THREE TERM RECURRENCE AND RESIDUE COMPLETENESS

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Abstract. Let \( w_n =qw_{n-1}+w_{n-2} \) and let \( K \) be the set of all \( m \)'s such that \( \{w_n\} \) modulo \( m \) is \( \mathbb{Z}_m \). The present article studies \( \{w_n\} \) modulo \( m \) and determines the set \( K \). In particular, (i) Pell numbers modulo \( m \) is residue complete if and only if \( m \in \{2,3^a,5^b\} \), and (ii) Pell-Lucas numbers modulo \( m \) is residue complete if and only if \( m = 3^n \)

1. Introduction

Given \( a,b \in \mathbb{Z}, q \in \mathbb{Z} - \{0\} \). Define the three term linear recurrence \( \{w_n(a,b,q)\} = \{w_n\} \) by the following.

\[
 w_0 = a, \quad w_1 = b, \quad w_n = qw_{n-1}+w_{n-2}.
\]

Let \( m \in \mathbb{N} \). The recurrence \( \{w_n\} \) modulo \( m \) is periodic (see Lemma 2.1). Let \( w(a,b,q,m) \) be a period. The length of \( w(a,b,q,m) \) is denoted by \( k(a,b,q,m) \). We say \( \{w_n\} \) modulo \( m \) is residue complete if \( x \in \{w_r\} \) modulo \( m \) for every \( x \in \mathbb{Z}_m \). Equivalently, \( x \in w(a,b,q,m) \) for every \( x \in \mathbb{Z}_m \). Let \( p \) be a prime divisor of such \( m \). Applying results of Bumby [2], Li [6], Schinzel [7], Somer [8], and Webb and Long [9]

\[
 p \in \Delta \cup \Omega,
\]

where \( \Delta = \{2,3,5,7\} \) and \( \Omega \) is the set of prime divisors of \( q^2 + 4 \). Note that 3 and 7 cannot be divisors of \( q^2 + 4 \). In the present article, we study \( \{w_n\} \) modulo \( p^e \), where \( p \in \Delta \cup \Omega \). We will give a series of lemmas that enables us to find the set \( K \) that consists of all the \( m \)'s such that \( \{w_n(a,b,q)\} \) modulo \( m \) is residue complete (see Discussion 5.3). In particular, one has (i) Pell numbers modulo \( m \) is residue complete if and only if \( m \in \{2,3^a,5^b\} \), and (ii) Pell-Lucas numbers modulo \( m \) is residue complete if and only if \( m = 3^n \) (see Proposition 5.1 and 5.2).

The period \( w(0,0,q,m) \) is called the trivial period. Throughout the article, the periods being considered are nontrivial periods.

2. Basic Properties about \( w_n \)

Let \( \sigma = \begin{bmatrix} q & 1 \\ 1 & 0 \end{bmatrix} \). Then \( \begin{bmatrix} qb+a & b \\ b & a \end{bmatrix} \sigma^n = \begin{bmatrix} w_{n+2} & w_{n+1} \\ w_{n+1} & w_n \end{bmatrix} \). \hspace{1cm} (2.1)

Lemma 2.1. The sequence \( \{w_n(a,b,q)\} \) (mod \( m \)) is periodic. Suppose \( \gcd(a^2+qab-b^2,m) = 1 \). Then the length of \( w(a,b,q,m) \) is the smallest positive integer \( k \) such that \( \sigma^k \equiv 1 \) (mod \( m \)).

Proof. In (2.1), there exists some \( r \in \mathbb{N} \) such that \( \sigma^r \equiv 1 \) (mod \( m \)). It follows from (2.1) that \( (w_1,w_0) = (b,a) \equiv (w_{r+1},w_r) \). Hence \( \{w_n\} \) modulo \( m \) is periodic. Let \( k \) be the length of \( w(a,b,q,m) \). Then \( k \) is the smallest positive integer such that \( (qb+a,b) = (w_2,w_1) \equiv (w_{k+2},w_{k+1}) \) as well as \( (b,a) = (w_1,w_0) \equiv (w_{k+1},w_k) \). As a consequence, identity (2.1) becomes

\[
 \begin{bmatrix} qb+a & b \\ b & a \end{bmatrix} \sigma^k = \begin{bmatrix} w_{k+2} & w_{k+1} \\ w_{k+1} & w_k \end{bmatrix} \equiv \begin{bmatrix} qb+a & b \\ b & a \end{bmatrix} \hspace{1cm} (2.2)
\]

In the case \( \gcd(a^2+qab-b^2,m) = 1 \), all the matrices in (2.2) are invertible modulo \( m \). Hence \( k \) is the smallest positive integer such that \( \sigma^k \equiv 1 \) modulo \( m \). \hspace{1cm} \( \square \)
Discussion 2.2. The smallest $k > 0$ such that $\sigma^k \equiv 1 \pmod{m}$ is called the order of $\sigma$ modulo $m$. Lemma 2.1 implies that if $\gcd(a^2 + qab - b^2, m) = 1$, then the length of $w(a, b, q, m)$ is the order of $\sigma$ modulo $m$ which is also the length of $w(0, 1, q, m)$. In general, the length of a period $w(a, b, q, m)$ is a divisor of the order of $\sigma$ modulo $m$.

Suppose that $\gcd(a^2 + qab - b^2, m) = 1$. Applying Lemma 2.1, the length of $w(a, b, q, m)$ is the order of $\sigma$ modulo $m$. Denoted by $k(m)$ the order of $\sigma$ modulo $m$.

**Lemma 2.3.** Suppose $\gcd(m, n) = 1$. Then $k(mn) = k(m)k(n)/\gcd(k(m), k(n))$. Further,

(i) Suppose that $\gcd(q, 5) = 1$, $q \not\equiv 7, 18 \pmod{25}$. Then $k(5^{n+1}) = 5k(5^n)$ for all $n \geq 1$.

(ii) If $q \equiv 1, 4 \pmod{5}$, then $k(5) = 20$. If $q \equiv 2, 3 \pmod{5}$, then $k(5) = 12$.

(iii) Suppose that $\gcd(q, 3) = 1$, $q \not\equiv 4, 5 \pmod{9}$. Then $k(3^n) = 8 \cdot 3^{n-1}$.

(iv) If $q \equiv 4, 5 \pmod{9}$, then $k(3) = k(9) = 8$. If $q \equiv 7, 18 \pmod{25}$, then $k(5) = k(5^2) = 12$.

**Proof.**

Direct calculation shows that $\sigma^4 \not\equiv 1 \pmod{3}$ and

$$\sigma^8 = I + 3A,$$

where $A = (a_{ij})$ is a two by two matrix such that $\gcd(3, a_{11}, a_{12}, a_{21}, a_{22}) = 1$. This implies that the order of $\sigma$ modulo 3 is 8. One may now apply (2.3) and induction to prove that $k(3^n) = 8k(3^{n-1})$. The remaining cases can be proved similarly. $\square$

**Definition 2.4.** Two periods $w(a, b, q, m)$ and $w(c, d, q, m)$ are called equivalent if one can be obtained from the other by a cyclic permutation. The set of all inequivalent periods is called a fundamental system modulo $m$. Denote this system by $FS(m)$.

**Lemma 2.5.** Suppose that $\gcd(q, m) = 1$. The total number of terms in $FS(m)$ is $m^2 - 1$.

**Proof.** Since $\gcd(q, m) = 1$, one sees easily that every period has length at least 3. Let $C = (c_1, c_2, \ldots, c_k)$ be a period of length $k$. $C$ gives $k$ adjacent pairs $(c_1, c_2), (c_2, c_3), \ldots, (c_{k-1}, c_k)$ and $(c_k, c_1)$. Since $k \geq 3$, the above pairs are all distinct from one another.

Let $(a, b)$ be a nonzero pair. By our definition of periods, $(a, b)$ must appear as adjacent terms in some periods. Since the fundamental system consists of inequivalent periods, the pair $(a, b)$ appears as adjacent terms exactly once in $FS(m)$. Our assertion now follows from the fact that there are exactly $m^2 - 1$ nonzero pairs in $\mathbb{Z}_m \times \mathbb{Z}_m$ and that each period $C$ of length $k$ contributes exactly $k$ adjacent pairs. $\square$

**Lemma 2.6.** Let $\{w_n\}$ be given as in (1.1). Then $w_{n+2}w_n - w_{n+1}^2 = w_n^2 + qw_nw_{n+1} - w_{n+1}^2 = (-1)^n(a^2 + qab - b^2)$ for all $n$. In particular, $\pm(a^2 + qab - b^2)$ is an invariant of $\{w_n(a, b, q)\}$.

**Proof.** The lemma follows by taking the determinants of the matrices given in (2.1). $\square$

Let $w_n = w_n(a, b, q)$. Suppose that $\{w_n\}$ is residue complete modulo $m$. Then $0 \in \{w_n\}$ modulo $m$. It follows that $w(a, b, q, m) = d \cdot w(0, 1, q, m)$ for some $d$. Applying Lemma 2.6, $a^2 + qab - b^2 \equiv \pm d^2$ modulo $m$. Since $1 \in d \cdot w(0, 1, q, m)$, we conclude that $\gcd(d, m) = 1$. It follows that $\gcd(a^2 + qab - b^2, m) = 1$. In summary, the following is true.

**Lemma 2.7.** The recurrence $\{w_n(a, b, q)\}$ modulo $m$ is residue complete if and only if $w(a, b, q, m) = d \cdot w(0, 1, q, m)$, $\gcd(a^2 + qab - b^2, m) = 1$ and $\{w_n(0, 1, q)\}$ modulo $m$ is residue complete. Suppose that $\{w_n(a, b, q)\}$ is residue complete modulo $m$. Then $\{w_n(a, b, q)\}$ is residue complete modulo $m$ for any $r \mid m$.

The following lemmas give some basic results about $\{w_r(a, b, q)\}$ when $p$ is a divisor $q^2 + 4$.

**Lemma 2.8.** Let $p$ be a prime divisor of $q^2 + 4$. Then $\{w(a, b, q)\}$ modulo $p$ is residue complete if and only if $\gcd(a^2 + qab - b^2, p) = 1$.

**Proof.** See Bumby [2], Somer [8], and Webb and Long [9]. $\square$

**Lemma 2.9.** Let $p$ be an odd prime. Suppose that $p \mid (q^2 + 4)$ and $\gcd(a^2 + qab - b^2, p) = 1$. Then the length of $w(a, b, q, p)$ is $4p$. 2
Proof. Apply results of Wyler [10] (see Lemma 2.2 and 2.4 of [6] also).

3. \( p \in \Omega \)

The main purpose of this section is to study \( \{w_r(a, b, q)\} \) modulo \( p \), where \( p \) is a divisor of \( q^2 + 4 \). Note that the proof of Lemma 3.1 can be applied to prove Lemma 4.3.

**Lemma 3.1.** Let \( p \) be an odd prime. Suppose that \( p|\gcd(q^2 + 4, m) \). Suppose further that \( k(pm) = pk(m) \) and \( \{w_r\} \) modulo \( m \) is residue complete. Then \( \{w_r\} \) modulo \( pm \) is residue complete.

**Proof.** Set \( k(m) = k \) and \( a^2 + qab - b^2 = D \). Since \( \{w_r\} \) modulo \( m \) is residue complete, \( \gcd(a^2 + qab - b^2, m) = 1 \) and for each \( A \in \mathbb{Z}_m \), \( w_n \equiv A \) modulo \( m \) for some \( n \). Since the length of a period of \( \{w_r\} \) modulo \( m \) is \( k \), one has \( w_n \equiv w_{n+k} \equiv \cdots \equiv w_{n+(p-1)k} \equiv A \) modulo \( m \). Hence

\[
\{w_n, w_{n+k}, \ldots, w_{n+(p-1)k}\} \equiv \{A + im : 0 \leq i \leq p-1\} \pmod{pm}. \tag{3.1}
\]

Set \( w_{n+1} \equiv B \pmod{m} \). Then

\[
\{w_{n+1}, w_{n+k+1}, \ldots, w_{n+(p-1)k+1}\} \equiv \{B + jm : 0 \leq j \leq p-1\} \pmod{pm}. \tag{3.2}
\]

Our goal is to show that members in (3.1) are distinct from one another modulo \( pm \). Suppose that two members in (3.1) are equal to each other modulo \( pm \). Without loss of generality, \( w_n \equiv w_{n+(p-1)k} \pmod{pm} \). Then \( w_n \equiv w_{n+(p-1)k} \equiv A + im \) modulo \( pm \) for some \( i \leq p-1 \). Since \( \pm D = \pm(a^2 + qab - b^2) \) is an invariant of \( \{w_r(a, b, q)\} \) (Lemma 2.6),

\[
w_n^2 - qw_{n+1}w_n - w_{n+1}^2 = \pm D, \quad w_{n+(p-1)k}^2 - qw_{n+(p-1)k+1}w_{n+(p-1)k} - w_{n+(p-1)k+1}^2 = \pm D. \tag{3.3}
\]

Since \( w_n \equiv w_{n+(p-1)k} \equiv A + im \) modulo \( pm \), equations in (3.3) take the following alternative forms modulo \( pm \)

\[
(A + im)^2 + qw_{n+1}(A + im) - w_{n+1}^2 \equiv \pm D, \quad (A + im)^2 + qw_{n+(p-1)k+1}(A + im) - w_{n+(p-1)k+1}^2 \equiv \pm D.
\]

It follows that \( Y = w_{n+1} \) and \( w_{n+(p-1)k+1} \) are solutions of the following equation modulo \( pm \)

\[
(A + im)^2 + qY(A + im) - Y^2 \equiv \pm D. \tag{3.4}
\]

Since \( w_{n+1} \) and \( w_{n+(p-1)k+1} \) are members in (3.2), they take the form \( B + mj \). Hence the \( j \)'s associated with \( w_{n+1} \) and \( w_{n+(p-1)k+1} \) are solutions for \( y \) of the following equation modulo \( pm \)

\[
(A + im)^2 + q(B + ym)(A + im) - (B + ym)^2 \equiv \pm D. \tag{3.5}
\]

Note that

\[
A^2 + qAB - B^2 \equiv \pm D \pmod{m}.
\]

This implies that \( A^2 + qAB - B^2 = mT \pm D \). An easy calculation shows that the left hand side of (3.5) takes the following form.

\[
L = m^2(-y^2 + qiy + i^2) - m(y(2B - qA) + mi(qB + 2A)) + mT \pm D. \tag{3.6}
\]

Equation (3.6) implies that (3.5) holds if and only if \( L \) is congruent to \( \pm D \) modulo \( pm \). Since \( p|m \) and \( \gcd(D, m) = 1 \), it is equivalent to

\[
-y(2B - qA) + i(qB + 2A) + T \equiv 0 \pmod{p}. \tag{3.7}
\]

However, \( 2B - qA \not\equiv 0 \pmod{p} \) since otherwise \( \pm D \equiv A^2 + qAB - B^2 \equiv 4^{-1}(q^2 + 4)A^2 \equiv 0 \pmod{p} \). A contradiction (\( p \) is a divisor of \( q^2 + 4 \)). Similarly, \( qB + 2A \not\equiv 0 \pmod{p} \). As a consequence, for each \( i \), there exists exactly one \( y = j \) such that (3.7) (as well as (3.5)) is true. Hence for each \( A + im \), there is a unique \( Y \) of the form \( B + ym \) such that (3.4) is true. Hence \( w_{n+1} \equiv w_{n+(p-1)k+1} \pmod{pm} \). This implies that \( \{w_n, w_{n+1} \equiv (w_{n+(p-1)kJ}, w_{n+(p-1)k+1}) \pmod{pm} \). In particular, the length of a period of \( \{w_r\} \) modulo \( pm \) is at most \((p-1)k \). This
is a contradiction. Hence the $p$ members in (3.1) are all distinct from one another modulo $pm$. Since $k(pm) = pk(m)$, $\{w_r\}$ modulo $m$ is residue complete and every member in $\{w_r\}$ modulo $m$ has $p$ pre-images in $\{w_r\}$ modulo $pm$, we conclude that $\{w_r\}$ modulo $pm$ is residue complete.

**Lemma 4.2.** Let $p$ be an odd prime. Suppose that $p|(q^2 + 4)$ and $\gcd(p, m) = 1$. Suppose further that $k(pm) = pk(m)$ and $\{w_r = w_r(a, b, q)\}$ modulo $m$ is residue complete. Then $\{w_r = w_r(a, b, q)\}$ modulo $pm$ is residue complete.

**Proof.** By Lemma 2.7, we may assume $\{w_r\} = \{w_r(0,1,q)\}$. Set $k(m) = k$. Since $pk(m) = k(pm) = k(p)k/\gcd(k(p), k)$ and $k(p) = 4p$ (Lemma 2.3 and 2.9), we conclude that $\gcd(k, 4p) = 4$. Since $\{w_n\}$ modulo $m$ is residue complete, for each $A \in \mathbb{Z}_m$, there exists some $w_n$ such that $A \equiv w_n$ modulo $m$. We now consider the set $X = \{w_n, w_{n+k}, w_{n+2k}, \ldots, w_{n+(p-1)k}\}$. Since the length of a period of $\{w_n\}$ modulo $m$ is $k$, $w_{n+rk} \equiv A \pmod m$ for $r = 0, 1, 2, \ldots, p-1$. We now consider $X$ modulo $p$. Since

$$\gcd(k, 4p) = 4, k(p) = 4p,$$

and $w_{y+4p} \equiv w_y \pmod p$ for all $y$, the set $X$ modulo $p$ takes the form $X = \{w_x, w_{x+4}, w_{x+8}, w_{x+12}, \ldots, w_{x+4(p-1)}\}$ for some $x$ in the range $0 \leq x \leq 3$. By our results in Appendix A, $X$ modulo $p$ is $\{0, 1, 2, \ldots, p-1\} = \mathbb{Z}_p$. As a consequence, members in $X$ are not congruent to one another modulo $pm$. In particular, every member in $\{w_r\}$ modulo $m$ has $p$ pre-images in $\{w_r\}$ modulo $pm$. Since $k(pm) = pk(m)$, $\{w_n\}$ modulo $pm$ is residue complete. \hfill \Box

4. $p \in \Delta$

$\Delta = \{2, 3, 5, 7\}$. We shall study $\{w_r\}$ modulo $p^e \ (p \in \Delta)$ in the following subsections. The main results can be found in lemmas 4.1-4.4.

**4.1.** We study $\{w_r\}$ modulo $2^e$. Note that in the case $q$ is even 2 is also a divisor of $q^2 + 4$.

**Lemma 4.1.** Let $w_n = w_n(0,1,q)$ be given as in (1.1). Then the following holds.

(i) Suppose that $q$ is odd. Then $\{w_n\}$ modulo 4 is residue complete and $\{w_n\}$ modulo $2^n$ is not residue complete for all $n \geq 3$.

(ii) Suppose that $q$ is even. Then $\{w_n\}$ modulo 2 is residue complete and $\{w_n\}$ modulo $2^n$ is not residue complete for all $n \geq 2$.

**Proof.** The lemma follows easily from (iii) and (iv) of the following, which can be verified by direct calculation. (iii) Suppose that $q$ is odd. Then $\{w_n\}$ modulo 4 is residue complete and $\{w_n\}$ modulo 8 is not residue complete. (iv) Suppose that $q$ is even. Then $\{w_n\}$ modulo 2 is residue complete and $\{w_n\}$ modulo 4 is not residue complete. \hfill \Box

**4.2.** We study $\{w_r\}$ modulo $3^e$. Note that 3 cannot be a divisor of $q^2 + 4$. The idea of the proof of Lemma 4.2 is mainly taken from Burr [3].

**Lemma 4.2.** If $3|q$, then $\{w_n\}$ modulo $3^n$ is not residue complete. Suppose $\gcd(a^2 + qab - b^2, 3) = \gcd(q, 3) = 1$. In the case $q \equiv 4,5 \pmod 9$, $w(a, b, q, 3^n)$ is residue complete if and only if $n = 1$. In the case $q \not\equiv 4, 5 \pmod 9$, $w(a, b, q, 3^n)$ is residue complete for all $n \geq 1$.

**Proof.** The first part of the lemma is clear. In the case $q \equiv 4, 5 \pmod 9$, by (iv) of Lemma 2.3, $w(a, b, q, 9)$ is not residue complete. We now assume $\gcd(q(a^2 + qab - b^2), 3) = 1$ and $q \not\equiv 4, 5 \pmod 9$. We shall first prove that $w(0,1,q,m)$ is residue complete. By Lemma 2.6, $\pm(a^2 + qab - b^2)$ is an invariant for $w(a, b, q, m)$. Since the invariant for $w(0,1,q,m)$ is $\pm 1$, our assertion is proved if the following two facts are verified.

(i) The fundamental system $FS(m)$ has only one period with invariant $\pm 1$ modulo $m$.

(ii) For any $a \in \mathbb{Z}_m$, there exists some $b$ such that $a^2 + qab - b^2 \equiv 1$ or $-1$ modulo $m$. 

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Proof of (i) for $m = 3^n$. Applying Lemma 2.3, the period $C = w(0, 1, q, 3^{n})$ has length $8 \cdot 3^{n-1}$. For each $r_i$ prime to 3 in the range $1 \leq r_i \leq 3^n/2$, define $C_i = \{ r_i x : x \in C \}$. The $C_i$’s are not equivalent to each other as their invariants $\pm r_i^2$ are distinct from each other modulo $m$. For each $D \in FS(3^{n-1})$, define $3 \cdot D = \{ 3d : d \in D \}$. They clearly form a subset of inequivalent periods for $FS(3^n)$. Let $\phi(x)$ be the Euler function. The total number of terms in $\{ C_i : 1 \leq i \leq 3^n/2 \} \cup \{ 3 \cdot D : D \in FS(3^{n-1}) \}$ is $8 \cdot 3^{n-1} \phi(3^n)/2 + 3^{2(n-1)} = 3^{2n} - 1$. Hence

$$FS(3^n) = \{ C_i : 1 \leq i \leq 3^n/2 \} \cup \{ 3 \cdot D : D \in FS(3^{n-1}) \}. \tag{4.1}$$

Since $x^2 \equiv 1 \pmod{3^n}$ if and only if $x \equiv \pm 1 \pmod{3^n}$, we conclude that the only period in (4.1) with invariant $\pm 1$ modulo $3^n$ is $C_1 = C = w(0, 1, q, 3^n)$.

Proof of (ii) for $m = 3^n$. We consider two cases: (i) $q$ is a multiple of 3, (ii) $\gcd(q, 3) = 1$.

In the first case, one sees easily that $a^2 + qax - x^2 \equiv -1 \pmod{3^n}$ is solvable. In the second case, $a^2 + qax - x^2 \equiv 1 \pmod{3^n}$ is solvable.

For any $a \in \mathbb{Z}_m$, by (ii), the invariant of $w(a, b, q, m)$ is $\pm 1$ for some $b$. By (i), $w(a, b, q, m) = w(0, 1, q, m)$. In particular, $a \in w(0, 1, q, m)$. Hence $w(0, 1, q, m)$ is residue complete. Consequently, all the $C_i$’s are residue complete. This completes the proof of the lemma. □

4.3. We study $\{ w_r \}$ modulo $5^a$. Note that 5 can be a divisor of $q^2 + 4$.

Lemma 4.3. Suppose that 5 is not a divisor of $q(q^2 + 4)$, $5|m$, and $a^2 + qab - b^2 = D \equiv \pm 1 \pmod{5}$. Suppose further that $k(5m) = 5k(m)$ and $\{ w_r \}$ modulo $m$ is residue complete. Then $\{ w_r \}$ modulo $5m$ is residue complete.

Proof. Our lemma can be proved by applying the proof of Lemma 3.1 step by step. The only difference is how the fact $2B - qA \not\equiv 0 \pmod{p}$ is being verified (see (3.7)). In Lemma 3.1, one has $2B - qA \equiv 0 \pmod{p}$ since otherwise $\pm D \equiv A^2 + qAB - B^2 \equiv 4^1(q^2 + 4)A^2 \equiv 0 \pmod{p}$, while in the present proof, one has $2B - qA \not\equiv 0 \pmod{5}$ since otherwise $\pm 1 \equiv \pm D \equiv A^2 + qAB - B^2 \equiv 4^{-1}(q^2 + 4)A^2 \pmod{5}$, where $\gcd(q, 5) = 1$. A contradiction. □

4.4 Throughout the section $w_n = w_n(0, 1, q)$. The purpose of this section is to study $\{ w_r \}$ modulo $7^a$. Note that 7 cannot be a divisor of $q^2 + 4$. The following is clear.

(i) $q^2 + 4$ is not a square modulo 7 if and only if $\{ w_r \}$ modulo 7 is residue complete.

(ii) $\{ w_r \}$ modulo 49 is not residue complete.

Note that in the first case $q \equiv 1, 3, 4, 6 \pmod{7}$. In summary, the following holds.

Lemma 4.4. The recurrence $\{ w_r = w_r(0, 1, q) \}$ modulo $7^a$ is residue complete if and only if $n = 1$ and $q \equiv 1, 3, 4, 6 \pmod{7}$.

5. The Main Results

Proposition 5.1. Pell numbers modulo $m$ is residue complete if and only if $m \in \{ 2a, 3b, 5^c \}$.

Proof. Recall first that $P_n = w_n(0, 1, 2)$. Suppose that $\{ P_n \}$ modulo $m$ is residue complete. By (1.2), the possible prime divisors of $m$ are 2, 3, 5, or 7. Direct calculation shows that $w(0, 1, 2, 7)$ is not residue complete. Hence $m$ is of the form $2a3^b5^c$ (Lemma 2.7). Direct calculation shows that $\{ P_n \}$ modulo $m$ is not residue complete if $m = 6, 10, 15$. Hence $m \in \{ 2a, 3b, 5^c \}$ (Lemma 2.7).

Direct calculation shows that $\{ w(0, 1, 2) \}$ modulo 5 is residue complete. By Lemma 4.3 and 2.3, $\{ w(0, 1, 2) \}$ modulo $5^c$ is residue complete. The proposition now can be proved by applying Lemma 4.1 and 4.2. □

Recall that Pell-Lucas numbers is given by $w(2, 2, 2)$. Similar to the above, the following can be proved as well.

Proposition 5.2. Pell-Lucas numbers modulo $m$ is residue complete if and only if $m = 3^n$. 5
Discussion 5.3. Suppose that \( \{w_n(a, b, q)\} \) modulo \( m \) is residue complete. Then \( \{w_n(0, 1, q)\} \) modulo \( m \) is residue complete (Lemma 2.7). By (1.2), the set of prime divisors of \( m \) is a finite set and \( m \) is of the form \( 2^a 3^b 5^c 7^d \prod p_i^{e_i} \), where \( p_i | (q^2 + 4) \). By Lemma 4.1, one has \( a \leq 2 \). By Lemma 4.4, one has \( d \leq 1 \). Similar to Proposition 5.1, one may apply lemmas 3.1, 3.2, 4.1-4.4 and some easy calculation to determine completely the set of such \( m \)'s (see Example 5.4). Note that the prime 5 is different from the other primes. We apply Lemma 3.1 and 3.2 if 5 is a divisor of \( q^2 + 4 \), and apply Lemma 4.3 if 5 is not a divisor of \( q^2 + 4 \).

In particular, results of Burr [3] for Fibonacci numbers \( (q^2 + 1 = 5) \) and results of Avila and Chen [1] for Lucas numbers \( (q^2 + 4 = 8) \) can be obtained by our method. It is our duty to point out that although our result is more general than Burr [3], the main idea is mainly taken from Burr [3].

Example 5.4. Define \( \{w_r\} \) by \( w_0 = 0, w_1 = 1, w_n = 3w_{n-1} + w_{n-2} \). Then \( q^2 + 4 = 13 \). A period of \( \{w_r\} \) modulo 13 has length 52 and is given by

\[
(0, 1, 3, 10, 7, 5, 9, 6, 1, 9, 2, 2, 8, 0, 8, 11, 2, 4, 1, 7, 9, 8, 7, 3, 3, 12, \\
0, 12, 10, 6, 8, 4, 7, 12, 4, 11, 11, 5, 0, 5, 2, 11, 9, 12, 6, 4, 5, 6, 10, 10, 1). 
\]

Suppose that \( \{w_r\} \) modulo \( m \) is residue complete. By (1.2), \( m \) takes the form \( 2^a 3^b 5^c 7^d 13^e \). It is clear that \( \{w_r = w_r(0, 1, 3)\} \) modulo 3 is not residue complete. Hence \( m \) takes the form \( 2^a 5^c 7^d 13^e \). By Lemma 4.1, \( a \leq 2 \). By Lemma 4.4, \( d \leq 1 \). Further,

(i) \( \{w_r\} \) modulo 4 (Lemma 4.1), \( 5^c \) (Lemma 4.3), 7 (Lemma 4.4), \( 13^e \) (Lemma 3.1) are residue complete.

Direct calculation shows that \( k(2) = 3, k(4) = 6, k(5^n) = 12 \cdot 5^n - 1, k(7) = 16, k(13^n) = 4 \cdot 13^n \).

(ii) \( \{w_r\} \) modulo 10, 28, 35 are not residue complete.

(iii) \( \{w_r\} \) modulo 14, 52 are residue complete.

Applying Lemma 3.1 and 3.2, we conclude that \( \{w_r\} \) modulo \( m \) is residue complete if and only if

\[
m \in \{13^b, 2 \cdot 13^b, 4 \cdot 13^b, 5^a 13^b, 7 \cdot 13^b, 14 \cdot 13^b\}. 
\]

6. Appendix A. Residue Completeness of Subsequences of \( \{w_r(0, 1, q)\} \)

Set \( F_0 = 0, F_1 = 1, F_n = qF_{n-1} + F_{n-2} \) and \( L_0 = 2, L_1 = q, L_n = qL_{n-1} + L_{n-2} \). One can show easily that

\[
F_{n+4} = L_4 F_n - F_{n-4}. 
\]

Define four recurrences \( A_n = F_{4n}, B_n = F_{4n+1}, C_n = F_{4n+2}, \) and \( D_n = F_{4n+3} \). It is clear all four recurrences satisfy the same recurrence

\[
w_n = L_4 w_{n-1} - w_{n-2}. 
\]

Let \( p \) be an odd prime divisor of \( q^2 + 4 \). Applying Lemma 2.8 and 2.9, \( \{F_n\} \) modulo \( p \) is residue complete and a period of \( \{F_n\} \) modulo \( p \) has length \( 4p \). Since \( p \) is a divisor of \( q^2 + 4 \), \( L_4 \equiv 2 \pmod{p} \). Hence (A.2) takes the following form.

\[
w_n = 2w_{n-1} - w_{n-2} \pmod{p}. 
\]

Consider \( \{A_n\} \) modulo \( p \). Apply the recurrence relation (A.3), our sequence modulo \( p \) takes the following form.

\[
A_0 = 0, A_1 \equiv -2q, A_2 \equiv -4q, \cdots, A_n \equiv -2nq, \cdots. 
\]

It is clear that \( \{A_n\} = \{F_0, F_4, F_8, \cdots, F_{4(p-1)}\} \equiv Z_p \pmod{p} \). Similarly, one can show \( \{B_n\} = \{C_n\} = \{D_n\} = Z_p \pmod{p} \).
7. APPENDIX B. \( u_n = qu_{n-1} - u_{n-2} \)

Let \( u_0 = a, u_1 = b, u_n = qu_{n-1} - u_{n-2} \). Similar to (2.1), one has the following matrix form.

\[
\begin{bmatrix}
q & 1 \\
-1 & 0
\end{bmatrix}, \quad \begin{bmatrix}
qb - a & b \\
b & a
\end{bmatrix}
\]

\[
\tau^n = \begin{bmatrix}
\ u_{n+2} & \ u_{n+1} \\
\ u_{n+1} & \ u_{n}
\end{bmatrix}. \tag{B.1}
\]

We shall first give an alternative proof of the following fact (see lemmas 2.3 and 2.4 of [6]).

**Lemma B1.** Let \( p \) be an odd prime and let \( k(p) \) be the length of a period of \( \{u_n\} \) modulo \( p \). Suppose that \( \gcd(p, -a^2 + qab - b^2) = 1 \). Then (i) \( k(p) \) is the order of \( \tau \) modulo \( p \); (ii) \( k(p) | (p + 1) \) if \( x^2 - qx + 1 \equiv 0 \) has two distinct roots in \( \mathbb{Z}_p \).

**Proof.** Since \( \gcd(p, -a^2 + qab - b^2) = 1, k(p) \) is the order of \( \tau \) modulo \( p \). Note that \( \tau \) is similar to a diagonal matrix \( \text{diag}(\lambda_1, \lambda_2) \), where the \( \lambda_i \)'s are the eigenvalues of \( \tau \). In the case \( x^2 - qx + 1 \equiv 0 \) has two distinct roots in \( \mathbb{Z}_p \), \( \lambda_i \in \mathbb{Z}_p^\times \). Since \( \mathbb{Z}_p^\times \) is of order \( p - 1 \), it follows that \( \tau \) is a divisor of \( p - 1 \). In the case \( x^2 - qx + 1 \) is irreducible over \( \mathbb{Z}_p \), since the norm of the \( \lambda_i \)'s is 1, the \( \lambda_i \)'s are in the kernel of the surjective norm map \( N : GF(p^2)^\times \to \mathbb{Z}_p^\times \), where \( GF(p^2) \) is a finite field of \( p^2 \) elements (see Chaps VIII, Ex 28 of [4]). Since the kernel of \( N \) is a group of order \( p + 1 \), the order of \( \tau \) is a divisor of \( p + 1 \).

It is clear that periods of length \( p - 1 \) or \( p + 1 \) cannot be residue complete. Hence \( \{u_n\} \) modulo \( p \) is residue complete only if \( x^2 - qx + 1 = 0 \) has repeated roots. Equivalently, \( p \) is a divisor of \( q^2 - 4 \). Let \( P \) be the set of all prime divisors of \( q^2 - 4 \). The above argument implies that \( \{u_n\} \) modulo \( m \) is residue complete only if \( m \) is of the form

\[
\prod_{p_i \in P} p_i^{e_i}.
\]

The sequence \( \{u_n\} \) is said to be uniformly distributed modulo \( m \) if every element \( a \) of \( \mathbb{Z}_m \) occurs in a period of \( \{u_n\} \) equally often. We now state without proof the main result of Bumby [2] (for the sequence \( \{u_n\} \)).

**Theorem B2.** (Bumby [2]) The sequence \( \{u_n\} \) is uniformly distributed modulo \( m \) if and only if it is uniformly distributed modulo all prime powers of \( m \). The sequence \( \{u_n\} \) modulo \( p \) (a prime) is uniformly distributed if and only if

\[
\begin{align*}
(i) \quad & \gcd(p, b - qa/2) = 1 \quad (i f q = \pm 2), \\
(ii) \quad & \gcd(p, \alpha(b - \alpha a)) = 1, \quad \gcd(p, \alpha - \alpha') \neq 1 \quad (i f q \neq \pm 2),
\end{align*}
\]

where \( \alpha \) and \( \alpha' \) are roots of \( x^2 - qx + 1 = 0 \) in \( \mathbb{C} \). Equivalently, \( p | (q^2 - 4) \) and \( \gcd(p, -a^2 + qab - b^2) = 1 \). Further, if \( \{u_n\} \) modulo \( p \) is uniformly distributed, then \( \{u_n\} \) is uniformly distributed modulo \( p^h \) with \( h > 1 \) if and only if \( (a) \ p > 3, \ (b) \ p = 3, \ and \ q^2 \neq 1 \pmod{9}, \ or \ (c) \ p = 2, \ q \equiv 2 \pmod{4} \).

In case (b) of the theorem, one can show easily that \( \{u_n\} \) modulo 9 is not residue complete if \( q^2 \equiv 1 \pmod{9} \). In case (c) of the theorem, one can show easily that \( \{u_n\} \) modulo 4 is not residue complete if \( q \equiv \pm 1 \pmod{4} \). As a consequence, we have the following corollary of Bumby’s Theorem.

**Corollary B2.** (Bumby [2]) The sequence \( \{u_n\} \) is residue complete modulo \( m \) if and only if it is residue complete modulo all prime powers of \( m \). The sequence \( \{u_n\} \) modulo \( p \) (a prime) is residue complete if and only if \( p | (q^2 - 4) \) and \( \gcd(p, -a^2 + qab - b^2) = 1 \). Further, if \( \{u_n\} \) modulo \( p \) is residue complete, then \( \{u_n\} \) is residue complete modulo \( p^h \) with \( h > 1 \) if and only if \( (a) \ p > 3, \ (b) \ p = 3, \ and \ q^2 \neq 1 \pmod{9}, \ or \ (c) \ p = 2, \ q \equiv 2 \pmod{4} \).
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