Tripartite entanglement dynamics in a system of strongly driven qubits

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Abstract

We study the dynamics of tripartite entanglement in a system of two strongly driven qubits individually coupled to a dissipative cavity. We aim to explain the previously noted entanglement revival between two qubits in this system. We show that the periods of entanglement loss correspond to the tripartite entanglement formation between the qubits and the cavity, and the recovery is associated with an inverse process. We demonstrate that the overall process of qubit–qubit entanglement loss is due to the coupling to the cavity, and further loss of multipartite entanglement is due to the coupling of the cavity to the external continuum, which explains the behaviour of the qubit–qubit entanglement loss reported previously.

1. Introduction

Entanglement is one of the key aspects distinguishing quantum from classical physics. Its fragility, due to inevitable coupling of a quantum system, such as qubits or photons, to an environment (classical or quantum), however, sets limits to its applicability to quantum information and quantum communication technologies. It is, therefore, very important to understand what entanglement is most vulnerable to and what processes can avert or undo entanglement loss.

Qubits are the fundamental building blocks of quantum information science, and the last two decades marked great developments both theoretically as well as experimentally [1, 2]. After many successes in that field, the current research frontier is the qubit–qubit entanglement, which requires either a direct coupling between qubits or an indirect one through an auxiliary system, for example, a resonator [3, 4]. As a result of that, the qubits can potentially entangle to, or disentangle from each other, depending on the system’s design, parameters, interaction time, or even the types of environments [5–17].

In this paper, we focus on the process of entanglement revival in a system of two qubits driven by a strong external, classical ac field and simultaneously coupled to a quantum resonator, which thus, indirectly couples the qubits. We previously discussed this phenomenon in [15] and demonstrated that entanglement does not only need to decay as the system evolves in time, but the system can also periodically regain some of its initial entanglement. Previously, it was shown for this system that the disentanglement between the qubits may be a consequence of the cavity dissipation [16, 17]. In this paper, we demonstrate that this is not the only mechanism leading to entanglement loss. Specifically, we show that the mere presence of a qubit–cavity coupling results in disentanglement in the subspace spanned by the qubits and that the further coupling of the cavity to the electromagnetic continuum leads to an overall tripartite entanglement decay. Therefore, the best way to understand the qubit entanglement dynamics is by looking at the phenomena from a larger, multipartite perspective. This has previously been studied in the case of non-interacting qubits in separate environments [18, 19], and with this work we aim to provide insight into entanglement transfer back and forth within a multipartite system subject to dissipation.

This paper is structured as follows. In section 2, we quantitatively introduce the system of strongly driven qubits. We derive the equations of motion and present their solutions. In section 3, we outline and discuss entanglement measures applicable to the tripartite analysis. In section 4, we quantify entanglement between individual subsystems in a dissipationless regime. Subsequently, in section 5, we study the effects of an imperfect cavity on entanglement formation, revival and loss among different subsystems. We close the paper with conclusions.
2. The model and its dynamics

We consider the system of two identical qubits externally driven by a classical field of the amplitude $A$ and the frequency $\omega_c$ strongly coupled to a single mode resonator [20] (see figure 1). This system is described by the following Hamiltonian,

$$
\hat{H} = \hat{H}_o + \hat{H}_d + \hat{H}_I,
$$

$$
\hat{H}_o = \frac{\Omega}{2} \sum_{j=1}^{2} \sigma_j^+ + \omega \hat{a}^\dagger \hat{a},
$$

$$
\hat{H}_d = A \sum_{j=1}^{2} (\hat{e}^{-i\omega_c t} \sigma_j^+ + \hat{e}^{i\omega_c t} \sigma_j^-),
$$

$$
\hat{H}_I = \sum_{j=1}^{2} g_j (\sigma_j^+ \hat{a} + \sigma_j^- \hat{a}^\dagger),
$$

(1)

where $\Omega$ represents the level spacing of each qubit, $\omega$ is the frequency of the resonator eigenmode and $g_j$ is the coupling strength of the Jaynes–Cummings type interaction between the $j^\text{th}$ qubit and the eigenmode of the resonator. Additionally, we use $\sigma_j^\pm (\hat{a}^\dagger, \hat{a})$ to denote the qubit (resonator) raising and lowering operators. Throughout this paper we chose units where $\hbar = 1$.

We work under the assumption that the qubits are driven very strongly (ensuring their greater resistance to a qubit decay rate $\gamma$), and are moderately coupled to the cavity mode, i.e. $A \gg \omega, \omega_c \gg g \gg \gamma$. Moreover, we consider the cavity dissipation rate $\kappa$ to be the dominant source of decoherence in the system. A realization of these conditions can be found for instance with superconducting qubits where $\omega \approx 5 \text{ GHz}$, $g \approx 100 \text{ MHz}$ and $\gamma \approx 1 \text{ MHz}$ [21].

The Hamiltonian in equation (1) is time-dependent. To suppress the time dependence, one can apply a number of unitary transformations [20]. First we go to the frame oscillating with the driving field frequency $\omega_c$ using $\hat{U} = \exp \left(-i\omega_c t (\hat{a}^\dagger \hat{a} + \sum_j \sigma_j^+/2)\right)$ and further to the interaction picture (IP) $\mathcal{V} = \exp[-i(H_o + H_d)t] \hat{H}_I \exp[i(H_o + H_d)t]$. Upon ignoring the quickly rotating terms $\propto \exp \left[\pm 2iAt\right]$ (the strong driving regime), the effective IP Hamiltonian becomes (see [15] for technical details)

$$
\mathcal{V} = \sum_{j=1}^{2} g_j \sigma_j^2 (\hat{a} e^{-i\delta t} + \hat{a}^\dagger e^{i\delta t}),
$$

(2)

where we set $\Omega = \omega_c$ and $\delta = \omega - \omega_c$. What we observe is that in the strong driving regime the coupling to the resonator reduces to the qubit state dependent bosonic displacement generator.

We model the evolution of the system with the Lindblad type master equation

$$
\dot{\rho} = \mathcal{L} (\rho) = -i [\mathcal{V}, \rho] + \kappa \mathcal{D} (\rho),
$$

(3)

where $\mathcal{D} (\rho) = \hat{a} \rho \hat{a}^\dagger - \frac{1}{2} \hat{a}^\dagger \hat{a} \rho - \frac{1}{2} \rho \hat{a}^\dagger \hat{a}$ is a Markovian dissipation operator. The system is initially in a direct product state of a cavity field coherent state $|\alpha\rangle$ and a Bell state of the qubits\(^1\),

$$
\Psi = \frac{1}{\sqrt{2}} \left(|++\rangle + |--\rangle\right),
$$

(4)

where we used the diagonal basis $|\pm\rangle = \frac{1}{\sqrt{2}} (|e\rangle \pm |g\rangle)$, which are the eigenstates of the Pauli $\sigma_z$ matrix. The solution to the Lindblad equation (3) is given in terms of the density operator $\rho_{ij;kl}$, where the Latin indices $i$ and $j$ ($k$ and $l$) stand for the $|\pm\rangle$ state of the first (second) qubit and the Greek indices indicate the state of the cavity. The solutions to equation (3) are found in [15]. The only non-zero entries of the density matrix $\rho_{ij;kl}$ with the initial condition given by equation (4) read

$$
\rho_{++;++} (t) = \frac{1}{2} |\alpha_+\rangle\langle \alpha_+|,
$$

$$
\rho_{++;+-} (t) = \frac{1}{2} \exp \left[ h_1 (t) + i h_2 (t) \right] \rho_{++;--} (t) \langle \alpha_- | \langle \alpha_- |, \rho_{--;+-} (t) = \frac{1}{2} |\alpha_-\rangle\langle \alpha_-|,
$$

(5)

Here we have defined

$$
\alpha_\pm = Ae^{\pm i\delta t} \pm f (t),
$$

$$
f (t) = \frac{g_j^2}{\kappa - i \delta} \left( e^{-i\delta t} - e^{-i\delta t} \right),
$$

$$
h_1 (t) = \frac{8 e^{-i\delta t} g_j^2 \kappa (\delta \sin \delta t - \kappa \cos \delta t)}{(\kappa^2 + \delta^2)^2} - \frac{8 e^{-i\delta t} g_j^2 \kappa^2 \cos \delta t}{(\kappa^2 + \delta^2)^2} - \frac{4 g_j^2 \kappa \delta}{\kappa^2 + \delta^2},
$$

$$
h_2 (t) = h_2 (0) = \frac{2e^{-2i\delta t} g_j (\kappa \alpha_+ + \kappa \delta)}{\kappa^2 + \delta^2} - \frac{2e^{-2i\delta t} g_j (\alpha_\delta - 3 \kappa \alpha_+ \cos \delta t + (3 \kappa \alpha_+ + 3 \kappa \alpha_+ + \kappa \delta) \sin \delta t)}{\kappa^2 + \delta^2},
$$

(6)

where $\alpha_\pm$ are the real and imaginary parts of the initial state of the cavity and $g = g_1 + g_2$ is the effective qubit–cavity coupling.

The coherent state considered here has a continuous, time-dependent amplitude. Such a state is represented by a vector spanning the whole of the infinite Fock space, making

\(^1\) We assume that such an entangled state is available. In our follow up work we show how one could produce an entangled state using a system of strongly driven qubits.
the qubit–cavity system $2 \times 2 \times \infty$ dimensional, and thus rendering some of the entanglement measures inapplicable. These different coherent states, however, can be written in bases found in [22]. Using the fact that every coherent state is a single-parameter state, we can recast the two coherent states $|\alpha\rangle$ in a two-dimensional form by means of orthogonalization through the Gram–Schmidt process,

$$
|\uparrow\rangle = |\alpha\rangle, \\
|\downarrow\rangle = \frac{1}{\sqrt{1 - |\chi|^2}} (|\alpha\rangle - \chi |\alpha\rangle), \\
\chi (t) = \langle \alpha | |\alpha\rangle .
$$

(7)

such that $(\uparrow | \downarrow) = 0$ and where the inverse transformation reads

$$
|\alpha\rangle = |\uparrow\rangle, \\
|\alpha\rangle = \sqrt{1 - |\chi|^2} |\downarrow\rangle + \chi |\uparrow\rangle.
$$

As a result, the system is now reduced to $2 \times 2 \times 2 = 8$ dimensions. In the last subspace definitions of the bases are time-dependent, but the resulting set of states $\{|\uparrow\rangle, |\downarrow\rangle\}$ is orthogonal at any point in time. In this form, we can easily use the established entanglement formalism.

The full system time-dependent density operator $\rho$ in the $C^2 \times C^2 \times C^2$ space spanned by the diagonal basis of the qubits and the orthogonalized coherent state basis reads

$$
\rho = \begin{pmatrix}
K & \cdots & L \\
\vdots & \ddots & \vdots \\
L^t & \cdots & M
\end{pmatrix},
$$

(8)

which is a sparse $8 \times 8$ matrix where the only non-zero elements are contained in the $2 \times 2$ blocks

$$
K = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}, \\
L = \begin{pmatrix}
e^{ih_1 + ih_2} x & e^{ih_1 + ih_2} \sqrt{1 - |x|^2} \\
e^{i(-h_1 + -ih_2)} & 0
\end{pmatrix},
$$

$$
M = \frac{1}{2} \begin{pmatrix}
|\chi|^2 & \chi \sqrt{1 - |\chi|^2} \\
\chi^* \sqrt{1 - |\chi|^2} & 1 - |\chi|^2
\end{pmatrix}.
$$

3. Entanglement measures

Establishing the degree of non-separability (entanglement) in a particular system, based on the form of its density operator, is not necessarily an easy task. Fortunately, the last two decades brought developments in the field of entanglement measures [23–25]. There it was shown that stepping beyond $2 \times 2$ physical systems, where one often uses concurrence, one has to account for a greater number of correlations between individual players in a multipartite physical system [27], and can choose from among others entanglement witnesses, negativity, or the three-tangle. We devote this section to briefly review some of the most important aspects which we will find useful in our subsequent analysis.

One of the first entanglement measures to be introduced and since then widely used for $2 \times 2$ dimensional systems is concurrence [23]. Its mathematical form is given by

$$
C = \max(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}),
$$

(9)

where $\lambda_i$ are the eigenvalues of $R = (\sigma_i \otimes \sigma_j) \rho^y (\sigma_i \otimes \sigma_j) \rho$, with $\lambda_1$ being the largest of them, $\sigma_i$ being the Pauli $\gamma$-matrix. The value of $C$ ranges from zero (no entanglement) to one (maximum entanglement). This measure, however, no longer suffices when dealing with systems involving more than two two-dimensional subsystems.

In order to study entanglement in the tripartite system, we use Horodeckis’ separability criterion [26] and stemming from it negativity [25] to quantify tripartite entanglement. Using the partial transposition (in the second subspace) defined by

$$
\rho^{p\pi_2} = \sum_{ijkl} \alpha_{ijkl} |j \rangle \langle k| \otimes |l\rangle \langle i|,
$$

this criterion states that the density operator of an entangled state upon transposition in one of the subspaces will have at least one negative eigenvalue. Negativity is then the sum of absolute values of negative eigenvalues of $\rho^{p\pi_2}$.

Thus when studying a tripartite system composed of three subsystems $A, B$ and $C$ (in this case $A$ and $B$ are the qubits and $C$ is the cavity, but the labelling is completely arbitrary), we can find the degree of entanglement between the combined bipartite subsystem $AB$ and subsystem $C$, by partially transposing the density operator $\rho_{ABC}$ of the system in the basis states that span the subsystem $C$, and later adding up all of the absolute values of the negative eigenvalues of $\rho_{ABC}^{p\pi_2}$. As a result we obtain negativity $\mbox{Neg} (AB|C)$, which when equal to zero corresponds to no (or bound i.e. a state with zero negativity that is not separable) entanglement and when equal to $\frac{1}{2}$ indicates maximum bipartite entanglement. To get the full picture of tripartite entanglement in this system we need to also calculate $\mbox{Neg} (AC|B)$ and $\mbox{Neg} (BC|A)$, where the partial transposition is made in the subsystem $B$ and $A$ basis, respectively.

Since the dimension of this system is larger than six, i.e. the limit imposed by Horodeskis’ separability criterion, we could encounter bound entanglement. We could avoid this subtlety by creating a map $C^2 \otimes C^2 \rightarrow C^2$ which transforms the entangled state $|+\rangle |+\rangle |-\rangle$ onto a superposition $|+\rangle |+\rangle |+\rangle$ reducing the dimensionality of the system, by removal of permanently empty rows and columns of the density operator. This will, however, prove to be unnecessary, as we will see later, the only time when negativity is strictly zero is at the expected periodically distributed points in time $\delta t = 2\pi n$ for integer $n$, when the qubits are completely disentangled from the cavity (see figures 2 and 5); something that can be easily seen without invoking any formalism of the entanglement measures. Thus the excessive dimensionality of our system poses no problems with regards to using negativity as an entanglement measure.

One drawback of the negativity is that it only provides information about entanglement of two parts of the system.
under partitioning of our choice and does not tell us anything about the total entanglement present. Adapting the approach of [28], we can use the sum of the bipartite entanglements
\[
\text{TE} = \text{Neg} \left( Q_1 |C\rangle |Q_2\rangle + \text{Neg} \left( Q_2 |C\rangle |Q_1\rangle \right) + \text{Neg} \left( Q_1 |Q_2\rangle |C\rangle \right),
\]
where we replace the arithmetic mean by a direct sum. As a result of that \(\text{TE} = 1\) becomes an easy reference point, since it is the value of how much entanglement was initially there (\(t = 0\)) in the system. In this way we obtain that \(0 \leq \text{TE} \leq \frac{1}{2}\), where the lower bound indicates no and the upper bound indicates maximal tripartite entanglement, that of, for example, the GHZ3 state
\[
|\text{GHZ}_3\rangle = \frac{1}{\sqrt{2}} \left( |000\rangle + |111\rangle \right),
\]
where all negativities \(\text{Neg} \left( Q_1 |Q_2\rangle |Q_1\rangle = \text{Neg} \left( Q_2 |Q_1\rangle |Q_2\rangle = \text{Neg} \left( Q_1 |Q_2\rangle |Q_2\rangle = \right.\frac{1}{2}.\]

The GHZ3 state shows a feature that will be important to our further discussions, namely tripartite entanglement sharing. In this state (as opposed to the W-state), the individual subsystems share bipartite entanglement but partial tracing over one of the subsystems (losing a qubit) results in a statistically mixed state (the W state results in a Bell state).

This result is known as the monogamy of entanglement, which states that a subsystem \(A\) maximally entangled to the second subsystem \(B\) cannot be simultaneously entangled with another subsystem \(C\). This has been first formulated by Coffman–Kundu–Wootters [24] in terms of the inequality
\[
C^2_{(A|BC)} > C^2_{(A|B)} + C^2_{(A|C)},
\]
where \(C^2\) are the tangles (concurrences squared). Here \(C_{(A|BC)}\) is found by reducing the dimensionality of the density operator \(\rho\). This is carried out by analysing the \(BC\) subspace and removing the rows and columns corresponding to the eigenvectors with zero eigenvalues. \(C_{(A|B)}\) and \(C_{(A|C)}\), on the other hand, are the concurrences of the bipartite subsystem obtained by partially tracing the total tripartite system over the subsystem \(C\) and \(B\), respectively.

The inequality (10) can be used to define a three-tangle given by the inequality mismatch
\[
\tau_{ABC} = C^2_{(A|BC)} - C^2_{(A|B)} - C^2_{(A|C)}.
\]
This new quantity tells us how much of the residual tripartite entanglement there is when all of the bipartite contributions are taken away. It is easy to see from the definition of concurrence that the three-tangle ranges from zero (no shared entanglement) to 1 (completely inseparable state of GHZ3 type).

From the solutions to equation (3), we see that the qubits entangle with the cavity, which for \(\delta, \kappa \to 0\) and \(t \to \infty\) leads to a perfectly entangled GHZ-like state, as the coherent state amplitudes undergo a shift in opposite directions,
\[
|\Psi, a \rangle (0) = |\Psi \rangle \otimes |a \rangle,
|\Psi, a \rangle (t) = \frac{1}{\sqrt{2}} \left( \left| + + \right\rangle |a_0 - 2igt\rangle + \left| - - \right\rangle |a_0 + 2igt\rangle \right)
= \frac{1}{\sqrt{2}} \left( \left| + + , A_+ \right\rangle + \left| - - , A_- \right\rangle \right).
\]
At finite times, this state is different from the GHZ state since \(\left| A_+ A_- \right\rangle \neq 0\). Upon taking the partial trace of \(\rho (t)\) over the cavity, we observe that the diagonal entries of the density operator \(\text{Tr}_c \left( |\Psi \rangle \langle \Psi | \right)\) are unchanged, but the entries \(\left| + + \right\rangle \left\langle - - \right|\) acquire time dependence \(\exp \left[ -2igt^2 \right]\), which mimics dephasing of the two-qubit state. This is because as \(t\) grows the state of the system more and more closely resembles the GHZ state, and taking the trace leaves the state in a completely mixed state to a larger extent. It is in continuous time analogue of the GHZ state formation from the original Bell state.

The effect of entanglement revival in this system is brought about by the presence of detuning between the cavity and the resonantly driven qubits. Since we would be interested in the periodic revival of entanglement for the remainder of our analysis we have to keep \(\delta \neq 0\). In what follows we will mainly focus on the dissipationless case as it provides very good insight into qualitative, as well as quantitative, aspects of qubit–cavity entanglement dynamics. Later we will study the effect a combination of dissipation and detuning has on the inter-qubit as well as the qubit–cavity entanglement.

4. Dissipationless cavities

In the closed system \((\kappa \to 0\) limit), the non-resonant interaction between the qubits and the cavity will result in the formation of a coherent state with an amplitude oscillating in time with frequency \(\delta\). Under these conditions, the complete state of the system is still represented by equation (5), where the previously defined expressions in equations (7) and (6) take the form \(h_1 = 0\) and
\[
\chi = \exp \left[ \frac{g}{\delta} (\cos \delta t - 1) \left( \frac{4g}{\delta} - 2abh + ia \sin 2\delta t - 1 \right) \right],
\]
where \(a\) and \(b\) are the real and imaginary parts of the initial coherent state amplitude, and the value of \(h_2 (t)\) will have no
effect on the result. Upon partially transposing the expression in equation (8) with respect to the cavity subspace we obtain

$$\rho^{pTC} = \left( \begin{array}{ccc} K & \ldots & L^T \\ \vdots & \ddots & \vdots \\ L^* & \ldots & M^T \end{array} \right),$$

where there is only one negative eigenvalue, and the negativity takes the form

$$\text{Neg}(Q_1Q_2|C) = \frac{1}{2} \left( 1 - e^{-\frac{4g^2\pi}{\delta^2}} \right)^{\frac{1}{2}}. \quad (12)$$

Taking the partial transposes in the individual qubit spaces, we obtain a symmetric result

$$\text{Neg}(Q_1C|Q_2)(t) = \text{Neg}(Q_2C|Q_1)(t) = \frac{1}{2}. \quad (13)$$

Note that the initial state of the cavity \( \alpha \) has no effect on the results. It is important to note that under dissipationless evolution, the entanglement between the two subsystems spanned by the joint qubit–cavity subspace and the other qubit does not change with time (i.e. there will be no bipartite entanglement variation between the two qubits), thus since \( \text{Neg}(Q_1Q_2|C) \geq 0 \), the total entanglement can only increase relative to its initial value.

The behaviour of \( \text{Neg}(Q_1Q_2|C) \) (see figure 2) displays periods of entanglement and disentanglement between the qubits and the cavity. This is due to the fact that for every period of length \( 2\pi/\delta \), the coherent state of the cavity returns to its initial state. Figure 2 also shows that the strength of the qubit–cavity entanglement formed depends on detuning. The coherent states under detuned driving of the qubits change their amplitudes to a limited extent. The values of the coherent state amplitude and phase follow a circular trajectory in a complex plane centred at \( \alpha_0 \pm g/\delta \) with periods \( \delta \) and radii \( g/\delta \), where \( \alpha_0 \) is the initial coherent state amplitude.

The creation of entanglement between the qubits and the cavity, however, bares consequences for the qubit–qubit subsystem. Previously in [15], we saw that qubits can undergo oscillations in their relative entanglement strengths (even if the qubit does not change with time (i.e. there will be no bipartite entanglement variation between the two qubits)), thus since \( \text{Neg}(Q_1Q_2|C) \geq 0 \), the total entanglement can only increase relative to its initial value.

The creation of entanglement between the qubits and the cavity, however, bares consequences for the qubit–qubit subsystem. Previously in [15], we saw that qubits can undergo oscillations in their relative entanglement strengths (even if we take the \( \kappa \to 0 \) limit of equation (9)). By considering the solutions in equation (5) and figure 2 we can see that throughout the evolution, the qubit–qubit subsystem oscillates between completely entangled and partially mixed states,

$$\text{Tr}_c(\rho) = |\Phi\rangle\langle\Phi| \leftrightarrow \rho_{Q_1Q_2},$$

$$\rho_{Q_1Q_2} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & \epsilon(t) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \epsilon(t)^* & 0 & 0 & 1 \end{pmatrix},$$

where \( \text{Tr}_c(\rho) \) denotes a partial trace over the cavity states and \( \epsilon(t) = \langle \alpha_+|\alpha_- \rangle \). This only has to do with the fact that the \( \text{Tr}_c(p_{++,---}) \) entries carry time dependence, while the populations i.e. the \( \text{Tr}_c(p_{++,+++}) \) and \( \text{Tr}_c(p_{---,--}) \) entries, are constant in time. This is because the effective Hamiltonian does not allow for the individual qubit state populations to change.

When just the two qubits are considered (trace over the cavity), the entanglement between them will undergo fluctuations. In line with the monogamy of entanglement, every time that the cavity almost completely entangles to the qubits, the qubits themselves must share very little bipartite entanglement, because now the system forms a tripartite entangled state as a whole.

Recalling that the qubit–qubit concurrence in the dissipationless case reads

$$C_{Q_1Q_2} = \exp \left[ \frac{4g^2}{\delta^2} (\cos \delta t - 1) \right],$$

we calculate the amount of tripartite-shared entanglement to find

$$C_{(C|Q_1Q_2)} = \sqrt{1 - \exp \left[ \frac{8g^2(\cos \delta t - 1)}{\delta^2} \right]},$$

which incidentally is just twice the negativity in equation (12) [29]. If we calculate the \( C_{(C|Q_1)} \) by taking the trace over the subsystem \( Q_2 \) (\( Q_1 \)), we find that they are equal to zero. Thus we obtain that

$$\tau_{CQ_1Q_2} = 1 - \exp \left[ \frac{8g^2(\cos \delta t - 1)}{\delta^2} \right].$$
Figure 4. The plot of Neg(\(Q_1C|Q_2\)) \((t)\) (red solid) and Neg(\(Q_1Q_2|C\)) \((t)\) (blue dashed) made for \(g_1 + g_2 = 200\text{ MHz}\), \(\delta = 100\text{ MHz}\) (first column), \(\delta = 300\text{ MHz}\) (second column) and \(\delta = 500\text{ MHz}\) (third column), and \(\kappa = 100\text{ MHz}\) (first row) and \(\kappa = 1\text{ MHz}\) (second row). Bottom-left: \((g = 50\text{ MHz}, \delta = 1\text{ GHz}, \kappa = 400\text{ MHz})\) we see that the Neg(\(Q_1Q_2|C\)) \((t)\) is much longer lived and eventually decays but always later than Neg(\(Q_1C|Q_2\)) \((t)\). We see that the entanglement generated between the cavity and the two qubits depends on a delicate balance between the detuning and the dissipation rate. Bottom-right: comparison between the qubit–qubit concurrence decay rate with and without decay. Plots made for \(g_1 + g_2 = 200\text{ MHz}\), \(\kappa = [500, 100, 50, 0]\text{ MHz}\) in blue, red, green, black (increasing dashing frequency), respectively. We see that larger cavity decay rates offer slower concurrence decays, however, due to further cavity–environment coupling asymptotically concurrence is zero.

Thus we see that for small detunings most of the entanglement is shared among the three entities and only when time \(t\) is close to an integer multiples of \(2\pi/\delta\) then the entanglement between the subsystem of the qubits and the cavity is lost, resulting in the recovery of the bipartite entanglement between the qubits (see figure 3).

The total entanglement \(TE\) will simply be Neg(\(Q_1Q_2|C\)) + 1, thus we can extend the conclusions above to the total entanglement in the system as their qualitative nature does not change. It is interesting to analyse the case when dissipation is present, which is the focus of the next section.

5. Dissipative cavity

Using the solutions of equation (5), and repeating the analysis presented above in the dissipative cavity case we find the negativities

\[
\text{Neg}(Q_1Q_2|C) = \frac{1}{2}e^{h_1} + \frac{1}{2}e^{h_2} + \sqrt{1 - 4e^{h_1}e^{h_2} - 2e^{h_1} + 2e^{h_2}},
\]

\[
\text{Neg}(Q_1C|Q_2) = \text{Neg}(Q_2C|Q_1) = \frac{1}{2}e^{h_1},
\]

with \(h_1 = h_1 (g, \delta)\) is given by equation (6) and \(\chi\) stems from the definition equation (7) and in this case reads

\[
|\chi|^2 = \exp\left\{\frac{4\kappa^2 \exp(\kappa t) (\cos(\delta t) - \cosh(\kappa t))}{\delta^2 + \kappa^2}\right\}.
\]

We can see that, as a result of cavity dissipation, the negativity \(\text{Neg}(Q_1C|Q_2)(t)\) (constant when \(\kappa = 0\)) becomes a nontrivial function of time whose plots are presented in figure 4 for different values of the dissipation rate \(\kappa\) and detuning \(\delta\). In addition to detuning, which creates a (dis)entanglement oscillation, there are other factors affecting the bi- and multipartite entanglement in the system. The coupling-to-dissipation ratio at resonance leads to a decay of qubit–qubit entanglement and the creation of qubit–cavity entanglement; detuning on the other hand, limits the qubit–qubit disentanglement, by means of impairing the qubit–cavity
entanglement as we have seen in the previous section in figure 2. Since the steady-state amplitude of the coherent state of the cavity is proportional to $g/(\kappa + i\delta)$, small detunings and a low dissipation rate facilitate the formation of coherent states of larger maximum amplitudes, which make the state resemble the GHZ state to a larger extent, thus disentangling the qubits (lowering the qubit–qubit concurrence). For large detunings, the qubits are coupled to the cavity field less, causing smaller coherent state amplitudes and reducing the disentanglement rate. A greater cavity decay rate again only amplifies this process. This explains the findings of the previous paper [15], and we see this process quite clearly here due to detuning, which marks the frequency of re- and disentanglement. For large detunings, the qubits are coupled to the cavity field less, causing smaller coherent state amplitudes and reducing the disentanglement rate. A greater cavity decay rate again only amplifies this process. This explains the findings of the previous paper [15], and we see this process quite clearly here due to detuning, which marks the frequency of re- and disentanglement. Additionally, we point out that looking at the whole.

The conclusion that we can draw from these results is that in either scenario there is some entanglement being formed between the qubits and the cavity and as a consequence of the monogamy of entanglement, the greater the degree of entanglement between the qubits and the cavity the less they can retain their inner-qubit entanglement. Additionally, a greater dissipation rate of the cavity weakens the qubit–cavity entanglement formation and strengthens the cavity–reservoir entanglement formation. Since the reservoir is traced out (in the procedure of derivation of the equations of motion), this leads to degradation of any entanglement in the system as a whole.

6. Conclusions
In this paper we have studied the dynamics of tripartite entanglement between two driven qubits non-resonantly coupled to a cavity. Using tripartite entanglement measures (negativity and three-tangle) we have shown that the previously reported entanglement loss followed by its revival is a consequence of an entanglement formation and subsequent disentanglement between the subsystem composed of qubits
and the subsystem spanned by the cavity. Additionally, further tripartite entanglement loss is due to the dissipation of the cavity, which can be seen to form a greater entangled state with the environment states, which has been traced over in a process of a derivation of the master equation. One can interpret this result as the cavity playing the role of an intermediate non-Markovian bath, which is further coupled to a Markovian one. This non-Markovian-like behaviour can be seen to be due to the presence of the external driving field and the cavity frequency mismatch $\delta$, which clocks the (dis)entanglement process. With this work we want to emphasize the danger of attributing all correlation losses to the dissipation alone as seen by the evolution of correlations in this system. Here the qubit–qubit entanglement is lost due to the qubit–cavity entanglement formation even for $\kappa = 0$, and it was only the formation of a larger system–environment entanglement that leads to tripartite intra-system entanglement loss.

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