Seven dimensional flat manifolds with cyclic holonomy

Rafał Lutowski∗ Institute of Mathematics
University of Gdańsk
ul. Wita Stwosza 57
80-952 Gdańsk, Poland
E-mail: rlutowski@mat.ug.edu.pl

October 20, 2011

Abstract
We classify (up to affine equivalence) all 7-dimensional flat manifolds with cyclic holonomy group.

1 Introduction
Let $M^n$ be a closed Riemannian manifold of dimension $n$. We shall call $M^n$ flat if, at any point, sectional curvature is equal to zero. Equivalently, $M^n$ is isometric to the orbit space $\mathbb{R}^n/\Gamma$, where $\Gamma$ is a discrete, torsion-free and co-compact subgroup of $O(n) \ltimes \mathbb{R}^n = \text{Isom}(\mathbb{R}^n)$. From the first Bieberbach theorem (see [1], [6], [8]) $\Gamma$ defines a short exact sequence of groups

$$0 \to \mathbb{Z}^n \to \Gamma \xrightarrow{\rho} G \to 0,$$

(1)

where $G$ is a finite group. $\Gamma$ is called a Bieberbach group and $G$ its holonomy group. Moreover, from second and third Bieberbach theorems (see [1], [6], [8]) there are only finite number of the isomorphism classes of Bieberbach

∗Supported by University of Gdańsk grant number BW - 5107-5-0345-0
groups of dimension \( n \) and two Bieberbach groups are isomorphic if and only if they are conjugate in the group \( \mathrm{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n \).

With support of a computer system CARAT ([5]) it is possible to give a complete list of all isomorphism classes of Bieberbach groups up to dimension 6. Moreover for a finite group \( G \) and a number \( n \), CARAT gives possibility for a classification (up to isomorphism) of all Bieberbach groups of a dimension \( n \) with a holonomy group \( G \). In this article the CARAT system is used to calculate a list of all isomorphism classes of 7-dimensional Bieberbach groups with cyclic holonomy group. The final list of 316 groups is presented on the www page (see [4]), where the method of exposition is borrowed from [3]. Our main motivation was a paper [7] about \( \eta \)-invariants of flat manifolds, where our results are applied.

A holonomy representation \( \phi : G \to \mathrm{GL}(n, \mathbb{Z}) \) of the Bieberbach group \( \Gamma \) (cf. [1]) is defined by the formula:

\[
\forall g \in G, \phi(g)(e_i) = \tilde{g} e_i \tilde{g}^{-1},
\]

(2)

where \( e_i \in \Gamma \) are generators of \( \mathbb{Z}^n \) for \( i = 1, 2, ..., n \), and \( \tilde{g} \) is an element of \( \Gamma \), such that \( p(\tilde{g}) = g \).

## 2 \( \mathbb{Q} \)-classes of holonomy representation

By [3], the possible orders of cyclic groups, that can be realized as holonomy groups of crystallographic groups in dimension 7 are: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 18, 20, 24, 30. By [3, Lemma 2.1], the degree of a matrix with \( n \)-th primitive root of 1 is not less than \( \varphi(n) \), where \( \varphi \) is the Euler’s function. Since for \( n = 15, 20, 24, 30 \) \( \varphi(n) > 7 \), then any matrix of order \( n \) and degree 7 must be taken as a direct sum of matrices, which orders are proper divisors of \( n \).

Let \( n \in \mathbb{N}, n > 1 \) and

\[
\Phi_n(x) = x^{\varphi(n)} + a_{\varphi(n)-1} x^{\varphi(n)-1} + \ldots + a_1 x + a_0
\]

be the cyclotomic polynomial of order \( n \) (see [2, page 137]). Since the char-
acteristic polynomial of the matrix

\[
A_n = \begin{bmatrix}
0 & 0 & \ldots & 0 & -a_0 \\
1 & 0 & \ldots & 0 & -a_1 \\
0 & 1 & \ldots & 0 & -a_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & -a_{\varphi(n)-1}
\end{bmatrix}
\]

is equal to \( \pm \Phi_n(x) \), then eigenvalues of \( A_n \) are primitive \( n \)-th roots of the unity.

Let \( G \) be a cyclic group of order \( n \), generated by an element \( g \). Then for each \( d \mid n \), \( \rho_d : G \to GL(\varphi(d), \mathbb{Z}) \), given by

\[
\rho_n(g) = A_d
\]

is an integral representation of \( G \), which is irreducible over \( \mathbb{Q} \). Moreover, by \cite{2} Corollary 39.5], these are all, up to equivalence, rational irreducible representation of \( G \).

From the above remarks, rational representations of a cyclic group of order \( n \) in dimension 7 are of the form

\[
\rho = \bigoplus_{d \mid n, d \leq 18} a_d \rho_d,
\]

where \( a_i \in \mathbb{N} \) and

\[
\sum_{d \mid n, d \leq 18} a_d \varphi(d) = 7,
\]

and \( \rho \) is faithful, if

\[
\text{LCM}\{d \mid n; a_d \neq 0\} = n.
\]

In the Table 1 we give a list of cyclotomic polynomials for given \( n \) and some remarks about the matrices \( A_n \). The relation \( \sim \) means "the same conjugacy class in \( \text{GL}(\varphi(n), \mathbb{Q}) \)".

3 Determination of Bieberbach groups

Let \( G \) be a cyclic group of order \( n \). From (3) we know, how to determine all equivalence classes of seven dimensional rational representation of \( G \). We
want to classify all Bieberbach groups with a holonomy group $G$. There are three steps:

1. Determine, up to equivalence, all faithful representations $\rho: G \to \text{GL}(7, \mathbb{Q})$;

2. Determine all integral representations (up to equivalence) of $G$ equivalent over $\mathbb{Q}$ to $\rho$;

3. For each representation $\tau: G \to \text{GL}(7, \mathbb{Z})$ from the previous point, determine all Bieberbach groups (up to isomorphism) with holonomy representation $\tau$.

To determine $\mathbb{Z}$-classes of faithful representations of cyclic group of prime order, we use [2, Theorem 74.3].

As mentioned before, the complete list of seven dimensional Bieberbach groups with a cyclic holonomy group is given in [4]. Let us give a short dictionary of tables. If a Bieberbach group $\Gamma$ has a name of the form

$$\frac{n}{n.a_1.f_1-b_1+\ldots+a_1.f_1-b_1.p.q.r},$$

then $n$ is the order of the holonomy group $G$, the $\mathbb{Q}$-class of a holonomy representation of $G$ is given by the representation

$$a_1\rho_{b_1} \oplus \ldots \oplus a_l\rho_{b_l}$$

| $n$ | $\varphi(n)$ | $\Phi_n(x)$ | Remarks |
|-----|-------------|-------------|--------|
| 2   | 1           | $x - 1$     |        |
| 3   | 2           | $x^2 + x + 1$ |        |
| 4   | 2           | $x^2 + 1$ |        |
| 5   | 4           | $x^4 + x^3 + x^2 + x + 1$ |        |
| 6   | 2           | $x^2 - x + 1$ | $A_6 \sim -A_3$ |
| 7   | 6           | $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ |        |
| 8   | 4           | $x^4 + 1$ |        |
| 9   | 6           | $x^6 + x^3 + 1$ |        |
| 10  | 4           | $x^4 - x^3 + x^2 - 1$ | $A_{10} \sim -A_5$ |
| 12  | 4           | $x^4 - x^2 + 1$ |        |
| 14  | 6           | $x^6 - x^5 + x^4 - x^3 + x^2 - 1$ | $A_{14} \sim -A_7$ |
| 18  | 6           | $x^6 - x^3 + 1$ | $A_{18} \sim -A_9$ |

Table 1: Cyclotomic polynomials for given numbers
(cf. (3)). Moreover \( f_i = \varphi(b_i) \), for \( i = 1, \ldots, l \); \( p,q \) is a symbol of the \( \mathbb{Z} \)-class of a holonomy representation and \( r \) is a number of the group \( \Gamma \). The numbers \( p, q, r \) are assigned by CARAT.

References

[1] L. S. Charlap, *Bieberbach Groups and Flat Manifolds*, Universitext, Springer-Verlag, New York, 1986

[2] C.W. Curtis, I. Reiner, *Representation theory of finite groups and associative algebras*. Pure and Applied Mathematics, Vol. XI Interscience Publishers, a division of John Wiley & Sons, New York-London 1962

[3] H. Hiller, *The Crystallographic Restriction in Higher Dimensions*, Acta Cryst. (1985), A41, 541–544

[4] R. Lutowski, *A list of 7-dimensional Bieberbach groups with cyclic holonomy*, available online, [http://rlutowsk.mat.ug.edu.pl/flat7cyclic/](http://rlutowsk.mat.ug.edu.pl/flat7cyclic/)

[5] J. Opgenorth, W. Plesken, T. Schulz, *CARAT – Crystallographic algorithms and tables*, Version 2.0, 2003, [http://wwwb.math.rwth-aachen.de/carat](http://wwwb.math.rwth-aachen.de/carat)

[6] A. Szczepański, *Geometry of the crystallographic groups*, book in preparation available on web [http://www.mat.ug.edu.pl/aszczepsa](http://www.mat.ug.edu.pl/aszczepsa)

[7] A. Szczepański, *Eta invariants for flat manifolds*, preprint 2010, submitted

[8] J. Wolf, *Spaces of constant curvature*, MacGraw Hill, New York-London-Sydney, 1967