FLUXBRANE AND S-BRANE SOLUTIONS WITH POLYNOMIALS RELATED TO RANK-2 LIE ALGEBRAS

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Composite fluxbrane and S-brane solutions for a wide class of intersection rules are considered. These solutions are defined on a product manifold $R_s \times M_1 \times \ldots \times M_n$ which contains $n$ Ricci-flat spaces $M_1, \ldots, M_n$ with 1-dimensional factor spaces $R_s$ and $M_1$. They are determined up to a set of functions obeying non-linear differential equations equivalent to Toda-type equations with certain boundary conditions imposed. Exact solutions corresponding to configurations with two branes and intersections related to simple Lie algebras $C_2$ and $G_2$ are obtained. In these cases, the functions $H_s(z), s = 1, 2$, are polynomials of degrees (3, 4) and (6, 10), respectively, in agreement with a conjecture put forward previously in Ref. [1]. The S-brane solutions under consideration, for special choices of the parameters, may describe an accelerating expansion of our 3-dimensional space and a small enough variation of the effective gravitational constant.

1. Introduction

In this paper, we deal with the so-called multidimensional fluxbrane solutions (see [1, 3–18] and references therein) that are in fact generalizations of the well-known Melvin solution [2]. (Melvin’s original solution describes the gravitational field of a magnetic flux tube.) In [1], a subclass of generalized fluxbrane solutions was obtained. These fluxbrane solutions are governed by functions $H_s(z) > 0$ defined on the interval $(0, +\infty)$ and obeying a certain set of second-order nonlinear differential equations,

$$\frac{d^2}{dz^2} \left( \frac{z}{H_s} \frac{d}{dz} H_s \right) = \frac{1}{4 B_s} \prod_{s' \in S} H_{s'}^{-A_{s,s'}}, \quad (1.1)$$

with the boundary conditions

$$H_s(+0) = 1, \quad (1.2)$$

$s \in S$ ($S$ is a non-empty set). In (1.1), all $B_s \neq 0$ are constants and $(A_{s,s'})$ is the so-called “quasi-Cartan” matrix $(A_{s,s} = 2)$ coinciding with the Cartan matrix when intersections are related to Lie algebras.

In [1], the following hypothesis was suggested: the solutions to Eqs. (1.1), (1.2) (if they exist) are polynomials when intersections correspond to semisimple Lie algebras. As pointed out in [1], this hypothesis could be readily verified for the Lie algebras $A_n, C_{n+1}$, $m = 1, 2, \ldots$, as was done for black-brane solutions from [22, 23].

In [1], explicit formulae for solutions corresponding to the Lie algebras $A_1 \oplus \ldots \oplus A_1$ and $A_2$ were presented.

In this paper, we consider generalized “flux-brane” solutions depending on the parameter $w = \pm 1$. For $w = +1$, these solutions are coinciding with flux-brane solutions from [1]. For $w = -1$, they describe special S-brane solutions with Ricci-flat factor spaces. (For general S-brane configurations see [19] and references therein.) Here we present new solutions with polynomials $H_s(z)$ related to the algebras $C_2$ and $G_2$.

The S-brane solutions corresponding to the Lie algebras $A_2$, $C_2$ and $G_2$, for special choices of the parameters, may describe an accelerating expansion of “our” 3-dimensional space with small enough variations of the effective gravitational constant [24] (for the case of the $A_2$ algebra see also [25]).

2. Flux- and S-brane solutions with general intersection rules

2.1. The model

We consider a model governed by the action

$$S = \int d^{D}x \sqrt{|g|} \left\{ R[g] - h_{\alpha \beta} g^{MN} \partial_{M} \varphi^{\alpha} \partial_{N} \varphi^{\beta} - \sum_{a \in \Delta} \frac{\theta_{a}}{N_{a}} \exp[2\lambda_{a}(\varphi)](F^{a})^{2} \right\}, \quad (2.1)$$

where $g = g_{MN}(x)dx^{M} \otimes dx^{N}$ is the metric, $\varphi = (\varphi^{a}) \in \mathbb{R}^{l}$ is a vector of scalar fields, $(h_{\alpha \beta})$ is a constant symmetric non-degenerate $l \times l$ matrix ($l \in \mathbb{N}$), $\theta_{a} = \pm 1$,

$$F^{a} = dA^{a} = \frac{1}{N_{a}} F_{M_{1} \ldots M_{n_{a}}} dz^{M_{1}} \wedge \ldots \wedge dz^{M_{n_{a}}} \quad (2.2)$$

is an $N_{a}$-form $(n_{a} \geq 1), \lambda_{a}$ is a 1-form on $\mathbb{R}^{l}: \lambda_{a}(\varphi) = \lambda_{a} \varphi^{a}, a \in \Delta, \alpha = 1, \ldots, l$. In (2.1), we denote $|g| = |\det(g_{MN})|$,

$$\left( F^{a} \right)^{2} = F_{M_{1} \ldots M_{n_{a}}} F^{N_{1} \ldots N_{a}} g^{M_{1} N_{1}} \ldots g^{M_{n_{a}} N_{n_{a}}}, \quad (2.3)$$

$a \in \Delta$, where $\Delta$ is some finite set.
2.2. “Flux-S-brane” solutions

Let us consider a family of exact solutions to the field equations corresponding to the action (2.1) and depending on one variable \( \rho \). These solutions are defined on the manifold

\[ M = (0, +\infty) \times M_1 \times M_2 \times \ldots \times M_n, \]

where \( M_1 \) is a one-dimensional manifold. The solutions read

\[ g = \left( \prod_{s \in S} H_s^{2h_s \cdot d(I_s)/(D-2)} \right) \left\{ w \rho \otimes d \rho + \left( \prod_{s \in S} H_s^{-2h_s} \rho^2 g^1 + \sum_{i=2}^n \left( \prod_{s \in S} H_s^{-2h_s \cdot \delta_{i,s}} \right) g_i^1 \right) \right\}, \]

\[ \exp(\varphi^\alpha) = \prod_{s \in S} H_s^{h_s \chi^s \cdot \Lambda^a_s}, \]

\[ F^a = \sum_{s \in S} \delta^a_s F^s, \]

where

\[ F^s = -Q_s \left( \prod_{s' \in S} H_{s'}^{-A_{s',s}} \right) \rho d \rho \wedge \tau(I_s), \]

\[ \tau(I_s) = \begin{cases} 1 & \text{for } s \in S_e, \\ 0 & \text{for } s \in S_m. \end{cases} \]

The functions \( H_s(z) > 0, \) \( z = \rho^2 \) obey Eqs. (1.1) with the boundary conditions (1.2).

In (2.5), \( g^i = g^{m_i,n_i}(y_i) dy^{m_i}_i \otimes dy^{n_i}_i \) is a Ricci-flat metric on \( M_i, \) \( i = 1, \ldots, n, \)

\[ \delta_{ij} = \sum_{s \in I} \delta_{ij}, \]

is the indicator of \( i \) belonging to \( I: \) \( \delta_{ij} = 1 \) for \( i \in I \) and \( \delta_{ij} = 0 \) otherwise.

The brane set \( S \) is, by definition, a union of two sets:

\[ S = S_e \cup S_m, \quad S = \cup_{a \in \Delta} \{a\} \times \{v\} \times \Omega_{a,v}, \]

\[ v = e, m \quad \text{and} \quad \Omega_{a,e}, \Omega_{a,m} \subset \Omega, \quad \text{where} \quad \Omega = \Omega(n) \]

is the set of all non-empty subsets of \( \{1, \ldots, n\} \). Any brane index \( s \in S \) has the form

\[ s = (a_s, v_s, I_s), \]

where \( a_s \in \Delta \) is the colour index, \( v_s = e, m \) is the electro-magnetic index, and the set \( I_s \in \Omega_{a_s,v_s} \) describes the location of the brane worldvolume.

The sets \( S_e \) and \( S_m \) define electric and magnetic branes, respectively. In (2.6),

\[ \chi_s = +1, -1 \]

for \( s \in S_e, \) \( S_m, \) respectively. In (2.7), the forms (2.8) correspond to electric branes and the forms (2.9) to magnetic branes; \( Q_s \neq 0, s \in S. \) In (2.9) and in what follows,

\[ \bar{I} \equiv I_0 \setminus I, \quad I_0 = \{1, \ldots, n\}. \]

All manifolds \( M_i \) are assumed to be oriented and connected, and the volume \( d_i \)-forms

\[ \tau_i \equiv \sqrt{|g^i(y_i)|} \, dy^1_i \wedge \ldots \wedge dy^d_i, \]

and the parameters

\[ \varepsilon(i) \equiv \text{sign} \det(g^i_{m,n}) = \pm 1 \]

are well defined for all \( i = 1, \ldots, n. \) Here \( d_i = \dim M_i, \)

\[ i = 1, \ldots, n, \quad D = 1+\sum_{i=1}^n d_i. \]

For any \( I = \{i_1, \ldots, i_k\} \in \Omega, i_1 < \ldots < i_k, \) we denote

\[ \tau(I) \equiv \tau_{i_1} \wedge \ldots \wedge \tau_{i_k}, \]

\[ M(I) \equiv M_{i_1} \times \ldots \times M_{i_k}, \]

\[ d(I) \equiv \dim M(I) = \sum_{i \in I} d_i, \]

\[ \varepsilon(I) \equiv \varepsilon(i_1) \ldots \varepsilon(i_k). \]

\( M(I_s) \) is isomorphic to a brane worldvolume (see (2.12)).

The parameters \( h_s \) appearing in the solution satisfy the relations

\[ h_s = K_s^{-1}, \quad K_s = B_{ss}, \]

where

\[ B_{ss'} = d(I_s \cap I_{s'}) + \frac{d(I_s)d(I_{s'})}{2-D} + \chi_s \chi_s' \lambda_{a,s} \lambda_{a,s'} h_{a,\beta}, \]

\[ s, s' \in S, \quad (h_{a,\beta}) = (h_{a,\beta})^{-1}. \] In (2.6), \( \chi_s = h_{a,\beta} \lambda_{a,s}. \)

We will assume that

\[ \text{(i) } \quad B_{ss} \neq 0, \]

for all \( s \in S, \) and

\[ \text{(ii) } \quad \det(B_{ss'}) \neq 0, \]

i.e. the matrix \( (B_{ss'}) \) is nondegenerate. In (2.8), there appears another nondegenerate matrix (the so-called “quasi-Cartan” matrix)

\[ (A_{ss'}) = (2B_{ss'}/B_{ss'}). \]

In (1.1),

\[ B_s = \varepsilon_s K_s Q_s^2, \quad s \in S, \]

where

\[ \varepsilon_s = (-\varepsilon[g])^{(1-\chi_s)/2} \varepsilon(I_s) \theta_{a,s}, \]

\[ s \in S, \quad \varepsilon[g] = \text{sign} \det(g_{MN}). \]

More explicitly, (2.27) reads:

\[ \varepsilon_s = \varepsilon(I_s) \theta_{a,s} \quad \text{for} \quad v_s = e \quad \text{and} \quad \varepsilon_s = -\varepsilon[g] \varepsilon(I_s) \theta_{a,s} \quad \text{for} \quad v_s = m. \]

Due to (2.8) and (2.9), the brane worldvolume dimension \( d(I_s) \) is determined as

\[ d(I_s) = N_{a,s} - 1, \quad d(I_s) = D - N_{a,s} - 1, \]

for \( s \in S_e, S_m, \) respectively. For an \( Sp \)-brane: \( p = p_s = d(I_s) - 1. \)
Restrictions on brane configurations. The solutions presented above are valid if two restrictions on the sets of branes are satisfied. These restrictions guarantee a block-diagonal form of the energy-momentum tensor and the existence of the sigma-model representation (without additional constraints) [20, 27]. These restrictions are:

\[(R1)\quad d(I \cap J) \leq d(I) - 2\]  
\[(R2)\quad d(I \cap J) \neq 0,\]

for any \(I, J \in \Omega_{a,v}, a \in \Delta, v = e, m\) (and here \(d(I) = d(J)\)).

The solu-
tions (3.2) take place when all \(B_s > 0\). Eq. (3.1) is satisfied when all \(\theta_a > 0, \varepsilon[g] = -1\) (e.g., when the metric \(g\) has a pseudo-Euclidean signature \((-,-,\ldots,+)\)) and

\[\varepsilon(I_s) = +1\]

for all \(s \in S\). (The relation (3.3) takes place for \(S\)-brane and fluxbrane solutions.) The second relation (3.2) takes place when all \(d(I_s) < D - 2\) and the matrix \((h_{a\beta})\) is positive-definite (i.e., there are no phantom scalar fields).

Let us consider the second-order differential equations (1.1) with the boundary conditions (1.2) for the functions \(H_s(z) > 0, s \in S\). We will be interested in analytical solutions to Eqs. (1.1) in some disc \(|z| < L\):

\[H_s(z) = 1 + \sum_{k=1}^{\infty} P_s^{(k)} z^k,\]

\[(3.4)\]

where \(P_s^{(k)}\) are constants, \(s \in S\). Substitution of (3.4) into (1.1) gives an infinite chain of relations for the parameters \(P_s^{(k)}\) and \(B_s\). The first relation in this chain

\[P_s \equiv P_s^{(1)} = \frac{1}{4} B_s = \frac{1}{2} K_s Q_s^2,\]

\[(3.5)\]

\(s \in S\), corresponds to the \(z^0\)-term in the decomposition of (1.1).

It can be shown that, for analytic functions \(H_s(z), s \in S\) (3.4) \((z = \rho^2)\), the metric (2.5) is regular at \(\rho = 0\) for \(w = +1\), i.e. in the fluxbrane case.

Let \((A_{ss'})\) be a Cartan matrix of a finite-dimensional semisimple Lie algebra \(G\).

It has been conjectured in [1] that there exist polynomial solutions to Eqs. (1.1), (1.2), having the form

\[H_s(z) = 1 + \sum_{k=1}^{n_s} P_s^{(k)} z^k,\]

\[(3.6)\]

where \(P_s^{(k)}\) are constants, \(k = 1, \ldots, n_s\). Here, \(P_s^{(n_s)} \neq 0, s \in S\), and

\[n_s = 2 \sum_{s' \in S} A^{ss'}\]

\[(3.7)\]

The integers \(n_s\) are components of the so-called twice dual Weyl vector in the basis of simple roots [26].

3.1. Examples of solutions for rank-2 Lie algebras

Consider configurations with two branes, i.e., \(S = \{s_1, s_2\}\).

3.1.1. Solutions in the \(A_1 \oplus A_1\) case

For the Lie algebra \(A_2 = sl(3)\) with the Cartan matrix

\[(A_{ss'}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},\]

\[(3.9)\]

we have [1] \(n_1 = n_2 = 2\):

\[H_s(z) = 1 + P_s z,\]

\[(3.8)\]

with \(P_s \neq 0\) satisfying (3.5).

3.1.2. Solutions in the \(A_2\) case

For the Lie algebra \(A_2 = sl(3)\) with the Cartan matrix \(A_{ss'}\)

\[(3.10)\]

where, here and in what follows, \(P_s\) obey Eq. (3.5), and

\[P_s^{(2)} = \frac{1}{4} P_1 P_2,\]

\[(3.11)\]

3.1.3. Solutions for the Lie algebra \(C_2\)

For the Lie algebra \(C_2 = so(5)\) with the Cartan matrix

\[(A_{ss'}) = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix},\]

\[(3.12)\]

we get from (3.7): \(n_1 = 3\) and \(n_2 = 4\). For the moduli functions we obtain

\[H_1 = 1 + P_1 z + \frac{1}{4} P_1 P_2 z^2 + \frac{1}{36} P_1^2 P_2 z^3,\]

\[(3.13)\]

\[H_2 = 1 + P_2 z + \frac{1}{2} P_1 P_2 z^2 + \frac{1}{9} P_1^2 P_2 z^3 + \frac{1}{144} P_1^3 P_2^2 z^4.\]

\[(3.14)\]
3.1.4. Solutions for the Lie algebra $G_2$

Consider now the exceptional Lie algebra $G_2$ with the Cartan matrix

$$
(A_{ss'}) = \begin{pmatrix}
2 & -1 \\
-3 & 2
\end{pmatrix}.
$$

We get from (3.7) $n_1 = 6$ and $n_2 = 10$. Calculations (using MATHEMATICA and MAPLE) give:

$$
H_1 = 1 + P_1 z + \frac{1}{4} P_1 P_2 z^2 + \frac{1}{18} P_1^3 P_2 z^3 + \frac{1}{144} P_1^3 P_2^2 z^4
+ \frac{1}{3600} P_1^3 P_2^2 z^5 + \frac{1}{129600} P_1^4 P_2^2 z^6,
$$

(3.15)

$$
H_2 = 1 + P_2 z + \frac{3}{4} P_1 P_2 z^2 + \frac{1}{3} P_1^2 P_2 z^3
+ \frac{1}{16} P_1^2 P_2 \left( \frac{1}{23} P_2 + \frac{1}{27} P_1 \right) z^4 + \frac{7}{600} P_1^3 P_2^2 z^5
+ \frac{1}{64} P_1^3 P_2^2 \left( \frac{1}{23} P_2 + \frac{1}{27} P_1 \right) z^6 + \frac{1}{10800} P_1^4 P_2^2 z^7
+ \frac{1}{172800} P_1^5 P_2^2 z^8 + \frac{1}{4656000} P_1^6 P_2^2 z^9
+ \frac{1}{46560000} P_1^7 P_2^2 z^{10}.
$$

(3.16)

$$
\text{The intersection rules are given by Eqs. (2.25).}
$$

We would like to outline some useful relations for the big numbers appearing in the denominators of the polynomial coefficients: $1296 = 6^4$, $1728 = 3(24^2)$, $4656 = 6^6$.

There are at least two ways of calculating the coefficients $P_s^{(k)}$ of the polynomials. The first one (performed by MAPLE) consists in a straightforward substitution of the polynomials (3.6) into the “master” equations (1.1). The second one (carried out with MATHEMATICA) uses recurrent relations for the coefficients $P_s^{(k+1)}$ as functions of other coefficients $P_s^{(1)}$, ..., $P_s^{(k)}$. These recurrent relations were obtained analytically by simply decomposing the “master” equations (1.1) into a power series in the parameters $z$ [28].

4. Conclusions

We have presented explicit formulae for fluxbrane and S-brane solutions governed by polynomials which correspond to Lie algebras: $A_1 \oplus A_1$, $A_2$, $C_2$ and $G_2$. The pairs of moduli functions ($H_1$, $H_2$) in these solutions are polynomials of degrees: $(1, 1)$, $(2, 2)$, $(3, 4)$ and $(6, 10)$, in agreement with a conjecture from Ref. [1]. The general S-brane solutions presented here, governed by polynomials, are new. The fluxbrane solutions related to Lie algebras $C_2$ and $G_2$ are new as well.

Acknowledgement

This work was supported in part by the Russian Foundation for Basic Research grant Nr. 05-02-17478 and by DFG grant Nr. 436 RUS 113/807/0-1.

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