An optimal system and invariant solutions of dark energy Models in cylindrically symmetric space-time

Anil Kumar Yadav¹ and Ahmad T Ali²,³

¹ Department of Physics, Galgotias College of Engineering and Technology, Knowledge Park - II, Greater Noida - 201306, India.
E-mail: abanilyadav@yahoo.co.in

² King Abdul Aziz University, Faculty of Science, Department of Mathematics, PO Box 80203, Jeddah, 21589, Saudi Arabia.
E-mail: atali71@yahoo.com

³ Mathematics Department, Faculty of Science, Al-Azhar University, Nasr city, 11884, Cairo, Egypt

Abstract

In this paper, we derive some new invariant solutions of dark energy models in cylindrically symmetric space-time. To quantify the deviation of pressure from isotropy, we introduce three different time dependent skewness parameters along the spatial directions. The matter source consists of dark energy which is minimally interact with perfect fluid. We use symmetry analysis method for solving the non-linear partial differential equations (NLPDEs) which is more powerful than the classical methods of solving NLPDEs. The geometrical and kinematical features of the models and the behaviour of the anisotropy of dark energy, are examined in detail.

PACS: 98.80.JK, 98.80.-k.

Keywords: Optimal system, Invariant solutions, Dark Energy, General Relativity.

1 Introduction

The most striking discovery of the modern cosmology is that the expansion of universe is accelerating at late time which is confirmed by the high red-shift supernovae experiments (Riess et al [1]; Perlmutter et al [2]; Bennet et al [3]; Padmanabham [4]). Subsequent observations including more detailed studied of supernovae and independent evidence from cluster of galaxies, LSS and CMBR confirmed and firmly established this remarkable findings. The recent observations also suggest that dark energy is contributing about 73 % of the total energy of the universe (Spregel et al [5]). There are various directions aimed to construct the viable dark energy models such as scalar field models, dark fluid with complicated equation of state parameter. However, the satisfactory explanation of dark energy origin is still unknown.

Many natural phenomena are described by a system of NLPDEs which is often difficult to be solved analytically, since there is no general theory for completely solving NLPDEs. One of the most useful techniques for finding exact solutions for the Einstein field equations described by a system of NLPDEs is the symmetry method [6, 7]. The existence of symmetries of differential equations under Lie group of transformations often allows those equations to be reduced to simpler equations. One of the major accomplishment of Lie was to identify that the properties of global transformations of the group are completely and uniquely determined by the infinitesimal transformations around the identity transformation. This allows the non-linear relations for the identification of invariance groups to be dealing with global transformation equations, we use differential operators, called the group generators, whose exponentiation generates the action of the group. The collection of these differential operators forms the basis for the Lie algebra. There is a one-to-one correspondence between the Lie groups and the associated Lie algebras. A basic problem concerning the group invariant solution is its classification. Since a Lie group usually contain infinitely many subgroups of the same dimensional, a classification of them up to some equivalence relation is necessary. Ovsianikov [8] given equivalent of two subalgebras of a given Lie algebra. Optimal system consists of representative elements of each equality class. Discussion on optimal systems can be found in ref. [9]. Also Ibragimov [10], in his paper has given some examples of optimal system.

The study of universe on the scale in which anisotropy and inhomogeneity are not ignored, cylindrically symmetric cosmological models play an important role. It has a significant contribution in understanding some essential features of the universe such as the formation of galaxies during the early stage of evolution.
of universe. Also the case of cylindrically symmetry is natural because of mathematical simplicity of field equations whenever there exists a direction in pressure and energy density are being equal. In the literature, Senovilla [11] obtained exact solution of Einstein’s field equation in cylindrically symmetric space-time. These are singularity free cosmological model and satisfying energy and causality condition. Later on, Davinich et AL [12] have established a link between the FRW model and the singularity free models. In our previous paper [9, 7], we have also investigated the new class of exact solution of NLP DEs by symmetry group analysis method in inhomogeneous space-time.

In this paper, we find one-dimensional optimal system for the Einstein field equations for the dark energy model and classify reductions obtained by using one-dimensional subalgebras. The paper is organized as follows: The metric and field equations are presented in section 2. Section 3 and 4 deal with the symmetry analysis method and optimal system respectively. The invariant solution of field equations are given in section 5. Section 6 deals with the physical and geometrical properties of the models. Finally the results are discussed in section 7.

2 The metric and field equations

The space-time is given by

$$ds^2 = A^2(dx^2 - dt^2) + B^2dy^2 + C^2dz^2$$

(1)

The metric potentials $A$, $B$ and $C$ are function of $x$ and $t$ The Einstein’s field equation is given by

$$R^i_j - \frac{1}{2}g^i_j R = -T^{(m)i}_j - T^{(de)i}_j$$

(2)

where $T^{(m)i}_j$ and $T^{(de)i}_j$ are energy momentum tensor of perfect fluid and dark energy respectively. These are given by

$$T^{(m)i}_j = \text{diag}[-\rho^{(m)}, p^{(m)}] , p^{(m)} = \frac{m}{2A^2}$$

(3)

and

$$T^{(de)i}_j = \text{diag}[-\rho^{(de)}, p^{(de)}] , p^{(de)} = \frac{m}{2A^2}$$

(4)

where $\rho^{(m)}$ and $\rho^{(de)}$ are, respectively the pressure and energy density of the perfect fluid component; $\rho^{(de)}$ is the energy density of the DE components; $\delta(t)$, $\gamma(t)$ and $\eta(t)$ are skewness parameters along $x$, $y$ and $z$ axis respectively, which modify equation of state parameter of dark energy.

In comoving coordinate system, the field equation [2], for the inhomogeneous space-time [11], read as

$$\frac{1}{A^2} \left[ \frac{\dot{B}}{B} - \frac{\dot{C}}{C} + \frac{\dot{A}}{A} \left( \frac{B'}{B} + \frac{C'}{C} \right) + A^2 \left( \frac{B''}{B} + \frac{C''}{C} \right) + \frac{B' C'}{BC} - \frac{B \dot{C}}{BC} \right] = p^{(m)} + (\omega + \delta) \rho^{(de)}$$

(5)

$$\frac{1}{A^2} \left[ -\frac{\ddot{A}}{A} + \frac{\dot{A}^2}{A^2} + \frac{A''}{A} - \frac{A'^2}{A^2} - \frac{\dot{C}}{C} + \frac{C''}{C} \right] = p^{(m)} + (\omega + \gamma) \rho^{(de)}$$

(6)

$$\frac{1}{A^2} \left[ -\frac{\ddot{A}}{A} + \frac{\dot{A}^2}{A^2} + \frac{A''}{A} - \frac{A'^2}{A^2} - \frac{\dot{B}}{B} + \frac{B''}{B} \right] = p^{(m)} + (\omega + \eta) \rho^{(de)}$$

(7)

$$\frac{1}{A^2} \left[ \frac{B''}{B} - \frac{C''}{C} + \frac{A'}{A} \left( \frac{B'}{B} + \frac{C'}{C} \right) + \frac{\dot{A}}{A} \left( \frac{B'}{B} + \frac{C'}{C} \right) - \frac{B' C'}{BC} + \frac{\dot{B}}{B} \frac{\dot{C}}{C} - \frac{A'}{A} \left( \frac{B'}{B} + \frac{C'}{C} \right) \right] = \rho^{(m)} + \rho^{(de)}$$

(8)

Here $A' = \frac{dA}{dt}$, $\dot{A} = \frac{dA}{dt}$ and so on.

The velocity field $u^i$ is ir-rotational. The scalar expansion $\Theta$, shear scalar $\sigma^2$, acceleration vector $\dot{u}_i$ and proper volume $V$ are respectively found from the following expressions [13, 14]:

$$\Theta = u^i_{;i} = \frac{1}{A} \left( \frac{C_i}{C} + \frac{B_i}{B} + \frac{A_i}{A} \right)$$

(10)
\[ \sigma^2 = \frac{1}{2} \sigma_{ij} \sigma^{ij} = \frac{\Theta^2}{3} - \frac{1}{A^2} \left( \frac{B_t C_t}{BC} + \frac{A_t C_t}{AC} + \frac{A_t B_t}{AB} \right), \]  
\hfill (11)

\[ \dot{u}_i = u_{ij} w^j = \left( \frac{A_x}{A}, 0, 0, 0 \right), \]  
\hfill (12)

\[ V = \sqrt{-g} = A^2 BC, \]  
\hfill (13)

where \( g \) is the determinant of the metric. The shear tensor is

\[ \sigma_{ij} = u_{i(j)} + u_{(i} u_{j)} - \frac{1}{3} \Theta (g_{ij} + u_i u_j). \]  
\hfill (14)

and the non-vanishing components of the \( \sigma^i_j \) are

\[
\begin{align*}
\sigma_1^1 &= \frac{1}{3} A \left( \frac{2 A_t}{A} - \frac{B_t}{B} - \frac{C_t}{C} \right), \\
\sigma_2^2 &= \frac{1}{3} A \left( \frac{2 B_t}{B} - \frac{C_t}{C} + \frac{A_t}{A} \right), \\
\sigma_3^3 &= \frac{1}{3} A \left( \frac{2 C_t}{C} - \frac{B_t}{B} + \frac{A_t}{A} \right), \\
\sigma_4^4 &= 0.
\end{align*}
\hfill (15)
\]

The Einstein field equations (12) - (16) constitute a system of five highly NLPDEs with six unknowns variables, \( A, B, C, p^{(m)}, \rho^{(m)} \) and \( \rho^{(de)} \). Therefore, one physically reasonable conditions amongst these parameters are required to obtain explicit solutions of the field equations. Let us assume that the expansion scalar \( \Theta \) in the model (10) is proportional to the eigenvalue \( \sigma_1 \) of the shear tensor \( \sigma^k_k \). Then from (10) and (15), we get

\[ \frac{2 A_t}{A} - \frac{B_t}{B} - \frac{C_t}{C} = 3 \gamma \left( \frac{A_t}{A} + \frac{B_t}{B} + \frac{C_t}{C} \right), \]  
\hfill (16)

where \( \gamma \) is a constant of proportionality. The above equation can be written in the form

\[ A_t = n \left( \frac{B_t}{B} + \frac{C_t}{C} \right), \]  
\hfill (17)

where \( n = \frac{1 + 3 \gamma}{2 - 3 \gamma} \). If we integrate the above equation with respect to \( t \), we can get the following relation

\[ A(x,t) = f(x) \left( B(x,t) C(x,t) \right)^n, \]  
\hfill (18)

where \( f(x) \) is a constant of integration which is an arbitrary function of \( x \). If we substitute the metric function \( A \) from (12) in the Einstein field equations, the equations (13)-(17) transform to the NLPDEs of the coefficients \( B \) and \( C \) only, as the following new form:

\[
\begin{align*}
E_1 &= (\omega_y - \omega_z) \left[ \left( \frac{f'}{f} \right)' - \frac{f'}{f} \left( \frac{B'}{B} + \frac{C'}{C} \right) - 2 n \left( \frac{B'^2}{B^2} + \frac{C'^2}{C^2} \right) - (2 n + 1) \frac{B' C'}{B C} - (2 n - 1) \frac{B C'}{B C} \right] \\
&\quad + \left( \omega_x - n \omega_y + (n - 1) \omega_z \right) \frac{\dot{B}}{B} - \left( \omega_x + (n - 1) \omega_y - n \omega_z \right) \frac{\dot{C}}{C} \\
&\quad - \left( \omega_x - (n + 1) \omega_y + n \omega_z \right) \frac{B''}{B} + \left( \omega_x + n \omega_y - (n + 1) \omega_z \right) \frac{C''}{C} = 0,
\end{align*}
\hfill (19)

\[ E_2 = \frac{\dot{B}}{B} + \frac{\dot{C}}{C} - \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \left[ 2 n \left( \frac{B'}{B} + \frac{C'}{C} \right) + \frac{f'}{f} \right] = 0, \]  
\hfill (20)
where
\[
\begin{align*}
 f^2 A^2 \rho^{(m)}(x, t) &= \left( \frac{f'}{f} \right)' + n \left( \frac{\dot{B}^2 - B'^2}{B^2} + \frac{\dot{C}^2 - C'^2}{C^2} \right) - (\lambda \omega_y - n) \left( \frac{\ddot{B} - B''}{B} \right) \\
&- (n - \lambda \omega_z) \left( \frac{\ddot{C} - C''}{C} \right),
\end{align*}
\]
and
\[
\begin{align*}
 f^2 A^2 \rho^{(de)}(x, t) &= \lambda \left[ \frac{\ddot{B} - B''}{B} + \frac{C'' - \ddot{C}}{C} \right],
\end{align*}
\]

3 Symmetry analysis method

In order to obtain an exact solutions of the Einstein field equations \([19, 23]\), enough to find a solution to the system of NLPDEs \([19, 20]\). The classical method for finding the solution is a separation method by taking \(B(x, t) = B_1(x) B_2(t)\) and \(C(x, t) = C_1(x) C_2(t)\) \([15, 16, 17]\). The symmetry analysis method is a powerful method which gives an invariant solutions. It is worth noting that: there are three major methods to compute Lie point symmetries. The first one uses prolonged vector fields (Lie group method) which we will use and explain it in details for this work. In the last century, the application of this method has been developed by a number of mathematicians. Ovsiannikov \([8]\), Olver \([9]\), Baumann \([18]\), Bluman, G.W. and Anco \([19]\) are some of the mathematicians who have enormous amount of studies in this field.

The second utilizes differential forms (wedge products) due to Cartan \([20]\). A differential geometric approach to invariance groups and solutions of PDEs was presented by Harrison and Estabrook \([21]\). They showed how to derive infinitesimal symmetries using Cartan’s exterior differential calculus. Edelen developed the theory of the differential form method extensively and wrote some computer programs using this technique \([22, 23]\). Suhubi \([24]\) developed a differential geometric method to find a set of explicit determination equations whose solutions determine the components of the isovector fields. The isovector fields, which are the infinitesimal generators of geometric transformations with suitable algebraic invariance properties, are then used to obtain invariant solutions of several PDEs which written in the balance form. This method is applied for some equations such as: the heat equation \([24]\), the generalized K-dV-Burger type equation \([26]\), the vacuum Maxwell equations \([27]\), the Einstein vacuum equations \([28, 29, 30]\), the Biot’s equations for one-dimensional linear poroelasticity \([31]\).

The third one uses the notation of "formal symmetry" \([32, 33, 34]\). Although restricted to evolution systems with two independent variables, this method provides a very quick way to compute canonical generalized symmetries. Due to its limited scope we will not elaborate on that technique. For the first method, we write

\[
\begin{align*}
 x_i^* &= x_i + \epsilon \xi_i(x, u_\beta) + \mathcal{O}(\epsilon^2), \\
u_\alpha^* &= u_\alpha + \epsilon \eta_\alpha(x, u_\beta) + \mathcal{O}(\epsilon^2), \\
i, j, \alpha, \beta &= 1, 2,
\end{align*}
\]

as the infinitesimal Lie point transformations. We have assumed that the system \([19, 20]\) is invariant under the transformations given in Eq. (24). The corresponding infinitesimal generator of Lie groups (symmetries) is given by

\[
X = \sum_{i=1}^{2} \xi_i \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{2} \eta_\alpha \frac{\partial}{\partial u_\alpha},
\]

where \(x_1 = x, x_2 = t, u_1 = B\) and \(u_2 = C\). The coefficients \(\xi_1, \xi_2, \eta_1\) and \(\eta_2\) are the functions of \(x, t, B\) and \(C\). These coefficients are the components of infinitesimals symmetries corresponding to \(x, t, B\) and \(C\) respectively, to be determined from the invariance conditions:

\[
Pr^{(2)} X \left( E_m \right) \big|_{E_m=0} = 0,
\]
where $E_m = 0$, $m = 1, 2$ are the system \([19-20]\) under study and $\Pr^{(2)}$ is the second prolongation of the symmetries $X$. Since our equations \([19-20]\) are at most of order two, therefore, we need second order prolongation of the infinitesimal generator in Eq. \([20]\). It is worth noting that, the 2-th order prolongation is given by:

$$\Pr^{(2)} X = \sum_{i=1}^{2} \xi_i \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{2} \eta_\alpha \frac{\partial}{\partial u_\alpha} + \sum_{i=1}^{2} \sum_{\alpha=1}^{2} \eta_{\alpha,i} \frac{\partial}{\partial u_{\alpha,i}} + \sum_{j=1}^{2} \sum_{i=1}^{2} \sum_{\alpha=1}^{2} \eta_{\alpha,i,j} \frac{\partial}{\partial u_{\alpha,i,j}}, \tag{27}$$

where

$$\eta_{\alpha,i} = D_i (\eta_\alpha) - \sum_{j=1}^{2} u_{\alpha,j} D_i (\xi_j), \quad \eta_{\alpha,i,j} = D_j (\eta_{\alpha,i}) - \sum_{k=1}^{2} u_{\alpha,k} D_j (\xi_k). \tag{28}$$

The operator $D_i$ is called the total derivative (Hach operator) and taken the following form:

$$D_i = \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{2} u_{\alpha,i} \frac{\partial}{\partial u_\alpha} + \sum_{j=1}^{2} \sum_{\alpha=1}^{2} u_{\alpha,j,i} \frac{\partial}{\partial u_{\alpha,j}}, \tag{29}$$

where $u_{\alpha,i} = \frac{\partial u_\alpha}{\partial x_i}$ and $u_{\alpha,i,j} = \frac{\partial^2 u_\alpha}{\partial x_i \partial x_j}$.

Expanding the system of Eqs. \([20]\) along with the original system of Eqs. \([19-20]\) to eliminate $B_{xx}$ and $B_{xt}$ while we set the coefficients involving $C_x$, $C_t$, $C_{xx}$, $C_{xt}$, $C_{tt}$, $B_x$, $B_t$, $B_{tt}$ and various products to zero give rise the essential set of over-determined equations. Solving the set of these determining equations, the components of symmetries takes the following form:

$$\xi_1 = a_1 x + a_2, \quad \xi_2 = a_1 t + a_3, \quad \eta_1 = a_4 B, \quad \eta_2 = a_5 C, \tag{30}$$

such that the following conditions must be satisfied

$$f(x) = a_6 (a_1 x + a_2)^{a_7} \quad \omega_z = (1 + a_8) \omega_x - a_8 \omega_y, \tag{31}$$

where $a_i$, $i = 1, 2, \ldots, 8$ are an arbitrary constants.

The characteristic equations associated to the general symmetries \([31]\) are given by:

$$\frac{dx}{a_1 x + a_2} = \frac{dt}{a_1 t + a_3} = \frac{dB}{a_4 B} = \frac{dC}{a_5 C}. \tag{32}$$

4 Optimal system

The general Lie point symmetries \([24]\) becomes

$$X = (a_1 x + a_2) \frac{\partial}{\partial x} + (a_1 t + a_3) \frac{\partial}{\partial t} + a_4 B \frac{\partial}{\partial B} + a_5 C \frac{\partial}{\partial C}. \tag{33}$$

Consequently, the non-linear Einstein field equations \([19-20]\) admits the 5-dimensional Lie algebra spanned by the independent symmetries shown below:

$$X_1 = x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial t}, \quad X_4 = B \frac{\partial}{\partial B}, \quad X_5 = C \frac{\partial}{\partial C}. \tag{34}$$

The forms of the symmetries $X_i$, $i = 1, \ldots, 5$ suggest their significations: $X_2$, $X_3$ generate the symmetry of space translation. $X_1$, $X_4$, $X_5$ are associated with the scaling transformations. When the Lie algebra of these symmetries is computed, the only non-vanishing relations are:

$$[X_1, X_2] = -X_2, \quad [X_1, X_3] = -X_3. \tag{35}$$

It is well known that reduction of the independent variables by one is possible using any linear combinations of the generators of symmetries \([34]\). We will construct a set of minimal combinations known as optimal system \([9, 8]\). An optimal system of a Lie algebra is a set of $l$-dimensional subalgebra such that every $l$-dimensional is equivalent to a unique element of the set under some element of the adjoint representation. The adjoint representation of a Lie algebra \($\{X_i, i = 1, \ldots, 5\}$) is constructed using the formula:

$$\text{Ad}(\exp[\varepsilon X_i]) X_j = \sum_{k=0}^{\infty} \varepsilon^k \frac{k!}{k!} (\text{Ad}(X_i))^k X_j = X_j - \varepsilon [X_i, X_j] + \frac{\varepsilon^2}{2} [X_i, [X_i, X_j]] - \ldots. \tag{36}$$

In order to find the optimal system of the Einstein field equations \([19-20]\), first the following adjoint table is constituted as the following:
Using simplification procedure in [9, 8], we acquire an optimal system of one-dimensional subalgebras to be those spanned by:

\[
\begin{align*}
X^{(1)} &= X_1 + a_4 X_4 + a_5 X_5, \\
X^{(2)} &= X_2 + a_3 X_3 + a_4 X_4 + a_5 X_5, \\
X^{(3)} &= X_3 + a_4 X_4 + a_5 X_5, \\
X^{(4)} &= X_4 + a_5 X_5, \\
X^{(5)} &= X_5.
\end{align*}
\]  

(37)

## 5 Invariant solutions

If we considered the symmetries \(X^{(3)}\) or \(X^{(4)}\) or \(X^{(3)}\), then \(a_1 = a_2 = 0\). From equation (31) this leads to \(f(x) = 0\). So, that, we shall analyse the invariant solutions associated with the optimal systems of symmetries \(X^{(1)}\) and \(X^{(2)}\) only as the following:

**Solution (I):** The symmetries \(X^{(1)}\) has the characteristic equations:

\[
\frac{dx}{x} = \frac{dt}{t} = \frac{dB}{a_4 B} = \frac{dC}{a_5 C}.
\]  

(38)

Then the similarity variable and the similarity transformations takes the form:

\[
\xi = \frac{t}{x}, \quad B(x, t) = x^a \Psi(x), \quad C(x, t) = x^b \Phi(x),
\]  

(39)

where \(a = a_4\) and \(b = a_5\) are an arbitrary constants. In this case, we have

\[
f(x) = c x^d, \quad \omega_z = (1 + q) \omega_x - q \omega_y,
\]  

(40)

where \(c = a_6, d = a_7\) and \(q = a_8\) are an arbitrary constants. Substituting the transformations \([9]\) in the field Eqs. (19) - (20) lead to the following system of ordinary differential equations:

\[
2 n \xi^2 \left[ \frac{\Psi'}{\Phi} + \frac{\Phi'}{\Phi} \right]^2 = \xi \left[ \frac{\Psi''}{\Psi} + \frac{\Phi''}{\Phi} \right] + \alpha_2 \frac{\Psi'}{\Psi} + \alpha_1 \frac{\Phi'}{\Phi},
\]  

(41)

\[
(1 + q) \left[ 2 n \xi^2 \left( \frac{\Psi'^2}{\Psi^2} + \frac{\Phi'^2}{\Phi^2} \right) + \left[ \xi^2 - 1 + 2 n (\xi^2 - 1) \right] \frac{\Psi'\Phi'}{\Psi \Phi} \right] + \alpha_3 = \xi \left[ \alpha_2 \frac{\Psi'}{\Psi} + \alpha_1 \frac{\Phi'}{\Phi} \right] + \left[ q + \xi^2 + n(1 + q)(\xi^2 - 1) \right] \frac{\Psi''}{\Psi} + \left[ 1 + q \xi^2 + n(1 + q)(\xi^2 - 1) \right] \frac{\Phi''}{\Phi},
\]  

(42)

where

\[
\begin{align*}
\alpha_a &= 1 + d + 2 n(a + b) - a, \\
\alpha_b &= 1 + d + 2 n(a + b) - b, \\
\alpha_1 &= a + d + 2 n(1 + a + b) + \left[ a + d + 2(1 - b) + 2 n(1 + a + b) \right] q, \\
\alpha_2 &= b + d + 2 n(1 + a + b) + 2(1 - a) + \left[ b + d + 2 n(1 + a + b) \right] q, \\
\alpha_3 &= (1 + b)(d + n b) + \left[ d + b(1 + d + n + b(n - 1)) \right] q + a^2 \left[ n(1 + q) - 1 \right] + a \left[ 1 + [d + n + b(2 n + 1)](1 + q) \right].
\end{align*}
\]  

(43)

The equations \((41)\) and \((42)\) are non-linear ordinary differential equations (NLODEs) which is very difficult to solve. However, in a special cases, we can find a solution. Now, we suppose the following conditions:

\[
\frac{\Psi'}{\Psi} = \beta_1, \quad \frac{\Phi'}{\Phi} = \beta_2.
\]  

(44)
where $\beta_1$ and $\beta_2$ are an arbitrary constants. By integration the above equations, we get the following solution:

$$\Psi(\xi) = \beta_3 \exp[\beta_1 \xi], \quad \Phi(\xi) = \beta_4 \exp[\beta_2 \xi], \quad \text{(45)}$$

where $\beta_3$ and $\beta_4$ are an arbitrary constants of integration. Substitute (45) in (41), we have the following condition:

$$[2(n-1)(\beta_1^2 + \beta_2^2) + 4n \beta_1 \beta_2] \xi = \alpha_a \beta_1 + \alpha_b \beta_2. \quad \text{(46)}$$

The coefficients of $\xi$ and the absolute value must be equal zero. Solving the two resulting conditions with respect to $n$ and $d$, we have:

$$n = \frac{\beta_1^2 + \beta_2^2}{2(\beta_1 + \beta_2)^2}, \quad d = -\frac{(1 + a)\beta_1^2 - (a + b - 2)\beta_1 \beta_2 + (1 + b)\beta_2^2}{(\beta_1 + \beta_2)^2}. \quad \text{(47)}$$

Substituting from (47) and (45) into (42), we find the following condition:

$$\beta_5 \xi^2 + \beta_6 \xi + \beta_7 = 0, \quad \text{(48)}$$

where

$$\begin{align*}
\beta_5 &= (\beta_1 + \beta_2)^2 \left[1 - q \left(\beta_1^2 - \beta_2^2\right) - 2(1 + q)\beta_1 \beta_2\right], \\
\beta_6 &= 2(\beta_1 + \beta_2) \left[\beta_1^2 \left[2 + b - a + q(a + b)\right] + 2\beta_1 \beta_2(b + a q) + \beta_2^2 \left[b + a + q(2 + a - b)\right]\right], \\
\beta_7 &= \beta_2^2 \left[2 + b(b - 1)(q - 1) + 2q + a^2(q + 3) + a(1 + 3q - 2b(1 + q))\right] \\
&\quad + 2\beta_1 \beta_2 \left[(q - 1)(b^2 - b - a^2 + a) + 2(q + 1)(1 - 2ab)\right] \\
&\quad + \beta_1^2 \left[2(1 - q)(a - 1) + 2(q + 1)(1 - ab) + b[3 + b + q(1 + 3b)]\right] - \beta_5.
\end{align*} \quad \text{(49)}$$

The condition (45) leads to $\beta_5 = \beta_6 = \beta_7 = 0$. Solving it with respect to $q$, $a$ and $b$, we have

$$q = \frac{\beta_1^2 - 2\beta_1 \beta_2 - \beta_2^2}{\beta_1^2 + 2\beta_1 \beta_2 - \beta_2^2}, \quad a = -\frac{(\beta_1 + \beta_2)(\beta_1^3 - 3\beta_1^2 \beta_2 + \beta_1 \beta_2^2 + \beta_2^3)}{(\beta_1 + \beta_2)^2}, \quad b = \frac{(\beta_2^2 - \beta_1^2)(\beta_1^2 + 2\beta_1 \beta_2 - \beta_2^2)}{(\beta_1 + \beta_2)^2}. \quad \text{(50)}$$

Now, by using (50), (47), (45), (40) and (39), we can find the solution if Einstein field equations as the following:

$$\begin{align*}
A(x, t) &= \left(\frac{\gamma_1}{x}\right) \exp\left[\frac{n(\beta_1 + \beta_2) t}{x}\right], \quad B(x, t) = \gamma_2 x^a \exp\left[\frac{\beta_1 t}{x}\right], \\
C(x, t) &= \gamma_3 x^b \exp\left[\frac{\beta_2 t}{x}\right], \quad \omega_x(t) = 2 \frac{(\beta_1^2 - \beta_2^2) \omega_x(t) - (\beta_1^2 - 2\beta_1 \beta_2 - \beta_2^2) \omega_y(t)}{\beta_1^2 + 2\beta_1 \beta_2 - \beta_2^2},
\end{align*} \quad \text{(51)}$$

where $\gamma_1 = c (\beta_3 \beta_4)^n$, $\gamma_2 = \beta_3$, $\gamma_3 = \beta_4$, $\beta_1$ and $\beta_2$ are an arbitrary constants, while $\omega_x$ and $\omega_y$ are an arbitrary functions of $t$. It is observed from equations (51), the line element (11) can be written in the following form:

$$ds^2 = \left(\frac{\gamma_1}{x}\right)^2 \exp\left[\frac{2n(\beta_1 + \beta_2) t}{x}\right] (dx^2 - dt^2) + \gamma_2^2 x^{2a} \exp\left[\frac{2\beta_1 t}{x}\right] dy^2 + \gamma_3^2 x^{2b} \exp\left[\frac{2\beta_2 t}{x}\right] dz^2. \quad \text{(52)}$$

Remark: In the above solution, we can replace $t$ by $t + \delta_1$ and $x$ by $x + \delta_2$ without loss of generality, where $\delta_1$ and $\delta_2$ are an some arbitrary constants.

Solution (II): The symmetries $X^{(2)}$ has the characteristic equations:

$$\frac{dx}{1} = \frac{dt}{a_3}, \quad \frac{dB}{a_4 B} = \frac{dC}{a_5 C}. \quad \text{(53)}$$

Then the similarity variable and the similarity transformations takes the form:

$$\xi = at - bx, \quad B(x, t) = \Psi(\xi) \exp[ct], \quad C(x, t) = \Phi(\xi) \exp[dt], \quad \text{(54)}$$
where \( a_3 = \frac{b}{a} \), \( c = \frac{a_4}{a_3} \) and \( d = \frac{a_5}{a_3} \) are an arbitrary constants. In this case, we have

\[
f(x) = p, \quad \omega_z = (1 + q) \omega_x - q \omega_y,
\]

where \( p = a_6 \) and \( q = a_8 \) are an arbitrary constants. Substituting the transformations (53) in the field Eqs. (54), (55), we can get the following system of ordinary differential equations:

\[
2 n a \left[ \frac{\Psi'}{\Phi} + \frac{\Phi'}{\Phi} \right]^2 + \frac{\alpha_c \Psi'}{\Phi} + \frac{\alpha_d \Phi'}{\Phi} = a \left[ \frac{\Psi''}{\Phi} + \Phi'' \right],
\]

\[
\alpha_1 + (1 + q) \left[ 2 n b^2 \left( \frac{\Psi'^2}{\Phi^2} + \frac{\Phi'^2}{\Phi^2} \right) + \frac{\alpha_2 \Psi'}{\Phi} + \frac{\alpha_3 \Phi'}{\Phi} + \frac{\alpha_4 \Phi''}{\Phi} + \frac{\alpha_5 \Phi''}{\Phi} \right] = 0,
\]

where

\[
\begin{align*}
\alpha_a &= a^2 \left[ n (1 + q) - 1 \right] - b^2 \left[ n (1 + q) + q \right], \\
\alpha_b &= a^2 \left[ n (1 + q) - q \right] - b^2 \left[ n (1 + q) + 1 \right], \\
\alpha_c &= 2 n (c + d) - c, \\
\alpha_d &= 2 n (c + d) - d, \\
\alpha_1 &= (c + d) \left[ c \left(n (1 + q) - q \right) + d \left(n (1 + q) - 1 \right) \right], \\
\alpha_2 &= 2 n (a^2 + b^2) + b^2 - a^2, \\
\alpha_3 &= a \left[ (1 + q) \alpha_d - 2 c q \right], \\
\alpha_4 &= a \left[ (1 + q) \alpha_c - 2 d \right].
\end{align*}
\]

The equations (56) and (57) are NLODEs which is very difficult to solve. However, in a special cases, we can find a solution. If we take \( q = -1 \), the equation (57) convert to the following simple form:

\[
\left( \frac{a^2 - b^2}{2 a} \right) \left( \frac{\Psi''}{\Phi} + \Phi'' \right) + \frac{c \Psi'}{\Phi} - \frac{d \Phi'}{\Phi} + \frac{c^2 - d^2}{2} = 0.
\]

The above equation can be integrated when \( b = \pm a \) and we get the following solution:

\[
\Phi(\xi) = \beta_1 \Psi(\xi) \exp \left[ \frac{\left( c^2 - d^2 \right)}{2 a d} \xi \right],
\]

where \( \beta_2 \) is an arbitrary constants of integration. Substitute (60) in (56), we have the following equation:

\[
4 a^2 d (c + d) \frac{\Psi'''}{\Phi} + 4 a c \left[ c^2 + d^2 - 2 n (c + d)^2 \right] \frac{\Psi'}{\Phi} + 4 a^2 \left[ c^2 - c d - 2 n (c + d)^2 \right] \frac{\Psi'^2}{\Phi^2} = 2 n (c + d)^2 \left( c^2 - d^2 \right) + d^4 - c^4.
\]

Under the transformation

\[
\Psi'(\xi) = \Psi(\xi) \Omega(\xi)
\]

the above equation becomes:

\[
4 a^2 d (c + d) \Omega' = \left[ 2 n (c + d)^2 - c^2 - d^2 \right] \left(c - d + 2 a \Omega \right) \left(c + d + 2 a \Omega \right),
\]

By integrating the above equation, we have:

\[
\Omega(\xi) = \frac{d}{a - \beta_2 \exp \left[ \frac{2 n (c + d)^2 - c^2 - d^2}{a (c + d)} \right] \xi} - \frac{c - d}{2 a}.
\]

where \( \beta_2 \) is an arbitrary constant of integration. Substituting from (64) into (62) and integrating the resulting equation, we have:

\[
\Psi(\xi) = \beta_3 \exp \left[ \frac{d - c}{2 a} \xi \right] \left(a - \beta_2 \exp \left[ \frac{2 n (c + d)^2 - c^2 - d^2}{a (c + d)} \right] \xi \right) \frac{d (c + d)}{c^2 + d^2 - 2 n (c + d)^2}.
\]
where $\beta_3$ is an arbitrary constant of integration. Therefore, from (65) and (66) we get:

$$
\Phi(\xi) = \beta_1 \beta_3^{c/d} \exp \left[ \left( \frac{c-d}{2a} \right) \xi \right] \left( a - \beta_2 \exp \left[ \gamma_4 (t-x) \right] \right) \frac{c (c+d)}{c^2 + d^2 - 2 n (c+d)^2}.
$$

(66)

Now, by using (66), (65), (55) and (54), we can find the solution if Einstein field equations as the following:

$$
\begin{align*}
A(x,t) &= \gamma_1 \exp \left[ n (c+d) t \right] \left( a - \beta_2 \exp \left[ \gamma_4 (t-x) \right] \right) \frac{\gamma_4}{n (c+d)}, \\
B(x,t) &= \gamma_2 \exp \left[ \frac{(c+d) t + (c-d) x}{2} \right] \left( a - \beta_2 \exp \left[ \gamma_4 (t-x) \right] \right) \frac{d}{\gamma_4}, \\
c(x,t) &= \gamma_3 \exp \left[ \frac{(c+d) t - (c-d) x}{2} \right] \left( a - \beta_2 \exp \left[ \gamma_4 (t-x) \right] \right) \frac{c}{\gamma_4}, \\
\omega_z(t) &= \omega_y(t),
\end{align*}
$$

(67)

where $\gamma_1 = p n^{(c+d)/d}$, $\gamma_2 = \beta_3$, $\gamma_3 = \beta_3^{c/d}$, $c$, $d$, $a n$ and $\beta_2$ are an arbitrary constants, while $\omega_x$ and $\omega_y$ are an arbitrary functions of $t$ such that $\gamma_4 = \frac{2 n (c+d)^2 - c^2 - d^2}{c + d}$. It is observed from equations (67), the line element (11) can be written in the following form:

$$
\begin{align*}
\text{ds}_2^2 &= \gamma_1^2 \exp \left[ 2 n (c+d) t \right] \left( a - \beta_2 \exp \left[ \gamma_4 (t-x) \right] \right) \frac{2 n (c+d)}{\gamma_4} (dx - dt)^2, \\
&+ \gamma_2^2 \exp \left[ (c+d) t + (c-d) x \right] \left( a - \beta_2 \exp \left[ \gamma_4 (t-x) \right] \right) \frac{2 d}{\gamma_4} dy^2, \\
&+ \gamma_3^2 \exp \left[ (c+d) t - (c-d) x \right] \left( a - \beta_2 \exp \left[ \gamma_4 (t-x) \right] \right) \frac{2 c}{\gamma_4} dz^2.
\end{align*}
$$

(68)

6 Physical and geometrical properties of the models

For the Model (52):

The expressions of $p^{(m)}$, $\rho^{(m)}$ and $\rho^{(de)}$ for the model (52), are given by:

$$
p^{(m)}(x,t) = \frac{1}{\gamma_1^2 x^2 (\omega_x - \omega_y)} \left[ \beta_2^2 t^2 + 2 (1 - b + n) \beta_2 + n \beta_1 \right] x t + \left[ 1 + b(b-1) - \beta_2^2 \right] x^2 \omega_x \\
- \left[ \left( \beta_1^2 + q \beta_2^2 \right) t^2 + 2 \left( \beta_1 (1 - a + n) + n \beta_2 + q [\beta_2 (1 - b + n) + n \beta_1] \right) x \right. \\
+ \left( 1 + a(a - 1) - \beta_2^2 + q [1 + b(b-1) - \beta_2^2] \right) x^2 \left( \omega_y \left( \frac{x}{1+q} \right) \right) \exp \left[ - \frac{2 n (\beta_1 + \beta_2) t}{x} \right],
$$

(69)
The expansion scalar, which determines the volume behavior of the fluid, is given by:

\[ \sigma = \frac{1}{(1 + q) \gamma^2 x^2 (\omega_x - \omega_y)} \left[ (1 + q) (\omega_x - \omega_y) \left[ \left[ (\beta_1 + \beta_2)^2 - \beta_1 \beta_2 \right] t^2 \right. \right. \]

\[ + \left. \left. \left( \beta_1 \left[ 1 - b - 2a + 2n(a+b) \right] + \beta_2 \left[ 1 - 2b - a + 2n(a+b) \right] \right) x t \right. \right. \]

\[ + \left. \left. \left( a^2 + ab + b^2 - \beta_1 \beta_2 - n \left( \beta_1 + \beta_2 \right)^2 \right) x^2 \right] - G(x, y) \right] \exp \left[ - \frac{2 n (\beta_1 + \beta_2) t}{x} \right], \]  

(70)

The deceleration parameter is given by [13, 14]

\[ \rho^{(dc)}(x, t) = \left[ \frac{G(x, t)}{(1 + q) \gamma^2 x^2 (\omega_x - \omega_y)} \right] \exp \left[ - \frac{2 n (\beta_1 + \beta_2) t}{x} \right], \]

(71)

where

\[ G(x, t) = (\beta_1^2 - \beta_2^2)^2 t^2 - 2 [(a - 1)\beta_1 + (1 - b)\beta_2] x t - [a - a^2 - b + b^2 + \beta_1^2 - \beta_2^2] x^2. \]

The volume element is

\[ V = \gamma_3^2 x a^2 \exp \left[ \frac{(2 n + 1) (\beta_1 + \beta_2) t}{x} \right]. \]

(72)

The expansion scalar, which determines the volume behavior of the fluid, is given by:

\[ \Theta = \left( \frac{3 \beta_1^2 + 4 \beta_1 \beta_2 + 3 \beta_2^2}{2 \gamma_1 (\beta_1 + \beta_2)} \right) \exp \left[ - \frac{n (\beta_1 + \beta_2) t}{x} \right], \]

(73)

The non-vanishing components of the shear tensor, \( \sigma^i_j \), are:

\[ \sigma^1_1 = - \frac{4 \beta_1 \beta_2}{9 \beta_1^2 + 12 \beta_1 \beta_2 + 9 \beta_2^2}, \]

(74)

\[ \sigma^2_2 = \frac{3 \beta_1^2 + 2 \beta_1 \beta_2 - 3 \beta_2^2}{9 \beta_1^2 + 12 \beta_1 \beta_2 + 9 \beta_2^2}, \]

(75)

\[ \sigma^3_3 = \frac{3 \beta_2^2 - 2 \beta_1 \beta_2 - 3 \beta_1^2}{9 \beta_1^2 + 12 \beta_1 \beta_2 + 9 \beta_2^2}. \]

(76)

The shear scalar is:

\[ \frac{\sigma^2}{\Theta^2} = \frac{3 \beta_1^4 - 2 \beta_1^2 \beta_2^2 + 3 \beta_2^4}{3 (3 \beta_1^2 + 4 \beta_1 \beta_2 + 3 \beta_2^2)^2}. \]

(77)

The acceleration vector is given by:

\[ \dot{u}_i = \left( \frac{\beta_1^2 + \beta_2^2}{2 (\beta_1 + \beta_2) x^2 - \frac{1}{x}} \right) (1, 0, 0). \]

(78)

The deceleration parameter is given by [13, 14]

\[ q = -3 \Theta^2 \left( \Theta^2 u^i + \frac{1}{3} \Theta^2 \right) \]

\[ = - \left( \frac{\beta_1 (3 \beta_1^2 + 4 \beta_1 \beta_2 + 3 \beta_2^2)}{4 \gamma_1^2 (\beta_1 + \beta_2)^4} \right) \exp \left[ - \frac{4 n (\beta_1 + \beta_2) t}{x} \right]. \]

(79)

For the Model (68):

The expressions of \( p^{(m)} \), \( \rho^{(m)} \) and \( \rho^{(dc)} \) for the model (68), are given by:

\[ p^{(m)}(x, t) = \frac{e^{-2 n (c+d) t}}{\gamma_1^2 (\omega_y - \omega_x)} \left[ a - \beta_2 e^{4 \gamma_4 (t-x)} \right]^{2 n (c+d) / \gamma_4 - 2} \left[ c d \left( a^2 - \beta_2^2 e^{2 \gamma_4 (t-x)} \right) \omega_x \right. \]

\[ + \left. \left( a^2 \left[ (n-1) (c+d)^2 + c d \right] - a \beta_2 (c+d) \gamma_4 e^{4 \gamma_4 (t-x)} + \beta_2^2 \left[ n (c+d)^2 + c d \right] e^{2 \gamma_4 (t-x)} \right) \omega_y \right]. \]

(80)
\[ \rho^{(m)}(x, t) = \frac{e^{-2n(c+d)t}}{\gamma_1^4} \left[ a - \beta_2 e^{\gamma_4(t-x)} \right]^{2n(c+d)/\gamma_4 - 2} \frac{1}{\omega_y - \omega_x} \left[ a^2 \left[ n (c+d)^2 - c^2 - d^2 \right] - a \beta_2 (c + d) \gamma_4 e^{\gamma_4(t-x)} + n \beta_2^2 (c + d)^2 e^{2\gamma_4(t-x)} \right] - a \beta_2 (c + d) \gamma_4 e^{\gamma_4(t-x)} + n \beta_2^2 (c + d)^2 e^{2\gamma_4(t-x)} \right] \]

\[ \rho^{(de)}(x, t) = \frac{e^{-2n(c+d)t}}{\gamma_1^2 (\omega_x - \omega_y)} \left[ a - \beta_2 e^{\gamma_4(t-x)} \right]^{2n(c+d)/\gamma_4 - 2} \frac{a^2 \left[ n (c+d)^2 - c^2 - d^2 \right] - a \beta_2 (c + d) \gamma_4 e^{\gamma_4(t-x)} + n \beta_2^2 (c + d)^2 e^{2\gamma_4(t-x)} \right] - a \beta_2 (c + d) \gamma_4 e^{\gamma_4(t-x)} + n \beta_2^2 (c + d)^2 e^{2\gamma_4(t-x)} \right] \]

The volume element is

\[ V = \gamma_1^2 \gamma_3^2 e^{(2n+1)(c+d)t} \left[ a - \beta_2 e^{\gamma_4(t-x)} \right]^{-(2n+1)(c+d)/\gamma_4} \]

The expansion scalar, which determines the volume behavior of the fluid, is given by:

\[ \Theta = \frac{a (n + 1) (c + d) e^{n(c+d)t}}{\gamma_1} \left[ a - \beta_2 e^{\gamma_4(t-x)} \right]^{n(c+d)/\gamma_4 - 1} \]

The non-vanishing components of the shear tensor, \(\sigma_i^j\), are:

\[ \frac{\sigma_1^1}{\Theta} = \frac{2n - 1}{3(n+1)} \]

\[ \frac{\sigma_2^2}{\Theta} = \frac{a (1 - 2n) (c + d) - 3 \beta_2^2 (c - d) e^{\gamma_4(t-x)}}{6 a (n + 1) (c + d)} \]

\[ \frac{\sigma_3^3}{\Theta} = \frac{a (1 - 2n) (c + d) + 3 \beta_2^2 (c - d) e^{\gamma_4(t-x)}}{6 a (n + 1) (c + d)} \]

The shear scalar is:

\[ \frac{\sigma^2}{\Theta^2} = \frac{1}{12} \left( \frac{1 - 2n}{1 + n} \right)^2 + \left( \frac{\beta_2 (c - d) e^{\gamma_4(t-x)}}{2 a (n + 1) (c + d)} \right)^2 \]

The acceleration vector is given by:

\[ \dot{u}_i = \frac{a \beta_2 (c + d)}{\beta_2 - a e^{\gamma_4(x-t)}} \left( 1, 0, 0 \right) \]

The deceleration parameter is given by:

\[ q = \left( \frac{a^3 (1 + n)^3 (c + d)^3}{\gamma_1^4} \right) e^{-4n(c+d)t} \frac{a (2n - 1) (c + d) - 3 \beta_2 \gamma_4 e^{\gamma_4(x-t)}}{\gamma_1^4} \left[ a - \beta_2 e^{\gamma_4(x-t)} \right]^{4n(c+d)/\gamma_4 - 4} \]

7 Conclusion

In this paper, we have investigated an optimal system and invariant solutions of dark energy models in cylindrically symmetric space-time. Generally, the models represent shearing, expanding and non-rotating universe in which flow vector is geodetic. On the basis of optimal systems of symmetries \(X^{(1)}\) and \(X^{(2)}\), we obtained two models \([62]\) and \([63]\) respectively. For model \([62]\), our study reveals:

- \(\frac{\sigma}{\dot{q}}\) = constant forever, therefore the model does not approach isotropy.
- The declaration parameter is negative, therefore it represents the model of accelerating universe.
• As $t \to \infty$, $V \to \infty$ but $\rho^{(dc)} \to 0$ hence volume increases with grow of time while dark energy density decreases.

For model \([68]\), we have following observations

• $\lim_{t \to 0} \sigma \theta = \text{constant}$.

• The deceleration parameter is positive hence it represents the model of decelerating universe.

Thus, in our analysis, model \([52]\) is most suitable model of universe matches with observations.

References

[1] Riess AG et al, Astron J. 116, 1009 (1998).
[2] S perlmutter et al, Astrophys. J. 517, 565 (1999).
[3] Bennet CL et al, Astrophys. J. 148 1 (2003).
[4] Padmanabham T, Phys Rep. 380 235 (2003).
[5] Spregel, DN et al, Astrophys. J. Suppl. 170, 377 (2007).
[6] Ali AT, Yadav AK and Mahmoud SR, Astrophys Space Science 349(1) 539 (2014).
[7] Ali AT and Yadav AK, Int J Theor Phys http://link.springer.com/article/10.1007/s10773-014-2049-1.
[8] Ovsiannikov LV, ”Group Analysis of Differential Equations”, Academic Press, NY, 1982.
[9] Olver PJ, ”Application of Lie Groups to differential equations in graduate texts in mathematics”, Vol. 107, Second edition, Springer, New York, 1993.
[10] Ibragimov NH, ”Transformation groups applied to mathematical physics”, Reidel, Dordrecht, 1985.
[11] Senovilla JMM, Phys. Rev. Lett. 64, 2219 (1990).
[12] Dadhich N, Tiikekar R, Patel L, Curr. Sci 65, 694 (1993).
[13] Feinstein A and Ibanez J, Class Quantum Grav 10, L227 (1993).
[14] Raychaudhuri AK, ”Theoretical Cosmology”, Oxford, p.80, 1979.
[15] Ali AT, and Rahaman F, Int. J. Theor. Phys. 52, 4055 (2013).
[16] Pradhan A, Yadav AK, Singh RP and Singh VK, Astrophys. Space Sc. 312, 145 (2007).
[17] Yadav AK, Yadav VK and Yadav L, Int. J. Theor. Phys. 48, 568 (2009).
[18] Baumann G, ”Symmetry Analysis of Differential Equations with Mathematica”, Telos, Springer Verlag, New York, 2000.
[19] Bluman GW and Anco SC, ”Symmetry and Integration Method for Differential Equations”, Springer Verlag, New York, 2002.
[20] Cartan E, ”Les Systemes Differentiels Exterieur et leurs Applications Geometrique”, Hermann, Paris, 1946.
[21] Harrison BK and Estabrook FB, J Math Phys 12 653 (1971).
[22] Edelen DGB, ”Programs for Calculation of Isovector Fields in the REDUCE 2 Environment”, Center for the Application of Mathematics, Lehigh University, 1981.
[23] Edelen DGB, ”Applied Exterior Calculus” John Wiley and Sons, New York, 1985.
[24] Suhubi ES, Int J Ingng Sci 29 133 (1991).
[25] Harrison BK, Proc. Second Int. Conf. Symmetry in Nonlinear Mathematical Physics 1 21 (1997).

[26] Bhutani OP and Singh K, Int J Engng Sci 38 1741 (2000).

[27] Harrison BK, SIGMA 1 1 (2005).

[28] Ali AT, Phys Scr 79(3) 035006 (2009).

[29] Ali AT, Phys Scr 87(1) 015002 (2013).

[30] Attallah SK, El-Sabbagh MF and Ali AT, Commun Nonlinear Sci Numer Simulat 12(7) 1153 (2007).

[31] Ou MG, Appl Anal 86(12) 1509 (2007).

[32] Bocharov AV, "DEliA: A System of Exact Analysis of Differential Equations Using S. Lie Approach", Academy of Science, Pereslavl-Zalessky, U.S.S.R., 1989.

[33] Bocharov, AV, SIGSAM Bulletin 24 37 (1990).

[34] Mikhailov AV, Shabat AB and Sokolov VV, "What is Integrability", Springer Lecture Notes in Nonlinear Dynamics, Ed.: Zakharov, V.I. Springer Verlag, New York, 1990.