Gyrohydrodynamics: Relativistic spinful fluid with strong vorticity

Zheng Cao¹, Koichi Hattori²,³, Masaru Hongo⁴,⁵, Xu-Guang Huang⁶,⁷, and Hidetoshi Taya⁵

¹Physics Department and Center for Particle Physics and Field Theory, Fudan University, Shanghai 200438, China
²Zhejiang Institute of Modern Physics, Department of Physics, Zhejiang University, Hangzhou, Zhejiang 310027, China
³Research Center for Nuclear Physics, Osaka University, 10-1 Mihogaoka, Ibaraki, Osaka 567-0047, Japan
⁴Department of Physics, Niigata University, Niigata 950-2181, Japan
⁵RIKEN iTHEMS, RIKEN, Wako 351-0198, Japan
⁶Key Laboratory of Nuclear Physics and Ion-beam Application (MOE), Fudan University, Shanghai 200433, China
⁷Shanghai Research Center for Theoretical Nuclear Physics, NSFC and Fudan University, Shanghai 200438, China

We develop a relativistic (quasi-)hydrodynamic framework, dubbed the gyrohydrodynamics, to describe fluid dynamics of many-body systems with spin under strong vorticity based on entropy-current analysis. This framework generalizes the recently-developed spin hydrodynamics to the regime where the spin density is at the leading order in derivatives but suppressed by another small parameter, the Planck constant \( \hbar \), due to its quantum nature. Our analysis shows that the complete first-order constitutive relations of gyrohydrodynamics involve fourteen transport coefficients and are highly anisotropic.

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Introduction.— Hydrodynamics is a low-frequency and long-wavelength effective theory for interacting many-body systems [1]. When incorporated with relativity, it has been applied very successfully in astrophysics, cosmology, and relativistic heavy-ion collisions. For example, relativistic hydrodynamics provides an important foundation for the understanding of the core-collapse supernova explosions, the structures and bulk properties of neutron stars and white dwarfs, the dynamics of accretion discs around compact objects, and the merging process of binary neutron stars and black holes; see, e.g., Refs. [2–6]. In high-energy heavy-ion collisions, relativistic hydrodynamics has become the ‘standard model’ to describe the evolution of quark-gluon plasma created during the collisions; see e.g., Refs. [7–9].
It is noteworthy that rotation of fluids, whether it is global or local (local rotation is characterized by fluid vorticity), plays a crucial role in all of the systems mentioned above. One of the critical consequences of rotation is spin polarization when the constituents of fluids are spinful. In particular, the strongest fluid vorticity has recently been detected in heavy-ion collisions [10]. Measurements of hyperon polarization [10–12] and vector-meson spin alignment [13–15] in heavy-ion collisions have indicated strong spin polarization of quarks and/or gluons in the quark-gluon plasma, supporting earlier theoretical ideas proposed in, e.g., Refs. [16–21]. As such, it is desirable to develop a relativistic hydrodynamic framework that deals appropriately with the spin degrees of freedom and the vorticity generation. Such a framework may pave the way to quantitative understanding of spin transports in quark-gluon plasma, supernova explosions, and so on. Specifically, it may be used to resolve the so-called sign problem of the local spin polarization, that is, the discrepancy between the experimental data for azimuthal-angle dependence of hyperon polarization and the theoretical calculations based on thermal vorticity (see below for definition); see reviews [22–26] and references therein for more details.

There have been great efforts on developing the framework of spin hydrodynamics [27–54]. In most of these developments, the spin and vorticity are considered to be of similar magnitudes as the shear viscous correction to the hydrodynamics, meaning that they are assumed to be $O(\partial^1)$ in terms of the derivative expansion. This is possible, since vorticity is the skew gradient of velocity while the shear tensor is the symmetric gradient of velocity. Thus, one expects that their strength may be comparable in magnitude. However, there is a significant difference between the two: the shear tensor vanishes at the global equilibrium, whereas vorticity does not. This implies that, on the physical ground, these two can be regarded to be at different orders in derivatives in and near equilibrium (cf. Ref. [37] for another power counting scheme based on the same consideration). Furthermore, the discovery of strong spin polarization in the quark-gluon plasma also demands development of a hydrodynamic theory with high spin density and vorticity.

Motivated by these observations, we will investigate a scenario in which vorticity is counted as zeroth order in derivatives, whereas other types of gradients of thermodynamic quantities are counted as first order in derivatives. The spin polarization induced by the zeroth-order vorticity is also regarded as the zeroth order in derivatives, but we assume it to be suppressed by another small parameter, i.e., the Planck constant $\hbar$ due to its quantum nature. We dub the resultant framework gyrohydrodynamics.

As we will demonstrate, the constitutive relations in gyrohydrodynamics have much richer structure than those in spin hydrodynamics [32, 34, 35, 43, 45, 46, 51–53]. Firstly, the leading-order constitutive relation describes the anisotropic pressure induced by strong vorticity. Besides, we find fourteen anisotropic transport coefficients: three bulk, four shear, four rotational viscosities, and three charge conductivities. Among these transport coefficients, five correspond to Hall-like transports, which do not induce the entropy production. The appearance of the anisotropic transports is analogous to magnetohydrodynamics, but there are no Hall transports in magnetohydrodynamics in the strict hydrodynamic limit [55, 56].

We stress that spin is not a conserved quantity due to the inherent spin-orbit coupling in relativistic systems. As a result, spin density is not a strict hydrodynamic variable.
Therefore, spin hydrodynamics and thus gyrohydrodynamics should be regarded as quasi-
hydrodynamics [57] or a hydro+ theory [58] valid in a regime where the relaxation time of
spin density towards its local equilibrium value is much longer than that of other non-
hydrodynamic modes [45]. To keep the notations simple, however, we will continue calling
spin density a hydrodynamic variable and gyrohydrodynamics a hydrodynamic theory.

Throughout the paper, we use the mostly-plus metric $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$, totally
anti-symmetric tensor $\epsilon^{\mu\nu\rho\sigma}$ with $\epsilon^{0123} = +1$, and notations $X^{(\mu\nu)} = \frac{1}{2}(X^{\mu\nu} + X^{\nu\mu})$ and
$X^{[\mu\nu]} = \frac{1}{2}(X^{\mu\nu} - X^{\nu\mu})$ for an arbitrary tensor $X^{\mu\nu}$.

Entropy current analysis for rotating spinful fluids. — The hydrodynamic equations follow
from Ward-Takahashi identities derived from symmetries of underlying microscopic theories.
In this Letter, we consider charged relativistic fluids that enjoy the translational, Lorentz,
and vector $U(1)$ symmetries. These symmetries lead to the equations of motion,

$$
\partial_{\mu} \Theta^{\mu}_\nu = 0, \quad \partial_{\mu} \Sigma^{\mu}_{\nu\rho} = -2\Theta_{[\nu\rho]}, \quad \partial_{\mu} J^{\mu} = 0,
$$

(1)

where $\Theta^{\mu\nu}$, $\Sigma^{\mu\nu}_{\rho\sigma}$, and $J^{\mu}$ are the energy-momentum tensor, the spin current, and the charge
current, respectively. Note that the second equation follows from the total angular moment-
conservation $\partial_{\mu} J^{\mu}_{\nu\rho} = 0$ and a decomposition of the total angular momentum tensor
$J^{\mu}_{\nu\rho} = L^{\mu}_{\nu\rho} + \Sigma^{\mu}_{\nu\rho}$ with $L^{\mu}_{\nu\rho} = x_{\nu} \Theta^{\mu}_{\rho} - x_{\rho} \Theta^{\mu}_{\nu}$ being the orbital angular momentum. We
choose the spin current $\Sigma^{\mu}_{\nu\rho}$ to be totally anti-symmetric, motivated by Dirac fermions as
the microscopic carrier of spin.

We then introduce our dynamical variables — the energy density $e$, the fluid-four velocity
$u^{\mu}$, and the spin density $\sigma^{\mu\rho}$ — by

$$
\Theta^{(\mu\nu)} u_{\nu} = -e u^{\mu}, \quad u_{\mu} \Sigma^{\mu}_{\nu\rho} = -\sigma_{\nu\rho}, \quad J^{\mu} u_{\mu} = -n,
$$

(2)

where we employed the Landau frame in defining the fluid velocity and the energy density $[1]$
and normalize the fluid velocity as $u^{\mu} u_{\mu} = -1$. The spin density $\sigma^{\mu\nu}$ only has three dynamical
degrees of freedom (rather than six), as the totally anti-symmetric property of the spin current
forces $\sigma_{\mu\nu}$ to satisfy the Frenkel condition $\sigma_{\mu\nu} u^{\nu} = 0$ in addition to the anti-symmetric condition
$\sigma_{\nu\rho} = -\sigma_{\rho\nu}$. Consequently, three components of the spin non-conservation law,

$$
u^{\rho} \partial_{\mu} \Sigma^{\mu}_{\nu\rho} = -2\Theta_{[\nu\rho]} u^{\rho},
$$

(3)

give constraint equations rather than dynamical ones. This becomes manifest in the fluid
rest frame specified as $u^{\mu} = (1, \mathbf{0})$, where Eq. (3) does not have any time derivative. In
the following, for notational simplicity we also use the dual variable $\sigma^{\mu} \equiv \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} u_{\nu} \sigma_{\rho\sigma}$ (or
$\sigma^{\mu\nu} = -\epsilon_{\mu\nu\rho\sigma} u^{\rho} \sigma^{\sigma}$), which satisfies $\sigma^{\mu} u_{\mu} = 0$.

We introduce entropy density $s$ as a function of our dynamical variables $e, \sigma^{\mu\nu}, n$, and
define the conjugate variables — the inverse temperature $\beta = 1/T$, chemical potential $\mu$, and
spin potential $\mu_{\nu\rho}^{\mu} = -\epsilon_{\mu\nu\rho\sigma} u^{\sigma}$, as

$$
T ds = de - \frac{1}{2} \nu^{\rho} d\sigma_{\nu\rho} - \mu dn \Leftrightarrow \beta \equiv \frac{\partial s}{\partial e}, \quad \beta \mu^{\nu\rho} \equiv -2\frac{\partial s}{\partial \sigma_{\nu\rho}}, \quad \beta \mu \equiv -\frac{\partial s}{\partial n}.
$$

(4)

Since the spin density $\sigma_{\nu\rho}$ is transverse to the fluid velocity $u^{\mu}$, it is sufficient to assume that the
spin potential $\mu^{\nu\rho} = -\mu^{\mu\nu}$ and its dual $\mu^{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} u^{\rho} \mu^{\sigma}$ satisfy $\mu_{\nu\rho}^{\mu} = 0 = \mu^{\mu\nu} u_{\nu}$.

We will also use the thermal vector $\beta^{\mu} \equiv \beta u^{\mu}$ in the subsequent calculations for brevity.

To find the constitutive relations of strongly rotating spinful fluids, we rely on a power
counting scheme different from that for the conventional hydrodynamics. We count vorticity,
the other gradients, and spin density all differently: Spatial components of the (thermal) vorticity \( \omega_{\mu\nu} \equiv \partial_{[\mu} \beta_{\nu]} \) (or its dual vector \( \omega^\mu \equiv \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} u_\nu \omega_{\rho\sigma} \)) and the spin potential \( \mu^{\nu\rho} \) are counted as \( O(\partial^0) \), while their difference \( (\omega_{\mu\nu} - \beta_{\mu\nu}) \) and the other gradients are counted as \( O(\partial^1) \). Besides, we assume that the spin density is suppressed by another small parameter \( \hbar \) as \( \sigma_{\nu\rho} = O(\hbar) \), while the vorticity-induced part of the spin potential is not. This designed power counting scheme is motivated by two observations: (i) vorticity belongs to a family of non-dissipative gradients that does not produce entropy, and (ii) the spin operator is accompanied by the Planck constant since spin is a quantum object. As the spin potential is not completely fixed by the vorticity, the three spin densities enjoy their intrinsic dynamics.

In our power counting scheme, the zeroth-order quantities that one can use for the tensor decomposition are \( u^\mu, \tilde{\omega}^\mu = \omega^\mu / \sqrt{\omega^\nu \omega_\nu} \), \( \epsilon^{\mu\nu\rho\sigma} \) and \( \eta^{\mu\nu} \). Assuming the parity symmetry and noticing the totally anti-symmetric property of the spin current, one may parametrize the (non-)conserved currents as

\[
\Theta^{\mu\nu} = (e + p_1) u^\mu u^\nu + p_1 \eta^{\mu\nu} + p_2 \tilde{\omega}^\mu \tilde{\omega}^\nu + p_3 \epsilon^{\mu\nu\rho\sigma} u^\rho \delta s^\sigma - \delta q^\nu u^\nu + \delta \Theta^{\mu\nu},
\]

\[
\Sigma^{\mu\nu\rho} = \epsilon^{\mu\nu\rho\lambda} (\sigma_\lambda + u_\lambda \delta \sigma),
\]

\[
J^\mu = u^\mu + \delta J^\mu,
\]

where \( \delta \Theta^{\mu\nu}, \delta q^\nu, \) and \( \delta J^\mu \) satisfy \( \delta \Theta^{\mu\nu} u_\nu = 0 = u_\mu \delta \Theta^{\mu\nu} \) and \( \delta q^\nu u_\mu = 0 = \delta J^\mu u_\mu \). Note that the constraint equation (3) with the above parametrization relates \( \delta q^\mu \) with the spin density \( \sigma^{\nu\rho} \) as

\[
\delta q^\nu = - \frac{1}{2} \epsilon^{\mu\nu\rho\lambda} u_\rho \partial_\lambda (\sigma_\mu + u_\mu \delta \sigma).
\]

In the following, we will express the so-far undetermined quantities in Eq. (5) in terms of our conjugate variables \( \beta^\mu, \mu^{\nu\rho}, \) and \( \mu \). Those quantities are \( p_a (a = 1, 2, 3) \) at \( O(1) \) and \( \delta \Theta^{\mu\nu}, \delta \sigma, \) and \( \delta J^\mu \) at \( O(\partial) \). Then, the equations of motion (1) serve as a closed set of partial differential equations.

To obtain the constitutive relations, we use the second law of local thermodynamics: There exists an entropy current \( s^\mu \) satisfying \( \partial_\mu s^\mu \geq 0 \) for any configuration of \( \beta^\mu, \mu^{\nu\rho}, \) and \( \mu \). Expressing the entropy current as \( s^\mu = su^\mu + \delta s^\mu \) with the first-order correction \( \delta s^\mu \), one obtains constraints resulting from the condition \( \partial_\mu s^\mu \geq 0 \) up to the second order in derivatives \( \partial \) and spin \( \hbar \). This enables us to relate \( p_a (a = 1, 2, 3) \) to the thermodynamic quantities and determine the tensor structures of \( \delta \Theta^{\mu\nu}, \delta \sigma, \) and \( \delta J^\mu \), and \( \delta s^\mu \). We first insert Eq. (5) into the equations of motion (1). Contracting them with \( \beta_\nu, \frac{1}{2} \beta \mu^{\nu\rho}, \) and \( \beta_\mu \), we have

\[
\beta D e = - \beta (e + p_1) \theta - p_2 \tilde{\omega}^\mu \tilde{\omega}_\nu \partial_\mu \beta^\nu - p_3 \epsilon^{\mu\nu\rho\sigma} \partial_\rho \beta^\sigma - \delta \Theta^{(\nu\rho)} \partial_\mu \beta_\nu - \delta \Theta^{[\nu\rho]} \partial_\mu \beta_\nu
\]

\[
- \delta q_\nu (D \beta^{\nu\rho} + \delta^{\nu\rho} \beta) + \partial_\mu (\beta \delta q^\mu),
\]

\[
- \frac{1}{2} \beta \mu^{\nu\rho} D \sigma_{\nu\rho} = \mu^\mu \sigma_\mu \partial_\mu \beta^\nu - \frac{1}{2} \delta \sigma \epsilon^{\mu\nu\rho\lambda} u_\lambda \partial_\mu (\beta \mu^{\nu\rho}) + \partial_\mu (\beta \beta^{\mu\nu} \delta \sigma) + \left[ p_3 \beta \mu^{\nu\rho} \epsilon^{\nu\rho} + \beta \mu^{\nu\rho} \epsilon^{\nu\rho} \right] \delta \Theta^{[\nu\rho]},
\]

\[
- \beta \mu D n = \beta \mu n \theta - \delta J^\mu \partial_\mu (\beta \mu) + \partial_\mu (\beta \mu \delta J^\mu),
\]

where we introduced \( D \equiv u^\mu \partial_\mu \) and \( \theta \equiv \partial_\mu u^\mu \) and eliminated \( \delta q^\mu \) using the constraint equation (6). Applying the chain rule for \( s(x) = s(e(x), \sigma_{\nu\rho}(x), n(x)) \) and using Eq. (7),
we rewrite the entropy production rate as
\[
\partial_\mu s^\mu = s\theta + \beta D e - \frac{1}{2}\beta \mu^\nu \rho D\sigma_{\nu \rho} - \beta \mu D n + \partial_\mu \delta s^\mu \\
= [s - \beta (e + p_1 - \mu n)] \theta - (p_2 \delta^{\mu \rho} \sigma_{\mu \rho} - \mu^\nu \sigma^\nu) \partial_\mu \beta_\nu - p_3 \epsilon^{\mu \nu} (\partial_\mu \beta_\nu - \beta \mu_{\mu \nu}) \\
- \delta \Theta (\mu \nu) \partial_\mu \beta_\nu - \delta \Theta (\nu \mu) (\partial_\mu \beta_\nu - \beta \mu_{\mu \nu}) - \sigma \delta \partial_{\mu} (\beta \mu_{\mu} - \omega^\nu) - \delta \mu_{\mu \nu} \partial_\mu (\beta \mu_{\mu}) \\
+ \partial_\mu (\delta s^\mu + \beta \mu_{\mu} \sigma + \beta \mu \delta J^\nu) + \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} u_\rho \partial_\mu \sigma_\sigma (\partial_\nu \beta + D \beta_\nu),
\]
(8)
where we used \( \omega^\mu D u_\mu = \partial_{\mu \nu} \omega^\mu - T \omega^\mu \partial_\mu \beta_\nu \), following from a Bianchi-like identity \( \partial_\mu (\epsilon^{\mu \nu \rho \sigma} \omega_{\rho \sigma}) = 0 \), and introduced \( \partial_{\mu \nu} \equiv (\delta_{\mu} + u_\mu \nu) \partial_\nu \). We also retained all the terms here, i.e., Eq. (8) is exact and does not depend on the power counting scheme at this stage.

To implement our power counting scheme in a clear manner, we decompose \( \mu^\mu \) and \( \sigma_\nu \) in terms of the order-one vector \( \hat{\omega}^\mu \) as
\[
\mu^\mu = \mu_{\parallel} \hat{\omega}^\mu + \mu_{\perp}^{\mu}, \quad \sigma_\nu = \sigma_{\parallel} \hat{\omega}^\nu + \sigma_{\perp}^{\nu},
\]
(9)
where \( \sigma_{\parallel} = O(h^1) \) and \( \sigma_{\perp}^{\mu} = O(h^1 \theta^1) \) according to our power counting scheme. On the other hand, the magnitude of the spin potential reads \( \mu_{\parallel} = O(1) \) and \( \mu_{\perp}^{\mu} = O(\theta^1) \) without the suppression by \( h \) because the spin potential induced by vorticity is not suppressed according to our assumption. We then obtain the divergence of the entropy current up to \( O(h^1 \theta^2, \theta^3) \) as
\[
\partial_\mu s^\mu = [s - \beta (e + p_1 - \mu n)] \theta - (p_2 - \mu \parallel \sigma_{\parallel}) \hat{\omega}^\mu \hat{\omega}^\nu \partial_\mu \beta_\nu - p_3 \epsilon^{\mu \nu} (\partial_\mu \beta_\nu - \beta \mu_{\mu \nu}) \\
- \delta \Theta (\mu \nu) \partial_\mu \beta_\nu - \delta \Theta (\nu \mu) (\partial_\mu \beta_\nu - \beta \mu_{\mu \nu}) - \delta \mu_{\mu \nu} \partial_\mu (\beta \mu_{\mu}) \\
+ \partial_\mu (\delta s^\mu + \beta \mu_{\mu} \delta \sigma + \beta \mu \delta J^\nu) + O(h^1 \theta^2, \theta^3).
\]
(10)
Note that here one can safely drop the term \( \delta \mu \partial_{\mu \nu} (\beta \mu_{\mu} - \omega^\mu) \) in Eq. (8), which is higher-order in derivatives.

**Constitutive relations of rotating spinful fluids.**— We derive the constitutive relations up to the first order in derivatives \( \partial \) and spin \( h \) by requiring that the divergence of the entropy current (10) respects the second law of thermodynamics.

Let us first consider the leading-order constitutive relation for the energy-momentum tensor. The first line of Eq. (10) results from the non-dissipative terms, which we require to vanish so that \( \partial_\mu s^\mu = 0 \) at the leading order. We find
\[
e + p_1 = Ts + \mu n, \quad p_2 = \mu \parallel \sigma_{\parallel}, \quad p_3 = 0,
\]
(11)
which relates the parameters \( p_a \) (\( a = 1, 2, 3 \)) with the thermodynamic quantities. Equation (11) indicates that the leading-order energy-momentum tensor has anistropic pressure induced by vorticity as
\[
\Theta^{\mu \nu}_{(0)} = eu^\mu u^\nu + p_{\perp} \Sigma^{\mu \nu} + p_{\parallel} \hat{\omega}^\mu \hat{\omega}^\nu,
\]
(12)
where we introduced \( p_{\perp} \equiv p_1 \) and \( p_{\parallel} \equiv p_1 + \mu \parallel \sigma_{\parallel} \) and the projection tensor \( \Sigma^{\mu \nu} = \eta^{\mu \nu} + u^\mu u^\nu - \hat{\omega}^\nu \hat{\omega}^\nu \). Note that one can express the first relation of Eq. (11) as \( e + p_{\parallel} = Ts + \mu n + \mu \parallel \sigma_{\parallel} \) that captures the contribution of spin to the thermodynamic relation. The constitutive relation (12) bears similarity with that for relativistic magnetohydrodynamics in which the direction of magnetic field plays the role of \( \hat{\omega}^\mu \) and magnetization of the fluid induces difference between \( p_{\perp} \) and \( p_{\parallel} \) [55, 56, 59–61].
We turn to the first-order corrections to the constitutive relations. Requiring the local second law of thermodynamics \(\partial_\mu s^\mu \geq 0\), we find
\[
\begin{align*}
\delta \Theta^{(\mu \nu)} &= -T \eta^{\mu \nu \rho \sigma} \partial_\rho \beta_\sigma, \\
\delta \Theta^{[\mu \nu]} &= -T \eta_{s}^{\mu \nu \rho \sigma} (\partial_\rho \beta_\sigma - \beta_{\mu \rho \sigma}) , \\
\delta J^\mu &= -T \kappa^{\mu \nu} \partial_\nu (\beta \mu), \\
\delta s^\mu &= -\beta_\mu \delta \sigma - \beta_\mu \delta J^\mu,
\end{align*}
\]
with the two viscous tensors \(\eta^{\mu \nu \rho \sigma}\) and \(\eta_{s}^{\mu \nu \rho \sigma}\), and conductivity tensor \(\kappa^{\mu \nu}\). These tensors are decomposed by the use of the order-one spatial tensors \(\hat{\omega}^\mu, \Xi^{\mu \nu}, \) and \(\epsilon^{\mu \nu}\). The transverse, longitudinal, and Hall-like components read
\[
\eta^{\mu \nu \rho \sigma} = \zeta_\perp \Xi^{\mu \nu} \Xi^{\rho \sigma} + \zeta_\parallel \hat{\omega}^\mu \hat{\omega}^\nu \hat{\omega}^\rho \hat{\omega}^\sigma + \zeta_\times (\hat{\omega}^\mu \hat{\omega}^\nu \Xi^{\rho \sigma} + \Xi^{\mu \nu} \hat{\omega}^\rho \hat{\omega}^\sigma)
\]
\[
+ \eta_\perp \frac{1}{2} (\Xi^{\mu \rho} \Xi^{\nu \sigma} + \Xi^{\mu \sigma} \Xi^{\nu \rho} - \Xi^{\mu \nu} \Xi^{\rho \sigma}) + \eta_\parallel (\hat{\omega}^\mu \Xi^{\nu (\rho, \sigma)} + \hat{\omega}^\nu \Xi^{\mu (\rho, \sigma)})
\]
\[
+ \eta_{H\perp} (\hat{\omega}^\rho \epsilon^{\nu (\rho, \sigma)}) + \eta_{H\parallel} (\hat{\omega}^\mu \epsilon^{\nu (\rho, \sigma)}),
\]
\[
\eta_{s}^{\mu \nu \rho \sigma} = \frac{1}{2} \eta_{s\perp} (\Xi^{\mu \rho} \Xi^{\nu \sigma} - \Xi^{\mu \sigma} \Xi^{\nu \rho}) + \eta_{s\parallel} (\hat{\omega}^\mu \Xi^{\nu (\rho, \sigma)} - \hat{\omega}^\nu \Xi^{\mu (\rho, \sigma)})
\]
\[
+ \eta_{sH\perp} (\hat{\omega}^\rho \epsilon^{\nu (\rho, \sigma)}) + \eta_{sH\parallel} (\hat{\omega}^\mu \epsilon^{\nu (\rho, \sigma)}),
\]
\[
\kappa^{\mu \nu} = \kappa_\perp \Xi^{\mu \nu} + \kappa_\parallel \hat{\omega}^\mu \hat{\omega}^\nu + \kappa_H \epsilon^{\mu \nu}.
\]
The bulk viscosity \(\zeta_\times\) captures the non-equilibrium correction to the anisotropic pressures \(p_\perp\) and \(p_\parallel\) in response to expansion or compression in the parallel and perpendicular directions, respectively. Those two processes are reciprocal to one another, and we required them to satisfy Onsager’s reciprocal relation [62] to respect the second law. As a result, we find fourteen transport coefficients: three bulk viscosities \(\zeta_\parallel, \zeta_\perp, \zeta_\times\), four shear viscosities \(\eta_\parallel, \eta_\perp, \eta_{H\perp}, \eta_{H\parallel}\), four rotational viscosities \(\eta_{s\parallel}, \eta_{s\perp}, \eta_{sH\perp}, \eta_{sH\parallel}\), and three conductivities \(\kappa_\perp, \kappa_\parallel, \kappa_H\). To respect the second law, the dissipative transport coefficients must satisfy a set of inequalities,
\[
\begin{align*}
\zeta_\perp &\geq 0, \quad \zeta_\parallel &\geq 0, \quad \zeta_\times - \zeta_\times^2 &\geq 0, \\
\eta_\perp &\geq 0, \quad \eta_\parallel &\geq 0, \quad \eta_{s\perp} &\geq 0, \quad \eta_{s\parallel} &\geq 0, \\
\kappa_\perp &\geq 0, \quad \kappa_\parallel &\geq 0.
\end{align*}
\]
On the other hand, the signs of the five coefficients \(\eta_{H\perp}, \eta_{H\parallel}, \eta_{sH\perp}, \eta_{sH\parallel}\), and \(\kappa_H\) are not constrained by the second law, as they do not contribute to the entropy production just like the Hall transports in relativistic magnetohydrodynamics [55, 56, 59–61]. Those non-dissipative transports arise due to strong vorticity, which can be interpreted as an analogue of strong magnetic fields [63]: The Coriolis force does not do work, similarly to the Lorentz force by magnetic fields. One clear difference from the Hall transports in magnetohydrodynamics is that those transport coefficients in gyrohydrodynamics are even under charge conjugation and can have non-vanishing values for neutral fluids.

**Summary and Outlook.**—In this Letter, we have reported the phenomenological derivation of the constitutive relations for spinful fluids with strong vorticity. It is formulated in the “spin hydrodynamic regime,” where spin density enjoys its intrinsic dynamics as the energy-momentum and vector charge densities do. We have found the leading and next-to-leading order constitutive relations that capture the spin dynamics intertwined with rotating fluid
dynamics as well as the anisotropic pressure and transport phenomena induced by the strong vorticity. We have obtained fourteen transport coefficients: three bulk viscosities $\zeta_\parallel, \zeta_\perp, \zeta_\times$, four shear viscosities $\eta_\parallel, \eta_\perp, \eta_{H\parallel}, \eta_{H\perp}$, four rotational viscosities $\eta_{s\parallel}, \eta_{s\perp}, \eta_{sH\parallel}, \eta_{sH\perp}$, and three conductivities $\kappa_\perp, \kappa_\parallel, \kappa_H$. A subset of the transport coefficients should satisfy the positivity constraint (15), while there are no such constraints on the Hall-like transport coefficients.

The entropy-current analysis leaves the fourteen transport coefficients as phenomenological parameters. The theoretical evaluation of these parameters can be carried out on the basis of either the linear-mode analysis or the Green-Kubo formula [64–68], both of which are regarded as the matching conditions for the (quasi-)hydrodynamic description. Deriving such matching conditions for the fourteen transport coefficients and evaluating them with microscopic theories (such as quantum chromodynamics) are important outlooks in line with the present Letter.

We investigated the formulation with the totally anti-symmetric spin current that is motivated by the canonical spin current of Dirac fermions, while previous works do not necessarily assume this tensor structure (e.g., Ref. [32]). One may always redefine the spin current and energy-momentum tensor operators — or change what one calls spin current and energy-momentum tensor operators — without modifying the form of their equations of motion via the so-called pseudo-gauge transformations [69–71] (see, e.g., Refs. [37, 72, 73] for recent discussion). This may lead to another definition of a spin current operator which is not necessarily totally anti-symmetric. It may be worth investigating what the resulting hydrodynamic equation looks like with a different choice of the spin and energy-momentum tensor. We leave all these issues as future works.

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