LOOKBACK OPTION PRICING PROBLEM OF MEAN-REVERTING STOCK MODEL IN UNCERTAIN ENVIRONMENT

Miao Tian\textsuperscript{1}, Xiangfeng Yang\textsuperscript{2∗} and Yi Zhang\textsuperscript{3}
\textsuperscript{1}School of Mathematics, Renmin University of China, Beijing 100872, China
\textsuperscript{2}School of Information Technology & Management, University of International Business & Economics, Beijing 100029, China
\textsuperscript{3}School of Economics & Management, Beijing University of Chemical Technology, Beijing 100029, China

(Communicated by Changjun Yu)

\textbf{Abstract.} A lookback option is an exotic option that allows investors to look back at the underlying prices occurring over the life of the option, and to exercise the right at assets optimal point. This paper proposes a mean-reverting stock model to investigate the lookback option in an uncertain environment. The lookback call and put options pricing formulas of the stock model are derived, and the corresponding numerical algorithms are designed to compute the prices of these two options.

1. \textbf{Introduction.} The real world is always in the state of indeterminacy, especially in the field of finance. For modeling indeterminacy, there exist two mathematical systems, one is probability theory \[6\], which is interpreted as frequency, and the other is uncertainty theory \[7\], which is interpreted as personal belief degree. Many researchers pointed out that uncertain differential equation in uncertainty theory is possible to model the actual financial market instead of the stochastic differential equation in probability theory. Interested readers may consult Liu \[10\] for the differences between the uncertain differential equation and stochastic differential equation. An uncertainty theory was founded by Liu \[7\] based on normality, duality, subadditivity, and product axioms and was refined by Liu \[9\] to formulate a branch of mathematics. At the same time, Liu \[9\] founded an uncertain calculus to deal with the integral and differential of an uncertain process. Based on it, uncertain differential equation theory was first applied into finance by Liu \[9\] and has been the potential mathematical foundation of finance theory. For a more detailed exposition of the uncertain differential equation, the readers may consult Yao’s recent book \[24\].

Uncertain differential equations have solved many derivative pricing problems in financial situations. Liu \[9\] presented a kind of uncertain stock model, called Liu’s stock model, and then provided European option pricing formulas. Besides, Chen

\textsuperscript{2010 Mathematics Subject Classification.} Primary: 91G99; Secondary: 91G80.

\textsuperscript{Key words and phrases.} Uncertainty theory, uncertain differential equation, mean-reverting stock model, lookback option.

\textsuperscript{The second author is supported by the Program for Young Excellent Talents in UIBE (No.18YQ06).}

\textsuperscript{∗ Corresponding author: Xiangfeng Yang.}
[1] derived the American option pricing formulas, Zhang and Liu [26] proposed the Geometric average Asian option. While Liu’s stock model describes stock prices in the short-run adequately, it can’t describe stock prices in the long-run, as the stock prices fluctuate around some average price in the long-run. Peng and Yao [12] provided an uncertain mean-reverting stock model to describe the stock price in the long term. Yao [20] explored the no-arbitrage determinant theorem on the uncertain mean-reverting stock model. Tian et al. [14, 15] investigated the barrier option and equity warrants pricing problems based on the uncertain mean-reverting stock model. After that, Chen et al. [3] proposed an uncertain stock model with periodic dividends. Yu [25] studied the stock model with jumps, and Ji and Zhou [5] presented the option pricing formulas based on the uncertain stock model with jumps. Yao [23] investigated the uncertain contour process and applied it to the stock model with a floating interest rate. Sun and Su [13] presented a mean-reverting stock model with a floating interest rate and derived the European and American options pricing formulas. Yang et al. [19] investigated the Asian-barrier option pricing problem in the framework of uncertainty theory. Except for finance theory, uncertain differential equations have also been used in other areas extensively. For example, they have been used to uncertain optimal control with application to a portfolio selection model by Zhu [27], to the differential game with applications to capitalism and resource extraction problem by Yang and Gao [16, 17], to heat conduction problem by Yang and Yao [18], and to currency models by Liu et al. [11].

Lookback option is a type of path-dependent exotic option, and it offers the payoff that depends on an optimal value of underlying asset price occurring over the life of the option. The option allows the holder to look back over time to determine the payoff. Gao et al. [4] proposed the lookback option pricing formulas based on the uncertain exponential Ornstein-Uhlenbeck model, which does not take a mean-reversion into consideration. As is known to us, the stock price fluctuates around a certain price in the long run periodically, so we proposed lookback option pricing formulas based on the uncertain mean-reverting stock model.

The rest of this paper is set out as follows. Sections 2 and 3 recall some preliminary concepts in uncertainty theory and introduce the mean-reverting stock model, respectively. Section 4 derives lookback call option pricing formula with the fixed strike price, and a numerical algorithm is designed to calculate the price of the lookback call option. Section 5 derives lookback put option. Finally, Section 6 provides the conclusions in this paper.

2. Preliminary. In this section, it will introduce some basic definitions and theorems in uncertainty theory. Uncertainty theory was founded by Liu [7] and refined by Liu [9].

**Definition 2.1.** (Liu [7]) Let \( \mathcal{L} \) be a \( \sigma \)-algebra on a non-empty set \( \Gamma \). A set function \( \mathcal{M} : \mathcal{L} \rightarrow [0, 1] \) is called an uncertain measure if it satisfies the following axioms.

- **Axiom 1:** (Normality Axiom) \( \mathcal{M}(\Gamma) = 1 \) for the universal set \( \Gamma \).
- **Axiom 2:** (Duality Axiom) \( \mathcal{M}(\Lambda) + \mathcal{M}(\Lambda^c) = 1 \) for any even \( \Lambda \).
- **Axiom 3:** (Subadditivity Axiom) For every countable sequence of events \( \Lambda_1, \Lambda_2, \ldots \), we have

\[
\mathcal{M}\left(\bigcup_{i=1}^{\infty} \Lambda_i\right) \leq \sum_{i=1}^{\infty} \mathcal{M}(\Lambda_i).
\]
Axiom 4: (Product Axiom) (Liu [9]) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{ \prod_{k=1}^{\infty} \Lambda_k \right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where $\Lambda_k$ are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

An uncertain variable $\xi$ is a function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers.

**Definition 2.2.** (Liu [9]) The uncertain variables $\xi_1, \xi_2, \ldots, \xi_m$ are said to be independent if

$$\mathcal{M}\left\{ \bigcap_{i=1}^{m}\{\xi_i \in B_i\}\right\} = \bigwedge_{i=1}^{m} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_m$ of real numbers.

The uncertainty distribution $\Phi$ of an uncertain variable $\xi$ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}, \forall x \in \mathbb{R}$$

for any real number $x$. If the uncertainty distribution $\Phi(x)$ is a continuous and strictly increasing function with respect to $x$ at which $0 < \Phi(x) < 1$, and

$$\lim_{x \to -\infty} \Phi(x) = 0, \quad \lim_{x \to +\infty} \Phi(x) = 1,$$

then it is called regular. In this case, the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of $\xi$. Furthermore, the expected value can be calculated as

$$E[\xi] = \int_{0}^{1} \Phi^{-1}(\alpha)d\alpha.$$

An uncertain process, as a sequence of uncertain variables indexed by the time, is used to model the evolution of uncertain phenomena.

**Definition 2.3.** (Liu [9]) An uncertain process $C_t$ is said to be a Liu process if

(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
(ii) $C_t$ has stationary and independent increments,
(iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with an uncertainty distribution

$$\Phi_t(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3t}}\right)\right)^{-1}.$$  \hspace{1cm} (1)

There are big differences between the Wiener process and Liu process. The first, almost all sample paths of Wiener process $\{W_t, t > 0\}$ are continuous but not differentiable, but almost all sample paths of Liu process are Lipschitz continuous, so the Liu process is more smooth. The second, every increment $W_{s+t} - W_s$ of Wiener process is a random normal distribution with a probability distribution $N(0, \sigma^2 t)$, but the every increment $C_{s+t} - C_s$ is a normal uncertain variable with an uncertainty distribution (1).

**Definition 2.4.** (Liu [8]) Suppose that $C_t$ is a Liu process, and $f$ and $g$ are two real functions. Then

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t$$

is called an uncertain differential equation. A solution is an uncertain process $X_t$ that satisfies the above equation identically in $t$. 
Theorem 2.5. (Chen and Liu [2]) Let \( u_{1t}, u_{2t}, v_{1t}, v_{2t} \) be integrable uncertain processes. Then the linear uncertain differential equation
\[
dX_t = (u_{1t}X_t + u_{2t})dt + (v_{1t}X_t + v_{2t})dC_t
\]
has a solution
\[
X_t = U_t \cdot V_t
\]
where
\[
U_t = \exp \left( \int_0^t u_{1s}ds + \int_0^t v_{1s}dC_s \right), \quad V_t = \left( X_0 + \int_0^t \frac{u_{2s}}{U_s}ds + \int_0^t \frac{v_{2s}}{U_s}dC_s \right).
\]

Definition 2.6. (Yao and Chen [22]) The \( \alpha \)-path (\( 0 < \alpha < 1 \)) of an uncertain differential equation
\[
dX_t = f(t, X_t)dt + g(t, X_t)dC_t
\]
with an initial value \( X_0 \) is a deterministic function \( X_\alpha_t \) with respect to \( t \) that solves the corresponding equation
\[
dX_\alpha_t = f(t, X_\alpha_t)dt + |g(t, X_\alpha_t)| \Phi^{-1}(\alpha)dt, \quad X_\alpha_0 = X_0
\]
where \( \Phi^{-1}(\alpha) \) is the inverse uncertainty distribution of standard normal uncertain variable, i.e.,
\[
\Phi^{-1}(\alpha) = \sqrt{\frac{3}{\pi}} \ln \frac{\alpha}{1-\alpha}, \quad 0 < \alpha < 1.
\]

Theorem 2.7. (Yao and Chen [22]) Assume that \( X_t \) and \( X_\alpha_t \) are the solution and \( \alpha \)-path of the uncertain differential equation
\[
dX_t = f(t, X_t)dt + g(t, X_t)dC_t,
\]
respectively. Then
\[
\mathcal{M}\{X_t \leq X_\alpha_t, \forall t\} = \alpha, \quad \mathcal{M}\{X_t > X_\alpha_t, \forall t\} = 1 - \alpha.
\]

Theorem 2.8. (Yao and Chen [22]) Let \( X_t \) and \( X_\alpha_t \) be the solution and \( \alpha \)-path of the uncertain differential equation
\[
dX_t = f(t, X_t)dt + g(t, X_t)dC_t,
\]
respectively. Then the solution \( X_t \) has an inverse uncertainty distribution
\[
\Psi_t^{-1}(\alpha) = X_\alpha_t.
\]

Theorem 2.9. (Yao [21]) Let \( X_t \) and \( X_\alpha_t \) be the solution and \( \alpha \)-path of the uncertain differential equation
\[
dX_t = f(t, X_t)dt + g(t, X_t)dC_t,
\]
respectively. Then for any time \( s > 0 \) and strictly increasing (decreasing) function \( J(x) \), the supremum
\[
\sup_{0 \leq t \leq s} J(X_t)
\]
has an inverse uncertainty distribution
\[
\Phi_s^{-1}(\alpha) = \sup_{0 \leq t \leq s} J(X_\alpha_t) \quad (\Phi_s^{-1}(\alpha) = \sup_{0 \leq t \leq s} J(X_{1-\alpha}^t)).
\]
3. Mean-reverting uncertain stock model. The uncertain differential equations are commonly used in financial markets. The Liu’s stock model [9] assumed that the stock price $X_t$ follows uncertain differential equation,

$$\begin{aligned}
\begin{cases}
    dX_t &= \mu X_t dt + \sigma X_t dC_t \\
    dY_t &= rY_t dt
\end{cases}
\end{aligned}$$

(2)

where $X_t$ is the stock price, $Y_t$ is the bond price, $r$ is the riskless interest rate, $\mu$ is the stock drift, $\sigma$ is the stock diffusion, and $C_t$ is a Liu process.

The stock price fluctuates periodically around a certain price in the long term, so the mean-reverting stock model via uncertainty theory is introduced. The mean-reverting uncertain stock model for the long term [12] is written as follows,

$$\begin{aligned}
\begin{cases}
    dX_t &= (m - aX_t)dt + \sigma X_t dC_t \\
    dY_t &= rY_t dt
\end{cases}
\end{aligned}$$

(3)

where $r > 0, m > 0, a > 0$ and $\sigma > 0$ are constants.

**Theorem 3.1.** Suppose that the stock price follows the model $dX_t = (m - aX_t)dt + \sigma X_t dC_t$ where $X_t$ represents the stock price at time $t$. Then

$$X_t = \exp(-at + \sigma C_t) \left( X_0 + m \int_0^t \exp(as - \sigma C_s) ds \right).$$

**Proof.** By Theorem 2.5, we have

$$U_t = \exp \left( \int_0^t -ads + \int_0^t \sigma ds \right) = \exp(-at + \sigma C_t).$$

The stock price $X_t$ is

$$\begin{aligned}
    X_t &= U_t \cdot \left( X_0 + \int_0^t \frac{m}{U_s} ds + \int_0^t 0 dC_s \right) \\
    &= U_t \cdot \left( X_0 + \int_0^t \frac{m}{\exp(-as + \sigma C_s)} ds \right) \\
    &= \exp(-at + \sigma C_t) \left( X_0 + m \int_0^t \exp(as - \sigma C_s) ds \right).
\end{aligned}$$

The Theorem is thus proved. \qed

**Theorem 3.2.** Suppose that the stock price follows the model $dX_t = (m - aX_t)dt + \sigma X_t dC_t$ where $X_t$ represents the stock price at time $t$. Then the inverse uncertainty distribution of $X_t$ is

$$\Phi^{-1}(\alpha) = X_0 \cdot \exp \left( (\sigma \Phi^{-1}(\alpha) - a) t \right) + \frac{m}{a - \sigma \Phi^{-1}(\alpha)} \left[ 1 - \exp \left( (\sigma \Phi^{-1}(\alpha) - a) t \right) \right]$$

where

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}.$$
Proof. According to the Yao-Chen formula (Theorem 2.7), the \( \alpha \)-path of \( X_t \) is the solution of ordinary differential equation

\[
dX_t^\alpha = (m - aX_t^\alpha)dt + \sigma X_t^\alpha \Phi^{-1}(\alpha)dt
\]

where

\[
\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}.
\]

Therefore,

\[
X_t^\alpha = X_0 \cdot \exp \left( \left( \sigma \Phi^{-1}(\alpha) - a \right) t \right) + \frac{m}{a - \sigma \Phi^{-1}(\alpha)} \left[ 1 - \exp \left( \left( \sigma \Phi^{-1}(\alpha) - a \right) t \right) \right].
\]

From Theorem 2.8, we get the inverse uncertainty distribution \( \Phi^{-1}_t(\alpha) = X_t^\alpha \). \( \square \)

4. **Lookback call option pricing formula with the fixed strike.** A lookback call option offers the holder the right to sell a certain asset at the highest price during a certain period. Suppose that a lookback call option has a fixed strike price \( K \) and an expiration time \( T \). And \( X_t \) is the price of the time \( t \), then the payoff from selling a lookback call option is

\[
\left( \max_{0 \leq t \leq T} X_t - K \right)^+.
\]

Considering the time value of money resulted from the stock, the present value of this payoff is

\[
\exp(-rT) \left( \max_{0 \leq t \leq T} X_t - K \right)^+.
\]

Hence the price of lookback call option should be the expected present value of the payoff, that is, option has a price

\[
f_{\text{call}} = \exp(-rT)E \left[ \left( \max_{0 \leq t \leq T} X_t - K \right)^+ \right].
\]

**Theorem 4.1.** Suppose that a lookback call option for the stock model (3) has a strike price is \( K \) and an expiration time \( T \). Then the lookback call option pricing formula is

\[
f_{\text{call}} = \exp(-rT) \int_0^1 \max_{0 \leq t \leq T} \left( X_0 \cdot \exp \left( \left( \sigma \Phi^{-1}(\alpha) - a \right) t \right) + \frac{m}{a - \sigma \Phi^{-1}(\alpha)} \left[ 1 - \exp \left( \left( \sigma \Phi^{-1}(\alpha) - a \right) t \right) \right] - K \right)^+ \, d\alpha
\]

where

\[
\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}.
\]

Proof. According to Theorem 3.2, we know the \( \alpha \)-path of \( X_t \) is

\[
X_t^\alpha = X_0 \cdot \exp \left( \left( \sigma \Phi^{-1}(\alpha) - a \right) t \right) + \frac{m}{a - \sigma \Phi^{-1}(\alpha)} \left[ 1 - \exp \left( \left( \sigma \Phi^{-1}(\alpha) - a \right) t \right) \right],
\]

and

\[
\left( \max_{0 \leq t \leq T} X_t - K \right)^+ = \left( \max_{0 \leq t \leq T} (X_t - K) \right)^+ = \max_{0 \leq t \leq T} (X_t - K)^+.
\]
From Theorem 2.9, we can get that the inverse uncertainty distribution function of \( \exp(-rT) \max_{0 \leq t \leq T} (X_t^\alpha - K)^+ \) is

\[
\Psi^{-1}(\alpha) = \exp(-rT) \max_{0 \leq t \leq T} (X_t^\alpha - K)^+.
\]

Therefore,

\[
f_{\text{call}} = \int_0^1 \Psi^{-1}(\alpha) d\alpha.
\]

The lookback call option pricing formula is verified. □

According to Theorem 4.1, the algorithm to calculate the lookback call option price of the stock model (3) is designed as below.

**Step 0:** Set \( \alpha_i = i/N \) and \( t_j = jT/M \), \( i = 1, 2, \ldots, N - 1 \), \( j = 1, 2, \ldots, M \), where \( N \) and \( M \) are two large numbers.

**Step 1:** Set \( i = 0 \).

**Step 2:** Set \( i \leftarrow i + 1 \).

**Step 3:** Set \( j = 0 \), and \( G_{t_0}^{\alpha_i} = 0 \).

**Step 4:** Set \( j \leftarrow j + 1 \).

**Step 5:** Calculate the stock price of the time \( t_j \)

\[
G_{t_j}^{\alpha_i} = X_0 \cdot \exp \left( \left( \frac{\sqrt{3}\sigma}{\pi} \ln \frac{\alpha_i}{1 - \alpha_i} - a \right) t_j \right) + \frac{m}{a - \frac{\sqrt{3}\sigma}{\pi} \ln \frac{\alpha_i}{1 - \alpha_i}} \left[ 1 - \exp \left( \left( \frac{\sqrt{3}\sigma}{\pi} \ln \frac{\alpha_i}{1 - \alpha_i} - a \right) t_j \right) \right].
\]

If \( G_{t_j}^{\alpha_i} \geq G_{t_{j-1}}^{\alpha_i} \) and \( j < M \), then return to **Step 4**. If \( G_{t_j}^{\alpha_i} \geq G_{t_{j-1}}^{\alpha_i} \) and \( j = M \), then jump to **Step 7**.

**Step 6:** Set \( G_{t_j}^{\alpha_i} \leftarrow G_{t_{j-1}}^{\alpha_i} \), if \( j < M \), then return to **Step 4**.

**Step 7:** Calculate \( G_{t_j}^{\alpha_i} - K \).

**Step 8:** Set \( G^{\alpha_i} \leftarrow \max \left( 0, G_{t_j}^{\alpha_i} - K \right) \). If \( i < N - 1 \), then return to **Step 2**.

**Step 9:** Calculate the lookback call option price is

\[
f_{\text{call}} \leftarrow \exp(-rT) \frac{1}{N - 1} \sum_{i=1}^{N-1} G^{\alpha_i}.
\]

**Example 4.1.** Assume the initial value of the stock price is \( X_0 = 30 \), and other parameters of the stock price are \( r = 0.08 \), \( K = 31 \), \( a = 0.06 \), \( m = 1.8 \) and \( \sigma = 0.75 \).

Then the price of the stock price with a maturity data \( T = 0.25 \) is \( f_{\text{call}} \approx 1.3426 \).

Then, we give the curve graphs of lookback call option pricing formula with different parameters as follows. Figure 1(a) shows that the price \( f_{\text{call}} \) is a decreasing function to the strike \( K \) when other parameters remain unchanged. That is because, the higher is the striking price, the less likely is it to be executed, and the smaller pricing is the option. Figure 1(b) shows that the price \( f_{\text{call}} \) is an increasing function to the starting value \( X_0 \) when other parameters remain unchanged. That is because, the higher is the spot price, the higher is the option price. Figure 1(c) shows that the price \( f_{\text{call}} \) is an increasing function to the interest rate \( \sigma \) and Figure 1(d) shows that the price \( f_{\text{call}} \) is also an increasing function to the maturity data \( T \) when other parameters remain unchanged. These laws are in line with common sense in financial markets.
5. Lookback put option pricing formula with the fixed strike. Suppose that a lookback put option has a strike price $K$ and an expiration time $T$. Then the payoff from buying a lookback put option is the value of

$$\left(K - \min_{0 \leq t \leq T} X_t\right)^+$$

over the time interval $[0, T]$. Considering the time value of money resulted from the bond, the present value of this payoff is the value of

$$\exp(-rT) \left(K - \min_{0 \leq t \leq T} X_t\right)^+.$$

Hence the price of lookback put option should be the expected present value of the payoff, that is,

$$f_{\text{put}} = \exp(-rT)E \left[\left(K - \min_{0 \leq t \leq T} X_t\right)^+\right].$$

**Theorem 5.1.** Suppose that a lookback put option for the stock model (3) has a strike price is $K$ and an expiration time $T$. Then the lookback put option pricing
LOOKBACK OPTION PRICING PROBLEM OF MRSM IN UNCERTAIN ...

formula is

\[ f_{\text{put}} = \exp(-rT) \int_0^1 \max_{0 \leq t \leq T} \left( K - X_0 \cdot \exp \left( (\sigma \Phi^{-1}(\alpha) - a) t \right) \right. \]

\[ - \frac{m}{a - \sigma \Phi^{-1}(\alpha)} \left[ 1 - \exp \left( (\sigma \Phi^{-1}(\alpha) - a) t \right) \right] + \left. d\alpha \right. \]

where

\[ \Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}. \]

Proof. We easily know \( \exp(-rT) \left( K - \min_{0 \leq t \leq T} X_t \right) \) is a decreasing function of \( X_t \), and we have

\[ \left( K - \min_{0 \leq t \leq T} X_t \right)^+ = \left( K + \max_{0 \leq t \leq T} (-X_t) \right)^+ = \max_{0 \leq t \leq T} (K - X_t)^+. \]

From Theorem 2.9, we can get that the inverse uncertainty distribution function of

\[ \exp(-rT) \max_{0 \leq t \leq T} (K - X_t)^+ \]

is

\[ \Psi^{-1}(\alpha) = \exp(-rT) \max_{0 \leq t \leq T} (K - X_t^{1-\alpha})^+. \]

From Theorem 3.2, we obtain that the \( \alpha \)-path is

\[ X_t^{1-\alpha} = X_0 \cdot \exp \left( \left( \frac{\sqrt{3} \sigma}{\pi} \ln \frac{1 - \alpha}{\alpha} - a \right) t \right) \]

\[ + \frac{m}{a - \sqrt{3} \sigma \pi \ln \frac{1 - \alpha}{\alpha}} \left[ 1 - \exp \left( \left( \frac{\sqrt{3} \sigma}{\pi} \ln \frac{1 - \alpha}{\alpha} - a \right) t \right) \right]. \]

Therefore,

\[ f_{\text{put}} = \exp(-rT) E \left[ \max_{0 \leq t \leq T} (K - X_t)^+ \right] \]

\[ = \exp(-rT) \int_0^1 \Psi^{-1}(\alpha) d\alpha \]

\[ = \exp(-rT) \int_0^1 \max_{0 \leq t \leq T} \left( K - X_0 \cdot \exp \left( (\sigma \Phi^{-1}(1-\alpha) - a) t \right) \right. \]

\[ - \frac{m}{a - \sigma \Phi^{-1}(1-\alpha)} \left[ 1 - \exp \left( (\sigma \Phi^{-1}(1-\alpha) - a) t \right) \right] + \left. d\alpha \right. \]

\[ = \exp(-rT) \int_0^1 \max_{0 \leq t \leq T} \left( K - X_0 \cdot \exp \left( (\sigma \Phi^{-1}(\alpha) - a) t \right) \right. \]

\[ - \frac{m}{a - \sigma \Phi^{-1}(\alpha)} \left[ 1 - \exp \left( (\sigma \Phi^{-1}(\alpha) - a) t \right) \right] + \left. d\alpha \right. \]

The lookback put option pricing formula with the fixed strike is verified. \qed

According to Theorem 5.1, the algorithm to calculate the lookback put option price of the stock model (3) is designed as below.

**Step 0:** Set \( \alpha_i = i/N \) and \( t_j = jT/M, i = 1, 2, \ldots, N - 1, j = 1, 2, \ldots, M \), where \( N \) and \( M \) are two large numbers.

**Step 1:** Set \( i = 0 \).
Step 2: Set $i \leftarrow i + 1$.
Step 3: Set $j = 0$, and $G_{t_0}^{\alpha_i} = 0$.
Step 4: Set $j \leftarrow j + 1$.
Step 5: Calculate the stock price of the time $t_j$

\[
G_{t_j}^{\alpha_i} = X_0 \cdot \exp \left( \left( \frac{\sqrt{3} \sigma}{\pi} \ln \frac{1 - \alpha_i}{\alpha_i} - a \right) t_j \right)
\]

\[
+ \frac{m}{a - \sqrt{3} \sigma \ln \frac{1 - \alpha_i}{\alpha_i}} \left[ 1 - \exp \left( \left( \frac{\sqrt{3} \sigma}{\pi} \ln \frac{1 - \alpha_i}{\alpha_i} - a \right) t_j \right) \right].
\]

If $G_{t_j}^{\alpha_i} \leq G_{t_{j-1}}^{\alpha_i}$ and $j < M$, then return to Step 4. If $G_{t_j}^{\alpha_i} \leq G_{t_{j-1}}^{\alpha_i}$ and $j = M$, then jump to Step 7.

Step 6: Set $G_{t_j}^{\alpha_i} \leftarrow G_{t_{j-1}}^{\alpha_i}$, if $j < M$, then return to Step 4.
Step 7: Calculate $K - G_{t_j}^{\alpha_i}$.
Step 8: Set $G^{\alpha_i} \leftarrow \max \left( 0, K - G_{t_j}^{\alpha_i} \right)$. If $i < N - 1$, then return to Step 2.
Step 9: Calculate the lookback put option price is

\[
f_{\text{put}} \leftarrow \exp(-rT) \frac{1}{N - 1} \sum_{i=1}^{N-1} G^{\alpha_i}.
\]

Example 5.1. Assume the initial value of the stock price is $X_0 = 30$, and other parameters of the stock price are $a = 0.06$, $K = 29$, $r = 0.08$, $m = 1.8$ and $\sigma = 0.75$. Then, the price of the stock price with a maturity data $T = 0.25$ is $f_{\text{put}} \approx 1.0542$.

Then, we give the curve graphs of lookback put option pricing formulas with different parameters as follows. Figure 2(a) shows that the price $f_{\text{put}}$ is an increasing function to the strike $K$ when other parameters remain unchanged. That is because, the higher is the striking price, the more likely is it to be executed, and the higher pricing is the option. Figure 2(b) shows that the price $f_{\text{put}}$ is a decreasing function with respect to the starting value $X_0$ when other parameters remain unchanged. That is because, the higher is the spot price, the smaller is the option price. Figure 2(c) shows that the price $f_{\text{put}}$ is an increasing function with respect to the interest rate $\sigma$ and Figure 2(d) shows that the price $f_{\text{put}}$ is also an increasing function with respect to the maturity data $T$ when other parameters remain unchanged. These laws are in line with common sense in financial markets.

6. Conclusions. This paper mainly investigated the lookback option pricing problem for the mean-reverting stock model in an uncertain environment. It considered the lookback call option and lookback put option, and derived their options pricing formulas with the fixed strike. Subsequently, the corresponding numerical methods were designed to calculate the price of the lookback options, and two numerical experiments were performed. Future research could consider the barrier option pricing problem based on the mean-reverting uncertain stock model.

REFERENCES

[1] X. Chen, American option pricing formula for uncertain financial market, Int. J. Oper. Res. (Taichung), 8 (2011), 27–32.
[2] X. Chen and B. Liu, Existence and uniqueness theorem for uncertain differential equations, Fuzzy Optim. Decis. Mak., 9 (2010), 69–81.
[3] X. Chen, Y. Liu and D. A. Ralesu, Uncertain stock model with periodic dividends, Fuzzy Optim. Decis. Mak., 12 (2013), 111–123.
Lookback Option Pricing Problem of Mrsm in Uncertain ...

Figure 2. Lookback put option price $p_{put}$ with different parameters

(a) Price with respect to $K$
(b) Price with respect to $X_0$
(c) Price with respect to $\sigma$
(d) Price with respect to $T$

[4] Y. Gao, X. Yang and Z. Fu, Lookback option pricing problem of uncertain exponential Ornstein-Uhlenbeck model, *Soft Computing*, 22 (2018), 5647–5654.
[5] X. Ji and J. Zhou, Option pricing for an uncertain stock model with jumps, *Soft Computing*, 19 (2015), 3323–3329.
[6] A. Kolmogorov, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Springer-Verlag, Berlin-New York, 1973.
[7] B. Liu, Uncertainty theory. An introduction to its axiomatic foundations, in *Studies in Fuzziness and Soft Computing*, 154, Springer-Verlag, Berlin, 2004.
[8] B. Liu, Fuzzy process, hybrid process and uncertain process, *Journal of Uncertain Systems*, 2 (2008), 3–16.
[9] B. Liu, Some research problems in uncertainty theory, *Journal of Uncertain Systems*, 3 (2009), 3–10.
[10] B. Liu, Toward uncertain finance theory, *J. Uncertain. Anal. Appl.*, 1 (2013).
[11] Y. Liu, X. Chen and D. A. Ralescu, Uncertain currency model and currency option pricing, *International Journal of Intelligent Systems*, 30 (2015), 40–51.
[12] J. Peng and K. Yao, A new option pricing model for stocks in uncertainty markets, *Int. J. Oper. Res. (Taichung)*, 8 (2011), 18–26.
[13] Y. Sun and T. Su, Mean-reverting stock model with floating interest rate in uncertain environment, *Fuzzy Optim. Decis. Mak.*, 16 (2017), 235–255.
[14] M. Tian, X. Yang and Y. Zhang, Barrier option pricing problem of mean-reverting stock model in uncertain environment, *Math. Comput. Simulation*, 166 (2019), 126–143.
[15] M. Tian, X. Yang and S. Kar, Equity warrants pricing problem of mean-reverting model in uncertain environment, *Phys. A*, 531 (2019), 9 pp.
[16] X. Yang and J. Gao, Uncertain differential games with application to capitalism, *J. Uncertain. Anal. Appl.*, 1 (2013), Art. 17.
[17] X. Yang and J. Gao, Linear-quadratic uncertain differential games with application to resource extraction problem, *IEEE Transactions on Fuzzy Systems*, 24 (2016), 819–826.
12 MIAO TIAN, XIANGFENG YANG AND YI ZHANG

[18] X. Yang and K. Yao, Uncertain partial differential equation with application to heat conduction, Fuzzy Optim. Decis. Mak., 16 (2017), 379–403.
[19] X. Yang, Z. Zhang and X. Gao, Asian-barrier option pricing formulas of uncertain financial market, Chaos Solitons Fractals, 123 (2019), 79–86.
[20] K. Yao, No-arbitrage determinant theorems on mean-reverting stock model in uncertain market, Knowledge Based Systems, 35 (2012), 259–263.
[21] K. Yao, Extreme values and integral of solution of uncertain differential equation, J. Uncertain. Anal. Appl., 1 (2013), Art. 2.
[22] K. Yao and X. Chen, A numerical method for solving uncertain differential equations, J. Intell. Fuzzy Systems, 25 (2013), 825–832.
[23] K. Yao, Uncertain contour process and its application in stock model with floating interest rate, Fuzzy Optim. Decis. Mak., 14 (2015), 399–424.
[24] K. Yao, Uncertain Differential Equations, Springer Uncertainty Research, Springer-Verlag, Berlin, 2016.
[25] X. Yu, A stock model with jumps for uncertain markets, Internat. J. Uncertain. Fuzziness Knowledge-Based Systems, 20 (2012), 421–432.
[26] Z. Zhang and W. Liu, Geometric average asian option pricing for uncertain financial market, Journal of Uncertain Systems, 8 (2014), 317–320.
[27] Y. Zhu, Uncertain optimal control with application to a portfolio selection model, Cybernetics and Systems, 41 (2010), 535–547.

Received February 2019; revised January 2020.

E-mail address: tianmiao@ruc.edu.cn
E-mail address: yangxf@uibe.edu.cn
E-mail address: ethanzhang@ruc.edu.cn