A NON-ARCHIMEDEAN APPROACH TO K-STABILITY, I:
METRIC GEOMETRY OF SPACES OF TEST CONFIGURATIONS AND
VALUATIONS
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Abstract. For any polarized variety \((X, L)\), we show that test configurations and, more gen-
erally, \(\mathbb{R}\)-test configurations (defined as finitely generated filtrations of the section ring) can be
analyzed in terms of Fubini–Study functions on the Berkovich analytification of \(X\) with respect
to the trivial absolute value on the ground field. Building on non-Archimedean pluripotential
theory, we describe the (Hausdorff) completion of the space of test configurations, with respect
to two natural pseudo-metrics, in terms of plurisubharmonic functions and measures of finite
energy on the Berkovich space. We also describe the Hausdorff quotient of the space of all
filtrations, and compare various induced metrics on the space of valuations of linear growth.

Contents

Introduction 1
1. Background 8
2. Homogenization and the Fubini–Study operator 14
3. Spectral analysis 21
4. Non-Archimedean pluripotential theory 29
5. The Darvas metric 33
6. The regularized Fubini–Study operator 40
7. The space of valuations of linear growth 42
Appendix A. Test configurations, integral closure and homogenization 45
Appendix B. The toric case 47
References 48

Introduction

The notion of K-stability was introduced in complex differential geometry as a conjectural—and
now partially confirmed—algebro-geometric criterion for the existence of special Kähler
metrics. Lately, it has also become a subject in its own respect. In a series of two (largely
independent) papers of which this is the first, we show how global pluripotential theory over a
trivially valued field, as developed in [BoJ21], can be used to study K-stability.

Let \(X\) be a projective variety (reduced and irreducible) of dimension \(n \geq 1\) over an alge-
braically closed field \(k\) (of arbitrary characteristic for the moment), and \(L\) an ample \(\mathbb{Q}\)-line
bundle on \(X\). The definition of K-stability of the polarized variety \((X, L)\), as given by Don-
aldson [Don02], involves the sign of an invariant attached to (ample) test configurations for
\((X, L)\). As shown in [BHJ17, BoJ21], test configurations can be alternatively understood in
terms of (rational) Fubini–Study functions on the Berkovich analytification \(X^{an}\), and uniform
K-stability becomes a linear growth condition for the non-Archimedean K-energy on the set of such functions.

Filtrations of the section ring of $(X, L)$ provide another, widely used description of test configurations; more precisely, the latter correspond to $\mathbb{Z}$-filtrations of finite type $[\text{WN12, Sze15}].$ Recent works related to the Hamilton–Tian conjecture $[\text{CSW18, DeSz20, HL20, BLXZ21}]$ have emphasized the importance of considering more general $\mathbb{R}$-test configurations, defined as $\mathbb{R}$-filtrations of finite type, and the first objective of this paper is to show that these can again be understood as (real) Fubini–Study functions on $X^{\text{an}}.$

On the other hand, Chi Li’s recent breakthrough on the Yau–Tian–Donaldson conjecture for cscK metrics $[\text{Li20}]$ (based in part on the first version of the present paper) involves a stronger form of uniform K-stability, formulated as a linear growth condition for the K-energy on the space of functions of finite energy on $X^{\text{an}}.$ The latter are obtained as limits of Fubini–Study functions, and are the central topic of pluripotential theory on $X^{\text{an}}$ $[\text{BoJ21}].$ Building on the latter technology, the second objective of this paper is to show how functions and measures of finite energy can be described using the completion of the space of test configurations with respect to a natural metric, leading to a picture that is quite similar to the well-developed complex analytic case $[\text{Dar15a, DLR20}].$

**From test configurations to Fubini–Study functions.** Denote by $\mathcal{N}_{\mathbb{R}}$ the space of (decreasing, left-continuous, separated, exhaustive) filtrations of the section algebra $R^{(r)} = R(X, rL)$ for $r$ sufficiently divisible. It is convenient to view these as norms $\chi: R^{(r)} \rightarrow \mathbb{R} \cup \{+\infty\},$ for which we use ‘additive’ terminology, see $[\text{IL}].$ A norm $\chi \in \mathcal{N}_{\mathbb{R}}$ is of finite type if the associated graded algebra is finitely generated. For any subgroup $\Lambda \subset \mathbb{R},$ let $\mathcal{N}_{\Lambda} \subset \mathcal{N}_{\mathbb{R}}$ be the set of norms with values in $\Lambda \cup \{+\infty\},$ and denote by

$$\mathcal{T}_{\mathbb{R}} \subset \mathcal{N}_{\mathbb{R}} \quad \text{and} \quad \mathcal{T}_{\Lambda} := \mathcal{T}_{\mathbb{R}} \cap \mathcal{N}_{\Lambda}$$

the subsets of norms of finite type.

As is by now well-known (see $[\text{WN12, Sze15, BHJ17}]),$ the Rees construction provides a 1–1 correspondence between $\mathcal{T}_{\mathbb{Z}}$ and the set of (ample) test configurations for $(X, L).$ In line with $[\text{DeSz20}],$ we view $\mathcal{T}_{\mathbb{R}}$ as the space of $\mathbb{R}$-test configurations. Any $\chi \in \mathcal{T}_{\mathbb{R}}$ lies in $\mathcal{T}_{\Lambda}$ for some finitely generated subgroup $\Lambda \simeq \mathbb{Z}^{r}.$ Geometrically, $\chi$ can be interpreted as a degeneration of $(X, L)$ to a polarized projective scheme $(X_0, L_0)$ with an action of the torus $\text{Spec } k[\Lambda] \simeq \mathbb{G}_{m}^{r},$ obtained as the Proj of the associated graded algebra. The scheme $X_0$ is reduced iff $\chi$ is homogeneous, i.e. $\chi(s^d) = d\chi(s)$ for all $d \in \mathbb{N}.$

The space $\mathcal{N}_{\mathbb{R}}$ comes equipped with a non-decreasing family $(d_p)_{1 \leq p \leq \infty}$ of natural pseudo-metrics. By $[\text{BE21}],$ the space $\mathcal{N}_{\mathbb{R}}(V)$ of norms on any finite dimensional vector space $V$ is indeed endowed with a metric $d_p$ for any $p \in [1, \infty],$ the distance between two norms being the $L^p$-length of their relative spectrum, defined by joint diagonalization in some basis. For $p = 2,$ this is the classical Tits metric of $\mathcal{N}_{\mathbb{R}}(V)$ as a Euclidean building, whose relevance to K-stability was already emphasized in $[\text{Oda15, Cod19}].$

Any $\chi \in \mathcal{N}_{\mathbb{R}}$ restricts to a norm on $R_m := H^0(X, mL)$ for all $m$ sufficiently divisible, and we define the pseudometric $d_p$ on $\mathcal{N}_{\mathbb{R}}$ by setting

$$d_p(\chi, \chi') := \limsup_{m} m^{-1} d_p(\chi|_{R_m}, \chi'|_{R_m}),$$

where the limsup is actually a limit for $p \leq \infty,$ by $[\text{BC11, CM15, BHJ17}].$ The $L^p$-norm of a test configuration in $\mathcal{T}_{\mathbb{Z}},$ as in $[\text{Don02, Sze15, BHJ17}],$ can be computed using $d_p.$

The pseudo-metric $d_p$ is not a metric, even after restriction to $\mathcal{T}_{\Lambda},$ and our first main result describes the Hausdorff quotient of $(\mathcal{T}_{\Lambda}, d_p)$ as a natural space of functions on the Berkovich
analytification of $X$ (with respect to the trivial absolute value on $k$). Recall that the latter is a compact Hausdorff topological space $X^{an}$, whose elements are semivaluations $v$ on $X$, i.e. $\mathbb{R}$-valued valuations on the function field of some subvariety of $X$, trivial on $k$. The space $X^{an}$ contains as a dense subset the space $X^{\text{div}}$ of divisorial valuations on $X$, induced by a prime divisor on a birational model of $X$.

For any $v \in X^{an}$ and any section $s$ of a line bundle on $X$, we can define $v(s) \in [0, +\infty]$ by trivializing the line bundle at the center of $v$, and setting $|s|(v) := e^{-v(s)}$ defines a continuous function $|s|: X^{an} \to [0, 1]$. Given a subgroup $\Lambda \subset \mathbb{R}$, a $\Lambda$-Fubini–Study function for $L$ is a function $\varphi \in C^0 = C^0(X^{an})$ of the form

$$\varphi = \frac{1}{m} \max_j \{|s_j| + \lambda_j\},$$

where $m \geq 1$ is sufficiently divisible, $(s_j)$ is a finite set of sections of $mL$ without common zeroes, and $\lambda_j \in \Lambda$.

The set $H_\Lambda \subset C^0$ of $\Lambda$-Fubini–Study functions is stable under max and the action of $\Lambda$ by translation, and satisfies $H_\Lambda = H_{\Lambda^0}$. It is related to the space $T_\Lambda$ of $\Lambda$-test configurations by the Fubini–Study operator, a surjective map

$$FS: T_\Lambda \rightarrow H_\Lambda$$

that associates to $\chi \in T_\Lambda$ the Fubini–Study function $FS(\chi) = m^{-1} \max_j \{|s_j| + \chi(s_j)\}$, where $(s_j)$ is any $\chi$-orthogonal basis of $R_m$ for $m$ sufficiently divisible. Viewed as a map from (usual) test configurations to Fubini–Study functions, $FS: T_Z \rightarrow H_Z = H_Q$ is compatible with the one constructed and studied in [BHJ17, BoJ21] (see Appendix A).

**Theorem A.** For any polarized variety $(X, L)$, any subgroup $\Lambda \subset \mathbb{R}$ and $p \in [1, \infty]$, the Fubini–Study operator identifies the Hausdorff quotient of the pseudo-metric space $(T_\Lambda, d_p)$ with $H_\Lambda$. For $p = \infty$, the induced metric $d_\infty$ on $H_\Lambda$ further coincides with the supnorm metric.

It is enough to prove this for $\Lambda = \mathbb{R}$. Let us first describe the case $p = \infty$. The restrictions $\chi|_{R_m}$ of any norm $\chi \in N_\mathbb{R}$ generate a sequence of canonical approximants $\chi_m \in T_\mathbb{R}$, which allows us to extend the Fubini–Study operator to a map

$$FS: N_\mathbb{R} \rightarrow \mathcal{L}^\infty$$

into the space of bounded functions on $X^{an}$, by setting $FS(\chi) := \lim_m FS(\chi_m)$. On the other hand, any $\varphi \in \mathcal{L}^\infty$ defines an infimum norm $IN(\varphi) \in N_\mathbb{R}$, the avatar of the usual supnorm $\sup_{X^{an}} |s| e^{-m\varphi}$ on $R_m$ in our additive terminology. This defines an operator

$$IN: \mathcal{L}^\infty \rightarrow N_\mathbb{R}^\text{hom}$$

into the space of homogeneous norms. Using standard but nontrivial results in non-Archimedean geometry, we show that:

- the composition $IN \circ FS: N_\mathbb{R} \rightarrow N_\mathbb{R}^\text{hom}$ coincides with the homogenization operator, which corresponds to the spectral radius construction in the usual ‘multiplicative’ terminology;
- homogenization maps $T_\mathbb{R}$ onto $T_\mathbb{R}^\text{hom} := T_\mathbb{R} \cap N_\mathbb{R}^\text{hom}$;
- on $T_\mathbb{R}$, both the Fubini–Study operator and the pseudo-metric $d_\infty$ factor through homogenization.

These results imply that $FS: (T_\mathbb{R}, d_\infty) \rightarrow (H_\mathbb{R}, d_\infty)$ is a surjective isometry, which restricts to an isometric isomorphism $(T_\mathbb{R}^\text{hom}, d_\infty) \simeq (H_\mathbb{R}, d_\infty)$, where $d_\infty$ on $H_\mathbb{R}$ is the supnorm metric; this settles Theorem A for $p = \infty$. 


For any \( p \in [1, \infty) \), we have \( d_1 \leq d_p \leq d_\infty \) as pseudo-metrics on \( \mathcal{T}_R \subset \mathcal{N}_R \). By the previous step, \( d_\infty \) factors through the Fubini–Study operator. Thus \( d_p \) descends to a pseudo-metric on \( \mathcal{H}_R \), and Theorem A asserts that it is a metric. It is enough to prove this for \( p = 1 \), which is accomplished via an explicit expression for \( d_1 \) in terms of the Monge–Ampère energy, analogous to the known expression for the Darvas metric in the complex analytic case [Dar15b].

Our approach is based on the close relation of the \( d_1 \)-pseudometric on \( \mathcal{N}_R \) to the volume of a norm \( \chi \in \mathcal{N}_R \), defined as the limit
\[
\text{vol}(\chi) = \lim_{m \to \infty} m^{-1} \text{vol}(\chi |_{R_m}) \in \mathbb{R},
\]
where \( \text{vol}(\chi |_{R_m}) \) is the barycenter of the spectrum of \( \chi |_{R_m} \). Indeed, for all \( \chi, \chi' \in \mathcal{N}_R \), we have
\[
d_1(\chi, \chi') = \text{vol}(\chi) + \text{vol}(\chi') - 2 \text{vol}(\chi \wedge \chi'),
\]
with \( \chi \wedge \chi' \in \mathcal{N}_R \) the pointwise min of \( \chi \) and \( \chi' \). When \( \chi \in \mathcal{T}_R \) corresponds to a test configuration \( (X, \mathcal{L}) \to \mathbb{A}^1 \), with canonical compactification \( (\bar{X}, \bar{\mathcal{L}}) \to \mathbb{P}^1 \), it was proved in [BHJ17], using the Riemann–Roch formula, that
\[
\text{vol}(\chi) = \frac{(\mathcal{L}^n + 1)}{(n+1)(L^n)},
\]
where the right-hand side is also, by definition, the Monge–Ampère energy \( E(\varphi) \) of \( \varphi := \text{FS}(\chi) \in \mathcal{H}_Q \). Setting \( E(\varphi) := \sup \{ E(\psi) \mid \varphi \geq \psi \in \mathcal{H}_Q \} \) defines a functional \( E: C^0 \to \mathbb{R} \), and an approximation argument based on Okounkov bodies leads to the key formula
\[
d_1(\chi, \chi') = E(\varphi) + E(\varphi') - 2 E(\varphi \wedge \varphi')
\]
for all \( \chi, \chi' \in \mathcal{H}_R \), where \( \varphi = \text{FS}(\chi), \varphi' = \text{FS}(\chi') \). The right-hand side thus defines a pseudo-metric \( d_1 \) on \( \mathcal{H}_R \), and a result of [BoJ21] allows us to show that it separates points, thereby finishing the proof of Theorem A.

**Functions and measures of finite energy.** By Theorem A, the Hausdorff completion of \( (\mathcal{T}_R, d_p) \) can be identified with the completion of the metric space \( (\mathcal{H}_R, d_p) \). When \( p = \infty \), this is simply the closure of \( \mathcal{H}_R \subset C^0 \) in the topology of uniformy convergence, which is by definition the space CPSH of continuous \( L \)-psh functions (in line with [Zha95, Gub98]).

Our next goal is to describe the completion of \( (\mathcal{H}_R, d_1) \). The answer relies on global pluripotential theory over a trivially valued field, as developed in [BoJ21] (inspired in part by the discretely valued case studied in [BFJ16]). Let us briefly described the salient points of this theory.

Inspired by the complex analytic case, we define an \( L \)-psh function \( \varphi: X^\an \to \mathbb{R} \cup \{-\infty\} \) as an upper semicontinuous (usc) function that can be written as the limit of a decreasing sequence (or net) in \( \mathcal{H}_R \) (or \( \mathcal{H}_Z = \mathcal{H}_Q \)), excluding \( \varphi \equiv -\infty \). Such functions are uniquely determined by their restrictions to \( X^\div \subset X^\an \), which are further finite valued, and we equip the space PSH of \( L \)-psh functions with the topology of pointwise convergence on \( X^\div \). By Dini’s Lemma, the space of continuous \( L \)-psh functions CPSH considered above can be described as CPSH = PSH \cap C^0.

The Monge–Ampère energy \( E \) admits a unique usc, monotone increasing extension
\[
E: \text{PSH} \to \mathbb{R} \cup \{-\infty\},
\]
given by \( E(\varphi) = \inf \{ E(\psi) \mid \varphi \leq \psi \in \text{PSH} \} \), and the space of \( L \)-psh functions of finite energy is defined as
\[
\mathcal{E}^1 := \{ E > -\infty \} \subset \text{PSH}.
\]
A function in $\mathcal{E}^1$ is thus a decreasing limit of functions in $\mathcal{H}_Q$ with bounded energy. The space $\mathcal{E}^1$ is endowed with the strong topology, defined as the coarsest refinement of the subspace topology from $\text{PSH} \supset \mathcal{E}^1$ for which $E: \mathcal{E}^1 \to \mathbb{R}$ is continuous. Any decreasing net in $\mathcal{E}^1$ is strongly convergent, and $\mathcal{H}_Q$ is thus dense in $\mathcal{E}^1$ in the strong topology.

To each $\varphi \in \mathcal{E}^1$ is associated a Monge–Ampère measure $\text{MA}(\varphi)$, a (Radon) probability measure on $X^{\text{an}}$ that integrates functions in $\mathcal{E}^1$. When $\varphi \in \mathcal{H}_Q$, $\text{MA}(\varphi)$ has finite support in $\text{div}$, and can be described using intersection numbers computed on the central fiber of an associated test configuration. The Monge–Ampère operator $\varphi \mapsto \text{MA}(\varphi)$ is continuous on $\mathcal{E}^1$, and characterized as the derivative of $E$, i.e.

$$
\frac{d}{dt} \bigg|_{t=0} E((1-t)\varphi + t\psi) = \int_{X^{\text{an}}} (\psi - \varphi) \text{MA}(\varphi)
$$

for all $\varphi, \psi \in \mathcal{E}^1$. It takes its values in the space $\mathcal{M}^1$ of measures of finite energy, i.e. Radon probability measures $\mu$ on $X^{\text{an}}$ for which the Legendre transform

$$
E^\vee(\mu) := \sup_{\varphi \in \mathcal{E}^1} \{ E(\varphi) - \int \varphi d\mu \} \in [0, +\infty]
$$

is finite. In analogy to the complex analytic case [BBGZ13], the variational approach of [BoJ21] shows that $\mu = \text{MA}(\varphi)$ with $\varphi \in \mathcal{E}^1$ iff $\varphi$ achieves the supremum that defines $E^\vee(\mu)$.

The space $\mathcal{M}^1$ also comes with a strong topology, the coarsest refinement of the weak topology of measures such that $E^\vee: \mathcal{M}^1 \to \mathbb{R}_{\geq 0}$ is continuous. A key result of [BoJ21] shows that the Monge–Ampère operator induces a topological embedding with dense image

$$
\text{MA}: \mathcal{E}^1 / \mathbb{R} \hookrightarrow \mathcal{M}^1
$$

(with respect to the strong topologies), which is further onto iff continuity of envelopes holds for $(X, L)$.

The latter important property has several equivalent formulations, including the compactness of the quotient space $\text{PSH} / \mathbb{R}$, a fundamental fact in the setting of compact complex manifolds. It guarantees in particular that the usc regularization of the bounded function $\text{FS}(\chi)$ is $L$-psh for any norm $\chi \in \mathcal{N}_R$, thereby defining a regularized Fubini–Study operator

$$
\text{FS}^*: \mathcal{N}_R \to \mathcal{E}^\infty
$$

into the subspace $\mathcal{E}^\infty \subset \mathcal{E}^1$ of bounded $L$-psh functions. Continuity of envelopes was established (for any $L$) when $X$ is smooth and char $k = 0$, using multiplier ideals, and we conjecture that it holds as long as $X$ is normal (or merely unibranch, which is in turn a necessary condition).

With these preliminaries in hand, we can now state:

**Theorem B.** For any polarized variety $(X, L)$, the following holds:

(i) there exists a unique metric $d_1$ on $\mathcal{E}^1$ that defines the strong topology and extends the metric $d_1$ on $\mathcal{H}_R \subset \mathcal{E}^1$;

(ii) there exists a unique metric $d_1$ on $\mathcal{M}^1$ that defines the strong topology and induces the quotient metric of $d_1$ on $\mathcal{E}^1 / \mathbb{R} \hookrightarrow \mathcal{M}^1$;

(iii) the metric space $(\mathcal{M}^1, d_1)$ is complete, while $(\mathcal{E}^1, d_1)$ is complete iff continuity of envelopes holds for $(X, L)$;

(iv) if continuity of envelopes holds, then the regularized Fubini–Study operator identifies the Hausdorff quotient of $(\mathcal{N}_R, d_1)$ with the space $(\mathcal{E}^\infty, d_1)$ of $L$-psh functions regularizable from below.
As a result, \( \mathcal{M}^1 \) is the Hausdorff completion of \( (\mathcal{T}_R/\mathbb{R}, d_1) \), while \( \mathcal{E}^1 \) is the Hausdorff completion of \( (\mathcal{T}_R, d_1) \) iff continuity of envelopes holds, e.g. when \( X \) is smooth and \( \text{char } k = 0 \) (see also [DX20] for an approach based on geodesic rays, when \( k = \mathbb{C} \)).

In (iv) a function \( \varphi \in \mathcal{E}^\infty \) is regularizable from below if it is the limit in PSH of an increasing net in CPSH (or, equivalently, \( \mathcal{H}_R \)). Inspired by a result of Bedford in the complex analytic case, it was proved in [BoJ21] that a bounded \( L \)-psh function is regularizable from below iff its discontinuity locus is pluripolar. There are inclusions

\[
\text{CPSH} \subset \mathcal{E}_1^\infty \subset \mathcal{E}^\infty,
\]

where the first one is strict as soon as \( n \geq 2 \), and the second one is always strict. Theorem B (iv) thus reveals that norms/filtrations on the section ring of \( L \), while simple to define from an algebraic point of view, actually correspond to a rather delicate class of functions from an analytic perspective, which is advantageously replaced by the larger class \( \mathcal{E}^1 \) of functions of finite energy, or the smaller class CPSH of continuous \( L \)-psh functions. In particular, uniform K-stability with respect to filtrations, \( \mathcal{E}^1 \) or CPSH are equivalent (see [Li20], based on the first version of this paper).

We call the metric \( d_1 \) on \( \mathcal{E}^1 \) the Darvas metric; its complex analytic analogue, introduced by T. Darvas [Dar15b], plays a crucial role in global pluripotential theory, and in particular in the variational approach to the Yau–Tian–Donaldson conjecture [BBJ21, Li19, Li20]. The space \( \mathcal{E}^1 \) is studied over more general non-Archimedean fields in [Reb20], where it is shown that \( (\mathcal{E}^1, d_1) \) is a geodesic metric space. We expect \( (\mathcal{M}^1, d_1) \) to also be a geodesic space. In analogy with the complex analytic case [Dar15a] [DLR20], we also expect that, for any \( p \in [1, \infty) \), the completion of \( (\mathcal{H}_R, d_p) \) can be identified with the space

\[
\mathcal{E}^p := \{ \varphi \in \mathcal{E}^1 \mid \varphi \in L^p(\text{MA}(\varphi)) \},
\]

assuming continuity of envelopes.

We observe that the quotient metric \( d_1 \) is equivalent to the \( L^1 \)-norm of test configurations (see Remark 5.11). Uniform K-stability can thus be studied by viewing the non-Archimedean K-energy as a functional on \( \mathcal{M}^1 \). A key advantage of this approach is that the topological space \( \mathcal{M}^1 \), as opposed to \( \mathcal{E}^1 \), turns out to be independent of the choice of ample \( \mathbb{Q} \)-line bundle \( L \) (while the metric \( d_1 \) does depend on \( L \)). This makes \( \mathcal{M}^1 \) amenable for studying questions in K-stability when \( L \) is allowed to vary, a direction that is pursued in the companion paper [BoJ21b].

Among other things, the proof of Theorem B is based on a precise comparison between \( d_1, d_1 \) and quasi-metrics on \( \mathcal{E}^1 \) and \( \mathcal{M}^1 \) studied in [BoJ21], using estimates that ultimately derive from the Hodge Index Theorem.

**Valuations of linear growth.** The set \( X^\text{val} \) of valuations on the function field of \( X \), trivial on \( k \), is a dense subset of \( X^\text{an} \). Following [BKMS15], we say that \( v \in X^\text{val} \) is of linear growth if there exists \( C > 0 \) such that \( v(s) \leq Cm \) for all nonzero sections \( s \in R_m = H^0(X, mL) \) with \( m \) sufficiently divisible; the smallest such constant \( C \) is then denoted \( T(v) \). In terms of pluripotential theory, the set \( X^\text{lin} \subset X^\text{val} \) of valuations of linear growth coincides with the set of points \( v \in X^\text{an} \) that are non-pluripolar, i.e. such that every \( \varphi \in \text{PSH} \) is finite at \( v \); in particular, it contains the set \( X^\text{div} \) of divsorial valuations.

Like \( \mathcal{M}^1 \), the set \( X^\text{lin} \) of valuations of linear growth does not depend on \( L \); for any \( v \in X^\text{an} \) we have indeed \( v \in X^\text{lin} \iff \delta_v \in \mathcal{M}^1 \), and the corresponding embedding \( X^\text{lin} \hookrightarrow \mathcal{M}^1 \) allows us to restrict the \( d_1 \)-metric to \( X^\text{lin} \). On the other hand, any \( v \in X^\text{lin} \) defines a (homogeneous) norm \( \chi_v \in \mathcal{N}_R \), simply by setting \( \chi_v(s) := v(s) \), yielding an injection \( X^\text{lin} \hookrightarrow \mathcal{N}_R \) which endows
$X^{\text{lin}}$ with the restriction of the pseudometric $d_\rho$ for any $p \in [1, \infty]$. Denoting by $v_{\text{triv}} \in X^{\text{div}}$ the trivial valuation, we have in particular

$$d_\infty(v, v_{\text{triv}}) = T(v), \quad d_1(v, v_{\text{triv}}) = S(v)$$

where $S(v) := \text{vol}(\chi_v)$ is the expected vanishing order of $L$ along $v$, widely used in relation to the stability threshold/δ-invariant [Fuj19a Li17 BLJ20]. The invariant $d_\rho(v, v_{\text{triv}})$ with $v \in X^{\text{div}}$ also appears (under a slightly different guise) in [Zha20].

**Theorem C.** The metrics $d_1$ and $d_\rho$, $1 \leq p \leq \infty$, on $X^{\text{lin}}$ are equivalent and complete, and they are independent of $L$ up to bi-Lipschitz equivalence. For any $v \in X^{\text{lin}}$, we further have

$$d_1(v, v_{\text{triv}}) \leq d_1(v, v_{\text{triv}}) = S(v) = E^v(\delta_v).$$

Completeness with respect to $d_\infty$ was already observed in [BoJ21], and the key point is thus to show $d_\infty \leq C d_1$, which is done by invoking inequalities involving Monge–Ampère integrals, as in the proof of Theorem A (see §7.2 for details). The last point is a consequence of the formula $\text{vol}(\chi) = E(\text{FS}(\chi))$, that we establish for all norms $\chi \in \mathcal{N}_2$ via an approximation argument based on Okounkov bodies.

In [BoJ21] we use the space $\mathcal{M}^1$ to analyze K-stability. When $X$ is a Fano variety, restricting to Dirac masses $\delta_v \in \mathcal{M}^1$, with $v$ in $X^{\text{lin}}$ or $X^{\text{div}}$, recovers the valuative criterion of K-stability of Fano varieties due to Fujita and Li [Fuj19a Li17].

An interesting type of valuations $v \in X^{\text{lin}}$ are those for which the associated filtration $\chi_v$ is of finite type. If $v \in X^{\text{div}}$, this means $v$ is ‘dreamy’ in the sense of K. Fujita [Fuj19a], associated to a test configuration with irreducible special fiber. While valuations $v \in X^{\text{lin}}$ with $\chi_v$ of finite type play a crucial role in recent work on K-stability of Fano varieties [BLX19 BLZ19 BLXZ21 BX20 LX20 HL20], their role in the general polarized case is less clear (although see [Del20]). The condition of $\chi_v$ being of finite type is quite subtle and in particular depends on the ample $\mathbb{Q}$-line bundle $L$. For this reason we believe that it is useful to study K-stability using functionals on $X^{\text{lin}}$ or $\mathcal{M}^1$, without any finite type assumption.

**Organization.** After giving some background in §2 we study homogenization and the related Fubini–Study and infimum norm operators in §3, proving part of Theorem A. In §4 we make a spectral analysis of norms on the section ring of $(X, L)$, building upon [CM15 BE21]. After that we give additional background on non-Archimedean pluripotential theory from [BoJ21]; in particular we revisit the spaces used in Theorem B. In §5 we study the Darvas metrics on $\mathcal{E}^1$ and $\mathcal{M}^1$, and prove the remaining part of Theorem A and the main part of Theorem B. The rest of Theorem B is proved in §6 dedicated to the regularized Fubas metrics on $X^{\text{lin}}$ or $\mathcal{M}^1$, without any finite type assumption.

Finally, Appendix A revisits the relation between test configurations and Fubini–Study functions, and Appendix B provides some remarks on the toric case.

**Notation and conventions.**

- We work over an algebraically closed field $k$, of arbitrary characteristic unless otherwise specified.
- For $x, y \in \mathbb{R}_+$, $x \lesssim y$ means $x \leq C_n y$ for a constant $C_n > 0$ only depending on $n$, and $x \approx y$ if $x \lesssim y$ and $y \lesssim x$. Here $n$ will be the dimension of a fixed variety $X$ over $k$.
- A pseudo-metric on a set $Z$ is a function $d: Z \times Z \to \mathbb{R}_+$ that is symmetric, vanishes on the diagonal, and satisfies the triangle inequality. It is a metric if it further separates points.
The Hausdorff quotient of a pseudo-metric space \((Z, d)\) is the metric space \((Z_H, d_H)\) where \(Z_H\) is the quotient of \(Z\) by the equivalence relation \(x \sim y \Leftrightarrow d(x, y) = 0\), and \(d_H\) is the induced metric. The map \((Z, d) \rightarrow (Z_H, d_H)\) is the unique isometric map of \((Z, d)\) onto a metric space, up to unique isomorphism.

The Hausdorff completion of a pseudo-metric space \((Z, d)\) is the complete metric space \((\hat{Z}, \hat{d})\) defined as the completion of the Hausdorff quotient \((Z_H, d_H)\). It comes with an isometric map \((Z, d) \rightarrow (\hat{Z}, \hat{d})\) with dense image, which is universal with respect to maps into complete metric spaces.

A quasi-metric on \(Z\) is function \(d : Z \times Z \rightarrow \mathbb{R}_+\) that is symmetric, vanishes precisely on the diagonal, and satisfies the quasi-triangle inequality

\[d(x, y) \leq C (d(x, z) + d(z, y))\]

for some constant \(C > 0\). A quasi-metric space \((Z, d)\) comes with a Hausdorff topology, and even a uniform structure. In particular, Cauchy sequences and completeness make sense for \((Z, d)\). Such uniform structures have a countable basis of entourages, and are thus metrizable, by general theory.

- We use the standard abbreviations \(usc\) for ‘upper semicontinuous’, \(lsc\) for ‘lower semicontinuous’, \(wlog\) for ‘without loss of generality’, and \(iff\) for ‘if and only if’.
- A net is a family indexed by a directed set. On many occasions we shall consider nets indexed by \(m_0\mathbb{Z}_{\geq 1}\) for some \(m_0 \geq 1\), and ordered by divisibility.

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1. Background

In the entire paper, \((X, L)\) denotes a projective variety (reduced and irreducible) endowed with an ample \(\mathbb{Q}\)-line bundle. We review a number of basic facts about norms/filtrations and Berkovich anaytification, referring for instance to [BeJ21, BoJ21] for more details.

1.1. Norms on a vector space. As in [BT72] we will use ‘additive’ terminology, so by a norm on a \(k\)-vector space \(V\) we mean a function \(\chi : V \rightarrow \mathbb{R} \cup \{+\infty\}\) such that

- \(\chi(v) = +\infty\) iff \(v = 0\);
- \(\chi(\lambda v) = \chi(v)\) for \(a \in k^\times\) and \(v \in V\); and
- \(\chi(v + w) \geq \min\{\chi(v), \chi(w)\}\) for all \(v, w \in V\).

Note that \(\|\cdot\|_\chi := e^{-\chi(\cdot)}\) is then a non-Archimedean norm on \(V\) with respect to the trivial absolute value on \(k\) in the usual (‘multiplicative’) sense [BGR].

Setting \(F^\lambda V := \{\chi \geq \lambda\}\) for \(\lambda \in \mathbb{R}\) yields a 1–1 correspondence between norms on \(V\) and (non-increasing, left-continuous, exhaustive and separated) filtrations of \(V\). We also write \(F^{>\lambda} V := \bigcup_{\lambda>\lambda} F^\lambda V\), and define the associated graded space as the \(\mathbb{R}\)-graded vector space

\[\text{gr}_\chi V := \bigoplus_{\lambda \in \mathbb{R}} F^\lambda V/F^{>\lambda} V.\]

Each norm \(\chi\) on \(V\) turns it into a (Hausdorff) topological vector space, in which \((F^{m\varepsilon} V)_{m\in \mathbb{N}}\) forms a countable basis of (open and closed) neighborhood of 0, for any \(\varepsilon > 0\). The normed space \((V, \chi)\) admits a completion \(\hat{V}\), a complete topological vector space containing \(V\) as a...
dense subspace, whose topology is defined by a (unique) norm on \( \hat{V} \) extending \( \chi \). The inclusion \( V \hookrightarrow \hat{V} \) induces an isomorphism

\[
gr_{\chi} V \cong gr_{\chi} \hat{V}.
\]

We denote by \( N_{R}(V) \) the set of norms on \( V \). It has a distinguished element \( \chi_{\text{triv}} \), the trivial norm, such that \( \chi_{\text{triv}}(v) = 0 \) for all \( v \neq 0 \), and it admits a scaling action by \( R_{>0} \) and a partial ordering defined by \( \chi \leq \chi' \) iff \( \chi(v) \leq \chi'(v) \) for all \( v \). Any two elements \( \chi, \chi' \in N_{R}(V) \) admit an infimum \( \chi \wedge \chi' \in N_{R}(V) \), defined pointwise by

\[
(\chi \wedge \chi')(v) := \min\{\chi(v), \chi'(v)\}.
\]

For any subgroup \( \Lambda \subset R \), we denote by \( N_{\Lambda}(V) \) the set of norms with values in \( \Lambda \cup \{+\infty\} \). Thus

\[
\{\chi_{\text{triv}}\} = N_{\{0\}}(V) \subset N_{\Lambda}(V) \subset N_{R}(V).
\]

A norm \( \chi \in N_{R}(V) \) lies in \( N_{\Lambda}(V) \) iff the \( \Lambda \)-grading of \( gr_{\chi} V \) reduces to a \( \Lambda \)-grading.

Assume now that \( V \) is finite dimensional. Any norm \( \chi \) on \( V \) admits an orthogonal basis \( (e_{i}) \), i.e. a basis of \( V \) such that \( \chi(\sum a_{i} e_{i}) = \min_{a_{i} \neq 0} \chi(e_{i}) \) for all \( a_{i} \in k \). Up to reordering, an orthogonal basis is simply a compatible basis for the flag of linear subspaces underlying the filtration defined by \( \chi \), and elementary linear algebra thus implies that any two norms \( \chi, \chi' \) on \( V \) admit a common orthogonal basis.

In particular, all norms on \( V \) are equivalent, which means in our additive terminology that \( \chi - \chi' \) is bounded on \( V \setminus \{0\} \) for all \( \chi, \chi' \in N_{R}(V) \). The classical Goldman–Iwahori metric on \( N_{R}(V) \) is defined by

\[
d_{\infty}(\chi, \chi') = \sup_{v \in V \setminus \{0\}} |\chi(v) - \chi'(v)|.
\]

The metric space \( (N_{R}(V), d_{\infty}) \) is complete, but not locally compact as soon as \( \dim V \geq 2 \). For any \( \chi \in N_{R}(V) \), we also set

\[
\lambda_{\min}(\chi) := \min_{v \in V \setminus \{0\}} \chi(v), \quad \lambda_{\max}(\chi) := \sup_{v \in V \setminus \{0\}} \chi(v).
\]

Thus

\[
d_{\infty}(\chi, \chi_{\text{triv}}) = \max\{\lambda_{\max}(\chi), -\lambda_{\min}(\chi)\}.
\]

Any norm \( \chi \) on \( V \) induces a norm on the dual space and on all tensor powers, in such a way that the bases canonically induced by any orthogonal basis of \( V \) remain orthogonal. If \( \pi : V \to V' \) is a surjective linear map, then \( \chi \) also induces a quotient norm \( \chi' \) on \( V' \), such that \( \chi'(v') = \max\{\chi(v) \mid \pi(v) = v'\} \) for all \( v' \in V' \).

1.2. Norms on a graded algebra. Let now \( R = \bigoplus_{m \in \mathbb{N}} R_{m} \) be a graded \( k \)-algebra. It comes with an action of \( k^{\times} \) for which \( a \cdot s = a^{m}s \) for \( a \in k^{\times} \) and \( s \in R_{m} \). We write \( N_{R}(R) \) for the set of vector space norms \( \chi : R \to R \) that are

- superadditive, i.e. \( \chi(f \cdot g) \geq \chi(f) + \chi(g) \) for \( f, g \in R \);
- \( k^{\times} \)-invariant, i.e. \( \chi(a \cdot f) = \chi(f) \) for \( a \in k^{\times} \) and \( f \in R \); this is equivalent to \( \chi \) being compatible with the grading of \( R \), that is, \( \chi(\sum m s_{m}) = \min_{m} \chi(s_{m}) \) where \( s_{m} \in R_{m} \);
- linearly bounded, i.e. there exists \( C > 0 \) such that \( |\chi| \leq C m \) on \( R_{m} \setminus \{0\} \) for all \( m \geq 1 \).

Norms in \( N_{R}(R) \) are in 1–1 correspondence with graded, linearly bounded filtrations of \( R \) as in [BC11] [Szé15]. Each \( \chi \in N_{R}(R) \) defines a graded algebra

\[
gr_{\chi} R = \bigoplus_{m \in \mathbb{N}} gr_{\chi} R_{m} = \bigoplus_{(m, \lambda) \in \mathbb{N} \times \mathbb{R}} F^{\lambda} R_{m}/F^{\lambda+1} R_{m}.
\]
Lemma 1.3. canonical surjective map $S$ of degree 1, thus (ii) holds. We say that a norm $\lambda$ on $R$ is generated in degree 1 iff, given a $\lambda$-orthogonal basis $(s_i)$ of $R_1$, any $s \in R_m$ can be written as $s = \sum_{|\alpha|=m} c_\alpha \prod_i s_i^{\alpha_i}$ with $c_\alpha \in k$ and $\chi(s) = \min_{c_\alpha \neq 0} \sum_i \alpha_i \chi(s_i)$.

Example 1.1. Consider the ring $k[T] = k[T_1, \ldots, T_N]$ with the usual grading. For each $\lambda \in \mathbb{R}^N$, the monomial valuation
\[
v_\lambda(\sum_\alpha c_\alpha T^\alpha) = \min_{c_\alpha \neq 0} (\alpha, \lambda) = \min_{\alpha} \{v_{\text{triv}}(c_\alpha) + (\alpha, \lambda)\}
\]
defines a norm on the graded algebra $k[T]$. The completion of $(k[T], v_\lambda)$ is the algebra $k[T; \lambda]$ of all series $\sum_\alpha c_\alpha T^\alpha \in k[T]$ with $\lim_\alpha (v_{\text{triv}}(c_\alpha) + (\alpha, \lambda)) = +\infty$, whose norm is still defined by (1.5). In multiplicative notation, $k[T; \lambda]$ is the polydisc algebra $k\{r^{-1}T\}$, with $r_j = e^{-\lambda_j}$, a building block of Berkovich spaces [Berk90, Berk93].

Definition 1.2. We say that a norm $\chi \in N_\mathbb{R}(R)$ is generated in degree 1 if $R$ is generated in degree 1 and, for any $m \geq 1$, the restriction $\chi|_{R_m}$ is the quotient norm of $S^m(\chi|_{R_1})$ under the canonical surjective map $S^mR_1 \to R_m$.

Concretely, $\chi$ is generated in degree 1 iff, given a $\chi$-orthogonal basis $(s_i)$ of $R_1$, any $s \in R_m$ can be written as $s = \sum_{|\alpha|=m} c_\alpha \prod_i s_i^{\alpha_i}$ with $c_\alpha \in k$ and $\chi(s) = \min_{c_\alpha \neq 0} \sum_i \alpha_i \chi(s_i)$.

Lemma 1.3. For any subgroup $\Lambda \subset \mathbb{R}$ and $\chi \in N_\Lambda(R)$, the following conditions are equivalent:

(i) $\chi$ is generated in degree 1;
(ii) $\text{gr}_\chi R = \bigoplus_{m \in \mathbb{N}} \text{gr}_\chi R_m$ is generated in degree 1;
(iii) there exists $\lambda \in \Lambda^N$ and a surjective map of graded $k$-algebras $\pi: k[T_1, \ldots, T_N] \to R$ with respect to which $\chi$ is the quotient norm of $v_\lambda$, as in Example 1.1.

When this holds, we further have $\chi \in N_{\Lambda'}(R)$ for some finitely generated subgroup $\Lambda' \subset \Lambda$.

Proof. Assume (i), and choose a $\chi$-orthogonal basis $(s_i)_{1 \leq i \leq N}$ of $R_1$. Pick $s \in R_m \setminus \{0\}$ and write as above $s = \sum_{|\alpha|=m} c_\alpha \prod_i s_i^{\alpha_i}$ with $\lambda := \chi(s) = \min_{c_\alpha \neq 0} \sum_i \alpha_i \chi(s_i)$.

Define $A$ as the set of $\alpha$ achieving the minimum and set $s' := \sum_{\alpha \in A} c_\alpha \prod_i s_i^{\alpha_i}$. Then $s - s' \in F^{>\lambda}R_m$, so $s = s'$ in $\text{gr}_\chi R_m$. This shows that $S^m \text{gr}_\chi R_1 \to \text{gr}_\chi R_m$ is surjective, and hence (i)$\Rightarrow$(ii). If we define $\pi: k[T] \to R$ by $\pi(T_i) = s_i$, then it is clear that $\chi$ is the quotient norm of $v_\lambda$, hence (i)$\Rightarrow$(iii).

Conversely, any quotient of a norm generated in degree 1 is plainly generated in degree 1 as well; hence (iii)$\Rightarrow$(i). Assume now (ii), and pick again a $\chi$-orthogonal basis $(s_i)$ of $R_1$. Each $s \in R_m \setminus \{0\}$ can then be written as $s = \sum_{|\alpha|=m} a_\alpha \prod_i s_i^{\alpha_i} + s'$ where $a_\alpha \in k^\times$, $\sum_i \alpha_i \chi(s_i) = \lambda := \chi(s)$ for all $\alpha$ and $s' \in F^{>\lambda}R_m$. Repeating the procedure with $s'$ in place of $s$ and using the fact that $\lambda \mapsto F^\Lambda R_m$ jumps only finitely many times, we end up with a decomposition $s = \sum_{|\alpha|=m} c_\alpha \prod_i s_i^{\alpha_i}$ such that $\lambda = \min_{c_\alpha \neq 0} \sum_i \alpha_i \chi(s_i)$. This proves that $\chi$ is generated in degree 1, thus (ii)$\Rightarrow$(i).
Finally, if \( \chi \) is generated in degree 1, the \( \Lambda \)-grading of the finitely generated algebra \( \text{gr}_\chi R \) reduces to a \( \Lambda \)-grading for some finitely generated subgroup \( \Lambda' \subset \Lambda \), which yields \( \chi \in N_{\Lambda'}(R) \).

\[ \square \]

1.3. Norms on section rings and \( \mathbb{R} \)-test configurations. Recall that \( L \) is an ample \( \mathbb{Q} \)-line bundle on a projective variety \( X \). For any \( r \in \mathbb{N} \) such that \( rL \) is an actual line bundle, we write \( R_r := \text{H}^0(X, rL) \), and denote by

\[
R^{(r)} = R(X, rL) = \bigoplus_{m \in \mathbb{N}} R_{rm}
\]

the section ring of \( rL \), which is generated in degree 1 for all \( r \) sufficiently divisible. If \( r \) divides \( r' \), then we have a restriction map \( N_\mathbb{R}(R^{(r)}) \to N_\mathbb{R}(R^{(r')}) \), and we set

\[
N_\mathbb{R} = N_\mathbb{R}(X, L) := \lim_{r \to \infty} N_\mathbb{R}(R^{(r)}).
\]

The set \( N_\mathbb{R} \) inherits a partial order with finite infima, and commuting actions of \( \mathbb{R}_{>0} \) (by scaling) and \( \mathbb{R} \) (by translation).

An element \( \chi \in N_\mathbb{R} \) is represented by a norm on some \( R^{(r)} \), two such norms being identified if they coincide on some further Veronese subalgebra; for convenience, we simply refer to \( \chi \) as a norm. For all \( m \) sufficiently divisible, we denote by \( \chi|_{R_m} \in N_\mathbb{R}(R_m) \) the restriction of \( \chi \) to \( R_m \).

Remark 1.4. To define \( \chi|_{R_m} \), one needs to choose a representative of \( \chi \). But any other choice leads to the same norms \( \chi|_{R_m} \in N_\mathbb{R}(R_m) \) for \( m \) sufficiently divisible, and the choice of representative can thus safely be ignored.

The Goldman–Iwahori metric \([1.2]\) induces a pseudo-metric \( d_\infty \) on \( N_\mathbb{R} \) by setting

\[
d_\infty(\chi, \chi') := \lim_{m \to \infty} m^{-1} d_\infty(\chi|_{R_m}, \chi'|_{R_m}) \in \mathbb{R}_{\geq 0}
\]

The limsup is taken with respect to the partial ordering on \( \mathbb{Z}_{>0} \) by divisibility, and it is finite by linear boundedness of \( \chi, \chi' \). This pseudo-metric is not a metric (see, however Proposition \([2.5]\]):

Example 1.5. Define a norm \( \chi \) on \( R = \mathbb{K}[T_1, T_2] \) (with its usual grading) by \( \chi(s) = -1 \) if \( s \in T_1 \mathbb{K}[T] \) and \( \chi(s) = 0 \) otherwise. Then \( d_\infty(\chi|_{R_m} \cdot \chi_{\text{triv}}|_{R_m}) = 1 \) for \( m \geq 1 \), so \( d_\infty(\chi, \chi_{\text{triv}}) = 0 \).

We also introduce

\[
\lambda_{\text{max}}(\chi) := \lim_{m \to \infty} m^{-1} \lambda_{\text{max}}(\chi|_{R_m}),
\]

where \( \lambda_{\text{max}}(\chi|_{R_m}) \) is defined by \([1.3]\) and the limit exists because \( m^{-1} \lambda_{\text{max}}(\chi|_{R_m}) \) is increasing with respect to divisibility. Note that

\[
\chi \geq \chi_{\text{triv}} \iff d_\infty(\chi, \chi_{\text{triv}}) = \lambda_{\text{max}}(\chi).
\]

Definition 1.6. We say that a norm \( \chi \in N_\mathbb{R} \) is of finite type if it is represented by a norm on some \( R^{(r)} \) whose associated graded algebra \( \text{gr}_\chi R^{(r)} \) is of finite type.

By Lemma \([1.3]\), a norm \( \chi \in N_\mathbb{R} \) is of finite type iff it is represented by a norm on some \( R^{(r)} \) that is generated in degree 1. We denote by

\[
\mathcal{T}_\mathbb{R} \subset N_\mathbb{R}
\]

the set of such norms. In line with \([\text{DeSz20}]\), we interpret the elements of \( \mathcal{T}_\mathbb{R} \) as \( \mathbb{R} \)-test configurations. This is justified by the Rees construction, which sets up a 1–1 correspondence between the subset

\[
\mathcal{T}_\mathbb{Z} := N_\mathbb{Z} \cap \mathcal{T}_\mathbb{R}
\]
of \( \mathbb{Z} \)-valued norms in \( T_\mathbb{R} \) and the set of (usual) ample test configurations for \( (X, L) \), see Appendix A. More generally, for any subgroup \( \Lambda \subset \mathbb{R} \) we introduce the space of \( \Lambda \)-test configurations

\[
T_\Lambda := \mathcal{N}_\Lambda \cap T_\mathbb{R}.
\]

By Lemma 1.3, we have

\[
T_\mathbb{R} = \bigcup_{\Lambda \subset \mathbb{R} \text{ finitely generated}} T_\Lambda.
\]

**Definition 1.7.** The canonical approximants of a norm \( \chi \in \mathcal{N}_\mathbb{R} \) is the sequence \( \chi_r \in T_\mathbb{R} \) defined for \( r \in \mathbb{Z}_{\geq 1} \) sufficiently divisible by letting \( \chi_r \) be the (class of the) norm on \( R^{(\nu)} \) generated in degree 1 by \( \chi_r \).

If \( r \) divides \( r' \) then \( \chi_r \leq \chi_{r'} \leq \chi \). As in Remark 1.4, this construction is not entirely canonical, as it depends on the choice of a representative of \( \chi \), but this can be ignored as any other choice leads to the same approximants \( \chi_r \) for \( r \) sufficiently divisible.

A norm \( \chi \in \mathcal{N}_\mathbb{R} \) is of finite type iff \( \chi = \chi_r \) for all sufficiently divisible \( r \). Note also that

\[
\chi \in \mathcal{N}_\Lambda \implies \chi_r \in T_\Lambda.
\]

### 1.4. The Berkovich analytification

By a valuation on \( X \) we mean a real-valued valuation \( v: k(X)^\times \to \mathbb{R} \), trivial on \( k \). We denote by \( X^\text{val} \) the space of valuations, endowed with the topology of pointwise convergence on \( k(X)^\times \). The trivial valuation \( v_{\text{triv}} \in X^\text{val} \) is defined by \( v_{\text{triv}}(f) = 0 \) for all \( f \in k(X)^\times \).

By Berkovich, the space \( X^\text{val} \) admits a natural compactification \( X^\text{an} \), which as a set equals \( X^\text{an} = \prod Y^\text{val} \) with \( Y \) ranging over all subvarieties of \( X \). We somewhat imprecisely refer to the points on \( X^\text{an} \) as semivaluations on \( X \). The support of a semvaluation in \( Y^\text{val} \subset X^\text{an} \) is the subvariety \( Y \).

By the valuative criterion of properness, each valuation \( v \in X^\text{val} \) admits a center \( c_X(v) \in X \), characterized as the unique (scheme) point \( \xi \in X \) such that \( v \geq 0 \) on the local ring \( O_{X,\xi} \) and \( v > 0 \) on its maximal ideal. This applies to semivaluations as well, replacing \( X \) with a closed subvariety, and thus defines a map \( c_X: X^\text{an} \to X \) (which turns out to be anticontinuous, i.e. the preimage of an open subset is closed).

The space \( X^\text{an} \) comes with a natural action of \( \mathbb{R}_{>0} \) by scaling \( (t,v) \mapsto tv \). This induces an action \( (t, \varphi) \mapsto t \cdot \varphi \) on functions \( \varphi \) on \( X^\text{an} \) by setting

\[
(t \cdot \varphi)(v) := t \varphi(t^{-1}v),
\]

whose fixed points are functions that are homogeneous, i.e. \( \varphi(tv) = t \varphi(v) \) for all \( t \in \mathbb{R}_{>0} \) and \( v \in X^\text{an} \).

The set \( X^\text{an} \) is also endowed with a partial order relation, for which \( v \geq v' \) iff \( c_X(v) \) is a specialization of \( c_X(v') \) and \( v \geq v' \) pointwise on the local ring at \( c_X(v) \). The trivial valuation satisfies \( v \geq v_{\text{triv}} \) for all \( v \in X^\text{an} \).

A (rational) divisorial valuation \( v \) on \( X \) is a valuation of the form \( v = t \text{ord}_E \), where \( E \) is a prime divisor on a normal, projective birational model \( X' \to X \) and \( t \in \mathbb{Q}_{>0} \). The center \( c_X(v) \) is then the generic point of the image of \( E \) in \( X \). For convenience, we also count the trivial valuation \( v_{\text{triv}} \) as divisorial, i.e. we allow \( t = 0 \) above. The set \( X^\text{div} \) of divisorial valuations is dense in \( X^\text{an} \).
1.5. **Semivaluations and line bundles.** A semivaluation \( v \in X^\text{an} \) can be naturally evaluated on a section \( s \in H^0(X, M) \) of any line bundle \( M \) on \( X \), by defining \( v(s) \) as the value of \( v \) on the local function corresponding to \( s \) in any local trivialization of \( M \) at \( c_X(v) \). Thus \( v(s) \in [0, +\infty), v(s) > 0 \) iff \( s \) vanishes at \( c_X(v) \), and \( v(s) = \infty \) iff \( s \) vanishes along the support of \( v \). Further, \( v \in X^\text{val} \) iff \( v(s) < +\infty \) for all \( s \in H^0(X, M) \setminus \{0\} \) and all line bundles \( M \). We define a continuous function \(|s| : X^\text{an} \to [0, 1]\) by setting
\[
|s|(v) := \exp(-v(s)).
\] (1.11)

Now suppose \( L \) is an (ample) line bundle. The \( \mathbb{Z} \)-grading of \( R = R(X, L) \) defines an action of \( \mathbb{G}_m \) on \( Y := \text{Spec } R \). We have a natural surjective \( \mathbb{G}_m \)-invariant morphism \( \pi : Y \setminus \{o\} \to X \), where the vertex \( o \) of the affine cone \( Y \) is the point defined by the maximal ideal \( \bigoplus_{m > 0} R_m \). For any \( \xi \in X \), the fiber \( \pi^{-1}(\xi) \) contains a unique \( \mathbb{G}_m \)-invariant point defined by the homogeneous prime ideal generated by all sections \( s \in R_m, m \geq 1 \) that vanish at \( \xi \).

By general properties of the analytification functor in \([\text{Berk90}]\), the \( \mathbb{G}_m \)-action on \( Y \) induces an action of \( \mathbb{G}_m(k) = k^\times \) on \( Y^\text{an} \), and \( \pi \) induces a surjective \( k^\times \)-invariant map \( \pi^\text{an} : Y^\text{an} \setminus \{w_o\} \to X^\text{an} \), where \( w_o \in Y^\text{an} \) satisfies \( w_o = +\infty \) on \( \bigoplus_{m > 0} R_m \). A semivaluation \( w \in Y^\text{an} \) is \( k^\times \)-invariant iff \( w(\sum_m s_m) = \min_m w(s_m) \), where \( s_m \in R_m \).

It is easy to see \([\text{Li17}, \text{§4.2}]\) that if \( v \in X^\text{an} \), then the set of \( k^\times \)-invariant points in \((\pi^\text{an})^{-1}(v)\) is of the form \( \{w_{v,\lambda}\}_{\lambda \in \mathbb{R}} \), where \( w_{v,\lambda} \) is defined by
\[
w_{v,\lambda}(s) = \min_m \{v(s_m) + \lambda m\} \quad \text{for any } s = \sum_m s_m \in R,
\] (1.12)
and where the value \( v(s_m) \) is defined at the top of of this section. Note that \( w_{v,\lambda} \) is centered at the cone point \( o \) iff \( \lambda > 0 \).

1.6. **Valuations of linear growth.** Following \([\text{BKMS15}]\), we define the maximal vanishing order of \( L \) at \( v \in X^\text{an} \) as
\[
T(v) := T_L(v) = \sup m^{-1}v(s),
\] (1.13)
where the supremum is over \( m \) sufficiently divisible and \( s \in R_m \setminus \{0\} \). We say that \( v \) has linear growth if \( T(v) < +\infty \); this notion is independent of \( L \). The set \( X^\text{lin} \subset X^\text{an} \) of valuations of linear growth satisfies \( X^\text{div} \subset X^\text{lin} \subset X^\text{val} \).

Any \( v \in X^\text{lin} \) defines a norm \( \chi_v \in \mathcal{N}_R \) given by \( \chi_v(s) := v(s) \). Such norms will be studied in \( \text{§7.1} \).

**Example 1.8.** If \( \pi : X' \to X \) is a proper birational morphism, with \( X' \) normal, and \( E \subset X' \) is a prime divisor which is \( \mathbb{Q} \)-Cartier, then \([\text{BKMS15} \text{ Theorem 2.24}]\) yields
\[
T_L(\text{ord}_E) = \sup\{t > 0 \mid \pi^*L - tE \text{ big}\}.
\]

**Remark 1.9.** Consider a valuation \( v \in X^\text{val} \), and its extensions \( w_{v,\lambda} \) as \( k^\times \)-invariant valuations on the affine cone \( Y = \text{Spec } R^{(r)} \) given by (1.12). Then, for any \( \lambda > 0 \), \( v \) is of linear growth iff \( w_{v,\lambda} \leq C_\lambda \text{ord}_o \) on \( R^{(r)} \) for some \( C_\lambda > 0 \), where
\[
\text{ord}_o(f) := \max\{j \geq 0 \mid f \in m_o^j\},
\]
which may fail to be a valuation in general (see for instance \([\text{BF14} \text{ §4.1}]\)).
1.7. Fubini–Study functions. A Fubini–Study function (for $L$) is a function $\varphi \in C^0 = C^0(X^{an})$ of the form

$$\varphi = \frac{1}{m} \max_{j} \{ \log |s_j| + \lambda_j \}, \quad (1.14)$$

with $m \geq 1$ such that $mL$ is a (globally generated) line bundle, $(s_j)$ a finite set of $R_m$ without common zeros, and $\lambda_j \in \mathbb{R}$. Recall that $(1.14)$ means $\varphi(v) = \frac{1}{m} \max_j \{ \lambda_j - v(s_j) \}$ for all $v \in X^{an}$, see (1.11).

**Remark 1.10.** The function $\varphi$ defines a continuous metric $| \cdot | e^{-m \varphi}$ on the Berkovich analytification of $mL$. This metric is the pullback of a standard (non-Archimedean) Fubini–Study (or Weil) metric on $O(1)$ under the morphism $X \to \mathbb{P}^N$ defined by $(s_j)_{0 \leq j \leq N}$, which explains the chosen terminology.

If the $\lambda_j$ in $(1.14)$ can be chosen in a subgroup $\Lambda \subset \mathbb{R}$, we say that $\varphi$ is a $\Lambda$-Fubini–Study function, and write $\mathcal{H}_\Lambda = \mathcal{H}_\Lambda(L) \subset C^0$ for the set of such functions. Thus

$$\{0\} = \mathcal{H}_\{0\} \subset \mathcal{H}_\Lambda \subset \mathcal{H}_\mathbb{R}.$$

Note that

$$\mathcal{H}_\Lambda = \mathcal{H}_{\mathbb{Q}\Lambda} \quad (1.15)$$

and $\mathcal{H}_\Lambda(rL) = r\mathcal{H}_\Lambda(L)$ for any $r \in \mathbb{Q}_{>0}$. The set $\mathcal{H}_\Lambda$ is stable under finite max and under the action of $\Lambda$ by translation.

Recall the action $(1.10)$ of $\mathbb{R}_{>0}$ on functions on $X^{an}$. If $\varphi$ is given by $(1.14)$ and $t \in \mathbb{R}_{>0}$, then

$$t \cdot \varphi = \frac{1}{m} \max_{j} \{ \log |s_j| + t\lambda_j \}.$$

Thus $\mathcal{H}_\mathbb{R}$ is stable under the action of $\mathbb{R}_{>0}$, while $\mathcal{H}_\Lambda$ is stable under the action of the stabilizer $\{ t \in \mathbb{R}_{>0} | t\Lambda \subset \Lambda \}$. In particular, $\mathcal{H}_{\mathbb{Q}}$ is stable under the action of $\mathbb{Q}_{>0}$.

2. Homogenization and the Fubini–Study operator

In this section we study the homogenization of a norm, and the related Fubini–Study and infimum norm operators. We show that homogenization preserves norms of finite type, establish a 1–1 correspondence between homogeneous norms of finite type and Fubini–Study functions, and we prove Theorem A in the case $p = \infty$.

In what follows, $C^\infty$ denotes the space of bounded functions on $X^{an}$, endowed with its usual supnorm metric $d_\infty(\varphi, \varphi') := \sup_{X^{an}} | \varphi - \varphi' |$.

2.1. Homogenization. In this section, $R = \bigoplus_{m \in \mathbb{N}} R_m$ denotes any reduced graded $k$-algebra.

**Definition 2.1.** We say that a norm $\chi \in \mathcal{N}_R(R)$ is homogeneous if $\chi(f^d) = d\chi(f)$ for all $f \in R$ and $d \in \mathbb{N}$.

In multiplicative terminology, this means that $\| \cdot \|_\chi = e^{-\chi}$ is power-multiplicative, see [BGR]. It is easy to see that a norm $\chi \in \mathcal{N}_R(R)$ is homogeneous iff the associated graded algebra $\text{gr}_\chi R$ is reduced. We denote by

$$\mathcal{N}_R^{\text{hom}}(R) \subset \mathcal{N}_R(R)$$

the set of homogeneous norms on $R$. For any Veronese subalgebra $R^{(r)} = \bigoplus_{m \in \mathbb{N}} R_{rm}$, $r \geq 1$, the restriction map $\mathcal{N}_R(R) \to \mathcal{N}_R(R^{(r)})$ induces a bijection

$$\mathcal{N}_R^{\text{hom}}(R) \cong \mathcal{N}_R^{\text{hom}}(R^{(r)}). \quad (2.1)$$
Any norm $\chi \in \mathcal{N}_R(R)$ is dominated by a minimal homogeneous norm, namely its homogenization $\overline{\chi}$, defined by

$$\overline{\chi}(f) := \sup_{d \geq 1} \frac{1}{d} \chi(f^d) = \lim_{d \to \infty} \frac{1}{d} \chi(f^d),$$

where the second equality holds by superadditivity of $d \mapsto \chi(f^d)$ and Fekete’s Lemma. It is indeed easy to check that (2.2) defines a vector space norm on $R$ that is superadditive, $k^\times$-invariant, linearly bounded and homogeneous, i.e. an element $\overline{\chi} \in \mathcal{N}_R^{\text{hom}}(R)$.

**Remark 2.2.** Note that $\|\cdot\|_{\overline{\chi}} = e^{-\overline{\chi}(\cdot)}$ is the spectral radius (semi)norm of $\|\cdot\|_\chi$ in the (multiplicative) terminology of \cite{Berk90}.

Using standard but nontrivial results on $k$-affinoid algebras, we prove:

**Theorem 2.3.** Let $\chi \in \mathcal{N}_R(R)$ be a norm generated in degree 1, with homogenization $\overline{\chi}$. Then:

(i) there exists $C > 0$ such that $\chi(f) \leq \overline{\chi}(f) \leq \chi(f) + C$ for all $f \in R$;

(ii) the $k$-algebra $\text{gr}_R \tilde{R}$ is finitely generated;

(iii) if $\chi \in \mathcal{N}_\Lambda(R)$ for a subgroup $\Lambda \subset \mathbb{R}$, then $\overline{\chi} \in \mathcal{N}_{\text{QA}}(R)$.

**Proof.** Pick a surjective map of graded algebras $\pi: k[T] = k[T_1, \ldots, T_N] \to R$ and $\lambda \in \mathbb{R}^N$ such that $\chi$ is the quotient norm of $v_\lambda$ (see Lemma 1.3). As in Example 1.1 the completion of $k[T]$ with respect to $v_\lambda$ is the polydisc algebra $k\{T; \lambda\}$, and $\pi$ induces a surjection $k\{T; \lambda\} \to \tilde{R}$ onto the completion of $R$, whose norm is the quotient of the norm $v_\lambda$ of $k\{T; \lambda\}$.

As a consequence, $\tilde{R}$ is a $k$-affinoid algebra in the sense of \cite{Berk90}, corresponding geometrically to the affinoid domain $Y_{\text{an}} := \text{Spec } \mathbb{R} \to \mathbb{A}_k^N = \text{Spec } k[T]$, where $\mathbb{D}(r) \subset \mathbb{A}_k^{N,\text{an}}$ is the closed polydisc of polyradius $r = (e^{-\lambda_1}, \ldots, e^{-\lambda_N})$.

Since $R$ is assumed to be reduced, it follows from the non-Archimedean GAGA principle that $\tilde{R}$ is reduced as well (see \cite[Théorème 3.3]{Duc09}), and (i) is now a consequence of \cite[Proposition 2.1.4 (ii)]{Berk90}, which states (in multiplicative terminology) that the spectral radius (semi)norm of any reduced $k$-affinoid algebra is equivalent to the given norm.

Next, note that $\text{gr}_R \tilde{R} \simeq \text{gr}_R \tilde{R}$ (see (1.1)) coincides, by definition, with the graded reduction of $R$ in the sense of Temkin \cite[§3]{Tem04}. By \cite[Proposition 3.1]{Tem04}, the surjection $k\{T; \lambda\} \to \tilde{R}$ therefore induces a finite morphism $\text{gr}_{v_\lambda} k\{T; \lambda\} \to \text{gr}_R \tilde{R}$. Since $\text{gr}_{v_\lambda} k\{T; \lambda\} = \text{gr}_{v_\lambda} k[T]$ is of finite type over $k$, this yields (ii).

Finally suppose that $\chi \in \mathcal{N}_\Lambda(R)$ for a subgroup $\Lambda \subset \mathbb{R}$. In this case, we can pick $\lambda \in \Lambda^N$, so $k\{T; \lambda\}$ and $\tilde{R}$ are both $\Lambda$-strict $k$-affinoid algebras. By \cite[3.1.2.1 (iv)]{Tem15}, we thus have $\overline{\chi}(\tilde{R} \setminus \{0\}) \subset \mathcal{N}_{\text{QA}}$, which proves (iii). \hfill $\square$

### 2.2. Homogenization of norms on section rings.

Returning to the setting of a polarized variety $(X, L)$ and its space of norms $\mathcal{N}_R$, we introduce:

**Definition 2.4.** Consider a norm $\chi \in \mathcal{N}_R$. Then:

(i) we say that $\chi$ is homogeneous if it admits a homogeneous representative on $R^{(r)} = R(X, rL)$ for some $r$;

(ii) the homogenization of $\chi$ is the norm $\overline{\chi} \in \mathcal{N}_R$ induced by the homogenization of any representative of $\chi$.

By (2.2), $\overline{\chi}$ is well-defined; it satisfies $\overline{\chi} \geq \chi$, and is characterized as the smallest homogeneous norm with this property. We denote by

$$\mathcal{N}_R^{\text{hom}} \subset \mathcal{N}_R$$
the subset of homogeneous norms. By (2.1), we have
\[ N_{\mathbb{R}}^{\text{hom}} \cong N_{\mathbb{R}}^{\text{hom}}(R^{(r)}) \]
for any \( r \) such that \( rL \) is a line bundle. Recall that \( N_{\mathbb{R}} \) is equipped with a pseudo-metric \( d_{\infty} \), see (1.7).

**Proposition 2.5.** The restriction of \( d_{\infty} \) to \( N_{\mathbb{R}}^{\text{hom}} \) is a metric, and \( (N_{\mathbb{R}}^{\text{hom}}, d_{\infty}) \) is further complete.

**Proof.** After replacing \( L \) with a multiple, we may assume that \( L \) is a line bundle, and hence \( N_{\mathbb{R}}^{\text{hom}} \cong N_{\mathbb{R}}^{\text{hom}}(R) \) with \( R = R(X, L) \). Using that \( (N_{\mathbb{R}}(R_{m}), d_{\infty}) \) is a complete metric space for each \( m \), it is easy to see that \( N_{\mathbb{R}}(R) \hookrightarrow \prod_{m} N_{\mathbb{R}}(R_{m}) \) is complete with respect to the metric \( \tilde{d}_{\infty}(\chi, \chi') := \sup_{m \geq 1} m^{-1} d_{\infty}(\chi|_{R_{m}}, \chi'|_{R_{m}}) \).

It is also clear that \( N_{\mathbb{R}}^{\text{hom}}(R) \) is closed in \( N_{\mathbb{R}}(R) \), so that \( (N_{\mathbb{R}}^{\text{hom}}(R), \tilde{d}_{\infty}) \) is a complete metric space. Now if \( \chi, \chi' \in N_{\mathbb{R}}^{\text{hom}}(R) \) then
\[ m \mapsto m^{-1} d_{\infty}(\chi|_{R_{m}}, \chi'|_{R_{m}}) = m^{-1} \sup_{s \in R_{m} \setminus \{0\}} |\chi(s) - \chi'(s)| \]
is increasing with respect to divisibility. This implies that \( d_{\infty} = \tilde{d}_{\infty} \) on \( N_{\mathbb{R}}^{\text{hom}}(R) \), and the result follows.

The homogenization map
\[ H: N_{\mathbb{R}} \rightarrow N_{\mathbb{R}}^{\text{hom}} \]
is defined by \( H(\chi) := \overline{\chi} \). It is equivariant for the actions of \( \mathbb{R}_{>0} \) and \( \mathbb{R} \), and satisfies
\[ H(\chi \wedge \chi') = H(\chi) \wedge H(\chi') \text{ for all } \chi, \chi' \in N_{\mathbb{R}}. \quad (2.3) \]
By (2.2), we also trivially have
\[ \lambda_{\max}(\chi) = \lambda_{\max}(\overline{\chi}). \quad (2.4) \]

**Lemma 2.6.** The homogenization \( H: N_{\mathbb{R}} \rightarrow N_{\mathbb{R}}^{\text{hom}} \) maps \( T_{\mathbb{R}} \) onto \( T_{\mathbb{R}}^{\text{hom}} \). Further, the restriction of \( d_{\infty} \) to \( T_{\mathbb{R}} \) factors through \( H: T_{\mathbb{R}} \rightarrow T_{\mathbb{R}}^{\text{hom}} \).

**Proof.** The first point follows from Theorem 2.3 (ii), while the second point amounts to \( d_{\infty}(\chi, \overline{\chi}) = 0 \) for all \( \chi \in T_{\mathbb{R}} \), which is a consequence of the estimate in Theorem 2.3 (i).

For any subgroup \( \Lambda \subset \mathbb{R} \) we set
\[ N_{\Lambda}^{\text{hom}} := N_{\Lambda} \cap N_{\mathbb{R}}^{\text{hom}} \quad \text{and} \quad T_{\Lambda}^{\text{hom}} := T_{\Lambda} \cap N_{\mathbb{R}}^{\text{hom}} = T_{\mathbb{R}} \cap N_{\Lambda}^{\text{hom}}. \]

Theorem 2.3 implies
\[ H(T_{\Lambda}) \subset T_{\mathbb{Q}_{\Lambda}}^{\text{hom}}, \quad (2.5) \]
and the homogenization map \( H: T_{\Lambda} \rightarrow T_{\mathbb{Q}_{\Lambda}}^{\text{hom}} \) turns out to be onto (see Corollary 2.13). For \( \Lambda = \mathbb{Z} \), it is closely related to integral closure (see Appendix A for a detailed discussion).
2.3. **The Fubini–Study operator.** Assume first that $L$ is a globally generated line bundle. To any norm $\chi$ on $R_1 = H^0(X, L)$, we associate a function on $X^\text{an}$ by setting

$$FS_L(\chi) := \sup_{s \in R_1 \setminus \{0\}} \{\log |s| + \chi(s)\},$$

i.e. $FS_L(\chi)(v) = \sup_{s \in R_1 \setminus \{0\}} \{\chi(s) - v(s)\}$ for $v \in X^\text{an}$. Given a $\chi$-orthogonal basis $(s_i)$ of $R_1$, one easily checks that

$$FS_L(\chi) = \max_i \log |s_i| + \chi(s_i),$$

see [BE21, Lemma 7.17]. This implies

$$\chi \in \mathcal{N}_\Lambda(R_1) \implies FS_L(\chi) \in \mathcal{H}_\Lambda.$$

for any subgroup $\Lambda \subset \mathbb{R}$.

**Lemma 2.7.** Assume that $L$ is a line bundle, and let $\chi$ be a norm on $R = R(X, L)$. For each $m \geq 1$ we then have

$$FS_{m L}(\chi|_{R^m}) \geq m FS_L(\chi|_{R_1}),$$

where equality holds if $\chi$ is generated in degree 1.

**Proof.** For each $s \in R_1 \setminus \{0\}$ we have $\chi(s^m) \geq m \chi(s)$, and the inequality follows, by (2.6). Assume that $\chi$ is generated in degree 1. To get equality, we need to show

$$v(s) \geq \chi(s) - m FS_L(\chi_1)(v)$$

for all $s \in R_m \setminus \{0\}$ and $v \in X^\text{an}$. To see this, pick an orthogonal basis $(s_i)$ of $R_1$, and write $s = \sum_{|\alpha| = m} c_{\alpha} \prod_i s_i^{\alpha_i}$ with $\chi(s) = \min_{\alpha \neq 0} \sum_i c_{\alpha} \chi(s_i)$ for some $\alpha$ with $c_{\alpha} \neq 0$. Then

$$FS_L(\chi)(v) = \max_i \{\chi(s_i) - v(s_i)\},$$

and hence

$$v(s) \geq \min_{c_{\alpha} \neq 0} \sum_i \alpha_i v(s_i) \geq \min_{c_{\alpha} \neq 0} \sum_i \alpha_i \left(\chi(s_i) - FS_L(\chi_1)(v)\right) = \chi(s) - m FS_L(\chi_1)(v),$$

which concludes the proof. \qed

Returning to the general case of a $\mathbb{Q}$-line bundle, pick $\chi \in \mathcal{N}_\mathbb{R}$, and set

$$FS_m(\chi) := m^{-1} FS_{m L}(\chi|_{R^m})$$

for $m$ sufficiently divisible. By Lemma 2.7, $FS_m(\chi)$ is an increasing function of $m$ with respect to divisibility, and is further uniformly bounded, by linear boundedness of $\chi$. We may thus introduce:

**Definition 2.8.** The Fubini–Study operator $FS: \mathcal{N}_\mathbb{R} \to L^\infty$ takes a norm $\chi \in \mathcal{N}_\mathbb{R}$ to the bounded function $FS(\chi): X^\text{an} \to \mathbb{R}$ defined as the pointwise limit

$$FS(\chi) := \lim_{m} FS_m(\chi) = \sup_m FS_m(\chi).$$

The bounded function $FS(\chi)$ is lsc (lower semicontinuous), being a supremum of continuous functions; it is however not continuous in general. Note also that the canonical approximants $\chi_m \in \mathcal{T}_\mathbb{R}$ of any norm $\chi \in \mathcal{N}_\mathbb{R}$ satisfy

$$FS(\chi_m) = FS_m(\chi)$$

for all $m$ sufficiently divisible, by Lemma 2.7. In particular, if $\chi \in \mathcal{T}_\mathbb{R}$ then the limit in Definition 2.8 is stationary.
The Fubini–Study operator $FS: \mathcal{N}_R \rightarrow \mathcal{L}^\infty$ is increasing with respect to the partial orderings on $\mathcal{N}_R$ and $\mathcal{L}^\infty$, and equivariant under the actions of $\mathbb{R}$ and $\mathbb{R}_>0$. It is also $1$-Lipschitz with respect to the $d_\infty$-(pseudo)metrics, i.e.

\[ d_\infty(FS(\chi), FS(\chi')) \leq d_\infty(\chi, \chi') \text{ for all } \chi, \chi' \in \mathcal{N}_R. \]  

(2.11)

The next result shows that $FS$ is invariant under homogenization.

**Proposition 2.9.** For any $\chi \in \mathcal{N}_R$ we have $FS(\chi) = FS(\chi)\overline{\chi}$.

**Proof.** That $FS(\chi) \leq FS(\chi)$ is clear, since $\chi \leq \overline{\chi}$. Now let $v \in X^{an}$ and $\varepsilon > 0$. Successively pick $m \geq 1$ sufficiently divisible such that $FS(\chi)(v) \leq FS(\chi)(v) + \varepsilon$, then $s \in R_m \setminus \{0\}$ such that $m FS_m(\chi)(v) \leq (\chi - v(s) + m\varepsilon$, and finally $r \geq 1$ such that $\chi(s) \leq \chi(s) + m\varepsilon$, see (2.2). Then

\[ FS(\chi)(v) \leq \frac{1}{mr}(\chi(s') - v(s')) + 3\varepsilon \leq FS_{rm}(\chi)(v) + 3\varepsilon \leq FS(\chi)(v) + 3\varepsilon, \]

completing the proof. \qed

The Fubini–Study operator relates norms of finite type and Fubini–Study functions, as follows:

**Proposition 2.10.** For any subgroup $\Lambda \subset \mathbb{R}$, we have

\[ FS(T_\Lambda) = \mathcal{H}_\Lambda = \mathcal{H}_{ QA} = FS(T_{QA}^{hom}). \]

**Proof.** If $\chi \in T_\Lambda$ then $\chi = \chi_m$ for $m$ sufficiently divisible, and hence $FS(\chi) = FS(\chi_m) = FS_m(\chi) \in \mathcal{H}_\Lambda$ by (2.9). Conversely, pick $\varphi \in \mathcal{H}_\Lambda$, and write $\varphi = r^{-1} \max_i \{\log |s_i| + \lambda_i\}$ with $r \geq 1$, a finite family $(s_i)_{1 \leq i \leq N}$ in $R^{(r)}$ without common zeros, and $\lambda_i \in \Lambda$. After replacing each $s_i$ with a large power, we may assume wlog that $rL$ is a globally generated line bundle. After enlarging the family $(s_i)$ and choosing the corresponding $\lambda_i \ll 0$ in $\Lambda$, we may further assume that $(s_i)$ spans $R^{(r)}$. Consider the surjective map $k^N \rightarrow R^{(r)}$ that takes the canonical basis $(e_i)$ to $(s_i)$. Denote by $\chi_0$ the norm on $k^N$ that is diagonal in $(e_i)$, with $\chi_0(e_i) = \lambda_i$, and let $\chi_r \in \mathcal{N}_R(R^{(r)})$ be the quotient norm. It is then easy to check (see [BE21, Lemma 7.17]) that

\[ \varphi = r^{-1} \max_j \{\log |s_j| + \lambda_j\} = r^{-1} FS_{rL}(\chi_r) \]

By Lemma 2.7, the norm $\chi \in \mathcal{N}_A(R^{(r)})$ generated in degree 1 by $\chi_r$ satisfies $FS(\chi) = \varphi$, which proves $FS(T_\Lambda) = \mathcal{H}_\Lambda$. By Proposition 2.9 and (2.9), we infer

\[ \mathcal{H}_\Lambda = FS(T_\Lambda) = FS(H(T_\Lambda)) \subset FS(T_\Lambda^{hom}) \subset \mathcal{H}_{QA}, \]

which concludes the proof since $\mathcal{H}_{QA} = \mathcal{H}_\Lambda$, see (1.15). \qed

### 2.4 The infimum norm and homogenization.

Next we define an operator

\[ \text{IN}: \mathcal{L}^\infty \rightarrow \mathcal{N}_R^{hom} \]

that to a bounded function $\varphi$ on $X^{an}$ attaches a homogeneous norm, the **infimum norm** $\chi = \text{IN}(\varphi)$. For any $m \in \mathbb{N}$ such that $mL$ is a line bundle, it is defined on $R_m$ by

\[ \chi(s) = \inf_{v \in X^{an}} \{v(s) + m\varphi(v)\} = \inf_{X^{an}} \{m\varphi - \log |s|\}. \]  

(2.12)

In ‘multiplicative’ terminology, this is simply the usual supremum norm

\[ \|s\|_\chi = \sup_{X^{an}} |s| e^{-m\varphi}. \]
The operator \( \text{IN} = \text{IN}_L \) is increasing, and equivariant for the actions of \( \mathbb{R} \) and \( \mathbb{R}_{>0} \). It is also easy to see that
\[
\text{IN}(\varphi \wedge \varphi') = \text{IN}(\varphi) \wedge \text{IN}(\varphi')
\]
and
\[
d_{\infty}(\text{IN}(\varphi), \text{IN}(\varphi')) \leq d_{\infty}(\varphi, \varphi')
\]
for any \( \varphi, \varphi' \in \mathcal{L}^\infty \). The following key result relates homogenization and infimum norms, as follows:

**Theorem 2.11.** For any \( \chi \in \mathcal{N}_\mathbb{R} \), we have \( \text{IN}(\text{FS}(\chi)) = \overline{\chi} \).

The proof uses the Berkovich maximum modulus principle as well as the remarks in §1.5.

**Proof.** After passing to a multiple, we may assume that \( L \) is a line bundle and \( \chi \) is a norm on \( R = R(X, L) \). Let \( M \) be the Berkovich spectrum of the normed ring \( (R, \chi) \), i.e. the set of semivaluations \( w: R \to \mathbb{R} \cup \{+\infty\} \) such that \( w \geq \chi \). Geometrically, \( M \) sits as a compact subset of the analytification \( Y^\text{an} \) of the affine cone \( Y = \text{Spec} R \), and is obtained as the image of the unit disc bundle in the total space of \( L^\vee \), i.e. the blowup of \( o \in Y \) (compare [Fan20]).

Since homogenization corresponds to the spectral radius seminorm construction (see Remark 2.2), the Berkovich maximum modulus principle [Berk90, Theorem 1.3.1] (applied to the completion of \( (R, \chi) \)) yields \( \overline{\chi}(f) = \min_{w \in M} w(f) \) for any \( f \in R \), where the infimum is attained by compactness of \( M \). In particular, for any \( s \in R_m \setminus \{0\}, m \geq 1 \), we have \( \overline{\chi}(s) = \min_{w \in M} w(s) \).

Let \( M^\text{inv} \) be the set of \( k^\times \)-invariant semivaluations in \( M \). We have a projection \( p: M \to M^\text{inv} \) defined by
\[
p(w)(\sum_m s_m) = \min_m w(s_m)
\]
where \( s_m \in R_m \). Thus \( p(w) \leq w \), so in the formula for \( \overline{\chi}(s) \), it suffices to take the infimum over \( w \in M^\text{inv} \). As in §1.3 consider the projection \( \pi^\text{an}: Y^\text{an} \setminus \{w_o\} \to X^\text{an} \), where \( w_o \) is the trivial semivaluation at the vertex of the cone; this satisfies \( w_o(f) = +\infty \) for \( f \in R \setminus \bigoplus_{m>0} R_m \) and \( w_o(f) = 0 \) for \( f \in R \setminus \bigoplus_{m>0} R_m \). For any \( v \in X^\text{an} \), the set of \( k^\times \)-invariant points in \( (\pi^\text{an})^{-1}(v) \) is of the form \( \{w_{v,\lambda}\}_{\lambda \in \mathbb{R}} \), where \( w_{v,\lambda} \) is the unique \( k^\times \)-invariant point such that \( w_{v,\lambda}(s) = v(s) + m\lambda \) for \( s \in R_m, m \geq 1 \), see (1.12).

It follows that \( M^\text{inv} \subset M \subset Y^\text{an} \) is the set of semivaluations \( w_{v,\lambda} \), where \( v \in X^\text{an} \) and \( v(s) + m\lambda \geq \chi(s) \) for all \( s \in R_m, m \geq 1 \). Note that this condition on \( \lambda \) means precisely that \( \lambda \geq \text{FS}(\chi)(v) \). Altogether, this means that if \( s \in R_m, m \geq 1 \), then
\[
\overline{\chi}(s) = \inf\{v(s) + m\lambda \mid v \in X^\text{an}, \lambda \geq \text{FS}(\chi)(v)\} = \inf\{v(s) + m\text{FS}(\chi)(v) \mid v \in X^\text{an}\} = \text{IN}(\text{FS}(\chi))(s),
\]
which completes the proof. \( \square \)

**Corollary 2.12.** The Fubini–Study operator defines an isometric embedding
\[
\text{FS}: (\mathcal{N}_R^\text{hom}, d_{\infty}) \hookrightarrow (\mathcal{L}^\infty, d_{\infty}),
\]
with \( \text{IN}: (\mathcal{L}^\infty, d_{\infty}) \to (\mathcal{N}_R^\text{hom}, d_{\infty}) \) as a left-inverse.

**Proof.** By Theorem 2.11 \( \text{IN} \) is a left-inverse of \( \text{FS} \) on \( \mathcal{N}_R^\text{hom} \). Further, both \( \text{FS}: (\mathcal{N}_R, d_{\infty}) \to (\mathcal{L}^\infty, d_{\infty}) \) and \( \text{IN}: (\mathcal{L}^\infty, d_{\infty}) \to (\mathcal{N}_R^\text{hom}, d_{\infty}) \) are 1-Lipschitz, by (2.11) and (2.14), respectively. They must therefore be isometries, and the result follows. \( \square \)
This implies in turn:

**Corollary 2.13.** For any subgroup \( \Lambda \subset \mathbb{R} \), \( H \colon \mathcal{T}_\Lambda \to \mathcal{T}_{\Lambda}^{\text{hom}} \) is onto.

**Proof.** By Proposition 2.10 both \( \mathcal{T}_\Lambda \to \mathcal{H}_\Lambda \) and \( \mathcal{T}_{\Lambda}^{\text{hom}} \to \mathcal{H}_{\Lambda} = \mathcal{H}_\Lambda \) are onto, and the latter is also injective, by Corollary 2.12. By Proposition 2.9 the composition of \( H \colon \mathcal{T}_\Lambda \to \mathcal{T}_{\Lambda}^{\text{hom}} \) with \( \mathcal{T}_{\Lambda}^{\text{hom}} \to \mathcal{H}_\Lambda \) coincides with the surjective map \( \mathcal{T}_\Lambda \to \mathcal{H}_\Lambda \), and \( H \colon \mathcal{T}_\Lambda \to \mathcal{T}_{\Lambda}^{\text{hom}} \) is thus surjective as well. \( \square \)

**2.5. The Fubini–Study envelope.** As in [BE21, Definition 7.21] and [BoJ21, §4.5], we define the **Fubini–Study envelope** of a bounded function \( \varphi \in \mathcal{L}^\infty \) as the pointwise supremum

\[
Q(\varphi) = \sup\{\psi \in \mathcal{H}_\mathbb{R} \mid \psi \leq \varphi\}. \tag{2.15}
\]

Since any \( \psi \in \mathcal{H}_\mathbb{R} \) is a uniform limit of functions in \( \mathcal{H}_\mathbb{Q} \), one can replace \( \mathcal{H}_\mathbb{R} \) with \( \mathcal{H}_\mathbb{Q} \) in this definition. We note that \( Q_{rL}(r \varphi) = r Q_L(\varphi) \) for any \( r \in \mathbb{Q}_{>0} \), and refer [4.4] for more information.

We view the next result as a ‘dual version’ of Proposition 2.9.

**Proposition 2.14.** For any \( \varphi \in \mathcal{L}^\infty \) we have \( \text{IN}(\varphi) = \text{IN}(Q(\varphi)) \).

This is in fact a special case of [BE21, Lemma 7.23] but we repeat the simple argument for the convenience of the reader.

**Lemma 2.15.** If \( \varphi \in \mathcal{L}^\infty \) and \( s \in R_m \) with \( m \) sufficiently divisible, then \( \log |s| \leq m \varphi \) iff \( \log |s| \leq m Q(\varphi) \).

**Proof.** We may assume \( m = 1 \). Since \( Q(\varphi) \leq \varphi \), we only need to prove the direct implication. For \( t \in \mathbb{R} \), set \( \psi_t = \max\{\log |s| - t\} \). Then \( \psi_t \in \mathcal{H}_\mathbb{R} \), and \( \psi_t \leq \varphi \) for \( t \gg 0 \) since \( \varphi \) is bounded. Thus \( \psi_t \leq Q(\varphi) \) by the definition of \( Q \), so \( \log |s| \leq \psi_t \leq Q(\varphi) \). \( \square \)

**Proof of Proposition 2.14** Pick \( s \in R_m \) with \( m \) sufficiently divisible. We must prove that \( \lambda := \inf_{X^{an}}(m \varphi - \log |s|) \) equals \( \lambda' := \inf_{X^{an}}(m Q(\varphi) - \log |s|) \). Since \( Q(\varphi) \leq \varphi \) we have \( \lambda' \leq \lambda \). The reverse inequality follows from Lemma 2.15 applied to the bounded function \( \varphi - m^{-1} \lambda \).

We similarly have a dual version of Theorem 2.11.

**Proposition 2.16.** For any \( \varphi \in \mathcal{L}^\infty \), we have \( \text{FS}(\text{IN}(\varphi)) = Q(\varphi) \).

**Proof.** After passing to a multiple, we may assume that \( L \) is a line bundle, so that \( \chi := \text{IN}(\varphi) \) is a norm on \( R = R(X, L) \). For all \( m \geq 1 \) and \( s \in R_m \setminus \{0\} \), we have \( \log |s| \leq m \varphi - \chi(s) \) on \( X^{an} \), by definition of the infimum norm. By Lemma 2.15 this yields \( \log |s| \leq m Q(\varphi) - \chi(s) \), and hence

\[
\text{FS}(\chi) = \sup\{m^{-1}(\log |s| + \chi(s)) \mid m \geq 1, s \in R_m \setminus \{0\}\} \leq Q(\varphi).
\]

For the reverse inequality, pick any \( \psi \in \mathcal{H}_\mathbb{R} \) with \( \psi \leq \varphi \). Since \( \text{FS}(\mathcal{T}_\mathbb{R}) = \mathcal{H}_\mathbb{R} \) (see Proposition 2.10), there exists \( m \geq 1 \) and a norm \( \chi' \) on \( R_m \) such that

\[
m \psi = \text{FS}_{mL}(\chi') = \sup_{s \in R_m \setminus \{0\}} \{\log |s| + \chi'(s)\}.
\]

Since \( \psi \leq \varphi \), this gives \( \log |s| + \chi'(s) \leq m \varphi \), i.e. \( \chi' \leq \chi \mid_{R_m} \) on \( R_m \). As a result,

\[
\psi = m^{-1} \text{FS}_{mL}(\chi') \leq m^{-1} \text{FS}_{mL}(\chi \mid_{R_m}) = \text{FS}_m(\chi) \leq \text{FS}(\chi),
\]

which completes the proof. \( \square \)
Combining Theorem 2.11 and Proposition 2.10 with Propositions 2.9 and 2.14, we obtain

**Corollary 2.17.** We have $\text{FS} \circ \text{IN} \circ \text{FS} = \text{FS}$ on $\mathcal{N}_\mathbb{R}$, and $\text{IN} \circ \text{FS} \circ \text{IN} = \text{IN}$ on $\mathcal{L}_\infty$.

2.6. **Proof of Theorem A, case** $p = \infty$ We have the next result is a more precise version of Theorem A in the case $p = \infty$.

**Theorem 2.18.** For any subgroup $\Lambda \subset \mathbb{R}$, the operators

$$\text{FS}: (\mathcal{T}_\Lambda, d_{\infty}) \rightarrow (\mathcal{H}_\Lambda, d_{\infty}), \quad \text{IN}: (\mathcal{H}_\Lambda, d_{\infty}) \hookrightarrow (\mathcal{T}_\Lambda, d_{\infty})$$

are isometries such that $\text{FS} \circ \text{IN} = \text{id}$. When $\Lambda = \mathbb{Q}\Lambda$ is divisible, they further restrict to an isometric isomorphism $(\mathcal{T}^{\text{hom}}_\Lambda, d_{\infty}) \simeq (\mathcal{H}_\Lambda, d_{\infty})$.

**Proof.** For all $\chi, \chi' \in \mathcal{T}_\Lambda$ we have

$$d_{\infty}(\chi, \chi') = d_{\infty}(\text{FS}(\chi), \text{FS}(\chi')) = d_{\infty}(\text{FS}(\chi), \text{FS}(\chi'))$$

where the first equality holds by Lemma 2.6, the second one by Corollary 2.12, and the third one by Proposition 2.9. By Proposition 2.16, $\text{IN}$ is a right-inverse of $\text{FS}$, and hence necessarily an isometric embedding, since $\text{FS}$ is an isometry. When $\Lambda = \mathbb{Q}\Lambda$ is divisible, $\text{FS}: \mathcal{T}^{\text{hom}}_\Lambda \rightarrow \mathcal{H}_\Lambda$ is onto, by Proposition 2.10, and injective, by Corollary 2.12. The last point follows.

□

3. Spectral analysis

In this section we define a volume function $\text{vol}: \mathcal{N}_\mathbb{R} \rightarrow \mathbb{R}$ as well as pseudo-metrics $d_p$, $p \in [1, \infty)$, on the space $\mathcal{N}_\mathbb{R}$ of norms on section rings of multiples of $L$. Much of the material is studied for more general non-Archimedean ground fields in [CM15, BE21], but we present the details for the convenience of the reader.

3.1. **The finite-dimensional case.** We first describe the space $\mathcal{N}_\mathbb{R}(V)$ of non-Archimedean norms on a $k$-vector space $V$ of dimension $N < \infty$, essentially following [MFK, BE21].

Pick a norm $\chi \in \mathcal{N}_\mathbb{R}(V)$, and a $\chi$-orthogonal basis $(e_j)_{1 \leq j \leq N}$ of $V$. After permutation, we may assume that the sequence $\lambda_j(\chi) := \chi(e_j)$, $j = 1, \ldots, N$ satisfies

$$\lambda_1(\chi) \geq \cdots \geq \lambda_N(\chi).$$

It is then independent of the choice of orthogonal basis and is called the spectrum of $\chi$, i.e. the ‘jumping values’ of the associated filtration in the terminology of [BC11]. In terms of (1.3) we have

$$\lambda_1(\chi) = \lambda_{\max}(\chi), \quad \lambda_N(\chi) = \lambda_{\min}(\chi).$$

The volume of $\chi$ is defined as the mean value of its spectrum, i.e.

$$\text{vol}(\chi) := \frac{1}{N} \sum_j \lambda_j(\chi).$$

For any basis $(e_j)$ of $V$ we have $\text{vol}(\chi) \geq N^{-1} \sum_j \chi(e_j)$ with equality iff $(e_j)$ is orthogonal.

More generally, any two norms $\chi, \chi'$ admit a common orthogonal basis $(e_j)$. The relative spectrum of $\chi$ with respect to $\chi'$ is the sequence

$$\lambda_1(\chi, \chi') \geq \cdots \geq \lambda_N(\chi, \chi')$$

obtained by reordering $(\chi(e_j) - \chi'(e_j))_{1 \leq j \leq N}$, and the spectral measure of $\chi$ with respect to $\chi'$ is the corresponding probability measure

$$\sigma(\chi, \chi') := \frac{1}{N} \sum_j \delta_{\lambda_j(\chi, \chi')}.$$
Its barycenter satisfies
\[
\int_\mathbb{R} \lambda \, d\sigma(\chi, \chi') = \frac{1}{N} \sum_j \lambda_j(\chi, \chi') = \text{vol}(\chi) - \text{vol}(\chi').
\]  
(3.1)

When \(\chi' = \chi_{\text{triv}}\) is the trivial norm, we simply write
\[
\sigma(\chi) = \sigma(\chi, \chi_{\text{triv}}) = \frac{1}{N} \sum_j \delta_{\lambda_i(\chi)},
\]
and call it the spectral measure of \(\chi\).

To any basis \(e = (e_i)\) of \(V\) is associated an apartment \(A_e \subset N_\mathbb{R}(V)\), defined as the set of norms diagonalized in this basis. We then have a canonical parametrization
\[
\iota_e : \mathbb{R}^N \xrightarrow{\sim} A_e,
\]
and a Gram–Schmidt retraction
\[
\rho_e : N_\mathbb{R}(V) \to A_e.
\]
The map \(\iota_e\) sends \((\lambda_j) \in \mathbb{R}^N\) to the unique \(\chi \in A_e\) such that \(\chi(e_i) = \lambda_i\), while \(\rho_e\) sends a norm \(\chi\) to the unique norm \(\rho_e(\chi) \in A_e\) such that
\[
\rho_e(\chi)(e_i) = \sup_{a \in \mathbb{K}^N} \chi(e_i + \sum_{j<i} a_j e_j).
\]
i.e. the norm induced, via the basis \(e\), from the natural subquotient norm of the graded object of the complete flag defined by \(e\). By additivity of the volume in exact sequences, we have
\[
\text{vol}(\rho_e(\chi)) = \text{vol}(\chi),
\]  
(3.2)

see [BE21, Lemma 2.12]. Since any two norms \(\chi, \chi'\) can be jointly diagonalized, they both belong to some common apartment \(A_e\). Each \(A_e\) is trivially preserved by the translation action of \(\mathbb{R}\), the scaling action by \(\mathbb{R}_{>0}\), and by the operation \((\chi, \chi') \mapsto \chi \wedge \chi'\). Moreover,
\[
\chi \leq \chi' \implies \rho_e(\chi) \leq \rho_e(\chi').
\]

Generalizing the classical construction of the Tits metric on the Euclidean building \(N_\mathbb{R}(V)\) (see for instance [MFK, §2.2]), it is shown in [BE21, Theorem 3.1] that each \(G_N\)-invariant norm \(\tau\) on \(\mathbb{R}^N\) induces a unique metric \(d_\tau\) on \(N_\mathbb{R}(V)\) such that \(\iota_e : (\mathbb{R}^N, \tau) \to (N_\mathbb{R}(V), d_\tau)\) is an isometry for any basis \(e\). It has the property that \(\rho_e : N_\mathbb{R}(V) \to A_e\) is a contraction. All metrics on \(N_\mathbb{R}(V)\) obtained this way are equivalent. They turn \(N_\mathbb{R}(V)\) into a metric space that is complete, but not locally compact.

In particular, for each \(p \in [1, \infty]\) we define a metric \(d_p\) on \(N_\mathbb{R}(V)\) by setting for any two norms \(\chi, \chi'\) with relative spectrum \((\lambda_i) = (\lambda_i(\chi, \chi'))\)
\[
d_p(\chi, \chi') := (N^{-1} \sum_i |\lambda_i|^p)^{1/p}
\]  
(3.3)

for \(p \in [1, \infty)\), and
\[
d_\infty(\chi, \chi') := \max_i |\lambda_i|.
\]

Thus \(d_p(\chi, \chi')\) is the \(L^p\)-norm of \(\lambda\) with respect to the spectral measure \(\sigma(\chi, \chi')\). Note that
\[
d_1 \leq d_p \leq d_\infty^p \leq d_\infty
\]  
(3.4)
on $\mathcal{N}_R(V)$ for $p \in (1, \infty)$. The metric $d_2$ is the Tits metric mentioned above, while $d_\infty$ coincides with the Goldman–Iwahori metric \textsuperscript{1,2}. Our main interest lies in the metric $d_1$, which is closely related to the volume:

**Lemma 3.1.** For all $\chi, \chi' \in \mathcal{N}_R(V)$ and $p \in [1, \infty)$ we have

$$d_p(\chi, \chi')^p = d_p(\chi, \chi \wedge \chi')^p + d_p(\chi \wedge \chi', \chi')^p.$$ \hspace{1cm} (3.5)

For $p = 1$, we further have

$$d_1(\chi, \chi') = \text{vol}(\chi) + \text{vol}(\chi') - 2\text{vol}(\chi \wedge \chi').$$ \hspace{1cm} (3.6)

**Proof.** The first assertion follows from the fact that the minimum $\chi \wedge \chi'$ of two norms in an apartment $\mathbb{A}_e \simeq \mathbb{R}^N$ is computed component-wise, and the trivial identity

$$\sum_i |\lambda_i - \lambda'_i |^p = \sum_i |\lambda_i - \min\{\lambda_i, \lambda'_i\} |^p + \sum_i |\min\{\lambda_i, \lambda'_i\} - \lambda'_i |^p.$$ 

for all $\lambda, \lambda' \in \mathbb{R}^N$. On the other hand, it follows from \textsuperscript{3.1} that $\chi \geq \chi' \implies d_1(\chi, \chi') = \text{vol}(\chi) - \text{vol}(\chi')$, and \textsuperscript{3.6} follows.

The volume function is trivially 1-Lipschitz with respect to $d_1$, i.e.

$$|\text{vol}(\chi) - \text{vol}(\chi')| \leq d_1(\chi, \chi')$$ \hspace{1cm} (3.7)

for all $\chi, \chi' \in \mathcal{N}_R(V)$. This is also the case for the min operator:

**Lemma 3.2.** Let $\chi_i, \chi'_i, i = 1, 2$, be norms on $V$. Then

$$d_1(\chi_1 \wedge \chi_2, \chi'_1 \wedge \chi'_2) \leq d_1(\chi_1, \chi'_1) + d_1(\chi_2, \chi'_2).$$ \hspace{1cm} (3.8)

**Proof.** First assume $\chi_i \geq \chi'_i$, $i = 1, 2$. Pick a basis $e$ such that $\chi'_1, \chi'_2 \in \mathbb{A}_e$. Lemma \textsuperscript{3.1} together with \textsuperscript{3.2} show that replacing $\chi_i$ by $\rho_e(\chi_i)$, $i = 1, 2$ does not change the right-hand side of \textsuperscript{3.8}. As for the left-hand side, $\rho_e : \mathcal{N}_R(V) \to \mathbb{A}_e$ being order preserving implies

$$\rho_e(\chi_1) \wedge \rho_e(\chi_2) \geq \rho_e(\chi_1 \wedge \chi_2) \geq \chi'_1 \wedge \chi'_2,$$

which shows that the left-hand side of \textsuperscript{3.8} can only increase upon replacing $\chi_i$ by $\rho_e(\chi_i)$, $i = 1, 2$, using again \textsuperscript{3.2} and \textsuperscript{3.6}.

As a result, we may in fact assume that all four norms belong to $\mathbb{A}_e$. Write $\chi_i(e_j) = \lambda_{i,j}$ and $\chi'_i(e_j) = \lambda'_{i,j}$ for $1 \leq j \leq N$ and $i = 1, 2$. Then $\lambda_{i,j} \geq \lambda'_{i,j}$ for all $i, j$, and we must prove that

$$\sum_j \lambda_{1,j} \wedge \lambda_{2,j} - \sum_j \lambda'_{1,j} \wedge \lambda'_{2,j} \leq \sum_j (\lambda_{1,j} - \lambda'_{1,j}) + \sum_j (\lambda_{2,j} - \lambda'_{2,j});$$

this is straightforward.

Finally consider arbitrary norms. Set $\chi''_i = \chi_i \wedge \chi'_i$ for $i = 1, 2$. By \textsuperscript{3.5} we have

$$d_1(\chi_1 \wedge \chi_2, \chi'_1 \wedge \chi'_2) = d_1(\chi_1 \wedge \chi_2, \chi''_1 \wedge \chi''_2) + d_1(\chi'_1 \wedge \chi'_2, \chi''_1 \wedge \chi''_2)$$

and $\chi_i, \chi'_i \geq \chi''_i$, for $i = 1, 2$, so \textsuperscript{3.8} follows from what precedes, together with \textsuperscript{3.5}. \qed
3.2. **Spectral measures and volume.** Now we return to the setting of a projective variety $X$ and an ample $\mathbb{Q}$-line bundle $L$ on $X$. The following equidistribution result is a special case of a result of Chen–Micali [CM15], which deals with general non-Archimedean fields.

**Theorem 3.3.** For any two norms $\chi, \chi' \in \mathcal{N}_\mathbb{R}$, the scaled spectral measures
\[
(1/m) \ast \sigma(\chi|_{R_m}, \chi'|_{R_m})
\]
have uniformly bounded support, and they admit a weak limit.

The limit is taken with respect to the partial order by divisibility. If $L$ is an actual line bundle and $\chi, \chi'$ are norms on $R(X, L)$, then the limit also exists as $m \to \infty$ in the usual total ordering.

**Definition 3.4.** For any $\chi, \chi' \in \mathcal{N}_\mathbb{R}$, the spectral measure of $\chi$ with respect to $\chi'$ is the compactly supported (Borel) probability measure on $\mathbb{R}$ defined as
\[
\sigma(\chi, \chi') := \lim_{m \to \infty} (1/m) \ast \sigma(\chi|_{R_m}, \chi'|_{R_m}).
\]

The spectral measure of $\chi$ is $\sigma(\chi) := \sigma(\chi, \chi_{\text{triv}})$, and the volume of $\chi$ is the barycenter
\[
\text{vol}(\chi) = \int_{\mathbb{R}} \lambda \, d\sigma(\chi).
\]

By (3.1), we have
\[
\text{vol}(\chi) = \lim_{m \to \infty} m^{-1} \text{vol}(\chi|_{R_m}). \tag{3.9}
\]

The existence of the spectral measure $\sigma(\chi)$ (called the limit measure of the corresponding filtration in [BHJ17 §5.1]) follows from [BC11 Theorem A]. As we shall see, a simple trick borrowed from [CM15] reduces the proof of Theorem 3.3 to this special case $\chi' = \chi_{\text{triv}}$.

**Proof of Theorem 3.3.** The uniform boundedness part is a direct consequence of the linear boundedness condition that we impose on norms in $\mathcal{N}_\mathbb{R}(R)$. Set $N_m := \dim R_m$, denote by $(\lambda_{m,j})_{1 \leq j \leq N_m}$ the spectrum of $\chi|_{R_m}$ with respect to $\chi'|_{R_m}$, and set
\[
\sigma_m := (1/m) \ast \sigma(\chi|_{R_m}, \chi'|_{R_m}) = \frac{1}{N_m} \sum_{j=1}^{N_m} \delta_{m^{-1} \lambda_{m,j}}.
\]

As is well-known, in order to prove convergence of $\sigma_m$, it suffices to show that
\[
\int_{\mathbb{R}} \max\{\lambda, c\} \, d\sigma_m = \frac{1}{mN_m} \sum_{j=1}^{N_m} \max\{\lambda_{j,m}, mc\}
\]
converges for all $c \in \mathbb{R}$, see [CM15 Proposition 5.1]. But $(\max\{\lambda_{j,m}, cm\})_j$ is the spectrum of $\chi|_{R_m}$ with respect to $\chi'|_{R_m} \wedge (\chi|_{R_m} - cm)$. Replacing $\chi'$ with $\chi' \wedge (\chi - c)$ (where $\mathbb{R}$ acts by translation according to (1.4)), we are reduced to proving that the barycenter
\[
\left(mN_m\right)^{-1} \sum_{j} \lambda_{m,j}
\]
of the measure $\sigma_m$ converges. Now, this barycenter is the difference of the barycenters of $(1/m) \ast \sigma(\chi|_{R_m})$ and $(1/m) \ast \sigma(\chi'|_{R_m})$, each of which admits a limit by [BC11 Theorem A], and we are done. \[\square\]
The proof of Theorem 3.3 shows that
\[ \int \max\{\lambda, c\} \, d\sigma(\chi, \chi') = \text{vol}(\chi) - \text{vol}(\chi' \land (\chi - c)) \] (3.10)
for all \( \chi, \chi' \in \mathcal{N}_\mathbb{R} \) and \( c \in \mathbb{R} \).

Some further properties of the spectral measure \( \sigma(\chi) \) are described by the following result, which is a direct consequence of [BC11], see [BHJ17, Theorem 5.3].

**Theorem 3.5.** For any \( \chi \in \mathcal{N}_\mathbb{R} \), the support of \( \sigma(\chi) \) is a compact interval with upper bound \( \lambda_{\text{max}}(\chi) \). Further, \( \sigma(\chi) \) is absolutely continuous with respect to Lebesgue measure, except perhaps for a point mass at \( \lambda_{\text{max}}(\chi) \).

The next result shows how spectral measures behave under operations on norms. It follows from elementary computations of spectra in joint orthogonal bases; the details are left to the reader.

**Proposition 3.6.** Let \( \chi, \chi' \in \mathcal{N}_\mathbb{R} \), and pick \( c \in \mathbb{R} \), \( t \in \mathbb{R}_{>0} \). Then:

\begin{enumerate}[(i)]
\item \( \sigma(\chi, \chi') \) is the pushforward of \( \sigma(\chi, \chi') \) under \( \lambda \mapsto -\lambda \);
\item \( \sigma(\chi + c, \chi') \) is the pushforward of \( \sigma(\chi, \chi') \) under \( \lambda \mapsto \lambda + c \);
\item \( \sigma(\chi, \chi' \land \chi') \) is the pushforward of \( \sigma(\chi, \chi') \) under \( \lambda \mapsto \max\{\lambda, 0\} \);
\item \( \sigma(t\chi, t\chi') \) is the pushforward of \( \sigma(\chi, \chi') \) by \( \lambda \mapsto t\lambda \).
\end{enumerate}

**Remark 3.7.** In [BLXZ21, Theorem 3.3] (which appeared after the first version of this article was posted), the authors construct a natural joint spectral measure \( \rho(\chi, \chi') \) on \( \mathbb{R}^2 \) associated to any pair \( \chi, \chi' \in \mathcal{N}_\mathbb{R} \), that encodes the asymptotic behavior of the spectra of \( \chi \) and \( \chi' \) in jointly orthogonal bases. The spectral measure \( \sigma(\chi, \chi') \) is the pushforward of \( \rho(\chi, \chi') \) under the map \( \mathbb{R}^2 \rightarrow \mathbb{R} \) given by \( (\lambda, \lambda') \mapsto \lambda - \lambda' \).

### 3.3. The \( d_p \)-pseudometrics and asymptotic equivalence

Pick \( p \in [1, \infty) \) and \( \chi, \chi' \in \mathcal{N}_\mathbb{R} \). By definition, we have
\[ d_p(\chi_{|R_m}, \chi'_{|R_m})^p = \int_{\mathbb{R}} |\lambda|^p \, d\sigma(\chi_{|R_m}, \chi'_{|R_m}) \].

Theorem 3.3 thus shows that the limit
\[ d_p(\chi, \chi') := \lim_{m \to \infty} m^{-1} d_p(\chi_{|R_m}, \chi'_{|R_m}) \]
exists in \([0, +\infty)\), and coincides with the \( L_p \)-norm of the identity with respect to the spectral measure \( \sigma(\chi, \chi') \), i.e.
\[ d_p(\chi, \chi')^p = \int_{\mathbb{R}} |\lambda|^p \sigma(\chi, \chi'). \] (3.11)

It is clear that \( (d_p)_{1 \leq p < \infty} \) is a non-decreasing family of pseudo-metrics on \( \mathcal{N}_\mathbb{R} \). For \( p = 1 \), (3.6) further yields
\[ d_1(\chi, \chi') = \text{vol}(\chi) + \text{vol}(\chi') - 2 \text{vol}(\chi \land \chi') \]. (3.12)

For any \( p \in [1, \infty] \), we have \( d_1 \leq d_p \leq d_\infty \), but the analogue of (3.11) fails for the pseudo-metric \( d_\infty \) (see, however Lemma 3.12).

**Example 3.8.** Assume \( (X, L) = (\mathbb{P}^1, \mathcal{O}(1)) \), and define a norm \( \chi \) on \( R = R(X, L) = k[T_1, T_2] \)
by \( \chi(s) = -m \) for \( s \in R_m \) with \( T_1 \not\mid s \) and \( \chi(s) = 0 \) otherwise. Then \( \chi \in \mathcal{N}_{\text{hom}} \), \( \sigma(\chi_{|R_m}) = \frac{1}{m+1} \delta_m + \frac{m}{m+1} \delta_0 \), hence \( \sigma(\chi) = \lim_{m \to \infty} (1/m, \sigma(\chi_{|R_m}) = \delta_0 \). However, \( d_\infty(\chi_{|R_m}, \chi_{\text{triv}}_{|R_m}) = m \) for all \( m \); hence \( d_\infty(\chi, \chi_{\text{triv}}) = 1 \) (compare [Szé13, p.458]).
The pseudo-metric $d_1$ defines a (non-Hausdorff) topology on $\mathcal{N}_R$, which is strictly coarser than the $d_p$-topology for any $p > 1$. However, (3.4) remains valid on $\mathcal{N}_R$, and shows that the $d_p$-topologies with $p < \infty$ all agree on $d_\infty$-bounded subsets of $\mathcal{N}_R$. In particular, they share the same pairs of non-separated points, which gives rise to:

**Definition 3.9.** We say that two norms $\chi, \chi' \in \mathcal{N}_R$ are asymptotically equivalent, and write $\chi \sim \chi'$, if the following equivalent conditions hold:

1. $d_1(\chi, \chi') = 0$;
2. $d_p(\chi, \chi') = 0$ for all $p \in [1, \infty)$;
3. $\sigma(\chi, \chi') = \delta_0$.

The equivalence between (i)—(iii) follows from (3.11). While we trivially have $d_\infty(\chi, \chi') = 0 \implies \chi \sim \chi'$, the converse fails in general, see Example 3.8.

As we next show, spectral measures are continuous with respect to the $d_1$-topology.

**Theorem 3.10.** Consider nets $(\chi_i), (\chi'_i) \in \mathcal{N}_R$, converging respectively to $\chi, \chi' \in \mathcal{N}_R$ in the $d_1$-topology. Then $\sigma(\chi_i, \chi'_i) \rightharpoonup \sigma(\chi, \chi')$ weakly.

*Proof.* Set $\sigma_i := \sigma(\chi_i, \chi'_i)$ and $\sigma := \sigma(\chi, \chi')$. As in the proof of Theorem 3.3, it suffices to prove that

$$\int \max\{\lambda, c\} \, d\sigma_i = \text{vol}(\chi_i) - \text{vol}(\chi'_i \wedge (\chi_i - c))$$

converges to

$$\int \max\{\lambda, c\} \, d\sigma = \text{vol}(\chi) - \text{vol}(\chi' \wedge (\chi - c))$$

for all $c \in \mathbb{R}$, see (3.10). This follows immediately from the Lipschitz property of the volume and min operators, see (3.7) and Lemma 3.2.

**Corollary 3.11.** For any $\chi, \chi' \in \mathcal{N}_R$, the quantities

$$\sigma(\chi, \chi'), \lambda_{\max}(\chi) \text{ and } \text{vol}(\chi)$$

only depend on the asymptotic equivalence classes of $\chi, \chi'$. Further,

$$\chi_i \sim \chi'_i, \ i = 1, 2 \implies \chi_1 \wedge \chi_2 \sim \chi'_1 \wedge \chi'_2.$$

*Proof.* The first claim follows directly from Theorem 3.10. It implies the second one, as $\lambda_{\max}(\chi)$ can be reconstructed from $\sigma(\chi) = \sigma(\chi, \chi_{\text{triv}})$, by Theorem 3.5. Finally, the Lipschitz properties of the volume and the min operator (see (3.7) and Lemma 3.2) carry over to $\mathcal{N}_R$, which takes care of the last two claims.

Following [Fuj19b, Zha20], we finally show:

**Lemma 3.12.** If $\chi \in \mathcal{N}_R$ satisfies $\chi \geq \chi_{\text{triv}}$, then

$$\lambda_{\max}(\chi) = d_\infty(\chi, \chi_{\text{triv}}) \leq C_{n,p} d_p(\chi, \chi_{\text{triv}})$$

where $C_{n,p} := (n + p)^{1/p} \leq (n + n)^{1/n}$ satisfies $C_{n,p} \to 1$ as $p \to \infty$.

*Proof.* The first equality is (1.9). To prove the inequality, we argue as in [Zha20 §5]. Set $\lambda_{\max} := \lambda_{\max}(\chi)$. By [BLL17, Theorem 5.3], $f(\lambda) := \sigma([\lambda, +\infty))^{1/n}$ is concave on $(-\infty, \lambda_{\max}(\chi))$. 


Since \( \chi \geq \chi_{\text{triv}} \), the support of \( \sigma = \sigma(\chi) \) is contained in \([0, \lambda_{\text{max}}]\), and hence \( f(0) = 1 \). For \( \lambda \in [0, \lambda_{\text{max}}] \), we thus have \( f(\lambda) \geq 1 - \frac{\lambda}{\lambda_{\text{max}}} \), which yields

\[
d_p(\chi, \chi_{\text{triv}})^p = \int_0^{\lambda_{\text{max}}} \lambda^p \, d\sigma = p \int_0^{\lambda_{\text{max}}} \chi^{p-1} f(\lambda)^n \, d\lambda \\
\geq p \int_0^{\lambda_{\text{max}}} \lambda^{p-1} \left(1 - \frac{\lambda}{\lambda_{\text{max}}}\right)^n \, d\lambda = \lambda_{\text{max}}^p p \int_0^1 t^{p-1}(1-t)^n \, dt = \frac{p!n!}{(n+p)!} \lambda_{\text{max}}^p,
\]

and the result follows. \( \square \)

### 3.4. The space of norms modulo translation.

For any \( p \in [1, \infty] \), the additive action of \( \mathbb{R} \) on \( \mathcal{N}_\mathbb{R} \) preserves the pseudo-metric \( d_p \), which thus induces a quotient pseudo-metric \( d_p^\mathbb{R} \) on the space of norms modulo translation \( \mathcal{N}_\mathbb{R}/\mathbb{R} \), such that

\[
d_p^\mathbb{R}(\chi, \chi') = \inf_{c \in \mathbb{R}} d_p(\chi, \chi' + c)
\]

for \( \chi, \chi' \in \mathcal{N}_\mathbb{R} \). This supremum is actually achieved:

**Lemma 3.13.** For any \( \chi, \chi' \in \mathcal{N}_\mathbb{R} \), there exists \( c \in \mathbb{R} \) such that \( d_p^\mathbb{R}(\chi, \chi') = d_p(\chi, \chi' + c) \).

In particular, \( d_p^\mathbb{R}(\chi, \chi') = 0 \) iff \( \chi, \chi' \) are asymptotically equivalent modulo translation, i.e. \( \chi \sim \chi' + c \) for some \( c \in \mathbb{R} \), uniquely determined by \( c = \text{vol}(\chi) - \text{vol}(\chi') \). When \( \chi' = \chi_{\text{triv}} \) we say that \( \chi \) is asymptotically constant.

**Proof.** Since the spectral measure \( \sigma := \sigma(\chi, \chi') \) has compact support, the function

\[
c \mapsto d_p(\chi, \chi' + c) = \int_{\mathbb{R}} |\lambda + c| \, d\sigma
\]

is (Lipschitz) continuous and tends to \(+\infty\) as \(|c| \to \infty\). The result follows. \( \square \)

**Definition 3.14.** Following Donaldson, for each \( p \in [1, \infty] \), we define the \( L^p \)-norm \( \|\chi\|_p \) of a norm \( \chi \in \mathcal{N}_\mathbb{R} \) using the \( p \)-th central moment of its spectral measure, i.e.

\[
\|\chi\|_p^p = \int \left| \lambda - \bar{\lambda} \right|^p \, d\sigma(\chi),
\]

where \( \bar{\lambda} = \int \lambda \, d\sigma(\chi) = \text{vol}(\chi) \) is its barycenter.

**Remark 3.15.** This definition extends the notion in [Don02] of the \( L^p \)-norm of a test configuration, see [BHJ17, Remark 6.10].

**Proposition 3.16.** Given a norm \( \chi \in \mathcal{N}_\mathbb{R} \), the following are equivalent:

- (i) \( \chi \) is asymptotically constant;
- (ii) \( \|\chi\|_p = 0 \) for some \( p \in [1, +\infty) \);
- (iii) \( \|\chi\|_p = 0 \) for all \( p \in [1, +\infty) \).

**Proof.** If \( \chi \sim \chi_{\text{triv}} + c \) with \( c \in \mathbb{R} \), then \( c = \text{vol}(\chi) \), by \( (3.7) \). The rest is straightforward. \( \square \)
3.5. Convergence of the canonical approximants. The next result strengthens the approximation result proved in [BC11] Theorem 1.14 (see also [Bon14, Théorème 3.15] and [Cod19]). A version valid for arbitrary non-Archimedean fields is given in [Reb21, Theorem 4.5.4] (which appeared after the first version of the current paper).

Recall that the canonical approximants $\chi_r \in T_R$ of a norm $\chi \in \mathcal{N}_R$, which are defined for $r$ sufficiently divisible, satisfy $\chi_r \leq \chi$ and form an increasing net with respect to divisibility.

**Theorem 3.17.** For any $\chi \in \mathcal{N}_R$ and $p \in [1, \infty)$, we have $\chi_r \to \chi$ in the $d_p$-topology.

For $p = \infty$, the result fails in general. Indeed, $d_\infty(\chi_r, \chi) \to 0$ implies $\text{FS}(\chi) \in C^0$, by (2.11), which is contradicted by $\chi = \text{IN}(\varphi)$ for a discontinuous function $\varphi \in \mathcal{E}_1^\infty$, see Theorem 6.1 and Remark 4.2.

By continuity of spectral measures (see Theorem 3.10), this implies:

**Corollary 3.18.** For all $\chi, \chi' \in \mathcal{N}_R$ we have $\lim_r \sigma(\chi_r, \chi'_r) = \sigma(\chi, \chi')$.

**Proof of Theorem 3.17.** Since $\chi_r \leq \chi$ is an increasing net, $d_\infty(\chi_r, \chi)$ is decreasing, and hence uniformly bounded. In view of (3.4), it is thus enough to show the result for $p = 1$, i.e.

$$d_1(\chi_r, \chi) = \text{vol}(\chi) - \text{vol}(\chi_r) = \int_R \lambda(\sigma(\chi) - \sigma(\chi_r))$$

tends to 0. Since $(\chi_r)$ is $d_\infty$-bounded, the support of $\sigma(\chi_r)$ is further uniformly bounded, and it will thus suffice to show $\sigma(\chi_r) \to \sigma(\chi)$ weakly. To prove this, we may, after replacing $L$ by a multiple, assume that $R = R(X, L)$ is generated in degree 1 and that $\chi$ is defined on $R$. We recall the description of the spectral measure $\sigma(\chi)$ from [BC11]. The classical Hilbert–Serre theorem guarantees the existence of the volume

$$V = \text{vol}(R) := \lim_m \frac{n!}{m^n} \dim R_m,$$

and we clearly have $\text{vol}(R(r)) = r^n V$. More generally, for each $\lambda \in \mathbb{R}$, consider the graded subalgebra $R^\lambda \subset R$ with graded pieces

$$R^\lambda_m := \{ s \in R_m \mid \chi(s) \geq m\lambda \}.$$

Using Okounkov bodies [LM09, KK12], one shows that the volume

$$\text{vol}(R^\lambda) := \lim_m \frac{n!}{m^n} \dim R^\lambda_m$$

exists, and [BC11, Theorem 1.11] yields $\sigma(\chi) = -dg$ in the sense of distributions, where $g: \mathbb{R} \to [0, 1]$ is the nonincreasing function defined by

$$g(\lambda) := V^{-1} \text{vol}(R^\lambda).$$

For each $r$, we similarly have a family of graded subalgebras $R^{\lambda, (r)} \subset R^{(r)}$ with graded pieces

$$R^{\lambda, (r)}_m = \{ s \in R_{rm} \mid \chi(s) \geq rm\lambda \},$$

and $\sigma(\chi_r) = -dg_r$ with $g_r: \mathbb{R} \to [0, 1]$ defined by

$$g_r(\lambda) := (r^n V)^{-1} \text{vol}(R^{\lambda, (r)}).$$

Since the degree 1 part of $R^{\lambda, (r)}$ is equal to $R^\lambda$, Lemma 3.19 below yields $g_r \to g$ pointwise on $\mathbb{R}$. By dominated convergence, it follows that $g_r \to g$ in $L^1_{\text{loc}}(\mathbb{R})$, and hence $-dg_r = \sigma(\chi_r) \to -dg = \sigma(\chi)$ weakly. 

\[\square\]
Lemma 3.19. Let $S \subset R$ be a graded subalgebra, and suppose we are given, for each $r$ divisible enough, a graded subalgebra $T(r) \subset S^{(r)}$ such that $T(r)_1 = S^{(r)}_1 = S_r$. Then $\lim r^{-n} \text{vol}(T(r)) = \text{vol}(S)$.

Proof. We use Okounkov bodies, following [Bon14]. Set $K := k(X)$, and pick a valuation $\nu: K^\times \to \mathbb{Z}^n$ of maximal rational rank, equal to $n$ (e.g. associated to a flag of subvarieties as in [LM09]). Set $\Gamma_m := \nu(S_m \setminus \{0\})$ and $\Gamma(r)_m := \nu(T(r)_m \setminus \{0\})$ for $m \geq 1$. Let $\Delta(S)$ and $\Delta(T(r))$ be the closed convex hull inside $\mathbb{R}^n$ of $\bigcup_m m^{-1}\Gamma_m$ and of $\bigcup_m m^{-1}\Gamma(r)_m$, respectively. Then $\text{vol}(S) = n! \text{vol}(\Delta(S))$ and $\text{vol}(T(r)) = n! \text{vol}(\Delta(T(r)))$, so it suffices to prove that $\lim n! \text{vol}(\Delta(T(r))) = \text{vol}(\Delta(S))$.

Since $T(r)_m \subset S_m$, we get $\Gamma(r)_m \subset \Gamma_m$ for all $r, m$, and hence $r^{-1}\Delta(T(r)) \subset \Delta(S)$ for all $r$. If $\text{vol}(\Delta(S)) = 0$, we are done, so we may assume $\Delta(S)$ has nonempty interior. Pick compact subsets $A$ and $B$ of $\mathbb{R}^n$ with $A \subset B \subset \Delta(S)$. It suffices to prove that $r^{-1}\Delta(T(r)) \supset A$ for $r$ sufficiently divisible. Now $r^{-1}\mathbb{Z}^n \cap B = r^{-1}\Gamma_r \cap B$, see [Bon14] Lemma 1.13. If $\Delta_r$ is the convex hull of $r^{-1}\Gamma_r$, it follows that $\Delta_r \supset A$. But $T(r)_1 = S_r$, so $\Gamma(r)_1 = \Gamma_r$, and hence $r^{-1}\Delta(T(r)) \supset \Delta_r \supset A$, which completes the proof. □

4. Non-Archimedean pluripotential theory

The goal of this section is to summarize results from [BoJ21] that are relevant to our later purposes.

4.1. L-psh functions. An $L$-psh function $\varphi: X^\an \to [-\infty, +\infty)$ is defined as the pointwise limit of all decreasing net in $\mathcal{H}_Q$ (or $\mathcal{H}_Z$), excluding $\varphi \equiv -\infty$. We denote by $\text{PSH} = \text{PSH}(L)$ the set of all $L$-psh functions. If $\varphi \in \text{PSH}$, then $\varphi + c \in \text{PSH}$ for all $c \in \mathbb{R}$. If $\varphi, \psi \in \text{PSH}$, then $\max\{\varphi, \psi\} \in \text{PSH}$. If $(\varphi_j)_j$ is a decreasing net in $\text{PSH}$, and $\varphi$ is the pointwise limit of $(\varphi_j)_j$, then $\varphi \in \text{PSH}$, or $\varphi \equiv -\infty$. We can thus describe $\text{PSH}$ as the smallest class of functions which is invariant under max, translation by a constant, decreasing limits, and contains all functions of the form $m^{-1}\log |s|$ with $m$ sufficiently divisible and $s \in R_m \setminus \{0\}$

By Dini’s Lemma, the set

$$\text{CPSH} := \text{PSH} \cap C^0$$

of continuous $L$-psh functions is the closure of $\mathcal{H}_Q$ (or $\mathcal{H}_Z$) in $C^0$ (in line with the definition of a semipositive (continuous) metric in [Zha93, Gub98, CM18]). By Theorem 2.18, CPSH is thus the Hausdorff completion of $(\mathcal{T}_Z, d_\infty)$.

The set $\text{PSH}$ is stable under convex combinations, and under the action $(t, \varphi) \mapsto t \cdot \varphi$ of $\mathbb{R}_{>0}$ on functions, see [I.10]. If $v, v' \in X^\an$ and $v \leq v'$, then $\varphi(v) \geq \varphi(v')$ for all $\varphi \in \text{PSH}$. In particular,

$$\sup \varphi := \sup_{X^\an} \varphi = \varphi(\nu_{\text{triv}})$$

for all $\varphi \in \text{PSH}$.

A subset $E \subset X^\an$ is pluripolar if $E \subset \{\varphi = -\infty\}$ for some $L$-psh function $\varphi$. This condition is independent of the choice of ample $\mathbb{Q}$-line bundle $L$, and $X^\lin$ is precisely the set of non-pluripolar points $v \in X^\an$, i.e. such that every $\varphi \in \text{PSH}$ is finite-valued on $v$. Moreover, the invariant (1.13) satisfies

$$T(v) = \sup_{\varphi \in \text{PSH}} \{\sup \varphi - \varphi(v)\},$$

see [BoJ21] Lemma 4.23. Since every divisorial valuation has linear growth, $L$-psh functions are finite-valued on $X^\div$. The restriction map $\text{PSH} \to \mathbb{R}^{X^\div}$ is further injective [BoJ21].
Corollary 4.30, and we endow PSH with the induced topology of pointwise convergence on $X^{\text{div}}$. This is in fact equivalent to pointwise convergence on $X^{\text{lin}}$ [BoJ21 Theorem 8.3].

Note that since $H_{\mathbb{R}}(rL) = rH_{\mathbb{R}}(L)$ we have $\text{PSH}(rL) = r\text{PSH}(L)$ for any $r \in \mathbb{Q}_{>0}$. To study $\text{PSH}(L)$ we may therefore in practice assume that $L$ is an ample line bundle and that $R(X, L)$ is generated in degree 1.

We refer to [BoJ21 Example 4.7] for a concrete description of $L$-psh functions on curves. See also Appendix B below for the toric case.

4.2. Monge–Ampère operator and energy. As a special case of the theory of Chambert-Loir and Ducros [Cha06, CD12], there is a mixed Monge–Ampère operator, which associates to any tuple $(\varphi_1, \ldots, \varphi_n)$ of continuous $L$-psh functions a Radon probability measure

$$\text{MA}(\varphi_1, \ldots, \varphi_n)$$

on $X^{\text{an}}$, and depends continuously on the $\varphi_i$ with respect to uniform convergence.

By continuity, the Monge–Ampère operator is uniquely determined by its restriction to the dense subset $H_{\mathbb{Q}} \subset \text{CPSH}$ of rational Fubini–Study functions. To each such function $\varphi \in H_{\mathbb{Q}} = H_{\mathbb{Z}}$ corresponds a test configuration $(X, \mathcal{L})$ (see Appendix A), and the Monge–Ampère operator can also be defined in terms of intersection theory on the central fiber $X_0$, which is the approach taken in [BHJ17, BoJ21]. In particular, if $\varphi_i \in H_{\mathbb{Q}}$, $1 \leq i \leq n$, then $\text{MA}(\varphi_1, \ldots, \varphi_n)$ is an atomic measure supported on $X^{\text{div}}$.

In the notation of [BoJ21 §5.3], itself inspired by the complex analytic setting, we have

$$\text{MA}(\varphi_1, \ldots, \varphi_n) = (L^n)^{-1}(\theta + \text{dd}^c \varphi_1) \wedge \cdots \wedge (\theta + \text{dd}^c \varphi_n),$$

where $\theta = c_1(L)$.

We will use notation such as $\text{MA}(\varphi^{(j)}, \psi^{(n-j)})$, with $j$ copies of $\varphi$ and $n-j$ copies of $\psi$, and write $\text{MA}(\varphi) = \text{MA}(\varphi^{(n)})$. We then have $\text{MA}(0) = \delta_{\text{triv}}$.

The Monge–Ampère energy $E: \text{CPSH} \to \mathbb{R}$ is the primitive of the Monge–Ampère operator in the sense that

$$\left. \frac{d}{dt} \right|_{t=0} E((1-t)\varphi + t\psi) = \int (\psi - \varphi) \text{MA}(\varphi)$$

for $\varphi, \psi \in \text{CPSH}$, normalized by $E(0) = 0$. As such, $E$ is monotone increasing, i.e. $\varphi \geq \psi \implies E(\varphi) \geq E(\psi)$. Integration along line segments yields

$$E(\varphi) - E(\psi) = \frac{1}{n+1} \sum_{j=0}^{n} \int_{X^{\text{an}}} (\varphi - \psi) \text{MA}(\varphi^{(j)}, \psi^{(n-j)}),$$

for $\varphi, \psi \in \text{CPSH}$, and hence

$$E(\varphi) = \frac{1}{n+1} \sum_{j=0}^{n} \int_{X^{\text{an}}} \varphi \text{MA} \left( \varphi^{(j)}, 0^{(n-j)} \right).$$

If $\varphi \in H_{\mathbb{Z}} = H_{\mathbb{Q}}$ is represented by a test configuration $(X, \mathcal{L})$, then

$$E(\varphi) = \frac{(\bar{\mathcal{L}}^{n+1})}{(n+1)(L^n)},$$

where $(\bar{X}, \bar{\mathcal{L}}) \to \mathbb{P}^1$ is the canonical compactification of $(X, \mathcal{L}) \to \mathbb{A}^1$.

The functional $E$ is concave on $\text{CPSH}$, which amounts to

$$E(\varphi) - E(\psi) \leq \int (\varphi - \psi) \text{MA}(\psi).$$
for all $\varphi, \psi \in \text{CPSH}$, by (4.1). Combined with (4.2), this implies
\[ \varphi \geq \psi \implies E(\varphi) - E(\psi) \approx \int (\varphi - \psi) \text{MA}(\psi). \tag{4.5} \]
In addition to $E$, we introduce the translation invariant functional
\[ I(\varphi, \psi) := \int (\varphi - \psi) (\text{MA}(\psi) - \text{MA}(\varphi)) \geq 0, \tag{4.6} \]
which satisfies the quasi-triangle inequality
\[ I(\varphi_1, \varphi_2) \lesssim I(\varphi_1, \varphi_3) + I(\varphi_3, \varphi_2). \]
We also set
\[ I(\varphi) := I(\varphi, 0) := \sup \varphi - \int \varphi \text{MA}(\varphi). \]
The Monge–Ampère operator is homogeneous with respect to the action of $\mathbb{R}_{>0}$ on continuous $L$-psh functions $\varphi$, in the sense that $\text{MA}(t \cdot \varphi) = t \cdot \text{MA}(\varphi)$ for all $t > 0$. Similarly, we have $E(t \cdot \varphi) = t E(\varphi)$, $I(t \cdot \varphi) = t I(\varphi)$.

4.3. Functions and measures of finite energy. The Monge–Ampère energy admits a unique non-decreasing, usc extension $E$: $\text{PSH} \to \mathbb{R} \cup \{-\infty\}$, given for $\varphi \in \text{PSH}$ by
\[ E(\varphi) := \inf \big\{ E(\psi) \mid \varphi \leq \psi \in \text{CPSH} \big\}. \tag{4.7} \]
We denote by
\[ \mathcal{E}^1 := \{ \varphi \in \text{PSH} \mid E(\varphi) > -\infty \} \]
the set of $L$-psh functions of finite energy. In other words, functions in $\mathcal{E}^1$ are decreasing limits of nets $\varphi_i \in \mathcal{H}_Q$ with energy $E(\varphi_i)$ uniformly bounded below.

The weak topology of $\mathcal{E}^1$ is its subspace topology from $\text{PSH}$, and the strong topology on $\mathcal{E}^1$ is the coarsest refinement of the weak topology for which $E$ becomes continuous.

For a decreasing or increasing net $(\varphi_j)$ in $\mathcal{E}^1$, strong and weak convergence coincide, i.e. $\varphi_j \to \varphi$ strongly in $\mathcal{E}^1$ iff $\varphi_j \to \varphi$ pointwise on $X^{\text{div}}$, see Example 9.2 and Theorem 9.5 in [BoJ21], respectively.

Denote by $\mathcal{M}$ the space of Radon probability measures on $X^{\text{an}}$, endowed with the weak topology. The main point in introducing the strong topology is that the mixed Monge–Ampère operator $\text{MA}$, a priori only defined as a map $(C^0 \cap \text{PSH})^n \to \mathcal{M}$, admits a (unique) extension $((\mathcal{E}^1)^n) \to \mathcal{M}$ that is continuous in the strong topology. Further,
\[ (\varphi_0, \varphi_1, \ldots, \varphi_n) \mapsto \int \varphi_0 \text{MA}(\varphi_1, \ldots, \varphi_n) \]
is finite-valued and (strongly) continuous on tuples in $\mathcal{E}^1$. In particular, the functional $I$ from §4.2 extend continuously to $\mathcal{E}^1$, and it induces a quasi-metric on $\mathcal{E}^1 / \mathbb{R}$ that defines the strong topology.

The energy of a probability measure $\mu \in \mathcal{M}$ on $X^{\text{an}}$ is defined by
\[ E^\vee(\mu) := \sup \big\{ E(\varphi) - \int \varphi \, d\mu \big\}, \tag{4.8} \]
where the supremum can be restricted to functions in $\mathcal{H}_Q$, by approximation. This defines a convex, lsc function $E^\vee : \mathcal{M} \to [0, +\infty]$. We denote by
\[ \mathcal{M}^1 := \{ \mu \in \mathcal{M} \mid E^\vee(\mu) < +\infty \} \]
the set of measures of finite energy. It comes with a strong topology, defined as the coarsest
refinement of the weak topology of measures in which $E^\vee$ is continuous. The topological space $\mathcal{M}^1$ does not depend on $L$.

By (4.4), for any $\varphi \in \mathcal{E}^1$, the measure $\mu = MA(\varphi)$ has finite energy, and $\varphi$ achieves the supremum in (4.8), i.e. $E^\vee(\mu) = E(\varphi) - \int \varphi MA(\varphi)$. Conversely, a measure $\mu \in \mathcal{M}^1$ satisfies $\mu = MA(\varphi)$ with $\varphi \in \mathcal{E}^1$ iff $\varphi$ achieves the supremum in (4.8). By a main result of [BoJ21], the Monge–Ampère operator induces a topological embedding with dense image

$$MA : \mathcal{E}^1/\mathbb{R} \hookrightarrow \mathcal{M}^1,$$

with respect to the strong topology on both sides.

4.4. Envelopes. Given a function $\varphi : X^{\text{an}} \to \mathbb{R} \cup \{\pm \infty\}$ we define the \textit{psh envelope} pointwise as

$$P(\varphi) := \sup \{ \psi \in \text{PSH} \mid \psi \leq \varphi \}.$$

Note that the \textit{Fubini–Study envelope} in (2.15) can be written

$$Q(\varphi) = \sup \{ \psi \in H_Q \mid \psi \leq \varphi \} = \sup \{ \psi \in \text{CPSH} \mid \psi \leq \varphi \}.$$

in both cases the convention $\sup \emptyset = -\infty$ applies. Clearly $Q(\varphi) \leq P(\varphi) \leq \varphi$, and either $\inf \varphi = -\infty \equiv Q(\varphi)$, or $Q(\varphi)$ is (finite-valued and) lsc. In the latter case,

$$Q(\varphi) = P(\varphi^*) \quad (4.9)$$

where $\varphi^*$ is the lsc regularization of $\varphi$. In particular, $Q(\varphi) = P(\varphi)$ when $\varphi$ is continuous.

The functions $P(\varphi)$ and $Q(\varphi)$ are not psh in general. For any $\varphi \in C^0$, we have

$$P(\varphi) = Q(\varphi) \iff P(\varphi) \in C^0 \iff P(\varphi) \in \text{PSH}.$$

We say that \textit{continuity of envelopes} holds (for $L$) if these properties are satisfied for all $\varphi \in C^0$. It is proved in [BoJ21, Theorem 4.53] that continuity of envelopes holds for any ample $L$ if $X$ is smooth and $k$ has characteristic zero, or in any characteristic if $\dim X \leq 2$ [GJKM17].

One checks that continuity of envelopes holds iff, for any family $(\varphi_\alpha)$ of $L$-psh functions that is uniformly bounded above, if we set $\varphi = \sup \alpha \varphi_\alpha$, then the usc regularization $\varphi^*$ is psh. A deeper result states that the set $\{ \varphi < \varphi^* \}$ is then further pluripolar. In particular, $\varphi = \varphi^*$ on $X^{\text{div}}$.

For the next result, see [BoJ21, Corollary 4.58].

\textbf{Lemma 4.1.} Assume that continuity of envelopes holds for $(X, L)$, and consider a usc function $\varphi : X^{\text{an}} \to \mathbb{R} \cup \{-\infty\}$. Then:

(i) either $P(\varphi) \in \text{PSH}$ or $P(\varphi) \equiv -\infty$;

(ii) if $\varphi$ is the pointwise limit of a decreasing net of usc functions $\varphi_i : X^{\text{an}} \to \mathbb{R} \cup \{-\infty\}$, then $P(\varphi_i) \searrow P(\varphi)$ pointwise on $X^{\text{an}}$.

Denote by $\mathcal{E}^\infty \subset \mathcal{E}^1$ the space of bounded $L$-psh functions. A function $\varphi \in \mathcal{E}^\infty$ is \textit{regularizable from below} if there exists an increasing net $(\varphi_j)_j$ in CPSH that converges to $\varphi$ in PSH (i.e. pointwise on $X^{\text{div}}$). Such a net can then be chosen in $H_Q$, and converges strongly to $\varphi$. We write

$$\mathcal{E}_1^\infty \subset \mathcal{E}^\infty$$

for the space of $L$-psh functions regularizable from below. If continuity of envelopes holds, then a bounded function $\varphi \in \mathcal{E}^\infty$ lies in $\mathcal{E}_1^\infty$ iff its discontinuity locus $\{ \varphi^* < \varphi \}$ is pluripolar [BoJ21, Theorem 11.23].
Theorem 11.19, it follows that $P(\phi)$.

For any bounded function $\phi \in L^\infty$, denote by $Q^*(\phi) := Q(\phi)^*$ the usc regularization of $Q(\phi)$.

Lemma 4.3. Assume continuity of envelopes. Then:

(i) $Q^*: L^\infty \to L^\infty$ is a projection operator onto $E^\infty$

(ii) for all $\phi, \psi \in E^\infty$ we have $Q^*(\phi \wedge \psi) = P(\phi \wedge \psi)$.

Proof. (i) follows from [BoJ21, Theorem 11.23]. Pick $\phi, \psi \in E^\infty$. By (4.9) we have $Q(\phi \wedge \psi) = P((\phi \wedge \psi)_+ \wedge \psi)$. Since $\phi, \psi$ are regularizable from below, their discontinuity locus is pluripolar, i.e. $\phi_\ast = \phi$, $\psi_\ast = \psi$, and hence $\phi \wedge \psi_\ast = \phi \wedge \psi$, outside a pluripolar set. By [BoJ21, Theorem 11.19], it follows that $P(\phi \wedge \psi_\ast)^* = P(\phi \wedge \psi)^*$, which coincides with $P(\phi \wedge \psi)$ since $\phi \wedge \psi$ is usc (see Lemma 4.1). This proves (ii).

5. The Darvas metric

In this section, we study the metrics on the spaces $H_\mathbb{R}$, $E^1$ and $M^1$ induced by the $d_1$-pseudometric of $N_\mathbb{R}$, and prove the main part of Theorem B.

5.1. The Darvas metric on $H_\mathbb{R}$. By Theorem 2.18 the operators

$$FS: (T_\mathbb{R}, d_\infty) \to (H_\mathbb{R}, d_\infty), \quad IN: (H_\mathbb{R}, d_\infty) \to (T_\mathbb{R}, d_\infty)$$

are isometries such that $FS \circ IN = id$.

For any $p \in [1, \infty]$, the pseudo-metric $d_p$ on $T_\mathbb{R} \subset N_\mathbb{R}$ satisfies $d_p \leq d_\infty$; it is thus constant along the fibers of $FS$, and hence descends to a pseudo-metric $d_p$ on $H_\mathbb{R}$, such that

$$FS: (T_\mathbb{R}, d_p) \to (H_\mathbb{R}, d_p), \quad IN: (H_\mathbb{R}, d_p) \to (T_\mathbb{R}, d_p)$$

are isometries. Theorem A asserts that $d_p$ is a metric on $H_\mathbb{R}$. Since $d_p \geq d_1$, this follows from the following more precise result.

Theorem 5.1. The pseudo-metric $d_1$ on $H_\mathbb{R}$ is a metric, uniquely characterized by

$$\varphi \geq \psi \implies d_1(\varphi, \psi) = E(\varphi) - E(\psi);$$

$$d_1(\varphi, \psi) = \inf\{d_1(\varphi, \tau) + d_1(\tau, \psi) \mid \tau \in H_\mathbb{R}, \tau \leq \varphi \wedge \psi\},$$

for all $\varphi, \psi \in H_\mathbb{R}$.

Our key tool is the following general result, that will also later be used to analyze the regularized Fubini–Study operator.

Lemma 5.2. For any $\chi \in N_\mathbb{R}$ we have $vol(\chi) = E(FS(\chi))$.

Here the function $FS(\chi)$ is bounded and lsc, but not $L$-psh in general. As in [BoJ21, §6.2], we use the extended Monge–Ampère energy, defined for arbitrary functions $\varphi: X^{an} \to \mathbb{R} \cup \{\pm \infty\}$ by setting

$$E(\varphi) := \sup\{E(\psi) \mid \psi \in PSH, \psi \leq \varphi\} \in \mathbb{R} \cup \{\pm \infty\}. \quad (5.3)$$

Trivially, $E(\varphi) = E(P(\varphi))$. If $\varphi$ is lsc and bounded below, then $P(\varphi) = Q(\varphi)$, see (4.9), and hence

$$E(\varphi) = E(P(\varphi)) = E(Q(\varphi)) = \sup\{E(\psi) \mid \psi \in H_\mathbb{R}, \psi \leq \varphi\}. \quad (5.4)$$

A Dini-type argument (see [BoJ21, Proposition 6.9]) further yields:
Lemma 5.3. The functional \( \varphi \mapsto E(\varphi) \) is continuous along increasing nets of bounded-below lsc functions.

Proof of Lemma 5.2. Replacing \( \chi \) by its round-down \( \lfloor \chi \rfloor \) does not change \( \text{FS}(\chi) \) or \( \text{vol}(\chi) \), so we may assume \( \chi \in \mathcal{N} \). By Theorem 3.17, the canonical approximants \( \chi_r \in \mathcal{T} \) satisfy \( \text{vol}(\chi_r) \to \text{vol}(\chi) \). On the other hand, \( \text{FS}(\chi_r) = \text{FS}(\chi) \) increases pointwise to \( \text{FS}(\chi) \) (see (2.10)), and hence \( E(\text{FS}(\chi_r)) \to E(\text{FS}(\chi)) \), by Lemma 5.3.

We are thus reduced to the case \( \chi \in \mathcal{T} \), which is a consequence of [BHJ17]. Indeed, \( \chi \) corresponds to an ample test configuration \((\mathcal{X},\mathcal{L})\) for \((X,L)\) under the Rees correspondence (see Appendix A). By [BHJ17, Proposition 3.12], the spectral measure \( \sigma(\chi) \) coincides with the Duistermaat–Heckman measure \( \text{DH}(\mathcal{X},\mathcal{L}) \), and passing to the barycenters yields

\[
\text{vol}(\chi) = \frac{(\mathcal{L}^{n+1})}{(n+1)(\mathcal{L}^n)}.
\]

by [BHJ17] Lemma 7.3. By (4.3), the right-hand side is also equal to \( E(\text{FS}(\chi)) \), and the result follows.

Proof of Theorem 5.1. We first prove that \( d_1 \) satisfies (5.1) and (5.2), which will take care of uniqueness. Pick \( \varphi,\psi \in \mathcal{H}_R \), and set \( \chi := \text{IN}(\varphi), \chi' := \text{IN}(\psi) \). Then \( \text{FS}(\chi) = \varphi, \text{FS}(\chi') = \psi \), and Lemma 5.2 implies \( \text{vol}(\chi) = E(\varphi), \text{vol}(\chi') = E(\psi) \). If \( \varphi \geq \psi \), then \( \chi \geq \chi' \), and (3.12) yields

\[
d_1(\varphi,\psi) = d_1(\chi,\chi') = \text{vol}(\chi) - \text{vol}(\chi') = E(\varphi) - E(\psi),
\]

which proves (5.1). Next, pick \( \varepsilon > 0 \). Applying Theorem 3.17 to \( \chi \wedge \chi' \) yields \( \chi'' \in \mathcal{T} _R \) such that \( \chi'' \leq \chi \wedge \chi' \) and \( d_1(\chi'',\chi \wedge \chi') \leq \varepsilon \). If we set \( \tau := \text{FS}(\chi'') \in \mathcal{H}_R \), then \( \tau \leq \varphi,\psi \), and

\[
d_1(\varphi,\tau) + d_1(\tau,\psi) = d_1(\chi,\chi'') + d_1(\chi'',\chi') \\
\leq d_1(\chi,\chi \wedge \chi') + d_1(\chi \wedge \chi',\chi') + 2\varepsilon \\
= d_1(\chi,\chi') + 2\varepsilon = d_1(\varphi,\psi) + 2\varepsilon,
\]

where we have used (3.12). This proves (5.2). Assume now \( d_1(\varphi,\psi) = 0 \). By (5.1) and (5.2), there exists a sequence \( (\tau_k) \) in \( \mathcal{H}_R \) such that \( \tau_k \leq \varphi,\psi \) and \( E(\tau_k) \to E(\varphi) \) and \( E(\tau_k) \to E(\psi) \). By [BoJ21, Proposition 9.7], it follows that \( (\tau_k) \) converges to both \( \varphi \) and \( \psi \) in \( \mathcal{E}^1 \), and hence \( \varphi = \psi \), since the topology is separated. This proves, as desired, that \( d_1 \) is a metric on \( \mathcal{H}_R \) (this conclusion alternatively follows from (5.9) below).

5.2. The Darvas metric on \( \mathcal{E}^1 \). We next prove that the metric \( d_1 \) on \( \mathcal{H}_R \) canonically extends to \( \mathcal{E}^1 \), yielding an analogue in our context of the metric introduced by Darvas [Dar15b] in the complex analytic setting.

Theorem 5.4. There exists a unique metric \( d_1 \) on \( \mathcal{E}^1 \) that defines the strong topology and restricts to the previous metric \( d_1 \) on \( \mathcal{H}_R \). Further:

(i) for all \( \varphi,\psi \in \text{CPSH} \), we have

\[
d_1(\varphi,\psi) = E(\varphi) + E(\psi) - 2E(P(\varphi \wedge \psi)); \tag{5.5}
\]

(ii) the metric space \( (\mathcal{E}^1, d_1) \) is complete iff continuity of envelopes holds for \((X,L)\);

(iii) if continuity of envelopes holds, then (5.5) remains valid for all \( \varphi,\psi \in \mathcal{E}^1 \), and \( P(\varphi \wedge \psi) \in \mathcal{E}^1 \).
Recall that continuity of envelopes holds whenever \( X \) is smooth and \( \text{char}(k) = 0 \), and fails if \( X \) is not unibranch. We refer to the metric \( d_1 \) on \( E^1 \) as the Darvas metric. By [Reb20], \((E^1, d_1)\) is also a geodesic metric space.

Our strategy to extend \( d_1 \) to \( E^1 \) is to compare it to the functional

\[
I(\varphi, \psi) := I(\varphi, \varphi \vee \psi) + I(\varphi \vee \psi, \psi)
\]

It was indeed proven in [BoJ21, §9.1] that \( I \) is a quasi-metric on \( E^1 \) that defines the strong topology, and further satisfies

\[
I(\varphi, \psi) = I(\varphi, \varphi \vee \psi) + I(\varphi \vee \psi, \psi) \tag{5.6}
\]

Set

\[
I(\varphi) := I(\varphi, 0), \quad d_1(\varphi) := d_1(\varphi, 0).
\]

By the quasi-triangle inequality, (5.6) implies

\[
I(\varphi \vee \psi) \lesssim \max\{I(\varphi), I(\psi)\}. \tag{5.7}
\]

**Lemma 5.5.** The quasi-metrics \( d_1 \) and \( I \) on \( H_{\mathbb{R}} \) are Hölder comparable, in the sense that

\[
d_1(\varphi, \psi) \lesssim I(\varphi, \psi)^\alpha \max\{I(\varphi), I(\psi)\}^{1-\alpha} \tag{5.8}
\]

\[
I(\varphi, \psi) \lesssim d_1(\varphi, \psi)^\alpha \max\{d_1(\varphi), d_1(\psi)\}^{1-\alpha}, \tag{5.9}
\]

with \( \alpha := 1/2^n \). In particular, \( d_1(\varphi) \approx I(\varphi) \).

**Proof.** Assume \( \varphi \geq \psi \). Then (5.1) and (4.5) show that

\[
d_1(\varphi, \psi) = E(\varphi) - E(\psi) \approx \int (\varphi - \psi) MA(\psi),
\]

and hence

\[
I(\varphi, \psi) = \int (\varphi - \psi)(MA(\psi) - MA(\varphi)) \lesssim d_1(\varphi, \psi).
\]

Now we can write

\[
I(\varphi, \psi) = \int (\varphi - \psi) MA(\psi) + \int (\varphi - \psi)(MA(0) - MA(\varphi)). \tag{5.10}
\]

As a special case of [BoJ21, Lemma 5.28] we also have the estimate

\[
\left| \int (\varphi - \psi)(MA(0) - MA(\varphi)) \right| \lesssim I(\varphi, \psi)^\alpha \max\{I(\varphi), I(\psi)\}^{1-\alpha}.
\]

Since \( I \leq I \), this yields on the one hand

\[
d_1(\varphi, \psi) \lesssim I(\varphi, \psi) + I(\varphi, \psi)^\alpha \max\{I(\varphi), I(\psi)\}^{1-\alpha}
\]

\[
\lesssim I(\varphi, \psi)^\alpha \max\{I(\varphi), I(\psi)\}^{1-\alpha}
\]

\[
\lesssim I(\varphi, \psi)^\alpha \max\{I(\varphi), I(\psi)\}^{1-\alpha}.
\]

Since \( I \leq d_1 \), we get on the other hand

\[
I(\varphi, \psi) \lesssim d_1(\varphi, \psi) + d_1(\varphi, \psi)^\alpha \max\{d_1(\varphi), d_1(\psi)\}^{1-\alpha}
\]

\[
\lesssim d_1(\varphi, \psi)^\alpha \max\{d_1(\varphi), d_1(\psi)\}^{1-\alpha}.
\]

This proves (5.8), and (5.9) when \( \varphi \geq \psi \).
Now consider arbitrary \( \varphi, \psi \in \mathcal{H}_R \). To prove (5.8), set \( \sigma := \varphi \vee \psi \in \mathcal{H}_R \). From what precedes and (5.7), we have

\[
d_1(\varphi, \sigma) \leq \bar{I}(\varphi, \sigma)^\alpha \max\{\bar{I}(\varphi), \bar{I}(\psi)^{1-\alpha}\}
\]

and

\[
d_1(\psi, \sigma) \leq \bar{I}(\psi, \sigma)^\alpha \max\{\bar{I}(\varphi), \bar{I}(\psi)^{1-\alpha}\},
\]

which together with the triangle inequality for \( d_1 \) yields (5.8).

The proof of (5.9) is similar. By (5.2) we can pick \( \tau \in \mathcal{H}_R \) with \( \tau \leq \varphi, \psi \) such that

\[
\max\{d_1(\tau, \varphi), d_1(\tau, \psi)\} \leq d_1(\varphi, \psi),
\]

and hence

\[
d_1(\tau) \leq \max\{d_1(\varphi), d_1(\psi)\}.
\]

Since \( \tau \leq \varphi, \psi \), the first step yields

\[
\bar{I}(\varphi, \tau) \leq d_1(\varphi, \tau)^\alpha \max\{d_1(\varphi), d_1(\tau)\}^{1-\alpha} \leq d_1(\varphi, \psi)^\alpha \max\{d_1(\varphi), d_1(\psi)^{1-\alpha}\},
\]

\[
\bar{I}(\psi, \tau) \leq d_1(\psi, \tau)^\alpha \max\{d_1(\psi), d_1(\tau)\}^{1-\alpha} \leq d_1(\varphi, \psi)^\alpha \max\{d_1(\varphi), d_1(\psi)^{1-\alpha}\},
\]

and the quasi-triangle inequality for \( \bar{I} \) yields (5.9).

**Proof of Theorem 5.4** Since \( \mathcal{H}_R \) is dense in \( \mathcal{E}_1 \), uniqueness is clear. Given \( \varphi, \psi \in \mathcal{E}_1 \), pick sequences \( (\varphi_i), (\psi_i) \) in \( \mathcal{H}_R \) converging strongly to \( \varphi \) and \( \psi \), respectively (for example, we can use decreasing sequences). Thus

\[
\lim_i \bar{I}(\varphi_i, \varphi) = \lim_i \bar{I}(\psi_i, \psi) = 0.
\]

Using (5.8) this implies that \( (d_1(\varphi_i, \psi_i))_i \) is a Cauchy sequence, so that \( d_1(\varphi, \psi) := \lim_j d_1(\varphi_j, \psi_j) \) exists. It is easy to see that it does not depend on the choice of sequence \( (\varphi_i) \) and \( (\psi_i) \), and that the extension is a pseudo-metric on \( \mathcal{E}_1 \). Further, the estimates of Lemma 5.5 still hold for \( \varphi, \psi \in \mathcal{E}_1 \). In particular, \( d_1(\varphi, \psi) = 0 \iff \bar{I}(\varphi, \psi) = 0 \iff \varphi = \psi \), so \( d_1 \) is a metric on \( \mathcal{E}_1 \). These estimates also show that \( d_1 \) and \( \bar{I} \) share the same Cauchy sequences in \( \mathcal{E}_1 \), so that \( (\mathcal{E}_1, d_1) \) is complete iff \( (\mathcal{E}_1, \bar{I}) \) is complete. By [BoJ21, Theorem 9.9], this is also equivalent to continuity of envelopes for \( (X, L) \), which proves (ii).

Next, pick \( \varphi, \psi \in \mathcal{H}_R \). By (5.1) and (5.2), we have

\[
d_1(\varphi, \psi) = \inf \left\{ d_1(\varphi, \tau) + d_1(\tau, \psi) \mid \tau \in \mathcal{H}_R, \tau \leq \varphi \wedge \psi \right\}
\]

\[
= E(\varphi) + E(\psi) - 2 \sup \{ E(\psi) \mid \psi \in \mathcal{H}_R, \psi \leq \varphi \wedge \psi \}
\]

\[
= E(\varphi) + E(\psi) - 2 E(\rho(\varphi \wedge \psi)),
\]

see (5.4). Since \( d_1 \leq d_\infty \), all terms are continuous with respect to uniform convergence, and the identity therefore remains valid on CPSH, which yields (i).

Finally, assume that continuity of envelopes holds. Pick \( \varphi, \psi \in \mathcal{E}_1 \), set \( \rho := \rho(\varphi \wedge \psi) \), and choose decreasing nets \( (\varphi_i), (\psi_i) \) in \( \mathcal{H}_R \) converging to \( \varphi, \psi \). By Lemma 4.1 we either have \( \rho \in PSH \) or \( \rho \equiv -\infty \), and \( \rho(\varphi_i \wedge \psi_i) \) decreases pointwise to \( \rho \). Since (5.5) holds for \( \varphi_i, \psi_i \in \mathcal{H}_R \), it also holds for \( \varphi, \psi \), by continuity of \( E \) along decreasing nets. In particular, \( E(\rho) \) is finite, and hence \( \rho \in \mathcal{E}_1 \). This proves (iii).

As in [DDL18, Theorem 3.7], we next provide a comparison of the Darvas metric \( d_1 \) on \( \mathcal{E}_1 \) with the functional \( I_1: \mathcal{E}_1 \times \mathcal{E}_1 \to \mathbb{R}_{\geq 0} \) defined by

\[
I_1(\varphi, \psi) := \int |\varphi - \psi| (\text{MA}(\varphi) + \text{MA}(\psi)).
\]

This functional is obviously symmetric, and it separates points, as a consequence of the Domination Principle (see [BoJ21, Corollary 7.6]). For all \( \varphi, \psi \in \mathcal{E}_1 \), we further have

\[
I_1(\varphi, \psi) = I_1(\varphi, \varphi \vee \psi) + I_1(\varphi \vee \psi, \psi).
\]

As with \( I \) and \( \bar{I} \), this follows from the Locality Principle (see [BoJ21, Theorem 5.35, Proposition 5.40]).

**Theorem 5.6.** For all \( \varphi, \psi \in \mathcal{E}_1 \) we have \( d_1(\varphi, \psi) \approx I_1(\varphi, \psi) \).
The proof relies on the following analogue of \cite{DDL18} Lemma 3.8.

**Lemma 5.7.** If $\varphi, \psi \in \mathcal{E}^1$ and $\rho := \frac{1}{2}(\varphi + \psi)$, then $d_1(\varphi, \psi) \approx d_1(\varphi, \rho) + d_1(\rho, \psi)$.

**Proof.** By approximation, we may assume $\varphi, \psi \in \mathcal{H}_R$. Pick any $\tau \in \mathcal{H}_R$ with $\tau \leq \varphi \wedge \psi$. Then $\tau \leq \varphi, \psi, \rho$, and (5.1) yields

$$d_1(\varphi, \rho) + d_1(\rho, \psi) \leq d_1(\varphi, \tau) + d_1(\psi, \tau) + 2d_1(\rho, \tau)$$

$$= (E(\varphi) - E(\tau)) + (E(\psi) - E(\tau)) + 2(E(\rho) - E(\tau))$$

$$\approx \int (\varphi - \tau) MA(\tau) + \int (\psi - \tau) MA(\tau) + 2 \int (\rho - \tau) MA(\tau)$$

$$= 2 \int (\varphi - \tau) MA(\tau) + 2 \int (\psi - \tau) MA(\tau)$$

$$\approx (E(\varphi) - E(\tau) + E(\psi) - E(\tau)) = d_1(\varphi, \tau) + d_1(\psi, \tau).$$

Here the first inequality is simply the triangle inequality for $d_1$, whereas the third and fifth lines follow from (4.5). By (5.2), the infimum over $\tau$ of the right-hand side equals $d_1(\varphi, \psi)$ and we are done. □

**Proof of Theorem 5.6.** Since $d_1(\varphi, \psi)$ and $I_1(\varphi, \psi)$ are both continuous along decreasing nets, we may assume wlog $\varphi, \psi \in \mathcal{H}_R$. Let us start by proving $d_1(\varphi, \psi) \lesssim I_1(\varphi, \psi)$. By (5.11) and the triangle inequality for $d_1$, it suffices to consider the case $\varphi \geq \psi$. But in this case (4.5) yields

$$d_1(\varphi, \psi) = E(\varphi) - E(\psi) \approx \int (\varphi - \psi) MA(\psi) \leq I_1(\varphi, \psi)$$

It remains to prove $d_1(\varphi, \psi) \gtrsim I_1(\varphi, \psi)$. Set $\rho := \frac{1}{2}(\varphi + \psi) \in \mathcal{H}_R$, so that Lemma 5.7 gives $d_1(\varphi, \psi) \approx d_1(\varphi, \rho) + d_1(\rho, \psi)$. Pick $\varepsilon > 0$. By (5.2), we can find $\sigma, \tau \in \mathcal{H}_R$ such that $\sigma \leq \varphi \wedge \rho$, $\tau \leq \rho \wedge \psi$, and

$$d_1(\varphi, \rho) \geq d_1(\varphi, \sigma) + d_1(\sigma, \rho) - \varepsilon \quad \text{and} \quad d_1(\rho, \psi) \geq d_1(\rho, \tau) + d_1(\tau, \psi) - \varepsilon,$$

and hence

$$d_1(\varphi, \psi) \gtrsim d_1(\sigma, \rho) + d_1(\rho, \tau) - 2\varepsilon.$$

As $\sigma \leq \rho$, we have

$$d_1(\sigma, \rho) = E(\rho) - E(\sigma) \geq \int (\rho - \sigma) MA(\rho) \geq 2^{-n} \int (\rho - \sigma)(MA(\varphi) + MA(\psi)),$$

where the last inequality follows by expanding $MA(\rho) = MA(\frac{1}{2}(\varphi + \psi))$. Combining this with the analogous lower bound on $d_1(\rho, \tau)$ yields

$$d_1(\varphi, \psi) \gtrsim \int (2\rho - \sigma - \tau)(MA(\varphi) + MA(\psi)) - 2\varepsilon.$$

We conclude by noting that $2\rho - \sigma - \tau \geq \frac{1}{2} |\varphi - \psi|$ and letting $\varepsilon \to 0$. □

**5.3. The Darvas metric on $\mathcal{M}^1$.** A main result of \cite{BoJ21} guarantees that the Monge–Ampère operator induces a topological embedding with dense image

$$MA : \mathcal{E}^1/\mathbb{R} \hookrightarrow \mathcal{M}^1,$$
where $\mathcal{E}^1$ and $\mathcal{M}^1$ are both equipped with the strong topology. In particular, the quotient topology of $\mathcal{E}^1/\mathbb{R}$ is Hausdorff. Since the action of $\mathbb{R}$ on $\mathcal{E}^1$ by translation preserves $d_1$, the topology of $\mathcal{E}^1/\mathbb{R}$ is defined by the quotient metric

$$d_1(\varphi, \psi) = \inf_{c \in \mathbb{R}} d_1(\varphi + c, \psi). \quad (5.12)$$

Note also that the isometric surjection $\mathbb{S} \ni (\mathcal{E}^1/\mathbb{R}, d_1) \to (\mathcal{N}, d_1)$, being equivariant with respect to the action of $\mathbb{R}$, induces an isometric surjection

$$\mathbb{S} \ni (\mathcal{E}^1/\mathbb{R}, d_1) \to (\mathcal{H}/\mathbb{R}, d_1), \quad (5.13)$$

where $d_1$ respectively denotes the restriction of the quotient metric on $\mathcal{N}/\mathbb{R}$ and $\mathcal{E}^1/\mathbb{R}$.

**Theorem 5.8.** There exists a unique metric $d_1$ on $\mathcal{M}^1$ that defines the strong topology and restricts to the quotient metric $[5.12]$ on $\mathcal{E}^1/\mathbb{R} \to \mathcal{M}^1$. Furthermore, the metric space $(\mathcal{M}^1, d_1)$ is complete.

Note that this result holds with or without continuity of envelopes, in contrast with Theorem 5.4. As with the latter, the proof is based on a comparison of $d_1$ with the translation invariant functional $I: \mathcal{E}^1 \times \mathcal{E}^1 \to \mathbb{R}_{\geq 0}$.

**Lemma 5.9.** For all $\varphi, \psi \in \mathcal{E}^1$, there exists $c \in \mathbb{R}$ such that $d_1(\varphi, \psi) = d_1(\varphi + c, \psi)$ and $|c| \leq \max\{I(\varphi), I(\psi)\}$.

Note that this provides another reason why $[5.12]$ is a metric on $\mathcal{E}^1/\mathbb{R}$.

**Proof.** Set $M := \max\{I(\varphi), I(\psi)\}$ and $d(c) := d_1(\varphi + c, \psi)$, a continuous function of $c \in \mathbb{R}$, so that

$$d_1(\varphi, \psi) = \inf\{d_1(\varphi + c, \psi) \mid d(c) \leq 2d(0)\}.$$ 

By Lemma 5.5 we have $d(0) \leq M$, and $|c| \leq d(c)^{1-\alpha} \max\{|c|, M\}$ for all $c \in \mathbb{R}$. Thus $d(c) \leq 2d(0) \implies |c| \leq M$, and the result follows since $c \mapsto d(c)$ achieves its minimum on any compact set. \qed

**Lemma 5.10.** The quasi-metrics $d_1$ and $I$ on $\mathcal{E}^1/\mathbb{R}$ are Hölder comparable, i.e.

$$d_1(\varphi, \psi) \lesssim I(\varphi, \psi) \max\{I(\varphi), I(\psi)\}^{1-\alpha}, \quad (5.14)$$

and

$$I(\varphi, \psi) \lesssim d_1(\varphi, \psi) \max\{d_1(\varphi), d_1(\psi)\}^{1-\alpha}. \quad (5.15)$$

for all $\varphi, \psi \in \mathcal{E}^1$, with $\alpha := 1/2^n$. In particular, $d_1(\varphi) \approx I(\varphi)$.

**Proof.** By translation invariance of $I$ and $d_1$, we may assume $\sup \varphi = \sup \psi = 0$. Then $[5.14]$ follows directly from $[5.8]$, since $d_1(\varphi, \psi) \leq d_1(\varphi, \psi)$. By Lemma 5.9 we can find $c \in \mathbb{R}$ such that $d_1(\varphi, \psi) = d_1(\varphi + c, \psi)$ and $|c| \leq \max\{I(\varphi), I(\psi)\}$. By $[5.9]$ we infer

$$I(\varphi, \psi) \leq I(\varphi + c, \psi) \lesssim d_1(\varphi, \psi) \max\{I(\varphi), I(\psi)\}^{1-\alpha}. \quad (5.16)$$

In particular, $I(\varphi) \lesssim d_1(\varphi) I(\varphi)^{1-\alpha}$, hence $I(\varphi) \lesssim d_1(\varphi)$, and $[5.15]$ follows. \qed

**Remark 5.11.** By [BILJ17, Theorem 7.9], Lemma 5.10 implies that $d_1$ is equivalent to the $L^1$-norm of test configurations, i.e. $|||I|||_1 \approx d_1(\varphi)$ if $\chi \in \mathcal{T}$ and $\varphi = \mathbb{S}(\chi)$.  

**Proof of Theorem 5.8.** Since $\mathcal{E}^1/\mathbb{R} \to \mathcal{M}^1$ has dense image, uniqueness is clear. By [BoJ21, Theorem 7.11], the strong topology of $\mathcal{M}^1$ is defined by a certain quasi-metric $I'$, that further satisfies $I'(\mathcal{MA}(\varphi), \mathcal{MA}(\psi)) \approx I(\varphi, \psi)$. Using the estimates of Lemma 5.10 and arguing just as
in the proof of Theorem 5.4 we infer the existence of an extension of \( d_1 \) to a pseudo-metric on \( \mathcal{M}^1 \) such that

\[
d_1(\mu, \delta_{\text{triv}}) \approx I^\vee(\mu, \delta_{\text{triv}}) \approx E^\vee(\mu)
\]

for all \( \mu, \nu \in \mathcal{M}^1 \) (recalling that \( \text{MA}(0) = \delta_{\text{triv}} \)). This shows that \( d_1 \) separates points, and hence is a metric on \( \mathcal{M}^1 \), which further shares the same convergent and Cauchy sequences as \( I^\vee \). It thus defines the strong topology of \( \mathcal{M}^1 \), and \( (\mathcal{M}^1, d_1) \) is complete, because \( (\mathcal{M}^1, I^\vee) \) is complete by [BoJ21, Theorem 7.13]. □

Combining the above estimates with a key estimate for Monge–Ampère integrals from [BoJ21], we get the following Hölder continuity property:

**Theorem 5.12.** There exist \( \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}_{>0} \), only depending on \( n \), such that \( \sum_i \alpha_i = 1 \) and

\[
\left| \int |\varphi - \varphi'| (\mu - \mu') \right| \lesssim d_1(\varphi, \varphi')^{\alpha_1} d_1(\mu, \mu')^{\alpha_2} M^{\alpha_3}
\]

for all \( \varphi, \varphi' \in E^1 \) and \( \mu, \mu' \in \mathcal{M}^1 \), where \( M := \max\{I(\varphi), I(\varphi'), E^\vee(\mu), E^\vee(\mu')\} \).

**Proof.** By [BoJ21 Theorem 7.3], we have

\[
\left| \int (\varphi - \varphi')(\mu - \mu') \right| \lesssim I(\varphi, \varphi')^\alpha I^\vee(\mu, \mu')^\beta M^{\gamma - \alpha}.
\]

Injecting (5.15) and (5.19) yields

\[
\left| \int (\varphi - \varphi')(\mu - \mu') \right| \lesssim d_1(\varphi, \varphi')^{\alpha_1} d_1(\mu, \mu')^{\alpha_2} M^{\alpha_3}
\]

with \( \alpha_1, \alpha_2, \alpha_3 \) as above. Next, write

\[|\varphi - \varphi'| = 2(\tau - \varphi') + (\varphi' - \varphi)
\]

with \( \tau := \varphi \vee \varphi' \in E^1 \). On the one hand, we have \( I(\tau) \lesssim M \). On the other hand, Theorem 5.6 and (5.11) yield

\[d_1(\tau, \varphi') \approx I_1(\tau, \varphi') \leq I_1(\varphi, \varphi') \approx d_1(\varphi, \varphi').
\]

Applying (5.21) to \( \tau, \varphi' \) and \( \varphi, \varphi' \) now yields (5.20). □

### 5.4. The space \( \mathcal{M}^1 \) as a Hausdorff completion.

Putting together (5.13) and Theorems 5.4 and 5.8 shows that the composition \( \text{MA} \circ \text{FS}: \mathcal{T}_R \to \mathcal{M}^1 \) descends to an isometry with dense image

\[
\text{MA} \circ \text{FS}: (\mathcal{T}_R/\mathbb{R}, d_1) \to (\mathcal{M}^1, d_1).
\]

Since \( (\mathcal{M}^1, d_1) \) is complete, this map realizes it as the Hausdorff completion of the pseudo-metric space \( (\mathcal{T}_R/\mathbb{R}, d_1) \).

**Theorem 5.13.** The map (5.22) admits a unique extension to an isometry with dense image

\[
(\mathcal{N}_R/\mathbb{R}, d_1) \to (\mathcal{M}^1, d_1),
\]

that realizes the Hausdorff completion.

**Proof.** By Theorem 3.17 \( \mathcal{T}_R \) is dense in \( (\mathcal{N}_R, d_1) \). As a consequence, \( \mathcal{T}_R/\mathbb{R} \) sits as a dense subspace of \( (\mathcal{N}_R/\mathbb{R}, d_1) \), and the rest follows formally from the fact that (5.22) is an isometry with dense image and \( (\mathcal{M}^1, d_1) \) is complete. □
Corollary 5.14. The fibers of the induced map $\mathcal{N}_R \to M^1$ consist precisely of asymptotic equivalence classes modulo translation.

6. The regularized Fubini–Study operator

In this section we assume that continuity of envelopes holds for $(X, L)$. Recall that this is true when $X$ is smooth and $\text{char } k = 0$ (or $\dim X \leq 2$). We introduce the regularized Fubini–Study operator, conclude the proof of Theorem B, and also discuss the relation to the homogeneous decomposition of psh functions.

6.1. Definition and main theorem. Recall that for each $\chi \in \mathcal{N}_R$, $\text{FS}(\chi)$ is a bounded lsc function on $X_{\text{an}}$, obtained as the pointwise limit of $\text{FS}_m(\chi) = \text{FS}(\chi_m) \in \mathcal{H}_R$, an increasing net with respect to divisibility. By continuity of envelopes, the usc regularization $\text{FS}^\star(\chi) := \text{FS}(\chi)^\star$ is thus psh, and regularizable from below, i.e. $\text{FS}^\star(\chi) \in \mathcal{E}_1^\infty$. We obtain a map

$$\text{FS}^\star : \mathcal{N}_R \to \mathcal{E}_1^\infty,$$

the regularized Fubini–Study operator, which is increasing and equivariant for the actions of $\mathbb{R}$ and $\mathbb{R}_{>0}$.

The next result proves in particular (iv) of Theorem B.

Theorem 6.1. The regularized Fubini–Study operator is a surjective isometry

$$\text{FS}^\star : (\mathcal{N}_R, d_1) \to (\mathcal{E}_1^\infty, d_1),$$

whose fibers consist of asymptotic equivalence classes. Furthermore,

$$\text{IN} : (\mathcal{E}_1^\infty, d_1) \hookrightarrow (\mathcal{N}_R, d_1)$$

is a section of $\text{FS}^\star$ such that, for any $\chi \in \mathcal{N}_R$, $\tilde{\chi} := \text{IN}(\text{FS}^\star(\chi))$ is the largest norm in the class of $\chi$.

The composition $\text{MA} \circ \text{FS}^\star : \mathcal{N}_R \to M^1$ coincides with the map in Corollary 5.14 which is thus refined by Theorem 6.1 in the presence of continuity of envelopes.

Lemma 6.2. If $\varphi \in \mathcal{L}^\infty$, then $\text{vol}(\text{IN}(\varphi)) = E(Q(\varphi))$.

Proof. We have $\text{vol}(\text{IN}(\varphi)) = E(\text{FS}(\text{IN}(\varphi))) = E(Q(\varphi))$, where the first equality follows from Lemma 5.2 and the second from Proposition 2.16. □

Lemma 6.3. For each $\varphi \in \mathcal{L}^\infty$ we have

$$E(\varphi) = E(P(\varphi)) = E(P^\star(\varphi)), \quad E(Q(\varphi)) = E(Q^\star(\varphi)).$$

Proof. We can find an increasing net $(\psi_i)$ of bounded $L$-psh functions such that $P(\varphi) = \lim_i \psi_i$ pointwise. On the one hand, we have, for each $i$, $E(\psi_i) \leq E(\text{FS}(\text{IN}(\varphi))) \leq E(P^\star(\varphi))$. Since $P(\varphi)$ coincides with $P^\star(\varphi) \in \text{PSH}$ on $X_{\text{div}}$, we have on the other hand $\psi_i \to P^\star(\varphi)$ in PSH, and hence strongly in $\mathcal{E}_1^1$, since $(\psi_i)$ is an increasing net. Thus $E(\psi_i) \to E(P^\star(\varphi))$. This proves the first point, and the second one follows since $Q(\varphi) = P(\varphi)$. □

Lemma 6.4. For any $\chi \in \mathcal{N}_R$ we have $E(\text{FS}^\star(\chi)) = E(\text{FS}(\chi)) = \text{vol}(\chi)$.

Proof. By Corollary 2.17 we have $\text{FS}(\chi) = Q(\text{FS}(\chi))$. Lemma 6.3 thus yields the first claimed equality, while the second one is Lemma 5.2. □

Lemma 6.5. If $\chi \in \mathcal{N}_R$, then $\tilde{\chi} := \text{IN}(\text{FS}^\star(\chi))$ is asymptotically equivalent to $\chi$. 

Proof. By Lemma 6.5.
Proof. Since \( \chi \leq \overline{\chi} = \text{IN}(\text{FS}(\chi)) \leq \overline{\chi} \), we have \( d_1(\chi, \overline{\chi}) = \text{vol}(\overline{\chi}) - \text{vol}(\chi) \). Now Lemma 6.2 and Lemma 6.4 yield \( \text{vol}(\overline{\chi}) = E(\text{FS}^*(\chi)) = \text{vol}(\chi) \); hence the result.

Proof of Theorem 6.1. We first show that \( \text{FS}^*: (\mathcal{N}_R, d_1) \to (\mathcal{E}_1^\infty, d_1) \) is an isometry. Pick \( \chi, \chi' \in \mathcal{N}_R \), and set \( \varphi := \text{FS}^*(\chi), \varphi' := \text{FS}^*(\chi') \). Then
\[
d_1(\chi, \chi') = \text{vol}(\chi) + \text{vol}(\chi') - 2\text{vol}(\chi \wedge \chi')
\]
and
\[
d_1(\varphi, \varphi') = \text{E} (\varphi) + \text{E} (\varphi') - 2 \text{E} (\varphi \wedge \varphi'),
\]
by (3.12) and Theorem 5.4(iii), respectively. By Lemma 6.4 we have on the one hand \( \text{vol}(\chi) = \text{E}(\varphi) \), \( \text{vol}(\chi') = \text{E}(\varphi') \). Since \( \varphi, \varphi' \) are regularizable from below, Lemma 1.3 yields on the other hand \( \text{P}(\varphi \wedge \varphi') = Q^*(\varphi \wedge \varphi') \), which implies
\[
\text{E}(\text{P}(\varphi \wedge \varphi')) = \text{E}(Q^*(\varphi \wedge \varphi')) = \text{E}(Q(\varphi \wedge \varphi')),
\]
by Lemma 6.3. By Lemma 6.2, the right-hand side is equal to
\[
\text{vol}(\text{IN}(\varphi \wedge \varphi')) = \text{vol}(\text{IN}(\varphi) \wedge \text{IN}(\varphi')),
\]
see (2.13). Finally, Lemma 6.5 shows \( \text{IN}(\varphi) \sim \chi, \text{IN}(\varphi') \sim \chi' \), which imply
\[
\text{IN}(\varphi) \wedge \text{IN}(\varphi') \sim \chi \wedge \chi', \quad \text{vol}(\text{IN}(\varphi) \wedge \text{IN}(\varphi')) = \text{vol}(\chi \wedge \chi'),
\]
by Corollary 3.11. We conclude that \( \text{E}(\text{P}(\varphi \wedge \varphi')) = \text{vol}(\chi \wedge \chi') \), which proves, as desired, that \( \text{FS}^* \) is an isometry.

For each \( \varphi \in \mathcal{E}_1^\infty \), Proposition 2.16 yields \( \text{FS}^*(\text{IN}(\varphi)) = \text{Q}^*(\varphi) = \varphi \) (see Lemma 4.3), which proves that \( \text{IN} \) is a section of \( \text{FS}^* \) (and hence automatically an isometry). Assume finally that \( \chi' \in \mathcal{N}_R \) is asymptotically equivalent to \( \chi \). Then \( \text{FS}^*(\chi') = \text{FS}^*(\chi) \), hence
\[
\chi' \leq \overline{\chi} = \text{IN}(\text{FS}(\chi')) \leq \text{IN}(\text{FS}^*(\chi)) = \text{IN}(\text{FS}^*(\chi)).
\]
This proves that \( \overline{\chi} \) is the largest norm in the fiber of \( \chi \), and concludes the proof.

Example 6.6. The norm \( \overline{\chi} = \text{IN}(\text{FS}^*(\chi)) \) may be strictly larger than the homogenization \( \overline{\chi} = \text{IN}(\text{FS}(\chi)) \). Consider indeed \((X, L) = (\mathbb{P}^1, \mathcal{O}(1)) \). Let \( \chi \in \mathcal{N}_R \) be the norm in Example 3.8 and let \( p \in \mathbb{P}_k^1 \subset \mathbb{P}_k^1 \) be the point defined by \( T_2 = 0 \). Then \( \chi \) is homogenous, and one checks that \( \varphi = \text{FS}(\chi) \) is given by \( \varphi(p) = -1 \) and \( \varphi \equiv 0 \) on \( \mathbb{P}_k^1 \setminus \{v_{p}\} \). Thus \( \text{FS}^*(\chi) \equiv 0 \), and \( \overline{\chi} = \text{IN}(\text{FS}^*(\chi)) \) is the trivial norm.

6.2. Homogeneous decomposition and base ideals. As in §6.1, we assume that continuity of envelopes holds for \((X, L) \). We will use Theorem 6.1 to provide an alternative description of the homogeneous decomposition (see [Bra12] §10.2) for functions in \( \mathcal{E}_1^\infty \).

Recall that a function \( \varphi \in \text{PSH} \) is homogeneous if \( \varphi(tv) = t \varphi(v) \) for \( t \in \mathbb{R}_{>0} \) and \( v \in X^\text{an} \), i.e. \( t \cdot \varphi = \varphi \).

Example 6.7. Any ideal \( b \subset \mathcal{O}_X \) defines a homogeneous function \( \log |b|: X^\text{an} \rightarrow [-\infty, 0] \), defined by \( \log |b|(v) := -v(b) \) for \( v \in X^\text{an} \). If \( L \otimes b \) is globally generated, then \( \log |b| \) is \( L \text{-psh} \).

Example 6.8. More generally, any graded sequence of ideals \( b_* = (b_m) \) (defined for \( m \) sufficiently divisible) yields a homogeneous function \( \log |b| := \lim_m m^{-1} \log |b_m| \), i.e.
\[
\log |b_*|(v) = -\inf_m m^{-1} v(b_m)
\]
for all \( v \in X^\text{an} \), compare [ELMNP06, JIM12]. If \( mL \otimes b_m \) is globally generated for all \( m \) sufficiently divisible, then \( m^{-1} \log |b_m| \) is \( L \text{-psh} \), and the use regularization \( \log^* |b_*| \) is \( L \text{-psh} \), thanks to continuity of envelopes.
In [BoJ21] Theorem 10.6] we proved that every $\varphi \in \text{PSH}$ admits a homogeneous decomposition

$$\varphi = \sup_{\lambda < \sup \varphi} \{ \phi^\lambda + \lambda \}, \quad (6.1)$$

where, for each $\lambda < \sup \varphi$, $\phi^\lambda$ is a homogeneous $L$-psh function defined by

$$\phi^\lambda = \inf_{t>0} \{ t \cdot \varphi - t\lambda \}.$$

Now suppose $\varphi = \text{FS}^*(\chi)$, where $\chi \in \mathcal{N}_R$ is a norm. Then $\sup \varphi = \lambda_{\max}(\chi)$. For $\lambda < \lambda_{\max}(\chi)$ and $m$ sufficiently divisible, let $b_m^\lambda \subset \mathcal{O}_X$ be the base ideal of $\{ \chi \geq \lambda m \} \subset R_m$. This defines a graded sequence of ideals $b^\lambda = (b_m^\lambda)_m$.

**Proposition 6.9.** For each $\lambda < \lambda_{\max}(\chi)$, we have $\phi^\lambda = \log^* |b^\lambda|$.

The proof relies on the following characterization of the homogeneous decomposition $[6.1]$.

**Lemma 6.10.** Let $\varphi \in \text{PSH}$. Suppose we are given a family $(\tau^\lambda)_{\lambda < \sup \varphi}$ of homogeneous $L$-psh functions such that, for each $v \in X_{\text{lin}}$, we have that $\lambda \mapsto \tau^\lambda(v)$ is concave, and $\varphi(v) = \sup_{\lambda} \{ \tau^\lambda(v) + \lambda \}$. Then $\tau^\lambda = \phi^\lambda$ for all $\lambda$.

**Proof.** Fix $v \in X_{\text{div}}$. The functions $h_1(\lambda) := \tau^\lambda(v)$ and $h_2(\lambda) = t^\lambda(v)$ on $(-\infty, \sup \varphi)$ are concave, so it suffices to show that they have the same Legendre transforms. The latter are naturally defined on $\mathbb{R}_{>0}$. Now, if $t > 0$, then $t^{-1} \cdot v \in X_{\text{lin}}$, and

$$\sup_{\lambda} \{ h_1(\lambda) + t\lambda \} = t \sup_{\lambda} \{ \tau^\lambda(t^{-1} \cdot v) + \lambda \} = t \varphi(t^{-1} \cdot v) = (t \cdot \varphi)(v),$$

and similarly for $h_2$. This implies $h_1 = h_2$ and completes the proof. \qed

**Proof of Proposition 6.9.** The function $\tau^\lambda := \log^* |b^\lambda|$ is $L$-psh and homogeneous for $\lambda < \lambda_{\max}(\chi)$. For $v \in X_{\text{lin}}$, we can drop the usc regularization, and write

$$\tau^\lambda(v) = \sup \{ -m^{-1}v(s) \mid s \in R_m, \chi(s) \geq m\lambda \}.$$ 

This formula shows that $\lambda \mapsto \tau^\lambda(v)$ is concave on $(-\infty, \lambda_{\max})$ for any $v \in X_{\text{lin}}$. We also have

$$\sup_{\lambda < \lambda_{\max}} \{ \tau^\lambda(v) + \lambda \} = \sup \{ -m^{-1}v(s) + \chi(s) \mid s \in R_m \} = \text{FS}(\chi)(v) = \text{FS}^*(\chi)(v),$$

and we conclude using Lemma 6.10. \qed

7. The space of valuations of linear growth

In this section, we return to the setting of an arbitrary polarized variety $(X, L)$ (for which continuity of envelopes is not assumed), and prove Theorem C.

7.1. **Invariants.** Each valuation $v \in X_{\text{lin}}$ of linear growth determines a homogeneous norm $\chi_v \in \mathcal{N}_{\text{hom}}$, defined by

$$\chi_v(s) := v(s)$$

for $s \in R_m$. If $v \in X_{\text{div}}$, then $\chi_v \in \mathcal{N}_{\text{hom}}$. The spectrum of $\chi_v|_{R_m}$ is the vanishing sequence of $R_m$ with respect to $v$ as defined in [BKMS15], i.e. the (finite) set of values of $v$ on nonzero elements of $R_m$, counted with multiplicity. As in [BlJ20], we introduce:

**Definition 7.1.** The expected vanishing order of $v \in X_{\text{lin}}$ (for $L$) is defined as

$$S(v) = S_L(v) := \text{vol}(\chi_v).$$
In other words, $S(v) = \lim_m m^{-1} \text{vol}(\chi_v|_{R_m})$ is the limit of the scaled mean value of $v$ on $R_m \setminus \{0\}$, which explains the chosen terminology. See also [MR15].

Next we consider the bounded lsc function

$$\varphi_v := FS(\chi_v) \in L^\infty.$$  

Note that $\chi_v \geq \chi_{\text{triv}}$, hence $\varphi_v \geq 0$. Concretely, $\varphi_v$ is the limit of the increasing net $(\varphi_{v,m})_m$, where $\varphi_{v,m} = FS_m(\chi_v) \in H$ satisfies

$$\varphi_{v,m}(w) = \frac{1}{m} \max \{v(s) - w(s) \mid s \in R_m \setminus \{0\}\}. \quad (7.2)$$

As $\chi_v$ is homogeneous, Theorem 2.11 implies

$$\text{IN}(\varphi_v) = \chi_v.$$  

The next result covers part of Theorem C in the introduction.

**Theorem 7.2.** For any $v \in X^\text{lin}$, we have

$$T(v) = \sup_{X^\text{lin}} \varphi_v = \lambda_{\text{max}}(\chi_v) \quad \text{and} \quad S(v) = E^\vee(\delta_v) = E(\varphi_v). \quad (7.3)$$

Here the Monge–Ampère energy $E(\varphi_v)$ is defined as in (5.3).

**Proof.** We have $\sup \varphi_{v,m} = m^{-1}\lambda_{\text{max}}(\chi_v|_{R_m})$, so by taking the limit over $m$ we see that $\sup \varphi_v = \lambda_{\text{max}}(\chi_v)$. Moreover, $\{1.13\}$ yields $T(v) = \sup_m m^{-1}\lambda_{\text{max}}(\chi|_{R_m}) = \lambda_{\text{max}}(\chi_v)$.

It remains to prove the last two equalities in (7.3). That $S(v) = E(\varphi_v)$ is a special case of Lemma 5.2. By Lemma 5.3, we have on the other hand $E(\varphi_v) = \lim_m E(\varphi_{v,m})$. The equality $E^\vee(\delta_v) = E(\varphi_v)$ is thus a consequence of [BoJ21, Lemma 8.16], but we repeat the simple argument for the convenience of the reader. Since $\varphi_{v,m} \in H$ satisfies $\varphi_{v,m}(v) = 0$, we have $E(\varphi_{v,m}) \leq E^\vee(\delta_v)$, and hence $E(\varphi_v) \leq E^\vee(\delta_v)$. Conversely, pick $\varphi \in H$, so that

$$\varphi = m^{-1} \max \{\log |s_i| + \lambda_i\}$$

with $m$ sufficiently divisible $(s_i)$ a finite set of $R_m$ without common zeroes, and $\lambda_i \in \mathbb{R}$. For each $i$ we have

$$-v(s_i) + \lambda_i = \log |s_i|(v) + \lambda_i \leq m\varphi(v).$$

This yields $\varphi \leq \varphi_{v,m} + \varphi(v)$, and hence $E(\varphi) - \varphi(v) \leq E(\varphi_{v,m})$, which yields

$$E^\vee(\delta_v) = \sup_{\varphi \in H} (E(\varphi) - \varphi(v)) \leq \lim_m E(\varphi_{v,m}) = E(\varphi_v),$$

and we are done. \qed

As a consequence, we obtain the following comparison between the invariants $S$ and $T$, originally proved in [Fuj19b, §2] for divisorial valuations.

**Proposition 7.3.** We have $\frac{1}{n+1} T(v) \leq S(v) \leq \frac{n}{n+1} T(v)$ for any $v \in X^\text{lin}$.

**Proof.** By [BoJ21, Proposition 8.1], the inequalities hold with $S(v)$ replaced by $E^\vee(\delta_v)$, so the result follows from Theorem 7.2. \qed

**Proposition 7.4.** If we assume continuity of envelopes, then $\varphi_v \in \text{CPSH}$ for every $v \in X^\text{lin}$. Moreover, $\varphi_v(0) = 0$ and $\text{MA}(\varphi_v) = \delta_v$.

**Proof.** This follows from [BoJ21, Proposition 9.18]. \qed
7.2. Comparison of metrics. The embedding $X^\text{lin} \hookrightarrow \mathcal{N}_R$ defined by $v \mapsto \chi_v$ endows $X^\text{lin}$ with pseudo-metrics $d_p$, $1 \leq p \leq \infty$. The (pseudo-)metric $d_\infty$ was in fact defined in §8.3 of [BoJ21], where it was shown to be complete and independent of $L$ up to bi-Lipschitz equivalence. Strictly speaking the definition there is slightly different, but Lemma 8.6 and (8.4) in loc. cit. show that the two metrics coincide, and that the formula

$$d_\infty(v, w) = \lim_{m} \max \{ \varphi_{v,m}(w), \varphi_{w,m}(v) \} = \max \{ \varphi_v(w), \varphi_w(v) \} \quad (7.4)$$

holds for any $v, w \in X^\text{lin}$. We also have $d_\infty(v, \nu_{\text{triv}}) = T(v)$, see [BoJ21] Proposition 8.5.

Further, since $\chi_v \geq \chi_{\text{triv}}$ we have

$$d_1(v, \nu_{\text{triv}}) = d_1(\chi_v, \chi_{\text{triv}}) = \text{vol}(\chi_v) = E^\nu(\delta_v),$$

where the last equality follows from Theorem 7.2.

On the other hand, the embedding $X^\text{lin} \hookrightarrow \mathcal{M}_1$ defined by $v \mapsto \delta_v$ endows $X^\text{lin}$ with a metric $d_1$.

**Theorem 7.5.** For any $p \in [1, \infty]$ we have

$$d_1 \leq d_1 \leq d_p \leq d_\infty \leq d_1$$

on $X^\text{lin}$. In particular, they are all equivalent metrics, that turn $X^\text{lin}$ into a complete metric space.

By [BoJ21] Theorem 8.13], these metrics are also Lipschitz equivalent to the quasi-metric $(v, w) \mapsto I^\nu(\delta_v, \delta_w)$.

**Proof.** The inequality $d_1 \leq d_1$ follows from Theorem 5.13 and it remains to show $d_\infty \leq d_1$. Since $\mu_m$ and $\mu'_m$ converge strongly, $I(\varphi_m) \approx E^\nu(\mu_m)$ and $I(\varphi'_m) \approx E^\nu(\mu'_m)$ are uniformly bounded. By Lemma 5.9 we can thus find $c_m = O(1)$ such that

$$d_1(\varphi_m, \varphi'_m) = d_1(\varphi_m + c_m, \varphi'_m) \approx \int |\varphi_m + c_m - \varphi'_m| (\mu_m + \mu'_m) =: A_m$$

see Theorem 5.6. Setting

$$B_m := \int |\varphi_m + c_m - \varphi_m| (\delta_v + \delta_w),$$

we have on the other hand $A_m = B_m + o(1)$, by Theorem 5.12. Now $\varphi_m(v) = \varphi'_m(w) = 0$ implies

$$B_m = |\varphi_m(w) + c| + |\varphi'_m(v) - c| \geq \varphi_m(w) + \varphi'_m(v) \geq \max \{ \varphi_m(w), \varphi'_m(v) \},$$

by the triangle inequality. Since $d_1(\varphi_m, \varphi'_m) \to d_1(v, w)$ and

$$\max \{ \varphi_m(w), \varphi'_m(v) \} \to \max \{ \varphi_v(w), \varphi_w(v) \} = d_\infty(v, w),$$

we conclude, as desired, $d_1(v, w) \geq d_\infty(v, w)$. 

By Theorem 7.5, we have $d_p(v, w) \approx d_\infty(v, w)$ for all $v, w \in X^\text{lin}$. When $w = \nu_{\text{triv}}$, Lemma 3.12 yields a more precise estimate

$$d_p(v, \nu_{\text{triv}}) \leq d_\infty(v, \nu_{\text{triv}}) \leq C_{n,p} d_p(v, \nu_{\text{triv}}) \quad (7.5)$$

with $C_{n,p} \to 1$ as $p \to \infty$. When $v$ is a divisorial valuation, $d_p(v, \nu_{\text{triv}})^p$ is the invariant $S^p(L, v)$ considered by K. Zhang in [Zha20], and (7.5) then follows from [Zha20] Corollary 1.4, whose proof (itself based on [Fuj19b]) we simply borrowed to get Lemma 3.12.
Appendix A. Test configurations, integral closure and homogenization

The goal of this appendix is to recall the correspondence between ample test configurations and integral norms of finite type [WN12 BHJ17], and to use this and [BoJ21] to describe the relationship between homogenization of norms and integral closure of test configurations.

A.1. The Rees correspondence. A test configuration \((X, L)\) for \((X, L)\) consists of: a flat projective morphism \(\pi: X \rightarrow \mathbb{A}^1\); a \(\mathbb{Q}\)-line bundle \(L\) on \(X\); a \(\mathbb{G}_m\)-action on \((X, L)\) that makes \(\pi\) equivariant; and a \(\mathbb{G}_m\)-equivariant isomorphism

\[(X, L)|_{\mathbb{G}_m} \simeq (X, L) \times \mathbb{G}_m.\]  

(A.1)

We denote by \(\varpi\) the coordinate on \(\mathbb{A}^1 = \text{Spec } k[\varpi]\) and \(\mathbb{G}_m = \text{Spec } k[\varpi^{\pm}].\)

Example A.1. The trivial test configuration \((X_{\text{triv}}, L_{\text{triv}})\) is defined by \(X_{\text{triv}} = X \times \mathbb{A}^1, L_{\text{triv}} = p_1^* L.\)

As originally pointed out in [WN12], to any test configuration \((X, L)\) is associated an integral norm \(\chi_L \in N_X\), defined on \(R_m = H^0(X, mL)\) for any \(m \in \mathbb{N}\) such that \(mL\) is a line bundle, as follows. Consider the embedding \(H^0(X, mL) \hookrightarrow R_m \otimes k[\varpi^\pm]\) induced by (A.1). This yields a decomposition

\[H^0(X, mL) = \bigoplus_{\lambda \in \mathbb{Z}} \varpi^{-\lambda} F^\lambda R_m,\]  

(A.2)

corresponding to the weight decomposition with respect to the \(\mathbb{G}_m\)-action, where

\[F^\lambda R_m = \{ s \in R_m \mid \varpi^{-\lambda} s \in H^0(X, mL) \}\]  

(A.3)

is a \(\mathbb{Z}\)-filtration of \(R_m\), and we define \(\chi_L\) as the associated norm. By flat base change, one easily checks:

Lemma A.2. If \((X_d, L_d)\) denotes the base change of \((X, L)\) with respect to \(\varpi \mapsto \varpi^d, d \geq 1,\) then \(\chi_{L_d} = d \chi_L.\)

A test configuration \((X, L)\) is ample if \(L\) is ample. For \(r\) sufficiently divisible, \(R(X, rL)\) is then generated in degree 1, which shows that \(\chi_L \in N_X\) is of finite type. Note further that

\[R(X_0, rL) \simeq \text{gr}_{\chi_L} R^{(r)}\]  

(A.4)

for \(r\) sufficiently divisible.

Denoting by \(\mathcal{T}\) the set of ample test configurations, we get a map \(\mathcal{T} \rightarrow \mathcal{T}_\mathbb{Z}\). A map in the reverse direction is indeed provided by the Rees construction. Given \(\chi \in \mathcal{T}_\mathbb{Z}\), pick \(r \geq 1\) such that \(\chi\) is represented by an integral norm on \(R^{(r)} = R(X, rL)\) generated in degree 1, with associated filtration \((F^\lambda R^{(r)})_{\lambda \in \mathbb{Z}}\). The Rees algebra

\[\mathcal{R} := \bigoplus_{\lambda \in \mathbb{Z}} \varpi^{-\lambda} F^\lambda R^{(r)}\]

is a graded \(k[\varpi]\)-algebra (with respect to the \(\mathbb{N}\)-grading inherited from that of \(R^{(r)}\)), generated in degree 1, and we set \(X := \text{Proj } \mathcal{R}\) and \(L = r^{-1} \mathcal{O}_X(1)\). This yields a map \(\mathcal{T}_\mathbb{Z} \rightarrow \mathcal{T}\) which is an inverse of the previous one (see [BHJ17 Proposition 2.15]). We shall refer to the 1-1 map

\[\mathcal{T} \simeq \mathcal{T}_\mathbb{Z}\]  

(A.5)

so defined as the Rees correspondence.

By (A.4), the central fiber \(X_0\) of an ample test configuration \((X, L)\) is reduced iff \(\text{gr}_{\chi_L} R^{(r)}\) is reduced for \(r\) sufficiently divisible, which holds iff \(\chi_L\) is homogeneous. The Rees correspondence...
thus induces a bijection between $\mathcal{T}_E^\text{hom}$ and the set of ample test configurations with reduced central fiber.

A.2. Integral closure and homogenization. We use [BoJ21, §1.4] as a reference. We say that a test configuration $(\mathcal{X}, \mathcal{L})$ is integrally closed if $\mathcal{X}$ is integrally closed in the generic fiber of $\pi$. When $\mathcal{X}$ is normal, this amounts to $\mathcal{X}$ being normal. Any test configuration $(\mathcal{X}, \mathcal{L})$ admits an integral closure $(\mathcal{X}, \mathcal{L})$, with $\mathcal{X}$ the integral closure of $\mathcal{X}$ in its generic fiber and $\mathcal{L}$ the pull-back of $\mathcal{L}$.

Under the Rees correspondence, homogenization now admits the following geometric description:

**Proposition A.3.** For any ample test configuration $(\mathcal{X}, \mathcal{L})$, the homogenization of the corresponding norm $\chi_{\mathcal{L}} \in \mathbb{T}_x$ satisfies $\chi_{\mathcal{L}} = d^{-1} \chi_{\mathcal{L}}$, for $d$ sufficiently divisible.

As above, $(\mathcal{X}_d, \mathcal{L}_d)$ denotes the base change of $(\mathcal{X}, \mathcal{L})$ with respect to $\varpi \mapsto \varpi^d$, and $(\mathcal{X}_d, \mathcal{L}_d)$ is its integral closure.

**Remark A.4.** Proposition A.3 can be used to provide a more elementary proof of Theorem 2.3 in the case of rational norms.

**Lemma A.5.** For any test configuration $(\mathcal{X}, \mathcal{L})$, we have $\chi_{\mathcal{L}} \leq \chi_{\mathcal{L}} \leq \chi_{\mathcal{L}}$, and $\chi_{\mathcal{L}} = \chi_{\mathcal{L}}$ iff $\mathcal{X}$ is reduced.

**Proof.** Pick $s \in R_m \setminus \{0\}$ with $m$ sufficiently divisible. Since $\mathcal{L}$ is the pullback of $\mathcal{L}$ to $\mathcal{X}$, $\chi_{\mathcal{L}}(s) \leq \chi_{\mathcal{L}}(s) =: \mu$ follows directly from A.3. By A.3, $\sigma := \varpi^{-s} \in \mathbb{H}(\mathcal{X}, m\tilde{\mathcal{L}})$ is integral over $\mathcal{X}$, and hence $\sigma^d + \sum_{i=1}^d \sigma_i \sigma^{d-i} = 0$ for some $d \geq 1$ and $\sigma_i \in \mathbb{H}(\mathcal{X}, i\tilde{\mathcal{L}})$. By A.2, we have a Laurent expansion $\sigma_i = \sum_{\lambda \in \mathbb{Z}} \sigma_{i, \lambda} \varpi^{\lambda}$ with $\sigma_{i, \lambda} \in \mathbb{H}(\mathcal{X}, m\mathcal{L})$ such that $\chi_{\mathcal{L}}(\sigma_{i, \lambda}) \geq \lambda$, and tracing the coefficient of $\varpi^{-d\mu}$ yields $s^d + \sum_{i=1}^d \sigma_{i, \mu} s^{d-i} = 0$. Since $\chi_{\mathcal{L}}(\sigma_{i, \mu}) \geq \chi_{\mathcal{L}}(\sigma_{i, \mu}) \geq i\mu$, we infer

$$d \chi_{\mathcal{L}}(s) = \chi_{\mathcal{L}}(s^d) \geq \min_{1 \leq i \leq d} \{i\mu + (d-i) \chi_{\mathcal{L}}(s)\},$$

hence $\chi_{\mathcal{L}}(s) \geq \mu = \chi_{\mathcal{L}}(s)$. We thus have $\chi_{\mathcal{L}} \leq \chi_{\mathcal{L}} \leq \chi_{\mathcal{L}}$. Now $\mathcal{X}$ is reduced iff $\mathcal{X}$ is homogeneous, and the final point follows since $\chi_{\mathcal{L}}$ is the smallest homogeneous norm dominating $\chi_{\mathcal{L}}$. $\square$

**Proof of Proposition A.3.** By BoJ21 Corollary 2.23, the central fiber of $(\mathcal{X}_d, \mathcal{L}_d)$ is reduced for $d$ sufficiently divisible. Then $\chi_{\mathcal{L}} = \chi_{\mathcal{L}}$, by Lemma A.5. By Lemma A.2, we have on the other hand $\chi_{\mathcal{L}} = d \chi_{\mathcal{L}}$, hence $\chi_{\mathcal{L}} = d \chi_{\mathcal{L}}$, and we are done. $\square$

For any integrally closed test configuration $\mathcal{X}$ for $X$, the local ring of $\mathcal{X}$ at the generic point of any irreducible component $E$ of $\mathcal{X}_0$ is a DVR, which defines a divisorial valuation $\text{ord}_E$ on $\mathcal{X}$; we denote by

$$b_E := \text{ord}_E(\mathcal{X}_0) = \text{ord}_E(\varpi) \quad (A.6)$$

the multiplicity of $\mathcal{X}_0$ along $E$. By A.1 we have a function field extension $k(\mathcal{X}) \hookrightarrow k(\mathcal{X})$, and the restriction of $b_E^{-1} \text{ord}_E$ to $k(\mathcal{X})$ is a divisorial valuation $v_E \in X^{\text{div}}$. Conversely, any element of $X^{\text{div}}$ can be geometrically realized in this way.

Any (not necessarily ample) test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, \mathcal{L})$ defines a function $\varphi_{\mathcal{L}} \in C^0(X^\text{an})$, whose restriction to the dense subset $X^{\text{div}} \subset X^\text{an}$ is given as follows. Pick $v \in X^{\text{div}}$, and choose an integrally closed test configuration $\mathcal{X}'$ for $X$ such that $v = v_E$ is associated to an irreducible component $E \subset \mathcal{X}'$ and such that the canonical $G_m$-equivariant birational maps
\(\mu \colon X' \to X\) and \(\rho : X' \to X_{\text{triv}}\) are morphisms. Then \(\mu^*L - \rho^*L_{\text{triv}} = D\) for a \(\mathbb{Q}\)-Cartier divisor \(D\) supported on \(X'_0\), and

\[
\varphi_L(v_E) = b_E^{-1} \text{ord}_E(D),
\]

By \cite{bojko2021} §2.4, \((X, L) \mapsto \varphi_L\) maps \(\mathcal{T}\) onto \(\mathcal{H}_Q\), and restricts to a bijection

\[
\mathcal{T}^\text{int} \xrightarrow{\sim} \mathcal{H}_Q.
\]  

The link between this approach and the Rees correspondence is provided by the Fubini–Study operator:

**Lemma A.6.** For any ample test configuration \((X, L)\) for \((X, L)\) we have

\[
\varphi_L = \text{FS}(\chi_L) = \text{FS}(\chi_L).
\]

**Proof.** The second equality follows from Proposition \ref{prop:fs} and we focus on the first one. After passing to a multiple of \(L\), we may assume that \(L\) is a globally generated line bundle such that \(R(X, L)\) is generated in degree \(1\), so that \(\chi_L\) is a rational norm on \(R = R(X, L)\) generated in degree \(1\). Denote by \(\chi\) its restriction to \(H^0(X, L)\). By density of \(X^\text{div}\), the desired result boils down to \(\text{FS}_L(\chi)(v) = \varphi_L(v)\) for \(v \in X^\text{div}\), i.e. \(\varphi_L(v) \geq \chi(s) - v(s)\) for all \(s \in H^0(X, L) \setminus \{0\}\), with equality for some \(s\), cf. \cite{fulman1993}.

Pick an integrally closed test configuration \(X'\) for \(X\) such that \(\varphi_L\) is associated to an irreducible component \(E \subset X'_0\) and such that the canonical \(G_m\)-equivariant birational maps \(\mu : X' \to X\) and \(\rho : X' \to X_{\text{triv}}\) are morphisms. Write \(\mu^*L - \rho^*L_{\text{triv}} = D\) for a Cartier divisor \(D\) supported on \(X'_0\), so that \(\varphi_L(v) = b_E^{-1} \text{ord}_E(D)\). Consider any nonzero section \(\sigma \in H^0(X, L)\), and write \(\sigma = \sum_{\lambda \in \mathbb{Z}} \sigma_{\chi^{-\lambda}}\) with \(\sigma_{\lambda} \in H^0(X, L)\) such that \(\chi(\sigma_{\lambda}) \geq \lambda\), see \cite{fulman1993}. Viewed as a rational section of \(\rho^*L_{\text{triv}} = \mu^*L - D\), \(\sigma\) satisfies

\[
-\text{ord}_E(D) \leq \text{ord}_E(\sigma) = \min_{\lambda} \{\text{ord}_E(\sigma_{\lambda}) - \lambda \text{ord}_E(\varpi)\},
\]

by \(G_m\)-invariance of \(E\). Dividing by \(b_E = \text{ord}_E(\varpi)\), we infer \(\varphi_L(v) \geq \max_{\lambda} \{\lambda - \text{ord}_E(\varpi)\}\), and equality holds iff \(\sigma\), now viewed as a section of \(L\), does not vanish at the generic point \(\eta\) of \(\mu(E)\).

Choosing \(\sigma := \varpi^{-\chi(s)}s\) with \(s \in H^0(X, L) \setminus \{0\}\), this proves on the one hand \(\varphi_L(v) \geq v(s) - \chi(s)\). On the other hand, since \(L\) is globally generated, we can pick \(\sigma \in H^0(X, L)\) nonzero at \(\eta\). The above discussion yields \(\varphi_L(v) = \lambda - v(\sigma_{\lambda})\) for some \(\lambda\), necessarily equal to \(\chi(\sigma_{\lambda})\) since \(\chi(\sigma_{\lambda}) \geq \lambda\) and \(\varphi_L(v) \geq \chi(\sigma_{\lambda}) - v(\sigma_{\lambda})\) by the previous step. We are thus done.

Denote by \(\mathcal{T}^\text{int}_\mathbb{Z} \subset \mathcal{T}_\mathbb{Z}\) the subset corresponding to integrally closed test configurations under the Rees correspondence. Any test configuration with reduced central fiber is integrally closed, and hence

\[
\mathcal{T}^\text{hom}_\mathbb{Z} \subset \mathcal{T}^\text{int}_\mathbb{Z} \subset \mathcal{T}_\mathbb{Z}.
\]

By Lemma A.6 and (A.8), the Fubini–Study operator restricts to a bijection \(\text{FS} : \mathcal{T}^\text{int}_\mathbb{Z} \xrightarrow{\sim} \mathcal{H}_Q\). Since it factors as \(H: \mathcal{T}^\text{int}_\mathbb{Z} \to \mathcal{T}^\text{hom}_\mathbb{Z}\) followed by \(\text{FS}: \mathcal{T}^\text{hom}_\mathbb{Z} \xrightarrow{\sim} \mathcal{H}_Q\) (see Corollary 2.13), we infer:

**Corollary A.7.** Homogenization induces a bijection \(H : \mathcal{T}^\text{int}_\mathbb{Z} \xrightarrow{\sim} \mathcal{T}^\text{hom}_\mathbb{Z}\).
functions, and a dual canonical embedding $N_\mathbb{R} \to T^{\text{val}}$ onto the set of $T(k)$-invariant valuations, such that $v(u) = \langle v, u \rangle$ for all $v \in N_\mathbb{R}$ and $u \in M \hookrightarrow k(T)^\times$.

A polarized toric variety $(X, L)$ is determined by a convex polytope $P \subset M_\mathbb{R}$ with vertices in $M$. It defines a convex PL function $f_P : N_\mathbb{R} \to \mathbb{R}$ given by $f_P(v) := \sup_{u \in P} \langle v, u \rangle$.

The set of $T(k)$-eigensections of $R_m = H^0(X, mL)$ can be identified with $mP \cap \mathbb{R}$. Let $Q \subset \mathbb{R} \times M_\mathbb{R}$ be the closed convex cone generated by $\{1\} \times P$. The set $N_\mathbb{R}^{\text{tor}}$ of $(T(k)$-invariant norms on $R(X, L)$ is then in 1-1 correspondence with functions $\chi : Q \cap (\mathbb{N} \times M) \to \mathbb{R}$ satisfying $\chi(u_1 + u_2) \geq \chi(u_1) + \chi(u_2)$. The homogenization $\overline{\chi} : Q \cap (\overline{\mathbb{Q}} \times N_\mathbb{Q}) \to \mathbb{R}$ defined by $\overline{\chi}(u) = \lim_m m^{-1} \chi(mu)$ extends uniquely to a homogeneous concave function on $Q$, and then restricts to a concave function on $P \simeq \{1\} \times P$.

The volume of $\chi \in N_\mathbb{R}^{\text{tor}}$ is now given by $\text{vol}(\chi) = \int_P \overline{\chi}$, where the integral is with respect to Lebesgue measure, and the $d_p$-pseudometric on $N_\mathbb{R}^{\text{tor}}$ is induced by the norm on $L^p(P)$.

The space of $T(k)$-invariant $L$-psh functions on $X^{\text{an}}$ can be identified with the set of convex functions $f : N_\mathbb{R} \to \mathbb{R}$ satisfying $f \leq f_P + O(1)$, and the Fubini–Study and infimum norm operators are both given as Legendre transforms.

The set $N_\mathbb{R}$ can be viewed as a the set of $T(k)$-invariant valuations on $X$ (or $T$), and the norm $\chi_p$ associated to $v \in N_\mathbb{R}$ is simply the corresponding linear function on $P$. This gives a concrete interpretation of the metrics $d_p$ and $d_1$ on $N_\mathbb{R} \subset X^{\text{lin}}$.

References

[Berk90] V. G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*. Mathematical Surveys and Monographs, 33. American Mathematical Society, Providence, RI, 1990.

[Berk93] V. G. Berkovich. *Étale cohomology for non-Archimedean analytic spaces*. Publ. Math. Inst. Hautes Études Sci. 78 (1993), 5–161.

[BBGZ13] R. J. Berman, S. Boucksom, V. Guedj and A. Zeriahi. *A variational approach to complex Monge–Ampère equations*. Publ. Math. Inst. Hautes Études Sci. 117 (2013), 179–245.

[BBJ21] R. J. Berman, S. Boucksom and M. Jonsson. *A variational approach to the Yau–Tian–Donaldson conjecture*. J. Amer. Math. Soc. 34 (2021), 605–652.

[BJJ20] H. Blum, M. Jonsson. *Thresholds, valuations, and K-stability*. Adv. Math. 365 (2020), 107062.

[BLX19] H. Blum, Y. Liu, C. Xu. *Openness of K-semistability for Fano varieties*. arXiv:1907.02408.

[BLXZ21] H. Blum, Y. Liu, C. Xu, Z. Zhuang. *The existence of the Kähler–Ricci soliton degeneration*. arXiv:2103.15278.

[BLZ19] H. Blum, Y. Liu, C. Zhou *Optimal destabilization of K-unstable Fano varieties via stability thresholds*. arXiv:1907.05399.

[BX20] H. Blum, C. Xu. *Uniqueness of K-polystable degenerations of Fano varieties*. Ann. of Math. (2) 190 (2020), 609–656.

[BGR] S. Bosch, U. Güntzer and R. Remmert. *Non-Archimedean Analysis*. Springer-Verlag, Berlin, Heidelberg, 1994.

[Bou14] S. Boucksom. *Corps d’Okounkov*. Exp. No. 1059. Astérisque 361 (2014), 1–41.

[BC11] S. Boucksom and H. Chen. *Okounkov bodies of filtered linear series*. Compos. Math. 147 (2011), 1205–1229.

[BE21] S. Boucksom, D. Eriksson. *Spaces of norms, determinant of cohomology and Fekete points in non-Archimedean geometry*. Adv. Math. 378 (2021), 107501.

[BFJ14] S. Boucksom, C. Favre and M. Jonsson. *A refinement of Izumi’s theorem*. Valuation theory in interaction, 55–81. EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich (2014).

[BFJ16] S. Boucksom, C. Favre and M. Jonsson. *Singular semipositive metrics in non-Archimedean geometry*. J. Algebraic Geom. 25 (2016), 77–139.

[BHJ17] S. Boucksom, T. Hisamoto and M. Jonsson. *Uniform K-stability, Duistermaat–Heckman measures and singularities of pairs*. Ann. Inst. Fourier 67 (2017), 743–841.

[BoJ21] S. Boucksom and M. Jonsson. *Global pluripotential theory over a trivially valued field*. arXiv:1801.08229v3.

[BoJ21b] S. Boucksom and M. Jonsson. *A non-Archimedean approach to K-stability, II*. In preparation.
[LM09] R. Lazarsfeld and M. Mustaţă. Convex bodies associated to linear series. Ann. Sci. Éc. Norm. Supér. (4), 42 (2009), 783–835.

[Li17] C. Li. K-semistability is equivariant volume minimization. Duke Math. J. 166 (2017), 3147–3218.

[Li19] C. Li. G-uniform stability and Kähler-Einstein metrics on singular Fano varieties. arXiv:1907.09399.

[Li20] C. Li. Geodesic rays and stability in the cscK problem. arXiv:2001.01366. To appear in Ann. Sci. Éc. Norm. Sup.

[LX20] C. Li and C. Xu. Stability of valuations and Kollár components. J. Eur. Math. Soc. Published online, 2020-05-11.

[MR15] D. McKinnon and M. Roth. Seshadri constants, diophantine approximation, and Roth’s theorem for arbitrary varieties. Invent. Math. 200 (2015), 513–583.

[MFK] D. Mumford, J. Fogarty and F. Kirwan. Geometric invariant theory. Third edition. Ergebnisse der Mathematik und ihrer Grenzgebiete (2), vol 34. Springer-Verlag, Berlin, 1994.

[Oda15] Y. Odaka. On parametrization, optimization and triviality of test configurations. Proc. Amer. Math. Soc. 143, 25–33.

[Reb20] R. Reboulet. Plurisubharmonic geodesics in spaces of non-Archimedean metrics of finite energy. arXiv:2012.07972.

[Reb21] R. Reboulet. The asymptotic Fubini–Study operator over general non-Archimedean fields. Math. Z. (2021), https://doi.org/10.1007/s00209-021-02770-2.

[Szé15] G. Székelyhidi. Filtrations and test-configurations. With an appendix by S. Boucksom. Math. Ann. 362 (2015), 451–484.

[Tem04] M. Temkin. On local properties of non-Archimedean analytic spaces II. Israel J. Math. 140 (2004), 1–27.

[Tem15] M. Temkin. Introduction to Berkovich analytic spaces. Berkovich spaces and applications, 3–66. Lecture Notes in Mathematics, 2119. Springer, 2015.

[WN12] D. Witt Nyström. Test configurations and Okounkov bodies. Compos. Math. 148 (2012), 1736–1756.

[Xia19] M. Xia, Mabuchi geometry of big cohomology classes with prescribed singularities. arXiv:1907.07234v1.

[Zha20] K. Zhang. Valuative invariants with higher moments. arXiv:2012.04856v2.

[Zha95] S.-W Zhang. Positive line bundles on arithmetic varieties. J. Amer. Math. Soc. 8 (1995), 187–221.

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