A nonlinear preconditioner for experimental design problems

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Abstract We address the slow convergence and poor stability of quasi-newton sequential quadratic programming (SQP) methods that is observed when solving experimental design problems, in particular when they are large. Our findings suggest that this behavior is due to the fact that these problems often have bad absolute condition numbers. To shed light onto the structure of the problem close to the solution, we formulate a model problem (based on the $A$-criterion), that is defined in terms of a given initial design that is to be improved. We prove that the absolute condition number of the model problem grows without bounds as the quality of the initial design improves. Additionally, we devise a preconditioner that ensures that the condition number will instead stay uniformly bounded. Using numerical experiments, we study the effect of this reformulation on relevant cases of the general problem, and find that it leads to significant improvements in stability and convergence behavior.

Keywords Sequential Quadratic Programming · Preconditioning · Design of Experiments

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1 Introduction

One of the most important aspects in model based optimization and investigation of real world processes is the estimation of parameters appearing in a model. Typically, one estimates these parameters solving a regression problem based on data collected from one or more experiments. Optimal experimental design is the task of choosing the best experimental setup from a set of possible ones, and according to a predefined criterion. As this task is bound to be constrained in non-trivial ways, and is formulated in terms of the optimality conditions of an underlying regression problem, it presents a rich class of challenging optimization problems. It is also a practically relevant class,
as the solution of these problems leads to important improvements in the efficiency of research and development [320].

A standard setting is the estimation of parameters using weighted least-squares regression. It is then natural to rate the experiments according to the quality of the Fisher information matrix, or of its inverse, the variance-covariance matrix. A variety of criteria to rate this quality exists, and they are traditionally named after letters of the alphabet. In this article we will focus mainly on the so called A-criterion, which is defined as the trace of the variance-covariance matrix of the regression. For a full list and further discussion we refer to the classic texts [210].

The experimental design problems we are concerned with here have the following form. We consider the nonlinear regression problem

\[ \hat{p} = \arg\min_p \sum_{i=1}^m |F(p, q, t_i) - \mu_i|^2 \]  

where \( F \) is a nonlinear function depending on parameters \( p \), controls \( q \), and measurement points \( t_i \). Often these points refer to measurement times, but our scope is not restricted to this case. The values \( \mu_i \) represent the results of measurements, and \( F \) represents the model under study. If the measurement errors are independent, normally distributed with variance one and zero mean, the parameters obtained from (1) will be a random variable which, to a first approximation, is drawn from a multivariate normal distribution centered around the true parameter values \( p^* \), and with variance-covariance matrix \( \Sigma = (J^TJ)^{-1} \), where the matrix \( J \) is given by

\[
J_{ij} = \frac{\partial}{\partial p_j} F(p^*, q, t_i).
\]

The experiment design problem we consider is the problem of finding controls \( q \), and a subset of measurement points such that the trace of the variance-covariance matrix is minimal, at least for the parameters \( p \) that we believe to be plausible when designing the experiment.

Our basic problem is thus, given a set of feasible controls \( \Theta \), and a measurement budget \( m_{\text{max}} < m \), find a set \( M \subset \{1, 2, \ldots, m\} \) with cardinality \( \#M = m_{\text{max}} \), and a \( q \in \Theta \) such that

\[
\text{Tr}\left((J[M](q)^T J[M](q))^{-1}\right)
\]

is minimal. Here, we have made explicit the dependence on \( q \), and have denoted by \( J[M] \) the matrix \( J \) after deleting all rows whose index is not in the set \( M \).

There are a couple of approaches in the literature to solve this problem [15][10][6][8]. One important approach, and the one we will consider here, consists in using a relaxed formulation to obtain a constrained minimization problem in continuous variables, and then to apply a modern optimization method, typically in the form of a sequential quadratic programming (SQP) [15][20][14] algorithm to solve it. This approach was pioneered by [10][14][5], and the formulation has the following form.

Define \( W : \mathbb{R}^m \to \mathbb{R}^{m\times m} \) through \( W(w) := \text{diag}(w) \), a diagonal matrix whose entries are the elements of the vector of weights \( w \). We define the set of admissible weights

\[
\Omega(m_{\text{max}}, m) := \{w \in [0, 1]^m | \sum_{i=1}^m w_i = m_{\text{max}}\}
\]
The problem now is
\[ \min_{w,q} \operatorname{Tr} \left( \left[ J^T (q) W (w) J (q) \right]^{-1} \right) \]  \hspace{1cm} (2a)
subject to
\[ q \in \Theta, \quad w \in \Omega (m_{\max}, m). \]  \hspace{1cm} (2b)

The form of this problem is such that it can be given to an SQP solver as it is. It has the important advantage that the optimization of the controls and of the weights occurs at the same time. On the other hand, it has the disadvantage of not necessarily yielding integer solutions, although practical experience shows that either the solution is integer, or only a few weights are not, which can then be remedied by using a rounding technique [22]. The understanding of this phenomenon is incomplete, but see [23] for an important contribution on this issue. Alternatively, the weights in (2) can be interpreted as a required precision of the measurements, i.e. as a measure of the maximum variance of the error which is acceptable.

In practice, it turns out that solving the problem (2) in this way leads to poor convergence. Typical behavior can be seen in Figure 1, where we have plotted the length of the search direction obtained from the quadratic problem against the iteration number (the precise description of the numerical experiment can be found in section 3). It is important to remark that this difficulty appears also when there are no external controls, i.e. when the vector of controls \( q \) is empty.

In what follows, we will attempt to shed some light on the question as to why this behavior emerges, and in particular onto how to solve it. To this end, inspired by the “test equation” [3] used in the theory of stiff ordinary differential equations, we tackle the generality of the setting by developing a model problem. In this model problem we discover arbitrarily bad absolute conditioning under fairly generic conditions. It turns out that this can be corrected by introducing a simple but nonobvious transformation. This transformation can also be applied to (2), yielding a problem for which the SQP method converges in much less iterations than for the original one.

While the connection between absolute condition numbers and the difficulty of solving an optimization problem has been made before [24,25], its role in the convergence
behavior of SQP methods has not, to our knowledge, been investigated. Thus, while we
cannot provide a complete theoretical justification for the slow convergence of SQP
methods when solving (2), we provide below what we consider to be strong evidence for
the hypothesis that it is due to bad absolute condition numbers. At the very least we
provide, in the form of a left preconditioner, an effective way to significantly accelerate
the numerical solution of experimental design problems.

2 The model problem and its conditioning

We are led to our model problem by removing elements of the full problem (2). The
first simplification is the removal of the external controls. As a consequence, the matrix
J is fixed, and we have a problem in a form that is addressed by classical texts (see e.g.
[6]). Even this simplified problem is difficult to analyze, as the assumptions imposed
on J in this theory are rather weak. Informally speaking, once it has full rank, J
can be any old matrix. Since this class of problems is very general, one can expect
to find counterexamples to any generalization of behavior observed in practice. Our
approach will thus be to restrict ourselves to a model problem that is endowed with
enough structure to understand its badly-conditioned nature, and to devise a method
of curing it.

This problem is as follows. Given initial information in the form of a variance-
covariance matrix Σ (which we, for simplicity, will consider to be a multiple α
of the identity matrix) for the parameters of interest, our task is to choose between
two additional observations using a relaxed formulation. The idea of this problem is to
model an advanced stage of our experimental design, in which, for the sake of argument,
all weights except for two are known to be either one or zero. The role of the parameter
α is to model the quality of the initial design that is the starting point of our model
problem. The smaller the α, the better the design that is to be improved.

The optimization problem we will study is thus

$$
\min_{w_1,w_2} \text{Tr} \left( \left[ \alpha^{-1}I + w_1 v_1^T + w_2 v_2^T \right]^{-1} \right)
$$

subject to

$$
w_1 + w_2 = 1 \quad \text{and} \quad 0 \leq w_i \leq 1, \ i = 1, 2. \quad (3b)
$$

**Theorem 1** Suppose that $v_1^Tv_2 = 0$ and $\|v_1\| = \|v_2\| = 1$. Then the solution of
problem (3) is $w_1 = w_2 = 1/2$, and the absolute condition number of the solution is

$$
\kappa_{\text{abs}} = \left( \frac{1}{2} + \frac{1}{2} \alpha \right)^3 / 2\alpha^3.
$$

The use of the absolute condition number is due to the fact that we are searching
for the zero of a derivative. The error in the “data” is thus a perturbation of the zero,
which is only meaningful in an absolute sense.

**Proof** We write first $w_2 = 1 - w_1$ to eliminate the equality constraint, and thus obtain
a problem in one variable $w$. The function we want to minimize is then

$$
f(w) := \text{Tr} \left( \left[ \alpha^{-1}I + w v_1^T + (1 - w) v_2^T \right]^{-1} \right). \quad (5)
$$
The orthogonality of \( v_1 \) and \( v_2 \) simplifies the application of the Sherman-Morrison formula to obtain

\[
f(w) := 2\alpha - \frac{w\alpha^2}{1 + w\alpha} - \frac{(1 - w)\alpha^2}{1 + (1 - w)\alpha}.
\]

(6)

Now we look for \( w^* \) such that \( f'(w^*) = 0 \). Straight-forward algebraic arguments yield \( w^* = 1/2 \).

To investigate the effect of perturbations, we choose \( \epsilon > 0 \), and define the solution mapping \( g : (-\epsilon, \epsilon) \to \mathbb{R} \) by \( f'(g(\epsilon)) = \epsilon \). Of course, \( g(0) = w^* \), and the condition number of the problem \( f'(w) = 0 \) is given by

\[
\kappa_{\text{abs}} = \left| \frac{g'(0)}{g(0)} \right|.
\]

Using implicit differentiation on \( f'(g(\epsilon)) - \epsilon = 0 \) we obtain the expression

\[
\kappa_{\text{abs}} = \frac{1}{|f''(w^*)||w^*|}.
\]

(7)

To finish the proof, we only need to compute \( f''(w) \) and verify the expression.

Theorem 1 suggests that as the optimization progresses and our experimental design becomes better and better, that is, \( \alpha \) becomes smaller, the absolute condition number of the problem will increase. It will behave as \( \alpha^{-3} \) for small \( \alpha \). Informally speaking, the bottom of the valley in which the solution lies will become very flat, making it harder and harder to choose between two additional rows of roughly the same size. The model problem thus predicts the stagnation in the solution process, which is precisely what often occurs in practice.

At least for the model problem, there is a surprisingly simple remedy in the form of a left preconditioner. A left preconditioner for a given problem is a diffeomorphism in the dependent variables that preserves the solution, and at the same time it improves the condition number. Its name comes from the fact that it appears to the left of the function that defines the unpreconditioned problem. The definition we use here is an extension of the definition that is commonly used in linear algebra (see e.g. [23]) to a nonlinear setting. The next theorem suggests a possible choice of a left preconditioner for the model problem.

**Theorem 2** The problem

\[
\min_{w_1, w_2} - \left\{ \text{Tr} \left( \left[ \alpha^{-1} I + w_1 v_1 v_1^T + w_2 v_2 v_2^T \right]^{-1} \right) \right\}^{-2}
\]

subject to

\[
w_1 + w_2 = 1 \quad \text{and} \quad 0 \leq w_i \leq 1, \; i = 1, 2.
\]

(8a)

has the same minimum as \( (3) \). For every \( \alpha > 0 \) the absolute condition number of the minimum is \( \kappa_{\text{abs}} = 2 \).

**Proof** Repeat, with \( \tilde{f} := -f^{-2} \), the calculation of the condition number of Theorem 1.
In light of the above, a possible way to achieve better convergence when solving (2) is to take inspiration in Theorem 2 and choose the left preconditioner

$$h : (0, \infty) \to (-\infty, 0) \quad h(z) := -z^{-2}.$$ 

As a consequence, we propose (and recommend) to solve the following problem instead of (2):

$$\min_{w, q} - \left\{ \text{Tr} \left( J_T(q) W(w) J(q) \right)^{-1} \right\}^{-2} \quad \text{(9a)}$$

subject to the constraints

$$w \in \Omega(m_{\text{max}}, m) \quad \text{and} \quad q \in \Theta. \quad \text{(9b)}$$

As we will see in the next section, this modification has the desired effect of accelerating the convergence of SQP methods when solving experimental design problems.

3 Numerical experiments

In what follows, we will compare experimentally how well the SQP solver behaves when solving the original problem (2) versus the preconditioned problem (9). In the first experiment, we tackle the task of optimizing a design when given a prior information, but without external controls $q$. In the second experiment, we keep $m_{\text{max}}$ constant and vary the number of candidate measurements to observe how the reformulation affects performance as the size of the problem changes. The third numerical experiment is a full nonlinear experimental design problem with external controls on a model defined through a system of ordinary differential equations. With it, we intend to illustrate the effect of the reformulation on practical problems.

To make the results representative and reproducible, we programmed the SQP solver as it is described in [19], with an augmented Lagrangian penalty function as described in [25]. This results in a reasonably robust and effective solver. The quadratic problems are solved using QPOPT [9], and the Hessian is approximated using damped BFGS updates [19]. To avoid scaling issues as much as possible, we use as an initial Hessian approximation a diagonal matrix with the absolute values of the diagonal of the exact Hessian. This retains the diagonal of the exact Hessian in the unpreconditioned (convex) case, and assures positive definiteness in the preconditioned case.

We developed a Radau IIa solver of order 5 based on the ideas in [12] that uses internal numerical differentiation [2] to compute accurate sensitivities of the differential equation (here: 1st to 3rd order). All derivatives of algebraic functions, including the inversion of the information matrix, were computed using automatic differentiation [11, 10], with a Common Lisp AD package [5] extended for higher derivatives. The inversion of the information matrix was done by directly computing $J_T W(w) J$ and applying a Cholesky decomposition. To cope with stability issues, we used quad-double arithmetic [14]. This also ensures we can use the full range $[0, 1]$ for the weights [17].

The problems without external controls are linear, and thus are defined through the matrix $J$, which we generate randomly. We do this by filling a matrix with random entries uniformly distributed in $[-1, 1]$, performing a singular value decomposition, and substituting its singular values with exponentially decaying ones chosen to obtain a condition number of $10^4$. 

3.1 Effect of prior information

In this first experiment, we want to observe the behavior of SQP methods on the sampling design problem for choosing 20 out of 50 possible measurement points:

$$\min_{w \in \Omega(20,50)} \operatorname{Tr} \left( \left[ \alpha^{-1} I + J^T W(w) J \right]^{-1} \right)$$

with and without using the preconditioner. The matrix $J \in \mathbb{R}^{50 \times 7}$ is chosen at random as described before, but we additionally normalize all its rows in the euclidean norm. We thus obtain a nontrivial problem that is similar to our model, and where we can observe the effect of prior information $\alpha I$ directly.

In Figure 2 we summarize the results of solving problem (10) with and without preconditioning for 200 different matrices and 11 different values of $\alpha$, chosen equidistantly on a logarithmic scale in $[10^{-6}, 1]$. As $\alpha$ decreases, and thus the prior information becomes better and better, we see that the iteration count of the preconditioned variant stabilizes to around 40, while the sensitivity to the initial guess is about the same for each $\alpha$. This distance was estimated by comparing the solution we obtained using two different starting guesses for the iteration. One that has the first 20 components equal to 1, and the rest is zero, and one starting guess that is the reverse. Since the problem is convex, we can use this sensitivity as a proxy for the distance to the exact solution. This value tends to be a lot larger for the unpreconditioned problem, which also becomes very hard to solve for small $\alpha$. On the right of Figure 2 we have plotted the percentage of problems that could not be solved because the quadratic solver reached its default iteration limit, something that never occurred with the preconditioned formulation.

3.2 Effect of problem size

Now we consider matrices $J$ of size $50n \times 7$ for $n = 1, 2, \ldots, 10$. For each size we generated 200 matrices, and solved the problem

$$\min_{w \in \Omega(20,50n)} \operatorname{Tr} \left( \left[ J^T W(w) J \right]^{-1} \right)$$
with and without preconditioner. We plot average iteration counts in Figure 3 with
error bars giving the standard deviations. Again we observe a significant improvement,
on average, of the iteration counts for each problem. We also observe that, for the
unpreconditioned problem, the iteration count increases with the problem size $n$,
which is likely due to the fact that the value of the minimum becomes smaller with
matrix size, so that the effect predicted by Theorem 1 increases. Whether the iteration
counts increase or not for the preconditioned variant is not possible to tell conclusively
from this experiment, but we conjecture that it does.

3.3 Full nonlinear problem

Our original motivation was to find an advantageous formulation for full nonlinear
experimental design as it is relevant for applications in engineering. Thus we consider
in our next experiment an experimental design problem defined on a system of nonlinear
differential equations. For simplicity, we choose the FitzHugh-Nagumo [7,18] model,
which is given by the system

$$\begin{align*}
\dot{x}_1 &= x_1 - x_1^3 - x_2 + I \\
\dot{x}_2 &= a \cdot (x_1 + b + cx_2)
\end{align*}$$

(12) $x_1(t_0) = x_{0,1},$  \hspace{1cm} (13) $x_2(t_0) = x_{0,2}.$

The task is to find a experimental design to estimate the four fixed parameters $z = 0.25,$
$a = 0.02, b = 0.7, \text{ and } c = -0.8$ using the values of $I,$ $x_{0,1}, \text{ and } x_{0,2}$ as controls, which
are expected to satisfy the constraints

$$-5 < x_{0,1} < 5, \quad -5 < x_{0,2} < 5, \quad -1 < I < 0.5.$$  \hspace{1cm} (14)

At most 30 measurements should be taken. A measurement is the reading of the value
of any of the two variables at any of the times $t_i = 5i, i = 1, 2, \ldots, 100.$

The initial guess for the weights is that all of them are equal. We choose the initial
guess for the optimal controls randomly to be able to assess typical behavior as much
Table 1: Performance statistics for the FitzHugh-Nagumo example. The values of $k$ denote iteration counts, the subscripts $u$ and $p$ indicate that they refer to unpreconditioned and preconditioned variants. The angled brackets indicate average, and $\sigma$ stands for the standard deviation of the speed-up factor $\langle k_u/k_p \rangle$.

| score | $\langle k_p \rangle$ | $\langle k_u \rangle$ | $\langle k_u/k_p \rangle$ | $\sigma$ |
|-------|-----------------------|-----------------------|-----------------------------|---------|
| 5/0   | 46.0                  | 260.8                 | 3.9                         | 1.7     |

Fig. 4: Optimal design for the FitzHugh-Nagumo system.

4 Final remarks and outlook

In this article, we have studied the convergence problems of SQP methods when solving experimental design problems. Our findings suggest that bad absolute condition numbers are indeed the cause of slow convergence.

We work around the generality of the problem by studying a carefully chosen model problem involving the $A$-criterion. From the understanding gained about the conditioning of this model problem we derive a transformation that guarantees constant absolute condition numbers in this particular case.

We then provide strong experimental evidence that this transformation yields significant improvements in the convergence behavior of SQP methods when applied to relevant settings, where we observe that the improvement is more significant for larger...
problems. The transformation itself does not introduce any noticeable additional computational cost.

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