Long Range Scattering and Modified Wave Operators for the Maxwell-Schrödinger System
I. The case of vanishing asymptotic magnetic field

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Abstract

We study the theory of scattering for the Maxwell-Schrödinger system in space dimension 3, in the Coulomb gauge. In the special case of vanishing asymptotic magnetic field, we prove the existence of modified wave operators for that system with no size restriction on the Schrödinger data and we determine the asymptotic behaviour in time of solutions in the range of the wave operators. The method consists in partially solving the Maxwell equations for the potentials, substituting the result into the Schrödinger equation, which then becomes both nonlinear and nonlocal in time, and treating the latter by the method previously used for the Hartree equation and for the Wave-Schrödinger system.

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1 Introduction

This paper is devoted to the theory of scattering and more precisely to the existence of modified wave operators for the Maxwell-Schrödinger (MS) system in 3 + 1 dimensional space time. This system describes the evolution of a charged nonrelativistic quantum mechanical particle interacting with the (classical) electromagnetic field it generates.

We write that system in Lorentz covariant notation: greek indices run from 0 to 3, latin indices run from 1 to 3, indices are raised and lowered with the metric tensor $g_{\mu\nu} = (1, -1, -1, -1)$ so that, for any vector field $v = (v^\mu)$, $v_0 = v^0$ and $v_j = -v^j$, and we use the standard summation convention on repeated indices. The MS Lagrangian density can be written as

$$L = -(1/2) F_{\mu\nu} F^{\mu\nu} - \text{Im} \bar{u} D_0 u + (1/2) D_j u D^j u \quad (1.1)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad , \quad D_\mu = \partial_\mu + i A_\mu$$

and $u$ and $A_\mu$ are respectively a complex scalar valued and a real vector valued function defined in $\mathbb{R}^{3+1}$. Here $\{F_{0j}\}$ are the components of the electric field and $\{F_{ij}\}$ are the components of the magnetic field. The associated variational system is

$$\begin{cases}
i \partial_0 u = (1/2) D_j D^j u + A_0 u \\
\partial^\nu F_{\nu\mu} = J_\mu
\end{cases} \quad (1.2)$$

where the current density $J_\mu$ is given by

$$J_0 = |u|^2 \quad , \quad J_j = - \text{Im} \bar{u} D_j u \quad (1.3)$$

and satisfies the local conservation condition $\partial_\mu J^\mu = 0$. Formally the $L^2$-norm is conserved as well as the energy

$$E(u, A) = \int dx \left\{ (1/2) \left( \sum_{\mu<\nu} F_{\mu\nu}^2 + \sum_j |D_j u|^2 \right) + A_0 |u|^2 \right\} \quad . \quad (1.4)$$

The system (1.2) is gauge invariant and we shall study it in the Coulomb gauge which experience shows to be the most convenient one for the purpose of analysis.

The MS system (1.2) in $\mathbb{R}^{3+1}$ is known to be locally well posed in sufficiently regular spaces in the Lorentz gauge $\partial_\mu A^\mu = 0$ [14] and to have global weak solutions.
in the energy space in various gauges including the Lorentz and Coulomb gauges. However that system is so far not known to be globally well posed in any space whatever the gauge is.

A large amount of work has been devoted to the theory of scattering for nonlinear equations and systems centering on the Schrödinger equation, in particular for nonlinear Schrödinger (NLS) equations, Hartree equations, Klein-Gordon-Schrödinger (KGS) systems, Wave-Schrödinger (WS) systems and Maxwell-Schrödinger (MS) systems. As in the case of the linear Schrödinger equation, one must distinguish the short range case from the long range case. In the former case, ordinary wave operators are expected and in a number of cases proved to exist, describing solutions where the Schrödinger function behaves asymptotically like a solution of the free Schrödinger equation. In the latter case, ordinary wave operators do not exist and have to be replaced by modified wave operators including a suitable phase in their definition. In that respect, the MS system in $IR^{3+1}$ belongs to the borderline (Coulomb) long range case, because of the $t^{-1}$ decay in $L^\infty$ norm of solutions of the wave equation. Such is the case also for the Hartree equation with $|x|^{-1}$ potential, for the WS system in $IR^{3+1}$ and for the KGS system in $IR^{2+1}$.

Whereas a well developed theory of long range scattering exists for the linear Schrödinger equation (see [1] for a recent treatment and for an extensive bibliography), there exist only few results on nonlinear long range scattering. The existence of modified wave operators in the borderline Coulomb case has been proved first for the NLS equation in space dimension $n = 1$ [17], then for the NLS equation in dimensions $n = 2, 3$ and for the Hartree equation in dimension $n \geq 2$ [2], for the derivative NLS equation in dimension $n = 1$ [10], for the KGS system in dimension 2 [18] and for the MS system in dimension 3 [19]. All those results are restricted to the case of small data. In the case of arbitrarily large data, the existence of modified wave operators has been proved for a family of Hartree type equations with general (not only Coulomb) long range interactions [3] [4] [5] by a method inspired by a previous series of papers by Hayashi et al [8] [9] on the Hartree equation. Part of the results have been improved as regards regularity [15] [16]. Finally the existence of modified wave operators without any size restriction on the data has been proved for the WS system in dimension $n = 3$ [6], by an extension of the method of [3] [4].

The present paper is devoted to the extension of the results of [3] to the MS system in the Coulomb gauge, and in particular to the proof of the existence of modified wave operators for that system without any size restriction on the Schrödinger data,
in the special case of vanishing asymptotic magnetic field, namely in the special case
where the asymptotic state for the vector potential is zero. The method consists in
replacing the Maxwell equation for the vector potential by the associated integral
equation and substituting the latter into the Schrödinger equation, thereby obtain-
ing a new Schrödinger equation which is both nonlinear and nonlocal in time. The
latter is then treated as in [4], namely $u$ is expressed in terms of an amplitude $w$
and a phase $\varphi$ satisfying an auxiliary system similar to that introduced in [4]. Wave
operators are constructed first for that auxiliary system, and then used to construct
modified wave operators for the original system (1.2). The detailed construction is
too complicated to allow for a more precise description at this stage, and will be
described in heuristic terms in Section 2 below.

The results of this paper improve over those of [19] by the fact that we do not re-
quire smallness conditions on the Schrödinger asymptotic data. On the other hand,
in contrast with [19], we restrict our attention to the case of vanishing asymptotic
data for the vector potential in order to avoid the difficulties coming from the dif-
ference of propagation properties of the Wave and Schrödinger equations that occur
both in [6] and in [19]. In a subsequent paper we hope to extend the results of the
present one to the general case.

We now give a brief outline of the contents of this paper. A more detailed de-
scription of the technical parts will be given at the end of Section 2. After collecting
some notation and preliminary estimates in Section 3, we study the asymptotic dy-
namics for the auxiliary system in Section 4 and we construct the wave operators
for that system by solving the local Cauchy problem at infinity associated with it in
Section 5, which contains the main technical results of this paper. We finally come
back from the auxiliary system to the original one and construct the modified wave
operators for the latter in Section 6, where the final result is stated in Proposition
6.1.

We conclude this section with some general notation which will be used freely
throughout this paper. We denote by $\| \cdot \|_r$, the norm in $L^r \equiv L^r(\mathbb{R}^3)$ and we define
$\delta(r) = 3/2 - 3/r$. For any interval $I$ and any Banach space $X$, we denote by $C(I, X)$
the space of strongly continuous functions from $I$ to $X$ and by $L^\infty(I, X)$ (resp.
$L^\infty_{loc}(I, X)$) the space of measurable essentially bounded (resp. locally essen-
tially bounded) functions from $I$ to $X$. For real numbers $a$ and $b$, we use the notation
$a \lor b = \text{Max}(a, b)$, and $a \land b = \text{Min}(a, b)$. In the estimates of solutions of the relevant
equations, we shall use the letter $C$ to denote constants, possibly different from an
estimate to the next, depending on various parameters, but not on the solutions themselves or on their initial data. We shall use the notation \( C(a_1, a_2, \cdots) \) for estimating functions, also possibly different from an estimate to the next, depending in addition on suitable norms \( a_1, a_2, \cdots \) of the solutions or of their initial data. Additional notation will be given in Section 3.

## 2 Heuristics and formal computations

In this section, we discuss in heuristic terms the construction of the modified wave operators for the MS system as it will be performed in this paper and we derive the equations needed for that purpose.

We first recast the MS system in the Coulomb gauge \( \partial_t A^j = 0 \) in the form that will be used later on. We introduce the noncovariant notation

\[
\partial_t = \partial_0 \quad , \quad \nabla = \{\partial_j\} \quad , \quad A = \{A^j\} \quad , \quad J = \{J^j\}
\]

so that

\[
\{D_j\} = \nabla - iA \quad , \quad J \equiv J(u, A) = \text{Im } \bar{u}(\nabla - iA)u
\]

and the Coulomb gauge condition becomes \( \nabla \cdot A = 0 \). With that notation, the MS system in the Coulomb gauge becomes

\[
i\partial_t u = -(1/2)(\nabla - iA)^2 u + A_0 u \quad (2.1)
\]

\[
\Delta A_0 = -J_0 \quad (2.2)
\]

\[
\Box A + \nabla (\partial_t A_0) = J \quad (2.3)
\]

where \( \Box = \partial_t^2 - \Delta \). We replace that system by a formally equivalent one in the following standard way. We solve (2.2) for \( A_0 \) as

\[
A_0 = -\Delta^{-1} J_0 = (4\pi|x|)^{-1} * |u|^2 \equiv g(u) \quad (2.4)
\]

where \( * \) denotes the convolution in \( \mathbb{R}^3 \), so that by the current conservation \( \partial_t J_0 + \nabla \cdot J = 0 \),

\[
\partial_t A_0 = \Delta^{-1} \nabla \cdot J \quad . \quad (2.5)
\]

Substituting (2.4) into (2.1) and (2.3) into (2.5), we obtain the new system

\[
i\partial_t u = -(1/2)(\nabla - iA)^2 u + g(u)u \quad (2.6)
\]

\[
\Box A = PJ \equiv P \text{ Im } \bar{u}(\nabla - iA)u \quad (2.7)
\]
where $P = 1 - \nabla \Delta^{-1} \nabla$ is the projector on divergence free vector fields. The system (2.6) (2.7) is the starting point of our investigation. We want to address the problem of classifying the asymptotic behaviours in time of the solutions of the system (2.6) (2.7) by relating them to a set of model functions $V = \{ v = v(v_+) \}$ parametrized by some data $v_+$ and with suitably chosen and preferably simple asymptotic behaviour in time. For each $v \in V$, one tries to construct a solution $(u, A)$ of the system (2.6) (2.7) defined at least in a neighborhood of infinity in time and such that $(u, A)(t)$ behaves as $v(t)$ when $t \to \infty$ in a suitable sense. We then define the wave operator as the map $\Omega : v_+ \to (u, A)$ thereby obtained. A similar question can be asked for $t \to -\infty$. We restrict our attention to positive time. The more standard definition of the wave operator is to define it as the map $v_+ \to (u, A)(0)$, but what really matters is the solution $(u, A)$ in the neighborhood of infinity in time, namely in some interval $[T, \infty)$. Continuing such a solution down to $t = 0$ is a somewhat different question which we shall not touch here, especially since the MS system is not known to be globally well posed in any reasonable space.

In the case of the MS system, which is long range, it is known that one cannot take for $V$ the set of solutions of the linear problem underlying (2.6) (2.7), namely

$$
\begin{align*}
\left\{ 
\begin{array}{ll}
    i\partial_t u &= -(1/2) \Delta u \\
    \Box A &= 0
\end{array}
\right.
\end{align*}
$$

(2.8)

and one of the tasks that will be performed in this paper will be to construct a better set $V$ of model asymptotic functions. The same situation prevails for long range Hartree equations and for the WS system and we refer to Sections 2 of [3] [4] [6] for more details on that point.

Constructing the wave operators essentially amounts to solving the Cauchy problem with infinite initial time. The system (2.6) (2.7) in this form is not well suited for that purpose, and we now perform a number of transformations leading to an auxiliary system for which that problem can be handled. We first replace the equation (2.7) by the associated integral equation namely

$$
A = A_0^\infty + A_1(u, A)
$$

(2.9)

where

$$
A_0^\infty = \dot{K}(t)A_+ + K(t)\dot{A}_+
$$

(2.10)

$$
A_1(u, A) = -\int_t^\infty dt' \ K(t - t')PJ(u, A)(t')
$$

(2.11)
with
\[ \omega = (-\Delta)^{1/2}, \quad K(t) = \omega^{-1} \sin \omega t, \quad \dot{K}(t) = \cos \omega t. \]

In particular, \( A_0^\infty \) is a solution of the free (vector valued) wave equation with initial data \((A_+, \dot{A}_+)\) at \( t = 0 \), and \((A_+, \dot{A}_+)\) is naturally interpreted as the asymptotic state for \( A \). In order to ensure the condition \( \nabla \cdot A = 0 \), we assume that \( \nabla \cdot A_+ = \nabla \cdot \dot{A}_+ = 0 \).

We next perform the same change of variables as for the Hartree equation and for the WS system. That change of variables is well adapted to the study of the asymptotic behaviour in time. The unitary group
\[ U(t) = \exp(i(t/2)\Delta) \] (2.12)
which solves the free Schrödinger equation can be written as
\[ U(t) = M(t) \, D(t) \, F \, M(t) \] (2.13)
where \( M(t) \) is the operator of multiplication by the function
\[ M(t) = \exp\left(\frac{ix^2}{2t}\right), \] (2.14)
\( F \) is the Fourier transform and \( D(t) \) is the dilation operator
\[ (D(t) \, f)(x) = (it)^{-n/2} \, f(x/t) \] (2.15)
normalized to be unitary in \( L^2 \). We shall also need the operator \( D_0(t) \) defined by
\[ (D_0(t)f)(x) = f(x/t). \] (2.16)
We parametrize the Schrödinger function \( u \) in terms of an amplitude \( w \) and of a real phase \( \varphi \) as
\[ u(t) = M(t) \, D(t) \, \exp[-i\varphi(t)]w(t). \] (2.17)
Correspondingly we change the variable for the vector potential from \( A \) to \( B \) according to
\[ A(t) = t^{-1} \, D_0(t) \, B(t) \] (2.18)
and similarly for \( A_0^\infty \) and \( A_1 \).

We now perform the change of variables (2.17) (2.18) on the system (2.6) (2.7). Substituting (2.17) (2.18) into (2.6) and commuting the Schrödinger operator with \( MD \), we obtain
\[ \left\{ i\partial_t + (2t^2)^{-1}(\nabla - iB)^2 + t^{-1}x \cdot B - t^{-1} \, g(w) \right\}(\exp(-i\varphi)w) = 0 \] (2.19)
where $g$ is defined by (2.4). Expanding the derivatives, using the condition $\nabla \cdot B = 0$ and introducing the notation $s = \nabla \varphi$, we rewrite (2.19) as

$$
\left\{ i\partial_t + \partial_t \varphi + (2t^2)^{-1} \Delta - it^{-2} \left( (s + B) \cdot \nabla + (1/2)(\nabla \cdot s) \right) - (2t^2)^{-1} (s + B)^2 
+ t^{-1} x \cdot B - t^{-1} g(w) \right\} w = 0 .
$$

We next turn to the Maxwell equation (2.7). Substituting (2.17) (2.18) into the definition of $J$, we obtain

$$
J(t) = t^{-3} D_0(t) \left\{ x|w|^2 + t^{-1} \Im \bar{w} \nabla w - (s + B)|w|^2 \right\}
= t^{-3} D_0(t) \left( M_a + t^{-1} M_b \right)
$$

where

$$
M_a = x|w|^2 , \quad M_b = \Im \bar{w} \nabla w - (s + B)|w|^2 .
$$

Using the identity

$$
- \int_t^\infty dt' K(t - t') t'^{-3-j} D_0(t') PM(t') = t^{-1-j} D_0(t) F_j(M)
$$

where $F_j$ is defined by

$$
F_j(M) = \int_1^\infty d\nu \omega^{-1} \sin(\omega(\nu - 1)) \nu^{-3-j} D_0(\nu) M(\nu t) ,
$$

we rewrite (2.7) as

$$
B = B_0^\infty + B_1
$$

with

$$
B_1 = B_a + B_b \quad \text{(2.27)}
B_a \equiv B_a(w) = F_0(M_a) \quad \text{(2.28)}
B_b = t^{-1} F_1(M_b) . \quad \text{(2.29)}
$$

This completes the change of variables for the system (2.6) (2.7). Now however we have parametrized $u$ in terms of an amplitude $w$ and a phase $\varphi$ and we have only one equation for the two functions $(w, \varphi)$. We arbitrarily impose a second equation, namely a Hamilton-Jacobi (or eikonal) equation for the phase $\varphi$, thereby splitting the equation (2.20) into a system of two equations, the other one of which being
a transport equation for the amplitude \( w \). There is a large amount of freedom in the choice of the equation for the phase. The role of the phase is to cancel the long range terms in (2.20) coming from the interaction. All the interaction terms in (2.20) having an explicit \( t^{-2} \) factor are expected to be short range and are therefore left in the equation for \( w \). Such is also the case for the contribution of \( B_b \) to \( x \cdot B \) because of the \( t^{-1} \) factor in (2.29). The term \( t^{-1}g(w) \) is clearly long range (of Hartree type), and is therefore included in the \( \varphi \) equation. The term \( t^{-1}(x \cdot B_a) \) is also long range, but since it is less regular than the previous one, it is convenient to split \( x \cdot B_a \) into a short range and a long range part, namely

\[
x \cdot B_a = (x \cdot B_a)_S + (x \cdot B_a)_L
\]

(2.30)
in the following way. We take \( 0 < \beta < 1 \) and we define

\[
\begin{cases}
(x \cdot B_a)_S = F^{-1} \chi(|\xi| > t^{\beta}) \ F(x \cdot B_a) \\
(x \cdot B_a)_L = F^{-1} \chi(|\xi| \leq t^{\beta}) \ F(x \cdot B_a)
\end{cases}
\]

(2.31)
where \( \chi(P) \) is the characteristic function of the set where \( P \) holds. (In practice we shall need \( 0 < \beta < 1/2 \) in most of the applications). Finally the contribution of \( B^\infty_0 \) to \( B \) should be considered short range. The Hamilton-Jacobi equation for the phase is then taken to be

\[
\partial_t \varphi = (2t^2)^{-1} s^2 + t^{-1}g(w) - t^{-1}(x \cdot B_a)_L(w)
\]

(2.32)

We are now in a position to introduce the auxiliary system which will be used to study the MS system. From now on, we restrict our attention to the case \( B^\infty_0 = 0 \), namely to the case where the asymptotic state \( (A_+, \dot{A}_+) \) for the vector potential is zero. This is the technical meaning of the expression “vanishing asymptotic magnetic field” used in the title and in the introduction of this paper.

The equation for \( w \) is obtained by substituting (2.32) into (2.20). The resulting equation contains \( \varphi \) only through \( s = \nabla \varphi \). The same property holds for the RHS of (2.32) and for (2.20)-(2.29). It is then convenient to replace (2.32) by its gradient so as to obtain a final system containing only \( s \). The Maxwell equation now has \( B = B_1 \). Furthermore we shall regard (2.28) as the definition of \( B_a \) as a function of \( w \), we shall take \( B_b \) as the dynamical variable and we shall regard (2.27) (with \( B = B_1 \)) as a change of variable from \( B \) to \( B_b \). The actual equation is then (2.29) regarded as an equation for \( B_b \). We finally introduce the notation

\[
Q(s, w) = s \cdot \nabla w + (1/2)(\nabla \cdot s)w
\]

(2.33)
for the transport term. We then obtain the auxiliary system in the following form:

\[
\begin{aligned}
\partial_t w &= i(2t^2)^{-1}\Delta w + t^{-2}Q(s + B, w) - i(2t^2)^{-1}(2B \cdot s + B^2)w \\
+ it^{-1}((x \cdot B_a)s + x \cdot B_b)w &\equiv L(w, s, B_b)w \\
\partial_t s &= t^{-2}s \cdot \nabla s + t^{-1}\nabla g(w) - t^{-1}\nabla(x \cdot B_a) \equiv L(w, s, B_b)w
\end{aligned}
\]  
(2.34)

with \( B = B_a + B_b \) and \( B_a \) defined by (2.28) (2.22) (2.25). The phase \( \varphi \) is regarded as a derived quantity to be recovered from (2.32). The linear operator \( L(w, s, B_b) \) is defined in an obvious way by (2.34). Its dependence on \( s, B_b \) is explicit, while its dependence on \( w \) occurs through \( B_a \). Since \( B_a \) and a fortiori \( x \cdot B_a \) contains \( xw \), it will be useful to have an explicit evolution equation for \( xw \). From (2.34) we obtain immediately

\[
\partial_t xw = L(w, s, B_b)xw - it^{-2}\nabla w - t^{-2}(s + B)w .
\]  
(2.36)

The Cauchy problem for the system (2.34) (2.35) with initial data \((w, s) (t_0) = (w_0, s_0)\) for some time \( t_0 \) is no longer a usual PDE Cauchy problem since \( B_a \) depends nonlocally in time and since (2.35) is an integral equation in time. A convenient way to handle that difficulty is to replace that problem by a partly linearized version thereof, namely

\[
\begin{aligned}
\partial_t w' &= L(w, s, B_b)w' \\
\partial_t s' &= t^{-2}s \cdot \nabla s' + t^{-1}\nabla g(w) - t^{-1}\nabla(x \cdot B_a) \equiv L(w, s, B_b)w' \\
B_b' &= t^{-1} F_1 \left( \text{Im} \ w \nabla w - (s + B)|w|^2 \right)
\end{aligned}
\]  
(2.37)

still with \( B = B_a + B_b \) and \( B_a = B_a(w) \). Correspondingly, the evolution equation for \( xw' \) becomes

\[
\partial_t xw' = L(w, s, B_b)xw' - it^{-2}\nabla w' - t^{-2}(s + B)w' .
\]  
(2.39)

Solving (2.37) with suitable initial data for given \((w, s, B_b)\), together with (2.38), defines a map \( \Gamma : (w, s, B_b) \rightarrow (w', s', B_b') \) and solving (2.34) (2.35) reduces to finding a fixed point of \( \Gamma \), which in favourable cases can be done for instance by contraction.

The first problem that we shall consider is whether the auxiliary system (2.34) (2.35) defines a dynamics for large time. This will be the subject of Section 4 below.
In particular we shall prove that the Cauchy problem for that system is locally well
posed in a neighborhood of infinity in time, namely that \((2.34) (2.35)\) with initial
data \((w, s)(t_0) = (w_0, s_0)\) has a unique solution defined in \([T, \infty)\) for some \(T \leq t_0,\)
with \(t_0\) and \(T\) suitably large depending on \((w_0, s_0)\), and with continuous dependence
on the data. Those results will be obtained through the use of the linearized system
\((2.37)(2.38)\) by following the method sketched above. In addition, we shall derive
some asymptotic properties of the solutions thereby obtained. In particular, for
those solutions, \(w(t)\) tends to a limit \(w_+\) when \(t \to \infty\).

The previous results are insufficient to construct the wave operators, namely
to solve the Cauchy problem for \((2.34) (2.35)\) with infinite initial time because (i)
\(T \to \infty\) when \(t_0 \to \infty\) and (ii) the solutions are not estimated uniformly in \(t_0,\)
so that it is not possible to perform the limit \(t_0 \to \infty\) in those results. In order
to construct the wave operators, we follow instead the procedure explained at the
beginning of this section. We choose an asymptotic function \(v\) and we look for
solutions that behave asymptotically as \(v\) when \(t \to \infty\). The asymptotic \(v\) will be
taken as a pair \((W, S)\) with \(S = \nabla \phi\) and with \(W(t)\) tending to a limit \(w_+\) as \(t \to \infty\).
This will provide the asymptotic form of \((w, s)\). No asymptotic form is needed
for \(B_b\), because \(B_b\) tends to zero at infinity. In order for \((W, S)\) to be an adequate
asymptotic function, it has to be an approximate solution of the system \((2.34)\). This
is achieved by solving that system approximately by iteration and taking for \((W, S)\)
an iterate of suitable order. In the present case, the first iteration turns out to be
sufficient, and we shall take accordingly

\[
\begin{cases}
W(t) = U^*(1/t)w_+ \\
S(t) = \int_1^t dt' t'^{-1} (\nabla g(W) - \nabla(x \cdot B_a)_{L}(W))
\end{cases}
\]  

\((2.40)\)

for some given \(w_+\) and for \(t \geq 1\). Actually, at the cost of some loss of regularity, we
could also make the simpler choice

\[
\begin{cases}
W(t) = w_+ \\
S(t) = \int_1^t dt' t'^{-1} (\nabla g(w_+) - \nabla(x \cdot B_a)_{L}(w_+))
\end{cases}
\]  

\((2.41)\)

The latter \(S(t)\) can be explicitly computed :

\[
S(t) = \ln t \, \nabla g(w_+) - \ln \left(t(\omega \lor 1)^{-1/\beta} \lor 1\right) \nabla(x \cdot B_a)(w_+)  
\]  

\((2.42)\)
where the second \( \ell n \) is defined in an obvious way in Fourier transformed variables.

In order to solve the system (2.34) (2.35) with \((w, s)\) behaving asymptotically as a given \((W, S)\) (possibly but not necessarily (2.40)), we make a change of variables from \((w, s)\) to \((q, \sigma)\) defined by

\[
(q, \sigma) = (w, s) - (W, S) .
\]  

Substituting (2.43) into (2.34) (2.35) will yield a new auxiliary system for the variables \((q, \sigma, B)\). For that purpose we introduce the following additional notation. We define

\[
g(w, w) = (4|\pi|x|^{-1}) \ast \text{Re } \bar{w}_{1} w_{2} 
\]  

and

\[
B_{a}(w_{1}, w_{2}) = F_{0}(x \text{ Re } \bar{w}_{1} w_{2})
\]  

so that \(g(w) = g(w, w)\) and \(B_{a}(w) = B_{a}(w, w)\). We next define \(B_{*} = B_{a}(W)\) and

\[
G \equiv G(q, W) = B_{a}(w) - B_{*} = B_{a}(q, q + 2W) .
\]  

We finally define the remainders

\[
\begin{align*}
R_{1}(w, s, B_{b}) &= -\partial_{t} w + L(w, s, B_{b}) w \\
R_{2}(w, s) &= -\partial_{s} s + t^{-2} s \cdot \nabla s + t^{-1} \nabla g(w) - t^{-1} \nabla (x \cdot B_{a}) L(w) \\
R_{3}(w, s, B_{b}) &= -B_{b} + t^{-1} F_{1}(\text{Im } \bar{w} \nabla w - (s + B)|w|^{2})
\end{align*}
\]  

so that the system (2.34) (2.35) can be rewritten as \(R_{i} = 0, i = 1, 2, 3\), and for general \((w, s, B_{b})\), the remainders measure its failure to satisfy that system. Using the previous notation, we can rewrite (2.34) (2.35) in the new variables as follows

\[
\begin{align*}
\partial_{t} q &= L(w, s, B_{b}) q + t^{-2} Q(\sigma + G + B_{b}, W) \\
- &- it^{-2} (B \cdot \sigma + (G + B_{b})(S + (B + B_{*})/2)) W \\
+ &+ it^{-1} ((x \cdot G)_{s} + x \cdot B_{b}) W + R_{1}(W, S, 0) \equiv L(w, s, B_{b}) q + \tilde{R}_{1} \\
\partial_{s} \sigma &= t^{-2} (s \cdot \nabla \sigma + \sigma \cdot \nabla S) + t^{-1} \nabla g(q, q + 2W) - t^{-1} \nabla (x \cdot G) L \\
+ &+ R_{2}(W, S) \equiv t^{-2} s \cdot \nabla \sigma + \tilde{R}_{2}
\end{align*}
\]  

so that

\[
B_{b} = t^{-1} F_{1}(\text{Im } \bar{w} \nabla q + \bar{q} \nabla W) - (s + B) \left( |q|^{2} + 2 \text{Re } q W \right) \\
- (\sigma + G + B_{b}) |W|^{2} \right) + R_{3}(W, S, 0) \equiv \tilde{R}_{3} .
\]  

(2.50)
For the same reason as above, we also need the evolution equation for $xq$. We define the linear operator $L_1$ by rewriting the evolution equation for $q$ in the form

$$\partial_t q = L(w, s, B_b)q + L_1 W + R_1(W, S, 0).$$  \hfill (2.51)

The evolution equation for $xq$ then becomes

$$\partial_t xq = L(w, s, B_b)xq + L_1 xW - it^{-2} \nabla q - t^{-2}(s + B)q$$

$$- t^{-2}(\sigma + G + B_b)W + xR_1(W, S, 0).$$  \hfill (2.52)

We shall also need to rewrite the partly linearized system (2.37) (2.38) in terms of the new variables $(q, \sigma)$ defined by (2.43) and $(q', \sigma')$ defined similarly by

$$(q', \sigma') = (w', s') - (W, S).$$  \hfill (2.53)

The system (2.37) (2.38) then becomes

$$\begin{cases}
\partial_t q' = L(w, s, B_b)q' + \tilde{R}_1 \\
\partial_t \sigma' = t^{-2} s \cdot \nabla \sigma' + \tilde{R}_2
\end{cases}$$

$$B'_b = \tilde{R}_3$$  \hfill (2.54)

where $\tilde{R}_i, i = 1, 2, 3,$ are defined by (2.49) (2.50), and the equation (2.38) becomes

$$\partial_t xq' = L(w, s, B_b)xq' + L_1 xW - it^{-2} \nabla q' - t^{-2}(s + B)q'$$

$$- t^{-2}(\sigma + G + B_b)W + xR_1(W, S, 0).$$  \hfill (2.56)

The main technical result of this paper is the construction of solutions $(q, \sigma, B_b)$ of the auxiliary system (2.43) (2.50) defined for large time and tending to zero at infinity in time. That construction is performed by solving the Cauchy problem for the linearized system (2.54) (2.55), first for finite initial time $t_0$, and then for infinite initial time by taking the limit $t_0 \to \infty$. One then proves the existence of a fixed point for the map $\Gamma : (q, \sigma, B_b) \to (q', \sigma', B'_b)$ thereby defined by a contraction method, as mentioned above. With that result available, it is an easy matter to construct the modified wave operator $\Omega$ for the MS system in the form (2.6) (2.7). We start from the asymptotic state $u_+$ for $u$ and we define $w_+ = Fu_+$. The asymptotic state $(A_+, \dot{A}_+)$ for $A$ is taken to be zero. We define $(W, S)$ by (2.40). We solve the system (2.49) (2.50) for $(q, \sigma, B_b)$ as indicated above. Through (2.43), this yields a
solution \((w, s, B_b)\) of the auxiliary system (2.34) (2.35) defined for large time. From \(s\) we reconstruct the phase \(\varphi\) by using (2.32). We finally substitute \((w, \varphi, B_b)\) into (2.17) (2.18) with \(B = B_a + B_b\) and \(B_a\) defined by (2.28). This yields a solution \((u, A)\) of the system (2.6) (2.7) defined for large time. The modified wave operator is the map \(u_+ \to (u, A)\) thereby obtained.

The main result of this paper is the construction of \((u, A)\) from \(u_+\), as described above, together with the asymptotic properties of \((u, A)\) that follow from that construction. It will be stated below in full mathematical detail in Proposition 6.1. We give here only a heuristic preview of that result, stripped from most technicalities.

**Proposition 2.1.** Let \(k > 3/2, 0 < \beta < 1/2\) and let \(\alpha > 1\) be such that \(\beta(\alpha+1) \geq 1\). Let \(u_+\) be such that \(w_+ = Fu_+ \in H^{k+\alpha+1}\) and \(xw_+ \in H^{k+\alpha}\). Define \((W, S)\) by (2.40). Then

1. There exists \(T = T(w_+), 1 \leq T < \infty\), such that the auxiliary system (2.34) (2.35) has a unique solution \((w, s, B_b)\) in a suitable space, defined for \(t \geq T\), and such that \((w-W, s-S, B_b)\) tends to zero in suitable norms when \(t \to \infty\).

2. There exist \(\varphi\) and \(\phi\) such that \(s = \nabla \varphi, S = \nabla \phi, \phi(1) = 0\) and such that \(\varphi-\phi\) tends to zero in suitable norms when \(t \to \infty\). Define \((u, A)\) by (2.17) (2.18) with \(B = B_a + B_b\) and \(B_a\) defined by (2.28). Then \((u, A)\) solves the system (2.6) (2.7) for \(t \geq T\) and \((u, A)\) behaves asymptotically as \(MD \exp(-i\phi)W, t^{-1}D_0 B_a(W)\) in the sense that the difference tends to zero in suitable norms (for which each term separately is \(O(1)\)) when \(t \to \infty\).

We now describe the contents of the technical parts of this paper, namely Sections 3-6. In Section 3, we introduce some notation, define the relevant function spaces and collect a number of preliminary estimates. In Section 4, we study the Cauchy problem for large time for the auxiliary system (2.34) (2.35). We solve the Cauchy problem with finite initial time for the linearized system (2.37) (Proposition 4.1), we prove a number of uniqueness results for the system (2.34) (2.35) (Proposition 4.2), we solve the Cauchy problem for the system (2.34) (2.35) for large but finite \(t_0\) in the special case \(A_0^\infty = 0\) (Proposition 4.3) and we finally derive some asymptotic properties of the solutions thereby obtained (Proposition 4.4). In Section 5, we study the Cauchy problem at infinity for the auxiliary system (2.34) (2.35) in the difference form (2.49) (2.50). We prove the existence of solutions first for the linearized system (2.54) (2.53), for \(t_0\) finite (Proposition 5.1) and infinite (Proposi-
tion 5.2), and then for the nonlinear system (2.49) (2.50) for \( t_0 \) infinite (Proposition 5.3). Finally in Section 6, we construct the modified wave operators for the system (2.6) (2.7) from the results previously obtained for the system (2.49) (2.50) and we derive the asymptotic estimates for the solutions \((u, A)\) in their range that follow from the previous estimates (Proposition 6.1).

3 Notation and preliminary estimates

In this section, we define the function spaces where we shall study the auxiliary system (2.34) (2.35) and we collect a number of estimates which will be used throughout this paper. In addition to the standard Sobolev spaces \( H^k \), we shall use the associated homogeneous spaces \( \dot{H}^k \) with norm \( \| u; \dot{H}^k \| = \| \omega^k u \|_2 \), where \( \omega = (-\Delta)^{1/2} \) and the spaces

\[ K^k = \dot{H}^1 \cap \dot{H}^k , \]

where it is understood that \( \dot{H}^1 \subset L^6 \). We shall use the notation

\[ |w|_k = \| w; H^k \| \quad , \quad |s|_k = \| s; K^k \| . \quad (3.1) \]

We shall look for solutions of the auxiliary system in spaces of the type \( C(I, X^k) \) where \( I \) is an interval and

\[ X^k = \{ (w, s, B) : w \in H^{k+1}, \ xw \in H^k, \ s \in K^{k+2}, \ B \in K^{k+1}, \ x \cdot B \in K^{k+1} \} . \quad (3.2) \]

For the needs of this paper, we could have replaced \( \dot{H}^1 \) in the definition of \( K^k \) by \( \dot{H}^{k_0} \) for some \( k_0 \) with \( 1/2 < k_0 < 3/2 \). We have chosen \( k_0 = 1 \) for simplicity.

We shall use extensively the following Sobolev inequalities, stated here in \( \mathbb{R}^n \), but to be used only for \( n = 3 \).

**Lemma 3.1.** Let \( 1 < q, r < \infty, 1 < p \leq \infty \) and \( 0 \leq j < k \). If \( p = \infty \), assume that \( k - j > n/r \). Let \( \sigma \) satisfy \( j/k \leq \sigma \leq 1 \) and

\[ n/p - j = (1 - \sigma)n/q + \sigma(n/r - k) . \]

Then the following inequality holds

\[ \| \omega^j u \|_p \leq C \| u \|^{1-\sigma}_q \| \omega^k u \|^{\sigma} . \quad (3.3) \]
The proof follows from the Hardy-Littlewood-Sobolev (HLS) inequality ([11], p. 117) (from the Young inequality if \( p = \infty \)), from Paley-Littlewood theory and interpolation.

We shall also use extensively the following Leibnitz and commutator estimates.

**Lemma 3.2.** Let \( 1 < r, r_1, r_3 < \infty \) and

\[
1/r = 1/r_1 + 1/r_2 = 1/r_3 + 1/r_4.
\]

Then the following estimates hold

\[
\| \omega^m (uv) \|_r \leq C (\| \omega^m u \|_{r_1} \| v \|_{r_2} + \| \omega^m v \|_{r_3} \| u \|_{r_4})
\]

(3.4)

for \( m \geq 0 \),

\[
\| [\omega^m, u]v \|_r \leq C (\| \omega^m u \|_{r_1} \| v \|_{r_2} + \| \omega^{m-1} v \|_{r_3} \| \nabla u \|_{r_4})
\]

(3.5)

for \( m \geq 1 \), where \([ , , \]\) denotes the commutator, and

\[
\| [\omega^m, u]v \|_2 \leq C \| F(\omega^m u) \|_1 \| v \|_2
\]

(3.6)

for \( 0 \leq m \leq 1 \).

**Proof.** The proof of (3.4) (3.5) is given in [12] [13] with \( \omega \) replaced by \( < \omega > \) and follows therefrom by a scaling argument. The proof of (3.6) follows from

\[
|F([\omega^m, u]v)(\xi)| = \left| \int d\eta |\xi|^m - |\eta|^m|\hat{u}(\xi - \eta)\hat{v}(\eta)| \right|
\leq \int d\eta |\xi - \eta|^m|\hat{u}(\xi - \eta)| \| \hat{v}(\eta) \|
\]

and from the Young inequality and the Plancherel theorem.

We shall also need the following consequence of Lemma 3.2.

**Lemma 3.3.** Let \( m \geq 0 \) and \( 1 < r < \infty \). Then the following estimate holds

\[
\| \omega^m(e^\varphi - 1) \|_r \leq \| \omega^m \varphi \|_r \exp (C \| \varphi \|_{\infty})
\]

(3.7)
**Proof.** For any integer \( n \geq 2 \), we estimate

\[
a_n \equiv \| \omega^m \varphi^n \|_r \leq C \left( \| \omega^m \varphi \|_r \| \varphi \|_\infty^{n-1} + \| \omega^m \varphi^{n-1} \|_r \| \varphi \|_\infty \right)
\]

\[
= C \left( a_1 b^{n-1} + a_{n-1} b \right)
\]

by (3.4), where \( b = \| \varphi \|_\infty \) and we can assume \( C \geq 1 \) without loss of generality. It follows easily from that inequality that

\[
a_n \leq n(Cb)^{n-1} a_1
\]

for all \( n \geq 1 \), from which (3.6) follows by expanding the exponential.

\( \Box \)

We shall apply Lemma 3.2 in the form of Lemma 3.4 below which we state for clarity in general dimension \( n \) and with general \( k_0 < n/2 \) and \( K^m = \dot{H}^m \cap \dot{H}^{k_0} \).

**Lemma 3.4.** Let \( \tilde{m} > n/2 \). The following inequalities hold:

1. Let \( 0 \leq m \leq \tilde{m} \). Then

\[
\| fg; \dot{H}^m \| \leq C_{\tilde{m}} \| f \|_{\dot{H}^{\tilde{m}}} \| g; \dot{H}^m \| \quad \text{for } m < n/2 ,
\]

\[
\leq C_{\tilde{m}} \| f \|_m \| g \|_m \quad \text{for } n/2 \leq m \leq \tilde{m} , \tag{3.8}
\]

\[
|fg|_m \leq C_{\tilde{m}} |f|_m |g|_m \quad \text{for } k_0 \leq m \leq \tilde{m} , \tag{3.9}
\]

\[
|fg|_m \leq C_{\tilde{m}} |f|_m |g|_m \quad \text{for } 0 \leq m \leq \tilde{m} . \tag{3.10}
\]

2. Let \( 0 \leq m \leq \tilde{m} + 1 \). Then

\[
\| [\omega^m, f] \nabla g \|_2 \leq C \| \nabla f \|_{\tilde{m}} \| \omega^m g \|_2 \quad \text{for } 0 \leq m < n/2 + 1 , \tag{3.11}
\]

\[
\leq C \| \nabla f \|_{\tilde{m}} \| \nabla g \|_{m-1} \quad \text{for } n/2 + 1 \leq m \leq \tilde{m} + 1 , \tag{3.12}
\]

\[
\leq C \| \nabla f \|_{\tilde{m}} \| g \|_{m} \quad \text{for } k_0 \leq m \leq \tilde{m} + 1 , \tag{3.13}
\]

\[
\leq C \| \nabla f \|_{\tilde{m}} \| g \|_{m} \quad \text{for } 0 \leq m \leq \tilde{m} + 1 . \tag{3.14}
\]

3. Let \( n \geq 3 \) and \( m \geq k_0, 1 < m \leq \tilde{m} + 1 \). Then

\[
\| [\omega^m, f] g \|_2 \leq C |f|_{\tilde{m}} |g|_{\tilde{m}} . \tag{3.15}
\]

**Proof.** Part (1). By Lemma 3.2, we estimate

\[
\| \omega^m fg \|_2 \leq C \left( \| f \|_\infty \| \omega^m g \|_2 + \| \omega^m f \|_{n/3} \| g \|_r \right) \tag{3.16}
\]
with $0 < \delta = \delta(r) \leq n/2$.

For $m < n/2$, we choose $\delta = m$ and continue (3.16) by
\[
\cdots \leq C \left( \| f \|_\infty + \| \omega^{n/2} f \|_2 \right) \| \omega^m g \|_2 \leq C|f|_m \| \omega^m g \|_2
\]
by Sobolev inequalities, which yields (3.8) in this case.

For $m > n/2$, we choose $\delta = n/2$ and continue (3.16) by
\[
\cdots \leq C \left( \| f \|_\infty \| \omega^m g \|_2 + \| \omega^m f \|_2 \| g \|_\infty \right) \leq C|f|_m |g|_m
\]
which yields (3.8) since $m \leq \bar{m}$.

For $m = n/2$, we take $k_0 \vee (n - \bar{m}) \leq \delta < n/2$ and estimate the last term in (3.16) by
\[
C \| \omega^{n-\delta} f \|_2 \| \omega^{\delta} g \|_2
\]
by Sobolev inequalities, which yields again (3.8) in that case.

Finally (3.9) and (3.10) are immediate consequences of (3.8).

Part (2). For $m > 1$, we estimate by Lemma 3.2
\[
\| [\omega^m, f] \nabla g \|_2 \leq C \left( \| \nabla f \|_\infty \| \omega^m g \|_2 + \| \omega^m f \|_{n/\delta} \| \nabla g \|_r \right) \tag{3.17}
\]
with $0 < \delta = \delta(r) \leq n/2$.

For $m < n/2 + 1$, we choose $\delta = m - 1$ and continue (3.17) by
\[
\cdots \leq C \left( \| f \|_\infty + \| \omega^{n/2} \nabla f \|_2 \right) \| \omega^m g \|_2
\]
by Sobolev inequalities, which yields (3.11) in this case.

For $m > n/2 + 1$, we choose $\delta = n/2$ and continue (3.17) by
\[
\cdots \leq C \left( \| \nabla f \|_\infty \| \omega^m g \|_2 + \| \omega^m f \|_2 \| \nabla g \|_\infty \right) \leq C|\nabla f|_{m-1} |\nabla g|_{m-1}
\]
which yields (3.12) since $m \leq \bar{m} + 1$.

For $m = n/2 + 1$, we take $k_0 \vee (n - \bar{m}) \leq \delta < n/2$ and estimate the last term in (3.17) by
\[
C \| \omega^{n-\delta} \nabla f \|_2 \| \omega^{\delta} \nabla g \|_2
\]
by Sobolev inequalities, which yields again (3.12) in this case.

For $0 \leq m \leq 1$, (3.12) follows immediately from Lemma 3.2 and from the inclusion $K^m \subset \mathcal{F}(L^1)$.
Finally (3.13) and (3.14) are immediate consequences of (3.11) (3.12).

Part (3). By Lemma 3.2, we estimate
\[
\| [\omega^m, f]g \|_2 \leq C \left( \| \omega^m f \|_2 \| g \|_\infty + \| \nabla f \|_{n/\delta} \| \omega^{m-1} g \|_r \right)
\]
(3.18)
with \(0 \leq \delta = \delta(r) < n/2\). We estimate the last term in (3.18) by Sobolev inequalities as
\[
C \| \omega^{1+n/2-\delta} f \|_2 \| \omega^{m-1+\delta} \|_2 \leq C \| \omega \|_{m} \| g \|_{\bar{m}}
\]
provided
\[
1 + n/2 - m \leq \delta \leq 1 + n/2 - k_0
\]
\[
k_0 - m + 1 \leq \delta \leq \bar{m} + 1 - m
\]
and the various conditions on \(\delta\) are compatible for \(m \geq k_0\) and \(1 < m \leq \bar{m} + 1\). This proves (3.15).

We next give some estimates of the various components of \(B_1\), defined by (2.27)-(2.29) and (2.31). It follows immediately from (2.25) and from the dilation identity
\[
\| \omega^m (x \cdot B_a)_S \|_2 \leq t^{\beta(m-p)} \| \omega^p (x \cdot B_a)_S \|_2 \leq t^{\beta(m-p)} \| \omega^p (x \cdot B_a) \|_2
\]
(3.19)
for \(m \leq p\), and similarly
\[
\| \omega^m (x \cdot B_a)_L \|_2 \leq t^{\beta(m-p)} \| \omega^p (x \cdot B_a)_L \|_2 \leq t^{\beta(m-p)} \| \omega^p (x \cdot B_a) \|_2
\]
(3.20)
for \(m \geq p\).

We next estimate \(F_j(M)\) defined by (2.25). From (2.25) and from the dilation identity
\[
\| \omega^m D_0(\nu) f \|_2 = \nu^{-m+3/2} \| \omega^m f \|_2
\]
(3.21)
it follows immediately that
\[
\| \omega^{m+1} F_j(M) \|_2 \leq C I_{m+j} (\| \omega^m M \|_2)
\]
(3.22)
for \(j = 0, 1\), where
\[
(I_m(f))(t) = \int_{1}^{\infty} d\nu \, \nu^{-m-3/2} f(\nu t)
\]
(3.23)
or equivalently
\[
(I_m(f))(t) = t^{m+1/2} \int_{t}^{\infty} dt' \, t'^{-m-3/2} f(t')
\]
(3.24)
for $t > 0$. In particular for $m \geq 1$ and $j = 0, 1$,
\[
|F_j(M)|_m \leq CI_j(|M|_m) .
\]
(3.25)

We next estimate $x \cdot F_j(M)$. Using the commutation relation
\[
[x, P] = (n - 1)\Delta^{-1}\nabla ,
\]
easily proved in Fourier transformed variables, and estimating
\[
|\sin((\nu - 1)\omega)| \leq \nu \omega \vee 1 ,
\]
we obtain
\[
\| \omega^{m+1} x \cdot F_j(M) \|_2 \leq C I_{m-1+j} (\| \omega^m (x \otimes M) \|_2 + \| M \|_2) ,
\]
and in particular for $m \geq 1$,
\[
|x \cdot F_1(M)|_{m+1} \vee |\nabla x \cdot F_0(M)|_m \leq I_0 (|x \otimes M|_m + |M|_m) .
\]
(3.28)

The estimates (3.25) (3.28) will be the main tools used to estimate $B_a$ and $B_b$ given by (2.28) (2.29).

4 Cauchy problem and asymptotics for the auxiliary system

In this section, we study the Cauchy problem for the auxiliary system (2.34)(2.35) for large but finite initial time, and we derive asymptotic properties in time of its solutions.

The basic tool of this section consists of a priori estimates for suitably regular solutions of the linearized system (2.37) (2.38). Those estimates can be proved by a regularisation and limiting procedure and hold in the integrated form at the available level of regularity. For brevity, we shall state them in differential form and we shall restrict the proof to the formal computation.

We first estimate a single solution of the linearized system (2.37) (2.38) at the level of regularity where we shall eventually solve the auxiliary system (2.34) (2.35).

Lemma 4.1. Let $k > 3/2$, let $T \geq 1$, $I = [T, \infty)$ and let $(w, s, B_b) \in \mathcal{C}(I, X^k)$ with $|w|_k \vee |xw|_k \in L^\infty(I)$. Let
\[
a = \| |w|_k \vee |xw|_k; L^\infty(I) \| .
\]
(4.1)
Let $I' \subset I$ be an interval, let $(w', s')$ be a solution of the system \((2.37)\) with $(w', s', 0) \in C(I', X^k)$ and let $B'_b$ be defined by \((2.38)\). Let $0 \leq \theta \leq 1$ and $k \leq \ell \leq k + 2$. Then the following estimates hold:

\[
|B_a|_{k+1} \leq C \ I_0 \left(|w|_{k} \ |xw|_{k}\right) \leq C \ a^2, \tag{4.2}
\]

\[
\|x \cdot B_a; \dot{H}^{k+1}\| \leq |\nabla (x \cdot B_a)|_{k} \leq C \ I_0 \left(|xw|_{k} (|xw|_{k} + |w|_{k})\right) \leq C \ a^2 \tag{4.3}
\]

for all $t \in I$.

\[
|\partial_t |w'|_{k+\theta}| \leq C \ t^{-2} \left(|\nabla (s + B)|_{k} \ |w'|_{k+\theta} + |\nabla \cdot s|_{k+\theta} |w'|_{k}\right) + \left(|s|_{k+\theta} + |B|_{k+\theta}\right) |B|_{k+\theta} + C \ t^{-1 - \beta(1-\theta)} \ |x \cdot B_a; \dot{H}^{k+1}| \ |w'|_{k}
\]

\[+ C \ t^{-1} |x \cdot B_b|_{k+\theta} |w'|_{k} \equiv M_4(\theta, w') \tag{4.4}
\]

\[
|\partial_t |xw'|_{k}| \leq M_4(0, xw') + C \ t^{-2} \left(|w'|_{k+1} + |s + B|_{k} |w'|_{k}\right), \tag{4.5}
\]

\[
|\partial_t |s'|_{\ell}| \leq C \ t^{-2} \left(|s|_{k+1} \ |s'|_{\ell} + \chi(\ell \geq k + 1) |s|_{\ell} \ |s'|_{k+1}\right)
\]

\[+ C \ t^{-1} |w|_{k} \ |w|_{\ell-1} + C \ t^{-1 + \beta(\ell-k)} |\nabla (x \cdot B_a)|_{k} \tag{4.6}
\]

for all $t \in I'$,

\[
|B'_b|_{k+\theta} \leq C \ t^{-1} \ I_1 \left(|w|_{k} (|w|_{k+\theta} + |s + B|_{k} |w|_{k})\right) \tag{4.7}
\]

\[
|x \cdot B'_b|_{k+\theta} \leq C \ t^{-1} \ I_0 \left(|w|_{k} + |xw|_{k} (|w|_{k+\theta} + |s + B|_{k} |w|_{k})\right) \tag{4.8}
\]

for all $t \in I$.

**Remark 4.1.** The statements on $B'_b$ are non empty only in so far as the integrals over $\nu$ in the RHS of \((1.7)\) \((1.8)\) are absolutely convergent. This requires suitable assumptions on the behaviour of $(w, s, B)$ at infinity in time. Such assumptions will be made in due course. The only assumption of this type made so far is \((4.1)\) which ensures \((4.2)\) \((4.3)\), thereby making \((4.4)\) \((4.5)\) \((4.6)\) into non empty statements.

**Proof.** \((4.2)\) and \((4.3)\) follow immediately from \((3.25)\) \((3.28)\) and from \((3.10)\) with $m = \bar{m} = k$. 

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We next estimate $w'$ in $H^{k+\theta}$. It is clear from (2.37) that $\|w'\|_2 = \text{const}$. Let $m = k + \theta$. We estimate by a standard energy method

\[
|\partial_t \| \omega^m w' \|_2| \leq t^{-2} \left\{ \| [\omega^m, s + B] \cdot \nabla w' \|_2 + \| [\omega^m, \nabla \cdot s] w' \|_2 
+ \| [\omega^m, 2B \cdot s + B^2] w' \|_2 \right\} + t^{-1} \| [\omega^m, (x \cdot B_a) s] w' \|_2 
+ t^{-1} \| [\omega^m, x \cdot B_b] w' \|_2.
\]  

(4.9)

We estimate the first norm in the RHS by (3.14) with $\bar{m} = k$ and the other norms by (3.15) with $\bar{m} = k$. Furthermore, by (3.19)

\[
\|(x \cdot B_a)s\|_{k+\theta} \leq t^{-\beta(1-\theta)} \| x \cdot B_a ; \hat{H}^{k+1} \|
\]  

(4.10)

so that the RHS of (1.9) is estimated by that of (1.4), which together with $L^2$ norm conservation, yields (1.4).

We next estimate $xw'$ in $H^k$, starting from (2.39). We obtain

\[
|\partial_t \| \omega^k x w' \|_2| \leq \text{terms containing } xw' 
+ t^{-2} \left( \| \omega^{k+1} w' \|_2 + \| \omega^k((s + B)w') \|_2 \right)
\]  

(4.11)

where the terms containing $xw'$ are obtained from the RHS of (1.9) by replacing $w'$ by $xw'$ and taking $m = k$. Those terms are estimated in the same way as before. Estimating the last norm in (4.11) by (3.13) and using the obvious $L^2$ estimate

\[
|\partial_t \| xw' \|_2| \leq t^{-2} (\| \nabla w' \|_2 + \| (s + B) \|_\infty \| w' \|_2)
\]  

(4.12)

yields (4.13).

We next estimate $s'$. From (2.37) we obtain

\[
|\partial_t \| \omega^\ell s' \|_2| \leq t^{-2} \left\{ \| [\omega^\ell, s] \nabla s' \|_2 + \| \nabla \cdot s) \omega^\ell s' \|_2 \right\} 
+ C \ t^{-1} \| [\omega^{\ell-1}|w|^2 \|_2 + t^{-1} \| \omega^{\ell+1}(x \cdot B_a) \|_2.
\]  

(4.13)

We estimate the first norm in the RHS by (3.11) (3.12) with $m = \ell$ and $\bar{m} = k$ if $\ell \leq k + 1$, while for $\ell \geq k + 1$

\[
\| [\omega^\ell, s] \nabla s' \|_2 \leq C \left( \| \nabla s \|_\infty \| \omega^\ell s' \|_2 + \| \omega^\ell s \|_2 \| \nabla s' \|_\infty \right)
\leq C \left( |s|_{k+1} |s'|_\ell + |s|_\ell |s'|_{k+1} \right)
\]  

(4.14)

by a direct application of Lemma 3.2. The last two norms in the RHS of (4.13) are estimated as

\[
\| \omega^{\ell-1}|w|^2 \|_2 \leq C \| w \|_\infty \| \omega^{\ell-1} w \|_2 \leq C |w|_k \ |w|_{\ell-1}
\]  

(4.15)
by Lemma 3.2, and
\[ \| \omega^{\ell+1}(x \cdot B_a) \|_2 \leq t^{\beta(\ell-k)} \| \omega^{k+1}(x \cdot B_a) \|_2 \]
by (3.20). Together with the simpler estimate
\[ |\partial_t \| \nabla s' \|_2 | \leq t^{-2} \| \nabla s \|_\infty \| \nabla s' \|_2 + C t^{-1} \| w \|_4^2 + t^{-1} \| \nabla^2 (x \cdot B_a) \|_2 \ (	ext{4.16}) \]
those estimates yield (4.10).

We finally estimate \( B'_b \). From (3.25) with \( j = 1 \) and (3.28), we obtain
\[ |B'_{b,k+\theta} | \leq C t^{-1} I_1 (|M_b|_{k+\theta-1}) \]  
(4.17)
\[ |x \cdot B'_{b,k+\theta} | \leq C t^{-1} I_0 (|xM_b|_{k+\theta-1} + |M_b|_{k+\theta-1}) \]  
(4.18)
from which (4.7) (4.8) follow by (3.9) (3.10).

\[ \square \]

We next estimate the difference of two solutions of the linearized system (2.37) (2.38) corresponding to two different choices of \((w, s, B_b)\). We estimate that difference at a lower level of regularity than the solutions themselves.

**Lemma 4.2.** Let \( k > 3/2 \), let \( T \geq 1 \), \( I = [T, \infty) \) and let \((w_i, s_i, B_{b_i}) \in C(I, X^k), i = 1, 2 \) with \( |w_i|_k \vee |xw_i|_k \in L^\infty(I) \). Let
\[ a = \text{Max}_i \| |w_i|_k \vee |xw_i|_k; L^\infty(I) \| . \]  
(4.19)
Let \( I' \subset I \) be an interval, let \((w_i', s_i') \) be solutions of the system (2.37) associated with \((w_i, s_i, B_{b_i}) \) with \((w_i', s_i', 0) \in C(I', X^k) \) and let \( B'_{b_i} \) be defined by (2.38) in terms of \((w_i, s_i, B_{b_i})\). Define \((w_\pm, s_\pm, B_{b_\pm}) = (1/2)((w_1, s_1, B_{b_1}) \pm (w_2, s_2, B_{b_2})) \) and similarly for the primed quantities and for \( B_a, B, B \cdot s \) and \( B^2 \). Let
\[ 1/2 < k' \leq k - 1 \ , \ 0 \leq \theta \leq 1 \ , \ 1 \vee k' \leq \ell \leq k' + 2 . \]  
(4.20)
Then the following estimates hold:
\[ |B_{a_\pm}|_{k'+1} \leq C I_0 (|xw_+|_k |w_-|_{k'}) , \]  
(4.21)
\[ \| x \cdot B_{a_\pm}; \dot{H}^{k'+1} \| \leq C I_{k'-1} ((|xw_+|_k + (|w_+|_k)|xw_-|_{k'}) , \]  
(4.22)
\[ |\partial_t w'_k| \leq C \{ t^{-2} \left( |\nabla s_+|_k + |\nabla B_+|_k + |(B s)_+|_k + |(B^2)_+|_k \right) \\
+ t^{-1-\beta(k-k')} \| x \cdot B_{a_+} ; \hat{H}^{k+1} \| + t^{-1} \| x \cdot B_{b_+} ; \hat{H}^{k+1} \| \}| w'_k, \]

\[ \partial_t |x w'_k| \leq \text{Idem} (\theta = 0, w' \rightarrow x w') \]

\[ + C t^{-2} \left\{ |w'_k|_{k+1} + \|(s + B)_+|_k \| w_- |_{k'} + \|(s + B)_-|_{k'+1} \| w'_k \right\}, \quad (4.24) \]

\[ \partial_t |s'_k| \leq C t^{-2} \left\{ |\nabla s_+|_k \| s'_k + |s_\ell| \| \nabla s'_k + |\chi (\ell \geq k) \| s'_k \| \nabla s'_k |_{\ell} \right\} \]

\[ + C t^{-1} |w_+|_k \| w_- |_{\ell-1} + C t^{-1+\beta(k-k')} \| \nabla (x \cdot B_{a_-}) ; \hat{H}^{k'} \| \nabla (x \cdot B_{b_-}) ; \hat{H}^{k'1} \|, \quad (4.25) \]

\[ |B'_{b_-}|_{k'+1} \leq t^{-1} I_0 \left( |M_{b_-}|_{k'} \right), \quad (4.26) \]

\[ |x \cdot B'_{b_-}|_{k'+1} \leq t^{-1} I_0 \left( |x M_{b_-}|_{k'} + |M_{b_-}|_{k'} \right), \quad (4.27) \]

where \( M_{b_-} = (M_{b1} - M_{b2}) / 2 \), \( M_b \) is defined by \((2.23)\), and

\[ |M_{b_-}|_{k'} \vee |x \cdot M_{b_-}|_{k'} \leq C | < x > w_+ |_k \left\{ |w_-|_{k'+1} + |s_+ + B_+|_k \| w_- |_{k'} \right\} \]

\[ + |s_- + B_-|_{k'+1} \| w_+ \|_k \} + C | < x > w_- |_k \| s_- + B_-|_{k'+1} \| w_- |_k \}. \quad (4.28) \]

**Proof.** The estimates \((4.21) (4.22)\) follow immediately from \((2.28)\), from \((3.25)\) \((3.28)\) and from \((3.10)\) with \( m = k' \) and \( \bar{m} = k \).

We next estimate \((w'_s, s'_s)\). Taking the difference of the systems \((2.37)\) for \((w'_s, s'_s)\), we obtain the following system for \((w'_s, s'_s)\) :

\[ \partial_t w'_s = i(2t^2)^{-1} \Delta w'_s + t^{-2} \left( Q(s_+ + B_+, w'_s) + Q(s_- + B_-, w'_s) \right) \]

\[ - i(2t^2)^{-1} \left( (2(B \cdot s)_+ + (B^2)_+) w'_s + (2(B \cdot s)_- + (B^2)_-) w'_s \right) \]

\[ + it^{-1} \left( ((x \cdot B_{a_+})_s + x \cdot B_{b_+}) w'_s + ((x \cdot B_{a_-})_s + x \cdot B_{b_-}) w'_s \right) \]

\[ \partial_t s'_s = t^{-2} \left( s_+ \cdot \nabla s'_s + s_- \cdot \nabla s'_s \right) + t^{-1} \nabla g(w_+, w_-) - t^{-1} \nabla (x \cdot B_{a_-})_L. \quad (4.29) \]

Let \( m = k' + \theta \), so that \( 1/2 < m \leq k \). We estimate \( \omega^m w'_s \) by the same method as in Lemma 4.1, using \((3.14)\) for the term \((s + B)_+ \cdot \nabla w'_s \) with \( \bar{m} = k \), and using \((3.8)\) for all the other terms, with \( \bar{m} = k \) or \( \bar{m} = k' + 1 \), depending on whether the estimated quantity is of + or - type. We obtain
\[ |\partial_t \| \omega^m w'_- \|_2 | \leq C \left\{ t^{-2} \left( |\nabla (s + B)_+|_k + |\nabla \cdot s_+|_k + (B \cdot s)_+ + (B^2)_+|_k \right) + t^{-1} \left( |(x \cdot B_{a_+})_s|_{k'+1} + |x \cdot B_{b_+}|_k \right) \right\} |w'_-|_m \]

\[ + C \left\{ t^{-2} \left( |(s + B)_-|_{k'+1} |w'_+|_{m+1} + |\nabla \cdot s_-|_m |w'_+|_k + (B \cdot s)_- + (B^2)_-|_{k'+1} |w'_+|_m \right) + t^{-1} \left( |(x \cdot B_{a_-})_s|_m |w'_+|_k + |x \cdot B_{b_-}|_{k'+1} |w'_+|_m \right) \right\}. \tag{4.31} \]

We next estimate

\[ |(x \cdot B_{a_+})_s|_{k'+1} \leq t^{-\beta(k-k')} \| x \cdot B_{a_+}; \hat{H}^{k+1} \|, \]

\[ |(x \cdot B_{a_-})_s|_m \leq t^{-\beta(1-\theta)} \| x \cdot B_{a_-}; \hat{H}^{k'+1} \| \]

by (3.19). We estimate \( \partial_t \| w'_- \|_2 \) in a simpler way as above by taking \( m = 0 \) and omitting the terms with \( w'_- \) in (4.31). Collecting the previous estimates yields (4.23).

We now turn to the proof of (4.24), namely to the estimate of \( xw'_- \). From (2.39) we obtain

\[ \partial_t xw'_- = \text{Idem}(w' \to xw') - it^{-2}\nabla w'_- - t^{-2} \left( (s + B)_+ w'_- + (s + B)_- w'_+ \right) \tag{4.32} \]

where Idem denotes the RHS of (4.29). The estimate (4.24) then follows from (4.23) with \( \theta = 0 \) and \( w' \) replaced by \( xw' \) and from an easy estimate of the additional terms using (3.10) with \( m = k' \) and \( \bar{m} = k \) or \( k' + 1 \).

We now turn to the proof of (4.25), namely to the estimate of \( s'_- \). We estimate

\[ \partial_t \| \omega^\ell s'_- \|_2 \leq t^{-2} \left\{ \| \omega^\ell, s_+ \cdot \nabla s'_- \|_2 + \| \nabla \cdot s_+ \|_\infty \| \omega^\ell s'_- \|_2 \right\} + C t^{-1} \| \omega^{\ell-1} (\bar{w}_+ w_-) \|_2 + t^{-1} \| \omega^{\ell+1} (x \cdot B_{a_-})_L \|_2. \tag{4.33} \]

We estimate the first norm in the RHS by (3.11) (3.12) with \( m = \ell \) and \( \bar{m} = k \). We estimate the third norm by (3.8) with \( m = \ell \) and \( \bar{m} = k \) for \( \ell \leq k \), while for \( \ell \geq k \)

\[ \| \omega^\ell (s_- \nabla s'_+) \|_2 \leq C \left( \| \omega^\ell s_- \|_2 \| \nabla s'_+ \|_\infty + \| s_- \|_\infty \| \omega^\ell \nabla s_+ \|_2 \right) \]

\[ \leq C \left( \| s\ell \|_k \| s'_+ \|_k + \| s_- \|_k \| \nabla s'_+ \|_k \right) \]

by a direct application of Lemma 3.2. We next estimate

\[ \| \omega^{\ell-1} (\bar{w}_+ w_-) \|_2 \leq C |w_+|_k |w_-|_{\ell-1} \]

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by (3.10) with \( m = \ell - 1 \) and \( \bar{m} = k \), and we estimate the last norm in (4.33) by (3.20). Together with the special case \( \ell = 1 \), the previous estimates yield (4.25).

Finally (4.26), (4.27) are special cases of (3.25), (3.28), while (4.28) follows from repeated use of (3.10) with \( m = k' \) and \( \bar{m} = k \) or \( k' + 1 \).

\[ \square \]

We now begin to study the auxiliary system (2.34), (2.37) and its linearized version (2.37), (2.38). The first step is to solve the linear system (2.37) globally in time.

**Proposition 4.1.** Let \( k > 3/2 \), let \( T \geq 1 \), \( I = [T, \infty) \) and let \( (w, s, B_0) \in C(I, X^k) \) with \( |w|_k \vee |sx|_k \in L^\infty(I) \). Let \( t_0 \in I \) and \( (w_0^\prime, s_0^\prime,0) \in X^k \). Then the system (2.37) has a unique solution \((w', s')\) in \( I \) such that \((w', s',0) \in C(I, X^k) \) and \((w', s')(t_0) = (w_0^\prime, s_0^\prime)\). That solution satisfies the estimates (4.24), (4.25) for all \( t \in I \). Two such solutions \((w_i^\prime, s_i^\prime)\) associated with \((w_i, s_i)\), \( i = 1, 2 \), satisfy the estimates (4.26), (4.27) for all \( t \in I \).

**Proof.** We first prove the existence of a unique solution \((w', s') \in C(I, Y^k)\) where \( Y^k = H^{k+1} \oplus K^{k+2} \). The proof proceeds in the same way as that of Proposition 4.1 of [3], through a parabolic regularization and a limiting procedure. We define \( U_1(t) = U(1/t) \) and \( \bar{w}'(t) = U_1(t)w'(t) \). We first consider the case \( t \geq t_0 \). The system (2.37) with a parabolic regularization added is rewritten in terms of the variables \((\bar{w}', s')\) as

\[
\begin{align*}
\partial_t \bar{w}' &= \eta \Delta \bar{w}' + U_1 \left( L - i(2t^2)^{-1} \Delta \right) U_1^\dagger \bar{w}' \equiv \eta \Delta \bar{w}' + G_1(\bar{w}') \\
\partial_t s' &= \eta \Delta s' + t^{-2} s \cdot \nabla s' + t^{-1} \nabla g(w) - t^{-1} \nabla (x \cdot B_a)_L \\
&\equiv \eta \Delta s' + G_2(s')
\end{align*}
\]

where \( L \) is defined in (2.34) and where the parametric dependence of \( L, G_1, G_2 \) on \((w, s, B_0)\) has been omitted. The Cauchy problem for that system can be recast into the integral form

\[
\begin{pmatrix}
\bar{w}' \\
s'
\end{pmatrix}(t) = V_\eta(t-t_0) \begin{pmatrix}
\bar{w}'_0 \\
s'_0
\end{pmatrix} + \int_{t_0}^{t} dt' V_\eta(t-t') \begin{pmatrix}
G_1(\bar{w}') \\
G_2(s')
\end{pmatrix}(t')
\]

(4.34)

where \( V_\eta(t) = \exp(\eta t \Delta) \). The operator \( V_\eta(t) \) is a contraction in \( Y^k \) and satisfies the bound

\[ \| \nabla V_\eta(t); L(Y^k) \| \leq C(\eta t)^{-1/2}. \]
From those facts and from estimates on $G_1$, $G_2$ similar to and mostly contained in those of Lemma 4.1, it follows by a contraction argument that the system (4.34) has a unique solution $(\bar{w}_i', s''_i) \in \mathcal{C}([t_0, t_0 + T_1], Y^k)$ for some $T_1 > 0$ depending only on $|w_0'|_{k+1}$, $|s'_0|_{k+2}$ and $\eta$. That solution satisfies the estimates (4.4) and (4.6) and can therefore be extended to $[t_0, \infty)$ by a standard globalisation argument using Gronwall’s inequality.

We next take the limit $\eta \to 0$. Let $\eta_1, \eta_2 > 0$ and let $(w_i', s'_i) = (w_{\eta_i}', s_{\eta_i}')$, $i = 1, 2$ be the corresponding solutions. Let $(w'_-, s'_-) = (1/2)(w_1' - w_2', s_1' - s_2')$. By estimates similar to, but simpler than those of Lemma 4.2, since in particular $(w_-, s_-, B_{b_+}) = 0$, we obtain

\[
\begin{cases}
\partial_t \| w_-'' \|_2^2 \leq |\eta_1 - \eta_2| (\| \nabla w'_1 \|_2^2 + \| \nabla w'_2 \|_2^2) \\
\partial_t \| \nabla s_-'' \|_2^2 \leq |\eta_1 - \eta_2| (\| \nabla^2 s'_1 \|_2^2 + \| \nabla^2 s'_2 \|_2^2) + C t^{-2} \| \nabla s \|_\infty \| \nabla s_-'' \|_2 .
\end{cases}
\]

Those estimates imply that $(w'_i, s''_i)$ converges in $L^2 \oplus \dot{H}^1$ uniformly in time in the compact subintervals of $[t_0, \infty)$, to a solution of the original system. It follows then by a standard compactness argument using the estimates (4.4) (4.6) that the limit belongs to $\mathcal{C}([t_0, \infty), Y^k)$. This completes the proof for $t \geq t_0$. The case $t \leq t_0$ is treated similarly.

We next show that $xw' \in \mathcal{C}(I, H^k)$. For that purpose we choose a function $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n, \mathbb{R}^+)$ with $0 \leq \psi \leq 1$, $\psi(x) = 1$ for $|x| \leq 1$, $\psi(x) = 0$ for $x \geq 2$, and we define $\psi_R$ by $\psi_R(x) = \psi(x/R)$. Clearly $\psi_R xw' \in \mathcal{C}(I, H^{k+1})$ and $\psi_R xw'$ satisfies the equation

\[
\begin{align*}
\partial_t \psi_R xw' & = L \psi_R xw' - it^{-2} (\nabla (\psi_R x)) \cdot \nabla w' - i(2t^2)^{-1} (\Delta (\psi_R x)) w' \\
& \quad - t^{-2} (s + B) \cdot (\nabla (\psi_R x)) w' .
\end{align*}
\tag{4.35}
\]

Using Lemma 4.1, more precisely (4.3) and the fact that the operator of multiplication by a function $\varphi_R(x) = \varphi(x/R)$ for $\varphi \in \mathcal{C}_0^\infty$ is a bounded operator in $H^m$ for all $m \geq 0$ uniformly in $R$ for $R \geq 1$, we estimate

\[
|\partial_t |\psi_R xw'|_k| \leq M_4(0, \psi_R xw') + C t^{-2} \left( |w'|_{k+1} + R^{-1} |w'|_k \right) + C t^{-2} |s + B|_k |w'|_k .
\tag{4.36}
\]

Integrating (4.36) between $t_0$ and $t$ and using Gronwall’s inequality, we obtain

\[
|\psi_R xw'|_k \leq C(t) \left( 1 + |\psi_R xw'_0|_k \right) \leq C(t) \left( 1 + |xw'_0|_k \right) .
\]
so that $\psi_R xw'$ is bounded in $H^k$ uniformly in $R$. This implies that $xw' \in L^\infty_{loc}(I, L^2)$ and that $\psi_R xw'$ tends to $xw'$ strongly in $L^2$ pointwise in $t$ when $R \to \infty$. Moreover, it follows from (4.35) that $xw'$ is weakly continuous in $L^2$ as a function of $t$. Together with (4.36), this implies that $xw' \in C(I, H^k)$ by standard compactness arguments.

We now turn to the study of the auxiliary system (2.34) (2.35). The main results will be the existence and uniqueness of solutions of that system, defined in a neighborhood of infinity in time and with suitable bounds at infinity, and some asymptotic behaviour of those solutions. The bounds on the solutions at infinity will be essentially dictated by the existence result (Proposition 4.3 below), and for simplicity we shall mostly restrict our attention to solutions satisfying those bounds, although more general solutions could be considered in the uniqueness and asymptotic behaviour results. We shall thus consider solutions $(w, s, B_b) \in C(I, X^k)$ for some interval $I = [T, \infty)$ such that

\begin{align*}
\Lambda(t) &\equiv |w|_k \vee |xw|_k \vee (\ell n t)^{-1}|w|_{k+1} \vee (\ell n t)^{-1}|s|_k \vee t^{-\beta} |s|_{k+1} \\
&\vee t^{-2\beta} |s|_{k+2} \vee |B_b|_k \in L^\infty(I). \tag{4.37}
\end{align*}

We first state the uniqueness result.

**Proposition 4.2.** Let $k > 3/2$, $0 < \beta < 1/2$ and $I = [T, \infty)$.

1. Let $t_0 \in I$ and $(w_0, s_0, 0) \in X^k$. Then for $t_0$ sufficiently large, the auxiliary system (2.34) (2.35) has at most one solution $(w, s, B_b) \in C(I, X^k)$ satisfying (4.37) and $(w, s)(t_0) = (w_0, s_0)$.

2. Let $(w_i, s_i, B_{b_i})$ $i = 1, 2$, be two solutions of the auxiliary system (2.34) (2.35) in $C(I, X^k)$ satisfying (4.37) and such that for some $\varepsilon > 0$

\begin{equation}
|s_{-1}|_{k+1} \vee t^{2\beta+\varepsilon} (|w_{-1}|_k \vee |xw_{-1}|_{k-1}) \to 0 \text{ when } t \to \infty. \tag{4.38}
\end{equation}

Then $(w_1, s_1, B_{b_1}) = (w_2, s_2, B_{b_2})$.

**Remark 4.2.** The condition $t_0$ sufficiently large in Part (1) takes the form

\begin{equation}
t_0 \geq \| 1 + \Lambda; L^\infty([t_0, \infty)) \|^N \tag{4.39}
\end{equation}

for some $N > 0$. For a given solution $(w, s, B_b)$, the RHS of (4.39) is decreasing in $t_0$ while the LHS is increasing, and Part (1) supplemented by (4.39) gives a lower
Proof. The proof relies on Lemma 4.2, and we first recast the estimates of that lemma in a simplified form for solutions satisfying (4.37). We consider two solutions \((w_i, s_i, B_{bi})\) (\(i = 1, 2\)) of the system (2.34) (2.35) satisfying (4.37) and we define

\[ A(t_1) = \max_{i=1,2} \| A_i(t_1); L^\infty([t_1, \infty)) \| \]  

for \(t_1 \geq T\), where \(A_i\) are the quantities defined by (4.37) for the two solutions. We define \((w_-, s_-, B_{b-})\) as in Lemma 4.2, and

\[
\begin{cases}
  y = |w_{-}|_{k-1} \vee |xw_{-}|_{k-1} , & y_1 = |w_{-}|_{k} , \\
  z_j = |s_{-}|_{k+j-1} , & j = 1, 2 .
\end{cases}
\]  

(4.41)

We rewrite the estimates (4.21)-(4.28) for general \(t \in [t_1, \infty)\) for some \(t_1 \geq T\). The + quantities are estimated by (4.37) (4.40) supplemented by (4.2) (4.3) (4.7) (4.8) as regards \(B_a\) and \(B_b\). This produces overall constants depending polynomially on \(A(t_1)\), which we omit for brevity, but the occurrence of which should be kept in mind for subsequent arguments. In terms having the same dependence on the dynamical variables, we keep only the terms with the leading behaviour in \(t\), namely we use

\[ t^{1-\beta} \geq \ell n t \geq 1 .\]

We take \(k' = k - 1\) and rewrite (4.21)-(4.28) as follows

\[ |B_{a-}|_{k} \leq I_0(y) , \]

(4.42)

\[ \| x \cdot B_{a-}; \dot{H}^k \| \leq I_{k-2}(y) , \]

(4.43)

\[ |\partial_t y| \leq t^{-1-\beta} y + t^{-2} \left( z_1 + |B_{-}|_{k} \ell n t \right) + t^{-1-\beta} I_{k-2}(y) + t^{-1} |x \cdot B_{b-}|_{k} + t^{-2} y_1 , \]

(4.44)

\[ |\partial_t y_1| \leq t^{-1-\beta} y_1 + t^{-2} \left( z_1 \ell n t + z_2 + |B_{-}|_{k} \ell n t \right) + t^{-1} I_{k-2}(y) + t^{-1} |x \cdot B_{b-}|_{k} , \]

(4.45)

\[ |\partial_t z_1| \leq t^{-2+\beta} z_1 + t^{-1} y + t^{-1+\beta} I_m(y) , \]

(4.46)

\[ |\partial_t z_2| \leq t^{-2+\beta} (z_2 + t^\beta z_1) + t^{-1} y_1 + t^{-1+2\beta} I_m(y) \]

(4.47)

with \(m = (k - 2) \land 0\),

\[ |B_{b-}|_{k} \leq t^{-1} I_1 \left( y_1 + y \ell n t + z_1 + |B_{-}|_{k} \right) , \]

(4.48)
\[ |x \cdot B_{a-}|_k \leq t^{-1} I_0 \left( y_1 + y \ln t + z_1 + |B_{-}|_k \right). \] (4.49)

The system (4.42)-(4.49) will be the starting point for the proof of Proposition 4.2.

**Part (1).** We first prove uniqueness for \( t \geq t_0 \). Note that this region is autonomous in the sense that the equations in this region involve the dynamical variables in this region only, since the integrals (2.25) occurring in (2.28)-(2.29) are taken for \( \nu \geq 1 \).

We define
\[
Y = \| y; L^\infty([t_0, \infty)) \|, \quad Y_1 = \| y_1 (\ln t)^{-1}; L^\infty([t_0, \infty)) \| \] (4.50)
which are finite by (4.37) and we prove that those quantities are zero by integrating (4.44)-(4.47) with initial condition \((y, y_1, z_1, z_2)(t_0) = 0\).

In keeping with the previous simplification, we perform that computation up to constants (depending on \( A(t_0) \)) and under conditions that \( t_0 \) is sufficiently large in the sense of (4.39) when needed. We furthermore eliminate the diagonal terms in (4.46)-(4.47) by exponentiation, namely by using the fact that
\[
\partial_t y \leq fy + g \Rightarrow y(t) \leq \exp \left( \int_{t_0}^t f \right) \left( y(t_0) + \int_{t_0}^t g \right) \] (4.51)
for integrable \( f \). Using (4.50) and integrating (4.46) (4.47) (and using \( z_1 \leq z_2 \) and \( 2\beta < 1 \)), we obtain
\[
z_1 \leq t^\beta Y, \quad z_2 \leq (\ln t)^2 Y_1 + t^{2\beta} Y. \] (4.52)

Substituting (4.50) (4.52) into (4.42) (4.48) (4.49) yields
\[
|B_{a-}|_k \lor \| x \cdot B_{a-}; \dot{H}^k \| \leq Y, \] (4.53)
\[
|B_{b-}|_k \lor |x \cdot B_{b-}|_k \leq t^{-1} \ln t \left( Y_1 + Y \right) + t^{-1+\beta} Y + t^{-1} \| |B_{-}|_k; L^\infty([t_0, \infty)) \| \leq t^{-1} \ln t Y_1 + t^{-1+\beta} Y \] (4.54)
for \( t_0 \) sufficiently large, so that one can assume
\[
|B_{-}|_k \leq Y. \] (4.55)

Substituting (4.50) (4.52)-(4.55) into (4.44) (4.45) yields
\[
\partial_t y \leq \left( t^{-1-\beta} + t^{-2+\beta} + t^{-2} \ln t \right) Y + t^{-2} \ln t \; Y_1, \] (4.56)
\[ \partial_t y_1 \leq \left( t^{-2+\beta} \ln t + t^{-2} \ln t + t^{-2+2\beta} + t^{-1} \right) Y + t^{-2} (\ln t)^2 Y_1 \]
\[ \leq t^{-1} Y + t^{-2} (\ln t)^2 Y_1. \]  
(4.57)

Integrating (4.56) (4.57) in \([t_0, t]\) and comparing with (4.50) yields
\[
\begin{aligned}
Y &\leq t_0^{-\beta} Y + t_0^{-1} \ln t_0 Y_1 \\
Y_1 &\leq Y + t_0^{-1} \ln t_0 Y_1
\end{aligned}
\]  
(4.58)

which implies \(Y = Y_1 = 0\) for \(t_0\) sufficiently large.

We next prove uniqueness for \(t \leq t_0\). Since we have already proved uniqueness for \(t \geq t_0\), \((y, y_1, z_1, z_2)\) vanish for \(t \geq t_0\), and the region \(t \leq t_0\) also becomes autonomous in the previous sense, which it would not have been if treated first. Now however we are in a standard situation where the variables \(\{y_i\} = \{y, y_1, z_1, z_2, I_0(|B_\cdot|)\} \) satisfy a system of inequalities of the type
\[
|\partial_t y_i| \leq \sum_j f_{ij} y_j + g_i \int_t^{t_0} dt' \sum_j h_{ij}(t') y_j(t')
\]  
(4.59)

for \(t \leq t_0\). This can be reduced to the case of a single function \(\bar{y} = \sum y_i\) namely
\[
|\partial_t \bar{y}| \leq f \bar{y} + g \int_t^{t_0} dt' h(t') \bar{y}(t')
\]  
(4.60)

with
\[
f = \text{Max}_j \sum_i f_{ij}, \quad g = \text{Max}_i g_i, \quad h = \text{Max}_j \sum_i h_{ij}.
\]

One can then eliminate \(f\) by exponentiation, in very much the same way as in (4.51). Integrating (4.60) with \(f = 0\) and \(\bar{y}(t_0) = 0\) yields
\[
\bar{y}(t) \leq G(t) \int_t^{t_0} dt' h(t') \bar{y}(t') \equiv G(t) \bar{Y}(t)
\]
with \(G(t) = \int_t^{t_0} dt' g(t')\) and therefore
\[
|\partial_t \bar{Y}| \leq h \bar{Y} \bar{Y}
\]
with \(\bar{Y}(t_0) = 0\), which implies \(\bar{Y} = 0\) and therefore \(\bar{y} = 0\) for all \(t \leq t_0\).

Part (2). We proceed in the same way as in Part (1) for \(t \geq t_0\). Let \(\lambda = 2\beta + \varepsilon\), so that \(2\beta < \lambda < 1\) and define
\[
\begin{aligned}
Y(t) &= \| \cdot \lambda y(\cdot); L^\infty([t, \infty)) \| \\
Y_1(t) &= \| \cdot \lambda y_1(\cdot); L^\infty([t, \infty)) \|
\end{aligned}
\]  
(4.61)
We take $t_1$ and $t_0$ such that $T \leq t_1 < t_0 < \infty$, with $t_1$ sufficiently large, and we estimate $(y, y_1, z_1, z_2)$ for $t \in [t_1, t_0]$ by integrating (4.44)-(4.47) between $t$ and $t_0$ with final data $(y_0, y_{10}, z_{10}, z_{20})$ at $t_0$. Let $Y = Y(t_1)$, $Y_1 = Y_1(t_1)$. From (4.42) (4.43) (4.61) we obtain

$$|B_{a-}[k] \cap |x \cdot B_{a-} k \cdot \hat{H}^k| \leq t^{-\lambda} Y$$

for $t \geq t_1$. Integrating (4.48) (4.49) as before, we now obtain

$$\begin{cases} 
    z_1 \leq z_{10} + t_\beta Y \\
    z_2 \leq z_{20} + t_\lambda Y_1 + \varepsilon^{-1} t^{-\varepsilon} Y
\end{cases}$$

for $t \geq t_1$, with $t_\beta = t \wedge t_0$. (Note that we need an estimate of $z_1$ for $t \geq t_0$ for substitution in (4.48) (4.49)). Substituting (4.61) (4.62) (4.63) into (4.48) (4.49) yields

$$|B_{b-}[k] \cap |x \cdot B_{b-} k \cdot \hat{H}^k| \leq t^{-1} \left\{ t^{-\lambda} Y_1 + Y_1 \ln t + z_{10} + t_\beta Y + t^{-\lambda} Y + I_0 \left( |B_{b-}[k]| \right) \right\} \leq t^{-1} \left( t^{-\lambda} Y_1 + z_{10} + t_\beta Y \right)$$

for $t_1 \leq t \leq t_0$ and $t_1$ sufficiently large. Substituting (4.61)-(4.64) into (4.44) (4.43) yields

$$|\partial t y| \leq \left( t^{-1-\beta-\lambda} + t^{-2+\beta-\lambda} \right) Y + t^{-2} z_{10} + t^{-2-\lambda} Y_1 \leq t^{-1-\beta-\lambda} Y + t^{-2} z_{10} + t^{-2-\lambda} Y_1$$

since $2\beta < 1$,

$$|\partial t y_1| \leq \left( t^{-1-\beta-\lambda} + t^{-2-\lambda} \right) Y_1 + t^{-2} (z_{10} \ln t + z_{20}) + \left( t^{-2+\beta-\lambda} + \varepsilon^{-1} t^{-2-\varepsilon} + t^{-1-\lambda} \right) Y \leq t^{-1-\beta-\lambda} Y_1 + t^{-2} (z_{10} \ln t + z_{20}) + t^{-1-\lambda} Y.$$ 

Integrating (4.65) (4.66) between $t$ and $t_0$ yields

$$\begin{cases} 
    y \leq y_0 + t^{-\beta-\lambda} Y + t^{-1} z_{10} + t^{-1-\lambda} Y_1 \\
    y_1 \leq y_{10} + t^{-\beta-\lambda} Y_1 + t^{-1} (z_{10} \ln t + z_{20}) + t^{-\lambda} Y
\end{cases}$$

(4.67)

Substituting the result into the definition (4.61) and using the fact that $t_0^0 Y_0 \leq Y(t_0)$, $t_0^0 y_{10} \leq Y_1(t_0)$ by definition, we obtain

$$\begin{cases} 
    Y \leq Y(t_0) + t_1^{-1+\lambda} z_{10} + t_1^{-\beta} Y + t_1^{-1} Y_1 \\
    Y_1 \leq Y_1(t_0) + t_1^{-1+\lambda} (z_{10} \ln t_1 + z_{20}) + Y + t_1^{-\beta} Y
\end{cases}$$

(4.68)
Proposition 4.3. Let \( k > 3/2 \) and \( 0 < \beta < 1/2 \). Let \((w_0, s_0, 0) \in X^k\). Then

1. There exists \( T_0 < \infty \), depending on \((w_0, s_0)\), such that for all \( t_0 > T_0 \), there exists \( T < t_0 \) such that the auxiliary system (2.34) (2.35) has a unique solution \((w, s, B_b) \in C(I, X^k)\), where \( I = [T, \infty) \), satisfying (4.37) and \((w, s)(t_0) = (w_0, s_0)\).

The dependence of \( T_0 \) (resp. \( T \)) on \((w_0, s_0)\) (resp. and on \( t_0 \)) can be formulated more precisely as follows. For any set \( \mathcal{A} = \{a, a_1, b_0, b_1, b_2\} \) of five positive numbers, there exists \( T_0 = T_0(\mathcal{A}) \) and for any \( t_0 > T_0 \), there exists \( T = T(t_0, \mathcal{A}) < t_0 \), increasing in \( t_0 \) and such that \( T(T_0(\mathcal{A}), \mathcal{A}) = T_0(\mathcal{A}) \), such that the previous statement holds for such \( T_0, t_0, T \) for all \((w_0, s_0, 0) \in X^k\) such that

\[
\begin{align*}
|w_0|_k &\leq a \quad |w_0|_{k+1} \leq a_1 \ln t_0 \\
|s_0|_{k+j} &\leq b_j (\ln t_0 + t_0^{j\beta}) \quad j = 0, 1, 2 .
\end{align*}
\] (4.70)

The solution \((w, s, B_b)\) satisfies the estimates

\[
\begin{align*}
|w|_k &\leq C a \quad |w|_{k+1} \leq C(a_1 + a^3)\ln t_+ , & (4.71) \\
|s|_{k+j} &\leq C (b_j + a^2) (\ln t_+ + t_+^{j\beta}) \quad j = 0, 1, 2 .
\end{align*}
\] (4.72)
\begin{equation}
|B_{b}^{i_{k+1}} \vee |x \cdot B_{b}\rangle_{k+1} \leq C \left(aa_{1} + a^{2}b_{0} + a^{4}\right) t^{-1} \ln t_{+} \tag{4.73}
\end{equation}
for all \( t \geq T \), with \( t_{+} = t \vee t_{0} \).

In addition \( w \in L^{\infty}(I, H^{k+\theta}) \) for \( 0 \leq \theta < 1 \).

(2) The map \((w_{0}, s_{0}) \rightarrow (w, s, B_{b})\) is continuous for fixed \( t_{0} \), on the bounded sets of \( X^{k} \), from the norm of \((w_{0}, s_{0}, 0) \in X^{k-1}\) to the norm of \((w, s, B_{b})\) in \( L^{\infty}(J, X^{k-1})\) for any interval \( J \subset I \), and in the weak-* sense to \( L^{\infty}(J, X^{k})\).

\textbf{Proof.} Part (1). The proof consists in exploiting the estimates of Lemmas 4.1 and 4.2 in order to show that the map \( \Gamma : (w, s, B_{b}) \rightarrow (w', s', B'_{b}) \) defined by Proposition 4.1 with \((w', s')(t_{0}) = (w, s)(t_{0})\) and by \([2.38]\) is a contraction of a suitable set \( \mathcal{R} \) of \( \mathcal{C}(I, X^{k}) \) for a suitably time rescaled norm of \( L^{\infty}(I, X^{k-1})\). More precisely, let \( I = [T, \infty) \) and \( t_{0} \in I \). For \((w, s, 0) \in \mathcal{C}(I, X^{k})\), we define
\begin{equation}
y = |w|_{k} \vee |xw|_{k} \ , \quad y_{1} = |w|_{k+1} \tag{4.74}
\end{equation}
and we define \( \mathcal{R} \) by
\begin{equation}
\mathcal{R} = \left\{(w, s, B_{b}) \in \mathcal{C}(I, X^{k}) : (w, s)(t_{0}) = (w_{0}, s_{0}), y \leq Y, y_{1} \leq Y_{1} \ln t_{+}, \right. \\
z_{j} \leq Z_{j} \left(\ln t_{+} + t_{+}^{\beta}\right), j = 0, 1, 2, |B_{b}^{i_{k+1}} \vee |x \cdot B_{b}\rangle_{k+1} \leq N t^{-1} \ln t_{+} \right\} \tag{4.75}
\end{equation}
for some positive constants \((Y, Y_{1}, Z_{j}, N)\) to be chosen later. Actually, those constants will turn out to take the form that appears in \([1.71]-[1.73]\). The proof will require various lower bounds on \( T \) and \( t_{0} \), depending on \((Y, Y_{1}, Z_{j}, N)\). Since those constants will take the form that appear in \([1.71]-[1.73]\), the lower bounds on \( T \) and \( t_{0} \) will eventually be expressed in terms of \((a, a_{1}, b_{j})\), thereby taking the form stated in the proposition.

We first show that the set \( \mathcal{R} \) is mapped into itself by \( \Gamma \). Let \((w, s, B_{b}) \in \mathcal{R} \). From \([4.2]-[1.3]\) it follows that
\begin{equation}
|B_{a}^{i_{k+1}} \vee |\nabla(x \cdot B_{a})|_{k} \leq C I_{0}(y^{2}) \leq C Y^{2}. \tag{4.76}
\end{equation}
We now get rid of the variable \( B_{b} \). From \([1.7] \ (4.8) \ (1.70]\) it follows that
\begin{align}
|B'_{b}^{i_{k+1}} \vee |x \cdot B'_{b}\rangle_{k+1} &\leq C t^{-1} I_{0} \left(yy_{1} + y^{2} \left(z_{0} + I_{0}(y^{2}) + |B_{b}^{i_{k}}|\right)\right) \\
&\leq C t^{-1} \ln t_{+} \left(YY_{1} + Y^{2} \left(Z_{0} + Y^{2} + Nt^{-1}\right)\right) \leq Nt^{-1} \ln t_{+} \tag{4.77}
\end{align}
with

\[ N = C \left( YY_{1} + Y^{2}(Z_{0} + Y^{2}) \right) \]  

(4.78)

provided \( T \) is sufficiently large, namely \( T \geq CY^{2} \). Therefore under that condition, if we choose \( N \) in (4.75) as given by (4.78), the condition on \( B_{b} \) in (4.75) is automatically reproduced by \( \Gamma \), and it remains only to show that \( \Gamma \) reproduces the conditions on \((w, s)\).

In order to proceed, we furthermore impose the condition

\[ |B_{b}|_{k+1} \leq Y^{2} \]  

(4.79)

which together with (4.76) ensures that

\[ |B_{b}|_{k+1} \leq C Y^{2}. \]  

(4.80)

That condition is ensured by taking \( T \) and \( t_{0} \) sufficiently large so that

\[ t (\ln t_{+})^{-1} \geq N/Y^{2} = C \left( Y_{1}/Y + Z_{0} + Y^{2} \right) \]  

(4.81)

for all \( t \geq T \), a condition which can be rewritten as

\[ t_{0} \geq T \geq NY^{-2} \ln t_{0} = C \left( Y_{1}/Y + Z_{0} + Y^{2} \right) \ln t_{0}, \]  

(4.82)

where we have included the condition \( t_{0} \geq T \) for completeness. Let now \((w', s')\) be the solution of the system (2.37) with initial data \((w', s')(t_{0}) = (w_{0}, s_{0})\) obtained by using Proposition 4.1. We define

\[
\begin{aligned}
  y' &= |w'|_{k} \vee |xw'|_{k}, \quad y'_{1} = |w'|_{k+1} \\
  z'_{j} &= |s'|_{k+j}, \quad j = 0, 1, 2.
\end{aligned}
\]  

(4.83)

From Lemma 4.1, more precisely from (4.4)-(4.6), we obtain

\[ |\partial t y'| \leq E \ y' + t^{-2} \ y'_{1}, \]  

(4.84)

\[ |\partial t y'_{1}| \leq E \ y'_{1} + C \ t^{-2} z_{2} \ y' + C \ t^{-1} |\nabla(x \cdot B_{a})|_{k} \ y', \]  

(4.85)

\[ |\partial z'_{j}| \leq C \ t^{-2} \left( z_{1} z'_{j} + z_{j} z'_{1} \right) + C \ t^{-1} y(y + \delta_{j2} y_{1}) + C \ t^{-1+j\beta} |\nabla(x \cdot B_{a})|_{k}, \]  

(4.86)

where \( \delta_{j2} \) is the Kronecker symbol and where

\[ E = C \ t^{-2} \left( z_{1} + |B'_{k+1}| \right) \left( 1 + |B|_{k+1} \right) + C \ t^{-1+j\beta} |\nabla(x \cdot B_{a})|_{k} + C \ t^{-1} |x \cdot B_{b}|_{k+1}. \]  

(4.87)
We want to integrate (4.84)-(4.86) between $t_0$ and $t$ with appropriate initial data at $t_0$. For that purpose, we first eliminate the diagonal terms in (4.84) (4.85) by exponentiation according to (4.51). By (4.75), (4.76), (4.80) we estimate

$$E \leq \bar{E} = C t^{-2} \left( Z_1 t_+^\beta + Y^2 \right) (1 + Y^2) + C t^{-1-\beta} Y^2 + C N t^{-2} \ell n t_+$$

and therefore by integration

$$\left| \int_{t_0}^{t} dt' \; E(t') \right| \leq C (t \wedge t_0)^{-1} \left( t_0^\beta Z_1 + Y^2 \right) (1 + Y^2) + C (t \wedge t_0)^{-1} \ell n t_0 \; N \leq C$$

where we have used the estimates

$$\left| \int_{t_0}^{t} dt' \; t'^{-2} t_+^{\beta} \right| \leq (1 - \beta)^{-1} (t \wedge t_0)^{-1} t_0^\beta,$$  

(4.90)

and where the last inequality is achieved for $T$ and $t_0$ sufficiently large, namely

$$t_0 \geq T \geq t_0^\beta Z_1 (1 + Y^2) + Y^2 + Y^{2/\beta} + N \ell n t_0.$$  

(4.92)

We next estimate $y'$ and $y_1'$ by integrating (4.84) (4.85) between $t_0$ and $t$ with initial conditions $y'(t_0) \leq a$ and $y_1'(t_0) \leq a_1 \ell n t_0$ according to (4.90) and with $E$ exponentiated to a constant under the condition (4.92). We obtain

$$\begin{align*}
\left\{ \begin{array}{l}
y' \leq C \; a + C \int t^{-2} \; y_1' \\
y_1' \leq C \; a_1 \ell n t_0 + C \; Z_2 \int t^{-2} \; t_+^{2\beta} y' + C \; Y^2 \int t^{-1} \; y'
\end{array} \right.
\end{align*}$$

(4.93)

by the use of (4.75) (4.76) and with the short hand notation

$$\int f(t) = \left| \int_{t_0}^{t} dt' \; f(t') \right|.$$  

Let now

$$Y' = \| y'; L^\infty([T, \infty)) \| \; , \; \quad Y_1' = \| (\ell n t_+)^{-1} y_1'; L^\infty([T, \infty)) \|.$$  

(4.94)

(Strictly speaking, we should use a bounded interval $[T, T_1]$ with $T_1$ large instead of $[T, \infty)$, check that the subsequent estimates are uniform in $T_1$, and take the limit $T_1 \to \infty$ at the end. We omit that step for simplicity). From (4.93) we obtain

$$\begin{align*}
\left\{ \begin{array}{l}
y' \leq C \; a + C \; Y_1'(t \wedge t_0)^{-1} \ell n t_0 \\
y_1' \leq C \; a_1 \ell n t_0 + C \; Z_2 \; Y'(t \wedge t_0)^{-1} t_0^{2\beta} + C \; Y^2 \; Y' \; \ell n t_+
\end{array} \right.
\end{align*}$$

(4.95)
by integration and by the use of (4.90), (4.91), and therefore
\[
\begin{align*}
Y'' &\leq C a + C Y_1' T^{-1} \ln t_0 \\
Y_1' &\leq C a_1 + C Z_2 Y' T^{-1} t_0^{2\beta} + C Y^2 Y'
\end{align*}
\] (4.96)
which implies \(Y' \leq Y\) and \(Y_1' \leq Y_1\) provided we take
\[
Y = C a, \quad Y_1 = C(a_1 + a^3)
\] (4.97)
and provided we take \(T\) and \(t_0\) sufficiently large so that
\[
t_0 \geq T \geq C(a_1/a + a^2)\ln t_0 \vee C a Z_2 t_0^{2\beta}/(a_1 + a^3) .
\] (4.98)
This shows that the conditions on \((y, y_1)\) in (4.75) are preserved by \(\Gamma\).

We finally estimate \(z_j'\), \(j = 0, 1, 2\), by integrating (4.86) between \(t_0\) and \(t\) with initial condition \(z_j'(t_0) \leq b_j(\ln t_0 + i_0^{2\beta})\) according to (4.70). Using (4.75) and exponentiating the diagonal term with \(z_1 z_j'\) to a constant under the condition
\[
t_0 \geq T \geq C Z_1 t_0^{2\beta} ,
\] (4.99)
we obtain
\[
\begin{align*}
z_j' &\leq C b_j (\ln t_0 + i_0^{2\beta}) + C Z_j \int t^{-2} (\ln t + i_0^{2\beta}) z_1' \\
&\quad + C Y^2 (\ln t + i_0^{2\beta}) + C Y Y_1 \delta_{j_2}(\ln t_+)^2 .
\end{align*}
\] (4.100)
We let now
\[
Z_j' = \| (\ln t + i_0^{2\beta})^{-1} z_j'; L^\infty([T, \infty)) \|
\] (4.101)
and obtain from (4.100)
\[
Z_j' \leq C b_j + C Z_j Z_1' T^{-1} i_0^{2\beta} + C Y^2 + C Y Y_1(\ln t_0)^2 t_0^{-2\beta}
\]
which implies \(Z_j' \leq Z_j, j = 0, 1, 2\) provided we take
\[
Z_j = C (b_j + a^2)
\] (4.102)
and provided \(T\) and \(t_0\) are sufficiently large so that (4.99) holds and in addition
\[
t_0^{2\beta} \geq C (a_1/a + a^2) (\ln t_0)^2 .
\] (4.103)

We have proved that \(\Gamma\) maps \(\mathcal{R}\) into itself provided \((Y, Y_1, Z_j, N)\) are chosen according to (4.97) (4.102) (4.78) and provided \(t_0\) and \(T\) are taken sufficiently large.
according to (4.82) (4.92) (4.98) (4.99) (4.103). The latter conditions clearly take the form stated in the proposition.

We next show that the map $\Gamma$ is a contraction in $\mathcal{R}$ for a suitably time rescaled norm of $L^{\infty}(I, X^{k-1})$. Let $(w_i, s_i, B_{bi}) \in \mathcal{R}$ and let $(w'_i, w''_i, B'_{bi}) = \Gamma(w_i, s_i, B_{bi}), \ i = 1, 2$. As in Lemma 4.2, we define $(w_-, w'_-, B_{b-}) = (1/2)((w_1, s_1, B_{b1}) - (w_2, s_2, B_{b2}))$ and similarly for the primed quantities. Furthermore, we define

$$
\bar{m} = \sqrt{\text{norm of } L^2 B},
\text{ and similarly for the primed quantities.}
$$

From Lemma 4.2 with $k' = k - 1$ and from the fact that the + quantities are estimated by the definition (4.75) of $\mathcal{R}$ and by (4.76), we obtain

$$
|B_{a-}|_k \leq C Y I_0(y_-),
$$

$$
\| x \cdot B_{a-}; \tilde{H}^k \| \leq C Y I_{k-2}(y_-),
$$

$$
|\partial y'_-| \leq \tilde{E} y'_- + C t^{-2} \left\{ \left( 1 + Y^2 \right) z_1 - + \left( 1 + Z_0 \ln t_+ + Y^2 \right) (Y I_0(y_-) + n_-) \right\}
+ C t^{-1-\beta} Y^2 I_{k-2}(y_-) + C t^{-1} Y^2 I_{k-2}(y_-) + C t^{-1} Y n_- + t^{-2} y'_1 - ,
$$

$$
|\partial y'_{1-}| \leq E y'_{1-} + C t^{-2} \left\{ \left( Y_1 \ln t_+ + Y^3 \right) z_1 - + Y z_2 - + \left( Y_1 \ln t_+ + Y Z_0 \ln t_+ + Y^3 \right) (Y I_0(y_-) + n_-) \right\}
+ C t^{-1} Y^2 I_{k-2}(y_-) + C t^{-1} Y n_- \quad (4.108)
$$

where $E$ is defined by (4.88),

$$
|\partial z'_{j-}| \leq C t^{-2} \left( Z_1 t^\beta \left( z_j - + z'_j - \right) \right) + \delta_j z_2 \ t_+^{2\beta} z_1 - 
+ C t^{-1} Y \left( y_- + \delta_j y'_1 \right) + C t^{-1+j\beta} Y I_{m}(y_-) \text{ for } j = 1, 2, \quad (4.109)
$$

where $m = (k - 2) \wedge 0$,

$$
n'_- \leq C t^{-1} Y I_0 \left\{ y_{1-} + \left( Z_0 \ln t_+ + Y^2 \right) y_- + Y \left( z_1 - + Y I_0(y_-) + n_- \right) \right\} . \quad (4.110)
$$

For brevity, we continue the argument with a simplified version of the system (4.107)-(4.110) where we eliminate the diagonal terms and in particular $\tilde{E}$ by exponentiation according to (4.51), and where we eliminate the constants and the factors.

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\((Y, Y_1, Z_j, N)\). As a consequence we shall not be able to follow the dependence of the additional lower bounds of \(t_0\) and \(T\) on those factors. That dependence is of the same type as that encountered in the proof of stability of \(R\) under \(\Gamma\). Thus we rewrite (4.107)-(4.110) as

\[
|\partial_t y'| \leq t^{-2} \left\{ z_{1-} + \ln t_+ (I_0(y_-) + n_-) \right\} + t^{1-\beta} I_{k-2}(y_-) + t^{-1} n_- + t^{-2} y'_1, \quad (4.111)
\]

\[
|\partial_t y'_1| \leq t^{-2} \left\{ z_{2-} + \ln t_+ \left( z_{1-} + I_0(y_-) + n_- \right) \right\} + t^{-1} I_{k-2}(y_-) + t^{-1} n_- , \quad (4.112)
\]

\[
|\partial_t z'_j| \leq t^{-2} \left\{ t^\beta z'_j + \delta_j t^\beta z_{1-} \right\} + t^{-1} \left( y_- + \delta_j y_1 \right) + t^{1+j\beta} I_m(y_-), \quad (4.113)
\]

\[
n'_- \leq t^{-1} I_0 \left( y_{1-} + \ln t_+ y_- + z_{1-} + I_0(y_-) + n_- \right). \quad (4.114)
\]

We now define

\[
\begin{align*}
Y_- &= \| y_-; L^\infty([T, \infty)) \|, \quad Y_{1-} = \| (\ln t_+)^{-1} y_{1-}; L^\infty([T, \infty)) \|, \\
Z_{j-} &= \| t^\beta z_{j-}; L^\infty([T, \infty)) \|, \quad j = 1, 2, \\
N_- &= \| t t^\beta n_-; L^\infty([T, \infty)) \|.
\end{align*}
\]

and similarly for the primed quantities. Using those definitions and omitting the – indices in the remaining part of the contraction proof, we obtain from (4.111)-(4.114)

\[
\begin{align*}
\partial_t y' &\leq t^{-2} \left\{ Z_1 t^\beta_{1+} + Y \ln t_+ + N t^{-1} t^\beta_{1+} \ln t_+ \right\} + t^{1-\beta} Y + t^{-2} t^\beta_{1+} N + t^{-2} \ln t_+ Y_1', \\
\partial_t y'_1 &\leq t^{-2} \left\{ Z_2 t^\beta_{1+} + Z_1 t^\beta_{1+} \ln t_+ + Y \ln t_+ + N t^{-1} t^\beta_{1+} \ln t_+ \right\} + t^{-1} Y + t^{-2} t^\beta_{1+} N, \\
\partial_t z'_j &\leq t^{-2} \left\{ t^j \beta_{1+} (Z_j + Z_1) + t^{-1} (Y + \delta_j \ln t_+ Y_1) + t^{1+j\beta} Y \right\}, \\
n'_- &\leq t^{-1} \left\{ (Y + Y_1) \ln t_+ + Z_1 t^\beta_{1+} + N t^{-1} t^\beta_{1+} \right\}.
\end{align*}
\]

Integrating (4.111)-(4.118) between \(t_0\) and \(t\) with initial condition \((y', y'_1, z'_j)(t_0) = 0\), using (4.90) (4.91) and similar estimates, and omitting again some absolute constants, we obtain

\[
y' \leq (t \wedge t_0)^{-1} \left\{ Z_1 t^\beta_{1+} + Y \ln t_0 + N (t \wedge t_0)^{-1} t^\beta_{1+} \ln t_0 \right\} + (t \wedge t_0)^{-\beta} Y \\
+ (t \wedge t_0)^{-1} t^\beta_{1+} N + (t \wedge t_0)^{-1} \ln t_0 Y_1', \quad (4.120)
\]

\[
y'_1 \leq (t \wedge t_0)^{-1} \left\{ Z_2 t^\beta_{1+} + Z_1 t^\beta_{1+} \ln t_0 + Y \ln t_0 + N (t \wedge t_0)^{-1} t^\beta_{1+} \ln t_0 \right\} \\
+ \ln t_+ Y + (t \wedge t_0)^{-1} t^\beta_{1+} N, \quad (4.121)
\]

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\[ z_j' \leq (t \wedge t_0)^{-1} t_0^\beta j^\beta (Z_j + Z_1) + Y \ln t_0 + \delta_{j2} Y_1 (\ln t_0)^2 + t_0^\beta Y . \] (4.122)

Substituting (4.120)-(4.122) and (4.119) into the primed analog of the definition (4.115) (and with the \(-\) indices omitted), we obtain

\[
\begin{align*}
Y' &\leq T^{-1} \left\{ Z_1 t_0^\beta + Y \ln t_0 + N t_0^\beta + Y' (\ln t_0) \right\} + T^{-\beta} Y \\
Y_1' &\leq T^{-1} \left\{ Z_2 t_0^{2\beta} + Z_1 t_0^\beta + Y + N t_0^{3\beta} \right\} + Y \\
Z_j' &\leq T^{-1} t_0^\beta (Z_j + Z_1) + Y + \delta_{j2} Y_1 t_0^{-2\beta} (\ln t_0)^2 \\
N' &\leq (Y + Y_1) t_0^\beta \ln t_0 + Z_1 + T^{-1} N .
\end{align*}
\] (4.123)

Substituting \( Y_1' \) from the second inequality into the first one, we recast (4.123) into the form

\[
\begin{align*}
Y' &\leq \varepsilon (Y + Z_1 + Z_2 + N) \\
Y_1' &\leq Y + \varepsilon (Z_1 + Z_2 + N) \\
Z_j' &\leq Y + \varepsilon (Y_1 + Z_1 + Z_j) \\
N' &\leq Z_1 + \varepsilon (Y + Y_1 + N)
\end{align*}
\] (4.124)

where \( \varepsilon \) can be made arbitrarily small by taking \( t_0 \) and \( T \) sufficiently large. We then define

\[ X = Y + Z_1/4 + (Y_1 + Z_2 + N)/8 \] (4.125)

and similarly for the primed quantities. It then follows from (4.124) that

\[ X' \leq (1/2 + O(\varepsilon)) X \]

and therefore \( \Gamma \) is a contraction of \( \mathcal{R} \) in the norms defined by (4.104) (4.115) for \( T \) and \( t_0 \) sufficiently large. By a standard compactness argument, \( \mathcal{R} \) is easily shown to be closed for the latter norms. Therefore \( \Gamma \) has a unique fixed point in \( \mathcal{R} \).

The uniqueness of the solution in \( C(I, X^K) \) under the assumption (4.37) follows from Proposition 4.2, part (1). The estimates (4.71)-(4.73) follow from the definition (4.75) of \( \mathcal{R} \) and from the choices (4.97) (4.102) (4.78) of \( (Y, Y_1, Z_j, N) \). The dependence of \( t_0 \) and \( T \) on \( (w_0, s_0) \) stated in the proposition follows from that choice and from the fact that the lower bounds on \( t_0 \) and \( T \) are expressed in terms of those quantities, as explained above.

Finally, the fact that \( w \in L^\infty(I, H^{k+\theta}) \) for \( 0 \leq \theta < 1 \) follows immediately from (4.4) by substituting the estimates contained in the definition of \( \mathcal{R} \) into the RHS,
and integrating in time after exponentiation of the diagonal term. The crucial point is the fact that the contribution of the term in $x \cdot B_a$ is integrable in time for $\theta < 1$.

**Part (2).** Let $(w_i, s_i, B_{bi})$, $i = 1, 2$, be two solutions of the system (2.34) (2.35) with initial conditions $(w_i, s_i)(t_0) = (w_{i0}, s_{i0})$ as obtained in Part (1). In particular those solutions satisfy the estimates (4.71)-(4.73). We define $(y_-, y_{1-}, z_{j-}, n_-)$ by (4.104). By the same estimates as in the contraction proof, we obtain (4.111)–(4.114) with the primes omitted. We next define $(Y_-, Y_{1-}, Z_{j-}, N_-)$ by (4.115). Omitting again the $-$ indices as in the contraction proof, we obtain (4.116)–(4.119) with the primes omitted. Integrating the first three equations thereof between $t_0$ and $t$ with initial condition $(y, y_1, z_j)(t_0) = (y_0, y_{10}, z_{j0})$, we obtain

\[
\begin{align*}
    y &\leq y_0 + \text{RHS of (4.121)} \\
    y_1 &\leq y_{10} + \text{RHS of (4.121)} \\
    z_j &\leq z_{j0} + \text{RHS of (4.122)}
\end{align*}
\]

and therefore

\[
\begin{align*}
    Y &\leq y_0 \\
    Y_1 &\leq y_{10} (\ell n t_0)^{-1} \\
    Z_j &\leq z_{j0} t_0^{-j\beta} \\
    N &\leq \\
    \text{RHS of (4.123)}
\end{align*}
\]

which by the same argument as in the contraction proof, implies that $X$ defined by (4.125) satisfies

\[
X \leq C \left( y_0 + y_{10} (\ell n t_0)^{-1} + z_{10} t_0^{-\beta} + z_{20} t_0^{-2\beta} \right).
\] (4.126)

This proves the continuity of the map $(w_0, s_0) \to (w, s, B_b)$ from the norm of $(w_0, s_0, 0)$ in $X^k$ on the bounded sets of $X^k$ to the norm $(w, s, B_b)$ in $C(I, X^k)$ in the norms defined by (4.113), and a fortiori to the norm of $(w, s, B_b)$ in $L^\infty(J, X^{k-1})$ for $J \subset I$. The last continuity follows by a standard compactness argument.

We conclude this section by deriving asymptotic properties in time of the solutions of the auxiliary system (2.34) (2.35) obtained in the previous proposition. We prove in particular the existence of asymptotic states $(w_+, \sigma_+)$ for those solutions.
Proposition 4.4. Let $k > 3/2, 0 < \beta < 1/2, T \geq 1, I = [T, \infty)$ and let $(w, s, B_0) \in C(I, X^k)$ be a solution of the auxiliary system (4.24) (4.23) satisfying (4.27). Then

(1) There exists $w_+ \in H^k$ such that $xw_+ \in H^k$, $w(t)$ tends to $w_+$ strongly in $H^k$ and $xw(t)$ tends to $xw_+$ strongly in $H^{k'}$ for $0 \leq k' < k$ and weakly in $H^k$ when $t \to \infty$. Furthermore the following estimates hold

$$
|w_+|_k \vee |xw_+|_k \leq a_\infty = \lim_{t \to \infty} \text{sup} (|w(t)|_k \vee |xw(t)|_k),
$$

(4.127)

$$
|w(t) - U^\ast (1/t)w_+|_k \vee |w(t) - w_+|_k \leq C \ t^{-\beta},
$$

(4.128)

$$
|xw(t) - U^\ast (1/t)xw_+|_{k-1} \vee |xw(t) - xU^\ast (1/t)w_+|_{k-1} \leq C \ t^{-2\beta},
$$

(4.129)

$$
|xw(t) - xw_+|_{k-1} \leq C \left( t^{-2\beta} + t^{-1/2} \right).
$$

(4.130)

The constants $C$ in (4.128)-(4.130) depend on $\| \Lambda; L^\infty (I) \|$, where $\Lambda$ is defined by (4.37).

Assume in addition that $w \in L^\infty (I, H^{k+\theta})$ for all $0 \leq \theta < 1$, and define $(W, S)$ by (2.40). Then

(2) For all $0 \leq \theta < 1$, $w_+ \in H^{k+\theta}$ and $w(t)$ tends to $w_+$ strongly in $H^{k+\theta}$ when $t \to \infty$. Furthermore $xW(t) \in L^\infty (I, H^{k+\theta-1})$ and $xW(t)$ tends to $xw_+$ strongly in $H^{k+\theta-1}$ when $t \to \infty$.

(3) For all $j, 0 \leq j < 2$, $S \in C([1, \infty), K^{k+j})$ and $S$ satisfies the estimate

$$
|S|_{k+j} \leq C \ v \ t^{\beta(j+\varepsilon)}
$$

(4.131)

for any $\varepsilon > 0$.

Furthermore there exists $\sigma_+$ such that for all $0 \leq \theta < 1$, $\sigma_+ \in K^{k+\theta}$, $s - S$ tends to $\sigma_+$ strongly in $K^{k+\theta}$, and the following estimate holds :

$$
|s - S - \sigma_+|_{k+\theta} \leq C \ t^{-\beta(1-\theta)}.
$$

(4.132)

Proof. Part (1). Let $\tilde{w}(t) = U(1/t)w(t)$ and $\tilde{w}_0 = \tilde{w}(t_0)$ for some $t_0 \in I$. From (2.34) we obtain

$$
\partial_t (\tilde{w} - \tilde{w}_0) = U(1/t) \left\{ t^{-2}Q(s + B, w) - i(2\theta^2)^{-1}(2B \cdot s + B^2)w 
+ i \ t^{-1} (x \cdot B_a) w_i + x \cdot B_0 \right\}.
$$

(4.133)

By estimates similar to, but simpler than, those of Lemma 4.1, we estimate

$$
|\partial_t (\tilde{w} - \tilde{w}_0)|_k \leq C \left\{ t^{-2} \left( |s|_k + |B|_k \right) \right.
+ \left. \left( t^{-1-\beta} \ ||x \cdot B_a; \dot{H}^{k+1}|| + t^{-1}|x \cdot B_{b[k]}|w|_k \right) \right\}
\leq C \ t^{-1-\beta}
$$

(4.134)
by (4.2) (4.3) (4.7) (4.8) and (4.37), so that by integration

$$|\tilde{w}(t) - \tilde{w}(t_0)|_k \leq C (t \wedge t_0)^{-\beta}.$$  

(4.135)

This implies the existence of $w_+ \in H^k$ such that $w(t) \to w_+$ strongly in $H^k$, and the first estimate of (4.128). The second estimate follows from the first one and from the fact that

$$|(U(1/t) - \mathbb{1})w|_k \leq t^{-1/2}|w|_{k+1} \leq C t^{-1/2} \ln t$$  

(4.136)

by (4.37), and that $\beta < 1/2$.

Let similarly $\tilde{x}w(t) = U(1/t)xw(t)$ and $(\tilde{x}w)_0 = \tilde{x}w(t_0)$. From (2.36) we obtain

$$\partial_t(\tilde{x}w - (\tilde{x}w)_0) = U(1/t)\left\{t^{-2}Q(s + B,xw) - t^{-2}(s + B)w - it^{-2}\nabla w \right. 

- i(2t^{-1})^{-1}(2B \cdot s + B^2)xw + it^{-1}((x \cdot B_a)_s + x \cdot B_b)xw \right\}$$  

(4.137)

and by similar estimates as previously

$$|\partial_t(\tilde{x}w - (\tilde{x}w)_0)|_{k-1} \leq C\{t^{-2}[|w|_k + (|s|_k + |B|_k)(|xw|_k + |w|_{k-1} 

+ |B|_k |xw|_{k-1})] + t^{-1-2\beta} \| x \cdot B_a \| \dot{H}^{k+1} \| |xw|_k + t^{-1} |x \cdot B_b|_k |xw|_{k-1}) \right\} 

\leq C \left( t^{-2} \ln t + t^{-1-2\beta} \right)$$  

(4.138)

so that by integration

$$|\tilde{x}w(t) - \tilde{x}w(t_0)|_{k-1} \leq C (t \wedge t_0)^{-2\beta}$$  

(4.139)

which together with (4.135) implies that $xw_+ \in H^{k-1}$ and that the first estimate of (4.129) holds. The fact that $xw_+ \in H^k$ and that $xw_+$ satisfies (4.127) and the additional convergences of $xw$ to $xw_+$ follow therefrom by standard compactness and interpolation arguments.

The second estimate of (4.129) follows from the first one and from the identity

$$U^*(1/t)x = \left( x + it^{-1}\nabla \right) U^*(1/t)$$  

(4.140)

which implies

$$|U^*(1/t)xw_+ - xU^*(1/t)w_+|_{k-1} \leq t^{-1}|w_+|_k.$$  

(4.141)

Finally (4.130) follows from (4.129) and from

$$|(U^*(1/t) - \mathbb{1})xw_+|_{k-1} \leq t^{-1/2}|xw_+|_k \leq a_\infty t^{-1/2}.$$  

(4.142)
Part (2). The first statement follows from Part (1) by standard compactness and interpolation arguments. The second statement follows from the first one, and from (4.140) which implies
\[ |U^*(1/t)xw_+ - xW(t)|_{k+\theta-1} \leq t^{-1}|w_+|_{k+\theta}. \] (4.143)

Part (3). We estimate in the same way as in Lemma 4.1
\[ \partial_t|S|_{k+j} \leq C t^{-1}|W|_{k+j-1} |W|_k + C t^{-1+\beta(j+\varepsilon)} I_0 \left( |xW|_{k-\varepsilon}^2 \right) \] (4.144)
for \( 0 < \varepsilon < k - 3/2 \). The first statement and the estimate (4.131) then follow from Part (2) and from (4.144) by integration.

Let now \( q = w - W \) and \( \sigma = s - S \). From (2.34), we obtain
\[ \partial_t\sigma = t^{-2}s \cdot \nabla s + t^{-1}\nabla g(q, w + W) - t^{-1}\nabla(x \cdot B_a) \] (4.145)
and by estimates similar to those in Lemma 4.1
\[ \partial_t|\sigma|_{k+\theta} \leq C t^{-2} \left\{ |s|_k |s|_{k+2} + |s|_{k+1}^2 \right\} + C t^{-1} a|q|_k \]
\[ + C t^{-1+\beta(1+\theta)} a I_m (|xq|_{k-1}) \]
where \( a \) is defined by (4.1) and \( m = (k - 2) \wedge 0 \),
\[ \cdots \leq C \left( t^{-2+2\beta} \ell_n t + t^{-1-\beta} + t^{-1-\beta(1-\theta)} \right) \]
by (4.128) (4.129), and therefore by integration
\[ |\sigma(t) - \sigma(t_0)|_{k+\theta} \leq C(t \wedge t_0)^{-\beta(1-\theta)} \]
from which the result follows.

\[ \square \]

Remark 4.3. In Proposition 4.4, we have stated the asymptotic properties of \((w, s)\) that follow readily from the bounds on the solutions obtained in Proposition 4.3, expressed in terms of \((W, S)\) defined by (2.40). However part of the results hold under more general assumptions. For instance if we drop the assumptions on the higher norms \(|w|_{k+1}\) and \(|s|_{k+2}\), we still get the existence of a limit \(w_+\) of \(w(t)\) with \(w_+ \in H^k\) and \(xw_+ \in H^k\), with the estimate (4.129) for \(xw\) and a similar estimate for \(w\). On the other hand, we could have expressed the asymptotic properties of \((w, s)\) in terms of the simpler \((W, S)\) defined by (2.41). However the convergence properties of \(w\) would be weaker (compare (4.130) with (4.129)), thereby yielding weaker convergence properties of \(s - S\) in Part (3).
5 Cauchy problem at infinity for the auxiliary system

In this section, we construct the wave operators for the auxiliary system (2.34) (2.35) in the difference form (2.49) (2.50) for infinite initial time, for the choice of \((W, S)\) given by (2.41). In the same spirit as in Section 4, we first solve the linearized version (2.54) of the system (2.49), which together with (2.55) defines a map \(\Gamma : (q, \sigma, B_b) \rightarrow (q', \sigma', B'_b)\). We then show that this map is a contraction on a suitable set in suitable norms.

The basic tool of this section again consists of a priori estimates for suitably regular solutions of the linearized system (2.54) (2.55). We first estimate a solution of that system at the level of regularity where we shall eventually solve the auxiliary system (2.49) (2.50).

Lemma 5.1. Let \(k > 3/2, 0 < \beta < 1/2, T \geq 1, I = [T, \infty)\). Let \((W, S, 0) \in C(I, X^{k+1})\) with \((U(1/t)W, S, 0) \in C^1(I, X^k)\) and let \((q, \sigma, B_b) \in C(I, X^k)\), with \(W, xW, q, xq \in L^\infty(I, H^k)\). Let \(I' \subset I\) be an interval, let \((q', \sigma')\) be a solution of the system (2.34) with \((q', \sigma', 0) \in C(I', X^k)\) and define \(B'_b\) by (2.53). Let \(0 \leq \theta \leq 1\) and \(k \leq \ell \leq k + 2\). Then the following estimates hold:

\[
G|_{k+1} \leq CI_0(|q|_k |x(2W + q)|_k),
\]

\[
\|x \cdot G; \dot{H}^{k+1}\| \leq |\nabla(x \cdot G)|_k \leq C I_0 ((|xq|_k + |q|_k)(x|2W + q|_k)),
\]

\[
|\partial_t q'_{k+\theta}| \leq M_4(\theta, q') + C \left\{ t^{-2} \left( |\sigma|_{k+\theta+1} + |\sigma|_{k+\theta} |B|_{k+\theta} \right) 
+ |G + B_{b|_{k+\theta}} (1 + 2S + B + B_{s|_{k+\theta}}) + t^{-1-\beta(1-\theta)} \|x \cdot G; \dot{H}^{k+1}\| 
+ t^{-1}|x \cdot B_{b|_{k+\theta}}| W|_{k+\theta+1} + |R_1(W, S, 0)|_{k+\theta} \right\}
\equiv M_4(\theta, q') + M_5(\theta)|W|_{k+\theta+1} + |R_1(W, S, 0)|_{k+\theta}
\]

where \(M_4(\theta, \cdot)\) is defined by the RHS of (4.4),

\[
|\partial_t |xq'|_k| \leq M_4(0, xq') + M_5(0)|xW|_{k+1}
+ C t^{-2} \left( |q'|_{k+1} + |s + B|_k |q'|_k + |\sigma + G + B_{b|_k} W|_k \right) + |xR_1(W, S, 0)|_k,
\]

\[
\partial_t |\sigma'|_\ell \leq C t^{-2} \left\{ |s|_{k+1} |\sigma'|_\ell + \chi(\ell \geq k + 1) s|_{k+1} \right\} + C t^{-1} |W|_{k+1} + |q|_k |q|_{\ell-1} + C t^{-1+\beta(\ell-k)} |\nabla(x \cdot G)|_k + |R_2(W, S)|_\ell,
\]
\[ |B_{k+1}^j| \leq C t^{-1} I_1 \left( |q|_{k+1} |W|_{k+1} + |q|_k + |s + B_k^j| W|_k + |q|_k \right) + |\sigma + B_k + G_k^j| W|_k^2 \| R_3(W,S,0) \|_{k+1} . \] (5.6)

\[ |x B_{k+1}^j| \leq C t^{-1} I_0 \left( |q|_{k+1} + |x q|_k |W|_{k+1} + |x w|_k + |s + B_k^j| |x q|_k + |q|_k \right) (|W|_k + |q|_k) + |\sigma + B_k + G_k^j| (|x W|_k + |W|_k) |W|_k + |x \cdot R_3(W,S,0) \|_{k+1} . \] (5.7)

**Remark 5.1.** The boundedness assumptions in time of \( W \) and \( q \) ensure that the integrals \( I_0 \) occurring in (5.1) (5.2) are convergent. Furthermore, by estimates similar to but simpler than those of Lemma 4.1, one sees easily that the norms of the remainders \( R_1 \) and \( R_2 \) that occur in (5.3) (5.4) (5.5) are finite under the assumptions made on \((W,S)\). On the other hand (see Remark 4.1), the statements on \( B_k^j \) are non empty only in so far as the integrals over \( \nu \) in the RHS of (5.3) (5.7) are convergent. This requires additional assumptions on the behaviour of \((W,S)\) and of \((q,\sigma,B_k^j)\) at infinity in time, which will be made in due course.

**Proof.** The proof is very similar to that of Lemma 4.1. The estimates (5.1) (5.2) follow immediately from (3.25) (3.28) and from (3.10) with \( m = \bar{m} = k \).

We next estimate \( q' \) in \( H^{k+\theta} \), starting from (2.54). Let \( m = k + \theta \). We estimate \( \partial_t \| \omega^m q' \|_2 \) by an energy method in the same way as in (4.9). The terms containing \( q' \) are estimated in the same way as in (4.9), thereby yielding \( M_4(\theta,q') \), while the remaining terms are estimated with the help of (3.10) with \( \bar{m} = m \), supplemented by (3.19) for the term containing \( x \cdot G \). Together with an elementary estimate of \( \partial_t \| q' \|_2 \) (to which the terms containing \( q' \) make no contribution), this proves (5.3).

We next estimate \( x q' \) in \( H^k \), starting from (2.58). The terms containing \( x q' \) or \( x W \) explicitly yield the first two terms in the RHS of (5.4) by the special case \( \theta = 0 \) of the proof of (5.3), while the remaining terms are estimated by (3.10) with \( m = \bar{m} = k \). This proves (5.4).

We next estimate \( \sigma' \), starting with \( \partial_t \| \omega^f \sigma' \|_2 \). The term \( s \cdot \nabla \sigma' \) is estimated exactly as in the proof of (4.6). The term \( \sigma \cdot \nabla S \) is estimated directly by Lemma 3.2 as

\[
\| \omega^f (\sigma \cdot \nabla S) \|_2 \leq C \left( \| \omega^f \sigma \|_2 \| \nabla S \|_\infty + \| \sigma \|_\infty \| \omega^{f+1} S \|_2 \right) \leq C \left( |\sigma|_{l} |S|_{k+1} + |\sigma|_{k} |S|_{\ell+1} \right) .
\] (5.8)
The term containing $g$ is estimated by (3.10) with $(m, \bar{m}) = (k \land (\ell - 1), k \lor (\ell - 1))$ or $(\ell - 1, k + 1)$ as
\[
\| \omega^{\ell - 1}(q(q + 2W)) \|_2 \leq C|q|_{\ell - 1}(|q|_k + |W|_{k+1}).
\] (5.9)

The term containing $G$ is estimated by (3.20). Together with a simpler estimate of $\partial_t \| \nabla \sigma' \|_2$, the previous estimates yield (5.5).

Finally (5.6) (5.7) follow from (2.55) (3.25) (3.28) and from repeated use of (3.10) with $m = \bar{m} = k$.

We shall also need estimates for the difference of two solutions of the linearized system (2.54) (2.55) corresponding to two different choices of $(q, \sigma, B_b)$ but to the same choice of $(W, S)$. Those estimates will be provided by Lemma 4.2, since for such solutions $(q_- , \sigma_- ) = (w_-, s_-)$ and $(q'_- , \sigma'_- ) = (w'_-, s'_-)$ in the notation of that Lemma extended in an obvious way.

We now begin the study of the Cauchy problem for the auxiliary system (2.49) (2.50) and for that purpose we first study that problem for the linearized system (2.54). For finite initial time $t_0$, that problem is solved by Proposition 4.1. The following is a special case of that proposition and of Lemma 5.1

**Proposition 5.1.** Let $k > 3/2$, $0 < \beta < 1/2$, $T \geq 1$ and $I = [T, \infty)$. Let $(W, S, 0) \in C(I, X^{k+1})$ with $(W(1/t), S, 0) \in C^1(I, X^k)$ and let $(q, \sigma, B_b) \in C(I, X^k)$, with $W, xW, q, xq \in L^\infty(I, H^k)$. Let $t_0 \in I$ and $(q_0, \sigma_0', 0) \in X^k$. Then the system (2.54) has a unique solution $(q', \sigma')$ in $I$ such that $(q', \sigma', 0) \in C(I, X^k)$ and $(q', \sigma')(t_0) = (q_0, \sigma_0')$. That solution satisfies the estimates (5.3) (5.4) (5.5) for all $t \in I$, with $G$ estimated by (5.1) (5.2).

In order to study the Cauchy problem with infinite initial time, both for the linearized system (2.54) and for the nonlinear system (2.49) (2.50), we shall need stronger assumptions on the asymptotic behaviour in time of $(W, S)$. For simplicity, from now on we make the final choice of $(W, S)$ that will turn out to satisfy those assumptions. Thus we choose $(W, S)$ as explained in Section 2, namely
\[
\begin{align*}
W(t) &= U^*(1/t)w_+ \\
S(t) &= \int_1^t dt' \ t'^{-1} (\nabla g(W) - \nabla(x \cdot B_a)_L(W))
\end{align*}
\] (2.40) $\equiv$ (5.10)
for some fixed \( w_+ \in H^{k+\alpha+1} \) with \( xw_+ \in H^{k+\alpha} \) for some \( \alpha \geq 1 \) (we shall eventually need \( \alpha > 1 \)), and we define

\[
a_+ = |w_+|_{k+\alpha+1} \vee |xw_+|_{k+\alpha}. \tag{5.11}
\]

Using the fact that

\[
\partial_t + i(2t^2)^{-1}\Delta = U^*(1/t)\partial_t U(1/t) \tag{5.12}
\]

we recast the remainders that occur in the system (2.49) (2.50) into the form

\[
R_1(W, S, 0) = t^{-2}Q(S + B_s, W) - i(2t^2)^{-1}(2B_s \cdot S + B_s^2)W + it^{-1}(x \cdot B_s)gw \tag{5.13}
\]

\[
R_2(W, S) = t^{-2}S \cdot \nabla S \tag{5.14}
\]

\[
R_3(W, S, 0) = t^{-1}F_1 \left( \text{Im} \bar{W} \nabla W - (S + B_s)|W|^2 \right) \tag{5.15}
\]

where \( B_s = B_a(W) \) (see Section 2).

We shall need the following estimates.

**Lemma 5.2.** Let \( k > 3/2 \) and \( 0 < \beta < 1/2 \). Let \( w_+ \in H^{k+\alpha+1} \) with \( xw_+ \in H^{k+\alpha} \) for some \( \alpha \geq 1 \). Define \((W, S)\) and \( a_+ \) by (5.10) (5.11). Then \((W, S, 0) \in C([1, \infty), X^{k+\alpha})\) and the following estimates hold

\[
|xW(t)|_{k+\alpha} \leq |xw_+|_{k+\alpha+1} + t^{-1} |\nabla w_+|_{k+\alpha} \leq 2a_+, \tag{5.16}
\]

\[
|B_s|_{k+\alpha+1} \vee |\nabla(x \cdot B_s)|_{k+\alpha} \leq I_0(|xW|_{k+\alpha}(|xW|_{k+\alpha} + |W|_{k+\alpha})) \leq C a_+^2, \tag{5.17}
\]

\[
|S|_{k+j} \leq C a_+^2 \left( \ell(1 + t^{\beta(j-\alpha)}) \right) \text{ for } 0 \leq j \leq 2 + \alpha, \tag{5.18}
\]

\[
|R_1(W, S, 0)|_{k+\alpha} \leq C(a_+) \left( t^{-2} \ell(1 + t^{1-\beta(a+1-\theta)}) \right) \text{ for } 0 \leq \theta \leq 1, \tag{5.19}
\]

\[
|xR_1(W, S, 0)|_k \leq C(a_+) \left( t^{-2} \ell(1 + t^{1-\beta(a+1)}) \right), \tag{5.20}
\]

\[
|R_2(W, S)|_{k+j} \leq C a_+^4 t^{-2} \ell(1 + t^{\beta(j+1-\alpha)}) \text{ for } 0 \leq j \leq 1 + \alpha, \tag{5.21}
\]

\[
|R_3(W, S, 0)|_{k+1} \vee |x \cdot R_3(W, S, 0)|_{k+1} \leq C a_+^2 t^{-1}(1 + a_+^2 \ell(1 + t)). \tag{5.22}
\]

**Proof.** We first estimate \( x \cdot W \). From the commutation relation (4.140), it follows that

\[
U(1/t) x W(t) = xw_+ + it^{-1} \nabla w_+. \tag{5.23}
\]

which implies (5.16).

The estimate (5.17) follows immediately form (3.25) (3.28) (3.10) and (5.16).
We next estimate $S$. Let $\ell = k + j \geq k$. From (5.10) we obtain
\[ \| \omega^\ell S \|_2 \leq C \int_1^t dt' \left\{ t'^{-1} \| \omega^\ell-1 |W|^2 \|_2 + t'^{-1+\beta(\ell-k-\alpha)} \| \nabla (x \cdot B_*') \|_{k+\alpha} \right\} \] (5.24)
from which (5.18) follows by (3.10), (5.16), (5.17) and integration on time, provided $\ell - 1 \leq k + 1 + \alpha$ or equivalently $j \leq 2 + \alpha$.

We next estimate $R_1$. By repeated use of (3.10) and by (3.19), we obtain from (5.13)
\[ |R_1(W, S, 0)|_{k+\theta} \leq C t^{-2} \left\{ |S|_{k+\theta+1} + |B_*|_{k+\theta} \left( 1 + |S|_{k+\theta} + |B_*|_{k+\theta} \right) \right\} |W|_{k+\theta+1} + C t^{-1-\beta(\alpha+1-\theta)} \| x \cdot B_* \hat{H}^{k+1+\alpha} \| |w_+|_{k+\theta} \] (5.25)
and therefore by (5.17) (5.18) and with $\alpha \geq 1 \geq \theta$
\[ |R_1(W, S, 0)|_{k+\theta} \leq C t^{-2} a_+^3 \left( \ln t + t^{\beta(1+\theta-\alpha)} + a_+^2 (1 + \ln t) \right) + C t^{-1-\beta(\alpha+1-\theta)} a_+^3 \] (5.26)
which proves (5.19) since $\beta < 1/2$.

The proof of (5.20) follows from the fact that
\[ xR_1(W, S, 0) = L_0 xW - t^{-2}(S + B_*)W \] (5.27)
where the linear operator $L_0$ is defined by rewriting the RHS of (5.13) as $L_0 W$. The term $L_0 xW$ is estimated in the same way as in the proof of (5.19) with $\theta = 0$, and by using in addition (5.16), while the last term in (5.27) is estimated by using (5.17) (5.18) and (3.10).

We next estimate $R_2$. By a direct application of Lemma 3.2, we obtain from (5.14)
\[ \| \omega^\ell R_2(W, S) \|_2 \leq C t^{-2} \left( \| S \|_\infty \| \omega^{\ell+1} S \|_2 + \| \nabla S \|_3 \| \omega^\ell S \|_6 \right) \] (5.28)
for $\ell \geq 1$, which yields (5.22) by the use of (5.18).

Finally the estimate (5.22) follows readily from (3.25) (3.28) (3.10) and from (5.16) (5.17) (5.18).

Remark 5.2. If one makes the simpler choice (2.41) for $(W, S)$, Lemma 5.2 and its proof remain essentially unchanged, the only difference being that the proof of
(5.19) (5.20) now requires $\alpha \geq 2$ in order to estimate the contribution of $\Delta w_+$ to $R_1$.

We can now solve the Cauchy problem with infinite initial time for the linearized system (2.54) (2.55) for the previous choice of $(W,S)$.

**Proposition 5.2.** Let $k > 3/2$, $0 < \beta < 1/2$, $T \geq 1$ and $I = [T, \infty)$. Let $w_+ \in H^{k+\alpha+1}$ with $x w_+ \in H^{k+\alpha}$ for some $\alpha > 1$ with $\beta(\alpha + 1) \geq 1$. Define $(W,S)$ and $a_+$ by (5.10) (5.11). Let $(q, \sigma, B_b) \in C(I,X^k)$ satisfy

$$|q|_k \vee |x q|_k \leq Y t^{-1} \ln t,$$  
(5.29)

$$|q|_{k+1} \leq Y_1 \left(t^{-1} \ln t + t^{-\alpha \beta}\right),$$  
(5.30)

$$|\sigma|_{k+j} \leq Z_j t^{-1} \ln t \left(\ln t + t^{j\beta}\right) \text{ for } j = 0, 1, 2,$$  
(5.31)

$$|B_b|_{k+1} \vee |x \cdot B_b|_{k+1} \leq N t^{-1} \ln t,$$  
(5.32)

for some constants $(Y, Y_1, Z_j, N)$ and for all $t \in I$.

Then the linearized system (2.54) (2.55) has a unique solution $(q', \sigma', B'_b) \in C(I,X^k)$ satisfying

$$|q'|_k \vee |x q'|_k \leq Y' t^{-1} \ln t,$$  
(5.33)

$$|q'|_{k+1} \leq Y'_1 \left(t^{-1} \ln t + t^{-\alpha \beta}\right),$$  
(5.34)

$$|\sigma'|_{k+j} \leq Z'_j t^{-1} \ln t \left(\ln t + t^{j\beta}\right) \text{ for } j = 0, 1, 2,$$  
(5.35)

$$|B'_b|_{k+1} \vee |x \cdot B'_b|_{k+1} \leq N' t^{-1} \ln t,$$  
(5.36)

for some constants $(Y', Y'_1, Z'_j, N')$ depending on $(Y, Y_1, Z_j, N, a_+, T)$. The solution is actually unique in $C(I,X^k)$ under the condition that $(q', \sigma')$ tends to zero in $L^2 \oplus \dot{H}^1$ when $t \to \infty$.

**Remark 5.3.** Whereas the conditions $0 < \beta < 1/2$ and $\alpha > 1$ are used in an essential way in the proof of Proposition 5.2, the condition $\beta(\alpha + 1) \geq 1$ has been imposed for convenience only, in order to obtain rather simple and optimal decay properties for $(q', \sigma')$, and could be relaxed at the expense of a weakening of those properties. For given $\alpha > 1$, it can be achieved by taking $\beta$ sufficiently close to $1/2$. Its meaning is that for a given regularity of $w_+$, one should put a sufficiently large part of the interaction $B_1$ into the long range part $(x \cdot B_a)_L$ so as to obtain a
sufficiently good decay of the short range part \((x \cdot B_α)s\).

**Proof.** With Proposition 5.1 available, it is sufficient to prove Proposition 5.2 for \(T\) sufficiently large, depending possibly on \((Y, Y_1, Z_j, N, a_+).\) Furthermore it is sufficient to solve the system (2.54) for \((q', σ'),\) since \(B'_0\) is given by an explicit formula, namely (2.55). The proof consists in showing that the solution \((q'_0, σ'_0)\) of the linearized system (2.54) with initial data \((q'_0, σ'_0)(t_0) = 0\) for some finite \(t_0 ≥ T,\) obtained from Proposition 5.1, satisfies the estimates (5.33)-(5.35) uniformly in \(q\) I that solution converges on the compact intervals of \(l\) uniformly in suitable norms.

Let therefore \(T\) be sufficiently large, in a sense to be made clear below. We define

\[
\begin{align*}
y &= |q| + |xq|_k, \quad y_1 = |q|_{k+1}, \\
z_j &= |σ|_{k+j}, \quad j = 0, 1, 2,
\end{align*}
\]

and we first take \(T\) large enough so that (5.29)-(5.32) imply

\[
y \vee y_1 ≤ a_+ \quad \text{(5.38)}
\]

\[
z_j ≤ a_+^2 \ell n t \quad \text{for } j = 0, 1, 2, \quad |B_0|_{k+1} ≤ a_+^2 \quad \text{(5.39)}
\]

for \(t ≥ T.\) It follows from (5.1) (5.2) (5.29) (5.38) that

\[
|G|_{k+1} \vee |∇(x \cdot G)|_k ≤ C a_+ I_0(y) ≤ C a_+ (Y t^{-1} \ell n t \vee a_+).
\]

(5.40)

Let \(t_0 > T\) and let \((q'_0, σ'_0)\) be defined as above. We want to estimate \((q'_0, σ'_0)\) for \(T ≤ t ≤ t_0.\) We define

\[
\begin{align*}
y' &= |q'_0| + |xq'_0|_k, \quad y'_1 = |q'_0|_{k+1}, \\
z'_j &= |σ'_0|_{k+j}, \quad j = 0, 1, 2, \\
Y' &= \| t(\ell n t)^{-1} y' ; L^∞([T, t_0]) \|, \\
Y'_1 &= \| (t^{-1} \ell n t + t^{-αβ})^{-1} y'_1 ; L^∞([T, t_0]) \|, \\
Z'_j &= \| (t^{-1} \ell n t (\ell n t + t β))^{-1} z'_j ; L^∞([T, t_0]) \|.
\end{align*}
\]

(5.41)

(5.42)

(5.43)

We first estimate \(q'_0\) and more precisely \(y'\) and \(y'_1,\) starting from (5.3) (5.4). We estimate \((W, S, B_α)\) by (5.16)-(5.18), \((σ, B_0)\) by (5.39), \(x \cdot B_0\) by (5.32), \(G\) by (5.40) and \(R_1\) by (5.19) (5.20), with \(β(α + 1) ≥ 1.\) We obtain

\[
|∂_t y'| ≤ C \left\{ a_+^2 (1 + a_+^2) + N \right\} t^{-2} \ell n t + a_+^2 t^{-1-β} \} y'
\]

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We integrate (5.44) between $t$ and therefore by substituting (5.46) into (5.42) as
\[
\frac{\partial y'}{y'} \leq C \left\{ \left( a_+^2 (1 + a_+) + N \right) t^{-2} \ln t + C a_+ Y t^{-2-\beta} \ln t + t^{-2} y_1' + C(a_+) t^{-2} \ln t \right\},
\]
(5.44)
\[
+ C \left( a_+^3 (1 + a_+) + N a_+ \right) t^{-2} \ln t + C a_+ Y t^{-2} \ln t + C(a_+) \left( t^{-2} \ln t + t^{-1-\alpha\beta} \right).
\]
(5.45)
We integrate (5.44) (5.45) between $t$ and $t_0$, with $y'(t_0) = y'_1(t_0) = 0$, exponentiating the diagonal terms to a constant for $T$ sufficiently large depending on $(a_+, N)$ and substituting (5.42) into the remaining terms. We obtain
\[
\begin{align*}
\begin{cases}
y'(t) &\leq C a_+^2 Y^{-1-\beta} \ln t + CY'_1 \left( t^{-2} \ln t + t^{-1-\alpha\beta} \right) + (C N a_+ + C(a_+)) t^{-1} \ln t \\
y'_1(t) &\leq C a_+^2 (Y' + Y) t^{-1} \ln t + C N a_+ + C(a_+). \left( t^{-1} \ln t + t^{-\alpha\beta} \right)
\end{cases}
\end{align*}
\]
(5.46)
and therefore by substituting (5.46) into (5.42)
\[
\begin{align*}
\begin{cases}
Y' &\leq C a_+^2 Y t^{-\beta} + CY'_1 T^{-\theta} + C N a_+ + C(a_+) \equiv CY'_1 T^{-\theta} + A \\
Y'_1 &\leq C a_+^2 (Y' + Y) + C N a_+ + C(a_+) \equiv C a_+^2 Y' + A_1
\end{cases}
\end{align*}
\]
(5.47)
with $\theta = 1 \wedge \alpha\beta > \beta$. It follows from (5.47) that
\[
\begin{align*}
\begin{cases}
Y' &\leq C a_+^2 T^{-\theta} Y' + A + C T^{-\theta} A_1 \leq C a_+^2 Y T^{-\beta} + C N a_+ + C(a_+) \\
Y'_1 &\leq C a_+^2 T^{-\theta} Y'_1 + C a_+^2 A + A_1 \leq C a_+^2 Y + C N a_+ (1 + a_+^2) + C(a_+)
\end{cases}
\end{align*}
\]
(5.48)
for $T^\theta \geq T^\beta \geq C a_+^2$. Finally
\[
\begin{align*}
\begin{cases}
Y' &\leq C a_+^2 T^{-\beta} Y + (1 + N) C(a_+) \\
Y'_1 &\leq C a_+^2 Y + (1 + N) C(a_+)
\end{cases}
\end{align*}
\]
(5.49)
for $T$ sufficiently large depending on $(Y, Y_1, Z_j, N, a_+)$. This proves that $q'_0$ satisfies the estimates (5.33) (5.34) for $T \leq t \leq t_0$.

We next estimate $q''_0$, and more precisely $z_j'$, starting from (5.3), which we rewrite with the help of the definitions (5.37) (5.41) and of the estimates (5.18) (5.21) and (5.38)-(5.40) as
\[
|\partial_\ell z'_j| \leq C a_+^2 t^{-2} \left\{ \ln t (z_j + z'_j) + \delta_j t^{\beta(2-\alpha)} (z_1 + z'_1) \right\} + C t^{-1} a_+ (y + \delta_j y_1) + C^t^{-1+j\beta} a_+ I_0(y) + C a_+^4 t^{-2} \ln t \left( \ln t + t^{\beta(j+1-\alpha)} \right)
\]
(5.50)
for $j = 0, 1, 2$ with $\delta_{jz}$ the Kronecker symbol in order to include additional terms for $j = 2$. Using (5.29)-(5.31) and (5.43), we obtain from (5.50)

\[
|\partial_t z_j'| \leq C a_+^2 t^{-2} \ln t \ z_j' + C a_+^2 t^{-3} \ln t \left\{ \ln t (\ln t + t^{j\beta}) Z_j + \right.
\]

\[
+ \delta_{jz} t^{j(3-\alpha)} (Z_1 + Z_1') \left\} + C a_+ t^{-2} \left\{ Y \ln t + \delta_{jz} Y_1 (\ln t + t^{1-\alpha \beta}) \right\}
\]

\[
+ C a_+ t^{-2+j\beta} \ln t \ Y + C a_+^4 t^{-2} \ln t (\ln t + t^{2(j+1-\alpha)}).
\]

(5.51)

We integrate (5.51) between $t$ and $t_0$ with $z_j'(t_0) = 0$, exponentiating the diagonal terms to a constant for $T$ sufficiently large in the sense that

\[
T (\ln T)^{-1} \geq C a_+^2
\]

and we substitute the result into the definition (5.43), thereby obtaining

\[
Z_j' \leq C a_+^2 \left( T^{-1} \ln T \ Z_j + \delta_{jz} T^{-1+j(1-\alpha)} (Z_1 + Z_1') \right)
\]

\[
+ C a_+ \left( Y + \delta_{jz} T^{-2\beta} + T^{1-(\alpha+2)\beta} Y_1 \right) + C a_+^4
\]

so that for $\alpha \geq 1$ and $\beta(\alpha + 1) \geq 1$ and under the condition (5.52)

\[
\left\{ \begin{array}{l}
Z_j' \leq C a_+^2 T^{-1} \ln T \ Z_j + C a_+ Y + C a_+^4, \quad \text{for } j = 0, 1, \\
Z_2' \leq C a_+^2 T^{-1} \ln T (Z_2 + Z_1) + C a_+ (Y + t^{-\beta Y_1}) + C a_+^4
\end{array} \right.
\]

(5.53)

for $T$ sufficiently large. This proves that $\sigma_{t_0}'$ satisfies the estimate (5.35) for $T \leq t \leq t_0$.

We now prove that $(q_{t_0}', \sigma_{t_0}')$ tends to a limit when $t_0 \to \infty$. For that purpose we consider two solutions $(q_i', \sigma_i') = (q_{t_i}', \sigma_{t_i}')$, $i = 1, 2$, of the system (2.54) corresponding to the same choice of $(q, \sigma, B_0)$ and to $t_0 = t_i$, $i = 1, 2$, for $T \leq t_1 \leq t_2$. Let $(q_{-}', \sigma_{-}') = 1/2(q_{1}' - q_{2}', \sigma_{1}' - \sigma_{2}')$. For fixed $(q, \sigma, B_0)$, the inhomogeneous term in $q'$ in the equation for $q'$ is the same and therefore $q_{-}'$ satisfies an homogeneous linear equation which preserves the $L^2$ norm, so that

\[
\| q_{-}'(t) \|_2 = \| q_{-}'(t_1) \|_2 = 1/2 \| q_{2}'(t_1) \|_2 \leq Y' t_1^{-1} \ln t_1
\]

(5.54)

for $T \leq t \leq t_1$, by (5.33) applied to $q_2'$ at $t = t_1 \in [T, t_2]$. Similarly, $\sigma_{-}'$ satisfies the equation

\[
\partial_t \sigma_{-}' = t^{-2} (S + \sigma) \cdot \nabla \sigma_{-}'
\]

(5.55)
so that by an elementary subestimate of Lemma 4.2 and by (5.18) (5.39)
\[ \partial_t \| \nabla \sigma' \|_2 \leq C a_+^2 t^{-2} \ell n t \| \nabla \sigma' \|_2 \]  
(5.56)
and therefore under the condition (5.52)
\[ \| \nabla \sigma'_t (t) \|_2 \leq C \| \nabla \sigma'_t (t_1) \|_2 \leq C Z_0' t_1^{-\ell n t_1} \]  
(5.57)
for \( T \leq t \leq t_1 \) by (5.35) applied to \( \sigma'_2 \) at \( t = t_1 \).

From (5.54) (5.57), it follows that \((q'_0, \sigma'_0)\) converges to a limit \((q', \sigma') \in C(I, L^2 \oplus \dot{H}^1)\) uniformly on the compact subintervals of \( I \). From the uniform estimates (5.33)-(5.35) and from Lemma 5.1, it then follows by a standard compactness argument that \((q', \sigma', 0) \in C(I, X^k)\) and that \((q', \sigma')\) also satisfies the estimates (5.33)-(5.35).

Clearly \((q', \sigma')\) satisfies the system (2.54). This completes the existence part of the proof.

The uniqueness statement follows immediately from the \( L^2 \) norm conservation for the difference of the \( q' \) components and from (5.55) (5.57) for the difference of the \( \sigma' \) components of two solutions.

As mentioned at the beginning of the proof, the existence and uniqueness of \( B'_b \) follow from the fact that it is given by an explicit formula (2.55) and the estimate (5.36) follows immediately from (5.38) (5.39) with
\[ N' = C a_+^2 (1 + a_+^2) \]  
(5.58)
for \( T \) large enough to ensure (5.38) (5.39).

We now turn to the main result of this section, namely the fact that for \( T \) sufficiently large, depending on \( a_+ \), the auxiliary system (2.49) (2.50) with \((W, S)\) defined by (5.10) has a unique solution \((q, \sigma, B_b)\) defined for all \( t \geq T \) and decaying at infinity in the sense of (5.29)-(5.32). In the same spirit as for Proposition 4.3, this will be done by showing that the map \( \Gamma : (q, \sigma, B_b) \to (q', \sigma', B'_b) \) defined by Proposition 5.2 is a contraction in suitable norms.

**Proposition 5.3.** Let \( k > 3/2 \) and \( 0 < \beta < 1/2 \). Let \( w_+ \in H^{k+\alpha+1} \) with \( xw_+ \in H^{k+\alpha} \) for some \( \alpha > 1 \) with \( \beta(\alpha + 1) \geq 1 \). Define \((W, S)\) and \( a_+ \) by (5.10) (5.11).

Then

1. There exists \( T = T(k, \beta, \alpha, a_+) \), \( 1 \leq T < \infty \) such that the system (2.49) (2.50) has a unique solution \((q, \sigma, B_b) \in C(I, X^k)\), with \( I = [T, \infty) \), satisfying the
estimates (5.29)-(5.32) for some constants \((Y,Y_1,Z_j,N)\) depending on \((k,\beta,\alpha,a_+)\).

Furthermore

\[
|G|_{k+1} \lor |\nabla(x \cdot G)|_k \leq C a_+ Y t^{-1} \ln t
\]

(5.40) \equiv (5.59)

where \(G\) is defined by (2.40). The solution is actually unique in \(C(I,X^k)\) under the conditions that

\[
|q|_k \lor |q|_{k+1} \lor |\sigma|_{k+2} \lor |B_b|_k \in L^\infty(I)
\]

(5.60)

and

\[
|\sigma|_{k+1} \lor t^{2\beta+\varepsilon} (|q|_k \lor |xq|_{k-1}) \to 0 \text{ when } t \to \infty
\]

(5.61)

for some \(\varepsilon > 0\).

(2) The map \(w_+ \to (W+q,S+\sigma,B_b) \equiv (w,s,B_b)\) is continuous on the bounded sets of the norm (5.11) from the norm \(|w_+|_k \lor |xw_+|_{k-1}\) for \(w_+\) to the norm of \((w,s,B_b)\) in \(L^\infty(J,X^{k-1})\) and in the weak-* sense to \(L^\infty(J,X^k)\) for any interval \(J \subset I\).

**Proof.** Part (1). The proof consists in showing that the map \(\Gamma : (q,\sigma,B_b) \to (q',\sigma',B'_b)\) defined by Proposition 5.2 is a contraction of a suitable set \(\mathcal{R}\) of \(C(I,X^k)\), with \(I = [T,\infty)\), for \(T\) sufficiently large and for a suitably time rescaled norm of \(L^\infty(I,X^{k-1})\). We define \((y,y_1,z_j)\) by (5.37) and we define \(\mathcal{R}\) by

\[
\mathcal{R} = \{(q,\sigma,B_b) \in C(I,X^k) : y \leq Y t^{-1} \ln t, y_1 \leq Y_1 \left(t^{-1} \ln t + t^{-a_\beta}\right), z_j \leq Z_j t^{-1} \ln t \left(t^{2\beta} + t^{\varepsilon}\right), j = 0, 1, 2, |B_b|_{k+1} \lor |x \cdot B_b|_{k+1} \leq N t^{-1} \ln t\}
\]

(5.62)

for some constants \((Y,Y_1,Z_j,N)\) depending on \(a_+\), to be chosen later, and we take \(T\) large enough so that (5.38) (5.39) and therefore (5.40) hold for \((q,\sigma,B_b) \in \mathcal{R}\).

We first show that the set \(\mathcal{R}\) is mapped into itself by \(\Gamma\) for suitable \((Y,Y_1,Z_j,N)\) and sufficiently large \(T\). Let \((q',\sigma',B'_b) = \Gamma(q,\sigma,B_b)\) as defined by Proposition 5.2. As mentioned in the proof of that proposition, it follows from (5.6) (5.7) (5.22) (5.38)-(5.40) that \(B'_b\) satisfies (5.36) with \(N'\) defined by (5.58) so that if we define \(N\) by

\[
N = C a_+^2 (1 + a_+^2)
\]

(5.63)

then the condition on \(B_b\) contained in (5.62) is reproduced by \(\Gamma\). It remains to estimate \((q',\sigma')\). Now from the proof of Proposition 5.2, it follows that \((q',\sigma')\) satisfies the estimates (5.33) (5.34) (5.35) with \((Y',Y'_1,Z'_j)\) satisfying (5.49) (5.53), for \(T\) sufficiently large depending on \((Y,Y_1,Z_j,a_+)\). With \(N\) given by (5.63), it
follows immediately from (5.49) that one can choose \( Y \) and \( Y_1 \) depending on \( a_+ \) such that (5.49) implies \( Y' \leq Y \) and \( Y'_1 \leq Y_1 \) for \( T \) sufficiently large, namely \( T^\beta \geq 4a_+^2 \). Therefore the conditions on \( q \) in (5.62) are also reproduced by \( \Gamma \) for that choice. Finally, it follows from (5.53) that one can choose \( Z_j \), actually in the form

\[
Z_j = C a_+ Y + C a_+^4, \quad j = 0, 1, 2
\]

(5.64)

so that (5.53) implies \( Z'_j \leq Z_j \) for \( T \) sufficiently large, in fact for

\[
T(\ln T)^{-1} \geq C a_+^2 \quad , \quad T^\beta \geq Y_1 / Y.
\]

(5.65)

This completes the proof of the fact that \( \Gamma \) maps \( R \) into itself for \( (Y,Y_1,Z_j,N) \) chosen as above, depending on \( a_+ \), and for \( T \) sufficiently large, depending on \( a_+ \).

We next show that the map \( \Gamma \) is a contraction in \( R \) for a suitably weighted norm of \( L^\infty(I, X^{k-1}) \). Let \( (q_i, \sigma_i, B_{bi}) \in R \) and let \( (q'_i, \sigma'_i, B'_{bi}) = \Gamma(q_i, \sigma_i, B_{bi}), i = 1, 2 \). We define \( (q'_\pm, \sigma'_\pm, B_{b\pm}) = 1/2((q_1, \sigma_1, B_{b1}) \pm (q_2, \sigma_2, B_{b2})) \) and similarly for the primed quantities. Furthermore we define

\[
y_\pm = |q_\pm|_{k-1} \vee |xq_\pm|_{k-1}, \quad y_1 = |q_\pm|_k
\]

\[
z_j^- = |\sigma^-|_{k+j-1}, \quad j = 1, 2, \quad n^- = |B_{b^-}|_k \vee |x \cdot B_{b^-}|_k
\]

(5.66)

and similarly for the primed quantities and for \( B_a \) and \( B \).

We shall estimate \( (q_\pm, \sigma_\pm, B_{b\pm}) \) by Lemma 4.2 applied to the solutions \((w'_i, s'_i, B'_{bi}) = (W + q'_i, S + \sigma'_i, B'_{bi}) \) of the system (2.37) (2.38) associated with \((w_i, s_i, B_{bi}) = (W + q_i, S + \sigma_i, B_{bi}) \). As a consequence we shall apply Lemma 4.2 with \((w_-, s_-) = (q_-, \sigma_-), (w'_-, s'_-) = (q'_-, \sigma'_-) \) and \((w_+, s_+) = (W + q_+, S + \sigma_+), (w'_+, s'_+) = (W + q'_+, S + \sigma'_+) \), and for that purpose we shall use the estimates (5.16) (5.18) for \((W,S)\) and the fact that \((q_+, \sigma_+)\) and therefore \((q'_+, \sigma'_+)\) are not larger than \((W,S)\) in the sense of (5.38) (5.39). From the first part of the proof, it follows that \((q'_i, \sigma'_i)\) and therefore \((q'_+, \sigma'_+)\) and \((w'_+, s'_+)\) satisfy the same properties. Together with (5.17) (5.18) (5.30) (5.63) (5.40), this implies that the available estimates on \((w_+, s_+, B_{b+})\) and on \(B_{a+}, B_+\) are:

\[
\begin{align*}
|w_+|_k \vee |w_+|_{k+1} & \leq C a_+ \\
|s_+|_{k+j} & \leq C a_+^2 \left( \ell t + t^{(j-\alpha)\beta} \right) \\
|B_{a+}|_{k+j} \vee |x \cdot B_{a+}|_{k+j} & \leq C a_+^2 (1 + a_+^2) t^{-1} \ell nt \\
|B_+|_{k+j} \vee |\nabla(x \cdot B_{a+})|_k & \leq C a_+^2
\end{align*}
\]

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and that the same estimates hold for the primed quantities.

From Lemma 4.2 with $k' = k - 1$ and from the estimates (5.67) of the + quantities, we obtain (compare with (4.105)-(4.110))

$$|B_{a_-}| \leq C \ a_+ \ I_0(y_-) \ , \quad (5.68)$$

$$\| x \cdot B_{a_-} ; \dot{H}^k \| \leq C \ a_+ \ I_{k-2}(y_-) \ , \quad (5.69)$$

$$|\partial_t y'_-| \leq E_1 \ y'_- + C \ t^{-2} \ a_+ \ \left\{ (1 + a_+^2)z_{1-} + (1 + a_+^2 t \ln t)(a_+ \ I_0(y_-) + n_-) \right\}$$

$$+ C \ t^{-1-\beta} \ a_+^2 \ I_{k-2}(y_-) + C \ t^{-1} \ a_+ \ n_- + t^{-2} \ y'_1 \ , \quad (5.70)$$

$$|\partial_t y'_{1-}| \leq E_1 \ y'_{1-} + C \ t^{-2} \ a_+ \ \left\{ a_+^2 \ z_{1-} + z_{2-} + (1 + a_+^2 t \ln t)(a_+ \ I_0(y_-) + n_-) \right\}$$

$$+ C \ t^{-1} \ a_+^2 \ I_{k-2}(y_-) + C \ t^{-1} \ a_+ \ n_- \ , \quad (5.71)$$

where

$$E_1 = C \ a_+^2 \ \left\{ (1 + a_+^2)t^{-2}(\ln t + t^{-1-\beta}) \right\}$$

$$|\partial_t z'_{j-}| \leq C \ t^{-2} \ a_+^2 \ \left\{ \ln t(z_{j-} + z'_{j-}) + \delta_{j-} t^{(2-\alpha)\beta} \ z_{1-} \right\}$$

$$+ C \ t^{-1} \ a_+ \ (y_- + \delta_{j-} y_{1-}) + C \ t^{-1+j\beta} \ a_+ \ I_m(y_-) \quad \text{for } j = 1, 2 \ , \quad (5.72)$$

where $m = (k - 2) \lor 0$,

$$n'_- \leq C \ t^{-1} \ a_+ \ I_0 \left\{ y_{1-} + a_+^2 \ t \ln t \ y_- + a_+(z_{1-} + a_+ \ I_0(y_-) + n_-) \right\} \ . \quad (5.73)$$

In the same way as in the proof of Proposition 4.3, we continue the argument with a simplified version of the system (5.70)-(5.73) where we exponentiate the diagonal terms and in particular $E_1$ to a constant according to (4.51) and where we eliminate the constants and the factors containing $a_+$, with the consequence that we lose detailed control of the dependence of the lower bounds for $T$ on $a_-$. Thus we rewrite (5.70)-(5.73) as (compare with (4.111)-(4.114))

$$|\partial_t y'_-| \leq t^{-2} \ \left\{ z_{1-} + \ln t(I_0(y_-) + n_-) \right\} + t^{-1-\beta} \ I_{k-2}(y_-) + t^{-1} \ n_- + t^{-2} \ y'_1 \ , \quad (5.74)$$

$$|\partial_t y'_{1-}| \leq t^{-2} \ \left\{ z_{2-} + \ln t(I_0(y_-) + n_-) \right\} + t^{-1} \ I_{k-2}(y_-) + t^{-1} \ n_- \ , \quad (5.75)$$

$$|\partial_t z'_{j-}| \leq t^{-2} \ \left\{ \ln t(z_{j-} + \delta_{j-} t^{(2-\alpha)\beta} \ z_{1-}) + t^{-1} (y_- + \delta_{j-} y_{1-}) + t^{-1+j\beta} \ I_m(y_-) \right\} \ , \quad (5.76)$$

$$n'_- \leq t^{-1} \ I_0 \left\{ y_{1-} + \ln t \ y_- + z_{1-} + I_0(y_-) + n_- \right\} \ . \quad (5.77)$$
We now define

\[
\begin{align*}
Y_\ast &= \| t(\ell n t)^{-1}y_\ast; L^\infty([T, \infty)) \|, \quad Y_\ast = \| t(\ell n t)^{-1}y_\ast; L^\infty([T, \infty)) \|, \\
Z_{j\ast} &= \| t^{1-j\beta}(\ell n t)^{-1}z_{j\ast}; L^\infty([T, \infty)) \|, \quad j = 1, 2, \\
N_\ast &= \| t^{-2-\beta}(\ell n t)^{-1}n_\ast; L^\infty([T, \infty)) \|
\end{align*}
\]

and similarly for the primed quantities. It follows from (5.29)-(5.36) that all those quantities are finite. Using those definitions and omitting the - indices in the remaining part of the contraction proof, we obtain from (5.74)-(5.77)

\[
|\partial_t y'| \leq t^{-2} \ell n t \left\{ Z_1 t^{-1+\beta} + Y t^{-\beta} + N t^{-1+\beta} + Y_1' t^{-1} \right\}, \quad (5.79)
\]

\[
|\partial_t y'_1| \leq t^{-2} \ell n t \left\{ Z_2 t^{-1+2\beta} + Y + N t^{-1+\beta} \right\}, \quad (5.80)
\]

\[
|\partial_t z'| \leq t^{-2} \ell n t \left\{ Z_j t^{-1+j\beta} \ell n t + \delta_j Z_1 t^{-1+(3-\alpha)\beta} + \delta_j Y_1 + Y t^j \right\}, \quad (5.81)
\]

\[
n' \leq t^{-2} \ell n t \left\{ Y_1 + Y \ell n t + Z_1 t^\beta + N t^{-1+\beta} \right\}. \quad (5.82)
\]

Integrating (5.79)-(5.81) between \( t \) and \( \infty \) with initial condition \( (y', y'_1, z'_j)(\infty) = 0 \), and substituting the result and (5.82) into the primed analog of the definition (5.78) (with the - indices omitted), we obtain

\[
\begin{align*}
Y' &\leq T^{-1+\beta} Z_1 + T^{-\beta} Y + T^{-1+\beta} N + T^{-1} Y_1', \\
Y'_1 &\leq T^{-1+2\beta} Z_2 + Y + T^{-1+\beta} N, \\
Z_j' &\leq T^{-1} \ell n t T Z_j + \delta_j T^{-1-(\alpha-1)\beta} Z_1 + \delta_j T^{-2\beta} Y_1 + Y, \\
N' &\leq T^{-\beta} Y_1 + T^{-\beta} \ell n t Y + Z_1 + T^{-1} N.
\end{align*}
\]

Substituting \( Y_1' \) from the second inequality into the first one, we recast the system (5.83) into the form (4.124), from which the contraction property follows exactly as in the proof of Proposition 4.3. This proves that \( \Gamma \) has a unique fixed point in \( \mathcal{R} \).

The uniqueness of the solution in \( \mathcal{C}(I, X^k) \) under the conditions (5.60) (5.61) follow from Proposition 4.2, part (2). Note that the time decay (5.60) (5.61) required for uniqueness is weaker than that contained in (5.29)-(5.32).

**Part (2).** Let \( w_{+i}, i = 1, 2 \), satisfy the assumptions of the proposition with

\[
|w_{+i} x| k+\alpha \vee |w_{+i} x| k+\alpha+1 \leq a_+
\]

(5.84)
and define \((W_i, S_i)\) by (5.10). Let \((w_i, s_i, B_i) = (W_i + q_i, S_i + \sigma_i, B_i)\) be the two solutions of the system (2.34) (2.35) obtained in Part (1). Let \((W_-, S_-) = (1/2)(W_1 - W_2, S_1 - S_2)\), define \((w_-, s_-, B_-)\) as in Lemma 4.2 and define

\[
\begin{aligned}
    y_- &= |w_-|_{k-1} \lor |xw_-|_{k-1}, \quad y_1_- = |w_-|_k \\
    z_j_- &= |s_-|_k{+}j{-}1, \quad j = 1, 2, \quad n_- = |B_-|_k \lor |x \cdot B_-|_k.
\end{aligned}
\] (5.85)

We assume that \(w_{+1} - w_{+2}\) is small in the sense that

\[
|x(w_{+1} - w_{+2})|_{k-1} \lor |w_{+1} - w_{+2}| \leq \eta
\] (5.86)

and we want to show that \((w_-, s_-, B_-)\) is small by estimating \((y_-, y_1-, z_j-, n_-)\) in terms of \(\eta\). We first estimate \((W_-, S_-)\). From (4.140) it follows that

\[
|xW_-|_{k-1} \lor |W_-|_k \leq \eta + t^{-1} a_+ \leq 2\eta
\] (5.87)

for \(t \geq a_+ / \eta\). Furthermore, in the same way as in Lemma 4.2

\[
|S_-|_k{+}j{-}1 \leq C a_+ \eta t^{j\beta} \quad \text{for } j = 1, 2.
\] (5.88)

We now take \(t_0\) large in a sense to be specified below, and we estimate \((y_-, y_1-, z_j-, n_-)\) separately for \(t \geq t_0\) and for \(t \leq t_0\). For \(t \geq t_0\), using (5.87) (5.88) and (5.29)-(5.31), we obtain

\[
\begin{aligned}
    y_- \lor y_1_- &\leq 2\eta + Y t^{-1} \ell n t \leq C \eta, \\
    z_j_- &\leq C a_+ \eta t^{j\beta} + Z_j t^{-1} \ell n t (\ell n t + t^{(j{-}1)\beta}) \leq C a_+ \eta t^{j\beta}
\end{aligned}
\] (5.89)

for \(t_0\) large in the sense that

\[
\eta t_0(\ell n t_0)^{-1} = C(a_+ \geq Y \lor (Z_1 \lor Z_2) a_+^{-1} t_0^{-\beta} \ell n t_0).
\] (5.90)

(Remember that in this proposition, \(Y, Z_1, Z_2\) are functions of \(a_+\)). Using the fact that for \(t \geq t_0\), \(I_m(f)\) depends only on \(f\) restricted to \(t \geq t_0\), we obtain in addition

\[
n_- \leq C t^{-1} (\eta (1 + \ell n t + a_+ t^{\beta}) + n_-)
\]

and therefore

\[
n_- \leq C (1 + a_+) \eta t^{-1 + \beta}
\] (5.91)

for \(t \geq t_0\).
We next estimate \((y_-, y_{1-}, z_{j-}, n_-)\) for \(t \leq t_0\) and for that purpose, we use the system (5.74)-(5.77) with the primes omitted, since \((w_i, s_i, B_{bi})\) are solutions of the system (2.34) (2.35). We choose \(\lambda\) such that \(2\beta < \lambda < 1\) (in the same way as in the proof of Proposition 4.2, part (2)) and we define

\[
\begin{aligned}
Y_-' &= \| t^\lambda y_-; L^\infty([T, t_0]) \|, \quad Y_{1-}' = \| t^\lambda y_{1-}; L^\infty([T, t_0]) \|, \\
Z_{j-}' &= \| t^{\lambda-j\beta} z_{j-}; L^\infty([T, t_0]) \|, \quad j = 1, 2, \\
N_- &= \| t^{\lambda+1-\beta} n_-; L^\infty([T, t_0]) \|.
\end{aligned}
\]  

(5.92)

Substituting those definitions into (5.74)-(5.77), using the fact that

\[
I_m(y_-) \leq I_m \left( t^{-\lambda} Y_- + C\eta \right) \leq C \left( t^{-\lambda} Y_- + \eta \right)
\]  

(5.93)

and similar relations for \(y_{1-}, z_{j-}, n_-\), and omitting the \(-\) indices, we obtain

\[
\begin{aligned}
|\partial_t y| &\leq \eta \ t^{-1-\beta} + t^{-1-\lambda}\{\cdot\} \\
|\partial_t y_1| &\leq \eta \ t^{-1} + t^{-1-\lambda}\{\cdot\} \\
|\partial_t z_j| &\leq \eta \ t^{1+j\beta} + t^{-1-\lambda}\{\cdot\} \\
n_- &\leq \eta \ t^{1+\beta} + t^{-1-\lambda}\{\cdot\}
\end{aligned}
\]  

(5.94)

where the brackets in the RHS are the same as in (5.79)-(5.82). Integrating the first three inequalities of (5.94) between \(t\) and \(t_0\) for \(t \leq t_0\) with initial condition at \(t_0\) estimated by (5.89), and omitting again absolute constants, we obtain

\[
\begin{aligned}
y &\leq \eta + t^{-\lambda}\{\cdot\} \\
y_1 &\leq \eta \ell n t_0 + t^{-\lambda}\{\cdot\} \\
z_j &\leq \eta \ t_0^{j\beta} + t^{-\lambda}\{\cdot\} \\
n_- &\leq \eta \ t^{1+\beta} + t^{-1-\lambda}\{\cdot\}
\end{aligned}
\]  

(5.95)

where the brackets in the RHS are the same as in (5.94). We substitute (5.95) into the definitions (5.92) and obtain a system similar to (5.83), with however the primes omitted, and with an additional term bounded by \(\eta t_0^\lambda \ell n t_0\) in each of the RHS. Proceeding therefrom as in the contraction proofs of Proposition 4.3 and of Part (1) of this proposition, we obtain

\[
X \leq \eta \ t_0^\lambda \ell n t_0
\]  

(5.96)
where $X$ is defined by (4.125), so that by (5.90), $X$ tends to zero when $\eta$ tends to zero (actually as a power of $\eta$). This proves the norm continuity of the map $w_+ \to (w, s, B_b)$ from the norm $|w_+|_k \vee |xw_+|_{k-1}$ (see (5.86)) to the norm in $L^\infty(J, X^{k-1})$ for compact $J$. The last continuity follows from a standard compactness argument.

\[ \square \]

**Remark 5.4.** In part (2) of Proposition 5.3, we prove actually a stronger continuity than stated, namely a suitably weighted $L^\infty$ continuity in the whole interval $[T, \infty)$, as follows from (5.89) (5.91) for $t \geq t_0$ and (5.92) (5.96) for $t \leq t_0$ defined in terms of $\eta$ by (5.90).

### 6 Wave operators and asymptotics for $(u, A)$

In this section we complete the construction of the wave operators for the system (2.6) (2.7) in the special case of vanishing asymptotic magnetic field, and we derive asymptotic properties of solutions in their range. The construction relies in an essential way on Proposition 5.3. So far we have worked with the system (2.34) for $(w, s)$ and the first task is to reconstruct the phase $\phi$. Corresponding to $S$ defined by (2.40), we define

$$
\phi = \int_1^t dt' t'^{-1} (g(W) - (x \cdot B_a)_L(W))
$$

so that $S = \nabla \phi$. Let now $(q, \sigma, B_b)$ be the solution of the system (2.49) (2.50) obtained in Proposition 5.3 and let $(w, s) = (W + q, S + \sigma)$. We define $\psi$ by $\psi(\infty) = 0$ and

$$
\partial_t \psi = (2t^2)^{-1} |s|^2 + t^{-1} \{g(w) - g(W) - (x \cdot B_a)_L(w) + (x \cdot B_a)_L(W)\}
$$

or equivalently

$$
\psi = -\int_t^\infty dt' \left\{ (2t'^2)^{-1} |s(t')|^2 + t'^{-1} g(q, q + 2W) - t'^{-1} (x \cdot B_a)_L(q, q + 2W) \right\}
$$

which is tailored to ensure that $\nabla \psi = \sigma$, given the fact that $S$ and $\sigma$ are gradients. The integral converges in $\dot{H}^1$, as follows from (5.16) (5.18), from (5.27) (5.29) and
from the estimate
\[ \partial_t \| \sigma \|_2 \leq t^{-2} \| s \cdot \nabla s \|_2 + t^{-1} \| \nabla g(q, q + 2W) \|_2 \]
\[ + t^{-1} \| \nabla (x \cdot B_a)(q, q + 2W) \|_2 \leq t^{-2} \| s \|_\infty \| \nabla s \|_2 + C t^{-1} \| q \|_2 + I_{-1} \| xq \|_2 \]
\[ \leq C(a_+) t^{-2} (\ell n t)^2 \]  \hspace{1cm} (6.4)
so that
\[ \| \nabla \psi \|_2 = \| \sigma \|_2 \leq C(a_+) t^{-1} (\ell n t)^2 . \]  \hspace{1cm} (6.5)

Finally we define \( \varphi = \phi + \psi \) so that \( \nabla \varphi = s \).

We can now define the modified wave operators for the MS system in the form (2.6) (2.7) in the special case of vanishing asymptotic magnetic field. We start from the asymptotic state \( u_+ \) for \( u \) and we define \( w_+ = Fu_+ \). The asymptotic state \( (A_+, \dot{A}_+) \) for \( A \) is taken to be zero. We define \( (W, S) \) by (2.40). We solve the system (2.49) (2.50) for \( (q, \sigma, B_b) \) by Proposition 5.3. Through (2.43), this yields a solution \( (w, s, B_b) \) of the auxiliary system (2.34) (2.35). We reconstruct the phase \( \varphi = \phi + \psi \) with \( \phi \) and \( \psi \) defined by (6.1) (6.3). We finally substitute \( (w, \varphi, B_b) \) into (2.17) and (2.18) with \( B = B_a + B_b \) and \( B_a \) defined by (2.28). This yields a solution \( (u, A) \) of the system (2.6) (2.7) defined for large time. The modified wave operator is the map \( \Omega : u_+ \rightarrow (u, A) \) thereby obtained.

In order to state the regularity properties of \( u \) that follow in a natural way from the previous construction, we introduce appropriate function spaces. In addition to the operators \( M = M(t) \) and \( D = D(t) \) defined by (2.14) (2.15), we introduce the operator
\[ J = J(t) = x + it \nabla , \]  \hspace{1cm} (6.6)
the generator of Galilei transformations. The operators \( M, D, J \) satisfy the commutation relation
\[ i M D \nabla = J M D . \]  \hspace{1cm} (6.7)
For any interval \( I \subset [1, \infty) \) and any \( k \geq 0 \), we define the space
\[ X^k(I) = \left\{ u : D^*_M u \in C(I, H^{k+1}), D^*_M x u \in C(I, H^k) \right\} \]
\[ = \left\{ u : J(t) >^{k+1} u \text{ and } < J(t) >^k x u \in C(I, L^2) \right\} \]  \hspace{1cm} (6.8)
where \(<\lambda> = (1 + \lambda^2)^{1/2}\) for any real number or self-adjoint operator \(\lambda\) and where the second equality follows from (6.7).

We now collect the information obtained for the solutions of the system (2.6) (2.7) in the range of the modified wave operators and state the main result of this paper as follows.

**Proposition 6.1.** Let \(k > 3/2\), \(0 < \beta < 1/2\) and let \(\alpha > 1\) be such that \(\beta(\alpha+1) \geq 1\). Let \(u_+\) be such that \(w_+ = Fu_+ \in H^{k+1}\) and \(xw_+ \in H^{k+1}\). Define \((W, S)\) by (2.4) and \(a_+\) by (5.11). Then

1. There exists \(T = T(a_+), \ 1 \leq T < \infty\), such that the auxiliary system (2.34) (2.35) has a unique solution \((w, s, B_b) \in C(I, X^k)\) where \(I = [T, \infty)\), satisfying

   \[
   |w - W|_k \vee |x(w - W)|_k \leq C t^{-1} \ell n t \tag{6.9}
   \]

   \[
   |w - W|_{k+1} \leq C \left( t^{-1} \ell n t + t^{-\alpha\beta} \right) \tag{6.10}
   \]

   \[
   |s - S|_{k+j} \leq C t^{-1} \ell n t \left( \ell n t + t^{j\beta} \right) \quad \text{for} \ j = 0, 1, 2. \tag{6.11}
   \]

   \[
   |B_b|_{k+1} \vee |x \cdot B_b|_{k+1} \leq C t^{-1} \ell n t. \tag{6.12}
   \]

2. Let \(\phi\) and \(\psi\) be defined by (6.1) and (6.3) with \(q = w - W\), and let \(\varphi = \phi + \psi\). Let

   \[
   u = MD \exp(-i\varphi)w, \tag{2.17} \equiv (6.13)
   \]

   \[
   A = t^{-1} D_0 B \tag{2.18} \equiv (6.14)
   \]

with \(B = B_a + B_b\) and \(B_a\) defined by (2.28). Then \(u \in X^k(I), (A, \partial_t A) \in C(I, K^{k+1} \oplus H^k)\), \((u, A)\) solves the system (2.6) (2.7) and \(u\) behaves asymptotically in time as \(MD \exp(-i\varphi)w_+\) in the sense that \(u\) satisfies the following estimates.

\[
\| J(t) \| < |x|/t > (\exp(i\phi(t, x/t))u(t) - M(t) D(t) W(t)) \|_2 \leq C t^{-1}(\ell n t)^2, \tag{6.15}
\]

\[
\| J(t) \|^{k+1} (\exp(i\phi(t, x/t))u(t) - M(t) D(t) W(t)) \|_2 \leq C \left( t^{-1}(\ell n t)^2 + t^{-\alpha\beta} \right) \tag{6.16}
\]

\[
\| |x|/t > (u(t) - M(t) D(t) \exp(-i\phi(t) W(t)) \|_r \leq C t^{-1-\delta(r)}(\ell n t)^2 \tag{6.17}
\]

for \(2 \leq r \leq \infty\), with \(\delta(r) = 3/2 - 3/r\).

Furthermore \(A\) behaves asymptotically in time as \(t^{-1} D_0 B_a(W)\) in the sense that the following estimates hold

\[
|B - B_a(W)|_{k+1} \vee |\nabla x \cdot (B - B_a(W))|_k \leq C t^{-1} \ell n t \tag{6.18}
\]
where $A$ and $B$ are related by (6.14).

**Proof.** The proof follows from Proposition 5.3 supplemented with the reconstruction of $\varphi$ described above in this section, except for the estimates (6.15)-(6.17) on $u$. In particular the estimates (6.9)-(6.12) are the estimates (5.29)-(5.32) supplemented with (6.5), while (6.18) follows from (5.32) and (5.40).

We next prove the estimates (6.15)-(6.17) on $u$. From (6.13) with $\varphi = \phi + \psi$ and $\varphi = \phi + \psi$, it follows that

\[
\| | J |^m (\exp(iD_0\phi)u - MDW) \|_2 = \| \omega^m (\exp(-i\psi)w - W) \|_2 , \quad (6.19)
\]

\[
\| | J |^m ([x]/t)(\exp(iD_0\phi)u - MDW) \|_2 = \| \omega^m |x| (\exp(-i\psi)w - W) \|_2 . \quad (6.20)
\]

We next estimate for $0 \leq m \leq k + 1$

\[
\| \omega^m (\exp(-i\psi)w - W) \|_2 \leq C \| \omega^m (\exp(-i\psi) - 1) \|_{r_1} \| w \|_{r_2} 
+ C \| \exp(-i\psi) - 1 \|_{\infty} \| \omega^m w \|_2 + C \| \omega^m (w - W) \|_2 
\leq C \exp(C \| \psi \|_{\infty}) \| \omega^m \psi \|_{r_1} \| w \|_{r_2} + C (\| \psi \|_{\infty} |w|_m + |w - W|_m) \quad (6.21)
\]

with $1/r_1 + 1/r_2 = 1/2$, $r_1 < \infty$, by Lemmas 3.2 and 3.3, and similarly for $0 \leq m \leq k$

\[
\| \omega^m |x| (\exp(-i\psi)w - W) \|_2 \leq C \exp(C \| \psi \|_{\infty}) \| \omega^m \psi \|_{r_1} \| xw \|_{r_2} 
+ C (\| \psi \|_{\infty} |xw|_m + |x(w - W)|_m) . \quad (6.22)
\]

Taking $r_1 = 6$, $r_2 = 3$ for $m = 0$ and $r_1 = 2$, $r_2 = \infty$ for $m \geq 1$, using the Sobolev inequality

\[
\| \psi \|_{\infty} \leq C (\| \sigma \|_{L^2} \| \nabla \sigma \|_2)^{1/2}
\]

and using (6.9)-(6.11) yields (6.15)-(6.16).

The estimate (6.17) follows immediately from (6.15) and from the inequality

\[
\| f \|_r = t^{-\delta(r)} \| \mathbf{D}^* \mathbf{M}^* f \|_r \leq C t^{-\delta(r)} \| \omega^{\delta(r)} \mathbf{D}^* \mathbf{M}^* f \|_2 
= C t^{-\delta(r)} \| | J(t) |^{\delta(r)} f \|_2
\]

for $2 \leq r < \infty$ and from a similar inequality for $r = \infty$. \[ \square \]
Remark 6.1. The leading term in the asymptotic behaviour of $A$ is $t^{-1}D_0B_a(W)$. Replacing $W$ by $w_+$ as a first approximation, one obtains

$$A \sim t^{-1} D_0 B_a(w_+)$$

and since $B_a(w_+)$ is constant in time, that term spreads by dilation by $t$ and decays as $t^{-1}$ in $L^\infty$ norm. In the norms considered in (6.18), that term is $O(1)$, so that the remainder is smaller than the leading term by $t^{-1}\ell n t$. We have stated the remainder estimates in terms of $B$ rather than $A$ because they are simpler for $B$, since for $A$ the dilation $D_0$ induces a dependence of the time decay on the order of derivation. In fact (6.18) is equivalent to

$$\| \omega^m (A - t^{-1}D_0B_a(W)) \|_2 \lor \| \omega^m \nabla x \cdot (A - t^{-1}D_0B_a(W)) \|_2$$

$$\leq C t^{-m-1/2} \ell n t$$

(6.23)

for the relevant values of $m$, namely $1 \leq m \leq k+1$ for the first norm and $1 \leq m \leq k$ for the second one.

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