MARKOVIAN RETRIAL QUEUES WITH TWO WAY COMMUNICATION

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Abstract. In this paper, we first consider single server retrial queues with two way communication. Ingoing calls arrive at the server according to a Poisson process. Service times of these calls follow an exponential distribution. If the server is idle, it starts making an outgoing call in an exponentially distributed time. The duration of outgoing calls follows another exponential distribution. An ingoing arriving call that finds the server being busy joins an orbit and retries to enter the server after some exponentially distributed time. For this model, we present an extensive study in which we derive explicit expressions for the joint stationary distribution of the number of ingoing calls in the orbit and the state of the server, the partial factorial moments as well as their generating functions. Furthermore, we obtain asymptotic formulae for the joint stationary distribution and the factorial moments. We then extend the study to multiserver retrial queues with two way communication for which a necessary and sufficient condition for the stability, an explicit formula for average number of ingoing calls in the servers and a level-dependent quasi-birth-and-death process are derived.

1. Introduction. Recently, retrial queues are paid much attention because they have applications in performance analysis of various systems such as call centers, computer networks and telecommunication systems [3, 12, 17, 28]. Retrial queues are characterized by the fact that customers (i.e., calls) that cannot receive service upon arrival enter a virtual orbit and retry for service again after some random time. The arrival flow from the orbit makes the underlying Markov chain of retrial queues to be nonhomogeneous. As a result, analysis of retrial queues is much more difficult than that of the corresponding queueing models without retrials and explicit results are obtained only in a few special cases [3, 12, 23, 24].

Hypergeometric functions and their special versions play an important role in the derivation of analytical solutions for retrial queues. In fact, the stationary characteristics of the system state of the conventional M/M/1/1 retrial queue are expressed in terms of special hypergeometric functions [3, 12, 24]. A review of the

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existing literature shows that the hypergeometric functions are also a key tool to analyze the stationary characteristics (i.e., limiting probabilities of the system state and their partial generating functions) of a wide variety of retrial queues including single server queues with Bernoulli abandonment [12, 24], retrial queues of M/M/1/1 type with Bernoulli abandonment and feedback [9], single server retrial queues with orbital search and nonpersistent customers [19], the M/M/1/1 retrial queue with linear retrial policy [2] and the M/M/2/2 retrial queue [14]. Maybe, the latest example is due to Kim [16] who studies a single server retrial queue with collision and impatience using hypergeometric functions.

In most literature on retrial queues, the server only serves ingoing arriving calls. After serving a call, the server waits either for the next arrival of a primary call or for a retrial call. However, there exist real life situations where the servers have a chance to make outgoing phone calls. The most obvious application arises in daily life because everybody uses a phone line, or a mobile phone, to receive calls but also for making calls to outside. Moreover, in various service systems such as a call center, an operator not only serves ingoing calls but it also makes outgoing phone calls if he or she is free. While the server is busy, ingoing arriving calls cannot receive a service. We assume that these calls join an orbit and retry to occupy the server after some exponentially distributed time independently of other calls.

Nowadays, call center business is very important because it provides a channel for two way communication between companies and their customers [1, 18, 26]. Typically, there are two types of call centers: inbound and outbound call centers. The former is used for customer support where customers call from outside for some requests such as reservation of tickets and confirmation of credit card information or complaint about products, etc. [27]. On the other hand, the latter is used for telephone marketing where a telephone dialing system randomly makes directed calls to potential customers for advertising or selling new products [26]. Recently, modern call centers integrate both inbound and outbound functions to increase the productivity [7, 10]. These are referred to as blended call centers where an operator not only receives ingoing calls but also makes phone calls to customers, when he or she is idle.

Bhulai and Koole [7] propose a multiserver queueing model with infinite buffer for blended call centers for which optimal and nearly optimal policies are derived for the case where ingoing calls and outgoing calls follow the same exponential distribution and otherwise, respectively. Deslauriers et al. [10] develop five Markovian queueing models for blended call centers where ingoing and outgoing calls are distinguished and undistinguished. As is pointed out in [10], the models where ingoing and outgoing calls follow different distributions are more difficult than that with the same service time distribution for both types of calls. In these papers [7, 10], retrials are not taken into account.

Falin [11] derives integral formulae for the partial generating functions and explicit expressions for some expected performance measures of an M/G/1/1 retrial queue with two way communication in which ingoing calls and outgoing calls are assumed to follow the same service distribution. Choi et al. [8] extend Falin’s model to M/G/1/K retrial queues where ingoing and outgoing calls are also assumed to follow the same service time distribution. However, from an application point of view, this assumption is restrictive because ingoing calls and outgoing calls may have different service time distributions. Artalejo and Resing [5] obtain the first
partial moments for the M/G/1/1 retrial queue with different service time distributions of ingoing and outgoing calls by using a mean value analysis approach. It should be noted that the mean value analysis cannot be used to derive the stationary distribution as well as higher factorial moments.

In this paper, the term two way communication refers to the fact that the server is able to make outgoing calls while it is not engaged in conversation. There are a number of retrial models which are related to this definition of the two way communication feature. In fact, from an analytical point of view, the two way communication model can even be viewed as a particular case of other existing models which, at their origin, were designed for modelling other different queueing features. This is the case of the references [21, 6, 12].

Martin and Artalejo [21] consider an M/G/1/1 queue with two type of impatient units which can be seen as a retrial queue with two way communication. In [21], a blocked customer is stored in an orbit queue from which only the customer in the head of the queue can retry after an exponentially distributed time. Avrachenkov et al. [6] use matrix analytic methods to study a single server retrial queue with two classes of customers whose retrial behaviors and service time distributions are different. The arrivals occur according to a marked Markovian arrival process. There is no doubt that the consideration of generalized Markovian arrivals allowing correlation is an interesting goal. However, it should be noted that the methodology used in [6] does not yield explicit solutions. Falin and Templeton [12] present a preliminary analysis on multiclass M/G/1/1 retrial queues for which a system of equations for the average numbers of customers in the orbit is presented. The authors in [12] also point out some open problems for the model which need further investigation.

The existing bibliography on retrial queues is vast and rich. As a result, in addition to the above mentioned references, it would be possible to find other retrial models related to the two way communication queue under study here. In general, among the closest retrial variants, we mention multiclass, priority and impatient retrial models. For a general overview, the reader is referred to Section 2.3 in Artalejo and Gomez-Corral [3], as well as to the updated bibliography [4].

The first and main aim of this paper is to provide a more extensive analysis of the M/M/1/1 retrial queue with two way communication and different service time distributions of ingoing and outgoing calls. In particular, we provide explicit solutions for the joint stationary distribution of the state of the server and the number of customers in the orbit, the partial factorial moments and their generating functions. We also present recursive formulae for the stationary distribution and the partial factorial moments based on which both symbolic and numerical algorithms can be implemented. Furthermore, we derive some simple asymptotic formulae for the stationary distribution and the partial factorial moments.

The second aim of this paper is to discuss an extension to multiserver retrial queues with two way communication and different distributions of ingoing and outgoing calls for which we obtain some explicit results. In particular, we establish the necessary and sufficient condition for the stability of the system and derive an explicit formula for the average number of ingoing calls in the servers. In addition, we formulate the multiserver model by a level-dependent quasi-birth-and-death (QBD) process, which can be used for a numerical investigation. We hope that our model is useful for performance analysis of blended call centers.
The rest of the paper is organized as follows. Section 2 describes the model in detail. Section 3 is devoted to the main results of this paper in which an extensive study of the M/M/1/1 retrial queue with two way communication is presented. In Section 4, we discuss an extension to a multiserver retrial queue with two way communication and obtain some explicit results. Finally, we conclude our paper and present some future research topics in Section 5.

2. Model description and preliminaries. In this section, we present the mathematical description of the single server retrial queue with two way communication in detail and provide some preliminaries which will be used in the main results presented in Section 3. We separate the multiserver model to Section 4 because the methodology for the multiserver model is different from that for the single server model.

2.1. Queueing model. We consider a single server retrial queue with two way communication. Primary ingoing calls arrive at the server according to a Poisson process with rate $\lambda$. An ingoing call that sees the server being busy enters an orbit and retries to occupy the server after an exponentially distributed time with mean $1/\mu$. In addition, we assume that if the server is idle then it makes an outgoing call after an exponentially distributed time with mean $1/\alpha$. The service times of the ingoing and the outgoing calls are exponentially distributed with mean $1/\nu_1$ and $1/\nu_2$, respectively. See Figure 1 for transitions among states.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{transitions.png}
\caption{Transitions among states.}
\end{figure}
2.2. Markov chain and balance equations. Let \( S(t) \) denote the state of the server,
\[
S(t) = \left\{ \begin{array}{ll}
0, & \text{if the server is idle,} \\
1, & \text{if the server is providing an incoming service,} \\
2, & \text{if the server is calling outside,}
\end{array} \right.
\]
and let \( N(t) \) be the number of calls in the orbit at time \( t \). It is easy to see that \((S(t), N(t); t \geq 0)\) forms a Markov chain on the state space \( \{0, 1, 2\} \times \mathbb{Z}_+ \), where \( \mathbb{Z}_+ = \{0, 1, 2, \ldots\} \). In what follows, we consider the system under the stability condition which will be derived later. Furthermore, let
\[
\pi_{i,j} = \lim_{t \to \infty} \Pr(S(t) = i, N(t) = j), \quad i = 0, 1, 2, \quad j \in \mathbb{Z}_+,
\]
denote the joint stationary distribution of the system state.

The system of balance equations for \( \{\pi_{i,j}; i = 0, 1, 2, j \in \mathbb{Z}_+\} \) is given by
\[
\begin{align*}
(\lambda + \alpha + j\mu)\pi_{0,j} &= \nu_1 \pi_{1,j} + \nu_2 \pi_{2,j}, \\
(\lambda + \nu_1)\pi_{1,j} &= \lambda \pi_{0,j} + (j+1)\mu\pi_{0,j+1} + \lambda \pi_{1,j-1}, \\
(\lambda + \nu_2)\pi_{2,j} &= \alpha \pi_{0,j} + \lambda \pi_{2,j-1},
\end{align*}
\]
for \( j \in \mathbb{Z}_+ \), where \( \pi_{i,-1} = 0 \) (\( i = 1, 2 \)).

Let \( \Pi_i(z) \) denote the partial generating functions
\[
\Pi_i(z) = \sum_{j=0}^{\infty} \pi_{i,j} z^j, \quad i = 0, 1, 2, \quad |z| \leq 1.
\]

Multiplying (1)-(3) by \( z^j \) and taking the sum over \( j \) yields
\[
\begin{align*}
(\lambda + \alpha)\Pi_0(z) + \mu z \Pi'_0(z) &= \nu_1 \Pi_1(z) + \nu_2 \Pi_2(z), \\
(\lambda + \nu_1)\Pi_1(z) &= \lambda \Pi_0(z) + \mu \Pi'_0(z) + \lambda z \Pi_1(z), \\
(\lambda + \nu_2)\Pi_2(z) &= \alpha \Pi_0(z) + \lambda \Pi_2(z).
\end{align*}
\]
Summing up equations (4)-(6) and rearranging the result, we obtain
\[
\lambda(\Pi_1(z) + \Pi_2(z))(z-1) = \mu \Pi'_0(z)(z - 1).
\]
Dividing both sides of the above formula by \( (z - 1) \) yields
\[
\lambda(\Pi_1(z) + \Pi_2(z)) = \mu \Pi'_0(z).
\]
Note that equation (7) represents a balance between the flows coming into and out the orbit.

2.3. Hypergeometric functions. In this section, we give a brief summary on hypergeometric functions, which will be used to obtain our explicit results in the sequel. For a complex number \( x \), let
\[
(x)_j = \left\{ \begin{array}{ll}
1, & j = 0, \\
x(x+1) \cdots (x+j-1), & j \in \mathbb{N},
\end{array} \right.
\]
denote the Pochhammer symbol, where \( \mathbb{N} = \{1, 2, \ldots\} \). Then, for complex numbers \( a, b, c \) and \( z \), the hypergeometric function \( F(a, b; c; z) \) is defined as
\[
F(a, b; c; z) = \sum_{j=0}^{\infty} \frac{(a)_j(b)_j}{(c)_j j!} \frac{z^j}{j!}, \quad |z| \leq 1.
\]
It should be noted that
\[ F(a,1; 1; z) = \sum_{j=0}^{\infty} \frac{(-a)(-a-1) \cdots (-a-j+1)}{j!} (-z)^j = (1-z)^{-a}, \]
where the last equality follows from the generalized Newton binomial formula. This formula will be frequently used throughout the paper.

2.4. Asymptotic formulae.

**Proposition 2.1** (Theorem VI.12, p. 434 in [13]). Let \( a(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( b(z) = \sum_{n=0}^{\infty} b_n z^n \) denote two power series with radii of convergence \( r_a > r_b \geq 0 \), respectively. Assume that \( b(z) \) satisfies the test
\[ \lim_{n \to \infty} \frac{b_n}{b_{n-1}} = r_b. \]
Then the coefficients of the product \( g(z) = a(z)b(z) \) satisfy
\[ [z^n]g(z) \sim a(r_b) b_n, \quad n \to \infty, \]
provided that \( a(r_b) \neq 0 \), where \( [z^n]g(z) \) denotes the coefficient of \( z^n \) in the power series expansion of \( g(z) \) and \( x_n \sim y_n \) is defined by \( \lim_{n \to \infty} x_n/y_n = 1 \).

**Proposition 2.2** (See p. 377 in [13]). For any complex number \( a \) whose real part is positive, we have
\[ [z^n](1-z)^{-a} \sim \frac{n^{a-1}}{\Gamma(a)}, \quad n \to \infty, \]
where \( \Gamma(a) \) is the Euler Gamma function defined as
\[ \Gamma(a) = \int_{0}^{\infty} e^{-t} t^{a-1} dt. \]

**Corollary 2.1.** For any complex number \( a \) whose real part is positive and positive number \( \gamma \), we have
\[ [z^n](1-\gamma z)^{-a} \sim \frac{n^{a-1}\gamma^n}{\Gamma(a)}, \quad n \to \infty. \]

**Proof.** Let \( x = \gamma z \). We have
\[ [z^n](1-x)^{-a} = \gamma^n [x^n](1-x)^{-a} \sim \frac{n^{a-1}\gamma^n}{\Gamma(a)}, \quad n \to \infty, \]
where the \( \sim \) follows from Proposition 2.2.

**Proposition 2.3.** Let \( \{a_n, b_n, \tilde{a}_n, \tilde{b}_n; n \in \mathbb{Z}_+\} \) denote sequences of real numbers such that
\[ a_n \sim \tilde{a}_n, \quad b_n \sim \tilde{b}_n, \quad n \to \infty, \quad \lim_{n \to \infty} \frac{\tilde{a}_n}{\tilde{b}_n} = 0. \]
Then, we have \( a_n + b_n \sim \tilde{b}_n \) as \( n \to \infty \).

**Proof.** The result is straightforward from the definition. Indeed, we have
\[ \lim_{n \to \infty} \frac{a_n + b_n}{b_n} = \lim_{n \to \infty} \left( \frac{a_n}{\tilde{a}_n} + \frac{b_n}{\tilde{b}_n} \right) = 1. \]

We apply these three propositions to derive asymptotic results in Section 3.6.
3. **Main results.** In this section, we consider the main case where \( \nu_1 \neq \nu_2 \) and \( \nu_1 \neq \lambda + \nu_2 \). Later on, in Appendix A, we will derive analytical results for the special cases \( \nu_1 = \nu_2 \) and \( \nu_1 = \lambda + \nu_2 \).

3.1. **Generating functions.** First, we derive explicit expressions for the partial generating functions.

**Theorem 3.1.** Explicit expressions for the partial generating functions are as follows:

\[
\Pi_0(z) = \frac{1 - \rho}{1 + \sigma} \left( \frac{1 - \theta}{1 - \theta z} \right)^{D_1} \left( \frac{1 - \rho}{1 - \rho z} \right)^{D_2} = \pi_{0,0} F \left( \frac{D_1}{\mu}, 1; 1; \theta z \right) F \left( \frac{D_2}{\mu}, 1; 1; \rho z \right),
\]

(11)

\[
\Pi_1(z) = \left( \frac{\lambda + C_2}{\nu_1 - \lambda z} + \frac{C_1}{\lambda + \nu_2 - \lambda z} \right) \Pi_0(z),
\]

(12)

\[
\Pi_2(z) = \frac{\alpha}{\lambda + \nu_2 - \lambda z} \Pi_0(z),
\]

(13)

where

\[
C_1 = \frac{\lambda \alpha}{\nu_1 - (\lambda + \nu_2)}, \quad C_2 = -\frac{\lambda \alpha}{\nu_1 - (\lambda + \nu_2)},
\]

\[
D_1 = \frac{\alpha (\nu_1 - \nu_2)}{\nu_1 - (\lambda + \nu_2)}, \quad D_2 = \frac{\lambda (\nu_1 - (\lambda + \alpha + \nu_2))}{\nu_1 - (\lambda + \nu_2)},
\]

\[
\theta = \frac{\lambda}{\lambda + \nu_2}, \quad \rho = \frac{\lambda}{\nu_1}, \quad \sigma = \frac{\alpha}{\nu_2},
\]

and

\[
\pi_{0,0} = \frac{1 - \rho}{1 + \sigma} \left( 1 - \theta \right)^{D_1} \left( 1 - \rho \right)^{D_2}.
\]

(14)

**Remark 3.1.** We observe that \( \Pi_0(z) \) is expressed in terms of a product of two special hypergeometric functions. Note also that \( \rho \) and \( \sigma \) denote the traffic intensity of ingoing calls and the traffic intensity of outgoing calls, respectively.

**Proof.** Equation (13) immediately follows from (6). Substituting (7) into (4), we obtain

\[
(\lambda + \alpha) \Pi_0(z) + \lambda z (\Pi_1(z) + \Pi_2(z)) = \nu_1 \Pi_1(z) + \nu_2 \Pi_2(z).
\]

By combining the above equation and (13), we find that

\[
\Pi_1(z) = \left( \frac{\lambda}{\nu_1 - \lambda z} + \frac{\lambda \alpha}{(\lambda + \nu_2 - \lambda z) (\nu_1 - \lambda z)} \right) \Pi_0(z)
\]

\[
= \left( \frac{\lambda + C_2}{\nu_1 - \lambda z} + \frac{C_1}{\lambda + \nu_2 - \lambda z} \right) \Pi_0(z),
\]

(15)

where the following equality

\[
\frac{\lambda \alpha}{(\lambda + \nu_2 - \lambda z) (\nu_1 - \lambda z)} = \frac{C_1}{\lambda + \nu_2 - \lambda z} + \frac{C_2}{\nu_1 - \lambda z}
\]

has been used and we have assumed that \( \nu_1 \neq \lambda + \nu_2 \).

From (7), (12) and (13), we obtain

\[
\Pi_0(z) = \frac{\lambda}{\mu} (\Pi_1(z) + \Pi_2(z)) = \frac{\lambda}{\mu} \left( \frac{\alpha + C_1}{\lambda + \nu_2 - \lambda z} + \frac{\lambda + C_2}{\nu_1 - \lambda z} \right) \Pi_0(z).
\]

(16)
Since $D_1 = \alpha + C_1$ and $D_2 = \lambda + C_2$, formula (16) reduces to
\[
\frac{\Pi_0'(z)}{\Pi_0(z)} = \frac{\lambda}{\mu} \left( \frac{D_1}{\lambda + \nu_2 - \lambda z} + \frac{D_2}{\nu_1 - \lambda z} \right),
\] (17)
and solving the differential equation (17), we have
\[
\Pi_0(z) = \Pi_0(1) \left( \frac{\nu_2}{\lambda + \nu_2 - \lambda z} \right)^{\frac{\nu_1}{\mu}} \left( \frac{\nu_1 - \lambda}{\nu_1 - \lambda z} \right)^{\frac{\nu_2}{\mu}}.
\]
From (12) and (13), we obtain
\[
\Pi_1(1) = \frac{\lambda(\alpha + \nu_2)}{\nu_2(\nu_1 - \lambda)} \Pi_0(1), \quad \Pi_2(1) = \frac{\alpha}{\nu_2} \Pi_0(1).
\] (18)
Therefore, it follows from the normalizing condition
\[
\Pi_0(1) + \Pi_1(1) + \Pi_2(1) = 1
\] and (18) that
\[
\Pi_0(1) = \frac{1 - \rho}{1 + \sigma}, \quad \Pi_1(1) = \rho, \quad \Pi_2(1) = \frac{(1 - \rho)\sigma}{1 + \sigma}.
\] (19)

**Remark 3.2.** Equation (19) implies that the necessary and sufficient condition for the stability of the system is given by $\rho < 1$.

### 3.2. Stationary distribution.

Our goal in this section is to derive explicit expressions for $\{\pi_{i,j}; i = 0, 1, 2, j \in \mathbb{Z}_+\}$.

**Theorem 3.2.** Explicit expressions for the stationary distribution are given by
\[
\pi_{0,j} = \pi_{0,0} \sum_{k=0}^j \left( \frac{D_1}{\mu} \right)_k \frac{\theta^k}{k!} \left( \frac{D_2}{\mu} \right)_{j-k} \frac{\rho^{j-k}}{(j-k)!},
\] (20)
\[
\pi_{1,j} = \frac{1}{\lambda + \nu_1} \sum_{k=0}^j \left( \lambda \pi_{0,k} + (k+1)\mu \pi_{0,k+1} \right) \left( \frac{\lambda}{\lambda + \nu_1} \right)^{j-k}
\] = \frac{C_1}{\lambda + \nu_2} \sum_{k=0}^j \pi_{0,k} \theta^{j-k} + \left( \rho + \frac{C_2}{\nu_1} \right) \sum_{k=0}^j \pi_{0,k} \rho^{j-k},
\] (21)
\[
\pi_{2,j} = \frac{\alpha}{\lambda + \nu_2} \sum_{k=0}^j \pi_{0,k} \theta^{j-k},
\] (22)
for $j \in \mathbb{Z}_+$.

**Proof.** According to (11), we have
\[
\Pi_0(z) = \frac{1 - \rho}{1 + \sigma} \left( \frac{1 - \theta}{1 - \theta z} \right)^{\frac{\nu_1}{\mu}} \left( \frac{1 - \rho}{1 - \rho z} \right)^{\frac{\nu_2}{\mu}}
\] = \pi_{0,0} \left( 1 - \theta z \right)^{-\frac{\nu_1}{\mu}} \left( 1 - \rho z \right)^{-\frac{\nu_2}{\mu}}
\] = \pi_{0,0} \left( \sum_{j=0}^\infty \left( \frac{D_1}{\mu} \right)_j \frac{\theta^j}{j!} z^j \right) \left( \sum_{j=0}^\infty \left( \frac{D_2}{\mu} \right)_j \frac{\rho^j}{j!} z^j \right),
\]
whose inversion leads to expression (20).

It follows from (13) that

\[ \Pi_2(z) = \frac{\alpha}{\lambda + \nu_2} \Pi_0(z) \frac{1}{1 - \theta z} = \frac{\alpha}{\lambda + \nu_2} \left( \sum_{j=0}^{\infty} \pi_{0,j} z^j \right) \left( \sum_{j=0}^{\infty} \theta^j z^j \right), \] (23)

Therefore, by inverting (23), we obtain (22).

Finally, from (5), we have

\[ \Pi_1(z) = \frac{\lambda \Pi_0(z) + \mu \Pi_0'(z)}{\lambda + \nu_1 - \lambda z} \]
\[ = \frac{1}{\lambda + \nu_1} \left( \sum_{j=0}^{\infty} (\lambda \pi_{0,j} + (j + 1)\mu \pi_{0,j+1}) z^j \right) \left( \sum_{j=0}^{\infty} \left( \frac{\lambda}{\lambda + \nu_1} \right)^j z^j \right). \] (24)

Thus, we obtain the first equality of (21). Furthermore, it follows from (12) that \( \{\pi_{1,j}; j \in \mathbb{Z}_+\} \) can also be computed by using the second expression in (21).

3.3. Factorial moments. We now deal with the partial factorial moments \( \{M^i_k; i = 0, 1, 2, k \in \mathbb{Z}_+\} \) defined by

\[ M^i_k = \sum_{j=k}^{\infty} (j - k + 1)_k \pi_{i,j}, \quad i = 0, 1, 2, \quad k \in \mathbb{Z}_+. \]

The moments of order \( k = 0 \) are trivially given by

\[ M^0_0 = \Pi_0(1), \quad M^1_0 = \Pi_1(1), \quad M^2_0 = \Pi_2(1). \]

It is easy to see that

\[ \Pi_i(1 + z) = \sum_{k=0}^{\infty} \frac{M^i_k}{k!} z^k, \quad i = 0, 1, 2, \] (25)

and therefore \( M^i_k \) \( (k \in \mathbb{Z}_+) \) can be obtained from the coefficient of \( z^k \) in the series \( \Pi_i(1 + z) \). Using the same techniques as used in the proof of Theorem 3.2, we obtain the following results.

**Theorem 3.3.** The partial factorial moments are given by

\[ M^0_k = M^0_0 k! \sum_{j=0}^{k} \frac{(D_1/\mu)_j \hat{\theta}^j (D_2/\mu)_{k-j} \hat{\rho}^{k-j}}{j!}, \] (26)

\[ M^1_k = k! \sum_{j=0}^{k} \frac{\lambda M^0_j + \mu M^0_{j+1}}{\nu_1 j!} \hat{\rho}^{k-j} \]
\[ = \frac{1}{1 - \rho} \left( \rho + \frac{C_2}{\nu_1} \right) k! \sum_{j=0}^{k} \frac{M^0_j}{j!} \hat{\theta}^{j-k} + \frac{C_1}{\nu_2} k! \sum_{j=0}^{k} \frac{M^0_j}{j!} \hat{\theta}^{k-j}, \] (27)

\[ M^2_k = \sigma k! \sum_{j=0}^{k} \frac{M^0_j}{j!} \hat{\theta}^{k-j}, \] (28)

for \( k \in \mathbb{Z}_+ \), where

\[ \hat{\rho} = \frac{\rho}{1 - \rho}, \quad \hat{\theta} = \frac{\theta}{1 - \theta} = \frac{\lambda}{\nu_2}. \]
3.4. **Recursive formulae.** In the previous sections 3.2 and 3.3, we have derived explicit expressions for the stationary distribution and the partial factorial moments. However, since the obtained formulae involve the auxiliary quantities $C_1$, $C_2$, $D_1$ and $D_2$, which may be either positive or negative, a computational implementation based on these formulae may be numerically unstable. In order to resolve this problem, we next present simple recursive schemes to compute the stationary probabilities and their partial factorial moments. It should be noted that these recursive formulae can be used for both symbolic or numerical implementations.

3.4.1. **Recursive formulae for stationary distribution.**

**Theorem 3.4.** The stationary probabilities can be computed from the following recursive formulae:

\[
\pi_{0,j} = \frac{\lambda (\pi_{1,j-1} + \pi_{2,j-1})}{j\mu}, \quad j \in \mathbb{N}, \tag{29}
\]

\[
\pi_{2,j} = \frac{\alpha \pi_{0,j} + \lambda \pi_{2,j-1}}{\lambda + \nu_2}, \quad j \in \mathbb{Z}_+, \tag{30}
\]

\[
\pi_{1,j} = \frac{\lambda (\pi_{0,j} + \pi_{1,j-1} + \pi_{2,j})}{\nu_1 - \lambda}, \quad j \in \mathbb{Z}_+. \tag{31}
\]

We remember that $\pi_{0,0}$ was given in (14).

**Proof.** We first use (7) to derive the level-crossing equation (29). Equation (30) agrees with (3). Finally, formula (31) is obtained by combining (29) and (2). \qed

3.4.2. **Recursive formulae for factorial moments.**

**Theorem 3.5.** We have the following recursive formulae for the partial factorial moments:

\[
M_{k}^0 = \frac{\lambda (M_{k-1}^1 + M_{k-1}^2)}{\mu}, \quad k \in \mathbb{N}, \tag{32}
\]

\[
M_{k}^2 = \frac{\alpha M_{k}^0 + k\lambda M_{k-1}^2}{\nu_2}, \quad k \in \mathbb{N}, \tag{33}
\]

\[
M_{k}^1 = \frac{\lambda (M_{k}^0 + kM_{k-1}^1 + M_{k}^2)}{\nu_1 - \lambda}, \quad k \in \mathbb{N}. \tag{34}
\]

Expressions for the moments of order $k = 0$ were given in (19).

**Proof.** By differentiating formulae (7) and (6) $k$ times at $z = 1$, we obtain

\[
\mu M_{k+1}^0 = \lambda (M_k^1 + M_k^2), \quad k \in \mathbb{Z}_+, \tag{35}
\]

\[
(\lambda + \nu_2)M_k^2 = \alpha M_k^0 + \lambda (M_k^2 + kM_{k-1}^2), \quad k \in \mathbb{Z}_+. \tag{36}
\]

Equations (32) and (33) now follow from (35) and (36), respectively.

On the other hand, by summing up (5) and (7) and also differentiating the resulting equation $k$ times, we find that

\[
\nu_1 M_k^1 = \lambda (M_k^0 + M_k^1 + kM_{k-1}^1 + M_k^2). \tag{37}
\]

Therefore, equation (34) follows from (37). \qed
3.5. First moments and cost model.

3.5.1. First moments. In this section, for the sake of completeness, we summarize some simple formulae for the first moments of the number of customers in the orbit.

By combining (19) and (32), we have

\[ M_0^1 = \frac{\lambda}{\mu} (M_0^1 + M_0^2) = \frac{\lambda (\rho + \sigma)}{\mu (1 + \sigma)}. \]  

(38)

It follows from (33) and (38) that

\[ M_1^2 = \sigma \left( \frac{\lambda (\rho + \sigma) + \lambda (1 - \rho)}{\mu (1 + \sigma)} + \frac{\lambda (1 - \rho)}{\nu_2 (1 + \sigma)} \right). \]  

(39)

Finally, from (34), (38) and (39), we obtain

\[ M_1^1 = \rho^2 + \rho \left( \frac{\rho^2}{1 - \rho} + \frac{\lambda \sigma}{\nu_2 (1 + \sigma)} + \frac{\lambda (\rho + \sigma)}{\mu (1 - \rho)} \right). \]  

(40)

The above formulae (38)-(40) are consistent with those derived in the existing literature [5, 11].

3.5.2. Cost model. Let \( U \) denote the utilization of the server, i.e.,

\[ U = M_0^1 + M_0^2 = \frac{\rho + \sigma}{1 + \sigma}, \]

where (19) is used in the second equality.

From formulae (38)-(40), we also obtain the mean number of customers in orbit which is given by

\[ E[N] = \frac{\rho^2}{1 - \rho} + \frac{\lambda \sigma}{\nu_2 (1 + \sigma)} + \frac{\lambda (\rho + \sigma)}{\mu (1 - \rho)}. \]

From a management point of view, we need to minimize the idle ratio of the server, \( 1 - U \), but at the same time, from a quality of service (QoS) point of view, we also need to minimize \( E[N] \). Thus, our objective is to find an optimal \( \sigma \) which satisfies both these needs. For this purpose, we consider the following minimization problem:

\[ \min f(\sigma) = A (1 - U) + B E[N], \]

\[ \text{s.t. } \sigma \geq 0, \]

where \( A \) and \( B \) are positive costs and \( \rho, \mu, \lambda \) and \( \nu_2 \) are kept constant. Since \( \nu_2 \) remains constant, to minimize with respect to \( \sigma \) in fact amounts to minimize with respect to the outgoing call rate \( \alpha \).

Remark 3.3. Our motivation for the above choice of the cost function is that the outgoing call rate \( \alpha \) is directly under the control of the server. The chance for controlling other system parameters (e.g., the ingoing call rate, \( \lambda \), or the retrial rate, \( \mu \)) is much more reduced in practice. However, we may consider other static optimization problems by introducing in the cost function other performance measures (e.g., blocking probabilities, waiting time indicators). A more sophisticated dynamic approach based on Markov decision processes may also be considered but it is not the objective of this paper.
We now express $f(\sigma)$ as

$$f(\sigma) = \frac{C}{1 + \sigma} + D\sigma + E,$$

where

$$C = A(1 - \rho) - \frac{B\lambda}{\nu_2}, \quad D = \frac{B\lambda}{\mu(1 - \rho)},$$

$$E = B \left( \frac{\rho^2}{1 - \rho} + \frac{\lambda}{\nu_2} + \frac{\lambda\rho}{\mu(1 - \rho)} \right).$$

Thus, we have

$$f'(\sigma) = -\frac{C}{(1 + \sigma)^2} + D.$$

If $C \leq D$, then $f'(\sigma) \geq 0$. Therefore, $f(\sigma)$ is minimized at $\sigma^* = 0$. On the other hand, if $C > D$, then $f(\sigma)$ is minimized at

$$\sigma^* = \sqrt{\frac{C}{D}} - 1.$$

3.6. **Asymptotic analysis.** The limitations of the explicit formulae, as long as they are expressed as finite sums involving positive and negative quantities, were mentioned in Section 3.4. The alternative recursive scheme is helpful for computing the stationary probabilities but it does not provide explicit expressions. In what follows, we supplement the exact and recursive formulae by deriving simple asymptotic formulae both for the stationary distribution and the factorial moments. In addition to their inherent value as limiting results, the asymptotic formulae are simpler than their explicit counterparts (see Theorems 3.6 and 3.7). As a result, a sensitivity analysis of the main performance measures can be carried out easier when it is based on the asymptotic formulae rather than on the explicit ones.

3.6.1. **Asymptotic formulae for the stationary distribution.**

**Theorem 3.6.** We have the following asymptotic results for the stationary distribution. If $\rho > \theta$, then we have

$$\pi_{0,n} \sim \frac{\pi_{0,0}}{\Gamma \left( \frac{D_2}{\mu} \right)} \left( 1 - \frac{\theta}{\rho} \right)^{-\frac{D_1}{\mu}} n^{\frac{D_2}{\mu} - 1} \rho^n,$$  \hspace{1cm} (41)

$$\pi_{2,n} \sim \frac{\pi_{2,0}}{\Gamma \left( \frac{D_2}{\mu} \right)} \left( 1 - \frac{\theta}{\rho} \right)^{-\frac{D_1 + \mu}{\mu}} n^{\frac{D_2}{\mu}} \rho^n,$$  \hspace{1cm} (42)

$$\pi_{1,n} \sim \frac{\pi_{0,0} D_2}{\nu_1 \Gamma \left( \frac{D_2 + \mu}{\mu} \right)} \left( 1 - \frac{\theta}{\rho} \right)^{-\frac{D_1}{\mu}} n^{\frac{D_2}{\mu}} \rho^n.$$  \hspace{1cm} (43)
as $n \to \infty$. For the case $\theta > \rho$, we have

\begin{align}
\pi_{0,n} &\sim \frac{\pi_{0,0} \left(1 - \frac{\theta}{\rho}\right) - \frac{D_2}{\rho}}{\Gamma \left(\frac{D_1}{\mu}\right)} n^{\frac{D_1}{\mu} - 1} \theta^n, \\
\pi_{2,n} &\sim \frac{\pi_{2,0} \left(1 - \frac{\theta}{\rho}\right) - \frac{D_2}{\rho}}{\Gamma \left(\frac{D_1 + \mu}{\mu}\right)} n^{\frac{D_1}{\mu} - 1} \theta^n, \\
\pi_{1,n} &\sim \frac{\pi_{1,0} C_1 \left(1 - \frac{\theta}{\rho}\right) - \frac{D_2}{\rho}}{(\lambda + \nu_2)\Gamma \left(\frac{D_1 + \mu}{\mu}\right)} n^{\frac{D_1}{\mu} - 1} \theta^n,
\end{align}

as $n \to \infty$.

**Remark 3.4.** We observe that $\pi_{0,n}$ and $\pi_{2,n}$ have the same order in the case of $\rho > \theta$, while $\pi_{1,n}$ and $\pi_{2,n}$ have the same order in the case of $\rho < \theta$.

**Remark 3.5.** We confirm that our asymptotic results for the case $\rho > \theta$ are consistent with those derived by Kim et al. [15] for the conventional M/M/1/1 retrial queue without outgoing calls (i.e., $\alpha = 0$).

**Proof.** First, we derive asymptotic formulae for $\{\pi_{0,n}; n \in \mathbb{Z}_+\}$. It follows from (11) and (14) that

\[ \Pi_0(z) = \pi_{0,0}(1 - \theta z)^{-\frac{D_1}{\rho}}(1 - \rho z)^{-\frac{D_2}{\rho}} = \pi_{0,0}a(z)b(z), \]

where

\[ a(z) = (1 - \theta z)^{-\frac{D_1}{\rho}}, \quad b(z) = (1 - \rho z)^{-\frac{D_2}{\rho}}. \]

Let $r_a$ and $r_b$ denote the convergence radius of $a(z)$ and $b(z)$ respectively. For the case $\rho > \theta \iff \nu_1 < \lambda + \nu_2$, we see that $D_2 > 0$. On the other hand, $D_1$ may be either positive or negative. Therefore, $r_b = 1/\rho$ while $r_a = 1/\theta$ if $D_1 > 0$ and $r_a = \infty$ if $D_1 < 0$. In any case, $r_b < r_a$. Furthermore, the coefficients of $b(z)$ satisfy (8). Indeed,

\[ \frac{\left(\frac{D_2}{\rho}\right)_{n-1}}{(n-1)!} \div \frac{\left(\frac{D_2}{\rho}\right)_n}{n!} = \frac{n}{\frac{D_2}{\rho} + n - 1} \rho^n \to r_b = \frac{1}{\rho}, \quad n \to \infty. \]

According to Proposition 2.1, we have

\[ [z^n]a(z)b(z) \sim a(1/\rho)[z^n]b(z), \quad n \to \infty. \]

Furthermore, using Proposition 2.2 yields

\[ [z^n]b(z) \sim \frac{n^{\frac{D_2}{\rho} - 1} \rho^n}{\Gamma \left(\frac{D_2}{\rho}\right)}, \quad n \to \infty. \]

Equations (47), (48) and (49) imply (41).

Second, we derive asymptotic formulae for $\{\pi_{2,n}; n \in \mathbb{Z}_+\}$. From (13), we have

\[ \Pi_2(z) = \pi_{2,0}(1 - \theta z)^{-\frac{D_1 + \mu}{\rho}}(1 - \rho z)^{-\frac{D_2}{\rho}}. \]

Using the same arguments as in the derivation of (41), we can easily obtain (42).
Finally, we derive asymptotic formulae for \( \{\pi_{1,n}; n \in \mathbb{Z}_+\} \). It follows from (11) and (12) that

\[
\Pi_1(z) = \frac{D_2}{\nu_1} \pi_{0,0}(1 - \theta z)^{-\frac{\rho_1}{\rho}} (1 - \rho z)^{-\frac{\rho_2 + \mu}{\mu}} + \frac{C_1}{\lambda + \nu_2} \pi_{0,0}(1 - \theta z)^{-\frac{\rho_1 + \mu}{\rho}} (1 - \rho z)^{-\frac{\rho_2}{\mu}}.
\]

(50)

Applying the same techniques as used in the derivation of (41) for the two components in the right hand side of (50), we find that

\[
[z^n] \frac{D_2}{\nu_1} (1 - \theta z)^{-\frac{\rho_1}{\rho}} (1 - \rho z)^{-\frac{\rho_2 + \mu}{\mu}} \sim \frac{D_2}{\nu_1} \left(1 - \frac{\theta}{\rho}\right)^{-\frac{\rho_1}{\rho}} \frac{n^{\frac{\rho_2}{\rho}} \rho^n}{\Gamma\left(\frac{D_2 + \mu}{\mu}\right)},
\]

(51)

\[
[z^n] \frac{C_1}{\lambda + \nu_2} (1 - \theta z)^{-\frac{\rho_1 + \mu}{\rho}} (1 - \rho z)^{-\frac{\rho_2}{\mu}} \sim \frac{C_1}{\lambda + \nu_2} \left(1 - \frac{\theta}{\rho}\right)^{-\frac{\rho_1 + \mu}{\rho}} \frac{n^{\frac{\rho_2 - 1}{\rho}} \rho^n}{\Gamma\left(\frac{D_2}{\mu}\right)},
\]

(52)

as \( n \to \infty \). Therefore, from (51), (52) and Proposition 2.3, we obtain (43).

Asymptotic formulae (44), (45) and (46) for the case \( \theta > \rho \) can be derived following similar arguments to that used above for the case \( \rho > \theta \). Thus, we omit the proof.

\[\Box\]

3.6.2. Asymptotic formulae for the factorial moments. In what follows, we present asymptotic results for the factorial moments \( \{M^n_i; i = 0, 1, 2, n \in \mathbb{Z}_+\} \). First, we observe that \( \rho > \theta \iff \hat{\rho} > \hat{\theta} \) and \( \rho < \theta \iff \hat{\rho} < \hat{\theta} \).

We can now derive the following asymptotic results for the factorial moments using the same techniques as used in Section 3.6.1.

**Theorem 3.7.** If \( \rho > \theta \), then we have

\[
M^n_0 \sim \frac{M^n_0}{\Gamma\left(\frac{D_2}{\mu}\right)} \frac{1 - \frac{\theta}{\rho}}{\lambda + \nu_2} \frac{\rho_2 - 1}{\rho} n^{\frac{\rho_2}{\rho}} \hat{\rho}^n n! \sim \frac{M^n_0}{\Gamma\left(\frac{D_2}{\mu}\right)} \frac{1 - \frac{\theta}{\rho}}{\lambda + \nu_2} \frac{\rho_2 - 1}{\rho} \sqrt{2\pi n} n^{\frac{\rho_2}{\rho}} \hat{\rho}^n n!,
\]

(53)

\[
M^n_2 \sim \frac{M^n_2}{\Gamma\left(\frac{D_2 + \mu}{\mu}\right)} \frac{1 - \frac{\theta}{\rho}}{\lambda + \nu_2} \frac{\rho_2 - 1}{\rho} n^{\frac{\rho_2}{\rho}} \hat{\rho}^n n! \sim \frac{M^n_2}{\Gamma\left(\frac{D_2 + \mu}{\mu}\right)} \frac{1 - \frac{\theta}{\rho}}{\lambda + \nu_2} \frac{\rho_2 - 1}{\rho} \sqrt{2\pi n} n^{\frac{\rho_2}{\rho}} \hat{\rho}^n n!,
\]

(54)

\[
M^n_1 \sim \frac{M^n_1}{(\nu_1 - \lambda)\Gamma\left(\frac{D_2 + \mu}{\mu}\right)} \frac{1 - \frac{\theta}{\rho}}{\lambda + \nu_2} \frac{\rho_2 - 1}{\rho} n^{\frac{\rho_2}{\rho}} \hat{\rho}^n n! \sim \frac{M^n_1}{(\nu_1 - \lambda)\Gamma\left(\frac{D_2 + \mu}{\mu}\right)} \frac{1 - \frac{\theta}{\rho}}{\lambda + \nu_2} \frac{\rho_2 - 1}{\rho} \sqrt{2\pi n} n^{\frac{\rho_2}{\rho}} \hat{\rho}^n n!,
\]

(55)
as \( n \to \infty \). For the case \( \theta > \rho \), we have

\[
M_n^0 \sim \frac{M_0^0 \left(1 - \frac{\rho}{\theta}\right)^{-D_2}}{\Gamma \left(\frac{D_1}{\mu}\right)} n^{D_2 - 1} \tilde{\theta}^n n! \sim \frac{M_0^0 \left(1 - \frac{\rho}{\theta}\right)^{-D_2}}{e^n \Gamma \left(\frac{D_1}{\mu}\right)} \sqrt{2\pi n^{n+\frac{D_2}{\mu}+\frac{1}{2}} \tilde{\theta}^n}, \quad (56)
\]

\[
M_n^2 \sim \frac{M_0^2 \left(1 - \frac{\rho}{\theta}\right)^{-D_2}}{\Gamma \left(\frac{D_1+\mu}{\mu}\right)} n^{D_2} \tilde{\theta}^n n! \sim \frac{M_0^2 \left(1 - \frac{\rho}{\theta}\right)^{-D_2}}{e^n \Gamma \left(\frac{D_1+\mu}{\mu}\right)} \sqrt{2\pi n^{n+\frac{D_2}{\mu}+\frac{1}{2}} \tilde{\theta}^n}, \quad (57)
\]

\[
M_n^1 \sim \frac{M_0^1 C_1 \left(1 - \frac{\rho}{\theta}\right)^{-D_2}}{\nu_2 \Gamma \left(\frac{D_1+\mu}{\mu}\right)} n^{D_2} \tilde{\theta}^n n! \sim \frac{M_0^1 C_1 \left(1 - \frac{\rho}{\theta}\right)^{-D_2}}{\nu_2 e^n \Gamma \left(\frac{D_1+\mu}{\mu}\right)} \sqrt{2\pi n^{n+\frac{D_2}{\mu}+\frac{1}{2}} \tilde{\theta}^n}, \quad (58)
\]

as \( n \to \infty \).

**Remark 3.6.** Similar to the stationary distribution, we also observe that \( M_n^0 \) and \( M_n^2 \) have the same order in the case of \( \rho > \theta \), while the order of \( M_n^1 \) and of \( M_n^2 \) is the same in the case of \( \rho < \theta \).

**Proof.** From (11), (12) and (13), we observe that the structure of \( \Pi_i(1+z) \) \((i = 0, 1, 2)\) is similar to that of \( \Pi_i(z) \) where the poles are replaced by \( \hat{\rho}^{-1} \) instead of \( \rho^{-1} \) and \( \tilde{\theta}^{-1} \). Therefore, using the same arguments as used in the proof of Theorem 3.6 and the fact that

\[
M_n^i = n! [z^n] \Pi_i(1+z), \quad i = 0, 1, 2, \quad n \in \mathbb{Z}_+,
\]

yields the first \( \sim \) in (53) to (58). The second \( \sim \) of (53) to (58) follows from the first \( \sim \) and the Stirling formula:

\[
n! \sim \sqrt{2\pi n^{n+\frac{1}{2}} e^{-n}}, \quad n \to \infty.
\]

\[\square\]

### 3.6.3. Numerical validation of asymptotic formulae.

In this section, we present some numerical examples to show the tail asymptotic behavior of the join stationary distribution. We set \( \mu = 1, \nu_1 = 1 \) and \( \alpha = 1 \). Figure 2 represents \( \{\pi_{i,n}^0; i = 0, 1, 2, n \in \mathbb{Z}_+\} \) against \( n \) for the case \( \lambda = 0.9 \) and \( \nu_2 = 2.5 \) for which \( \rho > \theta \). Figure 3 shows \( \{\pi_{i,n}^2; i = 0, 1, 2, n \in \mathbb{Z}_+\} \) against \( n \) for the case \( \lambda = 0.1 \) and \( \nu_2 = 0.01 \) for which \( \theta > \rho \). In both figures, \( \{\pi_{i,n}^1; i = 0, 1, 2, n \in \mathbb{Z}_+\} \) computed by recursive formulae presented in Theorem 3.4 and those calculated by asymptotic formulae in Theorem 3.6 are plotted. We observe in both figures that the curves by asymptotic formulae are well fitted to those by the recursive formulae when the number of customers in the orbit is large. The observation suggests that these asymptotic formulae can be used to estimate exact values with high accuracy for the case of a large number of retrial customers.

We also observe from Figure 2 that \( \pi_{1,n} \) dominates \( \pi_{0,n} \) and \( \pi_{2,n} \) and that the curves of \( \pi_{0,n} \) and \( \pi_{2,n} \) are asymptotically parallel when \( n \) is large. These observations agree with the asymptotic results presented in (41), (42) and (43), where \( \rho^{-1} \) is the dominant pole. On the other hand, Figure 3 shows that the curves of \( \pi_{1,n} \) and \( \pi_{2,n} \) are asymptotically parallel and that \( \pi_{0,n} \) is dominated by \( \pi_{1,n} \) and \( \pi_{2,n} \) when \( n \) is large. These results are consistent with asymptotic formulae (44), (45) and (46), where \( \theta^{-1} \) is the dominant pole.
4. A multiserver retrial queue with two way communication.

4.1. Queueing model. In this section, we present a generalization of our single server model. In particular, we consider an M/M/c/c retrial queue with two way
communication, where definitions of \( \lambda, \alpha, \nu_1, \nu_2 \) and \( \mu \) are the same as those of the single server model. An arriving customer that finds all the servers being busy joins the orbit. The behavior of the servers in the M/M/c/c retrial queue is the same as that of the single server model, i.e., each idle server makes an outgoing call in an exponentially distributed time with mean \( 1/\alpha \). For this model, first we obtain an explicit formula for the average number of ingoing calls in the servers. Second, we establish a necessary and sufficient condition for the ergodicity. Third, we express the underlying Markov chain as a level-dependent quasi-birth-and-death (QBD) process.

4.2. Markov chain and ergodic condition. Let \( S_1(t), S_2(t) \) and \( N(t) \) denote the numbers of ingoing calls and outgoing calls in the servers and the number of ingoing calls in the orbit at time \( t \geq 0 \), respectively. It is easy to see that the process \( \{\chi(t); t \geq 0\} = \{(S_1(t), S_2(t), N(t)); t \geq 0\} \) forms a Markov chain on the state space \( S \) defined by

\[
S = \{(i, j, k); i = 0, 1, \ldots, c, j = 0, 1, \ldots, c - i, k \in \mathbb{Z}_+\}.
\]

**Lemma 4.1.** Let \( E[S_1] \) denote the average number of ingoing calls in the servers at the steady state. We have

\[
E[S_1] = \frac{\lambda}{\nu_1}.
\]

**Proof.** A rigorous proof of Lemma 4.1 is given in Appendix B. \( \square \)

**Remark 4.1.** This result can be obtained by appealing to a variant of Little’s formula for the servers [20], where \( E[S_1] \) is the long-run average of the ingoing calls in the servers, \( \lambda \) is the arrival rate of ingoing calls and, of course, \( 1/\nu_1 \) is the mean ingoing service time. Although the validity of Little’s law is almost universal, in Appendix B we give an ad hoc proof for our two way communication retrial queue. This particular proof provides further insight as far as several equations have a nice meaningful balance flow interpretation.

**Theorem 4.2.** The process \( \{\chi(t); t \geq 0\} \) is ergodic if and only if \( \lambda < cv_1 \).

**Proof.** A proof of Theorem 4.2 is presented in Appendix C. \( \square \)

4.3. Level-dependent QBD process. It is easy to see that \( \{\chi(t); t \geq 0\} \) forms a level-dependent QBD process, where \( N(t) \) and \( \{S_1(t), S_2(t)\} \) are referred to as the level and the phase, respectively. The infinitesimal generator of the process is given by

\[
Q = \begin{pmatrix}
Q_{0,0} & Q_{0,1} & O & O & \cdots \\
Q_{1,0} & Q_{1,1} & Q_{1,2} & O & \cdots \\
O & Q_{2,1} & Q_{2,2} & Q_{2,3} & \cdots \\
O & O & Q_{3,2} & Q_{3,3} & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix},
\]

where \( O \) denotes a matrix with an appropriate size whose entries are all zero, while \( Q_{k,k-1} \) (\( k \in \mathbb{N} \)), \( Q_{k,k} \) and \( Q_{k,k+1} \) (\( k \in \mathbb{Z}_+ \)) are explicitly given in Appendix D. Since the stability condition and explicit block matrices of the infinitesimal generator are explicitly given, we could apply several approximation methods [22, 25] in order to obtain numerical results. However, because the objective of this paper is to derive explicit results, we omit here a detailed numerical analysis.
5. Conclusion and future work. We have analyzed the M/M/1/1 retrial queue with two way communication in detail. In particular, we have derived explicit expressions for the stationary distribution and the partial factorial moments. We have also derived recursive formulae based on which both numerical and symbolic algorithms can be implemented. In addition, a cost model has been presented in order to find the optimal rate of outgoing calls. Furthermore, some simple asymptotic formulae for the stationary distribution and partial factorial moments have also been obtained.

As for the M/M/c/c retrial queue, we have established a necessary and sufficient condition for the stability and have derived an explicit formula for the average number of ingoing calls in the servers. Furthermore, we have presented a level-dependent QBD process of the model for which numerical analysis could be carried out by several methods presented in literature [22, 25].

For the future work, we pay attention to the consideration of impatient customers in multiserver retrial queues with two way communication. Another extension is the consideration of single server retrial queues with two way communication with MAP arrivals and more general service time distributions of ingoing and outgoing calls.

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Appendix A. Special cases. This section presents results for the special cases $\nu_1 = \nu_2$ and $\nu_1 = \lambda + \nu_2$. While the former corresponds just with the particular case where both the ingoing and the outgoing calls receive identical service times, the latter implies some minor mathematical differences.

A.1. The case $\nu_1 = \nu_2$. In the special case $\nu_1 = \nu_2$, the stationary probabilities $\{\pi_{0,j}; j \in \mathbb{Z}_+\}$ reduce to

$$\pi_{0,j} = \frac{(1 - \rho)^{\lambda + \alpha + 1}}{1 + \sigma} \left(\frac{\lambda + \alpha}{\mu}\right)^j j^j, \quad j \in \mathbb{Z}_+,$$

and their generating function is given by

$$\Pi_0(z) = \frac{1 - \rho}{1 + \sigma} \left(\frac{1 - \rho z}{1 - \rho}\right)^{-\lambda + \alpha \mu} = \pi_{0,0} F\left(\frac{\lambda + \alpha}{\mu}, 1; 1; \rho z\right),$$

leading to the factorial moments

$$M_k^0 = \frac{1 - \rho}{1 + \sigma} \left(\frac{\lambda + \alpha}{\mu}\right)^k \rho^k, \quad k \in \mathbb{Z}_+.$$

On the other hand, the probabilities $\pi_{1,j}$ and $\pi_{2,j}$ ($j \in \mathbb{Z}_+$) can be expressed in terms of $\pi_{0,j}$ by the same formulae as (21) and (22). It is easy to confirm that these results are consistent with those derived in Section 3 where $D_1 = 0$ due to $\nu_1 = \nu_2$, and also with those presented by Falin [11].
Remark A.1. We notice that the case $\nu_1 = \nu_2$, as it appears in the literature [11], does not distinguish if the service in progress corresponds either to an ingoing call or to an outgoing call. In this short section, we have showed that the results in this paper are helpful to keep knowledge of the identity of the call that is receiving service.

A.2. The case $\nu_1 = \lambda + \nu_2$. Finally, we consider the case $\nu_1 = \lambda + \nu_2$. It follows from (7), (13) and the first equality in (15) that the differential equation for $\Pi_0(z)$ is given by

$$
\frac{\Pi_0'(z)}{\Pi_0(z)} = \frac{\lambda}{\mu} \left( \frac{\lambda\alpha}{(\nu_1 - \lambda z)^2} + \frac{\lambda + \alpha}{\nu_1 - \lambda z} \right). \tag{59}
$$

A comparison with equation (17), for the case $\nu_1 \neq \lambda + \nu_2$, shows that the point $\rho^{-1}$ is now a pole of order 2 in the right hand side of (59) instead of order 1 as in the right hand side of (17). The solution of (59) is given by

$$
\Pi_0(z) = \Pi_0(1) \left( \frac{1 - \rho}{1 - \rho z} \right)^{\frac{\lambda + \alpha}{\mu}} \exp \left( \frac{\alpha\rho(1 - 1)}{\mu(1 - \rho z)} \right),
$$

where we have used (19) is used in the second equality and

$$
\pi_{0,0} = \left( \frac{1 - \rho}{1 + \sigma} \right)^{\frac{\lambda + \alpha}{\mu} + 1} \exp \left( -\frac{\alpha\rho}{\mu} \right). \tag{60}
$$

Remark A.2. We observe that the generating function is expressed in terms of a product of a special hypergeometric function and an exponential function.

The first partial moments for the number of customers in the orbit are given by

$$
M_0^0 = \frac{\hat{\theta}(\lambda(1 - \rho) + \alpha)}{\mu(1 + \sigma)},
$$

$$
M_1^1 = \frac{\hat{\theta}(1 + 2\sigma)}{1 + \sigma} + \frac{\lambda(1 - \rho) + \alpha}{\mu(1 - \rho)},
$$

$$
M_2^2 = \sigma \left( \frac{\hat{\theta}(1 - \rho)}{1 + \sigma} + \frac{\hat{\theta}(\lambda(1 - \rho) + \alpha)}{\mu(1 + \sigma)} \right).
$$

Remark A.3. It should be noted that Algorithm 1 and Algorithm 2 can be applied for the above special cases $\nu_1 = \nu_2$ and $\nu_1 = \lambda + \nu_2$, where $\pi_{0,0}$ is computed by (14) and (60), respectively.

Appendix B. Proof of Lemma 4.1. Let $\{\pi_{i,j,k}; (i, j, k) \in S\}$ denote the stationary distribution, i.e., we have

$$
\pi_{i,j,k} = \lim_{t \to \infty} \Pr(S_1(t) = i, S_2(t) = j, N(t) = k).
$$
The forward Kolmogorov equations of \( \{ \chi(t); t \geq 0 \} \) are given by

\[
(\lambda + (c - i - j)\alpha + i\nu_1 + j\nu_2 + k\mu)\pi_{i,j,k} \\
= \lambda\pi_{i-1,j,k} + (c - i - j + 1)\alpha\pi_{i,j-1,k} + (k + 1)\mu\pi_{i-1,j,k+1} \\
+ (i + 1)\nu_1\pi_{i+1,j,k} + (j + 1)\nu_2\pi_{i,j+1,k}, \quad i + j = 0, 1, \ldots, c - 1, \tag{61}
\]

\[
(\lambda + i\nu_1 + (c - i)\nu_2)\pi_{i,c-i,k} \\
= \lambda\pi_{i-1,c-i,k} + \alpha\pi_{i,c-i-1,k} + (k + 1)\mu\pi_{i-1,c-i,k+1} + \lambda\pi_{i,c-i,k-1}, \tag{62}
\]

where \( k \in \mathbb{Z}_+ \) and \( \pi_{i,j,k} = 0 \) if \((i, j, k) \notin \mathbb{S} \).

Adding (61) and (62) over \( i \) and \( j \) yields

\[
\sum_{i=0}^{c-1} \sum_{j=0}^{c-1-i} (\lambda + (c - i - j)\alpha + i\nu_1 + j\nu_2 + k\mu)\pi_{i,j,k} \\
+ \sum_{i=0}^{c-1} (\lambda + i\nu_1 + (c - i)\nu_2)\pi_{i,c-i,k} \\
= \lambda \sum_{i=0}^{c-1} \sum_{j=0}^{c-1-i} \pi_{i-1,j,k} + \lambda \sum_{i=0}^{c-1} \pi_{i-1,c-i,k} \\
+ \alpha \sum_{i=0}^{c-1} \sum_{j=1}^{c-i} (c - i - j + 1)\pi_{i,j-1,k} + \alpha \sum_{i=0}^{c-1} \pi_{i,c-i-1,k} \\
+ (k + 1)\mu \sum_{i=0}^{c-1} \sum_{j=0}^{c-1-i} \pi_{i-1,j,k+1} + (k + 1)\mu \sum_{i=0}^{c-1} \pi_{i-1,c-i,k+1} \\
+ \nu_1 \sum_{i=0}^{c-1} \sum_{j=0}^{c-1-i} (i + 1)\pi_{i+1,j,k} + \nu_2 \sum_{i=0}^{c-1} \sum_{j=0}^{c-1-i} (j + 1)\pi_{i,j+1,k} + \lambda \sum_{i=0}^{c-1} \pi_{i,c-i,k-1}. \tag{63}
\]

Deleting the same quantities in both sides of (63), we obtain

\[
\lambda \sum_{i=0}^{c} \pi_{i,c-i,k} + k\mu \sum_{i=0}^{c-1} \sum_{j=0}^{c-1-i} \pi_{i,j,k} = \lambda \sum_{i=0}^{c} \pi_{i,c-i,k-1} + \lambda \sum_{i=0}^{c-1} \sum_{j=0}^{c-1-i} \pi_{i,j,k+1},
\]

which implies

\[
\lambda \sum_{i=0}^{c} \pi_{i,c-i,k} = (k + 1)\mu \sum_{i=0}^{c-1} \sum_{j=0}^{c-1-i} \pi_{i,j,k+1}, \quad k \geq 0, \tag{64}
\]

because \( \pi_{i,c-i,-1} = 0 \). Equation (64) represents the balance between the flows coming into and out the orbit.

Let \( \pi_{i,j} \) denote a marginal distribution with respect to \( k \), i.e., \( \pi_{i,j} = \sum_{k=0}^{\infty} \pi_{i,j,k} \). Summing up (64) over \( k \) yields

\[
\lambda \sum_{i=0}^{c} \pi_{i,c-i,} = \mu \sum_{i=0}^{c-1} \sum_{j=0}^{c-1-i} \sum_{k=0}^{\infty} k\pi_{i,j,k}. \tag{65}
\]
Furthermore, summing up (61) and (62) over $k$ yields

$$(\lambda + (c - i - j)\alpha + i\nu_1 + j\nu_2)\pi_{i,j} + \mu \sum_{k=0}^{\infty} k\pi_{i,j,k}$$

$$= \lambda\pi_{i-1,j} + (c - i - j + 1)\alpha\pi_{i,j-1} + \mu \sum_{k=0}^{\infty} k\pi_{i-1,j,k}$$

$$+ (i + 1)\nu_1\pi_{i+1,j} + (j + 1)\nu_2\pi_{i,j+1}, \quad i + j = 0, 1, \ldots, c - 1,$$

$$\eqref{66}$$

$$(\lambda + i\nu_1 + (c - i)\nu_2)\pi_{i,c-i}. $$

$$= \lambda\pi_{i-1,c-i} + \alpha\pi_{i,c-i-1} + \mu \sum_{k=0}^{\infty} k\pi_{i-1,c-i,k} + \lambda\pi_{i,c-i}, \quad i = 0, 1, \ldots, c. \eqref{67}$$

Summing up (66) and (67) over $j$ yields

$$i = 0, 1, \ldots, c - 1$$

$$(i + 1)\nu_1 \sum_{j=0}^{c-i-1} \pi_{i+1,j} + \nu_2 \sum_{j=0}^{c-i-1} (j + 1)\pi_{i,j+1}. $$

Arranging this equation, we obtain

$$\lambda \sum_{j=0}^{c-i-1} \pi_{i,j} + \mu \sum_{j=0}^{c-i-1} \sum_{k=0}^{\infty} k\pi_{i,j,k},$$

$$= \lambda \sum_{j=0}^{c-i-1} \pi_{i-1,j} + \mu \sum_{j=0}^{c-i-1} \sum_{k=0}^{\infty} k\pi_{i-1,j,k} + (i + 1)\nu_1 \sum_{j=0}^{c-i-1} \pi_{i+1,j}. $$

$$\eqref{68}$$

for $i = 0, 1, \ldots, c - 1$. This equation implies that

$$\lambda \sum_{j=0}^{c-i-1} \pi_{i,j} + \mu \sum_{j=0}^{c-i-1} \sum_{k=0}^{\infty} k\pi_{i,j,k} = (i + 1)\nu_1 \sum_{j=0}^{c-i-1} \pi_{i+1,j},$$

because $\pi_{-1,j} = 0$ and $\pi_{-1,j,k} = 0$.

It should be noted that (68) represents the balance between the flows coming into and out the state $i$. Summing up (68) over $i$, we have

$$\lambda \sum_{j=0}^{\infty} \sum_{i=0}^{c-1} \pi_{i,j} + \mu \sum_{j=0}^{\infty} \sum_{i=0}^{c-1} \sum_{k=0}^{\infty} k\pi_{i,j,k} = \nu_1 \sum_{i=0}^{c-1} (i + 1) \sum_{j=0}^{c-1} \pi_{i+1,j}.$$

$$\eqref{69}$$
By substituting (65) into (69), we then obtain
\[
\lambda \sum_{i=0}^{c-1} \sum_{j=0}^{c-i-1} \pi_{i,j} + \lambda \sum_{i=0}^{c} \sum_{j=0}^{c-i} \pi_{i,c-i,j} = \nu_1 \sum_{i=0}^{c} \sum_{j=0}^{c-i} \pi_{i,j}.
\]
Therefore, Lemma 4.1 follows from (70) due to the fact that
\[
\lambda \sum_{i=0}^{c-1} \sum_{j=0}^{c-i-1} \pi_{i,j} + \lambda \sum_{i=0}^{c} \sum_{j=0}^{c-i} \pi_{i,c-i,j} = \lambda, \quad \sum_{i=0}^{c} \sum_{j=0}^{c-i} \pi_{i,j} = \mathbb{E}[S].
\]

Appendix C. Proof of Theorem 4.2. The necessary condition immediately follows from Lemma 4.1. We next show the sufficient condition; that is, if \( \lambda < c\nu_1 \) then \( \{\chi(t); t \geq 0\} \) is ergodic. To this end, we use the Foster’s criterion for the discrete time Markov chain \( \{\zeta_n; n \in \mathbb{Z}_+\} \) embedded at the transition epochs of \( \{\chi(t); t \geq 0\} \). The Foster’s criterion states that an irreducible and aperiodic Markov chain \( \{\zeta_n; n \in \mathbb{Z}_+\} \) is ergodic if there exists some non-negative function \( f(s) \) \((s \in \mathcal{S})\) and some positive \( \epsilon \) such that the mean drifts \( \phi(s) = \mathbb{E}[f(\zeta_{n+1}) - f(\zeta_n) | \mathcal{Z}_n = s] \leq -\epsilon \), for all \( s \in \mathcal{S} \), except for perhaps a finite number of states.

Let \( Q = (q_{(i,j,k),(i',j',k')}) \) \((i,j,k),(i',j',k') \in \mathcal{S}\) denote the infinitesimal generator of \( \{\chi(t); t \geq 0\} \). Its elements are given by
\[
q_{(i,j,k),(i',j',k')} = \begin{cases} 
\lambda, & (i',j',k') = (i+1,j,k), \\
(c-i-j)\alpha, & (i',j',k') = (i+1,j,k), \\
i\nu_2, & (i',j',k') = (i,j,k), \\
j\nu_2, & (i',j',k') = (i,j,k), \\
k\mu, & (i',j',k') = (i,j,k), \\
-\xi_{i,j,k}, & (i',j',k') = (i,j,k), \\
0, & \text{otherwise},
\end{cases}
\]
where \( q_{i,j,k} = \lambda + (c-i-j)\alpha + i\nu_1 + j\nu_2 + k\mu \), for \( i+j = 0,1,\ldots,c-1 \), and
\[
q_{(i,c-i,k),(i',j',k')} = \begin{cases} 
\lambda, & (i',j',k') = (i,c-i,k+1), \\
(c-i)\nu_2, & (i',j',k') = (i,c-i,k), \\
-\xi_{i,c-i,k}, & (i',j',k') = (i,c-i,k), \\
0, & \text{otherwise},
\end{cases}
\]
where \( q_{i,c-i,k} = \lambda + i\nu_1 + (c-i)\nu_2 \), for \( i = 0,1,\ldots,c \).

The transition probability matrix \( P = (p_{(i,j,k),(i',j',k')}) \) \((i,j,k),(i',j',k') \in \mathcal{S}\) is given either by
\[
p_{(i,j,k),(i',j',k')} = \frac{q_{(i,j,k),(i',j',k')}}{q_{i,j,k}},
\]
if \((i',j',k') \neq (i,j,k)\) or by 0 if \((i',j',k') = (i,j,k)\). We observe that
\[
\sup_{(i,j,k) \in \mathcal{S}} q_{i,j,k} = +\infty, \quad \inf_{(i,j,k) \in \mathcal{S}} q_{i,j,k} > \lambda > 0,
\]
which guarantees that \( \{\chi(t); t \geq 0\} \) is ergodic if the embedded chain is ergodic. To apply the Foster’s criterion to \( \{\zeta_n; n \in \mathbb{Z}_+\} \), we consider a test function \( f(i,j,k) = ai + bj + k \), where \( a \geq 0 \) and \( b \geq 0 \) will be appropriately determined later. Let \( \phi(i,j,k) \) denote the mean drift associated with \( f(i,j,k) \), so we have
\[
\phi(i, j, k) = \sum_{(i', j', k') \neq (i, j, k)} p_{(i, j, k), (i', j', k')} (f(i', j', k') - f(i, j, k)), \quad (i, j, k) \in S.
\]

We find that
\[
\phi(i, j, k) = \frac{a\lambda + b(c - i - j)\alpha - a\nu_1 - b\nu_2 + (a - 1)k\mu}{\lambda + (c - i - j)\alpha + i\nu_1 + j\nu_2 + k\mu}, \tag{71}
\]
for \(i + j = 0, 1, \ldots, c - 1\), and
\[
\phi(i, c - i, k) = \frac{\lambda - a\nu_1 - b(c - i)\nu_2}{\lambda + i\nu_1 + (c - i)\nu_2},
\]
for \(i = 0, 1, \ldots, c\). Choosing \(b = a\nu_1/\nu_2\), \(\phi(i, c - i, k)\) is reduced to
\[
\phi(i, c - i, k) = \frac{\lambda - ac\nu_1}{\lambda + i\nu_1 + (c - i)\nu_2}.
\]
From (71), we have \(\lim_{k \to \infty} \phi(i, j, k) = a - 1\) for \(i + j = 0, 1, \ldots, c - 1\). In order to get \(\phi(i, j, k) \leq -\epsilon\) except for a finite number of states, we choose an \(a\) such that
\[
\frac{\lambda}{c\nu_1} < a < 1.
\]
This is possible because \(\lambda < c\nu_1\). Furthermore, we have
\[
\phi(i, c - i, k) = \frac{\lambda - ac\nu_1}{\lambda + i\nu_1 + (c - i)\nu_2} \leq \frac{\lambda - ac\nu_1}{\lambda + c\max(\nu_1, \nu_2)}.
\]
Therefore, if we choose
\[
\epsilon = \min \left(\frac{1 - a}{2}, \frac{ac\nu_1 - \lambda}{\lambda + c\max(\nu_1, \nu_2)}\right),
\]
then there exists some integer \(k_0\) such that \(\phi(i, j, k) \leq -\epsilon\), for \(k \geq k_0\). Therefore, \(\{\zeta_n; n \in \mathbb{Z}_+\}\) is ergodic. This implies that \(\{\chi(t); t \geq 0\}\) is also ergodic.

**Appendix D. Block matrices of the infinitesimal generator.** The block matrices \(Q_{k,k-1} (k \in \mathbb{N})\), \(Q_{k,k} (k \in \mathbb{Z}_+)\) and \(Q_{k,k+1} (k \in \mathbb{Z}_+)\) are explicitly written as
follows:

\[
Q_{k,k-1} = \begin{pmatrix}
O & N_0^{(k)} & O & \cdots & O \\
O & O & N_1^{(k)} & \ddots & \\
\vdots & \ddots & \ddots & \ddots & O \\
O & \cdots & O & N_i^{(k)} & \\
O & \cdots & \cdots & O & O
\end{pmatrix},
\]

\[
Q_{k,k} = \begin{pmatrix}
A_{0,0}^{(k)} & A_{0,1} & O & \cdots & \cdots & O \\
A_{1,0}^{(k)} & A_{1,1} & A_{1,2} & \ddots & \ddots & \\
O & A_{2,1} & A_{2,2} & \ddots & \ddots & O \\
\vdots & \ddots & \ddots & \ddots & \ddots & O \\
O & \cdots & \cdots & A_{c-1,1} & A_{c-1,0} & A_{c,1} \\
O & \cdots & O & A_{c,2} & A_{c,1}
\end{pmatrix},
\]

\[
Q_{k,k+1} = \begin{pmatrix}
A_0 & O & O & \cdots & O \\
O & \Lambda_1 & O & \ddots & \\
\vdots & \ddots & \ddots & \ddots & O \\
O & \cdots & \cdots & \Lambda_{c-1} & O \\
O & \cdots & O & \Lambda_c
\end{pmatrix},
\]

where \(N_i^{(k)}\) \((i = 0, 1, \ldots, c-1)\), \(A_1\) \((i = 0, 1, \ldots, c)\), \(A_{i,2}\) \((i = 1, 2, \ldots, c)\), \(A_{i,3}\) \((i = 0, 1, \ldots, c)\) and \(A_{i,0}\) \((i = 0, 1, \ldots, c-1)\) are \((c-i+1)\times(c-i)\), \((c-i+1)\times(c-i+1)\), \((c-i+1)\times(c-i+2)\), \((c-i+1)\times(c-i+1)\) and \((c-i+1)\times(c-i)\) matrices, whose entries are given by

\[
N_i^{(k)}(j, j') = \begin{cases}
k\mu, & j' = j (j = 0, 1, \ldots, c-i-1), \\0, & \text{otherwise},
\end{cases}
\]

\[
A_1(j, j') = \begin{cases}
\lambda, & j' = j = c-i, \\0, & \text{otherwise},
\end{cases}
\]

\[
A_{i,2}(j, j') = \begin{cases}
iv_1, & j' = j (j = 0, 1, \ldots, c-i), \\0, & \text{otherwise},
\end{cases}
\]

\[
A_{i,3}(j, j') = \begin{cases}
(c-i-j)\alpha, & j' = j + 1 (j = 0, 1, \ldots, c-i-1), \\
-j\nu_2, & j' = j - 1 (j = 1, 2, \ldots, c-i), \\
-q_{i,j,k}, & j' = j (j = 0, 1, \ldots, c-i), \\
0, & \text{otherwise},
\end{cases}
\]

\[
A_{i,0}(j, j') = \begin{cases}
\lambda, & j' = j (j = 0, 1, \ldots, c-i-1), \\0, & \text{otherwise}.
\end{cases}
\]

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