Correlation Clustering with Asymmetric Classification Errors*

Jafar Jafarov†
University of Chicago

Sanchit Kalhan†
Northwestern University

Konstantin Makarychev†
Northwestern University

Yury Makarychev†
Toyota Technological Institute at Chicago

Abstract

In the Correlation Clustering problem, we are given a weighted graph $G$ with its edges labelled as “similar” or “dissimilar” by a binary classifier. The goal is to produce a clustering that minimizes the weight of “disagreements”: the sum of the weights of “similar” edges across clusters and “dissimilar” edges within clusters. We study the correlation clustering problem under the following assumption: Every “similar” edge $e$ has weight $w_e \in [\alpha w, w]$ and every “dissimilar” edge $e$ has weight $w_e \geq \alpha w$ (where $\alpha \leq 1$ and $w > 0$ is a scaling parameter). We give a $(3+2 \log_e (1/\alpha))$ approximation algorithm for this problem. This assumption captures well the scenario when classification errors are asymmetric. Additionally, we show an asymptotically matching Linear Programming integrality gap of $\Omega \left( \log \frac{1}{\alpha} \right)$.

1 Introduction

In the Correlation Clustering problem, we are given a set of objects with pairwise similarity information. Our aim is to partition these objects into clusters that match this information as closely as possible. The pairwise information is represented as a weighted graph $G$ whose edges are labelled as “positive/similar” and “negative/dissimilar” by a noisy binary classifier. The goal is to find a clustering $C$ that minimizes the weight of edges disagreeing with this clustering: A positive edge is in disagreement with $C$, if its endpoints belong to distinct clusters; and a negative edge is in disagreement with $C$ if its endpoints belong to the same cluster. We call this objective the MinDisagree objective. The MinDisagree objective has been extensively studied in literature since it was introduced by Bansal, Blum, and Chawla [2004] (see e.g., Charikar et al. [2003], Demaine et al. [2006], Ailon et al. [2008], Pan et al. [2015], Chawla et al. [2015]). There are currently two standard models for Correlation Clustering which we will refer to as (1) Correlation Clustering on Complete Graphs and (2) Correlation Clustering with Noisy Partial Information. In the former model, we assume that graph $G$ is complete and all edge weights are the same i.e., $G$ is unweighted. In the latter model, we do not make any assumptions on the graph $G$. Thus, edges can have arbitrary

*The conference version of this paper appeared in the proceedings of ICML 2020.
†Equal contribution. Jafar Jafarov and Yury Makarychev were supported in part by NSF CCF-1718820 and NSF TRIPODS CCF-1934843. Sanchit Kalhan and Konstantin Makarychev were supported in part by NSF TRIPODS CCF-1934931.
weights and some edges may be missing. These models are quite different from the computational perspective. For the first model, Ailon, Charikar, and Newman [2008] gave a 2.5 approximation algorithm. This approximation factor was later improved to 2.06 by Chawla, Makarychev, Schramm, and Yaroslavtsev [2015]. For the second model, Charikar, Guruswami, and Wirth [2003] and Demaine, Emanuel, Fiat, and Immorlica [2006] gave an $O(\log n)$ approximation algorithm, they also showed that Correlation Clustering with Partial Noisy Information is as hard as the Multicut problem and, hence, $O(\log n)$ is likely to be the best possible approximation for this problem. In this paper, we show how to interpolate between these two models for Correlation Clustering.

We study the Correlation Clustering problem on complete graphs with edge weights. In our model, the weights on the edges are constrained such that the ratio of the lightest edge in the graph to the heaviest positive edge is at least $\alpha \leq 1$. Thus, if $w$ is the weight of the heaviest positive edge in the graph, then each positive edge has weight in $[\alpha w, w]$ and each negative edge has weight greater than or equal to $\alpha w$. We argue that this model – which we call Correlation Clustering with Asymmetric Classification Errors – is more adept at capturing the subtleties in real world instances than the two standard models. Indeed, the assumptions made by the Correlation Clustering on Complete Graphs model are too strong, since rarely do real world instances have equal edge weights. In contrast, in the Correlation Clustering with Noisy Partial Information model we can have edge weights that are arbitrarily small or large, an assumption which is too weak. In many real world instances, the edge weights lie in some range $[a, b]$ with $a, b > 0$. Our model captures a larger family of instances.

Furthermore, the nature of classification errors for objects that are similar and objects that are dissimilar is quite different. In many cases, a positive edge $uv$ indicates that the classifier found some actual evidence that $u$ and $v$ are similar; while a negative edge simply means that the classifier could not find any such proof that $u$ and $v$ are similar, it does not mean that the objects $u$ and $v$ are necessarily dissimilar. In some other cases, a negative edge $uv$ indicates that the classifier found some evidence that $u$ and $v$ are dissimilar; while a positive edge simply means that the classifier could not find any such proof. We discuss several examples below. Note that in the former case, a positive edge gives a substantially stronger signal than a negative edge and should have a higher weight; in the latter, it is the other way around: a negative edge gives a stronger signal than a positive edge and should have a higher weight. We make this statement more precise in Section 1.1.

The following examples show how the Correlation Clustering with Asymmetric Classification Errors model can help in capturing real world instances. Consider an example from the paper on Correlation Clustering by Pan, Papailiopoulos, Oymak, Recht, Ramchandran, and Jordan [2015]. In their experiments, Pan et al. [2015] used several data sets including dblp-2011 and ENWiki-2013. In the graph $dblp-2011$, each vertex represents a scientist and two vertices are connected with an edge if the corresponding authors have co-authored an article. Thus, a positive edge with weight $w^+$ between Alice and Bob in the Correlation Clustering instance indicates that Alice and Bob are coauthors, which strongly suggests that Alice and Bob work in similar areas of Computer Science. However, it is not true that all researchers working in some area of computer science have co-authored papers with each other. Thus, the negative edge that connects two scientists who do not have an article together does not deserve to have the same weight as a positive edge, and thus can be modeled as a negative edge with weight $w^- < w^+$.

Similarly, the vertices of the graph ENWiki-2013 are Wikipedia pages. Two pages are connected

\[1\] These data sets are published by Boldi and Vigna [2004], Boldi et al. [2011, 2004, 2014]
with an edge if there is a link from one page to another. A link from one page to the other is a strong suggestion that the two pages are related and hence can be connected with a positive edge of weight $w^+$, while it is not true that two similar Wikipedia pages necessarily should have a link from one to the other. Thus, it would be better to join such pages with a negative edge of weight $w^- < w^+$.

Consider now the multi-person tracking problem. The problem is modelled as a Correlation Clustering or closely related Lifted Multicut Problem Tang et al. [2016, 2017] on a graph, whose vertices are people detections in video sequences. Two detections are connected with a positive or negative edge depending on whether the detected people have similar or dissimilar appearance (as well as some other information). In this case, a negative edge $(u, v)$ is more informative since it signals that the classifier has identified body parts that do not match in detections $u$ and $v$ and thus the detected people are likely to be different (a positive edge $(u, v)$ simply indicates that the classifier was not able to find non-matching body parts).

The Correlation Clustering with Asymmetric Classification Errors model captures the examples we discussed above. It is instructive to consider an important special case where all positive edges have weight $w^+$ and all negative edges have weight $w^-$ with $w^+ \neq w^-$. If we were to use the state of the art algorithm for Correlation Clustering on Complete Graphs on our instance for Correlation Clustering with Asymmetric Classification Errors (by completely ignoring edge weights and looking at the instance as an unweighted complete graph), we would get a $\Theta(\max(w^+/w^-, w^-/w^+))$ approximation to the MinDisagree objective. While if we were to use the state of the art algorithms for Correlation Clustering with Noisy Partial Information on our instance, we would get a $O(\log n)$ approximation to the MinDisagree objective.

**Our Contributions.** In this paper, we present an approximation algorithm for Correlation Clustering with Asymmetric Classification Errors. Our algorithm gives an approximation factor of $A = 3 + 2 \log_e 1/\alpha$. Consider the scenario discussed above where all positive edges have weight $w^+$ and all negative edges have weight $w^-$. If $w^+ \geq w^-$, our algorithm gets a $(3 + 2 \log_e w^+/w^-)$ approximation; if $w^+ \leq w^-$, our algorithm gets a 3-approximation.

**Definition 1.** Correlation Clustering with Asymmetric Classification Errors is a variant of Correlation Clustering on a Complete Graph. We assume that the weight $w_e$ of each positive edge lies in $[\alpha w, w]$ and the weight $w_e$ of each negative edge lies in $[\alpha w, \infty)$, where $\alpha \in (0, 1]$ and $w > 0$.

We note here that the assumption that the weight of positive edges is bounded from above is crucial. Without this assumption (even if we require that negative weights are bounded from above and below), the LP gap is unbounded for every fixed $\alpha$ (this follows from the integrality gap example we present in Theorem 1.3).

The following is our main theorem.

**Theorem 1.1.** There exists a polynomial time $A = 3 + 2 \log_e 1/\alpha$ approximation algorithm for Correlation Clustering with Asymmetric Classification Errors.

We also study a natural extension of our model to the case of complete bipartite graphs. That is, the positive edges across the bipartition have a weight between $[\alpha w, w]$ and the negative edges across the bipartition have a weight of at least $\alpha w$. Note that the state-of-the-art approximation algorithm for Correlation Clustering on Unweighted Complete Bipartite Graphs has an approximation factor of 3 (see Chawla et al. [2015]).
Theorem 1.2. There exists a polynomial time $A = 5 + 2\log_e 1/\alpha$ approximation algorithm for Correlation Clustering with Asymmetric Classification Errors on complete bipartite graphs.

Our next result shows that this approximation ratio is likely best possible for LP-based algorithms. We show this by exhibiting an instance of Correlation Clustering with Asymmetric Classification Errors such that integrality gap for the natural LP for Correlation Clustering on this instance is $\Omega(\log 1/\alpha)$.

Theorem 1.3. The natural Linear Programming relaxation for Correlation Clustering has an integrality gap of $\Omega(\log 1/\alpha)$ for instances of Correlation Clustering with Asymmetric Classification Errors.

Moreover, we can show that if there is an $o(\log(1/\alpha))$-approximation algorithm whose running time is polynomial in both $n$ and $1/\alpha$, then there is an $o(\log n)$-approximation algorithm for the general weighted case$^2$ (and also for the MultiCut problem). However, we do not know if there is an $o(\log(1/\alpha))$-approximation algorithm for the problem whose running time is polynomial in $n$ and exponential in $1/\alpha$. The existence of such an algorithm does not imply that there is an $o(\log n)$-approximation algorithm for the general weighted case (as far as we know).

We show a similar integrality gap result for the Correlation Clustering with Asymmetric Classification Errors on complete bipartite graphs problem.

Theorem 1.4. The natural Linear Programming relaxation for Correlation Clustering has an integrality gap of $\Omega(\log 1/\alpha)$ for instances of Correlation Clustering with Asymmetric Classification Errors on complete bipartite graphs.

Throughout the paper, we denote the set of positive edges by $E^+$ and the set of negative edges by $E^-$. We denote an instance of the Correlation Clustering problem by $G = (V, E^+, E^-)$. We denote the weight of edge $e$ by $w_e$.

1.1 Ground Truth Model

In this section, we formalize the connection between asymmetric classification errors and asymmetric edge weights. For simplicity, we assume that each positive edge has a weight of $w^+$ and each negative edge has a weight of $w^-$. Consider a probabilistic model in which edge labels are assigned by a noisy classifier. Let $C^* = (C^*_1, \ldots, C^*_n)$ be the ground truth clustering of the vertex set $V$. The classifier labels each edge within a cluster with a “+” edge with probability $p^+$ and as a “−” edge with probability $1 - p^+$; it labels each edge with endpoints in distinct clusters as a “−” edge with probability $q^-$ and as a “+” edge with probability $1 - q^-$. Thus, $(1 - p^+)$ and $(1 - q^-)$ are the classification error probabilities. We assume that all classification errors are independent.

We note that similar models have been previously studied by Bansal et al. [2004], Elsner and Schudy [2009], Mathieu and Schudy [2010], Ailon et al. [2013], Makarychev et al. [2015] and others. However, the standard assumption in such models was that the error probabilities, $(1 - p^+)$ and $(1 - q^-)$, are less than a half; that is, $p^+ > 1/2$ and $q^- > 1/2$. Here, we investigate two cases (i) when

---

$^2$The reduction to the general case works as follows. Consider an instance of Correlation Clustering with arbitrary weights. Guess the heaviest edge $e$ that is in disagreement with the optimal clustering. Let $w_e$ be its weight, and set $w = n^2w_e$ and $\alpha = 1/n^4$. Then, assign new weights to all pairs of vertices in the graph. Keep the weights of all edges with weight in the range $[\alpha w, w]$. Set the weights of all edges with weight greater than $w$ to $w$ and the weights of all edges with weight less than $\alpha w$ (including missing edges) to $\alpha w$. 

4
\( p^+ < \frac{1}{2} < q^- \) and (ii) when \( q^- < \frac{1}{2} < p^+ \). We assume that \( p^+ + q^- > 1 \), which means that the classifier is more likely to connect similar objects with a “+” than dissimilar objects or, equivalently, that the classifier is more likely to connect dissimilar objects with a “−” than similar objects. For instance, consider a classifier that looks for evidence that the objects are similar: if it finds some evidence, it adds a positive edge; otherwise, it adds a negative edge (as described in our examples dblp-2011 and ENWiki-2013 in the Introduction). Say, the classifier detects a similarity between two objects in the same ground truth cluster with a probability of only 30% and incorrectly detects similarity between two objects in different ground truth clusters with a probability of 10%. Then, it will add a negative edge between two similar objects with probability 10%! While this scenario is not captured by the standard assumption, it is captured by case (i) (here, \( p^+ = 0.3 < \frac{1}{2} < q^- = 0.9 \) and \( p^+ + q^- > 1 \)).

Consider a clustering \( C \) of the vertices. Denote the sets of positive edges and negative edges with both endpoints in the same cluster by In\(^+\) and In\(^−\), respectively, and the sets of positive edges and negative edges with endpoints in different clusters by Out\(^+\) and Out\(^−\), respectively. Then, the log-likelihood function of the clustering \( C \) is,

\[
\ell(G; C) = \log \left( \prod_{(u,v) \in \text{In}^+(C)} p^+ \times \prod_{(u,v) \in \text{In}^-(C)} (1 - p^+) \times \prod_{(u,v) \in \text{Out}^+(C)} (1 - q^-) \times \prod_{(u,v) \in \text{Out}^-(C)} q^- \right)
\]

\[
= \log \left( (p^+)^{\lvert \text{In}^+(C) \rvert} (1 - p^+)^{\lvert \text{In}^-(C) \rvert} \cdot (1 - q^-)^{\lvert \text{Out}^+(C) \rvert} q^-^{\lvert \text{Out}^-(C) \rvert} \right)
\]

\[
= |\text{In}^+(C)| \log p^+ + |\text{In}^-(C)| \log(1 - p^+) + |\text{Out}^+(C)| \log(1 - q^-) + |\text{Out}^-(C)| \log q^-
\]

\[
= \left( |E^+| \log p^+ + |E^-| \log q^- \right) - \left( |\text{Out}^+(C)| \log \frac{p^+}{1 - q^-} + |\text{In}^-(C)| \log \frac{q^-}{1 - p^+} \right).
\]

Let \( w^+ = \log \frac{p^+}{1 - q^-} \) and \( w^- = \log \frac{q^-}{1 - p^+} \). Then, the negative term \(- \left( |\text{Out}^+(C)| \log \frac{p^+}{1 - q^-} + |\text{In}^-(C)| \log \frac{q^-}{1 - p^+} \right)\) equals \( w^+ |\text{Out}^+(C)| + w^- |\text{In}^-(C)| \). Note that \( |\text{Out}^+(C)| \) is the number of positive edges disagreeing with \( C \) and \( |\text{In}^-(C)| \) is the number of negative edges disagreeing with \( C \).

Now observe that the first term in the expression above \(- \left( |E^+| \log p^+ + |E^-| \log q^- \right)\) does not depend on \( C \). It only depends on the instance \( G = (V, E^+, E^-) \). Thus, maximizing the log-likelihood function over \( C \) is equivalent to minimizing the following objective

\[
w^+ (#\text{ disagreeing “+” edges}) + w^- (#\text{ disagreeing “−” edges}).
\]

Note that we have \( w^+ > w^- \) when \( p^+ < \frac{1}{2} < q^- \) (case (i) above); in this case, a “+” edge gives a stronger signal than a “−” edge. Similarly, we have \( w^- > w^+ \) when \( q^- < \frac{1}{2} < p^+ \) (case (ii) above); in this case, a “−” edge gives a stronger signal than a “+” edge.

### 2 Algorithm

In this section, we present an approximation algorithm for Correlation Clustering with Asymmetric Classification Errors. The algorithm first solves a standard LP relaxation and assigns every edge a length of \( x_{uv} \) (see Section 2.1). Then, one by one it creates new clusters and removes them from the graph. The algorithm creates a cluster \( C \) as follows. It picks a random vertex \( p \), called a pivot, among yet unassigned vertices and a random number \( R \in [0, 1] \). Then, it adds the pivot \( p \) and all
Algorithm 1 Approximation Algorithm

input An instance of Correlation Clustering with Asymmetric Weights $G = (V, E^+, E^-, w_e)$.

Initialize $t = 0$ and $V_t = V$.

while $V_t \neq \emptyset$ do

Pick a random pivot $p_t \in V_t$.

Choose a radius $R$ uniformly at random in $[0, 1]$.

Create a new cluster $S_t$; add the pivot $p_t$ to $S_t$.

for all $u \in V_t$ do

if $f(x_{pu}) \leq R$ then

Add $u$ to $S_t$.

end if

end for

Let $V_{t+1} = V_t \setminus S_t$ and $t = t + 1$.

end while

output clustering $S = (S_0, \ldots, S_{t-1})$.

vertices $u$ with $f(x_{pu}) \leq R$ to $C$, where $f : [0, 1] \rightarrow [0, 1]$ is a properly chosen function, which we define below. We give a pseudo-code for this algorithm in Algorithm 1.

Our algorithm resembles the LP-based correlation clustering algorithms by Ailon et al. [2008] and Chawla et al. [2015]. However, a crucial difference between our algorithm and above mentioned algorithms is that our algorithm uses a “dependant” rounding. That is, if for two edges $pv_1$ and $pv_2$, we have $f(x_{pv_1}) \leq R$ and $f(x_{pv_2}) \leq R$ at some step $t$ of the algorithm then both $v_1$ and $v_2$ are added to the new cluster $S_t$. The algorithms by Ailon et al. [2008] and Chawla et al. [2015] make decisions on whether to add $v_1$ to $S_t$ and $v_2$ to $S_t$, independently. Also, the choice of the function $f$ is quite different from the functions used by Chawla et al. [2015]. In fact, it is influenced by the paper by Garg, Vazirani, and Yannakakis [1996].

2.1 Linear Programming Relaxation

In this section, we describe a standard linear programming (LP) relaxation for Correlation Clustering which was introduced by Charikar, Guruswami, and Wirth [2003]. We first give an integer programming formulation of the Correlation Clustering problem. For every pair of vertices $u$ and $v$, the integer program (IP) has a variable $x_{uv} \in \{0, 1\}$, which indicates whether $u$ and $v$ belong to the same cluster:

- $x_{uv} = 0$, if $u$ and $v$ belong to the same cluster; and
- $x_{uv} = 1$, otherwise.

We require that $x_{uv} = x_{vu}$, $x_{uu} = 0$ and all $x_{uv}$ satisfy the triangle inequality. That is, $x_{uv} + x_{vu} \geq x_{uw}$.

Every feasible IP solution $x$ defines a partitioning $S = (S_1, \ldots, S_T)$ in which two vertices $u$ and $v$ belong to the same cluster if and only if $x_{uv} = 0$. A positive edge $uv$ is in disagreement with this partitioning if and only if $x_{uv} = 1$; a negative edge $uv$ is in disagreement with this partitioning if and only if $x_{uv} = 0$. Thus, the cost of the partitioning is given by the following linear function:

$$\sum_{uv \in E^+} w_{uv}x_{uv} + \sum_{uv \in E^-} w_{uv}(1 - x_{uv}).$$
\[
\min \sum_{uv \in E^+} w_{uv} x_{uv} + \sum_{uv \in E^-} w_{uv} (1 - x_{uv}).
\]

subject to

\[
\begin{align*}
x_{uw} &\leq x_{uv} + x_{vw} & \text{for all } u, v, w \in V \\
x_{uv} &= x_{vu} & \text{for all } u, v \in V \\
x_{uu} &= 0 & \text{for all } u \in V \\
x_{uv} &\in [0, 1] & \text{for all } u, v \in V
\end{align*}
\]

Figure 1: LP relaxation

Algorithm 2 One iteration of Algorithm 1 on triangle \(uvw\)

1. Pick a random pivot \(p \in \{u, v, w\}\).
2. Choose a random radius \(R\) with the uniform distribution in \([0, 1]\).
3. Create a new cluster \(S\). Insert \(p\) in \(S\).
4. for all \(a \in \{u, v, w\} \setminus \{p\}\) do
   1. if \(f_\alpha(x_{pa}) \leq R\) then
      1. Add \(a\) to \(S\).
   end if
end for

We now replace all integrality constraints \(x_{uw} \in \{0, 1\}\) in the integer program with linear constraints \(x_{uw} \in [0, 1]\). The obtained linear program is given in Figure 1. In the paper, we refer to each variable \(x_{uw}\) as the length of the edge \(uv\).

3 Analysis of the Algorithm

The analysis of our algorithm follows the general approach proposed by Ailon, Charikar, and Newman [2008]. Ailon et al. [2008] observed that in order to get upper bounds on the approximation factors of their algorithms, it is sufficient to consider how these algorithms behave on triplets of vertices. Below, we present their method adapted to our settings. Then, we will use Theorem 3.1 to analyze our algorithm.

3.1 General Approach: Triple-Based Analysis

Consider an instance of Correlation Clustering \(G = (V, E^+, E^-)\) on three vertices \(u, v, w\). Suppose that the edges \(uv, vw, \) and \(uw\) have signs \(\sigma_{uw}, \sigma_{vw}, \sigma_{uw} \in \{\pm\}\), respectively. We shall call this instance a triangle \((u, v, w)\) and refer to the vector of signs \(\sigma = (\sigma_{vw}, \sigma_{uw}, \sigma_{uv})\) as the signature of the triangle \((u, v, w)\).

Let us now assign arbitrary lengths \(x_{uw}, x_{vw}, \) and \(x_{uw}\) satisfying the triangle inequality to the edges \(uv, vw, \) and \(uw\) and run one iteration of our algorithm on the triangle \(uvw\) (see Algorithm 2).
We say that a positive edge $uv$ is in disagreement with $S$ if $u \in S$ and $v \notin S$ or $u \notin S$ and $v \in S$. Similarly, a negative edge $uv$ is in disagreement with $S$ if $u, v \in S$. Let $\text{cost}(u, v \mid w)$ be the probability that the edge $(u, v)$ is in disagreement with $S$ given that $w$ is the pivot.

$$\text{cost}(u, v \mid w) = \begin{cases} 
\Pr(u \in S, v \notin S \text{ or } u \notin S, v \in S \mid p = w), & \text{if } \sigma_{uv} = "+"; \\
\Pr(u \in S, v \in S \mid p = w), & \text{if } \sigma_{uv} = "-". 
\end{cases}$$

Let $lp(u, v \mid w)$ be the LP contribution of the edge $(u, v)$ times the probability of it being removed, conditioned on $w$ being the pivot.

$$lp(u, v \mid w) = \begin{cases} 
x_{uv} \cdot \Pr(u \in S \text{ or } v \in S \mid p = w), & \text{if } \sigma_{uv} = "+"; \\
(1 - x_{uv}) \cdot \Pr(u \in S \text{ or } v \in S \mid p = w), & \text{if } \sigma_{uv} = "-". 
\end{cases}$$

We now define two functions $ALG^\sigma(x, y, z)$ and $LP^\sigma(x, y, z)$. To this end, construct a triangle $(u, v, w)$ with signature $\sigma$ edge lengths $x, y, z$ (where $x_{vw} = x, x_{uw} = y, x_{uw} = z$). Then,

$$ALG^\sigma(x, y, z) = w_{uv} \cdot \text{cost}(u, v \mid w) + w_{uw} \cdot \text{cost}(u, w \mid v) + w_{vw} \cdot \text{cost}(v, w \mid u);$$

$$LP^\sigma(x, y, z) = w_{uv} \cdot lp(u, v \mid w) + w_{uw} \cdot lp(u, w \mid v) + w_{vw} \cdot lp(v, w \mid u).$$

We will use the following theorem from the paper by Chawla, Makarychev, Schramm, and Yaraslavtsev [2015] (Lemma 4) to analyze our algorithm. This theorem was first proved by Ailon, Charikar, and Newman [2008] but it was not stated in this form in their paper.

**Theorem 3.1** (see Ailon et al. [2008] and Chawla et al. [2015]). Consider a function $f_\alpha$ with $f_\alpha(0) = 0$. If for all signatures $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ (where each $\sigma_i \in \{\pm\}$) and edge lengths $x, y,$ and $z$ satisfying the triangle inequality, we have $ALG^\sigma(x, y, z) \leq \rho LP^\sigma(x, y, z)$, then the approximation factor of the algorithm is at most $\rho$.

### 3.2 Analysis of the Approximation Algorithm

**Proof of Theorem 1.1.** Without loss of generality we assume that the scaling parameter $w$ is 1. We use different functions for $\alpha \leq 0.169$ and $\alpha \geq 0.169$. Let $A = 3 + 2 \log_e 1/\alpha$. For $\alpha \leq 0.169$, we define $f_\alpha(x)$ as follows (see Figure 2):

$$f_\alpha(x) = \begin{cases} 
1 - e^{-Ax}, & \text{if } 0 \leq x < \frac{1}{2} - \frac{1}{2A}; \\
1, & \text{otherwise}; 
\end{cases}$$

and, for $\alpha \geq 0.169$, we define $f_\alpha(x)$ as follows:

$$f_\alpha(x) = \begin{cases} 
0, & \text{if } x < \frac{1}{3}; \\
\frac{1 - \alpha}{3}, & \text{if } \frac{1}{3} \leq x < \frac{1}{2} - \frac{1}{2A}; \\
1, & \text{if } x \geq \frac{1}{2} - \frac{1}{2A}. 
\end{cases}$$

Our analysis of the algorithm relies on Theorem 3.1. We will show that for every triangle $(u_1, u_2, u_3)$ with edge lengths $(x_1, x_2, x_3)$ (satisfying the triangle inequality) and signature $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, we have

$$ALG^\sigma(x_1, x_2, x_3) \leq A \cdot LP^\sigma(x_1, x_2, x_3).$$

Therefore, by Theorem 3.1, our algorithm gives an $A$-approximation.
Without loss of generality, we assume that \( x_1 \leq x_2 \leq x_3 \). When \( i \in \{1, 2, 3\} \) is fixed, we will denote the other two elements of \( \{1, 2, 3\} \) by \( k \) and \( j \), so that \( j < k \). For \( i \in \{1, 2, 3\} \), let \( e_i = (u_j, u_k) \) (the edge opposite to \( u_i \)), \( w_i = w_{e_i}, x_i = x_{u_j u_k}, y_i = f_a(x_i) \), and
\[
t_i = A \cdot lp(u_j, u_k | u_i) - \text{cost}(u_j, u_k | u_i).
\]

Observe that (1) is equivalent to the inequality \( w_1 t_1 + w_2 t_2 + w_3 t_3 \geq 0 \). We now prove that this inequality always holds.

**Lemma 3.2.** We have
\[
w_1 t_1 + w_2 t_2 + w_3 t_3 \geq 0 \tag{2}
\]

We express each \( t_i \) in terms of \( x_i \)'s and \( y_i \)'s.

**Claim 3.3.** For every \( i \in \{1, 2, 3\} \), we have
\[
t_i = \begin{cases} A(1 - y_j)x_i - (y_k - y_j), & \text{if } \sigma_i = "+" \\ A(1 - y_j)(1 - x_i) - (1 - y_k), & \text{if } \sigma_i = "-" \end{cases}
\]

**Proof.** If \( \sigma_i = "+" \), then
\[
t_i = A \cdot lp(u_j, u_k | u_i) - \text{cost}(u_j, u_k | u_i)
= A x_{u_j u_k} \cdot \Pr(u_j \in S \mid p = u_i) - \Pr(u_j \in S, u_k \notin S \mid p = u_i)
= A x_i \cdot \Pr(f_a(x_k) \leq R \mid p = x_i) - \Pr(f_a(x_k) \leq R, f_a(x_j) \leq R)
= A x_i (1 - y_j) - (y_k - y_j),
\]

where we used that \( y_k = f_a(x_k) \geq f_a(x_j) = y_j \) (since \( x_k \geq x_j \) and \( f_a(x) \) is non-decreasing).

If \( \sigma_i = "-" \), then similarly to the previous case, we have
\[
t_i = A \cdot lp(u_j, u_k | u_i) - \text{cost}(u_j, u_k | u_i)
= A (1 - x_{u_j u_k}) \cdot \Pr(u_j \in S \mid p = u_i) - \Pr(u_j \in S, u_k \notin S \mid p = u_i)
= A (1 - x_i) \cdot \Pr(f_a(x_k) \leq R \mid p = x_i) - \Pr(f_a(x_k) \leq R, f_a(x_j) \leq R)
= A (1 - x_i) \cdot (1 - y_j) - (1 - y_k).
\]

We say that edge \( e_i \) **pays for itself** if \( t_i \geq 0 \). Note that if all edges \( e_1, e_2, e_3 \) pay for themselves then the desired inequality (2) holds. First, we show that all negative edges pay for themselves.

**Claim 3.4.** If \( \sigma_i = "-" \), then \( t_i \geq 0 \).

**Proof.** By Claim 3.3, \( t_i = A(1 - y_j)(1 - x_i) - 1 - y_k \). Thus, we need to show that \( A(1 - y_j)(1 - x_i) \geq 1 - y_k \). If \( x_k \geq \frac{1}{2} \), then \( y_k \leq 1 \), and the inequality trivially holds. If \( x_k < \frac{1}{2} \), then using \( x_j \geq x_k \), we get
\[
A > \frac{1}{1 - 2 x_k} \geq \frac{1}{1 - x_k - x_j} \geq \frac{1}{1 - x_i},
\]
here we used the triangle inequality \( x_k + x_j \geq x_i \). Thus
\[
A(1 - y_j)(1 - x_i) \geq A(1 - y_k)(1 - x_i) \geq 1 - y_k.
\]
We now show that for short edges $e_i$, it is sufficient to consider only the case when $\sigma_i = \text{"+"}$. Specifically, we prove the following claim.

**Claim 3.5.** Suppose that $x_i < \frac{1}{2} - \frac{1}{2A}$. If (2) holds for $\sigma$ with $\sigma_i = \text{"+"}$, then (2) also holds for $\sigma'$ obtained from $\sigma$ by changing the sign of $\sigma_i$ to $\text{"-"}$.

**Proof.** To prove the claim, we show that the value of $t_i$ is greater for $\sigma'$ than for $\sigma$. That is,

$$A(1 - y_j)x_i - (y_k - y_j) < A(1 - y_j)(1 - x_i) - (1 - y_k).$$

Note that the values of $t_j$ and $t_k$ do not depend on $\sigma_i$ and thus do not change if we replace $\sigma$ with $\sigma'$. Since $f_\alpha$ is non-decreasing and $x_j \leq x_k$, we have $y_j \leq y_k$. Hence,

$$x_i < \frac{1}{2} - \frac{1}{2A} = \frac{1}{2} + \frac{1}{2A} - \frac{1}{A} \leq \frac{1}{2} + \frac{1}{2A} - \frac{(1 - y_k)}{A(1 - y_j)}.$$

Thus,

$$2A(1 - y_j)x_i < A(1 - y_j) + 1 - y_j - 2(1 - y_k).$$

Therefore,

$$A(1 - y_j)x_i - (y_k - y_j) < A(1 - y_j)(1 - x_i) - (1 - y_k),$$

as required. \qed

Unlike negative edges, positive edges do not necessarily pay for themselves. We now prove that positive edges of length at least $1/A$ pay for themselves.

**Claim 3.6.** If $\sigma_i = \text{"+"}$ and $x_i \geq 1/A$, then $t_i \geq 0$.

**Proof.** We have,

$$t_i = A(1 - y_j)x_i - (y_k - y_j) \geq (1 - y_j) - (y_k - y_j) = 1 - y_k \geq 0.$$

We now separately consider two cases $\alpha \leq 0.169$ and $\alpha \geq 0.169$.

### 3.3 Analysis of the Approximation Algorithm for $\alpha \leq 0.169$

First, we consider the case of $\alpha \leq 0.169$.

**Proof of Lemma 3.2 for $\alpha \leq 0.169$.** We first show that if $x_3 < \frac{1}{2} - \frac{1}{2A}$, then all three edges $e_1$, $e_2$, and $e_3$ pay for themselves.

**Claim 3.7.** If $x_3 < \frac{1}{2} - \frac{1}{2A}$, then $t_i \geq 0$ for every $i$.

**Proof.** Since $x_3 < \frac{1}{2} - \frac{1}{2A}$, for every $i \in \{1, 2, 3\}$ we have $x_i < \frac{1}{2} - \frac{1}{2A}$ and thus $y_i \equiv f_{\alpha}(x_i) = 1 - e^{-Ax_i}$. We show that $t_i \geq 0$ for all $i$. Fix $i$. If $\sigma_i = \text{"-"}$, then, by Claim 3.4, $t_i \geq 0$. If $\sigma_i = \text{"+"}$, then

$$y_k - y_j = e^{-Ax_j} - e^{-Ax_k} = e^{-Ax_j} \left(1 - e^{-A(x_k - x_j)}\right) \leq e^{-Ax_j}A(x_k - x_j) \leq e^{-Ax_j}A = A(1 - y_j)x_i,$$

where the first inequality follows from the inequality $1 - e^{-x} \leq x$, and the second inequality follows from the triangle inequality. Thus, $t_i = A(1 - y_j)x_i - (y_k - y_j) \geq 0$. \qed
We conclude that if \( x_3 < \frac{1}{2} - \frac{1}{2A} \), then (2) holds. The case \( x_3 < \frac{1}{2} - \frac{1}{2A} \) is the most interesting case in the analysis; the rest of the proof is more technical. As a side note, let us point out that Theorem 1.1 has dependence \( A = 3 + 2 \log_e 1/\alpha \) because (i) \( f_\alpha(x) \) must be equal to \( C - e^{-Ax} \) or a slower growing function so that Claim 3.7 holds (ii) Theorem 3.1 requires that \( f_\alpha(0) = 0 \), and finally (iii) we will need below that \( 1 - f \left( \frac{1}{2} - \frac{3}{2A} \right) \leq \alpha \).

From now on, we assume that \( x_3 \geq \frac{1}{2} - \frac{1}{2A} \) and, consequently, \( y_3 = f(x_3) = 1 \). Observe that if \( x_1 \geq \frac{1}{4} \), then all \( x_i \geq \frac{1}{4} \) and thus, by Claims 3.4 and 3.6, all \( t_i \geq 0 \) and we are done. Similarly, if \( x_2 \geq \frac{1}{2} - \frac{1}{2A} \), then \( x_2 \geq \frac{1}{4} \) (since \( A \geq 3 \)). Hence, \( t_2 \geq 0 \) and \( t_3 \geq 0 \); additionally, \( y_2 = y_3 = 1 \). Thus \( t_1 = 0 \) and inequality (2) holds. Therefore, it remains to show that inequality (2) holds when

\[
\begin{align*}
x_1 &< \frac{1}{A}, \\
x_2 &< \frac{1}{2} - \frac{1}{2A}, \text{ and } x_3 \geq \frac{1}{2} - \frac{1}{2A}.
\end{align*}
\]

By Claim 3.5, we may also assume that \( \sigma_1 = "+" \) and \( \sigma_2 = "+" \). Since \( \alpha \leq 0.169 \), we have \( A > 5 \) and

\[
x_2 \geq x_3 - x_1 \geq \left( \frac{1}{2} - \frac{1}{2A} \right) - \frac{1}{A} > \frac{1}{A} \text{ and } x_3 \geq \frac{1}{2} - \frac{1}{2A} > \frac{1}{A}.
\]

Thus, by Claims 3.4 and 3.6, \( t_2 \geq 0 \) and \( t_3 \geq 0 \). Hence, \( w_2t_2 + w_3t_3 \geq \alpha(w_2 + w_3) \). Also, recall that \( e_1 \) is a positive edge and thus \( w_1 \leq 1 \). Therefore, it is sufficient to show that

\[
t_1 \geq -\alpha(t_2 + t_3).
\]

Now we separately consider two possible signatures \( \sigma = ("+", "+", "+") \) and \( \sigma = ("+", "+", "-") \).

**First, assume that** \( \sigma = ("+", "+", "+") \). We need to show that

\[
A(1 - y_2)x_1 - (1 - y_2) \geq \alpha \left( (1 - y_1) + (y_2 - y_1) - A(1 - y_1)x_2 - A(1 - y_1)x_3 \right).
\]

Here, we used that \( y_3 = 1 \). Note that \( x_2 \geq x_3 - x_1 \geq \frac{1}{2} - \frac{1}{2A} - \frac{1}{A} = \frac{1}{2} - \frac{3}{2A} \). Therefore,

\[
1 - y_2 \leq 1 - \left( 1 - e^{-A(\frac{1}{2} - \frac{3}{2A})} \right) = e^{-\frac{3}{2} - \log_e \frac{1}{A}} = e^{-\log_e \frac{1}{\alpha}} = \alpha.
\]

Thus, \( (1 - y_2) + \alpha(1 - y_1) + \alpha(y_2 - y_1) \leq \alpha y_2 + 2\alpha(1 - y_1) \). To finish the analysis of the case \( \sigma = ("+", "+", "+"), it is sufficient to show that

\[
\alpha y_2 + 2\alpha(1 - y_1) \leq A(1 - y_2)x_1 + \alpha A(1 - y_1)x_2 + \alpha A(1 - y_1)x_3.
\]

This inequality immediately follows from the following claim (we simply need to add up (4) and (5) and multiply the result by \( \alpha \)).

**Claim 3.8.** For \( c = 0.224 \), we have

\[
(2 - c)(1 - y_1) \leq A(1 - y_1)x_2; \text{ and } \quad y_2 + c(1 - y_1) \leq A(1 - y_1)x_3.
\]

**Proof.** Since \( c \geq 2 - \log_e \frac{1}{0.169} \geq 2 - \log_e \frac{1}{\alpha} \) (recall that \( \alpha \leq 0.169 \)), we have

\[
2 - c \leq \log_e \frac{1}{\alpha} = \frac{A}{2} - \frac{3}{2} \leq A x_2.
\]
Therefore, (4) holds. We also have,
\[ c \leq 0.169 + \log_e \frac{1}{0.169} + 1 - e \leq \alpha + \log_e \frac{1}{\alpha} + 1 - e. \]
Thus, \( e - \alpha \leq \frac{4}{3} - \frac{1}{2} - c \leq Ax_3 - c. \) Therefore,
\[ e^{-1} (Ax_3 - c) \geq 1 - \alpha e^{-1} = 1 - e^{-A\left(\frac{1}{2} - \frac{1}{2A}\right)} \geq y_2, \quad (6) \]
where we used that \( x_2 < \frac{1}{2} - \frac{1}{2A} \) and \( y_2 = f_\alpha(x_2) = 1 - e^{-Ax_2}. \) Observe that from inequalities (6) and \( x_1 < \frac{1}{A} \) it follows that
\[ y_2 \leq \left(1 - f\left(\frac{1}{A}\right)\right) (Ax_3 - c) \leq (1 - y_1) (Ax_3 - c), \]
which implies (5).

Now, assume that \( \sigma = (\text{“+”}, \text{“+”}, \text{“-”}). \) We need to prove the following inequality,
\[ (1 - y_2) + \alpha (1 - y_1 + 1 - y_2) \leq A (1 - y_2) x_1 + \alpha A (1 - y_1) (x_2 + 1 - x_3). \quad (7) \]
As before,
\[ (1 - y_2) + \alpha (1 - y_1 + 1 - y_2) \leq \alpha + \alpha (1 - y_1 + 1 - y_2) \leq \alpha + 2\alpha (1 - y_1). \quad (8) \]
On the other hand,
\[ A (1 - y_2) x_1 + \alpha A (1 - y_1) (x_2 + 1 - x_3) \geq \alpha A (1 - y_1) (1 - x_1 + x_1 + x_2 - x_3) \]
\[ \geq \alpha A (1 - y_1) (1 - x_1) \]
\[ \geq \alpha A (1 - y_1) \left(1 - \frac{1}{A}\right) \]
\[ = \alpha (1 - y_1) (A - 1) \quad (9) \]
where the second inequality is due to the triangle inequality, and the third inequality is due to \( x_1 < \frac{1}{A}. \) Finally, observe that \( 1 \leq 2e^{-1} \log_e \frac{1}{\alpha} = e^{-1} (A - 3) \leq (1 - y_1) (A - 3). \) We get,
\[ \alpha (1 - y_1) (A - 1) \geq \alpha + 2\alpha (1 - y_1). \quad (10) \]
Combining (8), (9), and (10), we get (7). This concludes the case analysis and the proof of Theorem 1.1 for the regime \( \alpha \leq 0.169. \)

3.4 Analysis of the Approximation Algorithm for \( \alpha \geq 0.169 \)
We now consider the case when \( \alpha \geq 0.169. \) Observe that for \( \alpha \geq 0.169 \)
\[ A = 3 + 2 \log_e (1/\alpha) \geq \frac{6\alpha + 3 - (1 - \alpha)^2}{3\alpha} \quad (11) \]
and
\[ \frac{1 - \alpha}{3} \leq \frac{2\alpha}{1 + \alpha}. \quad (12) \]
Proof of Lemma 3.2 for \( \alpha \geq 0.169 \). Observe that if \( x_1 \geq \frac{1}{4} \), then all \( x_i \geq 1/A \) and thus, by Claims 3.4 and 3.6, all \( t_i \geq 0 \) and we are done. Moreover, if \( x_3 < \frac{1}{4} \) then all \( x_i < 1/A \) implying \( y_i = 0 \) and thus, \( t_i \geq 0 \) for \( \sigma_i = "+" \). This combined with Claim 3.4 imply all \( t_i \geq 0 \) and we are done. Similarly, if \( x_2 \geq \frac{1}{2} - \frac{1}{2A} \), then \( x_2 \geq 1/A \) (since \( A \geq 3 \)). Hence, \( t_2 \geq 0 \) and \( t_3 \geq 0 \); additionally, we have \( y_2 = y_3 = 1 \). Thus, \( t_1 = 0 \) and we are done.

Therefore, we will assume below that

\[
x_1 < \frac{1}{A}, \quad x_2 < \frac{1}{2} - \frac{1}{2A}, \quad x_3 \geq \frac{1}{A}.
\]

Furthermore, by Claim 3.5, we may assume \( \sigma_1 = "+" \) and \( \sigma_2 = "+" \). We consider four cases:
(i) \( x_2 \geq 1/A, \quad x_3 \geq 1/2 - 1/(2A) \),
(ii) \( x_2 < 1/A, \quad x_3 \geq 1/2 - 1/(2A) \),
(iii) \( x_2 \geq 1/A, \quad x_3 < 1/2 - 1/(2A) \), and
(iv) \( x_2 < 1/A, \quad x_3 < 1/2 - 1/(2A) \).

Consider the case \( x_2 \geq \frac{1}{4}, \quad x_3 \geq \frac{1}{2} - \frac{1}{2A} \). Then \( y_1 = 0, \quad y_2 = (1-\alpha)/3, \quad y_3 = 1 \). By Claims 3.4 and 3.6, \( t_2, t_3 \geq 0 \), and \( e_2, e_3 \) pay for themselves. If \( t_1 \geq 0 \), we are done. So we will assume below that \( t_1 < 0 \). Then,

\[
w_1 t_1 + w_2 t_2 + w_3 t_3 \geq 1 \cdot t_1 + \alpha t_2 + \alpha t_3
\]

(13)

(recall that we assume that \( e_1 \) is a positive edge and thus \( w_1 \leq 1 \)).

Now we separately consider two possible signatures \( \sigma = ("+", "+", "+") \) and \( \sigma = ("+", "+", "+") \).

First, assume that \( \sigma = ("+", "+", "+") \). Because of (13), to prove (2) it is sufficient to show

\[
(1 - y_2) + \alpha + \alpha y_2 \leq A(1 - y_2)x_1 + \alpha Ax_2 + \alpha Ax_3
\]

(14)

From (11) it follows that

\[
1 + \alpha \leq \frac{(1-\alpha)^2}{3} + \alpha (A-1)
\]

which implies (15) due to \( x_3 \geq \frac{1}{2} - \frac{1}{2A} \)

\[
1 + \alpha \leq \frac{(1-\alpha)^2}{3} + 2\alpha Ax_3
\]

(15)

Observe that from (15) together with triangle inequality and \( y_2 = \frac{1-\alpha}{3} \leq 1 - \alpha \) it follows that

\[
1 + \alpha \leq (1-\alpha)y_2 + A(1 - y_2)x_1 - \alpha Ax_1 + \alpha Ax_2 + \alpha Ax_3
\]

which is equivalent to (14).

Now, assume that \( \sigma = ("+", "+", "-") \). Because of (13), to prove (2) it is sufficient to show

\[
(1 - y_2) + \alpha + \alpha (1 - y_2) \leq A(1 - y_2)x_1 + \alpha Ax_2 + \alpha A(1 - x_3)
\]

(16)

From (11) and \( y_2 = \frac{1-\alpha}{3} \) it follows that

\[
1 + 2\alpha \leq \frac{(1-\alpha)^2}{3} + \alpha A \leq y_2 (1 + \alpha) + \alpha A
\]

Since \( y_2 \leq 1 - \alpha \),

\[
(1 + 2\alpha) \leq (1 + \alpha)y_2 + A(1 - y_2)x_1 - \alpha Ax_1 + \alpha A,
\]

13
Hence, using the triangle inequality,

\[ 1 + 2\alpha \leq (1 + \alpha)y_2 + A(1 - y_2)x_1 - \alpha Ax_1 + \alpha A + \alpha Ax_1 + \alpha Ax_2 - \alpha Ax_3. \]

which is equivalent to (16).

**Consider the case** \( x_2 < \frac{1}{A}, \ x_3 \geq \frac{1}{2} - \frac{1}{2A} \). Then \( y_1 = y_2 = 0, \ y_3 = 1 \). Observe that \( t_3 \geq 0 \) and \( t_1, t_2 < 0 \). Then,

\[ w_1t_1 + w_2t_2 + w_3t_3 \geq 1 \cdot t_1 + 1 \cdot t_2 + \alpha t_3. \]  

(17)

(recall that we assume that \( e_1, e_2 \) are positive edges and thus \( w_1, w_2 \leq 1 \)). Furthermore, since \( x_3 \geq \frac{1}{2} - \frac{1}{2A} \) we have

\[ Ax_3 \geq A(1 - x_3) - 1. \]  

(18)

From (18), we get that if (2) holds for \( \sigma \) with \( \sigma_3 = "\text{−}" \), then (2) also holds for \( \sigma' \) obtained from \( \sigma \) by changing the sign of \( \sigma_3 \) to “+”. Thus without loss of generality \( \sigma_3 = "\text{−}" \) and we only need to consider \( \sigma = ("+", "+", "\text{−}" \)). Then, because of (17), to prove (2) it is sufficient to show

\[ 1 + 1 + \alpha \leq Ax_1 + Ax_2 + \alpha A(1 - x_3). \]  

(19)

From (11) it follows that

\[ A \geq \frac{5 + \alpha}{\alpha + 1}, \]

which is equivalent to

\[ 2 + \alpha \leq \alpha A + (1 - \alpha)\left(\frac{A}{2} - \frac{1}{2}\right). \]  

(20)

Observe that from (20) together with triangle inequality and \( x_3 \geq \frac{1}{2} - \frac{1}{2A} \) it follows that

\[ 2 + \alpha \leq \alpha A + (1 - \alpha)Ax_3 = Ax_3 + \alpha A(1 - x_3) \leq Ax_1 + Ax_2 + \alpha A(1 - x_3). \]

**Consider the case** \( x_2 \geq \frac{1}{A}, \ x_3 < \frac{1}{2} - \frac{1}{2A} \). Then \( y_1 = 0, \ y_3 = (1-\alpha)/3 \). By Claim 3.5 we only need to consider \( \sigma = ("+", "+", "+" \)). Then by Claim 3.6, \( t_2, t_3 \geq 0 \). Thus, if \( t_1 \geq 0 \) then \( w_1t_1 + w_2t_2 + w_3t_3 \geq 0 \). Let us assume that \( t_1 < 0 \). Since \( e_1 \) is a positive edge, we have \( w_1 \leq 1 \).

Thus,

\[ w_1t_1 + w_2t_2 + w_3t_3 \geq 1 \cdot t_1 + \alpha t_2 + \alpha t_3 \]

We need to show that the right hand side in the above inequality is non-negative. Replace \( t_1, t_2, \) and \( t_3 \) with the expressions from Claim 3.3. Now to obtain (2), it is sufficient to prove that

\[ y_3 - y_2 + \alpha y_3 + \alpha y_2 \leq A(1 - y_2)x_1 + \alpha Ax_2 + \alpha Ax_3 \]

(21)

Observe that since \( x_3 \geq \frac{1}{A} \) we have

\[ 2\alpha \leq (1 - \alpha)y_2 + 2\alpha Ax_3. \]  

(22)

Inequalities (22) and (12) imply

\[ (1 + \alpha)y_3 \leq (1 - \alpha)y_2 + 2\alpha Ax_3. \]  

(23)
Observe that from (23) together with triangle inequality and $y_2 \leq 1 - \alpha$ it follows that

\[(1 + \alpha)y_3 \leq (1 - \alpha)y_2 + A(1 - y_2)x_1 - \alpha Ax_1 + \alpha Ax_2 + \alpha Ax_3\]

which is equivalent to (21).

**Consider the case** $x_2 < \frac{1}{3}$, $x_3 < \frac{1}{2} - \frac{1}{3x_1}$. Then $y_1 = y_2 = 0$. By Claim 3.5 we only need to consider $\sigma = (\text{“+”}, \text{“+”}, \text{“+”})$. Then by Claim 3.6, $t_3 \geq 0$.

If $x_1 \geq y_3/A$ then $t_1, t_2 \geq 0$ and we are done. Thus we assume $x_1 < y_3/A$ which implies $t_1 < 0$. We consider two different regimes: (i) $x_2 \geq y_3/A$ and (ii) $x_2 < y_3/A$.

**First, assume that** $x_2 \geq y_3/A$ which implies $t_2 \geq 0$. Then,

\[w_1 t_1 + w_2 t_2 + w_3 t_3 \geq t_1 + t_2 + t_3\]  \(\text{(24)}\)

(recall that we assume that $e_1$ is a positive edge and thus $w_1 \leq 1$).

Because of (24), to prove (2) it is sufficient to show

\[y_3 + \alpha y_3 \leq Ax_1 + \alpha Ax_2 + \alpha Ax_3\]  \(\text{(25)}\)

Observe that by (12) and $y_3 = (1 - \alpha)/3$ we have

\[(1 + \alpha)y_3 \leq 2\alpha \leq 2\alpha Ax_3 \leq \alpha Ax_3 + \alpha Ax_1 + \alpha Ax_2 \leq Ax_1 + \alpha Ax_2 + \alpha Ax_3\]

where the second inequality follows from $x_3 \geq \frac{1}{3}$ and the third inequality follows from triangle inequality.

**Now, assume that** $x_2 < y_3/A$ which implies $t_2 < 0$. Then,

\[w_1 t_1 + w_2 t_2 + w_3 t_3 \geq t_1 + t_2 + t_3\]  \(\text{(26)}\)

(recall that we assume that $e_1, e_2$ are positive edges and thus $w_1, w_2 \leq 1$).

Because of (26), to prove (2) it is sufficient to show

\[2y_3 \leq Ax_1 + Ax_2 + \alpha Ax_3\]  \(\text{(27)}\)

Observe that by (12) and $x_3 \geq \frac{1}{A}$

\[2y_3 \leq \frac{4\alpha}{1 + \alpha} \leq 1 + \alpha \leq (1 + \alpha)Ax_3 \leq Ax_1 + Ax_2 + \alpha Ax_3\]

where the last inequality follows from triangle inequality.

This concludes the case analysis and the proof of Theorem 1.1 for the regime $\alpha \geq 0.169$. \(\Box\)

4 Better approximation for values of $\alpha$ appearing in practice

We note that the choice of function $f(x)$ in Theorem 1.1 is somewhat suboptimal. The best function $f_{opt}(x)$ for our analysis of Algorithm 1 can be computed using linear programming (with high precision). Using this function $f_{opt}$, we can achieve an approximation factor $A_{opt}$ better than the approximation factor $A_{thm} = 3 + 2\log_2 1/\alpha$ guaranteed by Theorem 1.1 (for $\alpha \neq 1$).\(^3\) While asymptotically $A_{thm}/A_{opt} \to 1$ as $\alpha \to 0$, $A_{opt}$ is noticeably better than $A_{thm}$ for many values of $\alpha$ that are likely to appear in practice (say, for $\alpha \in (10^{-8}, 0.1)$). We list approximation factors $A_{thm}$ and $A_{opt}$ for several values of $\alpha$ in Table 1; we also plot the dependence of $A_{thm}$ and $A_{opt}$ on $\alpha$ in Figure 3.

\(^3\)It is also possible to slightly modify Algorithm 1 so that it gets approximation $A_{opt}$ without explicitly computing $f$. We omit the details here.
Table 1: Approximation factors $A_{thm}$ and $A_{opt}$ for different $\alpha$-s.

| $\log_e 1/\alpha$ | $1/\alpha$ | $A_{thm}$ | $A_{opt}$ |
|-------------------|-----------|-----------|-----------|
| 0                 | 1         | 3         | 3         |
| 1.61              | 5         | 6.22      | 4.32      |
| 2.30              | 10        | 7.61      | 4.63      |
| 3.91              | 50        | 10.82     | 6.07      |
| 4.61              | 100       | 12.21     | 6.78      |
| 6.21              | 500       | 15.43     | 8.69      |
| 6.91              | 1000      | 16.82     | 9.62      |
| 8.52              | 5000      | 20.03     | 11.9      |
| 10                | 22026.5   | 23        | 14.2      |
| 15                | $3.3 \times 10^6$ | 33        | 22.6      |
| 20                | $4.9 \times 10^8$ | 43        | 31.3      |

5 Analysis of the Algorithm for Complete Bipartite Graphs

Proof of Theorem 1.2. The proof is similar to the proof of Theorem 1.1. Without loss of generality we assume that the scaling parameter $w$ is 1. Define $f(x)$ as follows

$$f(x) = \begin{cases} 
1 - e^{-Ax}, & \text{if } 0 \leq x < \frac{1}{2} - \frac{1}{2A} \\
1, & \text{otherwise}
\end{cases}$$

where $A = 5 + 2\log_e 1/\alpha$. Our analysis of the algorithm relies on Theorem 3.1. Since in the proof of Theorem 3.1, we assumed that all edges are present, let us add missing edges (edges inside parts) to the bipartite graph and assign them weight 0; to be specific, we assume that they are positive edges. (It is important to note that Theorem 3.1 is true even when edges have zero weights). We will still refer to these edges as ‘missing edges’.

We will show that for every triangle $(u_1, u_2, u_3)$ with edge lengths $(x_1, x_2, x_3)$ (satisfying the triangle inequality) and signature $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, we have

$$\text{ALG}^\sigma(x_1, x_2, x_3) \leq A \cdot \text{LP}^\sigma(x_1, x_2, x_3)$$

Therefore, by Theorem 3.1, our algorithm gives an $A$-approximation. In addition to Theorem 3.1 we use Claims 3.3, 3.4, 3.5, 3.6 and 3.7. Recall that proofs of these claims rely on $f$ being non-decreasing which is satisfied by the above choice. Observe that (28) is equivalent to

$$\sum_{i=1}^{3} w_i t_i \geq 0. \quad (29)$$

Observe that if $x_1 \geq \frac{1}{A}$, then all $x_i \geq \frac{1}{A}$ and thus, by Claims 3.4 and 3.6, all $t_i \geq 0$ and we are done. Similarly, if $x_2 \geq \frac{1}{2} - \frac{1}{2A}$, then $t_2 \geq 0$ and $t_3 \geq 0$; additionally, $y_2 = y_3 = 1$, thus $t_1 = 0$ and we are done. Furthermore, if $x_3 < \frac{1}{2} - \frac{1}{2A}$ then all $x_i < \frac{1}{2} - \frac{1}{2A}$ and thus, by Claim 3.7, all $t_i \geq 0$ and we are done. Therefore, we will assume below that $x_1 < \frac{3}{A}$, $x_2 < \frac{1}{2} - \frac{1}{2A}$, and $x_3 \geq \frac{1}{2} - \frac{1}{2A}$. Further, by the triangle inequality $x_2 \geq x_3 - x_1 \geq \frac{A-3}{2A} - x_1 \leq \frac{A-1}{2A} \leq x_3 \leq x_1 + x_2$. We have (here we use that $A \geq 5$),

$$x_1 \leq \frac{1}{A} \leq \frac{A-3}{2A} \leq \frac{A-1}{2A} - x_1 \leq x_2 < \frac{A-1}{2A} \leq x_3 \leq x_1 + x_2.$$
We will use below that
\[ e^{A(x_2-x_1)} \geq e^{A(\frac{A-1}{2A} - 2x_1)} = e^{2 + \log \frac{1}{\alpha} - 2Ax_1} = e^{2(1-Ax_1)/\alpha} \geq 1/\alpha. \]

By Claim 3.5, we may also assume that \( \sigma_1 = "+" \) and \( \sigma_2 = "+" \) (and since we assume that missing edges are positive). By Claims 3.4 and 3.6, \( t_2 \geq 0 \) and \( t_3 \geq 0 \) (edges \( e_2 \) and \( e_3 \) pay for themselves). If \( t_1 \geq 0 \), we are done. So we will assume below that \( t_1 < 0 \). Since \( G \) is a complete bipartite graph, a triangle \((u_1, u_2, u_3)\) contains either (i) no edges or (ii) two edges. In case (i) we have \( w_1 = w_2 = w_3 = 0 \) and (29) holds trivially. In case (ii) if \( e_1 \) is the missing edge then \( w_1 = 0 \) and since \( t_2, t_3 \geq 0, (29) \) holds trivially. It remains to consider three signatures \( \sigma = ("+", "+", "o") \), \( \sigma = ("+", "o", "+") \) and \( \sigma = ("+", "o", "-" ) \) where "o" denotes a missing edge (which by our assumption above is a positive edge).

**First, assume that \( \sigma = ("+", "+", "o")\).** By Claim 3.3, \( t_1 = A(1-y_2)x_1-(1-y_2) = -e^{-Ax_2}(1-Ax_1) \) and \( t_2 = A(1-y_1)x_2-(1-y_2) = e^{-Ax_1}(Ax_2 - 1) \). Since \( e_3 \) is missing, \( w_3 = 0 \). We have, \( w_1t_1 + w_2t_2 + w_3t_3 \geq t_1 + \alpha t_2 \) (here we used that \( t_1 \leq 0 \) and \( t_2 \geq 0 \)). So it suffices to prove that \( t_1 + \alpha t_2 > 0 \) or, equivalently, \( e^{Ax_2}(\alpha t_2 + t_1) \geq 0 \). Using that \( e^{A(x_2-x_1)} \geq 1/\alpha \) and \( x_2 \geq \frac{A-1}{2A} - x_1 \),

\[
e^{Ax_2}(\alpha t_2 + t_1) = \alpha e^{A(x_2-x_1)}(Ax_2-1)-(1-Ax_1) \geq \alpha \frac{1}{\alpha} \left(A(\frac{A-1}{2A} - x_1)-1\right)+Ax_1-1 = \frac{A-5}{2} > 0,
\]
as required.

**Now, assume that \( \sigma = ("+", "o", "+")\).** Now we have \( t_1 = -e^{-Ax_2}(1-Ax_1) \) (as before) and

\[
t_3 = A(1-y_1)x_3-(y_2-y_1) = e^{-Ax_3}(Ax_3-1)+e^{-Ax_2}.
\]

We prove that \( t_1 + \alpha t_2 \geq 0 \) or, equivalently, \( e^{Ax_2}(\alpha t_3 + t_1) \geq 0 \). Using that \( e^{A(x_2-x_1)} \geq 1/\alpha \) and \( x_3 \geq \frac{A-1}{2A} \), we get

\[
e^{Ax_2}(\alpha t_3 + t_1) = \alpha(e^{A(x_2-x_1)}(Ax_3-1)+1)-(1-Ax_1)
\]

\[
\geq (Ax_3-1)+\alpha-(1-Ax_1) > Ax_3-2 \geq \frac{A-1}{2} - 2 \geq 0,
\]
as required.

**Finally, assume that \( \sigma = ("+", "o", "-" )\).** Now we have \( t_1 = -e^{-Ax_2}(1-Ax_1) \) (as before) and \( t_3 = A(1-y_1)(1-x_3)-(1-y_2) = e^{-Ax_1}(1-x_3)-(1-x_2) \). As in the previous case, we prove that \( e^{Ax_2}(\alpha t_3 + t_1) \geq 0 \). We have,

\[
e^{Ax_2}(\alpha t_3 + t_1) = \alpha \left( e^{A(x_2-x_1)}(1-x_3)-1 \right)(1-Ax_1) \geq \alpha \left( e^{A(x_2-x_1)}(1-x_1-x_2)-1 \right)(1-Ax_1).
\]

Denote the expression on the right by \( F(x_1, x_2) \). We now show that for a fixed \( x_1 \), \( F(x_1, x_2) \) is an increasing function of \( x_2 \) when \( x_2 \in \left[\frac{A-1}{2A} - x_1, \frac{A-1}{2A} \right] \). Indeed, we have

\[
\frac{\partial F(x_1, x_2)}{\partial x_2} = \alpha A e^{A(x_2-x_1)}(1-x_1-x_2) - 1 \geq A e^{A(x_2-x_1)} \left( A \left( 1 - \frac{1}{A} - \frac{A-1}{2A} \right) - 1 \right) = A e^{A(x_2-x_1)} \frac{A-3}{2} > 0.
\]

17
We conclude that
\[
F(x_1, x_2) \geq F \left(x_1, \frac{A-1}{2A} - x_1 \right) = \left( \alpha \left( A e^{A(\bar{x}_2-x_1)} (1-x_1-\bar{x}_2) - 1 \right) - (1-Ax_1) \right) \bigg|_{\bar{x}_2=\frac{A-1}{2A} - x_1}
\]
\[
\geq \alpha \cdot A \cdot \frac{1}{\alpha} \left( 1 - \frac{A-1}{2A} \right) - \alpha - (1-Ax_1) = \frac{A+1}{2} - \alpha - 1 + Ax_1 \geq \frac{A+1}{2} - 2 > 0.
\]

This concludes the case analysis and the proof of Theorem 1.2. \(\square\)

6 Integrality Gap

In this section, we give a \(\Theta(\log 1/\alpha)\) integrality gap example for the LP relaxation presented in Section 2.1. Notice that in the example each positive edge has a weight of \(w^+\) and each negative edge has a weight of \(w^-\) with \(w^+ \geq w^-\).

Proof of Theorem 1.3. Consider a 3-regular expander \(G = (V, E)\) on \(n = \Theta((\alpha^2 \log^2 \alpha)^{-1})\) vertices. We say that two vertices \(u\) and \(v\) are similar if \((u,v) \in E\); otherwise \(u\) and \(v\) are dissimilar. That is, the set of positive edges \(E^+\) is \(E\) and the set of negative edges \(E^-\) is \(V \times V \setminus E\). Let \(w^+ = 1\) and \(w^- = \alpha\).

Lemma 6.1. The integrality gap of the Correlation Clustering instance \(G_{cc} = (V, E^+, E^-)\) described above is \(\Theta(\log 1/\alpha)\).

Proof. Let \(d(u,v)\) be the shortest path distance in \(G\). Let \(\varepsilon = 2/ \log_3 n\). We define a feasible metric LP solution as follows: \(x_{uv} = \min (\varepsilon d(u,v), 1)\).

Let \(LP^+\) be the LP cost of positive edges, and \(LP^-\) be the LP cost of negative edges. The LP cost of every positive edge is \(\varepsilon\) since \(d(u,v) = 1\) for \((u,v) \in E\). There are \(3n/2\) positive edges in \(G_{cc}\). Thus, \(LP^+ < 3n/\log_3 n\). We now estimate \(LP^-\). For every vertex \(u\), the number of vertices \(v\) at distance less than \(\ell\) is upper bounded by \(3^\ell\) because \(G\) is a 3-regular graph. Thus, the number of vertices \(v\) at distance less than \(1/2 \log_3 n\) is upper bounded by \(\sqrt{n}\). Observe that the LP cost of a negative edge \((u,v)\) (which is equal to \(\alpha(1-x_{uv})\)) is positive if and only if \(d(u,v) < 1/2 \log_3 n\). Therefore, the number of negative edges with a positive LP cost incident on any vertex \(u\) is at most \(\sqrt{n}\). Consequently, the LP cost of all negative edges is upper bounded by \(\alpha n^{3/2} = \Theta(n/ \log 1/\alpha)\). Hence,

\[LP \leq \Theta(n/ \log 1/\alpha) + 3n/ \log_3 n = \Theta(n/ \log 1/\alpha)\].

Here, we used that \(\log n = \Theta(\log 1/\alpha)\).

We now lower bound the cost of the optimal (integral) solution. Consider an optimal solution. There are two possible cases.

1. No cluster contains 90% of the vertices. Then a constant fraction of positive edges in the expander \(G\) are cut and, therefore, the cost of the optimal clustering is at least \(\Theta(n)\).

2. One of the clusters contains at least 90% of all vertices. Then all negative edges in that cluster are in disagreement with the clustering. There are at least \(\left(0.9n \over 2\right) - m = \Theta(n^2)\) such edges. Their cost is at least \(\Omega(\alpha n^2)\).
We conclude that the cost of the optimal solution is at least \( \Theta(n) \) and, thus, the integrality gap is \( \Theta(\log(1/\alpha)) \).

We note that in this example \( \log(1/\alpha) = \Theta(\log n) \). However, it is easy to construct an integrality gap example where \( \log(1/\alpha) \ll \Theta(\log n) \). To do so, we pick the integrality gap example constructed above and create \( k \gg n \) disjoint copies of it. To make the graph complete, we add negative edges with (fractional) LP value equal to 1 to connect each copy to every other copy of the graph. The new graph has \( kn \gg n \) vertices. However, the integrality gap remains the same, \( \Theta(\log 1/\alpha) \).

Now we give a \( \Theta(\log 1/\alpha) \) integrality gap example when \( G \) is a complete bipartite graph.

**Proof of Theorem 1.4.** The proof is very similar to that of Theorem 1.3. We start with a 3-regular bipartite expander \( G = (L, R, E) \) on \( n = \Theta((\alpha^2 \log^2 \alpha)^{-1}) \) vertices (e.g., we can use a 3-regular bipartite Ramanujan expander constructed by Marcus, Spielman, and Srivastava [2013]). Then we define a correlation clustering instance as follows: \( G_{cc} = (L, R, E^+, E^-) \) where \( E^+ = E \) and \( E^- = (L \times R) \setminus E \); let \( w^+ = 1 \) and \( w^- = \alpha \). The proof of Lemma 6.1 can be applied to \( G_{cc} \); we only need to note that if a cluster contains at least 90% of the vertices, then there are at least \( \Theta(n^2) \) edges of \( G_{cc} \) between vertices in the cluster. It follows that the integrality gap is \( \Omega(\log(1/\alpha)) \).

**References**

Nir Ailon, Moses Charikar, and Alantha Newman. Aggregating inconsistent information: ranking and clustering. *Journal of the ACM (JACM)*, 55(5):23, 2008.

Nir Ailon, Yudong Chen, and Huan Xu. Breaking the small cluster barrier of graph clustering. In *International Conference on Machine Learning*, pages 995–1003, 2013.

Nikhil Bansal, Avrim Blum, and Shuchi Chawla. Correlation clustering. *Machine learning*, 56(1-3): 89–113, 2004.

Paolo Boldi and Sebastiano Vigna. The WebGraph framework I: Compression techniques. In *Proc. of the Thirteenth International World Wide Web Conference*, pages 595–601, 2004.

Paolo Boldi, Bruno Codenotti, Massimo Santini, and Sebastiano Vigna. Ubicrawler: A scalable fully distributed web crawler. *Software: Practice & Experience*, 34(8):711–726, 2004.

Paolo Boldi, Marco Rosa, Massimo Santini, and Sebastiano Vigna. Layered label propagation: A multiresolution coordinate-free ordering for compressing social networks. In *Proceedings of the International Conference on World Wide Web*, pages 587–596, 2011.

Paolo Boldi, Andrea Marino, Massimo Santini, and Sebastiano Vigna. BUbiNG: Massive crawling for the masses. In *Proceedings of the Companion Publication of the International Conference on World Wide Web*, pages 227–228, 2014.

Moses Charikar, Venkatesan Guruswami, and Anthony Wirth. Clustering with qualitative information. In *IEEE Symposium on Foundations of Computer Science*. Citeseer, 2003.

Shuchi Chawla, Konstantin Makarychev, Tselil Schramm, and Grigory Yaroslavtsev. Near optimal LP rounding algorithm for correlation clustering on complete and complete \( k \)-partite graphs. In *Proceedings of the Symposium on Theory of Computing*, pages 219–228, 2015.
A Proof of Theorem 3.1

For the sake of completeness we include the proof of Theorem 3.1 (see Ailon et al. [2008] and Chawla et al. [2015]).

Proof of Theorem 3.1. Our first task is to express the cost of violations made by Algorithm 1 and the LP weight in terms of $ALG^\sigma(\cdot)$ and $LP^\sigma(\cdot)$, respectively. In order to do this, we consider the cost of violations made by the algorithm at each step.

Consider step $t$ of the algorithm. Let $V_t$ denote the set of active (yet unclustered) vertices at the start of step $t$. Let $w \in V_t$ denote the pivot chosen at step $t$. The algorithm chooses a set $S_t \subseteq V_t$ as a cluster and removes it from the graph. Notice that for each $u \in S_t$, the constraint imposed by each edge of type $(u, v) \in E^+ \cup E^-$ is satisfied or violated right after step $t$. Specifically, if $(u, v)$ is a positive edge, then the constraint $(u, v)$ is violated if exactly one of the vertices $u, v$ is
in $S_t$. If $(u, v)$ is a negative constraint, then it is violated if both $u, v$ are in $S_t$. Denote the weight of violated constraints at step $t$ by $ALG_t$. Thus,

$$ALG_t = \sum_{(u,v) \in E^+ \setminus V_t} w_{uv} \cdot 1 \left( u \in S_t, v \not\in S_t \text{ or } u \not\in S_t, v \in S_t \right) + \sum_{(u,v) \in E^- \setminus V_t} w_{uv} \cdot 1 \left( u \in S_t, v \in S_t \right).$$

Similarly, we can quantify the LP weight removed by the algorithm at step $t$, which we denote by $LP_t$. We count the contribution of all edges $(u, v) \in E^+ \cup E^-$ such that $u \in S_t$ or $v \in S_t$. Thus,

$$LP_t = \sum_{(u,v) \in E^+ \setminus V_t} w_{uv} x_{uv} \cdot 1 \left( u \in S_t \text{ or } v \in S_t \right) + \sum_{(u,v) \in E^- \setminus V_t} w_{uv} (1 - x_{uv}) \cdot 1 \left( u \in S_t \text{ or } v \in S_t \right).$$

Note that the cost of the solution produced by the algorithm is the sum of the violations across all steps, that is $ALG = \sum_{t} ALG_t$. Moreover, as every edge is removed exactly once from the graph, we can see that $LP = \sum_{t} LP_t$. We will charge the cost of the violations of the algorithm at step $t$, $ALG_t$, to the LP weight removed at step $t$, $LP_t$. Hence, if we show that $E[ALG_t] \leq \rho E[LP_t]$ for every step $t$, then we can conclude that the approximation factor of the algorithm is at most $\rho$, since

$$E[ALG] = E \left[ \sum_{t} ALG_t \right] \leq \rho \cdot E \left[ \sum_{t} LP_t \right] = \rho \cdot LP.$$

We now express $ALG_t$ and $LP_t$ in terms of $cost(\cdot)$ and $lp(\cdot)$ which are defined in Section 3.1. This will allow us to group together the terms for each triplet $u, v, w$ in the set of active vertices and thus write $ALG_t$ and $LP_t$ in terms of $ALG^\sigma(\cdot)$ and $LP^\sigma(\cdot)$, respectively.

For analysis, we assume that for each vertex $u \in V$, there is a positive (similar) self-loop, and thus we can define $cost(u, u \mid w)$ and $lp(u, u \mid w)$ formally as follows: $cost(u, u \mid w) = \Pr(u \in S, u \not\in S \mid p = w) = 0$ and $lp(u, u \mid w) = x_{uu} \cdot \Pr(u \in S \mid p = w) = 0$ (recall that $x_{uu} = 0$).

$$E[ALG_t \mid V_t] = \sum_{(u,v) \in E \setminus V_t} \left( \frac{1}{|V_t|} \sum_{w \in V_t} w_{uv} \cdot cost(u, v \mid w) \right) = \frac{1}{2|V_t|} \sum_{u,v,w \in V_t \setminus w \neq v} w_{uv} \cdot cost(u, v \mid w) \quad (30)$$

$$E[LP_t \mid V_t] = \sum_{(u,v) \in E \setminus V_t} \left( \frac{1}{|V_t|} \sum_{w \in V_t} w_{uv} \cdot lp(u, v \mid w) \right) = \frac{1}{2|V_t|} \sum_{u,v,w \in V_t \setminus w \neq v} w_{uv} \cdot lp(u, v \mid w) \quad (31)$$

We divide the expressions on the right hand side by 2 because the terms $cost(u, v \mid w)$ and $lp(u, v \mid w)$ are counted twice. Now adding the contribution of terms $cost(u, u \mid w)$ and $lp(u, u \mid w)$ (both equal to 0) to (30) and (31), respectively and grouping the terms containing $u, v$ and $w$ together, we get,

$$E[ALG_t \mid V_t] = \frac{1}{6|V_t|} \sum_{u,v,w \in V_t} \left( w_{uv} \cdot cost(u, v \mid w) + w_{uw} \cdot cost(u, w \mid v) + w_{uw} \cdot cost(w, v \mid u) \right)$$

$$= \frac{1}{6|V_t|} \sum_{u,v,w \in V_t} ALG^\sigma(x, y, z)$$

21
and

$$\mathbb{E}[LP_t \mid V_t] = \frac{1}{6|V_t|} \sum_{u,v,w \in V_t} \left( w_{uv} \cdot lp(u, v \mid w) + w_{uw} \cdot lp(u, w \mid v) + w_{wv} \cdot lp(w, v \mid u) \right)$$

$$= \frac{1}{6|V_t|} \sum_{u,v,w \in V_t} LP^\sigma(x, y, z)$$

Thus, if $ALG^\sigma(x, y, z) \leq \rho LP^\sigma(x, y, z)$ for all signatures and edge lengths $x, y, z$ satisfying the triangle inequality, then $\mathbb{E}[ALG_t \mid V_t] \leq \rho \cdot \mathbb{E}[LP_t \mid V_t]$, and, hence, $\mathbb{E}[ALG] \leq \rho \cdot \mathbb{E}[LP]$ which finishes the proof.
Figure 2: This plot shows functions $f_\alpha(x)$ used in the proof of Theorem 1.1 for $\alpha \in \{0.001, 0.01, 0.1\}$. Additionally, it shows optimal functions $f_{opt}(x)$ (see Section 4 for details). Note that every function $f_\alpha(x)$, including $f_{opt}(x)$, has a discontinuity at point $\tau = 1/2 - 1/2A$; for $x \geq \tau$, $f_\alpha(x) = 1$. 
Figure 3: Plots of approximation factors $A_{thm}$ and $A_{opt}$. 