On a higher order multi-term time-fractional partial differential equation involving Caputo-Fabrizio derivative

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Abstract

In the present work we discuss higher order multi-term partial differential equation (PDE) with the Caputo-Fabrizio fractional derivative in time. We investigate a boundary value problem for fractional heat equation involving higher order Caputo-Fabrizio derivatives in time-variable. Using method of separation of variables and integration by parts, we reduce fractional order PDE to the integer order. We represent explicit solution of formulated problem in particular case by Fourier series.

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1 Introduction

Consideration of new fractional derivative with non-singular kernel was initiated by M.Caputo and M.Fabrizio in their work [1]. Motivation came from application. Precisely, new fractional derivatives can better describe material heterogeneities and structures with different scales. Special role of spatial fractional derivative in the study of the macroscopic behaviors of some materials, related with nonlocal interactions, which are prevalent in determining the properties of the material was also highlighted. In their next work [2], authors represented some applications of the introduced fractional derivative. J.J.Nieto and J.Losada studied some properties of this fractional derivative naming it as Caputo-Fabrizio (CF) derivative [3]. Namely, they introduced fractional integral associated with the Caputo-Fabrizio derivative, applying it to the solution of linear and nonlinear differential equations involving CF derivative.

Later, many authors showed interest to this CF derivative and as a result, several applications of CF derivative were discovered. For instance,
in groundwater modeling [4, 5], in electrical circuits [6], in controlling the wave movement [7], in nonlinear Fisher’s reaction-diffusion equation [8], in modeling of a mass-spring-damper system [9] and etc. We note also other works related with CF derivative [10, 11, 12, 13, 14, 15].

Different methods were applied for solving differential equations involving CF derivative. Namely, Laplace transform, reduction to the integral equation and reduction to the integer order differential equation. Last two methods were used in the works [16, 17].

In this paper, we aim to show algorithm how to reduce initial value problem (IVP) for multi-term fractional DE with CF derivative to the IVP for integer order DE and using this result to prove a unique solvability of a boundary value problem (BVP) for partial differential equation (PDE) involving CF derivative on time-variable. First, we give preliminary information on CF derivative and then we formulate our main problem. Representing formal solution of the formulated problem by infinite series, in particular case, we prove uniform convergence of that infinite series.

2 Preliminaries

\[ CFD_{at}^{\alpha} g(t) = \frac{1}{1 - \alpha} \int_{a}^{t} g'(s)e^{-\frac{\alpha}{1-\alpha}(t-s)}ds \] (1)

is a fractional derivative of order \( \alpha \) (0 < \( \alpha \) < 1) in Caputo-Fabrizo sense [2]. We note that operator (2) is well defined by the set

\[ W^{\alpha,1} = \left\{ g(t) \in L^1(a, \infty) ; (f(t) - f_a(s))e^{-\frac{\alpha}{1-\alpha}(t-s)} \in L^1(a, t) \times L^1(a, \infty) \right\}, \]

whose norm is given by for \( \alpha \neq 1 \) by

\[ ||g(t)||_{W^{\alpha,1}} = \int_{a}^{\infty} |g(t)|dt + \frac{\alpha}{1-\alpha} \int_{-\infty}^{t} \int_{a}^{t} |g_s(s)|e^{-\frac{\alpha}{1-\alpha}(t-s)}dsdt, \]

where \( g_a(t) = g(t), t \geq a, g_a(t) = 0, -\infty < t < a \) [2]. Moreover, the following equality

\[ CFD_{at}^{\alpha+n} g(t) = CFD_{at}^{\alpha} (CFD_{at}^{n} g(t)) \]

is true [9].

3 Formulation of a problem and formal solution

Consider the following time-fractional partial differential equation

\[ \sum_{n=0}^{k} \lambda_n \cdot CFD_{0t}^{\alpha+n} u(t, x) - u_{xx}(t, x) = f(t, x) \] (2)
in a domain $\Omega = \{(t, x) : 0 < t < q, 0 < x < 1\}$. Here $\lambda_n$ are given real numbers, $f(t, x)$ is a given function, $k \in \mathbb{N}_0$, $q \in \mathbb{R}^+$.

**Problem.** To find a solution of Eq. (2) satisfying the following conditions:

- \[ u(t, x) \in C^2(\Omega), \quad \frac{\partial^n u(t, x)}{\partial t^n} \in W^{\alpha,1}(0, q); \quad (3) \]
- \[ u(t, 0) = u(t, 1) = 0, \quad \frac{\partial^i u(t, x)}{\partial t^i} \bigg|_{t=0} = C_i, \quad i = 0, 1, 2, \ldots, k, \quad (4) \]

where $C_i$ are any real numbers.

### 3.1 Solution of higher order multi-term fractional ordinary differential equation

We expand $u(t, x)$ to the Fourier series as

$$u(t, x) = \sum_{m=1}^{\infty} T_m(t) \sin m\pi x,$$  \hspace{1cm} (5)

where $T_m(t)$ are Fourier coefficients of $u(t, x)$.

Substituting representation (5) into Eq. (2) and considering initial conditions (4), we get the following initial value problem with respect to time-variable:

$$\begin{cases}
\sum_{n=0}^{k} \lambda_n \cdot CF_{\delta t}^{\alpha+n} T_m(t) + (m\pi)^2 T_m(t) = f_m(t), \\
T_m^{(i)}(0) = C_i, \quad i = 0, 1, 2, \ldots, k
\end{cases}$$  \hspace{1cm} (6)

where $f_m(t)$ are Fourier coefficients of $f(t, x)$.

Based on definition (1) and initial conditions (4), after applying of the formula for integration by parts, we rewrite fractional derivatives as follows:

$$CF_{\delta t}^{\alpha} T_m(t) = \frac{1}{1-\alpha} T_m(t) - \frac{\alpha}{(1-\alpha)^2} \int_0^t T_m(s) e^{-\frac{\alpha}{1-\alpha}(t-s)} ds - \frac{T_m(0)}{1-\alpha} e^{-\frac{\alpha}{1-\alpha} t},$$

$$CF_{\delta t}^{\alpha+1} T_m(t) = \frac{1}{1-\alpha} T_m'(t) - \frac{\alpha}{(1-\alpha)^2} T_m(t) + \frac{\alpha^2}{(1-\alpha)^3} \int_0^t T_m(s) e^{-\frac{\alpha}{1-\alpha}(t-s)} ds + \frac{\alpha T_m(0)}{(1-\alpha)^2} e^{-\frac{\alpha}{1-\alpha} t} - \frac{T_m'(0)}{1-\alpha} e^{-\frac{\alpha}{1-\alpha} t},$$

$$CF_{\delta t}^{\alpha+2} T_m(t) = \frac{1}{1-\alpha} T_m''(t) - \frac{\alpha}{(1-\alpha)^2} T_m'(t) + \frac{\alpha^2}{(1-\alpha)^3} T_m(t) - \frac{\alpha^3}{(1-\alpha)^4} \int_0^t T_m(s) e^{-\frac{\alpha}{1-\alpha}(t-s)} ds \quad \text{and so on.}$$
Continuing this procedure, we can find for \( n \geq 1 \) the following formula

\[
CF D^{n+1}_t T_m(t) = \frac{1}{1-\alpha} \left\{ \sum_{i=0}^{n} \left( -\frac{\alpha}{1-\alpha} \right)^i \left[ T_m^{(n-i)}(t) - T_m^{(n-i)}(0)e^{-\frac{\alpha}{1-\alpha}t} \right] + \left( -\frac{\alpha}{1-\alpha} \right)^{n+1} \int_0^t T_m(s)e^{-\frac{\alpha}{1-\alpha}(t-s)}ds \right\}.
\]

We substitute (7) into (6) and deduce

\[
\sum_{n=0}^{k} \lambda_n \left\{ \sum_{i=0}^{n} \left( -\frac{\alpha}{1-\alpha} \right)^i \left[ T_m^{(n-i)}(t) - T_m^{(n-i)}(0)e^{-\frac{\alpha}{1-\alpha}t} \right] + \left( -\frac{\alpha}{1-\alpha} \right)^{n+1} \int_0^t T_m(s)e^{-\frac{\alpha}{1-\alpha}(t-s)}ds \right\} + (m\pi)^2 T_m(t) = f_m(t).
\]

We multiply this equality by \( (1-\alpha)e^{\frac{\alpha}{1-\alpha}t} \):

\[
\sum_{n=0}^{k} \lambda_n \left\{ \sum_{i=0}^{n} \left( -\frac{\alpha}{1-\alpha} \right)^i \left[ T_m^{(n-i)}(t)e^{\frac{\alpha}{1-\alpha}t} - T_m^{(n-i)}(0) \right] + \left( -\frac{\alpha}{1-\alpha} \right)^{n+1} \int_0^t T_m(s)e^{\frac{\alpha}{1-\alpha}(t-s)}ds \right\} = (1-\alpha)e^{\frac{\alpha}{1-\alpha}t} f_m(t).
\]

Introducing new function as \( \tilde{T}_m(t) = T_m(t)e^{\frac{\alpha}{1-\alpha}t} \), we rewrite some items of (8): \[
T_m'(t)e^{\frac{\alpha}{1-\alpha}t} = \tilde{T}_m'(t) - \frac{\alpha}{1-\alpha}\tilde{T}_m(t),
\]

\[
T_m''(t)e^{\frac{\alpha}{1-\alpha}t} = \tilde{T}_m''(t) - \frac{2\alpha}{1-\alpha}\tilde{T}_m'(t) + \left( \frac{\alpha}{1-\alpha} \right)^2 \tilde{T}_m(t),
\]

...\[
T_m^{(n)}(t)e^{\frac{\alpha}{1-\alpha}t} = \sum_{j=0}^{n} (-1)^{n-j} \frac{n!}{j!(n-j)!} \left( \frac{\alpha}{1-\alpha} \right)^{n-j} \tilde{T}_m^{(j)}(t).
\]

We note that \( \tilde{T}_m^{(0)}(t) = \tilde{T}_m(t) \).

Considering (9), from (8) we deduce

\[
\sum_{n=0}^{k} \lambda_n \left\{ \sum_{i=0}^{n} \left( -\frac{\alpha}{1-\alpha} \right)^i \sum_{j=0}^{n-i} \frac{(n-i)!}{j!(n-i-j)!} \left( -\frac{\alpha}{1-\alpha} \right)^{n-i-j} \times \left[ \tilde{T}_m^{(j)}(t) - \tilde{T}_m^{(j)}(0) \right] + \left( -\frac{\alpha}{1-\alpha} \right)^{n+1} \int_0^t \tilde{T}_m(s)ds \right\} + (m\pi)^2 (1-\alpha)\tilde{T}_m(t) = (1-\alpha)e^{\frac{\alpha}{1-\alpha}t} f_m(t).
\]

Differentiating (10) once by \( t \), we will get \((k+1)\)th order differential equation. Using its general solution and applying initial conditions, one can get explicit form of functions \( T_m(t) \), consequently, formal solution of the formulated
problem, represented by infinite series [5]. Imposing certain conditions to
the given functions, we prove uniform convergence of infinite series, which
will complete the proof of a unique solvability of the formulated problem.

In the next section we will show complete steps in particular case. We
note that even this particular case was not considered before.

# 4 Particular case

In this subsection we will consider particular case \( k = 2 \), in order to show
complete steps. In this case, after differentiation of \( \text{(10)} \) once by \( t \), we
will get the following third order ordinary differential equation

\[
\dddot{T}_m(t) + A_1 \ddot{T}_m(t) + A_2 \dot{T}_m(t) + A_3 T_m = g_m(t)
\]  

where \( \dddot{T}_m(t) = T_m(t) e^{\alpha t} \),

\[
A_1 = - \frac{3 \alpha}{1 - \alpha} + \frac{\lambda_1}{\lambda_2}, \quad A_2 = 3 \left( - \frac{\alpha}{1 - \alpha} \right)^2 + \frac{\lambda_0 - 2 \alpha \lambda_1}{1 - \alpha} + \frac{(m \pi)^2}{\lambda_2} (1 - \alpha),
\]

\[
A_3 = \left( - \frac{\alpha}{1 - \alpha} \right)^3 + \frac{(- \frac{\alpha}{1 - \alpha})^2 \lambda_1 - \frac{\alpha}{1 - \alpha} \lambda_0}{\lambda_2},
\]

\[
g_m(t) = [\alpha f_m(t) + (1 - \alpha) f_m'(t)] e^{\alpha t}.
\]  

## 4.1 General solution

Characteristic equation of \( \text{(11)} \) is

\[
\mu^3 + A_1 \mu^2 + A_2 \mu + A_3 = 0,
\]

discriminant of which is

\[
\Delta_m = -4 A_3^3 A_3 + A_1 A_2^2 - 4 A_2^3 + 18 A_1 A_2 A_3 - 27 A_3^2.
\]

According to the general theory, form of solutions depend on the sign of the
determinant. Below we will give explicit forms of solutions case by case.

Case \( \Delta_m > 0 \). In this case, characteristic equation will have 3 different
real roots \( (\mu_1, \mu_2, \mu_3) \), hence based on general solution we can find explicit
form of \( T_m(t) \) as

\[
T_m(t) = C_1 e^{(\mu_1 - \frac{\alpha}{1 - \alpha})t} + C_2 e^{(\mu_2 - \frac{\alpha}{1 - \alpha})t} + C_3 e^{(\mu_3 - \frac{\alpha}{1 - \alpha})t} +
\]

\[
+ \left( \frac{\mu_2 \mu_3 - \mu_2 \mu_3 - \mu_1 \mu_3 + \mu_1 \mu_3 - \mu_1^2 \mu_2}{1 - \alpha} \right) \left[ e^{(\mu_1 - \frac{\alpha}{1 - \alpha})t} (\mu_3 - \mu_2) \times
\]

\[
\times \int (\alpha g_m(z) + (1 - \alpha) g_m'(z)) e^{\frac{\alpha}{1 - \alpha} z} \, dz + e^{(\mu_2 - \frac{\alpha}{1 - \alpha})t} (\mu_1 - \mu_3) \times
\]

\[
\times \int (\alpha g_m(z) + (1 - \alpha) g_m'(z)) e^{\frac{\alpha}{1 - \alpha} z} \, dz +
\]

\[
+ e^{(\mu_3 - \frac{\alpha}{1 - \alpha})t} (\mu_2 - \mu_1) \int (\alpha g_m(z) + (1 - \alpha) g_m'(z)) e^{(\frac{\alpha}{1 - \alpha} - \frac{\alpha}{1 - \alpha})z} \, dz.
\]
Case $\Delta_m < 0$. In this case, characteristic equation has one real ($\mu_1$) and two complex-conjugate roots ($\mu_2 = \mu_21 \pm i\mu_22$). Therefore, $T_m(t)$ will have a form

$$
T_m(t) = C_1 e^{(\mu_1 - \frac{\alpha}{1-\alpha})t} + (C_2 \cos \mu_22t + C_3 \sin \mu_22t)e^{(\mu_21 - \frac{\alpha}{1-\alpha})t} + \\
+ \frac{1}{\mu_22^2 - 3\mu_21\mu_21 + \mu_1^2}\left[e^{(\mu_1 - \frac{\alpha}{1-\alpha})t}\int (\alpha g_m(z) + (1 - \alpha)g_m'(z))e^{(\frac{\alpha}{1-\alpha} - \mu_1)z}dz + \\
+ \frac{1}{\mu_22} e^{(\mu_21 - \frac{\alpha}{1-\alpha})t}\cos \mu_22t \int (\mu_1 \sin \mu_22z - \mu_21 \sin \mu_22z - \mu_22 \cos \mu_22z) \times \\
(\alpha g_m(z) + (1 - \alpha)g_m'(z))e^{(\frac{\alpha}{1-\alpha} - \mu_21)z}dz + \\
+ \frac{1}{\mu_22} e^{(\mu_21 - \frac{\alpha}{1-\alpha})t}\sin \mu_22t \int (\mu_21 \cos \mu_22z - \mu_22 \sin \mu_22z - \mu_1 \cos \mu_22z) \times \\
(\alpha g_m(z) + (1 - \alpha)g_m'(z))e^{(\frac{\alpha}{1-\alpha} - \mu_21)z}dz)\right].
$$

(14)

Case $\Delta_m = 0$. This case will have 2 sub-cases:

a) 3 real roots, two of which are equal ($\mu_1 = \mu_2 = \mu_3$):

$$
T_m(t) = C_1 e^{(\mu_1 - \frac{\alpha}{1-\alpha})t} + C_2 te^{(\mu_1 - \frac{\alpha}{1-\alpha})t} + C_3 e^{(\mu_3 - \frac{\alpha}{1-\alpha})t} + \\
+ \frac{1}{(\mu_1^2 - 2\mu_1\mu_3 + \mu_3^2)}\left[e^{(\mu_1 - \frac{\alpha}{1-\alpha})t}\int (\mu_3z - 1 - \mu_1 z)(\alpha g_m(z) + (1 - \alpha)g_m'(z))e^{(\frac{\alpha}{1-\alpha} - \mu_1)z}dz + \\
+ e^{(\mu_1 - \frac{\alpha}{1-\alpha})t}\int (\alpha g_m(z) + (1 - \alpha)g_m'(z))e^{(\frac{\alpha}{1-\alpha} - \mu_3)z}dz\right].
$$

(15)

b) all 3 real roots are the same ($\mu_1 = \mu_2 = \mu_3$):

$$
T_m(t) = C_1 e^{(\mu_1 - \frac{\alpha}{1-\alpha})t} + C_2 te^{(\mu_1 - \frac{\alpha}{1-\alpha})t} + C_3 e^{(\mu_3 - \frac{\alpha}{1-\alpha})t} + \\
+ \frac{1}{2} e^{(\mu_1 - \frac{\alpha}{1-\alpha})t}\int z^2(\alpha g_m(z) + (1 - \alpha)g_m'(z))e^{(\frac{\alpha}{1-\alpha} - \mu_1)z}dz - \\
- \frac{1}{2} e^{(\mu_1 - \frac{\alpha}{1-\alpha})t}\int z(\alpha g_m(z) + (1 - \alpha)g_m'(z))e^{(\frac{\alpha}{1-\alpha} - \mu_3)z}dz + \\
+ \frac{1}{2} e^{(\mu_1 - \frac{\alpha}{1-\alpha})t}\int (\alpha g_m(z) + (1 - \alpha)g_m'(z))e^{(\frac{\alpha}{1-\alpha} - \mu_3)z}dz.
$$

(16)

Here $C_j$ ($j = 1, 2, 3$) are any constants, which will be defined using initial conditions.

4.2 Convergence part

We consider the case $\Delta_m > 0$ in details. Found solution we satisfy to the initial conditions [14]. Without losing generality, we assume that $\tilde{C}_i = 0$ ($i = 0, 2$). Regarding the $C_j$ we will get the following algebraic system of equations

\[
\begin{cases}
C_1 + C_2 + C_3 = -d_1 \\
C_1(\mu_1 - \frac{\alpha}{1-\alpha}) + C_2(\mu_2 - \frac{\alpha}{1-\alpha}) + C_3(\mu_3 - \frac{\alpha}{1-\alpha}) = -d_2 \\
C_1(\mu_1 - \frac{\alpha}{1-\alpha})^2 + C_2(\mu_2 - \frac{\alpha}{1-\alpha})^2 + C_3(\mu_3 - \frac{\alpha}{1-\alpha})^2 = -d_3
\end{cases}
\]

where

\[
d_1 = \frac{1}{(\mu_22^2 - 3\mu_21\mu_21 + \mu_1^2)} \left[(\mu_3 - \mu_2)\int t\alpha g_m(t) + \\
-(1 - \alpha)g_m'(t)dt|_{t=0} + (\mu_1 - \mu_3)\int t\alpha g_m(t) + (1 - \alpha)g_m'(t)dt|_{t=0} + \\
+(\mu_2 - \mu_1)\int t\alpha g_m(t) + (1 - \alpha)g_m'(t)dt|_{t=0}
\right]
\]

(17)
In general, we can write
\[
d_2 = \frac{1}{\mu_2 \mu_3^2 - \mu_3^2 \mu_1 + \mu_3 \mu_2^2 + \mu_2 \mu_1 - \mu_1 \mu_2}(\mu_1 - \frac{\alpha}{1-\alpha})\mu_3 - \mu_2) \int_0^t \alpha g_m(t) + (1 - \alpha)g'_m(t)dt |_{t=0} + \\
+ (\mu_3 - \mu_2)(\alpha g_m(0) + (1 - \alpha)g'_m(0)) + \mu_2 - \frac{\alpha}{1-\alpha}(\mu_1 - \mu_3) \int_0^t \alpha g_m(t) + (1 - \alpha)g'_m(t)dt |_{t=0} + \\
+ (\mu_1 - \mu_3)(\alpha g_m(0) + (1 - \alpha)g'_m(0)) + (\mu_3 - \frac{\alpha}{1-\alpha})(\mu_2 - \mu_1) \int_0^t \alpha g_m(t) + (1 - \alpha)g'_m(t)dt |_{t=0} + \\
+ (\mu_2 - \mu_1)(\alpha g_m(0) + (1 - \alpha)g'_m(0)) \right] \\
\tag{18}
\]

Solving this system, we get
\[
C_1 = -d_1 - \frac{-d_1(\mu_2 - \frac{\alpha}{1-\alpha}) + d_3}{\frac{\mu_1 \mu_2 - \mu_2^2 \mu_1 - \mu_3^2 - 2(\frac{\alpha}{1-\alpha})^2}{(\mu_2 - \frac{\alpha}{1-\alpha})(\mu_1 - \frac{\alpha}{1-\alpha}) - (\mu_2 - \frac{\alpha}{1-\alpha})^2}} - \\
- \frac{-d_1(\mu_1 - \frac{\alpha}{1-\alpha}) + d_3}{\frac{\mu_1 \mu_2 - \mu_2^2 \mu_1 - \mu_3^2 - 2(\frac{\alpha}{1-\alpha})^2}{(\mu_2 - \frac{\alpha}{1-\alpha})(\mu_1 - \frac{\alpha}{1-\alpha}) - (\mu_2 - \frac{\alpha}{1-\alpha})^2}} \\
\tag{19}
\]

In general, we can write
\[
|C_2| \leq M_1 |d_1| + M_2 |d_2| + M_3 |d_3|
\]

Hence, we need the following estimations in order to provide convergence of used series:
\[
|d_1| \leq \frac{1}{(m \pi)^2} |M_4 \int (\alpha f_{4,0}(z) + (1 - \alpha)f_{4,1}(z))e^{(\frac{\alpha}{1-\alpha} - \mu_1)z} dz |_{t=0} + \\
+ M_5 \int \alpha f_{4,0}(z) + (1 - \alpha)f_{4,1}(z) e^{(\frac{\alpha}{1-\alpha} - \mu_2)z} dz | + \\
+ M_6 \int \alpha f_{4,0}(z) + (1 - \alpha)f_{4,1}(z) e^{(\frac{\alpha}{1-\alpha} - \mu_3)z} dz |_{t=0} | \leq \frac{1}{(m \pi)^2} M_7 \\
\tag{20}
\]

\[
|d_2| \leq \frac{1}{(m \pi)^2} |M_8 e^{(\frac{\mu_1 - \alpha}{1-\alpha})z} \int (\alpha f_{4,0}(z) + (1 - \alpha)f_{4,1}(z)) e^{(\frac{\alpha}{1-\alpha} - \mu_1)z} dz |_{t=0} + \\
+ M_9 e^{(\frac{\mu_2 - \alpha}{1-\alpha})z} \int (\alpha f_{4,0}(z) + (1 - \alpha)f_{4,1}(z)) e^{(\frac{\alpha}{1-\alpha} - \mu_2)z} dz | + \\
+ M_{10} e^{(\frac{\mu_3 - \alpha}{1-\alpha})z} \int (\alpha f_{4,0}(z) + (1 - \alpha)f_{4,1}(z)) e^{(\frac{\alpha}{1-\alpha} - \mu_3)z} dz |_{t=0} + \\
+ M_{11} \int (\alpha f_{4,0}(z) + (1 - \alpha)f_{4,1}(z)) e^{(\frac{\alpha}{1-\alpha} - \mu_3)z} dz |_{t=0} | \leq \frac{1}{(m \pi)^2} M_{12} \\
\tag{21}
\]
$|d_3| \leq \frac{1}{(m\pi)^4} |M_{13} e^{(\mu_1 - \frac{i\alpha}{16})t} \int (\alpha f_{4,0}(z) + (1 - \alpha)f_{4,1}(z))e^{(\frac{\pi^2}{16} - \mu_1)z} \, dz|_t +$
\[+ M_{14} e^{(\mu_2 - \frac{i\alpha}{16})t} \int (\alpha f_{4,0}(z) + (1 - \alpha)f_{4,1}(z))e^{(\frac{\pi^2}{16} - \mu_2)z} \, dz|_t +$
\[+ M_{15} e^{(\mu_3 - \frac{i\alpha}{16})t} \int (\alpha f_{4,0}(z) + (1 - \alpha)f_{4,1}(z))e^{(\frac{\pi^2}{16} - \mu_3)z} \, dz|_t +$
\[+ M_{16} (\alpha f_{4,0}(z) + (1 - \alpha)f_{4,1}(z)) + M_{17} (\alpha f_{4,1}(z) + (1 - \alpha)f_{4,2}(z))| \leq \frac{1}{(m\pi)^4} M_{18}

Here $M_i$ $(i = 1, 18)$ are any positive constants,

\[f_m(t) = \int_0^1 f(t, x) \sin m\pi x \, dx = \frac{1}{(m\pi)^4} f_{4,0}(t), \quad (23)\]
\[f_m'(t) = \frac{1}{(m\pi)^4} \int_0^1 \frac{\partial}{\partial t} \left( \frac{\partial^4}{\partial x^4} f(t, x) \right) \sin m\pi x \, dx = \frac{1}{(m\pi)^4} f_{4,1}(t), \quad (24)\]
\[f_m''(t) = \frac{1}{(m\pi)^4} \int_0^1 \frac{\partial}{\partial t^2} \left( \frac{\partial^4}{\partial x^4} f(t, x) \right) \sin m\pi x \, dx = \frac{1}{(m\pi)^4} f_{4,2}(t), \quad (25)\]

\[f_{4,0}(t) = \int_0^1 \frac{\partial^4 f(t, x)}{\partial x^4} \sin m\pi x \, dx, \quad (26)\]
\[f_{4,1}(t) = \int_0^1 \frac{\partial^5 f(t, x)}{\partial t \partial x^4} \sin m\pi x \, dx, \quad (27)\]
\[f_{4,2}(t) = \int_0^1 \frac{\partial^6 f(t, x)}{\partial t^2 \partial x^4} \sin m\pi x \, dx. \quad (28)\]

We note that for above-given estimations, we need to impose the following conditions to the given function $f(t, x)$:

\[\frac{\partial f}{\partial t}|_{t=0} = 0, \quad \frac{\partial^2 f}{\partial t^2}|_{t=0} = 0, \quad \frac{\partial^3 f}{\partial t^3}|_{t=0} = 0, \quad \frac{\partial^4 f}{\partial t^4}|_{t=0} = 0,\]

\[f(t, 1) = f(t, 0) = 0, \quad \frac{\partial^2 f(t, 1)}{\partial x^2} = \frac{\partial^2 f(t, 0)}{\partial x^2} = 0. \quad (29)\]

Based on estimations $(17)$ - $(19)$, we obtain

\[|C_j| \leq \frac{M_{19}}{(m\pi)^4} \quad (30)\]

and considering $(13)$, finally we get

\[|T_m| \leq \frac{M_{20}}{(m\pi)^4}. \quad (31)\]

Taking $(5)$ into account, on can easily deduce that

\[|u(t, x)| \leq \frac{M_{31}}{(m\pi)^4}, \quad |u_{xx}(t, x)| \leq \frac{M_{32}}{(m\pi)^2}. \quad (32)\]
Another required estimation

\[ |CFD_{0t}^\alpha u(t, x)| \leq \frac{M_{23}}{(m\pi)^4} \]

can be deduced easily, as well.

**Theorem.** If \( f(t, x) \in C^2(\Omega), \frac{\partial^3 f(t, x)}{\partial t^3} \in C(\Omega \cup \{ t = 0 \}) \) together with (29), then problem (2)-(4), when \( k = 2 \) has a unique solution represented by (5), where \( T_m(t) \) are defined by (13)-(16) depending on the sign of \( \Delta_m \).

**Remark 1.** Similar result can be obtained for general case, as well.

**Remark 2.** We note that used algorithm allows us to investigate fractional spectral problems such

\[
\begin{cases}
CFD_{0t}^{\alpha+1} T(t) + \mu T(t) = 0, \\
T(0) = 0, \quad T(1) = 0,
\end{cases}
\]

reducing it to the second order usual spectral problem.

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