Non-existence of expansion-free dynamical stars with rotation and spatial twist

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Extending a previous work by the same authors, we investigate the existence of expansion-free dynamical stars with non-zero spatial twist and rotation and show that such stars cannot exist. Firstly, it is shown that a rotating expansion-free dynamical star with zero twist cannot exist. This is due to the fact that such stars cannot radiate and they are shear-free, in which case the energy density $\rho$ is time independent. Secondly, we prove that a non-rotating expansion-free dynamical star with non-zero spatial twist also cannot exist, as either the strong energy condition must be violated, i.e. $\rho + 3p < 0$, or the star must be shear-free in which case the star is static ($\Theta = \Omega = \Sigma = 0$). Finally, if we insist that the rotation and spatial twist are simultaneously non-zero, then the star cannot be shear-free in which case we obtain a quadratic polynomial equation in $\phi$ and $\Sigma$ with no real solutions. Therefore such stars cannot exist.

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I. INTRODUCTION

Models of radiating stars in general relativity play a central role in the study of gravitational collapse and the astrophysics of gravitating bodies. Physically relevant exact models were obtained by Tewari and Charan [1], Tewari [2], and Ivanov [3–5]. These examples provide interesting insights into the processes involved during stellar evolution. It has also been anisotropy and dissipative effects during gravitational collapse have influence on the collapse rate and temperature profiles in radiating stars by Reddy et al. [6]. There are classes of exact solutions to the Einstein’s field equations (EFEs) obtained, referred to as Euclidean stars, which have been shown to regain Newtonian stars within the appropriate limit [7–9]. In recent years the method of Lie analysis of differential equations using symmetry invariance has proved an invaluable and systematic tool in obtaining general categories of exact solutions to the boundary condition of radiating stellar objects [10–12]. There is an important class of radiating stars that has been introduced by Herrera et al. [13] which are expansion-free. Expansion-free dynamical models implies the existence of a cavity or void. One important feature of expansion-free models is that matter distributions with a vanishing expansion scalar have to be inhomogeneous. These physical features should have important astrophysical consequences for spherically symmetric distributions. Also, such radiating astrophysical model might offer a plausible explanation into the existence of voids that have been observed on the cosmological scales. Various authors have explored expansion-free dynamical models with different considerations. Some of the studies containing the description of physical properties of expansion-free dynamical radiating stars can be found in several works [14–16]. The peak in interest in these models laid in the fact that in such models is the possibility that they could help explain the existence of voids on cosmological scales. In 2008, Herrera and co-authors [13] studied such models with non-zero shear and showed that the appearance of a cavity (see reference [17] for more discussion), with matter which is anisotropic and dissipative, undergoing explosion is inevitable. The same authors followed this result by a 2009 paper in [18] in which they ruled out the Skripkin expansion-free dynamical model (see reference [19]) with constant energy density and isotropic pressure. Another study in [20] involved the study of models collapsing adiabatically, and showed that the instability was independent of the star’s stiffness. In particular, it was shown that the instability is entirely governed by the pressures and the radial profile of the energy density. In a recent work by Sherif et al. [21] the authors employed, for the first time, the 1+1+2 formalism (a semitetrad covariant method for analyzing the field equations) to study the properties of expansion free models. With emphasis on non-rotating and non-twisting stars the authors found that a necessary condition for the existence of such stars is that the star simultaneously accelerate and radiate. It was also shown in the same paper that these stars must possess a conformally flat geometry.

In this paper we study the required geometric and thermodynamic properties for the existence of a relativistic expansion-free dynamical star with at least the rotation or spatial twist being non-zero. This analysis falls in the scope of stability analysis of self-gravitating systems (some of the references are given in [22–24]). Our approach will be to fix either of the rotation or the spatial twist to zero and see whether indeed expansion-free dynamical models of such stars can exist. In particular we would like to know the restrictions that the addition of spatial twist or/and rotation induces on the geometric and matter quantities such as acceleration, heat flux,

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e.t.c. As with our previous work \cite{21}, we will make use of the equivalent forms of the field equations from the 1 + 1 + 2 semi-tetrad covariant formulation of general relativity \cite{27,32}. The semi-tetrad formalism has proven to be an extremely useful approach in displaying geometrical features in a transparent fashion which are generally very difficult to find using other approaches.

In section \ref{section1} we briefly introduce the 1 + 1 + 2 semi-tetrad formalism and provide a definition of locally rotationally symmetric (LRS) spacetimes. In section \ref{section2} we present the results of the paper, a complete analysis of the expansion-free model with rotation and spatial twist. We conclude with a discussion of the results in section \ref{section4}.

II. LOCALLY ROTATIONALLY SYMMETRIC SPACETIMES AND ITS 1 + 1 + 2 SEMITETRAD SPLITTING

We provide some background material in this section, covering the 1 + 1 + 2 semi-tetrad covariant formalism as well as notes on and calculations of useful quantities, utilized in this paper.

Stellar models that are rotating and twisting can be studied using the models of spacetimes known as \textit{locally rotationally symmetric} spacetimes \cite{33,34}. As such we will use this model here to investigate expansion-free and dynamic stellar models. LRS spacetimes, as well as derivatives of the unit vector fields \cite{29,32}.

To start with, let \((M, g_{ab})\) be a spacetime manifold, with associated metric tensor \(g_{ab}\). To any timelike congruence of an observer we may associate a unit vector field \(u^a\) tangent to the congruence which satisfies \(u^a u_a = -1\). One may then split \(M\) as follows: Given any 4-vector \(U^a\) in the spacetime, the projection tensor \(h^a_b = g^a_b + u_a u^b\), projects \(U^a\) onto the 3-space as

\[
U^a = U u^a + U^{(a)},
\]

where \(U\) is the scalar along \(u^a\) and \(U^{(a)}\) is the projected 3-vector \([11]\). The splitting splits \(g_{ab}\) into components associated with the \(u^a\) and spatial directions. This naturally gives rise to two derivatives:

- **The covariant time derivative** (or simply the dot derivative) along the observers' congruence. Given any tensor \(S^{a..b..c..d}\), we have \(\dot{S}^{a..b..c..d} \equiv u^a \nabla_c S^{a..b..c..d}\).

- **Fully orthogonally projected covariant derivative** \(\nabla\) with the tensor \(h_{ab}\), with the total projection carried out on all the free indices. Given any tensor \(S^{a..b..c..d}\), we have \(D_c S^{a..b..c..d} \equiv h^c_{..d} h^{..b..c..d} + h^c_{..d} h^{..a..b..c}, \nabla_{g..p..q}\).

This 1+3 splitting of the spacetime irreducibly splits the covariant derivative of \(u^a\) as

\[
\nabla_a u_b = -A_a u_b + \frac{1}{3} h_{ab} \Theta + \sigma_{ab}. \tag{2}
\]

In \((2)\), the vector \(A_a = \dot{u}_a\) is the acceleration vector, \(\Theta \equiv D_a u^a - \) the trace of the fully orthogonally projected covariant derivative of \(u^a - \) is the expansion and \(\sigma_{ab} = D_b u_a\) is the shear tensor (wherever used in this paper, angle brackets will denote the projected symmetric trace-free part of the tensor). In the particular case of LRS spacetimes, all vector and tensor quantities vanish identically (see reference \cite{29} for details).

The splitting further allows for the energy momentum tensor to be decomposed as

\[
T_{ab} = \rho u_a u_b + 2q_{(a} u_{b)} + p h_{ab} + \pi_{ab}, \tag{3}
\]

where \(\rho \equiv T_{ab} u^a u^b\) is the energy density, \(q_a = -h^b_{ab} T_{cd} u^c\) is the 3-vector defining the heat flux, \(p \equiv (1/3) h^{ab} T_{ab}\) is the isotropic pressure and \(\pi_{ab}\) is the anisotropic stress tensor.
If there is a preferred unit normal spatial direction \( e^a \) which is the case with LRS II spacetimes, the metric \( g_{ab} \) can be split into terms along the \( u^a \) and \( e^a \) directions (the vector field \( e^a \) splits the 3-space), as well as on the 2-surface, i.e.

\[
g_{ab} = N_{ab} - u_a u_b + e_a e_b, \quad (4)
\]

where the projection tensor \( N_{ab} \) projects any two vector orthogonal to \( u^a \) and \( e^a \) onto the 2-surface defined by the sheet \( (N_a{}^a = 2, u^a N_{ab} = 0, e^a N_{ab} = 0) \), and \( e^a \) is defined such that \( e^a e_a = 1 \) and it is orthogonal to \( u^a \), i.e. \( u^a e_a = 0 \). This is referred to as the \( 1 + 1 + 2 \) splitting. This splitting of the spacetime additionally gives rise to the splitting of the covariant derivatives along the \( e^a \) direction and on the 2-surface:

- The hat derivative is the spatial derivative along the vector field \( e^a \): Given a 3-tensor \( \psi_{a,b,c} \), we have \( \hat{\psi}_{a,b,c} \equiv e^f D_f \psi_{a,b,c} \).

- The delta derivative is the projected spatial derivative on the 2-surface (projection by the tensor \( N_a{}^f N_b{}^g N_c{}^h N_d{}^i D_f \psi_{j,g,h,i} \)).

The complete set of \( 1 + 1 + 2 \) covariant scalars fully describing the LRS class of spacetimes are 29:

\[
\{A, \Theta, \phi, \Sigma, \varepsilon, H, \rho, p, \Pi, Q, \Omega, \xi\}.
\]

The quantity \( \phi \equiv \delta e_a e^a \) is the shear expansion, \( \Sigma \equiv \sigma_{abc} e^a e^b \) is the scalar associated with the shear tensor \( \sigma_{abc} \), \( \varepsilon \equiv E_{abc} e^a e^b \) is the scalar associated with the electric part of the Weyl tensor \( E_{abc} \), \( \mathcal{H} \equiv H_{abc} e^a e^b \) is the scalar associated with the magnetic part of the Weyl tensor \( H_{abc} \), \( \Pi \equiv \pi_{abc} e^a e^b \) is the anisotropic stress scalar, \( Q \equiv -e^c T_{ab} u^b = \kappa_a e^a \) is the scalar associated to the heat flux vector \( \kappa_a \). The quantities \( \xi \) and \( \Omega \) are the spatial twist and rotation scalar respectively, which are defined by \( \xi = ((1/2) e^{ab} \delta_{abc} \) (where \( \delta_{abc} \equiv \varepsilon_{abc} e^c = u^b \eta_{abcd} e^d \) is the Levi Cita 2-tensor, the volume element of the 2-surface) and \( \Omega = e^a \omega_a \) (where \( \omega_a = \Omega e^a + \Omega^a \) is the rotation vector, with \( \Omega^a \) being the sheet component of \( \omega^a \)).

The full covariant derivatives of the vector fields \( u^a \) and \( e^a \) are given by 29:

\[
\nabla_a u_b = -A u_a e_b + e_a e_b \left( \frac{1}{3} \Theta + \Sigma \right), \quad (5a)
\]

\[
+ N_{ab} \left( \frac{1}{3} \Theta - \frac{1}{2} \Sigma \right), \quad (5b)
\]

\[
\nabla_a e_b = -A u_a u_b + \left( \frac{1}{3} \Theta + \Sigma \right) e_a u_b + \frac{1}{2} \phi N_{ab}. \quad (5c)
\]

We also note the useful expression

\[
\hat{u}^a = \left( \frac{1}{3} \Theta + \Sigma \right) e^a. \quad (6)
\]

Any given scalar \( \psi \) satisfies the commutation relation

\[
\hat{\psi} - \hat{\psi} = -A \psi + \left( \frac{1}{3} \Theta + \Sigma \right) \psi. \quad (7)
\]

We will utilize this relation throughout this work when seeking constraint equations. The field equations for LRS spacetimes are given as propagation and evolution of the covariant scalars 29:

- **Evolution**

\[
\frac{2}{3} \dot{\Theta} - \dot{\Sigma} = A \phi - \frac{1}{2} \left( \frac{2}{3} \Theta - \Sigma \right)^2 - 2\varepsilon + \frac{1}{2} \Pi - \frac{1}{3} (\rho + 3p), \quad (8a)
\]

\[
\dot{\phi} = \left( \frac{2}{3} \Theta - \Sigma \right) \left( A - \frac{1}{2} \phi \right) + 2\xi + Q, \quad (8b)
\]

\[
\dot{\xi} = \psi - \frac{1}{2} (\xi + \Omega), \quad (8c)
\]

\[
\dot{\Omega} = A \phi - \frac{2}{3} \Theta - \Sigma \Omega, \quad (8d)
\]

\[
\dot{\mathcal{H}} = 3\xi \mathcal{E} - \frac{3}{2} \left( \frac{2}{3} \Theta - \Sigma \right) \mathcal{H} + \frac{1}{2} \Omega Q, \quad (8e)
\]

\[
\dot{\mathcal{E}} - \frac{1}{3} \rho + \frac{1}{2} \Pi = \left( \frac{2}{3} \Theta - \Sigma \right) \left( \frac{3}{2} \mathcal{E} + \frac{1}{4} \Pi \right) \mathcal{E} + \frac{1}{2} \phi Q
\]

\[
+ 3\xi \mathcal{H} + \frac{1}{2} \psi \left( \frac{2}{3} \Theta - \Sigma \right) (\rho + p), \quad (8f)
\]

- **Propagation**

\[
\frac{2}{3} \dot{\Theta} - \dot{\Sigma} = \frac{3}{2} \phi \xi + 2\xi \Omega + Q, \quad (9a)
\]

\[
\dot{\phi} = \frac{1}{2} \phi^2 + \left( \frac{1}{3} \Theta + \Sigma \right) \left( \frac{2}{3} \Theta - \Sigma \right) + 2\xi^2
\]

\[
- \frac{2}{3} \rho - \varepsilon - \frac{1}{2} \Pi, \quad (9b)
\]

\[
\dot{\xi} = -\phi \xi + \left( \frac{1}{3} \Theta + \Sigma \right) \Omega, \quad (9c)
\]

\[
\dot{\Omega} = (A - \phi) \Omega, \quad (9d)
\]

\[
\dot{\mathcal{H}} = - \left( 3\varepsilon + \rho + p - \frac{1}{2} \Pi \right) \mathcal{H} - 3\phi \mathcal{H}
\]

\[
- Q \xi, \quad (9e)
\]

\[
\dot{\mathcal{E}} - \frac{1}{3} \rho + \frac{1}{2} \Pi = \frac{3}{2} \phi \left( \mathcal{E} + \frac{1}{2} \Pi \right) - \frac{1}{2} \left( \frac{2}{3} \Theta - \Sigma \right) Q
\]

\[
+ 3\Omega \mathcal{H}, \quad (9f)
\]
\[ \dot{A} - \dot{\Theta} = - (A + \phi) A - \frac{1}{3} \Theta^2 + \frac{3}{2} \Sigma^2 - 2 \Omega^2 + \frac{1}{2} (\rho + 3p) , \tag{10a} \]
\[ \dot{\rho} + \dot{Q} = - \Theta (\rho + p) - (2 A + \phi) Q - \frac{3}{2} \Sigma \Pi , \tag{10b} \]
\[ \dot{Q} + \dot{\Pi} = - \left( A + \frac{3}{2} \phi \right) \Pi - \left( \frac{4}{3} \Theta + \Sigma \right) Q - (\rho + p) A , \tag{10c} \]

\[ \mathcal{H} = 3 \Sigma \xi - (2 A - \phi) \Omega. \tag{11} \]

Let us now analyze the expansion-free dynamical models with rotation and spatial twist.

\[ A - \dot{\Theta} = -(A + \phi) A - \frac{1}{3} \Theta^2 + \frac{3}{2} \Sigma^2 - 2 \Omega^2 \]
\[ + \frac{1}{2} (\rho + 3p) , \]
\[ \dot{\rho} + \dot{Q} = - \Theta (\rho + p) - (2 A + \phi) Q - \frac{3}{2} \Sigma \Pi , \]
\[ \dot{Q} + \dot{\Pi} = - \left( A + \frac{3}{2} \phi \right) \Pi - \left( \frac{4}{3} \Theta + \Sigma \right) Q - (\rho + p) A , \]

\[ \mathcal{H} = 3 \Sigma \xi - (2 A - \phi) \Omega. \]

Let us now analyze the expansion-free dynamical models with rotation and spatial twist.

### III. RESULTS

In [21] we considered expansion-free dynamical stars that are non-rotating and non-twisting. It was shown that the existence of such models requires the star to simultaneously accelerate and radiate, in which case the star is necessarily conformally flat. We consider here the case in which at least either one of \( \Omega \) or \( \xi \) is non-vanishing. Thus we are considering the following three cases [34, 37, 38]:

1. \( \xi = 0; \Omega \neq 0 \): These models fall under the class of spacetimes known as LRS I spacetimes, with \( e^a \) hypersurface orthogonal and \( u^a \) twisting. A well know example is the Gödel solution.

2. \( \xi \neq 0; \Omega = 0 \): These models fall under the class of spacetimes known as LRS III spacetimes, with \( e^a \) twisting and \( u^a \) hypersurface orthogonal.

3. \( \xi \neq 0; \Omega \neq 0 \): These models, investigated in [38], have the property that the heat flux \( Q \) cannot be zero and specific energy conditions are required to be satisfied, i.e.

\[ -\frac{1}{2} (\rho + p + \Pi) < Q < \frac{1}{2} (\rho + p + \Pi) . \tag{12} \]

One therefore expects that an expansion-free dynamical model to exist in models with \( \xi \neq 0 \) and \( \Omega \neq 0 \). The star being dynamical implies that all of the thermodynamic quantities, including \( p, \rho, \Pi, Q \) e.t.c., are functions of time.

### A. Case 1: \( \xi = 0; \Omega \neq 0 \)

Let us start by considering the case of a rotating expansion-free star with no spatial twist. From [10c] we have

\[ 0 = \Sigma \Omega. \tag{13} \]

Since by assumption, \( \Omega \neq 0 \), we must have \( \Sigma = 0 \). Furthermore, from [8c] we obtain

\[ 0 = \left( A - \frac{1}{2} \phi \right) \Omega, \tag{14} \]

which, from [11], gives \( \mathcal{H} = 0 \), so that for such stars the Weyl tensor is purely electric. Using [8c] one has

\[ 0 = \Omega Q, \tag{15} \]

from which we obtain \( Q = 0 \). Therefore the star is not dynamical as the energy density is time independent, i.e. \( \dot{\rho} = 0 \) from [10b]. It is also not difficult to show that such stars will necessarily accelerate. To see this, assume \( A = 0 \). Then from [14], since by assumption \( \Omega \neq 0 \) we must have \( \phi = 0 \) as well. From [8a], [9b], [9c] and [10a] we obtain the constraints

\[ 0 = -2 \Omega^2 + \mathcal{E} - \frac{1}{3} (\rho + 3p) - \frac{1}{2} \Pi, \tag{16a} \]
\[ 0 = \mathcal{E} + \frac{2}{3} \rho + \frac{1}{2} \Pi, \tag{16b} \]
\[ 0 = 3 \mathcal{E} + \rho + p - \frac{1}{2} \Pi, \tag{16c} \]
\[ 0 = -2 \Omega^2 + \frac{1}{2} (\rho + 3p). \tag{16d} \]

Comparing [16b] and [16c] we obtain

\[ 0 = -\rho + p - 2 \Pi. \tag{17} \]

Substituting [17] and [16b] into [16a] we obtain the constraint

\[ 0 = -2 \Omega^2 - \frac{1}{2} (\rho + 3p) , \tag{18} \]

which upon comparing to [16d] gives

\[ 0 = \rho + 3p. \tag{19} \]

Therefore \( \Omega = 0 \) (from either [16d] or [19]), which contradicts the assumption that \( \Omega \neq 0 \). Hence we have \( A \neq 0 \). In summary,

**Theorem III.1** There cannot exist an expansion-free dynamical star with vanishing spatial twist and non-zero rotation.
Though these stars are not dynamical, we have enumerated several properties we expect such stars to have. In particular, the star is shear-free and accelerates without radiating.

B. Case 2: $\xi \neq 0; \Omega = 0$

Next, we consider non-rotating expansion-free dynamical stars with non-zero spatial twist. We state and prove the following

**Theorem III.2** There cannot exist an expansion-free dynamical star with vanishing rotation and non-zero spatial twist.

**Proof** To prove this, we will show that if such star is to exist, then the star will either be static or it will violate the strong energy condition (SEC). From (8d) we have

$$0 = A \xi,$$

so we must have $A = 0$ since by assumption $\xi \neq 0$. Using (10) we have

$$\Sigma^2 = -\frac{1}{3} (\rho + 3p).$$

Thus for such a star to exist we must have $\rho + 3p < 0$, except in the case that the star is shear-free, in which case the star is static ($\Omega = \Theta = \Sigma = 0$).

In fact in this case we have shown that even the expansion-free condition cannot hold, not only that it is not dynamical.

C. Case 3: $\xi \neq 0; \Omega \neq 0$

Finally, we consider the case of a simultaneously rotating and twisting expansion-free dynamical star. We start by taking the dot derivative of (8a) and the hat derivative of (10a) and obtain respectively

$$-\dot{\Sigma} + \dot{\hat{\Sigma}} = \phi \dot{\hat{A}} + A \dot{\phi} - \Sigma \dot{\sigma} - 4\Omega \dot{\Omega} + \dot{\Sigma} - \frac{1}{2} \dot{\Pi} - \frac{1}{3} \dot{\rho} - \dot{\rho}$$

$$-\dot{\Sigma} + \dot{\hat{\Sigma}} = -A^2 \phi - \frac{3}{2} A \phi^2 - A \left( \frac{2}{3} \rho + \mathcal{E} + \frac{1}{2} \Pi \right) + 3 \phi \Sigma^2$$

$$+ 2 \phi \Omega^2 + \frac{1}{2} \phi (\rho + 3p) + \frac{3}{2} \Sigma Q + 2 \Omega \Sigma \xi + 3 \Omega \mathcal{H}$$

$$- \frac{3}{2} \phi \left( \mathcal{E} + \frac{1}{2} \Pi \right) \left( \dot{p} + \dot{\Pi} \right),$$

$$-\dot{\Sigma} + \dot{\hat{\Sigma}} = -\frac{3}{2} A \Sigma^2 + \frac{3}{2} \phi \Sigma^2 + 6 \Omega \Sigma \xi + \frac{3}{2} \Sigma Q - \frac{3}{2} A \phi^2$$

$$+ 2 \phi \Omega^2 - \frac{3}{2} \phi \mathcal{E} + \frac{3}{4} \phi \Pi + \frac{1}{2} \phi (\rho + 3p) + 2 A \Omega^2$$

$$+ 2 A \xi^2 + \dot{Q}. (22a)$$

Taking the difference of (22a) and (22b) and using (10c) we obtain

$$-\dot{\Sigma} + \dot{\hat{\Sigma}} = -A^2 \phi + \frac{1}{3} A (\rho + 3p) - A \left( \mathcal{E} - \frac{1}{2} \Pi \right)$$

$$+ \frac{3}{2} \phi \Sigma^2 - 4 \Omega \Sigma \xi - 6 A \Omega^2 + \frac{1}{2} A \Sigma^2 + \Sigma Q (23)$$

Using the commutation relation in (7) on $\Sigma$ we have

$$-\dot{\Sigma} + \dot{\hat{\Sigma}} = -A \dot{\Sigma} + \Sigma \dot{\hat{\Sigma}}$$

$$-\dot{\Sigma} + \dot{\hat{\Sigma}} = -A \dot{\Sigma} + \Sigma \dot{\hat{\Sigma}}$$

$$= -A^2 \phi + \frac{1}{2} A \Sigma^2 + 2 A \Omega^2 - A \left( \mathcal{E} - \frac{1}{2} \Pi \right)$$

$$+ \frac{1}{3} A (\rho + 3p) + \frac{3}{2} \phi \Sigma^2 + 2 \Omega \Sigma \xi + \Sigma Q. (24)$$

Comparing (23) and (24) and using (11) we obtain the constraint

$$\left( \frac{14}{3} A - \phi \right) \Omega = \Sigma \xi. (25)$$
Now, taking the dot derivative of (9a) and the hat derivative of (9b) we obtain respectively
\[
\dot{\phi} = -\Sigma \dot{A} - A \dot{\Sigma} + \frac{1}{2} \Sigma \dot{\phi} + \frac{1}{2} \phi \Sigma + 2 \Omega \dot{\xi} + 2 \xi \dot{\Omega} + \dot{Q},
\]
\[
= A^2 \Sigma + \frac{5}{2} A \phi \Sigma - 2 \Sigma^3 + 4 \Sigma \Omega^2 - \frac{5}{6} \Sigma \rho - 3 \Sigma p + 4 \Lambda \Omega \xi - \Sigma \phi^2 + \Sigma \xi^2 + \frac{1}{2} (2 A - \phi) Q - 5 \phi \Omega \xi - \frac{1}{2} \Sigma E - \frac{1}{4} \Sigma \Pi + \dot{Q},
\]
and
\[
\dot{\dot{\phi}} = -\phi \ddot{\phi} - 2 \Sigma \dot{\Sigma} + 4 \xi \ddot{\xi} - \frac{2}{3} \rho - \left( \dot{\xi} + \frac{1}{2} \Pi \right)
\]
\[
= 3 A \phi \Sigma - \Sigma^3 - \frac{1}{2} \Sigma \phi^2 - 4 \phi \xi \xi - \frac{3}{2} \phi Q + \frac{1}{2} \Sigma E - 4 \Sigma \Omega^2 + 2 \Sigma \xi^2 - \frac{5}{4} \Sigma \Pi + 4 A \Omega \xi - \frac{1}{6} \Sigma \rho - 3 \Sigma p - 3 \xi \mathcal{H} - \rho.
\]
which implies
\[
\dot{\xi} - \dot{\xi} = -A \xi + \Sigma \xi
\]
\[
= -\frac{1}{2} \Sigma \xi - A^2 \Omega + \frac{1}{2} A \phi \Omega - \phi \Sigma \xi + \Omega \xi^2.
\]
Comparing (31) and (32) we obtain the constraint
\[
\left( A + \frac{3}{2} \phi \right) \Sigma \xi + \left( 2 A^2 + \frac{1}{2} \phi^2 + \rho + p \right) \Omega = \xi Q
\]
Let us now first prove the following proposition.

**Proposition III.3** An expansion-free dynamical star that is simultaneously rotating and twisting cannot be shear-free, if it exists.

**Proof** Here we assume the existence of such stars and show that if \( \Sigma = 0 \), then the weak energy condition (WEC) must be violated. We start by assuming that \( \Sigma = 0 \). Then from (23) we obtain (taking into account that \( \Omega \neq 0 \))
\[
A = \frac{3}{14} \phi,
\]
and therefore from (29) we have
\[
- \frac{1}{14} \phi \Omega \xi = 0.
\]
Since by assumption \( \xi \neq 0, \Omega \neq 0 \) we must have \( \phi = 0 \), which implies \( A = 0 \) as well. Now, from (9a) and (33) we have respectively
\[
0 = 2 \xi \Omega + Q,
\]
\[
(\rho + p) \Omega = \xi Q.
\]
Substituting (36a) into (36b) and again noting that \( \Omega \neq 0 \) we obtain the energy condition
\[
(\rho + p) = -2 \xi^2,
\]
which gives \( (\rho + p) < 0 \).\[\square\]
Finally, we state and prove the following

**Theorem III.4** There cannot exist an expansion-free dynamical star with both rotation and spatial twist non-vanishing.
Proof As has been shown in [38], any scalar $\psi$ in LRS spacetimes obtained via the $1 + 1 + 2$ decomposition satisfies the relation

$$\dot{\psi} \Omega = \dot{\psi} \xi. \quad (38)$$

Using (8c) and (9c) to substitute $\dot{\xi}$ and $\hat{\xi}$ for $\dot{\psi}$ and $\hat{\psi}$ respectively in (38) we obtain

$$-(2A - \phi) \Omega = \Sigma \xi, \quad (39)$$

which, upon comparing to (25) gives

$$\phi = \frac{10}{3} A. \quad (40)$$

Using (8d) and (9d) to substitute $\dot{\Omega}$ and $\hat{\Omega}$ for $\dot{\psi}$ and $\hat{\psi}$ respectively in (38) we obtain

$$-\phi \xi = \Omega \Sigma. \quad (41)$$

Substituting (40) into (39) (or equivalently (25)) we obtain

$$\frac{2}{5} \phi \Omega = \Sigma \xi. \quad (42)$$

It is clear that $\phi \neq 0$, for otherwise we would have $A = 0$ (from (40)), in which case from (25) (or alternatively (39)) we would have $\xi = 0$ (we have already shown that $\Sigma \neq 0$), contradicting the assumption that $\xi \neq 0$.

Now, multiply both (41) and (42) by $\Omega$ we obtain respectively

$$-\phi \Omega \xi = \Omega^2 \Sigma, \quad (43a)$$

$$\frac{2}{5} \phi \Omega^2 = \Omega \xi \Sigma, \quad (43b)$$

which we can rewrite as

$$\Omega \xi = -\frac{\Omega^2 \Sigma}{\phi}, \quad (44a)$$

$$\Omega \xi = \frac{2 \phi \Omega^2}{5 \Sigma}, \quad (44b)$$

since $\phi \neq 0, \Sigma \neq 0$. Equating (44a) and (44b) and simplifying we obtain

$$\left(\frac{2}{5} \phi^2 + \Sigma^2\right) \Omega^2 = 0. \quad (45)$$

Since by assumption $\Omega \neq 0$ we must have $(2/5) \phi^2 + \Sigma^2 = 0$, which is not possible over the set of real numbers $\mathbb{R}$ for non-zero $\phi$ and $\Sigma$.

IV. DISCUSSION

A previous paper by the same authors [21] studied expansion-free dynamical stars in the case that the rotation and spatial twist are zero. It was shown that these stars exist under the particular conditions that the stars radiate and accelerate, and are conformally flat. As with the case of [21] we have utilized the $1 + 1 + 2$ semi-tetrad covariant formalism to study such stars. In this paper, we have shown that there cannot exist an expansion-free dynamical star if at least one of the rotation or the spatial twist is non-vanishing. In the case that the spatial twist is zero and the star is rotating, the star can be expansion-free but both the heat flux and the shear vanishes, in which case the energy density is time independent. Thus such expansion-free stars are not dynamical. If the rotation is zero and the spatial twist is non-vanishing then the star cannot be expansion-free, since for this to happen we must have the star being static (in which case the star is not dynamical) or the SEC must be violated. Lastly it is shown that if we assume non-vanishing of both the rotation and the spatial-twist then the shear cannot be zero. Further analysis on the basis that the shear is non-zero, using both the commutation relation and a result relating the dot and hat derivatives of an arbitrary scalar [38], show that this leads to a quadratic polynomial equation in $\phi$ and $\Sigma$ with no real solution for non-zero $\phi$ and $\Sigma$. This result, we think, is a valuable contribution to the increasing literature on the expansion-free condition. This result has also severely restricted the prevalence of such stars.

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