LOWER BOUNDS FOR SYMBOLIC COMPLEXITY OF ICEBERG
DYNAMICAL SYSTEMS

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ABSTRACT. The symbolic complexity of an infinite word \( W \) is the function \( p_W(l) \) counting the number of different subwords in \( W \) of length \( l \in \mathbb{N} \). In this paper our main purpose is to study the complexity for a class of topological dynamical systems, called iceberg systems, given by the following symbolic procedure. Starting from a given finite word \( w_1 \) we construct a sequence of words

\[
    w_{n+1} = w_n \rho_{\alpha_n(1)}(w_{n}) \cdots \rho_{\alpha_n(q_n-1)}(w_{n}),
\]

where \( \rho_{\alpha_n}(w_n) \) is a cyclic rotations of the word \( w_n \), and consider an infinite word \( w_\infty \) extending each \( w_n \) to the right. It is shown that for iceberg systems given by the randomized parameters \( \alpha_n(j) \) the complexity function satisfies the estimate \( p_{w_\infty}(l) \gtrsim l^3 - \varepsilon \) for any \( \varepsilon > 0 \), and at the same time it is observed that this estimate represents up to a small correction the optimal lower bound for the complexity function, namely, \( p_{w_{n+1}}(l_n) \leq l_n^3 \) along the subsequence \( l_n = |w_n| + 1 \).

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1. INTRODUCTION

The symbolic complexity of an infinite word \( W \) is the function \( p_W(l) \) counting the total number of different subwords in \( W \) of length \( l \in \mathbb{N} \). In a series of recent investigations the concept of symbolic complexity is applied to the study of orbit structure both in topological dynamics and ergodic theory of measure preserving transformations (e.g. see [2, 3, 4, 5, 9, 10]). Further, a kind of invariant of measure-theoretic isomorphism extending the concept of symbolic complexity is introduced by S. Ferenczi in [6]. Another general approach to the complexity or orbit structure introduced by A. Vershik is based on the notion of scaling entropy [14].

Out investigation is motivated by the idea of applying the concept of complexity and coding arguments to the problems of isomorphism in ergodic theory of measure preserving transformations with zero entropy. We would like to start with mentioning the classical symbolic construction of a rank one dynamical system. Given a non-trivial finite word \( v_1 \) in a finite alphabet \( \mathbb{A} \), we define words

\[
    v_{n+1} = v_n 0^n 0^n 0^n \cdots 0^n v_n,
\]

where symbol “0” used to create small spacers between the copies of the word \( v_n \), and an infinite word \( v_\infty \) having each \( v_n \) as a prefix. Following the standard scheme we can consider the compact set \( K \) defined as the weak closure of all translates of \( v_\infty \) and endow \( K \) with the standard measure which is invariant under the shift transformation \( T \). The triple \( (T, K, \mu) \) is called rank one transformation.

Rank one transformation serves as a classical example of dynamical system with simple spectrum. In paper [12] a class of ergodic maps is introduced, which is similar to rank one systems and extends this class. Namely, generalizing the rule (1), while defining \( w_{n+1} \) we combine not only exact copies but any cyclic rotations of the previous word \( w_n \). It is observed that the measure preserving map associated with this new symbolic procedure, called iceberg transformation, has always 1/4-local rank and, under certain conditions, simple spectrum. It is not obvious that the new class is actually larger than the class of rank one transformations. At the same time, these two classes potentially could contain transformations having spectral types with similar properties. Thus, it is interesting to find an invariant helping to distinguish these classes, and, in fact, it can be easily deduced from the results of this paper that the typical complexity of iceberg systems is \( p_{w_\infty}(l) \gtrsim l^3 - \varepsilon \), hence, a wide class of iceberg systems is certainly goes beyond the class of rank one systems, since for any rank one map there exists a finite partition of the phase space generating a
word \( w_\infty \) with the property \( p_{w_\infty}(l_n) \lesssim \frac{1}{4} l_n^2 \) (see [7]) in contradiction with the given lower bound, proving the existence of iceberg systems which are not rank one. Though, one can observe that this approach is essentially non-spectral, hence, there are no obstacles to existence of a pair \((T_1, T_2)\) of spectrally isomorphic maps, where \( T_1 \) is rank one and \( T_2 \) is an iceberg map, but not rank one. Thus, this class of symbolic systems, more general than rank one, can be considered as a source of new examples demonstrating that rank is not a spectral invariant.

It is interesting to compare the lower bound for symbolic complexity of iceberg systems with the asymptotics of the Pascal adic transformation complexity, \( p(l) \sim l^3/6 \) established by X. Mélá and K. Petersen [10].

2. Iceberg dynamical systems

We start with a series of constructions and definitions.

Definition. Consider a finite alphabet \( \mathbb{A} \) and a word \( w = a_1 \ldots a_N \) in \( \mathbb{A} \). We use the notation \( |w| \) for the length of the word \( w \). Given an integer number \( n \), \( 0 < \alpha < |w| \), let us define the cyclic rotation \( \rho_\alpha(w) \) of the word \( w \) as follows

\[
\rho_\alpha(w) = w_2 w_1, \quad \text{whenever} \quad w = w_1 w_2, \quad |w_1| = \alpha.
\]

Observe that if \( b \in \mathbb{A} \) is a letter and \( u \) is a word then \( \rho_1(bu) = ub \) and \( \rho_\alpha = (\rho_1)^\alpha \). Let us extend this definition to all values of \( \alpha \in \mathbb{Z} \) setting

\[
\rho_0(w) = w \quad \text{and} \quad \rho_\alpha(w) = \rho_{\alpha+k|w|}(w), \quad k \in \mathbb{Z}.
\]

The next definition is the starting point of the construction of iceberg systems. We start with defining an infinite word \( w_\infty \), then we consider the closure \( K_{w_\infty} \) of the sequence \( \{w_n\} \) in weak topology, and finally using the standard procedure we endow the compact set \( K_{w_\infty} \) with an invariant measure \( \mu(w_\infty) \) and we come to an ergodic transformation \( T \) acting as left shift on the space \( (K_{w_\infty}, \mu(w_\infty)) \) (see [12]).

Definition. Let \( w_1 \) be a fixed finite word in \( \mathbb{A} \). We suppose that \( w_1 \) is non-trivial and contains a pair of different letters. Let us construct a sequence of words \( w_n \), such that each next word \( w_{n+1} \) in the sequence is a concatenation of cyclic rotations of \( w_n \), namely,

\[
(2) \quad w_{n+1} = \rho_{\alpha_n(0)}(w_n) \rho_{\alpha_n(1)}(w_n) \cdots \rho_{\alpha_n(q_n-1)}(w_n),
\]

where \( q_n \geq 2 \) is the number of entries \( \rho_{\alpha_n(j)}(w_n) \). Now suppose, for simplicity, that \( \alpha_0(0) = 0 \) for any \( n \) so that \( w_{n+1} \) extends \( w_n \) to the right, and after the infinite sequence of steps we come to an infinite word \( w_\infty \) such that any \( w_n \) is prefix of \( w_\infty \). Let \( K_{w_\infty} \) be the closure of \( \{w_n\} \) in weak topology, i.e. the set of all sequences \( x = (x_j)_{j \in \mathbb{Z}} \) such that any finite subword of \( x \) can be found as a subword of some \( w_n \). The compact set \( K_{w_\infty} \) is invariant under the left shift map

\[
(3) \quad T: (\ldots, x_0, x_1, \ldots, x_j, \ldots) \mapsto (\ldots, x_1, x_2, \ldots, x_{j+1}, \ldots).
\]

We call \((T, K_{w_\infty})\) an iceberg topological dynamical system (without spacers).

2.1. Symbolic complexity. Given an infinite word \( w \) let us consider the language \( \mathcal{L}(w) \) defined as the set of all finite subwords of the word \( w \). We can partition the set \( \mathcal{L}(w) \) into a sequence of subsets \( \mathcal{L}(w, l) \) according to the length of a subword,

\[
\mathcal{L}(w, l) = \{ u \in \mathcal{L}(w) : |u| = l \}.
\]

Symbolic complexity \( p_w(l) \) of the word \( w \) is the function counting the number of different subwords in \( w \) of length \( l \) or, in other term, the function measuring the complexity of \( \mathcal{L}(w) \),

\[
p_w(l) = \# \mathcal{L}(w, l),
\]

where \( \#A \) is the cardinality of \( A \).

Our paper is devoted to the proof of the following observation.
Theorem 1. Consider an iceberg system given by the independent uniformly distributed parameters $\alpha_n(j)$, and assume that $q_n \to \infty$ sufficiently fast, $q_n \geq h_n^\gamma$ with $\gamma \geq 1$, where $h_n = |w_n|$. Then almost surely
\begin{equation}
 p_{w_n}(l) \geq l^{3-\varepsilon}
\end{equation}
for any $\varepsilon > 0$ and $l \geq l_0(\varepsilon)$.

2.2. Measure-theoretic point of view. Throughout this paper we will continuously compare pure combinatorial observations with the corresponding effects in the context of measure-theoretic ergodic theory. Infinite words are interpreted in this case as discrete sample path of stationary random processes.

The concept of iceberg transformation can be extended if we endow the compact space $K_{w_n}$ with the natural invariant measure $\mu = \mu(w_\infty)$ generated by the set of empirical distributions $\mu_l$ defined as follows: given a finite word $w \in L(w_\infty, l)$, a word of length $l$, the value $\mu_l(w)$ is the asymptotic frequency of occurrence of $w$ inside the infinite word $w_\infty$. This procedure leads to an ergodic dynamical system given by the left shift map $T$ acting on the probability space $(K_{w_\infty}, B, \mu(w_\infty))$, where $B$ is the sigma-algebra of Borel sets. It is shown in paper [12] that an iceberg map can be obtained as a result of cutting-and-stacking construction in a way independent on the symbolic formalism we discuss (see fig. 1).

The main idea of our paper is to provide a combinatorial background to further study of a typical symbolic property of the iceberg construction can be formulated in an invariant form, as a kind of dynamical property concides with

\begin{equation}
 \bar{W}_j \cdot \cdots \bar{S}_2 \bar{S}_1 \bar{W}_j \cdots \end{equation}

where $\bar{W}_j$ concides with $W_\varepsilon$ up to at most $\varepsilon |W_\varepsilon|$ wrong letters.

Definition. Let $T$ be a measure preserving transformation. We say that $T$ admits iceberg approximation if for any finite measurable partition $P$ and any $\varepsilon > 0$ one can find a set $\Omega_\varepsilon$ of measure $1-\varepsilon$ such that for any $x \in \Omega_\varepsilon$ the orbit of $x$ represented in a form of an infinite word $(x_j)$, where $T^j x \in P_{\varepsilon(j)}$, satisfies the following property: $(x_j)$ is $\varepsilon$-covered by disjoint words $\bar{W}_j$ which are $\varepsilon$-close to some fixed word $W_\varepsilon$:
\begin{equation}
 (x_j) = \cdots \bar{W}_j S_1 \bar{W}_2 S_2 \cdots \bar{S}_{j-1} \bar{W}_j \cdots ,
\end{equation}

where $\bar{W}_j$ is $\varepsilon$-close to some fixed word $W_\varepsilon$.

Definition. We say that a measure preserving map $T$ on a probability space $(X, \mu)$ is of rank one if for any measurable partition $P = \{P_1, \ldots, P_m\}$ and any $\varepsilon > 0$ there exists a set $\Omega_\varepsilon$ of measure $1-\varepsilon$ such that for any $x \in \Omega_\varepsilon$ the orbit of $x$ represented in a form of an infinite word $(x_j)$ is $\varepsilon$-covered by words $\bar{W}_j$ which are $\varepsilon$-close to cyclic rotations of a fixed $W_\varepsilon$:
\begin{equation}
 (x_j) = \cdots \bar{W}_j S_1 \bar{S}_2 \bar{S}_1 \cdots \bar{S}_{j-1} \bar{W}_j \cdots ,
\end{equation}

where $\bar{d}(u,v)$ measures the fraction of non-matching letters in $u$ and $v$.

It is important to say that a priori we cannot skip spacers in this definition. Thus, generally speaking a common construction of an iceberg map should contain spacers as well, but, at the same time, it is interesting to observe that from the complexity point of view adding spacers between cyclic rotations we do not influence significantly to the asymptotics of $p_{w_\infty}$ (see the effect discussed below in remark 3).

3. Upper Bound for Symbolic Complexity Along a Subsequence

We start with the estimation of the symbolic complexity for the sequence of special kind $l_n = h_n + 1 = |w_n| + 1$. The idea of this calculation is to show that the lower bound discussed in the next section is, in a sense, optimal.

Lemma 1. $p_{w_{n+1}}(h_n + 1) \leq h_n^3$ for any iceberg system.

Proof. Let us consider a subwords $v$ of length $h_n + 1 = |w_n| + 1$ inside $w_{n+1}$, and consider $w_{n+1}$ as a sequence of rotations $\rho_{\alpha_n(j)}(w_n)$. Suppose that $v$ starts at position $k$ so that $v$ includes the letters...
Figure 1. Sample of an iceberg. A column is an ordinary Rokhlin tower corresponding to a cyclic rotation of the word “abcdef”. Transformation \( T \) lifts each set drawn as a square to the upper square. Whenever a point reaches the top set in a column, it is mapped by \( T \) to an arbitrary bottom set of another (or the same) column.

\[ \rho_{\alpha_n}(j-1)(w_n) \quad \rho_{\alpha_n}(j)(w_n) \quad \rho_{\alpha_n}(j+1)(w_n) \quad \rho_{\alpha_n}(j+2)(w_n) \]

\[ w_1 \quad w_2 \]

\[ \xi \]

Figure 2. Estimation of \( p_{w_{n+1}}(l) \) for \( l = |w_n| \). The rectangle restricts a subword \( v \) of length \( l \). The vertical lines mark up the boundaries of blocks \( \rho_{\alpha_n}(j)(w_n) \).

at positions \( k..k + h_n \) and it covers exactly two adjacent subwords: \( \rho_{\alpha_n}(j)(w_n) \) and \( \rho_{\alpha_n}(j+1)(w_n) \). It is easy to see that the subword \( v \) is uniquely determined by the value of the triple \((\bar{k}, \alpha_n(j), \alpha_n(j))\), where \( \bar{k} = k \mod h_n + 1 \), \( k \neq 0 \), and the total number of such triples is estimated as \( h_n^3 \). This idea is illustrated on fig. 2. Let us summarize the estimates in the following line:

\[ p_{w_{n+1}}(h_n) \leq h_n^3 - h_n^2 + h_n, \quad p_{w_{n+1}}(h_n + 1) \leq h_n^3. \]

Now let us consider an iceberg system with spacers given by the scheme

\[ w_{n+1} = w_n 0^{s_n,0} \rho_{\alpha_n}(1)(w_n) 0^{s_{n-1}} \cdots 0^{s_{n-q_n-2}} \rho_{\alpha_n}(q_n-1)(w_n) 0^{s_{n-q_n-1}}, \]

and observe that spacers do not affect to the asymptotics along the sequence \( h_n + 1 \). We write \( w_1 \preceq w \) if \( w_1 \) is a subword of \( w \).

Remark (Melting word effect). Consider a subword

\[ v = w_1 S w_2, \quad w_1 \preceq \rho_{\varphi_1}(w_n), \quad w_2 \preceq \rho_{\varphi_2}(w_n), \]

of length \( l_n = |w_n| + 1 \) that covers a part \( w_1 \) of the block \( \rho_{\alpha_n}(j)(w_n) \), a part \( w_2 \) of the block \( \rho_{\alpha_n}(j+1)(w_n) \), and a spacer \( S = 0^\nu \). Let us imagine that the spacer \( S \) is growing to the left and becomes \( \nu \) symbols larger. It can be observed that the growth of the spacer can be compensated by the opposite cyclic rotation in the
first block $\rho_{\alpha_n(j)}(w_n)$ by $-\nu$ positions. In other terms, the original word $v$ coincides up to $\nu$ letters with a new word $\tilde{v}$ given by the configuration
\[
\tilde{v} = \tilde{w}_1 \tilde{S} w_2, \quad \tilde{w}_1 \leq \rho_{\phi_1 - \nu}(w_n) \quad \text{and} \quad \tilde{S} = 0^{\nu + \nu},
\]
and a priori there are no reasons to say if $v$ coincide with $\tilde{v}$ or not? If we interpret the area covered by $v$ as a window we monitor the sequence $\rho_{\alpha_n(j)}(w_n)$ through then it looks like the block $\rho_{\phi_1}(w_n)$ is “melting” from the right side. We erase $\nu$ letters in the subword $w_1$ while extending the spacer. Furthermore, from the statistical point of view the words $v$ and $\tilde{v}$ are asymptotically $\tilde{d}$-close with $\mu$-probability $1 - \varepsilon_n$, where $\varepsilon_n \to 0$, if spacers are supposed to be asymptotically small.

Definition. Let us consider the order on finite words: $w_1 \geq w_2$ if $w_1$ differs from $w_2$ by replacing some letters to the spacer symbol “0”. We call saturated complexity of an infinite word $w$ the function $\bar{p}_w(l)$ counting the the total number of maximal points with respect to the order “$\geq$” in the set $\mathcal{L}(w, l)$ of subwords in $w$ of length $l$.

Theorem 2. $\bar{p}_{w_{n+1}}(h_n + 1) \leq h_n^3$ for any iceberg system with spacers.

This theorem follows directly from the discussion of the previous remark.

4. Lower bounds for symbolic complexity

Now we pass to the discussion of lower estimates for $p_{w_{n+1}}(l)$. The idea is to observe that generically, for iceberg systems with sufficiently rich combinatorial structure almost all configurations
\[
v = w_1 w_2 \leq \rho_{\phi_1}(w_n) \rho_{\phi_2}(w_n), \quad w_1 \leq \rho_{\phi_1}(w_n), \quad w_2 \leq \rho_{\phi_2}(w_n),
\]
and, respectively, almost all triples $(|w_1|, \phi_1, \phi_2)$ are observed inside the word $w_{n+1}$. The following lemma is a simple combinatorial observation.

Lemma 2. Let us consider two large intervals on the integer line $[k, k + m]$ and $[l, l + m]$ of the same length $m$, and assume that $l - k \neq 0$. There exists a subset $A \subset [k, k + m]$ satisfying conditions
\[
A \cap (l + k - A) = \emptyset \quad \text{and} \quad \#A \geq \frac{1}{2} m (1 + o(1))
\]
For the sequel it is convenient to find and fix a value $m_0$ such that $\#A \geq \frac{1}{3} m$ in the above lemma whenever $m \geq m_0$.

Let us consider an iceberg system given by the independent random cyclic rotation parameters $\alpha_n(j)$ uniformly distributed on the integer interval $[0, h_n)$, where $h_n = |w_n|$. Let us use notation $w \leq_0 w$ in the case if $u$ is a subword of some cyclic rotation of $w$, i.e. $u \leq_0 \rho_\phi(w)$ for some $\phi$. This new class of dynamical systems generalizes the class or randomized rank one dynamical systems introduced by D. Ornstein ([11], see also [1]).

Definition. We say that our symbolic system satisfies property $D_m(\beta)$ with $\beta > 0$ if any two subwords $u \leq_0 w_n$ and $v \leq_0 w_n$ of the same length $m \geq \beta |w_n|$ are always identical whenever they are equal. Here we use the word identical to define the case when two subwords has the same starting position in $w$, i.e., in fact, they corresponds to one entrance of a subword. For example, two entrances of the word $u = v = \text{cat}$ in a larger word $w = \text{littlecatandbigcat}$ starting at position 6, and position 15 are equal but not identical.

In the sequel we will use symbol $P(\Lambda)$ for the probability of the set $A$ according to the probability space hosting the random parameters $\alpha_n(j)$. It is convinient to consider two letter alphabet $\Lambda = \{0, 1\}$. Assume that the initial word in the iceberg construction $w_1$ contains $\nu \cdot |w_1|$ ones and $(1 - \nu) \cdot |w_1|$. It follows easily from the definition that any word $w_n$ contains exactly $\nu$-fraction of ones. The following lemma mainly concerns the $n$th step of the construction when we build the new word $w_{n+1}$ combining a sequence of $q_n$ random cyclic rotations of $w_n$. 


Lemma 3. Consider a pair of independent cyclic rotations, for example,

\[ W_1 = \rho_{\alpha_n(j)}(w_n) \quad \text{and} \quad W_2 = \rho_{\alpha_n(j+1)}(w_n) \]

and a pair of intervals \( I_1, I_2 \subset \{0, 1, \ldots, |w_n| - 1\} \). Let \( v_1 = W_1|_{I_1} \) and \( v_2 = W_2|_{I_2} \) be the two subwords corresponding to the subsets of indexes \( I_1 \) and \( I_2 \). Then the probability of matching \( v_1 = v_2 \) is less than \( 1 - 2\nu + 2\nu^2 \).

Proof. It is enough to show that the conditional probability of matching \( v_1 = v_2 \) with respect to the finite set of random parameters that influenced to the structure of \( w_n \). The probability of matching is less than the probability of matching of letters at only one position in \( I_1 \). The uniform random rotation generates at any position the random variable which is distributed as '0' with probability \( 1 - \nu \) and '1' with probability \( \nu \). Thus, the one-letter matching process is equivalent to matching of two independent variables given by the vector \( \{1 - \nu, \nu\} \).

Remark that this lemma can be significantly strengthened, and our strategy in this paper is to avoid complicated investigation of random processes including large deviation mechanism and to use only elementary probability technique.

Lemma 4. Assume that \( \beta > \max\{36, m_0 + 1\} \cdot q_n^{-1} \). Then

\[ \mathcal{P}(D_{n+1}(\beta) \mid D_n(1/2)) \geq 1 - q_n^3 \cdot h^{-\beta q_n/12}. \]

Proof. Suppose that \( D_n(1/2) \) is satisfied. Consider two subwords \( v_1 \) and \( v_2 \) of \( w_{n+1} \) (or its cyclic rotation, further we will omit such kind of remark) starting at position \( k_1 \) and \( k_2 \) respectively. Assume also that

\[ |v_1| = |v_2| = M \geq \beta |w_{n+1}| = \beta q_n h_n. \]

Both \( v_1 \) and \( v_2 \) are of the form:

\[ v_i = v_i^{(\text{head})} \rho_{\phi_n,1}(w_n) \cdots \rho_{\phi_n,\alpha_i}(w_n) v_i^{(\text{tail})}. \]

In other words, it starts and ends with two incomplete blocks, enclosing the sequence of complete blocks \( \rho_{\phi_n,j}(w_n) \). In order to understand if \( v_1 \) and \( v_2 \) are matched, \( v_1 = v_2 \), let us write both \( v_1 \) and \( v_2 \) starting at zero position. At this point we must consider two cases which are essentially different. The first case is related to the pair of subwords which are very “close”, that is the first positions \( y_1 \) and \( y_2 \) of \( v_1 \) and \( v_2 \) satisfy inequality

\[ |y_1 - y_2| \leq \frac{1}{2}|w_n|. \]

And the second case deals with the situation when matching \( v_1 \) and \( v_2 \) actually we match pairs of independently rotated blocks.

In the first case we can simply apply axiom \( D_n(1/2) \) to the situation when a word \( w = \rho_{\phi}(w_n) \) is matched with itselfs shifted by non-zero amount of positions, since the size of overlapping is equal or grather than \( |w_n| - |y_1 - y_2| \geq |w_n|/2 \).

Concerning the second case it can be easily seen that \( v_1 \) and \( v_2 \) contain a sequence of overlapping block pairs

\[ (\rho_{\alpha_n(j)}(w_n), \rho_{\alpha_n(j+s)}(w_n)), \quad j = j_0, \ldots, j_0 + m - 1, \]

satisfying the following conditions:

\[ |\rho_{\alpha_n(j)}(w_n) \cap \rho_{\alpha_n(j+s)}(w_n)| \geq \frac{1}{2}|w_n|, \quad m \geq \frac{1}{2} \frac{M}{|w_n|} \geq \frac{1}{2} \beta q_n. \]

In this formulas we shortly write \( j + s \) instead of \( j + s \) (mod \( q_n \)), where \( q_n \) is the total number of subblocks \( \rho_{\alpha_n(j)}(w_n) \) in the rotated word \( w_{n+1} \). Now let us apply lemma 2 to the intervals \([j_0, j_0 + m - 1]\) and \([j_0 + s, j_0 + s + m - 1]\) on \( \mathbb{Z} \) and find a set \( J \) such that \( |J| \geq 1/3 \cdot m \) and \( J \cap (s + J) = \emptyset \). Observe that the pair of random variables \( \alpha_n(j), \alpha_n(j+s) \) for \( j \in J \) are globally independent, since all sets \( \{j, j+s\}, j \in J \), are disjoint. Now applying axiom \( D_n(1/2) \) we come to the following conclusion. The only possibility for a pair of overlapping blocks \( \rho_{\alpha_n(j)}(w_n) \) and \( \rho_{\alpha_n(j+s)}(w_n) \) to match is to encounter the situation when
these two blocks are proper rotated and the word in the overlapping enter identically in both blocks, and the probability of such events is exactly $h_n^{-1}$. Thus, counting all the pairs of subwords $v_1$ and $v_2$ under testing, we have

$$P(D_{n+1}(\beta) \mid D_n(1/2)) \geq 1 - h_{n+1}^2 \cdot h_n^{-\# J} \geq 1 - h_{n+1}^3 \cdot h_n^{-\beta q_n/6} \geq 1 - q_n^3 \cdot h_n^{-\beta q_n/12}.$$  

Suppose that some decreasing sequence $\beta_n \to 0$ is given. Applying the previous lemma we can easily find a sequence $q_n$ such that for any $n$ the probability discussed in the lemma is positive and we can take for any step of the construction an appropriate configuration $(\alpha_n(0), \ldots, \alpha_n(q_n - 1))$ such that $D_n(\beta_n)$ is true for all $n$.

Though, remark that this reasoning still has a strong logical gap. In fact, to start this process we need a starting word $w_1$ satisfying $D_1(1/2)$ “a priori”, and actually we can do it by choosing such “never matching” word like

$$10, 100, 1001, \ldots,$$

but in view of further applications to the isomorphism problem we need a version of this lemma that can be applied to any starting word $w_1$.

**Lemma 5.** Suppose that $\beta > \max\{36, m_0 + 1\} \cdot q_n^{-1}$ and assume that the initial word of the construction contains $\nu$-fraction of symbol 1. Then

$$P(D_n(1/2)) \geq 1 - 2q_n^3 \cdot \eta^{-\beta q_n/12},$$

where $\eta = 1 - 2\nu + 2\nu^2$.

**Proof.** Let us examine once more the first case in the proof of lemma 4 taking into account that now we cannot say nothing about the matching of large blocks in $w_n$. An example of such word poorly amenable to decoding is $10101010 \ldots 10$.

Without loss of generality we can assume that in a pair of matched subwords $v_1$ and $v_2$ the second subwords starts just one position to the right of (the beginning of) $v_1$. In other words, if $y_2 = y_1 + 1$. The only information we have is that the last symbol of any full block $\rho_{\alpha_n(j)}(w_n)$ in $v_2$ is compared to the first symbol of the next full block $\rho_{\alpha_n(j+1)}(w_n)$ included in $v_1$, and the probability of this event is less than $\eta$ (see lemma 3). The additional summand to the probability of matching is estimated as

$$h_{n+1}^2 \cdot h_n^2 \cdot \eta^{-\# J} \leq q_n^2 \cdot \eta^{-\beta q_n/12}.$$  

Further, if $|y_2 - y_1| > |w_n|/2$ then the estimate is given by the value

$$h_{n+1}^3 \cdot h_n^2 \cdot \eta^{-\# J} \leq q_n^3 \cdot \eta^{-\beta q_n/12}$$

and the result follows. □

The idea of the forthcoming discussion is to examine, modulo axiom $D_n(\beta_n)$, how many subwords $v$ of length $h_n$ can appear asymptotically in the word $w_{n+1}$. Matching axioms $D_n$ imply that counting subwords is, in a sense, equivalent to counting configuration $(|w_1|, \phi_1, \phi_2)$ describing how $v$ covers two adjacent blocks $\rho_{\alpha_n(j)}(w_n) \rho_{\alpha_n(j+1)}(w_n) \leq w_{n+1}$. Let us consider triples $(\xi, \phi_1, \phi_2)$ indexing such kind of configurations. Recall that for any subword $v$ in $w_{n+1}$ of length $h_n$,

$$v \leq \rho_{\alpha_n(j)}(w_n) \rho_{\alpha_n(j+1)}(w_n).$$

Here $\phi_1 = \alpha_n(j)$ and $\phi_2 = \alpha_n(j+1)$ are the corresponding cyclic rotation parameters of the adjacent blocks covering $v$, and $\xi$ has the meaning of point separating these blocks inside $v$. Notice that any triple defines in a unique way a word $v$ that can occur as a subword of $w_{n+1}$. Let us denote this word as

$$v = V(\xi, \phi_1, \phi_2).$$

Our purpose is to find a set of configurations such that the corresponding words $V(\xi, \phi_1, \phi_2)$ are different.
Lemma 6. Consider two triples $(\xi, \phi_1, \phi_2)$ and $(\eta, \psi_1, \psi_2)$ and suppose that
\[ \beta_n h_n < \xi < \eta < (1 - \beta_n)h_n, \]
and the conditions of Lemma 6 are satisfied. Then $V(\xi, \phi_1, \phi_2) = V(\eta, \psi_1, \psi_2)$ if and only if for any interval among $[0, \xi)$, $[\xi, \eta)$ and $[\eta, h_n)$ the corresponding subwords are matched identically, in other words, if
\[ \eta - \xi = \psi_1 - \phi_1 = \psi_2 - \phi_2 = h_n + \psi_1 - \phi_2. \]

Observe that if the conditions of Lemma 6 are true then evidently $V(\xi, \phi_1, \phi_2) \neq V(\eta, \psi_1, \psi_2)$, except one case: $\phi_1 = \phi_2$ and $\psi_1 = \psi_2$ which means that the adjacent blocks in the iceberg construction are cyclically rotated by the same amount of positions. In this case the boundaries given by the positions $\xi$ and $\eta$ are transparent and cannot be recognized, since $\rho_\phi(wv) = \rho_\phi(w)\rho_\phi(v)$.

Proof of Lemma 6. The idea of this lemma is to require that all the intervals that appear while matching the words generated by these two configurations must be sufficiently long, so that we can apply property $D_n(\beta_n)$. In fact, for intervals $[0, \xi)$, $[\xi, \eta)$ and $[\eta, h_n)$ we know that the corresponding subwords in the blocks must be identically located subwords, and it is easy to represent this conclusion in terms of parameters $\xi$, $\eta$, $(\phi_1, \phi_2)$ and $(\psi_1, \psi_2)$.

Lemma 7. Suppose that all the triples $(\xi, \phi_1, \phi_2)$ are found for subwords in $w_{n+1}$ of length $h_n$. Then counting configurations with $\phi_1 \neq \phi_2$ we have
\[ p_{w_{n+1}}(h_n) \geq \frac{1 - 4\beta_n}{\beta_n} \cdot (h_n^2 - h_n) \]
if $\beta_n \geq h_n^{-1+\delta}$ and $0 < \delta < 1/2$.

Proof. Let us choose a progression $\kappa$, $\kappa + a$, $\ldots$, $\kappa + (m-1)a$ of length $m$, where
\[ m = \left\lfloor \frac{(1 - 2\beta_n)h_n}{a} \right\rfloor \geq \frac{(1 - 2\beta_n)h_n}{\beta_n h_n + 1} - 1 \geq \frac{(1 - 2\beta_n)(1 - \beta_n^{(1-\delta)/\delta})}{\beta_n} - 1 \geq \frac{1 - 4\beta_n}{\beta_n}, \]
since $(1 - \delta)/\delta \geq 1$ for $\delta \leq 1/2$. Any pair $(\tau_1, \tau_2)$ in the set of triples
\[ \Sigma = \{(\kappa + ja, \phi_1, \phi_2): \phi_1 \neq \phi_2, \ 0 \leq j < m\} \]
satisfies the conditions of Lemma 6 and in addition we can state that $V(\tau_1) \neq V(\tau_2)$, since $\phi_1 \neq \phi_2$. Finally, let us count the triples in $\Sigma$:
\[ \#\Sigma = m \cdot (h_n^2 - h_n) \geq \frac{1 - 4\beta_n}{\beta_n} \cdot (h_n^2 - h_n). \]

Theorem 3. Suppose that $\varepsilon > 0$ is given, and $q_n$ satisfies the asymptotics $q_n \sim h_n^{\gamma}$ with $\gamma > 1 - \varepsilon$. Then our iceberg system given by the sequence of i.i.d. random parameters $\alpha_n(j)$ with $|w_{n+1}| = q_n |w_n|$ almost surely generates an infinite word $w_\infty$ such that for $n \geq n_0$
\[ p_{w_\infty}(h_n) \geq p_{w_{n+1}}(h_n) \geq h_n^{3-\varepsilon} \]
for $n \geq n_0$.

Proof. Let us take $\beta_n = h_n^{-1+\delta}$ with $\max\{1 - \gamma, 0\} < \delta < \varepsilon$. In order to apply Lemma 5 and establish property $D_n(\beta_n)$ let us look at the probability in 5:
\[ P(D_n(1/2)) \geq 1 - 2q_n^3 \cdot \eta^{-\beta q_n/12} = 1 - E_n \]
and
\[ E_n = 2\eta^{\beta \gamma \log_n h_n - h_n^{(-1+\delta+\gamma)/12}}, \]
where \( \beta_n q_n = h_n^{-1+\delta+\gamma} \) and \(-1+\delta+\gamma > 0\). Since the series \( \sum_n \mathcal{E}_n \) converges, we can apply Borel–Cantelli lemma and see that almost surely there exists \( n_1 \) such that \( D_n(\beta_n) \) for \( n \geq n_1 \). The symbolic complexity is now estimated as follows:

\[
pw(h_n) \geq \frac{h_n^2}{\beta_n} = \text{const.} \cdot h_n^{3-\delta} \geq h_n^{3-\varepsilon},
\]

whenever \( n \geq n_0 \geq n_1 \).

This theorem can be strengthen as follows to get symbolic complexity arbitrary close to \( l^3 \).

**Theorem 4.** Consider a sequence \( q_n \sim h_n^{\gamma}, \gamma \geq 1 \), defining a set of random iceberg systems given by the independent and uniformly distributed random parameters \( \alpha_n(j) \). Then almost surely the symbolic complexity for this iceberg system satisfies the inequality

\[
pw(h_n) \geq h_n^{3-\varepsilon}, \quad n \geq n_0(\varepsilon).
\]

The only difference with the proof of the previous theorem is that we have to fix some maximal value \( \varepsilon_0 \) and a universal sequence \( \beta_n = h_n^{1+\delta_n} \), such that \( \delta_n \to 0 \) sufficiently slow.

It was observed by S. Ferenczi [5] that the lowest rate of growth for the complexity function concerning rank one systems can be seen along the subsequence \( h_n \). Thus, this case is concerned as the most difficult if we are interested in the estimate from below (see also lemma 1). Now, the above theorems can be easily extended to all lengths of subwords \( l \in \mathbb{N} \). The complexity outside the sequence \( h_n \) becomes even larger (cf. [5]), since we have more freedom combining a word from several rotated copies of \( w_n \). Then we come to the following result.

**Theorem 5.** Given a random iceberg systems with \( q_n \sim h_n^{\gamma}, \gamma \geq 1 \), the infinite word \( w_\infty \) generated by the system with probability one possesses almost cubic estimate for the complexity,

\[
pw(l) \geq l^{3-\varepsilon}, \quad n \geq n_0(\varepsilon),
\]

where the function \( n_0(\varepsilon) \) depends on the system.

5. Application to the calculation of rank

Using the effects discussed in section 4 we can show that certain iceberg systems are not rank one. We give only an idea of proof.

**Theorem 6.** Keeping the conditions of theorem 5 almost surely the iceberg system is not rank one.

Remark that the method we use is not applicable to the multiple rank property, for example, rank two.

**Sketch of the proof.** Suppose that our transformation \( T \) is rank one. Then there exists a partition \( P = \{B_0, B_1\} \) and an orbit \( x_k = T^k x_0 \) such that the complexity of the word \( (x_k) \) is quadratic along a sequence \( H_i \) (see [5]),

\[
p(x_k)(H_i) \leq \frac{1}{2} H_i^2.
\]

The most difficult point in the proof is to manage infinite words which are generated by arbitrary finite partition \( P \). In other words, the following argument is used to show that the complexity of \( (x_k) \) is \( l^{3-\varepsilon} \) providing a contradiction. Having small \( \zeta \), erasing \( \zeta \)-fraction of letters the following estimate remains true:

\[
pw(l) \geq l^{3-\varepsilon}.
\]

Indeed, our infitite word \( (x_k) \) is covered up to a small error \( \zeta_n \) by cyclic rotations of \( w_n \) and asymptotically it is easy to see from approximation that passing through the iceberg with index \( n \) names for all points (but \( \zeta_n \)-part) are \( \zeta_n \)-close.

Applying these arguments we must take into account that \( \zeta_n \) is small but it can go to zero arbitrary slowly. \( \square \)
6. Scaling approximation and Pascal adic transformation

Let us consider a tree associated with the Pascal triangle given by the vertex set $V = \{(k, n) : 0 \leq k \leq n\}$, and the set of admissible paths in this graph:

$$X = \{(y_n, n) : y_{n+1} = y_n \text{ or } y_{n+1} = y_n + 1\}.$$ 

Next, let us consider an order on $X$ induced by the natural order on the vertices on same level: $(k, n) \leq (k+1, n)$. Pascal adic transformation $T$ is the transformation on $X$ that maps a path $x$ to the smallest path greater than $x$ (up to a small negligible set). Pascal map $T$ becomes ergodic measure preserving transformation if we endow $X$ with the natural measure describing statistics of paths (see [13]).

We mention Pascal transformation in this paper by two reasons. First, it belongs to the class of measure preserving maps containing iceberg systems as well and based on the concept of scaling approximation introduced below in this section. The second reason is the following theorem establishing the exact cubic asymptotics for the symbolic complexity of $T$ (see [10]).

**Theorem 7.** (X. Mélá, K. Petersen) The symbolic complexity of the Pascal adic transformation $T$ satisfies $p(l) \sim \frac{1}{6} l^3$.

**Definition.** We say that a measure preserving invertible map satisfies the property of *scaling approximation of rank one* with a scaling function $\lambda(h)$ (or simply *scaling rank one*) if for any $\varepsilon > 0$ almost every orbit encoded with a finite measurable partition can be $\varepsilon$-covered by subwords $u_j$ of a fixed word $|W(\varepsilon)| = h$ such that the average length of $u_j$ is greater than $\lambda(h)(1 + o(1))$.

The property of *funny scaling approximation of rank one*, by analogy with the funny rank one property, is defined in the same manner but taking instead of subwords arbitrary restrictions $v_j = W(\varepsilon)_{I(j)}$ of the words $W(\varepsilon)$ considered as a function $W(\varepsilon) : [0..|W(\varepsilon)| - 1] \to \mathbb{A}$.

![Figure 3](image.png)

**Figure 3.** On this figure we put the collection of towers in the cutting-and-stacking procedure for Pascal adic map, drawn in such a way that all the elementary sets sharing the same level are always labeled with the same letter. Here the 9th step of the construction is shown, the vertical columns are the Rokhlin towers of height $C(n, k)$, where $n = 9$ and $k = 0, 1, \ldots, n$, and the common height of this generalized tower is equal to $h_9 = 2^8 = 256$. Red lines correspond to symbol ‘1’ and blue lines – to symbol ‘1’. **Note:** you can zoom this picture and see in details any part of the coding sequence $x_0, x_1, \ldots, x_{255}$.

**Theorem 8.** Any rank one (respectively rank $m$) transformation has the property of scaling approximation of rank one (rank $m$) with $\lambda(h) = 1$.

**Theorem 9.** Any iceberg map is of scaling rank one with $\lambda(h) = 1/2$. 
This theorem follows from the simple observation that any cyclic rotation of the word $w_n$ is a kind of interval exchange involving two parts, one of length $a|w_n|$ and another of length $(1-a)|w_n|$ which are compensated: $\frac{1}{2}(a + (1-a)) = \frac{1}{2}$.

**Observation 1.** In can be easily seen that if $\lambda(h) \geq \text{const} > 0$ then the map has positive local rank (see the scheme of the proof in [12]).

**Observation 2.** Pascal adic transformation admits scaling approximation of rank one with the scaling function

$$\lambda(h) = \frac{1}{\sqrt{\pi \log_2 h}}$$

The picture on figure 3 is the result of reconstruction of the usual cutting-and-stacking representation of the Pascal map, when the towers of height $C(n,k)$ are lifted so that the sets on the same level of the generalized tower are always marked by the same symbol. Here $C(n,k)$ are the binomial coefficients.

**Question 1.** What is the precise symbolic complexity of the iceberg system with random rotations?

**Question 2 (iceberg systems with multiple IET).** Let us consider a class of systems given by the more general construction: instead of cyclic rotation (which is in fact a kind of discrete two interval exchange map) we apply a random $r$-interval exchange transformation to the words $w_n$ (see [12]). What is the typical asymptotics for the symbolic complexity for this ensemble of symbolic systems?

**Question 3.** Is it true that the scaling function given by (7) is optimal for the Pascal adic transformation?

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