove the equivalence of (1) and (2). By Proposition 6.3, $\delta$ factors over $H$ if and only if $t$ is a zero of $J(H)$, and it factors over $Q$ if and only if $t$ is a zero of $J(Q)$. But $t$ is a zero of $J'(H)$ if and only if it is a zero of $J(Q)$ for any proper quotient $Q$ of $H$, which proves the proposition.

We will later let $H$ be one of the groups $A_4$, $S_4$, or $A_5$, which deals with the first kind of subgroups. We handle the dihedral groups in a slightly more general context.

**Lemma 6.2.** Let $t \in K^{2m}$ be a trace tuple. Let $\emptyset \neq J \subseteq \{1, \ldots, m\}$. If $t_I = 0$ for all $I \in \mathcal{I}_m$ with $|I \cap J|$ odd, then $t_I = 0$ for all $\emptyset \neq I \subseteq \{1, \ldots, m\}$ with $|I \cap J|$ odd.

**Proof.** Assume $I \notin \mathcal{I}_m$ with $|I \cap J|$ odd. We proceed by induction on $|I|$. We assume that $I \cap \{1, 2\} = \emptyset$; the other cases are analogous. Let $i$ be the minimum of $I$, and let $j := I - \{i\}$. By Proposition 2.7, $t_I = t_J = 1/\rho(t) (\lambda_0(t) t_J + \lambda_1(t) t_{JJ} + \lambda_2(t) t_{JJJ} + \lambda_{12}(t) t_{JJJJ})$. There are eight cases to consider; we give the proof for two of them, the other six are analogous. The first case is $1, 2, i \notin J$; the sets $j, \{1\} \cup j$, $\{2\} \cup j$, and $\{1, 2\} \cup j$ have odd intersection with $J$, thus $t_J = t_{JJ} = t_{JJJ} = t_{JJJJ} = 0$ by induction. The formula for $t_{JJ}$ shows that $t_{JJ} = 0$. The second case is $1 \in J$ but $2, i \notin J$; now $t_1 = t_2 = t_i = t_{12} = t_{11} = t_{2i} = t_{12i} = t_{JJ} = t_{JJJ} = 0$. By Proposition 2.4, $\lambda_1(t) = 0$, so $t_{JJ} = 0$.

Let $\Delta \in \mathcal{R}(F_m, K)$; then $\Delta$ is imprimitive if $K^{2 	imes 1} = V_1 \oplus V_2$ for one-dimensional subspaces $V_1, V_2 \subseteq K^{2 	imes 1}$ such that $\Delta$ permutes the $V_i$ transitively.

**Proposition 6.3.** Let $K$ be an algebraically closed field. Let $\Delta \in \mathcal{R}(F_m, K)$, and let $t$ be its trace tuple. Then $\Delta$ is imprimitive if and only if there exists $\emptyset \neq J \subseteq \{1, \ldots, m\}$ such that $t_I = 0$ for all $I \in \mathcal{I}_m$ with $|I \cap J|$ odd.

**Proof.** Let $\chi : F_m \to \mathbb{F}_q : w \mapsto \text{tr}(\Delta(w))$ be the character of $\Delta$. By [Jam14, Theorem 3.3], $\Delta$ is imprimitive if and only if there exists an epimorphism $\psi : F_m \to \{\pm 1\}$ such that $\psi(w) = 0$ implies $\chi(w) = 0$ for all $w \in F_m$. For $\emptyset \neq J \subseteq \{1, \ldots, m\}$ define an epimorphism $\psi_J : F_m \to \{\pm 1\}$ by $\psi_J(g_j) = -1$ if $j \in J$ and $\psi_J(g_j) = 1$ otherwise. This yields a bijection between the non-empty subsets of $\{1, \ldots, m\}$ and the epimorphisms of $F_m$ onto $\{\pm 1\}$. Let $A_i := \Delta(g_i)$ for $i \in \{1, \ldots, m\}$. We show that $\psi_J(w) = -1$ implies $\chi(w)$ for all $w \in F_m$ if and only if $t_I = 0$ for all $I \in \mathcal{I}_m$ with $|I \cap J|$ odd.

The condition is obviously necessary; we show that it is sufficient. By Lemma 6.2 we may assume that $t_I = 0$ for all $\emptyset \neq I \subseteq \{1, \ldots, m\}$ with $|I \cap J|$ odd. Let $w \in F_m$ with $\psi_J(w) = -1$. We prove $\chi(w) = 0$ by induction on $|w|$, proceeding along the lines of the proof of Theorem 2.2. Note that $\chi(w) = \epsilon_A(\tau(w))$, where $A = (\Delta(g_1), \ldots, \Delta(g_m))$. If $w$ is conjugate to $g_i^{-1} w'$ for some $i \in \{1, \ldots, m\}$ and some $w' \in F_m$ with $|w'| = |w| - 1$, then $\chi(w) = \chi(g_i w') - \chi(g_i) \chi(w')$. By induction, $\chi(w) = \chi(g_i w')$, since either $\psi_J(g_i w') = -1$ or $\psi_J(g_i) = -1$. Similar considerations apply to the other cases of the proof of Theorem 2.2, so we conclude $\chi(w) = 0$. 

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The definition of imprimitivity depends on the field of definition. By abuse of notation we call a representation imprimitive if it is imprimitive after field extension.

**Corollary 6.4.** Let
\[ \mathcal{D} := \bigcap_{0 \neq J \subseteq \{1, \ldots, m\}} \langle x_I \mid I \in \mathcal{I}_m \text{ with } |I \cap J| \text{ odd} \rangle \leq \Phi'_m. \]

Let \( P \in \text{Spec}(\Phi'_m) \), and let \( \Delta \in R(F_m, K) \) be a realization of \( t_P \), where \( K \) is the quotient field of \( \Phi_m/P \). Then \( \Delta \) is imprimitive if and only if \( \mathcal{D} \subseteq P \).

In other words, the imprimitive representations correspond to the elements of the closed subset
\[ V(\mathcal{D}) = \{ P \in \text{Spec}(\Phi'_m) \mid \mathcal{D} \subseteq P \} \]
of \( \text{Spec}(\Phi'_m) \).

The dihedral subgroups of \( \text{PSL}(2, q) \) are precisely the images of imprimitive subgroups of \( \text{SL}(2, q) \). Setting \( \mathfrak{A}_4 := J'(A_4) \), \( \mathfrak{S}_4 := J'(S_4) \), and \( \mathfrak{A}_5 := J'(A_5) \), we can formulate the main result of this section.

**Theorem 6.5.** Let \( G \) be a finitely presented group on \( m \) generators. The set of normal subgroups \( N \trianglelefteq G \) such that \( G/N \cong \text{PSL}(2, q) \) for some prime power \( q > 5 \) or \( G/N \cong \text{PGL}(2, q) \) for some prime power \( q > 4 \) and such that \( \langle g_1 N, g_2 N \rangle \) is absolutely irreducible is in bijection to the set of \( \Sigma_m \)-orbits of maximal ideals of
\[ Q(G) := V(I(G)) - V(\mathcal{D} \cap \mathfrak{A}_4 \cap \mathfrak{S}_4 \cap \mathfrak{A}_5) \subseteq \text{Spec}(\Phi'_m). \]

### 7 The PSL-PGL-decision

**Definition 7.1.** A finite group is of \( L_2 \)-type if it is isomorphic to \( \text{PSL}(2, q) \) for some \( q > 5 \) or to \( \text{PGL}(2, q) \) for some \( q > 4 \). A quotient of a finitely presented group is an \( L_2 \)-quotient if it is of \( L_2 \)-type.

Theorem 6.5 gives a characterization of \( L_2 \)-quotients purely in algebra-geometric terms. To decide whether an \( L_2 \)-quotient is isomorphic to \( \text{PSL}(2, q) \) or \( \text{PGL}(2, q) \) for some \( q \), we use arithmetic tools.

Let \( M \in Q(G) \) be a maximal ideal, and let \( t_M \) be the trace tuple defined by \( M \). Let \( \Delta: F_m \to \text{SL}(2, q) \) be a realization of \( t_M \). The field \( \Phi'_m/M \) is generated by \( t_M \), so \( \Phi'_m/M \) is the character field of \( \Delta \). Since representations over finite fields can be realized over the characteristic field, we may assume \( q = |\Phi'_m/M| \).

If \( q \) is not a square, then by Dickson’s classification \( \Delta \) induces an epimorphism onto \( \text{PSL}(2, q) \), and if \( q = r^2 \), then \( \Delta \) induces an epimorphism onto \( \text{PSL}(2, q) \) or \( \text{PGL}(2, r) \). We give a criterion to decide which case occurs. Note that \( \Delta \) induces a projective representation onto \( \text{PGL}(2, r) \) if and only if the image of \( \Delta \) is conjugate to a subgroup of \( \text{GL}(2, r)F_q^\ast \), where \( F_q^\ast \) is identified with scalar matrices.

**Proposition 7.2.** Let \( q = r^2 \) be a prime power. Let \( t \in F_q^\ast \) be a trace tuple and \( \Delta: F_m \to \text{SL}(2, q) \) a realization of \( t \), and let \( \alpha \) be a generator of \( \text{Gal}(F_q/F_r) \). The image of \( \Delta \) is conjugate to a subgroup of \( \text{GL}(2, r)F_q^\ast \) if and only if \( \sigma t = \sigma t \) for some \( \sigma \in \Sigma_m \).

**Proof.** Let \( \chi: F_m \to F_q: w \mapsto \text{tr}(\Delta(w)) \) be the character of \( \Delta \). By [Jam14, Theorem 4.1], the image of \( \Delta \) is conjugate to a subgroup of \( \text{GL}(2, r)F_q^\ast \) if and only if there exists \( \sigma \in \Sigma_m \) with \( \sigma \chi = \sigma \chi \). Using Lemma 5.2 and the construction of the \( \tau(w) \) in the proof of Theorem 2.2, we can show as in the proof of Proposition 6.3 that this is equivalent to \( \sigma t_I = \sigma t_I \) for all \( I \in \mathcal{I}_m \). \( \square \)

**Remark 7.3.** Let \( M \trianglelefteq \Phi'_m \) be a maximal ideal, and let \( \sigma \in \text{Stab}_{\Sigma_m}(M) \). Then \( \Phi'_m/M \to \Phi'_m/M: x_I + M \mapsto \sigma x_I + M \) defines a Galois automorphism.

**Corollary 7.4.** Let \( M \trianglelefteq \Phi'_m \) be a maximal ideal such that \( |\Phi'_m/M| = q = r^2 \) is a square. Let \( t = t_M \) be the trace tuple of \( M \), and let \( \Delta: F_m \to \text{SL}(2, q) \) be a realization of \( t_M \). The image of \( \Delta \) is conjugate to a subgroup of \( \text{GL}(2, r)F_q^\ast \) if and only if \( M \) has a non-trivial stabilizer in \( \Sigma_m \).

Together with Theorem 6.5 we get the following result.
Theorem 7.5. Let $G$ be a finitely presented group on $m$ generators. The set of normal subgroups $N \trianglelefteq G$ such that $G/N \cong \text{PSL}(2,q)$ for some odd $q > 5$ with $(g_1N,g_2N)$ absolutely irreducible is in bijection to the regular $\Sigma_m$-orbits of maximal ideals of $Q(G)$. The set of normal subgroups $N \trianglelefteq G$ such that $G/N \cong \text{PGL}(2,q)$ for some $q > 4$ with $(g_1N,g_2N)$ absolutely irreducible is in bijection to the $\Sigma_m$-orbits of maximal ideals of $Q(G)$ with a stabilizer of order 2.

When dealing with infinitely many $L_2$-quotients, the following reformulation in terms of trace tuples if often useful.

Corollary 7.6. Let $G$ be a finitely presented group on $m$ generators, and let $q = p^d$ be a prime power. If $q > 5$ is odd, then the set of normal subgroups $N \trianglelefteq G$ such that $G/N \cong \text{PSL}(2,q)$ with $(g_1N,g_2N)$ absolutely irreducible is in bijection to the regular $\Sigma_m \times \text{Gal}(\mathbb{F}_q)$-orbits of zeroes $t \in \mathbb{F}_q^m$ of $Q(G)$ with $\mathbb{F}_q = \mathbb{F}_p[t]$. If $q > 4$, then the set of normal subgroups $N \trianglelefteq G$ such that $G/N \cong \text{PGL}(2,q)$ with $(g_1N,g_2N)$ absolutely irreducible is in bijection to the $\Sigma_m \times \text{Gal}(\mathbb{F}_q)$-orbits of zeroes $t \in \mathbb{F}_q^m$ of $Q(G)$ with $\mathbb{F}_q = \mathbb{F}_p[t]$ having stabilizer of order 2.

Let $G/N_1$ and $G/N_2$ be $L_2$-quotients of $G$ with $N_1 \neq N_2$. What is the isomorphism type of $G/N_1 \cap N_2$? Clearly, if $G/N_1$ or $G/N_2$ is simple, then $G/N_1 \cap N_2 \cong G/N_1 \times G/N_2$. This leaves the case that both $G/N_1$ are non-simple, that is, $G/N_1 \cong \text{PGL}(2,q_i)$ for some prime powers $q_i$.

Proposition 7.7. Let $G$ be a finitely presented group on $m$ generators. Let $M_1$ and $M_2$ be maximal ideals of $Q(G)$ with stabilizers $(\sigma^{(1)}) \leq \Sigma_m$ of order 2, and let $N_1, N_2 \trianglelefteq G$ be normal subgroups corresponding to $M_1, M_2$ in the bijection of Theorem 7.3. Let $q_1, q_2$ be prime powers with $G/N_i \cong \text{PGL}(2,q_i)$. If $N_1 \neq N_2$, then

$$G/N_1 \cap N_2 \cong \begin{cases} \text{PGL}(2,q_1) \times \text{PGL}(2,q_2) & \text{if } \sigma^{(1)} = \sigma^{(2)}, \\ \text{PGL}(2,q_1) \times \text{PGL}(2,q_2) & \text{otherwise}. \end{cases}$$

Proof. Let $\delta_i : G \to \text{PSL}(2,q_i^2)$ be a realization of $M_i$; define $\delta_1 \times \delta_2 : G \to \text{PSL}(2,q_1^2) \times \text{PSL}(2,q_2^2) : g \mapsto (\delta_1(g), \delta_2(g))$. The image $H$ of $\delta_1 \times \delta_2$ is a subdirect product of $\text{PGL}(2,q_1) \times \text{PGL}(2,q_2)$. Since $N_1 \neq N_2$, this subdirect product is amalgamated either in $C_2$ or in the trivial group, and in the latter case the product is direct. There is a unique epimorphism $\varepsilon : \text{PGL}(2,q_i) \to C_2$, where $\varepsilon_i(\delta(g_i)) = 1$ if and only if $\delta_i(g_i) \in \text{PSL}(2,q_i)$. By the proof of [Lam14 Theorem 4.1], this is equivalent to $\sigma^{(1)} = 1$. Hence $\varepsilon_1(\delta_1(g_1)) = \varepsilon_2(\delta_2(g_2))$ if and only if $\sigma^{(1)} = \sigma^{(2)}$, which proves the proposition.

8 Arbitrary representations

Until now, we only considered representations $\Delta : F_m \to \text{SL}(2,K)$ such that $\Delta_{(g_1,g_2)}$ is absolutely irreducible. We now show how the case of arbitrary absolutely irreducible representations can be reduced to this one.

Proposition 8.1. Let $\Delta : F_m \to \text{SL}(2,K)$ be a representation. For $1 \leq i, j, k \leq m$ set $\Delta_{i,j} := \Delta_{(g_i,g_j)}$ and $\Delta_{i,j,k} := \Delta_{(g_i,g_j,g_k)}$. Then $\Delta$ is absolutely irreducible if and only if $\Delta_{i,j}$ with $1 \leq i < j \leq m$, $\Delta_{1,2i}$ with $3 \leq i \leq m$, or $\Delta_{2,i,j}$ with $3 \leq i < j \leq m$ is absolutely irreducible.

Proof. We generalize [Fal00 Lemma 3.4.4] and so strengthen [BH95 Proposition B.7]. We may assume that $K$ is algebraically closed, so absolute irreducibility coincides with irreducibility. Clearly if some restriction of $\Delta$ is irreducible, then $\Delta$ is irreducible. So assume now that all given restrictions are reducible. We show that $\Delta$ is reducible. Since $\Delta_{i,j}$ is reducible, $\Delta(g_i)$ and $\Delta(g_j)$ have a common eigenspace. If the minimal polynomial of some $\Delta(g_i)$ is not square-free, then $\Delta(g_i)$ has a unique eigenspace of dimension 1, which has to be a common eigenspace for all $\Delta(g_j)$. Thus $\Delta$ is reducible. So assume now that the minimal polynomials of all $\Delta(g_i)$ are square-free. We may further assume that all $\Delta(g_i)$ have two distinct eigenvalues; for if $\Delta(g_i)$ is a scalar matrix, then $\Delta$ is reducible and if only if $\Delta_{i_1,\ldots,i_m}$ is reducible. Let $E_i$ be the set of eigenspaces of $\Delta_i$. Let $E_i := \{ E_i | 1 \leq i \leq m \}$. By our hypothesis, $|E_i \cap E_j| \geq 1$ for all $i, j$. Note that $|E_i| = 2$, so if $|E| \geq 4$, then the $E_i$ must have a common element, that is, the matrices have a common eigenspace. The same is trivially true if $|E| \leq 2$. Assume now that $|E| = 3$. Consider first the case $E_1 \neq E_2$. Let $E_1 = \{ \langle v_1 \rangle, \langle v_2 \rangle \}$ and $E_1 \cap E_2 = \{ \langle v_1 \rangle \}$. We claim that $\langle v_1 \rangle$ is a common
Proof. This follows by Proposition 8.1 and Theorem 7.5.

Theorem 8.2. If $\Delta(g)$ is absolutely irreducible, then $\langle v_1 \rangle$ or $\langle v_2 \rangle$. In the first case, $\Delta(g_2 g)$ and $\Delta(g_2)$ have eigenspace $\langle v_1 \rangle$, so $\Delta(g)$ has eigenspace $\langle v_1 \rangle$, contradicting our assumption. In the second case, $\Delta(g_2 g)$ and $\Delta(g_1)$ have eigenspace $\langle v_2 \rangle$, so $\Delta(g_2)$ has eigenspace $\langle v_2 \rangle$, again a contradiction. Thus the assumption that $\langle v_1 \rangle$ is not an eigenspace of $\Delta(g)$ is impossible. We conclude the proof by showing that $E_1 = E_2$ is not possible. Since $|E| = 3$, there exist $i < j$ with $E = \{E_1, E_i, E_j\}$. All sets have at least one element in common, so we may assume $E_1 = \langle v_1 \rangle$, $E_i = \langle v_1 \rangle$, and $E_j = \langle v_2 \rangle$. Since $\Delta_{2,4}$ is reducible, $\Delta(g_2)$ and $\Delta(g_2 g)$ have a common eigenspace. Assume that this is $\langle v_1 \rangle$; then $\langle v_1 \rangle$ is also an eigenspace of $\Delta(g_1)$, a contradiction. If it is $\langle v_2 \rangle$, then $\langle v_2 \rangle$ is also an eigenspace of $\Delta(g_1)$, also a contradiction. Thus $E_1 = E_2$ is impossible.

Let

$$U_m := \{(i,j) \mid 1 \leq i < j \leq m\} \cup \{(1,2, j) \mid 3 \leq j \leq m\} \cup \{(2, i,j) \mid 3 \leq i < j \leq m\}.$$ 

For every $u = (v_1, v_2) \in U_m$, let $\alpha_u \in Aut(F_m)$ with $\alpha_u(g_{v_1}) = g_1$ and $\alpha_u(g_{v_2}) = g_2$, where $g_v := g_{v_1} \cdots g_{v_k}$ for $v = v_1 < \cdots < v_k$. Thus $\Delta: F_m \to SL(2,K)$ is absolutely irreducible if and only if $(\Delta \circ \alpha_u^{-1})(g_{v_1}, g_{v_2})$ is absolutely irreducible for some $u \in U_m$.

By abuse of notation, if $\alpha \in Aut(F_m)$ and $G$ is a group generated by elements $g_1, \ldots, g_m$, then we denote the automorphism of $G$ defined by $g_i \mapsto \alpha(g_i)$ for $1 \leq i \leq m$ again by $\alpha$. Fix a total order < on $U_m$. Set

$$I_u(G) := I(\alpha(G)) + \langle \rho(x_{v_1}, x_{v_2}, x_{v_1} x_{v_2}) \rangle \mid v \in U_m, v < u \rangle.$$ 

For a maximal ideal $M \in V(I_u(G))$ let $\tau_M$ be the trace tuple, and let $M_\Delta: F_m \to SL(2,q)$ be a realization of $\tau_M$, where $q = |\Phi_m[M]|$. The projective representation induced by $\Delta \circ \alpha_u$ factors over $G$; denote this projective representation by $\delta_{M,u}$, and define $N_M := ker(\delta_{M,u}) \leq G$. Note that $N_M$ is constant on the $\Sigma_m$-orbit of $M$.

Conversely, let $N \leq G$ such that $G/N$ is of $L_2$-type. Let $\delta: G \to PSL(2,q)$ with $ker(\delta) = N$, and let $\Delta: F_m \to SL(2,q)$ be a lift of $\delta$. Set $t = t_\Delta$ and $M_N := P_t$; then $M_N$ is a maximal $L_2$-ideal. Note that $M_N$ is only well-defined up to the action of $\Sigma_m$. If $u \in U_m$ is minimal such that $(\Delta \circ \alpha_u^{-1})(g_{v_1}, g_{v_2})$ is absolutely irreducible, then $M_N \in V(I_u(G))$.

For $u \in U_m$, set

$$Q_u(G) := V(I_u(G)) \cap (\mathcal{D} \cap \mathcal{A}_4 \cap \mathcal{S}_4 \cap \mathcal{A}_5).$$ 

We now present the main result.

Theorem 8.2. Let $G$ be a finitely presented group on $m$ generators. The maps $M \mapsto N_M$ and $N \mapsto M_N$ induce mutually inverse bijections between $\Sigma_m$-orbits of maximal ideals of $\bigcup_{u \in U_m} Q_u(G)$ and normal subgroups $N \leq G$ such that $G/N$ is of $L_2$-type (where $\bigcup$ denotes the disjoint union).

Proof. This follows by Proposition 8.1 and Theorem 7.5.

9 Subgroup tests

Proposition 8.1 allows us to test whether a realization $\Delta: F_m \to SL(2,K)$ of a prime ideal $P \leq \Phi_m'$ maps projectively onto $A_4$, $S_4$, or $A_5$, using the ideals $J'(A_4)$, $J'(S_4)$, and $J'(A_5)$. These ideals are easily computed if $m = 2$, since there are only 4 presentations of $A_4$ on two generators, 9 for $S_4$, and 19 for $A_5$; see [Pedersen, Lemmas 3.7–3.9]. However, this approach is no longer efficient if $m \geq 3$. For example, there are 65 presentations of $A_4$ on three generators, 420 for $S_4$, and 1688 for $A_5$.

In this section, we describe a more efficient test, using the absolutely irreducible subgroups of $A_4$, $S_4$, and $A_5$. Set $A_i := \Delta(g_i)$ and let $a_i \in PSL(2,K)$ be the projective image, for $1 \leq i \leq m$. We assume that $\langle A_1, A_2 \rangle$ is absolutely irreducible. Define $H := \langle a_1, \ldots, a_m \rangle$. If $H \cong A_4$, then $\langle a_1, a_2 \rangle \in \{V_4, A_4\}$; if $H \cong S_4$, then $\langle a_1, a_2 \rangle \in \{V_4, S_4, D_8, A_4, S_4\}$; and if $H \cong A_5$, then $\langle a_1, a_2 \rangle \in \{V_4, S_4, D_8, D_{10}, A_4, S_4, A_5\}$. It is easy to check whether $\langle a_1, a_2 \rangle \in \{V_4, S_4, D_8, D_{10}, A_4, S_4, A_5\}$; for example, $\langle a_1, a_2 \rangle = V_4$ if and only if $tr(A_1) = tr(A_2) = tr(A_1 A_2) = 0$. If $\langle a_1, a_2 \rangle$ is one of the seven groups, then we can always find matrices $B_1 = w_1(A_1, A_2), B_2 = w_2(A_1, A_2)$ such that $tr(B_1) = tr(B_2) = 0$ and $\langle w_1(a_1, a_2), w_2(a_1, a_2) \rangle$
is a dihedral group of order 4, 6, or 10. In the latter two cases we may also assume that \( \text{tr}(B_1 B_2) = 1 \) or \( \text{tr}(B_1 B_2) \) is a root of \( X^2 + X - 1 \), respectively.

For \( B = (B_1, B_2) \in \text{SL}(2, q)^2 \) and \( X \in \text{SL}(2, q) \) let

\[
\theta_B(X) := (\text{tr}(X), \text{tr}(B_1 X), \text{tr}(B_2 X), \text{tr}(B_1 B_2 X)) \in F_q^4.
\]

If \( (B_1, B_2) \) is absolutely irreducible, then \( X \) is uniquely determined by \( \theta_B(X) \), see Proposition 2.4.

We give details of an \( A_4 \)-test. Fix \( B \), and let \( b_i \in \text{PSL}(2, q) \) be the projective image of \( B_i \). Assume \( \langle b_1, b_2 \rangle \cong V_4 \); let \( \langle b_1, b_2 \rangle \leq \Gamma \leq \text{PSL}(2, q) \) with \( \Gamma \cong A_4 \) and let \( \bar{\Gamma} \leq \text{SL}(2, q) \) be the full preimage of \( \Gamma \). Now \( X \in \text{SL}(2, q) \) maps onto an element of \( \Gamma \) if and only if \( \theta_B(X) \in \theta_B(\bar{\Gamma}) = \{ \theta_B(Y) \mid Y \in \bar{\Gamma} \} \), thus for an effective subgroup test it is enough to compute the sets \( \theta_B(\bar{\Gamma}) \). The subgroups of \( \text{PSL}(2, q) \) isomorphic to \( A_4 \) are all conjugate in \( \text{PGL}(2, q) \), and \( \theta(\text{PSL}(2, q)) = \theta(\text{PGL}(2, q)) \) for all \( M \in \text{GL}(2, q) \), so it is enough to compute \( \theta_B(\bar{\Gamma}) \) for a fixed \( \Gamma \) and all \( \text{PGL}(2, q)(\bar{\Gamma}) \)-conjugacy classes of pairs \( B \in \bar{\Gamma} \) mapping onto generators for \( V_4 \). Finally, the subgroups \( \bar{\Gamma} \) are up to conjugation images of \( \text{SL}(2, 3) \leq \text{SL}(2, \mathbb{Z}[i]) \) modulo a prime ideal of \( \mathbb{Z}[i] \), so \( \theta_B(\bar{\Gamma}) \) can be computed uniformly for all prime powers \( q \) by a single computation over \( \mathbb{Z} \).

Summarizing, we get the following result.

**Proposition 9.1.** Let \( G = \text{PSL}(2, q) \) for an odd prime power \( q \), let \( a_1, a_2 \in G \) be generators of a Klein four group \( V \), and let \( z \in G \). Let \( A_i \in \text{SL}(2, q) \) be a preimage of \( A_i \), and let \( Z \in \text{SL}(2, q) \) be a preimage of \( z \).

There is a unique \( H \leq G \) isomorphic to \( A_4 \) which contains \( V \), and \( z \in H \) if and only if \( \theta_B(Z) \) is one of the \( 24 \) elements

\[
\Theta_4 := \{(\pm 2, 0, 0, 0), (0, \pm 2, 0, 0), (0, 0, \pm 2, 0), (0, 0, 0, \pm 2), (\pm 1, \pm 1, \pm 1)\}.
\]

**Proposition 9.2.** Let \( A_1, \ldots, A_m \in \text{SL}(2, q) \) such that \( \langle A_1, A_2 \rangle \) is absolutely irreducible. Let \( t = (\text{tr}(A_1), \text{tr}(A_2), \text{tr}(A_1 A_2)) \), and let \( a_i \in \text{PSL}(2, q) \) be the image of \( A_i \). Set \( B := (A_1, A_2) \). Then \( \langle a_1, \ldots, a_m \rangle \) is isomorphic to \( A_4 \) if and only if one of the following conditions is satisfied.

1. \( t = (0, 0, 0) \) and \( \theta_B(A_i) \in \Theta_4 \) for all \( 3 \leq i \leq m \), where \( B = (A_1, A_2) \), and at least one \( \theta_B(A_i) = (\pm 1, \pm 1, \pm 1) \).
2. \( t = (0, \pm 1, \pm 1) \) and \( \theta_B(A_i) \in \Theta_4 \) for all \( 3 \leq i \leq m \), where \( B = (A_1, A_2^{-1} A_1 A_2) \).
3. \( t = (\pm 1, 0, \pm 1) \) and \( \theta_B(A_i) \in \Theta_4 \) for all \( 3 \leq i \leq m \), where \( B = (A_2, A_1^{-1} A_2 A_1) \).
4. \( t = (\pm 1, \pm 1, 0) \) and \( \theta_B(A_i) \in \Theta_4 \) for all \( 3 \leq i \leq m \), where \( B = (A_1 A_2, A_2 A_1) \).
5. \( t = (\pm 1, \pm 1, \pm 1) \) with an even number of \(-1\)'s, and \( \theta_B(A_i) \in \Theta_4 \) for all \( 3 \leq i \leq m \), where \( B = (A_1 A_2^{-1}, A_2^{-1} A_1) \).

**Proof.** The only absolutely irreducible subgroups of \( A_4 \) are the Klein four group and \( A_4 \). If \( t = (0, 0, 0) \), then \( \langle a_1, a_2 \rangle \) is a Klein four group, and the claim follows by Proposition 9.1. If \( t = (0, \pm 1, \pm 1) \), then \( \langle a_1, a_2 \rangle = A_4 \), and \( a_1, a_2^{-1} a_1 a_2 \) generate the subgroup of order 4; again, the claim follows by Proposition 9.1. The other cases correspond to the other three presentations of \( A_4 \) and are handled similarly. \( \square \)

It is straightforward to give similar conditions for \( S_4 \) and \( A_5 \), utilizing the subgroups \( S_3 \) and \( D_{10} \) in addition to \( V_4 \).

## 10 \text{L}_2\text{-ideals}

**Definition 10.1.** An \( \text{L}_2\text{-ideal} \) is a prime ideal \( P \in \text{Spec}(\Phi_m) - V(\mathfrak{O} \cap \mathfrak{A}_4 \cap \mathfrak{G}_4 \cap \mathfrak{A}_5) \). Let \( P \cap \mathbb{Z} = \langle p \rangle \), and let \( d \) be the Krull dimension of \( P \).

1. If \( d = 0 \), that is, \( P \) is a maximal ideal, let \( k := \dim_{\mathbb{F}_p}(\Phi_m/P) \). Then \( P \) is of type \( \text{L}_2(p^k) \) if \( \text{Stab}_{\Sigma_m}(P) \) is trivial, and of type \( \text{PGL}(2, p^{k/2}) \) otherwise.
2. If \( d > 0 \) and \( p \neq 0 \), then \( P \) is of type \( L_2(p^{\infty^d}) \), or of type \( L_2(p^{\infty}) \) if \( d = 1 \).

3. If \( d = 1 \) and \( p = 0 \), let \( k := \dim_{\mathbb{Q}}(\Phi'_m \otimes_{\mathbb{Q}} P / P) \). Then \( P \) is of type \( L_2(\infty^k) \).

4. If \( d > 1 \) and \( p = 0 \), then \( P \) is of type \( L_2(\infty^{d-1}) \), or of type \( L_2(\infty) \) if \( d = 2 \).

For an \( L_2 \)-ideal \( P \) let \( t_P \) be the trace tuple, \( \Delta_P \) a realization of \( t_P \), and \( \delta_P \) the projective representation induced by \( \Delta_P \).

**Proposition 10.2.** Let \( P \) be an \( L_2 \)-ideal.

1. If \( P \) is of type \( L_2(p^k) \), then the image of \( \delta_P \) is isomorphic to \( L_2(p^k) \); if \( P \) is of type \( \text{PGL}(2,p^k) \), then the image of \( \delta_P \) is isomorphic to \( \text{PGL}(2,p^k) \).

2. If \( P \) is of type \( L_2(\infty^k) \), then every maximal \( L_2 \)-ideal containing \( P \) is of type \( L_2(p^\ell) \) or \( \text{PGL}(2,p^{\ell/2}) \) with \( \ell \leq k \). Moreover, the set of maximal elements of \( \text{V}(P) \) which are not \( L_2 \)-ideals is finite.

3. If \( P \) is of type \( L_2(p^{\infty^d}) \), then there are infinitely many \( k \in \mathbb{N} \) such that \( \text{V}(P) \) contains \( L_2 \)-ideals of type \( L_2(p^k) \). Moreover, the set of prime ideals in \( \text{V}(P) \) which are not \( L_2 \)-ideals form a closed set of dimension at most \( d - 1 \).

4. If \( P \) is of type \( L_2(\infty^{d-1}) \), then for all but finitely many primes \( p \) there exist infinitely many \( k \in \mathbb{N} \) such that \( \text{V}(P) \) contains \( L_2 \)-ideals of type \( L_2(p^k) \). Moreover, the set of prime ideals in \( \text{V}(P) \) which are not \( L_2 \)-ideals form a closed set of dimension at most \( d - 1 \).

**Proof.** First note that the set of prime ideals in \( \text{V}(P) \) which are not \( L_2 \)-ideals are precisely the elements of the set \( \text{V}(P) \cap \text{V}(\mathfrak{D} \cap \mathfrak{A}_1 \cap \mathfrak{A}_5) = \text{V}(P + \mathfrak{D} \cap \mathfrak{A}_1 \cap \mathfrak{A}_5) \). Since \( P \) does not contain \( \mathfrak{D} \cap \mathfrak{A}_1 \cap \mathfrak{A}_5 \), and \( P \) is prime, \( P \subseteq P + \mathfrak{D} \cap \mathfrak{A}_1 \cap \mathfrak{A}_5 \), so the Krull dimension of the latter ideal is smaller than that of \( P \). This settles all claims about the size of \( \text{V}(P) \).

We prove the other claims.

1. This follows by Theorem 7.5.

2. The first point follows since \( \dim_{\mathbb{F}_p}(\Phi'_m \otimes_{\mathbb{F}_p} P / P) \leq \dim_{\mathbb{Q}}(\Phi'_m \otimes_{\mathbb{Q}} P / P) \) for all primes \( p \), and the maximal ideals of \( \Phi'_m \) containing \( P \) and \( p \) are in bijection to the maximal ideals of \( \Phi'_m \otimes_{\mathbb{F}_p} P \) containing \( P \). For the second point, note that a set of dimension zero is finite.

3. Since \( \Phi'_m \) is finitely generated, there are only finitely many epimorphisms of \( \Phi'_m \) onto \( \mathbb{F}_p^k \) for every \( k \). But there are infinitely many primes containing \( P \).

4. In this case, \( (\Phi'_m / P) \otimes_{\mathbb{Q}} \mathbb{Q} \) has algebraically independent elements, so there are epimorphisms onto number fields of arbitrarily high degrees. Let \( \alpha : \Phi'_m \otimes_{\mathbb{Q}} \mathbb{Q} \rightarrow K \) be an epimorphism onto a number field \( K \) of degree \( k \) such that \( \alpha \) factors over \( \Phi'_m \otimes_{\mathbb{Q}} \mathbb{Q} \); set \( Q := \ker(\alpha|_{\Phi'_m}) \leq \Phi'_m \). Then \( Q \supseteq P \); if \( Q \) is an \( L_2 \)-ideal, then it is of type \( L_2(\infty^d) \). But the prime ideals which are not \( L_2 \)-ideals form a set of Krull-dimension \( d - 1 \), so this approach yields an \( L_2 \)-ideal for almost all \( Q \). The result now follows by part (2). \( \square \)

### 11 The algorithm

**Definition 11.1.** Let \( G = (g_1, \ldots, g_m \mid w_1(g_1, \ldots, g_m), \ldots, w_r(g_1, \ldots, g_m)) \) be a finitely presented group. Then \( \Sigma_m \) acts on the set \( \{\pm1\}^r \) of sign systems by

\[
\sigma s := (w_1(\sigma_1, \ldots, \sigma_m)s_1, \ldots, w_r(\sigma_1, \ldots, \sigma_m)s_r)
\]

for \( \sigma \in \Sigma_m \) and \( s \in \{\pm1\}^r \).

**Remark 11.2.** A prime ideal \( P \in \text{Spec}(\Phi'_m) \) contains \( I_1(G) \) if and only if \( \sigma P \) contains \( I_{(\sigma_1)}(G) \). Let \( T \) be the kernel of the action and \( S \) a set of representatives of the orbits; then the \( \Sigma_m \)-orbits of \( \text{V}(I(G)) \) are in bijection to the \( T \)-orbits of \( \text{V}(\bigcap_{s \in S} I_s(G)) \).
Remark 11.6. Let \( \eta \) be the \( \infty \)-th root of unity. Set \( A := U_m \) and \( R := \emptyset \).

1. Let \( u \) be the smallest element in \( A \). Let \( T \) be the kernel and \( S \) a set of representatives for the orbits of the action of \( S_m \) on the sign systems of \( \alpha_u(G) \).

2. Let \( P \) be the set of minimal elements in \( \bigcup_{\sigma \in S} \text{MinAss}(I_{\sigma}^u(G)) \), where \( \text{MinAss}(I) \) denotes the minimal associated prime ideals of \( I \). Remove from \( P \) all elements which contain one of the ideals \( \mathfrak{D}, \mathfrak{A}_4, \mathfrak{S}_4, \text{ or } \mathfrak{A}_5 \).

3. Choose a set \( P' \) of representatives of \( T \)-orbits on \( P \).

4. Add \( (P', u) \) to \( R \), and remove \( u \) from \( A \). If \( A \neq \emptyset \), go to step 2; otherwise return \( R \).

Remark 11.5. 1. The output of the algorithm describes the \( L_2 \)-quotients of \( G \) as follows. For every \( N \leq G \) with \( G/N \) of \( L_2 \)-type there exists \( u \in U_m \) and \( \sigma \in \Sigma_m \) such that \( M_{\sigma} \geq P \) for some \( P \). Conversely, if \( M \geq P \) is a maximal \( L_2 \)-ideal for some \( P \), then \( N_{M} \leq G \) with \( G/N_{M} \) of \( L_2 \)-type.

2. If all prime ideals returned by the algorithm are maximal, then \( G \) has only finitely many \( L_2 \)-quotients, and the normal subgroups \( N \leq G \) with \( G/N \) of \( L_2 \)-type are in bijection to the maximal ideals.

3. If the algorithm returns at least one prime ideal of positive Krull dimension, then \( G \) has infinitely many \( L_2 \)-quotients.

The algorithm has been implemented in MAGMA [BCP97].

Remark 11.4. 1. The ring \( \Phi'_m \) is very useful for the theoretical description of the algorithm. However, in practice the localization at \( \rho \) slows down computations considerably. Instead, we work with the preimage of \( I_{\sigma}^u(G) \) in \( \mathbb{Z}[x_j \mid \emptyset \neq J \subseteq \{1, \ldots, m\}] \), and remove all prime components containing \( \rho \).

2. The implementation uses Gröbner bases to handle the ideals \( I_{\sigma}^u(G) \). However, Gröbner basis computations over the integers can be very slow, especially as \( m \) grows. The algorithm in [Jam11] to compute the minimal associated primes of an ideal replaces Gröbner basis computations over the integers by several Gröbner basis computations over prime fields, resulting in a much faster algorithm.

11.1 Adaptation to Coxeter groups

Coxeter groups are a special class of finitely presented groups, where the only relations are \((g, g_j)^{c_{ij}} = 1\) for a symmetric matrix \( C = (C_{ij}) \in (\mathbb{Z} \cup \{\infty\})^{m \times m} \) with 1’s along the diagonal (if \( C_{ij} = \infty \), then we simply omit the relation). We call \( C \) a Coxeter matrix and denote the finitely presented group by \( G_C \).

We are often only interested in smooth quotients of Coxeter groups, that is, those for which the images also have the prescribed orders (unless the prescribed order is \( \infty \)). In this case, the \( L_2 \)-quotient algorithm can be simplified, which also results in a considerable speed-up of the computation. This is based on the following.

For \( n \in \mathbb{N} \) let \( \zeta_n \in \mathbb{C} \) be a primitive \( n \)-th root of unity. Set \( \eta_n := \zeta_n + \zeta_n^{-1} \), and let \( \Psi_n \in \mathbb{Z}[T] \) be the minimal polynomial of \( \eta_n \). For convenience, we define \( \Psi_\infty := 0 \).

Remark 11.6. Let \( A \in \text{SL}(2, K) \) where \( k \) is a field of characteristic \( p \geq 0 \), and let \( n \in \mathbb{N} \).

1. If \( p = 0 \) or \( (n, p) = 1 \), then \( \Psi_n(\text{tr}(A)) = 0 \) if and only if \( |A| = n \).

2. If \( n = p \), then \( \Psi_n(\text{tr}(A)) = 0 \) if and only if \( |A| \in \{1, p\} \).

3. If \( n = 2p \neq 4 \), then \( \Psi_n(\text{tr}(A)) = 0 \) if and only if \( |A| \in \{2, 2p\} \).
For a Coxeter matrix \( C \in (\mathbb{Z} \cup \{\infty\})^{m \times m} \) set
\[
I(C) := \langle x_1, \ldots, x_m \rangle + \langle \Psi_{2C_{ij}}(x_{ij}) \mid 1 \leq i < j \leq m \text{ with } C_{ij} \text{ even} \rangle \\
+ \langle \Psi_{C_{ij}}(x_{ij}) \Psi_{2C_{ij}}(x_{ij}) \mid 1 \leq i < j \leq m \text{ with } C_{ij} \text{ odd} \rangle \leq \Phi',
\]
where \( x_{ij} = r^{-1}(\lambda_i x_j + \lambda_j x_i + \lambda_k x_{2j} + \lambda_{12} x_{12j}) \).

**Remark 11.7.** Let \( a_1, a_2 \in L_2(q) \) with \(|a_1| = |a_2| = 2\) and \(|a_1 a_2| \neq 1\). Then \( \langle a_1, a_2 \rangle \) is absolutely irreducible if and only if \( \langle q, |a_1 a_2| \rangle = 1 \).

**Theorem 11.8.** Let \( C \in (\mathbb{Z} \cup \{\infty\})^{m \times m} \) be a Coxeter matrix.

1. Let \( q = p^d \), and let \( \Delta: F_m \to \text{SL}(2,q) \) be a representation which induces a smooth projective representation \( \delta: \text{GC} \to \text{PSL}(2,q) \) such that \( \delta(G_C) \) is of \( L_2 \)-type. Let \( t := t_\Delta \) and \( P := P_t \). If \( |\delta(q_1, q_2)| \neq p \), then \( P \supseteq I(C) \).

2. Let \( M \supseteq I(C) \) be a maximal \( L_2 \)-ideal and \( \Delta = \Delta_M: F_m \to \text{SL}(2,q) \) a realization. Then \( \Delta \) induces a projective representation \( \delta: \text{GC} \to \text{PSL}(2,q) \) such that \( \delta(G_C) \) is of \( L_2 \)-type. If \( \langle q, 2C_{ij} \rangle = 1 \) for all \( 1 \leq i < j \leq m \), then \( \delta \) is smooth.

**Proof.** This follows easily by the preceeding remarks.

This can be easily turned into an algorithm. We leave the details to the reader.

### 11.2 Computing realizations

The \( L_2 \)-quotient algorithm returns a set of \( L_2 \)-ideals, which contain a lot of information, for example, the isomorphism types and number of \( L_2 \)-images. However, in certain cases one will want to compute an explicit epimorphism \( G \to \text{PSL}(2,q) \) encoded by an \( L_2 \)-ideal. We now present an algorithm to accomplish this. This algorithm works for representations of arbitrary degree, so we present it in this generality.

**Proposition 11.9.** Let \( G \) be a finitely generated group, and let \( \chi: G \to K \) be the character of an absolutely irreducible representation \( \Delta \) of degree \( n \). There is a probabilistic algorithm with input \( \chi \) and \( n \) which constructs an extension field \( L/K \) of degree at most \( n \) and a representation \( \Delta': G \to \text{GL}(n, L) \), such that \( \Delta' \) is equivalent to \( \Delta \). If \( K \) is finite, then we can choose \( L = K \).

**Proof.** We assume first that \( G = F_m \) is a free group on \( g_1, \ldots, g_m \). We first find words \( w_1, \ldots, w_n \in F_m \) such that \( \langle \Delta(w_1), \ldots, \Delta(w_n) \rangle \) is a basis of \( K^n \). Let \( W_i := \{ w \in F_m \mid |w| \leq i \} \), where \( |w| \) denotes the length of the word \( w \). For \( X \subseteq K^n \) denote by \( (X)_K \) the \( K \)-span of \( X \). Note that \( \langle \Delta(W_{i+1}) \rangle_K = \langle \Delta(W_i) \rangle_K \) for some \( i \) implies \( \langle \Delta(W_j) \rangle_K = \langle \Delta(W_i) \rangle_K \) for all \( j \geq i \). In particular, the chain
\[
\langle \Delta(W_0) \rangle_K \subseteq \langle \Delta(W_1) \rangle_K \subseteq \cdots
\]
stabilizes after at most \( n^2 \) steps, so \( \Delta(W_{n^2-1}) \) is a generating set of \( K^{n \times n} \). Let \( C \) be a subset of \( W_{n^2-1} \) of \( n^2 \) elements; define the matrix \( \Sigma := (\chi(v, w))_{v,w} \), where \( v \) and \( w \) run through \( C \). Since the trace bilinear form \( S: K^{n \times n} \times K^{n \times n} \to K: (V, W) \mapsto \text{tr}(VW) \) is non-degenerate, \( \Delta(C) \) is a basis of \( K^{n \times n} \) if and only if \( \Sigma \) is non-singular. By running through all \( n^2 \)-element subsets of \( W_{n^2-1} \) we can find the \( w_1, \ldots, w_n \).

Now let \( V := K^{n \times 1} \) be the \( KF_m \)-module induced by \( \Delta \). We first construct the \( KF_m \)-module \( V^n = V \oplus \cdots \oplus V \subseteq K^{n \times n} \). To determine the action of \( F_m \) on \( K^{n \times n} \), it is enough to determine values \( \lambda^i_{jk} \in K \) such that \( \Delta(q) \Delta(w_j) = \sum_k \lambda^i_{jk} \Delta(w_k) \), where \( 1 \leq i \leq m \) and \( 1 \leq j, k \leq n^2 \). Since \( S \) is non-degenerate, each \( \lambda^i_{jk} \) is uniquely determined by the \( n^2 \) equations
\[
\chi(g_i, w_j, w_k) = S(\Delta(g_i) \Delta(w_j), \Delta(w_k)) = S\left( \sum_k \lambda^i_{jk} \Delta(w_k), \Delta(w_k) \right) = \sum_{k=1}^{n^2} \lambda^i_{jk} \chi(w_k, w_k),
\]
where \( 1 \leq i \leq n^2 \). By solving the linear equations, we can construct the \( KF_m \)-module \( V^n \subseteq K^{n \times n} \).

Let \( \Gamma: F_m \to \text{GL}(K^{n \times n}) \) be the representation on \( V^n \subseteq K^{n \times n} \). We denote the extensions of \( \Delta \) and \( \Gamma \) to the group algebras again by \( \Delta \) and \( \Gamma \), respectively. Let \( v = (v_1, \ldots, v_n) \in K^{n \times n} \), where the \( v_i \) are
the columns of $v$. Then $\Gamma(a)v = (\Delta(a)v_1, \ldots, \Delta(a)v_n)$ for $a \in KF_m$. In particular, $\Gamma(a)$ and $\Delta(a)$ have the same minimal polynomial, and if $c \in K[x]$ is the characteristic polynomial of $\Delta(a)$, then $c^m$ is the characteristic polynomial of $\Gamma(a)$.

We now use an adaptation of GLGO06 to find a simple factor of the $KF_m$-module $K^{n \times n}$. If $K$ is finite, choose random elements $a \in KF_m$ until $\Gamma(a)$ has an eigenspace of dimension $n$. Since the image of $\Delta$ is isomorphic to $K^{n \times n}$, this terminates with high probability by a result of Holt and Rees (see HH94, Section 2.3). Set $L := K$, and let $\lambda \in L$ be an eigenvalue of $\Gamma(a)$ of multiplicity $n$. If $K$ is infinite, then choose random $a \in KF_m$ until the characteristic polynomial of $\Gamma(a)$ is an $n$-th power of a separable polynomial (that is, the characteristic polynomial of $\Delta(a)$ is separable). The characteristic polynomial of a matrix is inseparable if and only if its discriminant is zero, so the set of matrices with inseparable characteristic polynomial is Zariski dense in $K^{n \times n}$. Thus the matrices with separable characteristic polynomial are Zariski dense in $K^{n \times n}$. Since the image of $\Delta$ is isomorphic to $K^{n \times n}$, the probability of finding a suitable $a$ is very high. Let $L/K$ be a field extension such that the characteristic polynomial has a root $\lambda$ in $L$.

Let $v \in L^{n \times n}$ be an eigenvector of $\Gamma(a)$ with eigenvalue $\lambda$. Then

$$\Gamma(a)v = (\Delta(a)v_1, \ldots, \Delta(a)v_n) = \lambda v = (\lambda v_1, \ldots, \lambda v_n).$$

We may assume without loss of generality that $v_1$ is non-zero. Since the $\lambda$-eigenspace of $\Delta(a)$ is one-dimensional, there exist $\xi_1, \ldots, \xi_n \in L$ such that $v_i = \xi_i v_1$ for $i > 1$. Thus $v = (v_1, \xi_2 v_1, \ldots, \xi_n v_1)$ and $\Gamma(a)v = (\Delta(a)v_1, \xi_2 \Delta(a)v_1, \ldots, \xi_n \Delta(a)v_1)$, so $LF_m v = \lambda v$ is isomorphic to $L \otimes K V$. Now choose $w_1, \ldots, w_n \in F_m$ such that $B := (\Gamma(w_1)v_1, \Gamma(w_2)v_1, \ldots, \Gamma(w_n)v_1)$ is a basis of $LF_m v$. For every generator $g_i$ of $F_m$ let $\Delta'(g_i)$ be the representation matrix of $g_i$ on $LF_m v$ with respect to $B$. By construction, $\Delta'$ is equivalent to $\Delta$. This concludes the proof if $G = F_m$ is a free group.

Now assume that $G$ is an arbitrary finitely generated group generated by $m$ elements, and let $v : F_m \to G$ be an epimorphism. Let $\hat{\Delta} := \Delta \circ v$ and $\hat{\chi} := \chi \circ v$. We construct an extension field $L/K$ and a representation $\hat{\Delta}'$ such that $\hat{\Delta} \sim \hat{\Delta}'$. But then $\hat{\Delta}' : G \to \text{GL}(n, F)$ defined by $\hat{\Delta}'(g) := \hat{\Delta}'(g)$, where $\hat{g} \in F_m$ with $v(\hat{g}) = g$ is arbitrary, is a representation of $G$, equivalent to $\Delta$.

In our special setting, we can use the trace polynomials to compute all character values. Furthermore, we always assume that $\Delta_{g_1, g_2}$ is absolutely irreducible, so we can choose $(w_1, \ldots, w_i) = (1, g_1, g_2, g_i g_2)$ in the first part of the algorithm.

12 Examples

For the results in this section we use our implementation of the $L_2$-quotient algorithm in Magma [BCP97].

12.1 Groups with finitely many $L_2$-quotients

In [Cox39], Coxeter defines three families of presentations:

$$(\ell, m| n, k) = \langle a, b | a^\ell, b^n, (ab)^n, (a^{-1}b)^k \rangle,$$

$$(\ell, m| n, q) = \langle a, b | a^\ell, b^m, (ab)^n, [a, b]^q \rangle,$$

$$G^{m,n,p} = \langle a, b | a^m, b^n, c^p, (ab)^2, (ac)^2, (bc)^2, (abc)^2 \rangle.$$

These groups have been intensively studied, see [EJ08] for an overview. After recent work of Havas and Holt [HH10], only for four of these groups is it not known whether they are finite or infinite, namely (3,4,9;2), (3,4,11;2), (3,5,6;2), and $G^{3,7,19}$. We study these groups and their low-index subgroups [Sim94] using the $L_2$-quotient algorithm.

Proposition 12.1. Let $G = (3,4,9;2)$. Then $G$ has seven conjugacy classes of subgroups of index $\leq 50$. For $1 \leq i \leq 50$ let $H_i \leq G$ with $[G : H_i] = i$, if such a group exists. The only $L_2$-quotient of $H_i$ for $i \in \{1,3,4,12\}$ is $L_2(89)$; the group $H_5$ has a quotient $L_2(89) \times (\text{PGL}(2,5), \lambda_{12}) \times \text{PGL}(2,5)$; and $H_{30}$ and $H_{36}$ have a quotient $L_2(89) \times \text{PGL}(2,5)$.

Let $G = (3,5,6;2)$. Then $G$ has two conjugacy classes of subgroups of index $\leq 50$, a group of index 3 and $G$ itself. Both groups have the single $L_2$-quotient $L_2(61)$. 16
The groups $(3,4,11;2)$ and $G^{3,7,19}$ do not have non-trivial subgroups of index $\leq 50$. Both groups have a single $L_2$-quotient, namely $(3,4,11;2)$ has $L_2(769)$, and $G^{3,7,19}$ has $L_2(113)$.

The next result concerns a question of Conder [Con92], asking whether a group has non-trivial finite quotients.

**Proposition 12.2.** The group

$$G = \langle A, B, C, D, E, F \mid A^3, B^3, C^2, D^2, E^2, F^2, (AC)^3, (AD)^3, (AE)^3, (AF)^3, (BC)^3, (BD)^3, (BE)^3, (BF)^3, (ABA^{-1}C)^2, (ABA^{-1}D)^2, (A^{-1}BAE)^2, (A^{-1}BAF)^2, (BAB^{-1}C)^2, (B^{-1}ABD)^2, (BAB^{-1}E)^2, (B^{-1}ABF)^2 \rangle$$

has no quotients isomorphic to $L_2(q)$ or $\text{PGL}(2,q)$ for any prime power $q$.

### 12.2 Groups with $L_2$-ideals of type $L_2(\infty^k)$

If the algorithm returns an ideal of type $L_2(\infty^k)$, then the group has infinitely many $L_2$-quotients, finitely many in every characteristic. Using algebraic number theory, the precise quotient types can be determined as already outlined in [PF09, Example 8.1]. We illustrate the process by relaxing the conditions of the Coxeter presentation $G^{3,7,19}$.

**Proposition 12.3.** Let $G = \langle a, b, c \mid a^3, b^3, (ab)^2, (ac)^2, (bc)^2, (abc)^2 \rangle$. Then $G$ has finitely many $L_2$-quotients in every characteristic.

More precisely, let $K/\mathbb{Q}$ be the splitting field of $X^6 - 4X^4 + 3X^2 + 1$ with Galois group $I = \text{Gal}(K/\mathbb{Q}) \cong (1,4),(1,2,3)(4,5,6) = C_2 \wr C_3$. For a prime $p \neq 2,7$ denote by $\varphi_p \in \Gamma$ the Frobenius automorphism mod $p$. The $L_2$-quotient in characteristic $p$ is

1. $L_2(p)^3$ if $\varphi_p = 1$;
2. $L_2(p)^2 \times \text{PGL}(2,p)$ if $\varphi_p \sim (1,4)$;
3. $L_2(p) \times \text{PGL}(2,p) \wr C_2$ if $\varphi_p \sim (1,4)(2,5)$;
4. $\text{PGL}(2,p) \wr C_2 \times \text{PGL}(2,p) \wr C_2$ if $\varphi_p \sim (1,4)(2,5)(3,6)$;
5. $L_2(p^3)$ if $\varphi_p \sim (1,2,3)(4,5,6)^{\pm 1}$;
6. $\text{PGL}(2,p^3)$ if $\varphi_p \sim (1,2,3,4,5,6)^{\pm 1}$.

Moreover, $G$ has quotients $L_2(2^5)$ and $\text{PGL}(2,7)$.

In this case, we do not need the precise conjugacy type of the Frobenius automorphism, the decomposition of $X^6 - 4X^4 + 3X^2 + 1$ is enough. For example, taking $p = 6537$, we see that $X^6 - 4X^4 + 3X^2 + 1 \in \mathbb{F}_p[X]$ has two irreducible factors of degree 3; this shows that $G$ has a quotient $L_2(6537^3)$. Taking $p = 8388617$ we see that $X^6 - 4X^4 + 3X^2 + 1 \in \mathbb{F}_p[X]$ has two linear factors and two factors of degree 2; this shows that $G$ has quotient $L_2(8388617) \times \text{PGL}(2,8388617) \wr C_2 \times \text{PGL}(2,8388617)$, that is, there is precisely one $N \leq G$ with $G/N \cong L_2(8388617)$, precisely two $N \leq G$ with $G/N \cong \text{PGL}(2,8388617)$, and no other $N \leq G$ with $G/N \cong L_2(8388617^k)$ or $G/N \cong \text{PGL}(2,8388617^k)$ for some $k \in \mathbb{N}$.

**Proof.** The algorithm returns the single $L_2$-ideal $P = \langle x_1 + 1, x_2^3 + x_2^2 - 2x_2 - 1, x_3^2 + x_3^2 - 3, x_{12}, x_{13}, x_{23}, x_{123} \rangle$ of type $L_2(\infty^6)$. The zeroes are

$$t = (-1, -\xi^4 + 3\xi^2 - 1, \xi, 0, 0, 0) \in \mathbb{F}_q,$$

where $\xi$ is a root of $X^6 - 4X^4 + 3X^2 + 1$. We assume $\mathbb{F}_q = \mathbb{F}_p[\xi]$. Let $\delta: G \to \text{PSL}(2,q)$ be a realization of $t$. There is no characteristic such that $-\xi^4 + 3\xi^2 - 1 = 0$ or $\xi = 0$, so $\Delta$ is never imprimitive, by Proposition 6.3. Furthermore, $\xi$ is never a root of $\Phi_k$ for $k \in \{3,4,5,6,8,10\}$, so $|\delta(v)| > 5$ for all $q$ (see Remark 11.6), hence the image of $\delta$ cannot be $A_4$, $S_4$, or $A_5$. Thus $\text{im}(\delta) \in \{L_2(q), \text{PGL}(2,\sqrt{q})\}$. The precise isomorphism type depends on the action of the Galois group. Note that $a^t = a$ for a Galois automorphism $a$ and a non-trivial sign system $a$ if and only if $a = (1,1,-1)$ and $a(\xi) = -\xi$. The result for $p \neq 2,7$ now follows by Corollary 7.6 and the fact that the Galois automorphism in characteristic $p$ is determined by the Frobenius automorphism. For $p \in \{2,7\}$ the result can be verified directly. \qed
12.3 Groups with $L_2$-ideals of type $L_2(p^\infty)$

The other kind of $L_2$-ideals of Krull dimension 1 are the ones containing a prime $p$. They seem to occur far less frequently in practice than ideals of type $L_2(\infty^k)$. However, when they occur, we can again make precise statements about the quotients.

**Proposition 12.4.** Let $G = \langle a, b, c \mid a^3 = 1, [a, c] = [c, a^{-1}], aba = bab, abac^{-1} = caba \rangle$. There exist epimorphisms $G \to L_2(q)$ if and only if $q = 3^k$ for some $k \in \mathbb{N}$. Similarly, there exist epimorphisms $G \to PGL(2, q)$ if and only if $q = 3^k$ for some $k \in \mathbb{N}$.

**Proof.** The algorithm returns the single $L_2$-ideal

$$P = \langle 3, x_1 + 1, x_2 + 1, x_{12} - 1, x_{13} - x_3, x_{23} - x_3, x_{123} - x_3 x_{123} + 1 \rangle$$

of type $L_2(3^\infty)$, so $L_2$-quotients can only occur in characteristic 3, proving the ‘only if’ parts. It remains to show that every 3-power occurs. The zeroes of $P$ are the trace tuples of the form

$$t = (t_1, t_2, t_3, t_{12}, t_{13}, t_{23}, t_{123}) = (2, 2, \xi + \xi^{-1}, 1, \xi + \xi^{-1}, \xi + \xi^{-1}, \xi)$$

with $\xi \in \mathbb{F}_3$. Let $k = [\mathbb{F}_3[\xi] : \mathbb{F}_3]$, and let $\delta: G \to PSL(3, 3^k)$ be a realization of $t$. If $k = 2\ell$ and $\xi^{3^\ell} = -\xi$, then the Galois automorphism $\sigma = (x \mapsto x^{3^\ell})$ and the sign system $\sigma = (1, 1, -1)$ induce the same action on $t$, so the image of $\delta$ is $PGL(2, 3^\ell)$, by Proposition 7.2. Otherwise, the image is $L_2(3^k)$. \hfill $\square$

Variations of the presentation yield similar results. We omit the easy proof.

**Proposition 12.5.** Let $H = \langle a, b, c \mid [a, c][a^{-1}, c], [b, a]ba^{-1}, a^{-1}c^{-1}abac^{-1}a^{-1}b^{-1} \rangle$.

1. Let $G = H/\langle a^{5} \rangle H$. Then $L_2(q)$ and $PGL(2, q)$ are quotients of $G$ if and only if $q = 5^k$ for some $k \in \mathbb{N}$.
2. Let $G = H/\langle a^{7}, (ab^{-1})^8 \rangle H$. Then $L_2(q)$ and $PGL(2, q)$ are quotients of $G$ if and only if $q = 7^k$ for some $k \in \mathbb{N}$.
3. Let $G = H/\langle a^{11}, (ab^{-1})^5 \rangle H$. Then $L_2(q)$ and $PGL(2, q)$ are quotients of $G$ if and only if $q = 11^k$ for some $k \in \mathbb{N}$.
4. Let $G = H/\langle a^{19}, (ab^{-1})^9 \rangle H$. Then $L_2(q)$ is a quotient of $G$ if and only if $q = 19^k$ for some $k \in \mathbb{N}$ or $q = 37$; and $PGL(2, q)$ is a quotient of $G$ if and only if $q = 19^k$ for some $k \in \mathbb{N}$.

12.4 Coxeter groups

**Example 12.6.** Let

$$C := \begin{pmatrix} 1 & 8 & 3 & 2 \\ 8 & 1 & 5 & 5 \\ 3 & 5 & 1 & 13 \\ 2 & 5 & 13 & 1 \end{pmatrix} \in \mathbb{Z}^{4 \times 4}.$$  

Then $L_2(q)$ is a smooth quotient of $G_C$ if and only if $q$ is one of the five primes

$$79, \ 6449, \ 699127441, \ 8438303591453175937527551, \ 518103478579218726546844118197999.$$  

Similarly, $PGL(2, q)$ is a smooth quotient of $G_C$ if and only if $q$ is one of the six primes

$$11311, \ 28081, \ 68466319, \ 2400544249, \ 13345982337089, \ 408327690683773678271.$$  

**Definition 12.7.** A C-group representation of rank $m$ is a pair $C = (H, S)$ such that $S = \{a_1, \ldots, a_m\}$ is a generating set of involutions of $H$ which satisfy the intersection property

$$\langle a_i \mid i \in I \rangle \cap \langle a_j \mid j \in J \rangle = \langle a_k \mid k \in I \cap J \rangle$$  

for all $I, J \subseteq \{1, \ldots, m\}$. 

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Example 12.8. Let
\[ C := \begin{pmatrix} 1 & 2 & 3 & 3 \\ 2 & 1 & 3 & 4 \\ 3 & 3 & 1 & 2 \\ 3 & 4 & 2 & 1 \end{pmatrix} \in \mathbb{Z}^{4 \times 4}. \]

Then \( L_2(7) \) is the only smooth quotient of \( G_C \) of \( L_2 \)-type. A realization is given by
\[ S = \left\{ \begin{pmatrix} 0 & 1 \\ 6 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 6 \\ 5 & 5 \end{pmatrix}, \begin{pmatrix} 0 & 5 \\ 4 & 0 \end{pmatrix} \right\}, \]
and it is easy to check that this generating set satisfies the intersection property.

The intersection property can be checked easily if \( G_C \) only has finitely many \( L_2 \)-quotients. Infinitely
many quotients can be handled as well, but require a little more work, as shown in the following result.

Proposition 12.9. The only finite group of \( L_2 \)-type having a \( C \)-group representation of rank 4 such that
\((|a_1a_2|, |a_1a_3|, |a_1a_4|, |a_2a_3|, |a_3a_4|) = (2, 3, 2, 3, 3) \) is \( \text{PGL}(2, 5) \).

Proof. The algorithm returns only one \( L_2 \)-ideal \( P \) of type \( L_2(\infty^2) \). Let \( P \subseteq M \leq \Phi_m^p \) be a maximal ideal containing \( P \), and let \( t = t_M \in \mathbb{F}^5_p \) be the corresponding full trace tuple (see Theorem 3.6). Let
\( H = \langle a_1, \ldots, a_4 \rangle \leq \text{PSL}(2, q) \) be the image of the induced projective representation. Then
\[ t = (t_1, t_2, t_3, t_4, t_{1234}) = \left( 0, 0, 0, 0, -1, 0, -1, \frac{2}{3}, -1, \eta_4, \frac{4}{3} \eta_4, \eta_4, -\frac{2}{3} \eta_4, \frac{1}{3} \right), \]
where \( \eta_4^2 - 2 = 0 \). The induced trace tuple for \( H_1 = \langle a_2, a_3, a_4 \rangle \) is
\[ \theta := (t_2, t_3, t_4, t_{23}, t_{24}, t_{34}, t_{234}) = \left( 0, 0, 0, -1, \frac{2}{3}, -1, -1, \frac{2}{3} \eta_4 \right). \]

We determine the isomorphism type of \( H_1 \). By Proposition 6.3, \( H_1 \) is dihedral if and only if \( t_{234} = 0 \), that is, if and only if \( 2 \mid q \). The alternating group of degree 4 is not generated by involutions, and using the
methods of Section 6 it is easy to check that \( H_1 \cong S_4 \) if and only if \( 5 \mid q \); furthermore, \( H_1 \not\cong A_5 \) for all \( q \). So if \( (q, 30) = 1 \), then \( H_1 \) is of \( L_2 \)-type. More precisely, if \( X^2 - 2 \) has a solution mod \( p \), then \( \eta_4 \in \mathbb{F}_p^\times \), so \( H_1 = \text{PSL}(2, p) \). If \( X^2 - 2 \) has no solution mod \( p \), then \( \eta_4 \) is a generator of \( \mathbb{F}_p^\times / \mathbb{F}_p^\times \), and the Galois group acts by the automorphism \( \alpha \) which maps \( \eta_4 \) to \( -\eta_4 \). In particular, \( \ast \theta = \ast \theta \) for \( \sigma = (-1, -1, -1) \), so \( H_1 = \text{PGL}(2, p) \) by Proposition 7.2.

We now determine the isomorphism type of \( H \). If \( 2 \mid q \), then \( H \) is dihedral; in fact, in this case \( \eta_4 = 0 \), so \( t \in \mathbb{F}_5^5 \), that is, \( H = \text{PSL}(2, 2) = S_3 = H_1 \). If \( X^2 - 2 \) has a solution mod \( p \), then \( H = \text{PSL}(2, p) \); otherwise, \( \ast t = \ast t \) for \( s = (-1, -1, -1, -1) \) with \( \alpha \) as above, hence \( H = \text{PGL}(2, p) \). In any case, unless \( 5 \mid q \) we see \( H = H_1 \), so the generating set does not satisfy the intersection property. We compute the realization
\[ A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 4 \eta_4 + 2 \\ \eta_4 + 3 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \eta_4 + 3 \\ 2 \eta_4 + 2 & 0 \end{pmatrix} \] \( \in \text{SL}(2, 5^2) \)

of the unique trace tuple in characteristic 5, and it is easy to check that the induced projective tuple satisfies the intersection property.

In this way, the \( L_2 \)-quotient algorithm can be used in the classification of all \( C \)-group representations of \( L_2(q) \) and \( \text{PGL}(2, q) \) of rank 4 ([CJL13]).

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