Upcoming-wave equation of seismic migration

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Abstract. Upcoming-wave is the basis of several seismic migration methods that use the principle of 1-way-time by propagating only from the event point to the receiver. The migration method that uses this upcoming wave theory is the finite difference migration and Fourier transformation migration. In this paper, we will elaborate how the theories of these two migrations are related and described from scalar wave equations to various forms of upcoming wave approximations, both differential equation form of space-time (x-t) and frequency - wave number (f-k) form, along with their transformations.

Key words: upcoming wave, approximation, seismic migration

1. Introduction
Seismic data migration is one of the method to determine the exact location of the reflector by completing the wave equation using a variety of approaches. Generally, there are three migration methods based on wave equation solution techniques with different approaches [4],[16]: methods based on numerical approaches such as finite difference, diffraction summation method, known as Kirchhoff summation, and Fourier transformation method. The finite difference migration method was introduced by [2] by implementing downward continuation and upcoming waves as well as the principle of 1-way-time of wave. This method was later developed by [1] with the principle of a 2-way-time of wave known as the reverse time migration method. The migration method with diffraction summation was introduced by [9] and became a method of migration that was quite popular in various oil and gas companies. Fourier transformation method has a fairly rapid development due to the very efficient application of its computation. Developers of this method include [13], [5], and [12]. Table 1 provides an overview of the advantages and disadvantages of some of the migration methods according to [4].

In this paper we only focus on the basic theory of the migration method that uses the basic theory of upcoming wave. The upcoming wave, also referred to as a one-way wave, has also been investigated by several previous researchers who applied several methods to solve upcoming-wave equation such as [15] that applied forward modeling and Hartley's transformation, [14] who applied the spectral - finite difference method, and [7] implementing the pseudo-spectral method.
2. General equation of upcoming wave

An upcoming-wave is one of the wave models applied to the migration of seismic data where the characteristics of this wave are 1-way-time that propagates from the reflection point to the receiver. This wave has different dimension than scalar wave after going through a process of reflection. To determine the upcoming wave, first a time transformation is performed on the scalar wave function, then a re-substitution of the equation is performed. In theory, the main difference from this wave can be seen significantly on the wave function when the travel time defined. From the wave function, the upcoming wave can be determined again. The scalar wave equation that propagates subsurface of the earth from the source to the receiver [3], [11] is

\[(\nabla^2 \psi)^2 = \frac{\partial^2 \psi}{\partial t^2}\]  (1)

Where \(\nabla^2\) is the Laplace operator, \(v\) is a P-wave velocity, \(\psi\) is a wave function, and \(t\) is a travel time of wave propagation. The wave functions of the equation for 2D and 3D are respectively

\[
\psi(x, z, t) = Ae^{\left\{2j\omega t - jax \sin \theta - jax \cos \theta \right\}}
\]  (2)

\[
\psi(x, y, z, t) = Ae^{\left\{ \frac{2j\omega t - jax \sin \theta - \sqrt{2}jax \sin \theta - jax \cos \theta - 2jax \cos \theta}{v} \right\}}
\]  (3)

Where \(j = \sqrt{-1}\) and \(t\) is the overall wave travel times of the source – reflector point – receiver. The two wave functions above are determined using variable separation techniques and apply some of the conditions [3]. The \(t\) component of the two functions is the differentiating factor between scalar waves and the upcoming waves. If it is assumed that \(\tau\) is the travel time from the event point to the receiver, \(\tau = t - \frac{\tau}{v}\), then the results of the transformation from equations (2) and (3) are

\[
\xi(x, z, \tau) = Ae^{\left\{2j\omega \tau - jax \sin \theta - jax \cos \theta \right\}}
\]  (4)

\[
\xi(x, y, z, \tau) = Ae^{\left\{ \frac{2j\omega \tau - jax \sin \theta - \sqrt{2}jax \sin \theta - jax \cos \theta - 2jax \cos \theta}{v} \right\}}
\]  (5)
Equations (4) and (5) are upcoming wave functions where the components of \(x\) and \(y\) are surface components, while \(z\) in this equation there are two possibilities, subsurface cross section in time domain or depth domain. In the migration process, there are two domains produced as output, namely depth domain and time domain. Migration that applies the effect of thin-lens term, \(j\omega \left( \frac{1}{v(x, y)} - \frac{1}{v(z)} \right)\), is called depth migration [3] and migration that applies the diffraction term is called time migration [16]. When transformed into partial differential equations, for equations (4), the results are shown in equations (6) and (7). This transformation technique follows the transformation method by equalizing the variables in the scalar wave equation with the upcoming-wave equation [11].

\[
\frac{\partial^2 \xi}{\partial z \partial \tau} = \frac{v}{2} \left( \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial z^2} \right) \tag{6}
\]

\[
\frac{\partial^2 \xi}{\partial z \partial \tau} = \frac{v}{2} \left( \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2} \right) \tag{7}
\]

3. Approximation and transformation of the upcoming-wave equation
Claerbout [3] used this upcoming wave equation for the process of seismic data migration by applying downward continuation and the approximation conditions of wave angles when applied computational calculations. From this approximate condition, we can define the field area used to determine the true reflector point. The approximation commonly known of an upcoming wave is the approximation of 15°, 45°, 60°. In this paper, I will use the first (1st), second (2nd) and third (3rd) approximation names that represent 15°, 45°, and 60°. The higher the approximation is used, the wider the range of determining the actual reflector point, and the longer the calculation process is performed. Figure 1 shows the approximation waveform. The approximation of this wave can be obtained in various ways, such as by differentiating the wave equation, the basic approximation (1st) to the \(z\) domain, or by determining wave dispersion using the Muir’s square root expansion [3]. The initial step to do this method is to substitute equations (4) and (5) to the scalar wave equation by first reviewing the transformation.

Figure 1. The approximation model of wave angles, (a) [10], (b) [3]
For 2D obtained

\[
\frac{\partial^2 \psi}{\partial x^2} \approx \frac{\partial^2 \xi}{\partial x^2} = -\left(\frac{\omega \theta}{v}\right)^2 \xi(x, y, z, \tau) = -k_x^2 \xi(x, y, z, \tau) \tag{8a}
\]

\[
\frac{\partial^2 \psi}{\partial t^2} \approx \frac{\partial^2 \xi}{\partial t^2} = -(\omega)^2 \xi(x, y, z, \tau) = -\omega^2 \xi(x, y, z, \tau) \tag{8b}
\]

\[
\frac{\partial^2 \psi}{\partial z^2} \approx \frac{\partial^2 \xi}{\partial z^2} - \frac{2}{v} \frac{\partial^2 \xi}{\partial z \partial \tau} + \frac{1}{v^2} \frac{\partial^2 \xi}{\partial \tau^2} = -\left(\frac{\omega^2}{2v} - \frac{\omega}{v}\right)^2 \xi(x, y, z, \tau) = -k_z^2 \xi(x, y, z, \tau) \tag{8c}
\]

And for 3D

\[
\frac{\partial^2 \psi}{\partial x^2} \approx \frac{\partial^2 \xi}{\partial x^2} = -\left(\frac{\omega \theta}{v}\right)^2 \xi(x, y, z, \tau) = -k_x^2 \xi(x, y, z, \tau) \tag{9a}
\]

\[
\frac{\partial^2 \psi}{\partial y^2} \approx \frac{\partial^2 \xi}{\partial y^2} = -\left(\frac{\sqrt{3} \omega \theta}{v}\right)^2 \xi(x, y, z, \tau) = -k_y^2 \xi(x, y, z, \tau) \tag{9b}
\]

\[
\frac{\partial^2 \psi}{\partial t^2} \approx \frac{\partial^2 \xi}{\partial t^2} = -(2\omega)^2 \xi(x, y, z, \tau) = -4\omega^2 \xi(x, y, z, \tau) \tag{9c}
\]

\[
\frac{\partial^2 \psi}{\partial z^2} \approx \frac{\partial^2 \xi}{\partial z^2} - \frac{2}{v} \frac{\partial^2 \xi}{\partial z \partial \tau} + \frac{1}{v^2} \frac{\partial^2 \xi}{\partial \tau^2} = -\left(\frac{\omega}{v} (\theta^2 - 2)\right)^2 \xi(x, y, z, \tau) = -k_z^2 \xi(x, y, z, \tau) \tag{9d}
\]

From equations (8) and (9), if we only review the form of partial differential equations, the upcoming wave equation for the 1st approximation is

\[
\frac{\partial^2 \xi}{\partial \tau^2} = \frac{v}{2} \frac{\partial^2 \xi}{\partial x^2} \tag{1st} \tag{10a}
\]

Equation (10a) is a representation of [2] and [3] that directly applied the 1st approximation by declaring the wave angle was very small. The equation (10a) is also equivalent to the Schrodinger’s equation in quantum mechanics [8]. For each equations 2nd and 3rd are

\[
\frac{\partial^3 \xi}{\partial z \partial \tau^2} = \frac{v}{2} \left(\frac{\partial^3 \xi}{\partial x^3} + \frac{\partial^3 \xi}{\partial x^2 \partial \tau} \right) \tag{2nd} \tag{10b}
\]

\[
\frac{\partial^4 \xi}{\partial z^2 \partial \tau^3} = \frac{v}{2} \left(\frac{\partial^4 \xi}{\partial x^4} + \frac{v}{\partial x^2 \partial z} \frac{\partial^4 \xi}{\partial x^2 \partial \tau^2} + \frac{v^2}{4} \frac{\partial^4 \xi}{\partial x^4 \partial \tau^2} \right) \tag{3rd} \tag{10c}
\]

And For 3D

\[
\frac{\partial^2 \xi}{\partial z \partial \tau} = \frac{v}{2} \left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \right) \tag{1st} \tag{11a}
\]

\[
\frac{\partial^3 \xi}{\partial z \partial \tau^2} = \frac{v}{2} \left(\frac{\partial^3 \xi}{\partial x^3} + \frac{v}{\partial x^2 \partial z} \frac{\partial^3 \xi}{\partial x^2 \partial \tau} + \frac{\partial^3 \xi}{\partial z \partial \tau} + \frac{\partial^3 \xi}{\partial y^2 \partial \tau} \right) \tag{2nd} \tag{11b}
\]
The 3rd International Conference On Science

Phase-Shift method was later redeveloped into the Fourier Split-Step Migration Method [12].

Equation (8) and (9), when solved using the finite difference method, will be the basis for finite difference migration. The domain of this equation is space-time domain ($x-t$).

If equations (8) and (9) are substituted to equation (1), they are obtained for 2D and 3D respectively

\[ -k_x^2 - k_y^2 = -\frac{\omega^2}{v^2} \]  

(12)

\[ -k_x^2 - k_y^2 - k_z^2 = -\frac{4\omega^2}{v^2} \]  

(13)

Equation (12) and (13) are wave dispersion equations where $k$ is a wavenumber and $\omega$ is a frequency. To determine the upcoming-wave equation with an approximation of 1st, 2nd, 3rd from this equation, equation (12) and (13) applies Muir’s square root expansion then transformed back to the form of partial differential equations, so that for equation (12) is obtained

\[ k_x = \frac{\omega}{v} - \frac{2\nu k_x^2}{4\omega} \]  

(1st)  

(14a)

\[ k_x = \frac{\omega}{v} - \frac{2\omega\nu k_x^2}{4\omega^2 - v^2 k_x^2} \]  

(2nd)  

(14b)

\[ k_x = \frac{\omega}{v} - \frac{v^2 k_x^2}{8v - 4\left(\frac{v^2 k_x^2}{\omega^2}\right)} \]  

(3rd)  

(14c)

For equation (13)

\[ k_x = \frac{2\omega}{v} - \frac{v(k_x^2 + k_y^2)}{4\omega} \]  

(1st)  

(15a)

\[ k_x = \frac{2\omega}{v} - \frac{4\omega v(k_x^2 + k_y^2)}{16\omega^2 - v^2(k_x^2 + k_y^2)} \]  

(2nd)  

(15b)

\[ k_x = \frac{2\omega}{v} - \frac{v(k_x^2 + k_y^2)(16\omega^2 - v^2(k_x^2 + k_y^2))}{2\omega(36\omega^2 - 4v^2(k_x^2 + k_y^2))} \]  

(3rd)  

(15c)

If equations (14) and (15) are inverted where $j\nu$ corresponds to $\frac{\partial}{\partial z}$ based on the relationship of equation (8) and (9), it will produce the equation in Table 2. The equations in Table 2 are the equations that represent the basis of the Phase Shift migration method introduced by [5] and [6]. Stolt’s Migration [13] also has the same resemblance to the Phase Shift method where Fourier transformation is used to change the wave equation space-time domain ($x-t$) to the frequency–wavenumber domain ($f-k$). This Phase-Shift method was later redeveloped into the Fourier Split-Step Migration Method [12].
Table 2. Results of transformation from frequency – wavenumbers to differential equations

| 2D | Approx. | 3D |
|-----|---------|-----|
| \( \frac{\partial \xi}{\partial z} = j \frac{\omega - v k_z^2 / 2\omega}{v} \xi \) | \( 1^{st} \) | \( \frac{\partial \xi}{\partial z} = j \frac{2\omega - v(k_z^2 + k_y^2)}{v} \xi \) |
| \( \frac{\partial \xi}{\partial z} = j \frac{\omega - 2\omega v k_z^2}{v - 4\omega^2 - v^2 k_z^2} \xi \) | \( 2^{nd} \) | \( \frac{\partial \xi}{\partial z} = j \frac{2\omega - 4\omega v (k_z^2 + k_y^2)}{v - 16\omega^2 - v^2 (k_z^2 + k_y^2)} \xi \) |
| \( \frac{\partial \xi}{\partial z} = j \left( \frac{v^2 k_z^2}{\omega^2} - \frac{v^2 k_z^2}{\omega^2} \right) \xi \) | \( 3^{rd} \) | \( \frac{\partial \xi}{\partial z} = j \left( \frac{2\omega - v(k_z^2 + k_y^2) [16\omega^2 - v^2 (k_z^2 + k_y^2)]}{2\omega (36\omega^2 - 4v^2 [k_z^2 + k_y^2])} \right) \xi \) |

4. Conclusion

The upcoming wave equation can be obtained by determining the scalar wave function and defining travel time for the function. From this definition, we can obtain the upcoming-wave equation approximation of the partial differential form \((x-t)\) and the frequency form of the wave number \((f-k)\) along with their transformation. By applying this elaboration technique, we can obtain a higher upcoming wave approximation in accordance with the target calculation in the implementation of the migration methods with \(x-t\) and \(f-k\) domain.

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