A remark on a result of Helfgott, Roton and Naslund

Gyan Prakash
Harish-Chandra Research Institute
Chhatnag Road, Jhunsi
Allahabad 211019, India.
E-mail: gyan@hri.res.in

Abstract

Let \( F(X) = \prod_{i=1}^{k} (a_i X + b_i) \) be a polynomial with \( a_i, b_i \) being integers. Suppose the discriminant of \( F \) is non-zero and \( F \) is admissible. Given any natural number \( N \), let \( S(F,N) \) denotes those integers less than or equal to \( N \) such that \( F(n) \) has no prime factors less than or equal to \( N^{-1/(4k+1)} \). Let \( L \) be a translation invariant linear equation in \( 3 \) variables. Then any \( A \subset S(F,N) \) with \( \delta_F(N) := \frac{\text{card}(A)}{\text{card}(S(F,N))} \gg \epsilon, F, L \frac{1}{(\log \log N)^{1.1}} \) contains a non-trivial solution of \( L \) provided \( N \) is sufficiently large.

Given \( A \subset \mathbb{N} \) and a natural number \( N \) we set \( A(N) := A \cap [1, N] \). Given any natural number \( k \) we write \( \log_k N := \underbrace{\log \ldots \log}_{k\text{-times}} N \). Given a subset \( P_1 \) of the set of primes \( P \), we define the relative density \( \delta_{P}(N) = \frac{\text{card}(P_1(N))}{\text{card}(P(N))} \). In [4], Ben Green showed that any subset \( P_1 \) of the set of primes \( P \), with relative density \( \delta_{P}(N) \geq c(\log_5 N / \log_3 N)^{1/2} \) for some \( N \geq N_0 \), where \( c \) and \( N_0 \) are absolute constants, contains a non-trivial 3-term arithmetic progression. In [5], H. Helfgott and A. De. Roton improved this result to show that the same conclusion holds under the weaker assumption that the relative density \( \delta_{P}(N) \geq c(\log_3 N / \log_2 N)^{1/3} \). In [6], Eric Naslund, using a modification of the arguments of Helfgott and Roton, showed that the result holds under even a weaker assumption \( \delta_{P}(N) \geq c(\epsilon)(1 / \log_2 N)^{1-\epsilon} \), where \( \epsilon > 0 \) is any real number and \( c(\epsilon) > 0 \) is a constant depending only on \( \epsilon \). The purpose of this note is to observe that the arguments of Helfgott and Roton [5] and Eric Naslund [6] gives a more general result, namely Theorem 1.1 stated below.

Let \( F(X) = \prod_{i=1}^{k} (a_i X + b_i) \in \mathbb{Z}[X] \) with \( a_i \in \mathbb{Z} \setminus \{0\} \) and \( b_i \in \mathbb{Z} \). Moreover we suppose that

(i) the discriminant of \( F; \Delta(F) = \prod_{i=1}^{k} a_i \prod_{i \neq j} (a_i b_j - b_i a_j) \neq 0 \) and

(ii) is admissible that is to say that for all primes \( p \), there exists \( n \in \mathbb{Z} \) such that \( F(n) \neq 0 \) (mod \( p \)).

Hardy-Littlewood conjecture predicts that for any \( F \) as above, the number of integers \( n \leq N \) such that \( a_i n + b_i \) is prime for all \( i \) is asymptotically equal to \( c(F) \frac{N}{\log^k N} \). This is not known except the case when \( F \) is a linear polynomial. However using Brun’s sieve we know a lower bound for the number of \( n \leq N \) such that the number of prime factors of \( a_i n + b_i \) is at most \( 4k + 1 \). Given any real number \( z > 0 \), let \( P(z) = \prod_{p \leq z} p \) and we set

\[
S_F(N,z) = \{ n \leq N : \text{gcd} (F(n), P(z)) = 1 \}. \tag{1}
\]
Then using Brun’s sieve [2, see page number 78, (6.107)], we know the following lower bound
\[
\text{Card} \left( S_F(N, N^{1/(4k+1)}) \right) \geq c_1(F) \frac{N}{\log^k N},
\]  
(2)
where \( c_1(F) > 0 \) is a constant depending only upon \( F \). Given any \( A \subset S_F(N, N^{1/(4k+1)}) \)
we define the relative density \( \delta_F(N) \) of \( A \) to be
\[
\delta_F(N) = \frac{\text{Card}(A(N))}{\text{Card}(S_F(N, N^{1/(4k+1)}))}.
\]
(3)
For the brevity of notation, we shall also write \( \delta(N) \) or simply \( \delta \) to denote \( \delta_F(N) \).

Let \( s \geq 3 \) be a natural number and
\[
L := c_1x_1 + \cdots + c_sx_s = 0
\]
(4)
be a linear equation with \( c_i \in \mathbb{Z} \setminus \{0\} \). The linear equation \( L \) is said to be translation invariant if \( \sum_i c_i = 0 \). A solution \((x_1, \cdots, x_s)\) of \( L \) is said to be non-trivial if for some \( i, j \) we have \( x_i \neq x_j \).

**Theorem 0.1.** Let \( k \) be a natural number, \( F(X) = \prod_{i=1}^{k}(a_iX + b_i) \in \mathbb{Z}[X] \) with \( a_i \in \mathbb{Z} \setminus \{0\} \) and \( b_i \in \mathbb{Z} \) for all \( i \). Suppose that \( F \) is admissible and the discriminant of \( F \) is non-zero. Let \( L \) be a translation invariant linear equation as defined in (4) with \( s = 3 \). Then given any \( \epsilon > 0 \) there exists a constant \( c(F, L, \epsilon) > 0 \) and a natural number \( N(F, L, \epsilon) \) such that the following holds. Given any \( N \geq N(F, L, \epsilon) \), any set \( A \subset S_F(N, N^{1/(4k+1)}) \),
\[
\delta_F(N) \geq c(F, L, \epsilon) \frac{1}{(\log \log N)^{1-\epsilon}}.
\]
(5)
contains a non-trivial solution of \( L \).

**Remark 0.2.** (i) Let \( P \) be the set of primes. In the above theorem, taking \( F(n) = n \), \( L := x_1 + x_2 - 2x_3 = 0 \), and \( A \) to be a subset of primes \( P(N) \), with \( \text{Card}(A) \geq \delta \text{Card}(P(N)) \), with \( \delta \) satisfying (5), one recovers the result of Eric Naslund [6] stated above.

(ii) We say that a prime \( p \) is a Chen prime if \( p + 2 \) has at most two prime factors and also any prime factor of \( p + 2 \) is greater than \( p^{1/10} \). Green and Tao [3, Theorem 1.2] had shown that Chen primes contain a non-trivial 3-term arithmetic progression. In the above theorem taking \( F(X) = X(X + 2), L := x_1 + x_2 - 2x_3 = 0 \), we obtain any subset \( A \) of Chen primes with relative density \( \delta \), with \( \delta \) satisfying (5), contains a non-trivial 3-term arithmetic progression.

**Definition 0.3.** Let \( L \) be a translation invariant linear equation in \( s \) variables as in (4).

(i) Let \( h_L : (0, 1) \to \mathbb{R} \) be a non-negative function satisfying the following. Given any prime \( P \) and a set \( A \subset \mathbb{Z}/P\mathbb{Z} \) with \( \text{Card}(A) \geq \eta P \), the number of solutions of \( L \) in \( A \) is at-least \( h_L(\eta)P^{s-1} \).

(ii) Let \( g_L : \mathbb{N} \to \mathbb{R}_0^+ \) be a monotonically decreasing function with \( \lim_{N \to \infty} g_L(N) = 0 \) and satisfying the following properties. There exists a natural number \( N_0 \) such that given any \( N \geq N_0 \), any set \( A \subset [1, N] \) with \( |A| \geq g_L(N)N \) contains a non-trivial solution of \( L \). Given \( \eta > 0 \), let \( g^*_L(\eta) \) denotes the smallest natural number \( m \) such that \( g_L(m) \leq \eta \).
Remark 0.4. (i) Let $g_L$ be a function as in Definition 0.3. When the number of variables $s$ in $L$ is equal to $3$, then using an arguments due to Varnavides, it can be shown that the function $h_L(\eta) = \frac{\eta}{(2g_L^{s-1}(\eta/2))^s}$ is a function satisfying the properties as in Definition 0.3 (i). If we can find a similar relation between the function $g_L^s$ and $h_L$ for $s > 3$, then the result of Theorem 0.1 can be extended for $s > 3$ using the following result of Thomas Bloom.

(ii) In [1], Thomas Bloom showed that there exists an absolute constant $c > 0$ depending only on $L$ such that the function $g_L(N) = c\left(\frac{\log^5 \log N}{\log N}\right)^{s-2}$ satisfies the above properties. In this case $g_L^{s-1}(\eta) \leq \exp(c_1 \eta^{-1/(s-2)} \log \log(\frac{1}{\eta}))$ with $c_1 > 0$ being a constant depending only upon $L$.

Let $z = \frac{\log N}{3}$ and $M = \prod_{p \leq z} p$. For any $b \in \{0, 1, \ldots, M - 1\}$, we set

$$A_b = \{n : n \in A, n \equiv b \pmod{M}\}$$

We notice that

$$A_b \subset \{n \leq N/M : \gcd(F(n + bM), P(N^{1/(4k+1)}) = 1\}.$$ 

The following lemma is an easy consequence of W-trick due to Ben Green.

Lemma 0.5 (W-trick). There exists a $b_0 \in \{0, 1, \ldots, M - 1\}$ such that $\gcd(F(b_0), M) = 1$ and

$$\text{Card}(A_{b_0}) \geq c(F) \delta \log^k \log N \frac{N}{\log^k N} M,$$

where $c(F) > 0$ is a constant depending only upon $F$ and $\delta$ is as in (3).

Proof. Since $z \leq N^{1/(4k+1)}$, it follows that if $A_b = \emptyset$, then $F(b) \not\equiv 0 \pmod{p}$ for all $p \leq z$. Now for $p$ which does not divide $\Delta(F) \prod_{i=1}^k a_i$, the number of solutions $n \in \mathbb{Z}/p\mathbb{Z}$ of the equation $F(n) \equiv 0 \pmod{p}$ is equal to $k$. Let $\Delta'(F) = \Delta(F) \prod_{i=1}^k a_i$. Then using Chinese remainder theorem, it follows that the number of $b \in \{0, 1, \ldots, M - 1\}$ such that $A_b$ is not an empty set is at most $\frac{\prod_{p \leq z} (p-k) \prod_{p \mid \Delta'(F)} \frac{1}{p}}{\prod_{p \mid \Delta'(F)} (p-k)}$. Using this, the identity

$$\sum_{b=0}^{M-1} \text{card}(A_b) = \text{card}(A)$$

and (2), it follows that there exists a $b_0$ such that

$$\text{Card}(A_{b_0}) \geq c(F) \delta \prod_{p \leq z} (p-k)^{-1} \frac{N}{\log^k N} = c(F) \delta \prod_{p \leq z} \left(1 - \frac{k}{p}\right)^{-1} \frac{N}{M \log^k N},$$

where $c(F) = c_1(F) \prod_{p \mid \Delta'(F)} \frac{p-k}{p}$ with $c_1(F)$ as in (2). The lemma follows using this, and Mertens formula.

Let $b_0 \in \{0, 1, \ldots, M - 1\}$ be as provided by Lemma 0.5. Without any loss of generality, we may assume that $c_1, \cdots, c_r > 0$ and $c_{r+1}, \cdots, c_s < 0$. Let

$$c = c_1 + \cdots + c_r.$$
Let $P \in [cN/M, 2cN/M]$ be a prime and $A'$ denote the image of $A_{b_0}$ in $\mathbb{Z}/P\mathbb{Z}$ under the natural projection map. The set $A_{b_0}$ contains a non-trivial solution of $L$ if and only if $A'$ contains a non-trivial solution of $L$. We shall prove Theorem 0.3 by showing that $A'$ contains a non-trivial solution.

For any set $C \subset \mathbb{Z}/P\mathbb{Z}$, we set $d(C) = \frac{\text{card}(C)}{P}$ to denote the density of $C$ in $\mathbb{Z}/P\mathbb{Z}$. Given any set $C \subset \mathbb{Z}/P\mathbb{Z}$, let $f_C : \mathbb{Z}/P\mathbb{Z} \to \mathbb{R}^+$ be the function defined as $f_C(n) = \frac{1}{d_C} I_C(n)$. For any function $f : \mathbb{Z}/P\mathbb{Z} \to \mathbb{C}$, we set $\mathbb{E}(f) := \frac{1}{P} \sum_{n \in \mathbb{Z}/P\mathbb{Z}} f(n)$. Then we may verify that for any set $C$, we have $\mathbb{E}(f_C) = 1$. Given any integer $l \geq 1$, we write $\|f\|_l := (\mathbb{E}(|f|^l))^{1/l}$.

The Fourier transform of $f$ is a function $\hat{f} : \mathbb{Z}/P\mathbb{Z} \to \mathbb{C}$ defined as $\hat{f}(t) = \mathbb{E}(f(y) \exp(2\pi iyt))$. We also set

$$\Lambda_L(f) := \sum_{n_1, \ldots, n_s \in \mathbb{Z}/P\mathbb{Z}, \sum_i c_i n_i = 0} \prod_{i=1}^{s} f(n_i).$$

The following identity is easy to verify:

$$\Lambda_L(f) = P^{s-1} \sum_{t \in \mathbb{Z}/P\mathbb{Z}} \prod_{i=1}^{s} \hat{f}(c_i t).$$

Let $G$ be a finite commutative group. Given functions $f, g : G \to \mathbb{C}$, we define the convolution function $f * g : G \to \mathbb{C}$ as follows:

$$f * g(n) = \frac{1}{|G|} \sum_{y \in G} f(n - y)g(y). \quad (6)$$

**Proposition 0.6.** Let $A' \subset \mathbb{Z}/P\mathbb{Z}$ be as above and suppose $\delta > \log^{-100} P$. Let $B \subset [-\frac{P}{2}, \frac{P}{2}]$ with $\text{card}(B) \geq \log^{k+101} P$, then given any integer $l \geq 2$, we have

$$\Lambda_L(f_A' * f_B) \geq c_1 h_L(c_2 \delta^{\frac{1}{1-l}}) P^{s-1}, \quad (7)$$

where $\delta$ is as defined in (4) and $c_1, c_2 > 0$ are constant depending only upon $F, l$ and the linear equation $L$.

**Proposition 0.7.** Let $\epsilon_1, \epsilon_2 > 0$ be real numbers. Let $A' \subset \mathbb{Z}/P\mathbb{Z}$ be as above and let $S_{\epsilon_1} \subset \mathbb{Z}/P\mathbb{Z}$ be the set defined as $S_{\epsilon_1} = \text{Spec}_{\epsilon_1}(f_A') = \{t \in \mathbb{Z}/P\mathbb{Z} : |\hat{f}_A'(t)| > \epsilon_1\}$. Let $B \subset \mathbb{Z}/P\mathbb{Z}$ such that for every $t \in S = \bigcup_i \epsilon_1 c_i^{-1} S_{\epsilon_1}$, we have $|\hat{f}_B(t) - 1| \leq \epsilon_2$, then we have

$$|\Lambda_L(f_A') - \Lambda_L(f_A' * f_B)| \leq c(F) \frac{\epsilon_2 + \epsilon_1^{0.5}}{\delta^{0.5}} P^{s-1},$$

where $\delta$ is as defined in (4) and $c(F) > 0$ is a constant depending only upon $F$.

Let $G(X) = F(b + XM)$ be the polynomial with integer coefficients and let $S \subset \mathbb{Q}$ be the set of roots of $G$. For proving Proposition 0.6 we shall use the following result, which we prove using beta sieve.

**Proposition 0.8.** Let $h_1, h_2, \cdots, h_r$ be distinct integers with $|h_i| \leq N^{100}$ $\forall i$. Moreover suppose for $i \neq j$, we have $h_i - h_j \notin (S - S) \cap \mathbb{Z}$, where $S$ is the set of roots of the polynomial $G(X) = F(b + XM)$. Then we have

$$\text{Card}((A_{b_0} + h_1) \cap \cdots \cap (A_{b_0} + h_r)) \leq c(F, r) \frac{N \log kr}{M \log kr} \frac{z}{N}, \quad (8)$$

where $c(F, r) > 0$ is a constant depending only upon $F$ and $r$, and in particular does not depend upon $h_i$'s.
1 Proof of Proposition 0.6

In this section, we shall prove Proposition 0.6 using Proposition 0.8.

Given any \( f : \mathbb{Z}/P\mathbb{Z} \to \mathbb{R}^+ \), let \( D(f) \) be the subset of \( \mathbb{Z}/P\mathbb{Z} \) defined by \( D(f) := \{ n \in \mathbb{Z}/P\mathbb{Z} : f(n) > 1/2 \} \). The following two lemmas are easy to verify.

**Lemma 1.1.** Let \( f : \mathbb{Z}/P\mathbb{Z} \to \mathbb{R}^+ \) be a function with \( \mathbb{E}(f) = 1 \). Then we have
\[
\frac{1}{P} \sum_{n \in D(f)} f(n) \geq \frac{1}{2}. 
\]

**Proof.** The result follows by observing that \( \mathbb{E}(f) = \frac{1}{P} \sum_{n \notin D(f)} f(n) + \frac{1}{P} \sum_{n \in D(f)} f(n) \) and \( \frac{1}{P} \sum_{n \notin D(f)} f(n) \leq \frac{1}{P} \sum_{n \in \mathbb{Z}/P\mathbb{Z}} \frac{1}{2} \leq \frac{1}{2} \). \( \square \)

**Lemma 1.2.** For any \( f : \mathbb{Z}/P\mathbb{Z} \to \mathbb{R}^+ \) with \( \text{Card}(D(f)) \geq \eta P \), we have
\[
\Lambda_L(f) \geq \frac{1}{2^s} h_L(\eta) P^{s-1}. 
\]

For this we need the following result which follows using the arguments from [3] and [6].

**Theorem 1.3.** Let \( C \subset \mathbb{Z}/P\mathbb{Z} \) be a set with the following properties. There exists a subset \( S' \subset \mathbb{Z}/P\mathbb{Z} \) with \( S' = -S' \), \( 0 \in S' \) and \( \text{card}(S') \leq t \) such that given any integer \( l \geq 2 \) and \( h_1, \ldots, h_l \in \mathbb{Z}/P\mathbb{Z} \) with \( h_i - h_j \notin S' \) for \( i \neq j \), we have
\[
d \left( (C + h_1) \cap \cdots \cap (C + h_l) \right) \leq \frac{c(l)}{\beta^l} d(C)^l, \tag{9}
\]
for some \( \beta \leq 1 \) and where \( c(l) > 0 \) is a constant depending only upon \( l \). Then for any \( B \subset \mathbb{Z}/P\mathbb{Z} \) with \( \text{card}(B) \geq \frac{1}{n(C)} \), we have

(i) the cardinality of the set \( D := \{ n \in \mathbb{Z}/P\mathbb{Z} : f_C * f_B \geq 1/2 \} \) is at least \( c \beta^l(l-1) P \) and

(ii) and
\[
\Lambda_L(f_C * f_B) \geq \frac{1}{2^s} h_L \left( c \beta^l(l-1) \right) P^{s-1}, \tag{10}
\]
where \( c > 0 \) is a constant depending only upon \( t \) and \( l \).

First we prove Proposition 0.6 using Theorem 1.3 and Proposition 0.8.

**Proof of Proposition 0.6.** Let \( S \subset \mathbb{Q} \) be the set of roots of the polynomial \( G(X) = F(b + M X) \in \mathbb{Z}[X] \) as in Proposition 0.8 and \( \pi : \mathbb{Z} \to \mathbb{Z}/P\mathbb{Z} \) be the natural projection map. We shall prove the proposition by showing that the assumptions of Theorem 1.3 are satisfied with \( C = A' \) and \( S' = \pi((S - S) \cap \mathbb{Z}) \) and \( t = k^2 \) and \( \beta = \delta \).

Since \( \text{card}(B) \geq (\log P)^{k+101} \) and \( \delta \geq \frac{1}{(\log P)^{101} P} \), using Lemma 0.5 it follows that we have \( \text{card}(B)d(C) \geq 1 \).

We have \( S' = -S' \), \( 0 \in S' \) and
\[
\text{card}(S') \leq \text{card}(S - S) \leq \text{card}(S)^2 \leq k^2. 
\]

Let \( h_1, \ldots, h_l \in \mathbb{Z}/P\mathbb{Z} \) be such that for \( i \neq j \), we have \( h_i - h_j \notin S' \). The result follows by showing that (9) holds with \( \beta = \delta \), where \( \delta \) is as in (4). Given \( x \in \mathbb{Z}/P\mathbb{Z} \), let \( \bar{x} \) be
the integer in \([0, P]\) with \(\pi(x) = x\). By re-ordering \(h_i\)'s, if necessary, we may assume that \(\tilde{h}_1 > \tilde{h}_2 > \ldots > \tilde{h}_l\).

Given any \(n \in \cap_i(C + h_i)\), it follows that \(n - \tilde{h}_i \in A_{b_0}\) for all \(i\). Now we observe a relation between \(n - \tilde{h}_i\) and \(n - \tilde{h}_1\). For this note that for any \(i\), we have \(n - \tilde{h}_1 + \tilde{h}_1 - \tilde{h}_i \in [0, 2P)\).

If \(n - \tilde{h}_1 + \tilde{h}_1 - \tilde{h}_i \in [0, p)\) then we have \(n - \tilde{h}_i = n - \tilde{h}_1 + \tilde{h}_1 - \tilde{h}_i\) and if \(n - \tilde{h}_1 + \tilde{h}_1 - \tilde{h}_i \in [P, 2P)\), then we have \(n - \tilde{h}_i = n - \tilde{h}_1 + \tilde{h}_1 - \tilde{h}_i - P\). Using this it follows there exists \(j \leq l\) such that \(n - \tilde{h}_1 \in A^j\),

where \(A^j = \bigcap_{i=1}^j (A_{b_0} + \tilde{h}_i - \tilde{h}_1) \cap_{i=j+1}^l (A_{b_0} + \tilde{h}_i - \tilde{h}_1 + P)\). Therefore it follows that

\[
\text{Card}(\cap_i(C + h_i)) \leq \sum_{j=1}^r \text{Card}(A^j).
\]

Since the condition \(h_i - h_j \notin S'\) implies that for any \(m \in \mathbb{Z}\), we have \(\tilde{h}_i - \tilde{h}_j + mP \notin S - S\), using Proposition 0.8 it follows that for any \(j\), we have \(\text{Card}(A^j) \leq c(F, r)\frac{N \log kr}{M \log N}\) and hence

\[
d(\cap_i(C + h_i)) \leq c(F, r)\frac{\log kr \log N}{\log kr N}. \tag{11}
\]

Since \(\text{Card}(C) = \text{Card}(A_{b_0})\), using Lemma 0.5 we have

\[
\frac{\log kr \log N}{\log kr N} \leq \frac{c(F)d(C)}{\delta}. \tag{12}
\]

Therefore using (11) and (12), it follows that (9) holds with \(\beta = \delta\). Hence the result follows.

Now we shall prove Theorem 1.3. For this we use the following observation from [6].

**Proposition 1.4.** Let \(f : \mathbb{Z}/P\mathbb{Z} \to \mathbb{R}^+\) be a non negative real valued function with \(\mathbb{E}(f) = 1\). Then if \(\|f\|_l \leq \frac{c}{P}\) for some integer \(l \geq 2\), then we have

\[
\text{Card}(D(f)) \geq (c^{-1} \beta)^{l/(l-1)} P.
\]

**Proof.** Using Lemma 1.1 we have

\[
\frac{1}{P} \sum_{n \in D(f)} f(n) \geq 1/2.
\]

Moreover we have using Hölders inequality

\[
\frac{1}{P} \sum_{n \in D} f(n) \leq \|f\|_l \left(\frac{\text{card}(D)}{P}\right)^{1/q},
\]

where \(q > 1\) is a real number satisfying \(\frac{1}{l} + \frac{1}{q} = 1\). Hence we have \(\text{card}(D) \geq (c^{-1} \beta)^{l/(l-1)} P\) as claimed. \(\Box\)
Proposition 1.5. With the notations as in Theorem 1.3, we have

$$\|f_C * f_B\|_1 \leq \frac{c}{\beta},$$

where $c > 0$ is a constant depending only upon $t$ and $l$.

For proving Proposition 1.5 we first observe the following equality which is easy to verify:

$$\|f_C * f_B\|_1^l = \frac{1}{P \operatorname{card}(B)^l d(C)^l} \sum_{y_i \in B} \operatorname{card}((C - y_1) \cap \cdots \cap (C - y_l)).$$

Given $\tilde{y} = (y_1 \cdots y_l) \in B^l$, let $G(\tilde{y})$ be the graph with vertex set equal to $\{y_1 \cdots y_l\}$ and $y_i$ is joined by an edge to $y_j$ if and only if $y_i - y_j \in S'$, where $S'$ is as in Theorem 1.3. Let $C(G(\tilde{y}))$ denote the number of connected components of $G(\tilde{y})$. Given $G(\tilde{y})$ with $C(G(\tilde{y})) = r$, let $D(G(\tilde{y}))$ be a subset of $\{1, \cdots , l\}$ with $\operatorname{card}(D(G(\tilde{y}))) = r$ and for $i, j \in D(G(\tilde{y}))$ with $i \neq j$, we have $y_i$ and $y_j$ belongs to different connected components of $G(\tilde{y})$.

Lemma 1.6. Let $(y_1 \cdots y_l) \in B^l$ with $C(G(y_1 \cdots y_l)) = r$. Then we have

$$\operatorname{Card}((C - y_1) \cap \cdots \cap (C - y_l)) \leq \frac{c(r) P d(C)^r}{\beta^r},$$

where $c(r) > 0$ is a constant depending only upon $r$.

Proof. We have

$$\operatorname{Card}((C - y_1) \cap \cdots \cap (C - y_l)) \leq \operatorname{Card}((\cap_j \in D(G(\tilde{y}))(C - y_j))).$$

We have $\operatorname{card}(D(G(\tilde{y}))) = r$ and for $i, j \in D(G(\tilde{y}))$ with $i \neq j$, the element $y_i - y_j$ does not belong to $S'$, Therefore the result follows using (9). \qed

The following lemma is easy to verify.

Lemma 1.7. Let $\tilde{y} \in B^l$. If $y_i$ and $y_j$ belongs to the same connected components of $G(\tilde{y})$, then $y_i - y_j \in l S'$.

Using this we prove the following lemma.

Lemma 1.8. The number of $\tilde{y}_i \in B^l$ with $C(G(\tilde{y}_i)) = r$ is at-most $c(t, r) (\operatorname{card}(B))^r$ where $c(t, r) > 0$ is a constant depending only upon $r$ and $t$. We may take $c(t, r) = \binom{l}{r} (rt)^{2l^2}$.

Proof. Let $J$ be a subset of $\{1, \cdots , l\}$ with $\operatorname{card}(J) = r$. First we obtain an upper bound for the number of $\tilde{y}_i \in B^l$ such that $D(G(\tilde{y}_i)) = J$. For this we note that for any $i \in \{1, \cdots , l\} \setminus J$, there exists some $j \in J$ such that $y_i - y_j \in l S'$. Hence the number of such $\tilde{y}_i \in B^l$ is at-most $(r \operatorname{card}(l S'))^{l-1} \operatorname{card}(B)^r$. Since there are $\binom{l}{r}$ many different sets $J$ possible, the lemma follows. \qed
Proof of Proposition 1.5. Using (14), it follows that
\[ \|f_C \ast f_B\|_1 = \sum_{r=1}^l \frac{1}{P} \text{card}(B \setminus d(C)^r) \sum_{\tilde{y}_i \in B', C(G(\tilde{y}_i))=r} \text{Card}(\cap_{i=1}^l (C - y_i)). \]
Using this and Lemmas 1.6 and 1.8, we obtain that
\[ \|f_C \ast f_B\|_1 \leq \sum_{r=1}^l \frac{1}{\text{card}(B) \cdot d(C) - r \cdot \beta r \cdot c(r)c(t, r)}, \]
where \( c(r) \) is as in Lemma 1.6 and \( c(t, r) \) is as in Lemma 1.8. Since from assumption we have \( \text{card}(B) \cdot d(C) \geq 1 \) and \( \beta \leq 1 \), the result follows with \( c = \max_r c(r)c(t, r) \).

The claim (i) in Theorem 1.3 is an immediate consequence of Propositions 1.4 and 1.5. The claim (ii) in Theorem 1.3 follows using this and Lemma 1.2.

2 Proof of Proposition 0.8

We shall deduce Proposition 0.8 as an easy corollary of the following result.

Theorem 2.1. Let \( N' \) be a natural number and \( G(X) = \prod_{i=1}^m (e_i X + d_i) \in \mathbb{Z}[X] \) be a polynomial with \( e_i, d_i \in \mathbb{Z} \) and \( |e_i| + |d_i| \leq c_1 N'^{100} \). If \( \Delta(G) := \prod_i a_i \prod_{i \neq j} (e_i d_j - e_j d_i) \neq 0 \), then for any \( c < 1 \), we have
\[ \text{Card}\{n \leq N' : \gcd(G(n), P(N'^c)) = 1\} \leq c_2 \frac{N' \log m \log N'}{\log m \log N'}, \tag{15} \]
where \( c_2 = c_2(m, c_1) > 0 \) is a constant depending only upon \( m \) and \( c_1 \) and in particular does not depend upon \( N' \).

Proof of Proposition 0.8 Recall that with \( G(X) = F(b + MX) \), we have
\[ A_b \subset \{n \leq N/M : \gcd(G(n), P(N^{1/(4k+1)}) = 1\}. \]
Using this, it follows that
\[ \cap_i (A_b + h_i - h_1) \subset \{n \leq N/M : \gcd(H(n), P(N^{1/(4k+1)}) = 1\}, \]
where \( H(X) = \prod_{j=1}^l F'(X + h_1 - h_i) \). The assumption that \( h_i - h_j \notin (S - S) \) implies that the discriminant of \( G \) is non-zero. Using Theorem 2.1 with \( N' = \frac{N}{M} \) and \( G \) being the polynomial as above, we obtain that
\[ \text{card}(\cap_i (A_b + h_i - h_1)) \leq c_2 \frac{N \log kr \log N}{M \log kr N}, \]
where \( c_2 \) is a constant depending only upon \( l \) and \( F \). The result follows using this and the observation that \( \text{card}(\cap_i (A_b + h_i)) = \text{card}(\cap_i (A_b + h_i - h_1)) \). \( \square \)
Let $G \in \mathbb{Z}[X]$ be a polynomial of degree $m$. For any prime $p$, let $\nu_p$ denotes the number of $x \in \mathbb{Z}/p\mathbb{Z}$ such that $G(x) \equiv 0 \pmod{p}$. For any prime $p$ and integer $n$, we set $g(p) = \frac{\nu_p}{p}$. Then it is easy to verify that for any real numbers $1 \leq w \leq z$, we have
\[
\prod_{p \leq z} (1 - g(p))^{-1} \leq K \left( \frac{\log z}{\log w} \right)^m,
\]
where $K$ is an absolute constant. We also have
\[
\sum_{n \leq x, G(n) \equiv 0 \pmod{d}} 1 = xg(d) + r(d),
\]
with $|r(d)| \leq g(d)d$. Then we have the following result.

**Theorem 2.2.** [Z Theorem 6.9, page number 69] Let $z \geq 2$ and $D \geq z^{9m+1}$. Then we have
\[
\text{Card}\{n \leq x : \gcd(G(n), P(z)) = 1\} \leq (1 + K^{10e^{9m-s}}) x \prod_{p \leq z} (1 - g(p)) + \sum_{d \leq D} |r(d)|,
\]
where $s = \log D / \log z$.

**Proof of Theorem 2.1.** We have
\[
\sum_{d \leq D} |r(d)| \leq \sum_{d \leq D} g(d)d \leq D \prod_{p \leq D} (1 + g(p)) \ll D \log^m D.
\]
Now $g(p) = \frac{\nu_p}{p}$ for all $p$ not dividing $\Delta(G)$. From the assumption, we have $\Delta(G) = \prod_{i=1}^{m} e_i \prod_{i \neq j}(e_i d_j - e_j d_i) \leq c_1^{200} N^{200m}$. Therefore the number of primes dividing $\Delta(G)$ is at-most $c(m, c_1) \log N'$, where $c(m, c_1)$ is a constant depending only upon $m$ and $c_1$. Hence
\[
\prod_{p \leq z} (1 - g(p)) \leq \prod_{p \leq z} (1 - \frac{m}{p}) \prod_{p \leq c(m, c_1) \log N'} (1 - \frac{m}{p})^{-1} \leq c(m, c_1) \frac{\log^m \log N'}{\log^m z}.
\]
Therefore using Theorem 2.2 with $D = \frac{N'}{\log^{2m+1} + N'}$ and $z = D^{9m+1}$, we obtain the result if $c < \frac{1}{9m+1}$. The result for larger $c$ follows using this and observing that $\text{Card}\{n \leq N' : \gcd(G(n), P(z)) = 1\}$ is a decreasing function of $z$.

\[\square\]

### 3 Proof of Proposition 0.7

Let $G(X) = \prod_{i=1}^{m} (e_i X + d_i)$ be a polynomial with $e_i, d_i \in \mathbb{Z}$ and $\Delta(G) \neq 0$. Moreover we shall assume that $G$ is non-degenerate. The following result is a rewording of [Z Proposition 4.2].

**Proposition 3.1.** Let $R, N$ be large numbers such that $1 \ll R \ll N^{1/10}$ and let $G$ be a polynomial as above. Let $h : \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{C}$ be a function satisfying the following:
\[
h(n) \neq 0 \implies \gcd(G(n), P(R)) = 1,
\]
where $n \in [1, N]$. Then for any real number $l > 2$, we have
\[
\left( \sum_{t \in \mathbb{Z}/N\mathbb{Z}} |\hat{h}(t)|^l \right)^{2/l} \leq c(l, m) \frac{1}{\log^m R} \prod_{p} (1 - \frac{1}{p})^{-m} \frac{1}{N} \sum_{n} |h(n)|^2,
\]
where $c(l, m) > 0$ is a constant depending only upon $l$ and $m$. 

\[9\]
Applying this result with $G(X) = F(b_0 + MX)$, $N = P$ and $h = f_A'$, we obtain

**Corollary 3.2.** Given $c_1, \ldots, c_s \in (\mathbb{Z}/P\mathbb{Z})^*$ and $m_1, m_2, \ldots, m_s$ with $\sum_i m_i > 2$,

$$\sum_{t} \prod_{i=1}^{s} |\hat{f}_{A'}(c_it)|^{m_i} \leq \frac{c(F, \sum_i m_i)}{\delta^{\sum_i m_i}},$$

(21)

where $c(F, l) > 0$ is a constant depending only on $F$ and $l$.

**Proof of Proposition 0.7.** We have

$$|\Lambda_L(f_A) - \Lambda_L(f_A * B)| = P^{s-1} \left| \sum_{t} \prod_{i=1}^{s} \hat{f}_{A'}(c_i c_1^{-1} t) \left( 1 - \prod_{i=1}^{s} \hat{f}_B(c_i c_1^{-1} t) \right) \right|.$$  

(22)

Since for $t \in \bigcup_i c_i c_1^{-1} S_{c_1}$, we have $|\hat{f}_B(t) - 1| \leq \epsilon_2$, it follows that for $t \in S_{c_1}$, we have $|1 - \prod_i \hat{f}_B(c_i c_1^{-1} t)| \ll \epsilon_2$. Hence using this and (21) we have

$$\sum_{t \in S_{c_1}} \left| \sum_{i} \prod_{i=1}^{s} \hat{f}_{A'}(c_i c_1^{-1} t) \left( 1 - \prod_{i=1}^{s} \hat{f}_B(c_i c_1^{-1} t) \right) \right| \ll \epsilon_2 \sum_{t} \left| \prod_{i=1}^{s} \hat{f}_{A'}(c_i c_1^{-1} t) \right| \leq \epsilon_2 \frac{c(F, L)}{\delta^s}.$$  

(23)

For $t \notin LS_{c_1}$, we have $|f_{A'}(t)|^{1/2} \leq \epsilon_1 1/2$. Therefore the contribution in right hand side of (22) coming from such $t$ is at most and hence we using (21), we have

$$\epsilon_1^{1/2} \sum_{t \notin S_{c_1}} \left| \hat{f}(t)^{1/2} \prod_{i=2}^{s} \hat{f}_{A'}(c_i c_1^{-1} t) \right| \ll \epsilon_1^{1/2} \frac{c(F, L)}{\delta^s}.$$  

(24)

Using (22), (23) and (24), the result follows.

\[ \square \]

4 Relation between $g_L$ and $h_L$

When the number of variables $s$ in a translation invariant linear equation $L$ is 3, a relation between $g_L$ and $h_L$ follows from the following result.

**Theorem 4.1** (Varnavides theorem). Let $L$ be a translation invariant linear equation and $g_L, g_L^*$ are functions as defined in Definition 7.3. Let $\eta > 0$ and $D \subset \mathbb{Z}/P\mathbb{Z}$ with $\text{card}(D) \geq \eta P$. Then the number of solution of $L$ in $D$ is at least

$$\frac{\eta \cdot P(P-1)}{2 \left( g^{s-1}(\eta/2) \right)^2}.$$  

Proof. For the brevity of notation, we write $t$ to denote $g^{s-1}(\eta/2)$. Since the assumption implies that $D$ is non empty and hence contains at least one trivial solution of $L$, the result is true if $t \geq P$. Hence we may assume that $t < P$.

Given any $a \in \mathbb{Z}/P\mathbb{Z}$ and $d \in \mathbb{Z}/P\mathbb{Z} \setminus \{0\}$, let $I_{a,d} := \{a + d, \ldots, a + td\}$ be an arithmetic progression of length $t$. We say that $I_{a,d}$ is a “good” progression, if $\text{card}(D \cap I_{a,d}) \geq \frac{\eta}{2}$. We claim that if $I_{a,d}$ is good then $D' = D \cap I_{a,d}$ contains a non-trivial three term arithmetic progression. For this we first notice that since $P$ is prime and $d$ is a non zero element
of \(\mathbb{Z}/P\mathbb{Z}\), we have \(\text{card}(D') = \text{card}(\frac{D'-a}{d})\). Hence \(\frac{D'-a}{d} \subset [1, t]\) and contains at least \(\frac{2}{t} t\) elements. Therefore using the properties of \(g_L\) and definition of \(g_L^{*-1}(\eta/2)\), it follows that \(\frac{D'-a}{d}\) contains a non-trivial solution of \(L\), which proves the claim. Now we shall obtain a lower bound for the number of good \(I_{a,d}\).

Now for any fixed \(d_0\), we have the following identity:

\[
\sum_{a \in \mathbb{Z}/P\mathbb{Z}} \text{card}(D \cap I_{a,d_0}) = t \text{card}(D).
\]

This follows by observing that any \(c \in D\) belongs to exactly \(t\) many \(I_{a,d_0}\). From the above identity it follows that for any fixed \(d_0\), the number of good \(I_{a,d_0}\) is at least \(\frac{\text{card}(D)}{2}\) which by assumption is at-least \(\frac{\eta}{2} P\). Now varying \(d_0\), we obtain that the number of good \(I_{a,d}\) is at least \(\frac{\eta}{2} P\). The lemma follows using this and the observation that a given non-trivial solution of \(L\) can belong to at most \(t^2\) many good \(I_{a,d}\).

Using Theorem 4.1, we immediately obtain the following result.

**Corollary 4.2.** Let \(L\) be a translation invariant equation in \(s\) many variables and \(g_L\) be a function as satisfying the properties as in Definition 0.3. When \(s = 3\), then \(h_L(\eta) = \frac{\eta}{(2g_L^{*-1}(\eta/2))^2}\) is a function satisfying the properties as in Definition 0.3 (i).

As remarked earlier, Thomas Bloom [1] showed that there exists an absolute constant \(c > 0\) depending only on \(L\) such that the function \(g_L(N) = c \left(\frac{\log^5 \log N}{\log N}\right)^{s-2}\) satisfies the above properties. In this case \(g_L^{*-1}(\eta) \leq \exp(c_1 \eta^{-1/(s-2)} \log^6 \log(\frac{1}{\eta}))\) with \(c_1 > 0\) being a constant depending only upon \(L\). Therefore when \(s = 3\), there exists an absolute constant \(c > 0\) such that we may take

\[
h_L(\eta) = \exp\left(-c\eta^{-1} \log^6 \log \frac{1}{\eta}\right).
\]

(25)

**5 Proof of Theorem 0.1**

Let \(S\) be as in Proposition 0.7 and \(B \subset \mathbb{Z}/P\mathbb{Z}\) defined as

\[
B = \text{Bohr}(S, \epsilon_2) := \{x \in \mathbb{Z}/P\mathbb{Z} : \left|\exp\left(\frac{2\pi i xt}{P}\right) - 1\right| \leq \epsilon_2 \forall t \in S\}.
\]

Then \(B\) satisfies the assumptions in Proposition 0.7. We shall choose \(\epsilon_1\) and \(\epsilon_2\) in such a way that \(B\) also satisfies the assumptions in Proposition 0.6.

**Lemma 5.1** (Lemma 4.20 [7]). Given any set \(C \subset \mathbb{Z}/P\mathbb{Z}\) and any real number \(\epsilon > 0\), we have

\[
\text{card}(\text{Bohr}(C, \epsilon)) \geq (\epsilon)^{|C|} P.
\]

Moreover an immediate consequence of (21) is the following upper bound for the cardinality of \(S\):

\[
\text{Card}(S) \leq \frac{c_1^{-3} c(F, L)}{\delta^3}.
\]

Therefore we have \(\text{card}(B) \geq \log^{k+101} P\) and hence \(B\) satisfies the assumption of Proposition 0.6 provided, we have

\[
\frac{c_1^{-3} c(F, L)}{\delta^3} \log(\epsilon_2) \geq -\frac{\log P}{2}
\]

(26)
and \( P \) is sufficiently large. Therefore if (26) is satisfied, then using Propositions 0.6 and 0.7 we have

\[
\Lambda_L(f_{A'}) \geq c_1 h_L(c_2 \delta^{l/(l-1)}) P^{s-1} - c(F, L) \frac{\epsilon_2 + \epsilon_1^{0.5}}{\delta^s} P^{s-1}.
\]

Therefore choosing

\[
\epsilon_2 = \epsilon_1^{0.5} = \frac{\delta^s c_1 h_L(c_2 \delta^{l/(l-1)})}{c(F, L)}, \tag{27}
\]

we obtain

\[
\Lambda_L(f_{A'}) \geq c_1 h_L(c_2 \delta^{l/(l-1)}) P^{s-1}, \tag{28}
\]

where \( c_1 \) and \( c_2 \) are constants depending only upon \( F \) and the linear equation \( L \), provided our choice of \( \epsilon_1 \) and \( \epsilon_2 \) satisfies (26). Since \( s = 3 \), with the choice of \( h_L \) provided by (25), we have that for some \( c_1, c_2 > 0 \), we have

\[
\epsilon_1 = \exp \left( -c_1 \delta^{-l/(l-1)} \log^6 \log \frac{1}{\delta} \right), \quad \epsilon_2 = \exp \left( -c_1 \delta^{-l/(l-1)} \log^6 \log \frac{1}{\delta} \right).
\]

Therefore (26) holds using the assumed lower bound for \( \delta \), provided \( l \) is chosen sufficiently large depending on \( \epsilon \), where \( \epsilon \) is as in Theorem 0.1.

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