THE CHARACTER OF GOLDSTONE BOSONS

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A succinct review of the QCD gap equation and dynamical chiral symmetry breaking; their connection with Bethe-Salpeter equations and resolving the dichotomous nature of the pion; the calculation of the pion’s valence-quark distribution; and first results for the \( \pi \)-exchange contribution to the \( \gamma N \to \omega N \) cross-section, which is important in the search for missing nucleon resonances.

1 Introduction

In the strong interaction spectrum the pion is identified as both a Goldstone mode, associated with Dynamical Chiral Symmetry Breaking (DCSB), and a bound state composed of \( u \) and \( d \)-quarks. This dichotomy is remarkable because, while \( m_\rho/2 \sim m_N/3 \sim 350 \text{ MeV} =: M_q \), the constituent quark mass, \( m_\pi/2 \) is only \( \approx 0.2 M_q \); i.e., the pion is much lighter than comparable bound states. \( M_q \) is the “effective mass” of quarks in a hadron; and the ratio of this to the renormalisation-group-invariant \( u \)- and \( d \)-current-quark masses (\( \hat{m}_q \sim 10 \text{ MeV} \)) indicates the magnitude of the effect of nonperturbative dressing on light-quark propagation characteristics. The peculiar nature of the pion can be expressed by the question: “How does one form an almost-massless bound state from very massive constituents \( \text{without} \) fine-tuning?” Answering this question; i.e., resolving and understanding the dichotomy of the pion, is a key to unravelling the quark-gluon substructure of QCD’s Goldstone mode.

2 QCD’s Gap Equation

The insightful study of dynamical symmetry breaking can be pursued using a gap equation, which in QCD is the Dyson-Schwinger equation (DSE) for the dressed-quark propagator:

\[
S(p)^{-1} = Z_2 (i \gamma \cdot p + m_{\text{bare}}) + Z_1 \int_q \Lambda g^2 D_{\mu\nu}(p - q) \frac{\lambda^a}{2} \gamma_\mu S(q) \Gamma_\nu^a(q,p).
\]  

In this equation: \( D_{\mu\nu}(k) \) is the renormalised dressed-gluon propagator; \( \Gamma_\nu^a(q;p) \) is the renormalised dressed-quark-gluon vertex; \( m_{\text{bare}} \) is the \( \Lambda \)-dependent current-quark bare mass that appears in the Lagrangian; and
\[ \int_0^\Lambda \frac{d^4q}{(2\pi)^4} \] represents mnemonically a translationally-invariant regularisation of the integral, with \( \Lambda \) the regularisation mass-scale. In Eq. (1), \( Z_1(\zeta^2, \Lambda^2) \) and \( Z_2(\zeta^2, \Lambda^2) \) are the quark-gluon-vertex and quark wave function renormalisation constants, which depend on the renormalisation point, \( \zeta \), and the regularisation mass-scale, as does the mass renormalisation constant

\[ Z_m(\zeta^2, \Lambda^2) = Z_1(\zeta^2, \Lambda^2)/Z_2(\zeta^2, \Lambda^2), \]

(2)

with the renormalised mass given by \( m(\zeta) := m_{\text{bare}}(\Lambda)/Z_m(\zeta^2, \Lambda^2). \)

The solution of Eq. (1) has the form

\[ S(p)^{-1} = i\gamma \cdot p A(p^2, \zeta^2) + B(p^2, \zeta^2) = \frac{1}{Z(p^2, \zeta^2)} [i\gamma \cdot p + M(p^2, \zeta^2)], \]

(3)

where the functions \( A(p^2, \zeta^2), B(p^2, \zeta^2) \) express the effects of quark-dressing induced by the quark's interaction with its own gluon field. Equation (1) must be solved subject to a renormalisation [boundary] condition, and in QCD it is practical to impose the requirement that at a large spacelike \( \zeta^2 \)

\[ S(p)^{-1} \big|_{\rho^2=\zeta^2} = i\gamma \cdot p + m(\zeta). \]

(4)

The dressed-quark mass function: \( M(p^2, \zeta^2) = B(p^2, \zeta^2)/A(p^2, \zeta^2) \), is actually independent of the renormalisation point; i.e., it is a function only of \( p^2/\Lambda_{QCD}^2 \). At one loop order in perturbation theory the running mass

\[ m(\zeta) = M(\zeta^2) = \frac{\hat{m}}{\gamma_m \ln \left( \frac{\zeta^2}{\Lambda_{QCD}^2} \right)^{\gamma_m}}, \quad \gamma_m = 12/(33 - 2N_f), \]

(5)

where \( \hat{m} \) is the renormalisation point independent current-quark mass, and the mass renormalisation constant in Eq. (2) is \( Z_m(\zeta^2, \Lambda^2) = [\alpha(\Lambda^2)/\alpha(\zeta^2)]^{\gamma_m} \).

(At one-loop in Landau-gauge, \( Z_2 = 1 \).)

The chiral limit is unambiguously defined by \( \hat{m} = 0 \). In this case there is no perturbative contribution to the scalar piece of the quark self energy; i.e., \( B(p^2, \zeta^2) \equiv 0 \) at every order in perturbation theory.

3 Dynamical Chiral Symmetry Breaking

DCSB is the appearance of a \( B(p^2, \zeta^2) \neq 0 \) solution of Eq. (1) in the chiral limit, which is impossible in perturbation theory. Hence here, as in all studies of dynamical symmetry breaking, a nonperturbative analysis of the gap equation is required. For that one needs a systematic, symmetry-preserving truncation scheme, which goes beyond the weak coupling expansion that yields perturbation theory.
The leading term in one such scheme is the renormalisation-group-improved rainbow approximation to the gap equation:

\[
S(p)^{-1} = Z_2 (i\gamma \cdot p + m_{\text{bare}}) + \int_q \mathcal{G}((p-q)^2) D^\text{free}_{\mu\nu}(p-q) \frac{\lambda^a}{2} \gamma_\mu S(q) \frac{\lambda^a}{2} \gamma_\nu ,
\]

where \(D^\text{free}(k)\) is the free gauge boson propagator and \(\mathcal{G}(k^2)\) is an “effective coupling”. In QCD \(\mathcal{G}(k^2) = 4\pi\alpha(k^2)\) for \(k^2 \gtrsim 1\ \text{GeV}^2\), where \(\alpha(k^2)\) is the strong running coupling constant. However, the behaviour of \(\mathcal{G}(k^2)\) for \(k^2 < 1\ \text{GeV}^2\); i.e., at infrared length-scales \((\lesssim 0.2 \text{fm})\), is currently unknown.

The effective coupling is a phenomenological mnemonic for the contracted product of the dressed-gluon propagator and dressed-quark-gluon vertex, which forms the kernel in Eq. (1). The infrared strength of the interaction can then be characterised by the interaction-tension:

\[
\sigma^\Delta := \frac{1}{4\pi} \int_{\Lambda^2_{\text{QCD}}}^\Lambda^2 \frac{dk^2}{k^2} \frac{d^4 q}{(2\pi)^4} \mathcal{D}(q^2) \left[ \Delta(k^2) - \Delta(\Lambda^2_{\text{QCD}}) \right]
\]

with \(\Delta(k^2) = \mathcal{G}(k^2)/k^2\), and \(\Lambda_{\text{pQCD}} = 10 \Lambda_{\text{QCD}}\), the boundary above which perturbation theory is unquestionably valid.

Studies show that DCSB does not occur for \(\sigma^\Delta \lesssim 0.5\ \text{GeV}^2 \sim 9\Lambda^2_{\text{QCD}}\), which means that the existence of a dynamically generated, \(B \neq 0\) solution of Eq. (1) is impossible without a significant infrared enhancement of the effective interaction. Reproducing observable phenomena requires \(\sigma^\Delta \gtrsim 4\ \text{GeV}^2 \sim 70\Lambda^2_{\text{QCD}}\); i.e., a ten-fold enhancement over the critical value. The origin of this enhancement, whether in the gluon or ghost vacuum vacuum polarisation, or elsewhere, is currently unknown but is actively being sought (see, e.g., Refs. [9,10]). Nevertheless, its characterisation via a one-parameter model yields an efficacious description of light meson observables.

The dynamical generation of a dressed-quark mass in the chiral limit is necessarily accompanied by the formation of a vacuum quark condensate:

\[
- \langle \bar{q} q \rangle^0 = Z_4(\zeta^2, \Lambda^2) N_c \text{tr}_D \int_0^\Lambda \frac{d^4 q}{(2\pi)^4} \mathcal{S}_0(q, \zeta),
\]

where \(\text{tr}_D\) identifies a trace over Dirac indices only and the superscript “0” indicates the quantity was calculated in the chiral limit. This condensate is gauge-parameter and cutoff independent. In the model summarised in Ref. [11], an explanation of hadron observables requires \(\langle \bar{q} q \rangle^0_{\zeta=1\ \text{GeV}} = -(0.241 \ \text{GeV})^3\). The condensate is analogous to a Cooper pair density and this value corresponds to 1.8 fm\(^{-3}\). Considering, simply for illustration, the
condensate to be a sea of close-packed spheres then this density corresponds to a Cooper pair radius of 0.76 fm, which is just 15% more than the pion’s charge radius and 13% less than the nucleon’s radius. Clearly the scale set by this condensate is fundamental in QCD.

4 The Goldstone Boson

We began with a question: “How does one form an almost-massless bound state from very massive constituents without fine-tuning?” Bound states of a dressed-quark and dressed-antiquark; i.e., mesons, are described by the homogeneous Bethe-Salpeter equation (BSE). (The Bethe-Salpeter amplitude obtained as the solution of this equation is analogous to the wave function in quantum mechanics and provides similar insights into the nature of the bound state.) It was proven in Ref. that the flavour nonsinglet pseudoscalar meson BSE admits a massless solution if, and only if, the QCD gap equation supports DCSB. This is an expression of Goldstone’s theorem.

That proof establishes a number of corollaries. One is a collection of quark-level Goldberger-Treiman relations, which establish that the pseudoscalar meson Bethe-Salpeter amplitude necessarily has pseudovector components. It is these components that are responsible for the asymptotic $1/Q^2$ behaviour of the electromagnetic pion form factor. Another is a mass formula, valid for all flavour nonsinglet pseudoscalar mesons, independent of the current-quark mass of the constituents. This single mass formula unifies aspects of QCD’s light- and heavy-quark sectors, and provides a qualitative understanding of recent results from lattice simulations for the current-quark-mass-dependence of pseudoscalar meson masses. DCSB and the large value of the vacuum quark condensate also explain the large $\pi-\rho$ mass difference.

An heuristic demonstration of this is provided by the bosonisation procedure described in Ref. In the chiral limit, for the flavour nonsinglet mesons, there is an exact cancellation between the condensate-driven mass-term in the quark-loop piece of the effective action and that in the auxiliary field piece, while for the vector mesons, because $\{\gamma_5, \gamma_\mu\} = 0$, the terms add.

Confinement is the failure to observe coloured excitations in a detector. It is related to the effective interaction and interaction-tension described in Sec. If the interaction-tension is large enough to explain hadron observables then the analytic properties of the dressed-quark propagator are very different from those of a free fermion; e.g., the propagator no longer has a Källén-Lehmann representation. The absence of such a representation is a sufficient condition for confinement. From this perspective, the existence of flavour nonsinglet pseudoscalar Goldstone modes is a consequence of confine-
Table 1. Fit parameters for the calculated \( u \) valence-quark distribution function, Eq. (9).

| scale (GeV) | \( A_u \) | \( \eta_1 \) | \( \eta_2 \) | \( \epsilon_u \) | \( \gamma_u \) |
|------------|----------|-----------|-----------|---------|---------|
| 0.54       | 11.24    | 1.43      | 1.90      | 2.44    | 2.54    |
| 2.0        | 4.25     | 0.97      | 2.43      | 1.82    | 2.46    |

ment: one can have Goldstone modes without confinement, but one cannot have confinement in the chirally symmetric theory without the appearance of Goldstone modes.

5 Pion’s Valence-quark Distribution

The quark-gluon substructure of the pion is also expressed in its parton distribution functions. They can be measured in \( \pi N \) Drell-Yan and deep inelastic scattering (using the Sullivan process) but cannot be calculated in perturbation theory. The valence-quark distribution function has been calculated using the DSE model that unified the small- and large-\( Q^2 \) behaviour of the electromagnetic pion form factor. This covariant calculation of the relevant “handbag diagrams” indicates that, at a resolving scale of \( q_0 = 0.54 \, \text{GeV} = 1/(0.37 \, \text{fm}) \), valence-quarks with an active mass of 300 MeV carry 71\% of the pion’s momentum, and yields a numerical result for the distribution function that is pointwise well-described by

\[
x u_V^\pi(x; q_0) = A_u x^{\eta_1} (1 - x)^{\eta_2} (1 - \epsilon_u \sqrt{x} + \gamma_u x),
\]

with the parameter values listed in Table 1. This parametrisation, with the parameter values listed, provides an equally good fit to \( x u_V^\pi(x; 2 \, \text{GeV}) \) obtained by applying the three-flavour, leading-order, nonsinglet renormalisation group evolution equations. The low moments of the distribution calculated at this scale (2 GeV) are

\[
\begin{array}{ccc}
\langle x \rangle & \langle x^2 \rangle & \langle x^3 \rangle \\
\text{Calc.} & 0.24 & 0.098 & 0.049 \\
\text{Exp.} & 0.24 \pm 0.01 & 0.10 \pm 0.01 & 0.058 \pm 0.004 \\
\text{Latt.} & 0.27 \pm 0.01 & 0.11 \pm 0.3 & 0.048 \pm 0.020 \\
\end{array}
\]

with the experimental results from Ref. \cite{18} and the lattice results from Ref. \cite{22}.

The value of \( \eta_2 \simeq 2 \) obtained in the DSE calculation is consistent with a perturbative QCD analysis \( u_V(x) \sim (1 - x)^2 \) at \( x \simeq 1 \). That prediction, however, conflicts with the currently available Drell-Yan data \cite{23} which
Figure 1. \( \pi \)-exchange contribution to the \( \omega \)-photoproduction amplitude.

suggests that \( u^V_\pi(x) \approx (1 - x) \) for \( x \approx 1 \). This experimental result, if true, will be a profound challenge to QCD, even questioning whether the strong interaction Lagrangian contains a vector interaction.

We note that the agreement between the moments calculated using the DSE result and those inferred from experiment, see Eq. (10), indicates plainly that the low moments are insensitive to the large-\( x \) behaviour of the distribution; i.e., the valence region: even the sixth moments of these differently shaped distributions disagree by only 32\%. Hence pointwise calculations of the distribution functions are necessary to probe the valence region.

6 \( \omega \)-meson Photoproduction

Constituent quark models predict many nucleon resonances that are hitherto unobserved. That may be because they only couple weakly to the \( \pi N \) channel, which has been used to search for them. Experiments at facilities such as JLab are therefore seeking the “missing resonances” in electromagnetic reactions. Of these, \( \omega \)-photoproduction is particularly useful because its non-resonant reaction mechanisms are thought to be well understood. Hence a comparison between experimental data and the cross-section calculated using these mechanisms alone provides a sound base from which to search for nucleon resonance contributions.

Meson exchange models suggest that the leading contribution to the photoproduction cross-section at low energies and forward angles is given by the off-shell-\( \pi \)-exchange diagram depicted in Fig. 1. Here the filled circles represent meson-nucleon and meson-photon form factors, which are customarily parametrised in the calculation of observables.

However, these form factors can be calculated when one has a reliable quark-level description of hadrons. We illustrate that using the model of Ref. 24, with the Ansatz for the nucleon’s Fadde’ev amplitude updated as
Figure 2. π-exchange contribution to the $\gamma N \rightarrow \omega N$ cross-section obtained using the model of Refs. [24,25]. (Panels are labelled by the incident photon energy.) *, long-dashed line, our calculation (no parameters were varied to obtain this result); ○, short-dashed line, the meson-exchange model calculation of Ref. [23] (only the $t$-channel $\pi$-exchange contribution is shown); data from Ref. [26].

described in Ref. [25], to obtain a parameter-free prediction of the contribution from the $\pi$-exchange process to the $\omega$-photoproduction cross-section. The results are illustrated in Fig. 2. Much of the discrepancy at large angles owes to our neglect, in this preliminary calculation, of the contribution from $s$- and $u$-channel nucleon diagrams. Including them is part of an ongoing programme.
7 Epilogue

The DSEs are maturing as an efficacious nonperturbative tool in hadron physics and in this application they provide crucial information about the long-range part of the QCD interaction. The primary elements in the approach are the Schwinger functions: the dressed-propagators and -vertices. They can also be calculated using other techniques, such as effective field theory and lattice quantisation methods. There is therefore great scope for forging relationships between these approaches, thereby providing a tool whose domain of reliable application is larger than that of its individual parts. One concrete example is the confirmation in recent lattice-QCD simulations of the bulk features of the dressed-quark propagator that have long been suggested by DSE studies. Making that agreement quantitative will hone our understanding of the long-range part of the interaction and, e.g., make possible a DSE-built bridge between lattice-QCD and the $Q^2$-evolution of the electromagnetic pion form factor.

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