Maple procedures in teaching the canonical formalism of general relativity

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Abstract

We present some Maple procedures using the GrTensorII package for teaching purposes in the study of the canonical version of the general relativity based on the ADM formalism.

1 Introduction

The use of computer facilities can be an important tool for teaching general relativity. We have experienced several packages of procedures, (in REDUCE + EXCALC for algebraic programming and in Mathematica for graphic visualizations) which fulfill this purpose \cite{10}. In this article we shall present some new procedures in MapleV using GrTensorII package \cite{11} adapted for the canonical version of the general relativity (in the so called ADM formalism based on the 3+1 split of spacetime). This formalism is widely used \cite{8,9} in the last years as a major tool in numerical relativity for calculating violent processes as, for example the head-on collisions of black holes, massive stars or other astrophysical objects. Thus we used these computer procedures.
in the process of teaching the canonical formalism as an introductory part of a series of lectures on numerical relativity for graduated students. The next section of the article presents shortly the notations and the main features of the canonical version of the general relativity. Early attempts in using computer algebra (in REDUCE) for the ADM formalism can be detected in the literature ([3], [6], [7]). Obviously we used these programs in producing our new procedures for Maple + GrTensorII package, but because there are many specific features we shall present in some detail these procedures in the section 3 of the article. The last section of the article is dedicated to the conclusions pointed out by running the Maple procedures presented here and some future prospects on their usage toward the numerical relativity.

2 Review of the canonical formalism of general relativity

Here we shall use the specific notations for the ADM formalism [1], [2]; for example latin indices will run from 1 to 3 and greek indices from 0 to 3. The starting point of the canonical formulation of the general relativity is the (3+1)-dimensional split of the space-time produced by the split of the metric tensor:

\begin{equation}
(4) g_{\alpha \beta} = \begin{pmatrix}
(4) g_{oo} & (4) g_{oj} \\
(4) g_{io} & (4) g_{ij}
\end{pmatrix} = \begin{pmatrix}
N_k N^k - N^2 & N_j \\
N_i & g_{ij}
\end{pmatrix}
\end{equation}

where \( g_{ij} \) is the riemannian metric tensor of the three-dimensional spacelike hypersurfaces at \( t = \text{const.} \) which realize the spacetime foliation. Here \( N \) is the "lapse" function and \( N^i \) are the components of the "shift" vector [2].

The Einstein vacuum field equations now are (denoting by "." the time derivatives):

\begin{equation}
\dot{g}_{ij} = 2N g^{-1/2} [\pi_{ij} - \frac{1}{2} g_{ij} \pi^k_k] + N_{i/j} + N_{j/i}
\end{equation}

\begin{equation}
\dot{\pi}^{ij} = -Ng^{1/2}[R^{ij} - \frac{1}{2} g^{ij} R] + \frac{1}{2} Ng^{-1/2} g^{ij} [\pi^k_i \pi_{jl} - \frac{1}{2} (\pi^k_k)^2]
\end{equation}
\[ -2Ng^{-1/2}[\pi^{im} \pi^{jm} - \frac{1}{2} \pi^{jk} \pi^{km}] + g^{1/2}[N/ij - g^{ij} N/m] \]
\[ + [\pi^{ij} N^m]/m - N^i/m \pi^{mj} - N^j/m \pi^{mi} \]  

(3)

where \( \pi^{ij} \) are the components of the momenta canonically conjugate to the \( g_{ij} \)'s.

In the above formulas we denoted by \( \mathcal{J} \) the three-dimensional covariant derivative defined with \( g_{ij} \) using the components of the three-dimensional connection [2]:

\[ \Gamma^i_{jk} = \frac{1}{2}g^{im}(g_{mj,k} + g_{mk,j} - g_{jk,m}) \]  

(4)

The Ricci tensor components are given by

\[ R_{ij} = \Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^k_{ij} \Gamma^m_{km} - \Gamma^k_{im} \Gamma^m_{jk} \]  

(5)

The initial data on the \( t = \text{const.} \) hypersurface are not independent because they must satisfy the constraint equations, which complete the Einstein equations

\[ \mathcal{H} = -\sqrt{g}\{R + g^{-1} \frac{1}{2} (\pi^k)^2 - \pi^{ij} \pi_{ij}\} = 0 \]  

(6)

\[ \mathcal{H}^i = -2\pi^{ij}_j = 0 \]  

(7)

where \( \mathcal{H} \) is the super-hamiltonian, \( \mathcal{H}^i \) the super-momentum and \( g \) is the determinant of the three-dimensional metric tensor \( g_{ij} \).

The action functional in Hamiltonian form for a vacuum space-time can thus be written as [1], [2]:

\[ S = \int dt \int (\pi^{ij} g_{ij} - N\mathcal{H} - N_i \mathcal{H}^i) \omega^1 \omega^2 \omega^3 \]  

(8)

where the \( \omega^i \)'s are the basis one-forms. Thus the dynamic equations (3) and (5) are obtained by differentiating \( S \) with respect to the canonical conjugate pair of variables \( (\pi^{ij}, g_{km}) \).

3 Maple + GrTensorII procedures

Here we shall describe briefly the structure and the main features of the Maple procedures for the canonical formalism of the general
relativity as described in the previous section. Two major parts of
the programs can be detected: one before introducing the metric of
the spacetime used (consisting in several definitions of tensor objects
which are common to all spacetimes) and the second one, having line-
commands specific to each version.

The first part of the program starts after initialisation of the GrTen-
sorII package (grtw();) and has mainly the next lines:

```maple
> grdef('tr := pi{^i i}');
> grdef('ha0:=-sqrt(detg)*(Riccisclar+
(1/detg)*((1/2)*tr^2-2*pi{i j}*pi{ ^i ^j }}))';
> grdef('ha{ ^i }:=-2*(pi{ ^i ^j }-pi{ ^i j })*Chr{ p j ^p }');
> grdef('derge{ i j }:=2*N(x,t)*(detg)^(-1/2)*pi{ i j }-
(1/2)*g{ i j}*tr)+Ni{ i ;j } + Ni{ j ;i }');
> grdef('Ndd{ ^m j }:= Nd{ ^m ;j }');
> grdef('bum{ ^i ^j ^m}:=pi{ ^i ^j }*Ni{ ^m }');
> grdef('bla{ ^i ^j }:=bum{ ^i ^j ^m ;m }');
> grdef('derpi{ ^i ^j }:=
-N(x,t)*(detg)^(-1/2)*R{ ^i ^j }-(1/2)*g{ ^i ^j }*Riccisclar+
(1/2)*N(x,t)*(detg)^(-1/2)*pi{ ^k ^l }*pi{ k l }-
(1/2)*tr^2-2*N(x,t)*(detg)^(-1/2)*pi{ ^i ^m }*pi{ ^j m }-
(1/2)*pi{ ^i ^j }*tr+ (detg)^(-1/2)*Ndd{ ^i ^j }-g{ ^i ^j }*
Ndd{ ^m m } + bla{ ^i ^j } - Ni{ ^i ;m }*pi{ ^m ^j }-
Ni{ ^j ;m }*pi{ ^m ^i }');
```

Here \( ha_0 \) and \( ha{ ^i } \) represents the superhamiltonian and the
supermomentum as defined in eqs. (6) and (7) respectively and \( tr \)
is the trace of momentum tensor density \( \pi^{ij} \) - which will be defined
in the next lines of the program. Here \( N(x,t)\) represents the lapse
function \( N \). Also, \( derge{ i j } \) represents the time derivatives of the
components of the metric tensor, as defined in eq. (2) and \( derpi{ ^i ^j } \)
the time derivatives of the components of the momentum tensor
\( \pi^{ij} \) as defined in eq. (3).

The next line of the program is a specific GrTensorII command
for loading the spacetime metric. Here Maple loads a file (previously
generated) for introducing the components of the metric tensor as
functions of the coordinates. We also reproduced here the output of
the Maple session showing the metric structure of the spacetime we
introduced.

```maple
> qload('Cyl_dim');
```
Default spacetime = Cyl_din

For the Cyl_din spacetime:

Coordinates
x(up)

\[x = [x, y, z]\]

Line element

\[ds = \exp(\gamma(x, t) - \psi(x, t)) \, dx\]

\[+ R(x, t) \exp(-\psi(x, t)) \, dy + \exp(\psi(x, t)) \, dz\]

As is obvious we introduced above the metric for a spacetime with cylindrical symmetry, an example we used for teaching purposes being a well known example in the literature ([5]). In natural output this metric has the form:

\[
g_{ij} = \begin{pmatrix}
e^{\gamma - \psi} & 0 & 0 \\
0 & R^2 e^{-\psi} & 0 \\
0 & 0 & e^\psi
\end{pmatrix}
\]  

(9)

in cylindrical coordinates \(x, y, z\) with \(x \in [0, \infty), y \in [0, 2\pi), z \in (-\infty, +\infty)\) where \(R, \psi\) and \(\gamma\) are functions of \(x\) and \(t\) only.

After the metric of the spacetime is established the next sequence of the program just introduce the components of the momentum tensor \(\pi^{ij}\) as:

\[
> \text{grdef('Nd{ m } := [diff(N(x,t),x), 0, 0]');}
\]

\[
> \text{grdef('Ni{ i } := [N1(x,t), N2(x,t), N3(x,t)]');}
\]

\[
> \text{grdef('vi1{ i }:=[pig(x,t)*exp(\psi(x,t)-\gamma(x,t)),0,0]');}
\]

\[
> \text{grdef('vi3{ i } :=[0,0,exp(-\psi(x,t))*(pig(x,t)+(1/2)*R(x,t)*pir(x,t)+pip(x,t))]');}
\]

\[
> \text{grdef('vi2{ i }:=[0,\, (2*R(x,t))^(-1)*pir(x,t)*exp(\psi(x,t)),0]');}
\]

\[
> \text{grdef('pi{ i j } := vi1{ i }*kdelta{ j x}+vi2{ i }*kdelta{ j y }+vi3{ i }*kdelta{ j z}');}
\]

\[
> \text{grcalc(pi(up,up));}
\]

\[
> \text{grdisplay(pi(up,up));}
\]
Here \( N^i \) represents the shift vector \( N_i \) and the other objects \( (N_d, v_1, v_2 \) and \( v_3) \) represent intermediate vectors defined in order to introduce the momenum \( \pi^{ij} \) having the form:

\[
\pi^{ij} = \begin{pmatrix}
\pi_\gamma e^{\psi - \gamma} & 0 & 0 \\
0 & \frac{1}{2R} \pi R e^\psi & 0 \\
0 & 0 & e^{-\psi}(\pi_\gamma + \frac{1}{2} R \pi_R + \pi_\psi)
\end{pmatrix}
\]  

(10)

In the program we denoted \( \pi_\gamma, \pi_R \) and \( \pi_\psi \) with \( \pi_g, \pi_r \) and \( \pi_p \), respectively. The momentum components are introduced in order that the dynamic part of the action of the theory be in canonical form, that is: \( \dot{g}_{ij} \pi^{ij} = \pi_\gamma \dot{\gamma} + \pi_\psi \dot{\psi} + \pi_R \dot{R} \). The next lines of the program check if this condition is fulfilled:

```maple
> grdef('de1{ i } := [diff(grcomponent(g(dn,dn),[x,x]),t),0,0]');
> grdef('de2{ i } := [0,diff(grcomponent(g(dn,dn),[y,y]),t),0]');
> grdef('de3{ i } := [0,0,diff(grcomponent(g(dn,dn),[z,z]),t)]');
> grdef('ddg({ i j } := de1{ i }*kdelta{j $x}+de2{ i }*kdelta{j$y}+
    de3{ i }*kdelta{j$z}');
> grcalc(ddg(dn,dn));
> grdef('act := pi{ ^i ^j }*ddg{ i j }');
> grcalc(act); gralter(act,simplify); grdisplay(act);
```

By inspecting this last output from the Maple worksheet, the user can decide if it is necessary to redefine the components of the momentum tensor or to go further. Here the components of the momentum tensor were calculated by hand but, of course, a more experienced user can try to introduce here a sequence of commands for automatic calculation of the momentum tensor components using the above condition, through an intensive use of \texttt{solve} Maple command.

Now comes the most important part of the routine, dedicated to calculations of different objects previously defined:

```maple
> grcalc(ha0); gralter(ha0,simplify); grdisplay(ha0);
> grcalc(ha(up)); gralter(ha(up),simplify); grdisplay(ha(up));
> grcalc(derge(dn,dn)); gralter(derge(dn,dn),simplify); grdisplay(derge(dn,dn));
> d1 := exp(-psi(x,t))*grcomponent(derge(dn,dn),[z,z])+exp(psi(x,t))-
```
\begin{verbatim}
  gamma(x,t)*grcomponent(derge(dn,dn),[x,x]);
  simplify(d1);
  d2:=(1/(2*R(x,t)))*exp(psi(x,t))*grcomponent(derge(dn,dn),[y,y])+
  (1/2)*R(x,t)*exp(-psi(x,t))*grcomponent(derge(dn,dn),[z,z]);
  simplify(d2);
  d3:=exp(-psi(x,t))*grcomponent(derge(dn,dn),[z,z]);
  simplify(d3);
  grcalc(derpi(up,up)); gralter(derpi(up,up),simplify);
  grdisplay(derpi(up,up));
  f1 := exp(gamma(x,t)-psi(x,t))*grcomponent(derpi(up,up),[x,x])-
pig(x,t)*(d3-d1);
  simplify(f1);
  f2:= 2*R(x,t)*exp(-psi(x,t))*grcomponent(derpi(up,up),[y,y])+
  (1/R(x,t))*d2*pir(x,t)-pir(x,t)*d3;
  simplify(f2);
  f3 := exp(psi(x,t))*grcomponent(derpi(up,up),[z,z])+d3*(pig(x,t)+
  (1/2)*R(x,t)*pir(x,t)+pip(x,t))-f1-(1/2)*R(x,t)*f2-
  (1/2)*pir(x,t)*d2;
  simplify(f3);
\end{verbatim}

This is a simple series of alternation of \texttt{grcalc}, \texttt{gralter} and \texttt{grdisplay}
commands for obtaining the superhamiltonian, supermomentum and the dynamic equations for the theory. \texttt{d1} ... \texttt{d3} and \texttt{f1} ... \texttt{f3} are the time derivatives of the dynamic variables, \(\dot{\gamma}, \dot{R}, \dot{\psi}\) and \(\dot{\pi}_\gamma, \dot{\pi}_R, \dot{\pi}_\psi\) respectively. Denoting with \(^r\) the derivatives with respect to \(r\) we display here the results for the example used above (cylindrical gravitational waves):

\[
\mathcal{H}^0 = e^{\psi-\gamma}(2R'' - R'\gamma' + \frac{1}{2}(\psi')^2 R - \pi_\gamma R + \frac{1}{2R}(\pi_\psi)^2) = 0
\]

\[
\mathcal{H}^1 = \mathcal{H}^r = e^{\psi-\gamma}(-2\pi_\gamma' + \gamma'\pi_\gamma + R'\pi_R + \psi'\pi_\psi) = 0 \quad ; \quad \mathcal{H}^2 = \mathcal{H}^3 = 0
\]

\[
\dot{\gamma} = N^1\gamma' + 2N^1 - e^{\psi-\gamma} N\pi_R \quad ; \quad \dot{R} = N^1 R' - e^{\psi-\gamma} N\pi_\gamma
\]

\[
\dot{\psi} = N^1 \psi' + \frac{1}{R} e^{\psi-\gamma} N\pi_\psi \quad ;
\]

\[
\dot{\pi}_\gamma = N^1 \pi_\gamma' + N^1 \pi_\gamma - e^{\psi-\gamma}(R'\pi_R' + \frac{1}{2}R'\psi'N - \frac{1}{2}\psi'N\pi_R - \frac{1}{2}\psi'N\pi_\gamma - \frac{1}{4R}\pi_\gamma^2)
\]
\[ \dot{\pi}_R = N^1 \pi'_R + N'^{1'} \pi_R + e^{\frac{\gamma}{2R}}(\gamma'N' - 2N'' - 2N'\psi' + \frac{1}{2} \gamma' \psi'N - \psi''N - \psi'^2 + \frac{1}{2R} N \pi^2 \psi) \]

\[ \dot{\pi}_\psi = N^1 \pi'_\psi + N'^{1'} \pi_\psi + e^{\frac{\gamma}{2R}}(RN'\psi' - R''N + \frac{1}{2} R N' \gamma' + R' \psi'N - \frac{1}{2} \gamma' \psi' NR' \]

+ \psi''RN + \frac{1}{4} \psi'^2 R N + \frac{1}{2} N \pi R \pi \gamma - \frac{1}{4R} N \pi^2 \psi \]

These are the well-known results reported in (5) or (6).

One of the important goals of the canonical formalism of the general relativity (which constitutes the “kernel” of the ADM formalism) is the reductional formalism. Here we obtain the true dynamical status of the theory, by reducing the number of the variables through solving the constraint equations. This formalism is applicable only to a restricted number of space-time models, one of them being the above cylindrical gravitational waves model. Unfortunately only a specific strategy can be used in every model. Thus the next lines of our program must be rewritten specifically in every case. Here, for teaching purposes we present our example of cylindrical gravitational wave space-time model. Of course we encourage the student to apply his own strategy for other examples he dares to calculate.

In our example of cylindrical gravitational waves, the reductional strategy as described in (3) starts with the usual rescaling of \( \mathcal{H} \) and \( \mathcal{H}' \) to \( \mathcal{H} \) and \( \mathcal{H}' \) by

\[ \mathcal{H} = e^{\frac{\gamma}{2R}} \mathcal{H} \quad ; \quad \dot{N} = e^{\frac{\gamma}{2R}} \dot{N} \quad ; \quad \mathcal{H}' = e^{\gamma - \psi} \mathcal{H}' \quad ; \quad \dot{N}' = e^{\psi - \gamma} \dot{N}' \]

which produce the next sequence of Maple+GrTensorII commands:

\[ > \text{grdef('aha0:=sqrt(exp(gamma(x,t)-psi(x,t)))*ha0');} \]
\[ > \text{grdef('aha{ ^j } := exp(gamma(x,t)-psi(x,t))*ha{ ^j }');} \]
\[ > \text{grdef('an:=sqrt(exp(psi(x,t)-gamma(x,t)))*n(x,t)');} \]
\[ > \text{grdef('ani{ ^i } := exp(psi(x,t)-gamma(x,t))*ni{ ^i }');} \]

The canonical transformation to the new variables, including Kuchar’s “extrinsic time”, defined by:

\[ T = T(\infty) + \int_{\infty}^{R} (-\pi_\gamma)dr \quad , \quad \Pi_T = -\gamma' + [\ln ((R')^2 - (T')^2)]' \]

\[ R = R' \quad , \quad \Pi_R = \pi_R + [\ln (R' + T') - (T')']' \]

are introduced with:
\begin{align*}
\pi_{\text{g}}(x,t) &= -\text{diff}(T(x,t),x); \\
\pi_{\text{r}}(x,t) &= \pi_R(x,t) - \text{diff}(\ln((\text{diff}(R(x,t),x)+\text{diff}(T(x,t),x))/
\quad (\text{diff}(R(x,t),x)-\text{diff}(T(x,t),x)),x); \\
\end{align*}

and specific substitutions in the dynamic objects of the theory:

\begin{align*}
\text{grmap}(ha_0, \text{subs}, \text{diff}(\gamma(x,t),x) &= \\
&\text{diff}(\ln((\text{diff}(R(x,t),x))^2 - (\text{diff}(T(x,t),x))^2),x)-\pi_T(x,t),x'); \\
\text{grcalc}(ha_0); \text{gralter}(ha_0, \text{simplify}); \\
\text{grdisplay}(ha_0); \\
\text{grmap}(ha(\text{up}), \text{subs}, \text{diff}(\gamma(x,t),x) &= \\
&\text{diff}(\ln((\text{diff}(R(x,t),x))^2 - (\text{diff}(T(x,t),x))^2),x)-\pi_T(x,t),x'); \\
\text{gralter}(ha(\text{up}), \text{simplify}); \\
\text{grdisplay}(ha(\text{up})); \\
\text{grcalc}(aha_0); \\
\text{grmap}(aha_0, \text{subs}, \text{diff}(\gamma(x,t),x) &= \\
&\text{diff}(\ln((\text{diff}(R(x,t),x))^2 - (\text{diff}(T(x,t),x))^2),x)-\pi_T(x,t),x'); \\
\text{gralter}(aha_0, \text{simplify}, \text{sqrt}); \\
\text{grdisplay}(aha_0); \\
\text{grcalc}(aha(\text{up})); \\
\text{grmap}(aha(\text{up}), \text{subs}, \text{diff}(\gamma(x,t),x) &= \\
&\text{diff}(\ln((\text{diff}(R(x,t),x))^2 - (\text{diff}(T(x,t),x))^2),x)-\pi_T(x,t),x'); \\
\text{gralter}(aha(\text{up}), \text{simplify}); \\
\text{grdisplay}(aha(\text{up})); \\
\text{grmap}(act, \text{subs}, \text{diff}(\gamma(x,t),x) &= \\
&\text{diff}(\ln((\text{diff}(R(x,t),x))^2 - (\text{diff}(T(x,t),x))^2),x)-\pi_T(x,t),x'); \\
\text{grcalc}(act); \text{grdisplay}(act);
\end{align*}

Thus the action yields (modulo divergences):

\begin{align*}
S &= 2\pi \int_{-\infty}^{\infty} dt \int_{0}^{\infty} dr (\Pi_T \dot{T} + \Pi_R \dot{R} + \pi_{\psi} \dot{\psi} + \pi_{\chi} \dot{\chi} - \bar{N}_T \bar{H} - \bar{N}_I \bar{H}^I) \\
\end{align*}

where:

\begin{align*}
\bar{H} &= R' \Pi_T + T' \Pi_R + \frac{1}{2} R^{-1} \bar{\pi}_\psi^2 + \frac{1}{2} R \bar{\psi}^2 + \frac{1}{4} R^{-1} \bar{\pi}_\chi^2 + R \chi^2 \\
\bar{H}^I &= T' \Pi_T + R' \Pi_R + \psi' \bar{\pi}_\psi + \chi' \bar{\pi}_\chi
\end{align*}
Solving the constraint equations $\mathcal{H} = 0$ and $\mathcal{H}^1 = 0$ for $\Pi_T$ and $\Pi_R$ and imposing the coordinate conditions $T = t$ and $R = r$ we obtain finally:

$$S = 2\pi \int_{-\infty}^{+\infty} dT \int_{0}^{+\infty} dR [\pi_T \psi, T + \pi_R \chi, T - \frac{1}{2}(R^{-1} \pi_T^2 + R \psi, R + R \pi_R^2 + R^{-1} \chi^2)]$$

from the next sequence of program lines:

```plaintext
>R(x,t):=x; T(x,t):=t; grdisplay(aha0);
>solve(grcomponent(aha0),piT(x,t));
> piT(x,t):=-1/2*(x^2*diff(psi(x,t),x)^2+pip(x,t)^2)/x;
> eval(piR(x,t));
> piR(x,t):-diff(psi(x,t),x)*pip(x,t); piR(x,t);
> grdisplay(aha0); grdisplay(aha(up));
> piT(x,t);

$$\frac{2}{dx} \frac{2}{x} \frac{2}{x} \int psi(x, t) + pip(x, t)$$

$$- \frac{1}{2} \longleftarrow psi(x, t) \longleftarrow pip(x, t)$$

$$\frac{2}{dx} \frac{2}{x} \frac{2}{x}$$

$\piR(x,t)$;

$$\frac{-1}{dx} \frac{-1}{dx} psi(x, t) pip(x, t)$$

$\frac{-1}{dx} \frac{-1}{dx}$

> grcalc(act); grdisplay(act);

For the Cyl_dim spacetime:

$$act$$

$$\frac{-1}{dt} \frac{-1}{dt} psi(x, t) pip(x, t)$$

$$\frac{-1}{dt} \frac{-1}{dt}$$

> grdef('Action:=act+piT(x,t)*diff(T(x,t),t)+piR(x,t)*diff(R(x,t),t)');
> grcalc(Action);gralter(Action,factor,normal,sort,expand);
```

10
> grdisplay(Action);

For the Cyl_din spacetime:

\[
\text{Action} = - \frac{1}{2} x \left| \frac{\partial^2}{\partial x^2} \psi(x, t) \right| + \frac{1}{2} \left| \frac{\partial}{\partial t} \psi(x, t) \right| \pi_p(x, t)
\]

\[
\frac{2}{\pi_p(x, t)} - \frac{1}{2} \frac{1}{x}
\]

> grdef('Ham:=\pi_T(x,t)*diff(T(x,t),t)+\pi_R(x,t)*diff(R(x,t),t)');
> grcalc(Ham); galter(Ham,expand);
> grdisplay(Ham);

For the Cyl_din spacetime:

\[
\text{Ham} = - \frac{1}{2} x \left| \frac{\partial^2}{\partial x^2} \psi(x, t) \right| - \frac{1}{2} \frac{1}{x}
\]

4 Conclusions. Further improvements

We used the programmes presented above in the computer room with the students from the graduate course on Numerical Relativity. The main purpose was to introduce faster the elements of the canonical version of relativity with the declared objective to skip the long and not very straightforward hand calculations necessary to process an entire example of spacetime model. We encouraged the students to try to modify the procedures in order to compute new examples.

The major conclusion is that this method is indeed useful for an attractive and fast teaching of the methods involved in the ADM for-
malism. On the other hand we can use and modify these programs for obtaining the equations necessary for the numerical relativity. In fact we intend to expand our Maple worksheets for the case of axisymmetric model (used in the numerical treatment of the head-on collision of black-holes). Of course, for numerical solving of the dynamic equations obtained here we need more improvements of the codes for parallel computing and more sophisticated numerical methods. But this will be the object of another series of articles.

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