Some results of Fekete–Szegö type for Bavrin’s families of holomorphic functions in \( \mathbb{C}^n \)

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Abstract
In the paper there is considered a generalization of the well-known Fekete–Szegö type problem onto some Bavrin’s families of complex valued holomorphic functions of several variables. The definitions of Bavrin’s families correspond to geometric properties of univalent functions of a complex variable, like as starlikeness and convexity. First of all, there are investigated such Bavrin’s families which elements satisfy also a \((j, k)\)-symmetry condition. As application of these results there is given the solution of a Fekete–Szegö type problem for a family of normalized biholomorphic starlike mappings in \( \mathbb{C}^n \).

Keywords  Holomorphic functions of scv · \( n \)-circular domains in \( \mathbb{C}^n \) · Minkowski function · \((j, k)\)-symmetry · Fekete–Szegö type estimations

Mathematics Subject Classification  32A30 · 30C45

1 Introduction

By \( \mathbb{C}, \mathbb{R}, \mathbb{Z}, \mathbb{N}_0, \mathbb{N}_1, \mathbb{N}_2 \) let us denote the sets of complex numbers, real numbers, all integers, nonnegative integers, positive integers and the integers not smaller than 2, respectively. We say that a domain \( G \subset \mathbb{C}^n, n \in \mathbb{N}_1 \), is complete \( n \)-circular if \( z\lambda = (z_1\lambda_1, \ldots, z_n\lambda_n) \in G \) for each \( z = (z_1, \ldots, z_n) \in G \) and every \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \overline{U}^n \), where \( U \) is the unit disc \( \{ \zeta \in \mathbb{C} : |\zeta| < 1 \} \). From now on by \( G \) will be denoted a bounded complete \( n \)-circular domain in \( \mathbb{C}^n, n \in \mathbb{N}_1 \). Of course, only the open discs with the centre \( \zeta = 0 \) and the radius \( r > 0 \), are the bounded complete 1-circular domains \( G \subset \mathbb{C} \).

In our considerations the Minkowski function \( \mu_G : \mathbb{C}^n \to [0, \infty) \)

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\[ \mu_G(z) = \inf \left\{ t > 0 : \frac{1}{t} z \in G \right\}, z \in \mathbb{C}^n. \]

will be very useful. It is known (see e.g., [26]) that \( \mu_G \) is a norm in \( \mathbb{C}^n \) if \( G \) is a convex bounded complete \( n \)-circular domain. This function gives the possibility to redefine the domain \( G \) and its boundary \( \partial G \) as follows:

\[ G = \{ z \in \mathbb{C}^n : \mu_G(z) < 1 \}, \partial G = \{ z \in \mathbb{C}^n : \mu_G(z) = 1 \}. \]

Now, we recall some information about \( m \)-homogeneous polynomials. We say that a function \( Q_m : \mathbb{C}^n \to \mathbb{C}, m \in \mathbb{N}_1 \), is an \( m \)-homogeneous polynomial if

\[ Q_m(z) = L_m(z^m) = L_m(z, ..., z), z \in \mathbb{C}^n, \]

where \( L_m : (\mathbb{C}^n)^m \to \mathbb{C} \) is a bounded \( m \)-linear function (by \( Q_0 \) we note a complex constant). For this reason it is very natural to define (see [4]) the following generalization of the norm of \( m \)-homogeneous polynomials \( Q_m : \mathbb{C}^n \to \mathbb{C} \), i.e., the \( \mu_G \)-balance of such \( m \)-homogeneous polynomials

\[ \mu_G(Q_m) = \sup_{w \in \mathbb{C}^n \setminus \{ 0 \} } \frac{|Q_m(w)|}{(\mu_G(w))^m} = \sup_{v \in \partial G} |Q_m(v)| = \sup_{w \in G} |Q_m(w)|. \]

A simple kind of 1-homogeneous polynomials are the linear functionals \( J, I \in (\mathbb{C}^n)^* \) of the form

\[ J(z) = \sum_{j=1}^n z_j, z = (z_1, ..., z_n) \in \mathbb{C}^n, I(z) = (\mu_G(J))^{-1} J(z). \]

Note that for \( m \in \mathbb{N}_1 \), the mapping

\[ I^m : \mathbb{C}^n \to \mathbb{C}, I^m(z) = (I(z))^m, z \in \mathbb{C}^n, \]

is an \( m \)-homogeneous polynomial and \( \mu_G(I^m) = 1 \).

By \( \mathcal{F}(G, \mathbb{C}^m), m \in \mathbb{N}_1 \), let us denote the space of all functions \( f : G \to \mathbb{C}^m \), by \( \mathcal{H}_G \) the space of all holomorphic functions \( f \in \mathcal{F}(G, \mathbb{C}) \) and by \( \mathcal{H}_G(0), \mathcal{H}_G(1) \) the collection of all \( f \in \mathcal{H}_G \), normalized by \( f(0) = 0, f(0) = 1 \), respectively. Let us recall that every function \( f \in \mathcal{H}_G \) has a unique power series expansion

\[ f(z) = \sum_{m=0}^{\infty} Q_{f,m}(z), z \in G, \tag{1.1} \]

where \( Q_{f,m} : \mathbb{C}^n \to \mathbb{C}, m \in \mathbb{N}_0 \), are \( m \)-homogeneous polynomials determined uniquely by the \( m \)th Frechét differential \( D^m f(0) \) of \( f \) at zero via the formula

\[ Q_{f,m}(z) = \frac{1}{m!} D^m f(0)(z^m). \]

Many authors considered some problems connected with \( m \)-homogeneous polynomials in the power series expansion (1.1) of functions from different subfamilies of \( \mathcal{H}_G \) (see for instance [2, 5, 9, 13, 24]). In particular, in the case \( G \subset \mathbb{C}^2 \) Bavrin [2] gives the following sharp estimates

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\[
\sup_{z \in \mathcal{G}} \left| Q_{f,m}(z) \right| \leq 1 - \left| Q_{f,0} \right|^2, \quad m \in \mathbb{N}_1,
\]

for the homogeneous polynomials \( Q_{f,m} \) of functions belonging to the family
\[
\mathcal{B}_G = \{ f \in \mathcal{H}_G : |f(z)| < 1, z \in \mathcal{G} \}.
\]

Note, that the sharpness in the Bavrin’s result was understood as follows: There exists a bounded complete 2-circular domain \( \mathcal{G} \subset \mathbb{C}^2 \) and a function \( f \in \mathcal{B}_G \) which realizes the equality in the above inequality. Let us observe that in general case \( \mathcal{G} \subset \mathbb{C}^n, n \in \mathbb{N}_1 \), the above inequality takes currently the following form
\[
\mu_G(Q_{f,m}) \leq 1 - \left| Q_{f,0} \right|^2, \quad m \in \mathbb{N}_1,
\]
by the definition of the \( \mu_G \)-balance \( \mu_G(Q_m) \).

In the paper, we solve for \( \lambda \in \mathbb{C}, k \in \mathbb{N}_2 \), the problem of the sharp upper estimate
\[
\mu_G(Q_{f,2k} - \lambda(Q_{f,k})^2) \leq M(\lambda, k)
\]
for the pairs \( Q_{f,k}, Q_{f,2k} \) homogeneous polynomials of functions belonging to some Bavrin’s subfamilies of the families \( \mathcal{H}_{G_0}(0), \mathcal{H}_{G_1}(1) \) (the case \( k = 1 \) see the section Final Remarks). Moreover, here the sharpness is understood more generally. It means that for every bounded complete \( n \)-circular domain \( \mathcal{G} \subset \mathbb{C}^n \) there exists a function \( f \) which belongs to a mentioned Bavrin’s subfamily and realizes the equality in the above inequality.

Note that the afore-mentioned estimate is a generalization of the well known planar Fekete–Szegő [8] result onto the s.c.v. case.

In the sequel we use a special kind of functions symmetry. Let us observe that bounded complete \( n \)-circular domains \( \mathcal{G} \subset \mathbb{C}^n \) are \( k \)-symmetric sets, \( k \in \mathbb{N}_2 \), that is \( \varepsilon \mathcal{G} = \mathcal{G} \), where \( \varepsilon = \varepsilon_k = \exp \frac{2\pi i}{k} \) is a generator of the cyclic group of \( k \)th roots of unity. For \( k \in \mathbb{N}_2, j \in \mathbb{Z} \) we define the collections \( \mathcal{F}_{j,k}(\mathcal{G}, \mathbb{C}^m) \) of functions \( f \in \mathcal{F}(\mathcal{G}, \mathbb{C}^m) \), \((j,k)\)-symmetrical, i.e.,
\[
f(\varepsilon z) = \varepsilon^j f(z), \quad z \in \mathcal{G}.
\]
Let us observe that \( \mathcal{F}_{j,k}(\mathcal{G}, \mathbb{C}^m) \neq \mathcal{F}_{l,k}(\mathcal{G}, \mathbb{C}^m) \) for different \( j, l \in \{0, 1, ..., k - 1\} \). Moreover, the intersections \( \mathcal{F}_{j,k}(\mathcal{G}, \mathbb{C}^m) \cap \mathcal{F}_{l,k}(\mathcal{G}, \mathbb{C}^m) \) are the singleton \( \{0\} \) for such \( j, l \).

Now we present a functions decomposition theorem [19].

**Theorem A** For every function \( f \in \mathcal{F}(\mathcal{G}, \mathbb{C}^m) \) and every \( k \in \mathbb{N}_2 \) there exists exactly one sequence of functions \( f_{j,k} \in \mathcal{F}_{j,k}(\mathcal{G}, \mathbb{C}^m), j = 0, 1, ..., k - 1, \) such that
\[
f = \sum_{j=0}^{k-1} f_{j,k}.
\] (1.2)

Moreover,
\[
f_{j,k}(z) = \frac{1}{k} \sum_{l=0}^{k-1} \varepsilon^{-jl} f(\varepsilon^l z), \quad z \in \mathcal{G}.
\]

By the uniqueness of this decomposition, the functions \( f_{j,k}, j = 0, 1, ..., k - 1, \) will be called \((j,k)\)-symmetrical components of the function \( f \).
In the next sections of the paper will be very useful the fact that $f_{0,k} \in \mathcal{H}_G(1)$ for $f \in \mathcal{H}_G(1)$. Note that

$$f_{0,k}(z) = 1 + \sum_{m=1}^{\infty} Q_{f_{0,k},m}(z) = 1 + \sum_{s=1}^{\infty} Q_{f_{0,k},s}(z), z \in G.$$ 

Note that the above unique decomposition (1.2) of functions was used in [20] to solve some functional equations, in [21] to construction a semi power series and in [22] to obtain a uniqueness theorem of Cartan type for holomorphic mappings in $\mathbb{C}^n$.

We close this section with the following Golusin’s [10] result, very useful in the proof of the first result of Fekete–Szegö type for holomorphic functions of several complex variables.

**Lemma 1.1** Let $\Phi : U \to U$ be a holomorphic function of the form

$$\Phi(\zeta) = \sum_{v=0}^{\infty} a_v \zeta^v, \zeta \in U.$$ 

Then

$$|a_m| \leq 1 - |a_p|^2,$$

for every $m \in \mathbb{N}_1, p \in \mathbb{N}_0$, satisfying the condition $0 \leq 2p < m$. The estimates are optimal.

**Proof** Let us recall that the simplest case

$$|a_1| \leq 1 - |a_0|^2$$

follows from the well-known inequality

$$|\Phi'(\zeta)| \leq \frac{1 - |\Phi(\zeta)|^2}{1 - |\zeta|^2}, \zeta \in U.$$ 

For another $m, p$ we use a Krzyż’s idea [17, Chapt. 6.2] and some properties of $(j, k)$-symmetrical functions.

Let us take

$$\Psi(\zeta) = \zeta^{m-2p} \Phi(\zeta) = \sum_{s=0}^{\infty} a_s \zeta^{s+m-2p}, \zeta \in U.$$ 

Then $\Psi : U \to U$ and is holomorphic. Thus the $(0, m - p)$-part $\Psi_{0,m-p}$ of $\Psi$ transforms $U$ into itself, is holomorphic and

$$\Psi_{0,m-p}(\zeta) = \sum_{s=0}^{\infty} a_{p+s(m-p)} \zeta^{s+(m-p)}, \zeta \in U.$$ 

Hence, by the Schwarz Lemma (in version with the zero $\zeta = 0$ of multiplicity $m - p > 0$) it fulfils the inequality
Therefore, the function

\[ \Theta(\zeta) = \frac{1}{\zeta^{m-p}} \Psi_{0,m-p}(\zeta), \zeta \in U. \]

maps \( U \) into itself, is holomorphic and has the expansion

\[ \Theta(\zeta) = \sum_{s=0}^{\infty} a_{p+s(m-p)} \zeta^{s(m-p)}, \zeta \in U. \]

Consequently, replacing \( \zeta^{m-p} \in U \) by \( \bar{\xi} \in U \), we get that the function

\[ \sum_{s=0}^{\infty} b_s \bar{\xi}^s = \sum_{s=0}^{\infty} a_{p+s(m-p)} \bar{\xi}^s, \bar{\xi} \in U \]

transforms holomorphically \( U \) into itself. Hence, by the first part of the proof, we get the thesis.

Note that the equality in the inequality is attained by the functions \( F(\zeta) = \zeta^m, \zeta \in U \) and \( F(\zeta) = \zeta^p, \zeta \in U \). This completes the proof \( \square \)

2 Main results

We start this section with an \( n \)-dimensional Fekete–Szegö type theorem for bounded holomorphic functions on bounded complete \( n \)-circular domains in \( \mathbb{C}^n \). Note that it is a generalization of a 1-dimensional result given by Keogh and Merkes [14].

**Theorem 2.1** Let \( \varphi \) be a function from the family

\[ \mathcal{B}_{\varphi}(0) = \{ \varphi \in \mathcal{H}_{\varphi}(0) : |\varphi(z)| < 1, z \in \mathcal{G} \} \]

and has the form

\[ \varphi(z) = \sum_{j=1}^{\infty} Q_{\varphi,j}(z), z \in \mathcal{G}. \]

Then, for every \( k \in \mathbb{N}_1 \) and every \( \gamma \in \mathbb{C} \), there holds the inequality

\[ \mu_{\mathcal{G}} \left( Q_{\varphi,2k} - \gamma (Q_{\varphi,k})^2 \right) \leq \max\{1, |\gamma|\}. \tag{2.1} \]

The estimate is sharp.

**Proof** For every arbitrarily fixed \( z \in \mathcal{G} \) we define the function

\[ \Phi(\zeta) = \begin{cases} \frac{\varphi(z)}{\zeta}, & \zeta \in U \setminus \{0\} \\ Q_{\varphi,1}(z) = \lim_{z \to 0} \frac{\varphi(z)}{\zeta}, & \zeta = 0 \end{cases}. \]
Then $\Phi$ is holomorphic,

$$\Phi(\zeta) = \sum_{v=0}^{\infty} a_v \zeta^v, \zeta \in U,$$

for $a_{l-1} = Q_{\varphi,l}(z), l \in \mathbb{N}_1,$ and $|\Phi(0)| < 1.$ Moreover, applying a $C^2$-version of Schwarz Lemma, i.e., $|\varphi(z)| \leq \mu_G(z), z \in G,$ for $\varphi \in B_G(0)$, we conclude also that for $\zeta \in U \setminus \{0\}$

$$|\Phi(\zeta)| = \frac{|\varphi(\zeta)|}{|\zeta|} \leq \frac{\mu_G(\zeta)}{|\zeta|} = \frac{|\zeta| \mu_G(z)}{|\zeta|} < 1,$$

hence $\Phi : U \to U.$ Now let us observe that from the Lemma 1.1 we get

$$|a_{m-1}| \leq 1 - \left|a_{p-1}\right|^2,$$

for arbitrarily fixed $m, p \in \mathbb{N}_1,$ satisfying the condition $0 \leq 2(p-1) < m - 1.$ Thus

$$\left|Q_{\varphi,m}(z)\right| \leq 1 - \left|Q_{\varphi,p}(z)\right|^2, z \in G.$$

Therefore, for $z \in G$ and every $\gamma \in \mathbb{C}$

$$\left|Q_{\varphi,m}(z) - \gamma \left(Q_{\varphi,p}(z)\right)^2\right|$$

$$\leq \left|Q_{\varphi,m}(z)\right| + |\gamma| \left|Q_{\varphi,p}(z)\right|^2 \leq 1 - \left|Q_{\varphi,p}(z)\right|^2 + |\gamma| \left|Q_{\varphi,p}(z)\right|^2$$

$$= 1 + (|\gamma| - 1)\left|Q_{\varphi,p}(z)\right|^2 \leq \max\{1, |\gamma|\},$$

because

$$\begin{cases}
|\gamma| - 1 \left|Q_{\varphi,p}(z)\right|^2 \leq 0, & \text{if } |\gamma| < 1, \\
0 \leq (|\gamma| - 1)\left|Q_{\varphi,p}(z)\right|^2 \leq |\gamma| - 1, & \text{if } |\gamma| \geq 1.
\end{cases}$$

Consequently, by the arbitrariness of $z \in G,$ also

$$\sup_{z \in G} \left|Q_{\varphi,m}(z) - \gamma \left(Q_{\varphi,p}(z)\right)^2\right| \leq \max\{1, |\gamma|\}.$$

Putting in the above $p = k$ and $m = 2k, k \in \mathbb{N}_1,$ we obtain $m - 1 > 2(p - 1) \geq 1 \geq 0$ and

$$\sup_{z \in G} \left|Q_{\varphi,2k}(z) - \gamma \left(Q_{\varphi,k}(z)\right)^2\right| \leq \max\{1, |\gamma|\}.$$

Finally, by the fact that $Q_{\varphi,2k} - \gamma \left(Q_{\varphi,k}\right)^2$ is a $2k$-homogeneous polynomial for every $\gamma \in \mathbb{C}$ and by the definition of its $\mu_G$-balance, we get the statements of the Theorem 2.1.

Now, we will analyse the sharpness of the above estimate.

First, we prove that in the case $|\gamma| \geq 1$, the equality in (2.1) is attained by the function $\varphi = \tilde{\varphi} \in B_G(0), \tilde{\varphi} = I^k$, more precisely $\tilde{\varphi} = I^k|_G$, i.e., $\tilde{\varphi}(z) = I^k(z), z \in G$. Indeed, since $Q_{\tilde{\varphi},k} = I^k, Q_{\tilde{\varphi},2k} = 0$ and $\mu_G(I^k) = 1$, we have
\( \mu_{\gamma}(Q_{\varphi,2k} - \gamma (Q_{\varphi,k})^2) = \mu_{\gamma}(-\gamma (Q_{\varphi,k})^2) = |\gamma| \mu_{\gamma}(Q_{\varphi,k})^2 = |\gamma| = \max\{1, |\gamma|\} \).

Now, we show that in the case \(|\gamma| < 1\), the equality in (2.1) realizes the function \( \varphi = \hat{\varphi} \in B_{\mathcal{G}}(0), \hat{\varphi} = f^{2k} \), more precisely \( \hat{\varphi} = I_{|\gamma|}^{2k} \), i.e., \( \hat{\varphi}(z) = I_{|\gamma|}^{2k}(z), z \in \mathcal{G} \). Indeed, since \( Q_{\varphi,k} = 0, Q_{\varphi,2k} = I_{|\gamma|}^{2k}, \) we get
\[
\mu_{\gamma}(Q_{\varphi,2k} - \gamma (Q_{\varphi,k})^2) = \mu_{\gamma}(Q_{\varphi,k}) = 1 = \max\{1, |\gamma|\}.
\]
This completes the proof. \( \square \)

Note that Theorem 2.1 generalizes an earlier result given by authors in [7].

In the sequel we apply Theorem 2.1 to study two Bavrin's families \( \mathcal{M}_G, \mathcal{N}_G \) of functions \( f \in \mathcal{H}_G(1) \). These families are defined by the following family \( \mathcal{C}_G \),
\[
\mathcal{C}_G = \{ f \in \mathcal{H}_G(1) : \text{Re}(f) > 0, z \in \mathcal{G} \}
\]
and by the following Temljakov [29] linear operator \( \mathcal{L} : \mathcal{H}_G \rightarrow \mathcal{H}_G 
\]
\[
\mathcal{L} f(z) = f(z) + Df(z), z \in \mathcal{G},
\]
where \( Df(z) \) means the Fréchet derivative of \( f \) at the point \( z \). Note that the operator \( \mathcal{L} \) is invertible and
\[
\mathcal{L}^{-1} f(z) = \int_0^1 f(zt) dt, z \in \mathcal{G}.
\]
It is obvious also that for the transforms \( \mathcal{L} f, \mathcal{L} \mathcal{L} f \) of the functions \( f \in \mathcal{H}_G(1) \) we have
\[
\mathcal{L} f(z) = 1 + \sum_{m=1}^{\infty} Q_{\mathcal{L} f,m}(z) = 1 + \sum_{m=1}^{\infty} (m + 1) Q_{f,m}(z), z \in \mathcal{G}, \tag{2.2}
\]
\[
\mathcal{L} \mathcal{L} f(z) = 1 + \sum_{m=1}^{\infty} Q_{\mathcal{L} \mathcal{L} f,m}(z) = 1 + \sum_{m=1}^{\infty} (m + 1)^2 Q_{f,m}(z), z \in \mathcal{G}. \tag{2.3}
\]
Moreover,
\[
\mathcal{L}^{-1} f(z) = 1 + \sum_{m=1}^{\infty} Q_{\mathcal{L}^{-1} f,m}(z) = 1 + \sum_{m=1}^{\infty} \frac{1}{m + 1} Q_{f,m}(z), z \in \mathcal{G}. \tag{2.4}
\]

We say that a function \( f \in \mathcal{H}_G(1) \) belongs to the Bavrin’s family \( \mathcal{M}_G(N_G) \) if it satisfies the factorization
\[
\mathcal{L} f(z) = f(z) h(z), z \in \mathcal{G},
\]
\[
\mathcal{L} \mathcal{L} f(z) = \mathcal{L} f(z) h(z), z \in \mathcal{G}, \tag{2.5}
\]
together with a function \( h \in \mathcal{C}_G \) and the transform \( \mathcal{L} f, (\mathcal{L} \mathcal{L} f) \), respectively. Note that the families \( \mathcal{M}_G, \mathcal{N}_G \) correspond with the well-known families of normalized univalent starlike (convex) functions in the disc \( U \) [2] and the family \( \mathcal{M}_G \) can be used to construction biholomorphic starlike mappings in \( C^n \) (see [6, 18], compare also [12, 25]). Between functions
from $\mathcal{M}_G, \mathcal{N}_G$ there holds a relationship, corresponding to the well-known Alexander type connexion [1], for univalent starlike and convex mappings in the unit disc. Here, this relationship is the following: if $f \in \mathcal{N}_G$, then $Lf \in \mathcal{M}_G$ and conversely, if $f \in \mathcal{M}_G$, then $L^{-1}f \in \mathcal{N}_G$ [2].

We begin the presentation of some Fekete–Szegö type results in Bavrin’s families with the following theorem.

**Theorem 2.2** Let $G \subset \mathbb{C}^n$ be a bounded complete $n$-circular domain and let $f \in \mathcal{N}_G \cap \mathcal{F}_{0,k}(G,\mathbb{C}), k \in \mathbb{N}_2$. If the expansion of the function $f$ into a series of $m$-homogeneous polynomials $Q_{f,m}$ has the form (1.1), with $Q_{f,0} = 1$, then for the homogeneous polynomials $Q_{f,2k}, Q_{f,k}$ and every $\lambda \in \mathbb{C}$ there holds the following sharp estimate:

$$\mu_G\left(Q_{f,2k} - \lambda(Q_{f,k})^2\right) \leq \frac{1}{k(2k+1)} \max\left\{1, \left|\frac{4(2k+1)\lambda - (k+1)^2(2k+2)}{k(k+1)^2}\right|\right\}.$$  \hspace{1cm} (2.6)

**Proof** Let us recall that between the functions $p \in C_G$ and $\varphi \in B_G(0)$, there holds the following relation [2]:

$$\frac{p - 1}{p + 1} = \varphi \in B_G(0) \iff p \in C_G.$$  

Let $k \in \mathbb{N}_2$ be arbitrarily fixed and let the function $f$ belongs to $\mathcal{N}_G \cap \mathcal{F}_{0,k}(G,\mathbb{C})$. Then, by the definition of the family $\mathcal{N}_G$ and by the above relation between the families $C_G, B_G(0)$, we get

$$\frac{L}{L}f(z) - \frac{L}{L}f(z) = \varphi(z), z \in G.$$  \hspace{1cm} (2.7)

where $\varphi \in B_G(0) \cap \mathcal{F}_{0,k}(G,\mathbb{C})$. On the other hand, from (1.1) we have for $z \in G$:

$$\varphi(z) = \sum_{s=1}^{\infty} Q_{\varphi,s,k}(z),$$

$$f(z) = 1 + \sum_{s=1}^{\infty} Q_{f,s,k}(z)$$

and from (2.2), (2.3) also

$$Lf(z) = 1 + \sum_{s=1}^{\infty} (sk + 1)Q_{f,s,k}(z),$$

$$\frac{L}{L}f(z) = 1 + \sum_{s=1}^{\infty} (sk + 1)^2Q_{f,s,k}(z).$$

Inserting the above expansion of functions into (2.7), we receive after computations

$$\sum_{s=1}^{\infty} sk(1+sk)Q_{f,s,k}(z) = \left(\sum_{s=1}^{\infty} Q_{\varphi,s,k}(z)\right)\left(2 + \sum_{s=1}^{\infty} (sk + 1)(sk + 2)Q_{f,s,k}(z)\right).$$
Then, comparing the $m$-homogeneous polynomials of the same degree on both sides of the above equality, we can determine homogeneous polynomials $Q_{\varphi,k}$, $Q_{\varphi,2k}$, as follows

\[
\begin{align*}
Q_{\varphi,k} &= \frac{1}{2}k(k+1)Q_{f,k}, \\
Q_{\varphi,2k} &= k(2k+1)Q_{f,2k} - \frac{1}{4}k(k+1)^2(k+2)(Q_{f,k})^2.
\end{align*}
\]

(2.8)

Putting the above equalities into Theorem 2.1 and using the fact that the mapping $(Q_{f,k})^2$, is a 2$k$-homogenous polynomial, we obtain

\[
\mu_G\left[k(2k+1)Q_{f,2k} - \left(\frac{1}{4}k(k+1)^2(k+2) + \gamma \frac{1}{4}k^2(k+1)^2\right)(Q_{f,k})^2\right] \leq \max\{1, |\gamma|\}.
\]

Denoting

\[
\lambda = \frac{(k+1)^2}{4(2k+1)}(k+2+\gamma k),
\]

we obtain

\[
\mu_G\left(Q_{f,2k} - \lambda(Q_{f,k})^2\right) \leq \frac{1}{k(2k+1)} \max\left\{1, \left|\frac{4(2k+1)\lambda - (k+1)^2(k+2)}{k(k+1)^2}\right|\right\}.
\]

Now, we show the sharpness of our estimate. To do it, let us consider two cases.

At the beginning, we prove that, in the case

\[
\frac{4(2k+1)\lambda - (k+1)^2(k+2)}{k(k+1)^2} \geq 1
\]

the equality in (2.6) is attained by the function $f = L^{-1}\tilde{f}$, with

\[
\tilde{f}(z) = (1 - I^\ast(z))^{\frac{2}{\gamma}}, z \in \mathcal{G},
\]

(2.9)

where the branch of the function $(1 - \xi)^{\frac{2}{\gamma}}$ takes value 1 at the point $\xi = 0$. Indeed. Function $f$ belongs to $\mathcal{F}_{0,k}(\mathcal{G},\mathbb{C})$, because $\tilde{f} \in \mathcal{F}_{0,k}(\mathcal{G},\mathbb{C})$. On the other hand, the function $f = L^{-1}\tilde{f}$ belongs to $\mathcal{N}_G$, because the function $\varphi$ from (2.7) has the form $\varphi = \tilde{\varphi} = I^k$ and belongs to $\mathcal{B}_G(0)$. Therefore, we can write equalities (2.8) in the following form

\[
Q_{f,k} = \frac{2}{k(k+1)}I^k, Q_{f,2k} = \frac{k+2}{k^2(2k+1)}(I^k)^2.
\]

(2.10)

From this, by the case condition for $\lambda$, we have step by step:
\[
\mu_g(Q_{f,2k} - \lambda(Q_{f,k}))^2 = \mu_g \left[ \frac{(k + 1)^2(k + 2) - 4(2k + 1)\lambda}{k^2(k + 1)^2(2k + 1)} \right]^2 \\
= \left| \frac{(k + 1)^2(k + 2) - 4(2k + 1)\lambda}{k^2(k + 1)^2(2k + 1)} \right| \mu_g \left( t^{2k} \right) \\
= \frac{1}{k(2k + 1)} \max \left\{ 1, \left| \frac{4(2k + 1)\lambda - (k + 1)^2(k + 2)}{k(k + 1)^2} \right| \right\}.
\]

Now, we show that, in the case
\[
\left| \frac{4(2k + 1)\lambda - (k + 1)^2(k + 2)}{k(k + 1)^2} \right| < 1
\]
the equality in (2.6) realizes the function \( f = \mathcal{L}^{-1}\hat{f} \), with
\[
\hat{f}(z) = \left( 1 - t^{2k} \right)^{\frac{1}{\xi}}, \quad z \in \mathcal{G}, \quad (2.11)
\]
where the branch of the function \( (1 - \xi)^{\frac{1}{\xi}} \) takes value 1 at the point \( \xi = 0 \). The function \( f = \mathcal{L}^{-1}\hat{f} \) belongs to \( \mathcal{F}_{0,k}(G, \mathbb{C}) \), because \( \hat{f} \in \mathcal{F}_{0,k}(G, \mathbb{C}) \). Also, \( f \in \mathcal{N}_g \), because the function \( \varphi \) from (2.7) has the form \( \varphi = \hat{\varphi} = t^{2k} \) and belongs to \( \mathcal{B}_g(0) \). Therefore, we can write equalities (2.8) in the following form
\[
Q_{f,k} = 0, \quad Q_{f,2k} = \frac{1}{k(2k + 1)} t^{2k}.
\]
(2.12)

From this, by the case condition for \( \lambda \), we have:
\[
\mu_g(Q_{f,2k} - \lambda(Q_{f,k}))^2 = \mu_g \left( \frac{1}{k(2k + 1)} t^{2k} \right) = \frac{1}{k(2k + 1)} \mu_g (t^{2k}) \\
= \frac{1}{k(2k + 1)} \max \left\{ 1, \left| \frac{4(2k + 1)\lambda - (k + 1)^2(k + 2)}{k(k + 1)^2} \right| \right\}.
\]

This completes the proof. \( \square \)

We continue the presentation of some Fekete–Szegö type results in Bavrin’s families with the following theorem:

**Theorem 2.3** Let \( G \subset \mathbb{C}^n \) be a bounded complete n-circular domain and let \( k \in \mathbb{N}_2 \). If the expansion of the function \( f \in \mathcal{M}_g \cap \mathcal{F}_{0,k}(G, \mathbb{C}) \), into a series of m-homogenous polynomials \( Q_{f,m} \) has the form (1.1), with \( Q_{f,0} = 1 \), then for the homogeneous polynomials \( Q_{f,2k}, Q_{f,k} \) and \( \lambda \in \mathbb{C} \) there holds the following sharp estimate:
\[
\mu_g(Q_{f,2k} - \lambda(Q_{f,k}))^2 \leq \frac{1}{k} \max \left\{ 1, \frac{|2 + k - 4\lambda|}{k} \right\}. \quad (2.13)
\]

**Proof** Let \( k \in \mathbb{N}_2 \) be arbitrarily fixed. Then, it is obvious that \( \mathcal{L}^{-1}f \in \mathcal{F}_{0,k}(G, \mathbb{C}) \), because \( f \in \mathcal{F}_{0,k}(G, \mathbb{C}) \). Also the assumption that \( f \in \mathcal{M}_g \), by the relationship of the Alexander type, gives that \( \mathcal{L}^{-1}f \in \mathcal{N}_g \). Hence, we have that \( \mathcal{L}^{-1}f \in \mathcal{N}_g \cap \mathcal{F}_{0,k}(G, \mathbb{C}) \). On the other hand, its expansion into a series of \( m \)-homogenous polynomials \( Q_{f,m} \), by (2.4), has the form

\( \square \) Springer
\[ \mathcal{L}^{-1}f(z) = 1 + \sum_{s=1}^{\infty} \frac{1}{sk+1} Q_{f,sk}(z), z \in \mathcal{G}. \]

Thus, in view of (2.6) from Theorem 2.2, we get that for every \( \delta \in \mathbb{C} \)
\[
\mu_\mathcal{G}
\left( \frac{1}{2k+1} Q_{f,2k} - \delta \left( \frac{1}{k+1} Q_{f,k} \right)^2 \right)
\leq \frac{1}{k(2k+1)} \max \left\{ 1, \frac{4(2k+1)\delta - (k+1)^2(k+2)}{k(k+1)^2} \right\}.
\]

Hence,
\[
\mu_\mathcal{G}
\left( Q_{f,2k} - \delta \frac{2k+1}{(k+1)^2} (Q_{f,k})^2 \right)
\leq \frac{1}{k} \max \left\{ 1, \frac{4(2k+1)\delta - (k+1)^2(k+2)}{k(k+1)^2} \right\}
\]
and denoting \( \delta \frac{2k+1}{(k+1)^2} = \lambda \), finally
\[
\mu_\mathcal{G}
\left( Q_{f,2k} - \lambda (Q_{f,k})^2 \right)
\leq \frac{1}{k} \max \left\{ 1, \frac{4\lambda - (k+2)}{k} \right\},
\]
which is the same as (2.13).

It remains to show the sharpness of the estimate (2.13).

First, we prove that, in the case
\[
\frac{|2 + k - 4\lambda|}{k} \geq 1
\]
the equality in (2.13) is attained by the function \( f = \tilde{f} \) defined in (2.9). Of course \( \tilde{f} \in \mathcal{F}_{0,k}(\mathcal{G},\mathbb{C}) \) and, by the Alexander type relationship and the fact that \( \mathcal{L}^{-1}\tilde{f} \in \mathcal{N}_\mathcal{G} \) (see the proof of the sharpness in Theorem 2.2), the function \( \tilde{f} \) belongs to \( \mathcal{M}_\mathcal{G} \). Therefore, in view of (2.2) and (2.10), we achieve
\[
Q_{f,k} = Q_{\tilde{f},k} = \frac{2}{k} t^k, \quad Q_{f,2k} = Q_{\tilde{f},2k} = \frac{k+2}{k^2} (t^k)^2.
\]

From this (see the proof of the sharpness in Theorem 2.2) and by the case condition for \( \lambda \), we have step by step:
\[
\mu_\mathcal{G}
\left( Q_{f,2k} - \lambda (Q_{f,k})^2 \right)
= \mu_\mathcal{G}
\left( \frac{(k+2) - 4\lambda}{k^2} (t^k)^2 \right)
= \left| \frac{(k+2) - 4\lambda}{k^2} \right| \mu_\mathcal{G}(t^{2k}) = \frac{1}{k} \max \left\{ 1, \frac{|2 + k - 4\lambda|}{k} \right\}.
\]

Now, we show that, in the case
\[
\frac{|2 + k - 4\lambda|}{k} < 1
\]
the equality in (2.13) realizes the function \( f = \tilde{f} \) defined in (2.11). Of course \( \tilde{f} \in \mathcal{F}_{0,k}(\mathcal{G},\mathbb{C}) \) and, by the Alexander type relationship and the fact that \( \mathcal{L}^{-1}\tilde{f} \in \mathcal{N}_\mathcal{G} \) (see the proof of the
sharpness in Theorem 2.2), the function \( \hat{f} \) belongs to \( \mathcal{M}_G \). Therefore, in view of (2.2) and (2.12) we achieve

\[
Q_{f,k} = Q_{f,2k} = 0, \quad Q_{f,2k} = \frac{1}{k} f^{2k}.
\]

From this (see the proof of the sharpness in Theorem 2.2) and by the case condition for \( \lambda \), we conclude that:

\[
\mu \left( Q_{f,2k} - \lambda (Q_{f,k})^2 \right) = \mu \left( \frac{1}{k} f^{2k} \right) = \frac{1}{k} = \frac{1}{k} \max \left\{ 1, \frac{|2 + k - 4\lambda|}{k} \right\}.
\]

This completes the proof. \( \square \)

Now, we transfer the statement of Theorem 2.3, onto a family \( \mathcal{M}^k_G \cap \mathcal{F}_{0,k}(G,\mathbb{C}), k \in \mathbb{N}_2 \). Here, \( \mathcal{M}^k_G \) is defined by the factorization similar as for the elements from \( \mathcal{M}_G \). More precisely, the function \( f \) of the right hand side in (2.5) is replaced by a function from \( \mathcal{F}_{0,k}(G,\mathbb{C}), k \in \mathbb{N}_2 \), generated by \( f \). Formally, we say that a function \( f \in \mathcal{H}_G(1) \) belongs to \( \mathcal{M}^k_G, k \in \mathbb{N}_2 \) (see [3, 5]) if there exists a function \( h \in \mathcal{C}_G \) such that

\[
\mathcal{L} f(z) = f_{0,k}(z) h(z), z \in \mathcal{G},
\]

where \( f_{0,k} \in \mathcal{F}_{0,k}(G,\mathbb{C}) \) is the \((0,k)\)-symmetrical component of the function \( f \) in the decomposition (1.2) from Theorem A. This family for \( k = 2 \) corresponds to a well-known Sakaguchi family [27] of a complex variable functions; strictly speaking of functions univalent starlike with respect to two symmetric points. In the paper [5] it was shown that for \( k \in \mathbb{N}_2 \) the inclusions \( \mathcal{M}_G \subset \mathcal{M}^k_G, \mathcal{M}^k_G \subset \mathcal{M}_G \) do not hold, but

\[
\mathcal{M}_G \cap \mathcal{F}_{0,k}(\mathcal{G},\mathbb{C}) = \mathcal{M}^k_G \cap \mathcal{F}_{0,k}(\mathcal{G},\mathbb{C}).
\]

This identity and Theorem 2.3 implies directly the next result of Fekete–Szegő type in Bavrin’s families:

**Theorem 2.4** Let \( \mathcal{G} \subset \mathbb{C}^n \) be a bounded complete \( n \)-circular domain and let \( f \in \mathcal{M}^k_G \cap \mathcal{F}_{0,k}(\mathcal{G},\mathbb{C}), k \in \mathbb{N}_2 \). If the expansion of the function \( f \) into a series of \( m \)-homogeneous polynomials \( Q_{f,m} \) has the form (1.1), with \( Q_{f,0} = 1 \), then for the homogeneous polynomials \( Q_{f,2k}, Q_{f,k} \) and \( \lambda \in \mathbb{C} \) there holds the following sharp estimate:

\[
\mu \left( Q_{f,2k} - \lambda (Q_{f,k})^2 \right) \leq \frac{1}{k} \max \left\{ 1, \frac{|2 + k - 4\lambda|}{k} \right\}.
\]

The equality in the above inequality realize the same functions \( f = \tilde{f}, f = \hat{f} \) as in the previous Theorem 2.2.

## 3 Applications

In this section we apply Theorems 2.3 and 2.4, to obtain a Fekete–Szegő type results for two families of biholomorphic mappings in \( \mathbb{C}^n \). By \( \mathcal{S}^n(\mathbb{B}^n) \) let us denote the family of biholomorphic mappings \( F \in \mathcal{F}(\mathbb{B}^n, \mathbb{C}^n), F(0) = 0, DF(0) = I \) onto starlike domains
F(\mathbb{B}^n). For a wide collection of references in this area see the monographs [11, 16]. In a Kikuchi–Matsuno–Suffridge characterization [15, 23, 28] of the family $S^n(\mathbb{B}^n)$, the collection $P(\mathbb{B}^n)$ of all holomorphic mappings $H \in \mathcal{F}(\mathbb{B}^n, \mathbb{C}^n)$, $H(0) = 0$, $DH(0) = I$, such that $\Re \langle H(z), z \rangle > 0$, $z \in \mathbb{B}^n \setminus \{0\}$, plays the main role (here $\langle \cdot, \cdot \rangle$ means the Euclidean inner product). This characterization is included in the following theorem:

**Theorem B** A locally biholomorphic mapping $F \in \mathcal{F}(\mathbb{B}^n, \mathbb{C}^n)$ normalized by the conditions $F(0) = 0$, $DF(0) = I$, belongs to $S^n(\mathbb{B}^n)$, iff there exist a mapping $H \in P(\mathbb{B}^n)$ such that

$$F(z) = DF(z)H(z), \ z \in \mathbb{B}^n. \quad (3.1)$$

Let $\tilde{S}^n(\mathbb{B}^n)$ be the family of mappings $F \in S^n(\mathbb{B}^n)$ with the factorization

$$F(z) = zf(z), \ z \in \mathbb{B}^n, \quad (3.2)$$

where $f \in \mathcal{H}_{\mathbb{B}^n}(1)$.

In the paper [6] the authors considered a family of biholomorphic mappings $S^k(\mathbb{B}^n), k \in \mathbb{N}_2$, defined by an equation similar to (3.1). More precisely, the mapping $F$ of the left hand side in (3.1) is replaced by a function from $\mathcal{F}_{1,k}(\mathbb{B}^n, \mathbb{C}^n)$, generated by $F$.

Formally, we say that a locally biholomorphic mapping $F \in \mathcal{F}(\mathbb{B}^n, \mathbb{C}^n)$, normalized by $F(0) = 0$, $DF(0) = I$, belongs to the family $S^k(\mathbb{B}^n), k \in \mathbb{N}_2$, if it satisfies the equation

$$F_{1,k}(z) = DF(z)H(z), \ z \in \mathbb{B}^n,$$

where $H \in P(\mathbb{B}^n)$ and $F_{1,k} \in \mathcal{F}_{1,k}(\mathbb{B}^n, \mathbb{C}^n)$ is the $(1, k)$-symmetrical part of $F$ in the decomposition (1.2). Also in the paper [6] the authors proved, that for every $k \in \mathbb{N}_2$ any inclusions $S^k(\mathbb{B}^n) \subset S^n(\mathbb{B}^n), S^n(\mathbb{B}^n) \subset S^k(\mathbb{B}^n)$ do not holds. However, for the same $k$ there holds the following identity:

$$S^k(\mathbb{B}^n) \cap \mathcal{F}_{1,k}(\mathbb{B}^n, \mathbb{C}^n) = S^n(\mathbb{B}^n) \cap \mathcal{F}_{1,k}(\mathbb{B}^n, \mathbb{C}^n). \quad (3.3)$$

Let $\tilde{S}^k(\mathbb{B}^n), k \in \mathbb{N}_2$, be the family of mappings $F \in \tilde{S}^k(\mathbb{B}^n)$ with the factorization (3.2).

Now, we present the main theorem, in this section of the paper. It is a Fekete–Szegö type result for locally biholomorphic mappings in $\mathbb{C}^n$, compare [30].

**Theorem 3.1** For mappings $F \in \tilde{S}^k(\mathbb{B}^n) \cap \mathcal{F}_{1,k}(\mathbb{B}^n, \mathbb{C}^n), k \in \mathbb{N}_2$, the parameter $\lambda \in \mathbb{C}$ and points $z \in \mathbb{B}^n \setminus \{0\}$ there holds the following sharp estimate

$$\left| \frac{T_z(D^{2k+1}F(0)(z^{2k+1)}))}{(2k+1)!\|z\|^{2k+1}} - \lambda \left( \frac{T_z(D^{k+1}F(0)(z^{k+1)}))}{(k+1)!\|z\|^{k+1}} \right)^2 \right| \leq \frac{1}{k} \max \left\{ 1, \frac{|2k - 4\lambda|}{k} \right\}, \quad (3.4)$$

where $T_z \in (\mathbb{C}^n)^*, z \in \mathbb{B}^n \setminus \{0\}$, is arbitrary functional satisfying the conditions $\|T_z\| = 1$, $T_z(z) = \|z\|$.

**Proof** We start with a few facts, very useful in the proof:

1. A mapping $F$, satisfying (3.2), belongs to $\tilde{S}^k(\mathbb{B}^n)$, iff $f \in \mathcal{M}_{\mathbb{B}^n}^k$ (see [6, 18]).
2. A mapping $F$ of the form (3.2) belongs to $\mathcal{F}_{1,k}(\mathbb{B}^n, \mathbb{C}^n)$, iff $f \in \mathcal{F}_{0,k}(\mathbb{B}^n, \mathbb{C})$.
3. For $z \in \mathbb{B}^n$ there hold (see for instance [30]) the equalities:
\[
\frac{D^{2k+1}F(0)(z^{2k+1})}{(2k+1)!} = z^{k} \frac{D^{2k}f(0)(z^{2k})}{(2k)!},
\]
\[
\frac{D^{k+1}F(0)(z^{k+1})}{(k+1)!} = z^{k} \frac{D^{k}f(0)(z^{k})}{(k)!}.
\]

4. For \(\varepsilon \in \mathbb{C}\) the mapping \(Q_{f,2k} - \lambda Q_{f,k}^{2} \in \mathcal{F}(\mathbb{B}^{n}, \mathbb{C})\), is a 2kth homogeneous polynomial.

5. There hold the identity \(\mu_{\mathbb{B}^{n}}(\cdot) = || \cdot ||\) in \(\mathbb{C}^{n}\).

Applying the above facts and Theorem 2.5, we get step by step

\[
\left| \frac{T_{z}(D^{2k+1}F(0)(z^{2k+1})))}{(2k+1)!||z||^{2k+1}} - \lambda \left( \frac{T_{z}(D^{k+1}F(0)(z^{k+1})))}{(k+1)!||z||^{k+1}} \right)^{2} \right|
\]
\[
= \left| \frac{T_{z}(z)D^{2k}f(0)(z^{2k})}{(2k)!||z||^{2k+1}} - \lambda \left( \frac{T_{z}(z)D^{k}f(0)(z^{k})}{(k)!||z||^{k+1}} \right)^{2} \right|
\]
\[
= \left| \frac{D^{2k}f(0)(z^{2k})}{(2k)!||z||^{2k}} - \lambda \left( \frac{D^{k}f(0)(z^{k})}{(k)!||z||^{k}} \right)^{2} \right|
\]
\[
= \left| \frac{Q_{f,2k}(z)}{||z||^{2k}} - \lambda \left( \frac{Q_{f,k}(z)}{||z||^{k}} \right)^{2} \right|
\]
\[
\leq \sup_{z \in \mathbb{B}^{n}} \left| \frac{Q_{f,2k}(z) - \lambda (Q_{f,k}(z))^{2}}{||z||^{2k}} \right| = \mu_{\mathbb{B}^{n}}(Q_{f,2k} - \lambda (Q_{f,k})^{2})
\]
\[
\leq \frac{1}{k} \max \left\{ 1, \frac{2 + k - 4 \lambda}{k} \right\}.
\]

It remains to show the sharpness of the estimate (3.4).

To do it, observe first that there exist points \(z^{0} \in \mathbb{B}^{n}\setminus\{0\}\) such that \(|I(z^{0})||z^{0}|| = 1\). It follows from the maximum principle for the modulus of holomorphic functions of several complex variables, because ||f|| = \( \mu_{\mathbb{B}^{n}}(I) = 1\). We will show that the equality in (3.4) in such points are attained by the mappings \(\mathcal{F}, \mathcal{F}\) of the form (3.2), where \(f = \tilde{f}, f = \bar{f}\) are defined in \(\mathcal{G} = \mathbb{B}^{n}\) by (2.9), (2.11) in the cases \(|2 + k - 4 \lambda| \geq k\) and \(|2 + k - 4 \lambda| < k\), respectively. The mappings \(\mathcal{F}, \mathcal{F}\) belong to \(\mathcal{F}_{1,k}(\mathbb{B}^{n}, \mathbb{C}^{n})\), by the enumerated above fact 2, because \(\tilde{f}, \bar{f} \in \mathcal{F}_{0,k}(\mathbb{B}^{n}, \mathbb{C})\). Also \(\mathcal{F}, \mathcal{F} \in \mathcal{F}_{k}(\mathbb{B}^{n})\), by the enumerated above fact l and the relations \(\bar{f}, \bar{f} \in \mathcal{M}_{k}\).

First, we assume that \(|2 + k - 4 \lambda| \geq k\), i.e., \(f = \tilde{f}\) It is easy to check that in this case

\[
Q_{f,k} = Q_{\tilde{f},k} = \frac{2}{k} f^{k}, \quad Q_{f,2k} = Q_{\tilde{f},2k} = \frac{k + 2}{k} (f^{k})^{2}.
\]

For \(z = z^{0}\) and \(F = \tilde{F}\), i.e., \((f = \tilde{f})\), we obtain (for the first below equality see the previous part of the proof)
Theorem 3.2 For mappings $F \in \mathcal{S}^n(\mathbb{B}^n) \cap \mathcal{F}_{1,\lambda}(\mathbb{B}^n, \mathbb{C}^n), k \in \mathbb{N}_2$, points $z \in \mathbb{B}^n \setminus \{0\}$ and parameter $\lambda \in \mathbb{C}$, there holds the following sharp estimate
\[
\left| T_z(D^{2k+1}F(0)(z^{2k+1})) \right|_{(2k + 1)!\|z\|^{2k+1}} - \lambda \left( T_z(D^{k+1}F(0)(z^{k+1})) \right|_{(k + 1)!\|z\|^{k+1}}^2 \right) \\
= \frac{1}{\|z\|^{2k}} Q_{f,2k}(z) - \lambda \left( Q_{f,2k}(z) \right|_{k+1}^2 \\
= \frac{1}{\|z\|^{2k}}\left| k + 2 \left( f^k(z) \right) - \lambda \frac{4}{k^2} \left( f^k(z) \right) \right| = \left| \left( f^k(z) \right) \right|_{k+1}^2 \left| \frac{k + 2}{k^2} - \lambda \frac{4}{k^2} \right| \\
= \frac{1}{k} \max \left\{ 1, \frac{|2 + k - 4\lambda|}{k} \right\}.
\]

Now, we assume that $|2 + k - 4\lambda| < k$, i.e., $f = \hat{f}$. In this case it is easy to check that

$$Q_{f,k} = Q_{\hat{f},k} = 0, \quad Q_{f,2k} = Q_{\hat{f},2k} = \frac{1}{k} f^{2k}.$$ 

For $z = z^0$ and $F = \hat{F}$, i.e., $(f = \hat{f})$, we obtain similarly
\[
\left| T_z(D^{2k+1}F(0)(z^{2k+1})) \right|_{(2k + 1)!\|z\|^{2k+1}} - \lambda \left( T_z(D^{k+1}F(0)(z^{k+1})) \right|_{(k + 1)!\|z\|^{k+1}}^2 \right) \\
= \frac{1}{\|z\|^{2k}} \left| f^{2k}(z) \right| = \max \left\{ 1, \frac{|2 + k - 4\lambda|}{k} \right\}.
\]

The identity (3.3) and Theorem 3.1 implies the following result of Fekete–Szegö type for starlike biholomorphic mappings in $\mathbb{C}^n$.

Theorem 3.2 For mappings $F \in \mathcal{S}^n(\mathbb{B}^n) \cap \mathcal{F}_{1,\lambda}(\mathbb{B}^n, \mathbb{C}^n), k \in \mathbb{N}_2$, points $z \in \mathbb{B}^n \setminus \{0\}$ and parameter $\lambda \in \mathbb{C}$, there holds the following sharp estimate
\[
\left| T_z(D^{2k+1}F(0)(z^{2k+1})) \right|_{(2k + 1)!\|z\|^{2k+1}} - \lambda \left( T_z(D^{k+1}F(0)(z^{k+1})) \right|_{(k + 1)!\|z\|^{k+1}}^2 \right) \leq \frac{1}{k} \max \left\{ 1, \frac{|2 + k - 4\lambda|}{k} \right\},
\]

where $T_z \in (\mathbb{C}^n)^n, z \in \mathbb{B}^n \setminus \{0\}$, is arbitrary functional satisfying the conditions: $\|T_z\| = 1$, $T_z(z) = \|z\|$.

In the other way a similar theorem was proved by Xu [30].

4 Final remarks

It is possible to allow also $k = 1$ in definition of $(j,k)$-symmetrical, $j \in \mathbb{Z}$, functions, from $\mathcal{F}(G,\mathbb{C}^m)$. Then $\varepsilon = 1$ and we should take the convention $\mathcal{F}_{1,\lambda}(G,\mathbb{C}^m) = \mathcal{F}(G,\mathbb{C}^m)$ for $j \in \mathbb{Z}$.

Consequently, in the case $m = 1$
while in the case \( m = n \)

\[
\tilde{S}^n(\mathbb{B}^n) \cap \mathcal{F}_{1,1}(\mathbb{B}^n, \mathbb{C}^n) = \tilde{S}^n(\mathbb{B}^n) = \tilde{S}^1(\mathbb{B}^n) = \tilde{S}^1(\mathbb{B}^n) \cap \mathcal{F}_{1,1}(\mathbb{B}^n, \mathbb{C}^n).
\]

Therefore, for \( \lambda \in \mathbb{C} \), we obtain the following sharp estimates

\[
\mu_\mathcal{G}\left(Q_{f,2} - \lambda(Q_{f,1})^2\right) \leq \frac{1}{3} \max \{1, 3|\lambda - 1|\},
\]

\[
\mu_\mathcal{G}\left(Q_{f,2} - \lambda(Q_{f,1})^2\right) \leq \max \{1, 3|4\lambda|\},
\]

\[
\left| \frac{T_z(D^2F(0)(z^2))}{3||z||^3} - \lambda \left( \frac{T_z(D^2F(0)(z^2))}{2||z||^2} \right)^2 \right| \leq \max \{1, 3|4\lambda|\}, z \in \mathbb{B}^n \setminus \{0\},
\]

with \( T_z \in (\mathbb{C}^n)^*, \|T_z\| = 1, T_z(z) = ||z|| \), for the families \( \mathcal{N}_\mathcal{G}, \mathcal{M}_\mathcal{G} = \mathcal{M}_\mathcal{G}^1 \cap \mathcal{F}_{0,1}(\mathcal{G}, \mathbb{C}) \), respectively.

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