Finding Linear Dependencies in Integration-By-Parts Equations: A Monte Carlo Approach

Philipp Kant

aHumboldt-Universität zu Berlin, Institut für Physik, Newtonstraße 15, 12489 Berlin

Abstract

The reduction of a large number of scalar integrals to a small set of master integrals via Laporta’s algorithm is common practice in multi-loop calculations. It is also a major bottleneck in terms of running time and memory consumption. It involves solving a large set of linear equations where many of the equations are linearly dependent. We propose a simple algorithm that eliminates all linearly dependent equations from a given system, reducing the time and space requirements of a subsequent run of Laporta’s algorithm.

Keywords: Feynman Diagram Reduction, Laporta Algorithm, Redundancy, Dependent Systems of Linear Equations, Monte Carlo, Homomorphic Images

PACS: 12.38.Bx

PROGRAM SUMMARY

Program Title: ICE—the IBP Chooser of Equations
Available From: http://www.physik.hu-berlin.de/pep/tools
Programming language: Haskell
Computer: any system that hosts the Haskell Platform
Operating system: GNU/Linux, Windows, OS/X
Keywords: Multiloop Calculation, Laporta Algorithm, Integration-By-Parts
Classification: 4.4, 4.8, 5, 11.1
Nature of problem: find linear dependencies in a system of linear equations with multivariate polynomial coefficients. To be used on Integration-By-Parts identities before running Laporta’s Algorithm.
Solution method: map the system to a finite field and solve there, keeping track of the required equations.
Restrictions: typically less than the restrictions imposed by the requirement of being able to process the output with Laporta’s Algorithm.
Unusual features: complexity increases only very mildly with the number of kinematic invariants.
Running time: depends on the individual problem. Fractions of a second to a few minutes have been observed in tests.

Email address: philipp.kant@physik.hu-berlin.de (Philipp Kant)
1. Introduction

In multi-loop calculations, one often finds that the expression for a given Feynman diagram, after tensor decomposition, is given in terms of a very large number of integrals of the form

\[ I(\nu_1, \ldots, \nu_n) = \int d^{d}k_1 \cdots d^{d}k_n \frac{1}{D_1^{\nu_1} \cdots D_n^{\nu_n}}. \]  

(1)

Here, \( \nu_i \in \mathbb{Z} \) are called the indices of a given integral. The \( D_i \) are polynomials of total degree 2 in the loop momenta \( k_i \) and any external momenta and masses. Integrals with different indices satisfy a set of linear relations, and it is desirable to express a diagram using a minimal set of linearly independent integrals, the so-called master integrals.

One source of linear equations relating different integrals are the Integration-By-Parts (IBP) identities of [1, 2]. They are a consequence of translational invariance of the integral. Additional relations are obtained from Lorentz invariance (LI) [3]. Both IBP and LI equations relate an integral with indices \{\nu_i\} to integrals where some of the \{\nu_i\} are shifted. The coefficients are multivariate polynomials of total degree at most 1 in scalar products of external momenta, squared masses, and the space-time dimension \( d \).

Laporta [4] has given an algorithm that systematically solves IBP and LI identities to reduce a given set of integrals to a linearly independent set. Underlying the algorithm is the observation that allowing larger indices, the number of integrals grows slower than the number of IBP and LI identities relating these integrals with each other. At some point, the rank \( r \) of the system is sufficiently large that all integrals within a certain range of indices can be reduced to a small number of master integrals.

Laporta’s algorithm proceeds by defining an order on the set of integrals that corresponds roughly to the difficulty of calculating them. In each step of the algorithm, one equation is solved for the most “difficult” integral, and the equations solved in earlier steps are inserted. Finally, all integrals are expressed through a set of “simple” master integrals. The algorithm has become a standard procedure in higher order calculations, and several public implementations [5, 6, 7, 8, 9] are available.

There are two inconveniences that cause Laporta’s algorithm to be resource hungry. One is intermediate expression swell: starting with polynomial coefficients of low degree, the process of solving and substituting leads to equations over rational functions of high degree, and with large coefficients. Intermediate expressions are usually much larger than the final answer and can challenge the available memory and disk space. In order to mitigate the growth of coefficients and minimise memory usage, the intermediate expressions are regularly simplified, so that the overall running time of the algorithm is dominated by multivariate gcd calculations and rational function simplification.

This build-up of large intermediate expressions is amplified by the second problem: the number of IBP and LI equations relating a given set of integrals is much larger than their rank, the number of equations that are linearly independent. Consequently, much time is spent processing redundant information, effectively calculating a lot of zeros. Eliminating the redundancy in the linear system has the potential to reduce the demands on CPU time and memory.

The problem of identifying linearly dependent equations beforehand has seen some investigation. For instance, Lee [10] gives selection criteria based on the group structure of the IBP and LI identities. We follow a different approach and propose an algorithm that detects linear dependencies in a given set of IBP and LI equations, thus reducing the time and space...
requirements of a subsequent run of Laporta’s algorithm. Our algorithm is randomised in the Monte Carlo sense, i.e., it has deterministic running time and gives the correct answer with high probability.

2. The Algorithm

We now present an algorithm that removes any redundant equations from a system of linear equations with multivariate polynomial coefficients. In the case of Laporta’s algorithm, this can drastically reduce the size of the system, and thus the required CPU time and memory.

The basic idea is this: writing the system in matrix form, where each column corresponds to one integral and each row to a linear relationship between integrals, and solving by Gaussian elimination would reduce linearly dependent rows to zero during the forward elimination, allowing the identification and removal of redundant equations. But there would be no gain: determining the minimal set of equations would require the solution of the whole system in the first place.

However, the cost of Gaussian elimination can be reduced by mapping the coefficients homomorphically to a simpler domain. As long as the homomorphism does not reduce the rank of the system, one can still read off which equations are redundant. We follow the canonical choice of using \( \mathbb{F}_p \), the field of integer numbers modulo a prime \( p \). In this way, the Gauss algorithm does not suffer from intermediate expression swell, and no gcd calculations are necessary.\(^1\)

**Algorithm 1** Get a maximal linearly independent subset of a given system of linear equations over \( \mathbb{Z}[x_1, \ldots, x_s] \).

**Input:** \( A \), an \( n \times m \) matrix over \( \mathbb{Z}[x_1, \ldots, x_s] \).

**Output:** \( B \), an \( r \times m \) submatrix of \( A \) with linearly independent rows, where \( r \leq \text{rank} \ A \). With high probability, \( r = \text{rank} \ A \).

1: \( p \leftarrow \) a large prime number
2: \( A' \leftarrow A \mod p \in \left( \mathbb{F}_p[x_1, \ldots, x_s] \right)^{\times m} \)
   \( \triangleright \) Take the residue \( \mod p \) of every coefficient of every polynomial.
3: \( a_1, \ldots, a_s \leftarrow \) random points from \( \mathbb{F}_p \)
4: \( A'' \leftarrow A'(a_1, \ldots, a_s) \in \mathbb{F}_p^{\times m} \)
   \( \triangleright \) Evaluate every entry of \( A' \) at the point \( (x_1 = a_1, \ldots, x_s = a_s) \mod p \).
5: Perform forward Gauss elimination on \( A'' \). Before each step, perform a row permutation to get a non-zero pivot element. Let \( I = \{i_1, \ldots, i_t\} \) be the resulting permutation of rows, and \( r \) the number of non-zero rows after Gaussian elimination (i.e., the rank of \( A'' \)).
6: \( B \leftarrow \) the matrix consisting of rows \( i_1, \ldots, i_r \) of \( A \)

The resulting algorithm is depicted in Algorithm 1. The operation of taking the modulus of \( A \) in step 2 is meant to be element-wise: we take the modulus of each coefficient of each polynomial in the matrix. Likewise, the evaluation of the matrix in step 4 is meant as an evaluation (within \( \mathbb{F}_p \)) of every polynomial.

\(^1\)A pedagogical introduction to the technique of homomorphic images can be found, for example, in [11, 12].
It should be noted that in addition to identifying a maximal linearly independent set of equations, the algorithm also identifies the master integrals: any column that does not contain a pivot element corresponds to an integral that cannot be reduced with the given set of equations. Of these, some will be integrals with large indices that could be solved with additional equations, and the others will be the master integrals.

2.1. Simple Example

In order to illustrate the algorithm, we give a simple example. Consider

$$A = \begin{pmatrix} x & x + y & 1 & 0 \\ 5x & 3y & 0 & x \\ -4x & x - 2y & 1 & -x \\ 0 & x & y & 3x \\ x & 2x + y & y + 1 & 3x \end{pmatrix} \in \mathbb{Z}[x,y]^5 \times 4.$$  

Choosing \( p = 29 \) and evaluating at \( x = 6, y = 26 \) yields

$$A'' = \begin{pmatrix} 6 & 3 & 1 & 0 \\ 1 & 20 & 0 & 6 \\ 5 & 12 & 1 & 23 \\ 0 & 6 & 26 & 18 \\ 6 & 9 & 27 & 18 \end{pmatrix} \in \mathbb{Z}[x,y]^{5 \times 4}_{29}.$$  

Performing Gaussian forward elimination, we get

$$\begin{pmatrix} 6 & 3 & 1 & 0 \\ 0 & 5 & 24 & 6 \\ 0 & 24 & 5 & 23 \\ 0 & 6 & 26 & 18 \\ 0 & 6 & 26 & 18 \end{pmatrix} \rightarrow \begin{pmatrix} 6 & 3 & 1 & 0 \\ 0 & 5 & 24 & 6 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 6 & 3 & 1 & 0 \\ 0 & 5 & 24 & 6 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$  

Before the last step, we had to exchange the third and fourth row. This tells us that the first, second and fourth rows of \( A \) are linearly independent, and we return

$$B = \begin{pmatrix} x & x + y & 1 & 0 \\ 5x & 3y & 0 & x \\ 0 & x & y & 3x \end{pmatrix} \in \mathbb{Z}[x,y]^3 \times 4.$$  

2.2. Probability of Failure

The algorithm will always return \( r \) linearly independent rows of \( A \). This follows because the rows \( i_1, \ldots, i_r \) of \( A'' \) are linearly independent by construction, and \( A'' \) is obtained from \( A \) by evaluation and taking the residue modulo \( p \), which cannot remove a linear dependency. In unlucky cases, however, the rank can be decreased while going from \( A \) to \( A'' \). In this case, we erroneously discard too many equations and wind up with more master integrals than necessary. It is rather unlikely that this actually happens, and we can give an upper bound on the probability of running into such an unlucky case.

In order to derive this bound, let us consider a modification of the algorithm: instead of performing Gaussian elimination on \( A'' \), we perform fraction-free Gaussian elimination on the
original matrix $A$, and map the result to $\mathbb{F}_p$. The rank of the resulting matrix can only be
reduced if one of the pivot elements is mapped to zero by evaluation at $a_1, \ldots, a_n$ or by taking
the residue $\pmod{p}$. A bound on the probability for this can be derived using the Schwartz-
Zippel lemma [13, 14, 15]: the value of a non-zero polynomial of total degree $d$ at a point taken
randomly from a set of cardinality $p$ is zero with probability at most $\frac{d}{p}$.

The degree of the $ith$ pivot element in a fraction-free Gaussian elimination is bounded by
$i\delta$, where $\delta$ is the maximal degree of entries in the initial matrix (see, for example [16]). In the
case of IBP equations, the entries of the matrix $A$ are of degree at most one, so $\delta = 1$, and the
probability that none of the pivot elements are accidentally evaluated at a zero is at least

$$P(\text{success}) \geq \prod_{i=1}^{r} \left(1 - \frac{i}{p}\right). \quad (2)$$

In order to transfer this bound to our algorithm, where we take the residue and evaluate
before the elimination step, we have to make sure that these homomorphisms commute with
the operation of performing Gaussian elimination. Generally, this is not the case. However, if
a homomorphism does not map any of the pivot elements to zero — which is exactly the case
we considered — that homomorphism does commute with the elimination step [17].

By choosing a large prime $p$ in (2), we can get a high probability of finding the maximal
set of linearly independent equations. Of course, machine restrictions may prevent us from
choosing arbitrarily large primes, so for very large systems we might not be able to get (2) as
large as we would want. In such a case, it is always possible to run the algorithm multiple times,
keeping the result with the maximal rank, decreasing the probability of failure exponentially.

The estimate (2) seems to be a rather conservative bound. In testing the algorithm, we have
deliberately used rather small primes in order to increase the calculated maximal probability
of failure to nearly one, without observing an actual breakdown of the algorithm.

3. Implementation and Tests

To demonstrate the effectiveness of our algorithm, we have written an implementation
in Haskell [18]. The close resemblance of its terse syntax to mathematical notation, along
with its flexible type system makes it very convenient to express algorithms in this language.

We call our program ICE, the IBP Chooser of Equations. It is available for downloading at
http://www.physik.hu-berlin.de/pep/tools

In order to test our algorithm, we generate equations for some well-studied but non-trivial
diagrams and filter them with our program. The generation of the equations has been per-
formed with two codes, CRUSHER by Peter Marquard, and a program under development by
Johann Usovitsch.

The diagrams used in our tests are depicted in Figure 1. We generate equations to reduce
integrals with up to a certain number of dots (additional powers of propagators). We note the
number of generated equations and the number of equations our program selects as linearly
independent. We check our results by observing that the master integrals found by ICE co-
incide with those in the literature [22, 23, 24, 25]. The results are shown in the table below.
Depending on the individual system, we see that about one half up to three quarters of the
equations are eliminated.

---

2 Interesting applications of Haskell to scientific computing can be found, for example, in [19].

5
Figure 1: The three- and four-loop vacuum topologies BM (left) and H (right). The solid lines are massive, the dashed are massless. The names follow the notations in [22, 23, 24, 25].

| Topology | Dots | Equations | Independent Equations | Ratio |
|----------|------|-----------|-----------------------|-------|
| H        | 1    | 10464     | 5767                  | 0.55  |
| H        | 2    | 39600     | 18626                 | 0.47  |
| BM       | 3    | 3114      | 1148                  | 0.37  |
| BM       | 10   | 113571    | 28851                 | 0.25  |

3.1. Selecting Specific Equations

While the number of linearly independent equations is fixed for a given system, there is some arbitrariness in the choice of which equations to keep. Depending on the specifics of the implementation of Laporta’s algorithm, a clever selection can have a great impact on the running time of the reduction. In ICE, we try to minimise the number of entries below the diagonal of the resulting system, bringing the system as close to an upper triangular form as possible before solving any equations.

3.2. Optional Backwards Elimination

We have implemented the possibility to perform not only a forward, but also a backwards elimination. This allows to determine which master integrals appear in the reduction of each scalar integral.

4. Conclusions

The computational cost of Laporta’s algorithm is driven by two inconveniences: intermediate expression swell and a large amount of redundant information in the input. We have described a simple algorithm to deal with the latter problem by selecting a maximal linearly independent set of equations from the input. It is a Monte Carlo algorithm, i.e., it has deterministic running time and gives the correct answer with high probability. In particular, the cost of running the algorithm is virtually independent of the number of kinematic invariants in the problem: after evaluating the initial IBP equations at a random point, all arithmetic is performed in a finite field.

Our algorithm determines, en passant, a set of master integrals needed for the reduction of a specific class of diagrams. With some additional effort, it is also possible to determine which master integrals will appear in the result for specific integrals.

Recently, there has been some work on orthogonal methods to determine the set of master integrals without performing a full reduction [26, 27]. With such algorithms, it should be possible to guarantee success of our algorithm (at the possible expense of additional computing time due to repeated runs) by comparing the identified master integrals with their predicted number. In other words, the methods of [26, 27] deliver a criterion for the correctness of the result of our algorithm, so that it can be turned from a Monte Carlo to a Las Vegas algorithm.
5. Acknowledgements

We thank Peter Marquard and Johann Usovich for help in testing the algorithm, and Elisabeth Kant, Robert Schabinger, Matthias Steinhauser, A.V. and V.A. Smirnov, J. Bas Tausk, and Peter Uwer for valuable comments on the draft of this paper. This work was supported by the DFG through SFB/TR9.

References

[1] F. Tkachov, A Theorem on Analytical Calculability of Four Loop Renormalization Group Functions, Phys.Lett. B100 (1981) 65–68. doi:10.1016/0370-2693(81)90288-4
[2] K. Chetyrkin, F. Tkachov, Integration by Parts: The Algorithm to Calculate beta Functions in 4 Loops, Nucl.Phys. B192 (1981) 159–204. doi:10.1016/0550-3213(81)90199-1
[3] T. Gehrmann, E. Remiddi, Differential equations for two loop four point functions, Nucl.Phys. B580 (2000) 485–518. arXiv:hep-ph/9912329 doi:10.1016/S0550-3213(00)00223-6
[4] S. Laporta, High precision calculation of multiloop Feynman integrals by difference equations, Int.J.Mod.Phys. A15 (2000) 5087–5159. arXiv:hep-ph/0102033 doi:10.1016/S0217-7792(00)00199-1
[5] T. Gehrmann, E. Remiddi, Differential equations for two loop four point functions, Nucl.Phys. B580 (2000) 485–518. arXiv:hep-ph/9912329, doi:10.1016/S0550-3213(00)00223-6
[6] A. Smirnov, Algorithm FIRE – Feynman Integral REduction, JHEP 0810 (2008) 107. arXiv:0807.3243 doi:10.1088/1126-6708/2008/10/107
[7] C. Studerus, Reduce-Feynman Integral Reduction in C++, Comput.Phys.Commun. 181 (2010) 1293–1300. doi:10.1016/j.cpc.2010.03.012
[8] A. von Manteuffel, C. Studerus, Reduce 2 - Distributed Feynman Integral Reduction [arXiv:1012.4330]
[9] A. Smirnov, V. Smirnov, FIRE4, LiteRed and accompanying tools to solve integration by parts relation [arXiv:1302.5885]
[10] R. Lee, Group structure of the integration-by-part identities and its application to the reduction of multiloop integrals, JHEP 0807 (2008) 031. arXiv:0804.3008 doi:10.1088/1126-6708/2008/07/031
[11] R. Zippel, Effective polynomial computation, Vol. 241, Springer, 1993.
[12] J. V. Z. Gathen, J. Gerhard, Modern Computer Algebra, 2nd Edition, Cambridge University Press, New York, NY, USA, 2003.
[13] R. A. Demillo, R. J. Lipton, A probabilistic remark on algebraic program testing, Information Processing Letters 7 (4) (1978) 193 – 195. doi:http://dx.doi.org/10.1016/0020-0190(78)90067-4
[14] R. Zippel, Probabilistic algorithms for sparse polynomials, in: E. Ng (Ed.), Symbolic and Algebraic Computation, Vol. 72 of Lecture Notes in Computer Science, Springer Berlin Heidelberg, 1979, pp. 216–226. doi:10.1007/3-540-09519-5_73
[15] J. T. Schwartz, Fast probabilistic algorithms for verification of polynomial identities, Journal of the ACM (JACM) 27 (4) (1980) 701–717.
[16] K. Geddes, S. Czapor, G. Labahn, Algorithms for computer algebra, Kluwer Acad., Boston [u.a.], 1992.
[17] M. T. McClellan, The exact solution of systems of linear equations with polynomial coefficients, J. ACM 20 (4) (1973) 563–588. doi:10.1145/321784.321787
[18] S. Peyton Jones, Haskell 98 language and libraries: the revised report, Cambridge University Press, Cambridge U.K.; New York, 2003. URL: http://www.ams.org/sam/1998/1998021226143
[19] J. Karczmarczuk, Generating power of lazy semantics, Theor. Comput. Sci. 187 (1-2) (1997) 203–219. doi:10.1016/S0304-3975(97)00065-0
[20] J. Karczmarczuk, Scientific computation and functional programming, Computing in Science & Engineering 13 (3) (1999) 64–72.
[21] O. Lobachev, R. Loogen, Towards an implementation of a computer algebra system in a functional language, in: Intelligent Computer Mathematics, Springer, 2008, pp. 141–154.
[22] D. J. Broadhurst, Three loop on-shell charge renormalization without integration: Lambda-MS (QED) to four loops, Z. Phys. C54 (1992) 599–606. doi:10.1007/BF01594886
[23] L. V. Avalov, Recurrence relations for three loop prototypes of bubble diagrams with a mass, Comput.Phys.Commun. 98 (1996) 15–19. arXiv:hep-ph/9512422 doi:10.1016/0010-4656(96)00090-2
[24] M. Steinhauser, MATAD: A Program package for the computation of Massive TADpoles, Comput.Phys.Commun. 134 (2001) 335–364. arXiv:hep-ph/0009029 doi:10.1016/S0010-4656(00)00204-6

7
[25] Y. Schroder, A. Vuorinen. High-precision epsilon expansions of single-mass-scale four-loop vacuum bubbles, JHEP 0506 (2005) 051. arXiv:hep-ph/0503209, doi:10.1088/1126-6708/2005/06/051

[26] R. N. Lee, A. A. Pomeransky. Critical points and number of master integrals. arXiv:1308.6676

[27] R. M. Schabinger (speaker), A. von Manteuffel. The Two-Loop Analog of the Passarino-Veltman Result and Beyond, talk Presented at LoopFest XII, May 13-15 2013, Florida State University (2013).