Quantum Coding Theorem for Mixed States

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Abstract

We prove a theorem for coding mixed-state quantum signals. For a class of coding schemes, the von Neumann entropy $S$ of the density operator describing an ensemble of mixed quantum signal states is shown to be equal to the number of spin-$1/2$ systems necessary to represent the signal faithfully. This generalizes previous works on coding pure quantum signal states and is analogous to the Shannon’s noiseless coding theorem of classical information theory. We also discuss an example of a more general class of coding schemes which beat the limit set by our theorem.

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A key concept in classical information theory developed by Shannon [1] and others [2] is the entropy. For a discrete random variable (source) $A$, it is defined by

$$H(A) = - \sum_a p(a) \log_2 p(a). \quad (1)$$

Coding is an important issue in information theory. In particular, one may be interested in representing the messages produced by the source $A$ by a sequence of binary digits (bits) as short as possible. Suppose that $A$ emits a sequence of independent messages. If we allow ourselves to code entire blocks of independent messages together and tolerate an arbitrarily small error in the signals reconstructed from the coded version, it turns out that the mean number of bits per message needed can be arbitrarily made close to $H(A)$.

Recently, there has been much interest in the subject of quantum computation. Current investigations [3] include the physical implementation of quantum computers, quantum complexity theory, quantum teleportation and quantum coding. In quantum coding, Schumacher [4] and Jozsa and Schumacher [5] have considered the possibility that the signals are pure quantum states which are not necessarily orthogonal to one another. Suppose that a quantum source $A$ emits a sequence of independent signals, each of which is a pure state from the list $|a_1\rangle, \ldots, |a_m\rangle$ occurring with probabilities $p_1, \ldots, p_m$. We may associate the density matrix

$$\rho = \sum_{i=1}^m p_i |a_i\rangle\langle a_i| \quad (2)$$

to the source. By analogy with the classical measure of information, the bit, as a 2-state classical system, Schumacher used the term “qubit” (meaning quantum bit) for the quantum state storage capacity of a two-dimensional Hilbert space. Note that, unlike a classical bit which can only take on a value of either 0 or 1, the state of a qubit can be in some coherent superposition of 0 and 1. i.e. the state of a qubit $|u\rangle = \alpha |0\rangle + \beta |1\rangle$ where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$. Moreover, a qubit is capable of being entangled with the states of other qubits. For example, the state $\frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$ is allowed. The polarization of a single photon, for example, has a storage capacity of one qubit. We wish to encode the signals
with a least possible number of Hilbert space dimensions. Once again, block coding may be used and a small error may be allowed. In other words, we consider a $K$-blocked version $A_K$ of $A$. If $A$ has $m$ distinct signal states in a Hilbert space $H_n$ (of dimension $n$), then $A_K$ has $m^K$ signals in $H_{n^K}$ (of dimension $n^K$). In order to code the signals with a minimum number of Hilbert space dimensions, typically part of a system will be discarded during the coding. Therefore, the signal $|a_i\rangle$ is reconstituted as a mixed state with density matrix $W_i$. In Refs. [4] and [5] the concept of fidelity

$$ F = \sum_i p_i \langle a_i | W_i | a_i \rangle $$

was introduced. Notice that $\langle a_i | W_i | a_i \rangle$ is the probability that the state $W_i$ passes the yes/no test of being the state $a_i$. $0 \leq F \leq 1$ is the average probability of passing the test.

Analogous to the classical information theory, we introduce the von Neumann entropy

$$ S(\rho) = -\text{Tr}\rho \log_2 \rho. $$

The quantum noiseless coding theorem for pure states proved in Refs. [4] and [5] states the following. Given any quantum source with von Neumann entropy $S(\rho)$ and any $\epsilon, \delta > 0$,

(a) If $S(\rho) + \delta$ qubits are available per signal, then for each sufficiently large $N$, there exists a coding scheme with fidelity $F > 1 - \epsilon$ for signal strings of length $N$.

(b) If $S(\rho) - \delta$ qubits are available per signal, then any coding scheme for strings of length $N$ will have a fidelity $F < \epsilon$ for all sufficiently large $N$.

Therefore, the von Neumann entropy may be interpreted as the minimal number of qubits needed for reliable (almost noiseless) coding. This noiseless coding theorem works only for pure signal states. It is natural to generalize it to consider signals which are mixed states $\Pi_a$, with $\rho = \sum_a p(a) \Pi_a$. As noted in Refs. [4] and [5], it is not clear how to proceed. A naive generalization of the fidelity,

$$ F = \sum_a p(a) \text{Tr} \Pi_a W_a $$

is not close to unity even when $W_a = \Pi_a$ for all signals. To quantify the amount of distortion of a particular coding scheme, a notion of the distance between two mixed states is desired.
Such a concept has been introduced by Anandan [6] in the study of geometric phases. Let \( D \) denote the set of density operators representing the states of a given quantum system. (\( D \) consists of the set of Hermitian operators in the Hilbert space of this system with nonnegative eigenvalues and trace equal to 1.) \( D \) is a topological space with the pure states contained in its boundary. The set of pure states can be identified with the projective Hilbert space \( \mathcal{P} \). The inner product structure of a Hilbert space naturally induces a metric, namely the Fubini-Study metric on the projective Hilbert space \( \mathcal{P} \) which can be extended into the rest of \( D \). More concretely, the distance between two points \( p \) and \( p' \) in \( \mathcal{P} \) is defined by

\[
s(p, p')^2 = 4(1 - |\langle \psi | \psi' \rangle|^2) \tag{6}
\]

where \( |\psi \rangle \) and \( |\psi' \rangle \) are two normalized states contained in \( p \) and \( p' \). It is simple to check that \( s(p, p') \) satisfies all the axioms for a metric. Suppose that \( p \) and \( p' \) are separated by an infinitesimal distance \( ds \) in \( \mathcal{P} \):

\[
ds^2 = 4(1 - |\langle \psi | \psi' \rangle|^2) = 2\text{Tr}(\rho - \rho')^2 \tag{7}
\]

where the last equality follows from \( \text{Tr}(\rho^2) = \text{Tr}(\rho'^2) = 1 \). This defines a Riemannian metric on \( \mathcal{P} \), called the Fubini-Study metric. It is therefore reasonable to introduce a flat metric

\[
dS^2 = 2\text{Tr}(d\rho^2) \tag{8}
\]

on \( D \). When restricted to the pure states, it becomes the Fubini-Study metric.

Suppose that a quantum source produces a sequence of signals, each of which is a mixed state from the list \( \Pi_1, \cdots, \Pi_m \), with probabilities \( p_1, \cdots, p_m \) and that after coding, the signal \( \Pi_a \) is reconstituted as \( W_a \). Motivated by the above discussion, we define the distortion

\[
D = \sum_a p_a \text{Tr}(\Pi_a - W_a)^2. \tag{9}
\]

Notice that \( 0 \leq D \leq 2 \) and \( D = 0 \) if and only if \( \Pi_a = W_a \). This definition is reasonable because \( \Pi_a - W_a \) is the deviation of \( \Pi_a \) from \( W_a \). To obtain a real-valued function, we take the trace. However, \( \text{Tr}(\Pi_a - W_a) \) is identically zero. It is, therefore, natural to consider \( \text{Tr}(\Pi_a - W_a)^2 \) and take the ensemble average.
The ensemble of signals emitted by the source can be represented by the density operator

$$\rho = \sum_a p_a \Pi_a.$$  \hspace{1cm} (10)

Consider the following communication scheme discussed by Schumacher \cite{4}. Suppose that the signal is represented by a system $X$ which is composed of two subsystems, $C$ (for “channel”) and $E$ (for “extra”). Only the channel subsystem $C$ is transmitted to the receiver and the subsystem $E$ is simply discarded. To recover (some approximation of) the signal, we add to the channel system an auxiliary system $E'$ that is a copy of the discarded extra system $E$. Schumacher called such a communication scheme an approximate transposition via the limited channel $C$. For this type of communication schemes, we have the following theorem:

**Quantum Noiseless Coding Theorem for Mixed States.** For any quantum source which produces mixed signal states $\Pi_a$’s with probabilities $p_a$’s, define the von Neumann entropy $S(\rho)$ as in Eq. (2). For any $\epsilon, \delta > 0$,

(a) if $S(\rho) + \delta$ qubits are available per signal, then for each sufficiently large $N$, there exists a coding scheme with $D < \epsilon$.

(b) if $S(\rho) - \delta$ qubits are available per signal, then for a sufficiently large $N$, any approximate transposition coding scheme for a string of length $N$ has a distortion $D \geq \sum_a p_a \text{Tr} \Pi_a^2 - \epsilon$.

This implies that for a given quantum source, $D$ will not tend to zero unless at least $S(\rho)$ qubits are available per signal. Therefore, $S(\rho)$ may again be interpreted as the mean number of bits needed for the noiseless coding of a source which emits signals that are mixed states if an approximate transposition coding scheme is used.

To minimize our usage of resources, we would like to code signals on a $d$-dimensional subspace $\Lambda$ of $\mathcal{H}_n$. (In applying the following lemmas to prove the main theorem, we will use block coding. The signal states will therefore be $K$-blocks of signals.) Let $|b_1\rangle, |b_2\rangle, \cdots, |b_d\rangle$ be a basis of $\Lambda$ and $|b_{d+1}\rangle, |b_{d+2}\rangle, \cdots, |b_n\rangle$ a basis of $\Lambda^\perp$, the orthogonal complement of $\Lambda$. For each $a$, $\Pi_a$ can be diagonalized and expressed in terms of its eigenvectors $|a_i\rangle$ as
\[ \Pi_a = \sum_{a_i} q_{a_i} |a_i\rangle \langle a_i| = \sum_{a_i} q_{a_i} \Pi_{a_i} \]  

(11)

where \( \Pi_{a_i} = |a_i\rangle \langle a_i| \) and for each \( a \), \( \sum_{a_i} q_{a_i} = 1 \). Suppose that, with respect to the basis \( |b_1\rangle, |b_2\rangle, \ldots, |b_n\rangle \),

\[ \Pi_{a_i} = |a_i\rangle \langle a_i| = \begin{pmatrix} M_{a_i} & A_{a_i}^\dagger \\ A_{a_i} & N_{a_i} \end{pmatrix} \]  

(12)

where \( M_{a_i} \) is a \( d \times d \) matrix. We now introduce an explicit coding scheme based on \( \Lambda \). Let \( |0\rangle \) be an arbitrary state in \( \Lambda \) and \( P \) the projection into \( \Lambda \). For each \( \Pi_{a_i} = |a_i\rangle \langle a_i| \), we measure the observable \( P \) on \( |a_i\rangle \). If the result 0 is obtained, then \( |0\rangle \) is substituted for the post measurement state. In other words, we associate with each \( \Pi_{a_i} \) a density matrix

\[ W_{a_i} = \begin{pmatrix} M_{a_i} & 0 \\ 0 & 0 \end{pmatrix} + (1 - TrM_{a_i}) |0\rangle \langle 0|. \]  

(13)

**Lemma 1.** Suppose that the sum of the \( d \) largest eigenvalues of the density operator \( \rho \) is greater than \( 1 - \xi \). Let \( \Lambda \) be the span of the \( d \) eigenvectors of \( \rho \) corresponding to the \( d \) largest eigenvalues. Then the association \( \Pi_a = \sum_{a_i} q_{a_i} \Pi_{a_i} \leftrightarrow W_a = \sum_{a_i} q_{a_i} W_{a_i} \), defined by Eq. (13) has distortion \( D < 2 \xi \).

**Proof:** Note that \( f(X) = Tr(X^2) \) is a convex function. We have \( Tr[E(X)]^2 \leq E(TrX^2) \) where \( E(X) \) denotes the weighted mean of a variable \( X \). Denoting \( \Pi_a - W_a \) by \( X_a \) and \( \Pi_{a_i} - W_{a_i} \) by \( X_{a_i} \), the distortion

\[ D = \sum_a p_a Tr(X_a)^2 \]

\[ = \sum_a p_a Tr(\sum_{a_i} q_{a_i} X_{a_i})^2 \]

\[ \leq \sum_a \sum_{a_i} p_a q_{a_i} Tr(X_{a_i})^2. \]  

(14)

Here convexity of the function \( f(X) = Tr(X^2) \) has been used. Now let \( P \) denote the projection operator into \( \Lambda \), the space spanned by the \( d \) eigenvectors corresponding to the \( d \) largest eigenvalues of \( \rho \). By assumption,
\[ \text{Tr}(\rho P) > 1 - \xi. \quad (15) \]

Consider
\[
\sum \sum p_a q_{a_i} \text{Tr}[\Pi_{a_i}(1 - P)] \\
= \sum p_a \text{Tr}[\sum q_{a_i} \Pi_{a_i}(1 - P)] \\
= \sum p_a \text{Tr}[\Pi_a(1 - P)] \\
= \text{Tr}[\sum p_a \Pi_a(1 - P)] \\
= \text{Tr}[\rho(1 - P)] \\
< \xi. \quad (16)
\]

Notice that with Eqs. (14) and (16) we have essentially reduced the case of mixed signal states to that of pure signal states with a priori probabilities \( p_a q_{a_i} \). In what follows, we shall therefore consider the case of pure signal states only. For simplicity, we also suppress the index \( a \). Write \( |a_i\rangle \) in terms of its components in \( \Lambda \) and \( \Lambda^\perp \):
\[
|a_i\rangle = \alpha_i |l_i\rangle + \beta_i |m_i\rangle \quad (17)
\]
where \( \alpha_i, \beta_i \geq 0 \), \( \alpha^2 + \beta^2 = 1 \), \( |l_i\rangle \in \Lambda \) and \( |m_i\rangle \in \Lambda^\perp \). For \( \Pi_i = |a_i\rangle \langle a_i| \), we have
\[
\Pi_i = \alpha_i^2 |l_i\rangle \langle l_i| + \alpha_i \beta_i |l_i\rangle \langle m_i| + \alpha_i \beta_i |m_i\rangle \langle l_i| + \beta_i^2 |m_i\rangle \langle m_i|. \quad (18)
\]
\( \Pi_i \) is associated to
\[
W_i = \alpha_i^2 |l_i\rangle \langle l_i| + \beta_i^2 |0\rangle \langle 0|. \quad (19)
\]
It is then a simple exercise to check that
\[
\sum_i p_i \text{Tr}(\Pi_i - W_i)^2 = 2 \sum_i p_i \beta_i^2 \\
= 2 \sum_i \text{Tr}[\Pi_i(1 - P)] < 2\xi. \quad (20)
\]
This completes our proof of Lemma 1.

**Lemma 2.** Consider any coding scheme

\[ \Pi_i \leftrightarrow W_i \quad i = 1, \ldots, m \]  

(21)

where \( W_i \) is a density matrix supported on some \( d \)-dimensional subspace \( D \) of \( \mathcal{H}_n \). If the sum of the \( d \) largest eigenvalues of \( \rho \) is \( \eta \), then the distortion \( D \geq \sum_i p_i \text{Tr} \Pi_i^{2} - 2\eta \).

**Proof.** Let us denote the projection into \( D \) by \( P' \) and the projection into the space spanned by the \( d \) eigenvectors with the \( d \) largest eigenvalues by \( P \). By assumption, \( \sum_i p_i \text{Tr}[\Pi_i P'] \leq \text{Tr}[\rho P] = \eta \).

\[
D = \sum_i p_i \text{Tr}[\Pi_i - W_i]^2 \\
\geq \sum_i p_i \text{Tr} \Pi_i^2 - 2 \sum_i p_i \text{Tr} [\Pi_i W_i] \\
\geq \sum_i p_i \text{Tr} \Pi_i^2 - 2 \sum_i p_i \text{Tr} [\Pi_i P'] \\
\geq \sum_i p_i \text{Tr} \Pi_i^2 - 2\eta.
\]  

(22)

Having proved the two lemmas, we proceed to prove the main theorem. For this, we make use of the “Asymptotic Equipartition Property (AEP)” (an analog of the weak law of large numbers) in classical information theory. The weak law of large numbers states that for independent, identically distributed (i.i.d.) random variables, \( \frac{1}{N} \sum_{i=1}^{N} X_i \) is close to its expected value \( E(X) \) for a large \( N \). Functions of independent random variables are also independent random variables. Since \( X_i \)'s are i.i.d, so are \( \log_2 p(X_i) \)'s. Applying the weak law of large number to \( \log_2 p(X_i) \)'s, we obtain the AEP, which states that \( \frac{1}{N} \log_2 \frac{1}{p(X_1, X_2, \ldots, X_N)} \) is close to the entropy \( H \). Here \( X_1, X_2, \ldots, X_N \) are i.i.d. random variables and \( p(X_1, X_2, \ldots, X_N) = p(X_1)p(X_2) \cdots p(X_N) \) is the probability of the occurrence of the sequence \( X_1, X_2, \ldots, X_N \). Therefore, it is highly likely that the probability assigned to an observed sequence is close to \( 2^{-NH} \).

This enables us to divide the set of all possible sequences into two subsets, the set of “typical sequences”, where the sample entropy is close to the true entropy, and the atypical
set, which contains all other sequences. In classical noiseless coding theorem, we just choose our codewords in one-one correspondence with the typical set. In other words, we only code all the typical sequence. If an atypical sequence occurs, we accept failure. The important point is that the probability for a sequence to be in the atypical set is small as $N$ gets large.

Proof of the quantum noiseless theorem for mixed states. (a) Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of the density matrix $\rho$ of a quantum source $A$. Consider $\lambda_1, \lambda_2, \cdots, \lambda_n$ as the probabilities of a probability distribution $P$. The Shannon entropy $H(P)$ is the same as the von Neumann entropy $S(\rho)$. Note also that the $K$-blocked version $A_K$ of $A$ has a density matrix $\rho_K = \bigotimes^K \rho$. The AEP states that for sufficiently large $K$, there exists a set of $2^{KS+\delta}$ eigenvalues of $\rho_K$ with a sum of eigenvalues greater than $1 - \epsilon/2$. Therefore, the sum of the $2^{KS+\delta}$ largest eigenvalues must be larger than $1 - \epsilon/2$. By Lemma 1, there exists a coding scheme for $A_K$ which uses $K(S + \delta)$ qubits per signal for $A_K$ and has distortion $D < \epsilon$.

(b) Using the weak law of large numbers, it can be shown that, for all sufficiently large $K$, any subset of $P_K$ of size less than $2^{KS-\delta}$ has probability less than $\epsilon$. (See Ref. [5].) In particular, the sum of the $2^{KS-\delta}$ largest eigenvalues will still be less than $\epsilon$. By Lemma 2, we find that for all sufficiently large $K$, any coding scheme with $K(S - \delta)$ qubits per signal will have distortion $D \geq \sum_i p_i \text{Tr} \Pi_i^2 - 2\eta$.

This completes our proof of the noiseless coding theorem for mixed states.

Note that this theorem applies only to approximate transposition coding schemes. Is it possible to devise a more efficient coding scheme? The answer is yes [6]. Mixed state signals might be re-constituted from a compressed version by adjoining an ancilla in a standard state, and applying a measurement process. Suppose, for instance, that there are two signals $\rho_1$ and $\rho_2$ with probabilities $p_1$ and $p_2$ respectively, and that these signals live in a 4-dimensional space with supports in two 2-dimensional subspaces, which are orthogonal to each other. We can compress the data as follows: Measure the signal. Since the two signals have orthogonal supports, the measurement tells us with certainty which of the two signals we are given. Record the possible outcomes of our measurement by pure orthogonal (i.e. classical) states.
|1⟩ and |2⟩ occurring with probabilities \( p_1 \) and \( p_2 \). It follows from the Shannon’s classical noiseless coding theorem that the signal can further be compressed to the Shannon entropy \( H(p_1, p_2) \) qubits/signal. To reconstitute the signals, we simply decode (and decompose) the classical signal and represent each of the binary digit 0 or 1 in the resulting sequence by a density matrice \( \rho_1 \) or \( \rho_2 \) accordingly. But \( H(p_1, p_2) \) is less than \( S(p_1 \rho_1 + p_2 \rho_2) \) (if either \( \rho_1 \) or \( \rho_2 \) is a mixed state). The limit set by the mixed state coding theorem is, thus, beaten by the above method. A natural question to ask is: what is the information theoretic limit of the compression rate of mixed-state signals that no coding scheme can surpass?

Another point to note is that Shannon’s more important results deal with channels with noise. The information capacity of a noisy channel deserves further investigations.

After the completion of this work we learned that Jozsa [8] has proven essentially the same result, using Uhlmann’s transition probability formula [9] as a fidelity function [10]. We thank R. Jozsa for bringing his unpublished results to our attention. Helpful discussions with H. F. Chau, K. Y. Szeto and F. Wilczek are also gratefully acknowledged. This work was supported in part by DOE DE-FG02-90ER40542 and HKTIIT 92/93.002 .
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