Abstract—We propose and numerically validate a patch reflectarray modeling approach that describes each patch as a pair of polarizable magnetic dipoles. We introduce an extraction technique to obtain the effective polarizability of the patch dipoles via full-wave simulations on individual patches. This dipole framework serves as an alternative to the ray tracing model often used in reflectarray designs, in which rays are drawn from the feed point and scattered off of the patch elements. Whereas the ray tracing method approximates the feed as a point source or plane wave and solves the design problem in terms of phase delays, the dipole framework presented here can accurately predict beam patterns by accounting for both the magnitude and phase of arbitrary illumination patterns and empirically characterized patches. We illustrate this technique by applying it to two modulation strategies: a variable patch size reflectarray in which the phase can be continuously tuned (grayscale patch response), and a fixed patch size (binary patch response) in which on/off modulation is achieved through selective patch electrical shorting. Methods for incorporating these cases into the dipole design framework are discussed and the results compared to those from full wave simulation.

Index Terms—reflectarray, dipole, polarizability.

I. INTRODUCTION

BEAMFORMING antennas are a technological prerequisite for a variety of modern applications, with prevalent designs in the form of reflective structures such as parabolic dish antennas or collections of array elements. A reflectarray incorporates the response of both of these constituents in a planar reflection geometry, offering a convenient form factor in many scenarios [1], [2]. Reflectarrays become especially appealing at higher frequencies where the use of a free space feed eliminates many of the losses and design challenges associated with microstrip feed networks [3]. The beam quality achieved by reflectarray antennas as well as their compatibility with simple electronic tuning strategies [4] provide performance competitive with phased array architectures at substantially lower cost, weight, and power.

Whereas the physical realization of a reflectarray differs from that of a typical phased array, the design approach is necessarily similar. The fields for a desired beam back-propagated to the aperture plane must equate to the fields originating from the feed illumination and scattered by the reflectarray. Typically it is the phase that predominates the scattering characteristics, such that it is often a good approximation to consider only the phase distribution of the feed added to the phase distribution contributed by the reflectarray in the aperture plane. Patches situated on a ground plane are often employed to realize this phase distribution due to their well-understood electromagnetic response and relatively straightforward fabrication.

A standard approach to reflectarray design is to characterize the phase response of a small patch in a periodic array of identical elements through full wave simulation [5], [6]. Simulations can be performed for a variety of patch geometries (or other tuning mechanisms), yielding a characteristic S curve that describes the patch phase response versus geometry. The periodic boundary condition approximately incorporates the effects of mutual element coupling among elements, even though—in reality—the patch geometry will vary over the array. Many works have successfully demonstrated this numerical approach when the variation of patch geometry over the array is sufficiently slow or mutual coupling sufficiently weak. With the phase response of the patch specified in this way, array design may proceed by prescribing the patch distribution according to an approximate phase shift model, specifying a patch geometry at position \( r_i \) such that its complex response \( e^{j \psi_i} \), when added to the complex incident source field \( e^{-jkR_i} \), equals the phase distribution required to steer a beam in the direction \( \hat{R}_b \), \( e^{-jkR_b \cdot \hat{r}_i} \). This criterion leads to a phase distribution prescribed by [5], [3], [6]:

\[
\psi_i = k (R_i - \hat{R}_b \cdot \hat{r}_i) + 2\pi N
\]  

where \( k = 2\pi/\lambda \) is the free space wavenumber, \( R_i \) is the distance from the feed to the patch, and \( N \) is an integer. This design approach is simple and effective, yet the various approximations limit its accuracy and applicability. In particular, the model accounts only for the phase of illumination as well as that of the patch, and neglects the amplitude variation that is inherently coupled to patch phase response for different patch geometries. Additionally, the conventional model ignores the gain pattern of the patch, which can have a measurable impact for patches of sufficient electrical size. Finally, simplified source approximations often neglect practical illumination asymmetries that aid in reducing quantization lobes and artifacts arising from constrained reflection values [7], [8]. [9]. In some cases, these effects may not have noticeable impact on the design procedure, but will nevertheless degrade the model’s accuracy in predicting beam performance. Full-wave simulations can supplement or replace the conventional approach, but are computationally burdensome, especially for optimization iterations. An alternative approach, presented here, formulates

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a quasi-analytic description of patch reflectarrays that can provide more effective modelling capabilities with better accuracy and insight than the geometric optics approximations provide.

Motivated by the standard cavity model of microstrip patch antennas as a pair of polarizable magnetic dipoles, we may describe the collective array response through a discrete dipole formalism. If we appropriately structure the array so that interelement coupling can be assumed weak, then a holographic design procedure will inform the required phase distribution for arbitrary illumination. We will restrict our study to this weakly-coupled approximation, though non-negligible coupling can be efficiently accommodated by a coupled dipole approximation that has been extensively applied to systems of effective magnetic dipoles [10], [11], [12], [13]. Regardless of the extent of patch interactions, reflectarray design and analysis using this model proceeds from experimental or numerical characterization of a given patch geometry, or a range of such geometries. The patch response, described in this work by its magnetic polarizability, can vary continuously by sweeping the patch size or, for example, using electronic modulation such as varactor diodes [14], [15], [16], [17]. In this case, a reflectarray can achieve grayscale phase and amplitude tuning. Other reflectarray designs may exploit a discrete set of patch values or modulation states [18], [19], [20], [21], yielding quantized phase and amplitude values. At its lowest limit, a binary reflectarray leverages only two modulation levels (on/off) for beamforming, accessed for instance with PIN diodes [4], [22]. In this work, we will explore variable patch size reflectarrays for grayscale tuning, as well as a simple binary modulation strategy for static reflectarray design in which the patch is either radiating (on) or electrically shorted to ground and therefore non-radiating (off).

We will outline the polarizability description of patches in the following section, followed by a polarizability retrieval strategy for patch characterization in Section III. Section IV describes a holographic reflectarray design procedure formulated within the discrete dipole model. To remain consistent with common treatments of these structures [23], we will employ two distinct spherical coordinate systems describing either the patches individually or the reflectarray structure in its entirety. These coordinate systems are illustrated for each case in the following section. Finally, we will provide numerical results illustrating the implementation of this procedure and the effect of illumination diversity on beam performance by applying the discrete dipole methodology to grayscale and binary patch reflectarrays.

II. POLARIZABLE PATCH MODEL

A standard description of a ground plane-backed microstrip patch antenna treats the gap beneath the patch as a cavity with perfect electric conductors (PEC) as walls on the patch and ground plane surfaces, and perfect magnetic conductors (PMC) as the vertical walls over the patch “slots”. These boundary conditions give rise to a collection of cavity modes that may be excited inside the gap between the patch and ground plane. An incident electromagnetic field illuminating the patch can couple to these cavity modes to excite fields in the gap that in turn can be described by radiating effective magnetic surface currents over each of the four patch slot surfaces. If we assume that an incident magnetic field is directed along the \( \hat{z} \) axis and excites only the lowest order cavity mode, then the fields from the faces oriented along the \( xy \) plane (see Figure (a)) cancel, leaving two radiating slots described by magnetic surface currents \( \mathbf{J}_{m1} \) and \( \mathbf{J}_{m2} \). These magnetic surface currents can be defined according to surface equivalence principles by [23]

\[
\mathbf{J}_{mk} = \mathbf{E}_k \times \hat{n}
\]

where \( \mathbf{E}_k \) is the total electric field over the surface of slot \( k \). For sufficiently small patch dimensions relative to the wavelength, these magnetic currents can be approximated as point magnetic dipoles \( \mathbf{m}_k \) for \( k = 1, 2 \) that are related to the continuous magnetic surface currents by

\[
\mathbf{J}_{mk}(\mathbf{r}) = j \omega \mu_0 \mathbf{m}_k \delta(\mathbf{r} - \mathbf{r}_k).
\]

at angular frequency \( \omega \), with \( \mu_0 \) the magnetic permeability of free space.

As described in [24], these radiating slots form polarizable magnetic dipoles, and can be equivalently characterized by a magnetic polarizability tensor \( \tilde{\alpha}_k \) in which each nonzero component follows a Lorentzian resonance profile with resonance frequency \( \omega_0 \):

\[
[\tilde{\alpha}_k]_{ij} = \frac{F \omega^2}{\omega_0^2 - \omega^2 + j \gamma \omega}
\]

with oscillator strength \( F \) and damping factor \( \gamma \) related to the oscillator quality factor \( Q \) by \( \gamma = \omega_0 / Q \). The polarizability tensor \( \tilde{\alpha}_k \) is in turn related to the magnetic dipole moment

Fig. 1. Coordinate systems used in this analysis. (a) Conventional spherical coordinate system used to describe fields radiated by a grounded patch. (b) Conventional spherical coordinate system used to describe the far field of a radiating planar antenna.
through \( \mathbf{m}_k = \bar{\alpha}_k \mathbf{H}_i \), where \( \mathbf{H}_i \) is the superposition of the illuminating magnetic field with its reflection: \( \mathbf{H}_i = (1 - \Gamma) \mathbf{H}_0 \). Here, we take \( \Gamma \) to be the appropriate reflection coefficient for the dielectric-ground plane substrate and magnetic field polarization under a plane wave approximation, though an improved estimate may be obtained through more rigorous numerical methods. The slot’s magnetic polarizability can be considered the fundamental quantity characterizing the electromagnetic response of a patch with a given geometry, in a given electromagnetic environment. We will assume that the patches are geometrically symmetric in such a way that the polarizability of the two radiating slots \( \bar{\alpha}_1 \) and \( \bar{\alpha}_2 \) are equal.

### III. PATCH POLARIZABILITY RETRIEVAL

Using the description of a patch as a set of two radiating dipoles, the response of a patch reflectarray can be decomposed into the superposition of (1) the radiation from the collection of magnetic dipoles over the antenna with (2) the fields reflected from the ground plane. While modelling ground plane reflections is straightforward, determined by the illumination profile and the ground plane material properties, accurate modelling of the radiation from each patch dipole requires a method for characterizing the slot polarizabilities [25]. Once the patch polarizability is retrieved, patch far field contributions can be predicted through standard radiation integrals. In the following, we introduce and compare two methods for characterizing patch polarizabilities.

#### A. Polarizability Extraction by Surface Equivalence

The first method for quantitative polarizability extraction assumes that the magnetic surface currents \( J_{mk} \) can be calculated over the slot surfaces through, for instance, direct evaluation of Eq. (2) in numerical simulations. Then the magnetic dipole moment of the point magnetic dipole describing a single slot can be obtained upon integrating Eq. (3) over the slot surface as

\[
\mathbf{m}_k = \frac{1}{j\omega \mu_0} \int J_{mk}(\mathbf{r})d^2\mathbf{r}.
\]

Consistent with the geometry of Fig. 1(a) in which the slot dipoles are oriented in the \( z \) direction, we can then recover the \( z \) component of the polarizability corresponding to the \( k \)th slot as

\[
\alpha_{kz} = \frac{m_{kz}}{H_{iz}(\mathbf{r}_0)}
\]

where \( H_{iz}(\mathbf{r}_0) \) is the total (incident plus reflected) magnetic field evaluated at the center of the slot. Assuming patch symmetry, \( \alpha_{1z} = \alpha_{2z} \), and only one slot need be evaluated in this fashion.

#### B. Far-Field Polarizability Extraction

A second approach for polarizability extraction recovers the patch polarizability from far field electric field measurements and is thus amenable to experimental remote measurement techniques. The far field electric field radiated by a magnetic dipole oriented in the \( z \) direction is given by:

\[
E_\phi = -\frac{\omega k_0}{4\pi r} \alpha_z H_{iz}(\mathbf{r}_0) e^{-jkr} \sin \theta.
\]

Applying this to the two radiating dipoles comprising a single patch yields the total electric field radiated by the patch as

\[
E_{\phi,tot} = -\frac{\omega k_0}{4\pi r} e^{-jkr} \alpha_z (1 - \Gamma) \sin \theta 2\cos \left( \frac{kh}{2} \sin \theta \cos \phi \right) \\
\times \left[ H_{iz}(\mathbf{r}_1) e^{-jkw \sin \theta \sin \phi} + H_{iz}(\mathbf{r}_2) e^{jkw \sin \theta \sin \phi} \right],
\]

where the factor \( 2\cos \left( \frac{kh}{2} \sin \theta \cos \phi \right) \) is an array factor term resulting from the superposition of a single slot dipole with its image in the ground (PEC) plane. As defined previously, \( \Gamma \) is the reflection coefficient from the dielectric-ground plane substrate, which can be computed locally under a plane wave approximation or via more rigorous numerical methods, and \( W \) is the patch width separating the radiating dipoles. If the incident illumination is arranged so as to yield equal magnetic field values at the two slots \( (H_{iz}(\mathbf{r}_1) = H_{iz}(\mathbf{r}_2) = H_{iz}) \), e.g. for a normally incident plane wave, and if the substrate thickness \( h \) is small, then Eq. [8] reduces to

\[
E_{\phi,tot} = -\frac{\omega k_0}{\pi} e^{-jkr} \alpha_z (1 - \Gamma) H_{iz} \sin \theta \cos \left( \frac{kW}{2} \sin \theta \sin \phi \right)
\]

where now the term \( 2\cos \left( \frac{kW}{2} \sin \theta \sin \phi \right) \) gives the array factor for the two patch dipoles. Given a complex electric field measurement \( E_{\phi,tot} \), this expression can be inverted to obtain the slot polarizability \( \alpha_z \).

We compare these two polarizability extraction methods applied to a square patch of size 1.5 cm \( \times \) 1.5 cm in Fig. 1. In this example, the patch is separated from the ground plane by 0.25 mm of air, and the polarizability values obtained using the two methods show close agreement. The small discrepancies may arise from meshing errors in the distinct simulation methods.

Simulated results obtained using the surface equivalence polarizability retrieval method are given in Fig. 3 for square patches on 0.762 mm-thick Rogers 4350B substrate (\( \varepsilon_r = 3.48, \tan \delta = 0.0037 \)). Figures 3(a) and (b) illustrate the resulting polarizability for patches of varying size at a frequency of 10 GHz. As the patches range in length from 4 mm to 10 mm, the polarizability sweeps out a resonance that peaks at 7.4 mm. As demonstrated in many patch reflectarray examples,
and as we will similarly demonstrate under our polarizability framework in Section IV such a library of patch geometries can be used to design a grayscale reflectarray by properly selecting the patch geometry at each array position in order to conform to some design criteria.

The polarizability values for a 7.5 mm-wide (λ/4 at 10 GHz) square patch are plotted in Figs. 3(c) and (d) as a function of frequency. Here, in contrast to the grayscale values accessed in Figs. 3(a) and (b), we explore a simple method for achieving two binary patch states. The radiating state corresponds to the 7.5 mm-wide square patch with a polarizability that peaks just below 10 GHz. The patch can be made approximately non-radiating at 10 GHz by shorting it to ground, thus shifting its resonance far from the operating frequency. In this case, we connect a metallized via from the patch to the ground plane at the center of each radiating slot (see Fig. 10(b)). We refer to the radiating and non-radiating states of this binary design as $\alpha_{on}$ and $\alpha_{off}$, respectively. We will investigate the feasibility of utilizing such a binary scheme in later studies. In practice, dynamic control of patch response can be achieved through mechanical or electronic strategies [6], for example incorporating electronic tuning elements into a fixed patch geometry. These approaches can use PIN diodes for binary tuning or varactor diodes for grayscale tuning.

IV. HOLOGRAPHIC ARRAY DESIGN

The above procedures illustrate methods for extracting or characterizing the magnetic polarizabilities that define a patch with a given geometry. Through the characterization of patch responses over a range of geometries or tuning states, a complete design set can be obtained. Assuming such a set of values is known, an inverse design problem can be solved: given a desired output far-field beam, determine the required polarizabilities over an aperture. This is the analogue of the objective typically addressed through the simple phase mapping of Eq. (1) in conventional reflectarray design. Here, instead of describing the coupling of the incident fields to patches in terms of an abstract phase response, we use the polarizability description to arrive at an electromagnetic field theoretical solution.

The polarizability design framework adapted here to ground plane-backed patches has been largely developed and applied in the context of metamaterials and metasurfaces consisting of subwavelength elements [26]. For electrically small elements, the scattering is predominantly dipolar in nature, and higher order contributions may be neglected. These same arguments apply to the cavity model of a subwavelength patch on a thin dielectric substrate. If the scattering response of the metamaterial or patch element is sufficiently dipolar, one can avoid the computational complexity of finite element or moment methods that require computing field solutions over the entire aperture and surrounding space. Instead, one can rigorously predict scattering behavior from arbitrary collections of dipolar elements using a coupled dipole approximation [27]. Modelling a collection of dipoles in this way requires only the the scattering response (polarizability) of each element in addition to the domain propagation behavior described in terms of a Green’s function. The forward problem of computing the fields for a given set of polarizabilities has been validated for waveguide slots in [12], [25] and for more general metasurfaces and metamaterial structures in [25], [10], [28], [29].

The inverse problem of selecting the polarizabilities that achieve a desired output pattern is generally nonlinear due to mutual coupling effects. One approach to achieving a design solution is to incorporate coupling using the coupled dipole model and proceed iteratively to arrive at a satisfactory set of polarizabilities as in [13], [20]. Otherwise, if mutual coupling can be minimized through, for example, sufficient separation between the array elements, one can apply a Born approximation and neglect mutual coupling effects in the design procedure to arrive at a so-called “holographic” solution. The analysis of a beamforming metasurface in [31] follows this approximation. Furthermore, it is reported in [32] that mutual coupling indeed has little observed effect in the authors’ practical reflectarray design example. For these reasons, we will seek a holographic expression for the required polarizabilities in a patch reflectarray by neglecting mutual coupling between patches.

To design a beamforming reflectarray antenna, we seek an array of patches described by a spatial polarizability distribution which, when combined with ground plane reflections, transforms an incident illumination profile to a beam in a specified direction (see Fig. 4(a)). To achieve this result, the realized polarizability values will necessarily depend on the form of illumination, though the design approach can generally be applied for any illumination profile. A commonly used illumination source that we adopt for the results that follow is a horn antenna. Figure 4(b) shows a rendering of a 10 dBi X-band pyramidal horn antenna (PE9856B-10) along with a plot of its radiation pattern measured at 10 GHz. Using a standard analytical description of this pyramidal horn ([23], Chapter 13), the fields illuminating the reflectarray ($\mathbf{H}_i$) can be calculated numerically, as shown in Fig. 4(c) at 10 GHz and with the horn placed 20 cm from the reflectarray surface.
Combined with an expression for the output beam, one can then present the reflectarray design in terms of the required polarizabilities at each array position.

We start by expressing the far fields of our reflectarray antenna in terms of the electric and magnetic surface currents. The far-field magnetic field radiated by our reflectarray antenna can be decomposed into contributions from the magnetic and electric vector potentials \( A \) and \( F \), respectively:

\[
H = H_A + H_F = -\frac{j\omega}{\eta} \hat{k}_b \times A - j\omega F
\]  

(10)

where \( \hat{k}_b \) is the unit vector in the beam direction and \( \eta = \frac{\sqrt{\mu_0}}{\varepsilon_0} \) is the free space impedance. The magnetic and electric vector potentials are related, respectively, to the Fourier transforms of the electric and magnetic surface currents over the antenna surface:

\[
A = \frac{\mu_0 e^{-jkr}}{4\pi r} \int \int \mathcal{F} \{ J_e \} e^{jkr'} d^2r' = \frac{\mu_0 e^{-jkr}}{4\pi r} \mathcal{F} \{ J_e \}
\]  

(11)

\[
F = \frac{\varepsilon_0 e^{-jkr}}{4\pi r} \int \int \mathcal{F} \{ J_m \} e^{jkr'} d^2r' = \frac{\varepsilon_0 e^{-jkr}}{4\pi r} \mathcal{F} \{ J_m \}
\]  

(12)

where \( \mathcal{F} \{ \cdot \} \) denotes the two-dimensional Fourier transform over the aperture coordinates. Note that we are neglecting finite aperture effects for the sake of deriving a design expression, but such effects will be incorporated naturally upon numerical implementation. Inserting these definitions into Eq. (10) gives an expression relating the far-field magnetic field to the electric and magnetic surface currents over the antenna surface:

\[
H = -\frac{j\omega \mu_0 e^{-jkr}}{\eta 4\pi r} \mathcal{F} \{ \hat{k}_b \times J_e \} - \frac{j\omega \varepsilon_0 e^{-jkr}}{4\pi r} \mathcal{F} \{ J_m \}
\]  

(13)

\[
= -\frac{j\omega \varepsilon_0 e^{-jkr}}{4\pi r} \left[ \eta \mathcal{F} \{ \hat{k}_b \times J_e \} + \mathcal{F} \{ J_m \} \right].
\]

An ideal far-field beam should satisfy

\[
H = H_0 \delta(k - k_b).
\]  

(14)

That is, the magnetic field profile in the far field of the proposed infinite array should peak in the desired beam direction \( k_b \). Equating this desired response to the expression (13), taking an inverse Fourier transform, and using the Fourier transform relationship between a delta function and a plane wave yields a prescription for the magnetic and electric surface currents required to form a beam in the far field:

\[
J_m + \eta \hat{k}_b \times J_e = a H_0 e^{-jkr'}
\]  

(15)

where \( a = \frac{je^{jkr_0}}{\omega \mu_0 e^{-jkr}} \) is a constant incorporating the far field distance \( r \) and normalization constant \( \kappa \), and provides some scaling freedom in the holographic design procedure.

The magnetic surface currents \( J_m \) consist of the discrete patch dipoles:

\[
J_m = \frac{j\omega \mu_0}{A^2} \hat{\alpha} H_t
\]  

(16)

with patch spacing \( \Lambda \). The illuminating fields also excite electric surface currents \( J_e \) over the substrate that can be defined through surface equivalence principles as

\[
J_e = \hat{n} \times \left[ (1 - \Gamma) H_t \right] = \hat{n} \times H_t
\]  

(17)

where \( \hat{n} \) denotes the antenna surface normal. Finally, using these definitions for the electric and magnetic surface currents, one can show for a diagonal polarizability tensor that the \( j \)th component along the diagonal is given by

\[
\alpha_j = -\frac{j\Lambda^2}{\omega \mu_0} \frac{1}{\hat{\alpha} H_t \cdot j} \left[ (\hat{\alpha} H_t \cdot j) e^{-jkr} - \eta \left( (\hat{k}_b \cdot \hat{H}_t)(\hat{j} \cdot \hat{n}) - (\hat{k}_b \cdot \hat{n})(\hat{H}_t \cdot \hat{j}) \right) \right].
\]  

(18)

For the geometry shown in Fig. (1)b) with the magnetic field oriented in the \( y \) direction and the antenna in the \( xy \) plane, the required \( y \) component of the polarizability is thus given by

\[
\alpha_y = -\frac{j\Lambda^2 H_{ty}}{\omega \mu_0} \left[ \frac{\alpha H_{ty}}{H_{ty}} e^{-jkr} + \eta \hat{k}_b \cdot \hat{n} \right].
\]  

(19)

A. Lorentzian Polarizability and the Patch Phase Response

Equation (15) provides the physical foundation for beamforming with a reflectarray by decomposing a plane wave over the aperture into the superposition of electric currents arising from ground plane reflections and magnetic currents describing the radiating patches. The required patch geometries can then be prescribed by the corresponding polarizabilities given in Eq. (19). In view of the Lorentzian form of the patch polarizabilities (Eq. 4), one notices that the available patch slot contributions traverse only 180 degrees of phase (Fig. 3), in contrast to the full 360 degrees of phase shift accessible to reflection-mode patches presented in the literature [3]. Here, we examine Eq. (15) in more detail in order to reconcile this different behavior.

In order to simplify the analysis, we suppose that the antenna is aligned with the \( xy \) plane and illuminated by a magnetic field linearly polarized in the \( y \) direction, so that \( H_t = H_t(r')\hat{y} \) and \( \hat{n} = \hat{z} \). In addition, we assume that the radiating patch slots are similarly oriented in the \( y \) direction, resulting in a single nonzero element of the polarizability...
tensor $\alpha_y$. This geometry leads to electric and magnetic surface currents given by

$$\mathbf{J}_e = -\mathbf{H}_0 \hat{x}, \quad \mathbf{J}_m = \frac{j \omega \mu_0}{\Lambda^2} \alpha_y \mathbf{H}_0 \hat{y}. \tag{20a}$$

Further restricting our present analysis to a broadside beam with $\mathbf{H}_0 = \mathbf{H}_0 \hat{y}$, we have $k_0 = \hat{z}$, and the electric surface current term on the left hand side of Eq. (15) is given by $\eta k_0 \times \mathbf{J}_e = -\eta \mathbf{H}_0 \hat{y}$.

We will now confine the study to a single frequency and parameterize the slot polarizabilities in a more convenient form:

$$\alpha_y = \alpha_0 \left( -\frac{j + e^{-j \psi}}{2} \right), \tag{21}$$

where $\alpha_0$ is a constant magnitude and $\psi$ a phase ranging from 0 to 360 degrees. In the complex plane, Eq. (21) represents the constrained form of a Lorentzian resonator as a circle shifted 0 to 360 degrees at 0 to $2\pi$. In the complex plane, Eq. (21) represents the parameterized Lorentzian resonator.

Using Eq. (21) in Eq. (20b) gives

$$\mathbf{J}_m = \frac{j \omega \mu_0}{\Lambda^2} \left( -\frac{j + e^{-j \psi}}{2} \right) \mathbf{H}_0 \hat{y} \tag{22}$$

and the holographic design equation (15) becomes

$$\left[ (b - \eta) + j \alpha e^{-j \psi} \right] \mathbf{H}_0 = a \mathbf{H}_0 e^{-j k_0 r} \tag{24}$$

where $b = \frac{\omega k_0}{2 \Lambda^2}$ and $a$ is defined above. Examining Eq. (24), we see that the electric currents exactly compensate for the offset of the Lorentzian circle in the complex plane when $b = \eta$, or when

$$\alpha_{\text{opt}} = \frac{2 \Lambda^2}{k_0} \tag{25}$$

In this case, the combined electromagnetic patch response represented by the left hand side of Eq. (24) becomes proportional to $e^{-j \psi}$ and recovers a full 360 degrees of phase shift. Figure 5 illustrates the convergence to an ideal phase profile realized by the patch electromagnetic response $J_{em}$, where $J_{em}$ describes the left hand side of Eq. (24), as $\alpha_0$ is varied from $\alpha_{\text{opt}}/2$ to $\alpha_{\text{opt}}$. In this case, the accessible phase increases from 180 degrees for the case of $\alpha_{\text{opt}}/2$, represented by a circle in the lower half plane, to a full 360 degrees at $\alpha_{\text{opt}}$.

For the examples studied in the results that follow, $k_0$ and $\Lambda$ correspond to a frequency of 10 GHz and a lattice spacing of $\lambda/2$, and result in a polarizability magnitude of $\alpha_{\text{opt}} = 2.14 \times 10^{-4} \text{m}^3$ for optimal phase coverage. Using the substrate reflection coefficient $\Gamma$ and the approximation of an infinitesimal patch, the approximate effective polarizability of a single patch is comprised of contributions from two slot dipoles and two image dipoles, and so is found to be approximately $2(1 - \Gamma)$ times the polarizability of a single slot dipole. Then a more accurate comparison for concluding optimality in the case of a ground plane-backed patch is $|\alpha_{\text{opt}}/(2(1 - \Gamma))| = 5.43 \times 10^{-7} \text{m}^3$, which approaches the peak effective polarizability magnitude for the grayscale values shown in Fig. 3(a) given by $4.97 \times 10^{-7} \text{m}^3$. The retrieved grayscale polarizabilities from Figs. 3(a) and (b) are plotted in the complex plane in Fig. 6 parameterized by patch length, alongside the optimal complex polarizability corresponding to a single dipole, $\frac{1}{2(1 - \Gamma)} \frac{|\mathbf{H}_0|}{\lambda} \left( -\frac{j + e^{-j \psi}}{2} \right)$, indicating the potential for successful beam formation using the retrieved polarizability values.

V. RESULTS

To demonstrate application of our polarizability model in the design of reflectarrays, we first consider the case of varying patch lengths with polarizabilities given in Figs. 3(a) and (b). The reflectarray in Fig. 7 covers a 30 cm $\times$ 30 cm area, with patches spaced at 1.5 cm, or half the free-space wavelength. A pyramidal horn antenna positioned 20 cm from the reflectarray illuminates the antenna with its electric field polarized in the $\hat{x}$ direction, using the conventional spherical coordinate system illustrated in Fig. 1(b). The reflection coefficient $\Gamma$ was calculated according to 0.762 mm-thick Rogers 4350B substrate, and the polarizability distribution was selected in order to steer a beam to $(\theta_0, \phi_0) = (25^\circ, 0^\circ)$. The design procedure consists of first specifying the patch array positions. Each patch consists of two polarizable slots oriented in the $\hat{y}$ direction. Although we model each patch with two magnetic dipoles, our assumption of geometrically symmetric patches requires that both of these dipoles be identical. Therefore, in order to impose this equality constraint during the design stage, we approximate the patch as a single dipole positioned at the center of the patch. We then apply Eq. (19), here using a value of $H_{0y} = 1$, and $H_{ly}$ numerically calculated at all array...
Equation (19) yields unconstrained polarizability values at the patch locations that must then be constrained to the realizable polarizability values. For a grayscale demonstration, we take the values shown in Figs. 3(a) and (b) as our achievable design values, and map the unconstrained polarizabilities to a corresponding patch size according to a criterion of minimum length in the complex plane (26):

$$\alpha_i = \arg \min_{\alpha_c} ||\alpha_c - \alpha_{0,i}||$$

where $|| \cdot ||$ denotes the $l_2$-norm in the complex plane between the ideal polarizability prescribed for position $i$, $\alpha_{0,i}$, and the available patch polarizabilities $\alpha_c$. This constraint method is illustrated through a complex-plane representation in Fig. 8 where the blue dots correspond to the ideal polarizability values returned by Eq. (19), and the red dots indicate the achievable polarizabilities of the variable-length patches from Figs. 3(a) and (b). The extent of the ideal polarizabilities in the complex plane has been modified by adjusting the scaling constant $a$ to heuristically realize improved design performance. The inset reveals the constrained polarizability states and resulting mapping in more detail. Note that the ideal polarizability values must compensate for the varying magnitude of the feed horn, forming a seemingly irregular pattern in the complex plane. Were we to ignore this magnitude variation, the ideal polarizabilities would instead lie on a circle centered in the complex plane.

Before making this mapping, we note that the polarizability value recovered using Eq. (9) corresponds to that of a single slot. In order to properly make the correspondence between the prescribed polarizabilities at the patch locations and the slot polarizabilities of Eq. (19), we multiply the slot polarizability values by a factor of $2(1 - \Gamma)$.

$$2(1 - \Gamma)$$

This accounts for the presence of the two slot dipoles by an array factor approximation in the limit of an infinitesimally small patch $2\cos\left(\frac{2\pi}{2}\sin\theta\sin\phi\right) \rightarrow 2$, as well as the slot image dipoles by an additional factor of $(1 - \Gamma)$. Once the constrained design is achieved, we replace the approximate array factor by the true array factor $(1 - \Gamma)2\cos\left(\frac{2\pi}{2}\sin\theta\sin\phi\right)$ for beam prediction and analysis, again assuming a thin substrate. The magnetic surface currents can then be computed according to Eq. (16) using the illuminating magnetic field at each patch position, and the radiation pattern calculated from the combined magnetic and electric surface currents.
Figure 7(b) illustrates the far field pattern resulting from our grayscale reflectarray design according to radiation integrals numerically applied to the equivalent magnetic and electric surface currents of our polarizability model (PM), compared to that computed by the full-wave electromagnetic field solver CST, revealing that the polarizability model successfully predicts the main beam location as well as qualitative side lobe behavior. The illumination in the full-wave simulation is provided by a WR-90 pyramidal horn excited by a rectangular waveguide port and positioned 20 cm from the reflectarray surface. The directivity computed using the polarizability model is 29.7 dBi, compared with 29.6 dBi as predicted by CST. The polarizability model reveals a 3-dB beamwidth of 6.17° at a peak angle of φ = 25.14°, in close agreement with the beamwidth value of 6.48° at a peak angle of φ = 25° calculated by CST.

Figure 7 highlights the ability of the polarizability model to accurately design and model patch reflectarrays with reduced computational complexity. In this case, the polarizability model accommodates grayscale patch design while accurately incorporating both the amplitude and phase of the illuminating fields, as well as the amplitude and phase response of the patches themselves. These effects can be crucial in predicting performance under non-trivial illumination strategies. To demonstrate the effects of different illumination on beam performance and the ability of our polarizability model to capture these effects, Fig. 9 compares PM-predicted performance of a binary reflectarray under plane wave versus pyramidal horn illumination. The design utilizes a 50 cm × 50 cm aperture operating at 10 GHz. As above, the horn antenna is positioned 20 cm from the reflectarray, while the electric field polarization for both the plane wave and horn illumination points in the ẑ direction.

Figure 9(a) shows the complex-plane polarizability mapping between the unconstrained polarizability values obtained using Eq. (19) and the binary values corresponding to the radiating (on) and non-radiating (off) patches for normal plane wave illumination. The corresponding constrained values are shown in Fig. 9(c). For this demonstration, we take the values shown in Fig. 9(c) and (d) as our achievable design values, and map the unconstrained polarizabilities to these binary polarizabilities using a simple phase thresholding method. That is, for each patch position, we compare the phase of the unconstrained value obtained by Eq. (19) to the phase of the radiating patch. If the phase difference is less than a chosen threshold, here taken as 90°, then a radiating patch is placed at that location. Otherwise, a non-radiating, shorted patch is used. In the complex-plane representation of the polarizabilities, this 90° threshold maps the unconstrained values to the binary value residing in the same half-circle of the complex plane. The periodicity observed in the binary solution is known to result in strong quantization lobes [7, 8, 9], which are evident in the far-field radiation pattern illustrated in Fig. 9(g).

In contrast, the designed polarizability distribution obtained under pyramidal horn illumination similarly oriented with its magnetic field in the ẑ direction is depicted in Fig. 9(e) and its binary-constrained mapping in (f). The polarizability design corresponding to horn illumination exhibits noticeably reduced periodicity, which substantially improves the far field response (Fig. 9(e)) by eliminating quantization lobes.

In Fig. 10 we compare the beam pattern predicted by our polarizability model to that computed in the full-wave electromagnetic simulation software CST for a binary reflectarray. The designed reflectarray consists of the patches corresponding to the polarizability values reported in Fig. 3 and employed in the previous result of Fig. 9. Here, the patches are again spaced at half of the free space wavelength to cover a 30 cm × 30 cm area, and the polarizability distribution selected by Eq. (19) to steer a beam to (θ₀, φ₀) = (25°, 0°). After constraining the ideal polarizabilities to binary states using the previously described phase thresholding method, the designed geometry was modelled and simulated in the frequency domain to evaluate the far field response. The resonating and shorted patch geometries can be seen in detail in Fig. 10(a), which provides a closeup perspective of four of the modelled patches with the dielectric substrate removed. The resulting far-field radiation pattern is represented in Fig. 10(b) as a surface plot, which we may compare to the equivalent representation computed using the extracted patch polarizabilities and our polarizability model in Fig. 10(c). Figure 10(d) shows cross sections over
standard surface equivalence principles, or through numerical or experimental far-field measurements. Using these recovered polarizabilities, the patch reflectarray can be specified using a holographic inverse method. We have applied this method to the design of a variable patch size, grayscale state reflectarray, as well as that of a binary state reflectarray, and shown that the method results in accurate beam pattern predictions that agree well with full-wave simulation.

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