The Class of Self-Oscillating Systems with Analytically Defined Limiting Cycles

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Abstract. It is studied the model that generalizes the well-known equations of the nonlinear oscillation theory (Van der Pol and Rayleigh). The limit cycles of the model are determined by the total energy of the oscillation in the absence of dissipation / energy input into the system. In contrast to the traditional equations of self-oscillations, equations in the Lagrange form are used here. Phase portraits of limit cycles which can be used in engineering are constructed analytically and numerically.

1. Introduction.

The monograph [1] describes a solid-state wave gyroscope, the physical basis of which is a solid-state resonator made of high quality quartz. Undamped standing waves are excited in the resonator, which make it possible to transform this resonator into a gyroscope. A detailed description of the principle of operation is also given in a number of other publications [2–4]. Currently, wave gyroscopes are widely used in technology [5, 6]. The analysis of the properties of such oscillatory systems, the problems of determining their stability and control of parameters are being actively investigated at present [7–10]. The self-oscillating regime of the resonator was considered in detail earlier [11]. There are two thoroughly studied differential equations describing the emergence and existence of steady-state oscillations. These are the Van der Pol equation [12] and the Rayleigh equation [13]. Both of these equations are used to design and analyze the operation of various types of alternators used in electrical and radio engineering. The main mathematical and experimental results are in the instability of the trivial state (rest point) and the existence of a periodic solution with constant amplitude and period. This solution is called a limit cycle.

Consider the hybrid equation obtained from the Van der Pol and Rayleigh equations

\[ \dddot{u} - 2\varepsilon (1 - (1 - a)u^2 - au^2) \dddot{u} + u = 0 \] (1.1)

Here the parameter \( a \) can take on the values \( 0 \leq a \leq 1 \). For \( a = 0 \), equation (1.1) goes over into the van der Pol equation, and for \( a = 1 \), into the Rayleigh equation. If we put \( a = 1/2 \), then we get the equation proposed by Academician V.F. Zhuravlev

\[ \dddot{u} - 2\varepsilon (1 - \frac{1}{2}u^2 + \dot{u}^2) \dddot{u} + u = 0 \] (1.2)

which has an exact solution for any values of the parameter \( \varepsilon \).
2c \cos, 2\sin
\tag{1.3}
\end{equation}
with a period $T = 2\pi$. The limit cycle $u = \sqrt{2} \cos t$ is stable. Note also note that analytical results were obtained for the Van der Pol and Rayleigh equations in the cases when the parameter $\varepsilon \ll 1$.

Previously [14], a general equation was constructed that makes it possible to analytically find limit cycles, as well as to study their stability. Here will be developed the further generalization of the Van der Pol and Rayleigh equations to the case when the free oscillatory system is described by the Lagrangian equations.

2. Lagrangian description of the generalized model.

Let $u$ be a generalized coordinate and $L(u, \dot{u})$ a Lagrange function. The equation of free oscillation has the form
\begin{equation}
\frac{d}{dt} \frac{\partial L}{\partial \dot{u}} - \frac{\partial L}{\partial u} = 0 \tag{2.1}
\end{equation}
Equation (2.1) has the first integral (energy integral)
\begin{equation}
\dot{u} \frac{\partial L}{\partial \dot{u}} - L = E = \text{const} \tag{2.2}
\end{equation}
It is assumed that there exist periodic solutions of equation (2.1).

Let in a mechanical system acts nonlinear friction of the form
\begin{equation}
2\varepsilon \left( E_0 - \left( \dot{u} \frac{\partial L}{\partial \dot{u}} - L \right) \right) \dot{u}, \tag{2.2}
\end{equation}
which provides instability of the rest point and leads to the appearance of limit cycles. In technology, such a self-oscillation mechanism is implemented in the form of electromechanical devices of various types [15].

The equation of motion of a mechanical system with friction of the form (2.2) is written as
\begin{equation}
\frac{d}{dt} \frac{\partial L}{\partial \dot{u}} - \frac{\partial L}{\partial u} = 2\varepsilon \left( E_0 - \left( \dot{u} \frac{\partial L}{\partial \dot{u}} - L \right) \right) \dot{u} \tag{2.3}
\end{equation}
Multiplying (2.1) by $u$, we obtain the equation for the energy of the system
\begin{equation}
\frac{dE}{dt} = -2\varepsilon (E - E_0) \dot{u}^2 \tag{2.4}
\end{equation}
where $E = u \left( \frac{\partial L}{\partial \dot{u}} - L \right)$.

Equation (2.4) shows that if the initial data are such that $E > E_0$, then the energy turns out to be a non-increasing function of time and vice versa. Thus, the form of the limit cycle is determined by the condition $dE / dt = 0$ ($E = E_0$), which for the Lagrange function is written as
\begin{equation}
\dot{u} \frac{\partial L}{\partial \dot{u}} - L = E_0 \tag{2.5}
\end{equation}
For an oscillator described by the equation
\begin{equation}
\ddot{u} = -f(u), f(-u) = -f(u) \tag{2.6}
\end{equation}
with energy integral
\begin{equation}
E = \frac{\dot{u}^2}{2} + \Pi, \quad \Pi = \int_0^u f(u) du \tag{2.7}
\end{equation}
accounting nonlinear friction (2.2) leads to oscillation equation
\begin{equation}
\ddot{u} + 2\varepsilon (E - E_0) \dot{u} + f(u) = 0 \tag{2.8}
\end{equation}
where $\varepsilon$ has the meaning of the feedback coefficient, $E_0$ is a constant that shows how long the energy of the oscillator $E$ can grow so that the negative resistance decreases to zero. If condition (2.5) is
considered as a function on the phase plane \( (u, \dot{u}) \), whose graph forms a closed curve (a set of closed curves – a multiply connected region), then such a region will be the limit cycle of equation (2.8). Let us write the equation of the limit cycle in the explicit form
\[
\frac{\dot{v}^2}{2} + \Pi(u) = \frac{v^2}{2} + \int_{0}^{u} f(u) \, du = E_0, \text{ where } v = \dot{u}
\]

If we put \( v = 0 \), then the amplitude is determined by the equation
\[
\Pi = E_0 \quad (2.9)
\]

Here, by default, it is assumed that for a given value of \( E_0 \) there exist two finite roots of equation (2.9) \( u_1 \) and \( u_2 \). Then the oscillation period will be determined by the expression
\[
\int_{u_1}^{u_2} \frac{du}{\sqrt{2(E_0 - \Pi(u))}} = \frac{T}{2} \quad (1.11)
\]
The value of the period and other properties of the oscillations will obviously be determined by the value of the energy parameter \( E_0 \) and the form of the function \( f(u) \).

3. Examples.
   1. An example of a self-oscillating system with a symmetric limit cycle
   \[
   \ddot{u} + 2\varepsilon(E - E_0)\dot{u} + u^5 \exp(bu^4) = 0; \quad E_0 = 1, \quad \varepsilon = 1/8, \quad b = 2
   \]
   \[
   E = \frac{\dot{u}^2}{2} + \frac{1}{6b}\left(\exp(bu^4) - 1\right)
   \]

   \( E(0) > E > E_0 \)

   2. An example of a self-oscillating system with an asymmetric limit cycle
   \[
   \ddot{u} + 2\varepsilon(E - E_0)\dot{u} + u\exp(bu) = 0; \quad E_0 = 1, \quad \varepsilon = 1/8, \quad b = 0.98
   \]
   \[
   E = \frac{\dot{u}^2}{2} + \left(\frac{u}{b} - \frac{1}{b^2}\right)\exp(bu) + \frac{1}{b^2}
   \]

   \( E_0 > E > E(0) \)
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