Noncommutative probability in classical systems

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Abstract

Two examples of the situation when the classical observables should be described by a noncommutative probability space are investigated. Possible experimental approach to find quantum-like correlations for classical disordered systems is discussed. The interpretation of noncommutative probability in experiments with classical systems as a result of context (complex of experimental physical conditions) dependence of probability is considered.

1 Introduction

It is widely believed, that classical systems should be described by commutative, or Kolmogorovian, probability space. In the present paper we investigate the following question: is it possible to observe correlation of noncommutative observables in purely classical situation? This would mean, that the classical system will be described by noncommutative (or quantum) probability space.

The related subjects were investigated in Bohmian mechanics [1], [2] and in approach by t’Hooft [3], [4], where the properties of quantum system were discussed as a result of some underlying classical dynamics in the space of hidden classical parameters. In papers by Accardi and Regoli, see [5], violation of the Bell inequality in classical system was discussed.

Another related subject was discussed in papers [6], [7], where numerical simulation of discretized classical mechanics was performed, and quantum-like interference fringes were obtained.

In papers [8], [9] of one of the authors there was developed a contextual probabilistic approach to the statistical theory of measurements over quantum as well as classical physical systems. It was demonstrated that by taking into account dependence of probabilities on complexes of experimental physical conditions, physical contexts, we can derive quantum interference for probabilities of alternatives. Such a contextual derivation is not directly related to special quantum (e.g. superposition) features of physical systems. Those contextual models [10], [11] stimulated the present search for noncommutative structures in classical statistical physical models, e.g. disordered systems.
In the present paper we discuss two examples of classical statistical mechanical systems where we obtain the correlation functions in noncommutative probability space.

Our approach, in principle, may be compared with experiments in the following standard way. The traditional way to distinguish between classical and quantum system is the Bell inequality, satisfied by classical correlation functions. Thus if we would find violation of the Bell inequality, we will prove that the system is described by noncommutative probability space. However, our present theoretical considerations are still far from such experimental study.

The structure of the present paper is as follows.

In Section 2 we investigate the example of arising of noncommutative probability for classical observables as a result of time averaging.

In Section 3 we consider the correlation functions on noncommutative probability space for classical disordered system. The noncommutativity there will be a result of ensemble averaging. Also in this Section we discuss the considered examples from the point of view of the context dependent interpretation of noncommutative probability.

2 Noncommutative probability and time averaging

In the present section we discuss the following problem. Consider the dynamics of quantum system, described by some Hamiltonian $H_0$ and the algebra of observables $\mathcal{A}$. Let this (noncommutative) algebra of observables $\mathcal{A}$ contains some (commutative) classical subalgebra $\mathcal{C}$. This classical subalgebra is not conserved by time evolution, but for $X, Y \in \mathcal{C}$ the time evolutions $X(t) = e^{itH_0}Xe^{-itH_0}$ and $Y(t) = e^{itH_0}Ye^{-itH_0}$ will commute by definition.

Let us assume that the time evolution, defined by Hamiltonian $H_0$ is very fast, and in experiment we observe some time averaged observables. These time averaged observables, in general, already will not commute, since the classical subalgebra is not conserved by time evolution. This means that, in principle, we might expect that these time averaged operators for classical physical variables $X(t), Y(t)$ from the classical subalgebra $\mathcal{C}$ will have nonclassical correlations.

The natural example of this kind of behavior is observed in the quantum stochastic limit approach [10]. In this approach we consider the quantum system with the Hamiltonian in the form

$$H = H_0 + \lambda H_I$$

where $H_0$ is called the free Hamiltonian, $H_I$ is called the interaction Hamiltonian, and $\lambda \in \mathbb{R}$ is the coupling constant.

We investigate the dynamics of the system in the new slow time scale of the stochastic limit, taking the van Hove time rescaling [11]

$$t \mapsto t/\lambda^2$$

and considering the limit $\lambda \to 0$. In this limit [11] the free evolutions of the suitable collective operators

$$A(t, k) = e^{itH_0}A(k)e^{-itH_0}$$

will become quantum white noises:

$$\lim_{\lambda \to 0} \frac{1}{\lambda} A \left( \frac{t}{\lambda^2}, k \right) = b(t, k)$$
The convergence is understood in the sense of correlators. The \( \lambda \to 0 \) limit describes the time averaging over infinitesimal intervals of time and allows to investigate the dynamics on large time scale, where the effects of interaction with the small coupling constant \( \lambda \) are important.

For the details of the procedure see [10].

The collective operators describe joint excitations of different degrees of freedom in systems with interaction, and may have the form of polynomials over creations and annihilation of the field, or may look like combinations of the field and particles operators etc.

For example, for nonrelativistic quantum electrodynamics without the dipole approximation the collective operator is

\[
A_j(k) = e^{ikq} a_j(k)
\]

where \( a_j(k) \) is the annihilation of the electromagnetic (Bose) field with wave vector \( k \) and polarization \( j \), \( q = (q_1, q_2, q_3) \) is the position operator of quantum particle (say electron), \( qk = \sum_i q_i k_i \).

The nontrivial fact is that, after the \( \lambda \to 0 \) limit, depending on the form of the collective operator, the statistics of the noise \( b(t, k) \) depends on the form of the collective operator and may be nontrivial.

Consider the following examples.

1) We may have the following possibility

\[
[b_i(t, k), b_j(t', k')] = 2\pi \delta_{ij} \delta(t - t') \delta(k - k') \delta(\omega(k) - \omega_0)
\]

which corresponds to the quantum electrodynamics in the dipole approximation, describing the interaction of the electromagnetic field with two level atom with the level spacing (energy difference of the levels) equal to \( \omega_0 \). Here \( \omega(k) \) is the dispersion of quantum field.

In this case the quantum noise will have the Bose statistics, and different annihilations of the noise will commute

\[
[b_i(t, k), b_j(t', k')] = 0
\]

2) The another possibility is the relation

\[
b_i(t, k) b_j(t', k') = 2\pi \delta_{ij} \delta(t - t') \delta(k - k') \delta(\omega(k) + \varepsilon(p) - \varepsilon(p + k))
\]

which corresponds to the quantum electrodynamics without the dipole approximation [11]. Here \( \omega(k) \) and \( \varepsilon(p) \) are dispersion functions of the field and of the particle correspondingly.

In this case the quantum noise will have the quantum Boltzmann statistics [11], [12], [13], and different annihilations of the noise will not commute

\[
b_i(t, k) b_j(t', k') \neq b_j(t', k') b_i(t, k)
\]

The commutation relations of the types (2), (3) are universal in the stochastic limit approach (a lot of systems will have similar relations in the stochastic limit \( \lambda \to 0 \)).

Take two operators \( b_i(t, k) \) for the same time and fixed polarization and consider the combinations

\[
X(k) = b_i(t, k) + b_i^+(t, k), \quad X(k') = b_i(t, k') + b_i^+(t, k')
\]

which correspond to the coordinate operator.

Then for the case of quantum Boltzmann relations we have

\[
X(k)X(k') \neq X(k')X(k)
\]
Of course, we need some regularization of the product of generalized functions.

Operators $X(k)$ before the stochastic limit belonged to the classical subalgebra. More precisely, corresponding combinations $x(k)$ of interacting operators would belong to the classical subalgebra:

$$x(k) = A_i(k) + A_i^\dagger(k)$$

$$[x(k), x(k')] = 0$$

We proved, that for the case when after the stochastic limit the statistics of the field become quantum Boltzmannian, operators (4) will not commute even if we take them for equal time. This is not a mystery, since in the stochastic limit we work with the time averaged observables. But in real experiments we may observe the result of time averaging. The discussed example shows, that for quantum electrodynamics beyond the dipole approximation we may observe quantum correlations for time averaged observables in the classical subalgebra.

The considered in the present section situation is similar in some sense to the results of [8], [9]. In these papers the quantum like interference fringes were observed for the discretization of the classical dynamical system. The discretization is an analog, in some sense, of the time averaging procedure. Probably the results of [8], [9] might be possible to embed into the frameworks of the approach considered in the present section.

We would like to mention that since long time De Muynck, see e.g. [23], discuss analogy between thermodynamics and quantum mechanics. They considered in the EPR–Bohm framework quantum expectations as a kind of thermodynamic averages. This should induce violation of Bell’s inequality.

### 3 Noncommutative probability and disordered systems

In the present section we discuss the possibility of using of noncommutative probability to describe (classical) disordered systems, following [14], [15]. In these papers the new procedure, called the noncommutative replica procedure, which is an analog of the replica procedure of Edwards and Anderson [16], was proposed to describe the statistical mechanics of quenched disordered systems (for example, spin glasses).

We will not discuss here the standard replica approach, see for introduction to spin glasses and the replica method [16], [17], [18].

Consider the disordered system with Hamiltonian $H[\sigma, J]$ which depends on the random parameter $J$ which in the most interesting cases (for spin glasses for instance) is the large random $N \times N$ matrix with independent Gaussian matrix elements $J_{ij}$, considered in the thermodynamic $N \to \infty$ limit.

To describe the system with quenched disorder in [14], [15] it was to proposed to consider the state described by the noncommutative replica statistic sum

$$Z^{(p)} = \int \sum_{\{\sigma\}} \exp (-\beta H[\sigma, \Delta J]) \prod_{a=0}^{p-1} \exp \left( -\frac{1}{2} \sum_{i \leq j} J_{ij}^{(a)} \right) \prod_{i \leq j} dJ_{ij}^{(a)}$$

where $\Delta$ is the following coproduct operation

$$\Delta : J_{ij} \mapsto \frac{1}{\sqrt{p}} \sum_{a=0}^{p-1} J_{ij}^{(a)}$$
which maps the matrix element $J_{ij}$ into the linear combination of independent replicas $J_{ij}^{(a)}$, enumerated by the replica index $a$. This operation was called the quenching in [15].

In the large $N$ limit, by the Wigner theorem, see [19]–[22], the system of $p$ random matrices with independent variables will give rise to the quantum Boltzmann algebra with $p$ degrees of freedom with the generators $A_a$, $A_a^\dagger$, $a = 0, \ldots, p - 1$ and the relations

$$A_a A_b^\dagger = \delta_{ab}$$

These operators are the limits of the large random matrices

$$\lim_{N \to \infty} \frac{1}{N} J_{ij}^{(a)} = Q_a = A_a + A_a^\dagger$$

where the convergence is understood in the sense of correlators (as in the central limit theorem).

Then in the thermodynamic limit $N \to \infty$ the noncommutative replica procedure (6) will take the form of the following map of the quantum Boltzmann algebra with one degree of freedom into quantum Boltzmann algebra with $p$ degrees of freedom:

$$\Delta : Q \mapsto \frac{1}{\sqrt{p}} \sum_{a=0}^{p-1} Q_a$$

Note that different $Q_a$ do not commute. We see, that we again have obtained noncommutative probability in purely classical system.

Actually the picture is more complicated, compared to the discussed above. The correlations of the system in the noncommutative replica approach will be given by

$$\lim_{N \to \infty} \langle (\Delta J)^k \rangle = \langle \left( \frac{1}{\sqrt{p}} \sum_{a=0}^{p-1} Q_a \right)^k \rangle$$

where the state $\langle \cdot \rangle$ is generated by the noncommutative replica statistic sum (3). In principle, it is not clear, how to extract noncommutativity from this set of correlators, since different degrees $(\Delta Q)^k$ commute.

To distinguish noncommutative and commutative systems we have to consider the set of correlation functions which will be large enough. In the present case this set should contain correlations of different linear combinations of $Q_a$, more general than $\Delta Q$.

This problem may be discussed in the following way. The quenching (6) in principle may be related to particular way of preparation of the disordered system under consideration. If we will use different physical preparation of the disordered system, this may result in the different quenching procedure. The example of quenching different from (6) was discussed in [15]. This example has the following form

$$\Delta' : J \mapsto \frac{1}{\sqrt{p}} \sum_{a=0}^{p-1} c_a J_a; \quad (7)$$

where $c_a$ are real valued coefficients, which should satisfy the condition

$$\sum_{a=0}^{p-1} c_a^2 = p$$
Varying coefficients $c_a$ we will obtain different quenchings.

Then, using different physical preparations of the system, we measure the correlation functions which will correspond to different quenchings. After we may (at least in principle) use the Bell inequality to prove, do we really have noncommutative probability space which describes the behavior of the disordered system under investigation.

Actually the most natural example of this setup is the experiments with spin glasses, where the glass transition with different external magnetic field was investigated, and non trivial behavior of magnetization on preparation was observed [17], [18].

We propose to analyze these experiments taking into account the correlations between the systems with different preparations (i.e. freezed in the presence of different external magnetic fields, which in our approach should correspond to different quenchings), and to check the validity of the Bell inequalities.

This would help to check the validity of the noncommutative replica approach itself, since there is no direct way to introduce noncommutativity in the standard replica approach.

The discussed here experimental situation could also be described by using contextual probabilistic approach, see [3], [4]. In the contextual framework probabilities (which are interpreted as conventional ensemble probabilities) depend on physical contexts — complexes of experimental physical conditions. Mathematically this means that we could not use one fixed Kolmogorov probability space and we should work with a system of probability spaces depending on physical contexts. In our case various contexts are defined by choosing various external magnetic fields. We recall that by using contextual probabilistic approach it is possible to obtain ”quantum rule” for interference of probabilities of alternatives, see [4], [3]. Such an interference of probabilities is just another way to describe noncommutativity of probabilities (induced in the conventional formalism by using Hilbert space calculus). Therefore we could expect that, by taking into account contextuality of statistics for disordered systems freezed in the presence of different external magnetic fields, it would be possible to find experimental confirmations of the presence of noncommutative structure for classical disordered systems (in particular, spin glasses).

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