LAWS OF THE ITERATED LOGARITHM FOR A CLASS OF ITERATED PROCESSES

ERKAN NANE

Abstract. Let $X = \{X(t), t \geq 0\}$ be a Brownian motion or a spectrally negative stable process of index $1 < \alpha < 2$. Let $E = \{E(t), t \geq 0\}$ be the hitting time of a stable subordinator of index $0 < \beta < 1$ independent of $X$. We use a connection between $X(E(t))$ and the stable subordinator of index $\beta/\alpha$ to derive information on the path behavior of $X(E_t)$. This is an extension of the connection of iterated Brownian motion and ($\frac{1}{4}$)-stable subordinator due to Bertoin [7]. Using this connection, we obtain various laws of the iterated logarithm for $X(E(t))$. In particular, we establish law of the iterated logarithm for local time Brownian motion, $X(L(t))$, where $X$ is a Brownian motion (the case $\alpha = 2$) and $L(t)$ is the local time at zero of a stable process $Y$ of index $1 < \gamma \leq 2$ independent of $X$. In this case $E(\rho t) = L(t)$ with $\beta = 1 - 1/\gamma$ for some constant $\rho > 0$. This establishes the lower bound in the law of the iterated logarithm which we could not prove with the techniques of our paper [27]. We also obtain exact small ball probability for $X(E_t)$ using ideas from [2].

Key words and phrases. Local time Brownian motion, hitting time of stable subordinator, spectrally negative stable process, law of the iterated logarithm, Hirsch’s integral test, Kolmogorov’s integral test, small ball probability.
1. Introduction

Self-similar processes arise naturally in limit theorems of random walks and other stochastic processes, and they have been applied to model various phenomena in a wide range of scientific areas including telecommunications, turbulence, image processing and finance. In this paper, we will study a class of self similar processes that are non-Gaussian and non-Markovian.

To define the processes studied in this paper, let $X = \{X(t), t \geq 0\}$ be a Brownian motion or a spectrally negative stable Lévy process of index $1 < \alpha < 2$ started at 0. When $\alpha = 2$, we denote $X = W$. Let $E = \{E(t), t \geq 0\}$ be the hitting time of a stable subordinator of index $0 < \beta < 1$ independent of $X$. Our aim in this paper is to obtain various laws of the iterated logarithm (LIL) for the process $Z = \{Z(t) = X(E(t)), t \geq 0\}$. This process is self-similar with index $H = \beta/\alpha$.

Many authors have constructed and investigated various classes of non-Gaussian self-similar processes. See, for example, \cite{38} for information on self-similar stable processes with stationary increments. Let $W'$ be an independent copy of $W$. Define $B_1(t) = W(t), t \geq 0$ and $B_1(t) = W'(-t), t \leq 0$. Burdzy \cite{13, 14} introduced the so-called iterated Brownian motion (IBM), $A(t) = B_1(B(t))$, by replacing the time parameter in $B_1(t)$ by an independent one-dimensional Brownian motion $B = \{B(t), t \geq 0\}$. His work inspired many researchers to explore the connections between IBM (or other iterated processes) and PDEs, to establish potential theoretical results and to study its sample path properties, see \cite{1, 31, 32, 34, 35, 36, 40} and references therein.

The process $Z$ emerges as the scaling limit of a continuous time random walk with heavy-tailed waiting times between jumps \cite{5, 29}. Moreover, $Z$ has a close connection to fractional partial differential equations. Baeumer and Meerschaert \cite{3}, and Meerschaert and Sheffler \cite{29} showed that the process $Z$ can be applied to provide a solution to the fractional Cauchy problem. More precisely, they proved that, $u(t, x) = E_x[f(Z(t))]$ solves the following fractional in time PDE

\begin{equation}
\frac{\partial^{\beta}}{\partial t^{\beta}}u(t, x) = G_x u(t, x); \quad u(0, x) = f(x),
\end{equation}

where $G_x$ is the generator of the semigroup of $X_t$ and $\partial^{\beta}g(t)/\partial t^{\beta}$ is the Caputo fractional derivative in time, which can be defined as the inverse Laplace transform of $s^{\beta}\tilde{g}(s) - s^{\beta-1}g(0)$, where $\tilde{g}(s) = \int_0^\infty e^{-st}g(t)dt$ is the usual Laplace transform. Let $L(t)$ be the local time at zero of a strictly stable process $Y(t)$ of index $1 < \gamma < 2$ independent of $X$. The process, $W(L(t))$ is called local time Brownian motion in \cite{27}. Recently Baeumer, Meerschaert and Nane \cite{4} further established the equivalence of the governing PDEs of $X(L(t))$ and $X(|B(t)|)$ when $\beta = 1/2$. Here $B = \{B(t), t \geq 0\}$ is another Brownian motion independent of $X$ and $Y$. When $\gamma = 2$, the process $W(L(t))$ has also appeared in the works of Borodin \cite{10, 11}, Ikeda and Watanabe \cite{23}, Kasahara \cite{24}, and Papanicolaou et al. \cite{37}.
We are interested in sample path properties of $Z$. Since $Z$ is non-Gaussian, non-Markovian and does not have stationary increments, the existing theories on Markov and/or Gaussian processes can not be applied to $Z$ directly and some new tools need to be developed. In this regard, we mention that Csáki, Földes and Révész [18] studied the Strassen type LIL of $Z(t) = W(L(t))$ when $L(t)$ is the local time at zero of a symmetric stable Lévy process; Meerschaert, Nane and Xiao [27] established a large deviation result, and uniform modulus of continuity for $W(L(t))$. In particular they obtained upper bounds in the LIL and the modulus of continuity of $W(L(t))$. In this paper, we will establish LIL results for $Z$. This paper is a continuation of [18] and [27].

In section 2, we study small ball probability for $Z(t)$. Our results in Section 2 imply in particular that there can not be a Chung type LIL for $Z$ due to power decay in the small ball probability. In section 3, we study asymptotic properties of $Z = \{Z(t), t \geq 0\}$ by first establishing a connection to a stable subordinator of index $\beta/\alpha$, which is an extension of the connection of iterated Brownian motion and $1/4$-stable subordinator due to Bertoin [7] and then making use of limiting theorems on stable subordinators.

2. Small ball Probability

The study of small ball probability is useful for studying Chung type LIL. In this section we show that there cannot be a Chung type LIL for $Z$.

In what follows we will write $f(t) \approx g(t)$ as $t \to a$ to mean that for some positive $C_1$ and $C_2$, $C_1 \leq \liminf_{t \to a} f(t)/g(t) \leq \limsup_{t \to a} f(t)/g(t) \leq C_2$. We will also write $f(t) \sim g(t)$, as $t \to a$, to mean $\lim_{t \to a} f(t)/g(t) \to 1$.

Next we introduce the processes that will be studied in this paper. A Lévy process $X = \{X(t), t \geq 0\}$ with values in $\mathbb{R}$ is called strictly stable of index $\alpha \in (0, 2)$, $\alpha \neq 1$ if its characteristic function is given by

$$E[\exp(i\xi X(t))] = \exp \left(-t|\xi|^{\alpha} \frac{1 + i\nu \text{sgn}(\xi) \tan(\frac{\pi \alpha}{2})}{\chi}\right),$$

where $-1 \leq \nu \leq 1$ and $\chi > 0$ are constants. In the terminology of [38] Definition 1.1.6], $-\nu$ and $\chi^{-1/\alpha}$ are respectively the skewness and scale parameters of the stable random variable $X(1)$. When $\alpha = 2$ and $\chi = 2$, $X$ is a standard Brownian motion and, if $\chi = 1$, $X$ is a Brownian motion running at twice the speed of standard Brownian motion and we denote it by $W$. When $1 < \alpha < 2$ and $\nu = 1$, $X$ is called spectrally negative stable Lévy process.

In general, many properties of stable Lévy processes can be characterized by the parameters $\alpha, \nu$ and $\chi$. For a systematic account on Lévy processes we refer to [6].

Let $E(t)$ be the continuous inverse of a stable subordinator $D_t$ of index $0 < \beta < 1$:

$$E(t) = \inf\{s : D(s) > t\}, \ (t \geq 0).$$
$E(t)$ has also been called the hitting time of stable subordinator of index $0 < \beta < 1$ in the literature, see, for example, [5].

The next theorem is our first result on the small ball probability of $W(E(t))$.

**Theorem 2.1.** Let $W(t)$ be a Brownian motion. Let $E(t)$ be the continuous inverse of a stable subordinator $D(t)$ of index $0 < \beta < 1$. Then

$$\lim_{u \to 0} u^{-2} \mathbb{P} \left( \sup_{0 \leq t \leq 1} |W(E(t))| \leq u \right) = \frac{32 \Gamma(\beta) \sin(\beta \pi)}{\pi^4} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^3}. $$

**Proof.** From a well-known formula (see Chung [16]):

$$\mathbb{P} \left[ \sup_{0 \leq t \leq 1} |W(t)| \leq u \right] = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} \exp \left[ -\frac{(2k-1)^2 \pi^2}{8u^2} \right],$$

Since $E(t)$ is continuous and nondecreasing, we use conditioning to get

$$\mathbb{P} \left[ \sup_{0 \leq t \leq 1} |Z(t)| \leq u \right] = \mathbb{P} \left[ \sup_{0 \leq t \leq E(1)} |W(t)| \leq u \right]$$

$$= \mathbb{E} \left[ \mathbb{P} \left( \sup_{0 \leq t \leq E(1)} |W(t)| \leq \frac{u}{\sqrt{E(1)}} \right) \right]$$

$$= \mathbb{E} \left( \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} \exp \left[ -\frac{(2k-1)^2 \pi^2 E(1)}{8u^2} \right] \right) \right)$$

(2.2)

The Laplace transform of $E(t)$ ($E(t)$ inverse stable subordinator of index $\beta$) (cf. [22, Remark 2.8]) is

$$\mathbb{E}[\exp(-sE(t))] = E^\beta(-st^\beta) = \sum_{n=0}^{\infty} \frac{(-st^\beta)^n}{\Gamma(1 + n\beta)}$$

(2.3)

which is the Mittag-Leffler function evaluated at $-st^\beta$. And, by [26 equation (13)], for some $c > 0$

$$0 \leq E^\beta(-at^\beta) \leq \frac{c}{(1 + at^\beta)} \right).$$

(2.4)

Using Equation (2.4) and the fact that Mittag-Leffler function is completely monotone we obtain that the series in (2.2) is absolutely convergent, we get by dominated
convergence

\[ E \left( \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} \exp \left[ -\frac{(2k-1)^2\pi^2 E(1)}{8u^2} \right] \right) \]

\[ = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} E^\beta \left( -\frac{(2k-1)^2\pi^2}{8u^2} \right) \]

Therefore, using the fact that

(2.5) \[ E^\beta(-z) \sim cz^{-1} \text{ as } z \to \infty \]

with \( c = \frac{\Gamma(\beta)\sin(\beta\pi)}{\pi} \), see, Krägeloh \[26\] and taking the limit \( u \to 0 \) inside the sum using dominated convergence we obtain

(2.6) \[
\lim_{u \to 0} \frac{\mathbb{P}[ \sup_{0 \leq t \leq 1} |Z(t)| \leq u ]}{u^2} = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} \lim_{u \to 0} \frac{E^\beta(-\frac{(2k-1)^2\pi^2}{8u^2})}{u^2} \\
= \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} \frac{8c}{(2k-1)^2\pi^2} \\
\]

Thus

\[ 32c \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^3} \]

\[ \square \]

Remark 2.2. Since the exact small ball probability in Theorem \[2.1\] is a power decay, we do not expect to get a Chung type LIL for \( Z(t) \). We will obtain a Hirsch’s integral test below for \( Z(t) \) using another method.

Using the same conditioning method in Theorem \[2.1\] and ideas in Aurzada and Lifshits \[2\] and Nane \[33\], we also can get the following

Theorem 2.3. Let \( U = \{ U(t), t \geq 0 \} \) be a self similar process of index \( 0 < H < 1 \), and let \( E(t) \) be the inverse of a stable subordinator of index \( 0 < \beta < 1 \), independent of \( U \). Let \( \theta = 1/H > 0 \) and suppose

(2.7) \[ -\log \mathbb{P} \left( \sup_{0 \leq t \leq 1} |U(t)| \leq u \right) \sim ku^{-\theta}, \text{ as } u \to 0. \]

Then

\[ \mathbb{P} \left( \sup_{0 \leq t \leq 1} |U(E(t))| \leq u \right) \approx u^{1/H} \text{ as } u \to 0. \]
Remark 2.4. In this theorem we get upper and lower bounds for the small ball probability due to the lack of an explicit formula for the small ball probability for the process \( U \).

Proof. Let \( \nu > 0 \). By assumption, for all \( 0 < \epsilon < \epsilon_0(\nu) \),
\[
e^{-k(1+\nu)u^{-\theta}} \leq \mathbb{P}\left( \sup_{0 \leq t \leq 1} |U(t)| \leq u \right) \leq e^{-k(1-\nu)u^{-\theta}}
\]
This implies that there are constants \( c_1, c_2 > 0 \) that depend on \( \epsilon_0 \) such that for all \( u > 0 \),
\[
(2.8) \quad c_1 e^{-k(1+\nu)u^{-\theta}} \leq \mathbb{P}\left( \sup_{0 \leq t \leq 1} |U(t)| \leq u \right) \leq c_2 e^{-k(1-\nu)u^{-\theta}}
\]

Since \( E_t \) is continuous and nondecreasing, we use conditioning to get
\[
\mathbb{P}\left[ \sup_{0 \leq t \leq 1} |U(E(t))| \leq u \right] = \mathbb{P}\left[ \sup_{0 \leq t \leq E(1)} |U_t| \leq u, \right] = \mathbb{E} \left[ \mathbb{P}\left( \sup_{0 \leq t \leq E(1)} |U(t)| \leq \frac{u}{E(1)^H} |E(1)| \right) \right] \leq c_2 E \left[ \exp \left( -k(1-\nu)u^{-\theta} E(1) \right) \right].
\]
we use the upper bound in (2.8) to get the last inequality. We also obtain a lower bound for the small ball probability of \( U(E) \) using the lower bound in (2.8).

Now taking limit as \( u \to 0 \) in (2.9) using equations (2.3), (2.5) and then letting \( \nu \to 0 \) we get the upper bound for the small ball probability of \( U(E) \). With a similar argument we obtain the lower bound. \( \square \)

Fractional Brownian motion (fBm) is a centered Gaussian process \( W^H = \{W^H(t), t \in \mathbb{R}\} \) with \( W^H(0) = 0 \) and covariance function
\[
(2.10) \quad \mathbb{E}(W^H(s)W^H(t)) = \frac{1}{2} \left( |s|^{2H} + |t|^{2H} - |s-t|^{2H} \right),
\]
where \( H \in (0, 1) \) is a constant. \( W^H \) is a self-similar process of index \( H \). The small ball probability is given for \( W^H \) with \( \theta = 1/H \) in (2.7), see, for example, [38]. We obtain the small ball probability for local time fractional Brownian motion, \( W^H(L(t)) \), here \( L(t) \) is the Local time at zero of an independent strictly stable process of index \( 1 < \gamma \leq 2 \), as a consequence of the following corollary using the fact that \( E(\rho t) = L(t) \) with \( \beta = 1 - 1/\gamma \) for some constant \( \rho > 0 \), see, for example, [39].

Corollary 2.5. Let \( W^H = \{W^H(t), t \geq 0\} \) be a fractional Brownian motion with index \( 0 < H < 1 \), and let \( E(t) \) be the inverse of a stable subordinator of index \( 0 < \beta < 1 \), independent of \( W^H \). Then
\[
\mathbb{P}\left( \sup_{0 \leq t \leq 1} |W^H(E(t))| \leq u \right) \approx u^{1/H} \quad \text{as} \quad u \to 0.
\]
The process $W^H(E)$ emerges as the scaling limit of a correlated continuous time random walk with heavy-tailed waiting times between jumps \cite{28}.

Strictly stable process of index $0 < \alpha \leq 2$ is a self-similar process of index $1/\alpha$ and the small ball probability is given with $\theta = \alpha$ in \cite{27}, see, for example, \cite{7}. We obtain the small ball probability estimate of $Z$ as a consequence of

**Corollary 2.6.** Let $X$ be a strictly stable process of index $0 < \alpha \leq 2$ and let $E_t$ be the inverse of a stable subordinator of index $0 < \beta < 1$, independent of $X$. Then

$$\mathbb{P}\left(\sup_{0 \leq t \leq 1} |X(E(t))| \leq u\right) \approx u^\alpha \text{ as } u \to 0.$$  

**Remark 2.7.** Since the Laplace transform of the process $E(t)$ decays like $E(e^{-zE(1)}) \sim c(\beta)/z$ as $z \to \infty$ for which the power of $z$ is independent of the index $\beta$, the small ball probabilities does not depend on $\beta$ in the theorems in this section.

For other extensions of small ball probabilities for iterated (fractional) Brownian motions and iterated stable processes, see Aurzada and Lifshits \cite{2}.

### 3. Path regularity of $Z$

We will consider the process $Z = \{Z(t), t \geq 0\}$ defined by

$$Z(t) = X(E(t)), \forall t \geq 0,$$

where $X = W$ for $\alpha = 2$ and $X$ is the spectrally negative stable Lévy process with $1 < \alpha < 2$, $\nu = 1$.

Let $S(t) = \sup_{0 \leq s \leq t} X(s)$. Since $E(t)$ is an increasing and continuous process, we have

$$\bar{Z}(t) = \sup_{0 \leq s \leq t} Z(s) = S(E(t)).$$

Let $\sigma^\alpha$ be a stable subordinator of index $1/\alpha$, then $S$ is the continuous inverse $\sigma^\alpha$, see Bingham \cite{8} Proposition 1;.

$$S(t) = \inf\{s : \sigma^\alpha(s) > t\}, (t \geq 0)$$

more precisely,

$$\mathbb{E}\left(\exp(-s\sigma^\alpha(t))\right) = \exp(-t(s/c_1)^{1/\alpha}) \ (s \geq 0, t \geq 0),$$

where $c_1 = \chi^{-1} \sec(\pi - \frac{1}{2} \pi \alpha)$.

The inverse $D(t)$ of $E(t)$ is a stable subordinator of index $0 < \beta < 1$. Here $D$ has Laplace transform, $\mathbb{E}(e^{-sD(t)}) = e^{-ts^\beta}$. According to Bochner \cite{9}, $D \circ \sigma^\alpha$ is a stable subordinator with index $\frac{\beta}{\alpha}$, and more precisely,

$$\log\left(\mathbb{E}[e^{-sD\circ\sigma^\alpha(t)}]\right) = mt \int_0^\infty (e^{-sx} - 1)x^{-(1+\beta/\alpha)}dx = -c_1^{-1/\alpha} ts^{\beta/\alpha}$$
by integration by parts, we can show that

\[ m = c_1^{-1/\alpha}(\beta/\alpha)[\Gamma(1 - \beta/\alpha)]^{-1}. \]

Define

\[ \mu = (\Gamma(1 - \beta/\alpha)m)^{\alpha/(\alpha - \beta)}(\alpha - \beta)/\beta. \]

Let \( \lambda = (\beta/\alpha)/(1 - \beta/\alpha) = \beta/(\alpha - \beta) \). Plainly, \( D \circ \sigma^\alpha \) is the right-continuous inverse of \( S \circ E \).

**Proposition 3.1.** The right-continuous inverse of the supremum of \( Z(t) \) is

\[ \inf\{ t : Z(t) > \cdot \} = \sigma^Z(\cdot), \]

where \( \sigma^Z \) is a stable subordinator with Laplace exponent \( s \to c_1^{-1/\alpha}s^{\beta/\alpha} \).

We can now apply certain results in the literature on stable subordinators to derive asymptotic results about \( Z \).

Our first result on LIL is the following

**Theorem 3.2.** Let \( X \) be a spectrally negative stable Lévy process with \( 1 < \alpha \leq 2 \), \( \nu = 1 \), and let \( E(t) \) be the inverse to a stable subordinator of index \( 0 < \beta < 1 \) independent of \( X \). Then

\[ \limsup_{t \to 0^+ \text{ and } t \to \infty} \frac{Z(t)}{t^{\beta/\alpha}(\log |\log t|)^{(\alpha - \beta)/\alpha}} = (c(\beta, \alpha))^{\beta/\alpha} \text{ a.s.} \]

Both as \( t \to 0^+ \) and \( t \to \infty \). Here \( c(\beta, \alpha) = \mu^{1/\lambda} \).

When \( \alpha = 2 \), the upper bound in this large time LIL result was proved by Meerschaert, et al. [27] for the case \( E_{\rho t} = L_t \) is the local time at zero of a strictly stable process of index \( 1 < \gamma \leq 2 \) and in this case \( \beta = 1 - 1/\gamma \). Here \( \rho > 0 \) given by

\[ \rho^{-(1-1/\gamma)} = \pi^{-1}\Gamma(1 + 1/\gamma)\Gamma(1 - 1/\gamma)\chi^{-1}Re\{[1 + i\nu\tan(\pi\gamma/2)]^{-1/\gamma}\}, \]

see Stone [39]. Hence we obtain the following LIL for local time Brownian motion \( W(L(t)) \).

**Corollary 3.3.** Let \( W \) be a Brownian motion, and let \( L(t) \) be the local time at zero of a stable process of index \( 1 < \gamma \leq 2 \), independent of \( W \). Then

\[ \limsup_{t \to 0^+ \text{ and } t \to \infty} \frac{W(L(t))}{t^{(\gamma-1)/2\gamma}(\log |\log t|)^{(\gamma+1)/2\gamma}} = (\rho c(\beta = 1 - 1/\gamma, 2))^{(\gamma-1)/2\gamma} \text{ a.s.} \]

Both as \( t \to 0^+ \) and \( t \to \infty \).

**Remark 3.4.** Corollary 3.3 is an extension of the results in [27, 18]. Also the case \( \gamma = 2 \) gives the LIL for a version of iterated Brownian motion, \( W(|B(t)|) \), which was obtained in [13, 17], here \( B \) is a Browninan motion independent of \( W \).

**Proof of Theorem 3.2.** Applying the LIL for stable subordinators proved by Fristedt [19] (see also Breiman [12]) to \( D \circ \sigma^\alpha \), we have

\[ \liminf_{t \to 0^+ \text{ and } t \to \infty} \frac{D \circ \sigma^\alpha(t)}{t^{\alpha/\beta}(\log |\log t|)^{\beta/\alpha}/\beta} = \mu^{1/\lambda} = c(\beta, \alpha) \text{ a.s.} \]
both as \( t \to 0^+ \) and \( t \to \infty \).

Since \( S^\alpha \circ E(D \circ \sigma^\alpha_t) = t \), we deduce that the probability that

\[
t \leq S^\alpha \circ E(c t^{\alpha/\beta} (\log |\log t|)^{(\beta-\alpha)/\beta})
\]

infinitely often as \( t \to 0^+ \) (or as \( t \to \infty \)) equals 1 for \( c > c(\beta, \alpha) \) and 0 for \( c < c(\beta, \alpha) \). Recall that \( t \to t^{\beta/\alpha} (\log |\log t|)^{(\alpha-\beta)/\alpha} \) is an asymptotic inverse of \( t \to t^{\alpha/\beta} (\log |\log t|)^{(\beta-\alpha)/\beta} \), so

\[
\lim \sup \frac{S^\alpha \circ E(t)}{t^{\beta/\alpha} (\log |\log t|)^{(\alpha-\beta)/\alpha}} = (c(\beta, \alpha))^{\beta/\alpha} \text{ a.s.}
\]

both as \( t \to 0^+ \) and \( t \to \infty \).

One can replace \( S \circ E(t) \) with \( Z(t) \).

We next obtain the Kolmogorov’s integral test for \( Z(t) \).

**Theorem 3.5.** Let \( X \) be a spectrally negative stable Lévy process with \( 1 < \alpha \leq 2 \), \( \nu = 1 \), and let \( E(t) \) be the inverse to a stable subordinator of index \( 0 < \beta < 1 \) independent of \( X \). Then

\[
\mathbb{P}[Z(t) \geq t^{\beta/\alpha} f(t) \text{ infinitely often as } t \to \infty] = 0 \text{ or } 1,
\]

according as

\[
\int_1^\infty f(t)^{1/(\alpha-\beta)} \exp(-\mu[f(t)]^{\alpha/(\alpha-\beta)}) \frac{dt}{t}
\]

converges or diverges. A similar result holds for small times by replacing the integral over \((1, \infty)\) with the integral over \((0, 1)\).

**Proof.** Breiman [12] obtained a refinement of Fristedt’s LIL for stable subordinators. It states that if \( \varphi : (0, \infty) \to (0, \infty) \) is a decreasing function, then

(3.1) \quad \mathbb{P}[D \circ \sigma(t) \leq t^{\alpha/\beta} \varphi(t) \text{ infinitely often as } t \to \infty] = 0 \text{ or } 1

according as the integral

\[
I(\varphi) = \int_1^\infty [\varphi(t)]^{-\lambda/2} e^{-\mu[\varphi(t)]^{-\lambda}} \frac{dt}{t}
\]

converges or diverges. Here \( \lambda = \beta/(\alpha - \beta) \).

Let \( f : (0, \infty) \to (0, \infty) \) be an increasing function, then by first replacing \( t \) by \( t^{\beta/\alpha} f(t) \) in (3.1), we get that

\[
D \circ \sigma^\alpha(t^{\beta/\alpha} f(t)) \leq tf(t)^{\alpha/\beta} \varphi(t^{\beta/\alpha} f(t)) \text{ infinitely often as } t \to \infty
\]

and then taking \( S \circ E \) of both sides we get the probability that

\[
t^{\beta/\alpha} f(t) \leq S \circ E(tf(t)^{\alpha/\beta} \varphi(t^{\beta/\alpha} f(t))) \text{ as } t \to \infty
\]

equals 0 or 1 according as \( I(\varphi) < \infty \) or \( I(\varphi) = \infty \). If we now choose \( \varphi \) such that \( \varphi(t^{\beta/\alpha} f(t)) = f(t)^{-\alpha/\beta} \), we then see that

\[
\mathbb{P}[S \circ E(t) \geq t^{\beta/\alpha} f(t) \text{ infinitely often as } t \to \infty] = 0 \text{ or } 1,
\]
according as $I(\varphi) < \infty$ or $I(\varphi) = \infty$. An easy calculation shows that the latter alternative reduces to decide whether the integral (replace $\varphi(t)$ with $f(t)^{-\alpha/\beta}$)

$$\int_1^\infty f(t)^{1/(\alpha-\beta)} \exp(-\mu [f(t)]^{\alpha/(\alpha-\beta)}) \frac{dt}{t}$$

converges or diverges. We can replace $S \circ E$ by $X \circ E = Z$, which gives the Kolmogorov’s integral test for the upper functions of $Z$ for large times. A similar result holds for small times by replacing the integral over $(1, \infty)$ with the integral over $(0, 1)$.

$\square$

We next obtain Hirsch’s integral test for $Z(t)$

**Theorem 3.6.** Let $X$ be a spectrally negative stable Lévy process $1 < \alpha \leq 2$, $\nu = 1$, and let $E(t)$ be the inverse to a stable subordinator of index $0 < \beta < 1$ independent of $X$. Then

$$\liminf_{t \to \infty} Z(t)/\varphi(t) = 0 \text{ or } \infty$$

according as the integral $\int_1^\infty \varphi(t) t^{-(1+\beta/\alpha)} dt$ diverges or converges. A similar result holds for small times by replacing the integral over $(1, \infty)$ with the integral over $(0, 1)$.

**Proof.** Recall that if $f : (0, \infty) \to (0, \infty)$ is an increasing function, then

$$\limsup_{t \to \infty} D \circ \sigma^\alpha(t)/f(t) = 0 \text{ or } \infty \text{ a.s.}$$

according as $\int_1^\infty f(t)^{-\beta/\alpha} dt$ converges or diverges. See Theorem 11.2 on page 361 in Fristedt [20]. Consider first the case Limsup is equal to 0. Let $\epsilon > 0$. Then,

$$D \circ \sigma^\alpha(t) \leq \epsilon f(t) \text{ for all } t \text{ large a.s.}$$

Now taking $S \circ E$ of both sides we get

$$t \leq S \circ E(\epsilon f(t)) \text{ for all } t \text{ large a.s.}$$

Using scaling of $S \circ E$ we get

$$t \leq (\epsilon f(t))^{\beta/\alpha} t^{\beta/\alpha} S \circ E(t) \text{ for all } t \text{ large a.s.}$$

hence

$$t^{1+\beta/\alpha}(\epsilon f(t))^{-\beta/\alpha} \leq S \circ E(t) \text{ for all } t \text{ large a.s.}$$

We use a similar argument for the case Limsup is infinite. Hence, setting $\varphi(t) = t^{1+\beta/\alpha} f(t)^{-\beta/\alpha}$ we get

$$\liminf_{t \to \infty} S \circ E(t)/\varphi(t) = 0 \text{ or } \infty$$

according as the integral $\int_1^\infty \varphi(t) t^{-(1+\beta/\alpha)} dt$ diverges or converges. Again, there is a similar result for small times. $\square$

We next obtain a modulus of continuity result for the supremum process $\bar{Z}$. 

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10
Theorem 3.7. Let $X$ be a spectrally negative stable Lévy process with $1 < \alpha \leq 2$, $\nu = 1$, and let $E(t)$ be the inverse to a stable subordinator of index $0 < \beta < 1$ independent of $X$. Then

$$
\lim_{\epsilon \to 0^+} \sup_{0 \leq t \leq 1, 0 < h < \epsilon} \frac{\bar{Z}(t+h) - \bar{Z}(t)}{h^{\beta/\alpha} \log(1/h)^{(\alpha - \beta)/\alpha}} = d(\alpha, \beta) = (\alpha c_2(\beta/\alpha)/\beta)^{(1-\beta/\alpha)} \text{ a.s.}
$$

where $c_2 = c_2(\beta/\alpha) = (1 - \beta/\alpha)(\beta/\alpha)^{(\alpha - \beta)/\beta}$.

Proof. Since $\sigma Z$ is a stable subordinator of index $\beta/\alpha$, Theorem 1 in Hawkes [21] gives (here we change the Lévy measure of $\sigma^\alpha$ so that the Laplace exponent is $s \mapsto s^{\beta/\alpha}$)

$$
\lim_{\epsilon \to 0^+} \inf_{0 \leq t \leq 1, 0 < h < \epsilon} \frac{\sigma^Z(t+h) - \sigma^Z(t)}{h^{\beta/\alpha} \log(1/h)^{-(\alpha - \beta)/\beta}} = c_2{(\alpha-\beta)/\beta} \text{ a.s.}
$$

where $c_2 = c_2(\beta/\alpha) = (1 - \beta/\alpha)(\beta/\alpha)^{(\alpha - \beta)/\beta}$. Taking inverses as in Hawkes [21] we obtain

$$
\lim_{\epsilon \to 0^+} \sup_{0 \leq t \leq 1, 0 < h < \epsilon} \frac{\bar{Z}(t+h) - \bar{Z}(t)}{h^{\beta/\alpha} \log(1/h)^{(\alpha - \beta)/\alpha}} = d(\alpha, \beta) = (\alpha c_2(\beta/\alpha)/\beta)^{(1-\beta/\alpha)} \text{ a.s.}
$$

□

Remark 3.8. We can also deduce the analog of the Chung-Erdős-Sirao integral test for $\bar{Z}$ from Theorem 7.2.1 in Mijnheer [30]. We refer to Meerschaert, Nane and Xiao [27] for the modulus of continuity of $W(L(t))$, where $L(t)$ is the local time at zero of an independent strictly stable process $Y$ of index $1 < \gamma \leq 2$. We also refer to Khoshnevisan and Lewis [25] for the modulus of continuity of IBM $A(t)$.

Remark 3.9. Bertoin [7], remarked that similar results holds for $n$-iterated Brownian motions. In this regard we get similar LIL and integral tests for $n$-iterated $E(t)$ processes. We do not know whether this is the case for $n$-iterated spectrally negative Lévy processes.

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Erkan Nane, Mathematics and Statistics, Auburn University, 221 Parker Hall, Auburn, AL 36849
E-mail address: nane@auburn.edu