A horizontal Chern–Gauss–Bonnet formula on totally geodesic foliations

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Received: 8 July 2021 / Accepted: 8 January 2022 / Published online: 1 February 2022 © The Author(s) 2022

Abstract

Under suitable conditions, we show that the Euler characteristic of a foliated Riemannian manifold can be computed only from curvature invariants which are transverse to the leaves. Our proof uses the hypoelliptic sub-Laplacian on forms recently introduced by two of the authors in Baudoin and Grong (Ann Glob Anal Geom 56(2):403–428, 2019).

1 Introduction

The goal of the paper is to prove the following result:

Theorem 1.1 Let \( M \) be a smooth, connected, oriented and \( n + m \) dimensional compact manifold. We assume that \( M \) is equipped with a Riemannian foliation \( \mathcal{F} \) with bundle-like metric \( g \) and totally geodesic \( m \)-dimensional leaves. We also assume that the horizontal distribution \( \mathcal{H} = \mathcal{F}^\perp \) is bracket-generating and that there exists \( \varepsilon > 0 \) such that

\[
(\nabla_v J)w = -\frac{1}{2\varepsilon} [Jv, Jw]
\]

(1.1)

for any \( v, w \in T_x M, x \in M \), where \( \nabla \) is the Bott connection of the foliation and \( J \) is the tensor defined in (2.2). Denoting \( \chi(M) \) the Euler characteristic of \( M \):

- If \( n \) or \( m \) is odd, then \( \chi(M) = 0 \);

F. Baudoin: Author supported in part by the NSF Grant DMS 1901315. E. Grong: Author supported by grant from the Trond Mohn Foundation—Grant TMS2021STG02 (GeoProCo).

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If $n$ and $m$ are both even, then

$$\chi (\mathcal{M}) = \int_{\mathcal{M}} \hat{\omega}^H \wedge \left[ \det \left( \frac{\mathcal{F}}{\sinh(\mathcal{F})} \right)^{1/2} \right]_m.$$ 

Notations are further explained in Sect. 4, but we point out that a remarkable feature of that result is that the density $\hat{\omega}^H \wedge \left[ \det \left( \frac{\mathcal{F}}{\sinh(\mathcal{F})} \right)^{1/2} \right]_m$ essentially only depends on horizontal curvature quantities. Therefore, the theorem illustrates further the fact already observed in [4] that topological properties of $\mathcal{M}$ might be obtained from horizontal curvature invariants only provided that the bracket-generating condition of the horizontal distribution is satisfied; thus, in essence, the theorem is a sub-Riemannian result. We also note that the condition (1.1) is satisfied in a large class of examples including the H-type foliations introduced in [5], see Example 2.4.

The proof of Theorem 1.1 is based on the study of the heat semigroup generated by the hypoelliptic sub-Laplacian on forms recently introduced in [4]. The heat equation approach to Chern–Gauss–Bonnet type formulas (or index formulas) that we are using is of course not new: It was suggested by Atiyah–Bott [1] and McKean-Singer [16] and first carried out by Patodi [18] and Gilkey [12] and is by now classical, see the book [9]. However, a difficulty in our setting is that the sub-Laplacian on forms we consider is only hypoelliptic but not elliptic. To carry out the required small-time asymptotics analysis to obtain the horizontal Chern–Gauss–Bonnet formula, we will make use of the probabilistic Brownian Chen series parametrix method first introduced in [3] and which is easy to adapt to hypoelliptic situations, see [2].

The paper is organized as follows. In Sect. 2, we introduce the horizontal Laplacian on forms $\Delta H_{\epsilon}$ and prove that it is a self-adjoint operator if and only if the condition (1.1) is satisfied. In Sect. 3, we prove a McKean–Singer type formula for $\Delta H_{\epsilon}$, namely that for every $t > 0$,

$$\text{Str}(e^{t\Delta H_{\epsilon}}) = \chi (\mathcal{M}).$$

Finally, in Sect. 4 we study the small-time asymptotics of $\text{Str}(e^{t\Delta H_{\epsilon}})$ and conclude the proof of Theorem 1.1.

2 Preliminaries

In this section, we first recall the framework and notations of Baudoin and Grong [4] and the references therein to which we refer for further details. We then prove a necessary and sufficient condition for the form horizontal Laplacian of a totally geodesic foliation to be a symmetric operator.

2.1 Totally geodesic foliations

Let $(\mathcal{M}, g)$ be a smooth, oriented, connected, compact Riemannian manifold with dimension $n + m$. We assume that $\mathcal{M}$ is equipped with a foliation $\mathcal{F}$ with $m$-dimensional leaves. The distribution $\mathcal{V}$ formed by vectors tangent to the leaves is referred to as the set of vertical directions (or vertical subbundle). Define the horizontal subbundle $\mathcal{H} = \mathcal{V}^\perp$ as its orthogonal complement. We will always assume in this paper that the horizontal distribution $\mathcal{H}$ is
everywhere bracket-generating. The foliation is called Riemannian and totally geodesic if for any \( X \in \Gamma(\mathcal{H}), Z \in \Gamma(\mathcal{V}) \), the respective conditions are satisfied,

\[
(L_Z g)(X, X) = 0, \quad (L_X g)(Z, Z) = 0.
\]

Equivalently, we can describe these conditions using the Bott connection. Write \( \pi_\mathcal{H} \) and \( \pi_\mathcal{V} \) for the respective orthogonal projections to \( \mathcal{H} \) and \( \mathcal{V} \). Let \( \nabla^g \) be the Levi–Civita connection of \( g \). Introduce a new connection \( \nabla \) on \( T^M \) according to the rules,

\[
\nabla_X Y = \begin{cases} 
\pi_\mathcal{H}(\nabla_X^g Y) & \text{for any } X, Y \in \Gamma(\mathcal{H}), \\
\pi_\mathcal{V}(\langle [X, Y] \rangle) & \text{for any } X \in \Gamma(\mathcal{V}), Y \in \Gamma(\mathcal{H}), \\
\pi_\mathcal{V}([X, Y]) & \text{for any } X \in \Gamma(\mathcal{H}), Y \in \Gamma(\mathcal{V}), \\
\pi_\mathcal{V}(\nabla_X^g Y) & \text{for any } X, Y \in \Gamma(\mathcal{V}).
\end{cases}
\tag{2.1}
\]

We observe that \( \nabla \) preserves \( \mathcal{H} \) and \( \mathcal{V} \) under parallel transport. The foliation \( \mathcal{F} \) is then both Riemannian and totally geodesic if and only if \( \nabla^g = 0 \). For the rest of the paper, we will assume that \( \nabla \) is indeed compatible with the metric \( g \). The torsion \( T \) of \( \nabla \) is given by

\[
T(X, Y) = -\pi_\mathcal{V}[\pi_\mathcal{H}X, \pi_\mathcal{H}Y].
\]

Define a corresponding endomorphism valued one-form \( Z \mapsto J_Z \) by

\[
(Z, T(X, Y))_g, \quad X, Y, Z \in \Gamma(T^M).
\tag{2.2}
\]

Let \( g_\mathcal{H} \) and \( g_\mathcal{V} \) be the respective restrictions of \( g \) to \( \mathcal{H} \) and \( \mathcal{V} \). We then define the canonical variation \( g \) by \( g_\varepsilon = g_\mathcal{H} \oplus \frac{1}{\varepsilon} g_\mathcal{V}, \varepsilon > 0 \), and make the following observations:

(i) If \((\mathcal{M}, \mathcal{F}, g)\) is a Riemannian, totally geodesic foliation, then so is \((\mathcal{M}, \mathcal{F}, g_\varepsilon)\).

(ii) Although the Levi-Civita connection \( \nabla^{g_\varepsilon} \) of \( g_\varepsilon \) is different from the connection \( \nabla^g \) of \( g \), replacing \( \nabla^g \) with \( \nabla^{g_\varepsilon} \) in formula (2.1) will lead to exactly the same connection. In other words, when defining the Bott connection \( \nabla \), we obtain the same connection for any metric \( g_\varepsilon \) in the family of canonical variations.

(iii) For any fixed \( \varepsilon > 0 \), define a connection

\[
\hat{\nabla}_X^\varepsilon Y = \nabla_X Y + \frac{1}{\varepsilon} J_X Y.
\tag{2.3}
\]

This connection preserves \( \mathcal{H} \) and \( \mathcal{V} \) under parallel transport and is compatible with \( g_{\varepsilon'} \) for any \( \varepsilon' > 0 \). Furthermore, its torsion

\[
\hat{T}(X, Y) = T(X, Y) + \frac{1}{\varepsilon} J_X Y - \frac{1}{\varepsilon} J_Y X,
\]

is skew-symmetric with respect to \( g_\varepsilon \). Hence, if we consider its adjoint connection

\[
\nabla_X^\varepsilon Y = \hat{\nabla}_X^\varepsilon Y - \hat{T}(X, Y) = \nabla_X Y - T(X, Y) + \frac{1}{\varepsilon} J_Y X,
\tag{2.4}
\]

it will also be compatible with \( g_\varepsilon \). However, \( \mathcal{H} \) and \( \mathcal{V} \) are not parallel with respect to \( \nabla^\varepsilon \).

### 2.2 Horizontal Laplacian on forms

For the totally geodesic Riemannian foliation \((\mathcal{M}, \mathcal{F}, g)\), define its horizontal Laplacian on functions \( f \in C^\infty(\mathcal{M}) \) by

\[
\Delta_\mathcal{H} f = \text{tr}_\mathcal{H} \nabla_X df(X).
\tag{2.5}
\]
We note that since $\mathcal{H}$ is assumed to be bracket-generating, from Hörmander’s theorem, $\Delta_{\mathcal{H}}$ is a subelliptic operator. We also note that since $g_{\mathcal{H}}$ and the Bott connection are independent of $\varepsilon > 0$, the horizontal Laplacian is as well; that is, the choice of any metric $g_{\varepsilon}$ in the canonical variation family will not change $g_{\mathcal{H}}$, the Bott connection, or the horizontal Laplacian.

Consider now the totally geodesic Riemannian foliation $(M, F, g_{\varepsilon})$ for some fixed $\varepsilon > 0$. We want to extend the horizontal Laplacian on functions (2.5) to a differential operator on forms $\Delta_{\mathcal{H},\varepsilon}$ satisfying the following requirements:

(I) $\Delta_{\mathcal{H},\varepsilon} f = \Delta_{\mathcal{H}} f$ for any smooth function $f$;

(II) The operator $\Delta_{\mathcal{H},\varepsilon}$ is of Weitzenböck type, i.e., $\Delta_{\mathcal{H},\varepsilon} = L_{\mathcal{H},\varepsilon} + R_{\varepsilon}$ where $R_{\varepsilon}$ is a zero-order differential operator and

$$L_{\mathcal{H},\varepsilon} = \text{tr}_{\mathcal{H}} \nabla_{\varepsilon}^2,$$  

(2.6)

is the connection horizontal Laplacian of some connection $\nabla_{\varepsilon}$ compatible with $g_{\varepsilon}$;

(III) If $d$ is the exterior differential, then

$$[\Delta_{\mathcal{H},\varepsilon}, d] = 0.$$  

Given these requirements, there is an essentially unique extension of $\Delta_{\mathcal{H}}$ to forms, see [4,15] for details. We call $\Delta_{\mathcal{H},\varepsilon}$ the $\varepsilon$-horizontal Laplacian on forms. This operator can described as follows.

**Proposition 2.1** (Horizontal Laplacian on forms, see [4]) Consider the $\varepsilon$-horizontal divergence operator defined by

$$\delta_{\mathcal{H},\varepsilon} \eta = - \text{tr}_{\mathcal{H}} (\nabla_{\varepsilon}^\flat) (\eta, \cdot).$$

The operator

$$\Delta_{\mathcal{H},\varepsilon} = - \delta_{\mathcal{H},\varepsilon} d - d \delta_{\mathcal{H},\varepsilon}$$

is called the $\varepsilon$-horizontal Laplacian on forms, and it satisfies the requirements (I), (II), (III). In particular, this operator has Weitzenböck decomposition $\Delta_{\mathcal{H},\varepsilon} = L_{\mathcal{H},\varepsilon} + R_{\varepsilon}$ where $L_{\mathcal{H},\varepsilon}$ is defined as in (2.6) relative to $\nabla_{\varepsilon}$.

We can describe the zero order operator $R_{\varepsilon}$ can be made explicit, see [4]. For later use, we will prefer to write the operators using Fermion calculus, see Appendix A.1. Let $X_1, \ldots, X_n$ and $Z_1, \ldots, Z_m$ be local orthonormal bases of, respectively, $\mathcal{H}$ and $\mathcal{V}$. Define $a_i = iX_i$ and $b_r = iZ_r$ for the corresponding annihilation operators, with the dual operators $a_i^* = X^*_i \wedge$ and $b_r^* = Z^*_r \wedge$ acting by wedge products. The dual are here relative to the $L^2$ inner product with respect to the fixed metric $g$. Relative to the curvature tensor $\hat{R}^\varepsilon$ of $\nabla^\varepsilon$, write

$$\hat{R}^\varepsilon_{ijk} = \langle \hat{R}^\varepsilon (X_i, X_j)X_k, X_l \rangle g,$$  

(2.7)

and use similar notation for other tensors with indices $i, j, k, l$ denoting evaluations with respect to the basis of $\mathcal{H}$, indices $r, s$ with respect to the basis of $\mathcal{V}$. We emphasize that these indices are always defined relative to the fixed metric $g$. Then, $R_{\varepsilon}$ is given by

$$R_{\varepsilon} = \sum_{i,j,k=1}^n \hat{R}^\varepsilon_{ijk} a_k^* a_j + \sum_{i,k=1}^n \sum_{r=1}^m \hat{R}^\varepsilon_{irk} a_k^* b_r + \frac{1}{2} \sum_{i,j,k,l=1}^n \hat{R}^\varepsilon_{ijkl} a_k^* a_l^* a_j a_i$$

$$+ \sum_{i,j,k=1}^n \sum_{r=1}^m \hat{R}^\varepsilon_{ijk} a_k^* b_r a_i + \frac{1}{2} \sum_{i,j,k=1}^m \sum_{r,s=1}^m \hat{R}^\varepsilon_{rsi} a_i^* a_j^* b_r b_i.$$  

(2.8)
We want to give a formula for this operator that shows the dependence of $\varepsilon$ explicitly. Let $T$ and $R$ be the curvature of the Bott connection $\nabla$ and use indices after semi-colons to denote covariant derivatives with respect to this connection. Using Lemma A.2, Appendix, we can write

$$R \varepsilon = \frac{1}{2} \sum_{i,j,k,l=1}^{n} \left( R^{k}_{kji} + \frac{1}{\varepsilon} \sum_{r=1}^{m} T^{r}_{ik} T^{r}_{jk} \right) a^{*}_{i} a^{*}_{j} - \frac{1}{2} \sum_{i,j,k,l=1}^{n} \left( T^{r}_{ij} a^{*}_{r} b_{i} b_{j} + T^{r}_{kj} a^{*}_{r} b_{i} b_{k} \right).$$

(2.9)

2.3 Symmetry of the horizontal Laplacian

Consider the exterior algebra

$$\Omega = \Omega(M) = \bigoplus_{k=0}^{\dim M} \Omega^{k},$$

with the $L^{2}$-inner product from $g_{\varepsilon}$. When restricted to elements in $\Omega^{0} \oplus \Omega^{1}$, the operator $\Delta_{H, \varepsilon}$ is symmetric if and only if $H$ satisfies the Yang–Mills condition, i.e., if

$$\sum_{i=1}^{n} T^{0}_{ij;i} = 0, \quad \text{for any } j = 1, \ldots, n, r = 1, \ldots, m.$$

see [6]. Considering all forms, we have the following result.

**Proposition 2.2** The operator $\Delta_{H, \varepsilon}$ is symmetric with respect to the $L^{2}$-inner product of $g_{\varepsilon}$ if and only if

$$(\nabla_{v} J)_{w} = -\frac{1}{2\varepsilon} [J_{v}, J_{w}],$$

(2.10)

for any $v, w \in T_{x} M, x \in M$. In particular, $\nabla_{v} J = 0$ for any $v \in \mathcal{H}$.

We note that under the above condition, the expression of $\mathcal{R}_{\varepsilon}$ reduces to

$$\mathcal{R}_{\varepsilon} = \frac{1}{2} \sum_{i,j,k=1}^{n} \left( R^{k}_{kji} + \frac{1}{\varepsilon} \sum_{r=1}^{m} T^{r}_{ik} T^{r}_{jk} \right) a^{*}_{i} a^{*}_{j} a^{*}_{k} + \frac{1}{2} \sum_{i,j,k,l=1}^{n} \left( R^{j}_{ki} + \frac{1}{\varepsilon} \sum_{r=1}^{m} T^{r}_{ki} T^{r}_{ij} \right) a^{*}_{i} a^{*}_{j} a^{*}_{k}. $$

(2.11)

**Proof** $L_{H, \varepsilon}$ is symmetric by Grong and Thalmaier [15, Lemma A.1], so we only need to determine when $\mathcal{R}_{\varepsilon}$ is symmetric. We choose a local bases $X_{1}, \ldots, X_{n}$ and $Z_{1}, \ldots, Z_{m}$ of, respectively, $\mathcal{H}$ and $\mathcal{V}$. We consider the representation of $\mathcal{R}_{\varepsilon}$ as in (2.9). Then, for $\mathcal{R}_{\varepsilon}$ to be
symmetric, we must have

\[ 0 = \langle R_\varepsilon X^*_k \wedge Z^*_r, X^*_i \wedge X^*_j \rangle_\varepsilon - \langle R_\varepsilon X^*_i \wedge X^*_j, X^*_k \wedge Z^*_r \rangle_\varepsilon = \frac{1}{\varepsilon} T^r_{ij;k}, \]

\[ 0 = \langle R_\varepsilon Z^*_r \wedge Z^*_s, X^*_i \wedge X^*_j \rangle_\varepsilon - \langle R_\varepsilon X^*_i \wedge X^*_j, Z^*_r \wedge Z^*_s \rangle_\varepsilon = \frac{2}{\varepsilon} T^r_{ij} + \frac{1}{\varepsilon^2} \sum_{k=1}^{n} (T^r_{kj} T^s_{ik} - T^s_{kj} T^r_{ik}). \]

These equations are clearly equivalent to (2.10). If these hold, then \( R_\varepsilon \) reduces to the expression (2.11), which is symmetric by Lemma A.3 (i).

**Remark 2.3** If we assume that \( m = 1 \) (i.e., the leaves are one-dimensional), then it is immediate from the previous result that the following are equivalent:

(i) \( \Delta_{\mathcal{H},\varepsilon} \) is symmetric for some \( \varepsilon > 0 \).

(ii) \( \Delta_{\mathcal{H},\varepsilon} \) is symmetric for all \( \varepsilon > 0 \).

(iii) \( \nabla J = 0 \).

Recall that the statement \( \nabla J = 0 \) is equivalent to \( \nabla T = 0 \). For \( m > 1 \), the above statement remains true if we replace (i) by the following assumption

(i') \( \Delta_{\mathcal{H},\varepsilon} \) is symmetric at least two values \( \varepsilon > 0 \) and \( \varepsilon' > 0 \).

**Example 2.4** (H-type foliations) Following definitions given in [5], we say that a foliated Riemannian manifold \( (\mathbb{M}, \mathcal{F}, g) \) is of H-type if for every \( Z \in \Gamma(V) \), we have \( J^2_Z = -\|Z\|_V^2 \pi_{\mathcal{H}} \).

Expand the definition of \( J \) from taking values from \( V \) to its Clifford algebra \( \text{Cl}(V) \) by the rule \( J_1 = \pi_{\mathcal{H}} \) and iteratively \( J_{u \cdot v} = J_u J_v, u, v \in \text{Cl}(V) \). We then further say that the foliation is of horizontally parallel Clifford type if \( \nabla_X J = 0 \) for any horizontal vector fields \( X \in \Gamma(\mathcal{H}) \) and while for \( u, v \in V \).

\[ (\nabla_u J)_v \in J_{\text{Cl}(V)}. \]

It then turns out that for some \( \kappa \in \mathbb{R} \),

\[ (\nabla_u J)_v = -\kappa J_{u \cdot v + \langle u, v \rangle} = -\frac{\kappa}{2} [J_u, J_v]. \]

The number \( \kappa \) determines the Ricci curvature of \( \nabla \), see [5, Theorem 3.16]. We see that if we have an H-type Riemannian foliation \( (\mathbb{M}, \mathcal{F}, g) \) of horizontally parallel Clifford type, then \( \Delta_{\mathcal{H},\varepsilon} \) is symmetric with respect to \( g_\varepsilon \) for \( \varepsilon = \frac{1}{\kappa} \).

Finally, to conclude the section we point out the following result. For the definition of the Carnot–Carathéodory metric \( d_{cc} \) of the sub-Riemannian manifold \( (\mathbb{M}, \mathcal{H}, g_{\mathcal{H}}) \) and the tangent cone of a metric space, see, e.g., [13].

**Corollary 2.5** Assume that \( \Delta_{\mathcal{H},\varepsilon} \) is symmetric on forms for some fixed \( \varepsilon > 0 \). Then, the following holds:

(a) The horizontal bundle \( \mathcal{H} \) has step 2, that is \( \mathcal{H} + [\mathcal{H}, \mathcal{H}] = T \mathbb{M} \). In particular, the torsion \( T \) of the Bott connection \( \nabla \) will be surjective on \( V \).

(b) The tangent cones of the metric space \( (\mathbb{M}, d_{cc}) \) at any pair of points \( x, y \in \mathbb{M} \) are isometric.
**Proof** (a) Recall that if $\Delta_{\mathcal{H},\varepsilon}$ is symmetric on forms for some $\varepsilon > 0$, then in particular $\nabla_v J = 0$ for any $v \in \mathcal{H}$. We can rewrite it as $\nabla_v T = 0$ for any $v \in \mathcal{H}$ since $\nabla$ is compatible with $g$. Define $\mathcal{H}^2 = \mathcal{H} + [\mathcal{H}, \mathcal{H}]$ and let $X_1, X_2, X_3 \in \Gamma(\mathcal{H})$ be arbitrary. We first see that

$$T(X_2, X_3) = \nabla_{X_2}X_3 - \nabla_{X_3}X_2 - [X_2, X_3] = 0 \quad \text{mod} \ \mathcal{H}^2,$$

since $\nabla$ preserves $\mathcal{H}$. Furthermore, by the definition of the Bott connection

$$[X_1, [X_2, X_3]] = -[X_1, T(X_2, X_3)] \quad \text{mod} \ \mathcal{H}^2 = -\nabla_{X_1}T(X_2, X_3) \quad \text{mod} \ \mathcal{H}^2$$

$$= -T(\nabla_{X_1}X_2, X_3) - T(X_2, \nabla_{X_1}X_3) \quad \text{mod} \ \mathcal{H}^2 = 0 \quad \text{mod} \ \mathcal{H}^2.$$

It follows that $\mathcal{H}$ only generates $\mathcal{H}^2$. As we assumed that $\mathcal{H}$ is bracket generating, we have $\mathcal{H}^2 = TM$.

(b) Since both $\mathcal{H}$ and $\mathcal{H}^2 = \mathcal{H} + [\mathcal{H}, \mathcal{H}] = TM$ have constant rank, it follows by Mitchell [17] and Bellaïche [8] that the tangent cone at a point $x$ is a Carnot group $G_x$. Its Lie algebra $\mathfrak{g}_x$ is given by

$$\mathfrak{g}_x = \mathfrak{g}_{x,1} \oplus \mathfrak{g}_{x,2} = \mathcal{H}_x \oplus T_xM/\mathcal{H}_x,$$

where $TM/\mathcal{H}_x$ is the center, and for $X_x, Y_x \in \mathcal{H}_x = \mathfrak{g}_{x,1}$ the Lie bracket is defined as

$$[[X_x, Y_x]] = [X, Y]_x \quad \text{mod} \ \mathcal{H}_x.$$

where $X, Y$ are any pair of vector fields extending this vectors. The Carnot group $G_x$ is then the corresponding simply connected Lie group of $\mathfrak{g}_x$ with the sub-Riemannian structure given by left translation of $\mathfrak{g}_x = \mathcal{H}_x$ and its inner product.

If identify $\mathfrak{g}_x = \mathcal{H}_x \oplus T_xM/\mathcal{H}_x$ with $T_yM = \mathcal{H}_x \oplus \mathcal{V}_x$ through the map $v \mod \mathcal{H}_x \mapsto \pi_{\mathcal{V}_x}(v), v \in T_xM$, then the Lie bracket becomes,

$$[[v, w]] = -T(v, w), \quad v, w \in T_xM.$$

Let now $y$ be any other point and let $\gamma : [0, 1] \to \mathbb{M}$ be any horizontal curve from $x$ to $y$, which exists form our assumption that $\mathcal{H}$ satisfies the bracket-generating condition. Then, $\nabla_{\gamma(t)}T = 0$ for any $t \in [0, 1]$, so if we write

$$I_{\gamma,t} = I_{1} : T_x\mathbb{M} \to T_{\gamma(t)}\mathbb{M},$$

for the parallel transport map along $\gamma$, then this satisfies

$$I_{\gamma} \ T(u, v) = T(I_{\gamma}u, I_{\gamma}v), \quad v, w \in T_x\mathbb{M}.$$

As a consequence, $I_1 : \mathfrak{g}_x = T_x\mathbb{M} \to \mathfrak{g}_y = T_y\mathbb{M}$ is a Lie algebra isomorphism, which can be integrated to a Lie group isomorphism from $G_x$ to $G_y$. Since the parallel transport $I_1$ also maps $\mathcal{H}_x$ onto $\mathcal{H}_y$ isometrically, the induced map on Carnot groups is in fact a sub-Riemannian isometry.

\[\Box\]

3 Horizontal McKean–Singer theorem

We work on a totally geodesic foliation $(\mathbb{M}, \mathcal{F}, g)$ and assume that there is some $0 < \varepsilon < +\infty$ such that horizontal Laplacian $\Delta_{\mathcal{H},\varepsilon}$ is symmetric. From Proposition 2.2, this assumption is
equivalent to the fact that
\[
(\nabla_v J)_w = -\frac{1}{2\epsilon} [J_v, J_w].
\]

Since \(\Delta_{\mathcal{H},\epsilon}\) commutes with \(d\) on smooth forms and is symmetric, it also commutes on smooth forms with the codervivative \(\delta\), and thus, it also commutes with the Hodge–de Rham operator \(\Delta_\epsilon := -d\delta - \delta_d d\) on smooth forms. From Hodge theorem, the operator \(\Delta_\epsilon\) is elliptic with a compact resolvent and the space of \(L^2\)-forms can be decomposed as \(\bigoplus_{k=0}^{+\infty} E_{\lambda_k}\), where the \(E_{\lambda_k}\)'s are the eigenspaces of \(\Delta_\epsilon\). Those eigenspaces only contain smooth forms, therefore \(\Delta_{\mathcal{H},\epsilon}(E_{\lambda_k}) \subset E_{\lambda_k}\). This implies that \(\Delta_{\mathcal{H},\epsilon}\) is essentially self-adjoint and generates the semigroup:
\[
\exp(t\Delta_{\mathcal{H},\epsilon}) = \bigoplus_{k=0}^{+\infty} \exp(t\Delta_{\mathcal{H},\epsilon})|_{E_{\lambda_k}} \tag{3.1}
\]

By hypoellipticity (see [4, Lemma 4.9]), this semigroup has a smooth kernel \(p_{\mathcal{H},\epsilon}(t, x, y)\) and is a bounded trace class operator in \(L^2(\wedge \cdot M, g_\epsilon)\). Let us denote by \(E^+_{\lambda}(\Delta_{\mathcal{H},\epsilon})\) (resp. \(E^-_{\lambda}(\Delta_{\mathcal{H},\epsilon})\)) the space of harmonic even forms for \(\Delta_{\mathcal{H},\epsilon}\) (resp. the space of harmonic odd forms for \(\Delta_{\mathcal{H},\epsilon}\)).

The goal of the section is to prove the following theorem, which is an analogue for our horizontal Laplacian of the classical McKean–Singer formula found in [16]:

**Theorem 3.1** (Horizontal McKean-Singer formula) For every \(t > 0\),
\[
\text{Str}(\exp(t\Delta_{\mathcal{H},\epsilon})) := \int_M \text{Tr}(p^+_\mathcal{H}_{\epsilon}(t, x, x))d\mu(x) - \int_M \text{Tr}(p^-\mathcal{H}_{\epsilon}(t, x, x))d\mu(x)
\]
\[
= \dim E^+_{\lambda}(\Delta_{\mathcal{H},\epsilon}) - \dim E^-_{\lambda}(\Delta_{\mathcal{H},\epsilon})
\]
\[
= \chi(M)\]
where \(\chi(M)\) is the Euler characteristic of \(M\).

We turn to the proof of Theorem 3.1. We denote by
\[
D_\epsilon = d + \delta_\epsilon
\]
the Dirac operator of the metric \(g_\epsilon\). Observe that \(D_\epsilon\) commutes with \(\Delta_{\mathcal{H},\epsilon}\) since both \(d\) and \(\delta_\epsilon\) commute with it. The main idea to prove Theorem 3.1 is to introduce a deformation of \(\Delta_{\mathcal{H},\epsilon}\) as follows:
\[
\square_{\epsilon,\theta} = (1 - \theta)\Delta_{\mathcal{H},\epsilon} - \theta D_\epsilon^2, \quad \theta \in [0, 1].
\]

A first lemma is the following:

**Lemma 3.2** Let \(\lambda\) be a nonzero eigenvalue of \(\square_{\epsilon,\theta}\). Then, \(D_\epsilon : E^+_{\lambda}(\square_{\epsilon,\theta}) \to E^+_{\lambda}(\square_{\epsilon,\theta})\) is an isomorphism. Therefore, \(\dim E^+_{\lambda}(\square_{\epsilon,\theta}) = \dim E^-_{\lambda}(\square_{\epsilon,\theta})\).

**Proof** Let \(\lambda\) be a nonzero eigenvalue of \(\square_{\epsilon,\theta}\). The corresponding eigenspace \(E^\pm_{\lambda}(\square_{\epsilon,\theta})\) is finite-dimensional since \(e^{\square_{\epsilon,\theta}}\) is a compact operator for \(t > 0\). Moreover, since \(D_\epsilon\) commutes with \(\square_{\epsilon,\theta}\), \(D_\epsilon : E^\pm_{\lambda}(\square_{\epsilon,\theta}) \to E^\pm_{\lambda}(\square_{\epsilon,\theta})\) is well defined. Let now \(\alpha \in E^+_{\lambda}(\square_{\epsilon,\theta})\) such that \(D_\epsilon\alpha = 0\). One has then
\[
d\alpha = -\delta_\epsilon \alpha.
\]
This implies that
\[
\|d\alpha\|^2_{L^2(\wedge M, g_\epsilon)} = -\langle d\alpha, \delta_\epsilon \alpha \rangle_{L^2(\wedge M, g_\epsilon)} = 0,
\]
Lemma 3.3

For every $t > 0$, the map $\theta \mapsto \text{Str}(e^{t \Box_{e, \theta}})$ is continuous on $[0, 1]$.

Proof Let $q_{e, \theta}(t, x, y)$ be the heat kernel of $\Box_{e, \theta} = (1 - \theta)\Delta_{H, e} - \theta D_e^2$, $p_{H, \lambda}(t, x, y)$ be the heat kernel of $\Delta_{H, e}$ and $p_e(t, x, y)$ be the heat kernel of $-D_e^2$. Since $-D_e^2$ and $\Delta_{H, e}$ commute, we have

$$e^{t \Box_{e, \theta}} = e^{(1 - \theta)\Delta_{H, e}} e^{-t\theta D_e^2}.$$ 

Therefore:

$$q_{e, \theta}(t, x, y) = \int_M p_{H, \lambda}(t(1 - \theta), x, z) p_e(t\theta, z, y)dz$$

and the result easily follows since

$$\text{Str}(e^{t \Box_{e, \theta}}) = \int_M q_{e, \theta}(t, x, x)dx.$$
**Remark 3.4 (Dependence on the symmetry condition)** It would obviously be beneficial to prove the above statement without the assumption of symmetry on $\Delta_{\mathcal{H},\varepsilon}$. A semigroup approach to non-symmetric horizontal Laplacians has been used, see [15, Appendix A]. In the above proof, however, we really rely on the fact that $\Delta_{\mathcal{H},\varepsilon}$ commutes with the codifferential $\delta_{\varepsilon}$, and with the Laplace–Beltrami operator $-D_{\varepsilon}^2$. We can no longer use these properties if we remove the symmetry assumption.

### 4 Horizontal Chern–Gauss–Bonnet formula

As before, we consider the horizontal Laplacian

$$\Delta_{\mathcal{H},\varepsilon} = -d\delta_{\mathcal{H},\varepsilon} - \delta_{\mathcal{H},\varepsilon}d,$$

and assume that it is symmetric for a fixed $\varepsilon$. As seen earlier, $\Delta_{\mathcal{H},\varepsilon}$ satisfies the Weitzenböck identity

$$\Delta_{\mathcal{H},\varepsilon} = L_{\mathcal{H},\varepsilon} - R_{\varepsilon} = -(\nabla_{\mathcal{H}}^\varepsilon)^*\nabla_{\mathcal{H}}^\varepsilon - R_{\varepsilon}. \quad (4.1)$$

where the later equality follows from [15, Lemma 2.1]. The goal of the section is to compute the pointwise limit

$$\lim_{t \to 0} \text{Str} (p_{\mathcal{H},\varepsilon}(t, x, x))$$

and deduce from it our horizontal Chern–Gauss–Bonnet formula. The computation of that limit will be based on the probabilist method of Brownian Chen series (see [3,7]) which has the advantage of being easily adapted to subelliptic operators like $\Delta_{\mathcal{H},\varepsilon}$, see [2]. For convenience and to introduce notation, we include in Appendix A.2 the main elements of that theory.

A first step to implement the method in [2] is to study the small-time heat kernel asymptotics of a diffusion tangent to the scalar horizontal Laplacian $\Delta_{\mathcal{H}}$. Since we assume that $\Delta_{\mathcal{H},\varepsilon}$ is symmetric, from Corollary 2.5 one has $T\mathbb{M} = \mathcal{H} + [\mathcal{H}, \mathcal{H}]$, and thus the tangent diffusion will take its values in a two-step Carnot group [the so-called tangent cone, see Corollary 2.5(b)] for which an explicit formula for the heat kernel is known (see [10,11]). In a local horizontal frame $\{X_1, \ldots, X_n\}$ around $x_0$ write

$$V_t(x_0) = \sum_{i=1}^{n} \sqrt{2}X_i(x_0)B_t^i + \sum_{1 \leq i < j \leq n} \pi_{\mathcal{V}}([X_i, X_j](x_0)) \int_0^t B_s^i dB_s^j - B_s^i d B_s^j,$$

where $(B_t)_{t \geq 0}$ is a Brownian motion in $\mathbb{R}^n$. We note that $V_t(x_0)$ can be written in a basis free way as

$$\sqrt{2}B_t(x_0) - \int_0^t T(B_s(x_0), dB_s(x_0))$$

where $B_t(x_0) = \sum_{i=1}^{n} X_i(x_0)B_t^i$ is a standard Brownian motion in $\mathcal{H}_{x_0}$.

**Lemma 4.1** Let $x_0 \in \mathbb{M}$. For $t > 0$, let $d_t(x_0)$ be the density at 0 of the $T_{x_0}\mathbb{M}$ valued random variable $V_t(x_0)$. Then, when $t \to 0$,

$$d_t(x_0) \sim \frac{2^m}{(4\pi t)^{\frac{t}{2}+m}} \int_{V_{x_0}} \det \left( \frac{\sqrt{J_z^s J_z^t}}{\sinh \sqrt{J_z^s J_z^t}} \right)^{1/2} \, dz.$$
Proof The process $(V_t(x_0))_{t \geq 0}$ is the horizontal Brownian motion in the tangent cone $G_{x_0}$ which is a 2-step Carnot group when it is identified with $T_{x_0} \mathbb{M}$ using the group exponential map. The heat kernel of the horizontal Laplacian is known explicitly in 2-step Carnot groups (see [10,11]) which yields the small-time asymptotics.

Remark 4.2 We note that $d_t(x_0)$ is independent of $x_0$ because of Corollary 2.5(b).

In the sequel, we will use the notation $F_I$ (defined with respect to the connection $D = \nabla^\varepsilon$) and $\Lambda_I(B)_t$, as introduced and discussed in Appendix A.2.

Corollary 4.3 It will hold that as $t \to 0$

$$\frac{\text{Str}(p_{H^*,\varepsilon}(t, x_0, x_0)) - d_t(x_0)\mathbb{E}\left(\text{Str}\left(\exp\left(\sum_{I, d(I) \leq n+2m} \Lambda_I(B)_I\right)(x_0)\right) \mid B_1 = 0\right)}{\sqrt{\frac{2m}{2\pi} \int_{\mathbb{M}} \det\left(\frac{\sqrt{J_z^*J_z}}{\sinh\sqrt{J_z^*J_z}}\right)^{1/2}} dz},$$

where $d_t(x_0)$ is the density at 0 of $V_t(x)$, as in Lemma 4.1.

Proof Since $H^*$ is two-step bracket generating, the homogeneous dimension is $Q = \dim H^* + 2 \dim V = n + 2m$. Taking $N = n + 2m$ in Theorem A.1, and applying similar arguments as in the proof of Proposition 4.2 in [3], the corollary follows by recognizing that for $|I| > 2$, $X_I$ is a linear combination of $X_i$, $[X_j, X_k]$ so that when $t \to 0$ the density at 0 of

$$\sum_{I, d(I) \leq n+2m} \Lambda_I(B)_I X_I$$

is equivalent to $d_t(x_0)$ from the previous lemma.

Applying the previous results, we are now able to compute $\lim_{t \to 0} \text{Str}(p_{H^*,\varepsilon}(t, x_0, x_0))$. Choose local orthonormal bases $X_1, \ldots, X_n$ and $Z_1, \ldots, Z_m$ of, respectively, $H^*$ and $V$.

Lemma 4.4 The integral

$$J = J(x_0) = \frac{2^m}{(2\pi)^{\frac{n}{2} + m}} \int_{\mathbb{V}_0} \det\left(\frac{\sqrt{J_z^*J_z}}{\sinh\sqrt{J_z^*J_z}}\right)^{1/2} dz,$$

is a constant, so independent of the point $x_0 \in \mathbb{M}$ chosen. Furthermore, it holds that

$$\lim_{t \to 0} \text{Str}(p_{H^*,\varepsilon}(t, x_0, x_0)) = \begin{cases} \frac{J}{(\frac{n}{2} + m)!} \mathbb{E}\left(\text{Str}\left[A_{x_0}\right] \mid B_1 = 0\right), & \text{if } n \text{ is even} \\
0, & \text{if } n \text{ is odd}. \end{cases}$$

where the random variable $A_{x_0}$ is given by

$$A_{x_0} = -\frac{1}{2} \sum_{i,j,k,l=1}^n \left(R^j_{kli} + \frac{1}{\varepsilon} \sum_{r=1}^m T^r_{kli} T^r_{ij}\right) a_i^* a_j^* a_l a_k$$

$$+ \sum_{1 \leq i < j \leq n} \sum_{r,s=1}^m T^r_{ij} T^s_{ij} b_i^* b_j \int_0^1 B^r_i dB^r_i - B^s_i dB^s_i.$$ (4.2)
Proof First, observe that
\[ \mathcal{J}(x_0) = (2t)^{\frac{n}{2} + m} d_I(x_0), \]
and so the independence of \( \mathcal{J}(x_0) \) from \( x_0 \) follows from Corollary 2.5(b) as in Remark 4.2.

Consider the expansion
\[
\text{Str} \left[ \exp \left( \sum_{I, d(I) \leq n + 2m} \Lambda_I(B_I) \mathcal{F}_I \right)(x_0) \right] = \sum_{k \geq 0} \frac{1}{k!} \text{Str} \left[ \left( \sum_{I, d(I) \leq n + 2m} \Lambda_I(B_I) \mathcal{F}_I \right)^k \right](x_0).
\]

From the Weitzenböck identity (4.1), we have for \( i, j \in \{1, \ldots, n + m\} \) that
\[ \mathcal{F}_0 = -\mathcal{R}_e, \quad \mathcal{F}_i = 0, \quad \mathcal{F}_{(i,j)} = \hat{R}^e_i(Y_i, Y_j) \]
where \( \{Y_1, \ldots, Y_{n+m}\} \) form a local orthonormal frame and the \( \{c_i, e_i^{a} \}_{i=1}^{n+m} \) form the associated Fermion calculus of \( TM \). Equation (2.11) allows us to write
\[
\mathcal{R}_e = \sum_{i,j,k=1}^n (\hat{R}^e_i(X_i, X_k)X_j, X_i)g^{a}b^{a}a_i + \sum_{i,j,k,l} (\hat{R}^e_i(X_i, X_j)X_k, X_l)g^{a}b^{a}a_i a_k
\]
where \( \{a_i, b^a\} \) form the Fermion calculus for \( \mathcal{H} \).

Recalling equation (A.1) in the appendix, we see that the supertrace will vanish for any term that is not of full degree; from our expressions for \( \mathcal{F}_I \), it is thus clear that for \( k < \frac{n}{2} + m \)
\[
\text{Str} \left[ \left( \sum_{I, d(I) \leq n + 2m} \Lambda_I(B_I) \mathcal{F}_I \right)^k \right](x_0) = 0.
\]

Let us assume that \( n \) is even. Applying the scaling property of Brownian motion, when \( t \to 0 \) the term \( k = \frac{n}{2} + m \) will be dominant. More precisely,
\[
\mathbb{E} \left( \text{Str} \left[ \exp \left( \sum_{I, d(I) \leq n + 2m} \Lambda_I(B_I) \mathcal{F}_I \right)(x_0) \mid B_1 = 0 \right] \right) = \frac{1}{(\frac{n}{2} + m)!} \mathbb{E} \left( \text{Str} \left[ \left( \sum_{I, d(I) \leq n + 2m} \Lambda_I(B_I) \mathcal{F}_I \right)^{\frac{n}{2} + m} (x_0) \right] \mid B_1 = 0 \right) + O \left( t^{\frac{n}{2} + m + \frac{1}{2}} \right). \tag{4.3}
\]

Then, we have
\[
\mathbb{E} \left( \text{Str} \left[ \left( \sum_{I, d(I) \leq n + 2m} \Lambda_I(B_I) \mathcal{F}_I \right)^{\frac{n}{2} + m} (x_0) \right] \mid B_1 = 0 \right) = \mathbb{E} \left( \text{Str} \left[ (-t^{\frac{n}{2}} \mathcal{F}(x_0) + \sum_{1 \leq i < j \leq n} \sum_{s=1}^{l} \hat{R}^{\frac{s}{2} \times} b_i b_j f_i^s f_j^s B_i d B_j - B_i d B_j) \right)^{\frac{n}{2} + m} \right] \mid B_1 = 0 \right) + O \left( t^{\frac{n}{2} + m + \frac{1}{2}} \right). \tag{4.4}
\]

We can further simplify this expression using that by Lemma A.2, Appendix, we know that \( \hat{R}^{\frac{s}{2} \times} = R^{\frac{s}{2} \times} = T^{\frac{s}{2} \times} \). We also use (2.11) and the fact that only the last term in \( \mathcal{R}_e \) contributes to the supertrace. Combining Lemma 4.1, Corollary 4.3, and Eqs. (4.3) and (4.4), we apply the scaling property of Brownian motion again to find
\[
\text{Str}(p_{\mathcal{H}, e}(t, x_0, x_0)) = \frac{\mathcal{J}}{(\frac{n}{2} + m)!} \mathbb{E} \left( \text{Str} \left[ A^\frac{n}{2} + m \right] \right) \left( B_1 = 0 \right) + O \left( t^{\frac{1}{2}} \right).
\]

If \( n \) is odd, we get by similar arguments that
\[
\text{Str}(p_{\mathcal{H}, e}(t, x_0, x_0)) = O \left( t^{\frac{1}{2}} \right).
\]
Proposition 4.6

In what follows, we will introduce the tensor \( \mathcal{T} \) by
\[
\mathcal{T}(Y_1, Y_2) = \tilde{R}^e(\pi_H Y_1, Y_2)\pi_V = \pi_V \tilde{R}^e(\pi_H Y_1, Y_2).
\]

We observe that for any \( X_1, X_2 \in \Gamma(H) \) and \( Z \in \mathcal{V} \),
\[
\mathcal{T}(X_1, X_2)Z = (\nabla_Z T)(X_1, X_2) = \frac{1}{2e} (T(J_Z X_1, X_2) + T(X_1, J_Z X_2)),
\]
where the latter equality follows from the symmetry condition of \( \Delta \gamma_{e,t} \).

Example 2.4 (H-type foliation) We again consider the case of the of H-type foliations as in Example 2.4. We recall that in this case, we have that \( \Delta \gamma_{e,t} \) for \( e = \frac{1}{2} \). Let \( x \in \mathbb{M} \) be a fixed point and let \( CL(\mathcal{V}_x) \) be the Clifford algebra of the vertical space. We remark that in this case, for any \( u, v \in \gamma_{e,t} \) with \( v \in (\text{span}_{\xi \in CL(\mathcal{V}_x)} J_\xi u)^\perp \), we have \( \mathcal{T}(u, v) = 0 \). On the other hand, if \( v = J_\xi u \), then for any \( z \in \mathcal{V}_x \),
\[
\mathcal{T}(u, J_\xi u)z = k\pi_{\mathcal{V}_x}(z \cdot \zeta_{odd}),
\]
where \( \zeta_{odd} \) is the odd part of \( \zeta \) and \( \pi_{\mathcal{V}_x} CL(\mathcal{V}_x) \to \mathcal{V}_x \) is the projection to the first-order part.

We can use the above definition and the previous lemma to prove the following.

Proposition 4.6 Assume that \( n \) or \( m \) is odd, then
\[
\lim_{t \to 0} Str(p_{\gamma_{e,t}}(t, x, x)) dx = 0
\]
Assume that both \( n \) and \( m \) are even, then
\[
\lim_{t \to 0} Str(p_{\gamma_{e,t}}(t, x, x)) dx = \hat{\omega}^e_{\gamma_{e,t}} \det\left( \frac{\mathcal{T}}{\sinh(\mathcal{T})} \right)^{1/2} \]
where \([-]_m \) denotes the \( m \)-form part and \( \hat{\omega}^e_{\gamma_{e,t}} \) is the horizontal Euler form, locally defined as
\[
\hat{\omega}^e_{\gamma_{e,t}} = \frac{(-1)^{n/2}m!}{2n^2/2 + m!} \sum_{\sigma, \tau \in \mathcal{S}_n} \epsilon(\sigma) \epsilon(\tau) \prod_{i=1}^{n-1} \tilde{R}^e,_{\sigma(i)\sigma(i+1)} \tau(i) dx_{\gamma_{e,t}},
\]
In the above formula, \( \mathcal{S}_n \) is the set of the permutations of the indices \( \{1, \ldots, n\} \), \( \epsilon \) the signature of a permutation, \( \tilde{R}^e,_{i,j,k} \) is as in (2.7) and \( dx_{\gamma_{e,t}} \) the \( n \)-form \( X_1^* \wedge \cdots \wedge X_n^* \).

Proof We first assume that both \( n \) and \( m \) are even. It remains to compute \( E \left( \text{Str} \left[ A_{x_0}^{\frac{n}{2}+m} \right] B_1 = 0 \right) \).
Looking at (4.2), we have
\[
E \left( \text{Str} \left[ A_{x_0}^{\frac{n}{2}+m} \right] B_1 = 0 \right) = \text{Str} \left[ -\sum_{i,j,k,l} (\tilde{R}^e(X_i, X_j)X_k, X_l) g_{a_i^* a_j^* a_k a_l} \right]^{n/2} \left[ \sum_{1 \leq i < j \leq n} \mathcal{T}(X_i, X_j)(x_0) \int_0^1 B_i^j d B_j^i - B_i^j d B_j^i \right]^{m/2} \bigg| B_1 = 0 \right)
\]
The term \( \left( \sum_{i,j,k,l} (\tilde{R}^e(X_i, X_j)X_k, X_l) g_{a_i^* a_j^* a_k a_l} \right)^{n/2} \) is then analyzed as in the proof of Proposition 5.6 in [7] (see also Lemma 2.35 in [19]) and up to constant yields the horizontal Euler form \( \hat{\omega}^e_{\gamma_{e,t}} \). On the other hand, using again the formula for the supertrace, the term
\[
E \left[ \left( \sum_{1 \leq i < j \leq n} \mathcal{T}(X_i, X_j)(x_0) \int_0^1 B_i^j d B_j^i - B_i^j d B_j^i \right)^m \bigg| B_1 = 0 \right]
\]
can be replaced with
\[
m! \mathbb{E} \left[ \exp \left( \sum_{1 \leq i < j \leq n} \mathcal{F}(X_i, X_j)(x_0) \int_0^1 B^i_s d B^j_s - B^j_s d B^i_s \right) \right] B_1 = 0
\]
and is analyzed using the Lévy area formula as in the proof of Theorem 4.3 in [3]: it yields the top degree Fermionic piece of \( \det \left( \frac{\mathcal{F}}{\sinh(\mathcal{F})} \right)^{1/2} (x_0) \in \mathbf{End} (\wedge_{x_0} V^*) \) (Fermionic calculus is done here on \( V_{x_0} \)).

If \( n \) is even and \( m \) is odd, a similar analysis shows that
\[
\mathbb{E} \left( \text{Str} \left[ A_{x_0}^{\frac{n}{2}+m} \right] \middle| B_1 = 0 \right) = 0.
\]

Combining Theorem 3.1 and Proposition 4.6 finally yields our main theorem:

**Theorem 4.7** Assume that both \( n \) and \( m \) are even, then
\[
\chi(M) = \int_M \hat{\omega} \wedge \left[ \det \left( \frac{\mathcal{F}}{\sinh(\mathcal{F})} \right)^{1/2} \right]_m.
\]
Assume that \( n \) or \( m \) is odd, then \( \chi(M) = 0 \).

As a corollary, since \( \nabla J = 0 \) implies \( \mathcal{F} = 0 \), we obtain the following result:

**Corollary 4.8** Assume that \( \nabla J = 0 \), then \( \chi(M) = 0 \).

**Funding** Open access funding provided by University of Bergen (incl Haukeland University Hospital).

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**A Appendices**

**A.1 Fermion calculus and supertraces**

In this section, we recall some basic elements of Fermion calculus, see section 2.2.2 in [19] for more details. Let \( V \) be a \( d \)-dimensional Euclidean vector space. We denote \( V^* \) its dual and \( \wedge V^* = \bigoplus_{k \geq 0} \wedge^k V^* \), its exterior algebra. If \( u \in V^* \), we denote \( a_u^* \) the map \( \wedge V^* \rightarrow \wedge V^* \), such that \( a_u^*(\omega) = u \wedge \omega \). The dual map is denoted \( a_u \). Let now \( \theta_1, \ldots, \theta_d \) be an orthonormal basis of \( V^* \). We denote \( a_i = a_{\theta_i} \). If \( I \) and \( J \) are two words with \( 1 \leq i_1 < \cdots < i_k \leq d \) and \( 1 \leq j_1 < \cdots < j_l \leq d \), we denote
\[
A_{IJ} = a_{i_1}^* \cdots a_{i_k}^* a_{j_1} \cdots a_{j_l}.
\]
The family of all the possible $A_{IJ}$ forms a basis of the $2^d$-dimensional vector space $\text{End} (\wedge V^*)$.

If $A \in \text{End} (\wedge V^*)$, the supertrace $\text{Str}(A)$ is the difference of the trace of $A$ on even forms minus the trace of $A$ on odd forms. If $A = \sum_{I,J} c_{IJ} A_{IJ}$, then we have

$$\text{Str}(A) = (-1)^{d(d-1)/2} c_{[1,\ldots,d][1,\ldots,d]}.$$  \hspace{1cm} (A.1)

In this paper, $c_{[1,\ldots,d][1,\ldots,d]}$ will be called the top degree Fermionic piece of $A$ and $[A]_d := (-1)^{d(d-1)/2} c_{[1,\ldots,d][1,\ldots,d]} \theta_1 \wedge \cdots \wedge \theta_d$ the $d$-form part of $A$.

### A.2 The Brownian Chen series parametrix method

For the sake of completeness and to introduce some notations used in the paper, we reproduce here the essential ideas from [2,3,7] to which we refer for further details. Let $E$ be a finite-dimensional vector bundle over a compact manifold $M$ equipped with a connection $D$ and consider a second-order differential operator $L = D_0^2 + \sum_{i=1}^d D_i^2$ with $D_i = \mathcal{F}_i + D X_i$ for some smooth vector fields $X_i$ and potentials $\mathcal{F}_i$ on $E$. It is known that the differential equation

$$\frac{\partial \Phi}{\partial t} = L \Phi, \hspace{0.5cm} \Phi(0, x) = f(x)$$

has solution

$$\Phi(t, x) = (e^{tL} f)(x) = P_t f(x).$$

At strongly regular points $x_0 \in \mathbb{M}$, it is furthermore true that $P_t$ admits a smooth heat kernel $p_t(x_0, \cdot): \mathbb{R}_{\geq 0} \to \Gamma(\mathbb{M}, \text{Hom}(E))$

$$t \mapsto p_t(x_0, \cdot)$$

which is to say

$$(P_t f)(x_0) := (e^{tL} f)(x_0) = \int_{\mathbb{M}} p_t(x_0, y) f(y) \, dy.$$

We have a method of approximation for the heat kernel in this setting.

**Theorem A.1** Let $N \geq 1$ and define $(P_t^N f)(x) = \mathbb{E}(\Psi(1, x))$ where $\Psi(\tau, x)$ solves the random differential equation

$$\frac{\partial \Psi}{\partial \tau} = \sum_{I: d(I) \leq N} \Lambda_I(B)_{\tau} (D_I \Psi)(\tau, x), \hspace{0.5cm} \Psi(0, x) = f(x).$$  \hspace{1cm} (A.2)

where $I = (i_1, \ldots, i_k) \in \{0, \ldots, d\}^k$ is a word, $D_I = [D_{i_1}, \ldots, [D_{i_{k-1}}, D_{i_k}], \ldots]$, $d(I) = n(I) + k$ with $n(I)$ the number of 0’s in $I$, and the random coefficients are defined by

$$\Lambda_I(B)_{\tau} = 2^{d(I)/2} \sum_{\sigma \in \mathcal{S}_k} \frac{(-1)^{e(\sigma)}}{k^2 (e(\sigma))} \int_{\Delta^I(0, \tau)} \circ dB^{e^{-1}(I)}$$

where $(B_t)_{t \geq 0}$ is a standard Brownian motion in $\mathbb{R}^d$. Then,
For $k \geq 0$, define the norm
\[ \| f \|_k = \sup_{0 \leq l \leq k} \sup_{0 \leq i_1, \ldots, i_k \in \mathcal{M}} \| D_{i_1} \cdots D_{i_k} f(x) \|. \]

It will hold that for any $k \geq 0$
\[ \| P_t f - P^N_t f \|_k = O \left( t^{\frac{N+1}{2}} \right), \quad t \to 0 \]

$P^N_t$ admits a smooth kernel $p^N_t$ such that for $N \geq 2$
\[ p_t(x_0, x_0) = p^N_t(x_0, x_0) + O \left( t^{\frac{N+1-Q}{2}} \right), \quad t \to 0 \]

where $Q$ is the homogeneous dimension at $x_0$.

Write $\mathcal{F}_I = D_I - D_{X_I}$. For $N \geq 2$, it holds as $t \to 0$ that
\[ p^N_t(x_0, x_0) = d^N_t(x_0) \left( \exp \left( \sum_{I, d(I) \leq N} \Lambda_I(B)_t \mathcal{F}_I \right)(x_0) \right) \left( \sum_{I, d(I) \leq N} \Lambda_I(B)_t X_I(x_0) = 0 \right) + O \left( t^{\frac{N+1-Q}{2}} \right) \]

where $d^N_t(x)$ is the density at 0 of the random variable $\sum_{I, d(I) \leq N} \Lambda_I(B)_t X_I(x)$.

We refer to Baudoin [2] and Baudoin [7, Section 5.1] for the proofs and further details, but we remark that roughly the theorem says that in small time we can approximate the heat kernel of $\mathcal{L}$ by the kernel associated with solutions of Eq. (A.2), for which we will be able to say much more.

### A.3 Curvature of the connection $\hat{\nabla}^\varepsilon$

We want to give details on writing the curvatures of $\hat{\nabla}^\varepsilon$ in terms of the Bott connection $\nabla$.

#### Lemma A.2
Relative to the notation of (2.7) we have the following identities. Recall that $i, j, k, l$ denotes vector fields from a basis of $\mathcal{H}$, while indices $r, s$ denotes such elements from $\mathcal{V}$

(i) $R^l_{ijk} = R^l_{kli}$, $R^2_{ijsr_1} = R^2_{r_1sir_1}$,
(ii) $R^l_{ijr} = T^s_{ij;r}, R^l_{irk} = 0$, $R^2_{ijsr_1} = 0$,
(iii) $T^r_{ij;r} = 0$. Equivalently $(\nabla_Z J)Z = 0$ for any vector field $Z$ with values in $\mathcal{V}$.
(iv) $R^l_{ijk} = R^l_{ijk} + \frac{1}{\varepsilon} \sum_{s=1}^m T^s_{ij} T^s_{kl}$.
(v) $\hat{R}^l_{irk} = \frac{1}{\varepsilon} T^s_{kl;i}$.
(vi) $\hat{R}^l_{rsk} = 2 T^s_{kl;r} + \frac{1}{\varepsilon^2} \sum_{i=1}^n (T^s_{il} T^s_{ki} - T^s_{il} T^s_{ki})$

**Proof** From (2.3), we observe that
\[ \hat{R}^\varepsilon(X, Y)Z = R(X, Y)Z + \frac{1}{\varepsilon} (\nabla_X J)Y Z - \frac{1}{\varepsilon} (\nabla_Y J)X Z \]
\[ + \frac{1}{\varepsilon} J_T(X, Y)Z + \frac{1}{\varepsilon^2} [J_X, J_Y]Z. \]  

We will also use the first Bianchi identity for connections with torsion
\[ \bigcirc R(X, Y)Z = \bigcirc (\nabla_X T)(X, Y) + \bigcirc T(T(X, Y), Z), \]
where $\bigcirc$ denotes the cyclic sum. We furthermore observe the following identities.
(i) Since \( \langle T(Y_1, Y_2), Y_3 \rangle \) and \( T(T(Y_1, Y_2), Y_3) \) vanishes if \( Y_1, Y_2, Y_3 \) are either all vertical or all horizontal,

\[
\begin{align*}
(R(X_1, X_2)X_3, X_4)_g &= (R(X_3, X_4)X_1, X_2)_g, \\
(R(Z_1, Z_2)Z_3, Z_4)_g &= (R(Z_3, Z_4)Z_1, Z_2)_g,
\end{align*}
\]

for any \( X_i \in \Gamma(\mathcal{H}), Z_i \in \Gamma(\mathcal{V}), i = 1, 2, 3, 4. \)

(ii) From Grong [14, Appendix A], we know that for \( X_1, X_2 \in \Gamma(\mathcal{H}), Z_1, Z_2 \in \Gamma(\mathcal{V}), \)

\[ R(X_1, X_2)Z_1 = (\nabla_{Z_1}T)(X_1, X_2), \quad R(X_1, Z_1)X_2 = 0 \quad R(X_1, Z_1)Z_2 = 0. \]

(iii) Since \( \nabla \) is compatible with the metric then \( (\nabla_Z J)_Z = 0 \) for any \( Z \in \Gamma(\mathcal{V}) \), as for any \( X_1, X_2 \in \Gamma(\mathcal{H}), \)

\[ 0 = \langle Z, R(X_1, X_2)Z \rangle_g = \langle Z, \nabla_{Z}T)(X_1, X_2) \rangle_g = \langle X_2, (\nabla_Z J)_Z X_1 \rangle. \]

(iv) We observe first that from (A.3), for any \( X_1, X_2, X_3, X_4 \in \Gamma(\mathcal{H}) \)

\[
\langle \hat{R}^e(X_1, X_2)X_3, X_4 \rangle_g = (R(X_1, X_2)X_3, X_4)_g + \frac{1}{\varepsilon} \langle J_T(X_1, X_2)X_3, X_4 \rangle_g
\]

\[ \overset{(i)}{=} (R(X_3, X_4)X_1, X_2)_g + \frac{1}{\varepsilon} \langle T(X_1, X_2), T(X_3, X_4) \rangle_g. \]

(v) Next, for any \( X_1, X_2 \in \Gamma(\mathcal{H}), Z \in \Gamma(\mathcal{V}), \)

\[ \hat{R}^e(X_1, Z)X_2 \overset{(ii)}{=} \frac{1}{\varepsilon} (\nabla_{X_1} J)_Z X_2. \]

(vi) For the final property observe that

\[ R(Z_1, Z_2)X_1 \overset{(ii)}{=} R(Z_1, Z_2)X_1 = 0. \]

Hence,

\[
\hat{R}^e(Z_1, Z_2)X_1 \overset{(iii)}{=} \frac{1}{\varepsilon} (\nabla_{Z_1} J)Z_2 X_1 - \frac{1}{\varepsilon} (\nabla_{Z_2} J)Z_1 X_1 + \frac{1}{\varepsilon^2} [J_{Z_1}, J_{Z_2}]X_1
\]

\[ = \frac{2}{\varepsilon} (\nabla_{Z_1} J)Z_2 X_1 + \frac{1}{\varepsilon^2} [J_{Z_1}, J_{Z_2}]X_1. \]

\[ \square \]

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