ANALYTIC CONNECTIONS
ON RIEMANN SURFACES AND ORBIFOLDS

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Abstract. We give a differentially closed description of the uniformizing representation to the analytical apparatus on Riemann surfaces and orbifolds of finite analytic type. Apart from well-known automorphic functions and Abelian differentials it involves construction of the connection objects. Like functions and differentials, the connection, being also the fundamental object, is described by algorithmically derivable ODEs. Automorphic properties of all of the objects are associated to different discrete groups, among which are excessive ones. We show, in an example of the hyperelliptic curves, how can the connection be explicitly constructed. We study also a relation between classical/traditional ‘linearly differential’ viewpoint (principal Fuchsian equation) and uniformizing $\tau$-representation of the theory. The latter is shown to be supplemented with the second (to the principal) Fuchsian equation.

Contents
1 Introduction ................................................................. 2
2 Invariant quantities on $\mathcal{R}$ ........................................... 3
   2.1 Invariant counterparts of Fuchsian equations ...................... 3
   2.2 Automorphic property for a connection ............................ 5
3 Construction of connections .............................................. 6
   3.1 Zero genus orbifolds .................................................. 6
   3.2 Relation between arbitrary and zero genera ........................ 7
   3.3 Excessive automorphisms ............................................. 8
   3.4 Hyperelliptic case ..................................................... 9
4 Differential properties of connections ................................ 12
   4.1 ODEs for connections ................................................ 12
   4.2 Differentials, connections, and differential closedness .......... 14
References .................................................................. 16

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1. Introduction

Riemann surfaces are of fundamental importance to the mathematical physics because most effective part of the modern differential/integral calculus is related, one way or the other, with a complex analysis on a certain Riemann surface of (or not) a certain analytic function. Surfaces of a finite genus are distinctive in that the calculus is the best elaborated one with lot of applications. Strangely enough, the standard differential apparatus on such kind objects of higher genera \((g > 1)\) cannot be considered as completely closed; this remark requires some explanation.

Let \(\mathcal{R}\) be a finite genus Riemann surface determined by irreducible algebraic equation

\[
F(x, y) = 0. \tag{1}
\]

Uniformizing representation of \(\mathcal{R}\) is given by a pair of single-valued analytic functions \(x = \varphi(\tau), y = \psi(\tau)\), wherein the global uniformizer \(\tau\) belongs to the upper half-plane \(\mathbb{H}^+\), that is \(\Im(\tau) > 0\). Functions \(\varphi\) and \(\psi\) are the automorphic ones with respect to an infinite discrete Fuchsian group \(\mathfrak{G}_\mathcal{R}\) [8]:

\[
\varphi\left(\frac{\alpha \tau + \beta}{\gamma \tau + \delta}\right) = \varphi(\tau), \quad \psi\left(\frac{\alpha \tau + \beta}{\gamma \tau + \delta}\right) = \psi(\tau), \quad \forall \tau \in \mathbb{H}^+, \tag{2}
\]

where \(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathfrak{G}_\mathcal{R} \subset \text{PSL}_2(\mathbb{R})\). Complex analysis on \(\mathcal{R}\) includes Abelian differentials \(R(x, y)dx\), their integrals \(\int R(x, y)dx\), and the \(\mathcal{R}\) itself is completely determined by periods of Abelian integrals that are holomorphic (everywhere finite) [7]. We know also that if some function \(\psi(\tau)\) is a \(\tau\)-representation for any of the differentials above then its automorphic property is characterized by a weight-2 automorphic form [8, 9, 7]:

\[
\psi\left(\frac{\alpha \tau + \beta}{\gamma \tau + \delta}\right) = (\gamma \tau + \delta)^2 \cdot \psi(\tau). \tag{3}
\]

In the uniformization theory automorphic functions are described by the 2nd order linear ordinary differential equations (ODEs) of a Fuchsian class [8]

\[
\Psi_{xx} = \frac{1}{2} \mathcal{Q}(x, y) \Psi, \tag{4}
\]

where \(\mathcal{Q}\) is a rational function of its arguments. In general, this is a Fuchsian equation with algebraic coefficients. The uniformizing parameter \(\tau\) is then defined as a ratio

\[
\tau = \frac{\Psi_2(x)}{\Psi_1(x)} \tag{5}
\]

of the linearly independent solutions to Eq. (4) and inversion of this ratio determines one of the uniformizing functions: \(x = \varphi(\tau)\). It is well known that the theory is equivalent to the 3rd order ODE \(\{\tau, x\} = -\mathcal{Q}(x, y)\) containing no the auxiliary \(\Psi\)-function, where \(\{\tau, x\}\) is the standard notation for the Schwarz derivative [8]

\[
\{\tau, x\} := \frac{\tau_{xxx}}{\tau_x} - \frac{3}{2} \frac{\tau_{xx}^2}{\tau_x^2}. \tag{6}
\]

Since the theory is described by the third order ODEs, its complete data set is not exhausted by functions (2) and their first order differentials (3): the second order differentiation is missing. Alternatively, the \(\mathcal{R}\) may be thought of as a 1-dimensional complex
and all the objects above can be treated from the differential geometric viewpoint. Then functions (2)–(3) represent scalars and 1-forms and calculus should involve a covariant differentiation of these and other tensor fields. By this means, in order to close the complex analytic theory, we have to introduce (at least) a canonical bundle over our \( R \) and corresponding connection object \( \Gamma \). Partially, some ingredients of such a view on the theory have already been appeared in the literature. Dubrovin [6] gave a geometric treatment to the famous Chazy equation \( \pi \dddot{\eta} = 12i(2\eta\dddot{\eta} - 3\dddot{\eta}^2) \) when the group \( \mathfrak{S}_R \) is the genus zero full modular group \( \text{PSL}_2(\mathbb{Z}) =: \Gamma(1) \) and Hawley & Schiffer [13] introduced the connection \( \Gamma \) in the context of conformal mappings of planar domains and multi-connected representations of \( R \). The well-known modular forms [2] are the particular cases of automorphic forms when group is a subgroup of \( \Gamma(1) \). They possess interesting differential properties and some of them—ODEs for some low level groups \( \Gamma_0(N) \)—are constructed in [11]. It may be remarked here that even the theory of the \( \Gamma(1) \)-connection function, i.e., the Chazy–Weierstrass function \( \eta(\tau) \), is not restricted by the Chazy equation mentioned above. Recent work [14] provides an alternative theory (and nontrivial application) in the language of linear Fuchsian equations (4).

The known examples [6, 11, 2] are concerned only with the zero genus cases and general automorphic properties of bundles and connections on them, to our knowledge, are not considered in the literature. This is the subject matter of the present work. We give an analytically closed geometric description for the differential calculus on Riemann surfaces of finite analytic type (genus and number of punctures are finite) through the uniformizing \( \tau \)-representation for the connection objects \( \Gamma(\tau) \) and characterize their differential properties. More precisely, not only do functions and differentials satisfy some autonomic 3rd order ODEs (the known fact [8, 4, 2]), but connections also satisfy equations of such a kind. What is more, a remarkable property of (analytic) connections on \( R \)'s of arbitrary genera is the fact that all of them come from a trivial connection on certain orbifolds of the zero genus and satisfy autonomic ODEs. These ODEs are algorithmically derivable.

2. Invariant quantities on \( R \)

2.1. Invariant counterparts of Fuchsian equations. Let Fuchsian equation (4) determine, through its monodromical group \( \mathfrak{S}_x \), an exact representation of fundamental group \( \pi_1 \) of a certain orbifold or the \( R \) itself. However, from differential geometric viewpoint this equation is not well defined; it has no invariant (autonomic) form. Indeed, it contains explicitly the quantity \( x \) which, in a generic case \( F_y(x, y) \neq 0 \), is the standard usage for a local coordinate on \( R \). In turn the quantity \( \tau(x) \) coming from the definition (5) obviously does not produce the geometric object. However we may take 1-dimensionality of \( R \) into account and swap around the standard coordinate \( x \) and ‘object’ \( \tau \); thus \( x \) may be thought of as the scalar ‘field’ quantity \( x = x(\mathcal{P}) \) [12] being represented by function \( x = \mathcal{P}(\tau) \) on the universal cover \( \mathbb{H}^+ \). Then the automorphic property (2) becomes nothing but the \( \mathbb{H}^+ / \mathfrak{S} \)-factor topology reformulation to the property of \( x \) to be a scalar:

\[
\tilde{x}(\mathcal{P}) = x(\mathcal{P}), \quad \mathcal{P}
\begin{pmatrix}
\alpha \tau + \beta \\
\gamma \tau + \delta
\end{pmatrix} = \mathcal{P}(\tau); \tag{7}
\]
here $\tilde{x}(p)$ is a value of the quantity $x$ at point $p \in \mathbb{R}$ under the coordinate choice
\[ \tilde{\tau} = \frac{a \tau + b}{c \tau + d}, \tag{8} \]
and arbitrariness of the real numbers $(a, b, c, d)$ comes from well-known projective structure on $\mathbb{H}^+$. The second generator $y(p) = \psi(\tau)$ of the field of meromorphic functions on $\mathbb{R}$ is determined by the same properties as $(7)$. Therefore we should do an inverting the Schwarzian $(6)$ into the object $\{x, \tau\}$ and, denoting $[x, \tau] := -\{\tau, x\}$, introduce an invariant (coordinate-free) form of the principal equation $(4)$:
\[ [x, \tau] = Q(x, y), \tag{9} \]
where
\[ [x, \tau] := \frac{\dddot{x}}{\dot{x}^3} - \frac{3 \ddot{x}^2}{2 \dot{x}^4}, \tag{10} \]
and the dot above a symbol stands for a $\tau$-derivative. Proof uses the known property of the Schwarz derivative
\[ -\dot{x}^2 \{\tau, x\} = \{x, \tau\}. \]

Thus the 3rd order differential object $[x, \tau]$ represents, due to equation $(9)$, a scalar function on $\mathbb{R}$. It follows also that the nonlinear differential operator $z \mapsto [z, \tau]$ generates scalar objects from the scalar ones. Indeed, the well-known transformation rule for the Schwarz derivative of a function composition $\tau = f(\mu)$ and $\mu = g(z)$ $[8]$, i.e.,
\[ \{\tau, \mu\} d\mu^2 + \{\mu, z\} dz^2 = \{\tau, z\} dz^2, \tag{11} \]
being rewritten in terms of the $[x, \tau]$-objects, implies that if $z$ is any rational (scalar) function on $\mathbb{R}$, that is $z = R(x, y)$, then
\[ [z, \tau] = [R, x] + \frac{1}{R_x^2} Q(x, y); \]
this expression is again the rational function on $\mathbb{R}$.

As for the Abelian differentials, their two structure properties—analogs of $(7)$—have obviously the following form $[9]$:
\[ \tilde{\psi}(p) = \frac{d\tau}{d\tilde{\tau}} \cdot \psi(p), \quad \psi\left(\frac{\alpha \tau + \beta}{\gamma \tau + \delta}\right) = (\gamma \tau + \delta)^2 \cdot \psi(\tau), \]
where $\tilde{\psi}(p)$ represents the 1-form $\omega = \psi(p) d\tau$ in the coordinate $\tilde{\tau}$. Behavior of the higher order analytic differential $k$-forms $f(p) d\tau^k$ is defined in a similar manner:
\[ \tilde{f}(p) = \left(\frac{d\tau}{d\tilde{\tau}}\right)^k f(p), \quad f\left(\frac{\alpha \tau + \beta}{\gamma \tau + \delta}\right) = (\gamma \tau + \delta)^{2k} \cdot f(\tau), \]
Since the 1-dimensional complex analysis under construction deals with the analytic differentials and their powers, we have in fact to supplement the base $\mathbb{R}$ with a canonical (cotangent, holomorphic, and line) bundle $K$ $[1, 10]$.

*Greek symbols $\alpha, \beta, \ldots$ will be used for discrete group transformations and Latin $a, b, \ldots$ for coordinate changes.
2.2. Automorphic property for a connection. The transformation properties of geometric objects in cotangent bundles are completely determined by transformations in the base $\mathcal{R}$. Therefore we may take the known transformation law for a connection form $\Gamma_\alpha$ and adopt it to our 1-dimensional case $T^*(\mathcal{R})$; clearly, $\Gamma_\alpha$ must also respect the projective structure (8). For an arbitrary vector bundle with a structure group $G$ we have [5]

$$\tilde{\Gamma}_\alpha = \frac{\partial z^\beta}{\partial \tilde{z}^\alpha} \left( G \Gamma_\beta G^{-1} - \frac{\partial G}{\partial z^\beta} G^{-1} \right),$$

where the simultaneous transformations of coordinates in the base $z \mapsto \tilde{z}$ and in a fiber $\Psi \mapsto \tilde{\Psi}$ are carried out:

$$z \mapsto \tilde{z} = \tilde{z}(z), \quad \Psi \mapsto \tilde{\Psi} = G(\mathcal{P}) \Psi.$$

Of course, the covariant differentiation $\nabla$ is defined here by the standard rule:

$$\nabla_\alpha \Psi = \frac{\partial \Psi}{\partial z^\alpha} - \Gamma_\alpha \Psi.$$ 

Let us change notation for the old/new coordinates $(z, \tilde{z}) \rightarrow (\tau, \tilde{\tau})$ and take into account that for cotangent bundles we have to put $G(\mathcal{P}) = \frac{d\tau}{d\tilde{\tau}}$. Applying all this to the case under consideration, we get

$$\tilde{\Gamma}(\mathcal{P}) = \frac{d\tau}{d\tilde{\tau}} \bigg|_{\mathcal{P}} \cdot \Gamma(\mathcal{P}) + \left( \frac{d}{d\tilde{\tau}} \ln \frac{d\tau}{d\tilde{\tau}} \right)_{\mathcal{P}},$$

and the above mentioned projective structure leads to the following property.

**Proposition 1.** The transformation rule for an analytic connection on $T^*(\mathcal{R})$ realized on the universal cover $\mathbb{H}^+$ reads as

$$(ad - bc) \tilde{\Gamma} \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^2 \cdot \Gamma(\tau) + 2c(c\tau + d),$$

where $\{a, b, c, d\}$ are real.

In order to pass to the $\tau$-representation of our $\mathcal{R}$’s we now need to satisfy the factorization of the $\mathbb{H}^+$-topology with respect to some discrete group $\mathfrak{G}$ acting on $\mathbb{H}^+$. To put it differently, we have to obtain a property of the function $\Gamma(\tau)$ representing on $\mathbb{H}^+/\mathfrak{G}$ the connection object $\Gamma(\mathcal{P})$.

**Theorem 2.** Let $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \mathfrak{G}_\mathcal{R}$ be an exact matrix representation of $\pi_1(\mathcal{R})$ and $\Gamma(\tau)$ be a uniformizing representation of the connection object on $T^*(\mathcal{R})$. Then

$$(\alpha \delta - \beta \gamma) \Gamma \left( \frac{\alpha \tau + \beta}{\gamma \tau + \delta} \right) = (\gamma \tau + \delta)^2 \cdot \Gamma(\tau) + 2\gamma(\gamma \tau + \delta).$$

**Proof.** Rewrite the property (12) in form of ‘separated’ differentials:

$$\tilde{\Gamma}(\mathcal{P}) d\tilde{\tau} \big|_{\mathcal{P}} = \Gamma(\mathcal{P}) d\tau \big|_{\mathcal{P}} + (d \ln d\tau) \big|_{\mathcal{P}} - (d \ln d\tilde{\tau}) \big|_{\mathcal{P}}.$$

Hence

$$\Gamma(\mathcal{P}) d\tau \big|_{\mathcal{P}} + (d \ln d\tau) \big|_{\mathcal{P}} = \tilde{\Gamma}(\mathcal{P}) d\tilde{\tau} \big|_{\mathcal{P}} + (d \ln d\tilde{\tau}) \big|_{\mathcal{P}}.$$
and the quantity $\Gamma(\mathcal{P})d\tau|_{\mathcal{P}} + (d \ln d\tau)|_{\mathcal{P}}$ is thus a scalar invariant. Let points $\mathcal{P}$ and $\mathcal{Q}$ be equivalent with respect to group $\mathcal{G}_{\mathcal{A}}$, that is $\mathcal{P} \sim \mathcal{Q}$. We then may equate values of the latter scalar invariant taken at $\mathcal{P}$ and $\mathcal{Q}$:

$$\Gamma(\mathcal{P})d\tau|_{\mathcal{P}} + (d \ln d\tau)|_{\mathcal{P}} = \Gamma(\mathcal{Q})d\tau|_{\mathcal{Q}} + (d \ln d\tau)|_{\mathcal{Q}}.$$

Pass to the notation $\tau := \tau|_{\mathcal{P}}$ and $\tau := \tau|_{\mathcal{Q}}$; one gets

$$\Gamma(\mathcal{Q}) = \frac{d\tau}{d\tau} \cdot \Gamma(\mathcal{P}) + \frac{d}{d\tau} \ln \frac{d\tau}{d\tau}.$$

Substituting here the group transformations $\tau = \frac{\alpha \tau + \beta}{\gamma \tau + \delta}$, we arrive at formula (14). ■

Remark 1. Relations (13) and (14) are the properties in their own rights. The first one is local and defines the object $\Gamma(\mathcal{P})$: the two functions $\Gamma$, $\tilde{\Gamma}$ are evaluated at one point $\mathcal{P}$. The second property is nonlocal: one function representing the object itself is evaluated at two $\tau$-points. Obviously, the covariant differentiation of the $k$-differentials above is defined as follows [6]

$$\nabla f(\tau) = \frac{d}{d\tau} f(\tau) - k \Gamma(\tau) f(\tau).$$

In the case of modular subgroups of the group $\Gamma(1)$ the functions possessing the formal property (14) are sometimes called the quasi-modular forms [11, 2].

3. Construction of connections

3.1. Zero genus orbifolds. Let us consider first the case of genus zero. It is clear, that nontrivial theory appears only if the zero genus sphere $\mathcal{R} = \mathbb{CP}^1$ is endowed with points wherein this $\mathcal{R}$ ceases to be a manifold with a trivial fundamental group and each of these points determines a finite order element of the fundamental group $\pi_1$ (the conical singularity) or a group element of infinite order (the puncture). We thus have to consider an $N$-punctured sphere, which is an orbifold $\Xi$ with the generic group $\pi_1(\Xi) = \langle a_1, \ldots, a_{N-1} \rangle$, where $(a_s)^{p_s} = 1$ and integral $p_s$’s are formally allowed to be equal to $\infty$. In this case the theory is described by a Fuchsian equation with rational coefficients of the following form:

$$\Psi_{xx} = \frac{1}{2} \mathcal{Q}(x) \Psi \quad \text{ (15)}$$

$$= \frac{1}{4} \left\{ \frac{p_s^2 - 1}{(x - e_s)^2} + \cdots \right\} \Psi,$$

where $x \in \Xi = \mathbb{C} \{e_1, \ldots, e_{N-1}, \infty\}$ and accessory parameters (hidden in dots) have been chosen such that the monodromy representation of $\pi_1(\Xi)$ be a first kind Fuchsian group [8].

Since this base $\Xi$ is noncompact, every holomorphic vector bundle on it is trivial [9]. On the other hand, the map $x = \varphi(\tau)$ between $\Xi$ and its fundamental polygon for $\pi_1(\Xi)$ is single-valued in both directions; $\varphi$ is a Hauptmodul. Hence we may take $x$ as the global coordinate on $\Xi$, think of it as a flat one $\tilde{x} := x$, and consider in this (old) coordinate
the zero connection: \( \tilde{\Gamma}(p) \equiv 0 \). For the new (non-flat) coordinate \( \tau \) we therefore have, according to the law (12),

\[
0 = \frac{1}{\dot{x}} \Gamma(p) - \frac{d}{dx} \ln \dot{x} \quad \Rightarrow \quad \Gamma(\tau) = \frac{d}{d\tau} \ln \dot{\varphi}(\tau);
\]

whence it follows that such a \( \Gamma \) comes from a scalar function on \( \mathfrak{F} \). Since the difference of any two connections \( \Gamma - \Gamma' \) on \( T^*(\mathfrak{F}) \) is a differential, we obtain the following property.

**Proposition 3.** Uniformizing representations to the analytic everywhere holomorphic connections on \( T^*(\mathfrak{F}) \) for a zero genus orbifold \( \mathfrak{F} \) are determined by its Hauptmodul \( x = \varphi(\tau) \):

\[
\Gamma(\tau) = \frac{d}{d\tau} \ln \dot{\varphi}(\tau) + R(\varphi(\tau)) \dot{\varphi}(\tau),
\]

(16)

where \( R \) is a holomorphic function on \( \mathfrak{F} \).

Because we are interested in the *meromorphic* analysis on Riemann surfaces and orbifolds we handle only with meromorphic, i.e., Abelian differentials and, therefore, put the function \( R \) to be a rational one on \( \mathbb{C} \) with poles at points \( \{e_k, \infty\} \) at most. It defines a holomorphic differential \( R(x)dx \) on \( \mathfrak{F} \). If we allow for \( R(x) \) to have poles on \( \mathfrak{F} \) then one can speak of meromorphic connections.

Another kind arguments for construction of the connection above uses the fact that for a 1-dimensional case the curvature of an analytic connection is an identical zero and we may look for a covariantly constant section \( \psi \) (differential), that is

\[
\psi_{\tau} = \Gamma(\tau) \psi,
\]

where \( \Gamma(\tau) \) is as yet unknown. Differentials do certainly exist and all of them are generated from one of them, say, \( dx \) by the formula above \( R(x)dx \). Substituting here \( \psi = R(x)\dot{\varphi}(\tau) \) we arrive again at a formula of the form (16).

### 3.2. Relation between arbitrary and zero genera.

The transition from the previous case of \( g = 0 \) to the arbitrary genera is based on the following extension of a result implicitly formulated in [15].

**Theorem 4.** For a compact Riemann surface defined by an arbitrary algebraic curve (1) there exists a function field \( \mathbb{C}(x, y) \) generator pair \((z, w)\) such that one of the generators, e.g., \( z \) has a zero genus automorphism group \( \text{Aut}(z(\tau)) =: \mathfrak{G}_z \) and is determined by a Fuchsian equation with rational coefficients:

\[
[z, \tau] = \begin{vmatrix}
-\frac{3}{8} \sum_{s=1}^{2n+1} \frac{1}{(z - E_s)^2} - \frac{2nz^{2n-1} + A(z)}{(z - E_1) \cdots (z - E_{2n+1})}
\end{vmatrix},
\]

(17)

where \( A(z) \) is a properly chosen accessory polynomial of degree \( 2n - 2 \).

**Proof.** By an appropriate birational substitution \((x, y) \equiv (z, w)\) one can always transform the curve (1) into a nonsingular form \( \tilde{F}(z, w) = 0 \) having only the simple branch points \( z = E_s \). This means that multi-valued function \( w = w(z) \) has a local ramification structure of the form \((w - w_s)^2 = a_s(z - E_s) + \cdots \) for all \( E \)'s with \( a_s \neq 0 \) and holomorphic otherwise:
\[ w - w_0 = b_0(z - z_0) + \cdots \text{ under } z_0 \neq E_s. \] The number of \( E \)-points is always even (denote it as \( 2n + 2 \)) and we can put one of them at \( z = \infty \). Consider Fuchsian equation (17). The Klein–Poincaré theorem [8] states that there is a unique \( A(z) \)-polynomial such that this equation determines the globally single-valued on \( \mathbb{H}^+ \) analytic function \( z = z(\tau) \). The coefficient \( -\frac{3}{8} \) in (17) says that \( z(\tau) \) has the following local behavior in neighborhoods of \( E \)'s:
\[ \begin{align*}
z &= E + \tau^2 + \cdots \\
\text{At infinity we have the development} &:= \tau^{-2} + \cdots.
\end{align*} \]
Therefore function \( w(\tau) = w(z(\tau)) \) is everywhere single-valued as well. Eq. (17) has no other singularities except \( \{E_s\} \) and, hence, in neighborhoods of the regular points \((z_0, w_0) \in ˜F\) we have the developments
\[ \begin{align*}
z &= z_0 + A(\tau - \tau_0) + \cdots, \\
w &= w_0 + B(\tau - \tau_0) + \cdots,
\end{align*} \]
which are holomorphic. Since the pair \((z,w)\) has been obtained with the help of birational transformation, the functions \( z = R_1(x,y) \) and \( w = R_2(x,y) \) form just a different pair of generators of the function field: \( \mathbb{C}(x,y) = \mathbb{C}(z,w) \). We thus have a complete conformal purely hyperbolic \( \mathbb{H}^+ \)-image \((z(\tau), w(\tau))\) of (1): \( \mathcal{R} = \mathbb{H}^+ / \mathcal{G}_R \).

From the group-algebraic viewpoint this theorem means that the exact matrix representation of \( \pi_1(\mathcal{R}) = \mathcal{G}_R \) for a compact Riemann surface defined by (1) may be represented as an intersection of different monodromy group pairs
\[ \mathcal{G}_R = \mathcal{G}_x \cap \mathcal{G}_y = \mathcal{G}_z \cap \mathcal{G}_w = \cdots \] (18)
and one of the generators here (say, \( z \)) has a zero genus monodromy \( \mathcal{G}_z \); this is not obvious a priori. To put it differently, when describing purely hyperbolic higher genera Riemann surfaces the zero genus orbifolds do always appear and their structure properties are determined by ‘rational’ Fuchsian equations. (Recall that zero genus monodromy group may correspond both to a rational and to an algebraic Fuchsian equation). Hence, the function elements \( z \) with a zero genus automorphism \( \text{Aut}(z(\tau)) \) do certainly exist and there are no reasons to ignore such objects when constructing the effective theory of the \( \mathcal{R} \) itself. A simple explanation here is the fact that wider groups are easier described and this point should certainly be exploited in the theory.

Equation (17) determines the function \( z(\tau) \) which, besides being a defining Hauptmodul for a zero genus orbifold \( \mathcal{T}_z \), is a function element \( z = R_1(x,y) \) of the ‘nonzero genus field’ \( \mathbb{C}(x,y) \) of rational functions on (1). Hence it follows that whatever compact \( g > 1 \) Riemann surface \( \mathcal{R} \) may be there always exist associated zero genus orbifolds whose Hauptmoduln are elements of the function field \( \mathbb{C}(x,y) \). By this means we can construct connections on nonzero genera \( \mathcal{R} \)'s using connections on \( T^*(\mathcal{T}_z) \). The only thing we need is to satisfy the automorphic property (14) for the full group \( \mathcal{G}_R \).

3.3. Excessive automorphisms. Let us take a connection on \( T^*(\mathcal{T}_z) \) defined by formula (16)
\[ \Gamma(\tau) = \frac{d}{d\tau} \ln \dot z(\tau) \]
and add to it an arbitrary Abelian differential on the curve (1):
\[ \Gamma(\tau) = \frac{d}{d\tau} \ln \dot z(\tau) + R(z,w) \dot z(\tau). \] (19)

\( w - w_0 = b_0(z - z_0) + \cdots \text{ under } z_0 \neq E_s. \] The number of \( E \)-points is always even (denote it as \( 2n + 2 \)) and we can put one of them at \( z = \infty \). Consider Fuchsian equation (17). The Klein–Poincaré theorem [8] states that there is a unique \( A(z) \)-polynomial such that this equation determines the globally single-valued on \( \mathbb{H}^+ \) analytic function \( z = z(\tau) \). The coefficient \( -\frac{3}{8} \) in (17) says that \( z(\tau) \) has the following local behavior in neighborhoods of \( E \)'s:
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z &= z_0 + A(\tau - \tau_0) + \cdots, \\
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From the group-algebraic viewpoint this theorem means that the exact matrix representation of \( \pi_1(\mathcal{R}) = \mathcal{G}_R \) for a compact Riemann surface defined by (1) may be represented as an intersection of different monodromy group pairs
\[ \mathcal{G}_R = \mathcal{G}_x \cap \mathcal{G}_y = \mathcal{G}_z \cap \mathcal{G}_w = \cdots \] (18)
and one of the generators here (say, \( z \)) has a zero genus monodromy \( \mathcal{G}_z \); this is not obvious a priori. To put it differently, when describing purely hyperbolic higher genera Riemann surfaces the zero genus orbifolds do always appear and their structure properties are determined by ‘rational’ Fuchsian equations. (Recall that zero genus monodromy group may correspond both to a rational and to an algebraic Fuchsian equation). Hence, the function elements \( z \) with a zero genus automorphism \( \text{Aut}(z(\tau)) \) do certainly exist and there are no reasons to ignore such objects when constructing the effective theory of the \( \mathcal{R} \) itself. A simple explanation here is the fact that wider groups are easier described and this point should certainly be exploited in the theory.

Equation (17) determines the function \( z(\tau) \) which, besides being a defining Hauptmodul for a zero genus orbifold \( \mathcal{T}_z \), is a function element \( z = R_1(x,y) \) of the ‘nonzero genus field’ \( \mathbb{C}(x,y) \) of rational functions on (1). Hence it follows that whatever compact \( g > 1 \) Riemann surface \( \mathcal{R} \) may be there always exist associated zero genus orbifolds whose Hauptmoduln are elements of the function field \( \mathbb{C}(x,y) \). By this means we can construct connections on nonzero genera \( \mathcal{R} \)'s using connections on \( T^*(\mathcal{T}_z) \). The only thing we need is to satisfy the automorphic property (14) for the full group \( \mathcal{G}_R \).

3.3. Excessive automorphisms. Let us take a connection on \( T^*(\mathcal{T}_z) \) defined by formula (16)
\[ \Gamma(\tau) = \frac{d}{d\tau} \ln \dot z(\tau) \]
and add to it an arbitrary Abelian differential on the curve (1):
\[ \Gamma(\tau) = \frac{d}{d\tau} \ln \dot z(\tau) + R(z,w) \dot z(\tau). \] (19)
In this way we obtain the desired connection on $T^*(\mathbb{R})$ because automorphic property (14) for the object (19) does certainly hold for the group $\mathfrak{G}_\mathbb{R}$ if $R(z,w)$ is not a function of $z$ alone; all the transformations from group $\mathfrak{G}_z \cap \mathfrak{G}_w$ fit into the formula (14). This condition on function $R(z,w)$ is of course necessary but not sufficient because the rule (14) may take place for a group that may be wider than $\mathfrak{G}_\mathbb{R}$; just as group $\mathfrak{G}_z$ may be wider than group $\mathfrak{G}_\mathbb{R}$.

Such an ‘excessive’ extension may occur when constructing differentials and even functions: different differentials on one orbifold may possess the automorphic property (3)–(4) with respect to different groups forming towers of subgroups. For example, Whittaker’s family of curves $w^2 = z^{2g+1} + 1$ (the classical Weierstrass case $g = 1$ is not an exception) provides counterexamples when both the generators $z, w$ and the base differentials $dz$ and $dw$ have automorphic properties (2)–(3) with respect to groups that are larger than $\mathfrak{G}_\mathbb{R}$. In all these cases genera of $\mathfrak{G}_z$ and $\mathfrak{G}_w$ are equal to zero and groups $\mathfrak{G}_w$ are even the simple triangle ones:

$$[z, \tau] = -\frac{3}{8} \frac{z^{2g+1} - 4g(g + 1)}{(z^{2g+1} + 1)^2} z^{2g-1}, \quad [w, \tau] = -\frac{2g(g + 1)}{(2g + 1)^2} \frac{w^2 + 3}{(w^2 - 1)^2}.$$ 

These equations are solvable in terms of hypergeometric $\,_{2}\!F_{1}$-functions [16, 3]. Concerning functions, one view on this problem is a tessellation of a Fuchsian $\mathbb{R}$-polygon on the schlicht (univalent) domains of a given $u(\tau)$-function. Surprisingly, in such a formulation the problem is not elaborated even in the elliptic $g = 1$ case. In general we should involve, according to (18), automorphic objects with both the generating groups $\mathfrak{G}_z$ and $\mathfrak{G}_w$. Hence the problem above can be reduced to a problem of searching for an automorphic function $u(\tau) = R(z(\tau), w(\tau))$ whose automorphism group $\mathfrak{G}_u$ coincides with group $\mathfrak{G}_\mathbb{R} = \mathfrak{G}_z \cap \mathfrak{G}_w$. Complexity of the problem is related to the theory of Fuchsian equations with algebraic coefficients; the latter has almost not been developed.

**Remark 2.** To avoid lengthening terminology we shall apply the notion automorphism $\mathfrak{G}$ not only to functions (scalars) but to differentials and connections as well in the sense that the rules (3) and (14) respect the group $\mathfrak{G}$. This automorphism group is assumed, by definition, to be a maximal set of group elements under which the corresponding transformation laws hold.

**Proposition 5.** Uniformizing representations for meromorphic differentials on $\mathbb{R}$ and connections on $T^*(\mathbb{R})$ lift to differentials and connections for manifolds $\tilde{\mathbb{R}}$ that cover (finitely-sheeted) $\mathbb{R}$.

**Proof.** Fundamental group of a covering manifold is a subgroup of the one being covered [5].

Thus, since every group can be embedded in a larger group, the problem consists in elimination of the objects (functions, differentials, and connections) having wider automorphisms than exact representations of $\pi_1(\mathbb{R})$. Fortunately, there is an algorithmic solution in the case of hyperelliptic curves.

**3.4. Hyperelliptic case.** The aim of this section is to derive a class of algebraic dependencies (1) for which the connection object $\Gamma(\tau)$ is explicitly constructed and whose
automorphism coincides with group $\mathfrak{S}_\mathbb{R}$. By formula (19) any connection is built by a logarithmic derivative of a differential and we begin with differentials, namely, non-exact differentials because functions (exact differentials) may have wider automorphisms than $\mathfrak{S}_\mathbb{R}$.

Let $d\mathfrak{A} = R(x, y)du$, where $u$ is a holomorphic differential, be an Abelian differential whose $\tau$-representation $\hat{\mathfrak{A}}(\tau)$ (formula (3)) respects the group $\mathfrak{S}_\mathfrak{A} \supset \mathfrak{S}_\mathbb{R}$ with finite index $|\mathfrak{S}_\mathfrak{A} : \mathfrak{S}_\mathbb{R}| \neq 1$. In other words, assume that in addition to transformations $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathfrak{S}_\mathbb{R}$ there exists a transformation $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \notin \mathfrak{S}_\mathbb{R}$ such that

$$(\alpha \delta - \beta \gamma) R \left( \varphi \left( \frac{\alpha \tau + \beta}{\gamma \tau + \delta} \right), \psi \left( \frac{\alpha \tau + \beta}{\gamma \tau + \delta} \right) \right) \hat{u} \left( \frac{\alpha \tau + \beta}{\gamma \tau + \delta} \right) = (\gamma \tau + \delta)^2 R(\varphi(\tau), \psi(\tau)) \hat{u}(\tau).$$  \hspace{1cm} (20)

Let us analyze location of zeroes/poles of the differential because they are the well-defined and generate infinities of $\Gamma$. The latter in turn are the only well-defined objects for $\Gamma$. It follows that $\tau'$ may only be an image of another zero and analysis is complicated if $\hat{u}$ has several zeroes. On the other hand, if we take $\hat{u}$ having a single zero, its position should be preserved by the $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$-transformation and it must be an elliptic one. The existence of differentials with elliptic automorphisms on purely hyperbolic $\mathfrak{S}_\mathbb{R}$ follows from Theorem 4; whence it follows that the rational differential $R(z) \hat{z}$ has even the order two automorphisms at all the points $E$'s. It has however excessive zeroes so we drop out its rational part $R$ and, in order to reduce the zeroes, divide $\hat{z}$ by a function having simple zeroes at $E$'s, i.e., by $\prod_E \sqrt{z - E}$. One kind of algebraic curves for which this is possible suggests itself: this is the hyperelliptic class

$$w^2 = (z - E_1) \cdots (z - E_{2g+1}), \quad =: P(z)$$  \hspace{1cm} (21)

with parametrization $z = \varphi(\tau)$, $w = \psi(\tau)$. We put in this case

$$\hat{u} = \frac{\varphi}{\psi}$$

and this differential, being holomorphic, has the only zero at $\mathfrak{P}_0 = (\infty, \infty)$. Renormalizing the $\tau$-plane, we can assign $\tau_0 = 0$ to this point and impart the form $\tau \mapsto \varepsilon \tau$ to the elliptic transformation, that is $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, where $\varepsilon^n = 1$. The case under consideration corresponds to $n = 2$ and hence $z = \varphi(\tau)$ is an even function and $w = \psi(\tau)$ is an odd one:

$$\varphi(-\tau) = \varphi(\tau), \quad \psi(-\tau) = -\psi(\tau).$$
This is nothing but the hyperelliptic involution \((z, w) \mapsto (z, -w)\) (see also [15]). Now, let us build the following connection:

\[
\Gamma(\tau) = \frac{d}{d\tau} \ln \dot{u}(\tau) = \frac{d}{d\tau} \ln \frac{\dot{\varphi}}{\dot{\psi}}.
\]  

(22)

Compact Riemann surfaces are however purely hyperbolic manifolds but this connection admits our elliptic transformation in the sense that (22) transforms like differential \(\dot{\varphi}\):

\[
\dot{\varphi}(-\tau) = -\dot{\varphi}(\tau).
\]

Indeed,

\[
\Gamma(\tau) = \frac{\ddot{\varphi}}{\varphi} - \frac{\dot{\psi}}{\dot{\psi} - \varphi} = \frac{\dot{\varphi}}{\varphi} - \frac{1}{2}P'(\varphi)\frac{\dot{\varphi}}{\varphi}
\]

and this \(\Gamma(\tau)\) has the same parity as \(\dot{\varphi}\). Moreover, any connection built by arbitrary hyperelliptic differential respects a wider automorphism:

\[
\Gamma = \frac{d}{d\tau} \ln R(z) \frac{\dot{z}}{w} = \frac{d}{d\tau} \ln R(z) \dot{z} - \frac{1}{2} \frac{d}{d\tau} \ln w^2 = \frac{d}{d\tau} \ln R(z) \dot{z} - \frac{1}{2} \sum_E (z - E)^{-1} \cdot \dot{z},
\]

which is actually a connection on orbifold \(\mathbb{H}^+ / \mathcal{G}_z\). Thus, in order to avoid excessive elliptic automorphisms we should add to connection (22) any pole-free (nonzero) differential insensitive to permutation of sheets. Clearly, this is done by a holomorphic differential:

\[
\Gamma(\tau) = \frac{d}{d\tau} \ln \frac{\dot{\varphi}}{\dot{\psi}} + (c_1 + c_2 \varphi + \cdots + c_g \varphi^{g-1}) \frac{\dot{\varphi}}{\dot{\psi}}.
\]  

(23)

We thus have arrived at the following result.

**Theorem 6.** Let \(\mathcal{G}_\mathcal{R}\) be a matrix representation of \(\pi_1(\mathcal{R})\) for a hyperelliptic Riemann surface (21). Then connection \(\Gamma(\tau)\) defined by (23), subjected to a condition that at least one of \(c_j \neq 0\), has an automorphism \(\mathcal{G}_\mathcal{R}\) and represents a connection object on \(T^*(\mathcal{R})\) with a single pole at infinite point \(\mathcal{P}_0 = (\infty, \infty)\).

**Proof.** The transformation law (14), where \(\mathcal{G}_\mathcal{R} = \mathcal{G}_z \cap \mathcal{G}_w\), is satisfied by construction. Suppose that (23) respects a ‘larger’ factor topology on certain \(\mathbb{H}^+ / \mathcal{G}\):

\[
(\alpha \delta - \beta \gamma)\Gamma \left( \begin{pmatrix} \alpha \tau + \beta \\ \gamma \tau + \delta \end{pmatrix} \right) = (\gamma \tau + \delta)^2 \cdot \Gamma(\tau) + 2\gamma (\gamma \tau + \delta), \quad \left( \begin{pmatrix} \alpha \beta \\ \gamma \delta \end{pmatrix} \right) \in \mathcal{G} \supset \mathcal{G}_\mathcal{R}. \]

(24)

Owing to uniqueness of singularities of \(\Gamma(\tau)\), this transformation must be an elliptic one of the 2nd order and we have instead of (24)

\[-\Gamma(-\tau) = \Gamma(\tau).\]

However this rule is not satisfied by expression (23) because its two terms have opposite parities: the first one is odd and the first one is even, a contradiction; the transformation \(\left( \begin{pmatrix} \alpha \beta \\ \gamma \delta \end{pmatrix} \right)\) may only be an identical one.

**Remark 3.** Modifying the holomorphic differential

\[
\dot{u} = (z - E_k)^{g-1} \frac{\dot{z}}{w},
\]

we get connections with single singularities at points \(\mathcal{P}_k = (E_k, 0)\). Connections with single singularities (elementary, in terminology of [13]) may serve as building blocks for
construction of all the connections: by adding to them meromorphic differentials, one obtains connections with singularities at prescribed points. Moreover all the $\Gamma$’s have an invariant related to their poles. Indeed, integrating formula (12), we derive

$$
\int_{\partial R} \tilde{\Gamma}(\tilde{\tau}) d\tilde{\tau} = \int_{\partial R} \Gamma(\tau) d\tau + \int_{\partial R} d\ln \frac{d\tau}{d\tilde{\tau}} = \int_{\partial R} \Gamma(\tau) d\tau \quad \text{(invariant)},
$$

since $\frac{d\tau}{d\tilde{\tau}}$ is everywhere holomorphic without zeroes. Putting here $\Gamma = \frac{d}{d\tau} \ln \hat{u}(\tau)$ with arbitrary differential $\hat{u}$, one can compute the value of this invariant if $\mathcal{R}$ is a compact surface (not orbifold). In this case difference between number of zeroes and poles of any differential $\hat{u}$ is equal to $2g - 2$ and we obtain that

$$
\int_{\partial \mathcal{R}} \Gamma(\tau) d\tau = (2g - 2) \cdot 2\pi i.
$$

4. Differential properties of connections

4.1. ODEs for connections. This is a classical result by Hurwitz that the automorphic weight-2 forms satisfy the third order ODEs [4, 2]. These forms are the standard objects in the complex analysis on $\mathcal{R}$’s and in this section we shall show that the similar result takes place for any connection object.

Let us consider the finite genus hyperbolic Riemann surface (or orbifold) $\mathcal{R}$. Apart from algebraic equation (1) this object is described by the Fuchsian equation for a meromorphic function $u = R(x, y)$ on $\mathcal{R}$ whose uniformizing form $u(\tau)$ has an automorphism $\mathfrak{G}_u$ coinciding with $\pi_1(\mathcal{R})$. Corresponding equation (9) does certainly exist because $\pi_1$ for such an object can always be realized as a monodromy of the 2nd order linear Fuchsian ODE. We have

$$
[u, \tau] = \mathcal{Q}(x, y).
$$

Construct first a connection $\Gamma(\tau)$ on $T^*(\mathcal{R})$ with the help of an exact differential $\hat{u}$:

$$
\Gamma = \frac{d}{d\tau} \ln \hat{u}
$$

and define the corresponding covariant differentiation of $k$-differentials $\nabla = \partial_\tau - k\Gamma(\tau)$. Clearly, $\nabla u = \hat{u}$ and $\nabla^2 u \equiv 0$. Making use of the property

$$
[u, \tau] u^2 = \frac{d}{d\tau} \left( \frac{d}{d\tau} \ln \hat{u} \right) - \frac{1}{2} \left( \frac{d}{d\tau} \ln \hat{u} \right)^2,
$$

we observe that the object

$$
K := \hat{\Gamma} - \frac{1}{2} \Gamma^2
$$

is a 2-differential: $K = \mathcal{Q}(x, y)(\nabla u)^2$. Taking the $\nabla$-derivative, we have

$$
\nabla K = \nabla \mathcal{Q}(x, y) \cdot (\nabla u)^2.
$$

---

*The object $K$ is an analog of the standard curvature. Since $u$ may serve as a local coordinate on $\mathcal{R}$, this connection $\Gamma$ and its ‘curvature’ $K$ may be thought of as having almost everywhere zero values (in the flat coordinate $u$) or as having the $\delta$-function like distributions concentrated at a finite number of points wherein $\hat{u}(\tau) = \{0, \infty\}$. 
The quantities \( x(\tau) \) and \( y(\tau) \) represent the scalar objects on \( \mathcal{R} \) (not only on \( \mathbb{H}^+ / \mathcal{G}_x \) or \( \mathbb{H}^+ / \mathcal{G}_y \); Corollary 5), that is \( \dot{x} = \nabla x \) and \( \dot{y} = \nabla y \). Hence we derive

\[
\nabla K = (Q_x \dot{x} + Q_y \dot{y}) \cdot (\nabla u)^2 = F_y^{-1}(F_y Q_x - F_x Q_y) \dot{x} \cdot (\nabla u)^2,
\]

where \( \nabla K := \dot{K} - 2\Gamma K \). The definition \( u = R(x, y) \) gives

\[
\dot{x} = \frac{F_y}{F_y R_x - F_x R_y} \nabla u
\]

and, consequently,

\[
K = Q \cdot (\nabla u)^2, \quad \nabla K = \frac{F_y Q_x - F_x Q_y}{F_y R_x - F_x R_y} \cdot (\nabla u)^3.
\]

Elimination of \( \nabla u \) gives the identity

\[
\frac{(\nabla K)^2}{K^3} = \frac{(F_y Q_x - F_x Q_y)^2}{(F_y R_x - F_x R_y)^2} Q^{-3}.
\]

The second \( \nabla \)-derivative yields

\[
\nabla^2 K = \frac{F_y}{F_y R_x - F_x R_y} \frac{d}{dx} \frac{F_y Q_x - F_x Q_y}{F_y R_x - F_x R_y} \cdot (\nabla u)^4,
\]

where \( \nabla^2 K := \left( \frac{d}{d\tau} - 3\Gamma \right)(\dot{K} - 2\Gamma K) \). As before, elimination of \( \nabla u \) produces the second scalar identity

\[
\frac{\nabla^2 K}{K^2} = \frac{F_y Q^{-2}}{F_y R_x - F_x R_y} \frac{d}{dx} \frac{F_y Q_x - F_x Q_y}{F_y R_x - F_x R_y}.
\]

As a result we obtain three identities

\[
\frac{(\nabla K)^2}{K^3} = S(x, y), \quad \frac{\nabla^2 K}{K^2} = T(x, y), \quad F(x, y) = 0 \quad (26)
\]

with certain rational functions \( S \) and \( T \). Eliminating here the variable \( x \) followed by \( y \), one derives the equation \( L(K, \nabla K, \nabla^2 K) = 0 \), where \( K \) is understood to be expressed as (25) and \( \nabla K \) and \( \nabla^2 K \) as above. This is nothing but the 3rd order ODE satisfied by the connection object \( \Gamma \). By construction ODEs (26) and their polynomial consequences are invariant. We thus have arrived at the following result.

**Theorem 7.** Let \( \mathcal{R} \) be an arbitrary compact Riemann surface defined by equation (1) or an arbitrary orbifold whose compactification is equivalent to (1). Let \( \Gamma(\tau) \) be a uniformizing representation for a connection object \( \Gamma \) on \( T^*(\mathcal{R}) \). Then \( \Gamma(\tau) \) satisfies the algorithmically derivable 3rd order autonomic polynomial ODE

\[
\Xi(\tilde{\Gamma}, \dot{\Gamma}, \ddot{\Gamma}, \Gamma) = 0 \quad (27)
\]

whose general solution is given by the following single-valued analytic function:

\[
\Gamma = \frac{ad - bc}{(c\tau + d)^2} \Gamma \left( \frac{a\tau + b}{c\tau + d} \right) - 2c \frac{(ad - bc)}{c\tau + d}
\]

with free constants \( \{a, b, c, d\} \).
Proof. The last formula follows from the transformation law (13) and the only thing is left to be proved is that any other connection satisfies the certain ODE \((27)\).

Let \(\tilde{\Gamma}\) be such a connection: \(\tilde{\Gamma} = \Gamma - r(x, y) \dot{u}\). Redefine the objects used above:

\[
\tilde{\nabla} = \frac{d}{d\tau} - k\tilde{\Gamma}, \quad \tilde{K} := \tilde{\Gamma}_\tau - \frac{1}{2} \tilde{\Gamma}
\]

and take into account that \(u\), in contrast to the preceding, is no longer a flat coordinate:

\[
\tilde{\nabla}u = \dot{u}, \quad \tilde{\nabla}^2 u = r(x, y)(\tilde{\nabla}u)^2.
\]

It follows that

\[
\tilde{K} = \frac{d}{d\tau} (\Gamma - r \dot{u}) - \frac{1}{2} (\Gamma - r \dot{u})^2 = \dot{\Gamma} - \frac{1}{2} \Gamma^2 - r \dot{u} - \frac{1}{2} r^2 \dot{u}^2.
\]

Let prime ', as always in the sequel, stand for the total \(x\)-derivative:

\[
f' := f_x - \frac{F_x}{F_y} f_y.
\]

Since \(\dot{u} = R' \dot{x}\), expression (28) can be rewritten as follows

\[
\tilde{K} = \left( Q - \frac{r'}{R'} - \frac{1}{2} r^2 \right) (\tilde{\nabla}u)^2 =: Q(x, y) (\tilde{\nabla}u)^2.
\]

Hence we compute

\[
\tilde{\nabla}\tilde{K} = \frac{1}{R'} (Q' + 2r Q R') \cdot (\tilde{\nabla}u)^3, \quad \tilde{\nabla}^2 \tilde{K} = \frac{1}{R'} \left\{ \left( \frac{Q'}{R'} \right)' + 2Qr' + 5rQ' + 6r^2 QR' \right\} \cdot (\tilde{\nabla}u)^4
\]

and therefore

\[
\frac{(\tilde{\nabla}^2 \tilde{K})^2}{\tilde{K}^3} = \frac{(Q' + 2r Q R')^2}{Q^3 R^2}, \quad \frac{\tilde{\nabla}^2 \tilde{K}}{\tilde{K}^2} = \frac{1}{Q^2 R'} \left\{ \left( \frac{Q'}{R'} \right)' + 2Qr' + 5rQ' + 6r^2 QR' \right\}.
\]

As before, equation of the form (27) follows by elimination of the pair \((x, y)\).

4.2. Differentials, connections, and differential closedness. Since invariant ODEs follow from differential properties of automorphic functions, there should exist an equivalent ‘non-invariant’ description/explanation in the traditional language of linear ODEs.

Let us return to Eq. (4). It is clear, that its two linearly independent solutions

\[
\Psi_1(x) = \sqrt{x}, \quad \Psi_2(x) = \tau \sqrt{x}
\]

are not differentially closed. Therefore from differential viewpoint the complete and closed differential apparatus on \(\mathcal{R}\) must involve not only the principal equation (4):

\[
\Psi_{xx} = \frac{1}{2} Q(x, y) \Psi,
\]

but equation for a derivative \(\Phi\) of the \(\Psi\)-function:

\[
\Phi := \Psi_x \quad \Rightarrow \quad \Phi_{xx} - \ln' Q(x, y) \cdot \Phi_x - \frac{1}{2} Q(x, y) \Phi = 0.
\]
Clearly, this equation also belongs to a Fuchsian class and has its proper monodromy group. Since connections are defined up to differentials, we can set one of them as follows

\[ \Gamma(\tau) = \frac{d}{d\tau} \ln \dot{\varphi}(\tau) \implies \Gamma(\tau) = 2\Psi \dot{\psi} = 2 \frac{\dot{\psi}}{\Psi} \]

and, therefore, introduction of a connection and the \( \Psi \)-derivative are in fact the equivalent operations. Complete set of data for the theory can thus be written in both the \( x \)- and \( \tau \)-representation:

\[ \{ \Psi_1(x), \Psi_2(x), \Psi'_1(x) \} \iff \{ \varphi(\tau), \dot{\varphi}(\tau), \Gamma(\tau) \}. \]

In this context the above mentioned 3rd order ODEs satisfied by the 1- and 2-differential (say, \( \mathcal{K} \)) are also the direct and algorithmical consequences of the linear equations. Indeed, first we take the base differential \( \psi = \dot{x} \), that is \( \psi = \Psi_1^2 \). Taking into account that \( \frac{d}{dx} = \psi^{-1} \frac{d}{d\tau} \), we obtain the following \( \tau \)-equivalents of (4′)–(29):

\[ 2\ddot{\psi} \psi - 3\dot{\psi}^2 = 2 \mathcal{Q}(x,y) \psi^4, \quad \dddot{\psi} \psi^2 - 6 \dot{\psi} \ddot{\psi} \psi + 6 \dot{\psi}^3 = \mathcal{Q}'(x,y) \psi^6. \]  

(30)

Pass now to the arbitrary differential \( \psi = R^{-1}(x,y) \dot{x} \). Then these two expressions should be changed according to the rules

\[ \psi \to R \cdot \psi, \quad \frac{d}{d\tau} \psi \to RR' \cdot \psi^2 + R \cdot \dot{\psi}, \quad \frac{d}{d\tau} R \to RR' \cdot \psi. \]

As in the case of connections (Sect. 4.1), we get three polynomial equations

\[ S(\psi, \dot{\psi}, \ddot{\psi}; x, y) = 0, \quad T(\psi, \dot{\psi}, \dddot{\psi}; \varphi, \dot{\varphi}) = 0, \quad F(x,y) = 0 \]  

(31)

which are, by construction, are independent of coordinate choice (8). By elimination of \( (x, y) \) the differential \( \psi = \psi(\tau) \) satisfies a certain autonomic ODE of the form \( \Xi(\psi, \dot{\psi}, \ddot{\psi}, \dddot{\psi}) = 0 \) whose general solution is

\[ \psi = \frac{ad - bc}{(c\tau + d)^2} \psi \left( \frac{a\tau + b}{c\tau + d} \right), \]

where \( \psi(\tau) \) is any particular one.

Thus all the analytic geometric objects on arbitrary orbifolds of finite genus are constructed to be governed by invariant (autonomic) ODEs: scalars are described by Eq. (4), connections by Eq. (27), and 1-differentials by Eqs. (30) or (31).

It is worthy of special emphasis that we have nowhere used the fact that the group \( \mathfrak{G}_x \) must be of 1st Fuchsian kind acting on \( \mathbb{H}^+ \) (with unique accessory parameters in (4′)). All the statements above hold for \( \mathcal{Q} \)-functions with different values of accessory parameters determining the 2nd kind Fuchsian (Kleinian) groups acting in \( \mathbb{C} \) without invariant circle (groups of Schottky, Weber, and Burnside [8]).

*It is an interesting problem to study the extended set of Fuchsian equations (4′)–(29) in the framework of uniformization theory. This question, including a series of examples, will be the subject matter of a separate work. When \( \mathfrak{G}_{\mathbb{H}^+} \) is the zero genus group \( \Gamma(1) \) a closure of the ‘\( \Psi \)-theory’ into the ‘\( \Phi \)-one’ is considered in [14] in the context of the modelling selforganised patterns of vegetation.
Corollary 8. Let $\mathcal{R}$ be a hyperbolic Riemann surface or an orbifold of finite analytic type with uniformizing group $\mathfrak{G}_R$ of Fuchsian or Schottky type. Then the $\tau$-representations for meromorphic differentials and connection objects on $T^*(\mathcal{R})$ satisfy the algorithmically derivable autonomic ODEs of 3rd order.

Algorithmical constructions here are the same as in the case of $\mathbb{H}^+$ described above, except that the different values of accessory parameters in linear equations (4) will lead to different nonlinear autonomic equations for differentials and connections. It may be noted here that these ODEs carry all the information about group and, if group is $\mathfrak{G}_R$, about Riemann surface itself. They can serve as an alternative to the linear but non-autonomic Fuchsian ODEs and deserve to be further investigated in their own rights.

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