Biclique Coverings and the Chromatic Number

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Abstract

Consider a graph $G$ with chromatic number $k$ and a collection of complete bipartite graphs, or bicliques, that cover the edges of $G$. We prove the following two results:

- If the bicliques partition the edges of $G$, then their number is at least $2\sqrt{\log_2 k}$. This is the first improvement of the easy lower bound of $\log_2 k$, while the Alon-Saks-Seymour conjecture states that this can be improved to $k - 1$.

- The sum of the orders of the bicliques is at least $(1 - o(1))k \log_2 k$. This generalizes, in asymptotic form, a result of Katona and Szemerédi who proved that the minimum is $k \log_2 k$ when $G$ is a clique.

1 Introduction

It is a well-known fact that the minimum number of bipartite graphs needed to cover the edges of a graph $G$ is $\lceil \log \chi(G) \rceil$, where $\chi(G)$ is the chromatic number of $G$ (all logs are to the base 2). Call a complete bipartite graph a biclique. Two classical theorems study related questions. One is the Graham-Pollak theorem [1] which states that the minimum number of bicliques needed to partition $E(K_k)$ is $k - 1$. Another is the Katona-Szemerédi theorem [4], which states that the minimum of the sum of the orders of a collection of bicliques that cover $E(K_k)$ is $k \log k$. Both of these results are best possible.

An obvious way to generalize these theorems is to ask whether the same results hold for any $G$ with chromatic number $k$. 


**Conjecture 1 (Alon-Saks-Seymour)** The minimum number of bicliques needed to partition the edge set of a graph $G$ with chromatic number $k$ is $k - 1$.

Note that every graph has a partition of this size, simply by taking a proper coloring $V_1, \ldots, V_k$ and letting the $i$th bipartite graph be $(V_i, \cup_{j>i} V_j)$.

Another motivation for Conjecture 1 is that the non-bipartite analogue is an old conjecture of Erdős-Faber-Lovász. The Erdős-Faber-Lovász conjecture remains open although it has been proved asymptotically by Kahn [3]. Conjecture 1 seems much harder than the Erdős-Faber-Lovász conjecture, indeed, as far as we know there are no nontrivial results towards it except the folklore lower bound of $\log_2 k$ which doesn’t even use the fact that we have a partition. Our first result improves this to a superlogarithmic bound for $k$ large.

**Theorem 2** The number of bicliques needed to partition the edge set of a graph $G$ with chromatic number $k$ is at least $2\sqrt{2\log k (1+o(1))}$.

Motivated by Conjecture 1, we make the following conjecture that generalizes the Katona-Szemerédi theorem.

**Conjecture 3** Let $G$ be a graph with chromatic number $k$. The sum of the orders of any collection of bicliques that cover the edge set of $G$ is at least $k \log k$.

We prove Conjecture 3 asymptotically.

**Theorem 4** Let $G$ be a graph with chromatic number $k$, where $k$ is sufficiently large. The sum of the orders of any collection of bicliques that cover the edge set of $G$ is at least

$$k \log k - k \log \log k - k \log \log \log k.$$ 

The next two sections contain the proofs of Theorems 2 and 4.
The Alon-Saks-Seymour Conjecture

It is more convenient to phrase and prove our result in inverse form. Let $G$ be a disjoint union of $m$ bicliques $(A_i, B_i), 1 \leq i \leq m$. The Alon-Saks-Seymour conjecture then states that the chromatic number of $G$ is at most $m + 1$.

We prove the following theorem which immediately implies Theorem 2.

**Theorem 5** Let $G$ be a disjoint union of $m$ bicliques. Then $\chi(G) \leq m^{1 + \frac{\log m}{2}} (1 + o(1))$.

**Proof.** We will begin with a proof of a worse bound. We will first show that $\chi(G) \leq m^{\log m} (1 + o(1))$. A color will be an ordered tuple of length at most $\log m$, with each element a positive integer of value at most $m$. We will construct this tuple in stages. In the $i$th stage we will fill in the $i$th co-ordinate. Note that the length of the tuple may vary with vertices.

With each vertex $v$, at stage $i$, we will associate a set $S(i, v) \subset V(G)$. The set $S(i, v)$ will contain all vertices which have the same color sequence, so far, as $v$ (in particular, $v \in S(i, v)$ for all $i$).

A biclique $(A_j, B_j)$ is said to cut a subset of vertices $S$ if $S \cap A_j \neq \emptyset$ and $S \cap B_j \neq \emptyset$.

Consider two bicliques $(A_k, B_k)$ and $(A_l, B_l)$ from our collection. Since they are edge disjoint, $(A_l, B_l)$ cuts either $A_k$ or $B_k$, but not both.

Fix a vertex $v$. We set $S(0, v) := V(G)$. The assignment for the $i + 1$st stage is as follows. Suppose we have defined $S(i, v)$. Let $\mathcal{F}(i, v)$ denote the set of all bicliques that cut $S(i, v)$. For each biclique $(A_j, B_j) \in \mathcal{F}(i, v)$ for which $v \in A_j \cup B_j$, let $C_j$ be the set among $A_j, B_j$ that contains $v$ and let $D_j$ be the set among $A_j, B_j$ that omits $v$. For a vertex $v$, check if there is a biclique $(A_j, B_j) \in \mathcal{F}(i, v)$ such that $v \in A_j \cup B_j$ and

- The number of bicliques in $\mathcal{F}(i, v)$ that cut $C_j$ is at most the number that cut $D_j$.

If there is such a $j$, then the $i + 1$st co-ordinate of the color of $v$ is $j$ and $S(i + 1, v) = S(i, v) \cap C_j$. If there are many candidates for $j$, pick one arbitrarily.
If there is no such \((A_j, B_j)\), then the coloring of \(v\) ceases and the vertex will not be considered in subsequent stages. In other words, the final color of vertex \(v\) will be a sequence of length \(i\).

Note that in this process every vertex is assigned a color except vertices that were not assigned a color in the very first step. We will show below that if a vertex is assigned a color then this coloring is proper. The same argument shows that the vertices that do not get assigned a color in the first step form an independent set. These vertices are all assigned a special color which is swallowed up in the \(o(1)\) term.

The following technical lemma establishes the statements needed to prove correctness and a bound on the number of colors used.

**Lemma 6** For each vertex \(v\), the set \(S(i, v)\) is determined by the color sequence \(x_1, \ldots, x_i\) assigned to the vertex \(v\). Also, the number of bicliques that cut \(S(i, v)\) is at most \(m/2^i\).

**Proof.** The proof is by induction on \(i\). Both statements are trivially true for \(i = 0\). For the inductive step, assume that \(S(i - 1, v)\) is determined by \(x_1, \ldots, x_{i-1}\) and at most \(m/2^{i-1}\) bicliques cut \(S(i - 1, v)\). If \(v\) ceases to be colored then we are done by induction. Now suppose that \(v\) is colored with \(x_i = t\) in step \(i\). Then \((A_t, B_t) \in \mathcal{F}(i, v)\) and \(v \in A_t \cup B_t\). As before, define \(C_t\) and \(D_t\). Because \(v\) is colored in this step, the number of bicliques in \(\mathcal{F}(i, v)\) that cut \(C_t\) is at most the number which cut \(D_t\). As \(S(i, v) = C_t \cap S(i-1, v)\), we see that \(S(i, v)\) is determined by \(x_1, \ldots, x_{i-1}, t\). Also, since the number of bicliques that cut \(C_t\) is at most half the number that cut \(S(i-1, v)\) the second assertion follows.

We argue first that the coloring is proper. Assume for a contradiction that two adjacent vertices \(v\) and \(w\) are assigned the same color sequence. Suppose the sequence is of length \(i\). Then by the previous lemma \(S(i, v) = S(i, w)\). There has to be one biclique, say \((A_p, B_p)\), such that \(v \in A_p\) and \(w \in B_p\). If the number of bicliques in \(\mathcal{F}(i, v)\) that cut \(A_p\) is at most the number that cut \(B_p\) then \(v\) will be given a color in the \(i+1\)st step. Otherwise \(w\) will be colored. In any case, at least one of them will be given a color contradicting our assumption that both sequences are of length \(i\). This argument also shows that the vertices which were not assigned a color in
the first step form an independent set. The coloring stops when \( F(i, v) \) is empty for every vertex and that happens after \( \log m \) steps from the lemma.

A simple observation helps in reducing this bound by a square-root factor. At each stage, the colorings of the \( S(i, v) \)s are independent. Hence the colors only matter within the vertices in each of these sets. The number of bicliques that cut \( S(i, v) \) is at most \( m/2^i \). We renumber these bicliques from 1 to \( m/2^i \). Hence the labels in the \( i \)th stage will be restricted to this set. The total number of colors used, of length \( i \) is at most \( m \cdot m^2 \cdot \cdots \cdot m^{2i} \). The number for \( i < m \) is swallowed up in the \( o(1) \) term and the value for \( i = m \) simplifies to the main term in the bound given.

### 3 Generalizing the Katona-Szemerédi Theorem

In this section we prove Theorem 4. Given a graph \( G \), let \( b(G) \) denote the minimum, over all collections of bicliques that cover the edges of \( G \), of the sum of the orders of these bipartite graphs.

One proof of the Katona-Szemerédi theorem is due to Hansel [2] and the same proof yields the following lemma which is part of folklore.

**Lemma 7** Let \( G = (V, E) \) be an \( n \) vertex graph with independence number \( \alpha \). Then \( b(G) \geq n \log(n/\alpha) \).

In other words for any graph \( G \), \( \alpha(G) \geq \frac{n}{\chi(G)^2} \). Let \( k = \chi(G) \). We may assume that \( n \leq k \log k \), since we are done otherwise. Let \( G = G_0 \). Starting with \( G_0 \), repeatedly remove independent sets of size given by Hansel’s lemma as long as the number of vertices is at least \( k \). Let the graphs we get be \( G_0, G_1, \ldots, G_t \). Let \( |V(G_i)| = n_i \) and \( \beta = \max_i 2^{b(G_i)/n_i} \). Let this maximum be achieved for \( i = p \). From the definition, we see that \( n_{i+1} \leq n_i (1 - \frac{1}{2^{b(G_i)/n_i}}) \). Hence \( n_i \leq n(1 - 1/\beta)^i < ne^{1/\beta} < n2^{1/\beta} \) and together with \( n_t \geq k \) we obtain

\[
t \leq \beta \log(n/k).
\]

There are two cases to consider. First suppose that \( t \geq k/\log k \). Then from the above two inequalities we obtain

\[
2^{b(G_p)/n_p} \log(n/k) \geq k/\log k.
\]
Taking logs and using the facts that $n \leq k \log k$ and $n_p \geq k$ we get

$$b(G_p) \geq k(\log k - \log \log k - \log \log \log k).$$

We now consider the case that $t < k/\log k$. Let $G'$ be the graph obtained after removing an independent set from $G_t$. By definition of $t$ we have $|V(G')| < k$. Also $\chi(G') \geq k(1 - 1/\log k)$. Since the color classes of size one in an optimal coloring form a clique, this implies that $G'$ has a clique of size at least $k(1 - 2/\log k)$. Using the fact that $\log(1 - x) > -2x$ for $x$ sufficiently small and applying the Katona-Szemerédi theorem, we get $b(G') \geq k \log k - 3k$.

Note that in the proof $b(G_i)$ could use different covers, but with sizes smaller than the one induced by $b(G_0)$. One can get better lower order terms by adjusting the threshold between the two cases.

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