On long time behavior of the focusing energy-critical NLS on $\mathbb{R}^d \times T$ via semivirial-vanishing geometry

Yongming Luo *†

Abstract

We study the focusing energy-critical NLS

$$i\partial_t u + \Delta_{x,y} u = -|u|^\frac{4}{d-1} u$$

on the waveguide manifold $\mathbb{R}^d_x \times T_y$ with $d \geq 2$. We reveal the somewhat counterintuitive phenomenon that despite the energy-criticality of the nonlinear potential, the long time dynamics of (NLS) are purely determined by the semivirial-vanishing geometry which possesses an energy-subcritical characteristic. As a starting point, we consider a minimization problem $m_c$ defined on the semivirial-vanishing manifold with prescribed mass $c$. We prove that for all sufficiently large mass the variational problem $m_c$ has a unique optimizer $u_c$ satisfying $\partial_y u_c = 0$, while for all sufficiently small mass, any optimizer of $m_c$ must have non-trivial $y$-dependence. Afterwards, we prove that $m_c$ characterizes a sharp threshold for the bifurcation of finite time blow-up ($d = 2, 3$) and globally scattering ($d = 3$) solutions of (NLS) in dependence of the sign of the semivirial. To the author’s knowledge, the paper also gives the first large data scattering result for focusing NLS on product spaces in the energy-critical setting.

1 Introduction and main results

We study the focusing energy-critical nonlinear Schrödinger equation (NLS)

$$i\partial_t u + \Delta_{x,y} u = -|u|^\frac{4}{d-1} u$$ (1.1)

on the waveguide manifold $\mathbb{R}^d_x \times T_y$ with $d \geq 2$ and $T = \mathbb{R}/2\pi\mathbb{Z}$. The equation (1.1) serves as a toy model in various physical applications such as nonlinear optics and Bose-Einstein condensation. For a more comprehensive introduction on the physical background of (1.1), we refer to [49, 50, 38] and the references therein. From a mathematical point of view, the mixed type nature of the underlying domain also makes the analysis of (1.1) rather challenging and interesting. In a previous paper [45] the so-called semivirial-vanishing geometry has been introduced by the author to study the intercritical analogue of (1.1). The purpose of this paper is to reveal the interesting phenomenon that despite its energy-subcritical characteristic, the framework of semivirial-vanishing geometry indeed continues to work for the energy-critical model (1.1).

In recent years, there has been an increasing interest in studying dispersive equations on compact manifolds and product spaces. Among all, we underline that the first result for energy-critical NLS on tori might date back to Herr, Tataru and Tzvetkov [32], where the local well-posedness of the quintic NLS on $T^3$ was shown. As an application, the framework of $X^s$- and $Y^s$-spaces developed in [32] was later invoked to obtain local well-posedness results for energy-critical NLS on other different manifolds such as the 4D product space $\mathbb{R}^d \times T^{4-d}$ [33] and Zoll manifolds [31]. By appealing to the concentration compactness arguments initiated by Kenig and Merle [39] and using the global well-posedness results for the defocusing energy-critical NLS on $\mathbb{R}^3$ and $\mathbb{R}^4$ [21, 48] as Black-Box-Theories, Ionescu, Pausader and Staffilani showed that the defocusing energy-critical NLS is always globally well-posed on $T^3$, $\mathbb{R} \times T^4$ and on the three-dimensional hyperbolic space $\mathbb{H}^3$ [34, 35, 36]. Following the same ideas in [34, 35, 36], a corresponding large data well-posedness result for focusing energy-critical NLS on $T^4$ and $\mathbb{R} \times T^3$ has been recently established by Yu, Yue and Zhao [61, 59]. We also refer to [10, 8, 9, 12, 13, 14, 64] for further interesting results in this direction.

*Institut für Wissenschaftliches Rechnen, Technische Universität Dresden, Germany
†Email: yongming.luo@tu-dresden.de
Although the large data well-posedness results are already satisfactory to a certain extent, we are more interested in results concerning scattering or finite time blow-up phenomena which give a more accurate description on the long time dynamics of a solution. In general, however, the scattering results are more difficult to prove, as they demand a global control on the decay of the solutions. Moreover, while a solution of an NLS on \( \mathbb{R}^d \) with nonlinearity of intercritical growth shall be scattering in time in the energy space under certain circumstances (e.g. small initial data), a solution on \( \mathbb{T}^n \) does not scatter principally. The situation thus becomes more interesting when considering an NLS on the product space \( \mathbb{R}^d \times \mathbb{T}^n \).

We naturally ask whether the strong dispersion coming from the \( \mathbb{R}^d \)-side could ultimately guarantee the scattering of a solution of NLS on \( \mathbb{R}^d \times \mathbb{T}^n \). Motivated by the scattering results of NLS on \( \mathbb{R}^d \), we expect a solution to be scattering as long as

(i) The nonlinearity is at most energy-critical w.r.t. the space dimension \( d+n \).

(ii) The nonlinearity is at least mass-critical w.r.t. the space dimension \( d \).

This particularly requires \( n \in \{1,2\} \) (the case \( n = 0 \) reduces to the well-known \( \mathbb{R}^d \)-case). The case \( n = 2 \) is significantly more interesting and difficult than the case \( n = 1 \), as in the case \( n = 2 \) the nonlinearity is both mass- and energy-critical. The first breakthrough in this direction was made by Hani and Pausader [29], where the authors studied the defocusing quintic NLS on the waveguide manifold \( \mathbb{R} \times \mathbb{T}^2 \). In particular, the authors proved suitable Strichartz estimates on \( \mathbb{R} \times \mathbb{T}^2 \) which are given in terms of the \( X^s \)- and \( Y^s \)-spaces introduced in [32, 33] and particularly provide sufficiently strong dispersion that guarantee the scattering of a solution in the energy space with small initial data. Indeed, by making use of the Black-Box-Theory initiated in [34, 35, 36] the authors were also able to prove that a solution of the defocusing quintic NLS on \( \mathbb{R} \times \mathbb{T}^2 \) is always global and scattering. We shall also point out that the large data scattering result in [29] was originally conditional and based on a conjecture concerning the large data scattering result of the corresponding large scale resonant system, which was later confirmed by [19]. By making use of the idea from [29] the large data scattering problems for defocusing NLS on waveguide manifolds with algebraic nonlinearities have been completely resolved, see [29, 58, 19, 18, 63, 62]. We shall also underline the interesting paper by Tzvetkov and Visciglia [56], where the defocusing intercritical analogue of (1.1) on \( \mathbb{R}^d \times \mathbb{T} \) was investigated. Instead of using the concentration compactness principle, the large data scattering result given in [56] was proved by using the interaction Morawetz inequality originated in [21].

In particular, the scattering result given in [56] is available for all \( d \geq 1 \) and all intercritical nonlinearities which are not necessarily algebraic. We also refer to [55, 25, 53, 30, 3, 4, 20, 64, 59, 60] for further interesting works on scattering problems of dispersive equations on product spaces.

Despite the abundant results on the defocusing NLS on product spaces, similar large data scattering results for the focusing model are less well-known. The first result\(^1\) on large data scattering for focusing NLS on product space was recently given by the author [44], where the focusing cubic NLS on \( \mathbb{R}^2 \times \mathbb{T} \) was studied. A key new ingredient for showing the large data scattering result in [44] was a specialized Gagliardo-Nirenberg inequality on \( \mathbb{R}^2 \times \mathbb{T} \) which is particularly scale-invariant w.r.t. the \( x \)-variable and differs from the standard inhomogeneous ones. Nonetheless, the scattering threshold formulated in [44] might possibly be non-sharp, since we were unable to prove ground state solutions lying on the threshold and a criterium for a solution being blowing-up was also missing. The situation becomes much better when the nonlinearity is at least mass-supercritical. By introducing the so-called semivirial-vanishing geometry [45] the author was able to formulate a threshold which determines the bifurcation of finite time blow-up and globally scattering solutions of the focusing intercritical NLS on \( \mathbb{R}^d \times \mathbb{T} \) in dependence of the sign of the semivirial functional. Moreover, the given threshold is sharp due to the existence of ground state solutions lying on the threshold. Using a subtle scaling argument according to Terracini-Tzvetkov-Visciglia [54] we were also able to prove the existence of a critical value \( c_* \in (0, \infty) \) which sharply determines the \( y \)-dependence of the ground state solutions in dependence of the size of the mass \( c \).

In this paper we continue our study on the focusing energy-critical NLS (1.1) by appealing to the framework of semivirial-vanishing geometry. As a starting point, we shall firstly introduce some basic concepts of the underlying theory.

\(^1\)We shall point out that the large data scattering result of the corresponding large scale resonant system proved in [44] was independently shown in [17].
Semivirial-vanishing geometry

When considering an NLS on $\mathbb{R}^d$, a very important tool to study the long time dynamics of a solution is the celebrated Glassey’s virial identity. Consider for instance a solution $u$ of the NLS

$$i\partial_t u + \Delta u = -|u|^\alpha u \quad \text{on } \mathbb{R}^d. \quad (1.2)$$

We then define the virial action functional $V_{\mathbb{R}^d}(t)$ by

$$V_{\mathbb{R}^d}(t) := \int_{\mathbb{R}^d} |x|^2 |u(t,x)|^2 \, dx.$$ 

In [26], Glassey gave the following celebrated virial identity

$$\frac{d^2}{dt^2} V_{\mathbb{R}^d}(t) = 8 \hat{Q}(u(t)) := 8 \left( \|\nabla_x u(t)\|_{L^2(\mathbb{R}^d)}^2 - \frac{\alpha d}{2(\alpha + 2)} \|u(t)\|_{L^{\alpha + 2}(\mathbb{R}^d)}^{\alpha + 2} \right).$$

The quantity $\hat{Q}(u)$ is usually referred to as the virial of the solution $u$. Hence a virial bounded above by some negative number shall ultimately lead to a finite time blow-up. On the contrary, a positive virial might be a sign indicating global well-posedness or even scattering of a solution. In fact, this can be rigorously proved under certain circumstances, see for instance [57, 39]. Motivated by these heuristics, when considering an NLS on $\mathbb{R}^d \times \mathbb{T}$ it is therefore natural to define the similar quantity $\int_{\mathbb{R}^d \times \mathbb{T}} |(x,y)|^2 |u(t,x,y)|^2 \, dxdy$. However, due to the boundedness of $\mathbb{T}$ the previously defined quantity is in fact less helpful for obtaining results concerning the long time dynamics of the NLS. Indeed, by calculating the second time derivative explicitly we shall see that there are some boundary integral terms remaining which can not be eliminated even by invoking the periodicity of the solution. Alternatively, we shall consider the quantity

$$V_{\mathbb{R}^d \times \mathbb{T}}(t) := \int_{\mathbb{R}^d \times \mathbb{T}} |x|^2 |u(t,x,y)|^2 \, dxdy.$$ 

By formally taking the second time derivative we arrive at

$$\frac{d^2}{dt^2} V_{\mathbb{R}^d \times \mathbb{T}}(t) = 8 \hat{Q}(u(t)) := 8 \left( \|\nabla_x u(t)\|_{L^2(\mathbb{R}^d \times \mathbb{T})}^2 - \frac{\alpha d}{2(\alpha + 2)} \|u(t)\|_{L^{\alpha + 2}(\mathbb{R}^d \times \mathbb{T})}^{\alpha + 2} \right).$$

We shall simply refer $Q(u)$ to as the semivirial functional. At the first glance, the way we define $Q(u)$ is purely due to the issue that $\mathbb{T}$ is bounded. However, since generally we do not expect a nonlinear Schrödinger wave to be scattering along the torus side, it seems reasonable to consider the dispersive effects that are purely provided by the $\mathbb{R}^d$-side. Nevertheless, from Theorem 1.1 given below we shall see that albeit such heuristics might be true for solutions with large mass, in the small mass case the impact from the torus side must be taken into account in a non-trivial way. From now on we focus on the problem (1.1) and set

$$Q(u) = \|\nabla_x u\|_{L^2(\mathbb{R}^d \times \mathbb{T})}^2 - \frac{d}{d + 1} \|u\|_{L^{2+4/(d-1)}(\mathbb{R}^d \times \mathbb{T})}^{2+4/(d-1)}.$$

Motivated by Jeanjean’s seminal work [37] we define the variational problem $m_c$ on the semivirial-vanishing manifold with prescribed mass $c$ by

$$m_c := \inf_{u \in H^1(\mathbb{R}^d \times \mathbb{T})} \{ E(u) : M(u) = c, Q(u) = 0 \}, \quad (1.3)$$

where $M(u)$ and $E(u)$ denote the usual mass and energy of a function $u$:

$$M(u) := \|u\|_{L^2(\mathbb{R}^d \times \mathbb{T})}^2, \quad E(u) := \frac{1}{4} \|\nabla_{x,y} u\|_{L^2(\mathbb{R}^d \times \mathbb{T})}^2 - \frac{d - 1}{2(d + 1)} \|u\|_{L^{2+4/(d-1)}(\mathbb{R}^d \times \mathbb{T})}^{2+4/(d-1)}.$$

Main results

In order to formulate our main results, we firstly fix some notation. For a function $u \in H^1(\mathbb{R}^d)$, let the quantities $\hat{M}(u)$, $\hat{E}(u)$ and $\hat{Q}(u)$ be the mass, energy and virial of $u$ defined by (1.12), (1.13) and (1.15) below respectively. For $c \in (0, \infty)$ define also the variational problem $\hat{m}_c$ by

$$\hat{m}_c := \inf_{u \in H^1(\mathbb{R}^d)} \{ \hat{E}(u) : \hat{M}(u) = c, \hat{Q}(u) = 0 \}. \quad (1.4)$$

Our first main result reveals the relation between the quantities $m_c$ and $\hat{m}_c$. 

3
Theorem 1.1 (y-dependence of the ground states). Let $m_c$ and $\hat{m}_c$ be the quantities defined by (1.3) and (1.4) respectively. Then there exists some $c_\ast \in (0, \infty)$ such that

(i) For all $c \in (0, c_\ast)$ we have $m_c < 2\pi \hat{m}(2\pi)^{-1}c$.

(ii) For all $c \in [c_\ast, \infty)$ we have $m_c = 2\pi \hat{m}(2\pi)^{-1}c$. Moreover, if $c \in (c_\ast, \infty)$, then any minimizer $u_c$ of $m_c$ must satisfy $\partial_y u_c = 0$.

Here follow several comments on Theorem 1.1:

(i) Notice by assuming a function $u \in H^1(\mathbb{R}^d \times \mathbb{T})$ with $Q(u) = 0$ is independent of $y$ we see that $u \in H^1(\mathbb{R}^d)$ and $\hat{Q}(u) = 0$. Consequently we infer that $m_c \leq 2\pi \hat{m}(2\pi)^{-1}c$. In particular, from Theorem 1.1 (ii) it is immediate that for $c \in (c_\ast, \infty)$ the optimizers of $m_c$ coincide with the ones of $\hat{m}(2\pi)^{-1}c$. By a classical result from [28], the set of optimizers of $\hat{m}_c$ is given by

$$Z_c = \{e^{i\theta} P_c(\cdot + y) : \theta \in \mathbb{R}, y \in \mathbb{R}^d\},$$

where $P_c \in H^1(\mathbb{R}^d)$ is the unique radially symmetric and positive solution of

$$-\Delta_x P_c + \omega_c P_c = \frac{P_c}{\|P_c\|_{L^1}}$$

on $\mathbb{R}^d$

with $\hat{M}(P_c) = c$ and some positive $\omega_c > 0$ which is uniquely determined by the mass $c$. Hence up to symmetries, the ground states of $m_c$ for $c \in (c_\ast, \infty)$ are unique and independent of $y$.

(ii) Later we shall use the quantity $m_c$ to formulate a threshold for determining scattering and finite time blow-up solutions (Theorem 1.2 and 1.3). While Theorem 1.1 (ii) tells us that in the large mass case the balance point of linear dispersion and nonlinear effect is attained at the $\mathbb{R}^d$-ground states, Theorem 1.1 (i) indicates the interesting fact that in the small mass case, the threshold must display certain non-trivial y-dependence.

The proof of Theorem 1.1 follows the same strategy given in [54, 45]: For $\lambda > 0$ we introduce the rescaled energy $E_\lambda(u)$ defined by (1.10) below and consider the minimization problem

$$m_{1,\lambda} := \inf_{u \in H^1(\mathbb{R}^d \times \mathbb{T})} \{E_\lambda(u) : M(u) = 1, Q(u) = 0\}.$$

By appealing to a simple rescaling argument, proving Theorem 1.1 is essentially equivalent to showing the statements

$$\lim_{\lambda \to 0} m_{1,\lambda} = 2\pi \hat{m}(2\pi)^{-1}, \quad \lim_{\lambda \to 0} m_{1,\lambda} \leq 2\pi \hat{m}(2\pi)^{-1}.$$

While the statement $\lim_{\lambda \to 0} m_{1,\lambda} < 2\pi \hat{m}(2\pi)^{-1}$ can be proved by constructing some special test functions, the proof of the statement $\lim_{\lambda \to 0} m_{1,\lambda} = 2\pi \hat{m}(2\pi)^{-1}$ relies on some subtle coercivity arguments given by [54]. We shall also point out that despite the proof of Theorem 1.1 is almost identical to the one given for [45, Thm 1.2], we encounter the new difficulty that the concentration compactness arguments given by [54] for identifying a non-vanishing weak limit of a minimizing sequence of $m_c$ (resp. $m_{1,\lambda}$) are no longer valid in the energy-critical setting. The key observation here is that for any minimizing sequence $(u_n)_n$ with sufficiently large mass (resp. large $\lambda$), we are able to prove the fact that the zero Fourier coefficients $m(u_n) := (2\pi)^{-1} \int T u_n dy$ of $u_n$ w.r.t $y$ will be concentrating as $n \to \infty$. Since $m(u_n)$ is independent of $y$, we are then able to invoke the classical concentration compactness arguments applied on $\mathbb{R}^d$ to infer a non-vanishing weak limit of $u_n$, as desired. Again, the proof of the concentration effect of $m(u_n)$ relies on a subtle scale-invariant Gagliardo-Nirenberg inequality on $\mathbb{R}^d \times \mathbb{T}$ which follows the same fashion as the ones given in [44, 45]. It remains an open question whether ground states optimizers of $m_c$ exist for $c \in (0, c_\ast)$. We underline that the proof of showing the existence of ground states with large mass (Proposition 2.7) in fact works equally well for $m_c$ with arbitrary $c \in (0, \infty)$ as long as we know that a minimizing sequence has a non-vanishing weak limit. In general we conjecture that $m_c$ has no ground state optimizers when at least $c$ is sufficiently small.

Finally, we prove that the quantity $m_c$ characterizes a sharp threshold for the bifurcation of scattering and finite time blow-up solutions in dependence of the sign of the semivirial.
\textbf{Theorem 1.2} (Scattering below threshold). Let \(d = 3\) and let \(u \in X_{c,\text{loc}}(I)\) be a solution of (1.1) with maximal lifespan \(I \subset \mathbb{R}\) containing zero, where the space \(X_{c,\text{loc}}(I)\) is defined by (3.4). Assume that

\[ E(u) < m_{M(u)} \quad \text{and} \quad Q(u(0)) > 0. \]

Then \(u\) is a global and scattering in time solution of (1.1) in the sense that there exist \(\phi^\pm \in H^1(\mathbb{R}^3 \times \mathbb{T})\) such that

\[ \lim_{t \to \pm \infty} \|u(t) - e^{it\Delta_{x,y}}\phi^\pm\|_{H^1(\mathbb{R}^3 \times \mathbb{T})} = 0. \]  

\textbf{Theorem 1.3} (Finite time blow-up below threshold). Let \(d \in \{2, 3\}\) and let \(u \in X_{c,\text{loc}}(I)\) be a solution of (1.1) with maximal lifespan \(I \subset \mathbb{R}\) containing zero. Assume that

\[ |x|u(0) \in L^2(\mathbb{R}^d \times \mathbb{T}) \land E(u) < m_{M(u)} \land Q(u(0)) < 0. \]

Then \(u\) blows up in finite time.

The proofs of Theorem 1.2 and 1.3 rely on the classical concentration compactness arguments initiated by Kenig and Merle [39] and the virial arguments by Glassey [26] respectively. The restriction \(d < 4\) is due to the fact that at the moment the local well-posedness results and Strichartz estimates for (1.1) in higher dimensions \((d \geq 4)\) are still open problems (we encounter the difficulty that the nonlinearity is no longer algebraic in higher dimensions). Moreover, the further restriction \(d = 3\) given in Theorem 1.2 is attributed to the fact that the celebrated Kenig-Merle large data scattering result for focusing quintic NLS on \(\mathbb{R}^3\) [39] is only known to hold for radial initial data. Nevertheless, Theorem 1.2 extends straightforwardly to the case \(d = 2\) when the corresponding Black-Box-Theory on \(\mathbb{R}^3\) is also available for non-radial initial data, which is widely believed to be true.

We shall also emphasize that the main difficulty for proving Theorem 1.2 is to embed a Euclidean profile \(T_4^j \phi^j\) appearing in the linear profile decomposition (Lemma 3.17) into the Black-Box of the large data scattering result for focusing cubic NLS on \(\mathbb{R}^4\) given by Dodson [23]. To be more precise, we mainly need to solve the following two issues:

(i) While the semivirial \(Q(T_4^j \phi^j)\) of a Euclidean profile \(T_4^j \phi^j\) contains only the information of the partial kinetic energy \(\|\nabla T_4^j \phi^j\|_{L^2(\mathbb{R}^3 \times \mathbb{T})}^2\), in order to apply the Black-Box-Theory on \(\mathbb{R}^4\) we need to consider the full virial which contains the complete kinetic energy \(\|\nabla_x \phi^j\|_{L^2(\mathbb{R}^4)}^2\). At the first glance, this seems to be impossible since we are attempting to upgrade some degenerate information into a complete one in the absence of further useful conditions. We shall however see that the missing information concerning the energy \(\|\partial_x \phi^j\|_{L^2(\mathbb{R}^4)}^2\) is indeed deeply hidden in the seemingly unrelated condition \(E(u) < m_{M(u)}\), where the proof relies on some highly non-trivial variational estimates.

(ii) Another issue here is that the Schrödinger flow \(e^{it\Delta_{x,y}}\) does not necessarily preserve the Lebesgue norm in the case \(T_4^c(N_j^d)^2 \to \pm \infty\) (where \(T_4^c\) and \(N_j^d\) are the time translation and scaling parameters corresponding to the profile \(T_4^j \phi^j\)). In the defocusing case, this issue can be easily solved by invoking the Sobolev’s inequality \(\|T_4^j \phi^j\|_{L^4(\mathbb{R}^3 \times \mathbb{T})} \lesssim \|\phi^j\|_{H^1(\mathbb{R}^4)}\), as we demand no restriction on the initial data in order to apply the corresponding Black-Box-Theory on \(\mathbb{R}^4\). In the focusing case, however, any attempt involving a naive application of the Sobolev’s inequality might immediately violate the underlying variational structure and lead to a failure of embedding the Euclidean profile into the Black-Box of the scattering result on \(\mathbb{R}^4\). We shall prove that in this case the statement \(\|T_4^j \phi^j\|_{L^4(\mathbb{R}^3 \times \mathbb{T})} = o_\alpha(1)\) holds true and leads to the desired claim. Notice when replacing \(\mathbb{R}^3 \times \mathbb{T}\) to \(\mathbb{R}^4\), the claim follows directly from the dispersive estimate on \(\mathbb{R}^4\). In our case there will still be an \((N_j^d)^{-\frac{4}{d}}\)-decay missing after applying the dispersive estimate on \(\mathbb{R}^3\). We shall prove that the missing decay can be compensated by the torus side by appealing to the Sobolev’s inequality on \(\mathbb{T}\).

At the end of the introductory section, we remark that in the previous works [44, 45] and also in the present paper we have mainly dealt with the case \(n = 1\). We expect that the framework of semivirial-vanishing geometry should also work in the mass-critical case and improve the results given in [44]. Consequently, we expect that the so far developed theory should work equally well in the much harder double critical case \(n = 2\). These open problems shall provide some interesting topics for future research.
Outline of the paper

The rest of the paper is organized as follows: In Section 1.1 we list some notation and definitions that will be used throughout the paper. In Section 2 we prove Theorem 1.1. Finally, Theorem 1.2 and 1.3 are shown in Section 3.

1.1 Notation and definitions

We use the notation $A \lesssim B$ whenever there exists some positive constant $C$ such that $A \leq CB$. Similarly we define $A \gtrsim B$ and we use $A \sim B$ when $A \lesssim B \lesssim A$.

For simplicity, we ignore in most cases the dependence of the function spaces on their spatial domains and hide this dependence in their indices. For example $L^2_\omega = L^2(\mathbb{R}^d)$, $H^1_{x,y} = H^1(\mathbb{R}^d \times \mathbb{T})$ and so on. However, when the space is involved with time, we still display the underlying temporal interval such as $L^2_tL^2_\omega(I)$, $L^\infty_tL^2_\omega(\mathbb{R})$ etc. The norm $\| \cdot \|_p$ is defined by $\| \cdot \|_p := \| \cdot \|_{L^p_w}$. We shall also consider functions defined on $\mathbb{R}^{d+1}$. In this case, the $\mathbb{R}^{d+1}$-gradient is denoted by $\nabla_{\mathbb{R}^{d+1}}$.

The following quantities will be used throughout the paper: For $u \in H^1_{x,y}$, define

$$M(u) := \|u\|^2_2,$$
$$E(u) := \frac{1}{2}\|\nabla_{x,y}u\|^2_2 - \frac{d-1}{2(d+1)}\|u\|^{2+4/(d-1)}_2,$$
$$Q(u) := \|\nabla_{x,y}u\|^2_2 - \frac{d}{d+1}\|u\|^{2+4/(d-1)}_{2+4/(d-1)},$$
$$I(u) := \frac{1}{2}\|\partial_y u\|^2_2 + \frac{1}{2(d+1)}\|u\|^{2+4/(d-1)}_{2+4/(d-1)} = E(u) - \frac{1}{2}Q(u).$$

For $\lambda \in (0, \infty)$, define

$$E_\lambda(u) := \frac{\lambda}{2}\|\nabla_{y}u\|^2_2 + \frac{1}{2}\|\nabla_{x}u\|^2_2 - \frac{d-1}{2(d+1)}\|u\|^{2+4/(d-1)}_2,$$
$$I_\lambda(u) := \frac{\lambda}{2}\|\partial_y u\|^2_2 + \frac{1}{2(d+1)}\|u\|^{2+4/(d-1)}_{2+4/(d-1)}.$$

For $u \in H^1_{x,y}$, define

$$\tilde{M}(u) := \|u\|^2_{L^2_\omega},$$
$$\tilde{E}(u) := \frac{1}{2}\|\nabla_{x}u\|^2_{L^2_\omega} - \frac{d-1}{2(d+1)}\|u\|^{2+4/(d-1)}_{L^2_\omega},$$
$$\tilde{I}(u) := \frac{1}{2(d+1)}\|u\|^{2+4/(d-1)}_{L^2_\omega},$$
$$\tilde{Q}(u) := \|\nabla_{x}u\|^2_{L^2_\omega} - \frac{d}{d+1}\|u\|^{2+4/(d-1)}_{L^2_\omega}.$$

For $u \in H^1(\mathbb{R}^{d+1})$, define

$$E^*(u) := \frac{1}{2}\|\nabla_{\mathbb{R}^{d+1}}u\|^2_{L^2(\mathbb{R}^{d+1})} - \frac{d-1}{2(d+1)}\|u\|^{2+4/(d-1)}_{L^2(\mathbb{R}^{d+1})},$$
$$Q^*(u) := \|\nabla_{\mathbb{R}^{d+1}}u\|^2_{L^2(\mathbb{R}^{d+1})} - \|u\|^{2+4/(d-1)}_{L^2(\mathbb{R}^{d+1})}.$$

We also define the sets

$$S(c) := \{u \in H^1_{x,y} : M(u) = c\},$$
$$V(c) := \{u \in S(c) : Q(u) = 0\},$$
$$\hat{S}(c) := \{u \in H^1_{x,y} : \tilde{M}(u) = c\},$$
$$\hat{V}(c) := \{u \in \hat{S}(c) : \tilde{Q}(u) = 0\}.$$
and the variational problems
\[ m_c := \inf \{ E(u) : u \in V(c) \}, \quad (1.22) \]
\[ m_{1,\lambda} := \inf \{ E_{\lambda}(u) : u \in V(1) \}, \quad (1.23) \]
\[ \hat{m}_c := \inf \{ \hat{E}(u) : u \in \hat{V}(c) \}. \quad (1.24) \]

Finally, for a function \( u \in H_{x,y}^1 \), the scaling operator \( u \mapsto u^t \) for \( t \in (0, \infty) \) is defined by
\[ u^t(x, y) := t^{\frac{4}{d}} u(tx, ty). \quad (1.25) \]

The following well-known results concerning the variational problem \( \hat{m}_c \) will also be frequently invoked. We refer for instance to [16, 37, 6, 5] for details of the corresponding proofs.

**Lemma 1.4.** The following statements hold true:

(i) For any \( c > 0 \) the variational problem \( \hat{m}_c \) has an optimizer \( P_c \in \hat{S}(c) \). Moreover, \( P_c \) satisfies the standing wave equation
\[ -\Delta_x P_c + \omega_c P = |P_c|^{\frac{4}{d-4}} P_c \quad (1.26) \]
with some \( \omega_c > 0 \).

(ii) Any solution \( P_c \in H^1(\mathbb{R}^d) \) of (1.26) with \( \omega_c > 0 \) is of class \( W^{3,p}(\mathbb{R}^d) \) for all \( p \in [2, \infty) \).

(iii) Any solution \( P_c \in H^1(\mathbb{R}^d) \) of (1.26) satisfies \( \hat{Q}(P_c) = 0 \).

(iv) The mapping \( c \mapsto \hat{m}_c \) is strictly monotone decreasing and continuous on \( (0, \infty) \). Moreover, we have
\[ \lim_{c \to 0} \hat{m}_c = \infty \quad \text{and} \quad \lim_{c \to \infty} \hat{m}_c = 0. \]

## 2 \( y \)-dependence of the ground states

In this section we give the proof of Theorem 1.1. Similarly as in [54, 45] we shall firstly consider the auxiliary problem \( m_{1,\lambda} \) defined by (1.23). Following the same line as in [54, 45] we prove the following characterization of \( m_{1,\lambda} \) for varying \( \lambda \).

**Lemma 2.1.** Let \( \hat{m}_c \) be the quantity defined through (1.24). Then there exists some \( \lambda_* \in (0, \infty) \) such that

- For all \( \lambda \in (0, \lambda_*) \) we have \( m_{1,\lambda} < 2\pi \hat{m}_c(2\pi)^{-1} \).
- For all \( \lambda \in (\lambda_*, \infty) \) we have \( m_{1,\lambda} = 2\pi \hat{m}_c(2\pi)^{-1} \). Moreover, any minimizer \( u_{\lambda} \) of \( m_{1,\lambda} \) must satisfy \( \partial_y u_{\lambda} = 0 \).

The proof of Theorem 1.1 follows then from Lemma 2.1 by simple rescaling arguments. We underline that in comparison to the models studied in [54, 45], the new challenge here is to identify a non-vanishing weak limit of a minimizing sequence for the variational problem \( m_c \) (respectively \( m_{1,\lambda} \)), where the concentration compactness arguments in [54] fail in the energy-critical setting. The key observation is that for large mass \( c \) (respectively large \( \lambda \)) we are able to prove that the \( L^{2+4/(d-4)}_{x,y} \)-norm of a minimizing sequence \( (u_n)_n \) will concentrate to the zero Fourier coefficient \( m(u_n) := (2\pi)^{-1} \int \hat{Q}(u_n(x, y)) dy \) of \( u_n \) w.r.t. the \( y \)-direction. In this case we may appeal to the classical concentration compactness arguments on \( \mathbb{R}^d \) to identify a non-vanishing weak limit.

### 2.1 Some auxiliary preliminaries

As a starting point, we collect in this subsection some usefully auxiliary results from [45]. We shall simply omit the proofs and refer to [45] for further details.

**Lemma 2.2** (Scale-invariant Gagliardo-Nirenberg inequality on \( \mathbb{R}^d \times \mathbb{T} \)). There exists some \( C > 0 \) such that for all \( u \in H_{x,y}^1 \) we have
\[ \|u\|_{\frac{2(d+1)}{d-4}} \leq C \|\nabla_x u\|_{2}^{\frac{2d}{d+1}} (\|u\|_{2}^{\frac{2}{d-4}} + \|\partial_y u\|_{2}^{\frac{2}{d-4}}). \]
Lemma 2.3 (Lower and upper bound of $m_c$). For any $c \in (0, \infty)$ we have $m_c \in (0, \infty)$, where $m_c$ is defined by (1.22).

Lemma 2.4 (Property of the mapping $t \mapsto Q(u')$). Let $c > 0$ and $u \in S(c)$. Then the following statements hold true:

(i) $\frac{d}{dt} E(u') = t^{-1} Q(u')$ for all $t > 0$.

(ii) There exists some $t^* = t^*(u) > 0$ such that $u^* \in V(c)$.

(iii) We have $t^* < 1$ if and only if $Q(u) < 0$. Moreover, $t^* = 1$ if and only if $Q(u) = 0$.

(iv) Following inequalities hold:

$$Q(u') \begin{cases} > 0, & t \in (0, t^*), \\ < 0, & t \in (t^*, \infty). \end{cases}$$

(v) $E(u') < E(u^*)$ for all $t > 0$ with $t \neq t^*$.

Lemma 2.5 (Property of the mapping $c \mapsto m_c$). The mapping $c \mapsto m_c$ is continuous and monotone decreasing on $(0, \infty)$.

Lemma 2.6 (Characterization of a minimizer as a standing wave solution). For any $c \in (0, \infty)$ an optimizer $u$ of $m_c$ is a solution of

$$-\Delta_{x,y} u + \beta u = |u|^{\frac{p-1}{2}} u \quad \text{(2.1)}$$

with some $\beta \in \mathbb{R}$.

2.2 Existence of ground states with large mass

In order to initiate the proof of Lemma 2.1 we will need to prove that the variational problem $m_{1,\lambda}$ has an optimizer $u_{\lambda}$ for all sufficiently large $\lambda$. By rescaling, this is equivalent to show that the variational problem $m_c$ has an optimizer $u_c$ for all sufficiently large $c$. This statement will be given as Proposition 2.7 below. We also point out that Proposition 2.7 is of independent interest in the sense that its proof is indeed available for any mass as long as the underlying minimizing sequence has a non-vanishing weak limit (which at the moment is only known to be true for large mass).

Proposition 2.7 (Existence of ground states with large mass). There exist $c^* \in [0, \infty)$ such that for any $c \in (c^*, \infty)$ the minimization problem $m_c$ has a positive optimizer $u_c$. Moreover, $u_c$ solves the standing wave equation (2.1) with some $\beta = \beta_c > 0$.

Proof. We split our proof into five steps.

Step 1: Non-vanishing weak limit of a minimizing sequence

As a starting point, we firstly show that for all sufficiently large mass, a minimizing sequence $(u_n)_n \subset V(c)$ of $m_c$ shall always weakly converge (up to a subsequence and $\mathbb{R}^d$-translations) to a non-vanishing function $u$ in $H^1_{x,y}$. We start with showing that $(u_n)_n$ is a bounded sequence in $H^1_{x,y}$. Indeed, using Lemma 2.3 and the fact that $Q(u_n) = 0$ we infer that for all sufficiently large $n$

$$\infty > 2m_c \geq E(u_n) = E(u_n) - \frac{d-1}{2d} Q(u_n) = \frac{1}{2} \|\partial_y u_n\|^2_2 + \frac{1}{2d} \|\nabla_x u_n\|^2_2, \quad \text{(2.2)}$$

which in turn implies the $H^1_{x,y}$-boundedness of $(u_n)_n$. Next define $m(u) := (2\pi)^{-1} \int_T u(y) dy$. Using triangular inequality and $Q(u_n) = 0$ we obtain

$$\|m(u_n)\|_{2(\frac{d+1}{2})} \geq \|u_n\|_{2(\frac{d+1}{2})} - \|u_n - m(u_n)\|_{2(\frac{d+1}{2})} = (d + 1)^{\frac{d-1}{d}} \|\nabla_x u_n\|_{2} - \|u_n - m(u_n)\|_{2(\frac{d+1}{2})}. \quad \text{(2.3)}$$
To handle the term $\|u_n - m(u_n)\|_{L^2(\mathbb{R}^d)}$, we firstly recall the following well-known Sobolev’s inequality on $\mathbb{T}$ for functions with zero mean (see for instance [7]):

$$\|u - m(u)\|_{L^2(\mathbb{R}^d)} \lesssim \|u\|_{H^1_y}.$$  \hfill (2.4)

Writing $u$ into the Fourier series $u(x,y) = \sum_k e^{iky} u_k(x)$ w.r.t $y$ and followed by (2.4), Minkowski, Gagliardo-Nirenberg on $\mathbb{R}^d$ and Hölder we obtain

$$\|u - m(u)\|_{L^2(\mathbb{R}^d)} \lesssim \left(\sum_k \|u_k\|_{L^2}^2\right)^{\frac{d+1}{2(d+1)}} \lesssim \left(\sum_k \|u_k\|_{L^2}^2\right)^{\frac{d+1}{2(d+1)}} \lesssim \left(\sum_k \|u_k\|_{L^2}^2\right)^{\frac{d+1}{2(d+1)}} \lesssim \left(\sum_k \|u_k\|_{L^2}^2\right)^{\frac{d+1}{2(d+1)}} = \|\partial_y u\|_{L^2_y} \|\nabla_x u\|_{L^2_x}.$$  \hfill (2.5)

Thus (2.2), (2.3) and (2.5) imply that there exist some positive constants $C = C(d) > 0$ such that for all sufficiently large $n$

$$\|m(u_n)\|_{L^2(\mathbb{R}^d)} \gtrsim C \|\nabla_x u_n\|_{L^2_x}^{\frac{d+1}{2}} \left(1 - \|\partial_y u\|_{L^2_y} \|\nabla_x u_n\|_{L^2_x}^{\frac{d+1}{2}}\right).$$  \hfill (2.6)

By assuming that a function $u \in V(c)$ is independent of $y$ we infer that $m_c \leq 2\pi \tilde{m}(2\pi)^{-1}$. Combining with Lemma 1.4 (iv) we deduce $\lim_{c \to \infty} m_c = 0$. Thus using (2.2) we know that for all sufficiently large $c$ there exists some sufficiently large $N = N(c) \in \mathbb{N}$ such that $\|\partial_y u_n\|_{L^2_y} \|\nabla_x u_n\|_{L^2_x} \leq \frac{1}{2}$ for all $n \geq N$. On the other hand, by Lemma 2.2 and $Q(u_n) = 0$ we obtain

$$\|\nabla_x u_n\|_{L^2_x}^2 \geq \frac{d}{d+1} \left\|u_n\right\|_{L^2(\mathbb{R}^d)}^{2(d+1)/(d-1)} \lesssim \|\nabla_x u_n\|_{L^2_x}^{\frac{d+1}{2}},$$  \hfill (2.7)

and we conclude that $\liminf_{n \to \infty} \|\nabla_x u_n\|_{L^2_x} > 0$. Summing up we infer that

$$\liminf_{n \to \infty} \|m(u_n)\|_{L^2(\mathbb{R}^d)} \gtrsim \liminf_{n \to \infty} \|\nabla_x u_n\|_{L^2_x}^{\frac{d+1}{2}} > 0$$

for all sufficiently large $c$. Since $(u_n)_n$ is a bounded sequence in $H^1_{x,y}$, we know that $(m(u_n))_n$ is a bounded sequence in $H^1_{x}$. Notice also that the exponent $\frac{2(d+1)}{d-1}$ lies in the intercritical regime $\in (2, 2 + \frac{4}{d-2})$, thus by the classical concentration compactness arguments on $\mathbb{R}^d$ (see for instance [43]) we can find some $(x_n)_n \subset \mathbb{R}^d$ and $v \in H^1_{x}\setminus \{0\}$ such that

$$m(u_n)(x + x_n) \rightharpoonup v(x) \quad \text{weakly in } H^1_x.$$  

On the other hand, the sequence $(u_n(x + x_n, y))_n$ is also a bounded $H^1_{x,y}$-minimizing sequence of $m_c$. If we denote the weak $H^1_{x,y}$-limit (up to a subsequence) of $(u_n(x + x_n, y))_n$ by $u$, then $m(u) = v$. In particular we infer that $u \neq 0$, which in turn completes the proof of Step 1.

**Step 2: A Le Coz characterization of $m_c$**

Next, we shall give a different and much handier characterization for $m_c$ due to Le Coz [42] that is more useful for our analysis. Define

$$\tilde{m}_c := \inf\{I(u) : u \in S(c), Q(u) \leq 0\},$$  \hfill (2.8)

where $I(u)$ is the energy functional defined by (1.9). We aim to prove $m_c = \tilde{m}_c$. Let $(u_n)_n \subset S(c)$ be a minimizing sequence for the variational problem $\tilde{m}_c$, i.e.

$$I(u_n) = \tilde{m}_c + o_n(1), \quad Q(u_n) \leq 0 \quad \forall n \in \mathbb{N}.\hfill (2.9)$$

By Lemma 2.4 we know that there exists some $t_n \in (0,1]$ such that $Q(u_n^{t_n})$ is equal to zero. Thus

$$m_c \leq E(u_n^{t_n}) = I(u_n^{t_n}) \leq I(u_n) = \tilde{m}_c + o_n(1).$$

Sending $n \to \infty$ we infer that $m_c \leq \tilde{m}_c$. On the other hand,

$$\tilde{m}_c \leq \inf\{I(u) : u \in V(c)\} = \inf\{E(u) : u \in V(c)\} = m_c,$$

which completes the proof.
Step 3: Existence of a non-negative optimizer of $m_c$

Define

$$c^* := \inf \{ \hat{c} \in (0, \infty) : \text{ $m_c$ has a minimizing sequence with non-vanishing } H^1_{d, y} \text{-weak limit } \forall c \geq \hat{c} \}.$$ 

By Step 1 we know that $c^* \in [0, \infty)$ and from now on we shall fix some $c \in (c^*, \infty)$. Let $(u_n)_n \subset V(c)$ be a minimizing sequence of $m_c$ which also possesses a non-vanishing $H^1_{d, y}$-weak limit $u \neq 0$. Using diamagnetic inequality and Step 2, by replacing $u_n$ and $u$ to $|u_n|$ and $|u|$ respectively we may assume that all $u_n$ and $u$ are non-negative, $Q(u_n) \leq 0$ and $I(u_n)$ approaches $\tilde{m}_c$. By weakly lower semicontinuity of norms we deduce

$$M(u) := c_1 \in (0, c], \quad I(u) \leq \tilde{m}_c.$$  \hspace{1cm} (2.10)

We next show $Q(u) \leq 0$. Assume the contrary $Q(u) > 0$. By Brezis-Lieb lemma, $Q(u_n) \leq 0$ and the fact that $L^2_{d, y}$ is a Hilbert space we infer that

$$M(u_n - u) = c - c_1 + o_n(1),$$

$$Q(u_n - u) \leq -Q(u) + o_n(1).$$

Therefore, for all sufficiently large $n$ we know that $M(u_n - u) \in (0, c)$ and $Q(u_n - u) < 0$. By Lemma 2.4 we also know that there exists some $t_n \in (0, 1)$ such that $Q((u_n - u)^{t_n}) = 0$. Consequently, Lemma 2.5, Brezis-Lieb lemma and Step 2 yield

$$\tilde{m}_c \leq I((u_n - u)^{t_n}) < I(u_n - u) = I(u_n) - I(u) + o_n(1) = \tilde{m}_c - I(u) + o_n(1).$$

Sending $n \to \infty$ and using the non-negativity of $I(u)$ we obtain $I(u) = 0$. This in turn implies $u = 0$, which is a contradiction and thus $Q(u) \leq 0$. If $Q(u) < 0$, then again by Lemma 2.4 we find some $s \in (0, 1)$ such that $Q(u^s) = 0$. But then using Lemma 2.5, Step 2 and the fact $c_1 \leq c$

$$\tilde{m}_{c_1} \leq I(u^s) < I(u) \leq \tilde{m}_c \leq \tilde{m}_{c_1},$$

a contradiction. We conclude therefore $Q(u) = 0$

Thus $u$ is a minimizer of $m_{c_1}$. From Lemma 2.6 we know that $u$ is a solution of (2.1) and it remains to show that the corresponding $\beta$ in (2.1) is positive and $M(u) = c$.

Step 4: Positivity of $\beta$

First we prove that $\beta$ is non-negative. Testing (2.1) with $u$ and followed by eliminating $\|\nabla_x u\|_2^2$ using $Q(u) = 0$ we obtain

$$\|\partial_y u\|_2^2 + \beta M(u) = \frac{1}{d+1}\|u\|^{2(d+1)/(d-1)}_{2(d+1)/(d-1)}.$$  \hspace{1cm} (2.11)

Next, we define the scaling operator $T_\lambda$ by

$$T_\lambda u(x, y) := \lambda^{d-1} u(\lambda x, y).$$  \hspace{1cm} (2.12)

Then

$$\|T_\lambda (\nabla_x u)\|_2^2 = \lambda^n \|\nabla_x u\|_2^2,$$

$$\|T_\lambda u\|_{2(d+1)/(d-1)}^2 = \lambda^n \|u\|_{2(d+1)/(d-1)}^{2(d+1)/(d-1)},$$

$$Q(T_\lambda u) = \lambda Q(u),$$

$$\|T_\lambda (\partial_y u)\|_2^2 = \lambda^{-1} \|\partial_y u\|_2^2,$$

$$\|T_\lambda u\|_2^2 = \lambda^{-1} \|u\|_2^2.$$ 

Using Lemma 2.5 and the fact that $u_c$ is an optimizer of $m_c$ we infer that $\frac{d}{4} E(T_\lambda u_c)|_{\lambda = 1} \geq 0$, or equivalently

$$\|\partial_y u\|_2^2 \leq \frac{1}{d+1}\|u\|^{2(d+1)/(d-1)}_{2(d+1)/(d-1)}.$$  \hspace{1cm} (2.13)
Combining with (2.11) we deduce $\beta M(u) \geq 0$. Since $u \neq 0$, we conclude that $\beta \geq 0$. It is left to show that $\beta = 0$ leads to a contradiction, which completes the proof of Step 4. Assume therefore that $u$ satisfies the equation

$$-\Delta_{x,y} u = u^{\frac{d+3}{d-1}}.$$  \hspace{1cm} (2.14)

By the Brezis-Kato estimate [11] (see also [52, Lem. B.3]) and the local $L^p$-elliptic regularity (see for instance [52, Lem. B.2]) we know that $u \in W^{2,p}_{\text{loc}}(\mathbb{R}^{d+1})$ for all $p \in [1, \infty)$. Hence by Sobolev embedding we also know that $u$ and $\nabla u$ are of class $L^\infty_{\text{loc}}(\mathbb{R}^{d+1})$. Taking $\partial_j$ to (2.14) with $j \in \{1, \cdots, d+1\}$ we obtain

$$-\Delta_{x,y} \partial_j u = \frac{d+3}{d-1}(u^{\frac{d+3}{d-1}} \partial_j u) \in L^\infty_{\text{loc}}(\mathbb{R}^{d+1}).$$

Hence by applying the local $L^p$-elliptic regularity again we deduce $u \in W^{3,p}_{\text{loc}}(\mathbb{R}^{d+1})$ for all $p \in [1, \infty)$. Consequently, by Sobolev embedding we infer that $u \in C^2(\mathbb{R}^{d+1})$. Using strong maximum principle we also know that $u$ is positive. By [15], any positive $C^2$-solution of (2.14) must be of the form

$$u(x, y) = b \left(\frac{a}{1 + a^2 \left|\left(x, y) - (x_0, y_0)\right|^2\right)^{\frac{d+1}{d}}\right) \hspace{1cm} \text{with some } a, b > 0 \text{ and } (x_0, y_0) \in \mathbb{R}^{d+1}.$$  \hspace{1cm} (2.15)

However, in this case $u$ can not be periodic along the $y$-direction, which leads to a contradiction. This completes the proof of Step 4.

**Step 5: $M(u) = c$ and conclusion**

Finally, we prove $M(u) = c$. Assume therefore $c_1 < c$. By Lemma 2.5 and (2.10) we know that $m_{c_1}$ is a local minimizer of the mapping $c \mapsto m_c$, which in turn implies that the inequality in (2.13) is in fact an equality. Now using (2.13) (as an equality) and (2.11) we infer that $\beta M(u) = 0$, which is a contradiction since $\beta > 0$ and $u \neq 0$. We thus conclude $M(u) = c$. That $u$ is positive follows immediately from the strong maximum principle. This completes the desired proof.

2.3 Proof of Lemma 2.1

Before we finally give the proof of Lemma 2.1, we still need state some usefully auxiliary lemmas.

**Lemma 2.8.** For all sufficiently large $\lambda$ the minimization problem $m_{1,\lambda}$ has a positive minimizer.

**Proof.** It is easy to check that $u \mapsto T_c u$ (where $T_c u$ is given by (2.12)) defines a bijection between $V(c)$ and $V(1)$. Direct calculation also shows that $E(u) = c^{-1} E_c(T_c u)$, thus $m_c = c^{-1} m_{1,c}$. The desired claim follows from the existence claim of positive optimizers of $m_c$ for large $c$ deduced in Proposition 2.7.

**Lemma 2.9.** We have

$$\lim_{\lambda \to \infty} m_{1,\lambda} = 2\pi \tilde{m}_{(2\pi)^{-1}}. \hspace{1cm} (2.15)$$

Additionally, for all sufficiently large $\lambda$ let $u_{\lambda} \in V(1)$ be a positive optimizer of $m_{1,\lambda}$ which also satisfies

$$-\Delta x u_{\lambda} - \lambda \partial_y^2 u_{\lambda} + \beta u_{\lambda} = |u_{\lambda}|^{p-1} u_{\lambda} \text{ on } \mathbb{R}^d \times \mathbb{T} \hspace{1cm} (2.16)$$

for some $\beta_{\lambda} > 0$. Then

$$\lim_{\lambda \to \infty} \lambda \partial_y u_{\lambda} \|_2^2 = 0. \hspace{1cm} (2.17)$$

**Remark 2.10.** That the frequency exponent $\beta_{\lambda}$ is positive follows from Lemma 2.8 and Step 4 of the proof of Proposition 2.7.
Proof. By assuming that a candidate in $V(1)$ is independent of $y$ we already conclude

$$m_{1,\lambda} \leq 2\pi \tilde{m}_{(2\pi)^{-1}}.$$  \hfill (2.18)

Next we prove

$$\lim_{\lambda \to \infty} \|\partial_y u_\lambda\|_2^2 = 0.$$  \hfill (2.19)

Suppose that (2.19) does not hold. Then we must have

$$\lim_{\lambda \to \infty} \lambda \|\partial_y u_\lambda\|_2^2 = \infty.$$  \hfill (2.20)

Since $Q(u_\lambda) = 0$,

$$m_{1,\lambda} = E_\lambda(u_\lambda) - \frac{d-1}{2d} Q(u_\lambda) = \frac{\lambda}{2} \|\partial_y u_\lambda\|_2^2 + \frac{1}{2d} \|\nabla_x u_\lambda\|_2^2 \geq \frac{\lambda}{2} \|\partial_y u_\lambda\|_2^2 \to \infty$$

as $\lambda \to \infty$, which contradicts (2.18) and in turn proves (2.19). Using (2.18) and (2.20) we infer that

$$\|\nabla_x u_\lambda\|_2^2 \lesssim m_{1,\lambda} \leq 2\pi \tilde{m}_{(2\pi)^{-1}} < \infty.$$  \hfill (2.21)

Therefore $(u_\lambda)$ is a bounded sequence in $H^1_{x,y}$, whose weak limit is denoted by $u$. Since $\|\partial_y u_\lambda\|_2^2 \to 0$, using (2.6) and (2.7) we conclude that

$$\liminf_{\lambda \to \infty} \|m(u_\lambda)\|_{2+4/(d-1)} > 0.$$  \hfill (2.22)

Thus arguing as in Step 1 of Proposition 2.7 we may also assume that $u \neq 0$. By (2.19) we know that $u$ is independent of $y$ and thus $u \in H^1_{\tilde{m}}$. Moreover, using weakly lower semicontinuity of norms we know that $\tilde{M}(u) \in (0,(2\pi)^{-1})$. On the other hand, using $Q(u_\lambda) = 0$, (2.16) and $M(u_\lambda) = 1$ we obtain

$$\beta_\lambda = \frac{1}{d+1} \|u_\lambda\|_{2(d+1)/(d-1)} - \lambda \|\partial_y u_\lambda\|_2^2 \lesssim \|u_\lambda\|_{2(d+1)/(d-1)}.$$  \hfill (2.23)

Thus $(\beta_\lambda)_\lambda$ is a bounded sequence in $(0,\infty)$, whose limit is denoted by $\beta$. We now test (2.16) with $\phi \in C^\infty_c(\mathbb{R}^d)$ and integrate both sides over $\mathbb{R}^d \times T$. Notice particularly that the term $\int_{\mathbb{R}^d \times T} \partial^2_x u_\lambda \phi \, dx dy = 0$ for any $\lambda > 0$ since $\phi$ is independent of $y$. Using the weak convergence of $u_\lambda$ to $u$ in $H^1_{x,y}$, by sending $\lambda \to \infty$ we obtain

$$-\Delta_x u + \beta u = |u|^{\frac{4}{d-2}} u \quad \text{in } \mathbb{R}^d.$$  \hfill (2.24)

In particular, by Lemma 1.4 we know that $\tilde{Q}(u) = 0$ and consequently $\beta > 0$. Combining with weakly lower semicontinuity of norms we deduce

$$2\pi \tilde{E}(u) = 2\pi \tilde{F}(u) \leq \liminf_{\lambda \to \infty} I_\lambda(u_\lambda) = \liminf_{\lambda \to \infty} E_\lambda(u_\lambda) \leq 2\pi \tilde{m}_{(2\pi)^{-1}},$$

where $\tilde{E}(u)$, $\tilde{F}(u)$, $I_\lambda(u)$ and $E_\lambda(u)$ are the quantities defined by (1.13), (1.14), (1.11) and (1.10) respectively. However, by Lemma 1.4 the mapping $c \mapsto \tilde{m}_c$ is strictly monotone decreasing on $(0,\infty)$, from which we conclude that $\tilde{M}(u) = (2\pi)^{-1}$ and $u$ is an optimizer of $\tilde{m}_{(2\pi)^{-1}}$. Using the weakly lower semicontinuity of norms we obtain

$$m_{1,\lambda} = E_\lambda(u_\lambda) = E_\lambda(u_\lambda) - \frac{d-1}{2d} Q(u) = \frac{\lambda}{2} \|\partial_y u_\lambda\|_2^2 + \frac{1}{2d} \|\nabla_x u_\lambda\|_2^2 \
\geq \frac{1}{2d} \|\nabla_x u_\lambda\|_2^2 \geq \frac{2\pi}{2d} \|\nabla_x u_\lambda\|_{L^2_x}^2 + o_\lambda(1) = 2\pi \tilde{E}(u) + o_\lambda(1) \geq 2\pi \tilde{m}_{(2\pi)^{-1}} + o_\lambda(1).$$  \hfill (2.25)

Letting $\lambda \to \infty$ and taking (2.18) into account yield (2.15). Finally, (2.17) follows directly from the previous calculation by not neglecting $\lambda \|u_\lambda\|_2^2$ therein. This completes the desired proof. \hfill \Box

Lemma 2.11. Let $u_\lambda$ and $u$ be the functions given in the proof of Lemma 2.9. Then $u_\lambda \to u$ strongly in $H^1_{x,y}$.  

12
Proof. In fact, in the proof of Lemma 2.9 we see that all the inequalities involving the weakly lower semicontinuity of norms are in fact equalities. This particularly implies \(\|u_\lambda\|_{H^1_{x,y}} \to \|u\|_{H^1_{x,y}} = 2\pi \|u\|_{H^1_x}\) as \(\lambda \to \infty\), which in turn implies the strong convergence of \(u_\lambda\) to \(u\) in \(H^1_{x,y}\).

Lemma 2.12. There exists some \(\lambda_0\) such that \(\partial_\beta u_\lambda = 0\) for all \(\lambda > \lambda_0\).

Proof. Let \(w_\lambda := \partial_\beta u_\lambda\). Then taking \(\partial_\beta\)-derivative to (2.16) we obtain

\[
-\Delta_x w_\lambda - \lambda \partial_\beta^2 w_\lambda + \beta \lambda w_\lambda = \partial_\beta(\|u_\lambda\|^{\frac{1}{\beta-1}} u_\lambda) = \frac{d+3}{d-1}\|u_\lambda\|^{\frac{1}{\beta-1}} w_\lambda.
\]

(2.24)

Testing (2.24) with \(w_\lambda\) and rewriting suitably, we infer that

\[
0 = \|\nabla_x w_\lambda\|_2^2 + \lambda\|\partial_\beta w_\lambda\|_2^2 + \beta \lambda w_\lambda^2 - \frac{d+3}{d-1} \int_{\mathbb{R}^{d+1}} |u_\lambda|^{\frac{1}{\beta-1}} |w_\lambda|^2 \, dx dy
\]

\[
= (\lambda - 1)\|\partial_\beta w_\lambda\|_2^2 - \frac{d+3}{d-1} \int_{\mathbb{R}^{d+1}} |u_\lambda|^{\frac{1}{\beta-1}} |w_\lambda|^2 \, dx dy
\]

\[
+ \beta \lambda w_\lambda^2 + \|\nabla_x w_\lambda\|_2^2
\]

\[
- \frac{d+3}{d-1} \int_{\mathbb{R}^{d+1}} (|u_\lambda|^{\frac{1}{\beta-1}} - |u|^{\frac{1}{\beta-1}}) w_\lambda^2 \, dx dy.
\]

(2.25)

(2.26)

(2.27)

For (2.25), we firstly point out that by Lemma 1.4 (ii) and Sobolev embedding we have \(u \in L^\infty(\mathbb{R}^d)\). On the other hand, since \(\int_\mathbb{R} \lambda w_\lambda \, dx = 0\), we have \(\|w_\lambda\|_2 \leq \|\partial_\beta w_\lambda\|_2\). Summing up, we conclude that

\[
(2.25) \geq (\lambda - 1 - \frac{d+3}{d-1}\|u\|_{L^\infty_x}) \|\partial_\beta w_\lambda\|_2^2 \geq 0
\]

for all sufficiently large \(\lambda\). For (2.27), we discuss the cases \(\frac{d}{d+1} \leq 1\) and \(\frac{d}{d-1} > 1\) separately. For \(\frac{d}{d+1} \leq 1\), we estimate the second term in (2.27) using subadditivity of concave function, Hölder’s inequality, Lemma 2.11 and the Sobolev embedding \(H^1_{x,y} \to L^{2(d+1)/(d-1)}_{x,y}\):

\[
\int_{\mathbb{R}^{d+1}} (|u_\lambda|^{\frac{1}{\beta-1}} - |u|^{\frac{1}{\beta-1}}) |w_\lambda|^2 \, dx dy
\]

\[
\leq \int_{\mathbb{R}^{d+1}} |u_\lambda - u|^{\frac{1}{\beta-1}} |w_\lambda|^2 \, dx dy
\]

\[
\leq \|w_\lambda - u\|_{2+4/(d-1)}^2 \|w_\lambda\|_{2+4/(d-1)} \leq o(1) \|w_\lambda\|_{H^1_{x,y}}^2.
\]

(2.28)

The case \(\frac{d}{d-1} > 1\) can be similarly estimated as follows:

\[
\int_{\mathbb{R}^{d+1}} (|u_\lambda|^{\frac{1}{\beta-1}} - |u|^{\frac{1}{\beta-1}}) |w_\lambda|^2 \, dx dy
\]

\[
\leq \int_{\mathbb{R}^{d+1}} |u_\lambda - u| |w_\lambda|^2 (|u_\lambda|^{1/\beta - 1} + |u|^{1/\beta - 1}) \, dx dy
\]

\[
\leq \|u_\lambda - u\|_{2+4/(d-1)} \|u_\lambda\|_{2+4/(d-1)}^{-1} \|w_\lambda\|_{2+4/(d-1)} \leq o(1) \|w_\lambda\|_{H^1_{x,y}}^2.
\]

(2.29)

Therefore, (2.25), (2.27), the facts \(\beta_0 = \beta + o(1)\) and \(\beta > 0\) then imply

\[
0 \geq \|w_\lambda\|_{H^1_{x,y}}^2 (1 - o(1)) \geq \|w_\lambda\|_{H^1_{x,y}}^2
\]

for all \(\lambda > \lambda_0\) with some sufficiently large \(\lambda_0\). We therefore conclude that \(0 = w_\lambda = \partial_\beta u_\lambda\) for all \(\lambda > \lambda_0\).

Having all the preliminaries we are in a position to prove Lemma 2.1.

Proof of Lemma 2.1. Define

\[
\lambda_* := \inf\{\tilde{\lambda} \in (0, \infty) : m_{1,\lambda} = 2\pi \tilde{m}(2\pi{\tilde{m}}_{-1})^{-1} \forall \lambda \geq \tilde{\lambda}\}.
\]

13
By Lemma 2.12 we know that $\lambda_* < \infty$. Next we show $\lambda_* > 0$. It suffices to show
\[
\lim_{\lambda \to 0} m_{1,\lambda} < 2\pi \tilde{m}_{1(2\pi)^{-1}}.
\] (2.28)

To see this, we firstly define the function $\rho : [0, 2\pi] \to [0, \infty)$ as follows: Let $\alpha \in (0, \pi)$ such that $a > \pi - 3\pi \left(\frac{3}{3+4/(d-1)}\right)^{\frac{d+1}{3}}$. This is always possible for $a$ sufficiently close to $\pi$. Then we define $\rho$ by
\[
\rho(y) = \begin{cases} 
0, & y \in [0, a] \cup [2\pi - a, 2\pi], \\
(\pi - a)^{-1} \left(\frac{3}{3+4/(d-1)}\right)^{\frac{d+1}{3}} (y - a), & y \in [a, \pi], \\
(\pi - a)^{-1} \left(\frac{3}{3+4/(d-1)}\right)^{\frac{d+1}{3}} (2\pi - a - y), & y \in [\pi, 2\pi - a].
\end{cases}
\]

By direct calculation we see that $\rho \in H^1_y$ and
\[
2\pi > \|\rho\|_{L^2_y}^2 = \|\rho\|_{L_y^{2+4/(d-1)}}^2.
\]

Next, let $P \in H^1_y$ be an optimizer of $\tilde{m}_{\|\rho\|_{L_y^2}}$. Since by Lemma 1.4 the mapping $c \mapsto \tilde{m}_c$ is strictly decreasing on $(0, \infty)$ and $\|\rho\|_{L_y^2}^2 > (2\pi)^{-1}$, we infer that $\tilde{m}_{\|\rho\|_{L_y^2}} < \tilde{m}_{1(2\pi)^{-1}}$. Furthermore, by Lemma 1.4 (iii) we have $\|\nabla_x P\|_{L_y^2}^2 = \frac{d}{d+1} \|P\|_{L_y^{2+4/(d-1)}}^2$. Now define $\psi(x, y) := \rho(y) P(x)$. Then we conclude that $\psi \in H^1_{x,y}$, $M(\psi) = \|\rho\|_{L_y^2}^2 \tilde{m}(P) = 1$,
\[
Q(\psi) = \|\nabla_x \psi\|_{L_y^2}^2 = \frac{d}{d+1} \|\psi\|_{L_y^{2+4/(d-1)}}^2 = \frac{d}{d+1} \|\rho\|_{L_y^{2+4/(d-1)}}^2 \|P\|_{L_y^{2+4/(d-1)}}^2
\]
and
\[
E_* (\psi) := \frac{1}{2} \|\nabla_x \psi\|_{L_y^2}^2 - \frac{1}{2+4/(d-1)} \|\rho\|_{L_y^{2+4/(d-1)}} \|\rho\|_{L_y^{2+4/(d-1)}} \|P\|_{L_y^{2+4/(d-1)}}^2
\]
\[
= \frac{1}{2} \|\nabla_x \psi\|_{L_y^2}^2 - \frac{1}{2+4/(d-1)} \|\rho\|_{L_y^{2+4/(d-1)}} \|P\|_{L_y^{2+4/(d-1)}}^2
\]
\[
= \frac{1}{2} \|\nabla_x \psi\|_{L_y^2}^2 - \frac{1}{2+4/(d-1)} \|\rho\|_{L_y^{2+4/(d-1)}} \|P\|_{L_y^{2+4/(d-1)}}^2 < 2\pi \tilde{m}_{1(2\pi)^{-1}}.
\]
Consequently,
\[
\lim_{\lambda \to 0} m_{1,\lambda} \leq \lim_{\lambda \to 0} E_*(\psi) = E_* (\psi) < 2\pi \tilde{m}_{1(2\pi)^{-1}},
\] (2.29)
and (2.28) follows.

Next, since the mapping $\lambda \mapsto m_{1,\lambda}$ is monotone increasing and $\lambda_*$ is defined as an infimum, we know that $m_{1,\lambda} < 2\pi \tilde{m}_{(2\pi)^{-1}}$ for all $\lambda \in (0, \lambda_*).$ It is left to show the necessity of $y$-independence of the minimizer $u_\lambda$ for $\lambda > \lambda_*$. We borrow an idea from [22] to prove this claim. Assume the contrary that an optimizer $u_\lambda$ of $m_{1,\lambda}$ satisfies $\|\partial_y u_\lambda\|_{L_y^2}^2 \neq 0$. Let $\mu \in (\lambda_*, \lambda)$. Then
\[
2\pi \tilde{m}_{1(2\pi)^{-1}} = m_{1,\mu} \leq E_\mu(u_\lambda) = E_\lambda(u_\lambda) + \frac{\mu - \lambda}{2} \|\partial_y u_\lambda\|_{L_y^2}^2 < E_\lambda(u_\lambda) = m_{1,\lambda} = 2\pi \tilde{m}_{1(2\pi)^{-1}},
\]
a contradiction. This completes the desired proof.

2.4 Proof of Theorem 1.1

We now prove Theorem 1.1 by using Lemma 2.1 and a simple rescaling argument.
Proof of Theorem 1.1. From the proof of Lemma 2.8 we know that $m_c = c^{-1} m_{1,c^2}$. Using similar rescaling arguments we also know that $\tilde{m}_{(2\pi)}^{-1} c = c^{-1} 2 \pi \tilde{m}_{(2\pi)}^{-1}$. Set $c_* := \sqrt{\lambda_*}$, where $\lambda_*$ is the number given by Lemma 2.1. Then by Lemma 2.1 we know that

- For all $c \in (0, c_*)$
  \[ m_c = c^{-1} m_{1,c^2} < c^{-1} 2 \pi \tilde{m}_{(2\pi)}^{-1} = 2 \pi \tilde{m}_{(2\pi)}^{-1} c. \]
- For all $c \in (c_*, \infty)$
  \[ m_c = c^{-1} m_{1,c^2} = c^{-1} 2 \pi \tilde{m}_{(2\pi)}^{-1} = 2 \pi \tilde{m}_{(2\pi)}^{-1} c. \]

The statement concerning the $y$-independence of the minimizers follows also from Lemma 2.1 simultaneously. That $m_{c_*} = 2 \pi \tilde{m}_{(2\pi)}^{-1} c_*$ follows from the continuity of the mappings $c \mapsto m_c$ and $c \mapsto \tilde{m}_c$ deduced from Lemma 2.5 and Lemma 1.4 respectively. This completes the desired proof.

3 Scattering and finite time blow-up below threshold

In this section we give the proofs of Theorem 1.2 and 1.3. To begin with, we firstly give a quick recap of the large data scattering result for the focusing energy-critical NLS on $\mathbb{R}^4$. In the same subsection we also collect some useful energy-trapping results and a sharp Sobolev’s inequality on $\mathbb{R}^4 \times \mathbb{T}$ due to Yu-Yue-Zhao [59]. Throughout this section we also denote by $S = S_{d+1}$ the best constant of the Sobolev’s inequality on $\mathbb{R}^{d+1}$ defined by

\[
S := \inf_{u \in D^{1,2}(\mathbb{R}^{d+1})} \left( \|\nabla \partial_{\mathbb{R}} u\|_{L^2(\mathbb{R}^{d+1})}^2 / \|u\|_{L^{2+4/(d-1)}(\mathbb{R}^{d+1})}^2 \right) .
\]

3.1 Recap of the focusing energy-critical NLS on $\mathbb{R}^{d+1}$

Consider the focusing energy-critical NLS

\[
i \partial_t u + \Delta_{\mathbb{R}} u = - |u|^2 u \quad \text{on} \quad \mathbb{R}^4.
\]

We have the following large data scattering result for (3.1) due to Dodson [23]:

Theorem 3.1 (Threshold scattering of the focusing energy-critical NLS on $\mathbb{R}^4$, [23]). Assume that a function $\phi \in H^1(\mathbb{R}^4)$ satisfies

\[
E^*(\phi) < \frac{S^2}{4} \quad \text{and} \quad \|\nabla_{\mathbb{R}} \phi\|^2_{L^2(\mathbb{R}^4)} < S^2 ,
\]

where the energy functional $E^*$ is defined by (1.16). Then a solution $u$ of (3.1) with $u(0) = \phi$ is global and scattering in time.

We will also make use of the following energy-trapping result due to Kenig-Merle [39].

Lemma 3.2 (Energy-trapping, [39]). Let $\phi \in H^1(\mathbb{R}^{d+1})$. Assume that

\[
\|\nabla_{\mathbb{R}^{d+1}} \phi\|^2_{L^2(\mathbb{R}^{d+1})} < S_{d+1}^{\frac{4}{d+1}}
\]

and there exists some $\delta_0 \in (0, 1)$ such that

\[
E^*(\phi) \leq (1 - \delta_0) S_{d+1}^{\frac{4}{d+1}} / (d+1).
\]

Then there exists some $\delta = \delta(\delta_0) \in (0, 1)$ such that $\|\nabla_{\mathbb{R}^{d+1}} \phi\|^2_{L^2(\mathbb{R}^{d+1})} \leq (1 - \delta) S_{d+1}^{\frac{4}{d+1}}$.

In our context we indeed make use of a variational setting based on the semivirial functional $Q(u)$ and the $\mathbb{R}^{d+1}$-virial functional $Q^*(\phi)$ (defined by (1.17)) that seem irrelevant to the scattering scheme (3.2) at the first glance. The following lemma reveals the fact that the positivity of $Q^*(\phi)$ is a sufficient condition for (3.2).

Lemma 3.3. Assume that a function $\phi \in \dot{H}^1(\mathbb{R}^{d+1})$ satisfies $E^*(\phi) < S_{d+1}^{\frac{4}{d+1}}$. If $Q^*(\phi) > 0$, then (3.2) holds for $\phi$. 

15
Proof. Assume therefore the contrary that \( \|\nabla_{R^{d+1}} \phi\|_{L^2(\mathbb{R}^{d+1})}^2 \geq S^{\frac{d+1}{d+1}} \). Using \( Q^*(\phi) > 0 \) we obtain

\[
\frac{S^{\frac{d+1}{d+1}}}{d+1} > E^*(\phi) = \frac{1}{2} \|\nabla_{R^{d+1}} \phi\|_{L^2(\mathbb{R}^{d+1})}^2 - \frac{d-1}{2(d+1)} \|\phi\|_{L^{2+4/(d-1)}(\mathbb{R}^{d+1})}^{2+4/(d-1)} > \frac{\|\nabla_{R^{d+1}} \phi\|_{L^2(\mathbb{R}^{d+1})}^2}{d+1} \geq S^{\frac{d+1}{d+1}},
\]

a contradiction.

For the upcoming proofs we also need the following useful sharp Sobolev’s inequality on \( \mathbb{R}^d \times \mathbb{T} \) given by Yu-Yue-Zhao [59].

**Lemma 3.4** (Sharp Sobolev’s inequality on \( \mathbb{R}^d \times \mathbb{T} \), [59]). Let \( d = 3 \). Then there exists some \( c > 0 \) such that

\[
\|u\|_4 \leq S^{-\frac{2}{d}}\|\nabla_{x,y}u\|_2 + c\|u\|_2.
\]

Combining with Young’s inequality we deduce the following corollary of Lemma 3.4.

**Corollary 3.5.** Let \( d = 3 \). For any \( \varepsilon > 0 \) there exists some \( c_\varepsilon > 0 \) such that

\[
\|u\|_4 \leq (S^{-2} + \varepsilon)\|\nabla_{x,y}u\|_2^2 + C_\varepsilon\|u\|_2^2.
\]

### 3.2 A refined uniform bound for \( m_c \)

We note that in order to apply the scattering and energy-trapping arguments stated in last subsection, it is always assumed that the energy \( E^*(\phi) \) must be strictly less than \( S^{\frac{d+1}{d+1}}/(d+1) \). Such a typical upper bound of the energy functional, which arises often in problems involving energy-critical potentials, can not be obtained by solely using Lemma 2.3 concerning the size of \( m_c \). Indeed, we are able to prove that \( m_c \) will never exceed the threshold \( S^{\frac{d+1}{d+1}}/(d+1) \).

**Lemma 3.6** (A refined upper bound for \( m_c \)). We have \( m_c \leq S^{\frac{d+1}{d+1}}/(d+1) \) for any \( c \in (0, \infty) \).

**Proof.** For \( 0 < \varepsilon \ll 1 \), define

\[
v_\varepsilon(z) := \varphi(z) \cdot \left( \frac{\varepsilon}{\varepsilon^2 + |z|^2} \right)^{\frac{d-1}{2}},
\]

where \( \varphi \in C^\infty_c(\mathbb{R}^{d+1}; [0,1]) \) is a radially symmetric and decreasing cut-off function such that \( \varphi(z) \equiv 1 \) in \( |z| \leq 1 \) and \( \varphi(z) \equiv 0 \) in \( |z| \geq 2 \). Next, for \( t > 0 \) we define the operator \( S_t : H^1(\mathbb{R}^{d+1}) \to H^1(\mathbb{R}^{d+1}) \) by

\[
S_t u(z) := t^{\frac{d+1}{d+1}} u(tz).
\]

Particularly, we have \( \|S_t u\|_{L^2(\mathbb{R}^{d+1})} = \|u\|_{L^2(\mathbb{R}^{d+1})} \) for any \( t \in (0, \infty) \). Moreover, by [51, Lem. 5.3] we can find a unique \( t^*_\varepsilon = t^*_\varepsilon(v_\varepsilon) \in (0, \infty) \) such that

\[
Q^*(S^*_\varepsilon v_\varepsilon) = 0 \quad \text{and} \quad E^*(S^*_\varepsilon v_\varepsilon) = S^{\frac{d+1}{d+1}}/(d+1) + O(\varepsilon^{-d-1}).
\]

Next, we define the operator \( S^* : H^1(\mathbb{R}^{d+1}) \to H^1(\mathbb{R}^{d+1}) \) for \( s \in (0, \infty) \) by

\[
S^* u(z) := s^{\frac{d+1}{d+1}} u(sz).
\]

One easily verifies that

\[
Q^*(S^* S^*_\varepsilon v_\varepsilon) = Q^*(S^*_\varepsilon v_\varepsilon) = 0 \quad \text{and} \quad E^*(S^* S^*_\varepsilon v_\varepsilon) = E^*(S^*_\varepsilon v_\varepsilon)
\]

for all \( s \in (0, \infty) \), \( S^* S^*_\varepsilon v_\varepsilon(z) \) is supported in \( |z| \leq 1 \) for all sufficiently large \( s \) and

\[
\lim_{s \to \infty} \|S^* S^*_\varepsilon v_\varepsilon\|_{L^2(\mathbb{R}^{d+1})} = 0.
\]

Hence for all sufficiently large \( s \), writing \( z = z = (x, y) \) we may identify \( S^* S^*_\varepsilon v_\varepsilon \) as a function in \( H^1_{x,y} \) by extending \( S^* S^*_\varepsilon v_\varepsilon \) periodically modulo \( 2\pi \) along the \( y \)-direction. In particular, we have

\[
M(S^* S^*_\varepsilon v_\varepsilon) = \|S^* S^*_\varepsilon v_\varepsilon\|_{L^2(\mathbb{R}^{d+1})}^2 \quad \text{and} \quad E(S^* S^*_\varepsilon v_\varepsilon) = E^*(S^* S^*_\varepsilon v_\varepsilon).
\]
Since $S^sS_{t\varepsilon}v_\varepsilon$ is radially symmetric on $\mathbb{R}^{d+1}$, we also infer that
\[ Q(S^sS_{t\varepsilon}v_\varepsilon) = \frac{d}{d+1}Q^*(S^sS_{t\varepsilon}v_\varepsilon) = 0. \]
For a given $c \in (0, \infty)$, we choose some $s$ sufficiently large such that $M(S^sS_{t\varepsilon}v_\varepsilon) =: \tilde{c} \leq c$. Using the monotonicity of the mapping $c \mapsto m_c$ deduced from Lemma 2.5 we infer that
\[ m_c \leq m_s \leq E(S^sS_{t\varepsilon}v_\varepsilon) = E^*(S_{t\varepsilon}v_\varepsilon) = S^{d+1}/(d+1) + O(\varepsilon^{d-1}). \]
Since $\varepsilon$ can be chosen arbitrarily small, we conclude the desired claim. \(\square\)

### 3.3 Function spaces and Strichartz estimates

In this subsection we introduce the function spaces and Strichartz estimates for the model problem (1.1) (in the case $d = 3$) which were originated in [32, 33, 39, 63]. We begin with defining the Littlewood-Paley projectors. Let $\Phi \in C_0^\infty([0, 1])$ be radially symmetric and decreasing, $\Phi(t) \equiv 1$ for $|t| \leq 1$ and $\Phi(t) \equiv 0$ for $|t| \geq 2$. For $z \in \mathbb{R}^4$ let $\eta(z) := \Phi(z_1)\Phi(z_2)\Phi(z_3)\Phi(z_4)$. Then for $N > 0$ we define
\[ \eta_{\leq N}(z) := \eta(z/N), \quad \eta_N(z) := \eta_{\leq N}(z) - \eta_{\leq N/2}(z), \quad \eta_{> N}(z) := 1 - \eta_{\leq N/2}(z). \]
For a dyadic number $N \leq 1$ we define the Littlewood-Paley projector $P_{\leq N}$ by $F(P_{\leq N}) = \eta$. For $N \geq 2$ we similarly define
\[ F(P_{\leq N}) = \eta_{\leq N}, \quad F(P_N) = \eta_N, \quad F(P_{> N}) = \eta_{> N}. \]
Next, we introduce the spaces $X^s$ and $Y^s$. Denote by $C = (-1/2, 1/2)^4 \subset \mathbb{R}^4$ the unit cube in $\mathbb{R}^4$. For $z \in \mathbb{R}^4$ the translated cube $C_z$ is defined by $C_z := C + z$. Moreover, we define the projector $P_{C_z}$ by
\[ F(P_{C_z}u) := \chi_{C_z}F(u), \]
where $\chi_{C_z}$ is the characteristic function of $C_z$. For $s \in \mathbb{R}$ we then define the spaces $X_0^s(\mathbb{R})$ and $Y^s(\mathbb{R})$ through the norms
\[ \|u\|_{X_0^s(\mathbb{R})}^2 := \sum_{z \in \mathbb{Z}^4} \langle z \rangle^{2s}\|P_{C_z}u\|_{L_{x,y}^2(\mathbb{R}; L_{s,x}^2)}, \]
\[ \|u\|_{Y^s(\mathbb{R})}^2 := \sum_{z \in \mathbb{Z}^4} \langle z \rangle^{2s}\|P_{C_z}u\|_{L_{x,y}^1(\mathbb{R}; L_{s,x}^2)}, \]
where $L_{s,x}^2$ and $L_{s,x}^1$ are the standard atom spaces taking values in $L_{x,y}^2$ (see for instance [27] for their precise definitions). For any subinterval $I \subset \mathbb{R}$, the space $X^s(I)$ is defined through the norm
\[ \|u\|_{X^s(I)} := \inf \{ \|v\|_{X^s(\mathbb{R})} : v \in X_0^s(\mathbb{R}), v|_I = u|_I \}. \]
The space $Y^s(I)$ is similarly defined. We also define the space $X_0^s(\mathbb{R})$ by
\[ X_0^s(\mathbb{R}) := \{ u \in C(\mathbb{R}; H_{s,y}^1) : \phi_{-\infty} := \lim_{t \to -\infty} e^{-it\Delta_{x,y}}u(t) \text{ exists in } H_{s,y}^1, \phi_{-\infty} \in X_0^s(\mathbb{R}) \}. \]
The space $X^s(\mathbb{R})$ is equipped with the norm
\[ \|u\|_{X^s(\mathbb{R})}^2 := \|\phi_{-\infty}\|_{H_{s,y}^2}^2 + \|u(t) - e^{it\Delta_{x,y}}\phi_{-\infty}\|_{X_0^s(\mathbb{R})}^2 \sim \sup_{K \subset \mathbb{R} \text{ compact}} \|u\|_{X_0^s(K)}^2. \]

For any subinterval $I \subset \mathbb{R}$, we define the space $X^s(I)$ via the second definition of the $X^s$-norm running over all compact subsets $K$ of $I$. We also define the space $X^s_{c,\text{loc}}(I)$ by
\[ X^s_{c,\text{loc}}(I) := \bigcap_{J \subset I \text{ compact}} X^s(J). \]
For an interval $I = (a, b)$, the space $N^s(I)$ is defined through the norm
\[ \|u\|_{N^s(I)} := \| \int_a^b e^{i(t-\sigma)\Delta_{x,y}}u(\sigma) \, d\sigma \|_{X^s(I)}. \]
When $s = 1$, we simply write $X^1 = X, Y^1 = Y$ and so on. We record the following useful properties of the previously defined function spaces.
Moreover, for any smooth function \( g \) defined on \( I = [a, b] \) we have

\[
\|u\|_{N(I)} \lesssim \sup_{\|v\|_{Y^{-1}(I)} \leq 1} \int_{I \times (\mathbb{R}^3 \times T)} u(t, x, y) v(t, x, y) \, dx \, dy \, dt.
\]

Moreover, for any smooth function \( g \) defined on \( I = [a, b] \) we have

\[
\|g\|_{X(I)} \lesssim \|g(a)\|_{H^2_{t,x,y}} + \left( \sum_{N \geq 1} \|P_N (i \partial_t + \Delta_{x,y}) g\|_{L^2_t H^2_{t,x,y}(I)}^2 \right)^{1/2}.
\]

We shall also need the following Strichartz estimate on \( \mathbb{R}^3 \times T \):

**Lemma 3.9** (Strichartz estimate on \( \mathbb{R}^3 \times T \), [63, 3]). Let \( p \in (3, 6) \) and \( \frac{2}{q} + \frac{3}{p} = \frac{3}{2} \). Then for any \( \phi \in H^1_{t,x,y} \) we have

\[
\left( \sum_{\gamma \in \mathbb{Z}} \|e^{it\Delta_{x,y}} \phi\|_{L^q_t H^2_{t,x,y}(\gamma + 1, \gamma + 1)}^q \right)^{\frac{1}{q}} \lesssim \|\phi\|_{H^2_{t,x,y}}.
\]

Using the embedding \( \ell^q \hookrightarrow \ell^\bar{q} \) for \( \bar{q} > q \) we obtain immediately the following corollary of Lemma 3.9.

**Corollary 3.10.** Let \( p \in (3, 6) \) and \( \frac{2}{q} + \frac{3}{p} = 1 \). Then for any \( \phi \in H^1_{t,x,y} \) we have

\[
\left( \sum_{\gamma \in \mathbb{Z}} \|e^{it\Delta_{x,y}} \phi\|_{L^q_t H^2_{t,x,y}(\gamma + 1, \gamma + 1)}^q \right)^{\frac{1}{q}} \lesssim \|\phi\|_{H^2_{t,x,y}}.
\]

### 3.4 Small data and stability theories

We collect in this subsection the small data and stability theories for (1.1) that were originally given in [63], where the defocusing analogue of (1.1) was studied. We shall begin with defining the scattering \( Z \)-norm. For a time interval \( I \subset \mathbb{R} \), we define the space \( Z(I) \) via the norm

\[
\|u\|_{Z(I)} := \left( \sum_{N \geq 1} N^{2/p - p} \left( \sum_{\gamma \in \mathbb{Z}} \|\chi_I(t) P_N u\|_{L^q_t H^2_{x,y}(\gamma + 1, \gamma + 1)}^q \right)^{\frac{1}{q}} \right)^{\frac{1}{2}},
\]

where \( 2/q + 3/p = 1 \) with \( p \in (5, 11/2) \). The precise value of \( p \) is nonetheless of no importance. Notice also that by Corollary 3.10 we have for any \( \phi \in H^1_{t,x,y} \)

\[
\|e^{it\Delta_{x,y}} \phi\|_{Z(\mathbb{R})} \lesssim \|\phi\|_{H^2_{t,x,y}}.
\]

We are now ready to state the small data and stability results for (1.1) due to Zhao [63].

**Lemma 3.11** (Small data theory, [63]). Let \( I \) be a time interval containing zero. Suppose that a function \( u_0 \in H^1_{t,x,y} \) satisfies

\[
\|u_0\|_{H^1_{t,x,y}} \leq A.
\]

Then there exists some \( \delta = \delta(A) \) such that if

\[
\|e^{it\Delta_{x,y}} u_0\|_{Z(I)} \leq \delta,
\]

then (1.1) possesses a unique strong solution \( u \) in \( X_c(I) \) with \( u(0) = u_0 \). Moreover, if a solution \( u \in X_{c,\text{loc}}(I) \) satisfies

\[
\|u\|_{Z(I)} < \infty,
\]

then

(i) If \( I \) is finite, then \( u \) can be extended to some strictly larger interval \( J \) with \( I \subseteq J \in \mathbb{R} \).
Lemma 3.14. We have \(\|f_N\|^2 \simeq o_N(1)\) and
\[
\|\phi\|^2_{H^1(\mathbb{R}^4)} = \|f_N\|^2_{H^1_{L^2}} + o_N(1),
\]
\[
\|\phi\|^4_{L^4(\mathbb{R}^4)} = \|f_N\|^4_{L^4} + o_N(1)
\]
as \(N \to \infty\).

Remark 3.12. Using (3.5) we infer that (1.1) is always globally well-posed and scattering in time when \(\|u_0\|_{\dot{H}^1_{L^2}}\) is sufficiently small. \(\Box\)

Lemma 3.13 (Stability theory, [63]). Let \(I \subset \mathbb{R}\) be an interval containing zero and let \(\tilde{u} \in X(I)\) be an approximate solution of the perturbed NLS
\[
i\partial_t \tilde{u} + \Delta_{x,y} \tilde{u} = -|\tilde{u}|^2 \tilde{u} + \epsilon
\]
with some error term \(\epsilon\). Suppose also that there exists some \(M > 0\) such that
\[
\|\tilde{u}\|_{Z(I)} + \|\tilde{u}\|_{L_t^\infty H^1_{L^2}(I)} \leq M.
\]
Then there exists some positive \(\epsilon_0 = \epsilon_0(M) \ll 1\) such that if
\[
\|\tilde{u}(0) - u_0\|_{\dot{H}^1_{L^2}} + \|\epsilon\|_{N(I)} \leq \epsilon \ll \epsilon_0,
\]
then there exists a solution \(u \in X(I)\) of (1.1) with \(u(0) = u_0\) and
\[
\|u\|_{X(I)} \leq C(M) \quad \text{and} \quad \|u - \tilde{u}\|_{X(I)} \leq C(M)\epsilon.
\]

3.5 Linear profile decomposition

The present subsection is devoted to introducing a suitable profile decomposition for the model problem (1.1), which being a standard preliminary for a rigidity proof based on the concentration compactness arguments. We shall invoke the profile decomposition applied in [63] for the study of the defocusing analogue of (1.1), which follows the same fashion as the ones given in [36, 34, 35, 29].

We firstly fix some necessary notation. Let \(\eta \in C_0^\infty(\mathbb{R}^4; [0, 1])\) be the same auxiliary function given previously for constructing the Littlewood-Paley projectors (see the beginning of Section 3.3). For a function \(\phi \in \dot{H}^1(\mathbb{R}^4)\) and a number \(N \geq 1\) we define the function \(\phi_N\) by
\[
\phi_N(z) := N\eta(N^4 z)\phi(Nz).
\]

Let now \(\Psi : \{z \in \mathbb{R}^4 : |z| \leq 1\} \to \mathbb{R}^3 \times \mathbb{T}\) be the identity mapping. Then we define the function \(f_N(z)\) for \(z = (x, y) \in \mathbb{R}^3 \times \mathbb{T}\) by
\[
f_N(z) := \phi_N(\Psi^{-1}(z)).
\]
The definition of \(f_N\) is at the first glance somewhat misunderstanding, since \(\Psi^{-1}(z)\) is not well-defined for arbitrary points \(z \in \mathbb{R}^3 \times \mathbb{T}\). We shall make the following convention to clarify the definition of \(f_N\):
- For \(z \in \mathbb{R}^3 \times \mathbb{T}\), we identify \(z\) as the point locating at \(\mathbb{R}^3 \times [-\pi, \pi]\).
- For \(z \in \mathbb{R}^3 \times [-\pi, \pi]\), if \(|z| > 1\), then we simply set \(f_N(z) = 0\).

In other words, for all sufficiently large \(N\) (which will be the case for a Euclidean profile) \(f_N\) is nothing else but the periodic extension of \(\phi_N\) along the \(y\)-direction modulo \(2\pi\). Using Sobolev’s embedding and Hölder’s inequality one easily verifies that \(f_N \in \dot{H}^1_{L^2} \) and
\[
\limsup_{N \to \infty} \|f_N\|_{\dot{H}^1_{L^2}} \lesssim \|\phi\|_{\dot{H}^1(\mathbb{R}^4)}.
\]

For our purpose we will need the following stronger statement on the asymptotics of \(f_N\).

Lemma 3.14. We have \(\|f_N\|^2 \simeq o_N(1)\) and
\[
\|\phi\|^2_{\dot{H}^1(\mathbb{R}^4)} = \|f_N\|^2_{\dot{H}^1_{L^2}} + o_N(1),
\]
\[
\|\phi\|^4_{L^4(\mathbb{R}^4)} = \|f_N\|^4_{L^4} + o_N(1)
\]
as \(N \to \infty\).
Proof. We only prove the statements concerning the $L^2_{x,y}$ and $\dot{H}^1_{x,y}$-norms of $f_N$, the one for the $L^4_{x,y}$-norm can be deduced similarly. Notice that when $N$ tends to infinity the function $f_N$ concentrates to the zero point, any integration over $\mathbb{R}^3 \times \mathbb{T}$ can be replaced to an integration over $\mathbb{R}^4$ for all sufficiently large $N$. For $\|f_N\|_2^2$, using change of variable, Hölder and Sobolev’s embedding we obtain

$$\|f_N\|_2^2 = N^2 \int_{\mathbb{R}^4} |\eta(N^{1/2}z)\phi(Nz)|^2 \, dz = N^{-2} \int_{\mathbb{R}^4} |\eta(N^{-1/2}z)|^2 \, dz \lesssim N^{-2} \left( \int_{\mathbb{R}^4} |\eta(N^{-1/2}z)|^4 \, dz \right)^{1/2} \|\phi\|^2_{L^4(\mathbb{R}^4)} \lesssim N^{-1} \|\phi\|^2_{\dot{H}^1(\mathbb{R}^4)} = o(N^1).$$

Next, by product rule we know

$$\nabla_z \phi_N = N^{1/2} \nabla_z \eta(N^{1/2}z)\phi(Nz) + N^2 \eta(N^{1/2}z) \nabla_z \phi(Nz) =: I + II.$$

By dominated convergence theorem we already infer that $\|II\|^2_{L^2(\mathbb{R}^4)} \rightarrow \|\phi\|^2_{\dot{H}^1(\mathbb{R}^4)}$. For $I$, we choose some $\tilde{\phi} \in C_c^\infty(\mathbb{R}^4)$ such that $\|\phi - \tilde{\phi}\|_{\dot{H}^1(\mathbb{R}^4)} \leq \varepsilon$, where $\varepsilon > 0$ is some arbitrarily chosen constant. Then using Hölder

$$\|I - N^{1/2} \nabla_z \eta(N^{1/2}z)\tilde{\phi}(Nz)\|^2_{L^2(\mathbb{R}^4)} \lesssim \|\phi - \tilde{\phi}\|^2_{L^4(\mathbb{R}^4)} \leq \varepsilon^2.$$

Since $\varepsilon$ is arbitrarily chosen, it suffices to show

$$N^3 \int_{\mathbb{R}^4} \nabla_z \eta(N^{1/2}z)^2 |\tilde{\phi}(Nz)|^2 \, dz = o(N^1).$$

But using change of variable and the uniform boundedness of $\nabla_z \eta$ we obtain

$$N^3 \int_{\mathbb{R}^4} \nabla_z \eta(N^{1/2}z)^2 |\tilde{\phi}(Nz)|^2 \, dz \lesssim N^{-1} \int_{\mathbb{R}^4} |\tilde{\phi}(z)|^2 \, dz = o(N^1),$$

as desired. \qed

Next, for $(f,t_0,p_0) \in L^2_{x,y} \times \mathbb{R} \times (\mathbb{R}^3 \times \mathbb{T})$ and $\phi \in \dot{H}^1(\mathbb{R}^4)$ we define the operators

$$\pi_{p_0} f := f(z - p_0),$$

$$\Pi_{t_0,p_0} f := e^{-it_0 \Delta} f(z - p_0) = \pi_{p_0}(e^{-it_0 \Delta} f),$$

$$\mathcal{T}_N \phi := f_N(\phi) = \phi_N(\Psi^{-1}(z)).$$

We are now ready to introduce the concepts of Euclidean and scale-one profiles.

**Definition 3.15** (Frames and profiles). We define a frame $F$ to be a sequence $(N_n, t_n, p_n)_{n=1}^{\infty}$ in $2^{\mathbb{N}_0} \times \mathbb{R} \times (\mathbb{R}^3 \times \mathbb{T})$. We also define two special classes of frames:

- **A Euclidean frame** $F$ is a frame satisfying
  
  (i) $\lim_{n \to \infty} N_n = \infty$.
  
  (ii) $t_n \equiv 0$ for all $n \in \mathbb{N}$ or $\lim_{n \to \infty} |t_n N_n^2| = \infty$.

- **A scale-one frame** $F$ is a frame satisfying
  
  (i) $N_n \equiv 1$ for all $n \in \mathbb{N}$.
  
  (ii) $t_n \equiv 0$ for all $n \in \mathbb{N}$ or $\lim_{n \to \infty} |t_n| = \infty$.

Associated to each Euclidean or scale-one frame, we define a **profile** as follows:

- If $F$ is an Euclidean frame, then for $\phi \in \dot{H}^1(\mathbb{R}^4)$ we define the **Euclidean profile** $T_n \phi$ by
  
  $$T_n \phi := \Pi_{t_n,p_n} \mathcal{T}_n \phi.$$

- If $F$ is a scale-one frame, then for $\phi \in H^1_{x,y}$ we define the **scale-one profile** $T_n \phi$ by
  
  $$T_n \phi := \Pi_{t_n,p_n} \phi.$$
Remark 3.16. In the rest of the paper, a frame will always be referred to as a Euclidean or a scale-one frame. \[\square\]

We have the following linear profile decomposition for a bounded sequence in $H^1_{x,y}$ according to [63]. The version stated here is slightly different from the original one given in [63] and is better suited to our context.

Lemma 3.17 (Linear profile decomposition, [63]). Let $(\psi_n)_n$ be a bounded sequence in $H^1_{x,y}$. Then up to a subsequence, there exist nonzero $(\phi^j_j) \subset \dot{H}^1(\mathbb{R}^4) \cup H^1_{x,y}$, a sequence of frames $(N^j_j, p^j_j, p^j_j)_j$, a sequence of remainders $(w^j_j)_j \subset H^1_{x,y}$ and some number $K^* \in \mathbb{N} \cup \{\infty\}$ such that

(i) For any finite $1 \leq k \leq K^*$ we have the decomposition

$$\psi_n = \sum_{j=1}^k T^j_n \phi^j + w^k_n.$$  

(ii) The remainders $(w^j_j)_j$ satisfy

$$\lim_{j \to K^*} \lim_{n \to \infty} \|e^{it \Delta_{x,y}} w^j_n\|_{Z(\mathbb{R})} = 0.$$  

(iii) The frames are orthogonal in the sense that

$$|\log(N^j_j/N^k_k)| + |t_n - t_n(N^j_j)^2| + |p^j_j - p^j_j|N^j_j \to \infty$$  

as $n \to \infty$ for any $j \neq k$.

(iv) For any finite $1 \leq k \leq K^*$ and $D \in \{1, \partial_x, \partial_y\}$ we have the energy decompositions

$$\|D\psi_n\|^2 = \sum_{j=1}^k \|D(T^j_n \phi^j)\|^2_2 + \|Dw^k_n\|^2_2 + o_n(1),$$  

$$\|\psi_n\|^4 = \sum_{j=1}^k \|T^j_n \phi^j\|^4_4 + \|w^k_n\|^4_4 + o_n(1).$$  

We end this subsection with the following small scale approximation result for a Euclidean profile.

Lemma 3.18 (Small scale approximation). Let $F = (N_n, t_n, p_n)_n$ be a Euclidean frame and let $\phi \in \dot{H}^1(\mathbb{R}^4)$ satisfy (3.2). Let also $U_n$ be a solution of (1.1) with $U_n(0) = \Pi_{t_n, p_n} T_{N_n} \phi$. Then

(i) For all sufficiently large $n$ the solutions $U_n$ are global and scattering. Moreover, we have

$$\limsup_{n \to \infty} \|U_n\|_{X(\mathbb{R})} \lesssim e^{\gamma(\phi)} 1.$$  

(ii) Let $u$ be the global solution of (3.1) with $u(0) = \phi$ and let $\phi^\pm \in \dot{H}^1(\mathbb{R}^4)$ be the scattering data such that

$$\lim_{t \to \pm \infty} \|u(t) - e^{it \Delta_{x,y}} \phi^\pm\|_{H^1(\mathbb{R}^4)} = 0.$$  

For $x \in \mathbb{R}^3 \times T$ and $R > 0$ define

$$u_{n,R}(t, z) := \Pi_{p_n} [N_n \eta[N_n \Psi^{-1}(z)/R] u(N_n^2(t - t_n), N_n \Psi^{-1}(z))].$$  

Then

$$\lim_{T \to \infty} \lim_{R \to \infty} \lim_{n \to \infty} \|U_n - u_{n,R}\|_{X([t - t_n] \leq N_n^{-2} T)} = 0,$$  

$$\lim_{T \to \infty} \lim_{R \to \infty} \lim_{n \to \infty} \|U_n - \Pi_{t_n, p_n} T_{N_n} \phi^\pm\|_{X([t - t_n] \geq N_n^{-2} T)} = 0.$$  

\[\square\]
3.6 The MEI-functional and its properties

In this subsection we introduce the mass-energy-indicator (MEI) functional $\mathcal{D}$ and state some of its very useful properties which play a fundamental role for setting up an inductive hypothesis of a contradiction proof. The MEI-functional was firstly introduced in [41] for the study of the focusing-defocusing 3D cubic-quintic NLS and further applied in [1, 40, 46, 47, 44, 2, 45] for different models. Such inductive scheme is particularly useful when the inductive scheme is multidirectional (for instance in our case we need to consider the mass and energy separately). Since the proofs of the to be listed statements are identical to the ones given in [45], we shall simply omit the details here.

To begin with, we firstly define the domain $\Omega \subset \mathbb{R}^2$ by

$$\Omega := \left(-\infty, 0\right] \times \mathbb{R} \cup \left\{ (c, h) \in \mathbb{R}^2 : c \in (0, \infty), h \in (-\infty, m_c) \right\},$$

where $m_c$ is defined by (1.22). Then we define the MEI-functional $\mathcal{D} : \mathbb{R}^2 \to [0, \infty]$ by

$$\mathcal{D}(c, h) = \begin{cases} h + \frac{h + c}{\text{dist}((c,h), \partial \Omega)}, & \text{if } (c, h) \in \Omega, \\ \infty, & \text{otherwise.} \end{cases}$$

For $u \in H^1_{x,y}$, define $\mathcal{D}(u) := \mathcal{D}(M(u), E(u))$. We also define the set $A$ by

$$A := \{ u \in H^1_{x,y} : E(u) < m_{M(u)}, Q(u) > 0 \}.$$

By conservation of mass and energy we know that if $u$ is a solution of (1.1), then $\mathcal{D}(u(t))$ is a conserved quantity, thus in the following we simply write $\mathcal{D}(u) = \mathcal{D}(u(t))$ as long as $u$ is a solution of (1.1).

**Lemma 3.19 (Invariance of NLS-flow along $A$).** Let $u$ be a solution of (1.1) and assume that there exists some $t$ in the lifespan of $u$ such that $u(t) \in A$. Then $u(t) \in A$ for all $t$ in the maximal lifespan of $u$.

**Remark 3.20.** In view of Lemma 3.19 we will therefore write $u \in A$ for a solution $u$ of (1.1) if $u(t) \in A$ for some $t$ in the lifespan of $u$.

**Lemma 3.21 (Equivalence of $E(u)$ and $\|\nabla_{x,y} u\|_2^2$).** Let $u \in A$. Then

$$\frac{1}{d^2}\|\nabla_{x,y} u\|_2^2 \leq E(u) \leq \frac{1}{2}\|\nabla_{x,y} u\|_2^2$$

**Lemma 3.22 (Properties of the MEI-functional).** Let $u, u_1, u_2$ be functions in $H^1_{x,y}$. The following statements hold true:

(i) $u \in A \Rightarrow \mathcal{D}(u) \in (0, \infty)$.

(ii) Let $u_1, u_2 \in A$ satisfy $M(u_1) \leq M(u_2)$ and $E(u_1) \leq E(u_2)$, then $\mathcal{D}(u_1) \leq \mathcal{D}(u_2)$. If in addition either $M(u_1) < M(u_2)$ or $E(u_1) < E(u_2)$, then $\mathcal{D}(u_1) < \mathcal{D}(u_2)$.

(iii) Let $\mathcal{D}_0 \in (0, \infty)$. Then

$$m_{M(u)} - E(u) \gtrless_{\mathcal{D}_0} 1,$$

$$E(u) + M(u) \lesssim_{\mathcal{D}_0} \mathcal{D}(u)$$

uniformly for all $u \in A$ with $\mathcal{D}(u) \leq \mathcal{D}_0$.

3.7 Existence of a minimal blow-up solution

Having all the preliminaries we are now ready to construct a minimal blow-up solution of (1.1). Define

$$\tau(\mathcal{D}_0) := \sup \left\{ \|u\|_{Z(t_{\text{max}})} : u \text{ is solution of (1.1), } \mathcal{D}(u) \in (0, \mathcal{D}_0) \right\}$$

and

$$\mathcal{D}^* := \sup \{ \mathcal{D}_0 > 0 : \tau(\mathcal{D}_0) < \infty \}.\tag{3.8}$$

By Lemma 3.11, 3.21, 3.22 and Remark 3.12 we know that $\mathcal{D}^* > 0$. Therefore we may simply assume $\mathcal{D}^* < \infty$, relying on which we derive a contradiction. This in turn ultimately implies $\mathcal{D}^* = \infty$ and the
proof of Theorem 1.2 will be complete in view of Lemma 3.22. By the inductive hypothesis we can find a sequence \((u_n)_n\) which are solutions of (1.1) with \((u_n(0))_n \subset A\) and maximal lifespan \((I_n)_n\) such that
\[
\lim_{n \to \infty} \|u_n\|_{Z((\text{int}\, I_n, 0])} = \lim_{n \to \infty} \|u_n\|_{Z([0, \sup I_n])} = \infty ,
\]
\[
\lim_{n \to \infty} D(u_n) = D^*.
\]
Up to a subsequence we may also assume that
\[
(M(u_n), E(u_n)) \to (M_0, E_0) \quad \text{as} \quad n \to \infty.
\]
By continuity of \(D\) and finiteness of \(D^*\) we know that
\[
D^* = D(M_0, E_0), \quad M_0 \in [0, \infty), \quad E_0 \in [0, m_{M_0}).
\]
From Lemma 3.21 and 3.22 it follows that \((u_n(0))_n\) is a bounded sequence in \(H^1_{x,y}\), hence Lemma 3.17 is applicable to \((u_n(0))_n\): There exist nonzero \((\phi^j)_j \subset H^1(\mathbb{R}^4) \cup H^1_{x,y}\), a sequence of frames \((N^j_n, t^j_n, p^j_n)_{j,n}\), a sequence of remainders \((w^j_n)_j \subset H^1_{x,y}\) and some number \(K^* \in \mathbb{N} \cup \{\infty\}\) such that
\begin{enumerate}[i)]  
  
  \item For any finite \(1 \leq k \leq K^*\) we have the decomposition
  \[
  u_n(0) = \sum_{j=1}^{k} T^j_n \phi^j + w^k_n.\]
  
  \item The remainders \((w^j_n)_k,n\) satisfy
  \[
  \lim_{k \to K^*} \lim_{n \to \infty} \| e^{it\Delta_{x,y}} w^k_n \|_{Z(\mathbb{R}^4)} = 0.
  \]
  
  \item The parameters are orthogonal in the sense that
  \[
  |\log(N^j_n/N^k_n)| + |t^j_n - t^k_n| (N^j_n)^2 + |p^j_n - p^k_n| N^j_n \to \infty
  \]
  as \(n \to \infty\) for any \(j \neq k\).
  
  \item For any finite \(1 \leq k \leq K^*\) and \(D \in \{1, \partial_x, \partial_y\}\) we have the energy decompositions
  \[
  \|D(u_n(0))\|^2 = \sum_{j=1}^{k} \|D(T^j_n \phi^j)\|^2 + \|Dw^k_n\|^2 + o_n(1),
  \]
  \[
  \|u_n(0)\|^2 = \sum_{j=1}^{k} \|T^j_n \phi^j\|^4 + \|w^k_n\|^4 + o_n(1).
  \]
\end{enumerate}

We now define the nonlinear profiles as follows: Let \(1 \leq k \leq K^*\). If \((N^k_n, t^k_n, p^k_n)\) is a Euclidean frame, we define the nonlinear profile \(u^k_n\) to be the solution \(U_n\) given in Lemma 3.18 with \(U_n(0) = T^k_n \phi^k\). If \((N^k_n, t^k_n, p^k_n)\) is a scale-one frame, then
\begin{itemize}
  
  \item For \(t^k_n = 0\), we define \(u^k\) as the solution of (1.1) with \(u^k(0) = \phi^k\).
  
  \item For \(t^k_n \to \pm \infty\), we define \(u^k\) as the solution of (1.1) that scatters forward (backward) to \(e^{it\Delta_{x,y}} \phi^k\) in \(H^1_{x,y}\).
\end{itemize}

In both cases we define the nonlinear profiles \(u^k_n\) by
\[
 u^k_n := u^k(t - t^k_n, z - p^k_n).
\]
Then \(u^k_n\) is also a solution of (1.1). In all the cases we have for each finite \(1 \leq k \leq K^*\)
\[
\lim_{n \to \infty} \|u^k_n(0) - T^k_n \phi^k\|_{H^1_{x,y}} = 0.\]

In the following we establish a Palais-Smale type lemma which is essential for the construction of the minimal blow-up solution.
Lemma 3.23 (Palais-Smale-condition). Let \((u_n)_n\) be a sequence of solutions of (1.1) with maximal lifespan \(I_n\), \(u_n \in A\) and \(\lim_{n \to \infty} D(u_n) = D^+\). Assume also that there exists a sequence \((t_n)_n \subset \prod I_n\) such that

\[
\lim_{n \to \infty} \|u_n\|_{Z(\inf I_n, t_n)} = \lim_{n \to \infty} \|u_n\|_{Z(\sup I_n)} = \infty. \tag{3.17}
\]

Then up to a subsequence, there exists a sequence \((x_n)_n \subset \mathbb{R}^3\) such that \((u_n(t_n, \cdot + x_n, y))_n\) strongly converges in \(H^1_{x,y}\).

**Proof.** By time translation invariance we may assume that \(t_n \equiv 0\). Let \((u_{j,n}^i)_j\) be the nonlinear profiles corresponding to the linear profile decomposition of \((u_n(0))_n\). We divide the remaining proof into three steps.

**Step 1: Decomposition of energies of the linear profiles**

We firstly show that for a given nonzero linear profile \(\phi^j\) we have

\[
E(T_n^j \phi^j) > 0, \quad Q(T_n^j \phi^j) > 0 \tag{3.18}
\]

for all sufficiently large \(n = n(j) \in \mathbb{N}\). Since \(\phi^j \neq 0\) we know that \(T_n^j \phi^j \neq 0\) for all sufficiently large \(n\). Suppose now that (3.19) does not hold. Up to a subsequence we may assume that \(Q(T_n^j \phi^j) \leq 0\) for all sufficiently large \(n\). Recall the energy functional \(I\) defined by (1.9). Using (3.14) and (3.15) we infer that

\[
I(u_n(0)) = \sum_{j=1}^k I(T_n^j \phi^j) + I(u_n^k) + o_n(1). \tag{3.20}
\]

By the non-negativity of \(I\), (3.20) and (3.6) we know that there exists some sufficiently small \(\delta > 0\) depending on \(D^+\) and some sufficiently large \(N_1\) such that for all \(n > N_1\) we have

\[
\hat{m}_{M(T_n^j \phi^j)} \leq I(T_n^j \phi^j) \leq I(u_n(0)) + \delta \leq E(u_n(0)) + \delta \leq m_{M(u_n(0))} - 2\delta, \tag{3.21}
\]

where \(\hat{m}\) is the quantity defined by (2.8). By continuity of \(c \mapsto m_c\) we also know that for sufficiently large \(n\) we have

\[
m_{M(u_n(0))} - 2\delta \leq m_{M_0} - \delta. \tag{3.22}
\]

Using (3.14) we deduce that for any \(\varepsilon > 0\) there exists some large \(N_2\) such that for all \(n > N_2\) we have

\[
M(T_n^j \phi^j) \leq M_0 + \varepsilon.
\]

From the continuity and monotonicity of \(c \mapsto m_c\) and Step 2 in the proof of Proposition 2.7, we may choose some sufficiently small \(\varepsilon\) to see that

\[
\hat{m}_{M(T_n^j \phi^j)} = m_{M(T_n^j \phi^j)} \geq m_{M_0 + \varepsilon} \geq m_{M_0} - \frac{\delta}{2}. \tag{3.23}
\]

Now (3.21), (3.22) and (3.23) yield a contradiction. Thus (3.19) holds, which combining with Lemma 3.21 also yields (3.18). Similarly, for each \(j \in \mathbb{N}\) we have

\[
E(u_n^j) > 0, \quad Q(u_n^j) > 0 \tag{3.24}
\]

(3.25) for sufficiently large \(n\).

**Step 2: Applicability of Lemma 3.18**

Next, we show that for a Euclidean profile (say it is the \(j\)-th profile), the function \(\phi^j \in H^1(\mathbb{R}^4)\) given in the Euclidean profile satisfies

\[
E^*(\phi^j) \leq E_0 \in (0, S^2/4) \quad \text{and} \quad \|\nabla \phi^j\|_{L^2(\mathbb{R}^4)}^2 < S^2,
\]

where \(S^2\) is the quantity defined by (2.8).
which in turn implies that Lemma 3.18 is applicable for the $j$-th linear profile. That $E_0 \in (0, S^2/4)$ follows from Lemma 3.6, Lemma 3.22 and the fact that $D^* < \infty$. For $u \in H^1_{x,y}$, we define
\[ Q^*(u) := ||\nabla_{x,y} u||_2^2 - ||u||_4^4. \]
We aim to show that $Q^*(T_n^j \phi^i) > 0$ for all sufficiently large $n$. Assume that this is not the case. Up to a subsequence we may assume that $Q^*(T_n^j \phi^i) \leq 0$ for all sufficiently large $n$. From Step 1 we already know that $Q(T_n^j \phi^i) > 0$ for $n \gg 1$. Therefore for all $n \gg 1$ it is necessary that
\[ ||\partial_y (T_n^j \phi^i)||_2^2 - \frac{1}{4} ||T_n^j \phi^i||_4^4 < 0. \]
We now recall the scaling operator $T_n$ defined by (2.12). The term $T_n \lambda^n \phi^i$ is well-defined for all $\lambda \in (0, 1]$ and all sufficiently large $n$ since $T_n^j \phi^i$ is concentrating to zero with shrinking support as $n \to \infty$. Let $\lambda_* \in (0, 1)$ satisfy
\[ ||\partial_y (T_n \lambda^n \phi^i)||_2^2 = \frac{1}{4} ||T_n \lambda^n \phi^i||_4^4. \]
By direct calculation one easily sees that $\lambda_* = 2 ||T_n \phi^i||_4^{-2} ||\partial_y (T_n \phi^i)||_2 < 1$. Moreover, calculating the derivative of the mapping
\[ \lambda \mapsto g(\lambda) := \lambda^{-1} ||\partial_y T_n \phi^i||_2^2 + \frac{\lambda}{4} ||T_n \phi^i||_4^4 \]
we see that $g(\lambda)$ is monotone decreasing on $(0, \lambda_*)$ and increasing on $(\lambda_*, \infty)$. Next, we rewrite $E(T_n \lambda^n \phi^i)$ to
\[ E(T_n \lambda^n \phi^i) = \frac{1}{2} \theta(\lambda) + \frac{\lambda}{2} Q(T_n^j \phi^i). \]
Since $Q(T_n^j \phi^i) > 0$ (for $n \gg 1$), we conclude that $\lambda \mapsto E(T_n \lambda^n \phi^i)$ is monotone increasing on $(\lambda_*, 1)$. On the other hand, by definition of $\lambda_*$ we also know that
\[ Q^*(T_n \lambda \phi^i) = \lambda_* Q(T_n^j \phi^i) > 0. \]
Thus there exists some $\lambda_* \in (\lambda_*, 1)$ such that
\[ E(T_n \lambda \phi^i) \leq E(T_n^j \phi^i) \leq E_0 < S^2/4 \quad \text{and} \quad Q^*(T_n \lambda \phi^i) = 0. \]
Now let $\varepsilon > 0$ be given. By Corollary 3.5 and $Q^*(T_n \lambda \phi^i) = 0$ we obtain
\[ ||\nabla_{x,y} (T_n \lambda \phi^i)||_2^2 = ||T_n \lambda \phi^i||_4^4 \leq (S^{-2} + \varepsilon)||\nabla_{x,y} (T_n \lambda \phi^i)||_2^2 + C_4 ||T_n \lambda \phi^i||_4^4, \]
which in turn implies
\[
\left( S^{-2} + \varepsilon \right)||\nabla_{x,y} (T_n \lambda \phi^i)||_2^2 \
\geq 1 - \frac{C_4 ||T_n \lambda \phi^i||_4^4}{||\nabla_{x,y} (T_n \lambda \phi^i)||_2^2} = 1 - \frac{C_4 ||T_n \phi^i||_4^4}{\lambda_* ||\nabla_{x,y} \phi^i||_2^2}.
\]
But by Lemma 3.14 and the embedding $H^1_{x,y} \hookrightarrow L^4_{x,y}$ we know that
\[
\sup_{n \in \mathbb{N}} ||T_n \phi^i||_4 \leq 1, \\
\lim_{n \to \infty} ||T_n \phi^i||_2 = 0, \\
\lim_{n \to \infty} (||\nabla_{x,y} T_n \phi^i||_2^2 ||\partial_y T_n \phi^i||_2) = ||\nabla \phi^i||_2^2 ||\partial_y \phi^i||^2_{L^2(R^3)} > 0.
\]
Hence there exits some $J = J(\varepsilon)$ such that for all $n \geq J$
\[ (S^{-2} + \varepsilon)||\nabla_{x,y} (T_n \lambda \phi^i)||_2^2 \geq 1 - \varepsilon, \]
or equivalently
\[ \| \nabla_{x,y} (T_{\lambda, \mu} T_{h, \nu} \phi) \|_2^2 \geq \frac{1 - \varepsilon}{1 + \varepsilon S^2} S^2. \]

Now combining with \( Q^{**} (T_{\lambda, \mu} T_{h, \nu} \phi) = 0 \), we deduce
\[ \frac{S^2}{4} > E_0 \geq E(T_{\lambda, \mu} T_{h, \nu} \phi) = \frac{\| \nabla_{x,y} (T_{\lambda, \mu} T_{h, \nu} \phi) \|_2^2}{4} \geq \frac{1 - \varepsilon}{4(1 + \varepsilon S^2)} S^2. \]

But
\[ \lim_{\varepsilon \to 0} \frac{1 - \varepsilon}{4(1 + \varepsilon S^2)} S^2 = \frac{S^2}{4}. \]

Hence we can choose \( \varepsilon \ll 1 \) such that \( \frac{1 - \varepsilon}{4(1 + \varepsilon S^2)} S^2 \) lies between \( E_0 \) and \( S^2/4 \), which leads to a contradiction by taking \( n \) sufficiently large. We thus conclude that \( Q^{**} (T_{h, \nu} \phi) > 0 \) for all sufficiently large \( n \).

Notice that this is still not the desired claim. In the case \( t_n \equiv 0 \) the claim follows already from Lemma 3.3 and Lemma 3.14. It is therefore left to consider the case \( |t_n N_n^2| \to \infty \). The main issue here is that the Schrödinger group does not necessarily leave the Lebesgue norm invariant. To overcome this difficulty we shall appeal to the Sobolev’s inequality on \( T \) and the dispersive estimate on \( \mathbb{R}^3 \). For \( \varepsilon > 0 \) let \( \phi \in C_c^\infty(\mathbb{R}^4) \) such that \( \| \phi^1 - \varphi \|_{L^1(\mathbb{R}^4)} \leq \varepsilon \). W.l.o.g. we may assume that \( \varphi \) takes the form \( \varphi(x, y) = \varphi^1(x) \varphi^2(y) \) with \( \varphi^1 \in C_c^\infty(\mathbb{R}^3) \) and \( \varphi^2 \in C_c^\infty(\mathbb{R}) \). Indeed, the general form of \( \varphi \) should be a finite sum of atoms taking the form \( \varphi^1 \varphi^2 \), but once the claim for a single atom is proved, the general claim follows immediately by using the triangular inequality. Since spatial translations leave the Lebesgue norm invariant, we may also assume that \( p_n^1 \equiv 0 \). Next we recall that the function \( \eta \) is given as a product of the functions \( \Phi \):
\[ \eta(z) = \Phi(x_1) \Phi(x_2) \Phi(x_3) \Phi(y). \]

With slight abuse of notation we simply write \( \Phi(x) := \Pi_{j=1}^3 \Phi(x_j) \). Then
\[ e^{it \Delta_x} \eta((N_j)^{1/2} z) (N_j)^{1/2} \varphi(N_j x)) = e^{it \Delta_x} (\Phi((N_j)^{1/2} x) \varphi^1((N_j)^{1/2} x)) \times e^{it \Delta_x} (\Phi((N_j)^{1/2} y) \varphi^2((N_j)^{1/2} y)). \]

Consequently,
\[ \| T_n \varphi \|_4^4 = (N_j)^4 \| e^{it \Delta_x} (\Phi((N_j)^{1/2} x) \varphi^1((N_j)^{1/2} x)) \|_{L^4(\mathbb{R}^3)} \| e^{it \Delta_x} (\Phi((N_j)^{1/2} y) \varphi^2((N_j)^{1/2} y)) \|_{L^4(\mathbb{R}^3)}. \]

For \( I \), using the dispersive estimate of \( e^{it \Delta_x} \) on \( \mathbb{R}^3 \) we infer that
\[ I \lesssim |t_n|^{-3} \| \Phi((N_j)^{1/2} x) \varphi^1((N_j)^{1/2} x) \|_{L^4(\mathbb{R}^3)} \lesssim |t_n|^{-3} (N_j)^{-3} \| \varphi^1 \|_{L^4(\mathbb{R}^3)}^4. \]

For \( II \), we set \( u := \Phi((N_j)^{1/2} y) \varphi^2((N_j)^{1/2} y) \) and then decompose \( u \) into
\[ e^{it \Delta_x} u = m(u) + e^{it \Delta_x} (u - m(u)). \]

For \( m(u) \), we have
\[ \| m(u) \|_{L^4(\mathbb{R}^3)} \lesssim |m(u)|^4 \left( \int_{\mathbb{R}} |\varphi^2((N_j)^{1/2} y)| \, dy \right)^4 = (N_j)^{-4} \| \varphi^2 \|_{L^4(\mathbb{R}^3)}^4 \lesssim (N_j)^{-3} \| \varphi^2 \|_{L^4(\mathbb{R}^3)}^4. \]

For \( e^{it \Delta_x} (u - m(u)) \), using (2.4), the fact that \( e^{it \Delta_x} \) is an isometry on \( H^s_y \) with \( s \in \mathbb{R} \) and interpolation we obtain
\[ \| e^{it \Delta_x} (u - m(u)) \|_{L^4(\mathbb{R}^3)} \lesssim \| u \|_{L^2(\mathbb{R}^3)}^4 \lesssim (N_j)^{-1} \| \varphi^2 \|_{L^4(\mathbb{R}^3)}^4. \]

Summing up, we infer by combining \( t_n^1 (N_j)^{1/2} \to +\infty \) that
\[ \| T_n \varphi \|_4^4 \lesssim \| t_n^1 (N_j)^{1/2} \|_{L^4(\mathbb{R}^3)}^4 \| \varphi^2 \|_{L^4(\mathbb{R}^3)}^4 \to 0 \]
as \( n \to \infty \). Since \( \varepsilon \) can be chosen arbitrarily, we finally conclude \( \| T_n \varphi \|_4^4 \lesssim o_n(1) \). Combining now with Lemma 3.14, the fact that \( e^{it \Delta_x} \) is an isometry on \( H^s_y \) and the asymptotic positivity of the energies of linear profiles deduced in Step 1 we obtain
\[ E^*(\phi^j) \leq \frac{1}{4} \| \nabla_{x,y} \phi^j \|_{L^2(\mathbb{R}^4)}^2 = \lim_{n \to \infty} E(T_n^1 \phi^j) \leq E_0 < S^2/4. \]

Now by Lemma 3.2 we see that \( \phi^j \) satisfies (3.2) and the desired claim follows.
Step 3: Conclusion

The remaining proof follows essentially the same line as in the proof of [29, Prop. 7.1], where we should suitably replace the nonlinear estimates applied on $\mathbb{R} \times \mathbb{T}^2$ in [29] by the ones applied on $\mathbb{R}^3 \times \mathbb{T}$, the latter being established in [63]. Since the adaptation is straightforward and tedious, we shall only give here a sketch of the key arguments and refer to [29] and [63] for the full details. Using (3.14) and (3.15) we infer that for any finite $1 \leq k \leq K^*$

\[ M_0 = \sum_{j=1}^{k} M(T_n^j \phi^j) + M(w_n^k) + o_n(1), \]  \hfill (3.26)

\[ E_0 = \sum_{j=1}^{k} E(T_n^j \phi^j) + E(w_n^k) + o_n(1). \]  \hfill (3.27)

For (3.26) and (3.27) two different scenarios shall potentially take place: either

\[ \sup_{j \in \mathbb{N}} \lim_{n \to \infty} M(T_n^j P_n^j \phi^j) = M_0 \] and
\[ \sup_{j \in \mathbb{N}} \lim_{n \to \infty} E(T_n^j P_n^j \phi^j) = E_0, \]  \hfill (3.28)

or there exists some $\delta > 0$ such that

\[ \sup_{j \in \mathbb{N}} \lim_{n \to \infty} M(T_n^j P_n^j \phi^j) \leq M_0 - \delta \] or
\[ \sup_{j \in \mathbb{N}} \lim_{n \to \infty} E(T_n^j P_n^j \phi^j) \leq E_0 - \delta. \]  \hfill (3.29)

In the case (3.28), by the asymptotic positivity of the energies of the linear profiles deduced in Step 1 we know that there exists exactly one non-zero linear profile $\phi^1$ and

\[ u_n(0) = T_n^1 \phi^1 + w_n^1. \]

Particularly, from (3.26) and (3.27) it follows

\[ \lim_{n \to \infty} M(T_n^1 \phi^1) = M_0, \]  \hfill (3.30)

\[ \lim_{n \to \infty} E(T_n^1 \phi^1) = E_0, \]  \hfill (3.31)

\[ \lim_{n \to \infty} ||w_n^1||_2 = 0, \]  \hfill (3.32)

\[ \lim_{n \to \infty} E(w_n^1) = 0. \]  \hfill (3.33)

Combining with Lemma 3.22, (3.33) also implies

\[ \lim_{n \to \infty} ||\nabla_{x,y} w_n^1||_2 = 0, \]  \hfill (3.34)

thus together with (3.32) we deduce

\[ \lim_{n \to \infty} ||w_n^1||_{H^1_{x,y}} = 0. \]  \hfill (3.35)

If $\phi^1$ corresponds to a Euclidean profile (which is Case IIa in the proof of [29, Prop. 7.1]) then we are able to apply Lemma 3.18 (which is applicable by Step 2) to obtain

\[ \limsup_{n \to \infty} ||u_n||_{L^2(L)} < \infty, \]  \hfill (3.36)

which contradicts (3.17). Thus $\phi^1 \in H^1_{x,y}$ corresponds to a scale-one profile (which is Case IIc in the proof of [29, Prop. 7.1]) and

\[ u_n(0, z) = e^{it_n \Delta_x} \phi^1(z - p_n^1) + w_n^1. \]  \hfill (3.37)
Lemma 3.25. Let $u_c$ be the minimal blow-up solution given by Lemma 3.24. Then

(i) There exists a center function $x : \mathbb{R} \to \mathbb{R}^3$ such that for each $\varepsilon > 0$ there exists $R > 0$ such that

$$\int_{|x+x(t)| \geq R} |\nabla_{x,y} u_c(t)|^2 + |u_c(t)|^2 + |u_c(t)|^4 \, dx \, dy \leq \varepsilon \quad \forall t \in \mathbb{R}. \tag{3.39}$$

(ii) There exists some $\delta > 0$ such that $\inf_{t \in \mathbb{R}} Q(u_c(t)) = \delta$. 

Notice that since $T$ is compact, we may simply assume that $p_n^1 = (x_n^1, 0)$. If $t_n^1 \equiv 0$, then we are done. Otherwise $t_n^1 \to \pm \infty$. We show that this leads to a contradiction. It suffices to consider the case $t_n^1 \to \infty$, the case $t_n^1 \to -\infty$ can be dealt similarly. We have

$$\|e^{it\Delta_{x,y}} T_n^1 \phi^1\|_{Z(\inf t_n, 0)} \leq \|e^{it\Delta_{x,y}} T_n^1 \phi^1\|_{Z(-\infty, 0)} = \|e^{it\Delta_{x,y}} \phi^1\|_{Z(-\infty, -t_n)} \to 0$$

as $n \to \infty$. But then by Lemma 3.11 we reach the contradiction (3.36) again. This finishes the discussion of the case (3.28). If otherwise case (3.29) takes place, then Step 1 and Lemma 3.22 imply

$$\sup_{1 \leq j \leq K^*} \limsup_{n \to \infty} \mathcal{D}(T_n^j \phi^j) < \mathcal{D}^*.$$ 

This is exactly Case III in the proof of [29, Prop. 7.1]. Now arguing as in [29], by the inductive hypothesis (3.8), the stability Lemma 3.13 (setting $\tilde{u} = \sum_{j=1}^k u_j^k + e^{i\Delta_{x,y}} \mu^k$ and $u = u_n$ therein), the orthogonality condition (3.13) and the smallness condition (3.12) we arrive at the contradiction (3.36) again. This completes the desired proof.

Lemma 3.24 (Existence of a minimal blow-up solution). Suppose that $\mathcal{D}^* \in (0, \infty)$. Then there exists a global solution $u_c$ of (1.1) such that $\mathcal{D}(u_c) = \mathcal{D}^*$ and

$$\|u_c\|_{Z((-\infty, 0))} = \|u_c\|_{Z(0, \infty)} = \infty.$$ 

Moreover, $u_c$ is almost periodic in $H^1_{x,y}$ modulo $\mathbb{R}^3$-translations.

Proof. As discussed at the beginning of this section, under the assumption $\mathcal{D}^* < \infty$ one can find a sequence $(u_n)_n$ of solutions of (1.1) that satisfies the preconditions of Lemma 3.23. We apply Lemma 3.23 to infer that $(u_n(0))_n$ (up to modifying time and space translation) is precompact in $H^1_{x,y}$. We denote its strong $H^1_{x,y}$-limit by $\psi$. Let $u_c$ be the solution of (1.1) with $u_c(0) = \psi$. Then $\mathcal{D}(u_c(t)) = \mathcal{D}(\psi) = \mathcal{D}^*$ for all $t$ in the maximal lifespan $I_{\text{max}}$ of $u_c$ (recall that $\mathcal{D}$ is a conserved quantity).

We firstly show that $u_c$ is a global solution. It suffices to show that $s_0 := \sup I_{\text{max}} = \infty$, the negative direction can be similarly proved. If this does not hold, then by Lemma 3.11 there exists a sequence $(s_n)_n \subset \mathbb{R}$ with $s_n \to s_0$ such that

$$\lim_{n \to \infty} \|u_c\|_{Z((-\inf I_{\text{max}}, s_n))} = \lim_{n \to \infty} \|u_c\|_{Z(s_n, \sup I_{\text{max}})} = \infty.$$ 

Define $u_n(t) := u_c(t + s_n)$. Then (3.17) is satisfied with $t_n \equiv 0$. We then apply Lemma 3.23 to the sequence $(u_n(0))_n$ to conclude that there exists some $\varphi \in H^1_{x,y}$ such that, up to modifying the space translation, $u_c(s_n)$ strongly converges to $\varphi$ in $H^1_{x,y}$. But then using Strichartz we obtain

$$\|e^{it\Delta_{x,y}} u_c(s_n)\|_{Z(0, s_n)} = \|e^{it\Delta_{x,y}} \varphi\|_{Z(0, s_n)} + o_n(1) = o_n(1).$$ 

By Lemma 3.11 we can extend $u_c$ beyond $s_0$, which contradicts the maximality of $s_0$. Now by (3.9) and Lemma 3.13 it is necessary that

$$\|u_c\|_{Z((-\infty, 0))} = \|u_c\|_{Z(0, \infty)} = \infty. \tag{3.38}$$

We finally show that the orbit $\{u_c(t) : t \in \mathbb{R}\}$ is precompact in $H^1_{x,y}$ modulo $\mathbb{R}^3$-translations. Let $(\tau_n)_n \subset \mathbb{R}$ be an arbitrary time sequence. Then (3.38) implies

$$\|u_c\|_{Z((-\infty, \tau_n))} = \|u_c\|_{Z(\tau_n, \infty)} = \infty.$$ 

The claim follows by applying Lemma 3.23 to $(u_c(\tau_n))_n$. 

We end this section by establishing some useful properties of the minimal blow-up solution $u_c$. 

We then obtain

$$A_R(u_c(t)) = 4 \int \left( \partial_{x_j}^2 \chi \left( \frac{x}{R} \right) - 2 \right) |\partial_{x_j} u_c|^2 \, dx dy + 4 \sum_{j \neq k} \int_{R \leq |x| \leq 2R} \partial_{x_j} \partial_{x_k} \chi \left( \frac{x}{R} \right) \partial_{x_j} u_c \partial_{x_k} u_c \, dx dy$$

$$- \frac{1}{R^2} \int \Delta^2 \chi \left( \frac{x}{R} \right) |u_c|^2 \, dx dy - \int \left( \Delta \chi \left( \frac{x}{R} \right) - 6 \right) |u_c|^4 \, dx dy.$$
for some $C_1 > 0$. By Lemma 3.25 we know that there exists some $\delta > 0$ such that
\[ \inf_{t \in \mathbb{R}} (8Q(u_c(t))) \geq 8\delta =: 2\eta_1 > 0. \] (3.42)

From Lemma 3.25 it also follows that there exists some $R_0 \geq 1$ such that
\[ \int_{|x+x(t)| \geq R_0} |\nabla x,y u_c(t)|^2 + |u_c(t)|^2 + |u_c(t)|^4 \, dx dy \leq \frac{\eta_2}{C_1}. \]

Thus for any $R \geq R_0 + \sup_{t \in [t_0,t_1]} |x(t)|$ with some to be determined $t_0, t_1 \in [0,\infty)$, we have
\[ \partial_t^2 z_R(t) \geq \eta_1 \] (3.43)
for all $t \in [t_0, t_1]$. By Lemma 3.26 we know that for any $\eta_2 > 0$ there exists some $t_0 \gg 1$ such that $|x(t)| \leq \eta_2 t$ for all $t \geq t_0$. Now set $R = R_0 + \eta_2 t_1$. Integrating (3.43) over $[t_0, t_1]$ yields
\[ \partial_t z_R(t_1) - \partial_t z_R(t_0) \geq \eta_1 (t_1 - t_0). \] (3.44)

Using (3.40), Cauchy-Schwarz and Lemma 3.22 we have
\[ |\partial_t z_R(t)| \leq C_2 D^* R = C_2 D^* (R_0 + \eta_2 t_1) \] (3.45)
for some $C_2 = C_2(D^*) > 0$. (3.44) and (3.45) give us
\[ 2C_2 D^* (R_0 + \eta_2 t_1) \geq \eta_1 (t_1 - t_0). \]

Setting $\eta_2 = \frac{\eta_1}{4C_2}$, dividing both sides by $t_1$ and then sending $t_1$ to infinity we obtain $\frac{1}{2} \eta_1 \geq \eta_1$, which implies $\eta_1 \leq 0$, a contradiction. This completes the proof. \qed

### 3.9 Finite time blow-up below ground states

In the final subsection we give the proof of Theorem 1.3.

**Proof of Theorem 1.3.** The proof makes use of the classical Glassey’s virial arguments [26]. In the context of normalized ground states, we shall invoke the same idea from the proof of [5, Thm. 1.5] to show the claim. We firstly prove the following statement: for $\phi \in H_{x,y}^1$ satisfying $E(\phi) < m_{M(\phi)}$ and $Q(\phi) < 0$ one has
\[ Q(\phi) \leq E(\phi) - m_{M(\phi)}. \] (3.46)

Indeed, from Lemma 2.4 we know that there exists some $t^* \in (0,1)$ such that $Q(\phi^{t^*}) = 0$ and \[ \frac{d}{ds}(E(\phi^s))(1) = Q(\phi^s) \] for $s \in (t^*, 1)$. Then
\[ E(\phi) = E(\phi^1) = E(\phi^{t^*}) + \int_{t^*}^1 \frac{d}{ds}(E(\phi^s))(s) \, ds \geq E(\phi^{t^*}) + (1 - t^*) \frac{d}{ds}(E(\phi^s))(1) \]
\[ = E(\phi^{t^*}) + (1 - t^*)Q(\phi) > m_{M(\phi)} + Q(\phi), \]
which implies (3.46). Next, using the same arguments as in the proof of Lemma 3.19 we infer that $Q(\phi(t)) < 0$ for all $t$ in the maximal lifespan of $u$, thus also $Q(\phi(t)) \leq m_{M(u)} - E(u)$. We now define
\[ V(t) := \int |x|^2 |u(t)|^2 \, dx dy. \]
By using the same approximation arguments as in the proof of [16, Prop. 6.5.1] we know that $|x|u(t) \in L^2_{x,y}$ for all $t$ in the maximal lifespan of $u$. Direct calculation (which is similar to the one given in the proof of Theorem 1.2) yields
\[ \partial_t^2 V(t) = 8Q(u(t)) \leq 8(m_{M(u)} - E(u)) < 0. \]
This particularly implies that $t \mapsto V(t)$ is a positive and concave function simultaneously. Hence the function $t \mapsto V(t)$ can not exist for all $t \in \mathbb{R}$ and the desired claim follows. \qed
Acknowledgements
The author acknowledges the funding by Deutsche Forschungsgemeinschaft (DFG) through the Priority Programme SPP-1886 (No. NE 21382-1). The author is also grateful to Zehua Zhao for some stimulating discussions.

References

[1] Ardila, A. H. Scattering of the energy-critical NLS with dipolar interaction, 2020, 2010.16354.
[2] Ardila, A. H., and Murphy, J. The cubic-quintic nonlinear schrödinger equation with inverse-square potential, 2021.
[3] Barron, A. On global-in-time Strichartz estimates for the semiperiodic Schrödinger equation. Anal. PDE 14, 4 (2021), 1125–1152.
[4] Barron, A., Christ, M., and Pausader, B. Global endpoint Strichartz estimates for Schrödinger equations on the cylinder $\mathbb{R} \times \mathbb{T}$. Nonlinear Anal. 206 (2021), Paper No. 112172, 7.
[5] Bellazzini, J., and Jeanjean, L. On dipolar quantum gases in the unstable regime. SIAM J. Math. Anal. 48, 3 (2016), 2028–2058.
[6] Bellazzini, J., Jeanjean, L., and Luo, T. Existence and instability of standing waves with prescribed norm for a class of Schrödinger-Poisson equations. Proc. Lond. Math. Soc. (3) 107, 2 (2013), 303–339.
[7] Bényi, A., and Oh, T. The Sobolev inequality on the torus revisited. Publ. Math. Debrecen 83, 3 (2013), 359–374.
[8] Bourgain, J. Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations. Geom. Funct. Anal. 3, 2 (1993), 107–156.
[9] Bourgain, J. Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II. The KdV-equation. Geom. Funct. Anal. 3, 3 (1993), 209–262.
[10] Brézis, H., and Gallouet, T. Nonlinear Schrödinger evolution equations. Nonlinear Anal. 4, 4 (1980), 677–681.
[11] Brézis, H., and Kato, T. Remarks on the Schrödinger operator with singular complex potentials. J. Math. Pures Appl. (9) 58, 2 (1979), 137–151.
[12] Burq, N., Gérard, P., and Tzvetkov, N. Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds. Amer. J. Math. 126, 3 (2004), 569–605.
[13] Burq, N., Gérard, P., and Tzvetkov, N. Bilinear eigenfunction estimates and the nonlinear Schrödinger equation on surfaces. Invent. Math. 159, 1 (2005), 187–223.
[14] Burq, N., Gérard, P., and Tzvetkov, N. Multilinear eigenfunction estimates and global existence for the three dimensional nonlinear Schrödinger equations. Ann. Sci. École Norm. Sup. (4) 38, 2 (2005), 255–301.
[15] Caffarelli, L. A., Gidas, B., and Spruck, J. Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. Comm. Pure Appl. Math. 42, 3 (1989), 271–297.
[16] Cazenave, T. Semilinear Schrödinger equations, vol. 10 of Courant Lecture Notes in Mathematics. New York University, Courant Institute of Mathematical Sciences, New York: American Mathematical Society, Providence, RI, 2003.
[17] Cheng, X., Guo, Z., Hwang, G., and Yoon, H. Global well-posedness and scattering of the two dimensional cubic focusing nonlinear Schrödinger system, 2022, 2202.10757.
[18] Cheng, X., Guo, Z., Yang, K., and Zhao, L. On scattering for the cubic defocusing nonlinear Schrödinger equation on the waveguide $\mathbb{R}^2 \times \mathbb{T}$. Rev. Mat. Iberoam. 36, 4 (2020), 985–1011.
[19] Cheng, X., Guo, Z., and Zhao, Z. On scattering for the defocusing quintic nonlinear Schrödinger equation on the two-dimensional cylinder. *SIAM J. Math. Anal.* 52, 5 (2020), 4185–4237.

[20] Cheng, X., Zhao, Z., and Zheng, J. Well-posedness for energy-critical nonlinear Schrödinger equation on waveguide manifold. *J. Math. Anal. Appl.* 494, 2 (2021), Paper No. 124654, 14.

[21] Colliander, J., Keel, M., Staffilani, G., Takaoka, H., and Tao, T. Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$. *Ann. of Math.* (2) 167, 3 (2008), 767–865.

[22] de Laire, A., Gravejat, P., and Smets, D. Minimizing travelling waves for the gross-pitaevskii equation on $\mathbb{R} \times T$, 2022.

[23] Dodson, B. Global well-posedness and scattering for the focusing, cubic Schrödinger equation in dimension $d = 4$. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 26, 3 (2009), 917–941.

[24] Glassey, R. T. On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations. *J. Math. Phys.* 18, 9 (1977), 1794–1797.

[25] Herr, S. The quintic nonlinear Schrödinger equation on three-dimensional Zoll manifolds. *Amer. J. Math.* 135, 5 (2013), 1271–1290.

[26] Herr, S., Tataru, D., and Tzvetkov, N. Global well-posedness of the energy-critical defocusing nonlinear Schrödinger equations in curved spaces. *Anal. PDE* 5, 4 (2012), 705–746.

[27] Jeanjean, L. Existence of solutions with prescribed norm for semilinear elliptic equations. *Nonlinear Anal.* 28, 10 (1997), 1633–1659.

[28] Kengne, E., Vaillancourt, R., and Malomed, B. A. Bose–einstein condensates in optical lattices: the cubic–quintic nonlinear schrödinger equation with a periodic potential. *Journal of Physics B: Atomic, Molecular and Optical Physics* 41, 20 (2008), 205202.

[29] Kenig, C. E., and Merle, F. Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case. *Invent. Math.* 166, 3 (2006), 645–675.
[40] Killip, R., Murphy, J., and Visan, M. Scattering for the cubic-quintic NLS: crossing the virial threshold. *Siam J. Math. Anal.* 53, 5 (2021), 5803–5812.

[41] Killip, R., Oh, T., Pocovnicu, O., and Visan, M. Solitons and scattering for the cubic-quintic nonlinear Schrödinger equation on $\mathbb{R}^3$. *Arch. Ration. Mech. Anal.* 225, 1 (2017), 469–548.

[42] Le Coz, S. A note on Berestycki-Cazenave’s classical instability result for nonlinear Schrödinger equations. *Adv. Nonlinear Stud.* 8, 3 (2008), 455–463.

[43] Lions, P.-L. The concentration-compactness principle in the calculus of variations. The locally compact case. I. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 1, 2 (1984), 109–145.

[44] Luo, Y. Large data global well-posedness and scattering for the focusing cubic nonlinear Schrödinger equation on $\mathbb{R}^2 \times \mathbb{T}$, 2022.

[45] Luo, Y. Normalized ground states and threshold scattering for focusing NLS on $\mathbb{R}^d \times \mathbb{T}$ via semiviral-free geometry, 2022.

[46] Luo, Y. Sharp scattering for the cubic-quintic nonlinear Schrödinger equation in the focusing-focusing regime. *J. Funct. Anal.* 283, 1 (2022), Paper No. 109489, 34.

[47] Luo, Y. On sharp scattering threshold for the mass-energy double critical NLS via double track profile decomposition. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* (to appear).

[48] Ryckman, E., and Visan, M. Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equation in $\mathbb{R}^{1+4}$. *Amer. J. Math.* 129, 1 (2007), 1–60.

[49] Schneider, T. *Nonlinear Optics in Telecommunications*. Springer Science & Business Media, Berlin Heidelberg, 2013.

[50] Snyder, A., and Love, J. *Optical Waveguide Theory*. Springer Science & Business Media, Berlin Heidelberg, 2012.

[51] Soave, N. Normalized ground states for the NLS equation with combined nonlinearities: the Sobolev critical case. *J. Funct. Anal.* 279, 6 (2020), 108610, 43.

[52] Struwe, M. *Variational methods*, second ed., vol. 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1996. Applications to nonlinear partial differential equations and Hamiltonian systems.

[53] Tarulli, M., and Venkov, G. Scattering for systems of $N$ weakly coupled NLS equations on $\mathbb{R}^d \times M^2$ in the energy space. *Pliska Stud. Math.* 26 (2016), 239–252.

[54] Terracini, S., Tzvetkov, N., and Visciglia, N. The nonlinear Schrödinger equation ground states on product spaces. *Anal. PDE* 7, 1 (2014), 73–96.

[55] Tzvetkov, N., and Visciglia, N. Small data scattering for the nonlinear Schrödinger equation on product spaces. *Comm. Partial Differential Equations* 37, 1 (2012), 125–135.

[56] Tzvetkov, N., and Visciglia, N. Well-posedness and scattering for nonlinear Schrödinger equations on $\mathbb{R}^d \times \mathbb{T}$ in the energy space. *Rev. Mat. Iberoam.* 32, 4 (2016), 1163–1188.

[57] Weinstein, M. I. Nonlinear Schrödinger equations and sharp interpolation estimates. *Comm. Math. Phys.* 87, 4 (1982/83), 567–576.

[58] Yang, K., and Zhao, L. Global well-posedness and scattering for mass-critical, defocusing, infinite dimensional vector-valued resonant nonlinear Schrödinger system. *SIAM J. Math. Anal.* 50, 2 (2018), 1593–1655.

[59] Yu, X., Yue, H., and Zhao, Z. Global Well-posedness for the focusing cubic NLS on the product space $\mathbb{R} \times \mathbb{T}^3$. *SIAM J. Math. Anal.* 53, 2 (2021), 2243–2274.

[60] Yu, X., Yue, H., and Zhao, Z. On the decay property of the cubic fourth-order schrödinger equation, 2022.
[61] Yue, H. Global well-posedness for the energy-critical focusing nonlinear Schrödinger equation on $T^4$. *J. Differential Equations* 280 (2021), 754–804.

[62] Zhao, Z. Global well-posedness and scattering for the defocusing cubic Schrödinger equation on waveguide $\mathbb{R}^2 \times T^2$. *J. Hyperbolic Differ. Equ.* 16, 1 (2019), 73–129.

[63] Zhao, Z. On scattering for the defocusing nonlinear Schrödinger equation on waveguide $\mathbb{R}^m \times T$ (when $m = 2, 3$). *J. Differential Equations* 275 (2021), 598–637.

[64] Zhao, Z., and Zheng, J. Long time dynamics for defocusing cubic nonlinear Schrödinger equations on three dimensional product space. *SIAM J. Math. Anal.* 53, 3 (2021), 3644–3660.