Symmetric Space $\sigma$-model Dynamics: Current Formalism

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Abstract

After explicitly constructing the symmetric space sigma model lagrangian in terms of the coset scalars of the solvable Lie algebra gauge in the current formalism we derive the field equations of the theory.

1 Introduction

The low energy effective limit or the massless background coupling of the superstring theories are governed by the supergravity theories [1]. The scalar sectors of the supergravity theories reflect the global symmetry properties of these theories [2, 3, 4]. Also the restriction of the global symmetry group to the integers gives the U-duality symmetry of the corresponding superstring theory [5, 6]. The majority of the scalar sectors of the supergravities can be constructed as $G/K$ symmetric space sigma models [3, 4, 7, 8, 9, 10]. The symmetric space sigma model field equations in the internal metric formalism [7, 8] have been studied in [9, 10] and further in [11, 12]. In these works the field equations are obtained under a specific trace convention of the symmetry algebra representation. In [13] in the internal metric formalism, for a generic
trace convention the most general form of the field equations are obtained for the axion-dilaton parametrization of the symmetric space coset manifold. On the other hand in the vielbein formalism [2,3,4] without specifying a gauge which would parameterize the symmetric space coset manifold the derivation of the field equations is a standard task which results in a set of equations for the vielbein which is on the symmetric space $G/K$.

In this work we assume the solvable Lie algebra gauge [14] to parameterize the scalar coset manifold $G/K$. This parametrization contributes a simplification to the construction of the lagrangian in the current formalism which in this case can exactly be expressed in terms of the Cartan-form instead of only an abstractly defined piece of it [2,3,4]. Thus by using the Cartan-form which is derived in [12] we will explicitly construct the lagrangian in terms of the scalar fields which parameterize the symmetric space $G/K$. We will then derive the field equations of the theory for the coset scalars by directly varying the lagrangian which we have constructed in the solvable Lie algebra gauge.

Section two is reserved for the construction of the lagrangian explicitly in terms of the solvable Lie algebra gauge scalar fields. In section three we derive the field equations. We also shortly inspect the algebraic properties of certain matrix terms which appear in the field equations.

2 Lagrangian in the Solvable Lie Algebra Parameterization

The symmetric space sigma model (SSSM) is based on the coset manifold $G/K$ where $G$ is in general a non-compact real form of any other semi-simple Lie group and $K$ is a maximal compact subgroup of it [1]. The coset manifold $G/K$ is a Riemannian globally symmetric space for all the $G$-invariant Riemannian structures on it [15]. In this section we will present the construction of the SSSM lagrangian in the current formalism under the solvable Lie algebra gauge [14]. Now consider the set of $G$-valued maps $\nu(x)$. We assume that they transform as $\nu \to k(x)\nu g$, $\forall g \in G$, $k(x) \in K$. The maps $\nu(x)$ are from the $D$-dimensional spacetime into the group $G$. We further assume that the maps $\nu(x)$ correspond to a parametrization of the coset $G/K$. Thus their images are the representatives of the left cosets of $G/K$. In the solvable Lie

\footnote{We will consider the left cosets.}
algebra gauge \[14\] the parametrization \(\nu(x)\) of the coset \(G/K\) can be chosen as
\[
\nu(x) = e^{\phi^i(x)T_i}, \tag{2.1}
\]
where the basis \(\{T_i\}\) generates a solvable Lie subalgebra \(s\) of \(g\) which is the Lie algebra of \(G\) and \(\{\phi^i(x)\}\) are scalar fields on the \(D\)-dimensional spacetime. The solvable Lie algebra \(s\) emerges from the Iwasawa decomposition of \(g\) which reads \[15\]
\[
g = k \oplus s, \tag{2.2}
\]
where \(k\) is the Lie algebra of \(K\). Explicit basis constructions for \(s\) within the root space decomposition of \(g\) can be referred in \[10, 12, 15\]. In this work we will consider a generic basis \(\{T_i\}\) for an arbitrary Iwasawa decomposition \[2.2\]. One of the consequences of the Iwasawa decomposition \[2.2\] is the local diffeomorphism \[15\]
\[
\text{Exp} : s \longrightarrow G/K, \tag{2.3}
\]
from the \(\mathbb{R}^{\text{dim}s}\)-manifold \(s\) into \(G/K\) which enables the definition of the coset map \[2.1\]. In the vielbein formalism of the SSSM the construction of the lagrangian which is invariant under the right rigid action of \(G\) and the left local action of \(K\) is based on the introduction of the Cartan-form
\[
G' = \nu'^{-1}d\nu' = P + Q, \tag{2.4}
\]
where
\[
P = P^iF_i, \quad Q = Q^jK_j, \tag{2.5}
\]
\(\{K_j\}\) is a basis for \(k\) and \(\{F_i\}\) is the basis which generates the orthogonal complement of \(k\) in a vector space direct sum of \(g\). The fields \(\{P^i\}\) form a vielbein of the \(G\)-invariant Riemannian structures on \(G/K\) and \(\{Q^j\}\) can be considered as the connection one-forms of the gauge theory over the \(K\)-bundle. Further transformation properties of \(P\) and \(Q\) can be referred in \[3, 12\]. An invariant lagrangian under the above mentioned global and local transformations can be constructed as \[3, 4\]
\[
\mathcal{L} = \frac{1}{2} \text{tr}(\ast P \wedge P). \tag{2.6}
\]
Now we should remark an important aspect of the solvable Lie algebra gauge. In the general parametrization of the coset the coset generators need not
form a subalgebra \[^3\][4]. For this reason in general the Cartan-form (2.4) has components both in \(P\) and \(Q\) directions since the commutation of the coset generators may have components along the basis \(\{K_j\}\). On the other hand when we take the basis \(\{T_i\}\) which generates the solvable Lie algebra \(\mathfrak{s}\) to be the coset generators\(^2\), then as the basis elements \(\{T_i\}\) close on themselves we have a major simplification in the calculation of the Cartan-form (2.4).

On the other hand one can also construct the symmetric space sigma model in the internal metric formalism \[^7\][8][9][10][11][12]. In [13] the field equations of the internal metric formalism of the symmetric space sigma model are derived for the axion-dilaton parametrization. In this work bearing in mind the above discussion of the simplification of the calculation of the Cartan-form we will consider the lagrangian

\[
\mathcal{L} = \frac{1}{2} tr(*G' \wedge G').
\] (2.7)

This lagrangian that is based on the Cartan form (2.4) which is a Noether’s current\(^3\) can also be obtained from the one in [13] by local field redefinitions following the discussions of [12] however on its own right it formulates the theory for a generic solvable Lie algebra parametrization in the form (2.4). As we mentioned before the Cartan-form (2.4) is already calculated in [12] it reads

\[
G' = \hat{T} W \hat{d}\varphi,
\] (2.8)

where the \(\text{dim}\mathfrak{s} \times \text{dim}\mathfrak{s}\) matrix \(W\) is

\[
W = (I - e^{-M})M^{-1}.
\] (2.9)

In (2.9) we define the matrix \(M\) as

\[
M^\beta_\alpha = \varphi^i C^\beta_{i\alpha}.
\] (2.10)

Also the components of the row vector \(\hat{T}\) are \(T_i\) and \(\hat{d}\varphi\) is a column vector of the field strengths \(\{d\varphi^i\}\). We introduce the structure constants of \(\mathfrak{s}\) as

\[
[T_i, T_j] = C^k_{ij} T_k.
\] (2.11)

\(^2\)This will be our choice in this work.

\(^3\)Although it can be expressed in terms of the Cartan-form since its kinetic term is directly written via an internal metric we use the name of internal metric formalism for the lagrangian studied in [13]. On the other hand we call the lagrangian (2.7) to be the current formalism since it is directly constructed from the Cartan-form.
Now that we have the exact form of the Cartan-form at hand we can explicitly write down the lagrangian (2.7) in terms of the coset parameterizing scalars \{\varphi^i\}. Inserting (2.8) in (2.7) yields

\[
\mathcal{L} = \frac{1}{2} T_{np} W_n^m \ast d\varphi^l \wedge W_p^k d\varphi^k,
\]

where without specifying any trace convention we have introduced the generic trace convention coefficients

\[
T_{np} = tr(T_n T_p).
\]

3 The Field Equations

In this section starting from the lagrangian (2.12) we will derive the field equations of the coset scalars \{\varphi^i\} by direct variation. Beforehand we should examine the variation properties of the matrix \(W\). Firstly we observe that

\[
M' \equiv \frac{\partial M}{\partial \varphi^m} = C_m,
\]

where

\[
(C_m)_j^k = C_{mj}^k,
\]

is the matrix representative of the generator \(T_m\) in the adjoint representation of \(s\) that is induced by the basis \{\(T_i\)\}. Now the variation of \(e^{-M}\) with respect to the field \(\varphi^m\) yields [16, 17]

\[
\frac{\partial e^{-M}}{\partial \varphi^m} = -e^{-M} \left( \frac{e^{ad_M}}{ad_M} - I \right) (M')
\]

\[
= -e^{-M} (M' + \frac{1}{2!} [M, M'] + \frac{1}{3!} [M, [M, M']] + \cdots).
\]

Before going further we should inspect the structure of the commutation series in (3.3). For this purpose we will take a closer look at the properties of the solvable Lie algebra \(s\). The set

\[
\mathcal{D} s = \{ [X, Y] \},
\]

5
which is generated by all the elements \( X, Y \in s \) is an ideal of \( s \). Since an ideal is a subalgebra \( Ds = [s, s] \) is called the derived algebra of \( s \). The higher order derived algebras are defined inductively in the same way over one less rank derived algebra and they are denoted as \( D^n s \) where

\[
D^n s = D(D^{n-1} s),
\]

and \( D^0 s = s \). Since the Lie algebra \( s \) is solvable there exists an integer \( n \geq 0 \) such that \( D^n s = \{0\} \). As we have mentioned before one can construct a basis for \( s \) by using the root space decomposition of \( g \). From [10, 12] we have

\[
s = h_p \oplus n, \tag{3.6}
\]

where \( h_p \) is a subalgebra of the Cartan subalgebra of \( g \) and \( n \) is a nilpotent subalgebra which is generated by certain positive root generators. For the explicit construction of this decomposition we refer the reader to [10, 12, 15]. For our purposes in this work it suffices to observe that

\[
[h_p, h_p] = 0, \quad [n, n] \subset n, \quad [h_p, n] = n, \tag{3.7}
\]

from which we conclude that the first derived algebra of \( s \) is the nilpotent algebra \( n \) namely

\[
Ds = [s, s] = n. \tag{3.8}
\]

On the other hand the image of the nilpotent Lie subalgebra \( n \) in the adjoint representation \( ad(n) \) is also nilpotent [15] thus the central descending series

\[
\varphi^0 ad(n) \supset \varphi^1 ad(n) \supset \varphi^2 ad(n) \supset \cdots, \tag{3.9}
\]

terminates with \( \varphi^m ad(n) = \{0\} \) for some \( m \geq \dim(ad(n)) \) [15, 18, 19]. In (3.9) the ideals are defined as

\[
\varphi^{p+1} ad(n) = [ad(n), \varphi^p ad(n)], \tag{3.10}
\]

with \( \varphi^0 ad(n) = ad(n) \). From their definitions in (2.10) and (3.1) we deduce that \( M \) and \( M' \) lie in the adjoint representation of \( s \) that is induced by \( \{T_i\} \). From (3.8) we have

\[
Dad(s) = [ad(s), ad(s)] = ad(n). \tag{3.11}
\]

Now also bearing in mind that the derived algebra \( Dad(s) \) is an ideal we have

\[
[ad(s), Dad(s)] = Dad(s) = ad(n). \tag{3.12}
\]
Therefore we conclude that except the first one the rest of the terms in the series (3.3) all lie in $Dad(s) = ad(n)$. There always exists a basis $\{T_j\}$ which induces a matrix representation such that the matrix representatives of the nilpotent endomorphisms which form up the nilpotent algebra $ad(n)$ have null entries on and below the diagonal. Such a basis choice will contribute a major simplification to the calculation of (3.3). Even for a matrix representation induced by a generic basis the commutation terms in (3.3) will produce images which belong to a fixed generation of matrices due to the closure relations in (3.7) and the terminating central descending series structure in (3.9) of $ad(n)$. Thus in general although the series in (3.3) does not terminate after a finite number of terms the calculation of the series (3.3) will reduce to the calculation of the coefficient series of the periodically appearing matrices in the above mentioned set of finite number of generations. Now we are ready to vary the matrix function $W$. After some algebra we find

$$K_m \equiv \frac{\partial W}{\partial \varphi^m} = e^{-M}(e^{ad_M} - I)(M')M^{-1}$$
(3.13)

$$- W M' M^{-1},$$

where we have made use of the identity

$$\frac{\partial M^{-1}}{\partial \varphi^m} = -M^{-1}M'M^{-1}.$$  
(3.14)

Equipped with the above variation tools we can now directly vary the lagrangian (2.12) with respect to the scalar field $\varphi^m$. The variation yields

$$(-1)^{(D-1)}d(T_{np}(W^n_k W^p_m + W^n_m W^p_k) * d\varphi^k)$$
(3.15)

$$= T_{np}(K^n_m W^p_k + W^n_i K^p_m) * d\varphi^i \wedge d\varphi^k.$$  

These are the field equations for the solvable Lie algebra gauge scalars $\{\varphi^i\}$ of the current formalism of the symmetric space sigma model. We observe that the non-linearity of the theory is highly reflected in the matrix component coefficients in (3.15).
4 Conclusion

By adopting the solvable Lie algebra gauge to parameterize the coset manifold $G/K$ we have explicitly constructed the lagrangian of the symmetric space sigma model in the current formalism in terms of the coset scalar fields. The formulation makes use of the exact form of the Cartan-form whose components are derived as functions of the coset parameterizing scalar fields in $[12]$. Having expressed the lagrangian in terms of the scalar fields we have varied it to obtain the corresponding field equations.

The scalar fields in the supergravity theories are either elements of the building block multiplets or they arise as components of the dimensional reduction ansatz in the compactification scheme of higher dimensional theories. Therefore they have a more fundamental role than the vielbein. This fact justifies the importance of deriving the field equations explicitly in terms of the scalars. Our formulation is performed for a general symmetric space $G/K$ and it is purely in terms of the unspecified solvable Lie algebra structure constants. We have not also assumed a specific representation. Thus we have constructed a general formalism to derive the field equations of the symmetric space sigma model in the Noether’s current approach. Obtaining the field equations is a systematic task. For a specific example one only needs to identify the solvable Lie algebra which takes part in an Iwasawa decomposition of the global symmetry algebra and then to specify a basis for the solvable Lie algebra which gives the structure constants that are the keys of the formulation.

We observe that due to the matrix structure of the field equations there is a high degree of non-linearity. However one may express the field equations as a matrix equation and one may make use of the adjoint representation and perform certain field transformations to simplify the coefficients of the field strengths and the kinetic terms of the scalars which may abolish a degree of coupling in the equations. One may also work on the first-order formulation of the theory. An axion-dilaton parametrization of the coset manifold in the current formalism can separately be studied.

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