STRUCTURE OF ANN-CATEGORIES AND MAC LANE-SHKULA COHOMOLOGY

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Abstract. In this paper we study the structure of a class of categories having two operations which satisfy axioms analogous to that of rings. Such categories are called "Ann-categories". We obtain the classification theorems for regular Ann-categories and Ann-functors by using Mac Lane-Shukla cohomology of rings. These results give new interpretations of the cohomology groups $H^3(R, M)$ and $H^2(R, M)$ of the rings $R$.

1 Introduction and Preliminaries

Monoidal categories and symmetric monoidal categories were studied first by S. Mac Lane [8], J. Bémaou [1] and G. M. Kelly [3]. They are, respectively, categories $A$ together with a bifunctor $\otimes: A \times A \rightarrow A$ and a system of natural equivalences of associativity-unitivity, or a system of natural equivalences of associativity-unitivity-commutativity. A. Solian [14], H. X. Sinh [2] and K. H. Ulbrich [15], investigated $\otimes$-categories from the point of view of algebraic structure. They examined the monoidal categories whose all objects are invertible.

The problem of coherence always plays a fundamental role in the study of any class of $\otimes$-categories. From initial conditions, we have to prove that the morphisms generated by a given ones depend only on its source butt. The consideration of structures arose later in the papers of H. X. Sinh [2] and B. Mitchell [9]. Here we obtained deep results on the classification by the cohomology of groups.

By the other direction, M. Laplaza [4] considered the coherence of natural equivalences of distributivity in a category having two operations $\oplus$ and $\otimes$. In the papers of Laplaza, the distribution of monomorphisms together with the natural isomorphisms of the two symmetrical monoidal structures must satisfy 24 commutative diagrams, that form natural relations between them.

In this paper, we consider a class of Pic-categories (see H. X. Sinh [2]) in which the second operation and natural equivalences of distributivity are defined so that the analogous axioms of rings are verified. Such categories are called Ann-categories. Coherence for Ann-categories was shown in [11].

Throughout we define invariants of Ann-category basing on construction of reduced Ann-categories and pre-sticked of the type $(R, M)$. From this we obtain classification theorems for the regular Ann-categories and Ann-functors by using cohomology groups $H^3(R, M)$, $H^2(R, M)$ of the ring $R$. These theorems give a relation between the notion of Ann-category with the theory of cohomology of rings and the problem of extention of rings.

For convinience, the tensor product of two objects $A$ and $B$ is denoted by $AB$ instead of $A \otimes B$, but for the morphisms we still write $f \otimes g$ to avoid confusion with the composition of morphisms.

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The notions and results on monoidal categories are supposed to be familiar to the readers (see [3, 5, 8] for example).

Recall that a Pic-category is a symmetric monoidal category \( \mathcal{A} \) (or a \( \otimes \mathcal{A}U \)-category \( \mathcal{A} \)) in which every object is invertible and every morphism is an isomorphism (see [2]).

**Definition 1.1.** An Ann-category is a category \( \mathcal{A} \) together with

(i) Two bifunctors \( \oplus, \otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \).

(ii) A fixed object \( 0 \in \mathcal{A} \) with natural isomorphisms \( a^{\oplus}, c, g, d \) such that \( (\mathcal{A}, \oplus, a^{\oplus}, c, (0, g, d)) \) is a Pic-category.

(iii) A fixed object \( 1 \in \mathcal{A} \) with natural isomorphisms \( a, l, r \) such that \( (\mathcal{A}, \otimes, a, (1, l, r)) \) is a monoidal category (i.e. a \( \otimes \mathcal{A}U \)-category).

(iv) Two natural isomorphisms \( \mathcal{L}, \mathcal{R} \)

\[
\mathcal{L}_{A,X,Y} : A(X \oplus Y) \to AX \oplus AY
\]

\[
\mathcal{R}_{X,Y,A} : (X \oplus Y)A \toXA \oplus YA
\]

satisfying the following conditions

(Ann-1) For every object \( A \in \mathcal{A} \), the pair of \( \oplus \)-functors \( (L^A, \tilde{L}^A), (R^A, \tilde{R}^A) \) defined by

\[
\begin{align*}
L^A : X &\to AX \\
\tilde{L}_{X,Y}^A &\mathcal{L}_{A,X,Y} \\
R^A : X &\to AX \\
\tilde{R}_{X,Y}^A &\mathcal{R}_{X,Y,A}
\end{align*}
\]

are \( \oplus \mathcal{A}C \)-functors.

(Ann-2) For any \( A, B, X, Y \in \mathcal{A} \) the following diagrams are commutative

\[
\begin{array}{ccc}
A(B(X \oplus Y)) & \xrightarrow{id \otimes \mathcal{L}} & A(BX \oplus BY) & \xrightarrow{\mathcal{L}} & A(BX) \oplus A(BY) \\
\downarrow{a} & & \downarrow{a \oplus a} & & \\
(AB)(X \oplus Y) & & (AB)X \oplus (AB)Y
\end{array}
\]

\[
\begin{array}{ccc}
(X \oplus Y)(AB) & \xrightarrow{\mathcal{R}} & X(AB) \oplus Y(AB) \\
\downarrow{a} & & \downarrow{a \oplus a} \\
((X \oplus Y)A)B & \xrightarrow{\mathcal{R} \otimes id} & (XA \oplus YA)B & \xrightarrow{\mathcal{R}} & (XA)B \oplus (YA)B
\end{array}
\]
Definition 1.2. Let $A$ and $A'$ be Ann-categories. An Ann-functor from $A$ to $A'$ is a functor $F: A \to A'$ together with natural isomorphisms $\tilde{F}, \bar{F}$ such that: $(F, \tilde{F})$ is a $\oplus AC$-functor, $(F, \bar{F})$ is a $\otimes A$-functor and $\tilde{F}, \bar{F}$ are compatible with natural equivalences of distributivity in the sense that the following two diagrams are commutative.

If $F$ is an equivalence, then $(F, \tilde{F}, \bar{F})$ is called an Ann-equivalence.

Proposition 1.3. Let $A$ be an Ann-category and $A \in A$. Then there exist unique isomorphisms $\hat{L}^A: A \otimes 0 \to 0$, $\hat{R}^A: 0 \otimes A \to 0$ so that $(L^A, \hat{L}^A, \hat{R}^A)$ and $(R^A, \hat{R}^A, \hat{L}^A)$ are symmetrical monoidal functors ($\oplus ACU$-functor).
Proposition 1.4. In any Ann-category $\mathcal{A}$, the isomorphisms $\widehat{L}^A$, $\widehat{R}^A$ have the following properties:

(i) The family $\widehat{L}^- = \widehat{L}$ (resp. the family $\widehat{R}^- = \widehat{R}$) is a $\oplus$-morphism from the functor $(\widehat{R}^0, \widehat{L}^0)$ (resp. $(\widehat{L}^0, \widehat{R}^0)$) to the functor $(\theta : A \mapsto 0, \overline{\theta} = g_0^{-1})$ i.e. the following diagrams are commutative:

\[
\begin{array}{ccc}
A0 & \xrightarrow{f \otimes \text{id}} & B0 \\
\widehat{L}^A & \downarrow & \widehat{L}^B \\
0 & \downarrow & 0
\end{array}
\quad
\begin{array}{ccc}
(X \oplus Y)0 & \xrightarrow{R^0} & X0 \oplus Y0 \\
\widehat{L}^A \otimes Y & \downarrow & \widehat{L}^B \otimes \widehat{L}^Y \\
0 \oplus \overline{g_0} & \downarrow & 0 \oplus 0
\end{array}
\]

(resp. $\widehat{R}^B(id \otimes f) = \widehat{R}^A$ and $\widehat{R}^X \otimes Y = g_0(\widehat{R}^X \oplus \widehat{R}^Y)\widehat{L}^0$).

(ii) For any $A, B \in \mathcal{A}$, the following diagrams are commutative:

\[
\begin{array}{ccc}
X(0Y) & \xrightarrow{\theta} & (X0)Y \\
\widehat{L}^X \otimes Y & \downarrow & \widehat{L}^X \oplus \overline{\theta} \\
X0 & \downarrow & \overline{g_0} \oplus 0 Y
\end{array}
\quad
\begin{array}{ccc}
(X0)Y & \xrightarrow{\theta \otimes \text{id}} & X0 \\
\widehat{L}^X \oplus \overline{\theta} & \downarrow & \widehat{L}^X \oplus \overline{\theta} \\
\overline{g_0} \oplus 0 & \downarrow & \overline{g_0} \oplus 0
\end{array}
\]

and $\widehat{R}^{XY} = \widehat{R}^Y(\widehat{R}^X \otimes \text{id})a_0, X, Y$.

(iii) $L^1 = l_0$, $R^1 = r_0$.

2 The first two invariants of an Ann-category

Let $\mathcal{A}$ be an Ann-category. Then the set $\Pi_0(\mathcal{A})$ of the isomorphic classes of objects of $\mathcal{A}$ is a ring with the operations induced by the ones $\oplus$, $\otimes$ in $\mathcal{A}$, and $\Pi_1(\mathcal{A}) = \text{Aut}(0)$ is an abelian group with operation denoted by $+$.

The following two Theorems on the structure of the Ann-categories can be found in [12].

Theorem 2.1. $\Pi_1(\mathcal{A})$ is an $\Pi_0(\mathcal{A})$-bimodule where the left and right operations of the ring $\Pi_0(\mathcal{A})$ on the abelian group $\Pi_1(\mathcal{A})$ are defined respectively by

\[
su = \lambda_X(u), \quad us = \rho_X(u), \quad X \in s \in \Pi_0(\mathcal{A}), \quad u \in \Pi_1(\mathcal{A})
\]

in which $\lambda_X$, $\rho_X$ are the two maps $\text{Aut}(0) \to \text{Aut}(0)$ given by the following commutative diagrams:

\[
\begin{array}{ccc}
X0 & \xrightarrow{\lambda_X} & 0 \\
\downarrow & \downarrow & \downarrow \\
0X & \xrightarrow{\rho_X} & 0
\end{array}
\quad
\begin{array}{ccc}
X0 & \xrightarrow{\rho_X} & 0 \\
\downarrow & \downarrow & \downarrow \\
0X & \xrightarrow{\lambda_X} & 0
\end{array}
\]

the following theorem shows the invariableness of $\Pi_0(\mathcal{A})$-bimodule $\Pi_1(\mathcal{A})$. 

Proof. Since $(\mathcal{A}, \oplus)$ is a Pic-category, each $\oplus$-AC-functor is also a $\oplus$-ACU-functor. □
Theorem 2.2. Given two Ann-categories $\mathcal{A}$, $\mathcal{A}'$. Then any Ann-functor $(F, \hat{F}, \tilde{F})$: $\mathcal{A} \to \mathcal{A}'$ yields a ring homomorphism

$$F_0: \Pi_0(\mathcal{A}) \to \Pi_0(\mathcal{A}')$$

$$clX \mapsto clFX$$

and a group homomorphism

$$F_1: \Pi_1(\mathcal{A}) \to \Pi_1(\mathcal{A}')$$

$$u \mapsto \gamma_{F_0}(Fu)$$

having the properties

$$F_1(su) = F_1(s)F_0(u) \quad F_1(us) = F_0(u)F_1(s)$$

where $\gamma_A: \text{Aut}(0) \to \text{Aut}(0)$ is defined by $\gamma_A(u) = g_A(u \otimes id_A)g_A^{-1}$. Moreover, $F$ is an Ann-equivalence if and only if $F_0$, $F_1$ are isomorphisms.

Hence $\Pi_0(\mathcal{A})$ and $\Pi_1(\mathcal{A})$ are the first two invariants of an Ann-category.

3 Reduced Ann-categories

In preparing to define the third invariant of Ann-categories, we construct reduced Ann-categories. Let $\mathcal{A}$ be an Ann-category. The reduced category $\mathcal{S}$ is constructed from $\Pi_0(\mathcal{A})$ and $\Pi_1(\mathcal{A})$ as follows: its objects are the elements of $\Pi_0(\mathcal{A})$, its morphisms are the automorphisms of the form $(s,u)$ with $s \in \Pi_0(\mathcal{A})$, $u \in \Pi_1(\mathcal{A})$ i.e.

$$\text{Aut}(s) = \{s\} \times \Pi_1(\mathcal{A})$$

The composition law of morphisms is reduced by addition in $\Pi_1(\mathcal{A})$. We shall use the transmission of structures (see [10]) to change $\mathcal{S}$ into an Ann-category which is equivalent to $\mathcal{A}$. Choose for every $s \in \Pi_0(\mathcal{A})$ a representant $X_s \in \mathcal{A}$ such that $X_0 = 0$, $X_1 = 1$ and then, for every pair $s$, $t \in \Pi_0(\mathcal{A})$, two families of isomorphisms

$$\varphi_{s,t}: X_s \oplus X_t \to X_{s+t}, \quad \psi_{s,t}: X_sX_t \to X_{st}$$

such that

$$\varphi_{0,t} = g_{X_t}, \quad \varphi_{s,0} = d_{X_s}$$

$$\psi_{1,t} = 1_{X_t}, \quad \psi_{s,1} = r_{X_s}, \quad \psi_{0,t} = \widehat{R}_{X_t}, \quad \psi_{0,s} = \widehat{L}_{X_s}$$

Defining the functor $H: \mathcal{S} \to \mathcal{A}$ by $H(s) = X_s$, $H(s,u) = \gamma_{X_s}(u)$ and putting $\widehat{H} = \varphi^{-1}$, $\widehat{H} = \psi^{-1}$ we can use the theorem of transmission of structures (see [10]) to obtain $\mathcal{S}$ to be an Ann-category with the two operations in the explicit forms:

$$s \oplus t = s + t \quad \text{(sum in ring } \Pi_0(\mathcal{A}))$$

$$s \oplus t = (s, u + v)$$

$$s \otimes t = st \quad \text{(product in ring } \Pi_0(\mathcal{A}))$$

$$s \otimes t = (st, sv + ut)$$

and with the natural equivalences induced by that of $\mathcal{A}$. $\mathcal{S}$ is called the reduced Ann-category of $\mathcal{A}$. We now have:
Theorem 3.1. In the reduced Ann-category $S$ of $A$, the natural equivalences of unitivity of the two operations $\otimes$, $\odot$ are identities, and the natural equivalences $\xi$, $\eta$, $\alpha$, $\lambda$, $\rho$ induced from $a^+$, $c$, $a$, $\mathcal{L}$, $\mathcal{R}$ by $(H, \tilde{H}, \tilde{H})$ are functions having the values in $\Pi_1(A)$ and satisfying the following relations

1. $\xi(y, z, t) - \xi(x + y, z, t) + \xi(x, y + z, t) - \xi(x, y, z + t) + \xi(x, y, z) = 0$

2. $\xi(0, y, z) = \xi(x, 0, t) = \xi(x, y, 0) = 0$

3. $\xi(x, y, z) - \xi(x, z, y) + \xi(z, x, y) - \eta(x, z) + \eta(x + y, z) - \eta(y, z) = 0$

4. $\eta(x, y) + \eta(y, x) = 0$

5. $x\eta(y, z) - \eta(xy, xz) = \lambda(x, y, z) - \lambda(x, z, y)$

6. $\eta(x, y)z - \eta(xz, yz) = \rho(x, y, z) - \rho(y, x, z)$

7. $x\xi(y, z, t) - \xi(xy, xz, xt) = \lambda(x, z, t) - \lambda(x, y + z, t) + \lambda(x, y, z + t) - \lambda(x, y, z)$

8. $\xi(x, y, z)t - \xi(xt, yt, zt) = \rho(y, z, t) - \rho(x + y, z, t) + \rho(x, y + z, t) - \rho(x, y, t)$

9. $\rho(x, y, z + t) - \rho(x, y, z) - \rho(y, x, t) + \lambda(x, z, t) + \lambda(y, z, t) - \lambda(x + y, z, t) - \xi(xz, xt, yz, yt) + \xi(xz, xt, yz) - \eta(xz, yz) + \xi(xz + yz, xt, yt) - \xi(xz, yz, xt)$

10. $\alpha(x, y, z + t) - \alpha(x, y, z) - \alpha(x, y, t) = x\lambda(y, z, t) + \lambda(x, yz, yt) - \lambda(xy, z, t)$

11. $\alpha(x, y + z, t) - \alpha(x, y, t) - \alpha(x, z, t) = xp(y, z, t) - \rho(xy, xz, t) + \lambda(x, yt, zt) - \lambda(x, y, z)t$

12. $\alpha(x + y, z, t) - \alpha(x, z, t) - \alpha(y, z, t) = -\rho(x, y, z)t - \rho(xz, yz, zt) + \rho(x, y, zt)$

13. $x\alpha(y, z, t) - \alpha(xy, z, t) + \alpha(x, yz, t) - \alpha(x, y, z)t + \alpha(x, y, z)t = 0$

14. $\alpha(1, y, z) = \alpha(x, 1, z) = \alpha(x, y, 1) = 0$

15. $\alpha(0, y, z) = \alpha(x, 0, t) = \alpha(x, y, 0) = 0$

16. $\lambda(1, y, z) = \lambda(0, y, z) = \lambda(x, 0, z) = \lambda(x, y, 0) = 0$

17. $\rho(x, y, 1) = \rho(0, y, z) = \rho(x, 0, z) = \rho(x, y, 0) = 0$

for $x, y, z, t \in \Pi_0(A)$.

For the two choices of different representants $(X_s, \varphi, \psi)$, we can prove the followings:

Proposition 3.2. If $S$ with $(X_s, \varphi, \psi)$ and $S'$ with $(X'_s, \varphi', \psi')$ are two reduced Ann-categories of $A$, then there exists an Ann-equivalence $(F, \tilde{F}, \tilde{F})$: $S \to S'$, with $F = id$. 
If we substitute $\Pi_0(A)$ by a ring $R$ and $\Pi_1(A)$ by an $R$-bimodule $M$, we can construct an Ann-category $\mathcal{I}$ with the operations $\otimes, \otimes$ defined by the relations (3.1)-(3.4) and the natural equivalences

$$a^+ = \xi, c = \eta, a = \alpha, \mathcal{L} = \lambda, \mathcal{R} = \rho$$

satisfying the relations in the theorem 3.1. This Ann-category $\mathcal{I}$ is called an Ann-category of type $(R, M)$.

If the function $\eta$ satisfies the regular condition $\eta(x, x) = 0$, the family $(\xi, \eta, \alpha, \lambda, \rho)$ is a 3-cocycle of the ring $R$ with coefficients in the $R$-bimodule $M$ in the Mac Lane-Shukla sense (see theorem 4.3). In particular, when $\lambda = 0, \rho = 0, \xi = 0$ and hence $\alpha$ becomes a normal 3-cocycle of the $Z$-algebra $R$ in the Hochshild sense (see [10]).

Any ring $R$ with the unit $1 \neq 0$ may be considered as an Ann-category of the type $(R, 0)$. Hence we have proved the following theorem:

**Theorem 3.3.** Any Ann-category is an Ann-equivalence to an Ann-category of the type $(R, M)$.

### 4 Cohomology classification of the regular Ann-categories

According to theorem 3.3 we have only to consider the classification of the Ann-categories having the first two common invariants.

**Definition 4.1.** Let $R$ be a ring with unit, $M$ be an $R$-bimodule considered as a ring with the null multiplication. An Ann-category $\mathcal{A}$ is called having pre-stick of the type $(R, M)$ if there exists a pair of ring isomorphisms $(\epsilon_0, \epsilon_1)$

$$\epsilon_0 : R \longrightarrow \Pi_0(A), \quad \epsilon_1 : M \longrightarrow \Pi_1(A)$$

satisfying the conditions:

$$\epsilon_1(su) = \epsilon_0(s)\epsilon_1(u), \quad \epsilon_1(us) = \epsilon_1(u)\epsilon_0(s), \quad s \in R, u \in M.$$

A morphism between two Ann-categories $\mathcal{A}, \mathcal{A}'$ having the same pre-stick of the type $(R, M)$ is an Ann-functor $(F, \tilde{F}, \tilde{\epsilon}) : \mathcal{A} \longrightarrow \mathcal{A}'$ such that the following diagrams are commutative

$$
\begin{array}{ccc}
\Pi_0(A) & \xrightarrow{F_0} & \Pi_0(A') \\
\downarrow{\epsilon_0} & & \downarrow{\epsilon_0'} \\
R & \xrightarrow{\epsilon_1} & R'
\end{array} \quad \begin{array}{ccc}
\Pi_1(A) & \xrightarrow{F_1} & \Pi_1(A') \\
\downarrow{\epsilon_1} & & \downarrow{\epsilon_1'} \\
M & \xrightarrow{\epsilon_1} & M'
\end{array}
$$

in which $F_0, F_1$ are two ring morphisms induced from $(F, \tilde{F}, \tilde{\epsilon})$. It follows directly from the definition that $\tilde{\epsilon}$ is an equivalence.

The two Ann-categories $\mathcal{A}, \mathcal{A}'$ are called congruences if there exists a morphism $(F, \tilde{F}, \tilde{\epsilon})$ between them.

**Definition 4.2.** An Ann-category $\mathcal{A}$ having a natural equivalence $c$ of commutativity so that $c_{X, X} = id$ is called a regular Ann-category.

For the regular Ann-categories we can define its third invariant, that is an element of Mac Lane-Shukla cohomology group $H^3(R, M)$ of the ring $R$.

Recall that the cohomology of an algebra $\Lambda$ with coefficients in an
Theorem 4.3. A 3-cochain $f = \langle \zeta, \eta, \alpha, \lambda, \rho \rangle$ of the ring $R$ with coefficients in the $\Lambda$-bimodule $M$ is a 3-cocycle if and only if $(\zeta, \eta, \alpha, -\lambda, \rho)$ is a family of natural equivalences of a regular Ann-category of the type $(R, M)$.

Proof. The essence of the proof is to compute the group $Z^3(R, M)$ by choosing a convenient resolution of the ring $R$ (as a $\mathbb{Z}$-algebra), different from the two resolutions of Shukla and Mac Lane. For the additional structure of $R$, we consider the complex of abelian groups:

$$0 \longrightarrow B_4 \overset{d_4}{\longrightarrow} B_3 \overset{d_3}{\longrightarrow} B_2 \overset{d_2}{\longrightarrow} B_1 \overset{d_1}{\longrightarrow} B_0 \overset{\nu}{\longrightarrow} R \longrightarrow 0$$

in which

$$B_0 = \mathbb{Z}(\hat{R}), \quad B_1 = \mathbb{Z}(\hat{R} \times \hat{R}), \quad B_2 = \mathbb{Z}(\hat{R} \times \hat{R} \times \hat{R}) \oplus \mathbb{Z}(\hat{R} \times \hat{R})$$

$$B_3 = \mathbb{Z}(\hat{R} \times \hat{R} \times \hat{R} \times \hat{R}) \oplus \mathbb{Z}(\hat{R} \times \hat{R} \times \hat{R} \times \hat{R}) \oplus \mathbb{Z}(\hat{R} \times \hat{R}) \oplus \mathbb{Z}(\hat{R})$$

$$B_4 = \text{Kerd}_3, \quad \hat{R} = R \setminus \{0\}$$

($\mathbb{Z}(\hat{R}^i)$, $i = 1, 2, 3, 4$ are the free abelian groups generated by the set $\hat{R}^i$).

The morphisms are given by:

$$\begin{align*}
\nu[x] & = x, \quad x \in \hat{R} \\
\nu [x, y] & = [y] - [x + y] + [x] \\
d_1 [x, y] & = [x, z] - [x + y, z] + [x, y + z] - [x, y] \\
d_2 [x, y] & = [x, y] - [y, x] \\
d_3 [x, y, z, t] & = [y, z, t] - [x + y, z, t] + [x, y + z, t] - [x, y, z + t] + [x, y, z] \\
d_3 [x, y, z] & = [x, y, z] - [x, z, y] + [z, x, y] + [x + y, z] - [x, z] - [y, z] \\
d_3 [x, y] & = [x, y] + [y, x] \\
d_3 [x] & = [x, x]
\end{align*}$$
\(d_4 = i\) is the natural embedding.

We now define a distributive multiplication in \(B = \sum B_i\) such that \(B\) becomes a graded differential algebra over \(\mathbb{Z}\). A 3-cochain \(f\) is an element of a direct sum

\[
\Hom_\mathbb{Z}(B_2, M) \oplus \Hom_\mathbb{Z}(B_1 \otimes B_0, M) \oplus \Hom_\mathbb{Z}(B_0 \otimes B_1, M) \\
\oplus \Hom_\mathbb{Z}(B_0 \otimes B_0 \otimes B_0, M)
\]

This implies that \(f\) is defined by a family of mappings

\[
\begin{align*}
\zeta(x, y, z) &= f([x, y, z]) \\
\eta(x, y) &= f([x, y]) \\
\lambda(x, y, z) &= f([x] \otimes [y, z]) \\
\rho(x, y, z) &= f([x, y] \otimes [z]) \\
\alpha(x, y, z) &= f([x] \otimes [y] \otimes [z])
\end{align*}
\]

From the formula of differentiation of the above resolution we complete the proof.

**Theorem 4.4** (Classification theorem). There exists a bijection between the set of the congruence classes of pre-sticked regular Ann-categories of the type \((R, M)\) and the cohomology group \(H^3(R, M)\) of the ring \(R\), with coefficients in the \(R\)-bimodule \(M\).

**Proof.** Consider the resolution that is shown in the proof of the theorem 4.3. If \(f = <\zeta, \eta, \alpha, \lambda, \rho>\) is 3-coboundary, \(f = \delta g\), with \(g\) is a pair of mappings

\[
\begin{align*}
\mu &: B_1 \rightarrow M \\
\nu &: B_0 \otimes B_0 \rightarrow M
\end{align*}
\]

we have the following relations

\[
\begin{align*}
-\zeta(x, y, z) &= \mu(y, z) - \mu(x + y, z) + \mu(x, y + z) - \mu(x, y) \\
-\eta(x, y) &= \mu(x, y) - \mu(y, x) = \text{ant}\mu(x, y) \\
\alpha(x, y, z) &= x\nu(y, z) - \nu(xy, z) + \nu(x, yz) - \nu(x, y)z \\
-\lambda(x, y, z) &= \nu(x, y + z) - \nu(x, y) - \nu(x, z) + x\mu(y, z) - \mu(xy, xz) \\
\rho(x, y, z) &= \nu(x + y, z) - \nu(x, z) - \nu(y, z) - \mu(x, y)z + \mu(xz, yz)
\end{align*}
\]

These relations imply what we have to prove.

This theorem leads to the investigation of application of the Ann-category concept into the theory of ring extensions. The classification theorem in the general case is still an open problem.

5 **Ann-functors and low dimension cohomology groups of rings**

In this section given problem is that of finding whether there is Ann-functor between two Ann-categories and, if so, how many. Since each Ann-category is Ann-equivalent to one Ann-category of the type \((R, M)\) so the solution of problem for a class of Ann-categories of the type \((R, M)\) is enough.
Proof. If $f = (\zeta, \eta, \alpha, \lambda, \rho)$ is a 3-cocycle in $\mathbb{Z}^3(R, M)$ the structure $(\zeta, \eta, \alpha, -\lambda, \rho)$ of Ann-category $(R, M)$ is denoted by $\hat{f}$. Moreover, if

$$F = (F, \tilde{F}, \hat{F}) : (R, M, \hat{f}) \longrightarrow (R', M', \hat{f}')$$

is an Ann-functor, this functor is a pair of ring homomorphisms $(F_0, F_1)$ compatible with actions of bimodule. So sometimes $F$ is denoted by $(F_0, F_1)$. $R'$-bimodule $M$ may be changed into $R$-bimodule by the homomorphism $F_0$,

$$m'r = mF(r), \quad rm' = F(r)m', \quad r \in R, m' \in M'.$$

Because $f \in \mathbb{Z}^3(R, M)$ and $f' \in \mathbb{Z}^3(R', M')$, $F$ induces canonically 3-cocycles

$$f_*, f'^* \in \mathbb{Z}^3(R, M').$$

For example

$$\zeta_*(x, y, z) = F(\zeta(x, y, z))$$

$$\zeta'^*(x, y, z) = \zeta'(Fx, Fy, Fz).$$

Isomorphisms $\tilde{F}$, $\hat{F}$ are mappings $R \times R \longrightarrow M'$

$$\mu(x, y) = \tilde{F}_{x,y} : F(x + y) \longrightarrow Fx + Fy$$

$$\nu(x, y) = \hat{F}_{x,y} : F(xy) \longrightarrow (Fx)(Fy)$$

These mappings, according to definition, satisfy diagrams in definition 1.2. On the other hand, $< \mu, \nu >$ is a 2-cochain of ring cohomology. From a calculation of $H^3(R, M)$ we have

$$f_* - f'^* = \delta < \mu, \nu > \quad (5)$$

**Theorem 5.1.** Let $\mathcal{I} = (R, M, \hat{f}), \mathcal{I}' = (R', M', \hat{f}')$ be two regular Ann-categories and

$$F = (F_0, F_1) : \mathcal{I} \longrightarrow \mathcal{I}'$$

be a functor that satisfies the condition (5.1). Then $F$ is an Ann-functor if and only if $H_*(f) - H^*(f') = 0$ in $H^3(R, M')$. In this case, we can say that Ann-functor $(F, \tilde{F}, \hat{F})$ is induced by the functor $F$.

**Proof.** If $(F, \tilde{F}, \hat{F})$ is an Ann-functor with $\tilde{F} = \mu, \hat{F} = \nu$, the condition (5.1) gives equation

$$H_*(f) - H^*(f') = 0$$

Conversely, the equation $H_*(f) - H^*(f') = 0$ automatically implies $f_* - f'^* = \delta g$, there $g = < \mu, \nu >$ is a 2-cochain. Let $\tilde{F} = \mu, \hat{F} = \nu$, we have an Ann-functor $(F, \tilde{F}, \hat{F})$. \qed

**Definition 5.2.** An Ann-functor $F : (R, M, f) \longrightarrow (R', M', f')$ is called regular if $F$ satisfies condition $f_* = f'^*$.

In case there exists a regular Ann-functor $F$, we have the following theorem

**Theorem 5.3.** (i) There exists a bijection between the set of the congruence classes of regular Ann-functors induced by a pair $(F_0, F_1)$ and the cohomology group $H^3(R, M')$ of the ring $R$ with coefficients in the $R$-bimodule $M'$.

(ii) If $F : (R, M, f) \longrightarrow (R', M', f')$ is an Ann-functor, there exists a bijection $\text{Aut}(F) \longrightarrow \mathbb{Z}^1(R, M')$ between the group of automorphisms of Ann-functor $F$ and the group $\mathbb{Z}^1(R, M')$. 


Proof. (i) Let \((F, \bar{F}, \tilde{F})\) be a regular Ann-functor
\[
(F, \bar{F}, \tilde{F}) : (R, M, f) \rightarrow (R', M', f)
\]
Then
\[
f_* - f'^* = \delta <\mu, \nu >= 0
\]
where \(\bar{F} = \mu, \tilde{F} = \nu\). It means \(<\mu, \nu>\) is 2-cocycle.
Suppose that \((G, \bar{G}, \tilde{G})\) is another regular Ann-functor
\[
(G, \bar{G}, \tilde{G}) : (R, M, f) \rightarrow (R', M', f)
\]
and \(\alpha : F \rightarrow G\) is an Ann-morphism. Then, by definition, the following diagrams
are commutative
\[
\begin{align*}
F(x + y) & \xrightarrow{\bar{F}} Fx + Fy \\
\alpha_{x+y} \downarrow & \quad \alpha_{x+y} \\
G(x + y) & \xrightarrow{\bar{G}} Gx + Gy
\end{align*}
\]
\[
\begin{align*}
F(xy) & \xrightarrow{\bar{F}} (Fx)(Fy) \\
\alpha_{xy} \downarrow & \quad \alpha_{x}\alpha_{y} \\
G(xy) & \xrightarrow{\bar{G}} (Gx)(Gy)
\end{align*}
\]
where \(x, y \in R\). Also from the definition we have
\[
\alpha_x \otimes \alpha_y = (Fx)\alpha_y + \alpha_x(Fy) = x\alpha_y + \alpha_x y
\]
so
\[
\tilde{G}_{x,y} - \bar{F}_{x,y} = \alpha_x - \alpha_{x+y} + \alpha_y
\]
\[
\tilde{G}_{x,y} - \bar{F}_{x,y} = x\alpha_y - \alpha_{x+y} + \alpha_x y.
\]
Because \(g = <\bar{F}, \tilde{F}>\), \(g' = <\bar{G}, \tilde{G}>\) are 2-cocycles and \(\alpha\) is 1-cochain and by a
calculation of \(H^2(R, M)\) we have
\[
g' - g = \delta \alpha
\]  
(6)

Equation (6) proves the existance of a correspondence from a class of regular
Ann-functors \(\text{cls}(F, \bar{F}, \tilde{F})\) to a class of cohomologies \(g + B^2(R, M')\),
g \(= <\bar{F}, \tilde{F}>\). Moreover this correspondence is an injection. We now prove that it
is a projection. In fact, let \(g = <\mu, \nu>\) be any 2-cocycle. Then we can directly verify
that \((F, \mu, \nu)\) is a regular Ann-functor \((R, M, f)\) to \((R', M', f)\) corresponding
to 2-cocycle \(g\), proving (i).
(ii) Let
\[
F = (F, \mu, \nu) : (R, M, f) \rightarrow (R', M', f)
\]
be an Ann-functor and \(\alpha \in \text{Aut}(F)\). Then the equation (6) becomes \(\delta(\alpha) = 0\), i.e.
\(\alpha \in \mathbb{Z}^1(R, M')\), proving (ii).
6 Ann-category and theory of the extensions of rings

In this section, we establish a direct relation between theory of the extensions of rings and theory of Ann-categories. According to Mac Lane [7] we call a bimultiplication of a ring $A$ a pair of mappings $a \mapsto \sigma a, a \mapsto a \sigma$ of $A$ into itself which satisfy the rules

$$ \sigma(a + b) = \sigma a + \sigma b, \quad (a + b) \sigma = a \sigma + b \sigma $$

$$ \sigma(ab) = (\sigma a)b , \quad (ab) \sigma = a (b \sigma) $$

$$ a(\sigma b) = (a \sigma)b $$

for all elements $a, b \in A$. The sum $\sigma + \nu$ and the product $\sigma \nu$ of two bimultiplications $\sigma$ and $\nu$ are defined by the equations

$$ (\sigma + \nu)a = \sigma a + \nu a \quad , \quad a(\sigma + \nu) = a \sigma + a \nu $$

$$ (\sigma \nu)a = \sigma (\nu a) \quad , \quad a(\sigma \nu) = (a \sigma)\nu $$

for all $a$ in $A$.

The set of all bimultiplications of $A$ is a ring denoted by $M_A$. For each element $c$ of $A$, a bimultiplication $\mu_c$ is defined by

$$ \mu_c a = ca, \quad a \mu_c = ac, \quad a \in A $$

We call $\mu_c$ an inner bimultiplication. Clearly $\mu : A \to M_A$ is a ring homomorphism and the image $\mu A$ of this homomorphism is a two-sided ideal in $M_A$. The quotient ring $P_A = M_A/\mu A$ is called the ring of outer bimultiplications of $A$ and ring homomorphism $\theta : R \to P_A$ is called regular if $\theta(1) = 1$ and two any elements of $\theta(R)$ are permutable (the bimultiplications $\sigma$ and $\nu$ are called permutable if $\sigma(\nu a) = (\sigma a)\nu$ and $\nu(a \sigma) = (\nu a)\sigma$ for every $a$ in $A$). Then

$$ C_A = \{c \in A| ca = ac = 0, \forall a \in A\} $$

is called bicenter of $A$, and $C_A$ is a $R$-bimodule under the operations

$$ xc = (\theta x)c, \quad cx = c(\theta x), \quad c \in C_A, x \in A. $$

The ”Extension problem” of rings requires finding the exact sequence of rings

$$ 0 \to A \to S \to R \to 1 $$

induces homomorphism $\theta : R \to P_A$.

Let $\sigma : R \to M_A$ be a mapping such that $\sigma(x) \in \theta x, x \in R$ and $\sigma(0) = 0, \sigma(1) = 1$. Then we define two mappings

$$ f : R \times R \to A $$

$$ g : R \times R \to A $$

such that

$$ \mu f(x, y) = \sigma(x + y) - \sigma(x) - \sigma(x) $$

$$ \mu g(x, y) = \sigma(xy) - \sigma(x)\sigma(x) $$
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for all \( x, y \in R \). The ring structure of \( M_A \) implies mappings \( \zeta, \alpha, \lambda, \rho : M_A^3 \rightarrow C_A \) and \( \eta : M_A^2 \rightarrow C_A \)

\[
\begin{align*}
\zeta(x, y, z) &= f(x, y) - f(x + y, z) + f(x, y + z) - f(x, y) \\
\eta(x, y) &= f(x, y) - f(y, x) \\
\alpha(x, y, z) &= xg(y, z) - g(x, y, z) + g(x, y, z) - g(x, y)z \\
\lambda(x, y, z) &= xf(y, z) - f(xy, xz) + g(x, y + z) - g(x, y) - g(x, z) \\
\rho(x, y, z) &= f(x, y)z - f(x, yz) + g(x + y, z) - g(x, z) - g(y, z)
\end{align*}
\]

We call the family \((\zeta, \eta, \alpha, \lambda, \rho)\) of the above mappings an obstruction of the regular homomorphism \( \theta \). We can prove that if all these mappings are null, the homomorphism \( \theta : R \rightarrow P_A \) can be realized by a ring extension. It is the ring

\[
S = \{(a, r) \mid a \in A, r \in R \}
\]

with operations

\[
\begin{align*}
(a_1, r_1) + (a_2, r_2) &= (a_1 + a_2 + f(r_1, r_2), r_1 + r_2) \\
(a_1, r_1)(a_2, r_2) &= (r_1a_2 + a_1r_2 + g(r_1r_2), r_1r_2)
\end{align*}
\]

In the general case we have

**Proposition 6.1.** If \((\zeta, \eta, \alpha, \lambda, \rho)\) is an obstruction of the regular homomorphism \( \theta : R \rightarrow P_A \), it is a family of natural equivalences of Ann-categories of the type \((R, C_A)\).

**Proof.** We can verify directly that \( \zeta, \eta, \alpha, \lambda, \rho \) satisfy the relations in the proposition 3.1.

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