Exploring the Collective Behavior on the One-dimensional Quartic Map

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(November 19, 2018)

In this work we study numerically a lattice composed of two parameter single quartic maps with local diffusive coupling. We find large regions over the parameter space where the single quartic map is periodic and the coupled system is not synchronized. We explore further one point on this region to show that it presents a real non-trivial collective behavior.

I. INTRODUCTION

Since the last decade different systems composed of coupled map lattices (CML) have been studied as simple toy models used to unveil basic properties of spatially extended nonlinear dynamical systems. Conceived as a far from equilibrium system with many degrees of freedom such systems can also be used to construct a statistical mechanics for irreversible process.

In the beginning of the decade, studying systems of locally coupled logistic maps on spatial dimensions greater than one, Chaté and Manneville [2] found a temporal dynamics on spatial averages over the system emerging out of the local chaos, the so called non-trivial collective behavior (NTCB). They found that such behavior happens when the single logistic map parameter is set to any value beyond the chaotic onset, that is for \( \lambda > \lambda_c \). When \( \lambda < \lambda_c \) the whole system synchronizes to the local periodic behavior. So, the idea here is that the basic ingredients to find NTCB are local chaos competing with local sensibility to initial conditions (SIC).

Most works found in the literature [1] use the logistic map to describe the local dynamics.
A characteristic feature of the logistic map is the presence of a single attractor on the interval of convergence of its variable. On a CML composed of such units the effect of the local coupling is to average the variable (field) over the neighborhood producing a new value inside the basin of attraction of the very same attractor. The motivation of this work is to investigate a CML possessing two non-infinite attractors. In order to do that we use for the local dynamics the one-dimensional quartic map introduced by one of us in 1993 [3].

Starting from random initial conditions we ran a system composed of $N$ units whose dynamics at the site $i$ follows the quartic map,

$$y_{i}^{t+1} = ((x_{i}^{t+1})^2 - a)^2 - b,$$

where $x_{i}^{t}$ is a real variable representing some quantity of interest measured at time $t$; $a$ and $b$ are real parameters. After each generation $t$ the value of $y_{i}$ is averaged over the first neighbors,

$$x_{i}^{t} = (1 - \epsilon)y_{i}^{t} + \frac{\epsilon}{2D} \sum_{j=1}^{n} y_{j}^{t}$$

where $D$ is the spatial dimension of the system and $n$ is the number of nearest neighbors. The geometry used for the simulation presented here is of a square lattice of sizes up to $N = 512 \times 512$ units.

We first explored the system scanning the space of parameters $a$ and $b$ on the interval $\{-1, 2\}$ dividing it with a resolution of $512 \times 512$. For each couple of parameters a small lattice of $64 \times 64$ sites was iterated up to one thousand times departing from initial conditions randomly chosen from the basin of attraction of the uncoupled quartic map. From such experiment we could separate the set of parameters whose lattice rapidly synchronize from those which present either a long transient or a non-trivial collective behavior (NTCB). In figure 1 we show these results in connection with the Lyapunov exponents of the local map. In this figure the black regions show sets of parameters where the coupled system does not synchronize up to one thousand iterations and the local uncoupled map is periodic, the white color represents regions where the system does not synchronize and the local uncoupled map is chaotic, the dark gray shows the regions where the coupled system synchronizes to the
local uncoupled map and the pale gray color shows the orbits which are attracted to infinite.

The striking result is shown in the black region of figure 1, there we may have NTCB without chaos on the local uncoupled oscillator. In fact the absence of synchronization of the variable over the space may also be just a long transient. To clear out this we have particularly explored the point $a = 0.35, b = 1.35$ on the black region. The local uncoupled map has two fixed points, $x_1 = -1.227591, x_2 = -0.011399$ and negative Lyapunov exponents $L_1 = L_2 = -1.200$. The coupled system presents a period two NTCB, after a small transient the instantaneous average over the variable jumps alternatively from the interval $\{.125144, .815711\}$ to the interval $\{-1.349951, -1.245386\}$. See figure 2. We have performed $2.5 \times 10^5$ iterations with large lattices of $512 \times 512$ sites and up to $10^6$ interactions with smaller lattices of $128 \times 128$ sites without observing synchronization on the system. In figure 1 we show the images of a lattice of $512 \times 512$ sites in two consecutive steps after $2.5 \times 10^5$ iterations.

In order to confirm the NTCB for this point of the parameter space we have observed the dependence of the time fluctuation of the spatial average of the field variable with the lattice size during the same time interval. For large systems this average is expected to become more and more defined. In fact for increasing lattice sizes the fluctuation decreases with the $\sqrt{N}$ and the NTCB is better and better defined. Another test to the NTCB is to verify if it robust to changes on the parameters, so we ran the system on a small region of radius 0.01 around this point on the parameter space. We have found the same period two NTCB with small changes on the spatial averages showing that it is robust to small changes on parameters.

From the first work of Chaté and Manneville it was already clear that the NTCB existed for the logistic map on the so called periodicity windows appearing when $\lambda > \lambda_c$. A similar result has also been remarked by one of us in a collaboration with Chaté and Manneville when exploring the robusticity of the NTCB on the coupled Rössler system. In both cases, nevertheless, the NTCB for periodic parameters emerges on a small window on the parameter space surrounded by chaos. Here it emerges on the boarding region between
periodic and chaotic behavior.

From these results we can state that sensibility to initial conditions (SIC) on the local attractor is not a necessary condition to NTCB and we may find NTCB even when the system is periodic over large regions of the parameter space. Nevertheless the existence of spatial dispersion clearly indicates that the diffusive coupling, which forces the system to synchronize, is counterbalanced by some kind of SIC. What means that knowledge of the local attractor does not imply knowledge of the global one which emerges from the coupled system. We may try to generalize summing up the simulations presented here and the previous results. That is, for the NTCB to exist on a periodic local base we need either a basin of attraction more complex with two relevant parameters (and two attractors) or a small window of periodicity surrounded by chaos.

ACKNOWLEDGMENTS

We thank H. Chaté and A. Pikovsky for important discussions and suggestions. We acknowledge the Centro de Supercomputação of our university for the computing time on the CRAY YP-MP2 and the supporting Brazilian agencies CNPq and FAPERGS.

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Figure 1
Figure 2
Figure 3
**Figure Captions**

**Figure 1**
Image of parameter space. Black: region where the single quartic map is periodic and the coupled system does not synchronize. White: the single map is chaotic and the coupled system does not synchronize. Dark gray: coupled system synchronizes to the local uncoupled map. Pale gray: orbits attracted to infinite. The scale in both axes is in the interval $\{-1.0, 2.5\}$.

**Figure 2**
Image of the spatial variable ($x$) showing a period two NTCB for parameters $a=0.35$ and $b=1.35$. The gray level scale on left is in the interval $\{0.125144, 0.815711\}$ and on the right in the interval $\{-1.349951, -1.245386\}$.

**Figure 3**
Time fluctuation of the lattice spatial average, mean deviation for different lattice sizes.