SINGULAR PERIODIC SOLUTIONS FOR THE \textit{p}-LAPLACIAN IN A PUNCTURED DOMAIN

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Abstract. In this paper we are interested in studying singular periodic solutions for the \textit{p}-Laplacian in a punctured domain. We find an interesting phenomenon that there exists a critical exponent \(p_c = N\) and a singular exponent \(q_s = p - 1\). Precisely speaking, only if \(p > p_c\) can singular periodic solutions exist; while if \(1 < p \leq p_c\) then all of the solutions have no singularity. By the singular exponent \(q_s = p - 1\), we mean that in the case when \(q = q_s\), completely different from the remaining case \(q \neq q_s\), the problem may or may not have solutions depending on the coefficients of the equation.

1. Introduction. Let \(\Omega\) be a bounded domain of \(\mathbb{R}^N\) containing the origin with smooth boundary. We are concerned with the existence of positive periodic solutions of the following evolutionary \(p\)-Laplacian in the punctured domain

\[
\frac{\partial u}{\partial t} - \text{div}(|\nabla u|^{p-2}\nabla u) = m(x,t)u^q, \quad x \in \Omega', \quad t \in \mathbb{R},
\]

with the homogeneous Dirichlet boundary condition

\[
u(x,t) = 0, \quad x \in \partial\Omega, \quad t \in \mathbb{R},
\]

and the periodicity condition

\[
u(x,t) = \nu(x,t + \omega), \quad x \in \Omega, \quad t \in \mathbb{R},
\]

where \(\Omega' = \Omega \setminus \{0\}\), and \(p > 1\) and \(q \geq 0\) are constants. At the singular boundary point \(x = 0\), an additional boundary condition, namely

\[
u(0,t) = M(t), \quad t \in \mathbb{R},
\]

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may be required depending on the value of \( p \). Here, \( m(x, t) \) and \( M(t) \) are appropriately smooth and positive functions which are \( \omega \)-periodic in time with \( \omega > 0 \).

The problem about singular solutions is an important issue in the study of PDEs, which comes from many practical problems arising in physics, geometry, etc. Interest in the study of the removability of isolated singularities of differential equations has been aroused from various applications in mathematical physics in recent years, see [2, 8, 9, 14, 15, 17] and the references therein. To the best of our knowledge, it was Serrin [14, 15] who first considered the removable singularities of solutions of second-order partial differential equations having the form

\[
\text{div} \vec{A}(x, u, \nabla u) = B(x, u, \nabla u)
\]

in a punctured domain \( D \setminus \{0\} \), where the domain \( D \) is a connected open set in \( \mathbb{R}^N \). Serrin’s model contains the \( p \)-Laplacian as a typical form. For the case \( 1 < p \leq N \), Serrin showed that \( x = 0 \) is a removable singular point; while for the case \( p > N \), he pointed out that even though the solution is bounded, \( x = 0 \) can also be an unremovable singular point, which is completely different from the case \( 1 < p \leq N \). In other words, it is natural to consider the problem in a punctured domain in some cases. In [2], Brezis and Véron considered the equation

\[
-\Delta u + |u|^{q-1}u = 0, \quad x \in D \setminus \{0\},
\]  

with \( N > 2 \), and proved that for \( q \geq \frac{N}{N-2} \) any isolated singularity of (5) is removable, which is not true when \( 1 < q < \frac{N}{N-2} \). Later, Vázquez and Véron [17] extended the results of [2] by considering two types of typical differential operators, the \( p \)-Laplacian and the mean curvature operator, namely

\[
-\text{div} \left( \vec{A}(\nabla u) \right) + g(\cdot, u) = 0,
\]

with \( g \) satisfying some assumption of power-like growth. Recently, Liskevich and Skrypnik [8, 9] established the best possible conditions for isolated singularities to be removable for general quasilinear equations, supposing that the lower-order terms satisfy certain nonlinear Kato-type conditions and extending the results of Brezis and Véron. However, as far as we know, there are no results on periodic solutions of parabolic equations with singularity at the isolated singular point \( x = 0 \).

The comparison of time-periodic Dirichlet boundary value problem in punctured domain with corresponding problems in regular domains is meaningful. We list the following related problems, say, problem in regular domain \( \Omega \):

\[
\begin{align*}
\frac{\partial u}{\partial t} - \text{div}[|\nabla u|^{p-2}\nabla u] &= m(x, t)u^q, \quad x \in \Omega, \ t \in \mathbb{R}, \\
u(x, t) &= u(0, t + \omega), \quad x \in \Omega, \ t \in \mathbb{R},
\end{align*}
\]

\((P)\)

problem in punctured domain \( \Omega' \) with boundary condition at \( \{0\} \):

\[
\begin{align*}
\frac{\partial u}{\partial t} - \text{div}[|\nabla u|^{p-2}\nabla u] &= m(x, t)u^q, \quad x \in \Omega', \ t \in \mathbb{R}, \\
u(x, t) &= u(0, t + \omega), \quad x \in \Omega', \ t \in \mathbb{R},
\end{align*}
\]

\((P')\)
problem in punctured domain $\Omega'$ without boundary condition at $\{0\}$:  
\[ \begin{cases} 
\frac{\partial u}{\partial t} - \text{div}(|\nabla u|^{p-2}\nabla u) = m(x,t)u^q, & x \in \Omega', \ t \in \mathbb{R}, \\
u(x,t) = 0, & x \in \partial\Omega', \ t \in \mathbb{R}, \\
u(x,t) = u(x,t + \omega), & x \in \Omega, \ t \in \mathbb{R}, 
\end{cases} \quad (P'') \]
and problem in regular domain $\Omega_\varepsilon = \Omega \setminus \overline{B}_\varepsilon(0)$:
\[ \begin{cases} 
\frac{\partial u}{\partial t} - \text{div}(|\nabla u|^{p-2}\nabla u) = m(x,t)u^q, & x \in \Omega_\varepsilon, \ t \in \mathbb{R}, \\
u(x,t) = 0, & x \in \partial\Omega, \ t \in \mathbb{R}, \\
u(x,t) = u(x,t + \omega), & x \in \Omega, \ t \in \mathbb{R}, \\
u(x,t) = M(t), & x \in \partial B_\varepsilon, \ t \in \mathbb{R}. 
\end{cases} \quad (P_\varepsilon) \]

Detailed definitions of solutions for the above problems are listed in Section 2. For the sake of convenience, the sets of positive solutions are denoted by $U, U', U''$, with positive solutions being denoted by $u, u', u''$, respectively. Clearly, $U \subset U''$, $U' \subset U''$ for any $M(t)$, and $u \in U'$ for some $M(t)$. The limit as $\varepsilon \to 0$ of problems $P_\varepsilon$ varies with the exponent $p$. If $1 < p \leq N$, we will show that $U = U''$. Known results show that $U$ is empty for $1 < p < N$, $q \geq Np/(N - p) - 1$ and uniformly convex $\Omega$, while $U_\varepsilon$ is nonempty for any $\varepsilon > 0$ and $q \neq p - 1$. Thus, generally speaking, the solutions $u_\varepsilon \subset U_\varepsilon$ do not possess uniform estimates with respect to $\varepsilon$. If $p > N$ and $q \neq p - 1$, using the Sobolev-Poincaré-type inequality and the barrier function technique, we will prove that $U'$ is nonempty and $u_\varepsilon$ converges to $u'$ for any $M(t)$. Therefore, $u' \in U$ for some $M(t)$ and $u_\varepsilon$ converges to $u$ for those $M(t)$. Meanwhile, there exists infinitely many singular periodic solutions $u' \notin U$.

Our interest here lies in seeking those periodic solutions with singularity at the isolated singular point $x = 0$. For this purpose, it is worth mentioning that the exponent $p$ has a critical value $p_c = N$ in the sense that only if $p > N$ can those solutions exist. In other words, if $1 < p \leq N$, then the problem $(P'')$ is always well-posed with solutions having no singularity, see Theorem 2.5 below. So, throughout the subsequent discussion for singular periodic solutions, we shall always assume that $p > N$. Another interesting phenomenon is revealed for the other exponent $q$. Precisely speaking, the exponent $q$ has a singular value $q_\ast = p - 1$, since in this case, completely different from the remaining case $q \neq q_\ast$, the problem may or may not have solutions depending on the coefficients of the equation.

The rest part of this paper is organized as follows. In Section 2, we present some preliminaries and our main results. In Section 3, we prove the existence and nonexistence for the singular exponent $q = p - 1$. The existence for $q \neq p - 1$ is demonstrated in Section 4. In the last section, we show the non-singularity for $1 < p \leq N$.

2. Preliminaries and the main results. We fix some terminologies and notations which will be frequently used in this paper. Let $\tau \in \mathbb{R}$ be fixed and set  
\[ Q_\omega = \Omega \times (\tau, \tau + \omega), \quad \overline{m} = \sup_{(x,t) \in Q_\omega} m(x,t), \quad \underline{m} = \inf_{(x,t) \in Q_\omega} m(x,t). \]
Assume that $p > 1$, $q \geq 0$, $0 < \underline{m} \leq m(x,t) \leq \overline{m} < \infty$, and $0 < M(t) \in L^\infty(\tau, \tau + \omega)$. Denote by $E, E_0$ and $E_M^q$ the following reasonable solution spaces,
\[ E = \{u \in L^{q+1}(Q_\omega); \ u_t \in L^2(Q_\omega), \nabla u \in L^p(Q_\omega)\}, \]
\[ E^0 = \{ u \in E; u|_{\partial\Omega \times (\tau, \tau+\omega)} = 0 \}, \]
\[ E^0_M = \{ u \in E^0 \cap C(\bar{Q}_\omega); u(0,t) = M(t) \}. \]

We shall show that only in the case \( p > N \) can we impose the condition (4) for the periodic problem in the punctured domain. We first give the definitions of weak solutions for \( 1 < p \leq N \) and \( p > N \) respectively in the following distributional sense

\[ \int_{Q_\omega} \frac{\partial u}{\partial t} \varphi dx dt + \int_{Q_\omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx dt = \int_{Q_\omega} m(x,t) u^q \varphi dx dt, \quad (6) \]

where \( \varphi \) is a test function which will be specified later in different cases.

**Definition 2.1.** For \( 1 < p \leq N \), a function \( u \in E^0 \) is called a weak \( \omega \)-periodic solution (in \( \Omega' \times (\tau, \tau+\omega) \)) of the problem (1), (2)–(3) (problem (\( P' \))) if \( u \) is \( \omega \)-periodic in time and (6) holds for any function \( \varphi \in C^1_0(\Omega' \times (\tau, \tau+\omega)) \).

The condition \( \nabla u \in L^p(Q_\omega) \) might be stronger than \( \nabla u \in L^p(\Omega' \times (0,\omega)) \), but this definition still makes sense (at least for radially symmetric domains and solutions).

**Definition 2.2.** For \( p > N \), a function \( u \in E^0_M \) is called a weak \( \omega \)-periodic solution (in \( \Omega' \times (\tau, \tau+\omega) \)) of the problem (1), (2)–(4) (problem (\( P'' \))) if \( u \) is \( \omega \)-periodic in time and (6) holds for any function \( \varphi \in C^1_0(\Omega' \times (\tau, \tau+\omega)) \).

We also take an interest in the periodic problem in regular domains

\[ \frac{\partial u}{\partial t} - \text{div}(|\nabla u|^{p-2} \nabla u) = m(x,t) u^q, \quad x \in \Omega, \quad t \in \mathbb{R}, \quad (7) \]

Weak periodic solutions in a regular domain are defined as follows.

**Definition 2.3.** A function \( u \in E^0 \) is called a weak \( \omega \)-periodic solution (in \( \Omega \times (\tau, \tau+\omega) \)) of the problem (7), (2)–(3) (problem (\( P \))) if \( u \) is \( \omega \)-periodic in time and (6) holds for any function \( \varphi \in C^1_0(\Omega \times (\tau, \tau+\omega)) \).

By a singular periodic solution, we denote a solution of the periodic problem in a punctured domain (1), (2)–(3), with or without the condition (4), but not necessarily a solution of the periodic problem in a regular domain (7), (2)–(3). We state our main results here.

**Theorem 2.4.** For \( p > N \), there exists a singular exponent \( q_s = p - 1 \) such that

(i) when \( q = p - 1 \), the existence of positive periodic solutions of the problem (\( P' \)) in a punctured domain depends on the coefficient \( m(x,t) \);
(ii) when \( q \neq p - 1 \), the problem in a punctured domain \((P')\) admits at least one positive periodic solution.

**Theorem 2.5.** For \( 1 < p \leq N \), there does not exist a singular periodic solution of the problem \((P'')\) in a punctured domain.

Since the periodic problem in regular domains has been intensively studied (we refer the reader to \([1, 3, 4, 5, 12, 13, 19, 20]\)), we shall only consider the supercritical case \( p > N \) in the following sections and leave the proof of Theorem 2.5 in the last section.

We will use the upper and lower solution method to show the existence of positive periodic solutions of the problem \((1), (2)-(3)\) with \((4)\) in the case \( p > N \) and \( 0 \leq q < p - 1 \). Here, we first give the following definition of upper and lower solutions.

**Definition 2.6.** For \( p > N \), a function \( u \in E \) is called a weak \( \omega \)-periodic upper solution of the problem \((1), (2)-(4)\) provided that for any nonnegative function \( \varphi \in C(Q_\omega) \cap C^1(\Omega' \times (\tau, \tau + \omega)) \) with \( \varphi(x, t) = 0 \) for \( x \in \partial \Omega \cup \{0\} \), it holds that

\[
\begin{align*}
\int_{Q_\omega} \frac{\partial u}{\partial t} \varphi dxdt + \int_{Q_\omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dxdt & \geq \int_{Q_\omega} m(x, t) u^q \varphi dxdt, \\
u(x, t) & \geq 0, \quad x \in \partial \Omega, \quad t \in (\tau, \tau + \omega), \\
u(0, t) & \geq M(t), \quad t \in (\tau, \tau + \omega), \\
u(x, \tau + \omega) & \geq u(x, \tau + \omega), \quad x \in \Omega.
\end{align*}
\]

Replacing “\( \geq \)” by “\( \leq \)” in the above inequalities, it follows the definition of a weak lower solution.

It is clear that for \( p > N \), if \( u \in E \) is a weak upper solution as well as a weak lower solution, then \( u \) belongs to \( E^0_M \) and \( u \) is a weak periodic solution of the problem \((1), (2)-(4)\) as defined in Definition 2.2.

The following lemma will be used later to show the existence of weak solutions.

**Lemma 2.7.** Assume that \( p > N, q \geq 0, 0 < M(t) \in C^1([\tau, \tau + \omega]) \) and \( 0 < m(x, t) \in C^1(\overline{\Omega} \times [\tau, \tau + \omega]) \). Let \( \overline{u}, u \) with \( \overline{u} \geq u \geq 0 \) be a pair of bounded upper and lower solutions of the problem \((1), (2)-(4)\). Then the problem \((1), (2)-(4)\) admits a bounded positive weak solution \( u \in E^0_M \) with \( \underline{u} \leq u \leq \overline{u} \).

**Proof.** Define a function sequence \( \{u_n\}_{n=0}^\infty \) by the following iteration scheme

\[
\begin{align*}
\frac{\partial u_n}{\partial t} &= \text{div}(|\nabla u_n|^{p-2} \nabla u_n) + m(x, t) u_n^{q-1}, \quad x \in \Omega', \quad t \in (\tau, \tau + \omega), \\
u_n(x, t) &= 0, \quad x \in \partial \Omega, \quad t \in (\tau, \tau + \omega), \\
u_n(0, t) &= M(t), \quad t \in (\tau, \tau + \omega), \\
u_n(x, \tau + \omega) &= u_{n-1}(x, \tau + \omega), \quad x \in \Omega,
\end{align*}
\]

where \( u_0 = \underline{u} \). The domain \( \Omega' \) is not regular, since there exists a singular boundary point \( 0 \). The existence and uniqueness of solutions for the above problem \((8)\) can be deduced by a domain regularization approach, similar to the proof of Lemma 4.4 below. For the sake of convenience we omit the proof here. Thus, \( u_n \) is well defined. Then we have

\[
\underline{u} = u_0 \leq u_1 \leq u_2 \leq \cdots \leq u_{n-1} \leq u_n \leq \cdots \leq \overline{u}.
\]
In fact, by the iteration scheme and the definition of upper and lower solutions, we have
\[
\begin{aligned}
\frac{\partial u_1}{\partial t} - \text{div}(|\nabla u_1|^{p-2}\nabla u_1) &= m(x, t)u_1^q - \text{div}(|\nabla u_0|^{p-2}\nabla u_0), \\
&= 0, \quad x \in \Omega, \quad t \in (\tau, \tau + \omega), \\
u_1(x, t) &= 0 \geq u_0(x, t), \quad x \in \partial \Omega, \quad t \in (\tau, \tau + \omega), \\
u_1(0, t) &= M(t) \geq u_0(0, t), \quad t \in (\tau, \tau + \omega), \\
u_1(x, \tau) &= u_0(x, \tau + \omega) \geq u_0(x, \tau), \quad x \in \Omega.
\end{aligned}
\]

The comparison principle shows that \(u_1 \geq u_0\). Other order relations can be verified similarly. By the proof of the unique solvability of the problem (8) using the domain regularization approach (see the proof of Lemma 4.4 or the estimates (23) below), we see that
\[
\sup_{t \in (\tau, \tau + \omega)} \int_{\Omega} |\nabla u_n|^p \, dx \leq C, \quad \sup_{t \in (\tau, \tau + \omega)} \|\frac{\partial u_n}{\partial t}\|_{L^2(Q_\omega)} \leq C,
\]
where \(C\) is a constant depending on \(\|M(t)\|_{C^1([\tau, \tau + \omega])}, \|\pi\|_{L^\infty(Q_\omega)}\) and \(\bar{\omega}\). By the monotonicity of \(u_n\) with respect to \(n\), there exists a function \(u\) such that \(u_n(x, t)\) tends to \(u\) a.e. in \(Q_\omega\), \(u(x, \tau) = u(x, \tau + \omega), \underline{u} \leq u \leq \bar{u}\), and for any \(r \geq 1\),
\[
u_n \to u \text{ in } L^r(Q_\omega), \quad \nabla u_n \to \nabla u \text{ in } L^p(Q_\omega), \quad \frac{\partial u_n}{\partial t} \to \frac{\partial u}{\partial t} \text{ in } L^2(Q_\omega),
\]
as \(n \to \infty\), which implies that \(u \in E^0_M\) with \(\underline{u} \leq u \leq \bar{u}\) is a bounded weak periodic solution of the problem (1), (2)–(4). By the boundary condition \(u(0, t) = M(t) \geq \min M(t) > 0\), we see that there exists \(\varepsilon > 0\) and \(\delta > 0\) such that \(u(x, t) \geq \varepsilon\) for \(x \in B_\delta\) and \(t \in (\tau, \tau + \omega)\). The strong maximum principle of \(p\)-Laplacian on the regular domain \(\Omega \setminus B_\delta \times (\tau, \tau + \omega)\) implies that \(u > 0\), since \(u \equiv 0\) is not a solution. Thus, \(u\) is positive on \(Q_\omega\).

3. Existence and nonexistence for \(q = p - 1\). In this section, we consider the existence and nonexistence of singular periodic solutions for the singular case \(q = p - 1\). We shall show that the existence of positive periodic solutions depends on the value of \(m(x, t)\).

The following Sobolev-Poincaré-type inequality will be used frequently in this paper.

**Lemma 3.1.** If \(1 < p < N\) and \(1 < q \leq \frac{Np}{N-p}\), or if \(p = N\) and \(q > 1\), it holds that
\[
\|u\|_{L^p(\Omega)} \leq C_0\|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W^{1,p}_0(\Omega);
\]
while for \(p > N\) and \(q > 1\), it holds that
\[
\|u\|_{C^\alpha(\Omega)} \leq C_0\|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W^{1,p}_0(\Omega),
\]
where \(\alpha = 1 - \frac{N}{p}\) and \(C_0\) is a constant independent of \(u\).

We note that for functions \(u \in W^{1,p}(\Omega_\varepsilon), \Omega_\varepsilon = \Omega \setminus B_\varepsilon\) with \(B_\varepsilon \subset B_{\varepsilon_0} \subset \Omega\) and \(u(x) = 0\) for \(x \in \partial \Omega\), the above inequalities are valid for a constant \(C_0\) independent of \(\varepsilon < \varepsilon_0\).

For the evolutionary \(p\)-Laplacian, we present the following comparison principle and strong maximum principle.
Lemma 3.2. Suppose that $u, v \in E$ and $u, v$ are $\omega$-periodic with respect to $t$, such that
$$u_t - \text{div}(|\nabla u|^{p-2}\nabla u) \geq v_t - \text{div}(|\nabla v|^{p-2}\nabla v), \quad (x, t) \in Q_\omega$$
in the sense of distributions, and $u(x, t) \geq v(x, t)$ for $(x, t) \in \partial \Omega \times (0, \omega)$. Then $u(x, t) \geq v(x, t)$ for $(x, t) \in Q_\omega$.

Proof. For any $0 \leq \varphi \in L^2(0, \omega; H^1_0(\Omega))$, there holds
$$\iint_{Q_\omega} (u_t - v_t) \varphi \, dx \, dt + \iint_{Q_\omega} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) \cdot \nabla \varphi \, dx \, dt \geq 0.$$Since $u(x, t) \geq v(x, t)$ for $(x, t) \in \partial \Omega \times (0, \omega)$, we can choose $\varphi = (v - u)_+$ as the test function and get
$$\frac{1}{2} \iint_{Q_\omega} \frac{\partial u^2}{\partial t} \, dx \, dt + \iint_{Q_\omega} (|\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u) \cdot \nabla (v - u)_+ \, dx \, dt \leq 0.$$By the periodicity, we see that
$$\iint_{Q_\omega} \frac{|\nabla \varphi|^{p+2-p_+}}{(|\nabla u| + |\nabla v|)^{2-p_+}} \, dx \, dt \leq 0.$$Thus, $\nabla \varphi = 0$ and $\varphi = 0$ a.e. $Q_\omega$. \hfill $\square$

Lemma 3.3. Suppose that $B_\delta \subset \Omega$ and $D = \Omega \setminus \overline{B_\delta}$. Let $u \in C(\overline{D} \times [0, \omega])$ with $\nabla u \in C(\overline{D} \times [0, \omega])$, $u_t \in L^2(D \times (0, \omega))$, such that $u$ is $\omega$-periodic with respect to $t$,
$$u_t - \text{div}(|\nabla u|^{p-2}\nabla u) \geq 0, \quad (x, t) \in D \times (0, \omega)$$in the sense of distributions, and $u(x, t) = 0$ for $(x, t) \in \partial \Omega \times (0, \omega)$, $u(x, t) > 0$ for $(x, t) \in (D \cup \partial B_\delta) \times [0, \omega]$. Then for any $x_0 \in \partial \Omega$ and $t_0 \in [0, \omega]$, the outer normal derivative
$$\frac{\partial u}{\partial \nu}(x_0, t_0) < 0.$$Proof. We follow the approach of the strong maximum principle for uniformly parabolic operator \cite{[11]}. Since $\partial \Omega$ is smooth, there exists a ball $B_R(y) \subset D$ with $x_0 \in \partial B_R(y)$. Let $K = B_{R/2}(x_0) \cap B_R(y)$ and $\Gamma_1 = (\partial B_{R/2}(x_0) \cap \overline{B_R(y)}$, $\Gamma_2 = B_{R/2}(x_0) \cap \partial B_R(y)$. Thus, $\partial K = \Gamma_1 \cup \Gamma_2$, $u(x, t) > 0$ for $(x, t) \in \Gamma_1 \times [0, \omega]$, and $u(x, t) \geq 0$ for $(x, t) \in \Gamma_2 \times (0, \omega)$. There exists a constant $\varepsilon > 0$ such that $u(x, t) \geq \varepsilon$ for $(x, t) \in \Gamma_1 \times [0, \omega]$.

Define the following auxiliary function for $\alpha > 0$
$$v(x, t) = \varepsilon(e^{-\alpha|x-y|^2} - e^{-\alpha R^2}).$$Direct calculation shows that
$$v_t - \text{div}(|\nabla v|^{p-2}\nabla v) = (2\varepsilon)\alpha^{-p-1}e^{-\alpha/2}|x-y|^{p-2}(|x-y|^{p-2}N + p - 2 - 2\alpha(p-1)|x-y|^2).$$Thus, $v_t - \text{div}(|\nabla v|^{p-2}\nabla v) \leq 0$ for $(x, t) \in K \times (0, \omega)$ provided that $2\alpha(p-1)(R/2)^2 \geq N + p - 2$. Clearly, $u(x, t) \geq v(x, t)$ for $(x, t) \in \Gamma_2 \times (0, \omega)$, and $u(x, t) \geq \varepsilon \geq v(x, t)$ for $(x, t) \in \Gamma_1 \times (0, \omega)$. Lemma 3.2 implies that $u(x, t) \geq v(x, t)$ for $(x, t) \in K \times (0, \omega)$. Therefor, the conclusion holds for $t_0 \in \partial \Omega$. Since the auxiliary function $v(x, t)$ is independent of $t$ and $u(x, t)$ is time periodic, it follows the conclusion for $t_0 \in [0, \omega]$.
Lemma 3.4. Suppose that \( B_{\delta} \subset \Omega \) and \( D = \Omega \setminus \overline{B_{\delta}} \). Let \( u, v \in C(\overline{D} \times [0, \omega]) \) with \( \nabla u, \nabla v \subset C(\overline{D} \times [0, \omega]) \), such that \( u(x, t) = v(x, t) = 0 \) for \( (x, t) \in \partial \Omega \times [0, \omega] \), \( u(x, t) > 0 \) for \( (x, t) \in (D \cup \partial B_{\delta}) \times [0, \omega] \), and \( \partial u/\partial v < 0 \) for \( (x, t) \in \partial \Omega \times [0, \omega] \). Then there exists a constant \( K > 0 \), such that \( v(x, t) \leq Ku(x, t) \) for \( (x, t) \in \overline{D} \times [0, \omega] \).

Proof. Since \( \partial u/\partial v < 0 \) for \( (x, t) \in \partial \Omega \times [0, \omega] \) and \( \nabla u, \nabla v \subset C(\overline{D} \times [0, \omega]) \), there exist constants \( \eta > 0, \epsilon > 0 \), and \( \kappa > 0 \), such that \( \partial u/\partial v \leq -\epsilon, |\partial v/\partial u| \leq \kappa \), for \( (x, t) \in D_{\eta} \times [0, \omega] \), \( D_{\eta} = \{x \in D; \text{dist}(x, \partial \Omega) < \eta\} \). For any \( (x, t) \in D_{\eta} \times [0, \omega] \), we have

\[
|v(x, t)| = \left| \frac{\partial v(x_1, t)}{\partial \nu} \right| \cdot \text{dist}(x, \partial \Omega) \leq \kappa \cdot \text{dist}(x, \partial \Omega),
\]

and

\[
|u(x, t)| = \left| \frac{\partial u(x_2, t)}{\partial \nu} \right| \cdot \text{dist}(x, \partial \Omega) \geq \epsilon \cdot \text{dist}(x, \partial \Omega).
\]

It follows that \( v(x, t) \leq Ku(x, t) \) for \( (x, t) \in D_{\eta} \times [0, \omega] \) with \( K = \kappa/\epsilon \). Thus, for sufficiently large \( K > 0 \), the above inequality holds for all \( (x, t) \in \overline{D} \times [0, \omega] \). \( \square \)

For \( N \geq 2 \), consider the following eigenvalue problems

\[
\begin{align*}
-\text{div}(\nabla \varphi |^{p-2} \nabla \varphi) &= \lambda |\varphi|^{p-2} \varphi, \quad x \in \Omega', \\
\varphi(x) |_{\partial \Omega \setminus \{0\}} &= 0.
\end{align*}
\]

and

\[
\begin{align*}
-\text{div}(\nabla \varphi |^{p-2} \nabla \varphi) &= \lambda |\varphi|^{p-2} \varphi, \quad x \in \Omega, \\
\varphi(x) |_{\partial \Omega} &= 0.
\end{align*}
\]

By the variational approach (see for example [7]), we see that the above problems admit their first eigenvalues \( \lambda_1(\Omega') > \lambda_1(\Omega) > 0 \). Let \( \varphi_1 > 0 \) and \( \psi_1 > 0 \) with \( \|\varphi_1\|_{L^\infty} = \|\psi_1\|_{L^\infty} = 1 \) be their first eigenfunctions corresponding to \( \lambda_1(\Omega') \) and \( \lambda_1(\Omega) \), respectively. For the case \( N = 1, \Omega' \) is not connected and we shall assume that \( \Omega = (-R, R) \) with \( R > 0 \).

Theorem 3.5. Assume that \( p > N, q = p - 1, 0 < M(t) \in C^1([\tau, \tau + \omega]), \) and \( 0 < m(x, t) \in C^1([\tau, \tau + \omega]). \) Then the following conclusions hold:

(i) If \( N \geq 2 \) and \( m(x, t) \geq M > \lambda_1(\Omega') \), where \( \lambda_1(\Omega') > 0 \) is the first eigenvalue of the eigenvalue problem (9), then the problem (1), (2)–(4) admits no positive periodic solution;

(ii) If \( m(x, t) \leq \lambda_1(\Omega) \), where \( \lambda_1(\Omega) \) is the first eigenvalue of the eigenvalue problem (10), then there is a positive periodic solution to the problem (1), (2)–(4);

(iii) If \( N = 1, \Omega = (-R, R) \) with \( R > 0 \), \( M(t) \equiv M > 0 \) and \( m(x, t) \equiv m > 0 \), then the problem (1), (2)–(4) admits a positive periodic solution if and only if \( m < \lambda_1(\Omega') \), where \( \lambda_1(\Omega') = \lambda_1((0, R)) = \frac{\pi^2}{R^2} \).

Proof. First, we prove the nonexistence (i). Suppose that for \( N \geq 2 \) and \( m(x, t) \geq M > \lambda_1(\Omega') \), the problem (1), (2)–(4) admits a positive periodic solution \( u(x, t) \).

By the boundary condition \( u(0, t) = M(t) \), there exists \( \varepsilon > 0 \) and \( \delta > 0 \), such that \( u(x, t) \geq \varepsilon \) for \( (x, t) \in B_\delta \times (\tau, \tau + \omega) \). Applying Lemma 3.3 and Lemma 3.4, we see that there exists a constant \( \kappa > 0 \) such that \( u(x, t) \geq \kappa \varphi_1(x) \) for \( (x, t) \in (\Omega \setminus B_\delta) \times (\tau, \tau + \omega) \), where \( \varphi_1 \geq 0 \) with \( \|\varphi_1\|_{L^\infty} = 1 \) is the first eigenfunction of the problem (9). We can take \( \kappa > 0 \) sufficiently small so that \( u(x, t) \geq \kappa \varphi_1(x) \)
for \((x,t) \in Q_\omega\). Let \(\tilde{m}\) be a constant such that \(\underline{m} > \tilde{m} > \lambda_1(\Omega')\). Consider the following periodic problem

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \text{div}\left( |\nabla u|^{p-2} \nabla u \right) + \tilde{m} u^{p-1}, & \quad & (x,t) \in \Omega' \times \mathbb{R}, \\
u(x,t) &= 0, & \quad & x \in \partial\Omega, \ t \in \mathbb{R}, \\
u(0,t) &= 0, & \quad & t \in \mathbb{R}, \\
u(x,t) &= u(x,t + \omega), & \quad & x \in \Omega, \ t \in \mathbb{R}.
\end{aligned}
\]

(11)

We see that \(u(x,t)\) and \(\underline{u} = \kappa \varphi_1\) are a pair of upper and lower solutions of the periodic problem (11). Using the monotone method that is similar to the proof of Lemma 2.7, we find that the problem (11) admits a bounded periodic solution \(\hat{u}\) with \(0 < \kappa \varphi_1 \leq \hat{u} \leq u\). Since \(\tilde{m}\) is a constant, we assert that \(\hat{u}\) is a steady state. In fact, multiplying the first equation of (11) by \(u_t\) and integrating over \(Q_\omega\), we have \(\int_{Q_\omega} u_t^2 \, dx \, dt = 0\) by the periodicity. Thus, \(\hat{u}\) is independent of time. Therefore, \(\hat{u} > 0\) is an eigenfunction corresponding to \(\tilde{m} > \lambda_1(\Omega')\) of the elliptic eigenvalue problem (9). On the other hand, the first eigenvalue \(\lambda_1(\Omega')\) is the unique eigenvalue that corresponds to positive eigenfunctions for the eigenvalue problem (9). We arrive at a contradiction since we find another eigenvalue \(\tilde{m} > \lambda_1(\Omega')\) that corresponds to the positive eigenfunction \(\hat{u}\).

Next, we prove the existence (ii). Since \(m(x,t) \leq \lambda_1(\Omega)\), we see that \(\pi(x,t) = \kappa \varphi_1(x)\) and \(\underline{u} \equiv 0\) are a pair of upper and lower solutions to the problem (1), (2)–(4), provided \(\kappa \geq \|M(t)\|_{L^\infty}(\Omega)\), where \(\varphi_1 > 0\) with \(\|\varphi_1\|_{L^\infty}(\Omega) = 1\) is the first eigenfunction of the problem (10). Lemma 2.7 implies the existence of positive periodic solutions to the problem (1), (2)–(4).

Finally, we use the phase plane method to prove the case \(N = 1\). Since \(\Omega' = (-R,0) \cup (0,R)\), we only need to consider the half interval \((0,R)\). The first integral of \(-u'' = mu\) is

\[
\frac{1}{2} |u'|^2 + \frac{1}{2} mu^2 = E.
\]

Without loss of generality, a positive solution \(u(x)\) of \(-u'' = mu\) on \((0,R)\) with \(u(0) = M\) and \(u(R) = 0\) is equivalent to a trajectory starting from the positive \(u'-\)axis and finishing at some point on the line \(u = M\) with \(u > 0\) and the time

\[
T_1(E) = \int_0^M \frac{1}{\sqrt{2E - mu^2}} \, du = R,
\]

or

\[
T_2(E) = \int_0^U \frac{1}{\sqrt{2E - mu^2}} \, du + \int_U^M \frac{1}{\sqrt{2E - mu^2}} \, du = R,
\]

where \(E \geq \frac{1}{2} mM^2\) and \(mU^2 = 2E\). The range of \(T_1(E)\) for \(E \geq \frac{1}{2} mM^2\) is \((0,T_1(\frac{1}{2} mM^2))\), while the range of \(T_2(E)\) for \(E \geq \frac{1}{2} mM^2\) is \([T_1(\frac{1}{2} mM^2), 2T_1(\frac{1}{2} mM^2)]\). Thus, the problem (1), (2)–(4) admits a positive periodic solution if and only if

\[
R < 2T_1(\frac{1}{2} mM^2) = 2 \int_0^M \frac{1}{\sqrt{mM^2 - mu^2}} \, du = \frac{\pi}{\sqrt{m}}.
\]

That is \(m < \frac{\pi^2}{4R^2}\). The proof is completed. \(\square\)
4. Existence for $q \neq p - 1$. In this section, we first use the upper and lower solution method to show the existence of positive periodic solutions of the problem (1), (2)–(3) with (4) in the case of $0 \leq q < p - 1$. Then we use Leray-Schauder’s Fixed Point Theorem to prove the existence of positive periodic solutions in the case of $q > p - 1$.

**Theorem 4.1.** Assume that $p > N$, $0 \leq q < p - 1$, and $0 < m(x, t) \in C^1([\tau, \tau + \omega])$. Then the problem (1), (2)–(4) admits at least one bounded positive periodic solution $u \in E^0_M$.

**Proof.** Choose $R$ to be sufficiently large such that $\Omega \subset B_R/2$. Let $\lambda_1$ be the first eigenvalue of the $p$-Laplacian with the homogeneous Dirichlet boundary condition on $B_R$, and let $\varphi > 0$ with $\|\varphi\|_{L^\infty(B_R)} = 1$ be the eigenfunction corresponding to $\lambda_1$. That is, $\varphi$ satisfies
\[
\begin{aligned}
  \nabla (|\nabla \varphi|^{p-2}\nabla \varphi) &= \lambda_1 |\varphi|^{p-2}\varphi, \quad x \in B_R, \\
  \varphi(x) |_{\partial B_R} &= 0.
\end{aligned}
\]

The solvability of the above eigenvalue problem is well known, see for example [7]. Furthermore, there exists a constant $\delta > 0$ such that $\varphi \geq \delta$ for $x \in \Omega$. Then $u \equiv 0$ and $u = \kappa \varphi$ are a pair of weak periodic lower and upper solutions to the problem (1), (2)–(4) provided $\kappa = \max \left\{ \frac{(\pi/\lambda_1)^{1/(p-1-q)}}{\delta}, \frac{\|M(t)\|_{L^\infty}}{\delta} \right\}$.

Lemma 2.7 implies that the problem (1), (2)–(4) admits a periodic solution $u \in E^0_M$ with $\underline{u} \leq u(x, t) \leq \overline{u}$. \hfill $\Box$

Although the upper and lower solution method may be applicable for small $\|M(t)\|_{L^\infty}$ in the case of $q > p - 1$, we use the fixed point approach for the sake of consistency.

**Lemma 4.2 (The Leray-Schauder Fixed Point Theorem).** Let $E$ be a Banach space, and $T(u, \sigma)$ be a mapping from $E \times [0, 1]$ to $E$ satisfying:

1. $T$ is a completely continuous mapping;
2. $T(u, 0) = 0$, for any $u \in E$;
3. There exists a constant $C > 0$ such that $\|u\|_E \leq C$ for all $u \in E$ and $\sigma \in [0, 1]$ satisfying $u = T(u, \sigma)$.

Then the mapping $T(\cdot, 1)$ has a fixed point, that is, there exists a $u \in E$ such that $u = T(u, 1)$.

The domain $\Omega' = \Omega \setminus \{0\}$ is not regular, since there is a singular point 0 in the boundary $\partial \Omega' = \partial \Omega \cup \{0\}$. We need the following lemma concerned with the $C^\alpha$-norm.

**Lemma 4.3.** Suppose that $u \in C^\alpha(\overline{\Omega}\setminus\{0\}) \cap C(\overline{\Omega})$ and there exists a constant $C$ independent of $\varepsilon$ such that
\[
\|u\|_{C^\alpha(\overline{\Omega}\setminus\{0\})} \leq C, \quad \forall \varepsilon > 0.
\]

Then $u \in C^\alpha(\overline{\Omega})$. 

Proof. For any \( x_1, x_2 \in \overline{\Omega} \), if the segment \( x_1 x_2 \subset \overline{\Omega} \setminus \{0\} \), then there exists a constant \( \varepsilon > 0 \) such that \( x_1 x_2 \subset \overline{\Omega} \setminus B_\varepsilon \). Thus, 
\[
\frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\alpha} \leq \|u\|_{C^0(\overline{\Omega} \setminus B_\varepsilon)} \leq C.
\]

Otherwise, the segment \( x_1 x_2 \) lies across 0. For simplicity, we may assume that \( x_1, x_2 \) lie in the \( x^1 \)-axis and \( x_1 = (x_1^1, x') \), \( x_2 = (x_2^1, x') \) with \( x_1^1 < 0 < x_2^1 \) (the case \( x_1^1 = 0 \) or \( x_2^1 = 0 \) is simple). In the following equations we omit the variable \( x' \).

Choose \( \delta > 0 \) sufficiently small, then we have
\[
\frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\alpha} \leq \frac{|u(x_1) - u(-\delta)|}{|x_1 - x_2|^\alpha} + \frac{|u(\delta) - u(-\delta)|}{|x_1 - x_2|^\alpha} + \frac{|u(x_2) - u(\delta)|}{|x_1 - x_2|^\alpha}
\leq \frac{|u(x_1) - u(-\delta)|}{|x_1 - x_2|^\alpha} + \frac{|u(\delta) - u(-\delta)|}{|x_1 - x_2|^\alpha} + \frac{|u(x_2) - u(\delta)|}{|x_1 - x_2|^\alpha}
\leq 2C + \frac{|u(\delta) - u(-\delta)|}{|x_1^1 - x_2^1|^\alpha}.
\]

(12)

Letting \( \delta \to 0 \), by the continuity \( u \in C(\overline{\Omega}) \), we see that the conclusion holds. \( \square \)

For \( p > N \), define an operator \( G \) by
\[
G : L^\infty_w((\tau, \tau + \omega), C^0(\overline{\Omega})) \times [0, 1] \to L^\infty_w((\tau, \tau + \omega), C^0(\overline{\Omega})),
\]
\[
G(v, \sigma) = u,
\]
where \( 0 < \alpha < 1 - \frac{N}{p} \) is a constant, \( u \in L^\infty_w((\tau, \tau + \omega), C^0(\overline{\Omega})) \) with \( u_t \in L^2(Q_\omega) \) is the weak periodic solution of the following problem
\[
\begin{aligned}
&\frac{\partial u}{\partial t} - \text{div}(|\nabla u|^{p-2}\nabla u) = \sigma m(x, t)|v|^q, \quad x \in \Omega, \quad t \in (\tau, \tau + \omega), \\
u(x, t) = 0, \quad x \in \partial \Omega, \quad t \in (\tau, \tau + \omega), \\
u(0, t) = \sigma M(t), \quad t \in (\tau, \tau + \omega), \\
u(x, \tau) = u(x, \tau + \omega), \quad x \in \Omega.
\end{aligned}
\]

(13)

First we assert the unique solvability of the above problem (13).

**Lemma 4.4.** Assume that \( p > N, \ q \geq 0, \ 0 < M(t) \in C^1([\tau, \tau + \omega]), \) and \( 0 < m(x, t) \in C^1(\overline{\Omega} \times [\tau, \tau + \omega]) \). Then for any \( \sigma \in [0, 1], \ v \in L^\infty_w((\tau, \tau + \omega), C^0(\overline{\Omega})), \) problem (13) admits a unique weak solution \( u \in L^\infty_w((\tau, \tau + \omega), C^0(\overline{\Omega})) \) with \( u_t \in L^2(Q_\omega) \).

**Proof.** The uniqueness is trivial, thus we only prove the existence. For \( 0 < \varepsilon < \varepsilon_0 \) with \( B_{\varepsilon_0} \subset \subset \Omega \), consider the following regularized problem
\[
\begin{aligned}
&\frac{\partial u}{\partial t} - \text{div}(|\nabla u|^{p-2}\nabla u) = \sigma m(x, t)|v|^q, \quad x \in \Omega \setminus B_\varepsilon, \quad t \in (\tau, \tau + \omega), \\
u(x, t) = 0, \quad x \in \partial \Omega, \quad t \in (\tau, \tau + \omega), \\
u(x, \tau) = \sigma M(t), \quad x \in \partial B_\varepsilon, \quad t \in (\tau, \tau + \omega), \\
u(x, \tau) = u(x, \tau + \omega), \quad x \in \Omega.
\end{aligned}
\]

(14)

Let \( \hat{\psi} \in E^0 \) be the weak solution of the following \( p \)-Laplacian
\[
\begin{aligned}
&-\text{div}(|\nabla u|^{p-2}\nabla u) = \overline{m}(\|v\|_{L^\infty(Q_\omega)} + 1)^q, \quad x \in \Omega, \\
u(x) = 0, \quad x \in \partial \Omega.
\end{aligned}
\]

(15)
The existence and uniqueness of the above problem (15) is classical. It is clear that $0 < \psi \in L^\infty(Q_\omega)$. Take $\psi = \kappa \psi$ with $\kappa > 0$ sufficiently large so that $\psi(x, t) \geq M(t)$ for $(x, t) \in B_{\epsilon_0} \times (\tau, \tau + \omega)$. We see that $\psi_0 \equiv 0$ and $\psi$ are a pair of bounded lower and upper solutions of the problem (14). By means of the comparison principle, we conclude that problem (14) admits a unique weak solution $u^\epsilon$ such that

$$0 \leq u^\epsilon \leq \|\psi\|_{L^\infty(O)} \leq C,$$

where $C$ is independent of $\epsilon$ and depends on $\|M(t)\|_{L^\infty(\Omega \times (\tau, \tau + \omega))}$, $\|v\|_{L^\infty(\Omega \times (\tau, \tau + \omega))}$ and $\overline{\nu}$.

Now we show the uniform estimate for the $C^\alpha$-norm of $u^\epsilon$. Let $\xi(x) \in C^1(\overline{\Omega})$ such that $\xi(x) = 1$ for $x \in B_{\epsilon_0}$, $\xi(x) = 0$ for $x \in \partial \Omega$, $0 \leq \xi(x) \leq 1$ and $|\nabla \xi| \leq C$. Define $\Omega^\epsilon = \Omega \setminus B_\epsilon$ and $Q^\epsilon = (\Omega \setminus B_\epsilon) \times (\tau, \tau + \omega)$. Taking

$$\varphi(x, t) = u^\epsilon(x, t) - \sigma M(t)\xi(x)$$

as the test function in problem (14), we have

$$\iint_{Q^\epsilon} u^\epsilon_t(u^\epsilon - \sigma M(t)\xi(x)) \, dxdt + \iint_{Q^\epsilon} |\nabla u^\epsilon|^p \nabla (u^\epsilon - \sigma M(t)\xi(x)) \, dxdt \leq \iint_{Q^\epsilon} \overline{\nu} v^p \, dxdt \leq C.$$

By the periodicity of $u^\epsilon$, the first term in the above inequality satisfies that

$$\left| \iint_{Q^\epsilon} u^\epsilon_t(u^\epsilon - \sigma M(t)\xi(x)) \, dxdt \right| \leq \iint_{Q^\epsilon} |M(t)\xi(x)u^\epsilon_t| \, dxdt \leq \iint_{Q^\epsilon} |M'(t)\xi(x)u^\epsilon| \, dxdt \leq C.$$

We also have

$$\left| \iint_{Q^\epsilon} |\nabla u^\epsilon|^p \nabla (M(t)\xi(x)) \, dxdt \right| \leq \iint_{Q^\epsilon} |\nabla u^\epsilon|^p |M(t)||\nabla \xi| \, dxdt \leq C \left( \iint_{Q^\epsilon} |\nabla u^\epsilon|^p \, dxdt \right)^{p-1}. $$

Combining the last three inequalities, we find that

$$\iint_{Q^\epsilon} |\nabla u^\epsilon|^p \, dxdt \leq C + C \left( \iint_{Q^\epsilon} |\nabla u^\epsilon|^p \, dxdt \right)^{p-1},$$

which yields that

$$\iint_{Q^\epsilon} |\nabla u^\epsilon|^p \, dxdt \leq C,$$

where $C$ is independent of $\epsilon$. Taking

$$\varphi_t = u^\epsilon_t - \sigma M'(t)\xi(x)$$

as the test function yields

$$\int_{Q^\epsilon} u^\epsilon_t(u^\epsilon_t - \sigma M'(t)\xi(x)) \, dx + \int_{Q^\epsilon} |\nabla u^\epsilon|^p \nabla (u^\epsilon_t - \sigma M'(t)\xi(x)) \, dx \leq \int_{Q^\epsilon} \overline{\nu} v^p \, dxdt \leq \frac{1}{4} \int_{\Omega^\epsilon} |u^\epsilon|^2 \, dx + C.$$
We have the following estimates
\[
\left| \int_{\Omega^e} u_i^e M(t) \xi(x) \, dx \right| \leq \frac{1}{4} \int_{\Omega^e} \left| u_i^e \right|^2 \, dx + C,
\]
and
\[
\left| \int_{\Omega^e} \left| \nabla u^e \right|^{p-2} \nabla u^e \cdot \nabla \xi(x) M(t) \, dx \right| \leq C \left( \int_{\Omega^e} \left| \nabla u^e \right|^p \, dx \right)^{\frac{p-1}{p}}.
\]
Thus, the inequality (18) can be read as
\[
\int_{\Omega^e} \left| u_i^e \right|^2 \, dx + \frac{d}{dt} \int_{\Omega^e} \left| \nabla u^e \right|^p \, dx \leq C \left( \int_{\Omega^e} \left| \nabla u^e \right|^p \, dx \right)^{\frac{p-1}{p}} + C.
\]
Combining the estimate (17) and the periodicity of \( u^\varepsilon \), we deduce that
\[
\sup_{t \in [\tau, \tau + \omega]} \int_{\Omega^e} \left| \nabla u^\varepsilon \right|^p \, dx \leq C, \quad \int_{Q^e_{\omega}} \left| u_i^e \right|^2 \, dx \, dt \leq C,
\]
where \( C \) is a constant independent of \( \varepsilon \) and depends on \( \| M(t) \|_{C^1([\tau, \tau + \omega])}, \| v \|_{L^\infty(Q^e_{\omega})} \) and \( \overline{\mu} \).

For any \( 0 < \delta < \varepsilon_0 \), by the compact estimates (16), (17), (19) and the Sobolev-Poincaré-type inequality (3.1), we see that there exists a subsequence of \( \{ u^\varepsilon \} \) that converges to a function \( \omega \) that converges to a function \( \Omega \notin \Omega \) such that \( \| \nabla (\varepsilon \omega) \|_{L^p(\Omega \cup \{ 0 \})} \leq C \). The comparison principle implies that \( u_{\varepsilon}(x, t) \leq w(x, t) \) for \( x \in \Omega \setminus \Omega \) and \( t \in (\tau, \tau + \omega) \). Similarly, we can take \( w(x, t) = \sigma M(t) - A| x|^{7} \) with appropriately large \( A > 0 \) as a lower solution, and
\[
\omega(x) = \omega(x, t) \leq w(x, t).
\]
which follows from $u_\varepsilon(x,t) \geq \bar{w}(x,t)$ for $x \in \Omega \setminus B_\varepsilon$ and $t \in (\tau, \tau + \omega)$. Therefore, we have

$$|u_\varepsilon(x,t) - \sigma M(t)| \leq A|x|^{\gamma}, \quad x \in \Omega \setminus B_\varepsilon, \ t \in (\tau, \tau + \omega).$$

Consequently, the limit function $w(x,t)$ is continuous at $\{0\} \times (\tau, \tau + \omega)$,

$$|u(x,t) - \sigma M(t)| \leq A|x|^{\gamma}, \quad x \in \Omega \setminus \{0\}, \ t \in (\tau, \tau + \omega). \quad (22)$$

That is, $u(\cdot, t) \in C(\bar{\Omega})$. Combining with the estimate (20) and using Lemma (4.3), we see that $u(\cdot, t) \in C^\alpha(\bar{\Omega})$ and $u \in L_\infty^\infty((\tau, \tau + \omega), C^\alpha(\bar{\Omega}))$. The proof is completed. \hfill \Box

It is clear that $u \geq 0$. Now, we verify the compactness and continuity of the operator $G$.

**Lemma 4.5.** Assume that $p > N$, $q \geq 0$, $0 < M(t) \in C^1([\tau, \tau + \omega])$ and $0 < m(x, t) \in C^1(\bar{\Omega} \times [\tau, \tau + \omega])$. Then the operator $G$ is completely continuous and $G(v, 0) = 0$ for any $v \in L_\infty^\infty((\tau, \tau + \omega), C^\alpha(\bar{\Omega}))$.

**Proof.** By the uniform estimates (17), (19) and the continuity (22), we see that for any $\sigma \in [0, 1]$, $v \in L_\infty^\infty((\tau, \tau + \omega), C^\alpha(\bar{\Omega}))$,

$$\sup_{t \in (\tau, \tau + \omega)} \int_{\Omega} |\nabla u|^p \, dx \leq C, \sup_{t \in (\tau, \tau + \omega)} \|u_t\|_{L^2(Q_{\omega})} \leq C, \quad (23)$$

where $C$ is a constant depends on $\|M(t)\|_{C^1([\tau, \tau + \omega])}$, $\|v\|_{L_\infty^\infty(Q_{\omega})}$ and $\bar{\Omega}$. The compactness of the operator $G$ follows from the estimate (23) and compact embedding theorems. It is easy to obtain the continuity of $G$ by a similar procedure. The condition $G(v, 0) = 0$ for any $v \in L_\infty^\infty((\tau, \tau + \omega), C^\alpha(\bar{\Omega}))$ is easy to verify. \hfill \Box

Here we recall the following Liouville type results on the exterior domain $\mathbb{R}^N \setminus \{0\}$.

**Lemma 4.6 (16).** Assume that $q > p - 1$ and $p > N$, and $m(x)$ is an appropriately smooth function with $0 < \underline{m} \leq m(x) \leq \overline{m}$. Then the problem

$$\begin{cases}
\int_{\mathbb{R}^N} |\nabla u|^p - \int_{\mathbb{R}^N} m(x) u \varphi \, dx = 0, & \forall \varphi \in C_0^1(\mathbb{R}^N \setminus \{0\}), \\
0 < u \in W_{\text{loc}}^{1,p}(\mathbb{R}^N \setminus \{0\}) \cap L_\infty^\infty(\mathbb{R}^N \setminus \{0\}),
\end{cases} \quad (24)$$

has no solution.

It remains to verify the condition (3) in Lemma 4.2. That is, we reduce the problem of finding nontrivial solutions to the problem of establishing the following a priori estimate.

**Lemma 4.7.** Assume that $p > N$, $q > p - 1$, $0 < M(t) \in C^1([\tau, \tau + \omega])$, $0 < m(x, t) \in C^1(\bar{\Omega} \times [\tau, \tau + \omega])$ and $\Omega$ is convex. Then there exists a constant $C > 0$ such that

$$\sup_{t \in (\tau, \tau + \omega)} \|u(\cdot, t)\|_{C^\alpha(\bar{\Omega})} \leq C$$

for all $u \in L_\infty^\infty((\tau, \tau + \omega), C^\alpha(\bar{\Omega}))$ and $\sigma \in [0, 1]$ satisfying $u = G(u, \sigma)$.

**Proof.** According to the estimates given in (23), one only needs to verify the uniform boundedness of the norm $\|u\|_{L_\infty^\infty(Q_{\omega})}$, for which we shall give a proof by contradiction. Suppose there exists $\sigma_n$ and $u_n$ such that $u_n = G(u_n, \sigma_n)$ and

$$\|u_n\|_{L_\infty^\infty(Q_{\omega})} \to \infty, \quad \text{as } n \to \infty.$$
Let $\rho_n = \|u_n\|_{L^\infty(\Omega_n)} = u_n(x_n, t_n) \to \infty$. By the convexity of $\Omega$, there exists a $\delta_0 > 0$ such that $\text{dist}(x_n, \partial \Omega) \geq \delta_0$, see for example [6, 18]. Then there exists a subsequence, still denoted by $x_n$ for simplicity, such that $x_n \to x_0$, $t_n \to t_0$, with $\text{dist}(x_0, \partial \Omega) \geq \delta_0$.

In what follows we complete the proof by considering the cases $x_0 \neq 0$ and $x_0 = 0$ separately.

If $x_0 \neq 0$, let
\[ w_{n_j}(y, s) = \rho_n^{-q} u_n(\rho_n^{-(p-1)} y + x_0, t_j + j s), \quad \hat{m}_{n_j}(y, s) = m(\rho_n^{-(p-1)} y + x_0, t_j + j s), \]
and
\[ \Omega_n = \{ y; y = \rho_n^{-(p-1)} (x - x_0), x \in \Omega \}, \quad Q_{n_j} = \Omega_n \times \left( \frac{\tau - t_j}{j}, \frac{\tau + \omega - t_j}{j} \right). \]

Then $w_{n_j}$ with $\|w_{n_j}\|_{L^\infty(Q_{n_j})} = 1$ satisfies
\[ \rho_n^{1-q} \frac{\partial w_{n_j}}{\partial s} - j \text{div}(|\nabla w_{n_j}|^{p-2} \nabla w_{n_j}) = j \sigma_n \hat{m}_{n_j}(y, s) |w_{n_j}|^q. \]

Therefore, for any $\psi(y, s) \in C^1_\omega(Q_{n_j})$ with $\psi = 0$ on $\partial \Omega_n \cup \{ y; y = -\rho_n^{-(p-1)} x_0 \}$, we have
\begin{align*}
& \iint_{Q_{n_j}} \rho_n^{1-q} \frac{\partial w_{n_j}}{\partial s} \psi \, dy ds + j \iint_{Q_{n_j}} |\nabla w_{n_j}|^{p-2} \nabla w_{n_j} \cdot \nabla \psi \, dy ds \\
& = j \sigma_n \iint_{Q_{n_j}} \hat{m}_{n_j}(y, s) |w_{n_j}|^q \psi \, dy ds. \tag{25}
\end{align*}

Taking
\[ \psi = w_{n_j} - \sigma_n \rho_n^{-1} M(t_j + j s) \xi(\rho_n^{-(p-1)} y + x_0) = w_{n_j} - \sigma_n \rho_n^{-1} \hat{M}(s) \hat{\xi}(y), \]
where $\xi(x)$ is the function defined in the proof of Lemma 4.4, we have
\begin{align*}
& \iint_{Q_{n_j}} \rho_n^{1-q} \frac{\partial w_{n_j}}{\partial s} (w_{n_j} - \sigma_n \rho_n^{-1} \hat{M}(s) \hat{\xi}(y)) \, dy ds \\
& + j \iint_{Q_{n_j}} |\nabla w_{n_j}|^{p-2} \nabla w_{n_j} \cdot \nabla (w_{n_j} - \sigma_n \rho_n^{-1} \hat{M}(s) \hat{\xi}(y)) \, dy ds \\
& = j \sigma_n \iint_{Q_{n_j}} \hat{m}_{n_j}(y, s) |w_{n_j} - \sigma_n \rho_n^{-1} \hat{M}(s) \hat{\xi}(y)|^q \, dy ds.
\end{align*}

Similar to the estimate (17) in the proof of Lemma 4.4, we see that
\[ j \int_{Q_{n_j}} |\nabla w_{n_j}|^p \, dy ds \leq C j \int_{\Omega_n} \xi \, \omega_j \Omega_n \leq C |\Omega_n|, \tag{26} \]
which means that there exists $\delta_j \in \left[ \frac{\tau - t_j}{j}, \frac{\tau + \omega - t_j}{j} \right)$ such that
\[ \int_{\Omega_n} |\nabla w_{n_j}(y, \delta_j)|^p \, dy \leq C |\Omega_n|. \]

For any $s > \delta_j$, taking
\[ \psi = \chi(\delta_j, s) \left( \frac{\partial w_{n_j}}{\partial s} - j \sigma_n \rho_n^{-1} M'(t_j + j s) \xi(\rho_n^{-(p-1)} y + x_0) \right) \]
\[ = \chi(\delta_j, s) \left( \frac{\partial w_{n_j}}{\partial s} - j \sigma_n \rho_n^{-1} \hat{M}'(s) \hat{\xi}(y) \right) \]

in (25) yields
\[
\int_{\delta_j}^{s} \int_{\Omega_n} \rho_n^{1-q} \frac{\partial w_{nj}}{\partial s} \left( \frac{\partial w_{nj}}{\partial s} - j \sigma_n \rho_n^{-1} M'(s) \xi(y) \right) dyds \\
+ j \int_{\delta_j}^{s} \int_{\Omega_n} |\nabla w_{nj}|^{p-2} \nabla w_{nj} \cdot \nabla \left( \frac{\partial w_{nj}}{\partial s} - j \sigma_n \rho_n^{-1} M'(s) \xi(y) \right) dyds
\]
\[
=j \sigma_n \int_{\delta_j}^{s} \int_{\Omega_n} \tilde{m}_{nj}(y,s) |w_{nj}|^q \left( \frac{\partial w_{nj}}{\partial s} - j \sigma_n \rho_n^{-1} M'(s) \xi(y) \right) dyds. \tag{27}
\]
Similar to the proof of estimate (19), we have the following estimates
\[
\left| \int_{\delta_j}^{s} \int_{\Omega_n} \rho_n^{1-q} \frac{\partial w_{nj}}{\partial s} j \sigma_n \rho_n^{-1} M'(s) \xi(y) dyds \right|
\leq \frac{1}{2} \int_{\delta_j}^{s} \int_{\Omega_n} \rho_n^{1-q} \left| \frac{\partial w_{nj}}{\partial s} \right|^2 dyds + C \int_{\delta_j}^{s} \int_{\Omega_n} j^2 dyds,
\]
and
\[
\left| j \sigma_n \int_{\delta_j}^{s} \int_{\Omega_n} \tilde{m}_{nj}(y,s) |w_{nj}|^q \left( \frac{\partial w_{nj}}{\partial s} - j \sigma_n \rho_n^{-1} M'(s) \xi(y) \right) dyds \right|
\leq \frac{1}{2} \int_{\delta_j}^{s} \int_{\Omega_n} \rho_n^{1-q} \left| \frac{\partial w_{nj}}{\partial s} \right|^2 dyds + C \int_{\delta_j}^{s} \int_{\Omega_n} \rho_n^{q-1} j^2 dyds.
\]
Moreover, it follows from (26) that
\[
\left| j \int_{\delta_j}^{s} \int_{\Omega_n} |\nabla w_{nj}|^{p-2} \nabla w_{nj} \cdot \nabla \left( j \sigma_n \rho_n^{-1} M'(s) \xi(y) \right) dyds \right|
\leq j \left( \int_{\delta_j}^{s} \int_{\Omega_n} |\nabla w_{nj}|^p dyds \right)^{\frac{p-1}{p}} \left( \int_{\delta_j}^{s} \int_{\Omega_n} |\xi(y)| dydy \right)^{\frac{1}{p}} \leq C j |\Omega_n|.
\]
Then the equation (27) can be read as
\[
j \int_{\Omega_n} |\nabla w_{nj}(y,s)|^p dy \leq j \int_{\Omega_n} |\nabla w_{nj}(y,\delta_j)|^p dy + C j |\Omega_n| \rho_n^{-1} \leq C j |\Omega_n| \rho_n^{-1}.
\]
By the periodicity of \(w_{nj}\), the above inequality holds for all \(s \in \left( \frac{-t_j}{j}, \frac{t_j + \omega - t_j}{j} \right)\). That is
\[
\sup_{\Omega_n} \int_{\Omega_n} |\nabla w_{nj}(y,s)|^p dy \leq C |\Omega_n| \rho_n^{-1}. \tag{28}
\]
Moreover, for any \(\varphi \in C_0^1(\Omega_n \setminus \{y; y = -\rho_n \frac{e^{-(y-1)}}{p} x_0\})\), we have
\[
j \int_{Q_n} |\nabla w_{nj}|^{p-2} \nabla w_{nj} \cdot \nabla \varphi dyds = j \int_{Q_n} \tilde{m}_{nj}(y,s) |w_{nj}|^q \varphi dyds.
\]
By the Lebesgue differential theorem, there exists \(s_j \in \left( \frac{-t_j}{j}, \frac{t_j + \omega - t_j}{j} \right)\) such that
\[
\int_{\Omega_n} |\nabla w_{nj}|^{p-2} \nabla w_{nj}(y,s_j) \cdot \nabla \varphi dy = \int_{\Omega_n} \tilde{m}_{nj}(y,s_j) |w_{nj}(y,s_j)|^q \varphi dy.
\]
Then there exists a function \(w_n \in W^{1,p}(\Omega_n)\) with \(\|w_n\|_{L^\infty} = 1\) such that as \(j \to \infty\) (passing to a subsequence if necessary)
\[
\nabla w_{nj} \to \nabla w_n \text{ in } L^p(\Omega_n), \quad w_{nj} \to w_n \text{ in } L^r(\Omega_n) \text{ for any } r > 1,
\]
and
\[
\tilde{m}_{nj}(y,s_j) \to \tilde{m}_n(y) \text{ uniformly}, \quad \tilde{m}_n(y) \in C^\gamma(\Omega_n) \text{ for some } 0 < \gamma < 1,
\]
we obtain that
\[
\int_{\Omega_n} |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla \varphi \, dy = \int_{\Omega_n} \tilde{m}_n |w_n|^q \varphi \, dy.
\]  \tag{29}

Take \( \varphi = w_n \eta^{2p}(y) \), where
\[
\eta(y) = \begin{cases} 
1, & y \in B_R(0), \\
0, & y \in \mathbb{R}^N \setminus B_{2R}(0),
\end{cases}
\]
with \( 0 \leq \eta \leq 1 \) sufficiently smooth and \( |\nabla \eta| \leq \frac{C}{R} \). Then for sufficiently large \( n \), we have \( \rho_n^{-\frac{q}{q-p-1}} |x_0| > 2R \), which yields that \( B_{2R} \subset \Omega_n \). Thus, we have
\[
\int_{B_{2R}} \eta^{2p} |\nabla w_n|^p \, dy = -2p \int_{B_{2R}} \eta^{2p-1} w_n |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla \eta \, dy + \int_{B_{2R}} \tilde{m}_n |w_n|^q w_n \eta^{2p} \, dy
\]
\[
\leq \frac{1}{2} \int_{B_{2R}} \eta^{2p} |\nabla w_n|^p \, dy + \frac{C}{R^p} \int_{B_{2R}} \eta^p w_n^p \, dy + m \int_{B_{2R}} |w_n|^q w_n \eta^{2p} \, dy
\]
\[
\leq \frac{1}{2} \int_{B_{2R}} \eta^{2p} |\nabla w_n|^p \, dy + C R^{N-p} + C R^N.
\]

That is
\[
\int_{B_R} |\nabla w_n|^p \, dy \leq C \int_{B_{2R}} \eta^{2p} |\nabla w_n|^p \, dy \leq C R^N
\]  \tag{30}
for sufficiently large \( R > 0 \). Then there exists a function \( \hat{w} \in W^{1,p}_{\text{loc}}(\mathbb{R}^N) \) such that, passing to a subsequence if necessary, as \( n \to \infty \)
\[
\tilde{m}_n(y) \to \tilde{m}(y), \quad \text{uniformly on } B_R,
\]
\[
\nabla w_n \to \nabla \hat{w} \text{ in } L^p(B_R), \quad w_n \to \hat{w} \text{ in } L^r(B_r) \text{ for any } r > 1.
\]

Then we have
\[
\begin{cases} 
\int_{B_R} |\nabla \hat{w}|^{p-2} \nabla \hat{w} \cdot \nabla \varphi \, dy = \int_{B_R} \tilde{m}(y) \hat{w}^q \varphi \, dy \quad \text{for any } \varphi \in C^1_0(B_R), \\
\| \hat{w} \|_{L^\infty(B_R)} = 1, \quad \hat{w} > 0, \quad y \in B_R.
\end{cases}
\]

Taking the ball \( B_R \) larger and larger, and repeating the argument for the subsequence \( w_n \) obtained at the previous step, we get a Cantor diagonal subsequence, for simplicity we still denote it by \( w_k \), which converges in \( W^{1,p}_{\text{loc}}(\mathbb{R}^N) \) to a function \( w \in W^{1,p}_{\text{loc}}(\mathbb{R}^N) \) satisfying
\[
\begin{cases} 
\int_{\mathbb{R}^N} |\nabla w|^{p-2} \nabla w \cdot \nabla \varphi \, dy = \int_{\mathbb{R}^N} \tilde{m}(y) w^q \varphi \, dy \quad \text{for any } \varphi \in C^1_0(\mathbb{R}^N), \\
\| w \|_{L^\infty(\mathbb{R}^N)} = 1, \quad w > 0, \quad y \in \mathbb{R}^N.
\end{cases}
\]  \tag{31}

We can now conclude that (31) is a contradiction. Indeed, thanks to the Liouville type result, Lemma 4.6, we see that (31) has no solution.

It remains to prove the case \( x_0 = 0 \), in which the rescaling coincides with the singular point 0. Similar to the case \( x_0 \neq 0 \), instead of Eq. (29), we have
\[
\int_{\Omega_n} |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla \varphi \, dy = \int_{\Omega_n} \tilde{m}_n |w_n|^q \varphi \, dy,
\]
for any \( \varphi \in C^1(\Omega_n \setminus \{0\}) \). Taking
\[
\varphi = (w_n - \rho_n^{-1}\sigma_n \tilde{M})\eta^{2p}(y)
\]
as the test function, where \( \eta(y) \) is the function defined in the proof of (30) and \( \tilde{M} \)is the limit of \( M_j(s_j) \), we see that
\[
\int_{B_{2R}} \eta^{2p}|\nabla w_n|^p \, dy
= -2p \int_{B_{2R}} \eta^{2p-1}(w_n - \rho_n^{-1}\sigma_n \tilde{M})|\nabla w_n|^{p-2}\nabla w_n \cdot \nabla \eta \, dy
+ \int_{B_{2R}} \tilde{m}_n |w_n|^q(w_n - \rho_n^{-1}\sigma_n \tilde{M})\eta^{2p} \, dy
\leq \frac{1}{2} \int_{B_{2R}} \eta^{2p}|\nabla w_n|^p \, dy + C R^{N-p} + CR^N.
\]
That is
\[
\int_{B_R} |\nabla w_n|^p \, dy \leq CR^N.
\]
Then there exists a function \( \tilde{w} \in W^{1,p}_0(\mathbb{R}^N) \) such that, passing to a subsequence if necessary, as \( n \to \infty \)
\[
\tilde{m}_n(y) \to \tilde{m}(y), \quad \text{uniformly on } B_R,
\]
\[
\nabla w_n \to \nabla \tilde{w} \quad \text{in } L^p(B_R),\quad w_n \to \tilde{w} \quad \text{in } L^r(B_r) \quad \text{for any } r > 1.
\]
Then we have
\[
\begin{cases}
\int_{B_R} |\nabla \tilde{w}|^{p-2}\nabla \tilde{w} \cdot \nabla \varphi \, dy = \int_{B_R} \tilde{m}(y)\tilde{w}^{q}\varphi \, dy, & \text{for any } \varphi \in C^1_0(B_R \setminus \{0\}), \\
\|\tilde{w}\|_{L^\infty(B_R)} = 1, & \tilde{w} > 0, \quad y \in B_R.
\end{cases}
\]
Similarly, taking the ball \( B_R \) larger and larger and repeating the above argument, we get a Cantor diagonal subsequence, still denoted by \( w_k \), which converges in \( W^{1,p}_0(\mathbb{R}^N \setminus \{0\}) \) to a function \( w \in W^{1,p}_0(\mathbb{R}^N \setminus \{0\}) \)
satisfying
\[
\begin{cases}
\int_{\mathbb{R}^N} |\nabla w|^{p-2}\nabla w \cdot \nabla \varphi \, dy = \int_{\mathbb{R}^N} \tilde{m}(y)w^{q}\varphi \, dy, & \text{for any } \varphi \in C^1_0(\mathbb{R}^N \setminus \{0\}), \\
\|w\|_{L^\infty(\mathbb{R}^N)} = 1, & w > 0, \quad y \in \mathbb{R}^N.
\end{cases}
\]
Again, we can conclude that (32) is a contradiction by the Liouville-type result Lemma 4.6. The above contradictions imply that \( \|u_n\|_{L^\infty} \) is uniformly bounded. \( \square \)

Now we can apply Lemma 4.2 to show the following existence result.

**Theorem 4.8.** Assume that \( p > N, \ q > p - 1, \ 0 < M(t) \in C^1([\tau, \tau + \omega]), \ 0 < m(x, t) \in C^1(\Omega \times [\tau, \tau + \omega]) \) and \( \Omega \) is convex. Then the problem (1), (2)–(4) in a punctured domain admits at least one positive periodic solution.

**Proof.** This result follows from Lemma 4.7, Lemma 4.5, and the Leray-Schauder fixed point theorem Lemma 4.2. \( \square \)
5. **Non-singularity for** $1 < p \leq N$. We prove that for $1 < p \leq N$, the weak solution of the periodic problem in a punctured domain (1), (2)–(3), is actually a weak solution of the periodic problem in the regular domain (7), (2)–(3), and the condition (4) is not needed. That is to say, to prove Theorem 2.5, we only need to prove the following result.

**Proposition 1.** Suppose that $1 < p \leq N$ and $u$ is a weak solution of the periodic problem (1), (2)–(3). Then $u$ is a weak solution of the periodic problem (7), (2)–(3).

**Proof.** For any $\varepsilon > 0$, let $\xi_\varepsilon(x) \in C^1(\Omega)$ such that $\xi_\varepsilon = 0$ on $B_\varepsilon(0)$, $\xi_\varepsilon = 1$ on $\Omega \setminus B_{2\varepsilon}(0)$, $0 \leq \xi_\varepsilon \leq 1$ and $|\nabla \xi_\varepsilon| \leq \frac{C}{\varepsilon}$. Thus, for any $\varphi \in C^1_0(\Omega \times (\tau, \tau + \omega))$, we see that $\xi_\varepsilon \varphi \in C^1_0(\Omega' \times (\tau, \tau + \omega))$. From the definition of weak solutions, we have

$$
\iint_{Q_\omega} \frac{\partial u}{\partial t} \xi_\varepsilon \varphi \, dx \, dt + \iint_{Q_\omega} |\nabla u|^{p-2} \nabla u \cdot \nabla (\xi_\varepsilon \varphi) \, dx \, dt = \iint_{Q_\omega} m(x, t) u^q \xi_\varepsilon \varphi \, dx \, dt.
$$

That is

$$
\iint_{Q_\omega} \frac{\partial u}{\partial t} \xi_\varepsilon \varphi \, dx \, dt + \iint_{Q_\omega} |\nabla u|^{p-2} \nabla u \cdot (\xi_\varepsilon \nabla \varphi + \varphi \nabla \xi_\varepsilon) \, dx \, dt = \iint_{Q_\omega} m(x, t) u^q \xi_\varepsilon \varphi \, dx \, dt.
$$

(33)

Since $u \in E^0$ and $1 < p \leq N$, we have the following estimate

$$
\left| \iint_{Q_\omega} \varphi |\nabla u|^{p-2} \nabla u \cdot \nabla \xi_\varepsilon \, dx \, dt \right| 
\leq C \iint_{Q_\omega} |\nabla u|^{p-1} |\nabla \xi_\varepsilon| \, dx \, dt
$$

(34)

\[ C \left( \int_0^{\tau + \omega} \left( \int_{B_{2\varepsilon} \setminus B_\varepsilon} |\nabla u|^p \, dx \right)^{\frac{p-1}{p}} \right)^\frac{1}{\frac{p}{2} - 1} \left( \int_0^{\tau + \omega} \left( \int_{B_{2\varepsilon} \setminus B_\varepsilon} |\nabla \xi_\varepsilon|^p \, dx \right)^{\frac{p-1}{p}} \right)^\frac{1}{\frac{p}{2} - 1} 
\leq C \left( \int_0^{\tau + \omega} \left( \int_{B_{2\varepsilon} \setminus B_\varepsilon} |\nabla u|^p \, dx \right)^{\frac{p-1}{p}} \right)^\frac{1}{\frac{p}{2} - 1} 
\leq C \left( \int_0^{\tau + \omega} \left( \int_{B_{2\varepsilon} \setminus B_\varepsilon} |\nabla u|^p \, dx \right)^{\frac{p-1}{p}} \right)^\frac{1}{\frac{p}{2} - 1},
$$

where $C$ is a constant independent of $\varepsilon$. By the integrability of $u$, we see that

$$
\lim_{\varepsilon \to 0} \left| \iint_{Q_\omega} \varphi |\nabla u|^{p-2} \nabla u \cdot \nabla \xi_\varepsilon \, dx \, dt \right| = 0.
$$

Letting $\varepsilon \to 0$ in equation (33), by the dominated convergence theorem, we have

$$
\iint_{Q_\omega} \frac{\partial u}{\partial t} \varphi \, dx \, dt + \iint_{Q_\omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx \, dt = \iint_{Q_\omega} m(x, t) u^q \varphi \, dx \, dt.
$$

Thus $u$ is a weak solution of the periodic problem (7), (2)–(3).

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