Self-dual form of Ruijsenaars-Schneider models
and ILW equation with discrete Laplacian

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Abstract

We discuss a self-dual form of the Bäcklund transformations for the continuous (in
time variable) $\mathfrak{gl}_N$ Ruijsenaars-Schneider model. It is based on the first order
equations in $N + M$ complex variables which include $N$ positions of particles and $M$ dual variables.
The latter satisfy equations of motion of the $\mathfrak{gl}_M$ Ruijsenaars-Schneider model. In the
elliptic case it holds $M = N$ while for the rational and trigonometric models $M$ is not
necessarily equal to $N$. Our consideration is similar to the previously obtained results for
the Calogero-Moser models which are recovered in the non-relativistic limit. We also show
that the self-dual description of the Ruijsenaars-Schneider models can be derived from
complexified intermediate long wave equation with discrete Laplacian by means of the
simple pole ansatz likewise the Calogero-Moser models arise from ordinary intermediate
long wave and Benjamin-Ono equations.

1 Introduction

It was observed in \cite{8} that $N$-body classical Calogero-Moser model \cite{9} appears from $N$-soliton
solution of the Benjamin-Ono equation \cite{6} on the real line

\[ f_t + f f_x - \frac{1}{2} (H f)_{xx} = 0, \quad H f(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy, \quad x \in \mathbb{R}, \quad (1.1) \]

where $f$ is the principal value integral. Namely, (1.1) is fulfilled by the pole ansatz

\[ f(x, t) = \sum_{k=1}^{N} \frac{i}{x - q_k(t)} - \frac{i}{x - \bar{q}_k(t)}, \quad \text{Im}(q_k) < 0 \quad (1.2) \]

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when \( \{ q_j \} \) satisfy the first order equations
\[
\dot{q}_j = \sum_{k \neq j}^N \frac{\dot{q}_j - q_k}{\dot{q}_k - q_j}, \quad j = 1 \ldots N. \tag{1.3}
\]

After taking the time derivative of (1.3) it can be shown that the second order equations acquire the form of the Calogero-Moser equations of motion:
\[
\ddot{q}_j = \sum_{k \neq i}^N \frac{2}{(q_j - q_k)^3}, \quad j = 1 \ldots N. \tag{1.4}
\]

This trick was generalized by Wojciechowski in [29] to the Bäcklund transformations. Later, in [1, 2], it was referred to as the self-dual form of the Calogero-Sutherland model including harmonic and other [15] external potentials. From the point of view of the Benjamin-Ono equation (1.1) these models are related to the intermediate long wave (ILW) equation [13, 14, 4]
\[
f_t + \delta f_x + f f_x - \nu(T f)_{xx} = 0, \quad x \in \mathbb{R}, \tag{1.5}
\]
where \( \delta \) and \( \nu \) are constants and \( T \) is trigonometric or elliptic analogue of the Hilbert transformation \( H \) used in (1.1). (See Appendix B for a brief review.) For example, in the elliptic case [4, 17]
\[
T f(x) = \frac{i}{2\pi} \int_{-1/2}^{+1/2} E_1(y-x) f(y) dy, \tag{1.6}
\]
where \( E_1(z) \) is the first Eisenstein function (A.3).

We are going to consider complexified integrable many-body systems mentioned above, i.e., in our setting the positions of particles are complex numbers. For this purpose we need the complexified version of the ILW equation (1.5). It is written in terms of a pair of complex functions \( f^\pm(z) \) as follows [3, 11, 2]:
\[
f_t + f f_z + \frac{\nu}{2} \ddot{f}_{zz} = 0, \quad z \in \mathbb{C} \tag{1.7}
\]
with \( f = f^+ - f^- \) and \( \dot{f} = f^+ + f^- \). The reduction to the real case (1.5) is achieved by means of the Sokhotski-Plemelj formulae. The multi-pole ansatz for (1.7)
\[
f^+(z, t) = \nu \sum_{k=1}^N E_1(z - q_i(t)), \quad f^-(z, t) = \nu \sum_{\gamma=1}^M E_1(z - \mu_{\gamma}(t)) \tag{1.8}
\]
provides \( N + M \) first order equations (the Bäcklund transformations or the self-dual form of the Calogero-Sutherland model)
\[
\dot{q}_i = \nu \sum_{k \neq i}^N E_1(q_i - q_k) - \nu \sum_{\gamma=1}^M E_1(q_i - \mu_{\gamma}), \quad j = 1 \ldots N, \tag{1.9}
\]
\[
\dot{\mu}_\alpha = -\nu \sum_{\gamma \neq \alpha}^N E_1(\mu_\alpha - \mu_{\gamma}) + \nu \sum_{k=1}^N E_1(\mu_\alpha - q_k), \quad \alpha = 1 \ldots M.
\]
\[\text{The term } \delta^{-1} f_x \text{ can be eliminated by a shift of } f. \text{ It was saved for } \delta \to 0 \text{ limit which provides the KdV equation.}\]
In the elliptic case \( N = M \); in the trigonometric and rational cases \( N \) and \( M \) can be not equal to each other. In the latter cases the \( E_1(z) \) function in (1.9) should be replaced by \( \coth(z) \) and \( 1/z \) respectively, see (A.17).

By differentiating (1.9) with respect to the time variable and some tedious calculations it can be shown that both sets of variables \( q \) and \( \mu \) satisfy the Calogero-Moser equations of motion:

\[
\ddot{q}_i = \nu^2 \sum_{k \neq i}^N E'_2(q_i - q_k), \quad j = 1 \ldots N, \\
\ddot{\mu}_\alpha = \nu^2 \sum_{\gamma \neq \alpha}^M E'_2(\mu_\alpha - \mu_\gamma), \quad \alpha = 1 \ldots M,
\]

where \( E_2(z) = -\partial_z E_1(z) \), see (A.3), (A.4), (A.17). The derivation of (1.8)-(1.10) in the elliptic case was presented in [7].

Equations of type (1.9) are known to be embedded into discrete time dynamics [21, 27]. Then the two sets of variables \( \{q_i\} \) and \( \{\mu_i\} \) are related by a discrete time shift. The discrete equations of motion involve three sets of variables (related by two subsequent time shifts). We do not use this approach since we deal with only two sets of variables.

The purpose of the paper. First, we describe the self-dual form of the \( g_{1N} \) Ruijsenaars-Schneider models [23], which have the following equations of motion:

\[
\ddot{q}_i = \sum_{k \neq i}^N \dot{q}_i \dot{q}_k (2E_1(q_{ik}) - E_1(q_{ik} + \eta) - E_1(q_{ik} - \eta)), \quad i = 1 \ldots N, 
\]

where \( q_{ij} = q_i - q_j \). Hyperbolic and rational analogues of \( E_1(z) \) are given by \( \coth(z) \) and \( 1/z \) respectively, see (A.17). We claim that the described above construction for self-dual representation of the Calogero-Moser model is generalized to the Ruijsenaars-Schneider one.

**Theorem 1** Equations of motion (1.11) for the Ruijsenaars-Schneider model follow from the set of \( N + M \) equations

\[
\dot{q}_i = \prod_{k \neq i}^N \frac{\partial (q_i - q_k + \eta)}{\partial (q_i - q_k)}, \quad \prod_{\gamma = 1}^M \frac{\partial (q_i - \mu_\gamma - \eta)}{\partial (q_i - \mu_\gamma)}, \\
\dot{\mu}_\alpha = \prod_{\gamma \neq \alpha}^M \frac{\partial (\mu_\alpha - \mu_\gamma - \eta)}{\partial (\mu_\alpha - \mu_\gamma)}, \quad \prod_{k = 1}^N \frac{\partial (\mu_\alpha - q_k + \eta)}{\partial (\mu_\alpha - q_k)},
\]

where \( \partial(z) \) should be replaced by \( \sinh(z) \) and \( z \) in hyperbolic and rational cases respectively. The variables \( \{\mu_\alpha\} \) satisfy \( g_{1M} \) Ruijsenaars-Schneider equations of motion:

\[
\ddot{\mu}_\alpha = \sum_{\gamma \neq \alpha}^M \dot{\mu}_\alpha \dot{\mu}_\gamma (2E_1(\mu_{\alpha\gamma}) - E_1(\mu_{\alpha\gamma} + \eta) - E_1(\mu_{\alpha\gamma} - \eta)), \quad \alpha = 1 \ldots M.
\]

In the elliptic case \( N = M \), while in hyperbolic and rational cases \( N \) and \( M \) are arbitrary.

Note that equations (1.12) are well known in the theory of time discretization [22] (and/or Bäcklund transformations [16]) of the Ruijsenaars-Schneider model.\(^4\) Here we give a direct

\(^4\)We are grateful to Yu. Suris for drawing our attention to this.
proof (without usage of the discrete time dynamics) likewise it was presented in [29] for the Calogero-Moser models.

Let $N \geq M$ for definiteness. It was shown in [2] for the Calogero-Sutherland models that $M$ integrals of motion coincide, and other $N - M$ are equal to some constants. We give a proof of a similar result for the Ruijsenaars-Schneider models using determinant identities from [11, 5].

Next, we show that equations (1.12) follow from some multi-pole anzats for a pair of complex functions satisfying (complexified version of) the ILW equation with discrete Laplacian. The latter was suggested in [24, 25, 26]:

$$
\partial_t \log(F^+ (z) - F^- (z) + f_0) = F^+ (z) + F^- (z) - F^+ (z + \eta) - F^- (z - \eta).
$$

(1.14)

It can be reduced to the following equation for a single real function:

$$
f_t = f T f,
$$

(1.15)

where $f = f(x, t)$, $x \in \mathbb{R}$ and

$$
T f(x) = \frac{i}{2\pi} \int_{-1/2}^{1/2} \left( E_1(y - x + \eta) + E_1(y - x - \eta) - 2E_1(y - x) \right) f(y) dy.
$$

(1.16)

It should be mentioned that a relation between the Ruijsenaars-Schneider (and Calogero-Moser) models and ILW-Benjamin-Ono equations is known [3, 10] from the collective field theory description of integrable many-body systems, which is adapted to the $N \to \infty$ limit. Related algebraic structures and possible applications can be found in [7, 18, 19, 20].

The paper is organized as follows. In the next section we prove Theorem 1 and coincidence of (a part of) action variables for the Ruijsenaars-Schneider models (1.11) and (1.13). In Section 3 we review the ILW equation with discrete Laplacian following [24] and describe its relation to the self-dual form (1.12).

## 2 Self-dual form of Ruijsenaars-Schneider models

We are going to prove Theorem 1. Let us start with elliptic case, i.e. $M = N$. Its hyperbolic and rational counterparts are discussed in the end of the section. It is convenient to deal with the Kronecker function (A.2). Using (A.7), we rewrite (1.11) in the form

$$
\ddot{q}_i = \sum_{k \neq i} N \dot{q}_k \left( \frac{g(\eta, q_k - q_i)}{\phi(\eta, q_k - q_i)} - \frac{g(\eta, q_i - q_k)}{\phi(\eta, q_i - q_k)} \right).
$$

(2.1)

Also, rescale formally the time variable as

$$
t \to \frac{\vartheta(0)^{2N-1}}{\vartheta(\eta)^{N-1} \vartheta(-\eta)^N} t.
$$

(2.2)

This has no affect on equations (1.11) or (2.1) since they are homogeneous in $t$. At the same time (1.12) acquires the form

$$
\dot{q}_i = \prod_{k \neq i}^{N} \phi(\eta, q_i - q_k) \prod_{\gamma=1}^{N} \phi(-\eta, q_i - \mu_\gamma),
$$

(2.3)

$$
\dot{\mu}_\alpha = - \prod_{\beta \neq \alpha} \phi(-\eta, \mu_\alpha - \mu_\beta) \prod_{j=1}^{N} \phi(\eta, \mu_\alpha - q_j).
$$
For the proof of the theorem we need the following identity:

\[ \sum_{i=1}^{N} \ddot{q}_i = \sum_{i=1}^{N} \ddot{\mu}_i , \quad (2.4) \]

or, equivalently,

\[ \sum_{i=1}^{N} \left( \prod_{k \neq i}^{N} \phi(\eta, q_i - q_k) \prod_{k=1}^{N} \phi(-\eta, q_i - \mu_k) + \prod_{k \neq i}^{N} \phi(-\eta, \mu_i - \mu_k) \prod_{k=1}^{N} \phi(\eta, \mu_i - q_k) \right) = 0 . \quad (2.5) \]

It is a particular case of (A.9). Indeed, consider \( n = 2N - 1 \) in (A.9) or, what is more convenient, let the summation (and multiplication) index \( i \) in (A.9) to be \( i = 2, ..., 2N \). Substitute

\[ y_2 = \ldots = y_N = \eta, \quad y_{N+1} = \ldots = y_{2N} = -\eta, \quad \sum_{k=2}^{2N} y_k = -\eta , \]

\[ x_2 = q_1 - q_2, ..., x_N = q_1 - q_N, \quad x_{N+1} = q_1 - \mu_1, ..., x_{2N} = q_1 - \mu_N . \]

Then, using the property (A.5) we get (2.4) from (A.9) in the form \( \ddot{q}_1 = - \sum_{i=2}^{N} \ddot{q}_i + \sum_{i=1}^{N} \ddot{\mu}_i \). Consider derivative of the identity (2.5) with respect to \( q_i \). It reads as follows:

\[ \sum_{k \neq i}^{N} \ddot{q}_k \frac{g(\eta, q_i - q_k)}{\phi(\eta, q_i - q_k)} - \sum_{k \neq i}^{N} \ddot{\mu}_k \frac{g(-\eta, q_i - \mu_k)}{\phi(-\eta, q_i - \mu_k)} = 0 . \quad (2.7) \]

**Proof of Theorem** Compute \( \ddot{q}_i \) by differentiating the upper line of (2.3) by the time variable. This yields

\[ \frac{\ddot{q}_i}{\dot{q}_i} = \sum_{k \neq i}^{N} (\ddot{q}_k - \ddot{\mu}_k) \frac{g(\eta, q_i - q_k)}{\phi(\eta, q_i - q_k)} + \sum_{k=1}^{N} (\ddot{q}_k - \ddot{\mu}_k) \frac{g(-\eta, q_i - \mu_k)}{\phi(-\eta, q_i - \mu_k)} . \quad (2.8) \]

Equations of motion (2.1) appear by subtracting the l.h.s. of (2.7) from the r.h.s. of (2.8). Equations (1.13) follow from the symmetry of (1.12) under simultaneous exchange \( q \leftrightarrow \mu \) and \( \eta \leftrightarrow -\eta \).

The hyperbolic and rational cases are obtained as follows. All the proof is the same since the functions in each line of (A.17) satisfy the same identities. So that in the \( M = N \) case the statement is proved. In the case \( M \neq N \), say \( M < N \) we apply the argument called “dimensional reduction” in [2]: \( N - M \) coordinates \( \mu_\alpha, \alpha = M + 1...N \) may go to infinity in the rational or hyperbolic case, i.e. there is a limit \( |\mu_\alpha| \to \infty, \alpha = M + 1...N \) in which (1.12) with \( N = M \) turns into the same system of equations with \( M < N \). |}

For the non-relativistic limit

\[ \eta = \nu/c, \quad c \to \infty \quad (2.9) \]

of the Ruijsenaars-Schneider equations of motion (1.11), (1.13) to the Calogero-Moser equations (1.10) we put

\[ \dot{q}_i \to c + \dot{q}_i + O(1/c), \quad \dot{\mu}_\alpha \to c + \dot{\mu}_\alpha + O(1/c), \quad (2.10) \]

where in the r.h.s. the non-relativistic velocities are implied. In order to make the non-relativistic limit in (2.3) one should first come back to (1.12) (through the inverse rescaling of the time
variable \( \rho \) and make an additional rescaling \( t \to ct \) (this leads to multiplication of the r.h.s. of (1.12) by \( c \)). Then the limit (2.9), (2.10) of (1.12) reproduces the self-dual form of the elliptic Calogero-Moser model (1.9). For \( M \neq N \) in rational or trigonometric cases the “dimensional reduction” argument is used again. In this way we reproduce the self-dual representation of the Calogero-Moser models [8 1 2 7].

Let us discuss the relation between \( \text{gl}_N \) and \( \text{gl}_M \) Ruijsenaars-Schneider models in variables \( \{q\} \) (1.11) and \( \{\mu\} \) (1.13). For this purpose introduce the following pair of matrices used in [11] for the direct proof of the quantum-classical correspondence between the classical (rational) Ruijsenaars-Schneider model and generalized (XXX) quantum spin chains:

\[
L_{ij} = \frac{g \eta}{q_i - q_j + \eta} \prod_{k \neq j}^N \frac{q_j - q_k + \eta}{q_j - q_k} \prod_{\gamma = 1}^M \frac{q_j - \mu_\gamma - \eta}{q_j - \mu_\gamma}, \quad i, j = 1, \ldots, N
\]

and

\[
\tilde{L}_{\alpha \beta} = \frac{g \eta}{\mu_\alpha - \mu_\beta + \eta} \prod_{\gamma \neq \beta}^M \frac{\mu_\beta - \mu_\gamma - \eta}{\mu_\beta - \mu_\gamma} \prod_{k = 1}^N \frac{\mu_\beta - q_k + \eta}{\mu_\beta - q_k}, \quad \alpha, \beta = 1, \ldots, M,
\]

The following relation holds true for matrices (2.11) and (2.12):

\[
\det_{N \times N} (L - \lambda I) = (g - \lambda)^{N - M} \det_{M \times M} (\tilde{L} - \lambda I).
\]

Here \( I \) is the unity matrix. Set \( g = 1 \) and substitute the products in (2.11) and (2.12) from (1.12) (in the rational case \( \vartheta(z) \) is replaced by its argument \( z \)). Then \( L \) is the Lax matrix of \( \text{gl}_N \) Ruijsenaars-Schneider model while \( \tilde{L} \) is the Lax matrix of the \( \text{gl}_M \) Ruijsenaars-Schneider model. Relation (2.13) means that (for \( N \geq M \)) \( M \) action variables (eigenvalues of the Lax matrix) in both models coincide, and the other \( N - M \) of the first one take degenerated values (they equal 1, or \( c \), or \( \eta \) depending on normalization of the Lax matrix). This result is similar to the one derived in [2] for the Calogero-Moser-Sutherland case.

In the trigonometric case the analogues of relations (2.11)-(2.13) were used in [5] for the proof of the quantum-classical duality between the classical (trigonometric) Ruijsenaars-Schneider model and generalized (XXZ) quantum spin chain. Introduce \((N - M) \times (N - M)\) diagonal matrix

\[
S_{ij} = \delta_{ij} \exp\left(-(2j - 1 - (N - M)\eta)\right), \quad i, j = 1, \ldots, N - M
\]

and the pair of \( N \times N \) and \( M \times M \) matrices:

\[
L_{ij} = \frac{g \sinh \eta}{\sinh(q_i - q_j + \eta)} \prod_{k \neq j}^N \frac{\sinh(q_j - q_k + \eta)}{\sinh(q_j - q_k)} \prod_{\gamma = 1}^M \frac{\sinh(q_j - \mu_\gamma - \eta)}{\sinh(q_j - \mu_\gamma)}, \quad i, j = 1, \ldots, N
\]

\[
\tilde{L}_{\alpha \beta} = \frac{g \sinh \eta}{\sinh(\mu_\alpha - \mu_\beta + \eta)} \prod_{\gamma \neq \beta}^M \frac{\sinh(\mu_\beta - \mu_\gamma - \eta)}{\sinh(\mu_\beta - \mu_\gamma)} \prod_{k = 1}^N \frac{\sinh(\mu_\beta - q_k + \eta)}{\sinh(\mu_\beta - q_k)}, \quad \alpha, \beta = 1, \ldots, M
\]

Then the following identity is valid for (2.14), (2.15), (2.16):

\[
\det_{N \times N} (L - \lambda I) = \det_{(N - M) \times (N - M)} (gS - \lambda I) \det_{M \times M} (\tilde{L} - \lambda I).
\]
Again, we substitute the products (2.15), (2.16) from (1.12) (in the hyperbolic case \( \vartheta(z) \) is replaced by \( \sinh(z) \), see (A.17)). Then \( L \) and \( \tilde{L} \) are the Lax matrices of \( \mathfrak{gl}_N \) and \( \mathfrak{gl}_M \) Ruijsenaars-Schneider models respectively. And again, this means that \( M \) action variables in both models coincide, and the other \( N-M \) of the first one take degenerated values given by diagonal elements of the matrix \( S \). Notice that in \( N = M \) case similar result is easily obtained in the context of discrete time dynamics [22] (it comes immediately from the discrete Lax equation).

3 ILW equation and Ruijsenaars-Schneider model

A brief review of the Benjamin-Ono and ILW equations is given in Appendix B. Here we describe the construction of the (double periodic) ILW equation with discrete Laplacian [24] and its relation to the self-dual form of the Ruijsenaars-Schneider model.

3.1 ILW equation with discrete Laplacian

The periodic ILW equation with discrete Laplacian was proposed in [24]:

\[
\frac{f_t}{f} = f \mathcal{T} f , \tag{3.1}
\]

where \( f = f(x, t), x \in \mathbb{R} \) and

\[
\mathcal{T} f(x) = \frac{i}{2\pi} \int_{-1/2}^{+1/2} \left( E_1(y - x + \eta) + E_1(y - x - \eta) - 2E_1(y - x) \right) f(y) dy . \tag{3.2}
\]

The integral transformation (3.2) is defined for a periodic function \( f(x + 1, t) = f(x, t) \) on the real axis \( x \in \mathbb{R} \), and \( 0 < \text{Im}(\eta) < \text{Im}(\tau) \). The modular parameter of the elliptic curve is assumed to be purely imaginary: \( \text{Re}(\tau) = 0 \) (and \( \text{Im}(\tau) > 0 \)).

The last term in (3.2) is proportional to \( \mathcal{T} = (1/2i)T \) with \( T \) (B.8), (B.9) and normalization of the real half-period \( L = 1/2 \):

\[
\mathcal{T} f(x) = \frac{i}{2\pi} \int_{-1/2}^{+1/2} E_1(y - x)f(y)dy , \quad x \in \mathbb{R} . \tag{3.3}
\]

It was argued in [24] [25] that the integral operator (3.2) can be written in the following form:

\[
\mathcal{T} f(x) = F^+(x) + F^-(x) - F^+(x + \eta) - F^-(x - \eta) . \tag{3.4}
\]

In order to define \( F^\pm(x) \) let us denote by \( \tilde{f}(x) \) the zero mean part of \( f(x) \), i.e. \( \tilde{f}(x) = f(x) - f_0 \), where \( f_0 = \int_{-1/2}^{+1/2} f(x)dx \) is the zero Fourier component, so that \( \int_{-1/2}^{+1/2} \tilde{f}(x)dx = 0 \). Introduce also

\[
F(z) = \frac{1}{2\pi i} \int_{-1/2}^{+1/2} E_1(y - z)\tilde{f}(y)dy , \quad 0 < \text{Im}(z) < \text{Im}(\tau) . \tag{3.5}
\]

This function is periodic, \( F(z) = F(z + 1) \), holomorphic in the strip domain \( 0 < \text{Im}(z) < \text{Im}(\tau) \) and continuous up to its boundary. It can be \( \tau \)-periodically continued to piecewise\(^5\) holomorphic

\(^5\)The jumps of \( F(z) \) in \( \mathbb{C} \) take place on the lines \( \text{Im}(z) = k \text{Im}(\tau), k \in \mathbb{Z} \).
function on $\mathbb{C}$ due to the properties (A.6) and zero mean of $\hat{f}(x)$ (see [17]). Then, for

$$F^+(x) = \lim_{z \to x+0} F(z) ,$$

$$F^-(x) = \lim_{z \to x-0} F(z + \tau)$$

($x \in \mathbb{R}$) we have due to the Sokhotski–Plemelj formulae:

$$F^+(x) - F^-(x) = \hat{f}(x)$$

(3.7)

and

$$F^+(x) + F^-(x) = 2 \int_{-1/2}^{+1/2} E_1(y - x) \hat{f}(y) dy = -2 T \hat{f}(x) = -2 T[F^+ - F^-](x).$$

(3.8)

To summarize, the function $f(x)$ from (3.1) is represented as

$$f(x) = F^+(x) - F^-(x) + f_0$$

with (3.6), and the equation (3.1) is equivalent to

$$\partial_t \log(F^+(x) - F^-(x) + f_0) = F^+(x) + F^-(x) - F^+(x + \eta) - F^-(x - \eta),$$

(3.9)

where $f_0 = \int_{-1/2}^{+1/2} f(x) dx$. The functions $F^\pm (x \pm \eta)$ are analytical continuations of $F^\pm (x)$ (3.6).

The limit to the ILW equation is achieved as follows. Consider $T$ in the limit $\eta \to 0$. From (3.4) and (3.7), (3.8) we have

$$T f(x) = - \eta(F^+ - F^-) - \frac{\eta^2}{2}(F^+_{xx} + F^-_{xx}) + O(\eta^3) = - \eta f_x + \eta^2 T f_{xx} + O(\eta^3).$$

(3.10)

Let us make the substitution [24]: $f(x, t) = \nu + \eta u(y, t) + O(\eta^2)$ with $y = x - ct$, where $\nu$ and $c$ are some constants. Then (3.1) with (3.10) yields

$$\eta u_t - \nu c u_y = - \eta^2 \nu u_y + \eta^3 \nu T u_{yy} - \eta^3 uu_y + O(\eta^4).$$

(3.11)

Choose also $c = \nu \eta + a \eta^2 + O(\eta^3)$ with another constant $a$ and rescale $t \to \tilde{t} = t/\eta^2$. Then, taking the limit $\eta \to 0$, we obtain

$$u_{\tilde{t}} = au_y - uu_y + \nu T u_{yy}.$$

(3.12)

It is equation (1.5).

### 3.2 Ruijsenaars-Schneider models and discrete ILW equation

Let us consider the complexified version of (3.9) with a complex variable $x$. In [1] the self-dual form of the Calogero-Sutherland models were obtained by passing from equation (3.12) to the following one

$$u_t = - uu_z - \frac{\nu}{2} u_{zz},$$

(3.13)

written for a pair of independent complex functions $u = u^+ - u^-$ and $\tilde{u} = u^+ + u^-$. It was called bidirectional Benjamin-Ono equation. The relation of type (3.7) was treated as an additional

\[6\]In the notation of [1] $u^+ = u_1, u^- = -u_0$ and $\nu = -ig$.  

8
reduction. In a similar manner, the self-dual form of the elliptic Calogero-Moser model was described through the complexified periodic ILW equation [7].

In order to get the self-dual form for the Ruijsenaars-Schneider model (2.3), it is reasonable to study equation (3.1) with $T$ given by (3.4):

$$\partial_t \log (F^+(z) - F^-(z) + f_0) = F^+(z) + F^-(z) - F^+(z + \eta) - F^-(z - \eta).$$

Equation (3.14) generalizes (3.9) in the same way as (3.13) generalizes (3.12) (with $a = 0$). In what follows we deal with (3.14). It is written for two independent complex functions $F^+(z)$, $F^-(z)$, $z \in \mathbb{C}$ ($f_0$ is a constant). We are going to show that a natural multi-pole ansatz provides the Ruijsenaars-Schneider model in the form (2.3). Introduce

$$f(z) = \prod_{i=1}^{N} \phi(\eta, z - q_i) \prod_{\gamma=1}^{N} \phi(-\eta, z - \mu_\gamma).$$

The function $\phi(\eta, z)$ has simple pole at $z = 0$ with residue 1. Therefore, $f(z)$ has $2N$ simple poles at $\{q_i, i = 1...N\}$ and $\{\mu_\alpha, \alpha = 1...N\}$. The residues are

$$\text{Res}_{z=q_i} f(z) = \prod_{k \neq i}^{N} \phi(\eta, q_i - q_k) \prod_{\gamma=1}^{N} \phi(-\eta, q_i - \mu_\gamma)$$

and

$$\text{Res}_{z=\mu_\alpha} f(z) = \prod_{\beta \neq \alpha}^{N} \phi(-\eta, \mu_\alpha - \mu_\beta) \prod_{j=1}^{N} \phi(\eta, \mu_\alpha - q_j).$$

As a by-product, we have an alternative proof of (2.4): notice that $f(z)$ is double-periodic in $z$ due to (A.6) (here $M = N$ is necessary). Therefore, the sum of residues equals zero.

Due to the structure of poles the double-periodic function $f(z)$ (3.15) is represented also as

$$f(z) = f_0 + \sum_{k=1}^{N} E_1(z - q_k) \text{Res}_{z=q_i} f(z) + \sum_{\gamma=1}^{N} E_1(z - \mu_\gamma) \text{Res}_{z=\mu_\gamma} f(z)$$

with some constant $f_0$. Denote

$$F^+(z) = \sum_{k=1}^{N} E_1(z - q_k) \text{Res}_{z=q_i} f(z)$$

and

$$F^-(z) = - \sum_{\gamma=1}^{N} E_1(z - \mu_\gamma) \text{Res}_{z=\mu_\gamma} f(z).$$

**Proposition 3.1** The function (3.12) $f(z) = F^+(z) - F^-(z) + f_0$ with definitions (3.12), (3.20) satisfies (3.14) and provides equations of motion of the Ruijsenaars-Schneider models (2.3).

It should be noted that the form of (3.14) is much similar to the semi-discretized version of the KP equation [21] but with different meaning of variables. Such KP equation leads to the discrete Calogero-Moser model, while (3.14) provides the Ruijsenaars-Schneider model.
Proof: Indeed, differentiating \( f(z) \) (3.15) with respect to \( t \) and using (A.7) we obtain

\[
\frac{\partial_t f(z)}{f(z)} = -\sum_{i=1}^{N} \left( \dot{q}_i (E_1(z-q_i)+\eta) - E_1(z-q_i) \right) + \mu_i (E_1(z-\mu_i)-\eta) - E_1(z-\mu_i)).
\] (3.21)

For the r.h.s. of (3.14) with \( F^\pm(z) \) given by (3.19), (3.20) we have

\[
F^+(z) - F^+(z+\eta) + F^-(z) - F^-(z-\eta) = 
\sum_{i=1}^{N} \left( -(E_1(z-q_i)+\eta) - E_1(z-q_i) \right) \res f(z) + \left( E_1(z-\mu_i)-\eta - E_1(z-\mu_i) \right) \res f(z).
\] (3.22)

By comparing (3.21) and (3.22) we get:

\[
\res f(z) = \dot{q}_i,
\]
\[
\res f(z) = -\dot{\mu}_\alpha.
\] (3.23)

With (3.16), (3.17) these are the equations (2.3) ■.

Notice that the time variable in (3.14) is assumed to be rescaled as given in (2.2). Alternatively one could consider the ansatz (3.15) in a slightly different form:

\[
\tilde{f}(z) = \vartheta'(0) \prod_{i=1}^{N} \frac{\vartheta(z-q_k+\eta)}{\vartheta(z-q_k)} \prod_{\gamma=1}^{N} \frac{\vartheta(z-\mu_\gamma-\eta)}{\vartheta(z-\mu_\gamma)}.
\] (3.24)

In this case

\[
\tilde{f}(z) = \tilde{f}_0 + \sum_{k=1}^{N} E_1(z-q_k) \res \tilde{f}(z) - \sum_{\gamma=1}^{N} E_1(z-\mu_\gamma) \res \tilde{f}(z)
\]

and

\[
\tilde{F}^+(z) = \sum_{k=1}^{N} E_1(z-q_k) \res \tilde{f}(z), \quad \tilde{F}^-(z) = \sum_{\gamma=1}^{N} E_1(z-\mu_\gamma) \res \tilde{f}(z).
\]

The ansatz (3.24) leads to equations (1.12), and the additional rescaling of time variable is not needed.

4 Appendix A: elliptic functions

We use the odd theta-function with the modular parameter \( \tau \), \( \text{Im}(\tau) > 0 \) [12, 28]:

\[
\vartheta(z) = \sum_{k\in\mathbb{Z}} \exp \left( \pi i \tau (k + \frac{1}{2})^2 + 2 \pi i (z + \frac{1}{2})(k + \frac{1}{2}) \right),
\] (A.1)

the Kronecker function

\[
\phi(\eta, z) = \frac{\vartheta'(0) \vartheta(\eta + z)}{\vartheta(\eta) \vartheta(z)} = \phi(z, \eta)
\] (A.2)
and the Eisenstein functions

\[ E_1(z) = \partial'(z)/\partial(z), \quad E_2(z) = -\partial_z E_1(z). \]  \hfill (A.3)

The Eisenstein functions are simply related to the Weierstrass functions:

\[ \zeta(z) = E_1(z) - \frac{z}{3} \partial^m(0), \quad \varphi(z) = E_2(z) + \frac{1}{3} \partial^m(0). \]  \hfill (A.4)

The parities are:

\[ \vartheta(-z) = -\vartheta(z), \quad \phi(-\eta, -z) = -\phi(\eta, z), \quad E_1(-z) = -E_1(z), \quad E_2(-z) = E_2(z). \]  \hfill (A.5)

The behavior on the lattice of periods \( \Gamma = \mathbb{Z} + \tau \mathbb{Z} \) is

\[ \vartheta(z + 1) = -\vartheta(z), \quad \vartheta(z + \tau) = -\exp(-\pi i \tau - 2\pi i z)\vartheta(z), \]
\[ E_1(z + 1) = E_1(z), \quad E_1(z + \tau) = E_1(z) - 2\pi i, \]
\[ E_2(z + 1) = E_2(z), \quad E_2(z + \tau) = E_2(z), \]
\[ \phi(z + 1, u) = \phi(z, u), \quad \phi(z + \tau, u) = \exp(-2\pi i u)\phi(z, u). \]  \hfill (A.6)

The derivative of the Kronecker function with respect to the second argument is

\[ g(z, u) = \partial_u \phi(z, u) = \phi(z, u)(E_1(z + u) - E_1(u)). \]  \hfill (A.7)

The Fay trisecant identity for genus one reads

\[ \phi(z, q)\phi(w, u) = \phi(z - w, q)\phi(w, q + u) + \phi(w - z, u)\phi(z, q + u). \]  \hfill (A.8)

There are also higher \((n\text{-th order})\) identities:

\[ \prod_{i=1}^{n} \phi(x_i, y_i) = \sum_{i=1}^{n} \phi(x_i, \sum_{k=1}^{n} y_k) \prod_{j \neq i} \phi(x_j - x_i, y_j). \]  \hfill (A.9)

**Proof** is by induction in \(n\). Suppose \((A.9)\) is true. We then need to prove that

\[ \prod_{i=1}^{n+1} \phi(x_i, y_i) = \sum_{i=1}^{n+1} \phi(x_{i+1}, \sum_{k=1}^{n} y_k) \prod_{j \neq i}^{n+1} \phi(x_j - x_i, y_j). \]  \hfill (A.10)

For its l.h.s. we have

\[ \prod_{i=1}^{n+1} \phi(x_i, y_i) \stackrel{(A.9)}{=} \sum_{i=1}^{n} \phi(x_{n+1}, y_{n+1})\phi(x_i, \sum_{k=1}^{n} y_k) \prod_{j \neq i}^{n} \phi(x_j - x_i, y_j) \stackrel{(A.8)}{=} \]

\[ = \sum_{i=1}^{n} \phi(x_{n+1} - x_i, y_{n+1})\phi(x_i, \sum_{k=1}^{n+1} y_k) \prod_{j \neq i}^{n} \phi(x_j - x_i, y_j) + \]

\[ \text{...} \]
Consider the r.h.s. of (A.10). Let us write the first $n$ terms of the sum separately:

$$
\sum_{i=1}^{n+1} \phi(x_i, \sum_{k=1}^{n+1} y_k) \prod_{j \neq i} \phi(x_j - x_i, y_j) =
\sum_{i=1}^{n} \phi(x_i, \sum_{k=1}^{n} y_k) \prod_{j \neq i} \phi(x_j - x_i, y_j) + \phi(x_{n+1}, \sum_{k=1}^{n+1} y_k) \prod_{j=1}^{n} \phi(x_j - x_{n+1}, y_j)
$$

The first terms in (A.11) and (A.12) are the same. Therefore, we need to prove that the second terms are equal as well, i.e.

$$
\sum_{i=1}^{n} \phi(x_i - x_{n+1}, \sum_{k=1}^{n+1} y_k) \phi(x_{n+1}, \sum_{k=1}^{n+1} y_k) \prod_{j \neq i} \phi(x_j - x_i, y_j) = (A.12)
$$

After cancellation of the common factor $\phi(x_{n+1}, \sum_{k=1}^{n+1} y_k)$ we get (A.9) with $x_i \to x_i - x_{n+1}$.

The first Eisenstein function $E_1(z)$ possesses the following series representation:

$$
E_1(z|\tau) = \pi \cot(\pi z) + 4\pi \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^{2n}} \sin(2\pi n z), \quad q = \exp(\pi i \tau)
$$

(here $-\text{Im}(\tau) < \text{Im}(z) < \text{Im}(\tau)$). The first term in (A.13) regarded as a generalized function can be represented via

$$
\frac{1}{2} \cot\left(\frac{x}{2}\right) = \sum_{k=1}^{\infty} \sin(kx).
$$

Then (A.13) acquires the form

$$
E_1(z|\tau) = 2\pi \sum_{n=1}^{\infty} \frac{1 + q^{2n}}{1 - q^{2n}} \sin(2\pi n z) = -i\pi \sum_{n=0}^{\infty} \frac{1 + q^{2n}}{1 - q^{2n}} e^{2\pi i n z}.
$$

We also need the following modular transformation:

$$
E_1(z|\tau) = \frac{1}{\tau} E_1\left(\frac{z}{\tau} \mid -\frac{1}{\tau}\right) - 2\pi i \frac{z}{\tau}.
$$

Rational and hyperbolic analogues of the functions (A.1)-(A.3) are as follows:

| rational | $\vartheta(z)$ | $\phi(z, \eta)$ | $E_1(z)$ | $E_2(z)$ |
|----------|----------------|----------------|-----------|-----------|
| $z$      | $z$            | $\frac{1}{z} + \frac{1}{\eta}$ | $\frac{1}{z}$ | $\frac{1}{z^2}$ |

| hyperbolic | $\sinh(z)$ | $\coth(z) + \coth(\eta)$ | $\coth(z)$ | $\sinh^{-2}(z)$ |

They satisfy properties (A.5) and (A.7)-(A.9).
Appendix B: Benjamin-Ono and ILW equations

Here we review the Benjamin-Ono and ILW equations.

1. Rational case. Let $Hf$ be the Hilbert transform of the function $f(x,t)$ in the variable $x \in \mathbb{R}$:

$$Hf(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy.$$  \hfill (B.1)

The Benjamin-Ono equation [6] is as follows:  \hfill (B.2)

$$f_t + 2ff_x - (Hf)_{xx} = 0, \quad x \in \mathbb{R}.$$  \hfill (B.2)

2. Trigonometric case. Similarly, the periodic Benjamin-Ono equation [14] reads

$$f_t + 2ff_x + (Tf)_{xx} = 0, \quad x \in \mathbb{R}$$  \hfill (B.3)

with the integral transformation

$$Tf(x) = -\frac{1}{2L} \int_{-L}^{L} \cot(\pi \frac{L}{2L}(x-y)) f(y) dy.$$  \hfill (B.4)

When $L \to \infty$ we reproduce (B.2).

3. Hyperbolic case. The intermediate long wave (ILW) equation [13, 14] is

$$f_t + \delta^{-1}f_x + 2ff_x + (Tf)_{xx} = 0, \quad x \in \mathbb{R}, \quad \delta > 0$$  \hfill (B.5)

with

$$Tf(x) = -\frac{1}{2\delta} \int_{\mathbb{R}} \coth(\pi \frac{\delta}{2\delta}(x-y)) f(y) dy.$$  \hfill (B.6)

again reproduces (B.2) in the (deep water) limit $\delta \to \infty$. At the same time in the (shallow water) limit $\delta \to 0$ (B.3), (B.6) provides the KdV equation

$$f_t + 2ff_x + (\delta/3)f_{xxx} + O(\delta^3) = 0$$

due to

$$[T + \delta^{-1} \partial_x^{-1}] f(x) = -\frac{1}{2\delta} \int_{\mathbb{R}} \left( \coth(\pi \frac{\delta}{2\delta}(x-y)) - \text{sgn}(x-y) \right) f(y) dy = \frac{\delta}{3} f_x + O(\delta^3).$$  \hfill (B.7)

4. Elliptic case. The (double) periodic intermediate long wave (ILW) equation [4] (see also [17]) is (B.5), where the integral kernel is defined by the first Eisenstein function (A.3):

$$Tf(x) = \frac{1}{2L} \int_{-L}^{L} \tilde{T}(x-y, \delta, L) f(y) dy$$  \hfill (B.8)

and

$$\tilde{T}(x, \delta, L) = -\frac{1}{\pi} E_1(\frac{x}{2L} | \tau), \quad \tau = \frac{\delta}{L}.$$  \hfill (B.9)

\footnote{It is assumed that $f(x) \in L^p(\mathbb{R})$, $p > 1$ (as a function of the variable $x$).}

\footnote{The coefficient 2 in front of $ff_x$ differs from the normalization used in (1.1), (1.5). It is eliminated by the substitution $f \to f/2$. Following original papers, we keep this coefficient in the appendix.}
Plugging \( z = x/(2L) \) and \( q = e^{-\pi\delta/L} \) into (A.15), we get

\[
\tilde{T}(x, \delta, L) = -2 \sum_{n>0} e^{2\pi n\delta/L} \sin \frac{\pi n x}{L} = i \sum_{n \neq 0} \coth \frac{\pi n \delta}{L} \exp \frac{\pi n x}{L}.
\]  
(B.10)

Let \( f(x) \) be \( 2L \)-periodic function, \( f(x + 2L, t) = f(x, t) \). Substitution of its Fourier series

\[
f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k \exp \frac{\pi ikx}{L}, \quad \hat{f}_k = \frac{1}{2L} \int_{-L}^{L} f(x) \exp(-\frac{\pi ikx}{L}) \, dx
\]  
(B.11)

into (B.8) with \( \tilde{T} \) (B.10) yields

\[
T f(x) = i \sum_{n \neq 0} \coth \left( \frac{\pi n \delta}{L} \right) \hat{f}_n \exp \left( \frac{\pi inx}{L} \right).
\]  
(B.12)

Let us consider several limiting cases:

(a) the trigonometric limit (B.4) is achieved when \( L \) is fixed and \( \delta \to \infty \). Indeed, for (B.9) we get \( \tilde{T}(x, \infty, L) = -\cot(\frac{x\pi}{2L}) \) from (A.13) since \( q \to 0 \).

(b) the hyperbolic limit (B.6): \( \delta \) is fixed and \( L \to \infty \). In this case \( q \to 1 \) and we cannot use (A.13) immediately. Instead, we first perform the modular transformation (A.16). It yields

\[
\frac{1}{2L} \tilde{T}(x, \delta, L) = -\frac{1}{2\pi} E_1 \left( \frac{x}{2L} \bigg| \frac{i \delta}{L} \right) = -\frac{1}{2\pi i \delta} E_1 \left( \frac{x}{2i \delta} \bigg| \frac{i L}{\delta} \right) + \frac{x}{2\delta L}.
\]  
(B.13)

Now \( q' = \exp(-\pi L/\delta) \to 0 \) (when \( L \to \infty \)) and we may use (A.13):

\[
\left( \frac{1}{2L} \tilde{T} \right)(x, \delta, \infty) = -\frac{i}{2\delta} \cot \left( \frac{i \pi x}{2\delta} \right) = -\frac{1}{2\delta} \coth \left( \frac{\pi x}{2\delta} \right).
\]  
(B.14)

(c) The limit to the KdV equation is obtained similarly to (B.7): \( L \) is fixed and \( \delta \to 0 \). Consider (B.12) and use the Laurent series expression \( \coth(z) = 1/z + z/3 + O(z^3) \). It is then easy to see that \( (Tf)_{xx} = -\delta^{-1} f_x + (\delta/3) f_{xxx} + O(\delta^3) \).

It is not a coincidence that the limit to the KdV equation from the elliptic case is very similar to the one from the hyperbolic case (B.7). In fact, the transformations (B.6) and (B.8), (B.9) are identical for \( 2L \)-periodic functions since \( E_1(z) \) is an average of \( \coth(z) \) over one-dimensional lattice. The same answer (B.12) can be obtained from (B.6) as well. To see this, let \( f(x) = f(x + 2L) \) and \( \hat{f}_0 = 0 \) (the zero mean condition). Rewrite (B.6) in terms of the Fourier transform \( \hat{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} f(x) \, dx \) as

\[
T f(x) = i \int_{\mathbb{R}} \tilde{T}(\omega) \hat{f}(\omega) e^{i\omega x} \, d\omega, \quad \tilde{T}(\omega) = \coth(\omega \delta).
\]  
(B.15)

It follows from the periodicity of \( f(x) \) that it has Fourier series representation (B.11), and, therefore, \( \hat{f}(\omega) \) is a sum of delta-functions: \( \hat{f}(\omega) = \sum_{n=-\infty}^{\infty} \delta(n\pi/L - \omega) \). Plugging it into (B.15) we get (B.12) (the term \( n = 0 \) is excluded since \( \hat{f}_0 = 0 \)). That is, for a periodic function the transformations (B.6) and (B.8), (B.9) are identical.
Acknowledgments

A. Zotov acknowledges for hospitality Erwin Schrodinger International Institute for Mathematics and Physics (ESI), Vienna and the organizers of the Workshop “Elliptic Hypergeometric Functions in Combinatorics, Integrable Systems and Physics”, where a part of this work was done. The work of A. Zotov was also supported in part by RFBR grant 15-02-04175. The work of A. Zabrodin was funded by the Russian Academic Excellence Project ‘5-100’.

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