Busy period density of $M / C^b_2 / 1$ queueing system through lattice path approach: a special case

I Slamet$^1$, P N Hidayati$^2$, S Wibowo$^2$ and E Zukhronah$^1$

$^1$Department of Statistics, Universitas Sebelas Maret
Jl. Ir. Sutami 36A, Kentingan, Surakarta, 57126, Indonesia
$^2$Department of Mathematics, Universitas Sebelas Maret
Jl. Ir. Sutami 36A, Kentingan, Surakarta, 57126, Indonesia
E-mail: isnandarslamet@staff.uns.ac.id

Abstract. We consider $M / C^b_2 / 1$ queueing system which is a queueing system with arrival rate follows a Poisson distribution, service rate follows an exponential distribution which is approximated by Coxian 2-phase distribution, service is performed in $b$-size, and the number of service facility is one. This paper aims to derive the properties of $M / C^b_2 / 1$ queueing system when the system is in busy period using lattice path combinatorics approach. We consider a special case which is called pure incomplete busy period (PIBP). Through this approach, the queue system is represented in the form of lattice path. Furthermore, we have determined the number of lattice paths and finally, the probability function in the queue system $M / C^b_2 / 1$ in PIBP is derived.

1. Introduction
Queueing phenomenon often occurs in everyday life. Examples of the phenomenon include air traffic, road traffic, job-shop type system, and machine interference. Queue occurs if the rate of arrival is greater than the rate of service, so the customers have to wait until the previous customer has left the system finishing the service.

The queueing system is governed by the set of customers, services and rules of arrival and services of its customers [1-6]. Combinatorial approach has been applied to queueing systems for analyzing the performances of queueing systems under considerations [7].

Performance analysis could include the concern on the busy period [8-14], the probability of pure incomplete busy period (PIBP) [15,16], and the probability of the incomplete busy period (IBP) [15, 16]. Most queueing system research is devoted for busy period density studies. The busy queueing system is a queueing system in equilibrium state. The busy period can be represented by a lattice path from $(i_0,0)$ to $(m,m)$ which touches the line $Y = X$. Observing the queueing system from initial time $0$ to time $t$ continuously, the system can have busy and idle periods of services. During the duration of time $t$, system is said to as PIBP if the system can be represented by a lattice path from $(i_0,0)$ to $(m,n)$, where $m > n$ and always below the line $Y = X$. The case where the system undergoes at least one busy period during the time $t$ is called IBP. Therefore PIB can be represented by lattice path starting from $(i_0,0)$ to $(m,n)$, where $m > n$, not passing (touching or under) line $Y = X$.

The state of the queueing system that does not reach a state of equilibrium is called a transient state [1-6]. The equilibrium state is the state after $t$ time unit with the number of subscribers in the queueing system being stable, so the probability that $n$ customers at time $t$ becomes constant and independent of...
time [1-6]. Until 1995, most of the transient solutions available to queuing systems in the forms of Laplace-Stieltjes transforms. Other is in the form of integral transforms. As noted in [8-16]), these solutions are much complicated and hard. The technique of lattice path combinatorics has been set up so that the transient solutions is in explicit form. Further, transient queuing system analysis can be done using lattice path combinatorics approach [7].

Sen and Jain [7] have succeeded in deriving the transient solution for the $M / M / 1$ queuing system with the lattice path approach. In 2009 Borkakaty et al. [8-14] has derived the density function of busy periods on several queueing systems using lattice path approach where the arrival distribution and service time are approximated by Coxian 2-phase distribution, $C_2$. Furthermore, Slamet et al. [15-16] has derived a transient solution for PIBP on the $M / G / 1$ queuing system with $G$ approximated by a $C_2$, using the lattice path approach.

In this paper, we investigate the probability density of PIBP of the $M / G / 1$ queueing system in which the $G$ distribution is approximated by $C_2$ in the $b$-group service through lattice path combinatorics approach.

2. The $M / C_2^b / 1$ model

The queue model $M / C_2^b / 1$ is a queuing model in which the arrival rate is approximated by the Poisson distribution, the service rate is approximated by a 2-phase Coxian distribution with $b$-sized group service, and the number of service facilities is one, where

- $b$: batch size,
- $i_0$: initial number of customers in the system, $i_0 > 0$,
- $\lambda$: arrival rate of customers,
- $\mu_i$: service rate of server in phase $i$, $i = 1, 2$,
- $\alpha$: $P$ {a batch of $b$ customers goes to phase 2 of service after completing phase 1 of service},
- $\beta$: $P$ {a batch of $b$ customers departs from the system after completing phase 1 of service}, $\alpha + \beta = 1$.

The system starts initially non-empty. Service time distribution follows a Coxian $k$-phase distribution, $C_k$, as illustrated in Figure 1. A new customer enters phase 1 service only when the previous customer in service leaves the system after completing the service.

![Figure 1. Coxian k-phase distribution](image)

2.1 Transitions

Consider the time interval $(0, t)$. Let $T_0 = 0$ and $T_1, T_2, ...$ be the order of the time points. In these points, transitions of state can take place. Let $X_0, X_1, X_2, ...$ be the corresponding number of customers at the times. Assume $X_0 = i_0$ as an initial condition. Further, we find that $\{X_n\}$ is a Markov chain and its transition probability matrix $Q$, which for all $n \geq 0$; $u = 1, 2$, is given as

$$Q(i, j) = P(X_{n+1} = j | X_n = i)$$
\[
\begin{align*}
\frac{\lambda}{\lambda + \mu_u}, & \text{ the probability of an arrival where if a customer is } \\
& \text{ receiving service in phase } u (j = i + 1) \\
\frac{\beta \mu_1}{\lambda + \mu_1}, & \text{ the probability of a batch of } b \text{ customers leaving the system after } \\
& \text{ finishing service in phase 1 } (j = i - b) \\
\frac{\mu_2}{\lambda + \mu_2}, & \text{ the probability of a batch of } b \text{ customers leaving the system after } \\
& \text{ finishing service in phase 2 } (j = i - b) \\
\frac{\alpha \mu_1}{\lambda + \mu_1}, & \text{ the probability of a batch of } b \text{ customers entering phase 2 of } \\
& \text{ service after finishing service in phase 1 } (j = i)
\end{align*}
\]

2.2. Lattice Path Representation

We consider a specific sequence of arrivals, departures and entries into next phase that describes the system up to time \( t \). This sequence can be represented by the movement of a point along the lattice path in the XY-plane. Process sample paths can be represented as 2-dimensional lattice paths representing arrival of a customer during the service of phase 1 by a horizontal step, arrival of a customer during phase 2 service by a single dotted horizontal step, departure of a batch of \( b \) customers after phase 1 by a vertical step from unit \( b \), a batch of \( b \) customers enter phase 2 by a diagonal step of \( b\sqrt{2} \) units, and departure of a batch of \( b \) customers after phase 2 by a dotted vertical step of \( b \) units.

![Figure 2. A lattice path representation of \( M/C_b^2/1 \)](image)

Figure 2 shows a lattice path representing a sequence of events such as, for \( i_0 = 1 \), the server never empty until \((m, n) = (23, 22)\), since the lattice path does not touch the line \( Y = X \). A horizontal step will always be of size 1, a vertical step will be of size \( b \), and a service diagonal of size \( b\sqrt{2} \). Also, because the number of departures is not more than the number of arrivals at any time point, the lattice path never goes above the line \( Y = X \).
2.3 Counting Lattice Path

For counting of lattice paths, first, a lattice path will be transformed into a skeleton lattice path by removing all diagonal steps. Figure 3 is obtained from Figure 2 after removing all diagonals, so that it consists only of horizontal steps and vertical runs.

![Figure 3. Skeleton lattice path](image)

Borkakaty et al. [8] defines the horizontal run and vertical run as follows. Horizontal run is a sequence which consists of horizontal steps which are bounded by vertical steps. Likewise, vertical run is a sequence which consists of vertical steps which are bounded by horizontal steps. As an approximation of the service time, we consider 2-phase Coxian distribution. Based on this scenario, in process of inserting diagonals, we have to observe that:

a) Two or more consecutive diagonals can not occur.

b) A vertical dotted step expresses a vertical step followed by a diagonal step.

c) There should not either two or more vertical dotted steps.

d) There should not depart after completing service in phase 1 followed by departure after completing service in phase 2.

For horizontal and vertical runs, equal in number as well as in total length, the process of counting the number of possible lattice paths that can be done. This counting process can be generated by considering the above limits on diagonal insertion. In Figure 3, \( n' \) represents many departures in a busy period.

- \( r \) : number of horizontal runs and vertical runs (\( r \geq 1 \)),
- \( l_i \) : length (distance) of the \( i \)th horizontal run (\( i = 1, 2, ..., r \)),
- \( bL_i \) : length (distance) of the \( i \)th vertical run (\( i = 1, 2, ..., r \)),
- \( p \) : total number of diagonal (\( p \geq 0 \)),
- \( q \) : total number of diagonal inserted in horizontal runs,
- \( p - q \) : total number of diagonal inserted in vertical runs,
- \( i \) : \((i_1, i_2, ..., i_q)\), \( q \) horizontal runs in each of which a diagonal is inserted,
- \( \overline{l_i} \) : \((\overline{l}_i, \overline{l}_2, ..., \overline{l}_q)\), length of horizontal runs \( \overline{i} \),
- \( p_i \) : \((p_{i_1}, p_{i_2}, ..., p_{i_q})\), distances from extreme left end points where diagonals representing entry into phase 2 are inserted in horizontal runs \( \overline{i} \) (including vertices at both ends of the runs).

3. Results and Discussions

3.1. The case where all customers leaving the systems after phase 1 service

**Theorem 1.** Let \( i_0 \) be the initial number of customers in the system. Let \( t \) be the length of operation of the system. Let \( \rho_{i_0}^b \) denote the density that the system \( M / C_2^b / 1 \) in which service is provided in batch \( b \) and all customers leave the systems after phase 1.
Then we have

\[ f_{i_0}^b = \sum_{m=0}^{\infty} \sum_{n=0}^{m-i_0} \left( \binom{m+n-b-i_0}{n/b} - \binom{m+n-b-i_0}{m} \right) \]  

(1)

Proof. This term corresponds to the case where all customers leave the system after phase 1 service. The number of customers arrive in the system is \( m - i_0 \) and the number of customers completing the service is \( n \). As the result, for busy periods, the transition happens in \( m + \frac{n}{b} - i_0 \) transitions. Denote this number as \( N_1 \).

Let \( T_0 = 0 \) and \( T_1, T_2, \ldots \) be the order of the time points in which the transitions happens. We assume at time \( T_0 \), process starts and follows Poisson distribution with rate \( \lambda + \mu_1 \). An arrival may occur with probability \( \left( \frac{\lambda}{\lambda + \mu_1} \right) \) and a departure may occur with probability \( \left( \frac{\mu_1}{\lambda + \mu_1} \right) \). Therefore, the density function of \( T \) follows \( N_1 \)-Erlang with parameter \( \lambda + \mu_1 \) given by

\[ f_{N_1}(t) = \frac{e^{-(\lambda+\mu_1)t}(\lambda+\mu_1)^{N_1-1}}{\Gamma(N_1)}. \]

The density for this case becomes

\[ f_{i_0}^b = \sum_{m=0}^{\infty} \sum_{n=0}^{m-i_0} \left( \binom{m+n-b-i_0}{n/b} - \binom{m+n-b-i_0}{m} \right) \times \left( \frac{\lambda}{\lambda + \mu_1} \right)^{m-i_0} \times \left( \frac{\mu_1}{\lambda + \mu_1} \right)^{n} \times \frac{e^{-(\lambda+\mu_1)t}(\lambda+\mu_1)^{m+n-b-i_0}}{\Gamma(m+n-b-i_0)}, \]

where \( \binom{m+n-b-i_0}{n/b} - \binom{m+n-b-i_0}{m} \) is the total of lattice paths which starts from \((i_0, 0)\), ends in \((m, n)\), and always remains below the line \( Y = X \). Hence, the proof of (1).

3.2. The case where a batch of \( b \) customers depart from the system after completing service of phase 1

In this section, we consider the case where a batch of \( b \) customers depart from the system after completing service of phase 1. For derivation of the density of this case, structural property of the scenario is developed as in Theorem 2 below. Further, enumeration process which satisfies the structural property can be completed as in theorem 3.

**Theorem 2.** For non-negative integers \( i_0, m, n, p, q, r (r \geq 1), \frac{1}{b} k_j; L = (l_1, l_2, \ldots, l_r; L_1, L_2, \ldots, L_r) \), consider a lattice path starts from \((i_0, 0)\) and ends at \((m, n)\) where \( m > n \). Let \((LP_{0,1,0}^b)\), denote the corresponding number of lattice paths which remaining below the line \( Y = X \), each comprising of \( m - bp \) horizontal steps (including those from \((0, 0)\) to \((i_0, 0)\)), \( n-bp \) vertical steps and \( p \) diagonals, such that

(a) \( m - bp \) horizontal steps form \( r \) runs of length \( l_1, l_2, \ldots, l_r \), respectively, satisfying \( l_1 \geq 0, \)
(b) \( n-bp \) vertical steps form \( r \) runs of length \( L_1, L_2, \ldots, L_r \), respectively, satisfying \( L_1 \geq 0, \)
(c) \( l_1 \geq \max(i_0, L_1 + 1) \) and \( \sum_{i=1}^{u} l_i > b \sum_{i=1}^{u} L_i, u = 1, 2, \ldots, r - 1, \)
(d) \( q \) diagonals which represent \( b \) customers enter into phase 2 are inserted each in any \( q \) out of \( r \) horizontal runs with number \( l_1, l_2, \ldots, l_q \) on distance \( p_{l_1}, p_{l_2}, \ldots, p_{l_q} \) from the left-end point (including the two vertices at the ends of the runs).
(e) the last \( p - q \) diagonals which represent \( b(p - q) \) customers enter into phase 2 are inserted each at any \( \frac{n - bp}{b} - r \) vertices available along the vertical runs.

Then, for \( r \geq 1 \) and \( m > i_0 \), we find

\[
\left( L_{i_0, m, n, p, q; r, L} \right)^b = \sum R_7 \sum R_8 \left( \frac{n - bp}{p - q} - r \right)
\]

where

\[
R_7 = \left\{ (i_1, i_2, ..., i_q): 1 \leq i_1 < i_2 < \cdots < i_q \leq r \right\}
\]

\[
R_8 = \left\{ p_i = \left( p_{i_1}, p_{i_2}, ..., p_{i_q} \right): \Delta \leq p_{i} \leq \Delta_i, s = 1, 2, ..., q \right\}, \Delta = \begin{cases} 0, \text{ if } i_0 > 1 \\ i_0, \text{ if } i_0 = 1 \end{cases}
\]

**Proof.** For proving this theorem, we first get the skeleton path of the lattice path which is a lattice starts from \((i_0, 0)\) and ends in \((m - p, n - bp)\). This skeleton path consists of \( r \) horizontal as well as \( r \) vertical runs of lengths \( l_1 \) and \( L_i \) (\( i = 1, 2, ..., r \)), respectively. It is obvious that there will be one unique path with \( r \) runs, of each type of given fixed lengths. Now, insertion of a diagonal can be done as follows. Consider \( q \) diagonals which will be inserted into horizontal runs that numbers are \( i_1, i_2, ..., i_q \), respectively, of length \( l_{i_1}, l_{i_2}, ..., l_{i_q} \) at distances \( p_{i_1}, p_{i_2}, ..., p_{i_q} \) from the extreme left end points. The last \( p - q \) diagonals may be inserted in one of any \( p - q \) vertices out of \( \frac{n - bp}{b} - r \). There are \( \left( \frac{n - bp}{b} - r \right) \) possible ways to get this. By summing \( \left( \frac{n - bp}{b} - r \right) \) on all possibilities of \( q \)-tuples, \((i_1, i_2, ..., i_q)\) and \( p_{i_1}, p_{i_2}, ..., p_{i_q} \), we get (2).

**Corollary 1.** Let \( L_{i_0, m, n, p} \) the number of lattice path starts at \((i_0, 0)\) and ends in \((m, n)\), where \( m > n \). Let this lattice remaining below the line \( Y = X \) and each comprising of \( m - bp \) horizontal steps (including those from \((0, 0)\) to \((i_0, 0)\)), \( \frac{n - bp}{b} \) vertical steps, and \( p \) diagonals. By summing (2) on \( r, q \) and \( L \), we get

\[
L_{i_0, m, n, p} = \sum_{r \geq 1} \sum_{\Delta \geq 0} \left( \frac{n - bp}{b} - r \right) \sum_{p_{i_1}, p_{i_2}, ..., p_{i_q}} \left( p_{i_1}, p_{i_2}, ..., p_{i_q} \right) \Delta \leq p_{i} \leq \Delta_i, s = 1, 2, ..., q \}
\]

where

\[
R_4 = \left\{ r: 1 \leq r \leq \min \left( \frac{m - bp - i_0}{b} + 1, \frac{n - bp}{b} \right) \right\}
\]

\[
R_5 = \left\{ q: \max \left( 0, 2p - \frac{n}{b} + r \right) \leq q \leq \min (r, p) \right\}
\]

\[
R_6 = \left\{ L: l_1 \geq \max(i_0, L_1 + 1), l_1 + l_2 > L_1 + L_2, ..., l_1 + l_2 + \cdots + l_r, \sum_{i=1}^{r} l_i = m - bp, \sum_{i=1}^{r} l_i = n - bp \right\}
\]

**Theorem 3.** Let \( i_0 \) be the initial number of customers in the system. Let \( t \) be the length of operation of the system. Let \( f^2_{i_0}(t) \) denote the density that the system \( M / G_2 / 1 \) in which service is provided in batch \( b \) and all customers leave the systems after phase 1. Then we have

\[
f^2_{i_0}(t) = \sum_{(R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8)} \left( \frac{n - bp}{p - q} - r \right) \alpha \beta \mu_{i_0}^{n - bp - i_0} \mu_2^{n - bp} \exp(-\lambda + \mu_2) t
\]

\[
\times \sum_{a=0}^{\infty} \frac{(\mu_2 - 1)^a t^{m + n - bp - i_0 + a - 1}}{\Gamma(m + n - bp - i_0 + a - 1)}
\]

\[
\times \frac{\Gamma(m + n - bp - i_0 + a - 1)}{\Gamma(m + n - bp - i_0)}
\]

where
\[ R_1 = \{ m : \max (i_0, b+1) \leq m \leq \infty \}, \]
\[ R_2 = \{ n : b \leq n \leq m-1 \}, \]
\[ R_3 = \{ p : 0 \leq p \leq \min \left( \left\lfloor \frac{n}{2b} \right\rfloor, \left\lfloor \frac{m-i_0}{b} \right\rfloor \right) \} . \]

**Proof.** For proofing this theorem, we first consider \( t \) as the time needed by all customers in the system. Further, let \( t_1 \) as the time needed by all customers in phase 1 of service. Further, \( t-t_1 \) is the time that is needed by all customers in phase 2 of service. For fixed \( L \), we find \( m+\frac{n-bp}{b}-i_0 \) transitions happening in the system. These transitions refer to the number of transitions of the system when system is busy. These transitions consist of the number of customers arrive to the system, the number of customers departure from the system after completing the service in phase 1, the number of customers move to phase 2 to get the further service, and the number of customers departure from the system after completing the service in phase 2 which is \( m - bp - i_0, \frac{n-2bp}{b}, bp, \) and \( bp \), respectively.

Furthermore, in \( t_1 \), we will be able to count the transition which consists of the customers arrive in the system when customers receive service in phase 2, the number of customers move to phase 2 to get the further service, and the number of customers which is \( \sum (l_i s - p_i s) \) where its probability is given by

\[ f_{N_1}(t_1) = \frac{e^{-(\lambda+\mu_1)t_1(\lambda+\mu_1)^{N_1}}}{\Gamma(N_1)} . \]

For \( t_2 = t-t_1 \), we will be able to count the transition which consists of the customers arrive in the system when customers receive service in phase 2 and departures while customers are in phase 2 which is \( \sum (l_i s - p_i s) \) and \( p \), respectively. Hence, we find \( N_2 = \sum (l_i s - p_i s) + p \) where its probability is given by

\[ f_{N_2}(t_2) = \frac{e^{-(\lambda+\mu_2)t_2(\lambda+\mu_2)^{N_2}}}{{\Gamma(N_2)}} . \]

For all processes of service in phase 1 and phase 2, we get

\[ f_{i_0}^2(t) = \sum_{(R_1,R_2,R_3,R_4,R_5,R_6,R_7,R_8)} \left( \frac{n-bp}{b-p-q} \right) \times \int_0^{t_1} f_{N_1}(t_1) \left( \frac{\lambda}{\lambda+\mu_1} \right)^{N_1-n-bp} \left( \frac{\alpha u_1}{\lambda+\mu_1} \right)^p \left( \frac{\beta u_1}{\lambda+\mu_1} \right)^{n-bp} \right) \times f_{N_2}(t-t_2) \left( \frac{\lambda}{\lambda+\mu_2} \right)^{N_2-p} \left( \frac{\mu_2}{\lambda+\mu_2} \right)^p dt_1 . \]

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