Measurement-induced nonbilocal correlation based on Wigner-Yanase skew information

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Measurement-induced nonlocality (MIN) was proposed for measure the maximum global effect caused by locally invariant measurements. Similarly, the Measurement-induced nonbilocal correlation is a generalization of MIN can be used to measure the maximal global influence caused by the local measurement in the bilocal scenario. In this paper, we propose a new nonbilocal correlation measure based on the Wigner-Yanase skew information. The relationship between the MIN based on Wigner-Yanase skew information and our measure is discussed. We present an analytical expression of our measure for pure input states and also provide upper bounds for input general mixed states.

I. INTRODUCTION

One of the most important problem in quantum physics is to characterize the correlations between separated observers, and the original disscusion about correlations in the quantum physics can be traced back to EPR paradox[1]. The study of nonlocal correlation has fundamental implication for quantum information and other related fields. The latest progress in this field[2–7] is mainly focused on rely entanglement to refute local hidden variable theory, based on the famous Bell theorem[8, 9], to show the nonlocal correlations. However, it is well known that entanglement and nonlocality are different resources[10] and entanglement-separability verification is not the only way to explore the nonlocal correlation. An alternative trial way is to find a kind of metric to measure quantum correlation. The quantum discord as such a method was initially introduced by Olliver and Zurek[11] and by Henderson and Vedral[12], and deveoped in the last decade[13–18]. The measurement-induced nonlocality(MIN)[19] as another useful measure is different from, in some extend dual to, the geometric measure of quantum discord[13] and can be used to capture the genuine nonlocal effect of measurements on the state. The original proposed MIN is based on the Hilbert-Schmidt norm and now is extend to other measure[20–25].

The quantum nonlocality not only exist in the two-observer situation but also be found in multi-observer and multi-source scenarios. In particular, the simplest and nontrivial example is the entanglement-swapping experiment that is the foundation of more complicated quantum network and has be wildly studied[2, 3, 26–32]. There are some distinct features in this scenario, such as exist states that cannot vioalte the Clauser-Horne-Shimony-Holt (CHSH) inequality[33] but can violate the bilocal inequality[2, 3, 34], i.e., there exist nonbilocal correlations. To capture the nonbilocal correlations, there are some extensions of MIN applied to the bilocal scenario are proposed[35–38].

In this paper, we propose a kind of measurement-induced nonbilocality measure based on the Wigner-Yanase skew information. This paper is organized as follows. In Section 2, we brefily introduce two nonlocality measures, MIN and MIN based on Wigner-Yanase skew information(MINS)[22]. In Section 3, we define a new measure that is the measurement-induced nonbilocality based on Wigner-Yanase skew information (MINBS) that can be viewd as a kind of generalization of MINS, and we obatained the analytical solutions of the pure states and the tight upper bound of the mixed states. In Section 4, we calculate some examples to show this nonbilocality measure. At the end, we summarize our results in Section 5.

II. MEASUREMENT-INDUCED NONLOCALITY

In contrast with the Bell nonlocality, MIN is in some kind of more general correlation. For capture all the effects that can be induced by local measurements, MIN is defined as

\[ N(\rho_{ab}) \equiv \max_{\Pi_a} \| \rho_{ab} - \hat{\Pi}^a(\rho_{ab}) \|_2, \] (1)
where the maximum is taken over all the von Neumann measurements $\Pi^a = \{\Pi^a_k\}$ which do not disturb $\rho^a$ locally, that is $\sum_k \Pi^a_k \rho^a \Pi^a_k = \rho^a$, and $\|\cdot\|^2$ in the original proposal[19] is taken as the Hilbert-Schmidt norm $\|X\|^2 = trX^\dagger X$ the $\Pi^a(\rho_{a\overline{b}})$ denotes the post-measurement state, this quantity is an indicator of the global effect caused by locally invariant measurements.

In particular, as proposed by Piani for the geometric discord[16], the MIN also may change rather arbitrarily through some trivial and uncorrelated action of unmeasured party $b$. To see this regarding the state $\rho^{a\overline{bc}} = \rho^{a\overline{b}} \otimes \rho^c$ as a bipartite state; then

$$N(\rho^{a\overline{bc}}) = N(\rho^{a\overline{b}})tr(\rho^c)^2,$$

which, as long as $\rho^c$ is a mixed state, differs from the intuitive requirement $N(\rho^{a\overline{bc}}) = N(\rho^{a\overline{b}})$. To avert this issue, we can take measure that have contractivity such as the Wigner-Yanase skew information. The MIN based on Wigner-Yanase skew information is defined as[22]

$$N_s(\rho^{ab}) = \max_{\{\Pi^b_s\}} \{ \sum_{k=1}^m \mathcal{I}(\rho^{ab}, \Pi^b_k, \Pi^a \otimes I_n) \},$$

(2)

where $\mathcal{I}(\rho, K)$ is the skew information introduced by Wigner and Yanase[39] and $\mathcal{I}(\rho, K) = -\frac{1}{2}tr[\rho^2, K]^2$ where $K$ is a Hermitian observable and $[\cdot]$ denotes the commutator. That max is taken over all the von Neumann measurements which do not disturb $\rho^a$ locally.

### III. MEASUREMENT-INDUCED NONBILOCALITY BASED ON WIGNER-YANASE SKEW INFORMATION

In this section, we introduce our nonbilocality measure based on Wigner-Yanase skew information. For any arbitrary finite dimensional system $H = H_A \otimes H_B \otimes H_C \otimes H_D$ with $dim H_A = m, dim H_B = n, dim H_C = u$ and $dim H_D = v$.

Define the Measurement-induced nonbilocality based on Wigner-Yanase skew information as

$$\mathcal{N}^b_s(\rho_{AB} \otimes \rho_{CD}) = \max_{\{\Pi^b_{st}\}} \{ \sum_{s=1}^n \sum_{t=1}^n \mathcal{I}(\rho_{AB} \otimes \rho_{CD}, I_m \otimes \Pi^b_{st} \otimes I_v) \}. \tag{3}$$

Noting that this formula can be reduced to

$$\mathcal{N}^b_s(\rho_{AB} \otimes \rho_{CD}) = 1 - \min_{\{\Pi_{st}^b\}} \{ \sum_{s=1}^n \sum_{t=1}^n tr(\sqrt{\rho_{AB} \otimes \rho_{CD}(I_m \otimes \Pi_{st}^b \otimes I_v)} \sqrt{\rho_{AB} \otimes \rho_{CD}(I_m \otimes \Pi_{st}^b \otimes I_v)}) \}, \tag{4}$$

where the $\min$ is taken over all the von Neumann measurements which do not disturb $\rho^{bc}$ locally. There we list some basic properties of this measure.

(i) $\mathcal{N}^b_s(\rho_{AB} \otimes \rho_{CD}) = 0$ for all product states $\rho_{AB} = \rho_A \otimes \rho_B$ and $\rho_{CD} = \rho_C \otimes \rho_D$. But this measure is strictly positive for $\rho_{AB}, \rho_{CD}$ where at least one of the states is entanglement.

In particular, we have

(ii) $\mathcal{N}^b_s(\rho_{AB} \otimes \rho_{CD}) = 0$ for any quantum-classical state[10] $\rho_{AB} = \sum_f \rho^A_f \otimes \rho^B_f |f_B \rangle \langle f_B|$ and classical-quantum state $\rho_{CD} = \sum_f q_f |f_C \rangle \langle f_C| \otimes \rho^D_f$ whose marginal states $\rho_B$ and $\rho_A$ are both nondegenerate.

(iii) If $\rho_B, \rho_C$ have the spectral decomposition $\rho_B = \sum_i \lambda_i |i_B \rangle \langle i_B|, \rho_C = \sum_j \mu_j |j_C \rangle \langle j_C|$, respectively, and at least one of the states $\rho_B$ and $\rho_C$ is nondegenerate, then the von Neumann measurements that do not disturb $\rho_{BC}$ must have the form $\Pi_{BC} = \{ \Pi^B_t \otimes \Pi^C_i \}$, i.e., they are not the joint quantum measurements.

(iv) $\mathcal{N}^b_s(\rho_{AB} \otimes \rho_{CD})$ is locally invariant in the sense that $\mathcal{N}^b_s(\rho_{AB} \otimes \rho_{CD}) = \mathcal{N}^b_s(\rho_{AB} \otimes \rho_{CD})$ for any unitary operators $U_A, U_B, U_C, U_D$ acting on $H_A, H_B, H_C, H_D$, respectively.
(v) If $\rho_B$ and $\rho_C$ are both nondegenerate, then

$$N_s^b(\rho_{AB} \otimes \rho_{CD}) = 1 - \sum_{st} tr(\sqrt{\rho_{AB} \otimes \rho_{CD}}(I_m \otimes \Pi_{st}^{BC} \otimes I_v)\sqrt{\rho_{AB} \otimes \rho_{CD}}(I_m \otimes \Pi_{st}^{BC} \otimes I_v)).$$

(vi) There is a strong connection between MIN based on Wigner-Yanase skew information and our measure, that is $N_s^b(\rho_{BA} \otimes \rho_{AB}) \geq N_s(\rho_{AB})$ since

$$N_s^b(\rho_{BA} \otimes \rho_{AB}) = 1 - \sum_{ij} \lambda_i \lambda_j^2, \quad (5)$$

**Theorem 1.** If $\rho_{AB} \otimes \rho_{CD} = |\psi_{AB}\rangle \langle \psi_{AB}| \otimes |\phi_{CD}\rangle \langle \phi_{CD}|$ is pure, that is $|\psi_{AB}\rangle = \sum i \lambda_i |i_a i_b\rangle$ and $|\phi_{CD}\rangle = \sum_j \mu_j |j_c j_d\rangle$, then

$$N_s^b(\rho_{AB} \otimes \rho_{CD}) = 1 - \sum_{ij} \lambda_i \lambda_j^2.$$

**Proof** First, noting that any von Neumann measurements on $H_B \otimes H_C$ is expressed as $\Pi^{BC} = \{\Pi_{st}^{BC} \equiv U | s_i t_c \rangle \langle s_i t_c | U^{\dagger}\}$ and thus

$$\rho_{AB} \otimes \rho_{CD} = \sum_{i'i'j'} \lambda_i \lambda_{i'} \mu_{j} \mu_{j'} |i_a\rangle \langle i_a' | \otimes |i_b\rangle \langle i_b' | \otimes |j_c\rangle \langle j_c' | \otimes |j_d\rangle \langle j_d' |$$

and

$$\rho_{BC} = tr_{AB}(\rho_{AB} \otimes \rho_{CD}) = \sum_{ij} \lambda_i^2 \mu_j^2 |i_b j_c\rangle \langle i_b j_c|$$

$$\sqrt{\rho_{AB} \otimes \rho_{CD}(I_m \otimes \Pi_{st}^{BC} \otimes I_v)\sqrt{\rho_{AB} \otimes \rho_{CD}}(I_m \otimes \Pi_{st}^{BC} \otimes I_v)}$$

$$= \sum_{i'j'} \sqrt{\lambda_i \lambda_{i'} \mu_j \mu_{j'}} |i_a\rangle \langle i_a' | \otimes |i_b\rangle \langle i_b' | \otimes |j_c\rangle \langle j_c' | \otimes |j_d\rangle \langle j_d' | U | s_i t_c \rangle$$

$$\langle s_i t_c | U^{\dagger} \{ \sum_{uvu'} \sqrt{\lambda_u \lambda_v \mu_u \mu_{u'}} |u_a\rangle \langle u_a' | \otimes |u_b\rangle \langle u_b' | \otimes |v_c\rangle \langle v_c' | \otimes |v_d\rangle \langle v_d' | U | s_i t_c \rangle \langle s_i t_c | U^{\dagger}}$$

$$= \sum_{i'j'} \sum_{uvu'} \sqrt{\lambda_i \lambda_{i'} \mu_j \mu_{j'}} \lambda_u \lambda_v \mu_u \mu_{u'} |i_a\rangle \langle i_a' | \otimes |u_a\rangle \langle u_a' | \otimes |i_b j_c\rangle$$

$$\langle i' u' v_2 u' v_2' | U | s_i t_c \rangle \langle s_i t_c | U^{\dagger} | u_b v_c | \langle u_b' v_c' | U | s_i t_c \rangle \langle s_i t_c | U^{\dagger} \otimes |j_d\rangle \langle j_d' | v_d \langle v_d' |$$

from which we have

$$\sum_{s_i t_c} tr(\sqrt{\rho_{AB} \otimes \rho_{CD}}(I_m \otimes \Pi_{st}^{BC} \otimes I_v)\sqrt{\rho_{AB} \otimes \rho_{CD}}(I_m \otimes \Pi_{st}^{BC} \otimes I_v))$$

$$= \sum_{s_i t_c} \sum_{i'j'} \lambda_i \lambda_{i'} \mu_j \mu_{j'} |u_b v_c\rangle \langle u_b' v_c' | U | s_i t_c \rangle \langle s_i t_c | U^{\dagger} | u_b v_c | \langle u_b' v_c' | U | s_i t_c \rangle \langle s_i t_c | U^{\dagger} \otimes |i_b j_c\rangle$$

$$= \sum_{s_i t_c} (\lambda_i^2 \mu_j^2 |i_b j_c\rangle \langle i_b j_c|)$$

$$= \sum_{s_i t_c} ((s_i t_c | U^{\dagger} \rho_{BC} U | s_i t_c))^{2}$$
Since \( \Pi_{st}^{BC} = U \mid s_{t} \rangle \langle s_{t} \mid U \dagger \) leaves \( \rho_{BC} \) invariant, it follows that

\[
\rho_{BC} = \sum_{st} U \mid s_{t} \rangle \langle s_{t} \mid U \dagger \rho_{BC} U \mid s_{t} \rangle \langle s_{t} \mid U \dagger
\]

or, equivalently,

\[
\rho_{BC} = \sum_{st} (s_{t} \rangle \langle s_{t} \mid U \dagger \rho_{BC} U \mid s_{t} \rangle \langle s_{t} \mid U \dagger
\]

is a spectral decomposition of \( \rho_{BC} \).

Consequently,

\[
\sum_{sv} \text{tr}(\sqrt{\rho_{AB} \otimes \rho_{CD}}(I_{m} \otimes \Pi_{st}^{BC} \otimes I_{v})) \sqrt{\rho_{AB} \otimes \rho_{CD}}(I_{m} \otimes \Pi_{st}^{BC} \otimes I_{v})
\]

\[
= \sum_{sv} (s_{t} \rangle \langle s_{t} \mid U \dagger \rho_{BC} U \mid s_{t} \rangle \langle s_{t} \mid U \dagger
\]

\[
= \sum_{st} \lambda_{s}^{4} \mu_{t}^{4}
\]

The desired result is obtained. The optimum is achieved by any von Neumann measurement leaving \( \rho_{BC} \) invariant.

For mixed input states, there are some kinds of different situations and we listed as following:

Suppose the Hilbert spaces \( H_{A}, H_{B}, H_{C} \) and \( H_{D} \) are of dimensions \( \dim H_{A} = m, \dim H_{B} = n, \dim H_{C} = u \) and \( \dim H_{D} = v \), respectively. Let \( L(H_{i}) \) be the Hilbert space consisting of all linear operators on \( H_{i}(i = A, B, C, D) \), with the Hilbert-Schmidt inner product \( (x \mid y) \equiv \text{tr}(x \dagger y) \).

Let \( \{X_{i} : i = 0, 1, \ldots, m^{2} - 1\}, \{Y_{j} : j = 0, 1, \ldots, n^{2} - 1\}, \{Z_{k} : k = 0, 1, \ldots, u^{2} - 1\}, \{W_{l} : l = 0, 1, \ldots, v^{2} - 1\} \) be orthonormal Hermitian operator bases for \( L(H_{A}), L(H_{B}), L(H_{C}) \) and \( L(H_{D}) \), respectively, with \( X_{0} = 1^{A}/\sqrt{m}, Y_{0} = 1^{B}/\sqrt{n}, Z_{0} = 1^{C}/\sqrt{u} \) and \( W_{0} = 1^{D}/\sqrt{v} \). Then, general bipartite states \( \rho_{AB} \) and \( \rho_{CD} \) can always be represented as

\[
\sqrt{\rho_{AB}} = \sum_{ij} t_{ij}^{ab} X_{i} \otimes Y_{j}, \quad \sqrt{\rho_{CD}} = \sum_{kl} t_{kl}^{cd} Z_{k} \otimes W_{l}
\]

where \( t_{ij}^{ab} \equiv \text{tr}(\sqrt{\rho_{AB}}(X_{i} \otimes Y_{j})) \) and \( t_{kl}^{cd} \equiv \text{tr}(\sqrt{\rho_{CD}}(Z_{k} \otimes W_{l})) \). Let \( T_{ab} = (t_{ij}^{ab}), T_{cd} = (t_{kl}^{cd}) \), which may be regarded as some kind of correlation matrices for the state \( \rho_{AB} \) and \( \rho_{CD} \), respectively. Then, we have

\[
\sqrt{\rho_{AB} \otimes \rho_{CD}} = \sum_{i,j,k,l} t_{ij,k,l}^{abcd} X_{i} \otimes Y_{j} \otimes Z_{k} \otimes W_{l},
\]

\[
\sqrt{\rho_{BC,AB}} = \sum_{j,k,l} \left( \sum_{i} t_{ij,k,l}^{abcd} \right) Y_{j} \otimes Z_{k} \otimes X_{i} \otimes W_{l},
\]

where the matrix \( T_{bc,ad} = (t_{jk,il}^{bc,ad}) = (t_{ij,k,l}^{abcd}) = T_{ab} \otimes T_{cd} \).

Note that \( N_{s}^{b}(\rho_{AB} \otimes \rho_{CD}) = N_{s}(\rho_{BC,AD}) \), since

\[
\text{tr}(\sqrt{\rho_{AB} \otimes \rho_{CD}}(I_{m} \otimes \Pi_{st}^{BC} \otimes I_{v})) \sqrt{\rho_{AB} \otimes \rho_{CD}}(I_{m} \otimes \Pi_{st}^{BC} \otimes I_{v}) = \text{tr}(\sqrt{\rho_{BC,AD}}(I_{m} \otimes \Pi_{st}^{BC} \otimes I_{v})) \sqrt{\rho_{BC,AD}}(I_{m} \otimes \Pi_{st}^{BC} \otimes I_{v})
\]

\[
(8)
\]

**Theorem 2.** For \( \rho_{AB} \) and \( \rho_{CD} \) represented as eq. (6), we have

\[
N_{s}^{b}(\rho_{AB} \otimes \rho_{CD}) = 1 - \min_{F} F T_{bc,ad} T_{bc,ad}^{T} F^{\dagger} \leq 1 - \sum_{o=1}^{nu} t_{o},
\]

where \( F \equiv (f_{g (jk)}) \) is an \( nu \times n^{2}u^{2} \)-dimensional matrix with \( f_{g (jk)} = \text{tr}(\Pi_{g}^{BC} Y_{j} \otimes Z_{k}), (g = 0, 1, \ldots, nu - 1; (jk) = ju^{2} + k, j = 0, 1, \ldots, n^{2} - 1, k = 0, 1, \ldots, u^{2} - 1) \), \( T_{bc,ad} = (t_{g,h,k,l}^{bc,ad}) = T_{ab}^{g} \otimes T_{cd}^{h} \) is an \( n^{2}u^{2} \times m^{2}u^{2} \)-dimensional matrix, and \( \{t_{o} : o = 1, 2, \ldots, n^{2}u^{2}\} \) are the eigenvalues of the matrix \( T_{bc,ad} T_{bc,ad}^{T} \) listed in the increasing order.
By definition, we have the following theorem.

\[ \sum_{st} (\mathbb{I}_m \otimes \Pi^{BC}_{st} \otimes \mathbb{I}_v) \sqrt{\rho_{BC,AD}} (\mathbb{I}_m \otimes \Pi^{BC}_{st} \otimes \mathbb{I}_v) \]

\[ = \sum_{g} \sum_{ijkl} t^{ijkl}_{g} f_{g(jk)} (Y_j \otimes Z_k) \Pi^{BC}_g \otimes X_i \otimes W_l \]

\[ = \sum_{g} \sum_{ijkl} t^{ijkl}_{g} f_{g(jk)} \Pi^{BC}_g \otimes X_i \otimes W_l \]

and therefore

\[ \sum_{st} tr(\sqrt{\rho_{BC,AD}} (\mathbb{I}_m \otimes \Pi^{BC}_{st} \otimes \mathbb{I}_v) \sqrt{\rho_{BC,AD}} (\mathbb{I}_m \otimes \Pi^{BC}_{st} \otimes \mathbb{I}_v) \]

\[ = \sum_{g} \sum_{ijkl} t^{ijkl}_{g} f_{g(jk)} t^{ijkl}_{g} \]

\[ = \sum_{g} \sum_{ijkl} f_{g(jk)}^{bc,ad} t^{ijkl}_{g} \]

\[ = FT_{bc,ad} T_{bc,ad}^t \]

(10)

By definition, we have

\[ \mathcal{N}_s^h (\rho_{AB} \otimes \rho_{CD}) = 1 - \min_p FT_{bc,ad} T_{bc,ad}^t \leq 1 - \sum_{o=1}^{n} t_o, \]

(12)

where \( \{ t_o : o = 1, 2, \ldots, n^2 u^2 \} \) are the eigenvalues of the matrix \( T_{bc,ad} T_{bc,ad}^t \) listed in the increasing order.

In addition, with out lose of generality, if \( \rho_b \) is nondegenerate, then \( \Pi^{BC} = \{ \Pi^B \otimes \Pi^C \} = \{ | s_b \rangle \langle s_b | \otimes \Pi^C \} \) and we have the following theorem.

**Theorem 3.** If \( \rho_b \) is nondegenerate, then

\[ \mathcal{N}_s^h (\rho_{AB} \otimes \rho_{CD}) = 1 - tr BT_{ab}^t T_{ab} B^t \times \min_C tr CT_{cd} T_{cd}^t C^t \leq 1 - tr BT_{ab}^t T_{ab} B^t \times (\sum_{o'=1}^{u} t'_{o'}) \]

(13)

where \( B \equiv (b_{sj}) \) is an \( n \times n^2 \)-dimensional matrix with \( b_{sj} \equiv tr | s_b \rangle \langle s_b | Y_j, C \equiv (c_{lk}) \) is a \( u \times u^2 \)-dimensional matrix with \( c_{lk} \equiv tr \Pi^C_l Z_k \) and \( \{ t'_{o'} : o' = 1, 2, \ldots, u^2 \} \) are the eigenvalues of the matrix \( T_{cd} T_{cd}^t \) listed in increasing order.

In particular, if \( u=2 \), then

\[ \mathcal{N}_s^h (\rho_{AB} \otimes \rho_{CD}) = 1 - tr BT_{ab}^t T_{ab} B^t \times (||r_{cd}||^2 + r') \]

(14)

where \( r_{cd} = (t_{cd}^0, t_{cd}^1, \ldots, t_{cd}^{u-1}) \) and \( r' \) is the smallest eigenvalue of the \( 3 \times 3 \)-dimensional matrix \( RR^t \) with \( R = (t_{kl})_{k,l=1,2,3; f=0,1,\ldots,v^2-1} \).

**Proof** If \( \rho_b \) is nondegenerate, we have

\[ \mathcal{N}_s^h (\rho_{AB} \otimes \rho_{CD}) = 1 - \min_{\{ \Pi^C_g \}} \sum_{s=1}^{n} \sum_{l=1}^{u} tr(\sqrt{\rho_{AB} \otimes \rho_{CD}} (\Pi^B \otimes \Pi^C)^l) \sqrt{\rho_{AB} \otimes \rho_{CD}} (\Pi^B \otimes \Pi^C)^l \]

\[ = 1 - \min_{\{ \Pi^C_g \}} \sum_{g} tr(\sqrt{\rho_{AB}} (\Pi^B_g) \sqrt{\rho_{AB}} (\Pi^B_g)) tr(\sqrt{\rho_{CD}} (\Pi^C_g)) \sqrt{\rho_{CD}} (\Pi^C_g) \]

\[ = 1 - \sum_{g} tr(\sqrt{\rho_{AB}} (\Pi^B_g) \min_{\{ \Pi^C_g \}} \sum_{g'} tr(\sqrt{\rho_{CD}} (\Pi^C_{g'})) \sqrt{\rho_{CD}} (\Pi^C_{g'}) \]

\[ = 1 - tr BT_{ab}^t T_{ab} B^t \times \min_C tr CT_{cd} T_{cd}^t C^t \]

\[ \leq 1 - tr BT_{ab}^t T_{ab} B^t \times (\sum_{o'=1}^{u} t'_{o'}) \]
If $u=2$, the completeness relation $\sum_{i=0}^{1} \Pi_i^C = I^C$ implies that $c_{i+k} = -c_{i+k}(k = 1, 2, 3)$. Let $c = \sqrt{2}(c_{01}, c_{02}, c_{03})$, then from $\sum_{k=0}^{2} c_{i+k}^2 = 1$ and $c_{00} = c_{10} = \frac{1}{\sqrt{2}}$, we get $\|c\|=1.$

Then we have

$$C = (c_{i+k}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & c \\ 1 & -c \end{pmatrix}$$

and

$$T_{cd} = (t_{kl}) = \begin{pmatrix} r_{cd} \\ R \end{pmatrix}$$

with a $v^2$-dimenstional row vector $r_{cd} = (t_{00}, t_{01}, \ldots, t_{v^2-1})$, and a $3 \times v^2$-dimensional matrix $R = (t_{kl})_{k=1,2,3; l=0,1,\ldots,v^2-1}$. So we have

$$trCT_{cd}C^t = \|r_{cd}\|^2 + cRR^t r^t$$

**Theorem 4.** If the marginal states $\rho_a$ and $\rho_c$ are both nondegenerate, we straightforward have

$$\mathcal{N}_s^0(\rho_{AB} \otimes \rho_{CD}) = 1 - trBT_{ab}^t B^t \times trCT_{cd}^t C^t$$

where $C = (c_{i+k})$ is a $u \times u^2$-dimensional matrix with $c_{i+k} = tr \mid t_{kl} \rangle \langle t_{kl} \mid Z_k$.}

**IV. EXAMPLES**

In this part, we calculate MINBS for the pure states and mixed states, respectively.

**Example 1.** For any Bell states, such as $|\Phi_{AB}\rangle \otimes |\Phi_{CD}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_AB \otimes \frac{1}{2}(|00\rangle + |11\rangle)_CD$, we have

$$\mathcal{N}_s^0(\langle \Phi_{AB} | \otimes | \Phi_{CD} \rangle \langle \Phi_{CD} |) = 1 - 4 \times (\frac{1}{\sqrt{2}})^4 \times (\frac{1}{\sqrt{2}})^4 = \frac{3}{4}.$$ 

**Example 2.** Consider the separable states, for the classical seperable state $\rho_{AB} = \rho_{CD} = \rho = \frac{1}{2} |0\rangle \langle 0| \otimes |0\rangle \langle 0| + \frac{1}{2} |1\rangle \langle 1| \otimes |1\rangle \langle 1|$, we have

$$\sqrt{7} = \frac{1}{\sqrt{2}} |0\rangle \langle 0| \otimes |0\rangle \langle 0| + \frac{1}{\sqrt{2}} |1\rangle \langle 1| \otimes |1\rangle \langle 1| = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \otimes \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \sigma_3 \otimes \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}}$$

$$T_{ab} = T_{cd} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

We choose one of the most optimal von Neumann measurements as

$$\Pi^{BC} = \{ H^{\otimes 2} |00\rangle \langle 00|, H^{\otimes 2} |01\rangle \langle 01|, H^{\otimes 2} |10\rangle \langle 10|, H^{\otimes 2} |11\rangle \langle 11|, H^{\otimes 2} \},$$

where $H$ denotes the Hadamard gate matrix

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Through a straightforward calculation we have

$$\mathcal{N}_s^0(\rho \otimes \rho) = 1 - \min F T_{bc,ad}^t F^t = \frac{3}{4}.$$
Example 3. We consider the Bell-diagonal state $\rho_{ab} = \lambda_1 |\psi^+\rangle \langle \psi^+| + \lambda_2 |\psi^-\rangle \langle \psi^-| + \lambda_3 (|\phi^+\rangle \langle \phi^+| + |\phi^-\rangle \langle \phi^-|)$, where $|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ and $|\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$, $\sum_{i=0}^{3} \lambda_i = 1$, $\lambda_i \geq 0$; then by the operator functions,

$$\sqrt{\rho_{AB}} = \sqrt{\lambda_1} |\psi^+\rangle \langle \psi^+| + \sqrt{\lambda_2} |\psi^-\rangle \langle \psi^-| + \sqrt{\lambda_3} |\phi^+\rangle \langle \phi^+| + \sqrt{\lambda_4} |\phi^-\rangle \langle \phi^-|$$

Next we express the $\sqrt{\rho_{AB}}$ in the standard operator base $\{ \frac{1}{\sqrt{2}} I_i \sigma_i : i = 1, 2, 3 \}$

$$\sqrt{\rho_{AB}} = \frac{h_0}{2} I^a \otimes I^b + \frac{h_1}{2} \frac{\sigma_1}{\sqrt{2}} \otimes \frac{\sigma_1}{\sqrt{2}} + \frac{h_2}{2} \frac{\sigma_2}{\sqrt{2}} \otimes \frac{\sigma_2}{\sqrt{2}} + \frac{h_3}{2} \frac{\sigma_3}{\sqrt{2}} \otimes \frac{\sigma_3}{\sqrt{2}}$$

where

$$h_0 = \sqrt{\lambda_1} + \sqrt{\lambda_2} + \sqrt{\lambda_3} + \sqrt{\lambda_4}, \quad h_1 = \sqrt{\lambda_1} - \sqrt{\lambda_2} + \sqrt{\lambda_3} - \sqrt{\lambda_4},$$
$$h_2 = -\sqrt{\lambda_1} + \sqrt{\lambda_2} + \sqrt{\lambda_3} - \sqrt{\lambda_4}, \quad h_3 = \sqrt{\lambda_1} + \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4},$$

Thus we have

$$T_{ab} = \begin{pmatrix} \frac{h_0}{2} & 0 & 0 & 0 \\ 0 & \frac{h_1}{2} & 0 & 0 \\ 0 & 0 & \frac{h_2}{2} & 0 \\ 0 & 0 & 0 & \frac{h_3}{2} \end{pmatrix}$$

and

$$T_{aa,bb} = T_{ba}^d \otimes T_{ab} = diag \left( \frac{h_0^2}{4}, \frac{h_0 h_1}{4}, \frac{h_0 h_2}{4}, \frac{h_0 h_3}{4}, \frac{h_1^2}{4}, \frac{h_1 h_2}{4}, \frac{h_1 h_3}{4}, \frac{h_2^2}{4}, \frac{h_2 h_3}{4}, \frac{h_3^2}{4} \right)$$

According to the Theorem 2, and we choose the von Neumann measurement

$$\Pi^{BC} = \{ |\psi^+\rangle \langle \psi^+|, |\psi^-\rangle \langle \psi^-|, |\phi^+\rangle \langle \phi^+|, |\phi^-\rangle \langle \phi^-| \}$$

and we can get the matrix

$$F = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

and

$$tr(F T_{aa,bb}^d T_{aa,bb}^d F^d) = \frac{(h_0^4 + h_1^4 + h_2^4 + h_3^4)}{16}$$

So we get the following elegant result

$$\mathcal{N}^a_{a,b} (\rho_{ab} \otimes \rho_{ab}) = 1 - \frac{(h_0^4 + h_1^4 + h_2^4 + h_3^4)}{16}.$$ 

The maximum $\frac{2}{3}$ is taken in any one of the four Bell states just like illustrated in Example 1 and the minimum equals to zero when $\lambda_i = \frac{1}{4}$ that is a product state, this result coincidence with the property (i).

We point out that when $c_1 = \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4$ and $c_2 = -\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4$, $c_3 = \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4$ are all equals to a parameter $-v$, where $v \in [-\frac{1}{4}, 1]$, then the state $\rho_{ab}$ reduces to the two-qubit Werner state $\rho_{ab} = v |\phi^\mp\rangle \langle \phi^\pm| + \frac{1-v}{4} I_{aa,bb}$, thus can applied our results to this situation.
V. CONCLUSIONS

In this article, we have proposed a new form of measurement-induced nonbilocal correlation measure based on Wigner-Yanase skew information. Then we demonstrate there is a connection between MINS and our measure, and we have presented an analytical formulas of MINBS for pure input states and provide upper bounds for mixed input states. It’s interesting for the example we have proposed and it’s natural to ask whether there exist correlations can make our measure get one, i.e., have the maximal nonbilocal correlation. An interesting future work is to explore a effective way to measure the measurement-induced non N-local correlation for N larger than two.

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