A Tangential Markov Inequality on Exponential Curves

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Abstract

We show that on the curves \( y = e^{t(x)} \) where \( t(x) \) is a fixed polynomial, there holds a tangential Markov inequality of exponent four. Specifically, for the real interval \([a, b]\) there is a constant \( C \) such that

\[
\max_{x \in [a, b]} |d\frac{d}{dx}P(x, e^{t(x)})| \leq C(\deg(P))^4 \max_{x \in [a, b]} |P(x, e^{t(x)})|
\]

for all bivariate polynomials \( P(x, y) \).

1 Introduction

Recently [BLMT1, BLMT2, Ba1, Ba2, Br1, Br2, FN, RY] there has been considerable interest on extending the classical Markov/Bernstein inequalities bounding the derivatives of polynomials to higher dimensional cases. To be precise,
**Definition.** Suppose that \( M \) is a smooth (\( C^1 \)) manifold in \( \mathbb{R}^d \) (with or without boundary). We say that \( M \) admits a tangential Markov inequality of exponent \( \ell \) if there is a constant \( C > 0 \) such that for all polynomials \( P \in \mathbb{R}[x_1, \ldots, x_d] \) and points \( a \in M \)

\[ |D_T P(a)| \leq C (\deg(P))^{\ell} ||P||_M. \]

Here \( D_T P \) denotes any (unit) tangential derivative of \( P \) and \( ||P||_M \) is the supremum norm of \( P \) on \( M \).

One remarkable fact is that such tangential Markov inequalities of exponent one characterize \( M \) being algebraic.

**Theorem 1.1** ([BLMT1]). Suppose that \( M \) is a smooth (\( C^{\infty} \)) compact \( m \)-dimensional submanifold of \( \mathbb{R}^d \). If \( M \) admits a tangential Markov inequality of exponent \( \ell < 1 + 1/m \), then \( M \) is algebraic (i.e. a subset of an algebraic variety of the same dimension). Conversely, if \( M \) is algebraic, then it admits a tangential Markov inequality of exponent \( \ell = 1 \).

It is natural then to investigate the existence of a tangential Markov inequality of exponent \( \ell > 1 \) and its connection to the algebraicity of \( M \). For instance, in [BLMT2] it is shown that the non-singular part of the singular algebraic curve segment \( (x, x^r) \), \( 0 \leq x \leq 1 \), with \( r = q/p > 1 \) in lowest terms, admits a tangential Markov inequality of exponent \( \ell = 2p \) and this is best possible. However, not all analytic manifolds admit a tangential Markov inequality. In [BLMT1] there is given an example of an analytic curve in \( \mathbb{R}^2 \) which does not admit a tangential Markov inequality of any exponent \( \ell \). Specifically, the curve segment \( (x, f(x)) \), \( 0 \leq x \leq 1 \) with \( f(x) \) given by a gap series \( f(x) = \sum_{k=0}^{\infty} c_k x^{2k} \) with \( c_k \geq 0 \) and being convergent on \([0,1]\), has this property.

Thus the main question is to characterize manifolds which admit a tangential Markov inequality. Up till now it was not even clear whether or not the existence of a Markov inequality of some exponent meant that \( M \) was (possibly singular) algebraic. In this short note we show that this is not the case by showing that the analytic curve segments \( (x, e^{t(x)}) \), \( a \leq x \leq b \) do admit a tangential Markov inequality of exponent \( \ell = 4 \). Precisely, we prove
Theorem 1.2 Suppose that $t(x)$ is a fixed algebraic polynomial. Then for every interval $[a, b]$ there exists a constant $C = C(a, b)$ such that
\[
\max_{x \in [a, b]} \left| \frac{d}{dx} P(x, e^{t(x)}) \right| \leq C (\deg(P))^4 \max_{x \in [a, b]} |P(x, e^{t(x)})| \]
for all bivariate polynomials $P(x, y)$.

2 Siciak Type Extremal Functions

Suppose that $K \subset \mathbb{C}^d$ is compact and let
\[
\mathcal{P}_K = \{ P \in \mathbb{C}[z_1, \cdots, z_d] : \|P\|_K \leq 1 \text{ and } \deg(P) \geq 1 \}. 
\]
(Here $\|P\|_K$ denotes the uniform norm of $P$ on $K$.) The, by now classical, Siciak extremal function ([S], but see also the excellent monograph [K]), is defined, for $z \in \mathbb{C}^d$, as
\[
\Phi_K(z) = \sup \{ |P(z)|^{1/\deg(P)} : P \in \mathcal{P}_K \}.
\]
This function has proven to be a highly useful tool in the theory of analytic functions of several variables, and is one of the main ingredients in the proof of Theorem 1 above. However, the polynomials restricted to the curve $w = e^{t(z)}$ are not themselves polynomials and hence we need to generalize appropriately.

Suppose then that $\{V_n\}_{n \geq 1}$ is a collection of increasing subsets of $C(\mathbb{C}^d)$, i.e., $V_n \subset V_{n+1} \subset C(\mathbb{C}^d)$, $n = 1, 2, \cdots$, and let $V = \bigcup_{n=1}^{\infty} V_n$. For $f \in V$ we set
\[
\deg(f) = \inf \{ n : f \in V_n \}.
\]
Further, let
\[
\phi : \mathbb{Z}_+ \to \mathbb{R}
\]
be a given function. We define a generalized Siciak extremal function for the family $V$ (replacing polynomials) with exponent function $\phi$ (replacing $\deg(P)$) to be
\[
\Phi_K(z; V, \phi) = \sup_{f \in V, \|f\|_K \leq 1} |f(z)|^{1/\phi(\deg(f))} \quad (z \in \mathbb{C}^d).
\]
If there is no risk of confusion then we will drop the dependence on $V$ and $\phi$.

In our situation we take

$$V_n = \{ f(z) = P(z, e^{t(z)} : P \in \mathbb{C}[z_1, z_2] \text{ and } \deg(P) \leq n \} \subset C(\mathbb{C}^2) \quad (1)$$

and

$$\phi(n) = n^2. \quad (2)$$

**Remark.** We emphasize that in the classical case $\phi(n) = n^1$, but here we must, as we shall see, use the higher power $n^2$. This is a crucial point.

Now, in the course of the proof of Lemma 1, page 120, of [Bak], there is given an estimate on exponential polynomials (i.e. for $t(z) = z$) due to Tijdeman[T] which we may restate in the following form. For fixed $f \in V$ and $z_0 \in \mathbb{C}$ let $M(R) = \max_{|z-z_0| \leq R} |f(z)|$.

**Lemma 2.1** Suppose that $t(z) = z$. Then, there are constants $c_1, c_2 > 0$ such that for all $f \in V$ and $R_2 \geq R_1 > 0$,

$$M(R_2) \leq M(R_1)e^{c_1R_2n} \left(\frac{R_2}{R_1}\right)^{c_2\phi(n)}$$

where $n = \deg(f)$ and $\phi$ is given by (2).

**Proof.** See [Bak], §2 of Chapter 12. □

**Remark.** Inequalities of the form of Lemma 2.1 are classically referred to as Bernstein-Walsh inequalities. In case $R_2 = 2R_1$ they are also sometimes called doubling inequalities.

Taking $1/n^2$ powers of this estimate gives an immediate estimate for the Siciak extremal function for $K$ a disk.

**Corollary 2.2** Suppose that $t(z) = z$ and that $K = \{ z \in \mathbb{C} : |z - z_0| \leq R_1 \}$ (with $R_1 > 0$). Then, there is a constant $C = C(R_1, R_2) \leq e^{c_1R_2} \frac{R_2}{R_1}$ such that

$$\Phi_K(z; V, \phi) \leq C$$

for all $|z - z_0| \leq R_2$. 

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Baker proves Lemma 2.1 in order to exploit the classical Jensen inequality to obtain a bound on the valency of exponential polynomials (see [Bak, T] for details). Further, Roytvarf and Yomdin[RY] prove a converse to this for general analytic functions which they make use of in their theory of Bernstein classes.

Perhaps the most succinct statement of this converse is given by Brudnyi[Br2, Lemma 3.1] which we paraphrase as

**Lemma 2.3** Suppose that $f$ is analytic on $K = \{z \in \mathbb{C} : |z - z_0| \leq R_2\}$ and assumes no value there more than $p$ times. Then, for $0 < R_1 \leq R_2$ there is a constant $C = C(R_1, R_2)$ such that

$$M(R_2) \leq C^p M(R_1).$$

Thus to establish that the extremal function is bounded for polynomials restricted to the curve $w = e^{t(z)}$ we need only give an appropriate bound on the valency of such functions, which we do next.

**Theorem 2.4** Suppose that $t(z)$ is a fixed polynomial of degree $d$, and that $K = \{z \in \mathbb{C} : |z - z_0| \leq R\}$ with $R \geq R_0 > 0$. There is a constant $C = C(t, K, R_0)$ such that the valency in $K$ of every $f \in V_n$ is bounded,

$$p \leq C n^2.$$

More precisely, we have

$$p \leq A(dn^2 + dn \left(\frac{R}{R_0}\right)^d \sup_{|z - z_0| \leq R_0} |t(z)|)$$

where $d = \text{deg}(t)$ and $A \leq 4$ is an absolute constant.

**Proof.** For simplicity we take $z_0 = 0$. The idea of the proof is to reduce to the case of $t(z) = z$.

So suppose that $P(z, w)$ is a bivariate polynomial of degree at most $n$. Then

$$P(z, e^{t(z)}) = \sum_{k=0}^{n} a_k(z) e^{k t(z)}$$
for some polynomials $a_k(z)$ with $\deg(a_k) \leq n - k$. We would like to bound the number of zeros in $K$ of $P_c(z, e^{t(z)}) := P(z, e^{t(z)}) - c$. For $w \in \mathbb{C}$ let $z_1, z_2, \ldots, z_d$ denote the $d$ zeros of $t(z) - w$ (repeated according to multiplicity) and set

$$
\tilde{P}_c(w) = \prod_{j=1}^{d} P_c(z_j, e^{t(z_j)}).
$$

We claim that, in fact,

$$
\tilde{P}_c(w) = \sum_{k=0}^{nd} b_k(w) e^{kw}
$$

for some polynomials $b_k(w)$ with $\deg(b_k) \leq n$. To see this, first note that $t(z_j) = w$ for each $j$ and hence

$$
\tilde{P}_c(w) = \prod_{j=1}^{d} \left\{ \sum_{k=0}^{n} \tilde{a}_k(z_j) e^{kw} \right\}
$$

where $\tilde{a}_0(z) = a_0(z) - c$, $\tilde{a}_k(z) = a_k(z)$, $k \geq 1$. Upon expanding we see that

$$
\tilde{P}_c(w) = \sum_{k=0}^{nd} b_k(z_1, \ldots, z_d) e^{kw}
$$

where the $b_k$ are symmetric polynomials in the roots $z_1, \ldots, z_d$ of degree at most $nd$. But then the $b_k$ must be polynomials in the elementary symmetric functions $\sigma_i(z_1, \ldots, z_d)$, i.e., in the coefficients of $t(z) - w$. These are all constant, except for the constant coefficient which is a function of $w$. Further, the elementary symmetric function $\sigma_d(z_1, \ldots, z_d) = \prod_{j=1}^{d} z_j$ is already of degree $d$ and hence the degree of $b_k$, as a polynomial in $w$, must be at most $nd/d = n$.

To continue, for $R \geq R_0$ set

$$
R_1 := \left( \frac{R}{R_0} \right)^d \sup_{|z| \leq R_0} |t(z)|
$$

so that, by the classical Bernstein inequality,

$$
\sup_{|z| \leq R} |t(z)| \leq R_1.
$$
In other words, $|z| \leq R \implies |t(z)| \leq R_1$. Then, for $c \in \mathbb{C}$,

\[
\begin{align*}
\# \{ z : |z| \leq R \text{ and } P(z, e^{t(z)}) = c \} \\
\leq \# \{ z : |t(z)| \leq R_1 \text{ and } P(z, e^{t(z)}) = c \} \\
\leq \# \{ w : |w| \leq R_1 \text{ and } \tilde{P}_c(w) = 0 \} \\
\leq A(dn^2 + dnR_1)
\end{align*}
\]

by Lemma 1 of [Bak, p. 120]. The result follows. $\square$

In summary, Corollary 2.2 also holds for general polynomials $t(z)$.

**Corollary 2.5** Suppose that $K = \{ z \in \mathbb{C} : |z - z_0| \leq R_1 \}$ (with $R_1 > 0$). Then there is a constant $C = C(R_1, R_2)$ such that

\[
\Phi_K(z;V,\phi) \leq C
\]

for all $|z - z_0| \leq R_2$.

**Remark.** In the classical case the Siciak extremal function has been much studied in the context of plurisubharmonic functions. We do not know to what extent this carries over to our generalized Siciak extremal functions, but for our purposes here we need only know that it is bounded.

Corollary 2.5 gives a bound for the extremal function of a complex disk, but since we wish to prove a Markov inequality for a real segment of the curve $y = e^{t(x)}$, we will need to know that the Siciak extremal function is bounded for a real segment. Suppose then that $x_0 \in \mathbb{R}$ and let $B_R(x_0)$ be the complex disk of radius $R$ centred at $x_0$ and let $I_R(x_0)$ denote the real segment, centred also at $x_0$ with radius $R$.

**Theorem 2.6** Suppose that $V$ is a set of entire functions and is such that

\[
\Phi_{B_{R_1}(x_0)}(z;V,\phi) \leq C(R_1, R_2) \quad \forall |z - x_0| \leq R_2, \quad R_1 \leq R_2.
\]

Then there is an associated constant $C' = C'(R_1, R_2)$ such that

\[
\Phi_{I_{R_1}(x_0)}(z;V,\phi) \leq C'(R_1, R_2) \quad \forall |z - x_0| \leq R_2, \quad R_1 \leq R_2.
\]
Remark. Note, this holds for general $V$ and $\phi$. Of course, later we will apply it to generalized exponential polynomials and $\phi(n) = n^2$. Moreover, we will show that $\phi(n) = n^2$ is best possible.

Proof. The proof is an ingenious application, due to the second author A. Brudnyi[Br3], of a version of a generalization of a classical Lemma of Cartan, due to Levin[L], which we begin by quoting.

Theorem. ([L, p. 21]) Suppose that $f(z)$ is entire and such that $f(0) = 1$. Let $0 < \eta < 3e/2$ and $R > 0$. Then, inside the disk $|z| \leq R$, but outside a family of disks the sum of whose radii is not greater than $4\eta R$, we have

$$\ln |f(z)| > -H(\eta) \ln M(2eR)$$

for

$$H(\eta) = 2 + \ln \frac{3e}{2\eta}.$$ 

Now consider $g \in V_n$ and let $z^* \in \mathbb{C}$, $|z^* - x_0| = R_1$, be such that $|g(z^*)| = \max_{|z-x_0| \leq R_1} |g(z)|$.

We apply Cartan’s Lemma with $z^*$ the origin to

$$f(z) = \frac{g(z)}{g(z^*)}.$$ 

$R = 2R_1$ and $\eta = 1/16$. Then, there is a set of disks $\{D_i\}$ of radii $r_i$ such that $\sum_i r_i \leq 4\frac{1}{16} R = R_1/2 < R_1$ and that, inside the disk $B_{2R_1}(z^*) \supset B_{R_1}(x_0)$, but outside the disks $D_i$,

$$\ln |f(z)| > -(2 + \ln(24e)) \ln M_f(4eR_1). \quad (5)$$

But, since the sum of their radii is strictly less than $R_1$, the disks $D_i$ cannot possibly cover all of $I_{R_1}(x_0)$, i.e., there is a point $x \in I_{R_1}(x_0)$ such that (5) holds. For this point then, by the definition of $z^*$,

$$\ln \left( \frac{|g(x)|}{\max_{|z-x_0| \leq R_1} |g(z)|} \right) > -(2 + \ln(24e)) \ln \left( \frac{\max_{|z-z^*| \leq 4eR_1} |g(z)|}{\max_{|z-x_0| \leq R_1} |g(z)|} \right)$$
and hence
\[
\ln \left( \frac{\max_{|x-x_0| \leq R_1} |g(x)|}{\max_{|z-x_0| \leq R_1} |g(z)|} \right) > - (2 + \ln(24e)) \ln \left( \frac{\max_{|z-z^*| \leq 4eR_1} |g(z)|}{\max_{|z-x_0| \leq R_1} |g(z)|} \right).
\]
Reversing the sign in the equality, we get
\[
\ln \left( \frac{\max_{|z-x_0| \leq R_1} |g(z)|}{\max_{|z-x_0| \leq R_1} |g(x)|} \right) < (2 + \ln(24e)) \ln \left( \frac{\max_{|z-z^*| \leq (4e+1)R_1} |g(z)|}{\max_{|z-x_0| \leq R_1} |g(z)|} \right).
\]
Consequently,
\[
\frac{\max_{|z-x_0| \leq R_1} |g(z)|}{\max_{|z-x_0| \leq R_1} |g(x)|} \leq \left( \frac{\max_{|z-x_0| \leq (4e+1)R_1} |g(z)|}{\max_{|z-x_0| \leq R_1} |g(z)|} \right)^{2+\ln(24e)} \leq C(R_1, (4e + 1)R_1)^{(2+\ln(24e))\phi(n)}
\]
by the boundedness of the extremal function for complex disks. The result follows. \(\square\)

We will use the next result to show that, for the curve \(y = e^x\), the exponent function \(\phi(n) = n^2\) is best possible. First some notation. For \(f(x)\), analytic in a neighbourhood of the origin, let \(\text{ord}_0(f)\) denote the order of vanishing of \(f\) at \(x = 0\), i.e., so that
\[
f(x) = cx^{\text{ord}_0(f)} + \text{higher order terms}
\]
for some \(c \neq 0\), near \(x = 0\). For \(W\) a vector space of functions, analytic in a neighbourhood of the origin, we set
\[
\max\text{ord}_0(W) = \sup_{f \in V} \text{ord}_0(f),
\]
the maximum order of vanishing for functions in \(W\). (We take \(\text{ord}_0(0) = 0\).)

**Proposition 2.7** Let \(W\) be a vector space of functions, analytic in a neighbourhood of the origin, such that \(\max\text{ord}_0(W) < \infty\). Set
\[
M_W(r) = \sup_{f \in W} \frac{\max_{0 \leq x \leq 2r} |f(x)|}{\max_{0 \leq x \leq r} |f(x)|}.
\]
Then,
\[
\sup_{r \geq 0} M_W(r) \geq 2^{\max\text{ord}_0(W)}.
\]
Proof. For a fixed $0 \neq f \in W$, 

$$\lim_{r \to 0} \frac{\max_{0 \leq x \leq 2r} |f(x)|}{\max_{0 \leq x \leq r} |f(x)|} = 2^{\text{ord}_0(f)},$$

as is easy to see. Hence 

$$\sup_{r \geq 0} M_W(r) \geq \sup_{f \in W} 2^{\text{ord}_0(f)} = 2^{\text{maxord}_0(W)}.$$ 

\[\square\]

Now, since $y = e^x$ is not algebraic, $V_n = \{P(x, e^x) : \deg(P) \leq n\}$, is a vector space of dimension $N := \binom{n+2}{2}$.

Lemma 2.8

$$\text{maxord}_0(V_n) = N - 1.$$ 

Proof. First note that, setting $f_{ij}(x) = x^i e^{jx}$, $i + j \leq n$ positive integers, this is equivalent to the existence of constants $\alpha_{ij}$ such that 

$$f^{(s)}(0) = 0, \quad 0 \leq s \leq N - 2$$

and 

$$f^{(s)}(0) = 1, \quad s = N - 1,$$

for $f(x) = \sum_{ij} \alpha_{ij} f_{ij}(x)$. But this in turn is equivalent to solving, for the $\alpha_{ij}$, the linear system

$$\sum_{ij} \alpha_{ij} f^{(s)}_{ij}(0) = 0 \quad 0 \leq s \leq N - 2,$$

$$\sum_{ij} \alpha_{ij} f^{(s)}_{ij}(0) = 1, \quad s = N - 1.$$ 

This can be done as the corresponding $(N - 1) \times (N - 1)$ determinant $\det[f^{(s)}_{ij}(0)]$ is the Wronskian for the functions $\{f_{ij} = x^i e^{jx}\}_{i + j \leq n}$ which is non-zero due to the fact that these functions form a full set of linearly
independent solutions of a certain constant coefficient linear differential equation. □

Since $N = O(n^2)$, it follows from Proposition 2.7 and Lemma 2.8 that, indeed, $\phi(n) = n^2$ is best possible in order for the extremal function $\Phi_{[0,r]}(x; V, \phi)$ to be bounded on the interval $[0, 2r]$.

We now prove that a bounded extremal function implies the existence of a Markov type inequality.

**Theorem 2.9** Suppose that $[a, b] \subset \mathbb{R}$ and that the extremal function $\Phi_{[a,b]}(z; V, \phi)$ is bounded inside the ellipse $E_{a,b,c} \subset \mathbb{C}$ having foci at $a$ and $b$ and major axis $[a-c, b+c]$ for some $c > 1/(\phi(1))^2$. Then there is a Markov inequality, in the sense that there exists a constant $C > 0$ such that

$$\|f'(x)\|_{[a,b]} \leq C(\phi(n))^2\|f\|_{[a,b]}$$

for all $f \in V_n$.

**Proof.** We make use of the so-called relative extremal function of plurisubharmonic potential theory. $C$ will denote a generic constant; it need not have the same value in all of its instances.

Suppose that $\Omega \subset \mathbb{C}^d$ is an open set and that $E \subset \Omega$ is compact. Then the relative extremal function is defined as

$$u_{E,\Omega}(z) := \sup\{v(z) : v \in \mathcal{PSH}(\Omega), v \leq 1 \text{ on } \Omega, v \leq 0 \text{ on } E\}.$$  \hspace{1cm} (6)

Here $\mathcal{PSH}(\Omega)$ denotes the plurisubharmonic functions on $\Omega$ (see [K] for the precise definition of this class of functions). It is known that for $\Omega = E_{a,b,c} \subset \mathbb{C}$ and $E = [a, b] \subset \Omega$,

$$u_{E,\Omega}(z) = \frac{\ln |w/r + \sqrt{(w/r)^2 - 1}|}{\ln |R/r + \sqrt{(R/r)^2 - 1}|}$$  \hspace{1cm} (7)

where $w := z - (a+b)/2$, $r = (b-a)/2$ is the “radius” of the interval $[a, b]$ and $R = r + c$ is that of $[a - c, b + c]$.

Now, for non-constant $f \in V$, with the same meanings of $\Omega$ and $E$ as above, let

$$g(z) = \frac{\ln |f(z)| - \ln \|f\|_E}{\ln \|f\|_\Omega - \ln \|f\|_E}.$$
Then $g \in \mathcal{PSH}(\Omega)$ and, clearly, $g \leq 1$ on $\Omega$ and $g \leq 0$ on $E$. Hence $g$ is a competitor in the definition of the relative extremal function (6) and so $g(z) \leq u_{E,\Omega}(z)$. It follows that

$$\frac{\ln |f(z)| - \ln \|f\|_E}{\ln \|f\|_\Omega - \ln \|f\|_E} \leq u_{E,\Omega}(z)$$

and that

$$\ln \left( \frac{|f(z)|}{\|f\|_E} \right) \leq \ln \left( \frac{\|f\|_\Omega}{\|f\|_E} \right) u_{E,\Omega}(z)$$

$$\leq \ln \left( \sup_{z \in \Omega} \Phi_E(z; V, \phi)^{\phi(d(f))} \right) \ln \frac{|w/r + \sqrt{(w/r)^2 - 1}|}{|R/r + \sqrt{(R/r)^2 - 1}|}$$

$$\leq C\phi(d(f)) \ln |w/r + \sqrt{(w/r)^2 - 1}|, \, z \in \Omega \quad (8)$$

by the boundedness of the extremal function.

We now bound the derivative of $f \in V$ by means of the Cauchy Integral Formula. Suppose then that $x \in [a, b]$ and let $\Gamma_s \subset \mathbb{C}$ be the circle of radius $s$ centred at $x$. Then

$$f'(x) = \frac{1}{2\pi i} \int_{\Gamma_s} \frac{f(z)}{(z-x)^2} \, dz$$

so that

$$|f'(x)| \leq \frac{1}{s} \max_{z \in \Gamma_s} |f(z)|. \quad (9)$$

But then by (8), for $s$ sufficiently small so that $\Gamma_s \subset \Omega$,

$$|f'(x)| \leq \frac{1}{s} \max_{z \in \Gamma_s} |w/r + \sqrt{(w/r)^2 - 1}|^{C\phi(d(f))} \|f\|_E$$

where $w$ and $r$ have the same meanings as previously stated. Now, elementary calculations (cf. Lemma 1.1 of [BLMT1]) reveal that there is a constant $C$ such that for $z - x \leq s$,

$$|w/r + \sqrt{(w/r)^2 - 1}| \leq 1 + C \sqrt{s}.$$ 

Hence,

$$|f'(x)| \leq \frac{1}{s} (1 + C \sqrt{s})^{C\phi(d(f))} \|f\|_E$$
and the result follows by taking
\[ s = \frac{1}{(\phi(\deg(f)))^2}. \]
\[ \square \]

Putting together Corollary 2.5, Theorem 2.6 and Theorem 2.9, we obtain Theorem 1.2.

3 Concluding Remarks

Using methods similar to those presented in this paper, one can show that \( y = f(x) \) admits a tangential Markov inequality of some exponent (which can be effectively estimated) if \( f(x) = \sum_{i=1}^{k} p_i(x)e^{q_i(x)} \) is a generalized exponential polynomial. Here \( q_i, p_i \in \mathbb{R}[x] \) are fixed polynomials. We conjecture that, more generally, \( y = f(x) \) admits a tangential Markov inequality of some exponent if \( f(x) \) satisfies a linear differential equation

\[ y^{(d)} + a_1(x)y^{(d-1)} + \cdots + a_{d-1}(x)y' + a_d(x)y = 0 \]

with polynomial coefficients.

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