Bi-$\mathcal{PT}$ symmetry in nonlinearly damped dynamical systems and tailoring $\mathcal{PT}$ regions with position dependent loss-gain profiles

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We investigate the remarkable role of position dependent damping in determining the parametric regions of symmetry breaking in nonlinear $\mathcal{PT}$-symmetric systems. In the two dimensional cases of such position dependent damped systems, we unveil the existence of a class of novel bi-$\mathcal{PT}$-symmetric systems which have two fold $\mathcal{PT}$ symmetries. We analyze the dynamics of these systems and show how symmetry breaking occurs, that is whether the symmetry breaking of the two $\mathcal{PT}$ symmetries occurs in pair or occurs one by one. The addition of linear damping in these nonlinearly damped systems induces competition between the two types of damping. This competition results in a $\mathcal{PT}$ phase transition in which the $\mathcal{PT}$ symmetry is broken for lower loss/gain strength and is restored by increasing the loss/gain strength. We also show that by properly designing the form of the position dependent damping, we can tailor the $\mathcal{PT}$-symmetric regions of the system.

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I. INTRODUCTION

In recent times considerable interest has been shown in investigating systems which do not show $\mathcal{P}$ and $\mathcal{T}$ symmetries separately but which exhibit a combined $\mathcal{PT}$ symmetry. These $\mathcal{PT}$-symmetric systems have several intriguing features such as power oscillations [1], absorption enhanced transmission [2], double refraction, and non-reciprocity of light propagation [1]. Thus, these systems open up novel applications in optics [1], quantum optics [3,4], solid state physics [5], metamaterials [6,7], optomechanical systems [8,9], etc. The understanding of $\mathcal{PT}$-symmetric systems as non-isolated systems with balanced loss and gain has led to the exploration of these systems in mechanics as well as in electronics. Such observations of $\mathcal{PT}$-symmetric mechanical and electronic systems provide the simplest ground to experiment on these $\mathcal{PT}$-symmetric systems [10–14].

The above oscillator based $\mathcal{PT}$-symmetric systems are generically constructed by coupling an oscillator with linear loss to an oscillator with equal amount of linear gain [11–14]. Apart from the above type of systems, there exists a class of interesting dynamical systems with position dependent damping (or position dependent loss-gain profile) where the amount of damping depends on its displacement. Consequently one can have $\mathcal{PT}$-symmetric systems even with a single degree of freedom. In this case, the systems are invariant with respect to the $\mathcal{PT}$ operation defined by $\mathcal{P}$: $x \rightarrow -x$, $\mathcal{T}$: $t \rightarrow -t$, so that $\mathcal{PT}$: $x \rightarrow -x$, $t \rightarrow -t$ which we denote as the $\mathcal{PT}$–1 operation. As the position dependent damping term is found to be a nonlinear term in the evolution equation, we call this damping as nonlinear damping for simplicity. The main aim of this paper is to investigate the dynamics and underlying novel structures in these systems in comparison with the standard ones.

The recent explorations on the damping in systems with one or more atomic-scale dimensions have unveiled that the damping present in these systems is strongly position dependent [15–17]. Ref. [15] shows that this type of damping in mechanical resonators enhances the figure of merit of the system tremendously. In particular, with this type of damping, a quality factor of 100,000 has been achieved with graphene resonators. In addition, the famous van der Pol oscillator [15–19] and the Liénard type oscillator systems [20,21] also belong to this category.

Concerning the importance of these type of nonlinearly damped systems, we here focus on the $\mathcal{PT}$-symmetric cases of this category. The Hamiltonian structure [22] and quantization [23] of some of the nonlinearly damped $\mathcal{PT}$-symmetric systems with single degree of freedom have been studied recently, which show interesting symmetry breaking in these systems.

A proper coupling of two scalar nonlinearly damped $\mathcal{PT}$-symmetric systems can yield novel bi-$\mathcal{PT}$-symmetric systems which are invariant with respect to the $\mathcal{PT}$–1 ($x \rightarrow -x$, $y \rightarrow -y$, $t \rightarrow -t$) operation as well as with the $\mathcal{PT}$–2 operation which is defined as $\mathcal{PT}$–2: $x \rightarrow -y$, $y \rightarrow -x$, $t \rightarrow -t$. In this paper, we point out that the study of bi-$\mathcal{PT}$ symmetries in such coupled nonlinear damped systems can lead to interesting novel dynamical states of $\mathcal{PT}$ symmetry preserving and breaking types, besides oscillation death and bistable states.

A universal feature of the standard $\mathcal{PT}$-symmetric systems is that the $\mathcal{PT}$ symmetry is broken by increasing the loss/gain strength and is restored by reducing it. In contrast to this behavior, recently Liang et al. [24] have observed a reverse $\mathcal{PT}$ phase transition phenomenon in a lattice model known as $\mathcal{PT}$-symmetric Aubry-Andre model [25], in which the $\mathcal{PT}$ symmetry is broken for lower loss/gain strength and is restored for higher loss/gain strength. They observed this phenomenon only when two lattice potentials that introduce loss/gain in the system are applied simultaneously (which is not observed
when a single lattice potential is present). This type of inverse $\mathcal{PT}$ phase transition arises as a result of the competition between the two lattice potentials. Similarly, Miroshnichenko et al. [25] have studied the competing effect of linear and nonlinear loss-gain profile in discrete nonlinear Schrödinger system.

From a different point of view, in the present work, we add a linear damping in addition to the nonlinear damping and study the competing effects of the linear and nonlinear damping forces. With a single nonlinear damping, our system shows $\mathcal{PT}$ symmetry breaking like the classical $\mathcal{PT}$-symmetric systems, but as soon we add the linear damping to the nonlinear damping, we observe $\mathcal{PT}$ restoration at higher loss/gain strength similar to the case of Aubry-Andre model. Importantly, we illustrate that this competition among the damping terms in addition to the position dependent nature of damping aid in tailoring the $\mathcal{PT}$ regions of the system.

The organization of the paper is as follows, in section II we discuss the loss-gain profiles of the scalar $\mathcal{PT}$-symmetric and non-$\mathcal{PT}$-symmetric nonlinearly damped systems. In section III we consider a coupled system with a simple nonlinear damping $h(x, \dot{x}) = x\dot{x}$, which is also a bi-$\mathcal{PT}$-symmetric system. In section IV, in addition to the nonlinear damping, we introduce a linear damping in the system and show the occurrence of $\mathcal{PT}$ restoration at higher values of loss/gain strength. In section V we consider a general coupled system with linear and nonlinear damping and show the tailoring of $\mathcal{PT}$ regions in the system. In section VI we summarize the results of our work. In Appendices A and B we have illustrated the $\mathcal{PT}$-symmetric systems and non-$\mathcal{PT}$-symmetric systems with a single degree of freedom, respectively. In Appendices C, D and E we have presented the eigenvalues obtained through the linear stability analysis for the systems we considered.

II. NONLINEARLY DAMPED SYSTEMS-REVISITED

To start with, we analyze the loss-gain profiles of position dependent scalar nonlinearly damped systems. For this purpose, we first consider a system which is described by the second order nonlinear differential equation

$$\dot{x} + h(x, \dot{x}) + g(x) = 0. \quad \left( \frac{d}{dt} \right)$$

(1)

Here, $h(x, \dot{x}) = f(x)\dot{x}$ is the position dependent damping which we call for simplicity as the nonlinear damping term. Also, $f(x)$ is taken as a nonconstant function in $x$. The above equation can be considered as a dynamical system on its own merit, often with a nonstandard Hamiltonian description ([22]), or as a conservative nonlinear oscillator perturbed by a nonlinear damping force $h(x, \dot{x})$ which supplies or absorbs energy at different points in the $(x, \dot{x})$ phase space,

$$\ddot{x} + g(x) = -h(x, \dot{x}) = -f(x)\dot{x}. \quad (2)$$

The kinetic and the potential energies of the unperturbed particle are given respectively by

$$T(\dot{x}) = \frac{1}{2} \dot{x}^2; \quad V(x) = \int g(x)dx. \quad (3)$$

Thus the total energy of the particle in the potential $V(x)$ when $h(x, \dot{x}) = 0$ is

$$E = \frac{1}{2} \dot{x}^2 + \int g(x)dx. \quad (4)$$

The rate of change of energy of the particle is

$$\frac{dE}{dt} = \dot{x}(\dot{x} + g(x)). \quad (5)$$

From Eq. (1), we can write

$$\frac{dE}{dt} = -\ddot{x}h(x, \dot{x}) = -f(x)\dot{x}^2. \quad (6)$$

![FIG. 1: Loss-gain profiles $\frac{dE}{dt}$ of the systems given by (a) Eq. (7), (b) Eq. (8) and (c) Eq. (9) in the $(x, \dot{x})$ space: The pink shaded regions in the figures correspond to the regions in which $\frac{dE}{dt}$ is positive (or it denotes the region in which gain is present). Similarly, the gray shaded regions denote the regions in which $\frac{dE}{dt}$ is negative.]

If the quantity $\frac{dE}{dt} < 0$ (or $\ddot{x}h(x, \dot{x}) > 0$) in a region in $(x, \dot{x})$ phase space, then the energy is withdrawn from the system for the states lying in this region and the role of $h(x, \dot{x})$ is like a damping or loss term and if $\frac{dE}{dt} > 0$ (or $\ddot{x}h(x, \dot{x}) < 0$), then in the corresponding region the effect of $h(x, \dot{x})$ is like negative-damping or gain.

The above type of nonlinearly damped systems can be classified as (i) $\mathcal{PT}$-symmetric systems and (ii) non-$\mathcal{PT}$-symmetric systems depending on the form of $h(x, \dot{x})$, whereas all linearly damped systems are always non-$\mathcal{PT}$-symmetric. Here, the $\mathcal{PT}$-symmetric systems are those
systems that are invariant under the combined operation of $\mathcal{PT}$ (and not individual operation of $\mathcal{P}$ or $\mathcal{T}$): $x \rightarrow -x$, $t \rightarrow -t$. We denote this as $\mathcal{PT}^{-1}$ symmetry (in order to distinguish it from the additional $\mathcal{PT}$ symmetry in two dimensional systems). Then $\mathcal{PT}^{-1}$ symmetric systems belonging to (a) are those systems where $h(x, \dot{x})$ is a nonlinear function in $x$, $\dot{x}$ that is odd in $x$ as well as $\dot{x}$. In this article, we focus our attention towards the systems with $h(x, \dot{x}) = f(x)\dot{x}$, where $f(x)$ and $g(x)$ in (a) are odd functions. Systems of the form (a) which do not meet this requirement are non-$\mathcal{PT}$-symmetric. These non-$\mathcal{PT}$-symmetric systems are typically of two types, (i) systems exhibiting damped oscillations and (ii) systems admitting limit cycle oscillations. In the following we present specific examples of these three cases:

1. $\mathcal{PT}$-symmetric conservative system - Modified Emden Equation (MEE) [22, 27]:

$$\ddot{x} + a \dot{x} + \beta x^3 + \omega_0^2 x = 0$$  \hspace{1cm} (7)

2. Non-$\mathcal{PT}$-symmetric damped system [28]:

$$\ddot{x} + a \dot{x} + \beta x^3 + \omega_0^2 x = 0$$  \hspace{1cm} (8)

3. Limit cycle oscillator (van der Pol oscillator) [29]:

$$\ddot{x} + (\alpha^2 - 1) \dot{x} + \omega_0^2 x = 0.$$  \hspace{1cm} (9)

The system (7) is known as the modified Emden equation and is obviously invariant under the $\mathcal{PT}^{-1}$ operation. The $\mathcal{PT}$-symmetric nature of this system [22] and its quantization [23] have been studied for the specific case $\beta = \frac{\omega_0^2}{\alpha^2}$ which admits symmetry breaking for $\lambda < 0$. The $\mathcal{PT}$ symmetry breaking of the system is demonstrated clearly in Appendix A. The systems given in Eqs. (8) and (9) are examples of non-$\mathcal{PT}$-symmetric ones, as the damping term in these cases are found to be even functions of $x$. The system (8) admits damped oscillations, while the system (9) (the famous van der Pol oscillator) is found to have self sustained oscillations which is also noted in Appendix B.

Figure 1 shows the loss–gain profiles corresponding to Eqs. (7), (8) and (9), which are obtained by substituting the corresponding forms of $f(x)$ in Eq. (6). From the loss–gain profile (shown in Fig. 1(a)) corresponding to the $\mathcal{PT}^{-1}$ symmetric case (7), we can find that we have varying loss along the positive $x$– axis and varying gain along the negative $x$– axis. The amount of gain present for $x < 0$ is balanced by the amount of loss present for $x > 0$. Then from Figs. 1(b) and 1(c), we can see that in the case of non-$\mathcal{PT}$-symmetric systems, the loss and gain will not be balanced. In the case of the non-$\mathcal{PT}$ damped oscillator (8), from Fig. 1(b) we can find that loss is present everywhere in space. In the case of limit cycle oscillator (9), from Fig. 1(c), we can find that gain exists in the region $|x| < 1$ and loss exists in the region $|x| > 1$. This clearly shows that in this case, the amount of loss present in the $(x, \dot{x})$ space is not balanced by an equal amount of gain.

From Fig. 2, we can see that in the $\mathcal{PT}$-symmetric and limit cycle oscillator cases, there exists periodic and self sustained oscillations (Figs. 2(a), 2(e)), respectively, and in the non-$\mathcal{PT}$-symmetric damped oscillator case (Fig. 2(c)), we have damped oscillations. The corresponding rates of change of energy $\frac{dE}{dt}$ profiles are shown in Figs. 2(b), 2(d) and 2(f), respectively.

Comparing the periodic oscillations (Figs. 2(a) and 2(c)) corresponding to the $\mathcal{PT}$-symmetric oscillator case (Eq. (7)) and the limit cycle oscillator case (Eq. (9)), we can find that the $\mathcal{PT}$-symmetric system takes up different paths for different initial conditions but the limit cycle oscillator for different initial conditions tends to a particular path as time $t \rightarrow \infty$. The reason is that the balanced loss–gain profile (shown in Fig. 1(a)) of $\mathcal{PT}$-symmetric system allows it to have multiple paths along which net $\frac{dE}{dt}$ is zero. But in the case of limit cycle oscillator, Fig. 1(c) shows that the loss and gain are not balanced in the
case. Thus the paths along which total $\frac{dE}{dt}$ is zero are limited in this case. Consequently the phase space of limit cycle oscillators contains isolated paths only.

Now let us consider a system of coupled nonlinear damped oscillators (for simplicity we consider a linear coupling)

$$
\ddot{x} + h_1(x, \dot{x}) + h_2(x, \dot{x}) + g(x) + \kappa y = 0, \\
\ddot{y} + h_1(y, \dot{y}) - h_2(y, \dot{y}) + g(y) + \kappa x = 0,
$$

where $h_1(x, \dot{x}) = f_1(x) \dot{x}$ and $h_2(x, \dot{x}) = f_2(x) \dot{x}$ are the two position dependent nonlinear damping terms. Here, the functions $f_1(x)$ and $f_2(x)$ are chosen to be odd and even functions in $x$, respectively, and also the function $g(x)$ is chosen as odd. Consequently, the system becomes symmetric with respect to the $\mathcal{PT}$-2 operation (which is defined as $\mathcal{PT}$-2: $x \rightarrow -y$, $y \rightarrow -x$, $t \rightarrow -t$). Now, by making $f_2(x)$ to be zero, the system is symmetric with respect to both $\mathcal{PT}$-1 and $\mathcal{PT}$-2 operations. (Here $\mathcal{PT}$-1 corresponds to the operation $x \rightarrow -x$, $y \rightarrow -y$, $t \rightarrow -t$) Thus the system is bi-$\mathcal{PT}$-symmetric in this case.

Similar to the scalar case, we can consider the above system as a system of two coupled oscillators

$$
\ddot{x} + g(x) + \kappa y = 0, \\
\ddot{y} + g(y) + \kappa x = 0,
$$

acted upon by additional external forces $h_1(x, \dot{x})$ and $h_2(x, \dot{x})$. The total energy of the system (in the absence of nonlinear damping) is given by

$$
E = \frac{1}{2} \dot{x}^2 + \int g(x)dx + \frac{1}{2} \dot{y}^2 + \int g(y)dy + \kappa xy. 
$$

The rate of change of energy in the system due to the weak nonlinear damping term as specified by Eq. (10) is given by

$$
\frac{dE}{dt} = -\dot{x}[h_1(x, \dot{x}) + h_2(x, \dot{x})] - \dot{y}[h_1(y, \dot{y}) - h_2(y, \dot{y})].
$$

The above expression shows that similar to the scalar case, the coupled system (10) also has position dependent loss-gain profile. Further, the question whether a non-standard Hamiltonian description similar to the scalar case (Appendix A) exists for (10) has not yet been answered in the literature as far as the knowledge of the authors goes, though a class of such systems has recently been identified [30, 31].

### III. A BI-$\mathcal{PT}$-SYMMETRIC SYSTEM

As a simple case of the coupled nonlinear damped system (10), we consider a system of coupled modified Emden equations (MEE)

$$
\ddot{x} + \alpha xx + \beta x^3 + \omega_0^2 x + \kappa y = 0, \\
\ddot{y} + \alpha yy + \beta y^3 + \omega_0^2 y + \kappa x = 0.
$$

Here, $\alpha$ is the nonlinear damping coefficient, $\kappa$ is the coupling strength and $\omega_0$ is the natural frequency of the system when $\omega_0^2 > 0$. However, we will also consider the case $\omega_0^2 < 0$ corresponding to the double well potential. It is obvious that the system (14) admits a bi-$\mathcal{PT}$ symmetry. (i) It is invariant under the $\mathcal{PT}$-1 symmetry: $x \rightarrow -x$, $y \rightarrow -y$ and $t \rightarrow -t$. Eq. (14) is also invariant under (ii) $\mathcal{PT}$-2 symmetry: $x \rightarrow -y$, $y \rightarrow -x$ and $t \rightarrow -t$. Note that the above two symmetries also imply the symmetry $x(t) \rightarrow y(t)$.

Eq (14) can be rewritten as

$$
\dot{x} = x_1, \\
x_1 = -\alpha xx_1 - \beta x^3 - \omega_0^2 x - \kappa y, \\
\dot{y} = y_1, \\
y_1 = -\alpha yy_1 - \beta y^3 - \omega_0^2 y - \kappa x.
$$

The above set of dynamical equations (15) admit five symmetrical equilibrium points, $e_0, e_1, e_2, e_3$ and $e_4$:

(i) The trivial equilibrium point $e_0$: $(x^*, x_1^*, y^*, y_1^*) = (0, 0, 0, 0)$.

(ii) A symmetric pair of non-zero equilibrium points $e_{1,2}$: $(x^*, x_1^*, y^*, y_1^*) = (\pm a_1^*, 0, \mp a_1^*, 0)$, where $a_1^* = \sqrt{\frac{-\kappa - \omega_0^2}{\beta}}$.

(iii) Another pair of symmetric non-zero equilibrium points $e_{3,4}$: $(x^*, x_1^*, y^*, y_1^*) = (\pm a_2^*, 0, \pm a_2^*, 0)$, where $a_2^* = \sqrt{\frac{\kappa - \omega_0^2}{\beta}}$.

Besides the above five fixed points there exist four more asymmetric fixed points which turn out to be unstable in the parametric range of our interest. So we do not consider them in this paper further.

#### A. Case: $\Omega = \omega_0^2 > 0$

In analyzing (14), we first consider the case where $\Omega = \omega_0^2 > 0$. The existence of the above mentioned equilibrium points in different regions in the parametric space for this case is indicated in Table II (for our further studies we let $\beta > 0$ in Eq. (14) or (15)).

#### I. Linear Stability Analysis

Now, to explore the regions in which $\mathcal{PT}$ symmetries are found to be broken and unbroken, we first deduce the Jacobian matrix obtained from the linear stability analysis of the above system. It is given by

$$
J = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-c_21 & -\alpha x^* & -\kappa & 0 \\
0 & 0 & 0 & 1 \\
-\kappa & 0 & c_43 & -\alpha y^*
\end{bmatrix},
$$

where $c_{21} = -\alpha x_1^* - 3\beta x^2 - \omega_0^2$, $c_{43} = -\alpha y_1^* - 3\beta y^2 - \omega_0^2$ and $(x^*, x_1^*, y^*, y_1^*)$ are the equilibrium points of (15).
The eigenvalues of the above matrix determine the dynamical behavior of the system in the neighborhood of the equilibrium points qualitatively and the results will be helpful in identifying the broken and unbroken $PT$-symmetric regions of the system. In the unbroken $PT$ region, the solution, in addition to the evolution equation, replicates the symmetry of the system, while in the symmetry broken region it does not. In order that the solution to be symmetric under $PT$ operation, it should have a non-isolated periodic nature (due to the presence of the time reversal operator $T$ in the $PT$ operator). Thus, we look for the regions of the system parameters for which the equilibrium point is neutrally stable, that is the eigenvalues of the Jacobian matrix corresponding to the equilibrium point are pure imaginary. These regions give rise to unbroken $PT$-symmetric ranges. The eigenvalues of the linear stability matrix $J$ corresponding to different equilibrium points of the system are presented in the Appendix C, where the ranges of linear stability are also discussed.

Fixing the parameters $\alpha$, $\omega_0$, $\beta$ as $\alpha = 1.0$, $\omega_0 = 1.0$ and $\beta = 1.0$, we look for the ranges of $\kappa$ for which the $PT$ symmetry is unbroken. Fig. 3 shows the real parts of the eigenvalues of the equilibrium points $e_0$, $e_{1,2}$ and $e_{3,4}$ (given in Appendix C Eqs. (C1), (C2) and (C3)) under the variation of $\kappa$. Whenever the real parts of all the eigenvalues of $J$ ($Re[\mu]$) corresponding to an equilibrium point become zero, the latter is said to be neutrally stable. On the other hand, when all $Re[\mu]$’s corresponding to an equilibrium point are less then zero, it is said to be stable, while the equilibrium point is unstable in all the other cases. Using these criteria, we can identify the following facts, as depicted in Fig. 3.

- In the region $R_1$ (denoted in Fig. 3), where $\kappa < -\omega_0^2 = -1.0$, one can see that three branches appear for $e_0$, and a single branch appears each for $e_3$ and $e_4$. Among the four eigenvalues of $e_0$ (see Eq. (C1)), two are found to be pure imaginary, while the third one has a positive real part and the other has a negative real part. Thus, in the region $R_1$, there are three branches corresponding to $e_0$. In each of the cases of $e_3$ and $e_4$, all the eigenvalues have the same real parts (as seen from Eq. (C3)).

\[ \begin{array}{|c|c|c|c|} \hline \kappa & -\omega_0^2 & -\omega_0^2 \leq \kappa \leq \omega_0^2 & \kappa > \omega_0^2 \\ \hline \Omega = \omega_0^2 > 0 & e_0, e_3, e_4 & e_0 & e_0, e_1, e_2 \\ \hline \Omega = \omega_0^2 = 0 & e_0, e_3, e_4 & e_0 & e_0, e_1, e_2 \\ \hline \Omega = \omega_0^2 < 0 & e_0, e_3, e_4 & e_0, e_1, e_2 & e_0, e_1, e_2 \\ \hline \end{array} \]

TABLE I: Symmetric equilibrium points of \((15)\) in different regions of the \((\kappa, \omega)\) parametric space with $\beta > 0$ and $\Omega = \omega_0^2 > 0$, $\omega_0^2 = 0$ and $\omega_0^2 < 0$.

FIG. 3: Linear stability of equilibrium points of \((15)\) for $\Omega = \omega_0^2 > 0$ given in Table. I. Real parts of eigenvalues of $J$ given by Eq. \((16)\) are plotted as a function of $\kappa$ for the parameters $\alpha = 1.0$, $\beta = 1.0$ and $\omega_0 = 1.0$.

FIG. 4: Unbroken $PT−1$ and $PT−2$ symmetry for $\kappa = 0.15$, $\alpha = 1.0$, $\omega = 1.0$ and $\beta = 1.0$ (in the region $R_2$). (a) and (c): Phase portraits of $x(t), \dot{x}(t)$ and $y(t), \dot{y}(t)$. (b) and (d): Phase portraits of the $PT−1$ symmetric partners $(-x(-t), \dot{x}(-t))$ and $(-y(-t), \dot{y}(-t))$. Note that Figs. (d) and (b) correspond to the phase portraits of the $PT−2$ symmetric partners $(-y(-t), \dot{y}(-t))$ and $(-x(-t), \dot{x}(-t))$. 

Thus, $e_3$ and $e_4$ have a single branch each in Fig. 3. From the values of $Re[\mu]$ in the region $R_1$, we can find that among the equilibrium points $e_0$, $e_3$ and $e_4$, only $e_3$ is found to be stable. The stabilization of $e_3$ in the region gives rise to oscillation death. Here oscillation death in a system of coupled oscillators denotes the stabilization of the system to an inhomogeneous steady state due to the interaction of oscillators in the system. Since, none of the equilibrium points have pure imaginary eigenvalues, the $\mathcal{PT}$ symmetries (both $\mathcal{PT}^{-1}$ and $\mathcal{PT}^{-2}$) are broken in the region $R_1$.

- In the region $R_2$, where $-\omega_0^2 \leq \kappa \leq \omega_0^2$ (that is region $-1 \leq \kappa \leq 1$), the equilibrium points $e_3$ and $e_4$ disappear, and $e_0$ alone exists. The eigenvalues of the equilibrium point $e_0$ in this region are found to be pure imaginary (see also Eq. (C1)). The neutral stability of $e_0$ signals that the region is an unbroken $\mathcal{PT}$ region.

- For $\kappa > \omega_0^2 = 1$, in the region $R_3$, (defined by Eq. (C5)), $e_0$ loses its stability and gives rise to two new equilibrium points $e_1$ and $e_2$. These new equilibrium points are found to be neutrally stable, which again give rise to an unbroken $\mathcal{PT}$ region.

- For values of $\kappa$ in the region $R_4$ (beyond $R_3$), all the equilibrium points $e_0$, $e_1$ and $e_2$ are found to be unstable. Thus, both the $\mathcal{PT}^{-1}$ and $\mathcal{PT}^{-2}$ symmetries are found to be broken in the region.

In short, the $\mathcal{PT}$ symmetry is unbroken for the values of $\kappa$ in the regions $R_2$ and $R_3$, but this does not confirm which $\mathcal{PT}$ symmetry is unbroken in these regions, that is whether $\mathcal{PT}^{-1}$ or $\mathcal{PT}^{-2}$ or both. For this purpose we look at the nature of solutions of the system.

2. Classification of $\mathcal{PT}^{-1}$ and $\mathcal{PT}^{-2}$ regions

As a next step, in order to see which $\mathcal{PT}$ symmetry is unbroken in different regions of the parametric space for the system (14) or (15), we have presented the phase portraits of a typical pair of $\mathcal{PT}^{-1}$ and $\mathcal{PT}^{-2}$ symmetric partners in Figs. 4 and 5. The parameters with which we have plotted Fig. 4 lie in the region $R_2$. The phase portraits of $x$ and $y$ oscillators are given respectively in Figs. 4(a) and 4(c), while the phase portraits of the $\mathcal{PT}^{-1}$ symmetric partners are given in Figs. 4(b) and 4(d), respectively. Comparing the phase portraits of the $\mathcal{PT}^{-1}$ symmetric partners (that is comparing Fig. 4(a) with Fig. 4(b) and Fig. 4(c) with 4(d)), we can find that they look alike. This indicates that $\mathcal{PT}^{-1}$ symmetry is unbroken in the region $R_2$. Figures of $\mathcal{PT}^{-2}$ symmetric partners 4(a), 4(d) and 4(b), 4(c) also indicate that they are symmetric. Thus $\mathcal{PT}^{-2}$ symmetry is also unbroken. So, in the region $R_2$ both the $\mathcal{PT}^{-1}$ and $\mathcal{PT}^{-2}$ symmetries are unbroken.

Fig. 5 shows that by increasing $\kappa > 1$, a pitchfork type symmetry breaking bifurcation occurs, where the equilibrium point $e_0$ loses its stability and the new equilibrium points $e_1$ and $e_2$ become neutrally stable. Due to this symmetry breaking bifurcation, $\mathcal{PT}^{-1}$ symmetry breaks [22]. (For more details see Appendix C). This can be observed from Fig. 5 which is plotted for a set of parametric values lying in the region $R_3$. Fig. 5(a) shows that the $x(t)$ oscillator oscillates about the point $-\sqrt{\kappa - 1} = -\sqrt{0.1} < 0$, while Fig. 5(b) shows that the $-x(-t)$ oscillator oscillates about the point $\sqrt{\kappa - 1} = \sqrt{0.1} > 0$. This indicates $\mathcal{PT}^{-1}$ symmetry is broken in the region $R_3$. But, we can find from Figs. 5(a), 5(d) and also 5(b), 5(c) that $x(t)$ and $-y(-t)$ oscillators oscillate symmetrically. This indicates that $\mathcal{PT}^{-2}$ symmetry is still unbroken.

3. Dynamics in the $(\kappa, \alpha)$ parametric space

Next, we extend our study as a function of the damping parameter $\alpha$ also. Fig. 6 shows the broken and unbroken $\mathcal{PT}$-symmetric regions corresponding to system (14) in the $(\kappa, \alpha)$ parametric space. It shows that oscillation death appears in the region $\kappa < -1$ as seen earlier in Fig. 3 (as can be seen from Eq. (C8) in Appendix C). Looking at the region $-1 \leq \kappa \leq 1$ in Fig. 6 we can observe that the coupled nonlinearly damped system (14) like the scalar case (7) (see Appendix A), does not show any symmetry breaking on increasing $\alpha$ (see Eq. (C1) in Appendix C). This is in contrast to the systems with
linear damping which show symmetry breaking when the loss/gain strength is increased [11]. As mentioned in the previous subsection, in this region (that is the region $R_2$ seen in Fig. 3), both $PT-1$ and $PT-2$ symmetries are unbroken. Increasing $\kappa$ further ($\kappa > 1$), the system shows breaking of $PT-1$ symmetry (for the values of $\kappa > 1$ or in the region $R_3$ in Fig. 3) through a pitchfork bifurcation. In this region, $PT-2$ symmetry alone is unbroken. Fig. 6 shows that the $PT-2$ symmetry is unbroken only if $\alpha$ is small (from Eq. (C6) in Appendix C) and it is broken for increased $\alpha$ (Note that this type of symmetry breaking at higher values of loss/gain strength is a universal feature of all the $PT$-symmetric systems [11]). On further increasing $\kappa$, Fig. 6 shows that the $PT$ regions with respect to $\alpha$ get reduced.

4. Rotating wave approximation

In this section, we analyze the stability of the symmetric orbits centered around $e_0$ in the region $R_2$ using the well known rotating wave approximation. We consider periodic solutions for the system in the region $R_2$ to be of the form

$$x(t) = R_1(t)e^{i\omega t} + R_1^*(t)e^{-i\omega t},$$
$$y(t) = R_2(t)e^{i\omega t} + R_2^*(t)e^{-i\omega t},$$

where $\omega = \omega_0 - \Delta \omega$, and $\Delta \omega$ is a small deviation. Here, $R_1(t)$ and $R_2(t)$ are the slowly varying amplitudes with respect to a slow time variable. Substituting (17) in (14), and by rotating wave approximation, we obtain

$$\dot{R}_1 = \frac{1}{2i\omega}(-3\beta |R_1|^2 R_1 + (\omega^2 - \omega_0^2)R_1 - \kappa R_2),$$
$$\dot{R}_2 = \frac{1}{2i\omega}(-3\beta |R_2|^2 R_2 + (\omega^2 - \omega_0^2)R_2 - \kappa R_1).$$

Now, we separate the real and imaginary parts of the equation as $R_1 = a_1 + ib_1$, $R_2 = a_2 + ib_2$. We have steady periodic solutions when $\dot{a}_i = \dot{b}_i = 0$, $i = 1, 2$. Thus, the equilibrium points of the system represent steady periodic solutions. The system has five symmetric equilibrium points representing symmetric orbits, which are $E_{0;}(0, 0, 0, 0)$, $E_{1,2;}(0, \pm b_{11}^+, 0, \mp b_{11}^+)$, $E_{3,4;}(0, \pm b_{22}^+, 0, \pm b_{22}^+)$, where $b_{11}^+ = \sqrt{\frac{\omega_0^2 - \omega^2 - \kappa^2}{\kappa^2}}$ and $b_{22}^+ = \sqrt{\frac{\omega_0^2 - \omega^2 + \kappa^2}{\kappa^2}}$. The system also has asymmetric equilibrium points, which are $E_{5,6;}(0, \pm b_{33}^+, 0, \pm \sqrt{\frac{\omega_0^2 - \omega^2}{\kappa^2}} b_{33}^+ b_{44}^+)$.

Oscillation about $e_0$ Oscillation about $e_1,2$ PT− 1 Broken PT− 2 Broken PT− 1 Unbroken PT− 2 Unbroken

FIG. 6: Unbroken and broken $PT$ regions in the parametric space of $(\kappa, \alpha)$ for $\omega = \omega_0 > 0 = 1.0$ and $\beta = 1.0$. Here the light-gray shaded region denotes the region where both $PT$ symmetries are unbroken and the dark-gray shaded region denotes unbroken $PT-2$ symmetric region. The dark gray shaded regions are denoted as bistable regions in the sense that the equilibrium points $e_1$ and $e_2$ are neutrally stable in that region. The light blue shaded regions correspond to the oscillation death regions.
neutral stability of the equilibrium point will make the oscillation with frequency $\omega$ to be modulated by a slowly varying periodic amplitude. It indicates that the system shows beats type oscillations. As the equilibrium point $E_0$ is always neutrally stable, we have stable beats type periodic oscillations in the complete region $R_2$. However, the equilibrium points $E_{1,2}$ have two of their eigenvalues as zero, and so one needs to include higher order corrections to conclusively decide about their stability.

B. Case: $\Omega = \omega_0^2 = 0$

In this case, the existence of equilibrium points for different values of $\kappa$ is demonstrated in Table 4. The eigenvalues of $J$ with respect to $e_0$ (Eq. (C1)) clearly show that it is always unstable. The equilibrium points $e_1$ and $e_2$ are found to be neutrally stable for $\kappa > 0$ and for the values of $\alpha$ specified in (C6). The equilibrium points $e_3$ or $e_4$ stabilize for $\kappa < 0$ and give rise to oscillation death.

C. Case $\Omega = \omega_0^2 \leq 0$:

Next, we wish to show the unbroken and broken $\mathcal{PT}$ regions corresponding to the system (14) with $\Omega = \omega_0^2 < 0$ or the double well potential case. The equilibrium points at different values of $\kappa$ for this case are also given in Table 4. From the table, we can note that in contrast to the previous cases, in the region $-\omega_0^2 \leq \kappa \leq \omega_0^2$, the equilibrium points $e_{3,4}$ coexist with $e_{1,2}$. From the results of the linear stability analysis of this case (where $\Omega = \omega_0^2 < 0$), we can find that the equilibrium point $e_0$ (see Eq. (C1) Appendix C) completely loses its stability. Thus, when $\Omega < 0$, as in the scalar case, $\mathcal{PT}-1$ symmetry is always broken. The symmetric pair of equilibrium points $e_1$, $e_2$ and $e_3$, $e_4$ are still found to be stable in some regions in the $(\kappa, \alpha)$ parametric space. The region in which they are found to be neutrally stable or stable is given by Eqs. (C6) and (C8). Fig. 7 shows the unbroken $\mathcal{PT}-2$ symmetric regions and the oscillation death regions for $\omega_0^2 = -1$, $\beta = 1$. From the figure, we can observe that the $\mathcal{PT}-1$ symmetry is broken everywhere in the parametric space. The unbroken $\mathcal{PT}-2$ regions appear for small values of $\alpha$ and the symmetry is broken for higher values of $\alpha$.

Interestingly, in this case, there exists a region denoted by $R_0$ in Fig. 7 in which the $\mathcal{PT}-2$ symmetric region overlaps with the oscillation death region implying that the state will be initial condition dependent here. Also, this region can give rise to hysteresis type phenomenon, where an increase in $\alpha$ causes the transition from an unbroken $\mathcal{PT}$ region to a broken $\mathcal{PT}$ region, but by decreasing $\alpha$ the $\mathcal{PT}$ symmetry of the system may not become unbroken.

IV. NONLINEAR PLUS LINEAR DAMPING

Next we wish to investigate the effect of the introduction of a linear damping on the dynamics of the nonlinearly damped system (14). For this purpose, let us introduce the linear damping terms in addition to the nonlinear damping introduced in Eq. (14). Now, the system takes the form

$$\ddot{x} + \gamma \dot{x} + \alpha x \dot{x} + \beta x^3 + \omega_0^2 x + \kappa y = 0, $$
$$\ddot{y} - \gamma \dot{y} + \alpha y \dot{y} + \beta y^3 + \omega_0^2 y + \kappa x = 0,$$

where $\gamma$ is the linear loss/gain strength. Obviously, the added linear damping term in (23) breaks the $\mathcal{PT}-1$ symmetry. Thus the system is only symmetric with respect to the $\mathcal{PT}-2$ operation. Note that the equilibrium points of this system are the same as that of (14). The stability determining Jacobian matrix in this case becomes

$$J = \begin{bmatrix}
0 & 1 & 0 & 0 \\
\alpha x_1 & -\gamma - \alpha x^* & -\kappa & 0 \\
0 & 0 & 1 & 0 \\
-\kappa & 0 & c_{43} & \gamma - \alpha y^*
\end{bmatrix},$$

where $c_{21} = -\alpha x_1^* - 3\beta x^* x^2 - \omega_0^2$, $c_{43} = -\alpha y_1^* - 3\beta y^* y^2 - \omega_0^2$. The eigenvalues of this Jacobian matrix for different equilibrium points are given in Appendix D. For simplicity, we take $\beta = 1$ for further studies. As in Sec. III, we look for $\mathcal{PT}$ regions of (23) for the cases $\Omega = \omega_0^2 > 0$ and $\Omega = \omega_0^2 \leq 0$ respectively.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig7}
\caption{Phase diagram of (14) in $(\kappa, \alpha)$ parametric space for $\Omega = \omega_0^2 < 0$. Figure is plotted for $\Omega = -1.0$, $\beta = 1.0$, which shows the regions in which $\mathcal{PT}-2$ symmetry is unbroken (dark gray shaded region) and the region where oscillation death (light blue shaded region) occurs. One can clearly note from the figure that the $\mathcal{PT}-1$ symmetry is broken everywhere. In the region denoted by $R_0$, we can find that there exists oscillations about $e_{1,2}$ and oscillation death occurs about $e_3$.}
\end{figure}
FIG. 8: Linear stability of equilibrium points of \((23)\) for \(\Omega = \omega_0^2 > 0\) given in Table. \(1\) Real parts of eigenvalues of \(J\) given by Eq. \((24)\) are plotted as a function of \(\kappa\) for the parameters \(\gamma = 0.5, \alpha = 1.0, \beta = 1.0\) and \(\omega_0 = 1.0\).

A. Case: \(\Omega = \omega_0^2 > 0\)

To begin, we look for the \(\mathcal{PT}\) regions of the system with respect to \(\kappa\) for the case \(\Omega = \omega_0^2 > 0\). By fixing all the other parameters of the system as \(\alpha = 1.0, \gamma = 0.5, \beta = 1.0\) and \(\omega_0^2 = 1.0\) in \((23)\), Fig. 8 shows the plot of the real part of eigenvalues of \(J\) corresponding to the equilibrium points \(e_0, e_1, e_2, e_3\) and \(e_4\) as \(\kappa\) is varied. It is divided into seven regions \(S_1, S_2, \ldots, S_7\) along the \(\kappa\)-axis. For the system \((23)\), \(\mathcal{PT} - 2\) symmetry alone exists and the \(\mathcal{PT}\) regions correspond to the regions in which the \(\mathcal{PT} - 2\) symmetry is unbroken. The details are as follows.

- In the region \(S_1\) of Fig. 8 where \(\kappa < -\omega_0^2 = -1.0\), we can find that among the equilibrium points \(e_0, e_3\) and \(e_4\), only \(e_3\) is found to be stable which leads to oscillation death. As none of the equilibrium points admit pure imaginary eigenvalues, the \(\mathcal{PT}\) symmetry is broken in this region.

- The region corresponding to the values of \(\kappa\) between \(-\omega_0^2 < \kappa < 0\) \((-1 < \kappa < 1)\) is now divided into three regions, namely \(S_2, S_3\) and \(S_4\). In these regions as mentioned in Table. \(1\) the equilibrium point \(e_0\) alone exists.

(a) In the region \(S_2\), where \(-\omega_0^2 < \kappa \leq -\sqrt{4\omega_0^2 - (2\omega_0^2 - \gamma^2)^2}/4\) (that is \(-1 \leq \kappa \leq -0.484\)), we can note that the equilibrium point \(e_0\) is found to be neutrally stable (which can also be seen from Eq. \((D2)\)) and gives rise to an unbroken \(\mathcal{PT}\) region.

(b) In the region \(S_3\), where \(\kappa\) takes smaller values, \(-\sqrt{4\omega_0^2 - (2\omega_0^2 - \gamma^2)^2}/4 \leq \kappa \leq \sqrt{4\omega_0^2 - (2\omega_0^2 - \gamma^2)^2}/4\) (that is \(-0.484 \leq \kappa \leq 0.484\)), we can see that the equilibrium point \(e_0\) loses its stability (can be seen also from Eq. \((D2)\)) and the \(\mathcal{PT} - 2\) symmetry is broken now. As this \(\mathcal{PT} - 2\) sym-
FIG. 10: $\mathcal{PT}$–2 symmetry in the system \([23]\) for $\kappa = 0.5$, $\gamma = 0.2$, $\alpha = 1.0$, $\omega_0 = 1.0$ and $\beta = 1.0$. (a) and (c): Phase portraits of $(x(t), \dot{x}(t))$ and $(y(t), \dot{y}(t))$. (b) and (d): Phase portraits of the $\mathcal{PT}$–1 symmetric partners, $(-x(-t), \dot{x}(-t))$ and $(-y(-t), \dot{y}(-t))$. Note that Figs. (d) and (b) correspond to the phase portraits of the $\mathcal{PT}$–2 symmetric partners $(-y(-t), \dot{y}(-t))$ and $(-x(-t), \dot{x}(-t))$.

symmetry appears because of coupling (that is, the $\mathcal{PT}$–2 symmetry disappears when $\kappa = 0$) it will not be preserved for smaller values of $\kappa$.

(c) Now increasing $\kappa$, in the region $S_1$, for $\sqrt{\frac{4\alpha^2 - (2\omega_0^2 - \gamma^2)^2}{4}} \leq \kappa < \omega_0^2$, $e_0$ again becomes neutrally stable and gives rise to another unbroken $\mathcal{PT}$ region.

- For $\kappa > \omega_0^2 = 1$, there exists three regions which are designated as $S_5$, $S_6$ and $S_7$, identified from Eq. \([D4]\). In these regions, the equilibrium points $e_0$, $e_1$ and $e_2$ are found to exist (see Table \[I\]).

(a) In the region $S_5$, $(1.0 < \kappa < 1.28)$, the equilibrium point $e_0$ is found to be unstable, but $e_1$ and $e_2$ are found to be neutrally stable (can be seen also from Eq. \([D4]\)), thus giving rise to another unbroken $\mathcal{PT}$ region.

(b) In the region $S_6$, $(1.28 < \kappa < 4.4)$, in addition to $e_0$, $e_1$ also loses its stability (can be seen also from Eq. \([D4]\)). But $e_2$ is still neutrally stable, thus the region again corresponds to an unbroken $\mathcal{PT}$ region.

(c) On further increasing $\kappa$, for $\kappa > 4.4$, in the region $S_7$, all the equilibrium points $e_0$, $e_1$ and $e_2$ become unstable. Thus $\mathcal{PT}$ is broken for higher values of $\kappa$.

Figs. \[9\]a and \[9\]b show the time series plots of $x(t)$ and $y(t)$ for values of $\kappa = 0.01$ and $\gamma = 0.2$ (these parameteric values lie in the region $S_1$). Fig. \[9\] indicates that for finite values of $\kappa$, the $x$ oscillator has damped oscillations and the $y$ oscillator has growing oscillations. As the coupling strength $\kappa$ is small, the energy values between the $x$ and $y$ variables have not been equilibrated. Thus in the region $S_5$, $\mathcal{PT}$–2 symmetry is broken. By increasing the coupling strength to $\kappa = 0.5$, Fig. \[10\] clearly indicates the presence of $\mathcal{PT}$–2 symmetry and it also shows that the $\mathcal{PT}$–1 symmetry has been broken explicitly.

For $\alpha = 1.0$, $\omega_0 = 1.0$, and $\beta = 1.0$, the broken and unbroken regions in the $(\kappa, \gamma)$ parametric space of \([23]\) are indicated in Fig. \[11\]. By comparing Fig. \[11\] with Fig. \[9\] we can find the appearance of oscillation death for the values $\kappa < -1$ as in the previous case \([14]\). But in contrast to the previous case, the oscillation death regime disappears with an increase of $\gamma$. By increasing $\kappa$, the unbroken $\mathcal{PT}$ region appears in the range $-\omega_0^2 \leq \kappa \leq \omega_0^2$ (where $\omega_0 = 1.0$). In this region by increasing $\gamma$, the system shows symmetry breaking (see Eqs. \([D1]\) in Appendix \[D\]). But in the previous case \([14]\), we cannot find this type of behavior, where the $\mathcal{PT}$ symmetry is never broken by increasing the loss/gain strength $\alpha$ (see

FIG. 11: Broken and unbroken $\mathcal{PT}$ regions corresponding to the system \([23]\) in the $(\kappa, \gamma)$ parametric space for $\Omega = \omega_0^2 > 0$, which is plotted for $\alpha = 1.0$, $\omega_0 = \beta = 1.0$. The light blue shaded region corresponds to the oscillation death region. The light and dark gray shaded regions denote the unbroken $\mathcal{PT}$–2 region. The dark gray shaded region corresponds to the bistable region in the sense that the equilibrium points $e_1$ and $e_2$ are neutrally stable in the region.
For \( \kappa > 1 \) (the region in which \( \epsilon_1 \) and \( \epsilon_2 \) appear), Fig. 11 indicates that when \( \kappa \) is smaller than \( \approx 2.2 \), the \( \mathcal{PT} \) symmetry of the system is preserved for lower values of \( \gamma \) and it is broken for higher values of \( \gamma \). Increasing \( \kappa \) beyond \( \approx 2.2 \), the \( \mathcal{PT} \) symmetry of the system is broken for lower values of \( \gamma \), and on increasing \( \gamma \) the \( \mathcal{PT} \) symmetry is restored or it becomes unbroken for the values of \( \gamma \) mentioned in Eq. (D7). On further increasing \( \gamma \), the symmetry is again broken. Generally, in the standard type of \( \mathcal{PT} \)-symmetric systems, \( \mathcal{PT} \) is unbroken for lower values of \( \gamma \) and broken for higher values of \( \gamma \). Thus, this type of \( \mathcal{PT} \) restoration with the increase of loss/gain strength is unusual compared to the general \( \mathcal{PT} \)-symmetric systems, except for the case of Aubry- Andre model with two lattice potentials \( [24, 25] \). As mentioned in the introduction, the latter model is a lattice model in which the lattice potential is applied in such a way that each element of the lattice has different amount of loss and gain that makes the loss and gain present in the lattice to be position dependent. Then, the phenomenon of \( \mathcal{PT} \) restoration at higher values of loss/gain strength appears only when two such lattice potentials are applied simultaneously. The reason for this type of \( \mathcal{PT} \) restoration is the competition between the two applied potentials which introduces loss and gain in the system \( [24] \).

Similarly, in our case if a single damping is present in the system \( (14) \), we cannot observe such \( \mathcal{PT} \) restoration at higher loss/gain strength (see Fig. 6). But when two or more types of damping present in the system, as in the case of \( [23] \) (where linear and nonlinear dampings are present in the system) we can observe this type of \( \mathcal{PT} \) restoration (see Fig. 11). The above point will be further discussed in detail in the next section, where we will also show that by properly choosing the form of nonlinear damping, we can also tailor the \( \mathcal{PT} \) regions of the system in the parametric space. Fig. 11 shows that there exists bistable regions for finite values of \( \gamma \) and by increasing the coupling strength \( \kappa \) the bistable region disappears.

### B. Rotating wave approximation

Now, we look for the stability of the periodic orbits about \( e_0 \) in the region \(-\omega_0^2 \leq \kappa \leq \omega_0^2\). As we did in the previous case \( (14) \), we find that the amplitude equations are

\[
\dot{R}_1 = \frac{1}{2\omega}(\gamma \omega R_1^3 - 3\beta |R_1|^2 R_1 + (\omega^2 - \omega_0^2)R_1 - \kappa R_2),
\]
\[
\dot{R}_2 = \frac{1}{2\omega}(\gamma \omega R_2^3 - 3\beta |R_2|^2 R_2 + (\omega^2 - \omega_0^2)R_2 - \kappa R_1).
\]

Now, separating the real and imaginary parts of the equation as \( R_1 = a_1 + ib_1 \), \( R_2 = a_2 + ib_2 \), and from the linear stability analysis of the above equation, we can find that the system has an equilibrium point \((0, 0, 0, 0)\), whose eigenvalues are

\[
\lambda = \pm \frac{1}{2\omega} \sqrt{(-\kappa^2 + \gamma^2 \omega^2) - (\omega^2 - \omega_0^2)^2} \pm 2\sqrt{c_1} \quad (26)
\]

where \( c_1 = (\kappa^2 - \gamma^2 \omega^2)(\omega^2 - \omega_0^2)^2 \). The equilibrium points are found to be neutrally stable for \( -\frac{\sqrt{2}}{2} \leq \gamma \leq \frac{\sqrt{2}}{2} \). The linear stability discussed in the previous section tells that the equilibrium point \( e_0 \) can become neutrally stable in the region given by Eq. (D3) (see Appendix D) and the above stability analysis of periodic orbits in the region shows that the oscillations are found to be stable only for the values of \( \gamma \) mentioned above.

#### C. Case: \( \Omega = \omega_0^2 \leq 0 \)

By taking \( \Omega = \omega_0^2 \leq 0 \), the equilibrium point \( e_0 \) loses its stability (see Eq. (D1)). The equilibrium points \( e_{1,2} \) and \( e_{3,4} \) alone are found to be stable and the stable regions of these equilibrium points are given in Appendix D. Similar to the previous case, we have observed a region denoted by \( S_0 \) in Fig. 12 in which depending on the initial condition either oscillations with \( \mathcal{PT} -2 \) symmetry occur or the oscillation death occurs. As in the case where \( \Omega > 0 \), here also \( \mathcal{PT} \) restoration at higher loss/gain occurs.

### V. GENERAL CASE

In this section, we consider a more general coupled \( \mathcal{PT} \)-symmetric cubic anharmonic oscillator system with nonlinear damping. Here, we take the nonlinear damping
term \( h(x, \dot{x}) \) to be of the form \( f(x) \dot{x} \) so that the equation of motion will take the form

\[
\ddot{x} + \gamma \dot{x} + (-1)^n \alpha f(x) x + \beta x^3 + \omega_0^2 x + \kappa y = 0,
\]

\[
\ddot{y} - \gamma \dot{y} + \alpha f(y) \dot{y} + \beta y^3 + \omega_0^2 y + \kappa x = 0,
\]  
(27)

where \( n = 0 \) if \( f(x) \) is an odd function and \( n = 1 \) if \( f(x) \) is even. Thus the system is \( \mathcal{PT} \)-symmetric with respect to the \( \mathcal{PT} \)-2 operation. The novel bi-\( \mathcal{PT} \)-symmetric case arises when \( f(x) \) is odd and \( \gamma = 0 \). For all forms of \( f(x) \), the equilibrium points are found to be the same as that of (14). Now through the linear stability analysis let us find the unbroken and broken \( \mathcal{PT} \)-symmetric regions. The Jacobian matrix corresponding to (27) is

\[
J = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-c_{21} & -\gamma - (-1)^n \alpha f(x^*) & -\kappa & 0 \\
0 & 0 & 0 & 1 \\
-\kappa & 0 & c_{43} & \gamma - \alpha f(y^*)
\end{bmatrix},
\]  
(28)

where \( c_{21} = -\alpha f'(x^*) x_1^3 - 3\beta x^2 - \omega_0^2 \), \( c_{43} = -\alpha f''(y^*) y_1^3 - 3\beta y^2 - \omega_0^2 \). For simplicity, we consider the case of \( \omega_0 = 1, \beta = 1 \). The eigenvalues of this Jacobian matrix corresponding to odd and even \( f(x) \) cases of the system (27) about various equilibrium points are given in Appendix E.

A. Case: \( f(x) \) is odd

Considering the case where \( f(x) \) is an odd function, in the region \(-1 \leq \kappa \leq 1 \) (see Table I), in which the equilibrium point \( e_0 \) alone exists, the corresponding eigenvalues of \( J \) are the same as in (D1). In this region, we can find that the eigenvalues do not depend on \( \alpha \) but depends on \( \gamma \) (see Eq. (D1)). The region of unbroken \( \mathcal{PT} \) symmetry is confined to

\[
-\sqrt{2 - 2\sqrt{1 - \kappa^2}} \leq \gamma \leq \sqrt{2 - 2\sqrt{1 - \kappa^2}}. 
\]  
(29)

From the above, it is clear that when \( \gamma = 0 \) the \( \mathcal{PT} \) is always unbroken for all the values of \( \alpha \) in the region \(-1 \leq \kappa \leq 1 \). This indicates that in a purely nonlinearly damped system, we cannot observe any symmetry breaking while varying the nonlinear damping strength (\( \alpha \)) in this region. By varying \( \gamma \), we observe symmetry breaking for higher values of \( |\gamma| > \sqrt{2 - 2\sqrt{1 - \kappa^2}} \).

In the region \( \kappa > 1 \), where the non-trivial equilibrium points \( e_1 \) and \( e_2 \) come into action, we will show that by properly choosing the nonlinear damping we can tailor the \( \mathcal{PT} \) regions. In this regime, for the case in which \( f(x) \) is an odd function, the unbroken \( \mathcal{PT} \) region lies within the range of \( \gamma \) specified by (see Eq. (E3) in Appendix E)

\[
\pm \alpha f(\sqrt{\kappa - 1}) - \sqrt{a_1} \leq \gamma \leq \pm \alpha f(\sqrt{\kappa - 1}) + \sqrt{a_1},
\]  
(30)

where \( a_1 = (6\kappa - 4) - 4\sqrt{(2\kappa - 1)(\kappa - 1)} \). The presence of the term \( \alpha f(\sqrt{\kappa - 1}) \) in the above equation is found to be important. Because considering the case where \( \alpha f(\sqrt{\kappa - 1}) = 0 \), the \( \mathcal{PT} \) symmetry is unbroken for lower values of \( \gamma \) specified by \( |\gamma| < \sqrt{a_1} \) and is broken for the higher values of \( \gamma \) specified by \( |\gamma| > \sqrt{a_1} \). But, in the case where \( \alpha f(\sqrt{\kappa - 1}) \neq 0 \), for the values of \( \gamma \) defined by \( 0 < |\gamma| < \alpha f(\sqrt{\kappa - 1}) - \sqrt{a_1} \), the \( \mathcal{PT} \) symmetry is broken while it is unbroken for the values of \( \gamma \) defined by (30).

Thus, here the \( \mathcal{PT} \) symmetry breaking occurs at lower values of \( \gamma \) and the restoration of symmetry occurs by increasing \( \gamma \). We can also note that the term \( \alpha f(\sqrt{\kappa - 1}) \) depends on the form of \( f(x) \), which helps in tailoring \( \mathcal{PT} \) regions of the system.

In Fig. 13 we have presented the \( \mathcal{PT} \) regions of the system for the cases \( f(x) = x^3 \) and \( f(x) = \sin x \), which clearly show that the \( \mathcal{PT} \) regions can be tailored with the systems of the type (27) by properly choosing the form of \( f(x) \). From Fig. 13(b), we can note that by choosing \( f(x) \) to be a periodic one, we can observe \( \mathcal{PT} \) revivals.

In Fig. 13(c), we have shown the \( \mathcal{PT} \) regions of the system in the \((\gamma, \alpha)\) parametric space corresponding to \( f(x) = \sin x \) case, while the figure looks qualitatively the same for \( f(x) = x^3 \). The figure indicates that increasing \( \gamma \) (or \( \alpha \)) beyond a critical value, denoted as \( \gamma_c \) (or \( \alpha_c \)), the unbroken \( \mathcal{PT} \) region appears only when \( \alpha \) (or \( \gamma \)) is also sufficiently large.

B. Case: \( f(x) \) is even

The case of even \( f(x) \) can again be divided into two sub-cases: (i) \( f(0) = 0 \) and (ii) \( f(0) \) is a nonzero constant say, 1 (For the odd \( f(x) \) case, \( f(0) = 0 \) always and so there are no subcases.)

Case (i) \( f(0) = 0 \): Considering the case of \( f(x) \) with \( f(0) = 0 \) (Example: \( f(x) = x^2 \)), in the region \(-1 \leq \kappa \leq 1 \) where the equilibrium point \( e_0 \) alone exists (see Table I), the corresponding eigenvalues of \( J \) (given in (28)) are found to be the same as in (D1) and the unbroken \( \mathcal{PT} \) regions of the system are also the same as that of (29).

Case (ii) \( f(0) = 1 \): In this case, for example \( f(x) = \cos x \) or \( e^{-x^2} \), the eigenvalues of \( J \) are different from case (i) and they are given in (E4). In contrast to the previous cases, the eigenvalues of \( J \) corresponding to \( e_0 \) are found to depend on \( \alpha \), see Eq. (E4), and the \( \mathcal{PT} \) unbroken region can be given in terms of \( \gamma \) as

\[
\alpha - \sqrt{a_2} \leq \gamma \leq \alpha + \sqrt{a_2},
\]  
(31)

where \( a_2 = 2\omega_0^2 - 4(1 - \kappa^2) \). This equation indicates that the \( \mathcal{PT} \) symmetry is found to be broken for values of \( \gamma \) outside the range specified by (31) and \( \mathcal{PT} \) symmetry becomes unbroken by choosing \( \gamma \) within the range given in (31). Thus the \( \mathcal{PT} \) symmetry is broken for lower values of \( \gamma \), \( \gamma < \alpha - \sqrt{a_2} \) and restored at higher \( \gamma \), as in Eq. (31).

As \( \mathcal{PT} \) is broken for \( \gamma < \alpha - \sqrt{a_2} \), for \( \alpha > 0 \) the \( \mathcal{PT} \) regions preferentially exist for \( \gamma > 0 \) and found to
VI. CONCLUSION

In this work, we have brought out the nature of the novel bi-$\mathcal{PT}$ symmetry of certain nonlinear systems with position dependent loss-gain profiles. We have pointed out that the $\mathcal{PT}$-symmetric cases of this type of nonlinear systems with position dependent loss-gain profile occur even with a single degree of freedom. These scalar nonlinear $\mathcal{PT}$-symmetric systems are also found to show $\mathcal{PT}$ restoration at higher loss/gain strength.
symmetry breaking. By coupling two such scalar $\mathcal{PT}$-symmetric systems in a proper way, we have shown the existence of the novel bi-$\mathcal{PT}$-symmetric systems in two dimensions. We have also illustrated the phenomenon of symmetry breaking of the two $\mathcal{PT}$ symmetries in this bi-$\mathcal{PT}$-symmetric system. When this system is acted upon by a single nonlinear damping, we observed that for smaller coupling strengths, the coupled system shows no symmetry breaking while varying nonlinear loss/gain strength, whereas the coupled $\mathcal{PT}$-symmetric system with a linear damping [11] shows symmetry breaking by increasing loss/gain strength. By strengthening the coupling, this nonlinearly damped system shows symmetry breaking for higher loss/gain strength. Then, by applying the linear damping in addition to the nonlinear damping in a competing way, our results show that as in the $\mathcal{PT}$-symmetric Aubry-Andre model, $\mathcal{PT}$ restoration at higher values of loss/gain strength occurs. The advantage of having position dependent nonlinear damping with a competing linear damping is to help to tailor the $\mathcal{PT}$ regions of the system according to the needs by properly designing the nonlinear loss and gain profile. We have also observed $\mathcal{PT}$ revivals in the systems which have loss and gain periodically in space.

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Appendix A: $\mathcal{PT}$ symmetry breaking in MEE

The system mentioned in Eq. (1), namely
\[\ddot{x} + \omega_0^2 x + 2\beta x^3 + \alpha x = 0, \quad \lambda = \omega_0^2, \quad (A1)\]
is the simplest example for $\mathcal{PT}-1$ symmetric system. Eq. (A1) is known to admit a nonstandard conservative Hamiltonian description [22] and interesting dynamical properties. In particular, the specific choice $\beta = \frac{\omega_0^2}{\sqrt{3}}$ admits isochronous properties [22] (see below) and can be even quantized in momentum space, exhibiting $\mathcal{PT}$ symmetry and broken $\mathcal{PT}$ symmetry as shown by Chithiika Ruby et al. [23] recently. Eq. (A1) can be rewritten as
\[
\begin{align*}
\dot{x} & = x_1, \\
\dot{x}_1 & = -\alpha xx_1 - \beta x^2 - \lambda x.
\end{align*}
\]
\[\quad \quad (A2)\]

Now, let us look at the $\mathcal{PT}-1$ symmetry breaking property in the above system. As noted above, the $\mathcal{PT}$ is said to be unbroken, if the solution to (A1) or (A2) also replicates the symmetry; otherwise, the symmetry is broken. For this purpose, let us analyze the dynamical behavior of the system (A2) qualitatively through a linear stability analysis. This system has a trivial equilibrium point $E_0$: $(x^*, x_1^*) = (0, 0)$ and a pair of non-trivial symmetric equilibrium points $E_{1,2}$: $(\pm \sqrt{-\frac{\omega_0^2}{3}}, 0)$ (which exist only if $\lambda < 0$ or $\beta < 0$). In our following analysis, we take $\beta > 0$ and so $E_{1,2}$ exist only for $\lambda < 0$. The Jacobian matrix corresponding to the system (A2) is given by
\[
J = \begin{bmatrix}
0 & 1 \\
-\alpha x^*_1 - 3\beta x^* - \omega_0^2 & -\alpha x^*_1
\end{bmatrix}
\quad (A3)
\]

As the $\mathcal{PT}-1$ symmetry ($\mathcal{PT}-1$: $x \rightarrow -x$, $t \rightarrow -t$) primarily requires the solutions to be of nonisolated periodic nature in the unbroken $\mathcal{PT}$-symmetric regime, we look for the parametric regions in which the eigenvalues of $J$ are pure imaginary. The eigenvalues of $J$ corresponding to the equilibrium point $E_0$ are $\mu_{1,2} = \pm i \sqrt{\lambda}$. Similarly, the eigenvalues of $J$ corresponding to $E_1$ and $E_2$ are $\mu_{1,2}^{(1)} = \frac{1}{2\sqrt{2\beta}} (\pm \sqrt{\lambda} \pm \sqrt{\lambda (\alpha^2 - 8\beta)})$, $\mu_{1,2}^{(2)} = \frac{1}{2\sqrt{2\beta}} (\pm \sqrt{\lambda} \pm \sqrt{\lambda (\alpha^2 - 8\beta)})$ (where the equilibrium points $E_{1,2}$ exist only when $\lambda < 0$).

While looking at the eigenvalues of $J$ associated with the equilibrium points of $E_0$, $E_1$ and $E_2$, we note that $E_0$ alone has pure imaginary eigenvalues for $\lambda > 0$. This is illustrated in Fig. 15 for $\alpha = 2$, $\beta = 1$, where we find that the real part of the eigenvalues of $J$ corresponding to $E_0$ vanishes for $\lambda > 0$ and for $\lambda \leq 0$ it is either zero or nonzero. It implies that $\mathcal{PT}$ symmetry is unbroken for $\lambda > 0$ and broken for $\lambda \leq 0$. It may be noted that the other two equilibrium points exist only for $\lambda \leq 0$ and the associated eigenvalues of $J$ are either real or complex and not pure imaginary. So the stability of these equilibrium points does not contribute to the existence of $\mathcal{PT}$ symmetry.

Interestingly, the system (A2) is found to be integrable [22]. The explicit solution of this system for a particular parametric choice of $\beta = \frac{\omega_0^2}{\sqrt{3}}$ has been given in [22], where the system exhibits isochronous oscillations. For this case, the explicit solution for $\lambda > 0$ is given by
\[
x(t) = \frac{A \sin(\omega_0 t + \delta)}{1 - A \frac{\alpha}{3 \omega_0} \cos(\omega_0 t + \delta)}, \quad 0 \leq A \leq \frac{3 \omega_0}{\alpha}, \quad (A4)
\]
where $A, \delta$ are constants and $\omega_0 = \sqrt{\lambda}$. Eq. (A4) shows that the system has periodic type solution for $\lambda > 0$. Now let us check whether the $\mathcal{PT}$ symmetry is unbroken or broken using the solution given in (A4). Considering
\[
-x(t) = -\frac{A \sin(-\omega_0 t + \delta)}{1 - A \frac{\alpha}{3 \omega_0} \cos(-\omega_0 t + \delta)}, \quad 0 \leq A \leq \frac{3 \omega_0}{\alpha}, \quad (A5)
\]
we can rewrite

\[ -x(-t) = \frac{A \sin(\omega_0 t + \delta')} {1 - A \frac{\omega_0} {\omega_1} \cos(\omega_0 t + \delta')}, \quad 0 \leq A \leq 3 \frac{\omega_0} {\alpha} \]  

where \( \delta' = -\delta \). By comparing Eq. (A4) with (A6), we observe that the expressions of \( x(t) \) and \( -x(-t) \) are form invariant and hence the \( \mathcal{PT} \) symmetry in this region is unbroken. The solution (A4) is valid for all values of \( \alpha \). Thus, \( \mathcal{PT} \) symmetry remains unbroken by the variation of the loss/gain parameter \( \alpha \) for \( \lambda > 0 \).

Next, considering the solution of Eq. (A1) for the values of \( \lambda \leq 0 \),

\[ x(t) = \frac{3 \sqrt{\lambda} (I_1 e^{2\sqrt{\lambda}t} - 1)} {\alpha (I_1 I_2 e^{\sqrt{\lambda}t} + (I_1 e^{2\sqrt{\lambda}t} + 1))} \]  

where \( I_1 \) and \( I_2 \) are constants. Eq. (A7) shows that the system has decaying type solution for \( \lambda \leq 0 \) and that \( x(t) \neq -x(-t) \). We conclude that the \( \mathcal{PT}^{-1} \) symmetry is spontaneously broken for \( \lambda \leq 0 \).

The above results are illustrated in Figs. 16 and 17 respectively, for \( \lambda > 0 \) and \( \lambda \leq 0 \). Our analysis demonstrates the the \( \mathcal{PT}^{-1} \) symmetry breaking phase transition can be observed even in the case of systems with single degree of freedom of the type (A1).

The quantized version of the isochronous case has a spectrum \( E_n = (n + \frac{1}{2}) \hbar \sqrt{\lambda} \) with eigenfunctions having \( \mathcal{PT} \) symmetry and has a negative energy spectrum \( E_n = -(n + \frac{1}{2}) \hbar \sqrt{\lambda} \) having broken symmetry energy eigenfunctions.

\[ x(t) = R(t)e^{i\omega_0 t} + R^*(t)e^{-i\omega_0 t} \]  

### Appendix B: Non-\( \mathcal{PT} \)-symmetric oscillator

The non-\( \mathcal{PT} \)-symmetric oscillator given in Eq. (8), shows damped oscillations as given in Fig. 18 (a). But the linear stability analysis of this system indicates a different dynamical behavior. Note that this system has the same equilibrium points as that of (A1). The eigenvalues associated with the equilibrium point \( E_0 \), namely \( \pm i \sqrt{\lambda} \), show that it has periodic oscillations. But the numerical results show that it has damped oscillations. The apparent ambiguity can be removed using its amplitude equation. We assume

\[ x(t) = R(t)e^{i\omega_0 t} + R^*(t)e^{-i\omega_0 t} \]  

FIG. 16: Unbroken \( \mathcal{PT} \) symmetry for \( \lambda > 0 \): phase portraits of the isochronous system (A1) with \( \alpha = 3.0, \beta = 1.0, \lambda = 1.0 \). Figure shows that the phase portrait \((x(t) : \dot{x}(t))\) looks alike its \( \mathcal{PT} \) partner’s phase portrait \( \mathcal{PT}^{-1}(x(t) : \dot{x}(t)) \Rightarrow (-x(-t) : \dot{x}(-t)) \).

FIG. 17: Broken \( \mathcal{PT} \) symmetry for \( \lambda \leq 0 \): phase portraits of the system (A1) with \( \alpha = 3.0, \beta = 1, \lambda = -1.0 \), where the system has decaying type solution. Figs. (a) and (b) show that the phase portrait \((x(t) : \dot{x}(t))\) does not look alike its \( \mathcal{PT} \) partner phase portrait \((-x(-t) : \dot{x}(-t))\). Figs. (c) and (d) show the temporal behavior of the system.

FIG. 15: Plot of the real part of the eigenvalues of \( J \) associated with the equilibrium point \( E_0, E_1, E_2 \) of the system (A1) for the values of \( \alpha = 2 \) and \( \beta = 1 \).
where $R(t) = r(t)e^{i\delta(t)}$, $r(t)$ and $\delta(t)$ are slowly varying amplitude and phase. By differentiating we have

$$
\dot{x}(t) = (\dot{R}(t) + i\omega_0 R(t))e^{i\omega_0 t} + c.c.,
$$

$$
\ddot{x}(t) = (\dot{R}(t) + 2i\omega_0 \dot{R}(t) - \omega_0^2 R(t))e^{i\omega_0 t} + c.c.,
$$

where $c.c.$ denotes complex conjugate. As $R(t)$ is a slowly varying quantity, $\dot{R}(t) << \omega_0 R(t)$ and $\dot{R}(t) << \omega_0^2 \dot{R}(t)$. Thus, we use approximations like

$$
\dot{x}(t) = i\omega_0 R(t)e^{i\omega_0 t} + c.c.,
$$

$$
\ddot{x}(t) = (2i\omega_0 \dot{R}(t) - \omega_0^2 R(t))e^{i\omega_0 t} + c.c.
$$

Substituting (B3) and (B1) in (8), we get for the equation for amplitude ($r(t)$),

$$
\dot{r} = -\alpha \frac{r^3(t)}{2}.
$$

By solving the above, we get

$$
r(t) = \frac{1}{\sqrt{\alpha(t-t_0)}}, \quad t_0, \text{constant}
$$

This indicates that the amplitude of oscillation decreases due to the introduced nonlinear term. This is the reason why the system in (5) has damped oscillations.

On the other hand, the amplitude equation associated with $E_0$ corresponding to MEE (A1) is found to be

$$
\dot{r} = 0.
$$

Thus, $r(t) = \text{constant}$, in the case of MEE. Thus, it has periodic oscillations with constant amplitude.

Now considering the non-$\mathcal{PT}$-symmetric limit cycle oscillator equation given in (9), we see that it has an equilibrium point $E_0$ at $(0,0)$. The associated eigenvalues are

$$
\pm \sqrt{1 - 4\omega_0^2}.
$$

This shows that the system is unstable. But the amplitude equation of the system (obtained as in the previous case) is

$$
\dot{r} = \frac{-r^3 - r}{2}
$$

indicates that $\dot{r} = 0$ for $r = 1$. Thus the system exhibits limit cycle oscillations. Fig. 18(b) shows the limit cycle oscillation of (9). Although this system has an isolated periodic oscillation, it does not mean that the system is $\mathcal{PT}$-symmetric. To make this point clear, we have plotted the phase portrait ($x(t), \dot{x}(t)$) of the van der Pol oscillator in Fig. 19(a). Then we plot the phase portrait of its $\mathcal{PT}$-symmetric partner ($\mathcal{PT}x(t), \mathcal{PT}\dot{x}(t)$) ⇒ ($-x(-t), \dot{x}(-t)$) in Fig. 19(b). Comparing the phase portraits of the $\mathcal{PT}$ partners (Fig. 19(a) with 19(b)), we find that both are not the same. Thus, there is no $\mathcal{PT}$ symmetry in the system.

**Appendix C: Eigenvalues of Eq. (14)**

In this section, we present the eigenvalues of the Jacobian matrix $J$ (given in (16)) associated with the various equilibrium points. The eigenvalues of $J$ corresponding to the system (14) for the equilibrium point $e_0$ are

$$
\mu_j^{(0)} = \pm \sqrt{\omega_0^2 + \kappa}, \quad j = 1, 2, 3, 4.
$$

The eigenvalues are found to be pure imaginary when $\omega_0^2 \leq \kappa \leq -\omega_0^2$. In this range, for all values of the nonlinear damping coefficient $\alpha$, the eigenvalues are pure imaginary. This indicates that there is no symmetry breaking while increasing $\alpha$.

Now, we consider the equilibrium points $e_{1,2}$ which exist only for $\kappa > \omega_0^2$. The eigenvalues of (16) corresponding to $e_1$ and $e_2$ are the same and they are given by

$$
\mu_j^{(1,2)} = \pm \sqrt{ \frac{b_1 \pm \sqrt{b_1^2 - b_2}}{2}}; \quad j = 1, 2, 3, 4,
$$

where,

$$
b_1 = \left(\alpha \sqrt{\frac{\kappa - \omega_0^2}{\beta}}\right)^2 - \left(6\kappa - 4\omega_0^2\right),
$$

$$
b_2 = 16(2\kappa - \omega_0^2)(\kappa - \omega_0^2).
$$
For fixed values of $\alpha$ and $\beta$, these eigenvalues are found to be pure imaginary for the values of $\kappa$ in the range
\[
\omega_0^2 < \kappa \leq \frac{\alpha^2 - 6\beta\alpha^2 - 24\beta^2 - 4\alpha\beta\sqrt{2}\beta}{(\alpha^2 - 6\beta)^2 - 32\beta^2}. \tag{C5}
\]

Similarly, for a particular value of $\kappa$ in the range $\kappa > \omega_0^2$, the range of values of $\alpha$ for which the eigenvalues will be pure imaginary is given below,
\[
-\sqrt{\frac{\beta}{\kappa - \omega_0^2}} b_3 \leq \alpha \leq \sqrt{\frac{\beta}{\kappa - \omega_0^2}} b_3 \tag{C6}
\]
where
\[
b_3 = \sqrt{6\kappa - 4\omega_0^2 - \sqrt{b_2}} \tag{C7}
\]
with the values of $\kappa > \omega_0^2$.

The eigenvalues of $J$ corresponding to the equilibrium point $e_3$ (which exists when $\kappa < \omega_0^2$) are
\[
\mu_{1,2}^{(3)} = -\alpha \sqrt{-(\kappa + \omega_0^2) \pm \sqrt{[-(\alpha^2 + 8\beta)(\kappa + \omega_0^2)]}} \tag{C8}
\]
\[
\mu_{3,1}^{(3)} = -\alpha \sqrt{-(\kappa + \omega_0^2) \pm \sqrt{-\alpha^2(\kappa + \omega_0^2) + 8\beta(2\kappa + \omega_0^2)}}
\]
\[
\frac{2\sqrt{\beta}}{\kappa - \omega_0^2} \tag{C8}
\]

We can find from the above equation that these eigenvalues can never be pure imaginary if $\alpha \neq 0$. The equilibrium point $e_3$ is found to be stable and gives rise to oscillation death when $\alpha > 0$. The eigenvalues of $J$ corresponding to $e_1$ can be obtained by simply changing $\alpha \rightarrow -\alpha$ in Eq. (C8). One can check that its eigenvalues can never be pure imaginary for $\alpha \neq 0$ and that they can become stable when $\alpha < 0$.

**Appendix D: Eigenvalues of Eq. (23)**

In this appendix, we present the eigenvalues of $J$ given in (24) for the equilibrium points of the system (23). This system has the same set of equilibrium points as that of (14). The eigenvalues of $J$ for $e_0$ are
\[
\mu_{j}^{(0)} = \pm \sqrt{-(2\omega_0^2 - \gamma^2) \pm \sqrt{(2\omega_0^2 - \gamma^2)^2 + 4(\kappa^2 - \omega_0^4)}} \tag{D1}
\]
For the values of $\gamma$ in the range $-\sqrt{2\omega_0^2} < \gamma < \sqrt{2\omega_0^2}$, the eigenvalues are easily seen to be pure imaginary only for the values of $\kappa$ in the range
\[
-\omega_0^2 < \kappa \leq \frac{4\omega_0^4 + (2\omega_0^2 - \gamma^2)^2}{4} \quad \text{and} \quad \omega_0^2 > \kappa \geq \frac{4\omega_0^4 - (2\omega_0^2 - \gamma^2)^2}{4}. \tag{D2}
\]
For a particular values of $\kappa$ in the region $-\omega_0^2 \leq \kappa \leq \omega_0^2$, $e_0$ is neutrally stable for the values of $\gamma$ defined by
\[
-\sqrt{2\omega_0^2 - 2\sqrt{\omega_0^4 - \kappa^2}} \leq \gamma \leq \sqrt{2\omega_0^2 - 2\sqrt{\omega_0^4 - \kappa^2}}. \tag{D3}
\]
From the above relations, one can see that the increase in $\gamma$ beyond this range causes symmetry breaking in the system (in the region $-\omega_0^2 \leq \kappa \leq \omega_0^2$).

Then, the eigenvalues of $J$ for the equilibrium point $e_1$ are
\[
\mu_{j}^{(1)} = \pm \sqrt{b_1 \pm \sqrt{(b_1^2 - b_2)}} \tag{D4}
\]
where
\[
b_1 = \left(\alpha \sqrt{\frac{-\omega_0^2}{\beta} + \gamma}\right)^2 - (6\kappa - 4\omega_0^2), \tag{D5}
\]
and $b_2$ is given in (C4). The eigenvalues of $\alpha \rightarrow -\alpha$ in (D5) and (D6).

Then, considering the equilibrium point $e_3$ (which exist for $\kappa \leq -\omega_0^2$), its eigenvalues are the roots of the algebraic equation
\[
\mu_{j}^{(3)} = 4\alpha \sqrt{\frac{-(\kappa + \omega_0^2)}{\beta}) \mu_{j}^{(3)} + (-\alpha^2(\kappa + \omega_0^2) - \beta) - \gamma^2}
\]
\[
-(6\kappa + 4\omega_0^2)) \mu_{j}^{(3)} + \alpha \sqrt{\frac{-(\kappa + \omega_0^2)}{\beta}} - (6\kappa + 4\omega_0^2)) \mu_{j}^{(3)}
\]
\[
+ 4(2\kappa - \omega_0^2)(\kappa - \omega_0^2) = 0. \tag{D7}
\]
As the coefficients of $\mu_{j}^{(3)}$ and $\mu_{j}^{(3)}$ are non-zero for $\alpha \neq 0$, $\beta \neq 0$, the eigenvalues of the equilibrium point cannot take pure imaginary values. Similarly, the eigenvalue equation corresponding to the equilibrium point $e_4$ can be obtained by changing $\alpha \rightarrow -\alpha$ in (D7).

**Appendix E: Eigenvalues of Eq. (27)**

Now we consider the general case of Eq. (27), where we can choose $f(x)$ to be an odd or an even function. In this section, depending on the nature of $f(x)$ (odd or even), we have presented their corresponding eigenvalues.

### 1. Case: $f(x)$ - odd

In this case, the eigenvalues of $J$ corresponding to the equilibrium point $e_0$ are found to be the same as in (D1). The eigenvalues about the equilibrium point $e_1$ and $e_2$ are
\[
\mu_{j}^{(1-2)} = \pm \sqrt{b_1^{(1-2)} \pm \sqrt{(b_1^{(1-2)} - b_2)}} \tag{E1}
\]
where

$$\tilde{b}_1^{(1)} = +\alpha f \left( \sqrt{\frac{\kappa - \omega^2}{\beta}} + \gamma \right)^2 - (6\kappa - 4\omega^2),$$

$$\tilde{b}_1^{(2)} = -\alpha f \left( \sqrt{\frac{\kappa - \omega^2}{\beta}} + \gamma \right)^2 - (6\kappa - 4\omega^2), \quad \text{(E2)}$$

and $b_2$ is as given in (C4). The regions in which the eigenvalues of $c_1$ and $c_2$ are found to be pure imaginary are given respectively by

$$-\alpha f \left( \sqrt{\frac{\kappa - \omega^2}{\beta}} \right) - b_3 \leq \gamma \leq -\alpha f \left( \sqrt{\frac{\kappa - \omega^2}{\beta}} \right) + b_3 \quad \text{and}$$

$$+\alpha f \left( \sqrt{\frac{\kappa - \omega^2}{\beta}} \right) - b_3 \leq \gamma \leq +\alpha f \left( \sqrt{\frac{\kappa - \omega^2}{\beta}} \right) + b_3 \quad \text{(E3)}$$

where $b_3$ is given in (C7).

2. Case: $f(x)$ - even

Considering the case of even $f(x)$, the eigenvalues of $c_0$ are

$$\mu_j^{(0)} = \pm \sqrt{-c \pm \sqrt{c^2 - 4(\omega^2 - \kappa^2)}}. \quad \text{(E4)}$$

where $c = (2\omega^2 - (\gamma - \alpha f(0))^2)$. The eigenvalues in [E4] are found to be same as that of $[D1]$ when $f(0) = 0$. In the case $f(0) = 1$, thus the eigenvalues given in [E4] are different from that of $[D1]$. In contrast to the previous cases (Eq. (C1) and (D1)), the eigenvalues corresponding to $c_0$ are found to depend on $\alpha$ and the region in which the eigenvalues given in [E4] take pure imaginary values is

$$\alpha - \sqrt{2\omega^2 - \sqrt{4(\omega^2 - \kappa^2)}} \leq \gamma \leq \alpha + \sqrt{2\omega^2 - \sqrt{4(\omega^2 - \kappa^2)}}. \quad \text{(E5)}$$

The eigenvalues corresponding to both $c_1$ and $c_2$ are found to be the same and they are

$$\mu_j^{(1,2)} = -\sqrt{\tilde{b}_1^{(2)} + \sqrt{(\tilde{b}_1^{(2)})^2 - b_2}}. \quad \text{(E6)}$$

The eigenvalues corresponding to both $c_1$ and $c_2$ are found to be pure imaginary only when

$$\alpha f \left( \sqrt{\frac{\kappa - \omega^2}{\beta}} \right) - b_3 \leq \gamma \leq \alpha f \left( \sqrt{\frac{\kappa - \omega^2}{\beta}} \right) + b_3. \quad \text{(E7)}$$

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