Determination of torsional resonant frequencies of a tree

V Khegai*, V Savich, A Mihitarov
Department of Mechanics, Ukhta State Technical University, 13 Pervomayskaya Street, Komi Republic, 169300 Russian Federation

*Corresponding email: maxfks@yandex.ru

Abstract. This paper discusses the application of vibration, in particular torsional vibrations in the creation of machines for the uprooting of entire trees with the root system. However, the use of vibration can be effective only if the correct frequency response is selected depending on the frequency characteristics of the tree and the mechanism. On the basis of the created mathematical model, the spectrum of resonance frequencies of torsional vibrations of the tree is determined. The obtained results make it possible to select the frequency characteristics of the vibration device depending on the parameters of the tree.

1. Introduction
One of the important dynamic parameters of the system "machine-tree" is the vibration characteristics of the tree. A tree, being an elastic link with a distributed mass, has a very significant effect on the dynamics of the system [1-3].

Usually, in the study of the dynamics of the machine-tree system, the latter is considered as an object with discrete masses (or mass) interconnected by spring elements equivalent to the elastic characteristics of a tree trunk. Such a model only approximately reflects the dynamic characteristics of the tree.

2. Experimental part
Consider the task of determining the resonant frequencies of torsional vibrations of an uncut tree. Such a problem is relevant when uprooting whole trees with the root system using vibration, in particular, torsional vibrations. The task of determining the longitudinal resonant frequencies of tree oscillations are considered in [3, 4].

2.1. Methods
In the dynamic modeling of a system, we regard a tree as a flexible composite rod consisting of three heterogeneous sections, the lower end of which is rigidly embedded (Figure 1).

Each segment is characterized by the length $l_i$, the polar inertia moment of the cross section $J_{ip}$, the shear modulus $G_i$ and the propagation velocity of torsional disturbances $\lambda_i$ ($i = 1, 2, 3$). For each of the sections of the composite rod, we introduce our own coordinate axis $O_iS_i$ where $S_i\pi(0,l_i)$.

The angular displacement on the $i$-th segment is denoted by $\psi_i(s_i,t)$. In the calculations, the dissipation coefficient $\mu_i$ is assumed to be the same in all areas, that is, $\mu_1 = \mu_2 = \mu_3 = \mu$. 

2.2. Formula conclusions

The initial problem for torsional vibrations of a composite (inhomogeneous) rod will be written as [5, 6].

\[
\frac{\partial^2 \phi_i}{\partial t^2} + \mu \frac{\partial \phi_i}{\partial t} = \lambda_i^2 \frac{\partial^2 \phi_i}{\partial s_i^2}, \quad i = 1, 2, 3
\]

(1)

Border conditions:

\[
s_i = 0; \quad \phi_i = 0.
\]

(1.1)

\[
s_i = l_i, \quad \phi_1 = \phi_2, \quad \frac{\partial \phi_1}{\partial s_1} = \frac{\partial \phi_2}{\partial s_2} = \theta_1 = \frac{G_i l_{2y}}{G_i l_{1y}}.
\]

(1.2)

\[
s_2 = l_2, \quad \phi_2 = \phi_3, \quad \frac{\partial \phi_2}{\partial s_2} = \frac{\partial \phi_3}{\partial s_3} = \theta_2 = \frac{G_i l_{2y}}{G_i l_{2y}}.
\]

(1.3)

\[
s_3 = l_3: \quad \frac{\partial \phi_3}{\partial s_3} = 0.
\]

(1.4)

Bearing in mind that dissipative forces have little effect on the eigenfrequencies of the system [5-10] to simplify research, which, however, practically does not affect the final results, we accept the dissipation coefficients \( \mu = 0 \). Then the problem (1) takes the form:

\[
\frac{\partial^2 \phi_i}{\partial t^2} = \lambda_i^2 \frac{\partial^2 \phi_i}{\partial s_i^2}, \quad i = 1, 2, 3
\]

(2)

The boundary conditions of the problem (2) will remain unchanged, as in problem (1). In equations (2) make the change of variables

\[
\tau_i = \frac{\lambda_i l_i}{\delta_i}, \quad \delta_i = \frac{s_i}{l_i}, \quad i = 1, 2, 3.
\]

(3)

In the new variables, problem (2) takes the form

\[
\frac{\partial^2 \phi_i}{\partial \tau_i^2} = \lambda_i^2 \frac{\partial^2 \phi_i}{\partial s_i^2}, \quad i = 1, 2, 3.
\]

(4)

Border conditions:
\[
\xi_1 = 0: \varphi_1 = 0. \quad (4.1)
\]
\[
\xi_1 = 0, \xi_2 = 0: \varphi_1 = \varphi_2, \quad \frac{\partial \varphi_1}{\partial \xi_1} = \theta_1^* \frac{\partial \varphi_2}{\partial \xi_2}, \quad \theta_1^* = \frac{l_1}{l_2} \theta_2. \quad (4.2)
\]
\[
\xi_2 = 1, \xi_3 = 0: \varphi_2 = \varphi_3, \quad \frac{\partial \varphi_2}{\partial \xi_2} = \theta_2^* \frac{\partial \varphi_3}{\partial \xi_3}, \quad \theta_2^* = \frac{l_2}{l_3} \theta_3. \quad (4.3)
\]
\[
\xi_2 = 1, \xi_3 = 0: \varphi_3 = \varphi_3, \xi_3 = 1: \frac{\partial \varphi_3}{\partial \xi_3} = 0. \quad (4.4)
\]

The boundary condition 1 indicates the absence of angular displacement at the beginning of the first section, 2 and 3 express the equality of angular displacements and torques at the joints of the sections, and 4 indicates the zero torque at the end of the third section, i.e., when \( s_3 = l_3 \) or \( \xi_3 = 1 \).

The solution to problem (4) will be searched in the form:
\[
\varphi_i = A_i(\xi_i)e^{j\omega \xi_i}, \quad i = 1, 2, 3, \quad (5)
\]
where, \( j = \sqrt{-1} \)– the imaginary unit, \( \omega \) is a dimensionless constant parameter.

After substituting (5) into (4), we obtain ordinary differential equations:
\[
A_i'(\xi_i) + \omega^2 A_i(\xi_i) = 0, \quad i = 1, 2, 3. \quad (6)
\]

Border conditions:
\[
\xi_1 = 0: A_1(0) = 0. \quad (6.1)
\]
\[
\xi_1 = 1, \xi_2 = 0: A_1(1)e^{j\omega \xi_1} = A_2(0)e^{j\omega \xi_2}, \quad \left\{ \begin{array}{l}
A_1'(1)e^{j\omega \xi_1} = \theta_1^* A_2'(0)e^{j\omega \xi_2}, \\
\end{array} \right. \quad (6.2)
\]
\[
\xi_2 = 1, \xi_3 = 0: A_2(1)e^{j\omega \xi_2} = A_3(0)e^{j\omega \xi_3}, \quad \left\{ \begin{array}{l}
A_2'(1)e^{j\omega \xi_2} = \theta_2^* A_3'(0)e^{j\omega \xi_3}, \\
\end{array} \right. \quad (6.3)
\]
\[
\xi_3 = 1: A_3'(1) = 0. \quad (6.4)
\]

Boundary sections vary equally. Therefore, we have:
\[
\omega_1 \tau_1 = \omega_2 \tau_2 = \omega_3 \tau_3. \quad (7)
\]

Substituting the values of \( \tau_i \), here, according to (3), then reducing by the time \( t \), we get the following chain of equalities:
\[
\frac{\omega_1 \lambda_1}{l_1} = \frac{\omega_2 \lambda_2}{l_2} = \frac{\omega_3 \lambda_3}{l_3} = \Omega, \quad (8)
\]
where, \( \Omega \) is the circular frequency. Taking into account equalities (7), we rewrite the boundary conditions 2 and 3 of problem (6) in the form:
\[
A_1(0) = A_2(0), \quad A_1'(1) = \theta_1^* A_2'(0), \quad (9)
\]
at the junction of the second and third sections:

$$A_2'(0) = \theta_2' A_1(0)$$  \hspace{1cm} (10)

The solution to equation (6) is written as:

$$A_i(\xi_i) = C_{1,i} \sin \omega_i \xi_i + C_{2,i} \cos \omega_i \xi_i,$$  \hspace{1cm} (11)

where, \( C_{1,i} \) and \( C_{2,i} \) are arbitrary integration constants on the \( i \)-th segment of the rod.

Differentiate (11):

$$A'_i(\xi_i) = \omega_i (C_{1,i} \cos \omega_i \xi_i - C_{2,i} \sin \omega_i \xi_i) = 0.$$  \hspace{1cm} (12)

Bearing in mind equalities (11) and (12), relations (9) and (10) are represented as follows:

$$\begin{align*}
C_{1,1} \sin \omega_1 + C_{2,1} \cos \omega_1 &= C_{1,2}, \\
C_{1,1} \cos \omega_1 - C_{2,1} \sin \omega_1 &= \eta_1 C_{1,2}, \quad \eta_1 = \frac{\omega_1 \theta_1'}{\omega_1},
\end{align*}$$  \hspace{1cm} (13)

$$\begin{align*}
C_{1,2} \sin \omega_2 + C_{2,2} \cos \omega_2 &= C_{1,3}, \\
C_{1,2} \cos \omega_2 - C_{2,2} \sin \omega_2 &= \eta_2 C_{1,3}, \quad \eta_2 = \frac{\omega_2 \theta_2'}{\omega_2}.
\end{align*}$$  \hspace{1cm} (14)

We find from relations (13) \( C_{1,1}, C_{2,1} \) and consider their relationship:

$$\frac{C_{1,1}}{C_{2,1}} \sin \omega_1 + \eta_1 C_{1,2} \cos \omega_1 = \frac{C_{1,2}}{C_{2,1}} \cos \omega_1 + \eta_2 C_{1,2} \sin \omega_1$$  \hspace{1cm} (14.1)

We transform this expression into the following form:

$$\frac{C_{1,1}}{C_{2,1}} = \frac{\tan \omega_1 + \frac{c_{1,2}}{c_{2,2}} \eta_1}{1 - \frac{c_{1,2}}{c_{2,2}} \eta_1 \tan \omega_1}.$$  \hspace{1cm} (14.2)

It is easy to see that the recorded relationships can be represented in the form:

$$\frac{C_{1,1}}{C_{2,1}} = \tan (\omega_1 + \psi_1) = \infty,$$

$$\psi_1 = \arctan (\eta_1 \frac{c_{1,2}}{c_{2,2}}).$$  \hspace{1cm} (15)

Similarly, from relations (14) we find \( C_{1,2}, C_{2,2} \), and then their ratio:

$$\frac{C_{1,2}}{C_{2,2}} = \tan (\omega_2 + \psi_2)$$

$$\psi_2 = \arctan (\eta_2 \frac{c_{1,2}}{c_{2,2}}).$$  \hspace{1cm} (16)
Thus, relations (15) and (16) are obtained, expressing the ratio of the integration constants at the first junction of non-uniform sections of the rod through the ratio of similar constants at the second junction.

We use the relations (15) and (16) to find the resonant frequencies. For this purpose, we will move from the end of the third segment \( \xi_3 = 1 \).

According to the boundary condition 4 of problem (6) with \( \xi_3 = 1, \, A'_1(1) = 0 \). In this case, equation (12) gives:

\[
\frac{C_{1,3}}{C_{2,3}} = \tan \omega_h \tag{17}
\]

Then, successively substituting the relations of constant values \( \frac{C_i}{C_{2,i}} \) from the top of the third segment \( \xi_3 = 1 \) into relations (15) and (16), we will arrive at the beginning of the first \( \xi = 0 \):

\[
\begin{align*}
\xi_3 &= 1: \frac{C_{1,3}}{C_{2,3}} = \tan \omega_h, \\
\xi_2 &= 0, \, \xi_1 = 1: \frac{C_{1,2}}{C_{2,2}} = \tan (\omega_2 + \psi_2), \quad \psi_2 = \arctan \left( \frac{C_{1,3}}{C_{2,3}} \right), \\
\xi_1 &= 0, \, \xi_0 = 1: \frac{C_{1,1}}{C_{2,1}} = \tan (\omega_1 + \psi_1), \quad \psi_1 = \arctan \left( \frac{C_{1,2}}{C_{2,2}} \right), \\
\xi_0 &= 0: \frac{C_{1,1}}{C_{2,1}} = \infty.
\end{align*}
\tag{18}
\]

The last equality of system (18) is obtained from the first initial condition of the problem (6), \( A_1(0) = 0 \). Bearing in mind this equality from the third relation (18) we find:

\[
\tan (\omega_1 + \psi_1) = \infty \tag{19}
\]

Taking into account the equalities written above, we obtain the equation for determining the frequencies of torsional vibrations:

\[
\omega_1 + \psi_1 = \pi (m + 0.5), \quad m = 0, 1, 2, ..., \tag{20}
\]
\[ \psi_1 = \arctg(\eta_1 \tan(\omega_2 \psi_2)) \]  
(20.1)

\[ \psi_2 = \arctg(\eta_2 \omega_2) \]  
(20.2)

According to (8), the dimensionless frequency \( \omega_i \) is expressed in terms of the circular frequency \( \Omega \) as:

\[ \omega_i = \frac{l_i}{\lambda_i} \Omega, \quad i = 1, 2, 3. \]  
(21)

3. Results and Discussion

After substituting the values of \( \omega_1, \omega_2, \omega_3 \) and \( \eta_1, \eta_2 \) into equation (20) we finally have:

\[ \frac{l_i}{\lambda_i} \Omega + \arctg \left[ \frac{\lambda_1}{\lambda_2} \frac{G_2 I_{2p}}{G_1 I_{1p}} \tan \left[ \frac{l_2}{\lambda_2} \Omega + \arctg \left( \frac{\lambda_3}{\lambda_2} \frac{G_3 I_{2p}}{G_1 I_{1p}} \tan \frac{l_3}{\lambda_3} \Omega \right) \right] \right] = (m + 0.5)\pi, \quad i = 1, 2, 3 \]  
(22)

If all three dissimilar areas are made of the same material, then \( G_1 = G_2 = G_3 = G \), \( \lambda_1 = \lambda_2 = \lambda_3 = \lambda \). In this case, formula (22) will be rewritten in the form:

\[ \frac{l_i}{\lambda_i} \Omega + \arctg \left[ \frac{l_2}{l_1} \Omega \right] \arctg \left[ \frac{l_3}{l_2} \Omega \right] = (m + 0.5)\pi, \quad i = 1, 2, 3 \]  
(23)

4. Conclusion

Formulas (22) and (23) allow us to find the spectrum of the resonant frequencies of torsional vibrations of a composite rod consisting of three sections.

To determine the fundamental frequency of torsional oscillations, it is necessary in equations (22) or (23) to equate \( m = 0 \). In this case, we have:

\[ \frac{l_i}{\lambda_i} \Omega_1 + \arctg \left[ \frac{\lambda_1}{\lambda_2} \frac{G_2 I_{2p}}{G_1 I_{1p}} \tan \left[ \frac{l_2}{\lambda_2} \Omega_1 + \arctg \left( \frac{\lambda_3}{\lambda_2} \frac{G_3 I_{2p}}{G_1 I_{1p}} \tan \frac{l_3}{\lambda_3} \Omega_1 \right) \right] \right] = \frac{\pi}{2}, \]  
(24)

\[ \frac{l_i}{\lambda_i} \cdot \Omega_1 + \arctg \left( \frac{l_2}{l_1} \tan \left[ \frac{l_3}{l_2} \cdot \Omega_1 + \arctg \left( \frac{l_3}{l_2} \cdot \Omega_1 \right) \right] \right) = \frac{\pi}{2} \]  
(25)

5. References

[1] Gastev B G and Melnikov V I 1967 Basics of the dynamics of forest-carrying rolling stock (Moscow: Forestry) p 220
[2] Panovko G Ya 2008 Lectures on the basics of vibration machines and technologies (Moscow: Moscow State Technical University named after Bauman N E Press) p 192
[3] Khegai V K and Savich V L 2011 Determination of the resonant frequencies of the longitudinal oscillations of a tree News of the St. Petersburg Forestry Academy Vol. 194 (St. Petersburg: St. Petersburg Forestry Academy Press) p 59–67
[4] Zima I M 1964 Mechanization of agricultural work 2nd ed (Moscow: Forest industry) p 550
[5] Yunin E K and Khegai V K 2004 Dynamics of deep drilling (Moscow: Nedra) p 288
[6] Panovko Ya G 1971 Introduction to the theory of mechanical oscillations (Moscow: Science) p
239

[7] Biderman V L 1972 *Applied Theory of Mechanical Oscillations* (Moscow: Higher. School) p 416

[8] Babakov I M 2004 *Theory of oscillations: studies. manual for university students enrolled in tech. directions and specialties* 4th ed (Moscow: Drofa Tula printing house) p 592

[9] Svetlitsky V A 1973 *Collection of problems on the theory of oscillations* (Moscow: Higher. School) p 454

[10] Yablonsky A A 1975 *Course of the theory of oscillations* 3rd ed (Moscow: Higher. School) p 248