Group, geometry and algebra of nonextensive entropies in complex systems

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Abstract. The Abel entropy group and its matrix representation with the general law of nonextensive entropy composition and quadratic nonlinearity are defined. Four types of matrices for the corresponding parametrical entropies are given and geometries for their measures are determined. Algebraic representation of an entropy group is presented for this types of the general entropy classification in statistical thermodynamics of complex systems. The corresponding geometries are global Finsler ones. The well-known relations for the nonextensive entropies follow from the properties of the conformally generalized hypercomplex numbers.

1. The Abel entropy group
In many physical problems, equilibrium and nonequilibrium systems described by non-Gibbs and non-Gauss distributions of nonextensive (nonadditive) statistical mechanics and thermodynamics are investigated [1-3]. These investigations are based on the one-parameter entropy

\[ S = S_q = S_q \left[ N_{q^{-1}} (p) \right] \]

for classical systems with a real number \( q \).

The seminorm values \( N \) and average geometrical of distribution \( M \) are determined by expressions

\[ N = N_{q^{-1}} (p) = \left[ \int p'' dX \right]^{(q^{-1})}, \]

\[ M = \lim_{q \to 1} N_{q^{-1}} (p) = \exp \left( \int p \ln p dX \right), \quad \int p dX = 1. \]

Let us consider the Abel entropy group with the commutative law of composition of elements \( S_1 \circ S_2 = S_2 \circ S_1 \) and group axioms of associativity \( (S_1 \circ S_2) \circ S_3 = S_1 \circ (S_2 \circ S_3) \), unit element \( S \circ 0 = 0 \circ S = S \), and inverse element \( S \circ S^{-1} = S^{-1} \circ S = 0 \). The unit group element \( I \) corresponds to \( S = 0 \), and the inverse element is designated by \( S^{-1} \).

Let us present a method of derivation of the law of entropy composition [4]. We assume that the law of composition comprises only \( S_1 \) and \( S_2 \) and their product \( S_1 S_2 \) which specifies the quadratic nonlinearity.

Let us write down the law of entropy composition in the general form:
From the commutativity property and associativity condition, equate coefficients of arbitrary $S_2$, $S_1S_2$, $S_2S_3$, $S_3S_1$ and $S_1S_2S_3$, we obtain equalities

\[
\begin{align*}
g(F, S_1)w(S_1, S_2) &= w(S_1, G)g(S_2, S_1), \\
g(F, S_1)h(S_1, S_2) &= h(S_1, G)g(S_2, S_1), \\
h(F, S_1)w(S_1, S_2) &= w(S_1, G)h(S_2, S_1), \\
h(F, S_1)g(S_1, S_2) &= h(S_1, G)w(S_2, S_1), \\
h(F, S_1)h(S_1, S_2) &= h(S_1, G)h(S_2, S_3),
\end{align*}
\]

where $F = S_1 \circ S_2$ and $G = S_2 \circ S_3$. Using equation (4), we obtain law of composition

\[S = \frac{S_1 + S_2 + \varepsilon S_1 S_2}{1 + \omega S_1 S_2},\]

with the parameters $\varepsilon$ and $\omega$ independent of the entropy. From (5) we find the inverse element of the entropy group

\[S^{-1} = -\frac{S}{1 + \varepsilon S}, \quad (1 + \omega S S^{-1} - 1),\]

which is not equal to the opposite element ($-S$).

Let us present the most general properties of the law of entropy composition described by equation (5) that have not yet been considered. Using equation (6), we obtain the following equalities for $S$ and $S^{-1}$:

\[
\begin{align*}
\frac{1}{-S} - \frac{1}{S^{-1}} &= \varepsilon , \quad (1 + \varepsilon S)(1 + \varepsilon S^{-1}) = 1, \\
\frac{1}{1 + \omega SS^{-1}} &= \frac{1 + \varepsilon S}{\varepsilon S^2 - 1 + \varepsilon S^{-1} - \omega(s^{-1})^2},
\end{align*}
\]

For two entropy types $S_1$ and $S_2$, we have

\[
\begin{align*}
(1 + \varepsilon S - \omega S^2) &= \frac{1 + \varepsilon S_1 - \omega S_1^2}{(1 + \omega S_1 S_2)^2}, \\
\frac{S}{\sqrt{1 + \varepsilon S - \omega S^2}} &= \frac{S_1 + S_2 + \varepsilon S_1 S_2}{\sqrt{1 + \varepsilon S_1 - \omega S_1^2} \sqrt{1 + \varepsilon S_2 - \omega S_2^2}}, \\
1 + \varepsilon S_1 - \omega S_1 S &= \frac{1 + \varepsilon S_1 - \omega S_1^2}{1 + \omega S_1 S_2}, \quad 1 + \varepsilon S_2 - \omega S_2 S &= \frac{1 + \varepsilon S_2 - \omega S_2^2}{1 + \omega S_1 S_2}, \\
1 + \varepsilon S - \omega S^2 &= (1 + \varepsilon S_1 - \omega S_1 S)(1 + \varepsilon S_2 - \omega S_2 S) \\
1 + \varepsilon S_1 - \omega S_1 S &= \frac{1 + \varepsilon S_1 - \omega S_1^2}{1 + \varepsilon S_2 - \omega S_2^2}.
\end{align*}
\]

For the three and four entropy types, we have
Using expression (6) for the inverse element, from equation (5) we obtain $S_1$ and $S_2$:

$$S_i = S \circ S_i^{-1} = \frac{S - S_2}{1 + \varepsilon S_2 - \omega SS_2}, \quad S_2 = S_i^{-1} \circ S = \frac{S - S_1}{1 + \varepsilon S_1 - \omega SS_1}. \quad (10)$$

The following law of entropy composition is generally valid:

$$S^{(2 \omega)} = S_i^{(n)} \circ S_2^{(n)} = \frac{S^{(n)} + \varepsilon S_1^{(n)} S_2^{(n)}}{1 + \omega S_1^{(n)} S_2^{(n)}}, \quad (11)$$

where $S^{(n)} = S \circ S \circ \ldots \circ S$ is the composition of $n$ entropy types $S$.

The entropies of the common system $S$ and of two independent systems $S_1$ and $S_2$ are functions of the seminorm distributions of the examined systems, that is,

$$S = S(N), \quad S_1 = S(N_1), \quad S_2 = S(N_2). \quad (12)$$

$$N_1 = N_{q-1}(p_1) = \left[ \int p_1^q dX_1 \right]^{1/(q-1)}, \quad N_2 = N_{q-1}(p_2) = \left[ \int p_2^q dX_2 \right]^{1/(q-1)}, \quad (13)$$

where $p = p_1, p_2$ and $dX = dX_1 dX_2$. They have the multiplicativity property: $N = N_1 N_2$.

Differentiation of the seminorm of distribution (2) with allowance for equation (5) yields the equation

$$\left(1 + \varepsilon S - \omega S^2\right) \frac{d \ln N}{dS} = \lambda, \quad (14)$$

having the same form for all examined systems. Here $\lambda$ is an arbitrary constant which can depend on $\varepsilon$ and $\omega$.

Upon integration of equation (14) we obtain, depending on the determinant $D = \varepsilon^2 + 4 \omega$ of the quadratic equation $1 + \varepsilon S - \omega S^2 = 0$, four types of functions for four entropies considered in [5] and the corresponding types of matrix representations.

2. A matrix representation entropy group

We now assign the following representation of the Abel entropy group. A matrix $A(S)$ from a set of the second-order matrices is juxtaposed to each entropy $S$ of the set of entropies. All matrices juxtaposed to different entropies are different and hence, the group of matrices is isomorphic to the group being represented. The matrices form the Abel group for the binary multiplication operation – the conventional operation of matrix multiplication: $A(S) = A(S_1 \circ S_2) = A(S_1) A(S_2)$. The multiplication operation is commutative $A(S_1) A(S_2) = A(S_2) A(S_1)$ and associative $[A(S_1) A(S_2)] A(S_3) = A(S_1) [A(S_2) A(S_3)]$. The unit element in the condition
\( \mathbf{A} \mathbf{(S)} \mathbf{A}^{-1}(\mathbf{S}) = \mathbf{A}(\mathbf{S}) \mathbf{A}(\mathbf{S}) = \mathbf{A}(0) = \mathbf{E} \) is the unit matrix, and the inverse matrix has the form \( \mathbf{A}^{-1}(\mathbf{S}) = \mathbf{A}(\mathbf{S}) \).

**Type I.** The equation has different real roots, including the case of \( \omega = 0 \). Considering that \( \varepsilon = \varepsilon_1 - \varepsilon_2 \), \( \omega = \varepsilon_1 \varepsilon_2 \) and \( D = (\varepsilon_1 + \varepsilon_2)^2 \), we obtain matrix and entropy

\[
\mathbf{A}(\mathbf{S}) = \left( \begin{array}{cc}
\frac{1 - \varepsilon_2 S}{1 + \varepsilon_1 S} \\
\frac{1 + (\varepsilon_1 - \varepsilon_2)S}{\sqrt{(1 + \varepsilon_1 S)(1 - \varepsilon_2 S)}}
\end{array} \right),
\]

\[
\mathbf{A}(\mathbf{S}) = \left( \begin{array}{cc}
\frac{\varepsilon_1 \varepsilon_2 S}{\sqrt{(1 + \varepsilon_1 S)(1 - \varepsilon_2 S)}} \\
\frac{1}{\sqrt{(1 + \varepsilon_1 S)(1 - \varepsilon_2 S)}}
\end{array} \right),
\]

\[
S = 1 - \left[ \frac{N_{q-1}(p)}{\varepsilon_2 + \varepsilon_1} \right]^{(\varepsilon_1 + \varepsilon_2)/\lambda}.
\]

Entropy values lying within the limits \( -\varepsilon_1^{-1} < S < \varepsilon_2^{-1} \).

From equation (15), the following entropies follow:

a) \( \varepsilon_1 = k^{-1}(1 - q) \), \( \varepsilon_2 = 0 \), \( \lambda = k^{-1} \),

\[
S = \frac{k}{q - 1} \left[ 1 - \int p^q dX \right], \quad (\text{Havrda-Charvat, 1967; Daroczy, 1970}),
\]

b) \( \varepsilon_1 = k^{-1}(\beta - 1) \), \( q = 1/\beta \), \( \lambda = k^{-1} \),

\[
S = \frac{k}{1 - \beta} \left[ 1 - \left( \int p^{1/\beta} dX \right)^\beta \right], \quad (\text{Arimoto, 1971}),
\]

c) \( \varepsilon_1 = k^{-1}(1 - q) \), \( \varepsilon_2 = k^{-1}(1 - r) \), \( \lambda = k^{-1} \),

\[
S = \frac{k}{r - 1} \left[ 1 - \left( \int p^r dX \right)^{r-1} \right], \quad (\text{Sharma-Mittal, 1977}),
\]

\( d) \varepsilon_1 = 0, \varepsilon_2 = k^{-1}(1 - q) \), \( \lambda = k^{-1} \),

\[
S = \frac{k}{q - 1} \left[ 1 - \left( \int p^q dX \right)^{-1} \right], \quad (\text{Landsberg-Vedral, 1998}),
\]

e) \( \varepsilon_1 = \varepsilon_2 = k^{-1}(1 - q) \), \( \lambda = k^{-1} \),

\[
S = \frac{k}{r - 1} \left[ \frac{\int p^r dX - \left( \int p^q dX \right)^{-1}}{\int p^q dX + \left( \int p^q dX \right)^{-1}} \right] = \frac{k}{1 - q} \ln \left( \int p^q dX \right), \quad (\text{Zaripov, 2005}),
\]

f) \( \varepsilon_1 = k^{-1}(q - 1) \), \( \varepsilon_1 = k^{-1}(r - 1) \), \( \lambda = k^{-1} \),

\[
S = \frac{k}{1 - r} \left[ \frac{1 - \left( \int p^r dX \right)^{-1}/q}{1 + \left( \int p^q dX \right)^{-1}/q} \right] = \frac{k}{1 - r} \ln \left( \frac{1 - r}{1 - q} \int p^r dX \right), \quad (\text{Zaripov, 2005}).
\]

It is of interest to note that entropy Landsberg-Vedral is the inverse element in the Havrda-Charvat-Daroczy entropy group given by equation (6).
**Type II.** The equation has two equal real roots with\[ \omega = -\varepsilon^2, \quad \varepsilon = 2k^{-1}(q - 1), \quad \text{and} \quad D = 0. \]
Then we have the matrix and the entropy of the form
\[
A(S) = \exp \left( -\frac{\lambda S}{1 + \varepsilon S} \right) \times \begin{pmatrix}
1 + 2\varepsilon S & -\varepsilon^2 S \\
1 + \varepsilon S & 1 + \varepsilon S
\end{pmatrix},
\]
\[
S = -\frac{\lambda^{-1} \ln N_{q^{-1}}(p)}{1 + \varepsilon \lambda^{-1} \ln N_{q^{-1}}(p)} \frac{k}{1 - q} \left( \ln \int p^q dX \right) \left( 1 + \ln \int p^q dX \right), \quad \text{(Zaripov, 2005).}
\]

**Type III.** The equation has complex roots with \( \omega < 0 \) and \( D < 0 \), and we obtain
\[
A(S) = \exp \left( \frac{\lambda}{\sqrt{-\omega - \varepsilon^2 / 4}} \right) \arctg \left( S \frac{-\omega - \varepsilon^2 / 4}{1 + S \varepsilon / 2} \right) \times \begin{pmatrix}
1 + \varepsilon S & \omega S \\
\sqrt{1 + \varepsilon^2 S - \omega S^2} & \sqrt{1 - \varepsilon S \omega^2} \\
S & 1
\end{pmatrix},
\]
\[
S = \frac{\lambda}{\sqrt{-\omega - \varepsilon^2 / 4}} \frac{\arctg \left( \frac{-\omega - \varepsilon^2 / 4}{\lambda} \right) \ln \left[ N_{q^{-1}}(p) \right]}{\sqrt{\arctg \left( \frac{-\omega - \varepsilon^2 / 4}{\lambda} \right) \ln \left[ N_{q^{-1}}(p) \right]}} / 2.
\]
From equation (17) at \( \varepsilon = 0 \) and \( \omega = -k^{-2}(q - 1)^2 \), \( \lambda = -k^{-1} \) trigonometric entropy follows
\[
S = \frac{k}{1 - q} \left( \ln \left( \int p^q dX \right) \right), \quad \text{(Zaripov, 2005).}
\]

**Type IV.** The trinomial degenerates into unity at \( \varepsilon = \omega = 0 \), and we obtain the matrix
\[
A(S) = \exp \left( -\lambda S \right) \times \begin{pmatrix}
1 & 0 \\
S & 1
\end{pmatrix},
\]
\[
S = \frac{k}{1 - q} \left( \ln \left( \int p^q dX \right) \right). \quad \text{(Renyi, 1960).}
\]
In the limit \( q \to 1 \), we obtain entropy \( S = -k \int p \ln p dX \) (Boltzmann, 1892; Gibbs, 1902; Shannon, 1948) with the additive composition law.

We now assign the following algebraic representation of the Abel entropy group.

**3. Algebraic representation of entropy group. Finsler geometry**
Let us write down a conformally generalized hypercomplex number
\[
R = \varphi \left( S^0, S^i \right) \left( S^0 + S^i \varepsilon_1 \right)
\]
with real numbers of the dimensionless functions of entropies $S^0$ and $S^1$ (or components), basic elements $e_0 = 1$, $e_1$ and conformal multiplier $\varphi(S^0, S^1) = \varphi(S^1/S^0)$. The ratio of the functions gives dimensionless entropy: $S = S^1/S^0$. In equation (21), the basic element $e_0$ by which the number $S^0$ is multiplied has been omitted.

The law of basic element composition is defined in the general form as [6]

$$e_0 o e_1 = e_1 o e_0 = e_1, e_1^2 = \omega + \varepsilon e_1;$$

it has the properties of commutativity and associativity that represent a special case of the Clifford algebra.

Numbers with $\varphi(S^0, S^1) = 1$ and composition law (22) are called generalized hypercomplex numbers; they were studied in [6], [7], and so on. Depending on the discriminant $D = \varepsilon^2 + 4\omega = (\varepsilon - 2e_1)^2$ for the equation $e_1^2 - e_1 - \omega = 0$, numbers of three types can be identified [6]: hyperbolic with $D > 0$, parabolic with $D = 0$, and elliptic with $D < 0$, as well as the nonextensive entropies of three types. The fourth type of numbers is observed when $\varepsilon = \omega = 0$ and corresponds to the dual numbers with $e_1^2 = 0$.

The canonical form of the number with $\varepsilon = 0$ [6] is well-known for complex, dual, and double numbers at $\omega = -1, 0, 1$.

According to equation (21), let us write down the composition law for two conformally generalized hypercomplex numbers [8]

$$R = R_1 o R_2 = \varphi(S^0_1, S^1_1)(S^0_1 + S^1_1 e_1) =
\left[\varphi(S^0_1, S^1_1)(S^0_1 + S^1_1 e_1)\right] o \left[\varphi(S^0_2, S^1_2)(S^0_2 + S^1_2 e_1)\right] =
\varphi(S^0, S^1)\left[\left(S^0_1 S^0_2 + \omega S^1_1 S^1_2 + \varepsilon S^0_1 S^0_2 + \varepsilon S^1_1 S^1_2\right) e_1\right],$$

where the group properties of the function $\varphi(S^0, S^1)$ have been used:

$$\varphi(S) = \varphi(S_1 o S_2) = \varphi(S_1) \varphi(S_2),$$

$$\varphi^{-1}(S) = \varphi(S^{-1}), \varphi(S) \varphi(S^{-1}) = 1, \varphi(0) = 1.$$  

Then we obtain values of the components

$$S^0 = S^0_1 S^0_2 + \omega S^1_1 S^1_2,$$

$$S^1 = S^1_1 S^1_2 + S^0_1 S^0_2 + \varepsilon S^1_1 S^1_2.$$  

As expected, dividing expressions in equation (25) with allowance that $S_1 = S^0_1 / S^0_0$ and $S_2 = S^0_2 / S^0_0$ yields entropy composition law (5).

From composition law (23), we obtain the conjugate conformally generalized hypercomplex number

$$\overline{R} = \varphi(S^0, S^1)(S^0 + \varepsilon S^1, S^1 e_1),$$

which for $\varphi(S^0, S^1) = 1$ coincides with the well-known conjugate number [6]. Using conjugate number (26), we obtain the number modulus

$$|R| = (R o \overline{R})^{1/2} = \varphi(S^0, S^1)\left[\left(S^0\right)^2 + \varepsilon S^0 S^1 + \varphi(S^1)^2\right]^{1/2}$$
and the inverse number $R^{-1} = \frac{\bar{R}}{|R|^2}$ with $R \circ R^{-1} = 1$.

For geometrical representation of the entropy group functions, expression (27) with $|R| = 1$ represents a indicatrix or a curve with radius-vector $R(S^0, S^1)$ and unit length in the global Finsler structure of two-dimensional space of vectors $R(S^0, S^1)$. The angle in the given space is the additive dimensionless Renyi entropy $\alpha_q = (1 - q)^{-1} \ln \left( \int p^q \, dX \right) \ [9]$. Expression $|R(S^0, S^1)|$ is the metric function of the Finsler geometry [10].

The vectors of the examined two-dimensional geometries have the algebraic representation presented above.

According to composition law (25), the components of numbers are calculated in terms of matrices

$$e^{-\nu \alpha}(S^0, S^1) = e^{-\nu \alpha_1}(S^0_1 + \epsilon S^1_1, \omega S^1_1) e^{-\nu \alpha_2}(S^0_2, S^1_2),$$

(28)

where $\alpha_q = \alpha_{q_1} + \alpha_{q_2}$. Matrix

$$A(S^0, S^1) = e^{-\nu \alpha}(S^0 + \epsilon S^1, \omega S^1)$$

(29)

called the characteristic matrix and obtained by the well-known method according to the composition law of basic elements (22) was also introduced in [6] for the generalized hypercomplex numbers with $\nu = 0$. Transformation (28) between vectors of the product $R(S^0, S^1) = A(S^0, S^1)R(S^0_1, S^1_1)$ leaves the form-invariant the expression for the indicatrix of independent systems.

Thus, the Finsler geometry of three types corresponds to the conformally generalized hypercomplex numbers of three types, and the fourth type corresponds to the Galilean geometry. According to the general classification of entropy groups, we have for type I and type III the corresponding well-known expressions [9] of the function $\phi(S^0, S^1)$, and as a result, we obtain the conformally generalized hypercomplex numbers

$$R = \left( \frac{S^0 + \gamma S^1}{S^0 - \gamma S^1} \right)^{\nu/\nu_1} (S^0 + S^1 \epsilon_1),$$

(30)

$$\epsilon_1^2 = \epsilon_1 \epsilon_2 + (\epsilon_1 - \epsilon_2) \epsilon_1, \ \ (\omega = \epsilon_1 \epsilon_2, \ \epsilon = \epsilon_1 \ \epsilon_2),$$

$$R = \left\{ \exp \left[ \frac{2\nu}{\epsilon_1 + \epsilon_2} \arctan \left( \frac{S^1(S_1 + \epsilon_1)/2}{S^0 + S^1(S_1 + \epsilon_1)/2} \right) \right] \right\} (S^0 + S^1 \epsilon_1),$$

(31)

$$\epsilon_1^2 = -\frac{\epsilon_1^2 + \epsilon_2^2}{2} + (\epsilon_1 - \epsilon_2) \epsilon_1, \ \ \ (\omega = -\frac{\epsilon_1^2 + \epsilon_2^2}{2}, \ \ \epsilon = \epsilon_1 \ \ \epsilon_2).$$

The Galilean geometry corresponds to the conformally generalized dual number

$$R = \left[ \exp \left( \nu S^1 / S^0 \right) \right] (S^0 + S^1 \epsilon_1), \ \ e_1^2 = 0 \ (\omega = \epsilon = 0)$$

(32)

with the additive law of entropy composition

$$S = S_1 + S_2.$$  

(33)
Finally, after introduction of the angle $\alpha_q$, the conformally generalized hypercomplex number (21) is written as follows:

$$R = \left[ \exp\left(\nu \alpha_q \right) \right] \left( S^0 + S^i e_i \right).$$  \hspace{1cm} (34)

According to equation (21), the algebraic representation of the entropy group is given by the conformally generalized hypercomplex numbers of the form

$$R = \varphi(S) \frac{(1 + S e_i)}{\sqrt{1 + \varepsilon S - \omega S^2}}.$$  \hspace{1cm} (35)

The law of composition of two numbers is written in the form

$$R = R_1 \circ R_2 = \left[ \varphi(S_1) \frac{(1 + S_1 e_i)}{\sqrt{1 + \varepsilon S_1 - \omega S_1^2}} \right] \circ \left[ \varphi(S_2) \frac{(1 + S_2 e_i)}{\sqrt{1 + \varepsilon S_2 - \omega S_2^2}} \right]$$

$$= \varphi(S) \frac{(1 + S e_i)}{\sqrt{1 + \varepsilon S - \omega S^2}}.$$  \hspace{1cm} (36)

and yields the well-known equalities [9]

$$\frac{1}{\sqrt{1 + \varepsilon S - \omega S^2}} = \frac{1}{\sqrt{1 + \varepsilon S_1 - \omega S_1^2}} \cdot \frac{1}{\sqrt{1 + \varepsilon S_2 - \omega S_2^2}}$$

$$+ \omega \frac{S_1}{\sqrt{1 + \varepsilon S_1 - \omega S_1^2}} \cdot \frac{S_2}{\sqrt{1 + \varepsilon S_2 - \omega S_2^2}},$$

$$S = \frac{S_1 + S_2 + \varepsilon S_1 S_2}{\sqrt{1 + \varepsilon S_1 - \omega S_1^2} \sqrt{1 + \varepsilon S_2 - \omega S_2^2}}.$$  \hspace{1cm} (37)

The conjugate element and the modulus of the conformally generalized hyperbolic number have the following forms:

$$\bar{R} = \varphi(S) \left( 1 + \varepsilon S - \omega S^2 \right),$$

$$|R| = \varphi(S) \left( 1 + \varepsilon S - \omega S^2 \right)^{1/2}.$$  \hspace{1cm} (38)

4. The functions of hyperbolic entropy and Havrda-Charvat-Daroczy entropy

According to equation (30), we have the algebraic representation of functions of hyperbolic entropy

$$R = \left[ \frac{S^0 + (1 - q) S^1}{S^0 - (1 - q) S^1} \right]^{\nu/2 \left( 1 - q \right) \left( 1 - q \right) \alpha_q},$$

$$e_i^0 = (1 - q)^2, e_i^1 = (1 - q)^2 \varphi \left( \nu \alpha_q \right).$$  \hspace{1cm} (39)

and Havrda-Charvat-Daroczy entropy:

$$R = \left[ \frac{S^0 + (1 - q) S^1}{S^0 - (1 - q) S^1} \right]^{1 - \nu \alpha_q} \left( S^0 + S^i e_i \right),$$

$$e_i^0 = (1 - q)^2 e_i, e_i^1 = (1 - q) e_i, e_i^2 = (1 - q)^2 / 2.$$  \hspace{1cm} (40)
in type I and the corresponding indicatrix of the Finsler geometries for dimensional entropy functions have the following form:

\[
\frac{S^0 + (1 - q)S^1}{S^0 - (1 - q)S^1} \left[ \left( S^0 \right)^2 - (1 - q)^2 \left( S^1 \right)^2 \right] = 1, \quad r = \nu/(1 - q),
\]

\[
\frac{S^0 + (1 - q)S^1}{S^0 - (1 - q)S^1} \left[ \left( S^0 \right)^2 - (1 - q)S^0S^1 \right] = 1, \quad r = 2\nu/(1 - q),
\]

where \( r \) can have a constant value or depend on \( q \).

Let us consider the case with \( \nu = 0 \) and rewrite equation (42) in the form

\[
\left[ S^0 + (1 - q)S^1 \right]/2 - \left[ (1 - q)/2 \right] \left( S^0S^1 \right)^2 = 1.
\]

Then transition to the oblique-angled system of coordinates described by the system of equations

\[
S'' = S^0 + (1 - q)S^1/2 = \text{ch}\left[ (1 - q)\alpha_q/2 \right],
\]

\[
S'' = S^1 = 2\text{sh}\left[ (1 - q)\alpha_q/2 \right],
\]

leads equation (43) to the canonical form

\[
\left( S'' \right)^2 - \left( \frac{1 - q}{2} \right)^2 \left( S'' \right)^2 = 1,
\]

coinciding with expression (41) for the pseudo-Euclidean geometry. Thereby, the transition with linear transformations of the entropy functions and Havrda-Charvat-Daroczy entropy to form hyperbolic entropy proceeds by the dimensional formula

\[
S' = \frac{S}{2[1 + (1 - q)S]/2} = \frac{2k}{1 - q} \text{th} \left[ \frac{1}{2} \ln \left( \int p^q dX \right) \right].
\]

In the limit \( q \to 1 \), we obtain Boltzmann-Gibbs-Shannon entropy \( S' = -k \int p \ln p dX \).

In the algebraic approach, such transition corresponds to the linear transformations of the basic elements

\[
e' = e_0, \quad e'_1 = \frac{1}{2}[(1 - q)e_0 + _1].
\]

In this case, the composition law for new basic elements is written as follows:

\[
e_0 \circ e_i = e_i \circ e_0 = e_i, \quad e_0^2 = 1, \quad e_i^2 = \left( \frac{1 - q}{2} \right)^2.
\]

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