A NOTE ON TOROIDALIZATION:
THE PROBLEM OF RESOLUTION OF SINGULARITIES
OF MORPHISMS IN THE LOGARITHMIC CATEGORY

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§0. Introduction

0.1. Prologue. The purpose of this note is to discuss the following problem of “toroidalization”: Given a (proper) morphism \( f : X \to Y \) between nonsingular varieties, can we find sequences of blowups with smooth centers \( \pi_X : X' \to X \) and \( \pi_Y : Y' \to Y \) such that the induced map \( f' : X' \to Y' \) is a toroidal morphism (See §1 for the precise definition of a morphism to be toroidal.)?

\[
\begin{array}{ccc}
X & \xrightarrow{\pi_X} & X' \\
\downarrow f & & \downarrow f' \\
Y & \xleftarrow{\pi_Y} & Y'
\end{array}
\]

The problem of toroidalization evolved along with the problem of semi-stable reduction and with the problem of factorization of birational maps, among others. The motivation behind is that, via toroidalization, we should be able to reduce the original problem to a combinatorial one, utilizing the correspondence between the geometry of toric varieties (toroidal embeddings) and the combinatorics of convex bodies. As such, it may seem as though the problem of toroidalization is a mere working hypothesis. The theme of this note is, however, to recognize its intrinsic importance, by presenting and reformulating it as the problem of resolution of singularities of morphisms in the logarithmic category. (See §1 for the precise meaning.) This view point was first communicated to the author by Prof. Dan Abramovich, though apparently it had been known to the experts for a while (cf. Kato[20]Abramovich-Karu[4]).

0.2. A naive question. A naive question can be posed in the following way: Given a (proper and surjective) morphism \( f : X \to Y \) between nonsingular varieties, by choosing suitable “modifications” \( \pi_X : X' \to X \) and \( \pi_Y : Y' \to Y \), how “nice”
can one make the induced morphism \( f' : X' \to Y' \)? (We require that we choose modifications \( \pi_X \) and \( \pi_Y \) in such a way that \( f' \) is a morphism, which a priori is just a rational map.) Of course, in order to make this question meaningful, one has to make explicit the mathematical specifications of the words “modifications” and “nice”.

When \( \dim Y = 1 \) and we specify the morphism \( f' \) to be “nice” if every fiber is reduced with simple normal crossings, while the modification \( \pi_Y \) is restricted to a finite morphism (between nonsingular varieties) and the modification \( \pi_X \) is restricted to a sequence of blowups with smooth centers (possibly via normalization after base change), the question becomes the problem of semi-stable reduction (over a base of dimension one).

When \( \dim Y = 1 \) and we specify the morphism \( f' \) to be “nice” if every fiber has only simple normal crossings, while the modification \( \pi_Y \) is restricted to an isomorphism and the modification \( \pi_X \) is restricted to a sequence of blowups with smooth centers, the question becomes the problem of (embedded) resolution of hypersurface singularities.

When \( f \) is birational and we specify the morphism \( f' \) to be “nice” if it is an isomorphism, while the modifications \( \pi_X \) and \( \pi_Y \) are restricted to sequences of blowups with smooth centers, the question is the strong factorization conjecture of birational maps.

It should be warned, however, that the way we pose our question is very naive: for certain choices of the specifications of the word “modifications” or “nice”, we often obtain a question to which the answer is no in general. This can be observed, for example, when we specify the morphism \( f' \) to be “nice” if it is flat, while the modifications \( \pi_X \) and \( \pi_Y \) are restricted to sequences of blowups with smooth centers. This flattening process by blowups with smooth centers does not exist in general, even for a generically finite morphism between nonsingular surfaces \( f : X \to Y \), as is demonstrated by a famous example by Abhyankar[1] for the failure of simultaneous resolution of singularities. (The reader is also invited to see an easy toric example by T. Katsura in Oda[27].) It is, meanwhile, an interesting question whether the above flattening process by blowups with smooth centers exists or not, under the extra condition that the original \( f \) is a morphism with connected fibers. (The author does not know whether the answer is affirmative or negative.)

The problem of toroidalization appears as one form of the questions in the above setting, where we specify \( f' \) to be “nice” if \( f' \) is toroidal, while the modifications \( \pi_X \) and \( \pi_Y \) are restricted to sequences of blowups with smooth centers.

It should be mentioned and emphasized that Abramovich-Karu[4] succeeds in making the morphism \( f' \) toroidal, starting from a given proper morphism \( f \), but by not restricting the modifications \( \pi_X \) and \( \pi_Y \) to be only sequences of blowups with smooth centers.

0.3. **Algorithm.** The reformulation of toroidalization as log resolution, though simple and straightforward as it is, naturally leads one to the following algorithmic approach (See §2 for details.):

Step 1: Characterize the locus where the morphism is not toroidal as the logarithmic ramification locus, which can then be computed using the logarithmic differential forms.
TOROIDALIZATION

Step 2: Choose the center of blowup downstairs on $Y$, which should be contained in the image of the logarithmic ramification locus under $f$.

Step 3: Apply an algorithm for canonical principalization to the pull-back upstairs on $X$ of the defining ideal of the center chosen in Step 2 downstairs on $Y$.

If the induced morphism is toroidal, then we stop. If not, we go back to Step 1 with the original morphism replaced by the induced one.

Termination: Show that the algorithm terminates after finitely many steps.

In fact, in the case where $\dim X = \dim Y = 2$, there is essentially no choice involved in the above algorithm and hence is uniquely determined. Thus the only remaining problem is to show termination. We accomplish this goal by providing a local analysis of the algorithm and then showing that the logarithmic ramification locus decreases (under certain order, with respect to which the descending chain condition is satisfied).

A discussion of the higher dimensional case will be published elsewhere.

0.4. A possible approach to the factorization problem. The problem of toroidalization was conceived as a possible approach to the (strong) factorization conjecture of birational maps (between nonsingular varieties into a sequence of blowups and blowdowns with smooth centers) in two steps:

Step I. Given a proper birational morphism $f : X \to Y$ (if you start with a birational map $f : X \dasharrow Y$, then use the elimination of indeterminacy by Hironaka to reach the stage where $f$ can be assumed to be a morphism), apply toroidalization to make $f' : X' \to Y'$ toroidal.

Step II. Factor the toroidal birational morphism $f' : X' \to Y'$ into a sequence consisting of blowups with smooth centers immediately followed by blowdowns with smooth centers, by applying the toroidal version of a combinatorial algorithm for strong factorization of toric birational maps.

(Today the strong factorization conjecture remains open, while the weak factorization conjecture, which is strong enough for most of the applications, is a theorem achieving Steps I and II in a slightly weaker sense:

Step I: Given a proper birational morphism $f : X \to Y$, apply the theory of birational cobordism by Wlodarczyk[29] and torification by Abramovich-Karu-Matsuki-Wlodarczyk[6] to decompose $f$ as a composite of toroidal birational maps $f_i : X_i \dasharrow Y_i$ for $i = 0, ..., l$ (where $X = X_0, Y_i = X_{i+1}$ for $i = 0, ..., l-1$, and $Y_l = Y$).

Step II: Apply the toroidal version of the algorithm for weak factorization by Morelli (cf. Abramovich-Matsuki-Rashid[5]Matsuki[23]Wlodarczyk[28]) to each toroidal birational map $f_i : X_i \dasharrow Y_i$.

We fall short of realizing strong factorization, since in Step I the toroidal structures for $f_i$’s are not compatible with each other in general, and since in Step II we know only of the algorithm for weak (but not strong) factorization even for toric birational maps (in dimension three or more) for the moment.)

0.5. Work of S.D.Cutkosky. Even at the time when, inspired by the work of A.J. de Jong, we could only daydream of the above approach to the problem of factorization, S.D. Cutkosky[10] made a remarkable announcement of a proof for local factorization of birational maps in dimension 3. (The local version of the problem of factorization replaces a birational map between nonsingular varieties $X$ and $Y$ with birational local rings dominated by a fixed valuation, and replaces blowups along
smooth centers with monoidal transforms followed by localization subordinate to the valuation.) After Cutkosky gave a series of lectures on his breakthrough at Purdue University, we communicated our approach to global factorization via Steps I and II as above, suggesting that the local version of toroidalization may be called “monomialization” and that the first non-trivial and plausible test case of toroidalization should be checked when dim $X = 3$ and dim $Y = 2$.

In a sequence of papers [9][10][11], Cutkosky brilliantly establishes the monomialization in arbitrary dimension, thus completing Step I in the local case.

Recently Karu[19] succeeded in providing an algorithm for local strong factorization of toric (toroidal) birational maps in arbitrary dimension, extending the earlier work of Christensen[9] in dimension 3.

Thus the local strong factorization of birational maps, known also as the Abhyankar conjecture, is now a theorem according to the general scheme of establishing Steps I and II.

Moreover, Cutkosky[12] also establishes the global version, i.e., the toroidalization when dim $X = 3$ and dim $Y = 2$.

0.6. Contents of our paper. The simple purpose of this paper is to give a proof of toroidalization in dimension 2, guided by the principle toroidalization = log resolution, which has been the main point of our theme from the conception of the problem (cf. Abramovich-Karu-Matsuki-Wlodarczyk[6] Matsuki[24]). It is the belief (prejudice) of the author that only thorough this reformulation the ultimate solution would be achieved by finding a much-sought-for inductive structure needed to prove the general case of toroidalization.

The contents of the paper are as follows.

In §1, we present our formulation of the problem of toroidalization as the problem of resolution of singularities of morphisms in the logarithmic category. The key is the study of the behavior of the logarithmic differential forms. In §2, we present an essentially unique algorithm for toroidalization in the case where dim $X = \text{dim} Y = 2$. In §3, we present a proof for toroidalization in dimension 2. Several proofs are known for toroidalization in dimension 2 by now. However, all the existing ones use the strong factorization theorem of a (proper) birational morphism (via Castelnuovo’s contractibility criterion of a $(-1)$-curve) in the process of verification. This has a fatal defect: not only one cannot hope to extend the process to higher dimension where the factorization of a proper birational morphism into blowups with smooth centers fails to hold, but also it would bring a logical loop in our approach to solving the problem of factorization through toroidalization, even in dimension 2. They also use the properness assumption of $f$ in an essential way, and as a result, their analysis is local on $Y$ but global on $X$ in nature. In contrast, we provide a detailed but rather elementary analysis of the algorithm, which is completely local both on $Y$ and on $X$. As a consequence, we show toroidalization in dimension 2 without the properness assumption of $f$. We do not use the strong factorization theorem in the process. Therefore, establishing Steps I and II of the general scheme, we give a new proof of the strong factorization of birational maps in dimension 2.

The way we show that the algorithm for toroidalization terminates is also guided by the principle, rather than setting up invariants in an adhoc way. We show that the logarithmic ramification locus decreases (under certain order, with respect to
which the descending chain condition is satisfied) in the process.

0.7. Assumption on the base field. In this paper, we assume that the base field \( k \) is an algebraically closed field of characteristic zero. The assumption of \( k \) being algebraically closed is purely for the simplicity of the presentation: the “canonical” nature of our algorithm (and of the algorithm for principalization of ideals) implies that the process prescribed over \( k \) is equivariant under the action of the Galois group \( \text{Gal}(\overline{k}/k) \) even when \( k \) is not algebraically closed, and hence that it is actually defined over \( k \). Therefore, this assumption can be removed without much further ado. The assumption of \( k \) being of characteristic zero, however, is essential, because the problem of toroidalization as we formulate is too restrictive in positive characteristic (Some easy counter examples to this restrictive formulation can be found in positive characteristic. See, e.g. Cutkosky-Piltant[13].) and hence needs modification.

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§1. Formulation of toroidalization as resolution of singularities of morphisms in the logarithmic category

1.1. **Outline of this section.** In this section, we formulate the problem of toroidalization and the problem of resolution of singularities of morphisms in the logarithmic category, and show that a solution to the latter would imply a solution to the former. (It could be said that, modulo some minor technical points, the two problems are essentially equivalent.)

For this purpose, we define when a morphism $f : X \to Y$ between nonsingular varieties is toroidal, or in the presence of the prescribed structures as nonsingular toroidal embeddings, when a morphism $f : (U_X, X) \to (U_Y, Y)$ between nonsingular toroidal embeddings is toroidal (with respect to the given toroidal structures $(U_X, X)$ and $(U_Y, Y)$). We also define, in the presence of the logarithmic structures as nonsingular toroidal embeddings, when a morphism $f : (U_X, X) \to (U_Y, Y)$ in the logarithmic category is (log) smooth. (In order to make a clear distinction between the notion of a morphism being “smooth” in the usual category and the notion of a morphism being “smooth” in the logarithmic category, we always use the word “log smooth” for the latter.) One of the main points of the section is, though it is quite straightforward, to show that the condition of $f$ being toroidal is equivalent to $f$ being log smooth.

1.2. **Basic definitions.**

1.2.1. **Logarithmic category.** An object in the logarithmic category (of nonsingular toroidal embeddings) is a nonsingular toroidal embedding $(U_X, X)$, i.e., a pair consisting of a nonsingular variety $X$ and a dense open subset $U_X$ such that $D_X = X - U_X$ is a divisor with only simple normal crossings.

A divisor $D_X$ with only simple normal crossings consists of smooth irreducible components and satisfies the condition that at any closed point $p \in D_X$ there exists an open analytic neighborhood $p \in U_p \subset X$ with a system of regular parameters $(x_1, ..., x_d)$ of $\hat{O}_{X, p}$ such that

$$D_X \cap U_p = \{ \prod_{m \in M} x_m = 0 \}$$

for some subset $M \subset \{1, ..., d = \dim X\}$.

A morphism $f : (U_X, X) \to (U_Y, Y)$ in the logarithmic category is a morphism between nonsingular toroidal embeddings such that:

(a) $f : X \to Y$ is a dominant morphism between nonsingular varieties,

(b) $D_X = f^{-1}(D_Y)$,

(c) $f|_{U_X} : U_X \to U_Y$ is a smooth morphism.

1.2.2. **Definition of a morphism being toroidal.** Let $f : (U_X, X) \to (U_Y, Y)$ be a morphism in the logarithmic category of nonsingular toroidal embeddings. We say $f$ is toroidal if for any $p \in X$ with $q = f(p) \in Y$ there exist a toric (equivariant) morphism $\varphi : (T_V, V) \to (T_W, W)$ between nonsingular toric varieties with $T_V \subset V$ and $T_W \subset W$ indicating the tori, and points $p_V \in V$ and $q_W = \varphi(p_V) \in W$, such that we have isomorphisms of the completions of the local rings

$$\sigma : \hat{O}_{X, p} \sim \hat{O}_{V, p_V}$$

$$\tau : \hat{O}_{Y, q} \sim \hat{O}_{W, q_W}$$
with a commutative diagram
\[
\begin{array}{c}
\widehat{O}_{X,p} \xrightarrow{\sigma} \widehat{O}_{V,p} \\
\uparrow f^* \quad \uparrow \varphi^* \\
\widehat{O}_{Y,q} \xrightarrow{\tau} \widehat{O}_{W,q}
\end{array}
\]
which induces, via inclusions, another commutative diagram among the ideals defining the boundary divisors
\[
\begin{array}{c}
\widehat{I}_{D_{X,p}} \xrightarrow{\sim} \widehat{I}_{D_{V,q}} \\
\uparrow f^* \quad \uparrow \varphi^* \\
\widehat{I}_{D_{Y,q}} \xrightarrow{\sim} \widehat{I}_{D_{W,q}}
\end{array}
\]

We say that a morphism \( f : X \to Y \) between nonsingular varieties is toroidal if we can introduce the structures \((U_X, X)\) and \((U_Y, Y)\) of nonsingular toroidal embeddings, i.e., we can find divisors \( D_X = X - U_X \subset X \) and \( D_Y = Y - U_Y \subset Y \) with only simple normal crossings, such that \( f : (U_X, X) \to (U_Y, Y) \) is a toroidal morphism in the logarithmic category.

We also invite the reader to look atAbramovich-Karu\[4\] Kempf-Knudsen-Mumford-Saint-Donat\[21\] for reference.

1.2.3. Definition of a morphism being log smooth. Let \( f : (U_X, X) \to (U_Y, Y) \) be a morphism in the logarithmic category of nonsingular toroidal embeddings. We say \( f \) is log smooth if the natural homomorphism
\[
f^*\{\wedge^{\dim Y} \Omega^1_Y(\log D_Y)\} \wedge^{\dim X - \dim Y} \Omega^1_X(\log D_X) \to \wedge^{\dim X} \Omega^1_X(\log D_X)
\]
is surjective.

1.3. Equivalence of “toroidal” and “log smooth”.

Proposition 1.3.1. Let \( f : (U_X, X) \to (U_Y, Y) \) be a morphism in the logarithmic category of nonsingular toroidal embeddings. Then \( f \) is toroidal if and only if it is log smooth.

Proof. We only demonstrate a proof in the case where \( \dim X = \dim Y = 2 \). We make a further assumption that at the point \( p \in D_X \subset X \) of our concern two irreducible components of \( D_X \) meet and also that at \( q = f(p) \in D_Y \subset Y \) two irreducible components of \( D_Y \) meet. (In the terminology of \S3, the points \( p \) and \( q \) are of type \( 2_p \) and \( 2_q \).) Though simple, the consideration of this typical case captures the essence of the idea. (We refer the reader to Kato\[20\] for a general proof, though it is not so much more difficult to deal with the general case via local analysis than with the typical case.)

We choose a system of regular parameters \((x_1, x_2)\) of \( \widehat{O}_{X,p} \) in an analytic neighborhood \( U_p \) of \( p \) (resp. \((y_1, y_2)\) of \( \widehat{O}_{Y,q} \) in an analytic neighborhood \( U_q \) of \( q \)) such that it is compatible with the logarithmic structure, i.e., \( D_X \cap U_p = \{x_1 x_2 = 0\} \) (resp. \( D_Y \cap U_q = \{y_1 y_2 = 0\} \)).
By condition (b) $D_X = f^{-1}(D_Y)$ imposed on a morphism in the logarithmic category, we have

$$\begin{align*}
&f^*y_1 = u \cdot x_1^a x_2^b \\
f^*y_2 = v \cdot x_1^c x_2^d
\end{align*}$$

where $a, b, c, d \in \mathbb{Z}_{\geq 0}$ and $u, v \in \hat{\mathcal{O}}_{X, p}^\times$ are units.

We compute

$$f^*(d \log y_1 \wedge d \log y_2) = f^* \left( \frac{dy_1}{y_1} \wedge \frac{dy_2}{y_2} \right)$$

$$= d \log f^*y_1 \wedge d \log f^*y_2$$

$$= \left( \frac{du}{u} + a \frac{dx_1}{x_1} + b \frac{dx_2}{x_2} \right) \wedge \left( \frac{dv}{v} + c \frac{dx_1}{x_1} + d \frac{dx_2}{x_2} \right)$$

$$= r_{\log} \cdot \frac{dx_1 \wedge dx_2}{x_1 \cdot x_2}$$

where

$$r_{\log} = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \text{(higher terms i.e. } = 0 \text{ at } p).$$

Therefore, we conclude

- $f$ being log smooth
  $$\iff R_{\log} := \text{div}(r_{\log}) = 0$$
  $$\iff \exists (x_1, x_2) \& (y_1, y_2) \text{ s.t. } \begin{cases} f^*y_1 = u \cdot x_1^a x_2^b \\ f^*y_2 = v \cdot x_1^c x_2^d \end{cases}$$
  where $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0$ and $u, v \in \hat{\mathcal{O}}_{X, p}^\times$ are units.

- $\iff \exists (x_1, x_2) \& (y_1, y_2) \text{ s.t. } \begin{cases} f^*y_1 = x_1^a x_2^b \\ f^*y_2 = x_1^c x_2^d \end{cases}$ where $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0$

- $\iff f$ being toroidal.

It is obvious that the third condition implies the second, while one can see that the second implies the third by replacing $x_1$ with $u^a v^c \cdot x_1$ and $x_2$ with $u^b v^d \cdot x_2$, where the equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

has a (unique) solution $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in M_2(\mathbb{Q})$ because $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0$.

This completes the proof.

1.4. **Formulation of the main conjectures.**

1.4.1. **Toroidalization Conjecture.** Let $f : X \to Y$ be a dominant morphism between nonsingular varieties. Then there should exist sequences of blowups with smooth centers $\pi_X : X' \to X$ and $\pi_Y : Y' \to Y$ such that the induced map $f' : X' \to Y'$ is a toroidal morphism.
1.4.2. Resolution of singularities of morphisms in the logarithmic category. Let $f : (U_X, X) \to (U_Y, Y)$ be a morphism in the logarithmic category of nonsingular toroidal embeddings. Then there exist sequences of blowups with permissible centers $\pi_X : (U_X', X') \to (U_X, X)$ and $\pi_Y : (U_Y', Y') \to (U_Y, Y)$ such that the induced map $f' : (U_X', X') \to (U_Y', Y')$ is log smooth. (Note that a center is called permissible if it is smooth and locally defined by a part of a system regular parameters compatible with the log structure.)

1.5. Conclusion. An affirmative solution to the problem of resolution of singularities of morphisms in the logarithmic category implies an affirmative solution to the toroidalization conjecture.

In fact, let $f : X \to Y$ be a dominant morphism between nonsingular varieties. If $f$ is smooth, we can take $U_X = X$ and $U_Y = Y$ for the assertion of the toroidalization conjecture. If $f$ is not smooth, we look at the natural homomorphism

$$f^* \{ \wedge^{\dim Y} \Omega_Y^1 \} \wedge^{\dim X - \dim Y} \Omega_X^1 \to \mathcal{I}_R \otimes \wedge^{\dim X} \Omega_X^1 \hookrightarrow \wedge^{\dim X} \Omega_X^1$$

where $\mathcal{I}_R$ is the ideal defining the ramification locus $R$. We take a sequence of blowups with smooth centers $\tau_Y : \tilde{Y} \to Y$ so that $D_{\tilde{Y}} := \tau_Y^{-1}(f(R))$ is a divisor with only simple normal crossings. Then we take a sequence of blowups with smooth centers $\sigma_X : \tilde{X} \to X$ so that the induced rational map $\tilde{f} : \tilde{X} \to \tilde{Y}$ is a morphism. Finally, we take a sequence of blowups with smooth centers $\tau_X : \tilde{X} \to X$ so that $D_{\tilde{X}} := \tau_X^{-1} \tilde{f}^{-1} D_{\tilde{Y}}$ is a divisor with only simple normal crossings. We have now a morphism $\tilde{f} : (U_{\tilde{X}} := \tilde{X} \setminus D_{\tilde{X}}, \tilde{X}) \to (U_{\tilde{Y}} := \tilde{Y} \setminus D_{\tilde{Y}}, \tilde{Y})$ in the logarithmic category of nonsingular toroidal embeddings. Now it is clear that a solution for resolution of singularities of the morphism $\tilde{f}$ in the logarithmic category would imply a solution for toroidalization of the original morphism $f$ by construction.
§2. Algorithm for toroidalization

2.1. Algorithm for toroidalization in the case where $\dim X = \dim Y = 2$.

The purpose of this section is to describe an algorithm for toroidalization of a dominant morphism $f : X \to Y$ between nonsingular varieties in the case where $\dim X = \dim Y = 2$. We make a couple of remarks at the end of the section regarding possible algorithms in higher dimension.

2.1.1. Setting. As in the formulation of the problem of toroidalization in §1, let $f : (U_X, X) \to (U_Y, Y)$ be a morphism in the logarithmic category of nonsingular toroidal embeddings in the case where $\dim X = \dim Y = 2$, that is to say:

(a) $f : X \to Y$ is a dominant morphism between nonsingular varieties of $\dim X = \dim Y = 2$,
(b) $D_X = f^{-1}(D_Y)$, and
(c) the restriction of $f$ to the open subset $U_X$, denoted by $f|_{U_X}$, is a smooth morphism.

2.1.2. Description of the algorithm. Our algorithm for toroidalization proceeds as follows:

Step 1

We write down the natural homomorphism among the logarithmic differential forms

$$f^* \{ \wedge^{\dim Y} \Omega_Y^1(\log D_Y) \} \wedge^{\dim X - \dim Y} \Omega_X^1(\log D_X) \to \mathcal{I}_{R_{log}} \otimes \wedge^{\dim X} \Omega_X^1(\log D_X)$$

where $\mathcal{I}_{R_{log}}$ is the ideal defining the logarithmic ramification locus.

Note that in our case where $\dim X = \dim Y = 2$, or more generally when $f : X \to Y$ is a generically finite morphism, we have the usual logarithmic ramification formula between the log canonical divisors

$$K_X + D_X = f^*(K_Y + D_Y) + R_{log},$$

where $R_{log}$ is the logarithmic ramification divisor (cf. Iitaka[17]). The natural homomorphism above corresponds to the inclusion of line bundles

$$0 \to \mathcal{O}_X(f^*(K_Y + D_Y)) = \mathcal{O}_X(-R_{log} + K_X + D_X)$$

$$\hookrightarrow \mathcal{O}_X(K_X + D_X).$$

Step 2

We look at $f(R_{log})$, the image of the logarithmic ramification locus.

The special feature, which is a consequence of the assumption $\dim Y = 2$, is that $f(R_{log})$ consists of finitely many points.

Blowup $Y$ with center $f(R_{log})$ (or more precisely speaking, blowup $Y$ along the defining ideal $\mathcal{I}_{f(R_{log})}$ of the image of the logarithmic ramification locus with the reduced structure).
Apply an algorithm for canonical principalization (See Remark 2.2.2 for the precise meaning.) to the ideal \( f^{-1}(I_{f(R_{\text{log}})}) \cdot \mathcal{O}_X \):

\[
(U_X, X) = (U_{X_0}, X_0) \xrightarrow{\text{cp}_1} (U_{X_1}, X_1)
\]

\[
f = f_0 \downarrow f_1
\]

\[
(U_Y, Y) = (U_{Y_0}, Y_0) \xleftarrow{\text{bp}_1} (U_{Y_1}, Y_1)
\]

Note that by the universal property of blowup for principalization, the induced rational map \( f_1 \) is guaranteed to be a morphism.

If the induced morphism \( f_1 : (U_{X_1}, X_1) \to (U_{Y_1}, Y_1) \) is toroidal, we end the algorithm.

If the induced morphism \( f_1 : (U_{X_1}, X_1) \to (U_{Y_1}, Y_1) \) is not toroidal, then we go back to Step 1 of the algorithm with \( f := f_1 \) and repeat the process.

### 2.2. Remarks on the algorithm.

#### 2.2.1. A morphism is always toroidal over \( Y \) in codimension one.

It is straightforward to observe that, if \( \dim Y = 1 \), then \( R_{\text{log}} = \emptyset \) and hence also \( f(R_{\text{log}}) = \emptyset \).

In fact, let \( p \in D_X \subset X \) be an arbitrary closed point with \( q = f(p) \in D_Y \subset Y \) its image. Let \( f^* : \widehat{\mathcal{O}}_{Y,q} \to \widehat{\mathcal{O}}_{X,p} \) be the induced homomorphism between the completions of the local rings. We take some systems of regular parameters \((x_1, ..., x_d)\) of \( \widehat{\mathcal{O}}_{X,p} \) and \((y_1)\) of \( \widehat{\mathcal{O}}_{Y,q} \), compatible with the logarithmic structures. (That is to say, for some analytic neighborhoods \( U_p \) of \( p \) and \( U_q \) of \( q \), we have

\[
D_X \cap U_p = \{ \prod_{m \in M} x_m = 0 \}
\]

\[
D_Y \cap U_q = \{ y_1 = 0 \}
\]

for a subset \( M \subset \{1, ..., d\} \).

We observe

\[
f \text{ is toroidal } \iff \exists \ (x_1, ..., x_d) \ & \ (y_1) \ \text{ s.t.} \ \ f^* y_1 = x_1^{a_1} \cdots x_d^{a_d} = \prod x_i^{a_i} \text{ with } a_i \geq 0 \ \forall i \ \text{ and } a_{i_o} \neq 0 \text{ for some } i_o
\]

\[
\iff \exists \ (x_1, ..., x_d) \ & \ (y_1) \ \text{ s.t.} \ \ f^* y_1 = u \cdot x_1^{a_1} \cdots x_d^{a_d} = u \cdot \prod x_i^{a_i} \text{ with } a_i \geq 0 \ \forall i \ \text{ and } a_{i_o} \neq 0 \text{ for some } i_o,
\]

where \( u \) is a unit.

(It is obvious that the second condition implies the last, while one can see that the last condition implies the second by, e.g., replacing \( x_{i_o} \) with \( u^{1/i_o} \cdot x_{i_o} \), where \( u^{1/i_o} \in \widehat{\mathcal{O}}_{X,p} \) exists because of the assumption of the base field \( k \) being algebraically closed of characteristic zero.)
The last condition obviously holds by condition (b) \( D_X = f^{-1}(D_Y) \) imposed on a morphism in the logarithmic category.

Therefore, \( f \) is toroidal.

This observation immediately implies that, for an arbitrary dominant morphism \( f : (U_X, X) \to (U_Y, Y) \) in the logarithmic category, there is no locus of logarithmic ramification over a generic point of codimension one, i.e.,

\[
\text{codim}_Y f(R_{\log}) \geq 2.
\]

This is why \( f(R_{\log}) \) consists of finitely many points in the case where \( \dim Y = 2 \).

2.2.2. Canonical principalization. By an algorithm for canonical principalization, we mean a specific and fixed algorithm which, to a given ideal \( \mathcal{I} \) on a nonsingular toroidal embedding \((U_X, X)\) with support contained in the boundary divisor, i.e., \( \text{Supp} \mathcal{O}_X/\mathcal{I} \subset D_X = X \setminus U_X \), assigns a uniquely determined sequence of blowups

\[
(U_X, X) = (U_{X_0}, X_0) \overset{\pi_1}{\leftarrow} (U_{X_1}, X_1) \leftarrow \ldots
\]

\[
\ldots \overset{\pi_{i-1}}{\leftarrow} (U_{X_{i-1}}, X_{i-1}) \overset{\pi_i}{\leftarrow} (U_{X_i}, X_i) \leftarrow \ldots
\]

with permissible centers \( Y_{i-1} \subset D_{X_{i-1}} \subset X_{i-1} \) (The adjective “permissible” means, by definition, that \( Y_{i-1} \) is smooth and that for any closed point \( p \in Y_{i-1} \subset D_{X_{i-1}} \subset X_{i-1} \) there exists an analytic neighborhood \( U_p \) with a system of regular parameters \((x_1, \ldots, x_d)\) of \( \widehat{\mathcal{O}_{X_{i-1}, p}} \) such that

\[
D_{i-1} \cap U_p = \left\{ \prod_{m \in M} x_m = 0 \right\}
\]

\[
Y_{i-1} \cap U_p = \cap_{m \in N} \{ x_m = 0 \}
\]

for some subsets \( M, N \subset \{ 1, \ldots, d = \dim X \} \).

We require as a result of principalization that the ideal \( \pi^{-1}\mathcal{I} \cdot \mathcal{O}_{X_i} \) is principal (where \( \pi = \pi_1 \circ \cdots \circ \pi_1 \)), i.e., for any closed point \( p \in X_i \) there exists an analytic neighborhood \( U_p \) with a system of regular parameters \((x_1, \ldots, x_d)\) of \( \widehat{\mathcal{O}_{X_i, p}} \) such that

\[
D_i \cap U_p = \left\{ \prod_{m \in M} x_m = 0 \right\}
\]

\[
\pi^{-1}\mathcal{I} \cdot \mathcal{O}_{X_i}|_{U_p} = \left( \prod_{m \in L} x_m^{a_m} \right)
\]

for some subsets \( M, L \subset \{ 1, \ldots, d = \dim X \} \).

The adjective “canonical” is used to add the following two more requirements to the algorithm:

1. (Stability under pull-back by smooth morphisms) If \( f : (U_X, X) \to (U_Y, Y) \) is a morphism in the logarithmic category between nonsingular toroidal embeddings with \( f : X \to Y \) being a smooth morphism (in the usual sense), then the process for principalization of the ideal \( \mathcal{I}_X = f^*\mathcal{I}_Y \subset f^*\mathcal{O}_Y = \mathcal{O}_X \) specified by the algorithm
on \((U_X, X)\) should coincide with the pull-back (i.e., the one obtained by taking the Cartesian product “\(\times_Y X\)”) of the process for principalization of the ideal \(I_Y\) specified by the algorithm on \((U_Y, Y)\).

It should be noted that this requirement of the algorithm for principalization being stable under pull-back by smooth morphisms implies equivariance: if a group (finite or algebraic) acts on \((U_X, X)\) and the ideal \(I\) is invariant under the action, then the process of canonical principalization is equivariant, i.e., the action lifts to the blowups in the process. This can be seen easily by applying this requirement to the two smooth morphisms, \(pr: X \times G \to X\) the projection and \(\mu: X \times G \to X\) the multiplication map representing the action, and by observing that the process of canonical principalization on \(X \times G\) coincides both with the pull-back by \(pr\) and with the pull-back by \(\mu\).

2. (Analytic nature of the algorithm) Suppose that \((U_{X, \alpha}, X_{\alpha})\) with \(I_{\alpha}\) and \((U_{X, \beta}, X_{\beta})\) with \(I_{\beta}\) are given as above. Suppose that we have closed points \(p_{\alpha} \in X_{\alpha}\) and \(p_{\beta} \in X_{\beta}\) with an isomorphism between the completions
\[
\hat{O}_{X_{\alpha}, p_{\alpha}} = \mathcal{O}_{X_{\alpha}, p_{\alpha}} \otimes \hat{O}_{X_{\alpha}, p_{\alpha}} \xrightarrow{\sim} \mathcal{O}_{X_{\beta}, p_{\beta}} \otimes \hat{O}_{X_{\beta}, p_{\beta}} = \hat{O}_{X_{\beta}, p_{\beta}}
\]
which induces isomorphisms via inclusions
\[
\hat{I}_{\alpha} = I_{\alpha} \otimes \hat{O}_{X_{\alpha}, p_{\alpha}} \xrightarrow{\sim} I_{\beta} \otimes \hat{O}_{X_{\beta}, p_{\beta}} = \hat{I}_{\beta}
\]
\[
\hat{I}_{D_{\alpha}} = I_{D_{\alpha}} \otimes \hat{O}_{X_{\alpha}, p_{\alpha}} \xrightarrow{\sim} I_{D_{\beta}} \otimes \hat{O}_{X_{\beta}, p_{\beta}} = \hat{I}_{D_{\beta}}.
\]

Then the isomorphism
\[
(U_{X, \alpha}, X_{\alpha}) \times \text{Spec} \hat{O}_{X_{\alpha}, p_{\alpha}} \xrightarrow{\sim} (U_{X_{\beta}, X_{\beta}}) \times \text{Spec} \hat{O}_{X_{\beta}, p_{\beta}}
\]
lifts to the isomorphisms between the blowups in the process of principalization for the ideal \(I_{\alpha}\) on \((U_{X_{\alpha}, X_{\alpha}})\), pulled back by \(\text{Spec} \hat{O}_{X_{\alpha}, p_{\alpha}}\), and the blowups in the process of principalization for the ideal \(I_{\beta}\) on \((U_{X_{\beta}, X_{\beta}})\), pulled back by \(\text{Spec} \hat{O}_{X_{\beta}, p_{\beta}}\).

3. (Avoidance of blowing up the points where the ideal is already principal) If the ideal \(I\) is already principal at a closed point \(p \in X\), then the algorithm for canonical principalization leaves a neighborhood of \(p\) untouched.

An important consequence of requirements 1 and 2 is that if we apply the algorithm for canonical principalization to a toroidal ideal, then the blowups in the process of principalization are all toroidal as well as the ideals which are the total transforms of the original toroidal ideal.

In our special case where \(\dim X = \dim Y = 2\), there is a unique algorithm for canonical principalization, namely, at each stage on \(X_{i-1}\) we only blow up the points where the total transform \(I_{i-1}\) of the ideal is not principal.

2.2.3 Notation for the logarithmic ramification locus. Given a morphism \(f: (U_X, X) \to (U_Y, Y)\) in the logarithmic category, if we want to emphasize that the logarithmic ramification locus \(R_{\text{log}}\) is associated to the morphism \(f\), we write \(R_{\text{log}, f}\) instead of \(R_{\text{log}}\).
2.3. Further remarks on the algorithm

2.3.1. Case where \( \dim X > 2 \) and \( \dim Y = 2 \) Our algorithm formally makes sense even in the case \( \dim X > 2 \) as long as \( \dim Y = 2 \), adopting any algorithm for canonical principalization, such as the one prescribed by Bierstone-Milman[8] or Encinas-Villamayor[14][15] among others, satisfying the above requirements. However, the author cannot show that the algorithm terminates even in the case \( \dim X = 3 \).

The major difference between the case where \( \dim X > 2 \) and the case where \( \dim X = 2 \) is that in the latter the defining ideal for the logarithmic ramification locus is always principal whereas in the former may not be. In fact, even when \( \dim X > 2 \), if the defining ideal for the logarithmic ramification locus happens to be principal, a combinatorial analysis very similar to the one in the case where \( \dim X = 2 \) would yield toroidalization. Therefore, one could try to formulate an algorithm where we first aim at principalizing the defining ideal for the logarithmic ramification locus. An interpretation of the monumental work by Cutkosky[12] could be given from this point of view. Details will be published elsewhere.

2.3.2. Case where \( \dim Y > 2 \) We are completely at loss in the case where \( \dim Y > 2 \), even in the case where \( \dim X = \dim Y = 3 \) and hence the defining ideal for the logarithmic ramification locus is principal. The stalemate is a result of our (or rather, the author’s) lack of understanding of the expected inducational structure in the scheme of toroidalization, though the logarithmic category should be tailor-made for such an inducational scheme via adjunction.
§3. Local analysis in the case where dim $X = \dim Y = 2$

3.1. **Purpose of this section.** The purpose of this section is to present a detailed local analysis of what happens in Steps 1, 2, 3 of the algorithm 2.1 for toroidalization in the case where dim $X = \dim Y = 2$, under the same setting as in 2.1.1, i.e., $f : (U_X, X) \to (U_Y, Y)$ is a morphism in the logarithmic category of nonsingular toroidal embeddings in the case where dim $X = \dim Y = 2$ so that

(a) $f : X \to Y$ is a dominant morphism between nonsingular varieties of dim $X = \dim Y = 2$,
(b) $D_X = f^{-1}(D_Y)$, and
(c) the restriction of $f$ to the open subset $U_X$, denoted by $f|_{U_X}$, is a smooth morphism.

3.2. **Type and Case Descriptions.** Let $p \in D_X \subset X$ be a closed point and $q = f(p) \in D_Y \subset Y$ its image. In the following analysis, we always choose $q$ to be in the image of the logarithmic ramification locus, i.e., $q \in f(R_{\log})$.

We make our analysis sorting out the types and cases, based upon the following criteria:

- The morphism $f$ is toroidal or not at $p$.
  - We put the letter $T$ when it is toroidal, and the letter $N$ when not.
- Description of the boundary divisors at $p$ and $q$ and their behavior with respect to $f$.

We have the following type descriptions on the boundary divisors at $p$ and $q$ (We take some systems of regular parameters $(x_1, x_2)$ of $\hat{O}_{X, p}$ and $(y_1, y_2)$ of $\hat{O}_{Y, q}$, compatible with the logarithmic structures, in analytic neighborhoods $U_p$ of $p$ and $U_q$ of $q$, respectively):)

2$p$; The point $p$ is the intersection point of two irreducible components $G_1 = \{x_1 = 0\}$ and $G_2 = \{x_2 = 0\}$ with $G_1 \cup G_2 = D_X \cap U_p$ for some analytic neighborhood $U_p$ of $p$.

1$p$; The point $p$ belongs to only one irreducible component $G_1 = \{x_1 = 0\}$ with $G_1 = D_X \cap U_p$ for some analytic neighborhood $U_p$ of $p$, while $G_2 = \{x_2 = 0\} \not\subset D_X \cap U_p$.

2$q$; The point $q$ is the intersection point of two irreducible components $H_1 = \{y_1 = 0\}$ and $H_2 = \{y_2 = 0\}$ with $H_1 \cup H_2 = D_Y \cap U_q$ for some analytic neighborhood $U_q$ of $q$.

1$q$; The point $q$ belongs to only one irreducible component $H_1 = \{y_1 = 0\}$ with $H_1 = D_Y \cap U_q$ for some analytic neighborhood $U_q$ of $q$, while $H_2 = \{y_2 = 0\} \not\subset D_Y \cap U_q$. 
We have the following ten subcases (some of which, namely, Subcase 2\(_p\)1\(_q\)0 and Subcase 1\(_p\)2\(_q\)0, will be doomed impossible), where the third number indicates how many irreducible components of the boundary divisor map onto \(q\):

Subcase 2\(_p\)2\(_q\)0: Neither \(G_1\) nor \(G_2\) maps onto \(q\)
Subcase 2\(_p\)2\(_q\)1: Only one of \(G_1\) or \(G_2\) (say, \(G_1\)) maps onto \(q\)
Subcase 2\(_p\)2\(_q\)2: Both \(G_1\) and \(G_2\) maps onto \(q\)
Subcase 1\(_p\)2\(_q\)0: \(G_1\) does not map onto \(q\) (and hence maps onto \(H_1\) (or \(H_2\))
Subcase 1\(_p\)2\(_q\)1: \(G_1\) maps onto \(q\)
Subcase 1\(_p\)1\(_q\)1: \(G_1\) maps onto \(q\)
Subcase 2\(_p\)1\(_q\)0: Neither \(G_1\) nor \(G_2\) maps onto \(q\)
Subcase 2\(_p\)1\(_q\)1: Only one of \(G_1\) or \(G_2\) (say, \(G_1\)) maps onto \(q\)
Subcase 2\(_p\)1\(_q\)2: Both \(G_1\) and \(G_2\) maps onto \(q\)

3.3. Case by case analysis. Now we carry out a detailed analysis, in each (sub)case, of what happens when we blowup \(q\) in Step 2 and when we take the canonical principalization of \(f^{-1}I_q \cdot \mathcal{O}_X = f^{-1}m_q \cdot \mathcal{O}_X\) in Step 3.

We provide:

A) Coordinate Expression
B) Descriptions of the canonical principalization \(cp_1\) and the induced morphism \(f_1\)
C) Conclusion on the behavior of the logarithmic ramification divisor

It turns out that what weighs more, for the purpose of sorting out the subcases, is the type description of the point \(q\) than that of \(p\), and we regroup the subcases accordingly.

Remark on notation: When we write Subcases *2\(_q\) * T, we indicate that we are dealing with all the subcases of type 2\(_p\)2\(_q\)0, 2\(_p\)2\(_q\)1, 2\(_p\)2\(_q\)2, 1\(_p\)2\(_q\)0, 1\(_p\)2\(_q\)1, collectively, with the extra condition that \(f\) is toroidal at \(p\).

When we write 1\(_p\)2\(_q\)0 without mentioning whether \(f\) is toroidal or not at \(p\), we indicate that we are dealing with all the possibilities in the subcase without a priori putting the extra assumption on whether \(f\) is toroidal or not.
3.3.1) Subcases \( 2_q \ast \): These consist of the following subcases:

- Subcase \( 2_p 2_q 0 \)
- Subcase \( 2_p 2_q 1 \)
- Subcase \( 2_p 2_q 2 \)
- Subcase \( 1_p 2_q 1 \)

Subcase \( 1_p 2_q 0 \): This subcase does not happen. In fact, say \( G_1 \) maps onto \( H_1 \) (and hence does not map onto \( H_2 \)). Note that in a neighborhood \( U_p \) of \( p \) we have \( f^{-1}(H_2) \cap U_p \subset f^{-1}(D_Y) \cap U_p = D_X \cap U_p = G_1 \). Now since \( f^{-1}(H_2) \) is of pure codimension one, being the pull-back of a Cartier divisor, and since \( p \in f^{-1}(H_2) \), we conclude \( G_1 \cap U_p = f^{-1}(H_2) \cap U_p \). But this would imply either \( G_1 \) maps onto \( q \) or maps onto \( H_2 \), a contradiction!

3.3.1.A) Coordinate Expression

By the computation presented in the proof of Proposition 1.3.1, we conclude the following.

Subcases \( 2_p 2_q \ast T \): There exist systems of regular parameters \( (x_1, x_2) \) of \( \widehat{O}_{X,p} \) and \( (y_1, y_2) \) of \( \widehat{O}_{Y,q} \) compatible with the logarithmic structures such that

\[
\begin{align*}
    f^* y_1 &= x_1^a x_2^b \\
    f^* y_2 &= x_1^c x_2^d
\end{align*}
\]

where \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0 \).

Subcases \( 2_p 2_q \ast N \): There exist systems of regular parameters \( (x_1, x_2) \) of \( \widehat{O}_{X,p} \) and \( (y_1, y_2) \) of \( \widehat{O}_{Y,q} \) compatible with the logarithmic structures such that

\[
\begin{align*}
    f^* y_1 &= u \cdot x_1^a x_2^b \\
    f^* y_2 &= v \cdot x_1^c x_2^d
\end{align*}
\]

where \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0 \) with units \( u, v \in \widehat{O}_{X,p}^\times \).

Subcases \( 1_p 2_q \ast \): There exist systems of regular parameters \( (x_1, x_2) \) of \( \widehat{O}_{X,p} \) and \( (y_1, y_2) \) of \( \widehat{O}_{Y,q} \) compatible with the logarithmic structures, as described in 3.2, such that

\[
\begin{align*}
    f^* y_1 &= u \cdot x_1^a \\
    f^* y_2 &= v \cdot x_1^c
\end{align*}
\]

where \( a, c \in \mathbb{Z}_{>0} \) with units \( u, v \in \widehat{O}_{X,p}^\times \).

3.3.1.B) Descriptions of the canonical principalization \( c_p \) and the induced morphism \( f_1 \)

Subcases \( 2_p 2_q \ast T \): We observe that the ideal \( I_q = m_q = (y_1, y_2) \) is toroidal (in a neighborhood of \( q \)) in these subcases and that the ideal \( f^{-1}(y_1, y_2) \cdot O_{X,p} = (x_1^a x_2^b, x_1^c x_2^d) \) is also toroidal (in a neighborhood of \( p \)). Therefore, by remark 2.2.2, the blowup \( b_p : (U_{Y_1}, Y_1) \to (U_{Y_0}, Y_0) \) is toroidal over a neighborhood of \( q \), and so is the canonical principalization \( c_p : (U_{X_1}, X_1) \to (U_{X_0}, X_0) = (U_X, X) \) over a neighborhood of \( p \). Since a composite of toroidal morphisms is again toroidal and so is a composite of a toroidal morphism with the inverse of a toroidal birational morphism (assuming that the composite is again a morphism, which is
the case here), we conclude that \( f_1 = b p_1^{-1} \circ f_0 \circ c p_1 \) stays toroidal on the locus of \((U_{X_1}, X_1)\) which sits over a neighborhood of \(p\).

Subcases \(2p2q \ast N\): In these subcases, as we observed, we have

\[
\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0,
\]
that is to say, the vectors \((a, b), (c, d) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}\) are linearly dependent. (We may choose \((y_1, y_2)\), without loss of generality, so that \(a \geq c, b \geq d\). Note that \((a, b) \neq (0, 0)\) and \((c, d) \neq (0, 0)\).) This implies that the ideal \( f^{-1}(y_1, y_2) \cdot O_{X,p} = (x_1^a x_2^d) \) is already principal in a neighborhood of \(p\). Therefore, by remark 2.2.2 the canonical principalization \( c p_1 \) leaves a neighborhood of \(p\) untouched. Also, by remark 2.2.2, in a neighborhood of \(p\) the map \( f_1 = b p_1^{-1} \circ f_0 \circ c p_1 = b p_1^{-1} \circ f_0 \) is a morphism.

Subcases \(1p2q \ast\): In these subcases, the ideal \( f^{-1}(y_1, y_2) \cdot O_{X,p} = (x_1^a) \) (We may choose \((y_1, y_2)\) without loss of generality so that \(a \leq c\) in the coordinate expression.) is already principal in a neighborhood of \(p\). Therefore, by remark 2.2.2 the canonical principalization \( c p_1 \) leaves a neighborhood of \(p\) untouched. Also, by remark 2.2.2, in a neighborhood of \(p\) the map \( f_1 = b p_1^{-1} \circ f_0 \circ c p_1 = b p_1^{-1} \circ f_0 \) is a morphism.

3.3.1.C) Conclusion on the behavior of the logarithmic ramification divisor

In each of the subcases above, we conclude that over a neighborhood of the point \(q\), since \(b p_1\) is toroidal,

\[
K_{Y_1} + D_{Y_1} = b p_1^*(K_{Y_0} + D_{Y_0}) \quad \text{i.e.} \quad \wedge^{\dim Y_1} \Omega^{1}_{Y_1}(\log D_{Y_1}) = b p_1^* \wedge^{\dim Y_0} \Omega^{1}_{Y_0}(\log D_{Y_0}),
\]
and hence that the logarithmic ramification divisor stays intact, i.e.,

\[
R_{\log, f_0} = R_{\log, f_1}.
\]
3.3.2) Subcase $1_p \{ a \}$: $G_1$ does not map onto $q$ (and hence $G_1$ maps onto $H_1$)

3.3.2.A) Coordinate Expression

In this subcase, we show that $f$ is necessarily toroidal at $p$. (Although this is essentially due to Abhyankar’s lemma, we will not use it explicitly). We will show by some elementary calculation that there exist systems of regular parameters $(x_1, x_2)$ of $\overline{O}_{X,p}$ and $(y_1, y_2)$ on $\overline{O}_{Y,q}$, compatible with the logarithmic structures, such that

\[
\begin{cases}
  f^* y_1 = x_1^a \\
  f^* y_2 = x_2
\end{cases}
\]

using condition (c) $f|_{U_X}$ being smooth, imposed on a morphism in the logarithmic category (cf. 1.2.1).

First, we start with some systems of regular parameters as described in 3.2. Then since $f^{-1}(D_Y) = D_X$ where $D_Y \cap U_q = H_1 = \{y_1 = 0\}$ and $D_X \cap U_p = G_1 = \{x_1 = 0\}$, we have

\[
f^* y_1 = u \cdot x_1^a
\]

for some unit $u \in \overline{O}_{X,p}$. By replacing $x_1$ with $u^{1/a} \cdot x_1$, we may assume

\[
f^* y_1 = x_1^a.
\]

Set

\[
f^* y_2 = \sum_{i > 0, j \geq 0, \det \begin{bmatrix} a & 0 \\ i & j \end{bmatrix} = 0} \alpha_{ij} x_1^i x_2^j + \sum_{i \geq 0, j > 0, \det \begin{bmatrix} a & 0 \\ i & j \end{bmatrix} \neq 0} \alpha_{ij} x_1^i x_2^j,
\]

where there exists $j > 0$ with $\alpha_{0j} \neq 0$, since $x_1$ should not divide $f^* y_2$. As $G_1 = \{x_1 = 0\}$ does not map onto $q$ by the subcase assumption.

We set

\[
j_o := \min \{j; \alpha_{0j} \neq 0\}.
\]

The basic point of computing the ramification is the simple observation that

\[
d(x_1^a x_2^b) \wedge d(x_1^c x_2^d) = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot x_1^{a+c-1} x_2^{b+d-1} dx_1 \wedge dx_2.
\]

We compute

\[
f^* (dy_1 \wedge dy_2) = df^* y_1 \wedge df^* y_2
\]

\[
= d(x_1^a) \wedge \left( \sum_{i > 0, j \geq 0, \det \begin{bmatrix} a & 0 \\ i & j \end{bmatrix} = 0} \alpha_{ij} x_1^i x_2^j + \sum_{i \geq 0, j > 0, \det \begin{bmatrix} a & 0 \\ i & j \end{bmatrix} \neq 0} \alpha_{ij} x_1^i x_2^j \right)
\]

\[
= d(x_1^a) \wedge \left( \sum_{i \geq 0, j > 0, \det \begin{bmatrix} a & 0 \\ i & j \end{bmatrix} \neq 0} \alpha_{ij} x_1^i x_2^j \right)
\]

\[
= (\sum_{i \geq 0, j > 0, \det \begin{bmatrix} a & 0 \\ i & j \end{bmatrix} \neq 0} \det \begin{bmatrix} a & 0 \\ i & j \end{bmatrix} \cdot \alpha_{ij} x_1^{a+i-1} x_2^{j-1}) \cdot dx_1 \wedge dx_2
\]

\[
= x_1^{a-1} \cdot (\sum_{i \geq 0, j > 0, \det \begin{bmatrix} a & 0 \\ i & j \end{bmatrix} \neq 0} \det \begin{bmatrix} a & 0 \\ i & j \end{bmatrix} \cdot \alpha_{ij} x_1^{i} x_2^{j-1}) \cdot dx_1 \wedge dx_2.
\]
Therefore, if \( j_o > 1 \), then \( f \) ramifies along

\[
\left\{ \sum_{i \geq 0, j > 0} a_i 0, \det \begin{bmatrix} a & 0 \\ i & j \end{bmatrix} \neq 0 \cdot \det \begin{bmatrix} a & 0 \\ i & j \end{bmatrix} \cdot \alpha_{ij} x_1^i x_2^{j-1} = 0 \right\}
\]

other than along \( \{x_1 = 0\} \), a contradiction to condition \( f|_{U_X} \) being smooth! Therefore, we conclude

\( j_o = 1 \).

By replacing \( x_2 \) with \( f^* y_2 \), we obtain the desired systems of regular parameters.

Now since neither the divisor \( \{y_2 = 0\} \) nor \( \{x_2 = 0\} \) belongs to the boundary divisor defining the logarithmic structure, it is clear \( f \) is toroidal at \( p \) from the coordinate expression.

3.3.2.B) Descriptions of the canonical principalization \( \text{cp}_{1} \) and the induced morphism \( f_{1} \)

We consider the 1st blowup \( \text{cp}_{1,1} \) of the canonical principalization \( \text{cp}_{1} \), which is a sequence \( \text{cp}_{1} = \text{cp}_{1,1} \circ \cdots \circ \text{cp}_{1,l} \) of blowups specified by the canonical principalization algorithm.

Diagram 3.3.2.B.1.

Claim 3.3.2.B.2.

a) The rational map \( f_{1,1} = b^{-1}p \circ f_0 \circ \text{cp}_{1,1} \) is well-defined (regular), that is to say, the ideal \( (f_0 \circ \text{cp}_{1,1})^{-1}(y_1, y_2) \cdot \mathcal{O}_{X_{1,1}} \) is principal, except possibly at \( p_{1}' \).

b) At \( p_{1}' \), the morphism \( f \circ \text{cp}_{1,1} \) is in Subcase 1. The proof is as follows:

a) Observe that the morphisms \( f_0 \) and \( \text{cp}_{1,1} \) and the ideal \( (y_1, y_2) \) are toroidal, with respect to the modified logarithmic structures obtained by adding \( \{y_2 = 0\} \) and \( \{x_2 = 0\} \) to the original boundary divisors. From this it follows easily that \( f_{1,1} \) is regular, except possibly at \( p_{1} \) and/or \( p_{1}' \).
Now at $p_1$, we have a system of regular parameters $(\frac{x_1}{x_2}, x_2)$ with coordinate expression
\[
\begin{cases}
  (f_0 \circ c_{p_1,1})^* y_1 = (\frac{x_1}{x_2})^a \\
  (f_0 \circ c_{p_1,1})^* y_2 = x_2,
\end{cases}
\]
which immediately implies that the ideal $(f_0 \circ c_{p_1,1})^{-1}(y_1, y_2) \cdot \mathcal{O}_{X_{p_1,1}, p_1} = (x_2)$ is principal at $p_1$ and hence that $f_{1,1}$ is regular at $p_1$.

b) The verification for statement b) is immediate.

3.3.2.C) Conclusion on the behavior of the logarithmic ramification divisor

Claim 3.3.2.C.1.

a) The coefficient of $G_1$ in the logarithmic ramification divisor remains 0, i.e.,
\[
\nu_{G_1}(R_{\log,f_0}) = \nu_{G_1}(R_{\log,f_1}) = 0.
\]
b) The strict transform of the exceptional divisor $E_p$ for $c_{p_1,1}$, obtained by blowing up $p$, does not appear in $R_{\log,f_1}$, i.e.,
\[
\nu_{E_p}(R_{\log,f_1}) = 0.
\]

More generally, none of the irreducible components $E$ of the exceptional divisor for $c_{p_1}$ (i.e., none of the strict transforms of the exceptional divisors for $c_{p_1,1}, ..., c_{p_1,l}$) appear in $R_{\log,f_1}$, i.e.,
\[
\nu_E(R_{\log,f_1}) = 0.
\]

Proof.

a) The verification for statement a) is obvious.

b) Take the standard neighborhood of $q_1$ with a system of regular parameters $(y_1, y_2)$. Then observing
\[
\begin{cases}
  \nu_{E_p}(y_1) = a - 1 \geq 0 \\
  \nu_{E_p}(y_2) = 1 > 0,
\end{cases}
\]
we conclude that the generic point of $E_p$ maps into the standard neighborhood of $q_1$.

Observe that the morphisms $f_0, b_{p_1}, c_{p_1,1}$ are all toroidal with respect to the modified logarithmic structures obtained by adding $\{y_2 = 0\}$ and $\{x_2 = 0\}$ (and their pull-backs) to the original boundary divisors. The original logarithmic structures coincide with the modified ones in a neighborhood of the generic point of $E_p$ and in the standard neighborhood of $q_1$. Therefore, $f_{1,1}$ is toroidal in a neighborhood of the generic point of $E_p$ with respect to the original logarithmic structures, and hence
\[
\nu_{E_p}(R_{\log,f_1}) = 0.
\]

This proves the first part of statement b).

The second part of statement b) follows from Claim 3.3.2.B.2 b) and Claim 3.3.3.C.1 b).
3.3.3) Subcase \(1_p, 1_q\): \(G_1\) maps onto \(q\)

3.3.3.A) Coordinate Expression

First, we start with some systems of regular parameters as described in 3.2. Then since \(f^{-1}(D_Y) = D_X\) where \(D_Y \cap U_q = H_1 = \{y_1 = 0\}\) and \(D_X \cap U_p = G_1 = \{x_1 = 0\}\), we have

\[
f^*y_1 = u \cdot x_1^a \quad \text{for some unit } u \in \mathcal{O}_{X,p}^\times \quad \text{with} \quad a > 0.
\]

By replacing \(x_1\) with \(u^{1/a} \cdot x_1\), we may assume

\[
f^*y_1 = x_1^a.
\]

Set

\[
f^*y_2 = \sum_{i > 0, j \geq 0, \det \begin{bmatrix} a & 0 \\ i & j \end{bmatrix} = 0} \alpha_{ij} x_1^i x_2^j + \sum_{i \geq 0, j > 0, \det \begin{bmatrix} a & 0 \\ i & j \end{bmatrix} \neq 0} \alpha_{ij} x_1^i x_2^j,
\]

where no term of the form \(x_2^j = x_1^0 x_2^j\) \((i = 0, j > 0)\) appears, since \(x_1\) has to divide \(f^*y_2\).

Now since \(f|_{U_X}\) must be smooth by condition (c) imposed on a morphism in the logarithmic category (cf. 1.2.1) and by computing the ramification \(f^*(dy_1 \wedge dy_2)/dx_1 \wedge dx_2\) as in 3.3.2.A, we conclude that \(f^*y_2\) must be of the form

\[
f^*y_2 = \sum_{i \geq i_o, j > 0, \alpha_{ij} = 0} \alpha_{ij} x_1^i x_2^j + \sum_{i \geq i_o, j > 0, (i,j) \neq (i_o,1)} \alpha_{ij} x_1^i x_2^j
\]

with \(\alpha_{i_o,1} \neq 0\), where

\[
i_o = \min \{i; \alpha_{ij} \neq 0, \det \begin{bmatrix} a & 0 \\ i & j \end{bmatrix} \neq 0\}
\]

\[
i_s = \min \{i; \alpha_{ij} \neq 0, \det \begin{bmatrix} a & 0 \\ i & j \end{bmatrix} = 0\}.
\]

3.3.3.B) Descriptions of the canonical principalization \(\mathfrak{p}_1\) and the induced morphism \(f_1\)

Case \(i_s \leq i_o\): In this case, we have

\[
\begin{cases}
  f^*y_1 = x_1^a \\
  f^*y_2 = v \cdot x_1^{i_o}
\end{cases}
\quad \text{with } v \in \mathcal{O}_{X,p}^\times \quad \text{unit},
\]

which implies \(f^{-1}(y_1, y_2) \cdot \mathcal{O}_{X,p}^\times = (x_1^a, x_1^{i_o}) = (x_1^{\min\{a, i_o\}})\) is principal. Therefore, \(\mathfrak{p}_1\) is an isomorphism in a neighborhood of \(p\).

Case \(i_s > i_o\): In this case, we have

\[
\begin{cases}
  f^*y_1 = x_1^a \\
  f^*y_2 = x_1^{i_o} \cdot (\sum_{i \geq i_o, j > 0, \alpha_{ij} = 0} \alpha_{ij} x_1^{i-o} x_2^j + \alpha_{i_o,1} x_2 + \sum_{i \geq i_o, j > 0, (i,j) \neq (i_o,1)} \alpha_{ij} x_1^{i-o} x_2^j).
\end{cases}
\]

\[
\begin{cases}
  f^*y_1 = x_1^a \\
  f^*y_2 = x_1^{i_o} \cdot (\sum_{i \geq i_o, j > 0, \alpha_{ij} = 0} \alpha_{ij} x_1^{i-o} x_2^j + \alpha_{i_o,1} x_2 + \sum_{i \geq i_o, j > 0, (i,j) \neq (i_o,1)} \alpha_{ij} x_1^{i-o} x_2^j).
\end{cases}
\]
Replacing $x_2$ by $(\sum_{i\geq i_0, j>0} \alpha_{i,j} x_1^{i-i_0} + \alpha_{i,0} x_2 + \sum_{i\geq i_0, j>0, (i,j)\neq (i_0,1)} \alpha_{i,j} x_1^{i-i_0} x_2^j)$, we have
\[
\begin{cases}
  f^* y_1 = x_1^a \\
  f^* y_2 = x_1^c x_2
\end{cases} \quad \text{with} \quad c = i_0.
\]

Subcase $a < c+1$, i.e., $a \leq c$: In this subcase, the ideal $f^{-1}(y_1, y_2) \cdot \mathcal{O}_{X, p} = (x_1^a)$ is principal. Therefore, $c_1$ is an isomorphism in a neighborhood of $p$.

Subcase $a \geq c+1$: We consider the 1st blowup $c_{p_1,1}$ of the canonical principalization $c_p$, which is a sequence of blowups $c_p = c_{p_1,1} \circ \cdots \circ c_{p,l}$ specified by the canonical principalization algorithm.

Diagram 3.3.3.B.1.

**Claim 3.3.3.B.2.** The canonical principalization $c_p$ is an isomorphism in a neighborhood of $p$, except under the subcase $a \geq c+1$ in the case $i_0 > i_1$. In the exceptional subcase, we claim:

a) The rational map $f_{1,1} = b_{p-1} \circ f_0 \circ c_{p_1,1}$ is well-defined (regular), that is to say, the ideal $(f_0 \circ c_{p_1,1})^{-1}(y_1, y_2) \cdot \mathcal{O}_{X,1,1}$ is principal, except possibly at $p'_1$.

b) At $p'_1$, the morphism $f \circ c_{p_1,1}$ is in Subcase 1 $p'_1 \circ 1 q_1$.

**Proof.**

The first part of the claim is already verified. So we have only to prove the assertions under the subcase $a \geq c+1$ in the case $i_0 > i_1$.

a) Observe that the morphisms $f_0$ and $c_{p_1,1}$ and the ideal $(y_1, y_2)$ are toroidal, with respect to the modified logarithmic structures obtained by adding $\{y_2 = 0\}$ and $\{x_2 = 0\}$ to the original boundary divisors. From this it follows easily that $f_{1,1}$ is regular, except possibly at $p_1$ and/or $p'_1$.

Now at $p_1$, we have a system of regular parameters $(\frac{x_2}{x_2}, x_2)$ with coordinate expression
\[
\begin{cases}
  (f_0 \circ c_{p_1,1})^* y_1 = \left(\frac{x_1}{x_2}\right) x_2^a = \left(\frac{x_1}{x_2}\right)^a x_2^a \\
  (f_0 \circ c_{p_1,1})^* y_2 = \left(\frac{x_1}{x_2}\right)^c x_2 = \left(\frac{x_1}{x_2}\right)^c x_2^{c+1}
\end{cases}
\]
which immediately implies that the ideal \((f_0 \circ \text{cp}_{1,1})^{-1}(y_1, y_2) = ((\frac{d}{y_2})^c x_2^{c+1})\) is principal at \(p_1\) and hence that \(f_{1,1}\) is regular at \(p_1\).

b) The verification for statement b) is immediate.

3.3.3.C) Conclusion on the behavior of the logarithmic ramification divisor

Claim 3.3.3.C.1. a) The coefficient of \(G_1\) in the logarithmic ramification divisor strictly drops. More precisely,
\[
\nu_{G_1}(R_{\log,f_1}) = \begin{cases} 
  i_o - \min\{i_o, i_s\} & \text{if } a \geq \min\{i_o, i_s\} \\
  i_o - a & \text{if } a < \min\{i_o, i_s\} 
\end{cases} < i_o = \nu_{G_1}(R_{\log,f_0}).
\]

b) The canonical principalization \(\text{cp}_1\) is an isomorphism in a neighborhood of \(p\), except under the subcase \(a \geq c + 1\) in the case \(i_s > i_o\). In the exceptional subcase, the strict transform of the exceptional divisor \(E_p\) for \(\text{cp}_{1,1}\), obtained by blowing up \(p\), does not appear in \(R_{\log,f_1}\), that is to say, the coefficient of the strict transform of the exceptional divisor \(E_p\) is 0, i.e.,
\[
\nu_{E_p}(R_{\log,f_1}) = 0.
\]

More generally, none of the irreducible components \(E\) of the exceptional divisor for \(\text{cp}_1\) (i.e., none of the strict transforms of the exceptional divisors for \(\text{cp}_{1,1}\), etc.), appear in \(R_{\log,f_1}\), i.e.,
\[
\nu_{E}(R_{\log,f_1}) = 0.
\]

Proof.

a) Firstly we compute the coefficient \(\nu_{G_1}(R_{\log,f_0})\) of \(G_1\) in the logarithmic ramification divisor \(R_{\log,f_0}\).
\[
\nu_{G_1}(R_{\log,f_0}) = \nu_{G_1}\left(\frac{dy_1}{y_1} \wedge \frac{dy_2}{x_2} / \frac{dx_1}{x_2} \wedge \frac{dx_2}{x_2}\right) \\
= \nu_{G_1}\left(\frac{d(x_1^a)}{x_1} \wedge d \left(\sum_{i_o > 0, j > 0} \alpha_{ij} x_1^i x_2^j / \frac{dx_1}{x_2} \wedge \frac{dx_2}{x_2}\right)\right) \\
= \nu_{G_1}\left(\frac{d(x_1^a)}{x_1} \wedge d(x_1^i x_2) / \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2}\right) = i_o.
\]

Secondly we compute the coefficient of (the strict transform of) \(G_1\) in \(R_{\log,f_1}\).

Subcase \(a \geq \min\{i_o, i_s\}\): In this subcase, we have
\[
\begin{cases} 
  \nu_{G_1}(\frac{y_1}{y_2}) = a - \min\{i_o, i_s\} \geq 0 \\
  \nu_{G_1}(y_2) = \min\{i_o, i_s\} > 0,
\end{cases}
\]
which implies that the generic point of (the strict transform of) \(G_1\) maps under \(f_1\) into the standard neighborhood of \(q_1\) with a system of regular parameters \((\frac{x_1}{y_1}, y_2)\).
Therefore, we compute
\[
\nu_{G_1}(R_{\log,f_1}) = \nu_{G_1}\left(\frac{dy_1}{y_1} \wedge \frac{dy_2}{x_2} / \frac{dx_1}{x_2} \wedge \frac{dx_2}{x_2}\right) \\
= \nu_{G_1}\left(\frac{dy_1}{y_1} \wedge dy_2 / \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2}\right) - \nu_{G_1}(y_2) \\
= i_o - \min\{i_o, i_s\} < i_o.
\]
Subcase $a < \min\{i_o, i_s\}$: In this subcase, we have

$$\begin{cases} 
\nu_{G_1}(y_1) = a > 0 \\
\nu_{G_1}(\frac{y_2}{y_1}) = \min\{i_o, i_s\} - a > 0,
\end{cases}$$

which implies that the generic point of (the strict transform of) $G_1$ maps under $f_1$ into the standard neighborhood of $q_1'$ with a system of regular parameters $(y_1, \frac{y_2}{y_1})$.

Therefore, we compute

$$\nu_{G_1}(R_{\log, f_1}) = \nu_{G_1}\left(\frac{dy_1}{y_1} \land d\left(\frac{y_2}{y_1}\right) / d\left(\frac{x_1}{x_2}\right) \land dx_2/x_2\right)$$

$$= \nu_{G_1}\left(\frac{dy_1}{y_1} \land dx_2/x_1 \land dx_2/x_2\right) - \nu_{G_1}(y_1)$$

$$= i_o - a < i_o.$$

This completes the proof of the statement a).

b) We have only to consider the exceptional subcase $a \geq c + 1$ in the case $i_o > i_1$.

Take the standard neighborhood of $q_1$ with a system of regular parameters $(\frac{y_2}{y_1}, y_2)$. Then observing

$$\begin{cases} 
\nu_{G_1}(\frac{y_1}{y_2}) = a - c > 0 \\
\nu_{G_1}(y_2) = c > 0
\end{cases} \quad \& \quad \begin{cases} 
\nu_{E_p}(\frac{y_1}{y_2}) = a - (c + 1) \geq 0 \\
\nu_{E_p}(y_2) = c + 1 > 0,
\end{cases}$$

we conclude that the generic points of $G_1$ and $E_p$ maps into the standard neighborhood of $q_1$.

Observe that the morphisms $f_0, b_{p_1}, c_{p_{1,1}}$ are all toroidal with respect to the modified logarithmic structures obtained by adding $\{y_2 = 0\}$ and $\{x_2 = 0\}$ (and their strict transforms) to the original boundary divisors. The original logarithmic structures coincide with the modified ones in neighborhoods of the generic points of $G_1$ and $E_p$ and in the standard neighborhood of $q_1$. Therefore, $f_{1,1}$ is toroidal in neighborhoods of the generic points of $G_1$ and $E_p$ with respect to the original logarithmic structures, and hence

$$\nu_{E_p}(R_{\log, f_1}) = 0$$

and

$$\nu_{G_1}(R_{\log, f_1}) = 0 < c = \nu_{G_1}(R_{\log, f_0}).$$

This proves statement a) and the first part of statement b).

The second part of statement b) follows inductively from the first part of the statement b) and Claim 3.3.3.B.2 b).
3.3.4) Subcase $2_{p1}q_0$: Neither $G_1$ nor $G_2$ maps onto $q$ (and hence both $G_1$ and $G_2$ would map onto $H_1$)

This subcase does not happen. (Although this is again essentially due to Abhyankar’s lemma, we will not use it explicitly). We will show by some elementary calculation that, in this subcase, $f|_{U_X}$ cannot be smooth and hence that we violate condition (c) imposed on a morphism in the logarithmic category (cf. 1.2.1).

First, we start with some systems of regular parameters as described in 3.2. Then since $f^{-1}(D_Y) = D_X$ where $D_Y \cap U_q = H_1$ and $D_X \cap U_p = G_1 \cup G_2$, we have

$$f^*y_1 = u \cdot x_1^a x_2^b$$ for some unit $u \in \hat{O}_{X,p}$ with $a > 0, b > 0$.

By replacing $x_1$ and $x_2$ with the ones multiplied by some appropriate units, we may assume

$$f^*y_1 = x_1^a x_2^b$$ with $a > 0, b > 0$.

On the other hand, since neither $G_1$ nor $G_2$ maps onto $q$, neither $x_1$ nor $x_2$ divides $f^*y_2$ (Note that $y_2$ can be chosen in any way, irrelevant to the logarithmic structure, as long as $(y_1, y_2)$ form a system of regular parameters.). In the Taylor expansion of $f^*y^2 = \Sigma a_{ij}x^i_1 x^j_2$ we conclude that there exists $i > 0$ such that $a_{i0} \neq 0$ and that there exists $j > 0$ such that $a_{0j} \neq 0$.

We set

$$i_o := \min \{i; \alpha_{i0} \neq 0\} > 0$$

$$j_o := \min \{i; \alpha_{0j} \neq 0\} > 0.$$ 

Finally we compute

$$f^*(dy_1 \wedge dy_2) = df^*y_1 \wedge df^*y_2$$

$$= (ax_1^{a-1} x_2^b dx_1 + x_1^a b x_2^{b-1} dx_2)$$

$$\wedge (\alpha_{i_o i o} x_1^{i_o-1} dx_1 + \Sigma_{i > i_o} \alpha_{i0} i x_1^{i-1} dx_1$$

$$+ d(\Sigma_{i > j} \alpha_{ij} x_1^i x_2^j)$$

$$+ \Sigma_{j > j_o} \alpha_{0j} j x_2^{j-1} dx_2 + \alpha_{0j_o} j x_2^{j_o-1} dx_2)$$

$$= (-b \alpha_{i_o i o} x_1^{i_o} - \Sigma_{i > i_o} \alpha_{i0} i x_1^{i-o} \cdot x_1^{i_o}$$

$$+ x_1 x_2 \cdot h$$

$$+ a \alpha_{0j_o} j x_2^{j_o} + \Sigma_{j > j_o} a \alpha_{0j} j x_2^{j-o} \cdot x_2^{j_o}$$

$$\cdot x_1^{a-1} x_2^{b-1} dx_1 \wedge dx_2,$$

noting that

$$(ax_1^{a-1} x_2^b dx_1 + x_1^a b x_2^{b-1} dx_2) \wedge d(\Sigma_{i > j} \alpha_{ij} x_1^i x_2^j)$$

is divisible by $x_1^a x_2^b \cdot dx_1 \wedge dx_2$ and can be written as

$$x_1 x_2 \cdot h \cdot x_1^{a-1} x_2^{b-1} dx_1 \wedge dx_2$$ for some $h \in \hat{O}_{X,p}$.

But this implies that $f$ ramifies along

$$\{(-b \alpha_{i_o i o} x_1^{i_o} - \Sigma_{i > i_o} \alpha_{i0} i x_1^{i-o} \cdot x_1^{i_o} + x_1 x_2 \cdot h + a \alpha_{0j_o} j x_2^{j_o} + \Sigma_{j > j_o} a \alpha_{0j} j x_2^{j-o} \cdot x_2^{j_o}) = 0\}$$

other than possibly $\{x_1 = 0\}$ or $\{x_2 = 0\}$. This violates condition (c) $f|_{U_X} = x - (\{x_1 = 0\} \cup \{x_2 = 0\})$ being smooth imposed on a morphism in the logarithmic category.
3.3.5) Subcase 2_p,1_q: \( G_1 \) maps onto \( q \) but \( G_2 \) does not map onto \( q \)

3.3.5.A) Coordinate Expression

First, we start with some systems of regular parameters as described in 3.2. Then since \( f^{-1}(D_Y) = D_X \) where \( D_Y \cap U_q = H_1 \) and \( D_X \cap U_p = G_1 \cup G_2 \), we have

\[
f^*y_1 = u \cdot x_1^a x_2^b \quad \text{for some } \ u \in \mathcal{O}^{\times}_{X,p} \quad \text{with } \ a > 0, b > 0.
\]

By replacing \( x_1 \) and \( x_2 \) with the ones multiplied by some appropriate units, we may assume

\[
f^*y_1 = x_1^a x_2^b \quad \text{with } \ a > 0, b > 0.
\]

Set

\[
f^*y_2 = \sum_{i > 0, j > 0} \det \begin{bmatrix} a & b \end{bmatrix} \alpha_{ij} x_1^i x_2^j + \sum_{i > 0, j \geq 0, \det \begin{bmatrix} a & b \end{bmatrix} \neq 0} \alpha_{ij} x_1^i x_2^j,
\]

where no term of the form \( x_2^j = x_1^i x_2^j \) \((i = 0, j > 0)\) appears, since \( x_1 \) has to divide \( f^*y_2 \) as \( G_1 = \{x_1 = 0\} \) maps onto \( q \), and where there exists \( i > 0 \) with \( \alpha_{i0} \neq 0 \), since \( x_2 \) should not divide \( f^*y_2 \) as \( G_2 = \{x_2 = 0\} \) does not map onto \( q \).

Now since \( f \mid _{U_X} \) must be smooth by condition (c) imposed on a morphism in the logarithmic category (cf. 1.2.1) and by computing the ramification \( f^*(dy_1 \wedge dy_2)/dx_1 \wedge dx_2 \) as in 3.3.2.A, we conclude that \( f^*y_2 \) must be of the form

\[
f^*y_2 = \sum_{(i, j) \geq (i_0, j_0), \det \begin{bmatrix} a & b \end{bmatrix} = 0} \alpha_{ij} x_1^i x_2^j + \alpha_{i_0, 0} x_1^{i_0} + \sum_{i \geq i_0, j > 0, (i, j) \neq (i_0, 0), \det \begin{bmatrix} a & b \end{bmatrix} \neq 0} \alpha_{ij} x_1^i x_2^j,
\]

where

\[
i_0 = \min \{i; \alpha_{i0} \neq 0\}
\]

\[(i_s, j_s) = \min \{(i, j); \alpha_{ij} \neq 0, \det \begin{bmatrix} a & b \end{bmatrix} = 0\}.
\]

3.3.5.B) Descriptions of the canonical principalization \( cp_1 \) and the induced morphism \( f_1 \)

Case \( i_0 \leq i_s \) and \( i_0 \leq a \): In this case, we have

\[
\begin{aligned}
f^*y_1 &= x_1^a x_2^b \\
f^*y_2 &= v \cdot x_1^{i_0}
\end{aligned}
\]

with \( v \in \mathcal{O}^{\times}_{X,p} \), unit,

which implies \( f^{-1}(y_1, y_2) : \mathcal{O}^{\times}_{X,p} = (x_1^{i_0}) \) is principal. Therefore, \( cp_1 \) is an isomorphism in a neighborhood of \( p \).

Case Otherwise, i.e., \( i_0 > i_s \) or \( i_0 > a \): We consider the 1st blowup \( cp_{1,1} \) of the canonical principalization \( cp_1 \), which is a sequence of blowups \( cp_1 = cp_{1,1} \circ \cdots \circ cp_{1,l} \) specified by the canonical principalization algorithm.
Diagram 3.3.5.B.1.

Claim 3.3.5.B.2. The canonical principalization $c_{p_1}$ is an isomorphism in a neighborhood of $p$, except under the case $i_o > i_s$ or $i_o > a$. In the exceptional case, we claim:

a) The rational map $f_{1,1} = b p^{-1} \circ f_0 \circ c_{p_1,1}$ is well-defined (regular), that is to say, the ideal $(f_0 \circ c_{p_1,1})^{-1} (y_1, y_2) \cdot \mathcal{O}_{X_1,1}$ is principal, except possibly at finitely many points on $E_p$.

b) At $p_1$, the morphism $f \circ c_{p_1,1}$ is in Subcase $2_{p_1} 1_q 2$. At $p'_1$, the morphism $f \circ c_{p_1,1}$ is in Subcase $2_{p_1'} 1_q 1$. At any other point $p''_1 (\neq p_1, p'_1)$ on $E$, the morphism $f \circ c_{p_1,1}$ is in Subcase $1_{p''_1} 1_q 1$.

Proof.

Statements a) and b) are obvious.

3.3.5.C) Conclusion on the behavior of the logarithmic ramification divisor

Claim 3.3.5.C.1. a) The coefficient of $G_1$ in the logarithmic ramification divisor strictly drops, while the coefficient of $G_2$ in the logarithmic ramification divisor remains 0. More precisely,

$$\nu_{G_1}(R_{\log, f_1}) = \begin{cases} i_o - \min\{i_o, i_s\} & \text{if } a \geq \min\{i_o, i_s\} \\ i_o - a & \text{if } a < \min\{i_o, i_s\} \end{cases} < i_o = \nu_{G_1}(R_{\log, f_0})$$

and

$$\nu_{G_2}(R_{\log, f_1}) = \nu_{G_2}(R_{\log, f_0}) = 0.$$



b) The canonical principalization is an isomorphism in a neighborhood of $p$, except under the case $i_o > i_s$ or $i_o > a$. In the exceptional case, the coefficient in $R_{\log, f_1}$ of the strict transform of the exceptional divisor $E_p$ for $c_{p_1,1}$, obtained by blowing up $p$, is strictly smaller than the maximum of the coefficients of irreducible components in $R_{\log, f_0}$ in a neighborhood of $p$. More precisely,

$$\nu_{E_p}(R_{\log, f_1}) = \begin{cases} i_o - \min\{i_o, i_s + j_s\} & \text{if } a + b \geq \min\{i_o, i_s + j_s\} \\ i_o - (a + b) & \text{if } a + b < \min\{i_o, i_s + j_s\} \end{cases} < i_o = \nu_{G_1}(R_{\log, f_0}).$$
More generally, the coefficient in $R_{\log,f_1}$ of any irreducible component $E$ of the exceptional divisor for $c\mathbb{P}_1$, is strictly smaller than the maximum of the coefficients of irreducible components in $R_{\log,f_0}$ in a neighborhood of $p$, i.e.,

$$\nu_E(R_{\log,f_1}) < i_o = \nu_{G_1}(R_{\log,f_0}).$$

Proof.

a) Firstly we compute the coefficient $\nu_{G_1}(R_{\log,f_0})$ of $G_1$ in the logarithmic ramification divisor $R_{\log,f_0}$:

$$\nu_{G_1}(R_{\log,f_0}) = \nu_{G_1} \left( \frac{dy_1}{y_1} \wedge dy_2/\frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \right)$$

$$= \nu_{G_1} \left( \frac{d(x_1^a x_2^b)}{x_1^a x_2^b} \wedge d \left( \sum_{i \geq i_o, j \geq 0} \det \left[ \begin{array}{cc} a & b \\ i & j \end{array} \right] \neq 0 \right) \right) \left( \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \right)$$

$$= \nu_{G_1} \left( \frac{d(x_1^a x_2^b)}{x_1^a x_2^b} \wedge \frac{d(x_1^{i_o})}{x_1} \wedge \frac{dx_2}{x_2} \right) = i_o.$$

Secondly we compute the coefficient of (the strict transform of) $G_1$ in $R_{log,f_1}$.

Subcase $a \geq \min\{i_o, i_s\}$: In this subcase, we have

$$\nu_{G_1}(\frac{y_1}{y_2}) = a - \min\{i_o, i_s\} \geq 0$$

$$\nu_{G_1}(y_2) = \min\{i_o, i_s\} > 0,$$

which implies that the generic point of (the strict transform of) $G_1$ maps under $f_1$ into the standard neighborhood of $q_1$ with a system of regular parameters $(\frac{y_1}{y_2}, y_2)$. Therefore, we compute

$$\nu_{G_1}(R_{log,f_1}) = \nu_{G_1} \left( \frac{dy_1}{y_1} \wedge dy_2/\frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \right)$$

$$= \nu_{G_1} \left( \frac{dy_1}{y_1} \wedge dy_2/\frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \right) - \nu_{G_1}(y_2)$$

$$= i_o - \min\{i_o, i_s\} < i_o.$$
This completes the proof of statement a).

b) We have only to consider the exceptional case \( i_o > i_s \) or \( i_o > a \).

We compute the coefficient of (the strict transform of) the exceptional divisor \( E_p \) for \( cp_{1,1} \), obtained by blowing up \( p \), in the logarithmic ramification divisor \( R_{\log,f_1} \).

Subcase \( a + b \geq \min\{i_o, i_s + j_s\} \): In this subcase, we have

\[
\begin{align*}
\nu_{E_p}(y_1/y_2) &= a + b - \min\{i_o, i_s + j_s\} \\
\nu_{E_p}(y_2) &= \min\{i_o, i_s + j_s\} > 0,
\end{align*}
\]

which implies that the generic point of (the strict transform of) \( E_p \) maps under \( f_1 \) into the standard neighborhood of \( q_1 \) with a system of regular parameters \((y_1 y_2, y_2)\).

Therefore, we compute

\[
\nu_{E_p}(R_{\log,f_1}) = \nu_{E_p}\left(\frac{d(y_1/y_2)}{y_2} \wedge \frac{dy_2}{y_2} / \frac{d(y_1/y_2)}{y_2} \wedge \frac{dx_2}{x_2}\right) = \nu_{E_p}\left(\frac{dy_1}{y_1} \wedge dy_2 / \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2}\right) - \nu_{E_p}(y_2) = i_o - \min\{i_o, i_s + j_s\} < i_o.
\]

Subcase \( a + b < \min\{i_o, i_s + j_s\} \): In this subcase, we have

\[
\begin{align*}
\nu_{E_p}(y_1) &= a + b > 0 \\
\nu_{E_p}(y_2) &= \min\{i_o, i_s + j_s\} - (a + b) > 0,
\end{align*}
\]

which implies that the generic point of (the strict transform of) \( E_p \) maps under \( f_1 \) into the standard neighborhood of \( q_1 \) with a system of regular parameters \((y_1, y_2)\).

Therefore, we compute

\[
\nu_{E_p}(R_{\log,f_1}) = \nu_{E_p}\left(\frac{dy_1}{y_1} \wedge d(y_2/y_1) / \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2}\right) = \nu_{E_p}\left(\frac{dy_1}{y_1} \wedge dy_2 / \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2}\right) - \nu_{E_p}(y_1) = i_o - (a + b) < i_o.
\]

This completes the proof of the first part of statement b).

In order to verify the second part of statement b), we look at the points where \((f \circ cp_{1,1})^{-1}(y_1, y_2) \cdot \mathcal{O}_{X_{1,1}}\) is (possibly) not principal.

At \( p_1 \), the morphism \( f \circ cp_{1,1} \) is in Subcase 2, \( 1q2 \) with

\[
\begin{align*}
\nu_{G_1}(R_{\log,f \circ cp_{1,1}}) &= i_o \\
\nu_{E_p}(R_{\log,f \circ cp_{1,1}}) &= i_o.
\end{align*}
\]
Therefore, by Claim 3.3.6.C.1 b) we have for any irreducible component $E$, other
than (the strict transform of) $E_p$, which is exceptional for $cp_1$ and which maps onto
$p_1$ under $cp_{1,2} \circ \cdots \circ cp_{1,t}$

$$\nu_E(R_{\log,f_1}) < \max\{\nu_{G_1}(R_{\log,f_0 cp_{1,1}}), \nu_{E_p}(R_{\log,f_0 cp_{1,1}})\} = i_o = \nu_{G_1}(R_{\log,f_0}).$$

At $p'_1$, the morphism $f \circ cp_{1,1}$ is in Subcase 2$p'_11_q1$ with

$$\nu_{E_p}(R_{\log,f_0 cp_{1,1}}) = i_o.$$

Therefore, by the first part of statement b) and by induction on the length of the
sequence of blowups of points for $cp_1$, we have for any irreducible component $E$, other
than (the strict transform of) $E_p$, which is exceptional for $cp_1$ and which maps onto $p'_1$ under $cp_{1,2} \circ \cdots \circ cp_{1,t}$

$$\nu_E(R_{\log,f_1}) < \nu_{E_p}(R_{\log,f_0 cp_{1,1}}) = i_o = \nu_{G_1}(R_{\log,f_0}).$$

At $p''_1 \in E_p$, other than $p_1$ or $p'_1$, the morphism $(f \circ cp_{1,1})$ is in Subcase 1$p''_11_q1$.
Therefore, by Claim 3.3.3.C.1 b) we have for any irreducible component $E$, other
than (the strict transform of) $E_p$, which is exceptional for $cp_1$ and which maps onto
$p''_1$ under $cp_{1,2} \circ \cdots \circ cp_{1,t}$

$$\nu_E(R_{\log,f_1}) = 0 < \nu_{E_p}(R_{\log,f_0 cp_{1,1}}) = i_o = \nu_{G_1}(R_{\log,f_0}).$$
3.3.6) Subcase 2\textsubscript{p}1\textsubscript{q}2: Both \(G_1\) and \(G_2\) map onto \(q\)

3.3.6.A) Coordinate Expression

First, we start with some systems of regular parameters as described in 3.2. Then since \(f^{-1}(D_Y) = D_X\) where \(D_Y \cap U_q = H_1\) and \(D_X \cap U_p = G_1 \cup G_2\), we have

\[
f^*y_1 = u \cdot x_1^a x_2^b \text{ for some unit } u \in \hat{\mathcal{O}}_{X,p}^\times \text{ with } a > 0, b > 0.
\]

By replacing \(x_1\) and \(x_2\) with the ones multiplied by some appropriate units, we may assume

\[
f^*y_1 = x_1^a x_2^b \text{ with } a > 0, b > 0.
\]

Set

\[
f^*y_2 = \sum_{i>0,j>0, \det \begin{bmatrix} a & b \\ i & j \end{bmatrix} = 0} a_{ij} x_1^i x_2^j + \sum_{i>0,j>0, \det \begin{bmatrix} a & b \\ i & j \end{bmatrix} \neq 0} a_{ij} x_1^i x_2^j,
\]

where \(x_1x_2\) divides \(f^*y_2\), since both \(G_1\) and \(G_2\) maps onto \(q\).

Now since \(f|_{U_X}\) must be smooth by condition (c) imposed on a morphism in the logarithmic category (cf. 1.2.1) and by computing the ramification \(f^*(dy_1 \wedge dy_2)/dx_1 \wedge dx_2\) as in 3.3.2.A, we conclude that \(f^*y_2\) must be of the form

\[
f^*y_2 = \sum_{(i,j) \geq (i_o,j_o), \det \begin{bmatrix} a & b \\ i & j \end{bmatrix} = 0} a_{ij} x_1^i x_2^j
\]

\[+ \alpha_{i_o,j_o} x_1^{i_o} x_2^{j_o} + \sum_{i \geq i_o, j \geq j_o, (i,j) \neq (i_o,j_o), \det \begin{bmatrix} a & b \\ i & j \end{bmatrix} = 0} a_{ij} x_1^i x_2^j,
\]

with \(\alpha_{i_o,j_o} \neq 0\),

where

\[
i_o = \min \{i; \alpha_{ij} \neq 0, \det \begin{bmatrix} a & b \\ i & j \end{bmatrix} \neq 0\}
\]

\[
j_o = \min \{j; \alpha_{ij} \neq 0, \det \begin{bmatrix} a & b \\ i & j \end{bmatrix} \neq 0\}
\]

with \(\det \begin{bmatrix} a & b \\ i_o & j_o \end{bmatrix} \neq 0\), and

\[(i_s, j_s) = \min \{(i,j); \alpha_{ij} \neq 0, \det \begin{bmatrix} a & b \\ i & j \end{bmatrix} = 0\}.
\]

3.3.6.B) Descriptions of the canonical principalization \(cp_1\) and the induced morphism \(f_1\)

Case \(\left\{ \begin{array}{l} i_o \geq i_s \\ j_o \geq j_s \end{array} \right\}\) or \(\left\{ \begin{array}{l} i_o \leq i_s \\ j_o \leq j_s \end{array} \right\}\) or \(\left\{ \begin{array}{l} i_o \leq i_s \\ j_o \leq j_s \end{array} \right\}\) or \(\left\{ \begin{array}{l} i_o \geq i_s \\ j_o \geq j_s \end{array} \right\}\).

In this case, the ideal

\[
f^{-1}(y_1, y_2) \cdot \hat{\mathcal{O}}_{X,p} = (x_1^{i_o} x_2^{j_o}) \text{ or } (x_1^a x_2^b),
\]

or

\[
(x_1^{i_o} x_2^{j_o}),
\]

or

\[
(x_1^a x_2^b), \text{ respectively,}
\]
is principal. Therefore, \( cp_1 \) is an isomorphism in a neighborhood of \( p \).

Case Otherwise:

We consider the 1st blowup \( cp_{1,1} \) of the canonical principalization \( cp_1 \), which is a sequence of blowups \( cp_1 = cp_{1,1} \circ \cdots \circ cp_{1,l} \) specified by the canonical principalization algorithm.

Diagram 3.3.6.B.1.

**Claim 3.3.6.B.2.** The canonical principalization is an isomorphism in a neighborhood of \( p \), except for the case where \( f^{-1}(y_1, y_2) \cdot \mathcal{O}_{X,p} \) is not principal. In the exceptional case, we claim:

a) The rational map \( f_{1,1} = bp^{-1} \circ f_0 \circ cp_{1,1} \) is well-defined (regular), that is to say, the ideal \( (f_0 \circ cp_{1,1})^{-1}(y_1, y_2) : \mathcal{O}_{X_{1,1}} \) is principal, except possibly at finitely many points on \( E_p \).

b) At \( p_1 \) (resp. \( p'_1 \)), the morphism \( f \circ cp_{1,1} \) is in Subcase 2_{p_1,1_q1} (resp. 2_{p'_1,1_q1}). At any other point \( p''_1 \) on \( E_p \), the morphism \( f \circ cp_{1,1} \) is in Subcase 1_{p''_1,1_q1}.

**Proof.**

Statements a) and b) are obvious.

3.3.6.C) Conclusion on the behavior of the logarithmic ramification divisor

**Claim 3.3.6.C.1.** a) The coefficients of \( G_1 \) and \( G_2 \) in the logarithmic ramification divisor strictly drop. More precisely,

\[
\nu_{G_1}(R_{\log,f_1}) = \begin{cases} 
  i_o - \min\{i_o, i_s\} & \text{if } a \geq \min\{i_o, i_s\} \\
  i_o - a & \text{if } a < \min\{i_o, i_s\}
\end{cases} < i_o = \nu_{G_1}(R_{\log,f_0})
\]

\[
\nu_{G_2}(R_{\log,f_1}) = \begin{cases} 
  j_o - \min\{j_o, j_s\} & \text{if } b \geq \min\{j_o, j_s\} \\
  j_o - b & \text{if } b < \min\{j_o, j_s\}
\end{cases} < j_o = \nu_{G_2}(R_{\log,f_0}).
\]

b) The canonical principalization is an isomorphism in a neighborhood of \( p \), except for the case when \( f^{-1}(y_1, y_2) \cdot \mathcal{O}_{X,p} \) is not principal. In the exceptional case where \( f^{-1}(y_1, y_2) \cdot \mathcal{O}_{X,p} \) is not principal, the coefficient in \( R_{\log,f_1} \) of the strict
transform of the exceptional divisor $E_p$ for $c_{p,1}$, obtained by blowing up $p$, is strictly smaller than the maximum of the coefficients of the irreducible components in $R_{\log,f_0}$ in a neighborhood of $p$. More precisely,

$$\nu_{E_p}(R_{\log,f_1}) = \begin{cases} 
i_o + j_o - \min\{i_o + j_o, i_1 + j_1\} & \text{if } a + b \geq \min\{i_o + j_o, i_1 + j_1\} \\ i_o + j_o - (a + b) & \text{if } a + b < \min\{i_o + j_o, i_1 + j_1\} \end{cases} < \max\{i_o, j_o\} = \max\{\nu_{G_1}(R_{\log,f_0}), \nu_{G_2}(R_{\log,f_0})\}.$$ 

More generally, the coefficient in $R_{\log,f_1}$ of any irreducible component $E$ of the exceptional divisor for $c_{p,1}$, is strictly smaller than the maximum of the coefficients of the irreducible components in $R_{\log,f_0}$ in a neighborhood of $p$, i.e.,

$$\nu_E(R_{\log,f_1}) < \max\{\nu_{G_1}(R_{\log,f_0}), \nu_{G_1}(R_{\log,f_0})\} = \max\{i_o, j_o\}.$$ 

**Proof.**

a) Firstly we compute the coefficient $\nu_{G_1}(R_{\log,f_0})$ of $G_1$ in the logarithmic ramification divisor $R_{\log,f_0}$.

$$\nu_{G_1}(R_{\log,f_0}) = \nu_{G_1}(dy_1/y_1 \wedge dy_2/\frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2})$$

$$= \nu_{G_1}\left(\frac{d(x^a_1x^b_2)}{x^a_1x^b_2} \wedge d\left(\sum_{i \geq i_o, j \geq j_o} det\begin{bmatrix} a & b \\ i & j \end{bmatrix} \neq 0 \alpha_{ij}x^i_1x^j_2\right) / \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2}\right)$$

$$= \nu_{G_1}\left(\frac{d(x^a_1x^b_2)}{x^a_1x^b_2} \wedge d(x^a_1x^b_2)/\frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2}\right) = i_o.$$ 

Symmetrically, we compute the coefficient $\nu_{G_2}(R_{\log,f_0})$ of $G_2$ in the logarithmic ramification divisor $R_{\log,f_0}$ to be

$$\nu_{G_2}(R_{\log,f_0}) = j_o.$$ 

Secondly we compute the coefficient of (the strict transform of) $G_1$ in $R_{\log,f_1}$.

Subcase $a \geq \min\{i_o, i_s\}$: In this subcase, we have

$$\begin{cases} \nu_{G_1}(y_1/y_2) = a - \min\{i_o, i_s\} \geq 0 \\ \nu_{G_1}(y_2) = \min\{i_o, i_s\} > 0, \end{cases}$$

which implies that the generic point of (the strict transform of) $G_1$ maps under $f_1$ into the standard neighborhood of $q_1$ with a system of regular parameters $(\frac{y_1}{y_2}, y_2)$. Therefore, we compute

$$\nu_{G_1}(R_{\log,f_1}) = \nu_{G_1}\left(dy_1/y_1 \wedge dy_2/\frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2}\right)$$

$$= \nu_{G_1}\left(dy_1/y_1 \wedge dy_2/\frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2}\right) - \nu_{G_1}(y_2)$$

$$= i_o - \min\{i_o, i_s\} < i_o.$$
Subcase $a < \min \{ i_o, i_s \}$: In this subcase, we have
\[
\begin{align*}
\nu_{G_1}(y_1) &= a > 0 \\
\nu_{G_1}(\frac{y_2}{y_1}) &= \min \{ i_o, i_s \} - a > 0,
\end{align*}
\]
which implies that the generic point of (the strict transform of) $G_1$ maps under $f_1$ into the standard neighborhood of $q'_1$ with a system of regular parameters $(y_1, \frac{y_2}{y_1})$. Therefore, we compute
\[
\nu_{G_1}(R_{\log, f_1}) = \nu_{G_1} \left( \frac{dy_1}{y_1} \land d\left( \frac{y_2}{y_1} \right) \land \frac{d(\frac{x_1}{x_2})}{x_1} \land \frac{dx_2}{x_2} \right) \\
= \nu_{G_1} \left( \frac{dy_1}{y_1} \land dy_2 / \frac{dx_1}{x_1} \land \frac{dx_2}{x_2} \right) - \nu_{G_1}(y_1) \\
= i_o - a < i_o.
\]
Symmetrically, we compute
\[
\nu_{G_2}(R_{\log, f_1}) = \begin{cases} 
   j_o - \min \{ j_o, j_s \} & \text{if } b \geq \min \{ j_o, j_s \} \\
   j_o - b & \text{if } b < \min \{ j_o, j_s \}
\end{cases} < j_o = \nu_{G_2}(R_{\log, f_0}).
\]
This proves statement a).

b) We compute the coefficient of (the strict transform of) the exceptional divisor $E_p$ for $cp_{1,1}$, obtained by blowing up $p$, in the logarithmic ramification divisor $R_{\log, f_1}$.

Subcase $a + b \geq \min \{ i_o + j_o, i_s + j_s \}$: In this subcase, we have
\[
\begin{align*}
\nu_{E_p}(\frac{y_1}{y_2}) &= a + b - \min \{ i_o + j_o, i_s + j_s \} \\
\nu_{E_p}(y_2) &= \min \{ i_o + j_o, i_s + j_s \} > 0,
\end{align*}
\]
which implies that the generic point of (the strict transform of) $E_p$ maps under $f_1$ into the standard neighborhood of $q_1$ with a system of regular parameters $(\frac{y_1}{y_2}, y_2)$. Therefore, we compute
\[
\nu_{E_p}(R_{\log, f_1}) = \nu_{E_p} \left( \frac{dy_1}{y_1} \land d(\frac{y_2}{y_2}) / \frac{dx_1}{x_1} \land \frac{dx_2}{x_2} \right) \\
= \nu_{E_p} \left( \frac{dy_1}{y_1} \land dy_2 / \frac{dx_1}{x_1} \land \frac{dx_2}{x_2} \right) - \nu_{E_p}(y_2) \\
= i_o + j_o - \min \{ i_o + j_o, i_s + j_s \}.
\]
Now we claim that
\[
i_o + j_o - \min \{ i_o + j_o, i_s + j_s \} < \max \{ i_o, j_o \}
\]
unless $f^{-1}(y_1, y_2) \cdot \mathcal{O}_{X, p}$ is already principal.
In fact, if
\[ i_o + j_o \leq i_s + j_s, \]
then
\[ i_o + j_o - \min\{i_o + j_o, i_s + j_s\} = 0 < \max\{i_o, j_o\}. \]
Therefore, we may assume
\[ i_o + j_o > i_s + j_s, \]
in which case we have
\[ i_o + j_o - \min\{i_o + j_o, i_s + j_s\} = (i_o + j_o) - (i_s + j_s) \]
\[ = (i_o - i_s) + (j_o - j_s) \]
\[ = \begin{cases} 
(i_o - i_s) - (j_s - j_o) < i_o & \text{if } j_s \geq j_o \\
(j_o - j_s) - (i_s - i_o) < j_o & \text{if } i_s \geq i_o 
\end{cases} \]
or
\[ f^{-1}(y_1, y_2) \cdot \overline{O}_{X, p} = (x_1^a x_2^b, x_1^i x_2^j) = (x_1^{\min\{a, i\}}, x_2^{\min\{b, j\}}) \]
is principal if \( i_s < i_o \) & \( j_s < j_o \).

Subcase \( a + b < \min\{i_o + j_o, i_s + j_s\} \): In this subcase, we have
\[ \nu_{E_p}(y_1) = a + b > 0 \]
\[ \nu_{E_p}(\frac{y_2}{y_1}) = \min\{i_o + j_o, i_s + j_s\} - (a + b) > 0, \]
which implies that the generic point of (the strict transform of) \( E_p \) maps under \( f_1 \) into the standard neighborhood of \( q_1^\nu \) with a system of regular parameters \( (y_1, \frac{y_2}{y_1}) \).

Therefore, we compute
\[ \nu_{E_p}(R_{log, f_1}) = \nu_{E_p}(\frac{dy_1}{y_1} \wedge d(\frac{y_2}{y_1})(\frac{dy_2}{y_1}) \wedge \frac{dx_2}{x_2}) \]
\[ = \nu_{E_p}(\frac{dy_1}{y_1} \wedge dy_2 \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2}) - \nu_{E_p}(y_1) \]
\[ = i_o + j_o - (a + b). \]

Now we claim that
\[ i_o + j_o - (a + b) < \max\{i_o, j_o\} \]
unless \( f^{-1}(y_1, y_2) \cdot \overline{O}_{X, p} \) is already principal.

In fact, we have
\[ i_o + j_o - (a + b) = (i_o - a) + (j_o - b) \]
\[ = \begin{cases} 
(i_o - a) - (b - j_o) < i_o & \text{if } b \geq j_o \\
(j_o - b) - (a - i_o) < j_o & \text{if } a \geq i_o 
\end{cases} \]
or
\[ f^{-1}(y_1, y_2) \cdot \overline{O}_{X, p} = (x_1^a x_2^b) \]
principal if \( a < i_o \) & \( b < j_o \)
(Nota \( e a < i_s \) & \( b < j_s \)
by the subcase assumption \( a + b < \min\{i_o + j_o, i_s + j_s\} \).)
This proves the first part of statement b).

It remains to prove the second part of statement b).

Suppose \( p_k \in D_{X_{1,k}} \subset X_{1,k} \) is a closed point sitting over \( p = p_0 \in D_X = D_{X_{1,0}} \subset X = X_{1,0} \), i.e.,

\[
\text{cp}_{1,1} \circ \cdots \circ \text{cp}_{1,k}(p_k) = p.
\]

We set

\[
g_k = f \circ (\text{cp}_{1,1} \circ \cdots \circ \text{cp}_{1,k}).
\]

Suppose that \( g_k^{-1}(y_1, y_2) \cdot \mathcal{O}_{X_{1,k}, p_k} \) is not principal, and hence that we blowup \( p_k \) to obtain the exceptional divisor \( E_{p_k} \) for \( \text{cp}_{1,k+1} \), the \((k+1)\)-th stage of the canonical principalization \( \text{cp}_1 = \text{cp}_{1,1} \circ \cdots \circ \text{cp}_{1,k} \circ \text{cp}_{1,k+1} \circ \cdots \circ \text{cp}_{1,l} \).

At \( p_k \), the morphism \( g_k \) is either in Subcase 1 \( p_k 1_q \) 1 or in Subcase 2 \( p_k 1_q 2 \).

If \( g_k \) is in Subcase 1 \( p_k 1_q \) 1 at \( p_k \), then by Claim 3.3.3.C.1 we have

\[
\nu_E(R_{\log, f_1}) = 0 < \max\{i_o, j_o\}
\]

for (the strict transform of) any exceptional divisor \( E \) for \( \text{cp}_{1,k+1} \circ \cdots \circ \text{cp}_{1,l} \) which maps onto \( p_k \).

In order to analyze the case where \( g_k \) is in Subcase 2 \( p_k 1_q 2 \), we prove the following inductive lemma.

**Lemma 3.3.6.C.2.** Let \( p_k \in D_{X_{1,k}} \subset X_{1,k} \) be a point, sitting over \( p \), where \( g_k \) is in Subcase 2 \( p_k 1_q 2 \). Assume that the ideal \( g_k^{-1}(y_1, y_2) \cdot \mathcal{O}_{X_{1,k}, p_k} \) is not principal.

Suppose that we have the following two conditions:

(i) There exists a system of regular parameters \((x_{1,k}, x_{2,k})\) of \( \mathcal{O}_{X_{1,k}, p_k} \), compatible with the logarithmic structure, such that

\[
\begin{align*}
g_k^* y_1 &= x_{1,k}^{a_k} x_{2,k}^{b_k} \quad \text{with} \quad a_k > 0, b_k > 0 \\
\sum_{(i,j) \geq (i_{o,k}, j_{o,k}), \det \begin{bmatrix} a_k & b_k \\ i & j \end{bmatrix} = 0} \alpha_{ij} x_{1,k}^i x_{2,k}^j + \sum_{i \geq i_{o,k} > 0, j \geq j_{o,k} > 0, (i,j) \neq (i_{o,k}, j_{o,k}), \det \begin{bmatrix} a_k & b_k \\ i & j \end{bmatrix} \neq 0} \alpha_{ij} x_{1,k}^i x_{2,k}^j \\
&\quad \text{with} \quad \alpha_{i_{o,k}, j_{o,k}} \neq 0,
\end{align*}
\]

where

\[
i_{o,k} = \min \{i; \alpha_{ij} \neq 0, \det \begin{bmatrix} a_k & b_k \\ i & j \end{bmatrix} \neq 0\}
\]

\[
j_{o,k} = \min \{j; \alpha_{ij} \neq 0, \det \begin{bmatrix} a_k & b_k \\ i & j \end{bmatrix} \neq 0\}
\]

with \( \det \begin{bmatrix} a_k & b_k \\ i_{o,k} & j_{o,k} \end{bmatrix} \neq 0 \), and

\[
(i_{s,k}, j_{s,k}) = \min \{(i, j); \alpha_{ij} \neq 0, \det \begin{bmatrix} a_k & b_k \\ i & j \end{bmatrix} = 0\}.
\]
(ii) One of the following four holds:

\[(a) \quad \{ i_{o,k} > i_{s,k}, j_{o,k} < j_{s,k} \} \quad \text{with} \quad i_{o,k} - i_{s,k} < i_o \quad \text{and} \quad i_{o,k} - a_k < i_o \]

\[(b) \quad \{ i_{o,k} < i_{s,k}, j_{o,k} > j_{s,k} \} \quad \text{with} \quad j_{o,k} - j_{s,k} < j_o \quad \text{and} \quad j_{o,k} - b_k < j_o \]

\[(c) \quad \{ i_{o,k} \leq i_{s,k}, j_{o,k} \leq j_{s,k} \} \quad \text{with} \quad i_{o,k} - a_k < i_o \]

\[(d) \quad \{ i_{o,k} \leq i_{s,k}, j_{o,k} \leq j_{s,k} \} \quad \text{with} \quad j_{o,k} - b_k < j_o \]

Then, $E_{p_k}$ being the exceptional divisor for $c_{p_1,k+1}$, obtained by blowing up $p_k$, we have

$$\nu_{E_{p_k}}(R_{\log,f_1}) < \max\{i_o,j_o\}.$$ 

Moreover, let $p_{k+1} \in D_{X_{1,k+1}} \subset X_{1,k+1}$ be a point over $p_k$ where $g_{k+1}$ is in Subcase 2. Assume $g_{k+1}^{-1}(y_1,y_2) \cdot O_{X_{1,k+1},p_{k+1}}$ is not principal. Then we have the conditions (i) with coordinate expression $(\ast)_{k+1}$ and (ii) $(\ast)_{k+1}$ (derived inductively from the conditions (i) with coordinate expression $(\ast)_k$ and (ii)$_k$).

(Note that in the special case where in $g_0^k y_2$ there is no term $x_{1,0}^n x_{2,0}^m$ with $\det \begin{bmatrix} a_0 & b_0 \\ i & j \end{bmatrix} = 0$ and hence where subsequently in $g_0^k y_2$ there is no term $x_{1,k}^n x_{2,k}^m$ with $\det \begin{bmatrix} a_k & b_k \\ i & j \end{bmatrix} = 0$, we set $i_s,0 = i_s,k = j_s,0 = j_s,k = \infty$ by convention.)

Diagram 3.3.6.C.3.
Proof.

We compute the coefficient \( \nu_{E_p_k} (R_{\log, f_1}) \) of (the strict transform of) the exceptional divisor \( E_{p_k} \) for \( c_{p_1, k+1} \), obtained by blowing up \( p_k \), in the logarithmic ramification divisor \( R_{\log, f_1} \). (We note that the computation is identical to the one given for \( \nu_{E_p} (R_{\log, f_1}) = \nu_{E_p_0} (R_{\log, f_1}) \), except that at the end we compare \( \nu_{E_p_k} (R_{\log, f_1}) \) to \( \max \{ i_o, j_o \} \) but not to \( \max \{ i_o, j_o, j_s, s_k \} \).

Subcase \( a_k + b_k \geq \min \{ i_o, j_o, j_s, s_k \} \): In this subcase, we have

\[
\begin{cases}
\nu_{E_p_k} \left( \frac{y_1}{y_2} \right) = a_k + b_k - \min \{ i_o, j_o, j_s, s_k \} \geq 0 \\
\nu_{E_p_k} (y_2) = \min \{ i_o, j_o, j_s, s_k \} > 0,
\end{cases}
\]

which implies that the generic point of (the strict transform of) \( E_{p_k} \) maps under \( f_1 \) into the standard neighborhood of \( q_1 \) with a system of regular parameters \( (\frac{y_1}{y_2}, y_2) \).

Therefore, we compute

\[
\nu_{E_p_k} (R_{\log, f_1}) = \nu_{E_p_k} \left( \frac{d(y_1)}{\frac{y_1}{y_2}} \wedge \frac{dy_2}{y_2} \wedge \frac{dx_1, k}{x_2, k} \wedge \frac{dx_2, k}{x_2, k} \right) \\
= \nu_{E_p_k} \left( \frac{dy_1}{y_1} \wedge \frac{dy_2}{y_2} \wedge \frac{dx_1, k}{x_1, k} \wedge \frac{dx_2, k}{x_2, k} \right) - \nu_{E_p_k} (y_2) \\
= i_o, j_o - \min \{ i_o, j_o, j_s, s_k \}.
\]

Now we claim that

\[ i_o, j_o - \min \{ i_o, j_o, j_s, s_k \} < \max \{ i_o, j_o \} \]

unless \( g_k^{-1}(y_1, y_2) \cdot \mathcal{O}_{X_{1, k}, p_k} \) is already principal.

In fact, if

\[ i_o, j_o \leq i_s, s_k + j_s, \]

then

\[ i_o, j_o - \min \{ i_o, j_o, j_s, s_k \} = 0 < \max \{ i_o, j_o \}. \]

Therefore, we may assume

\[ i_o, j_o > i_s, s_k + j_s, \]

in which case we have

\[ i_o, j_o - \min \{ i_o, j_o, j_s, s_k \} = (i_o, j_o) - (i_s, s_k) \]

\[
= (i_o, j_o) - (i_s, s_k) + (j_s, j_o) - (j_s, j_o) < i_o \text{ if } j_s, j_o \geq j_o, j_s \]

\[
= (j_s, j_o) - (i_s, s_k) - (i_s, s_k) < j_o \text{ if } i_s \geq j_o, j_s \]

\[
= \begin{cases}
(i_o, j_o) - (j_s, j_o) & \text{if } j_s, j_o \geq j_o, j_s \\
(j_s, j_o) - (i_s, s_k) & \text{if } i_s \geq j_o, j_s
\end{cases}
\]

\[ g_k^{-1}(y_1, y_2) \cdot \mathcal{O}_{X_{1, k}, p_k} = (x_1, k, x_2, k, x_1, k, x_2, k) = (x_1, k, x_2, k, x_1, k, x_2, k) \]

principal if \( i_s, j_s < i_o, j_o \) & \( j_s, j_o < j_o, j_s \).
Remark that, under the assumption

\[ i_{o,k} + j_{o,k} > i_{s,k} + j_{s,k}, \]

when \( g_k^{-1}(y_1, y_2) \cdot \mathcal{O}_{X_1, k, p_k} \) is not principal, we have

\[
\begin{align*}
& j_{s,k} \geq j_{o,k} \implies (\alpha)_k \quad (\text{and hence } i_{o,k} - i_{s,k} < i_o) \\
& i_{s,k} \geq i_{o,k} \implies (\beta)_k \quad (\text{and hence } j_{o,k} - j_{s,k} < j_o).
\end{align*}
\]

Subcase \( a_k + b_k < \min\{i_{o,k} + j_{o,k}, i_{s,k} + j_{s,k}\} \): In this subcase, we have

\[
\begin{align*}
\nu_{E_{p_k}}(y_1) &= a_k + b_k > 0 \\
\nu_{E_{p_k}}\left(\frac{y_2}{y_1}\right) &= \min\{i_{o,k} + j_{o,k}, i_{s,k} + j_{s,k}\} - (a_k + b_k) > 0,
\end{align*}
\]

which implies that the generic point of (the strict transform of) \( E_{p_k} \) maps under \( f_1 \) into the standard neighborhood of \( q'_1 \) with a system of regular parameters \((y_1, \frac{y_2}{y_1})\).

Therefore, we compute

\[
\begin{align*}
\nu_{E_{p_k}}(R_{\log,f_1}) &= \nu_{E_{p_k}}\left(\frac{dy_1}{y_1} \wedge \frac{dy_2}{y_1} \frac{d(x_{1,k}^{1-x_{2,k}})}{x_{2,k}} \wedge \frac{dx_{2,k}}{x_{2,k}}\right) \\
&= \nu_{E_{p_k}}\left(\frac{dy_1}{y_1} \wedge \frac{dy_2}{y_1} \frac{dx_{1,k}}{x_{1,k}} \wedge \frac{dx_{2,k}}{x_{2,k}}\right) - \nu_{E_{p_k}}(y_1) \\
&= i_{o,k} + j_{o,k} - (a_k + b_k).
\end{align*}
\]

Now we claim that

\[ i_{o,k} + j_{o,k} - (a_k + b_k) < \max\{i_o, j_o\} \]

unless \( g_k^{-1}(y_1, y_2) \cdot \mathcal{O}_{X_1, k, p_k} \) is already principal.

In fact, we have

\[
\begin{align*}
i_{o,k} + j_{o,k} - (a_k + b_k) &= (i_{o,k} - a_k) + (j_{o,k} - b_k) \\
&= (i_{o,k} - a_k) - (b_k - j_{o,k}) < i_o & \text{if } b_k \geq j_{o,k} \\
&= (j_{o,k} - b_k) - (a_k - i_{o,k}) < j_o & \text{if } a_k \geq i_{o,k}
\end{align*}
\]

or

\[
\begin{align*}
g_k^{-1}(y_1, y_2) \cdot \mathcal{O}_{X_1, k, p_k} &= (x_{1,k}^{a_k} x_{2,k}^{b_k}) \quad \text{principal if } a_k < i_{o,k} & b_k < j_{o,k} \\
&= \text{by the subcase assumption } a_k + b_k < \min\{i_{o,k} + j_{o,k}, i_{s,k} + j_{s,k}\}.
\end{align*}
\]

Remark that, under the subcase assumption

\[ a_k + b_k < \min\{i_{o,k} + j_{o,k}, i_{s,k} + j_{s,k}\}, \]
when $g_k^{-1}(y_1, y_2) \cdot \mathcal{O}_{X_{1,k+1}}^*$ is not principal, we have

$$b_k \geq j_{o,k} \implies (\gamma)_k \quad \text{(and hence } i_{o,k} - a_k < i_o)$$

$$a_k \geq i_{o,k} \implies (\delta)_k \quad \text{(and hence } j_{o,k} - b_k < j_o).$$

Therefore, we conclude

$$\nu_{E_{p_k}}(R_{\log, f_1}) < \max\{i_o, j_o\}.$$ 

Finally, it remains to prove the assertions $(i)_{k+1}$ with $(*)_{k+1}$ and $(ii)_{k+1}$ on $p_{k+1} \in D_{X_{1,k+1}} \subset X_{1,k+1}$, which is a point over $p_k$ and where $g_{k+1}$ is in Subcase 2$_{p_{k+1}}$.  When we blow up $p_k$, there appear two points (lying over $p_k$) where $g_{k+1}$ is in Subcase 2$_{p_{k+1}}$, one having the standard neighborhood with a system of regular coordinates $(\text{the } X_{1,k+1})$ and the other having the standard neighborhood with a system of regular coordinates $(x_{1,k+1}, x_{2,k+1})$.  By symmetry, we have only to check the assertions on $p_{k+1}$ having the standard neighborhood with a system of regular coordinates $(x_{1,k+1}, x_{2,k+1}) = (\frac{x_{1,k}}{x_{2,k}^2}, x_{2,k})$.

Firstly, by substituting

$$\begin{cases} x_{1,k} = x_{1,k+1}x_{2,k+1} \\ x_{2,k} = x_{2,k+1} \end{cases}$$

into $(*)_k$, it is clear that the condition $(i)_{k+1}$ is satisfied with the system of regular coordinates $(x_{1,k+1}, x_{2,k+1})$ providing the coordinate expression $(*)_{k+1}$.

Secondly, we check the condition $(ii)_{k+1}$, assuming that $g_{k+1}^{-1}(y_1, y_2) \cdot \mathcal{O}_{X_{1,k+1}, p_{k+1}}^*$ is not principal.

Observe that

$$a_{k+1} = a_k,$$

$$b_{k+1} = a_k + b_k,$$

$$i_{o,k+1} = i_{o,k},$$

$$j_{o,k+1} = i_{o,k} + j_{o,k},$$

$$i_{s,k+1} = i_{s,k},$$

$$j_{s,k+1} = i_{s,k} + j_{s,k}.$$ 

We analyze what happens at the $(k+1)$-the stage in each of the possibilities $(\alpha)_k, (\beta)_k, (\gamma)_k, (\delta)_k$ at the $k$-th stage.

$(\alpha)_k$: Suppose that we are in the case $(\alpha)_k$.

If $j_{o,k+1} < j_{s,k+1}$, then, since $i_{o,k+1} = i_{o,k} > i_{s,k} = i_{s,k+1}$, we are in the case $(\alpha)_{k+1}$.  We also have

$$i_{o,k+1} - i_{s,k+1} = i_{o,k} - i_{s,k} < i_o$$

$$i_{o,k+1} - a_{k+1} = i_{o,k} - a_k < i_o.$$ 

If $j_{o,k+1} \geq j_{s,k+1}$, then, since $i_{o,k+1} = i_{o,k} > i_{s,k} = i_{s,k+1}$, we have

$$g_{k+1}^* y_2 = v_{k+1} \cdot x_{1,k+1}^{i_{s,k+1}} x_{2,k+1}^{j_{s,k+1}}$$

with $v_{k+1} \in \mathcal{O}_{X_{1,k+1}, p_{k+1}}^* \times \text{unit.}$
Therefore, the ideal
\[ g_{k+1}^{-1}(y_1, y_2) \cdot \mathcal{O}_{X_{1,k+1}} = (x_{1,k+1}^{a_{k+1}} x_{2,k+1}^{b_{k+1}}, x_{1,k+1}^{i_{o,k+1}+1} x_{2,k+1}^{j_{o,k+1}}) \]
\[ = (x_{1,k+1}^{\min\{a_{k+1}, i_{o,k+1}\}} x_{2,k+1}^{\min\{b_{k+1}, j_{o,k+1}\}}) \]
is principal.

(\beta)_k: Suppose that we are in the case (\beta)_k.

If \( j_{o,k+1} > j_{s,k+1} \), then since \( i_{o,k+1} = i_{o,k} < i_{s,k} = i_{s,k+1} \) we are in the case (\beta)_{k+1}. We also have

\[ j_{o,k+1} - j_{s,k+1} = (i_{o,k} + j_{o,k}) - (i_{s,k} + j_{s,k}) = (j_{o,k} - j_{s,k}) - (i_{s,k} - i_{o,k}) < (j_{o,k} - j_{s,k}) < j_o \]
\[ j_{o,k+1} - b_{k+1} = (i_{o,k} + j_{o,k}) - (a_k + b_k) = (j_{o,k} - b_k) - (a_k - i_{o,k}) < (j_{o,k} - b_k) < j_o. \]

(Note that under the case (\beta)_k, the inequality \( a_k \leq i_{o,k} \) would imply that \( g_{k}^{-1}(y_1, y_2) \cdot \mathcal{O}_{X_{1,k+1}} = (x_{1,k}^{a_k} x_{2,k}^{b_k}) \) is principal, being against the assumption. Thus we have \( a_k > i_{o,k} \).)

If \( j_{o,k+1} \leq j_{s,k+1} \), then since \( i_{o,k+1} = i_{o,k} < i_{s,k} = i_{s,k+1} \) we are in the case (\delta)_{k+1}, if \( g_{k+1}^{-1}(y_1, y_2) \cdot \mathcal{O}_{X_{1,k+1}} \) is not principal. We also have

\[ j_{o,k+1} - b_{k+1} = (i_{o,k} + j_{o,k}) - (a_k + b_k) = (j_{o,k} - b_k) - (a_k - i_{o,k}) < (j_{o,k} - b_k) < j_o. \]

(See the note above.)

(\gamma)_k: Suppose that we are in the case (\gamma)_k.

Since, under the case (\gamma)_k,
\[ i_{o,k} \leq i_{s,k} \quad \& \quad j_{o,k} \leq j_{s,k}, \]
we have
\[ i_{o,k+1} = i_{o,k} \leq i_{s,k} = i_{s,k+1} \]
\[ j_{o,k+1} = i_{o,k} + j_{o,k} \leq i_{s,k} + j_{s,k} = j_{s,k+1}. \]

Therefore, supposing \( g_{k+1}^{-1}(y_1, y_2) \cdot \mathcal{O}_{X_{1,k+1}} \) is not principal, we are in the case (\gamma)_{k+1}. We also have
\[ i_{o,k+1} - a_{k+1} = i_{o,k} - a_k < i_{o,k}. \]

(\delta)_k: Suppose that we are in the case (\delta)_k.

Since, under the case (\delta)_k,
\[ i_{o,k} \leq i_{s,k} \quad \& \quad j_{o,k} \leq j_{s,k}, \]
we have

\[ i_{o,k+1} = i_{o,k} \leq i_{s,k} = i_{s,k+1} \]
\[ j_{o,k+1} = i_{o,k} + j_{o,k} \leq i_{s,k} + j_{s,k} = j_{s,k+1}. \]

Therefore, supposing \( g_{k+1}^{-1}(y_1, y_2) \cdot \mathcal{O}_{X, k+1, p_{k+1}} \) is not principal, we are in the case \((\delta)_{k+1}\). We also have

\[ j_{o,k+1} - b_{k+1} = (i_{o,k} + j_{o,k}) - (a_k + b_k) \]
\[ = (j_{o,k} - b_k) - (a_k - i_{o,k}) < (j_{o,k} - b_k) < j_o. \]

Therefore, Lemma 3.3.6.C.2 is proved.

Now observe that at \( p_0 = p \) where \( g_0 = f \) is in Subcase 2.2, conditions \((i)_0\) with coordinate expression \((\ast)_0\) and \((ii)_0\) clearly hold. Therefore, by Lemma 3.3.6.C.2, at any point \( p_k \) (over \( p_0 = p \)) where the morphism \( g_k \) is in Subcase 2.2, and where the ideal \( g_k^{-1}(y_1, y_2) \cdot \mathcal{O}_{X, i, p_k} \) is not principal, we see that conditions \((i)_k\) with coordinate expression \((\ast)_k\) and \((ii)_k\) clearly hold, and that

\[ \nu_{E, p_k}(R_{\log, f_1}) < \max\{i_o, j_o\}. \]

This completes the proof of the second part of statement b).

This completes the local analysis of our algorithm 2.1 in the case where \( \dim X = \dim Y = 2. \)
4. Termination of the algorithm in the case where \( \dim X = \dim Y = 2 \)

4.1. **Purpose of this section.** The purpose of this section is, based upon the local analysis in §3, to observe that the logarithmic ramification divisor decreases in the process of the algorithm, under some partial order which we introduce among the Weil divisors (on possibly different ambient spaces) and with respect to which the descending chain condition is satisfied. Thus the algorithm must terminate after finitely many steps, ending with the logarithmic ramification divisor being zero, i.e., achieving toroidalization of the given morphism.

4.2. **Partial order among the Weil divisors.** Let \( R_1 = \Sigma a_i F_i \) and \( R_2 = \Sigma b_j G_j \) be Weil divisors on varieties \( X_1 \) and \( X_2 \), respectively, where the \( F_i \) and \( G_j \) are distinct irreducible components with the coefficients in decreasing order, i.e.,

\[
\begin{align*}
  a_1 &\geq a_2 \geq \cdots \geq a_{s-1} \geq a_s \\
b_1 &\geq b_2 \geq \cdots \geq b_{t-1} \geq b_t.
\end{align*}
\]

Then we define the partial order among the Weil divisors so that

\[ R_1 \geq R_2 \quad \text{if} \quad (a_1, a_2, \ldots, a_{s-1}, a_s) \geq (b_1, b_2, \ldots, b_{t-1}, b_t), \]

where the second inequality is given with respect to the lexicographical order.

It is clear that the set

\[
\{(a_1, a_2, \ldots, a_{s-1}, a_s)\}
\]

satisfies the descending chain condition (i.e., there is no infinite strictly decreasing sequence) and that so does the set of Weil divisors (possibly on different ambient spaces).

4.3. **Termination of the algorithm.**

**Theorem 4.3.1.** The algorithm 2.1 for toroidalization of a morphism \( f : (U_X, X) \to (U_Y, Y) \) in the logarithmic category, where \( \dim X = \dim Y = 2 \), terminates after finitely many steps.

More precisely, in the process of the algorithm

\[
\begin{align*}
  (U_X, X) &= (U_{X_0}, X_0) \xleftarrow{cp_1} (U_{X_0}, X_0) \xleftarrow{cp_2} \cdots \xleftarrow{cp_i} (U_{X_i}, X_i) \xleftarrow{\cdots} \\
  (U_X, X) &= (U_{X_0}, X_0) \xleftarrow{bp_1} (U_{X_0}, X_0) \xleftarrow{bp_2} \cdots \xleftarrow{bp_i} (U_{X_i}, X_i) \xleftarrow{\cdots},
\end{align*}
\]

(o) the logarithmic ramification divisor never increases (with respect to the order introduced in 4.2), i.e.,

\[ R_{\log, f_{i}} \geq R_{\log, f_{i+1}}. \]

(i) we have no infinite sequence of consecutive blowups

\[ bp_j \circ bp_{j+1} \circ \cdots \circ bp_{j+i} \circ \cdots \]

where the centers are only points \( q \) of type \( 2_q \).
(ii) every time we have a point \( q \) of type \( 1 \) in the center of \( \text{bp}_i \), the logarithmic ramification divisor strictly decreases (with respect to the order introduced in 4.2), i.e.,

\[ R_{\log, f_i} > R_{\log, f_{i+1}}. \]

Therefore, by the descending chain condition on the partial order, the algorithm must terminate after finitely many steps.

Proof.

(o) In each of the subcases from 3.3.1 through 3.3.6, we conclude in the subsection C that

either \( R_{\log, f_i} = R_{\log, f_{i+1}} \) or \( R_{\log, f_i} > R_{\log, f_{i+1}} \).

(i) We may assume \( j = 0 \) for the proof. Suppose that at the 0-th stage, the center of \( \text{bp}_1 \) consists only of points \( q \) of type 2. This means that \( f(R_{\log, f_0}) \) consists only of points of type 2.

Suppose \( p \notin R_{\log, f_0} \), i.e., \( f_0 \) is toroidal in a neighborhood of \( p \). Then by 3.3.1.B, over a neighborhood of \( p \), the morphism \( f_1 \) remains toroidal. That is to say, over this neighborhood of \( p \), there is no intersection with \( R_{\log, f_1} \).

Suppose \( p \in R_{\log, f_0} \). Then by the local analysis in 3.3.1, we have coordinate expression

\[
\begin{align*}
    f^* y_1 &= u \cdot x_1^{a_p} x_2^{b_p} \\
    f^* y_2 &= v \cdot x_1^{c_p} x_2^{d_p}
\end{align*}
\]

where \( \det \begin{bmatrix} a_p & b_p \\ c_p & d_p \end{bmatrix} = 0 \) with units \( u, v \in \widehat{\mathcal{O}}_{X_0, p} \),

for some systems of regular parameters \((x_1, x_2)\) and \((y_1, y_2)\) of \( \widehat{\mathcal{O}}_{X_0, p} \) and \( \widehat{\mathcal{O}}_{Y_0, p} \), compatible with the logarithmic structures.

Remark that the matrix

\[ \begin{bmatrix} a_p & b_p \\ c_p & d_p \end{bmatrix} \]

is independent of the choice of the systems of regular parameters, up to the column change and row change, and that the set

\[ \{ \begin{bmatrix} a_p & b_p \\ c_p & d_p \end{bmatrix} : p \in R_{\log, f_0} \} \]

is a finite set.

By the local analysis 3.3.1.B, the canonical principalization \( \text{cp}_1 \) is an isomorphism in a neighborhood of \( p \in R_{\log, f_0} \).

We have two possibilities: either

\( p \notin R_{\log, f_1} \)

or

\( p \in R_{\log, f_i} \) where \( f_1 \) is in Subcases \(* 2_q \) at \( p \) with the corresponding matrix

\[ \begin{bmatrix} a_p - c_p & b_p - d_p \\ c_p & d_p \end{bmatrix} \]

(under the convention \( a_p \geq c_p \) and \( b_p \geq d_p \)), since we assume that \( f_1(R_{\log, f_i}) \) consists of points \( q_1 \) of type \( 2_q \).

But from this it follows that the length of the sequence could at most be

\[ \max \{ a_p + b_p + c_p + d_p : p \in R_{\log, f_0} \}. \]
proving statement (i).

(ii) Statement (ii) follows from the local analysis in the subsection C of each of the subcases from 3.3.2. through 3.3.6.

Now termination of the algorithm after finitely many steps is an easy consequence of statements (i) and (ii), combined with the descending chain condition on the set of the Weil divisors under the order introduced in 4.2.

4.4. Corollaries

Corollary 4.4.1 (Resolution of singularities of morphisms in the logarithmic category in dimension 2). Let \( f: (U_X, X) \to (U_Y, Y) \) be a morphism in the logarithmic category of nonsingular toroidal embeddings with \( \dim X = 2 \). Then there exist sequences of blowups with permissible centers \( \pi_X: (U'_X, X') \to (U_X, X) \) and \( \pi_Y: (U'_Y, Y') \to (U_Y, Y) \) such that the induced map \( f': (U'_X, X') \to (U'_Y, Y') \) is log smooth.

Proof.

When \( \dim Y = 0 \), there is nothing to prove. When \( \dim Y = 1 \), it follows from 2.2.1 that \( f \) is already log smooth. When \( \dim Y = 2 \), the assertion is a consequence of Theorem 4.3.1.

Corollary 4.4.2 (Toroidalization in dimension 2). Toroidalization conjecture holds for a dominant morphism \( f: X \to Y \) between nonsingular varieties with \( \dim X = 2 \).

Proof.

This follows from 1.5 and Corollary 4.4.1.

4.5. Further remarks.

Corollary 4.5.1 (Strong factorization of birational maps in dimension 2). Strong factorization of birational maps in dimension 2 holds: Let \( \varphi: X \dasharrow Y \) be a proper birational map between nonsingular varieties in dimension 2. Then there exist a sequence of blowups with points as centers

\[
X = X_0 \xrightarrow{\phi_1} X_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_l} X_l
\]

and a sequence of blowdowns

\[
X_l = Y_m \xrightarrow{\psi_m} Y_{m-1} \xrightarrow{\psi_{m-1}} \cdots \xrightarrow{\psi_1} Y_1 \xrightarrow{\psi_1} Y_0 = Y
\]

such that

\[\varphi = \psi_1 \circ \cdots \circ \psi_m \circ \phi_l^{-1} \circ \cdots \circ \phi_1^{-1}.\]

Proof.

The whole point of mentioning this well-known fact is to emphasize the way we prove it, according to the general strategy Step I, which we have now completed proving toroidalization in dimension 2, and step II, which is a consequence of an easy combinatorial fact of geometry of convex bodies in dimension 2 (See
0.4 and Abramovich-Matsuki-Rashid[5]). We remark that the existing proofs for toroidalization use the factorization theorem of proper birational morphisms via Castelnuovo’s contractibility of a \((-1)\)-curve.

4.4.2. With the assumption of properness on \(f\).

Suppose that in the setting of 2.1.1 for a morphism \(f : (U_X, X) \to (U_Y, Y)\) in the logarithmic category, we put the properness assumption on the morphism \(f\).

Then we have much less possibilities in 3.3.5) Subcase 2\(_p\)1\(_q\)1 and 3.3.6) Subcase 2\(_p\)1\(_q\)2: Let \(U_q\) be an open neighborhood of \(q\) with a system of regular parameters \((y_1, y_2)\) with \(H_1 = \{y_1 = 0\}\). Then since \(f\) is smooth over \(U_q - D_Y = U_q - H_1\), Abhyankar’s lemma tells us that the normalization \(\mu: \tilde{U}_q \to U_q\) of \(U_q\) in the function field \(k(X)\) has a system of regular parameters \((\tilde{y}_1, \tilde{y}_2)\) such that \(\mu\) can be written in these coordinate systems

\[
\begin{align*}
\mu^* y_1 &= \tilde{y}_1^c \\
\mu^* y_2 &= \tilde{y}_2.
\end{align*}
\]

with \(b > 0\).

Now the factorization theorem of proper birational morphisms in dimension 2 tells us that \(g : f^{-1}(U_q) \to \tilde{U}_q\) is a sequence of blowups of points.

3.3.5) Subcase 2\(_p\)1\(_q\)1: Since \(G_2\) is not exceptional for \(f\) and hence not for \(g\), we see that there exists a system of regular parameters \((x_1, x_2)\) of \(\hat{O}_{X,p}\) such that

\[
\begin{align*}
g^* \tilde{y}_1 &= x_1^{a'} x_2 \\
g^* \tilde{y}_2 &= x_1
\end{align*}
\]

Combining the above two, we conclude

\[
\begin{align*}
f^* y_1 &= x_1^a x_2^b & \text{ with } a = a'e, b = c. \\
f^* y_2 &= x_1
\end{align*}
\]

3.3.6) Subcase 2\(_p\)1\(_q\)2: Since both \(G_1\) and \(G_2\) are exceptional for \(f\) and hence for \(g\), we see that there exists a system of regular parameters \((x_1, x_2)\) of \(\hat{O}_{X,p}\) such that we have either

\[
\begin{align*}
f^* y_1 &= x_1^a x_2^b \\
f^* y_2 &= x_1^c x_2^d
\end{align*}
\]

with \(\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0\)

where

\[
a \geq c \quad \& \quad b \geq d, \quad \text{or} \\
a \leq c \quad \& \quad b \leq d.
\]

or

\[
\begin{align*}
f^* y_1 &= u \cdot x_1^a x_2^b & \text{ with } \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0 \quad \text{and} \quad u, v \in \hat{O}_{X,p}^\times \quad \text{units.}
\end{align*}
\]

The conclusion is that, under the properness assumption on \(f\), in both 3.3.5) Subcase 2\(_p\)1\(_q\)1 and 3.3.6) Subcase 2\(_p\)1\(_q\)2 the ideal \(f^{-1}(y_1, y_2) : \hat{O}_{X,p}\) is already principal,
and hence that the canonical principalization is an isomorphism in a neighborhood of $p$.

Under the properness assumption on $f$, no new irreducible components appear in the logarithmic ramification divisor in the process. Moreover, identifying the strict transforms of the irreducible components in the logarithmic ramification divisors, we conclude (via the usual order among the Weil divisors)

3.3.1) Subcase $2 \ast \rightarrow R_{\log,f_0} = R_{\log,f_1}$
3.3.2) Subcase $1_p1_q0 \rightarrow R_{\log,f_0} = R_{\log,f_1}$
3.3.3) Subcase $1_p1_q1 \rightarrow R_{\log,f_0} > R_{\log,f_1}$
3.3.4) Subcase $2_p1_q0$: Impossible
3.3.5) Subcase $2_p1_q1 \rightarrow R_{\log,f_0} > R_{\log,f_1}$
3.3.6) Subcase $2_p1_q2 \rightarrow R_{\log,f_0} > R_{\log,f_1}$.

Therefore, the proof of Theorem 4.3.1 (ii) becomes much simpler under the properness assumption on $f$. 
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