COMPOSITION OPERATORS WITHIN SINGLY GENERATED COMPOSITION C*-ALGEBRAS

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Abstract. Let \( \varphi \) be a linear-fractional self-map of the open unit disk \( \mathbb{D} \), not an automorphism, such that \( \varphi(\zeta) = \eta \) for two distinct points \( \zeta, \eta \) in the unit circle \( \partial \mathbb{D} \). We consider the question of which composition operators lie in \( C^*(C_\varphi, K) \), the unital C*-algebra generated by the composition operator \( C_\varphi \) and the ideal \( K \) of compact operators, acting on the Hardy space \( H^2 \).

This necessitates a companion study of the unital C*-algebra generated by the composition operators induced by all parabolic non-automorphisms with common fixed point on the unit circle.

1. Introduction

Given any analytic self-map \( \varphi \) of the unit disk \( \mathbb{D} \) in the complex plane, one can form the composition operator \( C_\varphi : f \rightarrow f \circ \varphi \), which acts as a bounded operator on the Hardy space \( H^2 \). This paper is the second in a series of three investigating spectral theory in C*-algebras generated by certain composition and Toeplitz operators. In the first article [15], we studied \( C^*(T_z, C_\varphi) \), the unital C*-algebra generated by the unilateral shift \( T_z \) on \( H^2 \) and a single composition operator \( C_\varphi \) with \( \varphi \) satisfying

\[
\begin{cases}
\varphi \text{ is a linear-fractional self-map of } \mathbb{D} \text{ which is not an automorphism, and} \\
\varphi(\zeta) = \eta \text{ for distinct points } \zeta, \eta \text{ in the unit circle } \partial \mathbb{D}.
\end{cases}
\]

Throughout the current paper, \( \varphi \) will always have this meaning. The algebra \( C^*(T_z, C_\varphi) \) necessarily contains the ideal \( K \) of compact operators on \( H^2 \). The main result of [15] identifies \( C^*(T_z, C_\varphi)/K \) with a certain C*-algebra of \( 2 \times 2 \) matrix valued functions; see Theorem 4.12 of [15]. The case where \( \varphi \) is replaced by an automorphism of \( \mathbb{D} \), or even a discrete group of automorphisms, has been studied by Jury [12, 13], and has a rather different character.

The shift \( T_z \) does not appear to play a role in the questions we consider in this paper; accordingly we omit it and study \( C^*(C_\varphi, K) \), the unital C*-algebra generated by \( C_\varphi \), for \( \varphi \) as described above, and the compact operators. The composition \( \varphi \circ \varphi \) has sup-norm strictly less than 1, so that \( C_\varphi^2 = C_{\varphi \circ \varphi} \) is compact and non-zero. Since \( \varphi \) has no boundary fixed point, a theorem of Guyker [10] shows that \( C_\varphi \) is irreducible if and only if \( \varphi(0) \neq 0 \). It follows that when \( \varphi(0) \neq 0 \) the unital C*-algebra \( C^*(C_\varphi) \) generated by \( C_\varphi \) alone contains \( K \); see [5], p.74. We want our C*-algebras to always contain \( K \), and we indicate this by continuing to write \( C^*(C_\varphi, K) \) if the irreducibility criterion \( \varphi(0) \neq 0 \) is in doubt.

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Let $P$ denote the dense subalgebra of $C^*(C_\varphi, K)$ consisting of all finite linear combinations of the identity $I$, all words in $C_\varphi$ and $C_\varphi^*$, and all compact operators. An element $B$ in $P$ has a unique representation of the form

$$B = cI + f(C_\varphi C_\varphi) + g(C_\varphi C_\varphi^*) + C_\varphi p(C_\varphi^* C_\varphi) + C_\varphi^* q(C_\varphi C_\varphi^*) + K$$

where $f, g, p$ and $q$ are polynomials, $f(0) = g(0) = c$ is complex, and $K$ is compact. Let $s = 1/|\varphi'(\zeta)|$ and write $D$ for the $C^*$-algebra of continuous $2 \times 2$ matrix-valued functions $F$ on $[0, s]$ with $F(0)$ a scalar multiple of the identity, equipped with the supremum operator norm. It was shown in [15] that there is a unique *-homomorphism $\Psi$ of $C^*(C_\varphi, K)$ onto $D$ with $\ker \Psi = K$ and such that

$$\Psi(B) = \begin{bmatrix} c + g & rp \\ rq & c + f \end{bmatrix}$$

where $B$ is given by Equation (2) and $r(t) = \sqrt{t}$. Equivalently, we have a short exact sequence of $C^*$-algebras

$$0 \to K \to C^*(C_\varphi, K) \xrightarrow{\Psi} D \to 0$$

where $i$ is inclusion. For any operator $T$ on $H^2$ we write $\|T\|_e$ for the essential norm of $T$, that is, the distance from $T$ to the ideal $K$. We note that if $T$ is in $C^*(C_\varphi, K)$, then $\|T\|_e = \|\Psi(T)\|$. For bounded operators $A$ and $B$ on $H^2$, let us write $A \equiv B \pmod{K}$ if there exists a compact operator $K$ with $A = B + K$. In [15] the authors used C. Cowen's well-known adjoint formula [7] to show that if

$$\psi(z) = \frac{az + b}{cz + d}$$

is a linear-fractional self map of $\mathbb{D}$, not an automorphism but satisfying $|\psi(z_0)| = 1$ for some $z_0 \in \partial\mathbb{D}$, then

$$C_\varphi^* \equiv \frac{1}{|\psi'(z_0)|} C_{\sigma_\psi} \pmod{K}$$

where $\sigma_\psi$ is the so-called “Krein adjoint” of $\psi$,

$$\sigma_\psi(z) = \frac{\bar{a}z - \bar{c}}{-\bar{b}z + \bar{d}}.$$  

We denote the Krein adjoint of $\varphi$ itself simply by $\sigma$, which the reader can distinguish by context from the spectrum $\sigma(T)$ or essential spectrum $\sigma_e(T)$ of an operator $T$.

Now consider an operator $B$ in the dense subalgebra $P$, as expressed in Equation (2). Since $I = C_\varphi, C_\varphi^* C_\varphi \equiv sC_{\varphi \circ \sigma}$ (mod $K$) and $C_\varphi C_\varphi^* \equiv sC_{\sigma \circ \varphi}$ (mod $K$), where $s = 1/|\varphi'(\zeta)|$, we can rewrite Equation (2) as

$$B = cC_\varphi + A_1 + A_2 + A_3 + A_4 + K'$$

where $K'$ is compact and $A_1, A_2, A_3,$ and $A_4$ are finite linear combinations of composition operators whose associated self-maps are chosen, respectively, from the four lists

$$\{(\varphi \circ \sigma)_m\}, \{\sigma \circ \varphi\}_n, \{(\varphi \circ \sigma)_m \circ \varphi\}, \{(\sigma \circ \varphi)_m \circ \sigma\}.$$  

Here we write $(\psi)_n$ for the $n^{th}$ iterate of a self-map $\psi$ of $\mathbb{D}$, and let $n_k$ range over the positive integers for $k = 1, 2$, and over the non-negative integers for $k = 3, 4$. See [15] for further details. Thus $C^*(C_\varphi, K)$ is spanned, modulo the compacts, by
actual composition operators. This leads to our main question: for which analytic self-maps \( \psi \) of \( \mathbb{D} \) does \( C_\psi \) lie in \( C^* (C_\varphi, \mathcal{K}) \)?

In particular we will describe explicitly, in both function-theoretic and operator-theoretic terms, all linear-fractional composition operators lying in \( C^* (C_\varphi, \mathcal{K}) \). This description plays a role in the third paper of our series [16] which is devoted to spectral theory in Toeplitz-composition algebras with several generators. Along the way here we show that if \( C^*(\mathbb{P}_\gamma) \) denotes the unital \( C^* \)-algebra generated by composition operators induced by the parabolic non-automorphisms fixing \( \gamma \) in the unit circle, then there is a short exact sequence

\[
0 \to \mathcal{K} \to C^*(\mathbb{P}_\gamma) \to C([0,1]) \to 0
\]

of \( C^* \)-algebras, where \( C([0,1]) \) denotes the algebra of continuous functions on the unit interval.

2. Necessary conditions

If \( \psi : \mathbb{D} \to \mathbb{D} \) is analytic we write \( F(\psi) \) for the set of points \( \alpha \) in \( \partial \mathbb{D} \) at which \( \psi \) has a finite angular derivative in the sense of Caratheodory; see [8], [21]. In particular, if \( \alpha \) is in \( F(\psi) \), the nontangential limit \( \psi(\alpha) \) necessarily exists and has modulus one. We write \( \psi'(\alpha) \) for the angular derivative of \( \psi \) at \( \alpha \). Recall that this is the ordinary derivative if \( \psi \) extends analytically to a neighborhood of \( \alpha \) and \( |\psi(\alpha)| = 1 \).

It is well known that if \( C_\psi \) is compact on \( H^2 \), then \( F(\psi) \) is empty [20]. When \( C_\psi \) is considered as acting on the Bergman space in \( \mathbb{D} \), the converse assertion is true [17]. On the space \( H^2 \) considered here, however, “\( C_\psi \) is compact” is a strictly stronger requirement than “\( F(\psi) \) is empty”; see, for example, [8], [21]. Our first goal is to show that when \( C_\psi \) lies in \( C^*(C_\varphi, \mathcal{K}) \) these two conditions are equivalent.

First we recall that a linear-fractional self-map \( \rho \) of \( \mathbb{D} \) is \textit{parabolic} if \( \rho \) fixes one point \( \gamma \) in \( \partial \mathbb{D} \) and is conjugate, via the map \( (\gamma + z)/(\gamma - z) \), to translation in the right half-plane \( \Omega = \{ w : \text{Re } w > 0 \} \) by a complex number \( a \) with non-negative real part. We denote this parabolic map by \( \rho_a \), or by \( \rho_{\gamma,a} \) if the fixed point \( \gamma \) is not clear from the context. A linear-fractional map \( \rho \) with fixed point \( \gamma \) in \( \partial \mathbb{D} \) is parabolic provided \( \rho'(\gamma) = 1 \). Another representation of \( \rho_a \) will prove useful. The map \( \tau_a(z) = i(\gamma - z)/(\gamma + z) \) carries \( \mathbb{D} \) onto the upper half-plane \( \{ w : \text{Im } w > 0 \} \) and takes \( \gamma \) to 0, rather than infinity. We write \( u \) for the conjugate of \( \rho_a \) by \( \tau_\gamma : u = \tau_\gamma \circ \rho_a \circ \tau_\gamma^{-1} \). One readily computes that \( u(w) = iw/(i + wa) \), and so

\[
u''(0) = 2ia.
\]

Also important for us will be several lower bounds for the essential norm of a linear combination of composition operators. Given an analytic self-map \( \psi \) of \( \mathbb{D} \) and \( \alpha \) in \( F(\psi) \), we call \( D_1(\psi, \alpha) \equiv (\psi(\alpha), \psi'(\alpha)) \) the \textit{first order data vector} for \( \psi \) at \( \alpha \). If we have a finite collection of maps \( \psi_1, \psi_2, \ldots, \psi_n \) and \( \alpha \) lies in the union of the finite angular derivative sets \( F(\psi_1), F(\psi_2), \ldots, F(\psi_n) \), we define

\[
D_1(\alpha) = \{ D_1(\psi_j, \alpha) : 1 \leq j \leq n \text{ and } \alpha \in F(\psi_j) \},
\]

the set of possible first order data vectors at \( \alpha \) coming from that collection of maps. Theorem 5.2 in [14] states that if \( D_1(\alpha) \) is non-empty, then for any complex numbers \( c_1, \ldots, c_n \),
for somewhat larger class of maps is considered. For such \( \alpha \) that is,

\[
\|c_1 C_{\psi_1} + \cdots + c_n C_{\psi_n}\|_e^2 \geq \sum_{d \in D_1(\alpha)} \sum_{\alpha \in F(\psi_j)} c_j \frac{1}{|d_1|} \geq \sum_{j \in M_k(\alpha)} \sum_{\tau \in \partial D_{k-1}(\alpha) : \tau \cdot d = d} c_j \frac{1}{|d_1|},
\]

where \( d = (d_0, d_1) \).

There is a higher order version of this lower bound which works provided that for

the specific \( \alpha \) in \( F(\psi) \), \( \psi \) is analytically continuable across \( \partial D \) at \( \alpha \) and \( |\psi(e^{i\theta})| < 1 \) for \( e^{i\theta} \) near to, but not equal to, \( \alpha \). More detail can be found in [14], where a somewhat larger class of maps is considered. For such \( \alpha \), the curve \( \psi(e^{i\theta}) \), \( e^{i\theta} \) near \( \alpha \), automatically has positive and even order of contact 2

\[
\frac{1 - |\psi(e^{i\theta})|^2}{|\psi(\alpha) - \psi(e^{i\theta})|^{2m}}
\]

is bounded above and away from 0 for \( e^{i\theta} \) near \( \alpha \). For \( k \geq 1 \) the \( k \)th order data vector

\[
D_k(\psi, \alpha) = (\psi(\alpha), \psi'(\alpha), \cdots, \psi^{(k)}(\alpha))
\]

makes sense. Given a finite collection \( \psi_1 \cdots, \psi_n \) of such maps and \( k \geq 2 \), we write \( M_k(\alpha) \) for the set of integers \( j \), \( 1 \leq j \leq n \), for which \( F(\psi_j) \) contains \( \alpha \) and the order of contact of \( \psi_j \) at \( \alpha \) is at least \( k \). Further, put

\[
D_k(\alpha) = \{ D_k(\psi_j, \alpha) : j \in M_k(\alpha) \},
\]

the set of possible \( k \)th order data vectors at \( \alpha \) for associated orders of contact at
least \( k \). With this notation, Theorem 5.7 in [14] states that for any \( k \geq 2 \) and complex constants \( c_1 \cdots, c_n \),

\[
\|c_1 C_{\psi_1} + \cdots + c_n C_{\psi_n}\|_e^2 \geq \sum_{d \in D_{k-1}(\alpha)} \sum_{j \in M_k(\alpha)} c_j \frac{1}{|d_1|},
\]

where \( d = (d_0, d_1, \cdots, d_{k-1}) \).

For the case \( k = 2 \), we need a calculation which appears in the proof of a more
delicate version of the inequality \([10]\); see Lemma 5.9 in [14]. Given \( \psi \) as above,
with \( \alpha \) in \( F(\psi) \), convert it into a self-map \( u \) of the upper half-plane fixing the
origin by \( u = \tau_{\psi(\alpha)} \circ \psi \circ \tau_{\alpha}^{-1} \). Given our finite collection \( \psi_1 \cdots, \psi_n \), associate \( u_j \) to \( \psi_j \) in this manner. For \( D > 0 \) we write \( \Gamma_{\alpha,D} \) for the locus of the equation

\[
(1 - |z|^2)/(|\alpha - z|^2) = 4D, \quad \text{a circle internally tangent to } \partial D \text{ at } \alpha.
\]

We have

\[
\lim_{\Gamma_{\alpha,D}} \left< (\overline{c_1}C_{\psi_1}^* + \cdots + \overline{c_n}C_{\psi_n}^*) \frac{k_z}{\|k_z\|} \right>^2 = \sum_{d \in D_{k-1}(\alpha)} \sum_{j \in M_k(\alpha)} c_j \frac{1}{|d_1|} \left< \overline{c_j}k_{w_j}^+ \right>_{H^2_\Omega}^2,
\]

where \( k_z \) is the Szegö kernel for the Hardy space \( H^2 \) in the disk, \( H^2_\Omega \) is the Hardy
space of the right half-plane \( \Omega \), \( k_z^+(z) = 1/(\overline{w} + z) \) is its reproducing kernel, \( w_j = u_j(0)/2 - iDu_j(0) \), and the limit is taken as \( z \to \alpha \) along \( \Gamma_{\alpha,D} \). Since \( u_j(0) \)
necessarily has non-negative imaginary part, \( w_j \) is a complex number automatically
lying in \( \Omega \). For further discussion of this circle of ideas, see [14]. We note for future
reference that a non-automorphism linear-fractional self-map \( \psi \) of \( \mathbb{D} \) has order of contact two at the unique point in \( F(\psi) \).

Finally, we need a variant of a result of Berkson [4] and Shapiro and Sundberg [19], which states that if \( \psi_1, \ldots, \psi_n \) are distinct analytic self-maps of \( \mathbb{D} \) and \( J(\psi_i) = \{ e^{i\theta} : |\psi_i(e^{i\theta})| = 1 \} \), then for any complex constants \( c_1, \ldots, c_n \),

\[
\|c_1\psi_1 + \cdots + c_n\psi_n\|_\infty^2 \geq \frac{1}{2\pi} \sum_{j=1}^{n} |c_j|^2 |J(\psi_j)|.
\]

where \( |J| \) denotes the Lebesgue measure of \( J \); see Exercise 9.3.2 in [8].

**Theorem 1.** Let \( \psi \) be an analytic self-map of \( \mathbb{D} \) such that \( C_\psi \) lies in \( C^*(C_\varphi, K) \), where \( \varphi \) is as in [7]. If \( F(\psi) \) is empty, then \( C_\psi \) is compact.

**Proof.** Suppose that \( C_\psi \) lies in \( C^*(C_\varphi, K) \) and \( F(\psi) \) is empty. We want to show that \( C_\psi \) is compact, or equivalently, that the matrix function

\[
\Psi(C_\psi) = \begin{bmatrix} f_2 & f_3 \\ f_4 & f_1 \end{bmatrix}
\]

is identically zero on \([0, s]\). Given a small \( \epsilon > 0 \) (size to be specified later), there exists \( B \) in \( \mathcal{P} \) given by Equation (2) and equivalently by Equation (5), such that \( \|C_\psi - B\| < \epsilon \). If we write

\[
Y_1 = f(C_\psi C_\varphi), Y_2 = g(C_\psi C_\varphi), Y_3 = C_\psi p(C_\psi C_\varphi), Y_4 = C_\psi q(C_\psi C_\varphi),
\]

and \( Y = Y_1 + Y_2 + Y_3 + Y_4 \), it is clear that \( A_k \equiv Y_k \) (mod \( K \)) for each \( i \), and \( \Psi(Y) = \Psi(A) \), where \( A = A_1 + A_2 + A_3 + A_4 \). Now using the representation (5) for \( B \), we have

\[
\|C_\psi - cC_z - A\|_\epsilon < \epsilon.
\]

Since \( A \) is a finite linear combination of composition operators, we see from the inequality (11) that

\[
e^2 > \|C_\psi - cC_z - A\|_\epsilon^2 \geq \frac{|J(\psi)|}{2\pi} + |c|^2.
\]

From this we find that \( |c| < \epsilon \), and since \( \epsilon > 0 \) is arbitrary, \( |J(\psi)| = 0 \). In particular we have \( \|C_\psi - A\|_\epsilon < 2\epsilon \), hence

\[
\| \begin{bmatrix} f_2 - g & f_3 - rp \\ f_4 - rq & f_1 - f \end{bmatrix} \| = \| \Psi(C_\psi - Y) \| = \|C_\psi - A\|_\epsilon < 2\epsilon.
\]

It follows that \( |f_3(t) - \sqrt{t}p(t)| < 2\epsilon \) for \( 0 \leq t \leq s \), and similarly for the other three matrix entries.

We will show that \( f_3 \) vanishes identically on \([0, s]\). Suppose not, so that its supremum norm \( \|f_3\|_\infty \) is positive. Without loss of generality we may assume \( 8\epsilon < \|f_3\|_\infty \). It follows that there is a non-degenerate closed subinterval \( I \) of \([0, s]\), depending only on \( f_3 \) and not containing zero, with \( \sqrt{7}p(t) \geq \|f_3\|_\infty/2 \) for \( t \) in \( I \). Thus

\[
\int_I |p(t)|^2 dt \geq \frac{\|f_3\|_2^2 |I|}{4s}.
\]

We return to this inequality below.

Now we want to apply Equation (10) to the linear combination \( A \), which we write as

\[
A = c_1C_\psi + \cdots c_mC_{\psi_m}.
\]
Recall that the normalized Szego kernel functions \( k_z / \| k_z \| \) tend to zero weakly as \( |z| \to 1 \), and so
\[
\| T \|_e = \| T^* \|_e \geq \limsup_{|z| \to 1} \left\| T^* \left( \frac{k_z}{\| k_z \|} \right) \right\| \]
for any bounded operator \( T \) on \( H^2 \). The linear-fractional maps \( \psi_1, \ldots, \psi_m \) in Equation (13) are taken from the four lists in (10). The maps in each of these lists have a common angular derivative set (a singleton) and a single common first order data vector. For example, the maps \( \psi_i \) from the first list all have \( F(\psi_i) = \{ \eta \} \) and first order data vector \( D_1(\psi_i, \eta) = (\eta, 1) \), which we call \( d_1 \). The following table summarizes the corresponding information for each of the four lists:

| \( \psi_i \) chosen from | \( F(\psi_i) \) | unique first-order data vector |
|--------------------------|-----------------|-------------------------------|
| \( \{ (\varphi \circ \sigma)_{n_1} : n_1 \geq 1 \} \) | \( \{ \eta \} \) | \( d_1 = (\eta, 1) \) |
| \( \{ (\sigma \circ \varphi)_{n_2} : n_2 \geq 1 \} \) | \( \{ \zeta \} \) | \( d_2 = (\zeta, 1) \) |
| \( \{ (\varphi \circ \sigma)_{n_3} \circ \varphi : n_3 \geq 0 \} \) | \( \{ \zeta \} \) | \( d_3 = (\varphi(\zeta), \zeta) \) |
| \( \{ (\sigma \circ \varphi)_{n_4} \circ \sigma : n_4 \geq 0 \} \) | \( \{ \eta \} \) | \( d_4 = (\zeta, \sigma'(\eta)) \) |

Since \( C^*_\psi(k_z) = k_{\psi(z)} \), our hypothesis that \( F(\psi) \) is empty says exactly that
\[
\lim_{|z| \to 1} \left\| C^*_\psi \frac{k_z}{\| k_z \|} \right\|^2 = \lim_{|z| \to 1} \frac{1 - |z|^2}{1 - |\psi(z)|^2} = 0
\]
(see [3], p.132) and thus
\[
4\varepsilon^2 > \| C^*_\psi - A \|_e^2 \geq \limsup_{|z| \to 1} \left\| (C^*_\psi - A^*) \frac{k_z}{\| k_z \|} \right\|^2 \geq \lim_{\Gamma_{\zeta, D}} \left\| \left( \sum_{i=1}^m C^*_{\psi_i} + \cdots + C^*_{\psi_m} \right) \frac{k_z}{\| k_z \|} \right\|^2.
\]

We evaluate the limit on the right via Equation (11) with \( \alpha = \zeta \). Note that \( D_1(\zeta) = \{ d_2, d_3 \} \). Discarding the (necessarily non-negative) \( d_2 \) term from the right-hand side of Equation (10) yields
\[
4\varepsilon^2 > \lim_{\Gamma_{\zeta, D}} \left\| \left( c_1 C^*_{\psi_1} + \cdots + c_m C^*_{\psi_m} \right) \frac{k_z}{\| k_z \|} \right\|^2 \geq \sum_{\zeta \in F(\psi), D_1(\psi, \zeta) = d_3} \left\| c_k k^+_w \right\|^2_{H^2_z}.
\]

We can relabel \( \psi_1, \ldots, \psi_m \) so that the relevant \( \psi_i/s \) occur at the beginning, starting with \( i = 0 \). Then the right-hand side of (14) becomes
\[
\sum_{k=0}^n c_k k^+_w H^2_{w^+} = \sum_{i,j=0}^n c_i c_j \frac{1}{w^+_i + w^-_j} = \sum_{i,j=0}^n c_i c_j \int_0^1 \frac{t}{w^+_i + w^-_j - 1} dt = \int_0^1 \left( \sum_{k=0}^n c_k t^{w^+_k} \right)^2 t^{-1} dt.
\]
for appropriate $n \leq m - 1$.

Let us evaluate $e_k$ in terms of the polynomial $p$ occurring in the upper right entry of the matrix function $\Psi(A) = \Psi(Y)$. If

$$p(z) = \sum_{k=0}^{n} b_k z^k,$$

we have

$$Y_3 = C_{\varphi} p(C_{\varphi}^* C_{\varphi}) = C_{\varphi} p(s C_{\varphi} C_{\varphi})(\text{mod } K) = C_{\varphi} p(s C_{\varphi} \sigma) = \sum_{k=0}^{n} b_k s^k C_{(\varphi \circ \sigma)_k \circ \varphi}, = A_3,$$

so that, relabeling if necessary, $\psi_k = (\varphi \circ \sigma)_k \circ \varphi$ and $e_k = b_k s^k$ for $k = 0, 1, \cdots, n$.

Next we compute $w_k$ for $k = 0, 1, \cdots, n$. Let us convert $\psi_k$ into a self-map $U_k$ of the upper half-plane fixing the origin as described prior to Equation (10): $U_k = \tau_{\eta} \circ \psi_k \circ \tau_{\zeta}^{-1}$. We can do the same for the composition factors of $\psi_k = (\varphi \circ \sigma)_k \circ \varphi$. The map $\varphi \circ \sigma$ is a positive parabolic non-automorphism with fixed point $\eta \circ \rho_a$. Thus for $k \geq 1$, $(\varphi \circ \sigma)_k = \rho_{ka}$, and its half-plane transplant $u_k = \tau_{\eta} \circ \rho_{ka} \circ \tau_{\eta}^{-1}$ satisfies $u_k''(0) = 2i ka$ by Equation (7).

We write $v$ for the half-plane version of $\varphi : v = \tau_{\eta} \circ \varphi \circ \tau_{\zeta}^{-1}$. We have

$$U_k = \tau_{\eta} \circ \psi_k \circ \tau_{\zeta}^{-1} = (\tau_{\eta} \circ \rho_{ka} \circ \tau_{\eta}^{-1}) \circ (\tau_{\eta} \circ \varphi \circ \tau_{\zeta}^{-1}) = u_k \circ v.$$

Now $u_k'(0) = \rho_{ka}'(\eta) = 1$ and $v'(0) = |\varphi'(\zeta)| = \frac{1}{s}$, and we find

$$U_k'(0) = \frac{1}{s} \quad \text{and} \quad U_k''(0) = v''(0) + \frac{2ika}{s^2}.$$

From the discussion following Equation (10) we see that

$$w_k = \frac{1}{2s} - iDv''(0) + k \left( \frac{2aD}{s^2} \right).$$

To this point $D$ has been an arbitrary positive number. Let us choose $D$ so that $2aD/s^2 = 1$ and put $\mu = \frac{1}{2s} - iDv''(0)$, a complex number with positive real part. Thus $w_k = \mu + k$ and we can write the right hand side of Equation (15) as

$$\int_{0}^{1} \left| \sum_{k=0}^{n} b_k s^k \right|^2 t^{2Re(\mu)-1} dt = \int_{0}^{1} |p(st)|^2 t^{2Re(\mu)-1} dt \quad \text{and} \quad \frac{1}{s^{2Re(\mu)}} \int_{0}^{s} |p(t)|^2 t^{2Re(\mu)-1} dt.$$

We consider two cases: if $2Re(\mu) - 1 \geq 0$ then this last integral is at least

$$\frac{1}{s} \int_{I} |p(t)|^2 dt$$

where $t_0 > 0$ is the left-hand endpoint of $I$, and if $2Re(\mu) - 1 < 0$ it is at least

$$\frac{1}{s} \int_{I} |p(t)|^2 dt.$$
For small enough \(\epsilon > 0\), either case of this inequality, combined with the inequality (14) and Equation (15), is incompatible with the inequality (12), yielding the desired contradiction. It follows that \(f_3 \equiv 0\) on \([0, s]\). Entirely similar arguments show that \(f_1, f_2\) and \(f_4\) vanish identically on \([0, s]\), hence \(\Psi(C_\psi) = 0\).

With Theorem 1 in hand, we can present our necessary conditions for membership in \(C^*(C_\psi, K)\).

**Theorem 2.** Let \(\varphi\) be as in (14). Suppose \(\psi\) is an analytic self-map of \(\mathbb{D}\) with \(\psi\) lying in \(C^*(C_\psi, K)\) and \(C_\psi\) not compact. Then either \(\psi(z) = z\) or one of the following holds:

(a) \(F(\psi) = \{\zeta\}, \psi(\zeta) = \eta\) and \(\psi'(\zeta) = \varphi'(\zeta)\).
(b) \(F(\psi) = \{\zeta\}, \psi(\zeta) = \zeta\) and \(\psi'(\zeta) = 1\).
(c) \(F(\psi) = \{\eta\}, \psi(\eta) = \zeta\) and \(\psi'(\eta) = 1/\varphi'(\zeta)\).
(d) \(F(\psi) = \{\eta\}, \psi(\eta) = \eta\) and \(\psi'(\eta) = 1\).
(e) \(F(\psi) = \{\zeta, \eta\} \) with \(\psi(\zeta) = \eta, \psi'(\zeta) = \varphi'(\zeta), \psi(\eta) = \eta\) and \(\psi'(\eta) = 1\).
(f) \(F(\psi) = \{\zeta, \eta\} \) with \(\psi(\eta) = \zeta, \psi'(\eta) = 1/\varphi'(\zeta), \psi(\zeta) = \zeta\) and \(\psi'(\zeta) = 1\).

**Proof.** If \(\psi\) has no finite angular derivative, then Theorem 1 guarantees that \(C_\psi\) is compact. Thus we may assume \(F(\psi)\) is non-empty. We also assume \(\psi\) is not the identity, else there is nothing to prove. If \(C_\psi\) is in \(C^*(C_\psi, K)\), the density of the polynomial subalgebra \(P\) says that given \(\epsilon\), we may find a scalar \(c\) and a finite linear combination \(A\) of composition operators with associated maps from the lists (8) so that

\[
\|C_\psi - A - cC_\varphi\|_e < \epsilon.
\]

As in the beginning of the proof of Theorem 1 we may then conclude that \(\psi(e^{i\theta})|1\) a.e., \(|c| < \epsilon\), and that

\[
\|C_\psi - A\| < 2\epsilon.
\]

The self-maps of \(\mathbb{D}\) which define the composition operators in the linear combination \(A\) appear among those in Table I above, along with their angular derivative sets (all singletons) and first order data vectors. Suppose that \(\lambda\) is in \(F(\psi)\) and \(D_1(\psi, \lambda) = \textbf{d}\). If \(\lambda\) is not in \(\{\zeta, \eta\}\), then the inequality (8) gives

\[
\|C_\psi - A\|_e \geq \frac{1}{|\psi'(\lambda)|},
\]

contradicting (16). Similarly, if \(\lambda = \zeta\) and \(\textbf{d}\) is neither \textbf{d}_2\) nor \textbf{d}_3\) from Table I, or if \(\lambda = \eta\) and \(\textbf{d}\) is neither \textbf{d}_1\) nor \textbf{d}_4\) the inequality (8) and Table I again imply (14). It follows that if \(F(\psi)\) is a singleton, one of conditions (a)-(d) must hold.

The remainder of the proof considers the possibility that \(F(\psi) = \{\zeta, \eta\}\). The Julia-Caratheodory theory says a non-identity analytic self-map of \(\mathbb{D}\) cannot have fixed points at distinct points \(\zeta, \eta\) in \(\partial\mathbb{D}\) with derivative 1 at each. If we have both \(\psi'(\zeta) = \eta, \psi'(\zeta) = \varphi'(\zeta)\) and \(\psi'(\eta) = \zeta, \psi'(\eta) = 1/\varphi'(\zeta)\), then \(\psi \circ \psi\) fixes both \(\zeta\) and \(\eta\) with derivative 1 at each, so that \(\psi \circ \psi\) is the identity map, contradicting the observation above that \(|\psi(e^{i\theta})| < 1\) almost everywhere. Thus if \(F(\psi) = \{\zeta, \eta\}\), either (e) or (f) must hold, completing the proof. \(\square\)

We will see in Section 5 that there are indeed maps \(\psi\) of types (e) and (f) in Theorem 2 for which \(C_\psi\) belongs to \(C^*(C_\psi, K)\).
3. The $C^*$-algebra induced by parabolic non-automorphisms

Let us write $\mathcal{B}(H^2)$ for the algebra of bounded operators on $H^2$. A bounded operator $T$ on $H^2$ is essentially normal if $T$ and $T^*$ commute modulo $K$; normal operators and compact operators give trivial examples of essentially normal operators. The only normal composition operators $C_\psi$ are those of the form $\psi(z) = az, |a| \leq 1$. Bourdon, Levi, Narayan, and Shapiro \cite{1} showed that if $\psi$ is linear-fractional with $\|\psi\|_\infty = 1$ and not a rotation, then $C_\psi$ is essentially normal exactly when $\psi$ is a parabolic non-automorphism. Let us select a point $\gamma$ in $\partial\mathbb{D}$ and consider the set $\{\rho_a : \text{Re } a > 0\}$ of all parabolic non-automorphisms fixing $\gamma$. Here, as earlier, $a$ is the translation number for $\rho_a$. Any two of the maps $\rho_a$ commute under composition and in fact $\rho_a \circ \rho_b = \rho_{a+b}$, so $C_{\rho_a}$ and $C_{\rho_b}$ commute. One can easily check that the Krein adjoint of $\rho_a$ is $\rho_{\bar{a}}$. Since $\rho_a(\gamma) = 1$, it follows from Equation (4) that $C_{\rho_a} = C_{\rho_{\bar{a}}} + K$ for some compact operator $K$. A recent theorem of Montes-Rodríguez, Ponce-Escudero and Shkarin \cite{18} shows that $C_{\rho_a}$ is irreducible. Moreover $C^*\langle C_{\rho_a} \rangle$, the unital $C^*$-algebra generated by $C_{\rho_a}$, contains the commutator of $C_{\rho_a}$ and $C_{\rho_{\bar{a}}}$ which we know is compact but non-zero. Thus $C^*\langle C_{\rho_a} \rangle$ contains $K$ and $C^*\langle C_{\rho_a} \rangle/K$ is commutative. Now let $\mathbb{P}_\gamma$ denote the set of all composition operators $C_\rho$, where $\rho$, fixing $\gamma$, ranges over $\{\rho_a : \text{Re } a > 0\}$. We write $C^*\langle \mathbb{P}_\gamma \rangle$ for the unital $C^*$-algebra generated by the operators in $\mathbb{P}_\gamma$. Clearly $C^*\langle \mathbb{P}_\gamma \rangle$ contains $K$, and, by the above remarks, $C^*\langle \mathbb{P}_\gamma \rangle/K$ is also commutative. In this section we compute and apply the Gelfand representation of this quotient algebra.

We begin with two lemmas.

Lemma 1. For $a > 0$ there is an operator $A \geq 0$ and a compact operator $K$ with $C_{\rho_a} = A + K$.

Proof. Then

$$C_{\rho_a/2} = \frac{1}{2}(C_{\rho_a/2} + C_{\rho_a/2}^*) + \frac{1}{2}(C_{\rho_a/2} - C_{\rho_a/2}^*)$$

where $B$ is self-adjoint and $J$ is compact. Thus $C_{\rho_a} = C_{\rho_a/2}C_{\rho_a/2} = (B + J)^2 = B^2 + (BJ + JB + J^2)$. Since $B^2$ is positive and $BJ + JB + J^2$ is compact, we are done. \hfill $\Box$

Lemma 2. Let $a, b$ be positive with $b/a = m/n$, with $m$ and $n$ positive integers. Suppose $0 < \lambda \leq 1$ and there is a sequence $f_k$ of unit vectors in $H^2$ converging weakly to zero such that

$$\|(C_{\rho_a} - \lambda)f_k\| \to 0.$$ 

Then

$$\|(C_{\rho_a} - \lambda^m/n)f_k\| \to 0.$$ 

Proof. First observe that

$$C_{\rho_a}^m - \lambda^m = [C_{\rho_a}^{m-1} + \lambda C_{\rho_a}^{m-2} + \ldots + \lambda^{m-2}C_{\rho_a} + \lambda^{m-1}][C_{\rho_a} - \lambda].$$

In particular, $\|(C_{\rho_a}^m - \lambda^m)f_k\| \to 0$ as $k \to \infty$. Since $C_{\rho_a}^m = C_{\rho_a}^n$,

$$\|(C_{\rho_a}^m - \lambda^m)f_k\| \to 0.$$ 

Also note that we may factor $C_{\rho_a}^m - \lambda^m = C_{\rho_a}^n - (\lambda^m/n)^n$ as

$$[C_{\rho_a}^{n-1} + \lambda^{m/n}C_{\rho_a}^{n-2} + \ldots + (\lambda^{m/n})^{n-2}C_{\rho_a} + (\lambda^{m/n})^{n-1}][C_{\rho_a} - \lambda^m/n].$$
Apply Lemma 1 to $C_{p_n}$ to write

$$
C_{p_n}^{-1} + \lambda^{m/n} C_{p_n}^{-2} + \cdots + (\lambda^{m/n})^{n-2} C_{p_n} + (\lambda^{m/n})^{n-1} I = T + (\lambda^{m/n})^{n-1} I + K
$$

for some positive $T$ and compact $K$. We have

$$
\|(C_{p_n}^{-1} - \lambda) f_k\| = \|(T + (\lambda^{m/n})^{n-1} + K)(C_{p_n} - \lambda^{m/n}) f_k\|.
$$

Since $K$ is compact, $\|K(C_{p_n} - \lambda^{m/n}) f_k\| \to 0$ as $k \to \infty$. Since the left-hand side of Equation (18) goes to 0 as $k \to \infty$, we see, writing $c = (\lambda^{m/n})^{n-1}$ that

$$
\|(T + cI)(C_{p_n} - \lambda^{m/n}) f_k\| \to 0.
$$

But $\|(T + cI)(C_{p_n} - \lambda^{m/n}) f_k\|^2$ is equal to

$$
\|T(C_{p_n} - \lambda^{m/n}) f_k\|^2 + 2c(T(C_{p_n} - \lambda^{m/n}) f_k, (C_{p_n} - \lambda^{m/n}) f_k) + c^2\|(C_{p_n} - \lambda^{m/n}) f_k\|^2 \geq c^2\|(C_{p_n} - \lambda^{m/n}) f_k\|^2
$$

where the last inequality uses the positivity of $T$, so that $\|(C_{p_n} - \lambda^{m/n}) f_k\| \to 0$. $\square$

The essential spectrum $\sigma_e(T)$ of a bounded operator on $H^2$ is by definition the spectrum of the coset $[T]_e$ in $B(H^2)/\mathcal{K}$. We recall from [13] that if $a > 0$, $\sigma_e(C_{p_n}) = [0, 1]$. We will need the notion of joint essential spectrum, which is treated by Dash in [9]. If $a > 0$, the coset $[C_{p_n}]$ of $C_{p_n}$ modulo $\mathcal{K}$ will also be denoted by $x_a$. By either Lemma 1 or the discussion preceding it, $x_a$ is self-adjoint. Given $a$ and $b$, the joint essential spectrum $\sigma_e(C_{p_n}, C_{p_n})$ is defined to be the joint spectrum $\sigma(x_a, x_b)$ of the pair $x_a, x_b$ in the Calkin algebra $B(H^2)/\mathcal{K}$. This set coincides with the joint spectrum in the commutative unital subalgebra $C^*(x_a, x_b)$ generated by $x_a$ and $x_b$. If $\mathcal{M}$ is the maximal ideal space of this algebra, and $\hat{\cdot}$ denotes the Gelfand transform, then the map $\ell \mapsto (\hat{\sigma}_a(\ell), \hat{\sigma}_b(\ell))$ is a homeomorphism of $\mathcal{M}$ onto $\sigma(x_a, x_b)$. Let us assume that $a$ and $b$ are positive. A theorem of Dash [9] states, in this context, using $C_{p_n}^* \equiv C_{p_n}$ (mod $\mathcal{K}$) and similarly for $C_{p_n}$, that $(\lambda, \mu)$ lies in $\sigma_e(C_{p_n}, C_{p_n})$ if and only if there exists a sequence $\{f_k\}$ of unit vectors in $H^2$, converging weakly to zero, such that $\|(C_{p_n} - \lambda) f_k\|$ and $\|(C_{p_n} - \mu) f_k\|$ both tend to zero as $k \to \infty$.

**Corollary 1.** Suppose that $a, b$ are positive and $b/a$ is rational. Then

$$
\sigma_e(C_{p_n}, C_{p_n}) = \{(t^a, t^b) : 0 \leq t \leq 1\}.
$$

**Proof.** We know that

$$
\sigma_e(C_{p_n}, C_{p_n}) = \sigma_e(C_{p_n}) \times \sigma_e(C_{p_n}) = [0, 1] \times [0, 1].
$$

Let $0 < \lambda \leq 1$. Since $\lambda \in \sigma_e(C_{p_n})$, we may find unit vectors $f_k$ with $f_k \to 0$ weakly and $\|(C_{p_n} - \lambda) f_k\| \to 0$. By Lemma 2, $\|(C_{p_n} - \lambda^{b/a}) f_k\| \to 0$ as well, so that $(\lambda, \lambda^{b/a})$ is in $\sigma_e(C_{p_n}, C_{p_n})$ by Dash’s theorem. Setting $t^a = \lambda$ we have $\{(t^a, t^b) : 0 \leq t \leq 1\} \subset \sigma_e(C_{p_n}, C_{p_n})$. The set on the right is compact in $\mathbb{R}^2$, so it contains $(0, 0)$ as well, giving $\{(t^a, t^b) : 0 \leq t \leq 1\} \subset \sigma_e(C_{p_n}, C_{p_n})$. Conversely, if $(\lambda, \mu) \in \sigma_e(C_{p_n}, C_{p_n})$. Dash’s theorem gives the existence of a sequence of unit vectors $f_k$ converging weakly to 0 with both $\|(C_{p_n} - \lambda) f_k\|$ and $\|(C_{p_n} - \mu) f_k\|$ tending to zero as $k \to \infty$. If $\lambda > 0$, $\|(C_{p_n} - \lambda^{b/a}) f_k\| \to 0$ by Lemma 2. Thus $\mu = \lambda^{b/a}$ and $(\lambda, \mu) = (\lambda, \lambda^{b/a}) = (t^a, t^b)$ for some $t, 0 \leq t \leq 1$. If $\mu > 0$, the symmetric result says $\lambda = \mu^{a/b} > 0$, and, putting $\mu = t^b$, $(\lambda, \mu) = (\mu^{a/b}, \mu) = (t^a, t^b)$, again of the desired form. As for $(0, 0)$, we already know it lies in $\sigma_e(C_{p_n}, C_{p_n})$, and of course it has the form $(0^a, 0^b)$. $\square$
We will need the fact that on the domain $\Omega = \{a : \text{Re } a > 0\}$ the map $a \mapsto C_{\rho a}$ is a holomorphic function of $a$ in the operator norm topology; see for example the discussion in the proof of Theorem 6.1 in [6]. We continue to denote the coset of $C_{\rho a}$ by $x_a$ and to keep in mind that when $a > 0$, $\sigma(x_a) = [0, 1]$.

**Theorem 3.** There is a unique $*$-isomorphism $\Gamma : C([0, 1]) \to C^*(\mathbb{P}_\gamma)/\mathcal{K}$ such that $\Gamma(t^a) = [C_{\rho a}]$ for all $a \in \Omega$.

**Proof.** First consider $a = 1$ and $x_1 = [C_{\rho 1}]$. Since $\sigma(x_1) = [0, 1]$ we may define a $*$-isomorphism $\Gamma : C([0, 1]) \to C^*(x_1)$ by sending $p$ to $p(x_1)$ for any polynomial $p$. Fix any rational number $r > 0$. By Corollary 1

$$\sigma(x_1, x_r) = \sigma_r(C_{\rho 1}, C_{\rho r}) = \{(t, t') : 0 \leq t \leq 1\}.$$

The map $p(x_1, x_r) \mapsto p(z_1, z_2)$, where $p$ is a two-variable polynomial, extends to a unique $*$-isomorphism of $C^*(x_1, x_r)$ onto $C(\sigma_1, x_r)$). Since $\sigma(x_1, x_r)$ is homeomorphic to $[0, 1]$ via the map $t \mapsto (t, t')$, we see that $p(x_1, x_r) \mapsto p(t, t')$ defines a $*$-isomorphism of $C^*(x_1, x_r)$ onto $C([0, 1])$. Let $\tilde{\Gamma}$ denote the inverse of this map, that is $\tilde{\Gamma} : p(t, t') \mapsto p(x_1, x_r)$. Since polynomials in $t$ span $C([0, 1]), x_1$ generates the $C^*$-algebra $C^*(x_1, x_r)$ and $C^*(x_1) = C^*(x_1, x_r)$. It follows that $\tilde{\Gamma} = \Gamma$. Since $r$ is arbitrary in the set $\mathbb{Q}_+$ of positive rationals, we have shown that

$$C^*(\{x_r : r \in \mathbb{Q}_+\}) = C^*(x_1).$$

Moreover, $\Gamma(t^r) = x_r, r \in \mathbb{Q}_+$. It is easy to see that the map $a \mapsto t^a$ is a norm-holomorphic map of the right half plane into $C([0, 1])$ and thus that $a \mapsto t^a$ is norm-holomorphic from the right half-line to $\mathcal{B}(\mathcal{H}^2)/\mathcal{K}$, as is the function $a \mapsto x_a$.

We have seen that these functions agree on $\mathbb{Q}_+$, hence they must agree on the right half-plane $\Omega$. \qed

We record three immediate consequences.

**Corollary 2.** If $a_1, \cdots, a_n$ lie in the right half-plane $\Omega$, then

$$\sigma_e(C_{\rho a_1}, \cdots, C_{\rho a_n}) = \{(t^{a_1}, \cdots, t^{a_n}) : 0 \leq t \leq 1\}.$$

**Corollary 3.** If $\rho$ is a parabolic non-automorphism fixing $\gamma$, then $C^*(C_{\rho}) = C^*(\mathbb{P}_\gamma)$.

**Corollary 4.** If $\varphi$ is as in (1), then $\mathbb{P}_\zeta$ and $\mathbb{P}_\eta$ are both subsets of $C^*(C_{\varphi^*}, \mathcal{K})$.

4. LINEAR-FRACTIONAL MAPS

The goal of this section is to find all linear-fractional $\psi$ with $C_{\psi}$ in $C^*(C_{\varphi^*}, \mathcal{K})$, where $\varphi$ satisfies the conditions of (1). Since $C_{\psi}$ is compact if $\|\psi\|_{\infty} < 1$, our interest is in the case $\|\psi\|_{\infty} = 1$.

**Lemma 3.** If $\varphi$ is as in (1), $C^*(C_{\varphi^*}, \mathcal{K})$ contains $C_{\varphi}$ for all linear-fractional $\psi : \mathbb{D} \to \mathbb{D}$ with $\psi(\zeta) = \eta, \psi'(\zeta) = \varphi'(\zeta)$ and $\psi(\mathbb{D})$ properly contained in $\varphi(\mathbb{D})$.

**Proof.** Set $\tau = \varphi^{-1} \circ \psi$, noting that the hypothesis $\psi(D) \subset \varphi(\mathbb{D})$ means that $\tau$ is well-defined. Since this containment is proper, and $\tau'(\zeta) = 1$, $\tau$ is a parabolic non-automorphism with fixed point $\zeta$. Since $\varphi \circ \tau = \psi$, $C_{\psi} = C_{\tau} C_{\varphi}$. By Corollary 4 $C_{\tau} \in C^*(C_{\varphi^*}, \mathcal{K})$, from which the conclusion follows. \qed
Now consider the parabolic non-automorphism $\rho = \varphi \circ \sigma$. The unique fixed point for $\rho$ and its iterates $(\rho)_n$ is $\eta$. Fix an integer $n \geq 1$ and let $\varphi_1 = (\rho)_n \circ \varphi$. Clearly $\varphi_1(\mathbb{D})$ is properly contained in $\varphi(\mathbb{D})$. Note that $\varphi_1(\zeta) = \eta$ and, since $\rho(\eta) = 1$, $\varphi_1'(\zeta) = \varphi'(\zeta)$. It follows from Lemma 3 that $C^*(C_{\varphi_1}, K)$ is contained in $C^*(C_{\varphi}, K)$. Let $i$ denote the corresponding inclusion map. Since $i(K) = K$, $i$ induces a $*$-homomorphism

$$\hat{i}: C^*(C_{\varphi_1}, K)/K \to C^*(C_{\varphi}, K)/K$$

given by $\hat{i}([T]) = [T]$, where $[T]$ denotes the coset, modulo $K$ of the operator $T$. Note that $i$ is itself an inclusion. Also observe that the map $\Psi : C^*(C_{\varphi}, K) \to \mathcal{D}$ induces a $*$-isomorphism $\Phi : C^*(C_{\varphi}, K)/K \to \mathcal{D}$ given by $\Phi([T]) = \Psi(T)$. Let $\Phi_1$ denote the corresponding $*$-isomorphism $\Phi_1 : C^*(C_{\varphi}, K)/K \to \mathcal{D}$. Keep in mind that $\Phi_1$ should be defined by $\Phi_1([T]) = \Psi_1(T)$, where $\Psi_1 : C^*(C_{\varphi}, K) \to \mathcal{D}$ is associated to $\varphi_1$ as $\Psi$ is associated to $\varphi$. Thus if $B$ in $C^*(C_{\varphi_1}, K)$ is given by (2), but with $\varphi$ replaced by $\varphi_1$, then $\Psi_1(B)$ is given by (3). We have a commutative diagram

$$\begin{array}{ccc}
C^*(C_{\varphi_1}, K)/K & \xrightarrow{\hat{i}} & C^*(C_{\varphi}, K)/K \\
\Phi_1 \downarrow & & \Phi \downarrow \\
\mathcal{D} & \xrightarrow{\Lambda} & \mathcal{D}
\end{array}$$

where $\Lambda = \Phi \circ \hat{i} \circ \Phi_1^{-1}$. We seek to identify $\Lambda$ explicitly.

**Lemma 4.** For any element $F$ in $\mathcal{D}$,

$$(\Lambda F)(t) = F(t^{2n+1}/s^{2n}), \ 0 \leq t \leq s.$$  

**Proof.** For the purposes of the proof, we use $\Lambda$ to denote the map given by formula (19), and then show, with this redefinition, that it coincides with $\Phi \circ \hat{i} \circ \Phi_1^{-1}$, that is, that $\Lambda \circ \Phi_1 = \Phi \circ \hat{i}$. Recall that $C^*_\varphi \equiv sC_\sigma \mod K$ so that

$$C_{\varphi_1} = C_\varphi C_{(\rho)_n} = C_\varphi C_\sigma C_{\varphi_1} = \frac{1}{s^n} C_\varphi C_{\varphi_1}^n \mod K,$$

and, taking adjoints, $C_{\varphi_1}^* \equiv \frac{1}{s^n} C_{\varphi}^* C_{\varphi_1}^n$ modulo the compacts. Calculations using these two facts show that if we write $y = [C_{\varphi_1}]$ and $x = [C_\varphi]$, we have, for each non-negative integer $m$,

$$(y^* y)^m = \frac{1}{s^{2nm}} (x^* x)^{(2n+1)m},$$

$$(yy^*)^m = \frac{1}{s^{2nm}} (xx^*)^{(2n+1)m},$$

$$y(y^* y)^m = \frac{1}{s^{(2m+1)n}} x(x^*)^{(2n+1)m+n},$$

and

$$y^*(yy^*)^m = \frac{1}{s^{(2m+1)n}} x^*(xx^*)^{(2n+1)m+n}.$$
The left-hand sides in these four equations are elements in $C^*(C_{\varphi_1}, K)/K$, while the right-hand sides represent the same objects as elements of $C^*(C_{\varphi}, K)/K$. We first act on $y(y^*)^m$ by $\hat{i}$, followed by $\Phi$. We then act on $y(y^*)^m$ by $\Phi_1$, followed by $\Lambda$ (as defined by Equation (19)). As the reader can see from the following picture, we end up with a common result, the matrix function in the lower right-hand corner.

$$y(y^*)^m \xrightarrow{\hat{i}} \frac{1}{x(x^*)^m(x^*)^{2(n+1)m+n}} \-scaleobj{0.9}{\times} \xrightarrow{\Phi} \scaleobj{0.9}{\begin{bmatrix} 0 & \sqrt{t} \ t^m \ 0 & 0 \end{bmatrix}} \xrightarrow{\Lambda} \scaleobj{0.9}{\begin{bmatrix} 0 & \sqrt{t} (x^*)^{(2n+1)m+n} \\ 0 & 0 \end{bmatrix}} \xrightarrow{\Phi} \cdots$$

One can check that we also arrive at common values when $\Lambda \circ \Phi_1$ and $\Phi \circ \hat{i}$ act on $(y^*)^m$, and similarly for $(yy^*)^m$ and $y^*(yy^*)^m$. Since elements of the form $(y^*)^m$, $(yy^*)^m$, $y(y^*)^m$ and $y^*(yy^*)^m$, together with the identity, span $C^*(C_{\varphi_1}, K)/K$, we have $\Lambda \circ \Phi_1 = \Phi \circ \hat{i}$ as desired.

It is clear from (19) that $\Lambda$ is an automorphism of $\mathcal{D}$. It follows that $\hat{i}$ is an automorphism and thus that $i$ has range equal to all of $C^*(C_{\varphi_1}, K)$, that is,

$$C^*(C_{\varphi_1}, K) = C^*(C_{\varphi}, K).$$

More generally, we have the following result.

**Theorem 4.** Let $\psi$ be a linear-fractional map of $\mathbb{D}$, not an automorphism, with $\psi(\zeta) = \varphi(\zeta)$ and $\psi'(\zeta) = \varphi'(\zeta)$, where $\varphi$ is as in (7). Then $C^*(C_{\psi}, K) = C^*(C_{\varphi}, K)$.

**Proof.** The circles $\varphi(\partial \mathbb{D})$ and $\psi(\partial \mathbb{D})$ are both internally tangent to $\partial \mathbb{D}$ at $\eta$. If $\varphi(\mathbb{D})$ is a proper subset of $\psi(\mathbb{D})$, then

$$C^*(C_{\varphi}, K) \subset C^*(C_{\psi}, K)$$

by Lemma 3. Suppose on the other hand that $\psi(\mathbb{D}) \subset \varphi(\mathbb{D})$. If $a$ is the (necessarily positive) translation number of the parabolic map $\varphi \circ \sigma$ (so that $\varphi \circ \sigma = \rho_a$ in the terminology of Section 2), then $(\varphi \circ \sigma)^n = \rho_{na}$, and the radius of the disk $\rho_{na}(\mathbb{D})$ shrinks to zero as $n \to \infty$. Thus there exists $n$ with $\rho_{na}$ properly contained in $\psi(\mathbb{D})$. If $\varphi_1 = (\varphi \circ \sigma)^n \circ \varphi = \rho_{na} \circ \varphi$, then $\varphi_1(\mathbb{D})$ is also properly contained in $\psi(\mathbb{D})$. Since $\varphi_1(\zeta) = \eta = \psi(\zeta)$ and $\varphi'_1(\zeta) = \rho_{na}'(\eta) \varphi'(\zeta) = \varphi'(\zeta) = \psi'(\zeta)$, Lemma 3 implies that $C^*(C_{\varphi_1}, K)$ contains $C^*(C_{\varphi}, K)$, which by (20) coincides with $C^*(C_{\varphi}, K)$. The result is that (21) holds, whatever the relationship between the disks $\varphi(\mathbb{D})$ and $\psi(\mathbb{D})$. The statement of the theorem is symmetric in $\varphi$ and $\psi$, so symmetry implies that the containment reverse to that in (21) also holds, completing the proof.

**Theorem 5.** Suppose $\varphi$ is as in (7). Let $\psi$, not the identity, be any linear-fractional self-map of $\mathbb{D}$ with $\|\psi\|_{\infty} = 1$. Then $C_{\psi}$ is in $C^*(C_{\varphi}, K)$ if and only if $\psi$ is not an automorphism and one of the following conditions holds:

(a) $\psi(\zeta) = \eta$ and $\psi'(\zeta) = \varphi'(\zeta)$.
(b) $\psi(\zeta) = \zeta$ and $\psi'(\zeta) = 1$.
(c) $\psi(\eta) = \zeta$ and $\psi'(\eta) = 1/\varphi'(\zeta)$.
(d) $\psi(\eta) = \eta$ and $\psi'(\eta) = 1$. 
Proof. The “only if” statement follows immediately from Theorem 2 and the hypothesis that \( \psi \) is linear-fractional. Conversely, let \( \psi \) be a linear-fractional map which is not an automorphism. If \( \psi \) is parabolic with fixed point at either \( \zeta \) or \( \eta \), the result follows from Corollary 4; this handles the cases (b) and (d). If \( \psi \) is as in (a), then we have \( C_\psi \in C^*(C_\varphi, K) \) by Theorem 4. Finally, if \( \psi \) satisfies condition (c), then its Krein adjoint \( \sigma_\psi \) is a linear-fractional self-map of \( D \), not an automorphism, which satisfies condition (a), so that \( C_{\sigma_\psi} \in C^*(C_\varphi, K) \). Since \( C_\psi \equiv \sigma_\psi \) modulo the compacts, this completes the argument. \( \square \)

The maps satisfying (a)-(d) in Theorem 5 can be described more explicitly. Given a point \( \gamma \) on \( \partial D \), let us write \( \rho_{\gamma,a} \) for the unique parabolic map fixing \( \gamma \) with translation number \( a \). This will be a self-map of \( D \) when \( Re \ a \geq 0 \), but when \( Re \ a < 0 \), \( \rho_{\gamma,a} \) takes \( D \) onto a larger disk, whose boundary is externally tangent to \( \partial D \) at \( \gamma \). Clearly, the linear-fractional non-automorphisms of \( D \) satisfy condition (a), so that \( \rho_0 \) is an automorphism. If \( b \) is the unique positive number with \( \rho_{\gamma,b} (D) = \varphi(D) \), then \( \rho_{\gamma,-a} \circ \varphi \) is a non-automorphism self-map of \( D \) exactly when \( Re \ a < b \). Phrasing and summarizing, we conclude that the non-automorphisms \( \psi \) of \( D \) satisfying (a) are precisely the maps of the form \( \rho_{\gamma,a} \circ \varphi \), \( Re \ a > b \). Similarly, if \( c \) is the unique positive number with \( \rho_{\zeta,c} (D) = \sigma(D) \), then the non-automorphisms \( \psi \) satisfying (c) are exactly the maps of the form \( \psi = \rho_{\zeta,a} \circ \sigma \) with \( Re \ a > -c \). The next result shows that the positive translation numbers \( b \) and \( c \) are nicely related to each other, and to the translation numbers of the positive parabolic non-automorphisms \( \varphi \circ \sigma \) and \( \sigma \circ \varphi \).

Theorem 6. Let \( b \) and \( c \) be the unique positive numbers with \( \rho_{\eta,b} (D) = \varphi(D) \) and \( \rho_{\zeta,c} (D) = \sigma(D) \), respectively. We have \( c = |\varphi'(\zeta)| b \), and moreover, \( \varphi \circ \sigma = \rho_{\eta,2b} \) and \( \sigma \circ \varphi = \rho_{\zeta,2c} \).

Proof. Clearly there is no loss of generality in assuming that \( \zeta = 1 \). The non-affine linear-fractional self-maps of \( D \) which send 1 to \( \eta \in \partial D \) can be written in the form

\[
\varphi(z) = \eta \frac{(1 + s + sd)z + (d - s - sd)}{z + d},
\]

where \( s = |\varphi'(1)| \) and \( Re \ \frac{s - 1}{s + 1} \geq s \) (see [3]). A computation shows that \( \varphi'(1) = \eta s \) and \( \varphi''(1) = -2\eta s / (1 + d) \). The image of the unit circle under \( \varphi \) is a circle with curvature \( \kappa_1 = |\varphi'(1)|^{-1} Re [1 + \varphi''(1)/\varphi'(1)] = \frac{1}{2} Re [1 - \frac{2}{1 + d}] \). Since

\[
\sigma(z) = \frac{\eta(1 + s + sd)z - 1}{-\eta(d - s - sd)z + d},
\]

we find that \( \sigma'(\eta) = \overline{\eta} / s \) and

\[
\sigma''(\eta) = \frac{2\eta^2(d - s - sd)}{s^2(1 + d)}.
\]
so that the image of the unit circle under $\sigma$ is a circle with curvature
\[ \kappa_2 = s\text{Re} \left\{ 1 + \eta \frac{2\eta^2(d - s - sd)}{s^2(1 + d)} \right\} = 2 - 2\text{Re} \frac{1}{1 + d} - s. \]

A positive parabolic non-automorphism fixing 1 and corresponding to translation by $a$ has the form $((2 + d)z - 1)/(z + d)$ where $a = -2/(d + 1)$; by the above calculations the image of the unit circle under this map has curvature $1 + \text{Re} a$.

Thus, if $\varphi(1) = \eta$ and $|\varphi'(1)| = s$, the unique positive value $b$ such that the curvature of $\varphi(\partial \mathbb{D})$ is equal to the curvature of the image of $\partial \mathbb{D}$ under the positive parabolic map which corresponds to translation by $b > 0$ satisfies
\[ 1 + b = \frac{1}{s} \text{Re} \left( 1 - \frac{2}{1 + d} \right); \]
that is,
\[ b = \frac{1}{s} \text{Re} \left( 1 - \frac{2}{1 + d} \right) - 1. \]

Similarly, the curvature of the circle $\sigma(\partial \mathbb{D})$ is equal to the curvature of the circle which is the image of the unit circle under the positive parabolic non-automorphism corresponding to translation by $c$ precisely when $c = 1 - s - 2\text{Re} \frac{1}{1 + d}$. Thus $c = sb$.

This conclusion also holds when $\varphi$ is an affine map, $\varphi(z) = \eta(sz + 1 - s)$, where the computations are easier.

For the final statement, let $\psi = \rho_{\eta,b} \circ \varphi$, so that $\psi$ is an automorphism of $\mathbb{D}$ and $\varphi = \rho_{\eta,b} \circ \psi$. Since the Krein adjoint of an automorphism is its inverse, we have
\[ \sigma = \sigma_\varphi = \sigma_\psi \circ \sigma_{\rho_{\eta,b}} = \psi^{-1} \circ \rho_{\eta,b} = \psi^{-1} \circ \rho_{\eta,b} \]
and thus $\varphi \circ \sigma = \rho_{\eta,b} \circ \psi \circ \psi^{-1} \circ \rho_{\eta,b} = \rho_{\eta,2b}$. Similarly, $\sigma \circ \varphi = \rho_{1,2c} = \rho_{\xi,2c}$. \hfill $\square$

The remarks preceding Theorem [1] express the linear-fractional maps $\psi$ with $C_\psi$ belonging to $C^*(C_\varphi, \mathcal{K})$ in terms of $\varphi, \sigma, \rho_{\eta,a}$ and $\rho_{\zeta,a}$ for appropriate ranges of the translation numbers $a$. We describe below a corresponding operator-theoretic description of $C_\psi$ modulo $\mathcal{K}$, in terms of the polar factors of $C_\varphi$ and $C_\varphi^*$. In [15] it was shown that every operator $B$ in $C^*(C_\varphi, \mathcal{K})$ has a representation generalizing Equation [2] and having the form $B = T + K$ with $K$ compact and
\begin{equation}
T = cI + f(C_\varphi C_\varphi^*) + g(C_\varphi C_\varphi^*) + Uh(C_\varphi^* C_\varphi) + U^*k(C_\varphi^* C_\varphi^*),
\end{equation}
where $f$ and $h$ are continuous on $\sigma(C_\varphi^* C_\varphi)$, $g$ and $k$ are continuous on $\sigma(C_\varphi C_\varphi^*)$, all four functions vanish at zero, and $U$ is the partial isometry polar factor (which in this case is unitary) of $C_\varphi$. The restrictions of $f, g, h$ and $k$ to the interval $[0, s]$, which coincides with both of the essential spectra $\sigma_\psi(C_\varphi^* C_\varphi)$ and $\sigma_\varphi(C_\varphi C_\varphi^*)$, are uniquely determined by $B$. We call $T$ a \textit{distinguished representative} of the coset $[B]$, and recall from [15] that
\[ \Psi(B) = \Psi(T) = \begin{bmatrix} c + g & h \\ k & c + f \end{bmatrix}. \]

We start with the operator $(C_\varphi^* C_\varphi)^a$, defined by the self-adjoint functional calculus, where $\text{Re} a > 0$. Note that
\[ C_\varphi^* C_\varphi \equiv s C_\sigma C_\varphi \text{ (mod } \mathcal{K}) = s C_{\varphi \circ \sigma} = s C_{\rho_{\eta,2b}}. \]
where the last equality follows from Theorem 6. In Theorem [3] take \( \gamma = \eta \) and consider the *-isomorphism \( \Gamma \), here called \( \Gamma_\eta \) to emphasize the fixed point \( \eta \). We have

\[
[(C_\varphi^* C_\varphi)^s] = [C_\varphi^* C_\varphi]^s = s^a [C_{\rho_{\eta, a}}]^s = s^a \Gamma_\eta (t^{2ba})^s = s^a \Gamma_\eta (t^{2ba}) = s^a [C_{\rho_{\eta, a}}^s].
\]

The first and fourth equalities follow, respectively, from the facts that the coset map \( B \mapsto [B] \) and \( \Gamma_\eta \) are each *-homomorphisms. Relabeling \( 2ba \) as \( a \), we see that \( s^{-\frac{a}{2}} C_{\rho_{\eta, a}}^s \) is a distinguished representative of \( [C_{\rho_{\eta, a}}] \) for \( Re a > 0 \). A similar argument shows that the coset \( [C_{\rho_{c, a}}] \) has distinguished representative \( s^{-\frac{a}{2}} (C_\varphi^* C_\varphi)^s \) for \( Re a > 0 \).

Now consider \( \rho_{\eta, a} \circ \varphi \), which we know to be a self-map of \( \mathbb{D} \), but not an automorphism, when \( Re a > -b \). First we look at the case \( Re a > 0 \). We have

\[
C_{\rho_{\eta, a} \circ \varphi} = C_\varphi C_{\rho_{\eta, a}} = U(C_\varphi^* C_\varphi)^s C_{\rho_{\eta, a}} \equiv s^{-\frac{a}{2}} U(C_\varphi^* C_\varphi)^{\frac{1}{2} + \frac{a}{2s}} \pmod{K}
\]

by our above discussion. By the spectral theorem, \( (C_\varphi^* C_\varphi)^s \) is holomorphic for \( Re z > 0 \) in the weak operator topology, and therefore in the operator norm topology; see [13], Theorem 3.10.1. Thus the cosets \( [C_{\rho_{\eta, a} \circ \varphi}] \) and \( [s^{-\frac{a}{2}} U(C_\varphi^* C_\varphi)^{\frac{1}{2} + \frac{a}{2s}}] \) are both holomorphic \( B(H^2)/K \)-valued functions of \( a \), \( Re a > -b \), which agree on the subset \( \{ a : Re a > 0 \} \). Hence they agree on all of \( \{ a : Re a > -b \} \), showing that \( s^{-\frac{a}{2}} U(C_\varphi^* C_\varphi)^{\frac{1}{2} + \frac{a}{2s}} \) is a distinguished representative of \( [C_{\rho_{\eta, a} \circ \varphi}] \) when \( Re a > -b \).

An analogous statement holds for \( [C_{\rho_{c, a} \circ \sigma}] \) with \( Re a > -c \). The following table summarizes these conclusions.

| Condition on \( \psi \) in Theorem [5] | Linear-fractional \( \psi \) with \( C_\psi \) in \( C^*(C_\varphi, K) \) | Distinguished representative of \( [C_\psi] \) | Matrix function \( \Psi(C_\psi)(t), 0 \leq t \leq s \) |
|--------------------------------------|-------------------------------------------------|----------------------|-----------------------|
| (d) \( \rho_{\eta, a}, \ Re a > 0 \) | \( \rho_{\eta, a}, \ Re a > 0 \) | \( s^{-\frac{a}{2}} (C_\varphi^* C_\varphi)^{\frac{1}{2}} \) | \( \left[ \begin{array}{c} \frac{1}{s} \eta^s \ 0 \\ 0 \ 0 \end{array} \right] \) |
| (b) \( \rho_{\eta, a}, \ Re a > 0 \) | \( \rho_{\eta, a}, \ Re a > 0 \) | \( s^{-\frac{a}{2}} (C_\varphi^* C_\varphi)^{\frac{1}{2}} \) | \( \left[ \begin{array}{c} 0 \ 0 \\ 0 \ \left( \frac{1}{s} \eta^s \right) \end{array} \right] \) |
| (a) \( \rho_{\eta, a} \circ \varphi, \ Re a > -b \) | \( \rho_{\eta, a} \circ \varphi, \ Re a > -b \) | \( s^{-\frac{a}{2}} U(C_\varphi^* C_\varphi)^{\frac{1}{2} + \frac{a}{2s}} \) | \( \left[ \begin{array}{c} 0 \ \sqrt{t} \left( \frac{1}{s} \eta^s \right) \\ 0 \ 0 \end{array} \right] \) |
| (c) \( \rho_{\eta, a} \circ \sigma, \ Re a > -c \) | \( \rho_{\eta, a} \circ \sigma, \ Re a > -c \) | \( s^{-\frac{a}{2}} U(C_\varphi^* C_\varphi)^{\frac{1}{2} + \frac{a}{2s}} \) | \( \left[ \begin{array}{c} 0 \ 0 \\ \sqrt{t} \left( \frac{1}{s} \eta^s \right) \ 0 \end{array} \right] \) |

Given an operator \( B \) in \( C^*(C_\varphi, K), \sigma_c(B) \) and \( \|B\|_e \) coincide with \( \sigma(\Psi(B)) \) and \( \|\Psi(B)\|_e \), respectively. Thus, if \( B \) is a finite linear combination of composition operators \( C_\psi \) with \( \psi \)’s chosen from Column 2 in Table II, one can calculate \( \Psi(B) \) from Column 4 and in principle read off \( \sigma_c(B) \) and \( \|B\|_e \); see Theorem 4.17 in [13].

It is known [13] that the collection of linear-fractional composition operators \( C_\psi \) with \( \psi \) a non-automorphism having \( \|\psi\|_\infty = 1 \) is linearly independent modulo \( K \).

The following result shows that this remains true when \( K \) is replaced by the larger subspace \( C^*(C_\varphi, K) \) of \( B(H^2) \).
Theorem 7. Let $\varphi$ be as in (11). Suppose that $\beta_1, \cdots, \beta_q$ are distinct linear-fractional self-maps of $\mathbb{D}$ and that $a_1 \cdots a_q$ are non-zero complex numbers. If the linear combination $a_1 C_{\beta_1} + \cdots + a_q C_{\beta_q}$ lies in $C^*(C_\varphi, K)$, then so do the individual operators $C_{\beta_1}, \cdots, C_{\beta_q}$.

Proof. Let us discard those $C_{\beta_i}$’s which lie in $C^*(C_\varphi, K)$ and assume for the purpose of obtaining a contradiction that there are some left over. Relabel these as $C_{\beta_1}, \cdots, C_{\beta_r}$, let $a_1, \cdots, a_r$ be the corresponding constants, and put $T = a_1 C_{\beta_1} + \cdots + a_r C_{\beta_r}$, which lies in $C^*(C_\varphi, K)$. Here, none of the summands are compact, so $\|\beta_i\|_\infty = 1$ for $i = 1, \cdots, r$. Now we proceed almost as in the proof of Theorem 6 with $T$ playing the role of $C_\psi$. Given $\epsilon > 0$ there exists $A$ as in that proof and a complex $c$ such that $\|T - cC_z - A\|_\epsilon < \epsilon$. By the inequality (11)

$$\|T - cC_z - A\|^2 \geq |c|^2 + \frac{1}{2\pi} \sum_{i=1}^r |a_i|^2 |J(\beta_i)|$$

so that $|c| < \epsilon$, since each $|J(\beta_i)|$ must be zero, so $\beta_i$ is a non-automorphism. As earlier, we have $\|T - A\|_\epsilon < 2\epsilon$.

According to Corollary 5.17 in [14], the cosets $[C_{\beta_1}], \cdots, [C_{\beta_r}]$ are linearly independent in $\mathcal{B}(H^2)/K$. It follows that $T$ is not compact, so the matrix function

$$\Psi(T) = \begin{bmatrix} f_2 & f_3 \\ f_4 & f_1 \end{bmatrix}$$

is not identically zero on $[0, s]$. As in Theorem 6, we focus on $f_3$ and aim for a contradiction by assuming that $\|f_3\|_\infty > 0$. An appropriate choice of $\epsilon$ again yields the inequality (12), where $p$ has the same meaning as there. Again we write $A$ in the form (13), and thus have

$$4\epsilon^2 > \|T - A\|^2_\epsilon \geq \limsup_{|z| \to 1} \left\| (T^* - A^*) \frac{k_z}{\|k_z\|} \right\|^2$$

$$\geq \lim_{r, c, \psi} \left\| \left( C_{\psi}^* \sum_{i=1}^r a_i C_{\beta_i}^* - \sum_{i=1}^m C_{\psi}^* \right) \frac{k_z}{\|k_z\|} \right\|^2$$

$$\geq \left\| \sum_{\zeta \in F(\psi)} \frac{1}{D_{1(\psi, \zeta) \in F\psi}} \frac{k_{\psi_{\zeta}}^+}{H^2_{\infty}} \right\|^2$$

The rest of the argument follows that of Theorem 6 exactly, reaching the same contradiction. \hfill \Box

5. Non linear-fractional maps

In this section we explore maps $\psi$, satisfying either condition (e) or (f) of Theorem 2 for which $C_\psi$ lies in $C^*(C_\varphi, K)$; our main result shows that such maps exist. We begin with a lemma about finite Blaschke products.

Lemma 5. Given $\zeta, \eta$ distinct points on $\partial \mathbb{D}$, and positive numbers $t_1, t_2$, there exists a finite Blaschke product $B$ with the properties $B(\eta) = \eta, B(\zeta) = \eta, B'(t_1) = t_1$ and $|B'(\zeta)| = t_2$. Moreover, $B'(\zeta) = \eta t_2$. 

Proof. Clearly there is no loss of generality in taking $\eta = 1$. Initially we will also suppose that $\zeta = -1$; this condition will be removed at the end. A finite Blaschke product $B(z) = \prod \frac{a_n z}{\bar{a}_n z} (a_n - z)/(1 - \bar{a}_n z)$ will meet the conditions $B(1) = 1$, $B(-1) = 1$ if both
\[
\prod \frac{|a_n| a_n - 1}{a_n} = 1
\]
and
\[
\prod \frac{|a_n| a_n + 1}{a_n} = 1.
\]

It is easy to see that both of these conditions will be met if the zeros of $B$ are chosen to be a collection of conjugate pairs $\{a, \overline{a}\}$. The conditions $B'(1) = t_1, |B'(-1)| = t_2$ are satisfied if
\[
\sum \frac{1 - |a_n|^2}{|1 - a_n|^2} = t_1
\]
and
\[
\sum \frac{1 - |a_n|^2}{|1 + a_n|^2} = t_2
\]
respectively (see [2]).

Next observe that for any $t > 0$, $\{z : 1 - |z|^2 = t(1 - |z|^2)\}$ is a circle centered at $(t/(t + 1), 0)$ with radius $1/(t + 1)$ and $\{z : 1 - |z|^2 = t(1 + |z|^2)\}$ is a circle centered at $(-t/(t + 1), 0)$ with radius $1/(1 + t)$. As $t \to 0$, the centers of these circles approach 0 and the radii tend to 1. Thus given $t_1, t_2$ arbitrary positive numbers we may choose $m$ a positive integer sufficiently large so that the circles $\{z : 1 - |z|^2 = t_1/2^m |1 - |z|^2\}$ and $\{z : 1 - |z|^2 = t_2/2^m |1 + |z|^2\}$ intersect in a conjugate pair of points $a, \overline{a}$. Consider the Blaschke product $B(z)$ with a zero of order $2^{m-1}$ at $a$ and a zero of order $2^{m-1}$ at $\overline{a}$. Since the zeros occur at conjugate pairs, $B(1) = 1$ and $B(-1) = 1$. By construction
\[
(1 - |a|^2)/(|1 - a|^2) = (1 - |\overline{a}|^2)/(|1 - \overline{a}|^2) = \frac{t_1}{2^m},
\]
and
\[
(1 - |a|^2)/(|1 + a|^2) = (1 - |\overline{a}|^2)/(|1 + \overline{a}|^2) = \frac{t_2}{2^m},
\]
so that the zeros of $B$ satisfy Equations (24) and (25) as desired, and $B'(1) = t_1$, $|B'(-1)| = t_2$.

Now suppose $\zeta \in \partial \mathbb{D}$ is not equal to $−1$. Find a parabolic automorphism $\tau$ fixing 1, with derivative 1 there, and taking $\zeta$ to $−1$; a unique such $\tau$ exists since (purely imaginary) translations act transitively on the boundary of the right half-plane. Then for $B$ as constructed above, $B \circ \tau$ is a finite Blaschke product fixing 1, with derivative $t_1$ at 1, sending $\zeta$ to 1 and having derivative $|(B \circ \tau)'(\zeta)| = |B'(-1)||\tau'(\zeta)|$; since $|B'(-1)|$ can be arbitrarily prescribed and $\tau$ depends only on the value of $\zeta$, this means $|(B \circ \tau)'(\zeta)|$ can be chosen to be an arbitrary positive number. Finally observe that if $B$ is a Blaschke product with $B(\zeta) = \eta$ and $|B'(\zeta)| = s$, then we must have $B'(\zeta) = \eta \overline{s}$'s, since $\overline{\zeta}B(z)$ fixes $\zeta$, and hence has positive derivative there. $\square$

Theorem 8. Suppose that $\varphi$ is as in (1). There exist analytic self-maps $\psi_1$ and $\psi_2$ of $\mathbb{D}$, satisfying conditions (e) and (f) of Theorem 2 respectively, such that $C_{\psi_1}$ and
Suppose \( \gamma \) is in \( F(\psi) \) and \( \psi \) has order of contact exceeding two at \( \gamma \). Following Theorem 1, we let \( \epsilon > 0 \) and find a finite linear combination \( A \) of composition operators whose self-maps are chosen from the lists \( [3] \) such that \( \|C_\psi - A\| < \epsilon \). At the same time we apply the inequality \( [4] \) to the linear
combination $C_{\psi} - A$ at the point $\alpha = \gamma$. The maps in the lists (6), being linear-fractional non-automorphisms, all have order of contact two at the unique points in their angular derivative sets. Taking the left side of (9) to be $\|C_{\psi} - A\|_2^2$ and $k = 3$ on the right side, the sum on the right-hand side has only one term, namely $1/|\psi'(\gamma)|$, giving

$$\epsilon^2 > \|C_{\psi} - A\|_2^2 \geq \frac{1}{|\psi'(\gamma)|},$$

a contradiction. Thus $\psi$ must have order of contact two at $\gamma$.

Conversely, suppose $\psi$ satisfies (i) and (ii). If $F(\psi) = \{\zeta\}$, let $\beta$ be the unique linear-fractional map with $D_2(\beta, \zeta) = D_2(\psi, \zeta)$. Since $\psi$ has order of contact two at $\zeta$, the curvature of $\{\psi(e^{i\theta}) : e^{i\theta} \in \partial \mathbb{D}\}$ at $e^{i\theta} = \zeta$ exceeds unity, so that $\beta(\partial \mathbb{D})$, having the same curvature, is internally tangent to $\partial \mathbb{D}$ at $\psi(\zeta)$ and bounds a proper subdisk of $\mathbb{D}$. Thus $\beta$ is a non-automorphism of $\mathbb{D}$ satisfying (a) or (b) of Theorem 5, and $C_{\beta}$ lies in $C^*(C_{\varphi}, \mathcal{K})$. The same argument covers the case $F(\psi) = \{\eta\}$.

If $F(\psi) = \{\zeta, \eta\}$ we proceed as in the proof of Theorem 8 to produce linear-fractional non-automorphisms $\beta_1$ and $\beta_2$ of $\mathbb{D}$ with $C^*(C_{\varphi}, \mathcal{K})$ containing $C_{\beta_1}$ and $C_{\beta_2}$ and $C_{\psi} \equiv C_{\beta_1} + C_{\beta_2} \pmod{\mathcal{K}}$. □

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