Factorization of the Dirac equation and a graphene quantum dot

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Abstract. We consider a quantum dot described using a cylindrically symmetrical 2D Dirac equation. The potentials representing the quantum dot are taken to be of different types of potential configuration, scalar, vector, and pseudo-scalar to enable us to enrich our study. Using various potential configurations, we found that in the presence of a mass term, an electrostatically confined quantum dot can accommodate true bound states, which is in agreement with our previous work. The differential cross section associated with one specific potential configuration has been computed and discussed as a function of the various potential parameters.

Keywords: algebraic structures of integrable models, graphene (theory)

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1. Introduction

Recent technological advances in nano-fabrication have created a great deal of interest in the study of low dimensional quantum systems such as quantum wells, quantum wires, and quantum dots. In particular, there has been a considerable amount of work in recent years on confined semiconductor structures, which finds applications in electronic and optoelectronic devices. The two dimensional character of the system allows for electron confinement in one spatial direction for a specific potential configuration. Graphene \[1, 2\] has became one of the most important subjects in condensed matter research in the last few years. This is because of its exotic physical properties and the apparent similarity of its mathematical model to the one describing relativistic fermions in 2D. Graphene is a fascinating subject because its low energy quasiparticles are governed by a (2+1)-dimensional Dirac equation with the Fermi velocity \(v_F\). These unique and amazing properties make graphene one of the most promising materials for future nano-electronics devices \[3\]. Graphene quantum dots (artificial atoms) \[4–6\] have sparked intense research activities related to quantum information storage and processing using the spin information of confined electrons. Various methods were used to make quantum dots (QD) one of the most widely-used techniques using electrostatic gates \[5\].

On the other hand, the main features of the conductivity of doped single–layer graphene were analyzed and the models for different scattering mechanisms were presented by Guinea \[7\]. Many possible dependencies of the cross section on the Fermi wavelength were identified, depending on the type of scattering mechanism. Defects with an internal structure, such as ripples, showed non-monotonic dependencies, with maxima when the...
Fermi wavelength was comparable to the typical length scale of the defect. Furthermore, the electronic states of an electrostatically confined cylindrical graphene quantum dot and the electric transport through this device were studied theoretically within the continuum Dirac-equation approximation and compared with the numerical results obtained from a tight-binding lattice description by Pal et al [8]. A spectral gap, which may originate from strain effects, additional adsorbed atoms, or substrate-induced sublattice-symmetry breaking, allowed for both bound and scattering states. As long as the diameter of the dot was much larger than the lattice constant, the results of the continuum and the lattice model were in very good agreement. The influence of the dot-potential step, of the on-site disorder along the sample edges, of the uncorrelated short-range disorder potentials in the bulk, and of the random magnetic fluxes that mimic the ripple disorder, were all investigated. It was concluded that the quantum dots’ spectral and transport properties depend crucially on the specific type of disorder and in general, the peaks in the density of the bound states are broadened but remain sharp only in the case of edge disorder.

In this paper we use our recently developed formalism for the 2D Dirac equation [9] and apply our results to graphene based on the recent results reported in [7,8]. In particular, we study the energy spectrum of graphene QD in the presence of an electrostatic confining potential. One purpose of this research is to study the elastic scattering theory through radially-symmetric potentials and to evaluate the transport cross section, which is very valuable for the study of the transport properties of graphene [10]. In order to investigate the transport properties of the QD we add an environment to the isolated QD so that the exponentially decaying bound states are still finite when reaching the outer region. We explicitly investigate the electronic transport through a circular electrostatic potential in the presence of a constant mass term. We first obtain the asymptotic form of the 2D wave function, and we write them in terms of normalized spinor plane and cylindrical waves. Using the definition of the scattering matrix [7, 11] we calculate the differential cross section and then deduce the transport cross section, and finally we draw our conclusions.

The paper is organized as follows: section 2 summarizes our separation of variables approach for a 2D Dirac equation which was used in our previous work. In section 3, we give the solutions to the energy spectrum of the Dirac equation for two potential configurations. In section 4, we concentrate on graphene and assume the presence of a constant mass term that can be induced by different experimental methods [12]. We include a radially symmetric potential defining the QD and obtain the solution of the associated Dirac equation in various regions. The resulting energies of the bound states of the isolated QD are in good agreement with previous calculations of bound state energy [8,13]. We note that various methods have been used to analyze bound states in graphene [8, 13, 14]. In section 5, we study the transport properties for a graphene quantum dot by evaluating and studying the corresponding cross sections. A detailed investigation of the scattering processes in graphene will be performed for a specific potential configuration in section 6. We conclude our work in the last section.

2. Theoretical model

In this section we start by summarizing the main steps involved in the separation of variables approach we used for the 2D Dirac equation [9]. Consider the 2D Dirac equation
with an electromagnetic interaction through minimal coupling for a spin 1/2 particle of mass $m$ and charge $e$ in units ($\hbar = c = 1$)

$$
[\gamma^\mu (i\partial_\mu - eA_\mu) - (m + S)] \psi = 0
$$

(1)

where $\gamma^\mu \partial_\mu = \gamma^0 \partial_0 + \vec{\gamma} \cdot \vec{\nabla}$, $S$ is the scalar coupling and $A_\mu = (A_0, \vec{A})$ with $A_0$ is related to the electrostatic potential $\vec{E} = -\nabla A_0 = -\partial \vec{A} / \partial t$ and $\vec{A}$ is related to the magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$. The Dirac matrices $\gamma^\mu$ satisfy the algebra

$$
\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}
$$

(2)

with $\eta^{\mu\nu} = \text{diag}(1, -1, -1)$ and $\mu, \nu = 0, 1, 2$. In (2+1)-dimensions we select the following representation

$$
\gamma^0 = i \sigma_3, \quad \vec{\gamma} = i \vec{\sigma},
$$

where $\{\sigma_i\}_{i=1}^3$ are the $2 \times 2$ Pauli matrices

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

(3)

and then (1) can be written as follows

$$
i\gamma^0 \left( \frac{\partial}{\partial t} \psi \right) + i \vec{\gamma} \cdot \vec{\nabla} \psi - e \vec{\gamma} \cdot \vec{A} \psi - (m + S) \psi - e\gamma^0 A_0 = 0.
$$

(4)

Multiplying (4) by $\gamma^0$ and using the notation $\tilde{\alpha} = \gamma^0 \vec{\gamma}$, $\beta = \gamma^0$ to obtain

$$
i \frac{\partial}{\partial t} \psi = \left[ -i \tilde{\alpha} \cdot \vec{\nabla} + eA_0 + e \tilde{\alpha} \cdot \vec{A} + (m + S) \beta \right] \psi = H \Psi.
$$

(5)

In the forthcoming analysis we study different potential configurations in order to solve the above equation explicitly. For time-independent potentials, the two components' spinor wave function can be written as follows $\Psi(t, r, \theta) = e^{-i \epsilon t} \Psi(r, \theta)$ so that our previous equation becomes

$$
(H - \epsilon) \Psi(r, \theta) = 0.
$$

(6)

Knowing that in polar coordinates $\vec{\nabla} = \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta}$ and $\tilde{\alpha} = i \sigma_3 \sigma_\theta$ so that our Hamiltonian is now

$$
H = eA_0 + (m + S) \sigma_3 + \sigma_3 \sigma_r \partial_r + ie \sigma_3 \sigma_r A_r + \sigma_3 \sigma_\theta \frac{1}{r} \partial_\theta + ie \sigma_3 \sigma_\theta A_\theta.
$$

(7)

To proceed further we consider, along the line of our previous paper [15], a transformation $\Lambda(r, \theta)$ that transforms ($\sigma_r, \sigma_\theta$) into ($\sigma_1, \sigma_2$) and vice versa. Thus we require that

$$
\Lambda \sigma_r \Lambda^{-1} = \sigma_1, \quad \Lambda \sigma_\theta \Lambda^{-1} = \sigma_2
$$

(8)

which then turns out to have the following explicit form

$$
\Lambda(r, \theta) = \lambda(r, \theta) e^{\frac{1}{2} \sigma_3 \theta}
$$

(9)

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where \( \lambda(r, \theta) \) is a \( 1 \times 1 \) real function and the exponential is a \( 2 \times 2 \) unitary matrix. Then we can define the new Hamiltonian in matrix form as

\[
H = \begin{pmatrix}
  m + S + eA_0 & \partial_r - \frac{\lambda_r}{\lambda} + \frac{1}{2r} + iA_r \\
  -\partial_r + \frac{\lambda_r}{\lambda} - \frac{1}{2r} - ieA_r & -\frac{i}{r} \left( \partial_\theta - \frac{\lambda_\theta}{\lambda} \right) + eA_\theta \\
\end{pmatrix}.
\] (10)

One can show that the hermiticity of \( H \) requires \( \lambda = \sqrt{r} \) and reduces the Hamiltonian to

\[
H = \begin{pmatrix}
  m + S + eA_0 & \partial_r + iA_r - \frac{i}{r} \partial_\theta + eA_\theta \\
  -\partial_r - ieA_r - \frac{i}{r} \partial_\theta + eA_\theta & -m - S + eA_0 \\
\end{pmatrix}.
\] (11)

and we obtain the \( (2 + 1) \)-dimensional Dirac equation \( (H - \varepsilon)\chi = 0 \), or as an equivalent

\[
\begin{pmatrix}
  m + S + eA_0 - \varepsilon & \partial_r + iA_r - \frac{i}{r} \partial_\theta + eA_\theta \\
  -\partial_r - ieA_r - \frac{i}{r} \partial_\theta + eA_\theta & -m - S + eA_0 - \varepsilon \\
\end{pmatrix} \begin{pmatrix}
  \chi_+(r, \theta) \\
  \chi_-(r, \theta) \\
\end{pmatrix} = 0
\] (12)

where the transformed spinor wavefunction, \( \chi(r, \theta) = (\chi_+(r, \theta), \chi_-(r, \theta))^t \) and the superscript \( t \) stands for transpose of the spinor, is given by

\[
\chi(r, \theta) = \sqrt{r} e^{\frac{i}{2} \sigma_3 \theta} \Psi(r, \theta).
\] (13)

These will be used to explicitly determine the solutions of the energy spectrum, which will serve to deal with the different issues.

3. Potential configurations and solutions

In order to determine the energy spectrum, we use the potential configurations which were used in our previous work [9], which ensures the separation of the variables. Now we can write the spinor wave function as \( \chi_\pm(r, \theta) = \Phi(r) \pm F(\theta) \) where the subscripts stand for upper and lower spinor components. We consider the first potential configuration defined by

\[
A_0(\vec{r}) = V(r), \quad A_r(\vec{r}) = R(r), \quad A_\theta = W(r), \quad S = S(r).
\] (14)

We note that the radial part of the vector potential \( A_r \) can be gauged away in the above situations and hence will not be included in our future equations (12) becomes

\[
\begin{pmatrix}
  m + S + eV - \varepsilon & \partial_r + eW \\
  -\partial_r + eW & -m - S + eV - \varepsilon \\
\end{pmatrix} + \frac{1}{r} \begin{pmatrix}
  0 & -i\partial_\theta \\
  -i\partial_\theta & 0 \\
\end{pmatrix} \begin{pmatrix}
  \Phi_+ F_+ \\
  \Phi_- F_- \\
\end{pmatrix} = 0.
\] (15)

The angular component satisfies \( \partial_\theta F(\theta) = i\varepsilon_\theta F(\theta) \), giving the solution

\[
F(\theta) = e^{i\varepsilon_\theta \theta}.
\] (16)

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where the parameter $\varepsilon_\theta$ will be defined later by the boundary conditions. As a result, the following differential equation for each spinor component is obtained

$$\left[ \frac{d^2}{dx^2} - \frac{\mu_\pm^2 - \frac{1}{4}}{x^2} + \frac{\nu}{x} - \frac{1}{4} \right] \Phi_\pm = 0$$

for constant potentials $V(r)$, $W(r)$ and $S(r)$. Here we have defined the variable $x = 2\gamma r$ and the three quantities $\nu = -\frac{eW\varepsilon_\theta}{\gamma}$, $\mu_\pm^2 = \left( \varepsilon_\theta \mp \frac{1}{2} \right)^2$, $\gamma^2 = (m + S)^2 + e^2W^2 - (\varepsilon - eV)^2$.

The radial equation (17) is solved in terms of the Whittaker hypergeometric functions $M_{\nu, \mu\pm}(2\gamma r)$ and $W_{\nu, \mu\pm}(2\gamma r)$. The general solution takes the form

$$\Phi_\pm(r) = A_\pm M_{\nu, \mu\pm}(2\gamma r) + B_\pm W_{\nu, \mu\pm}(2\gamma r)$$

where $M_{\nu, \mu\pm}(2\gamma r)$ and $W_{\nu, \mu\pm}(2\gamma r)$ are given in terms of confluent hypergeometric functions [16]

$$M_{\nu, \mu\pm}(2\gamma r) = e^{-\gamma r}(2\gamma r)^{\mu\pm+1/2} F_1(1/2 + \mu_\pm - \nu, 1 + 2\mu_\pm, 2\gamma r)$$

$$W_{\nu, \mu\pm}(2\gamma r) = e^{-\gamma r}(2\gamma r)^{\mu\pm+1/2} U(1/2 + \mu_\pm - \nu, 1 + 2\mu_\pm, 2\gamma r).$$

The general solution to our original problem can be written as follows

$$\Psi(r, \theta) = \frac{1}{\sqrt{r}} e^{i(\varepsilon_\theta - \frac{i}{2} \sigma_3)\theta} \Phi(r).$$

On the other hand, the boundary condition on the total wave function $\psi(r, \theta) = \psi(r, \theta + 2\pi)$ requires that $e^{i(2\pi \varepsilon_\theta - \sigma_3 \pi)} = 1$ which gives the following quantization rule for the parameter $\varepsilon_\theta$

$$\varepsilon_\theta = \frac{k}{2}, \quad k = \pm 1, \pm 3, \pm 5, \ldots.$$  (22)

Hence the most general solution of our problem reads

$$\Psi_\pm(r, \theta) = \sum_{k, \pm} \frac{1}{\sqrt{r}} e^{i(k - \sigma_3)\theta} \left[ A_\pm M_{\nu, \mu\pm}(2\gamma r) + B_\pm W_{\nu, \mu\pm}(2\gamma r) \right]$$

which are also eigenfunctions of the total angular momentum

$$J_z = L_z + \frac{1}{2} \sigma_3 = -i\partial_\theta + \frac{1}{2} \sigma_3.$$  (24)

A second potential configuration can be considered in our present work. It is defined by the potential parameters

$$A_0(\vec{r}) = V(\vec{r}), \quad A_r(\vec{r}) = R(\vec{r}), \quad A_\theta = \frac{W(\theta)}{r}, \quad S = S(\vec{r}).$$  (25)

In this case (12) can be written as

$$\begin{pmatrix}
  m + S + eV - \varepsilon & \partial_r + ieR \\
  -\partial_r - ieR & -m - S + eV - \varepsilon
\end{pmatrix} + \frac{1}{r} \begin{pmatrix}
  0 & -i\partial_\theta + eW \\
  -i\partial_\theta + eW & 0
\end{pmatrix} \begin{pmatrix}
  \Phi_+ F_+ \\
  \Phi_- F_-
\end{pmatrix} = 0$$

(26)

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where the structure of the $\theta$-dependent spinor component is dictated by the angular operator $-\frac{i}{r}\partial_\theta + eW(\theta)$. Thus, we can factorize the angular part by requiring

$$ F_+(\theta) = F_-(\theta) = F(\theta), \quad [-i\partial_\theta + eW(\theta)] F = \varepsilon F \quad (27) $$

whose solution is

$$ F(\theta) = e^{i[\varepsilon_\theta - e\int W(\theta)d\theta]} \quad (28) $$

The periodicity of the total wave function requires that $\Psi(r, \theta) = \Psi(r, \theta + 2\pi)$ and gives the quantized quantities $\varepsilon_\theta$ which will depend on the shape of the non-central part of the potential function $W(\theta)$. The radial part of the wave function can be simplified by gauging away the spatial part of the vector potential, i.e. $eR(r)$ term, and reduces to

$$ \left( \frac{m + S + eV - \varepsilon}{-\partial_r + \frac{\varepsilon_\theta}{r}} - m - S + eV - \varepsilon \right) \left( \Phi_+ \right) = 0 \quad (29) $$

Making the change of variable $X = \alpha r$ with $\alpha^2 = (\varepsilon - eV)^2 - (m + S)^2$, $\mu_\pm = (\varepsilon_\theta + \frac{1}{2})^2$. After some algebra we obtained the second order differential equation

$$ \left[ \frac{d^2}{dX^2} - \frac{\varepsilon_\theta(\varepsilon_\theta + 1)}{X^2} + 1 \right] \Phi_\pm = 0 \quad (30) $$

where we assumed the potentials $V$ and $S$ to be constant in each region of space. This equation has some common features with the one associated with Bessel functions. To clarify this statement, let us write the solution of (30) as $\Phi_\pm(X) = \sqrt{X}F_{\mu_\pm}(X)$ to obtain

$$ \left[ \frac{d^2}{dX^2} + \frac{1}{X} \frac{d}{dX} - \frac{\mu_\pm^2}{X^2} + 1 \right] F_{\mu_\pm} = 0 \quad (31) $$

The remaining radial equation (31) is solved in terms of the Bessel functions. Hence, the general solution is a linear combination of the two independent Bessel functions. To simplify our task, we choose $W(\theta)$ in such way that $\int W(\theta)d\theta = g(\theta)$, with $g(\theta)$ being a periodic function of $\theta$ that implies $F(\theta) = e^{i(\varepsilon_\theta - eg(\theta))}$. The boundary condition on the total wave function $\psi(r, \theta) = \psi(r, \theta + 2\pi)$ using the fact that $g(\theta) = g(\theta + 2\pi)$ requires that $e^{i(2\pi\varepsilon_\theta - \sigma_3\pi)} = 1$ giving rise to the quantum number

$$ \varepsilon_\theta = \frac{k}{2}, \quad k = \pm 1, \pm 3, \pm 5, \cdots \quad (32) $$

which will be denoted as $\varepsilon_\theta = j$, with $j$ being a half integer. Finally, the most general solution is given by

$$ \psi_{\pm}(r, \theta) = \sum_{k, \pm} \sqrt{\alpha} e^{\frac{i}{2}(k-\sigma_3)\theta} \left[ A'_{\pm} J_{\mu_\pm}(\alpha r) + B'_{\pm} Y_{\mu_\pm}(\alpha r) \right]. \quad (33) $$

These results will be applied to graphene in the following sections, where we will set our units at $\hbar v_F = 1$ and consider the presence of an induced mass term $m$.

It is worth mentioning that the choice of the potential configurations are considered not only due to their separability properties, but also due to the fact that there is also a physical motivation. Indeed, for example the scalar potential $S(r)$ can be seen as a mass term that is very much needed for systems made of graphene. This term can be
created by introducing the underlying substrate. In addition, the vector potential $V(r)$ can be created by applying a local top gate voltage. Finally, the pseudo-scalar potential $W(r)$ can be produced by considering the physical system in the presence of an external magnetic field.

4. Graphene quantum dot

We use our previous work [9] and the above formalism to investigate an interesting case study on graphene. Indeed, we first show how our findings can model an isolated quantum dot and then study the physical properties of this quantum dot which might lead to interesting applications.

4.1. Isolated quantum dot

Here we consider graphene in the presence of a constant mass term $m$, that induces a gap of $2m$, which can be realized by different experimental methods in graphene systems [12, 17]. To realize a quantum dot in graphene we consider cylindrically symmetrical potentials associated with vector potential $V(r) = V_0 \Theta(R - r)$ and scalar potential $S(r) = S_0 \Theta(R - r)$, where $S_0$ and $V_0$ are real constants, $R$ represents the radius of the QD. We would like to study the potential existence of bound states within a quantum dot. The solutions to the Dirac equation, which describe the electronic states inside and outside a quantum dot, are given in terms of the Bessel function of the first and second kind $J_{\mu}(x), Y_{\mu}(x)$, the modified Bessel function $I_{\mu}(x), K_{\mu}(x)$, and the Hankel function of first and second kind $H^{(1,2)}_{\mu}(x)$. These solutions will be propagating if the wave vectors are real and decaying (exponentially decaying) if the wave vectors are imaginary. We summarize our findings in table 1, where we show the different domains chosen according to whether $\alpha$ (for $r < R$) and $\alpha'$ (for $r > R$) are purely imaginary or real and we give the suitable Bessel functions for each case, which describe the radial part of the electronic wavefunction inside and outside the QD.

We note that inside and outside the dot, the wave vectors are $\alpha^2 = (\varepsilon - V)^2 - (m + S)^2$ and $\alpha'^2 = \varepsilon^2 - m^2$, respectively. In domain I, one has $m + S_0 < |\varepsilon - V_0|$ and $m < \varepsilon$, so that both wave vectors $\alpha$ and $\alpha'$ are real, the wave function oscillates inside and outside the QD. In domain II, $m + S_0 < |\varepsilon - V_0|$ and $m > \varepsilon$, the wave vector $\alpha$ is real and $\alpha'$ is purely imaginary. In this region, we have true bound states that oscillate inside the QD and decay outside it. In domain III, we have $m + S_0 > |\varepsilon - V_0|$ and $m < \varepsilon$, which implies $\alpha$ is purely imaginary and $\alpha'$ is real, giving rise to the tunneling regime, that is, the wave function decays inside and oscillates outside the QD. Domain IV is characterized by $m + S_0 > |\varepsilon - V_0|$ and $m > \varepsilon$, which both give $\alpha$ and $\alpha'$ as being purely imaginary and hence the wave function decays inside and outside the QD. In domain II, the general solutions to the radial Dirac equation, which are regular at the origin and which decay exponentially as $r \to \infty$, are given in terms of Bessel function $A_{\pm} J_{|\frac{\varepsilon + V_0}{2}|}(\alpha r)$ inside the QD and $B_{\pm} K_{|\frac{\varepsilon + V_0}{2}|}(\eta r)$ outside the QD. We note that the two other functions diverge ($Y_{\mu}(x)$ for $r \to 0$ and $I_{\mu}(x)$ for $r \to \infty$).

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The general solutions to the Dirac equation, taking a positive value of \( j \), are given by

\[
\Psi(r, \theta) = \begin{cases} 
\sum_j A_\pm J_{|j+\frac{1}{2}|}(\alpha r) e^{i(j+\frac{1}{2}) \theta}, & r < R \\
\sum_j B_\pm K_{|j+\frac{1}{2}|}(\eta r) e^{i(j+\frac{1}{2}) \theta}, & r > R.
\end{cases}
\]  

\[
A_\pm = \frac{\sqrt{\varepsilon - V_0 + (m + S_0)}}{\sqrt{\varepsilon - V_0 - (m + S_0)}}, \quad B_\pm = \frac{\sqrt{m + \varepsilon}}{\sqrt{m - \varepsilon}}.
\]  

Then we obtain the following characteristic equation of the QD

\[
\xi_j J_{|j-\frac{1}{2}|}(\alpha R) K_{j+\frac{1}{2}}(\eta R) - \xi_{-j-\frac{1}{2}}(\alpha R) K_{j-\frac{1}{2}}(\eta R) = 0
\]  

where \( \xi = \sqrt{\varepsilon - V_0 \pm (m + S_0)} \). We can see that (36) is symmetrical with respect to the change in the sign of \( j \). Thus, we note that for the other valley \( K' \), the corresponding equation is obtained by replacing \( j \) by \( -j \) in (36). Therefore, below we only consider the positive values of \( j \), i.e. \( j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \).

In figure 1 we show the contour plot of the energy of the bound state in a graphene QD as function of the radius of the QD, for three different values of the half integer \( j(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}) \) and for the strength of the mass term \( m = 0.2 \) and vector potential \( V_0 = 0.8 \). We evaluate the characteristic equation (36) for two different values of the scalar potential \( S_0 \), \( (S_0 = 0.35 \) and \( S_0 = 0.5) \). We note that if we consider both valleys \( K \) and \( K' \), we find that each bound state is doubly degenerate. In figure 1(b), we can clearly see the effect of the scalar potential; in fact when we increase the value of the scalar potential the number of bound states decreases. We also show that when we increase the radius of the dot, the
Figure 1. The contour plot of the energy of the bound states in a graphene quantum dot as a function of the radius $R$ of the quantum dot, with $j = \frac{1}{2}, j = \frac{3}{2}, j = \frac{5}{2}, m = 0.2, V_0 = 0.8$ and $S_0 = 0.35$ for (a), $S_0 = 0.5$ for (b).

Figure 2. The contour plot of the energy of the bound states in a graphene quantum dot as a function of the potential strength $V_0$, with $j = \frac{1}{2}, j = \frac{3}{2}, j = \frac{5}{2}, m = 0.2, R = 5$ and $S_0 = 0.1$ for (a), $S_0 = 0.8$ for (b).

number of bound states increases mainly for large $R$. Finally, we observe that more bound states occur in the QD when its radius increases.

Figure 2 shows the contour plot of the energy of the bound state in a graphene QD as a function of the potential strength $V_0$ for three values of $j$ $\left(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}\right)$, the mass term used
is $m = 0.2$ and the radius of the QD is taken to be $R = 5$. To see the effect of the scalar potential we used two different values $S_0 = 0.1$, as shown in figure 2(a) and $S_0 = 0.8$, as shown in figure 2(b).

From the above computations we note that more bound states can be accommodated in a QD when either the strength of the confining potential or the radius of the QD increases. These results are in agreement with our previous work on bound state energies in radially symmetric graphene QD [8, 13]. In figure 3 we show six bound states for $V_0 = 0.6$ and $R = 30$: two for $j = \frac{1}{2}$, two for $j = \frac{3}{2}$ and two for $j = \frac{5}{2}$.

At this stage, let us consider the case of spin symmetric configuration with $V = S$. Within the framework of the Dirac equation, this configuration occurs when the difference between the Lorentz vector potential $V(r)$ and the Lorentz scalar potential $S(r)$ is constant, that is $\Delta(r) = V(r) - S(r) = \text{constant}$, which in this case this constant is taken to be zero. The near experimental realization of the spin symmetric potential configuration may explain the degeneracy in some heavy meson spectra [18]. In this case (36) becomes

$$\sqrt{m - \varepsilon} J_{j - \frac{1}{2}}(\alpha R)K_{j + \frac{1}{2}}(\eta R) - \sqrt{\varepsilon} - (m + 2S_0)J_{j + \frac{1}{2}}(\alpha R)K_{j - \frac{1}{2}}(\eta Rj) = 0. \quad (37)$$

Similar to what we have seen in (36) and (37) is also symmetrical with respect to the transformation $j \rightarrow -j$.

In figure 4 we show the solution to equation (37). We note that when $R = 8$ and $S_0 = 1.6$ we have eight bound states. Three for $j = \frac{1}{2}$, three for $j = \frac{3}{2}$, and two for $j = \frac{5}{2}$. We can also show that when we increase the radius of the dot or the strength of the scalar potential we accommodate more bound states.

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Figure 4. (a) The contour plot of the energy of the bound states in a graphene quantum dot as a function of the radius $R$. (b) The contour plot of the energy of the bound states in a graphene quantum dot as a function of the potential $S_0$. With $j = \frac{1}{2}$ (red line), $j = \frac{3}{2}$ (green line), $j = \frac{5}{2}$ (blue line), $m = 0.2$, and $S_0 = 1.5$ for (a), $R = 8$ for (b).

4.2. Specific potential configuration

We consider the first configuration discussed in section 3 and study graphene in the presence of a mass term $m$ that induces a gap value of $2m$. The solution to the Dirac equation in this case (23) is given in terms of the Whittaker hypergeometric functions $M_{\nu, \mu\pm}(2\gamma r)$ and $W_{\nu, \mu\pm}(2\gamma r)$. Then, the general solution, which is regular at the origin and decays exponentially at large $r$, can be written in terms of the confluent hypergeometric function (19) and (20) as

\[
(2\gamma)^{\mu\pm}\frac{1}{2}e^{-\gamma r}\mu\pm \left\{ \begin{array}{ll}
A_{\pm} & _1F_1\left(\frac{1}{2} + \mu\pm - \nu, 1 + 2\mu\pm, 2\gamma r\right), & r < R \\
B_{\pm} & U\left(\frac{1}{2} + \mu\pm - \nu, 1 + 2\mu\pm, 2\gamma r\right), & r > R 
\end{array} \right. 
\]  

(38)

where $\mu\pm = j \mp \frac{1}{2}$, $\nu = -\frac{W}{2}j$ and $\gamma^2 = (m + S)^2 + W^2 - (\varepsilon - eV)^2$.

At this stage let us investigate the bound states. For this purpose, we adopt the explicit potential configuration

\[
V(r) = \begin{cases} 
V_0 & r < R \\
0 & r > R 
\end{cases}, \quad W(r) = \begin{cases} 
W_0 & r < R \\
0 & r > R 
\end{cases}, \quad S(r) = \begin{cases} 
S_0 & r < R \\
0 & r > R 
\end{cases} 
\]

(39)

which will help us to simplify the above established formalism. The two matching conditions of the spinors at $r = R$ give

\[
\gamma^j e^{-\gamma R} A_+ _1F_1(j - \nu, 2j, 2\gamma R) = \gamma^{j'} e^{-\gamma' R} B_+ U(j, 2j, 2\gamma' R) 
\]

(40)

\[
\gamma^{j+1} e^{-\gamma R} A_- _1F_1(j + 1 - \nu, 2(j + 1), 2\gamma R) = \gamma^{j+1} e^{-\gamma' R} B_- U(j + 1, 2(j + 1), 2\gamma' R). 
\]

(41)

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Now, inside and outside the QD we have, respectively, the quantities $\gamma^2 = (m + S_0)^2 + W_0^2 - (\varepsilon - V_0)^2$ and $\gamma'^2 = m^2 - \varepsilon^2$. The condition for the existence of the bound states can be written as follows

$$\frac{\gamma \frac{A_+}{A_-}}{\gamma' \frac{B_+}{B_-}} = \frac{U(j + 1, 2(j + 1), 2\gamma R)}{U(j + 1, 2(j + 1), 2\gamma'R)} \quad (42)$$

where the ratios $\frac{A_+}{A_-}$ and $\frac{B_+}{B_-}$ are fixed by the coupled differential equation (17) using the general solutions for $r < R$ and $r > R$. After some straightforward but lengthy algebra we obtain

$$\frac{A_+}{A_-} = -2 \frac{\sqrt{(m + S_0)^2 + W_0^2 - (\varepsilon - V_0)^2}}{m + S_0 - \varepsilon + V_0} (2j + 1), \quad \frac{B_+}{B_-} = \frac{\sqrt{m + \varepsilon}}{\sqrt{m - \varepsilon}} \quad (43)$$

and (42) becomes

$$\xi_1 F_1(j - \nu, 2j, 2\gamma R) U(j + 1, 2(j + 1), 2\gamma R) + \zeta_1 F_1(j + 1 - \nu, 2(j + 1), 2\gamma'R) \times U(j, 2j, 2\gamma'R) = 0 \quad (44)$$

where $\xi = 2(2j + 1)(m - \varepsilon)$, $\zeta = (m + S_0 - (\varepsilon - V_0))$, $\gamma = \sqrt{(m + S_0)^2 + W_0^2 - (\varepsilon - V_0)^2}$, $\gamma' = \sqrt{m^2 - \varepsilon^2}$ and $\nu = -\frac{W_0}{\gamma}j$.

In figure 5 we show the energy of a QD as a function of the dot radius $R$. Evaluating (44), we show the states only for $j = \frac{1}{2}$, $j = \frac{3}{2}$, $j = \frac{5}{2}$. We use $m = 0.6$, $V_0 = 1.6$, $S_0 = 0.2$ and $W_0 = 0$ in figure 5(a) $W_0 = 0.5$ in figure 5(b). We can clearly see the effect of the pseudoscalar potential, its presence leading to a reduction in the number of bound states. In the situation where $W_0 = 0$ and $R = 5$ we have eight bound states: three with $j = \frac{1}{2}$, three with $j = \frac{3}{2}$, and two with $j = \frac{5}{2}$. On the other hand, for $W_0 = 0.5$ and for the same
Figure 6. The contour plot of the energy of the bound states in a graphene quantum dot as a function of the potential \( V_0 \) with \( j = \frac{1}{2}, j = \frac{3}{2}, j = \frac{5}{2} \), \( m = 0.6, S_0 = 0.2, R = 5 \) and \( W_0 = 0 \) for (a), \( W_0 = 0.5 \) for (b).

value of \( R \) (\( R = 5 \)), we have only seven bound states: three with \( j = \frac{1}{2} \), two with \( j = \frac{3}{2} \), and two with \( j = \frac{5}{2} \). From figure 5 it is clearly seen that when we increase the radius of the QD, more bound states can be accommodated in the QD; the same remark was made for the first potential configuration. The presence of a pseudoscalar potential leads to a reduction in the number of bound states.

In figure 6 we show the evolution of the energy of the bound states as a function of the strength of the potential \( V_0 \). Similar to figure 5, we present the states only with \( j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \). We use \( m = 0.6, S_0 = 0.2, R = 5 \) and \( W_0 = 0 \) in figure 6(a) and \( W_0 = 0.5 \) in figure 6(b).

We note that when \( V_0 = 1.6 \) and \( W_0 = 0 \) we have eight bound states: three with \( j = \frac{1}{2} \), three with \( j = \frac{3}{2} \), and two with \( j = \frac{5}{2} \). On the other hand, when \( W_0 = 0.5 \), for the same \( V_0 = 1.6 \) we only have seven bound states: three with \( j = \frac{1}{2} \), two with \( j = \frac{3}{2} \), and two with \( j = \frac{5}{2} \). These results are in good agreement with figure 5. It can be clearly seen in figure 6 that when we increase the radius of the QD we can accommodate more bound states than in the first case. In conclusion, the presence of a pseudoscalar potential leads to a reduction in the number of bound states.

5. Transport properties of graphene quantum dot

In order to calculate the transport properties through a circular graphene quantum dot, we consider the spin symmetric potential configuration with \( V = S \) and use a radially symmetric potential profile

\[
S(r) = \begin{cases} 
U, & r < a \\
0, & a < r < b \\
S, & b < r.
\end{cases}
\]  

(45)
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This potential represents a quantum dot with a radius of \( a \) connected to an environment represented by the external region \( r > b \). The bound states of the quantum dot are allowed to leak through the junction region \( a < r < b \), to reach the environment at \( r > b \). The general solution to the Dirac equation is oscillatory for \( r < a \) and decays exponentially for \( b < r \), so that we can have incoming and outgoing waves in the intermediate region \((a < r < b)\) and define a scattering cross section. The radial solutions of the Dirac equation are given by

\[
\begin{align*}
A_{\pm} J_{\nu_{\pm}}(\beta r), & \quad r < a \\
A_{\pm} K_{\nu_{\pm}}(\eta r) + B_{\pm} I_{\nu_{\pm}}(\eta r), & \quad a < r < b \\
A_{\pm} H_{\nu_{\pm}}^{(1)}(\alpha r) + B_{\pm} H_{\nu_{\pm}}^{(2)}(\alpha r), & \quad b < r
\end{align*}
\]  

(46)

where \( \alpha = \sqrt{(\epsilon - (m + 2S))(m + \varepsilon)} \), \( \eta = \sqrt{m^2 - \varepsilon^2} \) and \( \beta = \sqrt{(\epsilon - (m + 2U))(m + \varepsilon)} \).

The Dirac equation

\[
H \Psi = \varepsilon \Psi,
\]

has a scattering solution and the corresponding asymptotic form can be written as \([19]\)

\[
\Psi(r, \theta) = \Psi_{in}(r, \theta) + \Psi_{out}(r, \theta)
\]

(47)

where \( \Psi_{in}(r, \theta) \) is the incoming plane wave in the \( x \)-direction and \( \Psi_{out}(r, \theta) \) is the scattered outgoing spherical wave. These solutions have asymptotic behavior as \( r \to \infty \) given by

\[
\Psi_{in}(r, \theta) = \frac{1}{\sqrt{2}} e^{i\alpha r \cos \theta} \left( \frac{\sqrt{m + \varepsilon}}{\sqrt{\varepsilon - (m + 2S)}} \right)
\]

(48)

\[
\Psi_{out}(r, \theta) = \frac{e^{i\alpha r}}{\sqrt{\pi \alpha r}} f(\theta) \left( \frac{\sqrt{m + \varepsilon}}{\sqrt{\varepsilon - (m + 2S)}} \right)
\]

(49)

where \( f(\theta) \) defines the scattering amplitude and \( \theta \) is the scattering angle. Next, we use the decomposition of the plane wave

\[
e^{i\alpha x} = \sum_{m=-\infty}^{\infty} i^m J_m(k_2 r)e^{im\theta}
\]

(50)

and the asymptotic forms of the Bessel functions at large \( r \)

\[
J_m(\alpha r) \approx \frac{1}{\sqrt{2\pi \alpha r}} \left( e^{i(\alpha r - \frac{\alpha \pi}{2} - \frac{\pi}{4})} + e^{-i(\alpha r - \frac{\alpha \pi}{2} - \frac{\pi}{4})} \right)
\]

(51)

\[
H_{m}^{(1,2)}(\alpha r) \approx \sqrt{\frac{2}{\pi \alpha r}} e^{\pm i(\alpha r - \frac{\alpha \pi}{2} - \frac{\pi}{4})}.
\]

(52)

In order to extract more information about the present system, let us study the differential and total cross sections. These are given by \([19]\)

\[
\frac{d\Lambda(\theta)}{d\theta} = \frac{|f_{\pm}(\theta)|^2}{2\pi \alpha}, \quad \Lambda = \oint |f_{\pm}(\theta)|^2 \frac{d\theta}{2\pi \alpha}
\]

(53)

where the scattering amplitude \( f_{\pm}(\theta) \) can be expanded in a Fourier series with the coefficient \( f_j \), such as

\[
f_{\pm}(\theta) = \frac{1}{\sqrt{2}} \sum_{j} e^{i((\theta \pm \frac{1}{2}) - \frac{\pi}{4})\theta} f_j
\]

(54)

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Combining all these results to write the asymptotic solutions as
\[ \Psi_{\pm}(r, \theta) = \sum_{m=-\infty}^{\infty} \frac{i^m}{\sqrt{2\pi \alpha r}} \left[ (1 + f_j) e^{i(\alpha r - \frac{m}{2} \pi - \frac{\pi}{4})} + e^{-i(\alpha r - \frac{m}{2} \pi - \frac{\pi}{4})} \right] \left( \frac{\sqrt{m+S}}{\sqrt{\varepsilon - (m+2S)}} \right) e^{im\theta}. \]  
(55)

Turning back to the solution in (46), the asymptotic form for the spinor wave function when \( b < r \) can be expressed as
\[ \Phi_j(r) \approx \sqrt{\frac{2}{\pi \alpha r}} \left( A_{2+} e^{i(\alpha r - \frac{m}{2} \pi - \frac{\pi}{4})} + B_{2+} e^{-i(\alpha r - \frac{m}{2} \pi - \frac{\pi}{4})} \right) \]  
(56)
where the parameters are defined by
\[ A_{2+} = \sqrt{m + \varepsilon S_j}, \quad A_{2-} = \sqrt{\varepsilon - (m+2S)S_j} \]  
(57)
\[ B_{2+} = \sqrt{m + \varepsilon}, \quad B_{2-} = \sqrt{\varepsilon - (m+2S)}. \]  
(58)

We have defined the scattering matrix \( S_j \) by \( S_j = f_j + 1 \). Applying the matching conditions at the boundaries \( r = a \) and \( r = b \) we obtain
\[ A_+ J_{j+\frac{1}{2}}(\beta a) = A_+ K_{j-\frac{1}{2}}(\eta a) + B_+ I_{j-\frac{1}{2}}(\eta a) \]
\[ A_- J_{j+\frac{1}{2}}(\beta a) = A_- K_{j+\frac{1}{2}}(\eta a) + B_- I_{j+\frac{1}{2}}(\eta a) \]
\[ A_+ K_{j-\frac{1}{2}}(\eta b) + B_+ I_{j-\frac{1}{2}}(\eta b) = A_+ H^{(1)}_{j+\frac{1}{2}}(ab) + B_+ H^{(2)}_{j+\frac{1}{2}}(ab) \]
\[ A_- K_{j+\frac{1}{2}}(\eta b) + B_- I_{j+\frac{1}{2}}(\eta b) = A_- H^{(1)}_{j+\frac{1}{2}}(ab) + B_- H^{(2)}_{j+\frac{1}{2}}(ab) \]

The relationship between the different coefficients reads
\[ \frac{A_{1+}}{A_{1-}} = \sqrt{\frac{m+\varepsilon}{m-\varepsilon}}, \quad \frac{B_{1+}}{B_{1-}} = -\sqrt{\frac{m+\varepsilon}{m-\varepsilon}}, \quad \frac{A_+}{A_-} = \frac{\sqrt{m+\varepsilon}}{\sqrt{\varepsilon - (m+2S)}}. \]  
(59)

Defining the matrix
\[ M^{(1,2)} = \begin{pmatrix} -\frac{\sqrt{m+\varepsilon}J_{j+\frac{1}{2}}(\beta a)}{\sqrt{m-\varepsilon}} & \frac{\sqrt{m+\varepsilon}K_{j-\frac{1}{2}}(\eta a)}{\sqrt{m-\varepsilon}} & 0 \\ -\frac{\sqrt{m-\varepsilon}K_{j-\frac{1}{2}}(\beta a)}{\sqrt{m+\varepsilon}} & \frac{\sqrt{m-\varepsilon}I_{j+\frac{1}{2}}(\eta a)}{\sqrt{m+\varepsilon}} & 0 \\ 0 & 0 & \frac{\sqrt{m+\varepsilon}H^{(1,2)}_{j+\frac{1}{2}}(\beta a)}{\sqrt{m-\varepsilon}} \end{pmatrix} \]  
(60)

After some straightforward algebra we obtain the scattering matrix as
\[ S_j = -\frac{\det M^2}{\det M^1}. \]  
(61)

In figure 7 we show the angular dependence of the differential cross section. The differential cross section shows a narrow maximum at \( \theta = 0 \) and has a zero minimum when \( \theta = \pi/2 \). In addition the differential cross section is no longer symmetrical with respect to the sign of the incident angle. In figure 7(a), we fix \( S \) at 0.2 and take two different
values for $U$, $U = 0.2$ corresponding to the green curve and $U = 0.8$ corresponding to the red one. One can see that the maximum value of the differential cross section increases when $U$ decreases to a value that is greater than 0.2. In figure 7(b), we fix $S = 0.8$ and take two different values of $U$, the orange line corresponds to $U = 0.2$ and the blue line corresponds to $U = 0.8$. This shows that the maximum of the differential cross section increases when the value of $U$ increases even for values which are still smaller than 0.8.

6. Scattering through quantum dot

Below we develop the scattering theory for the 2D Dirac fermions in the presence of an axially symmetric potential using the second potential configuration. The Dirac equation $H \Psi = E \Psi$ has scattering solutions that have the asymptotic form \[ \Psi(r, \theta) = \Psi_{in}(r, \theta) + \Psi_{out}(r, \theta). \] These solutions have the following asymptotic forms as $r \to \infty$

\[
\Psi_{in}(r, \theta) = \frac{1}{\sqrt{2}} e^{i \alpha r \cos \theta} \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

\[
\Psi_{out}(r, \theta) = \frac{e^{i \alpha r}}{\sqrt{-2i \alpha r}} \begin{pmatrix} f_+(\theta) \\ f_-(\theta) \end{pmatrix}
\]

where $\Psi_{in}(r, \theta)$ is the incoming plane wave in the $x$-direction and $\Psi_{out}(r, \theta)$ is the scattered outgoing spherical wave; $f_{\pm}(\theta)$ is the scattering amplitude and $\theta$ is the scattering angle. Furthermore $f_-(\theta) = e^{i \theta} f_+(\theta)$. The incoming particle current density is $\alpha$. The number of
particles leaving per unit time radially in the direction $\theta$ is $\frac{d\theta}{\pi} |f(\theta)|^2$. Thus the differential scattering cross section is
\[
\frac{d\sigma(\theta)}{d\theta} = \frac{|f_\pm(\theta)|^2}{\pi\alpha}.
\] (64)

For large $r$, the solution of the Dirac equation is given as a function of the Hankel functions of the first and second kind $H^{(1,2)}_{\mu}(\alpha r) = J_{\mu}(\alpha r) \pm i Y_{\mu}(\alpha r)$, where the upper sign holds for superscript 1 and the lower one for 2. Their asymptotic behavior when $\alpha r \to \infty$ is given by
\[
J_{\mu}(\alpha r) \approx \sqrt{\frac{2}{\pi \alpha r}} \cos \left( \alpha r - \frac{(2\mu + 1)}{4} \pi \right),
\] (65)
\[
Y_{\mu}(\alpha r) \approx \sqrt{\frac{2}{\pi \alpha r}} \sin \left( \alpha r - \frac{(2\mu + 1)}{4} \pi \right),
\] (66)
\[
H^{(1,2)}_{\mu}(\alpha r) \approx \sqrt{\frac{2}{\pi \alpha r}} e^{\pm i(\alpha r - \frac{(2\mu + 1)}{4} \pi)}.
\] (67)

We note that $\Psi_{\text{out}}$ for $r \to \infty$ indeed has the form of (67). Using the decomposition of the plane wave $[20]$
\[
e^{i\alpha r \cos \theta} = \sum_j i^{j+\frac{1}{2}} e^{i(j+\frac{1}{2})\theta} J_{j+\frac{1}{2}}(\alpha r)
\] (68)
gives
\[
\Psi_{\text{in}} = \left( \sum_j i^{j-\frac{1}{2}} J_{j-\frac{1}{2}}(\alpha r) e^{i(j-\frac{1}{2})\theta} \right) \left( \sum_j i^{j+\frac{1}{2}} J_{j+\frac{1}{2}}(\alpha r) e^{i(j+\frac{1}{2})\theta} \right),
\] (69)

for the incoming wave. We use the defining equation
\[
f_\pm(\theta) = \frac{1}{\sqrt{2}} \sum_j f_j e^{i(j+\frac{1}{2})\theta - i\frac{\pi}{4}}
\] (70)
so that the outgoing wave function when $\alpha r \to \infty$ has the following form
\[
\Psi_{\text{out}} = \left( \sum_j i^{j-\frac{1}{2}} f_j \left[ J_{j-\frac{1}{2}}(\alpha r) + i Y_{j-\frac{1}{2}}(\alpha r) \right] e^{i(j-\frac{1}{2})\theta} \right) \left( \sum_j i^{j+\frac{1}{2}} f_j \left[ J_{j+\frac{1}{2}}(\alpha r) + i Y_{j+\frac{1}{2}}(\alpha r) \right] e^{i(j+\frac{1}{2})\theta} \right).
\] (71)

We assume that the scattering defect has a finite radius of $R$, so, when $r > R$ we can write the wave functions as a superposition of terms such as
\[
\Psi_j = \left( \left[ J_{j-\frac{1}{2}}(\alpha r) + R_j Y_{j-\frac{1}{2}}(\alpha r) \right] e^{i(j-\frac{1}{2})\theta} \right) \left( \left[ J_{j+\frac{1}{2}}(\alpha r) + R_j Y_{j+\frac{1}{2}}(\alpha r) \right] e^{i(j+\frac{1}{2})\theta} \right)
\] (72)

where $R_j$ is the reflection coefficients and the complex number $i$ is introduced for further convenience. From (69), (71) and (72) we can write
\[
\Psi_{\text{in}} + \Psi_{\text{out}} = \sum_j \beta_j \Psi_j
\] (73)
and one can deduce the expression of \( f_j \)

\[
 f_j = \frac{R_j}{1 - R_j}. \tag{74}
\]

The expression of the scattering amplitude can then be written in the compact form of

\[
 f_±(\theta) = \sum_j \frac{R_j}{1 - R_j} e^{i(j+\frac{1}{2})\theta - i\frac{\pi}{4}}. \tag{75}
\]

We note that the back-scattering amplitude vanishes \( f(\pi) = 0 \), which is a consequence of the pseudospin conservation for chiral scattering [21] and is related to the Klein paradox [22].

Now we consider a circular potential barrier in graphene and choose \( V(r) = V_0 \Theta(R-r) \) and \( S(r) = S_0 \Theta(R - r) \) where \( \Theta \) is the heaviest step function. Using the boundary conditions, with continuity of the eigenspinors at \( r = R \), we obtain

\[
 J_{j-\frac{1}{2}}(\alpha R) + R_j Y_{j-\frac{1}{2}}(\alpha R) = T_j J_{j-\frac{1}{2}}(\alpha' R) \tag{76}
\]

\[
 J_{j+\frac{1}{2}}(\alpha R) + R_j Y_{j+\frac{1}{2}}(\alpha R) = T_j J_{j+\frac{1}{2}}(\alpha' R) \tag{77}
\]

where \( \alpha = \sqrt{(\varepsilon - V_0)^2 - (\Delta + S_0)^2} \) and \( \alpha' = \sqrt{\varepsilon^2 - \Delta^2} \). The reflection coefficient is then given by

\[
 R_j = -\frac{J_{j-\frac{1}{2}}(\alpha R) J_{j+\frac{1}{2}}(\alpha' R) - J_{j+\frac{1}{2}}(\alpha R) J_{j-\frac{1}{2}}(\alpha' R)}{Y_{j-\frac{1}{2}}(\alpha R) J_{j+\frac{1}{2}}(\alpha' R) - Y_{j+\frac{1}{2}}(\alpha R) J_{j-\frac{1}{2}}(\alpha' R)}. \tag{78}
\]

Similarily for the transmission coefficient

\[
 T_j = \frac{Y_{j+\frac{1}{2}}(\alpha R) J_{j-\frac{1}{2}}(\alpha R) - Y_{j-\frac{1}{2}}(\alpha R) J_{j+\frac{1}{2}}(\alpha R)}{Y_{j+\frac{1}{2}}(\alpha R) J_{j-\frac{1}{2}}(\alpha' R) - Y_{j-\frac{1}{2}}(\alpha R) J_{j+\frac{1}{2}}(\alpha' R)}. \tag{79}
\]

We have the relations \( J_{-n}(x) = (-1)^n J_n(x) \) and \( Y_{-n}(x) = (-1)^n Y_n(x) \) [20]. If we take \( n = j - \frac{1}{2}, n \) as an integer, we get \( R_n = R_{-n+1} \), so the back-scattering amplitude vanishes, and \( T_n = T_{-n-1} \), that is \( n \leftrightarrow -n-1 \), which means \( f_n = f_{-n-1} \). Thus, (64) can be rewritten in the compact form

\[
 \frac{d\sigma(\theta)}{d\theta} = \frac{1}{\pi\alpha} \left| \sum_{n=0}^{\infty} f_n \cos (n + \frac{1}{2})\theta \right|^2. \tag{80}
\]

The angular dependence of the cross section is shown in figure 8, where we have the cross section as a function of the incident angle \( \theta \), which shows a narrow maximum at \( \theta = 0 \).

In figure 8 we plot the dependence of the differential cross section on the incident angle \( \theta \) choosing different values of the parameters. We show that the curve is symmetrical around the axis \( \theta = 0 \). In addition, it can be clearly seen that the differential cross section exhibits a narrow maximum around \( \theta = 0 \) and vanishes when \( \theta \) becomes \( \pm \pi \). The differential cross section shows resonances associated with the QD quasi-bound states [21].

\[
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\]
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Figure 8. Angular dependence of the differential cross section, $d\sigma(\theta)/d\theta$, in nanometers, for the radius $R = 15\,\text{nm}$ with the vector potential $V_0 = 40\,\text{meV}$. The green line corresponds to $\varepsilon = 20\,\text{meV}$, the scalar potential $S_0 = 8\,\text{meV}$ and with the mass term $m = 7\,\text{meV}$. The red line corresponds to $\varepsilon = 15\,\text{meV}$, $S_0 = 5\,\text{meV}$ and with the mass term $m = 2.5\,\text{meV}$.

7. Conclusion

We have studied the Dirac equation in $(2 + 1)$-dimensions, where we included all the types of potential coupling: vector, pseudoscalar and scalar. We used the method of separating the variables into polar coordinates to obtain the general spinor eigenfunctions and the associated energy spectra for two special potential configurations. The obtained results were used to investigate an interesting system made of graphene. This concerns an isolated quantum dot (QD), which was realized by considering a cylindrically symmetric potential associated with the vector potential $V(r) = V_0\Theta(R - r)$ and scalar potential $S(r) = S_0\Theta(R - r)$.

Subsequently, we have discussed the behaviors of the eigenstates occurring in domains (I, II, III, IV) according to the nature of the wave vectors $\alpha$ and $\alpha'$ (real or imaginary). The dependence of the bound states on the electrostatic potential and size of the QD were presented numerically. We also showed the influence of the different potential couplings: vector, pseudo-scalar, and scalar on the QD. For both potential configurations, we showed that when the radius of the QD or the strength of the confining potential is increased, more bound states can be accommodated in the QD. These results are in agreement with the previous calculations of bound-state energies [8,13].

Finally, by adding an environment to the isolated QD we developed a scattering theory for the 2D Dirac fermions in the presence of the axially symmetric potential. We computed and numerically discussed the angular dependence of the differential cross section. We showed that it is no longer symmetrical with respect to the sign of the incident angle; it has a maximum angle of around $\theta = 0$ and exhibits resonances associated with quantum dot quasi-bound states. Moreover, we also found that the variation in potential strength changes the maximum of the differential cross section.

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