Casimir force between surfaces close to each other

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Abstract

Casimir interactions (due to the massless scalar field fluctuations) of two surfaces which are close to each other are studied. After a brief general presentation of the technique, explicit calculations are performed for specific geometries.

I. Introduction

Experiments to observe and measure Casimir forces have so far been performed with the geometrical setups involving two (actually disconnected) surfaces [1]. The original parallel plate Casimir interaction is exact for infinite plane surfaces [2], which in practice means that valid for planes very close to each other. Effect for the parallel plane geometry were first verified in 1958 [3]. Recently the experiment for this geometry was improved to a high precision [4]. The Casimir experiments other than the above mentioned ones have been performed for a sphere close to a plane configuration [5]: which do not give rise the precise alignment problem of the parallel planes. Note that the calculation for the sphere-plane geometry gets closed to be exact if the radius of the sphere is small compared to the distance to the plane [6]. Sphere-sphere geometry has also been studied subject to the similar approximation as the sphere-plane problem [7]. The interaction of two co-centric spheres has recently been addressed [8].

Single cavity experiments so far have not been realized [8], which we think would be very interesting: For example inserting the data from quantum

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dots (i.e. radius \( \simeq 10^{-7} \text{ cm} \)) into the theoretical expression for the vacuum energy of a spherical cavity capable of confining electromagnetic field [1], one gets (in \( \hbar = c = 1 \text{ units} \)) \( 0.5 \cdot 10^6 \text{ cm}^{-1} = 10 \text{ eV} \) for the Casimir energy which is of appreciable magnitude [3]. This is comparable to the total energy between the parallel plates of the latest experiment [4], i.e. \( E = \frac{\pi^2}{720d^3}(\text{Area of Plates}) \simeq \frac{\pi^2}{720}(5 \cdot 10^{-3})^3 \cdot (2 \cdot 2) \text{ cm}^{-1} \simeq 10^6 \text{ cm}^{-1} \).

The purpose of the present work is to study new two surfaces geometries. We calculate the Casimir forces resulting from the vacuum fluctuations of massless scalar fields between surfaces close to each other. For massive fields, for any realistic experimental setups the Casimir energies are extremely small. The expressions always involve a factor \( e^{-\mu \Delta} \), where \( \mu \) is the mass and \( \Delta \) is the separation; which for electron and for nanometer distances is \( e^{-2.5 \cdot 10^{10} \cdot 10^{-7}} \simeq e^{-2500} \); thus it is practically zero.

After a brief outline of our approach to the surface surface interactions in Section 2, we proceed with specific examples, that is co-axial cylinders, co-centric tori, co-centric spheres and co-axial conical surfaces. The case of co-axial cylinders may offer an experimental test in the light of the recent advances in stable metal nanotubes [10].

II. Casimir energy for the region between two boundaries which are close to each other

We first choose the suitable spatial curvilinear coordinates \( \eta^j, j = 1, 2, 3 \) for the geometry we deal with. The corresponding Minkowski metric and the Klein-Gordon operator are then

\[
ds^2 = dt^2 - g_{ij}d\eta^id\eta^j \tag{1}
\]

and ( \( g \equiv \det(g_{ij}) \))

\[
\Delta = \frac{\partial^2}{\partial t^2} - \frac{1}{\sqrt{g}} \frac{\partial}{\partial \eta^i} \sqrt{g} \frac{\partial}{\partial \eta^i} \tag{2}
\]

The Green function is

\[
G = \sum_{\lambda_1, \lambda_2, \lambda_3} \frac{e^{i\omega(\lambda)(t-t')}}{2\omega(\lambda)} \Phi_{\omega(\lambda)}(\eta')\Phi_{\omega(\lambda)}(\eta), \tag{3}
\]
where $\Phi_{\omega(\lambda)}(\eta)$ and $\omega^2(\lambda)$ are the eigenfunctions and eigenvalues of the equation for the massless scalar field

$$-\frac{1}{\sqrt{g}} \frac{\partial}{\partial \eta^i} g^{ij} \sqrt{g} \frac{\partial}{\partial \eta^j} \Phi_{\omega(\lambda)}(\eta) = \omega^2(\lambda) \Phi_{\omega(\lambda)}(\eta).$$

(4)

(For massive scalar field one only changes $\omega^2$ by $\omega^2 + \mu^2; \text{ with } \mu \text{ being the mass} \). We assume that the above equation is separable in the spatial coordinates $\eta^j$. Here $\eta$ and $\lambda$ stand for the collection of the coordinates $\eta^j$ and the corresponding quantum numbers $\lambda_j$ (which are specified by the boundary conditions) respectively. The functions $\Phi_{\omega(\lambda)}(\eta)$ are normalized with respect to the norm

$$\|\Phi\|^2 = \int_A d^3 \eta \sqrt{g} |\Phi(\eta)|^2,$$

(5)

where $A$ is the domain of the coordinates $\eta^j$. The vacuum energy density can be then obtained by calculating the coincidence limit derivatives as:

$$T = \text{Reg} \left[ \lim_{t, \eta^j \to t', \eta'^j} \left( \frac{\partial^2}{\partial t \partial t'} + g^{ij} \frac{\partial^2}{\partial \eta^i \partial \eta'^j} \right) G(\eta, \eta') \right].$$

(6)

"Reg" stands for regularization. In the specific examples it means that we have to subtract the terms (in the Plana sum formulas to be employed over the modes) corresponding to the vacuum energy of the free space, the boundary energy etc. To calculate the Casimir energy one needs the eigenvalues of the problem. The eigenvalues $\omega^2(\lambda)$ depend on three quantum numbers $\lambda_j$ corresponding to the degrees of freedoms in directions $\eta^j$ in which we assume that the equation (4) can be separated. We further assume that after the separation of variables the eigenvalue equations in coordinates $\eta^1, \eta^2$ can be trivially solved, and the corresponding quantum numbers $\lambda_1, \lambda_2$ are easily obtained. This assumption does not introduce a strong restriction. In fact many problems in the literature are of that type. For example when one studies the Casimir energy inside a spherical cavity, only nontrivial problem is the radial equation in which one has to deal with the roots of the Bessel functions to impose the boundary condition [1].

In this work we employ an approximation method to calculate the nontrivial spectral parameter $\lambda_3$, which is valid if the problem involves two boundaries in direction $\eta^3$, which are close to each other.
After the separation, the problem in hand in $\eta_3$ can be converted into the Schrödinger form

$$[-\frac{d^2}{d(\eta_3)^2} + W_{\lambda_1\lambda_2}(\eta_3)]\Phi_{\lambda_3}(\eta_3) = E(\lambda)\Phi_{\lambda_3}(\eta_3).$$

(7)

The form of the ”potential” $W_{\lambda_1\lambda_2}(\eta_3)$ and the relation between $\omega^2(\lambda)$ and $E(\lambda)$ depend on the choice of coordinate systems. The explicit examples are given in the following sections. The boundary conditions we wish to impose for the type of geometries under investigations are

$$\Phi_{\lambda_3}(\eta_0^3) = 0, \quad \Phi_{\lambda_3}(\eta_1^3) = 0,$$

(8)

where $\eta_0^3 < \eta_1^3$. In practice these boundary conditions require dealing with the roots of special functions which are quite involved. However if the boundaries are close to each other, instead of (7) we can employ the simpler Schrödinger equation

$$[-\frac{d^2}{d(\eta_3)^2} + V^0_{\lambda_1\lambda_2}(\eta_3)]\Phi^0_{\lambda_3}(\eta_3) = E^0(\lambda)\Phi^0_{\lambda_3}(\eta_3)$$

(9)

where the constant potential in the region is given by

$$V^0_{\lambda_1\lambda_2}(\eta_3) = \begin{cases} \infty, & \eta_3 = \eta_0^3, \quad \eta_3 = \eta_1^3 \\ W_{\lambda_1\lambda_2}(\sqrt{\frac{\eta_0^3}{\eta_1^3}}), & \eta_3 \in (\eta_0^3, \eta_1^3) \end{cases}$$

(10)

The eigenvalue equation (9) has the following solutions

$$E^0(\lambda) = (\frac{\pi\lambda_3}{\Delta})^2 + W^0_{\lambda_1\lambda_2}$$

(11)

and

$$\Phi^0_{\lambda_3}(\eta_3) = \sqrt{\frac{2}{\Delta}} \sin(\frac{\pi\lambda_3}{\Delta}),$$

(12)

where $\Delta = \eta_1^3 - \eta_0^3$ and $\lambda_3 = 1, 2, \ldots$. The system given by (11) is a good approximation if the condition

$$\max_{\eta_3 \in (\eta_0^3, \eta_1^3)} |W_{\lambda_1\lambda_2}(\eta_3) - W^0_{\lambda_1\lambda_2}| \ll \min_{\lambda_3} |E^0(\lambda)|$$

(13)

is satisfied.
In the following sections we apply this approximation method to the specific geometries.

III. Casimir energy in the region between two close co-axial cylinders.

In the cylindrical coordinates, i.e. with the metric
\[
ds^2 = dt^2 - dz^2 - dr^2 - r^2 d\phi^2
\] (14)
the eigenvalue problem we have to solve is
\[
- \left[ \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \right] \Phi = \omega^2 \Phi
\] (15)

After solving for the trivial coordinates \( z \) and \( \phi \) we have
\[
\Phi = \frac{e^{ipz+im\phi}}{2\pi \sqrt{r}} v_{nm}(r).
\] (16)

Here \( v_{nm}^p(r) \) are the normalized wavefunctions corresponding to the radial equation
\[
[- \frac{d^2}{dr^2} + \frac{m^2 - 1/4}{r^2}] v_{nm} = \mu_{nm}^2 v_{nm},
\] (17)
with
\[
\omega_{nm} = \sqrt{p^2 + \mu_{nm}^2}.
\] (18)
The quantum number \( n \) should be determined from the boundary conditions on the co-axial cylinders with the radii \( r_0 < r_1 \):
\[
v_{nm}(r_0) = 0, \quad v_{nm}(r_1) = 0.
\] (19)
The solution of (17) satisfying the boundary condition at \( r_0 \) is given in terms of the Bessel functions as
\[
v_{nm}(r) = \sqrt{\mu_{nm} r} \frac{J_m(\mu_{nm} r_0) N_m(\mu_{nm} r) - J_m(\mu_{nm} r) N_m(\mu_{nm} r_0)}{\Omega_{nm}},
\] (20)
where \( \Omega_{nm} \) to be obtained from the normalization
\[
\int_{r_0}^{r_1} dr r |v_{nm}(r)|^2 = 1.
\] (21)
In practice however the above integral is very difficult to calculate for arbitrary values of \( r_0 \) and \( r_1 \). The spectrum \( \mu_{nm} \) should be determined from the boundary condition at \( r_1 \) which is quite involved equation. However if the cylindrical surfaces are close to each other we can rely on the approximation method summarized in the previous section. Instead of the eigenvalue problem (17) we consider the following one

\[
[-\frac{d^2}{dr^2} + V(r)]v^0_{nm} = (\mu^0_{nm})^2 v^0_{nm}
\]  

with the constant potential

\[
V(r) = \begin{cases} 
\infty, & r = r_0, \ r = r_1 \\
\frac{m^2 - \frac{1}{4}}{r_0 r_1}, & r \in (r_0, r_1)
\end{cases}
\]  

The above equation is then trivially solved as

\[
v^0_{nm} = \sqrt{\frac{2}{\Delta}} \sin(\mu^0_{nm}(r - r_0)); \quad \Delta \equiv r_1 - r_0
\]  

with the spectrum

\[
\mu^0_{nm} = \sqrt{\frac{\pi^2 n^2}{\Delta^2} + \frac{m^2 - \frac{1}{4}}{r_0 r_1}}; \quad n = 1, 2, 3, \ldots
\]  

For the present specific case the condition (13) is valid for \( \Delta \ll r_0 \). The Green function of the system is then easy to deal with:

\[
G = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dp e^{i\omega^0_{pnm}(t-t') + ip(z-z') + im(\phi-\phi')} \frac{v^0_{nm}(r) v^0_{nm}(r')}{8\pi^2 \omega^0_{pnm}}
\]  

where

\[
\omega^0_{pnm} = \sqrt{p^2 + (\mu^0_{nm})^2}.
\]  

We insert the above Green function into the coincidence limit formula

\[
T = Reg \left[ \frac{1}{2} \lim_{t, r, z, \phi \to t', r', z', \phi'} (\partial_t \partial_{t'} + \partial_r \partial_{r'} + \partial_z \partial_{z'} + \frac{1}{r^2} \partial_{\phi} \partial_{\phi'}) G \right],
\]  

where ”Reg” stands for the usual regularization which will be defined explicitly. The total vacuum energy per unit height is

\[
E = \int_0^{2\pi} d\phi \int_{r_0}^{r_1} r dr T = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \sum_{m=-\infty}^{\infty} Reg \left[ \sum_{n=1}^{\infty} \omega^0_{pnm} \right].
\]
To perform summations we use the Plana formula [1]:

$$\sum_{n=0}^{\infty} F(n) = F(0) \frac{2}{2} + \int_0^{\infty} dn F(n) + i \int_0^{\infty} dt \frac{F(it) - F(-it)}{e^{2\pi t} - 1}. \quad (30)$$

In the application of the above formula to the summation over the radial quantum number $n$, the $n = 0$ term and the second term, corresponding to the surface singularity and free space divergence respectively should be subtracted. Thus the regularization stands for (in $n$ summation )

$$\text{Reg} \left[ \sum_{n=1}^{\infty} F(n) \right] = \sum_{n=1}^{\infty} F(n) + F(0) \frac{2}{2} - \int_0^{\infty} dn F(n). \quad (31)$$

Going back to (29) we first perform the sum over $m$. Note that since the argument of the square root in (29) is always positive we can replace it by the absolute value

$$E = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \int_{-\infty}^{\infty} dm \text{ Reg } \sum_{n=1}^{\infty} \sqrt{|p^2 + \left( \frac{\pi n}{\Delta} \right)^2 + \frac{m^2 - 1/4}{R^2}| + }$$

$$+ \int_{-\infty}^{\infty} \frac{dp}{2\pi} \text{ Reg } \sum_{n=0}^{\infty} \text{ Reg } \sum_{m=1}^{\infty} \sqrt{\left| m^2 + p^2 + \left( \frac{\pi n}{\Delta} \right)^2 - \frac{1}{4R^2} \right|} =$$

$$= 2\pi R E(\Delta, \frac{1}{2R}) + E_1, \quad (32)$$

where $R^2 = r_0 r_1$. Here

$$E(\Delta, \frac{1}{2R}) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d^2 \vec{k}}{(2\pi)^2} \text{ Reg } \sum_{n=1}^{\infty} \sqrt{\left| \left| \vec{k} \right|^2 + \left( \frac{\pi n}{\Delta} \right)^2 - \frac{1}{4R^2} \right|} \quad (33)$$

with $\vec{k}$ being 2-dim. vector $\vec{k} = (p, \frac{m}{R})$; and,

$$E_1 = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \text{ Reg } \left[ \sum_{n=1}^{\infty} \text{ Reg } \left[ \sum_{m=1}^{\infty} \left| \left| m^2 + p^2 + \left( \frac{\pi n}{\Delta} \right)^2 - \frac{1}{4R^2} \right| \right] \right]. \quad (34)$$

It is obvious that employment of $\text{Reg} \sum_{m=1}^{\infty}$ does not mean that there is an actual regularization in $m$ summation. It simply imply the usage of (31); for as it will be seen below that the first summation on the right hand side of that formula is exactly calculable.
To evaluate $E_1$, we first employ (31) in $n$ summation. The last term, i.e., the $\int_0^\infty dn$ integral term, becomes formally the same as $E$ of (33). Thus we can write

$$E_1 = -2\triangle E(\pi R, \frac{1}{2R}) + E_2 + E_3. \quad (35)$$

The terms $E_2$ and $E_3$ which come from the first and second terms of (31) are

$$E_2 = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \text{Reg} \left[ \sum_{m=1}^{\infty} \sqrt{\frac{m^2}{R^2} + p^2 - \frac{1}{4R^2}} \right]. \quad (36)$$

and

$$E_3 = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \sum_{n=1}^{\infty} \text{Reg} \left[ \sum_{m=0}^{\infty} \sqrt{\frac{m^2}{R^2} + p^2 + \left(\frac{\pi n}{\triangle}\right)^2 - \frac{1}{4R^2}} \right]. \quad (37)$$

To evaluate $E_2$, we write (31) as the last term of the right hand side of (30); then after suitable change of variables we arrive at

$$E_2 = -\frac{1}{8\pi^2 R^2} \left( \int_0^\infty \frac{x^3 dx}{\sqrt{1 + x^2}} \int_1^\infty dy \frac{\sqrt{y^2 - 1}}{e^{\pi xy} - 1} + \int_0^1 \frac{x^3 dx}{\sqrt{1 - x^2}} \int_1^\infty dy \frac{\sqrt{y^2 + 1}}{e^{\pi xy} - 1} \right) \quad (38)$$

We can easily estimate the upper limit of the above integrals. The first one is smaller than $\frac{1}{7200}$, while the second is smaller than $\frac{1}{6}$. Thus

$$|E_2| < \frac{1}{48\pi^2 R^2}. \quad (39)$$

To evaluate $E_3$, since $\triangle \ll R$, we can first approximate it as:

$$E_3 \simeq -\int_{-\infty}^{\infty} \frac{dp}{\pi} \sum_{m=1}^{\infty} \int_{R}^{\infty} dy \frac{\sqrt{m^2 - p^2 - \frac{\pi^2 n^2}{\triangle^2}}}{e^{2\pi m} - 1} \simeq$$

$$\simeq -\int_{-\infty}^{\infty} \frac{dp}{\pi} \sum_{m=1}^{\infty} \int_{R}^{\infty} dy \frac{m^2 - p^2 - \frac{\pi^2 n^2}{\triangle^2}}{e^{2\pi m} - 1} =$$

$$= -\frac{1}{2\pi^2} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \sum_{n=1}^{\infty} \frac{\partial K_0(2\pi R \sqrt{p^2 + \left(\frac{\pi n}{\triangle}\right)^2})}{\partial R}. \quad (40)$$

We use the formula [11]

$$\int_0^\infty d\lambda K_0(x\sqrt{\lambda^2 + b^2}) = \frac{\pi}{2x} e^{-xb}. \quad (41)$$
for the integration over $p$, then perform the summation over $n$. Finally we have

$$E_3 \simeq -\frac{1}{4\pi R\Delta} e^{-2\pi^2 \frac{R}{\Delta}}. \quad (42)$$

which is negligible small.

To calculate the main contribution (33) and the first term of of (35) we apply (30) and (31):

$$E(a, \mu) = -\int_{-\infty}^{\infty} d^2 \vec{k} \int_{\pi \sqrt{|k^2 - \mu^2|}}^{\infty} \frac{dn}{e^{2\pi n} - 1} \sqrt{\frac{\pi^2 n^2}{a^2} - k^2 + \mu^2} =$$

$$= -\int_{\mu}^{\infty} \frac{dk k}{2\pi} \int_{\frac{\pi}{2}}^{\infty} \frac{dn}{e^{2\pi n} - 1} \sqrt{\frac{\pi^2 n^2}{a^2} - k^2 + \mu^2}$$

$$- \int_{0}^{\mu} \frac{dk k}{2\pi} \int_{\frac{\pi}{2} \sqrt{\mu^2 - k^2}}^{\infty} \frac{dn}{e^{2\pi n} - 1} \sqrt{\frac{\pi^2 n^2}{a^2} - k^2 + \mu^2}$$

$$= -\frac{\pi^2}{1440a^3} - \frac{1}{32\pi^2 a^3} \int_{0}^{2\mu a} y^3 dy \int_{1}^{\infty} dx \sqrt{1 + x^2} \quad (43)$$

For (33) i. e., with $a = \Delta$ and $\mu = \frac{1}{2R}$, since $\Delta \ll R$ we have

$$E(\Delta, \frac{1}{2R}) \simeq -\frac{\pi^2}{1440\Delta^3} - \frac{1}{192R^2 \Delta} \quad (44)$$

Inspecting (43), for $a = \pi R$, $\mu = \frac{1}{2R}$ we see that its contribution is $10^{-2}\frac{\Delta^2}{R^2}$ times the second term in the above expression: thus it is also negligible. The final result for the Casimir energy between the close cylinders is then:

$$E = -\frac{\pi^3 R}{720\Delta^3} \left(1 + \frac{15}{2\pi^2} \frac{\Delta^2}{R^2}\right). \quad (45)$$

Note that the inclusion of the second term in the above expression does not contradict our approximation of (23), for the contribution of the first term after this approximation in the potential would be of the order $\frac{\Delta^3}{R^3}$. It is easy to check that in $\frac{R}{\Delta} \to \infty$ limit the above result becomes same as parallel plate energy.

Finally we like to remark that, for one-boundary geometries, for example for $D$-dimensional ball there are satisfactory techniques to deal with the problem involving the roots of Bessel functions [12]. We may hope that
these techniques may also be adopted for geometries with two-boundaries. For boundaries close to each other however, we can rely on the result of (45), for it gives the correct limit of parallel plates in \( \frac{R}{\Delta} \to \infty \) limit.

IV. Casimir energy in the region between two tori

Problem differs from the previous one by the boundary condition. Instead of (19), the solution of the e-value equation (15) should satisfy

\[
\Phi|_{r=r_0} = \Phi|_{r=r_1} = 0, \quad \Phi|_{z=0} = \Phi|_{z=L}
\]  

(46)

where \( L \) is the circumference of the tori. For \( \Delta \ll r_0 \) we have

\[
\Phi = \frac{e^{i2\pi k \phi + im \phi}}{2\pi \sqrt{\Delta}} \sqrt{2 \Delta} \sin\left(\frac{\pi n}{\Delta} (r - r_0)\right)
\]  

(47)

and

\[
\omega_{knm} = \sqrt{\left(\frac{2\pi k}{L}\right)^2 + \left(\frac{m}{R}\right)^2 + \left(\frac{\pi n}{\Delta}\right)^2 - \frac{1}{4R^2}};
\]  

(48)

where \( k, m \in \mathbb{Z} \) and \( n = 1, 2, 3, \ldots \); and \( R^2 = r_0 r_1 \) as in the previous section. The total energy between the close tori is then, after employment of Plana formulas

\[
E = \frac{1}{2} \sum_{m=-\infty}^{\infty} \text{Reg}\left[ \sum_{k=-\infty}^{\infty} \sum_{n=1}^{\infty} \omega_{knm} \right] = \sum_{m=-\infty}^{\infty} \int_{0}^{\infty} dk \ \text{Reg}\left[ \sum_{n=1}^{\infty} \omega_{knm} \right]
\]

\[
+ \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \text{Reg}\left[ \sum_{k=1}^{\infty} \omega_{knm} \right]
\]  

(49)

Note that unlike the previous case, since the degree of freedom along the tori ( i.e. along z-coordinate ) is also restricted, we have to perform regularizations for the \( k \)-summation too. The first term on the right hand side of the above equation is exactly the Casimir energy for the co-axial cylinders considered in the previous section. Thus, we rewrite (49) as

\[
E = LE_c + E'
\]  

(50)
In a fashion parallel to the evaluation of (40), taking the advantage of $R \gg \triangle$ we can evaluate $\text{Reg} \sum_{k=1}^{\infty}$ in $E'$:

$$E' = -\frac{1}{8\pi} \frac{\partial}{\partial L} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} K_0(2L \sqrt{\frac{m^2}{R^2} + \frac{\pi^2 n^2}{\triangle^2}}) \quad (51)$$

From

$$\sum_{m=-\infty}^{\infty} K_0(2L \sqrt{\frac{m^2}{R^2} + \frac{\pi^2 n^2}{\triangle^2}}) = 2 \int_0^{\infty} dm K_0(2L \sqrt{\frac{m^2}{R^2} + \frac{\pi^2 n^2}{\triangle^2}}) + 2\pi \int_0^{\pi R / \triangle} \frac{dm J_0(2L \sqrt{\frac{m^2}{R^2} + \frac{\pi^2 n^2}{\triangle^2}})}{e^{2\pi m} - 1}, \quad (52)$$

using the approximation $\frac{1}{e^{2\pi m} - 1} \approx e^{-2\pi m}$ for $m \geq \frac{\pi R}{\triangle}$ we arrive at $[11]$

$$\sum_{m=-\infty}^{\infty} K_0(2L \sqrt{\frac{m^2}{R^2} + \frac{\pi^2 n^2}{\triangle^2}}) = \pi R \left( e^{-2\pi L / \triangle} + e^{-2\pi \sqrt{\frac{\pi^2 R^2 + L^2}{\triangle^2}}} \right). \quad (53)$$

Thus we have

$$E' \simeq -\frac{R}{8} \frac{\partial}{\partial L} \left( \frac{1}{2L} e^{\frac{2\pi n}{\triangle}} - 1 \right) + \frac{1}{\sqrt{\pi^2 R^2 + L^2}} e^{\frac{2\pi n \sqrt{\pi^2 R^2 + L^2}}{\triangle}} - 1/ \quad (54)$$

Since $L > R \gg \triangle$ this contribution is negligible small in compared to $LE_c$.

V. Co-axial cylindrical boxes of finite height.

Instead of (46), the solution of the e-value equation (15) should satisfy

$$\Phi|_{r=r_0} = \Phi|_{r=r_1} = 0, \quad \Phi|_{z=0} = \Phi|_{z=L} = 0 \quad (55)$$

with $L$ being the height of the cylinders. For $\triangle \ll r_0$ we have

$$\Phi = \frac{\sin \left( \frac{\pi k}{L} z \right) e^{im\phi}}{\pi \sqrt{r_0 \pi}} \sqrt{\frac{2}{L\triangle}} \sin \left( \frac{\pi n}{\triangle} (r - r_0) \right) \quad (56)$$
and
\[ \omega_{knm} = \sqrt{(\frac{\pi k}{L})^2 + (\frac{m}{R})^2 + (\frac{\pi n}{\Delta})^2 - \frac{1}{4R^2}}; \tag{57} \]

where \( m \in \mathbb{Z} \) and \( n, k = 1, 2, 3, \ldots \); and \( R^2 = r_0 r_1 \) as in the previous section. The total energy between the close cylinders is then
\[
E = \frac{1}{2} \sum_{m=-\infty}^{\infty} \text{Reg}[\sum_{k=1}^{\infty} \sum_{m=-\infty}^{\infty} \omega_{knm}] = \frac{1}{2} \sum_{m=-\infty}^{\infty} \int_0^\infty dk \, \text{Reg}[\sum_{n=1}^{\infty} \omega_{knm}]
- \frac{1}{4} \sum_{m=-\infty}^{\infty} \text{Reg}[\sum_{n=1}^{\infty} \omega_{0nm}] + \frac{1}{2} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \text{Reg}[\sum_{k=1}^{\infty} \omega_{knm}] \tag{58}
\]

The first term on the right hand side of the above equality is equal to \( LE_c \), where \( E_c \) is the Casimir energy for the co-axial cylinders geometry given by (45). The third term for \( L \Delta \geq 1 \) can be explicitly calculated using its similarity with (51) of the previous section. Namely we have to multiply (51) by \( \frac{1}{2} \) and make a change \( L \rightarrow 2L \). Using (54) we arrive at
\[
E_2 \simeq -\frac{R}{32} \frac{\partial}{\partial L} \left( \frac{1}{4L} \frac{1}{e^{\frac{\pi}{L}} - 1} + \frac{1}{\sqrt{\pi^2 R^2 + 4L^2}} \frac{1}{e^{\frac{\sqrt{\pi^2 R^2 + 4L^2}}{L}} - 1} \right) \tag{59}
\]

Since \( R \gg \Delta \) we can neglect the second term of the above expression. For \( L > \Delta \) we have
\[
E_2 \simeq \frac{R}{32L} \left( \frac{\pi}{\Delta} + \frac{1}{4L} e^{-\frac{\pi L}{\Delta}} \right) \tag{60}
\]

which is small due to the exponential factor.

Let us consider the second term of (58). For the sake of simplicity we omit the factor \( -\frac{1}{4R^2} \) in the spectrum. Applying the Plana formula to the summation over \( m \) we get
\[
W = \frac{\zeta(3) R}{16\Delta^2} + W', \tag{61}
\]

in which
\[
W' \simeq \frac{1}{48R} - \frac{\Delta \zeta(3)}{8\pi R^2} \tag{62}
\]

term is very small. The main contributions to the total energy than come from the first terms of (58) and (61):
\[
E = -\frac{\pi^2 RL}{720\Delta^3} + \frac{R\zeta(3)}{16\Delta^2}. \tag{63}
\]
Inspecting the above result we observe that the energy is positive around \( L \leq \frac{3}{2} \triangle \) (within our approximation). Around this value of the height, the radial force \( F_{\text{rad}} = -\frac{\partial E}{\partial \triangle} \) is repulsive. The force on the axial direction \( F_{\text{axial}} = -\frac{\partial E}{\partial L} \) however, is repulsive for all values of \( L \), which forces the cylinders to become of infinite length. When \( L \) becomes longer than \( \frac{3}{2} \triangle \), the radial force too becomes attractive.

\[ \text{VI. Casimir energy between two close co-centric spheres.} \]

We employ the spherical coordinates
\[
ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2)
\]
and insert the solution in terms of the spherical harmonics
\[
\Phi = Y^l_m(\theta, \phi) \frac{v_n(r)}{r}; \quad l = 0, 1, 2, \ldots, -l \leq m \leq l
\]
into the Klein-Gordon equation (4). The resulting radial eigenvalue problem we have to deal is
\[
\left[ -\frac{d^2}{dr^2} + \frac{(l + \frac{1}{2})^2}{r^2} \right] v_n(r) = (\omega_n)^2 v_n(r)
\]
subject to the boundary conditions
\[
v_n(r_0) = 0, \quad v_n(r_1) = 0.
\]
Here \( r_0 < r_1 \) are the radii of the spheres and \( n \) is the radial quantum number to be determined by the boundary conditions. To satisfy the boundary conditions one has to deal with the roots of the radial wave function \( v_n(r) \) which as in the previous section are the Bessel functions (with \( m \) replaced by \( l + 1/2 \)). However since we are interested in \( \triangle \equiv r_1 - r_0 \ll r_0 \) limit, we can proceed as we have done in the previous section. For the radial wave functions and the eigenvalues we obtain
\[
v^0_n(r) = \sqrt{\frac{2}{\triangle r}} \sin(\omega^0_n(r - r_0)),
\]
\[(\omega_{ln}^0)^2 = \frac{\pi^2 n^2}{\Delta^2} + \frac{(l + \frac{1}{2})^2}{R^2}; \quad n = 1, 2, \ldots \] (69)

With the above approximated radial eigenfunctions and eigenvalues we can write the Green function as

\[G = \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{e^{i \omega_{ln}(t-t')}}{2 \omega_{ln}} v^0_{ln}(r) v^0_{ln}(r') Y^l_m(\theta, \phi) \overline{Y^l_m(\theta', \phi')} \] (70)

Integrating the vacuum energy density

\[T = \text{Reg} \lim_{t,r,\theta,\phi \to t',r',\theta',\phi'} \left[ \frac{1}{2} \frac{\partial_t \partial_t' + \partial_r \partial_r' + \frac{1}{r^2} \partial_\theta \partial_\theta' + \frac{1}{r^2 \sin^2 \theta} \partial_\phi \partial_\phi'}{1} \right] G \] (71)

over the volume between two co-centric spheres we get the total energy

\[E = \sum_{l=0}^{\infty} (l + \frac{1}{2}) \text{Reg} \left[ \sum_{n=1}^{\infty} \omega_{ln}^0 \right]. \] (72)

Applying the Plana formula to the \(n\) summation and dropping the \(n = 0\) term and the integration over \(n\) we get

\[E = -\frac{2\Delta}{\pi R^2} \int_1^{\infty} dn F(n) \] (73)

where

\[F(n) = \sum_{s=\frac{1}{2}}^{\infty} \frac{s^3}{e^{2\frac{\pi}{R} sn} - 1} \] (74)

To use the Plana formula \[\text{II}\]

\[\sum_{k=0}^{\infty} f(k + \frac{1}{2}) = \int_0^{\infty} dy f(y) - i \int_0^{\infty} dy \frac{f(iy) - f(-iy)}{1 + e^{2\pi y}} \] (75)

we have to get rid off the poles of the function \(F(n)\) at the imaginary axis \(2\frac{\Delta}{R} ns = 2i\pi m\). Thus we work with the function

\[F_\beta = \sum_{s=\frac{1}{2}}^{\infty} \frac{s^3}{e^{2\frac{\beta}{R} s}\pi s^{\beta} - 1} \] (76)

with \(\beta > 0\). Then \(\text{II}\) becomes

\[E = -\frac{\pi^3 R}{360 \Delta^3} + E' \] (77)
where
\[
E' = \lim_{\beta \to 0} \frac{1}{2\pi \Delta} \int_0^{\infty} \frac{dss^3}{e^{2\pi s} + 1} \int_2^\infty dx \sqrt{x^2 - \left( \frac{2\Delta}{R} \right)^2} \left( \frac{1}{e^{x(\beta+is)} - 1} + \frac{1}{e^{x(\beta-is)} - 1} \right)
\]
(78)

Using \( \frac{\Delta}{R} \ll 1 \) we get
\[
E' \simeq -\frac{\pi}{288\Delta}
\]
(79)

Thus the total energy in the region between the spheres is
\[
E = -\frac{\pi^3 R^2}{360\Delta^3} \left( 1 + \frac{5\Delta^2}{4\pi^2 R^2} \right)
\]
(80)

In \( \frac{R}{\Delta} \to \infty \) it is obvious that the above energy approaches the parallel plate formula.

VII. Casimir interactions of two close co-axial cones.

The geometry we like to present in this section is two cones with common axis at positive \( z \)-direction and appeces at the origin. By close cones we mean the appex angles \( \theta_1 \) and \( \theta_1 \) are close to each other, that is
\[
\Delta \equiv \theta_1 - \theta_0 \ll \sqrt{\sin \theta_0 \sin \theta_1} \equiv \Theta.
\]
(81)

In the above approximation the solutions we employe ( in spherical coordinates ) which vanishes at the surfaces \( \theta = \theta_0 \) and \( \theta = \theta_1 \) are
\[
\Phi_{\omega nm}(\omega r) = \sqrt{\frac{\omega}{r}} J_{\mu_{nm}}(\omega r) \frac{e^{im\phi}}{\sqrt{\pi \Delta}} \sin \left( \frac{\pi n}{\Delta} (\theta - \theta_0) \right),
\]
(82)

where
\[
\mu_{nm} = \sqrt{\left( \frac{\pi n}{\Delta} \right)^2 + \left( \frac{m}{\Theta} \right)^2}
\]
(83)

The energy \( \omega \) in (82) is continuous. The Green function is ( with the cut off factor \( \beta \) )
\[
G = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{m=-l}^{l} e^{-\beta \omega + i\omega (t-t')} \frac{\Phi_{\omega nm}(r, \theta, \phi) \Phi_{\omega nm}(r', \theta', \phi')}{2\omega}.
\]
(84)
Note that in this section we employ different regularization method than the previous ones. The cut off method is more suitable for the continuous energy spectra. Inserting the Green function of (84) into the coincidence limit formula and then integrating over \( \theta \) and \( \phi \), we arrive at the vacuum energy density at \( r \):

\[
E = \frac{1}{4\pi r} \text{Reg}_\beta \left[ \left( \frac{\partial^2}{\partial r^2} + 2 \frac{\partial^2}{\partial \beta^2} \right) \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{Q_{-1/2+\mu nm}(1 + \frac{\beta^2}{r^2})}{r} \right]
\]  

(85)

\( \text{Reg}_\beta \) stands for the cut off regularization, that is we pick the finite part of the expression in \( \beta \to 0 \) limit. In deriving (85) we used the formula [11]

\[
\int_0^\infty d\omega e^{-\beta \omega} (J_\nu(\omega r))^2 = \frac{1}{\pi r} Q_{\nu-1/2}(1 + \frac{\beta^2}{2r^2}),
\]  

(86)

where \( Q_\nu(x) \) is Legendre function of the second kind. We rewrite the expression (85) as

\[
E = \frac{1}{2\pi r^4} \text{Reg}_y \left[ \hat{O} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} Q_{-1/2+\mu nm}(1 + y^2) \right]
\]  

(87)

with

\[
y \equiv \frac{\beta}{\sqrt{2r}}, \quad \hat{O} \equiv 1 + 2y \frac{\partial}{\partial y} + \frac{y^2 + 1}{2} \frac{\partial^2}{\partial y^2},
\]  

(88)

Applying the Plana formula to the summation over \( m \) we arrive at

\[
E = E_0 + E_1,
\]  

(89)

where

\[
E_0 = \frac{1}{\pi r^4} \text{Reg}_y \left[ \hat{O} \sum_{n=1}^{\infty} \int_0^\infty dm Q_{-1/2+\mu nm}(1 + y^2) \right]
\]  

(90)

and

\[
E_1 = \frac{1}{r^4} \text{Reg}_y \left[ \hat{O} \sum_{n=1}^{\infty} \int_0^\infty \frac{dm \tanh \sqrt{\left( \frac{m \Theta n^2}{\Delta} \right)^2 - \left( \frac{m \Delta^2}{\Theta n^2} \right)}}{e^{2\pi m} - 1} P_{\frac{1}{2}+i\sqrt{\left( \frac{m \Theta n^2}{\Delta} \right)^2 - \left( \frac{m \Delta^2}{\Theta n^2} \right)}}(1 + y^2) \right]
\]  

(91)

Making use of \( \frac{\pi \Theta n}{\Delta} \gg 1, \tanh x \leq 1 \) and

\[
\text{Reg}_y[\hat{O} P_{-1/2+is}(1 + y^2)] = \frac{7}{8} - \frac{s^2}{2}
\]  

(92)
we get
\[ |E_1| \leq \frac{1}{r^4} \left| \sum_{n=1}^{\infty} \int_{\frac{\pi n}{2r}}^{\infty} dme^{-2\pi n} \left( \frac{7}{8} - \left( \frac{w}{2r} \right)^2 - \left( \frac{\pi n}{2r} \right)^2 \right) \right|. \] (93)
That is
\[ |E_1| \leq \frac{\Theta}{4\pi \triangle r^4} e^{-2\pi^2 \frac{\triangle}{r^2}}. \] (94)
Thus \( E_1 \) is negligible small. To evaluate \( E_0 \), we apply the Plana formula to the summation over \( n \) in (90). The formula we obtain is
\[ E_0 = \mathcal{E} + \frac{\Theta \triangle}{2\pi r^4} A - \frac{\Theta}{2\pi r^4} B, \] (95)
where
\[ \mathcal{E} = \frac{\Theta \triangle}{2\pi r^4} \int_0^\infty dx \int_x^\infty dy \frac{\tanh \sqrt{y^2 - x^2}}{e^{2\triangle y} - 1} \left( \frac{7}{4} + x^2 - y^2 \right) \] (96)
\[ A = \text{Reg}_{y} \left[ \tilde{Q} \int_0^\infty ds Q_{-\frac{1}{2} + s} (1 + y^2) \right] \] (97)
\[ B = \text{Reg}_{y} \left[ \tilde{Q} \int_0^\infty ds Q_{\frac{1}{2} + s} (1 + y^2) \right] \] (98)
Changing the variables \( y = t, \ x^2 = t^2 - k^2 \) (96) can be rewritten as
\[ \mathcal{E} = \frac{\Theta \triangle}{2\pi r^4} \int_0^\infty dt \int_0^1 \frac{dkk}{e^{2\triangle t} - 1} \int_0^1 \frac{dkk}{\sqrt{1 - k^2}} \tanh(kt) \left( \frac{7}{4} - k^2 t^2 \right) \] (99)
Inspecting the integrals over \( k \), that is the terms
\[ f_1(t) = \int_0^1 \frac{dkk}{\sqrt{1 - k^2}} \tanh(kt) \] (100)
and
\[ f_2(t) = \int_0^1 \frac{dkk^3}{\sqrt{1 - k^2}} \tanh(kt) \] (101)
we see that both approach very fast from the value \( f_1(0) = f_2(0) = 0 \) to their respective asymptotic values \( f_1(t \to \infty) = 1 \) and \( f_2(t \to \infty) = 0.66 = \frac{2}{3} \).
Let us treat the second term in (99) in detail. We approximate \( f_2(t) \) as
\[ f_2(t) \simeq \begin{cases} \frac{at}{2}, & t \in [0, b] \\ \frac{2}{3}, & t \in [b, \infty) \end{cases} \] (102)
where $a$ and $b$ are both of order 1. The second term in (99) then becomes

$$
\mathcal{E}_2 = -\frac{\Theta \Delta}{2 \pi r^4} \left( a \int_0^b \frac{dtt^4}{e^{2\Delta t} - 1} + \int_b^\infty \frac{dtt^3}{e^{2\Delta t} - 1} \right)
$$

Since $\Delta \ll 1$, we can approximate the denominator of the first integrand as $e^{2\Delta t} - 1 \simeq 2\Delta t$. In the second integral making the change of variables $2\Delta t = s$, we can replace the lower boundary as $2b\Delta \simeq 0$. Thus (103) becomes

$$
\mathcal{E}_2 \simeq -\frac{\Theta \Delta}{2 \pi r^4} \left( \frac{ab^4}{8\Delta} + \frac{1}{16\Delta^4} \int_0^\infty \frac{dss^3}{e^s - 1} \right) = -\frac{\Theta ab^4}{16\pi r^4} - \frac{\Theta \pi^3}{720r^4\Delta^3}.
$$

It is obvious that the first term is negligible compared to the second. Similar treatment shows that that the first term in (99) gives contributions of orders $O(\Delta)$ and $O(\frac{1}{\Delta})$ both are small. Inspecting (90) we see that the second and third terms in $E_0$ are also negligible. Thus the final result for our Casimir energy is:

$$
E \simeq \mathcal{E}_2 \simeq -\frac{\Theta \pi^3}{720r^4\Delta^3}.
$$

(105)

Note that the above ”density” is an expression obtained after integrating over $\Theta$ and $\phi$. If we divide (105) by the angular integral

$$
\int_{\theta_0}^{\theta_1} \sin \theta \int_0^{2\pi} d\phi \simeq 2\pi \Theta \Delta
$$

(106)

we obtain the energy density averaged over the angular variables:

$$
E \simeq \mathcal{E}_2 = -\frac{\pi^2}{1440r^4\Delta^4} + O(\Delta^{-3}).
$$

(107)

In small $\Delta$ limit the above result is in perfect agreement with the energy density in the region between two infinite planes with angle $\Delta$ between them (i.e. the wedge problem) [13]

$$
E = -\frac{1}{1440r^4\Delta^2} \left( \frac{\pi^2}{\Delta^2} - \frac{\Delta^2}{\pi^2} \right).
$$

(108)
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