Notes on Fragments of First-Order Concatenation Theory

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Abstract. We identify a number of decidable and undecidable fragments of first-order concatenation theory. We also give a purely universal axiomatization which is complete for the fragments we identify. Furthermore, we prove some normal-form results.

1 Introduction

1.1 The Purpose of These Notes

The purpose of this paper is to give full proofs of results published elsewhere.

1.2 First-order Concatenation theory

First-order concatenation theory can be compared to first-order number theory, e.g., Peano Arithmetic or Robinson Arithmetic. The universe of a standard structure for first-order number theory is the set of natural numbers. The universe of a standard structure for first-order concatenation theory is a set of strings over some alphabet. A first-order language for number theory normally contains two binary functions symbols. In a standard structure these symbols will be interpreted as addition and multiplication. A first-order language for concatenation theory normally contains just one binary function symbol. In a standard structure this symbol will be interpreted as the operator that concatenates two strings. A classical first-order language for concatenation theory contains no other non-logical symbols apart from constant symbols.

In this paper we extend concatenation theory with a binary relation symbol and introduce bounded quantifiers analogous to the bounded quantifiers $(\forall x \leq t)\phi$ and $(\exists x \leq t)\phi$ we know from number theory. Before we go on and state our main results, we will explain some notation and state a few basic definitions.

1.3 Notation and Basic Definitions

We will use 0 and 1 to denote respectively the bits zero and one, and we use pretty standard notation when we work with bit strings: $\{0,1\}^*$ denotes the set
of all finite bit strings; \(|b|\) denotes the length of the bit string \(b\); \((b)_i\) denotes the \(i^{th}\) bit of the bit string \(b\); and \(01^40^21\) denotes the bit string \(0111001\). The set \(\{0, 1\}^*\) contains the empty string which we will denote \(\varepsilon\).

Let \(L_{BT}\) denote the first-order language that consist of the constants symbols \(e, 0, 1\), the binary function symbol \(\circ\) and the binary relation symbol \(\subseteq\). We will consider two \(L_{BT}\)-structures named \(\mathcal{B}\) and \(\mathcal{D}\).

The universe of \(\mathcal{B}\) is the set \(\{0, 1\}^*\). The constant symbol 0 is interpreted as the string containing nothing but the bit 0, and the constant symbol 1 is interpreted as the string containing nothing but the bit 1, that is, \(0^\mathcal{B} = 0\) and \(1^\mathcal{B} = 1\). The constant symbol \(e\) is interpreted as the empty string, that is, \(e^\mathcal{B} = \varepsilon\). Moreover, \(\circ^\mathcal{B}\) is the function that concatenates two strings (e.g. \(01 \circ^\mathcal{B} 000 = 01000\) and \(\varepsilon \circ^\mathcal{B} \varepsilon = \varepsilon\)). Finally, \(\sqsubseteq^\mathcal{B}\) is the substring relation, that is, \(u \sqsubseteq^\mathcal{B} v\) if there exists bit strings \(x, y\) such that \(xuy = v\).

The structure \(\mathcal{D}\) is the same structure as \(\mathcal{B}\) with one exception: the relation \(u \sqsubseteq^\mathcal{D} v\) holds iff \(u\) is a prefix of \(v\), that is, \(u\) is a substring of \(v\) such that \(ux = v\). To improve the readability we will use the symbol \(\preceq\) in place of the symbol \(\sqsubseteq\) when we are working in the structure \(\mathcal{D}\). Thus, \(u \preceq v\) should be read as “\(u\) is a substring of \(v\)”, whereas \(u \preceq v\) should be read as “\(u\) is a prefix of \(v\)”. When we do not have a particular structure in mind, e.g. when we deal with syntactical matters, we will stick to the symbol \(\sqsubseteq\).

We introduce the bounded quantifiers \((\exists x \sqsubseteq t)\alpha\) and \((\forall x \sqsubseteq t)\alpha\) as syntactical abbreviations for receptively \((\exists x)[x \sqsubseteq t \land \alpha]\) and \((\forall x)[x \sqsubseteq t \rightarrow \alpha]\) \((x\) is of course not allowed to occur in the term \(t\)), and we define the \(\Sigma\)-formulas inductively by

- \(\alpha\) and \(\neg\alpha\) are \(\Sigma\)-formulas if \(\alpha\) is of the form \(s \sqsubseteq t\) or of the form \(s = t\) where \(s\) and \(t\) are terms
- \(\alpha \lor \beta\) and \(\alpha \land \beta\) are \(\Sigma\)-formulas if \(\alpha\) and \(\beta\) are \(\Sigma\)-formulas
- \((\exists x \sqsubseteq t)\alpha\) and \((\forall x \sqsubseteq t)\alpha\) and \((\exists x)\alpha\) are \(\Sigma\)-formulas if \(\alpha\) is a \(\Sigma\)-formula.

We assume that the reader notes the similarities with first-order number theory. The formulas that correspond to \(\Sigma\)-formulas in number theory are often called \(\Sigma_1\)-formulas or \(\Sigma_1^0\)-formulas. Next we introduce the biterals. The biterals correspond to the numerals of first-order number theory. Let \(b\) be a bit string. We define the bilateral \(\overline{b}\) by \(\overline{\varepsilon} = e\), \(\overline{b0} = \overline{b} \circ 0\) and \(\overline{b1} = \overline{b} \circ 1\).

A \(\Sigma\)-formula \(\phi\) is called a \(\Sigma_{n,m,k}\)-formula if it contains \(n\) unbounded existential quantifiers, \(m\) bounded existential quantifiers and \(k\) bounded universal quantifiers. A sentence is a formula with no free variables. The fragment \(\Sigma_{n,m,k}^\mathcal{B}\) \((\Sigma_{n,m,k}^\mathcal{D})\) is the set of \(\Sigma_{n,m,k}\)-sentences true in \(\mathcal{B}\) (respectively, \(\mathcal{D}\)).

To improve the readability we may skip the operator \(\circ\) in first-order formulas and simply write \(st\) in place of \(s \circ t\). Furthermore, we will occasionally contract quantifiers and write, e.g., \(\forall w_1, w_2 \sqsubseteq u[\phi]\) in place of \((\forall w_1 \sqsubseteq u)(\forall w_2 \sqsubseteq u)\phi\), and for \(\sim \in \{\preceq, \sqsubseteq, =\}\), we will sometimes write \(s \not\sim t\) in place of \(\neg s \sim t\).
1.4 Main Results and Related Work

We prove that the fragment $\Sigma_{0,m,k}^B$ is decidable (for any $m, k \in \mathbb{N}$), and we prove that $\Sigma_{1,2,1}^D$ and $\Sigma_{1,0,2}^D$ are undecidable. Furthermore, we prove that the fragments $\Sigma_{0,m,k}^D$ and $\Sigma_{n,m,0}^D$ are decidable (for any $n, m, k \in \mathbb{N}$), and we prove that $\Sigma_{3,0,2}^D$ and $\Sigma_{4,1,1}^D$ are undecidable. Our results on decidable fragments are corollaries of theorems that have an interest in their own right: We prove the existence of normal forms, and we give a purely universal axiomatization of concatenation theory which is $\Sigma$-complete.

Recent related work can be found in Halfon et al. [6], Day et al. [2], Ganesh et al. [3], Karhumäki et al. [8] and several other places, see Section 6 of [3] for further references.

The material in Section 8 of the textbook Leary & Kristiansen [9] is also related to the research presented in this paper. So is a series of papers that starts with Grzegorczyk [4] and includes Grzegorczyk & Zdanowski [5], Visser [16] and Horihata [7]. These papers deal with the essential undecidability of various first-order theories of concatenation. The relationship between the various axiomatizations of concatenation theory we find in these papers and the axiomatization we give below has not yet been investigated.

The theory of concatenation seems to go back to work of Tarski [14] and Quine [12], see Visser [16] for a brief account of its history.

2 $\Sigma$-complete Axiomatizations

Definition 1. The first-order theory $B$ contains the following eleven non-logical axioms:

1. $\forall x \ [(x = ex \land x = xe)]$
2. $\forall xyz \ [(xy)z = x(yz)]$
3. $\forall xy \ [(x \neq y) \rightarrow \ ((x0 \neq y0) \land (x1 \neq y1)) ]$
4. $\forall xy \ [(x0 \neq y1)]$
5. $\forall x \ [(x \sqsubseteq e \leftrightarrow x = e)]$
6. $\forall x \ [(x \sqsubseteq 0 \leftrightarrow (x = e \lor x = 0)) ]$
7. $\forall x \ [(x \sqsubseteq 1 \leftrightarrow (x = e \lor x = 1)) ]$
8. $\forall xy \ [(x \sqsubseteq 0y0 \leftrightarrow (x = 0y0 \lor x \sqsubseteq 0y \lor x \sqsubseteq y0)]$
9. $\forall xy \ [(x \sqsubseteq 0y1 \leftrightarrow (x = 0y1 \lor x \sqsubseteq 0y \lor x \sqsubseteq y1)]$
10. $\forall xy \ [(x \sqsubseteq 1y0 \leftrightarrow (x = 1y0 \lor x \sqsubseteq 1y \lor x \sqsubseteq y0)]$
11. $\forall xy \ [(x \sqsubseteq 1y1 \leftrightarrow (x = 1y1 \lor x \sqsubseteq 1y \lor x \sqsubseteq y1)]$

We will use $B_i$ to refer to the $i^{th}$ axiom of $B$.

A first-order theory is essentially undecidable if the theory—and every extension of the theory—is undecidable. Tarski [15] is a very readable introduction to the subject.
Theorem 2 (Σ-completeness of $B$). For any $\Sigma$-sentence $\phi$, we have
\[ B \models \phi \Rightarrow B \vdash \phi. \]

Proof. (Sketch) Prove (by induction on the structure of $t$) that there for any variable-free $L_{BT}$-term $t$ exists a biteral $b$ such that
\[ B \models t = b \Rightarrow B \vdash t = b. \] (1)

Prove (by induction on the structure of $b_2$) that we for any biterals $b_1$ and $b_2$ have
\[ B \models b_1 \not\sqsubseteq b_2 \Rightarrow B \vdash b_1 \not\sqsubseteq b_2. \] (2)

Use $B \vdash \forall x[x \neq e \land x \neq e]$ when proving (2). Furthermore, prove (by induction on the structure of $b_2$) that we for any biterals $b_1$ and $b_2$ have
\[ B \models b_1 \sqsubseteq b_2 \Rightarrow B \vdash b_1 \sqsubseteq b_2 \quad \text{and} \quad B \models b_1 \not\sqsubseteq b_2 \Rightarrow B \vdash b_1 \not\sqsubseteq b_2. \] (3)

It follows from (1), (2) and (3) that we have
\[ B \models \phi \Rightarrow B \vdash \phi. \] (4)

for any $\phi$ of one of the four forms $t_1 = t_2$, $t_1 \not= t_2$, $t_1 \sqsubseteq t_2$, and $t_1 \not\sqsubseteq t_2$ where $t_1$ and $t_2$ are variable-free terms.

Use induction on the structure of $b$ to prove the following claim:

If $\phi(x)$ is an $L_{BT}$-formula such that we have $B \models \phi(b) \Rightarrow B \vdash \phi(b)$ for any biteral $b$, then we also have
\[ B \models (\forall x \sqsubseteq b)\phi(x) \Rightarrow B \vdash (\forall x \sqsubseteq b)\phi(x) \]
for any biteral $b$.

Finally, prove (by induction on the structure of $\phi$) that we for any $\Sigma$-sentence $\phi$ have $B \models \phi \Rightarrow B \vdash \phi$. Use (1) in the base cases, that is, when $\phi$ is an atomic sentence or a negated atomic sentence. Use the claim and (1) in the case $\phi$ is of the form $(\forall x \sqsubseteq t)\psi$. The remaining cases are rather straightforward.

A detailed proof of Theorem 2 can be found in Section 5.

Definition 3. The first-order theory $D$ contains the following seven non-logical axioms:

- the first four axioms are the same as the first four axioms of $B$
5. $\forall x[x \leq e \leftrightarrow x = e]$]
6. $\forall xy[x \leq y \circ 0 \leftrightarrow (x = y \circ 0 \lor x \leq y)]$
7. $\forall xy[x \leq y \circ 1 \leftrightarrow (x = y \circ 1 \lor x \leq y)]$
We will use $D_i$ to refer to the $i^{th}$ axiom of $D$.

The proof of the next theorem can be found in Section 6. More material related to the theories $B$ and $D$ can be found in Chapter 8 of Leary & Kristiansen [9].

**Theorem 4 (Σ-completeness of $D$).** For any $Σ$-sentence $ϕ$, we have

$$D \models \phi \Rightarrow D \vdash \phi.$$ 

**Corollary 5.** The fragments $Σ^B_{0,m,k}$ and $Σ^D_{0,m,k}$ are decidable (for any $m, k \in \mathbb{N}$).

**Proof.** We prove that $Σ^B_{0,m,k}$ is decidable. Let $ϕ$ be a $Σ_{0,m,k}$-formula. The negation of a $Σ_{0,m,k}$-formula is logically equivalent to a $Σ_{0,k,m}$-formula (by De Morgan’s laws). We can compute a $Σ_{0,k,m}$-formula $ϕ'$ which is logically equivalent to $¬ϕ$. By Theorem 2, we have $B \vdash ϕ$ if $B \models \phi$, and we have $B \vdash ϕ'$ if $B \models ¬ϕ$. The set of formulas derivable from the axioms of $B$ is computably enumerable. Hence it is decidable if $ϕ$ is true in $B$. The proof that the fragment $Σ^D_{0,m,k}$ is decidable is similar. ⊓ ⊔

### 3 Normal Forms

Some of the lemmas below are based on results and proofs found in Senger [13] and Büchi & Senger [11]. They prove that any $Σ$-formula in the language $\{◦, 0, 1, e\}$ is equivalent in $B_{\{0,1,e\}}$ to a formula of the form $(∃v_0)\ldots(∃v_k)(s = t)$.

**Lemma 6.** Let $A ∈ \{B, D\}$, and let $s_1, s_2, t_1, t_2$ be $L_{BT}$-terms. We have

$$A \models (s_1 = t_1 ∧ s_2 = t_2) \iff s_10s_2s_1s_2 = t_10t_2t_1t_2.$$ 

**Proof.** Assume $s_10s_2s_1 = t_10t_2t_1$. Then $|s_10s_2| = |t_10t_2|$ and $|s_1| = |t_1|$. The proof splits into the two cases $|s_1| = |t_1|$ and $|s_1| ≠ |t_1|$. In the case when $|s_1| = |t_1|$, we obviously have $s_1 = t_1$ and $s_2 = t_2$. Assume $|s_1| ≠ |t_1|$. We can w.l.o.g. assume that $|s_1| < |t_1|$. This implies that

$$0 = (s_10s_2)|s_1|+1 = (t)|t_1|+1 = (s_11s_2)|s_1|+1 = 1.$$ 

This is a contradiction. This proves the implication from the right to the left. The converse implication is obvious. ⊓ ⊔

**Lemma 7.** Let $s_1, s_2, t_1, t_2$ be $L_{BT}$-terms. There exist $L_{BT}$-terms $s, t$ and variables $v_1, \ldots, v_k$ such that

$$D \models (s_1 ≤ t_1 ∨ s_2 ≤ t_2) \iff ∃v_1\ldots∃v_k[s = t].$$
Proof. Let \( x_1, \ldots, x_6 \) be variables that do not occur in any of the terms \( s_1, s_2, t_1, t_2 \). It is not very hard to see that the formula \( s_1 \preceq t_1 \lor s_2 \preceq t_2 \) is equivalent in \( \mathcal{D} \) to the formula

\[
\exists x_1 \ldots x_6 \left[ s_1 = x_1 x_2 \land t_1 = x_1 x_3 \land s_2 = x_4 x_5 \land t_2 = x_4 x_6 \land (x_2 = e \lor x_3 = e) \right]. \quad (*)
\]

Let \( \psi(u, w) \) be the formula

\[
\exists y_1 y_2 y_3 y_4 \left[ y_1 y_2 = 0 \land y_3 y_4 = 1 \land uy_1 wy_2 = wy_2 uy_1 \land uy_3 wy_4 = wy_4 uy_3 \right].
\]

We claim that

\[
\mathcal{D} \models (u = e \lor w = e) \iff (uw = wu \land \psi(u, w)) \quad (**)
\]

We prove (**). Assume that \( u = e \lor w = e \). Let us say that \( u = e \) (the case when \( w = e \) is symmetric). It is obvious that we have \( uw = wu \). Moreover, \( \psi(u, w) \) holds with \( y_1 = y_3 = e \), \( y_2 = 0 \) and \( y_4 = 1 \). This prove the left-right implication of (**).

To see that the converse implication holds, assume that \( \neg (u = e \lor w = e) \), that is, both \( u \) and \( w \) are different from the empty string. Furthermore, assume that \( uw = wu \). We will argue that \( \psi(u, w) \) does not hold: Since \( uw = wu \) and both \( u \) and \( w \) contain at least one bit, it is either the case that 0 is the last bit of both strings, or it is that case that 1 is the last bit of both strings. If 0 is the last bit of both, the two equations \( uy_3 wy_4 = wy_4 uy_3 \) and \( y_3 y_4 = 1 \) cannot be satisfied simultaneously. If 1 is the last bit of both, the two equations \( uy_1 wy_2 = wy_2 uy_1 \) and \( y_1 y_2 = 0 \) cannot be satisfied simultaneously. Hence we conclude that \( \psi(u, w) \) does not hold. This completes the proof of (**).

Our lemma follows from (*) and (**) by Lemma 6. \( \Box \)

Lemma 8. Let \( \mathfrak{A} \in \{ \mathfrak{B}, \mathfrak{D} \} \). Let \( s_1, s_2, t_1, t_2 \) be \( \mathcal{L}_{BT} \)-terms. There exist \( \mathcal{L}_{BT} \)-terms \( s, t \) and variables \( v_0, \ldots, v_k \) such that

(1) \( \mathfrak{A} \models (s_1 = t_1 \lor s_2 = t_2) \iff \exists v_0 \ldots v_k [s = t] \)

(2) \( \mathfrak{A} \models s_1 \neq t_1 \iff \exists v_0 \ldots v_k [s = t] \).

Proof. Observe that \( s_1 = t_1 \lor s_2 = t_2 \) is equivalent in \( \mathcal{D} \) to

\( (s_1 \preceq t_1 \land t_1 \preceq s_1) \lor (s_2 \preceq t_2 \land t_2 \preceq s_2) \)

which again is (logically) equivalent to

\( (s_1 \preceq t_1 \lor s_2 \preceq t_2) \land (s_1 \preceq t_1 \lor t_2 \preceq s_2) \land (t_1 \preceq s_1 \lor s_2 \preceq t_2) \land (t_1 \preceq s_1 \lor t_2 \preceq s_2) \).
By Lemma 6 and Lemma 7, it follows that (1) holds for the structure $D$. To see that (1) also holds for the structure $B$, observe that the relation $x \preceq_D y$ can be expressed in $B$ by the formula $\exists v[xv = y]$.

In order to see that (2) holds, observe that the formula $s \neq t$ is equivalent—in both $B$ and $D$—to the formula
\[
\exists x\forall y\left[\begin{array}{l}
(s = x0y \land t = x0z) \lor (s = x1y \land t = x1z)
\end{array}\right].
\]

Thus, (2) follows from (1) and Lemma 6.

Lemma 9. Let $s_1, t_1$ be $L_{BT}$-terms. There exist $L_{BT}$-terms $s, t$ and variables $v_1, \ldots, v_k$ such that
\[
\begin{align*}
(1) & \quad D \models s_1 \preceq t_1 \iff \exists v_1[s_1 v_1 = t_1] \\
(2) & \quad D \models s_1 \not\preceq t_1 \iff \exists v_1 \ldots v_k[s = t].
\end{align*}
\]

Proof. It is obvious that (1) holds. Furthermore, the formula $s_1 \not\preceq t_1$ is equivalent in $D$ to the formula
\[
(1 \preceq s_1 \land t_1 \neq s_1) \lor \exists xy[z\left[(t_1 = s0y \land s_1 = x1z) \lor (t_1 = s1y \land s_1 = x0z)\right]].
\]

Thus, (2) follows by Lemma 6, Lemma 8 and (1).

Comment: It is not known to us whether the bounded universal quantifier that appears in clause (2) of the next lemma can be eliminated.

Lemma 10. Let $s_1, t_1$ be $L_{BT}$-terms. There exist $L_{BT}$-terms $s, t$ and variables $v_1, \ldots, v_k$ such that
\[
\begin{align*}
(1) & \quad B \models s_1 \subseteq t_1 \iff \forall v_1[s_1 v_1 \subseteq t_1] \\
(2) & \quad B \models s_1 \not\subseteq t_1 \iff \exists v_1 \ldots v_k[s = t].
\end{align*}
\]

Proof. Cause (1) is trivial. Furthermore, observe that $s_1 \not\subseteq t_1$ is equivalent in $B$ to $\forall v[t_1 \alpha]$ where $\alpha$ is
\[
\exists x[t_1 x = vs_1 \land x \neq e] \lor \exists xyz[(t_1 = x0y \land vs_1 = x1z) \lor (t_1 = x1y \land vs_1 = x0z)].
\]

If we let $vs_1 \preceq t_1$ abbreviate $(\exists x)(vs_1 x = t)$, then $\alpha$ can be written as $vs_1 \preceq t_1$. Thus, (2) follows by Lemma 9.

Theorem 11 (Normal Form Theorem I). Any $\Sigma$-formula $\phi$ is equivalent in $D$ to a $L_{BT}$-formula $\phi'$ of the form
\[
\phi' \equiv (Q_1^{t_1} v_1) \ldots (Q_m^{t_m} v_m)[s = t]
\]
where $t_1, \ldots, t_m, s, t$ are $\mathcal{L}_{BT}$-terms and $Q_j^t v_j \in \{ \exists v_j, \exists v_j \subseteq t_j, \forall v_j \subseteq t_j \}$ for $j = 1, \ldots, m$. Moreover, if $\phi$ does not contain bounded universal quantifiers, then $\phi'$ does not contain bounded quantifiers.

**Proof.** We proceed by induction on the structure of $\phi$ (throughout the proof we reason in the structure $\mathcal{D}$). Suppose $\phi$ is an atomic formula or the negation of an atomic formula. If $\phi$ is of the form $s = t$, let $\phi'$ be $s = t$. Use Lemma 3 if $\phi$ is of the form $-s = t$. Use Lemma 3 if $\phi$ is of one of the forms $s \leq t$ and $-s \leq t$.

Suppose $\phi$ is of the form $\alpha \land \beta$. By our induction hypothesis, we have formulas

$$\alpha' \equiv \left( Q_1^t x_1 \right) \ldots \left( Q_k^t x_k \right) (s_1 = t_1)$$

and

$$\beta' \equiv \left( Q_1^{t'} y_1 \right) \ldots \left( Q_m^{t'} y_m \right) (s_2 = t_2)$$

which are equivalent to respectively $\alpha$ and $\beta$. Thus, $\phi$ is equivalent to a formula of the form $\left( Q_1^t x_1 \right) \ldots \left( Q_k^t x_k \right) \left( Q_1^{t'} y_1 \right) \ldots \left( Q_m^{t'} y_m \right) (s_1 = t_1 \land s_2 = t_2)$. By Lemma 4 we have a formula $\phi'$ of the desired form which is equivalent to $\phi$. The case when $\phi$ is of the form $\alpha \lor \beta$ is similar. Use Lemma 5 in place of Lemma 3.

The theorem follows trivially from the induction hypothesis when $\phi$ is of one of the forms $(\exists v)\alpha$, $(\forall v \leq t)\alpha$ and $(\exists v \leq t)\alpha$.

**Theorem 12 (Normal Form Theorem II).** Any $\Sigma$-formula $\phi$ is equivalent in $\mathcal{B}$ to a $\mathcal{L}_{BT}$-formula $\phi'$ of one of the forms

$$\phi' \equiv \left( Q_1^t v_1 \right) \ldots \left( Q_m^t v_m \right) (s = t) \quad \text{or} \quad \phi' \equiv (\exists v) \left( Q_1^{t'} v_1 \right) \ldots \left( Q_m^{t'} v_m \right) (s = t)$$

where $t_1, \ldots, t_m, s, t$ are $\mathcal{L}_{BT}$-terms and $Q_j^t v_j \in \{ \exists v_j \subseteq t_j, \forall v_j \subseteq t_j \}$ for $j = 1, \ldots, m$.

**Proof.** Proceed by induction on the structure of $\phi$. This proof is similar to the proof of Theorem 11. A formula of the form $(\forall x \subseteq t)(\exists y)\alpha$ is equivalent (in $\mathcal{B}$) to a formula of the form $(\exists z)(\forall x \subseteq t)(\exists y \subseteq z)\alpha$, a formula of the form $(\exists x \subseteq t)(\exists y)\alpha$ is equivalent to a formula of the form $(\exists y)(\exists x \subseteq t)\alpha$, and a formula of the form $(\exists x)(\exists y)\alpha$ is equivalent to a formula of the form $(\exists x)(\exists y \subseteq z)\alpha$. Thus, the resulting normal form will contain maximum one unbounded existential quantifier.

**Corollary 13.** The fragment $\Sigma^D_{n,m,0}$ is decidable (for any $n, m \in \mathbb{N}$).

**Proof.** By Theorem 11 any $\Sigma_{n,m,0}$-sentence is equivalent in $\mathcal{D}$ to a sentence of the normal form $\exists v_1 \ldots v_k [s = t]$ (regard the bounded existential quantifiers as unbounded). The transformation of a $\Sigma_{n,m,0}$-formula into an equivalent formula (in $\mathcal{D}$) of normal form is constructive. Makanin [10] has proved that it is decidable whether an equation on the form

$$a_n x_n \ldots a_1 x_1 a_0 = b_m y_m \ldots b_1 y_1 b_0$$

where $a_1, \ldots, a_n, b_1, \ldots, b_m \in \{0, 1\}^*$, has a solution in $\{0, 1\}^*$. It follows that the fragment $\Sigma^D_{n,m,0}$ is decidable.
We have not been able to prove that any $\Sigma_{n,m,0}$-sentence is equivalent in $\mathcal{B}$ to a sentence of the form $\exists v_1 \ldots \exists v_k [s = t]$. See the comment immediately before Lemma 10. Thus, we cannot use Makanin’s result to prove that the fragment $\Sigma_{n,m,0}^\mathcal{B}$ is decidable.

Open Problem: Is the fragment $\Sigma_{n,m,0}^\mathcal{B}$ decidable (for any $n, m \in \mathbb{N}$)?

4 Undecidable Fragments

Definition 14. Post’s Correspondence Problem, henceforth PCP, is given by

- Instance: a list of pairs $\langle b_1, b'_1 \rangle, \ldots, \langle b_n, b'_n \rangle$ where $b_i, b'_i \in \{0, 1\}^*$
- Solution: a finite nonempty sequence $i_1, \ldots, i_m$ of indexes such that $b_{i_1}b_{i_2}\ldots b_{i_m} = b'_{i_1}b'_{i_2}\ldots b'_{i_m}$.

We define the map $N : \{0, 1\}^* \to \{0, 1\}^*$ by $N(\varepsilon) = \varepsilon$, $N(0) = 01\overline{0}$, $N(1) = 0\overline{1}0$, $N(b) = N(b)N(0)$ and $N(b_1) = N(b)N(1)$.

It is proved in Post [11] that PCP is undecidable. The proof of the next lemma is left to the reader.

Lemma 15. The instance $\langle b_1, b'_1 \rangle, \ldots, \langle b_n, b'_n \rangle$ of PCP has a solution iff the instance $\langle N(b_1), N(b'_1) \rangle, \ldots, \langle N(b_n), N(b'_n) \rangle$ has a solution.

We will now explain the ideas behind our proofs of the next few theorems. Given the lemma above, it is not very hard to see that an instance $\langle g_1, g'_1 \rangle, \ldots, \langle g_n, g'_n \rangle$ of PCP has a solution iff there exists a bit string of the form

$$01^50N(a_1)01^40N(b_1)01^50\ldots01^40N(b_m)01^50$$

(*)

where

(A) $N(a_m) = N(b_m)$
(B) $N(a_1) = g_1$ and $N(b_1) = g'_1$ for some $1 \leq j \leq n$
(C) $N(a_{k+1}) = N(b_k)N(g_j)$ and $N(b_{k+1}) = N(b_k)N(g'_j)$ for some $1 \leq j \leq n$.

We also see that an instance $\langle g_1, g'_1 \rangle, \ldots, \langle g_n, g'_n \rangle$ of PCP has a solution iff there exists a bit string $s$ of the form (*) that satisfies

(a) there is $j \in \{1, \ldots, n\}$ such that $01^50N(g_j)01^40N(g'_j)01^50$ is an initial segment of $s$
(b) if

\[ 01^30N(a)01^40N(b)01^50 \]

is a substring of \( s \), then either \( N(a) = N(b) \), or there is \( j \in \{1, \ldots, n\} \) such that

\[ 01^30N(a)N(g_j)01^40N(b)N(g'_j)01^50 \]

is a substring of \( s \).

In the proof of Theorem 16 we give a formula which is true in \( \mathcal{D} \) iff there exists a string of the form (*) that satisfies (A), (B) and (C). In the proof of Theorem 17 we give formulas which are true in \( \mathcal{B} \) iff there exists a string of the form (*) that satisfies (a) and (b). In order to improve the readability of our formulas, we will write \( # \) in place of the biteral \( 01^50 \) and \( ! \) in place of the biteral \( 01^40 \).

**Theorem 16.** The fragment \( \Sigma^\mathcal{D}_{3,0,2} \) is undecidable.

**Proof.** Let \( \psi(x) \equiv (\forall z \preceq x)(z \uparrow \not\preceq x) \). Observe that \( \psi \) contains one bounded universal quantifier. Observe that \( \psi(b) \) is true in \( \mathcal{D} \) iff the bit string \( b \) does not contain 4 consecutive ones. Furthermore, let \( \phi_n(x_1, \ldots, x_n, y_1, \ldots, y_n) \equiv \)

\[
\begin{align*}
&\exists u \left( \left( \bigvee_{j=1}^n #x_j! y_j # \preceq u \right) \land \\
&\forall u \preceq u \left[ v # \not\preceq u \lor v # = u \lor \exists w_1, w_2 \left\{ v # w_1 \! w_2 # \preceq u \land \\
&\psi(w_1, w_2) \land \\
&w_1 = w_2 \lor \left( \bigvee_{j=1}^n v # w_1 \! w_2 \! x_j \! y_j \# \not\preceq u \right) \right\} \right] \right).
\end{align*}
\]

Let \( \langle g_1, g'_1 \rangle, \ldots, \langle g_n, g'_n \rangle \) be an instance of PCP. We have

\[
\mathcal{D} \models \phi_n(N(g_1), \ldots, N(g_n), N(g'_1), \ldots, N(g'_n))
\]

to there exists a bit string of the form (*) that satisfies (A), (B) and (C) iff the instance \( \langle g_1, g'_1 \rangle, \ldots, \langle g_n, g'_n \rangle \) has a solution. Furthermore, \( \phi_n \) is a \( \Sigma_{3,0,2} \)-formula.

It follows that the fragment \( \Sigma^\mathcal{D}_{3,0,2} \) is undecidable. \( \square \)

**Theorem 17.** The fragments \( \Sigma^\mathcal{B}_{1,2,1} \) and \( \Sigma^\mathcal{B}_{1,0,2} \) are undecidable.

**Proof.** Let \( \vec{x} = x_1, \ldots, x_n \), let \( \vec{y} = y_1, \ldots, y_n \) and let

\[
\alpha(\vec{x}, \vec{y}, z) \equiv \left( \bigvee_{j=1}^n #x_j ! y_j # \preceq z \land 0 # x_j ! y_j # \not\preceq z \land 1 # x_j ! y_j # \not\preceq z \right).
\]
Consider the $\Sigma_{1,2,1}$-formula $\psi_n(x, y) \equiv$

$$
(\exists u) \left( \alpha(x, y, u) \land 
(\forall v \subseteq u) \left[ \#v \# \not\subseteq u \lor \overline{1^5} \subseteq v \lor (\exists w_1, w_2 \subseteq u) \left\{ v = w_1!w_2 \lor \overline{1^5} \not\subseteq w_1 \land \overline{1^5} \not\subseteq w_2 \land \left( w_1 = w_2 \lor \left( \bigvee_{j=1}^{n} \#w_1x_j!w_2y_j \# \not\subseteq u \right) \right) \right\} \right] \right)$$

and consider the $\Sigma_{1,0,2}^1$-formula $\gamma_n(x, y) \equiv$

$$
(\exists u) \left( \alpha(x, y, u) \land (\forall w_1, w_2 \subseteq u) \left\{ \#w_1!w_2 \not\subseteq u \lor \overline{1^4} \subseteq w_1w_2 \lor \left( w_1 = w_2 \lor \left( \bigvee_{j=1}^{n} \#w_1x_j!w_2y_j \# \not\subseteq u \right) \right) \right\} \right).$$

Let $(g_1, g'_1), \ldots, (g_n, g'_n)$ be an instance of PCP. We have

$\mathfrak{B} \models \psi_n(N(g_1), \ldots, N(g_n), N(g'_1), \ldots, N(g'_n))$

iff

$\mathfrak{B} \models \gamma_n(N(g_1), \ldots, N(g_n), N(g'_1), \ldots, N(g'_n))$

iff there exists a bit string of the form (*) that satisfies (a) and (b) iff the instance $(g_1, g'_1), \ldots, (g_n, g'_n)$ has a solution. It follows that the fragments $\Sigma_{1,2,1}^\mathfrak{B}$ and $\Sigma_{1,0,2}^\mathfrak{B}$ are undecidable.

The proof of the next theorem is based on the following idea: The instance $(g_1, g'_1), \ldots, (g_n, g'_n)$ of PCP has a solution iff there exists a bit string of the form

$$
\underbrace{01^50N(a_1)01^40N(b_1)01^60N(a_2)}01^40N(b_2)\underbrace{01^70}_\ldots
\ldots 01^5+m-10N(a_m)01^40N(b_m)01^5+m0
$$

with the properties (A), (B) and (C) given above.

**Theorem 18.** The fragment $\Sigma_{4,1,1}^\mathfrak{D}$ is undecidable.

**Proof.** Let $1^k \equiv \overline{01^k0}$. The $\Sigma_{4,1,1}$-formula

$$
(\exists u) \left( \left( \bigvee_{j=1}^{n} 1^{x_j}1^y \right) \leq u \right) \land (\forall v \leq u) \left[ v1^y \not\subseteq u \lor v = 0 \lor \right.
$$

$$
(\exists w_1, w_2, y)(\exists z \leq v) \left\{ v = z0y1^w0^2w_11^w201y \land 1y = y1 \land \right.\n$$

$$
\left. \left( w_1 = w_2 \lor \left( \bigvee_{j=1}^{n} 1^w0^0w_1x_j1^w2y_j011y1^y \not\subseteq u \right) \right) \right\} \right)
$$

yields the desired statement. Note that $y$ is a solution of the equation $1y = y1$ iff $y \in \{1\}^*$. \qed
5 Proof of Theorem 2: $\Sigma$-Completeness of $B$

Lemma 19.

$B_1, B_2, B_4 \vdash \forall x \ [x0 \neq e \land x1 \neq e]$. 

Proof. We reason in an arbitrary model for $\{B_1, B_2, B_4\}$. Let $x$ be an arbitrary element in the universe. Assume $x0 = e$. Then $1(x0) = 1e$. By $B_1$, we have $1(x0) = 1$. By $B_2$, we have $(1x)0 = 1$. By $B_1$, we have $(1x)0 = e1$. This contradicts $B_4$. This proves that $x0 \neq e$. A symmetric argument shows that $x1 \neq e$. This proves that

$B_1, B_2, B_4 \vdash \forall x [x0 \neq e \land x1 \neq e]$.

The lemma follows by the Completeness Theorem for first-order logic. 

Lemma 20. For any variable-free $L_{BT}$-term $t$ there exists a biteral $b$ such that $B \vdash t = b$. Furthermore, we have

$\mathfrak{B} \models t_1 = t_2 \Rightarrow B \vdash t_1 = t_2$

for any variable-free $L_{BT}$-terms $t_1$ and $t_2$.

Proof. We proceed by induction on the structure of $t$ to show that there exists a biteral $b$ such that $B \vdash t = b$.

If $t \equiv e$, let $b \equiv e$. Then $B \vdash e = e$.

If $t \equiv 0$, let $b \equiv e \circ 0$. By $B_1$, we have $B \vdash 0 = e \circ 0$.

If $t \equiv 1$, let $b \equiv e \circ 1$. By $B_1$, we have $B \vdash 1 = e \circ 1$.

Suppose $t \equiv t_1 \circ t_2$. Furthermore, suppose there exist biterals $b_1$ and $b_2$ such that $B \vdash t_1 = b_1$ and $B \vdash t_2 = b_2$. Then $B \vdash t_1 \circ t_2 = b_1 \circ b_2$. We note that $b_1 \circ b_2$ is of the form

$((\ldots((e \circ c_1) \circ \ldots) \circ c_n) \circ ((e \circ d_1) \circ \ldots) \circ d_m)$

where each $c_j$ and each $d_j$ is 0 or 1. Let $\mathfrak{A} \models B$. Then $\mathfrak{A} \models t_1 \circ t_2 = b_1 \circ b_2$. By $B_2$ we have

$\mathfrak{A} \models b_1 \circ b_2 = ((\ldots((e \circ c_1) \circ \ldots) \circ c_n) \circ (e \circ d_1) \circ \ldots) \circ d_m)$.

By $B_1$ we have

$\mathfrak{A} \models (b_1 \circ b_2) = ((\ldots((e \circ c_1) \circ \ldots) \circ c_n) \circ d_1) \circ \ldots) \circ d_m)$.

Let

$b \equiv ((\ldots((e \circ c_1) \circ \ldots) \circ c_n) \circ d_1) \circ \ldots) \circ d_m)$.

Then $b$ is a biteral and $\mathfrak{A} \models b_1 \circ b_2 = b$. Since $\mathfrak{A} \models t_1 \circ t_2 = b_1 \circ b_2$, we have $\mathfrak{A} \models t_1 \circ t_2 = b$. Since $\mathfrak{A}$ is an arbitrary model for $B$, we have $B \models t_1 \circ t_2 = b$, and then, by the Completeness Theorem for first-order logic, we have $B \vdash t_1 \circ t_2 = b$. 

This proves that there for any variable-free term \( t \) there exists a biteral \( b \) such that \( B \vdash t = b \).

Let \( t_1 \) and \( t_2 \) be \( L_{BT} \)-terms such that \( \mathfrak{B} \models t_1 = t_2 \). Then there exist bitals \( b_1 \) and \( b_2 \) such that

\[
B \vdash t_1 = b_1 \text{ and } B \vdash t_2 = b_2.
\]

Since \( \mathfrak{B} \models B \), we have

\[
\mathfrak{B} \models t_1 = b_1 \text{ and } \mathfrak{B} \models t_2 = b_2.
\]

and thus we also have \( \mathfrak{B} \models b_1 = b_2 \). Since each element in \( \{0, 1\}^* \) is mapped to a unique biteral, it follows that \( b_1 \) is the same biteral as \( b_2 \). Thus, \( B \vdash b_1 = b_2 \).

\[\square\]

**Lemma 21.** We have

\[
\mathfrak{B} \models \neg b_1 = b_2 \Rightarrow B \vdash \neg b_1 = b_2
\]

for any bitals \( b_1 \) and \( b_2 \). Furthermore, we have

\[
\mathfrak{B} \models \neg t_1 = t_2 \Rightarrow B \vdash \neg t_1 = t_2
\]

for any variable-free \( L_{BT} \)-terms \( t_1 \) and \( t_2 \).

**Proof.** Let \( b_1 \) and \( b_2 \) be bitals such that \( \mathfrak{B} \models \neg b_1 = b_2 \). We proceed by induction on the structure of \( b_2 \) to show that \( B \vdash \neg b_1 = b_2 \).

If \( b_2 \equiv e \), then \( b_1 \equiv b \circ 0 \) or \( b_1 \equiv b \circ 1 \) for some biteral \( b \). In either case, by Lemma [19] we have \( B \vdash (\neg b \circ 0 = e) \land (\neg b \circ 1 = e) \).

Suppose \( b_2 \equiv t \circ 0 \). Furthermore, suppose by induction hypothesis that

\[
\mathfrak{B} \models \neg b = t \Rightarrow B \vdash \neg b = t \tag{IH}
\]

for any biteral \( b \). We proceed by induction on \( b_1 \). If \( b_1 \equiv e \), we have \( B \vdash \neg e = t \circ 0 \) by Lemma [19]. If \( b_1 \equiv b \circ 0 \), then \( \mathfrak{B} \models \neg b = t \). By (IH), we have \( B \vdash \neg b = t \). By \( B_3 \), we have \( B \vdash \neg b \circ 0 = t \circ 0 \). If \( b_1 \equiv b \circ 1 \), we have \( B \vdash \neg b \circ 1 = t \circ 0 \) by \( B_4 \).

This case when \( b_2 \equiv t \circ 0 \) is symmetric to the case when \( b_2 \equiv t \circ 1 \).

This proves that

\[
\mathfrak{B} \models \neg b_1 = b_2 \Rightarrow B \vdash \neg b_1 = b_2.
\]

\[\ast\]

Now, suppose \( t_1 \) and \( t_2 \) are variable-free \( L_{BT} \)-terms such that \( \mathfrak{B} \models \neg t_1 = t_2 \).

By Lemma [20] there exist bitals \( b_1 \) and \( b_2 \) such that \( B \vdash t_1 = b_1 \land t_2 = b_2 \). As \( \mathfrak{B} \models B \), we have \( \mathfrak{B} \models t_1 = b_1 \land t_2 = b_2 \). It follows that \( \mathfrak{B} \models \neg b_1 = b_2 \). By \( \ast \), we have \( B \vdash \neg b_1 = b_2 \), and thus we also have \( B \vdash \neg t_1 = t_2 \).

\[\square\]
Lemma 22. We have

\[ \mathcal{B} \models b_1 \subseteq b_2 \Rightarrow B \vdash b_1 \subseteq b_2 \]

for any biterals \( b_1 \) and \( b_2 \). Furthermore, we have

\[ \mathcal{B} \models t_1 \subseteq t_2 \Rightarrow B \vdash t_1 \subseteq t_2 \]

for any variable-free \( L_{BT} \)-terms \( t_1 \) and \( t_2 \).

Proof. We prove this lemma by induction on the structure of \( b_2 \).

If \( b_2 \equiv e \) and \( \mathcal{B} \models b_1 \subseteq b_2 \), then \( b_1 \) is \( e \). By \( B_5 \), we have \( B \vdash e \subseteq e \).

If \( b_2 \equiv e \circ 0 \) and \( \mathcal{B} \models b_1 \subseteq b_2 \), then \( b_1 \) is \( e \) or \( e \circ 0 \). In either case, by Lemma 20 and \( B_6 \), we have \( B \vdash b_1 \subseteq b_2 \).

If \( b_2 \equiv e \circ 1 \) and \( \mathcal{B} \models b_1 \subseteq b_2 \), then \( b_1 \) is \( e \) or \( e \circ 1 \). In either case, by Lemma 20 and \( B_7 \), we have \( B \vdash b_1 \subseteq b_2 \).

Suppose \( b_2 \equiv e \circ 0 \circ t \circ 0 \). Furthermore, suppose by induction hypothesis that we have for any biteral \( s \) have

\[ \begin{align*}
- \quad & \mathcal{B} \models s \subseteq e \circ 0 \circ t \Rightarrow B \vdash s \subseteq e \circ 0 \circ t \\
- \quad & \mathcal{B} \models s \subseteq t \circ 0 \Rightarrow B \vdash s \subseteq t \circ 0 .
\end{align*} \]

Let \( \mathcal{B} \models b_1 \subseteq b_2 \). Then we have

\[ \mathcal{B} \models b_1 = e \circ 0 \circ t \circ 0 \lor b_1 \subseteq e \circ 0 \circ t \lor b_1 \subseteq t \circ 0 . \]

By our induction hypothesis and Lemma 20, we have

\[ B \vdash b_1 = e \circ 0 \circ t \circ 0 \lor b_1 \subseteq e \circ 0 \circ t \lor b_1 \subseteq t \circ 0 . \]

By \( B_1 \) and \( B_8 \), we have \( B \vdash b_1 \subseteq e \circ 0 \circ t \circ 0 \).

The case when \( b_2 \equiv e \circ 0 \circ t \circ 1 \), the case when \( b_2 \equiv e \circ 0 \circ t \circ 0 \) and the case when \( b_2 \equiv e \circ 1 \circ t \circ 0 \) are handled similarly using \( B_9 \), \( B_{10} \) and \( B_{11} \), respectively, in place of \( B_8 \). This proves that we have

\[ \mathcal{B} \models b_1 \subseteq b_2 \Rightarrow B \vdash b_1 \subseteq b_2 . \]  \( (\ast) \)

for any biterals \( b_1, b_2 \).

Suppose \( t_1 \) and \( t_2 \) are variable-free \( L_{BT} \)-terms such that \( \mathcal{B} \models t_1 \subseteq t_2 \). By Lemma 20, there exists biteral \( b_1 \) and \( b_2 \) such that \( B \vdash t_1 = b_1 \land t_2 = b_2 \). Since \( \mathcal{B} \models B \), we also have \( \mathcal{B} \models t_1 = b_1 \land t_2 = b_2 \). Hence, \( \mathcal{B} \models b_1 \subseteq b_2 \). By \( (\ast) \), we have \( B \vdash b_1 \subseteq b_2 \), and thus we also have \( B \vdash t_1 \subseteq t_2 \). \( \square \)

Lemma 23. Let \( \phi(x) \) be an \( L_{BT} \) formula such that

\[ \mathcal{B} \models \phi(b) \Rightarrow B \vdash \phi(b) \]

for any biteral \( b \). Then

\[ \mathcal{B} \models (\forall x \subseteq b) \phi(x) \Rightarrow B \vdash (\forall x \subseteq b) \phi(x) \]

for any biteral \( b \).
Proof. We proceed by induction on $b$.

Let $b \equiv e$. By $B_5$, we have

\[
\mathfrak{B} \models (\forall x \subseteq e) \phi (x) \iff \mathfrak{B} \models \phi (e) \implies B \vdash \phi (e) \implies B \vdash (\forall x \subseteq e) \phi (x).
\]

Let $b \equiv e \circ 0$. By $B_1$ and $B_6$, we have

\[
\mathfrak{B} \models (\forall x \subseteq e \circ 0) \phi (x) \iff \mathfrak{B} \models \phi (e) \land \phi (0) \implies B \vdash \phi (e) \land \phi (0) \implies B \vdash (\forall x \subseteq e \circ 0) \phi (x).
\]

Let $b \equiv e \circ 1$. This case is symmetric to the case $b \equiv e \circ 0$. Use $B_7$ in place of $B_6$.

Let $b \equiv e \circ 0 \circ t \circ 0$. Suppose by induction hypothesis (IH) that

\[
- \mathfrak{B} \models (\forall x \subseteq e \circ 0 \circ t) \phi (x) \implies B \vdash (\forall x \subseteq e \circ 0 \circ t) \phi (x)
\]

\[
- \mathfrak{B} \models (\forall x \subseteq t \circ 0) \phi (x) \implies B \vdash (\forall x \subseteq t \circ 0) \phi (x).
\]

Then, by the assumption on $\phi$ given in our lemma, we have

\[
\mathfrak{B} \models (\forall x \subseteq e \circ 0 \circ t \circ 0) \phi (x)
\]

\[
\downarrow
\]

\[
\mathfrak{B} \models (\forall x \subseteq e \circ 0 \circ t) \phi (x) \land (\forall x \subseteq t \circ 0) \phi (x) \land \phi (e \circ 0 \circ t \circ 0)
\]

\[
\downarrow
\]

\[
B \vdash (\forall x \subseteq e \circ 0 \circ t) \phi (x) \land (\forall x \subseteq t \circ 0) \phi (x) \land \phi (e \circ 0 \circ t \circ 0)
\]

\[
\downarrow
\]

\[
B \vdash (\forall x \subseteq e \circ 0 \circ t \circ 0) \phi (x).
\]

The case when $b \equiv e \circ 0 \circ t \circ 1$, the case when $b \equiv e \circ 1 \circ t \circ 0$ and the case when $b \equiv e \circ 1 \circ t \circ 1$ are handled similarly using $B_9$, $B_{10}$ and $B_{11}$, respectively, in place of $B_8$. \hfill \Box

Lemma 24. We have

\[
\mathfrak{B} \models \neg b_1 \subseteq b_2 \implies B \vdash \neg b_1 \subseteq b_2
\]

for any biterals $b_1, b_2$. Furthermore, we have

\[
\mathfrak{B} \models \neg t_1 \subseteq t_2 \implies B \vdash \neg t_1 \subseteq t_2
\]

for any variable-free $L_{BT}$-terms $t_1, t_2$.

Proof. We proceed by induction on $b_2$.

If $b_2 \equiv e$ and $\mathfrak{B} \models \neg b_1 \subseteq e$, then $\mathfrak{B} \models \neg b_1 = e$. By Lemma 21, we have $B \vdash \neg b_1 = e$. By $B_5$, we have $B \vdash \neg b_1 \subseteq e$. \hfill \Box
If $b_2 \equiv e \circ 0$ and $\emptyset \models \neg b_1 \subseteq e \circ 0$, then $\emptyset \models \neg b_1 = e \land \neg b_1 = 0$. By Lemma [24] we have $B \models \neg b_1 = e \land \neg b_1 = 0$. By $B_6$, we have $B \models \neg b_1 \subseteq e \circ 0$.

If $b_2 \equiv e \circ 1$ and $\emptyset \models \neg b_1 \subseteq e \circ 1$, then $\emptyset \models \neg b_1 = e \land \neg b_1 = 1$. By Lemma [24] we have $B \models \neg b_1 = e \land \neg b_1 = 1$. By $B_7$, we have $B \models \neg b_1 \subseteq e \circ 1$.

Let $b_2 \equiv e \circ 0 \circ t \circ 0$. Suppose by induction hypothesis that we have

\[- \emptyset \models s \subseteq e \circ 0 \circ t \Rightarrow B \models \neg s \subseteq e \circ 0 \circ t\]
\[- \emptyset \models s \subseteq t \circ 0 \Rightarrow B \models \neg s \subseteq t \circ 0\]

for any biteral $s$. Let $\emptyset \models \neg b_1 \subseteq b_2$. Then

\[\emptyset \models \neg b_1 \subseteq e \circ 0 \circ t \land \neg b_1 \subseteq t \circ 0 \land \neg b_1 = e \circ 0 \circ t \circ 0.\]

By our induction hypothesis and Lemma [24] we have

\[B \models \neg b_1 \subseteq e \circ 0 \circ t \land \neg b_1 \subseteq t \circ 0 \land \neg b_1 = e \circ 0 \circ t \circ 0.\]

By $B_8$, we have $B \models \neg b_1 \subseteq e \circ 0 \circ t \circ 0$.

The case when $b \equiv e \circ 0 \circ t \circ 0, 1$, the case when $b \equiv e \circ 0 \circ t \circ 0$ and the case when $b \equiv e \circ 0 \circ t \circ 1$ are handled similarly using $B_9, B_{10}$ and $B_{11}$, respectively, in place of $B_8$. Thus, we conclude that we have

\[\emptyset \models \neg b_1 \subseteq b_2 \Rightarrow B \models \neg b_1 \subseteq b_2.\]  

(*)

for any biterals $b_1, b_2$.

Let $t_1$ and $t_2$ be variable-free $L_{BT}$-terms such that $\emptyset \models \neg t_1 \subseteq t_2$. By Lemma [20] we have biterals $b_1$ and $b_2$ such that $B \models t_1 = b_1 \land t_2 = b_2$. Since $\emptyset \models B$, we also have $\emptyset \models t_1 = b_1 \land t_2 = b_2$. Hence $\emptyset \models \neg b_1 \subseteq b_2$. By (*), we have $B \models \neg b_1 \subseteq b_2$, and thus $B \models \neg t_1 \subseteq t_2$. $\square$

We are now prepared to prove Theorem [2]. We proceed by induction on the structure of the $\Sigma$-sentence $\phi$.

If $\phi$ is an atomic formula or the negation of an atomic formula, then applications of Lemma [24] Lemma [24] Lemma [24] or Lemma [24] give

\[\emptyset \models \phi \Rightarrow B \models \phi.\]

Let $\phi \equiv \alpha \lor \beta$. Assume $\emptyset \models \alpha \lor \beta$. Then we have $\emptyset \models \alpha$ or $\emptyset \models \beta$. We can w.l.o.g. assume that $\emptyset \models \alpha$. By our induction hypothesis, we have $B \models \alpha$. Finally, as $\alpha \lor \beta$ follows logically from $\alpha$, we conclude that $B \models \alpha \lor \beta$.

The case when $\phi \equiv \alpha \land \beta$ is similar to the case when $\phi \equiv \alpha \lor \beta$.

Let $\phi \equiv (\exists x)\alpha(x)$. The induction hypothesis yields

\[\emptyset \models \alpha(t) \Rightarrow B \models \alpha(t)\]
for any variable-free term \( t \). Now assume that \( \mathfrak{B} \models (\exists x)\alpha(x) \). Then there exists a biteral \( b \) such that \( \mathfrak{B} \models \alpha(b) \). By our induction hypothesis, we have \( B \vdash \alpha(b) \). As \( \vdash (\exists x)\alpha(x) \) follows logically from \( \alpha(b) \), we have \( B \vdash (\exists x)\alpha(x) \).

Let \( \phi \equiv (\exists x \sqsubseteq t)\alpha(x) \) where \( t \) is a variable-free term. The induction hypothesis yields

\[
\mathfrak{B} \models \alpha(t) \Rightarrow B \vdash \alpha(t)
\]

for any variable-free term \( t \). Assume \( \mathfrak{B} \models (\forall x \sqsubseteq t)\alpha(x) \). By Lemma \ref{lem:20} there exists a biteral \( b \) such that \( B \vdash t = b \). Obviously, \( \mathfrak{B} \models (\forall x \sqsubseteq b)\alpha(x) \). By Lemma \ref{lem:23} and our induction hypothesis, we have \( B \vdash (\forall x \sqsubseteq b)\alpha(x) \). Finally, as \( B \vdash t = b \), we have \( B \vdash (\forall x \sqsubseteq t)\alpha(x) \).

This completes the proof of Theorem \ref{thm:2}.

6 Proof of Theorem \ref{thm:4}: \( \Sigma \)-Completeness of \( D \)

We now proceed to prove that \( D \) is \( \Sigma \)-complete. Recall that the first four axioms of \( D \) are the same as the first four axioms of \( B \).

**Lemma 25.** For any variable-free \( \mathcal{L}_{BT} \)-term \( t \) there exists a biteral \( b \) such that \( D \vdash t = b \). Furthermore, we have

\[
\mathfrak{D} \models t_1 = t_2 \Rightarrow D \vdash t_1 = t_2
\]

for any variable-free \( \mathcal{L}_{BT} \)-terms \( t_1 \) and \( t_2 \).

*Proof.* This proof is identical to the proof of Lemma \ref{lem:20} \( \square \)

**Lemma 26.** For any biterrals \( b_1 \) and \( b_2 \)

\[
\mathfrak{D} \models \neg b_1 = b_2 \Rightarrow D \vdash \neg b_1 = b_2
\]

Furthermore, for any variable-free \( \mathcal{L}_{BT} \)-terms \( t_1 \) and \( t_2 \)

\[
\mathfrak{D} \models \neg t_1 = t_2 \Rightarrow D \vdash \neg t_1 = t_2
\]

*Proof.* This proof is identical to the proof of Lemma \ref{lem:21} \( \square \)
Lemma 27. We have
\[ \mathcal{D} \models b_1 \leq b_2 \Rightarrow D \vdash b_1 \leq b_2 \]
for any bimiters \( b_1 \) and \( b_2 \). Furthermore, we have
\[ \mathcal{D} \models t_1 \leq t_2 \Rightarrow D \vdash t_1 \leq t_2 \]
for any variable-free \( \mathcal{L}_{BT} \)-terms \( t_1 \) and \( t_2 \).

Proof. We proceed by induction on \( b_2 \).
If \( b_2 \equiv e \) and \( \mathcal{D} \models b_1 \leq b_2 \), then \( b_1 \) is \( e \). By \( D_5 \), we have \( D \vdash e \leq e \).
Let \( b_2 \equiv t \circ 0 \). Assume \( \mathcal{D} \models b_1 \leq b_2 \). Then \( \mathcal{D} \models b_1 \leq t \lor b_1 = b_2 \). By the induction hypothesis and Lemma 25, we have \( D \vdash b_1 \leq t \lor b_1 = b_2 \). By \( D_6 \), we have \( D \vdash b_1 \leq b_2 \).
The case when Let \( b_2 \equiv t \circ 1 \) is similar to the case \( b_2 \equiv t \circ 0 \). Use \( D_7 \) in place of \( D_6 \).
Thus, we conclude that
\[ \mathcal{D} \models b_1 \leq b_2 \Rightarrow D \vdash b_1 \leq b_2 \]
holds for any bimiters \( b_1, b_2 \). It is easy to see that also the second part of the theorem holds (see the proof Lemma 22). \( \square \)

Lemma 28. Let \( \phi(x) \) be an \( \mathcal{L}_{BT} \)-formula such that we have
\[ \mathcal{D} \models \phi(b) \Rightarrow D \vdash \phi(b) \]
for any biteral \( b \). Then, we also have
\[ \mathcal{D} \models (\forall x \leq b)\phi(x) \Rightarrow D \vdash (\forall x \leq b)\phi(x) \]
for any biteral \( b \).

Proof. We prove the lemma by induction on \( b \).
Let \( b \equiv e \). We have
\[ \mathcal{D} \models (\forall x \leq e)\phi(x) \iff \mathcal{D} \models \phi(e) \Rightarrow D \vdash \phi(e) \Rightarrow D \vdash (\forall x \leq e)\phi(x) \]
The last implication holds by \( D_5 \).
Let \( b \equiv t \circ 0 \). Assume by induction hypothesis that
\[ \mathcal{D} \models (\forall x \leq t)\phi(x) \Rightarrow D \vdash (\forall x \leq t)\phi(x) \]
By the assumption on \( \phi \) and the induction hypothesis, we have
\[ \mathcal{D} \models (\forall x \leq t \circ 0)\phi(x) \iff \mathcal{D} \models (\forall x \leq t)[\phi(x)] \land \phi(t \circ 0) \]
\[ \Rightarrow D \vdash (\forall x \leq t)[\phi(x)] \land \phi(t \circ 0) \]
\[ \Rightarrow D \vdash (\forall x \leq t \circ 0)\phi(x) \]
The last implication holds by \( D_6 \).
The case \( b \equiv t \circ 1 \) is similar to the case \( b \equiv t \circ 0 \). Use \( D_7 \) in place of \( D_6 \). \( \square \)
Lemma 29. We have

$$\mathcal{D} \models \neg b_1 \preceq b_2 \Rightarrow D \vdash \neg b_1 \preceq b_2$$

for any biterals $b_1$ and $b_2$. Furthermore, we have

$$\mathcal{D} \models \neg t_1 \preceq t_2 \Rightarrow D \vdash \neg t_1 \preceq t_2$$

for any variable-free $L_{BT}$-terms $t_1$ and $t_2$.

Proof. We proceed by induction on $b_2$.

Let $b_2 \equiv e$. Assume $\mathcal{D} \models \neg b_1 \subseteq e$. Then $\mathcal{D} \models \neg b_1 = e$. By Lemma 26 we have $D \vdash \neg b_1 = e$. By $D_5$, we have $D \vdash \neg b_1 \subseteq e$.

Let $b_2 \equiv t \circ 0$. Assume $\mathcal{D} \models \neg b_1 \preceq b_2$. Then $\mathcal{D} \models \neg b_1 \preceq t \land \neg b_1 = t$. By the induction hypothesis and Lemma 26, we have $D \vdash \neg b_1 \preceq t \land \neg b_1 = t$. By $D_6$, we have $D \vdash \neg b_1 \preceq b_2$.

The case $b_2 \equiv t \circ 1$ is similar to the case $b_2 \equiv t \circ 0$. Use $D_7$ in place of $D_6$.

This proves that

$$\mathcal{D} \models \neg b_1 \preceq b_2 \Rightarrow D \vdash \neg b_1 \preceq b_2$$

holds for any biterals $b_1, b_2$. It is easy to see that also the second part of the theorem holds (see e.g. the proof Lemma 24). \hfill \Box

Theorem 4 is proved by induction on the structure of the $\Sigma$-sentence $\phi$. Proceed as in the proof of Theorem 2 (see Section 5) and use the lemmas above.

References

1. Büchi, J. R. and Senger, S.: Coding in the existential theory of concatenation. Arch. math. Logik 26 (1986/7), 101-106.
2. Day, J., Ganesh, V., He, P., Manea, F. and Nowotka, D.: The satisfiability of extended word equations: The boundary between decidability and undecidability. arXiv:1802.00523 (2018).
3. Ganesh, V., Minnes, M., Solar-Lezama, A. and Rinard, M. C.: Word equations with length constraints: What’s decidable? In: Biere A., Nahir A., Vos T. (eds) Hardware and Software: Verification and Testing. HVC 2012. Lecture Notes in Computer Science, vol 7857, pp. 209-226. Springer, Berlin, Heidelberg.
4. Grzegorczyk, A.: Undecidability without arithmetization. Studia Logica 79 (2005), 163-230.
5. Grzegorczyk, A. and Zdanowski, K.: Undecidability and concatenation. pp. 72-91 in “Andrzej Mostowski and Foundational Studies” (eds. by Ehrenfeucht et al.), IOS, Amsterdam, 2008.
6. Halfon, S., Schnoebelen, P. and Zetzsche G: Decidability, complexity, and expressiveness of first-order logic over the subword ordering. In Proc. LICS 2017. IEEE Computer Society, 112.
7. Horihata, Y.: *Weak theories of concatenation and arithmetic*. Notre Dame Journal of Formal Logic, 53 (2012), 203-222.
8. Karhumäki, J., Mignosi, F. and Plandowski, W.: *The expressibility of languages and relations by word equations*. Journal of the ACM 47 (2000), 483-505.
9. Leary, C. and Kristiansen, L.: *A friendly introduction to mathematical logic*. 2nd Edition, Milne Library, SUNY Geneseo, Geneseo, NY, 2015.
10. Makanin, G. S.: *The problem of solvability of equations in a free semigroup*. Mathematics of the USSR-Sbornik 32 (1977), 129-198.
11. Post, E. L.: *A variant of a recursively unsolvable problem*. Bulletin of the American Mathematical Society 52 (1946), 264-268.
12. Quine, W. V.: *Concatenation as a basis for arithmetic*. The Journal of Symbolic Logic 11 (1946), 105-114.
13. Senger, S.: *The existential theory of concatenation over a finite alphabet*. PhD dissertation, Purdue University (1982).
14. Tarski, A.: *Der Wahrheitsbegriff in den formalisierten Sprachen*. Studia Philosophica 1 (1935), 261-405.
15. Tarski, A.: *Undecidable theories*. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Company, Amsterdam, 1953. In collaboration with A. Mostowski and R. M. Robinson.
16. Visser, A.: *Growing commas. A study of sequentality and concatenation*. Notre Dame Journal of Formal Logic, 50 (2009), 61-85.