APPLYING GROMOV’S AMENABLE LOCALIZATION TO GEODESIC FLOWS

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Abstract. Let $M$ be a compact connected smooth Riemannian $n$-manifold with boundary. We combine Gromov’s amenable localization technique with the Poincaré duality to study the traversally generic geodesic flows on $SM$, the space of the spherical tangent bundle. Such flows generate stratifications of $SM$, governed by rich universal combinatorics. The stratification reflects the ways in which the geodesic flow trajectories interact with the boundary $\partial(SM)$. Specifically, we get lower estimates of the numbers of connected components of these flow-generated strata of any given codimension $k$. These universal bounds are expressed in terms of the normed homology $H_k(M;\mathbb{R})$ and $H_k(DM;\mathbb{R})$, where $DM = M \cup_{\partial M} M$ denotes the double of $M$. The norms here are the Gromov simplicial semi-norms in homology. The more complex the metric on $M$ is, the more numerous the strata of $SM$ and $S(DM)$ are. So one may regard our estimates as analogues of the Morse inequalities for the geodesics on manifolds with boundary.

It turns out that some close relatives of the normed homology spaces form obstructions to the existence of globally $k$-convex traversally generic metrics on $M$.

1. Introduction

This paper is an extension of [AK] and especially [K4]. As in the latter articles, it draws its inspiration from the papers of Gromov [Gr], [Gr1], where, among other things, the machinery of amenable localization has been developed. Here we combine Gromov’s amenable localization with the Poincaré duality (as in [K4]) to study the traversally generic geodesic flows on compact connected smooth Riemannian $n$-manifolds $M$ with boundary (see [K2] and Section 2 for the definition of traversally generic vector fields in general).

For a given Riemannian metric $g$ on $M$, we denote by $v^g$ the geodesic vector field on $SM$, the space of the unitary (in the metric $g$) spherical tangent bundle.

In [K6], we study metrics $g$ on $M$ such that $v^g$ on $SM$ is of the gradient type; that is, there exists a smooth function $F : SM \to \mathbb{R}$ such that $dF(v^g) > 0$. By Theorem 2.1 and Corollary 2.3 from [K6], such metrics form an open nonempty set $\mathcal{G}(M)$ in the space $\mathcal{M}(M)$ of all Riemannian metrics on $M$.

In [K6], Definition 2.3, we also introduced the notion of boundary generic metrics (see Definitions 2.2 and 3.1 below). For them, the boundary $\partial M$ is “generically curved”. We denote by $\mathcal{G}^\dagger(M)$ the space of boundary generic metrics of the gradient type. They form an open set in $\mathcal{M}(M)$. Conjecturally, $\mathcal{G}^\dagger(M)$ is dense in the space $\mathcal{G}(M)$.

Finally, we consider a subspace $\mathcal{G}^\dagger(M) \subset \mathcal{G}^\dagger(M)$, formed by the traversally generic metrics on $M$. Their definition is more elaborated and will be given in Section 2 below; in
short, a metric \( g \) is traversally generic if the geodesic vector field \( v^g \) is traversally generic with respect to \( \partial(SM) \) in the sense of [K2], Definition 3.2. Conjecturally, \( \mathcal{G}^\ddagger(M) \) is also dense in the space \( \mathcal{G}(M) \). It is proven to be open in \( \mathcal{M}(M) \).

For a given metric \( g \in \mathcal{G}^\ddagger(M) \), let \( \mathcal{T}(v^g) \) denote the space of trajectories of the \( v^g \)-generated flow on \( SM \); in other words, \( \mathcal{T}(v^g) \) is the space of geodesics on \( M \). We denote by \( \Gamma : SM \to \mathcal{T}(v^g) \) the obvious map. In general, \( \mathcal{T}(v^g) \) is not a manifold, but, for \( g \in \mathcal{G}^\ddagger(M) \), a compact CW-complex ([K5]).

For a smooth traversally generic vector field \( v \) on a compact \((d+1)\)-manifold \( X \) with boundary, the trajectory space \( \mathcal{T}(v) \) acquires a stratification \( \{\mathcal{T}(v, \omega)\}_\omega \) labeled by the combinatorial patterns of tangency \( \omega \) that belong to an universal poset \( \Omega^\bullet(d) \) (see [K2] and Section 2). It depends only on \( d \). Similarly, for a traversally generic vector field \( v^g \) on a compact \((2n-1)\)-manifold \( SM \), \( SM \)-dimensional space of geodesics \( \mathcal{T}(v^g) \) acquires a stratification \( \{\mathcal{T}(v^g, \omega)\}_\omega \) by the combinatorial patterns of tangency \( \omega \) that belong to the universal poset \( \Omega^\bullet(2n-2) \).

The more numerous the connected components of these stratifications are, the more complex the \( v^g \)-flow is, and the more complex (in relation to the boundary \( \partial M \)) the metric \( g \) on \( M \) is. So our goal here is to find some lower bounds of the numbers such connected components. The key tool, enabling such estimates, has been developed in [K4] for arbitrary traversally generic flows.

The \( \Omega^\bullet(2n-2) \)-stratification of the geodesic space \( \mathcal{T}(v^g) \) generates the stratification

\[
\{SM(v^g, \omega) = \text{def } \Gamma^{-1}(\mathcal{T}(v^g, \omega))\}_{\omega \in \Omega^\bullet(2n-2)}
\]

of \( SM \) and the stratification \( \{\partial(SM)(v^g, \omega) = \text{def } SM(v^g, \omega) \cap \partial(SM)\}_{\omega} \) of its boundary \( \partial(SM) \). These stratifications can be refined by considering the connected components of the sets \( \{\partial(SM)(v^g, \omega), SM^\circ(v^g, \omega)\}_{\omega} \), where \( SM^\circ(v^g, \omega) = \text{def } SM(v^g, \omega) \cap \text{int}(SM) \).

We consider an auxiliary closed manifold, the double \( D(SM) = \text{def } SM \cup_{\partial(SM)} SM \) of \( SM \). The double comes equipped with an involution \( \tau \) so that \( (D(SM))^\tau = \partial(SM) \) and \( D(SM)/\{\tau\} = SM \). We stratify \( D(SM) \) by the connected components of the sets:

\[
\{\partial(SM)(v^g, \omega), SM^\circ(v^g, \omega), \tau(SM^\circ(v^g, \omega))\}_{\omega \in \Omega^\bullet(2n-2)}
\]

All these \( v^g \)-induced stratifications of \( \mathcal{T}(v) \), \( SM \), and \( D(SM) \) are the foci of our investigation.

Let \( SM^\circ_j(g) \) denote the union of strata of codimension \( j \) in \( SM^\circ \). Similarly, let \( D(SM)_{-j}(g) \) denote the union of strata of codimension \( j \) in \( D(SM) \).

Let \( D \) stand for the Poincaré Duality operator (over the coefficient ring \( \mathbb{R} \) of real numbers or over the ring \( \mathbb{Z} \) of integers) on an oriented manifold with boundary. For each tranversally generic metric \( g \) on \( M \), we introduce the localized Poincaré Duality linear operators:

\[
L_j(g; \mathbb{R}) : H_j(D(SM); \mathbb{R}) \xrightarrow{\approx} H^{2n-1-j}(D(SM); \mathbb{R}) \xrightarrow{i_{\text{loc}}} H^{2n-1-j}(D(SM)_{-j}(g); \mathbb{R}),
\]
\[ M_j(g; \mathbb{R}) : H_j(SM; \mathbb{R}) \xrightarrow{\sim_{D}} H^{2n-1-j}(SM, \partial(SM); \mathbb{R}) \]

where \( i_{\text{loc}}^* \) are the natural homomorphisms in the cohomology, induced by the corresponding embeddings of spaces.

The source spaces of these operators come naturally equipped with the Gromov semi-norms \( \| \sim \|_\mathbb{A} \) (Gr). We denote by \( B_j^\mathbb{A}(\sim) \) the unit balls in these semi-norms.

It turns out that the \( g \)-dependent target spaces of the operators \( L_j(g; \mathbb{R}) \) and \( M_j(g; \mathbb{R}) \) also may be equipped with some norms \( \| \sim \|_\mathbb{A}^* \). Their definition depends only on the codimension \( j \) connected components of the \( \Omega_{(2n-2)}^* \)-stratifications of the corresponding spaces \( D(SM) \) and \( SM \), respectively. For each \( j \), the unit balls \( \Omega_j(D(SM), g) \) and \( \Omega_j(SM, g) \) in these norms are compact convex polyhedra, which depend on the geodesic flow, and thus on the metric \( g \). In fact, they are linear projections of some perfect polyhedra in the appropriate vector spaces (these perfect polyhedra are the duals of the hypercubes).

Theorem 3.2, our main result, describes the images, under the localized Poincaré Duality operators \( L_j(g; \mathbb{R}) \) and \( M_j(g; \mathbb{R}) \), of the unitary spheres \( \partial B_j^\mathbb{A}(\sim) \) in Gromov’s semi-norms. Specifically, we prove that these images are contained in the complement to the ball \( \lambda \cdot \Omega_j(D(SM), g) \) of a radius \( \lambda \geq 1 \) and the ball \( \mu \cdot \Omega_j(SM, g) \) of a radius \( \mu \geq 1 \), respectively. Both constants, \( \lambda \) and \( \mu \), are universal for all pairs \( (M, g) \), where \( \dim M = n \geq 3 \) and \( g \in G^1(M) \).

These claims require that the fundamental groups \( \pi_1(M) \) and \( \pi_1(DM) \) are quite big (non-amenable); for the amenable fundamental groups all our results are trivial!

Theorem 3.2 implies that the kernels of the localized Poincaré Duality operators \( L_j(g; \mathbb{R}) \) and \( M_j(g; \mathbb{R}) \) are contained in the spaces \( H_j^{\| \sim \|_\mathbb{A}=0}(SM) \) and \( H_j^{\| \sim \|_\mathbb{A}=0}(SM) \), respectively. This observation leads to Theorem 3.4. The latter claims that the ranks of the reduced homology groups \( H_j^\mathbb{A}(D(SM); \mathbb{R}) \) give a lower bound for the number of connected components of the codimension \( j \) strata \( \{ \partial(SM)(v^g, \omega), SM^o(v^g, \omega), \tau(SM^o(v^g, \omega)) \} \).

Similarly, the ranks of the reduced homology groups \( H_j^\mathbb{A}(SM; \mathbb{R}) \) give a lower bound for the number of connected components of the codimension \( j \) strata \( \{ SM^o(v^g, \omega) \cap \text{int}(SM) \} \).

Both claims may be regarded as vague analogues of the Morse inequalities for the spaces of geodesics.

\(^1\)The unit ball in a semi-norm may not be compact.
We say that a metric $g \in \mathcal{G}^{j}(M)$ is globally $j$-convex if the $v^g$-induced stratification of $T(v^g)$ has no strata of codimension $\geq j$. For example, if $\partial M$ is strictly convex in $g$, then $g$ is globally 3-convex.

By Corollaries 3.1 and 3.2 the non-triviality of the groups $H^j_\gamma(SM; \mathbb{R})$ and $H^j_{\partial \gamma}(SM; \mathbb{R})$ constitutes an obstruction to the existence of a global $j$-convex metric $g \in \mathcal{G}^{j}(M)$.

For $g \in \mathcal{G}^{j}(M)$, we are also investigating the connection of the localized Poincaré Duality operators with the inverse geodesic scattering problem.

The scattering map $C_{v^g} : \partial^1_+(SM) \to \partial^1_-(SM)$, generated by the $v^g$-flow, takes a domain $\partial^1_+(SM)$ in the boundary $\partial(SM)$ to the complementary domain $\partial^1_-(SM) \subset \partial(SM)$. Let us outline the construction of $C_{v^g}$. For any point $x \in \partial M$ and any unit tangent vector $w \in T_x M$ that points inside of $M$ or is tangent to $\partial M$, we consider the geodesic $\gamma \subset M$. Take the next along $\gamma$ point $x' \in \gamma \cap \partial M$ and register the unit tangent to $\gamma$ vector $w' \in T_{x'} M$. By definition, $C_{v^g}(x, w) = (x', w')$.

In fact, $\partial^1_+(SM)$ is diffeomorphic to $\partial^1_-(SM)$, the smooth topological types of both domains being $g$-independent. We stress that, with few exceptions, $C_{v^g}$ is a fundamentally discontinuous map! It turns out that $C_{v^g}$ allows for a reconstruction of the stratified topological type of $SM$ (see Theorem 3.3 from [K7]). This reconstruction is an instance of a more general phenomenon, which we call “Holographic Principle” (see Theorem 3.1 from [K5]). By applying the Holographic Principle to the geodesic flows, we prove in Theorem 3.3 that it is possible to reconstruct the localized Poincaré Duality operators from the scattering map $C_{v^g}$.

2. Basics of Traversally Generic Vector Fields

We start with presenting few basic definitions and facts related to the traversing and traversally generic vector fields, as they appear in [K1] - [K3]. For a model case of vector fields on surfaces, the reader could glance at [K6].

Let $X$ be a compact connected smooth $(n+1)$-dimensional manifold with boundary. A vector field $v$ is called traversing if each $v$-trajectory is either a closed interval with both ends residing in $\partial X$, or a singleton also residing in $\partial X$ (see [K1] for the details). In particular, a traversing field does not vanish in $X$. In fact, $v$ is traversing if and only if $v \neq 0$ and $v$ is of the gradient type ([K1], Corollary 4.1).

We denote by $V_{\text{trav}}(X)$ the space of traversing fields on $X$.

For traversing fields $v$, the trajectory space $\mathcal{T}(v)$ is homology equivalent to $X$ ([K3], Theorem 5.1).

In this paper, we consider an important subclass of traversing fields which we call traversally generic (see formula (2.5) below and Definition 3.2 from [K2]).

For a traversally generic field $v$, the trajectory space $\mathcal{T}(v)$ is stratified by closed subspaces, labeled by the elements $\omega$ of an universal poset $\Omega^*_{\{n\}}$ which depends only on $\dim(X) = n + 1$. The partial order “$\rightarrow$” in $\Omega^*_{\{n\}}$ mimics the bifurcations of real roots of real polynomials of degree $2n + 2$ (see [K3], Section 2, for the definition and properties of $\Omega^*_{\{n\}}$). The elements $\omega \in \Omega^*_{\{n\}}$ correspond to combinatorial patterns that describe the
way in which $v$-trajectories $\gamma \subset X$ intersect the boundary $\partial X$. Each intersection point $a \in \gamma \cap \partial X$ acquires a well-defined multiplicity $m(a)$, a natural number that reflects the order of tangency of $\gamma$ to $\partial X$ at $a$ (see [K1] and Definition 2.1 for the expanded definition of $m(a)$). So $\gamma \cap \partial X$ can be viewed as a divisor $D_\gamma$ on $\gamma$, an ordered set of points in $\gamma$ together with their multiplicities. Then $\omega$ is just the ordered sequence of multiplicities $\{m(a)\}_{a \in \gamma \cap \partial X}$, the order being prescribed by $v$.

The support of the divisor $D_\gamma$ is either a singleton $a$, in which case $m(a) \equiv 0 \mod 2$, or the minimum and maximum points of $\text{sup} \ D_\gamma$ have odd multiplicities, and the rest of the points have even multiplicities.

Let $m(\gamma) = \defsum \sum_{a \in \gamma \cap \partial X} m(a)$ and $m'(\gamma) = \defsum \sum_{a \in \gamma \cap \partial X} (m(a) - 1)$.

Similarly, for $\omega = \defsum (j_1, j_2, \ldots, j_i, \ldots)$, where $j_i \in \mathbb{N}$, we introduce the norm and the reduced norm of $\omega$ by the formulas:

$$|\omega| = \defsum \sum_i j_i \quad \text{and} \quad |\omega'| = \defsum \sum_i (j_i - 1).$$

Let $\partial_j X = \defsum \partial_j X(v)$ denote the locus of points $a \in \partial X$ such that the multiplicity of the $v$-trajectory $\gamma_a$ through $a$ at $a$ is greater than or equal to $j$.

We may embed the compact manifold $X$ into an open manifold $\hat{X}$ of the same dimension, so that $v$ extends smoothly to a non-vanishing vector field $\hat{v}$ in $\hat{X}$. We treat the extension $(\hat{X}, \hat{v})$ as “a germ at $(X, v)$”.

Now, the locus $\partial_j X = \defsum \partial_j X(v)$ has a description in terms of an auxiliary function $z : \hat{X} \to \mathbb{R}$ that satisfies the following three properties:

$$|\omega| = \defsum \sum_i j_i \quad \text{and} \quad |\omega'| = \defsum \sum_i (j_i - 1).$$

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Now, the locus $\partial_j X = \defsum \partial_j X(v)$ has a description in terms of an auxiliary function $z : \hat{X} \to \mathbb{R}$ that satisfies the following three properties:

$$\text{(2.3)}$$

- 0 is a regular value of $z$,
- $z^{-1}(0) = \partial X$, and
- $z^{-1}((-\infty, 0]) = X$.

In terms of $z$, the locus $\partial_j X = \defsum \partial_j X(v)$ is defined by the equations:

$$\{z = 0, \ \mathcal{L}_v z = 0, \ldots, \mathcal{L}_v^{(j-1)} z = 0\},$$

where $\mathcal{L}_v^{(k)}$ stands for the $k$-th iteration of the Lie derivative operator $\mathcal{L}_v$ in the direction of $v$ (see [K2]).

The pure stratum $\partial_j X^0 \subset \partial_j X$ is defined by the additional constraint $\mathcal{L}_v^{(j)} z \neq 0$. The locus $\partial_j X$ is the union of two loci: (1) $\partial_j^+ X$, defined by the constraint $\mathcal{L}_v^{(j)} z \geq 0$, and (2) $\partial_j^- X$, defined by the constraint $\mathcal{L}_v^{(j)} z \leq 0$. The two loci, $\partial_j^+ X$ and $\partial_j^- X$, share a common boundary $\partial_{j+1} X$. 
Definition 2.1. The multiplicity \( m(a) \), where \( a \in \partial X \), is the index \( j \) such that \( a \in \partial_j X^\circ \).

Definition 2.2. The vector field \( v \) on \( X \) is called boundary generic, if for all \( j \) and each point \( a \in \partial_j X^\circ \), there exists a neighborhood \( V_a \subset \hat{X} \) of \( a \) and local coordinates \((u, \vec{x}, \vec{y}) : V_a \to \mathbb{R} \times \mathbb{R}^{j-1} \times \mathbb{R}^{n-j} \) so that \( \partial X \) is given by the polynomial equation

\[
\varphi(u, \vec{x}) \overset{\text{def}}{=} u^j + \sum_{l=0}^{j-2} x_l u^l = 0
\]

of degree \( j = m(a) \) in \( u \), while \( X \) by the polynomial inequality \( \pm \varphi(u, \vec{x}) \leq 0 \). Each \( v \)-trajectory in \( V \) is obtained by freezing the \( \vec{x}, \vec{y} \) coordinates.

The characteristic property of traversally generic fields is that they admit special flow-adjusted coordinate systems, in which the boundary is given by quite special polynomial equations (see formula (2.5)), and the trajectories are parallel to one of the preferred coordinate axis (see [K2], Lemma 3.4). For a traversally generic \( v \) on a \((n+1)\)-dimensional \( X \), the vicinity \( U \subset \hat{X} \) of each \( v \)-trajectory \( \gamma \) of the combinatorial type \( \omega = (j_1, j_2, \ldots, j_i, \ldots) \) has a special coordinate system \((u, \vec{x}, \vec{y}) : U \to \mathbb{R} \times \mathbb{R}^{\mid\omega\mid} \times \mathbb{R}^{n-\mid\omega\mid} \).

In these coordinates, by Lemma 3.4 and formula (3.17) from [K2], the boundary \( \partial X \) is given by the polynomial equation:

\[
\varphi(u, \vec{x}) \overset{\text{def}}{=} \prod_{i=1}^{\mid\omega\mid-\mid\omega'\mid} \left( u - i \right)^{j_i} + \sum_{l=0}^{j_i-2} x_{i,l} (u - i)^l = 0
\]

of an even degree \( \mid\omega\mid \) in \( u \). Here \( i \in \mathbb{Z}_+ \) runs over the distinct roots of \( \varphi(u, \vec{0}) \), and the vector \( \vec{x} \overset{\text{def}}{=} \{x_{i,l}\}_{i,l} \) is sufficiently small. At the same time, \( X \) is given by the polynomial inequality \( \{\varphi(u, \vec{x}) \leq 0\} \). Each \( v \)-trajectory in \( U \) is produced by freezing all the coordinates \( \vec{x}, \vec{y} \), while letting \( u \) to be free.

We denote by \( \mathcal{V}^t(X) \) the space of traversally generic fields on \( X \). In fact, \( \mathcal{V}^t(X) \) is an open and dense (in the \( C^\infty \)-topology) subspace of \( \mathcal{V}_{\text{trav}}(X) \) (see [K2], Theorem 3.5).

We denote by \( X(v, \omega) \) the union of \( v \)-trajectories whose divisors are of a given combinatorial type \( \omega \in \Omega_{(n)} \). Its closure \( \bigcup_{\omega' \preceq \omega} X(v, \omega') \) is denoted by \( X(v, \omega_{\preceq \omega}) \).

Each pure stratum \( \mathcal{T}(v, \omega) \subset \mathcal{T}(v) \) is an open smooth manifold and, as such, has a “conventional” tangent bundle.

Definition 2.3. We say that a traversing field \( v \) on \( X \) is globally \( k \)-convex if \( m'(\gamma) < k \) for any \( v \)-trajectory \( \gamma \). In other words, all strata \( \mathcal{T}(v, \omega) \) of codimension \( \geq k \) are empty.
3. The localized Poincaré Duality for geodesic flows

We are now in position to apply Gromov’s amenable localization to the study of certain class of Riemannian metrics (called “traversally generic”) on smooth manifolds $M$ with boundary. We will focus on the tangency patterns, exhibited by the geodesic curves with respect to the boundary $\partial M$.

Consider the unit spherical fibration $SM \to M$, associated with the tangent bundle $TM \to M$ and a Riemannian metric $g$ on $M$.

**Definition 3.1.** We say that a Riemannian metric $g$ on $M$ is:

- of the gradient type if the geodesic vector field $v^g$ on $SM$ is traversing,
- boundary generic if the geodesic vector field $v^g$ on $SM$ is boundary generic (in the sense of Definition 2.2) with respect to the boundary $\partial(SM)$,
- traversally generic if the geodesic vector field $v^g$ on $SM$ is traversally generic with respect to $\partial(SM)^2$.

Since $v^g$ depends smoothly on $g$ and since the traversally generic fields form an open set in the space of all smooth vector fields on $M$ ([K7], Theorem 2.2), we conclude that the traversally generic metrics $M^\sharp(M)$ form an open set in the space $M(M)$ of all smooth Riemannian metrics on $M$. The question whether, for a given $M$, the space $M^\sharp(M)$ is nonempty remains wide open. We conjecture that $M^\sharp(M)$ is dense in the space of metrics of the gradient type.

For a smooth compact manifold $M$ with boundary, we form its double $DM = \text{def} \ M \cup \partial M$. We denote by $\rho_M : DM \to DM$ a smooth involution such that $DM/\rho_M = M$. Let $D(SM) = \text{def} SM \cup_{\partial M} SM$. Evidently, $\rho_M$ generates an involution $\rho_{SM} : D(SM) \to D(SM)$ such that $D(SM)/\rho_{SM} = SM$.

In the context of geodesic flows, we are going to reintroduce and interpret some constructions from [K4]. For a compact connected smooth Riemannian $n$-manifold $M$ with boundary, any traversally generic metric $g$, via its geodesic flow $v^g$, defines a stratification of the space $SM$ by the pure strata $\{SM(v^g, \omega)\}_{\omega \in \Omega^\bullet(2n-2)}$. Let $SM^\circ(v^g, \omega) = \text{def} SM(v^g, \omega) \cap \text{int}(SM)$. In turn, these strata generate the filtration

\[
\left\{SM_{-(k+1)}^\circ(g) = \text{def} \bigcup_{\omega \in \Omega^\bullet} SM^\circ(v^g, \omega) \right\}_k
\]

of $SM \setminus \partial(SM)$ and the filtration

\[
\left\{SM_{-(k+1)}(g) = \text{def} \left( \bigcup_{\omega \in \Omega^\bullet} SM^\circ(v^g, \omega) \right) \bigcup \left( \bigcup_{\omega \in \Omega^\bullet} \partial(SM) \cap SM(v^g, \omega) \right) \right\}_k
\]

\[\text{In particular, traversally generic metrics are boundary generic and of the gradient type.}\]
of $SM$. The filtration $\{SM_{-(k+1)}(g)\}$ induces the $\rho_{SM}$-equivariant filtration $\{D(SM)_{-(k+1)}(g)\}$ of the double $D(SM) \approx S(DM)$.

For a commutative ring $R$, we introduce the free $R$-modules:

$$C^{2n-1-j}_i(D(SM), g; R) = \text{def} \ H^{2n-1-j}(D(SM)_{-j}(g), D(SM)_{-(j+1)}(g); R)$$

and

$$C^{2n-1-j}_i(SM^\circ, g; R) = \text{def} \ H^{2n-1-j}(SM_{-j}(g), SM_{-(j+1)}(g) \cup (SM_{-j}(g) \cap \partial(SM)); R).$$

So each traversally generic metric $g$ on $M$ gives rise to a differential complex

$$C^*_i(D(SM), g; R) = \text{def} \ \left\{ 0 \to C^0_i(D(SM), g; R) \xrightarrow{\delta_0} C^1_i(D(SM), g; R) \xrightarrow{\delta_1} \ldots \xrightarrow{\delta_{2n-2}} C^{2n-1}_i(D(SM), g; R) \to 0 \right\},$$

(3.3)

where the differentials $\{\delta_j\}$ are the boundary homomorphisms from the long exact cohomology sequences of the triples

$$\{D(SM)_{-(j+1)}(g) \supset D(SM)_{-j}(g) \supset D(SM)_{-(j+1)}(g)\}_j.$$

Similarly, $g$ produces the differential complex

$$C^*_i(SM^\circ, g; R) = \text{def} \ \left\{ 0 \to C^0_i(SM^\circ, g; R) \xrightarrow{\delta_0} C^1_i(SM^\circ, g; R) \xrightarrow{\delta_1} \ldots \xrightarrow{\delta_{2n-2}} C^{2n-1}_i(SM^\circ, g; R) \to 0 \right\},$$

(3.4)

where the differentials $\{\delta_j\}$ are the boundary homomorphisms from the long exact homology sequences of the triples

$$\{SM_{-(j+1)}(g) \cup \partial(SM) \supset SM_{-j}(g) \cup \partial(SM) \supset SM_{-(j+1)}(g) \cup \partial(SM)\}_j.$$

We denote by $B^{2n-1-j}_i(SM^\circ, g; R) \subset C^{2n-1-j}_i(SM^\circ, g; R)$ the image of the differential $\delta^{2n-2}_{2n-2}$ from (3.3). Similarly, let $B^{2n-1-j}_i(D(SM), g; R) \subset C^{2n-1-j}_i(D(SM), g; R)$ stand for the image of the differential $\delta^{2n-2}_{2n-2}$ from (3.3).

**Conjecture 3.1.** Let $g$ be a traversally generic Riemannian metric on a compact connected smooth $n$-manifold $M$ with boundary. Then, for any commutative ring $R$, the metric $g$ generates the differential complexes $C_i^*(SM^\circ, g; R)$ and $C_i^*(D(SM), g; R)$ of free $R$-modules.

The homology groups of these differential complexes depend only on the connected component of the space of traversally generic metrics on $M$ to which $g$ belongs.

**Theorem 3.1.** Let $g$ be a traversally generic Riemannian metric on a compact connected smooth $n$-manifold $M$ with boundary.

The differential complexes $C_i^*(SM^\circ, g; R)$ and $C_i^*(D(SM), g; R)$ can be reconstructed from the geodesic scattering map $C_{\psi^\circ} : \partial^1_+(SM) \to \partial^1_-(SM)$. 
Proof. Let $\mathcal{F}(v^g)$ denote the 1-dimensional oriented foliation on $SM$, generated by the geodesic flow. Any traversally generic geodesic field $v^g$ is boundary generic and of the gradient type. If the geodesic field $v^g$ is boundary generic and of the gradient type, then by Theorem 3.3 from [K3], the scattering map $C_{v^g} : \partial_1^v(SM) \rightarrow \partial_1^v(SM)$ allows for a reconstruction of the pair $(SM, \mathcal{F}(v^g))$, up to a homeomorphism of $SM$ which is the identity on the boundary $\partial(SM)$ and which preserves the stratification $S_{v^g}(SM)$, whose strata are the connected components of the stratification $S_{v^g}(SM) = \{(\partial(SM)(v^g, \omega), SM^g(v^g, \omega))\}_{\omega}$. Therefore, the topological types of the stratifications $S_{v^g}(SM^g)$ and $S_{v^g}(D(SM))$ are determined by $C_{v^g}$. As a result, the differential complexes $C^j_\Omega(SM^g, g; R)$ and $C^j_\Omega(D(SM), g; R)$, whose construction depends only on the stratified topological types of the spaces $SM$ and $D(SM)$, can be reconstructed from the geodesic scattering map $C_{v^g}$. 

Abusing notations, we denote by $C^j_\Omega(D(SM), g; \mathbb{Z})$ the image of $C^j_\Omega(D(SM), g; \mathbb{Z})$ in the vector space $C^j_\Omega(D(SM), g; \mathbb{R})$, viewed as an integral lattice. Similarly, we consider the integral lattice $C^j_\Omega(SM^g, g; \mathbb{Z}) \subset C^j_\Omega(SM^g, g; \mathbb{R})$. The integral lattice $C^j_\Omega(SM, g; \mathbb{Z})$ comes equipped with a basis whose vectors correspond to the connected components of the strata $\{SM(v^g, \omega) \cap \text{int}(SM)\}_{\omega \in \Omega} | \omega' = j$. Similarly, the integral lattice $C^j_\Omega(D(SM), g; \mathbb{Z})$ comes equipped with a basis whose vectors correspond to the connected components of the strata $\{SM(v^g, \omega)\}_{\omega \in \Omega} | \omega' = j$, each stratum being considered twice, together with the connected components of the strata $\{SM(v^g, \omega) \cap \partial(SM)\}_{\omega \in \Omega} | \omega' = j - 1$.

Using these bases, we introduce the $l_1$-norms $\| \sim \|_{l_1}$ in the vector spaces $C^j_\Omega(SM^g, g; \mathbb{R})$ and $C^j_\Omega(D(SM), g; \mathbb{R})$ so that the basic vectors (which belong to the lattices $C^j_\Omega(SM^g, g; \mathbb{Z})$ and $C^j_\Omega(D(SM), g; \mathbb{Z})$) have lengths 1. The unit balls in these norms are the perfect polyhedra, dual to the hypercubes in the corresponding spaces.

The norms $\| \sim \|_{l_1}$ induce norms $\| \sim \|_{l_1}^*$ in the quotient spaces $C^j_\Omega(SM^g, g; \mathbb{R})/B^j_1(SM^g, g; \mathbb{R})$ and $C^j_\Omega(D(SM), g; \mathbb{R})/B^j_1(D(SM), g; \mathbb{R})$, respectively. Recall that the “quotient norm” of a given vector $\vec{V}$ in the quotient space is defined to be the infimum of the norms of all the vectors (in the original space) that represent $\vec{V}$.

We denote by $\diamondsuit^j_\Omega(SM^g, g)$ and $\diamondsuit^j_\Omega(D(SM), g)$ the unit balls in these quotient norms $\{ \| \sim \|_{l_1} = 1 \}$ under the quotient maps.

The manifolds $TM$ and $T(DM)$ are orientable; thus are the manifolds $SM$ and $D(SM)$. So the Poincaré Duality is available for their homology and cohomology with coefficients in $\mathbb{R}$ or $\mathbb{Z}$. As in [K4] (where we dealt with arbitrary traversally generic flows), for each $j$, we consider the localized Poincaré Duality operators over the coefficient rings $R = \mathbb{Z}, \mathbb{R}^3$.

$$L_j(g) : H_j(D(SM)) \xrightarrow{\text{loc}} H^{2n-1-j}(D(SM)) \xrightarrow{\text{loc}} H^{2n-1-j}(D(SM)_{-j}(g))$$

\footnote{We have suppressed in [SM] the dependence of homology and cohomology on the coefficients $R$.}
\[ \approx C_{ij}^{2n-1-j}(D(SM), g) / B_{ij}^{2n-1-j}(D(SM), g), \]

\[ \mathcal{M}_j(g) : H_j(SM) \xrightarrow{\approx} H^{2n-1-j}(SM, \partial(SM)) \xrightarrow{i^*_{\text{loc}}} H^{2n-1-j}(SM_j(g), SM_j(g) \cap \partial(SM)) \]

\[ \approx C_{ij}^{2n-1-j}(SM^\circ, g) / B_{ij}^{2n-1-j}(SM^\circ, g). \]

(3.5)

Here the natural homomorphisms \( i^*_{\text{loc}} \) are induced by the inclusions of the strata

\[ SM^\circ_j(g) \subset SM \text{ and } D(SM)_{-j}(g) \subset D(SM) \]

—thus the term “localized” in the names of the two operators.

The RHS isomorphisms “\( \approx \)” in (3.5) can be justified exactly by the same homological argument as in [K4], page 516; just replace an arbitrary traversally generic vector fields \( v \) on \( X \) with the geodesic traversally generic vector fields \( v^g \) on \( SM \).

\[ \mathcal{L}_j(g; \mathbb{R}) \]

maps the lattice \( H_j(D(SM); \mathbb{Z}) \) to the lattice \( H^{2n-1-j}(D(SM)_{-j}(g); \mathbb{Z}) \). Similarly, \( \mathcal{M}_j(g; \mathbb{R}) \) maps the lattice \( H_j(SM; \mathbb{Z}) \) to the lattice \( H^{2n-1-j}(SM^\circ_j(g); \mathbb{Z}) \).

We denote by \( \hat{B}_j^\Delta(D(SM)) \subset H_j(D(SM); \mathbb{R}) \) and \( \hat{B}_j^\Delta(SM) \subset H_j(SM; \mathbb{R}) \) the set of vectors whose simplicial semi-norms \( | \sim | \Delta \) do not exceed \( 1 \). Let \( \partial \hat{B}_j^\Delta(D(SM)) \) and \( \partial \hat{B}_j^\Delta(SM) \) denote the sets of vectors whose semi-norms are equal \( 1 \) (these spheres may not be compact).

Note that, for any pair of vectors \( V \) and \( W \) such that \( |W|_\Delta = 0 \), we get \( |V + W|_\Delta = |V|_\Delta \). Therefore, \( | \sim | \Delta \) becomes a norm on the quotient \( H^\Delta_*(\sim; \mathbb{R}) \) of the homology space \( H_*(\sim; \mathbb{R}) \) by the subspace \( H^\Delta_*(\sim; \mathbb{R}) \) of vectors whose simplicial semi-norms vanish.

So we may form the compact convex ball \( \hat{B}_j^\Delta(D(SM)) \subset H_j^\Delta(D(SM); \mathbb{R}) \), the image of \( \hat{B}_j^\Delta(D(SM)) \) under the quotient map \( H_j(D(SM); \mathbb{R}) \to H_j^\Delta(D(SM); \mathbb{R}) \); similarly, we may form the compact convex ball \( \hat{B}_j^\Delta(SM) \subset H_j^\Delta(SM; \mathbb{R}) \), the image of \( \hat{B}_j^\Delta(D(SM)) \) under the quotient map \( H_j(SM; \mathbb{R}) \to H_j^\Delta(SM; \mathbb{R}) \).

We will use the localized Poincaré Duality operators \( \mathcal{L}_j(g; \mathbb{R}) \) and \( \mathcal{M}_j(g; \mathbb{R}) \) from (3.5) to project linearly the unit balls \( \hat{B}_j^\Delta(D(SM)) \) and \( \hat{B}_j^\Delta(SM) \) on the \( g \)-dependent “screens”

\[ C_{ij}^{2n-1-j}(D(SM), g; \mathbb{R}) / B_{ij}^{2n-1-j}(D(SM), g; \mathbb{R}), \]

\[ C_{ij}^{2n-1-j}(SM^\circ, g; \mathbb{R}) / B_{ij}^{2n-1-j}(SM^\circ, g; \mathbb{R}) \], respectively. The screens are manufactured by various metrics \( g \in \mathcal{G}^\Delta(M) \).

The next theorem makes few claims about the geometry of these projections.

**Theorem 3.2.** Let \( M \) be a compact connected smooth \( n \)-manifold with boundary, where \( n \geq 3 \).
• Assume that, for each connected component of the boundary $\partial M$, the image of its fundamental group in $\pi_1(DM)$ is amenable. Let $j \in [0,n]$.

Then there is a universal constant $\lambda = \lambda(n, j) \geq 1$ such that, for every $M$ and every traversally generic Riemannian metric $g$ on $M$, the image of the unit sphere $\partial B_j^\Delta(D(SM))$ in the simplicial semi-norm, under the localized Poincaré Duality operator $L_j(g; \mathbb{R})$, is contained in the complement to the ball $\lambda \cdot \Delta_{13}^{2n-1-j}(D(SM), g)$ of radius $\lambda$.

• Assume that, for each connected component of the boundary $\partial M$, the image of its fundamental group in $\pi_1(M)$ is amenable. Let $j \in [0,n-1]$.

Similarly, there is a universal constant $\mu = \mu(n, j) \geq 1$ such that, for every $M$ and every traversally generic Riemannian metric $g$ on $M$, the image of the unit sphere $\partial B_j^\Delta(SM)$ in the simplicial semi-norm, under the localized Poincaré Duality operator $M_j(g; \mathbb{R})$, is contained in the complement to the ball $\mu \cdot \Delta_{13}^{2n-1-j}(SM, g)$ of radius $\mu$.

*Proof.* Consider the $(n-1)$-spherical fibrations $p : SM \to M$ and $q : D(SM) \to DM$. We denote by $\partial_\alpha M$ a typical connected component of $\partial M$.

Using the long homotopy sequences of the fibration $p$ and of its restriction to $\partial_\alpha M$, we notice that if the image of $\pi_1(\partial_\alpha M) \to \pi_1(M)$ is an amenable group, so is the image of $\pi_1(p^{-1}(\partial_\alpha M)) \to \pi_1(SM)$. Similarly, if the image of $\pi_1(\partial_\alpha M) \to \pi_1(DM)$ is an amenable group, so is the image of $\pi_1(q^{-1}(\partial_\alpha M)) \to \pi_1(D(SM))$.

For dimensional reasons, the spectral sequences of the fibrations $p$ and $q$ collapse. Thus $H_*(SM) \approx H_*(M) \oplus H_{*-n+1}(M)$, and $H_*(D(SM)) \approx H_*(DM) \oplus H_{*-n+1}(DM)$.

We claim that the simplicial semi-norms of all elements of the second factor vanish. Let us justify the claim.

Recall that an odd multiple $[f]$ of every integral homology class $[h] \in H_k(M)$ can be realized by a continuous map $f : N \to M$, where $N$ is a closed orientable $k$-manifold [CF]. Similarly, odd multiples $[F]$ of the elements $[H] \in H_{k+n-1}(SM)$ that form the second summand are realizable by cycles of the form $F : K \to SM$, where $K$ is the space of the spherical fibration $\pi : K \to N$, induced by $f$ from the fibration $SM \to M$. For $n \geq 3$, the projection $p$ induces an isomorphism $p_* : \pi_1(SM) \to \pi_1(M)$. Let $\alpha : M \to K(\pi_1(M), 1)$ be the classifying map (that induces an isomorphism of the fundamental groups). Then $\alpha \circ p : SM \to K(\pi_1(M), 1)$ is also the classifying map. Note that $(\alpha \circ \pi)_*([F]) = 0$ in $H_{k+n-1}(K(\pi_1(M), 1))$ since $K$ is the boundary of the disk bundle $\Pi : Cyl(\pi) \to N$, associated with $\pi$, and $\alpha \circ \pi$ extends to $\alpha \circ \Pi$. Since continuous maps of spaces that induce isomorphisms of the fundamental groups are isometries in the semi-norms $\| \sim \|_\Delta$, we conclude that $\| [F] \|_\Delta = 0$ and thus $\| [H] \|_\Delta = 0$.

Therefore we get the canonical isomorphism $p_* : H^\Delta_*(SM) \approx H^\Delta_*(M)$ of the two quotient spaces, viewed as normed spaces.

Similar arguments, applied to the spherical fibration $q : D(SM) \to DM$, lead to the isometry $q_* : H^\Delta_*(D(SM)) \approx H^\Delta_*(DM)$. 

Thus we proved that the hypotheses of Theorem 3.2 imply the validity of the hypotheses of Theorem 4.2 from [K4]. The latter theorem applies to any traversally generic vector field \( v \) on a connected compact smooth manifold \( X \) with boundary, in particular, it applies to any traversally generic geodesic vector field \( v^\phi \) on \( SM \). Theorem 4.2 from [K4] is stated in a slightly different and less geometrical form than Theorem 3.2 here: it claims the validity of Theorem 4.2 from [K4]. The latter theorem applies to any traversally generic vector field and of the gradient type) seems to be important.

We denote the universal constant \( \mu \) by the connected components of \( \{ E_\omega(\partial_u, \hat{\omega}) \} \hat{\omega} \in \omega \leq \hat{\omega} \), labeled by the elements \( \hat{\omega} \geq \omega \). They form the sub-poset \( \omega \leq \subset \Omega_\{2n-2\} \).

Let \( \partial E_\omega(\partial_u, \hat{\omega}) = \{ E_\omega(\partial_u, \hat{\omega}) \} \partial E_\omega \) and \( E_\omega(\partial_u, \hat{\omega}) = \{ E_\omega(\partial_u, \hat{\omega}) \} \Omega_\{2n-2\} \). We denote \( S^\{2n-2\} \) the stratification of the space \( E_\omega \) by the connected components of these strata.

The universal constant \( \mu \) is the maximum of the \( S^\{2n-2\} \)-stratified \( \| h \|_{\Delta} \) relative to (their boundaries) simplicial norms of small \( j \)-disks, each one being normal to a particular connected component of the \( 2n+1-j \)-dimensional strata \( \{ E_\omega(\partial_u, \hat{\omega}) \} \omega \leq \hat{\omega} \), the maximum being taken over all pairs \( \omega \leq \hat{\omega} \), where \( \omega \in \Omega_\{2n-2\} \).

The universal constant \( \lambda \) is the maximum of the \( S^\{2n-2\} \)-stratified \( \| h \|_{\Delta} \) relative to (their boundaries) simplicial norms of small \( j \)-disks, each one being normal to a particular connected component of the \( 2n+1-j \)-dimensional strata \( \{ E_\omega(\partial_u, \hat{\omega}) \} \partial E_\omega(\partial_u, \hat{\omega}) \} \) in the double \( D(E_\omega) \). The maximum is taken over all pairs \( \omega \leq \hat{\omega} \), where \( \omega \in \Omega_\{2n-2\} \).

**Example 3.1.** Let \((N, g)\) be a closed connected smooth \( n \)-manifold, \( n \geq 3 \). Let \( U \subset N \) be a codimension zero submanifold with a smooth boundary. Assume that \( U \) is contained in a topological \( n \)-ball. Put \( M = \{ N \setminus \text{int}(U) \} \) and let us assume that \( g|_M \) is traversally generic. Then the number of connected codimension \( n \) components of the \( S^\{2n-2\} \)-stratification of \( D(SM) \) exceeds \( \lambda^{-1} \cdot \| DM \|_{\Delta} \), where \( \lambda \geq 1 \) depends only on \( n \).
particular, when \(\| [DM] \|_\Delta \neq 0\) there exists a \((n - 1)\)-dimensional family of geodesics \(\gamma\) in \(M\) such that their reduced multiplicity \(m'(\gamma) = n\).

When \(U\) is a smooth \(n\)-ball, then \(DM = N \# N\), where "\#" stands for the connected sum. For \(n \geq 3\), we get \(\| [DM] \|_\Delta = 2 \cdot \| [N] \|_\Delta \) \((\text{Gr})\). If \(N\) is a closed hyperbolic manifold, then \(\| [N] \|_\Delta\) can be expressed as \(\text{vol}_{\text{hyp}}(N)/\text{vol}_{\text{hyp}}(\Delta^n)\), the normalized hyperbolic volume of \(N\). In such case, the number of codimension \(n\) connected components of the \(S_{v^q}(D(SM))\)-stratification of \(D(SM)\) exceeds \(2(\lambda \cdot \text{vol}_{\text{hyp}}(\Delta^n))^{-1} \cdot \text{vol}_{\text{hyp}}(N) > 0\). So, for a given family of closed hyperbolic \(n\)-manifolds \(\{N_k\}_{k \to \infty}\), the number of codimension \(n\) connected components of the \(S_{v^q}(D(SM))\)-stratification of \(D(SM_k)\) grows at least as fast as \(\{\text{vol}_{\text{hyp}}(N_k)\}_{k \to \infty}\).

Note that \(\| [D(SM)] \|_\Delta = 0\), since \(D(SM) \to DM\) is a spherical fibration. Therefore, Theorem 3.3.2 does not tell anything about the number of codimension \(2n - 1\) connected components of the \(S_{v^q}(D(SM))\)-stratification of \(D(SM)\). These arise from the finitely many geodesics \(\gamma\) that have the maximal reduced multiplicity \(m'(\gamma) = 2n - 2\) to the boundary \(\partial M\).

\[\text{Theorem 3.3.}\text{ For any metric } g \in G^l(M), \text{ the localized Poincaré Duality operators } \mathcal{L}_j(g; \mathbb{R}) \text{ and } \mathcal{M}_j(g; \mathbb{R}) \text{ in (3.2), as well as the simplicial semi-norms } \| \sim \|_\Delta \text{ in their source spaces, and the norms } \| \sim \|_{\text{Gr}} \text{ in their target spaces, can be reconstructed from the geodesic scattering map } C_{v^q} : \partial^k_\Delta(SM) \to \partial^k_\Delta(SM).\]

\[\text{Proof. By Theorem 3.3 from } [K7], \text{ the scattering map } C_{v^q} \text{ allows for a reconstruction of the pair } (SM, \mathcal{F}(v^q)), \text{ up to a stratification-preserving homeomorphism of } SM \text{ which is the identity on } \partial(SM). \text{ Therefore, with the help of the involution } \tau : D(SM) \to D(SM), \text{ we get also a reconstruction of the stratified topological type of the double } D(SM).\]

The localized Poincaré Duality operators \(\mathcal{L}_j(g; \mathbb{R})\) and \(\mathcal{M}_j(g; \mathbb{R})\) also depend only on the \(S_{v^q}(D(SM))\)-stratified and \(S_{v^q}(SM^\circ)\)-stratified topological types of \(D(SM)\) and \(SM^\circ\), respectively. Thus \(\mathcal{L}_j(g; \mathbb{R})\) and \(\mathcal{M}_j(g; \mathbb{R})\) can be recovered from \(C_{v^q}\).

The semi-norm \(\| \sim \|_\Delta\) of a given homology class is an invariant of the topological type of the underlying space. Therefore (in accordance with Theorem 3.5 from [K7]), the semi-norms \(\| \sim \|_\Delta\) on the source spaces of \(\mathcal{L}_j(g; \mathbb{R})\) and \(\mathcal{M}_j(g; \mathbb{R})\) can be recovered from \(C_{v^q}\). The norm \(\| \sim \|_{\text{Gr}}\) on the target spaces is defined also solely in terms of the stratified topological types of \(D(SM)\) and \(SM^\circ\), and therefore, by Theorem 3.3 from [K7], depends on the scattering map \(C_{v^q}\) only.

\[\text{Theorem 3.4. Let } M \text{ be a compact connected smooth } n\text{-manifold with boundary, where } n \geq 3.\]

- Assume that, for each connected component of the boundary \(\partial M\), the image of its fundamental group in \(\pi_1(M)\) is amenable. Let \(j \in [0, n - 1]\).

Then, for any traversally generic Riemannian metric \(g\) on \(M\), the number of \((2n - 1 - j)\)-dimensional connected components in the stratification \(\{SM^\circ(v^q, \omega)\}_\omega\) is greater than or equal to \(\text{rank}(H^j_\Delta(M))\).
• Assume that, for each connected component of the boundary $\partial M$, the image of its fundamental group in $\pi_1(DM)$ is amenable. Let $j \in [0,n]$.

Then, for any traversally generic Riemannian metric $g$ on $M$, the number of $(2n-1-j)$-dimensional connected components in the stratification $\{D(SM)(v^g, \omega)\}_\omega$ is greater than or equal to $\text{rank}(H_j^\Delta(DM))$.

**Proof.** By Theorem 3.2, the kernel $\ker(L_j(g))$ is contained in the subspace $H_j^{\{\|\sim\|\geq 0\}}(D(SM); \mathbb{R}) \subset H_j(D(SM); \mathbb{R})$.

Therefore, employing (3.5), we get

$$\text{rank}(H_j^\Delta(D(SM); \mathbb{R})) \leq \text{rank}(H_j(D(SM); \mathbb{R})/\ker(L_j(g))) = \text{rank}(\text{im}(L_j(g)))$$

$$\leq \text{rank}(C_{ij}^{2n-1-j}(D(SM), g)/B_{ij}^{2n-1-j}(D(SM), g)) \leq \text{rank}(C_{ij}^{2n-1-j}(D(SM), g)),$$

the number of codimension $j$ connected components in the stratification $\{D(SM)(v^g, \omega)\}_\omega$.

Similarly,

$$\text{rank}(H_j^\Delta(SM; \mathbb{R})) \leq \text{rank}(C_{ij}^{2n-1-j}(SM, g))$$

the number of codimension $j$ connected components in the stratification $\{SM^\circ(v^g, \omega)\}_\omega$. 

\[\square\]

The next corollary spells out the claims of Theorem 3.4 by applying the formula on the bottom of page 537 in [K4] to the settings of the theorem.

**Corollary 3.1.** Let $M$ be a compact connected smooth $n$-manifold with boundary, and $g$ a traversally generic metric on $M$. Let $n \geq 3$.

• Assume that, for each connected component of the boundary $\partial M$, the image of its fundamental group in $\pi_1(M)$ is amenable.

Then, for each $j \in [0,n-1]$, the space of geodesics $T(v^g)$ has the property

$$\sum_{\omega \in \Omega} |\omega'| \cdot \#(\pi_0(T(v^g, \omega))) \geq \text{rank}(H_j^\Delta(M)),$$

where $T(v^g, \omega)$ stands for the family of geodesics $\gamma \subset M$ whose intersections $\gamma \cap \partial M$ generate the combinatorial tangency pattern $\omega$, $\sup(\omega) := |\omega| - |\omega'|$, and $\pi_0(\sim)$ denotes the set of connected components of the appropriate space.

• Assume that, for each connected component of the boundary $\partial M$, the image of its fundamental group in $\pi_1(DM)$ is amenable.

Then, for each $j \in [0,n]$, the space of geodesics $T(v^g)$ has the property

$$\sum_{\omega \in \Omega} \sup(\omega) \cdot \#(\pi_0(T(v^g, \omega))) +$$

$$+ 2 \cdot \sum_{\omega \in \Omega} (\sup(\omega) - 1) \cdot \#(\pi_0(T(v^g, \omega))) \geq \text{rank}(H_j^\Delta(DM)). \quad \Diamond$$
Example 3.2. Consider a collection $\{\Sigma_k\}_{k \in [1,q]}$ of closed surfaces of genera $g_k \geq 2$. Let $N'_k = \Sigma'_k \times S^2$. We denote by $N'$ the connected sum of all $N'_k$'s. Let $M'$ be the compact 4-dimensional manifold, obtained from $N'$ by removing a number of smooth 4-balls $D^4$ and solid “tori” of the form $T^2 \times D^2$ and $S^1 \times D^3$, residing in $N'_k$. We assume that these domains do not intersect the surfaces $\{\Sigma'_k\}$. Finally, let $M$ be any smooth compact 4-dimensional manifold which is homotopy equivalent to $M'$ and such that $\partial M = \partial M'$. Let $H : M \to M'$ denote this homotopy equivalence. Let $\Sigma_k$ be the $H$-preimage of $\Sigma'_k$. We may assume that $H$ is transversal to $\bigsqcup_k \Sigma'_k \subset M'$.

Then $\text{rank}(H^2_\Delta(SM)) \geq q$ and $\text{rank}(H^2_\Delta(DM)) \geq 2q$.

By Theorem 3.4, for any traversally generic metric $g$ on $M$, the number of 5-dimensional connected components in the stratification $\{SM^\circ(v^g, \omega)\}_\omega$ of $SM^\circ$ is $q$ at least. These are connected components of the strata, indexed by $\omega$'s such that $|\omega'| = 2$. Such $\omega$'s belong to the list $\{(2211), (13), (3, 1)\}$.

Let us rephrase these conclusions in terms of $M$. For any traversally generic metric $g$ on $M$, there exists at least $q$ four-dimensional families of geodesics $\gamma$ in $M$, such that either $\gamma$ is quadratically tangent to $\partial M$ at a pair of distinct points, or $\gamma$ has a single tangency to $\partial M$ of order 3. Each family is continuously parametrized by an open smooth 4-manifold. Different families do not share same geodesics.

Next, let us interpret the claims of Theorem 3.2 in this setting. Take, for example, the 2-cycle $h = \sum_{k=1}^q [\Sigma_k]$. Its simplicial norm $||h||_\Delta = \sum_{k=1}^q (2g_k - 2)$.

For an universal constant $\mu \geq 1$ and any traversally generic metric $g$ on $M$, the number of 5-dimensional connected components in the stratification $\{SM^\circ(v^g, \omega)\}_\omega$ of $SM$, where $\omega = (1221), (13), (3, 1)$, is greater than or equal to $\mu^{-1} \sum_{k=1}^q (2g_k - 2)$. Although estimating $\mu$ from above may be a bit challenging, at least we know the rate of growth of the number of 5-dimensional connected components in the $S_{v^g}(SM^\circ)$-stratification, as $k \to \infty$ or as individual genus $g_k \to \infty$.

Revisiting Definition 2.3, the formulas from Corollary 3.1 have an instant implication.

Corollary 3.2. Under the hypotheses of Theorem 3.4, the non-triviality of the groups $H^\Delta_2(M)$ and/or $H^\Delta_2(DM)$ represents an obstruction to the existence of a globally $j$-convex and traversally generic metric on $M$.

The assumption that a metric $g$ on a given manifold $M$ is traversally generic, perhaps, could be relaxed. To extend all the results of this paper to boundary generic and traversing geodesic flows, we need to consider only such geodesic flows that match a finite list of $j$-priori fixed semi-local models (in the spirit of (2.5)) of the vicinity of every $v^g$-trajectory $\gamma$. The models should be determined by the combinatorics of tangency (like $\omega$) of $\gamma$ to $\partial(SM)$.

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6 The fundamental groups of the boundaries of these domains are amenable.
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