\( \mathcal{N} = 1 \) Conformal Superspace in Four Dimensions

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Abstract

We construct in detail a \( \mathcal{N} = 1, D = 4 \) superspace with the superconformal algebra as the structure group and discuss its relation to prior component approaches and the existing Poincaré superspaces.
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1 Introduction

The use of conformal techniques to address supergravity has a long history. Not all that long after Wess and Zumino discovered the superspace formulation of supergravity [1], Kaku, Townsend, and van Nieuwenhuizen, along with Ferrara and Grisaru, worked out the conformal structure of component supergravity and demonstrated that Poincare supergravity was a gauge-fixed version of conformal supergravity [2]. Howe first proposed superspace formulations of four-dimensional $\mathcal{N} \leq 4$ conformal supergravities by explicitly gauging $\text{SL}(2,\mathbb{C}) \times \text{U}(\mathcal{N})$ [3]. Work continued on conformal supergravity over the next few years (an excellent review [4] on the topic was written by Fradkin and Tseytlin) eventually culminating in the work of Kugo and Uehara, who not only popularized the conformal compensator approach to supergravity and matter systems [5] but also made a comprehensive analysis of the component transformation rules and spinorial derivative structure of $\mathcal{N} = 1$ conformal supergravity [6].

In large part, the results presented here are a superspace response to this last work. Here we will take a complementary approach, treating superspace as an honest supermanifold with a conformal structure. Unlike Howe, we will seek to gauge the entire superconformal algebra. We derive the necessary constraints on the superspace curvatures to reproduce a minimal set of component fields. Lastly, we will review how the various equivalent formalisms of superspace – the minimal Poincaré [7], the minimal Kähler [8], and even the new minimal Poincaré [9] – are all derivable from a conformal superspace under different gauge-fixing constraints. We emphasize that none of these results (except for our derivation of the minimal Kähler model) are new – only the framework.

This paper is divided into two sections. In the first, we discuss conformal representations of superfields on superspace and construct the constraints necessary for the existence of such a space. We also give the explicit form of all the curvatures from solving the Bianchi identities. In the second, we demonstrate how the auxiliary structure of $U(1)$ superspace is identical to a certain gauge-fixed version of conformal superspace. In addition, we explicitly construct the superspace of minimal supergravity, Kähler supergravity, and new minimal supergravity. Included in the appendix is an elementary review of the structure of global and local spacetime symmetry groups as well as the structure of actions over both the full manifold and submanifolds of such theories.

Throughout this paper we use the superspace notations and conventions of Binetruy, Girardi, and Grimm [8] (which are a slight modification of those of Wess and Bagger [10]) – with our own slight modification: we choose the superspace $U(1)$ connection to be Hermitian. That is, our connection $A_M$ here is equivalent to $-iA_M$ of [8]; similarly, our corresponding generator $A$ is equivalent to their $iA$. (The unfortunate coincidence of the generator and connection names will, we hope, not overly confuse the reader.)

Although the theory discussed here ought to be properly denoted “superconformal superspace,” this is an awkward term that we would like to avoid. Instead we use “conformal” when the subject is superspace. (Similarly, supertranslations on superspace are simply called translations.) When the component theory is under consideration, we restore the “super.”
2 Conformal superspace

In the ensuing section we describe the gauge structure, geometric constraints, and curvatures of conformal superspace, which is defined as a normal $\mathcal{N} = 1$ superspace with the structure group of the superconformal algebra. We discuss representations of that algebra, invariant actions and chiral submanifold actions. As usual, constraints must be imposed to eliminate unwanted fields; we will discuss their construction and solution. But the first place to start is at the component level, where conformal supergravity is well-known and its properties well-established.

Some use will be made throughout of results presented in the appendix. Specific references will be made when especially relevant.

2.1 Conformal supergravity at the component level

Conformal supergravity at the component level begins with the extension of the Poincaré to the super-Poincaré algebra by the addition of fermionic internal symmetries $Q_\alpha$. These anticommute to give spacetime translations:

$$\{Q_\alpha, \bar{Q}_\dot{\alpha}\} = -2i\sigma_{\alpha\dot{\alpha}}^a P_a$$

(2.1)

The rest of the super-Poincaré algebra is

$$[M_{ab}, M_{cd}] = \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac} + \eta_{ad}M_{bc}$$

$$[M_{ab}, P_c] = P_a\eta_{bc} - P_b\eta_{ac}$$

$$[M_{ab}, Q_\gamma] = (\sigma_{ab})^\gamma_{\beta}Q_\beta$$

(2.2)

The bosonic part of the algebra can be extended to include the conformal algebra. This requires the introduction of the conformal scaling operator $D$ and the special conformal operator $K_a$, which loosely speaking can be understood as a translation conjugated by inversions. A brief review of the conformal algebra is given in Appendix A.1.1.

These two generators can be added to the super-Poincaré algebra provided one also includes two new operators, the fermionic special conformal operator $S_\alpha$ (and its conjugate $\bar{S}_\dot{\alpha}$) and the chiral rotation operator $A$. (This last generator is the $U(1)$ R-symmetry.) It should be noted that the special conformal generators possess the same Lorentz transformation properties as the corresponding translation and supersymmetry generators, but have opposite weights under scalings and chiral rotations:

$$[D, P_a] = P_a, \quad [D, Q_\alpha] = \frac{1}{2}Q_\alpha, \quad [D, \bar{Q}_{\dot{\alpha}}] = \frac{1}{2}\bar{Q}_{\dot{\alpha}}$$

$$[D, K_a] = -K_a, \quad [D, S_\alpha] = -\frac{1}{2}S_\alpha, \quad [D, \bar{S}_{\dot{\alpha}}] = -\frac{1}{2}\bar{S}_{\dot{\alpha}}$$

$$[A, Q_\alpha] = -iQ_\alpha, \quad [A, \bar{Q}_{\dot{\alpha}}] = +i\bar{Q}_{\dot{\alpha}}$$

$$[A, S_\alpha] = +iS_\alpha, \quad [A, \bar{S}_{\dot{\alpha}}] = -i\bar{S}_{\dot{\alpha}}$$

$$[M_{ab}, K_c] = K_a\eta_{bc} - K_b\eta_{ac}$$

$$[M_{ab}, S_\gamma] = (\sigma_{ab})^\gamma_{\beta}S_\beta$$

(2.3)

---

1This operation is often called “dilatation.”
The special conformal generators have an algebra among each other that is similar to the supersymmetry algebra:

\[ \{S_\alpha, \bar{S}_{\dot{\alpha}}\} = +2i\sigma^a_{\alpha\dot{\alpha}}K_a \]  

(2.4)

Finally, the commutators of the special conformal generators with the translation and supersymmetry generators are

\[ [K_a, P_b] = 2\eta_{ab}D - 2M_{ab} \]

\[ [K_a, Q_\alpha] = i\sigma^{\alpha\beta\dot{\beta}} \bar{S}^{\dot{\beta}}, \quad [K_a, \bar{Q}^{\dot{\alpha}}] = i\bar{\sigma}^{\dot{\alpha}\beta} S_\beta \]

\[ [S_\alpha, P_a] = i\sigma^{\alpha\beta\dot{\beta}} \bar{Q}^{\dot{\beta}}, \quad [\bar{S}^{\dot{\alpha}}, P_a] = i\bar{\sigma}^{\dot{\alpha}\beta} Q_\beta \]

\[ \{S_\alpha, Q_\beta\} = 2D\epsilon_{\alpha\beta} - 2M_{\alpha\beta} - 3iA\epsilon_{\alpha\beta} \]

\[ \{\bar{S}^{\dot{\alpha}}, \bar{Q}^{\dot{\beta}}\} = 2D\epsilon^{\dot{\alpha}\dot{\beta}} - 2M^{\dot{\alpha}\dot{\beta}} + 3iA\epsilon^{\dot{\alpha}\dot{\beta}} \]  

(2.5)

All other commutators vanish.

We have made use of the convenient shorthand

\[ M_{\alpha\beta} = (\sigma_{ba}^{\epsilon})_{\alpha\beta} \quad M^{\dot{\alpha}\dot{\beta}} = (\bar{\sigma}_{ba}^{\epsilon})_{\dot{\alpha}\dot{\beta}} \]  

Where \( P_{\gamma} \equiv P_c\sigma_{c\gamma} \). The canonical decomposition of a vector into dotted and undotted spinors is accomplished via contraction with a sigma matrix.

Conformal supergravity in four dimensions is the gauge theory of the above algebra. The connection forms \( W_m^A \) can be collected with their generators \( X_A \) into the total connection form

\[ W_m = e_m^a P_a + \frac{1}{2} \psi_m^a Q_\alpha + \frac{1}{2} \omega_m^{ba} M_{ab} + b_m D + A_m A + f_m^a K_a + f_m^\alpha S_\alpha \]  

(2.6)

Here \( \alpha \) denotes both spinor chiralities (\( \alpha \) and \( \dot{\alpha} \)) and the spinor summation convention is that of [10]. In the local theory, the generator \( P_a \) is identified as the covariant derivative when acting on a covariant field\(^2\). The algebra of the \( P_a \) among themselves is altered by the introduction of curvatures. One finds on a covariant field \( \Phi \)

\[ [P_a, P_b]\Phi \equiv [\nabla_a, \nabla_b]\Phi = -R_{ab}\Phi \]  

(2.7)

where the curvatures are

\[ R_{nm} = T_{nm}^a P_a + T_{nm}^\alpha Q_\alpha + \frac{1}{2} R_{nm}^{ba} M_{ab} + H_{nm} D + F_{nm} A + R(K)_{nm}^a K_a + R(S)_{nm}^\alpha S_\alpha \]  

(2.8)

Here we are using \( T_{nm}^\alpha \) as the supersymmetry curvature (anticipating that in superspace this will become part of the torsion), \( H \) and \( F \) as the curvatures associated with scalings and

\(^2\) A covariant field \( \Phi \) transforms as \( \delta_\phi \Phi = g^A X_A \Phi \). This is linear in \( \Phi \) and involves no derivatives of the parameter \( g^A \).
chiral rotations, and \( R(K) \) and \( R(S) \) as the curvatures associated with special conformal and fermionic special conformal transformations. (The curvatures – with Lorentz form indices – are also covariant fields in the sense that a curvature transforms into another curvature.) The construction of a local gauge theory from a generic global theory is detailed in Appendix A.2.

Constraints are imposed on these curvatures in such a way as to eliminate the spin connection \( \omega_m^{\lambda a} \) and the special conformal connections \( f_m^a \) and \( f_m^\alpha \) in terms of the other fields. This procedure is summarized in the review literature \([4]\) but the details do not concern us here.

The transformation rules of the various gauge fields are straightforward to calculate and are given in \([4]\). For our purposes, the only ones which will matter are the supersymmetry transformations of the unconstrained fields:

\[
\begin{align*}
\delta Q e_m^a &= i(\xi \sigma^a \psi_m - \psi_m \sigma^a \bar{\xi}) \\
\delta Q \psi_m^a &= 2\nabla_m \xi^a \\
\delta Q b_m &= 2f_m^a \xi \Omega^a \\
\delta Q A_m &= -3if_m^a \xi_a + 3if_m^\alpha \bar{\xi}^\alpha
\end{align*}
\]

(2.9) (2.10) (2.11) (2.12)

The derivative \( \nabla_m \) is covariant with respect to spin, scalings, and chiral rotations and \( \xi^a \) is assumed to transform with opposite scaling and chiral weights as \( Q_\alpha \). The gravitino transformation rule is especially simple.

It is also useful to record the transformation rules of chiral matter coupled to conformal supergravity. For the chiral multiplet \( \Phi = (\phi, \psi, F) \),

\[
\begin{align*}
\delta Q \phi &= \sqrt{2}\xi \phi \\
\delta Q \psi &= \sqrt{2}\bar{\xi} F + i\sqrt{2}a^a \bar{\xi} \nabla_a \phi, \\
\delta Q F &= i\sqrt{2}(\xi \sigma^a \nabla_a \psi)
\end{align*}
\]

(2.13)

which is identical to the supersymmetry algebra except for the replacement of the regular derivative with the covariant one.

These sets of component transformation rules must be reproduced at the superfield level once we move to superspace; this will help us to find the proper constraints on the curvatures in superspace.

### 2.2 Conformal superspace and representations of the algebra

\( \mathcal{N} = 1 \) superspace is a manifold combining four-dimensional Minkowski coordinates \( x^m \) with four Grassmann coordinates \( \theta^a, \bar{\theta}_\dot{a} \) into a single supermanifold with coordinate \( z^M = (x^m, \theta^a, \bar{\theta}_\dot{a}) \). The superconformal algebra can be represented as a set of transformations on these coordinates. In differential form they read \([11]\)

\[
\begin{align*}
p_a &= \partial_a, \quad q_a = \partial_a - i(\bar{\theta} \sigma^a \epsilon)_{\dot{a}} \partial_{\dot{a}}, \quad \bar{q}^\dot{a} = \sigma^\dot{a} - i(\theta \sigma^a \epsilon)^\dot{a} \partial_a \\
\sigma_{ab} &= -x_{[a} \partial_{b]} + \theta \sigma_{ab} \partial_\theta + \bar{\theta} \sigma_{ab} \partial_{\bar{\theta}} \\
d &= x^m \partial_m + \frac{1}{2} \theta \partial_\theta + \frac{1}{2} \bar{\theta} \partial_{\bar{\theta}}, \quad \bar{a} = -i \theta \partial_\theta + i \bar{\theta} \partial_{\bar{\theta}} \\
s_a &= -2\theta^2 \partial_a + i(x_b - i\zeta_b)(\sigma_b \partial_\theta)_a - (x_b + i\zeta_b)(\theta \sigma_c \bar{\sigma}_\epsilon)_{\dot{a}} \partial_c \\
s^\dot{a} &= -2\bar{\theta}^2 \sigma^\dot{a} + i(x_b + i\zeta_b)(\bar{\sigma}_b \partial_{\bar{\theta}})^\dot{a} - (x_b - i\zeta_b)(\theta \sigma_a \bar{\sigma}_\epsilon)^\dot{a} \partial_c \\
k_a &= 2x_a(x \cdot \partial) - x^2 \partial_a - 2\zeta_a(\zeta \cdot \partial) + \zeta^2 \partial_a - (x_b + i\zeta_b)(\theta \sigma_a \bar{\sigma}_b \partial_\theta) - (x_b - i\zeta_b)(\bar{\theta} \sigma_a \sigma_b \partial_{\bar{\theta}})
\end{align*}
\]

(2.14)
where $\zeta^a \equiv \theta^i \sigma^a \bar{\theta}$. These operators can be found in several ways. The most straightforward is to write down the supersymmetry line element $ds^2 = \left(dx^\alpha + i\theta^i \sigma^\alpha d\bar{\theta} + i\bar{\theta}^a \sigma^a d\theta\right)^2$ and require that it be preserved up to a conformal factor. This yields the coordinate representations we have given above. The elements $p_a$, $q_\alpha$, $\bar{q}^{\dot{\alpha}}$, $m_{ab}$ and $s$ preserve the line element exactly; the others, $d$, $k_a$, $s_\alpha$ and $s^{\dot{\alpha}}$ preserve it only up to a conformal factor.

The field representation possesses the same algebra as the coordinate representation but with the opposite sign. We will be most interested in the field representation, which is the only sensible approach when the symmetry is made a local one.

As it will be useful to collect terms in a way which makes manifest the supersymmetry, we will denote by $P_A$ the set of generators $P_a$, $Q_\alpha$, and $\bar{Q}^{\dot{\alpha}}$; $P_A$ represents the super-translation generator on superspace. Similarly, the special conformal generators may be collected into a single $K_A$. The algebra as in Section 2.1 can then be written

\[
[D,P_A] = \lambda(A)P_A, \quad [A,P_A] = -i\omega(A)P_A \\
[D,K_A] = -\lambda(A)K_A, \quad [A,K_A] = +i\omega(A)K_A \\
[P_A,P_B] = -C_{AB}^CPC, \quad [K_A,K_B] = C_{AB}^CK_C \\
[K_A,P_B] = +2\tilde{\eta}_{AB}D - 2M_{AB} + 3iA\eta_{AB}\omega(A) - \frac{1}{2}K^C_{AB}C_{CBA} - \frac{1}{2}P^C_{AB}C_{CAB} \quad (2.15)
\]

The commutators and other objects are to be understood as carrying an implicit grading, explained further in Appendix B.

The various objects defined above are

\[
P_A = (P_a, Q_\alpha, \bar{Q}^{\dot{\alpha}}), \quad K_A = (K_a, S_\alpha, S^{\dot{\alpha}}) \\
M_{AB} = (M_{ab}, M_{\alpha\beta}, M_{\dot{\alpha}\dot{\beta}}) \\
\eta_{AB} = (\eta_{ab}, -\epsilon_{\alpha\beta}, -\epsilon^{\dot{\alpha}\dot{\beta}}), \quad \tilde{\eta}_{AB} = (\eta_{ab}, +\epsilon_{\alpha\beta}, +\epsilon^{\dot{\alpha}\dot{\beta}}) \quad (2.16)
\]

where mixed objects such as $M_{ab}$ and $\eta_{ab}$ are defined to be zero. Note that $\tilde{\eta}_{AB} = (-)^A\eta_{AB}$; this will be the origin of graded signs $(-)^A$ in subsequent formulae.

We also have the flat-space torsion tensor

\[
C_{AB}^C = -C_{BA}^C = \left\{ \begin{array}{ll}
-2i(\sigma^\alpha \epsilon_\alpha \beta) & \text{if } A = \alpha, B = \beta, C = c \\
0 & \text{otherwise}
\end{array} \right. \quad (2.17)
\]

as well as the numerical coefficients

\[
\lambda(A) = \left\{ \begin{array}{ll}
1 & \text{if } A = a \\
1/2 & \text{if } A = \alpha, \dot{\alpha}
\end{array} \right. \\
\omega(A) = \left\{ \begin{array}{ll}
+1 & \text{if } A = \alpha \\
-1 & \text{if } A = \dot{\alpha}
\end{array} \right. \quad (2.18)
\]

The tensor $C$, like all explicitly supersymmetric objects, possesses an implicit grading.\footnote{That is, we interpret its antisymmetry condition to mean $C_{ab}^C = -C_{ba}^C$ but $C_{\alpha\beta}^C = +C_{\beta\alpha}^C$. The implicit grading works by appending an extra sign whenever two fermionic objects (fields, indices, etc.) are permuted past each other.}

The matrix $\eta_{AB}$ is used to raise and lower indices; $\tilde{\eta}_{AB}$ is its transpose, and is equivalent to $\eta_{AB}(-)^{ab}$, the sign coming from the implicit grading.
The main limitation of this form is that the would-be super-rotation generator \( M_{AB} \) is highly constrained: only the boson-boson part \( M_{ab} \) is independent. The fermion-boson part \( M_{\alpha b} \) vanishes, and the fermion-fermion part \( M_{\alpha\beta} \) is just a left-handed projection of \( M_{ab} \). Nevertheless, we may write its commutator with \( P_A \) in the elegant form

\[
[M_{AB}, P_C] = P_{A\eta_BC} - P_{B\eta_AC}
\] (2.19)

where it is to be understood that the \( A, B, \) and \( C \) are all of the same type and the implicit grading is understood.

The representation theory of fields under the conformal group is discussed in [12] as well as in Appendix A.1.1 and is rather straightforward. The only difference from Poincaré representations is that we require primary fields \( \Phi \) to have constant conformal weight under \( D \) and to be annihilated by the special conformal generator \( K_a \).

We may extend that discussion to the superconformal group with little effort. Let \( \Phi \) be a primary superfield. It may or may not possess Lorentz indices, but we will suppress them for notational elegance. The action of the superconformal group is

\[
P_A \Phi = \nabla_A \Phi, \quad M_{ab} \Phi = S_{ab} \Phi
\]

\[
D \Phi = \Delta \Phi, \quad A \Phi = i w \Phi
\]

\[
K_A \Phi = 0
\]

The action of \( P_A \) is the statement that the translation generator acts as the covariant derivative. The matrix \( S_{ab} \) is appropriate for whatever representation of the Lorentz algebra \( \Phi \) belongs to. \( \Delta \) and \( w \) represent its conformal and chiral weights.

### 2.2.1 Primary chiral superfields

The superconformal algebra by itself does not itself tell us anything more about an arbitrary superfield than the conformal algebra tells us in spacetime. The advantage comes when restrictions are imposed. For example, the most important theoretical and phenomenological superfields are chiral ones. These obey a constraint \( \bar{\nabla}^{\dot{\alpha}} \Phi = 0 \), where again we are suppressing Lorentz indices which \( \Phi \) may possess. Requiring this to be superconformally invariant leads to a nontrivial condition on \( \Phi \):

\[
0 = \{ S^{\dot{\alpha}}, \bar{\nabla}^{\dot{\beta}} \} \Phi = \epsilon^{\dot{\alpha}\dot{\beta}} (2 D + 3 iA) \Phi - 2 M^{\dot{\alpha}\dot{\beta}} \Phi = \epsilon^{\dot{\alpha}\dot{\beta}} (2 \Delta - 3 w) \Phi - 2 M^{\dot{\alpha}\dot{\beta}} \Phi
\] (2.21)

The first term is antisymmetric in the indices, the second is symmetric. Therefore each must vanish separately. The first tells us \( 2 \Delta = 3w \), that is, the \( U(1) \) weight and scaling dimension of the field \( \Phi \) have a fixed ratio. The second term tells us that \( \Phi \), when decomposed into irreducible spinors, contains no dotted indices, since \( M^{\dot{\alpha}\dot{\beta}} \) acts only on these. Thus, \( \Phi_{\alpha\beta} \), \( \Phi_{\alpha\dot{\beta}} \), and \( \Phi_{\dot{\alpha}\dot{\beta}\gamma} \) are valid chiral superfields, but \( \Phi_{(\alpha\dot{\beta})} = \sigma_{\alpha\beta}^{\dot{c}} \Phi_c \) is not. (We will use the notation \( (\dot{\alpha}\dot{\gamma}) \) to denote a vector index contracted with a sigma matrix.)

### 2.2.2 Primary gauge vector superfields

The next most important superfield is the gauge vector superfield \( V_\alpha \). It is related to the chiral superfield \( W_\alpha \) by \( W_\alpha = P [\nabla_\alpha V] \) where \( P \) is the chiral projection operator. In flat
supersymmetry this condition reads $W_\alpha = -\frac{1}{2} \bar{D}^2 D_\alpha V$ where $D_A$ is the flat space covariant derivative; we will assume without (yet) a proof that this expression holds in the case of a nontrivial geometry simply by replacing $D_A$ with $\nabla_A$. If we demand that $W_\alpha$ be primary in addition to $V$ being primary, we can deduce a nontrivial condition on $V$. The primary condition is actually three: the vanishing of $K$, $S$, and $\bar{S}$ on $W_\alpha$. Since the anti-commutator of $S$ and $\bar{S}$ yields $K$, we only need to check that $S$ and $\bar{S}$ vanish. Consider $S$ first:

$$0 = -4S_\beta W_\alpha = S_\beta \bar{\nabla}^2 \nabla_\alpha V = \bar{\nabla}^2 S_\beta \nabla_\alpha V = \bar{\nabla}^2 (2D\epsilon_{\beta\alpha} - 2M_{\beta\alpha} - 3iA\epsilon_{\beta\alpha}) V$$

Since $V$ is real, its chiral weight vanishes. Furthermore, it is a scalar so $M$ annihilates it. We are left with the condition $DV = 0$, that is $V$ must have conformal dimension zero. This forces $W_\alpha$ to have conformal dimension $3/2$, which is sensible since it must possess the gaugino as its lowest component. The check that $S^\beta$ also annihilates $W_\alpha$ is straightforward; no further constraints are required. It therefore follows that $W_\alpha$ is conformally primary precisely when $V$ has conformal dimension zero.

### 2.2.3 Primary F-term superfields

The last special case we will discuss is where $V$ is a superfield and we demand that its chiral projection $U = \mathcal{P}[V]$ is primary. (This is of interest since if $V$ is a D-term then $U$ is the corresponding F-term.) Generalizing the flat space result gives $U = -\frac{1}{2} \bar{\nabla}^2 V$ (which we will show is the case later). We assume that $V$ is primary with conformal weight $\Delta$ and chiral weight $w$. Then the primariness of $U$ reduces to checking that $\bar{S}$ annihilates $U$, since it is clear that $S$ annihilates $U$. This is a simple exercise:

$$-4S^\beta U = -\{S^\beta, \bar{\nabla}^\alpha\} \nabla_\alpha V - \nabla_\alpha \{S^\beta, \bar{\nabla}^\alpha\} V$$

$$= -(2D\epsilon^{\beta\dot{\alpha}} - 2M^{\beta\dot{\alpha}} + 3iA\epsilon^{\beta\dot{\alpha}}) \bar{\nabla}_\dot{\alpha} V - \bar{\nabla}_\dot{\alpha} \left(2D\epsilon^{\beta\dot{\alpha}} - 2M^{\beta\dot{\alpha}} + 3iA\epsilon^{\beta\dot{\alpha}}\right) V$$

$$= (8 - 4\Delta + 6w) \bar{\nabla}^{\beta} V$$

It follows that $2\Delta - 3w = 4$ is the condition on $V$ so that $U$ is primary. If we also require that the antichiral projection of $V$ be primary, then we would find $2\Delta + 3w = 4$, concluding that $w = 0$ and $\Delta = 2$. This latter case is most important since we will see if $V$ is a D-term action, then $U$ is the F-term action equivalent to $V$.

### 2.3 Local superconformal symmetry

A space of local symmetries includes a rule for parallel transport, which allows one to compare group elements at neighboring points. One demands that the supertranslation generators $P_A$ act as parallel transport operators with the supervierbein $E_M^A$ as their corresponding gauge field. For each of the other generators $X_A$, one also introduces a gauge field $W_M^A$:

$$W_M^A X_A = E_M^A P_A + \frac{1}{2} \phi_M^{ba} M_{ab} + B_M D + A_M A + f_M^A K_A$$

(2.22)

In practice, it is useful to decompose the algebra into the generators of parallel transport and the other generators, which annihilate pure functions (i.e. scalar primary fields with vanishing chiral and scaling weights). We denote the latter group as $\mathcal{H}$, its generators by $X_\alpha$, and its gauge fields by $h_M^\alpha$. In this manner, the total gauge connection is simply

$$W_M^A X_A = E_M^A P_A + h_M^\alpha X_\alpha$$

(2.23)
The action of the generators on fields is defined by
\[\delta_G(\xi^M W^A_M X_A)\Phi \equiv L\Phi.\] (2.24)
(See Appendix A.2 for a deeper discussion of this.) For fields lacking Einstein indices, this reduces to the simpler
\[\xi^M W^A_M X_A \Phi = \xi^M \partial_M \Phi\] (2.25)
Since the action of the non-translation generators is defined already, this defines the action of \(P_A\). One finds the standard definition of the covariant derivative
\[\xi^A P_A \Phi = \xi^M \nabla_M \Phi = \xi^M (\partial_M - h_M a X_a) \Phi\] (2.26)
If \(\Phi\) possesses an Einstein index, then an additional transformation rule for that index is required. For example, on the vierbien,
\[\delta_P(\xi) E^A_M = \xi^N \nabla_N E^A_M + \partial_M \xi^N E^N_A;\] (2.27)
this rule can be used to define \(\delta_P\) on any other Einstein tensor.

The algebraic structure of conformal superspace is identical to the flat case except for the introduction of curvatures to the \([P, P]\) commutator. We include here the results quoted in the most supersymmetric language: \[^{5}\]
\[
\begin{align*}
[P_A, P_B] &= -T_{AB}^C P_C - \frac{1}{2} R_{AB}^{\, dc} M_{cd} - H_{AB} D - F_{AB} A - R(K)_{AB}^C K_C \\
[M_{ab}, M_{cd}] &= \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac} + \eta_{ad} M_{bc} \\
[D, P_A] &= +\lambda(A) P_A \\
[A, P_A] &= -i\omega(A) P_A \\
[K_A, K_B] &= +C_{AB}^C K_C \\
[D, K_A] &= -\lambda(A) K_A \\
[A, K_A] &= +i\omega(A) K_A \\
[K_A, P_B] &= +2\eta_{AB} D - 2M_{AB} + 3i A\eta_{AB} \omega(A) - \frac{1}{2} K^C C_{CBA} - \frac{1}{2} P^C C_{CAB} 
\end{align*}
\] (2.28)

### 2.4 Invariant superconformal actions

Superspace actions fall into two types. The first is the D-type Lagrangian, involving an integration over the full Grassmannian manifold. The local action reads
\[S_D = \int d^4 x e L_D = \int d^4 x d^4 \theta E V\] (2.29)
Here \(e = \det(e_m^a)\) is the normal four dimensional measure factor, while \(E = \det(E_M^A)\) is the appropriate generalization for a D-term. (Setting \(E = e = 1\) retrieves the global action.) Invariance requires \(X_2 V = -(\ldots)^A f_{2A}^A V\), which amounts to
\[
\begin{align*}
DV &= 2V, \quad AV = 0, \quad M_{ab} V = 0, \\
K_a V &= 0, \quad S_a V = 0, \quad \bar{S} a V = 0
\end{align*}
\]
\[^{5}\] We have adopted the notation \(R(K)_{AB}^C\) for the special conformal curvature. One could similarly write \(R_{AB}^{\, dc}\) as \(R(M)_{AB}^{\, dc}\) but we have chosen to use the conventional name for the Lorentz curvature.
\[^{6}\] This determinant becomes a super-determinant when the implicit grading is taken into account.
$V$ must have scaling dimension two; its chiral weight must vanish; it must be a Lorentz scalar; it must be superconformally primary. One should also in general check the action of $P_a$, $Q_\alpha$, and $\bar{Q}^{\dot{\alpha}}$ to ensure that it is translation invariant and supersymmetric. Each of these gives a set of derivative operations; since the entire space is integrated over, each of these vanishes. A review of this material is presented in Appendix A.2.2.

The second Lagrangian of concern is the F-type, which involves an integration over the chiral submanifold $\mathcal{M}$ corresponding to $\bar{\theta} = 0$ (or to any other constant $\bar{\theta}$ slice):

$$S_F = \int d^4x \ e \mathcal{L}_F = \int d^4x \ d^2\theta \mathcal{E} W$$

(We will for brevity’s sake write only the chiral part; but in physical theories one must of course include the antichiral part.) The chiral measure $\mathcal{E}$ is to be understood as $\text{det}(E_m^a)$ where $m$ is the Einstein index over the chiral coordinates $y$ and $\theta$ and $a = (a, \alpha)$. This is a loose definition since the chiral $y$ and $\theta$ need to be better defined. Appendix A.2.5 contains a discussion of this.

For this action to be invariant under the non-translation part of the gauge group, $W$ must obey

$$D W = 3W, \quad A W = 2iW, \quad M_{ab}W = 0$$
$$K_aW = 0, \quad S_\alpha W = 0, \quad \bar{S}^{\dot{\alpha}} W = 0$$

For invariance under $P$, $Q$, and $\bar{Q}$, $W$ must be chiral, $\nabla_{\dot{\alpha}} W = 0$. In addition, consistency of this result (i.e. $\{\nabla_{\dot{\alpha}}, \nabla_{\dot{\beta}}\} W = 0$) leads to the following conditions on torsions and curvatures:

$$T_{\dot{\alpha}\dot{\beta}}^\gamma = T_{\dot{\alpha}\dot{\beta}}^{-\gamma} = 0, \quad H_{\dot{\alpha}\dot{\beta}} + \frac{2i}{3} F_{\dot{\alpha}\dot{\beta}} = 0$$

These constraints are invariant under the action of $\mathcal{H}$, as is expected, and should be understood as the minimal set of constraints for a conformal superspace.

### 2.5 Constraints

Since every superfield contains sixteen independent components, the number of degrees of freedom represented by unconstrained gauge fields is staggering. The vierbein $E_M^A$ alone consists of 64 superfields, each possessing 16 independent components for a total of 1024 component fields. This problem can be circumvented by the imposition of certain constraints in superspace, followed by solving the Bianchi identities subject to these constraints. Conformal superspace must reduce to a Poincaré superspace upon the breaking of the conformal symmetry, so the constraints on its geometry ought to be more severe than those normally imposed. We will guess the constraints necessary by identifying the properties we would like to have. If this overconstrains the theory, so be it; the Bianchi identities will tell us if this occurs.

Perhaps the most fundamental constraint is the existence of chiral primary superfields, the absence of which would render any superspace theory practically useless. The chiral requirement, $\nabla_{\dot{\alpha}} \Phi = 0$, implies that the anticommutator $\{\nabla_{\dot{\alpha}}, \nabla_{\dot{\beta}}\} \Phi$ vanishes. (We have suppressed any Lorentz indices which $\Phi$ may possess.) This commutator in turn gives the following constraints:

$$T_{\dot{\alpha}\dot{\beta}}^c = T_{\dot{\alpha}\dot{\beta}}^{-\gamma} = 0, \quad H_{\dot{\alpha}\dot{\beta}} + \frac{2i}{3} F_{\dot{\alpha}\dot{\beta}} = 0, \quad R_{\dot{\alpha}\dot{\beta}}^{\gamma\delta} = 0$$

(2.32)
(The complex conjugates are implied for the existence of anti-chiral superfields.) All of these conditions except the last we already knew; the last is required if chiral superfields with undotted spinor indices (such as $W_\alpha$ and $W_{\alpha\beta\gamma}$) should exist.

If we consider the component level behavior, more constraints may be deduced. The component conformal supergravity multiplet for a chiral matter scalar, $\phi$, possesses the same transformation laws as in flat supersymmetry, only with the regular derivative replaced by a covariant one:

$$\delta_Q \phi = \sqrt{2} \xi \psi, \quad \delta_Q \psi = \sqrt{2} F + i \sqrt{2} \sigma^a \xi \nabla_a \phi, \quad \delta_Q F = i \sqrt{2} (\xi \sigma^a \nabla_a \psi)$$

(2.33)

These equations can be interpreted as superspace formulae with the superfields $\psi_\alpha \equiv \frac{1}{\sqrt{2}} \nabla_\alpha \phi$ and $F \equiv -\frac{1}{4} \nabla^2 \phi$, and the formal definition of $\delta_Q \equiv \xi \nabla_\alpha + \bar{\xi} \nabla^\alpha$. Requiring that this variation $\delta_Q$ act on each of the superfields as indicated by the component transformation rules leads to a number of further constraints on the superspace curvatures:

$$T_{\alpha\beta\gamma} = T_{\alpha\beta\gamma} = T_{\beta\alpha\gamma} = 0, \quad T_{\alpha\beta} = 2i \sigma^c T_c^\alpha$$

(2.34)

Other more complicated conditions are also implied, but they end up being satisfied automatically by the Bianchi identities, so we do not bother listing them here in detail.

We can further restrict the superspace structure by requiring the component transformation laws for the gravitino, $U(1)$ gauge field, and scaling gauge field to behave as in component conformal supergravity. For example, the gravitino ought to transform under supersymmetry into a covariant derivative of the supersymmetry parameter, $\delta_Q \psi_m = 2 \nabla_m \xi$, without the need for any explicit auxiliary fields as in (2.10). Since we already know the transformation law for the gravitino is

$$\delta_Q E_{m}^\alpha = \nabla_{m} \xi^c + E_{m}^c \xi^\beta T_{\beta c}^\alpha + E_{m}^c \xi^\dot{\beta} T_{\dot{\beta} c}^\alpha$$

(2.35)

we are left to conclude $T_{\beta c}^\alpha = 0$. (These are the torsion components which in the minimal multiplet give the superfields $R$ and $G_c$ whose lowest components are the supergravity auxiliaries $M$ and $b_m$.) A similar analysis using the $U(1)$ and scaling gauge fields using (2.11) and (2.12) tells us $F_{\beta c} = H_{\beta c} = 0$.

One can continue in this manner to bootstrap constraints which seem reasonable. The ones discussed above are nearly sufficient to completely determine a minimal superspace formulation of conformal supergravity. It turns out only one additional constraint is needed: $R(K)_{\alpha\beta} = 0$ and its conjugate.

We summarize here the constraints we take. For torsion we have

$$T_{\gamma\beta}^A = T_{\gamma\beta} = 0 \quad T_{\gamma\beta}^a = 2i \sigma^a \gamma^\beta \quad T_{\gamma\beta} = T_{\gamma\beta}^A = 0$$

(2.36)

These define all torsion except for $T_{\gamma\beta}^\alpha$ and $T_{\gamma\beta}^\dot{\alpha}$ which remain undetermined. For the Lorentz curvature, we have

$$R_{\alpha\beta} = R_{\alpha\beta}^c = R_{\alpha\beta}^d = 0$$

(2.37)
For the chiral curvature,
\[ F_{\alpha\beta} = F_{\dot{\alpha}\dot{\beta}} = 0 \]
\[ F_{\dot{\alpha}b} = F_{\dot{\beta}b} = 0 \] (2.38)

Similarly for the scaling curvature:
\[ H_{\alpha\beta} = H_{\dot{\alpha}\dot{\beta}} = 0 \]
\[ H_{\dot{\alpha}b} = H_{\dot{\beta}b} = 0 \] (2.39)

For the special conformal curvature, we take
\[ R(K)_{\alpha\beta}^C = R(K)_{\dot{\alpha}\dot{\beta}}^C = R(K)_{\alpha\dot{\beta}}^C = 0 \] (2.40)

This set of conditions for the curvatures is especially interesting for one particular reason: it includes the condition \( R_{\alpha} = 0 \) for all curvatures except for torsion, where we choose the flat result \( T_{\alpha\beta}^c = 2i\sigma_{\alpha\beta}^c \). This is consistent with making the following demands on the fermionic covariant derivatives:
\[
\{\nabla_\alpha, \nabla_\beta\} = \{\nabla_{\dot{\alpha}}, \nabla_{\dot{\beta}}\} = 0 \] (2.41)
\[
\{\nabla_\alpha, \nabla_{\dot{\beta}}\} = -2i\nabla_{\alpha\dot{\beta}} \] (2.42)

The first of these implies the existence of a gauge choice where \( \nabla_\alpha = \partial_\alpha \) and the second implies the conjugate; the third implies that no gauge exists where both these conditions hold. Moreover, in flat supersymmetry, the chiral projector \( \mathcal{P} \) is proportional to \( \bar{D}^2 \). The condition that it should be \( \bar{D}^2 \) in conformal supergravity is equivalent to the constraints \( \{\nabla_\alpha, \nabla_\beta\} = \{\nabla_{\dot{\alpha}}, \nabla_{\dot{\beta}}\} = 0 \).

These constraints may at first glance seem exceedingly restrictive, certainly more so than those assumed in deriving Poincaré supergravity. It helps to recall that each of these objects, the torsion and the other curvatures, are internally more complicated than their non-conformal cousins due to the presence of the extra gauge fields. We will find that it is these fields, in particular those associated with the special conformal generators, which allow us to reconstruct normal Poincaré supergravity with its relaxed constraints after gauge fixing.

### 2.6 Jacobi and Bianchi identities

The discussion of the Jacobi and Bianchi identities in an arbitrary theory is given in [A.2.1](#) and merely needs to be specialized here. The Jacobi identity serves to define the gauge transformation properties of the curvatures:
\[
D \ T_{CB}^A = (\Delta(C) + \Delta(B) - \Delta(A)) \ T_{CB}^A
\]
\[
D \ R(K)_{CB}^A = (\Delta(C) + \Delta(B) + \Delta(A)) \ R(K)_{CB}^A
\]
\[
D \ R_{DC}^{BA} = (\Delta(D) + \Delta(C)) \ R_{DC}^{BA}
\]
\[
D \ F_{BA} = (\Delta(B) + \Delta(A)) \ F_{BA}
\]
\[
D \ H_{BA} = (\Delta(B) + \Delta(A)) \ H_{BA}
\] (2.43)

These conditions alone are probably sufficient to define a conformal superspace with dynamical spin connection and torsion as well as their superpartners; we conjecture that the extra constraints are to eliminate the spin connection and its associated multiplet but as yet are unaware of any direct evidence for this.
(With the exception of the $K$-curvature, these are entirely straightforward.) The $U(1)$ transformations are similarly simple:

$$A T_{CB}^A = -i \left( w(C) + w(B) - w(A) \right) T_{CB}^A$$

$$A R(K)_{CB}^A = -i \left( w(C) + w(B) + w(A) \right) R(K)_{CB}^A$$

$$A R_{DC}^{BA} = -i \left( w(D) + w(C) \right) R_{DC}^{BA}$$

$$A F_{BA} = -i \left( w(B) + w(A) \right) F_{BA}$$

$$A H_{BA} = -i \left( w(B) + w(A) \right) H_{BA}$$

The transformations under $K_A$ are, however, less than obvious:\footnote{\begin{flushleft}Note that gradings arising from the order of the indices have been left off for simplicity of notation. To replace them, note the order of the indices on the left side of the equation and add appropriate gradings to arrive at the same order. Also, contracted indices must be placed next to each other with the raised index on the left. For example, in the first line, the order of indices on the left is $DCB$. If we replace the gradings, we would have $K_D T_{CB}^A = \frac{1}{2} \Delta T_{CB} F C_{DF} A + \frac{1}{2} C_{F D(C} \Delta T_{B) F} A$.\end{flushleft}}

$$K_D T_{CB}^A = \frac{1}{2} \Delta T_{CB} F C_{DF} A + \frac{1}{2} C_{F D(C} \Delta T_{B) F} A$$

$$K_D H_{CB} = -(-)^D 2 \Delta T_{CBD} + \frac{1}{2} C_{F D(C} \Delta T_{BD F}$$

$$K_D F_{CB} = -3iw(D) \Delta T_{CBD} + \frac{1}{2} C_{F D(C} \Delta T_{BD F}$$

$$K_D R(K)_{CB}^A = R(K)_{CB} F C_{FD} A + \frac{1}{2} C_{F D(C} R(K)_{BD} F A - \frac{1}{2} \Delta T_{CB} F C_{F} A_D$$

$$- \lambda(D) H_{CB} \delta D A + iw(D) F_{CB} \delta D A + R_{CBD} A \frac{1}{2} \left( K_D R_{CB} a \right) M_{aa'} - 2 \Delta T_{CB} AM_{AD} - \frac{1}{4} C_{F D(C} R_{B D} F a \delta M_{aa'}$$

The notation $[CB]$ in the above denotes graded antisymmetrization of the respective indices. The rule for the Lorentz curvature has been left in a form with the explicit Lorentz transformations are similarly simple:

$$\sum_{[DCB]} \nabla_D F_{CB} + T_{DC} F_{FB} - 3i R(K)_{DCB} w(B)$$

$$\sum_{[DCB]} \nabla_D H_{CB} + T_{DC} H_{FB} - 2 R(K)_{DCB} (-)^B$$

$$\sum_{[DCB]} \nabla_D T_{CB} A + T_{DC} F T_{FB} A - R_{DCB} A + \lambda(A) H_{DC} \delta B A + iw(A) F_{DC} \delta B A - \frac{1}{2} R(K)_{DC} F C_{FA} B$$

$$\sum_{[DCB]} \nabla_D R(K)_{CBA} + T_{DC} F R(K)_{FBA} - \frac{1}{2} R(K)_{DC} F C_{BA F}$$

$$\sum_{[FDC]} \nabla_F R_{DC} \alpha + T_{FD} H R_{HC} \beta A - \frac{1}{2} R(K)_{FD} (\tilde{e}_\alpha) \phi + 2 R(K)_{FD} (\epsilon_\alpha) \delta$$

(2.46)
The sum over $[DCB]$ denotes a sum over graded cyclic permutations of those indices. Also, the notation $\{\}$ on indices denotes symmetrization; for example, $X_{(\alpha}Y_{\beta)} \equiv X_\alpha Y_\beta + Y_\beta X_\alpha$. (The last identity involving the Lorentz curvature has been projected to the left-handed part of the Lorentz group. The right-handed part is found by complex conjugation.)

As in [10] the constraints we have chosen restrict our gauge potentials; we must ensure that the Bianchi identities are satisfied in order for these constraints to be valid. Though our constraints are stronger than in [10], our curvatures and Bianchi identities are more numerous. We avoid recounting the derivation in detail here (see Appendix D for that) and merely quote the result: every curvature either vanishes or is expressed in terms of a single chiral superfield $W_{\alpha \beta \gamma}$. It obeys

$$DW_{\alpha \beta \gamma} = \frac{3}{2}W_{\alpha \beta \gamma}, \quad AW_{\alpha \beta \gamma} = iW_{\alpha \beta \gamma}, \quad K_A W_{\alpha \beta \gamma} = 0$$

(2.47)

That is, $W_{\alpha \beta \gamma}$ possesses the same scaling and $U(1)$ weights as it does in Poincaré supergravity and is conformally primary. Furthermore, it is constrained by its own Bianchi identity

$$\nabla^\gamma_{\beta} \nabla^\phi_{\delta \gamma \beta} = -\nabla^\beta_{\gamma} \nabla^\phi_{\delta \gamma \beta}$$

(2.48)

The results for the curvatures follow below.

### 2.6.1 Torsion

The conformal torsion two-form is defined in terms of the gauge connections:

$$T^A = dE^A + \lambda(A)E^A B - iw(A)E^A A + E^B \phi_B A - \frac{1}{2} C^{ABC} E_B f_C$$

(2.49)

We group the various components in terms of their scaling dimension.

- **Dimension 0**

  $$T_{\gamma \beta}^a = 0, \quad T_{\gamma}^{\delta a} = 0$$

  (2.50)

  $$T_{\gamma}^{\delta a} = -2i(\sigma^a e)_\gamma^\beta$$

  (2.51)

- **Dimension 1/2**

  $$T_{\gamma \beta}^a = 0, \quad T_{\gamma}^a = 0$$

  (2.52)

- **Dimension 1**

  $$T_{\gamma \beta}^a = 0, \quad T_{\gamma}^{\delta a} = 0$$

  (2.53)

  $$T_{\gamma}^{\delta a} = 0, \quad T_{\gamma}^{\alpha} = 0$$

  (2.54)

  $$T_{\gamma}^a = 0$$

  (2.55)

- **Dimension 3/2**

  $$T_{\gamma \beta}^a \sim T_{(\gamma \delta)(\beta \delta)\alpha} = +2\epsilon_{\gamma \delta} W_{\gamma \beta \alpha}$$

  (2.56)

  $$T_{\gamma \beta}^a \sim T_{(\gamma \delta)(\beta \delta)\dot{\alpha}} = -2\epsilon_{\gamma \delta} \tilde{W}_{\gamma \beta \dot{\alpha}}$$

  (2.57)
2.6.2 Lorentz curvature

The conformal Lorentz curvature two-form is

\[ R^{ba} = d\phi^b - \phi^c\phi_c^a - 2E[^b f^a] - 4E^\beta f^a(\sigma^{ba} e)_{\alpha\beta} - 4E^\beta f_a(\sigma^{ba} e)^{\dot{\gamma}\dot{\beta}} \]  (2.58)

The notation \([b...a]\) denotes antisymmetrization of those indices; for example, \(X_{[bY_a]} = X_bY_a - X_aY_b\).

Because the form is valued in the Lorentz group, it may be canonically decomposed:

\[ R_{DC}^{\beta\alpha} \leadsto R_{DC(\beta\dot{\gamma})}^{\gamma\alpha} = 2\epsilon^{\dot{\gamma}\dot{\beta}}R_{DC\beta\alpha} - 2\epsilon^{\beta\alpha}R_{DC\gamma\dot{\dot{\alpha}}} \]  (2.59)

It is simplest to express the curvature results in terms of these components.

- **Dimension 1**

  \[ R_{\dot{\delta}\gamma\beta\alpha} = 0, \quad R_{\dot{\delta}\gamma\dot{\beta}\dot{\alpha}} = 0 \]  (2.60)

  \[ R_{\dot{\delta}\dot{\gamma}\beta\alpha} = 0, \quad R_{\dot{\delta}\dot{\gamma}\dot{\beta}\dot{\alpha}} = 0 \]  (2.61)

  \[ R_{\delta\dot{\gamma}\beta\alpha} = 0, \quad R_{\delta\dot{\gamma}\dot{\beta}\dot{\alpha}} = 0 \]  (2.62)

- **Dimension 3/2**

  \[ R_{\delta(\gamma\dot{\gamma})\beta\alpha} = 0, \quad R_{\delta(\gamma\dot{\gamma})\dot{\beta}\dot{\alpha}} = +4i\epsilon^{\gamma\dot{\beta}}\dot{W}_{\gamma\dot{\beta}\dot{\alpha}} \]  (2.63)

  \[ R_{\delta(\gamma\dot{\gamma})\dot{\beta}\dot{\alpha}} = 0, \quad R_{\delta(\gamma\dot{\gamma})\beta\alpha} = -4i\epsilon^{\dot{\delta}\dot{\gamma}}W_{\gamma\beta\alpha} \]  (2.64)

- **Dimension 2**

  \[ R_{(\delta\dot{\delta})(\gamma\dot{\gamma})\beta\alpha} = +2\epsilon^{\dot{\delta}\dot{\gamma}}\chi^{\delta\gamma\beta\alpha} - \frac{1}{4}\epsilon^{\dot{\delta}\dot{\gamma}}\sum_{(\delta\dot{\gamma})(\beta\alpha)}\epsilon_{\delta\beta}\nabla^\dot{\beta}W_{\dot{\gamma}\alpha} \]

  \[ = +\epsilon^{\dot{\delta}\dot{\gamma}}\nabla_{\{\dot{\beta}W_{\dot{\alpha}}\}^\dot{\gamma}} \]  (2.65)

  \[ R_{(\delta\dot{\delta})(\gamma\dot{\gamma})\dot{\beta}\dot{\alpha}} = -2\epsilon^{\delta\gamma}\chi_{\delta\dot{\gamma}\beta\dot{\alpha}} + \frac{1}{4}\epsilon^{\dot{\delta}\dot{\gamma}}\sum_{(\dot{\delta}\dot{\gamma})(\dot{\beta}\dot{\alpha})}\epsilon_{\dot{\delta}\dot{\beta}}\nabla^{\dot{\beta}}\dot{W}_{\dot{\gamma}\dot{\alpha}} \]

  \[ = -\epsilon_{\dot{\delta}\dot{\gamma}}\nabla_{\{\dot{\beta}W_{\dot{\alpha}}\}^\dot{\gamma}} \]  (2.66)

The totally symmetric symbol \(\chi\) is itself the spinorial curl of the superfield \(W\):

\[ \chi_{\delta\gamma\beta\alpha} = \frac{1}{4}(\nabla_{\delta}W_{\gamma\beta\alpha} + \nabla_{\gamma}W_{\delta\beta\alpha} + \nabla_{\delta}W_{\gamma\dot{\beta}\dot{\alpha}} + \nabla_{\alpha}W_{\gamma\beta\delta}) \]  (2.67)

\[ \chi_{\delta\dot{\gamma}\beta\dot{\alpha}} = \frac{1}{4}(\nabla_{\delta}W_{\gamma\dot{\beta}\dot{\alpha}} + \nabla_{\dot{\gamma}}W_{\delta\dot{\beta}\dot{\alpha}} + \nabla_{\dot{\beta}}W_{\gamma\dot{\beta}\dot{\alpha}} + \nabla_{\dot{\alpha}}W_{\gamma\dot{\beta}\dot{\dot{\alpha}}}) \]  (2.68)

2.6.3 Scaling and \(U(1)\) curvatures

The conformal field strengths for scalings and chiral rotations are

\[ H = dB + 2E^A F_A(-)^a \]  (2.69)

\[ F = dA + 3iE^A F_A w(A) \]  (2.70)
The special conformal curvatures are

- **Dimension 2**
  \[ H_{3\gamma} = F_{\delta\gamma} = 0 \] (2.71)
  \[ H_{3\dot{\gamma}} = F_{\delta\dot{\gamma}} = 0 \] (2.72)
  \[ H_{3\dot{\dot{\gamma}}} = F_{\delta\dot{\dot{\gamma}}} = 0 \] (2.73)

- **Dimension 3/2**
  \[ H_{\mu(\gamma\dot{\gamma})} = F_{\mu(\gamma\dot{\gamma})} = 0 \] (2.74)
  \[ H_{\dot{\mu}(\gamma\dot{\gamma})} = F_{\dot{\mu}(\gamma\dot{\gamma})} = 0 \] (2.75)

- **Dimension 1**
  \[ H_{\mu(\gamma\dot{\gamma})} = F_{\mu(\gamma\dot{\gamma})} = 0 \]
  \[ H_{\dot{\mu}(\gamma\dot{\gamma})} = F_{\dot{\mu}(\gamma\dot{\gamma})} = 0 \]

\[ H_{\mu(\gamma\dot{\gamma})} \sim H_{(\gamma\dot{\gamma})(\beta\dot{\beta})} = 2\epsilon_{\gamma\dot{\beta}}\tilde{H}_{\gamma\dot{\beta}} - 2\epsilon_{\gamma\dot{\beta}}\tilde{H}_{\gamma\dot{\beta}} \] (2.76)

\[ F_{\mu(\gamma\dot{\gamma})} \sim F_{(\gamma\dot{\gamma})(\beta\dot{\beta})} = 2\epsilon_{\gamma\dot{\beta}}\tilde{F}_{\gamma\dot{\beta}} - 2\epsilon_{\gamma\dot{\beta}}\tilde{F}_{\gamma\dot{\beta}} \] (2.77)

The components \( \tilde{H} \) and \( \tilde{F} \) are themselves related to the spinorial divergence of the superfield \( W \):

\[ \nabla^\gamma W_{\gamma\dot{\alpha}} = \frac{4i}{3} \tilde{F}_{\beta\dot{\alpha}} = +2\tilde{H}_{\beta\dot{\alpha}} \] (2.78)

\[ \nabla^\gamma W_{\gamma\dot{\alpha}} = \frac{4i}{3} \tilde{F}_{\beta\dot{\alpha}} = -2\tilde{H}_{\beta\dot{\alpha}} \] (2.79)

### 2.6.4 Special conformal curvature

The special conformal curvatures are

\[ R(K)^{\alpha} = df^{A} - \lambda(A)f^{A}B + iw(A)f^{A}A + f^{B}\phi_{B}^{A} + \frac{1}{2}C^{ABC}f_{C}E_{B} + \frac{1}{2}f^{B}f^{C}C_{CB}^{A} \]

We will group them by their form indices.

- **Fermion/fermion**
  \[ R(K)_{\gamma\beta\alpha} = 0, \quad R(K)_{\gamma\dot{\beta}\alpha} = 0, \quad R(K)_{\gamma\dot{\alpha}} = 0 \]
  \[ R(K)_{\gamma\beta\dot{\alpha}} = 0, \quad R(K)_{\gamma\dot{\beta}\alpha} = 0, \quad R(K)_{\gamma\dot{\alpha}} = 0 \]
  \[ R(K)_{\gamma\dot{\beta}\dot{\alpha}} = 0, \quad R(K)_{\gamma\dot{\alpha}} = 0, \quad R(K)_{\gamma\dot{\beta}} = 0 \]

- **Fermion/boson**
  \[ R(K)_{\alpha(\beta\dot{\beta})\gamma} = 0, \quad R(K)_{\dot{\alpha}(\beta\dot{\beta})\gamma} = 0 \]
  \[ R(K)_{\alpha(\beta\dot{\beta})\dot{\gamma}} = +i\epsilon_{\alpha\beta}\nabla^{\phi}W_{\phi\beta\dot{\gamma}}, \quad R(K)_{\dot{\alpha}(\beta\dot{\beta})\dot{\gamma}} = +i\epsilon_{\dot{\alpha}\beta}\nabla^{\phi}W_{\phi\beta\dot{\gamma}} \]
  \[ R(K)_{\alpha(\beta\dot{\beta})(\gamma\dot{\gamma})} = -2i\epsilon_{\alpha\beta}\nabla_{\gamma\phi}W^{\phi\beta\dot{\gamma}}, \quad R(K)_{\dot{\alpha}(\beta\dot{\beta})(\gamma\dot{\gamma})} = -2i\epsilon_{\dot{\alpha}\beta}\nabla_{\gamma\phi}W^{\phi\beta\dot{\gamma}} \]

- **Boson/boson**
  \[ R(K)_{\mu\nu} = -\frac{i}{3}\nabla_{\mu}F_{\nu}, \quad R(K)_{\mu\bar{\nu}} = +\frac{i}{3}\nabla_{\mu}F_{\nu} \]
  \[ R(K)_{\gamma\gamma(\beta\dot{\beta})(\alpha\dot{\alpha})} = -\epsilon_{\gamma\beta}\nabla_{\gamma\alpha}W_{\phi\beta\dot{\alpha}} - \epsilon_{\dot{\alpha}\beta}\nabla_{\gamma\alpha}W_{\phi\dot{\alpha}\beta} \]

where the chiral curvature \( F_{cb} \) has been used for notational simplicity.
2.7 Chiral projectors and component actions

One can use the details of Appendix A.2.5 specifically equation (A.86) to construct an explicit form for the chiral projector in conformal superspace:

$$\mathcal{P}[V] = \int d^2\theta \bar{\Sigma} V$$

(2.88)

where $\bar{\Sigma}$ is the superdeterminant constructed out of $E^{\hat{\mu}\hat{a}}$ in the gauge where $E_{m\hat{a}}$ and $E_{\mu\hat{a}}$ vanish. Let us explicitly construct the vierbein (and other connections) in this gauge.

Recall that the variation of the connections $W^{\hat{\mu}A}$ is

$$\delta_G W^{\hat{\mu}A} = \partial^{\hat{\mu}} g^A + W^{\hat{\mu}B} g^{C} f_{CB}^{A}.$$  

(2.89)

The gauge parameter $g^{A}$ is a superfield and so has a larger parameter space than what survives at the component level. In principle, every $\theta$ and $\bar{\theta}$-dependent part of $g^{A}$ can be exhausted to put the connections in a desirable form without affecting the component Lagrangian. We will here use the $\theta$-dependence of $g^{A}$ to fix $W^{\hat{\mu}A}$ to a specific form. (This will correspond to a chiral version of Wess-Zumino gauge. Later on we shall fix the $\theta$-dependence.)

Let $g^{A} = \bar{\theta}_{\mu} g^{\hat{\mu}A} + \frac{1}{2} \bar{\theta}^2 g_2^{A}$ where the functions $g^{\hat{\mu}A}$ and $g_2^{A}$ depend on $x$ and $\theta$ but not $\bar{\theta}$. It is immediately clear by inspection of the gauge transformation law that $g^{\hat{\mu}A}$ can be chosen to fix the gauge $W^{\hat{\mu}A}|_{\theta = 0} = \delta^{\hat{\mu}A}$, meaning the vierbein is gauged to $\delta^{\hat{\mu}A}$ at lowest component and all other gauge fields set to zero. The gauge connection $\theta$-expansion then becomes

$$W^{\hat{\mu}A} = \delta^{\hat{\mu}A} + \bar{\theta}_{\nu} W_{\nu}^{\hat{\mu}A} + \frac{1}{2} \bar{\theta}^2 W_{2}^{\hat{\mu}A}$$

(2.90)

for fields $W_{\nu}^{\hat{\mu}A}$ and $W_{2}^{\hat{\mu}A}$ which depend on only $x$ and $\theta$. The remaining gauge parameter $g_2^{A}$ can be used to eliminate the antisymmetric part of $W_{\nu}^{\hat{\mu}A}$, leaving $W_{\nu}^{\hat{\mu}A} = W_{\bar{\nu}}^{\hat{\mu}A}$. This exhausts our $\bar{\theta}$-dependent gauge freedom. The curvatures then uniquely determine the remaining bits of the connection. By taking the definition of the curvature $R$ and projecting to $\bar{\theta} = 0$, one finds $W_{\bar{\nu}}^{\hat{\mu}A} = \frac{1}{2} R^{\hat{\mu}A|\bar{\nu}}|_{\bar{\theta} = 0}$. The remaining component of $W$ is determined by taking the derivative of the curvature formula and projecting to $\bar{\theta} = 0$. One finds $W_{2}^{\hat{\mu}A} = -\frac{1}{3} \nabla_{\hat{a}} R^{\hat{\mu}A|\hat{a}}|_{\theta = 0} - \frac{1}{6} R_{\hat{\alpha}\hat{\beta}}^A f_{\hat{\alpha}\hat{\beta}}^{A} |_{\theta = 0}$. The gives the formula

$$W^{\hat{\mu}A} = \delta^{\hat{\mu}A} + \frac{1}{2} \bar{\theta}_{\hat{a}} R^{\hat{\mu}\hat{a}A}|_{\bar{\theta} = 0} - \frac{1}{6} \bar{\theta}^2 \nabla_{\hat{a}} R^{\hat{\mu}\hat{a}A}|_{\bar{\theta} = 0} - \frac{1}{12} \bar{\theta}^2 R_{\hat{\alpha}\hat{\beta}} f_{\hat{\alpha}\hat{\beta}}^{A} |_{\theta = 0}.$$  

(2.91)

Within conformal superspace, all of the $\bar{\theta}$-dependent terms vanish, giving

$$E^{\hat{\mu}A} = \delta^{\hat{\mu}A}, \quad h^{\hat{\mu}A} = 0$$

(2.92)

Therefore the chiral projector is simply defined as

$$\mathcal{P}[V] = \int d^2\theta V = -\frac{1}{4} \partial_{\mu} \partial^{\hat{\mu}} V = -\frac{1}{4} \nabla^2 V$$

(2.93)

where the last equality follows due to the simplicity of the connections in this gauge. Since the left and right sides of this equation transform the same way under gauge transformations, their equality in this special gauge implies their equality in any.
Since the result is suspiciously simple, we should check that this approach works for minimal supergravity where the chiral projector is known to be not so simple. There the vierbein should take the general form
\[ E^{\mu A} = \delta^{\mu A} - \frac{1}{12} \bar{\theta}^2 R_{\dot{\alpha}} \Gamma^{\mu}_{\alpha A} |_{\bar{\theta} = 0} \] (2.94)
since the relevant torsion components vanish. The only curvature in Poincaré superspace is the Lorentz curvature, and it is straightforward to evaluate the term appearing here. One finds
\[ E^{\mu A} = \delta^{\mu A} - \delta^{\mu A} \bar{\theta}^2 R \] (2.95)
for the vierbein (as well as a non-vanishing spin connection which we will ignore since it turns out not to matter). The chiral projection formula becomes
\[ \mathcal{P}[V] = \int d^2 \bar{\theta} (1 + 2 \bar{\theta}^2 R) V = 2RV - \frac{1}{4} \partial_{\mu} \partial^{\mu} V = -\frac{1}{4} (\bar{\nabla}^2 - 8R) V \] (2.96)
Here the spin connection is not zero but it contributes nothing when \( \bar{\nabla}^2 \) acts on a field without dotted indices, and so \( \bar{\nabla}^2 \) in this gauge is as simple in Poincaré superspace as it is in conformal superspace.

In either formalism, the conversion from a D to an F-term proceeds straightforwardly. Using (A.85), we find
\[ \int d^4 x d^2 \bar{\theta} E W = \int d^4 x d^2 \bar{\theta} \mathcal{P}[V]. \] (2.97)
where the second integration is understood to occur at \( \bar{\theta} = 0 \). Although the operations above were performed in a specific \( \bar{\theta} \) gauge, the final results have been written in a gauge-invariant manner. In fact, since the gauge-fixing procedure undertaken had no effect on the fields at \( \bar{\theta} = 0 \), the right hand side of the above equation must be independent of our gauge choices.

### 2.7.1 F-term integrations

We have shown that any D-term can be written as an F-term. It is still necessary to evaluate the component Lagrangian corresponding to an F-term. A chiral integral has the form
\[ \int d^4 x d^2 \bar{\theta} \mathcal{E} W, \] (2.98)
an integral over the superspace slice where \( \bar{\theta} = 0 \). \( W \) is a chiral superfield transforming under the gauge group in order to leave the full action invariant.

We can evaluate this integral by the method of gauge-fixing, much like how we derived the D to F integral conversion formula. The first step is to use the \( \theta \)-dependent part of the gauge transformations to fix the connections.\(^9\) In a way entirely analogous to what we did in the previous section, we may choose\(^10\)
\[ W_\mu^A = \delta_\mu^A + \frac{1}{2} \theta^\alpha R_{\alpha \mu}^A |_{\theta = 0} - \frac{1}{6} \theta^2 \bar{\nabla}^\alpha R_{\alpha \mu}^A |_{\theta = 0} + \frac{1}{12} \theta^2 R_{\mu}^{\alpha b} f_\alpha^{\beta \gamma} A^\gamma |_{\theta = 0}. \] (2.99)
\(^9\) It is useful to note that whether or not we gauge-fixed the \( \bar{\theta} \)-dependent part of the connections is irrelevant for evaluating an F-term as its integral occurs at \( \bar{\theta} = 0 \).
by exhausting the remaining \( \theta \)-dependence of \( \sigma^A \). Here the projection to \( \tilde{\theta} = 0 \) has also already been done, so we will avoid indicating it explicitly.

In conformal superspace, this expression is extremely simple. It gives

\[
E^A_\mu = \delta^A_\mu, \quad h_{\mu}^\alpha = 0
\]  
(2.100)

The \( F \)-term integration then becomes

\[
\mathcal{L}_F = \int d^4x \, d^2\theta \, e \, W = -\frac{1}{4} e \partial^\mu e \partial_\mu W - \frac{1}{2} \partial^\mu e \partial_\mu W - \frac{1}{4} (\partial^\mu \partial_\mu e)W
\]  
(2.101)

The first term is rather simple. In our gauge choice, it is easy to see that \( \nabla^\alpha \nabla_\alpha W = \partial^\alpha \partial_\alpha W \) when \( \theta = \tilde{\theta} = 0 \). The other terms are usually constructed in the literature from supersymmetric completion of this term; here we will evaluate them directly in this gauge. For example,

\[
\partial_\mu e = e(\partial_\mu E_\sigma a) e_a^m = e(\partial_\mu E_\sigma a + T_{\mu \sigma}^a) e_a^m = 0 + eT_{\mu \beta}^a E_{m} \beta e_a^m = ie(\sigma^a \bar{\psi}_a)_\mu
\]  
(2.102)

where we have used \( E^a_\mu | = 0 \) as well as the torsion constraint \( T_{\gamma \beta}^a = T_{\gamma b}^a = 0 \). This allows us to evaluate the second term of \( \mathcal{L}_F \); we find \( ie(\bar{\psi}_a \bar{\sigma}^a \psi_b) \nabla_\alpha W/2 \) (since \( \partial_\alpha W = \nabla_\alpha W \) at \( \theta = \tilde{\theta} = 0 \) in this gauge.)

The remaining third term is slightly more complicated. One begins with

\[
\partial^\mu \partial_\mu e = \partial^\mu (eT_{\mu \alpha}^a E_{m} \alpha e_a^m)
\]  
(2.103)

which gives the chiral Lagrangian

\[
\mathcal{L}_F = \int d^2\theta \, e \, W = e \left( -\frac{1}{4} \nabla^\alpha \nabla_\alpha W + \frac{1}{2} \bar{\psi}_a \sigma^a \bar{\psi}_b \right) W
\]  
(2.105)

where the projection to \( \theta = \tilde{\theta} = 0 \) is implicit.

Again, we may repeat this process for Poincaré superspace. One finds

\[
E^A_\mu = \delta^A_\mu - \delta^A_\mu \theta^2 \bar{R}
\]  
(2.106)

and for the \( F \)-term

\[
\mathcal{L}_F = \int d^2\theta \, e \left( 1 + 2\theta^2 \bar{R} \right) W = -\frac{1}{4} e \partial^\mu e \partial_\mu W - \frac{1}{2} \partial^\mu e \partial_\mu W - \frac{1}{4} (\partial^\mu \partial_\mu e)W + 2\bar{R}W
\]  
(2.107)

The first and second terms are evaluated as before. The third gains an extra contribution of \(-16e\bar{R}\) from (2.103) when the spinorial derivative hits the gravitino. This gives the chiral Lagrangian

\[
\mathcal{L}_F = \int d^2\theta \, e \, W = e \left( -\frac{1}{4} \nabla^\alpha \nabla_\alpha W + \frac{1}{2} \bar{\psi}_a \sigma^a \bar{\psi}_b \right) W + 6\bar{R}W
\]  
(2.108)

where the projection to \( \theta = \tilde{\theta} = 0 \) is implicit.

\[\text{This last gauge-fixing has an interesting effect on } \theta. \text{ Their Einstein index is now effectively a Lorentz index, since every Lorentz rotation which would alter the vierbein must be countered by a } P \text{-gauge (or coordinate) transformation. The } \theta \text{'s are therefore the same as the } \Theta \text{ variables of } [10]. \text{ Their } F\text{-terms are written } \int d^2\Theta \mathcal{E} \text{ where } \Theta \text{ is equivalent to } \theta \text{ and their } \mathcal{E} \text{ is half of ours when we go to this gauge.}\]
2.7.2 D-term integrations

Within conformal superspace, the F-term component Lagrangian is

\[ \mathcal{L}_F = \int d^2 \theta \mathcal{E} W = e \left( F + \frac{i\sqrt{2}}{2} (\bar{\psi}_a \sigma^a \rho) - (\bar{\psi}_a \sigma^{ab} \psi_b) W \right) \]  

where

\[ F \equiv -\frac{1}{4} \nabla^2 W \quad \text{and} \quad \rho_\alpha \equiv \frac{1}{\sqrt{2}} \nabla_\alpha W \]  

A D-term can be divided into two terms, one evaluated via a chiral integration and the other via an antichiral integration in order to give a manifestly Hermitian action:

\[ \int d^4 \theta EV = \frac{1}{2} \int d^2 \theta \mathcal{E} U + \frac{1}{2} \int d^2 \bar{\theta} \bar{\mathcal{E}} \bar{U} \]  

where \( U \equiv -\frac{1}{4} \nabla^2 V \) and \( \bar{U} \equiv -\frac{1}{4} \nabla^2 \bar{V} \) are the chiral and antichiral projections of \( V \). These two F-terms can then be evaluated using the F-term formula giving the general D-term formula

\[ \mathcal{L}_D = \int d^4 \theta EV = e \left( \frac{1}{2} (F + \bar{F}) + \frac{i\sqrt{2}}{4} (\bar{\psi}_a \sigma^a \rho + \psi_a \sigma^a \bar{\rho}) - \frac{1}{2} (\bar{\psi}_a \sigma^{ab} \psi_b) U - \frac{1}{2} (\psi_a \sigma_{ab} \psi_b) \bar{U} \right) \]  

where

\[ U \equiv -\frac{1}{4} \nabla^2 V, \quad F \equiv \frac{1}{16} \nabla^2 \nabla^2 V, \quad \text{and} \quad \rho_\alpha \equiv -\frac{1}{4\sqrt{2}} \nabla_\alpha \nabla^2 V \]  

The fields \( F \) are actually not quite independent fields. In terms of the D-term of \( V \), they are

\[ F = D + \frac{1}{2} \nabla_c \nabla^c V + \frac{i}{2} \nabla_c V^c \]  

\[ \bar{F} = D + \frac{1}{2} \nabla_c \nabla^c V - \frac{i}{2} \nabla_c V^c \]  

where\(^{11}\)

\[ D \equiv \frac{1}{16} \nabla^\alpha \nabla^2 \nabla_\alpha V = \frac{1}{16} \nabla_\alpha \nabla^2 \nabla^\alpha V \]  

\[ V_c \equiv -\frac{1}{2} \sigma^{\alpha \dot{\alpha}} \nabla_\alpha \nabla^\dot{\alpha} V \]  

The imaginary part of the fields \( F \) and \( \bar{F} \) is the divergence of the vector component of \( V \). When evaluating a D-term integral, it is occasionally useful to use the fields \( D \) rather than \( F \).

\(^{11}\)As in normal superspace, one must be careful to note that \( \nabla_c \) is covariant even with respect to supersymmetry. That is,

\[ \nabla_c = e_c^m \left( \nabla_m - \frac{1}{2} \psi_m \sigma_{a} \nabla_a \right) = e_c^m \left( \partial_m - \frac{1}{2} \psi_m \sigma_{a} \nabla_a - h_m \sigma \right) \]  

where \( \sigma \) denotes both spinor indices. In fact, were we to treat supersymmetry as a gauge theory in normal space with internal symmetry operators \( \hat{Q} \) which happened to include translations in their algebra, we would denote \( \frac{1}{2} \psi_m \sigma_{a} \) as the gauge field associated with the generator \( \hat{Q}_a \). Then the above formula is simply the covariant derivative. There is a further mild complication in conformal superspace: \( \nabla_c \) will include the gauge action of \( \hat{S}_{\alpha \dot{\alpha}} \); therefore, a superconformal covariant derivative includes not only terms higher in the multiplet (due to \( \hat{Q} \)), but also terms lower in the multiplet (due to \( \hat{S} \)).
2.8 Kähler structure of conformal superspace of chiral superfields

It turns out that the conformal superspace of an arbitrary set of scalar chiral superfields possesses a simple Kähler structure due to its relation to the Kähler manifold $\mathbb{C}P^n$.

Suppose we are furnished with a set of chiral primary superfields $\Phi_I$ where $I = 0, 1, \ldots, n$. Our action consists in general of a D-term and an F-term which respectively take the forms

$$L_D = -3 \int d^4 \theta E \bar{Z}(\Phi_I, \bar{\Phi}_I), \quad L_F = \int d^2 \theta E P(\Phi_I)$$

where $Z$ is some real non-negative function of the fields with $\Delta(Z) = 2$ and $P$ is some chiral function with $\Delta(P) = 3$ and $w(P) = 2$. (The assumption of non-negativity of $Z$ is ultimately for stability of the underlying Einstein-Hilbert term. The factor of 3 is for convenience.) We can take the $\Phi_I$ as parametrizing some complex manifold. In order for $Z$ to have a nonzero scaling weight, at least one of the $\Phi_I$ must have $\Delta_I \neq 0$. We will assume without loss of generality that this is $\Phi_0$ (by renaming the fields if necessary) and that $\Delta_0 = 1$ (by redefining $\Phi_0 \rightarrow (\Phi_0)^{1/\Delta_0}$ if necessary).

It is then possible to trade the fields $\Phi_j$ with $j \geq 1$ for projective fields $\xi_j$ which have zero weight. (The simplest way of doing this is by defining $\xi_j \equiv \Phi_j / \Phi_0^{\Delta_j}$.) Since the fields $\xi_j$ have vanishing scaling weight, the fields $Z$ and $P$ in this parametrization are restricted in their form to

$$Z = \Phi_0 \Phi_0 \exp (-K/3), \quad P = \Phi_0^3 W$$

where $K = K(\xi_j, \bar{\xi}_j)$ is some real function of the projective fields and $W = W(\xi_j)$ is some chiral function. (The choice of this definition for real $K$ is possible only if $Z$ is assumed to be non-negative.) It is obvious that both $Z$ and $P$, viewed as functions of the complex manifold spanned by the $\Phi_I$, are independent of the projective representation chosen. A different representation is induced on the projective coordinates by the mapping

$$\Phi_0 \rightarrow \Phi_0 \exp(F/3), \quad K \rightarrow K + F + \bar{F}, \quad W \rightarrow e^{-F} W$$

where $F = F(\xi_j)$ is a holomorphic function of the projective parameters. (For example, trading $\Phi_0$ for $\Phi_1$ as the field to project with is accomplished by choosing $F = 3 \log(\Phi_1/\Phi_0) = 3 \log(\xi_1).$) The above transformation law is simply a Kähler transformation, and the manifold under discussion is the complex projective space $\mathbb{C}P^n$, a simple example of a Kähler manifold.

The two actions then take the form

$$\mathcal{L}_D = -3 \int d^4 \theta E \bar{\Phi}_0 e^{-K/3} \Phi_0, \quad \mathcal{L}_F = \int d^2 \theta E \Phi_0^3 W$$

where $W$ is chiral and $K$ is real. The factor of $e^{-K/3}$ is reminiscent of $e^V$ for a theory with an internal $U(1)_K$ symmetry; this $U(1)_K$ is gauged not by an independent gauge multiplet but by the other chiral fields. We may make the $U(1)_K$ more manifest in the following manner. Define a new complex superfield $\Psi_0$ by

$$\Psi_0 = e^{-K/6} \Phi_0, \quad \bar{\Psi}_0 = e^{-K/6} \bar{\Phi}_0$$

\[\text{12}\] Since $\Phi_0$ has scaling weight 1 and chiral weight $2/3$ (their ratio is fixed at 3/2 for any primary chiral superfield) $P$ has the correct scaling and chiral weights for an F-term.
under which the actions become

\[-3 \int d^4 \theta \, E \, \bar{\Psi} \Psi_0, \quad \int d^2 \theta \, \bar{E} \, \Psi_0^3 e^{K/2} W\]

The new field \(\Psi_0\) and effective superpotential \(e^{K/2} W\) are the only objects (besides \(K\)) which transform under Kähler transformations:

\[\Psi_0 \rightarrow \exp \left( + \frac{i}{3} \text{Im} F \right) \Psi_0, \quad \Psi_0 \rightarrow \exp \left( - \frac{i}{3} \text{Im} F \right) \Psi_0 \quad (2.123)\]

\[e^{K/2} W \rightarrow \exp (-i \text{Im} F) \, e^{K/2} W, \quad e^{K/2} \bar{W} \rightarrow \exp (+i \text{Im} F) \, e^{K/2} \bar{W} \quad (2.124)\]

We normalize the generator of Kähler transformations, \(k\), by requiring the above Kähler transformation to correspond to \(\exp \left( -i \text{Im} F k / 2 \right)\). In this way the Kähler weights of \(\Psi_0\) and \(e^{K/2} W\) are set to be \(-2/3\) and \(2\), respectively:

\[k \Psi_0 = -i \frac{2}{3} \Psi_0, \quad k \, e^{K/2} W = +2 i e^{K/2} W\]

(Note that \(e^{K/2} W\) is chiral from the point of view of the Kähler covariant derivative, which carries a Kähler connection.) This normalization is purely a matter of convention; it is chosen so that \(e^{K/2} W\) possesses the same Kähler and \(U(1)\) weights.

The Kähler covariant derivative then takes the form

\[\nabla^K \equiv \nabla - \hat{A} k\]

where \(k\) is the generator of the Kähler transformations. The Kähler connection \(\hat{A}\) is defined in terms of the Kähler potential \(K\):

\[\hat{A}_\alpha = + \frac{i}{4} \nabla_\alpha K, \quad \hat{A}_{\bar{\alpha}} = - \frac{i}{4} \nabla_{\bar{\alpha}} K\]

\[\hat{A}_{\alpha \bar{\alpha}} = \frac{i}{2} \left( \nabla_\alpha \hat{A}_{\bar{\alpha}} + \nabla_{\bar{\alpha}} \hat{A}_\alpha \right) = \frac{1}{8} [\nabla_\alpha, \nabla_{\bar{\alpha}}] K \quad (2.126)\]

(In these formulae, the function \(K\) is a primary scalar superfield and is therefore invariant under all the generators of the superconformal algebra.) The definition of \(\hat{A}_{\alpha \bar{\alpha}}\) is conventional; it is chosen so that \(\{\nabla^K_\alpha, \nabla^K_{\bar{\alpha}}\} = -2i \nabla^K_{\alpha \bar{\alpha}}\).
3 Degauging to Poincaré

Poincaré superspace lacks the explicit scaling and conformal symmetries enjoyed by conformal superspace. It may also, depending on the flavor of supergravity chosen, lack the $U(1)$ R-symmetry. Converting conformal supergravity to one of the flavors of Poincaré supergravity must then involve some measure of gauge-fixing. We will demonstrate how this is accomplished by first reducing the conformal multiplet to a theory with an explicit $U(1)$ symmetry and a nonlinearly realized conformal symmetry. To guide our path, we first review in broad strokes how it works without supersymmetry; the details of this can be found in [4].

3.1 Review: Conformal gravity and the Einstein-Hilbert Lagrangian

Conformal gravity consists of the following gauge connections:

$$W_m = e_m^a P_a + \frac{1}{2} \omega_m^{ba} M_{ab} + b_m D + f_m^a K_a$$

We will denote by $\tilde{R}$ the curvatures of the conformal theory and by $R$ the Poincaré curvatures. One usually takes the constraint of vanishing conformal torsion (which is equivalent to vanishing Poincaré torsion) to determine the spin connection $\omega_m^{ba}$ in terms of the vierbein and the scaling gauge field $b_m$. One also would like to express the special conformal gauge field $f_m^a$ in terms of other fields; this can be done by taking the conformal Ricci tensor to vanish, $\tilde{R}_{mn} e^n = 0$. Having done so, one finds

$$f_m^a = -\frac{1}{4} \left( R_m^a - \frac{1}{6} e_m^a R \right)$$

where $R_m^a = R_{mn} e^n$ is the Poincaré Ricci tensor and $R = R_m^a e^a$ the Poincaré Ricci scalar. One further, for simplicity, usually adopts the $K$-gauge choice $b_m = 0$. (This is possible since $\delta_K(\epsilon) b_m = -2 e_m^a \epsilon^a$ allows one to gauge $b_m$ away.)

Having made these constraints and gauge choices, one then examines the simplest conformal action for a scalar field $\phi$ with $\Delta = 1$:

$$e^{-1} \mathcal{L} = \frac{1}{2} \phi \nabla^a \nabla_a \phi = -\frac{1}{2} \nabla^a \phi \nabla_a \phi - \frac{1}{2} T_{ba}^a \phi \nabla^b \phi - f_a^a \phi^2$$

(We have integrated the covariant d'Alembertian by parts.) The torsion term vanishes by assumption. The term involving $\nabla_a \phi$ also vanishes if we fix the remaining $D$-gauge by gauging $\phi$ to the constant $\phi_0$:

$$\nabla_a \phi_0 = e_a^m \partial_m \phi_0 = 0$$

(There is no scaling connection in the above expression since $b_m = 0$ has been chosen as our $K$-gauge.) This leaves for the Lagrangian

$$e^{-1} \mathcal{L} = \frac{1}{2} \phi_0 \nabla^a \nabla_a \phi_0 = -f_a^a \phi_0^2 = +\frac{1}{12} \mathcal{R} \phi_0^2$$

This is almost the Einstein-Hilbert term $-\mathcal{R}/2$ (in units where the reduced Planck mass is unity). We need only start with the wrong sign for the kinetic term and then choose $\phi_0^2 = 6$.

If we had started with a complex gauge field $\phi$, the Lagrangian would have been

$$e^{-1} \mathcal{L} = \phi^* \nabla^a \nabla_a \phi = -\nabla^a \phi^* \nabla_a \phi - 2 f_a^a |\phi|^2$$
We may gauge $|\phi| = \phi_0$ but not the phase of $\phi$, which we shall denote $\omega$. This gives

$$e^{-1} \mathcal{L} = \phi^a \nabla^a \nabla_a \phi = -\phi_0^2 \partial^m \omega \partial_m \omega + \frac{1}{6} R \phi_0^2$$  \hspace{1cm} (3.6)$$

Gauging $\phi_0^2 = 3$ and choosing to flip the sign of the Lagrangian gives back the Einstein-Hilbert term; unfortunately this also leaves an unstable kinetic term for $\omega$. A model with an additional gauged $U(1)$ symmetry would be able to dispense with this phase. The superconformal algebra has such a symmetry, and we will find it is the supersymmetric version of this model with a complex $\phi$ which reproduces the minimal version of Poincaré supergravity.

### 3.2 $U(1)$ superspace

In conformal gravity, the scaling gauge field $b_m$ was the only field that transformed under the special conformal symmetry; moreover, this symmetry was precisely enough to allow the choice $b_m = 0$. The latter property is also enjoyed in the superconformal case, even though not every other field is $K$-inert. It is here that we begin our gauge fixing procedure.

Recall that under the action of $K_A$ with parameter $\epsilon^A$, $\delta K B_M = -2\epsilon^A E_{M} A(-)^a$. If we choose $\epsilon^A = \eta^M E_{M} A(-)^a$, then we find $\delta K B_M = -2\eta_M$ and we can freely choose the gauge $B = 0$. The generator $D$ then drops out of the covariant derivative. We also have chosen a gauge for $K_A$ and so we ought not to consider $K_A$ a symmetry any longer. We denote this by the breakdown of the conformally covariant derivative $\nabla$ to the Poincaré derivative $\mathcal{D}$.

Since $K_A$ is no longer considered a symmetry, the fields $f^A_M$ are now auxiliary. In order to analyze the various properties of these objects, we must make use of the conformal curvatures. Most of these (torsion, Lorentz, and $U$ curvatures) are now auxiliary. In subsequent formulae, we will use the combination $f^A_B = E^A_B f^B_M$, which possesses scaling and $U(1)$ weights of $\lambda(A) + \lambda(B)$ and $-(w(A) + w(B))$, respectively.
3.2.1 Constraint analysis

We shall start with the scaling curvature:

\[ \tilde{H}_{BA} = (dB)_{BA} + 2f_{BA}(-)^a - 2f_{AB}(-)^b \]

Since \( B \) has been gauged away, the constraints on the \( H_{BA} \) give constraints on the fields \( f_{M}^A \). These are:

\[ \tilde{H}_{\beta\alpha} = 0 = \Rightarrow f_{\beta\alpha} = -f_{\alpha\beta} = -\epsilon_{\beta\alpha} \bar{R} \quad (3.12) \]
\[ \tilde{H}_{\dot{\beta}\dot{\alpha}} = 0 = \Rightarrow f_{\dot{\beta}\dot{\alpha}} = -f_{\dot{\alpha}\dot{\beta}} = +\epsilon_{\dot{\beta}\dot{\alpha}} R \quad (3.13) \]
\[ \tilde{H}_{\beta\dot{\alpha}} = 0 = \Rightarrow f_{\beta\dot{\alpha}} = -f_{\dot{\alpha}\beta} = -\frac{1}{2} G_{\beta\dot{\alpha}} \quad (3.14) \]
\[ \tilde{H}_{\dot{\beta}a} = 0 = \Rightarrow f_{\dot{\beta}a} = -f_{\beta\dot{a}} \quad (3.15) \]

The above serve as definitions of the fields \( R \) and \( G_c \). The complex conjugation properties of the above identities tell us \( \bar{R} = R^\dagger \) and \( G_c = (G_c)^\dagger \). The scaling weights of these objects are \( \Delta(R) = \Delta(\bar{R}) = 2 \) and \( \Delta(G_c) = 2 \); the \( U(1) \) weights are \( w(R) = -w(\bar{R}) = 2 \) and \( w(G_c) = 0 \).

The next set of constraints to analyze are those of the \( U(1) \) curvature. Recall

\[ \tilde{F}_{BA} = F_{BA} + 3if_{BA}w(A) - 3if_{AB}w(B) \]

which leads to

\[ \tilde{F}_{\beta\alpha} = 0 = \Rightarrow F_{\beta\alpha} = 0 \quad (3.16) \]
\[ \tilde{F}_{\dot{\beta}\dot{\alpha}} = 0 = \Rightarrow F_{\dot{\beta}\dot{\alpha}} = 0 \quad (3.17) \]
\[ \tilde{F}_{\beta\dot{\alpha}} = 0 = \Rightarrow F_{\beta\dot{\alpha}} = 6if_{\beta\dot{\alpha}} = -3iG_{\beta\dot{\alpha}} \quad (3.18) \]
\[ \tilde{F}_{\dot{\beta}a} = 0 = \Rightarrow F_{\dot{\beta}a} = -3if_{\dot{\beta}a} \quad (3.19) \]
\[ \tilde{F}_{\beta a} = 0 = \Rightarrow F_{\beta a} = +3if_{\beta a} \quad (3.20) \]

Now consider the torsion. Noting that

\[ \tilde{T}_{CB}^A = T_{CB}^A + \frac{1}{2} F_{CD}^D C_{DB}^A \]

one can see the only torsions which differ between the conformal and Poincaré cases are those with \( A \) fermionic and either \( C \) or \( B \) (or both) bosonic. Thus the constraints on the conformal torsions pass unchanged for the fermion/fermion form indices:

\[ \tilde{T}_{\gamma\beta}^A = 0 \quad \Rightarrow \quad T_{\gamma\beta}^A = 0 \quad (3.22) \]
\[ \tilde{T}_{\gamma\dot{\beta}}^A = 0 \quad \Rightarrow \quad T_{\gamma\dot{\beta}}^A = 0 \quad (3.23) \]
\[ \tilde{T}_{\gamma\dot{\alpha}} = 0 \quad \Rightarrow \quad T_{\gamma\dot{\alpha}} = 0 \quad (3.24) \]
\[ \tilde{T}_{\gamma\beta}^a = 2i\sigma^a_{\gamma\beta} \quad \Rightarrow \quad T_{\gamma\beta}^a = 2i\sigma^a_{\gamma\beta} \quad (3.25) \]

For the fermion/boson form indices, it is only slightly more complicated:

\[ \tilde{T}_{\gamma\beta}^a = 0 \quad \Rightarrow \quad T_{\gamma(\beta)a} = +i\epsilon_{\beta\alpha} G_{\gamma\dot{\beta}} \quad (3.26) \]
\[ \tilde{T}_{\gamma\dot{\beta}}^a = 0 \quad \Rightarrow \quad T_{\gamma(\dot{\beta})a} = -2i\epsilon_{\gamma\dot{\beta}} \epsilon_{\beta\alpha} R \quad (3.27) \]
\[ \tilde{T}_{\gamma a} = 0 \quad \Rightarrow \quad T_{\gamma a} = 0 \quad (3.28) \]
\[ \tilde{T}_{\dot{\gamma} a} = 0 \quad \Rightarrow \quad T_{\dot{\gamma} a} = 0 \quad (3.29) \]
The only other torsion constraint was $\ddot{T}_{cb}^a = 0$, which gives the same constraint on the Poincaré torsion

$$\ddot{T}_{cb}^a = 0 \implies T_{cb}^a = 0.$$  \hfill (3.30)

The Lorentz curvature is quite simple to analyze:

$$\ddot{\bar{R}}_{\delta\gamma\beta\alpha} = 0 \implies \bar{R}_{\delta\gamma\beta\alpha} = 4(\epsilon_{\delta\beta}\epsilon_{\gamma\alpha} + \epsilon_{\delta\alpha}\epsilon_{\gamma\beta})\bar{R}$$  \hfill (3.31)

$$\ddot{\bar{R}}_{\delta\gamma\beta\alpha} = 0 \implies \bar{R}_{\delta\gamma\beta\alpha} = 0$$  \hfill (3.32)

$$\ddot{\bar{R}}_{\delta\gamma\beta\alpha} = 0 \implies \bar{R}_{\delta\gamma\beta\alpha} = 0$$  \hfill (3.33)

$$\ddot{\bar{R}}_{\delta\gamma\beta\alpha} = 0 \implies \bar{R}_{\delta\gamma\beta\alpha} = 4(\epsilon_{\delta\beta}\epsilon_{\gamma\alpha} + \epsilon_{\delta\alpha}\epsilon_{\gamma\beta})\bar{R}$$  \hfill (3.34)

$$\ddot{\bar{R}}_{\delta\gamma\beta\alpha} = 0 \implies \bar{R}_{\delta\gamma\beta\alpha} = -\epsilon_{\delta\beta}\beta_{\alpha\gamma} - \epsilon_{\delta\alpha}\beta_{\gamma\beta}$$  \hfill (3.35)

$$\ddot{\bar{R}}_{\delta\gamma\beta\alpha} = 0 \implies \bar{R}_{\delta\gamma\beta\alpha} = -\epsilon_{\gamma\beta}\beta_{\delta\alpha} - \epsilon_{\gamma\alpha}\beta_{\delta\beta}$$  \hfill (3.36)

The remaining curvatures are:

$$\ddot{R}(K)_{\gamma\beta\alpha} = 0 \implies D_\alpha\dddot{R} = 0$$  \hfill (3.37)

$$\ddot{R}(K)_{\gamma\beta\alpha} = 0 \implies f_{(\gamma\beta\alpha)} + f_{(\gamma\beta\alpha)} = -\frac{i}{2}D_{(\gamma\beta)}\dddot{R}$$  \hfill (3.38)

$$\ddot{R}(K)_{\gamma\beta\alpha} = 0 \implies D_{(\gamma\beta)}(a\alpha) = +2iG_{(\gamma\beta)\alpha}\dddot{R}$$  \hfill (3.39)

$$\ddot{R}(K)_{\gamma\beta\alpha} = 0 \implies f_{(\gamma\beta\alpha)} - 2f_{(\gamma\beta\alpha)} = \frac{i}{2}D_{(\gamma\beta)}G_{(\alpha\beta)} - i\epsilon_{\gamma\beta}D_{(\gamma\beta)}\dddot{R}$$  \hfill (3.40)

$$\ddot{R}(K)_{\gamma\beta\alpha} = 0 \implies f_{(\gamma\beta\alpha)} = \frac{i}{2}D_{(\gamma\beta)}(a\alpha) + 2\epsilon_{\beta\alpha}\dddot{R} + \frac{1}{2}G_{(\alpha\beta)}G_{(\beta\alpha)}$$  \hfill (3.41)

(We have used the spinor notation $f_{(\gamma\beta)} = f_{(\gamma\beta\alpha)}G_{(\alpha)}$, as well as $f_{(\gamma\beta)} = f_{(\gamma\beta\alpha)}G_{(\alpha\beta)}$. The first condition indicates that $\dddot{R}$ is an antichiral superfield; its complex conjugate tells that $\bar{R}$ is chiral. The second and fourth equations can be combined to yield

$$3if_{(\gamma\beta\alpha)} + \frac{1}{2}D_{(\gamma\beta)}G_{(\alpha\beta)} + D_{(\gamma\beta)}G_{(\alpha\beta)} + \epsilon_{(\gamma\beta)}D_{(\gamma\beta)}\dddot{R}$$  \hfill (3.42)

as well as its conjugate

$$3if_{(\gamma\beta\alpha)} = -\frac{1}{2}D_{(\gamma\beta)}G_{(\alpha\beta)} - D_{(\gamma\beta)}G_{(\alpha\beta)} - \epsilon_{(\gamma\beta)}D_{(\gamma\beta)}R.$$  \hfill (3.43)

This result can be substituted into the third equation, yielding

$$D^2G_c = 4iD_c\dddot{R}, \quad D^2G_c = -4iD_cR$$  \hfill (3.44)

The result given for $f_{\beta\alpha}$ allows the determination of $F_{\beta\alpha}$:

$$F_{\beta\alpha} = -3if_{(\gamma\beta\alpha)} = -\frac{3}{2}D_{(\gamma\beta)}G_{(\alpha\beta)} - \epsilon_{\beta\alpha}\dddot{X}_{\alpha}$$  \hfill (3.45)

$$F_{\beta\alpha} = +3if_{(\gamma\beta\alpha)} = -\frac{3}{2}D_{(\gamma\beta)}G_{(\alpha\beta)} - \epsilon_{\beta\alpha}\dddot{X}_{\alpha}$$  \hfill (3.46)

where

$$X_{\beta} \equiv D_{\beta}R - D^\beta G_{\beta\beta}, \quad \dddot{X}_{\beta} \equiv D_{\beta}\dddot{R} - D^\beta G_{\beta\beta}$$  \hfill (3.47)
just as in $U(1)$ superspace. Furthermore, (3.44) implies (after some algebra) the chirality of $X_\alpha$:

$$D_\alpha X_\alpha = 0, \quad D_\bar{\alpha} X_\bar{\alpha} = 0 \quad (3.48)$$

Finally the fourth $R(K)$ constraint gives

$$f(\beta\dot{\beta})(\alpha\dot{\alpha}) = \frac{i}{2} D_\beta f(\beta\dot{\beta})(\alpha\dot{\alpha}) + 2\epsilon_{\beta\bar{\alpha}}\epsilon_{\dot{\beta}\dot{\alpha}} R\bar{R} + \frac{1}{2} G_{\alpha\beta} G_{\beta\dot{\alpha}}$$

$$- \frac{1}{12} [D_\beta, D_\dot{\beta}] G_{\alpha\dot{\alpha}} - \frac{1}{6} D_\beta D_\alpha G_{\alpha\beta} + \frac{1}{6} D_\dot{\beta} D_{\bar{\alpha}} G_{\bar{\beta}\bar{\alpha}}$$

$$- \frac{1}{12} \epsilon_{\beta\bar{\alpha}} \epsilon_{\dot{\beta}\dot{\alpha}} (D^2 R + \bar{D}^2 \bar{R}) + 2\epsilon_{\beta\bar{\alpha}} \epsilon_{\dot{\beta}\dot{\alpha}} R\bar{R} + \frac{1}{2} G_{\alpha\beta} G_{\beta\dot{\alpha}} \quad (3.49)$$

The special conformal gauge field $f_B A$ is now entirely specified in terms of superfields $R$ and $G_c$.

It is worth pausing a moment to take stock of our position. We have now checked that every constraint taken in conformal superspace reproduces (in the $B = 0$ gauge) a known result in $U(1)$ superspace; in particular, we have reproduced among our relations the constraint structure of $U(1)$ superspace. Since the $U(1)$ constraints uniquely specify $U(1)$ superspace, the gauge $B = 0$ of our constrained conformal superspace must correspond to the standard $U(1)$ superspace. All further checks are merely tests of consistency.

3.2.2 Some consistency checks

- Torsion
  The only torsion components we have not yet discussed are those which we did not constrain: $T_{\beta\bar{\alpha}}$. These also differ between conformal and Poincaré theories. Using

$$\tilde{T}_{cb}^\alpha = T_{cb}^\alpha + if_{[\beta\dot{\beta}]\bar{\alpha}}$$

one finds

$$T_{(\gamma\gamma)(\beta\dot{\beta})\alpha} = +2\epsilon_{\gamma\bar{\beta}} W_{\gamma\beta\alpha} + \epsilon_{\alpha\beta} \left( D_{\beta} G_{\gamma\dot{\gamma}} + \frac{2}{3} \epsilon_{\beta\dot{\gamma}} X_{\gamma} \right) - \epsilon_{\alpha\gamma} \left( D_{\gamma} G_{\beta\dot{\beta}} + \frac{2}{3} \epsilon_{\gamma\dot{\beta}} X_{\beta} \right) \quad (3.50)$$

$$T_{(\gamma\gamma)(\beta\dot{\beta})\bar{\alpha}} = -2\epsilon_{\gamma\beta} W_{\gamma\beta\bar{\alpha}} + \epsilon_{\bar{\alpha}\dot{\beta}} \left( D_{\dot{\beta}} G_{\gamma\dot{\gamma}} + \frac{2}{3} \epsilon_{\dot{\beta}\dot{\gamma}} X_{\gamma} \right) - \epsilon_{\bar{\alpha}\gamma} \left( D_{\gamma} G_{\beta\dot{\beta}} + \frac{2}{3} \epsilon_{\gamma\dot{\beta}} \bar{X}_{\dot{\beta}} \right) \quad (3.51)$$

This is equivalent to the corresponding formulae in equations (B-2.12) through (B-2.18) of [S]; therefore, the torsion of $U(1)$ supergravity is equivalent to the $B = 0$ gauge of conformal superspace.

- Lorentz curvatures
  The Lorentz curvatures in their canonically decomposed form are

$$\tilde{R}_{DC}^{\beta\alpha} = R_{DC}^{\beta\alpha} + 2\delta_{[D} f_{C]}^{\beta\alpha} (\epsilon_{\sigma\beta\alpha})^{\beta\alpha} + 2\delta_{[D} f_{C]}^{\beta\alpha} (-)^C \quad (3.52)$$

The case of purely fermionic form indices has already been dealt with. Turn next to the fermion/boson case:

$$\tilde{R}_{\delta(\gamma\gamma)\beta\alpha} = R_{\delta(\gamma\gamma)\beta\alpha} + \sum_{\beta\alpha} (-\epsilon_{\gamma\alpha} f_{\delta(\beta\dot{\gamma})} + 2\epsilon_{\delta\beta} f_{\alpha(\gamma\dot{\gamma})}) \quad (3.53)$$
Noting that $\ddot{R}_{\delta(\gamma)}\beta\alpha = 0$ and inserting the explicit expression for $f_{\beta(\alpha\dot{\alpha})}$, one finds

$$R_{\delta(\gamma)}\beta\alpha = +i \sum_{\beta\alpha} \left( \frac{1}{2} \epsilon_{\delta\gamma} D_{\beta} G_{\alpha\gamma} + \frac{1}{2} \epsilon_{\delta\beta} D_{\gamma} G_{\alpha\dot{\gamma}} - \epsilon_{\delta\beta} \epsilon_{\gamma\alpha} \bar{D}_{\dot{\gamma}} \bar{R} \right)$$

(3.54)

as in $U(1)$ superspace [8]. The other Lorentz curvature term we need to calculate is

$$R_{\delta(\gamma)}\beta\alpha = \ddot{R}_{\delta(\gamma)}\beta\alpha + \sum_{\beta\alpha} f_{\delta(\gamma)} \epsilon_{\gamma\alpha}$$

$$= -4i \epsilon_{\delta\gamma} W_{\gamma\beta\alpha} + \sum_{\beta\alpha} \epsilon_{\gamma\alpha} \left( \frac{i}{6} \bar{D}_{\delta} G_{\beta\gamma} + \frac{i}{3} \bar{D}_{\gamma} G_{\beta\dot{\gamma}} + \frac{i}{3} \epsilon_{\delta\gamma} \bar{D}_{\beta} \bar{R} \right)$$

$$= -4i \epsilon_{\delta\gamma} W_{\gamma\beta\alpha} + i \sum_{\beta\alpha} \epsilon_{\gamma\alpha} \left( \frac{1}{2} \bar{D}_{\delta} G_{\beta\gamma} + \frac{1}{3} \epsilon_{\delta\gamma} X_{\beta} \right)$$

which is also as in $U(1)$ superspace [8].

At the dimension 2 level, results are a bit more interesting. Using (3.9), one finds

$$R_{(\delta\delta)(\gamma\dot{\gamma})\beta\alpha} = \ddot{R}_{(\delta\delta)(\gamma\dot{\gamma})\beta\alpha} + \sum_{\beta\alpha} \left( f_{(\delta\delta)(\gamma\dot{\gamma})} \epsilon_{\gamma\alpha} - f_{(\gamma\gamma)(\delta\delta)} \epsilon_{\delta\alpha} \right)$$

(3.55)

Recall that

$$\ddot{R}_{(\delta\delta)(\gamma\dot{\gamma})\beta\alpha} = +2 \epsilon_{\delta\gamma} \chi_{\delta\gamma\beta\alpha} - \frac{1}{4} \epsilon_{\delta\gamma} \sum_{(\delta\gamma) (\beta\alpha)} \epsilon_{\delta\beta} D^\delta W_{\phi\gamma\alpha}$$

(3.56)

where

$$\chi_{\delta\gamma\beta\alpha} = \frac{1}{4} (D_{\delta} W_{\gamma\beta\alpha} + D_{\gamma} W_{\delta\beta\alpha} + D_{\beta} W_{\gamma\delta\alpha} + D_{\alpha} W_{\gamma\delta\beta}).$$

We would like to show that (3.55) reduces to

$$R_{(\delta\delta)(\gamma\dot{\gamma})\beta\alpha} = +2 \epsilon_{\delta\gamma} \chi_{\delta\gamma\beta\alpha} - 2 \epsilon_{\delta\gamma} \epsilon_{\beta\alpha} \psi_{\delta\gamma\beta\alpha}$$

(3.57)

where

$$\chi_{\delta\gamma\beta\alpha} = \chi_{\delta\gamma\beta\alpha} + (\epsilon_{\delta\beta} \epsilon_{\gamma\alpha} + \epsilon_{\delta\alpha} \epsilon_{\gamma\beta}) \chi$$

(3.58)

$$\psi_{\delta\gamma\beta\alpha} = \frac{1}{8} \sum_{\delta} \sum_{\gamma\beta\alpha} \left( G_{\delta\beta} G_{\gamma\alpha} - \frac{1}{2} [D_{\delta}, D_{\beta}] G_{\gamma\alpha} \right)$$

(3.59)

$$\chi = -\frac{1}{12} (D^2 R + \bar{D}^2 \bar{R}) + \frac{1}{48} [D^\alpha, D^\dot{\alpha}] G_{\alpha\dot{\alpha}} - \frac{1}{8} G^{\alpha\dot{\alpha}} G_{\alpha\dot{\alpha}} + 2 R \bar{R}$$

(3.60)

This is a straightforward (albeit tiresome) check. Some intermediate results help:

$$\sum_{\beta\alpha} f_{(\beta\dot{\alpha})}(\phi) = -D^\phi W_{\phi\beta\alpha}$$

(3.61)

$$\sum_{\beta\alpha} f_{(\beta\dot{\alpha})}(\psi) = 4 \psi_{\beta\alpha}$$

(3.62)

$$f_{(\phi)}(\phi) = 4 \chi$$

(3.63)

which allow the complete expression of the $f$ terms from (3.55) in terms of the relevant Poincaré quantities. For example, (3.61) allows for the cancellation of the similar $D^\phi W_{\phi\beta\alpha}$ terms in (3.56); the remaining terms involving $\psi$ and $\chi$ combine with $\chi_{\delta\gamma\beta\alpha}$ to give the Poincaré Lorentz curvature.
• Scaling and $U(1)$ curvatures

The only $U(1)$ curvature we haven’t discussed yet is $F_{ba}$, but this is the same in both
conformal and Poincaré theories. We have

$$F_{(\gamma \dot{\gamma})(\beta \dot{\beta})} = 2 \epsilon_{\gamma \dot{\gamma}} \dot{F}_{\gamma \beta} - 2 \epsilon_{\gamma \dot{\gamma}} \dot{F}_{\dot{\gamma} \beta}$$

where $D^\phi W_{\beta \dot{\beta} a} = \frac{i}{2} \dot{F}_{\dot{\beta} a}$. This is exactly as in [8] (aside from the extra factor of $i$).

For the scaling curvature,

$$H_{(\gamma \dot{\gamma})(\beta \dot{\beta})} = 2 \epsilon_{\gamma \dot{\gamma}} \dot{H}_{\gamma \beta} - 2 \epsilon_{\gamma \dot{\gamma}} \dot{H}_{\dot{\gamma} \beta}$$

where $D^\phi W_{\phi \beta a} = +2 \dot{H}_{\beta a}$. This is easily checked explicitly. Since $H_{(\gamma \dot{\gamma})(\beta \dot{\beta})} = 2 f_{(\beta \dot{\beta})(\alpha \dot{\alpha})} - 2 f_{(\alpha \dot{\alpha})(\beta \dot{\beta})}$, it follows that

$$\dot{H}_{\beta a} = -\frac{1}{2} \sum_{\beta \alpha} f_{(\beta \dot{\beta})(\alpha \dot{\alpha})} = \frac{1}{2} D^\phi W_{\phi \beta a},$$

as needed.

• Special conformal curvatures

These are by far the most complicated expressions remaining to check. The ones
remaining for us to examine are $R(K)_{\gamma b A}$ and $R(K)_{c b A}$, which amount to five extra
checks to perform. These give no extra insight or relations beyond those we already
have, so we will avoid evaluating them explicitly here.

3.2.3 Conformal symmetry of $U(1)$ superspace

If $U(1)$ superspace is indeed a gauge-fixed version of a fully conformal superspace, then it
must permit some form of scale transformation. This must be more than that of Howe and
Tucker [14] since those authors were restricted to a chiral parameter in order to preserve the
minimal torsion constraints. In fact, an unrestricted transformation does exist. Binetruy,
Girardi, and Grimm [8] showed that the minimal matter coupling $e^{-K/3}$ could be absorbed
into the frame of the vierbein provided the minimal superspace structure was enlarged
to include a $U(1)$ superconnection. This can be understood as an unconstrained scale transformation

They postulated a transformation for the vierbein

$$E'^A_M = E^B_M X^A_B$$

(3.64)

with a parameter $X^A_B$ of the form

$$X^A_B = \begin{pmatrix}
\delta^a_b X \bar{X} & X^a_b & X^a_{\dot{b}} \\
0 & \delta^a_{\beta} X & 0 \\
0 & 0 & \delta^a_{\dot{\beta}} \bar{X}
\end{pmatrix}$$

(3.65)

where

$$X^a_b \equiv \frac{i}{2} (\epsilon_{b\bar{b}})^a_{\bar{\alpha}} \bar{X}^{-1} \partial_{\bar{\alpha}} (X \bar{X}), \quad X^a_{\dot{b}} \equiv \frac{i}{2} (\epsilon_{\bar{b}})_{\dot{\alpha}}^a X^{-1} \partial_{\alpha} (X \bar{X})$$

(3.66)

---

13Enlarging the structure group is not the only way to do this. Instead, one may choose fewer torsion
constraints in Poincaré supergravity, which allow the superfield $T_{\alpha}$ in addition to $R$, $W_{\alpha \beta \gamma}$ and $G_c$. One can show (see for example [17]) that this $T_{\alpha}$ is essentially $A_{\alpha}$. 

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is required to preserve torsion constraints. Otherwise, the factors $X$ and $\bar{X}$ are totally unconstrained. By investigating the constraints of $U(1)$ superspace, they found the required transformation rules of the superfields

$$R' = (\bar{X})^{-2} \left( R - \frac{1}{8} (XX)^{-2} D^2 (X\bar{X})^2 \right)$$

(3.67)

$$G'_{\alpha\dot{\alpha}} = (X\bar{X})^{-1} \left( G_{\alpha\dot{\alpha}} - \frac{1}{2} [D_{\alpha}, \bar{D}_{\dot{\alpha}}] \log(X\bar{X}) + Y_{\alpha} \bar{Y}_{\dot{\alpha}} \right)$$

(3.68)

$$W'_{\alpha\beta\gamma} = (X\bar{X})^{-1} (\bar{X})^{-1} W_{\alpha\beta\gamma}$$

(3.69)

where $Y_{\alpha} \equiv D_{\alpha} \log(X\bar{X}).$ Although they restricted to the case where the $U(1)$ connection was initially zero, it is simple to extend to the case of a non-vanishing initial connection:

$$A'_M = A_M - \frac{1}{2} Z_M - \frac{3i}{2} E_M{}^\alpha Y_{\alpha} + \frac{3i}{2} E_{M\dot{\alpha}} \bar{Y}_{\dot{\alpha}} + \frac{3}{4} E_M{}^{\alpha\dot{\alpha}} Y_{\alpha} \bar{Y}_{\dot{\alpha}}$$

(3.70)

where $Z_M \equiv \partial_M \log(X\bar{X}).$ Without loss of generality, the superfields $X$ and $\bar{X}$ can be written

$$X = \exp(-\Lambda/2 + i\Omega), \quad \bar{X} = \exp(-\Lambda/2 - i\Omega),$$

(3.71)

for real superfields $\Omega$ and $\Lambda.$ The infinitesimal transformation rules are

$$\delta E_{m}{}^{a} = -\Lambda E_{m}{}^{a}$$

(3.72)

$$\delta E_{m}{}^{\alpha} = \left( -\frac{1}{2} \Lambda + i\Omega \right) E_{m}{}^{\alpha} - \frac{i}{2} (\epsilon_{m}){}^{\alpha} \partial^{\beta} \Lambda$$

(3.73)

$$\delta R = (\Lambda + 2i\Omega) R + \frac{1}{4} D^2 \Lambda$$

(3.74)

$$\delta G_{\alpha\dot{\alpha}} = \Lambda G_{\alpha\dot{\alpha}} + \frac{1}{2} [D_{\alpha}, \bar{D}_{\dot{\alpha}}] \Lambda$$

(3.75)

$$\delta W_{\alpha\beta\gamma} = \left( \frac{3}{2} \Lambda + i\Omega \right) W_{\alpha\beta\gamma}$$

(3.76)

$$\delta A_M = \partial_M \Lambda + \frac{3i}{2} E_M{}^\alpha D_{\alpha} \Lambda - \frac{3i}{2} E_{M\dot{\alpha}} \partial^{\dot{\alpha}} \Lambda$$

(3.77)

(Of the fields in the supervierbein, we have listed only those corresponding to the graviton and the gravitino. The other components of the supervierbein also transform, but they are unphysical so we'll ignore them for simplicity.) The above set of transformation rules is quite interesting. For the most part, they have the form of scale ($\Lambda$) and chiral ($\Omega$) transformations, with $A$ as the gauge field for the chiral transformations; however, for every term other than $E_{m}{}^{a}, W_{\alpha\beta\gamma},$ and $A_{\alpha\dot{\alpha}},$ there are modifications which depend on the derivative of the scale parameter $\Lambda.$

These extra modifications can be viewed as requirements forced by the torsion and curvature constraints of $U(1)$ superspace, but they can also be viewed as having a deeper geometrical origin. Our claim was that $U(1)$ superspace is a gauge-fixed version of conformal superspace. This is straightforward to see. The variation of the field $B_M$ under $D$ and $K$ transformations is

$$\delta B_M = \partial_M \Lambda - 2E_M{}^A \epsilon_A (-)^a$$
where $\epsilon^A$ is the parameter for $K$ transformations and $\Lambda$ that of $D$. If we demand that $B_M = 0$ remain fixed, then every scale transformation must be accompanied by a $K$-transformation with $\epsilon^A = (-)^a \frac{1}{2} D_A \Lambda$. It is this corresponding $K$-transformation which generates the additional derivatives of $\Lambda$.

Consider first the vierbein. Under a $K$-transformation, $\delta_K E_M^A = \frac{1}{2} E_M^C \epsilon^B C^A_{BC}$, which corresponds to

$$\delta_K E_m^a = 0$$

$$\delta_K E_m^a = -i \epsilon^a \bar{\sigma}_m = \frac{i}{2} D_\beta \Lambda \bar{\sigma}^\beta_m$$

for the graviton and gravitino, reproducing the additional terms exactly. Take the $U(1)$ connection next. Under a $K$-transformation, $\delta_K A_M = -3 i w(A) E_M^A \epsilon_A$. Plugging in for $\epsilon$ we find

$$\delta_K A_M = \frac{3i}{2} w(A) E_M^A D_A \Lambda$$

as expected.

The fields $R$ and $G_{\alpha\bar{\beta}}$ are a bit more complicated. Recall that they are themselves related to the $K$-gauge fields by $f_{\alpha\beta} = \epsilon_{\alpha\beta} R$ and $f_{\alpha\bar{\beta}} = -G_{\alpha\bar{\beta}}/2$. The rule for the transformation of $f_{M\bar{\beta}}$ is $\delta_K f_{M\bar{\beta}} = D_M \epsilon_{\beta} - i E_M^\beta \epsilon_{\bar{\beta}}$ which corresponds to

$$\delta_K G_{\alpha\bar{\beta}} = D_\alpha D_\beta \Lambda + i D_{\alpha\bar{\beta}} \Lambda = \frac{1}{2} [D_\alpha, D_\beta] \Lambda.$$

For $R$, using $\delta_K f_{\alpha \bar{\beta}} = \epsilon_{\alpha \bar{\beta}} \bar{D}^2 \Lambda/4$ gives

$$\delta_K R = \bar{D}^2 \Lambda/4$$

These are precisely the extra terms enforced by the torsion constraints.

Finally note that $W_{\alpha \beta \gamma}$ is a chiral primary superfield; thus it is inert under $K$ and so has no extra terms.

### 3.3 Old minimal supergravity

We break the explicit scale invariance of the superspace theory by following as closely as possible the non-supersymmetric case. There a compensating matter field $\Phi$ was introduced with unit scaling weight. The $D$-gauge was then used to scale $\Phi$ to a constant, explicitly breaking the scale invariance and collapsing the kinetic Lagrangian into the Einstein-Hilbert term.

An analogous procedure can be undertaken in superspace. We must make use of a compensating superfield, and the simplest one is a chiral field. We denote it $\Phi_0$, assume it to have a scaling weight of $\Delta(\Phi_0) = 1$ (and therefore a chiral weight of $\omega(\Phi_0) = 2/3$). The kinetic multiplet for $\Phi_0$ is just the superspace $D$-term

$$- 3 \int \tilde{E} \tilde{\Phi}_0 \Phi_0$$

(Here and in the following we use $\tilde{\cdot}$ over the measure to denote when we are in the conformal frame where the gauge is unfixed.) We would like to gauge $\Phi_0 = 1$. That converts the kinetic action into the supervolume, which reproduces the supersymmetrized Einstein-Hilbert term.
First let us note some things. After gauge-fixing \( \Phi_0 \) to a constant, we are left with an issue of consistency, the equation of chirality for \( \Phi_0 \):

\[
0 = \nabla_\alpha \Phi_0 = \left( D_\alpha - B_\alpha - \frac{2i}{3} A_\alpha \right) \Phi_0 \quad (3.79)
\]

We have explicitly used all of the \( K \)-gauge to fix \( B = 0 \). When \( \Phi_0 \) is gauged to a constant, \( A_\alpha = 0 \) follows. A corresponding analysis with \( \bar{\Phi}_0 \) leads us to conclude \( A_\alpha \) vanishes as well.

Using \( F_\alpha^\dot{\alpha} = \frac{1}{4} \nabla^2 \Phi_0 = \frac{1}{4} (D^2 - 8 \bar{R}) \Phi_0 = 2 \bar{R} \Phi_0 \) \((3.80)\)

The anti-chiral superfield \( \bar{R} \) is itself nothing more than the F-term of the chiral compensator, which is a well-known result. \(^{14}\)

### 3.3.1 The chiral compensator and super-Weyl transformations

The normal approach to conformal supergravity \(^{13}\) makes use of a chiral field \( \Phi_0 \), introduced as a book-keeping device, whose bosonic component is used to fix the normalization of the Einstein-Hilbert term while the rest of the components are set to zero. This is completely analogous to the theory discussed above, except in those formulations the compensator is fixed at the component level. This theory also possesses a residual “super-Weyl” symmetry.

Begin with a model where the only field with scaling or chiral weight is the compensator \( \Phi_0 \). It must therefore be employed to make the conformal D and F-terms invariant. These take the form

\[
\mathcal{L}_D = \int d^4 \theta \; \bar{\Phi}_0 \Phi_0 V, \quad \mathcal{L}_F = \int d^2 \theta \; \bar{\Phi} \Phi_0^3 W \quad (3.81)
\]

Although \( V \) and \( W \) are generic real and chiral superfields of vanishing scaling and chiral weights, they possess a residual symmetry:

\[
\Phi_0 \to \Phi_0 e^{2\Sigma}, \quad V \to e^{-2\Sigma-2\bar{\Sigma}} V, \quad W \to e^{-6\Sigma} W \quad (3.82)
\]

where \( \Sigma \) is a chiral field of vanishing scaling and chiral weights. If we work in the gauge where \( \Phi_0 = 1 \), the above redefinition of the chiral compensator must be compensated by an honest conformal transformation with a rescaling \( \Lambda = -\Sigma - \bar{\Sigma} \) and a \( U(1) \) rotation \( \Omega = \frac{2i}{3}(\Sigma - \bar{\Sigma}) \). This combined redefinition and conformal transformation is the super-Weyl transformation of Howe and Tucker \(^{13}\) which preserves the form of the minimal Poincaré torsion constraints. \( V \) transforms as a real super-Weyl field with weight 2, \( W \) as a chiral super-Weyl field of weight 3, and the superdeterminant of the vierbein, \( E \), as a real super-Weyl field with weight -2. (The transformation rules on the superfields \( R, G_c \), the graviton, and gravitino can be derived from \((3.72-3.76)\).)

\(^{14}\) It can be shown (see for example \(^{8}\)) that the theory above, with a remnant \( U(1) \) field, can be converted to the theory of Wess and Bagger, where the \( U(1) \) connection is entirely absent, by a simple modification of the torsion components.
The conformal transformations discussed in this article must be contrasted with these super-Weyl transformations. The former are unconstrained in superspace; the latter are highly constrained in superspace (the $\Sigma$ must be chiral) but correspond to unconstrained superconformal transformations at the component level.

### 3.3.2 Integral relations between various formulations

We have several types of integrals – D and F, gauge fixed and unfixed – that describe the same physics, and we should demonstrate how they are related to each other.

The F-term action in conformal superspace can be rewritten

$$
\int d^2 \theta \bar{E} \Phi_0 W = -\frac{1}{4} \int d^2 \theta \bar{E} \nabla^2 \left( \frac{\Phi_0 \Phi_0^3 W}{F} \right) \tag{3.83}
$$

where $\bar{F} \equiv -\frac{1}{4} \nabla^2 \bar{\Phi}_0$. (The equivalency follows since the only non-chiral term in the parentheses is $\bar{\Phi}_0$, whose derivatives are cancelled by the denominator.) This is equivalent to a D-term:

$$
-\frac{1}{4} \int d^2 \theta \bar{E} \nabla^2 \left( \frac{\Phi_0 \Phi_0^3 W}{F} \right) = \int d^4 \theta \tilde{E} \frac{\Phi_0 \Phi_0^3 W}{F} \tag{3.84}
$$

Now we gauge $\Phi_0$ to one. This leaves the inverse of the F-component of $\bar{\Phi}_0$, but this is nothing more than the chiral field $R$. Thus we find the following set of equalities:

$$
\int d^2 \theta \bar{E} \Phi_0 W = \int d^2 \theta E W = \frac{1}{2} \int d^4 \theta \frac{E}{R} W \tag{3.85}
$$

The term on the left is the expression for the chiral F-term in the presence of a conformal multiplet. The term in the middle is the chiral F-term after conformal gauge-fixing. The term on the right is the form of the chiral F-term used in $[8]$. Since the difference between the first and third terms is just a gauge-fixing, it should make no difference when we fix the gauge. Therefore if we were to evaluate the first term completely within conformal superspace and then gauge-fix, we would necessarily arrive at the same answer as the term on the right.$^{15}$

To address the D-term, first note that in conformal superspace one can easily convert a D to an F-term:

$$
\int d^4 \theta \bar{E} \bar{\Phi}_0 \Phi_0 V = \int d^2 \theta \bar{E} \Phi_0 \left( \bar{F} V - \frac{1}{2} \nabla_{\dot{\alpha}} \bar{\Phi}_0 \nabla^{\dot{\alpha}} V - \bar{\Phi}_0 \frac{1}{4} \nabla^2 V \right) \\
= \int d^2 \theta \bar{E} \left( 2R \bar{\Phi}_0 \Phi_0 V - \frac{\Phi_0}{2} \nabla_{\dot{\alpha}} \bar{\Phi}_0 \nabla^{\dot{\alpha}} V - \bar{\Phi}_0 \Phi_0 \frac{1}{4} \nabla^2 V \right) \tag{3.86}
$$

(Here $V$ has zero scaling weight.) Now, let us gauge fix $\Phi_0$ to unity and equate the first and final steps. We find

$$
\int d^4 \theta E V = -\frac{1}{4} \int d^2 \theta E (\bar{D}^2 - 8R) V \tag{3.87}
$$

$^{15}$ One may also note that the rather curious form of $1/2R$ as the term converting from an $F$ to a $D$-term can be understood as a delta function. In particular, using the result of Appendix A.2, the chiral delta function is of a general form $\Delta = X/P[X]$. For the case of $X = 1$, this gives $\Delta = -1/4(\bar{D}^2 - 8R)(1) = 1/2R.$
This tells us that the proper way in Poincaré superspace to convert a D to an F-term is through the use of the chiral Poincaré projector. This is actually quite intuitive if we use our other F to D-term conversion formula:

\[ \int d^4 \theta \mathcal{E} V = -\frac{1}{4} \int d^2 \theta (\mathcal{D}^2 - 8R) V = -\frac{1}{8} \int d^4 \theta \frac{E}{R} (\mathcal{D}^2 - 8R) V \] (3.88)

The equality of the first and third terms follows by integration by parts in Poincaré super-space.¹⁶

### 3.4 Kähler supergravity

A general set of chiral fields coupled to conformal supergravity generically has D and F-terms

\[ \mathcal{L}_D = -3 \int d^4 \theta \bar{\Phi}_0 e^{-K/3} \Phi_0, \quad \mathcal{L}_F = \int d^2 \theta \bar{\Phi}_0^3 W \] (3.89)

for chiral primary superfield \( \Phi_0 \) with \( \Delta = 1 \) and \( \omega = 2/3 \). \( K \) is real and \( W \) is chiral, both with vanishing scale and chiral weights. The actions are invariant under a Kähler transformation

\[ K \to K + F + \bar{F} \] (3.90)

\[ \Phi_0 \to \Phi_0 e^{+F/3}, \quad \bar{\Phi}_0 \to \bar{\Phi}_0 e^{+\bar{F}/3} \] (3.91)

\[ W \to e^{-F} W, \quad \bar{W} \to e^{-\bar{F}} W \] (3.92)

Here the superfields \( F \) and \( \bar{F} \) are chiral/antichiral respectively. \( K \) is a real function of Kähler chiral matter fields \( \xi^i \) and \( \bar{\xi}^i \) with vanishing conformal weight, and \( W \) is a function of only the chiral ones \( \xi^i \). In the language of complex manifolds, \( W \) is a holomorphic function and \( K \) a real function. The transformation fields \( F \) and \( \bar{F} \) are, respectively, holomorphic and antiholomorphic functions of the chiral and anti-chiral Kähler matter fields. Note that the Kähler transformation has no effect a priori on the supergravity sector.

There are two straightforward ways to accomplish a conformal gauge fixing. The first is to gauge \( \Phi_0 \) to one. As the Kähler transformations alter \( \Phi_0 \), a corresponding conformal transformation must compensate every Kähler transformation. This is the well-known Howe-Tucker transformation [14], which when combined with the given Kähler transformations of \( K \) and \( W \) render the D and F-terms invariant. Unfortunately, the D-term action then yields a non-canonical Einstein-Hilbert term. There are two traditional methods for dealing with this. One may rescale fields at the component level in a quite complicated fashion; this is the path taken in [10]. One may also leave \( \Phi_0 \) unscaled until the very end of the calculation; this is the chiral compensator approach popularized by Kugo and Uehara [13].

A newer method is that of Binetruy, Girardi, and Grimm [8]. They demonstrated that enlarging to \( U(1) \) superspace from a minimal Poincaré superspace allowed an arbitrary super-Weyl transformation to absorb the factor \( e^{-K/3} \) into \( E \). From our point of view, their approach has a very simple interpretation. Rather than scale \( \Phi_0 \) = 1, choose the gauge \( \Phi_0 = e^{K/6} \). The equation of chirality then reads \( 0 = \mathcal{D}_\alpha \Phi_0 = D_\alpha \Phi_0 - \frac{2}{3} A_\alpha \Phi_0 \) which implies \( A_\alpha = -\frac{1}{2} D_\alpha K \). The antichirality of \( \Phi_0 \) similarly implies \( A_\alpha = \frac{1}{2} D_\alpha K \). The Poincaré

¹⁶Note the significance of these steps. Within conformal superspace as in flat supersymmetry, one can convert from a D to an F-term, but the reverse is not an easily defined operation. Upon gauge-fixing to minimal Poincaré superspace, we gain the field \( R \) which allows us to do so.
constraint $F_{\alpha\dot{\alpha}} = -3iG_{\alpha\dot{\alpha}}$ then gives $A_{\alpha\dot{\alpha}}$. The entire connection is given in terms of $K$ and $G_{\alpha\dot{\alpha}}$:

$$A_{\alpha} = -\frac{i}{4}D_{\alpha}K, \quad A_{\dot{\alpha}} = -\frac{i}{4}D_{\dot{\alpha}}K$$

$$A_{\alpha\dot{\alpha}} = -\frac{3}{2}G_{\alpha\dot{\alpha}} + \frac{1}{8}[D_{\alpha}, D_{\dot{\alpha}}]K$$  \hspace{1cm} (3.93)$$

The imaginary part of the Kähler transformation now plays the role of the $U(1)$ R-symmetry; the real part is equivalent to a super-Weyl transformation and corresponds to a rescaling of $\Phi_0$.

Alternatively, one may absorb the Kähler potential into the fields $\Phi_0$ to define Kähler-covariant fields $\Psi_0$ as in (2.122). Then the gauge choice $\Psi_0 = 1$ gives

$$0 = \nabla_{\dot{\alpha}}^K \Psi_0 = -\frac{2i}{3}A_{\dot{\alpha}} + \frac{2i}{3}A_{\dot{\alpha}} \Rightarrow A_{\dot{\alpha}} = A_{\dot{\alpha}}$$  \hspace{1cm} (3.94)$$

where $A_{\dot{\alpha}}$ is the $U(1)$ connection and $A_{\dot{\alpha}} = \frac{1}{3}D_{\dot{\alpha}}K$ is the Kähler connection. We arrive at the same result as (3.93). The gauge $\Psi_0 = 1$ breaks one combination of the $U(1)$ and Kähler symmetries, leaving the combination where the $U(1)$ and Kähler transform together. Therefore an effective transformation on the matter fields (the Kähler transformation) has been extended to the entire frame of superspace (by merging it with the $U(1)$ R-symmetry).

### 3.5 New minimal supergravity

In both of the prior cases, we have used the simplest superfield, a chiral one with eight components, to gauge fix to Poincaré supergravity. Needless to say this is not the only choice. Another minimal choice would be a linear multiplet, which also contains eight components. We begin with a real linear superfield $L$, obeying

$$\nabla^2 L = \bar{\nabla}^2 L = 0$$  \hspace{1cm} (3.95)$$

From the superconformal algebra, we know that $L$ must possess a scaling weight of $\Delta(L) = 2$ and, by reality, a vanishing $U(1)$ weight. This latter feature will leave the $U(1)$ gauge symmetry unaffected by the gauge-fixing procedure.

Before fixing the gauge $L = 1$, one important feature of the linear multiplet must be discussed. Due to the linearity constraint, $[\nabla^2, \bar{\nabla}^2]L = 0$, which implies $\nabla^{\alpha\dot{\alpha}}[\nabla_\alpha, \nabla_{\dot{\alpha}}]L = 0$ -- the divergence of the vector component of $L$ vanishes. In global supersymmetry, this implies the vector component is the dual of a three-form, but in supergravity this statement is modified by terms involving the gravitino. The simplest way to derive this behavior is to consider the two-form potential $B_{MN}$, whose three-form field strength $H = dB$ obeys a Bianchi identity, $dH = 0$. Following [17] and [8], one chooses $H$ to obey the constraints

$$0 = H_{\gamma\dot{\beta}a} = H_{\gamma\beta a} = H_{\dot{\gamma}\dot{\beta}a}$$  \hspace{1cm} (3.96)$$

Then as a consequence of the Bianchi identities, one can show that

$$H_{\gamma\beta} = 2i\sigma_{\gamma\beta}L$$  \hspace{1cm} (3.97)$$

$$H_{\gamma ba} = 2(\sigma_{ba})_\gamma^\phi \nabla^\phi L, \quad H_{\dot{\gamma} ba} = 2(\bar{\sigma}_{ba})^{\dot{\gamma}}_\dot{\phi} \bar{\nabla}^{\dot{\phi}} L$$  \hspace{1cm} (3.98)$$

$$H_{c ba} = \epsilon_{cba}^d \Delta_d L$$  \hspace{1cm} (3.99)$$
where $L$ is a linear superfield and where we have defined

$$\Delta_{\alpha\dot{\alpha}} L \equiv -\frac{1}{2} [\nabla_\alpha, \bar{\nabla}_{\dot{\alpha}}] L. \tag{3.100}$$

It follows that the dual of the three form is

$$\frac{1}{3!} \epsilon_{pnm\ell} H_{nml} = e_a^p \Delta^a L - \frac{i}{2} \epsilon_{pnm\ell} (\psi_n \sigma_m \bar{\psi}_\ell)L + i(\psi_n \sigma^{np}) \phi \nabla_\phi L - i(\bar{\psi}_n \bar{\sigma}^{np})_{\dot{\phi}} \nabla_{\dot{\phi}} L$$

$$= \frac{1}{2} \epsilon_{pnm\ell} \partial_n B_{m\ell} \tag{3.101}$$

Let us now gauge fix $L = 1$. The equations of linearity become, in Poincaré superspace,

$$(\mathcal{D}^2 - 8 \bar{R}) L = (\bar{\mathcal{D}}^2 - 8 R) L = 0 \tag{3.102}$$

Since $L$ is a constant, the only way this can be satisfied is if $R = \bar{R} = 0$. From the relations relating $R$ to $G_c$, this necessarily implies $\mathcal{D}_c G^c = 0$. Noting that

$$-2 \Delta_{\alpha\dot{\alpha}} L = [\nabla_\alpha, \bar{\nabla}_{\dot{\alpha}}] L = [\mathcal{D}_\alpha, \mathcal{D}_{\dot{\alpha}}] L - 4 G_{\alpha\dot{\alpha}} L \tag{3.103}$$

and that both $\mathcal{D}_\alpha L$ and $\mathcal{D}_{\dot{\alpha}} L$ vanish (we have gauged $B$ to zero, and the $U(1)$ connection appears in neither expression since $L$ has no chiral weight), we derive that

$$\Delta_a L = 2 G_a \tag{3.104}$$

in the gauge where $L = 1$. It follows that

$$e_a^p G^a \big| = \frac{1}{4} \epsilon_{pnm\ell} \partial_n b_{m\ell} + \frac{i}{4} \epsilon_{pnm\ell} (\psi_n \sigma_m \bar{\psi}_\ell) \tag{3.105}$$

where $b_{m\ell}$ denotes the bosonic lowest component $B_{m\ell}$. 

The bosonic two-form $b_{m\ell}$ corresponds to three real bosonic components (after accounting for its gauge invariance). The superfield $R$ vanishes so no component field $M$ is generated. However, the $U(1)$ symmetry has not been broken, and so we will have in our off-shell spectrum the bosonic field $A_m$ which is the gauge field of the chiral gauge symmetry, giving three bosonic components. As in the (old) minimal model, we have introduced six extra bosonic degrees of freedom to close the supergravity algebra off-shell.

The immediate candidate for the simplest D-term action is

$$\int d^4 \theta \bar{E} L \tag{3.106}$$

However, using the D to F conversion in conformal superspace, this becomes

$$\int d^4 \theta \bar{E} L = -\frac{1}{4} \int d^2 \theta \mathcal{E} \nabla^2 L = 0. \tag{3.107}$$

The linearity condition tells us that this simple integral vanishes. This immediately implies (after gauging $L$ to one) that in the new minimal Poincaré superspace the integral of the supervolume vanishes: $\int d^4 \theta E = 0$. This is a well-known property of the new minimal model, and nothing more meaningful than the fact that $R = 0$ [15].
To derive the form of the new minimal supergravity action, we will use a duality transform (as discussed in [16]) to transform a chiral compensator to a linear one. The properly normalized Einstein-Hilbert action is derivable from

\[ -3 \int d^4 \theta \bar{E} \Phi_0 \bar{\Phi}_0 \]  

(3.108)

after fixing the gauge \( \Phi_0 = 1 \). This action can in turn be derived from the first-order action

\[ -3 \int d^4 \theta \bar{E} \left( X - L \log(X/\Phi_0 \bar{\Phi}_0) \right) \]  

(3.109)

where \( L \) is a linear superfield, \( X \) is an arbitrary real superfield of scaling weight 2, and \( \Phi_0 \) is some chiral superfield of scaling weight 1. (Although the theory seems to depend on \( \Phi_0 \), this is illusory since the components of \( \Phi_0 \) are modified by the redefinition \( \Phi_0 \rightarrow \Phi_0 e^{F/3} \) for chiral \( F \) under which the first-order action is invariant.) Since a linear superfield \( L \) can be written as \( L = \nabla^a \nabla^2 \Omega_\alpha + \text{h.c.} \) for \( \Omega_\alpha \) with \( \Delta = 1/2 \) and \( w = -1 \), an action of the form \( LZ \) has an \( L \) equation of motion which sets \( Z = \bar{S} + \bar{\bar{S}} \) for chiral field \( S \) of vanishing conformal weight. Thus varying \( L \) gives \( X = \Phi_0 \bar{\Phi}_0 \), up to chiral and antichiral fields which can be absorbed into a redefinition of \( \Phi_0 \). This in turn restores the original action. On the other hand, we may vary \( X \) to conclude \( X = L \), which gives the action

\[ -3 \int d^4 \theta \bar{E} \left( L - L \log(L/\Phi_0 \bar{\Phi}_0) \right) = \int d^4 \theta \bar{E} \, LV_R \]  

(3.110)

where we have defined \( V_R \equiv 3 \log(L/\Phi_0 \bar{\Phi}_0) \) and dropped the term linear in \( L \) since a linear superfield has vanishing D-term. \( V_R \) is a scalar field with vanishing conformal and chiral weights, although it does possess a symmetry \( V_R \rightarrow V_R - F - \bar{F} \) with chiral field \( F \).

The prior gauge choice \( \Phi_0 = \bar{\Phi}_0 = 1 \) which gave a properly normalized Einstein-Hilbert term here corresponds to \( L = 1 \). Choosing this gauge gives the simple action \( \int d^4 \theta E \, V_R \) where \( V_R = -3 \log(\Phi_0 \bar{\Phi}_0) \). It is fairly simple to see now what sort of object \( V_R \) is. Since we have gauge-fixed the scale symmetry in addition to fixing \( B = 0 \), the structure group of our space differs only from Poincaré supergravity by the presence of a \( U(1) \) R-symmetry. These fields \( \Phi_0 \) and \( \bar{\Phi}_0 \) are covariantly chiral with respect to a derivative containing the corresponding \( U(1) \) connection. Any \( U(1) \) theory of covariantly chiral superfields \( \Phi \) (\( D_\alpha \Phi = \bar{\Phi}_0 \)) may be related to a theory with Einstein chiral superfields \( \phi \) (\( D_\alpha \phi = E_\alpha^M \partial_M \phi \)) and a \( U(1) \) prepotential \( V \),

\[ \bar{\Phi} \Phi \rightarrow \bar{\phi} e^{-V/3} \phi \]  

By choosing \( F \) appropriately, one may eliminate \( \phi \), arriving at \( V_R = V \).

While this is the simplest explanation for what \( V_R \) is, it is somewhat unsatisfying since throughout this paper we have avoided discussing prepotentials. To arrive at the some point by a rather more circuitous route, one begins by partially fixing the \( U(1) \) gauge which at the moment is still a full superfield symmetry. We choose \( \Phi_0 = \bar{\Phi}_0 \); that is, set their relative phase to zero. The symmetry \( \Phi_0 \rightarrow \Phi_0 e^{F/3} \) must be compensated with a chiral rotation with parameter \( \Omega = \frac{i}{2}(F - \bar{F}) \). We have now fixed the unconstrained \( U(1) \) parameter to the imaginary part of a chiral parameter, and we see immediately that \( V_R \) transforms suspiciously as if it were the prepotential of such a chiral version of R-symmetry. If we evaluate the spinorial derivatives of \( V_R \), we find this is exactly so. Begin with

\[ D_\alpha V_R = -3 \frac{1}{\Phi_0} D_\alpha \Phi_0 = -3 \frac{D_\alpha \Phi_0}{\Phi_0} + 2i A_\alpha \]  

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and then note that since as functions $\Phi_0 = \bar{\Phi}_0$,

$$D_\alpha \Phi_0 = D_\alpha \bar{\Phi}_0 = -\frac{2i}{3} A_\alpha \Phi_0 = -\frac{2i}{3} A_\alpha \bar{\Phi}_0$$

where we have used the chirality condition of $\bar{\Phi}_0$. It follows that

$$D_\alpha V_R = 4i A_\alpha, \quad D_\alpha V_R = -4i \bar{A}_\alpha.$$

$V_R$ plays here the role of the $U(1)$ R-symmetry prepotential, and so the term $\int d^4 \theta E V_R$ is nothing more than the $U(1)$ Fayet-Iliopolous term.

From our point of view, evaluating the $D$-term of $V_R$ is particularly easy. One considers $V_R$ in its original form involving $\Phi$. One can evaluate the $D$-term component Lagrangian directly. After integrating a number of terms by parts, one arrives at

$$e^{-1} \int d^4 \theta E V_R = \frac{1}{2} D^\alpha X_\alpha - \frac{i}{2} (\bar{\psi}_m \sigma^m)_{\dot{\alpha}} X^{\dot{\alpha}} - \frac{i}{2} (\bar{\psi}_m \sigma^m)^{\alpha} X_\alpha$$

$$- \epsilon^{pnm\ell} (A_p + \frac{3}{2} e_p c G_c) \times (4 G^b e_b^p + i e^{pnm\ell} (\bar{\psi}_n \sigma_m \psi_{\ell}))$$

The combination $A_p + \frac{3}{2} e_p c G_c$ can be thought of as the $U(1)$ connection if one chooses to define the bosonic derivative so that $F_{\alpha \dot{\alpha}}$ vanishes. (Recall that $F_{\alpha \dot{\alpha}} = -3i G_{\alpha \dot{\alpha}}$ in our convention.)

Using the definition for the lowest component of $G_b$, one finds

$$e^{-1} \int d^4 \theta E V_R = \frac{1}{2} D^\alpha X_\alpha - \frac{i}{2} (\bar{\psi}_m \sigma^m)_{\dot{\alpha}} X^{\dot{\alpha}} - \frac{i}{2} (\bar{\psi}_m \sigma^m)^{\alpha} X_\alpha$$

$$- \epsilon^{pnm\ell} (A_p + \frac{3}{2} e_p c G_c) \partial_{\alpha} b_{\nu \mu \ell}$$

The Einstein-Hilbert action will be contained within $D^\alpha X_\alpha$ and the Rarita-Schwinger action within the terms involving $X_\alpha$ and $X^{\dot{\alpha}}$. The remaining term, while involving the gauge potential $A_p$ directly, is gauge invariant when integrated by parts.

Recall that $D^\alpha X_\alpha$ is as defined in $U(1)$ superspace and obeys the equality

$$D^2 R + \bar{D}^2 \bar{R} = -\frac{2}{3} R_{ba}^{ba} - \frac{2}{3} D^\alpha X_\alpha + 4 G_a G_a + 32 R \bar{R}$$

Since $R = 0$, this equation serves to define

$$\frac{1}{2} D^\alpha X_\alpha \equiv -\frac{1}{2} R_{ba}^{ba} + 3 G_a G_a$$

$$= -\frac{1}{2} R - i (\bar{\psi}_b \sigma_a T^{ab}) - i (\bar{\psi}_b \sigma_a T^{ab}) - \frac{i}{2} \epsilon^{k \ell m n} G_k \psi_{\ell} \sigma_m \psi_n + 3 G_a G_a$$

Using $(\bar{\psi}_m \sigma^m \bar{X}) = -2(\bar{\psi}_m \sigma^m \sigma^{cb} T_{cb})$ and its conjugate, it is straightforward to derive

$$e^{-1} \int d^4 \theta E V_R = -\frac{1}{2} R + \frac{1}{2} \epsilon^{k \ell m n} (\bar{\psi}_k \sigma_{\ell} D_m \psi_{n}) - \frac{1}{2} \epsilon^{k \ell m n} (\bar{\psi}_k \sigma_{\ell} D_m \bar{\psi}_{n}) - \epsilon^{pnm\ell} A'_p \partial_{\alpha} b_{\nu \mu \ell}$$

where

$$A'_m \equiv A_m + \frac{3}{4} e_m^a G_a$$

The calculation of this total expression can be simplified by noting that any terms which shift under the chiral transformation of $\Phi$, such as $D_\alpha \log \Phi$, must have vanishing coefficients.
and $\mathcal{D}'$ is defined with $A'$ as its $U(1)$ connection. (This latter definition corresponds to choosing $F_{\alpha\dot{\alpha}} = -\frac{3i}{2} G_{\alpha\dot{\alpha}}$ in defining the bosonic derivative.)

In pure new minimal supergravity, the equation of motion of the two-form enforces the $A'$ connection to (at least locally) be pure gauge, $A' = d\lambda$. The $A'$ equation of motion on the other hand gives

$$0 = \varepsilon^{k\ell mn} \left( \partial_\ell b_{mn} + i\psi_\ell \sigma_m \bar{\psi}_n \right)$$

Aside from the coupling of the gravitino to the field $A'$, the auxiliary sector is that of a simple abelian BF model with topological action $\int b \wedge dA'$ and no propagating degrees of freedom.

### 3.5.1 New minimal supergravity coupled to matter

For reference, we include here the simplest couplings of new minimal supergravity to chiral matter of vanishing $U(1)_R$ charge. (This last condition forbids a superpotential, so these models are quite simple ones.) One can derive these by performing a duality transformation from the Kähler multiplet, where $\Psi_0$ is covariantly chiral with respect to a $U(1)_K$. The modification consists simply of exchanging $\Phi_0$ with $\Psi_0$ in the definition of $V_R$, which essentially shifts $V_R$ to $V_R + K$. The kinetic matter coupling of new minimal supergravity is then

$$\int d^4 \theta E K \tag{3.114}$$

as in global supersymmetry. Evaluating this is straightforward. One simply replaces $X_\alpha$ and $A_m$ associated with $V_R$ with $X^K_\alpha$ and $A^K_m$. Provided we make the definitions

$$X^K_\alpha = -\frac{1}{8} \mathcal{D}^2 D^K_\alpha K, \quad X^K_\dot{\alpha} = -\frac{1}{8} \mathcal{D}^2 \bar{D}^{\dot{\alpha}} K \tag{3.115}$$

and

$$A^K_m = -\frac{1}{2} e_m^a \Delta_a K + \frac{i}{4} \psi_m^\alpha \mathcal{D}_\alpha K - \frac{i}{4} \psi_m^{\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}} K \tag{3.116}$$

one finds

$$\int d^4 \theta E K = -\frac{1}{2} \mathcal{D}^\alpha X^K_\alpha + \frac{i}{2} (\psi_\sigma X^K) + \frac{i}{2} (\bar{\psi}\sigma X^K) + \frac{1}{2} k^{\ell mn} A^K_k \partial_\ell b_{mn} \tag{3.117}$$

Unlike in old minimal supergravity, the presence of a Kähler potential does not lead to extra additions to the Einstein-Hilbert term. This is known to be altered when the chiral matter carries a $U(1)_R$ charge (see for example [18]).
4 Conclusion

We have constructed the fully conformal superspace and found its nonvanishing curvatures to be uniquely described in terms of a single superfield $W_{\alpha\beta\gamma}$. This is an unsurprising result since it was long known at the linearized level. Similarly we have demonstrated how $U(1)$ superspace is related to a certain gauge of the full conformal superspace; this too is unsurprising, as it was anticipated in [17]. Finally, the various Poincaré formulations of superspace have been explicitly constructed at the superspace level within the conformal framework in a way more clearly, we believe, than it had been done in the past. For example, the construction of [8] whereby the Kähler potential is absorbed into the supergravity multiplet seems especially simple in this approach.

Beyond notational and theoretical elegance, is there anything genuinely new this approach can offer? Perhaps. Noting that the old minimal to Kähler frame conversion was reproduced here quite easily, one may inquire whether in new minimal superspace there exists an analogous absorbing of the Kähler potential into the frame of superspace. For the simplest chiral models this is not necessary since it turns out the coupling of chiral matter does not affect the normalization of the Einstein-Hilbert term. However, when non-chiral matter is considered, the story becomes more complicated. In particular, one can show that the condition of $R = 0$ in the new minimal model is relaxed in the presence of certain types of matter, just as $X_\alpha = 0$ is broken in the minimal model when conversion to the Kähler frame is undertaken. We hope to analyze this situation further in the future.

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A Geometric preliminaries

A.1 Global spacetime symmetries

The global structure of the conformal symmetry groups of arbitrary manifolds (with or without torsion and Grassmann coordinates) benefits from first discussing a simple example: the conformal group on four dimensional Minkowski (or Euclidean) space.

A.1.1 The conformal group

The flat metric, $ds^2 = dx^m dx^n \eta_{mn}$, is preserved up to a conformal factor by the differential generators\(^{18}\)

$$p_a = \partial_a, \quad (1 + \xi \cdot p) x^m = x^m + \xi^m$$

$$m_{ab} = -x_a \partial_b + x_b \partial_a, \quad (1 + \frac{1}{2} \omega^{ba} m_{ab}) x^m = x^m - \omega^{mn} x_n$$

$$d = x \cdot \partial, \quad (1 + \lambda d) x^m = x^m + \lambda x^m$$

$$k_a = 2x_a x \cdot \partial - x^2 \partial_a, \quad (1 + \epsilon \cdot k) x^m = x^m + 2(\epsilon \cdot x)x^m - x^2 \epsilon^m$$

(A.1)

The special conformal generator $k_a$ can also be thought of as a spatial inversion, followed by a translation and then another spatial inversion.

These generators are represented on fields by the operators $P_a, M_{ab}, D,$ and $K_a$ with the following algebra:

$[M_{ab}, P_c] = P_a \eta_{bc} - P_b \eta_{ac}$, \quad $[M_{ab}, K_c] = K_a \eta_{bc} - K_b \eta_{ac}$ \quad (A.2)

$[M_{ab}, M_{cd}] = \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac} + \eta_{ad} M_{bc}$ \quad (A.3)

$[D, P_a] = P_a$, \quad $[D, K_a] = -K_a$ \quad (A.4)

$[K_a, P_b] = 2\eta_{ab} D - 2M_{ab}$ \quad (A.5)

where all other commutators vanish. The action of such generators on fields is defined by their action at the origin. One usually takes for conformally primary fields $\Phi$,

$$P_a \Phi(0) = \partial_a \Phi(0), \quad M_{ab} \Phi(0) = S_{ab} \Phi(0), \quad D \Phi(0) = \Delta \Phi(0), \quad K_a \Phi(0) = 0$$

(A.6)

Here $S_{ab}$ is a differential rotation matrix appropriate for whatever representation of the rotation group $\Phi$ belongs to, $\Delta$ is the conformal scaling dimension, and the vanishing of $K_a$ is called the primary condition. In order to discern the transformation rules at points beyond the origin, one must make use of the translation operator $e^{x \cdot P}$ to translate from the origin. This is formally a Taylor expansion:

$$\Phi(x) = e^{x \cdot P} \Phi(0) = \Phi(0) + x^a P_a \Phi(0) + \frac{1}{2} x^a x^b P_a P_b \Phi(0) + \ldots$$

$$= \Phi(0) + x^a \partial_a \Phi(0) + \frac{1}{2} x^a x^b \partial_a \partial_b \Phi(0) + \ldots$$

The operator $P_a$ acts only on the field $\Phi$, returning its derivative, and has no action on the coordinate $x$, which is here just a parameter. The same is true for the other operators.

\(^{18}\)The convention used here for the generators eliminates factors of $i$ in group elements while making most of the generators anti-Hermitian.
If \( g \) is any generator of the conformal algebra, the action of \( g \) on \( \Phi(x) \) can be calculated easily by making use of the translation operator:

\[
g\Phi(x) = e^{x^{\mu}P_{\mu}}e^{-x^{\mu}P_{\mu}}g e^{x^{\mu}P_{\mu}}\Phi(0) \equiv e^{x^{\mu}P_{\mu}}\tilde{g}(x)\Phi(0)
\]

where \( \tilde{g}(x) \equiv e^{-x^{\mu}P_{\mu}}g e^{x^{\mu}P_{\mu}} \) is an abbreviated notation for the translated \( g \). It follows that

\[
\tilde{P}_{a}(x) = P_{a}, \quad \tilde{D}(x) = D + x^{a}P_{a}, \quad \tilde{M}_{ab}(x) = M_{ab} - x_{[a}P_{b]}
\]

\[
\tilde{K}_{a}(x) = K_{a} + 2x_{a}D - 2x_{b}M_{ab} + 2x_{a}x_{b}P_{b} - x^{2}P_{a}
\]

(A.8)

If these operators are taken to act on a pure function, they reproduce the derivative representations (A.1). It should be noted that the algebra of the derivative representations differs by a sign from the algebra of the field representations; the former can be thought of as a left action on the group manifold with the latter corresponding to a right action which yields an opposite sign in the commutator.

On a more general field these expansions involve extra terms appropriate for \( \Phi \)'s representation. For a primary field,

\[
D\Phi(x) = \Delta \Phi + x^{a}\partial_{a}\Phi, \quad M_{ab}\Phi(x) = S_{ab}\Phi(x) - x_{[a}\partial_{b]}\Phi(x)
\]

\[
K_{a}\Phi(x) = (2x_{a}\Delta - 2x_{b}S_{ab} + 2x_{a}x_{b}\partial_{b} - x^{2}\partial_{a})\Phi(x)
\]

(A.9)

The algebraic relations are simply applied. For example,

\[
DP_{a}\Phi(x) = [D, P_{a}]\Phi(x) + P_{a}\left(\Delta + x^{b}P_{b}\right)\Phi(x) = (\Delta + 1)P_{a}\Phi(x) + x^{b}P_{b}P_{a}\Phi(x)
\]

from which one can define the intrinsic scaling dimension of \( \partial_{a}\Phi(x) \) as \( \Delta + 1 \). Similarly can one determine the behavior of the Lorentz rotation and special conformal generators:

\[
M_{bc}P_{d}\Phi(x) = \left(S_{bc}\delta_{d}^{a} + \eta_{a[c}\delta_{d]b}^{d}\right)\partial_{a}\Phi(x) - x_{[b}\partial_{c]}\partial_{d}\Phi(x)
\]

\[
= S'_{bc}\partial_{a}\Phi(x) - x_{[b}\partial_{c]}\partial_{d}\Phi(x)
\]

(A.10)

\[
K_{b}P_{a}\Phi(x) = (2\eta_{ba}\Delta - 2S_{ba})\Phi(x) + 2x_{b}(\Delta + 1)\partial_{a}\Phi(x)
\]

\[
-2x_{c}\left(S_{bc}\delta_{d}^{a} + \eta_{a[c}\delta_{d]b}^{d}\right)\partial_{d}\Phi(x) + (2x_{b}x_{c}\partial_{c} - x^{2}\partial_{b})\partial_{a}\Phi(x)
\]

\[
= \kappa_{ba}\Phi(x) + (2x_{b}\Delta - 2x_{c}S'_{bc} + 2x_{b}x_{c}\partial_{c} - x^{2}\partial_{b})\partial_{a}\Phi(x)
\]

(A.11)

Both have precisely the forms expected, where \( \Delta' \) and \( S'_{bc} \) are the conformal dimension and rotation matrix appropriate for \( \partial_{a}\Phi(x) \). The only interesting feature is that the special conformal generator removes the derivative; at the origin, \( K_{b}P_{a}\Phi(0) = \kappa_{ba}\Phi(0) = (2\eta_{ba}\Delta - 2S_{ba})\Phi(0) \). This same feature is found in the local theory.

The conformal group action we’ve discussed above involves transformations only on the fields, leaving the coordinate invariant. That is, the action of a differential generator \( g \) is

\[
x \rightarrow x, \quad \Phi \rightarrow \Phi'(x) = \Phi(x) + g\Phi(x)
\]

(A.12)

If we begin with the action \( S = \int d^{4}x \mathcal{L} \) (with the Lagrangian a function of fields and perhaps also the coordinate), the action of \( g \) is only on the fields:

\[
\delta_{g}S = \int d^{4}x \left( \frac{\delta\mathcal{L}}{\delta\Phi}g\Phi + \frac{\delta\mathcal{L}}{\delta\partial_{a}\Phi}g\partial_{a}\Phi \right)
\]

(A.13)
For the case where \( g = \xi \cdot P \), one finds \( g \Phi = \xi \cdot \partial \Phi \) and \( g \partial_a \Phi = \xi \cdot \partial \partial_a \Phi \). The term in parentheses is then equivalent to \( \frac{de}{dx} - \frac{\partial C}{\partial x} \). The first term vanishes as a total derivative; the second must also vanish, which tells that the Lagrangian cannot contain an explicit dependence on the coordinate. For the other choices of \( g \), the obvious results are recovered: the Lagrangian must have \( \Delta = 4 \), it must be a Lorentz scalar, and it must be conformally primary. The simplest conformal action involving a single primary scalar field of dimension one is \( \mathcal{L} = \phi \partial^2 \phi/2 - a \phi^4 \). (The only non-trivial check is to ensure the kinetic term vanishes at the origin under the action of the special conformal generator.)

The approach outlined above has the feature that it places all the transformation into the fields themselves. One often finds reference to a formalism where both the coordinates and the fields transform:

\[
x \rightarrow x', \quad \Phi(x) \rightarrow \Phi'(x')
\]  
(A.14)

For example, under translations and finite scalings, one would have

\[
x \rightarrow x' = x - a, \quad \Phi(x) \rightarrow \Phi'(x') = \Phi(x)
\]  
(A.15)

\[
x \rightarrow x' = e^{-\lambda}x, \quad \Phi(x) \rightarrow \Phi'(x') = e^{\Delta \lambda} \Phi(x)
\]  
(A.16)

The part of \( g \) which acts as a coordinate shift has been moved off the fields and onto the coordinate explicitly; the remaining action of \( g \) can be thought of as a generalized rotation operation, which vanishes if the field \( \Phi \) is a pure function. The main reason this approach is employed is that it allows conformal transformations on scalar fields (but only scalar fields) to be compactly written

\[
x \rightarrow x', \quad \phi(x) \rightarrow \phi'(x') = \left| \frac{\partial x'}{\partial x} \right|^{-\Delta/4} \phi(x).
\]  
(A.17)

where \( \Delta \) is the conformal scaling dimension of \( \phi \). Invariance of the action can then be checked in one step for all the elements of the conformal group. The \( \phi^4 \) term, for example, transforms as

\[
\int d^4x \phi(x)^4 \rightarrow \int d^4x' \phi'(x')^4 = \int d^4x' J \phi(x)^4 \text{ where } J = |\partial x'/\partial x|.
\]

Invariance is found for \( \Delta = 1 \).

### A.1.2 Constant torsion

We will ultimately be concerned with a theory containing torsion, so it is useful to review the effects torsion induces. Assume the manifold possesses translation generators \( P_a \) with nontrivial (but constant) torsion: \( [P_a, P_b] = -C_{ab}^c P_c \). All other points \( x \) relative to the privileged origin are defined by the condition \( f(x) = e^{x \cdot P} f(0) \) for pure functions \( f \).

By Taylor’s theorem, the \( P_a \) in the exponent is playing the same role as \( \partial_a \) and so they are equivalent when evaluated on the function at the origin. However, since the \( P_a \) do not commute, the operator \( e^{x \cdot P} \) acting on a function \( f(y) \) does not return \( f(x+y) \) since \( e^{x \cdot P} e^{y \cdot P} \neq e^{(x+y) \cdot P} \).

Now let \( \Phi \) be a field valued on the manifold. All covariant fields \( \Phi \) are simple representations of the translational isometries, obeying \( \Phi(x) = e^{x \cdot P} \Phi(0) \). There are three reasonable but inequivalent notions of differentiation, which we denote the normal, left, and right

\[ \text{The index contraction } x \cdot P \text{ should be understood as } x^m \delta_m^a P_a. \]  
We will shortly discover a nontrivial vierbein arising from the torsion, but it does not appear in the translation group element.
differentiation:
\[ \partial_a \Phi(x) \equiv \frac{\partial}{\partial x^a} [e^{x \cdot P} \Phi(0)] \] (A.18)

\[ D^{(L)}_a \Phi(x) \equiv P_a e^{x \cdot P} \Phi(0) \] (A.19)

\[ D^{(R)}_a \Phi(x) \equiv e^{x \cdot P} P_a \Phi(0) \] (A.20)

In each of these definitions, the operation on the left is some sort of derivative on the group translation element \( e^{x \cdot P} \) of the general form
\[ D^{(L)}_a = e^{(L)_a} m (x) \partial_m, \quad D^{(R)}_a = e^{(R)_a} m (x) \partial_m, \] (A.21)

where \( \partial_m \) is to be understood as a derivative on the group parameters \( x^m \) and \( e^{(L)_a} m (x) \) and \( e^{(R)_a} m (x) \) are functions of \( x \) chosen so that the definitions are satisfied. They are found most easily by differentiating with respect to \( x \) and moving all the \( P \)'s to the left or to the right:

\[ \partial_m e^{x \cdot P} = e^{(L)_m} a (x) P_a e^{x \cdot P}, \quad \partial_m e^{x \cdot P} = e^{x \cdot P} e^{(R)_m} a (x) P_a \]

It is interesting to note the group commutation rules of these various derivative operations, which follow directly from their definitions. The normal differentiation has trivial commutations, which follow directly from their definitions. The no rmal differentiation has trivial

\[ [D^{(L)}_a, D^{(L)}_b] \Phi(x) = [P_b, P_a] e^{x \cdot P} \Phi = +C_{ab}^c D^{(L)}_c \Phi(x) \] (A.23)

A similar calculation with the right differentiation operators shows that they preserve the order, and we find

\[ [D^{(R)}_a, D^{(R)}_b] \Phi(x) = -C_{ab}^c D^{(R)}_c \Phi(x) \] (A.24)

The left and right derivatives formally commute with each other since they naturally place their corresponding \( P_a \) generators on opposite sides of the translation group element:

\[ D^{(L)}_a D^{(R)}_b e^{x \cdot P} \Phi = D^{(L)}_a e^{x \cdot P} P_b \Phi = P_a e^{x \cdot P} P_b \Phi = D^{(R)}_b D^{(L)}_a e^{x \cdot P} \Phi \] (A.25)

While each of these is interesting, only the right derivative is translationally covariant:

\[ e^{x \cdot P} D^{(R)}_a \Phi(x_0) = e^{x \cdot P} e^{x_0 \cdot P} P_a \Phi = D^{(R)}_a \Phi(e^{x \cdot P} x_0). \] (A.26)

(It is a straightforward exercise to show that the other derivative operations do not obey this rule unless torsion vanishes.) Therefore we may identify \( D^{(R)}_a \equiv D_a \) as the covariant derivative, and \( e^{(R)_a m} \equiv e^m_a \) as the physical vierbein. It can be easily calculated by noting

\[ e^m_a P_a \equiv e^{-x \cdot P} \partial_m e^{x \cdot P} \]
The result is \[20\]
\[ e_m^a = \delta_m^a - \frac{1}{2} x^b C_{mb}^a + \frac{1}{12} x^b x^c C_{mb}^c C_{dc}^a + \ldots \] (A.27)
where the $C$’s are understood to all possess Lorentz indices. (That is, the only vierbein in the expression is on the left hand side, and so this is an explicit, if unclosed, expression for the vierbein.) The above expansion can be written in a matrix form. Define the function $f(u) = (e^n - 1)/u$; then $e = f(xC)$ where $(xC)_a^b \equiv x^c C_{ca}^b$. It follows that the inverse vierbein can be expanded using the reciprocal:

\[ e_a^m = (1/f(xC))_a^m = \delta_a^m + \frac{1}{2} x^b C_{ab}^m + \frac{1}{12} x^b x^c C_{ab}^c C_{dc}^m + \ldots \] (A.28)

This relation for the vierbein can be shown to obey $\partial_{[a} e_{m]}^a = e_n^c e_m^b C_{cb}^a$ which shows that the torsion $T_{nm}^a$, in this flat case, is given in the Lorentz frame by the coefficients $C_{cb}^a$.

The above formalism is necessary in order to describe global supersymmetry in superspace. Begin with a Grassmann manifold with four bosonic dimensions $x^a$ and four fermionic dimensions $\theta^a$ and $\theta_\bar{a}$. The translation isometries consist of the bosonic translations $P_a$ and the fermionic ones $Q_\alpha$ and $\bar{Q}_{\bar{\alpha}}$, with a torsion term \(\{Q_\alpha, Q_{\bar{\alpha}}\} = -2i\sigma_{\alpha\bar{\alpha}}^a P_a\). The torsion term here is found in the anticommutator of the fermionic $Q$’s. It is useful to think of this anticommutator as just a normal commutator but with fermionic objects; whenever fermionic objects pass through each other, a relative sign is introduced, creating the anticommutator from a commutator.

A superfield $\Phi(x, \theta, \bar{\theta})$ is defined by the action at the origin: $\Phi(x, \theta, \bar{\theta}) = e^{x \cdot P + \theta Q + \bar{\theta} \bar{Q}} \Phi$. Since $P$ commutes with $Q$ and $\bar{Q}$, this can be written as $\Phi(x, \theta, \bar{\theta}) = e^{\theta Q + \bar{\theta} \bar{Q}} \Phi(x)$. If we apply a theta derivative to this superfield, there are two avenues for simplification. One is to move the $Q$ that is brought down all the way to the left, and the other is to move it all the way to the right. These two calculations are straightforward and yield

\[
\partial_\alpha \Phi(x, \theta, \bar{\theta}) = \partial_\alpha e^{\theta Q + \bar{\theta} \bar{Q}} \Phi(x) = (Q_\alpha + i\sigma_{\alpha\bar{\alpha}}^a \bar{\theta}^\bar{\alpha} P_a) e^{\theta Q + \bar{\theta} \bar{Q}} \Phi(x) = \left( D_\alpha^{(L)} + i\sigma_{\alpha\bar{\alpha}}^a \bar{\theta}^\bar{\alpha} P_a \right) \Phi(x, \theta, \bar{\theta})
\]

and

\[
\partial_{\bar{\alpha}} \Phi(x, \theta, \bar{\theta}) = \partial_{\bar{\alpha}} e^{\theta Q + \bar{\theta} \bar{Q}} \Phi(x) = e^{\theta Q + \bar{\theta} \bar{Q}} (Q_\alpha - i\sigma_{\alpha\bar{\alpha}}^a \bar{\theta}^\bar{\alpha} P_a) \Phi(x, \theta, \bar{\theta}) = \left( D_\alpha^{(R)} - i\sigma_{\alpha\bar{\alpha}}^a \theta^\alpha P_a \right) \Phi(x, \theta, \bar{\theta})
\]

From these we see immediately that the various derivatives have the form

\[
\partial_\alpha \equiv \frac{\partial}{\partial \theta^\alpha}, \quad D_\alpha^{(L)} = \partial_\alpha - i\sigma_{\alpha\bar{\alpha}}^m \bar{\theta}^\bar{\alpha} \partial_m, \quad D_\alpha^{(R)} = \partial_\alpha + i\sigma_{\alpha\bar{\alpha}}^m \theta^\alpha \partial_m \quad (A.29)
\]

Note that in the literature [10], it is the right derivative which is $D_\alpha$, the supersymmetry-covariant derivative. The left derivative is often denoted $Q_\alpha$ and represents the supersymmetry isometry (it preserves the form of the vierbein), which is different from the supersymmetry-covariant derivative. We will discuss this further in the general context \[\text{A.2.3}\]

\[20\] This result can be generalized in the presence of local curvatures; see Appendix A.2.4.
A.1.3 General case

Let $G$ consist of the full set of symmetry transformations acting on fields on the manifold and $\mathcal{H}$ denote the subgroup spanned by all the elements aside from translations. In practice, these normally consist of rotational, conformal, and any Yang-Mills transformations.

The intrinsic action of $G = \exp g$ on $\Phi$ is defined by $G\Phi(0)$, its action at the origin. The action of $G$ elsewhere can always be reconstructed using the transformations:

$$G\Phi(x) \equiv Ge^{x \cdot P}\Phi(0) = e^{x \cdot P} \bar{G}(x)\Phi(0)$$

where $\bar{G}(x) \equiv e^{-x \cdot P}Ge^{x \cdot P}$. The product group element $e^{x \cdot P}\bar{G}$ can be rearranged into a part depending on $P$ and an element of $\mathcal{H}$:

$$G\Phi(x) = Ge^{x \cdot P}\Phi(0) = e^{\bar{x} \cdot P}H_G(x)\Phi(0) \tag{A.30}$$

where $H_G(x) \in \mathcal{H}$. All of the translations have been absorbed in a redefinition of $x \rightarrow \bar{x}$. On a pure function $f(x)$ this would give $Gf(x) = f(\bar{x})$, and so $\bar{x}$ can be thought of as the action of $G$ induced on $x$.

The differential version of (A.30) can be compactly written

$$g\Phi(x) = e^{x \cdot P}\bar{g}(x)\Phi(0) = e^{x \cdot P}(\xi^a_g(x)P_a + h_g(x))\Phi(0)$$

where we have separated $\bar{g}(x)$ into a part $\xi_g$ consisting only of translation generators and a part $h_g(x)$ consisting only of generators from $\mathcal{H}$. This formula can be further simplified by noting the first term involves the covariant derivative:

$$g\Phi(x) = \xi^a_g D_a\Phi(x) + e^{x \cdot P}h_g\Phi(0) = \xi^a_ge^a_m \partial_m\Phi(x) + e^{x \cdot P}h_g\Phi(0)$$

The action of $g$ thus induces a shift in the coordinate from $x^m$ to $\bar{x}^m = x^m + \xi^a_g(x)e^a_m(x)$.

A.2 Local (gauged) spacetime symmetries

In the preceding sections we have discussed the construction of representations of spacetime symmetry groups which act on fields. There were several unsatisfying elements to this treatment: we had to choose a preferred point, the origin; there existed two alternative methods of describing the transformations, either as just transforming the fields or transforming the fields and the coordinates; and there was no clear way to generalize to local transformations.

Each of these objections can be answered by proceeding to a local formulation for the manifold. Again let $\Phi(x)$ denote the field $\Phi$ at the point $x$ on the manifold. Let the symmetry group $G$ consist of generators $X_A$. The action of such symmetry transformations on a field $\Phi$ is local; they transform the field into other fields at the same spacetime point. That is, $\delta_X \Phi(x) = g^A(x)X_A\Phi(x)$, where $g^A(x)$ is the position-dependent transformation. Here we view $X_A$ as an operator and the product $X_A\Phi$ as a single object. If instead we view $\Phi$ as a column vector in its appropriate representation, then $X_A\Phi$ can be identified as $t_A\Phi$ where $t_A$ is a matrix appropriate to that representation. The latter objects $t_A$ are what are normally considered in treatments of Yang-Mills. It should be noted that their multiplication rule is backwards from that of the operators. That is, $X_A X_B \Phi = X_A (t_B \Phi) = \ldots$
$t_B X_A \Phi = t_B t_A \Phi$ since the operator $X_A$ passes through the matrix $t_B$. It follows that if the algebra of the operators is

$$[X_A, X_B] = -f_{AB}^C X_C$$

then the algebra of the matrices is $[t_A, t_B] = +f_{AB}^C t_C$.

The generators can be decomposed into the translation generators $P_a$ (more precisely, the generators of parallel transport) and the others $X_a$. The existence of purely scalar, non-constant fields annihilated by $X_a$ implies that the commutator of two such generators cannot give a $P$. In other words, $f_{cb}^a = 0$ by assumption. (Supersymmetry in normal space violates this assumption since two internal symmetries $Q$ anticommute to give a translation $P$. This is one advantage of using superspace instead.)

Associated with each generator is a gauge connection $W_m^A$, which can be similarly decomposed into the vierbein $e_m^a$ and the others $h_m^a$. This decomposition can be written

$$W_m^A X_A = e_m^a P_a + h_m^a X_a$$  \hspace{1cm} (A.31)

The nature of the connection is defined by its action on fields:

$$\Phi(x + dx) = (1 + dx^m W_m^A(x) X_A) \Phi(x)$$  \hspace{1cm} (A.32)

where $\Phi$ is a scalar on the manifold but possibly nontrivial in the tangent space. (That is, it may possess Lorentz indices but no Einstein ones.) This equation is equivalent to

$$\partial_m \Phi(x) = W_m^A X_A \Phi(x) = e_m^a P_a \Phi(x) + h_m^a X_a \Phi(x)$$  \hspace{1cm} (A.33)

which can be read as defining the action of $P_a$ as that of the covariant derivative:

$$e_m^a P_a \Phi(x) = \nabla_m \Phi(x) = (\partial_m - h_m^a X_a) \Phi(x)$$  \hspace{1cm} (A.34)

Since the vierbein is generally invertible, $P_a \Phi(x) = e_a^m \nabla_m \Phi(x) = \nabla_a \Phi(x)$. Since $P_a$ is equivalent to the covariant derivative, the algebra of the $P_a$'s generally develops additional local elements corresponding to the various curvatures associated with the manifold. That is, the statement

$$[\nabla_c, \nabla_b] \Phi = -R_{cb}^A X_A \Phi$$

becomes a property of the algebra itself, $[P_c, P_b] = -R_{cb}^A X_A$. This alteration of the algebra is the only formal consequence when passing from a global to a local theory. In the language of the algebra, $f_{cb}^A = R_{cb}^A$ become structure functions in a local theory and depend on the value of the connections. We will see shortly how this comes about.

Under a gauge transformation, $\partial_m (\delta_g \Phi) = (\delta_g W_m^A) X_A \Phi + W_m^A \delta_g X_A \Phi$, where $X_A \Phi$ is considered a single object, leading to the gauge transformation of the connections,

$$\delta_g W_m^A = \partial_m g^A + W_m^B g^C f_{CB}^A.$$  \hspace{1cm} (A.35)

A finite gauge transformation is found by exponentiating an element of the algebra. That is, for an element $G = \exp(g)$, $\Phi(x) \rightarrow \Phi'(x) = G(x) \Phi(x)$. Here $G$ is understood as a power series expansion in $g = g^A t_A$ where the matrices $t_A$ act only on the fields $\Phi$. The relation (A.33) can also be straightforwardly integrated using a path-ordered exponential in the matrix language:

$$\Phi(x) = \mathcal{P} \exp \left( \int_{x_0}^x W^A t_A \right) \Phi(x_0).$$  \hspace{1cm} (A.36)

$^{22}$ $P_a$ is the operator which was frequently denoted $\Pi_a$ in older literature, the kinematic momentum, as opposed to the canonical momentum.
This equation is strongly reminiscent of a Wilson line, but extended to the full symmetry group of the tangent space. It can be compactly written $\Phi(x) = U(x, x_0)\Phi(x_0)$ where $U(x, x_0)$ is the path-ordered exponential. A derivative yields $\partial_m \Phi(x) = W_m^A X_A \Phi(x) = W_m^A(x_0) \Phi(x)$. Under a gauge transformation,

$$\Phi(x) \rightarrow G(x)\Phi(x), \quad U(x, x_0) \rightarrow U'(x, x_0) = G(x)U(x, x_0)G(x_0)^{-1} \quad (A.37)$$

The integrated rule for the connections can be found by considering $x$ vanishingly near to $x_0$:

$$W(x) \rightarrow W'(x) = -GdG^{-1} + GWG^{-1} \quad (A.38)$$

In order for the relation (A.36) to be path-independent, any path beginning and ending on the same point must vanish, $U(x, x) = 0$. This is equivalent to the condition that the formal gauge curvature $F^A = dW^A - W^B W^C f_{CB}^A$ vanishes. It serves not as a restriction but as a definition of the covariant curvatures $R^A$. An explicit calculation of $F$ using $[P_c, P_b] = -R_{cb}^A X_A$ yields

$$R^A = dW^A - e^b h^c f^A_{cb} = -\frac{1}{2} h^b h^c f^A_{cb} \quad (A.39)$$

as the relation between the covariant curvature (what we normally mean when we say the “curvature”) and the gauge fields. Under a $P$-gauge transformation, the vierbein varies as a covariant Lie derivative:

$$\delta_P(\xi) e^m_a = \partial_m \xi^a + \xi^b R_{bm}^a - \xi^b h_m^c f^a_{cb} = \xi^a \nabla_n e^m_a + \partial_m \xi^n e^a_n \quad (A.40)$$

where $\xi^m \equiv \xi^a e^m_a$. One recovers the normal Lie derivative by making corresponding gauge transformations involving the gauge connections:

$$\mathcal{L}_\xi e^m_a = \left\{ \delta_P(\xi^m e^a_m) + \delta_H(\xi^m h^a_m) \right\} e^m_a = \delta_G(\xi) e^m_a = \xi^n \partial_n e^m_a + \partial_m \xi^n e^a_n \quad (A.41)$$

This rule can be generalized to any function with Einstein indices. Thus a gauge transformation with gauge parameter $\xi^a W^m_a$ is equivalent to a Lie derivative on the field in question. This is precisely the behavior expected of a diffeomorphism.

### A.2.1 Jacobi and Bianchi identities

The generators $X_A$ must obey the Jacobi identity:

$$0 = [X_C, [X_B, X_A]] + [X_A, [X_C, X_B]] + [X_B, [X_A, X_C]] \quad (A.42)$$

Assuming this is obeyed for the global theory, the consequences for the local theory are simple to derive. Only terms involving the curvatures will differ, so only two classes of Jacobi identity must be checked: those with two $P$’s and a generator of $\mathcal{H}$ and those with three $P$’s. Taking

$$0 = [X_{\underline{A}}, [P_c, P_b]] + [P_b, [X_{\underline{A}}, P_c]] + [P_c, [P_b, X_{\underline{A}}]] \quad (A.43)$$

---

23 This is the reverse of the usual approach, where one simply defines the covariant derivative and then calculates the curvatures. The condition $F = 0$ is then nothing more profound than the commuting of the coordinate derivatives.
one finds
\[ X_\mathbf{d} R_{cb}^A = -R_{cb}^F f_{F \mathbf{d} A} - f_{d[c}^f R_{fb]}^A - f_{d[f}^c \Delta R_{fb]}^A \]  
(A.44)

The term involving two \( f \)'s can be eliminated using the global Jacobi identity, giving
\[ X_\mathbf{d} R_{cb}^A = -\Delta R_{cb}^F f_{F \mathbf{d} A} - f_{d[f}^c \Delta R_{fb]}^A \]  
(A.45)

where \( \Delta R^A \) represents the difference between the curvature in the local theory and in the
global theory; in the cases we’ve discussed, the only curvature in the global theory is the
constant torsion tensor \( C \), so \( \Delta R_{cb}^a = R_{cb}^a \), but \( \Delta R_{cb}^a = T_{cb}^a - C_{cb}^a \).

The case of the three \( P \)'s is also interesting. The rules found there correspond to the
Bianchi identities for the covariant derivative. They read
\[ 0 = \sum_{[dcb]} \left( \nabla_d T_{cb}^a + T_{dc}^f T_{fb}^a + R_{dc}^f f_{b a}^a \right) \]  
(A.46)
\[ 0 = \sum_{[dcb]} \left( \nabla_d R_{cb}^a + T_{dc}^f R_{fb}^a + R_{dc}^f f_{b a}^a \right) \]  
(A.47)

A.2.2 Gauge invariant actions over the manifold

An action \( S \) in four dimensions is the integral of a Lagrangian density \( \mathcal{L}(x) \) over the manifold
using the general coordinate invariant measure \( d^4 x e \). The invariance of the action under a
non-translational symmetry \( g^b \) relates the transformation rule of \( \mathcal{L} \) to that of \( e \):
\[ \delta_g S = \int d^4 x e \left( \delta_g \mathcal{L} + \delta_g e_m^a e_a^m \mathcal{L} \right) = \int d^4 x e \left( g^b X_b \mathcal{L} + g^b f_{ba}^a \mathcal{L} \right) \]  
(A.48)

One concludes \( X_b \mathcal{L} = -f_{ba}^a \mathcal{L} \) as a condition for invariance. One can now check invariance
under a translational symmetry \( g^a = \xi^a \), using \( \xi^a \nabla_a = \xi^m \nabla_m \):
\[ \delta_P S = \int d^4 x e \left( e^m b \xi^m \nabla_m e_n^b \mathcal{L} + \partial_m \xi^m \mathcal{L} + \xi^m \nabla_m \mathcal{L} \right) = \int d^4 x \partial_m (\xi^m e \mathcal{L}) = 0 \]  
(A.49)

This is nothing more than the statement that \( \delta_P \) is equivalent to a general coordinate
transformation followed by gauge transformations, under which the action is inert.

A good example of the local approach is again offered by the conformal group in four
dimensions. The non-vanishing part of the conformal algebra is
\[ [M_{ab}, P_c] = P_a \eta_{bc} - P_b \eta_{ac}, \quad [M_{ab}, K_c] = K_a \eta_{bc} - K_b \eta_{ac} \]
\[ [M_{ab}, M_{cd}] = \eta_{be} M_{ad} - \eta_{ec} M_{bd} - \eta_{bd} M_{ac} + \eta_{ad} M_{bc} \]
\[ [D, P_a] = P_a, \quad [D, K_a] = -K_a \]
\[ [K_a, P_b] = 2 \eta_{ab} D - 2 M_{ab} \]  
(A.50)

Coupled to each of these generators is a gauge field,
\[ W_m = e_m^a P_a + \frac{1}{2} \omega_m^{ba} M_{ab} + b_m D + f_m^a K_a \]  
(A.51)

\footnote{This transformation rule can also be derived from the definition of the \( R \)'s in terms of the gauge
connections, but the above is the more straightforward path.}
such that the action of $P_a$ on physical fields is the covariant derivative; the other generators
are defined by their intrinsic behavior:

$$P_a \Phi = \nabla_a \Phi, \quad M_{ab} \Phi = S_{ab} \Phi, \quad D \Phi = \Delta \Phi, \quad K_a \Phi = 0$$  \hspace{1cm} (A.52)

(If $\Phi$ possesses any Einstein indices, we separate them out with the vierbein and treat only
the Lorentz-indexed field as the actual $\Phi$.) The difference between this and the approach
discussed in the global theory is that these are the behaviors of the generators at all points
on the manifold. The algebra of the generators allows one to calculate the transformation
behavior of any covariant derivative of $\Phi$ by using the algebra. For example,

$$D \nabla_a \Phi = DP_a \Phi = (\Delta + 1) \nabla_a \Phi \hspace{1cm} (A.53)$$
$$K_b \nabla_a \Phi = K_b P_a \Phi = (2 \eta_{ba} \Delta - 2 S_{ba}) \Phi \hspace{1cm} (A.54)$$
$$M_{bc} \nabla_a \Phi = M_{bc} P_a \Phi = \left(S_{bc} \delta^d_a + \eta_{a[c} \delta^d_{b]}\right) \nabla_d \Phi \hspace{1cm} (A.55)$$

Each of these generators acts locally with no derivative of its parameter.

The above relations can also be checked using the explicit definition of the covariant
derivative. For that calculation, one would need the transformation of the gauge connec-
tions. For completeness, consider the arbitrary gauge parameter

$$\Lambda^A X_A = \xi^a P_a + \frac{1}{2} \theta^{ba} M_{ab} + \lambda D + \epsilon^a K_a$$  \hspace{1cm} (A.56)

Under a gauge transformation with such a parameter, the gauge connections transform as

$$\delta G(\Lambda) e_m^a = \partial_m \xi^a + \xi^b \omega_{mb}^a + \xi^a b_m + \theta^{ab} e_{mb} - \lambda e_m^a$$  \hspace{1cm} (A.57)
$$\delta G(\Lambda) \omega_{mb}^a = \partial_m \theta^{ba} + \theta^{bc} \omega_{mc}^a - 2 \xi^{[b} f_m^{a]} - 2 \epsilon^{[b} e_{ma]}$$  \hspace{1cm} (A.58)
$$\delta G(\Lambda) b_m = \partial_m \lambda + 2 \xi^a f_{ma} - 2 \epsilon^a e_{ma}$$  \hspace{1cm} (A.59)
$$\delta G(\Lambda) f_m^a = \partial_m \epsilon^a + \epsilon^b \omega_{mb}^a - \epsilon^a b_m + \theta^{ab} f_{mb} + \lambda f_m^a$$  \hspace{1cm} (A.60)

Using these definitions, one can check, for example, that $\delta K(\epsilon) \nabla_a \Phi = (2 \epsilon_a \Delta - 2 \epsilon^b S_{ba}) \Phi$
which agrees with the result from the algebra.

If an action $S$ in conformally invariant, the Lagrangian must obey (using $X_{\frac{1}{2}} \mathcal{L} = - f_{2a} \mathcal{L}$)

$$D \mathcal{L} = 4 \mathcal{L}, \quad M_{ab} \mathcal{L} = 0, \quad K_a \mathcal{L} = 0$$  \hspace{1cm} (A.61)

just as in the global case. Take as an example the standard $\phi^4$ theory. It is interesting
to note that the conventional way of writing the kinetic term, $\nabla_a \phi \nabla_a \phi$, is not actually
inert under the special conformal transformations. Rather, one needs to use the covariant
d’Alembertian ($\nabla^2 \nabla_a$) to give a gauge-invariant action:

$$S = \int d^4 x \left(\frac{1}{2} \phi \nabla^a \nabla_a \phi - a \phi^4\right)$$  \hspace{1cm} (A.62)

It is straightforward to check that this action is inert under all the gauge transformations.
A more interesting question is to ask how the kinetic action differs from the conventional
form. A convenient starting point is the identity

$$\partial_m (ee_a^m \phi \nabla^a \phi) = \nabla_m (ee_a^m \phi \nabla^a \phi) + f_m^b K_b (ee_a^m \phi \nabla^a \phi)$$  \hspace{1cm} (A.63)
which follows since the expression in the parentheses is invariant under every gauge transformation except the special conformal one. The above expression can be easily evaluated to give
\[ \partial_m (c e_a^m \phi \nabla^a \phi) = e \left( \nabla_a (\phi \nabla^a \phi) + T_{ba}^a \phi \nabla^b \phi + 2 f_a^a \phi^2 \right) \] (A.64)

This allows one to integrate the action by parts:
\[ S = \int d^4x \, e \left( \frac{1}{2} \phi \nabla^a \nabla_a \phi - a \phi^4 \right) = \int d^4x \, e \left( -\frac{1}{2} \nabla^a \phi \nabla_a \phi - \frac{1}{2} T_{ba}^a \phi \nabla^b \phi - f_a^a \phi^2 - a \phi^4 \right) \] (A.65)

The trace of the torsion tensor usually vanishes in physically interesting theories, but the term involving the $K$-gauge field $f_a^a$ is physically of interest. In common theories of conformal gravity, it is related to the Ricci tensor and its trace is proportional to the Ricci scalar. In such theories, the Lagrangian above can be gauge fixed to yield the Einstein-Hilbert Lagrangian. (The quartic, if present, would give a cosmological constant.)

### A.2.3 Global representations from local ones

We have discussed two ways of implementing the spacetime symmetry group on the fields. The first involved a selection of a privileged point, the origin, at which we defined the intrinsic behavior of the fields; the behavior elsewhere was then calculated by composing the group element with the translation element. The action of group elements was taken not only on the fields but also on the translation element, leading to non-trivial transformation rules for the fields away from the origin. The second way involved defining gauge connection 1-forms everywhere; no privileged point was needed, nor was there any discussion of moving points on the manifold. The advantage of this latter formulation was that it was trivial to implement local group transformations. The global structure should be represented by the local one when restricted to global gauge transformations.

Begin with a vanishing $\mathcal{H}$-connection and a $P$-connection as defined in (A.27) relative to some origin point $0$. Construct a gauge transformation $\tilde{g}(x)$ which takes the value $g$ at the origin but elsewhere is such as to keep the connections invariant. That is, $\tilde{g}(x)$ obeys
\[ 0 = \delta \tilde{g} W_m^A = \partial_m \tilde{g}^A + e_m^b \tilde{g}^C f_{cb}^A \] (A.66)

This equation can be integrated to give $\tilde{g}(x) = e^{-x \cdot P} ge^{x \cdot P}$ where $x \cdot P \equiv x^m \delta_m^a P_a$. To prove this is correct, recall that to first order in $\xi$, $1 + \xi^m e_m^a P_a = e^{-x \cdot P} e^{(x + \xi) \cdot P}$. It follows then that
\[
-\xi^m e_m^b \tilde{g}^C f_{cb}^A = [\tilde{g}, \xi^m e_m^b P_b] = e^{-x \cdot P} ge^{(x + \xi) \cdot P} - e^{-x \cdot P} e^{(x + \xi) \cdot P} e^{-x \cdot P} ge^{x \cdot P} = e^{-x \cdot P} ge^{(x + \xi) \cdot P} + e^{- (x + \xi) \cdot P} ge^{x \cdot P} - 2 = \xi^m \partial_m \tilde{g}^A
\]

where the last two equalities hold only to first order in $\xi$. This gauge transformation, $\tilde{g}(x)$, is the transformation discussed in the global approach.

The general form of the locally invariant action $S = \int d^4x \, e \, \mathcal{L}$ obeying $X_{\mathcal{L}} = -f_{\mathcal{L}}^a \mathcal{L}$ implies that the globally invariant form must also have that form. In particular the global measure must be $d^4x \, e$ where $e$ is nontrivial in the case of a sufficiently complicated (but constant) torsion. (This is not normally an issue since even global supersymmetry has $E = 1$.) To prove this requirement, consider the global action $S = \int d^4x \, e \, \mathcal{L}$. Under
a global gauge transformation \( \tilde{g} \), the measure is invariant and the Lagrangian changes as
\[
\delta L = \tilde{g}^b X_b L + \tilde{g}^b P_b L.
\]
We can first replace \( \tilde{g}^b X_b \rightarrow -\tilde{g}^b f_b^a \) and then equate that quantity to
\[
e_a^m \partial_m \tilde{g}^a + \tilde{g}^b f_b^a
\]
using the differential equation for \( \tilde{g} \). Finally note that \( \tilde{g}^b P_b L = \tilde{g}^b e_b^m \partial_m L \) and we find
\[
\delta L = e_a^m (\partial_m \tilde{g}^a) L + \tilde{g}^b f_b^a L + \tilde{g}^b e_b^m \partial_m L = e_b^m \partial_m (\tilde{g}^b L) + \tilde{g}^b f_b^a L
\]
(A.67)

Here by \( f_b^a \) we mean the trace of the torsion tensor, equivalently written \( C_b^a \) or \( T_b^a \) (these are identical in the global theory). The first term can be integrated by parts (if the measure is \( e \)) to cancel the second, rendering the action invariant.

The \( \tilde{g} \)'s discussed here represent the isometries of the flat space – the transformations which leave invariant the form of the connections. Of particular interest is the case where \( g = g^a P_a \). There we find that \( \tilde{g} = \tilde{g}^a P_a \) (no \( H \) bits are generated since the commutator of two \( P \)'s is another \( P \) in the flat, ungauged space), with the interesting property that \( \tilde{g}^a \) preserves the form of the vierbein. These are precisely the translation isometries of the space; that is, they are the diffeomorphisms which preserve the vierbein. We may write them as a coordinate transformation:
\[
x^m \rightarrow x^m + \tilde{g}^a e_a^m, \quad e^a \rightarrow e^a, \quad D_a \rightarrow D_a
\]
(A.68)

Recall that the vierbein used here was the one associated with right differentiation. The action of left differentiation was an isometry which preserved the form of the vierbein 1-form \( e^a \) and the right derivative operator \( D_a \). We have recovered this isometry above; it represents the general form of the translation isometry of a flat space with torsion.

**A.2.4 Normal gauge**

In general relativity, there exists a preferred gauge for the metric, the choice of Riemann normal coordinates, which expands the metric in terms of the curvature and derivatives thereof. Similarly in Yang-Mills theories, there exists a preferred gauge, the Fock-Schwinger gauge, which gives the gauge connection in terms of the gauge curvature and derivatives thereof. It is possible to generalize both of these conditions to the sort of theory discussed here.

Recall that a field at a point \( x \) is related to the field at a fixed point \( x_0 \) by a Taylor expansion:
\[
\phi(x) = \exp \left( (x - x_0) \cdot \partial \right) \phi(x_0)
\]
\[
= \phi(x_0) + (x - x_0)^m \partial_m \phi(x_0) + \frac{1}{2} (x - x_0)^m (x - x_0)^n \partial_m \partial_n \phi(x_0) + \ldots
\]
(A.69)

On the other hand, the parallel transport of the field from \( x_0 \) with parameter \( y \) is
\[
\phi(x_0; y) = \exp (y^a P_a) \phi(x_0)
\]
\[
= \phi(x_0) + y^a \nabla_a \phi(x_0) + \frac{1}{2} y^a y^b \nabla_a \nabla_b \phi(x_0) + \ldots
\]
(A.70)

One can choose a gauge such that these coincide for \( x = y + x_0 \) for scalar fields; this generalizes Riemann normal coordinates for non-Riemannian geometries (for example, those with torsion). The further choice that these should coincide for all fields leads to a generalization of Fock-Schwinger gauge.
In principle, one can equate these series term-by-term to determine the gauge fields. A slightly simpler method is to note that $e_a^m \partial_m \phi - h_a^b X_b \phi$ is the covariant derivative; therefore one may equate

$$e_a^m(y) \frac{\partial}{\partial y^m} \exp(y^a P_a) \phi(x_0) - h_a^b(y) \exp(y^a P_a) X_b \phi(x_0) = \exp(y^a P_a) P_a \phi(x_0) \quad (A.71)$$

This can be rearranged to

$$\frac{\partial}{\partial y^m} e^{y^a P_a} \phi(x_0) = e_m^a e^{y^a P_a} \phi(x_0) + h_m^b(y) e^{y^a P_a} X_b \phi(x_0)
= e^{y^a P_a} \tilde{e}_m^a \phi(x_0) + e^{y^a P_a} \tilde{h}_m^b(y) X_b \phi(x_0) \quad (A.72)$$

where we have defined $\tilde{e}_m^a$ and $\tilde{h}_m^b$ by conjugation with $e^{y^a P_a}$. Multiplying by an overall factor gives

$$e^{-y^a P_a} \frac{\partial}{\partial y^m} e^{y^a P_a} \phi(x_0) = \tilde{e}_m^a P_a \phi(x_0) + \tilde{h}_m^b(y) X_b \phi(x_0) \quad (A.73)$$

The term on the left can be straightforwardly evaluated term by term:

$$e^{-y^a P_a} \frac{\partial}{\partial y^m} e^{y^a P_a} = \partial_m + P_m + \frac{1}{2} [P_m, y^a P_a] + \frac{1}{3!} L_{y^a P_a}^2 P_m - \frac{1}{4!} L_{y^a P_a}^3 P_m + \ldots$$

$$= \partial_m + P_m + \sum_{j=1}^{\infty} \frac{(-1)^j}{(j+1)!} Q_m(j) \quad (A.74)$$

where $L_{y^a P_a} f = [y^a P_a, f] = y^a [P_a, f]$ and $Q_m(j) = L_{y^a P_a}^j P_m$. In this expansion the $y^a$ are to be treated as group parameters, inert under the action of the generators, and the explicit derivative $\partial_m$ is with respect to the $y$ only. One may formally solve for the gauge fields by defining

$$\tilde{e}_m^a = \delta_m^a + \sum_{j=1}^{\infty} \frac{(-1)^j}{(j+1)!} Q_m^a(j)$$

$$\tilde{h}_m^b = \sum_{j=1}^{\infty} \frac{(-1)^j}{(j+1)!} Q_m^b(j) \quad (A.75)$$

where we have expanded $Q_m = Q_m^a P_a + Q_m^b X_b$. Then conjugating by the group element $\exp(y \cdot P)$ generates the actual gauge fields:

$$e_m^a = \delta_m^a + \sum_{j=1}^{\infty} \frac{(-1)^j}{(j+1)!} \sum_{k=0}^{\infty} \frac{1}{k!} L_{y^a P_a}^k Q_m^a(j)$$

$$h_m^b = \sum_{j=1}^{\infty} \frac{(-1)^j}{(j+1)!} \sum_{k=0}^{\infty} \frac{1}{k!} L_{y^a P_a}^k Q_m^b(j) \quad (A.76)$$

Note that the conjugation generates covariant derivatives of the listed terms; for example, $L_{y^a P_a} Q_m^a(j) = y^b \nabla_b Q_m^a(j)$.

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25In the case where there are no curvatures except for constant torsion, the above reduce to $h_m^b = 0$ and $e_m^a$ given by (A.27). Normal gauge is the appropriate generalization.
A.2.5 Gauge invariant actions over submanifolds

In the case of global supersymmetry, we know that it is natural to consider not only integrals over the entire superspace of coordinates \((x, \theta, \bar{\theta})\) but also integrals over a chiral superspace of coordinates \((y, \theta)\) where \(y = x + i\theta \sigma \bar{\theta}\). It is natural to think of the chiral superspace as lying on a submanifold characterized by a constant value of \(\bar{\theta}\). Then change in coordinates from \(x\) to \(y\) is naturally understood, since in those coordinates \(D \bar{\alpha} = \partial \bar{\alpha}\) and so chiral superfields (those annihilated by \(D \bar{\alpha}\)) naturally live on such a submanifold.

Let us take this point of view seriously and derive some useful results about actions on submanifolds. We will assume that the space under consideration is purely bosonic so that our geometric intuition can be trusted. Let the full manifold \(M\) be \(D\)-dimensional on which we may define the parallel transport operators \(P_A\), where \(A = 1, \ldots, D\). Let \(P\) be decomposed as \(P_A = (P_a, P_{\bar{a}})\) where \(a = 1, \ldots, D\) and \(\bar{a} = \mathcal{D} + 1, \ldots, D\). We will use Gothic indices \(a\) to denote the submanifold tangent space indices. Our object of interest is a submanifold \(\mathcal{M}\) of dimension \(\mathcal{D}\) defined so that \(P_{\bar{a}}\) annihilates the functions naturally integrated over \(\mathcal{M}\).

This can be made more concrete by choosing coordinates \(z^M = (z^m, \bar{\theta}^\mu)\) so that \(\mathcal{M}\) is parametrized by \(z^m\) with constant \(\bar{\theta}\); we will assume \(\bar{\theta} = 0\) for definiteness, but any constant will do. In this way the coordinates on \(M\) can be related nicely to the coordinates on \(\mathcal{M}\). Then the condition that \(P_{\bar{a}}\) annihilates the natural integrands on \(\mathcal{M}\) means \(P_{\bar{a}} = \partial / \partial \bar{\theta}^a\) when acting on pure functions, or, equivalently, that \(\mathcal{M}\) lies at a constant slice of \(\bar{\theta}\). This choice of coordinates has the benefit of simplifying calculations while unfortunately forcing a breakdown in manifest general coordinate invariance on \(M\); equivalently, this forces one to choose a certain \(P\)-gauge. We will therefore avoid making this explicit assumption until it is absolutely necessary.

Recall that an invariant integral on the whole manifold \(M\) is

\[
S = \int_M E^1 \wedge E^2 \wedge \ldots \wedge E^D \ V = \int d^Dz \ E \ V
\]

where \(E = \det(E_M^A)\) and \(V\) is an appropriate integrand to make the action gauge invariant. We have already shown that invariance under the non-translation symmetries \(\mathcal{H}\) requires \(\delta_g V = -g^{ba} f_{bA} A\), while invariance under \(P\) follows from general coordinate invariance. An invariant integral over \(\mathcal{M}\) can be very similarly defined:

\[
\mathcal{S} = \int_{\mathcal{M}} E^1 \wedge E^2 \wedge \ldots \wedge E^\mathcal{D} \ W = \int_{\mathcal{M}} \mathcal{E}^1 \wedge \mathcal{E}^2 \wedge \ldots \wedge \mathcal{E}^\mathcal{D} \ W = \int d^\mathcal{D}z \mathcal{E} \ W, \tag{A.78}
\]

where \(\mathcal{E} = \det(\mathcal{E}_m^a)\) is the volume measure and \(W\) is an appropriate integrand. The subvierbein form \(\mathcal{E}_m^a\) is taken to be identical to \(E_a^m\) when restricted to the manifold \(\mathcal{M}\). Invariance of this integral under the action of \(\mathcal{H}\) requires \(\delta_g W = -g^{ba} f_{bA}^a W\). (Note the trace of the structure constant is over the submanifold’s Lorentz indices.) However, since the integral is over a submanifold, it is not obviously taken into itself under \(P\)-gauge transformations.

We check first the requirement of \(P_{\bar{a}}\) invariance, which means essentially that such actions should not depend on the constant value of \(\bar{\theta}\) used to define \(\mathcal{M}\). The action varies as

\[
0 = \delta_{\mathcal{E}} \mathcal{S} = \int d^\mathcal{D}z \mathcal{E} \left( \xi^\bar{a} T_{a\bar{b}m} \mathcal{E}_b^m W + \xi^\bar{a} \nabla_{\bar{a}} W \right). \tag{A.79}
\]

\(^{26}\)Of course \(\bar{\theta}\) here is to be understood as a bosonic coordinate at the moment.

\(^{27}\)In the special coordinates where \(\mathcal{M}\) corresponds to \(\bar{\theta} = 0\), the vierbein obeys \(E_{\mu}^a|_{\mathcal{M}} = 0\). This condition is equivalent to the conditions \(\mathcal{E}_a^m = E_a^m|_{\mathcal{M}} = d\bar{\theta}^m \mathcal{E}_m^a\).
(The term $\mathcal{E}_b^m$ represents the inverse of the subvierbein. It does not necessarily correspond to $E_b^m$, since the inverse of a submatrix is not necessarily the submatrix of the inverse unless certain requirements are placed on the coordinates $z$ being used for the submanifold, or equivalently, the gauge choice for the vierbein.) Each term should vanish separately. Requiring the second term to vanish enforces the covariant constancy of $W$ in the direction of $P_\alpha$. Requiring consistency of $\nabla_\alpha W = 0$ with the algebra gives several additional constraints:

$$0 = [\nabla_\alpha, \nabla_\beta]W = -T_{\alpha\beta}^c \nabla_c W + R_{\alpha\beta}^c f_{\xi^0}^b W \tag{A.80}$$

$$0 = [X_\alpha, \nabla_\beta]W = -f_{\alpha\beta}^c \nabla_c W + f_{\alpha\beta}^c f_{\xi^0}^b W \tag{A.81}$$

(The second commutator vanishes since $\nabla_\beta X_\alpha W = -\nabla_\alpha f_{\xi^0}^b W = 0$.) From this simple result we learn $T_{\alpha\beta}^c = f_{\alpha\beta}^c = 0$ as well as $R_{\alpha\beta}^c f_{\xi_0}^b = f_{\alpha\beta}^c f_{\xi_0}^b = 0$. The other term in the variation of the subaction gives two new terms which must vanish:

$$T_{\alpha m}^b \mathcal{E}_b^m = T_{\alpha\gamma}^b E_{\gamma}^m \mathcal{E}_b^m + T_{\alpha b}^b$$

The first of these, $T_{\alpha\gamma}^b E_{\gamma}^m \mathcal{E}_b^m = 0$, is already a condition for the existence of a covariantly constant $W$. The second, $T_{\alpha b}^b = 0$, amounts to an additional constraint on the space.\footnote{These constraints are stricter than necessary. One could choose that $\nabla_\alpha W = -T_{\alpha m}^b \mathcal{E}_b^m W$, as opposed to requiring each term to separately vanish. We have chosen to separate them in the way we have since it makes sense that the conditions we want should be simple conditions on $W$, like chirality, and simple conditions on the geometry, like vanishing of certain torsions, as opposed to something more complicated relating the two.}

Next we check $P_\alpha$ invariance of the subaction. One finds

$$0 = \delta \xi \mathcal{S} = \int d^2 z \mathcal{E} \left( \nabla_m \xi^a \mathcal{E}_a^m W + \xi^a T_{mn}^m \mathcal{E}_b^m W + \xi^a \nabla_\alpha W \right). \tag{A.82}$$

Integrating the first term by parts gives

$$0 = \delta \xi \mathcal{S} = \int d^2 z \mathcal{E} \left( -\xi^a \mathcal{E}_a^m \nabla_m W + \xi^a \nabla_\alpha W - \xi^a \mathcal{E}_a^m T_{mn}^m \mathcal{E}_b^m W + \xi^a T_{mn}^m \mathcal{E}_b^m W \right). \tag{A.83}$$

Invariance holds under the same set of conditions. For example, $\mathcal{E}_a^m \nabla_m W = \mathcal{E}_a^m E_{\gamma}^B \nabla_B W = \mathcal{E}_a^m E_{\gamma}^b \nabla_b W = \nabla_\alpha W$ since $W$ is covariantly constant with respect to $P_\alpha$ and $E_{\gamma}^b$ is equivalent to $\mathcal{E}_m^b$. A similar argument demonstrates the cancellation of the torsion terms.

The constraints we have found are:

$$T_{\alpha \beta}^c = 0, \quad f_{\alpha \beta}^c = 0$$

$$R_{\alpha \beta}^c f_{\xi_0}^b = 0, \quad f_{\alpha \beta}^c f_{\xi_0}^b = 0$$

$$T_{\alpha b}^b = 0$$

The next question to consider is whether integrals over a manifold $M$ can be related to integrals over the submanifold $\mathfrak{M}$, and vice-versa. We will deal with $M \rightarrow \mathfrak{M}$ first and then consider the reverse.

**Case 1: $M \rightarrow \mathfrak{M}$**

Consider the integration of a function $V$ over the whole manifold: $\int_M d^2 z E V$. We would like to decompose it into an integral of some other function $W$ over the submanifold $\mathfrak{M}$. The most straightforward way to do this is to adopt the coordinates
(equivalently, choose the $P$-gauge) so that $z^M = (z^m, \tilde{\theta}^\mu)$ and $\mathcal{M}$ corresponds to $\tilde{\theta} = 0$. Note that it is rather trivial to choose $E_\mu^a|_{\mathcal{M}} = 0$; it can be shown that the conditions we derived for the invariance of the subactions over $\mathcal{M}$ allow us to extend this condition over all of $M$.

We then can assume a gauge choice where $E_\mu^a|_{\mathcal{M}} = 0$ everywhere, as well as the additional requirements $h_\mu^\alpha f_{\alpha a} = 0$. These two conditions mean that $\nabla_\alpha W = 0$ is equivalent to $\partial_\mu W = 0$. Given these, one may easily show that $E$ is itself independent of $\tilde{\theta}$:

$$\partial_\mu E = \partial_\mu E_n \varepsilon_a^n = \nabla_\mu E_n \varepsilon_a^n = T_\mu \varepsilon_a^n = 0$$

This is important since the gauge choice for the vierbein implies $E = E \tilde{\Sigma}$, where $\tilde{\Sigma} \equiv \det(E^{\mu \alpha})$. Then $E$ separates into a part $(E)$ independent of $\tilde{\theta}$ and another $(\Sigma)$ which is an appropriate density in $\tilde{\theta}$.

Under these assumptions, we find

$$\int_M d^D z E V = \int_{\mathcal{M}} d^D \tilde{\theta} \tilde{\Sigma} V$$

where

$$\mathcal{P}[V] \equiv \int d^d \tilde{\theta} \tilde{\Sigma} V.$$  

Note that $\mathcal{P}[V]$ is covariantly constant with respect to $P_\alpha$ for a quite trivial reason: by construction, $\mathcal{P}[V]$ is independent of $\tilde{\theta}$ and so $\partial_\mu \mathcal{P}[V] = 0$ in a gauge where $\partial_\mu = \nabla_\alpha$.

This operation can be extended to any gauge by first evaluating it in the special gauge used here and then transforming to the desired gauge using $\delta g \mathcal{P}[V] = -g^{b f b} \varepsilon_a^n \mathcal{P}[V]$.

**Case 2: $\mathcal{M} \to M$**

In principle an integral over a submanifold $\mathcal{M}$ can be defined by an integral over the whole manifold $M$ using an appropriate delta function $\Delta$. Then

$$\int_{\mathcal{M}} d^D \varepsilon W = \int_M d^D z E W \Delta$$

That both sides remain gauge invariant under $\mathcal{H}$ implies $\delta g \Delta = -g^{b f b} \varepsilon_a^n \Delta$. The simplest way to describe the constraints is to choose the coordinates $z$ to decompose as $z^M = (z^m, \tilde{\theta}^\mu)$ where the submanifold $\mathcal{M}$ lives at $\tilde{\theta}^\mu = 0$. In this special gauge, $\Delta$ takes the simple form

$$\Delta = \frac{\delta^d(\tilde{\theta})}{\tilde{\Sigma}}.$$  

This is not the only such $\Delta$ that will work; an entire family is permissible, of the form

$$\Delta = \frac{X}{\mathcal{P}[X]}.$$  

The choice $X = \delta^d(\tilde{\theta})$ reproduces the simplest example. If, however, $\mathcal{P}[1]$ is a simple enough object, the choice $X = 1$ becomes extremely attractive.

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29 The construction will be given when needed for the explicit case of $\mathcal{N} = 1$ superspace.

30 The above construction applies very nicely to Poincaré supergravity, where if one chooses $X = 1$, one finds $\Delta = 1/2R$. 

56
That both of these results should hold implies

\[ \int_M d^D z \ E \ V = \int_{\mathfrak{g}} d^D \theta \ \mathcal{E}[V] = \int_M d^D z \ E \mathcal{P}[V] \Delta \quad (A.90) \]

Since \( \Delta \) can be placed in the form \( X/\mathcal{P}[X] \), the equivalence of the first and third forms implies \( \mathcal{P} \) is a self-adjoint operation under the full integration.

While it is self-adjoint, \( \mathcal{P} \) is not actually a projector, as it is not idempotent (that is, \( \mathcal{P}^2 \neq \mathcal{P} \)). The true projector (in the special gauge) is \( \Pi \), which is defined by

\[ \Pi[V] \equiv \int \! d^d \theta \Sigma V \Delta. \quad (A.91) \]

This formula is a very complicated way of saying a simple thing: \( \Pi[V] \) is formally identical (in this gauge) to \( V|_{\bar{\theta} = 0} \) provided we use the simplest \( \Delta \). The advantage of the more cumbersome form \( X/\mathcal{P}[X] \) is that it can be extended to any other gauge since the gauge transformation properties of the various objects are well-defined.\(^{31}\)

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\(^{31}\)Applying this to the case of Poincaré supergravity, one finds \( \mathcal{P} = -\frac{1}{4}(D^2 - 8R) \) and \( \Pi = -\frac{1}{8\pi}(D^2 - 8R) \).
**B  Implicit grading**

We make use of the convention of [10] with respect to superspace indices and their contractions. Furthermore, we adopt an implicit grading scheme to avoid cumbersome notation. In any formula involving capital Roman superindices \((A, B, C, \ldots)\), an order is set by the uncontracted indices of the first term; all other terms, if not in the order given, must be supplemented with a grading to flip the indices to the appropriate order. In addition, all index contractions are to be done high to low between adjacent indices; any other configuration of indices must be swapped into this configuration.

A few examples help a good deal. First the commutator:

\[
[\nabla_B, \nabla_A] = -R_{BA}
\]

Expanding this out gives

\[
\nabla_B \nabla_A - \nabla_A \nabla_B = -R_{BA}
\]

The first term sets the order to be \(B\) then \(A\); the second term has the wrong order and so a grading must be inserted. The final result is

\[
\nabla_B \nabla_A - (-)^{AB} \nabla_A \nabla_B = -R_{BA}
\]

The commutator is really an anticommutator if both \(A\) and \(B\) are fermionic.

Next, a more involved example:

\[
V_C^B \nabla_B W_A + V_C^B \nabla_A W_B = F_{AB}^{BD} G_{CD}
\]

The first term sets the order: \(C\) then \(A\). The \(B\) contraction is properly done, so no grading is necessary for the first term. The second term has \(C\) and \(A\) in the correct order, but the \(B\) contraction is done through the \(A\). One must swap the \(A\) with either \(B\) to get an adjacent contraction, giving a grading \((-)^{AB}\). The third term on the right side has the \(B\) contraction done in the wrong order. This requires we place a grading of \((-)^B\). In addition, the \(D\) contraction is done through the index \(C\), giving a grading of \((-)^{CD}\). Finally, the overall order of indices is \(A\) then \(C\); swapping them to the correct order gives a grading \((-)^{AC}\). The final result with the gradings restored is

\[
V_C^B \nabla_B W_A + (-)^{AB} V_C^B \nabla_A W_B = (-)^{B+CD+AC} F_{AB}^{BD} G_{CD}.
\]

Now suppose \(G\) were a two-form. Then the form indices \(CD\) can be swapped at the cost of a sign if they are not both fermionic; this gives

\[
V_C^B \nabla_B W_A + (-)^{AB} V_C^B \nabla_A W_B = -(-)^{B+AC} F_{AB}^{BD} G_{DC}.
\]

We would have compactly written this without the gradings as

\[
V_C^B \nabla_B W_A + V_C^B \nabla_A W_B = -F_{AB}^{BD} G_{DC}.
\]

which is equal to the first equation, provided we take \(G_{DC} = -G_{CD}\) which is true modulo the grading.

The advantage of this notational method is that in any calculation involving superindices, one may naively treat them as if they were all regular bosonic indices. Then when one wishes to actually insert the components, the gradings can be added on the fly subject to the rules we have given.
C  Global superconformal transformations

In the literature on the conformal group, the generators on the fields in the global approach are given at an arbitrary point \( x \). For example, \( D \) is defined as \( \Delta + x \cdot \partial \). (See for example [12].) For completeness, we present the global superconformal generators in the same global picture.

The action of a generator \( g \) on a field \( \Phi \) may be defined at the origin. One takes the defining relations for a primary superfield \( \Phi \) as

\[
P_a \Phi(0) = \partial_a \Phi(0), \quad Q_\alpha \Phi(0) = D_\alpha \Phi(0), \quad \bar{Q}^{\dot{\alpha}} \Phi(0) = \bar{D}^{\dot{\alpha}} \Phi(0)
\]

\[
M_{ab} \Phi(0) = S_{ab} \Phi(0), \quad D \Phi(0) = \Delta \Phi(0), \quad A \Phi(0) = i \omega \Phi(0)
\]

\[
K_a \Phi(0) = 0, \quad S_a \Phi(0) = 0, \quad S^{\dot{\alpha}} \Phi(0) = 0 \quad (C.1)
\]

The action of the supersymmetry translation generators \( Q_\alpha \) at the origin are formally defined to be the same as \( D_\alpha \). This is certainly allowed by the discussion in Wess and Bagger since both are equivalent to \( \partial_\alpha \) there; however, it will soon be apparent that the intrinsic action of \( Q_\alpha \) on a field \( \Phi \) at the origin.

In order to find the action of \( g \) elsewhere, conjugation by the translation operator is necessary. That is, in order to calculate \( g \Phi(z) \), one must commute \( g \) past the translation element, \( g \Phi(z) = ge^{zP} \Phi(0) = e^{zP} g(z) \Phi(0) \) where \( \hat{g}(z) = e^{-zP} g(z) e^{zP} \), and the elements in the expansion of \( g' \) are to be taken to act on \( \Phi \) at the origin. One may calculate the effect of conjugation by the translation operator on each of the generators:

\[
\tilde{P}_a(z) = P_a
\]

\[
\tilde{Q}_\alpha(z) = Q_\alpha - 2i(\sigma^a \bar{\theta})_a P_a
\]

\[
\tilde{Q}^{\dot{\alpha}}(z) = \bar{Q}^{\dot{\alpha}} - 2i(\bar{\sigma}^a \theta)_{a} \bar{P}_a
\]

\[
\tilde{D}(z) = D + x^a P_a + \frac{1}{2} \theta Q + \frac{1}{2} \bar{\theta} \bar{Q}
\]

\[
\tilde{A}(z) = A - i \theta Q + i \bar{\theta} \bar{Q} - 2(\theta \sigma^a \bar{\theta}) P_a
\]

\[
\tilde{M}_{ab}(z) = M_{ab} - x_{[a} P_{b]} + (\theta \sigma_{ab} Q) + (\theta \bar{\sigma}_{ab} \bar{Q}) + \epsilon_{abcd} (\theta \sigma_{d} \bar{\theta})
\]

\[
\tilde{K}_a(z) = K_a + 2x_a D - 2x_b M_{ab} + i(\theta \sigma_a \bar{S}) + i(\bar{\theta} \sigma_a S) + 2x_a x_b P_b - x^2 P_a
\]

\[
\tilde{S}_\alpha(z) = S_\alpha + ix_a (\sigma_a \bar{Q})_{\alpha} - 2i \theta_b D + 3i \partial_a A + 2(2 \sigma^b \theta)_a M_{ab}
\]

\[
\tilde{S}^{\dot{\alpha}}(z) = S^{\dot{\alpha}} + ix_a (\bar{\sigma}_a \theta)_{\dot{\alpha}} - 2\bar{D}^{\dot{\alpha}} D - 3i \bar{D}^{\dot{\alpha}} A + 2(2 \sigma^b \theta)_{\dot{\alpha}} M_{ab}
\]

where \( \zeta^a \equiv \theta \sigma^a \bar{\theta} \).

The first set of definitions imply

\[
P_a \Phi(z) = \partial_a \Phi(z), \quad Q_\alpha \Phi(z) = D_\alpha \Phi(z) - 2i(\sigma^a \bar{\theta})_a \partial_a \Phi(z), \quad \bar{Q}^{\dot{\alpha}} \Phi(z) = \bar{D}^{\dot{\alpha}} \Phi(z) - 2i(\bar{\sigma}^a \theta)_{a} \partial_a \Phi(z)
\]

which is consistent with the standard definitions in the literature.
D Solution to the Bianchi identities

D.1 General solution to gauge constraints

The constraints chosen for conformal supergravity include a set of constraints we shall call the “gauge” constraints for their similarity to the constraints imposed on internal gauge theories in superspace:

\[ \{\nabla_\alpha, \nabla_\beta\} = \{\nabla_\dot{\alpha}, \nabla_\dot{\beta}\} = 0 \]

\[ \{\nabla_\alpha, \nabla_\dot{\alpha}\} = -2i\nabla_{\alpha\dot{\alpha}} \]

where \( \nabla_A \equiv E_A^M (\partial_M - h_{M}{}^{b}X_b) \) is the covariant derivative. Here \( X_b \) is any non-translation symmetry generator; for the conformal group it consists of scalings \( D \), chiral rotations \( A \), Lorentz rotations \( M_{ab} \), and the special conformal transformations \( K_C \). In principle, it may also include any internal symmetries (eg. Yang-Mills), but we will not be explicitly concerned with those here. Since they commute with the conformal group, it is quite easier to add these symmetries later when needed.

The gauge constraints enforce relationships between the various fermionic connections. One could attempt to solve these constraints in terms of prepotentials and then give all the connections and curvatures in terms of these prepotentials. In the case of internal symmetries, this is quite straightforward to do; one finds the prepotentials take the form of adjoint Hermitian superfields \( V = V^r X_r \), where \( X_r \) is the internal symmetry generator. These in turn possess a gauge invariance of the form \( V \rightarrow V + \Lambda + \bar{\Lambda} \) for chiral superfields \( \Lambda \). When the symmetry group fails to commute with translations, this approach is more difficult (though not impossible). Moreover, in practice one is only concerned with calculating the curvatures themselves. It turns out the simpler procedure is usually to derive the constraints the curvatures obey and to solve the curvatures in terms of some unconstrained superfields. In this latter procedure, one finds chiral gaugino superfields \( \mathcal{W} = \mathcal{W}^r X_r \) whose lowest components are the gauginos and which transform homogeneously under the gauge transformation. (These, of course, can be written in terms of the gauge prepotentials, but this is usually not necessary to do.) It is this latter procedure which we will follow here.

The starting point to deriving constraints on the curvatures is the Bianchi identity

\[ 0 = \sum_{[ABC]} [\nabla_A, [\nabla_B, \nabla_C]] \]

where the sum is over (graded) cyclic permutations of the indices. Both the permutation and the commutator carry an implicit grading which gives an extra sign whenever two fermionic indices are pushed past each other. We shall examine each case in turn, in a treatment roughly analogous to that of [10].

The case of \( \alpha\beta\gamma \) is trivial. All terms in the sum vanish.

The second case is \( \alpha\beta\dot{\gamma} \). The Bianchi identity reads

\[ 0 = [\nabla_\alpha, \{\nabla_\beta, \nabla_\dot{\gamma}\}] + [\nabla_\dot{\gamma}, \{\nabla_\alpha, \nabla_\beta\}] + [\nabla_\beta, \{\nabla_\dot{\gamma}, \nabla_\alpha\}] \]

\[ = -2i[\nabla_\alpha, \nabla_\beta]\dot{\gamma} + 0 - 2i[\nabla_\beta, \nabla_\alpha]\dot{\gamma} \]

\[ = +2iR_{\alpha(\beta\dot{\gamma})} + 2iR_{\beta(\alpha\dot{\gamma})} \]

This implies the curvature is antisymmetric in the undotted indices. We therefore may define the “gaugino” superfield \( \mathcal{W} \) by

\[ R_{\alpha(\beta\dot{\gamma})} = 2i\epsilon_{\alpha\beta} \mathcal{W}_\dot{\gamma}, \quad R_{\dot{\alpha}(\dot{\beta}\gamma)} = 2i\epsilon_{\dot{\alpha}\dot{\beta}} \mathcal{W}_\gamma \]

(D.1)
We have included the analogous formulae for the complex conjugate. Note that $W^\dagger_\beta = -W^{\dot{\beta}}$ under this definition.

The third case of interest is $\alpha\beta c$. One finds

$$0 = \{\nabla_\alpha, \{\nabla_\beta, \nabla_c\} \} + \{\nabla_c, \{\nabla_\alpha, \nabla_\beta\}\} - \{\nabla_\beta, \{\nabla_c, \nabla_\alpha\}\}$$

$$= -\{\nabla_\alpha, R_{\beta c}\} + 0 - \{\nabla_\beta, R_{\alpha c}\}$$

Writing $R$ in terms of $W$ and contracting with $\sigma^c_{\gamma\dot{\gamma}}$ gives

$$0 = -2i\epsilon_{\beta\gamma}\{\nabla_\alpha, W_{\dot{\gamma}}\} - 2i\epsilon_{\alpha\gamma}\{\nabla_\beta, W_{\dot{\gamma}}\}$$

A further contraction with $\epsilon^\beta\dot{\gamma}$ gives

$$0 = \{\nabla_\alpha, W_\alpha\} = \{\nabla_\dot{\alpha}, W_\alpha\}\quad\text{(D.2)}$$

where we have included the conjugate result as well. This generalizes the chirality condition of the normal Yang-Mills case, but this is not quite the conventional chirality. To wit,

$$0 = \{\nabla_\alpha, W_\alpha X_B\} = (\nabla_\alpha W_\alpha^B)X_B - W_\alpha^C f_{Ca}^B X_B$$

$W_\alpha$ is antichiral in the conventional sense only when the second term vanishes, which is the case when the symmetry group under consideration is internal (i.e., one that commutes with translations). Nevertheless, it is useful to retain the term “chiral” to describe $W_\alpha$ and “antichiral” for $W_\alpha$.

The fourth case of interest is $\alpha\dot{\beta}c$. We find

$$0 = \{\nabla_\alpha, \{\nabla_\dot{\beta}, \nabla_c\}\} + \{\nabla_c, \{\nabla_\alpha, \nabla_\dot{\beta}\}\} - \{\nabla_\dot{\beta}, \{\nabla_c, \nabla_\alpha\}\}$$

$$= -\{\nabla_\alpha, R_{\dot{\beta}c}\} - 2i\{\nabla_c, \nabla_{\alpha\dot{\beta}}\} - \{\nabla_\dot{\beta}, R_{\alpha c}\} = -\{\nabla_\alpha, R_{\dot{\beta}c}\} + 2iR_{\alpha c(\alpha\dot{\beta})} + \{\nabla_\dot{\beta}, R_{\alpha c}\}$$

which serves to define the bosonic curvature:

$$2iR_{b(\alpha\dot{\alpha})} = \{\nabla_\alpha, R_{\alpha\dot{\alpha}}\} + \{\nabla_\dot{\alpha}, R_{\alpha\dot{\alpha}}\}$$

Rewriting the right-hand side in terms of $W$ gives

$$R_{(\beta\dot{\beta})(\alpha\dot{\alpha})} = +\epsilon_{\dot{\alpha}\dot{\beta}}\{\nabla_\alpha, W_{\dot{\beta}}\} + \epsilon_{\alpha\beta}\{\nabla_\dot{\alpha}, W_\beta\}$$

The left-hand side is antisymmetric under interchange of the pairs $(\beta\dot{\beta})$ and $(\alpha\dot{\alpha})$ and so the right-hand side must be as well. It is easy to check that this requires the additional condition

$$\{\nabla_\alpha, W_\alpha\} = \{\nabla_\dot{\alpha}, W_\dot{\alpha}\}\quad\text{(D.3)}$$

This generalizes the analogous property for the Yang-Mills case much as the chirality condition has been generalized. Using this constraint one may rewrite the curvature in the manifestly antisymmetric form

$$R_{(\beta\dot{\beta})(\alpha\dot{\alpha})} = -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\{\nabla_\{\beta, W_\alpha\}\} - \frac{1}{2}\epsilon_{\alpha\beta}\{\nabla_\{\dot{\beta}, W_\dot{\alpha}\}\}$$

The remaining cases to check are $abc$ and $abc$. These turn out to follow from the previous conditions on $W$ (just as in the Yang-Mills case) and so we do not include them here. All other cases are conjugates of those given above, and so the constraints have been solved.
It is useful to derive how the symmetry generator $X_d$ acts on $\mathcal{W}_\beta$. In order to do this, it is helpful to have a set of constraints on the structure constants consistent with the Jacobi identities. The easiest way to proceed is from the general formula (A.44), specializing to the cases of $CB$ equal to $\gamma \beta$ and $\gamma \dot{\beta}$. For the first case, one finds

$$0 = \sum_{(\gamma \beta)} \left( -f_{d \gamma}^F R_{F \beta} - f_{d \gamma}^F f_{f \beta}^A X_A \right)$$  \hspace{1cm} (D.5)

where $R_{F \beta} = R_{F \beta}^A X_A$ where $X_A$ in this and the above formula consists of both the translations $P_A$ and the non-translation symmetries $X_a$. For the second case, one finds

$$0 = 2i f_{\gamma (\beta \beta)}^A X_A - f_{\gamma \beta}^D R_{D \beta} - f_{\gamma \beta}^D f_{\gamma \beta}^A X_A - f_{\gamma \beta}^D f_{\gamma \beta}^A X_A$$  \hspace{1cm} (D.6)

(We have relabelled $d$ to $c$ and $\gamma$ to $\beta$ since $\beta$ and $\dot{\beta}$ naturally go together to form a vector index.)

One set of additional constraints is also useful. For any theory in superspace, we would like to be able to write down chiral integrals; the existence of these implies the structure constant constraints (A.81)

$$f_{a \beta}^c = f_{a \beta}^\gamma = 0, \quad f_{a \beta}^c \left( f_{a \beta}^d + f_{a \beta}^\delta \right) = 0$$

as well as their complex conjugates

$$f_{a \beta}^c = f_{a \beta}^\gamma = 0, \quad f_{a \beta}^\gamma \left( f_{a \beta}^d - f_{a \beta}^\delta \right) = 0$$

Applying these constraints to (D.5) gives

$$0 = \sum_{(\gamma \beta)} \left( -f_{d \gamma}^F R_{F \beta} - f_{d \gamma}^F f_{f \beta}^A X_A \right) = - \sum_{(\gamma \beta)} f_{d \gamma}^F f_{f \beta}^A X_A$$  \hspace{1cm} (D.7)

which is a further constraint on the structure constants. Note that this constraint is equivalent to

$$f_{d \gamma}^F f_{f \beta}^A X_A = \frac{1}{2} \epsilon_{\gamma \beta} f_{d \gamma}^F f_{f \beta}^A X_A$$  \hspace{1cm} (D.8)

Applying the constraints to (D.6) gives $f_{\gamma \beta}^A$ in terms of $f_{\gamma \beta}^A$ and $f_{\gamma \beta}^A$:

$$f_{\gamma (\beta \beta)}^{(a \alpha)} = 2\epsilon_{\beta \alpha} f_{\gamma \beta}^{a \alpha} - 2\epsilon_{\beta \alpha} f_{\gamma \beta}^{\alpha a}$$  \hspace{1cm} (D.9)

$$f_{\gamma (\beta \beta)}^\alpha = -\frac{i}{2} f_{\gamma \beta}^d f_{\gamma \beta}^\alpha$$  \hspace{1cm} (D.10)

$$f_{\gamma (\beta \beta)}^{\alpha \dot{\alpha}} = -\frac{i}{2} f_{\gamma \beta}^d f_{\gamma \beta}^{\alpha \dot{\alpha}}$$  \hspace{1cm} (D.11)

$$f_{\gamma (\beta \beta)}^{\alpha a} = -\frac{i}{2} f_{\gamma \beta}^d f_{\gamma \beta}^{\alpha a} - \frac{i}{2} f_{\gamma \beta}^{\alpha a} f_{\gamma \beta}^d$$  \hspace{1cm} (D.12)

We are now in a position to derive the general gauge transformation property of $\mathcal{W}_\beta$. To proceed, first note that in principle $R_{\gamma (\beta \beta)} = f_{\gamma (\beta \beta)}^A X_A + \Delta R_{\gamma (\beta \beta)}$ where the first term on the right is a structure constant in the global theory and the second term is the local correction. (In practice, the first term usually vanishes.) It follows that a similar
decomposition of $W$ takes place, giving $W_{\beta}^A = f_{\beta}^A + \Delta W_{\beta}^A$. Since the first term is a structure constant, it necessarily is gauge invariant; we therefore need only calculate the gauge transformation of the local correction. Using equation (A.45), for the case of $CB = \gamma b$

gives

$$2i\epsilon_{\gamma\beta} X_{\dot{d}} W_{\beta}^A = -2i\epsilon_{\gamma\beta} \Delta W_{\beta}^F f_{F A}^A + 2if_{\dot{d}\gamma\beta} \Delta W_{\beta}^A - if_{\dot{d}(\beta\dot{\beta})(\gamma\dot{\gamma})} \Delta W_{\gamma}^A$$

(D.13)

Using (D.9) allows one to show the right-hand size is proportional to $\epsilon_{\gamma\beta}$. The final result is

$$X_{\dot{d}} W_{\beta}^A = -\Delta W_{\beta}^F f_{F A}^A - f_{\dot{d}\phi} \Delta W_{\beta}^A - f_{\dot{d}\gamma\dot{\gamma}} \Delta W_{\gamma}^A$$

The first term on the right hand size can be combined with the left-hand side to yield the compact formula

$$[X_{\dot{d}}, \Delta W_{\beta}] = -f_{\dot{d}\phi} \Delta W_{\beta} - f_{\dot{d}\gamma\dot{\gamma}} \Delta W_{\gamma}$$

(D.14)

The complex conjugate is

$$[X_{\dot{d}}, \Delta W_{\dot{\beta}}] = +f_{\dot{d}\phi} \Delta W_{\dot{\beta}} - f_{\dot{d}\gamma\dot{\gamma}} \Delta W_{\gamma}$$

(D.15)

We include the precise definition of the covariant derivative of the local gaugino superfields for completeness:

$$\nabla_C \Delta W_{\beta}^A = E C^M \partial_M \Delta W_{\beta}^A + h c \dot{d} \left( \Delta W_{\beta}^F f_{F A}^A - f_{\dot{d}\phi} \Delta W_{\beta}^A + f_{\dot{d}\gamma\dot{\gamma}} \Delta W_{\gamma}^A \right)$$

(D.16)

$$\nabla_C \Delta W_{\dot{\beta}}^A = E C^M \partial_M \Delta W_{\dot{\beta}}^A + h c \dot{d} \left( \Delta W_{\dot{\beta}}^F f_{F A}^A + f_{\dot{d}\phi} \Delta W_{\beta}^A + f_{\dot{d}\gamma\dot{\gamma}} \Delta W_{\gamma}^A \right)$$

(D.17)

(The covariant derivative of the constant part of $W$ vanishes.)

**D.2 Conformal supergravity solution**

From the result of the previous section, we may define maximal conformal supergravity as the theory with the Yang-Mills constraints

$$\{\nabla_\alpha, \nabla_\beta\} = \{\nabla_{\dot{\alpha}}, \nabla_{\dot{\beta}}\} = 0$$

$$\{\nabla_\alpha, \nabla_{\dot{\alpha}}\} = -2i\nabla_{a\dot{a}}.$$

It follows that the remaining curvatures are of the form

$$R_{\alpha(\beta\dot{\beta})} = 2i\epsilon_{\alpha\beta\dot{\beta}} W_{\dot{\beta}}$$

$$R_{\dot{\alpha}(\beta\dot{\beta})} = 2i\epsilon_{\dot{\alpha}\dot{\beta}\beta} W_{\beta}$$

$$R_{(\beta\dot{\beta})(\alpha\dot{\alpha})} = -\frac{1}{2}\epsilon_{\beta\dot{\beta}} \{\nabla_{(\beta}, W_{\alpha)}\} - \frac{1}{2}\epsilon_{\beta\alpha} \{\nabla_{(\dot{\beta}}, W_{\dot{\alpha)}\}$$

where the superfields $W$ obey the constraints

$$\{\nabla_{\dot{\alpha}}, W_{\alpha}\} = \{\nabla_{\alpha}, W_{\dot{\alpha}}\} = 0$$

$$\{\nabla_{\alpha}, W_{\alpha}\} = \{\nabla_{\dot{\alpha}}, W_{\dot{\alpha}}\}$$
The $\mathcal{W}$ here is understood as

$$\mathcal{W}_\alpha = \mathcal{W}(P)_\alpha^B P_B + \frac{1}{2} \mathcal{W}(M)_\alpha^{cb} M_{bc} + \mathcal{W}(D)_\alpha D + \mathcal{W}(A)_\alpha A + \mathcal{W}(K)_\alpha^B K_B$$

That is, there is a $\mathcal{W}$ associated with each symmetry in the conformal group. These $\mathcal{W}$ are not conformally primary but are rotated into each other by the action of the conformal group. In this case, the global theory is characterized by $\mathcal{W} = 0$ and so no decomposition of $\mathcal{W}$ into global and local parts is necessary.

The chirality condition $\{\nabla_\dot{\alpha}, \mathcal{W}_\alpha\} = 0$ reads

$$0 = \nabla_\dot{\alpha} \mathcal{W}(P)_\alpha^B \nabla_B - \mathcal{W}(P)_\alpha^C T_{C\dot{a}}^B \nabla_B + \mathcal{W}(M)_\alpha^{\dot{a}\beta} \nabla^{\beta} + \frac{1}{2} \mathcal{W}(D)_\alpha \nabla_\dot{\alpha} + i \mathcal{W}(A)_\alpha \nabla_\dot{\alpha}$$

$$0 = \nabla_\dot{\alpha} \mathcal{W}(M)_\alpha^{cb} M_{bc} - \frac{1}{2} \mathcal{W}(P)_\alpha^D R_{D\dot{a}}^{cb} M_{bc} - 2 \mathcal{W}(K)_\alpha^{\dot{a}\beta} M_\beta^{\dot{a}}$$

$$0 = \nabla_\dot{\alpha} \mathcal{W}(K)_\alpha^B K_B - \mathcal{W}(P)_\alpha^C R(K)_\alpha^B K_B + i \mathcal{W}(K)_\alpha^{(\dot{a}\beta)} S_\beta$$

$$0 = \nabla_\dot{\alpha} \mathcal{W}(D)_\alpha - \mathcal{W}(P)_\alpha^B R(D)_B \dot{\alpha} - 2 \mathcal{W}(K)_\alpha$$

$$0 = \nabla_\dot{\alpha} \mathcal{W}(A)_\alpha - \mathcal{W}(P)_\alpha^B R(A)_B \dot{\alpha} - 3i \mathcal{W}(K)_\alpha$$ (D.18)

For the last two equations we have omitted the generators $D$ and $A$ respectively. The curvatures in these expressions are defined in terms of $\mathcal{W}$; therefore, these formulae possess both linear and quadratic terms in $\mathcal{W}$.

The condition $\{\nabla^\alpha, \mathcal{W}_\alpha\} = \{\nabla_\dot{\alpha}, \mathcal{W}_\dot{\alpha}\}$ reads

$$\nabla^\alpha \mathcal{W}(P)_\alpha^B \nabla_B + \mathcal{W}(P)^{\alpha C} T_{C\dot{a}}^B \nabla_B - \mathcal{W}(M)^{\dot{a}\beta}_\alpha \nabla^{\beta} - \frac{1}{2} \mathcal{W}(D)^{\alpha \dot{a}} \nabla_\alpha + i \mathcal{W}(A)^{\alpha \dot{a}} \nabla_\alpha$$

$$= \nabla_\dot{\alpha} \mathcal{W}(P)^{\dot{a}B} \nabla_B + \mathcal{W}(P)^{\dot{a}C} T_{C\dot{a}}^{\dot{B}} \nabla_B - \mathcal{W}(M)^{\dot{a}\beta}_\dot{\alpha} \nabla^{\beta} - \frac{1}{2} \mathcal{W}(D)^{\dot{a} \dot{\alpha}} \nabla_\dot{\alpha} - i \mathcal{W}(A)^{\dot{a} \dot{\alpha}} \nabla_\dot{\alpha}$$ (D.19)

$$\frac{1}{2} \nabla^\alpha \mathcal{W}(M)^{\dot{a}cb}_\alpha M_{bc} + \frac{1}{2} \mathcal{W}(P)^{\alpha D} R_{D\dot{a}}^{\dot{a}cb} M_{bc} + 2 \mathcal{W}(K)^{\dot{a}\beta}_\alpha M_\beta^{\dot{a}}$$

$$= \frac{1}{2} \nabla_\dot{\alpha} \mathcal{W}(M)^{\dot{a}cb}_\dot{\alpha} M_{bc} + \frac{1}{2} \mathcal{W}(P)^{\dot{a}D} R_{D\dot{a}}^{\dot{a}cb} M_{bc} + 2 \mathcal{W}(K)^{\dot{a}\beta}_\dot{\alpha} M_\beta^{\dot{a}}$$ (D.20)

$$\nabla^\alpha \mathcal{W}(K)^{\alpha B}_\alpha K_B + \mathcal{W}(P)^{\alpha C} R(K)_\alpha^B K_B - i \mathcal{W}(K)^{\alpha \dot{a}}_\dot{\alpha} S_\beta$$

$$= \nabla_\dot{\alpha} \mathcal{W}(K)^{\dot{a}B}_\dot{\alpha} K_B + \mathcal{W}(P)^{\dot{a}C} R(K)_\dot{C}^{\dot{a}B} K_B - i \mathcal{W}(K)^{\dot{a}(\dot{a}\beta)}_\dot{\alpha} S_\beta$$ (D.21)

$$\nabla^\alpha \mathcal{W}(D)_\alpha + \mathcal{W}(P)^{\alpha B} R(D)_B \dot{\alpha} + 2 \mathcal{W}(K)^{\alpha \dot{a}}_\alpha = \nabla_\dot{\alpha} \mathcal{W}(D)^{\dot{a}}_\dot{\alpha} + \mathcal{W}(P)^{\dot{a}B} R(D)_B \dot{\alpha} + 2 \mathcal{W}(K)^{\dot{a} \dot{\alpha}}_\dot{\alpha}$$ (D.22)

$$\nabla^\alpha \mathcal{W}(A)_\alpha + \mathcal{W}(P)^{\alpha B} R(A)_B \dot{\alpha} - 3i \mathcal{W}(K)^{\alpha \dot{a}}_\alpha = \nabla_\dot{\alpha} \mathcal{W}(A)^{\dot{a}}_\dot{\alpha} + \mathcal{W}(P)^{\dot{a}B} R(A)_B \dot{\alpha} + 3i \mathcal{W}(K)^{\dot{a} \dot{\alpha}}_\dot{\alpha}$$ (D.23)

This is a very complicated structure that simplifies a great deal when we apply the further constraints of conformal supergravity. These are $F_{ab} = 0$, $H_{ab} = 0$, and $T_{\gamma}^b A = 0$ along with their complex conjugates. (In addition, we want $T_{cb}^a = 0$ but this turns out
to be a consequence of the other constraints.) These constraints clearly force $\mathcal{W}(A)_{\alpha}$, $\mathcal{W}(D)_{\alpha}$, and $\mathcal{W}(P)_{\alpha B}$ to zero. Since this set of constraints is conformally invariant (i.e. $S_{\gamma} \mathcal{W}(D)_{\beta} = +2 \mathcal{W}(P)_{\beta \gamma} = 0$), it follows that the covariant derivative of any of these also vanishes.

The only non-vanishing gaugino superfields are then $\mathcal{W}(M)$ and $\mathcal{W}(K)$. In terms of these, the chirality constraints (D.18) read

$$0 = \mathcal{W}(M)_{\alpha \dot{\alpha} \beta} \nabla^{\dot{\beta}}$$

$$0 = \frac{1}{2} \nabla_{\dot{\alpha}} \mathcal{W}(M)_{\alpha}^{cb} M_{bc} - 2 \mathcal{W}(K)_{\alpha \beta} M^{\dot{\beta} \dot{\alpha}}$$

$$0 = \nabla_{\dot{\alpha}} \mathcal{W}(K)_{\alpha}^{B} K_B + i \mathcal{W}(K)_{\alpha (\beta} S_{\beta}$$

$$0 = -2 \mathcal{W}(K)_{\alpha \dot{\alpha}}$$

$$0 = -3 i \mathcal{W}(K)_{\alpha \dot{\alpha}}$$

It follows that $\mathcal{W}(M)_{\alpha \beta \dot{\gamma}}$ and $\mathcal{W}(K)_{\alpha \dot{\alpha}}$ vanish. Furthermore, $\mathcal{W}(M)_{\alpha \beta \gamma}$ is chiral and $\nabla_{\dot{\alpha}} \mathcal{W}(K)_{\alpha}^{\beta} = -i \mathcal{W}(K)_{\alpha \dot{\alpha} \beta}$.

Considering the remaining constraints, we have (D.19)

$$-\mathcal{W}(M)_{\alpha}^{\alpha \beta} \nabla_{\beta} = -\mathcal{W}(M)_{\dot{\alpha} \beta} \nabla^{\dot{\beta}}$$

This implies that $\mathcal{W}(M)_{\alpha \gamma} = 0$. Therefore, $\mathcal{W}(M)_{\alpha \beta \gamma}$ is totally symmetric in its indices. Similarly for the conjugate.

Next is (D.20)

$$\frac{1}{2} \nabla^{\alpha} \mathcal{W}(M)_{\alpha}^{cb} M_{bc} + 2 \mathcal{W}(K)_{\alpha \beta} M_{\beta \alpha} = \frac{1}{2} \nabla^{\dot{\alpha}} \mathcal{W}(M)_{\dot{\alpha}}^{cb} M_{bc} + 2 \mathcal{W}(K)_{\dot{\alpha} \beta} M^{\dot{\beta} \dot{\alpha}}$$

which implies that $\mathcal{W}(K)_{\beta (\alpha \dot{\alpha})} = -\frac{1}{2} \nabla^{\alpha} \mathcal{W}(M)_{\alpha \beta \gamma}$. Since we already know that $\mathcal{W}(K)_{\beta (\alpha \dot{\alpha})} = i \nabla_{\dot{\alpha}} \mathcal{W}(K)_{\beta \alpha}$, it follows that $\mathcal{W}(K)_{\beta (\alpha \dot{\alpha})} = -\frac{1}{2} \nabla^{\dot{\alpha}} \mathcal{W}(M)_{\phi \beta \alpha}$.

Equation (D.21) implies

$$\nabla^{\alpha} \mathcal{W}(K)_{\alpha}^{B} K_B - i \mathcal{W}(K)_{\dot{\alpha}}^{\alpha \beta} S_{\beta} = \nabla^{\dot{\alpha}} \mathcal{W}(K)_{\dot{\alpha}}^{\beta B} K_B - i \mathcal{W}(K)_{\dot{\alpha}}^{\alpha (\beta \dot{\gamma})} S_{\beta}$$

which, when we insert our existing formulae, gives a new identity

$$\nabla^{\beta} \nabla^{\alpha} \mathcal{W}(M)_{\phi \beta \alpha} = \nabla^{\dot{\beta}} \nabla^{\dot{\alpha}} \mathcal{W}(M)_{\phi \dot{\beta} \dot{\alpha}}$$

Finally, we note that the final two constraints (D.22) and (D.23) give

$$+2 \mathcal{W}(K)^{\alpha}_{\dot{\alpha}} = 2 \mathcal{W}(K)_{\alpha}$$

and

$$-3i \mathcal{W}(K)^{\alpha}_{\dot{\alpha}} = +3i \mathcal{W}(K)_{\alpha}$$

which are satisfied trivially. (Both sides vanish.)

All of the curvatures are then specified in terms of a single totally symmetric chiral superfield $\mathcal{W}(M)_{\alpha \beta \gamma}$ as well as its conjugate, which together obey a Bianchi identity. Furthermore, from the transformation rules of the $\mathcal{W}$ found in the previous section, $\mathcal{W}(M)_{\alpha \beta \gamma}$ is conformally primary of scale dimension $3/2$ and $U(1)$ weight +1. To make contact with
the conventional normalizations and reality conditions, we define a new superfield $W_{\alpha \beta \gamma}$ via
\[ W(M)_{\alpha \beta \gamma} = -2W_{\alpha \beta \gamma} \quad \text{and} \quad W(M)_{\dot{\alpha} \dot{\beta} \dot{\gamma}} = +2W_{\alpha \beta \gamma} \]
and summarize our results in terms of it:

\begin{align*}
W(P)_{\alpha}^{B} &= W(P)_{\dot{\alpha}}^{B} = 0 \\
W(D)_{\alpha} &= W(D)_{\dot{\alpha}} = 0 \\
W(A)_{\alpha} &= W(A)_{\dot{\alpha}} = 0 \\
W(M)_{\alpha \beta \gamma} &= W(M)_{\dot{\alpha} \dot{\beta} \dot{\gamma}} = 0 \\
W(M)_{\alpha \beta \gamma} &= -2W_{\alpha \beta \gamma}, \quad W(M)_{\dot{\alpha} \dot{\beta} \dot{\gamma}} = +2W_{\dot{\alpha} \dot{\beta} \dot{\gamma}} \\
W(K)_{\alpha, \beta} &= \frac{1}{2} \nabla^{\alpha} \nabla_{\beta} W_{\phi \alpha \beta}, \quad W(K)_{\dot{\alpha}, \dot{\beta}} = \frac{1}{2} \nabla^{\dot{\alpha}} \nabla_{\dot{\beta}} W_{\phi \dot{\alpha} \dot{\beta}} \\
W(K)_{\alpha}^{\dot{\beta}} &= W(K)_{\dot{\alpha}}^{\beta} = 0 \\
W(K)_{\alpha}^{(\beta \dot{\beta})} &= \nabla_{\beta}^{\alpha} W_{\phi \alpha \beta}, \quad W(K)_{\dot{\alpha}}^{(\beta \dot{\beta})} = \nabla_{\beta}^{\dot{\alpha}} W_{\phi \dot{\alpha} \dot{\beta}}
\end{align*}

$W_{\alpha \beta \gamma}$ is a totally symmetric chiral primary superfield obeying a Bianchi identity

\[ \nabla^{\beta} \nabla^{\phi}_{\dot{\alpha}} W_{\phi \beta \alpha} = -\nabla^{\dot{\beta}} \nabla^{\phi}_{\alpha} W_{\phi \beta \dot{\alpha}} \]

From the above definitions of $W$ and of the curvatures $R$ in terms of $W$, one can quite easily derive the curvatures in terms of $W$. These are given fully in Section 2.6.
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