Patterns of i.i.d. Sequences and Their Entropy - Part II: Bounds for Some Distributions

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Abstract

A pattern of a sequence is a sequence of integer indices with each index describing the order of first occurrence of the respective symbol in the original sequence. In a recent paper, tight general bounds on the block entropy of patterns of sequences generated by independent and identically distributed (i.i.d.) sources were derived. In this paper, precise approximations are provided for the pattern block entropies for patterns of sequences generated by i.i.d. uniform and monotonic distributions, including distributions over the integers, and the geometric distribution. Numerical bounds on the pattern block entropies of these distributions are provided even for very short blocks. Tight bounds are obtained even for distributions that have infinite i.i.d. entropy rates. The approximations are obtained using general bounds and their derivation techniques. Conditional index entropy is also studied for distributions over smaller alphabets.

Index Terms: patterns, monotonic distributions, uniform distributions, entropy.

1 Introduction

Recent work (see, e.g., [1], [2], [6], [10], [13], [14]) has considered universal compression for patterns of independent and identically distributed (i.i.d.) sequences. The pattern of a sequence \( x^n \triangleq (x_1, x_2, \ldots, x_n) \) is a sequence \( \psi^n \triangleq \psi \triangleq \Psi (x^n) \) of pointers that point to the actual alphabet letters, where the alphabet letters are assigned indices in order of first occurrence. For example, the pattern of all sequences \( x^n = \text{lossless}, x^n = \text{sellsoll}, x^n = 12331433, \) and \( x^n = 76887288, \) which is alphabet independent, is \( \psi^n = \Psi (x^n) = 12331433. \) Capital \( \Psi(\cdot) \) denotes the pattern operator.

Patterns are interesting in universal compression with unknown alphabets, where the dictionary and the pattern of \( x^n \) can be compressed separately (see, e.g., [13]). Pattern entropy is also important in learning applications. Consider all the new species an explorer observes. The explorer can identify these species with the first time each was seen, and assign indices to species in order of first occurrence. The entropy of patterns can thus model uncertainty of such processes.

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Initial work on patterns \([5, 6, 10, 13, 14]\), focused on showing diminishing universal compression redundancy rates. The first results on pattern entropy in \([10, 12, 14]\); however, showed that for sufficiently large alphabets, the pattern block entropy must decrease from the i.i.d. one even more significantly than the universal coding penalty for coding patterns. Since \(\Psi(x^n)\) is the result of data processing, its entropy must be no greater than \(H_\theta(X^n)\). For alphabet size \(k\),

\[
nH_\theta(X) - \log [k!/(\max \{0,k-n\})!] \leq H_\theta(\Psi^n) \leq nH_\theta(X),
\]

where capital letters denote random variables, and \(\theta\) is the parameter vector governing the source. The bounds in (1) already show that for \(k = o(n)\) the pattern entropy rate equals the i.i.d. one \(^2\) for non-diminishing \(H_\theta(X)\). Subsequently to the results in \([12]\), it was independently shown in \([4, 7]\) that for discrete i.i.d. sources, the pattern entropy rate is equal to that of the underlying i.i.d. process.

In contrast with the block entropy, for smaller alphabets, the conditional next index entropy \(H_\theta(\Psi_\ell | \Psi^{\ell-1})\) is guaranteed to start increasing from \(H_\theta(X)\) after some time \(\ell > 1\). The gain (decrease) in block entropy is thus due to first occurrences of new symbols, and gains only occur before first occurrences become sparse. This observation, pointed out also in \([4, 7]\), gives rise to a possibility of diminishing \(o(1)\) overall per block decreases of \(H_\theta(\Psi^n)\) from \(nH_\theta(X)\).

In \([11]\), general tight upper and matching lower bounds on the block entropy were derived. This paper continues the work in \([11]\), and uses the bounds derived in \([11]\) and their derivation methods to provide very accurate approximations of the pattern block entropies for uniform and several monotonic i.i.d. distributions. The complete range of uniform distributions, from over fixed small alphabets, to over infinite alphabets, is studied. Monotonic distributions from slowly to fast decaying ones are considered. It is shown that the pattern entropy can be approximated even for slowly decaying monotonic distributions with infinite i.i.d. entropy rates. Then, small alphabets and their conditional next index entropies are studied.

The derivation methods are based on those in \([11]\). The probability space is partitioned into a grid of points. Between each two points, there is a bin. Symbols whose probabilities lie in the same bin can be exchanged in \(x^n\) to provide another sequence \(x'^n\) with \(\Psi(x'^n) = \Psi(x^n)\) and almost equal probability. Counting such sequences, packing low probability symbols into single point masses, leads to bounds on \(H_\theta(\Psi^n)\). Proper choices of grids are key for tightening bounds.

Section \(2\) gives some preliminaries. General bounds (somewhat modified) from \([11]\) are reviewed in Section \(3\). Next, Section \(4\) summarizes pattern entropies for different distributions. Finally, Sections \(5\) and \(6\) contain the proofs for uniform distributions and monotonic distributions, respectively.

### 2 Preliminaries

Let \(x^n\) be an \(n\)-tuple with components \(x_i \in \Sigma = \{1, 2, \ldots, k\}\) (where the alphabet is defined without loss of generality). The asymptotic regime is that \(n \to \infty\). However, the general bounds are stated also for finite \(n\). The alphabet size \(k\) may be greater than \(n\) or infinite. The vector \(1\)Logarithms are taken to base 2, here and elsewhere. The natural logarithm is denoted by \(\ln\).

\(^2\)For two functions \(f(n)\) and \(g(n)\), \(f(n) = o(g(n))\) if \(\forall c, \exists n_0\), such that, \(\forall n > n_0\), \(|f(n)| < c \cdot |g(n)|\); \(f(n) = O(g(n))\) if \(\exists c, n_0\), such that, \(\forall n > n_0\), \(0 \leq |f(n)| \leq c \cdot |g(n)|\); with inequalities it will be assumed that \(f(n) \geq 0\), but with equalities, negative \(f(n)\) are possible; \(f(n) = \Theta(g(n))\) if \(\exists c_1, c_2, n_0\), such that, \(\forall n > n_0\), \(c_1 g(n) \leq f(n) \leq c_2 g(n)\).
\( \theta \triangleq (\theta_1, \theta_2, \ldots, \theta_k) \) is the set of probabilities of all letters in \( \Sigma \). Assume, without loss of generality, that \( \theta_1 \leq \theta_2 \leq \cdots \leq \theta_k \). Boldface letters denote vectors, and capital letters will denote random variables. The probability of \( \psi^n \) induced by an i.i.d. source is

\[
P_\theta(\psi^n) = \sum_{y^n : \Psi(y^n) = \psi^n} P_\theta(y^n).
\]

(2)

The pattern sequence or block entropy of order \( n \) is

\[
H_\theta(\Psi^n) \triangleq - \sum_{\psi^n} P_\theta(\psi^n) \log P_\theta(\psi^n).
\]

(3)

Following [11] (but more generally), consider two different grids: \( \eta \) and \( \xi \). For simplicity of notation, we omit the dependence on \( n \) from definitions of grid points. Let \( \varepsilon_0, \varepsilon_1, \) and \( \varepsilon_2 \) be three numbers that satisfy \( \varepsilon_0 \geq \max(0, \varepsilon_1) \) and \( \varepsilon_2 \geq \max(0, \varepsilon_1) \). Define

\[
\eta'_b \triangleq \sum_{j=1}^{b} \frac{2(j - \frac{1}{2})}{n^{1+\varepsilon_2}} = \frac{b^2}{n^{1+\varepsilon_2}}.
\]

(4)

The grid \( \eta \triangleq (\eta_0, \eta_1, \ldots, \eta_{B_\eta}) \) is defined by \( \eta_0 = 0, \eta_1 = \frac{1}{n^{1+\varepsilon_2}}, \eta_2 = \frac{1}{n^{1+\varepsilon_1}}, b' \triangleq b + \lfloor n^{(\varepsilon_2-\varepsilon_1)/2} \rfloor - 2, \) and

\[
\eta_b = \eta_{b'} - \lfloor n^{(\varepsilon_2-\varepsilon_1)/2} \rfloor - 2 = \eta_{b'}, \quad b = 3, 4, \ldots, B_\eta.
\]

(5)

For some \( \varepsilon > 0 \), if \( \varepsilon_0 = \varepsilon, \varepsilon_1 = -\varepsilon \), and \( \varepsilon_2 = 2\varepsilon \), \( \eta \) reduces to the one defined in [11]. The more general definition here allows achieving tighter bounds for some specific distributions also for finite \( n \). The relation \( \varepsilon_1 = -\varepsilon \) will be assumed by definition, but the other two parameters will not be tied to \( \varepsilon \) (except by \( \varepsilon_b \geq -\varepsilon \)). The grid \( \xi \triangleq (\xi_0, \xi_1, \ldots, \xi_{B_\xi}) \) is defined by \( \xi_0 = 0 \), and for an arbitrarily small \( \varepsilon = -\varepsilon_1 > 0 \),

\[
\xi_b \triangleq \sum_{j=1}^{b} \frac{2(j - 0.5)}{n^{1-\varepsilon}} = \frac{b^2}{n^{1-\varepsilon}}, \quad b = 1, 2, \ldots, B_\xi.
\]

(6)

For both grids, \( \eta_{B_\eta + 1} = \xi_{B_\xi + 1} \triangleq 1 \), and thus \( B_\eta = \lfloor \sqrt{n^{1+\varepsilon_2}} \rfloor \) and \( B_\xi = \lfloor \sqrt{n^{1-\varepsilon}} \rfloor \). We also define the maximal indices \( A_\eta \) and \( A_\xi \) whose grid points do not exceed 0.5 for \( \eta \) and \( \xi \), respectively. Hence, \( A_\eta = \lfloor \sqrt{n^{1+\varepsilon_2}} / \sqrt{2} \rfloor \) and \( A_\xi = \lfloor \sqrt{n^{1-\varepsilon}} / \sqrt{2} \rfloor \).

Let \( \kappa_b, b = 0, 1, \ldots, B_\eta \); and \( \kappa_b, b = 0, 1, \ldots, B_\xi \) denote the numbers of symbols \( \theta_i \in (\eta_b, \eta_{b+1}] \) and \( \theta_i \in (\xi_b, \xi_{b+1}] \), respectively (in bin \( b \) of \( \eta \) and \( \xi \), respectively). Specifically, for given \( \varepsilon_0, \varepsilon \), and \( \varepsilon_2 \), \( k_0 \) and \( k_1 \) denote the cardinalities of \( \theta_i \leq 1/n^{1+\varepsilon_0} \), and \( \theta_i \in (1/n^{1+\varepsilon_0}, 1/n^{1-\varepsilon}) \), respectively. Define also \( k_0 \triangleq k_0 + k_1 \), thus \( k - k_0 \) is the cardinality of \( \theta_i > 1/n^{1-\varepsilon} \). Also, let \( k'_b, b = 1, 2, \ldots, B_\xi \) be zero if \( \kappa_b \) is zero, and otherwise, the number of symbols for which \( \theta_i \in (\xi_{b-1}, \xi_{b+2}] \), with the exception of \( k'_1 \), which will only count letters for which \( \theta_i \in (\xi_1, \xi_3] \). (There is clearly an overlap between adjacent counters \( k'_b \), which is needed for one of the lower bounds.) Now, let

\[
\varphi_b \triangleq \sum_{\theta_i \in (\eta_b, \eta_{b+1}]} \theta_i.
\]
be the total probability of bin $b$ of $\eta$. Specifically, $\varphi_0$, $\varphi_1$, and $\varphi_{01} \triangleq \varphi_0 + \varphi_1$ are defined with respect to (w.r.t.) bins 0, 1, and 01, respectively.

The mean occurrence count of letter $i$ in $X^n$ is given by $E_{\theta} N_x(i) = n\theta_i$, where $n_x(i)$ is the occurrence count of $i$ in $x^n$, $N_i(i)$ is its random variable, and $E_{\theta}$ is expectation given $\theta$. Let $K_b$ be a random variable counting the distinct symbols from bin $b$ of $\eta$ that occur in $X^n$. Let $K$ be the total distinct letters occurring in $X^n$. Then, let

$$L_b \triangleq E_{\theta} [K_b] = \sum_{i : \theta_i \in (\eta_b, \eta_{b+1}]} P_{\theta} (i \in X^n) = \sum_{\theta_i \in (\eta_b, \eta_{b+1}]} [1 - (1 - \theta_i)^n]$$

and also define $L \triangleq E_{\theta} [K]$ similarly. Substituting $(1 - \theta_i)^n = \exp \{n \ln(1 - \theta_i)\}$ and using Taylor series expansion,

$$k_b - \sum_{\theta_i \in (\eta_b, \eta_{b+1}[} e^{-n\theta_i} \leq L_b \leq k_b - \sum_{\theta_i \in (\eta_b, \eta_{b+1}[)} e^{-n(\theta_i + \theta_i^2)}.$$ (9)

Specifically, using Binomial expansion for bin $b = 0$,

$$n \varphi_0 - \left(\frac{n}{2}\right) \sum_{i=1}^{k_0} \theta_i^2 \leq L_0 \leq n \varphi_0 - \left(\frac{n}{2}\right) \sum_{i=1}^{k_0} \theta_i^2 + \left(\frac{n}{3}\right) \sum_{i=1}^{k_0} \theta_i^3.$$ (10)

Similar bounds can be obtained for bin $b = 1$ if $\varepsilon_1 \geq 0$ ($\varepsilon \leq 0$).

### 3 General Bounds

General bounds based on [11] are summarized here. First, for given $\varepsilon_0$, $\varepsilon$, and $\varepsilon_2$, that determine $\eta$ and $\xi$, define

$$H^{(01)}_{\theta} (X) \triangleq -\varphi_{01} \log \varphi_{01} - \sum_{i=k_{01}+1}^{k} \theta_i \log \theta_i,$$ (11)

$$H^{(0,1)}_{\theta} (X) \triangleq -\sum_{b=0}^{1} \varphi_b \log \varphi_b - \sum_{i=k_{01}+1}^{k} \theta_i \log \theta_i.$$ (12)

The i.i.d. entropies above pack low probabilities into one or two point masses. The following lower bounds were derived in [11].

**Theorem 1** Let $\varepsilon > 0, \varepsilon_0 \geq 0$, and define $\xi$ with (4). Define $Z^n \triangleq (Z_1, Z_2, \ldots, Z_n)$ by $Z_j = 0$ if $\theta_{X_j} \leq 1/n^{1-\varepsilon}$, and 1 otherwise. Let $k^-_{\varphi}$ be the count of letters $i$ such that $\theta_i \in (\varphi^-/n^{1-\varepsilon}, 1/n^{1-\varepsilon}]$ and $k^+_{\varphi}$ the count of letters $i$ with $\theta_i \in (1/n^{1-\varepsilon}, \varphi^+/n^{1-\varepsilon}]$, where $\varphi^-$, $\varphi^+$ are constants that satisfy $\varphi^+ > 1 > \varphi^- > 0$. Then,

$$H_{\theta} (\Psi^n) \geq nH^{(01)}_{\theta} (X) - S_1 + S_2 + S_3 - S_4$$ (13)
where

\[
S_1 \leq \log(k - k_{01})!
\]

\[
S_1 \leq (1 - \varepsilon_n) \left\{ \sum_{b=1}^{A_k} \log (\kappa_b!) + (k - k_{01}) \log 3 \right\} + \varepsilon_n \log(k - k_{01})! + h_2 [\min (\varepsilon_n, 0.5)]
\]

\[
S_1 \leq (1 - \varepsilon_n) \sum_{b=1}^{A_k} \log (\kappa_b!) + \varepsilon_n \log(k - k_{01})! + h_2 [\min (\varepsilon_n, 0.5)]
\]

\[
\varepsilon_n \triangleq \min \left\{ n \cdot (k - k_{01}) \cdot e^{-0.1n^2}, 1 \right\}
\]

where \( h_2(\alpha) \triangleq -\alpha \log \alpha - (1 - \alpha) \log(1 - \alpha) \),

\[
S_2 = \sum_{i=1}^{k_{01}} E_\theta [N_\epsilon(i) - P_\theta(i \in X^n)] \log \frac{\varphi_01}{\theta_i}
\]

\[
S_2 \geq \sum_{i=1}^{k_{01}} \left[ n\theta_i - 1 + e^{-n(\theta_i + \theta^2)} \right] \log \frac{\varphi_01}{\theta_i}
\]

\[
S_2 \geq \left( 1 - \frac{1}{3n^{\varepsilon_0}} - \frac{2}{n} \right) \sum_{i=1}^{k_{01}} \theta_i^2 \log \frac{\varphi_01}{\theta_i} + \sum_{i=k_{01}+1}^{k_{01}} \left[ n\theta_i - 1 + e^{-n(\theta_i + \theta^2)} \right] \log \frac{\varphi_01}{\theta_i}
\]

\[
S_3 \geq (\log e) \sum_{i=1}^{L_{01} - 1} (L_{01} - i) \frac{\theta_i}{\varphi_01}
\]

and

\[
S_4 \leq \min \left\{ nh_2(\varphi_01), (1 - \varepsilon'_n) \log \left( \frac{k_{01}^\pm + k_{01}^\pm}{k_{01}^\pm} \right) + \varepsilon'_n n + h_2 [\min (\varepsilon'_n, 0.5)] \right\}
\]

where

\[
\varepsilon'_n \triangleq n \cdot k_{01, > n^{-3}} \cdot e^{-f(\vartheta^-, \vartheta^+)n^c + \frac{\ln \vartheta^-}{2(\vartheta^- - 1)n^{1+c}}}
\]

\[
k_{01, > n^{-3}} \text{ denotes the total symbol count with } \theta_i > 1/n^3 \text{, and }
\]

\[
f(\vartheta^-, \vartheta^+) \triangleq \min \left\{ \frac{\vartheta^\pm - 1}{\ln \vartheta^\pm} \ln \frac{\vartheta^\pm - 1}{e \cdot \ln \vartheta^\pm} + 1 \right\}
\]

where the minimum is taken between the values of the expression for \( \vartheta^- \) and for \( \vartheta^+ \).

Fix \( \delta > 0 \), let \( n \to \infty \), and \( \varepsilon \geq (1 + \delta)(\ln \ln n)/\ln n \). Then, \( \varepsilon_n = o(1), \varepsilon'_n = o(1) \), and all terms but the leading ones in (15), (16) and in the second argument of the minimum in (22) are \( o(1) \).

Second order terms are described in Theorem 1 more explicitly than in [11] and some terms are tightened (in second order) to allow use of the theorem for practical \( n \) in Section 6 (derivations of the explicit terms do follow [11]). This is specifically for cases where very slow rates are obtained for the gaps between the i.i.d. and pattern entropies, such as the geometric distribution. Term \( S_1 \) is the decrease in \( H_\theta(\Psi^n) \) due to first occurrences of symbols with \( \theta_i > 1/n^{1-\varepsilon} \), which results from indistinguishability among indices of letters in the same bin \( b \geq 1 \) of \( \xi \). Term \( S_2 \) is the cost of re-occurrences of letters with “small” probabilities. Term \( S_3 \) is the penalty in first occurrences of
“small” probability symbols beyond a single point mass. The bound in (21) is under a worst case assumption. Term \( S_4 \) is a correction from separation between “small” and “large” probabilities. Specifically, for \( \vartheta^− = e^{-5.5} \approx 0.004 \) and \( \vartheta^+ = e^{1.4} \approx 4.06 \), \( f(\vartheta^−, \vartheta^+) > 0.5 \), and the last term of (23) is upper bounded by \( 2.77/n^{1+\varepsilon} \).

The following upper bounds generalize the derivations in [11]:

**Theorem 2** Let \( \varepsilon_0, \varepsilon_1, \varepsilon_2, \eta, b, \) and \( b' \) be as in (4)-(5), and let \( \varepsilon = -\varepsilon_1 \). Then,

\[
H_\theta (\Psi^n) \leq nH_\theta^{(0,1)} (X) - U + R'_1 + R'_0 \tag{25}
\]

\[
H_\theta (\Psi^n) \leq nH_\theta^{(01)} (X) - U + R'_{01} \tag{26}
\]

where \( U \geq 0 \), and also

\[
U \geq \sum_{b=2}^{A_n} \max \left\{ 0, \, L_b \log \frac{L_b}{e}, \, (1 - \min \{ 1, k_b e^{-n\eta_b} \}) \log (k_b!) \right\} - \frac{(2 + 1/b)^2}{n^{\varepsilon_2}} \sum_{b \geq 2, k_b > 1} k_b \tag{27}
\]

\[
R'_0 \leq (n \varphi_b - L_b) \log [\min \{ k_b, n \}] + n \varphi_b \cdot h_2 \left( \frac{L_b}{n \varphi_b} \right), \quad b = 0, 1, 01 \tag{28}
\]

where (28) decreases with \( L_b \) for \( b = 0 \) and also for \( b = 1, 01 \) if either \( \varepsilon \leq 0 \) or \( k_1, k_{01} \geq (1 + \delta) n^\delta \) for some \( \delta > 0 \), respectively. Also,

\[
R'_b \leq \left( \frac{n^2}{2} \sum_{i: \theta_i \in (\eta_b, \eta_b + 1]} \theta_i^2 \right) \log \frac{2e \cdot \varphi_b \cdot \min \{ k_b, n \}}{n \sum_{j: \theta_j \in (\eta_b, \eta_b + 1]} \theta_j^2} \tag{29}
\]

for \( b = 0 \), and also for \( b = 1 \), and \( b = 01 \) if \( \varepsilon \leq 0 \), where \( \eta_{01} \triangleq \eta_0 = 0 \), and \( \eta_{01+1} \triangleq \eta_2 \).

Fix \( \delta > 0 \), let \( n \to \infty \), and \( \varepsilon, \varepsilon_2 \geq (1 + \delta) (\ln \ln n)/\ln n \). Then, (27) is

\[
U \geq (1 - o(1)) \sum_{b=2}^{A_n} \log (k_b!). \tag{30}
\]

The bounds of (25)-(26) consist of 1) an i.i.d. entropy which packs low probabilities into one or two point masses, 2) a correction term \( U \), expressing the gain in first occurrences of symbols with \( \theta_i > 1/n^{1-\varepsilon} \), 3) losses in packing low probabilities into single point masses (\( R'_0 \) terms). Theorem 2 compacts the representation of several bounds in [11] by allowing negative \( \varepsilon \). This also generalizes the upper bounds in [11] because two separate bins with probabilities asymptotically smaller than \( 1/n \) can be created. This is useful in obtaining tighter bounds for fast decaying distributions, such as geometric distributions (see Section 4). The proof of the generalization is identical to the proof in [11]. Probability is sequentially assigned to the joint index-bin sequence \( (\psi^n, \beta^n) \). Repetitions are assigned the mean bin probability, and first occurrences of an index in a bin are assigned the remaining bin probability. In bins 0 and 1 (or bin 01), repetitions are assigned smaller probabilities (which are optimized), and first occurrences thus greater remaining bin probability. The average description length of this code bounds the pattern block entropy (see [11] for details). The bound in (27) uses the better decrease in the pattern entropy that can be obtained in each large probability bin. The second (second order) term is the quantization cost in all bins. The coefficient is tightened from [11] based on (9) in [11] to allow tighter bounds for finite \( n \).
4 Bounds for Some Distributions

4.1 Uniform Distributions

The pattern entropy is bounded below for the complete range of uniform distributions. Applications, as compression with words as the single alphabet unit, can have alphabets of $k = O(n)$ or larger. Pattern entropy for uniform distributions with $k = O(n)$ or larger is also interesting in applications of population estimation from limited observations (see, e.g., [3]). For uniform distributions, all symbol probabilities are in the same bin, (also, unlike other cases, $H_\theta (\Psi_t \mid \Psi^{t-1}) = H_\theta (\Psi_t \mid X^{\ell-1})$). This guarantees a maximal decrease of the pattern entropy from the i.i.d. one for alphabets of $k = o(n)$. For alphabets of $k = O(n)$, the analysis in [11] can be simplified to derive tighter bounds due to the simplicity of the uniform distribution. First, however, the bounds derived from the general bounds in Section 3 are given in the following corollary:

Corollary 1 Let $\theta_i = \theta \geq 1/n^{1-\varepsilon}$, for $i = 1, 2, \ldots, 1/\theta = k$. Then,
\[
n H_\theta (X) - \log (k!) \leq H_\theta (\Psi^n) \leq n H_\theta (X) - \left(1 - e^{-n/k}\right) \log (k!). \tag{31}
\]

Let $\theta_i = \lambda/n$, for $i = 1, 2, \ldots, n/\lambda = k$, and a fixed $\lambda > 0$. Then,
\[
H_\theta (\Psi^n) \leq \left(1 - \frac{1 - e^{-\lambda}}{\lambda}\right) n \log n + \frac{\log e}{2} \cdot \frac{(1 - e^{-\lambda})^2}{\lambda} \cdot n - O (\log n) \leq \left(1 - \frac{1 - e^{-\lambda}}{\lambda}\right) n \log \left[\min \left\{n, \frac{n}{\lambda}\right\}\right] + n \cdot h_2 \left(1 - \frac{e^{-\lambda}}{\lambda}\right). \tag{32}
\]

Let $\theta_i = 1/n^{\mu+\varepsilon}$, for $i = 1, 2, \ldots, n^{\mu+\varepsilon} = k$, and $\mu \geq 1$. Then,
\[
\left(1 - O \left(\frac{1}{n^{\mu-1+\varepsilon}} + \frac{1}{n}\right)\right) \frac{n^{2-\mu-\varepsilon}}{2} \log (en^{\mu+\varepsilon}) \leq H_\theta (\Psi^n) \leq \frac{n^{2-\mu-\varepsilon}}{2} \log (2 en^{\mu+\varepsilon}). \tag{33}
\]

Corollary 1 shows the decrease of the block entropy for a uniform i.i.d. distribution from the original process to its pattern. While the i.i.d. entropy is always $n \log k$, the pattern entropy behaves differently in three regions. For small $k = o(n)$, the decrease in the block entropy is only in the second order essentially by $\log (k/e)$ bits per probability parameter. In the other extreme $k \gg n$, the block entropy decreases in its first order rate by a factor of $2n^{\mu-1+\varepsilon}$ from the i.i.d. one. If $\mu \geq 2$, while both the i.i.d. entropy rate and block entropy diverge, the pattern entropy for the whole block diminishes. This is expected since for such distributions the only pattern one expects to observe is $\psi^n = 123 \ldots n$. In the middle range ($k = O(n)$), the decrease is in the first order coefficient. Specifically, for $\lambda = 1$, the bounds in (32) reduce to
\[
\left(1 - O \left(\frac{1}{n^{\mu-1+\varepsilon}} + \frac{1}{n}\right)\right) \frac{n^{2-\mu-\varepsilon}}{2} \log (en^{\mu+\varepsilon}) \leq H_\theta (\Psi^n) \leq \frac{n^{2-\mu-\varepsilon}}{2} \log (2 en^{\mu+\varepsilon}). \tag{34}
\]

which yield
\[
\frac{n}{e} \log n + \frac{\log e}{2} \left(1 - \frac{1}{e}\right)^2 \cdot n - O (\log n) \leq H_\theta (\Psi^n) \leq \frac{n}{e} \log n + n \cdot h_2 \left(\frac{1}{e}\right),
\]

Thus, the first order gain (decrease) from the i.i.d. entropy is $(1 - \frac{1}{e}) n \log n$ bits. The decrease is because not all letters occur in a sequence. The gain thus results from higher probabilities of
occurrence of new indices. The gaps between the lower and upper bounds in (32) and in (34)-(35) affect only second order terms. However, tighter bounds for the middle range of uniform distributions are possible. Due to the simplicity of the uniform distribution, some looser bounding steps that are necessary to produce general bounds can be avoided. Theorem 3 provides tighter bounds for the $\lambda/n$ uniform distribution.

**Theorem 3** Let $\theta_i = \lambda/n$, for $i = 1, 2, \ldots, n/\lambda = k$, and a fixed $\lambda > 0$. Then,

$$H_\theta(\Psi^n) \leq \left(1 - \frac{1 - e^{-\lambda}}{\lambda}\right) n \log \frac{n}{\lambda} + \frac{(e^\lambda - \lambda - 1) \log e}{\lambda e^\lambda} \cdot n - O(\log n) \leq$$

$$\left(1 - \frac{1 - e^{-\lambda}}{\lambda}\right) n \log \left[\min\left\{n, \frac{n}{\lambda}\right\}\right] + \frac{(1 - e^{-\lambda}) \log e}{\lambda} \cdot n +$$

$$\left\{\log \alpha + \frac{\alpha - 1}{\max(1, \lambda)} \log (\alpha - 1) - \left(\frac{\alpha - 1}{\max(1, \lambda)} + \frac{1}{\lambda}\right)\right\} \cdot \frac{(\alpha - 1) + \max(1, \lambda)}{\lambda} \left(1 - e^{-\lambda}\right) \cdot n + O(\log n),$$

where $\alpha \geq 1$ is a parameter which can be optimized to minimize the upper bound.

The bounds of (36) are tighter than those of (32). For a specific $\lambda$, the upper bound in (36) is optimized by taking $\alpha \geq 1$ that gives a minimum. Specifically, for $\lambda = 1$,

$$\frac{n}{e} \log n + 0.38n - O(\log n) \leq H_\theta(\Psi^n) \leq \frac{n}{e} \log n + 0.76n + O(\log n),$$

where the best choice of $\alpha$ in (36) leading to (37) is $\alpha \approx 1.93$. In general, the smaller is $\lambda$, the greater the optimal $\alpha$. Figure 1 shows the bounds of (32) and (36) on $H_\theta(\Psi^n)$ as function of $\lambda$. It demonstrates the gaps between the i.i.d. block entropy and the pattern entropy, which significantly increase the greater the alphabet is. The bounds of (36) almost meet for larger $\lambda$. 

Figure 1: Bounds on pattern per symbol entropy $H_\theta(\Psi^n)/n$ vs. $\lambda$ for uniform distributions with $\theta_i = \lambda/n, \forall i$ for $n = 1000$ symbols.
4.2 Monotonic Distributions

While there exist processes for which the i.i.d. entropy cannot be bounded, the pattern block entropy, while it still increases with \( n \) (giving an infinite entropy rate), can be explicitly bounded.

4.2.1 Slowly Decaying Distribution Over the Integers

Consider the distribution over the integers

\[
\tilde{\theta}_j = \frac{\alpha}{j (\log j)^{1+\gamma}}, \quad j = 2, 3, \ldots,
\]

where \( \gamma > 0 \) and \( \alpha \) is a normalizing factor. Approximating \( \sum \tilde{\theta}_j = 1 \) by integrals

\[
0.5 + \frac{1}{3(\log 3)^{1+\gamma}} + \frac{\ln 2}{\gamma (\log 3)^{1+\gamma}} \leq \alpha \leq 0.5 + \frac{\ln 2}{\gamma (\log 3)^{1+\gamma}}.
\]

The distribution in (38) is particularly interesting for \( 0 < \gamma \leq 1 \), where \( H_{\theta}(X) = \infty \). This was used to demonstrate several points in [4], [7]. In particular, in [4], it was used to show that there exist i.i.d. pattern processes with entropy whose order is greater than \( \Theta((n \log n)^{1-\delta}) \) for every \( \delta \); \( 0 < \delta < 1 \). Here, tight bounds approximate \( H_{\theta}(\Psi_n) \) for the distribution in (38) for every \( \gamma > 0 \), even for relatively small \( n \). While \( H_{\theta}(X) = \infty \) for \( 0 < \gamma \leq 1 \), for \( \gamma > 1 \), it is computed by

\[
H_{\theta}(X) = -\log \alpha + \sum_{j=2}^{\infty} \frac{\alpha}{j (\log j)^{1+\gamma}} + \sum_{j=3}^{\infty} \frac{\alpha(1 + \gamma) \log(j)}{j (\log j)^{1+\gamma}}.
\]

Lower bounding the two sums by integrals

\[
H_{\theta}(X) \geq -\log \alpha + \frac{\alpha \ln 2}{\gamma - 1} + \frac{\alpha(1 + \gamma)}{\gamma^2 (\log 3)^{1+\gamma}} (1 + \gamma \ln(\log 3)) \triangleq H_{\theta}(X),
\]

\[
H_{\theta}(X) \leq H_{\theta}(X) + \frac{\alpha}{2} + \frac{\alpha(1 + \gamma) \log(\log 3)}{3(\log 3)^{1+\gamma}}.
\]

Tighter bounds (on both \( \alpha \) and \( H_{\theta}(X) \)) can be obtained by numerically summing more components of the sum, and using the integral bounds only on partial sums. The pattern entropy is bounded as follows:

**Theorem 4** Let \( n \to \infty \). Then, for \( \theta \) in (38),

\[
\frac{H_{\theta}(\Psi^n)}{n} = \begin{cases} 
(1 + o(1)) \cdot \left\{ \frac{\alpha \ln 2}{1-\gamma} \left( \frac{\ln n}{\gamma} \right)^{1-\gamma} + \frac{\alpha(1+\gamma)}{\gamma^2} \left[ \frac{1+\gamma \ln \log n}{(\log 3)^{1+\gamma} \ln \log n} - \frac{1+\gamma \ln \log n}{(\log n)^{1+\gamma}} \right] \right\}, & \text{for } \gamma < 1, \\
(1 + o(1)) \cdot \left\{ \alpha(\ln 2) \ln n + 2\alpha \left[ \frac{1+\gamma \ln \log n}{(\log 3)^{1+\gamma}} - \frac{1+\gamma \ln \log n}{(\log n)^{1+\gamma}} \right] \right\}, & \text{for } \gamma = 1, \\
H_{\theta}(X) - (1 + o(1)) \left\{ \frac{\alpha \ln 2}{(\gamma - 1)(\log n)^{1+\gamma}} \right\}, & \text{for } \gamma > 1.
\end{cases}
\]

Theorem 4 shows that the per-symbol average \( H_{\theta}(\Psi^n)/n \) is still finite even when \( H_{\theta}(X) = \infty \). Specifically, for \( \gamma < 1 \) it is \( \Theta((\log n)^{1-\gamma}) \), and for \( \gamma = 1 \), it is \( \Theta(\log \log n) \). (For \( \gamma < 1 \), a looser lower bound of the same order of magnitude was independently shown in [4].) The bounds in (43) for \( \gamma \leq 1 \) include second order terms. For \( \gamma < 1 \), while asymptotically in \( n \) these terms are negligible,
they are not negligible for $\gamma \to 1$ (the $1/2$ factor in the logarithm of the first term), and for $\gamma \to 0$ (the last terms). Additional second order terms for $\gamma \leq 1$ that are negligible even in these cases are $-\log \alpha$, and for an upper bound, the last two terms of (42). For $\gamma > 1$, $H_\theta(\Psi^n)/n$ asymptotically equals $H_\theta(X)$ but decreases from $H_\theta(X)$ by $\Theta(1/(\log n)^{\gamma-1})$.

Figure 2 shows the asymptotic bounds of Theorem 4 in (43) as well as non-asymptotic bounds (which are derived in the proof of Theorem 4 in Section 6) for different $\gamma$ and $n$. Curves are shown for bounds of $H_\theta(\Psi^n)/n$ (left) and of $nH_\theta(X) - H_\theta(\Psi^n)$ (right). As Theorem 4 and Figure 2 show, for small $\gamma$, (38) decays very slowly. This results in infinite $H_\theta(X)$ for $\gamma \leq 1$, but also in a very significant decrease of $H_\theta(\Psi^n)$ from $nH_\theta(X)$, where specifically $H_\theta(\Psi^n)$ is finite even for $\gamma \leq 1$. While $H_\theta(X)$ in this region is dominated by small probabilities, $H_\theta(\Psi^n)$ is dominated by the larger ones. The decrease between the two is thus dominated by the fact that small probability symbols rarely repeat. As $\gamma$ increases, (38) decays faster, the process is dominated more by the larger probabilities, and the decrease from $nH_\theta(X)$ to $H_\theta(\Psi^n)$ becomes asymptotically negligible, yet still significant for practical $n$.

### 4.2.2 The Zipf Distribution - A Fast Decaying Distribution Over the Integers

Now, consider the Zipf (or zeta) distribution over the integers (see, e.g., [18], [19]) given by

$$\tilde{\theta}_j = \frac{1}{\zeta(1+\gamma) \cdot j^{1+\gamma}}, \quad j = 1, 2, \ldots,$$

where $\gamma > 0$, and $\zeta(1+\gamma)$ is the Riemann zeta-function (see, e.g., [3]), given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x-1} \, dx$$
Figure 3: Bounds on $nH_\theta(X) - H_\theta(\Psi^n)$ vs. $\gamma$ for $n = 10^3$ (left) and vs. $n$ for different values of $\gamma$ (right) for the Zipf distribution in (44). Subscript $\infty$ indicates an asymptotic bound of Theorem 5

for $s > 1$, where $\Gamma(s)$ is the Gamma function. Approximating $\sum \hat{\theta}_j = 1$ by integrals

$$\frac{\gamma^{2\gamma} + 1}{\gamma^{2\gamma}} \leq \zeta(1 + \gamma) \leq \frac{\gamma^{2\gamma+1} + \gamma + 2}{\gamma^{2\gamma+1}} \leq 1 + \gamma.$$  \hspace{1cm} (46)

The Zipf distribution is very common in natural language and rare event modeling. The pattern entropy for the Zipf distribution is thus specifically interesting in compressing patterns of a previously unobserved language. It can also be used for estimation of the number of letters or words in a language by applying methods such as in [8], but on a Zipf distribution instead of a uniform one. Unlike the distribution given in (38), for every $\gamma > 0$, the distribution in (44) has a fixed entropy rate. Bounding sums by integrals (separating leading terms)

$$\log \zeta(1 + \gamma) + \frac{(1 + \gamma)}{\zeta(1 + \gamma)} \left[ \frac{1}{2^{1+\gamma}} + \frac{\log (3^7 e)}{\gamma^2 3^\gamma} \right] \triangleq H_{\theta}(X) \leq H_\theta(X) \leq \tilde{H}_\theta(X) + \frac{(1 + \gamma) \log 3}{\zeta(1 + \gamma) \cdot 3^{1+\gamma}}.$$  \hspace{1cm} (47)

The pattern entropy is bounded as follows:

**Theorem 5** Let $n \to \infty$. Then, for $\theta$ in (44),

$$H_\theta(\Psi^n) = nH_\theta(X) - \Theta \left( n^{\frac{1}{1+\gamma}} \log n \right).$$  \hspace{1cm} (48)

More precisely,

$$nH_\theta(X) \leq \left( 1 + \frac{1}{\gamma} - \frac{1}{3(1+2\gamma)} \right) \frac{1}{(1 + \gamma) \cdot \zeta(1 + \gamma) \frac{1}{1+\gamma}} (1 + o(1)) \cdot n^{\frac{1}{1+\gamma}} \log n \leq$$

$$H_\theta(\Psi^n) \leq nH_\theta(X) - \left( 1 - \frac{1}{e} + \frac{1}{\gamma} - \frac{1}{2(1+2\gamma)} \right) \frac{1}{(1 + \gamma) \cdot \zeta(1 + \gamma) \frac{1}{1+\gamma}} (1 - o(1)) \cdot n^{\frac{1}{1+\gamma}} \log n.$$  \hspace{1cm} (49)

As $\gamma$ increases, (44) decays faster, and the decrease from $nH_\theta(X)$ to $H_\theta(\Psi^n)$ is more negligible, because fewer letters with large enough probabilities dominate the process. For small $\gamma$, $H_\theta(X)$ is
large and is dominated mainly by symbols with relatively small probabilities. Since such symbols
rarely repeat, \( H_\theta (\Psi^n) \) is closer to 0, and the decrease from \( nH_\theta (X) \) is thus very significant. This
behavior resembles that of uniform distributions with \( k \gg n \). The coefficients 1 in the lower bound
and \( 1 - 1/e \) in the upper bound reflect the effect of symbols with probabilities close to 1/\( n \) that may
or may not occur. The remaining coefficients reflect decrease in entropy due to very low probability
symbols, which are unlikely to occur in \( X^n \). Figure 3 shows the asymptotic bounds of Theorem 5 in
\((49)\) as well as non-asymptotic bounds (which are derived in the proof of Theorem 5 in Section 6) for
different \( \gamma \) and \( n \). The gaps between the asymptotic and non-asymptotic behaviors are greater for
smaller \( \gamma \) and smaller for greater \( \gamma \). For small \( n \), second order terms are more significant. However,
for larger \( n \), the gaps between the bounds become negligible. (Specifically, only for \( \gamma = 0.01 \), the
asymptotic curves do not overlap the non-asymptotic ones on the right graph. For such low \( \gamma \),
curves for lower and upper bounds do overlap.)

### 4.2.3 Geometric Distribution

The geometric distribution, which decays faster than the preceding distributions, is given by

\[
\hat{\theta}_j = p (1 - p)^{j-1}; \quad j = 1, 2, \ldots
\]

where \( 0 < p < 1 \). It has a fixed entropy rate \( H_\theta(X) = h_2(p)/p \), where \( h_2(p) \) is the binary entropy
function. Its pattern entropy is bounded as follows.

**Theorem 6** Fix \( p \). Let \( n \to \infty \) and let \( \delta > (\ln 20)/(\ln \ln n) \). Then, for \( \theta \) in \((50)\),

\[
H_\theta (\Psi^n) \leq nH_\theta(X) - \frac{(1 + \delta)^2 (\log \log n)^2}{2 \log \frac{1}{1-p}} - C_{L1}(p)(1 + \delta) \log \log n - C_{L2}(p) - O \left( \frac{1}{\log n} \right) \leq \frac{1}{2(2-p)p} \log \frac{2e(2-p)}{(1-p) \log \frac{1}{1-p}} + \frac{1}{(\log \log n)^2} + O \left( \frac{1}{(\log n)(\log \log n)^2} \right)
\]

where

\[
C_{L1}(p) = \frac{\log p}{\log(1-p)} + \frac{5 + 2p - 2.5p^2}{3p(2-p)}
\]

\[
C_{L2}(p) = \frac{5 + 5p - 4p^2}{3p(2-p)} \log \frac{1}{p} + \frac{(1-p)^2}{p^2} \left( \frac{1}{1-p} - \frac{2(p^2 - 2p + 0.5)}{3(2-p)^2} \right) \log \frac{1}{1-p} + \frac{b_{g,max}(p)}{\log \frac{1}{1-p}}
\]

and

\[
b_{g,max}(p) \triangleq \frac{2 + \frac{1}{\sqrt{1-p}}}{\sqrt{1-p} - 1}, \quad k_\bar{\theta}(p) + k_\bar{\bar{\theta}}(p) \leq \left[ \frac{6.9 \log e}{\log \frac{1}{1-p}} + 1 \right], \quad k_\bar{\theta}(p) \leq \left[ \frac{1.4 \log e}{\log \frac{1}{1-p}} + 1 \right].
\]
Figure 4: Bounds on $nH_{\theta}(X) - H_{\theta}(\Psi^n)$ vs. $p$ for $n = 10^3$ (left) and vs. $n$ for different values of $p$ (right) for the geometric distribution. Subscript $\infty$ indicates an asymptotic bound of Theorem 6, the subscript “simple” implies to an upper bound with $U \geq 0$ in [25].

Theorem 6 shows that $H_{\theta}(\Psi^n)$ diverges from $nH_{\theta}(X)$ by at most $\Theta(\log \log n)^2$, and if $p$ is smaller (for $n \rightarrow \infty$, $p \leq 0.69$), by at least $\Theta(1/(\log \log n))$. Due to the very slow rates, second order terms are necessary in (51) for more accurate approximations. The proof of Theorem 6 presented in Section 6 is used to obtain numerical bounds even for relatively small $n$. Figure 4 and Table 1 show the asymptotic bounds of Theorem 6 and the tighter non-asymptotic bounds for different $p$ and $n$. The small bounds are very sensitive to the $\varepsilon_b$ parameters, which are numerically chosen. Hence, at larger $p$, where the bounds are small, “ringing” appears due to quantization of $\varepsilon_b$. A larger choice of $\delta$ above $(\delta > (\ln(20/p^2))/(\ln \ln n))$ will eliminate the last three expressions in (53) of the asymptotic bound. However, it will not result in a tighter asymptotic curve.

Due to the fast decay of (50), the decrease of $H_{\theta}(\Psi^n)$ from $nH_{\theta}(X)$ is much smaller than in the preceding cases. Yet, for smaller $p$, (50) decays slower, and $nH_{\theta}(X) - H_{\theta}(\Psi^n)$, although negligible w.r.t. $nH_{\theta}(X)$ for sufficiently large $n$, is still large. Furthermore, it is not negligible w.r.t. $nH_{\theta}(X)$ for smaller $n$. Table 1 demonstrates that. For example, for $p = 0.01$, even for $n = 1000$, $nH_{\theta}(X) - H_{\theta}(\Psi^n)$ is over 10% of $nH_{\theta}(X)$. For $n = 10$, $H_{\theta}(\Psi^n) \leq 2.28$ while $nH_{\theta}(X) > 80$. On the other hand, for $p = 0.8$, $nH_{\theta}(X) - H_{\theta}(\Psi^n)$ is at most 18.66 for $n = 10^{10}$.

As shown in Figure 4 and Table 1, the bounds on $nH_{\theta}(X) - H_{\theta}(\Psi^n)$ are relatively insensitive to $n$ for greater values of $n$. This implies that the decrease in the entropy effectively occurs during the first indices. This is also implied by the diminishing decrease from $nH_{\theta}(X)$ on the right hand side of (51). While the true rate of $nH_{\theta}(X) - H_{\theta}(\Psi^n)$ may be between those of the lower and upper bounds, diminishing decrease of $H_{\theta}(\Psi^n)$ from $nH_{\theta}(X)$ is possible. Fast decaying distributions may effectively behave like distributions over small alphabets, and the gain in $H_{\theta}(\Psi^n)$ is only due to occurrences of new indices. Once these become sparse, we may have $H_{\theta}(\Psi^f | \Psi^{f-1}) > H_{\theta}(X)$, thus possibly decreasing the gap between $H_{\theta}(\Psi^n)$ and $nH_{\theta}(X)$ (as discussed in Subsection 4.3).
Table 1: Bounds on $H_\theta (\Psi^n)$ for different (finite) $n$.

| $p$ | $n$ | $nH_\theta(X)$ | UB($H_\theta[\Psi^n]$) | LB($H_\theta[\Psi^n]$) | $nH_\theta(X) - \text{UB}(H_\theta[\Psi^n])$ | $nH_\theta(X) - \text{LB}(H_\theta[\Psi^n])$ |
|-----|-----|----------------|--------------------------|--------------------------|------------------------------------------|------------------------------------------|
| 0.01 | 10$^1$ | 80.8 | 2.28 | 1.64 | 78.52 | 79.16 |
|     | 10$^2$ | 808 | 212.2 | 150.3 | 595.8 | 657.7 |
|     | 10$^3$ | 80.8 · 10$^2$ | 7011 | 5483 | 1069 | 2596 |
|     | 10$^4$ | 80.8 · 10$^3$ | 79335 | 75936 | 1458 | 4857 |
| 0.05 | 10$^5$ | 80.8 · 10$^4$ | 80.6 · 10$^4$ | 80.1 · 10$^4$ | 1561 | 6979 |
|     | 10$^{10}$ | 80.8 · 10$^9$ | 80.8 · 10$^9$ | 80.8 · 10$^9$ | 1561 | 12632 |
| 0.8 | 10$^1$ | 57.28 | 8.37 | 5.31 | 48.91 | 51.97 |
|     | 10$^2$ | 572.8 | 486.9 | 295.4 | 85.9 | 277.4 |
|     | 10$^3$ | 5728 | 5630 | 5124 | 98 | 604 |
|     | 10$^4$ | 57280 | 57182 | 56348 | 98 | 932 |
| 0.8 | 10$^5$ | 9.02 | 8.96 | 5.26 | 0.06 | 3.76 |
|     | 10$^2$ | 90.24 | 90.16 | 82.4 | 0.08 | 7.84 |
|     | 10$^3$ | 902.41 | 902.34 | 893.15 | 0.07 | 9.26 |
|     | 10$^{10}$ | 9.02 · 10$^9$ | 9.02 · 10$^9$ | 9.02 · 10$^9$ | 0.07 | 18.66 |

4.2.4 Linear Monotonic Distributions

The monotonic distributions considered above were all over infinite alphabets. Consider a monotonic distribution over a finite alphabet, whose probabilities increase linearly. An example of such a distribution is given by

$$\theta_i = \frac{2(i - 0.5)\lambda^2}{n^2}, \quad i = 1, 2, \ldots, k = \frac{n}{\lambda},$$

where $0 < \lambda < n; i$ is a parameter. This parametrization is very similar to that of the uniform distribution in Theorem 3, but here the distribution is monotonically increasing. For $\lambda = 1, k = n$, and $\theta_i < 2/n$ for all $i$. If $\lambda \gg 1, k = o(n)$, and if $\lambda \ll 1, k \gg n$. The i.i.d. entropy rate of (55) is

$$H_\theta (X) = \log \frac{\lambda}{n} + \log \frac{\sqrt{e}}{2} + O \left( \frac{\lambda \log n}{\lambda} \right)$$

where the last term is negligible unless $\lambda = \Theta(n)$ (i.e., $k = \Theta(1)$). The pattern entropy of the distribution in (55) is as follows:

**Theorem 7** Let $n \to \infty$, let $\delta > 0$ be fixed arbitrarily small. Then, for $\theta$ in (56)

$$H_\theta (\Psi^n) = \begin{cases} nH_\theta (X) - o(1), & \text{if } \lambda \geq n^{\frac{2}{3} + \delta} \\ nH_\theta (X) - (1 + o(1)) \frac{\lambda}{2} \log \frac{n}{\lambda}, & \text{if } \frac{4}{27} \leq \lambda \leq n^{\frac{2}{3} - \delta} \\ (1 + o(1)) C_\lambda \lambda n \log \frac{n}{\lambda}, & \text{if } \lambda \leq \frac{1}{2}, \end{cases}$$

where $(1 - 2\lambda/3) \cdot 2/3 \leq C_\lambda \leq 2/3$.

Figure 5 shows the bounds for two regions of $\lambda$, and compares them to the bounds in Corollary 1 of a uniform distribution. The curves include second order terms shown in the proof of Theorem 7.
in Section 6. Also, more complex bounds (not shown in Section 6 for brevity) obtained using Theorems 1 and 2 for the boundary between the last two regions are used. When \( k = o(n^{1/3}) \) (first region), there are no letters with very small probabilities. All letters are distributed away from each other, such that at most a single letter populates a bin. Hence, \( H_\theta (\Psi^n) \) hardly decreases from \( nH_\theta(X) \). When \( k = o(n) \) (and is in the second region), first occurrences of letters with large probabilities dominate the decrease from \( nH_\theta(X) \) to \( H_\theta (\Psi^n) \). The behavior is very close to that of Corollary 1. However, each parameter gains \( \log \left( \frac{n}{\lambda^3/2} \right) \) bits instead of \( \log k = \log(n/\lambda) \) (e.g., if \( \lambda = k = \sqrt{n} \), instead of 0.5 \( \log n \), the gain here is 0.25 \( \log n \). In the last region, \( H_\theta (\Psi^n) = \Theta \left( \frac{n^2}{k} \log n \right) \). This order of magnitude, again, equals that of a uniform distribution.

### 4.3 Small Alphabets

While \( H_\theta (\Psi^n) \leq nH_\theta(X) \), it is not guaranteed that \( H_\theta (\Psi_\ell | \Psi^{\ell-1}) \leq H_\theta (X^n) \), or even that \( H_\theta (\Psi_{n_0+1} | \Psi^{n_0}) \leq (n - n_0)H_\theta(X) \) for some \( n_0 < n \). Following the chain rule

\[
H_\theta (\Psi_{n_0+1} | \Psi^{n_0}) = H_\theta (\Psi^n) - H_\theta (\Psi^{n_0}) = [H_\theta (X^n) - H_\theta (X^n | \Psi^n)] - [H_\theta (X^{n_0}) - H_\theta (X^{n_0} | \Psi^{n_0})] = (n - n_0)H_\theta(X) + H_\theta (X^{n_0} | \Psi^{n_0}) - H_\theta (X^n | \Psi^n).
\] (58)

For a larger \( n > n_0 \), it is not guaranteed that \( H_\theta (X^n | \Psi^n) > H_\theta (X^{n_0} | \Psi^{n_0}) \). In fact, for a smaller alphabet and small \( n_0 \), the opposite may be true, because the longer pattern may have less uncertainty of which symbols correspond to which indices. This argument is in concert with the proof of Theorem 7 in [7] and Proposition 4 in [4], which show that for a smaller alphabet, as \( n \to \infty \), \( H_\theta (\Psi_{n+1} | \Psi^n) \geq (1 - o(1))H_\theta(X) \). This is true for \( n \to \infty \), as long as \( \theta_i > 1/n^{1-\varepsilon} \), \( \forall i \leq k \); for an arbitrarily small \( \varepsilon \).

Opposite behaviors, where \( H_\theta (\Psi^n) \leq nH_\theta(X) \) but \( H_\theta (\Psi_\ell | \Psi^{\ell-1}) > H_\theta (X^n) \) for \( \ell > n_0 \) for some \( n_0 > 1 \), occur for smaller alphabets because the decrease in the block entropy is dominated...
by first occurrences. Once the dominant symbols in the distribution occur, the remainder of $X^n$ consists mainly of reoccurrences, where no decrease in entropy is exhibited. Such a behavior can also extend to fast decaying distributions, that while still over infinite alphabets, may only have a small subset of the alphabet symbols that will effectively occur in a sequence, such as the geometric distribution. Figure 6 shows $H_\theta (\Psi_\ell | \Psi_{\ell-1}) - H_\theta (X)$ for a binary and a ternary alphabet. In the binary case, the decrease of $H_\theta (\Psi^n)$ from $nH_\theta (X)$ is the sole result of the first index, where $H_\theta (\Psi_1) = 0$. All remaining indices have $H_\theta (\Psi_\ell | \Psi_{\ell-1}) \geq H_\theta (X)$. Thus $nH_\theta (X) - H_\theta (\Psi^n)$ diminishes to 0 as $n > 1$. As shown in Figure 6 a ternary alphabet exhibits a similar behavior, except that $H_\theta (\Psi_\ell | \Psi_{\ell-1}) > H_\theta (X)$ for the first time at a larger $\ell$. The value of that $\ell$ depends on the parameters of $\theta$. Pattern entropies shown in Figure 6 were computed precisely using

$$H_\theta (\Psi^n) = -\sum_{n_x} \left(n_x(1), \ldots, n_x(k)\right) \prod_{i=1}^k \theta_i^{n_x(i)} \cdot \log \left\{ \sum_{\sigma(n_x)} \prod_{j=1}^k \theta_j^{[\sigma(n_x)](j)} \right\}$$

(59)

where $n_x \triangleq (n_x(1), n_x(2), \ldots, n_x(k))$ is the occurrence vector of the alphabet symbols in $x^n$, the outer sum is taken over all such vectors, and the inner sum is taken over all $k! / (k - |n_x|)!$ nonzero element permutations $\sigma(n_x)$ of the occurrence vector, where $|n_x|$ is the cardinality of nonzero components in $n_x$. Conditional entropies were then computed with the first equality in (55).

5 Uniform Distributions - Proofs

Proof of Corollary 1: Corollary 1 results directly from Theorems 1 and 2. The lower bound of (31) is that of (11), resulting also from (13) and (14). The upper bound follows directly from (25) or (26) with (27), where $R'_b = 0$, and the second term of (27) does not exist because there is no $\theta_i$ in a bin which differs from the average bin probability. The lower bound of (32) follows from (13).
with $H^{(01)}_\theta (X) = S_1 = S_4 = 0$. Then, from (19),

$$S_2 \geq \frac{n}{\lambda} \left( \lambda - 1 + e^{-\lambda - \frac{\lambda^2}{2n}} \right) \log \frac{n}{\lambda} \tab (a)$$

$$\geq \left( 1 - \frac{1 - e^{-\lambda}}{\lambda} \right) n \log \frac{n}{\lambda} - \lambda e^{-\lambda} \log \frac{n}{\lambda} \tab (60)$$

where (a) follows from $e^{-\lambda^2/n} \geq 1 - \lambda^2/n$. Then, from (21),

$$S_3 \geq \left( \log e \right) \frac{L_{01-1}}{n} \sum_{i=1}^{L_{01-1}} \left( \lambda - 1 \right) \frac{\lambda n}{n} = \frac{\lambda \log e}{2n} \left( L_{01-1}^2 - L_{01-1} \right) \tab (a)$$

$$\geq \frac{(1 - e^{-\lambda})^2 \log e}{2\lambda} \cdot n - \frac{(1 - e^{-\lambda}) \log e}{2} \tab (61)$$

where (a) follows from the lower bound in (9). Summing (60)-(61) yields the lower bound of (32).

The upper bound follows from (25), where only $R_1'$, upper bounded by (28), using the lower bound on $L_1$ in (9), is not zero. The lower bound in (33) follows from (60)-(61) with $\lambda = 1/n^{\mu - 1 + \epsilon}$. Expressing exponents by their Taylor series,

$$S_2 + S_3 \geq \left( 1 - \frac{\lambda}{3} \right) \frac{\lambda n}{2} \log n^{\mu + \epsilon} + (1 - \lambda) \frac{\lambda n}{2} \log e - \lambda \log n^{\mu + \epsilon} - \frac{\lambda \log e}{2} \tab (62)$$

The upper bound follows from (25), where only $R_0'$, which is bounded by (29), is not zero.

**Proof of Theorem 3:** For the lower bound

$$H_\theta (\Psi^n) \tab (a)$$

$$= \left( n - L_1 \right) \log \frac{n}{\lambda} - \sum_{j=1}^{n/\lambda} \sum_{m=0}^{n/\lambda} \log \left( 1 - \frac{m\lambda}{n} \right) \tab (b)$$

$$= \log \frac{n}{\lambda} \log \left( 1 - \frac{m\lambda}{n} \right) \tab (c)$$

$$\geq \log \frac{n}{\lambda} \log \left( 1 - \frac{m\lambda}{n} \right) \log \left( \frac{n}{\lambda} - L_1 \right) \tab (d)$$

$$\geq \left( 1 - \frac{1 - e^{-\lambda}}{\lambda} \right) n \log \frac{n}{\lambda} + \frac{\lambda e^{-\lambda} - \lambda}{n} \log e \cdot n - O \left( \log n \right) \tab (63)$$

Equality (a) computes the average cost of repetitions (the first term) and that of first occurrences (the second term). Then, rearrangement of the second term leads to (b) by using $1 - m\lambda/n = (\lambda/n) \cdot (n/\lambda - m)$ and $EK_1 = L_1$. Inequality (c) is by Jensen’s inequality. Next, (d) is obtained from (9), and finally, Stirling’s approximation

$$\sqrt{2\pi m} \left( \frac{m}{e} \right)^m \leq m! \leq \sqrt{2\pi m} \left( \frac{m}{e} \right)^m \cdot e^{1/(12m)} \tab (64)$$

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and Taylor expansion of \(e^{-\lambda^2/n} = 1 - O(1/n)\) are used to obtain (e), proving the lower bound.

To prove the upper bound, the pattern entropy is upper bounded by the average description length of a code that assigns probability \(\rho_1 = \lambda/(an)\), where \(\lambda \triangleq \max(\lambda, 1)\) and \(\alpha \geq 1\) is a parameter, to a repeated index, and the remaining yet unassigned probability to a new index.

Using this code,

\[
H_\theta(\Psi^n) \overset{(a)}{\leq} (n - L_1) \log \frac{an}{\lambda'} - \sum_{j=1}^{n/\lambda'} P_\theta(K_1 = j) \sum_{m=0}^{j-1} \log \left(1 - \frac{m\lambda'}{an}\right)
\]

\[
= (n - L_1) \log \frac{an}{\lambda'} + L_1 \log \frac{an}{\lambda'} - \sum_{j=1}^{n/\lambda'} P_\theta(K_1 = j) \log \left[\frac{\lambda'(\alpha - 1) + j}{\lambda'(\alpha - 1)}\right]!
\]

\[
\overset{(b)}{\leq} n \log \frac{an}{\lambda'} - \log \left[\frac{\lambda'(\alpha - 1) + L_1}{\lambda'(\alpha - 1)}\right]!
\]

\[
\overset{(c)}{\leq} \left(1 - \frac{1 - e^{-\lambda}}{\lambda}\right) n \log \frac{n}{\lambda'} + \frac{(1 - e^{-\lambda}) \log e}{\lambda} \cdot n + \log \left[\left(\alpha - 1\right) + \frac{\lambda'}{\lambda} \left(1 - e^{-\lambda}\right)\right] \cdot n + O(\log n).
\]

Inequality (a) is since the entropy is upper bounded by the average description length of the code, which consists of the cost of repetitions (first term) and first occurrences (second term). The bound in (b) is under the worst case assumption that all \(n/\lambda'\) symbols occurred. The first occurrence of the last new index is assigned probability \(1 - 1/\alpha + \lambda'/(an)\), and those of the preceding indices are assigned this probability plus increments of \(\lambda'/(an)\), depending on the occurrence time. This step produces a tighter bound than in (25). Next, (c) follows Jensen’s inequality and the concavity of \(- \log(x!))\). Finally, (d) follows Stirling’s approximation and the bound in (9) on \(L_1\).

\(\square\)

6 Monotonic Distributions - Proofs

6.1 Slowly Decaying Distribution Over the Integers

Proof of Theorem 4: Let \(j_0\) and \(j_1\) be the indices of the greatest \(\hat{\theta}_j \leq \eta_1, \eta_2\), respectively. Then, substituting \(\chi_b \triangleq \alpha n^{1+\epsilon_b} (\ln 2)^{1+\gamma}\), it can be verified that

\[
j_b = \left[\frac{\beta_b \chi_b}{(\ln \chi_b)^{1+\gamma}}\right], \quad \text{where} \quad \beta_b = \frac{1}{1 + \frac{\ln \beta_b}{\ln \chi_b} - \frac{(1+\gamma) \ln \ln \chi_b}{\ln \chi_b}}^{1+\gamma}, 
\]

\(b = 0, 1\).

The value of \(\beta_b\) can be found numerically. It is constant for large enough \(n\), and as \(n \rightarrow \infty\), it approaches 1. Thus \(j_b = O\left(n^{1+\epsilon_b}/(\log n)^{1+\gamma}\right)\). Using an integral to approximate a sum

\[
\varphi_0 = \sum_{j=j_0}^{\infty} \frac{\alpha}{j(\log j)^{1+\gamma}} = \frac{\alpha \ln 2}{\gamma (\log j_0)^{\gamma}} \left(1 + O\left(\frac{1}{j_0 \log(j_0)}\right)\right) = \frac{\alpha \ln 2}{\gamma (1+\varepsilon_0)^{\gamma} (\log n)^{\gamma}} (1 + o(1)).
\]
Similarly,
\[
\varphi_{01} = \frac{\alpha \ln 2}{\gamma (\log j_1)^\gamma} \left( 1 + O \left( \frac{1}{j_1 \log(j_1)} \right) \right) = \frac{\alpha \ln 2}{\gamma (1 - \varepsilon)\gamma (\log n)^\gamma} (1 + o(1)).
\]

Using Taylor series approximations
\[
\varphi_1 = \varphi_{01} - \varphi_0 = \frac{\alpha(\ln 2)(\varepsilon + \varepsilon_0)}{(\log n)^\gamma} (1 + O(\varepsilon + \varepsilon_0)) (1 + o(1)).
\]

To use (13), following (11), and selecting \( \varepsilon = O((\log \log n)/(\log n)) \), for \( \gamma < 1 \),
\[
H^0_{\vartheta}(X) = -\varphi_{01} \log \varphi_{01} - \sum_{j=2}^{j_1-1} \tilde{\theta}_j \log \tilde{\theta}_j
\]
\[
\overset{(a)}{=} -\varphi_{01} \log \varphi_{01} - (1 - \varphi_{01}) \log \alpha + \sum_{j=2}^{j_1-1} \frac{\alpha}{j(\log j)^\gamma} + \sum_{j=3}^{j_1-1} \frac{\alpha(1 + \gamma) \log \log j}{j(\log j)^{1+\gamma}}
\]
\[
\overset{(b)}{=} -\log \alpha + \frac{\alpha \ln 2}{1 - \gamma} (\log \frac{j_1}{2})^{1-\gamma} + \frac{\alpha(1 + \gamma)}{\gamma^2} \left[ \frac{1 + \gamma \ln \log 3}{(\log 3)^\gamma} - \frac{1 + \gamma \ln \ln j_1}{(\log j_1)^\gamma} \right] + O(1)
\]
\[
\overset{(c)}{=} (1 + o(1)) \cdot \left\{ \frac{\alpha \ln 2}{1 - \gamma} \left( \log \frac{n}{2} \right)^{1-\gamma} + \frac{\alpha(1 + \gamma)}{\gamma^2} \left[ \frac{1 + \gamma \ln \log 3}{(\log 3)^\gamma} - \frac{1 + \gamma \ln \ln n}{(\log n)^\gamma} \right] \right\}
\]

where (b) follows from approximating sums by integrals, substituting the value of \( \varphi_{01} \) from (68) and including terms equal to the last two of the upper bound in (12) in an \( O(1) \) term. Then, (c) follows from substituting \( j_1 \) from (66) with \( \varepsilon = \Theta((\log \log n)/(\log n)) \), and absorbing all second order terms. Note that second order terms resulting from \( j_1 \) in this step, which are absorbed in other terms, are negligible w.r.t. the terms expressed above even if \( \gamma \to 0 \) or \( \gamma \to 1 \). A similar derivation follows for \( \gamma = 1 \), except that the second term in step (b) is replaced by the proper value of the integral as shown in (43). In a similar manner, for \( \gamma > 1 \),
\[
H^0_{\vartheta}(X) = -\sum_{j=2}^{\infty} \tilde{\theta}_j \log \tilde{\theta}_j + \sum_{j=1}^{\infty} \tilde{\theta}_j \log \tilde{\theta}_j = H_\vartheta(X) - \sum_{j=1}^{\infty} \tilde{\theta}_j \log \frac{\varphi_{01}}{\tilde{\theta}_j}
\]
\[
\overset{(a)}{=} H_\vartheta(X) - \varphi_{01} \log \frac{\varphi_{01}}{\alpha} - \sum_{j=1}^{\infty} \frac{\alpha}{j(\log j)^\gamma} - \sum_{j=1}^{\infty} \frac{\alpha(1 + \gamma) \log \log j}{j(\log j)^{1+\gamma}}
\]
\[
\overset{(b)}{=} H_\vartheta(X) - \varphi_{01} \log \frac{\varphi_{01}}{\alpha} = \frac{\alpha(\ln 2)}{(\gamma - 1)(\log j_1)^{\gamma-1}} - \frac{\alpha(1 + \gamma)(1 + \gamma \ln \log j_1)}{\gamma^2(\log j_1)^\gamma} + O \left( \frac{1}{j_1(\log j_1)^\gamma} \right)
\]
\[
\overset{(c)}{=} H_\vartheta(X) - (1 + o(1)) \frac{\alpha \ln 2}{(\gamma - 1)(\log n)^{\gamma-1}}
\]

where (b) follows from approximating sums by integrals, and (c) from substituting \( j_1 \) from (66) and absorbing second order terms, realizing that the dominant decrease emerges from the third term.

Next, we lower bound the first sum in (20) and \( S_3 \) by 0. Then, choosing \( \varepsilon_0 = 0 \),
\[
S_2 \overset{(a)}{=} \sum_{j=j_1}^{j_0-1} \left( n \tilde{\theta}_j - 1 \right) \log \frac{\varphi_{01}}{\theta_j} \overset{(b)}{=} (n \varphi_1 - (j_0 - j_1)) \log (\varphi_{01} n^{1-\varepsilon}) \overset{(c)}{=} O \left( \frac{n \log \log n}{(\log n)^\gamma} \right)
\]
where \((b)\) follows from \(\tilde{\theta}_j \leq 1/n^{1-\varepsilon}\) in bin 1, and \((c)\) from \((66)\) and \((69)\) with the choice of \(\varepsilon\) and \(\varepsilon_0\) above. (Note that a tighter nontrivial bound for the second sum of \((20)\) can also be obtained, but has a negligible effect.) Next, using the trivial bound of \((14)\),
\[
S_1 \leq \log (j_1!) = O \left( \frac{n^{1-\varepsilon}}{(\log n)^\gamma} \right).
\]
Similarly, with a proper choice of constant for \(\varepsilon = \Theta((\log \log n)/(\log n))\), \(S_1 = O(j_1)\). Adding the bounds above for all terms of \((13)\) (normalizing \(S_1, S_2, \) and \(S_4\) by \(n\)) results in lower bounds satisfying \((43)\) for all regions of \(\gamma\), where, regardless of \(\gamma\), the expression is dominated by \(H^{(01)}_\theta(X)\).

The bounds obtained above are asymptotic. To derive the numerical bounds in Figure 2 for finite \(n\), steps \((a)\) of \((70)\), \((71)\), and \((72)\) are used to compute sums (where dominant components of the sums are added, and remaining, sometimes infinite, partial sums are approximated by integrals). The value of \(\varepsilon\) is numerically tested for different values, and \(\varepsilon_0 = 0\) is used. The precise expression in \((22)\) is computed for each \(\varepsilon\). Then, \(\varepsilon\) that gives the maximal bound for each \(\gamma\) and \(n\) is used. Roughly, \(\varepsilon \approx 1.7(\ln \ln n)/(\ln n)\) produced the tightest lower bounds.

Asymptotically, \((26)\) is sufficient to obtain an upper bound on \(H_\theta(\Psi^n)\). A choice of \(\varepsilon = 0\) yields identical bounds to those in \((70)-(71)\) for \(H^{(01)}_\theta(X)\). Then, the trivial bound \(U \geq 0\) is used. Finally, 
\[
\sum_{j=j_1}^\infty \tilde{\theta}_j^2 = \sum_{j=j_1}^\infty j^2(\log j)^{2+2\gamma} \leq \frac{\alpha^2}{(j_1+1)(\log j_1)^{2+2\gamma}} \overset{(a)}{=} O \left( \frac{1}{n(\log n)^{1+\gamma}} \right),
\]
where \((a)\) follows from an integral upper bound, and \((b)\) from \((66)\), yields, using \((29)\),
\[
R'_{01} = O \left( \frac{n}{(\log n)^\gamma} \right).
\]
Combining the terms of \((26)\) from \((70)-(71)\) and \((75)\) yields upper bounds satisfying \((43)\) dominated by \(H^{(01)}_\theta(X)\). The numerical upper bounds in Figure 2 can be obtained using these terms, where precise expressions from steps \((a)\) of \((70)\), \((71)\) and from \((29)\) are used to obtain \(H^{(01)}_\theta(X)\) and \(R'_{01}\), respectively. Slightly tighter bounds can be obtained using \((25)\), where \(R'_{01}^0\) and \(R'_{01}^1\) are bounded separately, and \(\varepsilon_0\) is numerically optimized to minimize the bound. (These are the bounds shown in Figure 2)

\[
\begin{align*}
6.2 \quad & \text{The Zipf Distribution} \\
\text{Proof of Theorem 5.} \quad & \text{For convenience, let } \alpha \overset{\Delta}{=} 1/\zeta(1+\gamma). \text{ Let } j_0 \text{ and } j_1 \text{ be the indices of the greatest } \tilde{\theta}_j \leq \eta_1, \eta_2, \text{ respectively. Then,} \\
& j_b = \left[ \frac{1}{\alpha^{1+\gamma}} \cdot n^{\frac{1+\varepsilon_0}{1+\gamma}} \right], \quad b = 0, 1. \tag{76}
\end{align*}
\]
Similarly, for \(b \geq 2\), define \(j_b = \max \left\{ 1, \left[ (\alpha \cdot n^{1+\varepsilon_2}/(b'+1)^2)^{1+\gamma} \right] \right\} \) as the index of the greatest \(\tilde{\theta}_j \leq \eta_{b+1} \) (where \(b'\) is as defined in \((41)-(51)\)). Note that \(j_b = 1\) for \(b' \geq \sqrt{\alpha n^{1+\varepsilon_2}} - 1\), \(k_b = j_{b+1} - j_b\), and some bins may be empty. From \((70)\) and bounding a sum by an integral,
\[
\varphi_0 = \sum_{j=j_0}^\infty \frac{\alpha}{j^{1+\gamma}} \left\{ \leq \frac{\alpha}{j_0^{1+\gamma}} + \int_{j_0}^\infty \frac{\alpha}{x^{1+\gamma}} \, dx \leq \frac{\alpha}{\gamma j_0} \left( 1 + \frac{\alpha}{j_0} \right) \right\}
\]
Similarly,
\[
\frac{\alpha}{\gamma j_1} \leq \varphi_{01} \leq \frac{\alpha}{\gamma j_1} \left(1 + \frac{\gamma}{j_1}\right), \quad \varphi_1 = \varphi_{01} - \varphi_0 \geq \frac{\alpha}{\gamma j_1} \left[1 - \left(\frac{j_1}{j_0}\right)^\gamma \left(1 + \frac{\gamma}{j_0}\right)\right].
\]  
(78)

From (76)-(78), it follows that
\[
n_0 \varphi_0 = \frac{\alpha}{\gamma} \cdot n^{\frac{1-\gamma \varphi_0}{1+\gamma}} + O \left(\frac{1}{n^\alpha}\right),
\]
\[
n_0 \varphi_{01} = \frac{\alpha}{\gamma} \cdot n^{\frac{1+\gamma}{1+\gamma}} + O \left(n^\varepsilon\right).
\]  
(79)

While \(k_0, k_{01} = \infty\), it follows from (76) that
\[
k_1 = j_0 - j_1 = j_0 \left[1 - O \left(n^{\frac{\varepsilon_0 + \varepsilon}{1+\gamma}}\right)\right].
\]  
(80)

The lower bound of (13) can be derived for the distribution in (44) by separately bounding its terms. First, \(S_3 \geq 0\). Then, \(nH_\theta^{(01)}(X) + S_2\) is lower bounded, and \(S_1\) and \(S_4\) upper bounded.
\[
nH_\theta^{(01)}(X) + S_2 \geq (a) \quad nH_\theta(X) - n \sum_{j=j_0}^\infty \tilde{\theta}_j \log \frac{\varphi_{01}}{\tilde{\theta}_j} - \sum_{j=j_1}^{j_0-1} \log \frac{\varphi_{01}}{\tilde{\theta}_j} + \left(1 - \frac{1}{3n^\varepsilon_0} \cdot \frac{2}{\varepsilon}ight) \sum_{j=j_0}^\infty \tilde{\theta}_j^2 \log \frac{\varphi_{01}}{\tilde{\theta}_j} + \left(n \varphi_0 + j_0 - j_1\right) \log \frac{\varphi_{01}}{\alpha} + (1 + \gamma) \sum_{j=j_1}^{j_0-1} \log j + \left(1 + \gamma\right) \alpha n \cdot \sum_{j=j_0}^\infty \log j \geq j_1 \log \frac{j_0}{j_1},
\]  
(81)

where (a) follows from lower bounding (20), the definition of \(H_\theta^{(01)}(X)\) in (11), and from combining terms. Note that the summand of (3) can be inserted to the summand of \(V_2\) above to provide a tighter numeric expression. Now,
\[
V_1 + V_2 \overset{(a)}{=} \left(n \varphi_0 + j_0 - j_1\right) \log \frac{\varphi_{01}}{\alpha} + (1 + \gamma) \sum_{j=j_1}^{j_0-1} \log j + \left(1 + \gamma\right) \alpha n \cdot \sum_{j=j_0}^\infty \log j \overset{(b)}{\leq} \left(n \varphi_0 + j_0 - j_1\right) \log \frac{\varphi_{01}}{\alpha} + (1 + \gamma) \log \frac{j_0!}{j_1!} + (1 + \gamma) \alpha n \left\{ \log j_0 \frac{1}{j_1} + \log j_0 \frac{1}{j_0} + \log e \right\} \overset{(c)}{\leq} \left(n \varphi_0 + j_0 - j_1\right) \log \frac{\varphi_{01} j_0^{1+\gamma}}{\alpha} + (1 + \gamma) \left[ n \varphi_0 \log e \frac{1}{\gamma} - (j_0 - j_1) \log e \right] + O \left(j_1 \log \frac{j_0}{j_1}\right)
\]

where (a) follows from the definitions of \(\varphi_0\) and \(\tilde{\theta}_j\), and (b) follows from bounding the sum in the last term by an integral. The lower bound on \(\varphi_0\) in (77) leads to (c). A choice of \(\varepsilon_0 = 0\) leads to the minimal tradeoff between \(n \varphi_0\) and \(j_0\) in the dominant term of (82). The smallest possible \(\varepsilon\) will minimize the bound in (82). By Theorem 1, this value is constrained to \(\varepsilon = \Theta \left(\frac{\log \log n}{\log n}\right)\) to guarantee sufficient rate of \(S_4\). Using (76) and (79) for \(j_0\) and \(\varphi_0\), respectively, this yields
\[
V_1 + V_2 \leq (1 + o(1)) \cdot \frac{\alpha^{1+\gamma}}{\gamma} \cdot n^{1+\gamma} \log n.
\]  
(83)

Bounding sum by an integral
\[
V_3 \geq \frac{\alpha^2 n^2 j_0}{2(1 + 2\gamma)^2 j_0^{2+2\gamma}} \cdot \log \frac{\varphi_{01} j_0^{1+\gamma}}{\alpha} + \frac{\alpha^2 n^2 (1 + \gamma)}{2(1 + 2\gamma)^2} \cdot \frac{j_0}{j_0^{2+2\gamma}} \cdot \log e.
\]  
(84)
Using the substitutions above for \( \epsilon_0 \) and \( \epsilon \), and plugging (83) and (84) into (81),

\[
nH_\theta^{(01)}(X) + S_2 \geq nH_\theta(X) - \left(1 + \frac{1}{\gamma} - \frac{1}{3(1 + 2\gamma)}\right) \frac{\alpha^{1+\gamma}}{1+\gamma} \left(1 + o(1)\right) n^{\frac{1}{1+\gamma}} \log n. \tag{85}
\]

Optimization that also includes the bound on \( V_3 \) in (84) yields a slightly greater optimal \( \epsilon_0 > 0 \) (roughly between 0.1 and 0.2) that produces the maximal overall lower bound on \( nH_\theta^{(01)}(X) + S_2 \). However, this bound, while more complex, only negligibly gains on the one in (85) with \( \epsilon_0 = 0 \).

Since \( j_1 = k - k_{01} \), using the simple bound in (14) on \( S_1 \), plugging \( \epsilon = \Theta \left( (\log \log n) / (\log n) \right) \)

\[
S_1 \leq \log (j_1!) = O \left( j_1 \log j_1 \right) = O \left( n^{\frac{1}{1+\gamma}} \log n \right) = o \left( n^{\frac{1}{1+\gamma}} \log n \right). \tag{86}
\]

In a similar manner, \( S_4 = o \left( n^{\frac{1}{1+\gamma}} \log n \right) \). Combining (85), (86), \( S_3 \geq 0 \), and the bound on \( S_4 \) into (13) yields the lower bound in (19).

The lower bound in (19) is asymptotic. To obtain precise curves as in Figure 3 for finite \( n \), \( j_0 \) and \( j_1 \) are computed with (76). Then, either (77)–(78) can be used to bound \( \varphi_0 \) and \( \varphi_{01} \), or they can be computed precisely substituting \( j_0 \) and \( j_1 \). Step (b) of (82) and (84) are used to provide a bound on \( nH_\theta^{(01)}(X) + S_2 \), and more precise bounds are obtained on \( S_1 \) and \( S_4 \). (Alternatively \( V_2 \) can be computed precisely as discussed following (81).) To obtain bounds on \( S_1 \), let \( \tau_b \triangleq \max \left\{ 1, \left[ (\alpha \cdot n^{1-\epsilon} / (b + 1)^2) \right]^{1+\gamma} \right\} \), \( b = 0, 1, \ldots \); be the index of the greatest \( \tilde{\theta}_j \), such that \( \tilde{\theta}_j \leq \xi_{b+1} \). Then, \( \kappa'_1 = \tau_0 - \tau_2 \), and

\[
\kappa'_b = \tau_{b-2} - \tau_{b+1} \leq \frac{\alpha^{1+\gamma} n^{\frac{1-\epsilon}{1+\gamma}} \cdot 6}{(b - 1)(1 + \gamma)} + 1; \quad b = 2, 3, \ldots. \tag{87}
\]

This implies that only for

\[
b \leq \left( \frac{6}{1+\gamma} \right)^{\frac{1+\gamma}{\gamma}} \cdot \alpha^{\frac{1}{1+\gamma}} \cdot n^{\frac{1-\epsilon}{1+\gamma}} + 1 = o \left( n^{\frac{1}{1+\gamma}} \right) \tag{88}
\]

there may be more than a single letter in the bins surrounding bin \( b \) resulting in nonzero summands in (16). Similar derivations can be performed to generate the elements of the sum in (15), and more precise bounds on \( S_4 \) using (22). Bounds are obtained for different values of \( \epsilon \), and the value that attains a maximum is used for every \( \gamma \) and \( n \). Note that \( S_4 \) trades off with \( V_1 + V_2 \) by requiring a greater \( \epsilon \) to guarantee that \( \epsilon'_n \) in (23) diminishes. The choice of \( \vartheta^+ \) and \( \vartheta^- \) also influences the tradeoff (a smaller \( \vartheta^+ - \vartheta^- \) decreases the dominant term of \( S_4 \) in (22)). Roughly, the optimal value of \( \epsilon \) leading to the curves in Figure 3 equals 1.75 to 2 times \((\ln \ln n) / (\ln n)\) for large enough \( n \). The curves in Figure 3 were produces with \( \vartheta^- = e^{-1.97} \), and \( \vartheta^+ = e^{0.98} \), that lead to \( f(\vartheta^-, \vartheta^+) > 0.2 \).

To derive a tight upper bound, (25) is used, where \( \eta \) is built with \( \epsilon = 0 \) and \( \epsilon_0, \epsilon_2 > 0 \). This is
necessary for a tight bound on \( R'_1 \) and a negligible one on \( R'_0 \). First,

\[
nH^{(0,1)}_\theta(X) = nH_\theta(X) + \sum_{j=1}^{j_0-1} n\theta_j \log \varphi_j - \sum_{j=j_0}^{\infty} n\theta_j \log \varphi_j
\]

\[
\leq (a) nH_\theta(X) - n\varphi_1 \log \frac{\varphi_1}{\alpha} - n\varphi_0 \log \frac{\varphi_0}{\alpha} - (1+\gamma)\alpha n \sum_{j=j_1}^{\infty} \frac{\log j}{\gamma^2 j}
\]

\[
\leq (b) nH_\theta(X) - n\varphi_1 \log \frac{\varphi_1}{\alpha} - n\varphi_0 \log \frac{\varphi_0}{\alpha} - (1+\gamma)\alpha n \left\{ \frac{\log j_1}{\gamma j_1} + \frac{\log e}{\gamma^2 j_1} \right\}
\]

\[
\leq (c) nH_\theta(X) - (1-o(1)) n\varphi_1 \log \left( n\varphi_1 e^{1+\gamma} \right) - n\varphi_0 \log \left( n\varphi_0 e^{1+\gamma} \right)
\]

\[
\leq (d) nH_\theta(X) - \frac{\alpha^{1+\gamma}}{\gamma(1+\gamma)} \cdot n\frac{1}{\gamma^{1+\gamma}} \cdot \log \frac{\alpha n}{\frac{1}{\gamma^{1+\gamma}}} + o \left( n\frac{1}{\gamma^{1+\gamma}} \log n \right)
\]

where \((a)\) follows from \((44)\) and the definition of \( \alpha \), \((b)\) follows from bounding a sum by an integral, \((c)\) follows from \((78)\) and \((76)\), and \((d)\) follows from \((79)\) absorbing second order terms.

To bound \( R'_1 \) and \( R'_0 \), similarly to \((77)-(78)\),

\[
\sum_{j=1}^{j_0-1} \theta_j^2 \leq \sum_{j=1}^{j_0-1} \frac{\alpha^2}{j^{2+2\gamma}} \leq \frac{1}{n^2} + \frac{\alpha^2 j_1}{(1+2\gamma)^2 j_1^{2+2\gamma}} \leq \frac{1}{n^2} + \frac{\alpha^{1+\gamma} n^{1+\gamma}}{(1+2\gamma)^2 n^{2+2\gamma}}
\]

\[
\sum_{j=j_0}^{\infty} \theta_j^2 \leq \frac{1}{n^{2+2\epsilon_0}} + \frac{\alpha^{1+\gamma} n^{1+\gamma}}{(1+2\gamma)^2 n^{2+2\epsilon_0}}.
\]

From \((29)\) and \((90)-(91)\), it follows (using \(k_1 \leq j_0 \) and \((78)\)) that

\[
R'_1 \leq \frac{(1+\epsilon_0) \alpha^{1+\gamma} \cdot n^{1+\gamma} \cdot \log n}{2(1+\gamma)(1+2\gamma)} + \frac{\alpha^{1+\gamma} \cdot n^{1+\gamma}}{2(1+2\gamma)} \cdot \log \frac{2e(1+2\gamma)\alpha^{1+\gamma}}{\gamma} + O(\log n)
\]

\[
R'_0 = O \left( \frac{n^{1+\gamma} \log n}{n^{2\epsilon_0}} \right).
\]

While \( R'_0 \) requires a greater \( \epsilon_0 \) to minimize its contribution to the bound, \( R'_1 \) requires a smaller \( \epsilon_0 \) (which implies that \( k_1 \) is smaller). Trading off, a choice of \( \epsilon_0 = \Theta((\log \log n)/\log n) \) is optimal.

Finally, for \( b \geq 2 \) of \( \eta \),

\[
k_b = j_{b-1} - j_b = (1 + o(1)) \alpha^{1+\gamma} \cdot n^{1+\gamma} \left( \frac{1}{b_1^{1+\gamma}} - \frac{1}{(b_1 + 1)^{1+\gamma}} \right)
\]

where \( b_1 \) is the index in \( \eta' \) as defined preceding \((5)\). Specifically, since \( \eta_2 = 1/n \), \( b_1 = n^{\gamma/2}(1+o(1)) \).

Following \((9)\) and \( \theta_j > \eta_2 \), we have \( L_b \geq (1-1/e)k_b \). Using \((27)\) and \( \epsilon_2 = \Theta((\log \log n)/\log n) \),

\[
U \geq (1 + o(1)) \cdot \sum_{b \geq 2} L_b \frac{L_b}{e} \geq (1 + o(1)) \cdot \left( 1 - \frac{1}{e} \right) \cdot \sum_{b \geq 2} k_b \frac{(1-1/e)k_b}{e}.
\]

Since \( b_0 \geq b_1 \to \infty \) as \( n \to \infty \),

\[
k_b \geq (1 + o(1)) \cdot \frac{\alpha^{1+\gamma} \cdot n^{1+\gamma}}{1+\gamma} \cdot \frac{1}{b_1^{1+\gamma}}.
\]
Plugging both (94) and (96) in (95),
\[
U \geq (1 + o(1)) \cdot \left( 1 - \frac{1}{e} \right) \cdot \left\{ \sum_{b \geq 2} k_b \log \frac{(1 - 1/e)\alpha_{1+b/n}^{1+\varepsilon_2} n^{1+\gamma}}{e(1+\gamma)} - \frac{(3 + \gamma)\alpha_{1+b/n}^{1+\varepsilon_2} n^{1+\gamma}}{(1+\gamma)^2} \cdot \sum_{b \geq 2} \frac{\log b}{b^{1+\gamma}} \right\}
\]
\[
\geq (1 + o(1)) \cdot \left( 1 - \frac{1}{e} \right) \cdot \frac{\alpha_{1+b/n}^{1+\varepsilon_2} n^{1+\gamma}}{1+\gamma} \log n - O \left( \varepsilon_2 n^{\frac{1}{1+\gamma}} \log n \right)
\]
\[
\equiv (1 + o(1)) \cdot \left( 1 - \frac{1}{e} \right) \cdot \frac{\alpha_{1+b/n}^{1+\varepsilon_2} n^{1+\gamma}}{1+\gamma} \log n
\]
(97)
where \(a\) follows from (94) and the telescopic property of \(k_b\) in \(b_0\), since \(b_0^2 = n^{\varepsilon_2}\) and \(\varepsilon_2 = \Theta((\log \log n)/\log n)\), and by approximating the second sum by an integral. Then, (b) follows again from the value of \(\varepsilon_2\).

Now, substituting (89), (92)-(93), and (97) in (25) yields the upper bound in (49). Again, for the numerical bounds in Figure 3, \(\varphi_0\) and \(j_b\) are computed and then used with step (b) of (89) and with (90)-(91) and (29). For a tighter bound on \(R'_1\), (28) can also be used directly where \(L_1\) is computed with (8). Then, \(U\) is bounded with (27) using (91) to compute \(k_b\) and (9) to compute \(L_b\). Finally, for each \(\gamma\) and \(n\), values of \(\varepsilon_0\) and \(\varepsilon_2\) that minimize the bound are chosen. The value of \(\varepsilon_0\) is large for smaller \(n\), and decreases with \(n\), roughly following the curve of \((\ln \ln n)/(\ln n)\). This concludes the proof of Theorem 5.

\[\square\]

### 6.3 Geometric Distribution

**Proof of Theorem 6.** Let \(j_0\) and \(j_1\) be the indices of the greatest \(\tilde{\theta}_j \leq \eta_1, \eta_2\), respectively. Then,
\[
\log \frac{pm^{1+\varepsilon_b}}{1-p} \leq j_b = \left\lceil \log \frac{pm^{1+\varepsilon_b}}{1-p} \right\rceil \leq \log \frac{pm^{1+\varepsilon_b}}{(1-p)^2}, \quad b = 0, 1.
\]
(98)

For \(b \geq 2\), define \(j_b = \max \{1, \lceil \log \left\{ pm^{1+\varepsilon_b} / \lceil (b' + 1)^2 (1-p) \rceil \rceil \rangle / \log \lceil -(1-p) \rceil \rceil \} \) as the index of the greatest \(\tilde{\theta}_j \leq \eta_{b+1}\) (where \(b'\) is as defined in (41)-(51)). Note that \(j_b = 1\) for \(b' \geq \sqrt{pm^{1+\varepsilon_2} - 1}\), \(k_b = j_{b-1} - j_b\), and some bins may be empty. From (98),
\[
\varphi_0 = \sum_{j=j_0}^{\infty} p(1-p)^{j-1} = (1-p)^{j_0-1}.
\]
(99)

Similarly,
\[
\varphi_0 = (1-p)^{j_1-1}, \quad \varphi_1 = \varphi_0 - \varphi_0 = \varphi_0 \left\{ 1 - (1-p)^{j_0-j_1} \right\}.
\]
(100)

From (98)-(100), it follows that
\[
\frac{1-p}{pm^{1-\varepsilon}} \leq \varphi_0 \leq \frac{1-p}{pm^{1+\varepsilon_0}} \leq \varphi_0 \leq \frac{1}{pm^{1+\varepsilon_1}} = \frac{1}{pm^{1-\varepsilon}},
\]
(101)

While \(k_0, k_0 = \infty\), if \(pm^{1+\varepsilon_1} > 1-p\), it follows from (98) that
\[
\log \frac{pm^{1+\varepsilon_1}(1-p)}{-\log(1-p)} \leq k_1 = j_0 - j_1 \leq \frac{\log pm^{1+\varepsilon_1}}{-\log(1-p)}.
\]
(102)
Similarly to (99), (100),
\[
\sum_{j=j_0}^{\infty} \tilde{\theta}_j^2 = \frac{p\tilde{\varphi}_0^2}{2-p}, \quad \sum_{j=j_1}^{\infty} \tilde{\theta}_j^2 = \frac{p\tilde{\varphi}_0^2}{2-p}, \quad \sum_{j=j_1}^{j_0-1} \tilde{\theta}_j^2 = \frac{p(\varphi_{01}^2 - \varphi_0^2)}{2-p} = \frac{p\varphi_{01}^2}{2-p} \left(1 - (1-p)^{2k_1}\right) \tag{103}
\]
where (a) follows from (99), (100) and (102).

Now, the lower bound of (13) can be derived by separately bounding its terms. First, \(nH_{\theta}^{(01)}(X) + S_2\) is lower bounded, and \(S_1\) and \(S_4\) upper bounded. Lower bounding (20),
\[
nH_{\theta}^{(01)}(X) + S_2 \geq nH_{\theta}^{(01)}(X) + \left(1 - \frac{1}{3n^\varepsilon_0} - \frac{2}{n}ight) \sum_{j=j_0}^{\infty} \tilde{\theta}_j^2 \log \frac{\varphi_{01}}{\theta_j} + \sum_{j=j_1}^{j_0-1} \left(\tilde{n}\tilde{\theta}_j - 1\right) \log \frac{\varphi_{01}}{\theta_j} \tag{104}
\]
where (a) follows from the definition of \(H_{\theta}^{(01)}(X)\) in (11) and from combining of terms. Each component \(V_\varepsilon\) is now bounded. By definition of \(\tilde{\theta}_j\),
\[
V_1 = n\varphi_0 \log \frac{p}{\varphi_{01}} + np(1-p) [\log (1-p)] \sum_{j=j_0}^{\infty} (j-1) (1-p)^{j-2}
\]
\[
\overset{(a)}{=} n\varphi_0 \log \frac{p}{\varphi_{01}} + n [\log (1-p)] \left\{ (j_0 - 1) (1-p)^{j_0 - 1} + (1-p)^{j_0} / \varphi_0 \right\}
\]
\[
\overset{(b)}{=} n\varphi_0 \log \frac{\varphi_0}{\varphi_{01}} - \frac{n\varphi_0 h_2(p)}{p}
\]
\[
\overset{(c)}{\geq} \frac{\varepsilon + \varepsilon_0}{pn^\varepsilon_0} \log n - \frac{h_2(p)}{p^2 n^\varepsilon_0} = - \left| O \left( \frac{\varepsilon + \varepsilon_0 \log n}{n^\varepsilon_0} \right) \right| \tag{105}
\]
where (a) is obtained by representing each term of the sum as a derivative of \((1-p)^{j-1}\) w.r.t. \((1-p)\), exchanging order of summation and differentiation, and computing a geometric series sum, (b) follows from (99), and (c) follows from the upper bounds of (101) because the expression decreases with \(\varphi_{01}\), and for \(\varphi_0 < \varphi_{01} / e\) also with \(\varphi_0\). From (101),
\[
V_2 = (j_0 - j_1) \log \frac{(1-p)^{j_1}}{p} + \sum_{j=j_1}^{j_0-1} j \log \frac{1}{1-p}
\]
\[
\overset{(a)}{=} k_1 \left( 0.5k_1 \log \frac{1}{1-p} + \log \frac{1}{p} - 0.5 \log \frac{1}{1-p} \right)
\]
\[
\overset{(b)}{\leq} \frac{\varepsilon_0 + \varepsilon}{-2 \log (1-p)} (\log n)^2 + \left( 0.5 + \frac{\log p}{2 \log (1-p)} \right) (\varepsilon_0 + \varepsilon) \log n + \log \frac{1}{p} \tag{106}
\]
where (a) follows from computing the sum in the second term and using the definition of \(k_1\) in (102), and (b) follows from the upper bound on \(k_1\) in (102). Applying similar techniques to those
in (105),

\[ V_3 = \frac{n^2}{2} \left\{ \frac{p_0^2}{2-p} \log \frac{\varphi_{01}}{p_0} + \frac{(1-p)^2}{(2-p)^2} \varphi_0^2 \log \frac{1}{1-p} \right\} \]

\[ \geq \frac{(1-p)^2}{2p(2-p)} \cdot \frac{1}{n^2 e} \left\{ (\varepsilon_0 + \varepsilon) \log n + \log \frac{1}{p} + \left( \frac{(1-p)^2}{p(2-p)} - 1 \right) \log \frac{1}{1-p} \right\} \]

\[ = O \left( \frac{(\varepsilon_0 + \varepsilon) \log n}{n^2 e} \right) \quad (107) \]

where (a) follows from the lower bounds in (101)-(102).

To bound \( S_1 \), let \( \tau_b \) be the index of the greatest \( \tilde{\theta}_j \), such that \( \tilde{\theta}_j \leq \xi_{b+1} \). Similarly to (98),

\[ \tau_b = \max \left\{ 1, \left[ \frac{\log \frac{p_{01}^{1-\varepsilon}}{(b+1)^2(1-p)}}{\log \frac{1}{1-p}} \right] \right\}, \quad b = 0, 1, \ldots \quad (108) \]

Hence, \( \kappa'_1 = \tau_0 - \tau_2 \leq -2(\log 3)/\log(1-p) + 1 \), and

\[ \kappa'_b = \tau_{b-2} - \tau_{b+1} \leq \frac{2 \log \frac{b+2}{b-1}}{\log \frac{1}{1-p}} + 1; \quad b = 2, 3, \ldots, \min \left( B_\xi, \sqrt{p_{01}^{1-\varepsilon}/(1-p) - 2} \right) . \quad (109) \]

For \( b \geq 2 \), \( \kappa'_b \geq \kappa'_{b+1} \). Hence, the maximum bound is obtained for \( b = 2 \), \( \kappa'_2 \leq -4/\log(1-p) + 1 \). Only as long as \( \kappa'_b \geq 2 \), elements of the sum in (16) are nonzero. This is only possible as long as

\[ b \leq \frac{2 + \frac{1}{\sqrt{1-p}}}{\sqrt{1-p} - 1} = \bar{b}_{g,\text{max}} . \quad (110) \]

Since \( k - k_{01} = j_1 - 1 \), from (17), \( \varepsilon_\infty \leq \min \{ 1, n_{j_1} e^{-0.1n^\varepsilon} \} \). Combining these bounds, using (16),

\[ S_1 \leq (1 - \varepsilon_\infty) \left\{ \log \left[ \frac{2 \log \frac{3}{\log \frac{1}{1-p}}}{\log \frac{1}{1-p}} \right] + \sum_{b=2}^{\bar{b}_{g,\text{max}}} \log \left[ \frac{2 \log \frac{b+2}{(b-1)\sqrt{1-p}}}{\log \frac{1}{1-p}} \right] \right\} \]

\[ + \varepsilon_\infty \log (j_1!) + h_2 \left[ \min (0.5, \varepsilon_\infty) \right] . \quad (111) \]

To guarantee that the last two terms diminish at \( O \left( (\log n)^2(\log \log n)/n \right) \) (since \( j_1 = O(\log n) \), \( \varepsilon \geq (1+\delta)(\log \ln n)/(\log n) \), where \( \delta > (\ln 20)/(\ln \ln n) \) must be used, and then, \( S_1 = O(1) \).

An upper bound on \( S_4 \) is derived similarly to that on \( S_1 \). Choosing \( \bar{\vartheta}^- = e^{-5.5} \) and \( \bar{\vartheta}^+ = e^{1.4} \),

\[ k^-_\vartheta + k^+_- \leq \left[ \frac{\log \frac{\bar{\vartheta}^+}{\log \frac{1}{1-p}}}{\log \frac{1}{1-p}} + 1 \right] = \left[ \frac{6.9 \log e}{\log \frac{1}{1-p}} + 1 \right] \quad (112) \]

\[ k^+_\vartheta \leq \left[ \frac{\log \frac{\bar{\vartheta}^+}{\log \frac{1}{1-p}}}{\log \frac{1}{1-p}} + 1 \right] = \left[ \frac{1.4 \log e}{\log \frac{1}{1-p}} + 1 \right] \quad (113) \]

\[ k_{\theta_{>n^{-3}}} = \left[ \frac{\log \frac{p_{01}^{3}}{\log \frac{1}{1-p}}}{\log \frac{1}{1-p}} \right] = O \left( \log n \right) . \quad (114) \]

Plugging these values in (22) with the choice of \( \varepsilon \) above yields \( S_4 = O(1) \), where all terms of (22) but the first diminish with \( n \). (The bound can be tightened by narrowing \([\vartheta^-, \vartheta^+]\). Such narrowing
is limited to decreasing \( f(\vartheta^-, \vartheta^+) \) in (23), such that it still produces diminishing terms in (22).}

Note that if \( \xi \) is redefined by \( \xi \triangleq \{0, 1/n^{1-\varepsilon}\} \bigcup \{\tilde{\vartheta}_j : \tilde{\vartheta}_j > 1/n^{1-\varepsilon}\} \), and \( \varepsilon \) is chosen above with \( \delta > (\ln(20/p^2)) / (\ln \ln n) \), a bound of \( S_1 = o(1) \) can be obtained. This means that for \( n \to \infty \) each letter of \( \vartheta \) is in a single bin by itself. A similar approach yields \( S_4 = o(1) \). This approach, however, results in a larger first term in an overall usually looser lower bound in (51).

Combining (104)-(107), (111), and (22) gives a lower bound on \( H_\vartheta (\Psi^n) \). Choosing \( \varepsilon_0 = 0 \) and \( \varepsilon = (1 + \delta)(\log \ln n) / (\log n) \) with \( \delta > (\ln 20) / (\ln \ln n) \) yields the lower bound of (51).

To numerically compute a lower bound for a finite \( n \) with parameters \( \varepsilon \) and \( \varepsilon_0 \), \( j_0 \) and \( j_1 \) are computed by (98). Then, (99), (100) are used to compute \( \varphi_0 \) and \( \varphi_{01} \). Step (b) of (105) and the first equality of (107) are used to compute \( V_1 \) and \( V_3 \), respectively. Instead of using (106), the summand of (8) is included in the summand of \( V_2 \) in (101), and \( V_2 \) is precisely computed. This is necessary for tighter bounds for very small \( n \) as shown in Table I. Bin count \( b_{g, \text{max}} \) used in (111) to bound \( S_1 \) must be taken as the minimum between its value in (110) and \( \min \{B_\xi, \sqrt{pm^{1-\varepsilon}} / (1 - p - 2)\} \).

Asymptotically, the bounds of (14) and (15) are looser than that of (16) because they produce bounds of \( O((\log n)(\log \log n)) \) and \( O(\log n) \) on \( S_1 \), respectively. However, for practical \( n \), using these bounds may sometimes produce tighter bounds. The tightest bound for \( S_1 \) among those resulting from (14)-(16) can be used for each \( p, \varepsilon \), and \( n \). The sum in (15) is bounded similarly to the sum in (111), where the ratio \( (b + 2) / (b - 1) \) in (111) is replaced by \( (b + 1) / b \) to bound \( \kappa_b \); \( b = 1, 2, \ldots, b_{g, \text{max}} = \min \{A_\xi, \sqrt{pm^{1-\varepsilon}} / (1 - p) - 1, 1 / ((1 - p)^{-0.5} - 1)\} \). Last, \( S_4 \) is bounded with (22), numerically computing (124)-(125). For given \( p \) and \( n \), \( \varepsilon \) and \( \varepsilon_0 \) are numerically optimized to give the tightest bound, resulting in the non-asymptotic curves in Figure 4 and the values in Table I. While asymptotically negligible, \( S_1 \) dominates the bound for small \( p \) and large \( n \). Using precise expressions instead of bounds on \( V_2 \) yields better bounds with larger \( \varepsilon_0 \). Parameter \( \varepsilon \) decreases with \( n \), roughly following the curve of \( 1.5(\ln \ln n) / (\ln n) \).

To derive a tight upper bound, (25) is used, where \( \eta \) is built with \( \varepsilon \leq 0 \) (\( \varepsilon_1 \geq 0 \)). Nonnegative \( \varepsilon_1 \) is necessary to obtain negligible \( R^c_0 \), yet reducing the rate of \( R^c_1 \). A simpler bound can be obtained by using \( U \geq 0 \). The remaining terms of (25) are bounded below. First,

\[
nH_{\vartheta}^{(0,1)} (X) = nH_\vartheta (X) + \sum_{j=j_1}^{j_0-1} n\tilde{\vartheta}_j \log \frac{\tilde{\vartheta}_j}{\varphi_1} + \sum_{j=j_0}^{\infty} n\tilde{\vartheta}_j \log \frac{\tilde{\vartheta}_j}{\varphi_{01}}
\]

\[
= nH_\vartheta (X) + n\varphi_1 \log \frac{p}{\varphi_1} + n\varphi_0 \log \frac{p}{\varphi_0} + np(1-p) \log(1-p) \sum_{j=j_1}^{\infty} (j-1)(1-p)^{j-2}
\]

\[
\overset{(a)}{=} nH_\vartheta (X) - \frac{n\varphi_0 h_2 (p)}{p} + n\varphi_1 \log \frac{\varphi_{01}}{\varphi_1} + n\varphi_0 \log \frac{\varphi_{01}}{\varphi_0}
\]

\[
= nH_\vartheta (X) - \frac{n\varphi_0 h_2 (p)}{p} + n\varphi_0 h_2 \left(\frac{\varphi_0}{\varphi_{01}}\right)
\]

\[
\overset{(b)}{=} nH_\vartheta (X) - \frac{(1-p)h_2 (p)}{p^2 n^{\varepsilon_1}} + n\varphi_0 \log \frac{\varphi_{01}}{\varphi_0}
\]

\[
\overset{(c)}{=} nH_\vartheta (X) - \frac{(1-p)h_2 (p)}{p^2 n^{\varepsilon_1}} + \frac{1}{pm^{\varepsilon_0}} \log \frac{en^{\varepsilon_0 - \varepsilon_1}}{1 - p}
\]

(115)

where (a) follows from the same reasons as (a)-(b) in (105), (b) follows from (101) and Taylor expansion on the last term, and (c) follows again from (101).
From (29) and (103),

\[ R_0' \leq \frac{p}{2(2-p)} \cdot n^2 \phi_0^2 \cdot \log \frac{2e(2-p)}{p \phi_0} \leq \frac{1}{2(2-p)p} \cdot \frac{1}{n^{2\xi_0}} \cdot \log \frac{2e(2-p)n^{1+\xi_0}}{1-p} = O \left( \frac{\log n}{n^{2\xi_0}} \right) \]  \hspace{1cm} (116)

where (a) again follows from (101). In a similar manner,

\[ R_1' \leq \frac{p}{2(2-p)} \cdot n^2 \cdot (\phi_0^2 - \phi_0^2) \cdot \log \frac{2e\phi_0 k_1(2-p)}{np(\phi_0^2 - \phi_0^2)} \leq \frac{p\phi_0^2 n^2}{2(2-p)} \cdot \log \frac{2e(2-p)k_1}{np\phi_0} \left( 1 - \frac{1}{(1-p)^2 n^2(\epsilon_0 - \epsilon_1)} \right) \]

\[ \leq \frac{1}{2(2-p)pn^{2\xi_1}} \cdot \left[ \epsilon_1 \log n + \log \log n + \log \frac{2e(2-p)(\epsilon_0 - \epsilon_1)}{(1-p) \log \frac{1}{1-p}} \right] + O \left( \frac{1}{n^{2\xi_0}} \right) \] \hspace{1cm} (117)

where (a) follows from \( \phi_0^2 - \phi_0^2 \leq \phi_0^2, \phi_1 \leq \phi_0, \) and since \( \phi_0/\phi_0 = (1-p)^{j_0-j_1} \leq 1/[(1-p)n^{\epsilon_0-\epsilon_1}] \) following (99)-(100) and (102), and (b) follows from bounding \( \phi_0 \) with (101) and \( k_1 \) with (102). Note that the logarithmic bound on \( k_1 \) reduces the rate of \( R_1' \). This is the reason that two separate bins with positive \( \epsilon_0 \) and \( \epsilon_1 \) are used. With proper choices of these parameters, \( R_0' \) becomes negligible, yet, bin 0 holds most symbols, leaving only a logarithmic number of symbols in bin 1.

Summing (115), (116) and (117), a parametric upper bound on \( H_\theta(\Psi^n) \) is obtained. Substituting a constant to \( \epsilon_0 \) and letting \( \epsilon_1 = (\log \log n)/(\log n) \), where \( \epsilon_0 - \epsilon_1 \leq 1 \), gives the upper bound of (51). The dominant terms are the first two of (115) and those of (117), and \( R_0' \) is negligible. The upper bound can be tightened by lower bounding \( U \) of (25) using (27). The limits of the sum and its elements can be lower bounded in a similar manner to the derivation for \( S_1 \) in (108)-(111). Since \( L_b \geq k_b(1-e^{-n\theta_j}) = O(1/n^{\epsilon_1}) = O(1/(\log \log n)) \) when \( k_b \geq 2 \) this does not change the rate of the bound. This additional term was used together with the last equality of (115) and the first inequality of (116) to produce the non-asymptotic bounds in Figure 4 where, again, \( \epsilon_b \) were numerically optimized. Instead of using (117), the value of \( R_1' \) was computed precisely with (28), where \( L_1 \) was computed with (8). This was necessary to achieve tight bounds for small \( n \) as shown in Table 1. The “simple” bound in Figure 4 does not include the \( U \) term. For very small \( p \), this term does generate more significant gain. For example, for \( n = 10^5 \), and \( p = 0.01 \), out of at least 1561 bits of decrease from \( nH_\theta(X) \), 1017 result from the term \( U \) (i.e., multiple letters in bins \( b > 1 \) of \( \eta \)). However, for greater \( p \) the gain from \( U \) diminishes, because very few bins \( b > 1 \) (if any) contain more than a single letter. \( \Box \)

### 6.4 Linear Monotonic Distributions

**Proof of Theorem 7** Let \( \epsilon_2 = \epsilon = \Theta((\log \log n)/\log n) \ll \delta/2 \). Let \( i_b \) be the smallest \( i \), such that \( \theta_i \geq \xi_b \), and \( \ell_b \) be the smallest \( i \), such that \( \theta_i \geq \eta_b \). Hence,

\[ i_b = \left\lceil \frac{b^2 n^{1+\epsilon}}{2\lambda^2} + \frac{1}{2} \right\rceil, \quad b = 1, 2, \ldots, \quad \ell_b = \left\lceil \frac{b^2 n^{1-\epsilon_2}}{2\lambda^2} + \frac{1}{2} \right\rceil, \quad b = 3, 4, \ldots \] \hspace{1cm} (118)
where $b'_b$ is the proper index in $\eta'$ corresponding to index $b$ in $\eta$ (as defined in (5)), $\ell_2 \triangleq i_1$, and $i_0 = \ell_1 = \lceil n^{1-\varepsilon_0}/(2\lambda^2) + 0.5 \rceil$. It follows that

$$
\varphi_{01} = \begin{cases} 
0; & \text{if } \lambda \geq \sqrt{n^{1+\varepsilon}}/\sqrt{3} \\
\frac{\lambda^2}{2\lambda^2} (i_1 - 1)^2 = \frac{n^{2\varepsilon}}{4\lambda^2} \left( 1 + O \left( \frac{\lambda^2}{n^{1+\varepsilon}} \right) \right); & \text{if } \frac{n^{\varepsilon}}{2} \leq \lambda < \sqrt{n^{1+\varepsilon}}/\sqrt{3} \\
1; & \text{if } \lambda < 0.5n^{\varepsilon}. 
\end{cases}
$$

(119)

In the first region, $\lambda \geq n^{2/3+\delta}$, implying $k \leq n^{1/3-\delta}$. Using the trivial upper bound $H_\theta (\Psi^n) \leq nH_\theta (X)$. From (119), $\varphi_{01} = 0$. Hence, $S_2, S_3, S_4 = 0$, and $H_\theta^{(01)} (X) = H_\theta (X)$. Only $S_1$ remains for using (13). Since $\kappa'_b = i_b + 2 - i_{b-1}$, let

$$
\kappa'_b = \frac{n^{1+\varepsilon}}{2\lambda^2} ((b + 2)^2 - (b - 1)^2) = \frac{3n^{1+\varepsilon}}{\lambda^2} \left( b + \frac{1}{2b} \right) 
$$

(120)

be the unrounded value computed to obtain $\kappa'_b$. We must have $\kappa'_b \geq 1$ so that a summand in the dominant sum of (16) is not zero. This implies that such summands only exist for $b \geq b'_{\min}$, where

$$
b'_{\min} \geq \frac{\lambda^2}{3n^{1+\varepsilon}} (1 + o(1)) \geq \frac{1}{3} \cdot n^{\frac{1}{3} + 2\delta - \varepsilon} (1 + o(1))
$$

(121)

where (a) follows from $\lambda \geq n^{2/3+\delta}$. However, for the maximal probability, $\theta_k = 2\lambda^2(k - 0.5)/n^2 \geq \lambda^2/n^{1-\varepsilon}$. Thus the maximal populated bin has index $b_{\max} \leq \sqrt{2\lambda^2/n^2} \ll \lambda^2/(3n^{1+\varepsilon})$, where the last relation follows from $\lambda \geq n^{2/3+\delta}$ and since $\varepsilon \ll \delta$. Using (16), this implies that $S_1 = o(1)$.

For the second region, $S_3 \geq 0$, and lower bounding (19),

$$
H_\theta^{(01)} (X) + S_2 \geq nH_\theta (X) - \sum_{i_1 = 1}^{i_1 - 1} \log \frac{\varphi_{01}}{\theta_i} = nH_\theta (X) - O (i_1 \log i_1).
$$

(122)

Similarly to (120), for large $b$,

$$
\kappa_b = i_{b+1} - i_b = (1 + o(1)) \frac{n^{1+\varepsilon}b}{\lambda^2}.
$$

(123)

Defining $\tilde{\kappa}_b$ similarly to $\kappa'_b$ but w.r.t. $\kappa_b$, and requiring $\tilde{\kappa}_b \leq 2$ for 0 terms in the sum of $S_1$ leads to $b_{\min} \leq 2\lambda^2/n^{1+\varepsilon}$, where $b_{\min}$ is defined as $b'_{\min}$ but w.r.t. $\kappa_b$. Using (15), the sum of $S_1$ is

$$
\sum_{b = 1}^{b_{\max}} \log (\kappa_b!) \overset{(a)}{=} \sum_{b = b_{\min}}^{b_{\max}} \kappa_b \log \frac{\kappa_b}{e} + \frac{1}{2} \sum_{b = b_{\min}}^{b_{\max}} \log \kappa_b + O (b_{\max})
$$

$$
\overset{(b)}{=} (1 + o(1)) \begin{cases} 
(k - i_{b_{\min}}) \log \frac{n^{1+\varepsilon}}{\lambda^2 e} + \frac{n^{1+\varepsilon}}{\lambda^2} \sum_{b = b_{\min}}^{b_{\max}} b \log b \end{cases}
$$

$$
\overset{(c)}{=} (1 + o(1)) \begin{cases} 
(k - i_{b_{\min}}) \log \frac{n^{1+\varepsilon}}{\lambda^2 e} + \frac{n^{1+\varepsilon}}{\lambda^2} \left( \frac{b_{\max}^2}{2} \log \frac{b_{\max}}{\sqrt{e}} - \frac{b_{\min}^2}{2} \log \frac{b_{\min}}{\sqrt{e}} \right) \end{cases}
$$

$$
\overset{(d)}{\leq} (1 + o(1)) \begin{cases} 
\frac{n}{\lambda} \log \frac{n^{1+\varepsilon}}{\lambda^2 e} + \frac{n}{\lambda} \log \frac{2\lambda}{\sqrt{en^{1+\varepsilon}}} \end{cases}
$$

$$
= (1 + o(1)) \frac{n}{\lambda} \log \frac{\sqrt{2} n^{1+\varepsilon}/2}{\lambda^{3/2} e^{3/2}}
$$

(124)
where $(a)$ follows from Stirling’s approximation in (64), $(b)$ from (123), $(c)$ from approximating the sum by an integral, and $(d)$ since $k = n/\lambda$, $i_{b_{\min}} \leq 2\lambda^2/n^{1+\varepsilon}$ (which follows from (115)), and $b_{\max} \leq \frac{\sqrt{2N}}{\varepsilon^2}$. The terms that result from $i_{b_{\min}}$ and the lower limit of the integral are of second order. (By definition of the region, $\lambda^3 \leq n^2 - 3\delta \ll n^{2+\varepsilon}/2$, which implies that $i_{b_{\min}} \leq 2\lambda^2/n^{1+\varepsilon} \ll n/\lambda = k$. The upper limit on $\lambda$ also results in $b_{\min} \ll b_{\max}$.)

By definition in Theorem 1 and from (118),

$$S_4 = O(i_1) = O\left(\frac{n^{1+\varepsilon}}{a}\right) \leq o\left(\frac{n}{\lambda}\right)$$

(125)

where $(a)$ follows from the choice of $\varepsilon$ and since $n^\varepsilon \ll n^\delta/2 \leq \lambda$. Following (125), the last term in (122) is $O(i_1 \log i_1) = o(S_1)$. Hence, combining all terms of (13)

$$(127)
\begin{align*}
H_\theta(\Psi^n) \geq nH_\theta(X) - (1 + o(1)) \cdot \frac{n}{\lambda} \log \frac{3\sqrt{2n^{1+\varepsilon}/2}}{\lambda^{3/2}e^{3/2}}
\end{align*}

(126)

where the additional 3 in the argument of the logarithm follows from the second term of (15). With a choice of $\varepsilon = \Theta((\log \log n)/(\log n))$, this leads to the lower bound of (57) in this region.

For an upper bound in the second region, $H_\theta^{(01)}(X) \leq H_\theta(X)$. Using (25),

$$R_{01}' = O\left(n\varnothing_0 \log n \right) = O\left(\frac{n^1+\varepsilon}{\lambda^2} \log n\right) \leq o\left(\frac{n}{\lambda}\right)$$

(127)

where $(a)$ follows from (119) and $(b)$ from $n^2 \varepsilon \ll \lambda$ in this region. A lower bound on $U$ is obtained following the same steps as (124), where $-\varepsilon_2$ replaces $\varepsilon$. Plugging $\varepsilon_2 = \Theta((\log \log n)/(\log n))$, using (26), yields the upper bound of (57).

For the third region, let $\varepsilon_0 = -\log(2\lambda)/(\log n)$. This leads to $\eta_1 = 2\lambda/n$. Hence, since $\theta_k \leq 2\lambda/n$, $H_\theta^{(01)} = S_1 = S_4 = U = 0$. With looser bounding, also $S_3 \geq 0$. From (20)

$$S_2 \geq (1 + o(1)) \cdot \left(1 - \frac{2\lambda}{3}\right) \cdot \frac{n^2}{2} \sum_{i=1}^{k} \theta_i^2 \log \frac{1}{\theta_i}
$$

$$= (1 + o(1)) \cdot \left(1 - \frac{2\lambda}{3}\right) \cdot \frac{2\lambda^4}{n^2} \sum_{i=1}^{k} (i - 0.5)^2 \log \frac{n^2}{2(i - 0.5)\lambda^2}
$$

$$\stackrel{(a)}{=} (1 + o(1)) \cdot \left(1 - \frac{2\lambda}{3}\right) \cdot \frac{2\lambda^4 k^3}{3n^2} \log \frac{e^{1/3} \eta_n^2}{2\lambda^2 k}
$$

$$\stackrel{(b)}{=} (1 + o(1)) \cdot \left(1 - \frac{2\lambda}{3}\right) \cdot \frac{2}{3} \lambda n \log \frac{e^{1/3} \eta_n}{2\lambda}
$$

(128)

where $(a)$ follows from approximating the sum by an integral and since $n \to \infty$, and $(b)$ from substituting $k = n/\lambda$. To use (26), approximating a sum by an integral

$$\sum_{i=1}^{k} \theta_i^2 = (1 + o(1)) \cdot \frac{4\lambda^4}{n^4} \cdot \frac{k^3}{3} = (1 + o(1)) \cdot \frac{4}{3} \cdot \frac{\lambda}{n}.
$$

(129)

It then follows using (29) that

$$R_{01}' \leq (1 + o(1)) \cdot \frac{2}{3} \lambda n \log \frac{3en}{2\lambda}.
$$

(130)

Since all other terms but $S_2$ for the lower bound in (13) and $R_{01}'$ for the upper bound in (26) are 0 or bounded by 0, both bounds are proved from (128) and (130) for the third region of (57). □
7 Summary and Conclusions

Tight bounds on the entropy of patterns of i.i.d. sequences were used to provide asymptotic and non-asymptotic approximations of the pattern block entropies for several distributions. The finite block pattern entropy was approximated for blocks of data generated by uniform distributions and monotonic distributions. Monotonic distributions studied include slowly decaying distributions over the integers, the Zipf distribution, the geometric distribution, and a linearly increasing distribution. Specifically, the pattern entropy was bounded for distributions that have infinite i.i.d. entropy rates. Conditional next index entropy was studied for distributions over small alphabets.

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