THE HOMOLOGY GROUPS OF CERTAIN MODULI SPACES OF
PLANE SHEAVES

MARIO MAICAN

Abstract. Using the Białynicki-Birula method, we determine the additive
structure of the integral homology groups of the moduli spaces of semi-stable
sheaves on the projective plane having rank and Chern classes $(5,1,4)$,
$(7,2,6)$, respectively, $(0,5,19)$. We compute the Hodge numbers of these moduli spaces.

1. Introduction

Let $r > 0$, $c_1$, $c_2$ be integers. Let $M(r,c_1,c_2)$ be the moduli space of Gieseker
semi-stable sheaves on $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$ of rank $r$ and Chern classes $c_1$, $c_2$. Let
$$
\Delta(r,c_1,c_2) = \frac{1}{r} \left( c_2 - \left( 1 - \frac{1}{r} \right) \frac{c_1^2}{2} \right) \quad \text{and} \quad \mu = \frac{c_1}{r}
$$
be the discriminant and slope of a sheaf giving a point in $M(r,c_1,c_2)$. The moduli
spaces $M(r,c_1,c_2)$ of dimension zero consist of a point, the isomorphism class of
a semi-exceptional bundle, (see [16, Section 16.1]). According to [8], there exists
a unique function $\delta: \mathbb{Q} \rightarrow \mathbb{Q}$ such that for all $r$, $c_1$, $c_2$, $M(r,c_1,c_2)$ has positive
dimension if and only if $\Delta(r,c_1,c_2) \geq \delta(\mu)$. (The function $\delta$ is positive and periodic
of period 1, see [16, Section 16.4].) In [8], $M(r,c_1,c_2)$ is said to have height zero if
$\Delta = \delta(\mu)$. It was proved in [8] that the moduli spaces of height zero are isomorphic
to moduli spaces of semi-stable Kronecker modules.

Let $q$, $m$, $n$ be positive integers. The group $\text{GL}(m,\mathbb{C}) \times \text{GL}(n,\mathbb{C})$ acts by
conjugation on the vector space $\text{Hom}(\mathbb{C}^m \otimes \mathbb{C}^q, \mathbb{C}^n)$, whose elements are called
Kronecker modules. The subset of semi-stable Kronecker modules admits a good
quotient, denoted $N(q,m,n)$, which, according to [8], is an irreducible projective
variety of dimension $qm - m^2 - n^2 + 1$, if non-empty. It is smooth at points given
by stable Kronecker modules so, in particular, $N(q,m,n)$ is smooth if $m$ and $n$
are coprime. The main result of [8] states that if $M(r,c_1,c_2)$ has height zero, then
there are an exceptional bundle $E$ and integers $m$, $n$ such that we have a canonical
isomorphism
$$
M(r,c_1,c_2) \simeq N(3 \text{rank}(E), m, n).
$$
Let $x_E$ denote the smallest real solution to the equation $x^2 - 3 \text{rank}(E)x + 1 = 0$. Let $\mu(E)$ denote the slope of $E$. Then
$$
\mu(E) - \frac{x_E}{\text{rank}(E)} < \mu < \mu(E) + \frac{x_E}{\text{rank}(E)}.
$$

2010 Mathematics Subject Classification. 14D20, 14-04.
Key words and phrases. Moduli of plane sheaves; Białynicki-Birula decomposition; Torus ac-
tions; Hodge numbers.
Composing with the isomorphism $N(q,m,n) \simeq N(q,n,qn-m)$ of $[8]$, we obtain a new moduli space of height zero, denoted $\Lambda^+ M(r,c_1,c_2)$, whose slope is smaller than $\mu$ if $\mu < \mu(E)$, respectively, larger than $\mu$ if $\mu > \mu(E)$. Iterating this process we obtain an infinite sequence of isomorphic moduli spaces. We say that $M$ is initial if it is not of the form $\Lambda^+ M(r,c_1',c_2')$ for any integers $c_1'$, $c_2'$. The classification of the moduli spaces of height zero and dimension up to 10 was carried out in $[8]$. We will see in Section 3.1 that there are no moduli spaces of height zero and dimension 11.

The first goal of this paper is to determine the additive structure of the homology groups of the moduli spaces of height zero and dimension up to 11. In Section 3.1 we will show that they are isomorphic to $N(3,4,3)$, which is a smooth projective variety. We will use the method of Białynicki-Birula $[1]$, $[2]$, which consists of analysing the fixed-point locus for the action of a torus on a smooth projective variety. We refer to $[6]$, Section 2] for a short introduction to the Białynicki-Birula theory. We fix a vector space $V$ over $\mathbb{C}$ of dimension 3 and we identify $\mathbb{P}^2$ with $\mathbb{P}(V)$. We fix a basis $\{X,Y,Z\}$ of $V$. We consider the action of $(\mathbb{C}^*)^3$ on $\mathbb{P}^2$ given by

$$t(x_0 : x_1 : x_2) = (t_0^{-1}x_0 : t_1^{-1}x_1 : t_2^{-1}x_2),$$

where $t = (t_0,t_1,t_2)$. The induced action on the symmetric algebra of $V^*$ is given by the formula

$$t X^i Y^j Z^k = t_1^i t_2^j t_3^k X^i Y^j Z^k.$$

In particular, we get an action of $(\mathbb{C}^*)^3$ on $\text{Hom}(\mathbb{C}^4, \mathbb{C}^3 \otimes V^*)$ by multiplication on $V^*$. This descends to an action on $N(3,4,3)$. Consider the torus

$$T = (\mathbb{C}^*)^3/\{(c, c, c) \mid c \in \mathbb{C}^*\}.$$

Note that the action of $(\mathbb{C}^*)^3$ on $\mathbb{P}^2$ and, also, on $N(3,4,3)$ factors through an action of $T$.

**Theorem 1.** The initial moduli spaces of height zero and dimension twelve are $M(5,-1,4)$, $M(7,-2,6)$, $M(5,1,4)$, $M(7,2,6)$, as well as the moduli spaces obtained from these by twisting with $\mathcal{O}(m)$, $m \in \mathbb{Z}$. These moduli spaces are isomorphic to $N(3,4,3)$. The $T$-fixed locus of $N(3,4,3)$ consists of 62 isolated points and 3 projective lines. Moreover, the integral homology groups of $N(3,4,3)$ have no torsion and its Poincaré polynomial is

$$P(x) = x^{24} + x^{22} + 3x^{20} + 5x^{18} + 8x^{16} + 10x^{14} + 12x^{12} + 10x^{10} + 8x^8 + 5x^6 + 3x^4 + x^2 + 1.$$

The Euler characteristic of $N(3,4,3)$ is 68 and its Hodge numbers satisfy the relation $h^{p,q} = 0$ if $p \neq q$.

Drézet $[9]$ and Ellingsrud and Strømme $[12]$ have given algorithms for computing the homology of $N(q,m,n)$ based on different methods. Moreover, according to $[9]$, the cohomology ring of $N(q,m,n)$ is generated by the Chern classes of certain universal vector bundles, if $m$ and $n$ are coprime. This implies the above statement about the Hodge numbers of $N(3,4,3)$.

Let $r > 0$ and $\chi$ be integers. Let $M_{p^2}(r,\chi)$ be the moduli space of Gieseker semi-stable sheaves on $\mathbb{P}^2$ having Hilbert polynomial $P(m) = rm + \chi$ (this is the same as $M(0, r, r+3)/2 - \chi$). The aim of the second part of this paper is to compute the Hodge numbers of $M_{p^2}(5,1)$. According to $[15]$, this is a smooth projective variety of dimension 26.
The study of the moduli spaces $\mathbb{M}_{2,2}(1, 1)$ is partly motivated by Gromov-Witten Theory. Let $X$ be a polarised Calabi-Yau threefold and fix $\beta \in H_2(X, \mathbb{Z})$. Let $\mathbb{M}_X(\beta)$ be the moduli space of semi-stable sheaves $\mathcal{F}$ on $X$ having Euler characteristic 1 and whose support has dimension 1 and class $\beta$. Consider the genus-zero Gromov-Witten invariant $N_\beta(X)$ of $X$ and the Donaldson-Thomas invariant $n_\beta(X) = \deg[M_X(\beta)]^{vir}$. Katz [13] conjectured the relation

$$N_\beta(X) = \sum_{k|\beta} n_{\beta/k}(X) k^{-3},$$

for which he found evidence by looking at contractible curves. Li and Wu [17] have proved the conjecture in some particular cases. Assume now that $X$ is the local $\mathbb{P}^2$, that is, the total space of $\mathcal{O}_{\mathbb{P}^2}(2)$. Then, as noted in [14], $\mathbb{M}_X(r) = \mathbb{M}_{\mathbb{P}^2}(r, 1)$, hence, up to sign, $n_r(X)$ is equal to the Euler characteristic of $\mathbb{M}_{\mathbb{P}^2}(1, 1)$. We refer to [6] for a brief survey on the present state of research into these moduli spaces.

The Poincaré polynomial of $\mathbb{M}_{2,2}(5, 1)$ has already been computed in [22] by means of a cellular decomposition and in [5] by the wall-crossing method. We will, instead, apply the Bialynicki-Birula method to the action on $\mathbb{M}_{2,2}(5, 1)$ induced by the action of $\mathbb{T}$ on $\mathbb{P}^2$. Concretely, let $\mu_t : \mathbb{P}^2 \to \mathbb{P}^2$ denote the map of multiplication by $t \in \mathbb{T}$ and let $[\mathcal{F}]$ denote the point in $\mathbb{M}_{2,2}(5, 1)$ determined by a sheaf $\mathcal{F}$ on $\mathbb{P}^2$.

The action of $\mathbb{T}$ on the moduli space is given by

$$t \cdot [\mathcal{F}] = [\mu_t^* \mathcal{F}],$$

To determine the torus fixed locus we will use the classification of semi-stable sheaves $\mathcal{F}$ on $\mathbb{P}^2$ with Hilbert polynomial $P(m) = 5m + 1$ provided at [19]. The same technique was used in [20] to study the homology of $\mathbb{M}_{2,2}(5, 3)$ and in [6] to study $\mathbb{M}_{2,2}(4, 1)$.

**Theorem 2.** The $T$-fixed locus of $\mathbb{M}_{2,2}(5, 1)$ consists of 1407 isolated points, 132 projective lines and 6 irreducible components of dimension 2 that are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The integral homology groups of $\mathbb{M}_{2,2}(5, 1)$ have no torsion and its Poincaré polynomial is

$$P(x) = x^{32} + 2x^{50} + 6x^{48} + 13x^{46} + 26x^{44} + 45x^{42} + 68x^{40} + 87x^{38} + 100x^{36} + 107x^{34} + 111x^{32} + 112x^{30} + 133x^{28} + 113x^{26} + 113x^{24} + 112x^{22} + 111x^{20} + 107x^{18} + 100x^{16} + 87x^{14} + 68x^{12} + 45x^{10} + 26x^8 + 13x^6 + 6x^4 + 2x^2 + 1.$$

The Euler characteristic of $\mathbb{M}_{2,2}(5, 1)$ is $1695$ and its Hodge numbers satisfy the relation $h^{p,q} = 0$ if $p \neq q$.

The proof of Theorem 2 relies to a large extent on the proof of Theorem 1. Indeed, the moduli spaces $\mathbb{M}_{2,2}(5, 1)$ and $N(3, 4, 3)$ are closely related. According to [19] Proposition 3.2.1], there is an open $T$-invariant subset $M_0 \subset \mathbb{M}_{2,2}(5, 1)$ that is isomorphic to an open subset inside a certain fibre bundle with base $N(3, 4, 3)$ and fibre $\mathbb{P}^1$. The projection to the base $M_0 \to N(3, 4, 3)$ is $T$-equivariant, surjective, and has surjective differential at every point. Determining $N(3, 4, 3)$ is, therefore, indispensable for determining $M_0^T$. Moreover, as $T$-modules, the tangent space at a fixed point of $N(3, 4, 3)$ is a direct summand of the tangent space at a fixed point of $M_0$ lying over it.

The paper is organised as follows. In Sections 2 and 4 we will determine the torus fixed points in $N(3, 4, 3)$, respectively, $\mathbb{M}_{2,2}(5, 1)$. In Sections 3 and 5 we will
describe the torus action on the tangent spaces at the fixed points. We will omit
the proofs of most propositions from Sections 4 and 5 because they are analogous
to results in [6] and [20].

In closing, we mention that the Betti numbers of $M_{\mathbb{P}^2}(5,1)$ are the same as the
Betti numbers of $M_{\mathbb{P}^2}(5,3)$, as computed in [22] and [20]. This raises the question
whether these two moduli spaces are isomorphic.

Acknowledgements

The author was supported by Consiliul Național al Cercetării Științifice (Romania),
grant PN II–RU 169/2010 PD–219. The author has benefitted from discussions with
Jinwon Choi. The referee pointed out several omissions in the Introduction, nu-
meros improvements to the presentation, especially in Section 2, and a significant
error in Section 3.1.

2. The torus fixed locus of $N(3,4,3)$

Consider the vector space

$W = \text{Hom}(4\mathcal{O}_{\mathbb{P}^2}(-2), 3\mathcal{O}_{\mathbb{P}^2}(-1)) = \text{Hom}(\mathbb{C}^4 \otimes V, \mathbb{C}^3)$.

Its elements are represented by $3 \times 4$-matrices $\varphi$ with entries in $V^\ast$. The reductive
group

$G = (\text{GL}(4, \mathbb{C}) \times \text{GL}(3, \mathbb{C})) / \mathbb{C}^\ast$

acts on $W$ by the formula $(g, h)\varphi = h\varphi g^{-1}$. Here $\mathbb{C}^\ast$ is embedded as the subgroup
of homotheties. According to King’s Criterion of Semi-stability [14], the set $W^\text{ss}$ of
semi-stable elements consists of those matrices that are not in the orbit of a matrix
having a zero-column, or a zero $1 \times 3$-submatrix, or a zero $2 \times 2$-submatrix. We
have a geometric quotient $W^\text{ss}/G$, which is the Kronecker moduli space $N(3,4,3)$.

The point $[\varphi]$ in the moduli space determined by $\varphi$ is $T$-fixed if and only if for each
t $t \in T$ there is $(g, h) \in G$ such that $t\varphi = (g, h)\varphi$. Let $\zeta_i$, $1 \leq i \leq 4$, denote the
maximal minor of $\varphi$ obtained by deleting the column $i$. Let $W_i \subset W^\text{ss}$, $i = 0, 1, 2$,
denote the subset given by the condition

$\text{deg}(\gcd(\zeta_1, \zeta_2, \zeta_3, \zeta_4)) = i$.

Its image $N_i$ in $N(3,4,3)$ is torus-invariant. Moreover, $N_0$ is an open subset, $N_1$ is a
locally closed subset and $N_2$ is a closed subset of $N(3,4,3)$. There are no semi-stable
matrices $\varphi$ whose maximal minors have a common factor of degree 3. Indeed, if $\varphi$
were such a matrix, then we might assume, after we perform elementary column
operations, that $\zeta_1 = 0$, $\zeta_2 = 0$, $\zeta_3 = 0$, $\zeta_4 \neq 0$. But

$\varphi \begin{bmatrix} \zeta_1 & -\zeta_2 & \zeta_3 & -\zeta_4 \end{bmatrix}^T = 0$,

showing that the last column of $\varphi$ is zero, which is contrary to semi-stability. Also,
the case when $\zeta_i$ are all zero is not feasible. Indeed, assume that all maximal minors
of $\varphi$ were zero. Let $\varphi_1$ denote the matrix obtained from $\varphi$ by deleting column $i$.
By hypothesis, $\varphi_1$ is not equivalent to a matrix having a zero row, a zero column,
or a zero $2 \times 2$-submatrix. In other words, $\varphi_1$ is semi-stable as a Kronecker module.

From the description of $N(3,3,3)$ found in [15], we know that

$\varphi_1 \sim \begin{bmatrix} Y & X & 0 \\
Z & 0 & X \\
0 & Z & -Y \end{bmatrix}$. 
Thus, there is a row vector $l_i$ of length 3 whose entries form a basis of $V^*$, such that $l_i \varphi_1 = 0$. Since $\varphi_1$ and $\varphi_2$ have two columns in common, it is easy to see that $l_i = l_j$ for all $1 \leq i < j \leq 4$. It follows that $l_1 \varphi = 0$, hence $\varphi$ has linearly dependent columns, contrary to semi-stability. The above discussion shows that

$$N_0 \cup N_1 \cup N_2 = N(3,4,3).$$

**Notations.**

\begin{align*}
N &= N(3,4,3); \\
S_3 &= \text{the group of permutations } \sigma \text{ of the variables } (X,Y,Z); \\
A_2 &= \text{the subgroup of } S_3 \text{ of even permutations}; \\
\varphi_\sigma &= \text{the matrix obtained from } \varphi \text{ after performing the permutation } \sigma; \\
U_{\varphi} &= \text{span}\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\} \subset S^3 V^*; \\
type(\varphi) &= (\text{rank} \varphi \mod (Y,Z), \text{rank} \varphi \mod (X,Z), \text{rank} \varphi \mod (X,Y)); \\
p_0 &= [1:0:0]; \\
p_1 &= [0:1:0]; \\
p_2 &= [0:0:1]; \\
p_{ij} &= \text{the double point supported on } p_i \text{ and contained in the line } p_ip_j; \\
q_i &= \text{the triple point supported on } p_i, \text{ that is not contained in a line}; \\
q_{ij} &= \text{the triple point supported on } p_i \text{ and contained in the line } p_ip_j.
\end{align*}

2.1. **Fixed points in $N_0$.** Consider the Hilbert scheme of zero-dimensional subschemes of $\mathbb{P}^2$ of length 6. Let $\text{Hilb}_{\mathbb{P}^2}^0(6)$ be the open subset of schemes that are not contained in a conic curve. According to [10, Propositions 4.5 and 4.6], there is a $T$-equivariant isomorphism

$$N_0 \simeq \text{Hilb}_{\mathbb{P}^2}^0(6), \quad [\varphi] \mapsto Z,$$

where the ideal sheaf of $Z$ is the cokernel of $\varphi^*: 3\mathcal{O}(-4) \to 4\mathcal{O}(-3)$. Equivalently, $Z$ is the zero-set of the ideal generated by the maximal minors $\zeta_1$, $\zeta_2$, $\zeta_3$, $\zeta_4$ of $\varphi$. Thus, up to equivalence, $\varphi$ is uniquely determined by its maximal minors. The classification of the $T$-invariant subschemes $Z \in \text{Hilb}_{\mathbb{P}^2}^0(6)$ is well-known. Note that $Z$ is supported on a subset of $\{p_0, p_{11}, p_{22}\}$, so it has one of the following types: $(2,2,2), (1,2,3), (1,1,4), (1,5), (2,4), (3,3), 6$.

2.1.1. **Type $(2,2,2)$.** There are two $T$-invariant subschemes $Z$ of type $(2,2,2)$, namely $\{p_{11}, p_{12}, p_{20}\}$ and $\{p_{02}, p_{21}, p_{10}\}$. The ideal of the former is

$$(Y^2, Z) \cap (Z^2, X) \cap (X^2, Y) = (XYZ, YZ^2, X^2Z, XY^2),$$

so it corresponds to the matrix

$$\alpha = \begin{bmatrix}
Z & X & 0 & 0 \\
X & 0 & Y & 0 \\
Y & 0 & 0 & Z
\end{bmatrix}$$

in $N_0$. The matrix corresponding to the other scheme is $\alpha_\sigma$, where $\sigma$ is the transposition $(X,Y)$. 
2.1.2. Type (1, 2, 3). Note that $Z$ cannot contain a subscheme of the form $q_{ij}$, otherwise $Z$ would be contained in the union of two lines. Thus, $Z$ contains a subscheme of the form $q_1$, say $q_2$. Note that $\{p_0, p_{12}, q_2\}$ is contained in $p_0 p_2 \cup p_1 p_2$. Thus, there are six schemes obtained from $Z = \{p_0, p_{10}, q_2\}$ by the action of $S_3$. Since

$$I(Z) = \langle Y, Z \rangle \cap \langle X^2, Z \rangle \cap \langle X^2 Y, Y^2 \rangle = \langle X Y Z, Y^2 Z, X^2 Z, X^2 Y \rangle,$$

we see that $Z$ corresponds to the matrix

$$\beta = \begin{bmatrix} Y & X & 0 & 0 \\ X & 0 & Y & 0 \\ X & 0 & 0 & Z \end{bmatrix}.$$

2.1.3. Type (3, 3). The two subschemes of $Z$ of length 3 cannot be each contained in a line. The scheme $\{q_{01}, q_{12}\}$ is unfeasible because it is contained in the conic curve $\{Z^2 = 0\}$. It follows that $Z = \{q_i, q_{jk}\}$, for distinct $i, j, k$. There are six such schemes obtained from $Z = \{q_{01}, q_{12}\}$ by the action of $S_3$. Since

$$I(Z) = \langle Y^3, Z \rangle \cap \langle X^2, XY, Y^2 \rangle = \langle Y^2 Z, XYZ, X^2 Z, Y^3 \rangle,$$

we deduce that $Z$ corresponds to the matrix

$$\gamma = \begin{bmatrix} Y & X & 0 & 0 \\ X & 0 & Y & 0 \\ 0 & Y & 0 & Z \end{bmatrix}.$$

2.1.4. Type (1, 1, 4). Let $r$ be the point of $Z$ of multiplicity 4, supported say at $p_2$. Note that $r$ cannot be contained in a line, otherwise $Z$ would be contained in the union of two lines. Thus, $I(r) \cap \text{span} \langle X^2, XY, Y^2 \rangle$ is a $T$-invariant subspace of dimension 2. This subspace must be generated by two invariant monomials. Note that $r$ cannot be contained in the conic curve $\{XY = 0\}$, otherwise $Z$ would be contained in the said conic. Thus, $I(r) = \langle X^2, Y^2 \rangle$; we denote this point by $r_2$. We obtain three schemes: $\{r_0, p_1, p_2\}, \{r_1, p_0, p_2\}, \{r_2, p_0, p_1\}$. The ideal of the latter is

$$\langle X, Z \rangle \cap \langle Y, Z \rangle \cap \langle X^2, Y^2 \rangle = \langle Y^2 Z, X^2 Z, XY^2, X^2 Y \rangle,$$

so it corresponds to the matrix

$$\delta = \begin{bmatrix} Y & Z & 0 & 0 \\ 0 & Y & X & 0 \\ 0 & 0 & Z & X \end{bmatrix}.$$

The other two matrices are $\delta_\sigma$, where $\sigma = \langle X, Z \rangle$, respectively, $\langle Y, Z \rangle$.

2.1.5. Type (2, 4). Let $r$ be the point of $Z$ of multiplicity 4. As before, $r$ cannot be contained in a line. Moreover, $r \neq r_2$, otherwise $Z$ would be contained in the conic curve $\{X^2 = 0\}$ or $\{Y^2 = 0\}$. Thus, $r = r_{20}$ or $r = r_{21}$, where $r_{20}$ is given by the ideal $\langle X^3, XY, Y^2 \rangle$ and $r_{21}$ is given by the ideal $\langle X^2, XY, Y^3 \rangle$. We get six schemes obtained from $Z = \{p_{10}, r_{20}\}$ by the action of $S_3$. Since

$$I(Z) = \langle X^2, Z \rangle \cap \langle X^3, XY, Y^2 \rangle = \langle Y^2 Z, XYZ, X^2 Y, X^3 \rangle,$$

we deduce that $Z$ corresponds to the matrix

$$\epsilon = \begin{bmatrix} X & Y & 0 & 0 \\ Z & 0 & X & 0 \\ 0 & 0 & Y & X \end{bmatrix}.
2.1.6. **Type (1, 5).** Let \( s \) be the point of \( Z \) of multiplicity 5, supported say on \( p_2 \). Note that \( s \) cannot be contained in a line or in the conic \( \{XY = 0\} \). It follows that \( s = s_{20} \) or \( s = s_{21} \), where \( s_{20} \) is given by the ideal \((X^3, X^2Y, Y^2)\) and \( s_{21} \) is given by the ideal \((X^2, XY^2, Y^3)\). Letting \( S_3 \) act on \( Z = \{ p_1, s_{20} \} \) we obtain six \( T \)-fixed schemes. The ideal of \( Z \) is

\[
(X, Z) \cap (X^3, X^2Y, Y^2) = (Y^2Z, XY^2, X^2Y, X^3),
\]

hence \( Z \) corresponds to the matrix

\[
\zeta = \begin{bmatrix} Y & X & 0 & 0 \\ X & 0 & Y & 0 \\ 0 & Z & 0 & X \end{bmatrix}.
\]

2.1.7. **Type 6.** There is only one \( T \)-invariant point of multiplicity 6 supported on \( p_2 \). Its ideal is \((X^3, X^2Y, XY^2, Y^3)\) and the associated matrix is

\[
\eta = \begin{bmatrix} Y & X & 0 & 0 \\ X & 0 & Y & 0 \\ 0 & 0 & X & Y \end{bmatrix}.
\]

The matrices corresponding to the other two points of multiplicity 6 are \( \eta_\sigma \), where \( \sigma = (X, Z) \), respectively, \((Y, Z)\).

2.2. **Fixed points in \( N_1 \).** Consider \( \varphi \in W \). Assume that \( \gcd(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = 1 \) for some \( l \in V^* \). Let \( l \subseteq \mathbb{P}^2 \) denote the line given by the equation \( l = 0 \). According to \cite{19} Proposition 3.3.5, \( \varphi \) is semi-stable if and only if \( \zeta_1, \zeta_2, \zeta_3, \zeta_4 \) are linearly independent, which we assume in the sequel. Consider the vector space \( U_\varphi = \text{span}(\zeta_1, \zeta_2, \zeta_3, \zeta_4) \subseteq S^3 V^* \).

**Proposition 2.2.1.** Assume that \( \varphi \) is \( T \)-invariant. Then \( l \in \{X, Y, Z\}\) and any monomial occurring in any \( \zeta_i \), \( 1 \leq i \leq 4 \), belongs to \( U_\varphi \). Moreover, there are distinct monomials \( \xi_1, \xi_2, \xi_3, \xi_4 \in S^3 V^* \) such that \( U_\varphi = \text{span}(\xi_1, \xi_2, \xi_3, \xi_4) \).

**Proof.** For every \( t \in T \), \( t \varphi \sim \varphi \), hence \( U_{t \varphi} = U_\varphi \), hence \( \text{span}(t l) = \text{span}(l) \). Thus, \( l \) is a monomial. Let \( \xi_1, \ldots, \xi_k \) be the distinct monomials occurring in \( \zeta_1 \). We can find \( t_1, \ldots, t_k \in T \) such that \( t_1 \xi_1, \ldots, t_k \xi_1 \) are linearly independent. Thus, each \( \xi_i \), \( 1 \leq i \leq k \), is a linear combination of \( t_1 \xi_1, \ldots, t_k \xi_1 \), so it belongs to \( U_\varphi \). This shows that exactly four monomials occur in \( \zeta_1, \zeta_2, \zeta_3, \zeta_4 \), and these monomials span \( U_\varphi \). \( \square \)

Arguing as in the proof of \cite{19} Proposition 3.3.6, we can show that, after performing elementary row and column operations, we may write

\[
\varphi = \begin{bmatrix} \nu & 0 & 0 \\ * & * & * \\ 0 & 0 & l' \end{bmatrix},
\]

where \( \nu \) has linearly independent maximal minors. Thus, \( \nu \) gives a point in \( N(3, 3, 2) \). We will examine several cases according to the different possibilities for \( \text{Coker}(\nu) \).

Assume first that the maximal minors of \( \nu \) have no common factor. This condition defines an open subset \( N_0(3, 3, 2) \subset N(3, 3, 2) \). Consider the Hilbert scheme of zero-dimensional subschemes of length 3 of \( \mathbb{P}^2 \) and let \( \text{Hilb}^0_{\mathbb{P}^2}(3) \) denote the open
subset of subschemes that are not contained in a line. According to [10] Propositions 4.5 and 4.6, we have an isomorphism
\[ N_0(3,3,2) \simeq \text{Hilb}_P^3(3), \quad [u] \mapsto Y, \]
where the ideal sheaf of \( Y \) is the cokernel of \( u^T: 2O(-3) \to 3O(-2) \). Equivalently, \( Y \) is the zero-set of the ideal generated by the maximal minors of \( u \). Thus, up to equivalence, \( u \) is uniquely determined by its maximal minors. Note that \( Y' = 1 \).

Without loss of generality we may assume that \( Y' = X \), the other cases being obtained by a permutation of the variables. We have an extension
\[ 0 \longrightarrow O_{\mathbb{L}}(-1) \longrightarrow \text{Coker}(\varphi) \longrightarrow O_Y \longrightarrow 0. \]

In the sequel \( \varphi \) will represent a \( T \)-fixed point in \( N(3,4,3) \).

2.2.2. The case when \( Y \) is the union of three non-colinear closed points. Let \( Y' \subset Y \) be the subscheme supported on \( \mathbb{P}^2 \setminus L \). For \( t \in T \) denote by \( \mu_t: \mathbb{P}^2 \to \mathbb{P}^2 \) the map of multiplication by \( t \). Note that \( \mu_t^* (\text{Coker}(\varphi)) \simeq \text{Coker}(\varphi) \), hence, taking into account the exact sequence from above, \( \mu_t^*(O_{Y'}) \simeq O_{Y'} \). We deduce that \( Y' = \{ p_0 \} \). The other two points of \( Y \) are given by the ideals \( (X,aY+bZ) \), respectively, \( (X,cY+dZ) \), where \( ad - bc \neq 0 \). It follows that
\[ I(Y) = (aY + bZ, cY + dZ) \cap (X,aY+bZ) \cap (X,cY+dZ) \]
\[ = (X(aY+bZ), X(cY+dZ), (aY+bZ)(cY+dZ)). \]

Thus, we may write
\[ \varphi = \begin{bmatrix} X & aY + bZ & 0 & 0 \\ l_1 & l_2 & l_3 & X \end{bmatrix}. \]

Since \( X \) divides \( \zeta_4 \), \( X \) also divides \( l_1 \), so, performing column operations, we may assume that \( l_1 = 0 \). Note that \( a, b, c, d \) cannot be all non-zero, otherwise \( U_\varphi \) would contain the set \( X(Y^2, YZ, XY, XZ, Z^2) \), which is contrary to Proposition 2.2.1.

Assume that \( b = 0 \). Thus, \( a \neq 0 \), \( d \neq 0 \) and
\[ \varphi \sim \begin{bmatrix} X & Y & 0 & 0 \\ 0 & cY + Z & 0 \end{bmatrix}. \]

If \( c \neq 0 \), then \( c = 0 \), otherwise \( X(Y^2, YZ, XY, XZ, Z^2) \subset U_\varphi \), contradicting Proposition 2.2.1. Analogously, \( f = 0 \). We obtain the matrix
\[ \theta = \begin{bmatrix} X & Y & 0 & 0 \\ 0 & Z & 0 \end{bmatrix}, \]
which clearly represents a \( T \)-fixed point. If \( c = 0 \), then, by semi-stability, \( f \neq 0 \), so
\[ \varphi \sim \begin{bmatrix} X & Y & 0 & 0 \\ 0 & cY + Z & 0 \end{bmatrix} \sim \begin{bmatrix} X & Y & 0 & 0 \\ 0 & 0 & 0 & Y \end{bmatrix} \sim \theta_{(Y,Z)}. \]

In conclusion, we have six \( T \)-fixed points \( \{ \theta_\sigma \} \), \( \sigma \in S_3 \). They are distinct because \( U_\theta = \text{span}(XYZ, X^2Z, X^2Y, XZ^2) \) is not fixed by any \( \sigma \in S_3 \).
2.2.3. The case when \( Y \) is the union of a closed point on \( L \) and a double point outside \( L \). Assume that \( Y = \{ p, q \} \), where \( p \) is a closed point and \( q \) is a double point. Arguing as at Section 2.2.2, we see that \( p \) and \( q \) cannot both in \( \mathbb{P}^2 \setminus L \). Assume that \( p \in L \) and \( q \in \mathbb{P}^2 \setminus L \). Then \( q \in \{ p_0, p_0 \} \), say \( q = p_0 \). Thus, \( p \) is given by the ideal \( (X, aY + Z) \) for some \( a \in \mathbb{C} \) and \( q \) is given by the ideal \( (Y, Z(aY + Z)) \). We have

\[
I(Y) = (X, aY + Z) \cap (Y, Z(aY + Z)) = (XY, Y(aY + Z), Z(aY + Z)),
\]
so we may write

\[
\varphi = \begin{bmatrix}
Y & Z & 0 & 0 \\
0 & X & aY + Z & 0 \\
l_1 & l_2 & l_3 & X
\end{bmatrix}.
\]

Denote \( l_1 = a_i X + b_i Y + c_i Z, i = 1, 2, 3 \). Since \( X \) divides \( Yl_2 - Zl_1 \), hence \( X \) divides \( Y(b_2 Y + c_2 Z) - Z(b_1 Y + c_1 Z) \), hence \( b_2 = 0, c_1 = 0, b_1 = c_2 \). Performing elementary row and column operations, we may write

\[
\varphi = \begin{bmatrix}
Y & Z & 0 & 0 \\
0 & X & aY + Z & 0 \\
0 & 0 & l_3 & X
\end{bmatrix}.
\]

Performing elementary operations on \( \varphi \) we may assume that \( a_3 = 0 \) and \( c_3 = 0 \). Note that \( b_3 \neq 0 \), otherwise the semi-stability of \( \varphi \) would get contradicted. Thus, we may write

\[
\varphi = \begin{bmatrix}
Y & Z & 0 & 0 \\
0 & X & aY + Z & 0 \\
0 & 0 & Y & X
\end{bmatrix}.
\]

Consider \( \psi_c = \begin{bmatrix}
Y & Z & 0 & 0 \\
0 & X & acY + Z & 0 \\
0 & 0 & Y & X
\end{bmatrix} \) for each \( c \in \mathbb{C}^\ast \). Note that \( \{ [t\varphi], t \in T \} = \{ [\psi_c], c \in \mathbb{C}^\ast \} \), hence \( \varphi \sim \psi_c \) for all \( c \in \mathbb{C}^\ast \). The orbits for the action of \( G \) on \( W^{\text{ss}} \) are closed because there are no properly semi-stable points. Thus, \( \varphi \sim \lim_{c \to 0} \psi_c = \iota \), where

\[
\iota = \begin{bmatrix}
Y & Z & 0 & 0 \\
0 & X & 0 & 0 \\
0 & 0 & Y & X
\end{bmatrix}
\]

clearly determines a \( T \)-fixed point in \( N \). Note that \( \iota \sim \theta_\sigma \) for all \( \sigma \in S_3 \) because \( \text{type}(\theta) = (2, 1, 2) \) whereas \( \text{type}(\iota) = (2, 2, 2) \). Thus, we obtain six new fixed points \( \{ \iota_\sigma \}, \sigma \in S_3 \). They are distinct because \( U_i = \text{span}(XZ^2, XYZ, X^2Y, XY^2) \) is not fixed by any \( \sigma \in S_3 \).

2.2.4. The case when \( Y \) is the union of a closed point outside \( L \) and a double point. Assume that \( Y = \{ p, q \} \), where \( p \in \mathbb{P}^2 \setminus L \) is a closed point and \( q \) is a double point. Then, as before, \( p = p_0 \) and \( \text{red}(q) \in L \). In fact, we will show that \( q \) is a subscheme of \( L \). Assume that the contrary is true. Then \( I(q) = ((aY + bZ)^2, X + aY + bZ) \), where \( a \) and \( b \) are not both zero, say \( a \neq 0 \). We have \( I(p) = (aY + bZ, Z) \), hence

\[
I(Y) = (Z(X + aY + bZ), (aY + bZ)(X + aY + bZ), (aY + bZ)^2).
\]

We may now write

\[
\varphi = \begin{bmatrix}
X + aY + bZ & aY + bZ & 0 & 0 \\
0 & Z & aY + bZ & 0 \\
\ast & \ast & \ast & X
\end{bmatrix}.
\]
Note that \( b = 0 \), otherwise \( X[Y^2, YZ, Z^2, XY, XZ] \subset U_\varphi \), contrary to Proposition \( 2.2.1 \). After we perform elementary row and column operations, we may write
\[
\varphi = \begin{bmatrix}
X + aY & Y & 0 & 0 \\
0 & Z & Y & 0 \\
cY + dZ & 0 & eY + fZ & X
\end{bmatrix}.
\]
Note that \( U_\varphi = \text{span}(XY^2, XZ^2, YZ^2, XYZ) \). Since
\[
\zeta_4 = eXYZ + fXZ^2 + (ae + d)Y^2Z + afYZ^2 + cY^3
\]
we deduce, by virtue of Proposition \( 2.2.1 \), that \( f = 0 \), \( ae + d = 0 \), \( c = 0 \). Thus, \( e \neq 0 \) and
\[
\varphi \sim \begin{bmatrix}
X + aY & Y & 0 & 0 \\
0 & Z & Y & 0 \\
0 & 0 & Y & X
\end{bmatrix} \sim \begin{bmatrix}
X & Y & 0 & 0 \\
0 & -aZ & Z & Y \\
0 & 0 & Z & 0 & X
\end{bmatrix}.
\]
For each \( c \in \mathbb{C}^* \) consider the morphism
\[
\psi_c = \begin{bmatrix}
X & Y & 0 & 0 \\
0 & -acZ & Z & Y \\
0 & 0 & -Z & 0 & X
\end{bmatrix}.
\]
Note that \( \psi = \lim_{c \to 0} \psi_c \) belongs to \( W^{ss} \) because \( U_\varphi = \text{span}(XY^2, X^2Y, XZ^2, XYZ) \) has dimension 4. By the argument at Section \( 2.2.3 \) we deduce that \( \varphi \sim \psi \). However, this is absurd, because \( \text{type}(\varphi) \neq \text{type}(\psi) \).

The above discussion shows that \( q \) is a subscheme of \( L \), so \( I(q) = (X, (aY + bZ)^2) \), where \( a, b \) are not both zero, say \( a \neq 0 \). We have
\[
I(Y) = (aY + bZ, Z) \cap I(q) = (XZ, X(aY + bZ)), (aY + bZ)^2),
\]
so we may write
\[
\varphi = \begin{bmatrix}
X & aY + bZ & 0 & 0 \\
0 & Z & aY + bZ & 0 \\
* & * & * & X
\end{bmatrix}.
\]
Note that \( X([aY^2, abYZ, bZ^2, aXY, XZ] \subset U_\varphi \), hence, by Proposition \( 2.2.1 \), \( b = 0 \), that is, \( q = p_{21} \). We may write
\[
\varphi = \begin{bmatrix}
X & Y & 0 & 0 \\
0 & Z & Y & 0 \\
0 & 0 & cY + dZ & X
\end{bmatrix}.
\]
Since \( X \) divides \( \zeta_4 \), \( X \) also divides \( l_1 \), hence, performing row and column operations, we may write
\[
\varphi = \begin{bmatrix}
X & Y & 0 & 0 \\
0 & Z & Y & 0 \\
0 & 0 & cY + dZ & X
\end{bmatrix}.
\]
Note that \( X[Y^2, XY, XZ, cYZ, dZ^2] \subset U_\varphi \), hence, in view of Proposition \( 2.2.1 \), \( c = 0 \) or \( d = 0 \). If \( d = 0 \), then
\[
\varphi \sim \begin{bmatrix}
X & Y & 0 & 0 \\
0 & Z & Y & 0 \\
0 & 0 & Y & X
\end{bmatrix} \sim \begin{bmatrix}
X & Y & 0 & 0 \\
0 & Z & 0 & X \\
0 & 0 & Y & X
\end{bmatrix} \sim \begin{bmatrix}
X & 0 & Y & 0 \\
0 & Y & 0 & X \\
0 & 0 & Z & X
\end{bmatrix} \sim \theta_{(Y,Z)}.
\]
If \( c = 0 \), then we obtain the matrix

\[
\kappa = \begin{bmatrix}
X & Y & 0 & 0 \\
0 & Z & Y & 0 \\
0 & 0 & Z & X
\end{bmatrix}
\]

representing a \( T \)-fixed point. Note that \( U_\kappa = \text{span}\{XY^2, X^2Y, X^2Z, XZ^2\} \neq U_{i_\sigma} \), hence \( [\kappa] \neq [i_\sigma] \) for all \( \sigma \in S_3 \). Moreover considering types we see that \( [\kappa] \neq [\theta_\sigma] \) for all \( \sigma \in S_3 \). Since \( \kappa \sim \kappa(y,z) \), we conclude that we get three new \( T \)-fixed points, \( [\kappa_\sigma], \sigma \in A_3 \).

2.2.5. The case when \( Y \) is the union of a closed point on \( L \) and a double point whose support is on \( L \). Assume that \( Y = \{p, q\} \), where \( p, \text{red}(q) \in L \). Thus,

\[
I(q) = (X^2, eX + aY + bZ), \quad I(p) = (X, cY + dZ),
\]

where \( ad - bc \neq 0 \). It follows that

\[
I(Y) = (X(eX + aY + bZ), (cY + dZ)(eX + aY + bZ), X^2),
\]

so we may write

\[
\varphi = \begin{bmatrix}
edX + aY + bZ & X & 0 & 0 \\
0 & cY + dZ & X & 0 \\
\ast & \ast & \ast & X
\end{bmatrix}.
\]

Note that \( X\{X^2, aXY, bXZ, acY^2, bdZ^2, (ad + bc)YZ\} \subset U_\varphi \), hence, by Proposition 2.2.1 \( a = 0 \) or \( b = 0 \). We may assume that \( b = 0 \), the case when \( a = 0 \) being obtained by a permutation of variables. Thus, \( d \neq 0 \). Assume first that \( c \neq 0 \). Then \( X\{X^2, XY, XZ, YZ, acY^2\} \subset U_\varphi \), forcing \( c = 0 \), in view of Proposition 2.2.1. We may write

\[
\varphi = \begin{bmatrix}
edX + Y & X & 0 & 0 \\
0 & Z & X & 0 \\
l_1 & l_2 & l_3 & X
\end{bmatrix}.
\]

Since \( X \) divides \( \z_4 \), we see that \( X \) divides \( l_3 \), so, performing row and column operations, we may assume that \( l_3 = 0 \) and \( l_2 = fY \). Since \( fXY^2 \) belongs to \( U_\varphi = \text{span}\{X^3, X^2Y, XZ, XYZ\} \), we see that \( f = 0 \), hence

\[
\varphi \sim \begin{bmatrix}
edX + Y & 0 & 0 \\
0 & Z & X \\
Z & 0 & 0 & X
\end{bmatrix} \sim \begin{bmatrix}
y & X & 0 & 0 \\
0 & Z & X \\
Z & 0 & 0 & X
\end{bmatrix} = \lambda.
\]

Clearly, \( \lambda \) represents a \( T \)-fixed point of \( N \) different from \( \theta_\sigma, i_\sigma, k_\sigma \) for all \( \sigma \in S_3 \) because type(\( \lambda \)) = \( (3, 1, 2) \) is different from the types of the other points. We obtain six new \( T \)-fixed points \( [\lambda_\sigma], \sigma \in S_3 \).

Assume now that \( e = 0 \). As \( b = 0 \), we have \( a \neq 0 \). Thus, we may write

\[
\varphi = \begin{bmatrix}
y & X & 0 & 0 \\
0 & cY + Z & X & 0 \\
l_1 & l_2 & l_3 & X
\end{bmatrix}.
\]

As before, \( X \) divides \( l_3 \), hence, performing row and column operations, we may write

\[
\varphi = \begin{bmatrix}
y & X & 0 & 0 \\
0 & cY + Z & X & 0 \\
gZ & fY & 0 & X
\end{bmatrix}.
\]
Assume first that \( c \neq 0 \). Note that \( X[X^2, XY, Y^2, YZ, gXZ] \subset U_\varphi \), hence, in view of Proposition 2.2.1 \( g = 0 \) and

\[
\varphi \sim \begin{bmatrix} Y & X & 0 & 0 \\ 0 & cY + Z & X & 0 \\ 0 & Y & 0 & X \end{bmatrix} \sim \begin{bmatrix} Y & X & 0 & 0 \\ 0 & Z & X & 0 \\ 0 & 0 & 0 & X \end{bmatrix} = \mu.
\]

Clearly, \([\mu] \) is \( T \)-fixed. Considering types we can see that \([\mu] \neq [\theta_\sigma], [\iota_\sigma], [\kappa_\sigma] \) for all \( \sigma \in S_3 \). Moreover, \([\mu] \neq [\lambda_{Y,Z}] \) because \( U_\mu \neq U_{\lambda_{Y,Z}} \). Thus, we obtain six new points \([\mu_\sigma], \sigma \in S_3 \). Finally, we assume that \( c = 0 \), so we may write

\[
\varphi = \begin{bmatrix} Y & X & 0 & 0 \\ 0 & Z & 0 & X \\ gZ & fY & 0 & X \end{bmatrix}.
\]

We have \( U_\varphi = \text{span}[X^3, X^2Y, XYZ, X(fY^2 - gXZ)] \), hence, by Proposition 2.2.1 \( f = 0 \) or \( g = 0 \). We obtain the fixed points \([\lambda] \), respectively, \([\mu] \).

2.2.6. The case when \( \mathcal{Y} \) is a triple point. Assume that \( \mathcal{Y} \) is a triple point that is not contained in a line. If \( \mathcal{Y} \subset \mathbb{F}^2 \setminus L \), then \( \mathcal{Y} \) is \( T \)-fixed, hence \( \mathcal{Y} = \{q_0\} \) and \( I(\mathcal{Y}) = \{Y^2, YZ, Z^2\} \). Recalling from the beginning of Section 2.2 that the generators of \( I(\mathcal{Y}) \) determine \( \nu \) up to equivalence, we may write

\[
\varphi = \begin{bmatrix} Y & Z & 0 & 0 \\ 0 & Y & Z & 0 \\ l_1 & l_2 & l_3 & X \end{bmatrix},
\]

where \( l_1, l_2, l_3 \in \mathbb{C}[Y, Z] \). Since \( X \) divides \( \zeta_4 \) and \( \zeta_4 \in \mathbb{C}[Y, Z] \), we deduce that \( \zeta_4 = 0 \), which is contrary to our assumption that \( \varphi \) give a point in \( N_1 \). This shows that \( \text{red}(\mathcal{Y}) \) is a point on \( L \), given by the ideal \( \langle X, aY + bZ \rangle \), where \( a, b \) are not both zero, say \( a \neq 0 \). The possible ideals defining \( \mathcal{Y} \) are

(i) \( \langle X^2, X(aY + bZ), (aY + bZ)^2 \rangle \),
(ii) \( \langle X^2, X(aY + bZ), (aY + bZ)^2 - XZ \rangle \),
(iii) \( \langle (cX + aY + bZ)^2, (cX + aY + bZ)X, X^2 - (cX + aY + bZ)Z \rangle \).

In the first case we may write

\[
\varphi = \begin{bmatrix} X & aY + bZ & 0 & 0 \\ 0 & X & aY + bZ & 0 \\ * & * & * & X \end{bmatrix}.
\]

Note that \( X[a^2Y^2, abYZ, b^2Z^2, aXY, bXZ, X^2] \subset U_\varphi \), hence, by Proposition 2.2.1 \( b = 0 \), that is \( \mathcal{Y} = \{q_2\} \). We may now write

\[
\varphi = \begin{bmatrix} X & Y & 0 & 0 \\ 0 & X & Y & 0 \\ l_1 & l_2 & l_3 & X \end{bmatrix}.
\]

By hypothesis \( X \) divides \( \zeta_4 \), hence \( X \) divides \( l_1 \). Performing elementary row and column operations, we may assume that

\[
\varphi = \begin{bmatrix} X & Y & 0 & 0 \\ 0 & X & Y & 0 \\ 0 & cZ & dZ & X \end{bmatrix}.
\]
Note that $X\{X^2, XY, XZ, Y^2, bZ^2\} \subset \mathcal{U}_\varphi$, hence, by Proposition 2.2.1, $c = 0$ or $d = 0$. In the case when $d = 0$ we obtain the fixed point $[\mu]$. In the case when $c = 0$ we obtain a matrix

$$\nu = \begin{bmatrix} X & Y & 0 & 0 \\ 0 & X & Y & 0 \\ 0 & 0 & Z & X \end{bmatrix}$$

that represents a $T$-fixed point in $N$. Considering types we see that $[\nu] \neq [\theta_\sigma], [\upsilon_\sigma], [\kappa_\sigma]$ for all $\sigma \in S_3$. Moreover, $[\nu] \neq [\lambda_{1, Y, Z}], [\mu]$ because $U_\nu \neq U_{\lambda_{1, Y, Z}}, U_\mu$. Thus, we obtain six new $T$-fixed points $[\nu_\sigma], \sigma \in S_3$.

Assume now that $\mathcal{V}$ has the ideal given at (ii). We may write

$$\varphi = \begin{bmatrix} aY + bZ & X & 0 & 0 \\ Z & aY + bZ & X & 0 \\ \ast & \ast & \ast & X \end{bmatrix}.$$ 

Note that $X\{X^2, XY, XZ, Y^2, bZ^2\} \subset \mathcal{U}_\varphi$, hence, by Proposition 2.2.1, $b = 0$ and, performing row and column operations, we may write

$$\varphi = \begin{bmatrix} aY & X & 0 & 0 \\ Z & aY & X & 0 \\ l_1 & l_2 & l_3 & X \end{bmatrix}. \quad \text{Moreover, } \varphi \sim \begin{bmatrix} aY & X & 0 & 0 \\ Z & aY & X & 0 \\ 0 & cY + dZ & 0 & X \end{bmatrix}.$$

because $X$ divides $l_3$, since $X$ divides $\xi_4$. Since $adXYZ \in \mathcal{U}_\varphi$, we deduce that $d = 0$ and we obtain the fixed point $[\nu]$.

Assume, finally, that $\mathcal{V}$ has the ideal given at (iii). Thus, we may assume that

$$\varphi = \begin{bmatrix} X & cX + aY + bZ & 0 & 0 \\ Z & X & cX + aY + bZ & 0 \\ \ast & \ast & \ast & X \end{bmatrix}.$$

Notice that $X\{X^2, Y^2, XY, YZ, bZ^2, cXZ\} \subset \mathcal{U}_\varphi$ hence, by Proposition 2.2.1, $b = 0, c = 0$, and we may write

$$\varphi = \begin{bmatrix} X & aY & 0 & 0 \\ Z & X & aY & 0 \\ l_1 & l_2 & l_3 & X \end{bmatrix}.$$

Since $X$ divides $\xi_4$, $X$ also divides $Zl_3 - aYl_1$, from which, as in Section 2.2.3 it follows that

$$\varphi \sim \begin{bmatrix} X & aY & 0 & 0 \\ Z & X & Y & 0 \\ 0 & Z & 0 & X \end{bmatrix}. \quad \text{For } e \in \mathbb{C}^* \text{ denote } \psi_e = \begin{bmatrix} X & aY & 0 & 0 \\ eZ & X & Y & 0 \\ 0 & Z & 0 & X \end{bmatrix}.$$

As in Section 2.2.4 the morphism $\psi = \lim_{e \to 0} \psi_e$ lies in $W^n$ and $\varphi \sim \psi$, which yields a contradiction.

2.2.7. The case when the maximal minors of $\nu$ have a common linear factor. Assume that $\varphi \in W_1$ represents a $T$-fixed point and that the maximal minors of $\nu$, denoted $\nu_1, \nu_2, \nu_3$, have a common linear factor. Then

$$\gcd(\nu_1, \nu_2, \nu_3) = l = \gcd(\xi_1, \xi_2, \xi_3, \xi_4).$$

As before, we may assume that $l$ is a monomial, say $X$, the other cases being obtained by a permutation of the variables. Performing column operations, we
may assume that the maximal minors of \( \upsilon \) are \( X^2, XY, XZ \). It is now easy to see that we may write

\[
\varphi = \begin{bmatrix}
Y & X & 0 & 0 \\
Z & 0 & X & 0 \\
\ast & \ast & \ast & \ast \\
\end{bmatrix}.
\]

Applying Proposition 2.2.1 we can easily see that \( l' \) is a monomial. Indeed, if, say, \( l' = aX + bY \) with \( a \neq 0, b \neq 0 \), then \( \{X^2, XY, XZ, Y^2, YZ\} \subset U_\varphi \). In the sequel we will assume that \( l' \in \{X, Y, Z\} \). When \( l' = Y \) or \( l' = Z \) we do not obtain any new fixed points in \( N \). To see this it is enough to consider only the case when \( l' = Y \), the other case being obtained by swapping \( Y \) and \( Z \). Thus, we consider the matrix

\[
\varphi = \begin{bmatrix}
Y & X & 0 & 0 \\
Z & 0 & X & 0 \\
aX + bZ & cZ & dZ & Y \\
\end{bmatrix}.
\]

As at Section 2.2.4 the matrix \( \psi = \lim_{e \to 0} \psi_e \) belongs to \( W^{\text{ss}} \) and \( \varphi \sim \psi \), which is absurd. The same argument will lead to a contradiction in the case when \( b \neq 0 \) and \( c \neq 0 \). Assume now that \( c \neq 0 \), \( d \neq 0 \), so we may write

\[
\varphi = \begin{bmatrix}
Y & X & 0 & 0 \\
Z & 0 & X & 0 \\
\ast & \ast & \ast & \ast \\
0 & cZ & Z & Y \\
\end{bmatrix}.
\]

We obtain the fixed points \( [\mu], [\theta_{Y,Z}], [\iota] \).

It remains to examine the case when \( l' = X \). We may write

\[
\varphi = \begin{bmatrix}
Y & X & 0 & 0 \\
Z & 0 & X & 0 \\
aY + bZ & cY + dZ & X \\
\end{bmatrix}.
\]

Note that \( \{X^2, XY, XZ, aY^2, (b + c)YZ, dZ^2\} \subset U_\varphi \), hence, by Proposition 2.2.1 precisely one among the numbers \( a, b + c, d \) is non-zero. When \( b + c \neq 0 \) we get the fixed point represented by the matrix

\[
\lambda(b : c) = \begin{bmatrix}
Y & X & 0 & 0 \\
Z & 0 & X & 0 \\
0 & bZ & cY & X \\
\end{bmatrix}.
\]
There is a morphism \( P \) to \( \mathcal{X} \) of \( \mathcal{Y} \) that is injective, hence it is an isomorphism, but we will not need these facts.

We obtain a set \( A \) of fixed points in \( \mathcal{N} \) parametrised by \((b : c) \in \mathbb{P}^1 \setminus \{(1 : -1)\}\). The above notation is justified because for \((b : c) = (1 : 0)\) we obtain the point \([\lambda]\) and for \((b : c) = (0 : 1)\) we obtain the point \([\lambda_{YZ}]\). If \((b : c) = (1 : -1)\) we obtain the \( \mathcal{T} \)-fixed point in \( \mathcal{N}_2 \) represented by the matrix

\[
\lambda(1 : -1) = \begin{bmatrix}
Y & X & 0 & 0 \\
Z & 0 & X & 0 \\
0 & Z & -Y & X
\end{bmatrix}.
\]

In Section 2.3 below we will see that this matrix is semi-stable. The point \([\lambda(1 : -1)]\) lies in the closure of \( A \), so we get a connected set

\[
\Lambda = \{[\lambda(b : c)] \mid (b : c) \in \mathbb{P}^1\} \subset \mathcal{N}^T.
\]

Clearly, \( \Lambda \) is not reduced to a point because it contains points from \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \). Our description of \( \mathcal{N}^T \) (including Section 2.3) shows that \( \dim(\mathcal{N}^T) \leq 1 \). Thus, \( \Lambda \) is a connected component of \( \mathcal{N}^T \) of dimension 1. By [3, Theorem 4.2], \( \Lambda \) is smooth. There is a morphism \( \mathbb{P}^1 \to \Lambda \) given by \((b : c) \mapsto [\lambda(b : c)]\), hence \( \Lambda \) is isomorphic to \( \mathbb{P}^1 \), and hence \( A \) is an affine line. (It can be shown that the map \( \mathbb{P}^1 \to \Lambda \) is injective, hence it is an isomorphism, but we will not need these facts.)

Clearly, \( \Lambda \) is fixed by the transposition \( (Y,Z) \), so we get three projective lines \( \Lambda_{\sigma} \), \( \sigma \in A_3 \), of \( \mathcal{T} \)-fixed points in \( \mathcal{N} \).

Assume now that \( a \neq 0 \), \( c = -b \) and \( d = 0 \). We may write

\[
\varphi = \begin{bmatrix} Y & X & 0 & 0 \\ Z & 0 & X & 0 \\ 0 & Y + bZ & -bY & X \end{bmatrix}.
\]

Denote \( \psi_e = \begin{bmatrix} Y & X & 0 & 0 \\ Z & 0 & X & 0 \\ 0 & eY + bZ & -bY & X \end{bmatrix} \) for \( e \in \mathbb{C}^* \). If \( b = 0 \), then \([\varphi] = [\psi]\). If \( b \neq 0 \), then, as before, \( \varphi \sim \lim \psi_e \), hence \([\varphi] = [\lambda(1 : -1)]\). Assume that \( a = 0 \), \( b = 0 \), \( c = 0 \) and \( d \neq 0 \). Then \([\varphi] = [\lambda_{YZ}]\). Assume, finally, that \( a = 0 \), \( c = -b \neq 0 \) and \( d \neq 0 \). Then, as before, \([\varphi] = [\lambda(1 : -1)]\).

2.3. Fixed points in \( \mathcal{N}_2 \). Consider \( \varphi \in W \). Assume that \( \gcd(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = q \) for some \( q \in \mathbb{S}^2 \mathcal{V}^* \). According to [19, Proposition 3.3.2], \( \varphi \) is semi-stable if and only if \( \varphi \) is equivalent to a morphism of the form

\[
\begin{bmatrix}
Y & X & 0 & l_1 \\
Z & 0 & X & l_2 \\
0 & Z & -Y & l_3
\end{bmatrix}.
\]

Moreover, \( U_{\varphi} = \langle qX, qY, qZ \rangle \) and \( q \) is a monomial. Assume that \( q = X^2 \). Then \( l_3 \neq 0 \) mod \( \langle Y, Z \rangle \) so, without loss of generality, we may assume that

\[
\varphi = \begin{bmatrix} Y & X & 0 & a_1X + b_1Y + c_1Z \\ Z & 0 & X & a_2X + b_2Y + c_2Z \\ 0 & Z & -Y & X \end{bmatrix}.
\]

Since \( \zeta_1 \) is divisible by \( X^2 \) we deduce that \( Y(b_2Y + c_2Z) - Z(b_1Y + c_1Z) = 0 \), hence

\[
\varphi \sim \begin{bmatrix} Y & X & 0 & a_1X \\ Z & 0 & X & a_2X \\ 0 & Z & -Y & X \end{bmatrix} \sim \begin{bmatrix} Y & X & 0 & 0 \\ Z & 0 & X & 0 \\ 0 & Z & -Y & X + a_2Y - a_1Z \end{bmatrix}.
\]
We say that two solutions \((r, m, n)\) and \((r', m', n')\) are equivalent if one of them can be obtained from the other by applying finitely many times the operation \(r\). Checking the cases when \(3rn \leq m \leq m'\) for all \((m', n')\) ~ \((m, n)\) and \(m \leq m'\) for all \((m', n)\) ~ \((m, n)\), we get the T-fixed points \([\lambda(1 : -1)]\) for \(\sigma = (X, Y)\), respectively, \((X, Z)\).

Assume now that \(q = XY\). Clearly, \(l_1 = 0 \mod (X, Y)\), hence we may write

\[
\varphi = \begin{bmatrix}
Y & X & 0 & 0 \\
Z & 0 & X & l_2 \\
0 & Z & -Y & 1_3
\end{bmatrix}.
\]

Since \(X\) divides \(Xl_1 + Yl_2\), we see that \(X\) divides \(l_2\). Since \(Y\) divides \(Xl_1 + Yl_2\), we see that \(Y\) divides \(l_3\). Thus, \(\varphi\) is equivalent to the matrix

\[
\xi = \begin{bmatrix}
Y & X & 0 & 0 \\
Z & 0 & X & 0 \\
0 & Z & -Y & Y
\end{bmatrix}
\]

giving a T-fixed point in \(N\). We get three new isolated T-fixed points \(\xi_\sigma\), \(\sigma \in A_3\).

3. Proof of Theorem 1

In Section 2 we found that the isolated points of \(N^T\) are \(\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta, \iota, \kappa, \mu, \nu, \xi\), and the points obtained from these by permutations of variables. Thus, \(N^T\) consists of 62 isolated points and 3 projective lines, namely \(A\) and two other lines obtained by permutations of variables. In Section 3.3 below we will examine the action of \(T\) on the tangent spaces at the T-fixed points. Before that, we will find the initial moduli spaces of height zero and dimension 12.

3.1. Diophantine equations. The condition that the Kronecker moduli space \(N(3r, m, n)\) have dimension 12 is equivalent to the equation

\[
3rmn - m^2 - n^2 = 11, \quad r, m, n \in \mathbb{N}.
\]

We fix \(r\) and solve for \((m, n)\). The set of solutions is preserved under the operations

\[
R, S: Z^2 \to Z^2, \quad R(m, n) = (n, 3rn - m), \quad S(m, n) = (m, n).
\]

We say that two solutions \((m, n)\) and \((m', n')\) are equivalent if one of them can be obtained from the other by applying finitely many times the operations \(R\) and \(S\). Let \((m, n)\) be a solution that is smallest in its equivalence class, in the sense that \(n \leq n'\), \(n \leq m'\) for all \((m', n')\) ~ \((m, n)\) and \(m \leq m'\) for all \((m', n)\) ~ \((m, n)\). Thus, \(n \leq m \leq 3rn/2\). Indeed, if \(3rn/2 < m\), then \((m', n) = (3rn - m, n)\) would be an equivalent solution for which \(m' < m\), contradicting the choice of \((m, n)\). Checking the cases when \(m = n\) or \(m = 3rn/2\) yields no solutions, so we may assume that \(n < m < 3rn/2\). Thus,

\[
11 > 3rmn - 2m^2 = (3rn - 2m)m \quad \text{forcing} \quad 2 \leq m \leq 10.
\]

Checking the cases when \(2m/(3r) < n < m \leq 10\) yields the solution \((r, m, n) = (1, 4, 3)\). We conclude that the solutions to equation 3.1.1 are of the form \((1, m, n)\), where \((m, n) \sim (4, 3)\). As a consequence, the moduli spaces of height zero and dimension 12 are isomorphic to \(N(3, 4, 3)\). Note that the exceptional bundle \(E\) is a line bundle.

Analogously, the solutions to the diophantine equation

\[
3rmn - m^2 - n^2 = 10, \quad r, m, n \in \mathbb{N}.
\]
are of the form \((4, m, n)\), where \((m, n) \sim (1, 1)\). It follows that the moduli spaces of height zero and dimension 11 are associated to exceptional bundles of rank 4. According to [21], there are no exceptional bundles of rank 4 on \(\mathbb{P}^2\). We conclude that there are no moduli spaces of height zero and dimension 11.

3.2. Initial moduli spaces. Let us determine the initial moduli spaces \(M\) of height zero and dimension \(d = 12\). We saw in Section 3.1 that the exceptional bundle \(E\) is a line bundle. Without loss of generality, we may assume that \(E = O_{\mathbb{P}^2}\). We first consider the case when \(\mu \leq \mu(E)\), the case when \(\mu \geq \mu(E)\) being dual. The condition that \(M\) be initial implies the inequalities

\[
\mu(E) - \frac{1}{3 \text{rank}(E)^2} < \mu \leq \mu(E), \quad \text{that is,} \quad -\frac{1}{3} < \mu \leq 0.
\]

By [8] Proposition 30, we have the inequalities

\[
\text{rank}(E)\sqrt{d - 1} \leq \mu \leq \text{rank}(E)^2\sqrt{d - 1},
\]

forcing \(4 \leq \mu \leq 9\). The condition that \(M\) have height zero is equivalent to the condition

\[
\Delta = \frac{1}{2}(\mu - \mu(E))(\mu - \mu(E) + 3) + 1 - \Delta(E) = \frac{1}{2}\mu(\mu + 3) + 1.
\]

The isomorphism \(M \simeq N(3, 4, 3)\) of [8] preserves stable points, hence \(M\) contains stable points. This leads to the formula \(d - 1 = r^2(2\Delta - 1)\) of [10, Corollary 14.5.4]. Combining with the equation above we obtain the diophantine equation

\[
11 = c_1^2 + 3rc_1 + r^2.
\]

Verifying this equation for \(r = 4, \ldots, 9\) and \(c_1 = [-r/3] + 1, \ldots, 0\) yields the solutions \((r, c_1) = (5, -1)\) and \((7, -2)\). We obtain the moduli spaces \(M(5, -1, 4)\) and \(M(7, -2, 6)\).

By duality, if \(\mu \geq \mu(E)\), we obtain the initial moduli spaces \(M(5, 1, 4)\) and \(M(7, 2, 6)\).

3.3. Torus representation of the tangent spaces. Let \(\varphi \in W_{\text{ms}}\) give a torus fixed point in \(N(3, 4, 3)\). Assume that there are morphisms of groups

\[
u: (\mathbb{C}^*)^3 \rightarrow \text{GL}(3, \mathbb{C}),
\]

\[
u = \begin{bmatrix}
u_1 & 0 & 0 & 0 \\
0 & \nu_2 & 0 & 0 \\
0 & 0 & \nu_3 & 0 \\
0 & 0 & 0 & \nu_4
\end{bmatrix},
\]

such that \(t \varphi = \nu(t)\varphi u(t)\) for all \(t \in (\mathbb{C}^*)^3\). In Table 1 below we give such morphisms \(u\) and \(v\) for all fixed points \(\varphi\) found in Section 3. According to [6, Formula (6.1.1)], the action of \((\mathbb{C}^*)^3\) on \(T_{(\varphi)}N\), denoted by \(*\), is given by

\[
(3.3.1) \quad t * [w] = [\nu(t)^{-1}(tw)u(t)^{-1}]
\]

and is induced by an action on \(W\) given by the same formula. According to [6, Formula (6.1.3)], the induced action on \(T_{(\varphi)}(G\varphi)\) is given by

\[
(3.3.2) \quad t * (A, B) = (u(t)A u(t)^{-1}, v(t)^{-1}B v(t)).
\]

Here we identify \(T_{(\varphi)}(G\varphi)\) with the tangent space of \(G\) at its neutral element

\[
T_e G = (\text{End}(4\mathcal{O}(-2)) \oplus \text{End}(3\mathcal{O}(-1)))/\mathbb{C}.
\]
whose vectors are represented by pairs \((A, B)\) of matrices. Both actions factor through \(T\). The list of weights for the action of \(T\) on \(T_{\varphi} N(3, 4, 3)\), denoted by \(\chi^*[\varphi]\), is obtained by subtracting the list of weights for the action of \(T\) on \(T_{\varphi}(G\varphi)\), obtained by means of \((3.3.2)\), from the list of weights for the action of \(T\) on \(T_{\varphi} W\), obtained using \((3.3.1)\):

\[
\chi^*[\varphi] = \{v_j^{-1} u_i^{-1} t_k \mid i = 1, 2, 3, 4, \ j = 1, 2, 3, \ k = 0, 1, 2\} \\
\setminus \{(u_i u_j^{-1} \mid i, j = 1, 2, 3, 4) \cup \{v_i^{-1} v_j \mid i, j = 1, 2, 3\} \setminus \{\chi_0\} \}.
\]

Here \(\chi_0\) is the trivial character of \(T\). It is convenient to use additive notation when dealing with characters. We replace the character \(t_3^i t_1^j t_2^k\) with the expression \(ix + jy + kz\), where \(i, j, k \in \mathbb{Z}\). In Table 2 below we give, in additive notation, the list of weights for each fixed point from Section 2. These lists are obtained with the aid of the Singular \[7\] program from Appendix A. The points \(b : c\) from Section 2.2.7 are ignored because the \(T\)-representation of the tangent space at a point is unchanged if the point varies in a connected component of \(N^T\), so it is enough to examine only one point on \(\Lambda\), say \([\lambda]\).

**Table 1**

| Fixed point | \((v_1, v_2, v_3)\) | \((u_1, u_2, u_3, u_4)\) |
|-------------|---------------------|-------------------------|
| \(\alpha\)  | \((z, x, y)\)       | \((0, x - z, y - x, z - y)\) |
| \(\beta\)   | \((y, x, z)\)       | \((0, x - y, z - x, y - x)\) |
| \(\gamma\)  | \((y, x, 2y - x)\) | \((0, x - y, y - x, -2y + z)\) |
| \(\delta\)  | \((y, 2y - z, 2y - x)\) | \((0, z - y, x - 2y + z, 2x - 2y)\) |
| \(\epsilon\) | \((x, z, -x + y + z)\) | \((0, y - x, x - z, 2x - y - z)\) |
| \(\zeta\)   | \((y, x, -x + y + z)\) | \((0, x - y, y - x, 2x - y - z)\) |
| \(\eta\)    | \((y, x, 2x - y)\)  | \((0, x - y, y - x, 2x - y)\) |
| \(\theta\)  | \((x, x, x - y + z)\) | \((0, y - x, z - x, y - z)\) |
| \(\iota\)   | \((y, x + y - z, x + 2y - z)\) | \((0, z - y, -x - y + 2z, -2y + 2z)\) |
| \(\kappa\)  | \((x, x - y + z, x - 2y + 2z)\) | \((0, y - x, -x + 2y - z, 2y - 2z)\) |
| \(\lambda\) | \((y, -x + y + z, z)\) | \((0, x - y, 2x - y - z, x - z)\) |
| \(\mu\)    | \((x, z, y)\)       | \((y - x, 0, x - z, y - x)\) |
| \(\nu\)    | \((x, 2x - y, 2x - y + z)\) | \((0, y - x, 2y - 2x, x - y + 2z)\) |
| \(\xi\)    | \((y, z, -x + y + z)\) | \((0, x - y, x - z, y - z)\) |

Let \(\lambda(\tau) = (\tau^{n_0}, \tau^{n_1}, \tau^{n_2})\) be a one-parameter subgroup of \(T\) that is not orthogonal to any non-zero character appearing in Table 2. The set of weights from Table 2 is contained on the set

\[\{ix + jy + kz \mid -3 \leq i, j, k \leq 3\},\]

so we can choose \(\lambda(\tau) = (1, \tau, \tau^4)\). For each \(T\)-fixed point \([\varphi]\) in \(N\) denote by \(p(\varphi)\) the number of characters \(\chi \in \chi^*[\varphi]\) satisfying the condition \(\langle \lambda, \chi \rangle > 0\). Using the procedure “positive-parts” from Appendix A we compute the list of numbers \(p(\varphi_{\sigma, \sigma})\), \(\sigma \in S_4\). The results are written Table 3 below.

We quote below the Homology Basis Formula [3, Theorem 4.4]. Let \(X_1, \ldots, X_m\) denote the irreducible components of \(N^T\). Denote \(p(i) = p(X_i) = p(\varphi)\) for some (or, in fact, any) point \([\varphi]\) \(\in X_i\). Then for \(0 \leq n \leq 2 \dim(N)\) we have the isomorphism

\[(3.3.3) \quad H_n(N, \mathbb{Z}) \cong \bigoplus_{1 \leq i \leq m} H_{n-2p(i)}(X_i, \mathbb{Z}).\]
Let $\Pi$ denote the set of isolated $T$-fixed points in $N$. From (3.3.3) we get the formula

$$P_N(x) = \sum_{[\varphi] \in \Pi} x^{2p[\varphi]} + \sum_{\sigma \in A_3} (x^2 + 1)x^{2p[\Lambda_\sigma]}.$$ 

Substituting the values of $p[\varphi]$ from Table 3 yields the expression for the Poincaré polynomial from Theorem 1.
According to [3] Section 4.2.8, Formula (3.3.3) respects the Hodge decomposition, that is, for $0 \leq p, q \leq \dim(N)$, we have the isomorphism
\begin{equation}
\text{H}^p(N, \Omega^q_N) \simeq \bigoplus_{1 \leq i \leq m} \text{H}^{p-i}(X_i, \Omega^q_{X_i}^{p-i}).
\end{equation}
This shows that $h^p(N) = 0$ for $p \neq q$, because the same is true of the Hodge numbers of all $X_i$ (which are points or projective lines).

4. The torus fixed locus of $M_{p^2}(5,1)$

For the convenience of the reader we recall from [19] the classification of semi-stable sheaves on $\mathbb{P}^2$ having Hilbert polynomial $5m + 1$. In $M_{p^2}(5,1)$ we have four smooth strata $M_0, M_1, M_2, M_3$. The stratum $M_0$ is open and consists of sheaves having a presentation of the form
\[ 0 \rightarrow 4\mathcal{O}(-2) \xrightarrow{\varphi} 3\mathcal{O}(-1) \oplus \mathcal{O} \rightarrow \mathcal{F} \rightarrow 0, \]
where $\varphi_{11}$ is semi-stable as a Kronecker module. We denote by $M_{01} \subset M_0$ the locally closed subset given by the condition that the greatest common divisor of the maximal minors of $\varphi_{11}$, denoted $\zeta_1, \zeta_2, \zeta_3, \zeta_4$, is a linear form. Likewise, $M_{02}$ is the locally closed subset given by the condition that $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ have a common factor of degree 2. The complement $M_0 \setminus (M_{01} \cup M_{02})$ is given by the condition that $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ have no common factor. In this case the zero-set of $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ is a zero-dimensional subscheme $Z \subset \mathbb{P}^2$ of length 6 that is not contained in a conic curve. Thus, $M_0 \setminus (M_{01} \cup M_{02})$ consists of all sheaves of the form $\mathcal{O}_Q(Z_2)$, where $Q \subset \mathbb{P}^2$ is a quintic curve, $Z$ is a subscheme of $Q$ as above and $\mathcal{O}_Q(Z_2) \subset \mathcal{O}_Q$ is its ideal sheaf. Here $Q$ is defined by the equation $\det(\varphi) = 0$. For a one-dimensional sheaf $\mathcal{F}$ on $\mathbb{P}^2$ we use the notation $\mathcal{F}^0$ to denote the dual sheaf $\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbb{P}^2})$.

The stratum $M_1$ is locally closed and has codimension 2. It consists of those sheaves given by exact sequences of the form
\[ 0 \rightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \xrightarrow{\varphi} 2\mathcal{O} \rightarrow \mathcal{F} \rightarrow 0, \]
where $\varphi_{12}$ and $\varphi_{22}$ are linearly independent. Let $M_{10} \subset M_1$ be the open subset (in the relative topology) given by the condition that $\varphi_{12}$ and $\varphi_{22}$ have no common factor. Clearly, $M_{10}$ consists of sheaves of the form $\mathcal{O}_Q(-X)(2)$, where $Q \subset \mathbb{P}^2$ is a quintic curve, $X$ is the intersection of two conic curves without common component, $X$ is contained in $Q$ and $\mathcal{O}_Q(-X) \subset \mathcal{O}_Q$ is its ideal sheaf. The complement $M_{11} = M_1 \setminus M_{10}$ is given by the condition that $\varphi_{12} = \lambda_1, \varphi_{22} = \lambda_2$ for some linear forms $\lambda_1, \lambda_2$.

The stratum $M_2$ is locally closed of codimension 3. The points $[\mathcal{F}]$ in $M_2$ are given by exact sequences of the form
\[ 0 \rightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-1) \oplus 2\mathcal{O} \rightarrow \mathcal{F} \rightarrow 0, \]
where $\varphi_{23}$ has linearly independent entries, $\varphi_{13} = 0$, $\varphi_{12} \neq 0$, and $\varphi_{11}$ is not divisible by $\varphi_{12}$.

The deepest stratum $M_3$ is closed of codimension 5 and is isomorphic to the universal quintic curve. The sheaves $\mathcal{F}$ giving points in $M_3$ are cokernels of the form
\[ 0 \rightarrow 2\mathcal{O}(-3) \xrightarrow{\varphi} \mathcal{O}(-2) \oplus \mathcal{O}(1) \rightarrow \mathcal{F} \rightarrow 0, \]
where $\varphi_{11}$ and $\varphi_{12}$ are linearly independent. Equivalently, these are the sheaves of the form $\mathcal{O}_Q(-P)(1)^2$, where $P$ is a closed point on a quintic curve $Q$.

Let $W_i$ be the set of morphisms $\varphi$ as above such that $\text{Coker} (\varphi)$ gives a point in $M_i$. The ambient vector space $W_i$ of morphisms of sheaves is acted upon by the group of automorphisms $G_i$. Thus

$$W_1 = \text{Hom}(4\mathcal{O}(-2), 3\mathcal{O}(-1) \oplus \mathcal{O}),$$

$$G_1 = (\text{Aut}(4\mathcal{O}(-2)) \times \text{Aut}(3\mathcal{O}(-1) \oplus \mathcal{O})) / \mathbb{C}^*$$

etc. Clearly, $W_i$ is $G_i$-invariant. As shown in [19], the canonical maps $W_i \to M_i$ are geometric quotient maps. In particular, the strata $M_i$ are smooth.

4.1. **Fixed points in $M_0$.** Given a morphism $\psi: 4\mathcal{O}(-2) \to 3\mathcal{O}(-1)$ and a monomial $d$ of degree 5 belonging to the ideal generated by the maximal minors of $\psi$, we denote by $M(\psi, d)$ the image in $M_{22}(5,1)$ of the set of morphisms $\varphi \in W_0$ for which $\varphi_{11} = \psi$ and $\text{det} (\varphi) = d$. If $[\text{Coker}(\varphi)]$ is a $T$-fixed point in $M_0$, then, obviously, $\varphi_{11}$ gives a $T$-fixed point in $N(3,4,3)$. The torus-fixed points in $N(3,4,3)$ have been classified in Section 2.

Consider first the torus action on $M_0 \setminus (M_{01} \cup M_{02})$. Clearly, $\mathcal{O}_Q(-2)(1)^2$ gives a $T$-fixed point precisely if $Q$ and $Z$ are $T$-invariant. Up to a permutation of variables, there are seven schemes $Z$ corresponding to the matrices $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$, $\zeta$, and $\eta$ from Section 2. For each fixed $Z$ there are twelve invariant quintics $Q$ containing it. In the sequel, we will concentrate on finding the fixed points in $M_{01} \cup M_{02}$. Thus, $\varphi_{11}$ is equivalent to one of the matrices $\theta, \iota, \kappa, \lambda(a:b), \mu, \nu, \zeta$ from Section 2. Among these, $\lambda(1:-1)$ and $\zeta$ correspond to points in $M_{02}$, while the other matrices correspond to points in $M_{01}$. We will examine only the case when $\varphi_{11}$ is equivalent to $\theta$, the other cases being analogous. Write $d = \text{det}(\varphi) = X^4Y^2Z^2$ and fix quadratic forms $q_1, q_2, q_3, q_4$ satisfying the equation

$$[ q_1 \quad q_2 \quad q_3 \quad q_4 ] \Theta = d,$$

where

$$\Theta = [ \begin{array}{cccc} -X Y Z & X^2 Y & X Z & -XZ^2 \end{array} ],$$

is the column vector of maximal minors of $\theta$. The set $M(\theta, d)$ is parametrised by morphisms of the form

$$\begin{bmatrix} X & Y & 0 & 0 \\ X & 0 & Z & 0 \\ 0 & Z & 0 & X \\ q_1' & q_2' & q_3' & q_4' \end{bmatrix},$$

where

$$[ q_1' \quad q_2' \quad q_3' \quad q_4' ] \Theta = d.$$

From the relation

$$[ q_1' - q_1 \quad q_2' - q_2 \quad q_3' - q_3 \quad q_4' - q_4 ] \Theta = 0,$$

we deduce that the row matrix above is a linear combination (with polynomial coefficients) of the rows of the matrix

$$\begin{bmatrix} X & Y & 0 & 0 \\ X & 0 & Z & 0 \\ 0 & Z & 0 & X \\ Z & 0 & 0 & -Y \end{bmatrix}.$$
obtained by adjoining a row to \( \theta \). It follows that \( M(\theta, d) \) is parametrised by the set \( A \) of morphisms of the form

\[
\varphi(u) = \begin{bmatrix}
X & Y & 0 & 0 \\
X & 0 & Z & 0 \\
0 & Z & 0 & X \\
q_1 + uZ & q_2 & q_3 & q_4 - uY
\end{bmatrix}, \quad u = aY + bZ, \quad a, b \in \mathbb{C}.
\]

The canonical map \( A \to M(\psi, d) \) is an isomorphism, which can be seen using an argument similar to that of [3] Proposition 5.1]. The induced action of \( T \) on \( A \) is given by \((t, u) \mapsto t_0^{1-i}t_1^{-i}t_2^{-k}(tu)\). Choosing coordinates \((a, b)\) we identify \( A \) with \( \mathbb{A}^2 \). The induced action of \( T \) on \( \mathbb{A}^2 \) is given by

\[
t(a, b) = (t_0^{1-i}t_1^{-i}t_2^{-k}a, t_0^{1-i}t_1^{-i}t_2^{-k}b).
\]

We get an isolated fixed point, namely \((0, 0)\), unless \((i, j, k) = (1, 2, 2)\) or \((1, 1, 3)\), in which case we get an affine line of fixed points. Summarising, we obtain the following proposition.

**Proposition 4.1.1.** Assume that \( \psi \) gives a \( T \)-fixed point in \( N(3, 4, 3) \). Let \( \zeta_1, \zeta_2, \zeta_3, \zeta_4 \) be its maximal minors. Then, for any monomial \( \psi \) of degree 5 belonging to the ideal \((\zeta_1, \zeta_2, \zeta_3, \zeta_4)\), the set of fixed points for the action of \( T \) on \( M(\psi, \psi) \) has precisely one irreducible component, which is either a point or an affine line. We have a line in the following cases:

| \( \psi \) | \( d \) |
|---|---|
| \( \theta \) | \( XYZ^3, XY^2Z^2 \) |
| \( t \) | \( XY^3Z, XY^2Z^2 \) |
| \( k \) | \( XY^2Z^2 \) |
| \( \lambda(a : b) \) | \( X^2Y^2Z, X^2YZ^2 \) |
| \( \mu \) | \( XY^3Z, XY^2Z^2 \) |
| \( \nu \) | \( X^2Y^2Z, X^2YZ^2 \) |
| \( \xi \) | \( X^3YZ, XY^3Z, XYZ^3, XY^2Z^2, X^2YZ^2 \) |

Note that the torus fixed lines in \( M(\lambda(a : b), X^2Y^2Z) \) sweep a surface isomorphic to \( \mathbb{P}^1 \times \mathbb{A}^1 \). Thus, \( M^5 \) has six irreducible components of dimension 2 obtained from the surface \( \Sigma_0 \) with parametrisation

\[
\lambda((a : b), c) = \begin{bmatrix}
Y & X & 0 & 0 \\
Z & 0 & X & 0 \\
0 & aZ & bY & X \\
0 & (1-c)YZ & cY^2 & 0
\end{bmatrix}, \quad (a : b) \in \mathbb{P}^1, \quad c \in \mathbb{A}^1,
\]

by permutations of variables.

### 4.2. Fixed points in \( M_1 \)

Assume that the point in \( M_1 \) represented by the morphism

\[
\varphi = \begin{bmatrix}
f_1 & q_1 \\
f_2 & q_2
\end{bmatrix}
\]

is fixed by \( T \). Then, since the fibres of the map \( W_1 \to M_1 \) are the \( G_1 \)-orbits, we deduce that for each \( t \in T \) there is \((g(t), h(t)) \in G_1 \) such that \( t\varphi = h(t)\varphi g(t) \). By the argument at [20] Section 2.1.1, we deduce that \( q_1 \) and \( q_2 \) are distinct monomials of degree 2. Moreover, \( d = \det(\varphi) \) is a monomial of degree 5 that varies in the ideal \((q_1, q_2)\). We denote by \( M(q_1, q_2, d) \) the image in \( M_{\mathbb{P}^2}(5, 1) \) of the set
of morphisms $\varphi \in W_1$ for which $\varphi_{12} = q_1$, $\varphi_{22} = q_2$ and $\det(\varphi) = d = X^3Y^2Z^k$. If $q_1$ and $q_2$ have no common factor, then, $M(q_1, q_2, d)$ consists of a single $T$-fixed point of the form $O_{Q(-X^2)}$. Assume next that $q_1 = l_1$, $q_2 = l_2$ for some linear forms $l, l_1, l_2 \in \langle X, Y, Z \rangle$. Fix monomials $f_1$ and $f_2$ such that $f_1q_2 - f_2q_1 = d$. It is easy to see that $M(\{l_1, l_2, d\})$ is parametrised by the set $A$ of morphisms of the form

$$\begin{bmatrix} f_1 + q_1 l_1 \\ f_2 + q_2 l_2 \end{bmatrix}$$

with $q$ a quadratic form in the two variables different from $l$. Up to a permutation of variables, there are only two cases to be considered: $(q_1, q_2) = (X^2, XY)$ or $(XZ, YZ)$. In the first case

$$\varphi = \begin{bmatrix} f_1 + qX \\ f_2 + qY \end{bmatrix}, \quad q = aY^2 + bZ^2 + cYZ.$$  

Choosing coordinates $(a, b, c)$, we identify $A$ with $\mathbb{A}^3$. We have

$$t\varphi = \begin{bmatrix} t_0^{-1}t_1^{-3}t_2^{-1}f_1 + (tq)t_0X \\ t_0^{-1}t_1^{-1}t_2^{-2}f_2 + (tq)t_1Y \end{bmatrix} \sim \begin{bmatrix} f_1 + t_0^{-2}t_1^{-1}t_2^{-1}k(tq)X \\ f_2 + t_0^{-2}t_1^{-1}t_2^{-1}k(tq)Y \end{bmatrix}.$$  

The induced action of $T$ on $A \cong \mathbb{A}^3$ is given by

$$t(a, b, c) = (t_0^{-3}t_1^{-2}t_2^{-1}a, t_0^{-1}t_1^{-1}t_2^{-2}b, t_0^{-1}t_1^{-1}t_2^{-2}c).$$  

We get an isolated fixed point unless $(i, j, k) = (2, 3, 0)$, or $(2, 1, 2)$, or $(2, 2, 1)$, in which case we get an affine line of fixed points. Assume now that

$$\varphi = \begin{bmatrix} f_1 + qX \\ f_2 + qY \end{bmatrix}, \quad q = aX^2 + bY^2 + cXY.$$  

We have

$$t\varphi = \begin{bmatrix} t_0^{-1}t_1^{-1}t_2^{-1}f_1 + (tq)t_0X \\ t_0^{-1}t_1^{-1}t_2^{-1}f_2 + (tq)t_1Y \end{bmatrix} \sim \begin{bmatrix} f_1 + t_0^{-1}t_1^{-1}t_2^{-1}k(tq)X \\ f_2 + t_0^{-1}t_1^{-1}t_2^{-1}k(tq)Y \end{bmatrix}.$$  

The induced action of $T$ on $A \cong \mathbb{A}^3$ is given by

$$t(a, b, c) = (t_0^{-1}t_1^{-1}t_2^{-1}a, t_0^{-1}t_1^{-1}t_2^{-2}b, t_0^{-1}t_1^{-1}t_2^{-2}c).$$  

We get an isolated fixed point unless $(i, j, k) = (3, 1, 1)$, or $(1, 3, 1)$, or $(2, 2, 1)$, in which case we get an affine line of fixed points. Summarising, we obtain the following proposition.

**Proposition 4.2.1.** Let $q_1$ and $q_2$ be distinct monomials of degree 2. Let $d$ be a monomial of degree 5 in the ideal $(q_1, q_2)$. Then the set of fixed points for the action of $T$ on $M(q_1, q_2, d)$ has precisely one irreducible component, which is either a point or an affine line.

4.3. **Fixed points in $M_2$.** Assume that the point in $M_{p,2}(5,1)$ represented by the morphism

$$\varphi = \begin{bmatrix} q \\ f_1 \\ f_2 \end{bmatrix}$$

is fixed by $T$. Here $\varphi$ satisfies the conditions from the beginning of this section, namely: $\det(\varphi) \neq 0$, $l \neq 0$, $l$ does not divide $q$, and $l_1, l_2$ are linearly independent. For each $t \in T$, there is $(g(t), h(t)) \in \mathbb{G}_2$ such that $t\varphi = h(t)\varphi g(t)$. As at [20]
Section 2.2], it is easy to see that \( l_1, l_2 \) are distinct monomials, and that \( l, q \) are monomials. Thus, we may write

\[
g(t) = \begin{bmatrix} h_{11}^{-1}(tl)/q & 0 & 0 \\ 0 & h_{11}^{-1}(tl)/l_1 & 0 \\ u_1 & u_2 & 1 \end{bmatrix}, \quad h(t) = \begin{bmatrix} h_{11} & 0 & 0 \\ v_1 (tl_1)/l_1 & 0 & 0 \\ v_2 & 0 & (tl_2)/l_2 \end{bmatrix}.
\]

Permuting, if necessary, rows two and three of \( \varphi \), we may assume that \( l \) is not a multiple of \( l_1 \). Moreover, we may assume that \( f_1, q_1 \) do not contain any monomial divisible by \( l_1 \), and that \( q_1, q_2 \) do not contain any monomial divisible by \( l \). From the relation

\[
tq_1 = v_1 h_{11}^{-1}(tl) + \frac{tl_1}{l_1} h_{11}^{-1}(tl_1) q_1 + u_2(tl_1)
\]

we obtain the relations

\[
(1) \quad tq_1 = \frac{tl_1}{l_1} h_{11}^{-1}(tl) q_1, \quad v_1 = -a(tl_1), \quad u_2 = ah_{11}^{-1}(tl) \quad \text{for some } a \in \mathbb{C}.
\]

From the relation

\[
ft_1 = v_1 h_{11}^{-1}(tl) + \frac{tl_1}{l_1} h_{11}^{-1}(tl_1) f_1 + u_1(tl_1)
\]

we obtain the relations

\[
(2) \quad tf_1 = \frac{tl_1}{l_1} h_{11}^{-1}(tl) f_1, \quad u_1 = ah_{11}^{-1}(tl).
\]

From the relation

\[
tq_2 = v_2 h_{11}^{-1}(tl) + \frac{tl_2}{l_2} h_{11}^{-1}(tl_2) q_2 + u_2(tl_2)
\]

we obtain the relations

\[
(3) \quad tq_2 = \frac{tl_2}{l_2} h_{11}^{-1}(tl) q_2, \quad v_2 = -a(tl_2).
\]

Finally, we have the relation

\[
ft_2 = v_2 h_{11}^{-1}(tl) + \frac{tl_2}{l_2} h_{11}^{-1}(tl_2) f_2 + u_1(tl_2).
\]

Substituting the values for \( u_1 \) and \( v_2 \) found above yields the relation

\[
(4) \quad tf_2 = \frac{tl_2}{l_2} h_{11}^{-1}(tl) f_2.
\]

From relations (1)–(4) we deduce that \( q_1, q_2, f_1, f_2 \) are monomials. In fact, \( h_{11}^{-1}(t) = t_i^j t_{1k}^l t_{2k}^k \) for some integers \( i, j, k \) satisfying the equation \( i + j + k = 0 \) and

\[
\begin{bmatrix} f_1 & q_1 \\ f_2 & q_2 \end{bmatrix} = \begin{bmatrix} c_{11} X^i Y^j Z^k q_1 l_1 \\ c_{12} X^i Y^j Z^k q_1 l_2 \\ c_{21} X^i Y^j Z^k q_2 l_1 \\ c_{22} X^i Y^j Z^k q_2 l_2 \end{bmatrix},
\]

where \( c_{rs} = 0 \) if the corresponding monomial has negative exponents.

Given \( l_1, l_2, l, q \) as above and a monomial \( d \) of degree 5, we denote by \( M(l_1, l_2, l, q) \) the image in \( M_{p_2}(5, l) \) of the set of morphisms \( \varphi \in W_2 \) for which \( \varphi_{11} = q, \varphi_{12} = l, \) and \( \varphi_{23} \) has entries \( l_1, l_2 \). We denote by \( M(l_1, l_2, l, q, d) \) the subset given by the additional condition \( \det(\varphi) = d \).
Proposition 4.3.1. Assume that \( l_1, l_2, l \) belong to \( \{X, Y, Z\} \), \( l_1 \neq l_2 \), \( q \) belongs to \( \{X^2, Y^2, Z^2, XY, XZ, YZ\} \), and \( l \) does not divide \( q \). Then, for any monomial \( d \) of degree 5 belonging to the ideal \((l_1, l_2, q, l_1, q_1, l_2)\), the set of fixed points for the action of \( T \) on \( M(l_1, l_2, l, q, d) \) has precisely one irreducible component, which is either a point or an affine line.

Proof. We will only examine the case when \( l_1 = X, l_2 = Y, l = Y, q = XZ \), all other cases being analogous. Consider, therefore, a morphism of the form

\[
\varphi = \begin{bmatrix}
XZ & Y & 0 \\
0 & Z^2 & X \\
cZ^2 & 0 & Y
\end{bmatrix}, \quad c \in \mathbb{C} \setminus \{-1\}.
\]

Assume now that \( c_{12} = 0 \) and \( c_{11} \neq 0 \). Then \( i = -2 \), hence \( c_{21} = 0, c_{22} = 0 \), and we obtain the fixed points

\[
\varphi_1(c) = \begin{bmatrix}
XZ & Y & 0 \\
f & 0 & X \\
0 & 0 & Y
\end{bmatrix}, \quad f \in \{Y^3, Y^2Z, YZ^2, Z^3\}.
\]

Assume next that \( c_{11} = 0, c_{12} = 0, c_{22} \neq 0 \). Then \( j = -2 \), hence \( c_{21} = 0 \), and we obtain the fixed points

\[
\varphi_1(c) = \begin{bmatrix}
XZ & Y & 0 \\
0 & 0 & X \\
0 & q & Y
\end{bmatrix}, \quad q \in \{X^2, XZ, Z^2\}.
\]

In the final case to examine, when \( \varphi_1(c) = \varphi_2(c) = \varphi_3(c) = 0 \), we obtain the fixed points

\[
\varphi_1(c) = \begin{bmatrix}
XZ & Y & 0 \\
0 & 0 & X \\
f & 0 & Y
\end{bmatrix},
\]

where \( f \) is any monomial of degree 3. For \( f = Z^3 \) we obtain a point in the moduli space, which we denote by \( \varphi_1(\infty) \). In conclusion, for the action of \( T \) on \( M(X, Y, Z, XZ) \), we have sixteen fixed isolated points and an affine line of fixed points, namely \( \{\varphi_1(c) \mid c \in \mathbb{P}^1 \setminus \{-1\}\} \). \( \square \)

4.4. Fixed points in \( M_3 \). Clearly, \( \mathcal{O}_Q(-P)(1)^0 \) gives a T-fixed point in \( M_{p^2}(5, 1) \) if and only if \( Q \) is T-invariant and \( P \) is T-fixed. Thus, the torus fixed points in \( M_3 \) are given by morphisms \( \varphi \in W_3 \) that are represented by matrices with entries monomials.

5. The torus representation of the tangent spaces at the fixed points of \( M_{p^2}(5, 1) \)

In the sequel, for each fixed point in \( M_k \) we will find a representative \( \varphi \in W_k \) for which there exist diagonal matrices \( u(t), \nu(t) \) with entries characters \( u_i(t), \nu_j(t) \).
of \((\mathbb{C}^*)^3\), such that \(t\varphi = v(t)\varphi u(t)\) for all \(t \in (\mathbb{C}^*)^3\). This allows us to apply the method of Section 3.3: the action of \(T\) on \(T\varphi W_k\) is given by (3.3.2), the action of \(T\) on \(T\varphi (G_k\varphi)\) is given by (3.3.2) and, as \(T\)-modules, the quotient \(T\varphi W_k / T\varphi (G_k\varphi)\) is isomorphic to \(T_{[\varphi]} M_{P^2} (5,1)\).

**Notations.**

\[
S^1 = \{X^iY^jZ^k \mid i,j,k \in \mathbb{Z}, \ i,j,k \geq 0, \ i+j+k = 1\}, \text{ where } l \geq 1;
\]
\[
\{x,y,z\} = \text{ the standard basis for the lattice of characters of } (\mathbb{C}^*)^3;
\]
\[
\chi_0 = \text{ the trivial character of } T;
\]
\[
s^l = \{ix+jy+zk \mid i,j,k \in \mathbb{Z}, \ i,j,k \geq 0, \ i+j+k = l\}, \text{ where } l \geq 1.
\]

In this section we will use additive notation to denote characters of \(T\) or of \((\mathbb{C}^*)^3\). Thus,
\[
\chi^*(T) = \{ix+jy+zk \mid i,j,k \in \mathbb{Z}, \ i+j+k = 0\}.
\]

We also adopt the following convention: whenever a monomial \(X^iY^jZ^k\) appears in a list of characters, it stands for the expression \(ix+jy+zk\).

5.1. **Fixed points in \(M_0\)**. The action of \(T\) on \(T\varphi W_0\) is given by (3.3.1). Let \(i, j, k\) be non-negative integers. Let \(w_{mn}^{ijk}\) denote the matrix having entry \(X^iY^jZ^k\) on position \((m,n)\) and entries zero everywhere else. Viewed as a tangent vector, \(w_{mn}^{ijk}\) is acted by \(T\) with weight \(-v_m - u_n + ix + jy + kz\). Now \(T\varphi W_0\) is the space of 4 \(X\) 4-matrices with entries linear forms on the first three rows and quadratic forms on the fourth row. It follows that the set
\[
\{w_{mn}^{ijk} \mid m = 1,2,3, n = 1,2,3,4, i+j+k = 1\} \cup \{w_{4n}^{ijk} \mid n = 1,2,3,4, i+j+k = 2\}
\]
forms a basis of the tangent space. The corresponding list of weights is organised in the following array:

\[
\begin{align*}
-v_1 - u_1 + s^1 & \quad -v_1 - u_2 + s^1 & \quad -v_1 - u_3 + s^1 & \quad -v_1 - u_4 + s^1 \\
-v_2 - u_1 + s^1 & \quad -v_2 - u_2 + s^1 & \quad -v_2 - u_3 + s^1 & \quad -v_2 - u_4 + s^1 \\
-v_3 - u_1 + s^1 & \quad -v_3 - u_2 + s^1 & \quad -v_3 - u_3 + s^1 & \quad -v_3 - u_4 + s^1 \\
-v_4 - u_1 + s^2 & \quad -v_4 - u_2 + s^2 & \quad -v_4 - u_3 + s^2 & \quad -v_4 - u_4 + s^2
\end{align*}
\]

The group acting by conjugation on \(W_0\) is
\[
G_0 = \langle \text{Aut}(4\mathcal{O}(−2)) \times \text{Aut}(3\mathcal{O}(−1) \oplus \mathcal{O}) \rangle / \mathbb{C}^*.
\]

Let \(e\) denote its neutral element. We identify canonically \(T\varphi (G_0\varphi)\) with the space
\[
T_e G_0 = \langle \text{End}(4\mathcal{O}(−2)) \oplus \text{End}(3\mathcal{O}(−1) \oplus \mathcal{O}) \rangle / \mathbb{C},
\]

which is represented by pairs \((A,B)\) of matrices with polynomial entries. The \(T\)-action on this tangent space is given by (3.3.2). Let \(A_{mn}^{ijk}\) and \(B_{mn}^{ijk}\) be the matrices defined in the same way as \(w_{mn}^{ijk}\). Now \(T\) acts on the tangent vector \((A_{mn}^{ijk}, 0)\) with weight \(u_m - u_n + ix + jy + kz\) and on \((0, B_{mn}^{ijk})\) with weight \(-v_m + v_n + ix + jy + kz\).

The subspace of diagonal matrices \(\{(cI, cI), c \in \mathbb{C}\}\) that we quotient out is acted upon trivially. A basis for \(\text{End}(4\mathcal{O}(−2))\) is given by
\[
\{A_{mn}^{000} \mid 1 \leq m,n \leq 4\}.
\]

Also,
\[
\{B_{mn}^{000} \mid 1 \leq m,n \leq 3\} \cup \{B_{4n}^{ijk} \mid 1 \leq n \leq 3, i+j+k = 1\} \cup \{B_{44}^{000}\}
\]
forms a basis of End(3O(−1) ⊕ O). To get the list of weights for T_e G_0 we add the two lists for the two bases above and we subtract {χ_0}. The result is expressed in the following tableau:

| \( \chi_0 \) | \( u_1 - u_2 \) | \( u_1 - u_3 \) | \( u_1 - u_4 \) |
| \( u_2 - u_1 \) | \( u_2 - u_3 \) | \( u_2 - u_4 \) |
| \( u_3 - u_1 \) | \( u_3 - u_2 \) | \( \chi_0 \) | \( u_3 - u_4 \) |
| \( u_4 - u_1 \) | \( u_4 - u_2 \) | \( u_4 - u_3 \) | \( \chi_0 \) |
| \( -v_1 + v_2 \) | \( -v_1 + v_3 \) |
| \( -v_2 + v_1 \) | \( \chi_0 \) | \( -v_2 + v_3 \) |
| \( -v_3 + v_1 \) | \( -v_3 + v_2 \) | \( \chi_0 \) |
| \( -v_4 + v_1 + s^1 \) | \( -v_4 + v_2 + s^1 \) | \( -v_4 + v_3 + s^1 \) |

According to Section 4.11 the fixed points in M_0 are represented by matrices \( \varphi \), where, modulo a permutation of variables, \( \varphi_{11} \) is one among the matrices \( \alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta, \iota, \kappa, \lambda, \mu, \nu, \xi \) from Section 2. Let \( \zeta_i \), \( 1 \leq i \leq 4 \), be the maximal minor of \( \varphi_{11} \) obtained by deleting column \( i \). If \( \varphi_{11} \neq \lambda(1 : -1) \) and \( \varphi_{11} \neq \xi \), then all \( \zeta_i \) are non-zero, and we may assume that \( \varphi \) has the form

\[
\varphi_{11}(d) = \begin{bmatrix}
    c_1 d / \zeta_1 & c_2 d / \zeta_2 & c_3 d / \zeta_3 & c_4 d / \zeta_4 \\
    \varphi_{11}
\end{bmatrix},
\]

where \( d \) is a monomial of degree 5 belonging to the ideal \( (\zeta_1, \zeta_2, \zeta_3, \zeta_4) \) and \( c_i = 0 \) if \( d \) is not divisible by \( \zeta_i \). If \( \varphi_{11} = \lambda(1 : -1) \) or \( \varphi_{11} = \xi \), then \( \zeta_4 = 0 \) and the other \( \zeta_i \) are non-zero. However, the torus representation of the tangent space at a point does not change when the point varies in a connected component of \( M_0(\mathbb{C}, 1) \). For this reason, we may ignore the case when \( \varphi_{11} = \lambda(1 : -1) \), in fact, we may restrict to the case when \( \varphi_{11} = \lambda \). Without proof, we claim that, for each feasible \( d \), there is a point in \( M(\xi, d) \) (notation as at Section 4.11) represented by a matrix of the form

\[
\xi(d) = \begin{bmatrix}
    c_1 d / \zeta_1 & c_2 d / \zeta_2 & c_3 d / \zeta_3 & c_4 d / \zeta_4 & 0
\end{bmatrix}.
\]

In conclusion, we will only consider matrices of the form \( \psi(d) \), where

\[
\psi \in \{ \alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta, \iota, \kappa, \lambda, \mu, \nu, \xi \}.
\]

The values of \( v_1, v_2, v_3, u_1, u_2, u_3, u_4 \) for \( \psi(d) \) are the same as the values for \( \psi \), as given in Table 1, Section 3.3. On a case-by-case basis we can show that the equation

\[
d - \zeta_1 - u_1 = d - \zeta_2 - u_2 = d - \zeta_3 - u_3 = d - \zeta_4 - u_4
\]

of characters (there being no last term if \( \psi = \xi \)) holds. This is the character \( v_4 \) of \( \psi(d) \). Its values are given in Table 4 below.

| Fixed point | \( v_4 \) |
|-------------|-------------|
| \( \alpha(d) \) | \( d - x - y - z \) |
| \( \gamma(d) \) | \( d - x - y - z \) |
| \( \epsilon(d) \) | \( d - 2x - y \) |
| \( \eta(d) \) | \( d - x - 2y \) |
| \( \iota(d) \) | \( d - x - 2z \) |
| \( \lambda(d) \) | \( d - 3x \) |
| \( \nu(d) \) | \( d - x - 2y \) |

| Fixed point | \( v_4 \) |
|-------------|-------------|
| \( \beta(d) \) | \( d - x - y - z \) |
| \( \delta(d) \) | \( d - 2x - z \) |
| \( \zeta(d) \) | \( d - 2x - y \) |
| \( \theta(d) \) | \( d - x - y - z \) |
| \( \kappa(d) \) | \( d - x - 2y \) |
| \( \mu(d) \) | \( d - 2x - y \) |
| \( \xi(d) \) | \( d - 2x - y \) |
5.2. Fixed points in $M_1$. By analogy with Section 5.1 the list of weights for the action of $T$ on $T_\varphi W_1$ is expressed by the table
\[
\begin{array}{ccc}
-v_1 - u_1 + s^3 & -v_1 - u_2 + s^2 \\
-v_2 - u_1 + s^3 & -v_2 - u_2 + s^2 \\
\end{array}
\]
and the list of weights for the action of $T$ on $T_\varphi(G_1\varphi)$ is represented by the array
\[
\begin{array}{ccc}
X_0 & X_0 & -v_1 + v_2 \\
u_2 - u_1 + s^3 & -v_2 + v_1 & X_0 \\
\end{array}
\]
Let $\mathcal{F} = \text{Coker}(\varphi)$. By analogy with [6, Proposition 6.2], we have the following description of the torus action on the normal space at $[\mathcal{F}]$.

**Proposition 5.2.1.** The normal space $N_{[\mathcal{F}]}$ to $M_1$ at $[\mathcal{F}]$ can be identified with
\[
H^0(\mathcal{F})^* \otimes H^1(\mathcal{F}).
\]
The torus acts on $N_{[\mathcal{F}]}$ with weights $u_1 + v_1 - x - y - z$ and $u_1 + v_2 - x - y - z$.

Recall from Proposition 4.2.1 that an irreducible component of $M_1^T$ is uniquely determined by $q_1$, $q_2$ and $d$. Thus, $\varphi$ has the form
\[
o(q_1, q_2, d) = \begin{bmatrix}
c_1 d/q_2 & q_1 \\
c_2 d/q_1 & q_2
\end{bmatrix}.
\]
The characters $u_1$, $u_2$, $v_1$, $v_2$ have to be chosen such that (using additive notation)
\[
\begin{align*}
v_1 + u_1 &= d - q_2, & v_1 + u_2 &= q_1, \\
v_2 + u_1 &= d - q_1, & v_2 + u_2 &= q_2.
\end{align*}
\]
Clearly, we may choose
\[
u_1 = d - q_1 - q_2, \quad u_2 = 0, \quad v_1 = q_1, \quad v_2 = q_2.
\]
With the aid of Proposition 5.2.1 we can determine which points in $M_1^T$ are accumulation points of lines or surfaces in $M_0^T$. Indeed, $[\mathcal{F}]$ lies in the closure of a positive-dimensional component of $M_0^T$ if and only if at least one among the weights
\[
\begin{align*}
u_1 + v_1 - x - y - z &= d - q_2 - x - y - z, \\
u_1 + v_2 - x - y - z &= d - q_1 - x - y - z
\end{align*}
\]
equals $X_0$. As $q_1 \neq q_2$, both weights cannot equal $X_0$ at the same time. Thus, there are two possible situations. If $[\mathcal{F}]$ is an isolated point for the action of $T$ on $M_1$ and one among the above weights is $X_0$, then $[\mathcal{F}]$ is the limit point of a line in $M_0^T$. If $[\mathcal{F}]$ belongs to a line of $M_1^T$ and one of the above weights is $X_0$, then $[\mathcal{F}]$ is the limit point of a surface in $M_0^T$. The first column of Table 5 below lists the pairs $(q_1, q_2)$ up to a permutation of variables. The second column contains the monomials $d = X_1^i Y^j Z^k$ of degree 5 that are in the ideal generated by $q_1$ and $q_2$. The third column contains the values of $d$, found in Section 4.2, for which $M(q_1, q_2, d)^T$ is a line. The fourth column lists the values of $d$ for which $o(q_1, q_2, d)$ is the limit point of a line in $M_0^T$. The last column lists the values of $d$ for which $o(q_1, q_2, d)$ is the limit point of a surface in $M_0^T$. 


5.3. Fixed points in \( M_2 \). Recall from Proposition 4.3.1 that an irreducible component of \( M_2 \) is uniquely determined by \( l_1, l_2, l, q \) and \( d \). Thus, \( \varphi \) has the form

\[
\pi(l_1, l_2, l, q, d) = \begin{pmatrix}
q & 1 & 0 \\
c_{11}d/ll_2 & c_{12}d/qll_2 & l_1 \\
c_{21}d/ll_1 & c_{22}d/qll_2 & l_2
\end{pmatrix}.
\]

The characters \( u_1, u_2, u_3, v_1, v_2, v_3 \) have to be chosen such that (adopting additive notation)

\[
\begin{align*}
v_1 + u_1 &= q, & v_1 + u_2 &= l, \\
v_2 + u_1 &= d - l - l_2, & v_2 + u_2 &= d - q - l_2, & v_2 + u_3 &= l_1, \\
v_3 + u_1 &= d - l - l_1, & v_3 + u_2 &= d - q - l_1, & v_3 + u_3 &= l_2.
\end{align*}
\]

Clearly, we may choose

\[
\begin{align*}
u_1 &= -d + q + l + l_1 + l_2, \\
u_2 &= d - q - l_1 - l_2, \\
u_3 &= d - l - l_1 - l_2, \\
u_1 &= -d + q + l + l_1 + l_2, \\
u_2 &= d - q - l_1 - l_2, \\
u_3 &= d - l - l_1 - l_2.
\end{align*}
\]

By analogy with Section 5.1, the list of weights for the action of \( T \) on \( T_\varphi W_2 \) is represented by the tableau

\[
\begin{align*}
-v_1 - u_1 + s^2 & -v_1 - u_2 + s^1 \\
-v_2 - u_1 + s^3 & -v_2 - u_2 + s^2 & -v_2 - u_3 + s^1 \\
-v_3 - u_1 + s^3 & -v_3 - u_2 + s^2 & -v_3 - u_3 + s^1
\end{align*}
\]

Observe that \( \varphi \) has a stabiliser of dimension one consisting of matrices of the form

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
cq & cl_1 & cl_2
\end{pmatrix},
\]

where \( c \in \mathbb{C} \). Thus, \( T_c \text{Stab}(\varphi) \) is spanned by the tangent vector

\[
s = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
q & l_1 & 0
\end{pmatrix},
\]

Table 5. Fixed points \( o(q_1, q_2, d) \) in \( M_1 \).

| \( [q_1, q_2] \) | \( d \) | Affine lines | Limit points of lines | Limit points of surfaces |
|-----------------|--------|----------------|------------------------|-----------------------------|
| \( (X^2, Y^2) \) | \( S^5 \setminus \{Z^5, Z^4X, Z^4Y, Z^3XY\} \) | \( X^3YZ \) | \( XY^3Z \) |
| \( (X^2, YZ) \) | \( S^5 \setminus \{Y^5, XY^4, Z^5, XZ^4\} \) | \( X^3YZ \) | \( XY^2Z^2 \) |
| \( (X^2, XY) \) | \( S^5 \setminus \{Z^5, XZ^4, YZ^4, Y^2Z^3, \} \) | \( X^2Y^3 \) | \( X^3YZ \) | \( X^2Y^2Z \) |
| \( (XZ, YZ) \) | \( S^5 \setminus \{X^5, X^4Y, X^3Y^2, \} \) | \( X^3YZ \) | \( XY^2Z \) | \( XY^2Z^2 \) |
Let $s_1$ be obtained by setting $l = l_1 = l_2 = 0$ in the above expression, let $s_2$ be obtained by setting $q = l_1 = l_2 = 0$, let $s_3$ be obtained by setting $q = 1 = l_2 = 0$, and let $s_4$ be obtained by setting $q = 1 = l_1 = 0$. The torus acts on $s_1$, $s_2$, $s_3$, $s_4$ with weights

$$u_3 - u_1 + q, \quad u_3 - u_2 + l, \quad v_1 - v_2 + l, \quad v_1 - v_3 + l.$$ 

Substituting the values for $u_1$, $u_2$, $v_1$, $v_2$ from above we get the same expression in all four cases, namely

$$-d + q + l + l_1 + l_2.$$

Since $s = s_1 + s_2 + s_3 + s_4$, this is the weight for the action of $T$ on $T_e \operatorname{Stab}(\varphi)$. To get the list of weights for the action of $T$ on $T_\varphi(G_2 \varphi)$ we need to subtract this weight from the list

$$\begin{align*}
\chi_0, 
\chi_0 - u_1 - s^1, 
\chi_0 - u_2 - s^1, 
\chi_0 - v_2 - v_3, 
\chi_0 - v_3 + v_2.
\end{align*}$$

**Proposition 5.3.1.** Let $\mathcal{F} = \operatorname{Coker}(\varphi)$. Let $N_{[\mathcal{F}]}$ be the normal space to $M_2$ at $[\mathcal{F}]$. The torus $T$ acts on $N_{[\mathcal{F}]}$ with weights

$$u_1 + v_2 - x - y - z, \quad u_3 - x - y - z, \quad -v_1 - u_3.$$

**Proof.** By analogy with [II Theorem 4.3.3], we can show that $M_1 \cup M_2$ is a locally closed smooth subvariety of $M_{G_2}([5,1])$ of codimension 2 that is the geometric quotient of an open subset $W \subset W_2$ modulo $G_2$. Moreover, $M_2$ has codimension 1 in $M_1 \cup M_2$. In fact, $W$ is the subset of injective morphisms that have semi-stable cokernel. The table of weights for the action of $T$ on $T_\varphi W$ is the same as the table for $T_\varphi W_2$ except that it contains the weight $-v_1 - u_3$ in the upper-right corner, accounting for the normal direction to $M_2$ inside $M_1 \cup M_2$. The other two weights account for the two normal directions to $M_1 \cup M_2$, as in Proposition 5.2.1. \qed

In view of the above proposition, $[\mathcal{F}]$ lies in the closure of a positive-dimensional component of $(M_0 \cup M_1)^T$ if and only if at least one among the weights

$$u_1 + v_2 - x - y - z = d - l - l_1 - x - y - z, 
\quad u_1 + u_3 - x - y - z = d - l - l_1 - x - y - z, 
\quad -v_1 - u_3 = d - q - l - l_1 - l_2$$

equals $\chi_0$. In Table 6 below, which is organised as Table 5, we have the information regarding the fixed points in $M_2$. We assume that $l_1 = X$, $l_2 = Y$, the other cases being obtained by a permutation of variables.

### 5.4 Fixed points in $M_3$.

By analogy with Section 5.1 the list of weights for the action of $T$ on $T_\varphi W_3$ reads

$$-v_1 - u_1 + s^1, \quad -v_1 - u_2 + s^1, \quad -v_2 - u_1 + s^4, \quad -v_2 - u_2 + s^4$$

and the list of weights for the action of $T$ on $T_\varphi(G_3 \varphi)$ is expressed in the tableau

$$\begin{array}{ccc}
\chi_0 & u_1 - u_2 & \chi_0 \\
u_2 - u_1 & \chi_0 & -v_2 + v_1 + s^3
\end{array}$$
Recall from Section 4.4 that the action of $T$ on $M_3$ has only isolated fixed points, given by morphisms of the form

$$\rho(l_1, l_2, d) = \left[ \begin{array}{cc} l_1 & l_2 \\ c_1 d/l_2 & c_2 d/l_1 \end{array} \right].$$

The characters $u_1, u_2, v_1, v_2$ have to be chosen such that (using additive notation)

$$v_1 + u_1 = l_1, \quad v_1 + u_2 = l_2, \quad v_2 + u_1 = d - l_2, \quad v_2 + u_2 = d - l_1.$$

Clearly, we may choose $u_1 = l_1, u_2 = l_2, v_1 = 0, v_2 = d - l_1 - l_2$.

**Proposition 5.4.1.** Let $F = \text{Coker}(\varphi)$. Let $N_{[F]}$ be the normal space to $M_3$ at $[F]$. Then we have a canonical isomorphism

$$N_{[F]} \cong H^0(F(-1))^* \otimes H^1(F(-1)).$$

Denote $\{l_3\} = \{X, Y, Z\} \setminus \{l_1, l_2\}$. Then the torus $T$ acts on $N_{[F]}$ with weights

- $d - l_1 - 3l_2 - l_3$,
- $d - 3l_1 - l_2 - l_3$,
- $d - l_1 - 2l_2 - 2l_3$,
- $d - 2l_1 - l_2 - 2l_3$.

**Proof.** Denote $G = F^\circ(1)$. According to [18 Lemma 3], dualising the resolution for $F$ yields the resolution

$$0 \rightarrow \mathcal{O}(-3) \oplus \mathcal{O} \xrightarrow{\psi} 2\mathcal{O}(1) \rightarrow G \rightarrow 0,$$

where

$$\psi = \left[ \begin{array}{cc} c_1 d/l_2 & l_1 \\ c_2 d/l_1 & l_2 \end{array} \right].$$
This allows us to use the argument at \[20\] Proposition 6.2. Applying the Ext\(_1(\mathcal{O}, \mathcal{O})\) functor to the canonical morphism \(H^0(\mathcal{G}) \otimes \mathcal{O} \to \mathcal{G}\) yields the linear map

\[
e: \text{Ext}^1(\mathcal{G}, \mathcal{G}) \to H^0(\mathcal{G})^* \otimes H^1(\mathcal{G}).
\]

Its kernel is a subspace of \(T_{[\mathcal{G}]} M^0_3 \subset T_{[\mathcal{G}]} M_{p^2}(5,4)\). Recall that \(T_{[\mathcal{G}]} M^0_3\) has codimension 5 in \(\text{Ext}^1(\mathcal{G}, \mathcal{G})\). Since \(\dim(H^0(\mathcal{G})^* \otimes H^1(\mathcal{G})) = 5\), we deduce that \(e\) is surjective and that \(\text{Ker}(e) = T_{[\mathcal{G}]} M^0_3\). Thus, we obtain a canonical isomorphism

\[
N_{[\mathcal{F}^0(1)]} \simeq H^0(\mathcal{F}^0(1))^* \otimes H^1(\mathcal{F}^0(1)).
\]

Using Serre Duality, as at \[20\] Proposition 3.3.1, yields the canonical isomorphism from the proposition. We have identifications

\[
H^0(\mathcal{G}) = H^0(2\mathcal{O}(1))/H^0(\mathcal{O}) = (V^* \oplus V^*)/\mathbb{C}(l_1, l_2).
\]

The vectors \((X,0), (Y,0), (Z,0), (0,X), (0,Y), (0,Z)\) form a basis of \(V^* \oplus V^*\). The calculations at \[3\] Proposition 6.2 show that these are eigenvectors for the action of \((\mathbb{C}^*)^3\), corresponding to the weights

\[
-u_1 + x, \quad -u_1 + y, \quad -u_1 + z, \quad -u_2 + x, \quad -u_2 + y, \quad -u_2 + z.
\]

The vector \((l_1, l_2)\) is acted on trivially. Moreover, \((\mathbb{C}^*)^3\) acts on \(H^1(\mathcal{G})\) with weight \(v_2 - x - y - z\). It follows that the list of weights for the action of \(T\) on \(N_{[\mathcal{G}]}\) is obtained by subtracting the weight

\[
v_2 - x - y - z = d - 2l_1 - 2l_2 - l_3
\]

from the list

\[
\begin{align*}
u_1 + v_2 &- 2x - y - z, \\
u_1 + v_2 &- x - 2y - z, \\
u_1 + v_2 &- x - y - 2z, \\
u_2 + v_2 &- 2x - y - z, \\
u_2 + v_2 &- x - 2y - z, \\
u_2 + v_2 &- x - y - 2z,
\end{align*}
\]

which is the same as the list

\[
\begin{align*}
d - 2l_1 &- 2l_2 - l_3, \\
d - l_1 &- 3l_2 - l_3, \\
d - 1l_1 &- 2l_2 - 2l_3, \\
d - 3l_1 &- l_2 - l_3, \\
d - 2l_1 &- 2l_2 - l_3, \\
d - 2l_1 &- l_2 - 2l_3.
\end{align*}
\]

In view of the fact that \(N_{[\mathcal{G}]}\) and \(N_{[\mathcal{F}]}\) are isomorphic as \(T\)-modules, this proves the proposition. \(\square\)

The weights for the action of \(T\) on \(N_{[\mathcal{F}]}\) are distinct, so at most one of them can be \(\chi_0\). This shows that no point of \(M^0_3\) lies in a two-dimensional component of \(M_{p^2}(5,1)^T\). The points of \(M^1_3\) lying on projective lines inside \(M_{p^2}(5,1)^T\) are listed in the third column of Table 7 below.
5.5. Proof of Theorem 2. The following lemma is probably well-known, but we need it in order to determine the structure of the irreducible components of dimension 2 of the torus fixed locus.

**Lemma 5.5.1.** Let $E$ be a vector bundle of rank 2 on $\mathbb{P}^n$. Let $s$ be a section of $\mathbb{P}(E)$. Assume that $\mathbb{P}(E) \setminus \{s\}$ is the trivial bundle on $\mathbb{P}^n$ with fibre $\mathbb{A}^1$. Then $\mathbb{P}(E)$ is the trivial bundle on $\mathbb{P}^n$ with fibre $\mathbb{P}^1$.

**Proof.** Tensoring, possibly, $E$ with a line bundle, we may assume that $s$ lifts to a global section $\sigma$ of $E$. The map $O \to E$ of multiplication with $\sigma$ is injective and its cokernel is a line bundle $\mathcal{L}$. The total space of $\mathcal{L}^*$ is isomorphic to $\mathbb{P}(E) \setminus \{s\}$, hence $\mathcal{L}^*$ is trivial. Clearly, $E \cong O \oplus \mathcal{L}$, hence $E$ is trivial, hence $\mathbb{P}(E)$ is trivial. \hfill $\square$

**Proposition 5.5.2.** Each irreducible component of dimension 2 of $M_{\mathbb{P}^2}(5,1)^T$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

**Proof.** Recall the surface $\Sigma_0$ from Section 3.1 and the line $\Lambda$ from Section 2.2.7. Denote $\Sigma = \Sigma_0$. An examination of Tables 5, 6 and 7 convinces us that

$$\bigcup_{\sigma \in S_3} \Sigma_\sigma \cap (M_1 \cup M_2 \cup M_3) = \bigcup_{\sigma \in S_3} \langle \sigma(X^2,XY,X^2Y^2Z) \rangle \cup \pi(X,Y,Z,X^2Y^2Z) \rangle.$$

It follows that $\Sigma = \Sigma \setminus \Sigma_0$ is a section for the map $\Sigma \to \Lambda$. Thus, $\Sigma = \mathbb{P}(E)$ for a vector bundle $E$ of rank 2 over the projective line $\Lambda$. The proposition follows from Lemma 5.5.1 in view of the fact that $\Sigma_0$ is isomorphic to the trivial bundle over $\Lambda$ with fibre $\mathbb{A}^1$.

Denote $M = M_{\mathbb{P}^2}(5,1)$. The first part of Theorem 2 concerning the structure of $M^T$ follows from Section 1 and Proposition 5.5.2.

Let $\lambda(t) = (\tau^{nt_0},\tau^{nt_1},\tau^{nt_2})$ denote a one-parameter subgroup of $T$ that is not orthogonal to any non-zero character $\chi$ for which there is $\langle \varphi \rangle \in M^T$ such that the eigenspace

$$\langle T_{\langle \varphi \rangle} M \rangle \chi = \{w \in T_{\langle \varphi \rangle} M \mid tw = \chi(t)w \text{ for all } t \in T \}$$

is non-zero. From the results in this section it follows that the set of such characters $\chi$ is contained in the set

$$\{ix + jy + kz \mid -7 \leq i, j, k \leq 7\}.$$ 

Thus, we can choose $\lambda(t) = (1,\tau,\tau^8)$. Recall from Section 3.3 that for a point $[\varphi] \in M^T$ the integer $p(\varphi)$ is the sum of the dimensions of the spaces $\langle T_{\langle \varphi \rangle} M \rangle \chi$ for which $\langle \lambda, \chi \rangle > 0$. If $[\varphi]$ varies in an irreducible component $X$ of $M^T$, then $p(\varphi)$ does not change, so we may define the integer $p(X)$. These integers can be computed with the help of the SINGULAR 4.0 program from Appendix 13. From (6.3.8) we deduce the formula

$$P_M(X) = \sum_{\dim(X) = 0} x^{2p(X)} + \sum_{\dim(X) = 1} (x^2 + 1)x^{2p(X)} + \sum_{\dim(X) = 2} (x^4 + 2x^2 + 1)x^{2p(X)},$$
where the summation is taken over all connected components \( X \) of \( M^T \). Substituting the values for \( p(X) \) yields the expression of \( P_M \) from Theorem 2. The final statement about the Hodge numbers follows, as in the case of \( N(3,4,3) \), from 3.3.4.

**Appendix A. Singular programs I**

```plaintext
ring r=0,(x,y,z),dp;
proc weight-decomposition(list u,list v)
  {list the_variables=(x,y,z); list w=list(); list g=list();
   int i,j,k,e;
   for (i=1; i<=size(u); i=i+1) {for (j=1; j<=size(v); j=j+1)
     {for (k=1; k<=3; k=k+1){w=w+list(-v[j]-u[i]+the_variables[k]);};};};
   for (i=1; i<=size(u); i=i+1) {for (j=1; j<=size(u); j=j+1)
     {g=g+list(u[i]-u[j]);};};
   for (i=1; i<=size(v); i=i+1) {for (j=1; j<=size(v); j=j+1)
     {g=g+list(v[i]-v[j]);};};};
g=delete(g,size(g));
   for (i=1; i<=size(g); i=i+1) {e=1; for (j=1; j<=size(w); j=j+1)
     {if (w[j]==g[i] and e==1) {w=delete(w,j); e=0;};};};
   return(w);};

proc positive_part(list v)
  {int i; int p=0; for(i=1;i<=12;i=i+1) {if(v[i]>0) {p=p+1;};};
   return(p);};

proc the_values(list w, list l)
  {int i; list v; v=list(); for(i=1; i<=12; i=i+1)
    {v=v+list((w[i]/x)*l[1]+(w[i]/y)*l[2]+(w[i]/z)*l[3]);};
   return(v);};

proc positive-parts(list w)
  {list d; d=list(); int i; list omega;
   omega=list(list(0,1,4),list(1,4,0),list(4,0,1), list(1,0,4),
   list(4,1,0), list(0,4,1)); for(i=1; i<=6; i=i+1)
     {d=d+list(positive_part(the_values(w,omega[i])))};
   return(d);};
```

**Appendix B. Singular programs II**

```plaintext
ring r=0,(x,y,z),dp;
int i,j; poly P, q, q1, q2, l1, 12, 1; P=0;
int points, lines; points = 0; lines = 0;

list s1, s2, s3, s4, s5, d;

s1=list(x,y,z);
s2=list(2x, 2y, 2z, x+y, x+z, y+z);
s3=list(3x, 3y, 3z, 2x+y, 2x+z, x+2y, y+2z, x+y+z);
s4=list(4x, 4y, 4z, 3x+y, 2x+2y, x+3y, 3x+z, 2x+2z, x+3z,
   3y+z, 2y+3z, x+2y+z, x+y+2z);
s5=list(5x, 5y, 5z, 4x+y, 3x+2y, 2x+3y, x+4y, 4x+z, 3x+2z, 2x+3z, x+4z,
   4y+z, 3y+2z, 2y+3z, y+4z, 3x+y+z, x+3y+z, x+y+3z,
   2x+2y+z, 2x+2y+2z, x+2y+2z);
```
proc add(poly p, list l)
{int i; list ll; ll=list();
for(i=1; i<=size(l); i=i+1){ll=ll+list(p+l[i]);};
return(ll);};

proc positive_part(list l)
{int i; int p; p=0; for (i=1; i<=size(l); i=i+1) {if (l[i]>0) {p=p+1;}}; 
return(p);};

proc values(list w, list l)
{int i; list v; v=list(); for (i=1; i<=size(w); i=i+1)
{v=v+list((w[i]/x)*l[1]+(w[i]/y)*l[2]+(w[i]/z)*l[3]);};
return(v);};

proc sub(list l, list ll)
{list lll; int i,j,e; lll=l; for (j=1; j<=size(ll); j=j+1)
{e=1; for(i=1; i<=size(lll); i=i+1)
{if (lll[i]==ll[j] and e==1) {lll=delete(lll,i); e=0;};};
return(lll);};

proc id0(poly a, poly b)
{return(sub(s5, (sub(s5,
add(a+x, s3)+add(a+y, s3)+add(b+x, s2)+add(b+y, s2))));};

proc id2(list l)
{list ll; ll = list(); int i; for (i=1; i<=size(l); i=i+1)
{ll = ll+ add(l[i], s3);};
return(sub(s5, (sub(s5, ll))));};

proc id3(list l)
{list ll; ll = list(); int i; for (i=1; i<=size(l); i=i+1)
{ll = ll+ add(l[i], s2);};
return(sub(s5, (sub(s5, ll))));};

proc point_2(list l)
{points=points+2; 
return(x'*(2*positive_part(values(l, list(0,1,8))))
+x'*(2*positive_part(values(l, list(1,0,8)))));};

proc point_3(list l)
{points=points+3;
return(x'*(2*positive_part(values(l, list(0,1,8))))
+x'*(2*positive_part(values(l, list(8,1,0))))
+x'*(2*positive_part(values(l, list(0,8,1)))));};

proc point_3_1(list l)
{points=points+3;
return(x'*(2*positive_part(values(l, list(0,1,8))))
+x'*(2*positive_part(values(l, list(1,0,8))))
+x'*(2*positive_part(values(l, list(8,1,0)))));};

proc point_3_2(list l)
{points=points+3;
return(x'*(2*positive_part(values(l, list(0,1,8))))
+x'*(2*positive_part(values(l, list(1,0,8))))
+x'*(2*positive_part(values(l, list(8,1,0)))));};
{points=points+3;
return(x^(2*positive_part(values(l, list(0,1,8))))
+x^(2*positive_part(values(l, list(1,8,0))))
+x^(2*positive_part(values(l, list(8,0,1)))));
}

proc point_6(list l)
{points=points+6;
return(x^(2*positive_part(values(l, list(0,1,8))))
+x^(2*positive_part(values(l, list(1,0,8))))
+x^(2*positive_part(values(l, list(8,1,0))))
+x^(2*positive_part(values(l, list(1,8,0))))
+x^(2*positive_part(values(l, list(0,8,1))))
+x^(2*positive_part(values(l, list(8,0,1)))));
}

proc line_3(list l)
{lines=lines+3;
return(((1+x^2)*x^(2*positive_part(values(l, list(0,1,8))))
+(1+x^2)*x^(2*positive_part(values(l, list(1,0,8))))
+(1+x^2)*x^(2*positive_part(values(l, list(8,1,0)))));
}

proc line_3_1(list l)
{lines=lines+3;
return(((1+x^2)*x^(2*positive_part(values(l, list(0,1,8))))
+(1+x^2)*x^(2*positive_part(values(l, list(1,8,0))))
+(1+x^2)*x^(2*positive_part(values(l, list(8,0,1)))));
}

proc line_3_2(list l)
{lines=lines+3;
return(((1+x^2)*x^(2*positive_part(values(l, list(0,1,8))))
+(1+x^2)*x^(2*positive_part(values(l, list(1,0,8))))
+(1+x^2)*x^(2*positive_part(values(l, list(1,8,0))))
+(1+x^2)*x^(2*positive_part(values(l, list(8,0,1)))));
}

proc line_6(list l)
{lines=lines+6;
return(((1+x^2)*x^(2*positive_part(values(l, list(0,1,8))))
+(1+x^2)*x^(2*positive_part(values(l, list(1,0,8))))
+(1+x^2)*x^(2*positive_part(values(l, list(8,1,0))))
+(1+x^2)*x^(2*positive_part(values(l, list(1,8,0))))
+(1+x^2)*x^(2*positive_part(values(l, list(0,8,1))))
+(1+x^2)*x^(2*positive_part(values(l, list(8,0,1)))));
}

proc surface_3(list l)
{return((1+2*(x^2)+x^4)*x^(2*positive_part(values(l, list(0,1,8))))
+(1+2*(x^2)+x^4)*x^(2*positive_part(values(l, list(1,8,0))))
+(1+2*(x^2)+x^4)*x^(2*positive_part(values(l, list(8,0,1)))));
}

proc w0(list u, list v)
{list ll; ll=list(); int i;
for(i=1; i<=4; i=i+1)
{ll=ll+add(-v[1]-u[i], s1)+add(-v[2]-u[i], s1)+add(-v[3]-u[i], s1)
+add(-v[4]-u[i], s2);};
return(ll);};
proc g0(list u, list v)
{list ll; int i,j;
for(i=1; i<=4; i=i+1){for(j=1; j<=4; j=j+1)
{ll=ll+list(u[i]-u[j]);};};
for(i=1; i<=3; i=i+1){for(j=1; j<=3; j=j+1)
{ll=ll+list(-v[i]+v[j]);};};
ll=ll+add(-v[4]+v[1], s1)+add(-v[4]+v[2], s1)+add(-v[4]+v[3], s1);
return(ll);};

proc m0(list u, list v)
{return(sub(w0(u,v), g0(u,v)));};

list alpha;
d=id3(list(x+y+z, y+2z, 2x+z, x+2y));
for(i=1; i<size(d); i=i+1)
{alpha = m0(list(0, x-z, y-x, z-y), list(z, x, y, d[i]-x-y-z));
P = P + point_2(alpha);};

list beta;
d=id3(list(x+y+z, 2y+z, 2x+z, 2x+y));
for(i=1; i<size(d); i=i+1)
{beta = m0(list(0, x-y, y-x, z-x), list(y, x, x, d[i]-x-y-z));
P = P + point_6(beta);};

list gamma;
d=id3(list(2y+z, x+y+z, 2x+z, 3y));
for(i=1; i<size(d); i=i+1)
{gamma = m0(list(0, x-y, y-x, x-2y+z), list(y, x, 2y-x, d[i]-x-y-z));
P = P + point_6(gamma);};

list delta;
d=id3(list(2y+z, 2x+z, x+2y, 2x+y));
for(i=1; i<size(d); i=i+1)
{delta = m0(list(0, z-y, x-2y+z, 2x-2y), list(y, 2y-z, 2y-x, d[i]-2x-z));
P = P + point_3(delta);};

list epsilon;
d=id3(list(2y+z, 2x+y, x+2y, 3x));
for(i=1; i<size(d); i=i+1)
{epsilon = m0(list(0, y-x, x-z, 2x-y-z), list(x, z, -x+y+z, d[i]-2x-y));
P = P + point_6(epsilon);};

list zeta;
d=id3(list(2y+z, x+2y, 2x+y, 3x));
for(i=1; i<size(d); i=i+1)
{zeta = m0(list(0, x-y, y-x, 2x-y-z), list(y, x, -x+y+z, d[i]-2x-y));
P = P + point_6(zeta);};

list eta;
d=id3(list(3x, 2x+y, x+2y, 3y));
for(i=1; i<size(d); i=i+1)
{eta = m0(list(0, x-y, y-x, 2y-2x), list(y,x,2x-y, d[i]-x-2y));
P = P + point_3(eta);};
list theta;
d = sub(id3(list(x+y+z, 2x+z, 2x+y, x+2z)), list(x+y+3z, x+2y+2z));
for (i=1; i<=size(d); i=i+1)
{theta = m0(list(0, y-x, z-x, y-z), list(x, x-y+z, d[i]-x-y-z));
P = P + point_6(theta);};
d = list(x+y+3z, x+2y+2z);
for (i=1; i<=size(d); i=i+1)
{theta = m0(list(0, y-x, z-x, y-z), list(x, x-y+z, d[i]-x-y-z));
P = P + line_6(theta);};
list kappa;
d = sub(id3(list(x+2y, 2x+y, 2x+z, x+2z)), list(x+2y+2z));
for (i=1; i<=size(d); i=i+1)
{kappa = m0(list(y-x, 0, x-z, x-y), list(x, z, y, d[i]-2x-y));
P = P + point_3_2(kappa);};
d = list(x+2y+2z);
for (i=1; i<=size(d); i=i+1)
{kappa = m0(list(y-x, 0, x-z, x-y), list(x, z, y, d[i]-2x-y));
P = P + line_3_2(kappa);};
list lambda;
d = sub(id3(list(3x, 2x+y, 2x+z, x+2z)), list(2x+2y+z, 2x+y+2z));
for (i=1; i<=size(d); i=i+1)
{lambda = m0(list(y-x, 0, x-z, x-y), list(x, z, y, d[i]-3x));
P = P + point_3_2(lambda);};
d = list(2x+2y+z, 2x+y+2z);
for (i=1; i<=size(d); i=i+1)
{lambda = m0(list(y-x, 0, x-z, x-y), list(x, z, y, d[i]-3x));
P = P + surface_3(lambda);};
list mu;
d = sub(id3(list(3x, 2x+y, x+y+z, x+2y)), list(x+3y+z, x+2y+2z));
for (i=1; i<=size(d); i=i+1)
{mu = m0(list(y-x, 0, x-z, x-y), list(x, z, y, d[i]-2x-y));
P = P + point_6(mu);};
d = list(x+3y+z, x+2y+2z);
for (i=1; i<=size(d); i=i+1)
{mu = m0(list(y-x, 0, x-z, x-y), list(x, z, y, d[i]-2x-y));
P = P + line_6(mu);};
list nu;
d = sub(id3(list(x+2y, 2x+y, 3x, 2x+z)), list(2x+2y+z, 2x+y+2z));
\begin{verbatim}
for(i=1; i<=size(d); i=i+1)
{nu = m0(list(0, y-x, 2y-2x, -x+2y-z), list(x, 2x-y, 2x-2y+z, d[i]-x-2y));
P = P + point_6(nu);}
d=list(2x+2y+z, 2x+y+2z);
for(i=1; i<=size(d); i=i+1)
{nu = m0(list(0, y-x, 2y-2x, -x+2y-z), list(x, 2x-y, 2x-2y+z, d[i]-x-2y));
P = P + line_6(nu);}
list xi;
d=sub(id3(list(2x+y, x+2y, x+y+z)),
    list(3x+y+z, x+3y+z, x+y+3z, x+2y+2z, 2x+y+2z));
for(i=1; i<=size(d); i=i+1)
{xi = m0(list(0, x-y, x-z, x-z), list(y, z, -x+y+z, d[i]-2x-y));
P = P + point_3_2(xi);}
d=list(3x+y+z, x+3y+z, x+y+3z, x+2y+2z, 2x+y+2z);
for(i=1; i<=size(d); i=i+1)
{xi = m0(list(0, x-y, x-z, x-z), list(y, z, -x+y+z, d[i]-2x-y));
P = P + line_3_2(xi);}
proc w1(list u, list v)
{return(add(-v[1]-u[1], s3)+add(-v[1]-u[2], s2)+add(-v[2]-u[1], s3)+
    add(-v[2]-u[2], s2));}
proc g1(list u, list v)
{return(list(0,0,0,v[1]-v[2],v[2]-v[1]) + add(u[2]-u[1], s1));}
proc m1(list u, list v)
{return(sub(w1(u, v), g1(u, v)) + list(u[1]+v[1]-x-y-z, u[1]+v[2]-x-y-z));}
list omicron;
q1=2x; q2=2y;
d=sub(id2(list(q1,q2)), list(3x+y+z, x+3y+z));
for(i=1; i<=size(d); i=i+1)
{omicron=m1(list(d[i]-q1-q2, 0), list(q1, q2));
P = P + point_3(omicron);}
q1=2x; q2=y+z;
d=sub(id2(list(q1,q2)), list(3x+y+z, x+2y+2z));
for(i=1; i<=size(d); i=i+1)
{omicron=m1(list(d[i]-q1-q2, 0), list(q1, q2));
P = P + point_3_1(omicron);}
q1=2x; q2=x+y;
d=sub(id2(list(q1,q2)), list(2x+3y,2x+y+2z,2x+2y+z,3x+y+z));
for(i=1; i<=size(d); i=i+1)
{omicron=m1(list(d[i]-q1-q2, 0), list(q1, q2));
P = P + point_6(omicron);}
d=list(2x+3y,2x+y+2z);
for(i=1; i<=size(d); i=i+1)
{omicron=m1(list(d[i]-q1-q2, 0), list(q1, q2));
P = P + line_6(omicron);}
q1=x+z; q2=y+z;
\end{verbatim}
d = sub(id2(list(q1, q2)), list(3x+y+z, x+3y+z, 2x+2y+z, 2x+y+2z, x+2y+2z));
for (i = 1; i <= size(d); i = i + 1)
{
omicron = m1(list(d[i] - q1 - q2, 0), list(q1, q2));
P = P + point_3(omicron);
d = list(3x+y+z, x+3y+z, 2x+2y+z);
for (i = 1; i <= size(d); i = i + 1)
{
omicron = m1(list(d[i] - q1 - q2, 0), list(q1, q2));
P = P + line_3(omicron);
}
}

proc w2(list u, list v)
{return(add(-v[1]-u[1], s2)+add(-v[2]-u[1], s3)+add(-v[3]-u[1], s3)
 +add(-v[1]-u[2], s1)+add(-v[2]-u[2], s2)+add(-v[3]-u[2], s2)
 +add(-v[2]-u[3], s1)+add(-v[3]-u[3], s1));};

proc g2(list u, list v)
{return(sub(list(0, 0, 0, 0, 0, 0, v[2]-v[3], v[3]-v[2])
 +add(u[2]-u[1], s1)+add(u[3]-u[1], s2)+add(u[3]-u[2], s1)
 +add(v[1]-v[2], s1)+add(v[1]-v[3], s1), list(v[1]-v[2]+x));};

proc m2(list u, list v)
{return(sub(w2(u, v), g2(u, v))+list(u[1]+v[2]-x-y-z,
 u[1]+v[3]-x-y-z, -v[1]-u[3]));};

list pi;
l = y; q = 2z;
d = sub(id0(l, q), list(2x+2y+z, x+3y+z, x+2y+2z));
for (i = 1; i <= size(d); i = i + 1)
{pi = m2(list(d[i] - l - x - y, d[i] - q - x - y, 0), list(-d[i] + q + x + y, x, y));
P = P + point_6(pi);};
l = y; q = 2x;
d = sub(id0(l, q), list(2x+y+2z, 2x+2y+z, x+3y+z, 3x+2y));
for (i = 1; i <= size(d); i = i + 1)
{pi = m2(list(d[i] - l - x - y, d[i] - q - x - y, 0), list(-d[i] + q + x + y, x, y));
P = P + point_6(pi);};
d = list(2x+y+2z);
for (i = 1; i <= size(d); i = i + 1)
{pi = m2(list(d[i] - l - x - y, d[i] - q - x - y, 0), list(-d[i] + q + x + y, x, y));
P = P + line_6(pi);};
l = y; q = x+z;
d = sub(id0(l, q), list(x+y+3z, 2x+2y+z, x+3y+z));
for (i = 1; i <= size(d); i = i + 1)
{pi = m2(list(d[i] - l - x - y, d[i] - q - x - y, 0), list(-d[i] + q + x + y, x, y));
P = P + point_6(pi);};
d = list(x+y+3z);
for (i = 1; i <= size(d); i = i + 1)
{pi = m2(list(d[i] - l - x - y, d[i] - q - x - y, 0), list(-d[i] + q + x + y, x, y));
P = P + line_6(pi);};
l = z; q = 2x;
d = sub(id0(l, q), list(2x+y+2z, x+2y+2z, 3x+y+z));
for (i = 1; i <= size(d); i = i + 1)
\[ p_i = 2(\text{list}(d[i]-l-x-y, d[i]-q-x-y, 0), \text{list}(-d[i]+q+l+x+y, x, y)); \]
\[ P = P + \text{point}_6(p_i); \]

\[ l = z; q = x+y; \]
\[ d = \text{sub}(\text{id} 0(l, q), \text{list}(2x+y+2z, x+2y+2z, 2x+2y+z)); \]
\[ \text{for} (i=1; i<=\text{size}(d); i=i+1) \]
\[ p_i = 2(\text{list}(d[i]-l-x-y, d[i]-q-x-y, 0), \text{list}(-d[i]+q+l+x+y, x, y)); \]
\[ P = P + \text{point}_3(p_i); \]

\[ \text{proc} w3(\text{list} u, \text{list} v) \]
\[ \{ \text{return}(\text{add}(-v[1]-u[1], s1)+\text{add}(-v[1]-u[2], s1) \]
\[ +\text{add}(-v[2]-u[1], s4)+\text{add}(-v[2]-u[2], s4)); \} \]

\[ \text{proc} g3(\text{list} u, \text{list} v) \]
\[ \{ \text{return}(\text{list}(0, 0, u[1]-u[2], u[2]-u[1])+\text{add}(v[1]-v[2], s3)); \} \]

\[ \text{proc} m3(\text{list} u, \text{list} v) \]
\[ \{ \text{return}(\text{sub}(w3(u, v), g3(u, v)) \]
\[ +\text{sub}(\text{list}(u[1]+v[2]-2x-y-z, u[1]+v[2]-x-2y-z, u[1]+v[2]-x-y-2z, \]
\[ u[2]+v[2]-2x-y-z, u[2]+v[2]-x-2y-z, u[2]+v[2]-x-y-2z), \text{list}(v[2]-x-y-z))); \} \]

\[ \text{list} rho; \]
\[ d = \text{sub}(s5, \text{list}(5z, x+3y+z, 2x+2y+z, 3x+y+z, x+2y+2z, 2x+y+2z)); \]
\[ \text{for}(i=1; i<=\text{size}(d); i=i+1) \]
\[ \{ \text{rho} = \text{sub}(w3(\text{list}(x, y), \text{list}(0, d[i]-x-y)), g3(\text{list}(x, y), \text{list}(0, d[i]-x-y))) \]
\[ +\text{list}(d[i]-3x-y-z, d[i]-2x-2y-z, d[i]-x-3y-z, d[i]-2x-y-2z, d[i]-x-2y-2z); \]
\[ P = P + \text{point}_3(\text{rho}); \}

\text{References}

[1] A. Bialynicki-Birula. Some theorems on actions of algebraic groups. Ann. Math. \textbf{98} (1973), 480–497.
[2] A. Bialynicki-Birula. On fixed points of torus actions on projective varieties. Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys. \textbf{22} (1974), 1097–1101.
[3] J. Carrell. Torus actions and cohomology. Algebraic quotients. Torus actions and cohomology. The adjoint representation and the adjoint action. Encycl. Math. Sci. 131(2). Springer Verlag, Berlin, 2002.
[4] J. Choi. Enumerative invariants for local Calabi-Yau threefolds. Doctoral dissertation, U. Illinois at Urbana-Champaign, May 2012.
[5] J. Choi, K. Chung. Moduli spaces of $\alpha$-stable pairs and wall-crossing on $\mathbb{P}^2$. arXiv:1210.2499
[6] J. Choi, M. Maican. Torus action on the moduli spaces of plane sheaves. arXiv:1304.4871
[7] W. Decker, G.-M. Greuel, G. Pfister, H. Schönenmann. SINGULAR 3-1-6 — A computer algebra system for polynomial computations. http://www.singular.uni-kl.de (2012).
[8] J.-M. Drézet. Fibres exceptionnels et variétés de modules de faisceaux semi-stables sur $\mathbb{P}_2(\mathbb{C})$. J. reine angew. Math. \textbf{380} (1987), 14–58.
[9] J.-M. Drézet. Cohomologie des variétés de modules de hauteur nulle. Math. Ann. \textbf{281} (1988), 43–85.
[10] J.-M. Drézet. Variétés de modules alternatives. Ann. Inst. Fourier \textbf{49} (1999), 57–139.
[11] J.-M. Drézet, M. Maican. On the geometry of the moduli spaces of semi-stable sheaves supported on plane quartics. Geom. Dedicata \textbf{152} (2011), 17–49.
[12] G. Ellingsrud, S.A. Strømme. On the Chow ring of a geometric quotient. Ann. Math. \textbf{130} (1989), 159–187.
[13] S. Katz. Genus zero Gopakumar-Vafa invariants of contractible curves. J. Differ. Geom. \textbf{79} (2008), 185–195.
[14] A. King. Moduli of representations of finite dimensional algebras. Q. J. Math. Oxf. II Ser. \textbf{45} (1994), 515–530.
[15] J. Le Potier. *Faisceaux semi-stables de dimension 1 sur le plan projectif*. Rev. Roumaine Math. Pures Appl. **38** (1993), 635–678.

[16] J. Le Potier. *Lectures on vector bundles*. Cambridge Studies in Advanced Mathematics 54. Cambridge University Press, Cambridge, 1997.

[17] J. Li, B. Wu. *Note on a conjecture of Gopakumar-Vafa*. Chin. Ann. Math., Ser. B **27** (2006), 219–242.

[18] M. Maican. *A duality result for moduli spaces of semistable sheaves supported on projective curves*. Rend. Sem. Mat. Univ. Padova **123** (2010), 55–68.

[19] M. Maican. *On the moduli spaces of semi-stable plane sheaves of dimension one and multiplicity five*. Ill. J. Math. **55** (2011), 1467–1532.

[20] M. Maican. *On the homology of the moduli space of plane sheaves with Hilbert polynomial 5m + 3*. arXiv:1305.5511

[21] A. N. Rudakov. *The Markov numbers and exceptional bundles on \( \mathbb{P}^2 \)*. Math. USSR, Izv. **32** (1989), 99–112.

[22] Y. Yuan. *Moduli spaces of semistable sheaves of dimension 1 on \( \mathbb{P}^2 \)*. arXiv:1206.4800

Institute of Mathematics of the Romanian Academy, Calea Grivitei 21, Bucharest 010702, Romania

E-mail address: mario.maican@imar.ro