Analytical results for Scaling Properties of the Spectrum of the Fibonacci Chain

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Abstract

We solve the approximate renormalisation group found by Qiu Niu and Franco Nori \cite{2} for a quasiperiodic tight-binding hamiltonian on the Fibonacci chain. This enables us to characterize analytically the spectral properties of this model.

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In the past few years, many models have shown that a particle moving on an infinite chain and subjected to a quasiperiodic (QC) or incommensurate (IC) modulation can exhibit critical localization properties. Depending on the QC (IC) modulation strength, in contrast to the case of a disordered modulation, the particle wavefunction is not necessarily strongly localized (Insulating); it can also be extended (Metallic) as in a translationally invariant system or in a critical localization regime at the Metal–Insulator transition. Typical models where these peculiar localization properties occur are Harper like models (IC) or tight-binding hamiltonians associated with a quasiperiodic sequence such as the Fibonacci sequence (QC). In general, these models have two important parameters: an irrational \( \omega \) which is responsible for the absence of periodicity, and a parameter \( K \) which determines the strength of the (QC) or (IC) modulation. The usual procedure to characterize the localization properties of these hamiltonians \( H_{\omega,K} \) is to study the scaling of the spectral properties of a sequence of periodic hamiltonians \( H_{\omega_n,K} \) which converge to \( H_{\omega,K} \) as \( n \) goes to infinity. If the widths of the \( q_n \) individual bands that compose the spectrum of \( H_{\omega_n,K} \), decrease exponentially with the period \( (q_n) \) of \( H_{\omega_n,K} \) then, \( H_{\omega,K} \) is in a insulating regime and has a Pure Point spectrum. In the case when the widths decrease only inversely proportionally to \( q_n \), then \( H_{\omega,K} \) is in the metallic regime and has a Absolutely Continuous (AC) spectrum. In contrast with these two cases, the critical localization regime of \( H_{\omega,K} \) is expected to have a Singular Continuous (SC) spectrum with multifractal (MF) properties in the spectral measure. In fact, in this latter case, the scaling found by many numerical simulations is that the bandwidths decrease like \( \sim q_n^{-1/\alpha} \) where the exponent \( \alpha < 1 \) varies from band to band so that there are typically \( \sim q_n^{+g(\alpha)} \) bands with the same exponent. Due to these peculiarities, the critical regime has been extensively studied by both numerical simulations and analytic techniques. However, as yet, a analytical quantitative determination of the exponents \( \alpha \) and \( g(\alpha) \), has not been achievable for even one irrational \( \omega \).

The purpose of this paper is to show a QC tight-binding model for which we are able to analytically characterize all the spectral properties. Our work starts from the approximate renormalisation group (RG) found by Niu and Nori for a QC model on the Fibonacci
chain. By reformulating and solving the RG of Niu and Nori, we derive constructive and transparent recurrence schemes for both the energy levels and the bandwidths. From these two schemes we deduce new recurrence relations for the spectral measure, the large time average return probability of particle defined in [3], the spectrum Lebesgue measure, the MF partition function [4] and the bandwidth distribution. For most of these relations, a natural fixed point solution is a power law, either in size or time. By comparing with the fixed point equation of the MF partition function, it appears that the exponents associated to these power laws, are related to a subset of the anomalous dimensions which characterize the MF properties of the spectral measure. A direct calculation of the function $g(\alpha)$ vs $\alpha$ confirms these MF properties. To complete the analysis of the spectral properties, we also study the gap properties. We find that there are two types of gaps: transient and stable. For the first type, their properties are like those of the bandwidths. In contrast, for the stable gaps, the distribution of their width ($g$) is a stable power law $P(g) \sim g^{-(1 + D_F)}$ where $D_F$ is the Hausdorff dimension of the spectrum measure.

These results complete and correct a previous MF analysis of the spectral measure of this model [5]; our work also unifies many partial results obtained by other methods, for both this model [6] [7] [8] and the Harper model [9].

We consider the tight–binding hamiltonian $H_n$ defined on approximant of period $F_n$ of the Fibonacci chain by the following equation:

$$H_n = \sum_{i=1}^{F_n} V_i c_i^\dagger c_i + t_{i,i+1} c_i^\dagger c_{i+1} + t_{i-1,i} c_i^\dagger c_{i-1}$$  \hspace{1cm} (1)

The on site potential $V_i$ is taken to be uniform ($V_i = V$). In contrast, the hopping amplitude $t_{i,i+1}$ from site $i$ to site $i+1$ is given by $t_{i,i+1} = t_w (1 - \chi(\omega_n i)) + t_s \chi(\omega_n i)$, where $\omega_n = \frac{F_{n-1}}{F_n}$ tends to the golden mean $\omega = \frac{\sqrt{5} - 1}{2}$ ($F_{n+1} = F_n + F_{n-1}$ and $F_n \simeq \omega^n$). The characteristic function $\chi(\omega i)$ takes the value 0 or 1 according to the Fibonacci sequence and correspondingly, the bond $t_{i,i+1}$ will take the value $t_w$ or $t_s$. For finite $n$ the density of weak bonds ($t_w$) is $\omega_n$ and tends to $\omega$ in the quasiperiodic limit. For the strong bonds ($t_s$) the density is $\omega_s^2 = \frac{F_{n-2}}{F_n}$ and tends to $\omega^2$. The periodicity of the hamiltonian $H_n(V, t_s, t_w)$ allows us to define Bloch
boundary conditions of the form \( c_{j+F_n} = e^{ik}c_j \). For a fixed \( k \), the energy spectrum of \( H_n(V, t_s, t_w) \), which we define as \( W_n(V, t_w, t_s) \), consists of \( F_n \) levels \( E_n^i(k) \) \((i = 1, ..., F_n\) and \( E_n^i \leq E_n^{i+1} \) by convention). Varying \( k \) from 0 to \( \pi \) allows the association of an energy band of width \( \Delta_n^i = |E_n^i(\pi) - E_n^i(0)| \) to each of these levels.

Using a perturbative approach, Niu and Nori have shown that in the strong modulation regime \((t_w/t_s \ll 1)\) the spectrum \( W_n(0, t_w, t_s) \) is the union of three sub–spectra \( W_{n-2}(V^+, t_s^+, t_w^+) \), \( W_{n-3}(V^0, t_s^0, t_w^0) \) and \( W_{n-2}(V^-, t_s^-, t_w^-) \), which correspond to three sub–hamiltonians with periods \( F_{n-2}, F_{n-3}, F_{n-2} \) and renormalized parameters \( V^\pm, t_s^\pm, t_w^\pm \) respectively. A representation of this perturbative RG, with the explicit value of the renormalized parameters, is schematically given by the following relation (2).

\[
W_n(0, t_w, t_s) \rightarrow \begin{cases} 
W_{n-2}(t_s, \frac{t_w}{2}, \frac{t_w^2}{2t_s}) \\
W_{n-3}(0, \frac{t_w^2}{t_s}, \frac{t_w^3}{t_s}) \\
W_{n-2}(-t_s, -\frac{t_w}{2}, -\frac{t_w^2}{2t_s})
\end{cases}
\tag{2}
\]

In principle, the scheme (2) simplifies the problem, since it relates the spectral properties of a hamiltonian of period \( F_n \) to those of three sub–hamiltonians of smaller period. However, it is clear that upon \( l \) iterations of (2), the difficulty which is initially due to the large period \( F_n \), is replaced by the problem of an increasing number \((3^l)\) of different hamiltonians to be solved.

As we now describe, there is some properties of both the hamiltonian and the RG (2) allow us to overrule this difficulty. Firstly, it is clear that the spectrum of \( H_n(0, t_s, t_w) \) is independent of the sign of \( t_s \) and \( t_w \), thus we have \( W_n(0, t_s, t_w) = W_n(0, |t_s|, |t_w|) \) \[10\]. Secondly, we see that the spectrum \( W_{n-2}(V^\pm, t_s^\pm, t_w^\pm) \) is just uniformly translated from \( W_{n-2}(0, t_s^\pm, t_w^\pm) \) by a factor \( V^\pm \). Thirdly, the renormalized parameters have the following property:

\[
|t_s^\pm| = zt_s , \quad |t_w^\pm| = zt_w , \quad |t_s^0| = \bar{z}t_s , \quad |t_w^0| = \bar{z}t_w
\]

\[
z = \frac{t_w}{2t_s} \ll 1 \quad \bar{z} = \frac{t_w^2}{t_s^2 A} \ll 1
\tag{3}
\]

Combining these properties and using (2), we deduce the following new renormalisation scheme:
We see that if $t_s$ is sufficiently strong, the spectrum $W_{n-2}(0,t_s,t_w)$, which is contracted by a factor $z$ and centered around $\pm t_s$, is not mixed with the spectrum $W_{n-3}(0,t_s,t_w)$, which is contracted by $\bar{z}$. More precisely, if we call $\Delta_n$ the distance between the lowest and highest energy levels of $H_n$, then, this non–overlapping condition becomes $(z\Delta_{n-2} + \bar{z}\Delta_{n-3}) \leq 2t_s$. Under this condition, relation (4) gives the following recurrence scheme between the energy levels $E_n^i(0,t_s,t_w)$, $E_{n-2}^i(0,t_s,t_w)$ and $E_{n-3}^i(0,t_s,t_w)$:

$$W_n(0,t_s,t_w) \rightarrow \begin{vmatrix} -t_s + zW_{n-2}(0,t_s,t_w) \\ \bar{z}W_{n-3}(0,t_s,t_w) \\ +t_s + zW_{n-2}(0,t_s,t_w) \end{vmatrix}$$

(4)

Similarly, the associated recurrence for the bandwidths is given by:

$$\begin{align*}
\Delta_n^i &= z\Delta_{n-2}^i \\
\Delta_n^{i+F_{n-2}} &= \bar{z}\Delta_{n-3}^i \\
\Delta_n^{i+F_{n-1}} &= t_s + z\Delta_{n-2}^i
\end{align*}$$

(5)

(6)

We now give the quantitative consequences of the last three relations (4,5,6) upon the spectral properties. From the recurrence scheme (5) we see that we can assign a set of indices $\{+,0,-\}$ to each individual energy level, according to the path of bifurcations of that level. Therefore a typical level $E_n^i$ has $n_+, n_0$ and $n_-$ indices $+,0,-$; with the constraint $2(n_+ + n_-) + 3n_0 = n(\pm1)$. As examples, the lowest level is indexed by $\{-,-,-\}$, $(n_- = [n/3], n_+ = n_0 = 0)$ and the highest is indexed by $\{+,+,+\}$. From this indexation and relation (6), we see that the band associated with a level $E_n^i(n_+, n_0, n_-)$ has a width $\Delta_n^i(p,q) \approx z^p\bar{z}^q$ with $p = (n_+ + n_-)$ and $q = n_0$. Consequently, the number of bands of width $\Delta_n(p,q)$ is given by $N_n(p,q) = 2^p \binom{p+q}{p}$. These last two results allow us to calculate the exponents $\alpha$ and $g(\alpha)$ defined in the introduction ($\Delta_n^i = F_n^{-1/\alpha_i}$ and $N_n(p,q) = F_n^{g(\alpha)}$).
As shown in relation (7), in the quasiperiodic limit \((n \to \infty)\), these exponents are function of the parameter \(x = p/n\) which varies continuously in \([0, 1/2]\).

\[
\alpha(x) = \ln \omega / (x \ln z/\bar{z}^{2/3} + \ln \bar{z}^{1/3})
\]

\[
g(\alpha(x)) = (x \ln 3x/2 - (1 + x) \ln (1 + x)^{1/3} + (1 - 2x) \ln (1 - 2x)^{1/3})/\ln \omega
\]

From this last relation we can obtain two interesting properties of the spectrum. Firstly, we observe that when \(z = \bar{z}^{2/3}(t_w/t_s = 1/8)\), the exponent \(\alpha\) is independent of \(x\). As a consequence the spectrum is a pure fractal with Hausdorff dimension \(D_F = \alpha = \ln \omega^3/\ln \bar{z}\).

In contrast, when for example \(z > \bar{z}^{2/3}\), we see that the exponent \(\alpha\) varies between \(\alpha_{\text{min}} = \ln \omega^3/\ln \bar{z}\) and \(\alpha_{\text{max}} = \ln \omega^2/\ln z\). As these two exponents correspond to the bandwidth associated with the central and edge levels respectively, this property allows us to compare their values with the exact analytical result of Kohmoto [7] [8]. From this we observe that our shrinking factors \(z\) and \(\bar{z}\) are just the dominant terms in a \((t_w/t_s)\) series expansion of the two more exact values \(z_{ex} = 2/(\sqrt{(J - 1)^2 - 4 + J - 1})\) and \(\bar{z}_{ex} = 1/(\sqrt{1 + 4(I + 1)^2 + 2(I + 1)})\) where \(I = \frac{1}{4}(\frac{t_w}{t_s} - \frac{t_s}{t_w})^2\) and \(J = 3 + \sqrt{25 + 16I}\).

So far, we have only analyzed the properties of individual bandwidths and levels. However it is also possible and very instructive to study integrated quantities. We start with the spectral measure and a physical quantity closely related to it. The spectral measure \(d\mu_n(E)\) of the hamiltonian \(H_n\) is \(d\mu_n(E) = \rho_n(E)dE\) where \(\rho_n(E) = \frac{1}{F_n} \sum_{i=1}^{F_n} \delta(E - E_i^n)\) is the density of states. From this definition and relation (3) we deduce the following recurrence:

\[
d\mu_n(E) = \omega_n^2 d\mu_{n-2} \left(\frac{E + t_s}{z}\right) + \omega_n^3 d\mu_{n-3} \left(\frac{E}{\bar{z}}\right) + \omega_n^2 d\mu_{n-2} \left(\frac{E - t_s}{z}\right)
\]

As an example of the application of relation (8), we calculate the large time average return probability defined by \(p_n(t) = |\int_{-\infty}^{+\infty} e^{-iEt} d\mu_n(E)|^2 = 2\pi \tilde{\mu}(t) \tilde{\mu}^*(t)\). Using (8) we firstly deduce that \(\tilde{\mu}_n(t) = 2\omega_n^2 \cos t_s t \tilde{\mu}_{n-2}(zt) + \omega_n^3 \tilde{\mu}_{n-3}(\bar{z} t)\). Now, in the large time limit, we have on average \(\langle \cos t_s t \rangle \sim 0\) and \(\langle \cos^2 t_s t \rangle \sim 1/2\), and from this we immediately get:

\[
p_n(t) = 2\omega_n^4 p_{n-2}(zt) + \omega_n^3 p_{n-3}(\bar{z} t)
\]
In the limit \((n \to \infty)\), we see that a fixed point solution of relation (9) is 
\[ p^* (t) \sim t^{-\gamma} \]
where the exponent \(\gamma\) is determined by 
\[ 2\omega^4 z^{-\gamma} + \omega^6 \bar{z}^{-\gamma} = 1. \]
As we now show, this exponent \(\gamma\) is one of the anomalous dimensions \(D_q\) that characterize the MF properties of the spectral measure. In our case, these non-trivial dimensions \(D_q\) are defined by the requirement that the partition function \(\Gamma_n(q, \tau = (q - 1)D_q) = F_n^{-q} \sum_{i=1}^{F_n} (\Delta^i_n)^{-\tau}\) be stationary in the limit \(n \to \infty\). Using relation (6) we get the following recurrence for the \(\Gamma_n(q, \tau)\) [5]:
\[ \Gamma_n(q, \tau) = 2\omega^2 q z^2 \Gamma_{n-2}(q, \tau) + \omega^3 q \bar{z} \Gamma_{n-3}(q, \tau) \]  (10)
The stationary constraint then gives a self-consistent equation for the \(D_q\)
\[ 2\omega^2 q z(1-q)D_q + \omega^3 q \bar{z}(1-q)D_q = 1 \]  (11)
From this last relation, we immediately see that the exponent \(\gamma\) previously defined is in fact equal to \(D_2\). A second consequence of (11) is that the Hausdorff dimension \(D_F = D_0\) is the solution of \(2z^{D_F} + \bar{z}^{D_F} = 1\). To see further the use of relation (11) and the role of the \(D_q\) we calculate two other quantities of interest: the Lebesgue measure \(B_n = \sum_{i=1}^{F_n} \Delta^i_n\); and the number of bands of width between \(\Delta\) and \(\Delta + d\Delta\), \(dN_n(\Delta) = \sum_{i=1}^{F_n} \delta(\Delta - \Delta^i_n)d\Delta\). Using relation (6) we can easily deduce a recurrence relation for each of these quantities:
\[ B_n = 2zB_{n-2} + \bar{z}B_{n-3} \]  (12)
\[ dN_n(\Delta) = 2dN_{n-2}(\frac{\Delta}{z}) + dN_{n-3}(\frac{\Delta}{\bar{z}}) \]  (13)
The first of these equations was partially guessed in [6] for a model on the Fibonacci chain; a very similar relation was also derived for the case of the Harper model in [9]. From equation (12), we can show that the large \(n\) behavior of \(B_n\) is 
\[ B_n \sim B_0 F_n^{-\delta} \]
where the exponent \(\delta\) is related to the anomalous dimensions by \(D_{-\delta} = \frac{1}{1+\delta}\). Now, considering (13), we see that in the limit \(n \to \infty\), a possible invariant form is given by 
\[ dN^*(\Delta) = \Delta^{-(1+\beta)}d\Delta \]
with \(\beta = D_F\). In view of relation (6), this simple invariant solution is quite surprising and indeed its sense is not very clear. In particular, if instead of \(dN_n(\Delta)\) we look at the distribution of
bandwidths \( W_n(\Delta) = \frac{dN_n(\Delta)}{d\Delta} \), then the corresponding recurrence relation allows the two possible invariant distributions \( W^*(\Delta) = \delta(\Delta) \) and \( W^*(\Delta) = \Delta^{-1} \), whose form do not coincide with \( W^*(\Delta) = \frac{dN^*(\Delta)}{d\Delta} = \Delta^{-(1+D_F)} \) \([12]\).

To complete the study of the spectral properties, we also look at the statistical properties of the gaps. Roughly speaking a gapwidth is the distance between two levels. In consequence, we might expect their distribution to be quite similar to that of the bandwidths. However, there is an important difference; the number of gaps of a chain \( F_n \) is \( F_n - 1 \), thus, \( 2(F_{n-2} - 1) + F_{n-3} - 1 = F_n - 3 < F_n - 1 \)! Looking at figure \([1]\) the last inequality means that if we take only gaps coming from \( zW_{n-2}(\pm t_s, t_s, t_w) \) and \( \bar{z}W_{n-3}(0, t_s, t_w) \), we miss two gaps which are in fact precisely the biggest. Taking this into account, we see that the number of gaps of width between \( g \) and \( g + dg \), \( dN_n(g) = \sum_{i=1}^{F_n} \delta(g - g_n^i)dg \), obeys the following recurrence \((n \geq 3)\):

\[
dN_n(g) = 2dN_{n-2}(\frac{g}{z}) + dN_{n-3}(\frac{g}{\bar{z}}) + 2\delta(g - g_n^0)dg
\]

where the last term re–introduces the two largest gaps of width \( g_n^0 \) at each iteration. A further refining of our description requires two other important remarks. The first is that the initial conditions are \( dN_0(g) = 0 \), \( dN_1(g) = 0 \) and \( dN_2(g) = \delta(g - g_0)dg \). The second is that in the limit \( n \to \infty \) the width \( g_n^0 \) tends to a fixed value \( g^* = t_A - \frac{1}{2}(\bar{z}\Delta^* + z\Delta^*) = t_A \frac{1 - z^2}{(1-z)} \) where \( \Delta^* \) is the width of the spectrum in this limit. This last remark allows to replace the previous recurrence for \( dN_n(g) \) by the following effective equation which describes the infinite size behavior more effectively:

\[
dN_n(g) \simeq 2dN_{n-2}(\frac{g}{z}) + dN_{n-3}(\frac{g}{\bar{z}}) + 2\delta(g - g^*)dg
\]

As shown in table \([1]\) the iteration of relation \((15)\), with the previous initial conditions, produces two kinds of gaps. The first kind corresponds to what we call the transient gaps. For a chain \( F_n \), these gaps have widths of the form \( g = z^p\bar{z}^q g_0 \) with \( (2p+3q = n - 2) \) and \( N_n(p, q) = 2^p \binom{p+q}{p} \) (last two columns of table \([1]\)). These transient gaps are created by iteration of the initial condition \( dN_2(g) = \delta(g - g_0)dg \) and their effective recurrence does not
contain the last term of (15). For these reasons their distribution is strictly similar to that of bandwidths and in particular the sums of their widths decreases like the spectrum Lebesgue measure \( B_n \). In contrast, the second kind of gaps are those created by the presence of the last term in relation (15) (first two columns of table I). As can be seen in table I, if a gap of this kind opens for, say a chain \( F_p \), it persists for longer chains; it is stable. For a such a chain \( F_n \), these stable gaps have widths of the form \( g = z^p \bar{z}^q g^* \) with \( N_n(p, q) = 2^p \binom{p+q}{p} \) but now \( 2p + 3q \) takes all values between 0 and \( n - 3 \). Due to this difference the distribution of the stable gaps differs from that of the transient gaps in the following way: in a similar manner to (13), the absence of the last term yields an invariant solution to (15) of the form \( dN^*(g) = g^{-(1+D_F)} dg \). However, similarly to bandwidths (12), for the transient gaps, \( dN_n(g) \) tends to \( dN^*(g) \). In contrast when the last term is present, that is for the stable gaps, then, the function \( dN_n(g) \) really tends to \( dN^*(g) = g^{-(1+D_F)} dg \) over the whole interval \([0, g^*]\). Due to this property, in that case, we can also define a distribution which is of the form \( P^*(g) = \frac{dN^*(g)}{dg} = g^{-(1+D_F)} \). An additional property of the stable gaps concerns the value \( (G_n) \) of the sum of their widths. From (13) we see that \( G_n \) obeys a recurrence relation \( G_n = 2z G_{n-2} + \bar{z} G_{n-3} + 2g^* \), and from this we can deduce that in contrast to \( B_n \), \( G_n \) does not decrease with \( F_n \) but tends to a value \( G^* \) which is exactly equal to the spectrum width \( \Delta^* \).

In conclusion, we have described as completely as possible the statistical properties of the energy spectrum of a tight-binding hamiltonian on the Fibonacci chain. We have compared several of the new predictions with numerical computations with satisfactory results. As the text has made clear, our results, qualitatively and quantitatively complete and unify previous works on similar models.

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REFERENCES

[1] for reviews see H. Hiramoto and M. Kohmoto, Int. J. Mod. Phys. **B6**, 281 (1992), J. B. Sokoloff, Phys. Rep. **126**, 189 (1985) and also references therein.

[2] Q. Niu and F. Nori, Phys. Rev. Lett. **57**, 2057(1986), Phys. Rev. **B 42** 10329 (1990)

[3] T. Geisel, R Ketzmerick, and G. Petschel, Phys. Rev. Lett. **66** 1651 (1991) and Phys. Rev. Lett. **69** 695 (1992)

[4] T. C. Halsey et al, Phys. Rev. **A 33**, 1141 (1986)

[5] W. M. Zheng Phys. Rev. **B 35** 1467 (1987)

[6] M. Kohmoto Phys. Rev. Lett. **51**, 1198 (1983)

[7] M. Kohmoto and Y. Oono phys. Lett. **A102** 4 145 (1984)

[8] M. Kohmoto, B. Sutherland and C. Tang, G. Phys. Rev. **B 35** 1024 (1987)

[9] S. C. Bell and R. B. Stinchcombe J. Phys. A: Math. Gen **22** 717 (1989)

[10] For finite chain, this is only valid for the Bloch phase $k = \pi/2$ ($\sim$ band centers).

[11] This indexation allows us to find the positions of the two edges energy levels in the quasiperiodic limit: $(E_{\text{max},\text{min}} = \pm \frac{1}{1-z}t_s)$. This enables us to define an exact non–overlapping condition of the form: $\frac{z+\bar{z}}{1-z^2} \leq 1$.

[12] There are many other problems with distributions related to the bandwidths and the way we take the limit $n \to \infty$. All these problems come from the fact that both the individual bandwidth and their degeneracy are too much strongly fluctuating variables. Due to this, even if $dN^*(\Delta)$ vs $\Delta$ is an invariant of (13), for any finite chain $F_n$ the function $dN_n(\Delta)$ does not tend to $dN^*(\Delta)$. In fact, only the function $g(\alpha)$ has a meaning.

[13] F. Piéchon in preparation
TABLES

TABLE I. Gapwidths obtained for the six first iterations of relation (15). Stable gaps are written in the first column. Transient gaps are in the third column.

| $F_n$ | $g/g^*$ | $N(g)$ | $g/g_0$ | $N(g)$ |
|-------|---------|--------|---------|--------|
| 2     | #       | #      | 1       | 1      |
| 3     | 1       | 2      | #       | #      |
| 5     | 1       | 2      | $z$     | 2      |
| 8     | $1 \, z$ | 2 $4 \, \bar{z}$ | 1      |
| 13    | $1 \, z \, \bar{z}$ | 2 $4 \, 2 \, z^2$ | 4      |
| 21    | $1 \, z \, \bar{z} \, z^2$ | 2 $4 \, 2 \, 8 \, z\bar{z}$ | 4      |
| 34    | $1 \, z \, \bar{z} \, z^2 \, z\bar{z}$ | 2 $4 \, 2 \, 8 \, 8 \, z^3 \, \bar{z}^2$ | 8 $1$ |
| 55    | $1 \, z \, \bar{z} \, z^2 \, z\bar{z} \, z^3 \, \bar{z}^2$ | 2 $4 \, 2 \, 8 \, 8 \, 16 \, 2 \, \bar{z}^2 \bar{z}$ | 12     |
FIGURES

FIG. 1. Spectrum of the approximant hamiltonian \( (H_n, F_n) \), \( n \leq 8 \) deduced from relation (5) for \( (t_s = 1, t_w = 0.5) \). The three initial conditions are, \( (H_0, F_0 = 1, t_i = t_s) \); \( (H_1, F_1 = 1, t_i = t_w) \) and \( (H_2, F_2 = 2, t_{2i} = t_s, t_{2i+1} = t_w) \). \( z \) and \( \bar{z} \) are the two shrinking factors. \( g_O \) and \( g^* \), are the initial transient gap and the maximum stable gap respectively.