Extra Spacetime Dimensions and Unification

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Abstract

We study the effects of extra spacetime dimensions at intermediate mass scales, as expected in string theories with large-radius compactifications, and focus on the gauge and Yukawa couplings within the Minimal Supersymmetric Standard Model. We find that extra spacetime dimensions naturally lead to the appearance of grand unified theories at scales substantially below the usual GUT scale. Furthermore, we show that extra spacetime dimensions provide a natural mechanism for explaining the fermion mass hierarchy by permitting the Yukawa couplings to receive power-law corrections. We also discuss how proton-decay constraints may be addressed in this scenario, and suggest that proton-decay amplitudes may be exactly cancelled to all orders in perturbation theory as a result of new Kaluza-Klein selection rules corresponding to the extra spacetime dimensions.

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1 Introduction

The appearance of extra spacetime dimensions at high energy scales is a generic feature of string theory. Typically these extra dimensions remain compactified at the Planck scale, but it is possible for new dimensions to have an effect below the Planck scale. In particular, large-radius compactification schemes have recently been discussed in a number of theoretical and phenomenological contexts [1]. Similarly, the effects of extra dimensions below the Planck scale have played a role in understanding the strong-coupling behaviour of string theory. In the Hořava-Witten scenario [2], the gravitational coupling feels the effect of an extra (bulk) dimension, while the gauge couplings, which are confined to the boundary, remain immune. A more phenomenological point of view, although still within the confines of string theory, has been considered in Ref. [3], where the low-energy consequences of a single large (TeV-scale) dimension have been studied. Again the string models were constructed in such a way as to shield the gauge couplings from the extra dimensions. More recently, there has even been a field-theoretic proposal for extra dimensions in the millimetre range [4]. Once again, the gauge couplings were shielded from the extra dimensions.

In this paper, by contrast, we will not shield the gauge and Yukawa couplings from the effects of the extra dimensions. Rather, we will focus on the effects that these extra dimensions have on the gauge and Yukawa couplings in the Minimal Supersymmetric Standard Model (MSSM), and discuss how these effects may be exploited for phenomenological purposes.

The appearance of extra spacetime dimensions at an intermediate mass scale affects the gauge and Yukawa couplings in a nontrivial way. These effects can be calculated in purely field-theoretic terms for large radii and there is no need to invoke string theory other than to provide the underlying motivation. Formally, the appearance of an extra spacetime dimension is equivalent to the introduction of an infinite number of Kaluza-Klein states. Of course, increasing the spacetime dimensionality in field theory makes the divergences worse and leads to a loss of renormalisability. However, at any particular scale, we can assume that the contributions from the Kaluza-Klein states with masses larger than the scale of interest are decoupled from the theory, giving rise to an approximately renormalisable field theory. In this approximation we can calculate and trust the corrections to the gauge and Yukawa couplings. We will find that for a minimal set of Kaluza-Klein states, corresponding to a Kaluza-Klein tower for the gauge and Higgs bosons of the MSSM, the power-law corrections to the gauge couplings preserve gauge coupling unification, but lower the unification scale considerably. This scale can naturally be interpreted as the fundamental mass scale of a further underlying theory (perhaps even a string theory). Moreover, the finite power-law corrections to the Yukawa couplings have the right sign and magnitude to cancel the tree-level terms. This can help to explain the hierarchical structure of the fermion Yukawa couplings.
A detailed phenomenological analysis of the effects of extra spacetime dimensions will be presented in Ref. [5]. In this paper, we will concentrate on presenting the basic ideas and outline the calculation of the effects of extra dimensions. We begin in Sect. 2 with a discussion of how the Kaluza-Klein states of the gauge and Higgs bosons affect the gauge couplings. Remarkably, the unification of the gauge couplings will remain intact, and the consequences for low-energy phenomenology will be discussed. In Sect. 3 we consider the corresponding scenario for the Yukawa couplings and discuss the consequences for the fermion mass hierarchy. Our conclusions and final comments will be presented in Sect. 4.

2 Extra dimensions and gauge coupling unification

We will assume that there exist \( \delta \equiv D - 4 \) extra spacetime dimensions of fixed radius \( R \), where \( R^{-1} \) exceeds presently observable energy scales. Thus \( \mu_0 \equiv R^{-1} \) is the corresponding mass scale above which extra spacetime dimensions effectively appear. In principle, every particle state in the MSSM of mass \( m_0 \) can have an infinite tower of Kaluza-Klein states with masses

\[
m_n^2 \equiv m_0^2 + \sum_{i=1}^{\delta} \frac{n_i^2}{R^2},
\]

where each state exactly mirrors the zero-mode MSSM ground state and \( n_i \in \mathbb{Z} \) are the corresponding Kaluza-Klein excitation numbers. In the following we can safely neglect the zero mode mass \( m_0 \) in (2.1) since \( R^{-1} \) is presumed to exceed presently observable energy scales. However, in order to define consistent Kaluza-Klein masses in higher dimensions we need to introduce additional particles into the spectrum. A minimal extension is to consider a Kaluza-Klein tower for the gauge and Higgs bosons and assume that Kaluza-Klein excitations of the chiral fermions are absent. It is straightforward to identify all the states, since the massive Kaluza-Klein states fall into representations of \( N = 2 \) supersymmetry. Thus, at each Kaluza-Klein mass level \( n \), the particle content of the MSSM is augmented by an \( N = 2 \) vector supermultiplet for each gauge group, and an \( N = 2 \) hypermultiplet for the two Higgs doublets. In component form we have

\[
V = \begin{pmatrix} A^{(n)}_\mu \\ \chi^{(n)}_\lambda \\ \phi^{(n)} \end{pmatrix}, \quad H = \begin{pmatrix} H_1^{(n)}_1 \\ H_2^{(n)}_1 \\ H_1^{(n)}_2 \\ H_2^{(n)}_2 \\ \psi^{(n)}_1 \\ \psi^{(n)}_2 \end{pmatrix}
\]

where all gauge indices have been suppressed. In terms of four-dimensional \( N = 1 \) multiplets, the \( N = 2 \) vector supermultiplet corresponds to an \( N = 1 \) vector multiplet and an \( N = 1 \) chiral multiplet in the adjoint representation of the gauge group. One of the real scalar fields in the chiral multiplet becomes the longitudinal component of the massive gauge boson, while the other real scalar field and the Weyl fermion
remain in the spectrum at the massive level. The $N = 2$ hypermultiplet represents
the two massive Higgs doublets.

The fact that the massive states fall into representations of $N = 2$ supersymmetry
is strictly true for $\delta = 1, 2$. For $\delta \geq 2$, the compact coordinates in string theory
are naturally complexified; for example, the $\delta = 6$ case implies the existence
of three complex “planes”. In general orbifold models, gauge couplings can receive
corrections only if some orbifold element leaves a plane invariant; they will then
depend on the moduli (radii and angles) corresponding to the invariant plane. There
are no nontrivial orbifold elements which simultaneously leave two planes invariant.
Thus, without loss of generality we can restrict to $\delta = 1$ or 2 in the following.

Notice that in order to have consistent Kaluza-Klein gauge boson masses, we
had to introduce an $N = 1$ chiral adjoint at each massive level. If there were a
Kaluza-Klein tower for the chiral fermions as well, then we would have needed to
introduce a set of mirror fermions in order to consistently define a Dirac mass in
higher dimensions. While this is certainly possible, we will only consider Kaluza-
Klein excitations for the non-chiral sector of the MSSM. (In string language, this is
equivalent to assuming that the non-chiral states arise in the twisted sectors of the
theory. This “minimal” extension of the MSSM to higher dimensions is therefore
similar to that considered in Ref. [3].) Thus, below the scale $\mu_0$, our scenario consists
of the regular MSSM states, with the gauge couplings running in the usual logarithmic
fashion. At the scale $\mu_0$, the effects of extra spacetime dimensions compactified on
a circle of radius $R$ become significant due to the appearance of massive gauge and
Higgs bosons. For $\mu \gg \mu_0$, excitations of many Kaluza-Klein modes become possible,
and the contributions of these Kaluza-Klein states must be included in all physical
calculations.

Let us now consider the effects of these Kaluza-Klein states on the gauge cou-
plings. In ordinary four-dimensional field theory, the one-loop corrections to the
gauge couplings $g_i$ are given by

$$
\alpha_i^{-1}(\mu) = \alpha_i^{-1}(M_Z) - \frac{b_i}{2\pi} \ln \frac{\mu}{M_Z},
$$

(2.3)

where $\alpha_i \equiv g_i^2/4\pi$, and the $b_i$ are the MSSM one-loop $\beta$-function coefficients

$$(b_1, b_2, b_3) = (33/5, 1, -3).$$

(2.4)

We will take the $Z$-mass $M_Z \equiv 91.17$ GeV as an arbitrary low-energy reference
scale and at this scale (and within the $\overline{\text{MS}}$ renormalisation group scheme), the gauge
couplings are given by

$$
\begin{align*}
\alpha_Y^{-1}(M_Z)_{\overline{\text{MS}}} & \equiv 98.29 \pm 0.13 \\
\alpha_2^{-1}(M_Z)_{\overline{\text{MS}}} & \equiv 29.61 \pm 0.13 \\
\alpha_3^{-1}(M_Z)_{\overline{\text{MS}}} & \equiv 8.5 \pm 0.5.
\end{align*}
$$

(2.5)
As is well-known, an extrapolation of these low-energy couplings according to (2.3) leads to the celebrated unification relation

\[ \alpha_1(M_{\text{GUT}}) = \alpha_2(M_{\text{GUT}}) = \alpha_3(M_{\text{GUT}}) \approx \frac{1}{24} \]  

at the unification scale \( M_{\text{GUT}} \approx 2 \times 10^{16} \text{ GeV} \).

Let us now consider the effects of the extra spacetime dimensions. The usual one-loop corrections to the gauge couplings arise from the vacuum polarisation diagram where zero-mode masses are included in the loops. To compute the effects of extra dimensions we need to include the massive Kaluza-Klein states in the loops as well. While the full calculation for the MSSM can be done, for the sake of clarity we will first outline an analogous but much simpler calculation involving the Kaluza-Klein states of a single Dirac fermion charged under a \( U(1) \) gauge group. We will then generalise the results to the full MSSM.

For a single Dirac fermion with Kaluza-Klein excitations, the vacuum polarisation contribution is given by

\[ \Pi_{\mu\nu}(k^2) = - \sum_{n_i=-\infty}^{\infty} g^2 \int_0^\infty \frac{d^4q}{(2\pi)^4} \text{Tr} \left( \gamma_\mu \frac{1}{\not{q} - m_n} \gamma_\nu \frac{1}{\not{k} + \not{q} - m_n} \right) \]  

where we have used the notation

\[ \sum_{n_i=-\infty}^{\infty} \equiv \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \cdots \sum_{n_\delta=-\infty}^{\infty} \]  

(2.8)

to represent a summation over all corresponding Kaluza-Klein excitations with masses \( m_n^2 \) given in (2.1). Here \( m_0 \) is the energy of the ground state, which we will henceforth take to be zero for simplicity. The summation over the Kaluza-Klein states can be performed with the aid of the Jacobi \( \vartheta_3 \) function,

\[ \vartheta_3(\tau) \equiv \sum_{n=-\infty}^{\infty} \exp(\pi i \tau n^2) \]  

(2.9)

where \( \tau \) is a complex parameter. Using standard techniques to simplify the integral and introducing a Schwinger proper time parameter \( t \) leads to the expression

\[ \Pi(0) = \frac{g^2}{12\pi^2} \int_0^\infty \frac{dt}{t} \left\{ \vartheta_3 \left( \frac{it}{\pi R^2} \right) \right\}^4 , \]  

(2.10)

where we have used the relation \( \Pi_{\mu\nu}(k^2) = (k_\mu k_\nu - g_{\mu\nu} k^2) \Pi(k^2) \). The contribution from the infinite tower of Kaluza-Klein states is contained in the Jacobi \( \vartheta_3 \) function.

At this step, we must introduce our infrared and ultraviolet regulators, along with their corresponding cutoffs. This can simply be done by introducing upper and lower cutoffs on the \( t \)-integration:

\[ \int_0^\infty dt \rightarrow \int_{r_\Lambda^{-2}}^{r_\Lambda^{-2}} dt . \]  

(2.11)
In string theory, the issue of choosing suitable regulators has been considered in Ref. [7]; however, this choice of regulator is sufficient for our purposes and will be discussed more fully in Ref. [5]. Here Λ is our ultraviolet cutoff, μ₀ is our infrared cutoff, and the numerical coefficient r (which ultimately relates these cutoff parameters to the underlying physical mass scales) is defined as

\[ r \equiv \pi \left( \frac{X_δ}{\delta} \right)^{-2/δ} \]  

(2.12)

where the numerical factor \( X_δ \) will be discussed below. The expression (2.10) contains the complete contribution from the infinite tower of Kaluza-Klein states as a function of the radius scale \( R \). In the \( R \to 0 \) limit, we find \( \vartheta_3 \to 1 \) and thus we obtain the usual logarithmic corrections to the gauge coupling:

\[
\Pi(0) = -\frac{g^2}{6\pi^2} \ln \frac{\Lambda}{\mu_0} = \frac{g^2b}{8\pi^2} \ln \frac{\Lambda}{\mu_0} .
\]  

(2.13)

In the above expression we have identified \( b = -4/3 \) as the β-function coefficient of our single Dirac fermion.

Let us now generalise the above result to the MSSM where we have assumed that only the non-chiral states have Kaluza-Klein excitations. Since at each Kaluza-Klein massive level we have an \( N = 2 \) vector multiplet and an \( N = 2 \) hypermultiplet, the β-function coefficients multiplying the Jacobi \( \vartheta_3 \) function are

\[
(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3) = (3/5, -3, -6) .
\]  

(2.14)

Thus the gauge couplings receive corrections of the form

\[
\alpha_i^{-1}(\Lambda) = \alpha_i^{-1}(\mu_0) - \frac{b_i - \hat{b}_i}{2\pi} \ln \frac{\Lambda}{\mu_0} - \frac{\hat{b}_i}{4\pi} \int_{r\mu_0^{-2}} r \left\{ \vartheta_3 \left( \frac{it}{\pi R^2} \right) \right\}^δ dt,
\]  

(2.15)

The result (2.15) gives the exact corrections of the MSSM gauge couplings in the presence of an infinite tower of Higgs and gauge-boson Kaluza-Klein states associated with \( δ \) extra dimensions compactified on circles of radius \( R \).

In the limit \( \Lambda R \gg 1 \), the exact result (2.15) reduces to the form

\[
\alpha_i^{-1}(\Lambda) = \alpha_i^{-1}(\mu_0) - \frac{b_i - \hat{b}_i}{2\pi} \ln \frac{\Lambda}{\mu_0} - \frac{\hat{b}_i X_δ}{2\pi δ} \left[ \left( \frac{\Lambda}{\mu_0} \right)^δ - 1 \right],
\]  

(2.16)

where we have used the relation

\[
\vartheta_3 \left( \frac{it}{\pi R^2} \right) \approx R \sqrt{\frac{\pi}{t}}
\]  

(2.17)

to perform the integral in (2.15). Note that in the \( \Lambda R \gg 1 \) limit, (2.16) agrees with the result of a full string calculation since the string winding states decouple and the
effective field theory can be safely used. The expression (2.16) is exactly the result
that one would have obtained by performing all loop integrals in a \((\delta+4)\)-dimensional
spacetime. This can be seen by noting that in \(D\) spacetime dimensions, the gauge
couplings \(\tilde{g}_i\) accrue a classical mass dimension
\[\tilde{g}_i = 2 - \frac{D}{2} \implies [\tilde{\alpha}_i^{-1}] = D - 4 = \delta.\] (2.18)
However, since our extra spacetime dimensions have a fixed radius \(R\), the four- and
\(D\)-dimensional gauge couplings are related to each other via
\[\alpha_i = R^{-\delta} \tilde{\alpha}_i.\] (2.19)
Thus, in the expression (2.16), the power-law behaviour can be viewed as the “classi-
cal scaling” that we expect the gauge couplings to experience due to their enhanced
classical mass dimensions when the spacetime dimension is bigger than four.

The fact that the infinite Kaluza-Klein summation is equivalent to performing
loop integrals in flat \(D\)-dimensional spacetime highlights the nonrenormalisable na-
ture of the theory. Physical parameters now depend on the cutoff scale \(\Lambda\) and the
normalisation of this scale is parametrised by the coefficient \(X_\delta\) that appears in (2.16).
Naïvely, we would expect \(X_\delta\) to essentially be a \(D\)-dimensional phase space factor
resulting from a \(D\)-dimensional loop integral. However, this would be an incorrect
assessment because one could alternatively interpret this same factor as a normali-
sation for the cutoffs that appear in the \(D\)-dimensional integrations. This makes it
obvious that within the context of our nonrenormalisable field theory, the coefficient
\(X_\delta\) is essentially cutoff- and regulator-dependent. It turns out [5] that given the
regulator chosen in (2.11), the appropriate value of \(X_\delta\) is:
\[X_\delta = \frac{\pi^{\delta/2}}{\Gamma(1 + \delta/2)} = \frac{2\pi^{\delta/2}}{\delta \Gamma(\delta/2)}\] (2.20)
where \(\Gamma\) is the Euler gamma function. Thus, \(X_0 = 1\) (as expected), while \(X_1 = 2,
X_2 = \pi, X_3 = 4\pi/3\), and so forth. Given this choice for \(X_\delta\), it is then legitimate [5] to
interpret \(\Lambda\) as the mass scale for new physics beyond our effective nonrenormalisable
theory.

We have seen that the contribution from the infinite tower of Kaluza-Klein states
is given by the exact expression (2.15), and reduces to the expression (2.16) in the
limit \(\Lambda R \gg 1\). In practice, the exact expression (2.15) can be approximated as
follows: at any scale below \(\mu_0\), we can replace (2.15) with the usual logarithmic
running (2.3), while for any scale above \(\mu_0\) we can use the expression (2.16). If we
now match these two solutions at the scale \(\mu_0\), this yields our final result valid for all
\(\Lambda \geq \mu_0\):
\[\alpha_i^{-1}(\Lambda) = \alpha_i^{-1}(M_Z) - \frac{b_i}{2\pi} \ln \frac{\Lambda}{M_Z} + \frac{\tilde{b}_i}{2\pi} \ln \frac{\Lambda}{\mu_0} - \frac{\tilde{b}_i X_\delta}{2\pi \delta} \left[ \left( \frac{\Lambda}{\mu_0} \right)^{\delta} - 1 \right].\] (2.21)
It is shown in Ref. [5] that this is an excellent approximation to the full result given in (2.15).

We can see from (2.21) that the dependence of the gauge couplings on the scale \( \Lambda \) drastically changes the normal one-loop running of the gauge couplings. Remarkably, however, there always exists a value of \( \Lambda \) such that the gauge couplings continue to unify! Moreover, this property is robust and does not depend on the number \( \delta \) of extra spacetime dimensions or where the scale \( \mu_0 \) of new dimensions appears.

This unification is illustrated in Fig. 1, where we have shown four representative cases. For \( \mu \leq \mu_0 \), we are plotting the usual running of the four-dimensional gauge couplings. For \( \mu \geq \mu_0 \), however, we are treating \( \mu \) as the cutoff \( \Lambda \) and plotting the values of the corrected gauge couplings as functions of this cutoff. We see that above the scale \( \mu_0 \), the appearance of extra spacetime dimensions accelerates the “running” of the gauge couplings, due to the power-law corrections, and remarkably they continue to unify. This unification property is directly related to the \( \beta \)-function coefficients (2.14) of the \( N = 2 \) matter content of the theory at each Kaluza-Klein mass level. It is noteworthy to point out that had we instead included mirror fermions and constructed Kaluza-Klein states for the chiral fermions, the \( \beta \)-function coefficients \( \tilde{b}_2 \) and \( \tilde{b}_3 \) would have changed signs and magnitudes, causing these couplings to become strong very quickly.

In our case, however, the \( \beta \)-functions have the correct sign and magnitude such that the gauge couplings unify at a value \( \alpha'_{\text{GUT}} \) that remains weak. In fact, \( \alpha'_{\text{GUT}} \) is even weaker than the value \( \alpha_{\text{GUT}} \) obtained in the usual scenario. This behaviour is plotted in Fig. 3. Thus, we see that our scenario naturally predicts the emergence of a \( D \)-dimensional GUT at the new lower scale \( M'_{\text{GUT}} \) for which the unified gauge coupling is always more perturbative than in the MSSM!

The extent to which one can realistically lower the unification scale \( M'_{\text{GUT}} \) below the usual value \( M_{\text{GUT}} \approx 2 \times 10^{16} \) GeV depends upon the crucial question of proton decay. In the usual grand unification scenario, proton decay is effectively mediated at low energies by dimension-six terms in the Lagrangian which are suppressed by inverse powers of the unification scale. In supersymmetric models, dimension-five operators may also be relevant but can be suppressed. Even though lowering the unification scale makes the proton-decay problem worse, our scenario does have a compensating factor since the unified gauge coupling \( \alpha'_{\text{GUT}} \) becomes weaker. Thus, assuming that dimension-six terms are the dominant effective proton-decay operators, we find that the amplitude for proton decay in our scenario will only be enhanced by a factor

\[
\frac{\alpha'_{\text{GUT}}}{\alpha_{\text{GUT}}} \left( \frac{M_{\text{GUT}}}{M'_{\text{GUT}}} \right)^2 .
\]

Furthermore, we have slightly more freedom in lowering the unification scale because

\* We are tempted to abbreviate this \( D \)-Dimensional GUT scenario as the DDG scenario, but modesty prevents us from doing so!
Figure 1: Unification of gauge couplings in the presence of extra spacetime dimensions. We consider four representative cases: $\mu_0 = 10^5$ GeV (top left), $\mu_0 = 10^8$ GeV (top right), $\mu_0 = 10^{11}$ GeV (bottom left), and $\mu_0 = 10^{15}$ GeV (bottom right). In each case we have taken $\delta = 1$. 
Figure 2: The unified coupling $(\alpha_{\text{GUT}}')^{-1}$ as a function of the unification scale $M_{\text{GUT}}'$. This curve is independent of the number of extra spacetime dimensions. The limit of the usual four-dimensional MSSM is indicated with a dot.

the usual value $M_{\text{GUT}} \approx 2 \times 10^{16}$ GeV oversatisfies the experimental bounds. Since $M_{\text{GUT}}'$ depends on $\mu_0$ and $\delta$, we can use the experimental bounds on the proton lifetime to derive a lower bound on $\mu_0$ for any $\delta$. Specifically, we obtain $\mu_0 \gtrsim 1 \times 10^{14}$ GeV for $\delta = 1$, and $\mu_0 \gtrsim 3 \times 10^{14}$ GeV for $\delta = 2$. Thus, as long as the scale $\mu_0$ of the extra dimensions is sufficiently large, the usual proton-decay bounds can be satisfied in each case.

While our scenario for extra dimensions can be imposed at energy scales that are close to the usual unification scale near $10^{16}$ GeV, we stress that it might nevertheless be possible to tolerate lower energy scales by finding an intrinsically higher-dimensional solution to the usual proton-decay problem. In particular, if we impose Kaluza-Klein selection rules on the baryon number-violating couplings, then we could suppress proton decay. For example, if the $X$ boson and coloured Higgs triplets in an $SU(5)$ GUT had only odd Kaluza-Klein excitations, then they could not couple at tree-level to the usual MSSM states (which have $n = 0$). Thus, all perturbative proton-decay diagrams would generally vanish, and general nonrenormalisable operators would have additional suppression due to the presence of the large radius. Alternatively, we can avoid the proton-decay problem altogether by shifting the string
scale to \( M'_{\text{GUT}} \) (e.g., via the Witten scenario \([3]\)), and by recalling the fact (see Ref. \([9]\) for a review) that string theory is generally consistent with gauge coupling unification regardless of whether any GUT gauge group appears. Indeed, one could even imagine shifting the string scale all the way down to the electroweak scale \([10]\). A detailed discussion of these mechanisms will be presented in Ref. \([5]\).

3 Extra dimensions and the fermion mass hierarchy

Given that extra spacetime dimensions induce power-law corrections for the gauge couplings, it is natural to ask whether the fermion mass hierarchy might also be explained in our scenario. Unlike the usual logarithmic corrections in four-dimensional field theory, the power-law corrections that arise from extra dimensions can dramatically affect the fermion Yukawa couplings.

Let us first recall how the Yukawa couplings \( y_F \) (with \( F = e, \mu, \tau, u, d, s, c, b, t \)) run within the usual four-dimensional MSSM. If we define \( \alpha_F \equiv y_F^2/4\pi \) in analogy with the gauge couplings \( \alpha_i \), then the Yukawa coupling one-loop RGE’s in the MSSM have the form

\[
\frac{d}{d \ln \mu} \alpha_F^{-1}(\mu) = -\frac{b_F(\mu)}{2\pi} .
\]  

\[ (3.1) \]

Indeed, the only difference relative to the gauge couplings is that the one-loop \( \beta \)-function “coefficients” \( b_F(\mu) \) are not constants, but instead depend on the scale \( \mu \), since there are now many different couplings which can contribute to the one-loop \( \beta \)-function. For example, within the usual MSSM, \( b_t(\mu) \) is given by:

\[
b_t \equiv 6 + \frac{1}{\alpha_t} \left( \alpha_b + 3\alpha_u + 3\alpha_c - \frac{16}{3} \alpha_3 - 3\alpha_2 - \frac{13}{15} \alpha_1 \right) ,
\]  

\[ (3.2) \]

and each of the other \( b_F(\mu) \) has a similar form.

Let us now consider how the evolution of Yukawa couplings is modified in the presence of extra spacetime dimensions. Just as for the gauge couplings, we shall assume that a certain number \( \delta \) of extra spacetime dimensions appear at an energy scale \( \mu_0 \equiv R^{-1} \). Below the scale \( \mu_0 \), the Yukawa couplings run according to \((3.1)\). Note that the running implied by \((3.1)\) is solely a property of the four-dimensional renormalisable field theory. Above the scale \( \mu_0 \), where we have a nonrenormalisable theory, the Yukawa couplings instead receive finite one-loop corrections whose magnitudes depend upon the cutoff scale \( \Lambda \). By comparison with our prior results for the gauge couplings, the extra dimensions will induce corrections for the Yukawa couplings of the form

\[
\alpha_F^{-1}(\Lambda) = \alpha_F^{-1}(\mu_0) - \frac{\tilde{b}_F(\mu_0)}{2\pi} \ln \frac{\Lambda}{\mu_0} - \frac{\tilde{b}_F(\mu_0)}{2\pi} \frac{X_\delta}{\delta} \left[ \left( \frac{\Lambda}{\mu_0} \right)^\delta - 1 \right] .
\]  

\[ (3.3) \]

where we have matched onto the usual logarithmic running at \( \Lambda = \mu_0 \) and \( \tilde{b}_F \) represent the contributions from the states that have Kaluza-Klein excitations. The form of
the above solution is qualitatively very similar to that for the gauge couplings. The only difference is that since the \( \beta \)-function coefficients for the Yukawa couplings are not pure numbers, the \( \beta \)-functions must be evaluated at the fixed scale \( \mu_0 \) at which we are evaluating our fixed nonrenormalisable one-loop corrections. Moreover, \( X_\delta \) takes the same values as for the gauge couplings. These issues will be discussed more fully in Ref. \[5\].

The computation of the coefficients \( \tilde{b}_F \) proceeds in the usual way by considering the one-loop anomalous dimensions of the fermions and Higgs fields, with massive Kaluza-Klein states present in the loops. In our scenario, there are no contributions from the anomalous dimensions of the Higgs fields because the non-chiral Kaluza-Klein excitations fall into \( N = 2 \) representations. This is consistent with the fact that for \( N = 2 \) hypermultiplets there is no wavefunction renormalisation. Thus the only contribution can come from the anomalous dimensions of the fermions.

Na"ıvely one would expect that in the computation of the anomalous dimensions of the fermions, massive Kaluza-Klein states would not be present in the loops due to Kaluza-Klein momentum conservation. However, since the chiral fermions do not have Kaluza-Klein excitations in our scenario, the interactions between the fermions and the gauge-boson or Higgs Kaluza-Klein states occur only at orbifold fixed points where there is no Kaluza-Klein momentum conservation. Thus, in the minimal scenario, power-law corrections occur only for the fermion wavefunction renormalisation factors \( Z_F \) and \( Z_\mathcal{F} \) where

\[
\alpha_F(\Lambda) = Z_H^{-1} Z_F^{-1} Z_\mathcal{F}^{-1} \alpha_F(\mu_0) \tag{3.4}
\]

and \( Z_H \) is the Higgs field wavefunction renormalisation factor. Specifically we have

\[
Z_F = 1 - 2 \tilde{\gamma}_F(\mu_0) \frac{X_\delta}{\delta} \left[ \left( \frac{\Lambda}{\mu_0} \right)^{\delta} - 1 \right] \tag{3.5}
\]

and similarly for \( Z_\mathcal{F} \), where \( \tilde{\gamma}_F(\mu_0) \) is the anomalous dimension of the fermion \( F \) evaluated at the scale \( \mu_0 \). The wavefunction renormalisation factor \( Z_H \) receives only logarithmic corrections. If we linearise \( \text{(3.4)} \) then we obtain \( \text{(3.3)} \) with \( \tilde{b}_F(\mu_0) = 4\pi(\tilde{\gamma}_F(\mu_0) + \tilde{\gamma}_\mathcal{F}(\mu_0)) \). In the MSSM, the anomalous dimensions of the fermions are almost always dominated by the gauge couplings and we generically find that \( \tilde{\gamma}_F(\mu_0) < 0 \). Thus, for large \( \Lambda \), we obtain \( \alpha_F(\Lambda) \to 0 \), and the hierarchy between Yukawa couplings is affected only by factors of order one.

In order to obtain \( \tilde{\gamma}(\mu_0) > 0 \) we need to increase the strength of the non-gauge couplings. This can be done by introducing a four-dimensional chiral singlet field \( S \) with a superpotential term \( W_S = \lambda_S H_1 H_2 S \). Since at the massive level there is no Kaluza-Klein momentum conservation for a coupling of this form, \( Z_H \), will receive power-law corrections of the form \( \text{(3.3)} \) where \( \tilde{\gamma}_{H_i}(\mu_0) = \alpha_S/4\pi = \lambda_S^2/16\pi^2 \). Thus, we see that the \( \tilde{\gamma}_{H_i} \) are always positive and universal.
An immediate consequence of this fact is that the one-loop power-law corrections to $Z_{H_i}$ have the proper magnitudes and signs to bring all of the Yukawa couplings simultaneously to a common Landau pole scale. This can be explicitly seen in Fig. 3 where we have plotted the solution (3.4) using the universal anomalous dimension coefficients $\tilde{\gamma}_{H_i}(\mu_0)$ for $\alpha_S \simeq 1/4$. It is clear that above the scale $\mu_0$, the power-law term coming from the Kaluza-Klein states dominates the evolution, and the Yukawa couplings tend towards a common large Yukawa coupling (e.g., towards a common Landau pole defined by the equation $Z_{H_i} = 0$). Note that because the power-law corrections for each fermion are not coupled to those of the other fermions, as would have been the case for the usual renormalisation group equations, each fermion independently tends towards a Landau pole. Moreover, for appropriate values of the coupling $\lambda_S$, this Yukawa “unification” scale can naturally be associated with the scale $M'_{GUT}$ at which the gauge couplings unify.

Figure 3: The evolution of the Yukawa couplings $\alpha_{F}^{-1} = 4\pi/y_{F}^2$ within the MSSM, assuming the presence of a single extra dimension at $\mu_0 = 10^8$ GeV. We have taken $m_t = 180$ GeV and $\tan \beta = 3$ as a representative case. Note that we are plotting the Yukawa couplings on a logarithmic scale in order to display them all simultaneously. It is evident that all of the Yukawa couplings approach a common Landau pole which precisely agrees with the scale at which the gauge couplings unify.

Even though the Yukawa couplings tend towards a common Landau pole, there
still exists a hierarchy between any finite value of the Yukawa couplings if we only consider the power-law corrections. This stems from the universal coefficients $\tilde{\gamma}_H$. However, the effect of the logarithmic terms in $Z_H$ is to slightly shift the position of the Landau pole between the up- and down-type fermions. This is because $Z_{H_1}$ and $Z_{H_2}$ receive different logarithmic corrections. Thus, it is possible for up-type and down-type Yukawa couplings to pairwise cross and thereby completely eliminate the hierarchy between pairs of Yukawa couplings. It is also clear that the flavour-dependent corrections in $Z_F$ and $Z_\mathcal{F}$ do not affect the position of the Landau pole since this is controlled by $Z_H$. Therefore, the superpotential $W_S$ cannot affect the hierarchy of the fermions within the up- or down-type sector.

In order to simultaneously reduce the hierarchy for all fermions we need to introduce a flavour-dependent coupling. This can be simply achieved by introducing an MSSM singlet field $\Phi$ which couples to the Yukawa-coupling term with a generic superpotential of the form

$$W = \hat{y}_F \Phi^{n_F} \mathcal{F} H$$

(3.6)

where $\hat{y}_F$ is a dimensionful coupling and $\Phi$ has an associated Kaluza-Klein tower of massive states. Although this is reminiscent of the Froggatt-Nielsen scenario [11], we shall see that its implementation is different. Again, since only a Dirac fermion mass can be consistently defined in higher dimensions, one needs to introduce a conjugate superfield $\overline{\Phi}$ (with no couplings to the MSSM fermions) and arrange both superfields into an $N = 2$ hypermultiplet. The effect of $\Phi$ is to change the exponent of the power-law dependence for the $\beta$-function coefficients of the Yukawa couplings in a flavour-dependent way.

To see how this works, let us consider a single Yukawa coupling. Using dimensional analysis we can form an effective dimensionless Yukawa coupling $Y_F(\mu) \equiv \Lambda^{n_F} \hat{y}_F(\mu)$. At the scale $\mu_0$, the physical Yukawa coupling is $y_F(\mu_0) \equiv \hat{y}_F(\Phi)^{n_F} = Y_F(\mu_0)(\mu_0/\Lambda)^{n_F}$, where we have assumed that $\Phi$ decouples below the scale $\mu_0$. In order to see the effects of the higher dimensions we can compute the power-law corrections arising from the superpotential (3.6). The dominant power-law term comes from either $Z_F$ or $Z_\mathcal{F}$, and for a single effective dimensionless Yukawa coupling we obtain

$$\frac{1}{\alpha_{F}^{-1}(\Lambda)} \simeq \alpha_{F}^{-1}(\mu_0) - c_F \left[ \left( \frac{\Lambda}{\mu_0} \right)^{(n_F+1)\delta} - 1 \right]$$

(3.7)

where $\alpha_{F}^{-1} \equiv 4\pi/Y_F^2$ and $c_F > 0$ is a flavour-dependent constant. The power-law term comes from the $n_F$ copies of the $\Phi$ Kaluza-Klein states in $\delta$ extra dimensions and quickly drives the Yukawa coupling to a Landau pole. If the Yukawa coupling is close to its Landau pole scale, then $1/Y_F(\Lambda) \to 0$ and we obtain $Y_F^2(\mu_0) \sim (\mu_0/\Lambda)^{(n_F+1)\delta}$. Thus, $Y_F(\mu_0)$ receives an additional suppression from the extra $\delta$ dimensions, which arises from the Yukawa coupling being near a Landau pole. Including this extra suppression yields $y_F^2(\mu_0) \sim (\mu_0/\Lambda)^{\Delta_F}$, where $\Delta_F \equiv \delta + n_F(2+\delta)$. Thus, the effect of the extra spacetime dimensions is to increase the exponent of the power-law corrections.
by an amount \((n_F + 1)\delta\). In this way the hierarchy of the Yukawa couplings can be explained without the large values of \(n_F\) that are needed in the usual Froggatt-Nielsen scenario. A complete analysis including all Yukawa couplings will be presented in Ref. [5].

4 Conclusions and future prospects

The appearance of extra spacetime dimensions at high energy scales can have dramatic consequences on low-energy parameters. In particular, we have seen that the gauge and Yukawa couplings receive power-law corrections which arise from an infinite tower of Kaluza-Klein states. Alternatively, these power-law corrections may be interpreted as the classical scaling of dimensionful gauge and Yukawa couplings. Remarkably, in our scenario, we find that the gauge couplings continue to unify independently of the number of extra spacetime dimensions or the scale at which they are introduced. This leads to a \(D\)-dimensional GUT scenario in which the value of the unified gauge coupling is even more perturbative than in the MSSM. Our scenario may be safely implemented near scales of \(10^{14}\) GeV, where proton-decay constraints are satisfied. However, there exists the possibility of invoking Kaluza-Klein selection rules to forbid proton-decay processes altogether. This is an inherently higher-dimensional solution to the proton-decay problem and may even allow the appearance of new dimensions near the TeV scale. Furthermore, the Yukawa couplings receive power-law corrections of the right sign and magnitude to substantially ameliorate the Yukawa coupling hierarchy. Indeed, in our scenario, the Yukawa couplings all tend to unify at a scale which can be made to coincide with the scale of gauge coupling unification.

Our scenario clearly raises a number of intriguing questions. First, in this paper we have merely presented a general scenario by which the appearance of extra spacetime dimensions can preserve gauge coupling unification and also simultaneously ameliorate the fermion mass hierarchy. However, it would be interesting to construct an explicit model in which our mechanism is realised, and in which proton decay is suppressed as a result of Kaluza-Klein selection rules. This could be done either through field theory, or through an explicit string construction making use of large-radius compactifications and associating the string scale with our unification scale. It may also be possible to realise such constructions via suitable brane configurations.

Another closely related issue concerns the unification of the gauge couplings with the gravitational coupling. It is clear that the extra spacetime dimensions will also accelerate the “running” of the gravitational coupling, in much the same way as occurs in the Hořava-Witten scenario [3]. It is therefore possible that extra large spacetime dimensions can induce a complete unification of gauge and gravitational couplings — all without making recourse to the strong-coupling dynamics of string theory.

Finally, we remark that it may not even be necessary to assume the existence of supersymmetry in order for our scenario to achieve gauge coupling unification. This
issue will be discussed further in Ref. [5]. Moreover, supersymmetry is not as essential for solving the gauge hierarchy problem as it is within the MSSM if new spacetime dimensions populate the desert between the electroweak scale and the usual GUT scale. Thus, our scenario may make it possible to achieve gauge coupling unification, Yukawa unification, and also stabilise the Higgs mass, all without supersymmetry.

Thus, we see that that extra large spacetime dimensions are naturally consistent with an intermediate-scale grand unified theory, and may also ultimately help to explain the fermion mass hierarchy. Moreover, extra large spacetime dimensions are a natural mechanism whereby the phenomenological predictions of string theory might be shifted downwards in energy scale so that they might be more directly observed. These issues, as well as other phenomenological consequences of our scenario, will be discussed further in Ref. [5].

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