ASYMPTOTICS OF THE LEBOWITZ–RUBINOW–ROTHENBERG MODEL OF POPULATION DEVELOPMENT

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Abstract. We study a mathematical model of cell populations dynamics proposed by J. Lebowitz and S. Rubinow, and analysed by M. Rotenberg. Here, a cell is characterized by her maturity and speed of maturation. The growth of cell populations is described by a partial differential equation with a boundary condition. In the first part of the paper we exploit semigroup theory approach and apply Lord Kelvin’s method of images in order to give a new proof that the model is well posed. A semi-explicit formula for the semigroup related to the model obtained by the method of images allows two types of new results. First of all, we give growth order estimates for the semigroup, applicable also in the case of decaying populations. Secondly, we study asymptotic behavior of the semigroup in the case of approximately constant population size. More specifically, we formulate conditions for the asymptotic stability of the semigroup in the case in which the average number of viable daughters per mitosis equals one. To this end we use methods developed by K. Pichór and R. Rudnicki.

1. Introduction. In the Lebowitz–Rubinow–Rotenberg (LRR) model of cell populations dynamics [18, 23, 24] a cell is characterized by two variables, its maturity and speed of maturation. We assume that the maturity is a real number $x$ that belongs to the interval $I := (0, 1)$ and the speed of maturation $v$ belongs to the set $V := (a, b)$, where $a$ and $b$ are nonnegative real numbers such that $a < b < +\infty$. Growth of the cells’ population density is governed by the partial differential equation

$$\frac{\partial f}{\partial t} = -v \frac{\partial f}{\partial x},$$

(1.1)

where $f = f(x, v, t)$ with $t \geq 0$ is the cells’ density at $(x, v)$ at time $t$. In this model a cell starts maturing at $x = 0$ and divides reaching $x = 1$, and the boundary condition

$$v f(0, v, t) = p \int_V w k(w, v) f(1, w, t) \, dw$$

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describes the reproduction rule. Here \( k \) satisfies

\[
\int_V k(w, v) \, dv = 1
\]

for any \( w \in V \), and \( V \ni v \mapsto k(w, v) \) is the probability density of the daughter’s maturation velocity conditional on \( w \) being the velocity of the mother. Furthermore, it is assumed that \( p \geq 0 \) is the average number of viable daughters per mitosis. However, see [9], it may be also important to consider the case where there are cells that degenerate in the sense that their daughters inherit mother’s velocity. Such situation is described by the boundary condition

\[
f(0, v, t) = qf(1, v, t),
\]

where \( q \geq 0 \) is the average number of viable daughters per mitosis. Therefore, we combine these two cases and assume that the reproduction rule is characterized by the boundary condition

\[
vf(0, v, t) = p \int_V wk(w, v)f(1, w, t) \, dw + qvf(1, v, t), \quad v \in V.
\] (1.2)

It is well known, see [13, II.1.2], that the well-posedness of the problem (1.1)-(1.2) may be rephrased in terms of the semigroups theory as follows: The problem is well-posed if and only if the operator

\[
f \mapsto -v \frac{\partial f}{\partial x}
\]

with domain consisting of functions that are absolutely continuous with respect to \( x \) and satisfy (1.2) is the generator of a strongly continuous semigroup in the space of absolutely integrable functions.

The LRR model was studied by many authors, see for example [9, 17, 19, 27]. In the first part of this paper, in Section 2, we give a new proof of the generation theorem of Boulanouar [9, Theorem 2.2, Theorem 3.1]. To this end we use Lord Kelvin’s method of images. (For detailed introduction to the method of images see [6] and references given there. More examples may be found in [5, 7, 8].) As a by-product we obtain a semi-explicit formula for the semigroup \( T = \{ T(t), t \geq 0 \} \) related to the LRR model (see (2.24) and (2.26)). Moreover, this formula allows us to provide, in Section 3, growth/decay estimates for the LRR semigroup. In particular, in Theorem 3.5, we obtain new estimates in the case of decaying populations. Here we should point out that our formula (2.26) represents essentially the same information as Boulanouar’s formula (3.19) in [9]. However, we think that the method of images is a natural, straightforward approach, which allows for a more global point of view and gives a new insight into the model. In particular this leads to asymptotics results that are complementary to those of Boulanouar (see below).

Section 4 is devoted to the asymptotic behavior of the semigroup. Boulanouar proved [9, Theorem 6.1] that if \( p + q > 1 \), then properly rescaled LRR semigroup converges in the uniform topology to a rank one projection under some conditions on \( k \). (For precise statement, see Theorem 4.1.) We study the model in the case \( p + q \leq 1 \); this case is also of biological interest since in multicellular organisms the number of cells does not grow in an unrestricted way, as in the case \( p + q > 1 \). We note (see Remark 4.12) that the methods introduced by Boulanouar do not work in the case \( p + q = 1 \) and \( a = 0 \) – the most interesting case in our analysis.

We start by noting that in the case \( p + q = 1 \) the LRR semigroup is composed of Markov operators – this remark allows us to use the tools of the rich theory of
Markov semigroups (see for example [16, 21]). Then, using results of Pichór and Rudnicki [21], we prove that, for a fairly large class of kernels $k$, there is an invariant density $f_*$ for the LRR semigroup such that for all other densities $f$ we have
\[
\lim_{t \to \infty} \|T(t)f - f_*\| = 0,
\]
in an appropriate $L^1$-type norm. Interestingly, in this case there is a direct formula connecting $f_*$ with the stationary density for the kernel $k$. Moreover, we show that if $p + q < 1$, then the operators forming the LRR semigroup converge to zero in the strong operator topology; if, additionally, $a > 0$, the same is true in the operator norm. The last two statements are reflections of the fact that in the case $p + q < 1$ the cell population gradually dies out. In the case $a > 0$ all parts of the population die out uniformly fast. In the case $a = 0$, cells that mature very slowly survive much longer than the remaining cells and so the population dies out non-uniformly.

Our main result, combining Theorems 4.3 and 4.10 may be rephrased as follows.

**Theorem 1.1.** Let $T$ be the semigroup related to the LRR model.

(i) Suppose that $p > 0$ and $p + q = 1$. Assume that there exists a unique, up to an equivalence class, stationary density for the kernel $k$, that is, a nonnegative function $f_\diamond$ on $V$ with $\int_V f_\diamond = 1$, satisfying
\[
f_\diamond(v) = \int_V k(w, v)f_\diamond(w) \, dw
\]
for almost every $v \in V$. If $f_\diamond$ is strictly positive almost everywhere and the function
\[
v \mapsto v^{-1}f_\diamond(v)
\]
is integrable on $V$, then (1.3) holds with $f_*$ defined as
\[
f_*(v) := \frac{v^{-1}f_\diamond(v)}{\int_V w^{-1}f_\diamond(w) \, dw}, \quad v \in V.
\]

(ii) Suppose that $p + q < 1$. Then
\[
\lim_{t \to \infty} \|T(t)f\| = 0
\]
holds for any $f$ that is integrable on $I \times V$. Moreover, if $a > 0$, then
\[
\lim_{t \to \infty} \|T(t)\| = 0.
\]

In the last part, in Section 5, we discuss relations between Boulanouar’s assumptions [9, Theorem 6.1] on $k$ with these in Theorem 1.1. Moreover, in Remark 5.2, we compare asymptotic results obtained recently by Mokhtar-Kharroubi and Rudnicki [20] (see also [25, Chapter 6.3.5]) with part (i) of Theorem 1.1.

**2. Generation theorem.** As in Introduction we consider $I := (0, 1)$ as a measure space with the Lebesgue measure, denoted $\mu_L$, and fix real numbers $a, b$ such that $0 \leq a < b < +\infty$. We also let $V \subseteq (a, b)$ and introduce a $\sigma$-finite measure $\nu$ on $V$. From the biological point of view the most interesting cases are when $V$ equals $(a, b)$ or is its discrete subset (the underlying measure $\nu$ being the Lebesgue measure or the counting measure, respectively). However, in our generation theorem we do not need to assume that, and we can work in the abstract setup. Generalizations of the LRR model in the discrete case are discussed in [2, 3].

We denote
\[
\Omega := I \times V,
\]
see Figure 1, and introduce $L^1(\Omega)$ as the space of (equivalence classes of) absolutely integrable real functions on $\Omega$ with respect to the product measure $\mu = \mu_L \times \nu$. We also denote the standard $L^1$-norm by $\| \cdot \|_{L^1(\Omega)}$. Furthermore, we let $W^1(\Omega)$ to be the space of (equivalence classes of) functions $f \in L^1(\Omega)$ satisfying:

(i) for almost every $v \in V$ the function $I \ni x \mapsto f(x, v)$ is weakly differentiable,

(ii) the function $\Omega \ni (x, v) \mapsto v\partial_x f(x, v)$ belongs to $L^1(\Omega)$, where $\partial_x f$ is the weak derivative of $f$ with respect to $x$.

By the Sobolev embedding theorem, if $f \in W^1(\Omega)$, then for almost every $v \in V$ the function $I \ni x \mapsto f(x, v)$ has a unique representant which is continuous up to the boundary of $I$. Hence, in particular we can speak about $f(0, v)$ or $f(1, v)$.

Let $k: V \times V \to [0, +\infty)$ be a nonnegative $(\nu \times \nu)$-measurable real function such that

$$\int_V k(w, v) \nu(\mathrm{d}v) = 1, \quad w \in V. \quad (2.1)$$

Then we define the operator $A$ in $L^1(\Omega)$ by

$$Af(x, v) := -v\partial_x f(x, v), \quad (x, v) \in \Omega. \quad (2.2)$$

We let the domain $D(A)$ of $A$ to be composed of functions $f \in W^1(\Omega)$ satisfying the boundary condition

$$vf(0, v) = p\int_V w k(w, v) f(1, w) \nu(\mathrm{d}w) + qvf(1, v) \quad (2.3)$$

for almost every $v \in V$, where $p, q$ are fixed nonnegative real numbers such that $p + q > 0$. For simplicity of notation, for a given $v \in V$ we introduce the measure $\ell(\cdot, v) = \ell_\nu(\cdot, v)$ on $V$ by the formula

$$\ell(\mathrm{d}w, v) := pw^{-1} k(w, v) \nu(\mathrm{d}w) + q\delta_v(\mathrm{d}w), \quad (2.4)$$

Figure 1. The set $\Omega$. 
where \( \delta_v \) is the Dirac measure at \( v \). Then we may rewrite (2.3) in the form
\[
f(0,v) = \int_V f(1,w) \ell(dw,v).
\] (2.5)

The aim of this section is to prove the following result.

**Theorem 2.1.** The operator \( A \) generates a strongly continuous semigroup in \( L^1(\Omega) \).

To prove this theorem we can use the Lord Kelvin method of images. Indeed, formula (2.2) indicates that for a fixed \( v \in V \) the desired semigroup should resemble a translation semigroup. Hence, we would like to define
\[
T(t) = \{ T(t), t \geq 0 \}
\] in \( L^1(\Omega) \) by
\[
T(t)f(x,v) = \tilde{f}(x-tv,v), \quad t \geq 0, \ (x,v) \in \Omega, \ f \in L^1(\Omega),
\] (2.6)

where \( \tilde{f} \) is a function defined on
\[
\tilde{\Omega} := J \times V
\] for \( J := (-\infty,1) \). Since \( T(0)f \) equals \( f \), it follows that \( \tilde{f} \) must be an extension of \( f \). Moreover, because every semigroup leaves the domain of its generator invariant, given \( f \in D(A) \) we are looking for \( \tilde{f} : \tilde{\Omega} \to \mathbb{R} \) such that

(1) the restriction of \( \tilde{f} \) to \( \Omega \) equals \( f \), that is, \( \tilde{f}\Omega = f \),

(2) if \( f \in D(A) \), then \( T(t)f \) given by (2.6) belongs to \( D(A) \) for \( t \geq 0 \).

Construction of such extension \( \tilde{f} \) of \( f \) is the main part of the method of images.

**Lemma 2.2.** Given \( f \in D(A) \), if there exists \( \tilde{f} : \tilde{\Omega} \to \mathbb{R} \) satisfying (E1) and (E2), then it is uniquely determined.

**Proof.** Let \( f \in D(A) \). Condition (E2) implies in particular that \( \tilde{f} \) must be chosen in such a way that \( T(t)f \) given by (2.6) satisfies the boundary condition (2.5). Hence, we must have
\[
\tilde{f}(-tv,v) = \int_V \tilde{f}(1-tw,w) \ell(dw,v)
\]
for \( t \geq 0 \) and almost every \( v \in V \). If we denote \( x = -tv \), this may be rewritten as
\[
\tilde{f}(x,v) = \int_V \tilde{f}(1+xwv^{-1},w) \ell(dw,v), \quad x \leq 0, \ v \in V.
\] (2.8)

We set \( \Omega_0 := \Omega \) and
\[
\Omega_i := \{(x,v) \in \mathbb{R}^2 : v \in V, -ivb^{-1} < x \leq -(i-1)vb^{-1} \}, \quad i \geq 1,
\] (2.9)
see Figure 2.

For \( w \in V \) and \( j \geq 1 \) a little bit of algebra shows that
\[
(x,v) \in \Omega_j \text{ implies } (1+xwv^{-1},w) \in \bigcup_{i=0}^{j-1} \Omega_i.
\] (2.10)

Therefore, \( \tilde{f} \) is determined by induction: Having determined it on \( \bigcup_{i=0}^j \Omega_i \), \( j \geq 0 \) for \( (x,v) \in \Omega_{j+1} \) we determine \( \tilde{f}(x,v) \) by (2.8). This completes the proof.  

The reasoning presented in the proof of Lemma 2.2 suggests the following definition.
Figure 2. \( \tilde{\Omega} \) is the union of \( \Omega_i \)'s.

**Definition 2.3.** For \( f \in L^1(\Omega) \) we call \( \tilde{f} : \tilde{\Omega} \to \mathbb{R} \) satisfying

\[
\tilde{f}(x, v) = f(x, v)[x > 0] + \int_V \tilde{f}(1 + xwv^{-1}, w) \ell(dw, v)[x \leq 0]
\]

for almost every \((x, v) \in \tilde{\Omega},\) the *boundary extension* of \( f.\)

Here and subsequently we use the Iverson bracket notation [14, p. 24], that is, if \( P \) is a statement that can be true or false, then

\[
[P] = \begin{cases} 1, & P \text{ is true}, \\ 0, & \text{otherwise}. \end{cases}
\]

We note that the boundary extension is unique up to an equivalence class. We also stress that we do not assume that \( f \) belongs to \( D(A) \) in order to define \( \tilde{f}.\) However, what is crucial, boundary extensions of functions from the domain of \( A \) possess an important property which we describe in Lemma 2.8.

Now we need to find a suitable Banach space in which all extensions live. We introduce the product measure \( \mu_L \times \nu \) on \( \tilde{\Omega} \) (recall (2.7) for the definition of \( \tilde{\Omega} \)). We denote this product measure by \( \mu, \) as on \( \Omega. \) Given \( \omega \geq 0 \) we let \( L^1_\omega(\tilde{\Omega}) \) to be the space of equivalence classes of \( \mu \)-measurable functions \( f \) on \( \tilde{\Omega}, \) such that

\[
\|f\|_{L^1_\omega(\tilde{\Omega})} := \sup_{j \geq 0} e^{-\omega j} \|f\|_{L^1(\Gamma_j)} < +\infty,
\]

where

\[
\Gamma_j := \bigcup_{i=0}^j \Omega_i, \quad j \geq 0
\]

with \( \Omega_i \)'s defined as in (2.9). Here, we naturally set \( \|f\|_{L^1_\omega(\Gamma_j)} := \int_{\Gamma_j} |f| \, d\mu. \) It is easy to check that \( \| \cdot \|_{L^1_\omega(\tilde{\Omega})} \) is a norm on \( L^1_\omega(\tilde{\Omega}), \) and that \( L^1_\omega(\tilde{\Omega}) \) equipped with this norm is a Banach space.

**Lemma 2.4.** Let \( g \) be a \( \nu \)-integrable function defined on \( V. \) Then

\[
\int_V \int_V vw^{-1} g(w) \ell(dw, v) \nu(dv) = (p + q) \int_V g(w) \nu(dw).
\]
Figure 3. The set $\bigcup_{i=1}^{4}(1 + \Omega_i)$ is shaded.

Proof. The conclusion follows by (2.4), the Fubini theorem and (2.1).

Lemma 2.5. Assume that

$$\omega > \max\{\log(p + q), 0\}. \tag{2.13}$$

Then for $f \in L^1(\Omega)$ its boundary extension $\tilde{f}$ belongs to $L^1_\omega(\tilde{\Omega})$ and there exists $M_\omega > 0$ such that

$$\|\tilde{f}\|_{L^1_\omega(\tilde{\Omega})} \leq M_\omega \|f\|_{L^1(\Omega)}, \quad f \in L^1(\Omega). \tag{2.14}$$

Proof. Let $\omega > 0$, $f \in L^1(\Omega)$, and $\tilde{f}$ be its boundary extension. For $i \geq 1$, $v \in V$ we denote

$$\Omega_{i,v} := \{x \in \mathbb{R}: -ivb^{-1} < x \leq -(i - 1)v b^{-1}\}.$$  

Since $x \in \Omega_{i,v}$ implies $x \leq 0$, it follows by (2.11) that

$$\int_{\Omega_i} |\tilde{f}| \, d\mu = \int_V \int_{\Omega_{i,v}} |\tilde{f}(x,v)| \, dx \, \nu(dv)$$

$$\leq \int_V \int_{\Omega_{i,v}} |\tilde{f}(1 + x w v^{-1}, w)| \, dx \, \ell(dw, v) \, \nu(dv).$$

Changing variables $x \mapsto 1 + x w v^{-1}$ leads to

$$\int_{\Omega_i} |\tilde{f}| \, d\mu \leq \int_V \int_{1 + \Omega_{i,v}} v w^{-1} |\tilde{f}(x,w)| \, dx \, \ell(dw, v) \, \nu(dv)$$

$$= (p + q) \int_V \int_{1 + \Omega_{i,v}} |\tilde{f}(x,w)| \, dx \, \nu(dw),$$

where $1 + \Omega_{i,v}$ is the algebraic sum of $\{1\}$ and $\Omega_{i,v}$, with the last equality resulting from the Fubini theorem by Lemma 2.4. Thus

$$\int_{\Omega_i} |\tilde{f}| \, d\mu \leq (p + q) \int_{1 + \Omega_i} |\tilde{f}| \, d\mu, \quad i \geq 1, \tag{2.15}$$

where $1 + \Omega_i$ is the algebraic sum of $\{1\} \times V$ and $\Omega_i$. Furthermore, we have

$$1 + \Gamma_j \subset \Gamma_{j-1}, \quad j \geq 1;$$

see Figure 3 or use (2.10) with $w := v$. Combining this with (2.15), for $j \geq 1$ we
obtain
\[
\int_{\Gamma_j} |\tilde{f}| \, d\mu = \|f\|_{L^1(\Omega)} + \int_{\Gamma_j \setminus \Omega} |\tilde{f}| \, d\mu \leq \|f\|_{L^1(\Omega)} + (p + q) \int_{\Gamma_{j-1}} |\tilde{f}| \, d\mu.
\]
Hence, induction shows that
\[
\int_{\Gamma_j} |\tilde{f}| \, d\mu \leq \|f\|_{L^1(\Omega)} + \sum_{i=0}^{j-1} (p + q)^i, \quad j \geq 1.
\]
This implies that if \( p + q < 1 \), then for \( \omega \geq 0 \) we have
\[
\|\tilde{f}\|_{L^1_\omega(\tilde{\Omega})} \leq \frac{1}{1 - p - q} \|f\|_{L^1(\Omega)}. \tag{2.17}
\]
On the other hand, if \( p + q > 1 \) and \( \omega \geq \log(p+q) \), then we have \( \sup_{j \geq 0} e^{-\omega j}(p+q)^j \leq 1 \) and by (2.16) it follows that
\[
\|\tilde{f}\|_{L^1_\omega(\tilde{\Omega})} \leq \frac{p + q}{p + q - 1} \|f\|_{L^1(\Omega)}.
\]
Finally, if \( p + q = 1 \) and \( \omega > 0 \), then again by (2.16)
\[
\|\tilde{f}\|_{L^1_\omega(\tilde{\Omega})} \leq \sup_{j \geq 0} (j+1)e^{-\omega j} \|f\|_{L^1(\Omega)} \leq \frac{1}{\omega} e^{\omega-1} \|f\|_{L^1(\Omega)},
\]
which proves (2.14). \( \Box \)

We denote by \( \mathbb{Y} \) the set of all boundary extensions, that is,
\[
\mathbb{Y} := \{ \tilde{f} : f \in L^1(\Omega) \}.
\]
It is clear from (2.11) that for \( f, g \in L^1(\Omega) \) and \( \alpha \in \mathbb{R} \) we have \( \tilde{\alpha f} = \alpha \tilde{f} \) and \( \tilde{f} + \tilde{g} = \tilde{f + g} \). This implies that \( \mathbb{Y} \) is a linear space and in view of Lemma 2.5 we have \( \mathbb{Y} \subseteq L^1_\omega(\tilde{\Omega}) \) provided that (2.13) holds. Here and subsequently we fix such \( \omega \), that is,
\[
\omega > \max\{ \log(p+q), 0 \}.
\]
We endow \( \mathbb{Y} \) with \( \| \cdot \|_{L^1_\omega(\tilde{\Omega})} \) norm, and define the extension operator
\[
E : L^1(\Omega) \to \mathbb{Y}
\]
by
\[
Ef := \tilde{f}.
\]

In order to set up notation, given Banach spaces \( X, Y \) we let \( \mathcal{L}(X,Y) \) to be the space of bounded linear operators \( X \to Y \) with standard operator norm \( \| \cdot \|_{\mathcal{L}(X,Y)} \). If \( Y = X \) we write \( \mathcal{L}(X,X) = \mathcal{L}(X) \).

**Proposition 2.6.** The operator \( E \) is an isomorphism between the spaces \( L^1(\Omega) \) and \( \mathbb{Y} \). Moreover
\[
\|E\|_{\mathcal{L}(L^1(\Omega),L^1_\omega(\tilde{\Omega}))} \leq M_\omega, \tag{2.18}
\]
and
\[
\|E^{-1}\|_{\mathcal{L}(L^1_\omega(\tilde{\Omega}),L^1(\Omega))} = 1, \tag{2.19}
\]
where \( M_\omega \) is the constant from (2.14).
Proof. Inequality (2.18) follows directly from Lemma 2.5. On the other hand, since 
\[ E\hat{f} = 0 \implies 0 = \hat{f}_\Omega = f, \]  
the operator \( E \) is one-to-one. Moreover, the inverse \( E^{-1} \) is the restriction operator, that is, \( E^{-1}f = \hat{f}_\Omega \). Hence

\[ \|E^{-1}f\|_{L^1(\Omega)} = \|\hat{f}\|_{L^1(\Omega)} = \|f\|_{L^1(\Gamma_0)} \leq \|f\|_{L^1(\tilde{\Omega})}, \quad f \in L^1_\omega(\tilde{\Omega}). \]

This shows (2.19) and completes the proof. \( \square \)

Let now \( \tilde{T} = \{\tilde{T}(t), t \geq 0\} \) be the family of operators in \( L^1_\omega(\tilde{\Omega}) \) given by

\[ \tilde{T}(t)f(x,v) := f(x - tv, v), \quad t \geq 0, \quad (x,v) \in \tilde{\Omega}, \quad f \in L^1_\omega(\tilde{\Omega}). \]  \( \text{(2.20)} \)

A standard reasoning shows that \( \tilde{T} \) is a strongly continuous semigroup and its generator \( \tilde{A} \) is given by

\[ \tilde{A}f(x,v) := -v\partial_x f(x,v), \quad (x,v) \in \tilde{\Omega}, \quad f \in L^1_\omega(\tilde{\Omega}), \]

with domain

\[ D(\tilde{A}) := W^1_\omega(\tilde{\Omega}), \]

where \( W^1_\omega(\tilde{\Omega}) \) is the space of (equivalence classes of) functions \( f \in L^1_\omega(\tilde{\Omega}) \) satisfying:

(i) for almost every \( v \in V \) the function \( J \ni x \mapsto f(x,v) \) is weakly differentiable,

(ii) the function \( \Omega \ni (x,v) \mapsto v\partial_x f(x,v) \) belongs to \( L^1_\omega(\tilde{\Omega}) \).

Analogously as in the case of \( W^1(\Omega) \), if \( f \in W^1_\omega(\tilde{\Omega}) \), then for almost every \( v \in V \) the function \( J \ni x \mapsto f(x,v) \) may be uniquely extended to a continuous function on \( (\infty,1] \).

Lemma 2.7. The space \( \mathbb{Y} \) is invariant for the semigroup \( \tilde{T} \).

Proof. Fix \( t \geq 0 \). Let \( f \in L^1(\Omega) \), and let \( \tilde{f} \in \mathbb{Y} \) be its boundary extension. We prove that \( \tilde{T}(t)\tilde{f} \) is the boundary extension of \( g \in L^1(\Omega) \) defined by

\[ g(x,v) = \tilde{f}(x - tv, v), \quad (x,v) \in \Omega. \]

We proceed by induction, showing \( \tilde{g} = \tilde{T}(t)\tilde{f} \) on each \( \Gamma_j \), \( j \geq 0 \) (see (2.12)). Let \( (x,v) \in \tilde{\Omega} \) and recall that by (2.20), \( \tilde{T}(t)\tilde{f}(x,v) = \tilde{f}(x - tv, v) \). If \( (x,v) \in \Omega = \Gamma_0 \), then by (2.11) we have

\[ \tilde{g}(x,v) = g(x,v) = \tilde{T}(t)\tilde{f}(x,v). \]

Fix \( j > 0 \) and assume that \( \tilde{g}(x,v) = \tilde{T}(t)\tilde{f}(x,v) \) for \( (x,v) \in \Gamma_j \). If \( (x,v) \in \Omega_{j+1} = \Gamma_{j+1} \setminus \Gamma_j \), then for each \( w \in V \) it follows that \( 1 + xv^{-1}, w \) \( \in \Gamma_j \) by (2.10). Therefore, by (2.11),

\[ \tilde{g}(x,v) = \int_V \tilde{g}(1 + xwv^{-1}, w) \ell(dw,v) \]

\[ = \int_V \tilde{T}(t)\tilde{f}(1 + xwv^{-1}, w) \ell(dw,v) \]

\[ = \int_V \tilde{f}(1 + (x - tv)wv^{-1}, w) \ell(dw,v) \]

\[ = \tilde{f}(x - tv, v) \]

for almost every \( (x,v) \in \Gamma_{j+1} \setminus \Gamma_j \). This shows

\[ \tilde{g}(x,v) = \tilde{T}(t)\tilde{f}(x,v), \quad (x,v) \in \Gamma_{j+1}, \]

which completes the proof. \( \square \)
By Lemma 2.7 the part $\hat{A}_v$ of $\hat{A}$ in $\mathbb{V}$, that is, the operator defined as

$$\hat{A}_v := \hat{Af}, \quad D(\hat{A}_v) := \{f \in D(\hat{A}) \cap \mathbb{V}: \hat{Af} \in \mathbb{V}\},$$

generates a strongly continuous semigroup $\{\hat{T}_v(t), t \geq 0\}$ in $\mathbb{V}$ given by

$$\hat{T}_v(t)f(x, v) := f(x - tv, v), \quad t \geq 0, \quad (x, v) \in \Omega, \quad f \in \mathbb{V},$$

see for example [12, Corollary II.2.3].

**Lemma 2.8.** Let $f \in L^1(\Omega)$. We have $f \in D(A)$ if and only if $\hat{f} \in D(\hat{A}_v)$.

**Proof.** Assume first that $\hat{f} \in D(\hat{A}_v)$. Of course $f = \hat{f} \mid \Omega \in W^1(\Omega)$. We need to show that (2.5) holds. Let $U$ be a measurable subset of $V$ such that $\nu(V \setminus U) = 0$ and $\hat{f} |_{\Omega \times U}$ is weakly differentiable with respect to $v$. Without loss of generality we may assume that the function $[0, 1] \ni x \mapsto f(x, v)$ is continuous for every $v \in U$ and $f(x, v) = \hat{f}(x, v)$ for $(x, v) \in [0, 1] \times U$. Thus, for $v \in U$ by (2.11) we have

$$f(0, v) = \hat{f}(0, v) = \int_V \hat{f}(1, w) \ell(dw, v) = \int_V f(1, w) \ell(dw, v),$$

proving that $f \in D(A)$.

On the other hand, let $f \in D(A)$. The Hille-Yosida theorem implies that there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ the operator

$$\lambda - \hat{A}_v: D(\hat{A}_v) \rightarrow L^1(\Omega)$$

is bijective. Let $\lambda > \lambda_0$ and set

$$F(x, v) := \lambda f(x, v) + v \partial_x f(x, v), \quad (x, v) \in \Omega.$$

Then $F \in L^1(\Omega)$ since $f \in D(A)$, and hence there exists $g \in D(\hat{A}_v)$ satisfying

$$\lambda g - \hat{A}_v g = \hat{F}.$$  \hfill (2.21)

By the first part of the proof it follows that $E^{-1}g \in D(A)$. Therefore, letting

$$h := E^{-1}g - f$$

we see that $h \in D(A)$ and (2.21) gives

$$\lambda h(x, v) + v \partial_x h(x, v) = 0, \quad x \in I$$  \hfill (2.22)

for almost every $v \in V$. Thus, if we fix such $v \in V$, then by [15, Corollary 3.1.6] the function $I \ni x \mapsto h(x, v)$ is in fact a classical solution to (2.22). This implies

$$h(x, v) = c_{\lambda, v}e^{-\lambda x v^{-1}}, \quad x \in I$$

for a constant $c_{\lambda, v}$. However, since (2.5) holds for $h$, we get

$$c_{\lambda, v} = \int_V c_{\lambda, w}e^{-\lambda w^{-1}} \ell(dw, v).$$

Let

$$C_\lambda := \int_V |c_{\lambda, v}| \nu(dv).$$

We note that $C_\lambda$ is finite. Indeed, we have

$$\int_V |c_{\lambda, w}|e^{-\lambda w^{-1}} \ell(dw, v) = \int_V \int_I |c_{\lambda, w}|e^{-\lambda w^{-1}} dx \ell(dw, v) \leq \int_V \int_I |h(x, w)| dx \ell(dw, v),$$
the inequality resulting from $e^{-\lambda w^{-1}} < e^{-\lambda v^{-1}}$ for $x \in I$ and $w \in V$. Then, by the Fubini theorem,
\[
C_\lambda \leq \int_I \int_V \int_V v|h(x, w)| \ell(dw, v) \nu(dv) \, dx
\]
\[
= (p + q) \int_I \int_V w|h(x, w)| \nu(dw) \, dx
\]
\[
\leq (p + q) \int_I \int_V \|h\|_{L^1(\Omega)} \, dx,
\]
the equality being a consequence of Lemma 2.4 with $g(w) = g_x(w) := w|h(x, w)|$; this $g$ is $\nu$-integrable for almost every $x$ because $h \in L^1(\Omega)$. Since $e^{-\lambda w^{-1}} < e^{-\lambda v^{-1}}$ for $w \in V$,
\[
C_\lambda \leq e^{-\lambda b^{-1}} \int_V \int_V v|c_{\lambda, w}| \ell(dw, v) \nu(dv) = (p + q)e^{-\lambda b^{-1}} C_\lambda
\]
by Lemma 2.4 with $g(w) := w|c_{\lambda, w}|$; this $g$ is $\nu$-integrable because $C_\lambda$ is finite. This is true for all $\lambda > \lambda_0$ which leads to $C_\lambda = 0$ for $\lambda > \lambda'_0 := \max\{\lambda_0, b \ln(p + q)\}$. Hence $c_{\lambda, v} = 0$ for almost every $v \in V$, provided $\lambda > \lambda'_0$. Thus $h = 0$ and $f = E^{-1} g$. Finally, by the uniqueness of boundary extensions,
\[
\tilde{f} = EE^{-1} g = g \in D(\tilde{A}_V),
\]
which completes the proof. \hfill \Box

Now we are ready to prove the generation theorem.

**Proof of Theorem 2.1.** Let $T = \{T(t), t \geq 0\}$ be the family of linear operators in $L^1(\Omega)$ defined by
\[
T(t) := E^{-1}\tilde{T}_V(t)E, \quad t \geq 0.
\]
(2.23)
Then, see for example [4, 7.4.22], $T$ is a strongly continuous semigroup in $L^1(\Omega)$ similar to $\{\tilde{T}_V(t), t \geq 0\}$. Moreover, its generator is the operator $E^{-1}\tilde{A}_V E$ with domain $E^{-1}D(\tilde{A}_V)$, which equals $D(A)$ by Lemma 2.8. If $f \in D(A)$, then by (2.11) it follows that
\[
\partial_x \tilde{f}(x, v) = \partial_x f(x, v)
\]
for almost every $(x, v) \in \Omega$. Therefore
\[
(E^{-1}\tilde{A}_V E)f = Af, \quad f \in D(A).
\]
This shows that $A$ is the generator of $T$ and the theorem follows. \hfill \Box

We call the semigroup generated by $A$ the **LRR semigroup** and in what follows denote it by $T = \{T(t), t \geq 0\}$. By (2.23) we have
\[
T(t)f(x, v) = \tilde{f}(x - tv, v), \quad t \geq 0, \quad (x, v) \in \Omega, \quad f \in L^1(\Omega),
\]
(2.24)
as conjectured in (2.6). Furthermore, we introduce
\[
x_{wvt} := 1 + xw^{-1} - tw;
\]
(2.25)
for a cell characterized by a pair $(x, v) \in \Omega$ at time $t \geq 0$, $x_{wvt}$ is the maturity parameter of a potential mother of the cell with maturation speed $w$ at time 0, under proviso that $x_{wvt} \in I$. Relations (2.11) and (2.24) imply that given $t \geq 0$ and $f \in L^1(\Omega)$,
\[
T(t)f(x, v) = f(x - tv, v)[x > tv] + \int_V \tilde{f}(x_{wvt}, w) \ell(dw, v)[x \leq tv]
\]
(2.26)
for almost every \((x, v) \in \Omega\). In particular, let \(t \in [0, b^{-1}]\). We have \(x_{\text{wet}} > 1 - tw > 1 - wb^{-1} > 0\) for \((x, v) \in \Omega\) and \(w \in V\). Hence \(f(x_{\text{wet}}) = f(x_{\text{wet}})\), and we may rewrite the above relation in the form

\[
T(t)f(x, v) = f(x - tv, v)[x > tv] + \int_V f(x_{\text{wet}}, w) \ell(dw, v)[x \leq tv]
\]

(2.27)

for almost every \((x, v) \in \Omega\), provided that \(t \in [0, b^{-1}]\).

It is worth noting that the extension \(\tilde{f}\) is nonnegative provided that \(f\) is nonnegative, and hence \(T(t)\) is a positive operator for \(t \geq 0\). We use this fact later on.

3. Growth estimates. In this section we estimate the growth of the LRR semigroup \(T\). We still consider the general case, where \(V \subseteq (a, b)\) and \(\nu\) is any \(\sigma\)-finite measure on \(V\). However, we also need to assume here, which is not particularly restrictive, that \(\nu(V \cap U) > 0\) for any open interval \(U \subseteq (a, b)\).

Let \(T^* = \{T^*(t) : t \geq 0\}\) be the dual semigroup of \(T\), see [12, I.5.14]. First we find an explicit formula for the adjoint

\[
T^*(t) : L^\infty(\Omega) \to L^\infty(\Omega)
\]

of \(T(t)\) for \(t \in [0, b^{-1}]\), where \(L^\infty(\Omega)\) is the space of (equivalence classes of) essentially bounded functions on \(\Omega\) with standard essential supremum norm \(\| \cdot \|_{L^\infty(\Omega)}\).

We denote

\[
x_{\text{wet}}^* := (x + tv - 1)wv^{-1};
\]

for a cell characterized by a pair \((x, v) \in \Omega\) at time 0, \(x_{\text{wet}}^*\) is the maturity parameter of a potential daughter of the cell with maturation speed \(w\) at time \(t \geq 0\), under proviso that \(x_{\text{wet}}^* \in I\). Note that \(x_{\text{wet}}^* < 0\) reflects the fact that the cell characterized by \((x, v)\) at time 0 will not mature fast enough to divide before time \(t\). Analogously, \(x_{\text{wet}}^* \geq 0\) means that the cell will divide before time \(t\).

To simplify notation, let

\[
k^*(w, v) := k(v, w), \quad w, v \in V,
\]

and for \(v \in V\) let \(\ell^*(v, \cdot) = \ell_v^*(v, \cdot)\) be the measure on \(V\) defined by

\[
\ell^*(v, dw) := pk^*(w, v) \nu(dw) + q\delta_v(dw).
\]

Lemma 3.1. For \(t \in [0, b^{-1}]\) the adjoint operator \(T^*(t)\) of \(T(t)\) is given by

\[
T^*(t)\varphi(x, v) = \varphi(x + tv, v)[x_{\text{wet}}^* < 0] + \int_V \varphi(x_{\text{wet}}^*, w) \ell^*(v, dw)[0 \leq x_{\text{wet}}^*]
\]

(3.1)

for \(\mu\)-almost every \((x, v) \in \Omega\).

Proof. Let \(\varphi \in L^\infty(\Omega)\) and fix \(t \in [0, b^{-1}]\). By the definition of the adjoint operator

\[
\int_\Omega fT^*(t)\varphi \, d\mu = \int_\Omega \varphi T(t)f \, d\mu, \quad f \in L^1(\Omega).
\]

(3.2)

By (2.27) and (2.4), we have

\[
\int_\Omega \varphi T(t)f \, d\mu = I_1 + I_2 + I_3,
\]

(3.3)

where

\[
I_1 = \int_I \int_I \varphi(x, v)f(x - tv)[x > tv] \, dx \, \nu(dv),
\]

\[
I_2 = \int_I \int_I \varphi(x, v)f(x - tv)[x \leq tv] \, dx \, \nu(dv),
\]

\[
I_3 = \int_I \int_I \varphi(x, v)f(x - tv) \ell(dw, v)[x \leq tv] \, dx \, \nu(dv),
\]
Changing variables $x \mapsto x - tv$ we obtain

\[ I_1 = \int_V \int_I \varphi(x + tv, v) f(x, v) [x_{wvt} > 0] \, dx \, d\nu(dv), \]

since $[0 < x + tv < 1] [x + tv > tv] = [0 < x < 1] [x_{wvt} < 0]$ for $(x,v) \in \Omega$. Similarly, changing variables $x \mapsto x_{wvt}$, or equivalently $x_{wvt} \mapsto x$,

\[ I_2 = p \int_V \int_I k(w, v) \varphi(x_{wvt}, v) f(x, w) [0 \leq x_{wvt}] \, dx \, d\nu(dv), \]

and

\[ I_3 = q \int_V \int_I \varphi(x_{wvt}, v) f(x, v) [0 \leq x_{wvt}] \, dx \, d\nu(dv), \]

since $[0 < x_{wvt} < 1] [x_{wvt} \leq tv] = [0 < x < 1] [0 < x_{wvt}]$ for $(x,v) \in \Omega$ and $w \in V$. By (3.2), (3.3), and the definition of $k^*$, changing the order of integration in $I_2$, we obtain (3.1). \qed

We know that

\[ \|T(t)\|_{L^1(\Omega)} = \|T^*(t)\|_{L^{\infty}(\Omega)}, \quad t \geq 0. \tag{3.4} \]

Since $T(t)$ is a positive operator for $t \geq 0$, the same is true for $T^*(t)$. Hence, for $\varphi \in L^\infty(\Omega)$ such that $\|\varphi\|_{L^\infty(\Omega)} \leq 1$ we have

\[ -T^*(t)1_\Omega \leq -T^*(t)|\varphi| \leq T^*(t)\varphi \leq T^*(t)|\varphi| \leq T^*(t)1_\Omega, \]

where $1_\Omega$ is the indicator function of $\Omega$. This leads to

\[ \|T^*(t)\|_{L^1(\Omega)} = \sup_{\varphi \in L^\infty(\Omega)} \|T^*(t)\varphi\|_{L^\infty(\Omega)} = \|T^*(t)1_\Omega\|_{L^\infty(\Omega)}, \quad t \geq 0, \]

and by (3.4) we obtain

\[ \|T(t)\|_{L^1(\Omega)} = \|T^*(t)1_\Omega\|_{L^\infty(\Omega)}, \quad t \geq 0. \tag{3.5} \]

Finally, by (3.1) it follows that

\[ T^*(t)1_\Omega(x,v) = [x_{wvt} < 0] + (p + q) [0 \leq x_{wvt}] \tag{3.6} \]

for $t \in [0,b^{-1}]$ and $\mu$-almost every $(x,v) \in \Omega$.

**Lemma 3.2.** We have

\[ \|T(t)\|_{L^1(\Omega)} = \max\{1, p + q\}, \quad t \in (0, b^{-1}]. \]

**Proof.** Let $t \in (0, b^{-1}]$. Equality (3.6) shows that

\[ \|T^*(t)1_\Omega\|_{L^\infty(\Omega)} = \max\{1, p + q\} \]

because the sets $\{ (x,v) \in \Omega : x < 1 - tv \}$ and $\{ (x,v) \in \Omega : 1 - tv \leq x \}$ are both of positive $\mu$-measure. By (3.5) this is the desired conclusion. \qed

Recall that a positive linear operator $S : L^1(\Omega) \to L^1(\Omega)$ is a *Markov operator if $\|Sf\|_{L^1(\Omega)} = 1$ provided that $f \in L^1(\Omega)$ is nonnegative and $\|f\|_{L^1(\Omega)} = 1$. In case $p + q = 1$ we may improve Lemma 3.2 as follows.
Lemma 3.3. Let \( t \in [0, b^{-1}] \) and assume that \( p + q = 1 \). Then \( T(t) \) is a Markov operator.

Proof. Recall that the operator \( T(t) \) is positive. Let \( f \in L^1(\Omega) \) be nonnegative and such that \( \|f\|_{L^1(\Omega)} = 1 \). Then by (3.2),
\[
\|T(t)f\|_{L^1(\Omega)} = \int_{\Omega} 1_{\Omega} T(t)f \, d\mu = \int_{\Omega} f T^*(t)\mathbb{1}_{\Omega} \, d\mu.
\]
However, by (3.6) we have \( T^*(t)\mathbb{1}_{\Omega} = \mathbb{1}_{\Omega} \), thus \( \|T(t)f\|_{L^1(\Omega)} = \|f\|_{L^1(\Omega)} \), which completes the proof. \( \square \)

Theorem 3.4. Let \( t \geq 0 \).

(i) If \( p + q > 1 \), then
\[
\|T(t)\|_{L(L^1(\Omega))} \leq (p + q)^{\lceil tb \rceil},
\]
where \( \lceil tb \rceil \) is the smallest integer larger than or equal to \( tb \).

(ii) If \( p + q = 1 \), then \( T(t) \) is a Markov operator.

(iii) If \( p + q < 1 \), then
\[
\|T(t)\|_{L(L^1(\Omega))} \leq 1.
\]

Proof. Let \( n := \lceil tb \rceil \) and \( s := t/n \). Hence \( s \in (0, b^{-1}] \). By the semigroup property and Lemma 3.2 we have
\[
\|T(t)\|_{L(L^1(\Omega))} \leq \|T(s)\|_{L(L^1(\Omega))} \leq \max\{1, p + q\}^n,
\]
which proves (i) and (iii).

In order to show (ii) we fix \( n \) and \( s \) as above. Then by Lemma 3.3, \( T(s) \) is a Markov operator, and so is \( T(t) = [T(s)]^n \) as a power of a Markov operator. \( \square \)

Theorem 3.5. Assume that \( p + q < 1 \).

(i) If \( a = 0 \), then
\[
\|T(t)\|_{L(L^1(\Omega))} = 1, \quad t \geq 0.
\]

(ii) If \( a > 0 \), then
\[
\|T(t)\|_{L(L^1(\Omega))} = 1, \quad 0 \leq t < a^{-1},
\]
and
\[
\|T(t)\|_{L(L^1(\Omega))} \leq (p + q)^{\lceil ta \rceil}, \quad t \geq a^{-1},
\]
where \( \lceil ta \rceil \) is the largest integer less than or equal to \( ta \).

Before we prove Theorem 3.5 we state a couple of auxiliary results.

Lemma 3.6. For \( p + q < 1 \) we have
\[
\|T^*(t)\|_{L(L^\infty(\Omega))} \geq \text{ess sup}_{(x,v) \in \Omega} [x^*_v x^*_t < 0], \quad t > 0.
\]

Proof. For each \( t_1 \in [0, b^{-1}] \) by (3.6) we have
\[
T^*(t_1)\mathbb{1}_{\Omega}(x,v) = [x^*_v x^*_t < 0] + (p + q)[0 \leq x^*_v x^*_t] \geq [x^*_v x^*_t < 0]
\]
for \( \mu \)-almost every \( (x,v) \in \Omega \). Therefore, given \( t_2 \in [0, b^{-1}] \), by the semigroup property for \( T^* \), the positivity of \( T^*(t_2) \), and (3.1),
\[
T^*(t_1 + t_2)\mathbb{1}_{\Omega}(x,v) \geq T^*(t_2)[x^*_v x^*_t < 0]
\]
\[
\geq [(x + t_2 v)^*_v x^*_t < 0] x^*_v x^*_t < 0
\]
\[
= [x^*_v x^*_t < 0]
\]
for $\mu$-almost every $(x, v) \in \Omega$.

For $t > 0$ choose $N \geq 0$, $s \in (0, b^{-1}]$ such that

$$t = Nb^{-1} + s.$$  

Inequality (3.11) implies

$$T^*(t) \mathbb{1}_\Omega(x, v) = T(Nb^{-1} + s) \mathbb{1}_\Omega(x, v) \geq [x^*_{vvt} < 0]$$

for $\mu$-almost every $(x, v) \in \Omega$ by induction. Hence (3.10) follows by (3.5).

Lemma 3.7. If $p + q < 1$ and $a > 0$, then

$$T^*(t) \mathbb{1}_\Omega(x, v) \leq (p + q)^j, \quad t > 0$$

for $\mu$-almost every $(x, v) \in \Omega$, where

$$j = j(x, v, t) := \min\{i \geq 0 : x^*_{vvt(ia^{-1})} < 0\}.$$  

Note that for $(x, v) \in \Omega$, in view of the interpretation of $x^*_{vvt(ia^{-1})}$ from the beginning of this section, $j$ is the least nonnegative integer such that a cell characterized by a pair $(x, v)$ at time 0 will not divide before time max\{$t - ja^{-1}, 0\}$.

Proof. For $i \geq 0$ and $t > 0$ define $\varphi_{i,t}: \Omega \to \mathbb{R}$ by

$$\varphi_{i,t}(x, v) := [(i - 1)va^{-1} \leq x^*_{vvt} < i va^{-1}], \quad (x, v) \in \Omega,$$

and $\psi_{i,t,w}: \Omega \to \mathbb{R}$ for $w \in V$ by

$$\psi_{i,t,w}(x, v) := [(i - 1)va^{-1} + vw^{-1} \leq x^*_{vvt} < i va^{-1} + vw^{-1}], \quad (x, v) \in \Omega.$$  

Also, denote

$$r := p + q.$$  

Step 1. Inequality (3.12) is equivalent to

$$T^*(t) \mathbb{1}_\Omega \leq \sum_{i=0}^{+\infty} r^i \varphi_{i,t}, \quad t > 0.$$  

Indeed, let $t > 0$. The functions $\varphi_{i,t}$ for $i \geq 0$ are indicator functions of disjoint sets whose union equals $\Omega$, since

$$x^*_{vvt} = x + tv - 1 > -1 > -va^{-1}, \quad (x, v) \in \Omega.$$  

Hence, for fixed $(x, v) \in \Omega$, exactly one term in the series is nonzero at $(x, v)$, that is, there exists a unique $m \geq 0$ such that $\varphi_{m,t}(x, v) = 1$. For $i \geq 0$ we have

$$\varphi_{i,t}(x, v) = [-va^{-1} \leq x^*_{vvt} - i va^{-1} < 0] = [-va^{-1} \leq x^*_{vvt(ia^{-1})} < 0],$$

thus taking $i = m$ we see that $j \leq m$, where $j$ is defined by (3.13). In the case $m = 0$, clearly $j = m$. On the other hand, if $m \geq 1$, then $\varphi_{i,t}(x, v) = 0$ for $0 \leq i < m$, which by (3.16) implies $j \geq m$. Hence $j = m$, and finally

$$\sum_{i=0}^{+\infty} r^i \varphi_{i,t}(x, v) = r^m = r^j$$

as desired.

Step 2. Let $t > 0$ and $w \in V$. We estimate the sum $\sum_{i=0}^{+\infty} r^i \psi_{i,t,w}(x, v)$ under proviso $x^*_{vvt} \geq 0$. To this end we fix $(x, v) \in \Omega$ such that $x^*_{vvt} \geq 0$, and for $i \geq 0$ we define

$$\Psi_{i,t,w}(x, v) := [iva^{-1} \leq x^*_{vvt} < i va^{-1} + vw^{-1}].$$
Assuming that real numbers $\alpha$, $\beta$, $\gamma$ and $\delta$ satisfy
\[ \alpha \leq \beta \leq \gamma \leq \delta, \]  
(3.17)
we have
\[ [\alpha \leq y < \beta] + r[\beta \leq y < \delta] \leq [\alpha \leq y < \gamma] + r[\gamma \leq y < \delta], \quad y \in \mathbb{R}, \]  
(3.18)
since $r < 1$. Taking
\[ \alpha :=iva^{-1}, \quad \beta :=iva^{-1} + vw^{-1}, \quad \gamma :=iva^{-1} + va^{-1}, \quad \delta := (i + 1)va^{-1} + vv^{-1}, \]
condition (3.17) holds for $i \geq 0$ because $0 < w^{-1} < a^{-1}$. Hence, by (3.18) with $y := x_{\text{vet}}$ we obtain
\[ r^i\Psi_{i,t,w}(x,v) + r^{i+1}\psi_{i+1,t,w}(x,v) \leq r^i\varphi_{i+1,t}(x,v) + r^{i+1}\Psi_{i+1,t,w}(x,v), \quad i \geq 0. \]
Summing this inequality for $i \geq 0$ we get
\[ \sum_{i=0}^{+\infty} r^i\Psi_{i,t,w}(x,v) + \sum_{i=1}^{+\infty} r^i\psi_{i,t,w}(x,v) \leq \sum_{i=0}^{+\infty} r^i\varphi_{i+1,t}(x,v) + \sum_{i=1}^{+\infty} r^i\Psi_{i+1,t,w}(x,v); \]
note that here all sums are finite since $\varphi_{i,t}$'s, $\psi_{i,t,w}$'s and $\Psi_{i,t,w}$'s are indicator functions of disjoint sets. Thus
\[ \Psi_{0,t,w}(x,v) + \sum_{i=1}^{+\infty} r^i\psi_{i,t,w}(x,v) \leq \sum_{i=0}^{+\infty} r^i\varphi_{i+1,t}(x,v). \]
Recall that $x_{\text{vet}}^* \geq 0$, therefore
\[ \psi_{0,t,w}(x,v) = [-va^{-1} + vv^{-1} \leq x_{\text{vet}}^* < 0] + [0 \leq x_{\text{vet}}^* < vv^{-1}] = \Psi_{0,t,w}(x,v). \]
Hence finally
\[ \sum_{i=0}^{+\infty} r^i\psi_{i,t,w}(x,v) \leq \sum_{i=0}^{+\infty} r^i\varphi_{i+1,t}(x,v) \]  
(3.19)
provided that $t > 0$, $w \in V$, and $(x,v) \in \Omega$ satisfies $x_{\text{vet}}^* \geq 0$.

**Step 3.** Let $s > 0$ and set
\[ t := b^{-1} + s. \]
We find a formula for $T^*(b^{-1})\varphi_{i,s}$. For $(x,v) \in \Omega$ we have $(x + vb^{-1})^*_{\text{vet}} = x_{\text{vet}}^*$ and
\[ (x_{\text{vet}}^*)_{wes} = (x + vb^{-1})wv^{-1} + sw - 1 = x_{\text{vet}}^*wv^{-1} - 1. \]
Therefore,
\[ \varphi_{i,s}(x + vb^{-1}, v) = \varphi_{i,t}(x,v), \quad i \geq 0, \]
and
\[ \varphi_{i,s}(x_{\text{vet}}^*, w) = \psi_{i,t,w}(x,v), \quad i \geq 0, \quad w \in V. \]
Combining these relations with (3.1), we have
\[ T^*(b^{-1})\varphi_{i,s}(x,v) = \varphi_{i,t}(x,v)[x_{\text{vet}}^* < 0] + \int_V \psi_{i,t,w}(x,v) \ell^*(v,dw)[0 \leq x_{\text{vet}}^* - 1] \]  
(3.20)
for $i \geq 0$ and $\mu$-almost every $(x,v) \in \Omega$.

**Step 4.** We show that (3.14) holds for every $t \in ((n - 1)b^{-1}, nb^{-1}]$, $n \geq 1$ by induction on $n$. For $t \in (0,b^{-1}]$ we have
\[ x_{\text{vet}}^* = x + tv - 1 < 1 + vb^{-1} - 1 = vb^{-1} < va^{-1}, \quad (x,v) \in \Omega, \]
thus $[0 \leq x_{\text{vet}}^*] = [0 \leq x_{\text{vet}}^* < va^{-1}] = \varphi_{1,t}(x,v)$. Similarly, by (3.15),
\[ [x_{\text{vet}}^* < 0] = \varphi_{0,t}(x,v), \quad (x,v) \in \Omega. \]
Hence, by (3.6),
\[ T^*(t) \mathbb{1}_\Omega = \varphi_{0,t} + r \varphi_{1,t} = \sum_{i=0}^{+\infty} r^i \varphi_{i,t}. \]
This shows that (3.14) holds for \( t \in (0, b^{-1}] \).

In order to perform the inductive step let \( n \geq 1 \) and assume that (3.14) holds for each \( t \in ((n-1)b^{-1}, nb^{-1}] \). Fix \( t \in (nb^{-1}, (n+1)b^{-1}] \), and choose \( s \in ((n-1)b^{-1}, nb^{-1}] \) such that
\[ t = b^{-1} + s. \]
Since \( r < 1 \) we have \( \sum_{i=0}^{+\infty} r^i \varphi_{i,t} (x,v) \leq 1 \) for \( (x,v) \in \Omega \) (recall that exactly one term of the series is nonzero). Therefore \( \sum_{i=0}^{+\infty} r^i \varphi_{i,t} \) is an element of \( L^\infty(\Omega) \). Hence, by (3.14) with \( t \) replaced by \( s \), and by the fact that \( T^*(b^{-1}) \) is a positive and bounded operator,
\[ T^*(t) \mathbb{1}_\Omega = T^*(b^{-1}) T^*(s) \mathbb{1}_\Omega \leq \sum_{i=0}^{+\infty} r^i T^*(b^{-1}) \varphi_{i,s}. \]
Combining this with (3.20), we obtain
\[ T^*(t) \mathbb{1}_\Omega (x,v) \leq [x_{v,e}^* < 0] \sum_{i=0}^{+\infty} r^i \varphi_{i,t} (x,v) \]
\[ + [0 \leq x_{v,e}^*] \int_{V} \sum_{i=0}^{+\infty} r^i \psi_{i,t,w} (x,v) \ell^*(v, dw) \]
for \( \mu \)-almost every \((x,v) \in \Omega \).

Denote
\[ \chi(x,v) := [x_{v,e}^* < 0], \quad (x,v) \in \Omega. \]
For \((x,v) \in \Omega\), if \( x_{v,e}^* \geq 0 \), or equivalently \( 1 - \chi(x,v) = 1 \), then \( x_{v,e}^* = x_{v,e}^* + sv \geq 0 \). Hence, by (3.21) and (3.19), we obtain
\[ T^*(t) \mathbb{1}_\Omega \leq \chi \sum_{i=0}^{+\infty} r^i \varphi_{i,t} + (1 - \chi) \int_{V} \ell^*(v, dw) \sum_{i=0}^{+\infty} r^i \varphi_{i+1,t} \]
\[ = \chi \sum_{i=0}^{+\infty} r^i \varphi_{i,t} + (1 - \chi) \sum_{i=0}^{+\infty} r^{i+1} \varphi_{i+1,t} \]
\[ = \chi \varphi_{0,t} + \sum_{i=1}^{+\infty} r^i \varphi_{i,t}, \]
where in the first equality we used \( \int_{V} \ell^*(v, dw) = r \) for \( v \in V \). However, we see that \( \chi \varphi_{0,t} = \varphi_{0,t} \), since \( x_{v,e}^* \leq x_{v,e}^* \) for each \((x,v) \in \Omega \). Therefore (3.14) follows, and the proof is complete. \( \square \)

**Proof of Theorem 3.5.** For part (i) and (3.8) we argue as follows. Fix \( t > 0 \) and suppose that \( a = 0 \) or \( 0 < a < t^{-1} \). Then the set \( \{(x,v) \in \Omega: x_{v,e}^* < 0\} \) is of positive \( \mu \)-measure. Indeed, the set \( U := V \cap (a, t^{-1}) \) is of positive \( \nu \)-measure, and for \( v \in U \) the Lebesgue measure of the set of \( x \in I \) satisfying \( x + tv < 1 \) is positive, being equal \( 1 - tv > 0 \). Combining this with Lemma 3.6, we have \( \|T^*(t)\|_{L(L^1(\Omega))} \geq 1 \). Hence, by (3.4),
\[ \|T(t)\|_{L(L^1(\Omega))} \geq 1. \]
Therefore have note that to hold. To simplify notation we let measure and assume that \( m^{-1} \leq t < (m+1)a^{-1} \), and let \((x, v) \in \Omega\). Then

\[
x^*_v = x + tv - 1 \geq x + mva^{-1} - 1 > mva^{-1} - 1 > (m - 1)va^{-1},
\]

since \( va^{-1} > 1 \). This implies that

\[
x^*_{v(t - ia - 1)} = x^*_v - ia^{-1} > 0, \quad 0 \leq i < m - 1,
\]
hence

\[
j(x, v, t) = \min\{i \geq 0 : x^*_{v(t - ia - 1)} < 0\} \geq m. \quad \text{Thus, since} \quad p + q < 1, \quad \text{Lemma 3.7 implies}
\]

\[
\|T^*(t)1_\Omega\|_{L^\infty(\Omega)} \leq (p + q)^m = (p + q)^{\lfloor t \rfloor_a}
\]

and the proof is complete by (3.5). \( \square \)

A natural question arises whether the estimates in Theorem 3.4 (i) and Theorem 3.5 (ii) are optimal. As we shall see, the answer is generally in negative; the growth (resp. decay) is in fact slower (resp. faster) than Theorem 3.4 (i) and Theorem 3.5 (ii) may suggest. Unfortunately, the exact growth and decay rates seem to depend in crucial way on an interplay of parameters \( a, b \) and kernel \( k \), and thus an explicit formula, if it exists, evades us. We can show, however, that equality in (3.7) holds rather seldom (a similar argument applies to (3.9)). For the sake of this argument, we restrict ourselves to the case \( V = (a, b) \) with the Lebesgue measure and assume that

\[
r := p + q > 1.
\]

We begin by finding necessary and sufficient conditions for

\[
\|T(2b^{-1})\|_{L^1(\Omega)} = r^2
\]

to hold. To simplify notation we let \( s := b^{-1} \) and \( t := 2s \). Fix \((x, v) \in \Omega, w \in V\). Note that \( x^*_{\text{wus}} = x^*_v - sv, (x + sv)^*_{\text{wus}} = x^*_v \) and

\[
(x^*_{\text{wus}})_{\text{wus}} = (x + sv - 1)wv^{-1} + sw - 1 = (x + 2sv - 1)wv^{-1} - 1 = x^*_v wv^{-1} - 1.
\]

Therefore

\[
[x^*_{\text{wus}} < 0] = [x^*_v < 0],
\]

\[
[sv \leq x^*_v < sv + 1] = [0 \leq x^*_v < sv],
\]

\[
[sv \leq x^*_v < sv + 1] = [0 \leq x^*_v < sv],
\]

\[
[sv \leq x^*_v < sv + 1] = [0 \leq x^*_v < sv].
\]

Hence, using the semigroup property for \( T^*(t) = T^*(s + s) \), by (3.1) and (3.6) we have

\[
T^*(t)1_\Omega(x, v) = [x^*_v < 0] + p \int_0^{2s} k^*(w, v)[sv \leq x^*_v < sv]dw + q[sv \leq x^*_v < 1] + q[sv \leq x^*_v < 1] + qr[1 \leq x^*_v] + qr[1 \leq x^*_v]
\]

(3.23)
for almost every \((x, v) \in \Omega\).

Let \(v \in V\) and denote
\[
I_v := \{ x \in I : x^*_v \geq sv \} = \{ x \in I : x \geq 1 - sv \}.
\]

If \(x \in I_v\) and \(y \in I \setminus I_v\), then by (3.23) we have
\[
T^*(t) \mathbb{1}_\Omega(y, v) \leq r \leq T^*(t) \mathbb{1}_\Omega(x, v).
\]

Hence, since the Lebesgue measure of \(I_v\) is positive, being equal \(sv\), we have
\[
\|T^*(t)\|_{L(L^\infty(\Omega))} = \text{ess sup}_{v \in V, x \in I_v} T^*(t) \mathbb{1}_\Omega(x, v).
\]

Define
\[
c(x, v) := \max \left\{ a, \frac{v}{x^*_v} \right\}, \quad v \in V, \; x \in I_v;
\]

note that \(a \leq c(x, v) \leq b\). Then, using (3.23), for almost every \((x, v) \in \Omega\) such that
\(x \in I_v\) we obtain
\[
T^*(t) \mathbb{1}_\Omega(x, v) = C(x, v) + D(x, v),
\]

where
\[
C(x, v) := p \int_a^{c(x, v)} k^*(w, v) \, dw + pr \int_{c(x, v)}^b k^*(w, v) \, dw
\]

and
\[
D(x, v) := q[sv \leq x^*_v < 1] + qr[1 \leq x^*_v].
\]

Let
\[
d := \max \{ a, t^{-1} \} = \max \{ a, b/2 \},
\]

and observe that \(a \leq d < b\). For \(v \in V\) the set
\[
\{ x \in I : x^*_v \geq 1 \} = \{ x \in I : x \geq 2 - tv \}
\]

is of positive Lebesgue measure if and only if \(tv > 1\), or equivalently \(v \in (d, b)\). Consequently, if \(p = 0\) then (3.22) holds by (3.24) and (3.25). Suppose now that
\(p > 0\) and let \(v \in V\). As a function of \(x \in I_v = [1 - sv, 1]\), \(x \mapsto x^*_v\) is increasing and converges to \(tv\) as \(x \to 1^+\). This implies that \(I_v \ni x \mapsto c(x, v)\) is decreasing and converges to \(d\) as \(x \to 1^+\). Hence
\[
\text{ess sup}_{v \in V, x \in I_v} C(x, v) = pr
\]

if and only if the following condition holds:

\((V_\varepsilon)\) For each \(\varepsilon > 0\) there exists a Lebesgue measurable subset \(U\) of \(V\) with a positive measure such that
\[
\int_d^b k^*(w, v) \, dw > 1 - \varepsilon, \quad v \in U.
\]

Therefore, again by (3.24) and (3.25), if \(q = 0\), then (3.22) is equivalent to \((V_\varepsilon)\). Finally, if \(p, q > 0\), then (3.22) holds if and only if \((V_\varepsilon)\) holds with the additional requirement \(U \subseteq (d, b)\).

Note that by the definition of \(k^*\) and (2.1) condition \((V_\varepsilon)\) is trivially satisfied (for any \(U \subseteq V\) with positive Lebesgue measure) provided that \(d = a\), which is equivalent to \(b/2 \leq a\).

Continuing the procedure described above for \(t := nb^{-1}, \; n \geq 2\) we can generalize this result. To this end, we fix \(n \geq 2\) and introduce condition:
For each $\varepsilon > 0$ there exists a Lebesgue measurable subset $U$ of $V$ with a positive measure such that

$$\int_{d_n}^{b} k^*(w, v) \, dw > 1 - \varepsilon, \quad v \in U,$$

where

$$d_n := \max \left\{ a, \frac{nb}{n+1} \right\}, \quad n \geq 2.$$

Then we can check that the following result holds.

**Proposition 3.8.** Assume that $p+q > 1$, and $V = (a, b)$ with the Lebesgue measure. For $n \geq 2$ we have

$$\|T(nb^{-1})\|_{L(L^1(\Omega))} = (p + q)^n$$

if and only if

(i) $p = 0$, or
(ii) $q = 0$ and condition $(V\varepsilon)$ holds, or
(iii) condition $(V\varepsilon)$ holds with additional requirement $U \subseteq (d_n, b)$.

Note that if $p = 0$, then all cells degenerate and the structure of the population does not change in time, and so the model is quite uninteresting. On the other hand, condition $(V\varepsilon)$ is very restrictive. Thus the proposition shows that the cases when equality in Theorem (3.4) (i) holds are rather rare. For example, assuming $p, q > 0$, condition (iii) is satisfied for any $t = nb^{-1}$, $n \geq 2$, provided that

$$k^*(w, v) := \frac{1}{b - v}[v < w < b], \quad v, w \in U \cap V,$$

where $U \subset \mathbb{R}^2$ is a neighbourhood of $(b, b)$. For such $k^*$, $k$ does not even satisfy assumptions of Boulanouar’s theorem [9, Theorem 6.1], and seems to be rather uninteresting biologically. For this would mean that if a mother cell matures sufficiently fast, then all its daughter cells would need to mature even faster.

4. **Asymptotic behavior.** This section is devoted to the asymptotic behavior of the LRR semigroup. Throughout this section we assume that $V = (a, b)$, and $\nu$ is the Lebesgue measure. This is, in particular, crucial for Lemma 4.8. At first we state Boulanouar’s result.

Let $\omega_0(T)$ be the growth bound (or type) of the semigroup $T$ defined as

$$\inf \{ \omega \in \mathbb{R} : \sup_{t \geq 0} \| e^{-\omega t}T(t) \|_{L(L^1(\Omega))} < +\infty \},$$

or equivalently

$$\lim_{t \to +\infty} t^{-1} \log \| T(t) \|_{L(L^1(\Omega))}.$$
Then there exists a rank one projection \( P \) on \( L^1(\Omega) \) such that

\[
\lim_{t \to +\infty} e^{-\omega_0(T)t}T(t) = P
\]

in the operator norm topology.

We concentrate on the case \( p+q \leq 1 \) and study convergence of the LRR semigroup in strong topology as \( t \to \infty \), as opposed to the operator norm topology spoken of in Boulalour’s result.

Let us begin by recalling some classical notions for Markov and substochastic semigroups, see for example [16] or [21].

Let \((X, \mathcal{A}, m)\) be a \(\sigma\)-finite measure space and consider the space \(L^1(X, \mathcal{A}, m) = L^1(X)\) of (equivalence classes of) absolutely integrable functions on \(X\) with respect to \(m\). A linear operator \(S : L^1(X) \to L^1(X)\) is called substochastic if \(S\) is a positive contraction, that is, given \(f \in L^1(X)\) we have \(Sf \geq 0\) if \(f \geq 0\), and \(\|Sf\|_{L^1(X)} \leq \|f\|_{L^1(X)}\), where \(\cdot\) is the standard \(L^1\)-norm. Moreover, if \(SD_{L^1(X)} \subseteq D_{L^1(X)}\), where \(D_{L^1(X)}\) is the set of densities in \(L^1(X)\), that is,

\[
D_{L^1(X)} := \{f \in L^1(X) : f \geq 0, \|f\|_{L^1(X)} = 1\},
\]

then \(S\) is called a Markov (or stochastic) operator. If a substochastic operator \(S\) can be written in the form

\[
Sf(x) = \int_X h(x, y)f(y)m(dy) + Pf(x), \quad x \in X, \ f \in L^1(X),
\]

where \(h : X \times X \to \mathbb{R}\) is a measurable nonnegative function satisfying

\[
\int_X \int_X h(x, y) m(dy) m(dx) > 0,
\]

and \(P\) is a positive linear operator on \(L^1(X)\), then \(S\) is called partially integral. A strongly continuous semigroup \(S = \{S(t), t \geq 0\}\) in \(L^1(X)\) is said to be substochastic (resp. Markov) if \(S(t)\) is a substochastic (resp. Markov) operator for every \(t \geq 0\). Furthermore, \(S\) is partially integral if there exists \(t_0 > 0\) such that \(S(t_0)\) is partially integral. If \(f_* \in D_{L^1(X)}\) and

\[
S(t)f_* = f_*
\]

for all \(t \geq 0\), then \(f_*\) is called an invariant density for the semigroup. Finally, if \(f_*\) is an invariant density for \(S\) and for all \(f \in D_{L^1(X)}\) we have

\[
\lim_{t \to +\infty} \|S(t)f - f_*\|_{L^1(X)} = 0,
\]

then \(S\) is said to be asymptotically stable.

We use the following result of Pichór and Rudnicki [21, Proposition 2].

**Theorem 4.2.** Let \(S = \{S(t), t \geq 0\}\) be a partially integral substochastic semigroup in \(L^1(X)\). Assume that there is a unique, up to an equivalence class, invariant density for \(S\). If the invariant density is almost everywhere strictly positive, then the semigroup \(S\) is asymptotically stable.

Applying Theorem 4.2 to the LRR semigroup \(T\), we will obtain the main result of this paper, that is, Theorem 4.3. First, we state the crucial assumptions. Let \(K\) be the operator in \(L^1(V)\) defined by

\[
Kf(v) := \int_V k(v, w)f(w)dw, \quad v \in V, \ f \in L^1(V).
\]
A function $f_0 \in D_{L^1(V)}$ is called a *stationary density* for the kernel $k$ if
\[ Kf_0 = f_0. \]

We assume the following.

(H3) There exists a unique, up to an equivalence class, stationary density $f_0$ for the kernel $k$, and $f_0$ is strictly positive almost everywhere on $V$.

(H4) The function $V \ni v \mapsto v^{-1}f_0(v)$ belongs to $L^1(V)$.

**Theorem 4.3.** Suppose that $p > 0$ and $p + q = 1$. If conditions (H3)–(H4) hold, then the LRR semigroup $T$ is asymptotically stable with a unique, up to an equivalence class, invariant density $f_*$ given by
\[ f_*(x, v) = \frac{v^{-1}f_0(v)}{\int_V w^{-1}f_0(w) \, dw}, \quad (x, v) \in \Omega, \]
where $f_0$ is the stationary density for $k$.

**Remark 4.4.** If $a > 0$, then the function $V \ni v \mapsto v^{-1}$ is bounded, and in Theorem 4.3 we may omit condition (H4).

**Remark 4.5.** If condition (H3) holds, then (H4) is equivalent to
\[(H4)' \quad \text{The function } V \ni v \mapsto v^{-1}k(w, v) \text{ belongs to } L^1(\Omega) \text{ for almost every } w \in V.\]

Indeed, assume that $f_0$ is a strictly positive stationary density for the kernel $k$. Then by the Fubini theorem we have
\[ \int_V v^{-1}f_0(v) \, dv = \int_V f_0(w) \int_V v^{-1}k(w, v) \, dv \, dw. \]
Hence the left-hand side is finite if and only if $\int_V v^{-1}k(w, v) \, dv$ is finite for almost every $w \in V$.

In order to prove Theorem 4.3 we need a couple of lemmas.

**Lemma 4.6.** If $f_* \in L^1(\Omega)$ is an invariant density for the LRR semigroup, then for almost every $v \in V$ the function $I \ni x \mapsto f_*(x, v)$ is constant.

**Proof.** Assume that $f_*$ is an invariant density for $T$. Then $T(t)f_* - f_* = 0$ for $t \geq 0$, hence $f_* \in D(A)$ and $Af_* = 0$, where $A$ is the generator of $T$ given by (2.2). Therefore for almost every $v \in V$ the function $I \ni x \mapsto f_*(x, v)$ is weakly differentiable and $\partial_x f_* = 0$. This completes the proof.

**Lemma 4.7.** Let $p + q = 1$ and assume that $f_0 \in D_{L^1(V)}$ is a unique, up to an equivalence class, stationary density for the kernel $k$. If $f_0$ satisfies condition (H4), then there exists a unique, up to an equivalence class, invariant density $f_* \in D_{L^1(\Omega)}$ for the LRR semigroup and
\[ f_*(x, v) = \frac{F_0(v)}{\|F_0\|_{L^1(V)}}, \quad (x, v) \in \Omega, \]
(4.2)
for almost every $(x, v) \in \Omega$, where $F_0 \in L^1(V)$,
\[ F_0(v) := v^{-1}f_0(v), \quad v \in V. \]
Then by the semigroup property

Assume that Lemma 4.8.

Proof. First we prove that \( f_* \) defined by (4.2) is indeed an invariant density for \( T \). By (2.4) and since \( Kf_\phi = f_\phi \) we have

\[
\int_V F_\phi(w) \ell(dw, v) = pv^{-1}Kf_\phi(v) + qF_\phi(v) = F_\phi(v)
\]

for almost every \( v \in V \). Hence, for \( t \in [0, b^{-1}] \) by (2.27) we obtain

\[
\|F_\phi\|_{L^1(V)}T(t)f_\phi(x, v) = F_\phi(v)[x > tv] + \int_V F_\phi(w) \ell(dw, v)[x \leq tv] = F_\phi(v)
\]

for almost every \((x, v) \in \Omega\). In other words \( T(t)f_* = f_* \) for \( t \in [0, b^{-1}] \). If \( t > b^{-1} \), then we find a positive integer \( n \) such that \( s := t/n \leq b^{-1} \). Then \( T(s)f_* = f_* \), and by the semigroup property \( T(t)f_* = [T(s)]^n f_* = f_* \). Therefore \( f_* \) is an invariant density for \( T \).

For the uniqueness part, assume that \( f_1^* \) and \( f_2^* \) are invariant densities for \( T \). Let \( i \in \{1, 2\} \). By (2.27), equality \( T(t)f_i^* = f_i^* \), \( t \geq 0 \) implies that

\[
f_i^*(x, v) = f_i^*(x - tv, v)[x > tv] + \int_V f_i^*(x_{new}, v) \ell(dw, v)[x \leq tv]
\]

for \( t \in [0, b^{-1}] \) and almost every \((x, v) \in \Omega\). Using Lemma 4.6, this is true if and only if \( f_i^*(x, v) = \int_V f_i^*(x, w) \ell(dw, v) \) for almost every \((x, v) \in \Omega\), which by (2.4) we may rewrite as

\[
f_i^*(x, v) = p \int_V wv^{-1}k(w, v)f_i^*(x, w)\nu(dw) + qf_i^*(x, v).
\]

Since \( 1 - q = p \), this is equivalent to

\[
v f_i^*(x, v) = \int_V k(w, v)w f_i^*(x, w) dw. \tag{4.3}
\]

Let \( F_i^* \in L^1(\Omega) \) be given by

\[
F_i^*(x, v) := v f_i^*(x, v), \quad (x, v) \in \Omega,
\]

and define \( f_\phi^* \in L^1(V) \) by

\[
f_\phi^*(v) := \frac{F_i^*(x, v)}{\|F_i^*\|_{L^1(\Omega)}}, \quad (x, v) \in \Omega;
\]

this definition make sense since by Lemma 4.6, \( f_\phi^* \) does not depend on \( x \), and thus the same is true for \( F_i^* \). By (4.3), \( f_\phi^* \) is a stationary density for the kernel \( k \). Hence, by uniqueness assumption, we have

\[
\|F_i^*\|_{L^1(\Omega)} = \frac{F_\phi^*}{\|F_\phi^*\|_{L^1(\Omega)}}.
\]

Then \( \|F_\phi^*\|_{L^1(\Omega)}f_i^* = \|F_i^*\|_{L^1(\Omega)}f_i^* \). Integrating this relation over \( \Omega \), we obtain

\[
\|F_\phi^*\|_{L^1(V)} = \|F_\phi^*\|_{L^1(V)},
\]

since \( \|f_\phi^*\|_{L^1(\Omega)} = 1 \). Thus finally \( f_\phi^* = f_i^* \). \qed

**Lemma 4.8.** Assume that \( p > 0 \). The operator \( T(2b^{-1}) \) is partially integral provided that

\[
\int_V \int_{\max\{a, b/2\}}^{b} k(w, v) \, dv \, dw > 0. \tag{4.4}
\]
Proof. To simplify notation let, as before, \( s := b^{-1} \) and \( t := 2s \). By (2.27),

\[ T(s) = Q_1 + P_1 \]

where \( P_1 \) is a bounded positive operator in \( L^1(\Omega) \) and

\[ Q_1 f(x, v) = p \int_V w v^{-1} k(w, v) f(x_{wus}, w) \, dw [x < sv], \quad (x, v) \in \Omega, f \in L^1(\Omega). \]

Operator \( Q_1 \) describes a subpopulation of \( (x, v) \)-type cells, with \( x < sv \), alive at time \( b^{-1} \), which are non-degenerate daughters of cells from generation 0 (see definition (2.25)). It follows that

\[ T(t) = Q + P \]

where \( Q = (Q_1)^2 \) and \( P \) is another bounded positive operator in \( L^1(\Omega) \). Since

\[ (x_{wus})_{wus} = 1 + (1 + xwv^{-1} - sw)uw^{-1} - su = x_{uv} + uw^{-1}, \]

we have

\[ Qf(x, v) = p^2 \int_V \int_{x_{uv} + ub^{-1}} u v^{-1} k(u, v) k(u, v) f(x_{uv}, u) [x_{wus} < sw][x < sv] \, du \, dw \]

for \( (x, v) \in \Omega \). Furthermore, a little bit of algebra shows that if \( x_{wus} < sw \) for some \( w \in V \), which is the same as \( x_{uv} < 0 \), then \( x < sv \). Thus, \( [x_{wus} < sw][x < sv] \) in the formula above may be replaced by \( [x_{uv} < 0] \). Applying the Fubini theorem and substituting

\[ y = y(u) := x_{uv} + uw^{-1}, \]

in (4.5) we get

\[ Qf(x, v) = p^2 \int_V \int_{x_{uv} + ub^{-1}} y v^{-1} k(\hat{y}, v) k(u, \hat{y}) f(y, u) [y_{yv} < 0] \, dy \, du, \]

(4.6)

where \( \hat{y} = u(y - x_{uv})^{-1} \). If \( a = 0 \), then by convention we take \( a^{-1} := +\infty \). For \( x \in I \) and \( u, v \in V \), we have

\[ (x_{uv} + ub^{-1}, x_{uv} + ua^{-1}) \cap \{y \in \mathbb{R} : y_{yv} < 0\} \subseteq I. \]

Indeed, \( x_{uv} + ub^{-1} = x_{wub^{-1}} > 1 - ub > 0 \), and if \( y_{yv} < 0 \), then \( \hat{y}(t - xv^{-1}) > 1 \) which implies \( y < u(t - xv^{-1}) + x_{uv} = 1 \). Therefore, we may rewrite (4.6) as

\[ Qf(x, v) = \int_V \int_{x_{uv} + ub^{-1}} h(x, v, y, u) f(y, u) \, dy \, du, \]

for

\[ h(x, v, y, u) := p^2 \hat{y}^2 v^{-1} k(\hat{y}, v) k(u, \hat{y}) [y \in \Lambda_{yv}][y_{yv} < 0] \geq 0, \]

where \( (x, v), (y, u) \in \Omega \) and \( \Lambda_{yv} = (x_{uv} + ub^{-1}, x_{uv} + ua^{-1}) \). Hence, we are left with proving that \( \int_{\Omega} \int_{\Omega} h \, d\mu \, d\mu > 0 \). We have

\[ \int_{\Omega} \int_{\Omega} h \, d\mu \, d\mu = \int_{\Omega} Q_\Omega \, d\mu. \]

Let

\[ d := \max\{a, t^{-1}\} = \max\{a, b/2\} \]
and note that \( x \in I \) and \( x_{wvt} < 0 \) is equivalent to \( d < w < b \) and \( 0 < x < tv - vw^{-1} \). Thus by (4.5) we obtain (recall that \( [x_{wvt} < sv][x < sv] \) may be replaced by \( [x_{wvt} < 0] \))

\[
\frac{1}{p^2} \int \int \int h \, d\mu \, d\mu = \int \int \int \int u v^{-1} k(u, w)k(w, v) [x_{wvt} < 0] \, du \, dw \, dx \, dv
\]

\[
= \int \int \int \int u v^{-1} k(u, w)k(w, v) \, dx \, dv \, du
\]

\[
= \int \int \int \int u k(u, w)k(w, v)(t - w^{-1}) \, dv \, du
\]

\[
= \int \int \int \int u k(u, w)(t - w^{-1}) \, dw \, du,
\]

the second equality resulting from the Fubini theorem, and the third from (2.1). Since the function \( V \times (d, b) \ni (u, w) \mapsto u(t - w^{-1}) \) is strictly positive, the integral \( \int \int \int \int u k(u, w)(t - w^{-1}) \, dw \, du \) is positive if and only if condition (4.4) holds.

**Lemma 4.9.** If there exits a strictly positive stationary density \( f_\diamond \in D_{L^1(V)} \) for the kernel \( k \), then

\[
\int \int k(w, v) \, dv \, dw > 0
\]

for every Lebesgue measurable subset \( U \) of \( V \) with positive measure.

**Proof.** Assume, contrary to our claim, that

\[
\int \int k(w, v) \, dv \, dw = 0. \tag{4.7}
\]

for a set \( U \) of positive measure. However, for almost every \( v \in V \) we have

\[
f_\diamond(v) = \int k(w, v)f_\diamond(w) \, dw.
\]

Integrating this over \( U \),

\[
\int f_\diamond(v) \, dv = \int \int k(w, v)f_\diamond(w) \, dv \, dw \tag{4.8}
\]

by the Fubini theorem. Since \( k \) is nonnegative, (4.7) implies \( k = 0 \) almost everywhere on \( V \times U \). This means that

\[
\int f_\diamond(v) \, dv = 0
\]

by (4.8), which contradicts strict positivity of \( f_\diamond \). This completes the proof.

Now we are ready to prove the main theorem.

**Proof of Theorem 4.3.** We know by Theorem 3.4 (ii) that the LRR semigroup \( T \) is a Markov semigroup, hence in particular substochastic. By Lemma 4.7 there exists a unique invariant density \( f_\diamond \) for \( T \). Moreover, by (4.2), \( f_\diamond \) is strictly positive since \( f_\diamond \) is. Furthermore, Lemma 4.9 with \( U = (\max\{a, b/2\}, b) \) implies that \( T \) is partially integral by Lemma 4.8. Thus, we may apply Theorem 4.2 which completes the proof.
With the described approach asymptotic stability of the LRR semigroup is more difficult to prove if $p + q > 1$. In this case the semigroup $T$ needs not be substochastic and we should first rescale it, taking

$$S(t) := e^{-\omega_0(T)t}T(t), \quad t \geq 0.$$  

Then the semigroup $S = \{S(t), t \geq 0\}$ is bounded and its generator equals $A - \omega_0(T)$, where $A$ is the generator of $T$. Arguing as in the proofs of Lemma 4.6 and Lemma 4.7, $f_\ast \in D_{L^1(\Omega)}$ is an invariant density for $S$ if and only if

$$v(1 - qe^{-\omega_0(T)v^{-1}})f_\ast(x, v) = p \int_V wk(w, v)e^{-\omega_0(T)w^{-1}}f_\ast(x, w) dw$$  

for almost every $(x, v) \in \Omega$. The problem here is that, as discussed at the end of Section 3, in most cases we do not have an explicit formula for $\omega_0(T)$.

Finally, let us state a result describing the asymptotic behavior of the LRR semigroup in the case $p + q < 1$.

**Theorem 4.10.** Suppose that $p + q < 1$.

(i) If $a = 0$, then $T$ converges strongly to zero, that is,

$$\lim_{t \to +\infty} T(t)f = 0, \quad f \in L^1(\Omega).$$  

(ii) If $a > 0$, then $T$ converges to zero in the operator norm, that is,

$$\lim_{t \to +\infty} \|T(t)\|_{L(L^1(\Omega))} = 0.$$

Note that in the case $p + q < 1$ the cell population gradually dies out. Theorem 3.5 (i) reflects the fact that in the case $a = 0$ there exist cells that mature with arbitrarily slow speed and thus at any time $t \geq 0$ there may be cells that did not yet divide. In the case $a > 0$ all parts of the population die out uniformly fast.

**Proof.** Part (ii) follows directly by Theorem 3.5 (ii), and hence we assume that $a = 0$. Since $\|T(t)\|_{L(L^1(\Omega))} \leq 1$ for each $t \geq 0$ by Theorem 3.4 (iii), it is sufficient to show that

$$\lim_{n \to +\infty} T(nb^{-1})f = 0, \quad f \in L^1(\Omega). \quad (4.9)$$

For $n \geq 1$ and $f \in L^1(\Omega)$ by (2.24) we have

$$\|T(nb^{-1})f\|_{L^1(\Omega)} \leq \int_V \int_I |\tilde{f}(x - nb^{-1}, v)| \, dv \, dx = \int_V \int_{-nb^{-1}}^{1-nvb^{-1}} |\tilde{f}(x, v)| \, dx \, dv. \quad (4.10)$$

However, by (2.17) with $\omega = 0$,

$$\int_\Omega |\tilde{f}| \, d\mu < +\infty,$$

provided $p + q < 1$. Thus (4.10) and the Lebesgue dominated convergence theorem imply (4.9).

**Remark 4.11.** The proof of Theorem 4.10 (i) still works in the case where $(V, \nu)$ is any measure space with $V \subseteq (a, b)$ and $a \geq 0$. This is obvious since (2.17) holds in this general setup. Hence, if $p + q < 1$, then the LRR semigroup converges strongly to zero even if we do not assume that $V = (a, b)$ with the Lebesgue measure.
Remark 4.12. The methods introduced by Boulanouar [9] do not work in the case \( p + q \leq 1 \). In particular, assume that \( p + q = 1, p > 0, \) and \( a = 0 \). The proof of the main theorem in [9] is based on Webb's theorem (see [28] and [10, Theorems 9.10 and 9.11]). In order to use Webb's result Boulanouar proves that the spectral bound \( s(T) \) of the LRR semigroup is strictly greater than the growth bound \( \omega_0(T_q) \), where \( T_q \) is the LRR semigroup for \( p := 0 \) and fixed \( q \). However, by [9, proof of Lemma 5.1], if \( p + q = 1 \), then we have \( s(T) = 0 \). On the other hand, if additionally \( a = 0 \), then \( \omega_0(T_q) = 0 \) by Theorem 4.10 (i), since \( q < 1 \).

5. Discussion of assumptions. In our last theorem we discuss the relation between conditions (H1)–(H2) used in Theorem 4.1 and (H3)–(H4) used in Theorem 4.3. As a consequence of this result, we see in particular that Boulanouar’s assumptions are stronger than our condition (H3). In this context, it becomes clear that it is (H4) that is crucial for our analysis. As we have seen, (H4) is an assumption of existence and uniqueness of an invariant density for the LRR semigroup. See the upcoming [22] for a way to deduce existence and uniqueness of an invariant density from properties of a semigroup.

Theorem 5.1. For the kernel \( k \) the following is true.

(i) If conditions (H1)–(H2) hold, then so does (H3), but not necessarily (H4).

(ii) If condition (H3) holds, then so does (H1), but not necessarily (H2) even if we additionally assume (H4).

Proof. For part (i), in order to show that (H1) and (H2) imply (H3) we first note that the operator \( K \) defined by (4.1) is weakly compact provided (H1) and (H2) hold. Indeed, let \( S_V \) be the closed unit sphere in \( L^1(V) \). We will show that \( KS_V \) is relatively weakly compact, that is, the weak closure of \( KS_V \) is compact in the weak topology of \( L^1(V) \). By the Dunford-Pettis theorem [1, Theorem 5.2.9] this holds if and only if \( KS_V \) is composed of uniformly integrable functions. Let \( U \) be a measurable subset of \( V \). For \( f \in S_V \) by (H2) we have

\[
\int_U |Kf(v)| \, dv \leq \| k \|_{L^\infty(V \times V)} \int_U \int_V |f(w)| \, dw \, dv = \| k \|_{L^\infty(V \times V)} \mu_L(U),
\]

where \( \| k \|_{L^\infty(V \times V)} \) is the essential bound of \( k \) on \( V \times V \). This implies uniform integrability of functions belonging to \( KS_V \), and completes the proof of weak compactness of \( K \). Hence \( K^2 \) is a compact operator by [11, Corollary VI.8.13], the result also due to Dunford and Pettis. Therefore, by (H1) and the Jentzsch theorem [26, V.6.6], for the spectral radius \( r(K) \) of \( K \) there exists a unique density \( f_o \in D_{L^1(V)} \) satisfying \( Kf_o = r(K)f_o \), and moreover \( f_o > 0 \) almost everywhere on \( V \). However, by (2.1) it follows that \( r(K) = 1 \), and thus (H3) holds.

To see that (H1) and (H2) do not imply (H4) we take \( V := (0, 1) \), and let \( k \) be equal identically 1 on \( V \times V \). Then (H1) and (H2) are satisfied and \( f_o \) equaling identically 1 on \( V \) is the stationary density for \( k \), and yet \( \int_V v^{-1} f_o(v) \, dv = +\infty \).

For part (ii), assume that (H3) holds, and let \( f_o \) be the unique stationary density for \( k \). Suppose, contrary to our claim, that there exists a measurable set \( U \subset V \), such that \( \mu_L(U), \mu_L(V \setminus U) > 0 \) and

\[
\int_{V \setminus U} \int_U k(w, v) \, dv \, dw = 0. \tag{5.1}
\]

Let \( f \in L^1(V) \) be defined as \( f := 1_U f_o \), where \( 1_U \) is the indicator function of \( U \). Then
almost everywhere on $V$, since $K$ is a positive operator. Moreover
\[ Kf(v) = \int_V k(w, v) f(w) \, dw = \int_U k(w, v) f_\circ(w) \, dw, \]
and hence
\[ \int_{V \setminus U} Kf(v) \, dv = 0, \]
since by (5.1) we have $k = 0$ almost everywhere on $(V \setminus U) \times U$. Because $Kf$ is nonnegative, this means that $Kf = 0$ almost everywhere on $V \setminus U$. Thus, by (5.2), we obtain $Kf \leq f$ almost everywhere on $V$. However, by (2.1) and the Fubini theorem,
\[ \|Kf\|_{L^1(V)} = \|f\|_{L^1(V)}, \]
which implies $Kf = f$ almost everywhere on $V$. Then $g := f/\|f\|_{L^1(V)}$ is a stationary density for $k$, and $g \neq f_\circ$, which contradicts (H3). This contradiction proves that (H3) implies (H1).

Finally, to show that (H3) and (H4) do not imply (H2), let $V := (0, 1)$, and
\[ k(w, v) := \frac{3}{4} \cdot \frac{v}{\sqrt{1-v}}, \quad w, v \in V. \]
Then $k$ satisfies (2.1). Moreover, $f_\circ \in L^1(V)$ defined by
\[ f_\circ(v) := \frac{3}{4} \cdot \frac{v}{\sqrt{1-v}}, \quad v \in V, \]
is a stationary density for $k$. Next, if $f$ is a stationary density for $k$, then
\[ f(v) = \int_V k(w, v) f(w) \, dw = \frac{3}{4} \cdot \frac{v}{\sqrt{1-v}} \int_V f(w) \, dw = \frac{3}{4} \cdot \frac{v}{\sqrt{1-v}}, \quad v \in V. \]
Hence, $f_\circ$ is a unique stationary density for $k$, and it is strictly positive, that is, (H3) holds for $k$. Furthermore, it is easy to check that (H4) is satisfied, however $k$ is not essentially bounded and (H2) fails to hold.

\begin{remark}
As mentioned in Introduction asymptotic stability of the LRR semigroup in the case $p > 0$, $p + q = 1$ was investigated recently by Mokhtar-Kharroubi and Rudnicki [20]. The authors consider one-dimensional stochastic billiards, a generalization of the LRR model, and show in particular (see [20, Sections 4–6] and [25, Section 6.3.5]) that the LRR semigroup is asymptotically stable provided (H1), (H2) and (H4)' hold. By Theorem 5.1 and Remark 4.5 these conditions imply that (H3) and (H4) hold, which guarantee that assumptions of Theorem 4.3 are satisfied. In [20] the authors also use Theorem 4.2, but the other part of the reasoning is different than in this paper.

\end{remark}

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