WEAK INVARIANCE PRINCIPLE FOR THE LOCAL TIMES OF GIBBS-MARKOV PROCESSES

MICHAEL BROMBERG
SCHOOL OF MATHEMATICAL SCIENCES, TEL AVIV UNIVERSITY. TEL AVIV 69978, ISRAEL.

Abstract. The subject of this paper is to prove a functional weak invariance principle for the local time of a process generated by a Gibbs-Markov map. More precisely, let \((X, \mathcal{B}, m, T, \alpha)\) be a mixing, probability preserving Gibbs-Markov map and let \(\varphi \in L^2(m)\) be an aperiodic function with mean 0. Set \(S_n = \sum_{k=0}^n X_k\) and define the hitting time process \(L_n(x)\) be the number of times \(S_k\) hits \(x \in \mathbb{Z}\) up to step \(n\). The normalized local time process \(l_n(x)\) is defined by

\[
l_n(t) = \frac{L_n(\lfloor \sqrt{n}x \rfloor)}{\sqrt{n}}, \quad x \in \mathbb{R}.
\]

We prove under that \(l_n(x)\) converges in distribution to the local time of the Brownian Motion. The proof also applies to the more classical setting of local times derived from a subshift of finite type endowed with a Gibbs measure.

1. Introduction

Let \((X, \mathcal{B}, m, T, \alpha)\) be a mixing, probability preserving Gibbs-Markov map on a standard probability space. Let \(\varphi \in L^2(m)\) be an integer valued function with mean 0. We assume that \(\varphi\) is a uniformly Lipschitz continuous function on the partition \(\beta = T\alpha\), i.e. \(D_\beta f := \sup_{a \in \beta} D_a f < \infty\), where \(D_a f = \sup_{x,y \in a} \frac{|f(x)-f(y)|}{d(x,y)}\) is the Lipschitz norm on \(a\) and \(d(\cdot, \cdot)\) is the complete metric on \(X\).

In what follows, convergence in distribution of random variables \(X_n\) taking values in some standard probability space \(\Omega\) to a limit \(X\), means that for every bounded and continuous \(f : \Omega \to \mathbb{R}\), \(E(f(X_n)) \to E(f(X))\), where \(E(\cdot)\) denotes expectation. In this case, we write \(X_n \overset{d}{\to} X\).

Let \(S_n(x) := \sum_{k=0}^{n-1} \varphi(T^k(x))\) and \(\omega_n(t) = \frac{S_{\lfloor nt \rfloor}}{\sqrt{n}}\), where \(\lfloor x \rfloor\) is the integral value of \(x\), \(t \in [0, 1]\). The central limit theorem for \(S_n\) states that \(\frac{S_n}{\sqrt{n}}\) converges in distribution to the Gaussian distribution \(\mathcal{N}(0, \sigma^2)\), where \(\sigma^2 = \lim_{n \to \infty} \frac{\text{Var}(S_n^2)}{n}\) is the asymptotic variance of \(S_n\). The stronger, functional CLT states that the random functions \(\omega_n(\cdot)\) converge in distribution to \(\omega(\cdot)\), where \(\omega(\cdot)\) is the Brownian motion satisfying \(E(\omega(t)) = 0, \text{Var}(\omega(t)) = \sigma^2 t\). Here, convergence in distribution is of random variables taking values on the Skorokhod space \(D[0, 1]\) of functions on \([0, 1]\) that are continuous from the right with finite limits on the left (cadlag functions).

We wish to establish a distributional invariance principle for the local time of the sequence \(\omega_n\). To make this precise, define the occupation times of a function \(f \in D[0, 1]\) by

\[
\nu_f(A) = \int_0^1 1_A(f(t)) \, dt, \quad A \in B(\mathbb{R}).
\]
Recall that the occupation measure of the Brownian motion is almost surely, absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R} \). The (random) density function with respect to the occupation measure, which we denote by \( l(\cdot) \), is the local time of the Brownian motion.

We define the local time of \( \omega_n \) at the point \( x \) by

\[
l_n(x) = \frac{\# \{ 0 \leq k \leq n : S_k = \lfloor \sqrt{n}x \rfloor \}}{\sqrt{n}}.
\]

\( l_n \) is the normalized number of visits to the point \( \lfloor \sqrt{n}x \rfloor \) by the process \( \{S_k\} \) up to time \( n \). It may be roughly viewed as the density function of the atomic occupation measure \( \omega_n \). In fact (as section ... shows) the differences \( \nu_{\omega_n}[a,b] - \int_0^b l_n(x) \, dx \) converge in distribution to 0.

Existence of local time for the Brownian motion ensures that \( \nu_{\omega_n}(A) \xrightarrow{d} \nu_\omega(A) \) for every \( A \in \mathcal{B}(\mathbb{R}) \) with boundary of Lebesgue measure 0. We wish to establish the convergence in distribution of the corresponding local times.

Since the local time of the Brownian motion is an a.s continuous function, we may consider \( l_n \) and \( l \) as a family of random variables taking values in the space \( D \) of cadlag functions on \( \mathbb{R} \) (see [3]).

**Theorem 1.** Let \( \varphi \in L^2(m) \) with \( \sup_{a \in \beta} D_{a}\varphi < \infty \), \( m(\varphi) = 0 \). If \( \varphi \) is aperiodic (see definition [5]), then \( l_n(\cdot) \xrightarrow{d} l(\cdot) \).

To prove the theorem, we prove tightness of the sequence \( l_n \) in section [4] and then identify \( l \) as the only possible limit point for \( l_n \) in section [5].

### 2. Characteristic Function Operators

Throughout this section, let \((X,\mathcal{B},m,T,\alpha)\) be a mixing, probability preserving Gibbs-Markov map. For a measurable partition \( \beta \) of \( X \), denote by \( L_{p,\beta} \) the intersection of \( L^p \) with the space of all functions with a finite Lipchitz norm, i.e. \( L_{p,\beta} = \{ f \in L^p(m) \mid D_{\beta} f < \infty \} \) (see introduction for the definition of the Lipchitz norm).

Throughout the rest of this section \( \beta \) denotes the partition \( T\alpha \).

Consider \( T \) as an operator on \( L^\infty(m) \) defined by \( Tf = f \circ T \). Then the transfer operator \( \hat{T} : L^1(m) \to L^1(m) \) is the pre-dual of \( T \), uniquely defined by the equation

\[
\int f \cdot g \circ T \, d\mu = \int \hat{T} f \cdot g \, d\mu \quad \forall f \in L^1, g \in L^\infty.
\]

Recall that an operator \( S \) on a Banach space \( B \) is called quasi-compact if there exist \( S \)-invariant closed subspaces \( F,H \) such that:

1. \( F \) is finite dimensional and \( B = F \oplus H \).
2. \( T \) is diagonalizable when restricted to \( F \) with all eigenvalues having modulus equal to the spectral radius of \( T \), denoted by \( \rho(T) \).
3. When restricted to \( H \), the spectral radius of \( T \) is strictly less than \( \rho(T) \).

**Theorem 2.** \( \hat{T} \) is a quasi-compact operator on the space \( L := L_{\infty,\beta} \). Moreover, \( \hat{T} f = m(f) + Q f \), where \( m(f) \) is interpreted as a constant function on \( X \) and \( \rho(Q) < 1 \).

For a measurable function \( \varphi : X \to \mathbb{R} \), the characteristic function operators \( P_t : L^1(m) \to L^1(m) \), \( t \in \mathbb{R} \) are defined by

\[
(2.1) \quad P_t f = \hat{T} \left( e^{it\varphi} f \right).
\]

If \( \varphi \in L_{2,\beta} \) then \( P_t \) is a twice continuously differentiable function of \( t \).
Restricting $P_t$ to act on $L$ and using the implicit function theorem (see [7]) together with the quasi-compactness of $\hat{T}$ on $L$, we may obtain the Taylor’s expansion of the operator $P_t$ near 0. In case $m(\varphi) = 0$ (as we assume for our purposes), in a sufficiently small neighborhood of 0, $P_t : L \rightarrow L$ is of the form

$$P_t = \lambda_t \pi_t + N_t$$

where $\lambda_t$ is an eigenvalue with absolute value not exceeding 1, $\pi_t$ is a projection onto a one dimensional vector space generated by an eigenfunction $v_t$ and $\rho(N_t) < q < 1$ for some constant $q$. Moreover, $v_t$, $\pi_t$, $\lambda_t$ are twice continuously differentiable functions of $t$ and the Taylor’s expansions for $\lambda_t$ and $\pi_t$ are given by

$$\lambda_t = 1 - \sigma t^2 + o(t^2)$$
$$\pi_t = m + \eta_t$$
$$v_t = 1 + O(t)$$

(2.2)

(2.3)

where $\|\eta_t\| = O(t)$.

Dividing the eigenfunctions $v_t$ by $m(v_t)$ which do not vanish in a neighborhood of $t$ (and multiplying $\pi_t$ by the same value) we may assume that $m(v_t) = 1$.

We also need the fact that $v_0'$ is a purely imaginary function. To see this note that the equality

$$P_t v_t = \lambda_t v_t$$

implies

$$P_t' v_0 + P_0 v_0' = \lambda_t v_0 + \lambda_0 v_0'$$

Since $v_0 = 1$ we obtain

$$P_t' 1 = (I - P_0) v_0'$$

Now, $P_0'(1) = \hat{T} \, (i \varphi)$ is purely imaginary, since $\hat{T}f$ is real if $f$ is real. Moreover, $m(\varphi) = 0$ implies $m(P_0' 1) = 0$.

By corollary 3.6 in [7], the equation $P_0' 1 = (I - \hat{T}) f$, $m(f) = 0$ has a unique solution. Since $v_0'$ is the solution to this equation ($m(v_t) \equiv 1 \Rightarrow m(v_t') \equiv 0$) it follows that $v_0'$ is purely imaginary.

In what follows, we restrict $P_t$ to act on $L$.

**Definition 3.** A measurable function $\varphi : X \rightarrow \mathbb{Z}$ is aperiodic if there is no non-trivial character $\gamma \in \hat{\mathbb{Z}}$, such that $\gamma \circ \varphi$ is $T$-cohomologous to a constant, i.e. the only solution to the equation

$$e^{i\lambda \varphi} = \frac{f \circ T}{f}$$

with $f : X \rightarrow \mathbb{T}$ measurable, is $t \in 2\pi \mathbb{Z}$, $f \equiv 1$, $\lambda = 1$, $f \equiv 1$. $\varphi$ is periodic if it is not aperiodic.

**Remark 4.** If $\varphi$ is aperiodic then the characteristic function operator $P_t$ defined by (2.1) has spectral radius strictly less than 1 for all $t \notin 2\pi \mathbb{Z}$. By continuity of $P_t$, this implies that in every compact set $K \subseteq \mathbb{R} \setminus 2\pi \mathbb{Z}$, there exists a constant $q_K < 1$, such that $\|P_t^n\| \leq q_K^n$ for all sufficiently large $n$.

### 3. Probability Estimates

Throughout this section we assume that the conditions of theorem [1] hold (hence, all results of the previous section also hold).

**Proposition 5.** There exists a constant $C$ such that for any $x \in \mathbb{Z}$, $\sqrt{n} \cdot m(S_n = x) < C$. 

Proof. By the inversion formula for Fourier transform and definition of the characteristic function operator,

\[ m(S_n = x) = \text{Re} \int_{[-\pi, \pi]} m(e^{itS_n}) e^{-itx} dt \]

\[ = \text{Re} \int_{[-\pi, \pi]} m(P_t^n) e^{-itx} dt. \]

By (2.2) there exist a \( 0 < \delta < \pi \) such that \( P_t = \lambda_t \pi_t + N_t \) where \( \lambda_t = 1 - \sigma t^2 + \epsilon(t) \), where \( |\epsilon(t)| \leq \epsilon t^2 \) for some \( \epsilon \) satisfying \( c := \sigma - \epsilon > 0 \), and the spectral radius of \( N_t \) satisfies \( \rho(N_t) \leq q < 1 \) for all \( t \in (-\delta, \delta) \). Write \( C_\delta = (-\delta, \delta) \) and \( C_\delta = [-\pi, \pi] \setminus (-\delta, \delta) \). Then

\[ \text{(3.1)} \]

\[ \text{Re} \int_{[-\pi, \pi]} m(P_t^n) e^{-itx} dt \leq \int_{C_\delta} \| P_t^n \|_L dt + \int_{C_\delta} \| P_t^n \|_L dt. \]

Now, by remark \[ \sup_{t \in C_\delta} |P_t^n| |L| \] exponentially tends to 0. Hence, the second term on the right side of the above inequality multiplied by \( \sqrt{n} \) tends to 0 as \( n \) tends to \( \infty \) and in particular, is uniformly bounded. To bound the first term, write

\[ \| P_t^n \|_L \leq |\lambda|^n + \| N_t^n \| \leq (1 - c t^2)^n + c^n \]

for some constant \( c \), which exists since \( \rho(N_t) \leq q \) on \( C_\delta \). Then

\[ \int_{C_\delta} \| P_t^n \|_L dt \leq \int_{C_\delta} (1 - c t^2)^n dt + 2c^n \]

and by applying the substitution \( t = \frac{\sqrt{n}}{c} \), we obtain

\[ \int_{C_\delta} (1 - c t^2)^n dt = \frac{1}{\sqrt{n}} \int_{(-\sqrt{n}, \sqrt{n})} \left( 1 - \frac{c y^2}{n} \right)^n \leq \frac{1}{\sqrt{n}} \int_{-\infty}^{\infty} e^{-cy^2} dy \]

Since, the last integral converges, \( \sqrt{n} \int_{C_\delta} (1 - c t^2)^n dt \) is uniformly bounded by a constant. Since, the second term on the right hand side of the inequality \[ \text{(3.1)} \] tends to 0 exponentially fast, this completes the proof. \( \square \)

Remark 6. Note that during the proof, we showed that \( \sqrt{n} \int_{C_\delta} |\lambda|^n dt \) is uniformly bounded by a constant. Essentially the same proof may be used to show that \( n \int_{C_\delta} |t\lambda^t| dt \) is uniformly bounded by a constant. We use both these facts in the proof of the next proposition.

Proposition 7. For all \( x, y \in \mathbb{Z} \), there exists a constant \( C \) such that \( \sum_{n=1}^{\infty} |m(S_n = x) - m(S_n = y)| \leq C |x - y| \).

Proof. By the inversion formula,

\[ |m(S_n = x) - m(S_n = y)| = \left| \text{Re} \int_{[-\pi, \pi]} m(P_t^n) (e^{itx} - e^{ity}) dt \right|. \]

By (2.2) there exist a \( 0 < \delta < \pi \) such that \( P_t = \lambda_t \pi_t + N_t \) where \( |\lambda_t| \leq 1 - c t^2 \) for some positive constant, the spectral radius of \( N_t \) satisfies \( \rho(N_t) \leq q < 1 \) for all \( t \in (-\delta, \delta) \), and \( \pi_t = m + \eta_t \) with \( \|\eta_t\| \leq \tilde{c} t \) for some \( \tilde{c} \geq 0 \). Write \( C_\delta = (-\delta, \delta) \) and \( C_\delta = [-\pi, \pi] \setminus (-\delta, \delta) \). Then

\[ \left| \text{Re} \int_{[-\pi, \pi]} m(P_t^n) (e^{itx} - e^{ity}) dt \right| \leq \left| \text{Re} \int_{C_\delta} m(P_t^n) (e^{itx} - e^{ity}) dt \right| + \left| \int_{C_\delta} m(P_t^n) dt \right|. \]

As in the proof of proposition \[ \square \] the second term on right side of the above inequality tends to 0 exponentially fast and therefore, its sum over \( n \) converges. Thus, it is sufficient to bound the first term. Use the expansion of the characteristic function operator to get
| Re \int_{C_1} m(P^n_t \mathbf{1}) (e^{itx} - e^{ity}) dt | \leq | \int_{C_1} \lambda^n_t m(\pi_t \mathbf{1}) (e^{itx} - e^{ity}) dt | + 2 \cdot \int_{C_1} \|N_t\|^n dt |.

Since \( \rho(N_t) \leq q < 1 \), the sum over \( n \) of the second term on the right hand side is finite. We turn to analyze the first term.

\[
\left| \int_{C_1} \lambda^n_t m(\pi_t \mathbf{1}) (e^{itx} - e^{ity}) dt \right| = \left| \int_{C_1} (\Re \lambda^n_t m(\pi_t \mathbf{1})) (\cos tx - \cos ty) dt \right| + \left| \int_{C_1} (\Im \lambda^n_t m(\pi_t \mathbf{1})) (\sin tx - \sin ty) dt \right|
\]

(3.2)

Since \( |\Re \lambda^n_t| \leq |\lambda^n_t| \leq 1 - ct^2 \), and \( \|\pi_t\|_L = 1 \), we have

\[
\sum_{n=1}^{\infty} \left| \int_{C_1} (\Re \lambda^n_t m(\pi_t \mathbf{1})) (\cos tx - \cos ty) dt \right| \leq \sum_{n=1}^{\infty} \int_{C_1} (1 - ct^2)^n |\cos tx - \cos ty| dt
\]

\[
= \int_{C_1} \frac{1}{ct^2} |\cos tx - \cos ty| dt 
\]

\[
\leq C_1 |x - y|
\]

for some constant \( C_1 \).

Estimating the sum over the second term in (3.2) is more difficult since \( \sin tx - \sin ty \) is of order \( t \) instead of \( t^2 \). We start by using \( \pi_t = m + \eta_t \) to obtain

\[
\left| \int_{C_1} (\Im \lambda^n_t m(\pi_t \mathbf{1})) (\sin tx - \sin ty) dt \right| \leq \int_{C_1} |\Im \lambda^n_t| |\sin tx - \sin ty| dt + \int_{C_1} |\lambda^n_t \hat{c}t| |\sin tx - \sin ty| dt.
\]

Using \( |\lambda_t| \leq 1 - ct^2 \) we can estimate the second term on the right hand side of the above inequality.

\[
\sum_{n=1}^{\infty} \int_{C_1} |\lambda^n_t \hat{c}t| |\sin tx - \sin ty| dt \leq \hat{c} \int_{C_1} \frac{1}{ct} |\sin tx - \sin ty| dt \leq C_2 |x - y|.
\]

The estimation of the first term on the right hand side of (3.3) will take up the rest of the proof.

We first note that \( |\Im \lambda^n_t| \leq n |\lambda_t|^{n-1} |\Im \lambda_t| \). Then

\[
|\Im \lambda_t| = |m(\Im P_t \psi_t)| \leq |m(\Im P_t \mathbf{1})| + |m(\Im P_t \psi_t)|,
\]

where \( \psi_t = 1 - \nu_t \). By definition of the characteristic function operator, and the fact the \( \hat{T} f \) is real if \( f \) is real,

\[
|m(\Im P_t \psi_t)| \leq \left| m \left( \hat{T} (\cos t\varphi \Im \psi_t) \right) \right| + \left| m \left( \hat{T} (\sin t\varphi \Re \psi_t) \right) \right|.
\]

Since \( m(\psi_t) = 0 \), \( m \circ \hat{T} = m \), \( |1 - \cos t\varphi| \leq t^2 \varphi^2 \), \( |\psi_t| = o(|t|) \) and by the positivity of the transfer operator,

\[
\left| m \left( \hat{T} (\cos t\varphi \Im \psi_t) \right) \right| = \left| m \left( (\cos t\varphi - 1) \Im \psi_t \right) \right| \leq m \left( t^2 \varphi^2 |\Im \psi_t| \right) \leq C_3 |t|^3
\]

where we have used the finiteness of the second moment of \( \varphi \).
Finally, since $\psi_0 = 0$, $\text{Re} \psi_0' = 0$ and $\psi_t$ is twice continuously differentiable,

$$\left| m \left( \tilde{T} (\sin t \varphi \text{Re} \psi_t) \right) \right| \lesssim C_4 |t|^3.$$ 

Therefore,

$$\sum_{n=1}^{\infty} \int_{C_\delta} n |\lambda_t|^n |m(\text{Im} P_t \psi_t)| |\sin t x - \sin t y| dt \lesssim \sum_{n=1}^{\infty} \int_{C_\delta} n \left( 1 - ct^2 \right)^{n-1} \left( C_3 + C_4 \right) |t|^3 |\sin t x - \sin t y| dt$$

$$\lesssim \int_{C_\delta} \frac{1}{c t^4} \left( C_3 + C_4 \right) |t|^3 |\sin t x - \sin t y| dt \lesssim C_5 |x - y|$$

Finally, since $m(\varphi) = 0$ and $m \circ \tilde{T} = m$

$$|m(\text{Im} P_t 1)| = |m(\sin t \varphi)| = |m(\sin t \varphi - t \varphi)|$$

We split the last integral into parts where $|t \varphi| \leq 1$ and $|t \varphi| > 1$ to obtain

$$|m(\text{Im} P_t 1)| \lesssim m \left( 1_{\{|t \varphi| \leq 1\}} (\sin t \varphi - t \varphi) \right) + m \left( 1_{\{|t \varphi| > 1\}} \right) + m \left( 2 |t \varphi| 1_{\{|t \varphi| > 1\}} \right)$$

Thus, summing over $n$ and again using $|\lambda_t| \leq (1 - c t^2)$ we have

$$\sum_{n=1}^{\infty} \int_{C_\delta} n |\lambda_t|^n |m(\text{Im} P_t 1)| |\sin t x - \sin t y| dt \lesssim \int_{C_\delta} \frac{1}{c t^4} m \left( |t \varphi|^3 1_{\{|t \varphi| \leq 1\}} \right) |\sin t x - \sin t y| dt +$$

$$+ 2 \int_{C_\delta} \frac{1}{c t^4} m \left( |t \varphi| 1_{\{|t \varphi| > 1\}} \right) |\sin t x - \sin t y| dt$$

Bounding $|\sin t x - \sin t y|$ by $|t (x - y)|$ and changing the order of integration in the first term gives

$$\int_{C_\delta} \frac{1}{c t^4} m \left( |t \varphi|^3 1_{\{|t \varphi| \leq 1\}} \right) |\sin t x - \sin t y| dt \lesssim m \left( |t \varphi| \int_{|t \varphi| > 1} |x - y| dt \right)$$

$$\lesssim m \left( 2 |t \varphi|^2 \right) |x - y| \lesssim C_6 |x - y|$$

Changing the order of integration in the second term of [3.3] and using the fact the the integrand is an even function of $t$, gives

$$2 \int_{C_\delta} \frac{1}{c t^4} m \left( |t \varphi| 1_{\{|t \varphi| > 1\}} \right) |\sin t x - \sin t y| dt \lesssim 4 m \left( |t \varphi| \int_{|t \varphi| > 1} \frac{1}{t^2} |x - y| dt \right)$$

$$\lesssim C_7 |x - y|.$$ 

This completes the proof.

**Proposition 8.** For any $\epsilon > 0$, $1 < \alpha < 2$ there exists a constant $C$ such that for all $x, y \in \mathbb{R}$, $\frac{1}{\sqrt{n}} \leq |x - y| \leq 1$ and $n \in \mathbb{N}$, $\text{P} \left( |l_n(x) - l_n(y)| > \epsilon \right) \leq C \frac{|x - y|^\alpha}{\epsilon^\alpha}.$

To prove this estimate, let $L_n(x) = \# \{ 1 \leq k \leq n | S_k = x \}$, $x \in \mathbb{Z}$. Then by definition $l_n(x) = \frac{L_n(\sqrt{n}x)}{\sqrt{n}}$. It is enough to prove
Proposition 9. For any $1 < \alpha < 2$ there exists a constant $C$ such that for all $x, y \in \mathbb{Z}$ and $n \in \mathbb{N}$, $m \left( (L_n (x) - L_n (y))^6 \right) \leq C \cdot \left( (\sqrt{n} |x - y|)^3 + \sqrt{n}^4 |x - y| \log n + \sqrt{n}^5 \log n^2 \right)$. 

To see that proposition 8 follows from this, note that

$$m \left( (L_n (x) - L_n (y))^6 \right) = \frac{1}{\sqrt{n}} m \left( (L_n \left( (\sqrt{n}x) \right) - L_n \left( (\sqrt{n}y) \right))^6 \right)$$

$$\leq \frac{1}{n^3} C \left( n^3 |x - y|^3 + |x - y|^2 n^2 \log n + n^2 \log n^2 \right)$$

$$\leq C \left( |x - y|^3 + \frac{1}{n} |x - y|^2 \log n + \frac{\log n}{n |x - y|^\alpha} \right)$$

$$\leq \tilde{C} |x - y|^\alpha$$

for any $1 < \alpha < 2$ ($\tilde{C}$ of course depends on $\alpha$). The last inequality holds since $\frac{1}{\sqrt{n}} \leq |x - y| \leq 1$. Proposition 8 now follows from Markov’s inequality.

We turn to the proof of proposition 9. Using definition of $L_n (x)$ and writing $\psi(z) = 1_{\{x\}} - 1_{\{y\}}$, we obtain

$$m \left( (L_n (x) - L_n (y))^{2p} \right) = m \left( \left( \sum_{k=1}^n 1_{\{x\}} (S_k) - 1_{\{y\}} (S_k) \right)^{2p} \right) = \sum_{i \in I} \left( \prod_{l=1}^{2p} \psi (S_{i_l}) \right)$$

where $I$ is the set of all tuples of length $2p$ of integers between 1 and $n$. Clearly, it is enough to prove the estimate for the case where the coordinates in $I$ are not decreasing. Therefore, we denote $J = \{(j_1, ..., j_{2p}) | j_1, ..., j_{2p} \in \{1, ..., n\} \}$ and estimate

$$\sum_{j \in J} m \left( \prod_{l=1}^{2p} \psi (S_{j_l}) \right).$$

Fix $j \in J$, and let $\tilde{k} = (j_1, j_2 - j_1, ..., j_{2p} - j_{2p-1})$. Then

$$\prod_{l=1}^{2p} \psi (S_{i_l}) = \sum_{z_1, z_2, ..., z_{2p}} m \left( \prod_{l=1}^{2p} \psi (z_l) 1_{\{z_l - z_{l-1}\}} (S_{k_l}) \right)$$

where the sum goes over all $\tilde{z} = \{(z_1, ..., z_{2p}) | z_i \in \{x, y\}, i = 1, ..., 2p \}$ and $z_0 = 0$. Summing over $z$’s having even subscripts we obtain

$$\prod_{l=1}^{2p} \psi (S_{i_l}) = \sum_{z_1, z_2, ..., z_{2p-1}} m \left( \psi (z_1) 1_{\{z_1\}} (S_{k_1}) \left( \prod_{l=1}^{p-1} \psi (z_{2l-1}) h (l, z_{2l-1}, z_{2l+1}) \right) h (p, z_{2p-1}) \right)$$

where

$$h (l, u, v) = 1_{\{x-u\}} (S_{k_{2l+1}}) 1_{\{v-x\}} (S_{k_{2l+1}}) - 1_{\{y-u\}} (S_{k_{2l+1}}) 1_{\{v-y\}} (S_{k_{2l+1}})$$

and

$$h (l, u) = 1_{\{x-u\}} (S_{2l}) - 1_{\{y-u\}} (S_{2l}).$$

We can now take absolute values and write

$$(3.5) \quad \prod_{l=1}^{2p} \psi (S_{i_l}) \leq \sum_{z_1, z_2, ..., z_{2p}} m \left( 1_{\{z_1\}} (S_{k_1}) \left( \prod_{l=1}^{p-1} h (l, z_{2l-1}, z_{2l+1}) \right) h (p, z_{2p-1}) \right).$$
Adding and subtracting $1_{(x-u)}(S_{k_2})1_{(u-v)}(S_{k_2+1})$ from $h(l,u,v)$ we have

$$h(l,u,v) = 1_{(x-u)}(S_{k_2})\left(1_{(v-x)}(S_{k_2+1}) - 1_{(v-y)}(S_{k_2+1})\right)$$

and the spectral radius of $N_t$ satisfies $\rho(N_t) \leq q < 1$ for all $t \in (-\delta, \delta)$. We denote $C_\delta = (-\delta, \delta)$ and

At this point we use the inversion formula for the Fourier transform to estimate (3.5). To do this, let $\hat{t} = (t_1, \ldots, t_{2p})$ and write

$$\hat{h}_1(l, u, v) = e^{it_2(x-u)} \left(e^{-it_{2l+1}(z_{2l+1}-x)} - e^{-it_{2l+1}(z_{2l+1}-y)}\right)$$

$$\hat{h}_2(l, u, v) = \left(e^{it_2(y-u)} - e^{it_2(y-v)}\right) e^{it_{2l+1}(v-y)}$$

$$\hat{h}(l, u) = e^{it_2(x-u)} - e^{it_2(y-u)}.$$

Fix $z_1, z_3, \ldots, z_p$. Then by the inversion formula

$$m \left(1_{\{z_i\}}(S_{k_i}) \left(\prod_{l=1}^{p-1} h(l, z_{2l-1}, z_{2l+1})\right) h(p, z_{2p-1})\right) =$$

$$\text{Re} \int_{[-\pi,\pi]^{2p}} m \left(e^{i(t.S_k)}\right) e^{it_1z_1} \left(\prod_{l=1}^{p-1} \hat{h}_1(l, z_{2l-1}, z_{2l+1})\right) \hat{h}(p, z_{2p-1}) dt_1 \ldots dt_{2p}$$

$$\text{Re} \int_{[-\pi,\pi]^{2p}} m \left(e^{i(t.S_k)}\right) e^{it_1z_1} \left(\prod_{l=1}^{p-1} \hat{h}_1(l, z_{2l-1}, z_{2l+1})\right) \hat{h}(p, z_{2p-1}) dt_1 \ldots dt_{2p}$$

The next proposition completes the proof:

**Proposition 10.** Let $x,y \in \mathbb{Z}$, $\tilde{z} = (z_1, \ldots, z_p)$, $\tilde{w} = (w_1, \ldots, w_p)$ be two vectors with integer coordinates such that $z_i - w_i = x - y$ and let $\tilde{k}$ be a $p$-tuple of nonnegative integers. Also, let $\tilde{\xi}$ be a vector with the $i$-th coordinate being equal to either $e^{it_iz_i} - e^{it_1w_i}$ or $e^{it_iz_i}$. Denote by $J$ the set of coordinates $1 \leq i \leq p$ such that the $\xi_i = e^{it_iz_i} - e^{it_1w_i}$ and by $\tilde{J}$ the set of coordinates $1 \leq i \leq p$ with $\xi_i = e^{it_iz_i}$. Then

$$\sum_{1 \leq k_1 \leq \ldots \leq k_p \leq n} \left| \text{Re} \int_{[-\pi,\pi]^{2p}} m \left(e^{i(t.S_k)}\right) \prod_{l=1}^{p} \xi_l dt_1 \ldots dt_p \right| \leq C_p \sqrt{|x-y|^{\#J}} + \sqrt{|x-y|^{\#J+1}} \log n + \sqrt{n}^{p-2} (\log n)^2$$

where $C_p$ is a constant.

**Proof.** We may assume that $|x-y| \leq \sqrt{n}$ (since certainly $|x-y|$ is bounded by constant times $\sqrt{n}$). By definition of the characteristic function operator

$$m \left(e^{i(t.S_k)}\right) = m \left(\prod_{l=1}^{p} P_{t_l}^{k_1} \ldots P_{t_{p-1}}^{k_{p-1}} 1\right).$$

By (2.2) there exist a $0 < \delta < \pi$ such that $P_t = \lambda_t m + \lambda_t \eta_t + N_t$ where $\lambda_t \leq 1 - ct^2$ for some positive constant $c$, $\|\eta_t\| \leq c |t|$ and the spectral radius of $N_t$ satisfies $\rho(N_t) \leq q < 1$ for all $t \in (-\delta, \delta)$. We denote $C_\delta = (-\delta, \delta)$ and
\( C \delta = [-\pi, \pi] \setminus (-\delta, \delta). \) Thus,

\[
\text{Re} \int_{[-\pi, \pi]^p} m \left( e^{i(t, S_k)} \right) \prod_{l=1}^{p} \xi_l dt_1 \ldots dt_p = \text{Re} \int_{[-\pi, \pi]^{p-1} C \delta} \left[ \int \chi^{k_p}_{t_p} m \left( P_{t_p-1}^{k_p-1} \ldots P_{t_1}^{k_1} \right) \prod_{l=1}^{p} \xi_l dt_1 \ldots dt_p \right] + \text{Re} \int_{[-\pi, \pi]^{p-1} C \delta} \left[ \int m \left( \chi^{k_p}_{t_p} \eta_p P_{t_p-1}^{k_p-1} \ldots P_{t_1}^{k_1} \right) \prod_{l=1}^{p} \xi_l dt_1 \ldots dt_p \right] + \text{Re} \int_{[-\pi, \pi]^{p-1} C \delta} \left[ \int m \left( \lambda_{t_p}^{k_p} P_{t_p-1}^{k_p-1} \ldots P_{t_1}^{k_1} \right) \prod_{l=1}^{p} \xi_l dt_1 \ldots dt_p \right]
\]

We handle each of the terms on the right hand side separately. Since \( \int_{[-\pi, \pi]^{p-1}} m \left( P_{t_p-1}^{k_p-1} \ldots P_{t_1}^{k_1} \right) \prod_{l=1}^{p-1} \xi_l dt_1 \ldots dt_p \) is a difference of inverse Fourier transforms and therefore real,

\[
\text{Re} \int_{[-\pi, \pi]^{p-1} C \delta} \left[ \int \chi^{k_p}_{t_p} m \left( P_{t_p-1}^{k_p-1} \ldots P_{t_1}^{k_1} \right) \prod_{l=1}^{p} \xi_l dt_1 \ldots dt_p \right] = \left( \text{Re} \int_{[-\pi, \pi]^{p-1}} m \left( P_{t_p-1}^{k_p-1} \ldots P_{t_1}^{k_1} \right) \prod_{l=1}^{p-1} \xi_l dt_1 \ldots dt_p \right) \cdot \text{Re} \int_{C \delta} \left[ \int \chi^{k_p}_{t_p} \xi_\delta dt_p \right]
\]

If \( p \in J \), by the proof of the potential kernel estimate (proposition \( \square \)) \( \sum_{1 \leq k_p \leq n} \left| \text{Re} \int_{C \delta} \chi^{k_p}_{t_p} \xi_\delta dt_p \right| \leq C_1 |x - y| \).

Otherwise by remark \( \square \) this term is bounded by \( C_1 \sqrt{k_p} \) and \( \sum_{1 \leq k_p \leq n} \frac{C_1 \sqrt{k_p}}{k_p} \leq C_2 \sqrt{n} \). To estimate

\[
\text{Re} \int_{[-\pi, \pi]^{p-1}} m \left( P_{t_p-1}^{k_p-1} \ldots P_{t_1}^{k_1} \right) \prod_{l=1}^{p-1} \xi_l dt_1 \ldots dt_p
\]

we may use the induction hypothesis. Combining the two estimates we obtain

\[
\sum_{1 \leq k_1, \ldots, k_p \leq n} \left| \text{Re} \int_{[-\pi, \pi]^{p-1} C \delta} \left[ \int \chi^{k_p}_{t_p} m \left( P_{t_p-1}^{k_p-1} \ldots P_{t_1}^{k_1} \right) \prod_{l=1}^{p} \xi_l dt_1 \ldots dt_p \right] \right| \leq C_2 |x - y|^J \sqrt{n}^{#J} + \sqrt{n}^{#J+1} |x - y|^J \log n + \sqrt{n}^{p-2} (\log n)^2.
\]
We now turn to the second term in $(3.6)$. Expanding one more term in the integral we get

\[ \text{Re} \int_{[\pi, \pi]} \int_{C_{\delta}} m \left( \lambda_{t_p} \eta_p, P_{t_{p-1}}^{k_p-1} \cdots P_{t_1}^{k_1} \right) \prod_{l=1}^{p} \xi_l \, dt_p \cdots \, dt_1 \]

(3.7)

\[ = \text{Re} \int_{[\pi, \pi]} \int_{C_{\delta}} m \left( \lambda_{t_p} \eta_p \lambda_{t_{p-1}}^{k_{p-1}} \cdots \lambda_{t_1}^{k_1} \right) \prod_{l=1}^{p} \xi_l \, dt_p \cdots \, dt_1 \]

\[ + \text{Re} \int_{[\pi, \pi]} \int_{C_{\delta}} m \left( \lambda_{t_p} \eta_p \lambda_{t_{p-1}}^{k_{p-1}} \cdots \lambda_{t_1}^{k_1} \right) \prod_{l=1}^{p} \xi_l \, dt_p \cdots \, dt_1 \]

\[ + \text{Re} \int_{[\pi, \pi]} \int_{C_{\delta}} m \left( \lambda_{t_p} \eta_p \lambda_{t_{p-1}}^{k_{p-1}} \cdots \lambda_{t_1}^{k_1} \right) \prod_{l=1}^{p} \xi_l \, dt_p \cdots \, dt_1 \]

\[ + \text{Re} \int_{[\pi, \pi]} \int_{C_{\delta}} m \left( \lambda_{t_p} \eta_p \lambda_{t_{p-1}}^{k_{p-1}} \cdots \lambda_{t_1}^{k_1} \right) \prod_{l=1}^{p} \xi_l \, dt_p \cdots \, dt_1 \]

Since $\| \eta \| \leq c \| t \|$, by remark 12 we get that $\int_{C_{\delta}} \left\| \lambda_{t_p} \eta_p \right\| \, dt_p \leq \frac{C_{\delta}}{k_p}$, and hence $\sum_{k_p=1}^{n} \int_{C_{\delta}} \lambda_{t_p}^{k_p} \eta_p \, dt_p \leq \log n$. Thus, using also $\int_{C_{\delta}} \lambda_{t_p}^{k_p-1} \, dt_p \leq \frac{C_{\delta}}{\sqrt{k_p-1}}$

\[ \sum_{1 \leq k_1, \ldots, k_p \leq n} \left| \text{Re} \int_{[\pi, \pi]} \int_{C_{\delta}} m \left( \lambda_{t_p}^{k_p} \eta_p \lambda_{t_{p-1}}^{k_{p-1}} \cdots \lambda_{t_1}^{k_1} \right) \prod_{l=1}^{p} \xi_l \, dt_p \cdots \, dt_1 \right| \]

\[ \leq \sqrt{n} \log n \sum_{1 \leq k_1, \ldots, k_p \leq n} \left| \int_{[\pi, \pi]} m \left( P_{t_{p-2}}^{k_{p-2}} \cdots P_{t_1}^{k_1} \right) \prod_{l=1}^{p} \xi_l \, dt_p \cdots \, dt_1 \right| \]

\[ \leq C_2 \left( \sqrt{n}^{J+1} |x-y|^{J-2} \log n + \sqrt{n}^{p-2} (\log n)^2 \right) . \]

Using the same method and $\int_{[\pi, \pi]} \| P_k \| \, dt \leq \frac{C_{\delta}}{\sqrt{n}}$ we obtain

\[ \sum_{1 \leq k_1, \ldots, k_p \leq n} \left| \text{Re} \int_{[\pi, \pi]} \int_{C_{\delta}} m \left( \lambda_{t_p}^{k_p} \eta_p \lambda_{t_{p-1}}^{k_{p-1}} \cdots \lambda_{t_1}^{k_1} \right) \prod_{l=1}^{p} \xi_l \, dt_p \cdots \, dt_1 \right| \leq C_3 \left( \sqrt{n}^{p-2} (\log n)^2 \right) . \]

Keeping in mind that $\| P_k \|$ for $t \in \mathcal{C}_{\delta}$ and $\| N_t \|$ for $t \in C_{\delta}$ uniformly tend to 0 with an exponential rate we obtain the bound $C_3 \left( \sqrt{n}^{p-2} (\log n)^2 \right)$ for the third and fourth term on the right hand side of $(3.7)$ (actually we obtain a better bound, but we do not use it).

Combination of the estimates above proves the result.
4. Tightness of \( l_n \) in \( D \).

A sequence \( \{X_n\} \) of random variables taking values in a standard Borel Space \((X, \mathcal{B})\) is called tight if for every \( \epsilon > 0 \) there exists a compact \( K \subset X \) such that for every \( n \in \mathbb{N} \),

\[
P_n(K) > 1 - \epsilon,
\]

where \( P_n \) denotes the distribution of \( X_n \). By Prokhorov’s Theorem relative compactness of \( t_n(x) \) in \( D \) is equivalent to tightness. Therefore we are interested in characterizing tightness in \( D \).

For \( x(t) \) in \( D_{[-h,h]} \subseteq [-h,h] \) set

\[
\omega_x(T) = \sup_{s,t \in T} |x(s) - x(t)|
\]

and

\[
\omega_x(\delta) := \sup_{|s-t| < \delta} |x(s) - x(t)|.
\]

\( \omega_x(\delta) \) is called the modulus of continuity of \( x \). Due to the Arzela - Ascoli theorem, it plays a central role in characterizing tightness in the space \( C[-h,h] \) of continuous functions on \([-h,h]\), with a Borel \( \sigma \)-algebra generated by the topology of uniform convergence.

The function that plays in \( D_{[-h,h]} \) the role that the modulus of continuity plays in \( C[-h,h] \) is defined by

\[
\omega'_x(\delta) = \inf_{\{t_i\}} \max_{1 \leq i \leq v} \omega((t_i, t_{i+1})),
\]

where \( \{t_i\} \) denotes a \( \delta \) sparse partition of \([-h,h]\), i.e. \( \{t_i\} \) is a partition \(-h = t_1 < t_2 < \ldots < t_{v+1} = h \) such that \( \min_{1 \leq i \leq v} |t_{i+1} - t_i| > \delta \). It is easy to check that if \( \frac{1}{h} > \delta > 0 \), and \( h \geq 1 \),

\[
\omega'_x(\delta) \leq \omega_x(2\delta).
\]

For details see [3, Sections 12 and 13]. The next theorem is a characterization of tightness in the space \( D \).

**Theorem 11.** [3, Lemma 3, p.173] (1) The sequence \( l_n \) is tight in \( D \) if and only if its restriction to \([-h,h]\) is tight in \( D_{[-h,h]} \) for every \( h \in \mathbb{R}_+ \).

(2) The sequence \( l_n \) is tight in \( D_{[-h,h]} \) if and only if the following two conditions hold:

(i) \( \forall x \in [-h,h], \lim_{a \to \infty} \limsup_{n \to \infty} m(||l_n(x)|| \geq a) = 0 \).

(ii) \( \forall \epsilon > 0, \lim_{\delta \to 0} \limsup_{n \to \infty} m(\omega'_n(\delta) \geq \epsilon) = 0 \).

**Remark 12.** See [3, Thm. 13.2] and the Corollary that follows. Conditions (i) and (ii) of the previous theorem imply that

\[
\lim_{a \to \infty} \limsup_{n \to \infty} m \left[ \sup_{x \in [-m,m]} |l_n(x)| \geq a \right] = 0.
\]

**Proposition 13.** The sequence \( \{l_n\}_{n=1}^\infty \) is tight.

**Proof.** We prove that condition (2i) holds.

Fix \( \epsilon > 0, x \in \mathbb{R} \). Since the Brownian Motion \( \omega(t) \) satisfies

\[
\lim_{M \to \infty} P \left( \sup_{t \in [0,1]} |\omega(t)| > M \right) = 0
\]

and \( \omega_n \) converges in distribution to \( \omega \) there are \( M, n_0 \) such that for all \( n > n_0 \),
\[
m\left(\sup_{t \in [0,1]} |W_n(t)| > M\right) < \epsilon.
\]

By definition of \(t_n(x)\), it follows that if \(|x| > M\), \(n > n_0\),
\[
m(|l_n(x)| > 0) < \epsilon.
\]

Now, if \(|x| \leq M\), by proposition [13],
\[
m\left(|\ln(x) - l_n(M + 1)| > a\right) \leq m\left(|\ln(x) - l_n(M + 1)| > a\right) + m\left(|\ln(x) - l_n(M + 1)| > a\right)
\]

and the last expression can be made less than \(2\epsilon\) for sufficiently large \(a\).

To prove condition 2(ii) WLOG we may assume that \(m \geq 1\). Since \(\omega'_x(\delta) \leq \omega_x(2\delta)\), it is sufficient to prove that the stronger condition

\[
\forall \epsilon > 0, \lim_{\delta \to 0} \limsup_{n \to \infty} m\left[\sup_{x,y \in [-h,h],|x-y| < \delta} |l_n(x) - l_n(y)| \geq \epsilon \right] = 0
\]

holds.

Let \(\epsilon > 0, 1 < \alpha < 2\). By proposition [13] there exists \(C > 0\) such that for all \(x, y: \frac{1}{\sqrt{n}} \leq |x - y| \leq 1\)

\[
P^\mu\left(|l_n(x) - l_n(y)| > \epsilon\right) \leq \frac{C}{\epsilon^6} |x - y|^{\alpha}.
\]

Let \(\delta > 0\) and \(n > \delta^{-2}\), notice that \(l_n\) is constant on segments of the form \([\frac{j}{\sqrt{n}}, \frac{j+1}{\sqrt{n}}]\), hence

\[
m\left(\sup_{x,y \in [-h,h],|x-y| < \delta} |l_n(x) - l_n(y)| \geq 4\epsilon\right) \leq \sum_{|k\delta| \leq h} m\left(\sup_{k\delta \sqrt{n} \leq j \leq (k+1)\delta \sqrt{n}} |l_n(k\delta) - l_n\left(\frac{j}{\sqrt{n}}\right)| \geq \epsilon \right).
\]

By [3] Theorem 10.2 it follows from (4.2) that there exists \(C_2 > 0\) such that

\[
m\left(\sup_{k\delta \sqrt{n} \leq j \leq (k+1)\delta \sqrt{n}} |l_n(k\delta) - l_n\left(\frac{j}{\sqrt{n}}\right)| \geq \epsilon \right) \leq \frac{C_2}{\epsilon^6} \delta^\alpha.
\]

Therefore,

\[
m\left(\sup_{x,y \in [-h,h],|x-y| < \delta} |l_n(x) - l_n(y)| \geq 4\epsilon\right) \leq \frac{2C_2m}{\epsilon^6} \delta^{a-1} \delta \to 0.
\]

\[
\square
\]

5. **Identifying \(l\) as the Limit of A Convergent Subsequence of \(\{l_n\}_{n \in \mathbb{N}}\).**

**Proposition 14.** Assume that the sequence \(\{X_n\}\) satisfies the assumptions of theorem [4]. Let \(l_{n_k}\) be some subsequence of \(l_n\) that converges in distribution to some limit \(q\). Then \(q \overset{d}{=} l\).
Proof. Let \( G_k = \{ a_1, b_1, \ldots, a_k, b_k : a_i < b_i, i = 1, \ldots, k \} \). For \( g \in G_k \) define the transformation \( \pi_g : D \to \mathbb{R}^k \) by \( \pi_g(l) = \left( \int_{a_1}^{b_1} l(x) \, dx, \ldots, \int_{a_n}^{b_n} l(x) \, dx \right) \). Clearly, \( \mathcal{G} = \bigcup_{k=1}^{\infty} \{ \pi_g^{-1}([c_1, d_1] \times \ldots \times [c_k, d_k]) : g \in G_k, c_i, d_i \in \mathbb{R}, c_i < d_i, i = 1, \ldots, n \} \) is a \( \pi \)-system, i.e. closed under finite intersections. Moreover, \( \mathcal{G} \) generates the Borel \( \sigma \)-algebra of \( D \). It follows that if \( \pi_g(q) = \pi_g(l) \) for every \( g \in \bigcup_{k=1}^{\infty} G_k \) then \( q \equiv l \). Hence, to prove the theorem it is enough to show that if \( l_{n_k} \xrightarrow{d} q \) then \( \pi_g(l_{n_k}) \xrightarrow{d} \pi_g(l) \) for every \( g \in \bigcup_{k=1}^{\infty} G_k \). To this purpose we first prove that for \( g = [a_1, b_1] \times \ldots \times [a_k, b_k] \),

\[
\pi_g(l_n) - \left( \int_0^1 1_{[a_1, b_1]}(l_n) \, dt, \ldots, \int_0^1 1_{[a_k, b_k]}(l_n) \, dt \right) \xrightarrow{d} 0. \tag{5.1}
\]

Then we prove that

\[
\left( \int_0^1 1_{[a_1, b_1]}(l_n) \, dt, \ldots, \int_0^1 1_{[a_k, b_k]}(l_n) \, dt \right) \xrightarrow{d} \left( \int_0^1 1_{[a_1, b_1]}(l) \, dt, \ldots, \int_0^1 1_{[a_k, b_k]}(l) \, dt \right). \tag{5.2}
\]

5.1 and 5.2 imply that \( \pi_g(l_{n_k}) \xrightarrow{d} \pi_g(l) \), thus proving the proposition (see Billingsley).

We now prove 5.1. By straightforward calculations using definitions, we have

\[
\left| \int_0^1 1_{[a, b]}(\omega_n(t)) \, dt - \int_a^b l_n(x) \, dx \right| \leq \frac{|\sqrt{\pi_n} + 1|}{\sqrt{\pi_n} |\pi_n|} \int_0^1 l_n(x) \, dx + \frac{|\sqrt{\pi_n} + 1|}{\sqrt{\pi_n} |\pi_n|} \int_0^1 l_n(x) \, dx. \tag{5.3}
\]

Now,

\[
m \left( \left| \int_0^1 l_n(x) \, dx \right| > \epsilon \right) \leq m \left( \sup_{x \in [a-1, b+1]} |l_n(x)| > M \right) + m \left( \left| \int_0^1 l_n(x) \, dx \right| > \epsilon, \sup_{x \in [a-1, b+1]} |l_n(x)| \leq M \right).
\]

The second summand on the right side of the above inequality tends to 0 since the integral is less than \( \frac{M}{\sqrt{n}} \). The first summand is arbitrarily close to 0 for \( M, n \) large enough, by Remark 12. Same reasoning applied to both summands of equation 5.3 gives

\[
\left| \int_0^1 1_{[a, b]}(\omega_n(t)) \, dt - \int_a^b l_n(x) \, dx \right| \xrightarrow{d} 0. \tag{5.4}
\]

5.1 now follows from

\[
m \left( \left\| \pi_g(l_n) - \left( \int_0^1 1_{[a_1, b_1]}(l_n) \, dt, \ldots, \int_0^1 1_{[a_k, b_k]}(l_n) \, dt \right) \right\| > \epsilon \right) \leq \sum_{k=1}^{n} m \left( \left| \int_0^1 1_{[a_k, b_k]}(\omega_n(t)) \, dt - \int_a^b l_n(x) \, dx \right| > \frac{\epsilon}{k} \right)
\]

and 5.3.
We turn to the proof of 5.2. Since \( \omega_n \rightarrow^d \omega \), it is enough to show that the transformation \( \omega \rightarrow \int_0^1 1_{(a,b)}(\omega(t)) \, dt \) is continuous in the Skorokhod topology on \( D[0,1] \) at almost all sample points of the Brownian motion \( \omega \). This is proved in [10, Section 2].

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