A Note on the Limiting Spectral Distribution of a Symmetrized Auto-Cross Covariance Matrix

Zhidong Bai and Chen Wang

Abstract

In Jin et al. (2014), the limiting spectral distribution (LSD) of a symmetrized auto-cross covariance matrix is derived using matrix manipulation, with finite \((2 + \delta)\)-th moment assumption. Here we give an alternative method using a result in Bai and Silverstein (2010), in which a weaker condition of finite 2nd moment is assumed.

1 Introduction

Consider a large dimensional dynamic \(k\)-factor model with lag \(q\) taking the form of

\[
R_t = \sum_{i=0}^{q} \Lambda_i F_{t-i} + e_t, \quad t = 1, \ldots, T
\]

where \(\Lambda_i\)'s are \(N \times k\) non-random matrices with full rank. For \(t = 1, \ldots, T\), \(F_t\)'s are \(k\)-dimensional vectors of independent identically distributed (i.i.d.) standard complex components and \(e_t\)'s are \(N\)-dimensional vectors of i.i.d. complex components with mean zero and finite second moment \(\sigma^2\), independent of \(F_t\). This can also be considered as a type of information-plus-noise model (Dozier and Silverstein, 2007a, b; Bai and Silverstein, 2012) where the information comes from the summation part and the noise is \(e_t\)'s. Here both \(k\) and \(q\) are fixed but unknown, while both \(N\) and \(T\) tend to \(\infty\) proportionally.

Under this high dimensional setting, an important statistical problem is the estimation of \(k\) and \(q\) (Bai and Ng, 2002; Harding, 2012). To this objective, the following two variables are defined for fixed non-negative integer \(\tau\), namely:

\[
\Phi_N(\tau) = \frac{1}{2T} \sum_{j=1}^{T} (R_j R_{j+\tau}^* + R_{j+\tau} R_j^*)
\]

and

\[
M_N(\tau) = \sum_{j=1}^{T} (\gamma_j \gamma_{j+\tau}^* + \gamma_{j+\tau} \gamma_j^*),
\]

where \(\gamma_j = \frac{1}{\sqrt{2T}} e_j\) and \(*\) denotes the conjugate transpose.

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Note that when $\tau = 0$, we have $M_N(\tau) = \frac{1}{T} \sum_{j=1}^{T} e_j e_j^*$, which is a sample covariance matrix, whose LSD follows MP law (Marčenko and Pastur, 1967) with density
\[
f_c(x) = \frac{1}{2\pi c x} \sqrt{(b_c - x)(x - a_c)}, x \in [a_c, b_c]
\]
and a point mass $1 - 1/c$ at the origin if $c > 1$. Here $c = \lim_{N \to \infty} N/T$, $a_c = (1 - \sqrt{c})^2$ and $b_c = (1 + \sqrt{c})^2$.

Moreover, if we write
\[
\Lambda = (\Lambda_0, \Lambda_1, \ldots, \Lambda_q)_{N \times k(q+1)},
\]
then the covariance matrix of $R_t$ will be similar to
\[
\begin{pmatrix}
\sigma^2 I + \Lambda^* \Lambda & 0 \\
0 & \sigma^2 I
\end{pmatrix},
\]
with the size of the upper block and lower block $k(q + 1)$ and $N - k(q + 1)$, respectively. Thus, we have a spiked population model (Johnstone, 2001; Baik and Silverstein, 2006; Bai and Yao, 2008). In fact, under certain conditions, $k(q + 1)$ can be estimated by counting the number of eigenvalues of $\Phi_N(0)$ that go beyond the certain phase transition point. Therefore, it remains to estimate one of $k$ and $q$. To this end, it is necessary to investigate the LSD of $M_N(\tau)$ for at least one $\tau \geq 1$. As such, Jin et al. (2014) has established the following result.

**Theorem 1.1** (Theorem 1.1 in Jin et al. (2014)) Assume:

(a) $\tau \geq 1$ is a fixed integer.

(b) $e_k = (\varepsilon_{1k}, \ldots, \varepsilon_{Nk})^t$, $k = 1, 2, ..., T + \tau$, are $N$ dimensional vectors of independent standard complex components with $\sup_{1 \leq i \leq N, 1 \leq t \leq T + \tau} E|\varepsilon_{it}|^{2+\delta} \leq M < \infty$ for some $\delta \in (0, 2]$, and for any $\eta > 0$,
\[
\frac{1}{\eta^{2+\delta} NT} \sum_{i=1}^{N} \sum_{t=1}^{T+\tau} E(|\varepsilon_{it}|^{2+\delta} I(|\varepsilon_{it}| \geq \eta T^{1/(2+\delta)}) = o(1). \tag{1}
\]

(c) $N/(T + \tau) \to c > 0$ as $N, T \to \infty$.

(d) $M_N = \sum_{k=1}^{T} (\gamma_k \gamma_{k+\tau}^* + \gamma_{k+\tau} \gamma_k^*)$, where $\gamma_k = \frac{1}{\sqrt{2T}} e_k$.

Then as $N, T \to \infty$, $F_{M_N} \overset{D}{\to} F_c$ a.s. and $F_c$ has a density function given by
\[
\phi_c(x) = \frac{1}{2\pi c} \sqrt{\frac{y_0^2}{1+y_0} - (\frac{1-c}{|x|} + \frac{1}{\sqrt{1+y_0}})^2}, |x| \leq a,
\]
where
\[
a = \begin{cases} 
\frac{(1-c)\sqrt{1+y_0}}{y_0-1}, & c \neq 1, \\
\frac{1}{2}, & c = 1,
\end{cases}
\]
$y_0$ is the largest real root of the equation:

$$y^3 - \frac{(1-c)^2 - z^2}{z^2}y^2 - \frac{4}{z^2}y - \frac{4}{z^2} = 0$$

and $y_1$ is the only real root of the equation:

$$(1 - c^2)^2 - y^2 + y - 1 = 0$$

(2)

such that $y_1 > 1$ if $c < 1$ and $y_1 \in (0, 1)$ if $c > 1$. Further, if $c > 1$, then $F_c$ has a point mass $1 - 1/c$ at the origin.

In Jin et al. (2014), the key step of the proof of Theorem 1.1 is to establish that the Stieltjes transform $m$ of $F_c$ satisfies

$$(1 - c^2 m^2(z))(c + czm(z) - 1)^2 = 1,$$

(3)

from which four roots are obtained:

$$m_1(z) = \frac{(1-c) + \sqrt{1 + y_0}}{2c} + \sqrt{(1-c) - \frac{1}{\sqrt{1+y_0}})^2 - \frac{y_0^2}{1+y_0}}$$

$$m_2(z) = \frac{(1-c) + \sqrt{1 + y_0}}{2c} - \sqrt{(1-c) - \frac{1}{\sqrt{1+y_0}})^2 - \frac{y_0^2}{1+y_0}}$$

$$m_3(z) = \frac{(1-c) - \sqrt{1 + y_0}}{2c} + \sqrt{(1-c) + \frac{1}{\sqrt{1+y_0}})^2 - \frac{y_0^2}{1+y_0}}$$

$$m_4(z) = \frac{(1-c) - \sqrt{1 + y_0}}{2c} - \sqrt{(1-c) + \frac{1}{\sqrt{1+y_0}})^2 - \frac{y_0^2}{1+y_0}}.$$

Here $y_0$ is the largest real root of the equation:

$$f(y) := y^3 - \frac{(1-c)^2 - z^2}{z^2}y^2 - \frac{4}{z^2}y - \frac{4}{z^2} = 0.$$

Note that all the three roots of $f(y) = 0$ give the same set of $m_i$’s, up to a permutation order, and our choice $y_0$ as the largest real root is only for the sake of simplicity.

For the four $m_i$’s, after some justification, we have

$$m(z) = \begin{cases} 
  m_1(z), & z < 0, \\
  m_3(z), & z > 0.
\end{cases}$$

The density function is then derived using the inversion formula of the Stieltjes transform.

Figures 1 and 2 display the density functions $\phi_c(x)$ with $c < 1$ and $c > 1$, respectively. From these two figures, it is shown that as $c$ increases, the support of $\phi_c(x)$ gets wider, and $\phi_c(x)$ achieves the maximum at $x = 0$ which is sharper as $c$ gets closer to 1.
Figure 1: Density functions $\phi_c(x)$ of the LSD of $M_N$ with $c = 0.2$ (the solid line), $c = 0.5$ (the dashed line) and $c = 0.7$ (the dotted line).

Figure 2: Density functions $\phi_c(x)$ of the LSD of $M_N$ with $c = 1.5$ (the solid line), $c = 2$ (the dashed line) and $c = 2.5$ (the dotted line). Note that the area under each density function curve is $1/c$.

The goal of the paper is devoted to giving a more direct method of deriving (3), by using Theorem 4.1 in Bai and Silverstein (2010). It is worth noting that for our method to work, we only require the finiteness of the 2nd moment of the underlying random variable, which is weaker
than the finite \((2 + \delta)\)-th moment requirement in Jin et al. (2014). Once (3) is obtained, Theorem 1.1 will follow by employing the technique in Jin et al. (2014) and thus will not be presented again.

2 Notation

Before proceeding, it is necessary to rewrite \(M_N(\tau)\) into another form. For any \(\tau \geq 1\) fixed, write

\[
M_N(\tau) = \sum_{k=1}^{T} (\gamma_k \gamma_{k+\tau}^* + \gamma_{k+\tau} \gamma_k^*)
\]

\[
= \frac{1}{2T} \sum_{k=1}^{T} (\mathbf{e}_k \mathbf{e}_{k+\tau}^* + \mathbf{e}_{k+\tau} \mathbf{e}_k^*)
\]

\[
= \frac{1}{T} \begin{pmatrix} \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_{T+\tau-1}, \mathbf{e}_{T+\tau} \end{pmatrix}
\begin{pmatrix} 0 & \cdots & \frac{1}{2} & \cdots & 0 \\ \vdots & \ddots & 0 & \ddots & \vdots \\ \frac{1}{2} & 0 & \ddots & 0 & \frac{1}{2} \\ \vdots & \ddots & 0 & \ddots & \ddots \\ 0 & \cdots & \frac{1}{2} & \cdots & 0 \end{pmatrix}
\begin{pmatrix} \mathbf{e}_1^* \\ \mathbf{e}_2^* \\ \vdots \\ \mathbf{e}_{T+\tau-1}^* \\ \mathbf{e}_{T+\tau}^* \end{pmatrix}
\]

\[
= \frac{1}{T} \mathbf{X}_T \mathbf{C}_{T,\tau} \mathbf{X}_T^*,
\]

where the two bands of \(\frac{1}{2}\)'s are \(\tau\)-distance from the main diagonal.

3 A Useful Lemma

**Lemma 3.1** As \(n \to \infty\), the empirical spectral distribution (ESD) of \(\mathbf{C}_{n,\tau}\) tends to \(H\), which is an Arcsine distribution with density function

\[
H'(t) = \frac{1}{\pi \sqrt{1-t^2}}, \quad t \in (-1, 1).
\]

**Proof.** Fix \(n\) and let \(\lambda\) be an eigenvalue of \(\mathbf{C}_{n,\tau}\).

Define \(D_n = D_{n,\tau} = \det(\lambda \mathbf{I} - \mathbf{C}_{n,\tau})\) (for simplicity, we omit \(\tau\) from the subscript).

When \(n < \tau\), all the entries of \(\mathbf{C}_{n,\tau}\) are 0 and hence we have \(D_n = \lambda^n\). For \(n \geq \tau\), expand along the first row, and we have \(D_n = \lambda D_{n-1} + \frac{(-1)^{\tau}}{2} \tilde{D}_{n-1}\).

Expand along the first column of the matrix wrt \(\tilde{D}_{n-1}\), and we have \(\tilde{D}_{n-1} = \frac{(-1)^{\tau-1}}{2} D_{n-2}\).

Therefore, for \(n \geq \tau\), we have

\[
D_n = \lambda D_{n-1} - \frac{1}{4} D_{n-2}
\]
Solve the characteristic equation \( x^2 = \lambda x - \frac{1}{4} \) and we have \( \lambda_{1,2} = \frac{\lambda \pm \sqrt{\lambda^2 - 1}}{2} \).

Thus, we have, for \( n \geq \tau \),
\[
D_n = \lambda^\tau (a \lambda_1^{n-\tau} + b \lambda_2^{n-\tau}),
\]
where \( a, b \) can be determined by \( D_\tau \) and \( D_{\tau+1} \), i.e.
\[
\lambda^\tau = \lambda^\tau (a + b),
\]
\[
\lambda^{\tau+1} - \frac{\lambda^{\tau-1}}{4} = \lambda^\tau (a \lambda_1 + b \lambda_2).
\]

Substitute \( a, b \) into the equation \( D_n = 0 \) and use the facts that \( \lambda_1 + \lambda_2 = \lambda \) and \( \lambda_1 \lambda_2 = \frac{1}{4} \), we have
\[
\frac{\lambda^{\tau-1}(\lambda_1^{n-\tau+2} - \lambda_2^{n-\tau+2})}{\lambda_1 - \lambda_2} = 0.
\]

Therefore, if \( \lambda \neq 0 \), we must have
\[
\left( \frac{\lambda_2}{\lambda_1} \right)^{n-\tau+2} = 1, \quad \lambda_1 \neq \lambda_2,
\]
from which we obtain \( \frac{\lambda_2}{\lambda_1} = \cos \frac{2k\pi}{n-\tau+2} + i \sin \frac{2k\pi}{n-\tau+2}, \) \( k = 1, 2, \ldots, n - \tau + 1 \) (\( k = 0 \) corresponds to the case \( \lambda_1 = \lambda_2 \) and thus is rejected).

Hence, among the \( n \) eigenvalues of \( C_{n,\tau} \), \( \tau - 1 \) of them are 0 and the rest \( n - \tau + 1 \) are
\[
\lambda = \lambda_1 + \lambda_2 = \cos \frac{k\pi}{n - \tau + 2}, \quad k = 1, 2, \ldots, n - \tau + 1.
\]

Define a uniform random variable \( K \) taking values in \( \{1, 2, \ldots, n - \tau + 1\} \). Then we have
\[
P(\lambda \leq t) = \frac{\tau - 1}{n} I_{[0,\infty)}(t) + \frac{n - \tau + 1}{n} P\left( \frac{K}{n - \tau + 2} \geq \frac{\cos^{-1}(t)}{\pi} \right)
\]
\[
\rightarrow 1 - \frac{\cos^{-1}(t)}{\pi} =: H(t), \quad t \in (-1, 1)
\]

since \( \frac{K}{n - \tau + 2} \xrightarrow{D} \text{Uniform}(0, 1) \) as \( n \to \infty \).

Taking the derivative, we have
\[
H'(t) = \frac{1}{\pi \sqrt{1 - t^2}}, \quad t \in (-1, 1).
\]

The proof of the lemma is complete.

### 4 Derivation of the Stieltjes Transform

To derive the Stieltjes transform, we mainly use Theorem 4.1 in Bai and Silverstein (2010).
**Theorem 4.1** (Theorem 4.1 in Bai and Silverstein (2010)) Suppose that the entries of $X_n$ ($p \times n$) are independent complex random variables satisfying

$$
\frac{1}{\eta^2 np} \sum_{jk} E(|x_{ij}^{(n)}|^2 I(|x_{ij}^{(n)}| \geq \eta \sqrt{n})) \to 0.
$$

(4)

and that $T_n$ is a sequence of Hermitian matrices independent of $X_n$ and that the empirical spectral distribution (ESD) of $T_n$ tends to a non-random limit $H$ in some sense (in probability or a.s.). If $p/n \to y \in (0, \infty)$, then the ESD of the product $S_n T_n$ tends to a nonrandom limit $F$ in probability or almost surely (accordingly), where $S_n = \frac{1}{n}X_nX_n^*$.

**Remark 4.1** Note that the eigenvalues of the product matrix $S_n T_n$ are all real although it is not symmetric, because the whole set of eigenvalues is the same as that of the symmetric matrix $S_n^{1/2} T_n S_n^{1/2}$.

**Remark 4.2** Note that condition (4) can be implied by condition (1), so Theorem 4.1 is applicable to our case. In addition, the $(2 + \delta)$-th moment assumption can be weakened to the 2nd moment condition.

Also, according to (4.4.4) in Bai and Silverstein (2010),

$$
\frac{1}{z} m_F\left(\frac{1}{z}\right) = \frac{1}{y} - 1 + \frac{1}{2\pi iy} \oint_{|\zeta| = \rho} \log \left(1 - z\zeta^{-1} + zy\zeta^{-1} + \zeta^{-2}ym_H\left(\frac{1}{\zeta}\right)\right) d\zeta.
$$

(5)

Replacing $z^{-1}$ by $z$, we have

$$
z m_F(z) = \frac{1}{y} - 1 + \frac{z}{2\pi iy} \oint_{|\zeta| = \rho} \log \left(z - \zeta^{-1} + y\zeta^{-1} + \zeta^{-2}ym_H\left(\frac{1}{\zeta}\right)\right) d\zeta,
$$

(6)

where we have used the fact that the integral for $\log z$ with respect to $\zeta$ on the contour $|\zeta| = \rho$ is 0.

Next, set $\psi(u) = -\frac{1}{u} + y \int_{1+\tau_0}^u dH(t)$. Then (6) becomes

$$
z m_F(z) = \frac{1}{y} - 1 + \frac{z}{2\pi iy} \oint_{|\zeta| = \rho} \log(z - \psi(-\zeta)) d\zeta
$$

$$
= \frac{1}{y} - 1 - \frac{z}{2\pi iy} \oint_{|\zeta| = \rho} \frac{\zeta}{z - \psi(-\zeta)} d\psi(-\zeta)
$$

$$
= \frac{1}{y} - 1 - \frac{z}{2\pi iy} \int_\psi^{-1}(s) \frac{\psi^{-1}(s)}{z - s} ds.
$$

(7)

When $\zeta$ is in the contour $|\zeta| = \rho$ with $\rho \in (0, 1/\tau_0)$, where $\tau_0$ is a truncation point of eigenvalues of $T_n$ as defined in Section 4.3.1 in Bai and Silverstein (2010) and here we can take $\tau_0 = 1 + \varepsilon$ for some $\varepsilon > 0$ by Lemma 3.1 we have $\psi^{-1}(s) = -\zeta$ being bounded. Therefore, we have $s = z$ as the only pole.
Moreover, as the contour \( C \) is the image of the contour \( |\zeta| = \rho \) under the map \( \zeta \mapsto \psi(-\zeta) \) and note that \( \zeta \) lies on a small circle enclosing the origin. Hence, \( C \) encloses the whole complex plane except a small region containing the origin. Also, by Silverstein and Bai (1995), for each \( z \in \mathbb{C}^+ \equiv \{ z \in \mathbb{C} : \Im(z) > 0 \} \), there exists a unique solution \( \xi \in \mathbb{C}^+ \) such that \( z = \psi(-\xi) \). By taking \( \tau_0 \) large enough, we have \( s = z \) in the contour \( C \). Therefore,

\[
zm_F(z) = \frac{1}{y} - 1 + \frac{z}{y}\psi^{-1}(z),
\]

or equivalently,

\[
z = \psi(ym_F + \frac{y - 1}{z}). \tag{8}
\]

Note that

\[
\psi(u) = \frac{-1}{u} + y \int \frac{t}{1+tu} dH(t)
\]

\[
= \frac{y-1}{u} - \frac{y}{\pi u} \int_{-1}^{1} \frac{1}{(1+tu)\sqrt{1-t^2}} dt
\]

\[
= \frac{y-1}{u} - \frac{y}{2\pi u} \int_{0}^{2\pi} \frac{1}{1+u\cos \theta} d\theta
\]

\[
= \frac{y-1}{u} - \frac{y}{2\pi u i} \oint_{|s|=1} \frac{2}{us^2 + 2s + u} ds.
\]

The integrand has two poles at \( s_1 = \frac{-1 + \sqrt{1-u^2}}{u} \) and \( s_2 = \frac{-1 - \sqrt{1-u^2}}{u} \). As \( s_1s_2 = 1 \), we must have one of them is inside the contour and the other is outside. Therefore, we have

\[
\psi(u) = \frac{y-1}{u} - \frac{y}{2\pi u i} \oint_{|s|=1} \frac{2}{us^2 + 2s + u} ds
\]

\[
= \frac{y-1}{u} \pm \frac{2y}{u^2(s_1 - s_2)}
\]

\[
= \frac{y-1}{u} \pm \frac{y}{u\sqrt{1-u^2}}
\]

where the choice of + or − sign is determined by which of \( s_{1,2} \) is inside the contour. Substitute the above expression into (8), and we have

\[
m^2_F z^2 [1 - (ym_F + \frac{y - 1}{z})^2] = 1.
\]

Note that in our question, \( c = \lim_{N \to \infty} N/T = 1/y \). Therefore, the Stieltjes transform \( \tilde{m} \) of the LSD of \( \tilde{M}_N = \frac{1}{N} X_T^* X_T C_{T,r} \) satisfies

\[
\tilde{m}^2 z^2 [1 - (\frac{\tilde{m}}{c} + \frac{1/c - 1}{z})^2] = 1.
\]
Next, the Stieltjes transform $m$ of the LSD of $M_N = \frac{1}{N}X_T C_{T,T} X_T^*$ satisfies $m = \frac{\bar{m}}{c} + \frac{1/c - 1}{z}$ and therefore,

$$(cm - \frac{1 - c}{z})^2 z^2 (1 - m^2) = 1.$$ 

Finally, the Stieltjes transform $m$ of LSD of $M_N = \frac{1}{T}X_T C_{T,T} X_T^*$ satisfies

$$m(z) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\lambda_i - z} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\frac{N}{T} \lambda_i - z} = \lim_{N \to \infty} \frac{1}{cN} \sum_{i=1}^{N} \frac{1}{\lambda_i - \frac{1}{cz}} = \frac{1}{c} m\left(\frac{z}{c}\right).$$

Substituting back to the above equation, we have

$$(czm(z) + c - 1)^2 (1 - c^2 m^2(z)) = 1,$$

which is the same as (3).

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KLASMOE and School of Mathematics and Statistics, Northeast Normal University, P.R.C,
Changchun 130024
Email: baizd@nenu.edu.cn

Department of Statistics and Applied Probability, National University of Singapore, Singa-
pore 117546
Email: wangchen2351@gmail.com