Abstract. We use entropy theory as a new tool to study sectional hyperbolic flows in any dimension. We show that for $C^1$ flows, every sectional hyperbolic set $\Lambda$ is entropy expansive, and the topological entropy varies continuously with the flow. Furthermore, if $\Lambda$ is Lyapunov stable, then it has positive entropy; in addition, if $\Lambda$ is a chain recurrent class, then it contains a periodic orbit. As a corollary, we prove that for $C^1$ generic flows, every Lorenz-like class is an attractor. We also show that non-trivial chain recurrent classes for generic $C^1$ star flows satisfy a dichotomy: either it has zero entropy, or it is isolated. As a result, $C^1$ generic star flows have only finitely many Lyapunov stable chain recurrent classes.

1. Introduction

About half a century ago, Lorenz published his famous article [22] in which he used computer-aided numerical simulation to study the following system, which is
now known as the Lorenz equations:

\[
\begin{align*}
\dot{x} &= -\sigma x + \sigma y & \sigma = 10 \\
\dot{y} &= ry - xz & r = 28 \\
\dot{z} &= -bz + xy & b = 8/3.
\end{align*}
\]

Numerical simulations for an open neighborhood of the chosen parameters suggested that almost all points in phase space tend to a chaotic attractor.

An attractor is a bounded region in phase space, invariant under time evolution, to which the forward trajectories of most (positive probability) or, even, all nearby points converge. What makes an attractor chaotic is the fact that trajectories converging to the attractor are sensitive with respect to initial data: trajectories of any two nearby points diverge under time evolution.

Lorenz equations prove to be very resistant to rigorous mathematical analysis, from both conceptual (existence of the equilibrium accumulated by regular orbits prevents the attractor to be hyperbolic) as well numerical (solutions slow down as they pass near the equilibrium, which means unbounded return times and, thus, unbounded integration errors) point of view. Based on numerical experiments, Lorenz conjectured that the flow generated by the equations (1) presents a volume zero chaotic attractor which is robust (it persists under small perturbation of the parameters). This attractor is called a Lorenz attractor and it has a butterfly shape and displays an extremely rich dynamical properties. It is robust in the sense that nearby flows also possess an attractor with similar properties. Part of the reason for the richness of the Lorenz attractor is the fact that it has an equilibrium or singularity, i.e., a point where the vector field vanishes, that is accumulated by regular orbits (orbits through points where the corresponding vector field does not vanish) which prevents the flow from being uniformly hyperbolic.

In the seventies, a geometric Lorenz model for this attractor was proposed in \cite{14, 1, 15}. These models are flows in three dimensions for which one can rigorously prove the existence of a chaotic attractor that contains an equilibrium point of the flow, which is an accumulation point of typical regular solutions.

Finally, the above stated existence of a chaotic attractor for the original Lorenz system was not proved until the year 2000, when Tucker did so with a computer-aided proof \cite{28, 29}. To more on this, the interested reader can consult \cite{30} and references therein.

In order to describe the hyperbolicity of a Lorenz flow, or more generally, invariant sets for a three-dimensional flow that contain equilibria, Morales, Pacífico and Pujals \cite{24} proposed the notion of singular hyperbolicity, which requires the flow to have a one-dimensional uniformly contracting direction, and a two-dimensional sub-bundle containing the flow direction, on which the flow is volume expanding. It is shown in \cite{26} that every robust attractor of a three-dimensional flow must be singular hyperbolic. Later, Araújo et al proved in \cite{3} that every singular hyperbolic attractor for 3-flows is expansive. The key technique in their proof is the linearization near singularities, and being an attractor allows them to take a collection of cross sections, thus reduce the flow to a two-dimensional return map.

The notion of singular hyperbolicity was later generalized to sectional hyperbolicity for high dimensional flows, see \cite{17, 23}. More precisely:

**Definition 1.** A compact invariant set $\Lambda$ of a flow $X$ is called sectional hyperbolic, if it admits a dominated splitting $E^s \oplus F^{cu}$, such that $E^s$ is uniformly contracting, and $F^{cu}$ is sectional-expanding: there are constants $C, \lambda > 0$ such that for every $x \in \Lambda$ and any subspace $V_x \subset F^{cu}_x$ with $\dim V_x \geq 2$, we have

$$|\det D\phi_t(x)|_{V_x} | \geq C e^{\lambda t} \text{ for all } t > 0.$$
We will call $\lambda$ the sectional volume expanding rate on $F^{cu}$.

**Definition 2.** For $\epsilon > 0, T > 0$, a finite sequence $\{x_i\}_{i=0}^n$ is called an $(\epsilon, T)$-chain if there exists $\{t_i\}_{i=0}^{n-1}$ such that $t_i > T$ and $d(\phi_{t_i}(x_i), x_{i+1}) < \epsilon$ for all $i = 0, \ldots, n-1$. We say that $y$ is chain attainable from $x$, if there exists $T > 0$ such that for all $\epsilon > 0$, there exists an $(\epsilon, T)$-chain $\{x_i\}_{i=0}^n$ with $x_0 = x$ and $x_n = y$. It is straightforward to check that chain attainability is an equivalent relation on the set $CR(X) = \{x : x$ chain attainable from $x\}$. Each equivalent class under this relation is then called a chain recurrent class. A chain recurrent class $C$ is said to be non-trivial, if it is not a singularity or a periodic orbit.

**Definition 3.** A compact invariant set $\Lambda$ a Lorenz-like class, if it contains both singularities and regular points, and satisfies the following conditions:

(a) $\Lambda$ is a chain recurrent class.

(b) $\Lambda$ is Lyapunov stable, i.e., there is a sequence of compact neighborhoods $\{U_i\}$ such that:
   - $U_1 \supset U_2 \supset \cdots$ and $\bigcap_i U_i = \Lambda$;
   - for each $i \geq 1$, $\phi_t(U_{i+1}) \subset U_i$ for any $t \geq 0$.

(c) $\Lambda$ is sectional-hyperbolic.

Recall that a $C^1$ flow is a star flow if for any nearby flow, its critical elements, i.e., singularities and periodic orbits, are all hyperbolic. Recently, [11] proves that for generic star flows, every non-trivial Lyapunov stable chain recurrent class is Lorenz-like.

Note that in the definition of a Lorenz-like class we neither assume the singularities to be hyperbolic nor the class to be isolated. As a result, tools developed for three-dimensional singular hyperbolic flows become invalid for higher dimensional sectional hyperbolic flows: non-isolated invariant set makes it difficult to take cross sections, and non-hyperbolic singularities make linearization impossible. Also recall that even when the singularities are hyperbolic, one still need extra assumptions on the eigenvalues of the singularity in order to use linearization. To overcome these difficulties, we will look at the time-one map of the flow, and use the fake foliations developed in [19] to studied the expanding property in a neighborhood of the singularities.

On the other hand, entropy theory for discrete and continuous time systems (without singularities) have been developed for over 50 years and is proven to be quite successful. The entropy expansiveness, first introduced by Bowen in [6], is one of the key reasons that hyperbolic systems have nice properties in regard to entropy view point. In particular, Bowen proves that if the system is robust entropy expansive, then the metric entropy is upper semi-continuous at the invariant measure, and the topological entropy varies upper semi-continuously with the system. As an important corollary, there always exists an equilibrium state for every continuous potential.

In this paper, we will provide a new method to study sectional hyperbolic flows, which is based on the new development of entropy theory in [19]. For simplicity, we will assume for now that the singularities are hyperbolic. However, this assumption is not essential and can be removed with only a slight modification to the proof. We will deal with the case of non-hyperbolic singularities in the appendix. See Theorems [II] and [I].

**Definition 4.** Let $\Lambda$ be a compact invariant set for a $C^1$ flow $\phi_t$ and $\delta > 0$. We say that $\phi_t$ is robustly $\delta$-entropy expansive near $\Lambda$ if there is a neighborhood $U$ of $\phi_t$ in the $C^1$ topology and a neighborhood $U$ of $\Lambda$, such that for every $\psi_t \in U$, the maximal invariant set of $\psi_t$ in $U$ is $\delta$-entropy expansive.
Theorem A. Let $\Lambda$ be a compact invariant set that is sectional hyperbolic for a $C^1$ flow $\phi_t$, with all the singularities in $\Lambda$ hyperbolic. Then $\phi_t$ near $\Lambda$ is robustly $\delta$-entropy expansive.

We also obtain continuity of the topological entropy for sectional hyperbolic invariant sets:

Theorem B. Let $\Lambda$ be a sectional hyperbolic compact invariant set for a $C^1$ flow $\phi_t$, with all singularities in $\Lambda$ hyperbolic. Then there is a neighborhood $U$ of $\Lambda$, such that $h_{top}(\cdot|\tilde{\Lambda})$ is continuous at $\phi_t$, where $\tilde{\Lambda}$ is the maximal invariant set in $U$. More precisely, let $\phi^n_t$ be a sequence of $C^1$ flows with $\phi^n_t \xrightarrow{n \to \infty} \phi_t$ in $C^1$ topology, and denote by $\tilde{\Lambda}_n$ the maximal invariant set of $\phi^n_t$ in $U$, then

$$\lim_{n \to \infty} h_{top}(\phi^n_t|\tilde{\Lambda}_n) = h_{top}(\phi_t|\tilde{\Lambda}).$$

Furthermore, if $h_{top}(\phi_t|\Lambda) > 0$ then there are periodic orbits arbitrarily close to $\Lambda$. When $\Lambda$ is a chain recurrent class, such periodic orbits are indeed contained in $\Lambda$.

Note that in Theorems A and B we do not need $\Lambda$ to be Lyapunov stable. If $\Lambda$ is Lyapunov stable, then we get positive topological entropy:

Theorem C. Let $\Lambda$ be a compact invariant set that is sectional hyperbolic for a $C^1$ flow $\phi_t$, with all the singularities in $\Lambda$ hyperbolic. Furthermore, assume that $\Lambda$ is Lyapunov stable. Then we have $h_{top}(\phi_t|\Lambda) \geq \lambda > 0$, where $\lambda$ is the sectional volume expanding rate on $F_{cu}$.

We apply the previous theorem to Lorenz-like classes and obtain the following corollary:

Corollary D. Let $\Lambda$ be a Lorenz-like class for a $C^1$ flow $\phi_t$ with all singularities hyperbolic. Then $\Lambda$ is robustly $\delta$-entropy expansive, has positive topological entropy, and contains a periodic orbit.

Recall that a property is said to be $C^1$ generic if it holds on a residual set under $C^1$ topology. With the help of the periodic orbit obtained in Corollary D, we show that Lorenz-like classes are indeed attractors.

Corollary E. $C^1$ generically, every Lorenz-like class is an attractor and contains a periodic orbit.

We finish this section with the following dichotomy on the chain recurrent classes for generic $C^1$ star flows. Recall that a $C^1$ flow is called a star flow, if all the critical elements (singularities and periodic orbits) are hyperbolic. Star flows without singularity was proven to be Axiom A in [10]. For star flows with singularities, the situation becomes quite difficult. It is shown in [11] that for $C^1$ generic star flows, every Lyapunov stable chain recurrent class is Lorenz-like.

Theorem F. There is a residual set $\mathcal{R}$ of $C^1$ star flows, such that for every $X \in \mathcal{R}$ and every non-trivial chain recurrent class $C$ of $X$, we have

1. if $h_{top}(\phi_t|C) > 0$, then $C$ contains some periodic point $p$ and is isolated;
2. if $h_{top}(\phi_t|C) = 0$, then $C$ is sectional hyperbolic for $X$ or $-X$, and has no periodic orbit.

Note that in the second case, $C$ cannot be Lyapunov stable due to Corollary D. Moreover, the beautiful example constructed by Bonatti and da Luz [5], which has two singularities with different indices robustly contained in the same chain recurrent class, belongs to the first case and is isolated.

Corollary G. $C^1$ generic star flows have only finitely many Lyapunov stable chain recurrent classes.
Let us quickly prove this corollary. By [20], $C^1$ generic star flows have only finitely many periodic sinks. As a result, if $X$ has infinitely many Lyapunov stable chain recurrent classes $\{C_n\}$, then we may assume that $C_n$ are non-trivial and approach a chain recurrent class $C$. Note that $C$ cannot be trivial since trivial chain recurrent classes, i.e., periodic orbits and singularities, are all hyperbolic and isolated. Therefore $C$ is non-trivial and sectional hyperbolic with zero topological entropy due to Theorem [F]. Denote by $\lambda_C > 0$ the volume expanding rate on $F_{cu}$, then by continuity, nearby classes $C_n$ have volume expanding rate $\lambda_{C_n} > \lambda_C/2$. Apply Theorem [C] we see that $h_{top}(\phi_t|U_n) > \lambda_C/2$ within some neighborhoods $U_n$ of $C_n$. Then Theorem [B] shows that the topological entropy on $C$ must be at least $\lambda_C/2 > 0$, which is a contradiction.

Now let us explain how the entropy theory is used in the proof. In Section 3.1, we study the time-one map of the flow $f = \phi_1$ in a neighborhood of the sectional hyperbolic set $\Lambda$, which has a dominated splitting $E^s \oplus F^c u$. This enables us to use ‘fake foliations’, which are invariant under the map $f$, but are generally not preserved by the flow. However, it has been established that the infinite Bowen-balls are contained in those fake-foliations (Lemma 3.3), and the flow saturates the fake foliation for points in the infinite Bowen balls (Corollary 3.4). Using the fake foliations, we establish a local product structure near singularities (Lemma 3.7), which allows us to use the center foliation near singularities and establish some expanding property near neighborhoods of singularities without using linearization.

Note that our proof for the entropy expansiveness relies heavily on the sectional hyperbolic splitting. On the other hand, the example of Bonatti and da Luz does not admit a sectional hyperbolic splitting. It will be a challenging problem to obtain the entropy expansiveness for such classes.

In Section 3.2, we prove Theorem [C] by showing that the time-one map $f$ on a neighborhood of $\Lambda$ have positive topological entropy, using the volume expansion rate on the $F_{cu}$ bundle as a lower bound. Then Theorem [C] will follow by taking a sequence of such neighborhoods shrinking to $\Lambda$ and using the upper semi-continuity of metric entropy.

In Section 4, we prove Theorem [B] using a similar argument as Katok [16] and a shadowing lemma of Liao [20], which allows the pseudo orbit to pass near singularities. The proof of Corollary [D] and [E] is at the end of Section 4. In the last section, we prove Theorem [F] using the extended linear Poincaré flow on the Grassmannian manifold developed by Liao [20, 21], see also [17].

Finally, in the appendix we will revisit the proof of Theorem [A] without assuming that singularities are hyperbolic.

2. Preliminaries

Throughout this paper, $X$ will be a vector field that is $C^1$ on a $d$-dimensional compact manifold $M$. Denote by Sing($X$) (sometimes we also write Sing($\phi_t$)) the set of singularities of $X$, $\phi_t$ the flow generated by $X$, and $f = \phi_1$ the time-one map of $\phi_t$. We will write $\Phi_t$ for the tangent flow, i.e., $\Phi_t = D\phi_t : TM \to TM$.

2.1. Dominated splitting. Let $g \in \text{Diff}^1(M)$ be a diffeomorphism on $M$. We say that $g$ has a dominated splitting $E \oplus F$, if $TM$ can be decomposed into continuous, $Dg$ invariant subbundles $E$ and $F$, such that for some $L > 0$, we have

$$\frac{||Dg^L_x(u)||}{||u||} \leq \frac{1}{2} \frac{||Dg^L_x(v)||}{||v||}$$

for every $x \in M$ and every non-zero vectors $u \in E(x), v \in F(x)$. The dominated splitting on an invariant set $\Lambda$ can be defined in a similar way, with $TM$ replaced by $T_\Lambda M$. 
For \( a > 0 \) and \( x \in M \), a \((a,F)\)-cone on the tangent space \( T_xM \) is defined as
\[
C_a(F_x) = \{v : v = v_E + v_F \text{ where } v_E \in E, v_F \in F \text{ and } \|v_E\| < a\|v_F\|\} \cup \{0\}.
\]

When \( a \) is sufficiently small, the cone field \( C_a(F_x), x \in M \), is forward invariant by \( Dg \), i.e., there is \( \lambda < 1 \) such that for any \( x \in M \), \( Dg_a(C_a(F_x)) \subset C_{\lambda a}(F_{g(x)}) \). Similarly, we can define the \((a,E)\)-cone \( C_a(E_x) \), which is backward invariant by \( Dg \). When no confusing is caused, we call the two families of cones by \( F \) cones and \( E \) cones.

The images of the cones under the exponential map are also forward or backward invariant. To be more precise, fix \( \varepsilon_0 > 0 \) small enough, such that the exponential map is well-defined on the \( \varepsilon_0 \) ball in the tangent space. We denote by \( C^M_a(F_x) \) the image of \( C_a(F_x) \) under the exponential map restricted to the set \( B_{\varepsilon_0}(0) \cap C_a(F_x) \subset T_xM \) and call \( C^M_a(F_x) \) a local \( F \) cone in \( B_{\varepsilon_0}(x) \). Then for any \( x \in M \), we have:
\[
g(C^M_a(F_x) \cap B_{\varepsilon_0/\|g\|_{C^1}}(x)) \subset C^M_{\lambda a}(F_{g(x)}).
\]

In the same way we can define \( C^M_a(E_x) \).

**Definition 5.** Let \( D \) be a \( C^1 \) disk with dimension \( \dim F \). We say \( D \) is:

- tangent to \( F \) cone if for any \( x \in D \), \( T_xD \subset C_a(F_x) \);
- tangent to local \( F \) cone at \( x \) if \( D \subset C^M_a(F_x) \);
- tangent to local \( F \) cone if for any \( y \in D \), we have \( D \subset C^M_a(F_y) \).

\( D \) is tangent to local \( F \) cone implies that it is tangent to \( F \) cone. Conversely, if \( D \) is tangent to \( F \) cone, then it can be divided into finitely many sub-disks, each of which is tangent to local \( F \) cone.

**Remark 2.1.** Topologically, for a small enough, the local cones \( C^M_a(E_x) \) and \( C^M_a(F_x) \) are transverse to each other, that is, \( C^M_a(E_x) \cap C^M_a(F_x) = \{x\} \).

**Remark 2.2.** Suppose \( D \) is a disk with dimension \( \dim F \) and transverse to \( E \) bundle, then there is \( n > 0 \) sufficiently large, such that \( g^n(D) \) is tangent to \( F \) cone. Hence, it can be divided into finitely many connected pieces: \( g^n(D) = \bigcup_{i=1}^k D_i \), such that each piece \( D_i \) is tangent to local \( F \) cone.

The proof of the next lemma is simple and thus omitted.

**Lemma 2.3.** There is a constant \( K > 0 \) such that for every \( x \in M \) and any disk \( D \subset B_{\varepsilon_0}(x) \) tangent to local \( F \) cone, we have \( \text{vol}(D) < K \).

By the forward invariance of \( F \) cone field, we have

**Lemma 2.4.** For every \( x \in M \), \( \varepsilon < \frac{\text{vol}\,B_{\varepsilon_0}}{\|g\|_{C^1}} \) and \( n > 0 \), if \( D \subset B_{\varepsilon}(x) \) is tangent to local \( F \) cone (at \( x \)), with \( g^i(D) \subset B_{\varepsilon}(g^i(x)) \) for every \( 0 \leq i \leq n \), then \( g^n(D) \) is tangent to local \( F \) cone at \( g^n(x) \).

One can easily check that if \( g \) has dominated splitting on an invariant set \( \Lambda \) instead of \( M \), then the invariant cone fields can be extended to an attracting neighborhood of \( \Lambda \). One can define the local cones \( \{C^M_a\} \) in the same way, and the above lemmas hold for points in the neighborhood of \( \Lambda \).

### 2.2. Entropy for continuous maps

In this subsection \( g : M \to M \) will be a continuous map and \( K \) a subset of \( M \) not necessarily invariant. For \( \varepsilon > 0 \) and \( n \geq 1 \), we consider the dynamical ball of radius \( \varepsilon > 0 \) and length \( n \) around \( x \in M \):
\[
B_n(x,\varepsilon) = y \in M : d(g^j(x), g^j(y)) < \varepsilon \text{ for every } 0 \leq j \leq n - 1.
\]

This is also called \((n, \varepsilon)\)-Bowen ball and plays an important role in the study of topological entropy.
Lemma 2.5. Suppose \( D \) is a disk with dimension \( \dim F \) and tangent to local \( F \) cone. Then for any \( x \in D \) and \( \varepsilon < \varepsilon_0/\|g\|_{C^1} \), one has
\[
\text{vol}(g^n(D \cap B_{\varepsilon_0}(x, \varepsilon))) \leq K.
\]

Proof. This Lemma follows easily from Lemma 2.4 and the observation that \( g^n(B_{\varepsilon_0}(x, \varepsilon)) \subset B_{\varepsilon_0}(g^n x) \). \( \square \)

A set \( E \subset M \) is \((n, \varepsilon)\)-spanning for \( K \) if for any \( x \in K \), there is \( y \in E \) such that \( d(g^i(x), g^i(y)) < \varepsilon \) for all \( 0 \leq i \leq n - 1 \). In other words, the dynamical balls \( B_n(g, \varepsilon), y \in E \) cover \( K \). Let \( r_n(K, \varepsilon) \) denote the smallest cardinality of any \((n, \varepsilon)\)-spanning set, and
\[
r(K, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log r_n(K, \varepsilon).
\]
The topological entropy of \( g \) on \( K \) is then defined as
\[
h_{\text{top}}(g, K) = \lim_{\varepsilon \to 0} r(K, \varepsilon),
\]
and the topological entropy of \( g \) is defined as \( h_{\text{top}}(g) = h_{\text{top}}(g, M) \).

For each \( x \in M \) and \( \varepsilon > 0 \), let \( B_\infty(x, \varepsilon) = \{ y : d(g^n(x), g^n(y)) < \varepsilon \text{ for } n \in \mathbb{Z} \} \) be the two-sided \((\infty, \varepsilon)\)-Bowen ball at \( x \). The map \( g \) is \( \varepsilon \)-entropy expansive if
\[
\sup_{x \in M} h_{\text{top}}(g, B_\infty(x, \varepsilon)) = 0.
\]
In other words, the \((\infty, \varepsilon)\)-Bowen ball has zero entropy for all \( x \).

It is well known (see for example [6]) that if \( g \) is \( \varepsilon \)-entropy expansive, then the topological entropy “stabilizes” at \( \varepsilon \), that is, \( h_{\text{top}}(g) = r(M, \varepsilon) \).

Next we consider the metric entropy of an invariant measure. Let \( \mu \) be an invariant measure and \( A \) a finite measurable partition. The metric entropy of \( \mu \) corresponding to the partition \( A \) is defined as
\[
h_{\mu}(A) = -\lim_{n \to \infty} \frac{1}{n} \sum_{B \in A_{n-1}^0} \mu(B) \log \mu(B),
\]
where \( A_{n-1}^0 \) is the \((n-1)\)th joint of \( A \):
\[
A_{n-1}^0 = A \vee g^{-1} A \vee \cdots \vee g^{-(n-1)} A.
\]
The metric entropy of an invariant measure \( \mu \) is defined as
\[
h_{\mu} = \sup_{A \text{ is a finite partition}} \{ h_{\mu}(A) \}.
\]
By the variational principle, \( h_{\text{top}}(g) = \sup_{\mu \in M_{inv}(g)} h_{\mu} \), where \( M_{inv}(g) \) denotes the space of invariant probabilities of \( g \). If \( g \) is \( \varepsilon \)-entropy expansive, then for every finite partition \( A \) with \( \text{diam} A < \varepsilon \), we have \( h_{\mu} = h_{\mu}(A) \).

In general, for maps with finite differentiability, metric entropy is not necessarily upper semi-continuous with respect to the invariant measures, and the topological entropy may not be achieved by any metric entropy, see for example [3, 27]. However, if \( g \) is \( \varepsilon \)-entropy expansive (or asymptotically \( h \)-expansive), then the metric entropy \( h_{\mu} \) is upper semi-continuous with respect to \( \mu \). As a result, \( g \) admits a measure of maximal entropy.

We finish this subsection by stating the following lemma. Recall that the \( \varepsilon_0 \) is taken such that the exponential map is well-defined within \( \varepsilon_0 \) balls, and \( K \) is the constant given by Lemma 2.3.

**Lemma 2.5.** Suppose \( D \) is a disk with dimension \( \dim F \) and tangent to local \( F \) cone. Then for any \( x \in D \) and \( \varepsilon < \varepsilon_0/\|g\|_{C^1} \), one has
\[
\text{vol}(g^n(D \cap B_{\varepsilon_0}(x, \varepsilon))) \leq K.
\]
2.3. Ergodic theory for flows. In this section we state some results on the ergodic theory for flows, which will be used later. Throughout this section, $\Lambda$ denotes a compact invariant set of the flow $\varphi_t$ with singularity, and $\mu$ is a non-trivial invariant measure of $\varphi_t$, i.e., $\mu(\text{Sing}(\varphi_t)) = 0$.

A dominated splitting for a flow $\varphi_t$ is defined similarly to the case of diffeomorphisms. The set $\Lambda$ admits a dominated splitting $E \oplus F$ if this splitting is invariant for $\Phi_t$, and there exist $C > 0$ and $\lambda < 1$ such that for every $x \in \Lambda$, and every pair of unit vectors $u \in E_x$ and $v \in F_x$, one has

$$\| (\Phi_t)_x (u) \| \leq C \lambda^t \| (\Phi_t)_x (v) \| \text{ for } t > 0.$$ 

Note that in the above definition, the assumption on the invariance of the splitting is not necessary. The next lemma states the relation between dominated splitting for the flow and its time-one map.

**Lemma 2.6.** $E \oplus F$ is a dominated splitting for the flow $\varphi_t|_\Lambda$ if and only if it is a dominated splitting for the time-one map $f|_\Lambda$. Moreover, if $\varphi_t|_\Lambda$ is transitive, then we have either $X|_{\Lambda \setminus \text{Sing}(X)} \subset E$ or $X|_{\Lambda \setminus \text{Sing}(X)} \subset F$.

**Proof.** The proof of the ‘only if’ part is trivial. Now suppose $E \oplus F$ is a dominated splitting for $f|_\Lambda$. In order to show that it is a dominated splitting for $\varphi_t$, we only need to prove that it is invariant under $\varphi_t$. By the commutative property between $f$ and $\varphi_t$, it is easy to see that for any $t$, $\varphi_t(E) \oplus \varphi_t(F)$ is also a dominated splitting for $f$. Because the dominated splitting is unique once the dimension is fixed, we conclude that the splitting $E \oplus F$ is invariant for $\varphi_t$. Therefore, $E \oplus F$ is also a dominated splitting for $\varphi_t|_\Lambda$.

Now suppose $\varphi_t|_\Lambda$ is transitive. Take $x \in \Lambda \setminus \text{Sing}(X)$ such that $\text{Orb}^+(x)$ is dense in $\Lambda$. If $X(x) \notin E_x \cup F_x$, then for $t$ sufficient large, $X(\varphi_t(x))$ is close to $F(\varphi_t(x))$, by the domination between $E$ and $F$. We take $t_0$ large such that $\varphi_{t_0}(x)$ is close to $x$, then $X(\varphi_{t_0}(x))$ is close to $X(x)$ and $F(\varphi_{t_0}(x))$ is close to $F_x$, which implies that $X(x)$ is arbitrarily close to $F_x$, a contradiction. This shows that $X(x) \in E_x \cup F_x$. Because $\text{Orb}^+(x)$ is dense, by the continuation of flow direction and the sub-bundles $E$ and $F$, if $X(x) \in E_x$, we must have $X|_{\Lambda \setminus \text{Sing}(X)} \subset E$. The same argument applies if $X(x) \in F_x$. The proof is complete.

**Remark 2.7.** If the dominated splitting $E \oplus F$ is sectional hyperbolic, then $E$ is uniformly contracting by definition. Since the flow speed $\| X(x) \|$ is bounded and thus cannot be backward exponentially expanding, we must have $X|_{\Lambda \setminus \text{Sing}(X)} \subset F$. For more detail, see Lemma 3.10.

**Definition 6.** The topological entropy (resp. metric entropy) of a continuous flow is the topological entropy (resp. metric entropy) of its time-one map. A flow is $\epsilon$-entropy expansive if its time-one map is $\epsilon$-entropy expansive.

**Lemma 2.8.** Let $\mu$ be an ergodic invariant measure of $\varphi_t$, and $\bar{\mu}$ be an ergodic component of $\mu$ for the time-one map $f$. Then $h_\mu(\varphi_t) = h_{\bar{\mu}}(f)$.

**Proof.** Observe that $\bar{\mu} = (\varphi_t)_* \bar{\mu}$ is also an $f$-invariant measure and

$$\mu = \int_{[0,1]} \bar{\mu}_t \, dt.$$ 

On the other hand, $h_{\bar{\mu}_t}(f) = h_{\bar{\mu}}(f)$ due to the following observation: for any partition $\mathcal{A} = \{A_1, \ldots, A_k\}$, write $\mathcal{A}_t = \{\varphi_t(A_1), \ldots, \varphi_t(A_k)\}$, then $\bar{\mu}(A_t) = \bar{\mu}_t(\varphi_t(A_1))$. Since the metric entropy is an affine function respect to the invariant measures, we get

$$h_{\bar{\mu}}(\varphi_t) = h_{\mu}(f) = \int_{[0,1]} h_{\bar{\mu}_t}(f) \, dt = \int_{[0,1]} h_{\bar{\mu}}(f) \, dt = h_{\bar{\mu}}(f).$$
As a corollary of the previous lemma, we state the following two results regarding entropy expansiveness in flow version:

**Lemma 2.9.** [6] If \( \phi_t \) is entropy expansive then the metric entropy function is upper semi-continuous. In particular, there exists a measure of maximal entropy.

**Lemma 2.10.** [13] (Lemma 2.3) Let \( U \) be a \( C^1 \) open set of flows which are \( \varepsilon \)-entropy expansive for some \( \varepsilon > 0 \). Then the topological entropy varies in an upper semi-continuous manner for flows in \( U \).

The linear Poincaré flow \( \psi_t \) is defined as following: denote the normal bundle of \( \phi_t \) over \( \Lambda \) by

\[
N_{\Lambda} = \bigcup_{x \in \Lambda \setminus \operatorname{Sing}(X)} N_x,
\]

where \( N_x \) is the orthogonal complement of the flow direction \( X(x) \), i.e.,

\[
N_x = \{ v \in T_{\phi_t(x)} M : v \perp X(x) \}.
\]

Denote the orthogonal projection of \( T_{\phi_t(x)} M \) to \( N_x \) by \( \pi_x \). Given \( v \in N_x \) for a regular point \( x \in M \setminus \operatorname{Sing}(X) \) and recall that \( \Phi_t \) is the tangent flow, we can define \( \psi_t(v) \) as the orthogonal projection of \( \Phi_t(v) \) onto \( N_{\phi_t(x)} \), i.e.,

\[
\Psi_t(v) = \pi_{\phi_t(x)}(\Phi_t(v)) = \Phi_t(v) - \frac{<\Phi_t(v), X(\phi_t(x))>}{\|X(\phi_t(x))\|^2} X(\phi_t(x)),
\]

where \(<.,.>\) is the inner product on \( T_{\phi_t(x)} M \) given by the Riemannian metric. The following is the flow version of Oseledets theorem:

**Proposition 2.11.** For \( \mu \) almost every \( x \), there exist \( k = k(x) \in \mathbb{N} \) and real numbers

\[
\hat{\lambda}_1(x) > \cdots > \hat{\lambda}_k(x)
\]

and a \( \Psi_t \) invariant measurable splitting on the normal bundle:

\[
N_x = \hat{E}_x^1 \oplus \cdots \oplus \hat{E}_x^k,
\]

such that

\[
\lim_{t \to \pm \infty} \frac{1}{t} \log \|\Psi_t(v_t)\| = \hat{\lambda}_i(x) \text{ for every non-zero } v_t \in \hat{E}_x^i.
\]

Now we state the relation between Lyapunov exponents and the Oseledets splitting for \( \phi_t \) and for \( f \):

**Theorem 2.12.** For \( \mu \) almost every \( x \), denote by \( \lambda_1(x) > \cdots > \lambda_k(x) \) the Lyapunov exponents and

\[
T_x M = E_x^1 \oplus \cdots \oplus E_x^k
\]

the Oseledets splitting of \( \mu \) for \( f \). Then

\[
N_x = \pi_x(E_x^1) \oplus \cdots \oplus \pi_x(E_x^k)
\]

is the Oseledets splitting of \( \mu \) for \( \phi_t \). And the Lyapunov exponents of \( \mu \) (counting multiplicity) for the flow \( \phi_t \) is the subset of the exponents of \( f \) obtained by removing one of the zero exponent which comes from the flow direction.

**Definition 7.** \( \mu \) is called a hyperbolic measure for the flow \( \phi_t \) if it is an ergodic measure of \( \phi_t \) and all the exponents are non-vanishing. In other words, if we view \( \mu \) as an invariant measure for the time-one map \( f \), then \( \mu \) has exactly one exponent which is zero, given by the flow direction. We call the number of the negative exponents of \( \mu \), counting multiplicity, its index.
3. Positive topological entropy and Entropy expansiveness

In this section we prove Theorem A and Theorem C.

3.1. Entropy expansiveness. We will prove Theorem A using a new criterion for entropy expansiveness, given in [19] as Proposition 2.4. For that purpose, we make the following definition:

Definition 8. For \( \varepsilon > 0 \), we say \( g \) is \( \varepsilon \)-almost entropy expansive if for every \( g \)-invariant, ergodic measure \( \mu \) and for \( \mu \) almost every point \( x \), we have

\[
h_{\text{top}}(g, B_\infty(x, \varepsilon)) = 0.
\]

Then [19][Proposition 2.4] states that:

Lemma 3.1. \( g \) is \( \varepsilon \)-almost entropy expansive if and only if it is \( \varepsilon \)-entropy expansive.

In this section we assume that \( \Lambda \) is sectional hyperbolic for \( C^1 \) flow \( \phi_t \) and \( f = \phi_1 \) is the time-one map. We also assume, as in Theorem A, that all the singularities in \( \Lambda \) are hyperbolic. We take \( U \) a small neighborhood of \( \Lambda \), such that the maximal invariant set \( \tilde{\Lambda} \) for \( \Phi_t|_U \) is also sectional hyperbolic: on \( T_1 M \) there is a dominated splitting \( E^s \oplus F^{cu} \) such that \( \Phi_t \) on \( E^s \) is uniformly contracting. Moreover, there is \( 0 < \lambda_0 < 1 \) such that for any subspace \( V_x \subset F^{cu}_x \) with dimension at least 2, we have

\[
det(Df|_{V_x}) > \frac{1}{\lambda_0}.
\]

Enlarging \( \lambda_0 \) if necessary, we can assume that the above inequality holds for any two-dimensional subspace \( V_x \) in the cone \( C_a F^{cu} \), for \( a \) small enough. Since all the singularities in \( \Lambda \) are hyperbolic and thus isolated, we can take \( U \) small enough so that

\[
(2) \quad \text{Sing}(\phi_t|_U) = \text{Sing}(\phi_t|_{\Lambda}).
\]

Note that the for a \( C^1 \) flow \( \phi'_t \) close to \( \phi_t \), the maximal invariant set of \( \phi'_t \) in \( U \) is still sectional hyperbolic, with all singularities hyperbolic. Below we will only show the \( \delta \)-entropy expansiveness for the flow \( \phi_t \). For the robustness, one can easily check that the choice of \( \delta \) depends only on the fake foliation which is continuous with respect to the system (see [19]), the flow speed, the hyperbolicity of the singularities and the volume expanding rate \( \lambda_0 \), thus can be made uniform for nearby flows.

3.1.1. Structure of the proof and choice of parameters. Before getting into details, we briefly explain the structure of our proof of Theorem A. First we introduce the fake foliations for maps with dominated splitting, and show that the infinite Bowen ball of every point \( x \in \tilde{\Lambda} \) is contained in the fake \( cu \) foliation (Lemma 3.3). As we will see later, these fake foliations are \( f \)-invariant but generally not \( \phi_t \) invariant for non-integers \( t \). In particular, the \( cu \) fake leaves are not saturated by the flow orbits. However, there is a weak form of saturation for points in the infinite Bowen ball, as observed in Corollary 5.4.

In view of Lemma 3.1 we only need to show that there is \( \delta > 0 \), such that for every invariant ergodic measure \( \mu \), the \( (\infty, \delta) \)-Bowen ball at \( \mu \)-typical points have zero entropy. We prove this by building up sufficient expanding property for points in the \( F^{cu} \) leaves.

For the convenience of our reader, we provide a list of parameters that will be used in this section:

1. \( \varepsilon_0 \): the scale within which the exponential map is well-defined.
2. \( \lambda_0 < 1 \): \( \frac{1}{\lambda_0} \) is the volume expanding rate on \( F^{cu} \).
Lemma 3.3. For every given as in Lemma 3.2. we consider the (direction, for orbits that pass through the ε. Roughly speaking, Lemma 3.7 build up the expanding property on the normal flow. This is because the fake foliations depend on the extension of the dynamics in the structure of the fake foliations, and the uniform contracting on the infinite Bowen ball. The proof easily follows from the local product structure, and this structure is preserved as long as they stay in a neighborhood of singularities infinitely often. We will construct a local product-like structure in small neighborhoods of singularities, and establish expanding behavior for each time a point gets close to a singularity (Lemma 3.7). This method allows us to bypass linearization.

From now on, to simplify notation, we will write $x_t = \phi_t(x)$. In particular,

$$x_n = f^n(x) = \phi_n(x).$$

3.1.2. Fake foliations and infinite Bowen ball. The following lemma is borrowed from [19]Lemma 3.3 (see also [7]Proposition 3.1), which shows that one can always construct local fake foliations. Moreover, these fake foliations have local product structure, and this structure is preserved as long as they stay in a neighborhood.

Lemma 3.2. Let $K$ be a compact invariant set of $f$. Suppose $K$ admits a dominated splitting $T_K M = E^1 \oplus E^2 \oplus E^3$. Then there are $\rho > r_0 > 0$, such that the neighborhood $B_\rho(x)$ of every $x \in K$ admits foliations $F^s_x$, $F^u_x$, $F^c_x$, $F^{12}_x$ and $F^{23}_x$, such that for every $y \in B_\rho(x)$ and $s \in \{1, 2, 3, 12, 23\}$:

(i) $F^s_y$ is $C^1$ and tangent to the respective cone.

(ii) Forward and backward invariance: $f(F^s_x(y, r_0)) \subset F^s_{f(\alpha)}(f(y))$, and $f^{-1}(F^s_x(y, r_0)) \subset F^s_{f^{-1}(\alpha)}(f^{-1}(y))$.

(iii) $F^s_x$ and $F^u_x$ sub-foliate $F^{12}_x$, $F^s_x$ and $F^u_x$ sub-foliate $F^{23}_x$.

Now we take $K = \tilde{A}$, and consider the fake foliations $F^s$ and $F^c$ given by the dominated splitting $E^s \oplus F^c$. Note that the forward and backward invariance above may not hold for the flow, i.e., the fake foliation may not be preserved by $\phi_t$ when $t \notin \mathbb{Z}$. Moreover, the flow orbits may not even locally saturate $F^c$ leaves. This is because the fake foliations depend on the extension of the dynamics in the tangent bundle (see [7]Proposition 3.1), which is in general not preserved by the flow.

The next lemma is taken from [19]Theorem 3.1 and gives an important observation on the infinite Bowen ball. The proof easily follows from the local product structure of the fake foliations, and the uniform contracting on $F^s$. Let $r_0 > 0$ be given as in Lemma 3.2.

Lemma 3.3. For every $x \in \tilde{A}$, $B_\infty(x, r_0) \subset F^c_x$. 

As an immediately corollary of Lemma 3.3, we obtain a type of weak saturated property for cu fake leaf $F^\infty_x(x)$:

**Corollary 3.4.** There is $r_1 > 0$ such that for any $y \in B_\infty(x, r_0/2)$, $y_t \in F^\infty_x(x)$ for $|t| \leq r_1$.

**Proof.** Denote $D_0 = \max_{x \in M} \|X(x)\|$ and $r_1 = \frac{r_0}{2D_0}$. Then for any $|t| \leq r_1$ and every $n \in \mathbb{Z}$, the segment of flow orbit between $f^n(y_t)$ and $f^n(y)$ has length bounded by $r_1D_0 \leq r_0/2$. Hence, for every $n \in \mathbb{Z}$,

$$d(f^n(x), f^n(y)) \leq d(f^n(x), f^n(y_t)) + d(f^n(y_t), f^n(y)) \leq r_0,$$

which shows that $y_t \in B_\infty(x, r_0) \subset F^\infty_x(x)$ for $|t| \leq r_1$.

\[\square\]

### 3.1.3. Expanding property on cu leaves

In this section we will present two lemmas (3.6 and 3.7) which establish the expanding property between different flows lines for points in $B_\infty(x, \delta)$, for some $\delta \ll r_0$. The proof of these lemmas will be postponed to the end of this section. First we introduce the following definition:

**Definition 9.** For any $x \in U \setminus \text{Sing}(X)$ and $0 < \delta < \epsilon_0$, write $B_\delta^+(x) = \exp_x(N_x \cap B_\delta(0))$ for the image of local $N_x$ under the exponential map. Denote by $d^*_x$ the distance between $x$ and $y$ in the submanifold $B_\delta^+(x) \cap F^\infty_x(x)$. Also write $P_x$ for the projection along the flow:

$$P_x : B_\delta(x) \to B_\delta^+(x),$$

which is well-defined in a small neighborhood of $x$, and send every point $y$ along the flow direction to the normal plane $B_\delta^+(x)$. For points $y$ in this neighborhood, write $t_x(y)$ the time for which

$$y_{t_x(y)} = P_x(y).$$

Note that $t_x$ becomes unbounded as $x$ gets closer to some singularity. On the other hand, for every fixed $\epsilon > 0$, we can take $\delta$ small enough (depending on $\epsilon$), such that for every $x \in B_\epsilon(\text{Sing}(X))$, $t_x$ is uniformly small inside $B_\delta(x)$. In view of Corollary 3.4, we take $\delta$ small such that $|t_x(y)| < r_1$ for $y \in B_\delta(x)$ and $x \notin B_\epsilon(\text{Sing}(X))$. As a result, for $y \in B_\infty(x, \delta)$, $P_x(y)$ is still contained in $F^\infty_x(x)$. The proof of the next lemma is straight forward and thus omitted.

**Lemma 3.5.** For any $\epsilon > 0$ and $L_0 > 1$, there is $\delta > 0$ such that for every $x \in \Lambda \setminus B_\epsilon(\text{Sing}(X))$ and $y \in B_\delta^+(x) \cap F^\infty_x(x)$, we have

$$d(x, y) \leq d^*_x(x, y) \leq L_0d(x, y).$$

The following lemma considers points $x$ whose orbit stays $\epsilon$-away from all singularities:

**Lemma 3.6.** For $\epsilon > 0$ and $1 < b_0 < \frac{1}{\lambda_{10}}$, there is $\delta' > 0$ such that for any $x$ satisfying $(x_t)_{t \in [0, 1]} \subset \Lambda \setminus B_\epsilon(\text{Sing}(X))$, and for any $y \in B_\infty(x, \delta')$:

$$d^*_x(x_1, P_{x_1}(y_1)) > b_0 \frac{\|X(x)\|}{\|X(x_1)\|} d^*_x(x, P_x(y)).$$

In other words, Lemma 3.6 states that for points $y \in B_\infty(x, \delta')$, one sees an expansion by a factor of $b_0\|X(x)\|$ along the normal direction under the iteration of $f$, as long as the orbit $\{\phi_t(x)\}_{t \in [0, 1]}$ stays $\epsilon$ away from all singularities.

To estimate the expanding property for points travelling near a singularity, we take $\delta_0 > 0$ small enough (the choice of $\delta_0$ will be made clear in Remark 3.11). For each $\sigma \in \text{Sing}(\phi_\lambda)$ and $\delta \ll \delta_0$, we consider the set:

$$J_\delta(\sigma) = \{ y : y \in B_{\delta_0}(\sigma) \text{ and } \|X(y)\| \leq \delta \}.$$
Next lemma establishes the expanding within $B_\infty(x, \delta'')$ for points travelling through the $\varepsilon$ neighborhood of a singularity.

**Lemma 3.7.** For every $L > 1$ and $\delta_1 > 0$ small enough, there are constants $\varepsilon$, $\delta'' > 0$ with $\varepsilon < \delta_1$, such that for each singularity $\sigma$, $x \in B_{\delta_\varepsilon}(\sigma) \cap \Lambda$ and $T > 0$ that satisfies

- $\{x\} \cup \{T\} = B_{\delta_\varepsilon}(\sigma)$ and $\{x\} \cap B_{\varepsilon}(\sigma) \neq \emptyset$;
- $x, xT \notin J_\delta(\sigma), \{x\} \cap J_\delta(\sigma) \neq \emptyset$ and $\{x\} \cap J_\delta(\sigma) \neq \emptyset$,

then for every $y \in B_{\infty}(x, \delta'')$, we have

$$d^*_x(xT, P_{xT}(yT)) > Ld^*_x(x, P_x(y)).$$

Roughly speaking, this lemma states the following: given $\delta_1 > 0$, one can always take $\varepsilon < \delta_1$ small enough, such that if the orbit of $x$ starts with $\|X(x)\| \approx \delta_1$, get $\varepsilon$-close to a singularity, then leave the $\varepsilon$-neighborhood of the singularity to a position $x_T$ where $\|X(x_T)\| \approx \delta_1$ (note that $\|X(x)\|$ and $\|X(x_T)\|$ are “comparable”, and the flow speed in $B_\varepsilon(\sigma)$ are much smaller than $\delta_1$), then on the orbit segment from $x$ to $x_T$, one picks up an expanding factor of $L > 1$ on the normal direction inside the $F^c$ leaf.

3.1.4. Measures $\varepsilon$–away from singularities. For now we will assume that Lemma 3.6 and 3.7 hold. Fix $L > 1$, $\delta_1 > 0$ and take $\varepsilon$ according to Lemma 3.7 and consider ergodic measures whose supports are $\varepsilon$ away from any singularity. We will prove that for $\delta' > 0$ given by Lemma 3.6 and for almost every points of such measures, the $(\infty, \delta')$-Bowen ball is degenerate. Recall that all the singularities of $\phi_t$ in $U$ are exactly those contained in $\Lambda$, all of which are hyperbolic, and thus isolated. Write $\operatorname{Sing}(\phi_t|\Lambda) = \{\sigma_i\}_{i=1}^k$.

**Proposition 3.8.** There is $\delta'$, $K_1 > 0$, such that if $\mu$ is a non-trivial invariant, ergodic measure which satisfies

$$\mu(B_{\varepsilon}(\sigma_i)) = 0 \text{ for all } i = 1, 2, \ldots, k,$$

then for every $x \in \operatorname{supp} \mu$, $B_{\infty}(x, \delta')$ is a flow segment with length bounded by $K_1$.

**Proof.** Since $\operatorname{supp} \mu$ is $\varepsilon$ away from all singularities, so are all points $x \in \operatorname{supp} \mu$. For every $x \in \operatorname{supp} \mu$, apply Lemma 3.6 on $\{x_i : i \in \mathbb{N}\}$ yields

$$d^*_x(x_n, P_x(y_n)) > b_0^\varepsilon \|X(x_n)\| \|X(x)\| d^*_x(x, P_x(y)),$$

for every $y \in B_{\infty}(x, \delta')$ and every $n > 0$.

The term $\|X(x_n)\| / \|X(x)\|$ is bounded from above and below since $\{x_i\}$ stays away from singularities. $d^*_x(x_n, P_x(y_n))$ is also bounded since $y \in B_{\infty}(x, \delta')$. Therefore $d^*_x(x, P_x(y))$ must be 0, so is $d(x, P_x(y))$, which shows that $y$ is indeed contained in the local orbit of $x$. One can take $K_1 > 0$ uniform in $x \in \operatorname{supp} \mu$, such that every connected component of $\operatorname{Orb}(x) \cap B_{\delta'}(x)$ has length bounded by $K_1$. This concludes the proof of Proposition 3.8.

3.1.5. Measures near singularities and proof of Theorem 4. We have shown that if $\mu$ is a measure whose support is $\varepsilon$ away from singularities, then every point $x$ in the support of $\mu$ has degenerate infinite Bowen ball. It remains to consider measures whose support intersects with the $\varepsilon$ neighborhood of some singularity.

Recall that $\delta'$ is given by Lemma 3.6 and $\delta''$ by Lemma 3.7. The main proposition in this subsection is the following:

**Proposition 3.9.** Let $\delta = \min\{\delta', \delta''\}$. For every invariant, ergodic measure $\mu$ on $\Lambda$ with $B_\varepsilon(\operatorname{Sing}(X)) \cap \operatorname{supp} \mu \neq \emptyset$ and for $\mu$ almost every point $x$, the infinite Bowen ball $B_{\infty}(x, \delta)$ is a segment of the orbit of $x$. 
Once we prove Proposition 3.9, Theorem A will follow from Lemma 3.1, Proposition 3.8 and 3.9 with \( \delta = \min\{\delta', \delta''\} \). Note that \( \delta' \) and \( \delta'' \) only depends on the hyperbolicity of the singularities and the sectional volume expanding rate, and thus can be made continuous with respect to the flow \( X \). This gives the robust \( \delta \)-entropy expansiveness.

**Proof of Proposition 3.9.** Recall that \( 1 < b_0 < \frac{1}{\alpha_p'} \). Let

\[
D' = \min\{\frac{\|X(x)\|}{\|X(y)\|} : \text{there are singularities } \sigma, \sigma' \text{ such that } x \in \{\phi_t(\partial(J_\delta(\sigma)))\}_{t \in [0,1]} \text{ and } y \in \{\phi_t(\partial(J_\delta(\sigma')))\}_{t \in [-1,0]}\).
\]

Note that \( D' < 1 \). Let \( L = \frac{b_0}{D'} > 1 \) and \( \varepsilon, \delta' \) be given by Lemma 3.7. Let \( \delta'' \) be given by Lemma 3.6 using the \( \varepsilon \) above, and take \( \delta = \min\{\delta', \delta''\} \). We verify that \( \delta \) satisfies Proposition 3.9.

Let \( \mu \) be a non-trivial ergodic measure on \( \hat{\Lambda} \) such that \( B_\varepsilon(Sing(X)) \cap \text{supp} \mu \neq \emptyset \). Let \( x \) be a typical point of \( \mu \). Then the orbit of \( x \) must visit the \( \varepsilon \) neighborhood of singularities infinitely many times. The idea of the proof is very simple: we use Lemma 3.7 to get expanding for each time the orbit travels through the \( \varepsilon \) neighborhood, and use Lemma 3.6 to control the expanding in-between.

To this end we define a sequence \( 0 \leq T_1 \leq T_2 < T_2 < T_2 < \ldots, \) such that for each \( n \), we have

1. The time interval \([T_n, T_n']\) contains the orbit segment travelling through the \( \varepsilon \)-neighborhood: \( \{x_t\}_{T_n,T_n'} \subset B_{b_0}(\sigma) \) and \( \{x_t\}_{T_n,T_n'} \cap B_\varepsilon(\sigma) \neq \emptyset \), for some singularity \( \sigma \).
2. The start and end point of \( \{x_t\}_{[T_n,T_n']n} \) have “comparable” flow speed: \( x_{T_n}, x_{T_n'} \notin J_\delta'(\sigma), \{x_t\}_{[T_n,T_n'+1]n} \cap J_\delta(\sigma) \neq \emptyset \) and \( \{x_t\}_{[T_n',T_n'+1]n} \cap J_\delta(\sigma) \neq \emptyset \).
3. The time interval \([T_n', T_{n+1}]\) are spent outside \( \varepsilon \)-neighborhood of singularities: \( \{x_t\}_{[T_n,T_{n+1}]} \cap B_\varepsilon(Sing(X)) = \emptyset \).

By Lemma 3.6, we obtain for \( y \in B_{\infty}(x, \delta) \),

\[
d^*_{x_{T_{n+1}'}}(x_{T_{n+1}'}, P_{x_{T_{n+1}'}}(y_{T_{n+1}})) > b_0^{T_{n+1}'-T_n'} \frac{\|X(x_{T_{n+1}'})\|}{\|X(x_{T_{n+1}})\|} d^*_{x_{T_n}}(x_{T_n}, P_{x_{T_n}}(y_{T_n})),
\]

Applying Lemma 3.9 on the time interval \([T_n, T_n']\) yields

\[
d^*_{x_{T_n}}(x_{T_n}, P_{x_{T_n}}(y_{T_n})) > b_0 D' \frac{\|X(x_{T_n})\|}{\|X(x_{T_n})\|} d^*_{x_{T_n}}(x_{T_n}, P_{x_{T_n}}(y_{T_n})).
\]

Inductively we get:

\[
d^*_{x_{T_n}}(x_{T_n}, P_{x_{T_n}}(y_{T_n})) > b_0 D' \frac{\|X(x_{T_n})\|}{\|X(x_{T_n})\|} d^*_{x_{T_n}}(x_{T_n}, P_{x_{T_n}}(y_{T_n})),
\]

but the left hand side is bounded since \( y \in B_{\infty}(x, \delta) \), thus \( d^*_{x_{T_n}}(x_{T_n}, P_{x_{T_n}}(y_{T_n})) = 0 \), that is \( y \) belongs to the local orbit of \( x \).

This proves Proposition 3.9 and Theorem A now follows from Lemma 3.1, Proposition 3.8 and 3.9. \( \square \)

**3.1.6. Proof of Lemma 3.6 and 3.7**

**Proof of Lemma 3.6** The idea of the proof is very simple: we consider the ‘parallelogram’ generated by \( X(x) \) and the vector joining \( x \) and \( P_x(y) \), whose area is approximately \( \|X(x)\| \cdot d_x^*(x, P_x(y)) \). Then we compare it with the ‘parallelogram’
generated by $X(x_1)$ and the vector joining $x_1$ and $P_{x_1}(y_1)$, which has area approximately $\|X(x_1)\| \cdot d^*_{x_1}(x_1, P_{x_1}(y_1))$. The expanding factor $b_0$ is then given by the sectional hyperbolicity on $F^{cu}$.

To this end write $C = \left(\frac{1}{\lambda_0 b_0}\right)^{1/6}$ and take $0 < t_0 < \min\{r_0/2, r_1/2\}$ small enough, such that for any 2-dimensional subspace $\Sigma$ in the tangent space, we have $1/C < \text{Jac}(\Phi_t|_\Sigma) < C$ for any $|t| < t_0$.

For two vectors $u, v \in T_x M$, denote by $P[u, v]$ the parallelogram defined by these two vectors and $A(u, v)$ its area. For $\delta > 0$ sufficiently small, for $y \in B_\infty(x, \delta)$, we have $|t_x(y)|, |t_{x_1}(y_1)| < t_0 < r_1$. Then by Corollary 3.4, $P_x(y) \in F^{cu}_x(x) \cap B_{\infty}^{++}(x)$ and similar relation holds for $P_{x_1}(y_1)$.

There is $0 < \varepsilon_1 \ll r_0/2$ depending on $\varepsilon$ and $b_0$, such that for any $z \in B_{\infty}^+(x_1)$, the following conditions are satisfied:

1. $\frac{1}{C} < \frac{\|X(z)\|}{\|X(x_1)\|} < C$.
2. For $v \in T_z B_{\infty}^+(x_1)$, we have $\frac{1}{C}\|v\|\|X(z)\| \leq A(v, X(z)) \leq C\|v\|\|X(z)\|$.

We may further suppose that $\delta$ is sufficiently small, such that $d^*_x(x, P_x(y)) < \varepsilon_1, d^*_x(x_1, P_{x_1}(y_1)) < \varepsilon_1$, and by Lemma 3.5

\[ d(x, P_x(y)) < d^*_x(x, P_x(y)) < C d(x, P_x(y)), \]

and the same holds for $x_1$ and $y_1$.

![Diagram](image)

Take a curve $l_1 \subseteq B_{\infty}^+(x_1) \cap F_{x_1}^{cu}(x_1)$ which links $x_1$ and $P_{x_1}(y_1)$ with length$(l_1) = d^*_x(x_1, P_{x_1}(y_1))$, and write $l_0 = f^{-1}(l_1)$ and $l = P_x(l_0)$, we may suppose $|t_x(l)| \leq t_0$ and $l \subseteq B_{\infty}^+(x)$. Then $l$ is a curve contained in $B_{\infty}^+(x)$ which connects $x$ and $P_x(y)$.

We note that although $l_0 \subseteq F_x^{cu}(x)$ by the local invariance of fake leaves, $l$ is not necessarily contained in $F_x^{cu}(x)$ (recall that the saturation property only applies to points in the Bowen ball). Denote $H$ the smooth holonomy map between $l_1$ and $l$ which is induced by flow, then $H(P_{x_1}(y_1)) = P_x(y)$. We claim that

\[ \|DH\| \leq \lambda_0 C^5 \frac{\|X(x_1)\|}{\|X(x)\|}, \]

which implies that

\[ d(x, P_x(y)) \leq \text{length}(l) \leq \lambda_0 C^5 \frac{\|X(x_1)\|}{\|X(x)\|} d^*_x(x_1, P_{x_1}(y_1)). \]

Then the lemma follows by changing $d(x, P_x(y))$ to $d^*_x(x, P_x(y))$, resulting in an extra power of $C$, and our choice of $C = \left(\frac{1}{\lambda_0 b_0}\right)^{1/6}$.

It remains to prove this claim.

For any $z_1 \in l_1$, denote by $v_1$ a tangent vector of $l_1$, write $z_0 = f^{-1}(z_1)$ and $z = \phi_{t_{x_1}}(z_0) \in l$. Also write $v_0 = DH^{-1}(v_1)$. 

LORENZ-LIKE FLOWS 15
Then by sectional-hyperbolic,
\[ A(\Phi_{t_0}(v_1), X(z_0)) \leq \lambda_0 A(v_1, X(z_1)). \]
Because \(|t_0|_0 < t_0\), by the assumption on \(t_0\),
\[
A(\Phi_{t_0}(z_0) (P[\Phi_{-1}(v_1), X(z_0)])) \leq C\lambda_0 A(v_1, X(z_1)).
\]
Since \(\Phi_{t_0}(z_0) \Phi_{-1}(v_1) = \Phi_{-1+t_0}(v_1)\) and \(\Phi_{t_0}(z_0)(X(z_0)) = X(z)\), we have
\[
A(\Phi_{t_0}(z_0) (P[\Phi_{-1}(v_1), X(z_0)])) = A(\Phi_{-1+t_0}(v_1), X(z)).
\]
Note that \(v_0 = DH^{-1}(v_1)\) is the projection of \(\Phi_{-1+t_0}(v_1)\) along \(X(z)\) on \(T_xB^+(x)\), combine equations (4), (5), we obtain
\[
A(v_0, X(z)) = A(\Phi_{-1+t_0}(v_1), X(z)) \leq C\lambda_0 A(v_1, X(z_1)).
\]
By the assumptions (1) and (2) above on \(\varepsilon, 1\), we get
\[
C^{-2}||v_0||X(x)|| \leq C^3\lambda_0 ||v_1||||X(x_1)||,
\]
which implies the desired claim:
\[
\|DH\| \leq \frac{v_0}{v_1} \leq C^3\lambda_0 \frac{\|X(x_1)\|}{\|X(x)\|}.
\]

To prove Lemma 3.10 we start by showing that there is a finer dominated splitting on \(\text{Sing}(\phi_t|U)\). Recall that we take \(U\) small enough, such that \(\text{Sing}(\phi_t|U) = \text{Sing}(\phi_t|\Lambda) = \{\sigma_1, \ldots, \sigma_k\}\).

**Lemma 3.10.** For every \(j = 1, \ldots, k\), \(Df|_{F^c\nu}\) has exactly one eigenvalue with norm less than one. As a result, there is a hyperbolic splitting \(E^s \oplus E^c \oplus E^u\) on \(\text{Sing}(\phi_t|U)\), with \(E^c \oplus E^u = F^c\nu\).

**Proof.** The proof is quite standard. Suppose that for some \(\sigma \in \text{Sing}(\phi_t|U)\), all the eigenvalues of \(Df|_{F^c\nu}\) are positive (recall that we assume all singularities to be hyperbolic), we claim that \(\Lambda \cap W^s(\sigma) \setminus \{\sigma\} \neq \emptyset\).

To prove this claim, we take \(x_n \in \Lambda\) and \(t_n \to \infty\), such that \(\phi_{t_n}(x_n) \to \sigma\) as \(n \to \infty\). Fix some \(\varepsilon > 0\) small enough so that there is no singularity in \(B_\varepsilon(\sigma)\) other than \(\sigma\), and let \(y^n = \phi_{t_n}(x_n)\) where \(t_n < t_n\) is the last time the orbit of \(x_n\) enters \(B_\varepsilon(\sigma)\). Taking subsequence if necessary, we may assume that \(y^n \to y^0 \in \Lambda \setminus \text{Sing}(X)\). Since \(\phi_t(y^0) \in B_\varepsilon(\sigma)\) for all \(t > 0\), this shows that \(y^0 \in W^s(\sigma) \setminus \{\sigma\}\).

By the invariance of \(W^s(\sigma)\), \(\phi_t(y^0) \in W^s(\sigma)\) for all \(t \in \mathbb{R}\); in particular, \(X(y^0)\) is contained in the \(E^s\) cone. Fix some \(\delta > 0\) small enough and consider the orbit segment \(l = \{\phi_t(y^0)\}_{t \in [-\delta, \delta]}\), it follows that \(l\) is tangent to \(E^c\) cone. Since \(f^{-1}\) expands vectors in \(E^s\) cone, we have \(\text{length}(f^{-n}(l))/\text{length}(l) \to \infty\), which contradicts the a priori estimate:
\[
\text{length}(f^{-n}(l)) \leq \frac{\max\{\|X(x)\| : x \in \Lambda\}}{\min\{\|X(y)\| : y \in l\}} \cdot \text{length}(l),
\]
which is bounded.

**Remark 3.11.** In the previous section we defined the fake foliations \(\mathcal{F}^s, \mathcal{F}^c\nu\) using the dominated splitting \(E^s \oplus E^c\nu\) on \(\Lambda\). These two foliations are defined around the \(r_0\) ball at every \(x \in \Lambda\), in particular, around singularities. On the other hand, the hyperbolic splitting \(E^c \oplus E^s \oplus E^u\) (extended to a small neighborhood of singularities) gives fake foliations \(\mathcal{F}^i\), \(i = s, c, cs, cu\), which only exists near a neighborhood of singularities. We take \(\delta_0 > 0\) small enough such that for every \(\sigma \in \text{Sing}(\phi_t|\Lambda)\),

- \(\sigma\) is the only singularity within \(B_{\delta_0}(\sigma)\).
• $\delta_0 \ll r_0/2$, such that both the fake foliations $F^s, F^{cu}$ and the (also fake) foliations $F_S^i$, $i = s, c, u, cs, cu$ are well-defined within $B_{\delta_0}(\sigma)$.

The reason that we still need fake foliations $F^{cu}$ for points in $B_{\delta_0}(\sigma)$ is due to Lemma 3.3.

Proof of Lemma 3.7 First let us quickly explain the structure of the proof. The fake foliations $F^s$ and $F^u_S$ gives a local product structure within $B_{\delta_0}(\sigma)$, which allows us to consider the $u$ and $c$ distance between two points. If we take such points close to $W^u(\sigma)$ (that is, their orbits are about to leave $J^{s, u}\delta_1(\sigma)$), then the $u$-distance will be contracted by $\phi_{-t}$ exponentially fast, as the $E^u(\sigma)$ bundle is uniformly expanding. On the other hand, the $c$-distance seems to be expanding under backward iteration, due to the $E^c(\sigma)$ bundle having negative exponent. However, when projected to the normal plane $N_x$, the $c$-segment will be contracting under $\phi_t$ because of the sectional hyperbolicity. This shows that the distance on the normal plane is contracted under backward iteration.

From now on, without loss of generality we will assume that $T$ is an integer.

To prove expanding for $y \in B_\infty(x, \delta') \subset F^u_x(x)$, one of the main difficulties is that, although we have the foliation $F_S^u$ within $B_{\delta_0}(\sigma)$, $F^u_S$ may not sub-foliate $F^{cu}$ since $F^u_S$ and $F^{cu}$ are given by different dominated splittings (and extended to neighborhoods in different ways). To solve this issue, we blend these two families of fake foliations and define

$$F^c_x(z) = F^u_S(z) \cap F^{cu}_x(x), \text{ for every } z \in F^{cu}_x(x).$$

$F^c$ is a new 1-dimensional center fake foliation, which is locally invariant for $f$ since both $F^u_S(z)$ and $F^{cu}_x(x)$ are invariant, and sub-foliates $F^{cu}$.

Next we construct a local product structure around $x_n = f^n(x)$ in the following way. We take $D^u$ with $x \in D^u \subset F^{cu}$ a disk with dimension $\dim E^u_x$, and tangent to the $E^u$ cone. For positive integers $n \leq T$, Denote by

$$D^u_0 = D^u, \text{ and } D^u_n \text{ the component of } f^n(D^u) \cap F^{cu}_x(x_n) \text{ containing } x.$$
Then each $D_n^u$ is still tangent to the $E^u$ cone. For each $z \in F_{x_n}^{cu}(x_n)$, there is a unique transverse intersection between $D_n^u$ and $F^c(z)$, which we write

$$[z, x_n] = F^c(z) \pitchfork D_n^u.$$  

Note that this local product structure is preserved by $f$, due to the invariance of $F^c$ and the definition of $D_n^u$. To be more precise, for each $y \in B_{\infty}(x, r_0)$ we have

$$f^n([y, x_n]) = [y_n, x_n], \quad \text{for } 0 \leq n \leq T.$$  

Recall that $P_z$ is the projection along to flow to $B_{\perp}(\delta z)$, which is well-defined for $\delta$ small enough. Furthermore, we can make $\delta$ small such that for some $r_\delta > 0$, the time for the projection, which we denoted by $t_z$, satisfies $|t_z|_{B_{\perp}\delta} \ll r_\delta \ll r_1$ for every $z \in B_{\delta}(\sigma) \setminus J_\delta(\sigma)$.

To simply notation, for $y \in B_{\infty}(x, \delta)$ we take $\tilde{y} = y_{t_0}$ for some $|t_0| < r_\delta$, such that $f^T(\tilde{y}) = \tilde{y}_T = P_{x_T}(y_T)$. As a result of Corollary 3.4, we have

$$\tilde{y} \in F_{x_T}^{cu}(x) \cap B_{\infty}(x, \delta + r_\delta D_0),$$

where $D_0 = \max \|X\|$ as before.

Let $\tilde{z} = [\tilde{y}, x]$, then the invariance of $[\cdot, \cdot]$ gives $\tilde{z}_n = f^n(\tilde{z}) = [\tilde{y}_n, x_n]$ for all $n \leq T$. By the local product structure, we take:

- $l_T^c \subset F^c(\tilde{y}_T)$ joining $\tilde{y}_T$ and $\tilde{z}_T$;
- $l_T^u \subset D_T^u$ the shortest curve (in submanifold metric) connecting $\tilde{z}_T$ and $x_T$.

Then $l_T = l_T^c \cup l_T^u$ is a piecewise smooth curve connecting $x_T$ and $\tilde{y}_T$. Now using the invariance of $F^c$ and the definition of $D_n^u$, we can write

- $l_0^c = f^{-T}(l_T^c) \subset F^c(\tilde{y})$ a curve joining $\tilde{y}$ and $\tilde{z}$;
- $l_0^u = f^{-T}(l_T^u) \subset D_0^u$ a curve connecting $\tilde{z}$ and $x$.

and $l_0 = l_0^c \cup l_0^u$ is a piecewise smooth curve connecting $x$ and $\tilde{y}$. By the (uniform) transversality between $F^c(\tilde{y}_n)$ and $D_n^u$ and the uniform contracting of $f^{-1}$ within $E^u$ cone, there exists $C > 1$ and $\lambda_1 > 1$ such that

$$\max\{\text{length}(l_0^u), \text{length}(l_0^c)\} \leq Cd(x_T, \tilde{y}_T),$$

and

$$\text{length}(l_0^u) \leq C(\delta + r_\delta D_0), \quad \text{and } \text{length}(l_0^c) \leq \lambda_1^{-T}\text{length}(l_T^c).$$

Now we shrink $\delta$ one last time, denote by $\delta''$, such that:

(a) near $x$  

(b) near $x_T$
(1) \( P_x(l_0) = \tilde{l}_0 \) is well-defined, with \(|t_2| |t_0| \leq t_0 \ll t_1 \), where \( t_0 \) is small enough such that \( \frac{1}{2} |\text{Jac}(\Phi_t|_x)| \leq C < 0 \leq t < t_0 \) and any two-dimensional subspace \( \Sigma \subset T_x M \).

(2) \( \delta \) satisfies Lemma 3.5

\[
d(x, y) \leq d^*_{x_2}(x, y) \leq C d(x, y),
\]
and the same holds for \( x_T \) and \( y_T \).

Let \( \tilde{v}_0 = P_x(l_0^u) \) and \( \tilde{y}_0 = P_x(l_0^s) \).

Then we have

\[
dx^*(x, P_x(y)) \leq C d(x, P_x(y)) \leq C (\text{length}(\tilde{v}_0^u) + \text{length}(\tilde{y}_0^s)).
\]

It remains to estimate the length of \( \tilde{v}_0^u \) and \( \tilde{y}_0^s \).

Note that \( l_0^u \) is tangent to the \( E^u \) cone, and thus transverse to the flow direction (which is tangent to \( E^s \oplus E^c \) cone if \( \varepsilon \) is small). On the other hand, \( l_0^s \) is perpendicular to the flow directions. Since both \( l_0^u \) and \( l_0^s \) are transverse to the flow directions, there is \( C_1 > 0 \) such that

\[
\frac{1}{C_1} \leq \frac{\text{length}(\tilde{v}_0^u)}{\text{length}(\tilde{y}_0^s)} \leq C_1.
\]

This together with (8) shows that

\[
\text{length}(\tilde{v}_0^u) \leq C_2 \lambda_1^{-T} \text{length}(\tilde{y}_0^s).
\]

Next we estimate length(\( \tilde{y}_0^s \)) using the sectional hyperbolicity.

Let \( H \) be the holonomy map from \( \tilde{y}_0^s \) to \( \tilde{v}_0^u \) induced by the flow.

Similar to the \( l_0^u \) case, \( \tilde{y}_0^s \) is perpendicular to the flow direction, while \( l_0^s \) is tangent to the \( E^c \) cone (well-defined in a neighborhood of \( \sigma \)) but the flow direction is tangent to \( E^u \) cone for \( \varepsilon \) small. This shows that

\[
\frac{1}{C_1} \leq \frac{\text{length}(\tilde{y}_0^s)}{\text{length}(\tilde{y}_0^s)} \leq C_1.
\]

Recall that \( \mathcal{P}[u, v] \) is the parallelogram generated by vectors \( u \) and \( v \). If we take \( v \) a tangent vector of \( \tilde{y}_0^s \) at \( z \in \tilde{y}_0^s \) and consider \( H \mathcal{P}[v, X(z)] = \mathcal{P}[DH^{-1}v, X(H^{-1}(z))] \), sectional hyperbolicity of \( f \) implies that

\[
\|DH^{-1}v\|\|X(H^{-1}(z))\| \leq C_3 \lambda_1^{-T} \|v\|\|X(z)\|,
\]

which shows that

\[
\text{length}(\tilde{y}_0^s) \leq C_3 \lambda_1^{-T} \frac{\|X(x_T)\|}{\|X(x)\|} \text{length}(\tilde{y}_0^s).
\]

Combine this with (7), (9) and (10), we have

\[
dx^*(P_x(y)) \leq C (\text{length}(\tilde{v}_0^u) + \text{length}(\tilde{y}_0^s))
\]
\[
\leq C(C_2 \lambda_1^{-T} \text{length}(\tilde{v}_0^u) + C_3 \lambda_1^{-T} \|X(x_T)\| \text{length}(\tilde{y}_0^s))
\]
\[
\leq C_4 \lambda_1^{-T} (\frac{D_0}{\delta_1} + 1) d(x_T, y_T)
\]
\[
\leq C_4 \lambda_1^{-T} (\frac{D_0}{\delta_1} + 1) d^*_x(x_T, P_x(y_T)).
\]

Since \( T \to \infty \) as \( \varepsilon \to 0 \), for any given \( L > 1 \), we can take \( \varepsilon \) small enough such that \( C_4 \lambda_1^{-T} (\frac{D_0}{\delta_1} + 1) < \frac{1}{L} \). This finishes the proof of Lemma 3.7. \( \square \)
3.2. Positive entropy: proof of Theorem C. In this subsection we will prove Theorem C. The proof consists of two steps. First we prove that in every small neighborhood $U$ of $Λ$, the topological entropy of $f|U$ is bounded from below by the volume expanding rate along $F_{cu}$ bundle. Then we take a sequence of such neighborhoods shrinking to $Λ$, and use the upper semi-continuity of metric entropy to obtain an lower bound for $h_{top}(f|_{\Lambda})$.

First we introduce the volume expansion rate on a bundle.

Definition 10. Let $D$ be a disk tangent to the $F$ cone, then the volume expansion of $D$, which we denote by $v_F(D)$, is defined by:

$$\limsup_{n} \frac{1}{n} \log(\text{vol}(g^n(D))).$$

The volume expansion $v_F$ of bundle $F$ is defined by:

$$v_F = \sup\{v_F(D) : D \text{ is tangent to the } F \text{ cone}\}.$$ 

The positivity for the topological entropy of $f|U$ relies on the following theorem:

Theorem 3.12. Suppose $g$ is a diffeomorphism which admits a dominated splitting $E \oplus F$. Then $h_{top}(g) \geq v_F$.

Theorem 3.12 was first stated in [19], where the bundle $F$ is required to be uniformly expanding. The main reason is that, in the proof of [19][Proposition 2], they need $f^n(D \cap B_n(x, \varepsilon))$ to be ‘almost’ a ball, which occurs only when the bundle $F$ is uniformly expanding. In general, this set is not even connected.

Proof of Theorem 3.12. For any $\delta > 0$, let $D$ be a disk which is tangent to the $F$ cone and satisfies $v_F(D) \geq v_F - \delta$, that is:

$$\limsup_{n} \frac{1}{n} \log(\text{vol}(g^n(D))) \geq v_F - \delta.$$ 

Take $\varepsilon > 0$ and $\Gamma_n = \{x_1, \ldots, x_{r_n(D, \varepsilon)}\}$ an $(n, \varepsilon)$-spanning set of $D$. By Lemma 2.5,

$$\text{vol}(g^n(D \cap B_n(x, \varepsilon))) \leq K.$$ 

Since that $\{B_n(x, \varepsilon)\}_{x \in \Gamma_n}$ is a cover of $D$, we get

$$\text{vol}(f^n(D)) \leq \sum_{i=1}^{r_n(D, \varepsilon)} \text{vol}(f^n(D \cap B_n(x_i, \varepsilon))) \leq Kr_n(D, \varepsilon).$$

Taking logarithm and divide by $n$, we obtain

$$\limsup_{n} \frac{1}{n} \log(r_n(D, \varepsilon)) \geq \limsup_{n} \frac{1}{n} \log(\text{vol}(f^n(D))/K)$$

$$= \limsup_{n} \frac{1}{n} \log(\text{vol}(f^n(D)))$$

$$\geq v_F - \delta,$$

which implies that $r(D, \varepsilon) \geq v_F - \delta$. Since $\delta > 0$ is arbitrary, we conclude the proof of Theorem 3.12. □

Remark 3.13. Similar to Lemma 2.3, 2.4 and 2.5, if $g$ has dominated splitting on an invariant set $A$, one can extend the invariant cone field $C_\delta(F)$ to an attracting neighborhood of $A$, such that the extension is still forward invariant. It is easy to check that the proof of Theorem 3.12 applies to points in the neighborhood.

Now we are ready to show that every Lorenz-like class has positive topological entropy.
Proof of Theorem 3. Recall that $f$ is the time-one map of a $C^1$ flow $\phi_t$, and $\Lambda$ is a compact invariant set of $\phi_t$ that is sectional hyperbolic. We take a forward invariant $F^{cu}$ cone on $\Lambda$ along the bundle $F^{cu}$ which is assumed to be sectional-expanding.

Let $\{U_i\}_{i=1}^\infty$ be a sequence of neighborhoods of $\Lambda$ as in Definition 3, i.e., they satisfy:

- $U_1 \supset U_2 \supset \cdots$ and $\bigcap_i U_i = \Lambda$;
- for each $i \geq 1$, $\phi_t(U_{i+1}) \subset U_i$ for any $t \geq 0$.

Assume $U_1$ is taken small enough, such that the $F^{cu}$ cone can be extended to $U_1$ (thus to $U_i$ for every $i \geq 1$) and is still forward invariant. Then for any disk $D \subset U_1$ which is tangent to $F$ cone, $f^n(D)$ is tangent to $F$ cone.

Since $F^{cu}|_\Lambda$ is volume expanding (which also holds in $U_i$), there exist $C > 0$ and $\lambda > 0$ such that

$$\text{vol}(f^n(D)) \geq Ce^{\lambda n}.$$ 

This implies $v_F(D) \geq \lambda$. Theorem 3.12 applied to $f|_{U_i}$ yields

$$h_{top}(f|_{U_i}) \geq \lambda.$$

By the variation principle, there are $f$ ergodic invariant measures $\mu_i$ supported within $U_i$, with

$$h_{\mu_i} \geq \lambda - \frac{1}{i}.$$ 

Passing to a subsequence, we get $\mu_i \to \mu_0$ in the weak-* topology, where $\mu_0$ must be supported on $\Lambda$. By Theorem 3.10, $f|_{U_i}$ is entropy expansive, thus the metric entropy is upper semi-continuous. This shows that

$$h_{\mu_0} \geq \lambda > 0.$$ 

We conclude that

$$h_{top}(\phi_t|_{\Lambda}) = h_{top}(f|_{\Lambda}) \geq h_{\mu_0} \geq \lambda > 0,$$

the proof is complete. \qed

4. Continuity of topological entropy

In this section we will prove Theorem 3 by showing that the support of every hyperbolic measure can be approximated by horseshoes with large entropy.

Let $\Lambda$ be a sectional hyperbolic compact invariant set for a $C^1$ flow $\phi_t$, and $U$ be a neighborhood of $\Lambda$. Denote by $\Lambda_n$ and $\tilde{\Lambda}$ the maximal invariant set of $\phi^n_t$ and $\phi_t$ in $U$, respectively. If $h_{top}(\phi_t|_{\Lambda_n}) = 0$ (recall that in order to get positive entropy, we need $\Lambda$ to be Lyapunov stable), then $\lim_{n \to \infty} h_{top}(\phi^n_t|_{\tilde{\Lambda}_n}) \geq 0 = h_{top}(\phi_t|_{\tilde{\Lambda}})$ holds trivially. Therefore, we can assume that

$$h_{top}(\phi_t|_{\tilde{\Lambda}}) > 0.$$ 

Note that by Theorem 3, there is a neighborhood $U$ of $\Lambda$, such that $\phi_t$ is entropy expansive in $U$. If we denote by $f = \phi_1$ the time-one map of $\phi_t$ as before, then $f$ is entropy expansive and satisfies $h_{top}(f|_{\tilde{\Lambda}}) > 0$. We can apply Lemma 2.9 to $\phi_t$ and obtain a measure of maximal entropy, which we denote by $\mu_{\phi_t}$. Since supp $\mu_{\phi_t} \subset U$, it follows that $\phi_t$ on supp $\mu_{\phi_t}$ is sectional hyperbolic. In particular, $\mu_{\phi_t}$ is a hyperbolic measure.

The next theorem shows that every hyperbolic measure $\mu$ of $\phi_t$ with positive entropy can be approximated by horse-shoes with entropy close to $h_{top}(\phi_t|\text{supp}\mu)$.

**Theorem 4.1.** Let $\mu$ be a hyperbolic, ergodic measure of $C^1$ flow $\phi_t$ with positive entropy. Assume that there is a dominated splitting $E \oplus F$ on supp $\mu$, such that $\dim E$ is the (stable) index of $\mu$. Then for every $\varepsilon > 0$, there is a hyperbolic set $\Lambda_\varepsilon$ in a small neighborhood of supp $\mu$, uniformly away from singularities and contains some periodic orbit, with $h_{top}(\phi_t|_{\Lambda_\varepsilon}) > h_{top}(\phi_t|\text{supp}\mu) - \varepsilon$. 

For $\varepsilon$ small enough, the hyperbolic set given by the above theorem must be contained in $U$ and outside a neighborhood (with uniform size) of $\text{Sing}(X)$. Since the topological entropy of a hyperbolic set varies continuously, it follows that $h_{top}(\psi_{t})$ is lower semi-continuous at $\phi_{t}$. The upper semi-continuity follows from Theorem A and Lemma 2.10. When $\Lambda$ is a chain recurrent class, Lemma 4.5(f) below shows that $\Lambda$ contains a periodic orbit. This concludes the proof of Theorem B, leaving only the proof of Theorem 4.1.

The proof of Theorem 4.1 uses a similar argument of Katok in [16] for diffeomorphisms. Note however, that the original argument of [16] cannot be directly applied to flows even if the flow is uniformly hyperbolic without singularity. The main obstruction is due to the shadowing lemma for flow only allows one to compare the pseudo-orbit and the shadowing orbit up to a change of time. We overcome this issue using a shadowing lemma by Liao [20]. See Lemma 4.5 below, in particular item (d).

We organize this section in the following way. In 4.1 we establish the scaled linear Poincaré flows. In 4.2 we will introduce Liao’s shadowing lemma, which allows us to shadow pseudo-orbit that passes through neighborhoods of singularities and estimates the time difference between the pseudo-orbit and the shadowing orbit. Finally, we will prove Theorem 4.1 in Section 4.2.

4.1. Liao’s theory on the scaled linear Poincaré flows. Starting from now, $\mu$ will be a non-trivial hyperbolic ergodic measure with a dominated splitting $E \oplus F$ on $\text{supp} \mu$, such that $\dim E$ is the index of $\mu$.

Recall that for a regular point $x$ and $v \in T_{x}M$, the linear Poincaré flow $\psi_{t} : N_{x} \rightarrow N_{\phi_{t}(x)}$ is the projection of $\Phi_{t}(v)$ to $N_{\phi_{t}(x)}$, where $N_{x}$ is the orthogonal complement of $X(x)$. The scaled linear Poincaré flow, which we denote by $\psi_{*}^{t}$, is defined as

\[
\psi_{*}^{t}(v) = \frac{\|X(x)\|}{\|X(\phi_{t}(x))\|} \psi_{t}(v) = \frac{\psi_{t}(v)}{\|\Phi_{t} \cdot_{X(x)}\|}.
\]

Since $\|\Phi_{t}\|$ is bounded away from zero as long as $t$ remains bounded, and the flow direction corresponds to a zero Lyapunov exponent, we get:

**Lemma 4.2.** $\psi_{*}^{t}$ is a bounded cocycle over $N_{\Lambda}$ in the following sense: for any $\tau > 0$, there is $C_{\tau} > 0$ such that for any $t \in [-\tau, \tau]$,

\[
\|\psi_{*}^{t}\| \leq C_{\tau}.
\]

In particular, for every non-trivial ergodic measure $\mu$, the cocycles $\psi_{t}$ and $\psi_{*}^{t}$ have the same Lyapunov exponents and Oseledets splitting.

Recall that $\pi : T_{x}M \rightarrow N_{x}$ is the orthogonal projection along flow direction.

**Lemma 4.3.** We have $X|_{\text{supp} \mu \setminus \text{Sing}(X)} \subset F$. Furthermore, $\pi(E) \oplus \pi(F)$ is also a dominated splitting on $N_{\Lambda \setminus \text{Sing}(X)}$ for both $\psi_{t}$ and $\psi_{*}^{t}$, which is also the Oseledets splitting for $(\psi_{t}, \mu)$ corresponding to the negative exponents and positive exponents.

**Proof.** Since $\mu$ is hyperbolic with index $\dim E$, $\phi_{t}$ has precisely $\dim E$ many negative exponents, and a vanishing exponent given by the flow direction. Since $E \oplus F$ is dominated, Lyapunov exponents on $F$ must be larger than those in $E$, thus non-negative. It then follows that $E$ is the Oseledets splitting corresponding to the negative exponents, and $X|_{\text{supp} \mu \setminus \text{Sing}(X)} \subset F$.

Next we will show the second part of this lemma only for $\psi_{t}$. The result for $\psi_{*}^{t}$ will then follow from Lemma 4.2.

Take any $x \in \text{supp} \mu$ and unit vectors $u \in \pi(E_{x})$, $v \in \pi(F_{x})$. Since $X|_{\text{supp} \mu \setminus \text{Sing}(X)} \subset F$, we have $v \in \pi(F_{x}) \subset F_{x}$. Let $u' = \pi^{-1}u \in E_{x}$ and $v' = \Phi_{-t} \circ \psi_{t}(v) \in F_{x}$. Since
\[ E \oplus F \text{ is dominated for } \psi_t, \text{ we must have} \]
\[
\frac{\|\Phi_t(u')\|}{\|\Phi_t(v')\|} < C \lambda^t \frac{\|u'\|}{\|v'\|},
\]
for some \( C > 0 \) and \( \lambda \in (0, 1) \).

Since \( E \oplus F \) is dominated and \( X|_{\text{supp } \mu \setminus \text{Sing}(X)} \subset F \), the angle between \( E \) and \( X \) must be away from zero. On the other hand, since \( N_x \) is the orthogonal complement of \( X \), the angle between \( E \) and \( N_x \) must be away from \( \pi/2 \). This show that there exists some constant \( C_1 \) independent of \( x \), such that for all unit vectors \( u \in \pi(E_x) \), \( \|u'\| = \|\pi^{-1} u\| < C_1 \).

This shows that \( \Phi_t(v') = \psi_t(v) \). It follows that
\[
\|\Phi_t(u')\| < C C_1 \lambda^t \frac{1}{\|v'\|}. \tag{13}
\]

From the definition of \( \psi_t \) and \( \pi \), we have
\[
\psi_t(u) = \pi \circ \Phi_t(u) = \pi \circ \Phi_t(u') \quad \text{and} \quad \pi(v') = v,
\]
thus
\[
\|\psi_t(u)\| \leq \|\Phi_t(u')\| \quad \text{and} \quad \|v'\| \geq \|v\| = 1. \tag{14}
\]

Combine (13) and (14), we get
\[
\frac{\|\psi_t(u)\|}{\|\psi_t(v)\|} < C C_1 \lambda^t.
\]

Finally, we use the fact that \( \pi(v') = v \) to get \( \|v'\| \geq \|v\| = 1 \), therefore
\[
\frac{\|\psi_t(u)\|}{\|\psi_t(v)\|} < C C_1 \lambda^t.
\]

This shows that \( \pi(E) \oplus \pi(F) \) is a dominated splitting for \( \psi_t \).

Since \( \mu \) has index \( \dim E = \dim(\pi(E)) \), the argument used at the beginning of this proof shows that \( \pi(E) \oplus \pi(F) \) is indeed the Oseledets splitting for \( \psi_t \) corresponding to negative and positive exponents. \( \square \)

Next we describe the hyperbolicity for the scaled linear Poincaré flow \( \psi_t^* \).

**Definition 11.** For \( T_0 > 0, \lambda \in (0, 1) \), the orbit segment \( \{\phi_t(x)\}_{0 \leq t \leq T} \) is called \((\lambda, T_0)^*\) quasi-hyperbolic with respect to a splitting \( N_x = E_x^T \oplus F_x^N \) and the scaled linear Poincaré flow \( \psi_t^* \), if there exists a partition
\[
0 = t_0 < t_1 < \cdots < t_l = T, \quad \text{where } t_{i+1} - t_i \in [T_0, 2T_0],
\]
such that for \( k = 1, \ldots, l - 1 \), we have
\[
\prod_{i=0}^{k-1} \|\psi_{t_{i+1} - t_i}^*|_{\psi_t^*(E_x^N)}\| \leq \lambda^k; \prod_{i=k}^{l-1} m(\psi_{t_{i+1} - t_i}^*|_{\psi_t^*(E_x^N)}) \geq \lambda^{-(l-k)},
\]
and
\[
\frac{\|\psi_{t_{i+1} - t_i}^*|_{\psi_t^*(E_x^N)}\|}{m(\psi_{t_{i+1} - t_i}^*|_{\psi_t^*(E_x^N)})} \leq \lambda^2.
\]

**Definition 12.** For \( T_0 > 0, \lambda \in (0, 1) \), an orbit segment \( \phi_{[0,T]}(x) \) is called \((\lambda, T_0)^\star\) forward contracting for the bundle \( E \subset N_x \), if there exists a partition
\[
0 = t_0 < t_1 < \cdots < t_n = T, \quad \text{where } t_{i+1} - t_i \in [T_0, 2T_0],
\]
such that for all \( k = 1, \ldots, n - 1 \),
\[
\prod_{i=0}^{k-1} \|\psi_{t_{i+1} - t_i}^*|_{\psi_t^*(E)}\| \leq \lambda^k. \tag{15}
\]
An orbit segment \( \phi_{-T,0}(x) \) is called \((\lambda, T_0)\)-backward contracting for the bundle \( E \subset N_x \), if it is backward contracting for the flow \(-X\).

A point \( x \) is called a \((\lambda, T_0)\)-forward hyperbolic time for the bundle \( E \subset N_x \), if the infinite orbit \( \phi_{[0, +\infty)}(x) \) is \((\lambda, T_0)\)-forward contracting. In this case the partition is taken to be

\[
0 = t_0 < t_1 < \cdots < t_n < \ldots, \quad \text{where } t_{i+1} - t_i \in [T_0, 2T_0],
\]

and \( [15] \) is stated for all \( k \in \mathbb{N} \). \( x \) is called a \((\lambda, T_0)\)-backward hyperbolic time for the bundle \( E \subset N_x \), if it is a forward hyperbolic time for \(-X\). \( x \) is called a two-sided hyperbolic time, if it is both a forward and backward hyperbolic time.

By the classic work of Liao [20], there exists \( \delta > 0 \) such that if \( x \) is a backward hyperbolic time, then \( x \) has unstable manifold with size \( \delta \|X(x)\| \). Similarly, if \( x \) is a forward hyperbolic time then it has stable manifold with size \( \delta \|X(x)\| \). In both cases, we say that \( x \) has unstable/stable manifold up to the flow speed.

The next lemma can be seen as a \( C^1 \) version of the Pesin theory for flows.

**Lemma 4.4.** For almost every ergodic component \( \bar{\mu} \) of \( \mu \) with respect to \( f = \phi_1 \), there are \( L', \eta, T_0 > 0 \) and a compact set \( \Lambda_0 \subset \text{supp} \mu \setminus \text{Sing}(X) \) with positive \( \bar{\mu} \) measure, such that for every \( x \) satisfying \( f^n(x) \in \Lambda_0 \) for \( n > L' \), the orbit segment \( \{\phi_t(x)\} \in [0, n] \) is \((\eta, T_0)^*\) quasi-hyperbolic with respect to the splitting \( N_x = \pi(E_x) \oplus \pi(F_x) \) and the scaled linear Poincaré flow \( \psi^*_t \).

**Proof.** The proof is very standard. By Lemma 4.3 for \( \bar{\mu} \) almost every \( x \), \( \pi(E_x) \oplus \pi(F_x) \) is the Oseledets splitting of \( \psi^*_t \) corresponding to the negative and positive exponents. By the subadditive ergodic theorem, there is \( a < 0 \) such that for \( N_0 \) large enough, we have

\[
\frac{1}{N_0} \int \log \|\psi^*_{N_0}\|\, d\bar{\mu} < a \quad \text{and} \quad \lim_{t \to -\infty} \frac{1}{N_0} \int \log \|\psi^*_{-N_0}\|\, d\bar{\mu} < a.
\]

Let

\[
\bar{\mu} = \frac{1}{k_0} (\bar{\mu}_1 + \cdots + \bar{\mu}_{k_0})
\]

be the ergodic decomposition of \( \bar{\mu} \) with respect to \( f^{N_0} \). Change the order if necessary, we may assume that

\[
\frac{1}{N_0} \int \log \|\psi^*_{N_0}\|\, d\bar{\mu}_1 < a \quad \text{and} \quad \frac{1}{N_0} \int \log \|\psi^*_{-N_0}\|\, d\bar{\mu}_1 < a.
\]

By the Birkhoff ergodic theorem on \( f^{N_0} \), for \( \bar{\mu}_1 \) almost every \( x \),

\[
\lim_{m \to \infty} \frac{1}{mN_0} \sum_{i=0}^{m-1} \log \|\psi^*_{N_0}\|_{\pi(f_iN_0(x))} < a,
\]

and similarly on \( \pi(F) \):

\[
\lim_{m \to \infty} \frac{1}{mN_0} \sum_{i=0}^{m-1} \log \|\psi^*_{-N_0}\|_{\pi(f_i^{-1}N_0(x))} < a.
\]

Take \( n_x > 0 \) such that the above inequalities holds for all \( m > n_x \), and \( N_1 \) such that the set \( \Lambda' = \{x : n_x < N_1 \} \) has positive \( \bar{\mu}_1 \) measure. Let \( \Lambda_0 \subset \Lambda' \setminus \text{Sing}(X) \) be compact and has positive \( \bar{\mu}_1 \) measure. Then \( \bar{\mu}(\Lambda_0) > 0 \). By Lemma 4.2 we can take

\[
K = \max_{|t| < N_0, y \in \text{supp} \mu \setminus \text{Sing}(X)} \{\sup \{\psi^*_t|E_y\}; \sup \{\psi^*_t|F_y\} \}.
\]

Choose \( N_2 \) large enough such that

\[
\frac{N_2 + N_0}{N_0} a + 3K < b < 0.
\]
for some \( b < 0 \). We claim that for any sequence \( n_1 < n_2 < \ldots < n_l \) with \( N_2 \leq n_{i+1} - n_i \leq N_2 + N_0 \) for each \( i \), we have
\[
\frac{1}{l} \sum_{i=0}^{l-1} \log \| \psi_{n_{i+1} - n_i}^* \| < b < 0,
\]
and a similar inequality holds on \( \pi(F) \). The lemma will then follow from this claim and the domination between \( \pi(E) \) and \( \pi(F) \).

To prove this claim, for each \( i = 1, \ldots, l-1 \), write
\[
k_i = \left( \frac{n_i+1}{N_0} \right) - \left( \frac{n_i}{N_0} \right) - 1, \quad n_i' = (\lfloor \frac{n_i}{N_0} \rfloor + 1)N_0 \text{ and } n_{i+1}' = \left( \frac{n_{i+1}}{N_0} \right)N_0.
\]
Then we have \( n_i' \leq n_i \leq n_i' \), \( n_{i+1}' - n_i' = k_iN_0 \), and
\[
\psi_{n_{i+1} - n_i}^*|_{\pi(E_{jN_0}(x))} = \psi_{n_{i+1}' - n_i'}^*|_{\pi(E_{jN_0}(x))} \circ \psi_{n_i'}^*|_{\pi(E_{jN_0}(x))} \circ \psi_{n_i}^*|_{\pi(E_{jN_0}(x))}.
\]

Note that \( n_{i+1} - n_i' + 1 \leq N_0 \) and \( n_i' - n_i \leq N_0 \). By the choice of \( K \), we have
\[
\log \| \psi_{n_{i+1} - n_i}^* \pi(E_{jN_0}(x)) \| \leq 2K + \log \| \psi_{n_i'}^*|_{\pi(E_{jN_0}(x))} \| \leq 3K + \log \| \psi_{n_i'}^*|_{\pi(E_{jN_0}(x))} \|.
\]

Sum over \( i \), we obtain
\[
\frac{1}{l} \sum_{i=0}^{l-1} \log \| \psi_{n_{i+1} - n_i}^* \pi(E_{jN_0}(x)) \| \leq \frac{1}{l} \sum_{i=0}^{l-1} \log \| \psi_{n_i}' \| \pi(E_{jN_0}(x)) \| + 3K \leq \frac{n_{i+1}'}{lN_0}a + 3K \leq \frac{N_2 + N_0}{N_0}a + 3K < b < 0.
\]

\[\square\]

### 4.2. A shadowing lemma by Liao and proof of Theorem 4.1.

In this section we will introduce a shadowing lemma by Liao [20] for the scaled linear Poincaré flow.

**Lemma 4.5.** Given a compact \( \Lambda_0 \cap \text{Sing}(X) = \emptyset \) and \( \eta \in (0, 1), T_0 > 0 \), for any \( \varepsilon > 0 \) there exists \( \delta > 0 \), \( L > 0 \) and \( \delta_0 > 0 \), such that for any \( \eta, T_0 > \) quasi-hyperbolic orbit segment \( \{ \phi_t(x) \}_{0 \leq t \leq T} \) with respect to a dominated splitting \( N_x = E_x \oplus F_x \) and the scaled linear Poincaré flow \( \phi_t^* \), if \( x, \phi_T(x) \in \Lambda_0 \) with \( d(x, \phi_T(x)) < \delta \), then there exists a point \( p \) and a \( C^1 \) strictly increasing function \( \theta : [0, T] \to \mathbb{R} \), such that

1. (a) \( \theta(0) = 0 \) and \( |\theta'(t) - 1| < \varepsilon \);
2. (b) \( p \) is a periodic point with \( \phi_{\theta(T)}(p) = p \);
3. (c) \( d(\phi_t(x), \phi_{\theta(t)}(p)) \leq \varepsilon \| X(\phi_t(x)) \| \), for all \( t \in [0, T] \);
4. (d) \( d(\phi_t(x), \phi_{\theta(t)}(p)) \leq Ld(x, \phi_T(x)) \);
5. (e) \( p \) has stable and unstable manifold with size at least \( \delta_0 \).
6. (f) if \( \Lambda_0 \subset \Lambda \) for a sectional hyperbolic chain recurrent class \( \Lambda \), then \( p \in \Lambda \).

Furthermore, the result remains true with the same constants \( \delta, L \) and \( \delta_0 > 0 \) if \( \Lambda_0 \) is replaced by a subset of \( \Lambda_0 \).

**Remark 4.6.** By (a), the period of the shadowing orbit, \( \theta(T) \), satisfies \( \theta(T) \in [T(1 - \varepsilon), T(1 + \varepsilon)] \). However, using the fact that
\[
\sum_{i=0}^{l-1} d(\phi_{\theta(t)}(p), \phi_t(x)) \leq L^* d(x, \phi_T(x)).
\]
(note that the constant $L^*$ on the right hand does not depend on $T$), one can show the following modified version of (a):

(a') we have $\theta(0) = 0$ and $|\theta(t) - 1| < \varepsilon$; furthermore, there is a constant $C$ independent of $T$, such that $\theta(T) \in [T - C\varepsilon, T + C\varepsilon]$.

Now we are ready to prove Theorem 4.1.

Let $\tilde{\mu}$ be a typical ergodic component of $\mu$ with respect to $f$. By Lemma 2.5 $h_\mu(f) = h_{\tilde{\mu}}(f) > 0$. Let $\Lambda_0$ be the compact set with positive $\tilde{\mu}$ measure given by Lemma 4.4. Also let $L', T_0, \eta > 0$ be the constants given by the same lemma. Apply Lemma 4.5 with $\Lambda_0$ and $\eta, T_0$, for every $\varepsilon > 0$ we obtain $\delta, L$ and $\delta_0$.

Replace $\Lambda_0$ by a compact subset if necessary, we may assume that $\Lambda_0$ is away from singularities with diameter small enough, such that any two periodic points obtained by Lemma 4.5 are homoclinic related. Following the proof of [16] [Theorem 4.3], for every $\alpha, l > 0$ and $n \in \mathbb{N}$, there is a finite set $K_n = K_n(\alpha, l)$ with the following property:

- $K_n \subseteq \Lambda_0$;
- for $x, y \in K_n$, $d_n^l(x, y) = \max_{0 \leq j \leq n - 1} \{d(f^j x, f^j y)\} > \frac{1}{4}$;
- for every $x \in K_n$, there is an integer $m(x)$ with $n \leq m(x) \leq (1 + \alpha)n$, such that $f^m(x) \in \Lambda_0$ with $d(x, f^m(x)) < \frac{1}{4\alpha}$;
- $\lim_{n \to \infty} \lim_{n \to \infty} \frac{1}{n} \log \text{Card} K_n(\alpha, l) \geq h_{\tilde{\mu}}(f) - \alpha$.

We take $n, l$ large enough, such that $n > L', \frac{1}{2} < \delta$ and

$$\text{Card} K_n(\alpha, l) > \exp(n(h_{\tilde{\mu}} - \alpha)).$$

For every $x \in K_n$, by Lemma 4.5 the orbit segment $\{\phi_t(x)\}_{0 \leq t \leq m(x)}$ is shadowed by a periodic point $p_x$ with period no more than $n(1 + \varepsilon)(1 + \alpha)$. Item (d) in Lemma 4.5 guarantees that $d_n^l(p_x, p_y) \leq L_d(x, f^m(x)) < \frac{1}{4\alpha}$. As a result,

$$d_n^l(p_x, p_y) \geq d_n^l(x, y) - d_n^l(x, p_x) - d_n^l(y, p_y) > \frac{1}{2\alpha}.$$

Note that different $x, y \in K_n$ may be shadowed by the same periodic orbit $\text{Orb}(p)$.

When this happens, we must have $p_x = \phi_{t_{y,x}}(p_y)$ for some $t_{y,x}$. Then the estimate above implies that

$$t_{y,x} > \frac{1}{2\alpha D},$$

where $D = \max\{\|X\|\}$ is the maximum of the flow speed as before. Therefore, for each $x \in K_n$, the periodic orbit $\text{Orb}(p_x)$ can shadow no more than $2\alpha Dn(1 + \varepsilon)(1 + \alpha)$ different points in $K_n$.

As a result, there are at least

$$k_n = \frac{\exp(n(h_{\tilde{\mu}} - \alpha))}{2\alpha Dn(1 + \varepsilon)(1 + \alpha)}$$

many different periodic orbits, with periodic at most $n(1 + \varepsilon)(1 + \alpha)$. Since they are homoclinic related near $\Lambda_0$, we have a horse-shoe with topological entropy at least

$$\lim_{n \to \infty} \frac{1}{n(1 + \varepsilon)(1 + \alpha)} \log k_n = \lim_{n \to \infty} \frac{1}{n(1 + \varepsilon)(1 + \alpha)} \log \frac{\exp(n(h_{\tilde{\mu}} - \alpha))}{2\alpha Dn(1 + \varepsilon)(1 + \alpha)},$$

which converges to $\frac{h_{\tilde{\mu}} - \alpha}{1 + \alpha}$. Then Theorem 4.1 follows by taking $\alpha$ and $\varepsilon$ small enough.

Remark 4.7. A similar result for star flows can be found in [18]. Instead of using the shadowing lemma, they take a small neighborhood $N$ of $K_n$ and consider the Poincaré return map $P_x$ from the neighborhood of a point $x \in N$ to a neighborhood of $\phi_m(x) + \tau_n(x) \in N$. Then they show that for every $x, y \in K_n$, the connected
component of \( P_s(N) \) crosses the connected component of \( P_y^{-1}(N) \), thus giving a horse-shoe with \( \text{Card} \, K_n \) many components.

**Proof of Corollary D.** Recall that \( \Lambda \) is a Lorenz-like class if it is a sectional hyperbolic, Lyapunov stable chain recurrent class. Then Corollary D follows from Theorem A, C, B and Lemma 4.5(f). □

4.3. \( C^1 \) generic flows: proof of Corollary E. In this section, \( \Lambda \) will be a Lorenz-like class of a \( C^1 \) flow \( \phi_t \). Note that for every \( x \in \Lambda \), the unstable set of \( x \) is contained in \( \Lambda \).

We need the following generic properties for \( C^1 \) flows. The first property is the flow version of a famous property for \( C^1 \) generic diffeomorphisms, which can be found in [4].

**Proposition 4.8.** \( C^1 \) generically, every chain recurrent class \( C \) of \( \phi_t \) that contains a periodic point \( p \) coincides with the homoclinic class of \( \text{Orb}(p) \). In particular, \( C \) is transitive.

The next property is a simple application of the connecting lemma in [4], applied to a branch of stable manifold of the singularity and the unstable manifold of the periodic orbit \( \text{Orb}(p) \).

**Proposition 4.9.** Let \( C \) be a chain recurrent class for a \( C^1 \) generic flow \( \phi_t \), such that \( C \) contains a hyperbolic singularity \( \sigma \) and a hyperbolic periodic point \( p \). Assume that on \( \sigma \), the stable subspace \( E^{cs}_\sigma \) has a dominated splitting \( E^{cs}_\sigma = E^s_\sigma \oplus E^u_\sigma \), where \( E^s_\sigma \) is a 1-dimensional sub-bundle of \( E^{cs}_\sigma \). Then the strong stable manifold \( W^s(\sigma) \) divides the stable manifold \( W^{cs}(\sigma) \) into two branches \( W^{cs, +}(\sigma) \) and \( W^{cs, -}(\sigma) \); furthermore, if \( C \cap W^{cs, \pm}(\sigma) \setminus \{\sigma\} \neq \emptyset \), then \( W^{cs, \pm}(\sigma) \cap W^u(p) \neq \emptyset \).

**Proof of Corollary E.** Let \( R \) be the residual subset of flows that are Kupka-Smale, and satisfies the above properties. Then for every Lorenz-like class \( \Lambda \), by Theorem C and the variational principle, there is a hyperbolic measure \( \mu \) with positive entropy supported on \( \Lambda \). Apply Theorem 4.1, Lemma 4.5 and Proposition 4.8, we get that \( \Lambda \) is a homoclinic class of a periodic orbit \( p \) and is transitive. Since \( \Lambda \) is Lyapunov stable, to prove that \( \Lambda \) is an attractor, it suffices to show that it is isolated, i.e., it cannot be approximated by other chain recurrent classes.

Let \( \{U_n\} \) be the sequence of Lyapunov stable neighborhoods of \( \Lambda \). Suppose by contradiction that \( \Lambda \) is not isolated. Then one can find chain recurrent classes \( \Lambda \neq C_n \subset U_n \), with lim sup \( C_n \subset \Lambda \).

Every \( \phi_t \in R \) is Kupka-Smale, thus the singularities are all hyperbolic and isolated. Thus for \( n \) large, the singularities in \( U_n \) are precisely those in \( \Lambda \), as we have observed in [2, Section 3.1]. Since chain recurrent class is an equivalent class, we must have \( C_n \cap \Lambda = \emptyset \). We can therefore assume that for all \( n \), \( C_n \) does not contain singularities. Taking \( n \) large if necessary, we see that \( C_n \) are sectional hyperbolic without singularity, thus hyperbolic. As a result, there are hyperbolic periodic point \( p^n \in C_n \). Let \( \Lambda_0 \) be the Hausdorff limit of \( \text{Orb}(p^n) \), which is a compact, invariant and sectional hyperbolic subset of \( \Lambda \).

We claim that \( \Lambda_0 \) contains a singularity. If this claim is not true, then \( \Lambda_0 \) is hyperbolic. For \( n \) large enough, the hyperbolic sets \( C_n \) and \( \Lambda_0 \) must be homoclinic related; as a result, \( C_n \) and \( \Lambda_0 \) are in fact the same homoclinic class. This contradicts our assumption that \( C_n \neq \Lambda \).

Let \( \sigma \in \Lambda_0 \subset \Lambda \) be a singularity. By Lemma 3.10, we have a hyperbolic splitting \( E^s \oplus E^c \oplus E^u \) on \( T_0 M \) with \( \dim E^c = 1 \), and \( E^s \oplus E^c \) is the stable subspace of \( T_0 M \). As in Lemma 3.10 we have \( W^s(\sigma) \cap \Lambda = \{\sigma\} \). By Proposition 4.9, \( W^s(\sigma) \) divides \( W^{cs}(\sigma) \) into two branches, \( W^{cs, \pm}(\sigma) \).
The argument below is similar to the proof of Lemma 8.10. Since $\Lambda_0$ is the Hausdorff limit of $\text{Orb}(p^n)$, we may assume that $p^n \to \sigma$. Fix $\varepsilon > 0$ small, let $t_n < 0$ be the last time such that $\phi_{t_n}(p^n) \in \partial B_{\varepsilon}(\sigma)$. It is easy to see that $t_n \to -\infty$.

Let $z^n = \phi_{t_n}(p_n)$ and $z^n \to z$, then $z \in W^{cs}(\sigma)$.

We may assume that $z \in W^{cs,+}(\sigma)$. By Proposition 4.9, we can take $a \in W^{cs,+}(\sigma) \cap W^u(\text{Orb}(p))$, where $p$ is a periodic point in $\Lambda$. Take $s$ such that $a \in W^u(\phi_s(p))$ and a disk $D$ with $a \in D \subset W^u(\phi_s(p))$. Then by the $\lambda$-lemma, $\phi_t(D)$ approximated $W^u(\sigma)$, and $\{\phi_t(x) : t > 0, x \in D\}$ is a submanifold that is tangent to the $F^u$ bundle, with dimension dim $F^u$. Thus $W^s(z^n) \cap \{\phi_t(x) : t > 0, x \in D\} \neq \emptyset$.

On the other hand, $\phi_1(D)$, as a subset of $W^u(\phi_1(p))$ with $p \in \Lambda_0 \subset \Lambda$, must be contained in $\Lambda$. This shows that $W^s(z^n) \cap \Lambda \neq \emptyset$. In particular, for $n$ large, we have

$$d(\phi_t(p_n), \Lambda) \to 0 \text{ as } t \to \infty.$$ 

As $p_n \in C_n$, this shows that $C_n$ and $\Lambda$ are the same chain recurrent class, a contradiction.

$\Box$

5. $C^1$ Generic Star Flows

Let us briefly explain the structure of this section. In Section 5.1 we will establish the extended linear Poincaré flow studied by Liao [21] and Li [17], which is the linear Poincaré flow lifted to the Grassmannian manifold and extended to the singularities. In Section 5.2 we introduce the readers to the results in [11] on the structure of chain recurrent class for star flows. Most importantly, it is shown in [11] that for generic star flows, all the singularities in a chain recurrent class must be Lorenz-like or reverse Lorenz-like, and their stable indices can only differ by one. Section 5.3 contains detailed analysis for flow orbits near a hyperbolic singularities (Lemma 5.2 and 5.4) and estimates of the hyperbolicity for flow orbit approaching and leaving a Lorenz-like singularity (Lemma 5.7 and 5.8). Section 5.4 consists of the proof of Theorem 4.1.

5.1. Extended linear Poincaré flows. We first introduce the extended linear Poincaré flow, which is a useful tool developed by Liao [20, 21] and used by Li et al [17] to study singularities. Denote by

$$G^1 = \{L : L \text{ is a } 1\text{-dimensional subspace of } T_x M, x \in M\}$$

the Grassmannian manifold of $M$. Given a $C^1$ flow $\phi_t$, the tangent flow $\Phi_t$ acts naturally on $G^1$ by mapping each $L$ to $\Phi_t(L)$.

Write $\beta : G^1 \to M$ and $\xi : TM \to M$ the bundle projection. The pullback bundle of $TM$:

$$\beta^*(TM) = \{(L, v) \in G^1 \times TM : \beta(L) = \xi(v)\}$$

is a vector bundle over $G^1$ with dimension dim $M$. The tangent flow $\Phi_t$ lifts naturally to $\beta^*(TM)$:

$$\Phi_t(L, v) = (\Phi_t(L), \Phi_t(v)).$$

Recall that the linear Poincaré flow $\psi_t$ defined in Section 2 projects the image of the tangent flow to the normal bundle of the flow direction. The key observation here is that this projection can be defined not only on the bundle perpendicular to the flow, but to the orthogonal complement of any direction $L \in G^1$.

To be more precise, we write

$$\mathcal{N} = \{(L, v) \in \beta^*(TM) : v \perp L\}.$$
Then $\mathcal{N}$, consisting of vectors perpendicular to $L$, is a sub-bundle of $\beta^*(TM)$ over $G^1$ with dimension $\dim M - 1$. The extended linear Poincaré flow is then defined as

$$\psi_t : \mathcal{N} \rightarrow \mathcal{N}, \psi_t(L, v) = \pi(\Phi_t(L, v)),$$

where $\pi$ is the orthogonal projection from $\beta^*(TM)$ to $\mathcal{N}$.

If we consider the map

$$\zeta : \text{Reg}(X) \rightarrow G^1$$

that maps every regular point $x$ to the unique $L \in G^1$ with $\beta(L) = x$ and is parallel to the flow direction at $x$, then the extended linear Poincaré on $\zeta(\text{Reg}(X))$ coincides with the linear Poincaré flow defined earlier. On the other hand, given any invariant set $\Lambda$ of the flow $\phi_t$, consider the set:

$$\tilde{\Lambda} = \zeta(\Lambda \cap \text{Reg}(X)).$$

If $\Lambda$ contains no singularity, then $\tilde{\Lambda}$ can be seen as a natural copy of $\Lambda$ in $G^1$ equipped with the direction of the flow on $\Lambda$. If $\sigma \in \Lambda$ is a singularity, then $\tilde{\Lambda}$ contains all the direction in $\beta^{-1}(\sigma)$ that can be approximated by the flow direction at regular points in $\Lambda$. In other words, one replaces the singularity $\sigma$ by a subset of the sphere $\beta^{-1}(\sigma)$. The extended Poincaré flow restricted to $\tilde{\Lambda}$ can be seen as the continuous extension of the linear Poincaré flow on $\Lambda$.

5.2. Classification of chain recurrent classes and singularities for generic star flows. In this subsection we recap the main result in [11]. We begin with the following classification on the singularities.

**Definition 13.** Let $\sigma$ be a hyperbolic singularity contained in a non-trivial chain recurrent class $C(\sigma)$. Assume that the Lyapunov exponents of $\sigma$ are:

$$\lambda_1 \leq \cdots \leq \lambda_s < 0 < \lambda_{s+1} \leq \cdots \leq \lambda_{\dim M}.$$  

Write $\text{Ind}(\sigma) = s$ for the stable index of $\sigma$. We say that

- (1) $\sigma$ is Lorenz-like, if $\lambda_s + \lambda_{s+1} > 0$, $\lambda_{s-1} < \lambda_s$, and $W^{ss}(\sigma) \cap \{\sigma\} = \{\sigma\}$, where $W^{ss}(\sigma)$ is the stable manifold of $\sigma$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_{s-1}$; regular orbits in $C(\sigma)$ can only approach $\sigma$ along $E^{cu}(\sigma)$ cone;

- (2) $\sigma$ is reverse Lorenz-like, if it is Lorenz-like for $-X$; in this case, regular orbits in $C(\sigma)$ can only approach $\sigma$ along $E^{cs}(\sigma)$ cone.

Then it is shown in [11] that (all the labelling are with respect to [11]):

- for star flows, if a chain recurrent class $C$ is non-trivial, then every singularity in $C$ is either Lorenz-like or reverse Lorenz-like (Theorem 3.6);
• for generic star flows, if some periodic orbit $p$ is sufficiently close to $\sigma$, then:
  - when $\sigma$ is Lorenz-like, the index of the $p$ must be $\text{Ind}(\sigma) - 1$;
  - when $\sigma$ is reverse Lorenz-like, the index of the $p$ is $\text{Ind}(\sigma)$ (Lemma 4.4);

furthermore, the dominated splitting on $\sigma$ induced by such periodic orbits coincides with the hyperbolic splitting on $\sigma$ (proof of Theorem 3.7);
• every periodic orbit contained in a sufficiently small neighborhood of $C$ has the same index $\text{Ind}_p$ (Theorem 5.7);
• combine the previous two results, we see that all the singularity in $C$ has index either $\text{Ind}_p + 1$ (in which case it must be Lorenz-like) or $\text{Ind}_p$ (reverse Lorenz-like);
• if all the singularities in $C$ are Lorenz-like, then $C$ is sectional hyperbolic (Theorem 3.7);
• if $C$ contains singularity with different indices, then there is no sectional hyperbolic splitting on $C$; one of such examples was constructed by Bonatti and da Luz [5].

5.3. Flow orbit near singularities.

5.3.1. Flow orbit near hyperbolic singularities. In this section we will establish some geometric properties for flow orbit in a small neighborhood of a hyperbolic singularity. For this purpose, let $\sigma$ be a hyperbolic singularity with the hyperbolic splitting $E^s_{\sigma} \oplus E^u_{\sigma}$. Without loss of generality, we can think of $\sigma$ to be the origin in $\mathbb{R}^n$, and assume that $E^s_{\sigma}$ and $E^u_{\sigma}$ are perpendicular (which is possible if one changes the metric). In particular, we will assume that $E^s_{\sigma} = \mathbb{R}^s$ is the $s$-dimensional subspace of $\mathbb{R}^n$ with the last $\dim M - s$ coordinates being zero. Here $s = \dim E^s_{\sigma}$ is the stable index of $\sigma$. Similarly, $E^u_{\sigma}$ is the subspace of $\mathbb{R}^n$ where the first $s$ coordinates are zero.

Since the vector filed $X$ is $C^1$, we can take a neighborhood $U = B_r(\sigma)$ with $r$ small enough such that

• the flow in $U$ can be written as

$$\phi_t(x) = e^{At}x + C^1 \text{ small perturbation},$$

where $A$ is a matrix with non-zero eigenvalues;
• for $x \in U$, the tangent map $Df(x) = D\phi_t(x)$ are small perturbations of the hyperbolic matrix $e^{At}$, with eigenvalues bounded away from 1.

For each $x \in U$, denote by $x^u$ its distance to $W^u(\sigma)$ and $x^s$ its distance to $W^s(\sigma)$. Then for every $\alpha > 0$ small, we define the $\alpha$-cone on the manifold, denote by $D^u_{\alpha}(\sigma)$, $i = s, u$:

$$D^s_{\alpha}(\sigma) = \{x \in U : x^u < \alpha x^s\}, \quad D^u_{\alpha}(\sigma) = \{x \in U : x^s < \alpha x^u\}.$$

Note that the splitting $E^s_{\sigma} \oplus E^u_{\sigma}$ can be extended to $U$ in a natural way: for each $x \in U$, put $E^s(x)$ as the $s$-dimensional hyperplane that is parallel to $E^s_{\sigma}$. The same can be done for $E^u(x)$. This allows us to consider the $\alpha$-cones $C_{\alpha}(E^i)$ on the tangent bundle, as defined in Section 2. The next lemma easily follows from the smoothness of the vector field $X$:

**Lemma 5.1.** There exists $L \geq 1$, such that for all $\alpha > 0$ small enough,

1. for every $x \in D^s_{\alpha}(\sigma)$, we have $X(x) \in C_{L\alpha}(E^u)$;
2. for every $x \in U$, if $X(x) \in C_{\alpha}(E^u)$, we have $x \in D^u_{L\alpha}(\sigma)$.

Moreover, the same holds for $D^u_{\alpha}(\sigma)$ and $C_{\alpha}(E^s)$.

Let us fix some $\alpha > 0$ small enough that will be determined later. Note that If $x \in U \setminus (D^s_{\alpha}(\sigma) \cup D^u_{\alpha}(\sigma))$, we lose the control on the direction of $X(x)$. One can
think of the region $U \setminus (D^s_\alpha(\sigma) \cup D^u_\alpha(\sigma))$ to be the place where the flow is ‘making the turn’ from the $E^s$ cone to the $E^u$ cone. The next lemma states that the time that a orbit segment spend in this region is uniformly bounded. To this end, we write, for each $x \in U$,

$$t^+(x) = \sup \{t > 0 : \phi_{[0,t]}(x) \subset U\}, \quad t^-(x) = \sup \{t > 0 : \phi_{[-t,0]}(x) \subset U\}.$$ 

With slight abuse of notation, we will frequently drop the depends of $t^\pm(x)$ on $x$.

**Lemma 5.2.** For every $\alpha > 0$ small enough, there is $T^\alpha > 0$, such that for every $x \in U$, the set

$$T(x) := \{t \in (-t^-, t^+) : \phi_t(x) \notin D^s_\alpha(\sigma) \cup D^u_\alpha(\sigma)\}$$

has length bounded by $T^\alpha$.

**Proof.** By continuity, we can take $t_0 \in (-t^-, t^+)$ such that the point $x_0 = \phi_{t_0}(x)$ satisfies

$$x_0^u = x_0^w.$$ 

Note that $t_0 \subset T(x)$ if $\alpha < \frac{1}{2}$. On the other hand, by (16), there is $\lambda_0 > 1$ such that for each $x \in U$,

$$f(x)^u > \lambda_0 x^u, \quad f(x)^s < \lambda_0^{-1} x^s.$$

Apply this recursively on $x_0$, we obtain:

$$\frac{f^k(x_0)^u}{f_k(x_0)^w} > \lambda_0^{2k}.$$ 

In particular, if $k > -\frac{\log \alpha}{2 \log \lambda_0}$ then we must have $f^k(x_0) \in D^u_\alpha(\sigma)$. The same argument applied to $f^{-1}$ shows that if $k > -\frac{\log \alpha}{2 \log \lambda_0}$, then $f^{-k}(x_0) \in D^s_\alpha(\sigma)$. This shows that

$$|T(x)| \leq -\frac{\log \alpha}{\log \lambda_0},$$

concluding the proof of the lemma. \hfill \Box

Now let us look at this lemma from another perspective. Write for $i = s, u$,

$$\mathcal{L}^i(\sigma) = \{ L \in G^1 : \beta(L) = \sigma, L \text{ is parallel to } E^i \}.$$ 

then $\mathcal{L}^i$ are invariant under $\Phi_{1/|\beta^{-1}(\sigma)|}$ (note that this is the tangent flow on $G^1$). Furthermore, it is easy to see that $L^s$ is a repeller while $L^u$ is an attractor.

Next we take a sequence of points $\{x_i\} \subset U$ with $x_i \to \sigma$ as $i \to \infty$. To simply notation, we will write $t_i^\pm = t^\pm(x_i)$. Note that $t_i^\pm \uparrow +\infty$. For every $\varepsilon > 0$ small enough, the time that the orbit segments $\phi_{(-t^-_i, t^+_i)}(x_i)$ spend in the region $U \setminus B_\varepsilon(\sigma)$ is uniformly bounded in $i$. As a result, the empirical measures supported on these orbit segments behaves trivially:

$$\nu_i = \frac{1}{t_i^- + t_i^+} \int_{-t_i^-}^{t_i^+} \delta_{\phi_s(x_i)} \overset{i \to \infty}{\xrightarrow{\ast}} \delta_\sigma. \quad (17)$$

On the other hand, the map $\zeta : \text{Reg}(X) \to G^1$ defined earlier lifts any measure $\mu$ on $M$ with $\mu(\text{Sing}) = 0$ to a measure $\zeta_\ast(\mu)$ on $G^1$. Now consider the lift of the empirical measures:

$$\tilde{\nu}_i = \zeta_\ast(\nu_i). \quad (18)$$

If we take any weak*-limit $\tilde{\mu}$ of $\{\tilde{\nu}_i\}$, $\tilde{\mu}$ must be invariant under $\Phi_t$ and is supported on $L^s \cup L^u \subset \beta^{-1}(\sigma)$ since $L^s \cup L^u$ is the non-wandering set of $\Phi_{1/|\beta^{-1}(\sigma)|}$. Write $U^*_\alpha = \zeta(D^s_\alpha(\sigma))$ for $* = s, u$. Observe that by Lemma 5.1, $U^*_\alpha$ each contains a neighborhood of $L^s(\sigma)$ in $G^1$, $* = s, u$. Furthermore, we have $U^*_\alpha \cap U^*_\beta = \emptyset$. Combine this with Lemma 5.2 we obtain the following lemma:
Lemma 5.3. For all $\alpha > 0$ small, we have $\tilde{v}_i(U^t_{\alpha} \cup U^t_{\alpha}) \rightarrow 1$ as $i \rightarrow N$.

The next lemma states that the time that orbit segments $\phi_{(-t^+_i,t^-_i)}(x_i)$ spend in $D^s_{\alpha}(\sigma)$ and $D^u_{\alpha}(\sigma)$ are comparable:

Lemma 5.4. There is $a > 0$ independent of $\alpha$, such that for every sequence $\{x_i\} \subset U$ with $x_i \rightarrow \sigma$ and every weak*-limit $\tilde{\mu}$ of the empirical measure $\tilde{v}_i$ defined using (17) and (18), we have

$$\tilde{\mu}(U^t_{\alpha}) > a \text{ and } \tilde{\mu}(U^{t^*}_{\alpha}) > a.$$  

Proof. First, note that since the vector field is $C^1$, the flow speed is a Lipschitz function of $d(x,\sigma)$: there is $0 < C_1 < C_2$ such that

$$\frac{\|X(x)\|}{d(x,\sigma)} \in (C_1, C_2).$$

To simplify notation, we write

$$x_{e,i} = \phi_{-t^-_i}(x_i), \text{ and } x_{l,i} = \phi_{t^+_i}(x_i)$$

for the end points of $\phi_{(-t^-_i,t^+_i)}(x_i)$ that enter and leave the neighborhood $U$. By our construction, $x_{e,i}, x_{l,i} \in \partial U = \partial B_r(x)$. As a result, the ratio between $\|X(x_{e,i})\|$ and $\|X(x_{l,i})\|$ is in the interval $(\frac{C_1}{2}, \frac{C_2}{2})$ for every $i \in N$.

Denote by $t^*_i \in (-t^-_i, t^+_i)$ the time such that $x_i^1 = \phi_{t^*_i}(x_i)$ satisfies $(x_i^0)^* = (x_i^0)^u$.

We parse each orbit segment $\phi_{(-t^-_i, t^+_i)}(x_i)$ into three sub-segments:

- write $x_i^- = \phi_{t^*_i}(x_i)$ for the point on $\phi_{(-t^-_i, t^+_i)}(x_i)$ that is on the boundary of $D^s_{\alpha}(\sigma)$; then the orbit from $x_{e,i}$ to $x_i^-$ is contained in $D^s_{\alpha}(\sigma)$;

- write $x_i^+ = \phi_{t^*_i}(x_i)$ for the point on $\phi_{(-t^-_i, t^+_i)}(x_i)$ that is on the boundary of $D^u_{\alpha}(\sigma)$; then the orbit from $x_i^+$ to $x_{l,i}$ is contained in $D^u_{\alpha}(\sigma)$;

- the orbit segment from $x_i^-$ to $x_i^+$ is outside $D^s_{\alpha}(\sigma)$, $* = s, u$; by Lemma 5.2 $t^*_i - t^+_i \leq T^u$.

Note that $x_i^0$ is contained in the orbit segment from $x_i^-$ to $x_i^+$. Since the flow time from $x_i^0$ to $x_i^\pm$ is bounded and the flow is $C^1$, we obtain that

$$\frac{\|X(x_i^+)\|}{\|X(x_i^-)\|}\text{ is bounded from above and away from zero.}$$

For the orbit segment from $x_{e,i}$ to $x_i^-$, Lemma 5.1 shows that $X(x) \in C_{L_0}(E^s)$ for each $x$ in this orbit segment. Since the flow speed is uniformly exponentially contracting in $C_{L_0}(E^s)$ provided that $\alpha$ and $r$ are small enough, we see that the time length of this orbit segment satisfies

$$t^*_i + t^-_i = O(\log \frac{\|X(x_{e,i})\|}{\|X(x_i^-)\|}).$$

Similarly,

$$t^+_i - t^u_i = O(\log \frac{\|X(x_{l,i})\|}{\|X(x_i^+)\|}).$$

Then the ratio is

$$\frac{t^*_i + t^-_i}{t^+_i - t^u_i} = O\left(\frac{\log \|X(x_{e,i})\| - \log \|X(x_i^-)\|}{\log \|X(x_{l,i})\| - \log \|X(x_i^+)\|}\right) = O\left(\frac{\log \|X(x_i^-)\|}{\log \|X(x_i^+)\|}\right) = O(1),$$

where in the last equality we use the elementary fact that if $a_i \rightarrow 0, b_i \rightarrow 0$ such that $a_i/b_i$ is bounded from above and away from zero, then $\log a_i/\log b_i \rightarrow 1$.

Finally, note that even though the ratio $\frac{\|X(x_i^+)\|}{\|X(x_i^-)\|}$ depends on $\alpha$, $\frac{t^*_i + t^-_i}{t^+_i - t^u_i}$ only depends on the exponential contracting/expanding rate in $C_{L_0}(E^s)$, which can be made uniform for $\alpha$ small enough. This finishes the proof of the lemma. $\square$
We conclude this subsection with the following lemma, which will be used later
to create transverse intersection between the unstable manifold of a periodic orbit
and the stable manifold of the singularity $\sigma$. Recall that $s$ is the stable index of $\sigma$

**Lemma 5.5.** For each $\beta > 0$ small and $\delta > 0$, there is $\alpha > 0$ such that for every
point $x \in D^s_0(\sigma)$, let $W(x)$ be a $(\dim M - s)$-dimensional submanifold that contains
$x$ and is tangent to $C_\beta(E^u)$. If $\text{diam } W(x) > \delta \|X(x)\|$, then $W(x) \cap W^s(\sigma) \neq \emptyset$.

**Proof.** Since the flow speed at $x$ is a Lipschitz function of $d(x, \sigma)$, we see that
\[
\text{diam } W(x) > C\delta d(x, \sigma) > CC'\delta x^s
\]
for some $C, C' > 0$.

On the other hand, since $W(x)$ is tangent to the $\beta$-cone of $E^u$, there is $C'' > 0$ such that if $\text{diam } W(x) > C'' x^u = C'' d(x, W^s(\sigma))$, we must have $W(x) \cap W^s(\sigma) \neq \emptyset$.

Since $x^u < \alpha x^s$ in the cone $D^s_0$, the choice of $\alpha < CC'/C''$ guarantees that
\[
\text{diam } W(x) > CC'\delta x^s > C'' x^u.
\]
This concludes the proof of the lemma. \hfill \Box

5.3.2. Near Lorenz-like singularities. Now we turn our attention to Lorenz-like singularities. Assume that $\sigma$ is a Lorenz-like singularity contained in a non-trivial chain recurrent class $C = C(\sigma)$. The discussion below applies to reverse Lorenz-like singularities if one considers the flow $-X$.

Let $E^s_\sigma \oplus E^u_\sigma$ be the hyperbolic splitting on $T_xM$. We will write $E^s_\sigma$ the subspace of $T_xM$ corresponding to the exponents $\lambda_1, \ldots, \lambda_{s-1}$ and $E^u_\sigma$ the subspace of $T_xM$ corresponding to the exponent $\lambda_s$. Then $E^s_\sigma \oplus E^u_\sigma = E^\sigma_x$.

The discussion in the previous sub-section applies to $\sigma$ without any modification (note that this time, we change the notation of $E^s$ to $E^c$ and $L^s$ to $L^c$). Furthermore, we can think of $\sigma$ to be the origin in $\mathbb{R}^n$ with three bundles $E^c_\sigma, E^s_\sigma$ and $E^u_\sigma$ perpendicular to one another (which is possible if one changes the metric). These bundles can be naturally extended to $U = B_r(x)$ as before. As a result, the cone field $C_\sigma(E^\sigma)$ can be defined.

As discussed before, we have $W^{cs}(\sigma) \cap C(\sigma) = \emptyset$. Furthermore, it is shown in [11] and [17] that if the orbit segment is taken inside $C(\sigma)$ (or if the orbit segment belongs to a periodic orbit), then it can only approach the singularity $\sigma$ along the one-dimensional subspace $E^c_\sigma$ in the following sense: write
\[
L^c = \{ L \in G^1 : \beta(L) = \sigma, L \text{ is parallel to } E^c \},
\]
then $L^c \subset L^c$ consists of a single point in $G^1$. If we take $x_i \in C(\sigma)$ with $x_i \to \sigma$ and define the empirical measure $\nu_i$ and its lift $\tilde{\nu}_i$ according to [17] and [18], then any weak*-limit $\tilde{\mu}$ of $\{\tilde{\nu}_i\}$ must satisfy
\[
\text{supp } \tilde{\mu} = L^c \cup L^u.
\]

**Lemma 5.6.** Let $\{x_i\} \subset C(\sigma)$, then the conclusion of Lemma 5.3 and Lemma 5.4
remain true with $U^c(\alpha)$ replaced by a neighborhood $U^c(\alpha)$ of $L^c$ in $G^1$. The same
can be said if we take $\{p_i\}$ to be periodic points with $p_i \to \sigma$ but not necessarily in
$C(\sigma)$.

The proof remains unchanged and is thus omitted.

Now let us describe the hyperbolicity of the periodic orbits close to $\sigma$. Recall that for a regular point $x$, $N_x \subset T_xM$ is the orthogonal complement of the flow direction $X(x)$. Since $X$ is a star vector field, every periodic orbit is hyperbolic. It is proven in [11] Theorem 3.7 if $p_n$ is a sequence of periodic points near $C(\sigma)$ whose orbits get arbitrarily close to $\sigma$, then the dominated splitting on $\text{Orb}(p_n)$ extends to a dominated splitting on $\sigma$, which coincides with the splitting $E^s_\sigma \oplus E^u_\sigma$ where $E^s_\sigma = E^s_\sigma \oplus E^s_\sigma$. Since we assume that $E^{cs}$ is perpendicular to $E^c$, and
the flow direction is tangent to the cone $C_{L_{\alpha}}(E^c)$ as the orbit approaches the singularity $\sigma$, it follows that for $n$ large enough, the local stable manifold $W^s(p_n')$ for $p_n' \in \text{Orb}(p_n) \cap D^u_{\alpha}(\sigma)$, is tangent to a cone of $E^{ss}$. 

It is tempting to argue that for $n$ large, $W^s(p_n')$ must intersect transversally with $W^{cs}(\sigma)$. However, this is not necessarily the case: as $p$ gets closer to the singularity $\sigma$, the size of the invariant manifolds of $p$ will shrink. As a result, even if we have a sequence of periodic points $p_n \to p \in W^{cs}(\sigma)$, there is still no guarantee that the unstable manifold of $p_n$ will intersect with $W^{cs}(\sigma)$. To solve this issue, we use the hyperbolic times on $p_n$ as defined in Definition [12]. But before that, let us first estimate the hyperbolicity of the orbit segment inside $D^u_{\alpha}$ and $D^u_{\alpha}$.

Lemma 5.7. Let $\sigma$ be a Lorenz-like singularity, then for $\delta$ small enough, for every $x \in B_\delta(\sigma) \cap C(\sigma)$, the scaled linear Poincaré flow $\psi^*_t|_{E^s_c}$ is uniformly contracting along the orbit segment $\phi_{[0,T]}(x) \subset B_\delta(\sigma)$, where $E^{ss}_x$ is the stable subspace in $N_x$ for the scaled linear Poincaré flow. The same can be said for every periodic orbit close to $C(\sigma)$ but not necessarily contained in it.

Proof. We only need to estimate

$$\psi^*_t(v) = \frac{\|X(x)\|}{\|X(\phi_t(x))\|} \psi_t|_{E^s_c}(v),$$

for $v \in E^{ss}_x$.

We take $\varepsilon > 0$ small enough such that $\lambda_{-1} + 2\varepsilon < \lambda_s - \varepsilon < 0$ (recall that $\lambda^* < 0$ is the largest negative exponent of $\sigma$). If we take $\delta > 0$ small enough, then inside $B_\delta(\sigma)$ we have

$$\|\psi_t|_{E^s_c}(v)\| \leq e^{(\lambda_{-1} + \varepsilon)t} \|v\|.$$ 

On the other hand, for $\frac{\|X(x)\|}{\|X(\phi_t(x))\|}$, we have (in the worst case scenario, where the flow direction is tangent to the $E^c$ cone):

$$\|X(\phi_t(x))\| \geq e^{\lambda_s - \varepsilon} \|X(x)\|.$$ 

Indeed the flow speed is expanding while in the $D^u_{\alpha}$, while the orbit is neither in $D^u_{\alpha}$ nor in the $D^u_{\alpha}$, we lose all the estimate. However, the length of such orbit segment is uniformly bounded due to Lemma 5.2 and can be safely ignored.

As a result, we get

$$\|\psi^*_t(v)\| \leq e^{(\lambda_{-1} + \varepsilon - \lambda_s + \varepsilon)t} \|v\| \leq e^{-\varepsilon t} \|v\|,$$

and conclude the proof of the lemma. \(\square\)

On the unstable subspace $E^u_x \subset N_x$, the situation is different: when the orbit of $x$ exits a small neighborhood of $\sigma$, it can only do so along $D^u_{\alpha}(\sigma)$. As a result, one loses the hyperbolicity along the $E^c$ direction. On the other hand, when the orbit enters the neighborhood of $\sigma$, the orbit segment will be ‘good’ as long as the flow direction is tangent to the $E^c$ cone. This is summarized in the next lemma:

Lemma 5.8. Let $\sigma$ be a Lorenz-like singularity, then there exists $\lambda > 1, T_0 > 0, \alpha > 0$ and $\delta > 0$, such that if $x$ is a periodic orbit such that the orbit segment $\phi_{[0,T]}(x)$ is contained in $B_\delta(\sigma) \cap D^u_{\alpha}(\sigma)$, then the orbit segment $\phi_{[-T,0]}(\phi_T(x))$ is $(\lambda, T_0)$-backward contracting. If the orbit segment is in $D^u_{\alpha}(\sigma)$, then it does not have any sub-segment that is backward contracting.

Proof. 1. In $D^u_{\alpha}(\sigma)$.

To simplify notation we write $y = \phi_T(x)$. Take any $t \in [0, T]$, we will estimate

$$\psi^*_{\sigma}(v) = \frac{\|X(y)\|}{\|X(\phi_{-t}(y))\|} \psi_{-t}(v),$$

for $v \in E^u_y$. 

\[\]
Here recall that $\lambda_{s+1}$ is the smallest positive exponent of $\sigma$. Like in the previous lemma, we take $\varepsilon'>0$ small such that $\lambda_s + \varepsilon' < 0$, $\lambda_{s+1} - \varepsilon' > 0$, therefore $-\lambda_{s+1} + \lambda_s + 2\varepsilon' < 0$.

Then we take $\delta$ small enough such that
\[
\|\psi_{-t}(v)\| \leq e^{-(\lambda_{s+1} + \varepsilon')t}\|v\|,
\]
and
\[
\|X(y)\| \leq e^{(\lambda_s + \varepsilon')t}\|X(\phi_{-t}(y))\|
\]
since the orbit segment is in $D^c_n(\sigma)$. This gives
\[
\|\psi_{-t}^{*}(v)\| \leq e^{(-\lambda_{s+1} + \varepsilon + \lambda_s + \varepsilon')t}\|v\| = e^{(-\lambda_{s+1} + \lambda_s + 2\varepsilon')t}\|v\|,
\]
which shows that the orbit segment $\phi_{[-T,0]}(y)$ is backward contracting.

2. In $D^u_n(\sigma)$. We take any orbit segment $\phi_{[T,0]}(x)$ in $D^u_n(\sigma)$. Since the flow direction is almost parallel to $E^u$ if we take a small enough, we can take $v \in E^u_n$ such that $v$ is almost parallel to $E^c(\sigma)$. For such $v$ we have
\[
\|\psi_{-t}(v)\| \geq e^{-(\lambda_s + \varepsilon)t}\|v\|,
\]
and the flow direction satisfies
\[
\|X(\phi_{-t}(x))\| \leq e^{(-\lambda_{s+1} + \varepsilon)t}\|X(x)\|.
\]
This shows that
\[
\|\psi_{-t}^{*}(v)\| \geq e^{(-\lambda_{s+1} + \varepsilon + \lambda_s + \varepsilon')t}\|v\| = e^{(-\lambda_{s+1} + \lambda_s + 2\varepsilon')t}\|v\|.
\]
For $\varepsilon$ small enough, $-\lambda_s + \lambda_{s+1} - 2\varepsilon > 0$. As a result, $\phi_{-t}^{*}$ will never be contracting as long as the orbit segment is contained in $D^u_n(\sigma)$. 

Using the flow $-X$, we obtain a similar result for reverse Lorenz-like singularities.

Next, we introduce the main lemma in this section, which enables us to show that a periodic orbit $p$ is homoclinically related to a singularity.

Lemma 5.9. Let $\sigma$ be a Lorenz-like singularity, and $\{p_n\}$ a sequence of periodic points with $p_n \to \sigma$. For $\lambda \in (0,1), T_0 > 0$, assume that the set
\[
H_n = \{ t \in (-t^-_n, t^+_n) : \phi_t(p_n) \text{ is a } (\lambda, T_0)\text{-backward hyperbolic time.}\}
\]
has positive density: there exists $\nu > 0$ such that for every $n$, $\nu_n(\{\phi_t(p_n) : t \in H_n\}) > \nu > 0$,
where $t^+_n$ and $t^-_n$ are taken according to (17). Then there exists $N > 0$ such that
\[
W^u(\text{Orb}(p_n)) \cap W^{cs}(\sigma) \neq \emptyset, \text{ for all } n > N.
\]

Proof. First, note that the previous lemma remains true with the same $\lambda \in (0,1)$, when $\alpha > 0$ is replaced by $\alpha' < \alpha$.

According to the classic work of Liao [20], there is $\delta > 0$ such that for every $n$ and $t \in H_n$, $\phi_t(p_n)$ has unstable manifold $W^u(\phi_t(p_n))$ with size $\delta\|X(\phi_t(p_n))\|$, tangent to $C^\beta(E^u_n)$ for some $\beta > 0$ (in fact, tangent to $C^\beta(E^u_n)$ where $E^u_n$ is the unstable subspace in $N_n$); however, we may assume that $E^u_\sigma$ and $E^u_n$ are almost parallel as long as the flow orbit remains in the cone $D^c_0(\sigma)$. Let $\alpha' < \alpha$ be the size of the cone given by Lemma 5.1. It remains to show that $H_n \cap D^c_0(\sigma) \neq \emptyset$.

To this end, we parse the orbit segment $\phi_{(-t^-_n, t^+_n)}(p_n)$ into three consecutive parts like in the proof of Lemma 5.3:
\[
(-t^-_n, t^+_n) = (-t^-_n, t^+_n) \cup (t^-_n, t^+_n) \cup (t^-_n, t^+_n),
\]
such that:
- $t^-_n$ is the first time in $(-t^-_n, t^+_n)$ such that $\phi_{t^-_n}(p_n) \notin D^c_0(\sigma)$;
• $t^+_n$ is the first time in $(-t^-_n, t^+_n)$ such that $\phi_{t^+_n}(p_n) \in D^s_{\alpha}(\sigma)$.

In other words, the orbit segment in $(t^-_n, t^+_n)$ is ‘making the turn’. Then according to Lemma 5.2, $t^+_n - t^-_n$ is uniformly bounded by $T^{\alpha'}$. Therefore, we can take $n$ large enough such that

$$\nu_n(\{\phi_t(p_n) : t \in (t^-_n, t^+_n)\}) < \frac{a}{2}.$$  

On the other hand, Lemma 5.8 states that for every $t \in (t^+_n, t^-_n)$, $\phi_t(p_n)$ cannot be a backward hyperbolic time, since any sub-segment contained in $\phi_{(t^-_n, t^+_n)}(p_n)$ cannot be backward contracting. As a result, we have $H_n \cap (t^-_n, t^+_n) = \emptyset$. It then follows that

$$\nu_n(\{\phi_t(p_n) : t \in H_n \cap (t^-_n, t^+_n)\}) = \frac{a}{2}.$$  

In other words, there is a backward hyperbolic time $\phi_t(p_n)$ contained in $D^s_{\alpha}(\sigma)$. The proof of the lemma is finished. \hfill \Box

5.4. **Proof of Theorem F.** Let $C$ be a non-trivial chain recurrent class. If $C$ contains no singularity, then it must by hyperbolic ([13]); this leads to the first case of Theorem F. So we will assume that $C$ contains some singularity. In view of the discussion above, we consider two cases:

Case 1. All singularities in $C$ have the same index. Then they must be of the same type: either they are all Lorenz-like or all reverse Lorenz-like. By [11, Theorem 3.7], $C$ is sectional hyperbolic for $X$ or $-X$. We may assume that $C$ is sectional hyperbolic for $X$.

When $h_{\text{top}}(\phi|C) > 0$, Theorem F guarantees that there are periodic orbits $p$ arbitrarily close to $C$. Indeed, Lemma 4.5 shows that $p \in C$. Then the same proof as Corollary E shows that $C$ is isolated.

Case 2. $C$ contains singularities with different indices. Let $\sigma^+ \in C$ be a Lorenz-like singularity, and $\sigma^- \in C$ be reverse Lorenz-like. We have

$$\text{Ind}(\sigma^+) - 1 = \text{Ind}(\sigma^-) = \text{Ind}_p,$$

where $\text{Ind}_p$ is the index of the periodic orbits sufficiently close to $C$. Our strategy is similar to Case 1: first we shows that $C$ contains a periodic orbit $p$. By Proposition 1.8, $C$ is a homoclinic class. It then follows that $h_{\text{top}}(\phi_t|C) > 0$. Next, we prove that if $p'$ is a periodic point sufficiently close to $C$, then $p' \in C(\sigma)$. Then the same argument used in the proof of Corollary E shows that $C$ is isolated.

**Lemma 5.10.** $C$ contains a periodic point $p$.  

Proof. By \[11\], Lemma 2.1 which is originally due to Liao, there is $\lambda \in (0, 1)$, $T > 0$ such that for every periodic orbit $\gamma$ of $X$ with periodic $\pi(\gamma)$ longer than $T$, we have
\[
\frac{\pi(\gamma)}{T} \prod_{i=0}^{\lfloor \pi(\gamma)/T \rfloor - 1} \| \psi_T |_{N^s(\phi_t(x))} \| \leq \lambda^{\pi(\gamma)},
\]
and a similar estimate holds on $N^u$. Here $N^s \oplus N^u$ is the hyperbolic splitting on $N_x$ for the linear Poincaré flow $\psi_t$. In particular, the same proof as Lemma 1.4 shows that there are $(\lambda, T_0)$-backward hyperbolic times for $N^u$ along the orbit of $\gamma$. Moreover, such points have positive density along the orbit of $\gamma$ due to the Pliss Lemma.

Fix $\varepsilon > 0$ small enough. For each $n > 0$, consider the following property:

(P): there is a periodic orbit $p_n$ such that:

- the time that $\text{Orb}(p_n)$ spend inside $B_\varepsilon(\sigma^+)$ is at least $(1/2 - 1/n)\pi(p_n)$;
- the time that $\text{Orb}(p_n)$ spend inside $B_\varepsilon(\sigma^-)$ is at least $(1/2 - 1/n)\pi(p_n)$;
- the time that $\text{Orb}(p_n)$ spend outside $B_\varepsilon(\sigma^-) \cup B_\varepsilon(\sigma^+)$ is at most $1/n \pi(p_n)$.

Clearly this is an open property. On the other hand, note that $W^{cs}(\sigma^+)$ and $W^{cu}(\sigma^-)$ must have transverse intersection due to the Kupka-Smale theorem. Using the connecting lemma, we can create an intersection between $W^u(\sigma^+)$ and $W^s(\sigma^-)$, which gives a loop between $\sigma^+$ and $\sigma^-$. Then standard perturbation technique will allow one to create periodic orbits that satisfy the conditions above. Therefore the following property is generic:

(P'): there are periodic orbits $p_n$ arbitrarily close to both $\sigma^\pm$, such that the previous requirements on the time $\text{Orb}(p_n)$ spend inside and outside $B_\varepsilon(\sigma^\pm)$ hold.

As a result, passing to the generic subset where property (P') holds, we may assume that $X$ itself has periodic orbits $p_n$ satisfying the conditions above. We will show that for $n$ large enough, $W^s(\text{Orb}(p_n))$ has transverse intersection with $W^{cs}(\sigma^+)$. The same argument applied to the flow $-X$ shows the intersection between $W^s(\text{Orb}(p_n))$ and $W^{cu}(\sigma^-)$, thus $p_n$ is in $C$ for $n$ large enough.

Change to another point in $\text{Orb}(p_n)$ if necessarily, we may assume that $p_n \rightarrow \sigma^+$ such that $p_n$ lies on the boundary of $D_{\alpha'}^{cs}(\sigma^+)$, where $\alpha'$ is given in the proof of Lemma 5.9. We will show that $W^u(p_n)$ intersects transversally with $W^{cs}(\sigma)$ by proving that $p_n$ is a backward hyperbolic time. For this purpose, denote by $p_n^{\sigma^+} = \phi_{t_n}^\sigma(p_n)$.
where \( t_1^n < 0 \) is the largest real number such that \( p_n^{\sigma^+} \in B_x(\sigma^+) \), and 
\[
p_n^{\sigma^-} = \phi_n\varepsilon(p_n)
\]
where \( t_2^n = \sup\{t < t_1^n : \phi_t(p_n) \in B_x(\sigma^-)\} \). It follows from the conditions on \( p_n \) that \( t_1^n - t_2^n < \frac{1}{2}\pi(p_n) \). By the Pliss Lemma, The set of \((\lambda, T_0)\)-backward hyperbolic time \( p_n' \) on the orbit of \( p_n \) have density \( a > 0 \). Thus for \( n \) large enough, there must be hyperbolic times on the orbit segment inside \( B_x(\sigma^+) \).

If \( p_n' \) is a backward hyperbolic time contained in the orbit segment from \( p_n^{\sigma^+} \) to \( p_n \), note that the finite orbit segment from \( p_n^{\sigma^+} \) to \( p_n \) is contained in \( D^{\ast\ast}_n(\sigma^+) \), thus must be backward contracting due to Lemma 5.8. It then follows that \( p_n \) is a backward hyperbolic time.

If the hyperbolic times \( p_n' \) are contained in the orbit segment prior to \( p_n^{\sigma^-} \), i.e., inside \( B_x(\sigma^-) \), first note that thanks to Lemma 5.7 for the flow \(-X\), along the orbit segment from \( p_n' \) to \( p_n^{\sigma^-} \), the \( E^c_x \) bundle is uniformly expanding by the scaled linear Poincaré flow. This shows that \( p_n^{\sigma^-} \) itself must be a backward hyperbolic time.

Next, by Lemma 5.8 the orbit segment from \( p_n^{\sigma^+} \) to \( p_n \) is backward contracting, and the orbit segment from \( p_n^{\sigma^-} \) to \( p_n^{\sigma^+} \) has very small length comparing to the former. As a result, \( p_n \) is a backward hyperbolic time.

It then follow from both cases, that \( p_n \) is a backward hyperbolic time. As a result, \( p_n \) has unstable manifold with size \( \delta\|X(p_n)\| \). According to the choice of \( \alpha' \), such unstable manifold must intersect transversally with \( W^{ca}(\sigma^+) \).

The same argument applied to the flow \(-X\) shows that the stable manifold of \( \text{Orb}(p_n) \) intersects transversally with the unstable manifold of \( \sigma^+ \). It then follows that \( \text{Orb}(p_n) \) is contained in the chain recurrent class of \( \sigma^+ \), concluding the proof of this lemma.

This finishes the first part of the proof. It remains to show that \( C \) is isolated, which follows from the next lemma and the argument used in the proof of Corollary 1.

**Lemma 5.11.** There exists a neighborhood \( U \) of \( C \), such that every periodic orbit in \( U \) is indeed contained in \( C \).

**Proof.** We prove by contradiction. Assume that there is a sequence of periodic orbits \( \text{Orb}(p_n) \subset U_n \) with \( \cap_n U_n = C \), such that \( \text{Orb}(p_n) \) are not contained in \( C \). It is easy to see that the period of \( p_n \) must tend to infinity. Below we will show that the unstable manifold of \( \text{Orb}(p_n) \) intersect transversally with the stable manifold of some point in \( C \). Then the same argument applied to \(-X\) shows the stable manifold of \( \text{Orb}(p_n) \) intersect transversally with the unstable manifold of some point in \( C \), which means that \( p_n \notin C \).

By Theorem 5.7, the index of \( p_n \) coincides with \( \text{Ind}_p \) for all \( n \) large enough.

By Lemma 2.1, there exists \( \lambda \in (0, 1), T_0 > 0 \) such that \( \text{Orb}(p_n) \) contains \((\lambda, T_0)\)-backward hyperbolic times \( x_n \). Moreover, the collection of such points \( \Lambda_n = \{x_n \in \text{Orb}(p_n) : x_n \text{ is a backward hyperbolic time}\} \) have positive density in \( \text{Orb}(p_n) \) (independent of \( n \)) with respect to the empirical measure on \( \text{Orb}(p_n) \), thanks to the Pliss lemma.

Taking subsequence if necessary, We write \( \bar{C} \subset C \) the Hausdorff limit of \( \text{Orb}(p_n) \), and \( \Lambda \subset \bar{C} \) the Hausdorff limit of \( \Lambda_n \). We may also assume that the empirical measure \( \mu_n \) on \( \text{Orb}(p_n) \) converges to an invariant measure \( \mu \) supported on \( \bar{C} \). The backward hyperbolic times having uniform positive density implies that \( \mu(\Lambda) > 0 \).

**Case 1.** There is an ergodic component of \( \mu \), denote by \( \mu_1 \), with \( \mu_1(\Lambda) > 0 \) and \( \mu_1(\text{Sing}) = 0 \).
Then \( \mu_1 \) must be a non-trivial hyperbolic measure, thanks to [11, Theorem 5.6]. The same argument used in Lemma 4.5(f) shows that for \( n \) large enough, \( W^u(\text{Orb}(p_n)) \) has transverse intersection with the stable manifold of points in \( \text{supp} \tilde{\mu} \). Roughly speaking, every regular point in \( \Lambda \cap \text{supp} \tilde{\mu} \) have stable manifold, which must intersect transversely with the unstable manifold of the backward hyperbolic times \( x_n \in \Lambda_n \) (recall that such points have unstable manifold up to the flow speed; if we take them uniformly away from all singularities, then their unstable manifolds have uniform size).

**Case 2.** Every ergodic component \( \mu_1 \) of \( \mu \) with \( \mu_1(\Lambda) > 0 \) is supported on some singularity \( \sigma \).

**Subcase 1.** One of those \( \sigma \) is reverse Lorenz-like.

We use the argument in the proof of Corollary [2] Write \( \phi_t^{-X} \) for the time \( t \) map of the flow \(-X\). Note that \( \sigma \) is Lorenz-like, and points in \( \Lambda_n \) are forward hyperbolic times for the flow \(-X\), with stable manifold up to the flow speed.

Take \( q_n \in \Lambda_n \) forward hyperbolic times with \( q_n \to \sigma \), and \( t_n \downarrow -\infty \) such that \( z_n = \phi_{t_n}^{-X}(q_n) \in \partial B_\varepsilon(\sigma) \) for some fixed \( \varepsilon \) small enough.

Since \( \sigma \) is Lorenz-like for \(-X\), Lemma 5.7 shows that the orbit segment from \( z_n \) to \( q_n \) is forward contracting. As a result, \( z_n \) are also forward hyperbolic times for the flow \(-X\). Since \( z_n \) are uniformly away from \( \sigma \), \( W^u(z_n) \) has uniform size.

Recall that Lemma 5.8 shows that \( C \) contains a periodic orbit \( p \) and is a homoclinic class. Then the argument at the end of Corollary [2] shows that \( W^u(z_n) \) intersect transversally with \( W^u(p') \) for some \( p' \in \text{Orb}(p) \). Revert back to the flow \( X \), we see that \( W^u(z_n) \) intersect transversally with \( W^u(p') \).

**Subcase 2.** Every ergodic component \( \mu_1 \) of \( \mu \) with \( \mu_1(\Lambda) > 0 \) is supported on some Lorenz-like singularity \( \sigma \).

Note that \( \mu \) can be written as a combination of the following types of measures:

- non-trivial measures that have zero measure on \( \Lambda \);
- \( \delta_{\sigma^-} \) where \( \sigma^- \) is reverse Lorenz-like; such measure has zero mass on \( \Lambda \);
- \( \delta_{\sigma^+} \) where \( \sigma^+ \) is Lorenz-like.

In particular, the measure \( \mu_n|_{\Lambda_n} \) (recall that \( \mu_n(\Lambda_n) \) is bounded away from zero, since backward hyperbolic times have positive density) must converge to a convex combination of \( \delta_{\sigma^+} \) where \( \sigma^+ \) are the Lorenz-like singularities. Write \( \tilde{\mu}_n = \zeta_1(\mu_n|_{\Lambda_n}) \) the lift to \( G^1 \), then every limit of \( \tilde{\mu}_n \) is a convex combination of \( \delta_{\zeta_1} \).

Note that Lemma 5.8 states that there is no backward hyperbolic time inside \( D^u(\sigma^+) \). As a result, if we put \( \tilde{\Lambda} = \zeta(\Lambda) \) the lift of \( \Lambda \) to \( G^1 \), we must have \( \delta_{\zeta_1}(\tilde{\Lambda}) = 0 \) for every \( i \). It then follows that there exists a Lorenz-like singularity \( \sigma^+ \), such that \( \delta_{\zeta_1} \sigma^+ \tilde{\Lambda} > 0 \). In other words, backward hyperbolic times have positive density inside \( D^u_\ast(\sigma^+) \). By Lemma 5.9 \( W^u(\text{Orb}(p_n)) \) intersect transversally with \( W^u(\sigma^+) \).

The proof is now complete.

\[ \square \]

**Appendix A. On non-hyperbolic singularities**

Here we will demonstrate how to remove the assumption on the hyperbolicity of singularities in Theorem [A]. The theorem that we will prove is:

**Theorem H.** Let \( \Lambda \) be a compact invariant set that is sectional hyperbolic for a \( C^1 \) flow \( \phi_t \). Then there is a neighborhood \( U \) of \( \Lambda \), such that \( \phi_t|_U \) is entropy expansive.

Recall that Theorem [C] was proven using Theorem [A] and Theorem 3.12 where the later does not require any information on the singularity. This allows one to
easily get the following version of Theorem C without assuming the hyperbolicity of singularities:

**Theorem 1.** Let $\Lambda$ be a compact invariant set that is Lyapunov stable and sectional hyperbolic for a $C^1$ flow $\phi_t$. Then $h_{top}(\phi_t|_\Lambda) > 0$.

In order to prove Theorem 1, we use the same argument as in Section 3.1 by showing that all ergodic measures are $\epsilon$-almost entropy expansive. The singularities being hyperbolic or not does not affect the measures that are supported away from singularities. In other words, Proposition 3.8 remains valid.

To deal with measures whose support is close to some singularity, we need to establish the (topological) contracting property near singularities. This is done by a sequence of lemmas. Recall that $\tilde{\Lambda}$ is the maximal invariant set of $\phi_t$ in a small neighborhood $U$ of $\Lambda$ (when choosing $U$, there is no need to have $\text{Sing}(\phi_t|_U) = \text{Sing}(\phi_t|_{\Lambda})$). We refer the reader to the beginning of Section 3.1.5 for the meaning of symbols.

The first lemma is similar to Lemma 3.10. One can easily check that the proof of Lemma 3.10 applies with slight modification.

**Lemma A.1.** $f|_{\text{Sing}(X) \cap \tilde{\Lambda}}$ has a partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$, where $E^c$ is a one-dimensional sub-bundle of $E^\infty$ corresponding to an eigenvalue with norm at most one.

The next lemma describes the infinite Bowen ball at singularities.

**Lemma A.2.** [Theorem 3.1] For any singularity $\sigma \in \tilde{\Lambda}$, $B_\infty(\sigma, r_0/2)$ is a single point, or a 1-dimensional center segment with length bounded by $r_0$. In the first case the singularity $\sigma$ must be isolated. In the second case, the center segment consists of singularities and saddle connections. Moreover, there are only finitely many such singularities and center segments.

The finiteness of such singularities and center segments comes from the fact that every center segment must be contained in $B(\sigma, r_0)$, and there can only be one such segment in each $r_0$-ball.

This lemma allows one to write $\text{Sing}(X) \cap \tilde{\Lambda} \subset \{\sigma_1, \ldots, \sigma_m\} \cup I_1 \cup \cdots \cup I_k$, where $\{\sigma_1, \ldots, \sigma_m\}$ are isolated singularities, and $\{I_1, \ldots, I_k\}$ are center segments. Each $I_i$ is fixed by $f = \phi_1$ and is contained in $B_\infty(\sigma_i, r_0/2)$ for $\sigma_i \in I_i$. We may assume that for $i \neq j$, $I_i$ and $I_j$ intersect (if they intersect at all) at boundary points, which must be a singularity. Taking a double cover if necessary, we can assume that $E^c$ is orientable, which allows us to label the end points of $I_i$ as left extremal point $\sigma_i^-$ and right extremal point $\sigma_i^+$.

Next we describe the dynamics near each center segment. Recall that $F^c_\sigma$ are the fake foliations near $\sigma$, given by the dominated splitting $E^s \oplus E^u$. We denote for every sub-center segment $I \subset I_i$,

$$W^{uu}_\delta(I) = \bigcup_{x \in I} F^c_{\sigma_i}(x, \delta), \quad \mathcal{N}_\delta(I) = \bigcup_{y \in W^{uu}_\delta(I)} F^c_{\sigma_i}(y, \delta).$$

Then $\mathcal{N}_\delta(I)$ is a “box” containing $I$, with size $\delta$.

For each singularity $\sigma$, we can treat define $\mathcal{N}_\delta(\sigma)$ in a similar way. In this case, $\mathcal{N}_\delta(\sigma)$ is a co-dimensional one sub-manifold containing $\sigma$.

The next lemma states that for any non-trivial invariant measure $\mu$, generic point cannot approximate the interior of each segment $I_i$.

**Lemma A.3.** There are $\delta, r_2 > 0$ and finitely many center segments $I_{i,j} \subset I_i$ for $j = 1, \ldots, k_i$ with length($I_{i,j}$) < $r_2$ and $\text{Sing}(X) \cap I_i = \bigcup_{j=1}^{k_i} \text{Sing}(X) \cap I_{i,j}$, such that for any non-trivial invariant, ergodic measure $\mu$, we have $\mu(\mathcal{N}_\delta(I_{i,j})) = 0$ for every $I_{i,j}$. 
Proof. Note that $I_i$’s are normally hyperbolic sub-manifolds. By the stable manifold theorem, for each $x \in I_i$, $F^s_{\sigma_i}(x, r_0)$ is the local strong stable manifold of $x$. Similar result holds for the local strong unstable manifold. Moreover,

$$W^u_{\text{loc}}(I_i) = \bigcup_{x \in I_i} F^s_{\sigma_i}(x, r_0),$$

and the same holds for $W^u_{\text{loc}}(I_i)$.

Now we fix $r_2 > 0$ small enough, and divide $I_i$ into $k_i$ many sub-center segments $\{I_{i,j}\}$, each with length less than $r_2$, such that the boundary points of $I_{i,j}$ are singularities. In particular, each $I_{i,j}$ contains at least two singularities. Note that during this process, we may not have $\cup_{i=1}^{k_i} I_{i,j} = I_i$, especially if $I_i$ contains a saddle connection with length larger than $r_2$.

Suppose that there is a non-trivial ergodic measure $\mu$ and $\delta_n \to 0$ with

$$\mu(N_{\delta_n}(I_{i,j})) > 0.$$

Since $\mu$ is non-trivial, we must have $\mu(W^\ast_{\text{loc}}(I_i)) = \mu(W^u_{\text{loc}}(I_i)) = 0$. This allows us to take

$$x^n \in N_{\delta_n}(I_{i,j}) \setminus (W^\ast_{\text{loc}}(I_i) \cup W^u_{\text{loc}}(I_i)).$$

The negative iteration of $x^n$ must leave $N_{\delta_n}(I_{i,j})$. Take $t_n < 0$ the last time such that $\phi_{t_n}(x^n) \in \partial N_{\delta_n}(I_{i,j})$. Then $t_n \to -\infty$ as $n \to \infty$. We may suppose that

$$\phi_{t_n}(x^n) \to x^\ast \in W^\ast_{\text{loc}}(I_{i,j}) \setminus I_{i,j}.$$

If $r_2$ is taken small enough, then $x^\ast$ is close to the stable manifold of the singularity contained in $I_{i,j}$. By continuity, $X(x^\ast)$ and $X(\phi_{t_n}(x^n))$ are tangent to the $E^\ast$ cone. Now one can apply the standard argument at the end of the proof of Lemma 3.10 to get a contradiction.

For every $0 < \delta_1 < r_2$, if $\sigma$ is an end point of some segment $I_{i,j}$, then $N_{\delta}(\sigma)$ cuts the ball $B_{\delta_1}(\sigma)$ into two components, which we denote by $B^+_\delta(\sigma)$ and $B^-_\delta(\sigma)$. One of the components $B^\pm_\delta(\sigma)$ will intersect with $N_{\delta}(I_{i,j})$, in which case the half ball will have zero measure for any non-trivial ergodic measure $\mu$, according to the previous lemma. In this case we can write the other component (which does not intersect with $N_{\delta}(I_{i,j})$) as $B^\pm_{\delta}(\sigma)$. It is possible that there is another center segment $I_{i,j'}$, which intersect $I_{i,j}$ as $\sigma$. In this case, both components of $B_\delta(\sigma)$ must have zero measure for every non-trivial ergodic measure, due to the previous lemma. If this happens, we do not need to consider any of these two components. Otherwise, there is no singularity inside the half ball $B_{\delta_1}(\sigma)$, according to the construction of $I_{i,j}$.

If $\sigma$ is an isolated singularity, then $N_{\delta}(\sigma)$ cuts the ball $B_{\delta_1}(\sigma)$ into two components. Unlike the previous case, both of these two components may have positive measure for some measure $\mu$. In this case, it is convenient to treat the isolated singularity $\sigma$ as a trivial center segment, and denote the two component of $B_{\delta_1}(\sigma)$ as $B^\pm_{\delta_1}(\sigma)$ respectively.

To summarize, we get a finite subset $A \in \text{Sing}(X) \cap \tilde{A}$ (with each isolated singularity appears twice in $A$) and a collection of half balls $\{B^\pm_{\delta}(\sigma)\}_{\sigma \in A}$, each of which are singularity-free (of course, other than $\sigma$ itself) and may have positive measure for some invariant measure $\mu$. Furthermore, for every ergodic measure $\mu$, typical points of $\mu$ can only approximate a singularity by going through one of these half balls. Note that for $\sigma \in A$, $F^c_{\sigma_i}(\sigma, r_2)$ is also cut by $N_{\delta}(\sigma)$ into two branches, one of which intersects with the half ball $B^h_{\delta_1}(\sigma)$. We will denote

$$F^c_{\sigma_i}h(\sigma, \delta_1) = F^c_{\sigma_i}(\sigma, r_2) \cap B^h_{\delta_1}(\sigma).$$

The next lemma establishes the contracting property along $F^c_{\sigma_i}h(\sigma, r_2)$.
Lemma A.4. There is $\delta_1 > 0$, such that for every $\sigma \in A$, if $\mu(B_{\delta_1}^h(\sigma)) > 0$ for some non-trivial invariant measure $\mu$, then $F_{\sigma_1}^{c,h}(\sigma, \delta_1)$ is topologically contracting. More precisely, for every $x \in F_{\sigma_1}^{c,h}(\sigma, \delta_1)$ we must have $f^n x \to \sigma$ as $n \to +\infty$.

In other words, if a half ball $\mu(B_{\delta_1}^h(\sigma)) > 0$ can be ‘seen’ by some measure $\mu$, then the center direction must be topologically contracting.

Proof. Since $F_{\sigma_1}^{c,h}(\sigma, \delta_1)$ is invariant and contains no singularity, it must be topological contracting or expanding. If it is topological expanding, then $F_{\sigma_1}^{c,h}(\sigma, \delta_1)$ belongs to the unstable set of $\sigma$. We claim that there must be $\delta_2 < \delta_1$ such that $\mu(B_{\delta_2}^h(\sigma)) = 0$ for every non-trivial $\mu$. Since $A$ is a finite set, the lemma follows by shrinking $\delta_1$ a finitely number of times.

It remains to prove this claim. Assume by contradiction that there is a sequence $\delta_n \to 0$ and a measure $\mu$, such that $\mu(B_{\delta_n}^h(\sigma)) > 0$. Similar to the proof of the previous lemma, we can take $x^n \in B_{\delta_n}^h(\sigma) \setminus (W^u(\sigma) \cup F^s(\sigma))$.

Take $t_n < 0$ the last time that $\phi_{t_n}(x^n) \in \partial B_{\delta_n}^h(\sigma)$. Then $t_n \to -\infty$. Since $F_{\sigma_1}^{c,h}(\sigma, \delta_1)$ is topological expanding, we can take $\phi_{t_n}(x^n) \to x^* \in F^s(\sigma)$. The same argument in the proof of Lemma 3.10 will create a contradiction.

Thus far, we have shown that:

- there is a partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$ on the set of singularities;
- there are finitely many singularity-free half balls $\{B_{\delta_n}^h(\sigma)\}_{\sigma \in A}$, such that the orbit of every typical point $x$ (with respect to some non-trivial ergodic measure) can only approximate a singularity by going through these half balls;
- if a half ball $B_{\delta_1}^h(\sigma)$ can be “seen” by a non-trivial measure $\mu$, then the center direction of $\sigma$ must be topologically contracting.

In view of Remark 3.11, Lemma A.4 can be proven using the same argument. One only need to replace the foliation $F_S^{c,h}$ (given by the hyperbolic splitting on $\text{Sing}(X)$) by the fake foliation $F_S^{c,h}$, generated by the partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$ inside a neighborhood of the singularities. Then one can define the one-dimensional center fake foliation $\hat{F}^c$ as the intersection of $F_S^{c,h}$ and $F_S^{c,h}$, which will give a local product structure near the neighborhood of singularities. The rest of the proof of Lemma 3.7 remains unchanged.

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