An eigenvalue problem for the anisotropic \( \Phi \)-Laplacian

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Abstract

We study an eigenvalue problem involving a fully anisotropic elliptic differential operator in arbitrary Orlicz-Sobolev spaces. The relevant equations are associated with constrained minimization problems for integral functionals depending on the gradient of competing functions through general anisotropic \( N \)-functions. In particular, the latter need neither be radial, nor have a polynomial growth, and are not even assumed to satisfy the so called \( \Delta_2 \)-condition. The resulting analysis requires the development of some new aspects of the theory of anisotropic Orlicz-Sobolev spaces.

1 Introduction

In the present paper, we deal with the existence of solutions to a fully anisotropic eigenvalue problem having the form

\[
\begin{aligned}
-\text{div} (\Phi_{\xi} (\nabla u)) &= \lambda b(|u|) \text{sign } u & \text{in } \Omega \\
u &= 0 & \text{on } \partial \Omega,
\end{aligned}
\]

where \( \Omega \) is an open bounded subset in \( \mathbb{R}^n \), with \( n \geq 2 \), \( \lambda \) is a positive real parameter, \( \Phi : \mathbb{R}^n \to [0, \infty) \) is an \( N \)-function (see Section 2.1) and \( b : [0, \infty) \to [0, \infty) \) is an increasing, left-continuous function such that \( b(t) = 0 \) if and only if \( t = 0 \) and \( \lim_{t \to \infty} b(t) = +\infty \). Here, \( \Phi_{\xi} \) denotes the gradient of \( \Phi \). Let us emphasize that \( \Phi(\xi) \) neither necessarily depends on \( \xi \) through its length \( |\xi| \), nor necessarily has a power type behavior.

If the \( N \)-functions \( \Phi \) and \( B(t) = \int_0^{|t|} b(\tau) \, d\tau \) satisfy the so called \( \Delta_2 \)-condition, (1.2) represents the Euler-Lagrange equation associated with the following constrained minimization problem

\[
\inf \left\{ \int_{\Omega} \Phi(\nabla u) \, dx : u \in W_{0}^{1} L_{B, \Phi}(\Omega), \int_{\Omega} B(u) \, dx = r \right\},
\]

where \( r \) is any positive real constant, \( W_{0}^{1} L_{B, \Phi}(\Omega) \) is the anisotropic Orlicz-Sobolev space built upon \( \Phi \) and \( B \). We point out that neither \( \Phi \) nor \( B \) are required to fulfill the \( \Delta_2 \)-condition. Due to this fact, differentiability of the functionals appearing in (1.2) is not guaranteed. Hence, the equation in (1.1) cannot be derived via standard methods like constrained minimization or critical point techniques.

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The function $B$ will be subject to a sharp growth condition that follow from the anisotropic Sobolev inequality for $W^{1}_0L_{B,\Phi}(\Omega)$ proved in [C1]. For a comprehensive treatment of this matter, we refer the reader to Section 2.3 and Section 3.

Our aim is to show that for any $r > 0$ there exist $\lambda_r > 0$ and $u_r \in W^{1}_0L_{B,\Phi}(\Omega) \cap L^\infty(\Omega)$ such that $\int_\Omega B(u_r) \, dx = r$ and $u_r$ solves problem (1.1) with $\lambda = \lambda_r$.

Classical results in this line of investigations deal with the eigenvalue problem for $p-$Laplacian

$$
\begin{cases}
-\text{div} (|\nabla u|^{p-2}\nabla u) = \lambda |u|^{q-2} u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

with $1 < p < N$ and $1 < q < p^*$, where $p^*$ stands for the Sobolev conjugate of $p$. The equation in (1.3) is the Euler-Lagrange equation associated with the minimization problem (1.2) corresponding to the choice $\Phi(\xi) = \frac{1}{p} |\xi|^p$. Several results are available in the literature on existence and properties of eigenvalues and corresponding eigenfunctions to problem (1.3) (see, e.g., [D, FNSS, L]).

Isotropic eigenvalue problems and associated constrained minimization problems in the spirit of (1.1) and (1.2), respectively, with $\Phi(\xi) = \Phi(|\xi|)$ and $B(t) = \Phi(|\xi|)$, have been investigated in [MT]. Our contribution extends the results of [MT], not only in allowing for completely fully anisotropic differential operators, but also in admitting more general growths on the right-hand side $b(|u|) \text{sign} u$. In particular, the generality of the problems under consideration calls for the use and further development of the unconventional functional framework of anisotropic Orlicz and Orlicz-Sobolev spaces which are not necessarily reflexive (see, e.g., [BC, C1, C2, C3, Sc, Sk1, Sk2, Tr]).

Let us mention that elliptic equations and variational problems, whose growth is governed by an $n-$dimensional Young function $\Phi$, have been studied under diverse perspectives in [A, AdBF1, AdBF2, AC, ACCZ-G, BC, CGZ-G, Ch, C1, C2, C3, GWWZ].

The paper is organized as follows. Section 2 contains a background, as well as some new results, on anisotropic Orlicz and Orlicz-Sobolev spaces. The statements of our main results and some special instances are given in Section 3. The proofs of main results are presented in Section 4.

## 2 Functional setting

### 2.1 Young functions

Let $n \geq 1$. Let $\Phi : \mathbb{R}^n \to [0, +\infty]$ be an $n-$dimensional Young function, namely an even, convex function such that $\Phi(0) = 0$ and, for every $t > 0$, the set $\{\xi \in \mathbb{R}^n : \Phi(\xi) < t\}$ is bounded and contains an open neighborhood of 0. An $n-$dimensional Young function $\Phi$ is called an $n-$dimensional $N-$function if it is a finite valued function, vanishes only at 0 and the following additional conditions are in force

$$(2.1) \quad \lim_{|\xi| \to +\infty} \frac{\Phi(\xi)}{|\xi|} = +\infty$$

and

$$(2.2) \quad \lim_{|\xi| \to 0} \frac{\Phi(\xi)}{|\xi|} = 0.$$  

For $n = 1$, any 1-dimensional $N-$function $A : \mathbb{R} \to [0, +\infty)$ takes the form

$$(2.3) \quad A(t) = \int_0^{|t|} a(\tau) \, d\tau \quad \text{for } t \in \mathbb{R},$$

where $a : [0, \infty) \to [0, \infty)$ is a non-decreasing, right-continuous function, which is positive for $\tau > 0$ and satisfies conditions $a(0) = 0$ and $\lim_{\tau \to \infty} a(\tau) = \infty$. 

If $\Phi$ is an $n$-dimensional Young function, then
\begin{equation}
\Phi(h\xi) \leq |h|\Phi(\xi) \quad \text{for } |h| \leq 1 \text{ and } \xi \in \mathbb{R}^n.
\end{equation}

The Young inequality tells us that
\begin{equation}
\xi \cdot \xi' \leq \Phi(\xi) + \Phi^\ast(\xi') \quad \text{for } \xi, \xi' \in \mathbb{R}^n,
\end{equation}
where $\Phi^\ast$ is the Young conjugate function of $\Phi$ given by
\begin{equation}
\Phi^\ast(\xi') = \sup \{ \xi \cdot \xi' - \Phi(\xi) : \xi \in \mathbb{R}^n \} \quad \text{for } \xi' \in \mathbb{R}^n.
\end{equation}

Here, “$\cdot$” stands for scalar product in $\mathbb{R}^N$. We observe that under additional assumption (2.1) the function $\Phi^\ast$ is a finite-valued and hence an $n$-dimensional Young function. Note also that Young conjugation is involutive, i.e. $\Phi^{\ast\ast} = \Phi$. Moreover, $\Phi^\ast$ is an $N-$function, provided that $\Phi$ does.

An $n$-dimensional Young function $\Phi$ is said to satisfy the $\Delta_2^2$-condition near infinity, briefly $\Phi \in \Delta_2^2$ near infinity, if there exist constants $C > 2$ and $K \geq 0$ such that $\Phi(2\xi) \leq C \Phi(\xi)$ for $|\xi| > K$.

Let us consider a case when the $n$-dimensional $N-$function $\Phi$ is given by
\begin{equation}
\Phi(\xi) = \sum_{i=1}^n A_i(\xi_i) \quad \text{for } \xi \in \mathbb{R}^n,
\end{equation}
where $A_i$, for $i = 1, \ldots, n$, are 1-dimensional $N-$functions. A standard choice in (2.7) is $A_i(t) = |t|^{p_i}$ for some powers $1 < p_i < +\infty$, $i = 1, \ldots, n$. One can easily verify that in (2.7) every function $A_i \in \Delta_2$ near infinity if and only if $\Phi(\xi)$ does. An example of a function which does not satisfy the $\Delta_2$-condition is given by
\begin{equation}
\Phi(\xi) = \sum_{i=1}^n (e^{|\xi_i|^{\alpha_i}} - 1) \quad \text{for } \xi \in \mathbb{R}^n
\end{equation}
with $\alpha_i > 1$, for any $i = 1, \ldots, n$.

The following proposition is a special case of [Sk1, Th5.1].

**Proposition 2.1 [Equality cases in the Young inequality]** Let $\Phi$ be a differentiable $n-$dimensional Young function. Then, for any $\xi_0 \in \mathbb{R}^n$

$$\xi_0 \cdot \eta = \Phi(\xi_0) + \Phi^\ast(\eta)$$

if and only if $\eta = \Phi^\xi(\xi_0)$, where $\Phi^\xi$ denotes the gradient of $\Phi$.

Note that for $n = 1$, the differentiability assumption on $\Phi$ that appears in Proposition 2.1 is obviously verified.

Thanks to Proposition 2.1 in [BC, Proposition 6.7] the authors proved the following lemma.

**Lemma 2.2 [BC]** Let $\Phi$ be a differentiable $n-$dimensional Young function. Assume that (2.1) holds. Then
\begin{equation}
\Phi^\ast(\Phi^\xi(\xi)) \leq \Phi^\xi(\xi) \cdot \xi \leq \Phi(2\xi) \quad \text{for } \xi \in \mathbb{R}^N.
\end{equation}

Finally, we show a technical lemma which will be very useful in the sequel.

We say that two $n-$dimensional $N-$functions $\Phi$ and $\Psi$ are equivalent if there exist positive constants $k_1$ and $k_2$, depending only on $n$, such that
\begin{equation}
\Phi(k_1\xi) \leq \Psi(\xi) \leq \Phi(k_2\xi) \quad \text{for } \xi \in \mathbb{R}^n.
\end{equation}

We emphasize that $\Phi$ and $\Psi$ are equivalent if and only if $\Phi^\ast$ and $\Psi^\ast$ do.
Lemma 2.3 Given any \( n \)-dimensional \( N \)-function \( \Phi \), there exists another \( n \)-dimensional \( N \)-function which is differentiable and equivalent to \( \Phi \).

Proof. Theorem 26.3 in [Ro] states that the strictly convexity of an \( N \)-function guaranties the differentiability of its conjugate. Thus, it is enough to prove the existence of a strictly convex \( N \)-function equivalent to \( \Phi \). Let \( \Phi_- : \mathbb{R}^n \rightarrow [0, \infty) \) be the radial function defined as

\[
\Phi_-(\xi) = \sup\{ \Theta(\xi) \text{ s.t. } \Theta : \mathbb{R}^n \rightarrow [0, \infty) \text{ convex, radial and } \Theta(\xi) \leq \Phi(\xi) \}.
\]

By construction, \( \Phi_- \) is an \( N \)-function.

Fixed \( c > 0 \) and let \( g : [0, \infty) \rightarrow [0, \infty) \) be a strictly increasing function such that \( 0 < g(s) \leq c \) for \( s \geq 0 \). Then,

\[ G(t) = \int_0^t g(s) \, ds \]

is a strictly convex, increasing function and \( 0 < G(t) \leq ct \) for any \( t > 0 \). Set

\[ \Upsilon(\xi) = G(\Phi_-(\xi)) \]

Since \( \Phi_- \) is radial and \( G \) is strictly convex, it follows that also \( \Upsilon \) is strictly convex.

Then, \( \Phi + \Upsilon \) is an \( N \)-function, strictly convex and equivalent to \( \Phi \) because

\[
\Phi(\xi) \leq \Phi(\xi) + \Upsilon(\xi) \leq \Phi(\xi) + c\Phi_-(\xi) \leq (1 + c)\Phi(\xi) \leq \Phi((1 + c)\xi).
\]

\[ \square \]

2.2 Anisotropic Orlicz spaces

In this section we present Orlicz spaces built upon both a \( 1 \)-dimensional Young function (see, e.g., [Ad]) and \( n \)-dimensional Young functions (see, e.g., [BC, Sk1, Sk2, Sc]). For the convenience of the reader we give a brief background.

Let \( \Omega \) be a bounded measurable subset in \( \mathbb{R}^n \), with \( n \geq 2 \). The Orlicz space \( L_A(\Omega) \), associated with a \( 1 \)-dimensional Young function \( A \), is the set of all measurable functions \( g : \Omega \rightarrow \mathbb{R} \) such that the Luxemburg norm

\[
\|g\|_{L_A(\Omega)} = \inf \left\{ k > 0 : \int_{\Omega} A\left(\frac{|g(x)|}{k}\right) \, dx \leq 1 \right\}
\]

is finite. The functional \( \|g\|_{L_A(\Omega)} \) is a norm on \( L_A(\Omega) \), which makes the latter a Banach space.

Given two finite-valued \( 1 \)-dimensional Young functions \( A \) and \( D \), we say that \( A \ll D \), namely \( A \) increases essentially more slowly than \( D \) near infinity, if

\[
\lim_{t \to +\infty} \frac{A(\gamma t)}{D(t)} = 0 \quad \text{for every } \gamma > 0.
\]

Note that if \( A \ll D \), then

\[ L_D(\Omega) \hookrightarrow L_A(\Omega), \]

where the arrow “\( \hookrightarrow \)” stands for continuous embedding.

Let \( \Phi \) be an \( n \)-dimensional Young function. The anisotropic Orlicz class \( L_\Phi(\Omega; \mathbb{R}^n) \) is defined as

\[
L_\Phi(\Omega; \mathbb{R}^n) = \left\{ U : \Omega \rightarrow \mathbb{R}^n \text{ measurable s.t. } \int_{\Omega} \Phi(U) < +\infty \right\}.
\]

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Note that \( \mathcal{L}_\Phi(\Omega; \mathbb{R}^n) \) is a convex set of function and it need not be a linear space in general, unless \( \Phi \) satisfies the \( \Delta_2 \)-condition near infinity. The \textit{Orlicz space} \( L_\Phi(\Omega; \mathbb{R}^n) \) is the linear hull of \( \mathcal{L}_\Phi(\Omega; \mathbb{R}^n) \) and it is a Banach space with respect to the Luxemburg norm

\[
\|U\|_\Phi = \inf \left\{ k > 0 : \int_\Omega \Phi\left( \frac{U}{k} \right) \leq 1 \right\}.
\]

We stress that if two \( n \)-dimensional Young functions \( \Phi \) and \( \Psi \) are equivalent, then \( \| \cdot \|_\Phi \) and \( \| \cdot \|_\Psi \) are equivalent and then \( L_\Phi \) and \( L_\Psi \) are the same space. In particular \( L_\Phi(\Omega; \mathbb{R}^n) \subset L^1(\Omega) \) for any \( n \)-dimensional Young functions \( \Phi \). Let us denote by \( E_\Phi(\Omega; \mathbb{R}^n) \) the closure in \( L_\Phi(\Omega; \mathbb{R}^n) \) of the bounded measurable functions with compact support in \( \Omega \). In general

\[
E_\Phi(\Omega; \mathbb{R}^n) \subset L_\Phi(\Omega; \mathbb{R}^n) \subset L_\Phi(\Omega; \mathbb{R}^n),
\]

and the equality holds if and only if \( \Phi \) satisfies the \( \Delta_2 \)-condition near infinity (see [Sc, Corollary 5.1]).

From now on, let \( \Phi \) be an \( n \)-dimensional \( N \)-function. The following generalized Hölder inequality holds

\[
\int_\Omega U(x) \cdot V(x) \ dx \leq 2 \|U\|_\Phi \|V\|_\Phi,
\]

for every \( U \in L_\Phi(\Omega; \mathbb{R}^n) \) and \( V \in L_{\Phi^\ast}(\Omega; \mathbb{R}^n) \) (see [Sk2, Th 4.1]). The integral in (2.13) defines a linear and continuous functional on \( L_\Phi(\Omega; \mathbb{R}^n) \). The space \( L_\Phi(\Omega; \mathbb{R}^n) \) can be also endowed with the following Orlicz norm

\[
\|U\|_{\Phi} = \sup_{\int_\Omega \Phi(V(x)) \leq 1} \left| \int_\Omega U(x) \cdot V(x) \ dx \right|.
\]

If \( \Phi \) is differentiable, then the Luxemburg norm (2.12) and the Orlicz norm (2.14) are equivalent, i.e. \( \|U\|_\Phi \leq \|U\|_{\Phi} \leq 2\|U\|_\Phi \) (see [Sk2, Th 4.5]). The extra assumption on differentiability of \( \Phi \) can be dropped thanks to Lemma 2.3.

Combining the Orlicz norm and the Luxemburg norm together it is possible to get this sharp form of generalized Hölder inequality

\[
\int_\Omega U(x) \cdot V(x) \ dx \leq \|U\|_{\Phi} \|V\|_{\Phi^\ast},
\]

for every \( U \in L_\Phi(\Omega; \mathbb{R}^n) \) and \( V \in L_{\Phi^\ast}(\Omega; \mathbb{R}^n) \).

If \( A \) is a \( 1 \)-dimensional \( N \)-function, it is known that the dual space of \( E_A(\Omega) \) is isomorphic and homeomorphic to \( L_{A^\ast}(\Omega) \) (see [Ad, Theorem 8.18]). The analogue result holds for the anisotropic spaces.

**Proposition 2.4** Let \( \Phi \) an \( n \)-dimensional \( N \)-function. The dual space of \( E_\Phi(\Omega; \mathbb{R}^n) \) is isomorphic and homeomorphic to \( L_{\Phi^\ast}(\Omega; \mathbb{R}^n) \) and the duality pairing is given by

\[
<V, U> = \int_\Omega V(x) \cdot U(x) \ dx
\]

for \( V \in L_{\Phi^\ast}(\Omega; \mathbb{R}^n) \) and \( U \in E_\Phi(\Omega; \mathbb{R}^n) \).

**Remark 2.5** Note that if \( \Phi \in \Delta_2 \), then \( (L_\Phi(\Omega; \mathbb{R}^n))' = L_{\Phi^\ast}(\Omega; \mathbb{R}^n) \).
Proof of Proposition 2.4. We proceed by steps. First we show that any element \( V \in L_{\Phi^*}(\Omega; \mathbb{R}^n) \) determines a bounded linear functional defined as

\[
(2.16) \quad < l_V, U > = \int_{\Omega} U(x) \cdot V(x) dx
\]

for every \( U \in E_{\Phi}(\Omega; \mathbb{R}^n) \). Then it remains to be shown that every bounded linear functional on \( E_{\Phi}(\Omega; \mathbb{R}^n) \) is of form \( l_V \) for some such \( V \in L_{\Phi^*}(\Omega; \mathbb{R}^n) \). In order to this, we prove that any bounded linear functional \( l \) on \( E_{\Phi}(\Omega; \mathbb{R}^n) \) has the form \( (2.16) \) when we restrict ourselves to the set of simple functions, i.e. functions that assume a finite number of values. The density of this set in \( E_{\Phi} \) allows us to conclude the proof.

**Step 1.** \( l_V \) restricted on \( E_{\Phi}(\Omega; \mathbb{R}^n) \) belongs to \( (E_{\Phi}(\Omega; \mathbb{R}^n))^t \).

It follows by \( (2.13) \).

**Step 2.** The set of simple functions is dense in \( E_{\Phi} \).

Let us consider \( U \in L^\infty(\Omega; \mathbb{R}^n) \). By standard measure theory, there exists a sequence \( U_k \) of simple function such that

1. \( U_k \to U \) a.e. in \( \Omega \),
2. \( \|U_k\|_{L^\infty(\Omega; \mathbb{R}^n)} \leq \|U\|_{L^\infty(\Omega; \mathbb{R}^n)} \) for each \( k \).

We claim that \( \|U_k - U\|_\Phi \to 0 \). Indeed, by ii) we can choose \( C_1 \) s.t.

\[
\|U_k - U\|_{L^\infty(\Omega; \mathbb{R}^n)} \leq C_1.
\]

Given \( \epsilon > 0 \), set

\[
C_2(\epsilon) = \sup_{\|x\| \leq C_1} \Phi \left( \frac{\xi}{\epsilon} \right)
\]

and

\[
S_k = \{ x : \Phi \left( \frac{U_k(x) - U(x)}{\epsilon} \right) > \frac{1}{2|\Omega|} \}.
\]

Since i), the measure \( |S_k| \to 0 \) as \( k \to \infty \). Then, we can choose \( k \) sufficiently large such that \( |S_k| < \frac{1}{2C_2(\epsilon)} \). It follows that

\[
(2.17) \quad \int_{\Omega} \Phi \left( \frac{U_k - U}{\epsilon} \right) dx = \int_{S_k} \Phi \left( \frac{U_k - U}{\epsilon} \right) dx + \int_{\Omega \setminus S_k} \Phi \left( \frac{U_k - U}{\epsilon} \right) dx
\]

\[
\leq C_2(\epsilon) \frac{1}{2C_2(\epsilon)} + |\Omega| \frac{1}{2|\Omega|} \leq 1,
\]

that means \( \|U_k - U\|_\Phi \leq \epsilon \). By definition of \( E_{\Phi}(\Omega; \mathbb{R}^n) \), the result holds for \( U \in E_{\Phi}(\Omega; \mathbb{R}^n) \).

**Step 3.** Representation formula for \( l \in (E_{\Phi}(\Omega; \mathbb{R}^n))^t \) restricted to simple functions.

Let \( l \) be any continuous linear functional on \( E_{\Phi}(\Omega; \mathbb{R}^n) \) and let \( G \subset \Omega \) be a measurable set. Let us consider \( \overline{\chi}_G(x) = \sum_{h=1}^n \chi_G(x)e_h = \sum_{h=1}^n \chi_{G,h}(x) \), where \( \chi_{G,h}(x) = (0, \ldots, \chi_h^G(x), \ldots, 0) = \chi^G(x)e_h \) with \( \chi^G(x) \) the characteristic function of the set \( G \) and \( \{e_1, \ldots, e_n\} \) a base of \( \mathbb{R}^n \). For every \( h \in \{1, \ldots, n\} \), we have that \( \overline{\chi}_{G,h} \in E_{\Phi}(\Omega; \mathbb{R}^n) \) and \( \mu_h(G) = \langle l, \overline{\chi}_{G,h} \rangle > 0 \) is an absolutely continuous measure. Indeed,

\[
(2.18) \quad | \langle l, \overline{\chi}_{G,h} \rangle | \leq \|l\| \quad \|\overline{\chi}_{G,h}\|_\Phi
\]

with

\[
\|\overline{\chi}_{G,h}\|_\Phi = \inf \left\{ k : \int_{\Omega} \Phi \left( \frac{\overline{\chi}_{G,h}(x)}{k} \right) dx \leq 1 \right\} = \inf \left\{ k : |G\Phi \left( \frac{e_h}{k} \right) \leq 1 \right\}.
\]

Let us define the 1-dimensional Young function \( A_h : \mathbb{R} \to [0, \infty) \) as \( A_h(t) = \Phi(t e_h) \) for every \( t \in \mathbb{R} \). Then,

\[
(2.19) \quad \|\overline{\chi}_G\|_\Phi = \frac{1}{A_h^{-1} \left( \frac{1}{|G|} \right)}.
\]
The absolute continuity of measure $\mu_h$ follows by combing (2.18) and (2.19). By virtue of the Radon-Nikodym’s theorem, there exists a real valued function $V_h$ belonging to $L^1(\Omega)$ such that
\begin{equation}
\langle l, \chi_{G,h} \rangle = \int_G V_h(x)\chi_G(x)e_h \, dx \quad h = 1, \ldots, n.
\end{equation}
By (2.20),
\begin{equation}
\langle l, \chi_G \rangle = \sum_{h=1}^n \langle l, \chi_{G,h} \rangle = \int_G V(x)\chi_G(x) \, dx,
\end{equation}
where $V(\omega) = (V_1(x), \ldots, V_n(x))$. Moreover, if $U$ is a simple function defined as
\begin{equation}
U(\omega) = \sum_{j=1}^n U_j \chi_{G_j}(x),
\end{equation}
where $\{G_j\}$ are disjoint measurable subset of $\Omega$, by the linearity of $l$ and (2.21), we get
\begin{equation}
\langle l, U \rangle = \sum_{j=1}^n U_j \langle l, \chi_{G_j} \rangle = \sum_{j=1}^n U_j \int_{\Omega} V \chi_{G_j}(x) \, dx = \int_{\Omega} U \cdot V \, dx = \langle l_V, U \rangle,
\end{equation}
where $l_V$ is defined by (2.16).

**Step 4.** Function $V$ that appears in (2.23) belongs to $L_{\Phi_\bullet}(\Omega; \mathbb{R}^n)$.

Let $U \in E_\Phi(\Omega; \mathbb{R}^n)$. By Step 2, we know that there exists a sequence $U_h$ of simple functions such that $U_h \to U$ in $L_\Phi(\Omega; \mathbb{R}^n)$. This means that $U_h \to U$ almost everywhere and also the sequence $|U_h \cdot V| \to |U \cdot V|$ almost everywhere. Moreover, fixed some positive constant $K$, one can choose $h$ sufficiently large such that
\[ \|U_h\|_\Phi \leq \|U\|_\Phi + \|U_h - U\|_\Phi \leq \|U\|_\Phi + K, \]
for every $h \in \mathbb{N}$. Now, if $U_h \cdot V \geq 0$, on applying Fatou’s lemma, we get
\begin{equation}
\int_{\Omega} U \cdot V \, dx \leq \liminf_{h \to \infty} \int_{\Omega} U_h \cdot V \, dx \leq \liminf_{h \to \infty} \left| \int_{\Omega} U_h \cdot V \, dx \right| \leq \liminf_{h \to \infty} \|l\|_\Phi \|U_h\|_\Phi \leq (\|U\|_\Phi + K).
\end{equation}
On the other hand, if $U_h \cdot V \leq 0$, on applying Fatou’s lemma again, we get
\begin{equation}
\int_{\Omega} -U \cdot V \, dx \leq \liminf_{h \to \infty} \int_{\Omega} -U_h \cdot V \, dx \leq \liminf_{h \to \infty} \left| \int_{\Omega} -U_h \cdot V \, dx \right| \leq \liminf_{h \to \infty} \|l\|_\Phi \|U_h\|_\Phi \leq (\|U\|_\Phi + K).
\end{equation}
By (2.24) and (2.25), we deduce
\begin{equation}
\int_{\Omega} U \cdot V \, dx < +\infty
\end{equation}
for any $U \in E_\Phi(\Omega; \mathbb{R}^n)$. This means that if we choose any $U(x) = (U_1(x), \ldots, U_n(x))$ with
\[ U_i(x) = \frac{\partial \Phi_\bullet}{\partial e_i}(x) \quad i = 1, \ldots, n \]
then, by Proposition 2.1, we get
\[ \int_{\Omega} \Phi_\bullet(V) \, dx \leq \int_{\Omega} \Phi_\bullet(V) \, dx + \int_{\Omega} \Phi(U) \, dx = \int_{\Omega} U \cdot V \, dx. \]
We stress that the extra assumption on differentiability of $\Phi_\bullet$ required in Proposition 2.1 can be dropped thanks to Lemma 2.3. Finally since (2.26), it follows that $V \in L_{\Phi_\bullet}(\Omega; \mathbb{R}^n)$ and $l_V$ is linear bounded functional on $E_\Phi(\Omega, \mathbb{R}^n)$.

**Step 5.** Identification between $l$ and $l_V$.

We note that both the functionals $l_V$ defined as in (2.16) and $l$ assume the same values on the set of simple function. Since the last set is dense in $E_\Phi(\Omega, \mathbb{R}^n)$, they agree with $E_\Phi(\Omega, \mathbb{R}^n)$ and the proof is complete. \[ \Box \]
Here, we extend \([Go1, \text{Lemma 1}]\) to vector-valued functions that we will be useful in the following.

**Lemma 2.6** Let \(\Phi\) an \(n\)–dimensional \(N\)–function. For all \(V \in L_\Phi(\Omega; \mathbb{R}^n)\)

\[
\sup \left\{ \int_\Omega U \cdot V \, dx - \int_\Omega \Phi_*(U) \, dx : U \in L_\Phi^*(\Omega; \mathbb{R}^n) \right\} = \int_\Omega \Phi(V) \, dx
\]

(2.27)

\[
= \sup \left\{ \int_\Omega U \cdot V \, dx - \int_\Omega \Phi_*(U) \, dx : U \in E_\Phi^*(\Omega; \mathbb{R}^n) \right\}
\]

Proof. Since \(E_\Phi^*(\Omega; \mathbb{R}^n) \subset L_\Phi^*(\Omega; \mathbb{R}^n)\), we have only to prove that

\[
\sup \left\{ \int_\Omega U \cdot V \, dx - \int_\Omega \Phi_*(U) \, dx : U \in L_\Phi^*(\Omega; \mathbb{R}^n) \right\} \leq \int_\Omega \Phi(V) \, dx
\]

(2.28)

\[
\leq \sup \left\{ \int_\Omega U \cdot V \, dx - \int_\Omega \Phi_*(U) \, dx : U \in E_\Phi^*(\Omega; \mathbb{R}^n) \right\}.
\]

The left-hand side of (2.28) follows by applying Young inequality.

Let \(V \in L_\Phi^*(\Omega; \mathbb{R}^n)\). We define \(U_h = \Phi(V_h)\) by

\[
V_h = \begin{cases} 
V(x) & \text{for } x \text{ s.t. } |V(x)| \leq h \\
0 & \text{otherwise.}
\end{cases}
\]

By inequality (2.8), we get \(U_h \in L^\infty(\Omega; \mathbb{R}^n) \subset E_\Phi^*(\Omega; \mathbb{R}^n)\). By Proposition 2.1 we obtain \(\Phi(V_h) = U_h \cdot V_h - \Phi_*(U_h)\). So, by integrating on \(\Omega\), it follows that

\[
\int_\Omega \Phi(V_h) \, dx = \int_\Omega U_h \cdot V_h \, dx - \int_\Omega \Phi_*(U_h) \, dx = \int_\Omega U_h \cdot V \, dx - \int_\Omega \Phi_*(U_h) \, dx
\]

and then

\[
\int_\Omega \Phi(V_h) \, dx \leq \sup \left\{ \int_\Omega U \cdot V \, dx - \int_\Omega \Phi_*(U) \, dx : U \in E_\Phi^*(\Omega; \mathbb{R}^n) \right\}.
\]

By Fatou’s lemma, the left-hand side in (2.29) converges to \(\int_\Omega \Phi(V) \, dx < +\infty\) and (2.28) follows.

\[\square\]

### 2.3 Anisotropic Orlicz-Sobolev spaces

Let \(\Phi\) be an \(n\)–dimensional \(N\)–function. Let us define the Banach space \(W_0^{1,1}_\Phi(\Omega)\) (see \([BC]\)) as

\[
W_0^{1,1}_\Phi(\Omega) = \{ u : \Omega \to \mathbb{R} : \text{the continuation of } u \text{ by } 0 \text{ outside } \Omega \\
\text{is weakly differentiable and } \nabla u \in L_\Phi(\Omega; \mathbb{R}^n) \}
\]

equipped with the norm

\[
\|u\|_{W_0^{1,1}_\Phi(\Omega)} = \|\nabla u\|_{L_\Phi(\Omega; \mathbb{R}^n)}.
\]

(2.30)
We emphasize that the following anisotropic Sobolev type inequality holds for any functions in \( W_{1,0}^{1,0} L_{\Phi}(\Omega) \) (see \[\text{C1}\]). Assume that \( \Phi \) fulfils
\[
\int_0^\tau \left( \frac{\tau}{\Phi_o(\tau)} \right)^{\frac{1}{n-1}} d\tau < \infty,
\]
where \( \Phi_o : [0, \infty) \to [0, \infty) \) is a Young function satisfying
\[
|\{ \xi \in \mathbb{R}^n : \Phi(\xi) \leq t \}| = |\{ \xi \in \mathbb{R}^n : \Phi_o(|\xi|) \leq t \}| \quad \text{for } t \geq 0.
\]
Note that the function \( \xi \to \Phi_o(|\xi|) \) agrees with the spherically increasing symmetral of \( \Phi \).

We denote by \( \Phi_n : [0, \infty) \to [0, \infty) \) the optimal Sobolev conjugate of \( \Phi \) defined as
\[
\Phi_n(t) = \Phi_o(H^{-1}(t)) \quad \text{for } t \geq 0,
\]
where \( H : [0, \infty) \to [0, \infty) \) is given by
\[
H(t) = \left( \int_0^t \left( \frac{\tau}{\Phi_o(\tau)} \right)^{\frac{1}{n-1}} d\tau \right)^{\frac{n-1}{n}} \quad \text{for } t \geq 0,
\]
provided that the integral is convergent. Here, \( H^{-1} \) denotes the generalized left-continuous inverse of \( H \).

If
\[
\int_0^\infty \left( \frac{\tau}{\Phi_o(\tau)} \right)^{\frac{1}{n-1}} d\tau = \infty,
\]
then there exists a constant \( C_1 = C_1(n) \) such that
\[
\|u\|_{L_{\Phi_n}(\Omega)} \leq C_1 \|u\|_{W_{1,0}^{1,0} L_{\Phi}(\Omega; \mathbb{R}^n)}
\]
for every \( u \in W_{1,0}^{1,0} L_{\Phi}(\Omega) \) (see \[\text{C1} \text{ Theorem 1 and Remark 1}\]).

If
\[
\int_0^\infty \left( \frac{\tau}{\Phi_o(\tau)} \right)^{\frac{1}{n-1}} d\tau < \infty,
\]
then there exists a constant \( C_2 = C_2(\Phi, n, |\Omega|) \) such that
\[
\|u\|_{L_{\infty}(\Omega)} \leq C_2 C \|u\|_{W_{1,0}^{1,0} L_{\Phi}(\Omega; \mathbb{R}^n)}
\]
for every \( u \in W_{1,0}^{1,0} L_{\Phi}(\Omega) \) (see \[\text{C4} \text{ Theorem 3.5}\]).

We define the anisotropic Orlicz-Sobolev space \( W^{1,1} L_{B,\Phi}(\Omega) \) as
\[
W^{1,1} L_{B,\Phi}(\Omega) = \{ u \in L_B(\Omega) : u \text{ is weakly differentiable in } \Omega \text{ and } \nabla u \in L_{\Phi}(\Omega; \mathbb{R}^n) \}.
\]
The space \( W^{1,1} E_{B,\Phi}(\Omega) \) is defined accordingly by replacing \( L_B(\Omega) \) and \( L_{\Phi}(\Omega; \mathbb{R}^n) \) by \( E_B(\Omega) \) and \( E_{\Phi}(\Omega; \mathbb{R}^n) \), respectively. Both \( W^{1,1} L_{B,\Phi}(\Omega) \) and \( W^{1,1} E_{B,\Phi}(\Omega) \) can be identified to subspaces of the product \( L_B(\Omega) \times L_{\Phi}(\Omega; \mathbb{R}^n) \). The spaces \( W^{1,1} L_{B,\Phi}(\Omega) \) and \( W^{1,1} E_{B,\Phi}(\Omega) \) equipped with the norm
\[
\|u\|_{W^{1,1} L_{B,\Phi}(\Omega)} = \|u\|_{L_B(\Omega)} + \|\nabla u\|_{L_{\Phi}(\Omega; \mathbb{R}^n)}
\]
are Banach spaces (see \[\text{Ad} \text{ Theorem 3.2}\]).
The \( \sigma(L_B \times L_\Phi, L_\Phi \times E_{\Phi_n}) \)-closure of \( D(\Omega) \) in \( W^1L_{B,\Phi}(\Omega) \) is denoted by \( W^1_{0}L_{B,\Phi}(\Omega) \). Analogously, \( W^1_{0}E_{B,\Phi}(\Omega) \) stands for the closure of \( D(\Omega) \) in \( W^1L_{B,\Phi}(\Omega) \) with respect to the norm \( (2.39) \).

Given a function \( u \in W^1_{0}L_{B,\Phi}(\Omega) \), the function obtained by extending \( u \) outside \( \Omega \) by zero belongs to \( W^1L_{B,\Phi}(\mathbb{R}^n) \) and then
\[
(2.40) \quad W^1_{0}L_{B,\Phi}(\Omega) \subset W^1_{0}L_{\Phi}(\Omega).
\]

Both spaces, \( W^1_{0}L_{B,\Phi}(\Omega) \) and \( W^1_{0}L_{\Phi}(\Omega) \), are reflexive if and only if \( \Phi \in \Delta_2 \) near infinity. Embedding \( (2.40) \) yields directly that Sobolev type inequalities \( (2.35) \) and \( (2.37) \) hold for \( W^1_{0}L_{B,\Phi}(\Omega) \).

**Proposition 2.7** Let \( \Phi \) be an \( N \)-function. Assume that either \( (2.34) \) holds and \( B \precsim \Phi_n \) or \( (2.36) \) holds and \( B \) is anything. Then
\[
(2.41) \quad W^1_{0}L_{B,\Phi}(\Omega) \hookrightarrow \hookrightarrow E_B(\Omega).
\]

**Proof.** Arguing as in the proof of [CS, Theorem 2.1], we deduce that
\[
(2.42) \quad \left\{ u : \int_{\Omega} \Phi_n \left( \frac{|u(x)|}{\lambda} \right) dx < \infty \text{ for every } \lambda > 0 \right\} \subset \text{closure of } L^\infty(\Omega) \text{ in } L_{\Phi_n}(\Omega).
\]
Finally, observing that \( \int_0^{\infty} \Phi_n \left( \frac{|u(x)|}{\lambda} \right) dx < \infty \) for every \( \lambda > 0 \) whenever \( u \in W^1_{0}L_{B,\Phi}(\Omega) \) (see also [C0, Remark 7]), we conclude that
\[
(2.43) \quad W^1_{0}L_{B,\Phi}(\Omega) \subset E_{\Phi_n}(\Omega) \subset E_B(\Omega),
\]
thanks to \( B \precsim \Phi_n \).

Let \( \{u_h\}_h \) a bounded sequence in \( W^1_{0}L_{B,\Phi}(\Omega) \). Since the compact embedding (see [BC])
\[
(2.44) \quad W^1_{0}L_{B,\Phi}(\Omega) \to W^1_{0}L_{1}(\Omega) \subseteq L^1(\Omega),
\]
it follows that (up a subsequence) \( \{u_h\}_h \) converges in \( L^1(\Omega) \) and then in measure in \( \Omega \). The convergence in measure and the boundedness in \( L_{\Phi_n}(\Omega) \) of \( \{u_h\}_h \) (that follows using \( (2.35) \) ), combined with the assumption \( B \precsim \Phi_n \), yield that \( \{u_h\}_h \) converges in \( L_{B}(\Omega) \) (see [Ad, Theorem 8.22]). The embedding \( (2.20) \) and the closure of \( E_B(\Omega) \) conclude the proof.

\[\square\]

### 2.4 Complementary systems

Let \( X \) and \( K \) be real Banach spaces in duality with respect to continuous pairing \( \langle \cdot, \cdot \rangle \), and let \( X_0 \) and \( K_0 \) be subspaces of \( X \) and \( K \), respectively. Then \( (X, X_0; K, K_0) \) represent a so-called complementary system if, by means of \( \langle \cdot, \cdot \rangle \), the dual of \( X_0 \) can be identified to \( K \) and that of \( K_0 \) to \( K \).

Given a complementary system \( (X, X_0; K, K_0) \) and a closed subspace \( Y \) of \( X \), it is possible to construct a new complementary system imposing some restrictions on \( Y \). More precisely, set \( Y_0 = Y \cap X_0 \), \( Z = K/Y_0^\perp \) and \( Z_0 = \{ z + Y_0^\perp : z \in K_0 \} \subset Z \), where \( Y_0^\perp = \{ z \in K : \langle y, z \rangle = 0 \text{ for every } y \in Y_0 \} \).

The theory on complementary system has been investigated e.g. in [Go], and, for the convenience of the reader, we recall Lemma 1.2 contained in it. The relevant lemma provides conditions so that \( (Y, Y_0; Z, Z_0) \) is a complementary system generated by \( Y \) in \( (X, X_0; K, K_0) \).

**Lemma 2.8** The pairing \( \langle \cdot, \cdot \rangle \) between \( X \) and \( K \) induces a pairing between \( Y \) and \( Z \) if and only if \( Y_0 \) is \( \sigma(X, K) \) dense in \( Y \). In this case, \( (Y, Y_0; Z, Z_0) \) is a complementary system if \( Y \) is \( \sigma(X, K_0) \) closed, and conversely, when \( K_0 \) is complete, \( Y \) is \( \sigma(X, K) \) closed if \( (Y, Y_0; Z, Z_0) \) is a complementary system.
The topologies $\sigma(Y,Z)$ and $\sigma(Y,Z_0)$ are the weak topologies induced on $Y$ by $\sigma(X,K)$ and $\sigma(X,K_0)$, respectively, and $Z_0$ is the subspace of the dual space of $Y_0$ equals the set of those linear functionals on $Y_0$ which are $\sigma(X,K_0)$ continuous.

Here, our aim is to prove that $Y = W^1_0 L_{B,\Phi}(\Omega)$ generates a new complementary system in $(X,X_0;K,K_0) = (L_B \times L_{\Phi},E_B \times E_{\Phi};L_{B_*,\ast} \times L_{\Phi_{\ast}},E_{B_\ast} \times E_{\Phi_{\ast}})$. In order to do this we assume that $\Omega$ has the segment property, namely there exist a locally finite open covering $(\Omega_j)_j$ of $\partial\Omega$ and corresponding vectors $(y_j)_j$ such that $x + ty_j \in \Omega$ with $x \in \Omega \cap \Omega_j$ and $0 < t < 1$. This condition is essential in Lemma 2.9 below.

Let us verify that the conditions in Lemma 2.8 are fulfilled. First, $W^1_0 L_{B,\Phi}(\Omega)$ is $\sigma(L_B \times L_{\Phi},E_{B_\ast} \times E_{\Phi_{\ast}})$ closed thanks to the very definition of $W^1_0 L_{B,\Phi}(\Omega)$. Moreover, we have to verify that $Y_0 = W^1_0 L_{B,\Phi}(\Omega) \cap (E_B \times E_{\Phi})$ agrees with $W^1_0 E_{B,\Phi}(\Omega)$ and it is $\sigma(L_B \times L_{\Phi},L_{B_\ast} \times L_{\Phi_{\ast}})$ dense in $W^1_0 L_{B,\Phi}(\Omega)$.

**Lemma 2.9** If $\Omega$ has the segment property, then

(a) $W^1_0 E_{B,\Phi}(\Omega) = W^1_0 L_{B,\Phi}(\Omega) \cap (E_B \times E_{\Phi})$,

(b) $W^1_0 E_{B,\Phi}(\Omega)$ is $\sigma(L_B \times L_{\Phi},L_{B_\ast} \times L_{\Phi_{\ast}})$ dense in $W^1_0 L_{B,\Phi}(\Omega)$.

**Proof:** (a) To prove of (a) can be reduced to prove that $D(\Omega)$ is norm dense in $W^1_0 L_{B,\Phi}(\Omega) \cap (E_B \times E_{\Phi})$. By [Go] Theorem 1.3 and Corollary 1.10, it is enough to verify that $D(\Omega)$ is $\sigma(L_B \times L_{\Phi},L_{B_\ast} \times L_{\Phi_{\ast}})$ dense in $W^1_0 L_{B,\Phi}(\Omega)$. It follows by using an appropriate version of Lemmas 1.4 - 1.7 in [Go] applied to the $N$-dimensional Young function $\Phi$. In fact, one can easily verifies that those lemmas hold for vectorial functions, as well.

(b) Let us recall that, by definition, $D(\Omega)$ is dense in $W^1_0 E_{B,\Phi}(\Omega)$ with respect to the norm (2.12) and is $\sigma(L_B \times L_{\Phi},E_{B_\ast} \times E_{\Phi_{\ast}})$ dense in $W^1_0 L_{B,\Phi}(\Omega)$. Our goal is to prove that $W^1_0 E_{B,\Phi}(\Omega)$ is $\sigma(L_B \times L_{\Phi},L_{B_\ast} \times L_{\Phi_{\ast}})$ dense in $W^1_0 L_{B,\Phi}(\Omega)$, namely that for every $u \in W^1_0 L_{B,\Phi}(\Omega)$ there exists a sequence $(u_h)_h \subset W^1_0 E_{B,\Phi}(\Omega)$ such that

\[(2.45) \quad u_h \rightarrow u \quad \text{in} \quad \sigma(L_B \times L_{\Phi},L_{B_\ast} \times L_{\Phi_{\ast}}),\]

i.e.

\[(2.46) \quad \int_\Omega u_h \psi_1 \, dx + \int_\Omega \nabla u_h \cdot \psi_2 \, dx \rightarrow \int_\Omega u \psi_1 \, dx + \int_\Omega \nabla u \cdot \psi_2 \, dx \quad \forall (\psi_1,\psi_2) \in L_{B_\ast} \times L_{\Phi_{\ast}}.\]

Let us suppose by contradiction that there exists a function $\bar{u}$ in $W^1_0 L_{B,\Phi}(\Omega)$ such that, for every sequence $(u_h)_h \subset W^1_0 E_{B,\Phi}(\Omega)$,

\[(2.47) \quad \lim_{n \to \infty} u_h \neq \bar{u} \quad \text{in} \quad \sigma(L_B \times L_{\Phi},L_{B_\ast} \times L_{\Phi_{\ast}}).\]

On the other hand, by the very definition of $W^1_0 E_{B,\Phi}(\Omega)$, for every $u \in W^1_0 E_{B,\Phi}(\Omega)$ there exists a sequence $(u^k_{\bar{h}})_{k \in \mathbb{N}} \subset D(\Omega)$ such that $u^k_{\bar{h}} \rightarrow u$ in norm, and then

\[(2.48) \quad \lim_{k \to \infty} u^k_{\bar{h}} = u \quad \text{in} \quad \sigma(L_B \times L_{\Phi},L_{B_\ast} \times L_{\Phi_{\ast}}).\]

The statement follows observing that (2.48) does not agree with (2.47).

\[\square\]

Lemma 2.8 and Lemma 2.9 assure that $(W^1_0 L_{B,\Phi}(\Omega), W^1_0 E_{B,\Phi}(\Omega); W^{-1} L_{B_\ast,\Phi_{\ast}}(\Omega), W^{-1} E_{B_\ast,\Phi_{\ast}}(\Omega))$ is the complementary system generated by $W^1_0 L_{B,\Phi}(\Omega)$ in $(L_B \times L_{\Phi}, E_B \times E_{\Phi};L_{B_\ast} \times L_{\Phi_{\ast}}, E_{B_\ast} \times E_{\Phi_{\ast}})$, where

\[(2.49) \quad W^{-1} L_{B_\ast,\Phi_{\ast}}(\Omega) = \left\{ f \in D'(\Omega) : f = f_0 - \sum_{i=1}^N \frac{\partial}{\partial x_i} f_i, \ f_0 \in L_{B_\ast}(\Omega), (f_1, \ldots, f_n) \in L_{\Phi_{\ast}}(\Omega; \mathbb{R}^n) \right\} \]
and
\[ W^{-1}E_{B,\Phi}(\Omega) = \left\{ f \in D'(\Omega) : f = f_0 - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} f_i, \ f_0 \in E_{B,\Phi}(\Omega; \mathbb{R}), (f_1, \ldots, f_n) \in E_{\Phi}(\Omega; \mathbb{R}^n) \right\}. \]

### 3 Main results

Assume that \( \Omega \) is an open bounded subset of \( \mathbb{R}^n \), with \( n \geq 2 \), satisfying the segment property, and \( \Phi \) is a differentiable \( N \)–function fulfilling (2.31).

Our first main result concerns the existence of solutions to the following Dirichlet problem

\[
\begin{align*}
-\text{div} (\Phi \xi(\nabla u)) &= \lambda b(|u|) \text{sign} u & \text{in} \ \Omega \\
u &= 0 & \text{on} \ \partial \Omega,
\end{align*}
\]

where \( \Phi \xi \) denotes the gradient of \( \Phi \), \( \lambda > 0 \) and for \( t > 0 \) function \( b(t) \) is the derivative of an \( 1 \)-dimensional \( N \)-function \( B \) fulfilling some suitable assumptions.

**Definition 3.1** A function \( u \in W^1_0 L_B,\Phi(\Omega) \) is a weak solution of (3.1) if \( b(|u|) \in L_B(\Omega) \), \( \Phi \xi(\nabla u) \in L_{\Phi}(\Omega; \mathbb{R}^n) \) and

\[
\int_{\Omega} \Phi \xi(\nabla u) \cdot \nabla \varphi \, dx = \lambda \int_{\Omega} b(|u|) \text{sign} u \varphi \, dx
\]

for any \( \varphi \in W^1_0 L_B,\Phi(\Omega) \).

The following result holds

**Theorem 3.2** Let \( \Omega \) be an open bounded subset of \( \mathbb{R}^n \) satisfying the segment property. Let \( \Phi \) be a differentiable \( n \)-dimensional \( N \)-function fulfilling (2.31). Assume that \( B \) is a \( 1 \)-dimensional \( N \)-function such that \( B \ll \Phi_n \) if (2.34) holds or \( B \) is any if (2.36) is in force. Then, for any \( r > 0 \) there exist \( \lambda_r > 0 \) and \( u_r \in W^1_0 L_B,\Phi(\Omega) \cap L^\infty(\Omega) \) such that \( \int_{\Omega} B(u_r) \, dx = r \) and \( u_r \) solves problem (3.1) with \( \lambda = \lambda_r \).

Roughly speaking, there exists a pairs \((\lambda_r, u_r)\) which seems to solve an eigenvalue problem. Actually, Theorem 3.2 does not guarantee the existence of a solution to problem (3.1) for a fixed \( \lambda \). Indeed, the classical rescaling method fails due to the lack of homogeneity of the differential operator.

We are able to prove an existence result in \( W^1_0 L_\Phi(\Omega) \).

**Corollary 3.3** Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^n \). Let \( \Phi \) be a differentiable \( n \)-dimensional \( N \)-function fulfilling (2.31). Assume that \( B \) is a \( 1 \)-dimensional \( N \)-function such that \( B \ll \Phi_n \) if (2.34) holds or \( B \) is any if (2.36) is in force. Then, for any \( r > 0 \) there exist \( \lambda_r > 0 \) and \( u_r \in W^1_0 L_\Phi(\Omega) \cap L^\infty(\Omega) \) such that \( \int_{\Omega} B(u_r) \, dx = r \), \( b(|u_r|) \in L_B(\Omega) \), \( \Phi \xi(\nabla u_r) \in L_\Phi(\Omega; \mathbb{R}^n) \) and

\[
\int_{\Omega} \Phi \xi(\nabla u_r) \cdot \nabla \varphi \, dx = \lambda_r \int_{\Omega} b(|u_r|) \text{sign} u_r \varphi \, dx
\]

for any \( \varphi \in W^1_0 L_\Phi(\Omega) \cap L^\infty(\Omega) \).

**Remark 3.4** Note that any bounded Lipschitz domain in \( \mathbb{R}^n \) satisfies the segment property, also.
In order to establish our main result we consider the following constrained minimization problem

\[
(3.4) \quad c_r = \inf \left\{ \int_{\Omega} \Phi(\nabla u) \, dx : u \in W^1_0L_B(\Omega), \int_{\Omega} B(u) \, dx = r \right\}
\]

for any \( r > 0 \), where \( B \) is as above.

As already observed in the Introduction, differentiability of the functionals appearing in (3.4) is not guaranteed. Then we can not apply the standard method of Lagrange multipliers to obtain Theorem 3.2. However, the equation in (3.1) can be still regarded as the Euler-Lagrange equation associated with problem (3.4).

Our next result guarantees the existence of a minimizer of problem (3.4).

**Theorem 3.5** Let \( \Omega \) be an open bounded subset of \( \mathbb{R}^n \) satisfying the segment property. Let \( \Phi \) be an \( n \)-dimensional \( N \)-function fulfilling (2.31). Assume that \( B \) is a 1-dimensional \( N \)-function such that \( B \preccurlyeq \Phi_n \) if (2.34) holds or \( B \) is any if (2.36) is in force. Then, for any \( r > 0 \), minimization problem (3.4) has at least one minimizer \( u_r \in W^1_0L_B(\Phi)(\Omega) \).

We observe that since no \( \Delta_2 \)-condition is required on \( \Phi \), conditions \( \Phi_\xi(\nabla u_r) \in L_{\Phi_*}(\Omega; \mathbb{R}^N) \) and \( b(|u_r|) \in L_{B_*}(\Omega) \) does not necessary occur, then in general (3.2) is not well-defined. Nevertheless, we are able to prove the following result.

**Proposition 3.6** Under the same assumptions as in Theorem 3.5, if \( u_r \in W^1_0L_B(\Phi)(\Omega) \) is a minimizer of problem (3.4), then

(i) \( \Phi_\xi(\nabla u_r) \in L_{\Phi_*}(\Omega; \mathbb{R}^n) \);

(ii) \( b(|u_r|) \in L_{B_*}(\Omega) \).

### 3.1 Examples

In this Subsection, we specialize Theorem 3.2 to some classes of \( N \)-functions \( \Phi \), which govern the differential operator in the equation in (3.1), with a distinctive structure.

If \( \Phi \) is defined as in (2.7), problem (3.1) takes the form

\[
(3.5) \quad \begin{cases}
- \sum_{i=1}^n (A'_i(u_{x_i}))_{x_i} = \lambda b(|u|) \text{sign } u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( A_i \), for \( i = 1, \ldots, n \), are 1-dimensional \( N \)-functions. One has that (see [C1])

\[
(3.6) \quad \Phi_\circ(t) \approx \overline{A}(t) \quad \text{near infinity},
\]

where \( \overline{A} \) is the 1-dimensional \( N \)-function obeying

\[
(3.7) \quad \overline{A}^{-1}(\tau) = \left( \prod_{i=1}^n A_i^{-1}(\tau) \right)^{\frac{1}{n}}.
\]

Thus, our results about existence and boundedness of solutions to problem (3.5) follow from Theorem 3.2 on replacing \( \Phi_\circ \) by \( \overline{A} \) throughout.
Example 1. Let
\[ A_i(t) = \frac{1}{p_i}t^{p_i} \log^{\alpha_i}(c + t) \quad \text{for} \ t > 0, \]
where \( p_i > 1, \alpha_i \in \mathbb{R}, i = 1, \ldots, n, \) and \( c \) positive constant sufficiently large for all functions \( A_i(t) \) be convex. Let \( \overline{p} \) and \( \overline{\alpha} \) be defined as
\[ \frac{1}{\overline{p}} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{p_i} \quad \text{and} \quad \overline{\alpha} = \frac{\overline{p}}{n} \sum_{i=1}^{n} \alpha_i. \]
With this choice of \( A_i \), problem (3.1) agrees with
\[
\begin{aligned}
\sum_{i=1}^{n} \left( |u_{x_i}|^{p_i-2}u_{x_i} \log^{\alpha_i}(c + |\xi_i|) \right)_{x_i} &= \lambda b(|u|) \text{sign} u & \text{in} \ \Omega \\
\sum_{i=1}^{n} u &= 0 & \text{on} \ \partial \Omega.
\end{aligned}
\]
By (3.6) one has that
\[ \Phi_n(x) \approx \begin{cases} t^{\overline{p}} \left( \log(c + t) \right)^{\overline{\alpha}} & \text{if} \ \overline{p} < n \\ e^{-\frac{n}{\overline{p}}} & \text{if} \ \overline{p} = n, \overline{\alpha} < n - 1 \\ e^{\frac{n}{\overline{p}} - 1} & \text{if} \ \overline{p} = n, \overline{\alpha} = n - 1 \end{cases} \]
near infinity. When \( \overline{p} > n, \) or \( \overline{p} = n \) and \( \overline{\alpha} > n - 1, \) condition (2.36) holds.
Assume that
\[ B(t) \ll \begin{cases} t^{\overline{p}} \left( \log(c + t) \right)^{\overline{\alpha}} & \text{if} \ \overline{p} < n \\ e^{-\frac{n}{\overline{p}}} & \text{if} \ \overline{p} = n, \overline{\alpha} < n - 1 \\ e^{\frac{n}{\overline{p}} - 1} & \text{if} \ \overline{p} = n, \overline{\alpha} = n - 1 \end{cases} \]
and \( B(t) \) is any if \( \overline{p} > n \) or \( \overline{p} = n, \overline{\alpha} > n - 1. \)
Hence, thanks to Theorem 3.2, problem (3.8) admits a solution, namely, for any \( r > 0, \) there exist a constant \( \lambda > 0 \) and a solution function \( u \in W^1_0 L^p \log^2 L(\Omega) \) to problem (3.8) satisfying \( \int_{\Omega} B(u) \, dx = r. \)
Here, \( \overline{\alpha} \) stands for the vector \((\alpha_1, \ldots, \alpha_n)\).

Example 2. Let us consider now another particular case of the function (2.7) given by
\[ \Phi(\xi) = \sum_{i=1}^{n-1} \frac{1}{p_i} |\xi_i|^{p_i} + \left( e^{|\xi_n|^{\alpha}} - 1 \right) \quad \text{for} \ \xi \in \mathbb{R}^n, \]
where \( p_i > 1, \) for \( i = 1, \ldots, n \) and \( \alpha > 1. \) Note the \( \Phi \notin \Delta_2. \) Thus, problem (3.8) agrees with
\[
\begin{aligned}
\sum_{i=1}^{n-1} \left( |u_{x_i}|^{p_i-2}u_{x_i} \right)_{x_i} + \left( e^{u_{x_n}^{\alpha}} |u_{x_n}|^{\alpha-2}u_{x_n} \right) & = \lambda b(|u|) \text{sign} u & \text{in} \ \Omega \\
\sum_{i=1}^{n} u &= 0 & \text{on} \ \partial \Omega.
\end{aligned}
\]
One can verify via (3.6) and (3.7) that
\[ \Phi^{-1}_\alpha(s) \approx \left( \prod_{i=1}^{n} A_i^{-1}(s) \right)^{\frac{1}{n}} \approx \left( s^{\sum_{i=1}^{n-1} \frac{1}{p_i}} \left( \log(1 + s) \right)^{\frac{1}{n}} \right)^{\frac{1}{n}} \]
\[ = s^{\frac{1}{n} \sum_{i=1}^{n-1} \frac{1}{p_i}} \left( \log(1 + s) \right)^{\frac{1}{n}} \approx s^{\frac{1}{n}} \left( \log(1 + s) \right)^{\frac{1}{n}} \quad \text{near infinity}, \]

\[ (3.11) \]
\[ (3.12) \]
where \( \frac{1}{\beta} = \frac{1}{n} \sum_{i=1}^{n-1} \frac{1}{p_i} \). Then

(3.13) \[ \Phi_o(t) \approx t^{\beta} (\log(1 + t))^{-\frac{\beta}{n}} \] near infinity.

If \( \Phi_o \) verifies condition (2.34), namely if \( \frac{1}{n-1} \frac{1}{p_i} > -1 \), \( i.e. \) if \( \sum_{i=1}^{n-1} \frac{1}{p_i} > 1 \). Hence,

(3.14) \[ \Phi_n(t) \approx s^{\beta^*} (\log(1 + t))^{-\frac{\beta}{\alpha(n-\beta)}} \] near infinity.

Whereas, if \( \Phi \) fulfills condition (2.36), \( i.e. \) if \( \sum_{i=1}^{n-1} \frac{1}{p_i} \leq 1 \) then \( \Phi_n \) agrees with \( +\infty \) near infinity. By assuming that

(3.15) \[ B(t) \ll s^{\beta^*} (\log(1 + t))^{-\frac{\beta}{\alpha(n-\beta)}} \] \( \text{if} \ \sum_{i=1}^{n-1} \frac{1}{p_i} > 1 \)

and

(3.16) \[ B(t) \] is any if \( \sum_{i=1}^{n-1} \frac{1}{p_i} \leq 1 \).

Theorem 3.2 holds.

**Example 3.** We present now a possible instance of examples which generalize one from \([Tr]\)

provided by \( N \)-functions \( \Phi \) of the form

(3.17) \[ \Phi(\xi) = \sum_{k=1}^{K} A_k \left( \sum_{i=1}^{n} \alpha_i^k \xi_i \right) \] \( \text{for} \ \xi \in \mathbb{R}^n \),

where \( A_k \) are \( N \)-functions of one variable, and \( K \in \mathbb{N} \) and coefficients \( \alpha_i^k \in \mathbb{R} \) are arbitrary.

Thus, we consider, when \( n = 2 \), the \( N \)-function given by (see [ACCZ-G], Example 5)

\[ \Phi(\xi) = |\xi_1 - \xi_2|^p + |\xi_1|^q (\log(c + |\xi_1|))^{\alpha} \] \( \text{for} \ \xi \in \mathbb{R}^2 \),

where \( c \) is a sufficiently large constant for \( \Phi \) to be convex, \( p > 1 \) and either \( q \geq 1 \) and \( \alpha > 0 \), or \( q = 1 \) and \( \alpha > 0 \).

Hence, problem (3.8) becomes

(3.18) \[ \begin{cases} - \left[ (\Phi_{\xi_1}(u_{x_1}, u_{x_2}))_{x_1} + (\Phi_{\xi_2}(u_{x_1}, u_{x_2}))_{x_2} \right] = \lambda b(|u|) \text{sign} u & \text{in} \ \Omega \\ u = 0 & \text{on} \ \partial\Omega, \end{cases} \]

where

\[ \Phi_{\xi_1}(u_{x_1}, u_{x_2}) = p|u_{x_1} - u_{x_2}|^{p-2}(u_{x_1} - u_{x_2}) + q|u_{x_1}|^{q-2}u_{x_1} (\log(c + |u_{x_1}|))^{\alpha} + \alpha |u_{x_1}|^{q-1} \frac{u_{x_1}}{c + |u_{x_1}|} (\log(c + |u_{x_1}|))^{\alpha - 1}, \]

and

\[ \Phi_{\xi_2}(u_{x_1}, u_{x_2}) = -p|u_{x_1} - u_{x_2}|^{p-2}(u_{x_1} - u_{x_2}). \]

The Sobolev conjugate of \( \Phi \) takes the following values

(3.19) \[ \Phi_2(s) \approx \begin{cases} s^{\frac{2pq}{pq + p-q}} \log^{\frac{p-q}{pq}}(t) & \text{if} \ \ pq < p + q \\ \exp \left( t^{\frac{p-q}{pq}} \right) & \text{if} \ \ pq = p + q \ \text{and} \ pq < p + q \\ \exp(\exp(t^2)) & \text{if} \ \ pq = p + q \end{cases} \]
near infinity.
If $pq = p + q$ and $p \alpha > p + q$, or $pq > p + q$, then condition (2.36) holds.
By assumin that

$$B(t) \prec \prec \Phi_2(t),$$

where $\Phi_2$ is as in (3.19) and

$$B(t) \text{ is any if } pq = p + q \text{ and } p \alpha > p + q, \text{ or } pq > p + q,$$

then Theorem 3.2 holds.

**Example 4.** We conclude this subsection by showing a special instance with

$$\Phi(\xi) = \sum_{i=1}^{n} \left( e^{\lambda_1 |\alpha_i \xi_i| \frac{1}{\alpha_i}} - 1 \right) \text{ for } \xi \in \mathbb{R}^n,$$

where $\alpha_i > 1$.

The corresponding problems reads

$$- \sum_{i=1}^{n} \left( e^{2 \lambda u x_i |u_x| \alpha_i} - 2 \lambda u x_i \right) = \lambda b(|u|) \text{ sign } u \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial \Omega.$$

By using again (3.6) and (3.7), we have

$$\Phi^{-1}(s) \approx \left( \prod_{i=1}^{n} (\log(1 + s))^{\frac{1}{\alpha_i}} \right)^{\frac{1}{\sigma}} = (\log(1 + s))^{\frac{1}{\sigma} \sum_{i=1}^{n} \frac{1}{\alpha_i}} = (\log(1 + s))^{\frac{1}{\sigma}} \text{ near infinity},$$

where $\sigma$ is the harmonic average of $\alpha_i$ for $i = 1, \ldots, n$. Then,

$$\Phi(t) \approx e^{\sigma - 1} \quad \text{near infinity},$$

and condition (2.36) is always verified. Thus, Theorem 3.2 holds for any $N$-function $B$.

### 4 Proofs of main results

In this section, we provide the proof of our main results stated in section § 3. In order to prove Theorem 3.2, we first show the proof of the existence of a minimizer of constrained minimization problem (3.4). We focus us only on the case when $B \prec \prec \Phi_n$, since the other case runs easily.

**Proof of Theorem 3.5** Let us introduce the following functionals $F : W^{1}_{0} L_{B, \Phi}(\Omega) \to \mathbb{R}$ and $G : W^{1}_{0} L_{B, \Phi}(\Omega) \to \mathbb{R}$ defined as

$$F(u) = \int_{\Omega} \Phi(\nabla u) \ dx,$$

and

$$G(u) = \int_{\Omega} B(u) \ dx,$$

respectively. We observe that $F$ is a finite-valued functional on $W^{1}_{0} L_{B, \Phi}(\Omega)$ if and only if $\Phi$ fulfils the $\Delta_2$-condition. Whereas, $G(u)$ is always finite for every $u \in W^{1}_{0} L_{B, \Phi}(\Omega)$ for the compact embedding
stated in Proposition 2.7.
In order to prove the existence of a minimizer, we have to show first the continuity of $G$ and lower semicontinuity of $F$ with respect the topology $\sigma(W_0^1L_B,\Phi(\Omega),W^{-1}E_{B*,\Phi}(\Omega))$, where $W_0^1L_B,\Phi(\Omega)$ has to be understood as the dual space of $W^{-1}E_{B*,\Phi}(\Omega)$.

**Step 1.** $G$ is $\sigma(W_0^1L_B,\Phi(\Omega),W^{-1}E_{B*,\Phi}(\Omega))$ continuous.
It is enough to prove that

\[(4.3)\quad u_h \to u \text{ in } \sigma(W_0^1L_B,\Phi(\Omega),W^{-1}E_{B*,\Phi}(\Omega)),\]

then $G(u_h) \to G(u)$.

By \((4.3)\), it follows that $u_n$ is bounded in $W_0^1L_B,\Phi(\Omega)$. By the compact embedding of $W_0^1L_B,\Phi(\Omega)$ in $E_B(\Omega)$ (see Proposition 2.7), we have that $u_n$ converges to $u$ in norm in $E_B(\Omega)$. Since convergence in norm implies the mean convergence, we get $B(2(u_h-u)) \to 0$ in $L^1$. It follows that $u_h \to u$ a.e. in $\Omega$ and there exists (up a subsequence) a function $w \in L^1(\Omega)$ such that

$$B(2|u_h-u|) \leq w(x) \quad \text{a.e. in } \Omega.$$ 

Owing to the strictly monotonicity and convexity of function $B$, we obtain

$$B(u_h) \leq \frac{1}{2}B(u) + \frac{1}{2}w \quad \text{a.e. in } \Omega,$$

and then the statement of Step 1 follows thanks to Lebesgue’s dominate convergence theorem.

**Step 2.** $F$ is $\sigma(W_0^1L_B,\Phi(\Omega),W^{-1}E_{B*,\Phi}(\Omega))$ lower semicontinuous.
It suffices to prove that $F(u) \leq \liminf F(u_h)$ if \((4.3)\) holds.

Let us fix $\varepsilon > 0$. Since $\nabla u_h \in L_\Phi(\Omega;\mathbb{R}^n)$ for all $h$, by Lemma 2.6 there exists a function $W \in E_{\Phi}(\Omega;\mathbb{R}^n)$ such that

$$F(u_h) = \int_\Omega \Phi(\nabla u_h) \, dx \geq \int_\Omega \nabla u_h \cdot W \, dx - \int_\Omega \Phi_*(W) \, dx \quad \forall h \in \mathbb{N} \quad \text{and}$$

$$F(u) = \int_\Omega \Phi(\nabla u) \, dx \leq \int_\Omega \nabla u \cdot W \, dx - \int_\Omega \Phi_*(W) \, dx + \varepsilon,$$

namely

$$F(u_h) - F(u) \geq \int_\Omega \nabla u_h \cdot W \, dx - \int_\Omega \nabla u \cdot W \, dx - \varepsilon \quad \forall h \in \mathbb{N}. \quad (4.4)$$

By \((4.4)\) and \((4.3)\), we get

$$\liminf_{h} F(u_h) \geq F(u) - \varepsilon,$$

and the proof of Step 2 follows by the arbitrariness of $\varepsilon$.

**Step 3.** Existence of a minimizer of \((3.3)\).
Let $\{u_h\}_h \subset W_0^1L_B,\Phi(\Omega)$ be a minimizing sequence of \((3.3)\), i.e.

$$G(u_h) = \int_\Omega B(u_h) \, dx = r \quad \forall h \in \mathbb{N} \quad \text{and}$$

$$F(u_h) = \int_\Omega \Phi(\nabla u_h) \, dx \to c_r \quad \text{as } h \to \infty.$$

This means that $\{\nabla u_h\}_h$ is bounded in mean and then in norm in $L_\Phi(\Omega;\mathbb{R}^n)$. By Banach-Alaoglu’s Theorem, there exists (up a subsequence) $u_r \in W_0^1L_B,\Phi(\Omega)$ such that $u_h \to u_r$ in $\sigma(W_0^1L_B,\Phi(\Omega),W^{-1}E_{B*,\Phi}(\Omega))$. By Step 1 and Step 2, it follows

$$G(u_r) = r \quad \text{and} \quad F(u_r) \leq \liminf F(u_h) = c_r.$$

By definition of $c_r$, we conclude that $F(u_r) = c_r$. \(\square\)
Our next aim is to prove Proposition 3.6. To do this the next auxiliary lemmas will be critical.

Lemma 4.1 Let $U \in \mathcal{L}_\Phi(\Omega; \mathbb{R}^n)$. Then the following statements hold

(a) $\Phi_\xi((1-\varepsilon)U) \in L_{\Phi_*}(\Omega; \mathbb{R}^n)$ for all $\varepsilon \in (0, 1]$;

(b) $\Phi_\xi((1-\varepsilon)U + V) \in L_{\Phi_*}(\Omega; \mathbb{R}^n)$ for all $V \in E_\Phi(\Omega; \mathbb{R}^n)$ and for all $\varepsilon \in (0, 1]$.

Proof. The case $\varepsilon = 1$ is trivial, so let $\varepsilon \in (0, 1)$. Let $U \in \mathcal{L}_\Phi(\Omega; \mathbb{R}^n)$ and $V \in E_\Phi(\Omega; \mathbb{R}^n)$.

(a) By (2.8) and the convexity of $\Phi$, we get

$$\frac{\varepsilon}{1-\varepsilon} \Phi_\xi((1-\varepsilon)U) \leq \varepsilon U \cdot \Phi_\xi((1-\varepsilon)U) \leq \varepsilon U \cdot \Phi((1-\varepsilon)U) + \Phi((1-\varepsilon)U) \leq \Phi(U).$$

Then, since $U \in \mathcal{L}_\Phi(\Omega; \mathbb{R}^n)$, it follows $\Phi_\xi((1-\varepsilon)U) \in L^1(\Omega)$, namely $\Phi_\xi((1-\varepsilon)U) \in L_{\Phi_*}(\Omega; \mathbb{R}^n)$.

(b) Thanks to the convexity of $\Phi$, we have that

$$\Phi\left(\frac{1}{1-\varepsilon/2}((1-\varepsilon)U + V)\right) \leq \frac{1-\varepsilon}{1-\varepsilon/2} \Phi(U) + \left(1 - \frac{1}{1-\varepsilon/2}\right) \Phi\left(\frac{2V}{\varepsilon}\right).$$

Inequality (4.5) gives $\Phi\left(\frac{1}{1-\varepsilon/2}((1-\varepsilon)U + V)\right) \in L^1(\Omega, \mathbb{R}^n)$. Owing to (a), the statement (b) follows.

For convenience of the reader, we state Lemma 4.2 in [MT].

Lemma 4.2 Let $u, v \in E_B(\Omega)$, $u \neq 0$ and $\int_\Omega B(u)v\, dx \neq 0$. Then the condition

$$\int_\Omega B((1-\varepsilon)u + \delta v)\, dx = \int_\Omega B(u)\, dx$$

defines a continuously differentiable function $\delta = \delta(\varepsilon)$ in some interval $(-\varepsilon_0, \varepsilon_0)$ with $\varepsilon_0 > 0$. Moreover, $\delta(0) = 0$ and

$$\delta'(0) = \frac{\int_\Omega b(u)u\, dx}{\int_\Omega b(u)v\, dx},$$

where $b$ is the derivative of $B$.

Proof of Proposition 3.6. Part (i). Let $r > 0$ and $u_r \in W^1_0L_{B, \Phi}(\Omega)$ be a minimizer of (3.4). Suppose, by contradiction, that $\Phi_\xi(\nabla u_r) \notin L_{\Phi_*}(\Omega, \mathbb{R}^N)$. By Proposition 2.1 we get

$$\int_\Omega \Phi_\xi(\nabla u_r) \cdot \nabla u_r\, dx = \int_\Omega \Phi(\nabla u_r)\, dx + \int_\Omega \Phi_*\Phi_\xi(\nabla u_r)\, dx = +\infty.$$

Now we choose $v \in W^1_0E_{B, \Phi}(\Omega)$ such that $\int_\Omega b(u_r)u_r\, dx = \int_\Omega b(u_r)v\, dx$. By Lemma 4.2 and by (3.4), there exist $\varepsilon_0 \in (0, 1)$ and a function $\delta \in C^1(-\varepsilon_0, \varepsilon_0)$ fulfilling

$$\int_\Omega B((1-\varepsilon)u_r + \delta(\varepsilon)v)\, dx = \int_\Omega B(u_r)\, dx = r$$

for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $\delta(0) = 0$ and $\delta'(0) = 1$. Then, there exists $\varepsilon_1 \in (0, \varepsilon_0)$ such that $\delta(\varepsilon) \geq 0$ for all $\varepsilon \in [0, \varepsilon_1)$ and

$$|\delta'(\varepsilon)| \leq \frac{3}{2}$$

for all $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$. 

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By (4.9),

\[
|\delta(\varepsilon)| \leq \frac{3}{2|\varepsilon|} \quad \text{for all } \varepsilon \in [-\varepsilon_1, \varepsilon_1].
\]

Let us define the function \( \Psi : [0, \varepsilon_1] \to \mathbb{R} \) by

\[
\Psi(\varepsilon) = \int_{\Omega} \Phi(W_{\varepsilon}(x)) \, dx,
\]

where

\[
W_{\varepsilon}(x) = (1-\varepsilon) \nabla u_r(x) + \delta(\varepsilon) \nabla v(x) \quad \text{for } x \in \Omega.
\]

Since \( \Phi(\nabla W_{\varepsilon}) \cdot \nabla W_{\varepsilon} \geq 0 \) a.e. in \( \Omega \) and \( \Phi(\nabla W_{\varepsilon}) \cdot \nabla W_{\varepsilon} \to \Phi(\nabla u_r) \cdot \nabla u_r \) a.e. in \( \Omega \), then by Fatou’s Lemma and (4.7) we have

\[
\int_{\Omega} \Phi(\nabla W_{\varepsilon}) \cdot \nabla W_{\varepsilon} \, dx \to +\infty \quad \text{for } \varepsilon \to 0.
\]

Let \( \varepsilon \in (0, 1) \). By the convexity of \( \Phi \), estimate (4.10) and (2.4), it follows that

\[
\Phi(W_{\varepsilon}) \leq (1-\varepsilon) \Phi(\nabla u_r) + \varepsilon \Phi\left(\frac{\delta(\varepsilon)}{\varepsilon} \nabla v\right) \leq \Phi(\nabla u_r) + \frac{2\delta(\varepsilon)}{3} \Phi\left(\frac{3}{2} \nabla v\right)
\]

\[
\leq \Phi(\nabla u_r) + \Phi\left(\frac{3}{2} \nabla v\right) \in L^1(\Omega) \quad \text{for } \varepsilon \in (0, \varepsilon_1).
\]

Thanks to Lebesgue’s dominated convergence theorem, \( \Psi(\varepsilon) \) is continuous in \( (0, \varepsilon_1] \). It is easily to check the continuity in \( \varepsilon = 0 \), as well.

Let \( \varepsilon_2 \in (0, \varepsilon_1) \) be arbitrary and set \( U = \frac{2}{2-\varepsilon} [(1-\varepsilon) \nabla u_r + \delta(\varepsilon) \nabla v] \). First by (2.8) and convexity of \( \Phi \) we get

\[
\Phi(\varepsilon) (\Phi(\nabla u_r + \delta(\varepsilon) \nabla v)) = \Phi(\Phi(\nabla u_r + \delta(\varepsilon) \nabla v)) \leq \Phi\left(\left(1-\frac{\varepsilon}{2}\right) U\right) \cdot \left(1-\frac{\varepsilon}{2}\right) U
\]

\[
\leq \frac{2-\varepsilon}{\varepsilon} \left[ \Phi(\nabla u_r) + \varepsilon \Phi(\nabla v) + \varepsilon \Phi(\nabla u_r) \right]
\]

Young inequality (2.5), (4.14), (2.4), (4.9) and (4.10) yield

\[
\left| \frac{\partial}{\partial \varepsilon} \Phi(W_{\varepsilon}) \right| = \frac{1}{\varepsilon} \left[ \Phi\left(\left(1-\frac{\varepsilon}{2}\right) U\right) \cdot (\varepsilon \delta(\varepsilon) \nabla v - \varepsilon \nabla u_r) \right]
\]

\[
\leq \frac{1}{\varepsilon} \left[ \Phi\left(\Phi(\nabla u_r + \delta(\varepsilon) \nabla v)) \right) + \varepsilon \Phi(\nabla v) + \varepsilon \Phi(\nabla u_r) \right]
\]

\[
\leq \frac{1}{\varepsilon} \left[ \frac{2-\varepsilon}{\varepsilon} \Phi(\nabla u_r) + \frac{\delta(\varepsilon)}{2-\varepsilon} \Phi(\nabla v) + \varepsilon \Phi(\nabla u_r) \right]
\]

\[
\leq \frac{1}{\varepsilon_2} \left[ \frac{2-\varepsilon}{\varepsilon_2} \Phi(\nabla u_r) + \frac{\delta(\varepsilon)}{2\varepsilon} \Phi(\nabla v) + \frac{3}{2} \Phi(\nabla u_r) \right]
\]

\[
\leq \frac{1}{\varepsilon_2} \left[ \frac{2-\varepsilon}{\varepsilon_2} \Phi(\nabla u_r) + \frac{\delta(\varepsilon)}{2\varepsilon} \Phi(\nabla v) + \frac{3}{2} \Phi(\nabla u_r) \right] \in L^1(\Omega)
\]
for all $\varepsilon \in (\varepsilon_2, \varepsilon_1)$, because $v \in W_0^1 E_{B, \Phi}(\Omega)$ and $u_r$ is a minimizer of problem (3.4). Then we conclude that

$$
(4.16) \quad \Psi'(\varepsilon) = \int_{\Omega} \Phi_\xi(W_\varepsilon) \cdot (\delta'(\varepsilon) \nabla v - \nabla u_r) \, dx
$$

for all $\varepsilon \in (\varepsilon_2, \varepsilon_1)$. By the arbitrariness of $\varepsilon_2$, equality (4.16) holds for all $\varepsilon \in (0, \varepsilon_1)$.

We note that

$$
(4.17) \quad \Psi'(\varepsilon) = \int_{\Omega} \Phi_\xi(W_\varepsilon) \left( \delta'(\varepsilon) \nabla v - \frac{W_\varepsilon - \delta(\varepsilon) \nabla v}{1 - \varepsilon} \right) \, dx
$$

$$
= -\frac{1}{1 - \varepsilon} \int_{\Omega} \Phi_\xi(W_\varepsilon) \cdot W_\varepsilon \, dx + \left( \delta'(\varepsilon) + \frac{\delta(\varepsilon)}{1 - \varepsilon} \right) \int_{\Omega} \Phi_\xi(W_\varepsilon) \cdot \nabla v \, dx.
$$

Owing to Young inequality (2.5) and inequality (2.8), we have

$$
(4.18) \quad \Phi_\xi(\xi) \cdot 2\eta \leq \Phi_\ast(\Phi_\xi(\xi)) + \Phi(2\eta) \leq \xi \cdot \Phi_\xi(\xi) + \Phi(2\eta).
$$

Since $v \in W_0^1 E_{B, \Phi}(\Omega)$ by (4.17) and (4.18) we can deduce that

$$
(4.19) \quad \Psi'(\varepsilon) \leq \left( \frac{\delta'(\varepsilon)}{2} + \frac{\delta(\varepsilon)/2 - 1}{1 - \varepsilon} \right) \int_{\Omega} \Phi_\xi(W_\varepsilon) \cdot W_\varepsilon \, dx + \left( \delta'(\varepsilon) + \frac{\delta(\varepsilon)}{1 - \varepsilon} \right) \int_{\Omega} \Phi(2\nabla v) \, dx
$$

$$
\leq \left( \frac{3}{4} + \frac{\delta(\varepsilon)/2 - 1}{1 - \varepsilon} \right) \int_{\Omega} \Phi_\xi(W_\varepsilon) \cdot W_\varepsilon \, dx + C \quad \text{for } \varepsilon \in (0, \varepsilon_1),
$$

where $C$ is a positive constant independent of $\varepsilon$. The last estimate (4.19) and limit (4.12) imply

$$
\lim_{\varepsilon \to 0} \Psi'(\varepsilon) = -\infty.
$$

Then, there exists $\varepsilon_3 > 0$ such that $\Psi(\varepsilon_3) < \Psi(0)$. On setting $\hat{u}_r = (1 - \varepsilon_3) u_r + \delta(\varepsilon_3) v$, we have

$$
\int_{\Omega} \Phi(\nabla u_r) \, dx = \Psi(\varepsilon_3) < \Psi(0) = \int_{\Omega} \Phi(\nabla u_r) \, dx
$$

and

$$
\int_{\Omega} B(\hat{u}_r) \, dx = r,
$$

which is a contradiction. Hence, $\Phi_\xi(\nabla u_r) \in L_\Phi^\ast(\Omega; \mathbb{R}^n)$ and the proof of Part (i) is complete.

Part (ii) The idea of the proof of Part (ii) is similar to that of Part (i). For convenience of the reader, we give all details. Let $r > 0$ and let $u_r \in W_0^1 L_{B, \Phi}(\Omega)$ be a minimizer of problem (3.4). Thanks to embedding (2.41), $u_r \in E_{B}(\Omega)$. Let $v \in E_{B}(\Omega)$ such that $\int_{\Omega} b(u_r)v \, dx = \int_{\Omega} b(u_r)v \, dx$, and Lemma 4.1 guauntarizes that there exist $\varepsilon_0 \in (0, 1)$ and a function $\delta \in C^1((-\varepsilon_0, \varepsilon_0))$ satisfying (4.8). Moreover, $\delta(0) = 0$, $\delta(\varepsilon) \geq 0$ for all $\varepsilon \in [0, 1]$, and (4.10) and (4.11) hold. On setting

$$
(4.20) \quad \Lambda(\varepsilon) = \int_{\Omega} B(\omega_\varepsilon(x)) \, dx \quad \text{with } \varepsilon \in [0, \varepsilon_1],
$$

where $\omega_\varepsilon(x) = (1 - \varepsilon) u_r(x) + \delta(\varepsilon)v(x)$ for $x \in \Omega$, by (4.8), it follows that $\Lambda(\varepsilon) = r$ and then $\Lambda'(\varepsilon) = 0$ for every $\varepsilon \in [0, \varepsilon_1]$. Now we assume by absurdum that $b(u_r) \notin L_{B^\ast}(\Omega)$, i.e.

$$
(4.21) \quad \int_{\Omega} B_\ast(b(u_r)) \, dx = +\infty.
$$

Let $\varepsilon_2 \in (0, \varepsilon_1)$ be arbitrary. The monotonicity of $b$ and Lemma 4.1 for 1–dimensional Young function give

$$
(4.22) \quad \left| \frac{\partial}{\partial \varepsilon} B(\omega_\varepsilon(x)) \right| = \left| b((1 - \varepsilon) u_r(x) + \delta(\varepsilon)v(x)) (\delta'(\varepsilon)v(x) - u_r(x)) \right|
$$

$$
\leq \left| b((1 - \varepsilon) u_r(x) + \delta(\varepsilon)v(x)) \right| \left( \left| \delta'(\varepsilon) \right| |v(x)| + |u_r(x)| \right)
$$

$$
\leq \left| b((1 - \varepsilon) u_r(x) + \frac{3}{2} v(x)) \right| \left( \frac{3}{2} |v(x)| + |u_r(x)| \right) \in L^1(\Omega)
$$

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for any $\varepsilon \in (\varepsilon_2, \varepsilon_1)$. For the arbitrariness of $\varepsilon_2$, it follows

$$
\Lambda'(\varepsilon) = \int_\Omega b(\omega_\varepsilon)(\delta'(\varepsilon)v(x) - u_r(x)) \, dx
$$

for every $\varepsilon \in (0, \varepsilon_1)$. By using estimate $b(s)t \leq \frac{1}{2}b(s)s + b(2t)t$ for all $s, t \in \mathbb{R}$, we have

$$
\Lambda'(\varepsilon) = \int_\Omega b(\omega_\varepsilon)\left(\delta'(\varepsilon)v(x) + \frac{\delta(\varepsilon)v(x) - \omega_\varepsilon}{1 - \varepsilon}\right) \, dx
$$

$$
= -\frac{1}{1 - \varepsilon} \int_\Omega b(\omega_\varepsilon)\omega_\varepsilon \, dx + \int_\Omega \left(\delta'(\varepsilon) + \frac{\delta(\varepsilon)}{1 - \varepsilon}\right)b(\omega_\varepsilon)v(x) \, dx
$$

$$
\leq \left(\frac{\delta'(\varepsilon)}{2} + \frac{1}{1 - \varepsilon}\left(\frac{\delta(\varepsilon)}{2} - 1\right)\right) \int_\Omega b(\omega_\varepsilon)\omega_\varepsilon \, dx + \left(\delta'(\varepsilon) + \frac{\delta(\varepsilon)}{1 - \varepsilon}\right) \int_\Omega b(2v)\, dx
$$

for every $\varepsilon \in (0, \varepsilon_1)$. Young inequality and (2.6) yield

$$
\int_\Omega b(2v)\, dx \leq \int_\Omega B(v)\, dx + \int_\Omega B_*(b(2v))\, dx \leq \int_\Omega B(v)\, dx + \int_\Omega B(4v)\, dx < \infty.
$$

By Proposition 2.1 and (4.21), we get

$$
\int_\Omega b(u_r)u_r \, dx = \int_\Omega B(u_r)\, dx + \int_\Omega B_*(b(u_r))\, dx = +\infty.
$$

By the continuity of $b$, it follows that $b(\omega_\varepsilon)\omega_\varepsilon \rightarrow b(u_r)\omega_r$ a.e. in $\Omega$. Since $b(\omega_\varepsilon)\omega_\varepsilon \geq 0$ a.e. in $\Omega$, Fatou’s Lemma and (4.25) yield

$$
\lim_{\varepsilon \rightarrow 0^+} \int_\Omega b(\omega_\varepsilon)\omega_\varepsilon = \int_\Omega b(u_r)u_r \, dx = +\infty.
$$

Now we pass to the limit in (4.23) and, by (4.24) and (4.26), we have $\lim_{\varepsilon \rightarrow 0^+} \Lambda'(\varepsilon) = -\infty$ that is in contradiction with the fact that $\Lambda'(\varepsilon) = 0$.

This implies that $b(u_r) \in L_{B_*}(\Omega)$ and complete the proof of Part (ii).

\[\square\]

We are now in a position to accomplish the proof of Theorem 3.2. In what follows it is important to underline that $(W^{1}_0 L_{B, \Phi}(\Omega), W^{1}_0 E_{B, \Phi}(\Omega); W^{-1} L_{B_*, \Phi^*}(\Omega), W^{-1} E_{B_*, \Phi^*}(\Omega))$ is a complementary system (see Section 2.4) under our assumption on $\Omega$.

**Proof of Theorem 3.2.**

Let us define the functionals $dF$ ad $dG$ by

$$
\langle dF, v \rangle = \int_\Omega \Phi_\varepsilon(\nabla u_r) \cdot \nabla v \, dx
$$

and

$$
\langle dG, v \rangle = \int_\Omega \frac{b(|u_r|)}{|u_r|} u_r v \, dx
$$

for any $v \in W^{1}_0 E_{B, \Phi}(\Omega)$, where $u_r$ is a minimizer of problem (3.4). By Proposition 3.6, the previous functionals are well-defined. Set

$$
\text{Ker } dF = \{v \in W^{1}_0 E_{B, \Phi}(\Omega) : \langle dF, v \rangle = 0\}$$
On setting \( w \) it follows directly by \([A, \text{Theorem 4.1}]\).

If we prove that
\[
\text{Ker } dG = \{ v \in W^1_0 E_B, \Phi(\Omega) : \langle dG, v \rangle = 0 \}.
\]

Then Proposition 43.1 in \([Z]\) assures the existence of \( \lambda_r \in \mathbb{R} \), associated with the minimizer \( u_r \), such that \([3.2]\), for \( u = u_r \), holds for any test function \( \varphi \in W^1_0 E_B, \Phi(\Omega) \). Finally, the \( \sigma(W^1_0 L_B, \Phi(\Omega), W^{-1} L_B, \Phi(\Omega)) \)-density of \( W^1_0 E_B, \Phi(\Omega) \) in \( W^1_0 L_B, \Phi(\Omega) \) (see Lemma \([2.9]\) above) guarantees that \([4.29]\) it is enough to conclude.

Then our goal is to prove \([4.29]\), which will follow by the inclusion
\[
V_G := \left\{ v \in W^1_0 E_B, \Phi(\Omega) : \int_\Omega \frac{b(|u_r|)}{|u_r|} u_r v \, dx > 0 \right\} \subset \left\{ v \in W^1_0 E_B, \Phi(\Omega) : \int_\Omega \Phi(\nabla u_r) \cdot \nabla v \, dx > 0 \right\} := V_F.
\]

In order to verify the last inclusion, let us consider an arbitrary \( v \in V_G \). By Lemma \([1.2]\) there exist \( \varepsilon_0 \in (0, 1) \) and \( \delta \in C^1(-\varepsilon_0, \varepsilon_0) \) such that
\[
\int_\Omega B((1 - \varepsilon)u_r + \delta(\varepsilon)v) \, dx = \int_\Omega B(u_r) \, dx = r \quad \forall \varepsilon \in (-\varepsilon_0, \varepsilon_0).
\]

On setting \( w_\varepsilon = (1 - \varepsilon)u_r + \delta(\varepsilon)v \), the definition of \( u_r \) and \([4.30]\) assure that
\[
\int_\Omega \frac{\Phi(\nabla w_\varepsilon) - \Phi(\nabla u_r)}{\delta(\varepsilon)} \, dx \geq 0 \quad \forall \varepsilon \in (0, \varepsilon_1).
\]

By \([4.6]\), we have \( \delta'(0) > 0 \) and there exists \( \varepsilon_1 \in (0, \varepsilon_0) \) such that \( \frac{\delta'(0)}{2} < \delta'(\varepsilon) < 2\delta'(0) \) for all \( \varepsilon \in (-\varepsilon_1, \varepsilon_1) \). By integrating with respect to \( \varepsilon \), we obtain
\[
\frac{\delta'(0)}{2} < \frac{\delta(\varepsilon)}{\varepsilon} < 2\delta'(0) \quad \forall \varepsilon \in (0, \varepsilon_1).
\]

Since \( \frac{\delta(\varepsilon)}{\varepsilon} < 1 \), the convexity of \( \Phi \), \([2.4]\) and \([4.32]\) yield
\[
\frac{\Phi(\nabla w_\varepsilon) - \Phi(\nabla u_r)}{\delta(\varepsilon)} \leq \frac{(1 - \varepsilon)\Phi(\nabla u_r) + \varepsilon \Phi(\delta(\varepsilon)\nabla v) - \Phi(\nabla u_r)}{\delta(\varepsilon)}
\]
\[
\leq \frac{\varepsilon}{\delta(\varepsilon)} \Phi(\nabla u_r) + \frac{1}{3} \Phi(3\nabla v) \leq \frac{2}{\delta'(0)} \Phi(\nabla u_r) + \frac{1}{3} \Phi(3\nabla v).
\]

The rightmost side of \([4.33]\) belongs to \( L^1(\Omega) \) because \( v \in W^1_0 E_B, \Phi(\Omega) \) and \( u_r \) is the solution to \([3.1]\).

On recalling that \((H)\) holds and \( \nabla w_\varepsilon \to \nabla u_r \) a.e. in \( \Omega \) for \( \varepsilon \to 0^+ \), easily computation gives
\[
\lim_{\varepsilon \to 0^+} \frac{\Phi(\nabla w_\varepsilon) - \Phi(\nabla u_r)}{\delta(\varepsilon)} = \Phi(\nabla u_r) \cdot \nabla v - \Phi(\nabla u_r) \cdot \frac{\nabla u_r}{\delta'(0)} \quad \text{a.e. in } \Omega.
\]

Then, by Lebesgue’s dominate convergence theorem, it follows
\[
\lim_{\varepsilon \to 0^+} \int_\Omega \frac{\Phi(\nabla w_\varepsilon) - \Phi(\nabla u_r)}{\delta(\varepsilon)} \, dx = \int_\Omega \Phi(\nabla u_r) \cdot \nabla v - \Phi(\nabla u_r) \cdot \frac{\nabla u_r}{\delta'(0)} \, dx.
\]

Combining \([4.31]\) and \([4.34]\), since \( \lambda_r > 0 \) we have
\[
\int_\Omega \Phi(\nabla u_r) \cdot \nabla v \, dx \geq \frac{1}{\delta'(0)} \int_\Omega \Phi(\nabla u_r) \cdot \nabla u_r \, dx > 0,
\]

namely \( v \in V_F \). Then \( V_G \subset V_F \) follows by arbitrariness of \( v \in V_F \). Concerning the boundedness of \( u \), it follows directly by \([A, \text{Theorem 4.1}]\).

\[\square\]
We conclude proving Corollary \ref{cor:3.3}.

**Proof of Corollary \ref{cor:3.3}** \cite{ACCZ-G}. Theorem \ref{thm:3.2} guarantees existence of a function \( u \) in \( W^{1}_0L_{B,\Phi}(\Omega) \) such that \ref{thm:3.1} holds. Our aim is now to prove that this function \( u \) is actually a solution to problem \ref{prob:3.3} as stated in \ref{cor:3.3}. To do this, we first observe that by inclusion \ref{eq:2.40} the solution \( u \in W^{1}_0L_{\Phi}(\Omega) \). Next, by \cite{ACCZ-G}, Proposition 2.4, one has that, given any function \( \varphi \in W^{1}_0L_{\Phi}(\Omega) \cap L^{\infty}(\Omega) \), there exist a constant \( C = C(\Omega) \) and a sequence \( \{ \varphi_h \}_{h} \subset C^{\infty}_0(\Omega) \) such that

\begin{equation}
\varphi_h \to \varphi \quad \text{a.e. in } \Omega,
\end{equation}

\begin{equation}
\| \varphi_h \|_{L^{\infty}(\Omega)} \leq C \| \varphi \|_{L^{\infty}(\Omega)} \quad \text{for every } h \in \mathbb{N},
\end{equation}

\begin{equation}
\nabla \varphi_h \to \nabla \varphi \quad \text{modularly in } L^{\Phi}(\Omega; \mathbb{R}^n).
\end{equation}

Then, we have

\begin{equation}
\int_{\Omega} \Phi_{\xi}(\nabla u) \cdot \nabla \varphi_h \, dx = \lambda \int_{\Omega} b(|u|) \text{sign } u \varphi_h \, dx
\end{equation}

for any \( \varphi_h \in C^{\infty}_0(\Omega) \).

Condition \ref{eq:4.38} means that there exists a constant \( k > 0 \) such that

\[
\lim_{h \to \infty} \int_{\Omega} \Phi \left( \frac{\nabla \varphi_h - \nabla \varphi}{k} \right) \, dx = 0.
\]

Moreover, \cite{ACCZ-G}, Proposition 2.2 yields that, owing to conditions \ref{eq:4.38} and \( \Phi_{\xi}(\nabla u) \in L_{\Phi^*}(\Omega; \mathbb{R}^n) \), there exists a subsequence of \( \{ \nabla \varphi_h \}_{h} \), still indexed by \( h \), such that

\[
\lim_{h \to \infty} \int_{\Omega} \Phi_{\xi}(\nabla u) \cdot \nabla \varphi_h \, dx = \int_{\Omega} \Phi_{\xi}(\nabla u) \cdot \nabla \varphi \, dx.
\]

Therefore,

\[
\lim_{h \to \infty} \int_{\Omega} b(|u|) \text{sign } u \varphi_h \, dx = \int_{\Omega} b(|u|) \text{sign } u \varphi \, dx
\]

follows by the dominated convergence theorem coupling with \ref{eq:4.36}, \ref{eq:4.37} and the fact that \( b(|u|) \in L_{B^*}(\Omega) \), and hence in \( L^1(\Omega) \). Then, \( u \) is a solution in \( W^{1}_0L_{\Phi}(\Omega) \) in the sense of \ref{cor:3.3}.

\[\square\]

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