A Model for the Voltage Steps in the Breakdown of the Integer Quantum Hall Effect

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In samples used to maintain the US resistance standard the breakdown of the dissipationless integer quantum Hall effect occurs as a series of dissipative voltage steps. A mechanism for this type of breakdown is proposed, based on the generation of magneto-excitons when the quantum Hall fluid flows past an ionised impurity above a critical velocity. The calculated generation rate gives a voltage step height in good agreement with measurements on both electron and hole gases. We also compare this model to a hydrodynamic description of breakdown.

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In the integer quantum Hall effect (IQHE) regime [1], a two-dimensional electron fluid carries an almost dissipationless current and the ratio of the current, $I_x$, to the Hall voltage, $V_H$, is quantized in units of $e^2/h$. However, above a critical current, the dissipative voltage, $V_x$, measured along the direction of current flow, increases rapidly, leading to quantum Hall breakdown (QHBD). Several possible mechanisms for QHBD [2] have been proposed: avalanche heating [3]; percolation due to an increase of delocalized states [4]; quasi-elastic inter-Landau level scattering [5–7]; acoustic phonon emission due to intra-Landau level scattering [8]; formation of compressible metallic filaments [9]; and resonant impurity scattering of electrons [10]. For certain samples, including those used to maintain the US resistance standard at the National Institute of Standards and Technology (NIST), breakdown occurs as a series of up to twenty steps, in $V_x$, of roughly equal height $\Delta V_x \approx 5 mV$, see Fig. 1 [11,12]. This is fundamentally different from the case in which breakdown is observed as a single large increase of $V_x$ [3].

IQHE breakdown is not only of fundamental interest, but is also relevant to quantum metrology since a large value of $I_x$ can improve the measurement precision. In this work we develop a theoretical model to account for the dissipative steps observed by the NIST group [11,12] and others [13,14]. We show that, in the presence of charged impurity-induced disorder, the quantum Hall fluid (QHF) is unstable when the local fluid velocity exceeds a critical value. Under these conditions, magneto-exciton or electron-hole (e-h) pair excitations are generated spontaneously near an impurity. The voltage step height, $\Delta V_x$, is directly related to the rate of formation of the pairs, which we calculate using a parameter-free model. This type of excitation of the QHF is analogous to vortex-antivortex pair formation in classical or quantum fluid flow around an obstacle [15].

Earlier work [16–18] has formulated a method for calculating the excitation dispersion relation of e-h pairs generated by exciting an electron from an occupied Landau level $n$ to an unoccupied Landau level $(n+1)$, where $n$ is the Landau level index. By extending these models to include the electric field, arising from a charged impurity and the Hall voltage drop across the sample, we obtain a critical electric field at which it costs no energy to generate an e-h pair at a given wave-vector $Q$. We incorporate this into a calculation of the generation rate, $W$, of these pairs [19,20] due to a single charged impurity. This then gives a dissipative voltage increment, $\Delta V_x = hW/(2e)$.

Our starting point is the Fermi golden rule, from which we calculate the rate of generation of e-h pairs, in the Landau gauge, due to a single charged impurity in a 2DEG [19,20],

$$W_{n,(n+1)} = \frac{2\pi}{\hbar} \delta(\epsilon_n - \epsilon_{(n+1)}) \times \left| \int d^2 r \left[ \Xi_0(z) \right] \phi_n(r_\perp,k_z) \phi_{(n+1)}(r_\perp,k'_z)V(r) \right|^2,$$

where $V(r)$ is the impurity Coulomb potential, $\phi_n(r_\perp,k_z)$ is the electronic eigen-function in the $xy$ plane and $\Xi_0(z)$ is the envelope function of the first electronic sub-band. We assume that the $n$ lower Landau levels are filled and the $(n+1)$ level is empty. There is thus no static screening of the impurity charge [21]. This gives

$$W_{n,(n+1)} = \frac{2\pi}{\hbar} \delta(\epsilon_n - \epsilon_{(n+1)}) \int_{q_\perp} d^2 q_\perp \frac{S}{(2\pi)^2} \times$$

FIG. 1. Step like breakdown in the integer quantum Hall effect, from Ref. [12]. Each step contributes a multiple of $\approx 5 mV$ to the dissipative longitudinal voltage.
\[ \int_{r_{n+1}} d^2r \phi_n(r, k_x) \phi_{n+1}(r, k'_x) e^{iQr} F_1(q, z) \left| z_{n,n+1} \right|^2, \tag{2} \]

where

\[ F_1(q, z) = \frac{e^{2}}{2 Skq_{1}} \int_{-\infty}^{\infty} |\Xi_0(z)|^2 e^{-q_{1}|z-z_{1}|} dz. \tag{3} \]

Here \( \kappa \) is the dielectric constant (for GaAs \( \kappa = 0.11nFm^{-1} \)). If we assume that the impurity is at the center of the subband then,

\[ W_{n(n+1)} = \frac{2\pi}{\hbar} \left( \frac{e^{2}2\pi}{8\pi^2} \right)^2 (\delta(\epsilon_n - \epsilon_{n+1})) \int_{q_{1}} dky \frac{1}{q_{1}(1 + \frac{q^2}{\kappa})^2} j_{n(n+1)}^{k_x, k'_x} \left| z_{n,n+1} \right|^2, \tag{4} \]

where

\[ j_{n(n+1)}^{k_x, k'_x} = \int_{y} dy \Phi_n(y - Y_{k_x}) e^{iQy} \Phi_{n+1}(y - Y_{k'_x}), \tag{5} \]

\( \Phi_n(y - Y_{k_x}) \) is the simple harmonic oscillator solution to the Schrödinger equation centered on \( Y_{k_x} = l_y k_x - E_y m/eB^2 \), \( m \) is the mass of the electron, \( E_y \) is the y component of the electric field, \( l_y = \sqrt{\hbar/eB} \) is the magnetic length and \( B \) is the magnetic field. Calculating the transition rate out of state \( (n, k_x) \) we find that

\[ W_n = \sum_{k'_x} W_{n(n+1)} = \frac{2L_x}{2\pi} \int_{k_x} dk'_x W_{n(n+1)} = \frac{1}{L_x \hbar} \left( \frac{e^{2}}{4\pi\kappa} \right)^2 \frac{\partial k_x}{\partial \Delta \varepsilon} |_{\Delta \varepsilon = 0} \int_{q_{1}} dky \frac{1}{q_{1}(1 + \frac{q^2}{\kappa})^2} j_{n(n+1)}^{k_x, k'_x} \left| z_{n,n+1} \right|^2, \tag{6} \]

where [16–18]

\[ \Delta \varepsilon = \hbar \nu_e + \frac{e^{2}}{4\pi\kappa\hbar B} \Delta_n(n+1)[k'_{x} - k_{x}] - eE_y l_{B}^2 (k'_x - k_x), \tag{7} \]

\( \Delta_n(n+1)[k'_{x} - k_{x}] \) includes the exchange and Coulomb local-field corrections, which are independent of \( E_y \). To obtain Eq. (7) we have assumed \( e^{2}/\kappa a_{L} \ll \hbar \omega_c \); hence we neglect Landau level mixing. In Eq. (6) there is an implicit condition for \( k'_x \) which must be calculated. This condition can be obtained from Eq. (6) when \( \Delta \varepsilon = 0 \). If exchange and Coulomb interactions are omitted this condition is simply the same as for energy-conserving elastic inter-Landau level transitions. However, when interaction terms are included \( Q = k'_x - k_x \) must be evaluated numerically. Eq. (7) can be split into two components: the excitation interaction energy, given by the first two terms, which is independent of \( E_y \) and the electrostatic energy, \( eE_y l_{B}^2 Q \). In Fig. 2 the crossing point of the electrostatic energy (dashed line) and the excitation interaction energy (solid line) gives the value of the e-h separation \( (l_{B}^2/Q) \) for which it costs no energy to generate e-h pairs, i.e. \( \Delta \varepsilon = 0 \). To obtain the total rate of production of e-h pairs, we take Eq. (6) and sum over all initial states such that

\[ W = \sum_{k_x} W_{n} = \frac{2L_x}{2\pi} \int_{k_x} dk_x W_{n}. \tag{8} \]

The above equation gives us the generation rate of e-h pairs by a single charged impurity at given \( B \) and \( E_y \).

Consider a local region of the sample where \( E_y \) is large enough to create e-h pairs, due to scattering from a charged impurity, at a rate given by Eq. (8). Such regions can be expected to occur at high current, possibly near the sample edge where the Hall field is expected to be large [22]. A pair created close to an impurity will drift along the Hall bar at a velocity \( v \approx E_y/B \), so one can imagine, for a fixed generation rate, a stream of e-h pairs moving along the Hall bar. Then, for \( v = 2 \), we have a situation in which a small fraction of electrons in the lower Landau level \( (n = 0) \) have been replaced by holes and the previously empty upper Landau level \( (n = 1) \) contains some electrons. As the e-h pairs move away from the high field region, the spacing between the electron and hole in a pair will increase and most pairs will eventually ionise by acoustic phonon emission. Due to the absence of empty states into which the excited electron can relax and neglecting weak, second order Auger processes, we can assume that all the generated e-h pairs will eventually ionise and lead to a dissipative current \( i = eW \), flowing across the Hall voltage equipotentials. At \( v = 2 \) this gives a dissipative voltage

\[ V_x = \left( \frac{eW}{2e} \right). \tag{9} \]

We now compare the results of our model with breakdown measurements which show voltage steps. The NIST experiments on the US resistance standard samples [12] were carried out at \( v = 2 \) and \( B = 12.3T \). The experi-
The above calculated values are derived from a model which combines a calculation of the magneto-exciton dispersion with an impurity-related tunneling rate. We have included explicitly the mechanism for the tunneling between the Landau levels and exchange and Coulomb local field corrections.

An earlier paper by one of us [15] drew an analogy between the process described here and the formation of vortices behind an obstacle moving relative to a fluid (e.g. the von Karman vortex street in classical hydrodynamics). Using our model we now examine this analogy more closely. One of the key ideas in understanding the quantum Hall states was the appreciation by Laughlin [23] that they have low Coulomb energy because the particles are tied to zeros in the many-body wavefunction. Zhang, Hansson and Kivelson [24] effected such a binding of zeros to particles using a Chern-Simons construction. This leads to an effective field theory for the fractional quantum Hall states in which the quanta of flux of a fictitious magnetic field are tied to each particle. This has two effects: firstly, in the mean field approximation the fictitious flux cancels the real flux through the system; secondly, the charge-flux composites become bosonic. Hence, at the mean field level, the QHF in a field corresponding to integer filling is replaced by a system of composite bosons in zero magnetic field. Such a system forms a charged superfluid and one can view the quantum Hall state as a composite Bose condensate. From this, Stone [25] formulated an effective superfluid hydrodynamic model. The charged elementary excitations, e-h pairs, of the QHF appear naturally in this description as vortices in the order parameter for the composite boson superfluid. In this language, our model for the QHBD is the spontaneous creation of vortex-antivortex pairs when the QHF fluid velocity around an impurity reaches a critical value [26].

According to Stone [25] the equation of motion of the QHF is

\[ m^* [\dot{\mathbf{v}} - (\mathbf{v} \times \mathbf{\Omega})] = e (\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \nabla \left( \frac{m^*}{2} |\mathbf{v}|^2 + \mu \right), \]

\[ \text{(10)} \]

where \( \mathbf{v} \) is the velocity field, \( \mu \) is the local chemical potential containing all the interaction terms and \( \mathbf{\Omega} \) is the fluid vorticity given by

\[ \Omega_x = \Omega_y = 0 \quad \text{and} \quad \Omega_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}. \]

\[ \text{(11)} \]

Using Eq. (11), in conjunction with the continuity equation for the density of the QHF, and examining small perturbations in the velocity field, of the form

\[ v_x = \frac{E_y}{B} + \epsilon_1 \cos(Qx - \omega t) \]

\[ \text{(12)} \]

and

\[ v_y = \epsilon_2 \sin(Qx - \omega t) \]

\[ \text{(13)} \]
we find that when $\nabla \mu = 0$,
\[
\hbar \omega = \Delta \varepsilon = -eE_y l_B (l_B Q) + \hbar \omega_c.
\]  
(14)

This result is equivalent to Eq. (7) in the absence of interactions and for $\Delta \varepsilon = 0$ corresponds exactly to the elastic inter-Landau level tunneling condition [7]. Alternatively within this hydrodynamic model it corresponds to the condition required to generate a vortex-antivortex pairs at zero energy.

To make a direct comparison with our earlier quantum mechanical calculation we need to evaluate the dissipative voltage drop along the sample due to the generation of these vortex-antivortex pairs from a single impurity. For a specific system we would have to rely on a numerical simulation of Eq. (10). However, we can make considerable progress by implementing what we already know about fluid mechanics [27]. Consider an obstacle in the path of a fluid. At a low fluid velocity the flow around the obstacle is laminar. When the flow rate is increased vortex-antivortex pairs are formed in the vicinity of the obstacle. However, a vortex street is not formed until the flow is fast enough to free the vortex-antivortex pairs from the local flow field near the obstacle. In this steady state each vortex-antivortex pair moves away from the obstacle at a velocity which is governed by the background fluid velocity. This analogy suggests that the vortex-antivortex pair in a QHF moves away from the impurity at a velocity given by $E_y / B$. From classical hydrodynamics [27] it is also known that the distance between each vortex-antivortex (l) pair generated is approximately three times the separation between a single vortex and antivortex (d) (d/l = 0.28). Now consider two states for our fluid: firstly, the state where the fluid velocity is reached. Both of these calculations predict voltage steps which agree with experiment. We believe that the fluid model can be improved upon through numerical modeling and the inclusion of interactions, however, the simple analysis provided above gives a strong clue that future work in this direction could be very useful in making direct comparisons with experiments.

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