The Gravitational Energy of a Point Mass is Finite

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Abstract

We argue that our gravitation equations \cite{1} in the flat space-time lead to the finite proper gravitational energy of a point mass.

1 The Gravitation Equations

Perhaps \cite{2}, gravitation can be described by a tensor field $\psi_{\mu\nu}$ in flat space-time and the Lagrangian action for the test point masses $m$ in this field is of the form

$$L = -mc\left[g_{\alpha\beta}(\psi) \dot{x}^\alpha \dot{x}^\beta\right]^{1/2}, \quad (1)$$

where $\dot{x}^\alpha = dx^\alpha/dt$ and $g_{\alpha\beta}$ is a symmetric tensor whose components are the function of $\psi_{\alpha\beta}$.

The field $\psi_{\mu\nu}$ in flat space-time is an analog of the potential $A_\mu$ of the electromagnetic field. Therefore the field equations for $g_{\mu\nu}(\psi)$ must be invariant under the gauge transformations $\psi_{\mu\nu} \rightarrow \psi_{\mu\nu} + \phi_{\mu\nu}$. The simplest equations of such a kind were proposed in paper \cite{1}. The equations can be written in the form

$$B^\alpha_{\beta\gamma,\alpha} - B^\nu_{\beta\mu} B^\mu_{\gamma\nu} = 0, \quad (2)$$

where the comma denotes the covariant derivative with respect to the metric tensor $\eta_{\mu\nu}$ in pseudo-Euclidean space-time $E$.

In this equations the tensor $B^\alpha_{\beta\gamma}$ is given by

$$B^\gamma_{\alpha\beta} = \Pi^\gamma_{\alpha\beta} - \Pi^\gamma_{\alpha\beta}, \quad (3)$$

(Greek indices run from 0 to 3), where

$$\Pi^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} - (n + 1)^{-1} \left[\delta^\gamma_{\alpha} \Gamma^\epsilon_{\epsilon\beta} - \delta^\gamma_{\beta} \Gamma^\epsilon_{\epsilon\alpha}\right], \quad (4)$$
\[ \Pi_{\alpha\beta} = \Gamma_{\alpha\beta} - (n + 1)^{-1} \left[ \delta_{\alpha}^{\gamma} \Gamma_{\varepsilon\beta} - \delta_{\beta}^{\gamma} \Gamma_{\varepsilon\alpha} \right], \quad (5) \]

\( \Gamma_{\alpha\beta} \) are the Christoffel symbols in \( E \) and \( \Gamma_{\alpha\beta}^{\gamma} \) are the Christoffel symbols of the Riemannian space-time \( V \) of dimension \( n \), whose fundamental tensor is \( g_{\alpha\beta} \).

The tensor \( B_{\alpha\beta}^{\gamma} \) can be formed by replacing the ordinary derivatives in \( \Pi_{\alpha\beta} \) with the covariant ones in \( E \).

\( B_{\beta\gamma,\alpha} \) satisfies also the following identities:

\[ \ast_{\gamma} R_{\alpha\beta\delta} + \ast_{\delta} R_{\beta\alpha\gamma} + \ast_{\gamma} R_{\delta\alpha\beta} = 0, \quad (6) \]

where tensor \( R_{\alpha\beta\delta} \) is obtained from the Riemannian curvature tensor by replacing the Christoffel symbols with the Thomas symbols and by replacing ordinary derivative with the covariant one in \( E \).

The equations (2) and (6) are field equations for the tensor \( B_{\beta\gamma}^{\alpha} \).

Eqs. (2) are invariant under arbitrary transformations of the tensor \( g_{\alpha\beta} \) retaining invariant the equations of motion of a test particle, i.e., geodesic lines in \( V \). In other words, eqs. (2) are the geodesic-invariant. Thus, the tensor field \( g_{\alpha\beta} \) is defined up to geodesic mappings of space-time \( V \) (in the way analogous to the defining the potential \( A_{\mu} \) in electrodynamics up to gauge transformations). Therefore, additional conditions can be imposed on the tensor \( g_{\alpha\beta} \). In particular, if the tensor \( g_{\alpha\beta} \) satisfies the conditions

\[ Q_{\alpha} = \Gamma_{\alpha\sigma}^{\sigma} - \ast_{\alpha\sigma}^{\sigma} = 0, \quad (7) \]

then eqs (2) will be reduced to vacuum Einstein equations \( R_{\alpha\beta} = 0 \), where \( R_{\alpha\beta} \) is the Ricci tensor. Unlike the \( g_{\alpha\beta} \) or \( \Gamma_{\alpha\beta}^{\gamma} \), the tensor \( B_{\alpha\beta}^{\gamma} \) is invariant under the geodesic mappings of space-time \( V \) as well as strength tensor \( F_{\alpha\beta} \) in electrodynamics is invariant under the gauge transformations.

### 2 Spherically Symmetric Field

For a spherically symmetric field the nonzero components of the tensor \( B_{\alpha\beta}^{\gamma} \) in the spherical coordinate system are:

\[ B_{11}^{1} = A'/2A, \quad B_{22}^{1} = -r(A^{-1} - 1), \quad B_{10}^{0} = C'/2C, \]

\[ B_{33}^{1} = -r \sin^{2}(\theta)(A - 1), \quad B_{00}^{1} = CA'/2A, \]

where the functions \( A \), \( B \) and \( C \) are:

\[ A = (f')^{2}(1 - \beta/f)^{-1}, \quad B = f^{2}, \quad C = 1 - \beta/f \quad (8) \]

and

\[ f = (r^{3} + \beta^{2})^{1/3} \]
and \( f' = df/dr \). The constant \( \beta \) is not determined from the boundary conditions \(^1\). If we put \( \beta = \alpha \), where \( \alpha = 2GM/c^2 \), \( M \) is the attractive mass, \( G \) is the gravitational constant and \( c \) is the speed of light, then the solution (8) has no a physical singularity in the sense that this solution does not lead to the collapse. Some astrophysical consequences of this solution are considered in \(^3\).

### 3 The Gravitational Energy of a Point Mass

Eq. (2) without the nonlinear term is analogous to the electrodynamics equations \( F_{\beta,\gamma} = 0 \) for the strength tensor \( F_{\alpha\beta} \). Since the gravitational field must be self-interacting we can suppose that eq. (2) are of the form

\[
B_{\alpha\beta,\gamma}^\gamma = \kappa t_{\alpha\beta},
\]

where \( \kappa = 4\pi C/c^4 \) and

\[
t_{\alpha\beta} = \kappa^{-1} B_{\alpha\gamma} B_{\beta\delta}^\gamma
\]

is the energy-momentum tensor of gravitation field. The components of the 3-momentum density vector

\[
P_i = t_{0i} \quad (i = 1, 2, 3)
\]

are equal to zero for the solution (8).

Let us find the energy of a point masses

\[
\mathcal{E} = \int t^0_0 dV
\]

We have

\[
t_{00} = 2\kappa^{-1} B^1_{00} B^0_{01}
\]

and, therefore, in the used coordinates system

\[
\mathcal{E} = \int t^0_0 dV = - \frac{1}{8} \frac{\alpha^2 c^2}{\pi \gamma} J,
\]

where

\[
J = \int \frac{dV}{f^4} = \frac{4\pi}{3\beta} B(1, 1/3) = \frac{4\pi}{\beta},
\]

where

\[
B(z, w) = \int_0^\infty \frac{t^{z-1}}{(1+t)^{z+w}} dt
\]

is the B-function \(^4\). Using the equation

\[
B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)},
\]
where $\Gamma$ is $\Gamma$-function and setting $\beta = \alpha$, we obtain finally

$$\mathcal{E} = M c^2.$$  \hfill (18)

On the contrary, if we search the constant $\beta$ from the condition that gravitational energy of a point mass is given by equation (18), then we shall arrive at the conclusion that $\beta = \alpha$.

Certainly, the 4-divergency of the tensor $t_{\alpha\beta}$ must be equal to zero. It is likely that the 4-divergency of the tensor $B^\gamma_{\alpha\beta}$ is not equal to zero identically. However, it is should be observed that the conservation law for the energy-momentum tensor must not be is satisfied for an arbitrary field $B^\gamma_{\alpha\beta}$ (or $g_{\alpha\beta}$), and only for the fields which are solutions of eq. (2). For the received solution of the field equations the 4-divergency of the tensor $B^\gamma_{\alpha\beta}$ is indeed equal to zero. Therefore, the tensor $t_{\alpha\beta}$ for the spherical-symmetric field is a conservation value.

The gravitation energy of the point mass being finite, it is sufficient argument to suppose that the energy - momentum of gravitational field of an attracting mass is given by equation (10) up to a term, with the integral over the volume equal to zero.

References

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