An Efficient Hybrid Compact-WENO Scheme for Computational Aerodynamics

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Abstract. The hybrid compact-WENO scheme is an effective method to construct a high-resolution shock-capturing scheme for computational aerodynamics. Ren et al. (J. Comput. Phys. 192:365-386, 2003) presented a characteristic-wise hybrid compact-WENO scheme, which combines the high-resolution property of the compact scheme with the shock-capture ability of the WENO scheme. However, it needs to deal with the block-tridiagonal system of linear equations when applied to solve systems of hyperbolic conservation laws (such as Euler equations). Therefore, the scheme is computationally expensive, which limits its application in engineering practice. In this paper, we present a new hybrid compact-WENO scheme. This new scheme also uses the characteristic-wise method. However, it avoids solving the block-tridiagonal system of linear equations by cleverly utilizing the monotone-preserving limiter and the higher dissipative Lax-Friedrich flux splitting. As a result, the scheme has a much higher efficiency than Ren’s scheme. A series of numerical experiments verify the high-resolution property, excellent shock-capture ability, and high efficiency of the presented scheme. Therefore, it is suitable for engineering applications.

1. Introduction
Accurate numerical prediction of complex flow is an urgent problem in aircraft and spacecraft design. The flow fields involved in aerospace usually contain both strong shock waves and delicate, smooth structures. Therefore, the high-resolution shock-capturing numerical schemes are required.

WENO (weighted essentially non-oscillatory) schemes [1-4] are good choices because they possess ENO (essentially non-oscillatory) property for capturing shocks and achieve high-order accuracy in smooth regions. The characteristic of WENO schemes is that they use nonlinear weights for adaptive stencils. The nonlinear weights can automatically identify discontinuities and assign weights approaching zero to discontinuous substencils, thus allowing the schemes to have ENO property; and approximate so-called ideal weights in smooth regions, so that the schemes can achieve the optimal order of accuracy. Although WENO schemes have high-order accuracy in smooth regions, their spectral properties are not ideal for simulating multi-scale flows. Compact schemes [5], by contrast, have excellent spectral properties but produce severe numerical oscillations near discontinuities. The natural idea is to combine a compact scheme with a WENO scheme by a switch/weight function. This idea is the so-called hybrid scheme.

Pirozzoli [6] proposed the first hybrid compact-WENO scheme. In this scheme, the conservative compact scheme is used for smooth regions, and the WENO scheme is used for discontinuities. A simple switch function capable of detecting discontinuities is used to achieve the transition between
the two sub-schemes. In this way, the hybrid scheme inherits the high-resolution property of the compact scheme and the shock-capturing ability of the WENO scheme at the same time. However, there are still some defects in this scheme. Firstly, abrupt switching between the two sub-schemes results in numerical oscillations, which may spread and pollute the whole flow field. Secondly, for solving systems of hyperbolic conservation laws, the scheme in conjunction with the flux splitting technique is applied in a component by component manner. Although the component-wise method is quite simple and computational economical, its resolution and robustness are inferior to the characteristic-wise method. To overcome these defects, Ren et al. [7] improved the hybrid compact-WENO scheme in the following ways. Firstly, they replace the switch function with a continuous weight operator, thus achieves a smooth transition over the two sub-schemes. Secondly, the scheme solves the systems of hyperbolic conservation laws with the characteristic decomposition approach. Thirdly, the Roe type schemes are used, which have lower dissipation and higher resolution than schemes based on flux splitting. Numerical tests show that Ren's scheme has much better robustness and resolution than Pirozzoli's. Although Ren's scheme has excellent numerical performance, its high computational cost limits the application of Ren's scheme in engineering practice.

In this paper, we propose a new hybrid compact-WENO scheme. This new scheme still uses the characteristic-wise method. However, it avoids solving the block-tridiagonal system of linear equations by cleverly utilizing a flux-correction method and the high-dissipation Lax-Friedrich (HDLF) flux splitting [6]. As a result, the new scheme has the resolution and robustness of the characteristic-wise scheme, but only the computational cost of the component-wise scheme is needed to pay.

2. Numerical method

2.1. The conservative compact scheme
In this paper, we use Pirozzoli’s compact scheme [6] as the high-resolution sub-scheme,
\[
\frac{3}{10} \hat{f}_{j-1/2}^C + \frac{6}{10} \hat{f}_{j+1/2}^C + \frac{1}{10} \hat{f}_{j+3/2}^C = \frac{1}{12} f_{j-1} + \frac{19}{60} f_j + \frac{10}{30} f_{j+1},
\]
which was also used by Ren [7].

2.2. The WENO-Z scheme
Ren uses the WENO-JS scheme [2] as the shock-capturing sub-scheme in his hybrid scheme. In contrast, we adopt the WENO-Z scheme [4], which is less dissipative than WENO-JS. The fifth-order WENO scheme has three substencils and three corresponding low-order numerical fluxes:
\[
\begin{align*}
\hat{f}_{j+1/2}^0 &= \frac{1}{3} f_{j+2} - \frac{7}{6} f_{j+1} + \frac{11}{6} f_j, \\
\hat{f}_{j+1/2}^1 &= -\frac{1}{6} f_{j+1} + \frac{5}{6} f_j + \frac{1}{3} f_{j+1}, \\
\hat{f}_{j+1/2}^2 &= \frac{1}{3} f_j - \frac{5}{6} f_{j+1} - \frac{1}{6} f_{j+2}.
\end{align*}
\]
The high-order WENO flux is the weighted average of the above three low-order fluxes:
\[
\hat{f}_{j+1/2}^W = \sum_{k=0}^{2} \omega_k \hat{f}_{j+1/2}^k.
\]
The WENO-Z weights are given by
\[
\tau_5 = |\beta_2 - \beta_0|, \quad \alpha_k = d_k \left[ 1 + \left( \frac{\tau_5}{\epsilon + \beta_k} \right)^{\rho} \right], \quad \omega_k = \frac{\alpha_k}{\alpha_0 + \alpha_1 + \alpha_2}.
\]
\((d_0, d_1, d_2) = (1/10, 6/10, 3/10)\) are the ideal weights. The power parameter \(p \geq 1\) controls the numerical dissipation. We choose \(p = 1\) to minimize the numerical dissipation. The sensitivity parameter \(\varepsilon > 0\) is set to avoid denominator zero. We take \(\varepsilon = 10^{-40}\) to minimize its effect on the performance of the nonlinear weights.

2.3. The hybrid schemes

2.3.1. Ren’s hybrid scheme. Ren gave the weight operator for his hybrid scheme as follows:
\[
\sigma_{j+1/2} = \min \left( 1, \frac{r_{j+1/2}}{r_c} \right),
\]
\[
r_{j+1/2} = \min \left( r_j, r_{j+1} \right),
\]
\[
\hat{r}_j = \frac{2\Delta f_{j+1/2} \Delta f_{j-1/2} + \varepsilon}{\left( \Delta f_{j+1/2} \right)^2 + \left( \Delta f_{j-1/2} \right)^2 + \varepsilon},
\]
\[
\Delta f_{j+1/2} = (f_{j+1} - f_j), \quad \varepsilon = \frac{0.9r_c}{1 - 0.9r_c} \zeta^2, \quad \zeta = 10^{-3}.
\]
\(0 < r_c \leq 1.0\) is a parameter. Choosing a smaller value of \(r_c\), can reduce the usage of the WENO scheme, thus reduces the numerical dissipation and improves computational efficiency. After evaluating the weight of the compact scheme by Eq. (5), we can express the hybrid scheme as follows:
\[
\frac{3}{10} \sigma_{j+1/2} \hat{f}_{j+1/2}^{H} + \left[ \frac{6}{10} \sigma_{j+1/2} + (1 - \sigma_{j+1/2}) \right] \hat{f}_{j+1/2}^{H} + \frac{1}{10} \sigma_{j+1/2} \hat{f}_{j+1/2}^{H} = \frac{1}{30} f_{j-1} + \frac{19}{30} f_{j} + \frac{10}{30} f_{j+1} + (1 - \sigma_{j+1/2}) \hat{f}_{j+1/2}^{w}.
\]
(6)
The characteristic of the above scheme is that the two sub-schemes are coupled together.

2.3.2. A new hybrid scheme. We present a new hybrid scheme, which combines the two sub-schemes by the monotone-preserving limiter [8]. We first introduce the monotone-preserving limiter, which requires that the numerical flux \(\hat{f}_{j+1/2}\) must be within the interval \([f_j, f^{MP}]\). The upper bound \(f^{MP}\) is given by
\[
f^{MP} = f_j + \min \text{mod}(f_{j+1} - f_j, \alpha(f_j - f_{j-1})),
\]
where parameter \(\alpha \geq 2\) determines the CFL number,
\[
\text{CFL} \leq 1/(1 + \alpha)
\]
In this paper, we take \(\alpha = 4\) and \(\text{CFL} = 0.2\). The ‘\(\min \text{mod}\)’ function is defined as
\[
\min \text{mod}(x, y) = \frac{1}{2} \left[ \text{sgn}(x) + \text{sgn}(y) \right] \min(|x|, |y|).
\]
Now we give the evaluation steps of the hybrid scheme as follows:
1) The original numerical fluxes \(\hat{f}_{j+1/2}^{c}, \ j = 0, 1, \ldots, N - 1\) are evaluated by the conservative compact scheme (Eq. (1)) in conjunction with some appropriate boundary schemes.
2) The monotone-preserving limiter is used to test the original numerical fluxes one by one. In the test, \(\delta_{j+1/2} = (\hat{f}_{j+1/2}^{c} - f_j)(f_{j+1/2} - f^{MP})\) is evaluated.
3) If \(\delta_{j+1/2} < 0\), \(\hat{f}_{j+1/2}^{H} = \hat{f}_{j+1/2}^{c}\); otherwise, \(\hat{f}_{j+1/2}^{w}\) is calculated, and \(\hat{f}_{j+1/2}^{H} = \hat{f}_{j+1/2}^{w}\).
Unlike Ren’s scheme, the scheme presented is decoupled. That is to say, the two sub-schemes are used separately to get $\hat{f}^{C}_{j+1/2}$ and $\hat{f}^{W}_{j+1/2}$, then a selection process is applied to this two numerical fluxes. This new hybrid scheme is essentially a flux-correction method that uses the WENO fluxes to modify the fluxes of the compact scheme. The flux-correction method is different from the coupling solving of the two sub-schemes, so there is no abrupt switching between them.

2.4. The hybrid schemes for solving systems of hyperbolic conservation laws
When applied to solve systems of hyperbolic conservation laws, Ren’s scheme needs to deal with the block-tridiagonal system of linear equations. In contrast, the presented scheme accompanying with the HDLF flux splitting [6] avoids the computationally expensive block-tridiagonal system of linear equations. The principle is that when the HDLF flux splitting is adopted, the characteristic-wise linear, compact scheme is identical to the component-wise linear, compact scheme. Therefore, the local characteristic decompositions need to be implemented only in the flux-correction step.

3. Numerical tests
In this section, we apply the presented hybrid scheme to solve several benchmark problems that require solving one-dimensional or two-dimensional Euler systems. The basic settings of these problems, including the domains, initial conditions, boundary conditions, and final times, can be found in the relevant references. Therefore, we will not repeat them in this paper. We compare the numerical results and efficiency of the presented scheme with those of Ren’s scheme and WENO-Z scheme. The presented scheme is named HCW-M, and Ren’s scheme is named HCW-R. We use the third-order total variation diminishing Runge-Kutta (TVD-RK3) scheme [2] for time stepping.

3.1. One-dimensional Euler System

3.1.1. Lax problem [9]. This problem is solved in a grid with $N = 101$ points. The numerical results are show in Figure 1. Since HCW-R with $r_c = 0.3$ generates obvious numerical oscillation, we display the numerical result of HCW-R for $r_c = 0.4$ in the figure. It can be observed that the results of the two hybrid schemes are almost the same, and both of them are significantly better than the result of WENO-Z.

3.1.2. Interacting blast waves [10]. The grid number is $N = 401$. The numerical results are show in Figure 2. This problem poses a great challenge to the stability of the schemes. When using the HCW-R scheme, the calculation can only be carried out with $r_c = 1$. Therefore, we display the result of HCW-R for $r_c = 1$. It can be observed that the result of HCW-M is significantly better than that of HCW-R.
3.1.3. Osher-Shu problem [11]. The grid number is $N = 201$. The numerical results are shown in Figure 3. $r_c = 0.4$ in HCW-R. For this problem, the result of HCW-M is slightly better than that of HCW-R.

3.2. Two-dimensional Euler System

3.2.1. Two-dimensional Riemann problem [12]. Configuration 5 in the literature [12] is simulated here. We use a grid with $N = 401 \times 401$ points. The results are shown in Figure 4. $r_c = 0.4$ for HCW-R. The density contour is 20 equal parts between the maximum value and the minimum value. The hybrid schemes capture finer structures than the WENO-Z scheme. This improvement is due to the high-resolution property of the compact sub-scheme. The results of the two hybrid schemes are almost identical. However, the computational cost of HCW-M is much less than that of HCW-R (see Table 1).
Figure 4. Numerical solutions (density contours) of the two-dimensional Riemann problem

Table 1. The computational cost (CPU time in seconds) for the two-dimensional Riemann problem

| Scheme    | WENO-Z | HCW-R | HCW-M |
|-----------|--------|-------|-------|
|           | 765    | 4132  | 984   |

3.2.2. *Double Mach reflection problem* [10]. We use a grid with \( N = 961 \times 241 \) points. The results are show in Figure 5. When HCW-R with \( r_c = 0.3 \) is used, negative density occurs in the numerical solution, so the result of HCW-R for \( r_c = 0.4 \) is shown. Since the most complex structures are concentrated in the ‘roll-up’ region, we display the enlarge details of this region. The density contour is 50 equal parts between 2 and 22.

The results of the two hybrid schemes are significantly different. It seems that HCW-R has a better resolution since it captures more abundant vortex structures. However, at the bottom of the flow field and near the wall, the result of HCW-M is much finer and clearer than that of HCW-R. Figure 6 shows the results of WENO-Z and HCW-M with grid size \( N = 1601 \times 401 \), and Table 2 shows the computational costs (CPU time in seconds) of different schemes with different grid sizes. We can see that the computational cost of HCW-R using 961×241 meshes is even higher than that of HCW-M using 1601×401 meshes, and the result of the latter is significantly better than that of the former.
Figure 5. Numerical solutions (density contours) of the two-dimensional Double Mach reflection problem with grid size $N=961\times241$

![Figure 5](image1)

Figure 6. Numerical solutions (density contours) of the two-dimensional Double Mach reflection problem with grid size $N=1601\times401$

![Figure 6](image2)

Table 2. The computational cost (CPU time in seconds) for the two-dimensional Double Mach reflection problem

| grid size  | WENO-Z | HCW-R | HCW-M |
|------------|--------|-------|-------|
| 961\times241 | 3828   | 26146 | 4987  |
| 1601\times401 | 19468  | -     | 25501 |

3.3. Accuracy and efficiency test

In this test, we solve the two-dimensional Euler system. The domain is set as $[-\pi, \pi] \times [-\pi, \pi]$. The initial condition is $\rho(x, y, 0) = 1 + 0.2\sin(x+y)$, $u(x, y, 0) = 1$, $v(x, y, 0) = 1$, $\rho(x, y, 0) = 1$. Periodic boundaries are set. The exact solution of this problem is $\rho(x, y, t) = 1 + 0.2\sin(x+y-(u+v)t)$, $u(x, y, t) = 1$, $v(x, y, t) = 1$, $\rho(x, y, t) = 1$. We get the result at $t=0.2$ and evaluate the $L_1$ error of density $\rho$. Since in most applications of HCW-R, $r_c=0.4$, we set $r_c=0.4$ for HCW-R in this test.
Table 3 lists the $L_1$ error, order, and computational cost (CPU time in seconds) of different schemes with different grid sizes. The errors of the hybrid schemes are an order of magnitude lower than the WENO-Z scheme. The errors of HCW-M are smaller than those of HCW-R on coarse grids. However, as the grid refined, the errors of the two hybrid schemes tend to be consistent with that of the compact scheme.

It is more meaningful to compare the errors of the schemes at the same computational cost than to compare the errors of these schemes on the same grid. Therefore, we show the errors of different schemes as functions of CPU time in Figure 7. Although the error of HCW-R is much smaller than that of WENO-Z on the same grid, its computational cost is much higher. As a result, the efficiency of HCW-R is lower than that of WENO-Z. In contrast, HCW-M is much more efficient.

Table 3. $L_1$ error, order and computational costs (CPU time in seconds) for different schemes

| Grid size  | WENO-Z | Compact |
|-----------|--------|---------|
|           | $L_1$ error | $L_1$ order | CPU time | $L_1$ error | $L_1$ order | CPU time |
| 10×10     | 1.0913e-2  | -         | -        | 1.0526e-3  | -         | -        |
| 20×20     | 2.6975e-4  | 5.34      | -        | 2.8822e-5  | 5.19      | -        |
| 40×40     | 7.8673e-6  | 5.10      | 0.2969   | 8.5164e-7  | 5.08      | -        |
| 80×80     | 2.4024e-7  | 5.03      | 3.2969   | 2.5907e-8  | 5.04      | -        |
| 160×160   | 7.4217e-9  | 5.02      | 43.4063  | 7.9958e-10 | 5.02      | -        |

| Grid size  | HCW-R | HCW-M |
|-----------|-------|-------|
|           | $L_1$ error | $L_1$ order | CPU time | $L_1$ error | $L_1$ order | CPU time |
| 10×10     | 1.8183e-3  | -         | -        | 1.0638e-3  | -         | -        |
| 20×20     | 3.0188e-5  | 5.91      | 0.1406   | 2.8626e-5  | 5.22      | -        |
| 40×40     | 9.1960e-7  | 5.04      | 1.5469   | 8.5173e-7  | 5.07      | 0.3438   |
| 80×80     | 2.5907e-8  | 5.15      | 18.3438  | 2.5907e-8  | 5.04      | 4.2500   |
| 160×160   | 7.9958e-10 | 5.02      | 244.7188 | 7.9958e-10 | 5.02      | 54.0625  |

Figure 7. the errors of the different schemes as a function of CPU time

4. Conclusion
Ren’s scheme uses the characteristic-wise method, and its sub-schemes are coupled together. These characteristics make Ren’s scheme needs to solve the block-tridiagonal system of linear equations when applied to solve systems of hyperbolic conservation laws (such as Euler equations). Therefore, the computational cost is enormous, which limits the application of Ren’s scheme in engineering practice.
In this paper, we present a new hybrid compact-WENO scheme, which also uses the characteristic-wise method but has the following two characteristics. Firstly, since the scheme uses a flux-correction method, the two sub-schemes are implemented in a decoupling manner. Secondly, based on the HDLF flux splitting, the characteristic-wise linear, compact scheme is identical to the component-wise linear, compact scheme. In this way, the computational expensive block-tridiagonal system of linear equations is avoided. As a result, the new scheme has a much higher efficiency than Ren’s scheme. A series of numerical experiments verify the high resolution, excellent shock-capture ability, and high efficiency of the presented scheme. Therefore, it is suitable for engineering applications.

5. References
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