Singularity Analysis, Constraint Wrenches and Optimal Design of Parallel Manipulators

Nguyen Minh Thanh¹, Le Hoai Quoc² and Victor Glazunov³

¹Department of Automation, Hochiminh City University of Transport, ¹²Department of Science and Technology, People’s Committee of Hochiminh City, ³Mechanical Engineering Research Institute, Russian Academy of Sciences, ¹,²Vietnam ³Russia

1. Introduction

In recent years, numerous researchers have investigated parallel manipulators and many studies have been done on the kinematics or dynamics analysis. Parallel manipulators has been only mentioned in several books, as in (Merlet, 2006; Ceccarelli, 2004; Kong, & Gosselin, 2007; Glazunov, et al., 1991). Reference (Gosselin, & Angeles, 1990) has established singularity criteria based on Jacobian matrices when describing the various types of singularity. Then, in (Glazunov, et al. 1990) proposed other singularity criteria for consideration of these problems the screw theory based on the approach of the screw calculus, as in (Dimentberg, 1965). Those criteria are determined by the constraints imposed by the kinematic chains, as in (Angeles, 2004; Kraynev, & Glazunov, 1991), taking into account some problems the Plücker coordinates of constraint wrenches can be applied in (Glazunov, 2006; Glazunov, et al. 1999, 2007, 2009; Thanh, et al. 2009, 2010a). Dynamical decoupling allows increasing the accuracy for the parallel manipulators presented as in (Glazunov, & Kraynev, 2006; Glazunov, & Thanh, 2008). It is necessary to develop optimal structure have combined (Thanh, et al. 2008), as well as algorithms and multi-criteria optimization (Statnikov, 1999; Thanh, et al. 2010b) obtaining the Pareto set. It is very important to taking into account possible singularity configurations, to find out how they influence the characteristics of constraints restricting working space (Bonev, et al. 2003; Huang, 2004; Arakelian, et al. 2007).

The trend towards highly rapid manipulators due to the demand for greater working volume, dexterity, and stiffness has motivated research and development of new types of parallel manipulator (Merlet, 1991). This paper is focused the constraints and criteria existing in known parallel manipulators in form of a parallel manipulator with linear actuators located on the base.

2. Kinematic of parallel manipulator

In this section, let us consider a 6-DOF parallel manipulator with actuators situated on the base. The mechanical architecture of the considered robot is illustrated in Fig. 1.
The parallel manipulator as seen in Fig. 1 is composed of a mobile platform connected to a fixed base via six kinematic sub-chains (legs) comprising of one prismatic, one universal and one spherical pair (PUS pairs). Parameters of design of the platform and the base form an irregular hexagon positioned in the $(x-y)$ plane. $A_i, B_i$ $(i=1,\ldots,6)$ are coordinates of the points of the mobile platform (the output link) and of the base respectively. The points $A_1A_3A_5$ and $A_2A_4A_6$ make form equilateral triangles, the angle $\psi_p$ determines their location and $R_p$ is the radius of the circumscribed circle (Fig. 2, a). Similarly, the angle $\psi_b$ and the radius $R_b$ determines the location of the equilateral triangles $B_1B_3B_5$ and $B_2B_4B_6$ located on the base. Let the distance between the centers of the universal and spherical pairs $A_i$ and $C_i$ of the $i$-th leg be $l_i$. In addition, the generalized coordinates, which are equal to the distance between the points $B_i$ and $C_i$ are designated $\theta_i$. The radius-vectors of the points $A_i$ and $C_i$ are $r_i(x_{A_i}, y_{A_i}, z_{A_i})$ and $s_i(x_{C_i}, y_{C_i}, z_{C_i})$ respectively $(i=1,\ldots,6)$. We could note that the coordinates of the points $B_i$ and $C_i$ are $x_{C_i}=x_{B_i}, y_{C_i}=y_{B_i}, z_{C_i}=-\theta_i$.

![Parallel manipulator with linear actuators located on the base](image1)

![Parametrical and geometrical design of parallel manipulator](image2)
With this approach, the linear actuators can be firmly fixed on the base to reduce high acceleration movements because the power is not used to move heavy actuators but lightweight links. However, the obstacle is a smaller working space in comparison with a Stewart platform, due to the movement of the linear actuators. Moreover, forces acting on the actuator have a perpendicular component, whereas forces exerted upon Stewart actuators have a longitudinal component.

Let us consider an inverse kinematic problem of position of parallel manipulators, which has characteristic relation between the numbers of chains. The manipulator with six kinematic chains offers convenience in optimization of working space in terms of decreased rigidity and load-bearing capacity.

Likewise, the generalized coordinates of the \(i\)-th segment (the length) which are equal to the distance between the points \(A_i\) and \(B_i\), can be expressed as:

\[
f_i = \sqrt{(x_{Ai} - x_{Bi})^2 + (y_{Ai} - y_{Bi})^2 + (z_{Ai} - z_{Bi})^2}, \quad i = 1, \ldots, 6
\]

By geometrical method, the distance between the points \(C_i\) and \(C'_i\) (Fig. 2, b):

\[
g_i = \sqrt{f_i^2 - (z_{Ai})^2}, \quad i = 1, \ldots, 6
\]

when the length of the link \(l_i\) is known, the distance between the points \(A_i\) and \(C'_i\) (Fig. 2, b):

\[
h_i = \sqrt{l_i^2 - g_i^2}
\]

We could obtain the generalized coordinate \(\theta_i\) (Fig. 2, c) as follows:

\[
\theta_i = z_{Ai} - h_i
\]

It is the solution of the inverse kinematic problem. The inverse kinematics for parallel manipulator can be formulated to determine the required actuator heights for a given pose of the mobile platform with respect to the base. The pose consists of both position and orientation in the Cartesian system. Actuators are considered to act linearly in the vertical direction, parallel to the \(z\)-axis, in order to simplify the mathematics, although that needs not be the case.

### 3. Multi-criteria optimization

Influence of singularities on parameters of the working space of the parallel manipulator is a significant factor worth investigating. In these singularity configurations, the system is out of control and that greatly affects its functionality. It is necessary to determine the extent of the lack of control to see how that affects the parameters of the working space. These singularity configurations also affect the optimization results.

The constraint wrenches of zero pitch acting to the output link from the legs are located along the unit screws: \(E_i = e_i + \chi e_{oi}\) \((i=1, \ldots, 6)\) where \(e_i\) is the unit vector directed along the axis of the line \(C_iA_i\) of the corresponding leg, \(\chi\) is the Clifford factor, \(\chi^2 = 0\) (for a vector, \(e_i e_{oi} = 0\)). \(E_i\) consists of the unit vector \(e_i\) and its moment \(e_{oi} = i \times e_i\) corresponding with \(e_{oi} = s_{Cxi}e_{Cxi} - s_{Cyi}e_{Cyi} - s_{Czi}e_{Czi}\) and can be expressed by Plücker coordinates \(E_i = (x_i, y_i, z_i, x_{oi}, y_{oi}, z_{oi})\). These coordinates make form the \(6 \times 6\) matrix \((E)\):
Optimization of parameters of the parallel manipulator with linear actuators located on the base is considered. Let us take in account three criteria: working volume, dexterity and stiffness of parallel manipulator. The first criterion \( N_p \) is the quantity of the reachable points of the centre of the mobile platform. The second criterion \( N_c \) is the average quantity of orientations of the mobile platform in each reachable point. The third criterion \( D_e \) is the average module of the determinant \( \left| \det(E) \right| \) in each configuration. Determinant \( \left| \det(E) \right| \) constructed from coordinate axes of the drive kinematic couples is used as a third criterion of optimization. Since the value of this criterion is related to one of the important characteristics of the manipulator - its stiffness or load capacity. If determinant are more qualifiers, then the manipulator away from the singularity configuration and the stiffness of the above.

Let us consider optimization of the parameters of the manipulator for different values of the criterion of proximity to singular configurations, as well as the influence of this criterion in the optimization results. We set up four coefficients \( H_1, H_2, H_3 \) and \( H_4 \) expressed four parameters of optimization. The coefficient \( H_1 \) characterizes the length \( l_i \) of the links \( A_i C_i \) (\( i=1,\ldots,6 \)) (in Fig. 2, b). The coefficient \( H_2 \) characterizes the angle \( \varphi_p \) (Fig. 2, a) determining the location of the triangles \( A_1 A_3 A_5 \) and \( A_2 A_4 A_6 \) of the mobile platform. The coefficient \( H_3 \) characterizes the angle \( \varphi_p \) determining the location of the triangles \( B_1 B_3 B_5 \) and \( B_2 B_4 B_6 \) on the base. Moreover, the coefficient \( H_4 \) characterizes the relation between the radius \( R_p \) and the radius \( R_o \) of the circumscribe circles of the platform and of the base respectively.

The algorithm of determination of the Pareto-optimal solutions can be presented as follows:

**Step 1.** Establish the limits of the parameters of optimization.

\[
H_{1_{min}} \leq H_1 \leq H_{1_{max}} \quad H_{2_{min}} \leq H_2 \leq H_{2_{max}} \quad H_{3_{min}} \leq H_3 \leq H_{3_{max}} \quad H_{4_{min}} \leq H_4 \leq H_{4_{max}}.
\]

The number of steps of scanning in the space of parameters is \( n_p \). The limits of the scanned Cartesian coordinates of the centre of the moving platform and the limits of the scanned orientation angles of this platform in interval are \( x_{min} \leq x \leq x_{max} \), \( y_{min} \leq y \leq y_{max} \), \( z_{min} \leq z \leq z_{max} \), \( \alpha_{min} \leq \alpha \leq \alpha_{max} \), \( \beta_{min} \leq \beta \leq \beta_{max} \), \( \gamma_{min} \leq \gamma \leq \gamma_{max} \). As well as the number \( n_i \) of steps of scanning in the space of these coordinates and the limitation of changing of the generalized coordinates \( \theta_{i_{min}} \leq \theta_i \leq \theta_{i_{max}} \) (\( i=1,\ldots,6 \)). The limit of the determinant \( \left| \det(E) \right| \) \( \geq \epsilon \). At this step assume \( i=0 \), by this the parameters are \( H_{10} = H_{1_{min}} \), \( H_{20} = H_{2_{min}} \), \( H_{30} = H_{3_{min}} \), \( H_{40} = H_{4_{min}} \), \( N_p = N_c = D_e = 0 \).

**Step 2.** Determine the values of the criteria for all the values of the parameters.

1. Determine the parameters \( H_{1j}, \ldots, H_{4j} \) assume \( j=0 \), by this \( x_0 = x_{min} \), \( y_0 = y_{min} \), \( z_0 = z_{min} \), \( \alpha_0 = \alpha_{min} \), \( \beta_0 = \beta_{min} \), \( \gamma_0 = \gamma_{min} \).
2. Determine \( \theta_i \) (\( i=1,\ldots,6 \)) and \( \left| \det(E) \right| \); if all the \( \theta_{i_{min}} \leq \theta_i \leq \theta_{i_{max}} \) (\( i=1,\ldots,6 \)) and \( \left| \det(E) \right| \geq \epsilon \) then \( N_c = N_c + 1 \), \( D_e = D_e + \left| \det(E) \right| \); if \( x_j \neq x_{j-1} \) or \( y_j \neq y_{j-1} \) or \( z_j \neq z_{j-1} \) then \( N_p = N_p + 1 \).

\[
(E) = \begin{bmatrix}
    x_1 & y_1 & z_1 & x_1^0 & y_1^0 & z_1^0 \\
    x_2 & y_2 & z_2 & x_2^0 & y_2^0 & z_2^0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    x_n & y_n & z_n & x_n^0 & y_n^0 & z_n^0 \\
\end{bmatrix}
\]

\[
\epsilon > 0, D_e = D_{e+1} \]

\[
\left| \det(E) \right| \geq \epsilon
\]
2.3. \( j = j+1 \), if \( j \leq n_c \) then go back to 2.2.

2.4. Determine the criteria \( N_p = N_p, D_e = D_e / N_c, N_c = N_c / N_p \); assume \( N_p = N_c = D_e = 0 \).

2.5. \( i = i+1 \), if \( i \leq n_p \) then go back to 2.1.

**Step 3.** Determine the Pareto-optimal solutions (matrix \((Par)\)).

3.1. Assume \( i = 1 \) (by this \( N_p = N_{p1}, D_e = D_{e1}, N_c = N_{c1} \)), \( k = 0 \).

3.2. Determine \( N_p, D_e, N_c \); assume \( j = 1 \), the criteria of optimal solution \( K1 = 1, K2 = 0 \).

3.3. Determine \( N_p, D_e, N_c \); if \( N_p > N_{p1} \) or \( D_e > D_{e1} \) or \( N_c > N_{c1} \) then \( K2 = 1 \); if \( N_p = N_{p1} \) and \( D_e = D_{e1} \) and \( N_c = N_{c1} \) then \( K2 = 1 \).

3.4. If \( K2 \neq 1 \) then \( K1 = 0 \), \( K2 = 0 \), \( j = j+1 \); if \( j \leq n_p \) then go back to 3.3.

3.5. If \( K1 = 1 \) then \( k = k + 1 \), \( Par_{1k} = H_{11}, \ldots, Par_{7k} = N_{c1} \), \( i = i+1 \); if \( i \leq n_p \) then go back to 3.2.

Singularity of the manipulator is determined by closeness to zero of determinant of matrix \((E)\) of Plücker coordinates of unit wrenches. Let us fix certain value \( \epsilon > 0 \) as a criterion of singularity (the manipulator is in singular position if \( \left| \det(E) \right| \leq \epsilon \)). If \( \epsilon = 0 \) then the construction of the working space of the manipulator shows that the same results we can get without singularity constraint. Giving various values of the criterion of the singularity, we can get interval of the determinant of matrix \((E)\).

Further, analysis influences of the criterion of singularity \( \epsilon, \left| \det(E) \right| \leq \epsilon \) on the value of the working volume. With the value of the criterion of singularity is equal to \( \epsilon = 0.01 \) there exist 81 available solutions, but only 8 of them are Pareto-optimal. By the condition of the criterion of singularity is equal to \( \epsilon = 0.01 \) and the condition \( \epsilon = 0 \), there Pareto-optimal set consists of 6 and 29 solutions correspondingly. Therefore, the value of \( \epsilon \) influences on the results of optimization.

Value of the criterion that determines the proximity to singular configurations is equal to zero, we can assume that the constraints associated with the singularity in general, are not imposed in the analysis of each specific configuration. However, the criterion for determining the load capacity occurs. As a result, the number of Pareto-optimal variants varies very much. Here, 29 variants satisfy the conditions of Pareto set.

Limiting possible module of a determinant of matrix \((E)\) to singularity configurations changes the Plücker coordinates of the wrenches transmitted on the output link. Methodology for analyzing the singularities on optimization appearing in the parallel manipulator and their impact in the working space is proposed. The practical significance from the fact is the results obtained in this work increase the effectiveness of design automation.

### 4. Twist inside singularity

The approach based on matrix \((E)\) consisting of the Plücker coordinates of the constraint wrenches allows determining the twists of the platform inside singularity (Glazunov, 2006). Let us consider the increases of the Plücker coordinates of the unit screws \( E_i \) after an infinitesimal displacement \( S = (d\varphi, dr) = (dr_x, dr_y, dr_z, d\varphi_x, d\varphi_y, d\varphi_z)^T \) of the platform corresponding to displacement \( dr_i = (dx_{Ai}, dy_{Ai}, dz_{Ai})^T \) of the point \( A_i \) of the manipulator presented on the Fig. 1.

\[
\begin{align*}
    dx_{Ai} &= dr_x + d\varphi_y z_{Ai} - d\varphi_z y_{Ai} , \\
    dy_{Ai} &= dr_y + d\varphi_x z_{Ai} - d\varphi_z x_{Ai} , \\
    dz_{Ai} &= dr_z + d\varphi_x y_{Ai} - d\varphi_y x_{Ai} .
\end{align*}
\]
The generalized coordinate after mentioned infinitesimal displacement \((dx_i, dy_i, dz_i)\) is:

\[
\theta_i + d\theta_i = z_{Ai} + dz_{Ai} - \sqrt{l_i^2 - (x_{Ai} + dx_{Ai} - x_{Ci})^2 - (y_{Ai} + dy_{Ai} - y_{Ci})^2}
\]

(7)

After transformations the increase of the generalized coordinate is:

\[
d\theta_i = \left[\frac{(x_{Ai} - x_{Ci})dx_{Ai} + (y_{Ai} - y_{Ci})dy_{Ai} + (z_{Ai} - z_{Ci})dz_{Ai}}{(z_{Ai} - z_{Ci})}\right]
\]

(8)

The unit screw \(E_i\) can be rewritten as \(E_i + dE_i\) or as \(e_i + de_i\) and \(e_{oi} + de_{oi}\).

Using (6), (7), and (8) the coordinates of the \(de_i\) and \(de_{oi}\) can be expressed as:

\[
\begin{align*}
\frac{\partial x_i}{\partial \varphi_x} &= 0, \quad \frac{\partial x_i}{\partial \varphi_y} = \frac{z_{Ai}}{l_i}, \quad \frac{\partial x_i}{\partial \varphi_z} = -\frac{y_{Ai}}{l_i}, \quad \frac{\partial x_i}{\partial \varphi_r} = \frac{1}{l_i}, \quad \frac{\partial x_i}{\partial \varphi_y} = 0, \quad \frac{\partial x_i}{\partial \varphi_z} = 0, \\
\frac{\partial x_i^o}{\partial \varphi_x} &= \frac{(x_{Ai} - x_{Ci})(2y_{Ci} - y_{Ai})z_{Ai} - x_{Ai}(y_{Ai} - y_{Ci})}{z_{Ai} - z_{Ci}} \\
\frac{\partial x_i^o}{\partial \varphi_y} &= \frac{l_i}{(z_{Ai} - z_{Ci})}, \\
\frac{\partial x_i^o}{\partial \varphi_z} &= \frac{\left[(y_{Ai} - y_{Ci})x_{Ai} - (x_{Ai} - x_{Ci})y_{Ai}\right](2y_{Ci} - y_{Ai}) - x_{Ai}z_{Ci}}{(z_{Ai} - z_{Ci})}, \\
\frac{\partial x_i^o}{\partial \varphi_r} &= \frac{(x_{Ai} - x_{Ci})}{(z_{Ai} - z_{Ci})}, \\
\frac{\partial x_i^o}{\partial \varphi_r} &= \frac{(y_{Ai} - y_{Ci})(2y_{Ci} - y_{Ai})}{z_{Ci}}, \quad i = 1, \ldots, 6
\end{align*}
\]

Other partial derivatives also can be obtained from Eqs. (6), (7), and (8).

By means of the properties of linear decomposition of determinants \(d[\det(E)]\) can be obtained as the sum of 36 determinants (Glazunov, 2006). From this, \(d[\det(E)]\) can be presented as:
Using (10) the criterion of the singularity locus can be presented as $d[\det(E)]=0$. This condition imposes only one constraint. Therefore, there exist five twists of motions of the platform inside singularity.

For example, let us obtain five inside singularity of manipulator. Set up the coordinates of the vectors be $r_i$ are $r_1(-1, 0, 4)$, $r_2(-0.5, 1, 4)$, $r_3(0.5, 1, 4)$, $r_4(1, 0, 4)$, $r_5(0.5, -1, 4)$, $r_6(-0.5, -1, 4)$; $s_i$ are $s_1(-1.5, 0, 0.866)$, $s_2(-1, 1.5, 0.707)$, $s_3(1, 1.5, 0.707)$, $s_4(1.5, 0, 0.866)$, $s_5(1, -1.5, 0.707)$, $s_6(-1, -1.5, 0.707)$. From here, we can see the generalized coordinates as $(0.136, 0.305, 0.305, 0.136, 0.305, 0.305)$. Matrix $(E)$ is determined as:

$$
\begin{pmatrix}
0.158 & 0 & 0.988 & 0 & 1.618 & 0 \\
0.148 & -0.148 & 0.978 & 1.572 & 1.083 & -0.074 \\
-0.148 & -0.148 & 0.978 & 1.572 & -1.083 & 0.074 \\
-0.158 & 0 & 0.988 & 0 & -1.618 & 0 \\
-0.148 & 0.148 & 0.978 & -1.572 & -1.083 & -0.074 \\
0.148 & 0.148 & 0.978 & -1.572 & 1.083 & 0.074
\end{pmatrix}
$$

The determinant consisting of the Plücker coordinates of the unit screws is $\det(E) = 0$. Their partial derivatives are:

$$
\frac{\partial \det(E)}{\partial r_x} = -0.02, \quad \frac{\partial \det(E)}{\partial r_y} = 0, \quad \frac{\partial \det(E)}{\partial r_z} = 0,
$$

$$
\frac{\partial \det(E)}{\partial \phi_x} = 0, \quad \frac{\partial \det(E)}{\partial \phi_y} = 0.002, \quad \frac{\partial \det(E)}{\partial \phi_z} = 0
$$

Using the approach presented above, we find five independent twists inside singularity: $s_1(1, 0, 0, 0, 0, 0)$, $s_2(0, 0, 1, 0, 0, 0)$, $s_3(0, 0, 0, 0, 1, 0)$, $s_4(0, 0, 0, 0, 0, 1)$, $s_5(0, 4.433, 1, 0, 0, 0)$. The twist-gradient is calculated to be $\mathbf{s}^*(0.02, 0, 0, 0, 0.002, 0)$. This twist-gradient is practically important as it offers the highest speed.

### 5. Dynamical decoupling

In this section, let we consider the reduction of the dynamical coupling of the motors of the parallel manipulator with linear actuators located on the base. The basic idea is to represent the kinetic energy as the quadratic polynomial including only the squares of the generalized velocities (Glazunov, & Kraynev, 2006). The kinetic energy can be expressed by means of the matrix $(E)$.

Let $m$ be the mass and $J_x, J_y, J_z$ be the inertia moments of the platform. Assuming that the mass of the platform is much more than the masses of the legs and using the Eqs. (6)-(8), the kinetic energy $T$ can be expressed as follows (Dimentberg, 1965; Kraynev, & Glazunov, 1991):
\[ T = \frac{m}{2} \left[ \sum_{i=1}^{6} p_i^o \dot{\theta}_{Gii} \right]^2 + \left( \sum_{i=1}^{6} q_i^o \dot{\theta}_{Gii} \right)^2 + \left( \sum_{i=1}^{6} r_i^o \dot{\theta}_{Gii} \right)^2 \]  

\[ + \frac{J_x}{2} \left( \sum_{i=1}^{6} p_i \dot{\theta}_{Gii} \right)^2 + \frac{J_y}{2} \left( \sum_{i=1}^{6} q_i \dot{\theta}_{Gii} \right)^2 + \frac{J_z}{2} \left( \sum_{i=1}^{6} r_i \dot{\theta}_{Gii} \right)^2 \]  

where \( \dot{\theta}_i \) are the generalized velocities, \( p_i, q_i, r_i \) are the components of the matrix \( (E)^{-1} \), \( G_{ii} \) are the components of the diagonal matrix \( (G) \) (\( i=1,\ldots,6 \)).

The Lagrange equation of motion for a parallel manipulator can be written as:

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}_i} \right) - \frac{\partial T}{\partial \theta_i} = Q_i \]  

where \( Q_i \) are the generalized forces (\( i=1,\ldots,6 \)).

The dynamical coupling can be determined using the Eq. (11). The expression of each generalized force comprises all other generalized velocities and accelerations. In order to reduce the dynamical coupling we represent the kinetic energy as follows:

\[ T = \frac{m}{2} \left[ \sum_{i=1}^{6} \left( p_i^o \right)^2 \dot{\theta}_{Gii} + 2 \sum_{i=1, j=1, i \neq j}^{6} p_i^o p_j^o \dot{\theta}_{Gii} \dot{\theta}_{Gjj} \right] + \]  

\[ + \sum_{i=1}^{6} \left( r_i^o \dot{\theta}_{Gii} \right)^2 + 2 \sum_{i=1, j=1, i \neq j}^{6} r_i^o r_j^o \dot{\theta}_{Gii} \dot{\theta}_{Gjj} \right] + \frac{J_x}{2} \left( \sum_{i=1}^{6} \left( p_i \dot{\theta}_{Gii} \right)^2 + 2 \sum_{i=1, j=1, i \neq j}^{6} p_i p_j \dot{\theta}_{Gii} \dot{\theta}_{Gjj} \right] + \]  

\[ + \sum_{i=1}^{6} \left( r_i \dot{\theta}_{Gii} \right)^2 + 2 \sum_{i=1, j=1, i \neq j}^{6} r_i r_j \dot{\theta}_{Gii} \dot{\theta}_{Gjj} \]  

According to the Eq. (13) dynamical decoupling can be satisfied if the columns of the following matrix \( (D) \) are orthogonal:

\[ (D) = \begin{pmatrix} 
  p_1^o G_{11} \sqrt{m} & \cdots & p_6^o G_{66} \sqrt{m} \\
  q_1^o G_{11} \sqrt{m} & \cdots & q_6^o G_{66} \sqrt{m} \\
  r_1^o G_{11} \sqrt{m} & \cdots & r_6^o G_{66} \sqrt{m} \\
  J_x & \cdots & J_x \\
  J_y & \cdots & J_y \\
  J_z & \cdots & J_z 
\end{pmatrix} \]
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\[
(D) = \begin{pmatrix}
\sqrt{m} & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{m} & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{m} & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{EJ_x} & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{EJ_y} & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{EJ_z}
\end{pmatrix} \quad \quad (E)^{-1} \begin{pmatrix} G \end{pmatrix}
\]

\[
(D) = (M)(E)^{-1} \begin{pmatrix} G \end{pmatrix}
\]

Let the axes of the links \(A_iC_i\) be parallel to the axes of the links \(B_iC_i\) and the matrix \((G)\) is unit matrix. Then in order to satisfy the condition of orthogonal columns of the matrix \((D)\) the rows of the inverse matrix \((D)^{-1} = (E)(M)^{-1}\) are to be orthogonal. From this, the following matrix \((E)\) is proposed:

\[
(E) = \begin{pmatrix}
0 & 0 & 1 & 0 & -\sqrt{EJ_y/m} & 0 \\
0 & 0 & -1 & 0 & -\sqrt{EJ_y/m} & 0 \\
1 & 0 & 0 & 0 & 0 & -\sqrt{EJ_z/m} \\
-1 & 0 & 0 & 0 & 0 & -\sqrt{EJ_z/m} \\
0 & 1 & 0 & -\sqrt{EJ_x/m} & 0 & 0 \\
0 & -1 & 0 & -\sqrt{EJ_x/m} & 0 & 0
\end{pmatrix}
\]

The determinant of the matrix \((E)\) (15) can be written as:

\[
\det(E) = 8\sqrt{EJ_x EJ_y EJ_z}/m^3
\]

The matrix (15) corresponds to the Fig. 3. Here the center of the mass of the platform coincides with the center of the coordinate system \(xyz\) and the axes of the links of the legs

![Fig. 3. Parallel manipulator with dynamical decoupling](www.intechopen.com)
are parallel to the main central inertia axes of the platform. The proposed approach can be applicable for manipulators characterized by small displacements and high speeds. Moreover, this architecture causes partial kinematic decoupling because if the generalized coordinates corresponding to the opposite legs are equivalent then the moving platform keeps constant orientation.

6. Pressure angles

The parallel manipulators have singularity configurations in which there is an uncontrolled mobility because some of the wrenches acting on the output link are linearly dependent. The local criterion of singular configurations is the singular matrix of the screw coordinates of the wrenches, such as:

$$\det(E) = \varepsilon^*$$

(17)

where $\varepsilon^*$ is the preassigned minimal determinant value. The pressure angle of the linear dependent sub-chain is equal to $\pi/2$, as a reciprocal twist to five-member group of screws has a perpendicular moment at about any points of the axis. All stalled actuators but one the manipulator has $DOF=1$ and its output link can move along some twist $\Omega = \omega + \varepsilon^* \omega^0$ ($\varepsilon^2 = 0$) reciprocal to five-member group of the wrenches corresponding to stalled actuators. We can find this twist from the reciprocity condition:

$$\text{mom}(\Omega, R_i) = 0, \; i = 1, ..., 5$$

(18)

In general the six-member group of the unit wrenches of zero parameter $R_i(\tau_r, \tau_c)$ ($i=1, ..., 6$) is acting on the output link of the such manipulators, determinant composed of the screw coordinates of these wrenches as given in (5).

The velocity of any point $A_i$ ($i=1, ..., 6$) of the mobile platform can be found as a twist moment relative to this point:

$$V_{A_i} = \omega^0 + \omega \times r_{A_i}, \; i = 1, ..., 6$$

(19)

where $\omega^0$ is radius-vectors of the points $A_i$.

The pressure angle $\alpha_i$ for the stalled actuator $i$-th of the parallel manipulators (Fig.1) can be determined as:

$$\alpha_i = \arccos \left( \frac{V_{A_i} \cdot F_i}{|V_{A_i}| |F_i|} \right), \; i = 1, ..., 6$$

(20)

where $F_i$ is the force vector on the actuator axis. For normal functions of the manipulator it is necessary that working space be limited by positions:

$$\alpha_i \leq \alpha_{KP}, \; i = 1, ..., 6$$

(21)

where $\alpha_{KP}$ is maximum pressure angle is defined by friction coefficient.

The manipulator control system must be provided by algorithm testing the nearness to singular configurations based on the analysis of singular matrix (5) or on the pressure angle determination.
7. Manipulator for external conditions

In Fig. 4, (a, b) the six-DOFs parallel mechanisms and their sub-chain has a parallel connection of links and actuators are shown in (Glazunov, et al. 1999) which were invented by (Kraynev, & Glazunov, 1991). Such mechanisms may be utilized to manipulate the corrosive medium at all actuators that are located out of the working space. Existence of several sub-chains and many closed loops determine the essential complication of the mathematical description of these mechanisms. Screw calculus using screw groups is universal and effective for parallel mechanisms analysis. Here, 1 describes the fixed base, 2 describes the output link and 3 describes the actuators. Addition, $A_i$ expresses the spherical joint center situated on the fixed base; $B_j$ expresses the center of the spherical joints combined with translational; $C_j$ expresses the output link spherical joint centers; $l_i$, $d_j$ expresses the generalized coordinates and $s_j$, $f_j$ expresses the link lengths ($i=1,...,6; j=1,...,3$).

![Fig. 4. The six-DOFs parallel mechanisms](https://www.intechopen.com)
displacement. With the mechanisms as in (Fig. 3, a) the components of the wrench $R_i$ can be find as:

$$ r_i = \frac{1}{p_i} \left\{ a_i + \left[ \frac{s_j}{f_j} - \left( \frac{1}{f_j} + \frac{1}{s_j} \right) \left( a_i, b_i \right) b_j \right] d_j \right\}, \quad r_i^0 = \rho_{C_j} \times r_i \quad (22) $$

where $p_i$ - vector defining the wrench axis, $a_i$ - vector from point $A_i$ to point $C_j$, $b_i$ - vector from point $D_j$ to point $C_j$, and $\rho_{C_j}$ - radius vector point $C_j$.

Besides, with the mechanism as in (Fig. 4, b) the wrench axis ($j=1, \ldots, 3$) coincides with actuator axis. The wrench of the $i$-th stalled actuator ($i=1, \ldots, 3$) is given as

$$ r_i = \frac{1}{l_i} \left\{ a_i - \left( \frac{a_i, b_j}{d_j^2} \right) b_j \right\}, \quad r_i^0 = \rho_{C_j} \times r_i \quad (23) $$

The present approach may be applied for different types of mechanisms such as sub-chains with varied actuator connection using spherical pairs.

8. Conclusion

Thus in this paper various criteria of design and singularity analysis of parallel manipulators are presented. The constraint wrenches imposed to the platform by kinematic chains is proposed to rely on the screw theory by determinants of matrix consisting of the Plücker coordinates of the unit screws. Criteria for design and singularity analysis of parallel manipulators with linear actuators located on the base are presented. The kinematic criterion of singularity corresponds to linear dependence of wrenches supporting the mobile platform; the static criterion corresponds to the limitation of pressure angles. The dynamical decoupling allows increasing the accuracy, parametrical optimization allows designing the mechanisms with optimal working volume, dexterity and stiffness, determination of the twists inside singularity allows finding the differential conditions of singular loci. Furthermore, the use of screw groups in order to determination of the singular zones of the multi-DOFs parallel mechanisms that make form of continuous areas and manipulators for external conditions are expressed.

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