Abstract

The fractional Brownian motion can be considered as a Gaussian field indexed by $(t, H) \in \mathbb{R}_+ \times (0, 1)$. On compact time intervals, it is known to be almost surely jointly Hölder continuous in time and Lipschitz continuous in the Hurst parameter $H$. First, we extend this result to the whole time interval $\mathbb{R}_+$ and consider both simple and rectangular increments. Then we consider SDEs driven by fractional Brownian motion with contractive drift. We obtain that the solutions and their ergodic means are almost surely Hölder continuous in $H$, uniformly in time. The proofs are based on variance estimates of the increments of the fractional Brownian motion and fractional Ornstein-Uhlenbeck processes, multiparameter versions of the Garsia-Rodemich-Rumsey lemma and a combinatorial argument to estimate the expectation of a product of Gaussian variables.

Keywords and phrases: Fractional Brownian motion, Hurst sensitivity, Ergodic SDEs.

MSC2020 subject classification: 60G22, 60H10, 37H10.

1 Introduction

We consider the fractional Brownian motion (fBm) $B$ as a stochastic process of two parameters given by its Mandelbrot-Van Ness representation: $\forall t \in \mathbb{R}, \forall H \in (0, 1)$,

$$B^H_t = \frac{1}{\Gamma(H + \frac{1}{2})} \int_{\mathbb{R}} \left( (t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}} \right) dW_s,$$

where the integral is in the Wiener sense. For fixed $H$, this process appears in models of physics [6], biology [22], finance [9] to cite but a very few. Except when $H = \frac{1}{2}$ (Brownian case), $H$ usually encodes the long-range memory of the model or its roughness. It is therefore important to determine its value precisely and to evaluate the sensitivity of functionals of the model at hand, with respect to $H$. Concerning the determination of $H$, we refer for instance to [1, 7] and references therein about parametric estimation of $H$ in the setting of fractional stochastic differential equations (SDE). As for the sensitivity in the Hurst parameter, it has been studied in various situations and is an important topic in modeling: in [15, 16], the law of (multiple) integrals with respect to the fBm are proven to be continuous in $H$; in [21], the Hölder continuity in $H$ is studied for generalised fractional Brownian fields (over compact index sets); in [4], the solution to random differential equations driven by a 1D fBm is proven to be continuous with respect to $H$ when $H \in (\frac{1}{3}, \frac{1}{2})$; and in [10], the laws of quasilinear stochastic wave and heat equations with additive fractional noise are proven to be continuous in $H$. Finally in [23], the difference between functionals of a fractional SDE and its Markovian counterpart ($H = \frac{1}{2}$) are proven to be of order $|H - \frac{1}{2}|$, both for the law of the solution on a compact time interval, and for the law of a singular functional (namely the first hitting time).

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E.H. acknowledges the support of the Labex de Mathématique Hadamard.
In this paper, the main results concern the Hurst regularity of ergodic fractional SDEs. These results have direct applications to determine statistically the Hurst parameter from discrete observations, see the last paragraph of this introduction and our companion paper [12]. In particular, the Hölder continuity in $H$ of the ergodic mean of a fractional SDE (Theorem 4.5) is crucial in this statistical perspective.

To introduce our methodology, it is natural and interesting to start by discussing the joint time and Hurst regularity of the fBm on a non-compact time interval. In the case of ergodic SDEs, we will then elaborate on the arguments of the fBm case.

Almost sure regularity estimates in $t$ and $H$ can be obtained classically on compact intervals: considering the moments of the increments of $B$ in time, and applying Kolmogorov’s continuity theorem, one gets that $B$ admits a modification (that we still denote by $B$) which almost surely has a finite $(H - \varepsilon)$-Hölder norm on $[0, T]$. However this norm depends on $T$. One can also derive from (1.1) an upper bound on the moments of $B^H_t - B^H_t'$ (see [5, 21]) and deduce that $B$ is almost surely $(1 - \varepsilon)$-Hölder continuous in the Hurst parameter, uniformly in $t \in [0, T]$. Altogether we know that for $\varepsilon \in (0, 1)$ and a compact interval $\mathcal{H} \subset (0, 1)$, there exist a continuous modification of $B$ (still denoted by $B$ thereafter) and a random variable $C_T$ with finite moments such that almost surely
\[ \forall H, H' \in \mathcal{H}, \forall t, t' \in [0, T], \ |B^H_t - B^H_{t'}| \leq C_T \left( |t - t'|^{H \wedge H' - \varepsilon} + |H - H'|^{1 - \varepsilon} \right). \] (1.2)

We observe that the law of the iterated logarithm for fractional Brownian motion [19] shows that for fixed $H$, $B^H_t$ cannot grow much faster than $t^H$ at infinity. Hence we expect $C_T$ to grow slightly faster than $T^H$. For multiparameter processes, it is also natural and sometimes more useful to consider the rectangular increments (see e.g. [13, 17], as well as [25] for considerations on the joint time and space regularity of local times of Gaussian processes). For $B$, the rectangular increment between $(t, H)$ and $(t', H')$ reads
\[ \Box((t, H); (t', H')) B = B^H_t - B^H_{t'} + B^H_{t'} - B^H_t. \] (1.3)

Compared to (1.2), the upper bound on $|\Box((t, H); (t', H')) B|$ is now the product $C_T |t - t'|^{H \wedge H' - \varepsilon} |H - H'|^{1 - \varepsilon}$.

The results for these two types of increments (simple and rectangular) are presented respectively in Theorem 3.1 and Theorem 3.4 for parameters $H$ in a compact subset of $(0, 1)$ and $t$ on the whole half-line, hence making explicit the dependence of $C_T$ with respect to $T$ and providing sharp Hölder exponents. For the simple increments, this reads
\[ \forall t' \geq t \geq 0, \forall H, H' \in \mathcal{H}, \ |B^H_t - B^H_{t'}| \leq C (1 + t)^{\max(H) + \varepsilon} (1 \wedge |t - t'|^{\min(H)} + |H - H'|^{1 - \varepsilon}) \] (1.4)
almost surely, for some integrable random variable $C$. The proofs of these theorems rely on the introduction of an auxiliary process $X$ defined as
\[ X^H_t = (1 + t)^{-\max(H + \varepsilon)} B^H_1, \ t \in \mathbb{R}_+, \ H \in \mathcal{H}, \]
whose variance is bounded uniformly in $H$. Using multiparameter versions of the Garsia-Rodemich-Rumsey (GRR) lemma, the supremum of the increments of $X$ can be controlled on compacts by a quantity independent of time, which leads to the proof of Theorems 3.1 and 3.4.

In the second part of the paper, we study the fractional SDE $dY^H_t = b(Y^H_t) \, dt + dB^H_t$ and its ergodic mean. If $b$ is Lipschitz and bounded, the Hurst regularity of the solution behaves as in (1.4), but with an extra exponential factor in time (see Remark 4.1). Since we are interested in the long-time behaviour of solutions, it is natural to try to get rid of this exponential dependence in time. We first study this problem for the fractional Ornstein-Uhlenbeck (OU) process, see Proposition 4.2. Then assuming that the drift $b$ is contractive, we can control the trajectories of $Y^H$ by those of the OU process with the same parameter $H$. Then we obtain in Theorem 4.3 that almost surely, for all $t \geq 0$ and $H, H' \in \mathcal{H}$,
\[ |Y^H_t - Y^H_{t'}| \leq C (1 + t)^\varepsilon |H - H'|^{1 - \varepsilon}. \]
Finally we focus on the regularity in the Hurst parameter of the ergodic means of $Y$. In Theorem 4.5, we show that for $H$ a compact subset of $(0, \frac{1}{2})$ and $\beta = \frac{7}{8(1-\max(H))} - \frac{1}{2}$, the following holds almost surely: for all $t \geq 0$ and $H, H' \in H$,

$$\frac{1}{t+1} \int_0^{t+1} (Y^H_s - Y^{H'}_s)^2 \, ds \leq C |H - H'|^{\beta(1-c)}. \quad (1.5)$$

The main technical tool to reach this result is an upper bound on the increments in $t, H$ and $H'$ of the ergodic mean of the fractional OU process (Proposition 4.4). As an intermediate step, we relied on a Gaussian equality (Lemma A.6): if $U = (U_1, \ldots, U_n)$ is a centered Gaussian vector with $\mathbb{E}U_1^2 = \mathbb{E}U_2^2$, then $\mathbb{E}\prod_{i=1}^n (U_i^2 - \mathbb{E}U_i^2)$ can be expressed as a sum of products of the covariances $\mathbb{E}[U_i U_j]$ for $i \neq j$.

By similar computations, we derive in Theorem 5.2 a discrete analog of (1.5). Namely if $M^H$ denotes the solution of the Euler scheme associated to the SDE satisfied by $Y^H$ with time-step $\gamma > 0$, then almost surely, for all $N \in \mathbb{N}^*$ and $H, H' \in H$,

$$\frac{1}{N} \sum_{k=1}^N (M^H_{kh} - M^{H'}_{kh})^2 \leq C |H - H'|^{\alpha(1-c)}. $$

**Statistical applications.** In [12], these results are used to build an ergodic statistical estimator of $H$. In fact, assuming that the drift $b$ is Lipschitz and coercive, we know that the SDE has a unique invariant measure $\mu_H$ (see e.g. [3, Lemma 3(ii)]). Suppose that the solution $Y^H$ is observed at discrete times $\{kh, k \in \mathbb{N}\}$ for some time-step $h > 0$, then by [20, Proposition 3.3],

$$\lim_{n \to \infty} d\left(\frac{1}{n} \sum_{k=0}^{n-1} \delta_{Y^H_{kh}}, \mu_H\right) = 0,$$

where $d$ is any distance bounded by the Wasserstein distance. Assuming that the invariant measure identifies the Hurst parameter, it is then natural to consider the estimator

$$\hat{H}^n := \arg\min_{K \in (0,1)} d\left(\frac{1}{n} \sum_{k=0}^{n-1} \delta_{Y^H_{kh}}, \mu_K\right).$$

Studying the convergence of $\hat{H}^n$ therefore boils down to studying the argmin of the function $K \mapsto d\left(\frac{1}{n} \sum_{k=0}^{n-1} \delta_{Y^H_{kh}}, \mu_K\right)$. This requires in particular to prove some regularity properties on $K \mapsto \mu_K$.

In practice, $\mu_K$ is usually not known and can for instance be approximated by $\frac{1}{n} \sum_{k=0}^{n-1} \delta_{Y_{kh}}$. So it becomes necessary to know the regularity of ergodic means in the Hurst parameter. This is the main focus of Theorem 4.5 and Theorem 5.2. It is also worth mentioning that Proposition 3.5 is crucial in the analysis of $\hat{H}^n$. Ergodic estimators of $H$ offer a practical way of estimating the Hurst parameter, but we also refer to [7, 18] for alternative methods with vanishing time step (without ergodicity assumptions).

**Organisation of the paper.** In Section 2, we obtain sharp upper bounds on the moments of the simple and rectangular increments of both $B$ and $X$. Theorems 3.1 and 3.4 are stated and proven in Section 3. The regularity for solutions of fractional additive SDEs, Theorem 4.3, is presented in Section 4.1 and the regularity for ergodic means, Theorem 4.5, appears in Section 4.2. In Section 5, we consider discrete SDEs. The analysis in both Section 4 and 5 is based on a comparison with fractional OU processes, which are studied in Appendix A. We prove upper bounds on their moments in Lemma A.1, and on moments of ergodic means of both OU and discrete OU processes in Lemma A.10 and Lemma A.12.

**Notations.** We denote by $C$ a constant independent of time and the Hurst parameter. It may change from line to line.
2 Bounds on the variance of the increments of $B$ and $X$

We first study the regularity in $t$ and $H$ of the moments of the increments of $B$ and $X$. As mentioned in the introduction, this permits to introduce smoothly the methodology that is based on moment estimates and variations of the GRR lemma, and that will also be used and developed with further arguments in Sections 4 and 5.

2.1 Simple increments

**Proposition 2.1.** Let $H$ be a compact subset of $(0,1)$. There exists a constant $C$ such that for all $t' \geq t \geq 0$ and $H, H' \in H$:

$$\mathbb{E} \left( B_t^H - B_t^{H'} \right)^2 \leq C |t - t'|^{2H'} + C (t^{2H} \vee t^{2H'}) (\log^2(t) + 1) |H - H'|^2.$$  \hspace{1cm} (2.1)

*Proof.* Using $B_t^{H'}$ as a pivot term, we have

$$\mathbb{E} \left( B_t^H - B_t^{H'} \right)^2 \leq 2\mathbb{E} \left( B_t^H - B_t^{H'} \right)^2 + 2\mathbb{E} \left( B_t^{H'} - B_t^H \right)^2$$

$$= 2\mathbb{E} \left( B_t^H - B_t^{H'} \right)^2 + 2(t' - t)^{2H'}. \hspace{1cm} (2.2)$$

The estimation of $\mathbb{E}(B_t^H - B_t^{H'})^2$ has been done in a compact in [5] (see Lemma 3.2) and in a more abstract framework in [21] (see Corollary 3.4 for $t \in [0,1]$). For the sake of completeness, we provide here a different proof for $t \in \mathbb{R}_+$.

Using the Mandelbrot-Van Ness representation of the fBm $B$, we get

$$\mathbb{E} \left( B_t^H - B_t^{H'} \right)^2 = \mathbb{E} \left[ \left( \int_{-\infty}^0 (K_1(H, t, -s) - K_1(H', t, -s)) dW_s + \int_0^t (K_2(H, t, s) - K_2(H', t, s)) dW_s \right)^2 \right],$$

where the kernels $K_1$ and $K_2$ are given by

$$K_1(H, t, s) := \frac{(t - s)^{H-1/2} - (-s)^{H-1/2}}{\Gamma(H + 1/2)},$$

$$K_2(H, t, s) = \frac{(t - s)^{H-1/2}}{\Gamma(H + 1/2)}. \hspace{1cm} (2.3)$$

By the isometry property of the Wiener integral, we get

$$\mathbb{E} \left( B_t^H - B_t^{H'} \right)^2 = \int_0^\infty (K_1(H, t, s) - K_1(H', t, s))^2 ds + \int_0^t (K_2(H, t, s) - K_2(H', t, s))^2 ds.$$

Then with the notations $x_h := (H - H')h + H'$ and $\partial_H K_1$ (resp. $\partial_H K_2$) for the derivative of $K_1$ (resp. $K_2$) with respect to its first variable, we obtain

$$\mathbb{E} \left( B_t^H - B_t^{H'} \right)^2 \leq 2|H - H'|^2 \int_0^\infty \int_0^1 (\partial_H K_1(x_h, t, s))^2 ds dh \hspace{1cm} (2.4)$$

**Step 1: Uniform upper bound** on $\int_0^\infty (\partial_H K_1(x_h, t, s))^2 ds$.

First, the Gamma function is positive and bounded away from 0 on $[\frac{1}{2}, \frac{3}{2}]$. It follows that the derivative of $x \mapsto \frac{1}{\Gamma(x)}$ is also bounded on $[\frac{1}{2}, \frac{3}{2}]$.

Hence

$$\partial_H K_1(x, t, s) = \frac{(t + s)^{x-1/2} \log(t + s) - s^{x-1/2} \log(s)}{\Gamma(x + 1/2)} + ((t + s)^{x-1/2} - s^{x-1/2}) \left( \frac{1}{\Gamma} \right)' (x + 1/2),$$
so that

\[(\partial_t K_1(x,t,s))^2 \leq C \left( (t + s)^{x-1/2} \log(t + s) - s^{x-1/2} \log(s) \right)^2 + C \left( (t + s)^{x-1/2} - s^{x-1/2} \right)^2.\]

After integrating over \( s \) and applying the change of variables \( u = s/t \), we get

\[
\int_0^\infty (\partial_t K_1(x,t,s))^2 \, ds \leq C t^{2x} \int_0^\infty \left( (1 + u)^{x_h-1/2} - u^{x_h-1/2} \right)^2 \, du \\
+ C t^{2x} \int_0^\infty \left( (1 + u)^{x_h-1/2} - u^{x_h-1/2} \right) \log t + (1 + u)^{x_h-1/2} \log(1 + u) - u^{x_h-1/2} \log(u) \right)^2 \, du.
\]

It follows that

\[
\int_0^\infty (\partial_t K_1(x,t,s))^2 \, ds \leq C t^{2x}[\log^2(t) + 1] \\
\times \sup_{h \in \mathcal{H}} \left( \int_0^\infty \left( (1 + s)^{h-1/2} - s^{h-1/2} \right)^2 \, ds + \int_0^\infty \left( \log(1 + s)(1 + s)^{h-1/2} - \log(s)s^{h-1/2} \right)^2 \, ds \right) \\
\leq C t^{2x}[\log^2(t) + 1] \sup_{h \in \mathcal{H}} \left( \int_0^\infty \left( (1 + s)^{h-1/2} - s^{h-1/2} \right)^2 \, ds \\
+ \int_0^\infty \left( \log(1 + s) - \log(s) \right) (1 + s)^{2h-1} \, ds + \int_0^\infty \log(s)^2 \left( (1 + s)^{h-1/2} - s^{h-1/2} \right)^2 \, ds \right) \\
=: C t^{2x}[\log^2(t) + 1] \sup_{h \in \mathcal{H}} (I_1(h) + I_2(h) + I_3(h)).
\]

One can check that for any \( h \in (0,1) \), the integrals \( I_i(h), i=1,2,3 \) are finite. Besides, the mappings \( h \mapsto I_i(h), i=1,2,3 \) are continuous on the compact set \( \mathcal{H} \). Since \( x_1 \leq \max(H,H') \), we thus get

\[
\int_0^\infty (\partial_t K_1(x,t,s))^2 \, ds \leq C \left( t^{2H} \lor t^{2H'} \right) [\log^2(t) + 1]. \tag{2.5}
\]

**Step 2: Uniform upper bound on** \( \int_0^t (\partial_t K_2(x,t,s))^2 \, ds \). 

Recall the expression of \( K_2 \) in (2.3). Then we get

\[
\partial_t K_2(x,t,s) = \frac{(t-s)^{x-1/2} \log(t-s) + (t-s)^{x-1/2} \left( \frac{1}{\Gamma} \right)'(x+1/2)}{\Gamma(x+1/2)}
\]

and using the boundedness of \( \Gamma \) and \( \left( \frac{1}{\Gamma} \right)' \) on \([\frac{1}{2}, 1]\),

\[
\int_0^t (\partial_t K_2(x,t,s))^2 \, ds \leq C \int_0^t (t-s)^{x-1} \left( \log^2(t-s) + 1 \right) \, ds.
\]

We now make the change of variables \( u = s/t \) to get that

\[
\int_0^t (\partial_t K_2(x,t,s))^2 \, ds \leq C t^{2x} \int_0^1 (1 - u)^{2x-1} \left( \log^2(1 - u) + 1 \right) \, du.
\]

It follows that

\[
\int_0^t (\partial_t K_2(x,t,s))^2 \, ds \leq C \left( t^{2H} \lor t^{2H'} \right) [\log^2(t) + 1]. \tag{2.6}
\]
For any $t \in \mathbb{R}_+$, the bound (2.5) and (2.6) into (2.4) we get
\[
\mathbb{E} \left( B^H_t - B^H_t' \right)^2 \leq C \left( \left( t^2 H \lor t^2 H' \right) \left( \log^2(t) + 1 \right) |H - H'|^2 \right).
\]
Using the above bound in (2.2) gives the desired result.

As explained in the introduction, the bound from Proposition 2.1 is not sufficient to deduce regularity estimates of the fBm which are uniform in time (i.e. for all $t \in \mathbb{R}_+$). That is why we consider the process
\[
X^H_t = (1 + t)^{-\alpha} B^H_t, \quad t \in \mathbb{R}_+, \quad H \in (0, 1),
\]
for some $\alpha \geq 0$. The next proposition gives a bound similar to (2.1) for $X$, which will be useful to obtain almost sure regularity estimates of the fBm via the GRR Lemma (Theorem 3.1).

**Proposition 2.2.** Let $\mathcal{H}$ be a compact subset of $(0, 1)$. For $\alpha \geq 0$, consider the multiparameter process $X$ defined in (2.7). There exists a constant $C$ such that for all $t' \geq t \geq 0$ and $H, H' \in \mathcal{H}$,
\[
\mathbb{E} \left( X^H_t - X^{H'}_{t'} \right)^2 \leq C (1 + t)^{-2\alpha} \left( |t' - t|^{2H} + (t^{2H} \lor t^{2H'}) \left( \log^2(t) + 1 \right) |H - H'|^2 \right).
\]

**Proof.** First we use the pivot term $X^H_t$ to get
\[
\mathbb{E} \left( X^H_t - X^{H'}_{t'} \right)^2 \leq 2\mathbb{E} \left( X^H_t - X^H_t \right)^2 + 2\mathbb{E} \left( X^{H'}_{t} - X^{H'}_{t'} \right)^2
\]
\[
= 2(1 + t)^{-2\alpha} \mathbb{E} \left( B^H_t - B^H_t' \right)^2 + 2\mathbb{E} \left( X^{H'}_{t} - X^{H'}_{t'} \right)^2. \tag{2.8}
\]
For the first term in the right-hand side of the previous inequality, use Proposition 2.1. For the second term, introduce the pivot term $(1 + t')^{-\alpha} B^H_t$ to obtain
\[
\mathbb{E} \left( X^{H'}_{t} - X^{H'}_{t'} \right)^2 \leq 2 \left( (1 + t)^{-\alpha} - (1 + t')^{-\alpha} \right)^2 \mathbb{E} (B^H_t)^2 + 2(1 + t)^{-2\alpha} \mathbb{E} \left( B^H_t - B^H_t' \right)^2
\]
\[
= 2 \left( (1 + t)^{-\alpha} - (1 + t')^{-\alpha} \right)^2 t^{2H} + 2(1 + t)^{-2\alpha} |t - t'|^{2H'}.
\]
If $0 \leq t' - t < 1 + t$, we use the inequality $(1 + t)^{-\alpha} - (1 + t')^{-\alpha} \leq \alpha \frac{t' - t}{(1 + t)^{\alpha+1}}$ to get that for any $h \in [0, 1]$,
\[
(1 + t)^{-\alpha} - (1 + t')^{-\alpha} \leq \alpha^2 (t' - t)^{2h} \frac{(t' - t)^{2h}}{(1 + t)^{2\alpha+2}}
\]
\[
\leq \alpha^2 \frac{(t' - t)^{2h}}{(1 + t)^{2\alpha}}.
\]
Now if $t' - t \geq 1 + t$,
\[
(1 + t)^{-\alpha} - (1 + t')^{-\alpha} \leq (1 + t)^{-2\alpha} (1 + t)^{2h}
\]
\[
\leq (1 + t)^{-2\alpha} (t' - t)^{2h}.
\]
Hence for any $t' \geq t \geq 0$,
\[
(1 + t)^{-\alpha} - (1 + t')^{-\alpha} \leq (1 + t)^{2h} \leq (1 + t)^{-2\alpha} (1 + t)^{2h}
\]
\[
\leq (1 + t)^{-2\alpha} (t' - t)^{2h}. \tag{2.9}
\]
Thus applying (2.9) to $h = H'$, we get $\mathbb{E} \left( X^{H'}_{t} - X^{H'}_{t'} \right)^2 \leq C(t' - t)^{2H'} (1 + t)^{-2\alpha}$, and plugging this inequality in (2.8) gives the result.
2.2 Rectangular increments

For rectangular increments, we obtain results that are similar to Propositions 2.1 and 2.2. Recall that these increments are defined in (1.3).

**Proposition 2.3.** Let $\mathcal{H}$ be a compact subset of $(0,1)$. There exists a constant $C$ such that for all $t' \geq t \geq 0$ and $H, H' \in \mathcal{H}$,

$$
\mathbb{E} \left( \left\| (t', H') B - (t, H) B \right\|^2 \right) \leq C \left( |t'-t|^2 H^\vee |t'-t|^2 H' \right) \left( \log^2(|t'-t|) + 1 \right) |H - H'|^2.
$$

**Proof.** We will prove that for any $s \in \mathbb{R}$ and any $H, H' \in \mathcal{H}$, the laws of the following two-dimensional processes coincide:

$$
\left( B^H_t - B^H_s, B^{H'}_{t+s} - B^{H'}_s \right) \overset{(d)}{=} \left( B^H_t, B^{H'}_{t} \right)_{t \in \mathbb{R}}.
$$

(2.10)

Provided this equality holds, we get $\mathbb{E}(\nabla(t', H') B) = \mathbb{E}(B^{H'}_{t'-t} - B^H_{t'-t})^2$, so Proposition 2.1 then gives the desired result.

Hence it remains to prove (2.10). The processes on both side of (2.10) are centred Gaussian. Thus it suffices to prove that they have the same covariance matrix. The equality of the diagonal entries corresponds to a well-known property of the fBm. Hence we focus on the extra-diagonal entries. In view of (1.1), we get

$$
\mathbb{E} \left[ \left( B^H_{t+s} - B^H_s \right) (B^{H'}_{t+s} - B^{H'}_s) \right] = \frac{1}{\Gamma(H + 1/2) \Gamma(H' + 1/2)} \int_{\mathbb{R}} \left( (t + s - u)^{H'-\frac{1}{2}} - (s - u)^{H'-\frac{1}{2}} \right)
\times \left( (t' + s - u)^{H'-\frac{1}{2}} - (s - u)^{H'-\frac{1}{2}} \right) du.
$$

Apply the change of variables $v = u - s$ to get

$$
\mathbb{E} \left[ \left( B^H_{t+s} - B^H_s \right) (B^{H'}_{t+s} - B^{H'}_s) \right] = \frac{1}{\Gamma(H + 1/2) \Gamma(H' + 1/2)} \int_{\mathbb{R}} \left( (t - v)^{H'-\frac{1}{2}} - (-v)^{H'-\frac{1}{2}} \right)
\times \left( (t' - v)^{H'-\frac{1}{2}} - (-v)^{H'-\frac{1}{2}} \right) dv
= \mathbb{E} \left( B^H_t B^{H'}_{t'} \right),
$$

which proves that the covariances of the Gaussian processes involved in (2.10) are equal. \(\Box\)

Similarly to Proposition 2.2, we now give an upper bound on the variance of the rectangular increments of the process $X$.

**Proposition 2.4.** Let $\mathcal{H}$ be a compact subset of $(0,1)$. For $\alpha \geq 0$, consider the multiparameter process $X$ defined in (2.7). There exists a constant $C$ such that for all $t' \geq t \geq 0$ and $H, H' \in \mathcal{H}$,

$$
\mathbb{E} \left( \nabla(t', H') X \right)^2 \leq C(1 + |t'|)^{-2\alpha} |H - H'|^2 \left( |t'-t|^{2H} \vee |t'-t|^{2H'} \right) \left( 1 + \log^2(|t'-t|) + \log^2 t \right).
$$

**Proof.** First, notice that $\nabla(t', H') X = (1 + t')^{-\alpha} \nabla(t', H') B + ((1 + t)^{-\alpha} - (1 + t')^{-\alpha}) \left( B^H_t - B^{H'}_{t'} \right)$. Thus in view of Propositions 2.1 and 2.3, one gets

$$
\mathbb{E} \left( \nabla(t', H') X \right)^2 \leq C(1 + t')^{-2\alpha} \left( |t'-t|^{2H} \vee |t'-t|^{2H'} \right) \left( \log^2(|t'-t|) + 1 \right) |H - H'|^2
+ C \left( (1 + t)^{-\alpha} - (1 + t')^{-\alpha} \right)^2 \left( t^{2H} \vee t'^{2H'} \right) \left( \log^2(t) + 1 \right) |H - H'|^2.
$$

Using (2.9) with $h = H$ and $h = H'$, we obtain the desired result. \(\Box\)
3 Almost sure results on the whole half-line for the fBm

3.1 Statement of the results

Based on the results of the previous section and multiparameter versions of Garsia-Rodemich-Rumsey’s Lemma (Section 3.2), we obtain joint regularity estimates in $t \in \mathbb{R}_+$ and $H \in (0, 1)$ for the usual increments in Theorem 3.1, the rectangular increments in Theorem 3.4 and the supremum in $H$ in Proposition 3.5. The proofs are given respectively in Sections 3.3, 3.4 and 3.5.

**Theorem 3.1.** Let $\mathcal{H}$ be a compact subset of $(0, 1)$. Then for any $\varepsilon \in (0, 1)$ and any $p \geq 1$, there exists a positive random variable $C$ with a finite moment of order $p$ such that almost surely, for all $t' \geq t \geq 0$ and all $H, H' \in \mathcal{H}$,

$$|B_t^H - B_{t'}^{H'}| \leq C (1 + |t'|^{2p \min(H) + \max(H)}) \left(1 \wedge |t - t'|^{\min(H)} + |H - H'| \right)^{1-\varepsilon}.$$

**Remark 3.2.** As suggested in the introduction, this result permits to bound the random variable $C_T$ introduced in (1.2) by $C T^{\max(H) + 2\varepsilon}$.

**Remark 3.3.** Theorem 3.1 still holds in $\mathbb{R}_+$: under the assumptions of Theorem 3.1, we have for all $t' \leq t \leq 0$ and $H, H' \in \mathcal{H}$ that

$$|B_t^H - B_{t'}^{H'}| \leq C (1 + |t'|^{2p \min(H) + \max(H)}) \left(1 \wedge |t - t'|^{\min(H)} + |H - H'| \right)^{1-\varepsilon}.$$

This will be used in Lemma A.1.

**Theorem 3.4.** Let $\mathcal{H}$ be a compact subset of $(0, 1)$. Then for any $\varepsilon \in (0, 1)$ and any $p \geq 1$, there exists a positive random variable $C$ with a finite moment of order $p$ such that almost surely, for all $t' \geq t \geq 0$ and all $H, H' \in \mathcal{H}$,

$$|\square_{(t, H_i)} B| \leq C (1 + t')^{2p \min(H) + \max(H)} \left(1 \wedge |t - t'|^{\min(H)} + |H - H'| \right)^{1-\varepsilon}.$$

Finally, we state a result which slightly differs from the previous theorems, for it holds for all $t$, almost surely. However it is of the same nature and as Theorem 3.4, it is another consequence of Proposition 2.3. It is useful for example in the companion paper [12].

**Proposition 3.5.** Let $\mathcal{H}$ be a compact subset of $(0, 1)$ and let $q > 0$. There exists a constant $C$ such that for all $t', t \geq 0$ and all $H \in \mathcal{H}$,

$$\mathbb{E} \left( \sup_{H \in \mathcal{H}} |B_t^H - B_{t'}^{H}|^q \right) \leq C \left( |t - t'|^{q \min(H)} \vee |t - t'|^{q \max(H)} \right) [\log^q(\min(t', 1)) + 1].$$

3.2 Garsia-Rodemich-Rumsey’s lemmas

The Garsia-Rodemich-Rumsey lemma is usually stated for functions of one parameter [8]. A more general version is proven in [11, Lemma 2], from which we deduce the following lemma.

**Lemma 3.6.** Let $d \in \mathbb{N}^*$, $a, b \in (0, 1]$ and $c \in [0, a \wedge b]$. Define $\delta_{a,b,c}$ the distance on $\mathbb{R} \times (0, 1)^d$ given for any $x = (x_1, \ldots, x_{d+1})$, $y = (y_1, \ldots, y_{d+1}) \in \mathbb{R} \times (0, 1)^d$ by

$$\delta_{a,b,c}(x, y) = |x_1 - y_1|^a \wedge |x_1 - y_1|^b + \frac{1}{d} \sum_{i=2}^{d+1} |x_i - y_i|^c. \quad (3.1)$$

Let $f : \mathbb{R} \times (0, 1)^d \to \mathbb{R}$ be a continuous function (for this metric) and $K$ be a compact subset of $\mathbb{R} \times (0, 1)^d$. For any $p > 0$ satisfying $p > 2(a \wedge b)^{-1}(d + 1)$, there exists a constant $C > 0$ that depends only on $p$, $a$ and $b$, such that

$$\sup_{x, y \in K} \frac{|f(x) - f(y)|}{\delta_{a,b,c}(x, y)^{1 - p(d+1)}} \leq C \left( \int_K \int_K |f(z) - f(z')|^p \delta_{a,b,c}(z, z')^p \; dz \; dz' \right)^{\frac{1}{p}}.$$
Proof. Using the notations of [11, Lemma 2], we take \( \psi(u) = u^p, d(x,y) = \delta_{a,b,c}(x,y) \) and \( m \) the Lebesgue measure. We have to compute a lower bound on \( \sigma(r) := \inf_{x \in K} m(B(x,r)) \), where \( B(x,r) \) is the \( \delta_{a,b,c} \)-ball centred in \( x \) with radius \( r \). By the concavity of the function \( u \rightarrow u^a \wedge b \), we have that for all \( u, v \in [0,1], \)

\[
    u^a + v^b \leq u^{a \wedge b} + v^{a \wedge b} \leq 2^{1-a \wedge b} (u + v)^{a \wedge b} \leq 2(u + v)^{a \wedge b}.
\]

Moreover, if \( u \geq 1 \) and \( v \in [0,1] \), then \( u^c + v^c \leq u^c + v^c \leq 2(u + v)^{a \wedge b} \). Thus, for any \( r > 0 \), the set

\[
    A := \left\{ y \in K : \left( |x_1 - y_1| + \frac{1}{d} \sum_{i=2}^{d+1} |x_i - y_i| \right)^{a \wedge b} \leq \frac{r}{2} \right\}
\]

is included in \( B(x,r) \). Hence \( \sigma(r) \geq c_{a,b} \frac{d+1}{a \wedge b} \), for some constant \( c_{a,b} > 0 \). Using this inequality in the bound proposed in [11, Lemma 2] gives the desired result.

The Garsia-Rodemich-Rumsey lemma also has a version for the rectangular increments of two-parameter functions.

**Lemma 3.7.** Let \( d \in \mathbb{N}^* \). Let \( f : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R} \) be a continuous function. Let \( I \) be a product of \( d+1 \) closed intervals of \( \mathbb{R} \). Then for any \( p > 0 \) and any \( \theta_1, \theta_2, \eta > \frac{1}{p} \), there exists a constant \( C > 0 \) independent of \( I \) such that

\[
    \sup_{x,y \in I \setminus \{(z_1, \ldots, z_d) : (y_1, \ldots, y_d) \neq (z_1, \ldots, z_d) \}} \left\| \mathbb{E}_{(t,H)} \left( |x_1 - y_1|^\theta_1 \wedge |x_1 - y_1|^\theta_2 \right)^{\eta \wedge p} \right\| \leq C \left( \int \mathbb{E}_{(t,H)} \left( |z_1 - z'_1|^\theta_1 \wedge |z_1 - z'_1|^\theta_2 \right)^{\eta \wedge p} \right)^{\frac{1}{p}}.
\]

**Proof.** This is an application of Theorem 2.3 in [13]. More precisely, using the notations of [13], this result is obtained by taking \( n = d + 1 \), \( \psi(x) = x^p \), \( p_1(u) = u^{\theta_1 + 1/p} \wedge u^{\theta_2 + 1/p} \) and \( p_k(u) = u^{\theta_2 + 1/p} \) for \( k = 2, \ldots, d+1 \). The proof in [13] is done for \( I = [0,1]^n \) but it can be generalized to any product of closed intervals by normalizing the variable space.

### 3.3 Proof of Theorem 3.1

Denote \( a = \min(H) \), \( \bar{a} = \max(H) \) and let \( \alpha > 0 \), \( \eta \in (0,a) \) and \( p > \frac{1}{\bar{a} - \alpha} \vee \eta^{-1} \) (one can always choose \( p \) as large as necessary).

For any \( T \geq 1 \), let \( K_T := [0,T] \times H \). Recall that \( X_t^H = (1 + t)^{-\alpha} B_t^H \). The idea of the proof is to obtain an upper bound which is independent of \( T \) on

\[
    A_T := \mathbb{E}_{(t,H)} \left( \sup_{(t',H') \in K_T \setminus (t,H)} \frac{|X_t^H - X_{t'}^{H'}|^p}{(|t' - t|^\eta \wedge |t' - t|^a + |H - H'|)^{p-\frac{1}{p}}} \right).
\]

Provided \( A_T \) is bounded in \( T \in \mathbb{R}_+ \), we would then get by monotone convergence that

\[
    \sup_{(t,H),(t',H') \in \mathbb{R}_+ \times H \setminus (t,H) \neq (t',H')} \frac{|X_t^H - X_{t'}^{H'}|^p}{(|t' - t|^\eta \wedge |t' - t|^a + |H - H'|)^{p-\frac{1}{p}}} < \infty \text{ a.s.} \quad (3.2)
\]

The last step of the proof will then be to deduce from (3.2) the result on the increments of \( B \).

First, we prove (3.2). A consequence of Proposition 2.2 and Kolmogorov’s continuity criterion (see e.g. [17, Th. 2.3.1] in the multiparameter setting) is that the process \( \{X_t^H, t \in [0,T], H \in H \} \) has an almost sure continuous modification. We still denote by \( X_t^H \) this modification. Here the continuity is to be understood with respect to the distance \( \delta_{a,1,\eta} \), defined in (3.1). Hence by
Lemma 3.6 (applied here for \( a = \min(\mathcal{H}) \), \( b = 1 \), \( c = \eta \), \( d = 1 \) and \( p > \frac{2}{\alpha} \), there exists \( C > 0 \) which does not depend on \( T \) and \( \mathcal{H} \) such that

\[
A_T \leq C \int_0^T \int_0^T \int_H \int_H \mathbb{E}[(1 + s)^{-\alpha} B^h_s - (1 + s')^{-\alpha} B^h_{s'} | \mathcal{F}] \frac{p}{(s - s')^{\eta} / |s - s'|^{\alpha} + |h - h'|^{p}} \, dh' \, ds' \, ds .
\]

Since the random variable \((1 + s)^{-\alpha} B^h_s - (1 + s')^{-\alpha} B^h_{s'}\) is Gaussian,

\[
A_T \leq C \int_0^T \int_0^T \int_H \int_H \left( \mathbb{E}[(1 + s)^{-\alpha} B^h_s - (1 + s')^{-\alpha} B^h_{s'}]^2 \right)^{p/2} \frac{p}{(s - s')^{\eta} / |s - s'|^{\alpha} + |h - h'|^{p}} \, dh' \, ds' \, ds
\]

\[=: C (I_1 + I_2 + I_3), \quad (3.3)\]

where \( I_1 \) is the integral for \( s' \) between 0 and \((s - 1) \lor 0\); \( I_2 \) for \( s' \) between \((s - 1) \lor 0\) and \((s + 1) \lor T\); and \( I_3 \) for \( s' \) between \((s + 1) \lor T\) and \( T\).

Bound on \( I_1 \) and \( I_3 \). We only write the details for \( I_1 \), as \( I_3 \) can be treated similarly. In both cases one has \(|s' - s| \geq 1\), hence \(|s - s'|^{\eta} / |s - s'|^{\alpha} + |h - h'| \geq |s - s'|^{\eta} / |h|^{p}\). It comes

\[
I_1 \leq C \int_0^T \int_0^{s-1} \int_H \int_H \left( \mathbb{E}[(1 + s)^{-\alpha} B^h_s]^2 + \mathbb{E}[(1 + s')^{-\alpha} B^h_{s'}]^2 \right)^{p/2} \frac{p}{(s - s')^{\eta} / |s - s'|^{\alpha} + |h - h'|^{p}} \, dh' \, ds' \, ds
\]

\[
\leq C \int_1^T \int_1^{s-1} \int_H \int_H [(1 + s)^{-\alpha} s^\eta + (1 + s')^{-\alpha} (s')^\eta] \frac{p}{(s - s')^{\eta} / |s - s'|^{\alpha} + |h - h'|^{p}} \, dh' \, ds' \, ds.
\]

Now use that \( h \) and \( h' \) are smaller than \( \tilde{a} \) to get

\[
I_1 \leq C \int_1^T \int_1^{s-1} \left( (1 + s)^{-\alpha} s^\eta + (1 + s')^{-\alpha} ((s')^\eta \lor (s')^\eta) \right) \frac{p}{(s - s')^{\eta} / |s - s'|^{\alpha} + |h - h'|^{p}} \, dh' \, ds' \, ds.
\]

Using now that \( p\eta > 1 \),

\[
I_1 \leq C \int_1^T \left( (1 + s)^{-\alpha} s^\eta (1 - s^{1-p\eta}) + \int_0^{s-1} (1 + s')^{-\alpha} + \eta (s - s')^{-p\eta} \, ds' \right) \, ds.
\]

Now we have \( \int_1^T (1 + s)^{-\alpha} s^\eta (1 - s^{1-p\eta}) \, ds \leq \int_1^T (1 + s)^{p(\alpha - \eta)} \, ds \leq C(1 + T)^{p(\alpha - \eta) + 1} \) and \( \int_1^T \int_0^{s-1} (1 + s')^{-\alpha} + \eta (s - s')^{-p\eta} \, ds' \, ds = \int_0^{T-1} (1 + s')^{p(\alpha - \eta)} \int_0^T (s - s')^{-p\eta} \, ds' \). Hence

\[
I_1 \leq C (1 + T^{p(\alpha - \eta) + 1}). \quad (3.4)
\]

Proceeding similarly, the same bound holds for \( I_3 \).

Bound on \( I_2 \). Here we use \(|s - s'| \leq 1\) and Proposition 2.2 to obtain

\[
\mathbb{E}[(1 + s)^{-\alpha} B^h_s - (1 + s')^{-\alpha} B^h_{s'}]^p \leq C (1 + s \land s')^{p(\alpha - \eta)} (1 + \log^2 (s \land s'))^{\frac{p}{\alpha}} (s - s'|^{\alpha} + |h - h'|^{p}).
\]

Hence

\[
I_2 \leq C \int_0^T \int_0^{s+1} \int_H \int_H [(1 + s \land s')^{p(\alpha - \eta)} (1 + \log^2 (s \land s'))^{\frac{p}{\alpha}} \, dh' \, ds' \, ds
\]

\[
\leq C (T^{p(\alpha - \eta) + 1} (1 + \log^2 T)^{\frac{p}{\alpha}} + 1). \quad (3.5)
\]

One can now plug (3.4) and (3.5) into (3.3), and take \( \alpha > \frac{1}{p} + \tilde{a} \) to conclude that \( A_T \) is bounded uniformly in \( T \in \mathbb{R}_+ \). Hence (3.2) holds true.
Let $C$ denote the random variable

$$C := \sup_{(t,H),(t',H') \in \mathbb{R}_+ \times \mathcal{H} \atop (t,H) \neq (t',H')} \frac{|X_t^H - X_{t'}^{H'}|}{(|t' - t|^\alpha |t' - t|^{\alpha + |H - H'|})^{1 - \frac{1}{2p}}}.$$ 

Then $\mathbb{E}|C|^p \leq \sup_{T > 0} A_T < +\infty$. Let now $t' \geq t$ and observe that

$$(1 + t')^{-\alpha}|B_t^H - B_{t'}^{H'}| \leq |(1 + t')^{-\alpha} - (1 + t)^{-\alpha}|B_t^H| + |(1 + t)^{-\alpha}B_t^H - (1 + t')^{-\alpha}B_{t'}^{H'}|. \quad (3.6)$$

Apply (3.2) to $(1 + t)^{-\alpha}|B_t^H|$ (with the notations of (3.2), take $t' = 0$ and $H' = H$) so that for any $t \geq 0$,

$$(1 + t)^{-\alpha}|B_t^H| \leq C \left( t^\alpha + t^\alpha \right)^{1 - \frac{1}{2p}}.$$ 

Apply again (3.2) to $|(1 + t)^{-\alpha}B_t^H - (1 + t')^{-\alpha}B_{t'}^{H'}|$ so that (3.6) becomes

$$(1 + t')^{-\alpha}|B_t^H - B_{t'}^{H'}| \leq C \left( (1 + t)^{-\alpha} - (1 + t')^{-\alpha} \right) t^\alpha + |t' - t|^\alpha + |H - H'| \right)^{1 - \frac{1}{2p}}$$

Now (2.9) applied twice (first with $h = \eta (1 - \frac{1}{2p})$ and then with $h = a - \frac{1}{p}$) yields

$$(1 + t')^{-\alpha}|B_t^H - B_{t'}^{H'}| \leq C \left( (1 + t)^{-\alpha} (|t' - t|^\alpha + |H - H'|) \right)^{1 - \frac{1}{2p}}$$

Since we assumed $\eta < a$, it follows that

$$|B_t^H - B_{t'}^{H'}| \leq C(1 + t)^{\alpha} (|t' - t|^\alpha + |H - H'|) \right)^{1 - \frac{1}{2p}} \leq C(1 + t')^{\alpha + \eta} (1 + |t' - t|^\alpha + |H - H'|)|^{1 - \frac{1}{2p}}.$$ 

Besides, we have $1 - \frac{4}{ap} \geq 1 - \epsilon$ and $(1 + |t' - t|^\alpha + |H - H'|) \leq 2$, thus by setting $\eta = \epsilon a$ and $\alpha = a + \epsilon a$, we conclude that

$$|B_t^H - B_{t'}^{H'}| \leq C(1 + t')^{2\alpha + \epsilon} (1 + |t' - t|^\alpha + |H - H'|)|^{1 - \epsilon}.$$ 

### 3.4 Proof of Theorem 3.4

We follow the same approach as the proof of Theorem 3.1, and use the same notations. In particular, let $p > \frac{4}{ap}$, and let $\theta_1 = a - \frac{1}{p}$, $\theta_2 = 1 - \frac{1}{p}$ and $\eta \in (\frac{1}{p}, \theta_1)$. We apply Lemma 3.7 for the process $X_t^H$ defined on the compact set $\mathcal{K}_T$ and use the fact that $\square_{(t,H)} X$ is a Gaussian random variable to get that

$$B_T := \mathbb{E} \sup_{(t,H),(t',H') \in \mathcal{K}_T \atop t \neq t', H \neq H'} \left( |X_t^{(t',H')} - X_{t'}^{(t,H)}| \right)^p \leq C \int_0^T \int_0^T \int_0^T \int_0^T \left( |X_t^{(t',H')} - X_{t'}^{(t,H)}| \right)^p dh' dh' ds' ds \leq C \int_0^T \int_0^T \int_0^T \int_0^T \left( |X_t^{(t',H')} - X_{t'}^{(t,H)}| \right)^p dh' dh' ds' ds \leq C \int_0^T \int_0^T \int_0^T \int_0^T \left( |X_t^{(t',H')} - X_{t'}^{(t,H)}| \right)^p dh' dh' ds' ds =: C(J_1 + J_2 + J_3),$$

where $J_1$ is the integral for $s'$ between $0$ and $(s - 1) \lor 0$; $J_2$ for $s'$ between $(s - 1) \lor 0$ and $(s + 1) \land T$; and $J_3$ for $s'$ between $(s + 1) \land T$ and $T$. 

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Thus proceeding similarly, one obtains the same bound for $x$ variables. We bound $T$ when $s > -s'$ when $s > 1$ to deduce that

$$J_1 = C \int_1^T \int_0^{s-1} \int_{\mathcal{H}} \frac{(1 + s)^{-\rho_0} (\mathbb{E}[B_s^h - B_s^{h'}]^2)^{\frac{p}{2}} + (1 + s')^{-\rho_0} (\mathbb{E}[B_s^h - B_s^{h'}]^2)^{\frac{p}{2}}}{|s - s'|^{\eta p + 1}|h - h'|^{\theta_2 p + 1}} dh' dh ds' ds.$$  

In view of Proposition 2.1, we get

$$J_1 \leq C \int_1^T \int_0^{s-1} \int_{\mathcal{H}} \frac{(1 + s)^{-\rho_0} (s^{ph} \vee s'^{ph'}) (\log^p(s) + 1)}{|s - s'|^{\eta p + 1}|h - h'|^{\theta_2 p + 1 - p}} \, dh' dh ds' ds.$$  

Since $\theta_2 p + 1 - p = 0$, the integrals over $h$ and $h'$ are finite. Thus

$$J_1 \leq C \int_1^T \left( (1 + s)^{-\rho_0 + \rho a} (\log^p(s) + 1) + \int_0^{s-1} (1 + s')^{-\rho_0 + \rho a} (\log^p(s') + 1) |s - s'|^{-\eta p - 1} ds' \right) ds$$  

and using that $-\eta p - 1 < -2$, it comes that

$$J_1 \leq C(1 + (1 + T)^p(a - \alpha) + 1)(\log^p(T) + 1). \quad (3.8)$$  

Proceeding similarly, one obtains the same bound for $J_3$.

**Bound on $J_2$.** Here we use Proposition 2.4 and the fact that $|s - s'|^{\theta_1 p + 1} |s - s'|^{\eta p + 1} = |s - s'|^{\theta_1 p + 1}$ when $|s - s'| \leq 1$ to deduce that

$$J_2 \leq C \int_0^T \int_{(s-1) \vee 0}^{(s+1) \wedge T} \int_{\mathcal{H}} \frac{(1 + s \wedge s')^{-\rho_0} |s - s'|^{\rho_0 - \rho_0 + 1} (1 + \log^2 |s - s'| + \log^2 (s \wedge s'))^\frac{p}{2}}{|h - h'|^{\theta_2 p + 1 - p}} dh' dh ds' ds.$$  

Since $\theta_2 p + 1 - p = 0$, the integrals over $h$ and $h'$ are finite. Moreover, $pa - p\theta_1 - 1 = 0$, hence

$$J_2 \leq C \int_0^T \int_{s}^{(s+1) \wedge T} (1 + s)^{-\rho_0} (1 + |\log(|s - s'|)|^p + |\log(s)|^p) \, ds' ds$$

$$+ C \int_0^T \int_{(s-1) \vee 0}^{s} (1 + s')^{-\rho_0} (1 + |\log(|s - s'|)|^p + |\log(s')|^p) \, ds' ds$$  

$$=: C(J_{21} + J_{22}).$$  

We bound $J_{21}$ and skip the details for $J_{22}$, as similar computations work. Using the change of variables $x = s$, $y = s' - s$, we get

$$J_{21} \leq C \int_0^T \int_0^1 (1 + x)^{-\rho_0} (1 + |\log^p(y)| + |\log^p(x)|) \, dy dx$$

$$\leq C \int_0^T (1 + x)^{-\rho_0} (1 + |\log^p(x)|) \, dx.$$  

Thus $J_{21} \leq C(1 + T^{-\rho_0 + 1})(1 + \log^p(T))$ and the same bound holds for $J_{22}$, so that

$$J_2 \leq C(1 + (1 + T)^{-\rho_0 + 1}(1 + \log^p(T))). \quad (3.9)$$  

We can now plug (3.8) and (3.9) into (3.7). Choosing $\alpha > \frac{1}{p} + \bar{a}$, we obtain that $B_T$ is bounded uniformly in $T \in \mathbb{R}_+$. Hence we conclude that there exists a positive random variable $C$ that has a finite moment of order $p$ such that almost surely,

$$|\mathbb{E}_{(t, H')} X| \leq C |H - H'|^{\theta_2 - \frac{1}{p}} \left(|t - t'|^{\eta \frac{1}{p}} + |t - t'|^{\theta_1 - \frac{1}{p}}\right). \quad (3.10)$$
Now for $t' \geq t$, one has

$$(1 + t')^{-\alpha}\|\phi_{(t,H)}^{(t',H')}\|B \leq (1 + t)^{-\alpha} - (1 + t')^{-\alpha}\|B_t^H - B_t^{H'}| + \|\phi_{(t,H)}^{(t',H')}\|X.$$ 

To control the right-hand side of the previous inequality, it remains to use (2.9) and Theorem 3.1 for the first term of the sum, and (3.10) for the second term. Overall, it comes

$$\|\phi_{(t,H)}^{(t',H')}\|B \leq (1 + t)^\alpha \left( |H - H'|^{\alpha - \frac{2}{p}} (|t - t'|^{\frac{2}{p}} \wedge |t - t'|^{\frac{2}{p} - \frac{2}{p}}) \right).$$

Using $\eta < \theta_1$ and $t' \geq t$, we have

$$\|\phi_{(t,H)}^{(t',H')}\|B \leq (1 + t')^\alpha \left( |H - H'|^{\alpha - \frac{2}{p}} (1 \wedge |t - t'|^{\frac{2}{p} - \frac{2}{p}}) \right).$$

Setting $\alpha = \bar{a} + \frac{3}{p}$, $\eta = \frac{2}{p}$ and $\varepsilon = \frac{a}{\alpha p}$ yields the desired result.

### 3.5 Proof of Proposition 3.5

First, notice that it suffices to prove the result for $q$ large enough. Let $t, t' \geq 0$ and denote $\varphi(H) := B_t^H - B_t^{H'}$. Apply the classical Garsia-Rodemich-Rumsey’s lemma with the choice $\psi(x) = x^q$, $q > 2$ and $p(x) = x$, with the notations of [8]. We get

$$E \sup_{H, H' \in \mathcal{H}} \frac{|\varphi(H) - \varphi(H')|^q}{|H - H'|^{q - 2}} \leq C \int_{\mathcal{H}} \int_{\mathcal{H}} \frac{E|\varphi(h) - \varphi(h')|^q}{|h - h'|^{q - 2}} dh dh'. $$

$$\leq C \int_{\mathcal{H}} \int_{\mathcal{H}} \frac{\left( E|B_t^h - B_t^{h'} - B_t^{h'} + B_t^{h'}|^2 \right)^{q/2}}{|h - h'|^{q - 2}} dh dh'. $$

In view of Proposition 2.3, we further obtain

$$E \sup_{H, H' \in \mathcal{H}} \frac{|\varphi(H) - \varphi(H')|^q}{|H - H'|^{q - 2}} \leq C \left( |t' - t|^{q \min(H)} \vee |t' - t|^{q \max(H)} \right) \log^q(|t' - t|) + 1].$$

By fixing a particular $H_0 \in \mathcal{H}$, it follows that

$$E \sup_{H \in \mathcal{H}} |\varphi(H)|^q \leq C \left( |t' - t|^{q \min(H)} \vee |t' - t|^{q \max(H)} \right) \log^q(|t' - t|) + 1] + C E|\varphi(H_0)|^q$$

$$\leq C \left( |t' - t|^{q \min(H)} \vee |t' - t|^{q \max(H)} \right) \log^q(|t' - t|) + 1] + C |t' - t|^{q H_0}.$$ 

### 4 Regularity of ergodic fractional SDEs

#### 4.1 Regularity of the solutions

Let $B$ be an $\mathbb{R}^d$-valued fBm, i.e. an $\mathbb{R}^d$-valued process indexed by $(t, H) \in \mathbb{R}_+ \times (0, 1)$ with i.i.d. entries, each having the same law as (1.1). Consider the $\mathbb{R}^d$-valued SDE:

$$Y_t^H = Y_0 + \int_0^t b(Y_s^H) ds + B_t^H. \quad (4.1)$$

**Remark 4.1.** When $b$ is a Lipschitz function, this equation has a unique solution on $\mathbb{R}_+$. If in addition $b$ is bounded, then applying Grönwall’s inequality and Theorem 3.1 gives that almost surely, for all $t' \geq t \geq 0$ and all $H, H' \in \mathcal{H},$

$$|Y_t^H - Y_t^{H'}| \leq C (1 + t') \left( 1 \wedge |t - t'|^{\min(H)} \vee |H - H'| \right)^{1 - \varepsilon} e^{Ct}. $$

We seek to improve the previous bound when $b$ is dissipative, that is when
First, we analyse the fractional Ornstein-Uhlenbeck (OU) process. Then the main result about $|Y_t^H - Y_t^{H'}|$ is stated in Theorem 4.3 and is based on a comparison with the OU process. For $H \in (0,1)$, the fractional OU process is the solution to the following equation:

$$dU_t^H = -U_t^H dt + dB_t^H.$$  \hspace{1cm} (4.3)

This equation admits a unique stationary solution $\overline{U}^H$ which is given by $\overline{U}^H_t = \int^{-\infty}_t e^{-(t-s)} dB_s^H$. In view of this expression, we consider $\overline{U}$ as a random field indexed by $(t,H)$.

**Proposition 4.2.** Let $H$ be a compact subset of $(0,1)$. For $\varepsilon \in (0,1)$ and $p \geq 1$, there exists a random variable $C$ with a finite moment of order $p$ such that almost surely, for all $t' \geq t \geq 0$ and $H, H' \in H$,

$$|\overline{U}_t^H - \overline{U}_{t'}^{H'}| \leq C (1 + t')^{\varepsilon} \left(1 \wedge |t' - t|^{\min(H)} + |H - H'|^p\right)^{1-\varepsilon}.$$  

**Proof.** The scheme of proof is similar to Theorem 3.1: we introduce the auxiliary process

$$U_t^H = (1 + t)^{-\alpha} \overline{U}_t^H,$$  \hspace{1cm} (4.4)

for some $\alpha \in (0,1)$, compute the moments of $U_t^H - U_{t'}^{H'}$ and finally apply the GRR Lemma. First, we have from Lemma A.2 that for $p \geq 1$ and $t' \geq t$,

$$\mathbb{E}|U_t^H - U_{t'}^{H'}|^p \leq C(1 + t)^{-\rho\alpha} \left(1 \wedge |t - t'|^{p\min(H)} + |H - H'|^p\right).$$  \hspace{1cm} (4.5)

By Kolmogorov’s continuity theorem and the previous inequality, $U$ and $\overline{U}$ admit a continuous modification on any compact subset of $\mathbb{R}_+ \times H$, and we still denote by $\overline{U}$ and $\overline{U}$ these continuous modifications. Let $p > \frac{4}{\varepsilon \min(H)}$ and let $\eta \in \left(\frac{4}{p}, \min(H)\right)$. Apply Lemma 3.6 with $a = \min(H)$, $b = 1$, $c = \eta$ and $d = 1$: we get that for $T \geq 0$,

$$\mathbb{E}\sup_{t,t' \in [0,T] \atop H,H' \in H \atop H \neq H'} \frac{|U_t^H - U_{t'}^{H'}|^p}{(1 + |t - t'|^{\rho\alpha} + |H - H'|)^{p - 4/\eta}} \leq C \int_0^T \int_0^T \int_H \int_H \mathbb{E}|\overline{U}_s^H - \overline{U}_v^{H'}|^p \frac{dh'}{dh} \frac{ds'}{ds}.$$  

Where $I_1$ is the integral for $s' \in (0,1)$ and $I_2$ for $s'$ between $(s-1) \vee 0$ and $(s+1) \wedge T$; and $I_3$ for $s'$ between $(s+1) \wedge T$ and $T$.

Let us first bound $I_1$ and omit the similar proof for $I_3$. We have

$$I_1 \leq C \int_0^T \int_0^{(s-1)\vee 0} \mathbb{E}|U_s^H|^{p+H_v^{H'}(p)} |s - s'|^{-\eta p} ds' ds.$$  

Recalling that $\mathbb{E}|U_t^H|^p = (1 + t)^{-\rho\alpha} \mathbb{E}|\overline{U}_0^H|^p \leq C(1 + t)^{-\rho\alpha}$, we get

$$I_1 \leq C \int_0^T \int_0^{s-1} \left((1 + s)^{-\rho\alpha} + (1 + s')^{-\rho\alpha}\right)|s - s'|^{-\eta p} ds' ds \leq C \int_0^T \left((1 + s)^{-\rho\alpha} + (1 + s')^{-\rho\alpha}\right)s^{-\eta p} ds'.$$

Using that $\eta p > 1$, we get

$$I_1 \leq C(T^{-\rho\alpha+1}) + I_3 \leq C(T^{-\rho\alpha+1} + 1).$$
Choosing $\alpha \geq \frac{1}{p}$, it follows that $I_1 + I_3$ is bounded uniformly in $T$. Let us now bound $I_2$ from above. Using (4.5) gives

$$I_2 \leq C \int_0^T \int_{(s-1)\vee 0}^{s+1} (1 + s \wedge s')^{-p\alpha} \, ds \, ds'.$$

It follows that $I_2$ is bounded uniformly in $T$ since $\alpha \geq \frac{1}{p}$. Hence, by a monotone convergence argument, the random variable

$$C_1 = \sup_{t, t' \in \mathbb{R}_+ \atop (t, H, H') \in \mathcal{H}, (t, H) \neq (t', H')} \frac{|U_t^H - U_{t'}^H|}{(|t-t'|^\alpha |t-t'|^\alpha + |H-H'|)^{1-\frac{1}{ap}}},$$

has a finite moment of order $p$. Therefore, almost surely for all $t, t' \geq 0$ and $H, H' \in \mathcal{H}$,

$$|U_t^H - U_{t'}^H| \leq C_1 (|t-t'|^\alpha |t-t'|^\alpha + |H-H'|)^{1-\frac{1}{ap}}. \quad (4.6)$$

For $H' = \frac{1}{2}$, $t' = 0$ and $t \geq 1$, we have that

$$|U_t^H| \leq C_2 \left( t^\eta + |H - \frac{1}{2} \right)^{1-\frac{1}{ap}} + |U_0^H|$$

$$\leq C_2 (t + 1)^\eta. \quad (4.7)$$

where $C_2 := C_1 + |U_0^H|$ also has a finite moment of order $p$.

Now for $t' \geq t$, one has

$$(1 + t')^{-\alpha} |U_t^H - U_{t'}^H| \leq |(1 + t)^{-\alpha} - (1 + t')^{-\alpha} |U_t^H + |U_t^H - U_{t'}^H|. \quad (4.8)$$

In view of (4.7), $|U_t^H| \leq C_2(1 + t)^\eta \leq C_2(1 + t)^{1+\eta}$. Hence using (2.9) with $h = 1$ yields

$$|(1 + t)^{-\alpha} - (1 + t')^{-\alpha} |U_t^H| \leq C_2(t' - t)(1 + t)^{-\alpha + \eta}. \quad \text{If } t' - t \geq 1, \text{ it also comes directly that}$$

$$|(1 + t)^{-\alpha} - (1 + t')^{-\alpha} |U_t^H| \leq C_2(1 + t)^\eta. \quad \text{Thus we get that for any } t' \geq t,$$

$$|(1 + t)^{-\alpha} - (1 + t')^{-\alpha} |U_t^H| \leq C_2(1 + t)^\eta (1 \wedge (t' - t)). \quad (4.9)$$

It remains to plug (4.9) and (4.6) in (4.8) to obtain that

$$|U_t^H - U_{t'}^H| \leq C_2(1 + t)^{\alpha + \eta}(1 \wedge |t' - t|) + C_1 (|t-t'|^\alpha |t-t'|^\alpha + |H-H'|)^{1-\frac{1}{ap}}.$$

Since $C_2 \geq C_1$ and $p > \frac{\alpha}{\varepsilon}$, we have

$$|U_t^H - U_{t'}^H| \leq C_2(1 + t)^{\alpha + \eta}(1 \wedge |t-t'|^\alpha + |H-H'|)^{1-\varepsilon}.$$

Choosing $\alpha = \frac{\varepsilon}{\eta}$ (which satisfies the constraint $\alpha \geq \frac{1}{p}$) and $\eta = \frac{\varepsilon}{2}$ (which satisfies $\eta > \frac{1}{p}$) yields

$$|U_t^H - U_{t'}^H| \leq C_2(1 + t)^\varepsilon (1 \wedge |t-t'|^\alpha + |H-H'|)^{1-\varepsilon}.$$

\[ \square \]

**Theorem 4.3.** Let $\mathcal{H}$ be a compact subset of $(0, 1)$. For each $H \in \mathcal{H}$, let $Y^H$ be the solution to (4.1), with the drift $b$ satisfying (4.2).

(i) Let $\varepsilon \in (0, 1)$ and let $p \geq 1$. There exists a random variable $C$ with a finite moment of order $p$ such that almost surely, for all $t \geq 0$ and $H, H' \in \mathcal{H}$,

$$|Y_t^H - Y_t^{H'}| \leq C (1 + t)^\varepsilon |H - H'|^{1-\varepsilon}.$$
(ii) Let \( p \geq 1 \). There exists a constant \( C \) such that for all \( H, H' \in \mathcal{H} \),
\[
\sup_{t \geq 0} \mathbb{E} \left( |Y_t^H - Y_t^{H'}|^p \right) \leq C |H - H'|^p.
\]

**Proof.** The proof is based on a comparison with the Ornstein-Uhlenbeck process defined in (4.3) with initial condition \( Y_0 \). Recall that \( \overline{U}^H = \overline{U} \) denotes the stationary solution of (4.3). We will use the following set of notations: for \( H, H' \in (0, 1) \),
\[
\mathcal{B} = B^H - B^{H'}, \quad \mathcal{U} = U^H - U^{H'} \quad \text{and} \quad \overline{\mathcal{U}} = \overline{U}^H - \overline{U}^{H'}. \]

The first step of this proof is to establish the following inequality: with \( \kappa \) from (4.2),
\[
|Y_t^H - Y_t^{H'}|^2 \leq C \left( e^{-2pt} |\overline{U}_0|^2 + |\overline{U}_t|^2 + C \int_0^t e^{-\kappa(t-r)} |\overline{\mathcal{U}}_r|^2 dr \right). \tag{4.10}
\]

By the triangle inequality, it suffices to prove that
\[
|Y_t^H - Y_t^{H'} - \overline{\mathcal{U}}_t|^2 \leq C \int_0^t e^{-\kappa(t-r)} |\overline{\mathcal{U}}_r|^2 dr.
\]

We work first with the nonstationary process \( \mathcal{U} \). Use (4.1) and (4.3) to get that \( Y_t^H - Y_t^{H'} - \mathcal{U}_t = \int_0^t \left( b(Y_s^H) - b(Y_s^{H'}) + U_s^H - U_s^{H'} \right) ds \). Hence (4.2) yields
\[
\frac{d}{dt} |Y_t^H - Y_t^{H'} - \mathcal{U}_t|^2 = 2 \left( b(Y_t^H) - b(Y_t^{H'}) + \mathcal{U}_t \right) \left( b(Y_t^H) - b(Y_t^{H'}) + \mathcal{U}_t \right) \leq -2\kappa |Y_t^H - Y_t^{H'}|^2 - 2|\mathcal{U}_t|^2 + 2(1 + K)|Y_t^H - Y_t^{H'}||\mathcal{U}_t|.
\]

Apply Young’s inequality to get
\[
2(1 + K)|Y_t^H - Y_t^{H'}| |\mathcal{U}_t| \leq \kappa |Y_t^H - Y_t^{H'}|^2 + \frac{(1 + K)^2}{\kappa} |\mathcal{U}_t|^2.
\]

It follows that
\[
\frac{d}{dt} |Y_t^H - Y_t^{H'} - \mathcal{U}_t|^2 \leq -\kappa \left( |Y_t^H - Y_t^{H'}|^2 + |\mathcal{U}_t|^2 \right) + \left( \kappa - 2 + \frac{(1 + K)^2}{\kappa} \right) |\mathcal{U}_t|^2 \leq -\kappa |Y_t^H - Y_t^{H'} - \mathcal{U}_t|^2 + C |\mathcal{U}_t|^2.
\]

Hence Grönwall’s lemma followed by Jensen’s inequality give
\[
|Y_t^H - Y_t^{H'} - \mathcal{U}_t|^2 \leq C \int_0^t e^{-\kappa(t-r)} |\overline{\mathcal{U}}_r|^2 dr.
\]

Now let us establish a similar inequality but with the stationary OU process. We have
\[
|Y_t^H - Y_t^{H'}|^2 \leq C |\overline{\mathcal{U}}_t|^2 + C \int_0^t e^{-\kappa(t-r)} |\overline{\mathcal{U}}_r|^2 dr \leq C |\overline{\mathcal{U}}_t|^2 + C \int_0^t e^{-\kappa(t-r)} |\overline{\mathcal{U}}_t|^2 dr + C |\overline{\mathcal{U}}_t|^2 + C \int_0^t e^{-\kappa(t-r)} |\overline{\mathcal{U}}_t - \mathcal{U}_t|^2 dr.
\]

Notice that \( \mathcal{U}_t = e^{-t} Y_0 + \int_0^t e^{-(t-s)} dB_s^H \) and \( \overline{\mathcal{U}}_t = e^{-t} \overline{U}_0^H + \int_0^t e^{-(t-s)} dB_s^H \). Hence \( \mathcal{U}_t - \overline{\mathcal{U}}_t = e^{-t} \overline{U}_0^H - \overline{U}_0 = e^{-t} \overline{\mathcal{U}}_0 \) and we have (4.10).

**Proof of (i).** For \( \varepsilon \in (0, 1) \) and \( p \geq 1 \), we know thanks to Proposition 4.2 (for \( t = t' \)) that there exists a random variable \( C \) with a finite moment of order \( p \) such that
\[
|\overline{\mathcal{U}}_t| \leq C(t + \varepsilon)|H - H'|^{1-\varepsilon}.
\]

Using the previous inequality in (4.10) gives the result of (i).
Proof of (ii). To prove (ii), it suffices to combine (4.10) with Lemma A.1.

\section{4.2 Regularity of ergodic means}

First, observe that integrating the result of Theorem 4.3, there exists a random variable $C$ such that almost surely, for all $t > 0$ and $H, H' \in \mathcal{H}$, there is

$$\frac{1}{t} \int_0^{t+1} |Y^H_s - Y^H_{s'}| ds \leq C (1 + t)^\nu |H - H'|^{1-\nu}.$$  

This result is not optimal since for any $H, Y^H$ is ergodic and so the left hand side converges as $t \to \infty$. Thus in Theorem 4.5, we will follow a different approach to get rid of the term $(1 + t)^\nu$, at the price of a lower regularity in the Hurst parameter and values of $H$ smaller than $\frac{\nu}{2}$. We comment on this limitation in Remark 4.6. First, we need the following technical result.

\begin{proposition}
Let $\mathcal{H}$ be a compact subset of $(0, \frac{\nu}{2})$. Recall that $\mathcal{U}^H$ is the stationary Ornstein-Uhlenbeck. Let $\beta \in (0, \frac{\nu}{2} \max(H) \wedge \frac{\nu}{2})$. Let $\varepsilon \in (0, 1)$ and $p \geq 1$. There exists a random variable $C$ with finite moment of order $p$ such that almost surely, for all $t, t' \geq 0$ and $H, H', K, K' \in \mathcal{H}$,

$$\left| \frac{1}{t+1} \int_0^{t+1} \left| \mathcal{U}^H_s - \mathcal{U}^K_s \right|^2 ds - \frac{1}{t'+1} \int_0^{t'+1} \left| \mathcal{U}^{H'}_{s'} - \mathcal{U}^{K'}_{s'} \right|^2 ds \right| \leq C (1 + |t - t'|^{\frac{\nu}{2}}) (1 + |t - t'| + |H - H'|^{\beta} + |K - K'|^{\beta})^{1-\varepsilon}.$$

\end{proposition}

\begin{proof}
For $t \geq 0$ and $H, K \in \mathcal{H}$, define

$$V^{H,K}_{t+1} = \frac{1}{t+1} \int_0^{t+1} \left| \mathcal{U}^H_s - \mathcal{U}^K_s \right|^2 ds - \mathbb{E}|\mathcal{U}^H_0 - \mathcal{U}^K_0|^2,$$

and observe that

$$\left| \frac{1}{t+1} \int_0^{t+1} \left| \mathcal{U}^H_s - \mathcal{U}^K_s \right|^2 ds - \frac{1}{t'+1} \int_0^{t'+1} \left| \mathcal{U}^{H'}_{s'} - \mathcal{U}^{K'}_{s'} \right|^2 ds \right| \leq |V^{H,K}_{t+1} - V^{H',K'}_{t'+1}| + \mathbb{E}|\mathcal{U}^H_0 - \mathcal{U}^K_0|^2 - |\mathcal{U}^{H'}_0 - \mathcal{U}^{K'}_0|^2|.$$ 

By Lemma A.1, we have $\mathbb{E}|\mathcal{U}^H_0 - \mathcal{U}^{H'}_0|^2 \leq C |H - H'|^2$ and $\mathbb{E}|\mathcal{U}^K_0|^2 \leq C$. Hence

$$\mathbb{E}|\mathcal{U}^H_0 - \mathcal{U}^K_0|^2 - |\mathcal{U}^{H'}_0 - \mathcal{U}^{K'}_0|^2| \leq \mathbb{E} \left( |\mathcal{U}^H_0 - \mathcal{U}^{H'}_0|^2 - |\mathcal{U}^K_0 - \mathcal{U}^{K'}_0|^2 \right) \leq C \left( \mathbb{E}|\mathcal{U}^H_0 - \mathcal{U}^{H'}_0|^2 \right)^{\frac{1}{2}} + C \left( \mathbb{E}|\mathcal{U}^K_0 - \mathcal{U}^{K'}_0|^2 \right)^{\frac{1}{2}} \leq C (|H - H'| + |K - K'|).$$

Hence it remains to bound $|V^{H,K}_{t+1} - V^{H',K'}_{t'+1}|$. Let $T \geq 1$, $\beta < \frac{3}{4} \max(H) \wedge \frac{\nu}{2}$ and $p > \frac{6}{5}$. Apply Lemma 3.6 with $a = 1, b = \beta, c = \eta \in (\frac{1}{p}, \beta)$ and $d = 2$ to get that

$$\mathbb{E} \sup_{t, t' \in [0, T]} \left| V^{H,K}_{t+1} - V^{H',K'}_{t'+1} \right|^p \leq C \int_{[0, T]^2 \times \mathcal{H}^2} \mathbb{E}|V^{h,k}_{s+1} - V^{h',k'}_{s'+1}|^p \left( |s - s'|^{\eta} |h - h'|^{1/2} |K - K'|^{1/2} + \frac{1}{2} |h - h'|^{1/2} |k - k'|^{1/2} \right) dh' dk' ds' ds =: C A_T.$$ 

Decompose $A_T$ as

$$A_T = I_1 + I_2 + I_3,$$
where $I_1$ is the integral for $s'$ between 0 and $(s-1) \lor 0$; $I_2$ for $s'$ between $(s-1) \lor 0$ and $(s+1) \land T$; and $I_3$ for $s'$ between $(s+1) \land T$ and $T$.

For $I_1$, we have

$$I_1 \leq C \int_1^T \int_0^{s-1} \left( \mathbb{E}|V_{s',k}^h|^p + \mathbb{E}|V_{s',k'}^{h'}|^p \right) |s - s'|^{-\eta p} \, dh' \, dk' \, dh \, dk \, ds \, ds.$$

For $v \in (0,1)$, in view of Lemma A.8, we have

$$I_1 \leq C \int_1^T \int_0^{s-1} \left( (s + 1)^{-3(1-v)} (\log(s + 1)^3 + 1) + (s + 1)^8 (\max(H) - 1)^{(1-v)} \right) + (s' + 1)^{-3(1-v)} (\log(s' + 1)^3 + 1) + (s' + 1)^8 (\max(H) - 1)^{(1-v)} \right) (s - s')^{-\eta p} \, ds' \, ds.$$

Proceeding with the same computations for $I_3$ and using that $\eta p > 1$, we get

$$I_1 + I_3 \leq C \left( 1 + T^{8(\max(H) - 1)(1-v) + 1} + T^{-3(1-v) + 1} \log(T + 1)^3 \right).$$

For $I_2$, we use Lemma A.10 to get

$$I_2 \leq C \int_0^1 \int_0^{s+1} \left( (s \land s' + 1)^{-3(1-v)(1-\beta)} (\log(s \land s' + 1)^3 + 1) + (1 + s \land s')^{8(\max(H) - 1)(1-v)(1-\beta)} + (1 + s \land s')^{-p} \right) ds' \, ds \leq C \left( 1 + T^{8(\max(H) - 1)(1-v)(1-\beta) + 1} + T^{-3(1-v)(1-\beta) + 1} \log(T + 1)^3 \right).$$

Since $\beta < \frac{7 - 8 \max(H)}{8 - 8 \max(H)} \land \frac{3}{2}$ and $\max(H) < \frac{3}{2}$, we have

$$8(\max(H) - 1)(1-\beta) < -1 \quad \text{and} \quad -3(1-\beta) < -1.$$

Therefore, we fix $v$ small enough to ensure that

$$8(\max(H) - 1)(1-\beta)(1-v) < -1 \quad \text{and} \quad -3(1-\beta)(1-v) < -1.$$

For such $v$, we have $8(\max(H) - 1)(1-v) < -1$ and $-3(1-v) < -1$. Hence the quantities $I_1$, $I_2$ and $I_3$ are bounded uniformly in $T$. In view of (4.12), it follows that for any $T > 0$,

$$\mathbb{E} \sup_{t,t' \in [0,T]} \sup_{H,H',K,K' \in \mathcal{H}} \frac{|V_t^{H,K} - V_{t'}^{H',K'}|^p}{(|t - t'| \land |t - t'|^\eta + |H - H'|^\beta + |K - K'|^\beta)^{\eta - \frac{p}{2}}} \leq C.$$  

Thus by a monotone convergence argument, letting $T \to +\infty$, the random variable

$$C_1 := \sup_{t,t' \in \mathbb{R}_+, H,H',K,K' \in \mathcal{H}} \sup_{(t,H,K) \neq (t',H',K')} \frac{|V_t^{H,K} - V_{t'}^{H',K'}|}{(|t - t'| \land |t - t'|^\eta + |H - H'|^\beta + |K - K'|^\beta)^{1 - \frac{p}{2}}}$$

has a finite moment of order $p$. Since $p > \frac{6}{\beta \gamma}$ and

$$|t - t'| \land |t - t'|^\eta + |H - H'|^\beta + |K - K'|^\beta \leq (2 + 1 \lor |t - t'|^\eta),$$

we deduce that

$$|V_t^{H,K} - V_{t'}^{H',K'}| \leq C_1 (|t - t'| \land |t - t'|^\eta + |H - H'|^\beta + |K - K'|^\beta)^{1 - \frac{p}{2}} \leq (2 + 1 \lor |t - t'|^\eta)^{1 - \frac{p}{2}}.$$
Thus there exists $C$ with a finite moment of order $p$ such that
\[
|V_t^{H,K} - V_t^{H',K'}| \leq C(1 + |t - t'|^\eta(\rho + \gamma)) \left(|t - t'|^{\gamma} + |H - H'|^{\beta} + |K - K'|^{\beta}\right)^{1 - \varepsilon}.
\]

Now choosing $\eta = \frac{2}{p}$ (that satisfies the constraint $\eta \in \left(\frac{1}{p}, \beta\right)$), we get $\eta(1 - \frac{\eta}{p}) \leq \frac{2}{p} < \frac{\beta}{3}$. Hence $(1 + |t - t'|^{\eta(1 - \rho)}) \leq C(1 + |t - t'|^{\frac{2}{p}})$ and we get the result.

Using the previous proposition, we deduce the main result of this section.

**Theorem 4.5.** Let $\mathcal{H}$ be a compact subset of $(0, \frac{3}{2})$. For each $H \in \mathcal{H}$, let $Y^H$ be the solution of (4.1) with the drift $b$ satisfying (4.2). Let $\beta \in (0, \frac{7 - 8 \max(H)}{8(1 - \max(H))} \wedge \frac{2}{3})$ and $p \geq 1$. There exists a random variable $C$ with a finite moment of order $p$ such that almost surely, for all $t \geq 0$ and $H, K \in \mathcal{H},$
\[
\frac{1}{t + 1} \int_0^{t + 1} |Y_s^H - Y_s^K|^2 ds \leq C |H - K|^\beta.
\]

**Proof.** Integrating Inequality (4.10) over $[0, t + 1]$, we obtain
\[
\frac{1}{t + 1} \int_0^{t + 1} |Y_s^H - Y_s^K|^2 ds \leq \frac{C}{t + 1} \left(\left|U_0^H - U_0^K\right|^2 \right) \int_0^{t + 1} e^{-2s} ds + \int_0^{t + 1} \left|U_s^H - U_s^K\right|^2 ds + \int_0^{t + 1} \left|U_s^H - U_s^K\right|^2 dr ds
\]
\[
\leq \frac{C}{t + 1} \left(\left|U_0^H - U_0^K\right|^2 + \int_0^{t + 1} \left|U_s^H - U_s^K\right|^2 ds\right).
\]

The regularity of the first term in the previous inequality is given by Proposition 4.2 for $t = t' = 0$, and the regularity of the second one is given by Proposition 4.4 with $H' = K = K'$ and $t' = t$.

**Remark 4.6.** The restriction on the upper bound of $\mathcal{H}$ to $\frac{3}{2}$ and on the maximal Hölder regularity below $\frac{7 - 8 \max(H)}{8(1 - \max(H))} \wedge \frac{2}{3}$ comes from Lemma A.7. In this lemma, we developed the power 6 of $V_t^{H,K}$ into 6 multiple integrals which we could simplify into 3 multiple integrals using Gaussian calculus. This leads to a sharp bound on $\mathbb{E}|V_t^{H,K}|^6$. Although this method could theoretically be used to compute higher moments, it seems hardly tractable as more iterated integrals will appear. However we believe that this approach would allow to let $\mathcal{H}$ be any compact of $(0, 1)$ and $\beta$ any real in $(0, 1)$.

## 5 Regularity of discrete-time fractional SDEs

Let $\gamma \in (0, 1)$ and consider the $\mathbb{R}^d$-valued discrete Stochastic Differential Equation:
\[
\forall t \geq 0, \quad M_t^H = M_0 + \int_0^t b(M_{s_{\gamma}}) ds + B_t^H,
\]
where $b$ is a contracting drift which satisfies (4.2) and where $s_{\gamma} := \gamma \lfloor \frac{t}{\gamma} \rfloor$ is the leftmost approximation of $s$ in the discretisation $\{k\gamma, k \in \mathbb{N}\}$. In this section, we present a result similar to Theorem 4.5 for the process $M^H$. To that end, we first compare the process $M^H$ with the discrete Ornstein-Uhlenbeck process defined as
\[
\forall t \geq 0, \quad M_t^{0,H} = M_0 - \int_0^t M_{s_{\gamma}} ds + B_t^H.
\]
For $H, K \in \mathcal{H}$, we define the following processes
\[
\mathcal{M}_t = M_t^H - M_t^K, \quad \mathcal{M}_t^{0} = M_t^{0,H} - M_t^{0,K} \quad \text{and recall } \mathcal{U} = U^H - U^K, \quad \mathcal{U} = \mathcal{U}^H - \mathcal{U}^K.
\]

For the rest of this section, $C$ will denote a constant that depends only on $p, \kappa$ and $K$.\[\]
Comparison with the discrete OU process. First, notice that for $k \geq 1$,
\[
|M_{k\gamma} - M_{k\gamma}^0|^2 = |M_{(k-1)\gamma} - M_{(k-1)\gamma}^0|^2 + 2\gamma |M_{(k-1)\gamma}^0 - b(M_{(k-1)\gamma}^K)|^2 + 2\gamma^2 |M_{(k-1)\gamma}^0|^2 + 2\gamma^2 K^2 |M_{(k-1)\gamma}|^2 \
\leq |M_{(k-1)\gamma} - M_{(k-1)\gamma}^0|^2 + 2\gamma^2 |M_{(k-1)\gamma}^0|^2 + 2\gamma^2 K^2 |M_{(k-1)\gamma}|^2 \
+ 2\gamma |M_{(k-1)\gamma} - M_{(k-1)\gamma}^0 - b(M_{(k-1)\gamma}^K)|^2 + b(M_{(k-1)\gamma}^K)|^2 - b(M_{(k-1)\gamma}^K)|^2,
\]
where we used (4.2) and Young’s inequality. In order to treat the last term, we rewrite it as
\[
2\gamma |M_{(k-1)\gamma} - M_{(k-1)\gamma}^0 - b(M_{(k-1)\gamma}^K)|^2 + 2\gamma |M_{(k-1)\gamma} - M_{(k-1)\gamma}^0| |M_{(k-1)\gamma}^0 + 2\gamma |M_{(k-1)\gamma}^0| |M_{(k-1)\gamma}^0| - b(M_{(k-1)\gamma}^K)|^2.
\]
Now we invoke (4.2) and Young’s inequality again to bound the previous quantity by
\[
2\gamma - k |M_{(k-1)\gamma}|^2 + \frac{\varepsilon}{2} |M_{(k-1)\gamma}|^2 + \frac{1}{\varepsilon} |M_{(k-1)\gamma}|^2 + K^2 \frac{\varepsilon}{2} |M_{(k-1)\gamma}|^2,
\]
for some arbitrary $\varepsilon > 0$. Thus
\[
|M_{k\gamma} - M_{k\gamma}^0|^2 \leq |M_{(k-1)\gamma} - M_{(k-1)\gamma}^0|^2 + 2\left(-\kappa + \gamma (K^2 + 1)\right) |M_{(k-1)\gamma}|^2 \\
+ (2 + 2\gamma^2 + \frac{2\gamma}{\varepsilon}) |M_{(k-1)\gamma}|^2.
\]
We take $\varepsilon = \frac{\kappa}{1 + 2\gamma + K^2}$ to get
\[
|M_{k\gamma} - M_{k\gamma}^0|^2 \leq |M_{(k-1)\gamma} - M_{(k-1)\gamma}^0|^2 + (1 - \gamma (K^2 + 1)) |M_{(k-1)\gamma}|^2 + C |M_{(k-1)\gamma}|^2.
\]
Fix $\gamma_0 \in (0, 1)$ such that $\gamma < \gamma_0$, we have $0 < \gamma (K^2 + 1) > 1$. There is
\[
|M_{k\gamma} - M_{k\gamma}^0|^2 \leq \left(1 - \gamma (K^2 + 1)\right) |M_{(k-1)\gamma}|^2 + C |M_{(k-1)\gamma}|^2.
\]
By a direct induction, it comes that
\[
|M_{k\gamma} - M_{k\gamma}^0|^2 \leq C \sum_{j=0}^{k-1} (1 - \gamma (K^2 + 1)) |M_{(k-1)\gamma}|^2.
\]
It follows that for all $N \in \mathbb{N}^*$
\[
\frac{1}{N} \sum_{k=1}^{N} |M_{k\gamma}|^2 \leq \frac{1}{N} \sum_{k=1}^{N} \sum_{j=0}^{k-1} (1 - \gamma (K^2 + 1)) |M_{j\gamma}|^2 + \frac{1}{N} \sum_{k=1}^{N} |M_{k\gamma}|^2.
\]
Summing over $k$ first and recalling that $M_{0\gamma} = M_{0\gamma}$, we get
\[
\frac{1}{N} \sum_{k=0}^{N} |M_{k\gamma}|^2 \leq \frac{1}{N} \sum_{j=0}^{N} \frac{1 - (1 - \gamma (K^2 + 1)) |M_{j\gamma}|^2 + \frac{1}{N} \sum_{k=0}^{N} |M_{k\gamma}|^2}{\gamma (K^2 + 1)}
\]
\[
\leq \frac{1}{N} \sum_{k=0}^{N} |M_{k\gamma}|^2.
\]
Comparison of the discrete OU process and the OU process. Let us compare the two processes assuming they start at the same point.
\[
|M_{k\gamma}^0 - U_{k\gamma}|^2 = \left| (1 - \gamma) (M_{(k-1)\gamma}^0 - U_{(k-1)\gamma}) + \int_{(k-1)\gamma}^{k\gamma} (U_s - U_{(k-1)\gamma}) \, ds \right|^2
\leq (1 - \gamma)^2 |M_{(k-1)\gamma}^0 - U_{(k-1)\gamma}|^2 + \gamma \int_{(k-1)\gamma}^{k\gamma} 2|U_s|^2 \, ds + 2\gamma |U_{(k-1)\gamma}|^2
+ \varepsilon (1 - \gamma)^2 |M_{(k-1)\gamma}^0 - U_{(k-1)\gamma}|^2 + \frac{\gamma}{\varepsilon} \left( \int_{(k-1)\gamma}^{k\gamma} 2|U_s|^2 \, ds + 2\gamma |U_{(k-1)\gamma}|^2 \right),
\]
\[
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where we used Young’s and Jensen’s inequalities. Taking \(\varepsilon = \gamma\), we get
\[
|M_{t-k}^{0} - U_{k,\gamma}|^2 \leq (1 - \gamma)^2 |M_{t-k-1}^{0} - U_{k-1,\gamma}|^2 + C \left( \int_{(k-1)\gamma}^{k\gamma} 2|U_s|^2 ds + 2\gamma |U_{(k-1)\gamma}|^2 \right).
\]

It follows by induction that
\[
|M_{t-k}^{0} - U_{k,\gamma}|^2 \leq C \sum_{j=0}^{k-1} (1 - \gamma)^{k-1-j} \left( \int_{j\gamma}^{(j+1)\gamma} |U_s|^2 ds + \gamma |U_{j,\gamma}|^2 \right).
\]

Summing over \(k\) between 1 and \(N \geq 1\), we have
\[
\sum_{k=1}^{N} |M_{t-k}^{0} - U_{k,\gamma}|^2 \leq C \sum_{k=1}^{N} \sum_{j=0}^{k-1} (1 - \gamma)^{k-1-j} \left( \int_{j\gamma}^{(j+1)\gamma} |U_s|^2 ds + \gamma |U_{j,\gamma}|^2 \right) \leq 2 \sum_{k=0}^{N-1} |U_{k,\gamma}|^2 + 2 \int_{0}^{N\gamma} |U_s|^2 ds.
\]

It follows that
\[
\frac{1}{N} \sum_{k=1}^{N} |M_{t-k}^{0} - U_{k,\gamma}|^2 \leq C \frac{N-1}{N} \sum_{k=0}^{N-1} |U_{k,\gamma}|^2 + \frac{C}{N\gamma} \int_{0}^{N\gamma} |U_s|^2 ds.
\]

(5.3)

Recall that for all \(t \geq 0\) and \(H \in (0, 1)\), we have \(U_t - \overline{U}_t = e^{-t\overline{U}_0}\). Therefore
\[
\frac{1}{N} \sum_{k=0}^{N-1} |U_{k,\gamma}|^2 \leq C \left( \frac{1}{N} \sum_{k=0}^{N-1} |U_{k,\gamma}|^2 + |U_0|^2 \right).
\]

(5.4)

Finally, combining (5.4) with (5.3) and (5.2), we conclude that
\[
\frac{1}{N} \sum_{k=0}^{N} |M_{t,k}^{H} - M_{t,k}^{K}|^2 \leq C \left( \frac{1}{N} \sum_{k=0}^{N} |U_{t,k}^{H} - \overline{U}_{t,k}^{K}|^2 + |U_0^{H} - \overline{U}_0^{K}|^2 \right) + \frac{1}{N\gamma} \int_{0}^{N\gamma} |U_s^{H} - U_s^{K}|^2 ds.
\]

(5.5)

The regularity of the second term in the right hand side is given by Proposition 4.2 and the regularity of the third term is given by Theorem 4.5. To bound the first term, we need a discrete version of Theorem 4.5. We introduce the process
\[
\nu_t^{H,K} = \frac{1}{t + 1} \int_{0}^{t+1} (\overline{U}_{r}^{H} - \overline{U}_{r}^{K})^2 ds - E[(\overline{U}_0^{H} - \overline{U}_0^{K})^2], \quad t \geq 0, H,K \in \mathcal{H}.
\]

(5.6)

This process is continuous in time, \(H\) and \(K\), so the idea is to use again a GRR argument. In Lemma A.12, we prove upper bounds on the increments of \(\nu_t^{H,K}\) that are similar to those proved in Lemma A.10 (for its continuous-time version \(\nu_t^{H,K}\)). With those bounds in mind, the proof of the following Proposition is the same as that of Proposition 4.4.

Recall that \(\gamma_0 = \sup\{\gamma \in (0, 1) : \forall \xi \in (0, \gamma), 0 < \kappa \xi - 2K^2 \xi^2 < 1\}\).

**Proposition 5.1.** Let \(\gamma \in (0, \gamma_0)\) and \(\mathcal{H}\) be a compact subset of \((0, \frac{1}{\gamma})\). Let \(\beta \in (0, \frac{7 - 8 \max(\max(0, H, K), \frac{3}{2})}{8 \max(0, H, K)} \land \frac{3}{2})\), \(\varepsilon \in (0, 1)\) and \(p \geq 1\). There exists a random variable \(C\) with a finite moment of order \(p\) such that almost surely, for all \(t, t' \geq 0\) and all \(H, K, H', K' \in \mathcal{H}\),
\[
|\nu_t^{H,K} - \nu_{t'}^{H',K'}| \leq C (1 + |t' - t|^{\frac{\varepsilon}{2}}) \left( (1 \land |t - t'|^{1-\varepsilon}) + |H - H'|^{\beta(1-\varepsilon)} + |K - K'|^{\beta(1-\varepsilon)} \right).
\]

In view of (5.5) and using Proposition 4.2 and Proposition 5.1 with \(t = t'\) and \(H' = K = K'\), we deduce the following Theorem.
**Theorem 5.2.** Let $\gamma \in (0, \gamma_0)$ and $\mathcal{H}$ be a compact subset of $(0, \frac{\gamma}{2})$. Let $\beta \in (0, \frac{7-\max(\mathcal{H})}{3-\max(\mathcal{H})} \wedge \frac{3}{2})$ and $p \geq 1$. Recall that $M^H$ is defined in (5.1). There exists a random variable $C$ with a finite moment of order $p$ such that almost surely, for all $N \in \mathbb{N}^*$ and $H, K \in \mathcal{H}$,

$$
\frac{1}{N} \sum_{k=1}^{N} |M^H_k - M^K_k|^2 \leq C|H - K|^\beta.
$$

**A Long-time properties of the fractional Ornstein-Uhlenbeck process**

This section gathers results on the fractional Ornstein-Uhlenbeck process defined in (4.3), namely upper bounds on moments of their increments and on moments of their ergodic means.

**Lemma A.1.** Recall that $\overline{U}^H_t = \int^t_\infty e^{-(t-r)} dB^H_r$. Let $p \geq 1$ and $\mathcal{H}$ be a compact subset of $(0, 1)$. There exists a constant $C$ such that for all $H, H'$ in $\mathcal{H}$ and $t', t \geq 0$,

$$
\mathbb{E}|\overline{U}^H_t - \overline{U}^{H'}_{t'}|^p \leq C \left( 1 \wedge |t - t'|^{\min(\mathcal{H})} + |H - H'| \right)^p.
$$

**Proof.** Introducing the pivot term $\overline{U}^H_t$, we have

$$
\mathbb{E}|\overline{U}^H_t - \overline{U}^{H'}_{t'}|^p \leq C \left( \mathbb{E}|\overline{U}^H_t - \overline{U}^{H'}_{t'}|^p + \mathbb{E}|\overline{U}^H_t - \overline{U}^H_{t'}|^p \right)
=: C \left( A_1 + A_2 \right).
$$

(A.1)

Let us start by analyzing $A_1$. Without loss of generality, assume that $t' \geq t$. Since the process $\overline{U}^{H'}$ is solution to (4.3) and is stationary, we have using Jensen’s inequality that

$$
A_1 = \mathbb{E}\left| \int^t_{t'} \overline{U}^{H'}_s \, ds + B^H_{t'} - B^H_t \right|^p \leq C |t' - t|^{p-1} \int^t_{t'} \mathbb{E}|\overline{U}^{H'}_s|^p \, ds + C \mathbb{E}|B^H_{t'} - B^H_t|^p \leq C \left( (t' - t)^p + (t' - t)^{pH'} \right).
$$

So we get $A_1 \leq C |t' - t|^{pH'}$ when $|t' - t| \leq 1$. When $|t' - t| \geq 1$, we use the stationarity to get

$$
A_1 \leq C \mathbb{E}|\overline{U}^H_t|^p + C \mathbb{E}|\overline{U}^{H'}_{t'}|^p \leq C.
$$

Overall we obtain

$$
A_1 \leq C(1 \wedge |t' - t|)^{p \min(\mathcal{H})}. \tag{A.2}
$$

We now consider $A_2$. Recall that

$$
\overline{U}^H_t - \overline{U}^{H'}_{t'} = \int^t_{-\infty} e^{-(t-u)} \, dB^H_u - B^H_{t'}.
$$

Hence by integration-by-parts, in view of Remark 3.3, we get

$$
\overline{U}^H_t - \overline{U}^{H'}_{t'} = B^H_t - B^H_{t'} - \int^t_{-\infty} e^{-(t-u)} (B^H_u - B^H_{t'}) \, du = \int^t_{-\infty} e^{-(t-u)} (B^H_u - B^H_{t'} - B^H_{t} + B^H_{t'}) \, du.
$$
Therefore, using the previous equality and the Gaussian property of $\mathcal{U}_t^H - \mathcal{U}_t^{H'}$, we have

$$
\mathbb{E}|\mathcal{U}_t^H - \mathcal{U}_t^{H'}|^p \leq \left( \mathbb{E}|\mathcal{U}_t^H - \mathcal{U}_t^{H'}|^p \right)^{\frac{1}{p}}
$$

$$
\leq \left( \int_{-\infty}^{t} e^{-(t-u)} \sqrt{\mathbb{E}(B_t^H - B^H_u - B_t^{H'} + B_u^{H'})^2} \, du \right)^p
$$

$$
= \left( \int_{-\infty}^{t} e^{-(t-u)} \sqrt{\mathbb{E}(B_{t-u}^H - B_{t-u}^{H'})^2} \, du \right)^p,
$$

where the last equality comes from (2.10). We can now use Proposition 2.1 to conclude that

$$
A_2 \leq C|H - H'|^p. \tag{A.3}
$$

Hence, using (A.2) and (A.3) in (A.1) gives the result. \qed

**Lemma A.2.** Recall that the process $\mathbb{U}$ is defined in (4.4). Let $p \geq 1$ and $\mathcal{H}$ be a compact subset of $(0,1)$. There exists a constant $C$ such that for all $H, H' \in \mathcal{H}$ and $t' \geq t \geq 0$,

$$
\mathbb{E}|\mathbb{U}_t^H - \mathbb{U}_{t'}^{H'}|^p \leq C(1 + t)^{-p\alpha} \left( 1 \wedge |t - t'|^{\min(\mathcal{H})} + |H - H'| \right)^p.
$$

**Proof.** Introducing the pivot terms $(1 + t)^{-\alpha} \mathbb{U}_t^{H'}$ and $(1 + t')^{-\alpha} \mathbb{U}_{t'}^{H'}$, we have that

$$
\mathbb{E}|\mathbb{U}_t^H - \mathbb{U}_{t'}^{H'}|^p \leq C(1 + t)^{-p\alpha} \left( \mathbb{E}|\mathbb{U}_t^H - \mathbb{U}_t^{H'}|^p + \mathbb{E}|\mathbb{U}_t^{H'} - \mathbb{U}_{t'}^{H'}|^p \right)
$$

$$
+ (1 + t)^{-\alpha} - (1 + t')^{-\alpha})^p \mathbb{E}|\mathbb{U}_t^{H'}|^p.
$$

Using the inequality

$$
((1 + t)^{-\alpha} - (1 + t')^{-\alpha})^p \leq (1 + t)^{-p\alpha} (1 \wedge (t' - t)^p),
$$

the stationarity of $\mathbb{U}^{H'}$ and Lemma A.1, we get the desired result. \qed

**Lemma A.3.** Let $\mathcal{H}$ be a compact subset of $(0,1)$ and $B$ be a 1-dimensional fBm. Then there exists a constant $C$ such that for any $H, K \in \mathcal{H}$ and any $-\infty \leq a \leq 0 \leq c \leq d < \infty$,

$$
\left| \mathbb{E} \int_a^0 e^u \, dB_u^H \int_c^d e^v \, dB_v^K + \mathbb{E} \int_a^0 e^u \, dB_u^K \int_c^d e^v \, dB_v^H \right| \leq C \int_a^0 e^u \left( \int_c^d e^v (v - u)^{H+K-2} \, dv \right) \, du.
$$

**Proof.** Assume first that $c = 0$. By integration-by-parts, we get

$$
\mathbb{E} \int_a^0 e^u \, dB_u^H \int_0^d e^v \, dB_v^K = \mathbb{E} \left[ \left( - e^a B_a^H - \int_a^0 e^u B_u^H \, du \right) \left( e^d B_d^K - \int_0^d e^v B_v^K \, dv \right) \right]
$$

$$
= -e^{a+d} \mathbb{E} B_a^H B_d^K + e^a \int_0^d e^v \mathbb{E} B_a^H B_v^K \, dv - e^d \int_a^0 e^u \mathbb{E} B_u^H B_d^K \, du
$$

$$
+ \int_a^0 e^u \int_0^d e^v \mathbb{E} B_u^H B_v^K \, dv \, du.
$$

Integrating-by-parts with respect to $u$ yields

$$
\mathbb{E} \int_a^0 e^u \, dB_u^H \int_0^d e^v \, dB_v^K = e^d \int_a^0 e^u \partial_u \mathbb{E} B_u^H B_d^K \, du - \int_a^0 e^u \int_0^d e^v \partial_u \mathbb{E} B_u^H B_v^K \, dv \, du.
$$
After another integration-by-parts with respect to $v$, it now comes
\[ \mathbb{E} \int_a^0 e^u \, dB_u^H \int_0^d e^v \, dB_v^K = \int_a^0 e^u \int_0^d e^v \, \partial^2_{uv} \mathbb{E} B_u^H B_v^K \, dv \, du. \]

Hence one gets that
\[ \mathbb{E} \left[ \int_a^0 e^u \, dB_u^H \int_0^d e^v \, dB_v^K + \int_a^0 e^u \, dB_u^K \int_0^d e^v \, dB_v^H \right] = \int_a^0 \int_0^d e^{u+v} \, \partial^2_{uv} \mathbb{E} (B_u^H B_v^K + B_u^K B_v^H) \, dv \, du. \]

(A.4)

Now observe that
\[ \mathbb{E} (B_u^H B_v^K + B_u^K B_v^H) = -\mathbb{E} (B_u^H - B_u^K)(B_v^K - B_v^H) + \mathbb{E} B_u^H B_v^K + \mathbb{E} B_u^K B_v^H. \]

In view of the increment stationarity of $B$ (see (2.10)) and its integral representation (1.1), there is
\[ \mathbb{E}[(B_u^H - B_u^K)(B_v^K - B_v^H)] = \mathbb{E} B_{v-u}^K B_{v-u}^K = \frac{1}{\Gamma(H + \frac{1}{2}) \Gamma(K + \frac{1}{2})} \int_\mathbb{R} (v-u-s)^{H-\frac{1}{2}} (-s)^{K-\frac{1}{2}} (v-u-s)^{H-\frac{1}{2}} (-s)^{K-\frac{1}{2}} \, ds = (v-u)^{H+K} \mathbb{E} B_1^H B_1^K, \]

using the change of variables $\tilde{s} = \frac{v-u}{v-u}$ in the last equality. Since $\partial^2_{uv} \mathbb{E} B_1^H B_1^K = 0$, it follows that
\[ |\partial^2_{uv} \mathbb{E} (B_u^H B_v^K + B_u^K B_v^H)| \leq C (v-u)^{H+K-2}. \]

(A.5)

Plugging the previous inequality in (A.4) gives the result for $c = 0$.

For $c > 0$, we write
\[ \mathbb{E} \int_a^0 e^u \, dB_u^H \int_0^c e^v \, dB_v^K = \mathbb{E} \int_a^0 e^u \, dB_u^H \int_0^d e^v \, dB_v^K - \mathbb{E} \int_a^0 e^u \, dB_u^H \int_0^c e^v \, dB_v^K. \]

And we use (A.4) to get
\[ \mathbb{E} \left[ \int_a^0 e^u \, dB_u^H \int_0^d e^v \, dB_v^K + \int_a^0 e^u \, dB_u^K \int_0^d e^v \, dB_v^H \right] = \int_a^0 \int_0^d e^{u+v} \, \partial^2_{uv} \mathbb{E} (B_u^H B_v^K + B_u^K B_v^H) \, dv \, du - \int_a^0 \int_0^c e^{u+v} \, \partial^2_{uv} \mathbb{E} (B_u^H B_v^K + B_u^K B_v^H) \, dv \, du = \int_a^0 \int_0^d e^{u+v} \, \partial^2_{uv} \mathbb{E} (B_u^H B_v^K + B_u^K B_v^H) \, dv \, du. \]

Using again (A.5) in the previous equality yields the result. \qed

This following lemma is a generalisation of Theorem 2.3 in [2], in the sense that it allows for different Hurst parameters.

**Lemma A.4.** Let $\mathcal{H}$ be a compact subset of $(0,1)$. For any $H, K \in (0,1)$, the process $\mathbb{U}^H, \mathbb{U}^K$ is stationary. Besides, there exists a constant $C$ such that for any $H, K \in \mathcal{H}$, any $t \geq 0$ and $s \geq 0$,
\[ |\mathbb{E}((\mathbb{U}_t^H, \mathbb{U}_t^K) + (\mathbb{U}_{t+s}^H, \mathbb{U}_{t+s}^K))| = |\mathbb{E}((\mathbb{U}_0^H, \mathbb{U}_s^K) + (\mathbb{U}_s^H, \mathbb{U}_0^K))| \leq C \left(1 + s^{2 \max(\mathcal{H})-2}\right). \]

**Remark A.5.** Recall that the entries of the vector-valued process $B$ are independent. Hence it suffices to prove the previous result component by component. To avoid heavy notations, we proceed to the following proof in dimension $d = 1$. 

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Proof. First, we have by integration-by-parts that
\[ \mathbb{E}(U_t^H U_{t+s}^K) = \mathbb{E} \int_t^t e^{-(t-u)} dB_u^H \int_t^{t+s} e^{-(t+s-v)} dB_v^K \]
\[ = \int_{-\infty}^t e^{-(t-u)}(B_t^H - B_u^H) \, du \int_{-\infty}^{t+s} e^{-(t+s-v)}(B_{t+s}^K - B_v^K) \, dv. \]

Thus in view of (2.10),
\[ \mathbb{E}(U_t^H U_{t+s}^K) = \mathbb{E} \int_{-\infty}^t e^{-(t-u)} B_t^H \, du \int_{-\infty}^{t+s} e^{-(t+s-v)} B_{t+s}^K \, dv. \]

By the changes of variables \( \bar{u} = u - t \) and \( \bar{v} = v - t \), it follows that
\[ \mathbb{E}(U_t^H U_{t+s}^K) = \mathbb{E} \int_{-\infty}^0 e^{\bar{u}} B_{\bar{u}}^H \, d\bar{u} \int_{-\infty}^{s} e^{-(s-\bar{v})} B_{s-\bar{v}}^K \, d\bar{v}. \]

Hence by integration-by-parts again, we find
\[ \mathbb{E}(U_t^H U_{t+s}^K) = \mathbb{E} \int_{-\infty}^0 e^u dB_u^H \int_{-\infty}^{s} e^{-(s-v)} dB_v^K = \mathbb{E} U_0^H \tilde{U}_s^K, \]
which proves the claim that \( (U_t^H, \tilde{U}_s^K) \) is stationary since \( \tilde{U} \) is Gaussian.

We now bound \( \mathbb{E}(U_0^H \tilde{U}_s^H + U_0^H \tilde{U}_0^K) \); when \( s \in [0,1] \), this quantity is bounded. We now assume that \( s \geq 1 \) and use the following decomposition:
\[ \mathbb{E}(U_0^H \tilde{U}_s^K) = \mathbb{E} \int_{-\infty}^0 e^u dB_u^H \int_{-\infty}^{s} e^{-(s-v)} dB_v^K \]
\[ = e^{-s} \mathbb{E} \int_{-\infty}^0 e^u dB_u^H \int_{1}^{1} e^v dB_v^K + e^{-s} \mathbb{E} \int_{-\infty}^0 e^u dB_u^H \int_{1}^{s} e^v dB_v^K, \]
and similarly for \( \mathbb{E}(U_0^H \tilde{U}_0^K) \). The quantities \( |\mathbb{E} \int_{-\infty}^0 e^u dB_u^H \int_{1}^{1} e^v dB_v^K| = \| \mathbb{E} U_0^H \tilde{U}_1^K \| \) and \( |\mathbb{E} U_0^H \tilde{U}_0^K| \) do not depend on \( s \) and are bounded uniformly in \( H, K \in \mathcal{H} \). Thus by the previous remark and Lemma A.3 applied to \( a = -\infty \), \( c = 1 \) and \( d = s \), it comes
\[ |\mathbb{E}(U_0^H \tilde{U}_s^K + U_0^H \tilde{U}_0^K)| \leq C e^{-s} + C e^{-s} \int_{-\infty}^0 e^u \int_1^s e^v (v-u)^{H+K-2} dv \, du. \tag{A.6} \]

Use the changes of variables \( y = v - u \), \( z = -u \) and Fubini's theorem to get
\[ \int_{-\infty}^0 e^u \int_1^s e^v (v-u)^{H+K-2} dv \, du = \int_0^\infty e^{-2z} \int_{1+z}^{s+z} e^y y^{H+K-2} dy \, dz \]
\[ = \int_1^\infty e^y y^{H+K-2} \int_{(y-s)\vee 0}^{y-1} e^{-2z} dz \, dy \]
\[ = \frac{1}{2} \int_1^\infty e^y y^{H+K-2} (e^{-2((y-s)\vee 0)} - e^{-2(y-1)}) \, dy. \]

We now split the previous integral in three:
\[ \int_{-\infty}^0 e^u \int_1^s e^v (v-u)^{H+K-2} dv \, du = \frac{1}{2} \int_1^{s/2} e^y y^{H+K-2} (1 - e^{-2(y-1)}) \, dy \]
\[ + \frac{1}{2} \int_{s/2}^s e^y y^{H+K-2} (1 - e^{-2(y-1)}) \, dy + \frac{1}{2} \int_s^\infty e^{-y} y^{H+K-2} (e^{2s} - e^2) \, dy \]
\[ \leq \frac{1}{2} s^{s/2} + \frac{1}{2} (s^2)^{H+K-2} e^s + \frac{1}{2} s^{H+K-2} e^s. \]

Using the previous inequality in (A.6) and the inequality \( e^{-s/2} + e^{-s} \leq C s^{2\max(H) - 2} \) for \( s \geq 1 \) gives the desired result. \( \square \)
Lemma A.6. Let \( n \geq 2 \) and \( U = (U_1, \ldots, U_n) \) be a Gaussian vector with mean zero and \( \mathbb{E} U_i^2 = \mathbb{E} U_j^2 \) for any \( i, j \in \{1, \ldots, n\} \). Let \( P_n \) denote the set of partitions of \( \{1, 1, 2, 2, \ldots, n, n\} \) into distinct ordered pairs \( (i, j) \) (i.e., \( i < j \)) and \( \bar{P}_n \) denote the set of partitions of \( \{1, 2, 3, 3, \ldots, n, n\} \) into distinct ordered pairs. Then

\[
\mathbb{E} \left[ \prod_{i=1}^{n} (U_i^2 - \mathbb{E} U_i^2) \right] = \sum_{p \in \bar{P}_n} \alpha_{p,n} \prod_{\{i,j\} \in p} \mathbb{E}[U_i U_j] \tag{A.7}
\]

and

\[
\mathbb{E} \left[ U_1 U_2 \prod_{i=3}^{n} (U_i^2 - \mathbb{E} U_i^2) \right] = \sum_{p \in \bar{P}_n} \beta_{p,n} \prod_{\{i,j\} \in p} \mathbb{E}[U_i U_j], \tag{A.8}
\]

where \( \{\alpha_{p,n}\}_{p \in P_n} \) and \( \{\beta_{p,n}\}_{p \in \bar{P}_n} \) are constants independent of the law of \( U \).

Proof. The proof is by induction. We first handle the case \( n = 2 \), for which (A.8) trivially holds. As for (A.7),

\[
\mathbb{E} \left[ (U_1^2 - \mathbb{E} U_1^2)(U_2 - \mathbb{E} U_2^2) \right] = \mathbb{E}[U_1^2 U_2^2] - \mathbb{E}[U_1^2] \mathbb{E}[U_2^2].
\]

By the Feynman diagram formula [14, Theorem 1.28], we have

\[
\mathbb{E}[U_1^2 U_2^2] = 2 \mathbb{E}[U_1 U_2]^2 + \mathbb{E}[U_1^2] \mathbb{E}[U_2^2],
\]

and therefore \( \mathbb{E}[(U_1^2 - \mathbb{E} U_1^2)(U_2 - \mathbb{E} U_2^2)] = 2 \mathbb{E}[U_1 U_2]^2 \). Since \( P_2 = \{(1,2), (1,2)\} \), (A.7) holds.

Now let \( n \geq 2 \) and assume that (A.7) and (A.8) hold for any \( m \leq n \) and any centred Gaussian vector of size \( m \) such that \( \mathbb{E} U_i^2 = \mathbb{E} U_j^2 \) for any \( i, j \in \{1, \ldots, m\} \).

First, let us prove that (A.8) holds at rank \( n + 1 \). Using the Gaussian integration-by-parts formula (see e.g. [24, Lemma 2.1], applied to the function \( G(u_2, \ldots, u_{n+1}) := u_2 \prod_{i=3}^{n+1}(u_i^2 - \mathbb{E} U_i^2) \)), we get that

\[
\mathbb{E} \left[ U_1 U_2 \prod_{i=3}^{n+1} (U_i^2 - \mathbb{E} U_i^2) \right] = 2 \sum_{m=3}^{n+1} \mathbb{E}[U_1 U_m] \mathbb{E} \left[ U_2 U_m \prod_{k=3}^{n+1, k \neq m} (U_k^2 - \mathbb{E} U_k^2) \right] + \mathbb{E}[U_1 U_2] \mathbb{E} \left[ \prod_{k=3}^{n+1} (U_k^2 - \mathbb{E} U_k^2) \right]. \tag{A.9}
\]

For each \( m \in \{3, \ldots, n+1\} \), apply (A.8) at rank \( n \) to get that

\[
\mathbb{E} \left[ U_2 U_m \prod_{k=3}^{n+1, k \neq m} (U_k^2 - \mathbb{E} U_k^2) \right] = \sum_{p \in \bar{P}_n} \beta_{p,n} \prod_{\{i,j\} \in p} \mathbb{E}[\tilde{U}_i \tilde{U}_j],
\]

where \( \tilde{U}_1 = U_2, \tilde{U}_2 = U_m, \tilde{U}_i = U_i \) for \( 3 \leq i < m \) and \( \tilde{U}_i = U_{i+1} \) for \( m < i \leq n \). Moreover, applying (A.7) on the term \( \mathbb{E} \left[ \prod_{k=3}^{n+1} (U_k^2 - \mathbb{E} U_k^2) \right] \) permits to conclude that the following equality holds:

\[
\mathbb{E} \left[ U_1 U_2 \prod_{i=3}^{n+1} (U_i^2 - \mathbb{E} U_i^2) \right] = \sum_{p \in \bar{P}_{n+1}} \beta_{p,n+1} \prod_{\{i,j\} \in p} \mathbb{E}[U_i U_j], \tag{A.10}
\]

for some constants \( \{\beta_{p,n+1}\}_{p \in \bar{P}_{n+1}} \). Note that the sum is indeed over \( \bar{P}_{n+1} \), since in (A.9) the term \( \mathbb{E}[U_1 U_2] \) appears only once in \( \mathbb{E}[U_1 U_2] \mathbb{E} \left[ \prod_{k=3}^{n+1} (U_k^2 - \mathbb{E} U_k^2) \right] \) and covariances involving \( U_1 \) and \( U_2 \) also appear only once in \( \mathbb{E}[U_1 U_m] \mathbb{E}[U_2 U_m] \prod_{k=3}^{n+1, k \neq m} (U_k^2 - \mathbb{E} U_k^2) \).
Finally, let us prove that (A.7) holds at rank $n + 1$. Observe that

$$
\mathbb{E}\left[\prod_{i=1}^{n+1}(U_i^2 - \mathbb{E}U_i^2)\right] = \mathbb{E}\left[U_1 U_2 \prod_{i=2}^{n+1}(U_i^2 - \mathbb{E}U_i^2)\right] - \mathbb{E}[U_1^2] \mathbb{E}\left[\prod_{i=2}^{n+1}(U_i^2 - \mathbb{E}U_i^2)\right],
$$

then use again the Gaussian integration-by-parts ([24, Lemma 2.1] with the function $G(u_1, \ldots, u_{n+1}) = u_1 \prod_{i=2}^{n+1}(u_i^2 - \mathbb{E}U_i^2)$) to get that

$$
\mathbb{E}\left[U_1^2 \prod_{i=2}^{n+1}(U_i^2 - \mathbb{E}U_i^2)\right] = 2 \sum_{m=2}^{n} \mathbb{E}[U_1 U_m] \mathbb{E}\left[U_1 U_m \prod_{k=2, k \neq m}^{n+1}(U_i^2 - \mathbb{E}U_i^2)\right] + \mathbb{E}[U_1^2] \mathbb{E}\left[\prod_{i=2}^{n+1}(U_i^2 - \mathbb{E}U_i^2)\right].
$$

Thus we have

$$
\mathbb{E}\left[\prod_{i=1}^{n+1}(U_i^2 - \mathbb{E}U_i^2)\right] = 2 \sum_{m=2}^{n} \mathbb{E}[U_1 U_m] \mathbb{E}\left[U_1 U_m \prod_{k=2, k \neq m}^{n+1}(U_i^2 - \mathbb{E}U_i^2)\right].
$$

Apply now (A.10) on each term $\mathbb{E}[U_1 U_m \prod_{k=2, k \neq m}^{n+1}(U_i^2 - \mathbb{E}U_i^2)]$ to conclude that

$$
\mathbb{E}\left[\prod_{i=1}^{n+1}(U_i^2 - \mathbb{E}U_i^2)\right] = \sum_{p \in P_{n+1}} \alpha_{p,n+1} \prod_{(i,j) \in p} \mathbb{E}[U_i U_j],
$$

for some constants $\{\alpha_{p,n+1}\}_{p \in P_{n+1}}$. □

Recall that the process $\{V_t^{H,K}, t \geq 0, (H, K) \in (0,1)^2\}$ was defined in (4.11). In view of the stationarity of $\mathcal{U}$ (see Lemma A.4), we have

$$
V_t^{H,K} = \frac{1}{t+1} \int_0^{t+1} \left(|\mathcal{U}_s^H - \mathcal{U}_s^K|^2 - \mathbb{E}|\mathcal{U}_s^H - \mathcal{U}_s^K|^2\right) ds.
$$

**Lemma A.7.** Let $\mathcal{H}$ be a compact subset of $(0,1)$. There exists a constant $C$ such that for any $t \geq 0$ and any $H, K \in \mathcal{H}$,

$$
\mathbb{E}|V_t^{H,K}|^6 \leq \frac{C}{(t+1)^6} (I_1 + I_2),
$$

where

$$
I_1 := \left(\int_{[0,t+1]^6} \mathbb{E}\left[|\mathcal{U}_s^H - \mathcal{U}_s^K|^2 - \mathbb{E}|\mathcal{U}_s^H - \mathcal{U}_s^K|^2\right] ds dv\right)^3,
$$

$$
I_2 := \left(\int_{[0,t+1]^3} \mathbb{E}\left[|\mathcal{U}_v^H - \mathcal{U}_v^K|^2 - \mathbb{E}|\mathcal{U}_v^H - \mathcal{U}_v^K|^2\right]|\mathcal{U}_v^H - \mathcal{U}_v^K| dr dv\right)^2.
$$

**Proof.** As suggested in Remark A.5, it is enough to write the proof in dimension $d = 1$.

Let $t \geq 0$ and $H, K \in \mathcal{H}$. By Fubini’s theorem, we have

$$
\mathbb{E}|V_t^{H,K}|^6 = \frac{1}{(t+1)^6} \int_{[0,t+1]^6} \mathbb{E}\left[\prod_{i=1}^{6}\left(|\mathcal{U}_{s_i}^H - \mathcal{U}_{s_i}^K|^2 - \mathbb{E}|\mathcal{U}_{s_i}^H - \mathcal{U}_{s_i}^K|^2\right)\right] ds_1 \ldots ds_6.
$$

Apply Lemma A.6 with $U_i = \mathcal{U}_{s_i}^H - \mathcal{U}_{s_i}^K$ and set $C = \max_{p \in P_6} \alpha_{p,6}$ to get

$$
\mathbb{E}|V_t^{H,K}|^6 \leq C \sum_{p \in P_6} \frac{1}{(t+1)^6} \int_{[0,t+1]^6} \prod_{(i,j) \in p} \mathbb{E}|(\mathcal{U}_{s_i}^H - \mathcal{U}_{s_i}^K)(\mathcal{U}_{s_j}^H - \mathcal{U}_{s_j}^K)| ds_1 \ldots ds_6. \quad (A.11)
$$
Note that any $p$ in $P_6$ can be seen as a derangement (i.e., a permutation without fixed point) of \{1, \ldots, 6\} where each pair represents a number and its image. There are three types of derangements of \{1, \ldots, 6\}. The first type is a composition of three transpositions. For such $p$, there are transpositions $\tau_1$, $\tau_2$ and $\tau_3$ such that $p = \tau_1 \tau_2 \tau_3$, and by a slight abuse of notations, we denote by $(\tau_1, \tau_2^2)$ each pair of $\{1, \ldots, 6\}$ which is transposed by $p$. This yields

$$
\prod_{(i,j) \in p} E\left[ (\mathcal{U}_{s_i}^H - \mathcal{U}_{s_j}^K)(\mathcal{U}_{s_i}^H - \mathcal{U}_{s_j}^K) \right] = \prod_{i=1}^6 E\left[ (\mathcal{U}_{s_i}^H - \mathcal{U}_{s_i}^K)(\mathcal{U}_{s_{p(i)}}^H - \mathcal{U}_{s_{p(i)}}^K) \right] = \prod_{i=1}^3 E\left[ (\mathcal{U}_{s_{\tau_i}}^H - \mathcal{U}_{s_{\tau_i}}^K)(\mathcal{U}_{s_{\tau_i}^2}^H - \mathcal{U}_{s_{\tau_i}^2}^K) \right]^2.
$$

Thus for such $p$, the integral in (A.11) satisfies

$$
\frac{1}{(t+1)^6} \int_{[0,t+1]^6} \prod_{(i,j) \in p} E\left[ (\mathcal{U}_{s_i}^H - \mathcal{U}_{s_i}^K)(\mathcal{U}_{s_j}^H - \mathcal{U}_{s_j}^K) \right] ds_1 \ldots ds_6 = \left( \int_{[0,t+1]^2} E\left[ (\mathcal{U}_{s_i}^H - \mathcal{U}_{s_i}^K)(\mathcal{U}_{s_i}^H - \mathcal{U}_{s_i}^K) \right] ds \right)^3 = \mathcal{I}_1. \quad (A.12)
$$

The second type of derangement is when $p$ is composed of a 4-cycle $c$ and a transposition $\tau$. By a slight abuse of notations, we denote by $(c, c^2, c^3, c^4)$ the quadruple of \{1, \ldots, 6\} on which $p$ acts as a 4-cycle and $(\tau, \tau^2)$ the pair of \{1, \ldots, 6\} which is transposed by $p$. This yields

$$
\prod_{(i,j) \in p} E\left[ (\mathcal{U}_{s_i}^H - \mathcal{U}_{s_j}^K)(\mathcal{U}_{s_j}^H - \mathcal{U}_{s_j}^K) \right] = E\left[ (\mathcal{U}_{s_5}^H - \mathcal{U}_{s_5}^K)(\mathcal{U}_{s_6}^H - \mathcal{U}_{s_6}^K) \right] \prod_{i=1}^4 E\left[ (\mathcal{U}_{s_{ci}}^H - \mathcal{U}_{s_{ci}}^K)(\mathcal{U}_{s_{ci+1}}^H - \mathcal{U}_{s_{ci+1}}^K) \right],
$$

where $c^5$ is used to denote $c^4$. Then by Young's inequality,

$$
\prod_{(i,j) \in p} E\left[ (\mathcal{U}_{s_i}^H - \mathcal{U}_{s_i}^K)(\mathcal{U}_{s_j}^H - \mathcal{U}_{s_j}^K) \right] \leq \frac{1}{2} E\left[ (\mathcal{U}_{s_1}^H - \mathcal{U}_{s_1}^K)(\mathcal{U}_{s_2}^H - \mathcal{U}_{s_2}^K) \right] E\left[ (\mathcal{U}_{s_1}^H - \mathcal{U}_{s_1}^K)(\mathcal{U}_{s_3}^H - \mathcal{U}_{s_3}^K) \right] E\left[ (\mathcal{U}_{s_3}^H - \mathcal{U}_{s_3}^K)(\mathcal{U}_{s_4}^H - \mathcal{U}_{s_4}^K) \right] + \frac{1}{2} E\left[ (\mathcal{U}_{s_5}^H - \mathcal{U}_{s_5}^K)(\mathcal{U}_{s_6}^H - \mathcal{U}_{s_6}^K) \right] E\left[ (\mathcal{U}_{s_5}^H - \mathcal{U}_{s_5}^K)(\mathcal{U}_{s_6}^H - \mathcal{U}_{s_6}^K) \right] E\left[ (\mathcal{U}_{s_6}^H - \mathcal{U}_{s_6}^K)(\mathcal{U}_{s_5}^H - \mathcal{U}_{s_5}^K) \right].
$$

In each term of the above sum, no index appears twice, thus as in (A.12) we get that for such $p$,

$$
\int_{[0,t+1]^6} \prod_{(i,j) \in p} E\left[ (\mathcal{U}_{s_i}^H - \mathcal{U}_{s_i}^K)(\mathcal{U}_{s_j}^H - \mathcal{U}_{s_j}^K) \right] ds_1 \ldots ds_6 \leq \mathcal{I}_1. \quad (A.13)
$$

The last type of derangement $p$ is the composition of two 3-cycles. In this case, we get

$$
\int_{[0,t+1]^6} \prod_{(i,j) \in p} E\left[ (\mathcal{U}_{s_i}^H - \mathcal{U}_{s_i}^K)(\mathcal{U}_{s_j}^H - \mathcal{U}_{s_j}^K) \right] ds_1 \ldots ds_6 \leq \left( \int_{[0,t+1]^3} E\left[ (\mathcal{U}_{s_i}^H - \mathcal{U}_{s_i}^K)(\mathcal{U}_{s_i}^H - \mathcal{U}_{s_i}^K) \right] ds \right)^2 = \mathcal{I}_2. \quad (A.14)
$$

Thus plugging (A.12), (A.13) and (A.14) in (A.11) gives the result. \hfill \square

**Lemma A.8.** Let $\mathcal{H}$ be a compact subset of $(0,1)$, $p \geq 6$ and $v \in (0,1)$. There exists a constant $C$ such that for any $t \geq 0$ and any $H, K \in \mathcal{H}$,

$$
\mathbb{E}[V_t^{H,K}] \leq C \left( (t+1)^{-3(1-v)}(\log(t+1))^3 + 1 \right) + (t+1)^{8\max(\mathcal{H})-8(1-v)}.
$$
Proof. Let us first bound the term $I_1$ from Lemma A.7. Using Lemma A.4, we have
\[
|E(U^H_s - U^K_s, U^H_v - U^K_v)| \leq |E(U^H_s, U^H_v)| + |E(U^K_s, U^K_v)| + |E(U^H_s, U^K_v)| + |E(U^K_s, U^H_v)|
\]
\[
\leq C \left( 1 + |s - v|^{2\max(H) - 2} \right).
\]
Hence
\[
I_1 \leq C \left( \int_{[0,t+1]^2} 1 \wedge |s - v|^{4\max(H) - 4} ds \, dv \right)^3.
\]
The change of variables $x = s - v$ yields
\[
I_1 \leq C \left( \int_{-t}^{t+1} \int_{-t}^{t+1} 1 \wedge |x|^{4\max(H) - 4} dx \, dv \right)^3
\]
\[
\leq C (t + 1)^3 \left( \int_{-(t+1)}^{t+1} 1 \wedge |x|^{4\max(H) - 4} dx \right)^3
\]
\[
\leq C (t + 1)^3 \left( 1 + \int_{1}^{t+1} x^{4\max(H) - 4} dx \right)^3
\]
\[
\leq C (t + 1)^3 \left( 1 + \frac{1}{2 \max(H)} = \frac{1}{4} (\log(t+1) + 1) + \frac{3}{4} (t+1)^{4\max(H) - 3}\right)^3.
\]
Let us now provide an upper bound on $I_2$ from Lemma A.7. In view of (A.15), there is
\[
I_2 \leq C \left( \int_{[0,t+1]^3} 1 \wedge |s - v|^{2\max(H) - 2} \left( 1 \wedge |v - r|^{2\max(H) - 2} \right) \left( 1 \wedge |r - s|^{2\max(H) - 2} \right) dr \, ds \, dv \right)^2,
\]
and from the change of variables $x = s - v, y = v - r, z = r$, it comes
\[
I_2 \leq C \left( \int_{[-1,t+1]^3} 1 \wedge |x|^{2\max(H) - 2} \left( 1 \wedge |y|^{2\max(H) - 2} \right) \left( 1 \wedge |x - y|^{2\max(H) - 2} \right) dx \, dy \, dz \right)^2
\]
\[
= C (t + 1)^2 \left( \int_{[-1,t+1]^2} 1 \wedge |x|^{2\max(H) - 2} \left( 1 \wedge |y|^{2\max(H) - 2} \right) \left( 1 \wedge |x - y|^{2\max(H) - 2} \right) dx \, dy \right)^2.
\]
Now by symmetry, the previous integral over the domain $[-t - 1, 0]^2$ is equal to the integral over $[0,t+1]^2$, and the integral over $[-t - 1, 0] \times [0,t+1]$ is equal to the integral over $[0,t+1] \times [-t - 1, 0]$. Hence
\[
I_2 \leq 2C (J_1 + J_2),
\]
where
\[
J_1 := (t + 1)^2 \left( \int_{[0,t+1]^2} 1 \wedge |x|^{2\max(H) - 2} \left( 1 \wedge |y|^{2\max(H) - 2} \right) \left( 1 \wedge |x - y|^{2\max(H) - 2} \right) dx \, dy \right)^2,
\]
\[
J_2 := (t + 1)^2 \left( \int_{[0,t+1] \times [-t - 1, 0]} 1 \wedge |x|^{2\max(H) - 2} \left( 1 \wedge |y|^{2\max(H) - 2} \right) \left( 1 \wedge |x - y|^{2\max(H) - 2} \right) dx \, dy \right)^2.
\]
By a simple change of variable, we have
\[
J_2 = (t + 1)^2 \left( \int_{[0,t+1]^2} 1 \wedge |x|^{2\max(H) - 2} \left( 1 \wedge |y|^{2\max(H) - 2} \right) \left( 1 \wedge |x + y|^{2\max(H) - 2} \right) dx \, dy \right)^2.
\]
Notice that \(1 \wedge |x + y|^{2 \max(H) - 2} \leq 1 \wedge |x - y|^{2 \max(H) - 2}\), and therefore we have \(\mathcal{J}_2 \leq \mathcal{J}_1\). Thus \(\mathcal{I}_2 \leq 4C\mathcal{J}_1\). Now since the integrand in \(\mathcal{J}_1\) is symmetric in \(x\) and \(y\), we have

\[
\mathcal{J}_1 \leq C (t + 1)^2 \left( \int_0^{t+1} \int_0^x (1 \wedge |x|^{2 \max(H) - 2}) \left(1 \wedge |y|^{2 \max(H) - 2}\right) \left(1 \wedge (x - y)^{2 \max(H) - 2}\right) dy \, dx \right)^2
\]

\[
\leq C (t + 1)^2 \left(1 + \frac{t+2}{2} \int_0^x |x|^{2 \max(H) - 2} \left(1 \wedge |y|^{2 \max(H) - 2}\right) \left(1 \wedge (x - y)^{2 \max(H) - 2}\right) dy \, dx \right)^2.
\]

Denote \(K := \int_0^{t+2} \int_0^x |x|^{2 \max(H) - 2} \left(1 \wedge |y|^{2 \max(H) - 2}\right) \left(1 \wedge (x - y)^{2 \max(H) - 2}\right) dy \, dx\) and decompose this integral as \(K = K_1 + K_2 + K_3\), where \(K_1\) is the integral w.r.t \(y\) between 0 and 1, \(K_2\) is the integral w.r.t \(y\) between 1 and \(x - 1\) and \(K_3\) is the integral w.r.t \(y\) between \(x - 1\) and \(x\). There is

\[
K_1 = \int_2^{t+2} \int_0^1 x^{2 \max(H) - 2} (x - y)^{2 \max(H) - 2} dy \, dx
\]

\[
\leq C \left((t + 1)^4 \max(H) - 3 + \log(t + 1) + 1\right),
\]

where the logarithmic term appears when \(\max(H) = \frac{3}{4}\). As for \(K_2\),

\[
K_2 = \int_2^{t+2} \int_1^{x-1} x^{2 \max(H) - 2} y^{2 \max(H) - 2} (x - y)^{2 \max(H) - 2} dy \, dx.
\]

Note that for \(y \in [1, x - 1]\), one has either \(y^{2 \max(H) - 2} \leq \left(\frac{x - 1}{2}\right)^{2 \max(H) - 2}\) when \(y \geq \frac{x - 1}{2}\) or \((x - y)^{2 \max(H) - 2} \leq (x - 1)^{2 \max(H) - 2}\) when \(y \leq \frac{x - 1}{2}\). It follows that

\[
K_2 \leq C \int_2^{t+2} \int_1^{x-1} x^{2 \max(H) - 2} \left(\frac{x - 1}{2}\right)^{2 \max(H) - 2} dy \, dx
\]

\[
\leq C \left((t + 1)^4 \max(H) - 2 + \log(t + 1) + 1\right),
\]

where the logarithmic term appears when \(\max(H) = \frac{1}{2}\). Finally

\[
K_3 = \int_2^{t+2} \int_{x-1}^x x^{2 \max(H) - 2} y^{2 \max(H) - 2} dy \, dx
\]

\[
\leq C \left((t + 1)^4 \max(H) - 3 + \log(t + 1) + 1\right).
\]

Thus, we can bound \(\mathcal{I}_2\) as

\[
\mathcal{I}_2 \leq C \mathcal{J}_1
\]

\[
\leq C (t + 1)^2 \left(1 + K_1 + K_2 + K_3\right)^2
\]

\[
\leq C (t + 1)^2 \left(\log(t + 1) + 1\right)^2 + C (t + 1)^8 \max(H) - 2.\quad (A.17)
\]

Combining (A.16) and (A.17), and in view of Lemma A.7, we conclude that

\[
\mathbb{E}[V^H_K] \leq C \left((t + 1)^{-3} (\log(t + 1)^3 + 1) + (t + 1)^8 \max(H) - 8\right).\quad (A.18)
\]

It remains to compute the higher moments of \(V^H_K\). Let \(q = \frac{1}{1 - \nu}\) and \(r \geq 1\) such that \(\frac{1}{q} + \frac{1}{r} = 1\). By the Hölder inequality,

\[
\mathbb{E}[V^H_K]^p = \mathbb{E}[|V^H_K|^{\frac{p}{q}}] \mathbb{E}[|V^H_K|^{p - \frac{p}{q}}] \leq \left(\mathbb{E}[V^H_K]^q\right)^\frac{p}{q} \left(\mathbb{E}[|V^H_K|^{(p - \frac{p}{q})r}]\right)^{\frac{1}{r}}.
\]

Since \(p \geq 6\), we have \(r(p - \frac{p}{q}) \geq 1\). Therefore, by Jensen’s inequality and the stationarity of \(\mathcal{U}^H - \mathcal{U}^K\) (see Lemma A.4), we get \(\mathbb{E}[|V^H_K|^{(p - \frac{p}{q})r}] \leq C\). Finally, we conclude using (A.18). \(\square\)
Lemma A.10. Let $\mathcal{H}$ be a compact subset of $(0,1)$, $p \geq 6$ and $\nu, \beta \in (0,1)$. There exists a constant $C$ such that for all $H, H', K, K'$ in $\mathcal{H}$ and $t' \geq t \geq 0$,

$$\mathbb{E}|V_{t'}^{H,K} - V_{t'}^{H',K'}|^p \leq C \left( (t+1)^{-3(1-\nu)(1-\beta)}(\log(t+1)^3 + 1) + (t+1)^{8 \max(\mathcal{H}) - 8(1-\nu)(1-\beta)} \right) \times (|H - H'|^{p\beta} + |K - K'|^{p\beta}) + C (t+1)^{-p}|t-t'|^p.$$

Proof. In view of (4.11), we have

$$\mathbb{E}|V_{t'}^{H,K} - V_{t'}^{H',K'}|^p \leq C \mathbb{E}\left( \left| \frac{t+1}{t+1} \int_0^{t+1} |U_s^H - U_s^K|^2 - |U_s^{H'} - U_s^{K'}|^2 |ds \right|^p + C \int_0^{t+1} \mathbb{E}|U_s^H - U_s^K|^2 \mathbb{E}|U_s^{H'} - U_s^{K'}|^2 |ds \right|^p$$

$$\leq C \left( \frac{1}{t+1} \int_0^{t+1} \left( |U_s^H - U_s^{H'}|^2 + |U_s^K + U_s^{K'}|^2 \right)^{\frac{p}{2}} |ds \right)^p + C \left( \frac{1}{t+1} \int_0^{t+1} \left( |U_s^K - U_s^{K'}|^2 \right)^{\frac{p}{2}} |ds \right)^p$$

$$+ C \left( \mathbb{E}|U_s^K - U_s^{K'}|^2 \right)^{\frac{p}{2}} + C \left( \mathbb{E}|U_s^H - U_s^{H'}|^2 \right)^{\frac{p}{2}}.$$

Using Lemma A.1, it comes

$$\mathbb{E}|V_{t'}^{H,K} - V_{t'}^{H',K'}|^p \leq C(|H - H'|^{p\beta} + |K - K'|^{p\beta}).$$

Hence, interpolating between the previous inequality and the result of Lemma A.8, it follows that

$$\mathbb{E}|V_{t'}^{H,K} - V_{t'}^{H',K'}|^p \leq C \left( (t+1)^{-3(1-\nu)(1-\beta)}(\log(t+1)^3 + 1) + (t+1)^{8 \max(\mathcal{H}) - 8(1-\nu)(1-\beta)} \right) \times (|H - H'|^{p\beta} + |K - K'|^{p\beta}).$$

Moreover, we have for $t' \geq t$,

$$\mathbb{E}|V_{t'}^{H,K} - V_{t'}^{H',K'}|^p \leq C \left( \frac{1}{t'+1} \int_{t+1}^{t'+1} |U_s^H - U_s^K|^2 |ds \right)^p + C \left( \frac{t'-t}{(t+1)(t'+1)} \int_0^{t+1} |U_s^H - U_s^K|^2 |ds \right)^p$$

$$\leq C (t'+1)^{-p}|t'-t|^p |t-t'|^{p-1} \int_{t+1}^{t'+1} \mathbb{E}|U_s^H - U_s^K|^2 |ds$$

$$+ |t'-t|^p (t+1)^{-p-1} \int_0^{t+1} \mathbb{E}|U_s^H - U_s^K|^2 |ds,$$

using Jensen’s inequality. Since $\mathcal{U}$ is a stationary process, we conclude that

$$\mathbb{E}|V_{t'}^{H,K} - V_{t'}^{H',K'}|^p \leq C (t+1)^{-p}|t'-t|^p.$$

Combining (A.19) and (A.20), we get the desired result.

Lemma A.11. Let $\gamma \in (0,1)$ and $\mathcal{H}$ be a compact subset of $(0,1)$. Recall the process $V_{t}^{H,K}$ defined in (5.6). Let $p \geq 6$ and $\nu \in (0,1)$. There exists a constant $C$ such that for all $t \geq 0$ and $H, K \in \mathcal{H}$,

$$\mathbb{E}|V_{t}^{H,K}|^p \leq C \left( (t+1)^{-3(1-\nu)}(\log(t+1)^3 + 1) + (t+1)^{8 \max(\mathcal{H}) - 8(1-\nu)} \right).$$
Proof. The proof of Lemma A.8 can be transcribed to a discrete setting and the same computations can be done. First, we claim that Lemma A.7 still holds if the integral is replaced by a discrete sum, as it relies essentially on Lemma A.6 and properties of the integral that are also true for the sum. Hence

$$
\mathbb{E}[V_t^{H,K}]^6 \leq \frac{C}{(t+1)^6}(I_1^* + I_2^*),
$$

(A.21)

where

$$
I_1^* := \left( \int_{[0,t+1]^2} \left( \mathbb{E}(\mathcal{U}_{s_\gamma}^H - \mathcal{U}_{s_\gamma}^K, \mathcal{U}_{v_\gamma}^H - \mathcal{U}_{v_\gamma}^K) \right)^2 \, ds \, dv \right)^3,
$$

$$
I_2^* := \left( \int_{[0,t+1]^3} \mathbb{E}(\mathcal{U}_{s_\gamma}^H - \mathcal{U}_{s_\gamma}^K, \mathcal{U}_{v_\gamma}^H - \mathcal{U}_{v_\gamma}^K) \mathbb{E}(\mathcal{U}_{v_\gamma}^H - \mathcal{U}_{v_\gamma}^K, \mathcal{U}_{r_\gamma}^H - \mathcal{U}_{r_\gamma}^K) \mathbb{E}(\mathcal{U}_{r_\gamma}^H - \mathcal{U}_{r_\gamma}^K, \mathcal{U}_{s_\gamma}^H - \mathcal{U}_{s_\gamma}^K) \, dr \, ds \, dv \right)^2.
$$

To bound $I_1^*$, we proceed as in the proof of Lemma A.8. Using Lemma A.4, we have

$$
\left| \mathbb{E}(\mathcal{U}_{s_\gamma}^H - \mathcal{U}_{s_\gamma}^K, \mathcal{U}_{v_\gamma}^H - \mathcal{U}_{v_\gamma}^K) \right| \leq \left| \mathbb{E}(\mathcal{U}_{s_\gamma}^H, \mathcal{U}_{v_\gamma}^H) \right| + \left| \mathbb{E}(\mathcal{U}_{s_\gamma}^K, \mathcal{U}_{v_\gamma}^K) \right| + \left| \mathbb{E}(\mathcal{U}_{s_\gamma}^H, \mathcal{U}_{v_\gamma}^K) + (\mathcal{U}_{s_\gamma}^K, \mathcal{U}_{v_\gamma}^H) \right|
$$

$$
\leq C \left( 1 \wedge |s_\gamma - v_\gamma|^{2\max(H)-2} \right).
$$

It follows that

$$
I_1^* \leq C \left( \int_{[0,t+1]^2} 1 \wedge |s_\gamma - v_\gamma|^{4\max(H)-4} \, ds \, dv \right)^3 = C \left( \gamma^{-1(t+1)} \gamma \sum_{i,j=0}^{\gamma} 1 \wedge |i\gamma - j\gamma|^{4\max(H)-4} \right)^3
$$

and

$$
I_2^* \leq C \left( \gamma^{\gamma^{-1(t+1)} \gamma} \sum_{i,j,k=0}^{\gamma} (1 \wedge |i\gamma - j\gamma|^{2\max(H)-2})(1 \wedge |j\gamma - k\gamma|^{2\max(H)-2})(1 \wedge |k\gamma - i\gamma|^{2\max(H)-2}) \right)^2.
$$

Let $s \in [i\gamma, (i + 1)\gamma]$, $v \in [j\gamma, (j + 1)\gamma]$, $r \in [k\gamma, (k + 1)\gamma]$, and denote $\rho_1 = 4\max(H) - 4$ and $\rho_2 = 2\max(H) - 2$. If $|j\gamma - i\gamma| \geq 1 > \gamma$, then $|s - v| \leq |i\gamma - j\gamma| + \gamma \leq 2|j\gamma - i\gamma|$. While if $|j\gamma - i\gamma| < 1$ then $|s - v| < 1 + \gamma < 2$. In this case, we have $1 \wedge |i\gamma - j\gamma|^\rho_1 = 1 \wedge 2^{-\rho_1}|s - v|^\rho_1 \leq 2^{-\rho_1} (1 \wedge |s - v|^\rho_1)$. So overall we always have

$$
1 \wedge |s - v|^\rho_1 \geq 2^{\rho_1} (1 \wedge |i\gamma - j\gamma|^\rho_1).
$$

Arguing similarly for $|v - r|$ and $|r - s|$, we have

$$
(1 \wedge |s - v|^\rho_1)(1 \wedge |v - r|^\rho_2)(1 \wedge |r - s|^\rho_2) \geq 2^{\rho_1} (1 \wedge |i\gamma - j\gamma|^\rho_2)(1 \wedge |j\gamma - k\gamma|^\rho_2)(1 \wedge |k\gamma - i\gamma|^\rho_2).
$$

Summing (A.22) over $i$ and $j$, and (A.23) over $i$, $j$ and $k$, we get

$$
I_1^* \leq C \left( \int_{[0,t+1]^2} 1 \wedge |s - v|^{4\max(H)-4} \, ds \, dv \right)^3
$$

and

$$
I_2^* \leq C \left( \int_{[0,t+1]^3} (1 \wedge |s - v|^{2\max(H)-2})(1 \wedge |v - r|^{2\max(H)-2})(1 \wedge |r - s|^{2\max(H)-2}) \, ds \, dr \, dv \right)^2.
$$

We already bounded the right hand sides above in the proof of Lemma A.8 (see $I_1$ and $I_2$). Using those bounds, we have

$$
I_1^* \leq C \left( (t+1)^{3} \log((t+1)\gamma) + 1 \right)^3 + (t+1)_{\gamma}^{12 \max(H)-12}.
$$
and
\[ I_2^* \leq C \left( (t+1)^2 \log((t+1)\gamma) + 1 \right)^2 + (t+1)^{8\max(H) - 8}. \]

Since \( \frac{(t+1)^2}{2} \leq (t+1) \gamma \leq (t+1) \), we get
\[ I_1^* \leq C \left( (t+1)^3 \log(t+1) + (t+1)^{12\max(H) - 12} \right) \tag{A.24} \]
and
\[ I_2^* \leq C \left( (t+1)^2 \log(t+1)^3 + 1 \right) + (t+1)^{8\max(H) - 8}. \tag{A.25} \]

Injecting (A.24) and (A.25) in (A.21), we conclude that
\[ \mathbb{E}|V_t^{H,K}|^6 \leq C \left( (t+1)^{-3} \log(t+1)^3 + 1 + (t+1)^{8\max(H) - 8} \right). \]

Going from a bound on the moment of order 6 to a bound on any moment of order \( p \geq 6 \) can be done exactly as in the proof of Lemma A.8.

**Lemma A.12.** Let \( \gamma \in (0, 1) \) and \( H \) be a compact set of \( (0, 1) \). Let \( p \geq 6 \) and \( \nu, \beta \in (0, 1) \). There exists a constant \( C \) such that for all \( H, H', K, K' \) in \( H \) and \( t' \geq t \geq 0 \),
\[ \mathbb{E}|V_t^{H,K} - V_t^{H',K'}|^p \leq C \left( (t+1)^{-3} (1-\nu)(1-\beta) \log(t+1)^3 + 1 + (t+1)^{8\max(H) - 8} \right) \]
\[ \times \left( |H - H'|^{\nu \beta} + |K - K'|^{\beta} + (1 + t)^{-p}|t - t'|^p \right). \]

**Proof.** This lemma is a discrete version of Lemma A.10 and relies on Lemma A.11. The proof of Lemma A.10 can be completely transcribed to a discrete setting (it suffices to replace the dummy variable \( s \) by \( s_\gamma \) in the integrals).

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