Action of Groups on the Projective Plane over the Field \( \mathbb{GF}(41) \)

Emad Bakr Al-Zangana, Saja Abd Joudah
Department of Mathematics, College of Science–Mustansiriyah University, Iraq.
e.b.abdulkareem@uomustansiriyah.edu.iq, saja94.abd@gmail.com

Abstract. An \( k \)-arc is a set of \( k \) points of a projective plane such that some 2, but no 3 of them, are collinear. In this paper, an 5-arc of stabilizer group of type dihedral group of degree five, \( D_5 \) with five collinear points has been found in \( PG(2,41) \). From this 5-arc, a unique 6-arc of stabilizer group of type alternating group of degree five, \( A_5 \) with ten \( B \)-points is found. Finally, the effects of \( D_5 \) and \( A_5 \) on the points of \( PG(2,41) \) are discussed.

1. Introduction
Let \( GF(q) \) denote the Galois field of \( q \) elements and \( V(3, q) = \{(a_1, a_2, a_3)|a_i \in GF(q)\} \) be the respective vector space of row vectors of length three with entries in \( GF(q) \). Let \( PG(2, q) \) be the projective plane over the field \( GF(q) \). The subspace of \( PG(2, q) \) of dimension one is the zero set of the form of degree one

\[ V(aX_0 + bX_1 + cX_2) \]

which is called lines and the zero set of the form \( F \) of degree two:

\[ V(F) = V(aX_0^2 + bX_1^2 + cX_2^2 + dX_0X_1 + eX_0X_2 + fX_1X_2) \]

It is called plane quadric. The number of points dually; the number of lines in \( PG(2, q) \) is \( q^2 + q + 1 \). There are \( q + 1 \) points on each line dually; there are \( q + 1 \) lines passing through a point. For an introduction to coverings of vector spaces over finite fields and to the concept of projective plane, see [1, 2].

There are 1723 points and lines in \( PG(2,41) \), 42 points on every line and 42 lines passing through each point.
Let \( \alpha \) be the primitive element in \( GF(41) \). The following matrix:

\[
T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha^{39} & 1 & 1 \end{bmatrix}
\]

has been used to construct the points and lines of \( PG(2,41) \) by the following way:
The points are \( P(i) = [1,0,0]T^{i-1} \) and the lines are \( \ell_i = \ell_1 T^{i-1} , i = 1, ..., 41^2 + 41 + 1 \).
Where \( \ell_1 = \mathcal{V}(X_2) \) be the line passing through points \( P(X_0, X_1, X_2) \) with \( X_2 = 0 \). For detailed description of the representation of points and lines representations see [1, Chap. 4].

**Definition 1.1** [1]: A \( k \)-arc \( K \) or arc of degree 2 in \( PG(2, q) \) with \( k \geq 3 \) is a set of \( k \) points such that every lines meets \( K \) in at most two points and there is some lines meeting \( K \) in exactly two points.

**Definition 1.2** [1]: A line \( \ell \) of \( PG(2, q) \) is an \( i \)-secant of an \( k \)-arc \( K \) if \( |\ell \cap K| = i \). A 2-secant is called a bisecant, a 1-secant a unisecant and a 0-secant is an external line. The number of bisecants through a point \( Q \) out of \( K \) is called the index of \( Q \) with respect to \( K \).

Let \( c_i \) be the number of points of \( PG(2, q) \backslash K \) with index exactly \( i \). A point of index three is called a Brianchon point or \( B \)-point for short. The arc is called complete if \( c_0 = 0 \).

During this research, write \( ij \cdot kl \cdot mn = P_iP_j \cap P_kP_l \cap P_mP_n \) for \( B \)-point. Some finite groups that appear in this research are listed below.

- \( I \) = Trivial group.
- \( C_n \) = Cyclic group of order \( n \).
- \( S_n \) = Symmetric group of degree \( n \).
- \( A_n \) = Alternating group of degree \( n \).
- \( D_n \) = Dihedral group of order \( 2n = 〈r, s| r^n = s^2 = (rs)^2 = 1〉 \).

For details and full descriptions about above groups of order less than 32, see [3].

In [1], a full classification and details of arcs in \( PG(2,9) \) shows that there is a unique 5-arc with stabilizer group isomorphic to the dihedral group of degree five and a unique 6-arc with stabilizer group isomorphic to the alternating group of degree 5 with ten \( B \)-points. In [4], the same results have been obtained. In [5], theoretically existence of this 5-arc and 6-arc are proved. Also, there are many mathematicians study the effects of group on the projective plane as in [6]. In [7][8], the action of the groups \( D_5 \) and \( A_5 \) on the points of the projective plane over \( GF(q) \), \( q = 19,29,31 \) are studied.

The first aim of this paper is to compute the inequivalent 5-arcs and then show that there is a unique 5-arc in \( PG(2,41) \) with stabilizer group isomorphic to \( D_5 \) with five collinear points. The second goal is to find the 6-arcs and to show that there is a unique 6-arc stabilized by group isomorphic to \( A_5 \) with ten \( B \)-points. The third aim is to examine the action of the groups \( D_5 \) and \( A_5 \) on the points of \( PG(2,41) \).

### 2. Pentastigm with Collinearities of its Diagonal Points

**Definition 2.1** [1]: An \( k \)-stigm \( K \) in \( PG(2, q) \) is a set of \( k \) points, no three of which are collinear, together with the \( \frac{1}{2}k(k - 1) \) lines that are join of pairs of the points. The points and lines are called vertices and sides of \( K \), respectively. The intersection points of two sides of \( K \) which do not pass through the same vertex is called diagonal points.

During this research, the terminology pentastigm are used instead of a 5-stigm and write \( ij \cdot kl \) instead of \( P_iP_j \cap P_kP_l \).

Since the vertices of \( K \) form an \( n \)-arc, so, to construct a 5-stigm, started with unique projectively 4-arc, \( \Gamma_4 = \{U_0, U_1, U_2, U\} \) (standard frame) in \( PG(2, q) \) which has stabilizers group isomorphic to \( S_4 \), where

\[
U_0 = [1,0,0], \quad U_1 = [0,1,0], \quad U_2 = [0,0,1], \quad U = [1,1,1].
\]
By adding one point of index zero with respect to $\Gamma_{41}$, the 5-arcs have been constructed and the inequivalent ones have been found using mathematical programming language GAP.

**Theorem 2.2:** In $PG(2,41)$, there exist eighteen inequivalent 5-arcs through the frame $\Gamma_{41}$ with parameters $\{c_0, c_1, c_2\} = [1333,370,15]$ as summarized in (table 1).

| No. | 5-Arc | Stabilizer group types with its generators |
|-----|-------|------------------------------------------|
| 1   | $\mathcal{A}_1 = \Gamma_{41} \cup \{P(a^{13}, a^{12}, 1)\}$ | $I$ |
| 2   | $\mathcal{A}_2 = \Gamma_{41} \cup \{P(a^{17}, a^{22}, 1)\}$ | $C_2 = \langle \{[1,1,1], [0, a^{12}, 0], \{a, a^{37}, a^2, a^{20}\} \rangle$ |
| 3   | $\mathcal{A}_3 = \Gamma_{41} \cup \{P(a^{38}, a^4, 1)\}$ | $I$ |
| 4   | $\mathcal{A}_4 = \Gamma_{41} \cup \{P(a^{22}, a^{13}, 1)\}$ | $C_2 = \langle \{[1,1,1], [0, a^{36}, 0], \{a^2, a^{33}, a^{20}\} \rangle$ |
| 5   | $\mathcal{A}_5 = \Gamma_{41} \cup \{P(a^{35}, a^{37}, 1)\}$ | $I$ |
| 6   | $\mathcal{A}_6 = \Gamma_{41} \cup \{P(a^{38}, a^{11}, 1)\}$ | $I$ |
| 7   | $\mathcal{A}_7 = \Gamma_{41} \cup \{P(a^{32}, a^{23}, 1)\}$ | $I$ |
| 8   | $\mathcal{A}_8 = \Gamma_{41} \cup \{P(a^{11}, a^7, 1)\}$ | $C_2 = \langle \{[0,0,1], \{a^{20}, a^{20}, a^{20}\}, \{1, 0, 0\} \rangle$ |
| 9   | $\mathcal{A}_9 = \Gamma_{41} \cup \{P(a^6, a^{16}, 1)\}$ | $I$ |
| 10  | $\mathcal{A}_{10} = \Gamma_{41} \cup \{P(a^{26}, a^{13}, 1)\}$ | $C_2 = \langle \{0,0,0, a^{20}\}, \{0, a^{33}, 0\}, \{a^6, 0, 0\} \rangle$ |
| 11  | $\mathcal{A}_{11} = \Gamma_{41} \cup \{P(a^{35}, a^{11}, 1)\}$ | $C_2 = \langle \{a^6, 0, 0\}, \{a^{15}, a^{31}, a^{20}\}, \{a^{11}, a^{11}, a^{11}\} \rangle$ |
| 12  | $\mathcal{A}_{12} = \Gamma_{41} \cup \{P(a^{32}, a^{39}, 1)\}$ | $C_2 = \langle \{a^{39}, a^{39}, a^{39}\}, \{a^{12}, a^{19}, a^{20}\}, \{0,0,0, a^{21}\} \rangle$ |
| 13  | $\mathcal{A}_{13} = \Gamma_{41} \cup \{P(a^{25}, a^{14}, 1)\}$ | $C_2 = \langle \{a^5, a^{24}, a^{20}\}, \{0,0, a^{11}\}, \{0, a^{39}, 0\} \rangle$ |
| 14  | $\mathcal{A}_{14} = \Gamma_{41} \cup \{P(a^{36}, a^{15}, 1)\}$ | $I$ |
| 15  | $\mathcal{A}_{15} = \Gamma_{41} \cup \{P(a^{14}, a^{32}, 1)\}$ | $C_4 = \langle \{0,0,0,0\}, \{0, a^{32}, 0\}, \{a^{34}, a^{12}, a^{20}\} \rangle$ |
| 16  | $\mathcal{A}_{16} = \Gamma_{41} \cup \{P(a^6, a^{22}, 1)\}$ | $I$ |
| 17  | $\mathcal{A}_{17} = \Gamma_{41} \cup \{P(a^{39}, a, 1)\}$ | $D_5 = \langle r = \{a, 0, 0, 0, 0, 1, 1, 0\}, \{a^{19}, a^{21}, a^{20}\} \rangle$ |
| 18  | $\mathcal{A}_{18} = \Gamma_{41} \cup \{P(a^{34}, a^{22}, 1)\}$ | $C_2 = \langle \{a^{37}, 0, 0\}, \{a^{14}, a^2, a^{20}\}, \{a^{22}, a^{22}, a^{22}\} \rangle$ |

The condition to existence a pentastigm with five diagonal points are collinear in $PG(2,q)$ is that $x^2 - x - 1 = 0$ has solution in $F_q$ [6]. Therefore, the following corollary holds.

**Corollary 2.3:** If $q = 41$, the equation $x^2 - x - 1 = 0$ has two solutions $a^{21}, a^{39}$, that is; $PG(2,41)$ has a pentastigm with five collinear diagonal points.

**Theorem 2.4:** In $PG(2,41)$, the pentastigm which has the 5-arc $\mathcal{A}_{17}$ as vertices has five diagonal points which are collinear on the line $\ell_{1562} = V(a^{21}X_0 - X_1 + X_2)$ as shown below:

The diagonal points exactly are the points of index 2. Therefore, any 5-stigm has fifteen 15 diagonal. 

1. $01 \cdot 23 = P(1,1,0)$,
2. $01 \cdot 24 = P(1,0,1)$,

3. $02 \cdot 34 = P(0, a, 1)$,
4. $03 \cdot 12 = P(a^{19}, 1, 0)$,
5. $04 \cdot 13 = P(a^{38}, 1, 1)$,
6. $04 \cdot 23 = P(a, a, 1)$,
(3) \(01 \cdot 34 = P(0,1,1)\), 
(8) \(03 \cdot 14 = P(\alpha^{39},1,1)\), 
(13) \(12 \cdot 34 = P(0,\alpha^{39},1)\), 
(4) \(02 \cdot 13 = P(\alpha^{38},1,0)\), 
(9) \(03 \cdot 24 = P(1,\alpha,1)\), 
(14) \(13 \cdot 24 = P(1,\alpha^2,1)\), 
(5) \(02 \cdot 14 = P(\alpha^{39},0,1)\), 
(10) \(04 \cdot 12 = P(\alpha^{38},0,1)\), 
(15) \(14 \cdot 23 = P(\alpha^{39},\alpha^{39},1)\).

The only line which passes through the five diagonal points is 
\[\ell_{1562} = V(\alpha^{21}X_0 - X_1 + X_2)\]
And the points are given below.
\[
\begin{align*}
01 \cdot 34 &= P(0,1,1) = 1562, \\
02 \cdot 14 &= P(\alpha^{39},0,1) = 782, \\
03 \cdot 12 &= P(\alpha^{19},1,0) = 1240, \\
04 \cdot 23 &= P(\alpha,\alpha,1) = 322, \\
13 \cdot 24 &= P(1,\alpha^2,1) = 279.
\end{align*}
\]

3. Action of \(D_5\) on \(PG(2,41)\)

**Definition 3.1[1]:** A non-singular plane quadric is called a *Conic*.

In \(PG(2,q)\), with \(q \geq 4\), and there is a unique conic through a 5-arc [2]. Therefore, in \(PG(2,41)\), from the 5-arc \(\mathcal{A}_{17}\) there is a unique conic through it as given below.
\[
\mathcal{C}_{\mathcal{A}_{17}} = V(X_0X_1 + \alpha^{20}X_0X_2 - \alpha^{20}X_1X_2)
\]
\[
= \{\alpha^{21}(t^2 - \alpha^{13}t), \alpha^6(1 - \alpha^{27}t), \alpha^{13}t | t \in F_{41} \cup \{\infty\}\}.
\]

From (Table 1), the Dihedral group \(D_5\) generated by
\[
r = \begin{bmatrix}
\alpha & 0 & 0 \\
1 & 1 & 1 \\
\alpha^{19} & \alpha^{12} & \alpha^{20}
\end{bmatrix}, 
\quad s = \begin{bmatrix}
0 & \alpha^{22} & 0 \\
\alpha^{19} & \alpha^{21} & \alpha^{20} \\
1 & 1 & 1
\end{bmatrix}
\]
Which stabilized the 5-arc \(\mathcal{A}_{17}\) has the following effects on the points of \(PG(2,41)\).

1. Fixes the conic \(\mathcal{C}_{\mathcal{A}_{17}}\).
2. Acts transitively on \(\mathcal{A}_{17}\) since

\[
\begin{align*}
(U_0, rs) &\rightarrow U_1, \\
(U_0, s^3) &\rightarrow U_2, \\
(U_0, rs^4) &\rightarrow U_3
\end{align*}
\]

3. The elements of \(D_5\) divided into two classes according to fixing points of \(PG(2,41)\) by sending each point to itself as illustrated below.

**Class 1:** The five elements \(r, rs, rs^2, rs^3, rs^4\) of order two fixes 43 points if acts on \(PG(2,41)\) which is exactly line plus the diagonal point of \(\mathcal{A}_{17}\)

\[
\begin{array}{|c|}
\hline
\ell \setminus \{P_1, P_2, P_3, P_4, P_5\} \\
\hline
r & \ell_{320} \cup P(1,\alpha^4,1) \\
rs & \ell_{325} \cup P(\alpha^2,1,0) \\
rs^2 & \ell_{375} \cup P(\alpha,\alpha,1) \\
rs^3 & \ell_{807} \cup P(\alpha^{39},0,1) \\
rs^4 & \ell_{292} \cup P(0,1,1) \\
\hline
\end{array}
\]

and 37 points if acts on the points of index zero which is precisely line with five points excluded.
4. The lines $\ell_i, i = 3,292,320,375,807$ have the property that unisecant to $\mathcal{A}_{17}$ and bisecant to $C_{\delta_{17}}$.

**Class 2:** Each of the four elements $s, s^2, s^3, s^4$ of order five fixes three points one of the points is $P(a^{39}, a^{19}, 1)$ which is the intersection point of the five lines $\ell_i, i = 3,292,320,375,807$.

4. An 6-Arc with Stabilizer Group of Type $A_5$

The 6-arcs are computed after the action of the stabilizer groups on the points of index zeros. Then the in equivalent are founded as illustrated with details in the following theorem.

**Theorem 4.1:** In $PG(2,41)$, there are 2948 in equivalent 6-arcs through $I_{41}$. The stabilizers of the 6-arcs are partitions as given below.

| 2606: I | 191: $C_2$ | 100: $C_4$ | 8:$V_4$; 9:$C_4$ |
|---------|-----------|-----------|-----------------|
| 1:$C_3$ | 22:$S_3$  | 8:$A_4$; 1:$D_6$ | 1:$S_4$ |
| 1:$A_5$ |

The unique 6-arc $K$ with stabilizer group $A_5$ is just $\mathcal{A}_{17}$ union the intersection point of the lines $\ell_i, i = 3,292,320,375,807$. The arc $K$ in numeral form is $K = \{1,2,3,23,112,443\}$.

And its stabilizer group $A_5$ is given below.

$$a = \begin{bmatrix} a^{20} & 0 & 0 \\ 0 & 0 & a^{20} \\ 0 & a^{20} & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & a & 0 \\ a^{20} & a^{20} & a^{20} \\ a^{39} & a & 1 \end{bmatrix}. $$

There are fifteen ways to choose three bisecants no two of which intersect on $K$ as shown below.

(1) $12 \cdot 34 \cdot 56$
(2) $12 \cdot 35 \cdot 46$
(3) $12 \cdot 36 \cdot 45$
(4) $13 \cdot 24 \cdot 56$
(5) $13 \cdot 25 \cdot 46$
(6) $13 \cdot 26 \cdot 45$
(7) $14 \cdot 23 \cdot 56$
(8) $14 \cdot 25 \cdot 36$
(9) $14 \cdot 26 \cdot 35$
(10) $15 \cdot 23 \cdot 46$
(11) $15 \cdot 24 \cdot 36$
(12) $15 \cdot 26 \cdot 34$
(13) $16 \cdot 23 \cdot 45$
(14) $16 \cdot 24 \cdot 35$
(15) $16 \cdot 25 \cdot 34$

These possibilities divided into two parts:

**Part 1:** Ten possibilities forms $B$-points as shown below.
Let $K_{10} = \{4,485,486,498,947,1049,1265,1481,1561,1693\}$, set of B-points.

**Theorem 4.1:**
(i) The set $K_{10}$ form incomplete 10-arc.
(ii) The set $K_{10}$ partitions the 1723 lines of $PG(2,41)$ into 1348 external lines, 330 tangent lines and 45 bisecants lines.
(iii) $K_{10}$ has parameters $[c_0, c_1, c_2, c_3, c_4] = [372, 1020, 210, 90, 15, 6]$. 

**Part 2:** The remaining five possibilities constitute triangles, as shown below.
Where

\begin{align*}
P_1P_2 &= V(X_2) \\
P_1P_3 &= V(X_1) \\
P_1P_4 &= V(X_2 - X_1) \\
P_1P_5 &= V(\alpha X_2 - X_1) \\
P_1P_6 &= V(X_2 - \alpha X_1)
\end{align*}

\begin{align*}
P_3P_6 &= V(\alpha^{19}X_0 - \alpha^{38}X_1) \\
P_4P_5 &= V(\alpha^{21}X_0 - \alpha^{21}X_1 + \alpha^{20}X_2) \\
P_4P_6 &= V(\alpha^{22}X_0 - \alpha^{20}X_1 + X_2) \\
P_5P_6 &= V(\alpha^{20}X_0 - \alpha^{20}X_1 + X_2) \\
P_3P_5 &= V(X_0 - X_1) \\
P_2P_4 &= V(X_0 + X_2)
\end{align*}

And vertices of each triangle are given in numeral form.

Let the set of five triangles denote by \( M = \{I, II, III, IV, V\} \). The stabilizer group \( A_5 \) of \( K_{10} \) also fixes the set \( M \) of five triangles.

**Example 4.2:** The action of \( A_5 \) on \( I \) partition \( A_5 \) into five disjoint classes with twelve elements in each class as given below.

\[
\begin{array}{c|c}
\text{Class 1} &= \{g \in A_5 | gI = I\} \\
\hline
I &= \\
b \cdot bab &= \\
(abab^2)^3 \cdot (abab^2)^4 &= \\
a \cdot (abab^2)^3 &= \\
a \cdot (ab)^4 &= \\
b \cdot abab^2 &= \\
a \cdot (ab^2)^4 &= \\
a \cdot (bab)^2 &= \\
b \cdot (ab^2)^2 &= \\
ab \cdot (abab^2)^3 &= \\
(ab)^2 \cdot (ab^2)^2 &= \\
(ab)^3 \cdot (abab^2)^4 &= 
\end{array}
\]
Class $2 = \{geA_5|gI = \text{II}\}$

| Expression |
|------------|
| $b \cdot (ab)^3$ |
| $b^2 \cdot (bab)^2$ |
| $b \cdot (ab^2)^4$ |
| $b \cdot (bab)^2$ |
| $(ab)^2 \cdot (ab)^3$ |
| $(bab)^3 \cdot (ab)^2$ |
| $b \cdot a$ |
| $(ab)^2 \cdot (abab^2)^2$ |
| $(ab)^2 \cdot abab^2$ |
| $b^2 \cdot (abab^2)^3$ |
| $ab \cdot (ab^2)^2$ |
| $b \cdot (abab^2)^3$ |

Class $3 = \{geA_5|gI = \text{III}\}$

| Expression |
|------------|
| $(ab)^4 \cdot (abab^2)^3$ |
| $(ab)^2 \cdot (abab^2)^2$ |
| $(ab)^2 \cdot (abab^2)^3$ |
| $ab^2 \cdot (ab^2)^2$ |
| $(ab)^2 \cdot bab$ |
| $a \cdot (ab^2)^3$ |
| $a \cdot b$ |
| $a \cdot (bab)^3$ |
| $a \cdot (abab^2)^4$ |
| $b^2 \cdot (ab)^3$ |
| $b^2 \cdot abab^2$ |

Class $4 = \{geA_5|gI = \text{IV}\}$

| Expression |
|------------|
| $ab \cdot (ab^2)^3$ |
| $b \cdot (abab^2)^2$ |
| $(ab)^3 \cdot (bab)^2$ |
| $(ab)^4 \cdot (ab)^2$ |
| $a \cdot bab$ |
| $a \cdot b^2$ |
| $b^2 \cdot (abab^2)^2$ |
| bab $\cdot (ab^2)^3$ |
| $a \cdot (ab^2)^2$ |
| $b^2 \cdot (ab)^2$ |
| $a \cdot (ab)^3$ |
| $a \cdot abab^2$ |

Class $5 = \{geA_5|gI = \text{V}\}$

| Expression |
|------------|
| $b \cdot (ab^2)^3$ |
| $ab \cdot (abab^2)^2$ |
| $a \cdot (ab)^2$ |
| $a \cdot ab^2$ |
| $a \cdot (abab^2)^2$ |
\[ (ab^2)^3 \cdot (ab)^4 \]
\[ (ab)^4 \cdot (abab^2)^3 \]
\[ (ab)^2 \cdot (ab^2)^3 \]
\[ b \cdot (abab)^4 \]
\[ b \cdot (bab)^3 \]
\[ a \cdot (bab)^4 \]
\[ (ab)^2 \cdot (abab)^3 \]

**Theorem 4.3[5]:** When \( q = \pm 1 \pmod{10} \), then the set
\[ K^* = \{(1,0,1 - 2t), (1,0,2t - 1), (1,2t, 0), (1, -2t, 0), (0,1,2t), (0,1, -2t)\}, \]
when \( 4t^2 - 2t - 1 = 0 \), constitute a 6-arc fixed by \( A_5 \).

**Corollary 4.4:**
1. In \( GF(41) \), the equation \( 4t^2 - 2t - 1 = 0 \) has two solutions, \( \alpha^13 \) and \( \alpha^{35} \).
2. For \( t = \alpha^{13} \), the 6-arc \( K^* = \{(1, 0, \alpha^{21}), (1, 0, \alpha^9), (1, \alpha^{39}, 0), (1, \alpha^{19}, 0), (0,1,\alpha^{39}), (0,1, \alpha^{19})\} \)
is equivalent to the 6-arc \( K \) by the matrix transformation:
\[
T^* = \begin{bmatrix}
\alpha^{28} & \alpha^9 & 0 \\
\alpha^{26} & \alpha^{27} & \alpha^{14} \\
\alpha^7 & \alpha^8 & 0
\end{bmatrix}
\]

**5. Conclusion**
The effect of the stabilizer group \( D_5 \) of the unique 5-arc \( A_{17} \) on the points of \( PG(2,41) \) depends on the order of its elements as follows.

(i) Let \( G^2 \) be the set of the five elements of \( D_5 \) of order two and \( G^5 \) be the set of four elements of \( D_5 \) of order five.

(ii) Each element of \( G^2 \) fixes a subset of the plane \( PG(2,41) \) of length 43 by sending it to itself. These sets are exactly lines \( \ell_i^* \) with extra points \( P_i^* \), \( i = 1, \ldots, 5 \). The five extra points \( P_i^* \) are just the diagonal points of \( A_{17} \) and the lines \( \ell_i^* \) are bisecants of the conic \( C_{A_{17}} \) and unisecants of \( A_{17} \).

(iii) Each element of \( G^5 \) fix three points, one of the points, \( P^* \), is just the intersection point of the five lines \( \ell_i^* \).

**6. Open Problem**
Prove that, in \( PG(2,q) \), \( q \equiv \pm 1 \pmod{10} \), there is a unique 5-arc, \( D \), fixed by group of type \( D_5 \) and a unique 6-arc, \( \mathcal{R} \), with the following properties.

Let \( G^2 \) be a set of the five elements of \( D_5 \) of order two and \( G^5 \) be a set of four elements of \( D_5 \) of order five.

(i) Each element of \( G^2 \) fix \( q + 2 \) points of \( PG(2,q) \) by sending each point to itself which is line, \( \ell_i^* \), union diagonal point of \( D, P_i^* \).

(ii) Each element of \( G^2 \) fix \( q - 4 \) points of index zero with respect to \( D \) which is the line \( \ell_i^* \) excluded five points.

(iii) Each element of \( G^5 \) fix

(a) if \( q \equiv 1 \pmod{10} \) three points by sending each point to itself one of them is the common point of the five lines \( \ell_i^* \).

(b) if \( q \equiv -1 \pmod{10} \) one point by sending it to itself which is the common point of the five lines \( \ell_i^* \).

(iv) The five lines \( \ell_i^* \) are unisecant of \( D \) and bisecant of the unique conic through \( D \).
(v) The 6-arc \( \mathcal{R} \) is just \( \mathcal{D} \) union common point of the five lines \( \ell'_i \).
A special cases of this problem proved for \( q = 19, 29, 31, 41 \).

References
[1] Hirschfeld J W P 1998 Projective geometries over finite fields 2nd Edition Oxford
Mathematical Monographs the Clarendon Press Oxford University Press New York
[2] Gallier J 2013 Geometric methods and applications for computer science and engineering. 2nd Edition Springer New York Dordrecht Heidelberg London
[3] Thomas A D and Wood G V 1980 Group tables Shiva Mathematics Series. Ser. 2 Devon Print
Group Exeter Devon UK
[4] Sadeh A R 1984 Cubics surfaces with twenty seven lines over the eleven elements. Ph.D. Thesis, University of Sussex United Kingdom
[5] Storme L and Maldeghem V 1995 Primitive arcs in \( PG(2,q) \). J. Combin. Theory, Ser. A 69 pp 200-216.
[6] Lorimer P 1993 The groups \( A_7 \), \( A_8 \) and a projective plane of order 16 Australasian Journal of combinatorics 8 pp 45-51
[7] Al-Zangana E B 2011 The geometry of the plane of order nineteen and its application to error-correcting codes Ph.D Thesis University of Sussex United Kingdom
[8] Al-Zangana E B 2013 Groups effect of types \( D_5 \) and \( A_5 \) on the points of projective plane over \( F_q \), \( q = 29, 31 \) Ibn Al-Haitham Jour. for Pure & Appl. Sci. 26 3 pp 410-423.