Research Article

Variational Approach for the Variable-Order Fractional Magnetic Schrödinger Equation with Variable Growth and Steep Potential in $\mathbb{R}^{N^*}$

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Abstract

In this paper, we show the existence of solutions for an indefinite fractional Schrödinger equation driven by the variable-order fractional magnetic Laplace operator involving variable exponents and steep potential. By using the decomposition of the Nehari manifold and variational method, we obtain the existence results of nontrivial solutions to the equation under suitable conditions.

1. Introduction

In this paper, we investigate the existence of solutions of the following concave-convex fractional elliptic equation driven by the variable-order fractional magnetic Laplace operator involving variable exponents:

$$(-\Delta)^{\alpha}_{A}u + V_\lambda(x)u = f(x)\left|u\right|^{q(x)-2}u + g(x),$$

where $N \geq 1, s(\cdot) \colon \mathbb{R}^N \times \mathbb{R}^N \to (0, 1)$, is a continuous function, $(-\Delta)^{\alpha}_{A}$ is the variable-order fractional magnetic Laplace operator, the potential $V_\lambda(x) = \lambda V_+^\alpha(x) - V_-^\alpha(x)$ with $V^\pm = \max \{ \pm V, 0 \}$, $\lambda > 0$ is a parameter, and the magnetic field is $A \in C_0^\alpha(\mathbb{R}^N, \mathbb{R}^N)$ with $\alpha \in (0, 1]$. $p, q \in C(\mathbb{R})$ and $u : \mathbb{R}^N \to \mathbb{C}$. In [1], the fractional magnetic Laplacian has been defined as

$$(-\Delta)^{\alpha}_{A}u(x) = \lim_{r \to 0} \int_{B_r(x)} \frac{u(x) - e^{i(x-y) \cdot A(x+y/2)}u(y)}{|x-y|^{N+2\alpha}} dy,$$

for $x \in \mathbb{R}^N$. In [2], the variable-order fractional Laplace $(-\Delta)^{\alpha}_{A}$ is defined as for each $x \in \mathbb{R}^N$,

$$(-\Delta)^{\alpha}_{A}u(x) = \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2\alpha}} dy,$$

along any $u \in C_0^\alpha(\Omega)$. Inspired by them, we define the variable-order fractional magnetic Laplacian $(-\Delta)^{\alpha}_{A}$ as for each $x \in \mathbb{R}^N$,

$$(-\Delta)^{\alpha}_{A}u(x) = \lim_{r \to 0} \int_{B_r(x)} \frac{u(x) - e^{i(x-y) \cdot A(x+y/2)}u(y)}{|x-y|^{N+2\alpha}} dy.$$

Since $s(\cdot)$ is a function, magnetic field $A \in C_0^\alpha(\mathbb{R}^N, \mathbb{R}^N)$ with $\alpha \in (0, 1]$, we see that operator $(-\Delta)^{\alpha}_{A}$ is a variable-order fractional magnetic Laplace operator. Especially, when $s(\cdot) \equiv constant$, $(-\Delta)^{\alpha}_{A}$ reduces to the usual fractional magnetic Laplace operator. When $s(\cdot) \equiv constant$, $A = 0$, $(-\Delta)^{\alpha}_{A}$ reduces to the usual fractional Laplace operator. Very recently, when $A = 0$, $V^\alpha(x) = 0$ and $f(x), g(x) \equiv constant$, authors in [2] are given some sufficient conditions to ensure the existence of two different weak solutions, and used the variational method and the mountain pass theorem to obtain the two weak solutions of problem (5) which converge to two
solutions of its limit problems, and the existence of infinitely many solutions to its limit problem:

\[
\begin{aligned}
(-\Delta)^{s_i} u + \lambda V(x) u &= a(x)|u|^{q_i-2} u + b(x)|u|^{p_i-2} u \text{ in } \Omega, \\
u &= 0 \text{ in } \mathbb{R}^N \setminus \Omega.
\end{aligned}
\]  

(5)

In addition, authors studied the multiplicity and concentration of solutions for a Hamiltonian system driven by the fractional Laplace operator with variable-order derivative in [3]. For \( s(\cdot) = 1 \), \( p(x) \equiv \text{constant} \), and \( A = 0 \), in [4], authors obtained the multiplicity and concentration of the positive solution of the following indefinite semilinear elliptic equations involving concave-convex nonlinearities by the variational method:

\[
\begin{aligned}
\{ -\Delta u + V_{s}(x) u &= f(x)|u|^{q-2} u + g(x)|u|^{p-2} u \text{ in } \mathbb{R}^N, \\
u &\geq 0 \text{ in } \mathbb{R}^N.
\end{aligned}
\]  

(6)

For \( s(\cdot), p(x), q(x) \equiv \text{constant} \), and \( A = 0 \), in [5], under appropriate assumptions, Peng et al. obtained the existence, multiplicity, and concentration of nontrivial solutions for the following indefinite fractional elliptic equation by using the Nehari manifold decomposition:

\[
\begin{aligned}
(-\Delta)^{s} u + V_{s}(x) u &= a(x)|u|^{q-2} u + b(x)|u|^{p-2} u \text{ in } \mathbb{R}^N, \\
u &\geq 0 \text{ in } \mathbb{R}^N.
\end{aligned}
\]  

(7)

In [1], by using the Nehari manifold decomposition, authors studied the concave-convex elliptic equation involving the fractional order nonlinear Schrödinger equation:

\[
(-\Delta)^{s} u + V_{s}(x) u = f(x)|u|^{q-2} u + g(x)|u|^{p-2} u \text{ in } \mathbb{R}^N.
\]  

(8)

Some sufficient conditions for the existence of nontrivial solutions of equation (8) are obtained. Nevertheless, only a few papers see [6–12] deal with the existence and multiplicity of fractional magnetic problems. Some papers see [8, 13–16] deal with the solvability of Kirchhoff problems. Inspired by above, we are interested in the existence and multiplicity of solutions to problem (1) with variable growth and steep potential in \( \mathbb{R}^N \). As far as we know, this is the first time to study the multiplicity of nontrivial solutions of the indefinite fractional elliptic equation driven by the variable-order fractional magnetic Laplace operator with variable exponents and steep potential in \( \mathbb{R}^N \). This result was improved in the recent paper [1].

It is worth mentioning that in this paper, we not only obtain the existence and multiplicity results of nontrivial solutions of the variable-order fractional magnetic Schrödinger equation with variable growth and steep well potential in \( \mathbb{R}^N \) but also our main results are based on the study for the decomposition of Nehari manifolds. On the one hand, relative to [1], we extend the exponent to variable exponent, thus introducing the variable exponent Lebesgue space. In addition, compared with [2], we extend the range of \( p(x) \) to \( (2, \infty) \) and the research range from the bounded region \( \Omega \) to the whole space \( \mathbb{R}^N \). On the other hand, if we want to find the nontrivial solution of the equation (1) by the variational method, we need some geometry, such as a mountain structure and a link structure. However, the energy functional of equation (1) does not have the mountain structure. In order to overcome this obstacle, we seek another method, the Nehari manifold. By decomposing the Nehari manifold into three parts, we obtain the existence of nontrivial solutions of each part.

Inspired by the above works, we assume

(S1) There exist two constants \( 0 < s_0 < s_1 < 1 \) such that \( s_0 < s(x, y) < s_1 \) for all \( (x, y) \in \mathbb{R}^N \times \mathbb{R}^N \).

(S2) \( s(\cdot) \) is symmetric, that is, \( s(x, y) = s(y, x) \) for all \( (x, y) \in \mathbb{R}^N \times \mathbb{R}^N \).

(S3) \( V_{s}(\cdot) \) is a continuous function on \( \mathbb{R}^N \) and \( V_{s}(\cdot) \) is a nonempty and has finite measure. In addition, \( M^2 \{ V_{s}(\cdot) < k \} < 1 \), where \( |\cdot| \) is the Lebesgue measure and \( M \) is the best Sobolev constant (see Lemma 9).

(S4) \( \Omega = \{ x \in \mathbb{R}^N, V_{s}(x) = 0 \} \) is nonempty and has a smooth boundary with \( \Omega = \{ x \in \mathbb{R}^N, V_{s}(x) = 0 \} \).

(S5) There exists a constant \( \theta_0 > 1 \) such that

\[
\sup_{u \in D^{2,1}(\mathbb{R}^N)} \int_{\mathbb{R}^N} (u(x) - e^{(s-1)\frac{\|u(x)\|_{L^{2}(\mathbb{R}^N)}}{\|u(x)\|_{L^{s(\cdot)}(\mathbb{R}^N)}}})^2 dx + \lambda \int_{\mathbb{R}^N} |u(x)|^q dx \geq \theta_0,
\]  

for all \( \lambda > 0 \), where \( D^{2,1}(\mathbb{R}^N) \) is the Hilbert space related to the magnetic field \( A \) (see Section 2).

\[
\{ V_{s}(\cdot) \} \{ V_{s}(\cdot) > \theta_0 \} \{ 2A, A_{2}, A_{3}, A_{4} \} > 0
\]  

where

\[
A_{1} = \left( \frac{(\rho^{2} - 2)q - \rho^{2} + 2}{2(\rho^{2} - q^{2})} \right) \left( \frac{2 - q}{(\rho^{2} - q^{2})} \right) \left( \frac{\theta - 1}{\theta_{0}} \right) \left( \frac{2 - q}{\theta_{0}} \right) \left( \frac{\theta - 1}{\theta_{0}} \right)^{\frac{s}{2}} \left( \frac{2 - q}{\theta_{0}} \right)^{\frac{s}{2}}
\]

\[
A_{2} = \left( \frac{(\rho^{2} - 2)q - \rho^{2} + 2}{2(\rho^{2} - q^{2})} \right) \left( \frac{2 - q}{(\rho^{2} - q^{2})} \right) \left( \frac{\theta - 1}{\theta_{0}} \right) \left( \frac{2 - q}{\theta_{0}} \right) \left( \frac{\theta - 1}{\theta_{0}} \right)^{\frac{s}{2}} \left( \frac{2 - q}{\theta_{0}} \right)^{\frac{s}{2}}
\]

\[
A_{3} = \left( \frac{(\rho^{2} - 2)q - \rho^{2} + 2}{2(\rho^{2} - q^{2})} \right) \left( \frac{2 - q}{(\rho^{2} - q^{2})} \right) \left( \frac{\theta - 1}{\theta_{0}} \right) \left( \frac{2 - q}{\theta_{0}} \right) \left( \frac{\theta - 1}{\theta_{0}} \right)^{\frac{s}{2}} \left( \frac{2 - q}{\theta_{0}} \right)^{\frac{s}{2}}
\]
Theorem 3. Suppose that \( (\Omega, \nu) \) is a nonempty open set \( \Omega \subseteq \mathbb{R}^N \) such that \( g(x) > 0 \) in \( \Omega \). Then, equation (1) admits at least one nontrivial solution for all \( \lambda > 1/kM \mid \{V^* < k\} \) \].

Theorem 2. Under \((V_1)-(V_3), (H_1)-(H_2), (H_4)\), and \((H_2)\), there exists a nonempty open set \( \Omega \subseteq \mathbb{R}^N \) such that \( g(x) > 0 \) in \( \Omega \). Then, equation (1) admits at least one nontrivial solution for all \( \lambda > 1/kM \mid \{V^* < k\} \) \].

Theorem 3. Suppose that \((S_1),(S_2),(V_1)-(V_3), \text{ and } (H_1)-(H_4)\) are satisfied. Then, there exists \( \lambda^* > 0 \) such that for every \( \lambda > \lambda^* \), equation (1) has at least two nontrivial solutions.

Remark 4. Generally speaking, if \( s(\cdot) : \mathbb{R}^N \times \mathbb{R}^N \rightarrow (0,1) \) is a continuous function, magnetic field \( A \in C^{0,\alpha}(\mathbb{R}^N, \mathbb{R}^N) \) with \( \alpha \in (0,1] \), then the variable-order fractional magnetic Laplacian can be defined as for each given
\[ \| u \|_{L^p(\mathbb{R}^N, \mathbb{C})} = \| \cdot \|_{L^p(\mathbb{R}^N, \mathbb{R})}, \]

we know that in the variable exponent Lebesgue space, the Hölder inequality is still valid. For all \( u \in L^p(x)(\mathbb{R}^N, \mathbb{C}), \) \( v \in L^q(x)(\mathbb{R}^N, \mathbb{C}) \) with \( p(x) \in (1, \infty) \), the following inequality holds

\[
\int_{\mathbb{R}^N} |u(x)|^p |v(x)|^q \leq \left( \frac{1}{p(x)} + \frac{1}{q(x)} \right) \int_{\mathbb{R}^N} |u(x)|^{p(x)} \left( \int_{\mathbb{R}^N} |v(x)|^{q(x)} \right)^{\frac{1}{q(x)}} \leq 2 \| u \|_{L^{p(x)}(\mathbb{R}^N, \mathbb{C})} \| v \|_{L^{q(x)}(\mathbb{R}^N, \mathbb{C})}.
\]

(20)

Define

\[
D^{(\cdot)}(\mathbb{R}^N, \mathbb{C}) = \left\{ u \in L^2(\mathbb{R}^N, \mathbb{C}) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s_0}} \, dx \, dy < \infty \right\}.
\]

(21)

Equip \( D^{(\cdot)}(\mathbb{R}^N, \mathbb{C}) \) with the inner product

\[
\langle u, v \rangle_{D^{(\cdot)}} = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s_0}} \, dx \, dy + \int_{\mathbb{R}^N} u(x)v(x) \, dx,
\]

and the corresponding norm \( \| u \|^2_{D^{(\cdot)}} = \langle u, u \rangle_{D^{(\cdot)}}. \)

Especially, if \( s(\cdot) \equiv \text{constant} \), then the space \( D^{(s)}(\mathbb{R}^N, \mathbb{C}) \) is the usual fractional Sobolev space \( D^{s}(\mathbb{R}^N, \mathbb{C}) \).

**Lemma 5** (see [3] Lemma 5). Let \( p \in [2, 2_+^*], 2_+^* = 2N/N - 2s_0 \), if \( N > 2; 2_+^* = \infty \) if \( N \leq 2 \). The embedding \( D^{s}(\mathbb{R}^N, \mathbb{R}) \hookrightarrow D^{(\cdot)}(\mathbb{R}^N, \mathbb{R}) \hookrightarrow D^s(\mathbb{R}^N, \mathbb{R}) \hookrightarrow L^p(\mathbb{R}^N, \mathbb{R}) \) are continuous.

For each function \( u : \mathbb{R}^N \to \mathbb{C} \), set

\[
[u]_{D^{(\cdot)}}^2 = \int_{\mathbb{R}^N} \frac{|u(x) - e^{i(x-y)}A(x+y/2)|^2}{{|x-y|}^{N+2s_0}} \, dx \, dy,
\]

(23)

and the corresponding norm is defined as \( \| u \|_{D^{(\cdot)}}^2 = [u]_{D^{(\cdot)}}^2 \). Let \( D \) be the space of measurable functions \( u : \mathbb{R}^N \to \mathbb{C} \) such that \( \| u \|_{D^{(\cdot)}}^2 < \infty \); then, \( (D, \langle \cdot, \cdot \rangle_{D^{(\cdot)}}) \) is a Hilbert space. If we let \( D^{s(\cdot)}(\mathbb{R}^N, \mathbb{C}) \) as the closure of \( C_0^{s(\cdot)}(\mathbb{R}^N, \mathbb{C}) \) in \( D \), then \( D^{s(\cdot)}(\mathbb{R}^N, \mathbb{C}) \) is a Hilbert space.

**Lemma 6.** For each compact subset \( W \subset \mathbb{R}^N \), the embedding \( D^{(\cdot)}(\mathbb{R}^N, \mathbb{C}) \hookrightarrow D^{(\cdot)}(W, \mathbb{C}) \) is continuous.

**Proof.** Fixed any compact subset \( W \subset \mathbb{R}^N \), for any \( u \in D^{(\cdot)}(W, \mathbb{C}) \), we have

\[
\| u \|^2_{L^2(W, \mathbb{C})} = \int_W |u(x)|^2 \, dx + \int_W \int_W \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s_0}} \, dx \, dy
\]

\[
\leq \int_W |u(x)|^2 \, dx + 2 \int_W \int_W \frac{|u(x) - e^{i(x-y)}A(x+y/2)|^2 u(y)|^2}{|x-y|^{N+2s_0}} \, dx \, dy
\]

\[
+ 2 \int_W |u(x)|^2 \frac{|e^{i(x-y)}A(x+y/2) - 1|^2}{|x-y|^{N+2s_0}} \, dx \, dy
\]

\[
\leq 2 \int_W |u(x)|^2 + 2 \int_W \int_W \frac{|u(x) - e^{i(x-y)}A(x+y/2)|^2 u(y)|^2}{|x-y|^{N+2s_0}} \, dx \, dy
\]

\[
+ 2 \int_W |u(x)|^2 \frac{|e^{i(x-y)}A(x+y/2) - 1|^2}{|x-y|^{N+2s_0}} \, dx \, dy \leq 2 \| u \|^2_{D^{(\cdot)}} + 2J,
\]

(24)

where

\[
J = \int_W |u(y)|^2 \left( \int_{W \cap \{|x-y| > 1\}} \frac{|e^{i(x-y)}A(x+y/2) - 1|^2}{|x-y|^{N+2s_0}} \, dx \right) \, dy
\]

\[
+ \int_W |u(y)|^2 \left( \int_{W \cap \{|x-y| \leq 1\}} \frac{|e^{i(x-y)}A(x+y/2) - 1|^2}{|x-y|^{N+2s_0}} \, dx \right) \, dy
\]

\[
= J_1 + J_2.
\]

(25)

Since \( |e^{it} - 1| \leq 2 \), we have

\[
J_1 \leq 4 \int_W |u(y)|^2 \left( \int_{W \cap \{|x-y| > 1\}} \frac{1}{|x-y|^{N+2s_0}} \, dx \right) \, dy
\]

\[
\leq 4 \int_W |u(y)|^2 \left( \int_{W \cap \{|x-y| > 1\}} \frac{1}{|x-y|^{N+2s_0}} \, dx \right) \, dy
\]

\[
= 4 \int_W |u(y)|^2 \left( \int_{W \cap \{|x-y| > 1\}} \frac{1}{|x-y|^{N+2s_0}} \, dx \right) \, dy \leq C_1 \int_W |u(y)|^2 \, dy
\]

\[
\leq C_1 \int_{\mathbb{R}^N} |u(y)|^2 \, dy = C_1 \| u \|^2_{L^2(\mathbb{R}^N, \mathbb{C})}.
\]

(26)
\[ C_2 |x - y|^2, \text{ for } |x - y| \leq 1, x, y \in W. \] 

Thus, we obtain
\[
I_2 = \int_{W} |u(y)|^2 \left( \int_{W} \frac{|e^{i(x-y) \cdot A(\lambda^2/2)} - 1|^2}{|x-y|^N + 2(x,y)} \, dx \right) dy \\
\leq \int_{W} |u(y)|^2 \left( \int_{W} \frac{C_2}{|x-y|^N + 2(x,y)} \, dx \right) dy \\
\leq C_2 \int_{W} |u(y)|^2 \, dy \\
\leq C_3 \int_{\mathbb{R}^N} |u|^2 \, dy \leq C_3 \|u\|^2_{L^2(\mathbb{R}^N)}. 
\]

(27)

By (24)-(27), we can easily get that
\[
\|u\|^2_{D^0(\Omega, \mathbb{C})} \leq 2 \|u\|^2_{D_0(\Omega, \mathbb{C})} + 2C_1 \|u\|^2_{L^2(\mathbb{R}^N)} + 2C_2 \|u\|^2_{L^\infty(\mathbb{R}^N, \mathbb{C})} \leq C_3 \|u\|^2_{L^2(\mathbb{R}^N)}. 
\]

(28)

which implies that the embedding \( D^0(\Omega, \mathbb{C}) \) is continuously embedded into \( D^0(\mathbb{R}^N, \mathbb{C}) \).

Through the above lemma, we know that \( D^0(\mathbb{R}^N, \mathbb{C}) \hookrightarrow D^0(\mathbb{R}^N, \mathbb{C}) \), and from Theorem 2.1 of [2], we know that for \( \Omega \) a bounded subset of \( \mathbb{R}^N \) and \( p : \Omega \rightarrow [1, \infty) \) is continuous functions, \( D^0(\Omega, \mathbb{C}) \) is continuously embedded into \( L^p(\Omega, \mathbb{C}) \), so we seek another method to prove the size relationship between \( \int_{\mathbb{R}^N} |u(x)|^p \, dx \), \( \int_{\mathbb{R}^N} |u(x)|^q \, dx \), and \( \|u\|_p \).

**Lemma 7** (see [6] Lemma 10). For every \( u \in D^0(\mathbb{R}^N, \mathbb{C}) \), it holds \( \|u\|_p \leq M\|u\|_{D_0(\mathbb{R}^N, \mathbb{C})} \), for every \( u \in D^0(\mathbb{R}^N, \mathbb{C}) \). (29)

**Remark 8** (see [6] Remark 9). There holds
\[
|u(x) - e^{i(x-y) \cdot A(\lambda^2/2)}(y)| \geq \|u(x) - |u(x)| \|, \text{ for a.e. } x, y \in \mathbb{R}^N. 
\]

(30)

**Lemma 9.** Let \( p \in [2, 2^*_0] \), where \( 2^*_0 = 2N/2 - 3s_0 \) if \( N > 2 \);
\( 2^*_0 = \infty \) if \( N \leq 2 \). \( D^0(\mathbb{R}^N, \mathbb{C}) \) is continuously embedded into \( L^0(\mathbb{R}^N, \mathbb{C}) \). Moreover, if \( s_0 \in (1/2, 1) \), then \( D^0(\mathbb{R}^N, \mathbb{C}) \) can be continuously embedded into \( L^{2\infty}(\mathbb{R}^N, \mathbb{C}) \); that is, there exists a constant \( M > 0 \) such that
\[
\|u\|_{2\infty} \leq M\|u\|_{D_0(\mathbb{R}^N, \mathbb{C})}. 
\]

(31)

**Proof.** By Lemma 7, we know that for every \( u \in D^0(\mathbb{R}^N, \mathbb{C}) \), it holds \( \|u\|_{D^0(\mathbb{R}^N, \mathbb{C})} \). By Lemma 5, we know that for
\[
D^0(\mathbb{R}^N, \mathbb{R}) \hookrightarrow L^p(\mathbb{R}^N, \mathbb{R}) \text{ is continuous. In light of Remark 8, one has}
\]
\[
\|u\|^p_{L^p(\mathbb{R}^N, \mathbb{R})} = \|u\|^p_{L^p(\mathbb{R}^N, \mathbb{R})} \leq M\|u\|^p_{L^p(\mathbb{R}^N, \mathbb{R})} 
\]

(32)

From the above inequality, we immediately obtain the embedding \( D^0(\mathbb{R}^N, \mathbb{C}) \hookrightarrow L^p(\mathbb{R}^N, \mathbb{C}) \) which is continuous.

For \( \lambda > 0 \), define
\[
\langle u, v \rangle = \int_{\mathbb{R}^N} \left( u(x) - e^{i(x-y) \cdot A(\lambda^2/2)}(y) \right) \overline{\left( v(x) - e^{i(x-y) \cdot A(\lambda^2/2)}(y) \right)} \, dx dy
\]

(33)

Set \( E = \{ u \in D^0(\mathbb{R}^N, \mathbb{C}) : \int_{\mathbb{R}^N} |u|^2 \, dx < \infty \} \) be equipped with the inner product \( \langle u, v \rangle_E = \langle u, v \rangle \), (i.e., \( \lambda = 1 \) in \( \langle u, v \rangle \)). Obviously, \( \|u\|_E \leq \|u\|_1 \) for \( \lambda \geq 1 \). Set \( X_1 = (X_1, \|\cdot\|_1) \), Combining condition \( (V_4) \) and fractional Sobolev inequality, we could get
\[
\int_{\mathbb{R}^N} |u(x)|^2 \, dx \leq \int_{\{V^+ < k\}} |u(x)|^2 \, dx + \int_{\{V^+ \geq k\}} |u(x)|^2 \, dx \\
\leq M_1 \|u\|^2_{D_0(\mathbb{R}^N, \mathbb{C})} \|V^+ < k\| + \int_{\mathbb{R}^N} V^+ u^2 \, dx \\
= M_1 \|V^+ < k\| \left( \int_{\mathbb{R}^N} |u|^2 \, dx + |u|^2_{D_0(\mathbb{R}^N, \mathbb{C})} \right)
\]

(34)

which shows that
\[
\int_{\mathbb{R}^N} |u(x)|^2 \, dx \leq \frac{1}{1 - M_1 \|V^+ < k\| - M_1 \|u\|^2_{D_0(\mathbb{R}^N, \mathbb{C})}} \left( \sum_{\{V^+ < k\}} |u|^2_{D_0(\mathbb{R}^N, \mathbb{C})} + \int_{\mathbb{R}^N} V^+ u^2 \, dx \right) \\
\leq \max \left\{ M_1 \|V^+ < k\|, M_1 \|u\|^2_{D_0(\mathbb{R}^N, \mathbb{C})} \right\} \left( \frac{|u|^2_{D_0(\mathbb{R}^N, \mathbb{C})}}{1 - M_1 \|V^+ < k\|} + \int_{\mathbb{R}^N} V^+ u^2 \, dx \right)
\]

(35)
From the above inequality, it holds that
\[
[u]_{\mathcal{M}}^2 + \int_{\mathbb{R}^N} |u(x)|^2 dx \leq \left( 1 + \max \left\{ M^2 \mid \{ V^+ < k \} \right\} \right) \|u\|_{\mathcal{M}}^2,
\]
(36)
which shows that \( X \) is continuously embedded into \( \mathcal{D}_{\mathcal{M}}^{(1)}(\mathbb{R}^N, \mathbb{C}) \). Similarly, for all \( \lambda \geq 1/kM^2 \mid \{ V^+ < k \} \), there holds
\[
\int_{\mathbb{R}^N} |u(x)|^2 dx \leq \frac{1}{1-M^2 \mid \{ V^+ < k \} \mid} \cdot \left[ M^2 \mid \{ V^+ < k \} \mid \|u\|_{\mathcal{M}}^2 + \lambda \int_{\mathbb{R}^N} V^+ u^2 dx \right] = \frac{M^2 \mid \{ V^+ < k \} \mid \|u\|_{\mathcal{M}}^2}{1-M^2 \mid \{ V^+ < k \} \mid} \cdot \int_{\mathbb{R}^N} V^+ u^2 dx = \frac{M^2 \mid \{ V^+ < k \} \mid \|u\|_{\mathcal{M}}^2}{1-M^2 \mid \{ V^+ < k \} \mid} \cdot \int_{\mathbb{R}^N} V^+ u^2 dx
\]
(37)
where \( \theta = 1 - M^2 \mid \{ V^+ < k \} \mid M^2 \mid \{ V^+ < k \} \mid < 1 \). In addition, we have
\[
\int_{\mathbb{R}^N} |u(x)|^{p(x)} dx = \int_{\mathbb{R}^N} |u(x)|^{p(x)} \cdot |u(x)|^2 dx
\]
\[
\leq \max \left\{ \|u\|_{\mathcal{M}}^{p(x) - 2} \right\} \int_{\mathbb{R}^N} |u(x)|^2 dx
\]
\[
\leq \max \left\{ M^{p(x)} \left( \int_{\mathbb{R}^N} |u(x)|^2 dx \right)^{- \frac{p(x)}{2}} + \|u\|_{\mathcal{M}}^{p(x) - 2}, M^{p(x)} \right\}
\]
(38)
This together \( M^2 \mid \{ V^+ < k \} \mid < 1 \) yields that
\[
\int_{\mathbb{R}^N} |u(x)|^{p(x)} dx \leq \max \left\{ M^{p(x)} \left( \frac{M^2 \mid \{ V^+ < k \} \mid \|u\|_{\mathcal{M}}^2}{1-M^2 \mid \{ V^+ < k \} \mid} \right)^{- \frac{p(x)}{2}} + \|u\|_{\mathcal{M}}^{p(x) - 2}, M^{p(x)} \right\}
\]
\[
\cdot \left\{ \frac{M^2 \mid \{ V^+ < k \} \mid \|u\|_{\mathcal{M}}^2}{1-M^2 \mid \{ V^+ < k \} \mid} \right\} \cdot \left\{ \frac{1}{\theta^x \cdot M^{p(x)} \mid \{ V^+ < k \} \mid} \right\}
\]
\[
= \frac{1}{\theta^x \cdot M^{p(x)} \mid \{ V^+ < k \} \mid} \max \left\{ M^{p(x)} \|u\|_{\mathcal{M}}^2, \|u\|_{\mathcal{M}}^{p(x) - 2} \right\}
\]
(39)
For the sake of notational simplicity, we let \( \|u\|_{\mathcal{M}}^{2} = [u]_{\mathcal{M}}^2 + \int_{\mathbb{R}^N} V^+ u^2 dx \). Hence, by condition \( (V_q) \), we have
\[
\|u\|_{\mathcal{M}}^{2} \geq \|u\|_{\mathcal{M}}^{2} \geq \frac{\theta_0 - 1}{\theta_0} \|u\|_{\mathcal{M}}^{2} \quad \text{for all } \lambda \geq 0.
\]
(40)
Related to equation (1), we think the functional \( \Psi_\lambda : X_\lambda \to \mathbb{R} \),
\[
\Psi_\lambda(u) = \frac{1}{2} \|u\|_{\mathcal{M}}^2 - \frac{1}{2} \int_{\mathbb{R}^N} V^{-\frac{1}{2}} u^2 dx - \int_{\mathbb{R}^N} \left( \frac{f(x)}{\vartheta(x)} \|u\|_{p(x)}^{q(x)} + \frac{g(x)}{\rho(x)} \|u\|_{p(x)}^{r(x)} \right) dx
\]
(41)
In fact, we can easily verify that \( \Psi_\lambda \) is well-defined of class \( C^1 \) in \( X_\lambda \) and
\[
\langle \Psi'_\lambda(u), v \rangle = \langle u, v \rangle_\lambda - \mathfrak{R} \int_{\mathbb{R}^N} V^{-\frac{1}{2}} u\bar{v} dx - \mathfrak{R} \int_{\mathbb{R}^N} \left( \frac{f(x)}{\vartheta(x)} \|u\|_{p(x)}^{q(x)} - u \right) \bar{v} dx
\]
(42)
for all \( u, v \in X_\lambda \). Therefore, if \( u \in X_\lambda \) is a critical point of \( \Psi_\lambda \), then \( u \) is a solution of equation (1). Since the energy functional \( \Psi_\lambda \) is unbounded below on \( X_\lambda \), in order to overcome this problem, we use the Nehari manifold \( \mathcal{N}_\lambda = \{ u \in X_\lambda \setminus \{ 0 \} : \langle \Psi'_\lambda(u), u \rangle = 0 \} \) to study the energy functional. In addition, we also note that \( \mathcal{N}_\lambda \) contains every nonzero solution of equation (1). Especially, all critical points of the loci should be located in \( \mathcal{N}_\lambda \), and the local minimizers on \( \mathcal{N}_\lambda \) are usually critical points of \( \Psi_\lambda \).

3. Main Results

To start with, we can get an estimate of \( \Psi_\lambda \). Then, we will discuss some basic properties of \( \mathcal{N}_\lambda \). Finally, we prove Theorem 2 and Theorem 3 using the variational methods.

**Lemma 10.** \( \Psi_\lambda \) is coercive and bounded below on \( \mathcal{N}_\lambda \). Furthermore, one has
\[
\Psi_\lambda(u) \geq \max \left\{ - \frac{2-q^+}{2p^-} \frac{\theta_0 q^+}{\theta (\theta_0 - 1)(p^- - 2)} \right\} \|u\|_{\mathcal{M}}^{2} - \left( \frac{(p^- - q^+) \|f\|_{L^{2q(x)}(\mathbb{R}^N)}}{q^-} \right)^{\frac{2}{2-q^+}} - \left( \frac{\theta_0 q^+}{\theta (\theta_0 - 1)(p^- - 2)} \right)^{\frac{2}{2-q^+}} \|u\|_{\mathcal{M}}^{2} \right\}
\]
(43)

**Proof.** If \( u \in \mathcal{N}_\lambda \), in view of (37), (40), and H"older inequality, it gains
After observation, we can get that $t_0$, i.e., positive critical points of $	heta - 2 - u(\|x\|_2)\parallel u_{x_0} - \lambda$ is coercive and bounded below on $(\|x\|_2)$.

This map is divided into three parts $\max \left\{ \theta - 2 - \frac{\theta_0 q^+}{\theta_0 - 1} \parallel u_{x_0}^p, \frac{1}{\theta_0 - \theta_0 q^+} \parallel u_{x_0}^p \right\}$.

\begin{align*}
\Psi_\lambda(t) &= \Psi_\lambda(u) - \frac{1}{p} \left\langle \Psi'_\lambda(u), u \right\rangle \\
&= \left( \frac{1}{2} - \frac{1}{p} \right) \|u\|_{L^2(V)}^2 - \int_{\mathbb{R}^N} \left( \frac{1}{q} g(x) \right) |u(x)|^{q(x)} \, dx - \int_{\mathbb{R}^N} \frac{g(x)}{p(x)} |u(x)|^{p(x)} \, dx \\
&= - \int_{\mathbb{R}^N} \left( \frac{1}{q} - \frac{1}{p} \right) f(x) |u(x)|^{q(x)} \, dx - \left( 1 - \frac{1}{p} \right) \|u\|^2_{L^2(V)} \\
&= - \left( \frac{1}{q} - \frac{1}{p} \right) \|f\|_{L^{p,q}(\mathbb{R}^N)} \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right)^{\frac{p}{2}} - \left( \frac{1}{q} - \frac{1}{p} \right) \|\lambda\|_{L^{p,q}(\mathbb{R}^N)} \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right)^{\frac{p}{2}}.
\end{align*}

Therefore, $\Psi_\lambda$ is coercive and bounded below on $\mathcal{N}_\lambda$.

We know that $\mathcal{N}_\lambda$ is linked to the behavior of the function of the form $L_u(t): t \rightarrow \Psi_\lambda(tu)$ for $t > 0$. This map is called as the fiber map which can be traced back to basic works [1, 22, 23]. If $u \in X_\lambda$, then

\begin{align*}
L_u(t) &= \frac{t^2}{2} \|u\|_{L^2(V)}^2 - \int_{\mathbb{R}^N} \frac{f(x)}{q(x)} |tu(x)|^{q(x)} \, dx - \int_{\mathbb{R}^N} \frac{g(x)}{p(x)} |tu(x)|^{p(x)} \, dx \\
L'_u(t) &= t \|u\|_{L^2(V)}^2 - \int_{\mathbb{R}^N} f(x) |u(x)|^{q(x)-1} |u(x)|^{q(x)} \, dx - \int_{\mathbb{R}^N} g(x) |u(x)|^{p(x)-1} |u(x)|^{p(x)} \, dx; \\
L''_u(t) &= \|u\|_{L^2(V)}^2 - \int_{\mathbb{R}^N} f(x) |q(x)-1| |u(x)|^{q(x)} \, dx - \int_{\mathbb{R}^N} g(x)(p(x)-1) |u(x)|^{p(x)} \, dx.
\end{align*}

After observation, we can get that

\begin{align*}
tL'_u(t) &= \|tu\|_{L^2(V)}^2 - \int_{\mathbb{R}^N} f(x) |tu(x)|^{q(x)} \, dx - \int_{\mathbb{R}^N} g(x) |tu(x)|^{p(x)} \, dx
\end{align*}

and thus, for $u \in X_\lambda \setminus \{0\}$ and $t > 0$, $L'_u(t) = 0$ if and only if $t u \in \mathcal{N}_\lambda$, i.e., positive critical points of $L_u$ correspond points on the Nehari manifold. Especially, $L'_u(1) = 0$ if and only if $u \in \mathcal{N}_\lambda$. We found that $\mathcal{N}_\lambda$ can be divided into three parts corresponding local minimal, local maximum, and points of inflection. Based on the above, we can define

\begin{align*}
\mathcal{N}_\lambda^+ &= \left\{ u \in \mathcal{N}_\lambda : L'_u(1) > 0 \right\}; \\
\mathcal{N}_\lambda^0 &= \left\{ u \in \mathcal{N}_\lambda : L'_u(1) = 0 \right\}; \\
\mathcal{N}_\lambda^- &= \left\{ u \in \mathcal{N}_\lambda : L'_u(1) < 0 \right\}.
\end{align*}

For each $u \in \mathcal{N}_\lambda$, we can find that

\begin{align*}
L'_u(1) &= \|u\|_{L^2(V)}^2 - \int_{\mathbb{R}^N} (q(x)-1)f(x) |u(x)|^{q(x)} \, dx \\
&- \int_{\mathbb{R}^N} (p(x)-1)g(x) |u(x)|^{p(x)} \, dx.
\end{align*}

Now, we will deduce some results of $\mathcal{N}_\lambda^+, \mathcal{N}_\lambda^0$, and $\mathcal{N}_\lambda^-$. 

**Lemma 11.** Assume $u_0$ is a local minimizer of $\Psi_\lambda$ on $\mathcal{N}_\lambda$ and $u_0 \notin \mathcal{N}_\lambda^0$, then $\Psi'_\lambda(u_0) = 0$ in $X_\lambda^\prime$.

**Proof.** If $u_0$ is a local minimizer of $\Psi_\lambda$ on $\mathcal{N}_\lambda$, then $u_0$ is a solution of the optimization problem

\begin{equation}
\min_{\|u\|_{L^2(V)}^2} \Psi_\lambda(u) \text{ subject to } K(u) = 0,
\end{equation}

where $K(u) = \|u\|_{L^2(V)}^2 - \int_{\mathbb{R}^N} f(x) |u(x)|^{q(x)} \, dx - \int_{\mathbb{R}^N} g(x) |u(x)|^{p(x)} \, dx$. Consequently, by the theory of Lagrange multipliers, there exists $\nu \in \mathbb{R}$ such that $\Psi'_\lambda(u_0) = \nu K'(u_0)$. Therefore,

\begin{equation}
\left\langle \Psi'_\lambda(u_0), u_0 \right\rangle = \nu \left\langle K'(u_0), u_0 \right\rangle.
\end{equation}

It follows from $u_0 \in \mathcal{N}_\lambda$ that

\begin{equation}
\|u_0\|_{L^2(V)}^2 - \int_{\mathbb{R}^N} f(x) |u_0(x)|^{q(x)} \, dx - \int_{\mathbb{R}^N} g(x) |u_0(x)|^{p(x)} \, dx = 0.
\end{equation}

Thus,

\begin{align*}
\left\langle K'(u_0), u_0 \right\rangle &= 2\|u_0\|_{L^2(V)}^2 - \int_{\mathbb{R}^N} q(x) f(x) |u_0(x)|^{q(x)} \, dx \\
&- \int_{\mathbb{R}^N} g(x)(p(x)-1) |u_0(x)|^{p(x)} \, dx.
\end{align*}

If $u_0 \notin \mathcal{N}_\lambda^0$, then $\left\langle K'(u_0), u_0 \right\rangle \neq 0$. In view of (50), it gains $\nu = 0$. 
Lemma 12.

(1) \( \forall u \in \mathcal{A}_\lambda^+ \cup \mathcal{A}_\lambda^0 \), one has \( \int_{\mathbb{R}^N} f(x) |u|^{q(x)} \, dx > 0 \)

(2) \( \forall u \in \mathcal{A}_\lambda^+ \), one has \( \int_{\mathbb{R}^N} g(x) |u|^{p(x)} \, dx > 0 \)

Proof. By the definitions of \( \mathcal{A}_\lambda^+ \) and \( \mathcal{A}_\lambda^0 \), we can obtain

\[
0 \leq L_\lambda^+(1) = \|u\|_{\lambda,V}^2 - \int_{\mathbb{R}^N} (q(x) - 1) f(x) |u|^{q(x)} \, dx
- \int_{\mathbb{R}^N} (p(x) - 1) g(x) |u|^{p(x)} \, dx + \|u\|_{\lambda,V}^2
- \int_{\mathbb{R}^N} (q^* - 1) f(x) |u|^{q^*(x)} \, dx
- \int_{\mathbb{R}^N} (p^* - 1) g(x) |u|^{p^*(x)} \, dx < (2 - p^-) \|u\|_{\lambda,V}^2
- (q^-) \int_{\mathbb{R}^N} f(x) |u|^{q(x)} \, dx.
\]

(53)

It is easy to get that \( \int_{\mathbb{R}^N} f(x) |u|^{q(x)} \, dx > p^- - 2/p^- - q^- \|u\|_{\lambda,V}^2 \geq 0 \). It follows from the definition of \( \mathcal{A}_\lambda^+ \) that

\[
0 > L_\lambda^+(1) = \|u\|_{\lambda,V}^2 - \int_{\mathbb{R}^N} (q(x) - 1) f(x) |u|^{q(x)} \, dx
- \int_{\mathbb{R}^N} (p(x) - 1) g(x) |u|^{p(x)} \, dx \geq \|u\|_{\lambda,V}^2
- \int_{\mathbb{R}^N} (q^* - 1) f(x) |u|^{q^*(x)} \, dx
- \int_{\mathbb{R}^N} (p^* - 1) g(x) |u|^{p^*(x)} \, dx
= (2 - q^-) \|u\|_{\lambda,V}^2 - (p^- - q^-) \int_{\mathbb{R}^N} g(x) |u|^{p(x)} \, dx.
\]

(54)

which implies that \( \int_{\mathbb{R}^N} g(x) |u|^{p(x)} \, dx > 2 - q^- / p^- - q^- \|u\|_{\lambda,V}^2 \geq 0 \).

Lemma 13. Let the condition \((H_\lambda),(H_\lambda^0)\), and \((V_\lambda),(V_\lambda^0)\) are satisfied. Then, for all \( \lambda \geq 1/kM^2 \) \( \{ V < k \} \), one has \( \mathcal{A}_\lambda^0 = \emptyset \).

Proof. If the conclusion does not hold, then there exists \( \lambda \geq 1/kM^2 \) \( \{ V < k \} \), such that \( \mathcal{A}_\lambda^0 \neq \emptyset \). Then, for \( u \in \mathcal{A}_\lambda^0 \), by (40), (48), and the Hölder inequality, we have

\[
0 = L_\lambda^0(1) = \|u\|_{\lambda,V}^2 - \int_{\mathbb{R}^N} (q(x) - 1) f(x) |u|^{q(x)} \, dx
- \int_{\mathbb{R}^N} (p(x) - 1) g(x) |u|^{p(x)} \, dx < (2 - p^-) \|u\|_{\lambda,V}^2
- (q^-) \int_{\mathbb{R}^N} f(x) |u|^{q(x)} \, dx.
\]

(55)

This means that

\[
\frac{q^- - 1}{2} \frac{\|u\|_{\lambda,V}^2}{\|u\|_{\lambda,V}^2} \leq \frac{q^- - 1}{p^-} \|u\|_{\lambda,V}^2 \leq \frac{p^- - q^-}{p^-} \int_{\mathbb{R}^N} f(x) |u|^{q(x)} \, dx
< \frac{p^- - q^-}{p^-} \|f\|_{L^{q(x)}(\mathbb{R}^N)} \int_{\mathbb{R}^N} |u|^{p(x)} \, dx
< \frac{p^- - q^-}{p^-} \|f\|_{L^{q(x)}(\mathbb{R}^N)} \max \left\{ \frac{1}{q_\theta^{\gamma_1}}, \frac{1}{p_\theta^{\gamma_2}} \right\} \|u\|_{\lambda,V}^3.
\]

(56)

Thus, we have

\[
\|u\|_{\lambda} \leq \min \left\{ \frac{2q_\theta(p^- - q^-) \|f\|_{L^{q(x)}(\mathbb{R}^N)}^{1/\gamma_1}}{q^- (q^- - 1)(p^- - 2) \theta^{\gamma_2/2}}, \frac{2q_\theta(p^- - q^-) \|f\|_{L^{q(x)}(\mathbb{R}^N)}^{1/\gamma_1}}{q^- (q^- - 1)(p^- - 2) \theta^{\gamma_2/2}} \right\} \|u\|_{\lambda,V}^3.
\]

(57)

From (48), we seem to easily get that

\[
(2 - q^-) \|u\|_{\lambda,V}^2 - (p^- - q^-) \int_{\mathbb{R}^N} g(x) |u|^{p(x)} \, dx \leq \|u\|_{\lambda,V}^3
- \int_{\mathbb{R}^N} (q(x) - 1) f(x) |u|^{q(x)} \, dx - \int_{\mathbb{R}^N} (p(x) - 1) g(x) |u|^{p(x)} \, dx
= L_\lambda^0(1) = 0,
\]

(58)

which implies that

\[
\frac{2 - q^-}{p^- - q^-} \|u\|_{\lambda,V}^2 \leq \int_{\mathbb{R}^N} g(x) |u|^{p(x)} \, dx.
\]

(59)

Combining (39) and (40) with the Sobolev inequality, we have

\[
\frac{q^- - 1}{q_\theta(p^- - q^-)} \|u\|_{\lambda} \leq \frac{2 - q^-}{p^- - q^-} \|u\|_{\lambda,V}^2 \leq \int_{\mathbb{R}^N} g(x)|u|^{p(x)} \, dx
\]

\[
\leq \frac{\|g\|_{\infty}}{\theta^{\gamma_2/2} |\{ V < k \}|^{p^-/2}} \max \left\{ \|u\|_{\lambda}, M^{p^-} \right\} \|u\|_{\lambda}^3.
\]

(60)

This means that

\[
\|u\|_{\lambda} \geq \max \left\{ \frac{(\theta_\lambda - 1)(2 - q^-) \theta \land (p^+ + 2)}{|\{ V < k \}| \land (p^+ + 2)}, \frac{\theta_\lambda(p^- - q^-) \|g\|_{\infty}}{\theta^{\gamma_2/2} |\{ V < k \}|^{p^-/2}}, \frac{(\theta_\lambda - 1)(2 - q^-) \theta \land (p^+ + 2)}{|\{ V < k \}| \land (p^+ + 2)} \right\} \frac{\|u\|_{\lambda,V}^2}{\theta_\lambda(p^- - q^-) \|g\|_{\infty} M^{p^-}}.
\]

(61)
Hence, combining (57) and (61), we have
\[
|\{V^* < k\}| < \min \{A_1, A_2, A_3, A_4\} \leq \max \{A_1, A_2, A_3, A_4\},
\]
(62)
which is a contradictory with (V_5). Therefore, for all \(\lambda \geq 1/kM^2\), \(\{V^* < k\}\), one has \(\mathcal{Q}_\lambda = \emptyset\).

By Lemma 13, \(\lambda \geq 1/kM^2\), \(\{V^* < k\}\), we can easily get that \(\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-\) and define
\[
c_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda^+} \Psi_\lambda(u) \quad \text{and} \quad c_\lambda^- = \inf_{u \in \mathcal{N}_\lambda^-} \Psi_\lambda(u).
\]
(63)
Furthermore, we derive the following results.

**Lemma 14.** Under the condition (H_3), (H_4), and (V_1)-(V_5). Then, for all \(\lambda \geq 1/kM^2\), \(\{V^* < k\}\), there exists \(C_3\) such that \(c_\lambda^+ < 0 < C_3 < c_\lambda^-\). Particularly, \(c_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda^+} \Psi_\lambda(u)\).

**Proof.** Our proof is decoupled in the following two steps:

**Step 1.** We claim that there exist \(u \in \mathcal{N}_\lambda^+\) such that \(\Psi_\lambda(u) < 0\). Indeed, let \(v_0 \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda\), where \(t \in (0, 1)\) small enough. It follows from (48) that
\[
0 < L_{v_0}^t(1) = \|v_0\|_{L^2}^2 - \int_{\mathbb{R}^n} (q(x) - 1)f(x)|v_0|^qdx
- \frac{1}{p'} \int_{\mathbb{R}^n} (p(x) - 1)g(x)|v_0|^pdx - (q - p)\frac{1}{p'} \int_{\mathbb{R}^n} f(x)|v_0|^qdx.
\]
(64)
This shows that
\[
0 \leq \frac{p - 2}{p - q} \|v_0\|_{L^2}^2 < \int_{\mathbb{R}^n} f(x)\left|v_0\right|^qdx < t^2 \int_{\mathbb{R}^n} f(x)\left|v_0\right|^qdx.
\]
(65)
This yields at once that
\[
\int_{\mathbb{R}^n} f(x)\left|v_0\right|^qdx > 0.
\]
(66)
Similarly,
\[
0 < L_{v_0}^t(1) = \|v_0\|_{L^2}^2 - \int_{\mathbb{R}^n} (q(x) - 1)f(x)|v_0|^qdx
- \frac{1}{p'} \int_{\mathbb{R}^n} (p(x) - 1)g(x)|v_0|^pdx - (q - p)\frac{1}{p'} \int_{\mathbb{R}^n} f(x)|v_0|^qdx.
\]
(67)
From (67), we can easily get that
\[
\frac{2 - q^{-}}{p^{-} - q^{-}} \|v_0\|_{L^2}^2 > \int_{\mathbb{R}^n} g(x)|v_0|^pdx.
\]
(68)
Consequently, it derives from (67) and (68) that
\[
\Psi_\lambda(v_0) = \Psi_\lambda'\left(v_0, v_0\right) - \frac{1}{p} \left(\Psi_\lambda'^{'}(v_0), v_0\right) - \left(\frac{1}{2} - \frac{1}{p}\right) \|v_0\|_{L^2}^2 < \left(\frac{1}{2} - \frac{1}{p}\right) \|v_0\|_{L^2}^2
- \frac{1}{p} \left(\int_{\mathbb{R}^n} f(x)|v_0|^qdx - \frac{1}{p} \int_{\mathbb{R}^n} g(x)|v_0|^pdx\right)
+ \frac{1}{p} \left(\int_{\mathbb{R}^n} f(x)|v_0|^qdx\right) < \left(\frac{1}{2} - \frac{1}{p}\right) \|v_0\|_{L^2}^2
+ \frac{1}{p} \left(\int_{\mathbb{R}^n} f(x)|v_0|^qdx\right).
\]
(69)
Hence, \(c_\lambda^+ < 0\).

**Step 2.** We assert that there exist \(u \in \mathcal{N}_\lambda^-\) such that \(\Psi_\lambda(u) > 0\). In fact, let \(u \in \mathcal{N}_\lambda^- \subset \mathcal{N}_\lambda\). From (48), we seem to easily get that
\[
(2 - q^{-})\|u\|_{L^2}^2 - (p^{-} - q^{-}) \int_{\mathbb{R}^n} g(x)|u|^pdx \leq \|u\|_{L^2}^2
- \int_{\mathbb{R}^n} (q(x) - 1)f(x)|u|^qdx - \int_{\mathbb{R}^n} (p(x) - 1)g(x)|u|^pdx
= L_{u}^t(1) < 0,
\]
(70)
which implies that
\[
\frac{2 - q^{-}}{p^{-} - q^{-}} \|u\|_{L^2}^2 < \int_{\mathbb{R}^n} g(x)|u|^pdx.
\]
(71)
Combining (39), (40), and (71) with Sobolev inequality, we have
\[
\frac{(\zeta_0 - 1)(2 - q^+)}{\zeta_0(p^+ - q^+)} \| u^p_\lambda \|_{\lambda, V}^2 \leq 2 \frac{q^+}{p^+ - q^+} \| u^p_{\lambda, V} \|_{\lambda, V}^2 < \int_{\mathbb{R}^n} g(x) |u|^{p(x)} dx
\]

\[
\leq \frac{\| g \|_{\infty}}{\theta^{p/2}} \max \left\{ \| u^p_\lambda \|_{\lambda, V}, M^{p^+ - p} \| u^p_\lambda \|_{\lambda, V} \right\}
\]

(72)

and so

\[
\| u^p_\lambda \|_{\lambda, V} \geq \frac{\left\{ \left( \frac{(\zeta_0 - 1)(2 - q^+)\theta(p + 1/2)}{\zeta_0(p^+ - q^+)\theta} \left\| \frac{1}{\theta^{p/2}} \| u^p_\lambda \|_{\lambda, V}^2 \right\| \right\} - \left( \frac{1}{2 - q^+} - \frac{1}{p} \right) \| f \|_{L^2(\mathbb{R}^n)}^2 \right\}}{\left\{ \left( \frac{(\zeta_0 - 1)(2 - q^+)\theta(p + 1/2)}{\zeta_0(p^+ - q^+)\theta} \left\| \frac{1}{\theta^{p/2}} \| u^p_\lambda \|_{\lambda, V}^2 \right\| \right\} - \left( \frac{1}{2 - q^+} - \frac{1}{p} \right) \| f \|_{L^2(\mathbb{R}^n)}^2 \right\}} \max
\]

(73)

It follows from (44) that

\[
\Psi_\lambda(u) \geq \frac{p^+ - 3 \zeta_0 - 1}{2p^+} \| u^p_\lambda \|_{\lambda, V}^2 - \frac{1}{2p^+} \left( \frac{1}{2 - q^+} - \frac{1}{p} \right) \| f \|_{L^2(\mathbb{R}^n)}^2 \max
\]

\[\left\{ \frac{1}{\theta^{p/2}} \| u^p_\lambda \|_{\lambda, V}, M^{p^+ - p} \| u^p_\lambda \|_{\lambda, V} \right\} \right\} > \max
\]

(74)

Consequently, if \( \lambda > 1/kM^2 \) \{ \{ V^+ < k \} \}, then \( c_\lambda > C_5 \) for some \( C_5 > 0 \).

We note that if \( f, g \) and \( V_\lambda \) satisfy the hypotheses in Theorem 3, we can choose \( \varphi \in C_0^\infty(\Omega_\varphi, \mathcal{C}) \), such that \( L_\varphi(t) = \Psi_\lambda(t \varphi) = t^{p^+}/2\| \varphi \|_{\lambda, V}^2 - \int_{\Omega_\varphi} (f(x)|\varphi(q(x))|q(x)) dx - \int_{\Omega_\varphi} g(x)/p(x)|q(x)|/p(x) dx \) have \( t_0 > 0 \) and \( C_0 \) which are independent of \( \lambda \) that satisfy \( t_0 \varphi \in \mathcal{N}_\lambda \) for all \( \lambda > \lambda^* \) and

\[
\sup_{t \geq 0} L_\varphi(t) = L_\varphi(t_0) = C_0 > 0,
\]

(75)

which shows \( c_\lambda > C_0 \) for all \( \lambda > \lambda^* \).

**Lemma 15.** Assume that the conditions \( (H_1)-(H_3) \) and \( (V_1)-(V_3) \) hold, then there exists \( \lambda^* > 1/kM^2 \) \{ \{ V^+ < k \} \} such that \( \Psi_\lambda \) satisfies the \( (PS)_\varepsilon \) condition in \( X_\lambda \) for all \( c < C_0 \) and \( \lambda > \lambda^* \).

**Proof.** First, we assume \{ \( u_n \) \} is a \( (PS)_\varepsilon \) sequence with \( c < C_0 \). In view of Lemma 10, there exists a positive constant \( \tilde{C} \) related to \( \lambda \) such that \( \| u_n \|_{\lambda, V} \leq \tilde{C} \). Consequently, there is a sub-

sequence which is still denote as \{ \( u_n \) \} and \( u_0 \) in \( X_\lambda \) such that

\[
\begin{align*}
\quad\quad\quad\quad\quad\quad u_n & \rightarrow u_0 \text{ in } X_\lambda, \\
\quad\quad\quad\quad\quad\quad u_n & \rightarrow u_0 \text{ in } L^r(\mathbb{R}^n, \mathbb{C}), \text{ for } 2 \leq r \leq \infty,
\quad\quad\quad\quad\quad\quad g(x)|u_n|^{p(x)-2}u_n & \rightarrow g(x)|u_0|^{p(x)-2}u_0 \text{ in } L^p(\mathbb{R}^n, \mathbb{C}).
\end{align*}
\]

(76)

Besides, \( \Psi_\lambda'(u_0) = 0 \). Let \( v_n = u_n - u_0 \). Making use of the Vitali theorem, it holds that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^n} f(x)|v_n|^{q(x)} dx = 0.
\]

(77)

In fact, note that \( f \in L^{2-q}(\mathbb{R}^n, \mathbb{C}) \), for any \( 0 < \varepsilon < 1 \); then, there exists \( r(\varepsilon) > 0 \) such that for \( \zeta \in \mathbb{R}^n \) and \( r > r(\varepsilon) \),

\[
\int_{\mathbb{R}^n \setminus B_r(\zeta)} |f(x)|^{2-q(x)} dx < \varepsilon^2 r^2.
\]

(78)

For each \( \Omega_\varphi \subset B_r(\zeta) \), one has

\[
\begin{align*}
\int_{\Omega_\varphi} f(x)|v_n|^{q(x)} dx & \leq \max \left\{ \left( \int_{\Omega_{\varphi}} |f(x)|^{2-q(x)} dx \right)^{\frac{2-q}{2}} \right\} \max \left\{ \left( \int_{\Omega_{\varphi}} |v_n|^{2-q(x)} dx \right)^{\frac{2-q}{2}}, \left( \int_{\Omega_{\varphi}} |v_n|^{2-q(x)} dx \right)^{\frac{2-q}{2}} \right\} \\
& \leq \max \left\{ \left( \int_{\Omega_{\varphi}} |f(x)|^{2-q(x)} dx \right)^{\frac{2-q}{2}} \right\} \max \left\{ \left( \int_{\Omega_{\varphi}} |v_n|^{2-q(x)} dx \right)^{\frac{2-q}{2}}, \left( \int_{\Omega_{\varphi}} |v_n|^{2-q(x)} dx \right)^{\frac{2-q}{2}} \right\} \\
& \leq \tilde{C} \max \left\{ \left( \int_{\Omega_{\varphi}} |f(x)|^{2-q(x)} dx \right)^{\frac{2-q}{2}} \right\} \max \left\{ \left( \int_{\Omega_{\varphi}} |v_n|^{2-q(x)} dx \right)^{\frac{2-q}{2}}, \left( \int_{\Omega_{\varphi}} |v_n|^{2-q(x)} dx \right)^{\frac{2-q}{2}} \right\}.
\end{align*}
\]

(79)

It is easy to get that \( \{ f(x)|v_n|^{q(x)} \} \) is a equi-integrable on \( B_r(\zeta) \). Besides, \( f(x)|v_n|^{q(x)} \rightarrow 0 \), a.e., in \( B_r(\zeta) \). It follows from the Vitali theorem that

\[
\lim_{n \to \infty} \int_{B_r(\zeta)} f(x)|v_n|^{q(x)} dx = 0.
\]

(80)

Hence, there holds
\begin{align}
\int_{\mathbb{R}^N} f(x)|v_n|^r(x) \, dx = \int_{\mathcal{B}_0(0)} f(x)|v_n|^r(x) \, dx \\
+ \int_{\mathbb{R}^N \setminus \mathcal{B}_0(0)} f(x)|v_n|^r(x) \, dx \leq C_{10},
\end{align}
(81)
which implies that \( \lim_{n \to \infty} \int_{\mathbb{R}^N} f(x)|v_n|^r(x) \, dx = 0. \)

Next, we assert that \( u_n \to u_0 \) in \( X_1 \). In fact, by \((V_2)\), we obtain
\begin{align}
\int_{\mathbb{R}^N} v_n^2 \, dx = \int_{\{V \geq k\}} v_n^2 \, dx + \int_{\{V < k\}} v_n^2 \, dx \\
\leq \frac{1}{\lambda k} \int_{\{V \geq k\}} \lambda V^n v_n^2 \, dx \\
+ \frac{1}{\lambda k} \int_{\{V < k\}} v_n^2 \, dx \\
\leq \frac{1}{\lambda k} \|v_n\|_{L_q}^2 + o(1),
\end{align}
(82)
In light of the Hölder inequality with the Sobolev inequality, we have
\begin{align}
\int_{\mathbb{R}^N} |v_n|^{r+2} \, dx = \int_{\mathbb{R}^N} |v_n|^{r+2} \cdot |v_n|^2 \, dx \leq \max \left\{ \frac{\|v_n\|_{L_r}^2}{\lambda k}, \|v_n\|^2 \right\} \int_{\mathbb{R}^N} |v_n|^2 \, dx \\
\leq \max \left\{ \frac{\|v_n\|_{L_r}^2}{\lambda k}, \|v_n\|^2 \right\} \int_{\mathbb{R}^N} |v_n|^2 \, dx \\
\leq \frac{\|v_n\|_{L_r}^2}{\lambda k}, \quad \frac{\|v_n\|^2}{\lambda k},
\end{align}
(83)
By Bre'is-Lieb Lemma, we have
\begin{align}
\|u_n\|_{L^q}^2 = \|u_n - u_0\|_{L^q}^2 + \|u_0\|_{L^q}^2 + o(1).
\end{align}
(84)

By applying a Brézis-Lieb type result on variable exponent Lebesgue space (see [24]) and \((H_3)-(H_4)\), it is easy to obtain that
\begin{align}
\int_{\mathbb{R}^N} f(x)|u_n|^r(x) \, dx = \int_{\mathbb{R}^N} f(x)(u_n - u_0)^q(x) \, dx + \int_{\mathbb{R}^N} f(x)|u_0|^r(x) \, dx + o(1).
\end{align}
(85)
Similarly,
\begin{align}
\int_{\mathbb{R}^N} g(x)|u_n|^r(x) \, dx = \int_{\mathbb{R}^N} g(x)(u_n - u_0)^q(x) \, dx + \int_{\mathbb{R}^N} g(x)|u_0|^r(x) \, dx + o(1). \end{align}
(86)
Then, overall, we can get that \( \Psi_1(u_n) = \Psi_1(u_n) - \Psi_1(u_0) + o(1) \) and \( \Psi_1'(u_n) = o(1) \). Then, by virtue of \((77)\) and Lemma 10, we get that
\begin{align}
C_0 + C_7 + o(1) > c - \Psi_1(u_n) + o(1) = \Psi_1(u_n) - \frac{1}{p} \left< \left( \Psi_1'(u_n), v_n \right) \right>,
\end{align}
(87)
where
\begin{align}
C_7 = \min \left\{ \frac{2 - q^*}{2p} \left( \frac{\theta_0 q}{2q(\theta_0 - 1)(p - 2)} \right)^{\frac{1}{q^*}} \left( \frac{p - q^*}{q} \right)^{\frac{1}{q^*}}, \right.
\end{align}
(88)
Suppose by contradiction that \( \{v_n\} \) is not bounded in \( X_1 \). Then, there exists a subsequence still denoted by \( \{v_n\} \) such that \( \|v_n\|_{L^q} \to \infty \) as \( n \to \infty \). Hence, by virtue of \((87)\), we have
\begin{align}
C_0 + C_7 + o(1) \frac{1}{\|v_n\|_{L^q}^2} \|v_n\|^{r+2} \geq \frac{p - 2 \theta_0 - 1}{2p} \left( \frac{1}{q} - \frac{1}{p} \right) \|v_n\|^2 + o(1) \frac{1}{\|v_n\|_{L^q}^2},
\end{align}
(89)
which is contradictory since \( 1 < q^* \leq q^* < 2 < p^* \). Thus, \( \{v_n\} \) is bounded in \( X_1 \) for all \( \lambda > A' \geq 1/kM^2 \). [ \( V^* < k \) ] . That is, there exist a constant \( M_1 > 0 \) such that \( \|v_n\| \leq M_1 \). From
Assume that \( (f) \) exists a subsequence of \( \{u_n\} \) and \( \{v_n\} \) is a minimizer of \( (\mathcal{M}^\lambda) \), such that \( v_n \rightarrow v^* \) in \( X^\lambda \) sufficiently close to \( \lambda \), we can get that

\[
\left\| f\left( x \right) \right\|_{p(x)} dx = \frac{1}{\lambda^k} \max \left\{ (MM^\lambda_{-})^{p(-)} \right\} \left( \frac{\lambda k + 1}{\lambda k} \right)^{\epsilon_{\nu, \lambda}} \left\| v_n \right\|_{1, \lambda}^2 + o(1).
\]

(90)

Together \( \langle \mathcal{P}_{f}(v_n), v_n \rangle = o(1) \) with (77)-(90), there holds

\[
o(1) = \left\langle \mathcal{P}_{f}(v_n), v_n \right\rangle = \left\| v_n \right\|_{1, \lambda}^2 - \int_{\mathbb{R}^n} f\left( x \right) \left| v_n \right|^{p\left( x \right)} dx \\
\geq \frac{\theta_0 - 1}{\theta_0} \left\| v_n \right\|_{\lambda}^2 \cdot \left\| f \right\|_{\text{loc}}^\ast max \left( (MM^\lambda_{-})^{p(-)} \right) \left( \frac{\lambda k + 1}{\lambda k} \right)^{\epsilon_{\nu, \lambda}} \left\| v_n \right\|_{1, \lambda}^2 + o(1).
\]

(91)

We find that there exists \( \lambda^\ast \geq 1/kM^2 \) such that \( \{v_n \} \) large enough such that

\[
\left\| f \right\|_{\text{loc}}^\ast max \left( (MM^\lambda_{-})^{p(-)} \right) \left( \frac{\lambda k + 1}{\lambda k} \right)^{\epsilon_{\nu, \lambda}} < \frac{\theta_0 - 1}{\theta_0},
\]

(92)

for all \( \lambda > \lambda^\ast \). It follows from (91) that \( v_n \rightarrow 0 \) in \( X^\lambda \) for all \( \lambda > \lambda^\ast \).

**Theorem 16.** Assume that \( (H_1)-(H_2) \) and \( (V_1)-(V_5) \) hold, then there exists \( \lambda^\ast \geq 0 \) such that for every \( \lambda > \lambda^\ast \), \( \mathcal{P}_{f} \) has a minimizer \( u_{\lambda}^\ast \) in \( X^\lambda \) satisfying that

\[
\Psi_{\lambda}(u_{\lambda}^\ast) = c_{\lambda}^\ast = \inf_{u \in X^\lambda} \Psi_{\lambda}(u).
\]

(93)

(1) \( u_{\lambda}^\ast \) is a nontrivial solution of equation (1).

**Proof.** Combining Lemma 14 and the Ekeland variational principle in [25], there exists \( \{u_n\} \subset X^\lambda \) such that \( \{u_n\} \) is a \( (PS)_{c_{\lambda}^\ast} \) sequence for \( \mathcal{P}_{f} \). Furthermore, using Lemma 10, we can get that \( \{u_n\} \) is bounded in \( X^\lambda \). Consequently, there exists a subsequence of \( \{u_n\} \) (we still denote as \( \{u_n\} \) and \( u_{\lambda}^\ast \) in \( X^\lambda \) such that

\[
\left\| u_n \right\|_{1, \lambda}^2 \rightarrow \left\| u_{\lambda}^\ast \right\|_{1, \lambda}^2 \quad \text{in } X^\lambda,
\]

\[
\left\| u_n \right\|_{L^1_{\text{loc}}(\mathbb{R}^N, \mathcal{C})} \rightarrow u_{\lambda}^\ast, \quad \text{for } 2 \leq r \leq \infty.
\]

(94)

Besides, \( \mathcal{P}_{f}(u_{\lambda}^\ast) = 0 \). In view of Lemma 15, we know that \( u_n \rightarrow u_{\lambda}^\ast \) in \( X^\lambda \) and \( \mathcal{P}_{f}(u_{\lambda}^\ast) = c_{\lambda}^\ast \). In other words, \( u_{\lambda}^\ast \) is a solution of equation (1).

Now, we will check that \( u_{\lambda}^\ast \) is not a solution of equation (1). On the contrary, by combining (39), (H_4), the Egoroff theorem and the Hölder inequality, there holds \( \int_{\mathbb{R}^n} f\left( x \right) \left| u_n \right|^{p\left( x \right)} dx \rightarrow 0 \) as \( n \rightarrow \infty \), which shows that

\[
\left\| u_n \right\|_{1, \lambda}^2 = \frac{1}{\lambda^k} \int_{\mathbb{R}^n} f\left( x \right) \left| u_n \right|^{p\left( x \right)} dx + o(1).
\]

(95)

This is contradictory with \( \lim_{n \rightarrow \infty} \mathcal{P}_{f}(u_{\lambda}^\ast) = c_{\lambda}^\ast < 0 \). Hence, \( u_{\lambda}^\ast \) is a nontrivial solution of equation (1).

**Proof of Theorem 17.** The result of Theorem 2 is immediately available from Theorem 20.

**Theorem 18.** Assume that the conditions \( (H_3)-(V_2) \) and \( (V_1)-(V_5) \) are satisfied, then there exists \( \lambda^\ast \geq 0 \) such that for every \( \lambda > \lambda^\ast \), \( \mathcal{P}_{f} \) has a minimizer \( u_{\lambda}^\ast \) in \( X^\lambda \) satisfying that

\[
\Psi_{\lambda}(u_{\lambda}^\ast) = c_{\lambda}^\ast.
\]

(96)

(1) \( u_{\lambda}^\ast \) is a nontrivial solution of equation (1).

**Proof.** According to Lemma 16 and the Ekeland variational principle in [25], there exists \( \{u_n\} \subset X^\lambda \) such that \( \{u_n\} \) is a \( (PS)_{c_{\lambda}^\ast} \) sequence for \( \mathcal{P}_{f} \). Furthermore, using Lemma 10, we can get that \( \{u_n\} \) is bounded in \( X^\lambda \). Consequently, there exists a subsequence of \( \{u_n\} \) (we still denote as \( \{u_n\} \) and \( u_{\lambda}^\ast \) in \( X^\lambda \) such that

\[
\left\| u_n \right\|_{1, \lambda}^2 \rightarrow \left\| u_{\lambda}^\ast \right\|_{1, \lambda}^2 \quad \text{in } X^\lambda,
\]

\[
\left\| u_n \right\|_{L^1_{\text{loc}}(\mathbb{R}^N, \mathcal{C})} \rightarrow u_{\lambda}^\ast, \quad \text{for } 2 \leq r \leq \infty.
\]

(97)

Besides, \( \mathcal{P}_{f}(u_{\lambda}^\ast) = 0 \). In view of Lemma 15, we know that \( u_n \rightarrow u_{\lambda}^\ast \) in \( X^\lambda \) and \( \mathcal{P}_{f}(u_{\lambda}^\ast) = c_{\lambda}^\ast \). In other words, \( u_{\lambda}^\ast \) is a solution of equation (1).

Now, we will check that \( u_{\lambda}^\ast \) is not a solution of equation (1). On the contrary, by combining (39), (H_4), the Egoroff theorem and the Hölder inequality, there holds \( \int_{\mathbb{R}^n} f\left( x \right) \left| u_n \right|^{p\left( x \right)} dx \rightarrow 0 \) as \( n \rightarrow \infty \), which shows that

\[
\left\| u_n \right\|_{1, \lambda}^2 = \frac{1}{\lambda^k} \int_{\mathbb{R}^n} f\left( x \right) \left| u_n \right|^{p\left( x \right)} dx + o(1).
\]

(98)
This is contradictory with $\lim_{n \to \infty} \Psi_\lambda(u_n) = c_\lambda > 0$. Hence, $u^*_\lambda \neq 0$; that is, $u^*_\lambda$ is a nontrivial solution of equation (1).

**Proof of Theorem 19.** It derives from Theorem 20, Theorem 22, and Lemma 14 that equation (1) has two nontrivial solutions $u^*_\lambda$ and $u^*_{\lambda}$ such that $u^*_\lambda \in \mathcal{N}_\lambda^*$ and $u^*_{\lambda} \in \mathcal{N}_\lambda$ with $\Psi_\lambda(u^*_\lambda) = c_\lambda^* < 0 < C_5 < \Psi_\lambda(u^*_{\lambda}) = c_\lambda$.

**Theorem 20.** Assume that $(H_3)$, $(H_4)$, and $(V_1)$-$\cdot (V_5)$ hold, then for all $\lambda \geq 1/kM^2 \left\{ \{ V^+ < k \} \right\}$, $\Psi_\lambda$ has a minimizer $u^*_\lambda$ in $\mathcal{N}_\lambda$ satisfying that

$$\Psi_\lambda(u^*_\lambda) = c_\lambda^* = \inf_{u \in \mathcal{N}_\lambda} \Psi_\lambda(u) ;$$

(99)

(1) $u^*_\lambda$ is a nontrivial solution of equation (1).

**Proof.** Combining Lemma 14 and the Ekeland variational principle in [25], there exists $\{ u_n \} \subset \mathcal{N}_\lambda$ such that $\{ u_n \}$ is a $(PS)_{c_\lambda^*}$ sequence for $\Psi_\lambda$. Furthermore, using Lemma 10, we can get that $\{ u_n \}$ is bounded in $X_\lambda$. Consequently, there exists a subsequence of $\{ u_n \}$ (we still denote as $\{ u_n \}$) and $u^*_\lambda$ in $X_\lambda$ such that

$$u_n \rightharpoonup u^*_\lambda \text{ in } X_\lambda ,$$

$$u_n \to u^*_\lambda \text{ in } L^r_{\text{loc}}(\mathbb{R}^N), \text{ for } 2 \leq r \leq \infty .$$

Besides, $\Psi'_\lambda(u^*_\lambda) = 0$. In view of Lemma 15, we know that $u_n \to u^*_\lambda$ in $X_\lambda$ and $\Psi_\lambda(u^*_\lambda) = c_\lambda^*$. In other words, $u^*_\lambda$ is a solution of equation (1).

Now, we will check that $u^*_\lambda \neq 0$. On the contrary, by combining (40), (H_3), the Egoroff theorem, and the Hölder inequality, there holds $\int_{\mathbb{R}^N} f(x)|u_n|^{\phi(x)}dx \to 0$ as $n \to \infty$, which shows that

$$\|u_n\|_{\lambda, \nu}^2 = \int_{\mathbb{R}^N} g(x)|u_n|^{\phi(x)}dx + o(1)$$

(100)

Besides, $\Psi'_\lambda(u^*_\lambda) = 0$. In view of Lemma 15, we know that $u_n \to u^*_\lambda$ in $X_\lambda$ and $\Psi_\lambda(u^*_\lambda) = c_\lambda^*$. In other words, $u^*_\lambda$ is a solution of equation (1).

Now, we will check that $u^*_\lambda \neq 0$. Suppose the contrary, combining (39), (40), Egoroff theorem, and (H_4), there holds

$$\int_{\mathbb{R}^N} f(x)|u_n|^{\phi(x)}dx \to 0$$

(101)

This is contradictory with $\lim_{n \to \infty} \Psi_\lambda(u_n) = c_\lambda^* < 0$. Hence, $u^*_\lambda \neq 0$; that is, $u^*_\lambda$ is a nontrivial solution of equation (1).

**Proof of Theorem 21.** The result of Theorem 2 is immediately available from Theorem 20.

We note that if $f$, $g$ and $V_\lambda$ satisfy the hypotheses in Theorem 3, we can choose $\varphi \in C_0^\infty(\Omega_\lambda)$, such that $L_{\varphi}(t) = \Psi_\lambda(t \varphi) = t^{2/2} \| \varphi \|_{\lambda, \nu}^2 - \int_{\Omega_\lambda} f(x)(\varphi(x)|\varphi(x)|^{\phi(x)}dx - \int_{\Omega_\lambda} g(x)|\varphi(x)|dx(\varphi(x)/p(x))$ $|\varphi|^{\phi(x)}dx$ have $t_0 > 0$ and $C_8$ which are independent of $\lambda$ that satisfy $t_0 \varphi \in \mathcal{N}_\lambda$ for all $\lambda > \lambda^* \text{ and} \sup_{t \geq 0} L_{\varphi}(t) = L_{\varphi}(t_0) = C_8 > 0 ,$(102)

which shows $c_\lambda^* \leq C_8$ for all $\lambda > \lambda^*$. 

**Theorem 22.** Assume that the conditions $(H_3)$, $(H_4)$, and $(V_1)$-$\cdot (V_5)$ are satisfied, then for all $\lambda > 1/kM^2 \left\{ \{ V^+ < k \} \right\}$, $\Psi_\lambda$ has a minimizer $u^*_\lambda$ in $\mathcal{N}_\lambda$ satisfying that

$$\Psi_\lambda(u^*_\lambda) = c_\lambda ;$$

(103)

(1) $u^*_\lambda$ is a nontrivial solution of equation (1).

**Proof.** According to Lemma 14 and the Ekeland variational principle in [25], there exists $\{ u_n \} \subset \mathcal{N}_\lambda$ such that $\{ u_n \}$ is a $(PS)_{c_\lambda}$ sequence for $\Psi_\lambda$. Furthermore, using Lemma 10, we can get that $\{ u_n \}$ is bounded in $X_\lambda$. Consequently, there exists a subsequence of $\{ u_n \}$ (we still denote as $\{ u_n \}$) and $u^*_\lambda$ in $X_\lambda$ such that

$$u_n \rightharpoonup u^*_\lambda \text{ in } X_\lambda ,$$

$$u_n \to u^*_\lambda \text{ in } L^r_{\text{loc}}(\mathbb{R}^N), \text{ for } 2 \leq r \leq \infty .$$

(104)

Besides, $\Psi'_\lambda(u^*_\lambda) = 0$. In view of Lemma 15, we know that $u_n \to u^*_\lambda$ in $X_\lambda$ and $\Psi_\lambda(u^*_\lambda) = c_\lambda$. In other words, $u^*_\lambda$ is a solution of equation (1).

Now, we will check that $u^*_\lambda \neq 0$. Suppose the contrary, combining (39), (40), Egoroff theorem, and (H_4), there holds
\[
\int_{\mathbb{R}^n} g(x)|u_n|^p(x)\,dx \to 0 \text{ as } n \to \infty, \text{ which shows that }
\]
\[
\|u_n\|_{L^p}^p = \int_{\mathbb{R}^n} f(x)|u_n|^p(x)\,dx + o(1)
\]

\[
\Psi_A(u_n) = \frac{1}{2} \|u_n\|^2_{L^p} - \frac{1}{q} \int_{\mathbb{R}^n} f(x)|u_n|^q(x)\,dx - \frac{1}{p^q} \int_{\mathbb{R}^n} g(x)|u_n|^p(x)\,dx
\]
\[
\leq \frac{1}{2} \|u_n\|^2_{L^p} - \frac{1}{q} \int_{\mathbb{R}^n} f(x)|u_n|^q(x)\,dx + \frac{1}{p^q} \int_{\mathbb{R}^n} g(x)|u_n|^p(x)\,dx
\]
\[
= \frac{1}{2} \|u_n\|^2_{L^p} - \frac{1}{q^*} \|u_n\|^2_{L^q} + o(1) = q^* - 2 \frac{\|u_n\|^2_{L^p}}{2q^*} + o(1)
\]
\[
\leq (q^* - 2)(q^* - 1) \|u_n\|^2_{L^p} + o(1) \leq 0.
\]

(105)

This is contradictory with \( \lim_{n \to \infty} \Psi_A(u_n) = c_\lambda > 0 \). Hence, \( u_\lambda \neq 0 \); that is, \( u_\lambda \) is a nontrivial solution of equation (1).

**Proof of Theorem 23.** It derives from Theorem 20, Theorem 22, and Lemma 14 that equation (1) has two nontrivial solutions \( u_+ \) and \( u_- \) such that \( u_+ \in \mathcal{M}_A^+ \) and \( u_- \in \mathcal{M}_A^- \) with \( \Psi_A(u_+) = c_+ < 0 < C_\lambda < \Psi_A(u_-) = c_- \).

**Data Availability**
Not applicable data and material.

**Conflicts of Interest**
The authors declare that there is no conflict of interests regarding the publication of this paper.

**Authors’ Contributions**
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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