Multispike Solutions for a slightly subcritical elliptic problem with non-power nonlinearity

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Abstract. In this paper, we are concerned with the following elliptic equation
\[
\begin{aligned}
-\Delta u &= |u|^{4/(n-2)}u/[\ln(e + |u|)]^\varepsilon \text{ in } \Omega, \\
u &= 0 \text{ on } \partial\Omega,
\end{aligned}
\]
where \(\Omega\) is a smooth bounded open domain in \(\mathbb{R}^n\), \(n \geq 3\) and \(\varepsilon > 0\). Clapp et al. in Journal of Diff. Eq. (Vol 275) proved that there exists a single-peak positive solution for small \(\varepsilon\) if \(n \geq 4\).
Here we construct positive as well as changing sign solutions concentrated at several points at the same time.
Key words: critical Sobolev exponent, multispike blowing-up solution, Finite-dimensional reduction, subcritical nonlinearity.
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1 Introduction and results
Let us consider the nonlinear elliptic problem:
\[
(P_\varepsilon) \quad \begin{cases}
-\Delta u = f_\varepsilon(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]
where \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^n\), \(n \geq 3\) and
\[
f_\varepsilon(u) := \frac{|u|^{p-1}u}{[\ln(e + |u|)]^\varepsilon}, \quad \varepsilon \geq 0, \quad p = \frac{n+2}{n-2}.
\]
(1.1)
Here \(p + 1 = 2n/(n-2)\) is the critical Sobolev exponent for the embedding of \(H^1_0(\Omega)\) into \(L^{p+1}(\Omega)\). Recently, the nonlinearity \(f_\varepsilon\) has attracted a lot of attention (see for instance [7],[10],[13]). When \(\varepsilon = 0\), we get \(f_0(u) = |u|^{p-1}u\). In this case the variational problem corresponding to \((P_0)\) happens to be lacking of compactness and this is due to the noncompactness of the embedding of \(H^1_0(\Omega)\) into \(L^{p+1}(\Omega)\).
Despite that the nonlinearity $f_\varepsilon$ is very close to the critical growth, problem $(P_\varepsilon)$ is considered as a slightly subcritical one. Indeed, the nonlinearity $f_\varepsilon$ satisfies the condition
\[ \lim_{s \to \infty} \frac{f(x, s)}{|s|^{n-2}} = 0 \] uniformly with respect to $x \in \Omega$, \hspace{1cm} (1.2)

since $f_\varepsilon$ is defined by (1.1) and independent of the variable $x$. This new assumption, recently introduced by Harrabi in [10] guarantees the existence of solutions for problem $(P_\varepsilon)$. In fact, the Euler Lagrange functional associated to $(P_\varepsilon)$ satisfies Palais Smale condition and compactness is recovered thanks to (1.2). This is in same sense very similar to what happens in the subcritical regime where we usually use Ambrosetti-Rabinowitz conditions and the following subcritical polynomial growth condition
\[ f(x, s) \leq C(|s|^p + 1), \quad \text{for all } (x, s) \in \Omega \times \mathbb{R} \quad \text{for some } p \in [1, \frac{n+2}{n-2}], \]
to ensure the compactness of a bounded Palais Smale sequence. For more details of these aspects we refer to [10].

Recently, Clapp et al. constructed in [7] a single-peak solution for problem $(P_\varepsilon)$ and they proved that any $x_0$ non-degenerate critical point of the Robin function generates a family of solutions concentrating around $x_0$ as $\varepsilon$ goes to 0. In [13], the authors have analyzed the asymptotic behavior of radially symmetric solutions of $(P_\varepsilon)$ when $\Omega$ is a ball. The result of Clapp et al. [7] was a first step towards establishing the existence of blowing up solutions to problem $(P_\varepsilon)$. In this direction, our main result provides the existence of positive as well as changing sign solutions that blow up and/or blow down at different points in $\Omega$. In fact, problem $(P_\varepsilon)$ shares many aspects with the following nonlinear subcritical elliptic problem
\[ \begin{cases} -\Delta u = |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases} \] \hspace{1cm} (1.3)

where $\varepsilon > 0$. Thus one can expect the extension of existence type results obtained for (1.3) to our problem $(P_\varepsilon)$ as conjectured in [7]. But the non-power nonlinearity (1.1) gives rise to some technical difficulties. In this work, we were able to overcome these issues by some careful estimates.

Problem (1.3) has been investigated widely in the last decades, see for example [4, 5, 9, 11, 12, 14, 15, 16, 17]. In the sequel, we review some known facts related to (1.3). In [9] and [17], the asymptotic behavior of least energy solutions (hence positive) was analyzed as $\varepsilon \to 0$. It was proved that, a single peak solution blows up and concentrates at a critical point of the Robin’s function as $\varepsilon$ goes to 0. This study was completed in [4] in the case of general positive blowing up solutions. Conversely, by a finite reduction procedure, Bahri, Li and Rey proved the existence of a family of positive solutions of (1.3) concentrating at several points of $\Omega$. Later, Rey [18] extended the same results to dimension 3. Concerning changing sign solutions, Bartsch, Micheletti, and Pistoia [5] proved that the same method as Bahri-Li-Rey’s produces multipeak solutions blowing up or blowing down at different points characterized as solutions of a system of equations defined explicitly in terms of the gradients of the Green function and its regular part.

In [14], the authors proved that if $\Omega$ is symmetric with respect to the $x_1, \ldots, x_n$ axes, problem (1.3) has a sign-changing solution with the shape of a tower of bubbles with alternate signs, centered at the center of symmetry of the domain. Few years later, Musso and Pistoia [12] were able to extend such a result to a general domain. When the Laplacian operator in (1.3) is replaced by the biharmonic one, the existence of multipeak and bubble towers changing sign solutions for the counterpart of (1.3) have been studied in [6] and [8] respectively.

Going back to $(P_\varepsilon)$, the existence of multipeak solutions has not been studied yet and the main purpose of this paper is to focus on this issue. More precisely, we construct families of solutions for the equation, which blow-up positively or negatively and concentrate in $m$ different points of $\Omega$ as the parameter $\varepsilon$ goes to 0.

In order to state our main result, we introduce some notations.

The space $H^2_0(\Omega)$ is equipped with the norm $\| \cdot \|$ and its corresponding inner product $\langle \cdot, \cdot \rangle$ defined by
\[ \|u\|^2 = \int_{\Omega} |\nabla u|^2; \quad \langle u, v \rangle = \int_{\Omega} \nabla u \nabla v, \quad u, v \in H^2_0(\Omega). \]

For $a \in \Omega$ and $\lambda > 0$, let
\[ \delta_{(a, \lambda)}(y) = \frac{c_0 \lambda^{(n-2)/2}}{(1 + \lambda^2 |y-a|^2)^{(n-2)/2}}, \quad \text{where } c_0 := (n(n-2))^{(n-2)/4}. \] \hspace{1cm} (1.4)
The constant $c_0$ is chosen such that $\delta_{(a,\lambda)}$ is the family of solutions of the following problem

$$-\Delta u = u^{(n+2)/(n-2)}, \quad u > 0 \text{ in } \mathbb{R}^n. \quad (1.5)$$

Notice that the family $\delta_{(a,\lambda)}$ achieves the best Sobolev constant

$$S_n := \inf\{\|\nabla u\|^2_{L^2(\mathbb{R}^n)} \|u\|^2_{L^{(n+2)/(n-2)}(\mathbb{R}^n)} : u \not\equiv 0, \nabla u \in (L^2(\mathbb{R}^n))^n \text{ and } u \in L^{2n/(n-2)}(\mathbb{R}^n)\}. \quad (1.6)$$

We denote by $P\delta_{(a,\lambda)}$ the projection of $\delta_{(a,\lambda)}$ onto $H^1_0(\Omega)$, defined by

$$-\Delta P\delta_{(a,\lambda)} = -\Delta \delta_{(a,\lambda)} \text{ in } \Omega, \quad P\delta_{(a,\lambda)} = 0 \text{ on } \partial \Omega. \quad (1.7)$$

We will denote by $G$ the Green’s function and by $H$ its regular part, that is

$$G(x,y) = |x-y|^{2-n} - H(x,y) \quad \text{for } (x,y) \in \Omega^2,$$

and for $x \in \Omega$, $H$ satisfies

$$\begin{cases}
\Delta H(x,) = 0 & \text{in } \Omega, \\
H(x,y) = |x-y|^{2-n}, & \text{for } y \in \partial \Omega.
\end{cases}$$

Next we describe the solutions that we are looking for. Let $m$ be an integer and $(\gamma_1, \ldots, \gamma_m) \in \{-1,1\}^m$, we construct solutions of the form

$$u_\varepsilon = \sum_{i=1}^m \alpha_i \gamma_i P\delta_{(a_i,\lambda_i)} + v, \quad (1.8)$$

where $(a_1, \ldots, a_m) \in (0, +\infty)^m$, $(\lambda_1, \ldots, \lambda_m) \in (0, +\infty)^m$ and $(a_1, \ldots, a_m) \in \Omega^m$. The term $v$ has to be thought as a remainder term of lower order. Let

$$E_{(a,\lambda)} := \left\{ v \in H^1_0(\Omega) : \left( v, P\delta_{i} \right) = \left( v, \frac{\partial P\delta_{i}}{\partial \lambda_i} \right) = 0 \forall 1 \leq j \leq n, \forall 1 \leq i \leq m \right\}. \quad (1.9)$$

where $P\delta_{i} = P\delta_{(a_i,\lambda_i)}$ and $(a_i)_j$ is the $j^{th}$ component of $a_i$.

For $x = (a_1, \ldots, a_m) \in \Omega^m$ and $(\gamma_1, \ldots, \gamma_m) \in \{-1,1\}^m$, we denote by $M(x) = (m_{ij})_{1 \leq i,j \leq m}$ the matrix defined by

$$m_{ii} = H(a_i, a_i); \quad m_{ij} = -\gamma_i \gamma_j G(a_i, a_j), \quad i \neq j, \quad (1.10)$$

and by $\rho(x)$ its least eigenvalue. We also define

$$\mathbf{F}_x : (0, +\infty)^m \longrightarrow \mathbb{R} \quad \Lambda = (\Lambda_1, \ldots, \Lambda_m) \quad \mapsto \quad \frac{1}{2} \Lambda M(x)^t \Lambda - \ln \Lambda_1 \ldots \Lambda_m. \quad (1.10)$$

If $\rho(x) > 0$, $\mathbf{F}_x$ is strictly convex on $(0, +\infty)^m$, infinite on the boundary ; so $\mathbf{F}_x$ has in $(0, +\infty)^m$ a unique critical point $\Lambda(x)$, which is a minimum. On the subset of $\Omega^m$

$$\rho^+ = \{ x \in \Omega^m/\rho(x) > 0 \},$$

we define the function

$$\overline{\mathbf{F}}(x) = \mathbf{F}_x(\Lambda(x)) = \frac{m}{2} - \ln \Lambda_1(x) \ldots \Lambda_m(x), \quad (1.11)$$

whose differential is given by

$$\overline{\mathbf{F}}'(x) = \frac{1}{2} \Lambda(x) M'(x)^t \Lambda(x) = -\sum_{i=1}^m \frac{\Lambda_i'(x)}{\Lambda_i(x)}.$$

Now, we are able to state the following result:

**Theorem 1.1** Let $n \geq 4$ and $(\gamma_1, \ldots, \gamma_m) \in \{-1,1\}^m$. Let $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_m) \in \rho^+$ be a non-degenerate critical point of $\overline{\mathbf{F}}$. Then, there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, problem $(P_{\varepsilon})$ has a solution $(u_\varepsilon)$ of the form:

$$u_\varepsilon = \sum_{i=1}^m \alpha_i \varepsilon \gamma_i P\delta_{(a_i,\lambda_i,\varepsilon)} + v_\varepsilon, \quad (1.12)$$
where, as \( \varepsilon \to 0 \),
\[
\|v\| \to 0, \quad |\alpha_{i,\varepsilon} - 1| = O(\varepsilon \ln |\ln \varepsilon|), \quad \lambda_{i,\varepsilon} \frac{\varepsilon^{-2}}{\varepsilon^2} \left( \frac{\ln \varepsilon}{\varepsilon} \right)^\frac{1}{2} \to \pi \lambda_i(\pi), \quad |\alpha_{i,\varepsilon} - \pi_i| = O \left( \frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|} \right).
\]

Here, \( \pi \) is some positive constant. Moreover, we have \( |\nabla u_\varepsilon|^2 \to S_{m/2}^2 \sum_{i=1}^m \delta_{y_i} \) in \( D'(\Omega) \) when \( \varepsilon \to 0 \), where \( \delta_y \) denotes the Dirac mass at the point \( y \).

Let us remark that the solutions of \( (P_\varepsilon) \) we obtained in this paper do not converge to solutions of the limit problem \( (P_0) \) (even if \( (P_0) \) has solutions) since they converge weakly to zero as \( \varepsilon \to 0 \).

We think that the result holds true even for \( n = 3 \). The restriction on the dimension in our result comes from the bad estimate of the \( v \)-part of the solution in dimension three. To handle the case of three dimensional bounded domains, one can proceed as in [18] and that by improving the odd part of \( v \).

Our result Theorem 1.1 extends both existence results in [4] and [5] to a non-power nonlinearity.

Taking \( \gamma_i = 1 \) for each \( 1 \leq i \leq m \), a non-degenerate critical point \( \pi = (\pi_1, \ldots, \pi_m) \) of \( \bar{\Phi} \) in \( \rho^+ \) generates a family of positive solutions of \( (P_\varepsilon) \) which blow up and concentrate at \( m \)-different points. Indeed, since \( |u_\varepsilon|_{L^{p+1}} \) is small, arguing as in [4], the solution \( u_\varepsilon \) has to be positive in \( \Omega \). Such an assumption was firstly introduced in [4] to study the subcritical problem (1.3).

As in [7], the blow up rate of our solutions \( u_\varepsilon \) satisfies \( \|u_\varepsilon\|_\infty \sim \varepsilon (|\ln \varepsilon|^{-1})^{1/2} \) for both positive and changing sign solutions as \( \varepsilon \) goes to 0. Compared to multispikes solutions of (1.3), solutions given by Theorem 1.1 blow up faster. In fact, as described above, the concentration speed \( \lambda_i \)'s are of order \( (|\ln \varepsilon|^{-1})^{1/(n-2)} \) which differ from the usual power nonlinearity as stated in [4] and [5] where the \( \lambda_i \)'s are of order \( \varepsilon^{-1/(n-2)} \). Our choice will be justified by the expansion of \( \nabla I_\varepsilon(\alpha_i, \lambda_i, \partial P\delta_i/\partial \lambda_i) \) given in Proposition 2.11 where \( I_\varepsilon \) is the functional associated to \( (P_\varepsilon) \) introduced in (2.1).

The proof of our result is based on the finite reduction method introduced in [4]. This method has been widely used to study elliptic problems involving critical Sobolev exponent with small perturbations. The proofs of all existence results of blowing up or blowing down solutions as the parameter \( \varepsilon \) goes to zero, mentioned before, rely on this technique. We describe in the following our proof. Firstly, we show that given \( \underline{w} = \sum_{i=1}^m \alpha_i \gamma_i \bar{P} \delta_i \), the functional \( I_\varepsilon(\underline{w} + v) \) (introduced in (2.1)) has a unique minimum with respect to the variable \( v \in E_{\alpha,\lambda} \).

This leads to a finite dimensional reduced problem depending of the variables \( \alpha_i \), the concentration rates \( \lambda_i \) and the concentration points \( \alpha_i \). Our analysis requires a careful expansion of the gradient of the energy functional \( I_\varepsilon \). This will be developed in Section 2. We mention that we follow some ideas in [7] and we also improve some estimates and complete the analysis started in this work by taking into account the different interactions among the bubbles which depend of their respective signs.

Then by using a suitable change of variables, we obtain a system satisfied by the new variables. This system is explicitly given in terms of the geometry of the domain, namely Green and Robin functions applied to \( \pi \).

Lastly, the fact that \( \pi \) is a non-degenerate critical point of \( \bar{\Phi} \) allows us to conclude with a fixed point theorem. Finally, we mention that our analysis of the gradient which is developed in Section 2 was done in a general setting. We think that the obtained expansions will be useful to describe the blow up profile of positive and changing sign solutions, as \( \varepsilon \) goes to zero.

We also think that combining these expansions with some ideas in [12], one can construct bubble tower solutions. In fact, after the choice of suitable concentration rates and concentration points, the author in [12] applied similar finite dimensional reduction methods. We expect that similar arguments produce these kind of solutions for the problem \( (P_\varepsilon) \).

The paper is organized as follows: Section 2 is devoted to the technical framework which includes some asymptotic expansions of the gradient of the energy functional. In Section 3, we prove our main result.

Throughout this paper, we use the same \( c \) to denote various generic positive constants independent of \( \varepsilon \).
2 The Technical Framework

We introduce the general setting. For \( \varepsilon > 0 \), we define on \( H^1_0(\Omega) \) the functional

\[
I_\varepsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \int_\Omega F_\varepsilon(u), \text{ where } F_\varepsilon(s) = \int_0^s f_\varepsilon(t) dt.
\]

(2.1)

Note that each critical point of \( I_\varepsilon \) is a solution of \( (P_\varepsilon) \). Since our aim is to construct solutions of the form (1.8), we give the expansion of the gradient of \( I_\varepsilon \) for \( u = \sum_{i=1}^m \alpha_i \gamma_i P\delta_{I(a_\varepsilon, \lambda)} \) + \( v \) i.e. \( u \) belongs to a neighborhood of potential concentration sets. To do so we need some preliminary results. We start with [16, Proposition 1].

**Proposition 2.1** Let \( a \in \Omega \) and \( \lambda > 0 \) such that \( \lambda d := \lambda d(a, \partial \Omega) \) is large enough. For \( \varphi(a, \lambda) = \delta(a, \lambda) - P\delta(a, \lambda) \), we have the following estimates

\[
\begin{align*}
(a) \quad & 0 \leq \varphi(a, \lambda) \leq \delta(a, \lambda), \\
(b) \quad & \varphi(a, \lambda) = c_0 \frac{H(a, \lambda)}{\lambda^{n-2}} + f(a, \lambda),
\end{align*}
\]

where \( c_0 \) is defined in (1.4) and \( f(a, \lambda) \) satisfies

\[
\begin{align*}
f(a, \lambda) &= O\left(\frac{1}{\lambda^{n+2} d^n}\right), \\
\frac{1}{\lambda} \frac{\partial f(a, \lambda)}{\partial \lambda} &= O\left(\frac{1}{\lambda^{n+1} d^n}\right).
\end{align*}
\]

(c) \( |\varphi(a, \lambda)|_{2n/(n-2)} = O\left(\frac{1}{(\lambda d)^{(n-2)/2}}\right) \), \( \lambda \frac{\partial |\varphi(a, \lambda)|_{2n/(n-2)}}{\partial \lambda} = O\left(\frac{1}{(\lambda d)^{n/2}}\right) \), \( \|\varphi(a, \lambda)\| \leq \frac{1}{(\lambda d)^{(n-2)/2}} \), \( \lambda \frac{\partial \varphi(a, \lambda)}{\partial a} \) \( 2n/(n-2) \) = \( O\left(\frac{1}{(\lambda d)^n}\right) \),

where \( \|\cdot\| \) denotes the usual norm in \( L^q(\Omega) \) for each \( 1 \leq q \leq \infty \).

We also introduce the following lemmas which are used in several works. For the proof, we refer to [2].

**Lemma 2.2** Let \( a \in \Omega \) and \( \lambda > 0 \) be such that \( \lambda d := \lambda d(a, \partial \Omega) \) is very large. There hold

\[
\begin{align*}
\|P\delta(a, \lambda)\|^2 &= S_{n/2} - c_1 \frac{H(a, a)}{\lambda^{n-2}} + O\left(\frac{\ln(\lambda d)}{(\lambda d)^n}\right), \\
\left\langle P\delta(a, \lambda), \lambda \frac{\partial P\delta(a, \lambda)}{\partial \lambda} \right\rangle &= \frac{n-2}{2} c_1 \frac{H(a, a)}{\lambda^{n-2}} + O\left(\frac{\ln(\lambda d)}{(\lambda d)^n}\right), \\
\left\langle P\delta(a, \lambda), \lambda \frac{\partial P\delta(a, \lambda)}{\partial a} \right\rangle &= -c_1 \frac{\partial H(a, a)}{\lambda^{n-4}} + O\left(\frac{\ln(\lambda d)}{(\lambda d)^{n+1}}\right),
\end{align*}
\]

where \( \partial H/\partial a \) denotes the partial derivative of \( H \) with respect to the first variable, \( S_n \) is introduced in (1.6) and

\[
\tau_1 := c_0 \int_{\mathbb{R}^n} \frac{1}{(1 + |x|^2)^{(n+2)/2}} dx.
\]

Note that the constant \( S_n \) satisfies

\[
S_{n/2} = c_{2n} \int_{\mathbb{R}^n} \frac{1}{(1 + |x|^2)^n} dx.
\]

**Lemma 2.3** Let \( a \in \Omega \) and \( \lambda > 0 \) be such that \( \lambda d := \lambda d(a, \partial \Omega) \) is very large. There hold

\[
\begin{align*}
\int_\Omega P\delta^{n+1} &= S_{n/2} - \frac{2n}{n-2} c_1 \frac{H(a, a)}{\lambda^{n-2}} + O\left(\frac{\ln(\lambda d)}{(\lambda d)^n}\right) + (if \ n = 3) O\left(\frac{1}{(\lambda d)^2}\right), \\
\int_\Omega P\delta^{n+1} \lambda \frac{\partial P\delta}{\partial \lambda} &= 2 \left\langle P\delta(a, \lambda), \lambda \frac{\partial P\delta(a, \lambda)}{\partial \lambda} \right\rangle + O\left(\frac{\ln(\lambda d)}{(\lambda d)^n}\right) + (if \ n = 3) O\left(\frac{1}{(\lambda d)^2}\right), \\
\int_\Omega P\delta^{n+1} \lambda \frac{\partial P\delta}{\partial a} &= 2 \left\langle P\delta(a, \lambda), \frac{1}{\lambda} \frac{\partial P\delta(a, \lambda)}{\partial a} \right\rangle + O\left(\frac{\ln(\lambda d)}{(\lambda d)^n}\right).
\end{align*}
\]
In the sequel, we denote
\[ \varepsilon_{ij} := \left( \frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2 \right)^{(2-n)/2}. \] (2.3)

Note that, the variable \( \varepsilon_{ij} \) comes from the scalar product
\[ \int_{\mathbb{R}^n} \nabla \delta_{(a_i, \lambda_i)} \cdot \nabla \delta_{(a_j, \lambda_j)} = \int_{\mathbb{R}^n} \delta_{(a_i, \lambda_i)} \delta_{(a_j, \lambda_j)} = O(\varepsilon_{ij}) \] for \( i \neq j. \) (2.4)
(see [2] page 4). This condition means that the "interaction effect" between the \( P \delta_{(a_i, \lambda_i)} \)'s is negligible. For simplicity we shall write \( \delta_i \) for \( \delta_{(a_i, \lambda_i)} \) and \( P \delta_i \) for \( P \delta_{(a_i, \lambda_i)} \).

**Lemma 2.4** Let \( a_i, a_j \in \Omega \) and \( \lambda_i, \lambda_j > 0 \) be such that \( \lambda_k d_k := \lambda_k d(a_k, \partial \Omega) \) is very large for \( k = i, j \) and \( \varepsilon_{ij} \) is very small. There hold
\[ \langle P \delta_i, P \delta_j \rangle = \frac{\varepsilon_{ij}}{\lambda_j} + \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{(2-n)/2}} + O(R_1), \]
\[ \lambda_i \partial \delta_i \bigg| \partial \partial_a \bigg| \lambda_i = \frac{\varepsilon_{ij}}{\lambda_i} \left( \lambda_i \partial \delta_i \bigg| \partial \partial_a \bigg| \lambda_i = \frac{1}{\lambda_i \lambda_j}ight) + O(R_1), \]
\[ \langle P \delta_j, \frac{1}{\lambda_i} \partial \delta_i \partial a_i \rangle = \left( \langle P \delta_i, \frac{1}{\lambda_i} \partial \delta_i \partial a_i \rangle + O(R_1) \right) + \left( \frac{\varepsilon_{ij}}{\lambda_i} \left( \lambda_i \partial \delta_i \bigg| \partial \partial_a \bigg| \lambda_i = \frac{1}{\lambda_i \lambda_j} \right) + O(R_2) \right), \]
where \( \frac{\partial}{\partial a} \) (resp. \( \frac{\partial}{\partial \lambda} \)) denotes the derivative with respect to the first (resp. second) variable of the function \( (a, b) \rightarrow H(a, b), \)
\[ R_1 = \sum_{k=i, j} \ln(\lambda_k d_k) (\lambda_k d_k)^n + \varepsilon_{ij} \frac{n+1}{n} \ln(\varepsilon_{ij}^{-1}) \] and \( R_2 = \sum_{k=i, j} \ln(\lambda_k d_k) (\lambda_k d_k)^n + \lambda_i |a_i - a_j| \frac{n+1}{n} \).

**Lemma 2.5** Let \( a_i, a_j \in \Omega \) and \( \lambda_i, \lambda_j > 0 \) be such that \( \lambda_k d_k := \lambda_k d(a_k, \partial \Omega) \) is very large for \( k = i, j \) and \( \varepsilon_{ij} \) is very small. There hold
\[ \int_{\Omega} \partial P \delta_i \partial a_i = \langle P \delta_i, P \delta_i \rangle + O(R_1) + (if n = 3) \left( \frac{\varepsilon_{ij}^2}{\lambda_i} \ln(\varepsilon_{ij}^{-1}) \right)^2 + \frac{1}{\lambda_i \lambda_j \lambda_i}, \]
\[ \int_{\Omega} \partial P \delta_i \partial a_i = \langle P \delta_i, P \delta_i \rangle + O(R_1) + (if n = 3) \left( \frac{\varepsilon_{ij}^2}{\lambda_i} \ln(\varepsilon_{ij}^{-1}) \right)^2 + \frac{1}{\lambda_i \lambda_j \lambda_i}, \]
\[ \int_{\Omega} \partial P \delta_i \partial a_i = \langle P \delta_i, P \delta_i \rangle + O(R_1) + (if n = 3) \left( \frac{\varepsilon_{ij}^2}{\lambda_i} \ln(\varepsilon_{ij}^{-1}) \right)^2 + \frac{1}{\lambda_i \lambda_j \lambda_i}, \]
\[ \int_{\Omega} \partial P \delta_i \partial a_i = \langle P \delta_i, P \delta_i \rangle + O(R_1) + (if n = 3) \left( \frac{\varepsilon_{ij}^2}{\lambda_i} \ln(\varepsilon_{ij}^{-1}) \right)^2 + \frac{1}{\lambda_i \lambda_j \lambda_i}, \]
where \( R_1 \) is defined in Lemma 2.4.

To use the previous lemmas, we need some estimates of the nonlinearity \( f_\varepsilon \) and its derivatives. We start with the following result contained in [7].

**Lemma 2.6** 1. For any \( \varepsilon > 0 \), and any \( U \in \mathbb{R} \), we have \( |f_\varepsilon(U) - f_0(U)| \leq \varepsilon |U|^p \ln(e + |U|). \)
2. For \( \varepsilon \) small enough, and any \( U \in \mathbb{R}, \)
\[ |f_\varepsilon'(U)| \leq \varepsilon |U|^{p-1}, \] (2.5)
and
\[ |f_\varepsilon'(U) - f_0'(U)| \leq \varepsilon |U|^{p-1} \left( p \ln(e + |U|) + \frac{1}{\ln(e + |U|)} \right). \] (2.6)
3. There exists $c > 0$ such that, for $\varepsilon$ small enough and any $U, V \in \mathbb{R}$,
\[
|f'_\varepsilon(U + V) - f'_\varepsilon(U)| \leq \begin{cases} 
\frac{c|U|^{p-2} + |V|^{p-2}}{} & \text{if } n \leq 6, \\
\frac{c(V)^{p-1} + \varepsilon|U|^{p-1}}{} & \text{if } n > 6.
\end{cases}
\] (2.7)

We also have

**Lemma 2.7**

1. There exists $c > 0$ such that, for $\varepsilon$ small enough and any $U, V \in \mathbb{R}$
\[
|f'_\varepsilon(U + V) - f'_\varepsilon(U)| \leq c(|U|^{p-1} + |V|^{p-1})|V|, \quad \forall n \geq 3.
\] (2.8)

2. For $\varepsilon$ small enough, and any $U \in \mathbb{R}$,
\[|f''_\varepsilon(U)| \leq c|U|^{p-2}, \quad \forall n \geq 3.
\] (2.9)

**Proof.**

1. By the mean value theorem there exists some $t \in (0, 1)$ such that
\[f'_\varepsilon(U + V) - f'_\varepsilon(U) = f'_\varepsilon(U + tV)\varepsilon.
\]
Hence, using (2.5) and the fact that $p - 1 \geq 0$ we get the desired result.

2. Recall that
\[f'_\varepsilon(U) = \frac{|U|^{p-1}}{|\ln(e + |U|)|} \left(p - \frac{\varepsilon|U|}{(e + |U|) \ln(e + |U|)}\right).
\]
We see that
\[f''_\varepsilon(U) = \frac{\varepsilon|U|^{p-2}U}{|\ln(e + |U|)|} \left(\frac{|U| - \varepsilon \ln(e + |U|)}{(e + |U|)^2 \ln(e + |U|)^2}\right)
\]
\[+ \frac{|U|^{p-3}U}{|\ln(e + |U|)|} \left(p - 1 - \frac{\varepsilon|U|}{(e + |U|) \ln(e + |U|)}\right) \left(p - \frac{\varepsilon|U|}{(e + |U|) \ln(e + |U|)}\right).
\]
So, for $\varepsilon$ small enough, the desired result follows. \(\square\)

For $\eta > 0$, $m \in \mathbb{N}$ and $(\gamma_1, \ldots, \gamma_m) \in \{-1, 1\}^m$, let us define
\[V(m, \eta) = \left\{u \in H_0^1(\Omega) \mid \exists a_1, \ldots, a_m \in \Omega, \exists \lambda_1, \ldots, \lambda_m > \eta^{-1}, \exists a_1, \ldots, a_m > 0 \text{ with } \right\}
\]
\[\|u - \sum_{i=1}^m a_i \gamma_i P_{\delta_i(a_i, \lambda_i)}\| < \eta; \quad \|u\|_1 < \eta, \quad \lambda_i d(a_i, \partial \Omega) > \eta^{-1} \forall i, \quad \varepsilon_{ij} < \eta \forall i \neq j\}.
\]
We recall that we are looking for a solution of (P) in a small neighbourhood of \(\sum_{i=1}^m a_i \gamma_i P_{\delta_i(a_i, \lambda_i)}\). In the following, we will investigate the gradient of the functional $I_\varepsilon$ in $V(m, \eta)$.

**Proposition 2.8** Let $n \geq 3$ and $u = \sum_{j=1}^m \alpha_j \gamma_j P_{\delta_j} + v \in V(m, \eta)$. For each $i \in \{1, \ldots, m\}$, we have the following expansion
\[
\langle \nabla I_\varepsilon(u), P_{\delta_i}\rangle = \gamma_i \alpha_i (1 - \alpha_i^{p-1}) S_\alpha^{\alpha_i} + O(\varepsilon \ln(\ln(\lambda_i)) + \sum_{j \neq i} \frac{1}{(\lambda_j d_j)^{n-2}} + \sum_{j \neq i} \varepsilon_{ij} + ||v||),
\]
where $S_\alpha$ is the best Sobolev constant defined in (1.6), $d_j := d(a_j, \partial \Omega)$ and $\varepsilon_{ij}$ is introduced in (2.3).

**Proof.** Let $w := \sum_{j=1}^m \alpha_j \gamma_j P_{\delta_j}$. We have

\[
\langle \nabla I_\varepsilon(u), P_{\delta_i}\rangle = \langle u, P_{\delta_i}\rangle - \int_\Omega f_\varepsilon(u) P_{\delta_i}
\]
\[
= \langle u, P_{\delta_i}\rangle - \int_\Omega [f_\varepsilon(u) - f_\varepsilon(w)] P_{\delta_i} - \int_\Omega f_\varepsilon(w) P_{\delta_i}
\]
\[
= \langle w, P_{\delta_i}\rangle - A - B,
\] (2.10)
since \( v \in E_{(\alpha, \lambda)} \) where \( E_{(\alpha, \lambda)} \) is defined in (1.9).

Using (2.8), H"older's inequality and Sobolev embedding theorem, we obtain
\[
A = O \left( \int_\Omega (|u|^{p-1} + |v|^{p-1})|v|P\delta_i \right) = O(|v|). \tag{2.11}
\]

We compute \( B \).
\[
B = \int_\Omega \left[ f_\varepsilon(u) - f_\varepsilon(\alpha_i \gamma_i P\delta_i) \right] P\delta_i + \int_\Omega f_\varepsilon(\alpha_i \gamma_i P\delta_i) P\delta_i
= B_1 + B_2. \tag{2.12}
\]

Using again (2.8) and the fact that \( P\delta_i \leq \delta_j \), we obtain
\[
B_1 = O \left( \int_\Omega \left( \alpha_i^{p-1} P\delta_i^{p-1} + \left| \sum_{j \neq i} \alpha_j\gamma_j P\delta_j \right|^{p-1} \right) \sum_j \alpha_j P\delta_j \right) \\
= O \left( \sum_{j \neq i} \int_\Omega \delta_j^p \delta_j + \delta_i^p \right) \\
= O \left( \sum_{j \neq i} \varepsilon_{ij} \right), \tag{2.13}
\]

where we have used the facts that \( \int_\Omega \delta_i^p \delta_i = O(\varepsilon_{ij}) \) for all \( i \neq j \) and \( \alpha_j = O(1) \) for each \( j \).

Notice that
\[
0 \leq \ln(\varepsilon + \alpha_i P\delta_i) \leq c\ln(\ln \lambda_i). \tag{2.14}
\]

The first claim of Lemma 2.6 and (2.14) imply
\[
B_2 := \int_\Omega f_\varepsilon(\alpha_i \gamma_i P\delta_i) P\delta_i \\
= \int_\Omega f_0(\alpha_i \gamma_i P\delta_i) P\delta_i + \int_\Omega \left[ f_\varepsilon(\alpha_i \gamma_i P\delta_i) - f_0(\alpha_i \gamma_i P\delta_i) \right] P\delta_i \\
= \gamma_i \alpha_i^p \int_\Omega P\delta_i^{p-1} + O(\varepsilon \ln(\ln \lambda_i)). \tag{2.15}
\]

Combining (2.10)-(2.13), (2.15) and Lemmas 2.2-2.5, the proof of Proposition 2.8 follows. \( \square \)

Set
\[
\psi^0_{(\alpha, \lambda)} = \frac{\lambda \partial \delta_{(\alpha, \lambda)}}{\partial \lambda} \quad \text{and} \quad \psi^1_{(\alpha, \lambda)} = \frac{1}{\lambda} \frac{\partial \delta_{(\alpha, \lambda)}}{\partial a}.
\]

For \( \ell = 0, 1 \), we denote by \( P\psi^\ell_{(\alpha, \lambda)} \) the projection of \( \psi^\ell_{(\alpha, \lambda)} \) onto \( H_0^1(\Omega) \), defined by
\[
-\Delta P\psi^\ell_{(\alpha, \lambda)} = -\Delta \psi^\ell_{(\alpha, \lambda)} \quad \text{in} \ \Omega, \quad P\psi^\ell_{(\alpha, \lambda)} = 0 \quad \text{on} \ \partial \Omega.
\]

Thus, we have
\[
P\psi^0_{(\alpha, \lambda)} = \frac{\lambda \partial P\delta_{(\alpha, \lambda)}}{\partial \lambda} \quad \text{and} \quad P\psi^1_{(\alpha, \lambda)} = \frac{1}{\lambda} \frac{\partial P\delta_{(\alpha, \lambda)}}{\partial a}.
\]

For simplicity, we denote \( \psi^\ell_{(\alpha, \lambda)} \) by \( \psi^\ell_i \) for \( \ell = 0, 1 \) and for each \( i \). Recall that simple computations show that
\[
|P\psi^0_i| \leq cP\delta_i \leq c\delta_i \quad \text{and} \quad |\psi^1_i| \leq c\delta_i \quad \text{for} \ \ell = 0, 1. \tag{2.16}
\]

To give the asymptotic expansions of the scalar product of \( \nabla I_\varepsilon(u) \) with \( \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \) and \( \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial a_i} \) respectively, we need the following result.
Lemma 2.9 Let $\tau$ be a positive real number small enough. We have
\[
\int_{\Omega} \left[ f_\ell(\alpha, \gamma_i P\delta_i) - f_0(\alpha, \gamma_i P\delta_i) \right] P\psi_i^\ell \delta^p_{(0,1)}(y) \ln(\delta_{(0,1)}(y)) \psi_{(0,1)}^0(y) dy > 0 \quad \text{and} \quad \psi_{(0,1)}^0(y) = \frac{\alpha_0(\alpha - 2)}{2} \frac{1 - |y|^2}{(1 + |y|^2)^{n/2}}.
\]

Proof. Taylor's expansion with respect to $\varepsilon$ yields,
\[
\int_{\Omega} \left[ f_\ell(\alpha, \gamma_i P\delta_i) - f_0(\alpha, \gamma_i P\delta_i) \right] P\psi_i^\ell \delta^p_{(0,1)}(y) \ln(\delta_{(0,1)}(y)) \psi_{(0,1)}^0(y) dy = \mathcal{O}(\delta_i^{\ell-1} \ln(\ln \lambda_i)),
\]
where we have used (2.14).

Taking account of Proposition 2.1 (a), the mean value theorem and (2.14) yield
\[
P\delta^p_{(0,1)} \ln(\ln(e + \alpha_i P\delta_i)) = \delta^p_{(0,1)} \ln(\ln(e + \alpha_i \delta_i)) + O(\delta_i^{\ell-1} \ln(\ln \lambda_i)).
\]

Hence, by using Proposition 2.1, (2.16) and (2.18), we get
\[
\int_{\Omega} P\delta^p_{(0,1)} \ln(\ln(e + \alpha_i P\delta_i)) P\psi_i^\ell = \mathcal{O} \left( \ln \ln \lambda_i \int_{\Omega} \delta^p_{(0,1)} (\varphi_i + |\psi_i^\ell - P\psi_i^\ell|) \right)
\]
\[
= \int_{\Omega} \delta^p_{(0,1)} \ln(\ln(e + \alpha_i \delta_i)) \psi_i^\ell + O \left( \ln \lambda_i \int_{\Omega} \delta^p_{(0,1)} \ln(\ln(e + \alpha_i \delta_i)) \psi_i^\ell \right)
\]
\[
= \int_{\Omega} \delta^p_{(0,1)} \ln(\ln(e + \alpha_i \delta_i)) \psi_i^\ell + O \left( \ln \lambda_i \left( \int_{B_i} (\ln |\varphi_i - P\psi_i^\ell|_{L^\infty(B_i)}) \right) \right)
\]
\[
= \int_{\Omega} \delta^p_{(0,1)} \ln(\ln(e + \alpha_i \delta_i)) \psi_i^\ell + O \left( \ln \lambda_i \left( \int_{B_i} \frac{\ln |\varphi_i - P\psi_i^\ell|_{L^\infty(B_i)}}{\lambda_i^{n/2} d_i^{n-2}} \right) \right)
\]
\[
= \int_{\Omega} \delta^p_{(0,1)} \ln(\ln(e + \alpha_i \delta_i)) \psi_i^\ell + O \left( \ln \lambda_i \left( \int_{B_i} \frac{\ln |\varphi_i - P\psi_i^\ell|_{L^\infty(B_i)}}{\lambda_i^{n/2} d_i^{n-2}} \right) \right).
\]

where $B_i$ denotes the ball of center $a_i$ and radius $d_i/2$.

For $\ell = 1$, we have
\[
\int_{\Omega} \delta^p_{(0,1)} \ln(\ln(e + \alpha_i \delta_i)) \psi_i^\ell = \int_{B_i} \delta^p_{(0,1)} \ln(\ln(e + \alpha_i \delta_i)) \psi_i^\ell + \int_{B_i^c} \delta^p_{(0,1)} \ln(\ln(e + \alpha_i \delta_i)) \psi_i^\ell = \mathcal{O} \left( \ln \ln \lambda_i \right).
\]

since the function $\delta^p_{(0,1)} \ln(\ln(e + \alpha_i \delta_i)) \psi_i^\ell$ is antisymmetric with respect to $x - a_i$ in $B_i$. Thus, the desired result follows from (2.17), (2.19) and (2.20).

For $\ell = 0$, an improvement of Lemma B.2 of [7] is needed for our construction.

Lemma 2.10 For any $U > 0$ and $\lambda$ large enough, we have
\[
\ln(\ln(e + \lambda^{\frac{n-2}{2}} U) = \ln \ln(\lambda^{\frac{n-2}{2}}) + \frac{2 \ln U}{(n-2) \ln \lambda} + \left[ \ln \left( 1 + \frac{2 \ln(e^{1 - \frac{n-2}{2} \ln \lambda} + U)}{(n-2) \ln \lambda} \right) \right] - \frac{2 \ln U}{(n-2) \ln \lambda}
\]
and
\[
\lim_{\lambda \to \infty} (\ln \lambda)^2 \left[ \ln \left( 1 + \frac{2 \ln(e^{1 + \frac{n-2}{2} \ln \lambda} + U)}{(n-2) \ln \lambda} \right) \right] - \frac{2 \ln U}{(n-2) \ln \lambda} = \frac{2 \ln(U^2}{(n-2)^2}.
\]
Proof. We have that
\[ \ln \ln(e + \lambda - \frac{2}{\pi} \Omega) = \ln \left( \ln \left( \lambda - \frac{2}{\pi} \right) + \ln(e + \lambda - \frac{2}{\pi} \ln \lambda + U) \right) \]
\[ = \ln \ln(\lambda - \frac{2}{\pi}) + \frac{2 \ln U}{(n - 2) \ln \lambda} + \left[ \ln \left( 1 + \frac{2 \ln(e + \lambda - \frac{2}{\pi} \ln \lambda + U)}{(n - 2) \ln \lambda} \right) - \frac{2 \ln U}{(n - 2) \ln \lambda} \right]. \]
Set

\[ g(t) := \frac{1}{\pi^2} \ln \left( 1 + \frac{2t \ln(e + \lambda - \frac{2}{\pi} \ln \lambda + U)}{(n - 2) \ln \lambda} \right) - \frac{2 \ln U}{(n - 2) \ln \lambda}, \quad t > 0. \]

We have
\[ \lim_{t \to 0} g(t) = -\frac{2 \ln(U)^2}{(n - 2)^2}. \]

Taking \( t := \ln(\lambda)^{-1} \), we obtain the claim. \( \square \)

Let \( \Omega_{\lambda_i} := \lambda_i(\Omega - a_i) \). We recall that
\[ \delta_{(\alpha, \lambda_i)} = \lambda_i^{-\frac{n}{2}} \delta_{(0, 1)}(\lambda_i(\cdot - a_i)) \text{ and } \psi^0_{(\alpha, \lambda_i)} = \lambda_i^{-\frac{n}{2}} \psi^0_{(0, 1)}(\lambda_i(\cdot - a_i)). \]

Using Lemma 2.10 with taking \( U = \alpha_i \delta_{(0, 1)} \), we get
\[ \int_{\Omega} \delta_p^p \ln(e + \alpha_i \delta_{(0, 1)}(\lambda_i(\cdot - a_i))) \psi^0_{(\alpha, \lambda_i)} = \lambda_i^{-\frac{n}{2} + \frac{n}{2} - \frac{n}{2}} \int_{\Omega_{\lambda_i}} \delta^p_{(0, 1)}(y) \ln \ln(e + \lambda_i^{-\frac{n}{2}} \alpha_i \delta_{(0, 1)}(y)) \psi^0_{(0, 1)}(y) \, dy \]
\[ = \ln \ln(\lambda_i^{-\frac{n}{2}}) \int_{\Omega_{\lambda_i}} \delta^p_{(0, 1)}(y) \psi^0_{(0, 1)}(y) \, dy + \frac{2}{(n - 2) \ln \lambda_i} \int_{\Omega_{\lambda_i}} \delta^0_{(0, 1)}(y) \ln(\alpha_i \delta_{(0, 1)}(y)) \psi^0_{(0, 1)}(y) \, dy \]
\[ + \int_{\Omega_{\lambda_i}} \delta^0_{(0, 1)}(y) \left[ \ln \left( 1 + \frac{2 \ln(e^{1 - \frac{n}{2} \ln \lambda_i + \alpha_i \delta_{(0, 1)}(y)})}{(n - 2) \ln \lambda_i} \right) - \frac{2 \ln(\alpha_i \delta_{(0, 1)}(y))}{(n - 2) \ln \lambda_i} \right] \psi^0_{(0, 1)}(y) \, dy \]
\[ = A + B + C. \]

Recall that \( \int_{\mathbb{R}^n} \delta^p_{(0, 1)} \psi^0_{(0, 1)} = 0 \) and note that \( B(0, \lambda_i d_i) \subset \Omega_{\lambda_i} \). Thus
\[ A = \ln \ln(\lambda_i^{-\frac{n}{2}}) \int_{\mathbb{R}^n} \delta^p_{(0, 1)}(y) \psi^0_{(0, 1)}(y) \, dy - \ln \ln(\lambda_i^{-\frac{n}{2}}) \int_{\Omega_{\lambda_i}} \delta^0_{(0, 1)}(y) \psi^0_{(0, 1)}(y) \, dy \]
\[ = O \left( \frac{\ln \lambda_i}{(\lambda_i d_i)^n} \right) = O \left( \frac{\ln \lambda_i}{(\lambda_i d_i)^n} \right). \]

Concerning \( B \), we have
\[ B = \frac{2}{(n - 2) \ln \lambda_i} \int_{\mathbb{R}^n} \delta^p_{(0, 1)}(y) \left( \ln \alpha_i + \ln(\alpha_i \delta_{(0, 1)}(y)) \right) \psi^0_{(0, 1)}(y) \, dy \]
\[ = \frac{2}{(n - 2) \ln \lambda_i} \int_{\mathbb{R}^n} \delta^0_{(0, 1)}(y) \ln(\delta_{(0, 1)}(y)) \psi^0_{(0, 1)}(y) \, dy + O \left( \frac{1}{\ln \lambda_i} \int_{B(0, \lambda_i d_i)^c} \delta^0_{(0, 1)}(y)(\psi^0_{(0, 1)}(y) \, dy \right) \]
\[ = \Gamma_1 \frac{1}{\ln \lambda_i} + O \left( \frac{1}{\ln \lambda_i \ln(\lambda_i d_i)^{n - 2}} \right) \]
by using again \( \int_{\mathbb{R}^n} \delta^p_{(0, 1)} \psi^0_{(0, 1)} = 0 \) and since we have \( \ln \delta_{(0, 1)}(y) \leq c(|y|^2 + 1) \) for each \( y \in \mathbb{R}^n \).

Lastly we prove that \( C \) is a remainder term. Let \( \tau \) be a positive constant small enough. We split \( C \) into two parts:
\[ C = \int_{\Omega_{\lambda_i} \setminus \Omega_{\lambda_i}^C} \ldots + \int_{\Omega_{\lambda_i} \setminus \Omega_{\lambda_i}^C} \ldots = C_1 + C_2. \]
Note that $\Omega_\lambda \setminus \Omega_\lambda^{(1-\tau)/2} \subset B(0, \lambda d) \setminus B(0, \lambda^{(1-\tau)/2} d)$ where $d = \text{diam}(\Omega)$ and $d_i = d(a_i, \partial \Omega) > 0$. In $W := B(0, \lambda d) \setminus B(0, \lambda^{(1-\tau)/2} d)$ we have

$$\left| \ln \left( 1 + \frac{2 \ln(e^{1-\frac{n-2}{2} \ln \lambda_i} + \alpha_i \delta_{(0,1)})}{(n-2) \ln \lambda_i} \right) - \frac{2 \ln(\alpha_i \delta_{(0,1)})}{(n-2) \ln \lambda_i} \right| \leq \ln \frac{\ln(c + \ln \lambda_i^{\frac{n-2}{2}} \alpha_i \delta_{(0,1)})}{\ln \lambda_i} + \ln \ln \lambda_i^{\frac{n-2}{2}} + \frac{2 \ln(\alpha_i \delta_{(0,1)})}{(n-2) \ln \lambda_i} \leq c \ln \ln \lambda_i,$$

(2.27)

by using (2.22) and since $\frac{2 \ln(\alpha_i \delta_{(0,1)})}{(n-2) \ln \lambda_i}$ is bounded in $W$. Hence,

$$C_2 = O \left( \ln \ln \lambda_i \int_{B^{c}(0, \lambda^{(1-\tau)/2} d_i)} \delta_{(0,1)}^{\beta+1} \right) = O \left( \frac{\ln \ln \lambda_i}{\lambda^{(1-\tau)/2}} n^{\frac{n}{2}} d_i^n \right).$$

(2.28)

For each $y \in \Omega_\lambda^{(1-\tau)/2} \subset B(0, \lambda^{(1-\tau)/2} d)$, we have

$$\frac{2 \ln(e^{1-\frac{n-2}{2} \ln \lambda_i} + \alpha_i \delta_{(0,1)}(y))}{(n-2) \ln \lambda_i} \geq \frac{2 \ln(e^{\lambda \delta_{(0,1)}} + \alpha_i \lambda \delta_{(1,0)} \frac{(n-2)}{2} \ln \lambda_i)}{(n-2) \ln \lambda_i} \rightarrow -(1-\tau) > -1,$$

as $\lambda_i \to \infty$.

We deduce, for $\lambda_i$ large enough, that

$$\frac{2 \ln(e^{1-\frac{n-2}{2} \ln \lambda_i} + \alpha_i \delta_{(0,1)}(y))}{(n-2) \ln \lambda_i} \geq 1 + \frac{\tau}{2} > 1 \quad \forall y \in B(0, \lambda^{(1-\tau)/2} d).$$

(2.29)

Recall that we have

$$|\ln(1 + x) - x| \leq c e^x, \quad \text{for each } x \in [-\sigma, \infty) \text{ where } 0 < \sigma < 1.$$  

(2.30)

Using (2.29), (2.30) and the fact that $|\ln \delta_{(0,1)}| \leq \ln(\delta_{(0,1)})^2 + 1$, we obtain, for each $y \in B(0, \lambda^{(1-\tau)/2} d),$

$$\left| \ln \left( 1 + \frac{2 \ln(e^{1-\frac{n-2}{2} \ln \lambda_i} + \alpha_i \delta_{(0,1)})}{(n-2) \ln \lambda_i} \right) - \frac{2 \ln(\alpha_i \delta_{(0,1)})}{(n-2) \ln \lambda_i} \right| \leq c \left| \ln(e^{1-\frac{n-2}{2} \ln \lambda_i} + \alpha_i \delta_{(0,1)}) \right|^2 + \frac{2 \ln(e^{1-\frac{n-2}{2} \ln \lambda_i} + \alpha_i \delta_{(0,1)})}{(n-2) \ln \lambda_i} \leq c \frac{\ln(\delta_{(0,1)})^2 + 1}{\ln \lambda_i^2}. \quad \text{(2.31)}$$

(2.31) asserts that

$$\left| \ln(\lambda_i)^2 \delta_{(0,1)}^{\beta+1} \left[ \ln \left( 1 + \frac{2 \ln(e^{1-\frac{n-2}{2} \ln \lambda_i} + \alpha_i \delta_{(0,1)})}{(n-2) \ln \lambda_i} \right) - \frac{2 \ln(\alpha_i \delta_{(0,1)})}{(n-2) \ln \lambda_i} \right] \psi_0^0(1) \right| \leq c \delta_{(0,1)}^{\beta+1} \ln(\delta_{(0,1)})^2 + 1 =: h(y) \quad \text{in } \mathbb{R}^n.$$

Now using (2.21) and the previous uniform bounds, by applying the dominated convergence theorem, and since the function $h$ is integrable in $\mathbb{R}^n$, we deduce

$$C_1 = \frac{1}{\ln(\lambda_i)^2} (c' + o(1)) = O \left( \frac{1}{\ln(\lambda_i)^2} \right),$$

(2.32)

where $|c'| \leq c \int_{\mathbb{R}^n} \delta_{(0,1)}^2 |\psi_0^0(1)|^2 \psi_{0,1}^0 \, |\psi_0^0(1)|$. Choosing $\tau$ small enough and combining (2.17), (2.19), (2.23)-(2.26), (2.28) and (2.32), the desired result follows for $\ell = 1$. 

\qed
Proposition 2.11 Let $n \geq 3$ and $u = \sum_{j=1}^{m} \alpha_j \gamma_j P_{\delta_j} + v \in V(m, \eta)$. For each $i \in \{1, \ldots, m\}$, we have the following expansion
\[
\left\langle \nabla I_i(u), \lambda_i \frac{\partial P_{\delta_i}}{\partial \lambda_i} \right\rangle = \gamma_i \tau_i \frac{\alpha_i^2 \delta_i}{\ln \lambda_i} + (n - 2) \frac{\gamma_i \alpha_i}{\lambda_i} \left( 1 - 2 \alpha_i^{p-1} \right) \frac{H(a_i, a_i)}{\lambda_i^{n-2}} \left( \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} + \frac{n - 2}{2} \frac{H(a_i, a_i)}{(\lambda_i \lambda_j)^{(n-2)/2}} \right) \\
+ \nu \sum_{j \neq i} \gamma_j \alpha_j \left( 1 - \alpha_j^{p-1} - \alpha_i^{p-1} \right) \left( \lambda_i \frac{\partial \delta_j}{\partial \lambda_i} + \frac{n - 2}{2} \frac{H(a_i, a_i)}{(\lambda_i \lambda_j)^{(n-2)/2}} \right) \\
+ O \left( \frac{\epsilon}{\ln(\lambda_i)^{2n-4}} + \frac{1}{\lambda_i d_i^{2n-4}} + \sum_{j=1}^{m} \epsilon_j^2 \ln(\lambda_j)^{2} + \sum_{j=1}^{m} \ln(\lambda_j d_j) + (\text{if } n = 3) \sum_{j=1}^{m} \frac{1}{(\lambda_i \lambda_j)^{2}} \right),
\]
where $\tau_i$ is defined in (2.2) and $\tau$ is a positive constant small enough.

Proof. Recall that $P_{\psi_i}^0 = \lambda_i \frac{\partial P_{\delta_i}}{\partial \lambda_i}$ and $\psi_i := \sum_{j=1}^{m} \alpha_j \gamma_j P_{\delta_j}$. We have
\[
\left\langle \nabla I_i(u), P_{\psi_i}^0 \right\rangle = \langle u, P_{\psi_i}^0 \rangle - \int_{\Omega} f_i(u) P_{\psi_i}^0 \\
= \langle u, P_{\psi_i}^0 \rangle - \int_{\Omega} [f_i(u) - f_i(\underline{u}) - f'_i(\underline{u})] v P_{\psi_i}^0 - \int_{\Omega} f'_i(\underline{u}) v P_{\psi_i}^0 \\
= \langle u, P_{\psi_i}^0 \rangle - A - B - C
\]
(2.33)

since $v \in E(\alpha, \lambda)$. By the mean value theorem, there exists $\theta = \theta(x) \in (0, 1)$ such that
\[
A = \int_{\Omega} \left[ f'_i(\underline{u} + \theta v) - f'_i(\underline{u}) \right] v P_{\psi_i}^0.
\]

Using (2.7), (2.16), Holder’s inequality and Sobolev embedding theorem, we obtain
\[
|A| \leq c \left\{ \begin{array}{ll}
\int_{\Omega} (|u|^{p-2} + |v|^{p-2}) v^2 \delta_i & \text{if } n \leq 6, \\
\int_{\Omega} |v|^{p-1} + \epsilon |u|^{p-1} v P_{\delta_i} & \text{if } n > 6,
\end{array} \right.
\leq c \left\{ \begin{array}{ll}
\|v\|^2 & \text{if } n \leq 6, \\
\|v\|^{p} + \epsilon \|v\| & \text{if } n > 6.
\end{array} \right.
\]

We mention that the remainder term $\|v\|^{p}$ is bad when $n > 6$. Therefore, we need to be more precise for these dimensions. In fact, we devide the set $\Omega$ into three subsets and in each one, we will ameliorate this estimate.

Let $\Omega_1 := \{ x : \alpha_i \delta_i \geq 2 | \sum_{j \neq i} \alpha_j \gamma_j \delta_j \}$. Note that, in $\Omega \setminus \Omega_1$, it holds that $P_{\delta_i} \leq c \sum_{j \neq i} P_{\delta_j}$. Hence we get
\[
|A| \leq c \int_{\Omega \setminus \Omega_1} (|v|^{p-1} + \epsilon |u|^{p-1}) v P_{\delta_i} \leq c \sum_{j \neq i} \int_{\Omega \setminus \Omega_1} (|v|^{p} + \epsilon |u|^{p-1} |v|) \sqrt{\delta_j} \delta_i \leq (\|v\|^{p} + \epsilon \|v\|) \epsilon_{ij}^{1/2} \ln(\epsilon_{ij}^{1/2})^{(n-2)/(2n)}.
\]

For the integral over $\Omega_1$, we point out that in this set, it holds: $\alpha_i \delta_i/2 \leq |u| \leq 3 \alpha_i \delta_i/2$. Let $\Omega_2 := \{ x : |u| \leq 2\|v\| \}$, in $\Omega_1 \cap \Omega_2$, it holds: $P_{\delta_i} \leq c|v|$ and therefore
\[
|A| \leq c \int_{\Omega_1 \cap \Omega_2} (|v|^{p-1} + \epsilon |u|^{p-1}) v P_{\delta_i} \leq c \int |v|^{p+1} \leq c \|v\|^{p+1}.
\]

Finally, for the last one, which is integrating over $\Omega \cap (\Omega \setminus \Omega_1) =: \Omega_2$, observe that, for $x \in \Omega_2$, it holds:
\[
|u| \leq 2|u + \Theta v| \quad \text{for each } \Theta \in (0, 1).
\]
Thus, using the mean value theorem, (2.9), (2.16), Holder’s inequality and the fact that $p < 2$ for $n > 6$, we obtain
\[
|A| = \left| \int_{\Omega_2} [f'_i(u + \theta v) - f'_i(\underline{u})] v P_{\psi_i}^0 \right| = \left| \int_{\Omega_2} f''_i(u + \theta_1 \Theta v) \theta v^2 P_{\psi_i}^0 \right|, \text{ for some } \theta_1 \in (0, 1)
\leq c \int_{\Omega_2} |u + \theta_1 \Theta v|^{p-2} v^2 P_{\delta_i} \leq c \int_{\Omega_2} |u|^{p-2} v^2 P_{\delta_i} \leq c \int_{\Omega_2} v^2 P_{\delta_i} \leq c \|v\|^2.
\]
Lastly, we deduce

$$|A| \leq c \left\{ \begin{array}{ll}
\|v\|^2 & \text{if } n \leq 6, \\
\|v\|^2 + \varepsilon^2 + \varepsilon_n \ln(\varepsilon_n^{-1}) & \text{if } n > 6.
\end{array} \right.$$

(2.34)

To compute $B$, we write

$$B = \int_{\Omega} \left[ f_{\varepsilon}(\omega) - f_{\varepsilon}(\alpha_{i=1} \gamma_i \delta_{i}) - f_{\varepsilon}(\sum_{j \neq i} \alpha_{j=1} \gamma_j \delta_{j}) \right] P_{\psi_i}^0 + \int_{\Omega} f_{\varepsilon}(\alpha_{i=1} \gamma_i \delta_{i}) P_{\psi_i}^0$$

$$\quad + \int_{\Omega} f_{\varepsilon}(\sum_{j \neq i} \alpha_{j=1} \gamma_j \delta_{j}) P_{\psi_i}^0 + \int_{\Omega} f_{\varepsilon}(\alpha_{i=1} \gamma_i \delta_{i})(\sum_{j \neq i} \alpha_{j=1} \gamma_j \delta_{j}) P_{\psi_i}^0$$

$$= B_1 + B_2 + B_3 + B_4.$$

We start with the last integral. Using (2.6), (2.4) and (2.14), we get

$$B_4 := \int_{\Omega} f_{\varepsilon}(\sum_{j \neq i} \alpha_{j=1} \gamma_j \delta_{j}) P_{\psi_i}^0$$

$$\quad = \int_{\Omega} f_{\varepsilon}(\sum_{j \neq i} \alpha_{j=1} \gamma_j \delta_{j}) P_{\psi_i}^0 + \int_{\Omega} f_{\varepsilon}(\sum_{j \neq i} \alpha_{j=1} \gamma_j \delta_{j}) P_{\psi_i}^0$$

$$= \sum_{j \neq i} \alpha_{j=1} \gamma_j \alpha_{i=1} \gamma_i \int_{\Omega} P\delta_{i=1} - P\delta_{j=1} \psi_i^0 + O \left( \varepsilon \ln(\ln \lambda_{i=1}) \sum_{j \neq i} \varepsilon_{i=1} \right).$$

(2.36)

Using the first claim of Lemma 2.6, (2.16) and the fact that $\ln \ln(e + |\sum_{j \neq i} \alpha_{j=1} \gamma_j \delta_{j}|) = O(\ln \ln \lambda_{\max})$ where $\lambda_{\max} := \max(\lambda_1, \ldots, \lambda_m)$, we have

$$B_3 := \int_{\Omega} f_{\varepsilon}(\sum_{j \neq i} \alpha_{j=1} \gamma_j \delta_{j}) P_{\psi_i}^0$$

$$\quad = \int_{\Omega} f_{\varepsilon}(\sum_{j \neq i} \alpha_{j=1} \gamma_j \delta_{j}) P_{\psi_i}^0 + \int_{\Omega} f_{\varepsilon}(\sum_{j \neq i} \alpha_{j=1} \gamma_j \delta_{j}) P_{\psi_i}^0$$

$$\quad = \int_{\Omega} f_{\varepsilon}(\sum_{j \neq i} \alpha_{j=1} \gamma_j \delta_{j}) P_{\psi_i}^0 + O \left( \varepsilon \ln(\ln \lambda_{\max}) \sum_{j \neq i} \varepsilon_{i=1} \right).$$

(2.37)
Now, Lemma 2.9 implies that

\[
B_2 := \int_\Omega f_\epsilon(\alpha_i \gamma_i P \delta_i) P \psi_i^0 \\
= \int_\Omega [f_\epsilon(\alpha_i \gamma_i P \delta_i) - f_0(\alpha_i \gamma_i P \delta_i)] P \psi_i^0 + \int_\Omega f_0(\alpha_i \gamma_i P \delta_i) P \psi_i^0 \\
= \int_\Omega [f_\epsilon(\alpha_i \gamma_i P \delta_i) - f_0(\alpha_i \gamma_i P \delta_i)] P \psi_i^0 + \gamma_i \alpha_i^p \int \Omega P \delta_i^p P \psi_i^0 \\
= -\gamma_i \Gamma_i \frac{\alpha_i^p \epsilon_i}{\ln \lambda_i} + \gamma_i \alpha_i^p \int \Omega P \delta_i^p P \psi_i^0 + O \left( \epsilon^2 \ln(\ln \lambda_i)^2 + \frac{\epsilon}{\ln(\lambda_i)^2} + \frac{1}{(\lambda_i d_i)^{2n-4}} + \frac{1}{\chi_i^{(1-\gamma)n} d_i^{2n}} \right). \quad (2.38)
\]

In the sequel, we compute \(B_1\). Let \(\Omega_1 := \{ x : |\sum_{j \neq i} \alpha_j \gamma_j P \delta_j(x)| \leq \frac{4}{\epsilon} \alpha_i P \delta_i(x) \}\).

\[
B_1 := \int_\Omega \left[ f_\epsilon(u) - f_\epsilon(\alpha_i \gamma_i P \delta_i) - f_\epsilon(\sum_{j \neq i} \alpha_j \gamma_j P \delta_j) \right] P \psi_i^0 \\
= \int_{\Omega_1} \ldots + \int_{\Omega \setminus \Omega_1} \ldots \\
= B_{11} + B_{12}. \quad (2.39)
\]

Observe that, in \(\Omega_1\), it holds \(1/2 \alpha_i P \delta_i \leq |\alpha_i \gamma_i P \delta_i + \theta \sum_{j \neq i} \alpha_j \gamma_j P \delta_j| \leq 3/2 \alpha_i P \delta_i\) for each \(\theta \in (0, 1)\). By using the mean value theorem and (2.9), we have

\[
|B_{11}| \leq \int_{\Omega_1} \left| f_\epsilon(u) - f_\epsilon(\alpha_i \gamma_i P \delta_i) - f_\epsilon(\sum_{j \neq i} \alpha_j \gamma_j P \delta_j) \right| \delta_i + \int_{\Omega_1} |f_\epsilon(\sum_{j \neq i} \alpha_j \gamma_j P \delta_j)| \delta_i \\
\leq c \int_{\Omega_1} \left| f_\epsilon''(\alpha_i \gamma_i P \delta_i + \theta \sum_{j \neq i} \alpha_j \gamma_j P \delta_j) \right| \left( \sum_{j \neq i} \alpha_j P \delta_j \right)^2 \delta_i + \int_{\Omega_1} \left( \sum_{j \neq i} \alpha_j P \delta_j \right)^p \delta_i, \text{ for some } \theta \in (0, 1) \\
\leq c \int_{\Omega_1} \left( \sum_{j \neq i} \alpha_j P \delta_j \right)^2 \delta_i^{p-1} + c \sum_{j \neq i} \int_{\Omega_1} (\delta_j \delta_i)^{-1/2} \\
\leq c \sum_{j \neq i} \epsilon_{ij}^{-n/2} \ln(\epsilon_{ij}^{-1}) + (\text{if } n = 3) \ c \sum_{j \neq i} \epsilon_{ij}^2 \ln(\epsilon_{ij}^{-1})^{1/2}. \quad (2.40)
\]

Using again the mean value theorem, (2.5) and the fact that \(|\alpha_j - 1| < \eta\) for each \(j\), we obtain

\[
|B_{12}| \leq \int_{\Omega \setminus \Omega_1} \left( |f_\epsilon(\alpha_i \gamma_i P \delta_i)| + |f_\epsilon(\alpha_i \gamma_i P \delta_i)\| \sum_{j \neq i} \alpha_j \gamma_j P \delta_j | \right) P \delta_i + \int_{\Omega \setminus \Omega_1} |f_\epsilon(u) - f_\epsilon(\sum_{j \neq i} \alpha_j \gamma_j P \delta_j)| P \delta_i \\
\leq c \int_{\Omega \setminus \Omega_1} \alpha_i^p P \delta_i^{p+1} + \alpha_i^{p-1} P \delta_i^{p} \sum_{j \neq i} \alpha_j \gamma_j P \delta_j + \int_{\Omega \setminus \Omega_1} \left| f_\epsilon(\sum_{j \neq i} \alpha_j \gamma_j P \delta_j + \theta \alpha_i \gamma_i P \delta_i) \right| |P \delta_i^2|, \text{ for some } \theta \in (0, 1) \\
\leq c \sum_{j \neq i} \epsilon_{ij}^{-n/2} \ln(\epsilon_{ij}^{-1}) + c \int_{\Omega \setminus \Omega_1} \sum_{j \neq i} \alpha_j \gamma_j P \delta_j | P \delta_i^{p-1} P \delta_i^2 \\
\leq c \sum_{j \neq i} \epsilon_{ij}^{-n/2} \ln(\epsilon_{ij}^{-1}) + (\text{if } n = 3) \ c \sum_{j \neq i} \epsilon_{ij}^2 \ln(\epsilon_{ij}^{-1})^{1/2}. \quad (2.41)
\]

Lastly we estimate \(C\). Note that \(\ln(\ln(e + |u|)) = O(\ln \lambda_{\max})\). Using (2.6) and the fact that

\[
f_\epsilon'(u) = p \alpha_i^{p-1} P \delta_i^{p+1} + O(\sum_{j \neq i} P \delta_i^{p-1} \mathbf{1}_{P \delta_i \leq P \delta_j}) + P \delta_i^{p-2} P \delta_j \mathbf{1}_{P \delta_j \leq P \delta_i}),
\]
we get
\[ C := \int_\Omega f'_i(u) v \psi_i^0 \]
\[ = \int_\Omega f'_i(u) v \psi_i^0 + \int_\Omega \left( f'_i(u) - f'_i(u) \right) v \psi_i^0 \]
\[ = p\alpha_i^{p-1} \int_\Omega P\delta_i^{p-1} P\psi_i^0 v + O \left( \sum_{j \neq i} \int_{P\delta_i \subseteq P\delta_j} P\delta_j^{p-1} P\delta_i |v| + \int_{P\delta_j \subseteq P\delta_i} P\delta_i^{p-1} P\delta_j |v| + \|v\| \|\varepsilon\| \ln \lambda_{\max} \right) \]
\[ = C_1 + O \left( \sum_{j \neq i} \int_{P\delta_i \subseteq P\delta_j} P\delta_j^{p-1} P\delta_i |v| + \int_{P\delta_j \subseteq P\delta_i} P\delta_i^{p-1} P\delta_j |v| + \|v\| \|\varepsilon\| \ln \lambda_{\max} \right). \quad (2.42) \]

We need to estimate the following integral. For \( k \neq j \), we have
\[ \int_{P\delta_j \subseteq P\delta_k} P\delta_k^{p-1} P\delta_j |v| = O \left( \|v\| \left( \int_{P\delta_j \subseteq P\delta_k} \left( P\delta_k^{\frac{\alpha_j}{p+\frac{\alpha_j}{n}}} P\delta_j \right)^{\frac{n+2}{n+1}} \right) \right). \]

Observe that, for \( n \geq 6 \) and \( k \neq j \),
\[ \int_{P\delta_j \subseteq P\delta_k} \left( P\delta_k^{\frac{\alpha_j}{p+\frac{\alpha_j}{n}}} P\delta_j \right)^{\frac{n+2}{n+1}} \leq \int_{\Omega} (\delta_j \delta_i)^{\frac{n+2}{n+1}} = O \left( \varepsilon_{kj}^{\frac{n}{n+1}} \ln \left( \varepsilon_{kj}^{-1} \right) \right) \]
and for \( n \leq 5 \),
\[ \left( \int_{P\delta_j \subseteq P\delta_k} \left( P\delta_k^{\frac{\alpha_j}{p+\frac{\alpha_j}{n}}} P\delta_j \right)^{\frac{n+2}{n+1}} \right)^{\frac{n+2}{n+1}} \leq \left( \int_{\Omega} (\delta_i \delta_j)^{\frac{2n}{2n+2}} \delta_i^{\frac{\alpha_j}{p+\frac{\alpha_j}{n}}} \right)^{\frac{n+2}{n+1}} = O \left( \varepsilon_{kj}^{\frac{n}{n+1}} \ln \left( \varepsilon_{kj}^{-1} \right)^{\frac{n+2}{n+1}} \right). \]

Thus, we obtain
\[ C = C_1 + O \left( \varepsilon^2 \ln(\ln \lambda_{\max})^2 + \|v\|^2 + \sum_{j \neq i} \varepsilon_{ij}^{\frac{n+2}{n+1}} \ln \left( \varepsilon_{ij}^{-1} \right)^{\frac{n+2}{n+1}} + \sum_{j \neq i} \varepsilon_{ij}^2 \ln \left( \varepsilon_{ij}^{-1} \right)^{\frac{2(n-2)}{n+1}} \right). \quad (2.43) \]

Let \( B_i \) be the ball of center \( a_i \) and radius \( d_i/2 \). Concerning \( C_1 \), we have
\[ C_1 = \int_{B_i} \ldots + \int_{B_i} \ldots \]
\[ = p\alpha_i^{p-1} \int_{B_i} P\delta_i^{p-1} P\psi_i^0 v + O \left( \|v\| \frac{1}{(\lambda_i d_i)^{\frac{n+2}{n+1}}} \right) \]
\[ = p\alpha_i^{p-1} \int_{\Omega} \delta_i^{p-1} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} v + O \left( \int_{B_i} \delta_i^{p-1} \varphi |v| + \int_{B_i} \delta_i^{p-1} \lambda_i \frac{\partial \varphi}{\partial \lambda_i} |v| + \|v\| \frac{1}{(\lambda_i d_i)^{\frac{n+2}{n+1}}} \right) \]
\[ = O \left( \|v\| \left( \frac{1}{\lambda_i^{\frac{n+2}{n+1}} d_i^{n+2}} \left( \int_{B_i} \delta_i^{\frac{n+2}{n+1}} \frac{n+2}{n+1} + \frac{1}{(\lambda_i d_i)^{\frac{n+2}{n+1}}} \right) \right) \right) \]
\[ = O \left( \|v\| \left( \frac{1}{(\lambda_i d_i)^{\frac{n+2}{n+1}}} + (\text{if } n=6) \frac{\ln(\lambda_i d_i)}{(\lambda_i d_i)^2} + (\text{if } n < 6) \frac{1}{(\lambda_i d_i)^{n-2}} \right) \right), \quad (2.44) \]
where we have used Proposition 2.1, Holder inequality, the fact that \( v \in E_{a, \lambda} \) and the following computation
\[ \left( \int_{B_i} \delta_i^{\frac{\alpha_j}{p+\frac{\alpha_j}{n}}} \right)^{\frac{n+2}{n+1}} = O \left( \frac{1}{\lambda_i^{\frac{n+2}{n+1}}} (\text{if } n < 6) + \frac{\ln(\lambda_i d_i)}{\lambda^2} (\text{if } n=6) + \frac{d_i^{n-6}}{\lambda_i^2} (\text{if } n > 6) \right). \]

Combining (2.33)-(2.44) and Lemmas 2.2-2.5, the proof of Proposition 2.11 is completed. □
Proposition 2.12 Let $n \geq 3$ and $u = \sum_{i=1}^{m} \alpha_i \gamma_i P \delta_i + v \in V(m, \eta)$. For each $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, we have the following expansion

$$
\left( \nabla I_\epsilon(u), \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a, j)} \right) = \gamma_i \left( \alpha_i^p - \frac{\alpha_i}{2} \right) \frac{\gamma_i}{\lambda_i} \frac{\partial H(a_i, a_i)}{\partial (a, j)}
$$

$$
+ c \sum_{l=1, l \neq i}^{m} \gamma_i a_i \left( 1 - \alpha_i^p - 1 \right) \frac{1}{\lambda_i} \left( \frac{\partial \xi_{il}}{\partial (a, j)} - \frac{1}{(\lambda_i \lambda_l)^{(n-2)/2}} \frac{\partial H(a_i, a_l)}{\partial (a, j)} \right)
$$

$$
+ O \left( \sum_{l=1}^{m} \varepsilon^2 \ln(\lambda_i)^2 + \sum_{l=1}^{m} \varepsilon \ln(\lambda_i) + (\text{if } n = 3) \sum_{l=1}^{m} \frac{1}{\lambda_i} + \sum_{l \neq i} \lambda_i a_i - a_i \varepsilon \frac{1}{\lambda_i} \right)
$$

$$
+ \sum_{i \neq j} \varepsilon \frac{\lambda_i}{\lambda_j} \ln(\varepsilon^{-1}) + \sum_{i \neq j} \varepsilon \frac{\lambda_i}{\lambda_j} \ln(\varepsilon^{-1}) \frac{\ln(\lambda_i)}{\lambda_i} + \|v\|^2
$$

where $\frac{\partial H}{\partial (a, j)}$ denotes the partial derivative of $H$ with respect to the $j$-th component of the first variable.

The proof of Proposition 2.12 which we omit here is similar, up to minor modifications, to that of Proposition 2.11: arguing as previously, taking account of Lemma 2.9 and using Lemmas 2.2-2.5, we obtain this expansion.

## 3 Proof of Theorem 1.1

In this section, we restrict ourselves to the case $n \geq 4$. Now let

$$
M_\epsilon = \left\{ (\alpha, \lambda, a, v) \in (0, +\infty)^m \times (0, +\infty)^m \times \Omega_0^m \times H^1(\Omega) : |\alpha_i - 1| < \nu_0, \right.
$$

$$
\varepsilon \ln(\lambda_i) < \nu_0, \forall i, \frac{\lambda_i}{\lambda_j} < c, \right| a_i - a_j | > d'_0, \forall i, j \neq j; v \in E_{(a, \lambda)}, \|v\| < \nu_0 \right\}
$$

where $\nu_0$, $c$, $d_0$ and $d'_0$ are some suitable positive constants and $\Omega_0 = \{ x \in \Omega : d(x, \partial \Omega) > d_0 \}$. Let $(\gamma_1, \ldots, \gamma_m) \in \{-1, 1\}^m$, we define the function

$$
K_\epsilon : M_\epsilon \rightarrow \mathbb{R}; \quad (\alpha, \lambda, a, v) \mapsto I_\epsilon \left( \sum_{i=1}^{m} \alpha_i \gamma_i P \delta_i(a, a) + v \right). \tag{3.1}
$$

Note that the function $K_\epsilon$ depends on the choice of $(\gamma_1, \ldots, \gamma_m)$.

As described in the set $M_\epsilon$, the concentration speeds $\lambda_i$'s are comparable. For sake of simplicity, $O(f(\lambda))$ denotes any quantity dominated by $\sum_{i=1}^{m} f(\lambda_i)$. Observe also that the concentration points $a_i$'s are far away from the boundary and from each other. Hence, the parameter $\varepsilon_{ik}$ (introduced in (2.3)) satisfies

$$
\varepsilon_{ik} \leq \frac{c}{(\lambda_i \lambda_k |a_i - a_k|^2)^{\frac{n-2}{2}}} \leq \frac{c}{\lambda_i^{n-2}}, \forall i, k, i \neq k. \tag{3.2}
$$

Proposition 3.1 Let $(\alpha, \lambda, a, v) \in M_\epsilon$. $(\alpha, \lambda, a, v)$ is a critical point of $K_\epsilon$ if and only if $u = \sum_{i=1}^{m} \gamma_i \alpha_i \lambda \delta_i + v$ is a critical point of $I_\epsilon$, i.e. if and only if there exists $(A, B, C) \in \mathbb{R}^m \times \mathbb{R}^m \times (\mathbb{R}^n)^m$ such that the following holds:

$$
(E_\alpha) \quad \frac{\partial K_\epsilon}{\partial \alpha_i} = 0, \forall i \tag{3.3}
$$

$$
(E_\lambda) \quad \frac{\partial K_\epsilon}{\partial \lambda_i} = B \left( \frac{\partial^2 P \delta_i}{\partial \lambda_i^2} + \sum_{j=1}^{n} C_{ij} \left( \frac{1}{\lambda_i} \frac{\partial^2 P \delta_i}{\partial \lambda_i \partial \lambda_j}, v \right) \right), \forall i \tag{3.4}
$$

$$
(E_\alpha) \quad \frac{\partial K_\epsilon}{\partial a_i} = B \left( \frac{\partial^2 P \delta_i}{\partial \alpha_i^2} + \sum_{j=1}^{n} C_{ij} \left( \frac{1}{\lambda_i} \frac{\partial^2 P \delta_i}{\partial \alpha_i \partial \lambda_j}, v \right) \right), \forall i \tag{3.5}
$$

$$
(E_v) \quad \frac{\partial K_\epsilon}{\partial v} = \sum_{i=1}^{m} \left( A_i \frac{\partial P \delta_i}{\partial \alpha_i} + B_i \frac{\partial^2 P \delta_i}{\partial \lambda_i} + \sum_{j=1}^{n} C_{ij} \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial \alpha_j} \right). \tag{3.6}
$$
As usual in this type of problems, we first deal with the $v$-part of $u$. Namely, we have the following result.

**Proposition 3.2** There exists a smooth map which to any $(\varepsilon, \alpha, \lambda, a)$ verifying $\varepsilon$ is small enough and $(\alpha, \lambda, a, 0)$ in $M_{\varepsilon}$, associates $\overline{\varepsilon} \in E(\alpha, \lambda, a, 0)$, such that $(E_{0})$ is satisfied for some $(A, B, C) \in \mathbb{R}^m \times \mathbb{R}^n \times (\mathbb{R}^n)^m$. Such a $\overline{\varepsilon}$ is unique, minimizes $K_{\varepsilon}(\alpha, \lambda, a, v)$ with respect to $v$ in $\{v \in E(\alpha, \lambda, a, 0) \mid \|v\| < \nu_0\}$, and we have the following estimate

$$\|\overline{\varepsilon}\| \leq(3.7)$$

**Proof.** The proof of such a result follows the same ideas in [2] and [16]. In the following, we give a sketch of the proof. Let $(\alpha, \lambda, a, v) \in M_{\varepsilon}$, $u = \sum_{i=1}^{m} \alpha_i \gamma_i \delta_i + v$ and $\overline{u} = \sum_{i=1}^{m} \alpha_i \gamma_i \delta_i$. Using (2.1) and (3.1) and expanding $K_{\varepsilon}$ with respect to $v$, we obtain

$$K_{\varepsilon}(\alpha, \lambda, a, v) = \frac{1}{2} < u, u > - \int_{\Omega} F_{\varepsilon}(u)$$

$$= \frac{1}{2} \int_{\Omega} F_{\varepsilon}(u) + \frac{1}{2} \int_{\Omega} \left[ F_{\varepsilon}(u) - F_{\varepsilon}(u) - F_{\varepsilon}(u) v - \frac{1}{2} F_{\varepsilon}(u) v^2 \right]$$

$$= \frac{1}{2} \int_{\Omega} F_{\varepsilon}(u) v + \frac{1}{2} \int_{\Omega} \left[ F_{\varepsilon}(u) - F_{\varepsilon}(u) - F_{\varepsilon}(u) v - \frac{1}{2} F_{\varepsilon}(u) v^2 \right]$$

$$= \frac{1}{2} \int_{\Omega} F_{\varepsilon}(u) + \frac{1}{2} \int_{\Omega} \left[ F_{\varepsilon}(u) - F_{\varepsilon}(u) - F_{\varepsilon}(u) v - \frac{1}{2} F_{\varepsilon}(u) v^2 \right]$$

$$= \frac{1}{2} \int_{\Omega} F_{\varepsilon}(u) + \frac{1}{2} \int_{\Omega} \left[ F_{\varepsilon}(u) - F_{\varepsilon}(u) - F_{\varepsilon}(u) v - \frac{1}{2} F_{\varepsilon}(u) v^2 \right]$$

where $R_{\varepsilon, \alpha, \lambda, a} := \frac{1}{2} \int_{\Omega} (F_{\varepsilon}(u) - F_{\varepsilon}(u)) v^2 - \int_{\Omega} [F_{\varepsilon}(u) - F_{\varepsilon}(u) - F_{\varepsilon}(u) v - \frac{1}{2} F_{\varepsilon}(u) v^2]$ is a $C^2$ function satisfying

$$R_{\varepsilon, \alpha, \lambda, a}(v) = o(||v||^2), \quad R_{\varepsilon, \alpha, \lambda, a}(v) = o(||v||), \quad R_{\varepsilon, \alpha, \lambda, a}(v) = o(1)$$

uniformly with respect to $\varepsilon, \alpha, \lambda, a$, $(\alpha, \lambda, a, 0) \in M_{\varepsilon}$ and $\varepsilon$ small enough.

Indeed, using (2.6) and the fact that $\ln \ln(\varepsilon + |u|) \leq \ln \lambda$, it is easy to see that

$$\int_{\Omega} (F_{\varepsilon}(u) - F_{\varepsilon}(u)) v^2 = \int_{\Omega} (f_{\varepsilon}(u) - f_{\varepsilon}(u)) v^2 = O \left( \varepsilon \ln \lambda \int_{\Omega} |u|^{p-1} v^2 \right) = o(||v||^2)$$

and letting $\Omega_v = \{x : |u| < 2||v||\}$, by the mean value theorem there exists $\theta = \theta(x) \in (0, 1)$ such that

$$\int_{\Omega} \left[ F_{\varepsilon}(u) - F_{\varepsilon}(u) - F_{\varepsilon}(u) v - \frac{1}{2} F_{\varepsilon}(u) v^2 \right] = \frac{1}{6} \int_{\Omega \setminus \Omega_v} f_{\varepsilon}'(u + \theta v) v^3 + O(\int_{\Omega_v} |v|^{p+1})$$

$$= O \left( \int_{\Omega \setminus \Omega_v} |u|^{p-2} v^3 + \int_{\Omega_v} |v|^{p+1} \right)$$

$$= O \left( ||v||^{\inf(3, p+1)} \right) = o(||v||^2),$$

where we have used (2.9) and the facts that $|u| \leq 3||v||$ in $\Omega_v$ and $|F_{\varepsilon}(s)| \leq c|s|^{p+1}$ for each $s \in \mathbb{R}$.

Moreover, we know that the quadratic term in $v$, namely $v \rightarrow \int_{\Omega} |\nabla v|^2 - \frac{n+2}{n-2} \int_{\Omega} |u|^{p-1} v^2$ is coercive, with a modulus of coercivity bounded from below as $(\alpha, \lambda, a, 0) \in M_{\varepsilon}$ and $\varepsilon$ is sufficiently small for a proof of this fact, see [2, 3, 16].
Now we compute the linear part in \( v \). Using the first point of Lemma 2.6, we get
\[
\int_{\Omega} f_{\epsilon}(u)v = \int_{\Omega} f_{\epsilon}(u) - f_{0}(u)\ |v| + \int_{\Omega} f_{0}(u)v
\]
\[
= \int_{\Omega} f_{0}(u)v + O\left( \epsilon \int_{\Omega} |u|^p \ln(e + |u|)|v| \right)
\]
\[
= \int_{\Omega} f_{0}(u)v + O\left( \|v\|\epsilon \ln \lambda \right).
\]

We recall that
\[
\int_{\Omega} f_{0}(u)v = \int_{\Omega} |u|^{p-1}u |v| = O\left( \left( \frac{1}{\lambda^{n-2}} \text{ (if } n < 6) + \frac{\ln \lambda}{\lambda} \text{ (if } n = 6) + \frac{1}{\lambda^{n+2}} \text{ (if } n > 6) \right) \|v\| \right).
\]

The proof of the last estimate may be found in [2] and [16]. (3.9) and (3.10) assert that
\[
\int_{\Omega} f_{\epsilon}(u)v = O\left( \left( \epsilon \ln \lambda + \frac{1}{\lambda^{n-2}} \text{ (if } n < 6) + \frac{1}{\lambda^{n+2}} \text{ (if } n > 6) \right) \|v\| \right).
\]

Consequently, the implicit function theorem yields the conclusion of Proposition 3.2, together with estimate (3.7).

As mentioned earlier, we will follow the ideas introduced in [4] to construct a family of solutions of \((P_{\epsilon})\). In fact, the result of Theorem 1.1 will be obtained through a careful analysis of (3.3)-(3.6) on \( \mathcal{M}_c \). Once \( \mathcal{P} \) is defined by Proposition 3.2 which we denote by \( v \) for simplicity, we estimate the corresponding numbers \( A, B, C \) by taking the scalar product in \( H^1_0(\Omega) \) of \((v_{\lambda}, \lambda \partial P_{\delta_i}/\partial \lambda_i, \lambda^{-1} \partial P_{\delta_i}/\partial a_i)\) respectively. Thus we get a quasi-diagonal system whose coefficients are given by
\[
\langle P_{\delta_i}, P_{\delta_j} \rangle = S_{n}^{2} \delta_{ij} + O\left( \frac{1}{\lambda^{n-2}} \right),
\]
\[
\langle P_{\delta_i}, \lambda \frac{\partial P_{\delta_j}}{\partial \lambda_j} \rangle = O\left( \frac{1}{\lambda^{n-1}} \right),
\]
\[
\langle P_{\delta_i}, \frac{1}{\lambda} \frac{\partial P_{\delta_j}}{\partial a_j} \rangle = O\left( \frac{1}{\lambda^{n-1}} \right),
\]
\[
\langle \lambda \frac{\partial P_{\delta_i}}{\partial \lambda_i}, \lambda \frac{\partial P_{\delta_j}}{\partial \lambda_j} \rangle = C_{1} \delta_{ij} + O\left( \frac{1}{\lambda^{n-2}} \right),
\]
\[
\langle \lambda \frac{\partial P_{\delta_i}}{\partial \lambda_i}, \frac{1}{\lambda} \frac{\partial P_{\delta_j}}{\partial a_j} \rangle = O\left( \frac{1}{\lambda^{n-1}} \right),
\]
\[
\langle \lambda \frac{1}{\lambda}, \frac{\partial P_{\delta_i}}{\partial a_j} / \frac{\partial P_{\delta_j}}{\partial a_i} \rangle = C_{2} \delta_{ij} + O\left( \frac{1}{\lambda^{n}} \right)
\]
where \( \delta_{ij} \) and \( \delta_{hl} \) are the Krönecker symbol and \( C_{1}, C_{2} \) are positive constants.

The other hand side is given by
\[
\gamma_{i} \left( \frac{\partial K_{\epsilon}}{\partial v}, P_{\delta_i} \right) = \frac{\partial K_{\epsilon}}{\partial a_i},
\]
\[
\alpha_{i} \gamma_{i} \left( \frac{\partial K_{\epsilon}}{\partial v}, \lambda \frac{\partial P_{\delta_i}}{\partial \lambda_i} \right) = \lambda_{i} \frac{\partial K_{\epsilon}}{\partial \lambda_i},
\]
\[
\alpha_{i} \gamma_{i} \left( \frac{\partial K_{\epsilon}}{\partial v}, \frac{1}{\lambda} \frac{\partial P_{\delta_i}}{\partial a_i} \right) = \frac{1}{\lambda} \frac{\partial K_{\epsilon}}{\partial a_i},
\]
\[
(3.11)
\]

Using Proposition 2.8 and the fact that \( \frac{\partial K_{\epsilon}}{\partial v} = \gamma_{i} \langle \nabla I_{c}(u), P_{\delta_i} \rangle \) and taking account of (3.7),(3.2) and \( d_k = d(a_k, \partial \Omega) > d_0 \) for each \( k \), some computations yield to
\[
\frac{\partial K_{\epsilon}}{\partial a_i} = -(p-1)S_{n}^{2} \beta_{i} + V_{a_i}(\epsilon, \lambda, \alpha, a),
\]
\[
(3.12)
\]
where 
\[ \beta := (\beta_1, \ldots, \beta_m) = (\alpha_1 - 1, \ldots, \alpha_m - 1) \]
and \( V_{\alpha_i} \) is a smooth function that satisfies
\[
V_{\alpha_i}(\varepsilon, \alpha, \lambda, a) = O\left( \beta_i^2 + \varepsilon \ln(\ln \lambda) + \frac{1}{\lambda^{\alpha_i - 2}}(n \leq 5) + \frac{\ln \lambda}{\lambda^2}(n = 6) + \frac{1}{\lambda^{n-2}}(n \geq 7) \right). \tag{3.13}
\]
In the same way, from Proposition 2.11 we get
\[
\lambda_i \frac{\partial K_i}{\partial \lambda_i} = \Gamma_1 \left( \frac{\varepsilon}{\ln \lambda_i} - \frac{\varepsilon}{\ln \lambda_i} \right) \left( \frac{H(a_i, a_i)}{n^{n-2}} - \sum_{k \neq i} \frac{\gamma_{ik}}{\lambda_i \lambda_k} \frac{\partial G(a_i, a_k)}{\partial (a_i)} \right) + V_{\lambda_i}(\varepsilon, \alpha, \lambda, a), \tag{3.14}
\]
where \( V_{\lambda_i} \) is a smooth function satisfying
\[
V_{\lambda_i} = O\left( \varepsilon^2 \ln(\ln \lambda)^2 + \frac{\varepsilon}{\ln(\ln \lambda)^2} + \frac{1}{(1-r)\varepsilon} + \frac{1}{(1-r)^2} \right) \tag{3.15}
\]
and
\[
\Gamma_2 = \frac{(n-2)\sigma_1}{2} \text{ where } \sigma_1 \text{ is defined in (2.2).}
\]
Lastly, by using Proposition 2.12, we have
\[
\frac{1}{\lambda_i} \frac{\partial K_i}{\partial (a_i)_{j_1}} = \frac{\Gamma_3}{\lambda_i} \left( \frac{1}{\lambda_i^{n-2}} \frac{\partial H}{\partial (a_i)}(a_i, a_i) - \sum_{k \neq i} \frac{\gamma_k}{\lambda_i \lambda_k} \frac{\partial G}{\partial (a_i)}(a_i, a_k) \right) + V_{(a_i)_{j}}(\varepsilon, \alpha, \lambda, a), \tag{3.16}
\]
where \( \frac{\partial}{\partial a_i} \) and \( \frac{\partial}{\partial a_k} \) denote the derivatives with respect to the first variable and the second variable of the functions \((a, b) \rightarrow H(a, b)\) and \((a, b) \rightarrow G(a, b), V_{\alpha_i} \) is a smooth function such that
\[
V_{(a_i)_{j}}(\varepsilon, \alpha, \lambda, a) = O\left( \varepsilon^2 \ln(\ln \lambda)^2 + \frac{\varepsilon}{\ln(\ln \lambda)^2} + \frac{1}{(1-r)^2} \right) \tag{3.17}
\]
and
\[
\Gamma_3 = \frac{\sigma_2}{2}.
\]
Notice that these estimates imply
\[
\frac{\partial K_i}{\partial \alpha_i} = O\left( \frac{1}{\lambda^{n-2}} \ln(\ln \lambda) + \frac{1}{\lambda^{n-2}} \ln(\ln \lambda) \right) \text{ (if } n < 6 \right) + \frac{\ln \lambda}{\lambda^4} \right) \text{ (if } n = 6 \right) + \frac{1}{\lambda^{n+2}} \right) \text{ (if } n > 6 \right); \]
\[
\lambda_i \frac{\partial K_i}{\partial \lambda_i} = O\left( \frac{1}{\lambda^{n-2}} + \varepsilon^2 \ln(\ln \lambda)^2 \right); \]
\[
\frac{1}{\lambda_i} \frac{\partial K_i}{\partial (a_i)_{j}} = O\left( \frac{1}{\lambda^{n-1}} + \varepsilon^2 \ln(\ln \lambda)^2 \right). \]
The solution of the system in \( A, B \) and \( C \) shows that
\[
\begin{cases} 
A = O\left( \frac{1}{\lambda^{n-2}} + \frac{1}{\lambda^{n-2}} \ln(\ln \lambda) \right) \text{ (if } n < 6 \right) + \frac{\ln \lambda}{\lambda^4} \right) \text{ (if } n = 6 \right) + \frac{1}{\lambda^{n+2}} \right) \text{ (if } n > 6 \right), \\
B = O\left( \frac{1}{\lambda^{n-2}} + \frac{1}{\lambda^{n-2}} + \varepsilon^2 \ln(\ln \lambda)^2 \right), \\
C_{ij} = O\left( \frac{1}{\lambda^{n-1}} + \varepsilon^2 \ln(\ln \lambda)^2 + \frac{1}{\lambda^{n-2}} \right). 
\end{cases} \tag{3.18}
\]
This allows us to evaluate the right hand side in the equations \((E_{\lambda_i})\) and \((E_{\alpha_i})\), namely
\[
B_i \left( \lambda \frac{\partial^2 P_{\delta_i}}{\partial \lambda_i^2}, v \right) + \sum_{j=1}^n C_{ij} \left( \frac{1}{\lambda_i} \frac{\partial^2 P_{\delta_i}}{\partial (a_i)_j \partial \lambda_i}, v \right) = O\left( \frac{1}{\lambda_i} \left( \frac{1}{\lambda^{n-2}} + \varepsilon^2 \ln(\ln \lambda)^2 \right) \|v\| \right) \tag{3.19}
\]
and
\[
B_i \left( \lambda \frac{\partial^2 P_{\delta_i}}{\partial \lambda_i \partial (a_i)_j}, v \right) + \sum_{k=1}^n C_{ik} \left( \frac{1}{\lambda_i} \frac{\partial^2 P_{\delta_i}}{\partial (a_i)_k \partial (a_i)_j}, v \right) = O\left( \lambda_i \left( \frac{1}{\lambda^{n-2}} + \varepsilon^2 \ln(\ln \lambda)^2 \right) \|v\| \right), \forall j \tag{3.20}
\]
since
\[ \left\| \frac{\partial^2 P_{\delta_1}}{\partial \lambda_i^2} \right\| = O(\frac{1}{\lambda_i^2}), \quad \left\| \frac{\partial^2 P_{\delta_1}}{\partial \lambda_i \partial a_i} \right\| = O(1), \quad \left\| \frac{\partial^2 P_{\delta_1}}{\partial a_i^2} \right\| = O(\lambda_i^2). \] (3.21)

Now, we consider a point \( \overline{\mathbf{x}} = (\overline{x}_1, \ldots, \overline{x}_m) \in \Omega^m \) such that \( \rho(\overline{\mathbf{x}}) > 0 \) and \( \overline{\mathbf{x}} \) is a non-degenerate critical point of \( \overline{\mathbf{F}} \) where \( \overline{\mathbf{F}} \) is introduced in (1.11). We set
\[ \frac{1}{\lambda_i^{(n-2)/2}} = \left( \frac{(n-2)\Gamma_1}{\Gamma_2} \right)^{1/2} (\Lambda_i(\overline{\mathbf{x}}) + \zeta_i) \left( \frac{\varepsilon}{\ln \varepsilon} \right)^{1/2}, \quad i = 1, \ldots, m, \]
where \( \zeta_i \in \mathbb{R} \) and \( \xi_i \in \mathbb{R}^N \) are assumed to be small and \( (\Lambda_1(x), \ldots, \Lambda_m(x)) \) is defined as the minimum of \( \mathbf{F}_x \).

With these changes of variables and using (3.12) and (3.13), \( (E_{\alpha_i}) \) is equivalent to
\[ \beta_i = V_{\beta_i}(\varepsilon, \beta, \zeta, \xi) = O(\varepsilon \ln |\ln \varepsilon| + |\beta|^2). \] (3.22)

Now, using the changes of variables, an easy computation shows that
\[ \ln \lambda_i = \frac{|\ln \varepsilon|}{n-2} \left( 1 + O\left( \frac{|\ln \varepsilon|}{|\ln \varepsilon|} \right) \right) \quad \text{and} \quad \frac{1}{\ln \lambda_i} = \frac{n-2}{|\ln \varepsilon|} + O\left( \frac{|\ln \varepsilon|}{|\ln \varepsilon|} \right). \] (3.23)

Moreover, we have
\[ H(a_i, a_i) = H(\overline{\mathbf{x}}, \overline{\mathbf{x}}) + 2\frac{\partial H}{\partial a}(\overline{\mathbf{x}}, \overline{\mathbf{x}}) a_i + O(|a_i|^2) \] (3.24)
and
\[ G(a_i, a_j) = G(\overline{\mathbf{x}}, \overline{\mathbf{x}}) + \frac{\partial G}{\partial a}(\overline{\mathbf{x}}, \overline{\mathbf{x}}) a_i + \frac{\partial G}{\partial b}(\overline{\mathbf{x}}, \overline{\mathbf{x}}) a_j + O(|a_i|^2). \] (3.25)

Set \( \overline{\lambda}_i = \lambda_i(\overline{\mathbf{x}}) \) for each \( 1 \leq i \leq m \), we have
\[ 1 - H(\overline{\mathbf{x}}, \overline{\mathbf{x}}) \overline{\lambda}_i + \sum_{j \neq i} \gamma_{ij} G(\overline{\mathbf{x}}, \overline{\mathbf{x}}) \overline{\lambda}_i \overline{\lambda}_j = 0 \] (3.26)
since \( \overline{\mathbf{x}} = (\overline{\lambda}_1, \ldots, \overline{\lambda}_m) \) is a critical point of \( \mathbf{F}_x \).

(3.14), (3.15) and (3.19) imply that \( (E_{\lambda_i}) \) is equivalent, while using (3.23)-(3.26), to
\[ \left( 2H(\overline{\mathbf{x}}, \overline{\mathbf{x}}) \overline{\lambda}_i - \sum_{j \neq i} \gamma_{ij} G(\overline{\mathbf{x}}, \overline{\mathbf{x}}) \overline{\lambda}_j \right) \xi_i - \sum_{j \neq i} \gamma_{ij} G(\overline{\mathbf{x}}, \overline{\mathbf{x}}) \overline{\lambda}_i \zeta_j \]
\[ + \left( 2\frac{\partial H}{\partial a}(\overline{\mathbf{x}}, \overline{\mathbf{x}}) \overline{\lambda}_i - \sum_{j \neq i} \gamma_{ij} \frac{\partial G}{\partial a}(\overline{\mathbf{x}}, \overline{\mathbf{x}}) \overline{\lambda}_j \overline{\lambda}_j \right) \xi_i - \sum_{j \neq i} \gamma_{ij} \frac{\partial G}{\partial b}(\overline{\mathbf{x}}, \overline{\mathbf{x}}) \overline{\lambda}_i \overline{\lambda}_j \xi_j = V_\varepsilon(\varepsilon, \beta, \zeta, \xi), \] (3.27)
where
\[ V_\varepsilon(\varepsilon, \beta, \zeta, \xi) = O\left( \frac{|\ln \varepsilon|}{|\ln \varepsilon|} + |\beta|^2 + |\xi|^2 + \xi^2 \right). \]
We also have
\[ \frac{\partial H}{\partial a}(a_i, a_i) = \frac{\partial H}{\partial a}(\overline{\mathbf{x}}, \overline{\mathbf{x}}) + \frac{\partial^2 H}{\partial a^2}(\overline{\mathbf{x}}, \overline{\mathbf{x}}) a_i + \frac{\partial^2 H}{\partial a \partial b}(\overline{\mathbf{x}}, \overline{\mathbf{x}}) a_i + O(|a_i|^2) \] (3.28)
and
\[ \frac{\partial G}{\partial a}(a_i, a_j) = \frac{\partial G}{\partial a}(\overline{\mathbf{x}}, \overline{\mathbf{x}}) + \frac{\partial^2 G}{\partial a^2}(\overline{\mathbf{x}}, \overline{\mathbf{x}}) a_i + \frac{\partial^2 G}{\partial a \partial b}(\overline{\mathbf{x}}, \overline{\mathbf{x}}) a_j + O(|a_i|^2) \] (3.29)
Through (3.16), (3.17) and (3.20), and by using (3.28), (3.29), we get that \( (E_{a_i}) \) is equivalent to
\[ \frac{\partial H}{\partial a}(\overline{\mathbf{x}}, \overline{\mathbf{x}}) a_i - \sum_{j \neq i} \gamma_{ij} \frac{\partial G}{\partial a}(\overline{\mathbf{x}}, \overline{\mathbf{x}}) a_j \zeta_j + \left( \frac{\partial^2 H}{\partial a^2}(\overline{\mathbf{x}}, \overline{\mathbf{x}}) \overline{\lambda}_i + \frac{\partial^2 H}{\partial a \partial b}(\overline{\mathbf{x}}, \overline{\mathbf{x}}) \overline{\lambda}_i - \sum_{j \neq i} \gamma_{ij} \frac{\partial^2 G}{\partial a \partial b}(\overline{\mathbf{x}}, \overline{\mathbf{x}}) \overline{\lambda}_j \right) \xi_i \]
\[ - \sum_{j \neq i} \gamma_{ij} \frac{\partial^2 G}{\partial a \partial b}(\overline{\mathbf{x}}, \overline{\mathbf{x}}) \overline{\lambda}_i \overline{\lambda}_j \xi_j = V_\varepsilon(\varepsilon, \beta, \zeta, \xi), \] (3.30)
where
\[ V_\xi(\varepsilon, \beta, \zeta, \xi) = O\left( \frac{\ln |\ln \varepsilon|}{\ln \varepsilon} + |\beta|^2 + |\zeta|^2 + \xi^2 \right). \]
Furthermore, (3.22), (3.27) and (3.30) may be written as
\[
\begin{aligned}
\begin{cases}
\beta = V(\varepsilon, \beta, \zeta, \xi), \\
L(\zeta, \xi) = W(\varepsilon, \beta, \zeta, \xi),
\end{cases}
\end{aligned}
\] (3.31)
where \( L \) is a fixed linear operator of \( \mathbb{R}^{m(n+1)} \) defined by (3.27) and (3.30) and \( V, W \) are smooth functions satisfying
\[
\begin{aligned}
\begin{cases}
V(\varepsilon, \beta, \zeta, \xi) = O(\varepsilon \ln |\ln \varepsilon| + |\beta|^2), \\
W(\varepsilon, \beta, \zeta, \xi) = O\left( \frac{\ln |\ln \varepsilon|}{\ln \varepsilon} + |\beta|^2 + |\zeta|^2 + \xi^2 \right).
\end{cases}
\end{aligned}
\] (3.32)
Moreover, a simple computation shows that the determinant of \( L \) is proportional to the determinant of \( \tilde{F}'(\varpi) \), \( \varpi \) being a non-degenerate critical point of \( F \) by assumption, \( L \) is invertible, and Brouwer’s fixed point theorem ensures, provided that \( \varepsilon \) is small enough, the existence of a solution \((\beta_\varepsilon, \zeta_\varepsilon, \xi_\varepsilon)\) to (3.31), such that
\[ |\beta_\varepsilon| = O(\varepsilon \ln |\ln \varepsilon|), \quad |\zeta_\varepsilon| = O\left( \frac{\ln |\ln \varepsilon|}{\ln \varepsilon} \right) \quad \text{and} \quad |\xi_\varepsilon| = O\left( \frac{\ln |\ln \varepsilon|}{\ln \varepsilon} \right). \]
By construction, \( u_\varepsilon = \sum_{i=1}^{m} \alpha_{i,\varepsilon} \gamma_i P_{\delta_{i,\varepsilon}}(\Lambda_i, \zeta_\varepsilon, \xi_\varepsilon) + \varpi(\varepsilon, \alpha_\varepsilon, \lambda_\varepsilon, \alpha_\varepsilon) \in H_0^1(\Omega) \) with
\[ \alpha_{i,\varepsilon} = 1 + \beta_{i,\varepsilon}; \quad \frac{1}{\lambda_{i,\varepsilon}} = \left( \frac{(n-2)\Gamma_1}{\Gamma_2} \right)^{\frac{1}{n-2}} \left( \Lambda_i(\varpi) + \zeta_{i,\varepsilon} \right)^{-\frac{n-2}{2}} \left( \frac{\varepsilon}{\ln \varepsilon} \right)^{\frac{1}{n-2}} \lambda_{i,\varepsilon}, \]
is a critical point of \( I_\varepsilon \), whence
\[-\Delta u_\varepsilon = \frac{|u_\varepsilon|^{p-1}u_\varepsilon}{|\ln(\varepsilon + |u_\varepsilon|)|^s} \quad \text{in} \ \Omega.\]
The proof of Theorem 1.1 is thereby completed. \( \square \)

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