Looking for martingales associated to a self-decomposable law

F. Hirsch∗ M.Yor†

Abstract

We construct martingales whose 1-dimensional marginals are those of a centered self-decomposable variable multiplied by some power of time $t$. Many examples involving quadratic functionals of Bessel processes are discussed.

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1 Introduction, Motivation

1.1 Notation

We first introduce some notation which will be used throughout our paper.

If $A$ and $B$ are two random variables, $A \overset{d}{=} B$ means that these variables have the same law.

If $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ are two processes, $(X_t) \overset{(1.d)}{=} (Y_t)$ means that the processes $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ have the same one-dimensional marginals, that is, for any fixed $t$, $X_t \overset{d}{=} Y_t$.

If $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ are two processes, $(X_t) \overset{(d)}{=} (Y_t)$ means that the two processes are identical in law.

All random variables and processes which will be considered are assumed to be real valued.

1.2 PCOC's

In a number of applied situations involving randomness, it is a quite difficult problem to single out a certain stochastic process $(Y_t, t \geq 0)$, or rather its law, which is coherent with the real-world data.

In some cases, it is already nice to be able to consider that the one-dimensional marginals of $(Y_t)$ are accessible. The random situation being studied may suggest, for instance, that:

(i) there exists a martingale $(M_t)$ such that

$$(Y_t) \overset{(1.d)}{=} (M_t)$$

(this hypothesis may indicate some kind of “equilibrium” with respect to time),

(ii) there exists $H > 0$ such that

$$(Y_t) \overset{(1.d)}{=} (t^H Y_1)$$

(there is a “scaling” property involved in the randomness).

It is a result due to Kellerer [14] that (i) is satisfied for a given process $(Y_t)$ if and only if this process is increasing in the convex order, that is: it is integrable ($\forall t \geq 0$, $E[|Y_t|] < \infty$), and for every convex function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$,

$$t \geq 0 \mapsto E[\varphi(Y_t)] \in (-\infty, +\infty]$$

is increasing.

In the sequel, we shall use the acronym PCOC for such processes, since, in French, the name of such processes becomes: Processus Croissant pour l’Ordre Convexe.

A martingale $(M_t)$ which has the same one-dimensional marginals as a PCOC is said to be associated to this PCOC. Note that several different martingales may be associated to a given PCOC. We shall see several striking occurrences of this in our examples.

We also note that Kellerer’s work [14] does not contain a constructive algorithm for a martingale associated to a PCOC.
On the other hand, Roynette [23] has exhibited two large families of PCOC’s, denoted by (F1) and (F2): The family (F1) consists of the processes

\[
\left( \frac{1}{t} \int_0^t N_s \, ds, \ t \geq 0 \right),
\]

and the family (F2) consists of the processes

\[
\left( \int_0^t (N_s - N_0) \, ds, \ t \geq 0 \right),
\]

where \((N_s)\) above denotes any martingale such that:

\[
\forall t \geq 0 \ \mathbb{E}[\sup_{s \leq t} |N_s|] < \infty.
\]

1.3 Self-decomposability and Sato processes

It is a non-trivial problem to exhibit, for either of these PCOC’s, an associated martingale. We have been able to do so concerning some examples in (F1), in the Brownian context, with the help of the Brownian sheet ([9]), and in the more general context of Lévy processes, with the help of Lévy sheets ([10]). Concerning the class (F2), note that, considering a trivial filtration, it follows that \((tX)\), where \(X\) is a centered random variable, is a PCOC. Even with this reduction, it is not obvious to find a martingale which is associated to \((tX)\). In order to exhibit examples, we were led to introduce the class \((S)\) of processes \((Y_t)\) satisfying the above condition (ii) and such that \(Y_1\) is a self-decomposable integrable random variable. It is a result due to Sato (see Sato [24, Chapter 3, Sections 15-17]) that, if \((Y_t) \in (S)\), then there exists a process \((U_t)\) which has independent increments, is \(H\)-self-similar \((\forall c > 0, (U_{ct}) \overset{d}{=} (c^H U_t))\) and satisfies \(Y_1 \overset{d}{=} U_1\). This process \((U_t)\), which is unique in law, will be called the \(H\)-Sato process associated to \(Y_1\). Clearly, then \((U_t - \mathbb{E}[U_t])\) is a \(H\)-self-similar martingale which is associated to the PCOC \((V_t)\) defined by: \(V_t = Y_t - \mathbb{E}[Y_t]\). Moreover, \((V_t) \overset{1.d}{=} (t^H (Y_1 - \mathbb{E}[Y_1]))\).

We note that the self-decomposability property has also been used in Madan-Yor [18, Theorem 4,Theorem 5] in a very different manner than in this paper, to construct martingales with one-dimensional marginals those of \((tX)\).

1.4 Examples

We look for some interesting processes in the class \((S)\), in a Brownian framework.

Example 1 A most simple example is the process:

\[
Y_t := \int_0^t B_s \, ds, \ t \geq 0
\]

Then,

\[
\left( \int_0^t B_s \, ds \right) \overset{1.d}{=} \left( \int_0^t s \, dB_s \right)
\]

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and the RHS is a centered \((3/2)\)-Sato process. Moreover the process \((Y_t)\) obviously belongs to the class \((F2)\).

**Example 2** The process

\[ V_1(t) := \int_0^t (B_s^2 - s) \, ds, \quad t \geq 0 \]

and more generally the process

\[ V_N(t) := \int_0^t (R_N^2(s) - Ns) \, ds, \quad t \geq 0 \]

where \((R_N(s))\) is a Bessel process of dimension \(N > 0\) starting from 0, belongs to the family \((F2)\) and is 2-self-similar. We show in Section 4 that the centered 2-Sato process:

\[ \frac{N^2}{4} \int_0^\tau 1_{|B_s| \leq \frac{\sqrt{2} t}{N}} \, ds - \frac{Nt^2}{2}, \quad t \geq 0 \]

where \((\ell_s)\) is the local time in 0 of the Brownian motion \(B\), and

\[ \tau_t = \inf \{ s ; \ell_s > t \} \]

is a martingale associated to the PCOC \(V_N\).

**Example 3** We extend our discussion of Example 2 by considering, for \(N > 0\) and \(K > 0\), the process:

\[ V_{N,K}(t) := \frac{1}{K^2} \int_0^t s^{2(\frac{K-1}{K})} (R_N^2(s) - Ns) \, ds, \quad t \geq 0 \]

Then, in Section 5 a centered \((2/K)\)-Sato process (and hence a martingale) associated to the PCOC \(V_{N,K}\) may be constructed from the process of first hitting times of a perturbed Bessel process \(R_{K,1-\frac{N}{2}}\) as defined and studied first in Le Gall-Yor [16; 17] and then in Doney-Warren-Yor [6]. We remark that, if \(0 < K < 2\), then the process

\[ V_{N,K}(t^{\frac{K}{2-K}}), \quad t \geq 0 \]

belongs to \((F2)\).

**Example 4** In Section 6 we generalize again our discussion by considering the process

\[ V_N^{(\mu)}(t) := \int_{(0,\infty)} (R_N^2(ts) - Nts) \, d\mu(s), \quad t \geq 0 \]

for \(\mu\) a nonnegative measure on \((0, \infty)\) such that \(\int_{(0,\infty)} s \, d\mu(s) < \infty\). We show that \(V_N^{(\mu)}\) is a PCOC to which we are able to associate two very different martingales. The first one is purely discontinuous and is a centered 1-Sato process, the second one is continuous. The method of proof is based on a Karhunen-Loeve type decomposition (see, for instance, [5] and the references therein, notably Kac-Siegert [13]). For this, we need to develop a precise spectral study of the operator \(K^{(\mu)}\) defined on \(L^2(\mu)\) by:

\[ K^{(\mu)} f(t) = \int_{(0,\infty)} f(s)(t \wedge s) \, d\mu(s) \]
1.5 Organisation of the paper

We now present more precisely the organisation of our paper:
- in Section 2 we recall some basic results about various representations of self-decomposable variables, and we complete the discussion of Subsection 1.3 above;
- in Section 3 we consider the simple situation, as in Subsection 1.3, where \( Y_t = R^2_N(t) \), for \( R_N \) a Bessel process of dimension \( N \) starting from 0;
- the contents of Sections 4, 5, 6 have already been discussed in the above Subsection 1.4.

We end this introduction with the following (negative) remark concerning further self-decomposability properties for squared Bessel processes: indeed, it is well-known, and goes back to Shiga-Watanabe [25], that \( R^2_N(\cdot) \), considered as a random variable taking values in \( C(\mathbb{R}_+,\mathbb{R}_+) \) is infinitely divisible. Furthermore, in the present paper, we show and exploit the self-decomposability of \( \int_{(0,\infty)} R^2_N(s) \, d\mu(s) \) for any positive measure \( \mu \). It then seems natural to wonder about the self-decomposability of \( R^2_N(\cdot) \), but this property is ruled out: the 2-dimensional vectors \( (R^2_N(t_1), R^2_N(t_1 + t_2)) \) are not self-decomposable, as an easy Laplace transform computation implies.

2 Sato processes and PCOC’s

2.1 Self-decomposability and Sato processes

We recall, in this subsection, some general facts concerning the notion of self-decomposability. We refer the reader, for background, complements and references, to Sato [24, Chapter 3].

A random variable \( X \) is said to be self-decomposable if, for each \( u \) with \( 0 < u < 1 \), there is the equality in law:
\[
X \overset{d}{=} uX + \tilde{X}_u
\]
for some variable \( \tilde{X}_u \) independent of \( X \).

On the other hand, an additive process \((U_t, \ t \geq 0)\) is a stochastically continuous process with càdlàg paths, independent increments, and satisfying \( U_0 = 0 \).

An additive process \((U_t)\) which is \( H\)-self-similar for some \( H > 0 \), meaning that, for each \( c > 0 \), \( (U_{ct}) \overset{(d)}{=} (c^H U_t) \), will be called a Sato process or, more precisely, a \( H\)-Sato process.

The following theorem, for which we refer to Sato’s book [24, Chapter 3, Sections 16-17], gives characterizations of the self-decomposability property.

**Theorem 2.1.** Let \( X \) be a real valued random variable. Then, \( X \) is self-decomposable if and only if one of the following equivalent properties is satisfied:

1) \( X \) is infinitely divisible and its Lévy measure is \( \frac{h(x)}{|x|} \, dx \) with \( h \) increasing on \(( -\infty, 0)\) and decreasing on \(( 0, +\infty)\).
2) There exists a Lévy process \((C_s, s \geq 0)\) such that
\[
X \overset{d}{=} \int_0^\infty e^{-s} \, dC_s.
\]

3) For any (or some) \(H > 0\), there exists a \(H\)-Sato process \((U_t, t \geq 0)\) such that \(X \overset{d}{=} U_1\).

In 2) (resp. 3)) the Lévy process \((C_s)\) (resp. the \(H\)-Sato process \((U_t)\)) is uniquely determined in law by \(X\), and will be said to be associated with \(X\). We note that, if \(X \geq 0\), then the function \(h\) vanishes on \((-\infty, 0)\), \((C_s)\) is a subordinator and \((U_t)\) is an increasing process.

The relation between \((C_s)\) and \((U_t)\) was made precise by Jeanblanc-Pitman-Yor \([11, Theorem 1]\):

**Theorem 2.2.** If \((U_t)\) is a \(H\)-Sato process, then the formulae:
\[
C_s^{(-)} = \int_0^1 e^{-r} \, dU_r \quad \text{and} \quad C_s^{(+)} = \int_1^\infty e^r \, dU_r, \quad s \geq 0
\]
define two independent and identically distributed Lévy processes from which \((U_t, t \geq 0)\) can be recovered by:
\[
U_t = \int_0^\infty e^{-sH} \, dC_s^{(-)} \quad \text{if} \quad 0 \leq t \leq 1
\]
and
\[
U_t = U_1 + \int_0^{\log t} e^{sH} \, dC_s^{(+)} \quad \text{if} \quad t \geq 1.
\]

In particular, the Lévy process associated with the self-decomposable random variable \(U_1\) is
\[
C_s = C_{s/H}^{(-)}, \quad s \geq 0.
\]

### 2.2 Sato processes and PCOC’s

We recall (see Subsection 1.2) that a PCOC is an integrable process which is increasing in the convex order. On the other hand, a process \((V_t, t \geq 0)\) is said to be a \(1\)-martingale if there exists, on some filtered probability space, a martingale \((M_t, t \geq 0)\) such that \((V_t) \overset{(1,d)}{=} (M_t)\). Such a martingale \(M\) is said to be associated with \(V\). It is a direct consequence of Jensen’s inequality that, if \(V\) is a \(1\)-martingale, then \(V\) is a PCOC. As indicated in Subsection 1.2, the converse holds true (Kellerer [14]).

The following proposition, which is central in the following, summarizes the method sketched in Subsection 1.3.

**Proposition 2.3.** Let \(H > 0\). Suppose that \(Y = (Y_t, t \geq 0)\) satisfies:

(a) \(Y_1\) is an integrable self-decomposable random variable;

(b) \((Y_t) \overset{(1,d)}{=} (t^H Y_1)\).
Then the process
\[ V_t := Y_t - t^H \mathbb{E}[Y_1], \quad t \geq 0 \]
is a PCOC, and an associated martingale is
\[ M_t := U_t - t^H \mathbb{E}[Y_1], \quad t \geq 0 \]
where \((U_t)\) denotes the \(H\)-Sato process associated with \(Y_1\) according to Theorem 2.1.

3 About the process \((R_N^2(t), \ t \geq 0)\)

In the sequel, we denote by \((R_N(t), \ t \geq 0)\) the Bessel process of dimension \(N > 0\), starting from 0.

3.1 Self-decomposability of \(R_N^2(1)\)

As is well-known (see, for instance, Revuz-Yor [22] Chapter XI) one has
\[ \mathbb{E}[\exp(-\lambda R_N^2(1))] = (1 + 2\lambda)^{-N/2}. \]

In other words,
\[ R_N^2(1) \overset{d}{=} 2\gamma_{N/2} \]
where, for \(a > 0\), \(\gamma_a\) denotes a gamma random variable of index \(a\). Now, the classical Frullani’s formula yields:
\[ \frac{N}{2} \log(1 + 2\lambda) = \frac{N}{2} \int_0^\infty (1 - e^{-\lambda t}) \frac{e^{-t/2}}{t} \ dt. \]
Then, \(R_N^2(1)\) satisfies the property 1) in Theorem 2.1 with
\[ h(x) = \frac{N}{2} \mathbb{1}_{(0,\infty)}(x) e^{-x/2} \]
and it is therefore self-decomposable.

The process \(R_N^2\) is 1-self-similar and \(\mathbb{E}[R_N^2(1)] = N\). By Proposition 2.3, the process
\[ V_t^N := R_N^2(t) - tN, \quad t \geq 0 \]
is a PCOC, and an associated martingale is
\[ M_t^N := U_t^N - tN, \quad t \geq 0 \]
where \((U_t^N)\) denotes the 1-Sato process associated with \(R_N^2(1)\) by Theorem 2.1.

We remark that, in this case, the process \((V_t^N)\) itself is a continuous martingale and therefore obviously a PCOC. In the following subsections, we give two expressions for the process \((U_t^N)\). As we will see, this process is purely discontinuous with finite variation; consequently, the martingales \((V_t^N)\) and \((M_t^N)\), which have the same one-dimensional marginals, do not have the same law.
3.2 Expression of \((U_t^N)\) from a compound Poisson process

We denote by \((\Pi_s, s \geq 0)\) the compound Poisson process with Lévy measure:

\[
1_{(0, \infty)}(t) e^{-t} dt.
\]

This process allows to compute the distributions of a number of perpetuities

\[
\int_0^\infty e^{-\Lambda_s} d\Pi_s
\]

where \((\Lambda_s)\) is a particular Lévy process, independent of \(\Pi\); see, e.g., Nilsen-Paulsen [20]. In the case \(\Lambda_s = r s\), the following result seems to go back at least to Harrison [8].

**Proposition 3.1.** The Lévy process \((C^N_s)\) associated with the self-decomposable random variable \(R^2_N(1)\) in the sense of Theorem 2.1 is

\[
C^N_s = 2 \Pi_{N s/2}, \quad s \geq 0.
\]

**Proof**

a) First, recall that for a subordinator \((\tau_s, s \geq 0)\) and \(f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+\) Borel, there is the formula:

\[
\mathbb{E} \left[ \exp \left( - \int_0^\infty f(s) d\tau_s \right) \right] = \exp \left( - \int_0^\infty \Phi(f(s)) ds \right),
\]

where \(\Phi\) is the Lévy exponent of \((\tau_s, s \geq 0)\). Consequently, a slight amplification of this formula is:

\[
\mathbb{E} \left[ \exp \left( - \mu \int_0^\infty f(s) d\tau_{As} \right) \right] = \exp \left( - \int_0^\infty \Phi(\mu f(u/A)) du \right)
\]

for every \(\mu, A > 0\).

b) We set \(C^N_s = 2 \Pi_{N s/2}\). Then, as a consequence of the previous formula with \(\mu = 2, f(s) = \lambda e^{-s}, A = N/2\), we get:

\[
\mathbb{E} \left[ \exp \left( - \lambda \int_0^\infty e^{-s} dC^N_s \right) \right] = \exp \left( - \int_0^\infty \Phi(2 \lambda e^{-N u/2}) du \right)
\]

with, for \(x > 0\),

\[
\Phi(x) = \int_0^\infty (1 - e^{-t x}) e^{-t} dt = \frac{x}{1 + x}.
\]

c) We obtain by change of variable:

\[
\exp \left( - \int_0^\infty \Phi(2 \lambda e^{-N u/2}) du \right) = \exp \left( - \int_0^\infty \frac{2 \lambda e^{-N u/2}}{1 + 2 \lambda e^{-N u/2}} du \right)
\]

\[
= \exp \left( - N \int_0^{2 \lambda} \frac{1}{1 + x} dx \right)
\]

\[
= (1 + 2 \lambda)^{-N/2}.
\]
Consequently,
\[
\mathbb{E} \left[ \exp \left( -\lambda \int_0^\infty e^{-s} \, dc^N_s \right) \right] = (1 + 2\lambda)^{-N/2},
\]
which proves the result. □

By application of Theorem 2.2 we get:

**Corollary 3.1.** Let \( \Pi^{(+)} \) and \( \Pi^{(-)} \) two independent copies of the Lévy process \( \Pi \). Then

\[
U^N_t = 2 \int_{-\frac{N}{2}}^\infty e^{-2s/N} d\Pi^{(-)}_s \quad \text{if} \quad 0 \leq t \leq 1
\]

and

\[
U^N_t = U^N_1 + 2 \int_{\frac{N}{2}}^t e^{2s/N} d\Pi^{(+)}_s \quad \text{if} \quad t \geq 1.
\]

### 3.3 Expression of \((U^N_t)\) from the local time of a perturbed Bessel process

There is by now a wide literature on perturbed Bessel processes, a notion originally introduced by Le Gall-Yor [16, 17], and then studied by Chaumont-Doney [3], Doney-Warren-Yor [6]. We also refer the interested reader to Doney-Zhang [7].

We first introduce the perturbed Bessel process \((R_{1,a}(t), t \geq 0)\) starting from 0, for \( \alpha < 1 \), as the nonnegative continuous strong solution of the equation

\[
R_{1,a}(t) = B_t + \frac{1}{2} L_t(R_{1,a}) + \alpha M_t(R_{1,a})
\]

where \( L_t(R_{1,a}) \) is the semi-martingale local time of \( R_{1,a} \) in 0 at time \( t \), and

\[
M_t(R_{1,a}) = \sup_{0 \leq s \leq t} R_{1,a}(s),
\]

\((B_t)\) denoting a standard linear Brownian motion starting from 0. (The strong solution property has been established in Chaumont-Doney [3].)

It is clear that the process \( R_{1,0} \) is nothing else but the Bessel process \( R_1 \) (reflected Brownian motion).

We also denote by \( T_t(R_{1,a}) \) the hitting time:

\[
T_t(R_{1,a}) = \inf\{s; R_{1,a}(s) > t\}.
\]

We set \( L_{T_t}(R_{1,a}) \) for \( L_{T_t}(R_{1,a})(R_{1,a}) \).

Finally, in the sequel, we set

\[
\alpha_N = 1 - \frac{N}{2}.
\]

**Proposition 3.2.** For any \( \alpha < 1 \), the process \((L_{T_t}(R_{1,a}), t \geq 0)\) is a 1-Sato process, and we have

\[
(U^N_t)^{(d)} = (L_{T_t}(R_{1,a_N})�).
\]
Proof

By the uniqueness in law of the solution to the equation (1), the process $R_{1,\alpha}$ is $(1/2)$-self-similar. As a consequence, the process $(L_{T_t}(R_{1,\alpha})$, $t \geq 0)$ is $1$-self-similar.

On the other hand, the pair $(R_{1,\alpha}, M(R_{1,\alpha}))$ is strong Markov (see Doney-Warren-Yor [6, p. 239]). As

$$R_{1,\alpha}(u) = M_u(R_{1,\alpha}) = t \quad \text{if} \quad u = T_t(R_{1,\alpha}),$$

the fact that $(L_{T_t}(R_{1,\alpha}), t \geq 0)$ is an additive process follows from the strong Markov property.

Finally, we need to prove:

$$R^2_N(1) \overset{d}{=} L_{T_{1}}(R_{1,\alpha_N}).$$

For the remainder of the proof, we denote $R_{1,\alpha_N}$ by $R$, and $L_t(R_{1,\alpha_N})$, $T_t(R_{1,\alpha_N})$, $M_t(R_{1,\alpha_N})$ · · · are simply denoted respectively by $L_t$, $T_t$, $M_t$ · · · As a particular case of the “balayage formula” (Yor [26], Revuz-Yor [22, VI.4]) we deduce from equation (1), that:

$$\exp(-\lambda L_t)R_t = \int_0^t \exp(-\lambda L_s) \, dR_s = \int_0^t \exp(-\lambda L_s) \, dB_s + \frac{1 - \exp(-\lambda L_t)}{2\lambda} + \alpha_N \int_0^t \exp(-\lambda L_s) \, dM_s.$$

Hence,

$$\exp(-\lambda L_t)(1 + 2\lambda R_t) = 1 + 2\lambda \int_0^t \exp(-\lambda L_s) \, dB_s + 2\lambda \alpha_N \int_0^t \exp(-\lambda L_s) \, dM_s.$$

From this formula, we learn that the martingale

$$\left( \int_0^{u \wedge T_t} \exp(-\lambda L_s) \, dB_s, \ u \geq 0 \right)$$

is bounded; hence, by applying the optional stopping theorem, we get:

$$\mathbb{E}[\exp(-\lambda L_{T_t})](1 + 2\lambda t) = 1 + 2\lambda \alpha_N \mathbb{E} \left[ \int_0^{T_t} \exp(-\lambda L_s) \, dM_s \right]$$

$$= 1 + 2\lambda \alpha_N \int_0^t \mathbb{E}[\exp(-\lambda L_{T_u})] \, du,$$

by time-changing. Setting

$$\varphi_\lambda(t) = \mathbb{E}[\exp(-\lambda L_{T_t})],$$

we obtain:

$$\varphi_\lambda(t) = \frac{1}{1 + 2\lambda t} + \frac{2\lambda \alpha_N}{1 + 2\lambda t} \int_0^t \varphi_\lambda(u) \, du.$$
Consequently
\[ \phi_{\lambda}(t) = (1 + 2\lambda t)^{-N/2}. \]
Therefore,
\[ \mathbb{E}[\exp(-\lambda L_{T_1})] = (1 + 2\lambda)^{-N/2} = \mathbb{E}[\exp(-\lambda R_{N}^2(1))], \]
which proves the desired result.

\[\square\]

4 About the process \(\left( \int_0^t R_N^2(s) \, ds, \ t \geq 0 \right)\)

4.1 A class of Sato processes

Let \((\ell, t \geq 0)\) be the local time in 0 of a linear Brownian motion \((B_t, t \geq 0)\) starting from 0. We denote, as usual, by \((\tau, t \geq 0)\) the inverse of this local time:
\[ \tau_t = \inf\{s \geq 0; \ell_s > t\}. \]

**Proposition 4.1.** Let \(f(x,u)\) be a Borel function on \(\mathbb{R}^+ \times \mathbb{R}^+\) such that
\[ \forall t > 0 \quad \int_{\mathbb{R}^+ \times [0,t]} |f(x,u)| \, dx \, du < \infty. \tag{2} \]

Then the process \(A_{t}^{(f)}\) defined by:
\[ A_{t}^{(f)} = \int_0^{\tau_t} f(|B_s|, \ell_s) \, ds, \quad t \geq 0 \]
is an integrable additive process. Furthermore,
\[ \mathbb{E}[A_{t}^{(f)}] = 2 \int_{\mathbb{R}^+ \times [0,t]} f(x,u) \, dx \, du. \]

**Proof**

Assume first that \(f\) is nonnegative. Then,
\[ A_{t}^{(f)} = \sum_{0 \leq u \leq t} \int_{\tau_u}^{\tau_t} f(|B_s|, u) \, ds. \]

By the theory of excursions (Revuz-Yor [22, Chapter XII, Proposition 1.10]) we have
\[ \mathbb{E}[A_{t}^{(f)}] = \int_0^t du \int n(d\varepsilon) \int_0^{V(\varepsilon)} ds \ f(|\varepsilon|, u) \]

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where \( n \) denotes the Itô measure of Brownian excursions and \( V(\epsilon) \) denotes the life time of the excursion \( \epsilon \). The entrance law under \( n \) is given by:

\[
n(\epsilon_s \in dx; \ s < V(\epsilon)) = (2\pi s^3)^{-1/2} |x| \exp(-x^2/(2s)) \, dx.
\]

Therefore

\[
\mathbb{E}[A_t^{(f)}] = 2 \int_0^t du \int_0^\infty dx \, f(x,u).
\]

The additivity of the process \( A^{(f)} \) follows easily from the fact that, for any \( t \geq 0, (B_{\tau_t+s}, s \geq 0) \) is a Brownian motion starting from 0, which is independent of \( \mathcal{B}_{\tau_t} \) (where \( \mathcal{B}_u \) is the natural filtration of \( B \)).

\[\square\]

**Corollary 4.1.1.** We assume that \( f \) is a Borel function on \( \mathbb{R}_+ \times \mathbb{R}_+ \) satisfying (2) and which is \( m \)-homogeneous for \( m > -2 \), meaning that

\[
\forall a > 0, \forall (x,u) \in \mathbb{R}_+ \times \mathbb{R}_+, \quad f(ax,au) = a^m f(x,u).
\]

Then the process \( A^{(f)} \) is a \((m + 2)\)-Sato process.

**Proof**

This is a direct consequence of the scaling property of Brownian motion.

\[\square\]

### 4.2 A particular case

Let \( N > 0 \). We denote by \( A^{(N)} \) the process \( A^{(f)} \) with

\[
f(x,u) = \frac{N^2}{4} \mathbf{1}_{(x \leq \frac{2}{N} u)}.
\]

By Proposition 4.1, \( A^{(N)}_t \) is an integrable process and

\[
\mathbb{E}[A_t^{(N)}] = \frac{N t^2}{2}.
\]

We now consider the process \( Y_N \) defined by

\[
Y_N(t) = \int_0^t R_N^2(s) \, ds, \quad t \geq 0.
\]

**Theorem 4.2.** The process \( A^{(N)} \) is a 2-Sato process and

\[\left(Y_N(t) \right)^{(1,d)} = \left(A_t^{(N)} \right).\]
Proof

It is a direct consequence of Corollary 4.1.1 that $A^{(N)}$ is a 2-Sato process. By Mansuy-Yor [19, Theorem 3.4, p.38], the following extension of the Ray-Knight theorem holds: For any $u > 0$,

$$(L_{x_u}^{a-(2u/N)}, 0 \leq a \leq (2u/N)) \overset{(d)}{=} (R^2_N(a), 0 \leq a \leq (2u/N))$$

where $L_x^a$ denotes the local time of the semi-martingale $(|B_s| - \frac{2}{N} \ell_s, s \geq 0)$ in $x$ at time $t$. We remark that

$$s \in [0, \tau_t] \Rightarrow |B_s| - \frac{2}{N} \ell_s \geq -\frac{2t}{N}.$$ 

Therefore, the occupation times formula entails:

$$A^{(N)}_t = \frac{N^2}{4} \int_{-2t/N}^{0} L_{x_{\tau_t}}^a \, dx = \frac{N^2}{4} \int_0^{2t/N} L_{x_{\tau_t}}^{(2t/N)} \, dx.$$ 

Thus, by the above mentioned extension of the Ray-Knight theorem,

$$(A^{(N)}_t)^{(1,d)} \overset{(d)}{=} \left( \frac{N^2}{4} \int_0^{2t/N} R^2_N(s) \, ds \right).$$

The scaling property of $R_N$ also yields the identity in law:

$$(A^{(N)}_t)^{(1,d)} \overset{(d)}{=} \left( \int_0^t R^2_N(s) \, ds \right),$$

and the result follows from the definition of $Y_N$. 

\[ \square \]

We may now apply Proposition 2.3 to get:

**Corollary 4.2.1.** The process $V_N$ defined by:

$$V_N(t) = Y_N(t) - \frac{N t^2}{2}, \quad t \geq 0$$

is a PCOC and an associated martingale is $M_N$ defined by:

$$M_N(t) = A^{(N)}_t - \frac{N t^2}{2}, \quad t \geq 0.$$ 

Moreover, $M_N$ is a centered 2-Sato process.

### 4.3 Representation of $A^{(N)}$ as a process of hitting times

**Theorem 4.3.** The process $A^{(N)}$ is identical in law to the process

$$T_t(R_{1,\alpha_N}), \quad t \geq 0$$

where $R_{1,\alpha_N}$ denotes the perturbed Bessel process defined in Subsection 3.3 and

$$T_t(R_{1,\alpha_N}) = \inf\{s; R_{1,\alpha_N}(s) > t\}.$$
The proof can be found in Le Gall-Yor [17]. Nevertheless, for the convenience of the reader, we give again the proof below. A more general result, based on Doney-Warren-Yor [6], shall also be stated in the next section.

Proof
In this proof, we adopt the following notation: \((B_t)\) still denotes a standard linear Brownian motion starting from 0, \(S_t = \sup_{0 \leq s \leq t} B_s\) and \(\sigma_t = \inf\{s; B_s > t\}\). Moreover, for \(a < 1\) and \(t \geq 0\), we set:

\[
H^a_t = (B_t - a S_t), \quad X^a_t = \int_0^t 1_{(B_s > a S_s)} \, ds \quad \text{and} \quad Z^a_t = \inf\{s; X^a_s > t\}.
\]

Lemma 4.3.1. Let \(a < 1\). Then

\[
M_t(H^a) := \sup_{0 \leq s \leq t} H^a_s = (1 - a) S_t.
\]

Consequently, \(-a S_t = a M_t(H^a)\), with \(a = -a/(1 - a)\).

Proof
Since \(a < 1\), we have, for \(0 \leq s \leq t\),

\[
(B_s - a S_s) \leq (1 - a) S_s \leq (1 - a) S_t.
\]

Moreover, there exists \(s_t \in [0, t]\) such that \(B_{s_t} = S_t\) and therefore \(S_{s_t} = S_t\). Hence, \(B_{s_t} - a S_{s_t} = (1 - a) S_t\).

Lemma 4.3.2. Let \(a < 1\) and recall \(\alpha = -a/(1 - a)\). We set:

\[
D^a_t = H^{a}_{Z^a_t}, \quad t \geq 0.
\]

Then the processes \(D^a\) and \(R_{1,\alpha}\) are identical in law.

Proof

a) Since \(Z^a_t\) is a time of increase of the process

\[
X^a_s = \int_0^s 1_{(H^a_u > 0)} \, du, \quad s \geq 0,
\]

we get: \(D^a_t = H^{a}_{Z^a_t} \geq 0\). Moreover, since the process \((H^a)^+\) is obviously \(Z^a\)-continuous, the process \(D^a\) is continuous.

b) By Tanaka’s formula,

\[
(H^a_t)^+ = \int_0^t 1_{(H^a_u > 0)} \, d(B_u - a S_u) + \frac{1}{2} L_t(H^a)
\]
where \( L_t(H^a) \) denotes the local time of the semi-martingale \( H^a \) in 0 at time \( t \). If \( s > 0 \) belongs to the support of \( dS_s \), then \( B_s = S_s \) and, since \( a < 1, B_s - a S_s > 0. \) Therefore,

\[
(H_t^a)^+ = \int_0^t 1_{(B_s-a S_s, 0)} \ dB_s - a S_t + \frac{1}{2} L_t(H^a).
\]

By Lemma 4.3.1 \(-a S_t = \alpha M_t(H^a)\). Consequently,

\[
D_t^a = \int_0^t 1_{(B_s-a S_s, 0)} \ dB_s + \frac{1}{2} L_{Z_t^a}(H^a) + \alpha M_{Z_t^a}(H^a).
\]

By the Dubins-Schwarz theorem, the process

\[
\int_0^t 1_{(B_s-a S_s, 0)} \ dB_s, \quad t \geq 0
\]

is a Brownian motion.

On the other hand, it is easy to see that

\[
L_{Z_t^a}(H^a) = L_t(D^a) \quad \text{and} \quad M_{Z_t^a}(H^a) = M_t(D^a).
\]

Therefore, the process \( D^a \) is a continuous and nonnegative solution to equation (1).

\[\square\]

To conclude the proof of Theorem 4.3, we observe that by Lévy's equivalence theorem ([22 Theorem VI.2.3]), the process \( A^{(N)} \) is identical in law to the process

\[
\frac{N}{4} \int_0^{\sigma_t} 1_{(B_s, (1-\frac{2}{N})S_s)} \ ds, \quad t \geq 0.
\]

By the scaling property of \( B \), the above process has the same law as

\[
\int_0^{\sigma_{Nt/2}} 1_{(B_s, (1-\frac{2}{N})S_s)} \ ds = X_{\sigma_{Nt/2}}^{1-\frac{2}{N}}, \quad t \geq 0.
\]

Now,

\[
X_{\sigma_{Nt/2}}^{1-\frac{2}{N}} = \inf \{X_u^{1-\frac{2}{N}}; S_u > \frac{N t}{2}\}
\]

and, by Lemma 4.3.1

\[
S_u = \frac{N}{2} M_u(H^{1-\frac{2}{N}}).
\]

Thus,

\[
X_{\sigma_{Nt/2}}^{1-\frac{2}{N}} = \inf \{X_u^{1-\frac{2}{N}}; M_u(H^{1-\frac{2}{N}}) > t\}
\]

\[
= \inf \{v; M_v(D^{1-\frac{2}{N}}) > t\} = \inf \{v; D_v^{1-\frac{2}{N}} > t\}.
\]

The result then follows from Lemma 4.3.2.

\[\square\]
Corollary 4.3.1. The process

\[ T_t(R_{1,\alpha N}), \quad t \geq 0 \]

is a 2-Sato process and

\[ \left( \int_0^t R_N^2(s) \, ds \right)^{(1,d)} \equiv \left( T_t(R_{1,\alpha N}) \right). \]

5 About the process

\( \left( \frac{1}{K^2} \int_0^t s^{\frac{2(1-K)}{K}} R_N^2(s) \, ds, \quad t \geq 0 \right) \)

In this section we extend Corollary 4.3.1. We fix two positive real numbers \( N \) and \( K \). We first recall some important results on general perturbed Bessel processes \( R_{K,\alpha} \) with \( \alpha < 1 \).

5.1 Perturbed Bessel processes

We follow, in this subsection, Doney-Warren-Yor [6]. We first recall the definition of the process \( R_{K,\alpha} \) with \( K > 0 \) and \( \alpha < 1 \).

The case \( K = 1 \) was already introduced in Subsection 3.3. For \( K > 1 \), \( R_{K,\alpha} \) is defined as a continuous nonnegative solution to

\[ R_t = B_t + \frac{K-1}{2} \int_0^t \frac{1}{R_s} \, ds + \alpha M_t(R), \quad (3) \]

and, for \( 0 < K < 1 \), \( R_{K,\alpha} \) is defined as the square root of a continuous nonnegative solution to

\[ X_t = 2 \int_0^t \sqrt{X_s} \, dB_s + K t + \alpha M_t(X). \quad (4) \]

We note that, for any \( K > 0 \), \( (R_{K,0}(t)) \stackrel{(d)}{=} (R_K(t)) \). As in the case \( K = 1 \), for any \( K > 0 \), the pair \( (R_{K,\alpha}, M(R_{K,\alpha})) \) is strong Markov.

We denote, as before,

\[ T_t(R_{K,\alpha}) = \inf\{s; R_{K,\alpha}(s) > t\}. \]

The following theorem, due to Doney-Warren-Yor [6, Theorem 5.2, p. 246] is an extension of the Ciesielski-Taylor theorem and of the Ray-Knight theorem.

Theorem 5.1. 1)

\[ \int_0^\infty 1_{\{R_{K,\alpha}(s) \leq 1\}} \, ds \stackrel{d}{=} T_1(R_{K,\alpha}) \]

2)

\[ (I_{\infty}^a(R_{K,\alpha}), \ a \geq 0) \stackrel{(d)}{=} (\frac{a^{1-K}}{K} R_{2(1-a)}(a^K), \ a \geq 0) \]
5.2 Identification of the Sato process associated to $Y_{N,K}$

We denote, for $N > 0$ and $K > 0$, by $Y_{N,K}$ the process:

$$Y_{N,K}(t) = \frac{1}{K^2} \int_0^t s^{\frac{2(1-K)}{K}} R_N^2(s) \, ds, \quad t \geq 0.$$ 

We also recall the notation:

$$\alpha_N = 1 - \frac{N}{2}.$$

**Theorem 5.2.** The process

$$T_{t^{1/K}}(R_{K,\alpha_N}), \quad t \geq 0$$

is a $(2/K)$-Sato process and

$$(Y_{N,K}(t))^{(1,d)} = (T_{t^{1/K}}(R_{K,\alpha_N})).$$

**Proof**

In the following proof, we denote $R_{K,\alpha_N}$ simply by $R$, and we set $T_t$ and $M_t$ for, respectively, $T_t(R)$ and $M_t(R)$.

The first part of the statement follows from the $(1/2)$-self-similarity of $R$ and from the strong Markovianity of $(R, M)$, taking into account that, for any $t \geq 0$,

$$R_{T_t} = M_{T_t} = t.$$

By occupation times formula, we deduce from 1) in Theorem 5.1

$$\int_0^1 L_\infty^{x}(R_{K,\alpha_N}) \, dx \overset{d}{=} T_1.$$

Using then 2) in Theorem 5.1 we obtain:

$$\int_0^1 L_\infty^{x}(R_{K,\alpha_N}) \, dx \overset{d}{=} \int_0^1 \frac{x^{1-K}}{K} R_N^2(x^K) \, dx.$$

By change of variable, the last integral is equal to $Y_{N,K}(1)$, and hence,

$$Y_{N,K}(1) \overset{d}{=} T_1.$$

The final result now follows by self-similarity.

\[\square\]

**Corollary 5.2.1.** The process

$$V_{N,K}(t) := Y_{N,K}(t) - \frac{N}{2K} t^{2/K}, \quad t \geq 0$$

is a PCOC, and an associated martingale is

$$M_{N,K}(t) := T_{t^{1/K}}(R_{K,\alpha_N}) - \frac{N}{2K} t^{2/K}, \quad t \geq 0,$$

which is a centered $(2/K)$-Sato process.
Finally, we have proven, in particular, that for any \( \rho > -2 \) and any \( N > 0 \), the random variable

\[
\int_0^1 s^\rho R_N^2(s) \, ds
\]

is self-decomposable. This result will be generalized and made precise in the next section, using completely different arguments.

### 6 About the random variables \( \int R_N^2(s) \, d\mu(s) \)

In this section, we consider a fixed measure \( \mu \) on \( \mathbb{R}_+^+ = (0, \infty) \) such that

\[
\int_{\mathbb{R}_+^+} s \, d\mu(s) < \infty.
\]

#### 6.1 Spectral study of an operator

We associate with \( \mu \) an operator \( K(\mu) \) on \( E = L^2(\mu) \) defined by

\[
\forall f \in E \quad K(\mu)f(t) = \int_{\mathbb{R}_+^+} f(s)(t \wedge s) \, d\mu(s)
\]

where \( \wedge \) denotes the infimum.

**Lemma 6.1.** The operator \( K(\mu) \) is a nonnegative symmetric Hilbert-Schmidt operator.

**Proof**

As a consequence of the obvious inequality:

\[
(t \wedge s)^2 \leq ts,
\]

we get

\[
\int \int (t \wedge s)^2 \, d\mu(t) \, d\mu(s) \leq \left( \int_{\mathbb{R}_+^+} s \, d\mu(s) \right)^2,
\]

and therefore \( K(\mu) \) is a Hilbert-Schmidt operator.

On the other hand, denoting by \( (\cdot, \cdot)_E \) the scalar product in \( E \), we have:

\[
(K(\mu)f, g)_E = \mathbb{E} \left[ \int f(t) B_t \, d\mu(t) \int g(s) B_s \, d\mu(s) \right]
\]

where \( B \) is a standard Brownian motion starting from 0. This entails that \( K(\mu) \) is nonnegative symmetric. 

\( \Box \)
Lemma 6.2. Let $\lambda \in \mathbb{R}$. Then $\lambda$ is an eigenvalue of $K^{(\mu)}$ if and only if $\lambda > 0$ and there exists $f \in L^2(\mu)$, $f \neq 0$, such that:

i) 
$$\lambda f'' + f \cdot \mu = 0 \quad \text{in the distribution sense on } \mathbb{R}^*_+ \tag{5}$$

ii) $f$ admits a representative which is absolutely continuous on $\mathbb{R}_+$, $f'$ admits a representative which is right-continuous on $\mathbb{R}^*_+$;
(In the sequel, $f$ and $f'$ respectively always denote such representatives.)

iii) 
$$f(0) = 0 \quad \text{and} \quad \lim_{t \to \infty} f'(t) = 0.$$

Proof

Let $f \in L^2(\mu)$ and $g = K^{(\mu)} f$. We have, for $\mu$-a.e. $t > 0$,

$$g(t) = \int_0^t du \int_{(u, \infty)} f(s) \, d\mu(s). \tag{6}$$

Thus $g$ admits a representative (still denoted by $g$) which is absolutely continuous on $\mathbb{R}_+$ and $g(0) = 0$. Moreover, $g'$ admits a representative which is right-continuous on $\mathbb{R}^*_+$ and is given by:

$$g'(t) = \int_{(t, \infty)} f(s) \, d\mu(s). \tag{7}$$

In particular

$$|g'(t)| \leq t^{-1/2} \left[ \int_{(t, \infty)} f^2(s) \, d\mu(s) \int_{(t, \infty)} u \, d\mu(u) \right]^{1/2}. \tag{8}$$

Hence:

$$\lim_{t \to \infty} g'(t) = 0.$$

Besides, (7) entails:

$$g'' + f \cdot \mu = 0 \quad \text{in the distribution sense on } \mathbb{R}^*_+. \tag{6}$$

Consequently, 0 is not an eigenvalue of $K^{(\mu)}$ and the “only if” part is proven.

Conversely, let $f \in L^2(\mu)$, $f \neq 0$, and $\lambda > 0$ such that properties i),ii),iii) hold. Then

$$\lambda f'(t) = \int_{(t, \infty)} f(s) \, d\mu(s).$$

Hence

$$\lambda f(t) = \int_0^t du \int_{(u, \infty)} f(s) \, d\mu(s) = K^{(\mu)} f(t),$$

which proves the “if” part. 

□
We note that, since 0 is not an eigenvalue of $K^{(\mu)}$, $K^{(\mu)}$ is actually a positive symmetric operator. On the other hand, by the previous proof, the functions $f \in L^2(\mu), f \neq 0$, satisfying properties i),ii),iii) in the statement of Lemma 6.2 are the eigenfunctions of the operator $K^{(\mu)}$ corresponding to the eigenvalue $\lambda > 0$.

**Lemma 6.3.** Let $f$ be an eigenfunction of $K^{(\mu)}$. Then,

$$|f(t)| = o(t^{1/2}) \quad \text{and} \quad |f'(t)| = o(t^{-1/2})$$

when $t$ tends to $\infty$.

**Proof**

This is a direct consequence of (8).

\[ \square \]

**Lemma 6.4.** Let $f_1$ and $f_2$ be eigenfunctions of $K^{(\mu)}$ with respect to the same eigenvalue. Then,

$$\forall t > 0 \quad f_1'(t)f_2(t) - f_1(t)f_2'(t) = 0.$$

**Proof**

By (5),

$$\langle f_1'f_2 - f_1f_2' \rangle' = 0 \quad \text{in the sense of distributions on } \mathbb{R}_+.\]$$

By right-continuity, there exists $C \in \mathbb{R}$ such that

$$\forall t > 0 \quad f_1'(t)f_2(t) - f_1(t)f_2'(t) = C.$$

Letting $t$ tend to $\infty$, we deduce from Lemma 6.3 that $C = 0$.

\[ \square \]

**Lemma 6.5.** Let $f$ be a solution of (5) with $\lambda > 0$, and let $a > 0$. We assume as previously that $f$ (resp. $f'$) denotes the representative which is absolutely continuous (resp. right-continuous) on $\mathbb{R}_+$. If $f(a) = f'(a) = 0$, then, for any $t \geq a$, $f(t) = 0$.

**Proof**

This lemma is quite classical if the measure $\mu$ admits a continuous density with respect to the Lebesgue measure (see, for instance, [4]). The proof may be easily adapted to this more general case.

\[ \square \]

We are now able to state the main result of this section.

**Theorem 6.6.** The operator $K^{(\mu)}$ is a positive symmetric compact operator whose all eigenvalues are simple, i.e. the dimension of each eigenspace is 1.
Proof
It only remains to prove that the eigenvalues are simple. For this purpose, let \( \lambda > 0 \) be an eigenvalue and let \( f_1 \) and \( f_2 \) be eigenfunctions with respect to this eigenvalue. Let \( a > 0 \) with \( \mu(\{a\}) = 0 \). By Lemma 6.4
\[
f_1'(a) f_2(a) - f_1(a) f_2'(a) = 0.
\]
Hence, there exist \( c_1 \) and \( c_2 \) with \( c_1^2 + c_2^2 > 0 \) such that, setting \( f = c_1 f_1 + c_2 f_2 \), we have
\[
f(a) = f'(a) = 0.
\]
By Lemma 6.5, \( f(t) = 0 \) for any \( t \geq a \). But, since \( \mu(\{a\}) = 0 \), \( f' \) is also left-continuous at \( a \). Then, we may reason on \((0, a]\) as on \([a, \infty)\) and therefore we also have \( f(t) = 0 \) for \( 0 < t \leq a \). Finally,
\[
c_1 f_1 + c_2 f_2 = 0,
\]
which proves the result.
\[\square\]

In the following, we denote by \( \lambda_1 > \lambda_2 > \cdots \) the decreasing (possibly finite) sequence of the eigenvalues of \( K^{(\mu)} \). Of course, this sequence depends on \( \mu \), which we omit in the notation. The following corollary plays an essential role in the sequel.

Corollary 6.6.1. There exists a Hilbert basis \((f_n)_{n \geq 1}\) in \( L^2(\mu) \) such that
\[
\forall n \geq 1 \quad K^{(\mu)} f_n = \lambda_n f_n.
\]
Since \( K^{(\mu)} \) is Hilbert-Schmidt,
\[
\sum_{n \geq 1} \lambda_n^2 < \infty.
\]
It will be shown in Subsection 6.3 (see Theorem 6.7) that actually
\[
\sum_{n \geq 1} \lambda_n < \infty,
\]
i.e. \( K^{(\mu)} \) is trace-class.

6.2 Examples

In this subsection, we consider two particular types of measures \( \mu \).

6.2.1 \( \mu = \sum_{j=1}^{n} a_j \delta_{t_j} \)

Let \( a_1, \cdots, a_n \) positive real numbers and \( 0 < t_1 < \cdots < t_n \). We denote by \( \delta_t \) the Dirac measure at \( t \) and we consider, in this paragraph,
\[
\mu = \sum_{j=1}^{n} a_j \delta_{t_j},
\]
By the previous study, the sequence of eigenvalues of $K(\mu)$ is finite if and only if the space $L^2(\mu)$ is finite dimensional, that is if $\mu$ is of the above form. In this case, the eigenvalues of $K(\mu)$ are the eigenvalues of the matrix $(m_{i,j})_{1 \leq i,j \leq n}$ with

$$m_{i,j} = \sqrt{a_i a_j} t_{i,j}.$$ 

In particular, by the previous study, such a matrix has $n$ distinct eigenvalues, which are $> 0$.

6.2.2 $\mu = C \, t^\rho \, 1_{[0,1]}(t) \, dt$

In this paragraph, we consider

$$\mu = C \, t^\rho \, 1_{[0,1]}(t) \, dt$$

with $C > 0$ and $\rho > -2$. By Lemma 6.2, the eigenfunctions $f$ of $K(\mu)$ associated with $\lambda > 0$ are characterized by:

$$\lambda f''(x) + C x^\rho f(x) = 0 \quad \text{on} \quad (0, 1),$$

$$f(0) = 0, \quad f'(1) = 0.$$

We set $\sigma = (\rho + 2)^{-1}$ and $\nu = \sigma - 1$. For $a > -1$, we recall the definition of the Bessel function $J_a$:

$$J_a(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{a+2k}}{k! \Gamma(a + k + 1)}.$$

Then, the only function $f$ satisfying $\lambda > 0$ and $f(0) = 0$ is, up to a multiplicative constant,

$$f(x) = x^{1/2} J_\sigma \left( 2\sigma \sqrt{\frac{C}{\lambda}} x^ {1/2} \sigma \right).$$

We deduce from the equality, which is valid for $a > 1$,

$$a J_a(x) + x J'_a(x) = x J_{a-1}(x)$$

that $f'(1) = 0$ if and only if

$$J_\nu \left( 2\sigma \sqrt{\frac{C}{\lambda}} \right) = 0.$$

Denote by $(j_{\nu,k}, k \geq 1)$ the sequence of the positive zeros of $J_\nu$. Then the sequence $(\lambda_k, k \geq 1)$ of eigenvalues of $K(\mu)$ is given by:

$$2\sigma \sqrt{\frac{C}{\lambda_k}} = j_{\nu,k}, \quad k \geq 1$$

or, since $\sigma = \nu + 1$,

$$\lambda_k = 4 C (\nu + 1)^2 j_{\nu,k}^{-2}, \quad k \geq 1.$$

**Particular case** Suppose $\rho = 0$. Then $\nu = -1/2$ and

$$J_\nu(x) = J_{-1/2}(x) = \left( \frac{2}{\pi x} \right)^{1/2} \cos(x).$$

Hence,

$$\lambda_k = 4 C \pi^{-2} (2k-1)^{-2}, \quad k \geq 1.$$
6.3 Representation of $\int B_s^2 \, d\mu(s)$

We again consider the general setting defined in Subsection 6.1, the notation of which we keep. In this subsection, we study the random variable

$$Y^{(\mu)}_1 := \int B_s^2 \, d\mu(s).$$

The use of the operator $K^{(\mu)}$ and of its spectral decomposition in the type of study we develop below, is called the Karhunen-Loève decomposition method. It has a long history which goes back at least to Kac-Siegert [12 13]. We also refer to the recent paper [5] and to the references therein.

**Theorem 6.7.** The eigenvalues $(\lambda_k, \ k \geq 1)$ of the operator $K^{(\mu)}$ satisfy

$$\sum_{k \geq 1} \lambda_k = \int_{\mathbb{R}_+^*} t \, d\mu(t) \ (< \infty, \ by \ hypothesis).$$

Moreover, there exists a sequence $(\Gamma_n, \ n \geq 1)$ of independent normal variables such that:

$$Y^{(\mu)}_1 \overset{\text{d}}{=} \sum_{n \geq 1} \lambda_n \Gamma_n^2.$$

**Proof**

We deduce from Corollary 6.6.1 by the Bessel-Parseval equality, that:

$$Y^{(\mu)}_1 = \sum_{n \geq 1} \left( \int B_s f_n(s) \, d\mu(s) \right)^2 \text{ a.s.}$$

Taking expectations, we get

$$\int_{\mathbb{R}_+^*} t \, d\mu(t) = \sum_{n \geq 1} (K^{(\mu)} f_n, f_n)_{E} = \sum_{n \geq 1} \lambda_n.$$

We set, for $n \geq 1$,

$$\Gamma_n = \frac{1}{\sqrt{\lambda_n}} \int B_s f_n(s) \, d\mu(s).$$

Then $(\Gamma_n, \ n \geq 1)$ is a Gaussian sequence and

$$\mathbb{E} [\Gamma_n \Gamma_m] = \frac{1}{\sqrt{\lambda_n \lambda_m}} (K^{(\mu)} f_n, f_m)_{E} = \delta_{n,m}$$

where $\delta_{n,m}$ denotes Kronecker's symbol. Hence, the result follows. \qed
Corollary 6.7.1. The Laplace transform of \( Y_{1}^{(\mu)} \) is

\[
F_{1}^{(\mu)}(t) = \prod_{n \geq 1} (1 + 2t \lambda_{n})^{-1/2}.
\]

Proof
This is a direct consequence of the previous theorem, taking into account that, if \( \Gamma \) is a normal variable, then

\[ \Gamma^2 \overset{d}{=} 2 \gamma_{1/2}. \]

\[ \square \]

6.4 Representation of \( \int R_{N}^{2}(s) \, d\mu(s) \)

We now consider the random variable

\[ Y_{N}^{(\mu)} := \int R_{N}^{2}(s) \, d\mu(s). \]

A number of explicit computations of the Laplace transforms of these variables are found in Pitman-Yor ([21, Section 2]).

Theorem 6.8. There exists a sequence \( (\Theta_{N,n}, n \geq 1) \) of independent variables with, for any \( n \geq 1 \),

\[ \Theta_{N,n} \overset{d}{=} R_{N}^{2}(1) \overset{d}{=} 2 \gamma_{N/2} \]

such that

\[ Y_{N}^{(\mu)} \overset{d}{=} \sum_{n \geq 1} \lambda_{n} \Theta_{N,n}. \quad (10) \]

Moreover, the Laplace transform of \( Y_{N}^{(\mu)} \) is

\[
F_{N}^{(\mu)}(t) = \prod_{n \geq 1} (1 + 2t \lambda_{n})^{-N/2}. \quad (11)
\]

Proof
It is clear, for instance from Revuz-Yor ([22, Chapter XI, Theorem 1.7]), that

\[ F_{N}^{(\mu)}(t) = [F_{1}^{(\mu)}(t)]^{N}. \]

Therefore, by Corollary 6.7.1, formula (11) holds. Formula (10) then follows directly by the injectivity of the Laplace transform.

\[ \square \]
Corollary 6.8.1. The random variable $Y_N^{(\mu)}$ is self-decomposable. The function $h$, which is decreasing on $(0, \infty)$ and associated with $Y_N^{(\mu)}$ in Theorem 2.1, is

$$h(x) = \frac{N}{2} \sum_{n \geq 1} \exp \left( -\frac{1}{2\lambda_n} x \right).$$

Proof

We saw in Subsection 3.1 that $R^2_N(1)$ satisfies the property 1) in Theorem 2.1 with

$$h(x) = \frac{N}{2} 1_{(0,\infty)}(x) e^{-x/2}$$

and it is therefore self-decomposable. Using then the representation (10) of $Y_N^{(\mu)}$, we obtain the desired result.

\[ \square \]

As a consequence, following Bondesson [1], we see that $Y_N^{(\mu)}$ is a generalized gamma convolution (GGC) whose Thorin measure is the discrete measure:

$$\frac{N}{2} \sum_{n \geq 1} \delta_{1/2\lambda_n}.$$  

Particular case

We consider here, as in Section 5, the particular case:

$$\mu = \frac{1}{K^2} t^{2(1-K)} 1_{(0,1]}(t) \, dt.$$  

Then, $Y_N^{(\mu)}$ is the random variable $Y_{N,K}(1)$ studied in Section 5. As a consequence of Paragraph 6.2.2 with

$$C = \frac{1}{K^2} \quad \text{and} \quad \rho = \frac{2}{K} - 2,$$

we have

$$\lambda_k = j_{\nu,k}^{-2}, \quad k \geq 1$$

with $\nu = \frac{K}{2} - 1$. Moreover, by Theorem 5.2

$$Y_N^{(\mu)} \overset{d}{=} T_1(R_K).$$

It is known (see Kent [15] and, for instance, Borodin-Salminen [2] formula 2.0.1, p. 387)) that

$$\mathbb{E}[\exp(-t T_1(R_K))] = \frac{2^{-\nu} (\sqrt{2t})^{\nu}}{\Gamma(\nu + 1) \, I_\nu(\sqrt{2t})}$$

where $I_\nu$ denotes the modified Bessel function:

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{\nu+2k}}{k! \Gamma(\nu + k + 1)}.$$
We set:

\[ \tilde{I}_v(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{k! \Gamma(v+k+1)}. \]

Therefore, by formula (11) in the case \( N = 2 \), we recover the following representation:

\[ \tilde{I}_v(x) = \frac{1}{\Gamma(v+1)} \prod_{k \geq 1} \left(1 + \frac{x^2}{j_{v,k}}\right). \]

In particular (\( \nu = -1/2 \)),

\[ \cosh(x) = \prod_{k \geq 1} \left(1 + \frac{4x^2}{\pi^2 (2k - 1)^2}\right). \]

Likewise we obtain, for \( \nu = 1/2 \),

\[ \frac{\sinh(x)}{x} = \prod_{k \geq 1} \left(1 + \frac{x^2}{\pi^2 k^2}\right). \]

### 6.5 Sato process associated to \( Y^{(\mu)}_N \)

**Theorem 6.9.** Let \((U_t^N)\) be the 1-Sato process associated to \( R^2_N(1) \) (cf. Section 3). Then, the 1-Sato process associated to \( Y^{(\mu)}_N \) is \((U_t^{N,\mu})\) defined by:

\[ U_t^{(N,\mu)} = \sum_{n \geq 1} \lambda_n U_t^{N,n}, \quad t \geq 0 \]

where \((U_t^{N,n}, n \geq 1)\) denotes a sequence of independent processes such that, for \( n \geq 1 \),

\[ (U_t^{N,n}) \overset{d}{=} (U_t^N). \]

**Proof**

This is a direct consequence of Theorem 6.8. \(\square\)

**Corollary 6.9.1.** The process

\[ Y_t^{(N,\mu)} := \int_{\mathbb{R}_+^*} (R^2_N(t \ s) - N \ t \ s) \, d\mu(s), \quad t \geq 0 \]

is a PCOC and an associated martingale is

\[ M_t^{(N,\mu)} := U_t^{(N,\mu)} - N \ t \int_{\mathbb{R}_+^*} s \, d\mu(s), \quad t \geq 0. \]
The above martingale \((M_t^{(N,\mu)})\) is purely discontinuous. We also may associate to the PCOC \((V_t^{(N,\mu)})\) a continuous martingale, as we now state.

**Theorem 6.10.** A continuous martingale associated to the PCOC \((V_t^{(N,\mu)})\) is

\[
\sum_{n \geq 1} \lambda_n \left( (R^{(n)}_N(t))^2 - Nt \right), \quad t \geq 0
\]

where \((R^{(n)}_N(t))\), \(n \geq 1\) denotes a sequence of independent processes such that, for \(n \geq 1\),

\[
(R^{(n)}_N(t)) \overset{(d)}{=} (R_N(t)).
\]

**Proof**
This is again a direct consequence of Theorem 6.8.

We can also explicit the relation between \(U^{(N,\mu)}\) and \(U^{(N',\mu)}\). Let \(C^{(N,\mu)}\) (resp. \(C^{(N',\mu)}\)) be the Lévy process associated with \(Y^{(\mu)}_N\) (resp. \(Y^{(\mu)}_{N'}\)). We see, by Laplace transform, that

\[
(C^{(N',\mu)}, s \geq 0) \overset{(d)}{=} (C^{(N,\mu)}_{N'/s}, s \geq 0).
\]  \(12\)

Then, using the relations between the processes \(U\) and \(C\) given in Theorem 2.2, we obtain:

**Proposition 6.11.** We have:

\[
(U_t^{(N',\mu)}, 0 \leq t \leq 1) \overset{(d)}{=} \left( \int_{-\log t}^{\infty} e^{-s} \, dC^{(N',\mu)}_s, 0 \leq t \leq 1 \right).
\]

**Proof**
By Theorem 2.2,

\[
(U_t^{(N',\mu)}, 0 \leq t \leq 1) \overset{(d)}{=} \left( \int_{-\log t}^{\infty} e^{-s} \, dC^{(N',\mu)}_s, 0 \leq t \leq 1 \right).
\]

Therefore, we obtain by \(12\),

\[
(U_t^{(N',\mu)}, 0 \leq t \leq 1) \overset{(d)}{=} \left( \int_{-\log t}^{\infty} e^{-s} \, dC^{(N',\mu)}_s, 0 \leq t \leq 1 \right).
\]

or, after the change of variable: \(N'/s = u\),

\[
(U_t^{(N',\mu)}, 0 \leq t \leq 1) \overset{(d)}{=} \left( \int_{-\log t^{N'/N}}^{\infty} e^{-Nu/N'} \, dC^{(N,\mu)}_u, 0 \leq t \leq 1 \right).
\]

By Theorem 2.2,

\[
C^{(N,\mu)}_u = \int_{e^{-u}}^{1} \frac{1}{r} \, dU^{(N,\mu)}_r = \int_{0}^{u} e^v \, dU^{(N,\mu)}_{e^{-v}}.
\]
Therefore,

\[
(U_t^{(N', \mu)}, 0 \leq t \leq 1) \overset{(d)}{=} \left( \int_{-\log t^{N'/N}}^{\infty} e^{\frac{N'-N}{s} u} dU_{e^{-u}}^{(N, \mu)}, 0 \leq t \leq 1 \right)
\]

and, after the change of variable: \(e^{-u} = s\),

\[
(U_t^{(N', \mu)}, 0 \leq t \leq 1) \overset{(d)}{=} \left( \int_{0}^{t^{N'/N}} s^{\frac{N'-N}{s}} dU_{s^{N-N'}}^{(N, \mu)}, 0 \leq t \leq 1 \right).
\]

By a similar computation for \(t \geq 1\), we finally obtain the desired result.

\[\square\]

**Corollary 6.11.1.** For \(N > 0\) and \(K > 0\), we set, with the notation of Section 5,

\[T_t^{N, K} = T_t(R_{K, \mu(N)}), \quad t \geq 0.\]

Then, for \(N > 0\), \(N' > 0\) and \(K > 0\), for any \(t \geq 0\),

\[T_t^{N', K} \overset{d}{=} \int_{0}^{t^{N'/N}} s^{\frac{N-N'}{s}} dT_s^{N, K}.
\]

**Proof**

By Theorem 5.2, \((T_t^{N, K})_{t \geq 1/2}\) is the 1-Sato process associated with \(Y_{N}^{(\mu)}\) defined from

\[
\mu = \frac{1}{K^2} t^{\frac{2(1-K)}{K}} 1_{(0,1]}(t) \, dt.
\]

\[\square\]

**References**

[1] L. Bondesson. *Generalized gamma convolutions and related classes of distributions and densities*, Lect. Notes Stat. 76, Springer, 1992. [MR1224674]

[2] A.N. Borodin; P. Salminen. *Handbook of Brownian Motion - Facts and Formulae*, Birkhäuser, 1996. [MR1477407]

[3] L. Chaumont; R.A. Doney. Pathwise uniqueness for perturbed versions of Brownian motion and reflected Brownian motion. *Probab. Theory Relat. Fields*, 113-4 (1999), p. 519-534. [MR1717529]

[4] E.A. Coddington; N. Levinson. *Theory of Ordinary Differential Equations*, McGraw-Hill, 1955. [MR0069338]
[5] P. Deheuvels; G.V. Martynov. A Karhunen-Loeve decomposition of a Gaussian process generated by independent pairs of exponential random variables. *J. Funct. Anal.*, 255 (2008), 2363-2394. MR2473261

[6] R.A. Doney; J. Warren; M. Yor. Perturbed Bessel Processes. In *Séminaire de Probabilités XXXII*, Lect. Notes Math. 1686, Springer, 1998, p. 231-236. MR1655297

[7] R.A. Doney; T. Zhang. Perturbed Skorokhod equations and perturbed reflected diffusion processes. *Ann. Inst. Henri Poincaré, Probab. Stat.*, 41-1 (2005), p. 107-121. MR2110448

[8] J.M. Harrison. Ruin problems with compounding assets. *Stoc. Proc. Appl.*, 5 (1977), p. 67-79. MR0423736

[9] F. Hirsch; M. Yor. A construction of processes with one-dimensional martingale marginals, based upon path-space Ornstein-Uhlenbeck processes and the Brownian sheet. *J. Math. Kyoto Univ.*, 49-2 (2009), p. 389-417. MR2571849

[10] F. Hirsch; M. Yor. A construction of processes with one-dimensional martingale marginals, associated with a Lévy process, via its Lévy sheet. *J. Math. Kyoto Univ.*, 49-4 (2009), p. 785-815. MR2591117

[11] M. Jeanblanc; J. Pitman; M. Yor. Self-similar processes with independent increments associated with Lévy and Bessel processes. *Stoc. Proc. Appl.*, 100 (2002), p. 223-231. MR1919614

[12] M. Kac; A.J.F. Siegert. On the theory of noise in radio receivers with square law detectors. *J. App. Physics*, 18 (1947), p. 383-397. MR0020229

[13] M. Kac; A.J.F. Siegert. An explicit representation of a stationary Gaussian process. *Ann. Math. Statist.*, 26 (1947), p. 189-211. Also reprinted in *Mark Kac Selected Papers*, MIT Press, 1979. MR0021672

[14] H.G. Kellerer. Markov-Komposition und eine Anwendung auf Martingale. *Math. Ann.*, 14 (1971), p. 1-16. MR0355250

[15] J. Kent. Some probabilistic properties of Bessel functions. *Ann. Prob.*, 6 (1978), p. 760-770. MR0501378

[16] J.F. Le Gall; M. Yor. Excursions browniennes et carrés de processus de Bessel. *C.R. Acad. Sci. I*, 303 (1986), p. 73-76. MR0851079

[17] J.F. Le Gall; M. Yor. Enlacements du mouvement brownien autour des courbes de l’espace. *Trans. Amer. Math. Soc.*, 317 (1990), p. 73-76. MR0946219

[18] D. Madan; M.Yor. Making Markov martingales meet marginals: with explicit constructions. *Bernoulli*, 8-4 (2002), p. 509-536. MR1914701

[19] R. Mansuy; M. Yor. *Aspects of Brownian motion*. Springer Universitext, 2008. MR2454984

[20] T. Nilsen; J.Paulsen. On the distribution of a randomly discounted compound Poisson process. *Stoc. Proc. Appl.*, 61 (1996), p. 305-310. MR1386179
[21] J. Pitman; M. Yor. A decomposition of Bessel bridges. Zeitschrift für Wahr., 59 (1982), p. 425-457. MR0656509

[22] D. Revuz; M. Yor. Continuous martingales and Brownian motion, Springer, third edition, 1999. MR1725357

[23] B. Roynette. Personal communication.

[24] K. Sato. Lévy Processes and Infinitely Divisible Distributions. Cambridge University Press, 1999. MR1739520

[25] T. Shiga; S. Watanabe. Bessel diffusion as a one-parameter family of diffusion processes, Z.W., 27 (1973), p. 37-46. MR0368192

[26] M. Yor. Sur le balayage des semi-martingales continues. In Séminaire de Probabilités XIII, Lect. Notes Math. 721, Springer, 1979, p. 453-471. MR0544815