Maximal violation of Clauser-Horne-Shimony-Holt inequality for

two qutrits

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Bell-Clauser-Horne-Shimony-Holt inequality (in terms of correlation functions) of two qutrits is studied in detail by employing tritt er measurements. A uniform formula for the maximum value of this inequality for tritter measurements is obtained. Based on this formula, we show that non-maximally entangled states violate the Bell-CHSH inequality more strongly than the maximally entangled one. This result is consistent with what was obtained by Acín \textit{et al} [Phys. Rev. A \textbf{65}, 052325 (2002)] using the Bell-Clauser-Horne inequality (in terms of probabilities).

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\section{I. INTRODUCTION}

Bell inequality has come to be not only as a tool for exposing the weirdness of quantum mechanics, but also as a more powerful resource in a number of applications, such as in quantum communication. Bell-Clauser-Horne-Shimony-Holt (Bell-CHSH) inequality has been applied in communicating protocol (Ekert protocol) to detect the presence of the eavesdropper \cite{1}. Furthermore, it has been found that two entangled $N$-dimensional systems (quNits) generate correlations that are more robust against noise than those generated by two entangled qubits \cite{2–5}. It was suggested that the higher dimensional entangled systems

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may be much superior than two-dimensional systems in quantum communication. Naturally, the extension of the protocol (involving Bell-CHSH inequality) to higher dimension becomes an interesting problem. So, it is necessary and important to investigate the Bell inequality for higher dimensional systems.

In an interesting paper [6], by using the Bell-Clauser-Horne inequality (in terms of probabilities) [4,5], Acín et al. have shown that non-maximally entangled states violate the Bell-CHSH inequality more strongly than the maximally entangled one. Recently, a Bell-CHSH inequality (in terms of correlation functions) of two qutrits has been obtained [7] by searching the inequality which can give the minimal noise admixture $F_{thr}$ for the maximally entangled states. The minimal noise admixture $F_{thr}$ for the maximally entangled state of two qutrits has been obtained numerically by the method of linear optimization in [2] and analytically in [5,8]. The extension of the Bell-CHSH to higher dimension is a non-trivial and interesting problem. Actually, it has been applied to quantum cryptography [9]. In this paper, we study the Bell-CHSH inequality of two qutrits for tritter measurements by considering a class of pure states of two qutrits. A uniform formula of the maximum value of this inequality is obtained. Based on this formula, we find the states which give the maximum violation of the Bell-CHSH inequality. This result is consistent with what was obtained by Acín et al. [6].

II. THE INEQUALITY

Let us consider a gedanken experiment with two observers each measuring two observables on some state of two qutrits $\rho$. We denote the observables by $\hat{A}^i$ ($i = 1, 2$) for the first observer (Alice), $\hat{B}^j$ ($j = 1, 2$) for the second observer (Bob). The measurement of each observable yields three distinct outcomes which denote by $a_1^i, a_2^i, a_3^i$ for Alice’s measurement of the observable, and $b_1^j, b_2^j, b_3^j$ for Bob’s measurement of the observable. Specifically, the observables have the spectral decompositions: $\hat{A}^i = a_1^i \hat{P}_1^i + a_2^i \hat{P}_2^i + a_3^i \hat{P}_3^i$, and $\hat{B}^j = b_1^j \hat{Q}_1^j + b_2^j \hat{Q}_2^j + b_3^j \hat{Q}_3^j$, where $\hat{P}_l^i$ and $\hat{Q}_m^j$ ($l, m = 1, 2, 3$) are mutually orthogonal.
projectors respectively. The probability of obtaining the set of three numbers \((a^i_l, b^j_m)\) in a simultaneous measurement of observables \(\hat{A}^i\) and \(\hat{B}^j\) on the state \(\rho\) is denoted by \(P(a^i_l, b^j_m)\), which can be given by the standard formula

\[
P(a^i_l, b^j_m) = \text{Tr}(\rho \hat{P}^i_l \otimes \hat{Q}^j_m).
\]  

As introduced and used in Ref. [11], the correlation function \(Q(\vec{\varphi}^A_i, \varphi^B_j) (Q_{ij}\) for short) between Alice’s and Bob’s measurements is

\[
Q_{ij} = \sum_{l, m = 1}^3 \alpha^{l_i + m_j} P(a^i_l, b^m_j),
\]  

where \(\alpha = e^{i2\pi/3}\). Let us define the following quantity

\[
S = \text{Re}[Q_{11} + Q_{12} - Q_{21} + Q_{22}] + \frac{1}{\sqrt{3}}\text{Im}[Q_{11} - Q_{12} - Q_{21} + Q_{22}].
\]  

It can be shown [7], using the recently discovered Bell inequality for two qutrits [4], that according to local realistic theory \(S\) can not exceed 2, i.e. \(S \leq 2\) for local realistic theory. However, when using the quantum correlation function given in Eq. (2), \(S_{\text{max}}\) acquires the value \(\frac{2}{9}(6 + 4\sqrt{3}) \approx 2.87293\) for the state \(|\psi\rangle = \frac{1}{\sqrt{3}} \sum_i |i\rangle |i\rangle\), the maximally entangled state. Following [2], we define the threshold noise admixture \(F_{\text{thr}}\) (the minimal noise admixture fraction for \(|\psi\rangle\)) \(F_{\text{thr}} = 1 - 2/S_{\text{max}}\). Then for the maximally entangled two qutrits, we have \(F_{\text{thr}} = 0.30385\). For the maximally entangled two qubits, one has \(F_{\text{thr}} = 0.29289\). Obviously, entangled qutrits are more resistant to noise than entangled qubits [2,8].

As suggested in Ref. [7] and [4], the Bell-CHSH inequality for two qutrits can be expressed as

\[-4 \leq S \leq 2.\]  

On the other hand, the interesting thing is the maximal \(F_{\text{thr}}\) of two qutrits obtained in Ref. [10] by the numerical linear optimization method. The authors found that the optimal non-maximally entangled state of two qutrits is around 3% more resistant to noise than the maximally entangled one. The maximal \(F_{\text{thr}} = 0.31386\) for such state (a non-maximally
state). Similar result is obtained in Ref. [6]. Obviously, the maximal violation of the inequality should be 2.91485 for such non-maximally entangled states.

For simplicity, we consider such a gedanken experiment that Alice’s and Bob’s observables are defined by unbiased symmetric six-port beam-splitter on the state of two qutrits

$$|\psi\rangle = \frac{1}{\sqrt{3}} \sum_{i} a_i |i\rangle |i\rangle,$$  \hspace{1cm} (5)

with real coefficients $a_i$, the kets $|i\rangle$ ($i = 1, 2, 3$) denote the orthonormal basis states for the qutrit. The unbiased symmetric six-port beam-splitter, called tritter [12,13], is an optical device with three input and output ports. In front of every input port there is a phase shifter that changes the phase of the photon entering the given port. The observers select the specific local observables by setting appropriate phase shifts in the beams leading to the entry ports of the beam-splitters. Such process performs a unitary transformation between “mutually unbiased” bases in the Hilbert space [14–16]. The overall unitary transformation performed by such a device is given by

$$U_{ij} = \frac{1}{\sqrt{3}} \alpha^{(i-1)(j-1)} e^{i\varphi_{ij}}, \quad i, j = 1, 2, 3$$  \hspace{1cm} (6)

where $\alpha = e^{i\pi/3}$ and $j$ denotes an input beam to the device, and $i$ an output one; $\varphi_{ij}$ are the three phases that can be set by the local observer, denoted as $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ . The transformations at Alice’s side are denoted as $\vec{\varphi}^A = (\varphi_1^A, \varphi_2^A, \varphi_3^A)$, and $\vec{\varphi}^B = (\varphi_1^B, \varphi_2^B, \varphi_3^B)$ for Bob’s side.

The observables measured by Alice and Bob are now defined as follows. The set of projectors for Alice’s $i$-th measurement is given by $\hat{P}_i^l = U_A^+(\vec{\varphi}^A_i) |l\rangle \langle l| U_A(\vec{\varphi}^A_i)$ ($l = 1, 2, 3$), where $U_A(\vec{\varphi}^A_i)$ is the matrix of Alice’s unbiased symmetric six-port beam-splitter defined by Eq. (6). Bob’s $j$-th measurement is given by $\hat{Q}_m^j = U_B^+(\vec{\varphi}^B_j) |m\rangle \langle m| U_B(\vec{\varphi}^B_j)$ ($m = 1, 2, 3$). Then, from (1) and (2), the correlation function for state $|\psi\rangle$ reads

$$Q_{ij} = \sum_{n,k} \sum_{l_i, m_j} a_n a_k \alpha^{l_i+m_j} (\alpha^*)^{(n-1)(l_i+m_j-2)} \alpha^{(k-1)(l_i+m_j-2)} e^{i(\varphi_{n}^{A_i} + \varphi_{l}^{B_j} - \varphi_{k}^{A_i} - \varphi_{m}^{B_j})},$$  \hspace{1cm} (7)

This shows the results of the measurement obtained by Alice and Bob are strictly correlated.
In the following, we will investigate the Bell-CHSH inequality (4) for the tritter measurements and give analytical discussions of above results.

III. THE MAXIMAL VIOLATION

By substituting Eq. (7) into (3), after some elaborate, we obtain

\[ S = a_1 a_2 T_{12} + a_1 a_3 T_{13} + a_2 a_3 T_{23}, \] (8)

where

\[ T_{12} = \frac{1}{9}[3 \cos(\varphi_1^{A_2} - \varphi_2^{A_2} + \varphi_1^{B_1} - \varphi_2^{B_1}) - 3 \cos(\varphi_1^{A_1} - \varphi_2^{A_1} + \varphi_1^{B_1} - \varphi_2^{B_1}) \]

\[ -3 \cos(\varphi_1^{A_2} - \varphi_2^{A_2} + \varphi_1^{B_2} - \varphi_2^{B_2}) - \sqrt{3} \sin(\varphi_1^{A_2} - \varphi_2^{A_2} + \varphi_1^{B_1} - \varphi_2^{B_1}) \]

\[ +\sqrt{3} \sin(\varphi_1^{A_1} - \varphi_2^{A_1} + \varphi_1^{B_1} - \varphi_2^{B_1}) + 2\sqrt{3} \sin(\varphi_1^{A_1} - \varphi_2^{A_1} + \varphi_1^{B_2} - \varphi_2^{B_2}) \]

\[ +\sqrt{3} \sin(\varphi_1^{A_2} - \varphi_2^{A_2} + \varphi_1^{B_2} - \varphi_2^{B_2})], \] (9)

\[ T_{13} = \frac{1}{9}[3 \cos(\varphi_1^{A_1} - \varphi_3^{A_1} + \varphi_1^{B_1} - \varphi_3^{B_1}) - 3 \cos(\varphi_1^{A_2} - \varphi_3^{A_2} + \varphi_1^{B_1} - \varphi_3^{B_1}) \]

\[ +3 \cos(\varphi_1^{A_2} - \varphi_3^{A_2} + \varphi_1^{B_2} - \varphi_3^{B_2}) + \sqrt{3} \sin(\varphi_1^{A_1} - \varphi_3^{A_1} + \varphi_1^{B_1} - \varphi_3^{B_1}) \]

\[ -\sqrt{3} \sin(\varphi_1^{A_1} - \varphi_3^{A_1} + \varphi_1^{B_1} - \varphi_3^{B_1}) + 2\sqrt{3} \sin(\varphi_1^{A_1} - \varphi_3^{A_1} + \varphi_1^{B_2} - \varphi_3^{B_2}) \]

\[ +\sqrt{3} \sin(\varphi_1^{A_2} - \varphi_3^{A_2} + \varphi_1^{B_2} - \varphi_3^{B_2})], \] (10)

and

\[ T_{23} = \frac{1}{9}[3 \cos(\varphi_2^{A_1} - \varphi_3^{A_1} + \varphi_2^{B_1} - \varphi_3^{B_1}) - 3 \cos(\varphi_2^{A_2} - \varphi_3^{A_2} + \varphi_2^{B_1} - \varphi_3^{B_1}) \]

\[ +3 \cos(\varphi_2^{A_2} - \varphi_3^{A_2} + \varphi_2^{B_2} - \varphi_3^{B_2}) - \sqrt{3} \sin(\varphi_2^{A_1} - \varphi_3^{A_1} + \varphi_2^{B_1} - \varphi_3^{B_1}) \]
\[+\sqrt{3}\sin(\varphi_2^A - \varphi_3^A + \varphi_2^B - \varphi_3^B) - 2\sqrt{3}\sin(\varphi_2^A - \varphi_3^A + \varphi_2^B - \varphi_3^B)]\]

are three continuous functions of twelve angles \(\varphi^A_i\) and \(\varphi^B_j\) \((i, j = 1, 2)\). So, \(S\) is the continuous function of the twelve variables. The points which satisfy

\[\frac{\partial S}{\partial \varphi^A_i} = 0, \Lambda = A, B; \ i = 1, 2; \ \text{and} \ j = 1, 2, 3,\]

are the critical points of the function \(S\). According to the theory of extreme points of continuous functions, we know that the extreme points are belong to the critical points of the function. So, we can extract the maximum and minimum of \(S\) from the critical points by comparing the value of \(S\) among the critical points, since the maximum and minimum point must be one of extreme points.

On the other hand, we can know that \(|t_{12}| \leq \frac{4}{3}, |t_{13}| \leq \frac{4}{3}, |t_{23}| \leq \frac{4}{3}\). However, the above three formulae are strongly correlated, so \(t_{12}, t_{13}, \text{and} \ t_{23}\) can not reach their maximum value at the same time. It happens that when one of \(t_{12}, t_{13}, \text{and} \ t_{23}\) reaches its maximum value \(\frac{4}{3}\), the others can reach their sub-maximum value \(\frac{4}{3\sqrt{3}}\). If we consider \(t_{12}, t_{13}, \text{and} \ t_{23}\) as three coordinates, then they can form a complicated polyhedron. The polyhedral vertices are the points when \(t_{12}, t_{13}, \text{and} \ t_{23}\) reach their extreme values.

**Lemma** For the formula \(G = \sum_{i=1}^{N} \xi_i R_i\), where \(\xi_i\) are \(N\) real parameters, the maximum (minimum) points of \(G\) must on the boundary of the region formed by \(R_i\) for any \(\xi_i\).

**Proof:** Giving \(G^0 = \sum_{i=1}^{N} \xi_i R^0_i\), if \(R^0_i(i = 1, 2, \cdots, N)\) are in the inner region formed by \(R_i\), we can always have \(G = G^0 + \sum_{i=1}^{N} \xi_i \Delta R_i\), in which \(\Delta R_i\) are infinitesimal values satisfying \(\xi_i \Delta R_i > 0 \ (i = 1, 2, \cdots, N)\), so that \(G > G^0\); or \(\Delta R_i\) are infinitesimal values satisfying \(\xi_i \Delta R_i < 0 \ (i = 1, 2, \cdots, N)\), so that \(G < G^0\). So, we can know that the maximum (minimum) points of \(G\) can only find on the boundary.

**Theorem** The maximum and minimum value of \(S\) for a given state (5) must be found at the vertices of polyhedron formed by \(t_{ij} \ (i \neq j, i, j = 1, 2, 3)\).
Proof: We know that the maximum points of $S$ is belong to the critical points of $S$. For the critical points in the inner region formed by $(t_{12}, t_{13}, t_{23})$, from the Lemma we know that the value of such critical points must be less than some values of $S$ on the boundary, so they can not be the maximum points of $S$. For the same reason, if the critical point on the boundary (excepting for vertices), we can know that the value of $S$ on this point must be less than $S$ on one of the vertices on this boundary. Then, the maximum value of $S$ must be only found on the vertices of region formed by $(t_{12}, t_{13}, t_{23})$.

In analog to the above discussion, the minimum value of $S$ can also be found on the vertices.

To find out the maximum (minimum) value we have to calculate the vertices of the polyhedron formed by $t_{ij}$. For convenience, we denote $T_1$ as one of $\{t_{12}, t_{13}, t_{23}\}$, $T_2$ as one of $\{t_{12}, t_{13}, t_{23}\}/\{T_1\}$ and $T_3$ as one of $\{t_{12}, t_{13}, t_{23}\}/\{T_1, T_2\}$, where $\{\}/\{}$ means division of sets namely, if $T_1 = t_{12}$, then $T_2 \in \{t_{12}, t_{13}, t_{23}\}/\{t_{12}\} = \{t_{13}, t_{23}\}$, and so on. In the following, we list the vertices formed by the maximum and sub-maximum of $t_{ij}$ (it is enough),

\[
(|T_1|, |T_2|, |T_3|) = \left(\frac{4}{3}, \frac{4}{3\sqrt{3}}, \frac{4}{3\sqrt{3}}\right), \text{ for } T_1T_2T_3 > 0; \tag{13}
\]

and

\[
(|T_1|, |T_2|, |T_3|) = \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right), \text{ for } T_1T_2T_3 < 0. \tag{14}
\]

Comparing the value of $S$ among these points, we can obtain the maximum and minimum values of $S$ for the state (5). Assuming $\{K_i, \ (i = 1, 2, 3)\} = \{|a_1a_2|, |a_1a_3|, |a_2a_3|\}$ , where “$\ = $” means the equality of two sets, and $K_i$ are in decreasing order, i.e. $K_1 \geq K_2 \geq K_3$, let us define

\[
S_1(|\psi\rangle) = \frac{4}{3}K_1 + \frac{4}{3\sqrt{3}}(K_2 + K_3), \tag{15}
\]

and

\[
S_2(|\psi\rangle) = \frac{4}{3}(K_1 + K_2 - K_3), \tag{16}
\]
Then, we can know the maximum value of $S$ must be

$$S_{\text{max}}(|\psi\rangle) = \text{Max}(S_1(|\psi\rangle), S_2(|\psi\rangle)).$$  \hspace{1cm} (17)$$

From (15) and (16), we know that $S_2(|\psi\rangle) \geq S_1(|\psi\rangle)$ only for $\frac{K_2}{K_1} \leq 2 - \sqrt{3}$. If taking $\sum_i a_i^2 = 3$ into account, one can prove that when $\text{Max}(|a_1|, |a_2|, |a_3|) \geq \frac{\sqrt{6+3\sqrt{3}}}{2} = 1.67303$, $S_2(|\psi\rangle) \geq S_1(|\psi\rangle)$. Let us define $A_{\text{max}} = \text{Max}(|a_1|, |a_2|, |a_3|)$, finally we obtain that

$$S_{\text{max}}(|\psi\rangle) = \begin{cases} \frac{4}{3}K_1 + \frac{4}{3\sqrt{3}}(K_2 + K_3), & A_{\text{max}} \leq \frac{\sqrt{6+3\sqrt{3}}}{2}; \\ \frac{4}{3}(K_1 + K_2 - K_3), & A_{\text{max}} > \frac{\sqrt{6+3\sqrt{3}}}{2}. \end{cases}$$  \hspace{1cm} (18)$$

We can also prove that the minimum of $S$ is

$$S_{\text{min}}(|\psi\rangle) = -\frac{4}{3}(K_1 + K_2 + K_3).$$  \hspace{1cm} (19)$$

Obviously one can easily find that for maximally entangled state $|\psi\rangle = \frac{1}{\sqrt{3}} \sum_i |i\rangle |i\rangle$, (i.e., $a_i = 1$), we have $S_{\text{max}} = \frac{2}{9}(6 + 4\sqrt{3})$ and $S_{\text{min}} = -4$, which are the same as the results obtained in Refs. [4,6,10,17].

In Fig. 1, we give the comparison between the theoretical results and the numerical calculations obtained by multi random search optimization method, which shows a perfect agreement; (a) for $S_{\text{max}}$ and (b) for $S_{\text{min}}$, in which $a_1$ changes in region $[-\sqrt{3}, \sqrt{3}]$, $a_2 = \sqrt{(3 - a_1^2)\varepsilon}$ and $a_3 = \sqrt{(3 - a_1^2)(1 - \varepsilon)}$, $0 \leq \varepsilon \leq 1$. One can find some inflexion points in fig. 1(a), for example, at the point $a_1 = 1$ when $\varepsilon = 0.5$. These inflexion points are due to the discontinuous change of $K_1$, the maximum value among $|a_1a_2|, |a_1a_3|$ and $|a_2a_3|$, e.g., for $\varepsilon = 0.5$, $K_1 = a_2a_3 = \frac{(3-a_1^2)}{2}$ when $a_1 \leq 1$, but when $a_1 > 1$, $K_1 = a_1a_2 = a_1\sqrt{\frac{(3-a_1^2)}{2}}$. On the other hand, we can see from Fig.1(a) that the maximally entangled states are not the states that give the maximal violation of the Bell inequality.

Consider $a_i$ as variables, we can obtain the maximal value of $S_{\text{max}}$, denoted as $\bar{S}_{\text{max}}$, by calculating the extreme value of Eq. (18), after some elaboration, we get

$$\bar{S}_{\text{max}} = 1 + \sqrt{\frac{11}{3}},$$  \hspace{1cm} (20)$$

when
\[
\{ |a_1|, |a_2|, |a_3| \} = \left\{ \sqrt{\frac{3}{2}} \left( 1 - \sqrt{\frac{3}{11}} \right), \sqrt{\frac{3 - a_1^2}{2}}, \sqrt{\frac{3 - a_1^2}{2}} \right\}.
\]

One sees that for this value the threshold amount of noise is about \( F_{\text{thr}} = 0.3139 \), which is as the same as what has been obtained in recently calculation \([6,10,17]\). So, this result gives another evidence for the inequality \( (4) \).

On the other hand, we can also calculated the minimum value of \( S_{\text{min}} \), denoted as \( \bar{S}_{\text{min}} \),

\[ \bar{S}_{\text{min}} = -4, \text{ for } \{ |a_1|, |a_2|, |a_3| \} = \{1, 1, 1\}. \]  

Then, we can know that

\[ 0 \leq S_{\text{max}} \leq 1 + \sqrt{\frac{11}{3}}, -4 \leq S_{\text{min}} \leq 0. \]

Obviously, for tritter measurements, the left hand of the inequality \( (4) \) would never be violated, and the right hand only be violated by some of pure states. We can easily find the states that violate the inequality for tritter measurements from the formula \( (18) \). In Fig. 2, we show the states described by \((a_1, a_2 = \sqrt{(3 - a_1^2)} \varepsilon, a_3 = \sqrt{(3 - a_1^2)(1 - \varepsilon} )\) that violate the inequality for tritter measurements. The states which violate the inequality are in the shadow region; the states of which \( S_{\text{max}} = 2 \) are on the boundary of the shadow region; the states in other region can not violate the inequality for tritter measurements.

We should add here that some similar calculations as well as some equivalence results were made by Cereceda \([17]\) where the author compared some of the two-qutrit inequalities and investigated them in detail.

**IV. DISCUSSION**

In the above discussion we only concentrate on tritter measurements which can be easily carried out for nowadays technology \([12]\). By detail studying the Bell-CHSH inequality of two qutrits, we give formulae of the maximum and minimum values of this inequality, and obtain the states which give the maximal violation of the Bell-CHSH inequality. The maximal violation we obtained are the same as Refs. \([6,10]\).
Indeed, one should use general measurements to study the problem of maximizing the Bell violation for a state, or in other words, for some states the tritter measurements are not optimal.

So, some states that do not violate the inequality using tritter measurements, but may violate the inequality when general measurements are taken into account [17]. For example, for the state with $|a_1| = 1.56$ and $\varepsilon = 0.5$, $S_{\text{max}} = 1.964$ for tritter measurements, which does not violate the inequality; but if we employ the following measurements, $\hat{P}_l = U_A^+ (\tilde{\varphi}^A_l) |x_l\rangle \langle x_l| U_A (\tilde{\varphi}^A_l)$ ($l = 1, 2, 3$) and $\hat{Q}_m = U_B^+ (\tilde{\varphi}^B_m) |x_m\rangle \langle x_m| U_B (\tilde{\varphi}^B_m)$ ($m = 1, 2, 3$) where $|x_1\rangle = \frac{1}{\sqrt{2}}[|1\rangle + |2\rangle]$, $|x_2\rangle = \frac{1}{\sqrt{2}}[|1\rangle - |2\rangle]$ and $|x_3\rangle = |3\rangle$ are orthonormal basis, we can obtain $S_{\text{max}} = 2.0132$ (violates the inequality).

However, by employing tritter measurements, it can reveal many important properties of Bell inequality of entangled two qutrits. For instance, for the maximally entangled state $|\psi\rangle = \frac{1}{\sqrt{3}} \sum_i |i\rangle |i\rangle$ and the states that maximally violates the inequality, the tritter measurements are optimal, and based on such entangled qutrit pairs a cryptographic protocol has been presented more recently [9] by employing tritter measurements.

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VI. FIGURES CAPTION:

Fig. 1(a). The maximal value of the inequality for tritter measurements, $S_{\text{max}}$, for state given by Eq. (5), where $a_1$ changes in region $[-\sqrt{3}, \sqrt{3}]$, $a_2 = \sqrt{(3-a_1^2)\varepsilon}$ and $a_3 = \sqrt{(3-a_1^2)(1-\varepsilon)}$, $0 \leq \varepsilon \leq 1$. The solid lines are theoretical results, circles are numerical dates; dotted line shows the maximal value predicted by the local realistic theory, dashed line marks the value of the maximally entangled states. (b) The minimal value of the inequality, $S_{\text{min}}$.

Fig. 2. It shows the states that violate the inequality for tritter measurements. The states in shadow region violate the inequality.
Fig. 1(a)

$\varepsilon = 0.5$

$\varepsilon = 0.9$

$\varepsilon = 1.0$

Numerical
Theoretical

$S_{\text{max}}$

$a_1$

Fig. 1(a)
Fig. 1(b)

$S_{min}$ vs $a_1$

- $\varepsilon = 0.5$
- $\varepsilon = 0.9$
- $\varepsilon = 1.0$

Numerical
Theoretical
Fig. 2