Hopping Conductivity of a Nearly-1d Fractal: a Model for Conducting Polymers

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We suggest treating a conducting network of oriented polymer chains as an anisotropic fractal whose dimensionality $D = 1 + \epsilon$ is close to one. Percolation on such a fractal is studied within the real space renormalization group of Migdal and Kadanoff. We find that the threshold value and all the critical exponents are strongly nonanalytic functions of $\epsilon$ as $\epsilon \to 0$, e.g., the critical exponent of conductivity is $n_{\text{RT}} = 1 + 1/\epsilon$. The distribution function for conductivity of finite samples at the percolation threshold is established. It is shown that the central body of the distribution is given by a universal scaling function and only the low-conductivity tail of distribution remains $\epsilon$-dependent. Variable range hopping conductivity in the polymer network is studied: both DC conductivity and AC conductivity in the multiple hopping regime are found to obey a quasi-1d Mott law. The present results are consistent with electrical properties of poorly conducting polymers.

1. INTRODUCTION.

Charge transport in structures with fractional dimensionality has attracted a high degree of attention due to both its fundamental and applied interest \[1,2\]. Conductivity of a random fractal of resistors was studied in Ref. \[3\] as an example of critical phenomena. A large body of literature has been devoted to a study of the unusual dynamics of electrons in regular and random fractals \[4\]. The discussion of the questions and appropriate references may be found in recent reviews (see, e.g., \[5,6\]).

In the present paper we consider problems of percolation and hopping transport on nearly one-dimensional strongly anisotropic fractals \[7\] with dimensionality $D = 1 + \epsilon$. These fractals are expected to exhibit unique properties because their dimensionality at $\epsilon \ll 1$ is close to the low marginal one for the percolation transition. In contrast to isotropic fractals a small parameter $\epsilon$ enables us to obtain the exact solution. Moreover it becomes possible in the case of nearly-one-dimensional fractals to establish not only average characteristics but their entire distribution functions.

Another motivation for a study of quasi-one-dimensional fractals is recent experimental data on conducting polymers such as doped polyacetylene, polypyrrole and polyaniline \[8\]. In general, this class of polymers has a great variety of transport properties. In the undoped state these polymers are semiconductors with an energy gap of Peierls-Mott origin \[9\]. With doping the energy gap is suppressed quickly and for the highly doped case there is finite density of states at the Fermi level. The room temperature conductivity ($\sigma_{\text{RT}}$) of heavily doped sample may attain metallic values and temperature and frequency dependencies of conductivity may be close to metallic. The nature of the metallic phase in these samples is presently a subject of intensive study. One point is that the metallic state is dependent upon by strong interchain coupling \[10,12,13\].

In doped polymers with moderate $\sigma_{\text{RT}}$ (of the order of several hundreds \(S/cm\)) the conductivity, as a rule, decreases with decreasing temperature \[11\]. Because this decay follows a power law in a large temperature interval, presumably, these materials are near the metal-insulator transition which happens at the critical interchain coupling. Poorly conducting doped samples with $\sigma_{\text{RT}}$ of the order or less 1 \(S/cm\) have behavior that can be classified as “dielectric” \[10\]: it is similar to that observed in amorphous semiconductors. For such materials, DC conductivity is strongly dependent on temperature and, generally, follows: \(\sigma_{\text{DC}} \propto \exp(-T_0/T)^{1/2}\). It is noted that caution must be made to review a large temperature range in comparison of experimental conductivity with the model dependencies \[12\].

For a variable range hopping (VRH) mechanism of transport the temperature dependence of conductivity was initially derived \[13,14\] to be : \(\sigma_{\text{DC}} \propto \exp(-(T_0/T)^{1/(d+1)})\), where $d$ is system’s dimensionality. For $d = 1$ this formula reproduces the observable dependence $\sigma_{\text{DC}} \propto \exp(-(T_0/T)^{1/2})$. However this approach is not correct, since the 1d VRH \[15\] yields the Arrhenius law, $\sigma_{\text{DC}} \propto \exp(-T_0/T)$ with $T_0$ set by the highest barrier that occurs in the chain. The 1d VRH was modified for a quasi-1d system \[14,15\] to include weak hops between the nearest-neighbor, thereby avoiding the highest barriers. This approximation results in a quasi-1d VRH law, $\sigma_{\text{DC}} \propto \exp(-(T_0/T)^{1/2})$. The usual theory of hopping transport predicts, however, only a very weak
power temperature dependence for the frequency-dependent conductivity and the dielectric constant in two- and three-dimensional systems [19].

In the present work we exploit the specific structure of the polymer network to understand these peculiar features of conducting polymers. In stretched polyacetylene this network is formed by coupled polymer chains oriented along some direction. Electron micrographs show that in these substances polymeric chains are organized into fibrils [8], which may be distinctly seen to be subdivided into smaller ones [20]. In a non-fibrillar form of conducting polymers, like polyaniline, X-rays data reveal the existence of highly ordered “crystalline regions” with metallic properties [10–12]. Therefore the whole network of stretched polyaniline may be thought of as constructed from long one-dimensional polymer chains randomly coupled by metallic islands of various sizes. The volume fraction of metallic islands can be small.

We assume here that polymer structure represents a nearly one-dimensional fractal. That means a specific kind of polymer chain organization, defined in the following way: Choose a three-dimensional cube with the edge $L$. Chains, which are coupled within this cube, form a set of bundles disconnected from each other. If for large enough $L$ the cross-section of the maximum bundle is proportional to $L^\epsilon$, where $0 \leq \epsilon \leq 2$, then we shall call the system $d^* = 1 + \epsilon$-dimensional. Obviously $\epsilon = 0$ for purely one-dimensional systems (sets of uncoupled chains). Note, that if one assume chains to be connected either with a low concentration of uncorrelated interchain links, or with weak links (their resistivities being high compared to intrachain ones in our example), then we are dealing with a quasi–one-dimensional system [14], which is three-dimensional according to our definition.

The problem of electron localization in similar fractals was studied in Refs. [21,22]. It was found that even in the presence of a weak disorder all the electronic states remain localized as long as $\epsilon \leq 1$. Therefore, a mechanism of charge transport in the fractal with $\epsilon \ll 1$ is supposed to be variable range hopping (VRH). This assumption is in agreement with the experimental observations for poorly conducting highly doped polymers for which there is a finite density of states at the Fermi level [23].

The usual method to treat VRH models is the effective medium approximation [19], which gives wrong results in the nearly–1d case. For example, for the percolation model this method gives the threshold concentration of broken bonds $c_1 \approx \epsilon$, while $c_t \approx \exp(-1/\epsilon)$, as we shall see later. The results for critical exponents are also wrong in this approximation. To treat VRH in a nearly one-dimensional fractal we choose the following approach. We will first study the percolation problem in a nearly-1d fractal exactly. The VRH model is reduced to the percolation problem by constructing the effective percolation lattice [19,21,23].

In this way we have found that at low temperatures the VRH conductivity obeys a quasi-1d Mott law: $\sigma_{DC} \propto \exp[-(T_1/T_1)^{1/2}]$ but the characteristic temperature $T_1$ is greater than $T_0$ for 1d chain by a factor $1/\epsilon$. Similar temperature dependence is obtained for AC conductivity. These results can explain the observed temperature dependence of conductivity and dielectric constant in poorly conducting polymers. Additionally it was shown that there is the strong frequency dependence of conductivity in the region of extremely low frequencies. These peculiarities reflect the fact that in the random fractal with dimensionality close to one the low frequency conductivity is entirely controlled by the weak charge transfer between clusters. Each cluster is very dense and remains well isolated.

There exist several different problems related to the percolation. First, one can be interested in statistical properties of percolating media: distribution of connected clusters, probability of two or more points to be connected, etc. As it was shown by Fortuin and Kastelein [26], this problem can be reduced to the $q$-component Potts model in the limit $q \to 1$. Thus, the powerful set of field theory methods may be applied. This analogy, however, does not allow us to treat the conductivity of a percolating cluster. The evaluation of the conductivity exponent $\mu$, which describes the DC conductivity $\sigma$ behavior near percolation threshold:

$$\sigma \propto \left(\frac{c_t - c}{c_t}\right)^\mu,$$  \hspace{1cm} (1)

is much more complicated task than the “field-theoretic” ones, such as the exponents of correlation length and of infinite cluster capacity, etc.

Thus our first aim is to study the critical behavior of conductivity near the percolation threshold in a $d$–dimensional lattice, where $d$ is close to lower critical dimensionality, i.e. $d = 1 + \epsilon$, $\epsilon \ll 1$. The real space renormalization group of Migdal and Kadanoff (RGMK) [27,28], being exact at $d = 1$, may be expected to be the appropriate tool as $d$ tends to unity. This method was applied to the percolation problem several years ago by Kirkpatrick [4]. He had found critical exponents of correlation length and conductivity by using the RG equations for conductivity distribution truncated up to the first moment. Though he had not considered explicitly a case of nearly-1d system, this method, if properly applied, gives the right dependence of conductivity exponent on $\epsilon$ except for a pre-exponential factor.

We extend the RGMK method to consider the conductivities and resistivity distribution functions in random media. This enable us to derive the equation for the conductivity exponent $\mu$, which gives realistic values not only in the nearly-1d case. Moreover, explicit expressions for the distribution functions at the percolation threshold will
be obtained. To the best of our knowledge only numerical estimates of random conductivity momenta were available until this work (see, e.g. [31]).

Thorough investigation of the percolation problem and its various modifications is of interest both for its conceptual significance and due to its numerous applications [1,2,3,35]. Beside its application to random conducting media on which we shall concentrate here, the percolation approach was used, e.g., to treat the rigidity transition in random networks [44], and mechanical breakdown in solids [55]. Another application is the magnetic flux flow in type II superconductors [36]. In the case of magnetic vortices pinned by disorder, their motion just above the depinning threshold happens along a sparse (possibly, fractal) network of persistent channels. Directed percolation model is often applied now to describe a wide class of phenomena, in particular, self-organized criticality [37,38].

The paper is organized as follows: In Chapter II the notion of fractional dimensionality is introduced for the oriented chain arrays and illustrated by hierarchical structures. In Chapter III RGMK transformations for conductivity of disordered media and for connectivity of percolation system are derived, the latter one is studied in Chapter IV. The RG equation for the distribution function of conductivity at the percolation threshold is solved in Chapter V, and the explicit form of distribution function is found in Chapter VI. Using scaling relations, results for the AC conductivity of the percolating lattice near threshold are obtained in Chapter VII and they are applied to describe the temperature and frequency dependence of conductivity for the variable range hopping transport in Chapter VIII. Results are discussed in Chapter IX. Three Appendixes contain technical details.

II. HIERARCHICAL COUPLING OF ORIENTED CHAINS

The \((m,n)\) hierarchical structure (HS) is constructed through the infinite repetition of two successive steps (see Fig. 1): a) construction of \(n\)-chains, and b) construction of \(m\)-bundles. After every step the resulting construction may be treated as a new bond \((l\text{-th level bundles})\).

We use the following definition of dimensionality for a chain fractal considering an array of one-dimensional chains, connected in some regular or random fashion by transverse bonds of various lengths. In an \(L\)-size cube chains form a set of bundles, connected inside this cube. Within each bundle in the cube the chains are interconnected. There are no connection between the bundles within the \(L\)-size cube. If the number of chains in the maximum sized bundle scales as \(L^\epsilon\) for large enough \(L\), where \(0 \leq \epsilon \leq 2\), then we have \(D = 1 + \epsilon\) -dimensional network. Obviously \(\epsilon = 0\) for a purely one-dimensional systems (sets of disconnected chains). The characteristic feature of the fractals, constructed from oriented 1d chains is their self-similarity: the system at any scale looks like subdivided into bundles, which in turn are subdivided into smaller ones, etc.

In particular, the dimensionality \(D = 1 + \ln m / \ln n\) may be ascribed to the \((m,n)\) hierarchical structure, if in spatial dimension \(d \geq D\) we replace every site with \(2m^l\) bonds attached to it (\(l\) is the level of bundles attached at each side of this site) with \(m^l\) sites connected by transverse bonds of infinite strength (see example in Fig. 1).

Our hypothesis here is that oriented polymer network structures are of this type (with \(D = 1 + \epsilon\) close to 1, \(\epsilon \ll 1\)), at least in some wide enough interval of length scales, e.g. from the scale of fibrils (hundreds of nm) down to molecular scales. Transmission electron micrographs of fibrillar polyacetylene (see e.g. [20]) appear to support this hypothesis. Of course, the real structures are not regular ones, and the requirement of self-similarity here is to be treated in statistical sense. Nevertheless, we shall use the RGMK scheme, based on regular fractals (HS) for their analysis. In the case of conducting polymers such as nonfibrillar doped polyaniline and doped polypyrrole one may assume the polymer networks apparent fractality in some scale range to be caused by a dilute distribution of crystalline regions providing interchain links (fractality, generated by randomness [39]). Having in mind that the RGMK is exact in one dimension, one may hope to obtain meaningful results when the dimensionality is close to 1.

III. MIGDAL AND KADANOFF EQUATIONS

The renormalization group of Migdal and Kadanoff may be formulated in a quite simple phenomenological fashion. Suppose we have some random \(D\)-dimensional medium with fluctuating local conductivity/resistivity. Let us consider a \(\lambda\)-size cube within the medium. Its conductance is \(\sigma(\lambda)\lambda^{D-2}\), where the random conductivity \(\sigma(\lambda)\) is \(\lambda\)-dependent for strongly inhomogeneous systems. The distribution function for conductivity and for resistivity \(\rho(\lambda) = 1/\sigma(\lambda)\) are defined in the Laplace representation as:

\[
P(\eta, \lambda) = \langle \exp (-\eta \sigma(\lambda)) \rangle; \quad Q(s, \lambda) = \langle \exp (-s \rho(\lambda)) \rangle.
\] (2)
FIG. 1. Construction of hierarchical structure.

FIG. 2. Hierarchical structure of Fig. 1 depicted for a two-dimensional case.
If we change the size of the cube, $\lambda \to \lambda' = n\lambda$, we arrive at some new random variables $\sigma(\lambda')$, $\rho(\lambda')$ with distribution functions $P(\eta, \lambda')$, $Q(s, \lambda')$, respectively. The cube’s enhancement may be treated as $n$-times expansion in one (“longitudinal”) spatial direction, and in $D - 1$ other (“transverse”) ones. If one intends to treat these transformations as infinitesimal ones afterwards, the order of operations is not important.

The RGMK scheme is based upon two approximations: i) enhancing the size $n$ times in the longitudinal direction is treated as connection of $n$ resistors in series, and ii) in a similar way, the transverse cube’s enhancement is replaced by the parallel connection of $m = n^{D-1}$ elements. Thus we have:

$$\tilde{\rho}^{(n)}(\lambda) = \frac{1}{n} \sum_{i=1}^{n} \rho_i(\lambda), \quad \sigma(n\lambda) = \frac{1}{m} \sum_{i=1}^{m} \tilde{\sigma}_i^{(n)}(\lambda). \quad (3)$$

Here the tilde values refer to the rectangular element with the dimensions $n\lambda$ in the longitudinal direction and $\lambda$ in other ones. In both steps the resistivities $\rho_t$ and conductivities $\tilde{\sigma}_i^{(n)} = 1/\tilde{\rho}_i^{(n)}$ of the constituent component are supposed to be independent random variables, and, therefore, we have the distribution functions to be transformed in two steps simply as:

$$\hat{Q}(s, \lambda) = \left\langle \exp\left(-s\tilde{\rho}^{(n)}(\lambda)\right) \right\rangle = Q^n(s/n, \lambda), \quad P(\eta, n\lambda) = \hat{P}^m(\eta/m, \lambda), \quad m = n^{D-1}. \quad (4)$$

This transformation is exact for the $(m, n)$ hierarchical structure.

Equation (4) should be supplemented with the relation between conductivities and resistivities distribution functions (DF) in the Laplace representation. It may be easily derived from the definitions (2), using the following integral identity:

$$e^{-x/\alpha} = 1 - \sqrt{x} \int_{0}^{\infty} \frac{dy}{\sqrt{y}} J_1(2\sqrt{xy}) e^{-\alpha y},$$

where $J_1$ is the Bessel’s function. As a result, the relation between $Q(s, \lambda)$ and $P(\eta, \lambda)$ takes the form of Hankel’s transformation:

$$Q(s, \lambda) = 1 - \sqrt{s} \int_{0}^{\infty} \frac{d\eta}{\sqrt{\eta}} J_1(2\sqrt{\eta s}) P(\eta, \lambda), \quad (5)$$

the reverse relation is of the same form. This transformation has the following properties to be used later:

$$Q(0, \lambda) = 1 - P(+\infty, \lambda), \quad Q(+\infty, \lambda) = 1 - P(0, \lambda), \quad (6)$$

$$s \frac{\partial Q(s, \lambda)}{\partial s} = \sqrt{s} \int_{0}^{\infty} \frac{d\eta}{\sqrt{\eta}} J_1(2\sqrt{\eta s}) \frac{\partial P(\eta, \lambda)}{\partial \eta}, \quad (7)$$

$$s \frac{\partial^2 Q(s, \lambda)}{\partial s^2} = \sqrt{s} \int_{0}^{\infty} \frac{d\eta}{\sqrt{\eta}} J_1(2\sqrt{\eta s}) \eta P(\eta, \lambda). \quad (8)$$

Thus we have a closed set of equations for arbitrary rescaling factors $n$ and $m$. It appears to be more convenient to deal with an infinitesimal transformation by setting $n = 1 + \delta \lambda/\lambda$, $m = 1 + \epsilon \delta \lambda/\lambda$. From Eq.(4) we have the variation of distribution functions consisting of longitudinal and transverse parts: $\delta P = \delta_l P + \delta_t P$, $\delta Q = \delta_l Q + \delta_t Q$.

$$\delta_l Q(s, \lambda) = \left[-s \frac{\partial Q(s, \lambda)}{\partial s} + Q(s, \lambda) \ln Q(s, \lambda)\right] \frac{\delta \lambda}{\lambda};$$

$$\delta_l P(\eta, \lambda) = \epsilon \left[-\eta \frac{\partial P(\eta, \lambda)}{\partial \eta} + P(\eta, \lambda) \ln P(\eta, \lambda)\right] \frac{\delta \lambda}{\lambda}. \quad (9)$$

Using the relations (8), one may rewrite the first of these equations as:

$$\delta_l P(\eta, \lambda) = \left[\eta \frac{\partial P(\eta, \lambda)}{\partial \eta} - \int_{0}^{\infty} \frac{ds}{\sqrt{s}} J_1(2\sqrt{\eta s}) Q(s, \lambda) \ln Q(s, \lambda)\right] \frac{\delta \lambda}{\lambda} \quad (10)$$

Then, from $\delta P = \frac{\partial P}{\partial \lambda} \delta \lambda = \delta_l P + \delta_t P$, we have the following equation:
\[
\lambda \frac{\partial P(\eta, \lambda)}{\partial \lambda} = B\{P\}, \eta = (1 - \epsilon) \eta \frac{\partial P(\eta, \lambda)}{\partial \eta} + \epsilon P(\eta, \lambda) \ln P(\eta, \lambda) - \int_0^\infty \frac{ds}{\sqrt{s}} J_1(2\sqrt{ss}) Q(s, \lambda) \ln Q(s, \lambda).
\] (11)

This equation, combined with the Eq. (5), determines the evolution of the distribution function upon size scaling in a closed form.

This scheme also allows us to treat the percolation system, introducing the probability that the \(\lambda\)-sized cube is disconnected (i.e., has zero conductivity or infinite resistivity) \(c(\lambda)\). Taking into account the definitions of distribution functions (2), \(c(\lambda)\) may be written as:

\[
c(\lambda) = P(+0, \lambda) = 1 - Q(+\infty, \lambda).
\] (12)

Putting in Eq. (11) \(\eta = +\infty\) and using the formula (6), we have:

\[
\frac{dc}{d\lambda} = \epsilon c \ln c - (1 - c) \ln (1 - c).
\] (13)

The right hand side of this equation has three fixed points: two stable ones, \(c = 0\) and \(c = 1\), corresponding to connected and disconnected systems, respectively, and the unstable fixed point, \(c = c_t\), \(0 < c_t < 1\),

\[
\epsilon c_t \ln c_t = (1 - c_t) \ln (1 - c_t),
\] (14)

corresponding to the percolation threshold.

Now let us consider statistical properties of clusters for the percolation problem, i.e., the distribution of clusters over sizes and site numbers, existence and capacity of the infinite cluster, etc. We suppose every bond of a HS, to be either broken, with probability \(c\), or not, — with probability \(1 - c\). The statistics of percolating network is closely related to the thermodynamics of the \(q\)-states Potts model [26]. Namely, if we consider the partition function of the latter:

\[
Z_q = \exp(-\mathcal{H}_q^{(0)}),
\] (15)

\[
\mathcal{H}_q^{(0)} = K \sum_{\langle ij \rangle} (1 - \delta_{\eta_i, \eta_j}),
\] (16)

where variables \(\eta_i = 0, 1, \ldots, q - 1\), \(\delta\) is the Kronecker \(\delta\)-symbol. \(Z_q\) may be expressed in terms of the percolation model as:

\[
Z_q = \langle q^\Gamma \rangle.
\] (17)

Here \(\Gamma\) is the total number of connected clusters in the percolation model, and the average is over realizations with the broken bonds concentration \(c = \exp(-K)\).

To establish a further relationship, it is necessary to introduce external fields into the Hamiltonian of the Potts model:

\[
\mathcal{H}_q^{(1)} = \mathcal{H}_q^{(0)} + h_1 \sum_i (1 - \delta_{\eta_i, 0}),
\] (18)

which may be thought of as additional bonds with strength \(h_1\) between every site of a given lattice and some fictitious external one.

The statistical properties of the percolation model may be obtained from its “free energy”

\[
f(K, h_1) = -\frac{1}{N} \frac{\partial}{\partial q} \ln Z_q(K, h_1) \bigg|_{q=1} = -\frac{1}{N} \langle \Gamma \rangle_{h_1},
\] (19)

where \(N\) is the total number of sites. For an example, the order parameter

\[
P_1 \equiv 1 - \frac{\partial f}{\partial h_1} \bigg|_{h_1=0},
\] (20)
characterizes the “capacity” of the infinite cluster, i.e., the probability of a given site to belong to the infinite cluster. The second derivative is the “susceptibility”:

\[ \chi = - \frac{\partial^2 f}{\partial h^2} \bigg|_{h = 0}, \]

(21)

which gives the average number of sites in finite clusters [6].

To obtain the RG equation for \( f \) of a HS, one should sum over intermediate sites in bundles, thus, performing a transition from the initial Hamiltonian, containing variables of all sites, to the one, containing variables of sites of the next level (see Fig. 2). This transformation reproduces the structure of the initial Hamiltonian (18) with an additional term, corresponding to an extra external field \( h_2 \):

\[ H_q = H_q^{(1)} - h_2 \sum_{(ij)} \left[ \delta_{\eta_i,0} \delta_{\eta_j,0} - \delta_{\eta_i,\eta_j} - \frac{1}{2} (\delta_{\eta_i,0} + \delta_{\eta_j,0}) + 1 \right], \]

(22)

The last two terms in the sum are introduced for further convenience.

For the \((n,m)\) HS, due to its self-similarity, after the summation over variables at intermediate sites we have the following equality:

\[ Z_q(K, h, N) = \exp \left[ \frac{N}{mn} f_q^{(0)}(K, h) \right] Z_q \left( K', h', \frac{N}{mn} \right), \]

(23)

where \( h \equiv (h_1, h_2) \). Expressions for the parameters of the new Hamiltonian \( K' \) and \( h' \), and the function \( f_q^{(0)} \) may be found using the transfer matrix formalism. Introducing the transfer matrix for a bond:

\[ T^{\eta_1_{\eta_2}} = \exp \left[ -H(\eta_1, \eta_2) \right], \]

\[ H(\eta_1, \eta_2) = K (1 - \delta_{\eta_1,\eta_2}) + \frac{h_1}{2} (2 - \delta_{\eta_1,0} - \delta_{\eta_2,0}) - h_2 \left[ \delta_{\eta_1,0} \delta_{\eta_2,0} - \delta_{\eta_1,\eta_2} - \frac{1}{2} (\delta_{\eta_1,0} + \delta_{\eta_2,0}) + 1 \right], \]

(24)

it is easy to calculate the transfer matrix for the \((n,m)\) bundle:

\[ T'_{\eta_1_{\eta_2}} = \left( T^n \right)^m_{\eta_1_{\eta_2}} = \exp \left[ f_q^{(0)}(K, h) - H'(\eta_1, \eta_2) \right], \]

(25)

with \( H' \) having the same structure as \( H \), but with new parameters \( K' \) and \( h' \). Then we come to the following equation for the free energy of the Potts model:

\[ f_q(K, h) \equiv - \frac{1}{N} \ln Z_q = \frac{1}{mn} f_q(K', h') - \frac{1}{mn} f_q^{(0)}(K, h), \]

and for the percolation model:

\[ f(c, h) = \frac{1}{mn} f(c', h') - \frac{1}{mn} u(c, h), \]

(26)

where the variable \( c = \exp(-K) \) is introduced instead of \( K \), and \( u(c, h) = \partial f_q^{(0)}/\partial q \big|_{q=1} \).

Finally, after transition to infinitesimal transformations:

\[ n = 1 + \frac{d\lambda}{\lambda}, \quad m = 1 + \epsilon \frac{d\lambda}{\lambda}, \]

\[ c' = c + v_c \frac{d\lambda}{\lambda}, \quad h_{1,2} = h_{1,2} + v_{1,2} \frac{d\lambda}{\lambda}, \quad u = w \frac{d\lambda}{\lambda}, \]

(27)

we arrive at the following equation:

\[ v_c \frac{\partial f}{\partial c} + v_1 \frac{\partial f}{\partial h_1} + v_2 \frac{\partial f}{\partial h_2} - (1 + \epsilon) f = w, \]

(28)
where \(v_1, v_2\) and \(w\) are found in Appendix A. Eqs. (A4-A7). Setting \(h_{1,2} = 0\) in Eq. (28), and taking into account Eq. (A7), we have:

\[
\lambda \frac{df}{d\lambda} - (1 + \epsilon)f = w_0 = c + (1 - c) \ln(1 - c),
\]

where the independent variable \(\lambda\) is related to \(c\) by Eq. (13).

Equation (28) should be supplemented by the boundary conditions. Directly from the definition (19) we have: \(f = -1\) at \(c = 0\) and \(f = 0\) at \(c = 1\). Both Equations (28,29) have essentially two different solutions depending on \(c > c_t\) or \(c < c_t\), where \(c_t\) is the threshold concentration of broken bonds determined from Eq. (14).

### IV. PERCOLATION EXPONENTS

In the present section the critical exponents of connectivity will be obtained. Critical exponent \(\nu\) of the correlation length \(\xi:\)

\[
\xi \propto |c - c_t|^{-\nu},
\]

is determined by linearization of Eq. (13) near \(c = c_t:\)

\[
\nu^{-1} = \epsilon (1 + \ln c) + 1 + \ln(1 - c).
\]

Eqs. (13,29) can be solved analytically for \(\epsilon \ll 1\). In this case we have from Eq. (14):

\[
c_t = e^{-1/\epsilon}.
\]

After substituting Eq. (22) into Eq. (31), the critical exponent \(\nu\) reads:

\[
\nu \approx \frac{1}{\epsilon}.
\]

The procedure for the solution of Eqs. (13,29) is described in Appendix B. The “partition function” \(f(c)\) is found to be:

\[
f(c) = \begin{cases} 
-c, & c \gg c_t; \\
\frac{c^2}{2\pi} \left(2 \ln \frac{c_t}{c} \right)^{\ddagger + 1} \left(-\frac{1}{\epsilon} - 1, 2 \ln \frac{c_t}{c} \right) - \epsilon^{\ddagger + 1} \left(\ln \frac{c_t}{c} \right)^{\ddagger + 1}, & 1 \gg c > c_t; \\
\frac{c^2}{2\pi} \left(2 \ln \frac{c_t}{c} \right)^{\ddagger + 1} \Gamma \left(-\frac{1}{\epsilon} - 1, 2 \ln \frac{c_t}{c} \right), & c < c_t;
\end{cases}
\]

where \(\Gamma(a, x)\) is the incomplete \(\Gamma\)-function \([40], \gamma(a, x) = \Gamma(a) - \Gamma(a, x)\). It is curious that the “singular” part of the “free energy” to the left of the percolation threshold, \(c < c_t:\)

\[
f_s^{-1}(c) = -\frac{\sqrt{\pi}}{2} c_t \left(2\epsilon \right)^{\ddagger + \frac{1}{2}} \left(\ln \frac{c_t}{c} \right)^{\ddagger + 1}
\]

is strongly oscillating function of \(\epsilon\) at \(\epsilon \ll 1\). The critical exponent for the “specific heat” \(\alpha, f_s \propto |c - c_t|^{2 - \alpha}\) appears to be large negative (as usual in the percolation model, see, e.g. \([4]\)).

Differentiating Eq. (33) with respect to \(h_{1,2}\) and setting \(h_{1,2} = 0\) (see also Eqs. (A4-A7)) we get equations for the “order parameters” \(P_1\) (Eq. (22)) and \(P_2 \equiv -\partial f/\partial h_2|_{h=0}:\)

\[
\begin{align*}
\lambda \frac{dP_1}{d\lambda} &= \frac{1}{c} \left[2 - c + \frac{2}{c} (1 - c) \ln(1 - c) \right] cP_1 - P_2, \\
\lambda \frac{dP_2}{d\lambda} &= [c + \ln(1 - c)] P_1 + \left[1 - c + \frac{1}{c} (2 - c) \ln(1 - c) \right] P_2.
\end{align*}
\]

Boundary conditions: \(m_1|_{c=0} = 1, m_2|_{c=0} = 0, m_1|_{c=1} = m_2|_{c=1} = 0\) may be obtained directly from the definition of order parameters. At \(c > c_t\) we have the trivial solution of Eqs. (33), \(P_1 = P_2 = 0\). The lowest eigenvalue of the matrix in the right hand side of Eq. (33) at \(c = c_t\), which also is \(\beta/\nu\), may be easily found numerically for any dimensionality. The results are presented in Table I together with ones obtained by other methods.
For the nearly-1d system one can find the explicit form of order parameter dependence $P_1(c)$ and $P_1(\lambda)$. Because the region of our interest is $c < c_t \ll 1$, Eqs. (36) can be rewritten for small $c$ as:

$$\lambda \frac{dP_1}{d\lambda} = \frac{c^2}{3} P_1 - \frac{c}{3} P_2,$$

$$\lambda \frac{dP_2}{d\lambda} = -\frac{c^2}{2} P_1 + P_2.$$  \hspace{1cm} (37)

One can see from Eq. (38), that $P_2 = O(c^2 P_1)$, therefore one can neglect the second term in Eq. (37). As a result Eq. (37) directly is solved by the substitution $c = c_t \exp(-\lambda^t)$ (see Eq. (B1)) to read:

$$P_1 = \exp \left[ -\frac{c_t^2}{3\kappa} \int_{\kappa}^\infty \frac{dx}{x} e^{-2x} \right] = \exp \left[ \frac{c_t^2}{3\kappa} \exp(-2\lambda^t) \right],$$  \hspace{1cm} (39)

where $\exp$ is the integral exponent function. Taking into account the asymptotics of $\exp(-x) = \ln x - C + \ldots$ at small $x$, and the relation between $c$ and $\lambda$, near the percolation threshold we obtain:

$$P_1 = \phi \left( \frac{c_t - c}{c_t} \right)^\beta, \quad \beta = \frac{c_t^2}{3\kappa} = \frac{1}{3\kappa} \exp \left( -\frac{2}{c} \right).$$  \hspace{1cm} (40)

Thus the critical exponent $\beta$ of the infinite cluster capacity appears to be very small and a strongly nonanalytic function of $c$.

Using the scaling relations (13), the complete set of critical exponents for the connectivity problems may be expressed through $\nu$ and $\beta$ already obtained.

**V. CONDUCTIVITY EXPONENT**

Let us consider in more details the properties of the evolution functional $B$ in Eq. (11). If we assume, e.g., that the conductivity is 0, with probability $c$, and equals some finite value (say, 1), with probability $1 - c$, then its distribution function reads: $c + (1 - c) e^{-\eta}$. Choosing the distribution in Eq. (11) close to this form:

$$P(\eta) = c + (1 - c) e^{-\eta} + \psi(\eta),$$  \hspace{1cm} (41)

where $\psi(\eta)$ is the small correction. Linearizing the evolution operator $B$ with respect to $\psi$, we have:

$$B \{ P(\eta) \} = \epsilon (1 - c) \eta e^{-\eta} + \epsilon \left[ c + (1 - c) e^{-\eta} \right] \ln \left[ c + (1 - c) e^{-\eta} \right] +$$

$$(1 - c) \ln (1 - c) \left( 1 - e^{-\eta} \right) + \frac{d\psi}{d\eta} + (1 - \epsilon) \frac{d\psi}{d\eta} + \left[ 1 + \epsilon + \epsilon \ln \left[ c + (1 - c) e^{-\eta} \right] \right] \psi.$$  \hspace{1cm} (42)

Another approximation for $B$ is possible if we assume:

$$P(\eta) = c + (1 - c) \exp \left[ -\phi(\eta) \right],$$  \hspace{1cm} (43)

with $\phi(0) = 0$, $\phi(\eta) \to \pm \infty$ as $\eta \to \pm \infty$ rapidly enough (faster than $\pm \sqrt{|\eta|}$, as we shall see later).

The important point is also to assume analyticity of $P(\eta)$ and of $Q(s)$ within some stripe along the real axis. Using the relations:

$$J_1(z) = \frac{H_1^{(1)}(z) + H_1^{(2)}(z)}{2}, \quad H_1^{(1)}(ze^{i\pi}) = -H_1^{(2)}(z), \quad H_1^{(1)}(z) = -\frac{2i}{\pi z} \text{ as } z \to 0,$$

where $H_1^{(1,2)}$ are Hankel’s function of first and second kind, respectively, we may replace the integrals with the $J_1$-functions in Eqs. (13) with the ones containing $H_1^{(1)}$, along the contour $C$ shown on Fig. 3. Thus we obtain:

$$Q(s) = (1 - c) \left[ 1 - \sqrt{s} \int_0^\infty \frac{d\eta}{\sqrt{\eta}} J_1 \left( 2\sqrt{s\eta} \right) \exp(-\phi(\eta)) \right]$$

$$= - (1 - c) \frac{\sqrt{s}}{2} \int_C \frac{d\eta}{\sqrt{\eta}} H_1^{(1)} \left( 2\sqrt{s\eta} \right) \exp(-\phi(\eta)), \quad \hspace{1cm} (44)$$

$$= - (1 - c) \frac{\sqrt{s}}{2} \int_C \frac{d\eta}{\sqrt{\eta}} H_1^{(1)} \left( 2\sqrt{s\eta} \right) \exp(-\phi(\eta)), \quad \hspace{1cm} (45)$$

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where the contribution of the pole of \( H_1^{(1)}(2\sqrt{s\eta})/\sqrt{\eta} \) in Eq. (45) just reproduces the first term in the square brackets of Eq. (44). Assuming \(|s|\) to be large enough, one may replace \( H_1^{(1)} \) by its asymptotic expression:

\[
H_1^{(1)}(2\sqrt{s\eta}) \approx \pi^{-1/2} (s\eta)^{-1/4} \exp \left( -\frac{3i\pi}{4} + 2i\sqrt{s\eta} \right),
\]

and treat the integral (45) by the saddle point method. As a result, we get:

\[
Q(s) = (1 - c) \left[ \frac{\phi'(\eta_c)}{\phi'(\eta_c) + 2\eta_c\phi''(\eta_c)} \right]^{1/2} \exp \left( -\phi(\eta_c) + 2\eta_c\phi'(\eta_c) \right),
\]

where the saddle point should be determined from the equation:

\[
i\sqrt{\frac{s}{\eta_c}} - \phi'(\eta_c) = 0, \quad \text{or} \quad \eta_c (\phi'(\eta_c))^2 = -s.
\]

If we define:

\[
\chi(s) = \phi(\eta_c) - 2i\sqrt{s\eta_c} = \phi(\eta_c) - 2\eta_c\phi'(\eta_c),
\]

the following relations can be easily established:

\[
\chi'(s) = \frac{1}{\phi'(\eta_c)}, \quad s (\chi'(s))^2 = -\eta_c, \quad \phi(\eta_c) = \chi(s) - 2s\chi'(s),
\]

\[
\chi'(s) + 2s\chi''(s) = \frac{1}{\phi'(\eta_c) + 2\eta_c\phi''(\eta_c)}.
\]

Obviously, the transformation \( \phi(\eta) \leftrightarrow \chi(s) \) is symmetric, i.e., its reverse has the same functional form. With \( \chi(s) \) we may rewrite Eq. (46) as:

\[
Q(s) = (1 - c) \left[ 1 + 2s\chi''(s) \chi'(s) \right]^{1/2} \exp \left( -\chi(s) \right).
\]

Proceeding along the same line, one also can derive the following equality:

\[
\sqrt{\eta} \int_0^\infty \frac{ds}{\sqrt{s}} J_1(2\sqrt{s\eta}) Q(s) \ln Q(s) = (1 - c) \ln (1 - c) + \\
(1 - c) \left[ \chi(s_c) - \ln (1 - c) - \frac{1}{2} \ln \left( 1 + 2s_c\chi''(s_c) \chi'(s_c) \right) \right] \exp \left( -\chi(s_c) + 2s_c\chi'(s_c) \right) = \\
(1 - c) \left[ \phi(\eta) - 2\eta\phi'(\eta) + \frac{1}{2} \ln \left( 1 + 2\phi''(\eta) \phi'(\eta) \right) \right] \exp \left( -\phi(\eta) \right).
\]
Replacing $\phi(\eta) = -\ln [P(\eta) - c]$ back in Eq. (50), and substituting the result into Eq. (11), we get the evolution equation (11) in the differential form:

$$\lambda \frac{\partial P}{\partial \lambda} = B_1 \{ (P, \eta) = -(1 + \epsilon) \eta P' + \epsilon P \ln P - (1 - c) \ln (1 - c) + (P - c) \ln (P - c)$$

$$- \frac{1}{2} (P - c) \ln \left[ 1 - 2 \eta \frac{P''}{P'} - 2 \eta \frac{P'}{P - c} \right].$$

(51)

The derivation of Eq. (51) suggests the replacement of the evolution operator $B$ with its approximate form $B_1$ at least for large enough $\eta$. But, if the function $P(\eta)$ is analytic, then, taking into account $P(0) = 1$, it may be represented in the form (13) in some neighborhood of the point $\eta = 0$. If we plug Eq. (11) into Eq. (51), and linearize the resulting expression with respect to $\psi$, we arrive exactly at the same evolution operator as in Eq. (12), which was obtained by the linearization of the exact evolution operator $B$. This observation prompts us to enhance the region of validity of the approximate evolution equation (51) for everywhere in the complex plane $\eta$.

At the percolation threshold, $c = c_t$, the solution of the RG evolution equation can be taken in the form:

$$P(\eta, \lambda) = \tilde{P}(\eta \lambda^{-a}).$$

(52)

Here the critical index $a$ is related to the critical exponents of the conductivity $\mu$ and of the correlation length $\nu$ by the relation

$$a = \mu/\nu.$$

(53)

Indeed, according to Eqs. (12, 23), the average conductivity of the $\lambda$-sized cube in the critical regime, $c = c_t$, is

$$< \sigma(\lambda) > = \frac{dP(\eta, \lambda)}{d\eta} \big|_{\eta=0} = \sigma_0 \left( \frac{\lambda_0}{\lambda} \right)^a.$$

(54)

The same conductivity (24) is realized in the infinite disordered fractal with the correlation length $\xi$ equal to $\lambda$. Near the percolation threshold the correlation length $\xi$ is given by Eq. (30) and, therefore, according to Eq. (54) the fractal conductivity obeys the scaling law

$$< \sigma > \sim (c_t - c)^\mu, \quad \mu = a \nu.$$

With the scaling anzats (52), Eq. (11) becomes an ordinary differential one of the second order. It appears to be more convenient to use the function $\phi(x) = -\ln [\tilde{P}(x) - c_t] / (1 - c_t)$ instead of $\tilde{P}(x)$. Denoting $\phi_0 = -\ln [c_t/(1 - c_t)]$, we have:

$$\frac{1}{2} \ln \left[ 1 + 2x \frac{\phi''}{\phi'} \right] = (1 + \epsilon - a) x \phi' - \phi + \epsilon \left[ g(\phi) - g(-\phi_0) \right],$$

(55)

where we introduce:

$$g(\phi) \equiv (e^\phi + 1) \ln (1 + e^\phi).$$

(56)

An equation for $\phi_0$ which follows from Eq. (14) was used to derive Eq. (55). The latter may be reduced to the first order linear equation by introducing:

$$z(\phi) \equiv \exp \left[ -2 (1 + \epsilon - a) x \phi' \right],$$

(57)

and treating $\phi$ as independent variable:

$$\frac{dz}{d\phi} + (1 + \epsilon - a) z = -(1 + \epsilon - a) \exp \left\{ -2 \phi + 2 \epsilon \left[ g(\phi) - g(-\phi_0) \right] \right\},$$

(58)

where $a$ is related to the critical index of conductivity by Eq. (53).

One should require the solution of Eq. (58) $z(\phi) \to 0$ as $\phi \to +\infty$ faster than $\exp \left[ - (1 + \epsilon - a) \phi \right]$ to ensure the applicability of the saddle-point approximation. This selects a solution in the form

$$z(\phi) = (1 + \epsilon - a) e^{-(1+\epsilon-a)\phi} \int_{\phi}^{\infty} dy \exp \left\{ -(1 + \epsilon - a) y + 2 \epsilon \left[ g(y - \phi_0) - g(-\phi_0) \right] \right\}.$$
The normalization condition $\phi(0) = 0$ implies $z(0) = 1$, from which the equation for $a$ follows:

$$(1 + \epsilon - a) \int_0^\infty dy \exp \left\{ - (1 + \epsilon - a) y + 2\epsilon [g(y - \phi_0) - g(-\phi_0)] \right\} = 1.$$  \hspace{1cm} (60)

Comparing the values of $a$, obtained from Eq. (60) and by the numerical investigation of the original evolution equation (11), one can see that both methods give the same results at any dimensionality. This, together with the considerations presented above, prompts us to consider the saddle point solution as an exact one. Of course, the RGMK method itself is an approximate one.

Comparison of the numeric results for the critical exponent of conductivity $\mu$ is presented in Table I. In three dimensions we have from Eq. (60):

$$a \approx 1.891.$$  \hspace{1cm} (61)

For $\epsilon \ll 1$ from Eq. (60) it follows

$$a = \frac{1 + \epsilon}{\epsilon} \exp \left( - \frac{1 + \epsilon}{\epsilon} \right), \quad \mu = \frac{a}{\epsilon}.$$  \hspace{1cm} (62)

VI. DISTRIBUTION FUNCTION AT THE THRESHOLD

The function $\phi(x)$ may be determined as a reverse of the equation:

$$C x = \phi \exp \left[ - \int_0^\phi d\zeta \left( \frac{1 + \epsilon - a}{z(\zeta)} - \frac{1}{\zeta} \right) \right].$$  \hspace{1cm} (63)

The integration constant $C$ in Eq. (63) corresponds to arbitrary choice of the unit of conductivity, or, alternatively, of the length scale at the percolation threshold.

Thus the distribution function (DF) for conductivities which in the initial representation is defined as, $\Pi(\sigma, \lambda) = \langle \delta(\sigma_x - \sigma) \rangle$ at the percolation threshold takes the universal scaling form:

$$\Pi(\sigma, \lambda) = c_t + (1 - c_t)\bar{\Pi}(y),$$  \hspace{1cm} (64)

where $y$ is the conductivity in units of the average conductivity (54)

$$y = \frac{\sigma}{\sigma_{(\lambda)}} = \frac{\sigma}{\sigma_0} \left( \frac{\lambda}{\lambda_0} \right)^a,$$  \hspace{1cm} (65)

and the scaling function $\bar{\Pi}(y)$ may be expressed as the integral:

$$\bar{\Pi}(y) = \int_{-\infty}^{+i\infty} \frac{dx}{2\pi i} \exp \left[ xy - \phi(x) \right] = \frac{1}{y} \int_{0+i\infty}^{0+i\infty} \frac{d\phi}{2\pi i} \exp \left[ -\phi + yx(\phi) \right].$$  \hspace{1cm} (66)

The last equality was obtained through integration by parts.

However, it should be noted that additional unphysical contribution arises when evaluating the integral in Eq. (66). Namely, the function $x(\phi)$ is singular at $\phi = \tilde{\phi}_n = \phi_0 + i\pi(2n + 1)$, where $n$ is an integer number. These singularities result from the procedure of analytic continuation within the RGMK approach. It can be illustrated as follows: Let us assume the initial distribution of conductivities to be: $P_0(\eta) = c + (1 - c)e^{-\eta}$. After putting $m$ identically distributed conductors in parallel, the Laplace transform of DF for their sum, $P_1(\eta) = [c + (1 - c)e^{-\eta}]^m$, has $m$-th order zeroes at $\eta_m = \eta_0 + i\pi(2n + 1)$, where $\eta_0 = \ln[(1 - c)/c]$. These zeroes transform into singularities after analytic continuation to non-integer $m$. Thus the procedure of the transition from integer rescaling transformation (which is exact for a hierarchical structure) to the infinitesimal one is the source of the above singularities in Eq. (66). Since these singularities are artificial ones they should be merely discarded in the integral (66).
At large conductivities $y \gg 1$, shifting integration contour in Eq. (66) to the region $\Re \phi > \phi_0$, one has the following asymptotic expression for the DF:

$$\tilde{\Pi}(y \gg 1) = \frac{D_2}{y_2} \left( \frac{y}{y_2} \right)^{\frac{1}{(1-c_t)\gamma} - 1} \exp \left[ - \left( \frac{y}{y_2} \right)^{\frac{1}{(1-c_t)\gamma}} \right],$$

(67)

where:

$$D_2 = \frac{2\pi(1+\epsilon)(1+\epsilon-a)^{1/2}}{(1-c_t)a} \left[ \frac{1+\epsilon+a}{1+\epsilon-a} \right]^{\frac{2}{1+\epsilon-a}}, \quad y_2 = \frac{1+\epsilon}{1+\epsilon-a} \left( \frac{1+\epsilon-a}{a} \right)^{\frac{1}{1+\epsilon-a}} e^{A_2},$$

$$A_2 = 2(1+\epsilon-a) \int_0^\infty d\zeta \ln(-\zeta) \frac{d}{d\zeta} \ln z(\zeta).$$

(68)

Shifting the integration contour in Eq. (66) to the region $\Re \phi < \phi_0$, we arrive at the following expression for $\tilde{\Pi}(y)$ in the region of small $y$:

$$\tilde{\Pi}(y \ll 1) = \frac{D_1}{y_1} \left( \frac{y}{y_1} \right)^{\frac{1}{2(1-c_t)\gamma} + 1} \exp \left[ - \left( \frac{y}{y_1} \right)^{\frac{1}{2(1-c_t)\gamma}} \right],$$

(69)

with:

$$D_1 = \frac{2\pi(1+\epsilon-a)^{1/2}}{\epsilon-a} e^{-c_t \epsilon/(1-c_t)}, \quad y_1 = \frac{e^{-A_1}}{1+\epsilon-a} \left( \frac{\epsilon-a}{1+\epsilon-a} \right)^{\epsilon-a},$$

$$A_1 = 2(1+\epsilon-a) \int_0^\infty d\zeta \ln \epsilon \frac{d}{d\zeta} \ln z(\zeta).$$

(70)

More detailed results are available in the limit $\epsilon \ll 1$. At $\Re \phi < \phi_0 \simeq 1/\epsilon$ to first order of $c_t$ and of $a$ we get the following expression for $x(\phi)$:

$$\ln x(\phi) = \ln \phi + c_t \frac{e^\phi - 1 - \phi}{\phi} + \frac{a}{(1+\epsilon)^2} \int_0^\phi \frac{d\zeta}{\zeta^2} \left[ e^{(1+\epsilon)\zeta} - 1 - (1+\epsilon)\zeta \right].$$

(71)

Evaluating Taylor’s series of $\phi(x)$ at $x = 0$, the central momenta of conductivity are found to be of the order of $a$:

$$\frac{\langle \sigma - \langle \sigma \rangle \rangle^2}{\langle \sigma \rangle^2} = a + c_t, \quad \frac{\langle \sigma - \langle \sigma \rangle \rangle^3}{\langle \sigma \rangle^3} = -\frac{1}{2} (1+\epsilon)a + c_t, \ldots$$

(72)

On the other hand, using in Eq. (60) the asymptotics of $x(\phi)$ at $\Re \phi < \phi_0$ and $|\phi| \gg 1$ we have for large enough $y$ (see also Appendix [3]):

$$y\tilde{\Pi}(y) = e^{\phi_2} \frac{1+\epsilon}{a} WS(W),$$

(73)

where $\phi_2 = \frac{a}{(1+\epsilon)^2} - c_t$, and the new fluctuating variable was introduced:

$$W = e^{G_1} \frac{a}{1+\epsilon} y^{\frac{1}{1+\epsilon}}, \quad G_1 = 1 - \gamma - \ln(1+\epsilon) - \left( \frac{1+\epsilon}{a} \right) c_t \simeq 1 - \gamma.$$

(74)

$\gamma$ is Euler’s constant, and $S(W)$ is given by:

$$S(W) = \int^{\infty+\Delta}_{-\infty+\Delta} \frac{du}{2\pi i} W^u u.$$ 

(75)

An asymptotic expression for $S(W)$ may be easily obtained by the saddle-point method (Appendix [3]):

$$S(W) \approx \begin{cases} \exp(-W^{-1}W), & \text{as } W \gg 1; \\ \frac{\exp(W^{-1}W)}{\sqrt{2\pi W}}, & \text{as } W \ll 1. \end{cases}$$

(76)
Fig. 4 shows $WS(W)$ as a function of $\ln W$.

\begin{center}
\begin{figure*}
\includegraphics{fig4.png}
\end{figure*}
\end{center}

FIG. 4. Universal distribution of conductivity for the quasi-1D fractal at the percolation threshold (Eq. (73)). There is a sharp decay in the region of large conductivity and a long tail for large resistances (Eq. (74)).

The asymptote of $\bar{\Pi}(y)$ at small $y$ is given by Eqs. (69, 70). At $\epsilon \ll 1$ it may be reduced to:

$$y\bar{\Pi}(y) = \frac{y^{-\frac{1}{2}}}{\sqrt{2\pi \epsilon}} \exp \left( \frac{1}{2} - e^{-1}y^{-1/\epsilon} \right).$$

(77)

The two expressions (73,77) should be supplemented by one for the intermediate region, where the function $x(\phi)$ in the integral (66) can be expanded in $a$ and $c_t$. Here we have in the first order of $a$, $c_t$:

$$x(\phi) = \phi + \eta(\phi), \quad \eta(\phi) = c_t \left( e^\phi - 1 - \phi \right) + \frac{a \phi}{(1 + \epsilon)^2} \int_0^\phi d\zeta \left[ e^{(1+\epsilon)\zeta} - 1 - (1 + \epsilon) \zeta \right],$$

$$y\bar{\Pi}(y) = \delta (y - 1) + y \int_{-i\infty}^{i\infty} \frac{d\phi}{2\pi i} e^{(y-1)\phi} \eta(\phi).$$

Apart from the $\delta$-function term, this yields in the region $0 < y < 1$:

$$\bar{\Pi}(y) = \frac{a}{1 + \epsilon} \frac{1}{\Delta^2},$$

(78)

where $\Delta = 1 - y$.

To establish regions of validity for different expressions of DF, let us consider the region $\Delta \ll 1$. Here Eq.(77) turns into:

$$\bar{\Pi}(y) = \frac{1}{\sqrt{2\pi \epsilon}} \exp \left( \frac{1}{2} + \frac{\Delta}{2\epsilon} - e^{\Delta/\epsilon - 1} \right),$$

(79)

and Eq.(73) may be written, taking into account Eq.(76) at $W \ll 1$, as:

$$\bar{\Pi}(y) = \frac{\sqrt{2\pi}}{\epsilon} \frac{a}{1 + \epsilon} \left[ \ln \left( \frac{1+\epsilon + \frac{1+\epsilon \Delta}{\epsilon}}{\Delta + \frac{a^2}{\epsilon} \ln \frac{1+\epsilon}{\epsilon}} \right) \right]^2.$$  

(80)
Comparing Eq. (78) with Eqs. (80, 77), one can conclude that Eq. (73) is valid if $\Delta < \Delta_1 \sim a/\epsilon \ln(1/\epsilon)$, and Eq. (77) holds for $\Delta > \Delta_2 \sim \epsilon \ln(\epsilon/\epsilon)$. Fluctuations of conductivity appear to be distributed within narrow region of relative width $\Delta_1$, which ensures that not very high order central momenta of the conductivity to be small (see Eq. (72)). However, if expressed in terms of the universally fluctuating variable $W$, the distribution becomes smeared over a wide region with the lower cut-off $W_1 \sim a^n$, $p \approx 1 + 1/\ln(1/\epsilon)$.

The distribution function $S(W)$ arises naturally in a 1d chain of random resistors, if, to require a scaling form for the distribution function of $\lambda$-length chain: $Y(p, \lambda) = \bar{Y}(p\lambda^{-a})$, or $Q(s, \lambda) = Q(s\lambda^a)$ in the Laplace representation. Then from $Q(s, n\lambda) = Q^n(s/n, \lambda)$ it immediately follows: $Q(x) = \exp \left[-C x^{1/(1+a)}\right]$. Evaluating its inverse Laplace’s transform $Y(r)$, and assuming $a \ll 1$, which is true in the 1d case, we have after the proper rescaling of the integration variable:

$$r Y(r) = \frac{1}{a} W S(W), \quad W = ar^{-1/a}, \quad (81)$$

which is essentially the same formula as Eqs. (73, 74).

**VII. SCALING AND AC CONDUCTIVITY**

Exact results for AC-conductivity in disordered systems are available for a very limited class of models, mostly for 1d ones. The common method to study disordered hopping systems is the effective medium approximation (EMA), which gives qualitatively correct results for three-, two-, and even for one-dimensional systems. However, it fails for a nearly 1d system. In a percolation model, for an example, EMA gives threshold concentration value $c_t \propto \epsilon$ and completely wrong values of critical exponents. However, knowing the results for DC conductivity and topological properties of the percolation network, the qualitative behavior of low-frequency conductivity may be restored within the scaling hypothesis $[3, 12, 13]$

Namely, it should be assumed that the only length scale near the threshold is the correlation length $\xi \propto |\tau|^{-\nu}$, $\tau = (c_t - c)/c_t$. The second assumption is about the anomalous diffusion of a tracer placed onto the infinite cluster at the percolation threshold:

$$\langle r^2(t) \rangle_\infty \propto t^\zeta, \quad \zeta < 1. \quad (82)$$

$r(t)$ is the distance from tracer’s position at $t = 0$, $\langle \ldots \rangle_\infty$ means the average over the initial positions within the infinite cluster only. Above the threshold, when $c < c_t$, we have normal diffusion at sufficiently large times, when $\langle r^2(t) \rangle > \xi^2$:

$$\langle r^2(t) \rangle = D_\infty t, \quad (83)$$

with the diffusion constant $D_\infty$ connected with DC conductivity as:

$$\sigma_{DC} = \frac{e^2 n_e kT}{\xi} D_\infty. \quad (84)$$

$e$ and $n_e$ are charge and concentration of electrons, respectively. Near threshold we have $D_\infty \propto \tau^\mu$, $\mu$ is the critical exponent of the DC conductivity. Note, however, that in Eq. (83) the average is over the whole network, including finite clusters.

The relationship between exponents $\mu$ and $\zeta$ may be established in the following way: At some large enough time $t$ let us consider the $\lambda$-sized box, $\lambda = \langle r^2(t) \rangle^{1/2} \propto t^{\zeta/2}$ inside the system at the percolation threshold. Its conductivity is $\sigma(\lambda) \propto \lambda^{-\mu/\nu}$. At the same time, it may be expressed as $\sigma(\lambda) \propto P_\infty(\lambda) \lambda^{2/\nu}$, where $P_\infty(\lambda) \equiv P_1$ is the infinite cluster capacity for the system in which the correlation length equals $\lambda$. Since $P_\infty(\lambda) \propto \lambda^{-\beta/\nu}$, it follows $\sigma(\lambda) \propto \lambda^{\beta+2-2/\zeta}$. Thus we can conclude:

$$\zeta = \frac{2\nu}{2\nu + \mu - \beta}. \quad (85)$$

If we take into account the average over finite clusters, we obtain at the percolation threshold: $\langle r^2(t) \rangle \propto P_\infty(\lambda) \lambda^{2-\beta/\nu} t^{\bar{\zeta}}$, $\bar{\zeta}$ is the anomalous diffusion exponent including the contribution of finite clusters:

$$\bar{\zeta} = \left(1 - \frac{\beta}{2\nu}\right) \zeta = \frac{2\nu - \beta}{2\nu + \mu - \beta}. \quad (86)$$
Below the percolation threshold, at $c > c_t$, we have $\langle r^2(t) \rangle \sim \xi^2 \propto \tau^{-2\nu}$ at $t \to \infty$.

All the above may be summarized as:

$$\langle r^2(t) \rangle = \bar{\xi} G(\tau^{u/\mu}), \quad (87)$$

where:

$$u = 1 - \bar{\zeta} = \frac{\mu}{2\nu + \mu - \beta}, \quad (88)$$

with the scaling function $G(0)$ being some constant, $G(x) \propto x^\mu$ as $x \to +\infty$, and $G(x) \propto |x|^{-s}$,

$$s = 2\nu - \beta, \quad (89)$$

as $x \to -\infty$.

The conductivity may be expressed through $\langle r^2(S) \rangle$, the Laplace transform of $\langle r^2(t) \rangle$, as:

$$\sigma(\omega, \tau) = \frac{e^2 n_e}{kT} S^2 \langle r^2(S) \rangle, \quad (90)$$

where $S = -i\omega$. Using the Tauberian theorem for the Laplace transformation of power laws, from Eqs. $\langle87,90\rangle$ one can obtain:

$$\sigma(\omega, \tau) = \frac{e^2 n_e}{kT} S^u \bar{G} \left( \tau S^{-u/\mu} \right), \quad (91)$$

with scaling function $\bar{G}(x)$ having the same asymptotic properties as $G(x)$.

Thus we have at $c < c_t$ and $\omega \ll \tau^{s+\mu}$:

$$\sigma \propto \tau^\mu,$$

at $c > c_t$ and $\omega \ll \tau^{s+\mu}$:

$$\sigma \propto -i\omega \tau^{-s},$$

and at $\omega \gg \tau^{s+\mu}$ (in particular, at any $\omega$ if $c = c_t$):

$$\sigma \propto (-i\omega)^u.$$

The summary of the frequency dependence of AC conductivity is given by Fig. 5.
FIG. 5. Diagram of frequency-dependent conductivity in the quasi-1D fractal near the percolation threshold, $|\tau| \ll 1$. Here $s = 2\nu - \beta$ and $u = \mu / (2\nu + \mu - \beta)$ and $\nu$, $\beta$ and $\mu$ are the critical indexes of the correlation length, capacity of infinite cluster and conductivity; $\omega_c \sim |\tau|^{s+\mu}$ is the boundary frequency separated the critical region from the conducting and dielectric phases. For the quasi-1D fractal with the transverse dimensionality $\epsilon \ll 1$, $u \ll 1$, $\mu \ll 1$ and $s \gg 1$.

In a nearly 1d case the static conductivity exponent $\mu$ is given by Eq. (61), the frequency dependence exponent $u$ and the exponent $s$ of dielectric constant divergence may be written with Eqs. (33,40) as:

\[ s \approx \frac{2}{\epsilon}, \]  

(92)
Thus the exponent $s$ is very large but the exponent $u$ is very small. Therefore, in a dielectric phase the AC conductivity as a function of frequency demonstrates a step-type behavior. In the conducting state the frequency dependence of conductivity remains very weak.

**VIII. VARIABLE RANGE HOPPING**

In a conductor with localized carriers the charge transport is provided by the variable range hopping (VRH). The model may be formulated as follows. The phonon assisted hopping rate $w_{ij}$ from one localized state $j$ to the other $i$ per unit time, including Fermi occupation probabilities $p_i$, is approximated by the formula (94):

$$w_{ij}p_j (1 - p_i) = \omega_0 \exp (-2f_{ij}) ,$$

$$f_{ij} = \frac{|\epsilon_j - \epsilon_i| + |\epsilon_j - \epsilon_F| + |\epsilon_i - \epsilon_F|}{4kT} + \frac{|r_j - r_i|}{a} ,$$

where $\epsilon_i$ and $r_i$ are energies and position vectors of localized states respectively, and $a$ is their radius. We assume here the hopping motion to be along the chains. Assuming that the localized states near the Fermi level are distributed uniformly in space and energy, the distribution of random variables $f_{ij}$ is:

$$F(f) \equiv \text{Probability}(f_{ij} > f) = \exp \left[ - \left( \frac{f}{f_0} \right)^2 \right] ,$$

where:

$$f_0 = \left( \frac{T_0}{T} \right)^{1/2} , \quad kT_0 = \frac{1}{4N_F a} .$$

$N_F$ is the density of states at the Fermi level.

To study charge motion in a system with continuous distribution of hopping rates is a much more complicated problem than the one for a percolating system, where hopping rates are either 0, or some given finite value $w_0$. However, knowing the results for the conductivity of percolating system, qualitative conclusions can be obtained for the system with continuously distributed hopping rates. Namely, let us introduce some probe hopping rate $w_c$, replacing all hopping rates $w_{ij} < w_c$ with 0, and all $w_{ij} > w_c$ with $w_c$. Obviously the conductivity becomes lower than the initial one, but if to choose $w_c$ from the requirement to get a maximal conductivity, one can hope to obtain a good estimate for the original conductivity.

It is convenient to represent the probe value as $w_c = \exp(-2f_c)$. The corresponding broken bonds concentration $c$ is given by formula (95), i.e., $c = \exp \left[ - (f_c / f_0)^2 \right]$. The value of $f_c$ for the threshold concentration is $f_t = f_0 / \sqrt{\epsilon}$ or

$$f_t = \left( \frac{T_1}{T} \right)^{1/2} , \quad kT_1 = \frac{4kT_0}{\epsilon} = \frac{1}{\epsilon N_F a} .$$

Assuming the probe value $f_c = f_t + \delta$ to be close to the threshold one, $\tau = (\epsilon_t - c) / \epsilon_t \ll 1$, we have the relation

$$\delta = \frac{1}{2} \epsilon f_t \tau .$$

The scaling formula (91) for the conductivity of the percolation system now reads

$$\sigma(\omega, \tau) = \frac{e^2 n_e a^2}{kT} u^2 |\tau|^\mu w_c g \left[ \tau \left( \frac{S}{w_c} \right)^{-n/\mu} \right] ,$$

where $w_c = w_t \exp (-\epsilon f_t \tau)$, $w_t = \omega_0 \exp (-2f_t)$, $a_\parallel = 1/(N_F kT)$ is the hopping length, and the electron density $n_e = N_F kT$. Scaling function $g(x)$ has the following properties:
If the temperature is relatively high, $T_{\tau}$ where $\tau \equiv 1/T_\tau$ appears to be very narrow for the nearly 1d system. Its width may be written as:

$$g(x) \approx \begin{cases} A |x|^{-\mu} & \text{as } |x| \ll 1, \\ D + B_+ |x|^{-s-\mu} + \ldots & \text{as } x \gg 1, \\ B_- |x|^{-s-\mu} + \ldots & \text{as } x < 0, |x| \gg 1, \end{cases}$$

with coefficients $A, B_+, B_-$ and $D$ of the order of unity.

First let us consider DC conductivity. Obviously one should choose $\tau > 0$, and the expression to maximize the conductivity as a function of $\tau$ is $\tau^\mu \exp(-\epsilon_f \tau)$. The optimal value of probe parameter $\tau$ is very small: $\tau_{\text{DC}} = \mu/\epsilon_f$. Taking into account Eq. (99), we have:

$$\sigma_{\text{DC}} \sim \frac{e^2}{N_F(kT)^2} w_t, \quad w_t = \omega_0 \exp \left[-\left(\frac{T_1}{T}\right)^{1/2}\right]. \quad (100)$$

Thus the DC conductivity obeys a quasi-1d Mott’s law, but the characteristic temperature $T_1$ given by Eq. (97) is much greater than $T_0$ for VRH in a strictly 1d chain from Eq. (96).

The so-called “hydrodynamic region” of very low frequencies, where the conductivity’s frequency dependence is determined by expansion:

$$\sigma = \sigma_{\text{DC}} \left(1 - \frac{i\omega}{\omega_0} + \ldots\right), \quad \omega < \omega_h, \quad (101)$$

appears to be very narrow for the nearly 1d system. Its width $\omega_h$ may be estimated from the condition on the argument of the scaling function $g$ in Eq. (100) to be of the order of unity at $\tau = \tau_{\text{DC}}$ and $\omega = \omega_h$:

$$\omega_h \sim \tau_{\text{DC}}^2 w_t \sim \exp \left(-\frac{2\epsilon_f}{e^2}\right) f_t^{-2/\epsilon} w_t. \quad (102)$$

Within this region $|\omega| < \omega_h$ we get the effective value of $\tau_\epsilon$ to be dependent on frequency as $\tau_\epsilon = \tau_{\text{DC}}(1 - S/\omega_0 + \ldots)$. We suppose at further derivations $S = -i\omega$ to be real and positive, having in mind analytic continuation afterwards. At $S \sim \omega_h$, $\tau_\epsilon$ changes its sign, and now the conductivity is determined by the charge motion inside finite-size clusters. At $|\tau_\epsilon| \ll 1$, the size of effective clusters is large, i.e., the clusters contain many 1d chains. This frequency region is called multiple hopping one. Note, that in contrast to two- and three-dimensional systems for which the multiple hopping regime transforms at higher frequencies into the regime of pair hops, here the multiple hopping frequency region borders that of the one-dimensional hopping.

From properties of the scaling function $g$ in Eq. (100) one can conclude, that the conductivity is maximal if one chooses the probe value $\tau_\epsilon$ such that the argument of $g$ is of the order of unity. Taking into account explicit expressions for critical exponents in a nearly 1d case we have:

$$\tau' \exp \left(\frac{1}{2} \epsilon_f^2 f_t \tau'\right) \sim \left(\frac{S}{w_t}\right)^{\epsilon/2}, \quad (103)$$

where $\tau' \equiv -\tau_\epsilon$ and the conductivity may be estimated to be:

$$\sigma \sim \frac{e^2}{N_F(kT)^2} S \tau'^{-2/\epsilon} \sim \frac{e^2}{N_F(kT)^2} w_t \exp (\epsilon_f \tau' ). \quad (104)$$

One can see from Eq. (103), that the character of frequency dependence is determined by the parameter $\epsilon_f \equiv 2(T_2/T)^{1/2}$, where:

$$T_2 = \frac{1}{4} \epsilon_f^4 T_1. \quad (105)$$

If the temperature is relatively high, $T \gg T_2$, from Eq. (103) it follows $\tau' \sim (S/w_t)^{\epsilon/2}$, and the conductivity reads:

$$\sigma \sim \frac{e^2}{N_F(kT)^2} w_t \exp \left[\epsilon_f \left(\frac{S}{w_t}\right)^{\epsilon/2}\right]. \quad (106)$$

Thus the temperature dependence of conductivity at a given frequency is described by the quasi-1d Mott’s law $\exp - (T_1/T)^{1/2}$. The frequency region for application of Eq. (106) is determined by the requirement $\tau' \ll 1$, which may be written as:
At lower temperatures, \( T \ll T_2 \), Eq. (106) is valid as long as \( (1/2)\epsilon^2 f_t \tau' \ll 1 \), or:

\[
\frac{\epsilon}{2} \ln \frac{w_t}{S} \gg 1.
\] (107)

At higher frequencies the solution of Eq. (103) is given by the equation

\[
\tau' \sim \frac{1}{\epsilon f_t} \ln \frac{S}{\omega_1},
\]

where \( \omega_1 = (2/\epsilon^2 f_t)^{2/\epsilon} w_t = (T/T_2)^{2/\epsilon} w_t \), and the conductivity now is:

\[
\sigma \sim \frac{e^2}{N_F (kT)^2} S \left( \frac{\epsilon f_t}{\ln \frac{S}{\omega_1}} \right)^{2/\epsilon}.
\] (109)

This formula remains to be valid until \( \tau' \ll 1 \), i.e. if:

\[
\frac{\epsilon}{2} \ln \frac{S}{w_t} \ll \epsilon f_t.
\] (110)

If the frequency is higher than ones determined by Eqs. (107) or (110), the conductivity behavior becomes a 1d one (see, e.g., Ref. [14]).

IX. CONCLUSIONS.

As one naturally expects, the results for the percolation problem in nearly one dimension approach 1d ones as \( \epsilon \to 0 \). In particular, threshold concentration \( c_t \) (Eq. (32)) and critical exponents \( \beta \) and \( \mu \) (Eqs. (40, 61)) tend to zero. As a result the capacity and conductivity have a jump-like behavior as a function of concentration of broken bonds near a critical value. The reason is that in the limit of one dimension the infinite cluster arises at \( c = 0 \) and occupies immediately the whole system. The critical length exponent \( \nu \approx \epsilon^{-1} \) is, however, large, contrary to the 1d case, when \( \nu = 1 \), but this nearly 1d behavior of the correlation length can be observed in a very narrow range of concentrations, \( |c - c_t| \ll 1 \). Outside this region, when \( c_t < c \ll 1 \), critical length scales in a 1d manner, \( \xi = c^{-1} \).

The other surprising feature is the strongly nonanalytic behavior of both threshold concentration and of critical exponents, which points to a regular \( \epsilon \)-expansion near lower critical dimensionality \( d = 1 \) being rather impossible. Although the RGMK method becomes exact only in the limit \( \epsilon \ll 1 \) the comparison with numerical results (see Table 1) points out that the critical indexes obtained by this method proves realistic even for \( \epsilon = 1, 2 \).

All these features together teach us that the infinite cluster arises almost like a jump. The infinite cluster at the percolation threshold itself is a fractal with a number of dimensionalities \( D \) all of them less than the dimensionality of the original lattice itself. For example, fractal dimensionality:

\[
D_f = D - \frac{\beta}{\nu} \approx 1 + \epsilon - \frac{1}{3} \exp \left( -\frac{2}{\epsilon} \right),
\] (111)

characterizing the mass distribution within infinite cluster, is very close to the fractal dimension of the system itself \( D \), which means that infinite cluster at the threshold is “almost dense”.

Fracton, or spectral dimension \( \tilde{d} \):

\[
\tilde{d} = D_f - \frac{2\nu}{2\nu + \mu - \beta} \approx 1 + \epsilon - \frac{1}{2\epsilon} \exp \left( -1 - \frac{1}{\epsilon} \right),
\] (112)

was introduced to describe the behavior of random walk on the infinite cluster (it may also be used to describe, e.g. density of localized vibrational states, or fractons, etc.). Its closeness to \( D = 1 + \epsilon \) means that the diffusion on the infinite cluster at the threshold is almost normal. Respectively, the conductivity frequency dependence exponent \( \nu \), Eq. (13), is small. On the other hand, dielectric constant in the insulating phase, \( \epsilon' \propto |\tau|^{-s} \) diverges very strongly, nearly as \( \epsilon^2 \propto |\tau|^{-2\nu} \), but, similar to the correlation length \( \xi \), this divergence takes place in a narrow interval of concentrations of the order of \( c_t \).
The RGMK enables us to study not only the average characteristics of the system but also their fluctuations. These fluctuations become essential near the critical point when the correlation length $\xi$ becomes larger or comparable with the system size $\lambda$. In this case because of non-self-averaging a sample demonstrates individual characteristics corresponding to its specific disorder.

In the present work we have found the distribution of possible conductivities of samples in the critical regime. The average conductivity $\langle \sigma(\lambda) \rangle$ decays with the sample size $\lambda$ according to scaling law (54). All the fluctuations are found to obey the same scaling law (23). Thus the distribution of conductivity is the universal function, $\Pi(y)$, in units of the average conductivity, $y = \sigma_f / \langle \sigma(\lambda) \rangle$. In other words the above fractal dimensionalities of the percolating cluster do not vary with the fractal size, $\lambda$, as happens in multi-fractal systems [13]. This robustness comes from the additive laws (4) for classical charge transport.

The function $\Pi(y)$ represents the distribution of possible experimental deviations from the scaling law (54). It is shown, Eq. (23), that the central body of the distribution $\Pi(y)$ is concentrated in the narrow interval around the average value, i.e., $y = 1$, but it does not take the gaussian form. The distant tails of the distribution in the region of large conductivity $y \gg 1$ and for large resistivities $1/y \gg 1$ decay like a stretched exponent, Eqs. (55,60). In the limit $\epsilon \ll 1$ the shape of the distribution function (24) is consistent with the 1d scaling of the percolating cluster.

Returning to the variable range hopping in the chain fractal as a consequence of (i) 1d character of variable range hopping along the chains, and (ii) finite value of broken bonds threshold concentration $c_0$, the DC conductivity obeys a quasi-1d Mott’s law (100). But the characteristic temperature $T_1$ of this dependence is higher than the formal value of the characteristic temperature for 1d chain (remembering that Mott’s law is not valid for 1d systems), $T_0$, by a factor $1/\epsilon$. This increase can be understood through comparison with the quasi-1d model of weakly coupled metallic chains [4]. The variable range hopping conductivity of this model obeys the same law with the characteristic temperature $T^* = \frac{T_0}{2(d-1)}$, where $2(d-1)$ is the number of neighboring chains and $d-1$ is the transverse dimensionality of the quasi-1d system ($d = 3$). Taking $d-1 = \epsilon$ we formally reproduce $T_1$ for the nearly-1d fractal. Experimental temperature dependence of conductivity in the poorly conducting polymers very often follows a quasi-1d Mott’s law with substantially increased characteristic temperature $\bar{\Pi}^{(0)}(q,\epsilon)$, as expected for $\epsilon \ll 1$.

In 2d and 3d isotropic systems with VRH mechanism of charge transport the temperature dependences of the AC conductivity and of the dielectric constant are rather weak, but in nearly 1d systems there exists the region of frequencies and temperatures, where these dependencies are nearly the same as the quasi-1d Mott’s type (see Eq. 106), which continuously transforms into 1d dependence within a wide enough transient region, Eq. 109. Such a type of strong temperature dependence for both DC and AC conductivity and also dielectric constant is experimentally observed in conducting polymers with localized carriers [10,11].

The physical picture behind this dependence is the following. The low-dimensional random system can be separated in weakly coupled clusters within which carriers are confined. With increasing temperature, the size of clusters exponentially increases, as more space accessible for carriers due to thermal activation. As a result, the dielectric constant and the conductivity exhibit strong temperature dependencies. In contrast to the low-dimensional case, the clusters in two- and three-dimensional systems prove to be more effectively coupled. Therefore the large polarization of clusters does not happen because of transition of carriers between clusters. Thus our results support strongly the idea that even poorly conducting polymers represent low dimensional systems.

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APPENDIX A: EVALUATION OF COEFFICIENTS IN THE RG EQUATION FOR “FREE ENERGY”.

Comparing Eqs. (??) and (23), we have:

$$f_q^{(0)}(c, h) = \ln T_{00}, \quad u(c, h) = \frac{\partial}{\partial q} T_{00}^{(0)} \bigg|_{q=0}, \quad e' = e^{-K'}, \quad e'' = e^{-K''} = e^T_0 T_{00} L_0 T_{ll'}^{(0)},$$

$$e^{-K'} = \frac{T_{ll}^{(0)}}{T_{00}^{(0)}}, \quad e^{-K''} = \frac{T_{ll}^{(0)}}{T_{00}^{(0)}} \left(\frac{T_{ll'}^{(0)}}{T_{ll}^{(0)}}\right)^2,$$

where $l, l' = 1, \ldots, q - 1$, and the results are independent of the choice of $l \neq l'$. From Eq. (??) it follows that:
\[
(T^{n+1})_{00} = T_{00} (T^n)_{00} + (q - 1) T_{00} (T^n)_{t0} = (T^n)_{00} + (q - 1) e^{-(h_1 - h_2)/2} (T^n)_{t0},
\]
and, analogously,

\[
(T^{n+1})_{t0} = c e^{-(h_1 - h_2)/2} (T^n)_{00} + e^{-h_1} [1 + (q - 2) e^{h_2}] (T^n)_{t0},
\]

\[
(T^{n+1})_{ll} = c e^{-(h_1 - h_2)/2} (T^n)_{00} + e^{-h_1} (T^n)_{ll} + (q - 2) c e^{-h_1 + h_2} (T^n)_{ll'},
\]

\[
(T^{n+1})_{ll'} = c e^{-(h_1 - h_2)/2} (T^n)_{00} + c e^{-h_1 + h_2} (T^n)_{ll} + e^{-h_1} [1 + (q - 3) e^{h_2}] (T^n)_{ll'}.
\]

Here the symmetry property \(T_{n_2 n_1} = T_{n_1 n_2}\) was used. Introducing:

\[
\tilde{u}(n) = \frac{\partial}{\partial q} \left[ (T^n)_{00} \right]_{q=1}, \quad t_1(n) = \left. (T^n)_{t0} \right|_{q=1},
\]

\[
t_2(n) = \left. (T^n)_{ll} \right|_{q=1}, \quad t_3(n) = \left. (T^n)_{ll'} \right|_{q=1},
\]

and taking into account that at \(q = 1\) we have \(T_{00} = (T^n)_{00} = T'_0 = 1\), the following equations can be established to read:

\[
t_1(n + 1) = c e^{-(h_1 - h_2)/2} + e^{-h_1} (1 - c e^{h_2}) t_1(n),
\]

\[
t_2(n + 1) = c e^{-(h_1 - h_2)/2} t_1(n) + e^{-h_2} t_2(n) - c e^{-h_1 + h_2} t_3(n),
\]

\[
t_3(n + 1) = c e^{-(h_1 - h_2)/2} t_1(n) + c e^{-h_1 + h_2} t_2(n) + e^{-h_1} (1 - 2 c e^{h_2}) t_3(n),
\]

\[
(A2)
\]

\(\hat{u}(n + 1) = \tilde{u}(n) + c e^{-(h_1 - h_2)/2} t_1(n),\)

with initial conditions: \(t_1(0) = t_3(0) = \hat{u}(0) = 0, t_2(0) = 1\), which can be easily solved. After the transition to infinitesimal transformation, \(n = 1 + d\lambda/\lambda, m = 1 + e\lambda/\lambda\), we have, taking into account Eq. \((25)\):

\[
T'_{0l} = t'^{(n)}_{1l}(n) = c e^{-(h_1 - h_2)/2} \left\{ 1 + \frac{d\lambda}{\lambda} \left[ \epsilon \left( \ln c - \frac{h_1 - h_2}{2} \right) + \left( 1 - \frac{e^{h_1}}{e^{h_1} + c c e^{h_2} - 1} \right) \left[ \ln (1 - c e^{h_2}) - h_1 \right] \right] \right\},
\]

\[
T'_{ll} = t'^{(n)}_{2l}(n) = e^{-h_1} \left\{ 1 + \frac{d\lambda}{\lambda} \left[ e^{-h_1} + c c e^{h_2} \left( 1 - \frac{c e^{h_1}}{e^{h_1} + c c e^{h_2} - 1} \right) + \left( 1 - \frac{c^2 e^{2h_1 + h_2}}{(e^{h_1} + c c e^{h_2} - 1)^2} \right) \left[ \ln (1 - c e^{h_2}) - h_1 \right] \right] \right\},
\]

\[
T'_{ll'} = t'^{(n)}_{3l}(n) = c e^{-h_1 + h_2} \left\{ 1 + \frac{d\lambda}{\lambda} \left[ \epsilon \left( \ln c - h_1 + h_2 \right) + 1 - \frac{c e^{h_1}}{e^{h_1} + c c e^{h_2} - 1} + \left( 1 - \frac{c e^{2h_1}}{(e^{h_1} + c c e^{h_2} - 1)^2} \right) \left[ \ln (1 - c e^{h_2}) - h_1 \right] \right] \right\},
\]

\[
\frac{\partial}{\partial q} \ln T'_{00} \right|_{q=1} = \frac{d\lambda}{\lambda} \frac{c^2 e^{2h_2}}{(e^{h_1} + c c e^{h_2} - 1)^2} \left\{ 1 - \left( 1 - \frac{c e^{h_1}}{e^{h_1} + c c e^{h_2} - 1} \right) \left[ \ln (1 - c e^{h_2}) - h_1 \right] \right\}.
\]

From Eqs. \((A3)\), \((27)\) and \((A1)\) it immediately follows that:

\[
v_e = \epsilon c \ln c - c \left( 1 - \frac{c e^{h_1}}{e^{h_1} + c c e^{h_2} - 1} \right) - 
\]

\[
c \left[ 2 \frac{c e^{h_1}}{e^{h_1} + c c e^{h_2} - 1} - 1 - \frac{c e^{2h_1}}{(e^{h_1} + c c e^{h_2} - 1)^2} \right] \left[ \ln (1 - c e^{h_2}) - h_1 \right],
\]

\[(A4)\]
\[ v_1 = \epsilon h_1 - ce^{h_1} \left( 1 - \frac{ce^{h_1}}{e^{h_1} + ce^{h_2} - 1} \right) - \left[ 1 - \frac{c^2e^{2h_1+h_2}}{(e^{h_1} + ce^{h_2} - 1)^2} \right] \left[ \ln \left( 1 - ce^{h_2} \right) - h_1 \right], \]  
\( v_2 = \epsilon h_2 + (2 - ce^{h_2}) \left( 1 - \frac{ce^{h_1}}{e^{h_1} + ce^{h_2} - 1} \right) - \left[ 1 - \frac{2e^{h_1}}{e^{h_1} + ce^{h_2} - 1} \right] \left[ 1 - \frac{ce^{h_1}}{e^{h_1} + ce^{h_2} - 1} \right] \left[ \ln \left( 1 - ce^{h_2} \right) - h_1 \right] , \]  
\( w = \frac{e^{2e^{h_2}}}{(e^{h_1} + ce^{h_2} - 1)^2} \left[ 1 - \left( 1 - \frac{ce^{h_1}}{e^{h_1} + ce^{h_2} - 1} \right) \left[ \ln \left( 1 - ce^{h_2} \right) - h_1 \right] \right]. \)

**APPENDIX B: EVALUATION OF TOPOLOGICAL QUANTITIES AT} \( \epsilon \ll 1. \)**

At \( c \ll 1 \) Eq. (B3) may be rewritten as:

\[ \lambda \frac{dc}{d\lambda} = \epsilon c \ln \frac{c}{c_t} , \]

which can be easily solved to yield:

\[ c = c_t \exp \left[ \pm \left( \frac{\lambda}{\lambda_0} \right)^{\epsilon} \right] , \]

where \( \lambda_0 \) is the arbitrary positive integration constant. On the other hand, one can neglect first term in Eq. (B3) when \( c \gg c_t \), obtaining the solution:

\[ c = 1 - \exp(-\lambda/\lambda_1) , \]

with the other integration constant \( \lambda_1 \).

One can match plus sign solution (B1) with Eq. (B2) in the region \( c_t \ll c \ll 1 \). Setting \( \lambda_0 = \epsilon^{1/\epsilon} \lambda_0 \) in Eq. (B1), one can see, taking into account Eq. (B2), that to fulfill \( 1 \gg c \gg c_t \), one should require \( 0 < 1 - (\lambda/\lambda_0)^\epsilon \ll 1 \); therefore one can set \( (\lambda/\lambda_0)^\epsilon \approx 1 + \epsilon \ln(\lambda/\lambda_0) \), and formula (B1) turns into \( c \approx \lambda/\lambda_0 \). On the other hand, we have from Eq. (B2), that within the same region \( c \approx \lambda/\lambda_1 \), that is, \( \lambda_0 \approx \lambda_1 \), or \( \lambda_0 \approx \epsilon^{1/\epsilon} \lambda_1 \).

When \( c \ll 1 \), taking into account (B2), Eq. (B9) reads (we set the integration constant \( \lambda_0 = 1 \)):

\[ \lambda \frac{df}{d\lambda} - (1 + \epsilon) f = \frac{c_t^2}{2} \exp \left( \pm 2\lambda^\epsilon \right) . \]

At \( c < c_t \) (minus sign in the right hand side of Eq. (B3)), one should set \( f = 0 \) at \( \lambda \to \infty \), and the corresponding solution is:

\[ f(\lambda) = \frac{c_t^2}{2\epsilon} (2\lambda^\epsilon)^{\frac{1}{\epsilon} + 1} \Gamma \left( -\frac{1}{\epsilon} - 1, 2\lambda^\epsilon \right) \]

\[ = \frac{1}{2} \frac{c_t^2}{1 + \epsilon \sin \frac{\pi}{2} \Gamma \left( \frac{1}{\epsilon} \right) - \frac{c_t^2}{2} \sum_{n=0}^{\infty} \frac{(-2\lambda^\epsilon)^n}{n! \epsilon(n-1)}} , \]

where \( \Gamma(a, x) \) is the incomplete Gamma function [40], tending to 0 at \( x \to \infty \). Substituting in the first term in the right hand side of formula (B4) \( \lambda^\epsilon = \ln(c_t/c) \), and replacing \( \Gamma(1/\epsilon) \) with its asymptotic expression, one arrives at Eq. (B3). At \( 1 \gg c > c_t \), the general solution of Eq. (B3) with a plus sign is:
\[ f(\lambda) = -\frac{c_t^2}{2\epsilon} (-2\lambda^t)^{1+1} \gamma \left( -\frac{1}{\epsilon} - 1, -2\lambda^t \right) - A\lambda^{1+\epsilon}, \]

where \(\gamma(a, x) = \Gamma(a) - \Gamma(a, x)\) (note that \(x^{-a}\gamma(a, x)\) is analytic function of \(x\) at \(x = 0\)), and \(A\) is an integration constant to be determined through combining with the expression for the “free energy” at \(c \gg c_t\). The latter may be done, substituting Eq. (B4) with \(\lambda_1 = \epsilon^{1/\epsilon}\) into right hand side of Eq. (34).

Having in mind that solutions (B2) are to be matched with each other in lowest order on \(\epsilon\), and setting \(\epsilon = 0\) in the left-hand side of Eq. (29) too, we arrive at:

\[ f = \exp \left( -\epsilon^{1/\epsilon}\lambda \right) - 1 = -c. \]  

On the other hand, from Eq. (B3) we have within the same region, replacing the function \(\gamma\) with its asymptotics at a large value of its second argument \(1\):

\[ f \approx -\frac{c_t^2}{4\epsilon \lambda^t} \exp (2\lambda^t) - A\lambda^{1+\epsilon} \approx -\frac{c_t^2}{4\epsilon \ln \epsilon c_t} - A\epsilon^{-1/\epsilon - 1}c. \]  

Comparing expressions (B6) and (B7), one can conclude that \(A \approx \epsilon^{1/\epsilon + 1}\). Substituting this into (B3), replacing there \(\lambda^t\) with \(\ln(c/c_t)\), and combining with Eqs. (B4) and (B6), one arrives at Eq. (34).

**APPENDIX C: INVERSE LAPLACE TRANSFORMATION OF THE DISTRIBUTION FUNCTION.**

After some integrations by parts, Eq. (71) may be rewritten as:

\[ \ln x(\phi) = -\pi \left( 1 - \frac{a}{1 + \epsilon} \right) \ln (-\phi) + \frac{a}{1 + \epsilon} \left( 1 - \gamma - \ln (1 + \epsilon) + \frac{1}{(1 + \epsilon) \phi} \right) + \frac{\epsilon \phi - 1 - \phi}{\phi} + \frac{a}{1 + \epsilon} \int_{-\infty}^{\phi} d\zeta e^{(1+\epsilon)\zeta} \]

\[ \approx i\pi \left( 1 - \frac{a}{1 + \epsilon} \right) \ln (-\phi - \phi_1) + \frac{a}{1 + \epsilon} G_1 + c_t \frac{\epsilon \phi}{\phi} + \frac{a}{(1 + \epsilon)^2} e^{(1+\epsilon)\phi}. \]  

(C1)

\(\phi_1 = a/ (1 + \epsilon)^2\), \(G_1 = 1 - \gamma - \ln (1 + \epsilon) - (1 + \epsilon) c_t/a\). From the letter expression in Eq. (C1), which is the asymptotics as \(|\phi| \gg 1\), we have:

\[ x(\phi) = -e^{\pi i \gamma} G_1 (-\phi - \phi_1)^{1+1/\pi} + c_t e^\phi + \frac{a}{(1 + \epsilon)^2} e^{(1+\epsilon)\phi}. \]  

(C2)

In the evaluation of the inverse Laplace transformation (B6) one can neglect two last terms in Eq. (C2) if one considers the DF \(\bar{\Pi}(y)\) at \(y\) close enough to 1. Substituting \(x(\phi)\) into Eq. (B6), rescaling integration variable as:

\[ -\phi - \phi_1 = e^{G_1 y} \frac{1+\epsilon}{a} W u, \]

and taking into account \(a \ll 1\), which leads to the following expression in the exponent in Eq. (B6):

\[ -\phi + y x(\phi) = \phi_1 + \frac{1+\epsilon}{a} W u \left( 1 - u^{-\frac{\gamma}{1+\epsilon}} \right) \approx \phi_1 + W u \ln u, \]

one immediately arrives at the formulas (B3–B5).

To evaluate \(S(W)\), it is convenient to use the integration contour \(\Im (u \ln u) = 0\), or, introducing polar coordinates \(u = r \exp (i\theta)\):

\[ r = \exp (-\theta \cot \theta) . \]  

(C3)

Substituting Eq. (C3) into Eq. (B5), we have:

\[ S(W) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} V(\theta) e^{-\theta V(\theta)}, \quad V(\theta) = \frac{\theta}{\sin \theta} e^{-\theta \cot \theta} . \]  

(C4)
The value of the integral may be estimated by the saddle-point method, looking for the maxima of the expression 
\(\ln V(\theta) - WV(\theta)\). The stationary point equation is:

\[
V'(\theta) \left[ W - \frac{1}{V(\theta)} \right] = 0. \tag{C5}
\]

As \(W \gg 1\), the stationary point is \(\theta_s = 0\), and the asymptote of Eq. (C4) is:

\[
S(W) \approx (2\pi eW)^{-1/2} \exp \left( -W/e \right). \tag{C6}
\]

On the other hand, if \(W \ll 1\), we have two stationary points \(\pm \theta_s\), \(V(\pm \theta_s) = 1/W\), \(\theta_s = \pi - \delta\), and for \(\delta \ll 1\) the following equation may be obtained:

\[
\frac{\pi}{\delta} \exp \left( \frac{\pi}{\delta} - 1 \right) = \frac{1}{W},
\]

the solution for which is:

\[
\frac{\pi}{\delta} \approx \frac{\ln (e/W)}{\ln \ln (e/W)}.
\]

The asymptotics of \(S(W)\) turns out to be:

\[
S(W) \approx \frac{\sqrt{2\pi}}{eW} \left[ \frac{\ln \ln (e/W)}{\ln (e/W)} \right]^2. \tag{C7}
\]
|   | \( c_t \) | \( \nu \) | \( \beta \) | \( \mu \) |
|---|---|---|---|---|
| \( D \) | \( 1 + \epsilon \) | 2 | 3 | 4 | \( D \to \infty \) |
| RGMK | \( e^{-1/\epsilon} \) | 1/2 | 0.840 | 0.945 | 1 |
| Other | — | 1/2 | 0.751(BP) | 0.840(BP) | 1 |
| RGMK | \( 1/\epsilon \) | 1.629 | 1.219 | 1.092 | 1/2 |
| Other | — | 4/3=1.333 | 0.89 | 0.68 | 1/2 |
| RGMK | \( \frac{1}{4\epsilon}e^{-2/\epsilon} \) | 0.137 | 0.255 | 0.325 | \( \frac{1}{\nu} \) |
| Other | — | 5/36=0.139 | 0.40 | 0.65 | 1 |
| RGMK | \( \frac{1+\epsilon}{\epsilon^2} \exp\left(-\frac{1+\epsilon}{\epsilon}\right) \) | 1.333 | 2.330 | 3.230 | \( D - 1 \) |
| Other | — | 1.303 | 2.00 | 2.39 | 3 |

**TABLE I.** Threshold values \( c_t \) for bond (BP) and continuous (CP) percolation and critical exponents evaluated by the RGMK method of this work compared to the exact (if known) or best possible numeric values. Values for comparison were extracted from reviews [6,32], for the conductivity exponent \( \mu \) after Ref. [41].
[1] B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982).

[2] Fractals in Physics, *Essays in Honour of B. B. Mandelbrot*, ed. by J. Feder and A. Aharon (North Holland, Amsterdam, 1990).

[3] S. Kirkpatrick, Phys. Rev. B, 15, 1533 (1977).

[4] Fractals and Disordered Systems, ed. by A. Bunde and S. Havlin (Springer Verlag, Heidelberg, 1991).

[5] S. Havlin and D. Ben-Avraham, Adv. in Phys., 36, 695 (1987).

[6] T. Nakayama, K. Yakubo, and R. L. Orbach, Rev. Mod. Phys., 66, 381 (1994).

[7] A.N. Samukhin, V.N. Prigodin, and L. Jastrabík, Phys. Rev. Lett., 78, 376 (1997).

[8] J. Tsukamoto, Adv. Phys., 41, 509 (1992).

[9] See, for example, *Proceedings of the International Conference on Science and Technology of Synthetic Metals, Salt Lake City, 1996* [Synth. Met. 83-85 (1997)].

[10] Z. H. Wang, H. H. S. Javadi, A. Ray, A. J. MacDiarmid, and A. J. Epstein, Phys. Rev. B, 42, 5411 (1990).

[11] J. Joo, Z. Oblakowski, G. Du, J. P. Pouget, E. J. Oh, J. M. Weisinger, Y. Min, A. G. MacDiarmid, and A. J. Epstein, Phys. Rev. B, 49, 2977 (1994).

[12] R. Z. Kohlman and A. J. Epstein, *Handbook of Conducting Polymers*, ed. by T. R. Skotheim, R. L. Elsenbaumer, and J. R. Reynolds (Marcel Dekker, Inc.), 85 (1997).

[13] A. I. Shklovskii and A. L. Efros, *Electronic Processes in Non-crystalline Materials* (Clarendon Press, Oxford, 1979).

[14] J. Kurkijärvi, Phys. Rev. B, 8, 922 (1973).

[15] V. K. S. Shante, Phys. Rev. B, 16, 2597 (1977).

[16] H. Böttger and V. V. Bryksin, *Hopping Conduction in Solids* (Berlin, Akademie-Verlag, 1985).

[17] K. Araya, T. Micoh, T. Narahara, K. Akagi, and H. Shirakawa, Synth. Metals, 17, 247 (1987).

[18] B. Shapiro, Phys. Rev. Lett., 22, 823 (1982).

[19] A. Cohen, Y. Roth, and B. Shapiro, Phys. Rev. B, 38, 12125 (1988).

[20] J. Joo, V. N. Prigodin, Y. G. Min, A. G. MacDiarmid, and A. J. Epstein, Phys. Rev. B, 50, 12226 (1994).

[21] H. Böttger, V. V. Bryksin, and G. Yu. Yashin, J. Phys. C – Sol. St. Phys., 12, 3951 (1979).

[22] I. P. Zvyagin, Phys. Stat. Sol.(b), 97, 143 (1980).

[23] C. M. Fortuin and P. W. Kasteleijn, Physica, 57, 536 (1972).

[24] A.A. Migdal, Sov. Phys. JETP, 42, 413, 743 (1976).

[25] L.P. Kadanoff and A. Houghton, Phys. Rev. B, 11, 377 (1975).

[26] L.P. Kadanoff, Ann. Phys. (N.Y.), 100, 359 (1976).

[27] C. Tsallis and A. C. N. de Magalhães, Phys. Repts., 268, 305 (1996).

[28] G. G. Batrouni, A. Hansen, and B. Larson, cond-mat/9508011.

[29] M.B. Isichenko, Rev. Mod. Phys., 64, 961 (1992).

[30] D. Stauffer and A. Aharony, *Introduction to Percolation Theory*, (Taylor and Francis, London, 1994).

[31] C.F. Moukarzel, P.M. Duxbury, Phys. Rev. Lett., 75, 4055 (1995); C.F. Moukarzel, P.M. Duxbury, and P.L. Leath, Phys. Rev. E, 55, 5800 (1997).

[32] S. Zapperi, P. Ray, H. E. Stanley, and A. Vespignani, Proc. Mat. Res. Soc., to be published.

[33] J. Watson and D. S. Fisher, cond-mat/9610093, Phys. Rev. B, to be published.

[34] P. Grassberger, J. Stat. Phys., 79, 13 (1995).

[35] S. Maslov and Y-C. Zhang, Physica A, 223, 1 (1996).

[36] J. Hamburger, O. Bihun, and D. Anvir, Phys. Rev. E, 53, 3342 (1996).

[37] H. Bateman and A. Erdélyi, *Higher Transcendental Functions*, (Mc Grow Hill, NY, 1953).

[38] J.-M. Normand and H.J. Herrmann, Int. J. Mod. Phys. C, to be published.

[39] W. Schirmacher, J. Phys. A, 27, L727 (1994).

[40] C. Castellani and L. Peliti, J. Phys. A, 19, L429 (1986).

[41] J. Joo, S.M. Long, J.P. Pouget, E.J. Oh, A.G. MacDiarmid, and A.J. Epstein, Phys. Rev. B, 57, 9567 (1998).