A NON-VANISHING CRITERION FOR DIRAC COHOMOLOGY

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Abstract. This paper gives a criterion for the non-vanishing of the Dirac cohomology of \( L_S(Z) \), where \( L_S(\cdot) \) is the cohomological induction functor, while the inducing module \( Z \) is irreducible, unitarizable, and in the good range. As an application, we give a formula counting the number of strings in the Dirac series. Using this formula, we classify all the irreducible unitary representations of \( E_{6(2)} \) with non-zero Dirac cohomology. Our calculation continues to support Conjecture 5.7' of Salamanca-Ribas and Vogan [26]. Moreover, we find more unitary representations for which cancellation happens between the even part and the odd part of their Dirac cohomology.

1. Introduction

A formula for the Dirac cohomology of cohomologically induced modules has been given in Theorem B of [7]. However, even if the inducing irreducible unitary module \( Z \) has non-zero Dirac cohomology and lives in the good range, we do not know whether the cohomologically induced module \( L_S(Z) \) has non-zero Dirac cohomology or not. The first aim of this note is to fix this problem.

We start with a complex connected simple linear group \( G(\mathbb{C}) \) which has finite center. Let \( \sigma : G(\mathbb{C}) \to G(\mathbb{C}) \) be a real form of \( G(\mathbb{C}) \). That is, \( \sigma \) is an antiholomorphic Lie group automorphism such that \( \sigma^2 = \text{Id} \). Let \( \theta : G(\mathbb{C}) \to G(\mathbb{C}) \) be the involutive algebraic automorphism of \( G(\mathbb{C}) \) corresponding to \( \sigma \) via Cartan theorem (see Theorem 3.2 of [1]). Put \( G = G(\mathbb{C})^\sigma \) as the group of real points. Note that \( G \) must be in the Harish-Chandra class [11]. That is,

(a) \( G \) has only a finite number of connected components;
(b) the derived group \([G,G]\) has finite center;
(c) the adjoint action \( \text{Ad}(g) \) of any \( g \in G \) is an inner automorphism of \( g = (g_0)_\mathbb{C} \).

Denote by \( K(\mathbb{C}) := G(\mathbb{C})^\theta \), and put \( K := K(\mathbb{C})^\theta \). Denote by \( g_0 = \text{Lie}(G) \), and let

\[ g_0 = t_0 \oplus p_0 \]

be the Cartan decomposition corresponding to \( \theta \) on the Lie algebra level.

Choose a maximal torus \( T_f \) of \( K \). Let \( t_{f,0} = \text{Lie}(T_f) \) and put \( a_{f,0} = Z_{p_0}(t_{f,0}) \). Let \( A_f \) be the corresponding analytic subgroup of \( G \). Then \( H_f = T_f A_f \) is a \( \theta \)-stable fundamental Cartan subgroup of \( G \). As usual, we drop the subscript for the complexification. For instance,

\[ \mathfrak{h}_f = a_f \oplus t_f \]

is the Cartan decomposition of the complexified Lie algebra of \( H_f \).

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We fix a positive root system $\Delta^+(k, t_f)$ once for all. Denote by $\rho_K$ the half sum of roots in $\Delta^+(k, t_f)$. Then there are $s$ ways of choosing positive root systems of $\Delta(g, t_f)$ containing the fixed $\Delta^+(k, t_f)$. Here

$$s = \frac{\#W(g, t_f)}{\#W(k, t_f)},$$

where $W(k, t_f)$ (resp., $W(g, t_f)$) is the Weyl group of the root system $\Delta(k, t_f)$ (resp., $\Delta(g, t_f)$). We will refer to a $K$-type by one of its highest weights.

Recall that any $\theta$-stable parabolic subalgebra $q$ of $g$ can be obtained by choosing an element $H \in i t_{f, 0}$, and setting $q$ as the sum of non-negative eigenspaces of $\text{ad}(H)$. The Levi subalgebra $l$ of $q$ is the sum of zero eigenspaces of $\text{ad}(H)$. Note that $l$ and $q$ are both $\theta$-stable since $H$ is so. Put $L = N_G(q)$. Then $L \cap K$ is a maximal compact subgroup of $L$.

We choose a positive root system $\Delta^+(g, t_f)$ so that $\Delta(u, t_f) \subseteq \Delta^+(g, t_f)$. Let $\Delta^+(g, t_f)$ be the union of the fixed $\Delta^+(k, t_f)$ and $\Delta^+(p, t_f)$. We denote the half sum of roots in $\Delta(u)$ as $\rho(u)$. Put

$$\Delta(u \cap t) = \Delta(u) \cap \Delta^+(g, t_f), \quad \Delta(u \cap p) = \Delta(u) \cap \Delta^+(p, t_f),$$

and denote the half sum of roots in them by $\rho(u \cap t)$ and $\rho(u \cap p)$, respectively. Note that

$$(1) \quad \rho(u) = \rho(u \cap t) + \rho(u \cap p).$$

Put

$$\Delta^+(l, t_f) = \Delta(l, t_f) \cap \Delta^+(g, t_f), \quad \Delta^+(l \cap t, t_f) = \Delta(l, t_f) \cap \Delta^+(k, t_f).$$

Denote the half sum of roots in $\Delta^+(l \cap t, t_f)$ by $\rho_{L \cap K}$.

Cohomological induction functors lifts an $(l, L \cap K)$-module $Z$ to $(g, K)$-modules $\mathcal{L}_j(Z)$ and $\mathcal{R}_j(Z)$, where $j$ are some non-negative integers. The interesting thing usually happens at the middle degree $S := \dim (u \cap t)$. The results that we need about cohomological induction will be summarized in Theorem 2.2.

We will recall Dirac cohomology in Section 2. This notion was introduced by Vogan [30]. Motivated by his conjecture on Dirac cohomology proven by Huang and Pandžić [13], we say that a weight $\Lambda \in h_f^*$ satisfies the Huang-Pandžić condition (HP condition for short henceforth) if

$$(2) \quad \{ \delta - \rho_{\Lambda}^{(j)} \} + \rho_K = w \Lambda,$$

where $\delta$ is any highest weight of some $K$-type, $0 \leq j \leq s - 1$, and $w \in W(g, t_f)$. Note that if $\Lambda$ satisfies the HP condition, then it must be real in the sense of Definition 5.4.1 of [28]. That is, $\Lambda \in i t_{f, 0}^* + a_{f, 0}^*$.

**Theorem 1.1.** Let $G$ be a simple linear real Lie group in the Harish-Chandra class. Let $Z$ be an irreducible unitary $(l, L \cap K)$-module with infinitesimal character $\lambda_L \in i t_{f, 0}^*$ such that $\lambda_L$ is $\Delta^+(l \cap t, t_f)$-dominant. Assume that $\lambda_L + \rho(u)$ is good. That is,

$$\langle \lambda_L + \rho(u), \alpha \rangle > 0, \quad \forall \alpha \in \Delta(u, t_f).$$

Assume moreover that $\lambda_L + \rho(u)$ satisfies the HP condition. Then $H_D(\mathcal{L}_S(Z))$ is non-zero if and only if $H_D(Z)$ is non-zero.
Remark 1.2. (a) If $\lambda_L + \rho(u)$, a representative vector of the infinitesimal character of $L_\mathcal{S}(Z)$, does not satisfy the HP condition, then $H_D(L_\mathcal{S}(Z))$ must be zero in view of Theorem 2.1.

(b) If we further assume $G$ to be connected, then $L$ is connected as well. In this case, the proof will say that $\gamma_L \mapsto \gamma_L + \rho(u \cap p)$ is a multiplicity-preserving bijection from the $K_L$-types of $H_D(Z)$ to the $\tilde{K}$-types of $H_D(L_\mathcal{S}(Z))$. Here $K_L := K \cap L$, and $\tilde{K}_L$ is the pin double covering group of $K_L$. This completely extends Theorem 6.1 of [4] to real linear groups.

(c) Example 4.2 will tell us that the good range condition in the above theorem can not be weakened, say, to be weakly good.

Recall that in [5], a finiteness result has been given on the classification of $\hat{G}^d$—the set of all equivalence classes of irreducible unitary $(g, K)$-modules with non-zero Dirac cohomology. As suggested by Huang, we call the set $\hat{G}^d$ the Dirac series of $G$. Theorem 1.1 allows us to completely determine the number of strings in $\hat{G}^d$, see Section 5. Using this formula, we classify the Dirac series for the group $E_{6(2)}$ as follows.

Theorem 1.3. The set $\hat{E}_{6(2)}^d$ consists of 56 fully supported scattered representations (see Section 9) whose spin lowest $K$-types are all unitarily small, and 576 strings of representations. Each spin-lowest $K$-type of any Dirac series representation of $E_{6(2)}$ occurs with multiplicity one.

In the above theorem, the notion of fully supported scattered representation will be recalled in Section 2.3, that of spin-lowest $K$-type will be recalled in Section 6, and that of unitarily small $K$-type comes from [26]. We sort the statistic $\|\nu\|^2$ for these 56 fully supported scattered representations as follows:

$4.5, 8, (8.5)\underline{2}, 10\underline{2}, (10.5)\underline{4}, 13\underline{2}, 14\underline{2}, 15\underline{2}, 17\underline{4}, (17.5)\underline{4}, 18\underline{2}, 29\underline{2}, 29.5, 30\underline{2}, 42, 78$,

where $a\underline{b}$ means that the value $a$ occurs $k$ times. Note that the original Helgason-Johnson bound (see [14]) $\|\rho(G)\|^2 = 78$ is attained at the trivial representation, while the sharpened Helgason-Johnson bound (see [6]) 42 is attained at the minimal representation, namely, the first entry of Table 8. Numerically, one sees that there is still a remarkable gap between 302 and 42. We will pursue this later.

Among the above 56 fully supported scattered members of $\hat{E}_{6(2)}^d$, cancellation happens within the Dirac cohomology for 10 of them when passing to Dirac index. Recall that Dirac index of $\pi$ is defined as the following virtual $K$ module:

\begin{equation}
\text{DI}(\pi) = H_D^+(\pi) - H_D^- (\pi).
\end{equation}

Here $H_D^+(\pi)$ (resp., $H_D^- (\pi)$) is the even (resp., odd) part of $H_D(\pi)$. See [18].

The first instance of the cancellation phenomenon is recorded in Example 6.3 of [3] on $F_4\_s$, which disproves Conjecture 10.3 of [12]. It is worth noting that $F_4\_s$ is also a quaternionic real form, a notion raised by Wolf [31] in 1961. See also Appendix C of Knapp [15]. Let us sharpen Conjecture 10.3 of [12] to be the following one.

Conjecture 1.4. Further assume that $G$ is equal rank. Let $\pi$ be any irreducible unitary $(g, K)$ module such that $H_D(\pi)$ is non-zero. Then the Dirac index of $\pi$ must vanish if $\text{Hom}_{\tilde{K}}(H_D^+(\pi), H_D^- (\pi)) \neq 0$. 
The above conjecture asserts that there should be dichotomy among the spin LKTs whenever cancellation happens. By the way, our current calculations and those in [2, 3] lead us to make Conjecture 2.6, which asserts that the reverse direction of an old theorem of Vogan (namely, item (iv) of Theorem 2.2) should hold under certain additional restrictions.

The paper is organized as follows: necessary preliminaries will be collected in Section 2, the root system $\Delta(g, t_f)$ will be recalled in Section 3. We deduce Theorem 1.1 in Section 4, and give a formula counting the strings in $\hat{G}^d$ in Section 5. Theorem 1.3 will be proven in Section 6. The cancellation phenomenon will be studied in Section 7. The special unipotent representations of $E_6(2)$ with non-vanishing Dirac cohomology will be discussed in Section 8. All the fully supported scattered members of $\hat{E}_6(2)^d$ will be presented in Section 9 according to their infinitesimal characters.

2. Preliminaries

We continue with the notation in the introduction, and collect necessary preliminaries in this section. Note that $G$ must be linear.

2.1. Dirac cohomology of cohomologically induced modules. We fix a nondegenerate invariant symmetric bilinear form $B$ on $g_0$, which is positive definite on $p_0$ and negative definite on $k_0$. Its extensions/restrictions to $g$, $k_0$, $p_0$, etc., will also be denoted by the same symbol.

Fix an orthonormal basis $Z_1, \ldots, Z_n$ of $p_0$ with respect to the inner product induced by $B$. Let $U(g)$ be the universal enveloping algebra of $g$ and let $C(p)$ be the Clifford algebra of $p$ (with respect to $B$). The Dirac operator $D \in U(g) \otimes C(p)$ is defined by Parthasarathy [20] as

$$D = \sum_{i=1}^{n} Z_i \otimes Z_i.$$  

It is easy to check that $D$ does not depend on the choice of the orthonormal basis $\{Z_i\}_{i=1}^{n}$ and it is $K$-invariant for the diagonal action of $K$ given by adjoint actions on both factors.

Let $\tilde{K}$ be pin covering group of $K$. That is, $\tilde{K}$ is the subgroup of $K \times \text{Pin}(p_0)$ consisting of all pairs $(k, s)$ such that $\text{Ad}(k) = p(s)$, where $\text{Ad}: K \to O(p_0)$ is the adjoint action, and $p: \text{Pin}(p_0) \to O(p_0)$ is the pin double covering map. If $X$ is a $(g, K)$-module, and if Spin$_G$ denotes a spin module for $C(p)$, then $U(g) \otimes C(p)$ acts on $X \otimes \text{Spin}_G$ in the obvious fashion, while $\tilde{K}$ acts on $X$ through $K$ and on $\text{Spin}_G$ through the pin group $\text{Pin}(p_0)$. Now the Dirac operator acts on $X \otimes \text{Spin}_G$, and the Dirac cohomology of $X$ is defined as the $\tilde{K}$-module

$$H_D(X) = \text{Ker} D/(\text{Im} D \cap \text{Ker} D).$$

We embed $t_f^*$ as a subspace of $h_f^*$ by setting the linear functionals on $t_f$ to be zero on $a_f$. The following result slightly extends Theorem 2.3 of Huang and Pandžić [13] to disconnected groups.

**Theorem 2.1.** (Theorem A of [7]) Let $G$ be a real reductive Lie group in Harish-Chandra class. Let $X$ be an irreducible $(g, K)$-module with infinitesimal character $\Lambda$. Suppose that $\tilde{\delta}$ is an irreducible $\tilde{K}$-module in the Dirac cohomology $H_D(X)$ with a highest weight $\mu$. Then $\Lambda$ is conjugate to $\mu + \rho_K$ under the action of the Weyl group $W(g, h_f)$. 

A formula for the Dirac cohomology of cohomologically induced modules in the weakly good range has been given in [7]. See also [19]. To state it, assume that the inducing \((I, L \cap K)\)-module \(Z\) has infinitesimal character \(\lambda_L \in \mathfrak{i}t_{f,0}^*\) which is dominant for \(\Delta^+(I \cap t, t_f)\). We say that \(Z\) is weakly good if
\[
(\lambda_L + \rho(u), \alpha^\vee) \geq 0, \quad \forall \alpha \in \Delta(u, t_f).
\]

**Theorem 2.2.** ([29] Theorems 1.2 and 1.3, or [17] Theorems 0.50 and 0.51) Suppose the admissible \((I, L \cap K)\)-module \(Z\) is weakly good. Then we have

(i) \(\mathcal{L}_j(Z) = \mathcal{R}_j(Z) = 0\) for \(j \neq S\).
(ii) \(\mathcal{L}_S(Z) \cong \mathcal{R}_S(Z)\) as \((\mathfrak{g}, K)\)-modules.
(iii) if \(Z\) is irreducible, then \(\mathcal{L}_S(Z)\) is either zero or an irreducible \((\mathfrak{g}, K)\)-module with infinitesimal character \(\lambda_L + \rho(u)\).
(iv) if \(Z\) is unitary, then \(\mathcal{L}_S(Z)\), if nonzero, is a unitary \((\mathfrak{g}, K)\)-module.
(v) if \(Z\) is in good range, then \(\mathcal{L}_S(Z)\) is nonzero, and it is unitary if and only if \(Z\) is unitary.

It is worth noting that the reverse direction of Theorem 2.2(iv) is not true in general. However, we suspect that it may hold in certain special cases. See Conjecture 2.6.

Now we are able to state the aforementioned formula.

**Theorem 2.3.** (Theorem B of [7]) Suppose that the irreducible unitary \((I, L \cap K)\)-module \(Z\) has infinitesimal character \(\lambda_L \in \mathfrak{i}t_{f,0}^*\) which is weakly good. Then there is a \(K\)-module isomorphism
\[
(7) \quad H_D(\mathcal{L}_S(Z)) \cong \mathcal{L}_D^K(H_D(Z) \otimes \mathbb{C}_{-\rho(w_p)}).
\]

In the setting of the above theorem, it is clear that when \(H_D(\mathcal{L}_S(Z))\) is non-zero, then \(H_D(Z)\) must be non-zero. However, the other direction is unclear yet. Theorem 1.1 aims to fill this gap for linear groups.

### 2.2. A very brief introduction of the software atlas.

Let \(H(\mathbb{C})\) be a maximal torus of \(G(\mathbb{C})\). Its character lattice is the group of algebraic homomorphisms
\[
X^* := \text{Hom}_{\text{alg}}(H(\mathbb{C}), \mathbb{C}^\times).
\]
Choose a Borel subgroup \(B(\mathbb{C}) \supset H(\mathbb{C})\). In the software atlas [1, 32], an irreducible \((\mathfrak{g}, K)\)-module \(\pi\) is parameterized by a final parameter \(p = (x, \lambda, \nu)\) via the Langlands classification. See Theorem 6.1 of [1]. Here \(x\) is a \(K(\mathbb{C})\)-orbit of the Borel variety \(G(\mathbb{C})/B(\mathbb{C}), \lambda \in X^* + \rho\) and \(\nu \in (X^*)^{-\theta} \otimes_{\mathbb{Z}} \mathbb{C}\). In such a case, the infinitesimal character of \(\pi\) is
\[
(8) \quad \frac{1}{2}(1 + \theta)\lambda + \nu \in \mathfrak{h}^*,
\]
where \(\mathfrak{h}\) is the Lie algebra of \(H(\mathbb{C})\). Note that the Cartan involution \(\theta\) now becomes \(\theta_x\)—the involution of \(x\), which is given by the command \texttt{involution(x)} in \texttt{atlas}.

Among the three components of \(p = (x, \lambda, \nu)\), the KGB element \(x\) is hardest to understand. One can use the command \texttt{printKGB(G)} to see the rich information, and to identify which \(K(\mathbb{C})\)-orbit of the Borel variety it is. For our study, the most relevant knowledge is the following special case of Theorem 2.2, rephrased in the language of \texttt{atlas} by Paul [22].
Theorem 2.4. Let $p = (x, \lambda, \nu)$ be the atlas parameter of an irreducible $(\mathfrak{g}, K)$-module $\pi$. Let $S(x)$ be the support of $x$, and $q(x)$ be the $\theta$-stable parabolic subalgebra given by the pair $(S(x), x)$, with Levi factor $L(x)$. Then $\pi$ is cohomologically induced, in the weakly good range, from an irreducible $(\mathfrak{l}, L \cap K)$-module $\pi_{L(x)}$ with parameter $p_{L(x)} = (y, \lambda - \rho(u), \nu)$, where $y$ is the KGB element of $L(x)$ corresponding to the KGB element $x$ of $G$.

Let $l$ be the rank of $G$. atlas labels the simple roots of $\Delta^+(\mathfrak{g}, \mathfrak{h}_f)$ as \{0, 1, 2, \ldots, $l-1$\}, and the support of an KGB element $x$ is given by the command $\text{support}(x)$. We say that $x$ is fully supported if $\text{support}(x)$ equals to $[0, 1, 2, \ldots, l-1]$. Whenever $x$ is fully supported, we will have that $q(x) = \mathfrak{g}$. We say that the representation $p$ is fully supported if its KGB element $x$ is so.

Let us illustrate Theorem 2.4 and some basic atlas commands via a specific example. Some outputs will be omitted to save space.

Example 2.5. Firstly, let us input the linear split real form of $F_4$ into atlas.

\begin{verbatim}
G:F4_s
Value: connected split real group with Lie algebra 'f4(R)'

#KGB(G)
Value: 229

support(KGB(G,228))
Value: [0,1,2,3]
\end{verbatim}

This group has 229 KGB elements in total, and the last one, i.e., $\text{KGB}(G, 228)$, is fully supported. Indeed, as the following shows, there are 141 fully supported KGB elements in total.

\begin{verbatim}
set FS=## for x in KGB(G) do if #support(x)=4 then [x] else [] fi od
#FS
Value: 141
\end{verbatim}

Now let us look at a KGB element which is not fully supported.

\begin{verbatim}
set x21=KGB(G,21)
support(x21)
Value: [1]

set q21=Parabolic:(support(x21),x21)
is_parabolic_theta_stable(q21)
Value: true

set L21=Levi(q21)
L21
Value: connected quasisplit real group with Lie algebra 'sl(2,R).u(1).u(1).u(1)'
\end{verbatim}

The $\theta$-stable parabolic subalgebra $q21$ is the one defined by the KGB element $x21$, whose support consists of the second simple root. Since atlas labels the simple roots of $\Delta^+(\mathfrak{g}, \mathfrak{h}_f)$ opposite to that of Knapp [15], the second one is the $\alpha_3$ in Figure 1. Therefore, the Levi subgroup $L21$ of $q21$ has a simple factor $SL(2, \mathbb{R})$.

Now let us set up an irreducible unitary representation and illustrate Theorem 2.4.

\begin{verbatim}
set p=parameter(KGB(G,21),[0,1,0,1],[-1/2,1,-1,0])
is_unitary(p)
\end{verbatim}
Value: true
infinitsesimal_character(p)
Value: [ 0, 1, 0, 1 ]/1
set (Q,q)=reduce_good_range(p)
q
Value: final parameter(x=2,lambda=[-3,2,-4,0]/2,nu=[-1,2,-2,0]/2)
rho_u(Q)
Value: [ 3, 0, 4, 2 ]/2
lambda(p)-rho_u(Q)=lambda(q)
Value: true
Q=q21
Value: true

The last output says that the inducing module q does come from the $\theta$-stable parabolic subalgebra q21. To further identify q, let us start with the trivial representation of L21.

set t=trivial(L21)
t
Value: final parameter(x=2,lambda=[-1,2,-2,0]/2,nu=[-1,2,-2,0]/2)
goodness(t,G)
Value: "Good"

Now let us move down certain digits of t outside the support of L21 to get q.

set tm=parameter(x(t),lambda(t)-[1,0,1,0],nu(t))
tm=q
Value: true
goodness(q,G)
Value: "Weakly good"

Therefore, the inducing module q is a weakly good unitary character of L21.

theta_induce_irreducible(q,G)
Value: 1*parameter(x=21,lambda=[0,1,0,1]/1,nu=[-1,2,-2,0]/2) [57]

Thus doing cohomological induction from q recovers the original representation p. □

Since development of the software atlas is still quickly ongoing, we sincerely recommend the very helpful weakly seminar [33] to the reader.

2.3. **Scattered representations versus fully supported scattered representations.**

As defined in [5], a *scattered representation* is a member $\pi$ of $\hat{G}^d$ which can *not* be cohomologically induced from a member $\pi_L \in \hat{L}^d$ from the *good range*, where $L$ is the Levi subgroup of a proper $\theta$-stable parabolic subalgebra q of g.

We refer to the fully supported members of $\hat{G}^d$ as *fully supported scattered representations* of G. Note that fully supported scattered representations must be scattered representations. Indeed, let $\pi$ be an arbitrary member of $\hat{G}^d$ such that the KGB element $x$ of its atlas parameter $(x, \lambda, \nu)$ is fully supported, then $q(x) = g$. Since $q(x)$ is the minimum $\theta$-stable


Table 1. Dirac series of $SL(2, \mathbb{R})$, where $a$ runs over $\mathbb{Z}_{\geq 0}$

| $x$ | $\lambda = \Lambda$ | $\nu$ | spin LKT = LKT |
|-----|---------------------|------|---------------|
| 0   | $[a]$              | $[0]$| $[a + 1]$     |
| 1   | $[a]$              | $[0]$| $[-a - 1]$    |
| 2   | $[1]$              | $[1]$| $[0]$         |

parabolic subalgebra from which $\pi$ can be cohomologically induced from the weakly good range, we conclude that $\pi$ must be scattered.

However, scattered representations need not be fully supported. Perhaps the easiest example is the limit of holomorphic discrete series of $SL(2, \mathbb{R})$, whose atlas parameter is as follows:

final parameter $(x = 0, \lambda = [0]/1, \nu = [0]/1)$

It has zero infinitesimal character, and multiplicity-free $K$-types $1, 3, 5, 7, \ldots$. To make the story neat, we treat it as the starting point of the first row of Table 1. We treat the limit of anti-holomorphic discrete series of $SL(2, \mathbb{R})$ similarly, and summarize the results by saying that the Dirac series of $SL(2, \mathbb{R})$ consists of 1 fully supported scattered representations (the third row of Table 1, the trivial representation) and 2 strings of representations (the first row and the second row of Table 1).

In general, let $N_{FS}(G)$ denote the number of fully supported scattered representations of $G$. For instance, we have $N_{FS}(SL(2, \mathbb{R})) = 1$.

Supported by the calculations in [2, 3], we raise the following conjecture asserting that the reverse direction of Theorem 2.2(iv) should be true in certain special setting.

**Conjecture 2.6.** Let $\pi$ be any irreducible unitary $(\mathfrak{g}, K)$ module whose infinitesimal character $\Lambda$ meets the HP condition. Then $\pi|_{L(\nu)}$ must be unitary.

### 3. The root system $\Delta(\mathfrak{g}, \mathfrak{t}_f)$

This section should be well-known to the experts. We learn the content from [10], which might never be published. A good alternative reference is Steinberg [25].

We enumerate the simple roots for $\Delta^+(\mathfrak{g}, \mathfrak{t}_f)$ as follows:

- $\alpha_1, \ldots, \alpha_p$, (compact imaginary)
- $\beta_1, \ldots, \beta_q$, (non-compact imaginary)
- $\gamma_1, \ldots, \gamma_r, \theta(\gamma_1), \ldots, \theta(\gamma_r)$, (complex).

Note that each part above may be absent. We denote the corresponding fundamental weights by

$\varpi(\alpha_1), \ldots, \varpi(\alpha_p), \varpi(\beta_1), \ldots, \varpi(\beta_q), \varpi(\gamma_1), \ldots, \varpi(\gamma_r), \varpi(\theta(\gamma_1)), \ldots, \varpi(\theta(\gamma_r))$.

Let $\rho$ be the half sum of roots in $\Delta^+(\mathfrak{g}, \mathfrak{t}_f)$. For each root $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{t}_f)$, we denote by $\alpha^\vee$ its restriction to $\mathfrak{t}_f$, by $\alpha^\vee$ the coroot of $\alpha$. Note that $\theta(\gamma_j)^\vee = \theta(\gamma_j^\vee)$ for $1 \leq j \leq r$. We can label the simple roots $\gamma_1, \ldots, \gamma_r$ so that $\gamma_j, \theta(\gamma_j), \gamma_j^\vee$ and $\theta(\gamma_j^\vee)$ generate a subsystem of type $A_1 \times A_1$ for $2 \leq j \leq r$. However, when $j = 1$ the subsystem can be of type $A_2$. 

Collecting all these restricted roots $\bar{\alpha}$ for $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}_f)$, we get the root system $\Delta(\mathfrak{g}, \mathfrak{t}_f)$ which may not be reduced. Note that

$$\Delta_{\text{red}}(\mathfrak{g}, \mathfrak{t}_f) = \{ \bar{\alpha} \mid \bar{\alpha}/2 \notin \Delta(\mathfrak{g}, \mathfrak{t}_f) \}$$

is a reduced root system.

For any vector $\mu \in \mathfrak{t}_f^*$, we say that $\mu$ is integral for $\Delta(\mathfrak{g}, \mathfrak{t}_f)$ if the pairing of $\mu$ with each coroot for $\Delta(\mathfrak{g}, \mathfrak{t}_f)$ is an integer. Similarly, we say that $\mu$ is integral for $\Delta(\mathfrak{k}, \mathfrak{t}_f)$ if the pairing of $\mu$ with each coroot for $\Delta(\mathfrak{k}, \mathfrak{t}_f)$ is an integer. It is obvious that if $\mu$ is integral for $\Delta(\mathfrak{g}, \mathfrak{t}_f)$, then $\mu$ must be integral for $\Delta(\mathfrak{k}, \mathfrak{t}_f)$.

Restricting all the roots of $\Delta^+(\mathfrak{g}, \mathfrak{h}_f)$ to $\mathfrak{t}_f$, we get a positive root system $\Delta^+(\mathfrak{g}, \mathfrak{t}_f)$. Its simple roots are $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q, \gamma_1, \ldots, \gamma_r$.

Here $\gamma_j = \gamma_j + \theta(\gamma_j)$ for $1 \leq j \leq r$. Put

$$\varpi(\alpha_1), \ldots, \varpi(\alpha_p), \varpi(\beta_1), \ldots, \varpi(\beta_q), \varpi(\gamma_1) + \varpi(\theta(\gamma_1)), \ldots, \varpi(\gamma_r) + \varpi(\theta(\gamma_r)),$$

and

$$\varpi(\alpha_1), \ldots, \varpi(\alpha_p), \varpi(\beta_1), \ldots, \varpi(\beta_q), \varpi(\gamma_1) + \varpi(\theta(\gamma_1)), \ldots, \varpi(\gamma_r) + \varpi(\theta(\gamma_r)) \frac{1}{2}.$$

Note that $\mu$ is integral for $\Delta(\mathfrak{g}, \mathfrak{t}_f)$ if and only if the paring of $\mu$ with each coroot in (10) is an integer, if and only if $\mu$ is an integer combination of (11).

**Lemma 3.1.** Let $G$ be a simple linear real Lie group in the Harish-Chandra class. Let $\delta$ be the highest weight of any $K$-type. Then $\delta$ is integral for $\Delta(\mathfrak{g}, \mathfrak{t}_f)$.

**Proof.** We may and will assume that $G$ is simply connected. Let $X^*(H_f)$ be the lattice of rational characters of $H_f$. Define $X^*(T_f)$ similarly. Then as shown in [10],

$$X^*(T_f) = X^*(H_f)/\text{span} \{ \lambda - \theta(\lambda) \mid \lambda \in X^*(H_f) \}.$$

Since $G$ is simply-connected, the above denominator is

$$\text{span} \{ \varpi(\gamma_i) - \varpi(\theta(\gamma_i)) \mid 1 \leq i \leq r \}.$$

It follows that (11) is a basis for $X^*(T_f)$.

**Example 3.2.** Consider the linear split real form $F_4$. This group is equal rank, i.e., $\mathfrak{h}_f = \mathfrak{t}_f$. It is connected but not simply connected. Indeed,

$$K \cong \text{Sp}(3) \times \text{Sp}(1)/\{\pm 1\}.$$

Let us adopt the Vogan diagram for its Lie algebra as in [15], see Figure 1.
By choosing the Vogan diagram, we have actually fixed a positive root system $\Delta^+(\mathfrak{g}, t_f)$ with $\alpha_1, \ldots, \alpha_4$ being the simple roots. Then correspondingly a positive root system $\Delta^+(\mathfrak{k}, t_f)$ is fixed, see Figure 2, where the simple roots are

$$\gamma_1 = \alpha_1, \quad \gamma_2 = \alpha_2, \quad \gamma_3 = \alpha_3, \quad \gamma_4 = 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4.$$

![Diagram](image_url)

**Figure 2.** The Dynkin diagram for $\Delta^+(\mathfrak{k}, t_f)$

Let us denote by $\xi_1, \xi_2, \xi_3, \xi_4$ (resp., $\varpi_1, \varpi_2, \varpi_3, \varpi_4$) be the fundamental weights for $\Delta^+(\mathfrak{g}, t_f)$ (resp., $\Delta^+(\mathfrak{k}, t_f)$). Let $a, b, c, d$ be arbitrary non-negative integers. Then one calculates that

$$a\varpi_1 + b\varpi_2 + c\varpi_3 + d\varpi_4 = a\xi_1 + b\xi_2 + c\xi_3 + \left(\frac{a}{2} - b - \frac{3}{2}c + \frac{d}{2}\right)\xi_4.$$

Since $a\varpi_1 + b\varpi_2 + c\varpi_3 + d\varpi_4$ is the highest weight of a $K$-type if and only if $a + c + d$ is even, one sees from (12) that Lemma 3.1 holds.

However, if we pass to the universal covering group of the linear real split $F_4$, which is non-linear, then (12) says that Lemma 3.1 fails. \qed

4. Proof of the non-vanishing criterion

We collect elements of $W(\mathfrak{g}, t_f)$ moving the dominant Weyl chamber for $\Delta^+(\mathfrak{g}, t_f)$ within the dominant Weyl chamber for $\Delta^+(\mathfrak{k}, t_f)$ as $W(\mathfrak{g}, t_f)^1$. It turns out that the multiplication map

$$W(\mathfrak{k}, t_f) \times W(\mathfrak{g}, t_f)^1 \to W(\mathfrak{g}, t_f)$$

induces a bijection [16]. Therefore, the set $W(\mathfrak{g}, t_f)^1$ has cardinality $s$ defined in (1). Let us enumerate its elements as

$$W(\mathfrak{g}, t_f)^1 = \{w^{(0)} = e, w^{(1)}, \ldots, w^{(s-1)}\}.$$ 

Recall that the highest weights of $\text{Spin}_G$ as $\mathfrak{k}$-module are exactly

$$\rho^{(j)} = w^{(j)}\rho - \rho_K, \quad 0 \leq j \leq s - 1.$$ 

Proof of Theorem 1.1. As mentioned at the end of Section 2, it suffices to prove the "$\Leftarrow" direction. Assume that $H_D(Z) \neq 0$. Take any $\widetilde{L} \cap K$-type $\gamma_L$ in $H_D(Z)$. Here $\widetilde{L} \cap K$ stands for the pin covering group of $L \cap K$. By Theorem 2.1, there exists $w_1 \in W(\mathfrak{l}, t_f)$ such that

$$\gamma_L + \rho_{L \cap K} = w_1\lambda_L.$$ 

In particular, it follows that $w_1\lambda_L$ is dominant integral regular for $\Delta^+(\mathfrak{l} \cap \mathfrak{k}, t_f)$.

Put $\gamma_G := \gamma_L + \rho(\mathfrak{u} \cap \mathfrak{p})$. Then due to (2) and that $w_1\Delta(\mathfrak{u}) = \Delta(\mathfrak{u})$, we have

$$\gamma_G + \rho_K = \gamma_L + \rho_{L \cap K} + \rho(\mathfrak{u}) = w_1\lambda_L + \rho(\mathfrak{u}) = w_1(\lambda_L + \rho(\mathfrak{u})).$$

We claim that $\gamma_G + \rho_K$ is dominant integral regular for $\Delta^+(\mathfrak{k}, t_f)$. 

Hence

\[ \gamma \]

Let \( \Lambda \) be a simple linear Lie group in the Harish-Chandra class. Assume that \( \gamma \) is a non-negative integer combination of \( (11) \) under the action of \( f \). Thus \( \gamma \) is assumed further to meet the HP condition, there exist \( w \in W(\mathfrak{g}, t_f)^\perp \) and \( 0 \leq j \leq s - 1 \) such that

\[ \delta - \rho_n^{(j)} + \rho_K = w(\lambda_L + \rho(u)). \]

Note that

\[ \delta - \rho_n^{(j)} + \rho_K = \delta - \rho_n^{(j)} + \sum n_i \gamma_i + \rho_K = \delta - w^{(j)} \rho + 2\rho_K + \sum n_i \gamma_i, \]

where \( n_i \) are some non-negative integers, and \( \gamma_i \) are roots in \( \Delta^+(\mathfrak{k}, t_f) \). It follows from Lemma 3.1 and (18) that \( \{ \delta - \rho_n^{(j)} + \rho_K \} \) is integral for \( \Delta(\mathfrak{g}, t_f) \). Since \( W(\mathfrak{k}, t_f) \leq W(\mathfrak{g}, t_f) \), we conclude from (17) and (15) that \( w_1 \lambda_L + \rho(u) \) is integral for \( \Delta(\mathfrak{g}, t_f) \) as well. In particular, \( w_1 \lambda_L + \rho(u) \) is integral for \( \Delta(\mathfrak{k}, t_f) \).

Secondly, since \( \lambda_L + \rho(u) \) is assumed to be good, one sees that

\[ \langle w_1 \lambda_L + \rho(u), \alpha^\vee \rangle = \langle \lambda_L + \rho(u), w_1^{-1}(\alpha)^\vee \rangle > 0, \quad \forall \alpha \in \Delta^+(\mathfrak{k} \cap u, t_f). \]

Moreover, for any \( \alpha \in \Delta^+(\mathfrak{k} \cap u, t_f) \),

\[ \langle w_1 \lambda_L + \rho(u), \alpha^\vee \rangle = \langle \gamma_L + \rho(u), \alpha^\vee \rangle = \langle \gamma_L + \rho(u), \alpha^\vee \rangle > 0. \]

Therefore, \( w_1 \lambda_L + \rho(u) \) is dominant regular for \( \Delta^+(\mathfrak{k}, t_f) \).

To sum up, \( w_1 \lambda_L + \rho(u) \) is dominant integral regular for \( \Delta^+(\mathfrak{k}, t_f) \). Thus the claim holds.

Hence \( \gamma_G = w_1 \lambda_L + \rho(u) - \rho_K \) is dominant integral for \( \Delta^+(\mathfrak{k}, t_f) \). Thus \( \gamma_G = \gamma_L + \rho(u \cap p) \) occurs in \( H_D(\mathcal{L}_S(\mathbb{Z})) \) by Theorem 2.3.

**Remark 4.1.** Let \( G \) be a simple linear Lie group in the Harish-Chandra class. Assume that \( \Lambda \in \mathfrak{h}_f^+ \) satisfies the HP condition. Then the above proof says that \( \Lambda \) is conjugate to a vector in \( \mathfrak{t}_f^\perp \) which is a non-negative integer combination of \( (11) \) under the action of \( W(\mathfrak{g}, \mathfrak{h}_f) \).

Finally, let us present an example showing that Theorem 1.1 does not hold if the good range condition is loosen to be weakly good.

**Example 4.2.** Consider the representation \( \mathfrak{p} \) of \( \mathfrak{f}_4 \) studied in Example 2.5. As mentioned earlier, this is a weakly good \( A_q(\lambda) \) module. The bottom layer of \( \mathfrak{p} \) consists of the unique \( K \)-type

\[ \lambda + 2\rho(u \cap \mathfrak{p}) = w_2 + 8w_4, \]

which is also the unique lowest \( K \)-type of \( \mathfrak{p} \). This \( K \)-type has spin norm \( \sqrt{15} \), while \( \| \Lambda \| = \sqrt{11} \). Thus this unique bottom layer \( K \)-type can not contribute to \( H_D(\mathfrak{p}) \), which then must vanish by Proposition 4.5 of [7].

5. The number of strings in \( \hat{G}^d \)

As demonstrated in Section 2.3 and Example 4.2 of [3], we can use translation functor to merge any Dirac series representation which is not fully supported into a string. See Example 7.3 as well. This will allow us to present the Dirac series neatly, and in particular, allows us to count the number of strings in \( \hat{G}^d \). In other words, atlas teaches us that it is quite natural to arrange the Dirac series of \( G \) according to the support of their KGB elements.
For simplicity, we assume that $G$ is equal rank. Let $\{\xi_1, \ldots, \xi_l\}$ be the corresponding fundamental weights corresponding to the simple roots of $\Delta^+(g, h_f)$.

We assume that Conjecture 2.6 holds for $G$. We further assume that the following binary condition holds for $G$: Let $\pi$ with final atlas parameter $p = (x, \lambda, \nu)$ be an irreducible unitary representation. Let $\Lambda = \sum_{i=1}^l n_i \zeta_i$ be the infinitesimal character of $\pi$ which is integral (see Remark 4.1). Then each $n_i$ corresponding to a simple root in $\text{support}(x)$ is either 0 or 1. The binary condition should be closely related to Conjecture 5.7 of [26].

Let $S$ be any proper subset of the simple roots of $\Delta^+(g, h_f)$. We collect the dominant integral HP infinitesimal characters $\Lambda$ whose coordinates are 0 or 1 on the digits corresponding to $S$, and whose coordinates outside $S$ are 1 by $\Omega(S)$. Denote by $N(S)$ the number of Dirac series representations with infinitesimal character in $\Omega(S)$ and support $S$. Put

$$N_i = \sum_{\#S=i} N(S).$$

Then $N_0 + N_1 + \cdots + N_{l-1}$ is the number of strings in $\tilde{G}^d$.

6. DIRAC SERIES OF $E_{6(2)}$

In this section, we fix $G$ as the simple real exceptional linear Lie group $E_{6,4}$ in atlas. This connected group is equal rank. That is, $h_f = t_f$. It has center $\mathbb{Z}/3\mathbb{Z}$. The Lie algebra $g_0$ of $G$ is denoted as $\mathfrak{EI}$ in [15, Appendix C]. Note that

$$-\dim \mathfrak{t} + \dim \mathfrak{p} = -38 + 40 = 2.$$ 

Therefore, the group $G$ is also called $E_{6(2)}$ in the literature.

We present a Vogan diagram for $g_0$ in Fig. 3, where $\alpha_1 = \frac{1}{2}(1, -1, -1, -1, -1, -1, 1)$, $\alpha_2 = e_1 + e_2$ and $\alpha_i = e_{i-1} - e_{i-2}$ for $3 \leq i \leq 6$. By specifying a Vogan diagram, we have actually fixed a choice of positive roots $\Delta^+(g, t_f)$. Let $\zeta_1, \ldots, \zeta_6 \in t_f^*$ be the corresponding fundamental weights for $\Delta^+(g, t_f)$. The dual space $t_f^*$ will be identified with $t_f$ under the Killing form $B(\cdot, \cdot)$. We will use $\{\zeta_1, \ldots, \zeta_6\}$ as a basis to express the atlas parameters $\lambda, \nu$ and the infinitesimal character $\Lambda$. More precisely, in such cases, $[a, b, c, d, e, f]$ will stand for the vector $a\zeta_1 + \cdots + f\zeta_6$.

![Figure 3. The Vogan diagram for $\mathfrak{EI}$](image)

Put $\gamma_i = \alpha_{7-i}$ for $1 \leq i \leq 4$, $\gamma_5 = \alpha_1$, and

$$\gamma_6 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}).$$
which is the highest root of $\Delta^+(g, t_f)$. Then $\gamma_1, \ldots, \gamma_6$ are the simple roots of $\Delta^+(t, t_f) = \Delta(t, t_f) \cap \Delta^+(g, t_f)$. We present the Dynkin diagram of $\Delta^+(t, t_f)$ in Fig. 4. Let $\varpi_1, \ldots, \varpi_6 \in t^*_f$ be the corresponding fundamental weights.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig4}
\caption{The Dynkin diagram for $\Delta^+(t, t_f)$}
\end{figure}

Let $E_\mu$ be the $t$-type with highest weight $\mu$. We will use $\{\varpi_1, \ldots, \varpi_6\}$ as a basis to express $\mu$. Namely, in such a case, $[a, b, c, d, e, f]$ stands for the vector $a\varpi_1 + b\varpi_2 + c\varpi_3 + d\varpi_4 + e\varpi_5 + f\varpi_6$. For instance,

$$\beta := \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 = [0, 0, 1, 0, 0, 1]$$

and $\dim p = \dim E_\beta = 40$. The $t$-type $E_{[a,b,c,d,e,f]}$ has lowest weight $[-e, -d, -c, -b, -a, -f]$. Therefore, $E_{[a,b,c,d,e,f]}$ is the contragredient $t$-type of $E_{[a,b,c,d,e,f]}$. For $a, b, c, d, e, f \in \mathbb{Z}_{\geq 0}$, we have that $E_{[a,b,c,d,e,f]}$ is a $K$-type if and only if

$$a + c + e + f \text{ is even.} \tag{19}$$

Note that

$$\#W(g, t_f)^1 = \frac{51840}{1440} = 36.$$ 

In the current setting, the spin norm of the $t$-type $E_\mu$ specializes as

$$\|\mu\|_{\text{spin}} = \min_{0 \leq j \leq 35} \|\mu - \rho_n^{(j)}\| + \rho_K.$$

Note that $E_{\mu - \rho_n^{(j)}}$ is the PRV component [23] of the tensor product of $E_\mu$ with the contragredient $t$-type of $E_{\rho_n^{(j)}}$. Let $\pi$ be any infinite-dimensional irreducible $(g, K)$ module with infinitesimal character $\Lambda$. Put the spin norm of $\pi$ as

$$\|\pi\|_{\text{spin}} = \min \|\delta\|_{\text{spin}},$$

where $\delta$ runs over all the $K$-types of $\pi$. If $\delta$ attains $\|\pi\|_{\text{spin}}$, we will call it a spin-lowest $K$-type of $\pi$. If $\pi$ is further assumed to be unitary, Parthasarathy’s Dirac operator inequality [21] can be rephrased as

$$\|\pi\|_{\text{spin}} \geq \|\Lambda\|. \tag{20}$$

Moreover, as shown in [4], the equality happens in (20) if and only if $\pi$ has non-zero Dirac cohomology, and in this case, it is exactly the (finitely many) spin-lowest $K$-types of $\pi$ that contribute to $H_D(\pi)$.

Since $E_{6,q}$ is not of Hermitian symmetric type, results from [27] says that the $K$-type $V_{\delta + n\beta}$ must show up in $\pi$ for any non-negative integer $n$ provided that $V_{\delta}$ occurs in $\pi$. We call them the Vogan pencil starting from $V_{\delta}$. Now it follows from (20) that

$$\|\delta + n\beta\|_{\text{spin}} \geq \|\Lambda\|, \quad \forall n \in \mathbb{Z}_{\geq 0}. \tag{21}$$

In other words, whenever (21) fails, we can conclude that $\pi$ is non-unitary. Distribution of spin norm along Vogan pencils has been discussed in Theorem C of [5]. In practice, we will
take \( \delta \) to be a lowest \( K \)-type of \( \pi \) and use the corresponding Vogan pencil to do non-unitarity test.

This section aims to classify the Dirac series of \( E_{6(2)} \), see Theorem 1.3.

6.1. **Fully supported scattered representations of \( E_{6(2)} \).** This subsection aims to sieve out all the fully supported scattered Dirac series representations for \( E_{6(2)} \) using the algorithm in [5].

**Lemma 6.1.** Let \( \Lambda = a\zeta_1 + b\zeta_2 + c\zeta_3 + d\zeta_4 + e\zeta_5 + f\zeta_6 \) be the infinitesimal character of any Dirac series representation \( \pi \) of \( E_{6(2)} \) which is dominant with respect to \( \Delta^+(g, t_f) \). Then \( a, b, c, d, e, f \) must be non-negative integers such that \( a + c > 0, b + d > 0, c + d > 0, d + e > 0 \) and \( e + f > 0 \).

**Proof.** It follows from Remark 4.1 that \( a, b, c, d, e, f \) must be non-negative integers.

Now if \( a + c = 0 \), i.e., \( a = c = 0 \), then a direct check says that for any \( w \in W(g, t_f)^1 \), at least one coordinate of \( w\Lambda \) in terms of the basis \( \{\varpi_1, \ldots, \varpi_6\} \) vanishes. Therefore,

\[
\{ \mu - \rho_{(j)}^{(b)} \} + \rho_K = w\Lambda
\]

could not hold. This proves that \( a + c > 0 \), other inequalities can be similarly obtained. \( \Box \)

To obtain all the fully supported scattered Dirac series representations of \( E_{6(2)} \), now it suffices to consider all the infinitesimal characters \( \Lambda = [a, b, c, d, e, f] \) such that

- \( a, b, c, d, e, f \) are non-negative integers;
- \( a + c > 0, b + d > 0, c + d > 0, d + e > 0, e + f > 0 \);
- \( \min\{a, b, c, d, e, f\} = 0 \);
- there exists a fully supported KGB element \( x \) such that \( \|\Lambda - \theta_x\Lambda\| \leq \|2\rho\| \).

Let us collect them as \( \Phi \). It turns out that \( \Phi \) has cardinality 58061. There are 21 elements of \( \Phi \) whose largest entry equals to 1:

\[
[0, 0, 1, 1, 0, 1], [0, 0, 1, 1, 1, 1], [0, 0, 1, 1, 1, 0], [0, 1, 1, 0, 0, 1], [0, 1, 1, 1, 0, 1], [0, 1, 1, 1, 1, 0], [0, 1, 1, 1, 1, 1], [1, 0, 0, 0, 1, 1], [1, 0, 1, 0, 1, 1], [1, 0, 1, 1, 0, 1], [1, 0, 1, 1, 1, 1], [1, 1, 0, 1, 0, 1], [1, 1, 0, 1, 1, 0], [1, 1, 1, 0, 1, 1].
\]

A careful study of the irreducible unitary representations under the above 21 infinitesimal characters leads us to Section 9. Let \( p = (x, \lambda, \nu) \) be a fully supported scattered Dirac series representation of \( E_{6(2)} \) with infinitesimal character \( \Lambda \). It can happen that there exists another fully supported scattered Dirac series representation \( p' = (x', \lambda', \nu') \) of \( E_{6(2)} \) with infinitesimal character \( \Lambda' \) (resp. \( \lambda', \nu' \)) obtained from \( \Lambda \) (resp. \( \lambda, \nu \)) by exchanging its first and sixth, third and fifth coordinates. Moreover, the spin lowest \( K \)-types of \( p' \) are exactly the contragredient \( K \)-types of those of \( p \). Whenever this happens, we will fold \( p' \) by omitting \( \lambda', \nu' \) and the spin-lowest \( K \)-types of \( p' \). Instead, we only present \( x' \) in the bolded fashion in the last column along with \( p \). The reader can recover \( p' \) easily. For instance, let us come to the following representation in Table 8,

\[
p = \text{parameter}(KGB(G, 1624), [1, 1, 4, -1, 1, 1], [1, 1, 4, -3, 1, 1]).
\]
which has spin-lowest $K$-types $[0,3,0,0,0,0]$, $[0,3,1,0,0,1]$, $[0,3,2,0,0,2]$. The bolded $1623$ says that

$$p' = \text{parameter}(\text{KGB}(G, 1623), [1, 1, 1, -1, 4, 1], [1, 1, -3, 4, 1])$$

is also a fully supported scattered Dirac series representation. Moreover, its spin-lowest $K$-types are $[0, 0, 1, 0, 0, 0]$, $[0, 0, 0, 1, 0, 1]$, $[1, 0, 0, 1, 0, 2]$. For the other 58040 elements of $\Phi$, we use Parthasarathy’s Dirac operator inequality, and distribution of spin norm along the Vogan pencil starting from one lowest $K$-type to verify that there is no fully supported irreducible unitary representations with infinitesimal character $\Lambda$. This method turns out to be very effective in non-unitarity test. Indeed, it fails only on the following infinitesimal characters of $\Phi$:

$$[0, 0, 1, 0, 0, 2], [0, 2, 1, 0, 1, 0], [0, 2, 1, 0, 1, 1], [1, 0, 0, 1, 0, 2],$$
$$[1, 2, 1, 0, 1, 0], [2, 0, 0, 1, 0, 1], [2, 0, 0, 1, 0, 2], [2, 0, 0, 1, 1, 0].$$

However, a more careful look says that there is no fully supported irreducible unitary representation under them. Let us provide one example.

**Example 6.2.** Consider the infinitesimal character $\Lambda = [0, 0, 1, 1, 0, 2]$ for $E_6_q$.

```
G:E6_q
set all=all_parameters_gamma(G,[0,0,1,1,0,2])
#all
Value: 263
set i=0
void: for p in all do if #support(x(p))=6 then i:=i+1 fi od
i
Value: 110
```

There are 263 irreducible representations with infinitesimal character $\Lambda$, among which 110 are fully supported. A careful look at them says that the non-unitarity test using the pencil starting from one of the lowest $K$-types fails only for the following representation:

```
set p=parameter(KGB(G,1536),[0,0,3,0,1,1],[0,0,4,-2,0,2])
```

Indeed, it has a unique lowest $K$-type $\delta = [0, 3, 0, 0, 0, 0]$, and the minimum spin norm along the pencil $\{\delta + n\beta \mid n \in \mathbb{Z}_{\geq 0}\}$ is $\sqrt{47}$, while $\|\Lambda\| = 6$. Thus Dirac inequality does not work here. Instead, we check the unitarity of $p$ directly:

```
is_unitary(p)
Value: false
```

Thus there is no fully supported irreducible unitary representation with infinitesimal character $\Lambda$. □

6.2. **Conjecture 2.6 and binary condition for $E_{6(2)}$.** In this subsection, let us verify that $E_{6(2)}$ satisfies Conjecture 2.6 and the binary condition. Let $p = (x, \lambda, \nu)$ be any irreducible unitary representation with infinitesimal character $\Lambda = [a, b, c, d, e, f]$ meeting the requirements in Lemma 6.1. It suffices to check that the coordinates of $\Lambda$ within the support of $x$ are either 0 or 1, and that the inducing module $p_{L(x)}$ is unitary.
Example 6.3. Consider the case that \( \text{support}(x) = [0, 1, 2, 3, 4] \). There are 168 such KGB elements in total. We compute that there are 24109 infinitesimal characters \( \Lambda = [a, b, c, d, e, f] \) in total such that

- \( a, b, c, d, e \) are non-negative integers, \( f = 0 \) or 1;
- \( a + c > 0, b + d > 0, c + d > 0, d + e > 0, e + f > 0 \);
- \( \min\{a, b, c, d, e, f\} = 0 \);
- there exists a KGB element \( x \) with support \([0, 1, 2, 3, 4]\) such that \( \|\Lambda - \theta_x \Lambda\| \leq 2\rho \).

The reason that we can reduce the verification of the unitarity of \( p_{L(x)} \) to the cases \( f = 0 \) or 1 is due to the relation between translation functor and cohomological induction, see Theorem 7.237 of [17].

As in Section 6.1, we exhaust all the irreducible unitary representations under these infinitesimal characters with the above 168 KGB elements. It turns out that such representations occur only when \( a, b, c, d, e = 0 \) or 1. Then we check that each \( p_{L(x)} \) is indeed unitary. \( \square \)

All the other non fully supported KGB elements are handled similarly. Eventually we conclude that \( E_6(2) \) satisfies Conjecture 2.6 and the binary condition.

6.3. Number of strings in \( \widehat{E}_6(2)^d \). Thanks to Section 6.2, each representation in \( \widehat{E}_6(2)^d \) whose KGB element is not fully supported can be equipped into a string in the fashion of Example 7.3. Using the formula in Section 5, let us pin down the number of strings in \( \widehat{E}_6(2)^d \) in this subsection.

We compute that

\[
N([0, 1, 2, 3, 4]) = N([1, 2, 3, 4, 5]) = 45, \quad N([0, 2, 3, 4, 5]) = 29,
\]

and that

\[
N([0, 1, 3, 4, 5]) = N([0, 1, 2, 3, 5]) = 7, \quad N([0, 1, 2, 4, 5]) = 1.
\]

In particular, it follows that

\[
N_5 = 2 \times 45 + 29 + 2 \times 7 + 1 = 134.
\]

We also compute that

\[
N_0 = 36, \quad N_1 = 60, \quad N_2 = 80, \quad N_3 = 115, \quad N_4 = 151.
\]

Therefore, the total number of strings for \( E_6(2) \) is equal to

\[
\sum_{i=0}^{5} N_i = 576.
\]

To end up with this section, we mention that some auxiliary files have been built up to facilitate the classification of the Dirac series of \( E_6(2) \). They are available via the following link:

https://www.researchgate.net/publication/353352799_EII-Files
7. Cancellation in Dirac cohomology

It is interesting to note that cancellation continues to happen within the Dirac cohomology of some fully supported scattered members of \( \hat{E}_6(2) \) when passing to Dirac index. There are 10 such representations in total, and in each case the Dirac index vanishes. We will mark their KGB elements with stars (see Section 9).

Example 7.1. Consider the following representation \( \pi \)

\[
\text{final parameter}(x=1649, \lambda=[2,-2,0,4,-1,1]/1, \nu=[2,-3,0,5,-3,2]/2)
\]

It has infinitesimal character \([1,0,0,1,0,1]\), which is conjugate to \( \rho_K \) under the action of \( W(g,f) \). The representation \( \pi \) has four spin LKTs:

\[
[2,0,2,1,0,2] = \rho_n^{(25)}, [2,1,1,0,1,4] = \rho_n^{(13)}, [3,0,1,1,1,1] = \rho_n^{(27)}, [3,1,0,0,2,3] = \rho_n^{(16)}.
\]

Therefore, \( H_D(\pi) \) consists of four copies of the trivial \( \tilde{K} \)-type.

Note that \(-\rho_n^{(25)}, -\rho_n^{(13)}, -\rho_n^{(27)}, -\rho_n^{(16)}\) are the lowest weights of \( E_{\rho_n^{(22)}}, E_{\rho_n^{(15)}}, E_{\rho_n^{(26)}} \) and \( E_{\rho_n^{(18)}} \), respectively. Moreover, \( w^{(22)} = s_2s_4s_5s_6s_3s_4s_5s_1 \) and \( w^{(15)} = s_2s_4s_5s_6s_3s_4s_4 \) have even lengths, while \( w^{(26)} = s_2s_4s_5s_6s_3s_4s_5s_2s_1 \) and \( w^{(18)} = s_2s_4s_5s_6s_3s_4s_2 \) have odd lengths. Therefore, two trivial \( \tilde{K} \)-type live in the even part of \( H_D(\pi) \), while the other two live in the odd part of \( H_D(\pi) \). See Lemma 2.3 of [9]. As a consequence, the Dirac index of \( \pi \) vanishes.

Note that DI(\( \pi \)) can also be easily calculated by atlas using [18]:

\begin{verbatim}
G:6_e_q
set p=parameter (KGB (G)[1649],[3,-1,0,4,-3,3]/1,[2,-3,0,5,-3,2]/2)
show_dirac_index(p)
Dirac index is 0
\end{verbatim}

which agrees with the previous calculation. \( \square \)

Example 7.2. The first entry of Table 8 is the minimal representation of \( E_6(2) \). It has one LKT \([0,0,0,0,0,0]\) and four spin LKTs:

\[
[0,0,1,0,0,3], [0,0,2,0,0,4], [0,0,3,0,0,5], [0,0,4,0,0,6].
\]

We calculate that its Dirac index vanishes. \( \square \)

Example 7.3. Consider the following irreducible representation \( \pi_{0,0} \):

\begin{verbatim}
set p00=parameter(KGB(G)[851],[-1,1,1,1,1,-1]/1,[-2,1,1,0,1,-2]/1)
is_unitary(p00)
Value: true
infinitesimal_character(p00)
Value: [0, 1, 1, 0, 1, 0 ]/1
support(KGB(G,851))
Value: [1,2,3,4]
show_dirac_index(p00)
Dirac index is 0
\end{verbatim}

The representation \( \pi_{0,0} \) has two multiplicity-free spin LKTs: \([1,1,1,1,1]\) and \([2,0,2,0,2,2]\). Cancellation happens within \( H_D(\pi_{0,0}) \), resulting that DI(\( \pi_{0,0} \)) = 0.
set \((Q,q00)=\text{reduce_good_range}(p00)\)
goodness\((q00,G)\)
Value: "Weakly good"
Levi\((Q)\)
Value: connected quasisplit real group with Lie algebra 'so(4,4).u(1).u(1)'
is_unitary\((q00)\)
Value: true

Thus the representation \(\pi_{0,0}\) is cohomologically induced from the irreducible unitary representation \(q00\) of \(\text{Levi}(Q)\). Note that \(q00\) is actually the minimal representation of \(\text{Levi}(Q)\). Moreover, we compute that cancellation happens within the Dirac cohomology of \(q00\) and its Dirac index vanishes.

Now for any non-negative integers \(a, f\), we move the first and last coordinates of \(\lambda(q00)\) to \(a\) and \(f\), respectively. Then we arrive at the irreducible unitary representation \(qaf\) of \(\text{Levi}(Q)\). Doing cohomological induction from \(qaf\) will give us an irreducible unitary representation \(\pi_{a,f}\) of \(G\). By Theorem 2.3, \(\pi_{a,f}\) must have non-zero Dirac cohomology. Indeed, \(\pi_{a,f}\) has two multiplicity-free spin LKTs: \([1,a+1,1,f+1,1,1]\) and \([2,a,2,f,2,2]\).

Moreover, by Proposition 4.1 of [8], we have that \(\text{DI}(\pi_{a,f})=0\).

Although it is not easy to check that whether \(\pi_{0,0}\) is a scattered member of \(\hat{E}_{6(2)}^d\) or not, we can simply embed it into the string \(\pi_{a,f}\), where \(a, f\) run over \(\mathbb{Z}_{\ge 0}\).

\[\square\]

**Remark 7.4.** The Levi subgroup \(\text{Levi}(Q)\) is also of quaternionic type. The cancellation phenomenon seems to be closely related to quaternionic real forms.

### 8. Special unipotent representations

In the list [34] offered by Adams, the group \(E_6_q\) has 47 special unipotent representations. Ten of them are also fully supported with non-zero Dirac cohomology. We mark the nine non-trivial ones with the subscript ♣ in Section 9. It is interesting to note that except for the first entry of Table 8 (the minimal representation), the other eight non-trivial special unipotent representations are all \(A_q(\lambda)\) modules. A brief summary is given below.

| special unipotent representation | realization as an \(A_q(\lambda)\) module |
|----------------------------------|------------------------------------------|
| 1st entry of Table 6            | \(\text{KGP}(G,[0,1,2,3,4])[1]\), \(f\downarrow 6\), Weakly fair |
| 2nd entry of Table 6            | \(\text{KGP}(G,[0,1,2,3,4])[0]\), \(f\downarrow 6\), Weakly fair |
| 1st entry of Table 7            | \(\text{KGP}(G,[0,2,3,4,5])[1]\), \(b\downarrow 6\), None |
| 2nd entry of Table 7            | \(\text{KGP}(G,[0,2,3,4,5])[2]\), \(b\downarrow 6\), None |
| 3rd entry of Table 7            | \(\text{KGP}(G,[0,2,3,4,5])[2]\), \(b\downarrow 5\), Fair |
| 4th entry of Table 7            | \(\text{KGP}(G,[0,2,3,4,5])[1]\), \(b\downarrow 5\), Fair |
| 6th entry of Table 7            | \(\text{KGP}(G,[0,2,3,4,5])[0]\), \(b\downarrow 6\), None |
| 9th entry of Table 7            | \(\text{KGP}(G,[0,2,3,4,5])[0]\), \(b\downarrow 5\), Fair |

**Example 8.1.** Let us explain the second row of the above table.

\(G:E_6\_q\)
set \(P=\text{KGP}(G,[0,1,2,3,4])\)
#\(P\)
Value: 4
void: for i:4 do prints(P[i]," "is_parabolic_theta_stable(P[i]) od
([0,1,2,3,4], KGB element #1260) true
([0,1,2,3,4], KGB element #1434) true
([0,1,2,3,4], KGB element #1776) false
([0,1,2,3,4], KGB element #1790) false
set L=Levi(P[0])
set t=trivial(L)
set tm6=parameter(x(t),lambda(t)=[0,0,0,0,0,6],nu(t))
goodness(tm6,G)
Value: "Weakly fair"
theta_induce_irreducible(tm6,G)
Value: 1*parameter(x=1773,lambda=\[2,2,0,1,0,2\]/1,nu=\[3,3,-1,2,-1,3\]/2) [11]
Other rows are interpreted similarly.

9. Appendix

This appendix presents all the 55 non-trivial fully supported scattered representations in $E_6(2)$ according to their infinitesimal characters. Since each coordinate of any involved infinitesimal character is either 0 or 1, it follows from Lemma 2.2 of [8] that each spin-lowest $K$-type must be u-small.

### Table 2. Infinitesimal character $[1,0,0,1,1,1]$ and $[1,0,1,1,0,1]$

| #x   | $\lambda$      | $\nu$   | Spin LKTs | #x'  |
|------|-----------------|---------|-----------|------|
| 1686 | $[3,-4,-2,5,3,1]$ | $[1,-2,-2,3,1,1]$ | $[0,0,2,2,0,2]$ | 1687 |
| 1592 | $[1,-1,-1,3,1,2]$ | $[0,-2,-\frac{3}{2},2,0,\frac{3}{2}]$ | $[2,0,1,0,0,7]$ | 1612 |

### Table 3. Infinitesimal character $[1,1,0,1,1,1]$ and $[1,1,1,0,1,0]$

| #x   | $\lambda$      | $\nu$   | Spin LKTs | #x'  |
|------|-----------------|---------|-----------|------|
| 1539 | $[4,1,-3,3,1,1]$ | $[5,1,-4,1,1,1]$ | $[0,4,0,0,0,0], [0,4,1,0,0,1]$ | 1540 |
| 1415 | $[3,2,-1,1,1,1]$ | $[5,\frac{3}{2},-\frac{7}{2},0,2,0]$ | $[0,0,0,0,4,8], [0,0,1,0,4,9]$ | 1398 |

### Table 4. Infinitesimal character $[0,1,0,1,1,1]$ and $[1,1,0,1,0,1]$

| #x   | $\lambda$ | $\nu$ | Spin LKTs | #x'  |
|------|-----------|-------|-----------|------|
| 1761 | $[1,6,4,-3,3,1]$ | $[-1,2,2,-1,1,1]$ | $[0,0,2,1,0,2], [0,0,3,1,0,3]$ | 1763 |
| 1722 | $[-1,3,2,0,1,2]$ | $[-\frac{3}{2},2,\frac{3}{2},0,0,\frac{3}{2}]$ | $[0,1,0,0,0,6], [1,0,2,0,0,7]$ | 1728 |
| 874  | $[-1,1,4,-2,2,1]$ | $[-\frac{3}{2},1,\frac{3}{2},-\frac{3}{2},1,1]$ | $[0,2,1,0,3,4], [0,3,0,0,2,6]$ | 896  |
### Table 5. Infinitesimal character $[0, 1, 1, 0, 1, 0]$

| #x   | $\lambda$       | $\nu$       | Spin LKTs          |
|------|-----------------|-------------|--------------------|
| 1561 | $[-\frac{7}{2}, 1, \frac{7}{2}, -\frac{7}{2}, \frac{7}{2}, -\frac{7}{2}]$ | $[0, 0, 4, 0, 0, 2]$, $[0, 1, 0, 1, 0, 6]$ | $[0, 2, 0, 2, 0, 6]$, $[1, 1, 0, 1, 1, 8]$ |
| 1502 | $[-\frac{3}{2}, 0, \frac{3}{2}, -1, \frac{3}{2}, -\frac{3}{2}]$ | $[2, 0, 2, 0, 2, 0]$, $[2, 1, 0, 1, 2, 0]$ | $[3, 0, 0, 0, 3, 2]$ |

### Table 6. Infinitesimal character $[1, 1, 0, 1, 0, 1]$

| #x   | $\lambda$       | $\nu$       | Spin LKTs          | #x'         |
|------|-----------------|-------------|--------------------|-------------|
| 1787 | $[1, 1, 1, 4, -1, 1]$ | $[1, 1, 0, 1, 0, 1]$ | $[0, 0, 4, 0, 0, 4]$, $[0, 1, 2, 1, 0, 2]$ | $[0, 2, 0, 2, 0, 0]$ |
| 1773 | $[2, 2, 0, 1, 0, 2]$ | $[\frac{3}{2}, 0, -\frac{1}{2}, 1, -\frac{1}{2}, \frac{3}{2}]$ | $[0, 0, 1, 0, 0, 5]$, $[1, 0, 1, 0, 1, 7]$ | $[2, 0, 1, 0, 2, 9]$ |
| 1352 | $[1, 1, -1, 3, -1, 1]$ | $[1, 1, -\frac{5}{2}, \frac{7}{2}, -\frac{5}{2}, 1]$ | $[0, 0, 4, 0, 0, 4]$, $[0, 1, 1, 1, 0, 7]$ | $[0, 2, 0, 2, 0, 8]$ |
| 1269 | $[1, 1, -1, 3, -1, 1]$ | $[1, 0, -\frac{5}{2}, 4, -\frac{5}{2}, 1]$ | $[3, 0, 0, 0, 3, 0]$, $[3, 0, 1, 0, 3, 1]$ | $[3, 0, 1, 0, 3, 1]$ |
| 1166 | $[7, 3, -2, 1, -3, 6]$ | $[3, 1, -2, 1, -2, 3]$ | $[0, 2, 0, 2, 0, 0]$, $[1, 2, 0, 2, 1, 2]$ | $[0, 2, 0, 2, 0, 0]$ |
| 977  | $[2, 1, -1, 2, -1, 2]$ | $[\frac{3}{2}, 1, -\frac{5}{2}, 2, -\frac{5}{2}, \frac{3}{2}]$ | $[0, 1, 0, 1, 0, 12]$, $[1, 0, 0, 0, 1, 10]$ | $[0, 1, 0, 0, 1, 10]$ |
| 964  | $[3, 1, 0, 1, -1, 2]$ | $[3, 0, -2, 2, -\frac{5}{2}, \frac{5}{2}]$ | $[0, 0, 1, 1, 4, 3]$, $[0, 1, 0, 0, 5, 5]$ | $953$ |
TABLE 7. Infinitesimal character $[1,0,0,1,0,1]$

| #x | \lambda | \nu | Spin LKTs | #x' |
|----|--------|-----|----------|-----|
| 1787 | $[1,1,4,-1,2]$ | $[1,0,0,1,0,1]$ | $[1,0,3,0,1,1], [1,1,1,1,1,3]$ | $[2,0,1,0,2,5]$ |
| 1782 | $[1,0,1,4,-1,2]$ | $[1,0,0,1,0,1]$ | $[0,0,1,0,0,9], [1,0,3,0,1,1]$ | $[1,1,1,1,1,3], [2,0,1,0,2,5]$ |
| 1746 | $[3,0,-1,3,-1,3]$ | $[2, - \frac{1}{2}, - \frac{1}{2}, - \frac{1}{2}, 2, - \frac{1}{2}, 2]$ | $[0,0,4,0,0,0], [0,1,0,1,0,8]$ | $[0,2,0,2,0,4], [1,1,0,1,1,6]$ |
| 1726 | $[3,-1,0,2,0,3]$ | $[- \frac{2}{2}, -1,0,1,0, \frac{3}{2}]$ | $[2,0,2,0,2,0], [2,1,0,1,2,2]$ | $[3,0,0,0,3,4]$ |
| 1537 | $[3,-3,-2,6,-1,1]$ | $[1,-2,1,3,-1,1]$ | $[1,1,1,1,1,3], [2,1,0,1,2,2]$ | $[0,0,1,0,0,9]$ |
| 1377 | $[1,-2,1,5,-1,1]$ | $[0,-2,1,3,-1,0]$ | $[0,0,1,0,0,0], [0,1,0,1,0,8]$ | $[1,1,1,1,1,3]$ |
| 1268 | $[1,0,-2,5,-2,1]$ | $[1,0,-2,3,-2,1]$ | $[0,2,0,2,0,4], [1,1,1,1,1,3]$ | $[0,0,0,0,0,10]$ |
| 1267 | $[3,0,-2,3,-2,3]$ | $[2,0,-2,2,-2,2]$ | $[2,0,2,1,0,2], [2,1,1,0,1,4]$ | $[3,0,0,0,3,4]$ |
| 559 | $[2,-1,-2,4,-2,2]$ | $[1,- \frac{3}{2}, - \frac{3}{2}, - \frac{3}{2}, 2, 1]$ | $[1,0,1,1,2,4], [2,1,0,1,2,2]$ | $[2,1,1,0,1,4], [3,0,0,0,3,4]$ |
| 1649 | $[3,-1,0,4,-3,3]$ | $[1,- \frac{3}{2}, 0, \frac{5}{2}, - \frac{3}{2}, 1]$ | $[2,0,2,1,0,2], [2,1,1,0,1,4]$ | $[3,0,1,1,1,1], [3,1,0,0,2,3]$ |
| 1403 | $[3,-3,-1,4,0,1]$ | $[1,- \frac{3}{2}, - \frac{3}{2}, - \frac{3}{2}, 1]$ | $[0,2,0,3,3], [1,0,1,0,4,2]$ | $[0,0,2,0,3,3], [1,0,1,0,4,2]$ |
| 1205 | $[4,-3,-1,4,-1,1]$ | $[5, - \frac{3}{2}, - \frac{3}{2}, - \frac{3}{2}, 1]$ | $[0,1,2,0,2,2], [0,3,0,0,0,6]$ | $[1,0,1,1,2,4], [1,1,0,1,1,6]$ |
| 1130 | $[5,-2,-2,3,0,1]$ | $[3,-1,-2,2,-1,1]$ | $[4,0,0,2,0,0], [4,0,1,0,1,2]$ | $[4,0,0,2,0,0], [4,0,1,0,1,2]$ |
| 1129 | $[5,-2,-2,3,0,1]$ | $[3,-1,-2,2,-1,1]$ | $[0,0,2,0,3,3], [0,2,0,0,1,7]$ | $[0,0,2,0,3,3], [0,2,0,0,1,7]$ |
| 1128 | $[3,-1,-2,4,-3,3]$ | $[ \frac{3}{2}, - \frac{3}{2}, - \frac{3}{2}, 2, - \frac{3}{2}, 2]$ | $[0,1,0,0,5,1], [1,0,1,0,4,2]$ | $[0,1,0,0,5,1], [1,0,1,0,4,2]$ |

TABLE 8. Infinitesimal character $[1,1,1,0,1,1]$

| #x | \lambda | \nu | Spin LKTs | #x' |
|----|--------|-----|----------|-----|
| 1789 | $[1,1,2,0,2,1]$ | $[1,1,1,0,1,1]$ | $[0,0,0,0,0,0], n \beta, 1 \leq n \leq 4$ | |
| 1225 | $[1,4,1,-1,1,1]$ | $[1,\frac{5}{2},1,- \frac{2}{2},1,1]$ | $[0,0,3,0,0,7], [0,4,0,0,0,6]$ | $[0,1,2,1,0,8]$ |
| 1154 | $[1,3,2,-2,2,1]$ | $[1,4,3,-4,3,1]$ | $[4,0,0,0,4,0], [4,0,1,0,1,2]$ | $[4,0,0,0,4,0], [4,0,1,0,1,2]$ |
| 1624 | $[1,1,4,-1,1,1]$ | $[1,1,4,-3,1,1]$ | $[0,3,0,0,0,0], n \beta, 0 \leq n \leq 2$ | $[0,3,0,0,0,0], n \beta, 0 \leq n \leq 2$ |
| 1534 | $[2,2,2,-1,1,1]$ | $[1,1,2,2,2,2]$ | $[0,0,0,0,3,7], n \beta, 0 \leq n \leq 2$ | $[0,0,0,0,3,7], n \beta, 0 \leq n \leq 2$ |

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A complete list of special unipotent representations of real exceptional groups, see http://www.liegroups.org/tables/unipotentExceptional/.

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