Tight triangulations of closed 3-manifolds

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Abstract

It is well known that a triangulation of a closed 2-manifold is tight with respect to a field of characteristic two if and only if it is neighbourly; and it is tight with respect to a field of odd characteristic if and only if it is neighbourly and orientable. No such characterization of tightness was previously known for higher dimensional manifolds. In this paper, we prove that a triangulation of a closed 3-manifold is tight with respect to a field of odd characteristic if and only if it is neighbourly, orientable and stacked. In consequence, the Kühnel-Lutz conjecture is valid in dimension three for fields of odd characteristic.

Next let $F$ be a field of characteristic two. It is known that, in this case, any neighbourly and stacked triangulation of a closed 3-manifold is $F$-tight. For triangulated closed 3-manifolds with at most 71 vertices or with first Betti number at most 188, we show that the converse is true. But the possibility of an $F$-tight non-stacked triangulation on a larger number of vertices remains open. We prove the following upper bound theorem on such triangulations. If an $F$-tight triangulation of a closed 3-manifold has $n$ vertices and first Betti number $\beta_1$, then $(n - 4)(617n - 3861) \leq 15444\beta_1$. Equality holds here if and only if all the vertex links of the triangulation are connected sums of boundary complexes of icosahedra.

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1 Introduction

All simplicial complexes considered in this paper are finite and abstract. All homologies are simplicial homologies with coefficients in a field $F$. The vertex set of a simplicial complex $X$ will be denoted by $V(X)$. For $A \subseteq V(X)$, the induced subcomplex $X[A]$ of $X$ on the vertex set $A$ is defined by $X[A] := \{\alpha \in X : \alpha \subseteq A\}$. A simplicial complex $X$ is said to be a triangulated manifold if it triangulates a manifold, i.e., if the geometric carrier $|X|$ of $X$ is a topological manifold. A triangulated closed $d$-manifold $X$ is said to be $F$-orientable if $H_d(X; F) \neq 0$. So, for a field $F$ of characteristic two, any triangulated closed manifold is $F$-orientable.
Taking his cue from pre-existing notions of tightness in the theory of smooth and polyhedral embedding of manifolds in Euclidean spaces, Kühnel [12] introduced the following precise notion of tightness of a simplicial complex with respect to a field.

**Definition 1.1.** Let $X$ be a simplicial complex and $F$ be a field. We say that $X$ is **tight with respect to** $F$ (in short, $F$-tight) if (a) $X$ is connected, and (b) for every induced subcomplex $Y$ of $X$, the $F$-linear map $H_* (Y; F) \to H_* (X; F)$ (induced by the inclusion map $Y \hookrightarrow X$) is injective.

Recall that, if $X$ is a simplicial complex of dimension $d$, then its **face numbers** $f_i (X)$ are defined by $f_i (X) := \# \{ \alpha \in X : \dim (\alpha) = i \}$, $0 \leq i \leq d$. For $k \geq 2$, a simplicial complex $X$ is said to be $k$-**neighbourly** if any set of $k$ vertices of $X$ form a face, i.e., if $f_{k-1} (X) = \binom{f_0 (X)}{k}$. A 2-neighbourly simplicial complex is called **neighbourly**.

**Definition 1.2.** A simplicial complex $X$ is said to be **strongly minimal** if, for every triangulation $Y$ of the geometric carrier $|X|$ of $X$, we have $f_i (X) \leq f_i (Y)$ for all $i$, $0 \leq i \leq \dim (X)$.

Thus, a strongly minimal triangulation of a topological space, if it exists, is the most economical among all possible triangulations of the space. Unfortunately, there are very few criteria available in the literature which ensure strong minimality. The notion of tightness is of great importance in combinatorial topology because of the following tantalizing conjecture [13].

**Conjecture 1.3** (Kühnel-Lutz). Every $F$-tight triangulated closed manifold is strongly minimal.

Intuitively, $F$-tightness of a triangulated manifold $X$ means that all parts of $X$ are essential in order to capture the $F$-homology of the topological space $|X|$. In view of this intuition, Conjecture 1.3 appears to be entirely plausible. However, Example 6.3 below shows that this intuition is not correct for arbitrary simplicial complexes.

Recall the following from [15].

**Definition 1.4.** A triangulated manifold $\Delta$ of dimension $d + 1$ is said to be **stacked** if all its faces of codimension (at least) two are in the boundary $\partial \Delta$. A triangulated closed manifold $M$ of dimension $d$ is said to be **stacked** if there is a stacked triangulated manifold $\Delta$ of dimension $d + 1$ such that $M = \partial \Delta$.

In particular, a **stacked sphere** is a triangulated sphere which may be realized as the boundary of a stacked triangulated ball.

**Definition 1.5.** A triangulated manifold is said to be **locally stacked** if all its vertex links are stacked spheres or stacked balls.

Clearly, all stacked triangulated manifolds are locally stacked, but the converse is false (see Example 6.2 below). Due to [5, Theorem 2.24; case $k = 1$] we have the following.

**Proposition 1.6** (Bagchi-Datta). Let $M$ be a locally stacked $F$-orientable, neighbourly, triangulated closed 3-manifold. Then the following are equivalent

(i) $M$ is $F$-tight,

(ii) $M$ is stacked, and
(iii) \( \left( \frac{f_0(M)}{2} \right)^{-4} = 10\beta_1(M; \mathbb{F}) \).

Thus, all orientable, neighbourly, stacked triangulated closed 3-manifolds are \( \mathbb{F} \)-tight (for any field \( \mathbb{F} \)). Even more, we know from [6].

**Proposition 1.7** (Bagchi-Datta). Every \( \mathbb{F} \)-tight, locally stacked, triangulated closed manifold is strongly minimal.

The first main result of this paper (in Section 4) is a converse of Proposition 1.6 if a triangulated closed 3-manifold is tight with respect to a field of characteristic \( \neq 2 \), then it must be (orientable, neighbourly and) stacked. This result answers Question 4.5 of [8] affirmatively, in the case of odd characteristic. As a consequence of Proposition 1.7 it follows that the Kühnel-Lutz conjecture is true in a special case, namely, if \( \text{char}(\mathbb{F}) \neq 2 \) then any \( \mathbb{F} \)-tight triangulated closed 3-manifold is strongly minimal.

Let \( X_1 \) and \( X_2 \) be two triangulated \( d \)-manifolds intersecting in a common facet (\( d \)-face) \( \alpha \). That is, \( X_1 \cap X_2 = \tilde{\alpha} \), where \( \tilde{\alpha} \) denotes \( \alpha \) together with all of its subfaces. Then \( X_1 \# X_2 = (X_1 \cup X_2) \setminus \{ \alpha \} \) is said to be the connected sum of \( X_1 \) and \( X_2 \) along \( \alpha \) (for more general definition, see the end of Section 2). Let \( I = I_{72}^2 \) be the boundary complex of the icosahedron. Thus, \( I \) is a triangulated 2-sphere on 12 vertices. It is well known that \( I \) is the unique triangulation of \( S^2 \) in which each vertex is of degree 5. We introduce:

**Definition 1.8.** A triangulated 2-sphere is said to be icosian if it is a connected sum of finitely many copies of \( I \). A triangulated 3-manifold is said to be locally icosian if all its vertex links are icosian.

In [17], the third author proved the following interesting upper bound theorem for tight triangulations of odd dimensional manifolds.

**Proposition 1.9** (Spreer). Let \( M \) be an \((\ell - 1)\)-connected triangulated closed \((2\ell + 1)\)-manifold and \( \mathbb{F} \) be a field. If \( M \) is \( \mathbb{F} \)-tight, then

\[
\left( \frac{f_0(M)}{2} \right)^{-4} \left( \frac{f_0(M) - 1}{\ell + 1} \right) \left( \frac{f_{\ell + 1}(M) - 1}{\ell + 1} \right) \left( \frac{f_0(M) - 1}{\ell} \right) \leq \beta_1(M; \mathbb{F}).
\]

In Section 5, we consider fields of characteristic 2. According to Proposition 1.6 every neighbourly and stacked closed 3-manifold \( M \) is \( \mathbb{Z}_2 \)-tight. We do not know whether the converse is true or not. But in this paper we prove that if \( M \) is a \( \mathbb{Z}_2 \)-tight triangulated closed 3-manifold with \( f_0(M) \leq 71 \) or \( \beta_1(M; \mathbb{Z}_2) \leq 188 \), then \( M \) must be stacked and neighbourly (and therefore \( \left( \frac{f_0(M)}{2} \right)^{-4} = 10\beta_1(M; \mathbb{Z}_2) \)). We also show that, in general, each vertex link of a \( \mathbb{Z}_2 \)-tight closed 3-manifold must be a connected sum of \( I \)'s and \( S^2 \)'s. Further, we prove that any \( \mathbb{Z}_2 \)-tight triangulated closed 3-manifold \( M \) satisfies the following upper bound on \( f_0(M) \):

\[
(f_0(M) - 4)(617f_0(M) - 3861) \leq 15444\beta_1(M; \mathbb{Z}_2).
\]

Equality holds here if and only if \( M \) is locally icosian. In conjunction with the fact that \( \left( \frac{f_0(M)}{2} \right)^{-4} = 10\beta_1(M; \mathbb{Z}_2) \) when \( f_0(M) \leq 71 \), this inequality improves upon the upper bound of Proposition 1.9 in case \( \ell = 1 \). We also prove that, if there is a non-stacked \( \mathbb{F} \)-tight triangulated 3-manifold \( M \), then its integral homology group \( H_1(M; \mathbb{Z}) \) must have an element of order 2. The results of Section 5 were largely suggested by extensive machine computations using simpcomp [10]. Altogether, these results impose severe restrictions on the topology of 3-manifolds admitting tight triangulations (cf. Corollary 5.14).

In Section 6, we present some examples to show that the converses/generalizations of several results proved here are not true.
2 Preliminaries on stacked and tight triangulations

In this section, we gather together a few easy (and mostly known) consequences of tightness. For completeness, we include their proofs. We shall use:

Notation 2.1. If $x$ is a vertex of a simplicial complex $X$, then $X^x$ and $X_x$ will denote the antistar and the link (respectively) of $x$ in $X$. Thus,

$$X^x := \{ \alpha \in X : x \notin \alpha \} = X[V(X) \setminus \{x\}],$$

$$X_x := \{ \alpha \in X : x \notin \alpha, \alpha \cup \{x\} \in X \}.$$

We denote a face $\{u_1, \ldots, u_m\}$ in a simplicial complex by $u_1u_2 \cdots u_m$. If $X$ is a simplicial complex and $a \notin X$ is an element then the cone with apex $a$ and base $X$ is the simplicial complex $X \cup \{a \cup \{a\} : a \in X\}$ and is denoted by $a \ast X$. For a simplicial complex $X$ of dimension $d$, and for $0 \leq k \leq d$, the $k$-skeleton $\text{ske}_{k}(X)$ is defined by

$$\text{ske}_{k}(X) := \{ \alpha \in X : \dim(\alpha) \leq k \}.$$

Lemma 2.2. Every $\mathbb{F}$-tight simplicial complex is neighbourly.

Proof. Suppose, if possible, $x \neq y$ are two vertices of an $\mathbb{F}$-tight simplicial complex $X$ such that $xy$ is not an edge of $X$. Let $Y$ be the induced subcomplex of $X$ on the set $\{x, y\}$. Then $\beta_0(Y; \mathbb{F}) = 2 > 1 = \beta_0(X; \mathbb{F})$, so that $H_0(Y; \mathbb{F}) \rightarrow H_0(X; \mathbb{F})$ can not be injective. This is a contradiction since $X$ is $\mathbb{F}$-tight.

Lemma 2.3. Every induced subcomplex of an $\mathbb{F}$-tight simplicial complex is $\mathbb{F}$-tight.

Proof. Let $Y$ be an induced subcomplex of an $\mathbb{F}$-tight simplicial complex $X$. By Lemma 2.2, $X$ and hence also $Y$ are neighbourly. So, $Y$ is connected. Let $Z$ be an induced subcomplex of $Y$. Then $Z$ is an induced subcomplex of $X$ also. Since the composition of the linear maps $H_*(Z; \mathbb{F}) \rightarrow H_*(Y; \mathbb{F}) \rightarrow H_*(X; \mathbb{F})$ is injective, the first of them must be injective. Thus, $H_*(Z; \mathbb{F}) \rightarrow H_*(Y; \mathbb{F})$ is injective for all induced subcomplexes $Z$ of $Y$. Therefore, $Y$ is $\mathbb{F}$-tight.

Lemma 2.4. If $X$ is an $\mathbb{F}$-tight simplicial complex of dimension $d$, then $\text{ske}_{k}(X)$ is $\mathbb{F}$-tight for $1 \leq k \leq d$.

Proof. Since $k \geq 1$ and $X$ is neighbourly by Lemma 2.2, it follows that $\text{ske}_{k}(X)$ is neighbourly and hence connected. Let $Y$ be an induced subcomplex of $\text{ske}_{k}(X)$. Then $Y = \text{ske}_{k}(Z)$, where $Z$ is an induced subcomplex of $X$. Since $X$ is $\mathbb{F}$-tight, it follows that, for $0 \leq i \leq k - 1$, $H_i(Y) = H_i(Z) \rightarrow H_i(X) = H_i(\text{ske}_{k}(X))$ is injective. Clearly, $Z_k(Y) \subseteq Z_k(\text{ske}_{k}(X))$. Since both $Y$ and $\text{ske}_{k}(X)$ are of dimension $\leq k$, $B_k(Y) = 0 = B_k(\text{ske}_{k}(X))$. These imply that $H_k(Y) \rightarrow H_k(\text{ske}_{k}(X))$ is injective.

Lemma 2.5. Every $\mathbb{F}$-tight triangulation of a closed manifold is $\mathbb{F}$-orientable.

Proof. If $\text{char}(\mathbb{F}) = 2$ then there is nothing to prove. So, assume that $\text{char}(\mathbb{F}) \neq 2$. Let $X$ be an $\mathbb{F}$-tight triangulated closed $d$-manifold. We can assume that $d \geq 2$. Since $X$ is a connected triangulated closed $d$-manifold, it is easy to see ab initio that, for any proper subcomplex $Y$ of $X$, $H_d(Y; \mathbb{F}) = 0$. Now, fix a vertex $x$ of $X$, and consider the induced subcomplex $X^x$ of $X$ on the complement of $x$. Then $X = X^x \cup (x \ast X_x)$ and $X^x \cap (x \ast X_x) =$
orientable, Poincaré duality implies satisfies 
Hence [1, Corollary 1.8] implies that special case of [1, Corollary 1.8]. Suppose The inequality (as well as the fact that equality holds if Proof. \( \beta \)

\[
\sum_{x \in V(X)} \frac{1}{1 + f_0(x)} \], \quad i \geq 1.
\]

(Here \( \delta_{i1} \) is Kronecker’s symbol. Thus, \( \delta_{i1} = 1 \) if \( i = 1 \), and \( = 0 \) otherwise.) Notice that \( \mu_1(X; \mathbb{F}) \) is independent of the field \( \mathbb{F} \). Therefore, we will write \( \mu_1(X) \) for \( \mu_1(X; \mathbb{F}) \). We have:

**Lemma 2.6.** Let \( M \) be an \( \mathbb{F} \)-orientable, neighbourly triangulated closed 3-manifold. Then \( \beta_1(M; \mathbb{F}) \leq \mu_1(M) \). Equality holds here if and only if \( M \) is \( \mathbb{F} \)-tight.

**Proof.** The inequality (as well as the fact that equality holds if \( M \) is \( \mathbb{F} \)-tight) is a very special case of [1, Corollary 1.8]. Suppose \( \mu_1 = \beta_1 \). Since \( M \) is 2-neighbourly, it also satisfies \( \mu_0 = 1 = \beta_0 \). By [1, Theorem 1.6], \( \mu_2 = \mu_1 \) and \( \mu_3 = \mu_0 \). Also, since \( M \) is \( \mathbb{F} \)-orientable, Poincaré duality implies \( \beta_2 = \beta_1 \) and \( \beta_3 = \beta_0 \). Therefore, \( \mu_2 = \beta_2 \) and \( \mu_3 = \beta_3 \). Hence [1, Corollary 1.8] implies that \( M \) is \( \mathbb{F} \)-tight.

Recall that, a triangulated closed \( d \)-manifold \( X \) is called orientable if \( H_d(X; \mathbb{Z}) \neq 0 \). It follows from the universal coefficient theorem that, for a field \( \mathbb{F} \) of odd characteristic, a triangulated closed manifold is \( \mathbb{F} \)-orientable if and only if it is orientable.

**Corollary 2.7.** Let \( p \) be a prime and let \( M \) be an orientable triangulated closed 3-manifold. If \( M \) is \( \mathbb{Z}_p \)-tight but not \( \mathbb{Q} \)-tight, then \( p \) divides the order of the torsion subgroup of \( H_1(M; \mathbb{Z}) \).

**Proof.** The hypothesis and Lemma 2.6 imply that \( \beta_1(M; \mathbb{Q}) < \mu_1(M) = \beta_1(M; \mathbb{Z}_p) \). Hence the result follows from the universal coefficient theorem.
The following lemma is immediate from the definition of tightness.

**Lemma 2.8.** (a) A simplicial complex is tight w.r.t. a field of characteristic \( p \) if and only if it is \( \mathbb{Z}_p \)-tight. (b) A simplicial complex is tight w.r.t. a field of characteristic zero if and only if it is \( \mathbb{Z}_p \)-tight for all primes \( p \).

For any non-empty finite set \( \alpha, \overline{\alpha} \) will denote the simplicial complex whose faces are all the subsets of \( \alpha \). Thus, if \( \#(\alpha) = d+1 \), \( \overline{\alpha} \) is the standard triangulation of the \( d \)-ball (namely, it is the face complex of the geometric \( d \)-simplex). If \( \#(\alpha) = d+2 \), then the boundary \( \partial \overline{\alpha} \) of \( \overline{\alpha} \) is the standard triangulation of the \( d \)-sphere; it will be denoted by \( S^d_{d+2} \). (More generally, \( S^d_n \) usually denotes an \( n \)-vertex triangulation of the \( d \)-sphere.) From the definition of a stacked sphere, one can deduce the following (this also follows from \[3\] Lemmas 4.3 (b) & 4.8 (b)).

**Lemma 2.9.** A simplicial complex \( S \) is a stacked \( d \)-sphere if and only if \( S \) is a connected sum of finitely many copies of the \( (d+2) \)-vertex standard sphere \( S^d_{d+2} \).

Let \( X \) be a triangulated closed manifold and let \( \sigma \) and \( \tau \) be facets of \( X \). For a bijection \( \psi : \sigma \to \tau \), let \( X^\psi \) be the simplicial complex obtained from \( X \setminus \{\sigma, \tau\} \) by identifying \( v \) and \( \psi(v) \) for \( v \in \sigma \). If \( \text{lk}(v) \cap \text{lk}(\psi(v)) = \{0\} \) for each vertex \( v \in \sigma \), then \( X^\psi \) is a triangulated manifold. If \( \sigma \) and \( \tau \) belong to different connected components, say \( \sigma \in X_1, \tau \in X_2 \) and \( X = X_1 \sqcup X_2 \), then \( X^\psi \) is said to be the connected sum of \( X_1 \) and \( X_2 \) and is denoted by \( X_1 \#_p X_2 \). If \( \sigma \) and \( \tau \) belong to the same connected component of \( X \), then \( X^\psi \) is said to be obtained from \( X \) by a combinatorial handle addition. We know from \[3\] Proposition 2.10 (Datta-Murai). Let \( \Delta \) be a connected triangulated closed manifold of dimension \( d \geq 2 \). Then \( \Delta \) is stacked if and only if \( \Delta \) can be obtained from a stacked \( d \)-sphere by a sequence of successive combinatorial handle additions.

## 3 Induced surfaces in tight triangulations

Let \( X_x \) and \( X^x \) be as in Notation 2.1. If \( x \neq y \) are two vertices of a simplicial complex \( X \), then we shall also use notations such as \( X^x_y \) for \( (X^x)_y = (X_y)^x \). We say that two vertices in a simplicial complex \( X \) are adjacent (or, that they are neighbours) if they form an edge of \( X \). We now introduce:

**Notation 3.1.** If \( x \neq y \) are vertices of a simplicial complex \( X \), then \( c_X(x, y) \) will denote the number of distinct connected components \( K \) of \( X^x_y \) such that \( x \) is adjacent in \( X_y \) with some vertex in \( K \).

**Lemma 3.2.** If \( X \) is an \( \mathbb{F} \)-tight simplicial complex then for all \( x \in V(X) \), we have \( \beta_1(X; \mathbb{F}) = \beta_0(X_x; \mathbb{F}) + \beta_1(X^x; \mathbb{F}) \).

**Proof.** Clearly, \( X = X^x \cup (x \ast X_x) \) and \( X^x \cap (x \ast X_x) = X_x \). Therefore, the Mayer-Vietoris theorem yields the exact sequence (noting that the cone \( x \ast X_x \) is homologically trivial)

\[
H_1(X^x; \mathbb{F}) \to H_1(X; \mathbb{F}) \to \tilde{H}_0(X_x; \mathbb{F}) \to \tilde{H}_0(X^x; \mathbb{F}).
\]

Since \( X^x \) is an induced subcomplex of the \( \mathbb{F} \)-tight complex \( X \), the map \( H_1(X^x; \mathbb{F}) \to H_1(X; \mathbb{F}) \) is injective. Lemma 2.2 implies that \( X^x \) is connected and hence \( \tilde{H}_0(X^x; \mathbb{F}) = 0 \). So, we get the short exact sequence

\[
0 \to H_1(X^x; \mathbb{F}) \to H_1(X; \mathbb{F}) \to \tilde{H}_0(X_x; \mathbb{F}) \to 0.
\]

Hence the result. \( \square \)
Lemmas 2.2 and 2.3 certainly indicate that tightness is a severe structural constraint on a simplicial complex. So it is surprising that, beyond these two lemmas, no further structural (combinatorial) consequence of tightness seems to have been known. The following lemma establishes a strong structural restriction on the 2-skeleton of an $\mathbb{F}$-tight simplicial complex.

**Lemma 3.3.** Let $X$ be an $\mathbb{F}$-tight simplicial complex for some field $\mathbb{F}$. Then, for any two distinct vertices $x, y$ of $X$, we have $c_X(x, y) = c_X(y, x)$.

**Proof.** By Lemma 3.2, $\beta_1(X) = \tilde{\beta}_0(X_x) + \beta_1(X^x)$. Since $X^x$ is also tight by Lemma 2.3, applying Lemma 3.2 to the vertex $y$ of $X^x$, we get $\beta_1(X^y) = \tilde{\beta}_0(X^y_x) + \beta_1(X^{xy})$. Therefore, $\beta_1(X) = \tilde{\beta}_0(X_x) + \beta_0(X^y_x) + \beta_1(X^{xy})$. Interchanging the vertices $x$ and $y$ in this argument yields $\beta_1(X) = \tilde{\beta}_0(X_y) + \beta_0(X^y_y) + \beta_1(X^{yx})$. Since $X^{xy} = X^{yx}$, we get

\[
\tilde{\beta}_0(X^x_y) - \tilde{\beta}_0(X_y) = \tilde{\beta}_0(X^y_y) - \tilde{\beta}_0(X_x).
\]

But the two sides of this equation are just one less than $c_X(x, y)$ and $c_X(y, x)$. Hence the result. \hfill $\Box$

Notice that the graphs of graph theory are just the simplicial complexes of dimension $\leq 1$. In this paper, we do not use any non-trivial results from graph theory, but the language and the geometric intuition of graph theory will be useful. Recall that the degree of a vertex $v$ (denoted by $\deg(v)$) in a simplicial complex is the number of edges (1-faces) through $v$. A graph is said to be regular if all its vertices have the same degree. For $n \geq 3$, the cycle of length $n$ (in short, $n$-cycle) is the unique connected regular graph of degree two on $n$ vertices. It is the unique $n$-vertex triangulation of the circle $S^1$. An $n$-cycle with edges $a_1a_2, \ldots, a_{n-1}a_n, a_na_1$ will be denoted by $a_1a_2\cdots a_n a_1$. For $n \geq 1$, the path of length $n$ (the $n$-path) is the antistar of a vertex in the $(n+2)$-cycle. By an induced cycle (resp., path) in a simplicial complex $X$, we mean an induced subcomplex of $X$ which is a cycle (resp., path). Notice that, in particular, a 3-cycle is induced in $X$ if and only if it does not bound a triangle (2-face) in $X$. When $n \geq 4$, an $n$-cycle is induced in $X$ if and only if it is induced in the graph skel$_1(X)$. A connected acyclic graph is called a tree.

**Lemma 3.4.** Let the link of some vertex $x$ in a 2-dimensional $\mathbb{F}$-tight simplicial complex $X$ be a cycle. Then $X$ is a triangulation of a closed 2-manifold.

**Proof.** Let $C = X_y$ be a cycle. Fix a vertex $y \neq x$ of $X$. It suffices to show that the link $X_y$ is also a cycle. Note that, since $X$ is neighbourly (Lemma 2.2), $y$ is a vertex of $C$. Let $z$ and $w$ be the two neighbours of $y$ in $C$. It follows that $z$ and $w$ are the only two neighbours of the vertex $x$ in the graph $X_y$. Therefore, it suffices to show that $X^x_y$ is a path joining $z$ and $w$.

Since $X^x = C$ is a cycle and $X^y_x = C^y_x$ is a path, they are both connected. So, $c_X(x, y) = 1$. Therefore, by Lemma 3.3, $c_X(x, y) = 1$. That is, the vertices $z$ and $w$ (being the two neighbours of $x$ in $X_y$) belong to the same component of $X^y_x$. Thus, there is a path in $X^y_x$ joining $z$ to $w$. Let $P$ be a shortest path in the graph $X^y_x$ joining $z$ to $w$. Then, $P$ is an induced path in $X^y_x$. Take any vertex $v \neq x, y, z, w$ in $X$. (If there is no such vertex then $X^x_y$ is the edge $zw$, and we are done.) We will show that $v \in P$. Look at the induced subcomplex $Y = X^v$ of $X$. Then $Y_x = X^y_x = C^v_x$ is a path in which $y$ is an interior vertex. So, $Y^y_x = C^{vy}_x$ is the disjoint union of two paths. The vertex $z$ belongs to one of these two paths and $w$ belongs to the other. Therefore, $c_Y(y, x) = 2$. By Lemma 2.3, $Y$ is also $\mathbb{F}$-tight. So, by Lemma 3.3, $c_Y(x, y) = 2$. That is, the neighbours $z$ and $w$ of $x$ in $Y_y$ belong
to different components of \( Y^x_y \). Therefore, \( v \) belongs to the path \( P \) (or else \( P \) would be a path in \( Y^x_y \) joining \( z \) and \( w \)). Since \( v \neq z, w \) was an arbitrary vertex of \( Y^x_y \), this shows that the path \( P \) is a spanning path in \( Y^x_y \) (i.e., it passes through all the vertices). Since \( P \) is also an induced path in \( Y^x_y \), it follows that \( Y^x_y = P \) is a path joining \( z \) and \( w \).

Now, the following result is an easy consequence of Lemma 3.4. Notice that, in this result, there is no restriction on the dimension of \( M \), and \( M \) need not be a triangulated manifold.

**Theorem 3.5.** Let the simplicial complex \( M \) be tight with respect to some field. Let \( C \) be an induced cycle in the link of a vertex \( x \) in \( M \). Then the induced subcomplex of \( M \) on the vertex set of the cone \( x \circ C \) is a triangulated closed 2-manifold.

**Proof.** Let \( Y = M[V(C) \cup \{x\}] \) and let \( X = \text{ske}l_2(Y) \). Then, by Lemmas 2.3 and 2.4, \( X \) is \( \mathbb{F} \)-tight. Clearly, \( X \) is two dimensional and \( X_x = C \) is a cycle. Thus, by Lemma 3.4, \( X \) is a triangulated closed 2-manifold. To complete the proof, it suffices to show that \( \chi(Y) = 2 \), i.e., that \( \dim(Y) = 2 \). Suppose, if possible, \( \dim(Y) > 2 \). Take a 3-face \( \alpha \in Y \). Then the induced subcomplex of \( X = \text{ske}l_2(Y) \) on the vertex set \( \alpha \) is a 4-vertex triangulated 2-sphere. Since \( X \) is a connected and triangulated closed 2-manifold, it follows that \( X = S^2_4 \), and \( V(X) = \alpha \). Thus \( C \) is the 3-cycle on the vertex set \( \alpha \setminus \{x\} \). But, the 2-face \( \alpha \setminus \{x\} \) is in \( Y_x \subseteq M_x \). This contradicts the assumption that \( C \) is an induced cycle in the link \( M_x \).

**Corollary 3.6.** Let \( S \) be the link of some vertex in an \( \mathbb{F} \)-tight simplicial complex \( M \).

(a) If \( \text{char}(\mathbb{F}) = 2 \), then \( S \) has no induced cycle of length \( \equiv 1 \mod 3 \).

(b) If \( \text{char}(\mathbb{F}) \neq 2 \), then \( S \) has no induced cycle of length \( \equiv 0, 1, 4, 5, 7, 8, 9 \) or \( 10 \mod 12 \).

**Proof.** Let \( x \in V(M) \) and let \( C \) be an induced \( n \)-cycle in \( S = M_x \). By Theorem 3.5, the induced subcomplex \( X = M[V(C) \cup \{x\}] \) is an \((n+1)\)-vertex triangulated closed 2-manifold. By Lemma 2.2, \( X \) is neighbourly. So, it has \( n + 1 \) vertices, \( n(n + 1)/2 \) edges and hence \( n(n + 1)/3 \) triangles. So, 3 divides \( n(n + 1) \), i.e., \( n \neq 1 \mod 3 \). This proves part (a).

If \( \text{char}(\mathbb{F}) \neq 2 \) then, by Lemmas 2.3 and 2.5, \( X \) is an orientable triangulated 2-manifold. So, its Euler characteristic \( \chi(X) = (n + 1)(6 - n)/6 \) is an even number. Thus, \( n \neq 0, 1, 4, 5, 7, 8, 9 \) or \( 10 \mod 12 \). This proves part (b).

## 4 Odd characteristic

In this section, we prove that any triangulated 3-manifold is tight with respect to a field of odd characteristic if and only if it is neighbourly, orientable and stacked. For this, we first need some additional preliminary results on 2-sphere triangulations.

**Lemma 4.1.** Let \( S \) be a triangulation of \( S^2 \). If \( S \) has no induced cycle of length \( \leq 5 \) then \( S = S^2_4 \).

**Proof.** Let \( x \) be a vertex of minimum degree in \( S \). It is well known (and easy to prove) that the minimum degree of any triangulation of \( S^2 \) is at most five. So, \( \deg(x) = 3, 4 \) or \( 5 \).

If \( \deg(x) = 4 \) or \( 5 \), then \( S_x \) is a 4-cycle or a 5-cycle in \( S \), and hence it is not induced. So there are vertices \( y, z \) in \( S_x \) such that \( yz \) is an edge in \( S \) but not in \( S_y \). So \( x-y-z-x \) is an induced 3-cycle in \( S \), a contradiction. Thus, \( \deg(x) = 3 \). Say \( S_x \) is the 3-cycle \( x_1-x_2-x_3-x_1 \). Since this 3-cycle is not an induced cycle in \( S \), it follows that \( x_1x_2x_3 \in S \).
Then \((x \ast S_x) \cup \{x_1 x_2 x_3\} = S^2_4\). Since a triangulation of \(S^2\) cannot be a proper subcomplex of another triangulation of \(S^2\), it follows that \(S = S^2_4\). 

We now introduce:

**Definition 4.2.** A triangulated \(d\)-sphere \(S\) is said to be **primitive** if it can not be written as a connected sum of two triangulated \(d\)-spheres.

Clearly, every triangulated \(d\)-sphere is a connected sum of finitely many primitive \(d\)-spheres. The following lemma is more or less obvious.

**Lemma 4.3.** Let \(S\) be a triangulated \(d\)-sphere. Then \(S\) is primitive if and only if \(S\) has no induced subcomplex isomorphic to \(S^{d-1}_{d+1}\).

**Proof.** If \(S\) is not primitive, then \(S = S_1 \# S_2\), where \(S_1\) and \(S_2\) are triangulated \(d\)-spheres. Let \(\alpha\) be the unique common facet of \(S_1\) and \(S_2\). Then the boundary of \(\alpha\) is an induced \(S^{d-1}_{d+1}\) in \(S\).

Conversely, suppose \(S\) has an induced \(S^{d-1}_{d+1}\), say with vertex set \(\alpha\). This \(S^{d-1}_{d+1}\) divides \(S\) into two triangulated \(d\)-balls \(B_1\) and \(B_2\) such that \(S = B_1 \cup B_2\), \(\partial B_1 = \partial B_2 = B_1 \cap B_2 = S^{d-1}_{d+1}\). Put \(S_i = B_i \cup \{\alpha\}\), \(i = 1, 2\). Then \(S_1, S_2\) are triangulated \(d\)-spheres and \(S = S_1 \# S_2\). So, \(S\) is not primitive.

The following lemma and definition clarify the meaning of connected sums of several primitive triangulated spheres.

**Notation 4.4.** Let \(S\) be a triangulation of \(S^d\). Put \(A(S) := \{\alpha \subseteq V(S) : S'[\alpha] \cong S^{d-1}_{d+1}\}\), \(\overline{S} := S \cup A(S)\), and let \(B(S)\) be the collection of all the induced subcomplexes of the simplicial complex \(\overline{S}\) which are primitive triangulations of \(S^d\). Let \(T(S)\) be the graph with vertex set \(B(S)\) such that \(S_1, S_2 \in B(S)\) are adjacent in \(T(S)\) if \(S_1 \cap S_2 = \overline{\alpha}\) for some \(\alpha \in A(S)\).

**Lemma 4.5.** For any triangulated \(d\)-sphere \(S\) we have the following.

(a) Any two members of \(B(S)\) have at most one common facet; if they have a common facet \(\alpha\) then \(\alpha \in A(S)\).

(b) Each member of \(A(S)\) belongs to exactly two members of \(B(S)\).

(c) There is a natural bijection from \(A(S)\) onto the set of edges of \(T(S)\). It is given by \(\alpha \mapsto \{S_1, S_2\}\), where, for \(\alpha \in A(S)\), \(S_1\) and \(S_2\) are the two members of \(B(S)\) containing \(\alpha\).

(d) The graph \(T(S)\) is a tree. In consequence, \(#B(S) = 1 + #A(S)\).

**Proof.** Induction on \(m := 1 + #A(S)\). If \(m = 1\), \(A(S)\) is empty, so that \(S\) is primitive by Lemma 4.3. So, let \(m > 1\), and suppose that the result holds for all smaller values of \(m\). In this case, \(A(S)\) is non-empty. Take \(\alpha \in A(S)\). By the proof of Lemma 4.3, there are triangulated \(d\)-spheres \(S', S''\) such that \(S' \cap S'' = \overline{\alpha}, S = (S' \cup S'') \setminus \{\alpha\}\). Thus, \(S', S'' \subseteq S \cup \{\alpha\} \subseteq \overline{S}\). Hence it is easy to see that \(A(S) = A(S') \cup A(S'') \cup \{\alpha\}\), \(B(S) = B(S') \cup B(S'')\) and the graph \(T(S)\) is obtained from the disjoint union of the graphs \(T(S')\) and \(T(S'')\) by adjoining a single edge from a vertex of \(T(S')\) to a vertex of \(T(S'')\). Since, by induction hypothesis, the result is valid for \(S'\) and \(S''\), its validity for \(S\) follows. This completes the induction.
Definition 4.6. Let $S$ be a triangulated $d$-sphere. Any tree has a leaf (a vertex of degree one) and the deletion of a leaf from a non-trivial tree leaves a subtree. Therefore, the members of $\mathcal{B}(S)$ may be ordered as $S_1, \ldots, S_m$ in such a way that, for each $i$, $1 \leq i \leq m$, $S_i$ is a leaf of the induced subtree of $\mathcal{T}(S)$ on $\{S_1, \ldots, S_i\}$ (when $m \geq 2$, this ordering is not unique). For any such ordering, we write $S = S_1 \# \cdots \# S_m$, and say that $S$ is the connected sum of the $S_i$’s. Clearly, we have $S_1 \# \cdots \# S_m = (S_1 \# \cdots \# S_{m-1}) \# S_m$.

As a special case of [11, Theorem 8.5], Kalai proved a nice characterization of stacked 2-spheres. A triangulated 2-sphere $S$ is stacked if and only if $S$ has no induced cycle of length $\geq 4$. The following result is a dramatic improvement on this characterization.

Theorem 4.7. Let $S$ be a triangulated 2-sphere. Then $S$ is stacked if and only if it has no induced cycle of length 4 or 5.

Proof. Write $S = S_1 \# S_2 \# \cdots \# S_k$, where the $S_i$’s are primitive 2-spheres. If $S$ is stacked then, by Lemma 2.9, each $S_i$ is a copy of $S^2_4$. Let $C$ be an induced cycle of length $\geq 4$ in $S$. Since an induced cycle of length $\geq 4$ in a connected sum of two triangulated manifolds must be an induced cycle in one of the summands, it follows inductively that $C$ is an induced cycle in one of the $S_i$’s. Since $S^2_4$ has no induced cycle at all, it follows that $S$ has no induced cycle of length $\geq 4$. This proves the “only if” part.

For the converse, assume that $S$ has no induced cycle of length 4 or 5. It follows that no $S_i$ has any induced cycle of length 4 or 5. Being primitive, $S_i$ has no induced cycle of length 3 either (Lemma 4.3). Thus, by Lemma 4.1, each $S_i$ is a copy of $S^2_4$. Therefore, by Lemma 2.9, $S$ is stacked. This proves the “if” part.

Now, we are ready to prove one of the main results of this paper.

Theorem 4.8. A triangulated closed 3-manifold $M$ is tight with respect to some field $F$ with $\text{char}(F) \neq 2$ if and only if $M$ is orientable, neighbourly and stacked.

Proof. The “if” part follows from Proposition 1.6. To prove the “only if” part, let $M$ be an $F$-tight triangulated closed 3-manifold, $\text{char}(F) \neq 2$. By Lemmas 2.2 and 2.5, $M$ is neighbourly and orientable. It remains to show that $M$ must be stacked.

Corollary 3.6 (b) shows that, for each vertex $x$ of $M$, the vertex link $M_x$ is a triangulated 2-sphere with no induced 4-cycle or 5-cycle. So, Theorem 4.7 implies that each $M_x$ is a stacked 2-sphere. Thus, $M$ is locally stacked. The result now follows by Proposition 1.6.

Corollary 4.9. Let $M$ be a triangulated closed 3-manifold. If $M$ is tight w.r.t. a field $F$, where $\text{char}(F) \neq 2$, then $M$ is strongly minimal.

Proof. This result follows from Theorem 4.8 and Proposition 1.7.

5 Characteristic two

By Lemma 2.8, a simplicial complex is tight with respect to a field of characteristic two if and only if it is $\mathbb{Z}_2$-tight. So, without loss of generality, we take $F = \mathbb{Z}_2$ in this section. However, all the results apply equally well to arbitrary fields of characteristic two. Here, we characterize the links of $\mathbb{Z}_2$-tight triangulated 3-manifolds. This characterization is important since it leads to (a) severe restrictions on the size and topology of such a triangulation (see Theorem 5.1 and the tables at the end of this section) and (b) a polynomial time algorithm to decide tightness of 3-manifolds which is described in detail in [2].
By Proposition 1.6 all stacked and neighbourly triangulated closed 3-manifolds are $\mathbb{Z}_2$-tight. Here is a partial converse. (In case $M$ is orientable, this result follows from Corollary 2.7)

**Theorem 5.1.** Let $M$ be a $\mathbb{Z}_2$-tight triangulated closed 3-manifold. If the torsion subgroup of $H_1(M;\mathbb{Z})$ is of odd order (possibly trivial), then $M$ is stacked (and neighbourly).

**Proof.** If possible, assume that $M$ is not stacked. Then, by Proposition 1.6 $M$ is not locally stacked. So, there exists a vertex $v$ whose link $M_v$ is not a stacked 2-sphere. By Theorem 4.7 and Corollary 3.6(a), $M_v$ has an induced cycle $C$ of length 5. Then, by Theorem 3.5 the induced subcomplex $N := M[\{v\} \cup V(C)]$ of $M$ is a triangulated closed 2-manifold. Since $N$ is an induced subcomplex of $M$, by Lemmas 2.2 and 2.3 $N$ is neighbourly. Thus, $N$ is a 6-vertex neighbourly triangulated 2-manifold and hence is the 6-vertex triangulation $\mathbb{R}P^2$ of $\mathbb{R}P^2$. Take a non-triangle $abc$ of $N = \mathbb{R}P^2$ (there are 10 of them). Then $\alpha = ab + bc + ca$ can be viewed as an 1-cycle of $N$ with $\mathbb{Z}_2$-coefficient as well as with $\mathbb{Z}$-coefficient. In both view, it is not a boundary. However $2\alpha$ is a boundary with $\mathbb{Z}$-coefficient. Since the map $H_1(N;\mathbb{Z}_2) \to H_1(M;\mathbb{Z}_2)$, induced by the inclusion map $N \hookrightarrow M$, is injective, it follows that $[\alpha] \neq 0$ as an element of $H_1(M;\mathbb{Z}_2)$ and hence also as an element of $H_1(M;\mathbb{Z})$. But, $2[\alpha] = 0$ in $H_1(M;\mathbb{Z})$. So, $[\alpha]$ is an element of order 2 in $H_1(M;\mathbb{Z})$. This is a contradiction to the assumption on $H_1(M;\mathbb{Z})$.

**Lemma 5.2.** Let $S$ be a primitive triangulation of $S^2$. Suppose $S \neq S_1^2$ and $S$ has no induced 4-cycle. Then,

(a) All vertex links of $S$ are induced cycles in $S$.

(b) Any two adjacent vertices of $S$ have exactly two common neighbours in $S$.

**Proof.** Let $x \in V(S)$. Since $S \neq S_1^2$ is primitive, Lemma 4.3 implies that $S_x$ can not be a 3-cycle. Let the vertices $y, z$ of the cycle $S_x$ be neighbours in $S$. Since $S$ is primitive, the 3-cycle $x-y-z-x$ can not be induced in $S$. So, the triangle $xyz$ is in $S$. Therefore, $y$ and $z$ are neighbours in $S_x$. This implies that $S_x$ is induced in $S$. This proves part (a).

Let $x, y$ be adjacent vertices of $S$, and let $z$ be a common neighbour of $x$ and $y$. Then, as the 3-cycle $x-y-z-x$ is not induced in $S$, the triangle $xyz$ is in $S$. So, $z$ is the third vertex of one of the two triangles of $S$ through $xy$. Thus, $x$ and $y$ have exactly two common neighbours. This proves part (b).

**Lemma 5.3.** Let $S$ be a primitive triangulation of $S^2$ such that $S$ has no induced cycle of length $\equiv 1 \pmod{3}$. Then, either $S = S_1^2$ or all the vertices of $S$ have degree $2 \pmod{3}$.

**Proof.** By assumption and Lemma 5.2(a), $V(S) = V_0 \cup V_2$, where $V_i$ consists of the vertices of degree $i \pmod{3}$, $i = 0, 2$. If possible, let $x \in V_0$ and $y \in V_2$ be such that $xy$ is an edge of $S$. By Lemma 5.2, the cycles $S_x$ and $S_y$ are induced in $S$ (of length 0 (mod 3) and 2 (mod 3), respectively) with exactly two common vertices, say $u$ and $v$. Then $u$ and $v$ are the two neighbours of $x$ in $S_y$ and of $y$ in $S_x$.

Claim. No vertex in $V(S_x) \setminus V(S_y)$ is adjacent to any vertex in $V(S_y) \setminus V(S_x)$.

Indeed, if $a \in S_x$ is a neighbour of $b \in S_y$, then $S$ has the 4-cycle $x-a-b-y-x$. Since $S$ has no induced 3-cycle or 4-cycle, it follows that one of the triangles $xya$ and $xab$ is in $S$. Hence either $a$ or $b$ is in $S_x \cap S_y$. This proves the claim.

Therefore, if $C$ is the cycle obtained from $S_x \cup S_y$ by deleting the two vertices $x, y$ and the four edges $xu, xv, yu, yv$, then $C$ is an induced cycle in $S$ of length $\equiv 0 + 2 - 4 \equiv 1$.
(mod 3). This is a contradiction. Therefore, no vertex in $V_0$ is adjacent to any vertex in $V_2$. Since $S$ is connected and $V(S) = V_0 \cup V_2$, it follows that $V_0 = \emptyset$ or $V_2 = \emptyset$. If $V_2 = \emptyset$ then the degree of each vertex is 0 (mod 3) and hence $S$ has a vertex $z$ of degree 3. Since $S$ is primitive, $S_z$ bounds a triangle. Then, $S$ contains an $S^2_2$ and hence $S = S^2_4$. Otherwise, $V(S) = V_2$.

**Theorem 5.4.** Up to isomorphism, $S^2_4$ and $I^2_{12}$ are the only two primitive triangulations of $S^2$ with no induced cycle of length $\equiv 1 \pmod{3}$.

**Proof.** Clearly, $S^2_4$ has no induced cycle whatsoever. It is easy to see that all the induced cycles of $I^2_{12}$ are 5-cycles. (Indeed, these are precisely the twelve vertex links.) Thus, these two triangulations of $S^2$ are primitive with no induced cycle of length 1 (mod 3).

Conversely, let $S$ be a primitive triangulation of $S^2$ with no induced cycle of length 1 (mod 3). Assume $S \neq S^2_4$. By Lemma 5.3, the minimum degree of the vertices of $S$ is five. If all its vertices have degree 5, then $S = I^2_{12}$. So, suppose there is a vertex $u$ with deg$(u) > 5$. By Lemma 5.3, deg$(u) \geq 8$. So, $S_u$ is a cycle of length $\geq 8$. Therefore, we may choose vertices $v_1, v_2, w_1, w_2$ in $S_u$ such that $v_1, v_2$ are at distance 2, $w_1, w_2$ are at a distance 2, and dist$(v_i, w_j) \geq 2$, $i, j = 1, 2$, where all the distances are graphical distances measured along the cycle $S_u$. Let $D := (u * S_u) \cup (v_1 * S_{v_1}) \cup (v_2 * S_{v_2}) \cup (w_1 * S_{w_1}) \cup (w_2 * S_{w_2})$.

Figure 1 is a “picture” of $D$. The boundary of $D$ is the union of six paths in $S$ (drawn as circular arcs; edges of $S$ are drawn as straight line segments). Since $S$ has no induced cycle of length $\leq 4$, it is easy to see that these six paths intersect pairwise at most at common end points, and the six end points marked in Fig. 1 are distinct vertices of $S$. Thus, the boundary of $D$ is a cycle and hence $D$ is a disc.

![Figure 1: Disc $D$ in the proof of Theorem 5.3](image)

Fix an index $i \in \{1, 2\}$. Consider the boundary $C_i$ of the disc $(u * S_u)\cup(v_i * S_{v_i})\cup(w_i * S_{w_i})$. Then the vertex set of $C_i$ is the union of the vertex sets of $S_u$, $S_{v_i}$, $S_{w_i}$ minus the three vertices $u, v_i, w_i$. By Lemma 5.2 (b), $S_u$ and $S_{v_i}$ (as also $S_u$ and $S_{w_i}$) have exactly two common vertices. Since $D$ is a disc, it follows that $S_{v_i}$ and $S_{w_i}$ have a unique vertex (namely, $u$) in common. Also, by Lemma 5.3, each of $S_u$, $S_{v_i}$, $S_{w_i}$ have length 2 (mod 3). Therefore, the inclusion exclusion principle shows that the length of $C_i$ is $\equiv 2 + 2 + 2 - (2 + 2 + 1) - 3 \equiv 1 \pmod{3}$. Therefore, $C_i$ is not an induced cycle of $S$. Thus, there is an edge $a_i b_i$ in $S$ such that $a_i$ and $b_i$ are non-consecutive vertices in the cycle $C_i$. By the proof of Lemma 5.3, no vertex of $S_{v_i} \setminus S_u$ is adjacent in $S$ with any vertex of $S_u \setminus S_{v_i}$; also no vertex of $S_{w_i} \setminus S_u$ is adjacent in $S$ with any vertex of $S_u \setminus S_{w_i}$. Therefore, $a_i \in A_i^c$, $b_i \in B_i^c$ (see Fig. 1). That is, $a_i$ is an interior vertex of the path $A_i$ and $b_i$ is an interior vertex of the path $B_i$.
Now, consider the disc \( D' \) in \( S \) complementary to the disc \( D \) (i.e., \( D' := S[V(S) \setminus \{ u, v_1, v_2, w_1, w_2 \}] \)). Then, \( a_1b_1 \) and \( a_2b_2 \) are in \( D' \). Clearly, \( a_1b_1 \) separates \( A_2^0 \) from \( B_2^0 \) in \( D' \) (i.e., \( A_2^0 \) and \( B_2^0 \) are in different connected components of \( |D'| \setminus |a_1b_1| \)). Therefore, the geometric edges \( |a_1b_1| \) and \( |a_2b_2| \) intersect at an interior point. This contradicts the very definition of the geometric realization of a simplicial complex. Thus, there is no vertex \( u \) of \( S \) with \( \deg(u) > 5 \). This completes the proof. \( \square \)

**Corollary 5.5.** For triangulations \( S \) of \( S^2 \), the following conditions are equivalent:

(a) \( S \) has no induced cycle of length 1 (mod 3),

(b) all the induced cycles of \( S \) have length 3 and 5,

(c) \( S \) is a connected sum of \( S^2_4 \)'s and \( I^2_{12} \)'s.

(The statement in (c) includes the possibility that \( S \) is either icosian or stacked.)

**Proof.** Let \( S \) be as in (c). It follows that the only possible induced cycles in the summands are 5-cycles. Therefore, the only induced cycles in \( S \) are 3-cycles and 5-cycles. Thus, (c) \( \Rightarrow \) (b). Trivially, (b) \( \Rightarrow \) (a).

Now suppose (a) holds. Write \( S = S_1 \# S_2 \# \cdots \# S_k \), where each \( S_i \) is primitive. Since \( S \) has no induced cycle of length 1 (mod 3), it follows that no \( S_i \) has an induced cycle of length 1 (mod 3). Therefore, by Theorem 5.4, each \( S_i \) is \( S^2_4 \) or \( I^2_{12} \). Hence \( S \) is as in (c). Thus, (a) \( \Rightarrow \) (c). \( \square \)

Now, we can prove the second main result of this paper.

**Theorem 5.6.** Let \( M \) be a \( \mathbb{Z}_2 \)-tight triangulation of a closed 3-manifold. Then each vertex link of \( M \) is a connected sum of \( S^2_4 \)'s and \( I^2_{12} \)'s.

**Proof.** This is now immediate from Corollary 3.6 (a) and Corollary 5.5. \( \square \)

The following result provides a recursive procedure for the computation of the sigma-star vector.

**Theorem 5.7.** Let \( X_1 \) and \( X_2 \) be induced subcomplexes of a simplicial complex \( X \) and \( \mathbb{F} \) be a field. Suppose \( X = X_1 \cup X_2 \) and \( Y = X_1 \cap X_2 \). If \( Y \) is \( k \)-neighbourly, \( k \geq 2 \), then

\[
\sigma_i^*(X; \mathbb{F}) = \sigma_i^*(X_1; \mathbb{F}) + \sigma_i^*(X_2; \mathbb{F}) - \sigma_i^*(Y; \mathbb{F}) \quad \text{for} \quad 0 \leq i \leq k - 2.
\]

**Proof.** Let us write \( m_1 = f_0(X_1), \ m_2 = f_0(X_2), \ m = f_0(Y) \). Thus, \( f_0(X) = m_1 + m_2 - m \). For notational convenience, we write \( \tilde{\beta}_i(A) \) for \( \tilde{\beta}_i(X[A]; \mathbb{F}) \), \( A \subseteq V(X) \). Note that any subset \( A \) of \( V(X) \) can be uniquely written as \( A = A_1 \cup B \cup A_2 \), where \( A_1 \subseteq V(X_1) \setminus V(Y) \), \( B \subseteq V(Y) \) and \( A_2 \subseteq V(X_2) \setminus V(Y) \). Since \( Y \) is \( k \)-neighbourly, it follows from the exactness of the Mayer-Vietoris sequence that

\[
\tilde{\beta}_i(A_1 \cup B \cup A_2) = \tilde{\beta}_i(A_1 \cup B) + \tilde{\beta}_i(A_2 \cup B) - \tilde{\beta}_i(B) \quad \text{for} \quad 0 \leq i \leq k - 2.
\]

Therefore, we can compute
\[ \sigma^*_i(X; \mathbb{F}) = \frac{1}{m_1 + m_2 - m + 1} \sum_{A_1, A_2, B} \frac{\tilde{\beta}_i(A_1 \sqcup B) + \tilde{\beta}_i(A_2 \sqcup B) - \tilde{\beta}_i(B)}{\left( \frac{m_1 + m_2 - m}{\#(A_1 \sqcup A_2 \sqcup B)} \right)} \]

\[ = \frac{1}{m_1 + m_2 - m + 1} \left[ \sum_{A_1, B} \frac{\tilde{\beta}_i(A_1 \sqcup B)}{\#(A_1 \sqcup B) + \#(A_1)} \sum_{A_2} \frac{1}{\left( \frac{m_1 + m_2 - m}{\#(A_1) + \#(A_2)} \right)} \right. \]

\[ + \sum_{A_2, B} \tilde{\beta}_i(A_2 \sqcup B) \sum_{A_1} \frac{1}{\left( \frac{m_1 + m_2 - m}{\#(A_2) + \#(A_1)} \right)} - \sum_{B} \frac{\tilde{\beta}_i(B)}{\#(B) + \#(A_1 \sqcup A_2)} \left. \sum_{A_1, A_2} \frac{1}{\left( \frac{m_1 + m_2 - m}{\#(A_1 \sqcup A_2)} \right)} \right] \]

\[ = \frac{1}{m_1 + m_2 - m + 1} \left[ \sum_{k=0}^{m_1} \frac{\tilde{\beta}_i(A_1 \sqcup B)}{\#(A_1 \sqcup B) = k} \sum_{\ell=0}^{m_2-2m} \left( \frac{m_2 - m}{\ell + k} \right) \right. \]

\[ + \sum_{k=0}^{m_2} \frac{\tilde{\beta}_i(A_2 \sqcup B)}{\#(A_2 \sqcup B) = k} \sum_{\ell=0}^{m_1-m} \left( \frac{m_1 - m}{\ell + k} \right) \]

\[ - \sum_{k=0}^{m} \frac{\tilde{\beta}_i(B)}{\#(B) = k} \sum_{\ell=0}^{m_2-2m} \left( \frac{m_1 + m_2 - 2m}{\ell + k} \right) \]

\[ = \frac{1}{m_1 + 1} \sum_{k=0}^{m_1} \frac{\tilde{\beta}_i(A_1 \sqcup B)}{\#(A_1 \sqcup B) = k} + \frac{1}{m_2 + 1} \sum_{k=0}^{m_2} \frac{\tilde{\beta}_i(A_2 \sqcup B)}{\#(A_2 \sqcup B) = k} \]

\[ - \frac{1}{m + 1} \sum_{k=0}^{m} \frac{\tilde{\beta}_i(B)}{\#(B) = k} \]

\[ = \sigma^*_i(X_1; \mathbb{F}) + \sigma^*_i(X_2; \mathbb{F}) - \sigma^*_i(Y; \mathbb{F}). \]

\[ \square \]

In the penultimate step of the above computation, we have used the following well known identity to compute the inner sums. For any three non-negative integers \( p, q, r \), we have

\[ \sum_{i=0}^{p} \frac{\binom{p}{i}}{\binom{p+q+r}{i+r}} = \frac{p + q + r + 1}{q + r + 1} \times \frac{1}{\binom{q+r}{r}}. \]

As a particular case of Theorem 5.7 we have a formula for the sigma-star vector of a connected sum. (Since \( \sigma^*_0(X; \mathbb{F}) \) is independent of the field \( \mathbb{F} \), we denote it by \( \sigma^*_0(X) \).

**Corollary 5.8.** For any two triangulated d-spheres \( S_1, S_2 \) and any field \( \mathbb{F} \), we have

(a) \( \sigma^*_0(S_1 \# S_2) = \sigma^*_0(S_1) + \sigma^*_0(S_2) + \frac{1}{d+2} \), and

(b) \( \sigma^*_i(S_1 \# S_2; \mathbb{F}) = \sigma^*_i(S_1; \mathbb{F}) + \sigma^*_i(S_2; \mathbb{F}) \) for \( 1 \leq i \leq d-2 \).

**Proof.** Let \( S_1 \cap S_2 = \alpha \), where \( \alpha \) is a d-face. Then \( X := S_1 \# S_2, X_1 := S_1 \setminus \{\alpha\}, X_2 := S_2 \setminus \{\alpha\}, Y = \partial \alpha = S_{d-1} \) satisfy the hypothesis of Theorem 5.7 (with \( k = d \)), and trivially \( \sigma^*_0(S_{d+1}^{d-1}) = -1/(d+2), \sigma^*_i(S_{d+1}^{d-1}) = 0 \) for \( 1 \leq i \leq d-2 \). \( \square \)
**Notation 5.9.** For \( k, \ell \geq 0; (k, \ell) \neq (0, 0) \), by \( kI^2_1 \# \ell S^2_4 \) we denote a triangulated 2-sphere which can be written as a connected sum of \( k + \ell \) triangulated 2-spheres, of which \( k \) are copies of \( I^2_1 \) and the remaining \( \ell \) are copies of \( S^2_4 \) (in some order).

**Corollary 5.10.** Let \( k, \ell \geq 0 \) and \( (k, \ell) \neq (0, 0) \). Then \( \sigma^*_0(kI^2_1 \# \ell S^2_4) = \frac{617}{1716} k + \frac{1}{20} \ell - \frac{1}{4} \).

Proof. Trivially, \( \sigma^*_0(S^2_4) = -1/5 \). A computation shows that \( \sigma^*_0(I^2_1) = 47/429 \). Thus the result holds when \( k + \ell = 1 \). The general result follows by an induction on \( k + \ell \), where the induction leap uses Corollary \( 5.8 \) (with \( d = 2 \)). \( \square \)

Our next result lists a set of necessary conditions that a \( \mathbb{Z}_2 \)-tight triangulated 3-manifold must satisfy. Note that the statement of the main result of [17] implies that the inequality in part (a) of Theorem 5.11 holds, more generally, for all triangulations of closed 3-manifolds. Equality holds in this more general setting if and only if the triangulation is neighbourly and locally stacked. Therefore, Proposition 1.6 implies that equality holds if and only if the triangulation is \( \mathbb{Z}_2 \)-tight and stacked. Still we have included this inequality for the sake of completeness, and because its proof arises naturally in the course of proving the rest of the theorem.

**Theorem 5.11.** Let \( M \) be a \( \mathbb{Z}_2 \)-tight triangulated closed 3-manifold. Then the parameters \( n := f_0(M) \) and \( \beta_1 := \beta_1(M; \mathbb{Z}_2) \) must satisfy the following.

(a) \((n - 4)(n - 5) \equiv 20\beta_1 \pmod{776} \). Also, \((n - 4)(n - 5) \geq 20\beta_1 \), with equality if and only if \( M \) is stacked.

(b) \( 429(n - 4)(n - 5) - 776n\lfloor \frac{n - 4}{9} \rfloor \leq 8580\beta_1 \). Equality holds here if and only if each vertex link of \( M \) is a triangulated 2-sphere of the form \( \lfloor \frac{n - 4}{9} \rfloor I^2_1 \# (n - 4 \lfloor \frac{n - 4}{9} \rfloor) S^2_4 \).

Proof. By Theorem 5.6 for each \( x \in V(M) \) there are numbers \( k(x) \) and \( \ell(x) \) such that

\[
M_x = k(x)I^2_1 \# \ell(x)S^2_4.
\]

Since \( M \) is neighbourly by Lemma 2.2 equating the number of vertices in the two sides, we get \( n - 1 = 9k(x) + 3 + \ell(x) \). Hence \( \ell(x) = n - 4 - 9k(x) \), and we have \( 0 \leq k(x) \leq \lfloor \frac{n - 4}{9} \rfloor \). Equality holds in the lower bound if and only if \( M_x \) is stacked, and equality holds in the upper bound if and only if \( M_x \) is the connected sum of \( \lfloor \frac{n - 4}{9} \rfloor \) copies of \( I^2_1 \) and \( n - 4 - \lfloor \frac{n - 4}{9} \rfloor \) copies of \( S^2_4 \). Let us put \( k := \sum_{x \in V(M)} k(x) \). Then \( 0 \leq k \leq n \lfloor \frac{n - 4}{9} \rfloor \). The lower bound holds with equality if and only if \( M \) is locally stacked (hence, by Proposition 1.6 if and only if \( M \) is stacked) and the upper bound holds with equality if and only if each vertex link of \( M \) is the connected sum of \( \lfloor \frac{n - 4}{9} \rfloor \) copies of \( I^2_1 \) and \( n - 4 - \lfloor \frac{n - 4}{9} \rfloor \) copies of \( S^2_4 \). Then, by Corollary 5.10

\[
\sigma^*_0(M_x) = \frac{617}{1716} k(x) + \frac{1}{20} (\ell(x) - \frac{1}{4}) = \frac{617}{1716} k(x) + \frac{1}{20} (n - 4 - 9k(x)) - \frac{1}{4} = \frac{1}{20} n - \frac{194}{2145} k(x) - \frac{9}{20}. \tag{1}
\]

Now, Lemma 2.6 implies \( \beta_1 = \mu_1(M) := 1 + \sum_x \sigma^*_0(M_x) \). Therefore, adding (1) over all \( x \in V(M) \), we get

\[
429((n - 4)(n - 5) - 20\beta_1) = 776k. \tag{2}
\]

Since 776 is relatively prime to 429, the result follows from (2) and the above discussion on the bounds on \( k \). \( \square \)
Corollary 5.12 (An upper bound theorem for $\mathbb{Z}_2$-tight 3-manifolds). Let $M$ be a $\mathbb{Z}_2$-tight closed 3-manifold with $n := f_0(M)$, $\beta_1 := \beta_1(M; \mathbb{Z}_2)$. Then $(n - 4)(617n - 3861) \leq 15444\beta_1$. Equality holds here if and only if $M$ is locally icosian.

Proof. Since $\left\lfloor \frac{n - 4}{9} \right\rfloor \leq \frac{n - 4}{9}$, Theorem 5.11 (b) implies that $429(n - 4)(n - 5) - 776n(n - 4)/9 \leq 8580\beta_1$. This simplifies to the given inequality. Clearly, equality holds here if and only if $n \equiv 4 \pmod{9}$ and equality holds in Theorem 5.11 (b).

Corollary 5.13. Let $M$ be a $\mathbb{Z}_2$-tight triangulated closed 3-manifold. If $f_0(M) \leq 71$ or $\beta_1(M; \mathbb{Z}_2) \leq 188$, then $M$ is stacked.

Proof. Suppose, if possible, $M$ is not stacked. Then, by Theorem 5.11 (a), there is an integer $\ell \geq 1$ such that $(n - 4)(n - 5) = 776\ell + 20\beta_1$. Hence Corollary 5.12 implies

$$15444(n - 4)(n - 5) \geq 15444 \times 776\ell + 20(617n - 3861)(n - 4).$$

This inequality simplifies to

$$(n - 2)^2 \geq 3861\ell + 4. \quad (3)$$

Case 1. $n \leq 71$. Hence, by (3), $3861\ell + 4 \leq (71 - 2)^2 = 4761$. Thus, $\ell = 1$. Therefore, $(n - 4)(n - 5) = 776 + 20\beta_1$ and $(n - 2)^2 \geq 3861 + 4 > 62^2$. Thus, $n \geq 65$. But, $(n - 4)(n - 5) \equiv 776 \equiv -4 \pmod{20}$. Hence $n \equiv 12$ or $17 \pmod{20}$. But, $n \geq 65$. So, $n \geq 72$, a contradiction. So, the result is true in this case.

Case 2. $\beta_1(M; \mathbb{Z}_2) \leq 188$. Then, by Corollary 5.12, $n \leq 73$. Hence (3) yields $\ell = 1$. Thus $(n - 4)(n - 5) = 776 + 20\beta_1$, and therefore $n$ is congruent to 12 or 17 (mod 20). Hence $n \leq 72$, and if $n = 72$ then $\beta_1 = 189$. Therefore, $n \leq 71$. The result now follows by Case 1. □

Corollary 5.14. Let $M$ be a closed topological 3-manifold admitting a tight triangulation. If $\beta_1(M; \mathbb{Z}_2) < 189$, then $M$ is homeomorphic to one of the following manifolds

$$S^3, (S^2 \times S^1)^\#^k, (S^2 \times S^1)^\#^k,$$

where $k = 1, 12, 19, 21, 30, 63, 78, 82, 99, 154, 177$ or 183.

Proof. Let $X$ be an $n$-vertex tight triangulation of $M$. Then, by Corollary 5.13, Theorem 5.11 (a) and Proposition 2.10, $M$ is homeomorphic to $S^3$, $(S^2 \times S^1)^\#^k$ or $(S^2 \times S^1)^\#^k$, where $(n - 4)(n - 5) = 20k$. The result follows from this. □

The following table gives a list of small values for the parameters $(n, \beta_1)$ of a locally icosian $\mathbb{Z}_2$-tight triangulation of a closed 3-manifold. Indeed, the number $n$ of vertices in any such triangulation must be congruent modulo 15444 to one of the eight values of $n$ listed in this table.

| $n$   | 1408 | 3865 | 5269 | 8320 | 9724 | 12181 | 13585 | 15448 |
|-------|------|------|------|------|------|-------|-------|------|
| $\beta_1$ | 78625 | 595186 | 1106970 | 2762081 | 3773610 | 5922778 | 7367441 | 9527555 |

Table 1: Small feasible parameters for locally icosian tight 3-manifolds

The following tables list the small values for parameters $(n, \beta_1)$ satisfying the conclusion of Theorem 5.11. Table 2 for $(n, \beta_1)$ with strict inequality in part (b) of the theorem. Table 3 for $(n, \beta_1)$ with equality in part (c) of the theorem.
Table 2: Small feasible parameters for non-stacked \( \mathbb{Z}_2 \)-tight 3-manifolds

| \( n \) | 72 | 77 | 92 | 96 | 97 | 101 | 108 | 112 | 113 | 116 | 117 | 121 | 128 | 132 | 133 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( \beta_1 \) | 189 | 224 | 344 | 341 | 389 | 388 | 458 | 539 | 511 | 544 | 594 | 601 | 685 | 774 | 748 |

Table 3: Small feasible parameters for \( \mathbb{Z}_2 \)-tight 3-manifolds (with equality in Th. 5.11 (b))

| \( n \) | 825 | 1296 | 1408 | 1760 | 1881 | 1989 | 2145 | 2580 | 3168 | 3276 |
|---|---|---|---|---|---|---|---|---|---|---|
| \( \beta_1 \) | 26871 | 66637 | 78625 | 123049 | 140677 | 157336 | 183109 | 264924 | 399817 | 427582 |

6 Examples

In this section, we present some examples which help to put the results of this paper in proper perspective.

The first example shows that a strongly minimal and stacked triangulated closed 3-manifold need not be tight. So, the converse of Proposition 1.7 is not true.

Example 6.1 (Walkup [18]). Let \( J \) be the pure 4-dimensional simplicial complex with vertex set \( \mathbb{Z}_{10} = \mathbb{Z}/10\mathbb{Z} \) and an automorphism \( i \mapsto i + 1 \) (mod 10). Modulo this automorphism, there is only one representative facet (maximal face) of \( J \), namely 12345. The face vector of \( J \) is \((10, 40, 60, 40, 10)\). Each vertex link of \( J \) is an 8-vertex stacked 3-ball and hence \( J \) is a locally stacked triangulated 4-manifold with boundary. The boundary \( K = \partial J \) has face vector \((10, 40, 60, 30)\) and triangulates \( S^2 \times S^1 \). Since each 2-simplex of \( J \) is in the boundary, \( J \) is stacked and hence \( K \) is a stacked triangulated closed 3-manifold. Since \( K \) is not neighbourly, it is not tight with respect to any field.

Let \( Y \) be a triangulation of \( S^2 \times S^1 \). Since the only closed 3-manifolds with at most 9 vertices are \( S^3 \) and \( S^2 \times S^1 \) (see [4]), it follows that \( f_0(Y) \geq 10 \). Then, by [16, Theorem 5.2], \( f_1(Y) \geq 4f_0(Y) \geq 40 \). Since \( f_0(Y) - f_1(Y) + f_2(Y) - f_3(Y) = 0 \) and \( 2f_2(Y) = 4f_3(Y) \), it follows that \( f_3(Y) = f_1(Y) - f_0(Y) \geq 4f_0(Y) - f_0(Y) = 3f_0(Y) \geq 30 \). Hence \( f_2(Y) = 2f_3(Y) \geq 60 \). Thus, \( f_i(Y) \geq f_i(K) \) for \( 0 \leq i \leq 3 \). Therefore, \( K \) is strongly minimal.

The following example shows that a neighbourly and locally stacked triangulated closed 3-manifold need not be stacked. Equivalently, it need not be tight. Thus, the hypothesis ‘stacked’ in Theorem 4.8 can not be relaxed to ‘locally stacked’.

Example 6.2 (Lutz [14]). Let \( L \) be the pure 3-dimensional simplicial complex with vertex set \( \mathbb{Z}_{10} = \mathbb{Z}/10\mathbb{Z} \) and an automorphism \( i \mapsto i + 1 \) (mod 10). Modulo this automorphism, a set of representative facets of \( L \) are:

\[ 1236, 1237, 1257, 1368. \]

Then, each vertex link is a 9-vertex stacked 2-sphere and hence \( L \) is a locally stacked, neighbourly, triangulated closed 3-manifold. It triangulates \( S^2 \times S^1 \) and hence \( \beta_1(L; \mathbb{F}) = 1 \) for any field \( \mathbb{F} \). Therefore, by Proposition 1.6, \( L \) is not stacked and not tight w.r.t. any field. Since there are non-neighbourly 10-vertex triangulations of \( S^2 \times S^1 \) (cf. \( K \) in Example 6.1), \( L \) is not strongly minimal. By Propositions 1.7 and 1.6, this also shows that \( L \) is not stacked and is not \( \mathbb{F} \)-tight for any field \( \mathbb{F} \).
The following example shows that an arbitrary $F$-tight simplicial complex need not be strongly minimal (we do not know if it must be minimal in the sense of having the fewest number of vertices among all triangulations of its geometric carrier).

**Example 6.3.** Consider the neighbourly 2-dimensional simplicial complex $X$ on the vertex set $\{1, 2, 3, 4\}$ whose maximal faces are 123, 234 and 14. It is easy to see that $X$ is $F$-tight for any field $F$. Observe that $|X|$ is also triangulated by the simplicial complex $Z$ (on the same vertex set) whose maximal faces are 123, 14, 24. Thus, $X$ is not strongly minimal. Therefore, Conjecture [13] is not true for arbitrary simplicial complexes.

Recall that a $d$-dimensional simplicial complex is a pseudomanifold if (i) each maximal face is $d$-dimensional, (ii) each $(d-1)$-face is in at most two $d$-faces, and (iii) for any two $d$-simplices $\alpha$ and $\beta$, there exists a sequence $\alpha = \alpha_1, \ldots, \alpha_m = \beta$ of $d$-simplices such that $\alpha_i \cap \alpha_{i+1}$ is a $(d-1)$-face for $1 \leq i \leq m - 1$. We now extend the definitions of stackedness and locally stackedness to pseudomanifolds as follows.

**Definition 6.4.** A pseudomanifold $Q$ of dimension $d + 1$ is said to be stacked if all its faces of codimension (at least) two are in the boundary $\partial Q$. A pseudomanifold $P$ without boundary of dimension $d$ is said to be stacked if there is a stacked pseudomanifold $Q$ of dimension $d + 1$ such that $P = \partial Q$. A pseudomanifold is said to be locally stacked if all its vertex links are stacked pseudomanifolds (with or without boundaries).

**Example 6.5** (Emch [9]). Consider the 3-dimensional pseudomanifold $P$ with vertex set $\{1, \ldots, 8\}$ and the facet-transitive automorphism group $\text{PGL}(2, 7) = \langle(12345678), (132645), (16)(23)(45)(78)\rangle$. Modulo this group, a facet representative is 1235. The link of each vertex is isomorphic to the 7-vertex torus $T^2_7$. Its face vector is $(8, 28, 56, 28)$.

Since $P$ is a 3-neighbourly pseudomanifold with Euler characteristic 8, it follows that its integral homologies are torsion free and its vector of Betti numbers is $(1, 2)$. Now, if $Q$ is a pseudomanifold of dimension 4 such that $\partial Q = P$ and $\text{skel}_2(Q) = \text{skel}_2(P)$ then for any vertex $v$ of $Q$, we have $\partial(Q_v) = P_v$ and $\text{skel}_1(Q_v) = \text{skel}_1(P_v)$. Thus, $Q_v$ is a stacked 3-manifold whose boundary is the 7-vertex torus $P_v$. But it is easy to see that the 7-vertex torus bounds exactly three (distinct but isomorphic) stacked 3-manifolds, each with seven 3-faces. This implies that $f_4(v \ast Q_v) = f_3(Q_v) = 7$. Then, $f_4(Q) = (8 \times 7)/5$, which is not possible. Therefore, $P$ is not stacked. Thus, Theorems [4,8 and [5.1 are not true for 3-dimensional pseudomanifolds.

The next example disproves a putative generalization of Theorem 3.5 in which the induced cycle in the hypothesis is replaced by arbitrary manifolds without boundary.

**Example 6.6.** Consider the 6-vertex triangulation $\mathbb{RP}^2_6$ of $\mathbb{RP}^2$ whose facets are 123, 134, 145, 156, 235, 245, 246, 346, 356. Let $X = \{7 \ast \mathbb{RP}^2_6\} \cup \{123456\}$. So, $X$ is a 5-dimensional simplicial complex obtained from the cone $7 \ast \mathbb{RP}^2_6$ by adding a 5-face. The simplicial complex $X$ is isomorphic to an induced subcomplex of the 13-vertex triangulation $M$ of $SU(3)/SO(3)$ obtained by Lutz in [14]. (Indeed, for each vertex $v$ of $M$, $M[v]$ contains two induced $\mathbb{RP}^2$’s, say $M_1[A]$ and $M_2[B]$, where $V(M_v) = A \sqcup B$. Both the induced subcomplexes $M[\{v\} \cup A]$ and $M[\{v\} \cup B]$ are isomorphic to $X$.) Thus, $M$ is $\mathbb{Z}_2$-tight [13], $M_0[A] = \mathbb{RP}^2_6$ is a surface and yet the induced subcomplex $X = M[\{v\} \cup A]$ is not a pseudomanifold.
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References

[1] B. Bagchi, The $\mu$ vector, Morse inequalities and a generalized lower bound theorem for locally tame combinatorial manifolds, arXiv:1403.5675.

[2] B. Bagchi, B. A. Burton, B. Datta, N. Singh, J. Spreer, Efficient algorithms to decide tightness, in preparation.

[3] B. Bagchi, B. Datta, Lower bound theorem for normal pseudomanifolds, *Expositiones Math.* 26 (2008), 327–351.

[4] B. Bagchi, B. Datta, Minimal triangulations of sphere bundles over the circle, *J. Combin. Theory* (A) 115 (2008), 737–752.

[5] B. Bagchi, B. Datta, On $k$-stellated and $k$-stacked spheres, *Discrete Math.* 313 (2013), 2318–2329.

[6] B. Bagchi, B. Datta, On stellated spheres and a tightness criterion for combinatorial manifolds, *Euro. J. Combin.* 36 (2014), 294–313.

[7] B. A. Burton, B. Datta, N. Singh, J. Spreer, Separation index of graphs and stacked 2-spheres, arXiv:1403.5862.

[8] B. Datta, S. Murai, On stacked triangulated manifolds, arXiv:1407.6767.

[9] A. Emch, Triple and multiple systems, their geometric configurations and groups, *Trans. Amer. Math. Soc.* 31 (1929), 25–42.

[10] F. Effenberger, J. Spreer, *simpcomp – a GAP toolkit for simplicial complexes*, Version 1.5.4, 2011. http://www.igt.uni-stuttgart.de/LstDiffgeo/simpcomp. http://code.google.com/p/simpcomp/.

[11] G. Kalai, Rigidity and the lower bound theorem 1, *Invent. math.* 88 (1987), 125–151.

[12] W. Kühnel, *Tight Polyhedral Submanifolds and Tight Triangulations*, Lecture Notes in Mathematics 1612, Springer-Verlag, Berlin, 1995.

[13] W. Kühnel, F. Lutz, A census of tight triangulations, *Period. Math. Hungar.* 39 (1999), 161–183.

[14] F. H. Lutz, *Triangulated Manifolds with Few Vertices and Vertex-Transitive Group Actions*, Thesis (D 83, TU Berlin), Shaker Verlag, Aachen, 1999.

[15] S. Murai, E. Nevo, On $r$-stacked triangulated manifolds, *J. Alg. Combin.* 39 (2014), 373–388.

[16] I. Novik, E. Swartz, Socles of Buchsbaum modules, complexes and posets, *Adv. in Math.* 222 (2009), 2059–2084.

[17] J. Spreer, Necessary conditions for the tightness of odd-dimensional combinatorial manifolds, arXiv:1405.5962.

[18] D. W. Walkup, The lower bound conjecture for 3- and 4-manifolds, *Acta Math.* 125 (1970), 75–107.