Stationary Spinning Strings and Symmetries of Classical Spacetimes

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Abstract

We explore the symmetries of classical stationary spacetimes in terms of the dynamics of a spinning string described by a worldsheet supersymmetric action. We show that for stationary configurations of the string, the action reduces to that for a pseudo-classical spinning point particle in an effective space, which is a conformally scaled quotient space of the original spacetime. As an example, we consider the stationary spinning string in the Kerr-Newman spacetime, whose motion is equivalent to that of the spinning point particle in the three-dimensional effective space. We present the Killing tensor as well as the spin-valued Killing vector of this space. However, the non-generic supersymmetry corresponding to the Killing-Yano tensor of the Kerr-Newman spacetime is lost in the effective space.
I. INTRODUCTION

The equilibrium configurations of a Nambu-Goto string in a general stationary four-dimensional spacetime were first studied by Frolov et al. [1]. It was shown that for the stationary string, when the timelike Killing vector of the spacetime is supposed to be tangent to the string worldsheet, the Nambu-Goto action reduces to that for a classical test particle in an effective three-dimensional space. In other words, the problem of the stationary string motion in the original spacetime amounts to the study of geodesics of the effective space, which is one dimension fewer. Using this remarkable fact for the stationary string motion near a Kerr-Newman black hole, the authors of [1] demonstrated that the Nambu-Goto equation of motion is completely integrable. This was another manifestation of the Kerr-Newman spacetime-“miracle”, earlier discovered by Carter [2] for the integrability of the ordinary Hamilton-Jacobi equation in this spacetime. In both cases, the integrability originates from the hidden symmetries of the Kerr-Newman spacetime generated by a second rank Killing tensor [3].

In recent studies, it has been shown that these properties also survive in higher dimensions: The most general metrics for rotating black holes in all higher dimensions possess hidden symmetries, admitting the Killing and the Killing-Yano tensors [4, 5]. It has also been shown that the higher-dimensional black hole spacetimes admit the complete separation of variables in the Nambu-Goto equation for stationary string configurations [6, 7]. This fact is related to the existence of the Killing tensor just as in the case of four-dimensional Kerr-Newman spacetime [1]. The hidden symmetries of rotating charged black holes in five-dimensional minimal gauged supergravity [8] and the integrability of geodesics for the corresponding Hamilton-Jacobi and Nambu-Goto equations have been studied in [9, 10, 11].

An important manifestation of the hidden symmetries also occurred in separation of the Dirac equation in the spacetime of rotating black holes in four dimensions [12, 13]. It is the Killing-Yano tensor [14] which lies at the root of this separability. The authors of work [15] were able to verify this fact by constructing a linear differential operator, which depends on the Killing-Yano tensor and commutes with the Dirac operator. Later on, the systematic exploration of these aspects of the hidden symmetries, in the light of their relation to supersymmetry, was undertaken in [16]. In particular, it was shown that the existence of the second rank Killing-Yano tensor in the Kerr-Newman metric implies a new nontrivial
supersymmetry (“hidden” supersymmetry) in the motion of a pseudo-classical spinning point particle in this metric. The similar analysis of the hidden supersymmetry in the metrics of higher-dimensional rotating black holes is given in [17].

The purpose of this paper is to study the symmetries of stationary spacetimes in terms of the motion of a stationary spinning string described by a worldsheet supersymmetric action. In Sec.II we briefly recall the main facts about the action of a spinning string in a curved spacetime. In Sec.III we show that, when both the curved spacetime and the spinning string are stationary, the string action reduces to that for the pseudo-classical spinning point particle in the effective space, which is conformally related to the quotient space of the original spacetime. This precisely resembles the fact mentioned at the beginning for the stationary string motion governed by the usual bosonic Nambu-Goto action. In Sec.IV we give the basic equations describing the symmetries and conserved quantities in supersymmetric mechanics of the pseudo-classical spinning particle in the effective space. In Sec.V, as an illustrative example, we consider the motion of the stationary spinning string in the Kerr-Newman spacetime.

II. THE ACTION FOR A SPINNING STRING

As is known [18, 19, 20], the action for the spinning string is obtained by an extension of the action for a relativistic bosonic string, where in addition to the usual position variable \( x^\mu (\zeta^\alpha) \) one also introduces a fermion variable \( \psi^\mu (\zeta^\alpha) \). Here \( \zeta^\alpha = (\zeta^0, \zeta^1) \) are coordinates on the worldsheet of the string, the coordinate function \( x^\mu \) is a spacetime vector and \( \psi^\mu \) is a two-dimensional Majorana spinor on the worldsheet, which also behaves as a spacetime vector. In the following, for convenience of the notation, we omit all spinor indices. The spacetime index \( \mu \) runs over values \( (0, 1, \ldots, D - 1) \), where \( D \) is the dimension of the spacetime. In a general curved spacetime with metric \( g_{\mu\nu}(x) \) the action is given by

\[
S = -\frac{1}{2} \int d^2\zeta \sqrt{-G} \left( G^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu g_{\mu\nu} - i \bar{\psi}^A \rho^A D_\alpha \psi_A \right),
\]

where \( G \) is the determinant of the worldsheet metric \( G_{\alpha\beta} \). The description of spin degrees of freedom in the curved spacetime makes it necessary to introduce both a ‘vielbein’ \( e^A_\mu (x) \) and a ‘zweibein’ \( E^a_\alpha (\zeta) \) fields, such that

\[
g_{\mu\nu} = \eta_{AB} e^A_\mu e^B_\nu, \quad G^{\alpha\beta} = \eta^{ab} E^a_\alpha E^b_\beta, \quad \psi^\mu = e^A_\mu \psi^A, \quad \rho^\alpha = E^a_\alpha \rho^a,
\]

\( E^a_\alpha (\zeta) \) fields, such that
where the indices $A$ and $a$ are the local Lorentz indices, $\eta_{AB}$ is the Minkowski metric of the tangent space to the spacetime manifold, $\eta_{ab}$ is a two-dimensional Minkowski metric on the string worldsheet. The quantity $\rho^a$ defines the usual Dirac matrices in two dimensions

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

obeying the relation

$$\{\rho^a, \rho^b\} = -2\eta^{ab}$$

and $\bar{\psi} = \psi^\dagger \rho^0$. The covariant derivative is given by

$$D_\alpha \psi_A = \partial_\alpha \psi_A - \partial_\alpha x^\mu \omega_{\mu AB} \psi_B,$$

where $\omega_{\mu AB}$ is the spin connection on the spacetime manifold, while the spin connection on the worldsheet does not contribute to this expression.

The gauge invariance properties of the action (1) that is, its reparametrization symmetries and the local Weyl symmetry allow one to make a particular choice for the metric functions $G_{\alpha\beta}$. The most convenient choice is achieved in a conformal gauge, in which one has a flat metric on the worldsheet. In this gauge, the action (1) reduces to the form

$$S = \frac{-1}{2} \int d^2\zeta \left( \eta^{ab} \partial_a x^\mu \partial_b x^\nu g_{\mu\nu} - i \bar{\psi}^A \rho^a \partial_a \psi_A \right).$$

This action possesses a global worldsheet supersymmetry relating bosonic and fermionic coordinates (see for instance, Ref. [20] for details). Applying to this action the Noether procedure and using the transformations of the worldsheet supersymmetry, we obtain the two-dimensional energy momentum tensor

$$T_{ab} = g_{\mu\nu} \partial_a x^\mu \partial_b x^\nu - \frac{i}{2} \bar{\psi}^A \rho_{(a} D_{b)} \psi_A - \frac{\eta_{ab}}{2} \left( g_{\mu\nu} \partial^c x^\mu \partial_c x^\nu - \frac{i}{2} \bar{\psi}^A \rho^c D_c \psi_A \right)$$

and the supercurrent

$$Q_a = \frac{1}{2} \rho^b \rho_a \psi_\mu D_\mu x^a.$$

The round parentheses here and in the following denote symmetrization over the indices enclosed. The equations of motion for $x^\mu$ and $\psi^\mu$ derived from the action (6) are the familiar Nambu-Goto and Dirac equations in the curved background. They must be supplemented by the constraints that the energy momentum tensor (7) and the supercurrent (8) vanish on the worldsheet. The consistent derivation of these constraints requires the invariance of
the action for the spinning string under local supersymmetry as well. This is achieved by adding to the action (1) an auxiliary Rarita-Schwinger field $\chi_\alpha(\zeta)$, a fermionic partner to the zweibein field $E^a_\alpha(\zeta)$ [18, 20]. Large enough number of the local bosonic and fermionic symmetries of the resulting action enables one to fix the gauge $E^a_\alpha = \delta^a_\alpha$, $\chi_\alpha = 0$. With this gauge, we again arrive at the action of the form (6). Furthermore, the variation of the local supersymmetric action with respect to the fields $E^a_\alpha$ and $\chi_\alpha$, evaluated in this gauge, results in the constraint equations

$$T_{ab} = 0, \quad Q_a = 0,$$

which accompany the equations of motion for the physical fields $x^\mu$ and $\psi^\mu$. In the first case, we have two independent equations because of the traceless nature of the energy momentum tensor, while in the latter case there exists one independent equation due to the identity $\rho^a Q_a = 0$.

### III. STATIONARY SPINNING STRINGS

Let us now suppose that the spacetime metric $g_{\mu\nu}$ in the action (6) admits a timelike Killing vector $\xi = \xi^\mu \partial_\mu$. That is, the spinning string moves in a stationary curved spacetime $M$. In this case to proceed further it is useful to employ the formalism of Geroch [21] based on a foliation of the spacetime by its Killing trajectories. Assuming that the timelike Killing vector is not hypersurface orthogonal, one can consider a set of the Killing trajectories as a quotient space $\mathcal{M}$ of the spacetime $M$ ($\mathcal{M} = M/G_1$, where $G_1$ is a one-dimensional group generated by $\xi$). Then the metric of the quotient space $h_{\mu\nu}$ is related to the spacetime metric by

$$g_{\mu\nu} = h_{\mu\nu} + \frac{\xi_\mu \xi_\nu}{\xi^2},$$

where $\xi^2 = g_{00}$, and for the corresponding contravariant components we have

$$g^{\mu\nu} = \begin{pmatrix} \xi^{-2} (1 + h^{ij} \xi_i \xi_j / \xi^2) & -h^{ij} \xi_j / \xi^2 \\ -h^{ij} \xi_i / \xi^2 & h^{ij} \end{pmatrix}.$$  

We note that the metric $h_{ij}$ satisfies the completeness relation

$$h^{ik} h_{kj} = \delta_j^i, \quad i, j = 1, ..., D - 1.$$
The remarkable feature of this foliation is that there exists a uniquely “mirrored correspondence” between the differential geometries in the original spacetime and in its quotient space. The projection operator

$$h_{\nu}^{\mu} = \delta_{\mu}^{\nu} - \frac{\xi_{\mu} \xi^{\nu}}{\xi^2},$$  \hspace{1cm} (13)

provides this correspondence for any stationary tensor field, \((L_\xi T = 0, \text{where } L_\xi \text{ is the Lie derivative along } \xi)\) and its covariant derivative

$$\hat{T}^{\nu_1 \cdots \nu_n}_{\mu_1 \cdots \mu_m} = h_{\nu_1}^{\lambda_1} \cdots h_{\nu_m}^{\rho_m} T^{\lambda_1 \cdots \lambda_n}_{\rho_1 \cdots \rho_m}, \quad \hat{D}_{\gamma} \hat{T}^{\nu_1 \cdots \nu_n}_{\mu_1 \cdots \mu_m} = h_{\nu_1}^{\mu_1} \cdots h_{\nu_m}^{\mu_m} h_{\gamma}^{\lambda_1} D_{\lambda_1} \hat{T}^{\lambda_1 \cdots \lambda_n}_{\rho_1 \cdots \rho_m},$$  \hspace{1cm} (14)

where the covariant derivative operators \(\hat{D}_{\mu}\) and \(D_{\mu}\) are associated with metrics \(h_{\mu\nu}\) and \(g_{\mu\nu}\), respectively. Further details can be found in [21].

Next, we also suppose that not only the spacetime, but also the spinning string itself is stationary. For this time independent configuration, the embedding of the string worldsheet into the spacetime \(M\) can be parameterized by the equations

$$x^0 = \tau + f(\sigma), \quad x^i = x^i(\sigma),$$  \hspace{1cm} (15)

where we have used \(\zeta^0 = \tau\) and \(\zeta^1 = \sigma\). That is, the basis vector \(\partial_\tau x^\mu\) in the tangent plane to the worldsheet is parallel to the timelike Killing vector \(\xi^\mu\). As for the other tangent vector to the worldsheet, \(\eta^\mu = \partial_\sigma x^\mu\), we require it to lie in the tangent plane of the quotient space \(\mathcal{M}\), so that

$$\xi_\mu \eta^\mu = 0.$$  \hspace{1cm} (16)

Using constraints (15) and (16) along with (13) one can easily take the projection of the bosonic term in the action (6) onto the quotient space. We have

$$\eta_{ab} \partial_a x^\mu \partial_b x^\nu g_{\mu\nu} = -\xi^2 + h_{\mu\nu} \eta^\mu \eta^\nu.$$  \hspace{1cm} (17)

The spinor term in the action (6) can be written in the form

$$\overline{\psi}^\mu \rho^0 D_{\alpha} \psi_{\mu} = \xi^\mu \overline{\psi}^\mu \rho^0 D_{0} \psi_{\mu} + \eta^\nu \overline{\psi}^\mu \rho^1 D_{\nu} \psi_{\mu}.$$  \hspace{1cm} (18)

We require that the spinors obey the conditions

$$L_\xi \psi_{\mu} = 0, \quad \psi_{\mu} \xi^\mu = \xi^2 \Upsilon,$$  \hspace{1cm} (19)

where \(\Upsilon\) is a constant spinor. With these conditions we have the decomposition

$$\psi_{\mu} = \chi_{\mu} + \xi_{\mu} \Upsilon,$$  \hspace{1cm} (20)

where \(\chi_{\mu}\) is a spinor satisfying \(L_\xi \chi_{\mu} = 0\).
where \( \chi_\mu \) is a spinor field on the quotient space, \( \chi_\mu \xi_\mu = 0 \). Substituting this decomposition in equation (18) and taking into account the conditions (19) along with the mirrored correspondence (14), we obtain that

\[
\bar{\psi} \rho^a D_a \psi_\mu = \eta^\nu \bar{\chi}^\mu \rho^1 \hat{D}_\nu \chi_\mu + 2 \xi^\nu D_\nu \xi_\mu \bar{\chi}^\mu \rho^0 \Upsilon .
\]

(21)

Thus, the action (6) takes the form

\[
I = \frac{1}{2} \int d\sigma \left( -\xi^2 + h_{\mu\nu} \eta^\mu \eta^\nu - i\eta^\nu \bar{\chi}^\mu \rho^1 \hat{D}_\nu \chi_\mu - 2i \xi^\nu D_\nu \xi_\mu \bar{\chi}^\mu \rho^0 \Upsilon \right) .
\]

(22)

Similarly, the constraint equations accompanying this action are obtained by projecting equations (9) with (7) and (8) onto the quotient space. We have \( \hat{T}_{01} = \hat{T}_{10} \equiv 0 \) and

\[
\hat{T}_{00} = \frac{1}{2} \left( \xi^2 + h_{\mu\nu} \eta^\mu \eta^\nu \right) = 0 , \quad \hat{Q}_0 = \frac{1}{2} \left( \xi^2 \Upsilon + \rho^0 \rho^1 \chi_\mu \eta^\mu \right) = 0 .
\]

(23)

Next, we introduce an “effective” metric

\[
H_{\mu\nu} = -\xi^2 h_{\mu\nu} ,
\]

(24)

which is a conformally adjusted metric of the quotient space and a new variable \( d\sigma' = -\xi^2 d\sigma \). Then it is straightforward to show that the action in (22) can be put in the form

\[
I = \frac{1}{2} \int d\sigma' \left( H_{\mu\nu} \eta^\mu \eta^\nu - i\eta^\nu \bar{\chi}^\mu \rho^1 D'_\nu \chi_\mu \right) ,
\]

(25)

where \( \eta^\mu = \partial_{\sigma'} x^\mu \) and \( D'_\nu \) is the covariant derivative operator with respect to the metric \( H_{\mu\nu} \). In obtaining this expression we have used the relation

\[
\eta^\nu \bar{\chi}^\mu \rho^1 \left( D'_\nu - \hat{D}_\nu \right) \chi_\mu = 2 \xi^\nu D_\nu \xi_\mu \bar{\chi}^\mu \rho^0 \Upsilon ,
\]

(26)

which is derived by direct evaluating the actions of the covariant derivative operators in metrics \( H_{\mu\nu} \) and \( h_{\mu\nu} \), and making use of the second equation in (23).

We recall that in the quotient space \( \mathcal{M} \) there exist only the spatial components of fields and therefore one can introduce a vielbein field \( e_i^a \), such that

\[
H_{ij} = \delta_{ab} e_i^a e_j^b , \quad \chi_i = e_i^a \chi_a .
\]

(27)

where \( \delta_{ab} \) is a flat \((D - 1)\)-dimensional Euclidean metric. Choosing, for the sake of certainty, the spinors as

\[
\chi_a = \begin{pmatrix} 0 \\ \theta_a \end{pmatrix} , \quad \Upsilon = \begin{pmatrix} 0 \\ \Upsilon_0 \end{pmatrix} ,
\]

(28)
where $\theta_a$ is a Grassmann variable, it is easy to show that the action (25) reduces to the form

$$I = \int d\sigma \left( \frac{1}{2} H_{ij} \dot{x}^i \dot{x}^j + \frac{i}{2} \delta_{ab} \theta^a \frac{D\theta^b}{D\sigma} \right),$$

(29)

where the overdot means the derivative $d/d\sigma$ and to simplify the notation we have omitted primes, implying that all operations are taken with respect to the effective metric (24). This action looks precisely the same as that describing the dynamics of a pseudo-classical spinning point particle in a curved background \[16\]. However, in our case the background is a curved Euclidean space where the length parameter $\sigma$ plays the role of “time”-evolution.

The corresponding constraint equations are given by

$$H = \frac{1}{2} H_{ij} \dot{x}^i \dot{x}^j = 1, \quad Q = \dot{x}^i e^a_i \theta_a = -\Upsilon_0. \quad (30)$$

Thus, we conclude that the dynamics of a stationary spinning string in a $D$-dimensional stationary spacetime becomes equivalent to that of a pseudo-classical spinning point particle in a $(D-1)$-dimensional effective space whose metric is obtained by conformal adjusting the metric of the quotient space, as given in equation (24).

IV. SYMMETRIES AND CONSERVED QUANTITIES

The general theory of spacetime symmetries and conserved quantities in supersymmetric mechanics of pseudo-classical spinning point particles has been developed in \[16, 22\]. Our result in the previous section shows that this theory can be used to describe the symmetries of stationary spacetimes in terms of the motion of stationary spinning strings by adopting it to the effective (one dimension fewer) Euclidean space. Since in the latter case the metric involves a conformal factor, as in \[24\], not all nongeneric (hidden symmetries) of the original spacetime may survive in the effective space. That is, the explicit form of the conformal factor plays a crucial role for the existence of nongeneric symmetries in the motion of the stationary spinning string \[11\]. Following \[16\], we give here the basic equations describing the symmetries and conserved quantities which are admitted by the action (29). This action gives rise to the covariant momentum

$$\Pi_i = p_i + \frac{i}{2} \omega_{ab} \theta^a \theta^b = H_{ij} \dot{x}^j.$$

(31)
which enables one to pass to the Hamiltonian description of the theory in terms of covariant phase-space variables \((x^i, \Pi_i, \theta^a)\). The associated Hamiltonian has the form

\[
\mathcal{H} = \frac{1}{2} H^{ij} \Pi_i \Pi_j ,
\]

and one can define the evolution of any phase-space function \(\mathcal{J}(x, \Pi, \theta)\) in terms of its Poisson-Dirac bracket with this Hamiltonian. We have

\[
\frac{d\mathcal{J}}{d\sigma} = \{\mathcal{J}, \mathcal{H}\} .
\]

The Poisson-Dirac bracket of two general phase-space functions is given by

\[
\{F, G\} = D_i F \frac{\partial G}{\partial \Pi_i} - \frac{\partial F}{\partial \Pi_i} D_i G - R_{ij} \frac{\partial F}{\partial \Pi_i} \frac{\partial B}{\partial \Pi_j} + i(-1)^{\epsilon(F)} \frac{\partial F}{\partial \theta^a} \frac{\partial G}{\partial \theta_a} ,
\]

where \(\epsilon(F)\) stands for the Grassmann parity of \(F\), the phase-space covariant derivative operator is defined as

\[
D_i F = \partial_i F + \Gamma^k_{ij} \Pi_k \frac{\partial F}{\partial \Pi_j} + \omega^a_i \theta^b \frac{\partial F}{\partial \theta_a} ,
\]

and the spin-valued curvature tensor

\[
R_{ij} = \frac{i}{2} \theta^a \theta^b R_{abij} .
\]

Let us suppose that the set of phase-space functions \(\mathcal{J}(x, \Pi, \theta)\) describes all symmetries or associated conserved quantities of the system. Then from equation (33), it follows that they all must commute with the Hamiltonian (32). That is, we have the equation

\[
\{\mathcal{J}, \mathcal{H}\} = 0 ,
\]

Next, we expand the conserved quantities of the motion in terms of the covariant momentum as follows

\[
\mathcal{J} = \sum_{n=0}^{\infty} \frac{1}{n!} J^{(n)}_{i_1...i_n} (x, \theta) \Pi^{i_1} ... \Pi^{i_n} ,
\]

where the components \(J^{(n)}_{i_1...i_n} (x, \theta)\) can be thought of as Killing tensors generating the symmetries of the action (29). Substituting this expansion in equation (37), we obtain the chain of equations

\[
D_{i_1...i_n} J^{(n)}_{i_1...i_n} + \frac{\partial J^{(n)}_{i_1...i_n}}{\partial \theta^a} \omega^{a}_{i_{n+1}} \theta^b = R_{i_{n+1}j} J^{(n+1)}_{i_1...i_n} ,
\]

9
which couple the Killing tensors of different rank. In this sense, they are of generalized Killing equations \[22\]. These equations admit the most simple general solutions given by the second rank Killing tensor \(K_{ij} = H_{ij}\) and the Grassmann-odd Killing vector \(I^a = \theta^a\). In the first case we have the Hamiltonian \[32\], describing the translational symmetries along the evolution parameter \(\sigma\), while the later case gives rise to the supercharge \(Q = \Pi_a \theta^a\) (see also \[30\]), generating supersymmetry transformations. Along with their corresponding duals, these symmetries form the set of all generic symmetries which are actually built in the action \[29\]. In particular, the supercharge \(Q\) generates the supersymmetry transformations \[
\delta x^i = i\epsilon \{Q, x^i\} = -i\epsilon e^i_a \theta^a, \quad \delta \theta^a = i\epsilon \{Q, \theta^a\} = \epsilon e^a_i \dot{x}^i + \delta x^i \omega^a_{\phantom{a}b} \theta^b, \]
where \(\epsilon\) is the infinitesimal, Grassmann-odd parameter of the transformations. The corresponding superalgebra is given by \[
\{Q, \mathcal{H}\} = 0, \quad \{Q, Q\} = -2i\mathcal{H}. \]

The existence of nongeneric symmetries depends, in general, on the form of the metric \[24\] and in each particular case one needs to solve equations \[39\] explicitly. If one supposes that the theory also admits a nongeneric supersymmetry transformation \[
\delta x^i = -i\epsilon f^i_a \theta^a \equiv -i\epsilon J^{(1)i}, \]
it is then straightforward to show that this transformation is generated by the supercharge \(Q_f = f^i_a \Pi_i \theta^a + \frac{i}{6} C_{abc} \theta^a \theta^b \theta^c, \) where the tensors \(f^i_a\) and \(C_{abc}\) obey the conditions \[
D_i f_{ja} + D_j f_{ia} = 0, \quad D_i C_{abc} = -\left(R_{ijbc} f^j_a + R_{ijca} f^j_b + R_{ijab} f^j_c \right) \]
which are obtained from equations \[39\]. It is important to note that, unlike the case of two original supercharges \(Q\) in \[11\], the Poisson-Dirac bracket of two supercharges \(Q_f\) does not close on the Hamiltonian. Instead, it defines the corresponding Killing tensor, Killing vector and Killing scalar, thereby forming an exotic algebra \[16\]. On the other hand, if we require the tensor \(f_{ij} = f_{ia} e^a_j\) be antisymmetric, then the Poisson-Dirac bracket of \(Q_f\) with \(Q\) vanishes. In this case, the defining equation for the third rank tensor \(C_{abc}\) has the most simple form \[16\] \[
C_{abc} = -2\epsilon^i_a e^j_b e^k_c D_i f_{jk}, \]
and the superinvariant \(Q_f\) is determined by the second rank Killing-Yano tensor \(f_{ij}\).
V. STATIONARY SPINNING STRINGS IN THE KERR-NEWMAN SPACETIME

In this section, as an illustrative example, we consider the symmetries and conserved quantities in terms of the motion of a stationary spinning string near a black hole which is given by the Kerr-Newman spacetime metric

\[ ds^2 = -\frac{\Delta}{\rho^2} (dt - a \sin^2 \theta d\phi)^2 + \rho^2 \left( \frac{dr^2}{\Delta} + d\theta^2 \right) + \frac{\sin^2 \theta}{\rho^2} \left[ adt - (r^2 + a^2) d\phi \right]^2 , \]  

(46)

where the metric functions

\[ \Delta = r^2 + a^2 - 2Mr + Q^2 , \quad \rho^2 = r^2 + a^2 \cos^2 \theta \]  

(47)

and the parameters \( M, a = J/M \) and \( Q \) are the mass, rotation parameter and the electric charge of the black hole, respectively.

According to the theory developed above, the motion of the stationary spinning string becomes equivalent to that of a spinning point particle in the effective three-dimensional space with the metric (24). That is, we have the space interval

\[ dl^2 = \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \Delta \sin^2 \theta d\phi^2 , \]  

(48)

where

\[ \Sigma = \rho^2 - 2Mr + Q^2 . \]  

(49)

For this metric we can consider a conserved quantity of the form

\[ J = \frac{1}{2} K_{ij} \Pi^i \Pi^j + I_i \Pi^i . \]  

(50)

Then from equation (57) we obtain that \( K_{ij} \) is a Killing tensor and \( I_i = (i/2)I_{ab}\theta^a\theta^b \) is a spin-valued Killing vector, which are determined by the equations

\[ D_{(i} K_{ij)} = 0 , \quad D_{(i} I_{j)} = R_{abk(i} K_{j)k} . \]  

(51)

It is straightforward to check that the Killing tensor equation admits the solution

\[ K_{ij} dx^i dx^j = \Sigma \left( \frac{a^2 \sin^2 \theta}{\Delta} dr^2 + \Delta d\theta^2 \right) + \Delta \left( \Delta + a^2 \sin^2 \theta \right) \sin^2 \theta d\phi^2 . \]  

(52)

This expression agrees with that given in [1]. Next, it is convenient to pass to the local frame, which is given by the basis one-forms \( e^a = e^a_i dx^i \) of the metric (18), and put the second equation in (51) in the form

\[ D_{(a} F_{b) c} = Z_{abc} . \]  

(53)
where

\[ F_{bc} = \frac{1}{2} \varepsilon^e_{c} I_{bef}, \quad Z_{abc} = \frac{1}{2} \varepsilon^e_{c} R_{efd(a} K^d_{b)} \]  

(54)

and \( \varepsilon_{abc} \) is the usual Levi-Civita symbol. The nonvanishing components of the tensor \( Z_{abc} \) are given by

\[ Z_{312} = \frac{M^2 - Q^2}{\Sigma} \Delta, \quad Z_{231} = \frac{M^2 - Q^2}{\Sigma} a^2 \sin^2 \theta, \quad Z_{123} = -(Z_{312} + Z_{231}) \]  

(55)

Substituting these expressions in equation (53) it is straightforward to show that for the tensor

\[ F_{ab} = S_{ab} + A_{ab}, \]  

(56)

the symmetric part \( S_{ab} \) vanishes identically, while solving the remaining equations we find that the nonvanishing components of the antisymmetric tensor \( A_{ab} \) are given by

\[ A_{13} = \frac{a^2 \sin 2 \theta}{\sqrt{\Sigma}}, \quad A_{23} = -2 (r - M) \sqrt{\Delta / \Sigma}. \]  

(57)

Finally, using this expressions we obtain that the spin-valued Killing vector is given by

\[ I_i dx^i = \frac{ia^2 \sin 2 \theta}{\sqrt{\Delta}} \left( \theta^1 \theta^2 dr - \frac{\Delta \sin \theta}{\sqrt{\Sigma}} \theta^2 \theta^3 d\phi \right) - 2i (r - M) \left( \sqrt{\Delta} \theta^1 \theta^2 d\theta - \frac{\Delta \sin \theta}{\sqrt{\Sigma}} \theta^3 \theta^1 d\phi \right). \]  

(58)

It is now natural to ask whether the space (48) admits a nongeneric supercharge of the form (43). To answer this question, one needs to look for the Killing-Yano tensor \( f_{ij} \) of this space. Writing down explicitly the Killing-Yano equation

\[ D(i, f_{ij})_k = 0 \]  

(59)

in the background (48), we see that \( f_{12} = 0 \). Meanwhile, from the integrability conditions of the equations for the remaining components \( f_{13} \) and \( f_{23} \), we find that

\[ f_{13} = 0, \quad a^2 f_{23} = 0. \]  

(60)

It follows that a nontrivial solution exists only for the vanishing rotation parameter, \( a = 0 \). This solution has the simple form

\[ \frac{1}{2} f_{ij} dx^i \wedge dx^j = \left( \Delta \right)^{3/2} \sin \theta d\theta \wedge d\phi. \]  

(61)
We note that in three dimensions the Killing-Yano tensor can always be expressed as the dual of a one-form field which must be closed. In our case, it is the dual of the conformal Killing vector field $\Omega = \sqrt{\Delta} \, dr$ of the space (48). Evaluating now the nonvanishing components of the third rank tensor in (45), we obtain that

$$C_{123} = \frac{2(r - M)}{\sqrt{\Delta}}.$$  \hspace{1cm} (62)

With the expressions (61) and (62) the associated supercharge in (43) does not generate a new nongeneric supersymmetry. It is related to the conserved sum of the orbital and spin angular momenta of the spinning point particle. The Poisson-Dirac bracket of two such supercharges gives the Killing tensor (reducible) and the spin-valued Killing vector which agree precisely with the $a = 0$ limit of the expressions in (52) and (58).

VI. CONCLUSION

The main purpose of this paper was to show that in a stationary spacetime, the dynamics of stationary spinning strings governed by a worldsheet supersymmetric action becomes equivalent to that of pseudo-classical spinning point particles in the effective metric of the quotient space of the original spacetime. Employing the Geroch procedure of foliation of the spacetime by its Killing trajectories, we have shown that the action of a stationary spinning string reduces to the action of a spinning point particle in the quotient space with conformally scaled metric. This fact generalizes the similar result obtained earlier [1] for the stationary bosonic string dynamics and makes it possible to use the known general theory of spacetime symmetries in mechanics of spinning point particles [16]. In this framework, we have explored the symmetries of the Kerr-Newman spacetime in terms of the motion of the stationary spinning string. We have solved the equations for symmetries in the three-dimensional effective quotient space of the Kerr-Newman spacetime and presented the explicit expressions for the Killing tensor as well as for the spin-valued Killing vector. We have also shown that the effective space does not admit the Killing-Yano tensor, except in the case of vanishing rotation parameter. In the latter case, the associated supercharge, unlike the case of the original spacetime, does not generate a new nontrivial supersymmetry. It becomes related to the conserved sum of the orbital and spin angular momenta of the spinning point particle.
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