Synchronization as Aggregation: Cluster Kinetics of Pulse-Coupled Oscillators

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We consider models of identical pulse-coupled oscillators with global interactions. Previous work showed that under certain conditions such systems always end up in sync, but did not quantify how small clusters of synchronized oscillators progressively coalesce into larger ones. Using tools from the study of aggregation phenomena, we obtain exact results for the time-dependent distribution of cluster sizes as the system evolves from disorder to synchrony.

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In one of the first experiments on firefly synchronization, the biologists John and Elizabeth Buck captured hundreds of male fireflies along a tidal river near Bangkok and then released them at night, fifty at a time, in their darkened hotel room [1]. As they looked on in wonder, they observed that “centers of synchrony began to build up slowly among the fireflies on the wall. In one area we would notice that a pair had begun to pulse in unison; in another part of the room a group of three would be flashing together, and so on.” Synchronized groups continued to emerge and grow, until as many as a dozen fireflies were blinking on and off in concert. The Bucks realized that the fireflies were phase shifting each other with their flashes, driving themselves into sync.

Here we study stylized models of oscillators akin to the fireflies, in which synchrony builds up stepwise, in expanding clusters. By borrowing techniques used to analyze aggregation phenomena in polymer physics, materials science, and related subjects [2,3], we give the first analytical description of how these synchronized clusters emerge, coalesce, and grow. We hasten to add, however, that the models we discuss are not even remotely realistic descriptions of fireflies; they are merely intended as tractable first steps toward understanding how clusters evolve en route to synchrony.

Our work is part of a broader interdisciplinary effort [4,5]. Oscillators coupled by sudden pulses have been used to model sensor networks [6,10], earthquakes [11,12], economic booms and busts [13], firing neurons [14,15], and cardiac pacemaker cells [16]. Diverse forms of collective behavior can occur in these pulse-coupled systems, depending on how the oscillators are connected in space. Systems with local coupling often display waves [17,18] or self-organized criticality [11,19,20], with possible relevance to neural computation [15] and epilepsy [21]. In contrast, systems with global coupling, where every oscillator interacts equally with every other, tend to fall into perfect synchrony. Rigorous convergence results have been proven for this case [20,22,23]. But the techniques used previously have not revealed much about the transient dynamics leading up to synchrony—the opening and middle game, as opposed to the end game. Aggregation theory offers a new set of tools to explore this prelude to synchrony.

We introduce a toy model, which we call scrambler oscillators. It consists of $N \gg 1$ identical integrate-and-fire oscillators coupled all to all. Each oscillator has a voltage-like state variable $x$ that increases linearly according to $\dot{x} = 1$, rising from a baseline value of 0 to a threshold value of 1. Whenever any oscillator reaches threshold, it fires and does three things. (i) It kicks every other oscillator (and every cluster of oscillators) to a new random voltage, independently and uniformly—in this sense, it scrambles the other oscillators. (ii) It then “absorbs” any scrambled oscillators that lie within a distance $1/N$ of threshold, by bringing them to threshold and thereby synchronizing with them. To avoid the complications that would be caused by chain reactions of firings, we assume that the oscillators being brought to threshold do not get to fire until the next time they reach threshold. (iii) The oscillator that fired resets to $x = 0$ along with the oscillators it absorbed.

If a cluster of $j$ oscillators does the firing, the same rules apply, except that now any oscillators within a distance $j/N$ of threshold get absorbed. The assumed proportionality to $j$ is natural, if each member of the cluster contributes to the pulse strength. We study other plausible coupling rules in the Supplemental Material [29].

The motivation for this scrambler model is that it leads to the simplest possible mean-field approximation. In the infinite-$N$ limit, we would like clusters of every size to be uniformly distributed in voltage at all times. This convenient property would greatly ease the derivation of the rate equations for the cluster kinetics. As we will see below, the predictions that follow from this approximation agree reasonably well with simulations.

Assume the initial voltages $x_i$, for $i = 1, \ldots, N$, are independent and uniformly distributed. At first, nothing interesting happens. The oscillators increase their vol-
ages without interacting. But then one oscillator reaches threshold and fires. The remaining oscillators get scrambled, and perhaps some get absorbed. Then another oscillator fires, and so on. After this process has gone on for a while, the system has formed synchronized clusters of various sizes.

Let $N_j(t)$ denote the number of clusters of size $j$ at time $t$. Thus there are $N_1(t)$ singleton oscillators, $N_2(t)$ pairs of synchronized oscillators, $N_3(t)$ triplets, and so on. The $N_j$ are correlated random quantities. They are correlated because oscillators belonging to clusters of one size are unavailable to clusters of another size, and they are random because of the randomness in the initial conditions and the scrambling procedure. It does not seem feasible to understand the time-evolution of the $N_j$ unless they are so large that their fluctuations from one random realization to another are negligible.

So assume from now on that $N_j \gg 1$ for all $j$ and replace these random quantities by their ensemble averages. Let $c_j = N^{-1}(N_j)$ denote the average cluster densities. One hopes that relative fluctuations are small; more precisely, $N^{-1}N_j = c_j + O(N^{-1/2})$. An even stronger assumption is that the densities of different sub-populations are asymptotically uncorrelated: $N^{-2}N_iN_j = c_ic_j + O(N^{-1/2})$.

These $c_j$ allow us to define a natural disorder parameter, given by the total density $c(t) = \sum_j c_j(t)$. It measures the extent of the system’s fragmentation. To see this, note that at $t = 0$ each oscillator is alone; only clusters of size 1 exist. Accordingly $c_1(0) = 1$ and all other $c_j(0) = 0$ for $j > 1$. Hence $c(0) = 1$, correctly indicating that the system starts out maximally fragmented. At the opposite extreme, as $t \to \infty$ only one giant cluster of synchronized oscillators exists. The system is then minimally fragmented: $c(t) = 1/N \to 0$ as $N \to \infty$.

To derive a rate equation for the decline of $c(t)$, let $R_i$ be the rate at which clusters of size $i$ fire, for $i = 1, \ldots, N$, and let $L_i$ be the number of clusters lost to absorption in each such firing. Then $d = -\sum_i R_iL_i$.

To find $L_i$, recall that when a cluster of size $i$ fires, all the other clusters get assigned a new voltage uniformly at random. Moreover, any clusters assigned to the interval $[1-i/N, 1]$ get brought to threshold and absorbed. Since the voltages of these other clusters are uniformly distributed on $[0, 1]$, a fraction $i/N$ of them will be absorbed. There are $\sum_j N_j$ clusters in total. Hence the number absorbed is $L_i = (i/N) \sum_j N_j = i \sum c_j = ic$.

The rate $R_i$ takes more work to calculate. Since some clusters get absorbed, not every cluster gets the chance to fire. We must account for this depletion when calculating $R_i$. First consider the background rate of firing of clusters of size $i$ in the absence of absorptions. In other words, pretend for a moment that when an $i$-cluster fires, it simply scrambles every other cluster and restarts its own cycle without absorbing anyone. Call this background rate $R_i^0$. Since all oscillators move with velocity $v_i = x_i = 1$, and since the cluster density is $c_i$, the corresponding background rate of firing is $R_i^0 = c_iv_i = c_i$. Next, to find the actual $R_i$, we must subtract from $R_i^0$ the rate at which clusters of size $i$ are being absorbed and hence deprived of their chance at firing. Call this absorption rate $R_i^a$. Clusters of size $i$ are absorbed when clusters of size $j$ fire, for $j = 1, \ldots, N$, taking a fraction $j/N$ of the uniformly distributed $i$-clusters along with them. Since there are $N_j$ clusters of size $i$ and the $j$-clusters fire at rate $R_j$, the total rate at which $i$-clusters are being absorbed is given by $R_i^a = \sum_j (j/N)N_jR_j = \sum_j j\beta c_j = \beta c_i \sum_j R_j$.

Putting all this together gives $R_i = R_i^0 - R_i^a = c_i - c_i \sum_j j\beta c_j = c_i (1 - \sum_j j\beta c_j)$. Let $\beta = 1 - \sum_j j\beta c_j$. Note that $\beta$ is the same for all $i$, which enables it to be determined self-consistently, as follows. From $R_i = \beta c_i$ we obtain $\beta = 1 - \sum_j j\beta c_j = 1 - \sum_j j(\beta c_j)$. Now invoke the identity $\sum_j j\beta c_j = j(N_j/N) = 1$, which expresses conservation of oscillators. Solving for $\beta$ then gives $\beta = 1/2$ and therefore $R_i = c_i/2$.

Next, plug the expressions derived for $R_i$ and $L_i$ into the rate equation $\dot c = -\sum_i R_iL_i$. The result is $\dot c = -\sum_i (c_i/2)(ic) = -(c/2) \sum_i ic_i = -c/2$. Recalling that $c(0) = 1$, we conclude that

$$c(t) = \exp(-t/2).$$

Figure 1 shows this result matches simulations.

How do the individual cluster densities $c_i$ behave? To derive their rate equations, note that since the voltage space is the interval $[0, 1]$, a segment of length $N^{-1}$ contains on average $Nc \times N^{-1} = c$ clusters. In fact, the probability that it contains $n$ clusters (of any sizes) is given by the Poisson distribution: $\Pi_n = e^n c^{-n}/n!$. This is the mathematical expression of the assumption that clusters are distributed randomly without correlations.

With this in mind, let us solve for $c_i(t)$, the density of singletons. It is the easiest $c_j(t)$ to analyze, since it can only decrease. Two mechanisms cause the loss of
singletons: (i) a singleton fires and absorbs a cluster, and (ii) a cluster fires and absorbs \( p \geq 1 \) singletons.

Consider mechanism (i). From the expression for \( R_i \) obtained above, singletons fire at a rate \( R_1 = c_1/2 \). When they fire, they absorb any cluster lying in the voltage segment \([1 - 1/N, 1)\). According to the Poisson distribution, the probability that this segment contains one or more clusters is given by \( 1 - e^{-c} \). Hence singletons are lost by mechanism (i) at a rate \( (c_1/2) \left[ 1 - e^{-c(t)} \right] \).

The loss due to mechanism (ii) is handled similarly. Suppose \( p \) singletons lie in the interval \([1 - j/N, 1)\) at the moment when a cluster of size \( j \) fires, for \( j = 1, \ldots, N \). This event happens with probability \( e^{-jc}(jc_1)^p/p! \), and when it does, it consumes \( p \) singletons. The \( j \)-clusters fire at a rate \( R_j = c_j/2 \). Hence singletons are lost by mechanism (ii) at a rate

\[
\sum_{j \geq 1} \frac{c_j}{2} \sum_{p \geq 1} p \left( \frac{(jc_1)^p e^{-jc}}{p!} \right) = c_1/2. \tag{2}
\]

Summing the loss rates from (i) and (ii) gives

\[
\frac{dc_1}{dt} = -\frac{c_1}{2} (2 - e^{-c(t)}). \tag{3}
\]

This equation has a closed-form solution in terms of exponential integrals:

\[
c_1(t) = \exp(-t + Ei(-1) - Ei(-e^{-t/2})), \tag{4}
\]

where we have used the initial condition \( c_1(0) = 1 \). Figure 1 shows good agreement between the theoretical and numerical \( c_1(t) \).

For \( i > 1 \), the rate equation for \( c_i \) includes gain terms as well as loss terms. Clusters of size \( i \) \( > 1 \) can be created when two or more smaller clusters coalesce, or destroyed when they themselves coalesce with at least one other cluster. The loss term is a straightforward generalization of that for \( c_1 \), and is given by \( (c_i/2) \left[ 2 - e^{-ic(t)} \right] \).

To find the gain term, imagine that a cluster of size \( k \) fires. The segment \([1 - k/N, 1)\) may contain \( a_1 \) clusters of size 1, \( a_2 \) clusters of size 2, etc. This event happens with probability \( \left( \frac{k(a_1)}{a_1} \right)^1 \times \left( \frac{k(a_2)}{a_2} \right)^2 \times \cdots \) (where we are using the assumption that clusters of different sizes are independent as well as Poisson distributed). If the segment contains a combination of clusters such that \( k + a_1 + 2a_2 + 3a_3 + \cdots = i \), then a cluster of size \( i \) will form. We sum over all such combinations for a cluster of size \( k \) firing, and then sum over all \( k \), to get the rate at which clusters of size \( i \) are created:

\[
\frac{1}{i} \sum_{k=1}^{i-1} \frac{c_k}{2} e^{-kc} \sum_{a_1 + 2a_2 + \cdots = i-k} \left( \prod_{p \geq 1} \frac{(kcp)^{a_p}}{a_p!} \right). \tag{5}
\]

Combining the loss and gain terms, and transferring \( c_i e^{-ic} \) into the gain term, we finally obtain

\[
\dot{c}_i = -c_i + \frac{1}{i} \sum_{k=1}^{i} \frac{c_k}{2} e^{-kc} \sum_{a_1 + 2a_2 + \cdots = i-k} \left( \prod_{p \geq 1} \frac{(kcp)^{a_p}}{a_p!} \right). \tag{6}
\]

We see from the sum that the equations (6) are recursive. They can be solved one by one, though not analytically, so we resort to numerical integration. Figure 2 shows that the theoretical and simulated \( c_i \) agree.

Although we cannot find all the \( c_i(t) \) explicitly, we can get their moments \( M_n(t) = \sum_j j^n c_j(t) \). The first two moments are trivial: \( M_0 = \sum_j c_j(t) = c(t) \) and \( M_1 = \sum_j j c_j(t) = 1 \), from conservation of oscillators. The higher moments can be obtained from a generating function. Let \( G(z, t) = \sum_{k \geq 1} c_k(t) e^{kz} \). Then the infinite set of differential equations (6) transforms into

\[
\frac{\partial G(z, t)}{\partial t} + G(z, t) = \frac{1}{2} G(z - c(t) + G(z, t), t). \tag{7}
\]

This equation looks neat, but it is far from trivial, as the right-hand side involves \( G \) in a very nonlinear manner. Using the identity \( M_n(t) = \frac{\partial^n G}{\partial z^n} |_{z=0} \), we can however derive the following equations for the moments:

\[
\begin{align*}
M_2 & = \frac{3}{2} M_2 \\
M_3 & = \frac{7}{2} M_3 + 3(M_2)^2 \\
M_4 & = \frac{15}{4} M_4 + 16 M_2 M_3 + \frac{3}{2} (M_2)^3.
\end{align*} \tag{8}
\]

Like the \( c_i \) equations (6), these moment equations are recursive and can be solved in succession. We find

\[
\begin{align*}
M_2(t) & = e^{3t/2} \\
M_3(t) & = 7e^{7t/2} - 6e^{3t} \\
M_4(t) & = \frac{448}{5} e^{5t} - 128 e^{9t/2} + \frac{217}{5} e^{15t/4} - 4 e^{27t/8}.
\end{align*} \tag{9}
\]

Figure 3 plots theoretical and simulated values of the \( M_i \). The agreement is good for \( M_2 \) but worse for \( M_3 \) and \( M_4 \). This is to be expected. Each \( c_i(t) \) is a stochastic process, subject to fluctuations dominated by the chance formation of big clusters. Since \( M_n(t) = \sum_j j^n c_j(t) \), the
higher moments amplify these fluctuations more and are therefore noisier themselves.

Variants of the scrambler model can also be solved. For example, suppose that when a cluster of size $j$ fires, it absorbs all oscillators within a distance $kj/N$ of threshold, where $k > 0$ is a tunable coupling strength. Or suppose that the pulse strength is independent of the size $j$ of the firing cluster. Both these cases are discussed in [20]. Nothing qualitatively new happens, but there are some interesting quantitative differences, e.g., in the latter case the disorder parameter $c(t)$ decays algebraically rather than exponentially.

A more substantial modification would be to make the model deterministic. Scrambler oscillators respond randomly, but real oscillators respond deterministically, according to a phase-response curve [27]. This deterministic character is incorporated in Refs. [16, 22] by assuming that when a cluster of size $j$ fires, it adds a voltage pulse $je$ to every other oscillator, or pulls it up to threshold, whichever is less. For the case where $\epsilon = 1/N$ we show in [20] that this deterministic system can also be well approximated by the aggregation methods used above. The main new feature is that $c(t)$ and the cluster densities $c_i(t)$ become piecewise linear. But their overall shapes still resemble those seen in the scrambler model.

There are many avenues to explore in future work. For example, throughout our analysis each oscillator obeyed $\dot{x}_i = 1$. Such linear sawtooth waveforms are reasonable for the oscillators used in sensor networks [8], but not for neurons or cardiac pacemaker cells. Changing the concavity or shape of the waveform would make the analysis more difficult, and might even affect the cluster kinetics, because clusters would no longer be uniformly distributed as we have assumed throughout. Could different types of pulses be engineered to preserve a uniform distribution? Or perhaps different pulses would distribute the clusters in a nonuniform but still tractable manner. It would also be interesting to study cluster kinetics in oscillator systems with local coupling, network structure, heterogeneity, delays, and other realistic features.

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