O(3) Non-linear $\sigma$ model with Hopf term and
Higher spin theories

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Abstract
Following our earlier work we argue in detail for the equivalence of the nonlinear $\sigma$ model with Hopf term at $\theta = \pi/2s$ and an interacting spin-s theory. We give an ansatz for spin-s operators in the $\sigma$ model and show the equivalence of the correlation functions. We also show the relation between topological and Noether currents. We obtain the Lorentz and discrete transformation properties of the spin-s operator from the fields of the $\sigma$ model. We also explore the connection of this model with Quantum Hall Fluids.

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1. Introduction:

Interest in the $2 + 1$ dimensional $O(3)$ nonlinear $\sigma$ model (NLSM) with the Hopf term was roused when Wilzcek and Zee\cite{1} pointed out that the Hopf term\cite{2} causes the solitons to acquire fractional spin and statistics. The connection of the NLSM to the long wavelength fluctuations of antiferromagnets had been established by Haldane\cite{3}. Interesting possibilities of the model without the Hopf term has been discussed by Wen and Zee\cite{4}. The Hopf term has not been derived from a microscopic spin model so far\cite{5}. But it is expected on physical grounds that in some classes frustrated antiferromagnets the Hopf-term may actually be generated\cite{6}.

The topological features of this model gives the solitons the fractional statistics and spin. This model was analysed in a canonical framework by Bowick, Karabali and Wijayavardhana\cite{7} wherein they pointed out that addition of Hopf term adds a term to the angular momentum and changes the statistics of the solitons. Later the same model was analysed by Dzyaloshinsky, Polyakov and Wiegmann\cite{8} who showed that in the long wavelength limit the effect of Hopf term is to add a Chern-Simon’s term to the model. Further topological features of the model with torus and arbitrary genus Riemann surface boundary conditions gives the possibilities of fractional statistics only by using multi component wavefunctions\cite{9,10}. Following $CP_1$ formalism Polyakov\cite{11} then showed relativistic particles interacting with an abelian Chern Simons gauge fields (with $\theta = \pi$) are Dirac fermions in the long wavelength limit. This can easily be generalized to higher integer and half odd integer spins as well\cite{12}. Some of us then showed that the above result holds independent of long wavelength approximation\cite{13} and also in the presence of self interactions\cite{14}.

In an earlier paper\cite{15} we argued that this model for $\theta = \frac{\pi}{2s}$, $s = \frac{1}{2}, 1, \frac{3}{2}, \cdots$ is equivalent to an interacting spin $s$ theory. In particular when $s = \frac{1}{2}$ the theory is equivalent to Dirac fermions with four fermi interactions. In this paper we present this
argument in full detail and give further consequences.

In Sec. 2 we present a brief review of the topological features of the model and the origin of the fractional statistics and spin of the solitons. In Sec. 3 we present a local $CP_1$ model involving two gauge fields which is exactly equivalent to $O(3)$ $\sigma$ model with Hopf term. We present the route to higher spin formulation by formally integrating the gauge fields and $CP_1$ fields and show that the partition function is the same as obtained by integrating the higher spin fields in an interacting higher spin theory in Sec. 4. We also establish in Sec. 5 the equivalence by relating the correlation functions and in Sec. 6 bring out the relationship between topological current and the particle (Noether) current of the higher spin theory. Finally we derive the transformation properties of higher spin fields from the known transformation properties of the $CP_1$ fields and gauge fields in Sec. 7. In the final Sec. 8 we conclude with discussions of our work. Appendix A contains path integral representation for spin-s particles while in Appendix B we discuss the transformation properties of the spin-s fields.

2. Topological features of the model:

The model we consider is described by a three dimensional unit vector $n$ field with $n^a n^a = 1$ with $a = 1, 2, 3$ in 2+1 dimensions. It is described by the Euclidean action

$$S = g^2 \int \partial_\mu n^a \partial_\mu n^a + i \theta H(n^a)$$

Here $g$ is the coupling constant and the second term in the action is the Hopf invariant. At any instant of time for finite energy solutions the boundary conditions on the $n$ are such that

$$n \rightarrow (0, 0, 1)$$

Because of this the 2d plane becomes the compact $S^2$. Hence the configuration space for this field theory is

$$Q \equiv \text{Set of maps from } S^2 \rightarrow S^2$$
This space is not path connected as can be seen from the following:

\[ \pi_0(Q) \equiv \pi_2(S^2) = \mathcal{Z} \quad (2) \]

Hence there exists solitons in this model and soliton number is conserved. There exists the topological current given by

\[ j_{\mu}^{\text{top}} = \frac{1}{8\pi} \epsilon_{\mu\nu\lambda} \epsilon_{abc} n_a \partial_\nu n_b \partial_\lambda n_c \quad (3) \]

which is conserved. \( j_0^{\text{top}} \) is the soliton density.

If we consider \( \pi_1(Q) \) then it is easy to see that

\[ \pi_1(Q) \equiv \pi_3(S^2) = \mathcal{Z} \quad (4) \]

Hence the quantization is ambiguous \([14]\) upto a phase \( e^{i\theta} \). The effect of such an ambiguity can be incorporated in the action by adding the second term in (1), the Hopf term.

From the fact that \( j_\mu^{\text{top}} \) is conserved one can write down, atleast in zero soliton sector,

\[ j_\mu^{\text{top}} = \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda \quad (5) \]

Given such a vector potential \( A_\mu \), the Hopf term is given by

\[ H(n) = \int j_\mu^{\text{top}} A_\mu d^3x \quad (6) \]

Note that Hopf term (6) is non local since \( A_\mu \) defined in (5) has to be non local in the \( n \) fields.

For any configuration which evolves in time the Hopf invariant arises in the following way. A typical soliton configuration in the 2d plane be represented by a disc of radius \( R \) centered around some point, say origin. It is given by

\[ n \cdot \tau = \tau^3 \cos f(r) + \hat{x} \cdot \tau \sin f(r) \]

with

\[ f(0) = \pi; \quad f(R) = 0. \]
Such a soliton configuration in time generates a tube. In the zero soliton sector, a soliton and an anti soliton can be created and annihilated as shown in Fig.1.

![Fig.1](image1)

This configuration has Hopf invariant 0. But if we rotate the soliton by $2\pi$ before annihilating we get Hopf invariant 1 corresponding to the configuration in Fig.2.

![Fig.2](image2)

This accounts for the spin of the soliton. The amplitude for this process has a phase $\theta$ compared to that in Fig.1. If we take soliton anti soliton pairs created at two different locations exchange and further annihilation of the solitons give the configuration Fig.3.

![Fig.3](image3)
This can be isotopically mapped onto the previous one and thereby accounting for the statistics of the solitons. Hence the addition of the Hopf term leads to the change of spin and statistics of the solitons of the model. If we had considered the same process with solitons of soliton number $n_s$ the Hopf invariant for configurations in Fig.2 and Fig.3. is $n_s^2$ and hence the spin and statistics of soliton number $n_s$ object is

$$ \text{Spin} = n_s^2 \frac{\theta}{2\pi}, \quad \text{Statistics} = n_s^2 \theta $$

(7)

3. $CP_1$ formalism

We now present the $CP_1$ formalism in which $n$ fields are represented through $CP_1$ fields. This has the advantage that the Hopf term is local in $CP_1$ formalism. We use the fact

$$ S^2 = \frac{SU(2)}{U(1)} $$

and reexpress $n$ fields in terms of $SU(2)$ matrices $U$ through the definition

$$ U^\dagger \tau^3 U = n^a \tau^a $$

Here $\tau$ are Pauli matrices. For convenience we parametrize $U$ through two complex fields $z_1, z_2$.

$$ U = \frac{1}{2g} \begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix} $$

U being an unitary matrix we get,

$$ \sum_{\sigma=1}^2 |z_\sigma|^2 = 4g^2 $$

We also define

$$ L_\mu = i \partial_\mu U U^\dagger, \quad L_\mu^a = \frac{1}{2} tr \tau^a L_\mu $$

In terms of $L_\mu$, the elements of $SU(2)$ algebra, we get

$$ g^2 \partial_\mu n^a \partial_\mu n^a = 4g^2 (L_\mu^1 L_\mu^1 + L_\mu^2 L_\mu^2) = \sum |\partial_\mu z_\sigma|^2 - 4g^2 L_\mu^3 L_\mu^3. $$

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From the definition of $j^\mu_{top}$ we find that

$$j^\mu_{top} = \frac{1}{2\pi} \epsilon_{\mu\nu\lambda} \partial_\nu L^3_\lambda. \quad (8)$$

Hence the Hopf term becomes

$$H(n) = - \frac{1}{4\pi^2} \epsilon_{\mu\nu\lambda} L^3_\mu \partial_\nu L^3_\lambda. \quad (9)$$

The action of the non-linear sigma model can be rewritten in this $CP_1$ formalism as,

$$S_{CP_1} = \int |\partial_\mu z_\sigma|^2 - 4g^2 L^3_\mu L^3_\mu - i\theta \frac{4\pi^2}{\epsilon_{\mu\nu\lambda} L^3_\mu \partial_\nu L^3_\lambda}. \quad (10)$$

Next we introduce auxiliary fields $a_\mu$ and $b_\mu$ as follows:

$$\exp (4g^2 \int L^3_\mu L^3_\mu) = \int_{b_\mu} \exp (4g^2 (2b_\mu L^3_\mu - b_\mu b_\mu)) \quad \exp \left( \frac{i\theta}{4\pi^2} \int \epsilon_{\mu\nu\lambda} L^3_\mu \partial_\nu L^3_\lambda \right) = \int_{a_\mu} \exp \left( \frac{i\theta}{4\pi^2} (2\epsilon_{\mu\nu\lambda} a_\mu \partial_\nu L^3_\lambda - \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda) \right) \quad (11)$$

Notice that left hand side of the second equation is periodic in $\theta$ due to the integer valued Hopf invariant. But the periodicity of $\theta$ on the right hand side is not manifest. But if we make the shift

$$a_\mu \rightarrow a'_\mu = ca_\mu + (1 - c)L^3_\mu$$

where $c = \left( 1 + \frac{2n\pi}{\theta} \right)^\frac{1}{2}$ then this has the effect of replacing $\theta$ by $\theta + 2n\pi$ in the action for the $a_\mu$ field and one recovers periodicity.

We also have the constraint

$$z^\dagger z = 4g^2$$

This constraint is introduced through the lagrange multiplier $\lambda$ in the action as

$$\lambda (z^\dagger z - 4g^2)^2.$$ Using all these the $CP_1$ action is given by,

$$S_z = \int z^\dagger G z + S_\eta + S_{GF} \quad (12)$$
where

\[ G = -D_\mu D_\mu - 2i\eta\sqrt{\lambda} \]
\[ D_\mu = i\partial_\mu - A_\mu \]
\[ A_\mu = b_\mu + i\alpha\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda \]
\[ \alpha = \frac{\theta}{16\pi^2 g^2} \]

and

\[ S_{GF} = 4g^2 \int (-2i\alpha\epsilon_{\mu\nu\lambda} b_\mu \partial_\nu a_\lambda + \alpha^2 (\epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda)^2 + i\alpha \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda) \] (13)
\[ S_\eta = \int (\eta^2 - 8i\sqrt{\lambda}\eta g^2) \]

This theory defined by this action is formally exactly equivalent to NLSM in the limit \( \lambda \to \infty \) and we shall work with this action which we refer to as z-theory from now on.

4. Equivalence of partition functions:

First we analyze the partition function of the model. The \( z \) fields, occurring quadratically in the partition function, can be integrated out and we have,

\[ Z = \int e^{-S} = \int_{a_\mu, b_\mu, \eta} \exp \left\{ -2\ln \det G - S_{GF} - S_\eta \right\}. \] (14)

We can use the heat kernel representation of the logarithm of the determinant.

\[ -2 \ln \det G = 2 \int_{\frac{\Lambda}{\pi}}^\infty \frac{d\beta}{\beta} e^{-\beta G} \]
\[ = 2 \int_{\frac{\Lambda}{\pi}}^\infty \frac{d\beta}{\beta} \int_{x(r)} e^{-\int_0^\beta d\tau (\frac{1}{2}(\partial_\tau x^\mu)^2 + V(x) - i \oint A_\mu dx^\mu). \] (15)

where we have defined \( V(x) \equiv 2i\sqrt{\lambda}\eta \) and \( \Lambda \) is the ultraviolet cut off. We see that the dependence of the gauge field in the determinant operator, is of the form of Wilson loop, which is of course expected, due to the gauge invariant nature of the ‘Hamiltonian’ \( G \).
Expanding the \( \exp -2\ln G \) in power series, the partition can be written as

\[
Z = \sum_{n=0}^{\infty} \frac{Z_n[\eta]}{n!} e^{-S_\eta}
\]

where

\[
Z_n[\eta] = \prod_{i=1}^{n} \int_{\Lambda^{-2}}^{\infty} \frac{d\beta_i}{\beta_i} \int Dx_i e^{i \sum_{i=1}^{n} \int_0^{\beta_i} dr \left[ \frac{1}{2} (\partial_t x_i^\mu)^2 + V(x_i) \right]}
\times \int DaDb e^{-i \int C A dx_i - S_{GF}}
\]

This describes a grand canonical ensemble of particles, with \( Z_n \) interpreted as trajectories of \( z \) - particles. First we integrate over the gauge fields. The averaging over gauge fields of an arbitrary product of Wilson loops can be done as follows. Let \( C_i \) represent \( n \) distinct loops and \( C \equiv \bigcup C_i \) corresponding to propagation of \( n \) pairs of particles antiparticles created and destroyed after travelling time \( \beta_i \). The \( b_\mu \) integrals can now done to obtain

\[
\int_{b_\mu} e^{-i \int A_\mu dx^\mu - S_{GF}} = S(j_\mu^c - \frac{\theta}{2\pi^2} \epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda(x))
\times \exp \int_{x} \left[ 2 d_{\mu\nu} a_\mu j_\nu^c - 4 g^2 (d_{\mu\nu} a_\nu)^2 - 4 i g^2 a_\mu d_{\mu\nu} a_\nu \right]
\]

where

\[
\dot{j}_\mu = \sum_{i=1}^{n} \int \partial_\tau x_\mu^\nu \delta^3(x - x_\mu^\nu(\tau))
\]

and \( d_{\mu\nu} = \epsilon_{\mu\lambda\nu} \partial_\lambda \). Now the \( a_\mu \) integrals can be done easily. However, note that, at \( \theta = 0, j_\mu^C = 0 \). This means that the trajectory of the particle along a given curve, is accompanied by that of anti-particle, so that the total current is zero. Thus particles and antiparticles are confined and single \( z \) particle cannot propagate. The presence of the \( '\theta' \) term leads to the possibility of deconfinement of \( z \)-particles, similar to what happens in Chern Simons gauge theories coupled to matter-fields as discussed in [8]. When \( a_\mu \) integrals are also done, we obtain

\[
\exp \left( \int - \frac{i\pi^2}{\theta} j \cdot d^{-1} \cdot j + \frac{1}{16 g^2} j \cdot j \right)
\]
where \( j \cdot d^{-1} \cdot j \equiv j_\mu (\epsilon_{\mu\nu\lambda} \partial_\nu)^{-1} j_\lambda \). Hence we get,

\[
\langle \langle e^{-\oint A_\mu dx^\mu} \rangle \rangle = \exp \left( \frac{i\pi^2}{\theta} \left( \sum_{i=1}^{n} W(C_i) + \sum_{i\neq j}^{n} 2n_{ij} \right) \right)
\]

The first term in this is the well known integral and is given by:

\[
\exp \left( \frac{i\pi^2}{\theta} \left( \sum_{i=1}^{n} W(C_i) + \sum_{i\neq j}^{n} 2n_{ij} \right) \right) \tag{19}
\]

where \( W(C_i) \) is the writhe of the curve \( C_i \) and \( n_{ij} \) is the linking number of the curves \( C_i \) and \( C_j \). For \( \theta = \frac{\pi}{2s} \) the linking number term does not contribute. The writhe \( W(C_i) \) has the expression,

\[
W(C_i) = \frac{1}{2\pi} \Omega(C_i) + 2k + 1 \tag{20}
\]

where \( \Omega(C_i) \) is the solid angle subtended on a 2-sphere traced out by the unit tangent vector to \( C_i \) and \( 2k + 1 \) is an odd integer. Hence

\[
e^{-2\pi is} W(C_i) = (-1)^{2s} e^{-is} \Omega(C_i)
\]

Again linearizing the second term of Eq.(18) and combining the result of integrating \( z, a_\mu \) and \( b_\mu \) we can write the partition function as

\[
Z = \int_{\eta,v_\mu} e^{-S_\eta - \int_x v_\mu v_\mu - 2 \int d^4 x \left( \frac{i\beta \kappa}{\Lambda} \right) + V(x) + (-1)^{2s} \Omega} \tag{21}
\]

The addition of the Polyakov spin factor, \( e^{i s \Omega} \) to the free Bosonic path integral has been studied \[17\] well in recent times, and has been shown to describe spinning particles, with spin \( s \). We have here, a similar set up, except for the inclusion of background scalar \( (V(x)) \) and vector \( (v_\mu) \) fields. In the Appendix-A, we extend this proof, of showing its equivalence to spin-\( s \) particle, in the presence of external fields. Using the results of the appendix, we have

\[
Z_n[\eta] = 2^n \int_{v_\mu} e^{\int v_\mu^2 / 2 + \sum_{i=1}^{n} (-1)^{2s} \int_{\Lambda-2}^{\Lambda} d\Lambda / \Lambda \cdot e^{-L_i D(x)}}
\]

where

\[
L = \frac{\beta \kappa}{\kappa} \quad \kappa = \frac{\sqrt{\pi}}{4\Lambda}
\]
\[ D^{(s)} = sgn(\theta)(i\partial_{\mu} + i\frac{1}{2g}v_{\mu})\frac{T^{\mu}}{s} + M_s + \kappa V(x) \]

\[ M_s = \Lambda^2 \kappa \ln(2s + 1) \]

The mass term \((M_s)\), of \(O(\Lambda)\), arises from \(\epsilon\) the distance between adjacent \(\tau\)-slices, being equal to \(\frac{1}{\Lambda^2}\).

Doing the \(L\) integral, we get

\[ \mathcal{Z}_n = \int e^{\int_{v_{\mu}} v_{\mu}v_{\mu} 2^n (-)^{2s} \ln det(D^s)^n} \]

We can now substitute in Eq.(16) and sum the series to get

\[ \mathcal{Z} = \int e^{\int_{v_{\mu},\eta} v_{\mu}v_{\mu} - \int \eta^2 (-)^{2s+1} det(D^s)^2} \]

Depending on \(2s + 1\) being odd or even, this determinant can be written as functional integral over two-component complex \(c\)-number fields or Grassmanian fields \(\bar{\psi}_{m\sigma}, \psi_{m\sigma}(m = -s \cdots s, \sigma = 1, 2)\) respectively. The two-component isospin index \(\sigma\), comes from the two component nature of the original \(z\) fields. The \(v_{\mu}\) and \(\eta\) integrals can also be performed and we finally obtain for the partition function,

\[ \mathcal{Z} = \int \bar{\psi}_a \psi_a \exp \left\{ \int x \left[ \bar{\psi} \left( \frac{T^{\mu}}{s} i\partial_{\mu} + M \right) \psi + r_1 (\bar{\psi}T^{\mu}\psi)^2 + r_2 (\bar{\psi}\psi)^2 \right] \right\} \quad (22) \]

where

\[ M = sgn(\theta)(M_s - 8g^2\kappa\lambda) \]

\[ r_1 = \frac{1}{16g^2} \]

\[ r_2 = \lambda\kappa^2 \]

Here we have absorbed the factor \(sgn(\theta)\) with the fields. Thus it now appears with the mass term. Thus the partition function of the \(z\)-theory is equal to that of the above spin-\(s\) theory.
5. Equivalence of correlation functions:

In this section we explicitly construct nonlocal fields in the NLSM and prove that their two point correlation functions are exactly equal to the two point correlation functions of the $\psi$ and $\bar{\psi}$ fields in the spin-$s$ theory. The proof can be easily generalised to n-point functions following the methods in reference [13].

Our ansatz for spin-$s$ fields is as below,

$$\chi_{\sigma u}(x) = \zeta[c] \exp \left(i \int_c^x L_\mu^3 dx_\mu \right) \Delta(u_\mu - d_{\mu\lambda} L_\lambda^3(x)) z_\sigma(x)$$
(23)

$$\bar{\chi}_{\sigma u}(x) = \bar{\zeta}[c] \exp \left(-i \int_c^x L_\mu^3 dx_\mu \right) \Delta(u_\mu - d_{\mu\lambda} L_\lambda^3(x)) z^*_\sigma(x)$$
(24)

Here the line integral is over a fixed curve $c$ from some arbitrary point $x_0$ to $x$ and $\zeta[c]$ is a normalisation factor, which depends on the curve $c$ and will be chosen suitably. The $\Delta(u_\mu - d_{\mu\lambda} L_\lambda^3)$ term stands for an angular delta function, which can be got by integrating a Dirac delta function over the radial variables. That is,

$$\Delta(u_\mu - d_{\mu\lambda} L_\lambda^3) = \int_0^\infty u^2 du \delta(u_\mu - d_{\mu\lambda} L_\lambda^3) .$$

Now if we use the standard integral representation for the delta function, we can write,

$$\Delta(u_\mu - \epsilon_{\mu\nu\lambda} \partial_\nu L_\lambda^3) = \int \frac{d^3 k}{(2\pi)^3} \int_0^\infty u^2 du$$
$$\times \exp \left(ik_\mu(u_\mu - \epsilon_{\mu\nu\lambda} \partial_\nu L_\lambda^3) \right) .$$

The role of angular delta function in the ansatz is to relate the spin index of the $\chi$ field to the direction of the topological current at $x$. Now if we define

$$\tilde{j}_\mu^c = \int_c dx_\mu \delta(x - x(\tau)), \quad \text{then} \quad \int_c L_\mu^3 dx_\mu = \int d^3 x \tilde{j}_\mu^c L_\mu^3 .$$

Let us also define $\tilde{k}_\mu = \delta(x - x_i)k_\mu$ so that we can write $k_\mu \epsilon_{\mu\nu\lambda} L_\lambda^3$ as $\int d^3 x \tilde{k}_\mu \epsilon_{\mu\nu\lambda} L_\lambda^3$. Now we can rewrite $\chi_{\sigma u}$ as

$$\chi_{\sigma u} = \zeta[c] \int \frac{d^3 k}{(2\pi)^3} \exp(i k_\mu u_\mu) \exp(-ik_\mu \epsilon_{\mu\nu\lambda} \partial_\nu L_\lambda^3) \exp(i \int d^3 x \tilde{j}_\mu^c L_\mu^3) .$$
(25)
We will evaluate the two point correlation function with this ansatz (eq\[25\]) and show that it is exactly equal to the two point correlation function in the theory of interacting spin-$s$ fields.

\[
\langle \bar{\chi}_{\sigma_1 u_1} (x_1) \chi_{\sigma_2 u_2} (x_2) \rangle = \int Dz Dz^* \exp(-S_{CP}) \bar{\chi}_{\sigma_1 u_1} (x_1) \chi_{\sigma_2 u_2} (x_2) .
\] (26)

Introducing the $a_\mu$ and $b_\mu$ fields as before, we get,

\[
\langle \bar{\chi}_{\sigma_1 u_1} \chi_{\sigma_2 u_2} \rangle = \frac{1}{Z} \bar{\zeta}[c_{01}] \zeta[c_{02}] \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \int u_1^2 du_1 \int u_2^2 du_2 
\exp(-ik_{1\mu}u_{1\mu} + ik_{2\mu}u_{2\mu}) \int Dz Dz^* \delta (z^*_\sigma z_\sigma - 4g^2) z^*_\sigma_1 z_\sigma_2
\times \exp(-i \int d^3x (j_{01}^\mu - j_{02}^\mu) L_3^\mu)
\times \exp \left( i(k_{1\mu} - k_{2\mu}) \epsilon_{\mu\nu\lambda} \partial_\nu L_3^\lambda \right) \exp \left( - \int d^3x \partial_\mu z^*_\sigma \partial_\mu z_\sigma \right)
\int Da Db \ exp \left( 4g^2 \int d^3x (2b_\mu L_3^\mu - b_\mu b_\mu) \right)
\times \exp \left( \frac{i\theta}{4\pi} \int d^3x \ (2a_\mu d_\mu L_3^\lambda - a_\mu d_\mu a_\lambda) \right) .
\] (27)

Where $c_{01}$ and $c_{02}$ are curves starting from some point $x_0$ and ending at $x_1$ and $x_2$ respectively. We now make the shifts,

\[
b_\mu \rightarrow b_\mu + \frac{i}{8g^2} (j_{01}^\mu - j_{02}^\mu) ,
\]

and

\[
a_\mu \rightarrow a_\mu - \frac{2\pi^2}{\theta} (\bar{k}_{1\mu} - \bar{k}_{2\mu}) .
\]

The equation 27 can now be written as,

\[
\langle \bar{\chi}_{\sigma_1 u_1} \chi_{\sigma_2 u_2} \rangle = \bar{\zeta}[c_{01}] \zeta[c_{02}] \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \int u_1^2 du_1 \int u_2^2 du_2 e^{-ik_{1\mu}u_{1\mu} + ik_{2\mu}u_{2\mu}}
\int Dz Dz^* \delta (z^*_\sigma z_\sigma - 4g^2) z^*_\sigma_1 z_\sigma_2
\times \int Da Db e^{-\int d^3x (z^*_\mu (i\partial_\mu - A_\mu)(i\partial_\mu - A_\mu)z_\sigma - S_{GF})}.
\]
where we have dropped the terms that are zero for \( x_1 \neq x_2 \). We now introduce the \( \eta \) fields as before and do the \( z_\sigma \) integrals to get,

\[
\langle \bar{\chi}_{\sigma_1 u_1} \chi_{\sigma_2 u_2} \rangle = \bar{\zeta} [c_{01}] [c_{02}] \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} \int u_1^2 du_1 \int u_2^2 du_2 \\
\times \exp(-ik_{1\mu} u_{1\mu} + ik_{2\mu} u_{2\mu}) \\
\times \int D a D b D \eta \exp(-S_{GF} - S_\eta) \langle x_2 | \frac{1}{G} | x_1 \rangle [\det G]^{-2}
\]

(28)
apart from irrelevant constants. The propagator \( G^{-1} \) has a path integral representation as follows,

\[
\langle x_2 | \frac{1}{G} | x_1 \rangle = \langle x_2 | \int_{\frac{1}{\Lambda^2}}^{\infty} \frac{d\beta}{\pi^2} \left( \exp \left(-\beta G \right) \right) | x_1 \rangle \\
= \int_{0}^{\infty} d\beta \int D x e^{-\int_{0}^{\beta} d\tau (\frac{1}{4} \dot{x}^2 - 2i\eta \sqrt{\lambda})} e^{-i \int_{c_{12}} A_{\mu} dx_\mu}
\]

where the path integral is over an open path \( c_{12} \) from \( x_1 \) to \( x_2 \). For the determinant in (eq.28), as was done earlier, we have,

\[
[\det G]^{-2} = \exp(-2 \ln \det G) = \sum_{n=0}^{\infty} \frac{2^n}{n!} (-\text{tr} \ln G)^n
\]

where,

\[
-\text{tr} \ln G = \int_{1/\Lambda^2}^{\infty} \frac{d\beta}{\beta} \int D x e^{-\int_{0}^{\beta} d\tau (\frac{1}{4} \dot{x}^2 - 2i\eta \sqrt{\lambda})} e^{-i \int_{c_{12}} A_{\mu} dx_\mu}
\]

where the path integral is over all closed loops. Let us note that in both the above expressions the gauge field dependence comes through Wilson loop integrals. The next step is to do the gauge field integrals. For that we expand the “det” term as a series.

Now if we collect all the Wilson loop terms a typical term would be of the form,

\[
\exp \left(-i \int_{c_{12}} A_{\mu} dx_\mu\right) \exp \left(-i \left( \oint_{c_{1}} + \oint_{c_{2}} + \cdots + \oint_{c_{n}} \right) A_{\mu} dx_\mu\right)
\]

Let us note that except for the first integral (over \( c_{12} \)) all other terms are integrals over closed loops. Terms like these are to be substituted in the expression for the correlation
function and integrated over the gauge fields. That is our next step. Collecting all the terms involving \( a_\mu \) and \( b_\mu \) we get,

\[
\int \mathcal{D}a \mathcal{D}b \exp \left( - \int d^3x \left( 4g^2 \left( -2i\alpha b_\mu d_\mu a_\lambda + \alpha^2 (d_\mu a_\lambda)^2 + i\alpha a_\mu d_\mu a_\lambda \right) \right) \right) \\
\times \exp \left( -4g^2 \int d^3x \left( \frac{i}{4g^2} b_\mu (j_{\mu}^{c_{01}} - j_{\mu}^{c_{02}}) \right) \right) \\
\times \exp \left( -\frac{i\theta}{4\pi^2} \int d^3x \left( -\frac{4\pi^2}{\theta} (\tilde{k}_{1\mu} - \tilde{k}_{2\mu})\varepsilon_{\mu\nu\lambda}\partial_\nu a_\lambda \right) \right) \\
\exp \left( -i \int_{c_{12}} A_\mu dx_\mu \right) \exp \left( -i \left( \oint_{c_1} + \oint_{c_2} + \cdots + \oint_{c_n} \right) A_\mu dx_\mu \right) \\
\right)
\] (29)

Now let us introduce the following abbreviations. Let \( c = \cup c_i \) and \( c_{012} = c_{01} \cup c_{12} \cup c_{20} \) where \( c_{20} \) stands for \( c_{02} \) traversed in the opposite direction. Not that \( c_{012} \) stands for a closed curve consisting of two fixed curves \( (c_{01} \) and \( c_{20} \)) coming from the ansatz and a “fluctuating part” coming from the path integral for the matrix element of the propagator.

With this the above expression (29) can be simplified to:

\[
\int \mathcal{D}a \mathcal{D}b \exp \left( - \int d^3x \left( 4g^2 \left( -2i\alpha b_\mu d_\mu a_\lambda + \alpha^2 (d_\mu a_\lambda)^2 + i\alpha a_\mu d_\mu a_\lambda \right) \right) \right) \\
\times \exp \left( -i \int d^3x \left( b_\mu (j_{\mu}^{c_{01}} - j_{\mu}^{c_{02}}) \right) \right) \exp \left( +i \int d^3x \left( (\tilde{k}_{1\mu} - \tilde{k}_{2\mu})\varepsilon_{\mu\nu\lambda}\partial_\nu a_\lambda \right) \right) \\
\exp \left( -i \int d^3x j_{\mu}^{c_{12}} A_\mu \right) \exp \left( -i \int d^3x j_{\mu}^c A_\mu \right) \\
\right)
\]

where

\[
\begin{align*}
\hat{j}_i^c &= \sum_i \int_{c_i} dx_\mu^i \delta(x_\mu - x_\mu^i) \\
j_{\mu}^{c_{012}} &= j_{\mu}^{c_{01}} - j_{\mu}^{c_{02}} + j_{\mu}^{c_{12}} \\
j_{\mu}^{c_{12}} &= \int_{c_{012}} dx_\mu' \delta(x_\mu - x_\mu')
\end{align*}
\]

Now doing the \( b \) integrals (remembering that \( A_\mu = b_\mu + i\alpha d_\mu a_\nu \)) we get,

\[
\int \mathcal{D}a \delta \left( -8g^2 \alpha d_\mu a_\nu - j_{\mu}^{c_{012}} - j_{\mu}^c \right) \\
\times \exp \left( - \int d^3x \left( 4g^2 \left( +\alpha^2 (d_\mu a_\lambda)^2 + i\alpha a_\mu d_\mu a_\lambda \right) \right) \right)
\]
\[ \times \exp \left( i \int d^3 x \left( (\bar{\kappa}_{1\mu} - \bar{\kappa}_{2\mu})\epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda \right) \right) \]
\[ \exp \left( \alpha \int d^3 x j^{c_{12}} j_\mu a_\nu \right) \exp \left( \alpha \int d^3 x j^c_\mu d_{\mu\nu} a_\nu \right) \]

Now we can use the delta function connecting the "a" flux and the particle current to do the \( a_\mu \) integral. Solving the delta function condition for \( a_\mu \) we get,

\[ d_{\mu\nu} a_\nu = \frac{2\pi^2}{\theta} (j^{c_{012}} + j^c_\mu) \]

Doing the \( a_\mu \) integral we get,

\[ \exp \left( \frac{1}{16g^2} \int d^3 x \left( j^c_\mu + j^{c_{12}} \right)^2 \right) \]
\[ \exp \left( -\frac{i\pi^2}{\theta} \int d^3 x (j^{c_{012}} + j^c_\mu) [d^{-1}]_{\mu\nu} (j^{c_{012}} + j^c_\nu) \right) \]
\[ \times \exp \left( i \int d^3 x \left( (\bar{\kappa}_{1\mu} - \bar{\kappa}_{2\mu}) \frac{2\pi^2}{\theta} (j^{c_{012}} + j^c_\nu) \right) \right) \]

Of the terms above,

\[ \exp \left( -\frac{i\pi^2}{\theta} \int d^3 x (j^{c_{012}} + j^c_\mu) [d^{-1}]_{\mu\nu} (j^{c_{012}} + j^c_\nu) \right) \]
\[ = (-1)^2 s \exp(is\Omega[c_{012}])(-1)^2 s \exp(is\Omega[c]) \exp \left( -\frac{i\pi^2}{\theta} 2n[c_{012}, c] \right) \]

Here \( n[c_{012}, c] \) is the value of the Gauss integral for the curves \( c_{012} \) and \( c \) and is equal to an integer. Thanks to this the last term in the above expression can be dropped for the special values of \( \theta = \pi/2s \) that we have chosen. Also by introducing an auxiliary field \( v \) we can write,

\[ \exp \left( \frac{1}{16g^2} \int d^3 x \left( j^c_\mu + j^{c_{12}} \right)^2 \right) \]
\[ = \int Dv \exp \int d^3 x \left( -v_\mu v_\mu - \frac{1}{2g} (j^c_\mu + j^{c_{12}}) v_\mu \right) \]

As was argued earlier these ensure factorisation of Wilson loop integrals. So putting together all terms, we see that the effect of \( a_\mu \) and \( b_\mu \) integrals is that,

\[ \exp \left( -i \oint_a A_\mu dx_\mu \right) \rightarrow (-1)^2 s \exp (is\Omega[c]) \]
So the inverse of $G$ averaged over the gauge fields becomes,

$$
\langle x_2 | \frac{1}{G} | x_1 \rangle \rightarrow \langle x_2 | \int_{\frac{1}{\beta}}^{\infty} \frac{d\beta}{\beta} \left( \int D\!x \: e^{-\int_{0}^{\beta} d\tau \left( \frac{1}{4} \dot{x}^2 - 2i\eta \sqrt{\lambda} \right)} \right) (-1)^{2s} e^{is\Omega[c_{012}]} \exp \left( -\int_{0}^{\beta} d\tau \left( \frac{1}{4} \dot{x}^2 + 2i\eta \sqrt{\lambda} \right) \right) e^{-\frac{1}{2g} \int d^3x \: v_{\mu} j_{\mu}^{c12}} \rangle.
$$

and,

$$
-\text{tr} \ln G \rightarrow \int_{\frac{1}{\beta}}^{\infty} \frac{d\beta}{\beta} \left( \int D\!x \: \exp \left( -\int_{0}^{\beta} d\tau \left( \frac{1}{4} \dot{x}^2 - 2i\eta \sqrt{\lambda} \right) \right) (-1)^{2s} e^{is\Omega[c]} \right) - \exp \left( -\frac{1}{2g} \int d^3x \: v_{\mu} j_{\mu}^{c} \right).
$$

Now if we substitute all these in the expression for the correlation function (see Eq.(28)) the $k_1$, $k_2$, $u_1$, and $u_2$ integrals will give,

$$
\Delta \left( u_{1\mu} - j_{\mu}^{c012}(x_1) - j_{\mu}^{c}(x_1) \right) \Delta \left( u_{2\mu} - j_{\mu}^{c012}(x_2) - j_{\mu}^{c}(x_2) \right)
$$

If we assume the paths not to intersect

$$
j_{\mu}^{c}(x_1) = j_{\mu}^{c}(x_2) = 0
$$

Also we can write,

$$
\exp (is\Omega[c_{012}]) = \exp (is\Omega[c_{01}]) \exp (is\Omega[c_{12}]) \exp (-is\Omega[c_{02}])
$$

Now let us consider:

$$
\int_{\frac{1}{\beta}}^{\infty} \frac{d\beta}{\beta} \int D\!x \Delta \left( u_{1\mu} - j_{\mu}^{c012}(x_1) \right) \Delta \left( u_{2\mu} - j_{\mu}^{c012}(x_2) \right) \exp \left( -\int_{0}^{\beta} d\tau \left( \frac{1}{4} \dot{x}^2 - 2i\eta \sqrt{\lambda} \right) \right) (-1)^{2s} e^{is\Omega[c_{012}]} e^{-\frac{1}{2g} \int d^3x \: v_{\mu} j_{\mu}^{c12}}.
$$

If we now introduce velocity variables through delta functions etc. as was done in appendix A, we will get this equal to

$$
\langle x_2 | u_2 | \frac{1}{\text{sgn}(\theta) \sqrt{\pi}} \left( i\partial_{\mu} \left( \frac{1}{2g} v_{\mu} \right) \frac{T_0}{s} + \frac{\ln(2s+1)}{\epsilon} - 2i\eta \sqrt{\lambda} \right) | u_1 \rangle | x_1 \rangle
$$

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using the results proved earlier we have,

for integer $s$

$$ [\det G]^{-2} \to \left( \det \left( \frac{4}{\sqrt{\pi \epsilon}} \text{sgn}(\theta)(i\partial_\mu - \frac{1}{2g}v_\mu) \frac{T_\mu}{s} + \frac{\ln(2s + 1)}{\epsilon} - 2i\eta \sqrt{\lambda} \right) \right)^{-2} $$

and for half-odd integer $s$

$$ [\det G]^{-2} \to \left( \det \left( \frac{4}{\sqrt{\pi \epsilon}} \text{sgn}(\theta)(i\partial_\mu - \frac{1}{2g}v_\mu) \frac{T_\mu}{s} + \frac{\ln(2s + 1)}{\epsilon} - 2i\eta \sqrt{\lambda} \right) \right)^2 $$

So if we substitute everything in the expression for the 2-point correlation function we have:

$$ \langle \bar{\chi}_{\sigma_1u_1} \chi_{\sigma_2u_2} \rangle = \zeta[c_01] \zeta[c_02] \exp(is\Omega[c_01]) \exp(-is\Omega[c_02]) $$

$$ \int D\eta \exp(-S_\eta) \exp\left( -\int d^3x v_\mu v_\mu \right) \int D\bar{\psi} D\psi \bar{\psi}(x_1u_1)\psi(x_2u_2) $$

$$ \exp\left( -\int d^3x \bar{\psi}\left( \text{sgn}(\theta) \frac{4}{\sqrt{\pi \epsilon}} \left(i\partial_\mu - \frac{1}{2g}v_\mu\right) \frac{T_\mu}{s} + \frac{\ln(2s + 1)}{\epsilon} - 2i\eta \sqrt{\lambda} \right) \psi \right) $$

Choosing $\zeta[c_01] \equiv (\text{sgn}(\theta) \frac{\sqrt{\pi \epsilon}}{4})^2 \exp(is\Omega[c_01])$, redefining $\psi$ to absorb $\text{sgn}(\theta) \frac{1}{\sqrt{\pi \epsilon}}$ in the kinetic energy term and doing the $v_\mu$ and $\eta$ integrals, we obtain our result,

$$ \langle \bar{\chi}_{\sigma_1u_1}(x_1) \chi_{\sigma_2u_2}(x_2) \rangle = \int D\bar{\psi} D\psi \bar{\psi}_{\sigma_1u_1}(x_1)\psi_{\sigma_2u_2}(x_2)e^{-S_\psi[\bar{\psi},\psi]} \tag{30} $$

Where $S_\psi$ is the spin-s theory action defined earlier in equation 22. Thus we have established that the two point correlation functions of the $\chi$ operators in the NLSM as defined in equation 26 are equal to the two point functions of the $\psi$ operators in the spin-s theory.

6. Current Correlators

Having established the equivalence of two point functions now we shall derive certain relationships between the current-current correlation functions in the topological current in the NLSM and the Noether current in the spin-s theories. To do this we will consider the partition function of the NLSM with an external gauge field coupled to the topological
current, i.e. the generating functional for the topological current correlators. In the spin-s theory representation, it turns out to be related to the generating functional for the Noether current correlators. In this way we are able to relate the two correlators.

We have,
\[ j_{\mu}^{\text{top}}(x) \equiv \frac{1}{2\pi} \epsilon_{\mu \nu \lambda} \partial_{\nu} L^3_{\lambda} \, . \]

Now we couple an auxiliary field \( c_{\mu} \) to the topological current and write down the generating functional for the topological current correlators,
\[
Z[c_{\mu}] = \int Dz Dz^* e^{-SCP_1 + \frac{i}{2\pi} \int d^3x c_{\mu} \epsilon_{\mu \nu \lambda} \partial_{\nu} L^3_{\lambda}}. \tag{31}
\]

Now introduce the auxiliary \( a_{\mu} \) and \( b_{\mu} \) fields as before and make the change of variable,
\[ a_{\mu} \rightarrow a_{\mu} - \frac{2\pi^2}{i\theta} c_{\mu} \, . \]
we get,
\[
Z[c_{\mu}] = \int Dz Dz^* Da Db e^{-\tilde{S}_{CP_1} - \frac{\theta}{2\pi} \int d^3x \left( \frac{2\pi}{\theta} \epsilon_{\mu \nu \lambda} a_{\mu} \partial_{\nu} c_{\lambda} - (\frac{\theta}{2\pi})^2 \epsilon_{\mu \nu \lambda} c_{\mu} \partial_{\nu} c_{\lambda} \right)} \, .
\]

Now we can repeat the procedure in section 5. and get,
\[
Z[c_{\mu}] = \int D\bar{\psi} D\psi \ e^{-S_{\psi} - \int d^3x \left( 2sj_{\mu}^N c_{\mu} + \frac{s}{2} \epsilon_{\mu \nu \lambda} c_{\mu} \partial_{\nu} c_{\lambda} \right)} \, . \tag{32}
\]

We can now relate the current correlators by differentiating equations 31 and 32 with respect to \( c_{\mu} \) and then evaluating at \( c_{\mu} = 0 \). By differentiating once we get,
\[
\langle j_{\mu}^{\text{top}}(x) \rangle = 2s \langle j_{\mu}^N(x) \rangle \, . \tag{33}
\]

Thus we see that the soliton current is proportional to the the Noether current of spin-s theory. The relation also tells us that the spin-s particle corresponds to a soliton number \( 2s \) object. This is consistent with the spin of the particle being \( \frac{1}{2} (2s)^2 \frac{1}{2s} = s \) as expected from the NLSM for a soliton number \( 2s \) object (see equation 7). Similarly the statistics is also consistent. This relation also tells us that if the \( \psi \) particles were charged then the charge of the solitons is \( \frac{1}{2s} \) times the charge of a particle.
Next we can differentiate twice and obtain the following relation between the current-current correlators,

\[ \langle j^\text{top}_\mu(x)j^\text{top}_\nu(y) \rangle = (2s)^2 \langle j^N_\mu(x)j^N_\nu(y) \rangle + \frac{iS}{\pi} \epsilon_{\mu\nu\lambda} \partial_\lambda \delta^3(x - y) \]  
(34)

This then shows that the two currents are not exactly the same but differ in correlation functions involving products of currents at the same space-time points. The significance of the above relation is discussed later.

7. Lorentz and discrete transformations:

In this section we will show that our ansatz for spin-s fields has the correct transformation properties under Lorentz and discrete tranformations. We will do this by deriving the transformation properties of the correlation functions of the \( \chi \) fields and showing that it is the same as the transformation properties of the correlation functions of the spin-s fields. The transformation properties of the spin-s fields in the coherent state basis that we are using is not standard. We have therefore discussed them in detail in Appendix B.

1. Proper Lorentz rotations

In a Lorentz rotated frame, we would have \( x' = \Lambda x \) and \( u' = \Lambda u \). This implies

\[ c' = \Lambda c \quad \text{and} \quad C' = \Lambda C \]

where \( \Lambda c \) is the curve obtained by making Lorentz transformation to every point on \( c \). Under Lorentz transformations we also have,

\[ z'_\sigma(x') = z_\sigma(x) \]

\[ L^3_\mu(x') = (\Lambda^{-1}L^3(x))_\mu \]

\[ (dL^3)'_\mu(x') = (\Lambda dL^3(x))_\mu \]

The two point function in the Lorentz rotated frame is

\[ \langle \chi^\prime_{u'\sigma}[x', c']\bar{\chi}^\prime_{u'\sigma}[\bar{x}', \bar{c}'] \rangle. \]
Now we have

\[ \Omega[C', u^*] = \Omega[\Lambda C, \Lambda u^*] = \Omega[C, u^*]. \]

Also we have

\[
\int_c L^3_\mu dx'_\mu = \int_c L^3_\mu dx_\mu \\
\Delta \left( u' - (dL^3(x')) \right) = \Delta \left( \Lambda u - \Lambda dL^3(x) \right) = \Delta \left( u - (dL^3(x)) \right)
\]

and

\[
z'_\sigma(x') = z_\sigma(x) \quad z^*_\sigma(x') = z^*_\sigma(x).
\]

The action is invariant and

\[
\langle \psi'_{\omega'\theta}[x'c'] \bar{\psi}'_{\omega'\theta}([x'\bar{c}']) = \langle \psi_{\omega\theta}[x, c] \bar{\psi}_{\omega\theta}[\bar{x}, \bar{c}] \rangle.
\]

This then implies,

\[
\langle \psi'_{\omega'\theta}(x') \bar{\psi}'_{\omega'\theta}(\bar{x}') = \langle \psi_{\omega\theta}(x) \bar{\psi}_{\omega\theta}(\bar{x}) \rangle.
\]

This is the correct transformation of the spin-s fields under proper Lorentz transformations as has been shown in equation (B.7).

2. Parity

In a parity transformed frame, we have

\[
x' = -x \quad u' = -u \quad c' = -c \quad C' = -C
\]

\[
z'_\sigma(x') = z_\sigma(x) \quad z^*_\sigma(x') = z^*_\sigma(x)
\]

\[
L^3_\mu(x') = -L^3_\mu(x) \quad dL^3_\mu(x') = dL^3_\mu(x)
\]

But now the action is not invariant since the Hopf term changes sign. Thus in the parity transformed frame we have,

\[
\theta' = \frac{-\pi}{2s} = \frac{\pi}{2s'}
\]
This then implies that,

\[ s'\Omega[C', u'^*] = (-s)\Omega[-C, -u^*] = s\Omega[C, u^*] \]

\[ \Delta \left( u' - dL^3(x') \right) = \Delta \left( (-u) - dL^3(x) \right) \]

\[ \int_{x'} L^3_{\mu} dx^\mu = \int_{c} L^3_{\mu} dx^\mu \]

\[ \sqrt{\text{sgn}(\theta')} = i\sqrt{\text{sgn}(\theta)} \]

We then have,

\[ \langle \chi'_{u'\sigma}[x', c'] \bar{\chi}'_{\bar{u}'\bar{\sigma}}[\bar{x}', \bar{c}'] \rangle_{s'} = -\langle \chi_{-u'\sigma}[x, c] \bar{\chi}_{-\bar{u}\bar{\sigma}}[\bar{x}, \bar{c}] \rangle_{s} \]

This then implies

\[ \langle \psi'_{u'\sigma}[x', c'] \bar{\psi}'_{\bar{u}'\bar{\sigma}}[\bar{x}', \bar{c}'] \rangle_{s'} = -\langle \psi_{-u'\sigma}[x] \bar{\psi}_{-\bar{u}\bar{\sigma}}[\bar{x}] \rangle_{s_F} \]

where the subscripts refer to the action. \( S'_{F} \) differs from \( S_{F} \) in the sign of the mass term. This as shown in equation (B.8) is the correct parity transformation of the propagator.

3. Charge Conjugation

Under charge conjugation we have,

\[ x' = x \text{ and } c' = c^{-1} \]

where \( c^{-1} \) denotes the same path traversed in the opposite sense. This is because \( c \) is physically the world line of a particle and will thus reverse under charge conjugation. That is,

\[ u' = -u, \quad C' = -C^{-1}, \quad z'_{\sigma}(x') = z^{*}_{\sigma}(x), \quad z'^{*}_{\sigma}(x') = z_{\sigma}(x), \]

which imply that,

\[ L^3_{\mu}(x') = -L^3_{\mu}(x) \quad \text{and} \quad dL^3_{\mu}(x') = -dL^3_{\mu}(x). \]

Let us note that the action is invariant under this transformation.

We now have,

\[ \Omega[C', u'^*] = \Omega[-C^{-1}, -u^*] = \Omega[C, u^*] \]

\[ \Delta \left( u' - dL^3_{\mu}(x') \right) = \Delta \left( u - dL^3_{\mu}(x) \right) \]
With these we have,

\[
\langle \chi'_{u'\sigma}[x', c'] \bar{\chi'}_{u'\sigma}[\bar{x}, \bar{c}] \rangle = \langle \chi_{\bar{u}\sigma}[\bar{x}, \bar{c}^{-1}] \bar{\chi}_{u\sigma}[x, c^{-1}] \rangle
\]

Note that \( \bar{c}^{-1} \) starts at \( \infty \) and ends at \( \bar{x} \) and \( c^{-1} \) starts at \( x \) and ends at \( \infty \). So we have,

\[
\langle \psi'_{u'\sigma}(x') \bar{\psi'}_{u'\sigma}(x) \rangle = \langle \psi_{\bar{u}\sigma}(\bar{x}) \bar{\psi}_{u\sigma}(x) \rangle
\]

This, as shown in equation (B.9) is the correct transformation under charge conjugation.

8. Discussion and Conclusions:

To summarize, using the \( CP_1 \) formalism, we have shown that the z-theory defined in eq.(12) is formally exactly equivalent to the NLSM, in the limit \( \lambda \rightarrow \infty \). Then we showed that the partition function of the z-theory is exactly equivalent to the partition function of the spin-s theory defined in eq.(22). Next we constructed nonlocal operators in the NLSM and showed that the correlation functions of these operators in the z-theory are exactly equal to the correlation functions of the \( \psi \) operators in the spin-s theory. This established the exact equivalence between the z-theory and the spin-s theory. The above result then strongly suggests that there should be a nontrivial \( \lambda = \infty \) fixed point of the spin-s theory (and hence of the z-theory) near which the continuum theory should be exactly equivalent to the NLSM.

We now give some physical arguments for the plausibility of the existence of such a fixed point. The spin-s theory at large \( \lambda \) is a system of strongly interacting particles in two dimensions. For non-zero \( M_s \), the mass term, \( \bar{\psi}\psi \), breaks parity. Fractional Quantum Hall Effect (FQHE) systems are analogous non-relativistic systems. There again we have a system of strongly interacting particles in two dimensions where the interaction with the external magnetic field breaks parity. There are very good approximate solutions of the FQHE system \cite{18} from which we know that the system is characterised by fractionally charged anyonic excitations. These excitations are soliton like, in the sense that they are
extended, finite energy objects. Now this exactly the situation suggested by our result. Namely that the system of strongly interacting spin-s particles is equivalent to the NLSM in the continuum limit which has anyonic soliton excitations.

The analogy to FQHE systems can be made more precise following a very general analysis of such systems, dubbed as Quantum Hall fluids, by Frolich and Zee [19]. They classify these systems into universality classes characterised by three universal properties. (i) The Hall conductivity, \( \sigma_H \). (ii) The charge of the excitations, \( q \). (iii) The statistics of the excitations, \( \frac{\theta}{\pi} \). They have shown that, in the simplest case [19], the allowed values of these quantities are,

\[
\sigma_H = \frac{1}{p+1} \quad (35)
\]

\[
q = \frac{1}{p+1} \Phi \quad (36)
\]

\[
\frac{\theta}{\pi} = \frac{\Phi^2}{p+1} \quad (37)
\]

Where \( p+1 \) is an odd integer if the underlying particles are fermions an even integer if they are bosons. \( \Phi \) is the vorticity of the excitations. We will now show that the NLSM can be looked upon as a Quantum Hall fluid of the above type with \( p+1=s \), when the soliton number is identified with the vorticity \( \Phi \).

The fact that the statistics of a soliton number \( n_s \) object is \( n_s^2 \frac{\pi}{2s} \) has already been mentioned in equation (7). Thus our identifications are consistent with equation (37). Next, from the relation between the spin-s Noether current and the soliton current (33), as mentioned before, the charge of the soliton is \( \frac{1}{2s} \) if the charge of the spin-s particle is one. Thus the charge of the excitations is consistent with equation (36). Finally we come to the Hall conductivity. \( \sigma_H \) is related to the current-current correlator as follows,

\[
\sigma_H = \lim_{\omega \to 0} \frac{1}{\omega} \epsilon_{ij} < J_i(\omega, 0)J_j(-\omega, 0) > \quad (38)
\]

Note that here as in reference [19], by \( \sigma_H \) we mean the off-diagonal components of the
conductivity tensor at zero external field. From equation (34), it follows that,

\[ \sigma_H = \frac{1}{2s} + \lim_{\omega \to 0} \frac{1}{\omega} \epsilon_{ij} < j^\text{top}_i(\omega, 0) j^\text{top}_j(-\omega, 0) > \]

We will now argue that the second term in the RHS of the above equation is zero. This we will do by arguing that if we couple an external field to the topological current in the NLSM, then the induced Chern-Simons term will be zero. Consider the NLSM action in equation (31). Now introduce the auxilliary \( a_\mu \) fields as before to obtain the action,

\[ S[\hat{n}, a_\mu, c_\mu] = \int d^3 x \frac{1}{g^2} \partial_\mu \hat{n} \partial_\mu \hat{n} + j^\text{top}_\mu(2i\theta a_\mu + c_\mu) + i\theta \epsilon_{\mu \nu \lambda} a_\mu \partial_\nu a_\lambda \quad (39) \]

Now integrate over the \( \hat{n} \) fields to obtain the following form of the action,

\[ S[a_\mu, c_\mu] = F[2i\theta a_\mu + c_\mu] + i\theta \epsilon_{\mu \nu \lambda} a_\mu \partial_\nu a_\lambda \quad (40) \]

Now note that the part of the action in equation (39) that involves only the \( \hat{n} \) fields is parity invariant (parity is broken only by the Chern-Simons term). Therefore \( F \) will be a parity invariant functional and cannot contain any terms like \( \epsilon_{\mu \nu \lambda}(2i\theta a_\mu + A_\mu)\partial_\nu(2i\theta a_\lambda + A_\lambda) \). The action in equation (40) is then independent of \( c_\mu \) in the long wavelength limit. Hence there is no induced Chern-Simons term when the \( a_\mu \) fields are integrated out. It then follows that we have, \( \sigma_H = \frac{1}{2s} \). Thus our identifications are consistent with equation (35). This then establishes the fact that the NLSM is a Quantum Hall fluid of the type specified by equations (35), (36) and (37).

The equivalence of the spin-s theory and the NLSM is very convincing when viewed in the light of the above arguments. Namely, that the strongly interacting, parity non-invariant spin-s system should have a Quantum Hall Fluid phase, characterised by some universal properties. The corresponding NLSM, along with the relations between the current correlators that we have derived, is consistent with the general analysis of these universal properties done in ref.[19]. Also, we have shown the equivalence of the two theories in the formal \( \lambda \to \infty \) limit. Therefore the spin-s theory should have a \( \lambda = \infty \) fixed point which governs this phase of the system. However the \( \theta = 0 \) and \( \pi \) limits
(where the NLSM is parity invariant) are likely to be singular just like \( B = 0 \) limit in Quantum Hall systems\[^{20}\]. These two values of \( \theta \) are thus beyond the purview of the above discussion. Note that we had previously argued for different reasons that the \( \theta = 0 \) limit is likely to be singular (see the discussion after equation (17)). However the \( \theta = \pi \) point is different. Here our results indicate that the theory should be equivalent to a system of strongly interacting Dirac fermions though it is not a Quantum Hall fluid. Since the NLSM is parity invariant at this point, the mass term in the spin-\( \frac{1}{2} \) theory should become an irrelevant operator at the \( \lambda = \infty \) fixed point so that the continuum theory is parity invariant. This also shows that the usual large \( N \) expansion \[^{21}\] will not see this fixed point. This is understandable since it corresponds to generalizing \( CP_1 \) to \( CP_{N-1} \). This spoils the topological features of the \( N = 2 \) model which are crucial for the parity invariance. Finding a good expansion (that preserves the topology) to locate and analyse this fixed point is clearly an important problem.

Appendix A:

Sum over path representation of spin-\( s \) particle in external fields:

It is now well-known \[^{12}\] that addition of Polyakov spin factor to path integrals, for free spinless particles gives path integral representation for spinning particles. We now show that this proof can be extended to particle in the presence of background fields.

We start with the sum over path representation for spinless particle with spin-factor coupled to vector and scalar fields,

\[
K_{AV}(x, x|\beta) \equiv \int_{x(0)=x(\beta)} Dx e^{-\int_0^\beta \frac{1}{4} (\partial_s s)^2 + V(x(\tau)) + is\Omega[s]d\tau} - i \oint A_d dx^\mu \quad (A.1)
\]

Here \( K(x, x|\beta)_{AV} \) is the amplitude for the particle to make a closed curve in time \( \beta \).

We show that this is equal to,

\[
Tr \ e^{-L(D^{(s)} + \frac{\sqrt{s}}{4}V(x) + M_s)} \quad (A.2)
\]
where
\[ L = \frac{\beta}{\kappa} M_s = \Lambda^2\kappa \ln(2s + 1) \]
and
\[ D^{(s)} = \text{sgn}(\theta)(i\partial_\mu + A_\mu) \frac{T^\mu}{s} \]
with \( T^\mu \) being the generators of \( SU(2) \) in spin-\( s \) representation.

Starting with (A.1), we first change the variable in the integrand from \( \frac{dx^\mu}{d\tau} \) to \( u^\mu(\tau) \), using the identity,
\[ F\left[x, \dot{x}\right] = \int D^u \delta^{(3)}[\dot{x} - u] F[x, u] \tag{A.3} \]
\[ K(x, x|\beta) \]
\[ = \int \prod_\tau \frac{d^3 x(\tau)}{(4\pi \epsilon)^{3/2}} \int d^3 u(\tau) \delta^3(\dot{x}_\mu(\tau) - u_\mu(\tau)) e^{\frac{i}{\epsilon} [\frac{\dot{x}}{\epsilon^2} - iu.A + V[x]]} e^{is \int \Omega[\hat{u}]} \tag{A.4} \]
where \( \epsilon \) is the interval between two adjacent \( \tau \)-slices. Replace \( \prod_\tau \delta^3(\dot{x} - u) \) by
\[ \prod_\tau \int \frac{d^3 k(\tau)}{(2\pi)^3} e^{i\frac{k}{\epsilon^2} [\dot{x}(\tau + 1) - x(\tau) - e\cdot u(\tau)]} \tag{A.5} \]
The path-integral over the velocity variables are performed in radial and polar variables. The need for doing it this way will be seen later. The radial integral to be performed is,
\[ \int_0^\infty u^2 du e^{-i\hat{u} \cdot \vec{K} - \epsilon^2 u} \]
where \( \vec{K} = \vec{k} - \vec{A} \) Scaling \( u_\mu \to \sqrt{\epsilon}u_\mu \) and keeping only \( O(\sqrt{\epsilon}) \) term, this becomes,
\[ \frac{2\sqrt{\pi}}{\epsilon^{3/2}} e^{-i\sqrt{\pi} \hat{u} \cdot \vec{K}} \tag{A.6} \]
With this the measure for the remaining path integral to be performed over \( \vec{x}, \vec{k} \) and \( \hat{u} \), are
\[ \prod_\tau \frac{d^3 x(\tau)}{4\pi} \frac{d^3 k(\tau)}{(2\pi)^3} d\hat{u} \]
The path-integral to be performed is,
\[ \prod_{\tau=1}^N \int d^3 x(\tau) e^{\psi(x)} \int \frac{d^3 k(\tau)}{(2\pi)^3} \int d\hat{u} e^{\sum_{i=1}^N \{i\dot{w}_\tau \hat{u} \cdot \vec{K}(\tau)] + is\Omega[\hat{u}]\}} \tag{A.7} \]
where \( w^2 = \frac{4}{\sqrt{\pi}c} \). The need for performing the path integral over \( u^\mu \) in radial and angular variables is now clear: with the Polyakov spin-factor \( \Omega[\hat{u}] \), which is the solid angle subtended by the closed curve on the sphere \( S^2 \), traced by the unit velocity vector, having the geometrical meaning of providing the symplectic structure of \( SU(2) \) group the path integral over unit vector \( \hat{u} \) (with measure \( \frac{d\hat{u}}{4\pi} (2s + 1) \)), is the phase-space path integral for \( SU(2) \) group.

In general,
\[
\int_{\hat{u}(0)=\hat{u}(s)} d\hat{u} \frac{(2s + 1)}{4\pi} e^{isJ[\Omega[\hat{u}] + H[\hat{u}]]} d\tau
\]
\[= \quad Tr < \hat{u} | e^{iH[\hat{u}]} | \hat{u} > \]

where \( T \) are the generators in the spin-\( s \) representation of \( SU(2) \) group and \( |\hat{u}> \) is the \( SU(2) \) coherent state.

This relation can be proved in a straightforward way using the following properties of \( SU(2) \) coherent states:

\[
\int \frac{2s + 1}{4\pi} \int d\hat{u} |\hat{u} \rangle \langle \hat{u}| u u^* \rangle = \mathbb{I} ,
\]
\[
\langle uu^* | \gamma^\mu | uu^* \rangle = isu^\mu ,
\]
\[
\langle u_1 u^* | u_2 u^* \rangle = e^{is\Omega[u_1, u_2, u^*]} \left( \frac{1 + u_1 \cdot u_2}{2} \right)^s .
\]

where \( u^* \) refers to any fiducial point on \( S^2 \).

Using as the overlap between states \( |x(\tau) \hat{u}(\tau) > \) at \( \tau \) and \( \tau + 1 \) namely
\[
\int \frac{d^3k(\tau)}{(2\pi)^3} \frac{d\hat{u}(\tau)}{4\pi} (2s + 1) e^{ik \cdot (x(\tau+1) - x(\tau)) + is\Omega[\hat{u}]} \]

it follows easily, (A.7) is
\[
Tr < \hat{u} | e^{-sw^2 \left[ \frac{ln(2s+1)}{w^2} \right] - (D+ \frac{V(s)}{w^2})} | \hat{u} > . \quad (A.8)
\]

Note that when \( \theta = -\frac{\pi}{2s} \), then in all equations the solid angle term comes with a minus sign. To use the same conventions for the definitions of the coherent states, we make the change of variables \( \hat{u} \to -\hat{u} \) in equation (A.7) since \( \Omega[-\hat{u}] = -\Omega[\hat{u}] \). This results
in $K \to -K$ which leads to $D^{(s)} \to -D^{(s)}$. Therefore, in general, the operator that appears in equation (A.8) is, $D^{(s)} = \text{sgn}(\theta) (i \partial_{\mu} - A_{\mu}) \frac{T^\mu}{s}$. Thus with the definition $L = \beta w^2, \frac{1}{cw^2} \ln(2s+1) = M_s$ the claimed result (A.2) follows.

Appendix B:

Lorentz and Discrete Transformation Properties of spin-s Fields.

In this appendix we derive the transformation properties of the $\psi$ spin-s fields under Lorentz and the discrete transformations in the coherent state representation. i.e.,

$$
\psi_u(x) \equiv \langle uu^*|m\rangle \psi_m(x)
$$

$$
\bar{\psi}_u(x) \equiv \bar{\psi}_m(x)\langle m|uu^*\rangle
$$

1. Proper Lorentz rotations.

Under Lorentz transformations given by,

$$
x' = \Lambda x
$$

we have

$$
\psi'(x') = U^\dagger(\Lambda)\psi(x), \quad \bar{\psi}'(x') = \bar{\psi}(x)U(\Lambda),
$$

$$
U^\dagger(\Lambda)\gamma^\mu U(\Lambda) = \gamma^\nu\Lambda_{\nu\mu} \quad \text{and} \quad U(\Lambda)\gamma^\mu U^\dagger(\Lambda) = \Lambda_{\mu\nu}\gamma^\nu.
$$

Therefore in the coherent state basis we have,

$$
\psi_u(x) = \langle uu^*|m\rangle \psi_m(x)
$$

$$
= \langle uu^*|m\rangle U_{mm'}(\Lambda)\psi_{m'}(x')
$$

$$
= \langle uu^*|U(\Lambda)|m\rangle \psi_{m'}(x') .
$$

Now we will prove that,

$$
U^\dagger(\Lambda)|uu^*> = |\Lambda u, \Lambda u^*> .
$$
Therefore we have $\psi'(x') = \psi_u(x)$ where $u' = \Lambda u$.

Proof of $U^\dagger(\Lambda)|uu^*\rangle = |\Lambda u, \Lambda u^*\rangle$:

We have,

$$\int \frac{2s + 1}{4\pi} \, du U^\dagger(\Lambda)|uu^*\rangle\langle uu^*|U(\Lambda) = U^\dagger II U(\Lambda) = II,$$

$$\langle uu^*|U(\Lambda)\gamma^\mu U^\dagger(\Lambda)|uu^*\rangle = \Lambda_{\mu\nu} uu^* \gamma^\mu |uu^*\rangle = \Lambda_{\mu\nu} i u' = i s(\Lambda u)^\mu,$$

$$\langle u_1 u^*|U(\Lambda)U^\dagger(\Lambda)|u_2 u^*\rangle = e^{i\Omega[u_1 u_2 u^*]} \left(\frac{1 + u_1 \cdot u_2}{2}\right)^s.$$

But we have, $u_1 \cdot u_2 = (\Lambda u_1) \cdot (\Lambda u_2)$ and $\Omega[u_1 u_2 u^*] = \Omega[\Lambda u_1, \Lambda u_2, \Lambda u^*]$ so that we can write,

$$\langle u_1 u^*|U(\Lambda)U^\dagger(\Lambda)|u_2 u^*\rangle = e^{i\Omega[u_1 u_2 u^*]} \left(\frac{1 + \Lambda u_1 \cdot \Lambda u_2}{2}\right)^s.$$

That proves that $U^\dagger(\Lambda)|u, u^*\rangle = |\Lambda u, \Lambda u^*\rangle$.

2. Parity

Under $x' = -x$ we have $\psi'(x') = i\psi(x)$ and $\bar{\psi}'(x') = i\bar{\psi}(x)$. This implies that $\psi'_u(x') = i\psi_u(x)$ and $\bar{\psi}'_u(x') = i\bar{\psi}_u(x)$.

3. Charge conjugation

Here we need to distinguish between the cases of $s$ being integer and half-odd integer.

First we will consider the case of integer $s$ and then that of half-odd integer $s$.

**Case i) s=integer:**

We have under charge conjugation $x' = x$, $\psi'_m(x') = \bar{\psi}_m(x)$ and $\bar{\psi}'_m(x') = \psi_m(x)$. With these we get,

$$\psi_u(x) = \langle uu^*|m\rangle \bar{\psi}'_m(x') \equiv \bar{\psi}'_m(x')\langle m|U\rangle$$

and

$$\bar{\psi}_u(x) = \bar{\psi}_m\langle m|u u^*\rangle = \langle U|m\rangle \psi'_m(x')$$

where $\langle m|U\rangle \equiv \langle uu^*|m\rangle$. We will show that, $|U\rangle = |u, -u^*\rangle$. We then have

$$\psi'_u(x') = \bar{\psi}_u(x) \quad x' = x$$

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\[ \psi_u'(x') = \psi_u(x) \quad u' = u \]

Now we prove that, \(|U⟩ = |−u, −u^*⟩\) in the following way. We have

\[
\frac{2s + 1}{4\pi} \int \mathrm{d}u \langle n|U⟩⟨U|m⟩ = \frac{2s + 1}{4\pi} \int \mathrm{d}u \langle m|u^*⟩⟨u^*|n⟩ = \delta_{mn}. \tag{B.1}
\]

\[
⟨U|γ^µ|U⟩ = ⟨U|m⟩γ^µ⟨n|U⟩ = ⟨u^*|n⟩γ^µ_m n ⟨m|u u^*⟩ = ⟨u^*|⟨γ⟩^T|u u^*⟩. \tag{B.2}
\]

We have, for integer spins \((γ^µ)^T = −γ^µ\) so that we get, \(⟨U|γ^µ|U⟩ = −isu^µ\) Also we have

\[
⟨U_1|U_2⟩ = ⟨u_2 u^*|u_1 u^*⟩ = e^{isΩ[u_2,u_1,u^*]} \left(\frac{1 + u_1 \cdot u_2}{2}\right)^s. \tag{B.3}
\]

But we also we have

\[
Ω[u_2,u_1,u^*] = −Ω[u_1,u_2,u^*] = Ω[−u_1,−u_2,−u^*]
\]

Also since \(u_1 \cdot u_2 = (−u_1) \cdot (u_2)\) we can write:

\[
⟨U_1|U_2⟩ = e^{isΩ[−u_1,−u_2,−u^*]} \left(\frac{1 + (−u_1) \cdot (−u_2)}{2}\right)^s.
\]

Eq.(B.1),(B.2) and (B.3) proves that, \(|U⟩ = |−u, −u^*⟩\).

Case ii) \(s = \) half-odd integer spins:

We have \(x' = x\). Let

\[
ψ_{m'}(x') = \bar{ψ}_m(x)C_{mm'},
\]

\[
\bar{ψ}_{m'}(x') = C_{mm'}ψ_m(x),
\]

with \(C^2 = −1, C^* = C = −C^T\). Then we have

\[
ψ_u(x) = ⟨uu^*|m⟩ψ_m(x)
\]

\[
= −⟨uu^*|m⟩C_{mm'}\bar{ψ}_{m'}(x')
\]

\[
≡ \bar{ψ}_{m'}(x')⟨m'|U⟩,
\]

where,

\[
⟨m'|U⟩ ≡ −⟨uu^*|m⟩C_{mm'}.
\]

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Also we have,
\[ \bar{\psi}_u(x) = \bar{\psi}_m(x) \langle m | uu^* \rangle \]
\[ = -\psi'_m(x') C_{mm'} \langle m | uu^* \rangle \]
\[ = -\langle U | m' \rangle \psi'_m(x') . \]

Here we have used the fact that,
\[ \langle U | m' \rangle = -C_{m'm} \langle m | uu^* \rangle . \]

Now again we will prove that,
\[ | U \rangle = | -u, -u^* \rangle , \]
through the following steps. First we have,
\[ \frac{2s + 1}{4\pi} \int du \langle m | U | n \rangle = \frac{2s + 1}{4\pi} \int du (-) C_{mm'} \langle n' | uu^* \rangle \langle uu^* | m' \rangle C_{m'm} = \delta_{nm} \quad (B.4) \]
\[ \langle U | \gamma^\mu | U \rangle = -\langle uu^* | m' \rangle C_{m'm} \gamma^\mu_{nm} C_{nn'} \langle n' | uu^* \rangle = -\langle uu^* | C(\gamma^\mu)C|uu^* \rangle \\
= -is \gamma^\mu, \quad (B.5) \]
\[ \langle U_1 | U_2 \rangle = -\langle u_2 u^* | m' \rangle C_{m'm} C_{mm} \langle n | u_1 u^* \rangle = \langle u_2 u^* | u_1 u^* \rangle . \quad (B.6) \]

Now from Eq.(B.4),(B.5) and (B.6) it follows that \[ | U \rangle = | -u, -u^* \rangle . \] So we have
\[ \psi'_u(x') = \bar{\psi}_u(x) \quad x' = x \]
\[ \bar{\psi}'_u(x') = \psi_u(x) \quad u' = -u \]

Putting both results together, we have
\[ \psi'_u(x') = (-1)^{2s} \bar{\psi}_u(x) \quad x' = x \]
\[ \bar{\psi}'_u(x') = \psi_u(x) \quad u' = -u \]
4. Transformation of the propagator

We can now easily derive the transformation properties of the propagator by looking at the bilinears. We summarize the results in the next three equations.

i) Proper Lorentz transformations
Under \( x' = \Lambda x \) and \( u' = u \) we have,

\[
\psi'_{u_1'}(x'_1)\bar{\psi}'_{u_2'}(x'_2) = \psi_{u_1}(x_1)\bar{\psi}_{u_2}(x_2).
\]  

(B.7)

ii) Parity
In this case \( x' = -x \) and \( u' = u \) and we have,

\[
\psi'_{u_1'}(x'_1)\bar{\psi}'_{u_2'}(x'_2) = -\psi_{u_1}(x_1)\bar{\psi}_{u_2}(x_2).
\]  

(B.8)

iii) Charge conjugation
Under charge conjugation we have, \( x' = x, \ u' = -u \) and we get,

\[
\psi'_{u_1'}(x'_1)\bar{\psi}'_{u_2'}(x'_2) = \psi_{u_2}(x_2)\bar{\psi}_{u_1}(x_1).
\]  

(B.9)
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