Borel Spectrum of Operators on Banach Spaces

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Abstract. The paper investigates the variation of the spectrum of operators in infinite dimensional Banach spaces. In particular, it is shown that the spectrum function is Borel from the space of bounded operators on a separable Banach space; equipped with the strong operator topology, into the Polish space of compact subsets of the closed unit disc of the complex plane; equipped with the Hausdorff topology. Remarks and results are given when other topologies are used.

1. Preliminary

Let $X$ be an infinite dimensional Banach space. We denote by $T$ an arbitrary bounded operator on $X$ and by $I$ the identity operator on $X$. Let $\mathbb{D}$ be the closed unit disc of the complex plane $\mathbb{C}$. The restriction on $\mathbb{D}$ of the spectrum of an operator $T$, denoted by $\sigma(T)$, is defined as follows:

$$\sigma(T) := \{ \lambda \in \mathbb{D}; \ (T - \lambda I) \text{ is not invertible} \}.$$

Recently, essential spectra of some matrix operators on Banach spaces (see [3]) and spectra of some block operator matrices (see [5]) were investigated, with applications to differential and transport operators. This paper investigates the variations of the spectrum $\sigma(T)$ as $T$ varies over the space $L(X)$ of all bounded operator on the Banach space $X$. First, we introduce the sets and the topologies required for this study. We denote by

- $\mathcal{K}(\mathbb{D})$ the set of all compact subsets of the closed unit disc $\mathbb{D}$ of the complex plane $\mathbb{C},$
- $\sigma$ the spectrum function defined from $L(X)$ into $\mathcal{K}(\mathbb{D})$ that maps an operator $T$ to its spectrum $\sigma(T)$.

The set $\mathcal{K}(\mathbb{D})$ is endowed with the Hausdorff topology generated by the families of all subsets in one of the following forms

$$\{ F \in \mathcal{K}(\mathbb{D}); \ F \cap V \neq \emptyset \} \text{ and } \{ F \in \mathcal{K}(\mathbb{D}); \ F \subseteq V \}.$$
for $V$ an open subset of $\mathbb{D}$. Therefore, $K(\mathbb{D})$ is a Polish space, i.e., a separable metrizable complete space, since $\mathbb{D}$ is Polish (see [7], [8] or [2]). It is shown below that we can reduce the families that generate the above Hausdorff topology.

**Proposition 1.1.** Let $K(\mathbb{D})$ be the set of compact subsets of the closed unit disc $\mathbb{D}$. Then $K(\mathbb{D})$ equipped with the Hausdorff topology is a Polish space; where the Borel structure is generated by one of the following two families

$$
\left\{ \{ K \in K(\mathbb{D}) : K \cap V \neq \emptyset \}; V \text{ open in } \mathbb{D} \right\}
$$

$$
\left\{ \{ K \in K(\mathbb{D}) : K \subset V \}; V \text{ open in } \mathbb{D} \right\}.
$$

**Proof.** Let $V$ be an open subset of $D$. There exists a decreasing sequence of open subsets $(O_n)_{n \in \mathbb{N}}$ such that $V^c = \bigcap_{n \in \mathbb{N}} O_n$; for example $O_n = \{ x \in \mathbb{D} : \text{dist}(x, V^c) \leq \frac{1}{n} \}$. We have

$$
\{ K \in K(\mathbb{D}) : K \cap V \neq \emptyset \}^c = \bigcap_{n \in \mathbb{N}} \{ K \in K(\mathbb{D}) : K \subseteq O_n \}.
$$

On the other hand,

$$
\{ K \in K(\mathbb{D}) : K \subseteq V \}^c = \{ K \in K(\mathbb{D}) : K \cap V^c \neq \emptyset \}
$$

$$
= \{ K \in K(\mathbb{D}) : K \cap O_n \neq \emptyset, \forall n \in \mathbb{N} \}
$$

$$
= \bigcap_{n \in \mathbb{N}} \{ K \in K(\mathbb{D}) : K \cap O_n \neq \emptyset \}.
$$

Indeed, if for all $n \in \mathbb{N}$, there exists $x_n \in K \cap O_n$, then there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ that converges to $x \in K$, and $x \in \bigcap_{n} O_n$ since $(O_n)_n$ is decreasing. \hfill \Box

### 2. Norm Operator Topology and the Spectrum Function

We equip $L(X)$ with the canonical norm of operators defined by $\|T\| = \sup_{x \in B_X} \|T(x)\|$, where $B_X$ is the unit ball of $X$. Note that the map $\sigma : T \mapsto \sigma(T)$ is not continuous when $L(X)$ is endowed with its canonical norm. Indeed, the operators $T_n = (1 + \frac{1}{n})I$ converge to the identity $I$ while $\sigma(T_n) = \emptyset$ and $\sigma(I) = \{1\}$. However, we have the following result.

**Proposition 2.1.** Let $X$ be a Banach space, $(L(X), \|\|)$ the space of bounded operators equipped with the norm of operators, and $K(\mathbb{D})$ the set of compact subsets of the unit disc $\mathbb{D}$ equipped with the Hausdorff topology. Then the spectrum map

$$
\sigma : (L(X), \|\|) \longrightarrow K(\mathbb{D})
$$

$$
T \mapsto \sigma(T)
$$
is upper-semi continuous.

Proof. Let \( V \) be an open subset \( \mathbb{D} \). By proposition 1.1, it is only need to show that the set \( O_V = \{ T \in L(X); \sigma(T) \subseteq V \} \) is \( \| \cdot \| \)-open in \( L(X) \). Let \( T_0 \) be fixed in \( O_V \). Since \( \sigma(T_0) \cap \mathbb{D} \subseteq V \), then for all \( \lambda \in \mathbb{D} \setminus V \), the operator \( (T_0 - \lambda I)^{-1} \) is invertible and the map \( \lambda \in \mathbb{D} \setminus V \mapsto (T_0 - \lambda I)^{-1} \) is continuous (see [?]). It follows that 
\[
\sup_{\lambda \in \mathbb{D} \setminus V} \| (T - \lambda I)^{-1} \| < +\infty
\]
since \( \mathbb{D} \setminus V \) is compact. Put 
\[
\delta = \inf_{\lambda \in \mathbb{D} \setminus V} \frac{1}{\| (T_0 - \lambda I)^{-1} \|} > 0.
\]
Let \( T \in L(X) \) such that \( \| T - T_0 \| < \delta \). For any \( \lambda \in \mathbb{D} \setminus V \) we have 
\[
\| (T - \lambda I) - (T_0 - \lambda I) \| = \| T - T_0 \| < \frac{1}{\| (T_0 - \lambda I)^{-1} \|}.
\]
Thus, \( (T - \lambda I) \) is invertible and hence \( \lambda \notin \sigma(T) \). In other terms, \( \sigma(T) \subseteq V \) for all \( T \in L(X) \) with \( \| T - T_0 \| < \delta \). Therefore \( O_V \) is an open subset of \( (L(X), \| \cdot \|) \). \( \square \)

3. Strong Operator Topology and the Spectrum Function

Consider now \( L(X) \) equipped with the strong operator topology \( S_{op} \) (see [3]). In general, \( L(X) \) equipped with the strong operator topology is not a polish space (since it is not a Baire space). However, if \( X \) is separable, then \( (L(x), S_{op}) \) is a standard Borel space. Indeed, it is Borel-isomorph to a Borel subset of the Polish space \( X^\mathbb{N} \) equipped with the norm product topology via the map 
\[
\varphi : (L(X), S_{op}) \longrightarrow (X^\mathbb{N}, \mathcal{P})
\]
\[
T \longmapsto (Tz_n)_{n \in \mathbb{N}},
\]
where \( \{ z_n, n \in \mathbb{N} \} \) is a dense \( \mathbb{Q} \)-vector space in \( X \).

Let us check how this topology on \( L(x) \) affects the spectrum function.

**Theorem 3.1.** For any separable infinite dimensional Banach \( X \), the map 
\[
\sigma : L(X) \longrightarrow K(\mathbb{D})
\]
\[
T \longmapsto \sigma(T),
\]
which maps a bounded operator to its spectrum, is Borel when \( L(X) \) is endowed with the strong operator topology \( S_{op} \) and \( K(\mathbb{D}) \) with the Hausdorff topology.

**Proof.** As \( K(\mathbb{D}) \) is equipped with the Hausdorff topology, it follows from the proposition [1.3] that it is enough to show that for any open subset \( V \) of the disc \( \mathbb{D} \), the subset 
\[
E_V = \{ T \in L(X) : \sigma(T) \cap V \neq \emptyset \}
\]
is Borel in \((L(X), S_{op})\).

Let \(V\) be a fixed open subset of \(\mathbb{D}\). We have

\[
E_V = P_{L(X)}(\Omega),
\]

where \(P_{L(X)}\) stands for the canonical projection of \(L(X) \times \mathbb{D}\) onto \(L(X)\), and

\[
\Omega = \{(T, \lambda) \in L(X) \times V : \lambda \in \sigma(T)\}.
\]

By a descriptive set theory result from [9], to show that \(E_V\) is a Borel set it suffices to show that \(\Omega\) is a Borel set with \(K_\sigma\) vertical sections.

For \(T \in L(X)\), the vertical section of the set \(\Omega \subseteq L(X) \times \mathbb{D}\) along the direction \(T\) is given by

\[
\Omega(T) = \{\lambda \in \mathbb{D} : (T, \lambda) \in \Omega\} = \{\lambda \in \mathbb{D} : \lambda \in V \cap \sigma(T)\} = \sigma(T) \cap V.
\]

Thus, it is a \(K_\sigma\) of \(\mathbb{D}\).

Now, we need to prove that \(\Omega\) is a Borel set. Put

\[
\Delta = \{(T, \lambda) \in L(X) \times \mathbb{D} : \lambda \in \sigma(T)\}.
\]

Therefore

\[
\Omega = \Delta \cap L(X) \times V,
\]

Hence, to finish the proof, it is enough to prove the following claim.

**Claim:** \(\Delta\) is a Borel set of \(L(X) \times \mathbb{D}\).

First, note that \(\Delta = A \cup B\) with

- \(A = \{(T, \lambda) \in L(X) \times \mathbb{D} : T - \lambda I\text{ is not isomorph to its range }\}\)
- \(B = \{(T, \lambda) \in L(X) \times \mathbb{D} : (T - \lambda I)(X)\text{ is not dense in }X\}\).

Indeed, if \(T - \lambda I\) is an isomorphism onto its range, then \((T - \lambda I)(X)\) is a closed subspace that will be strict if \(\lambda \in \sigma(T)\), and thus not dense in \(X\).

On the other hand, since \(X\) is separable, there exists a countable and dense subset \(\mathcal{Y}\) in the sphere \(S_X\) of \(X\), and there exists a dense sequence \(\{x_n\}_{n \in \mathbb{N}}\) in \(X\).

Now, we will show that \(A\) and \(B\) are Borel sets. Let \((T, \lambda) \in L(X) \times \mathbb{D}\). From the definition of \(A\), we have \((T, \lambda) \in A\) if and only if

\[
\exists (z_n)_{n \in \mathbb{N}} \subseteq S_X : \lim_{n \to \infty} \|(T - \lambda I)z_n\| = 0.
\]

In other term, this is equivalent to

\[
\exists (z_n)_{n \in \mathbb{N}} \subseteq \mathcal{Y} : \forall k \geq 1 \exists N_k \in \mathbb{N} \forall n \geq N_k : \|(T - \lambda I)z_n\| < \frac{1}{k}.
\]
By choosing the subsequence \((z_{N_k})_{k \in \mathbb{N}}\) instead of \((z_n)_{n \in \mathbb{N}}\), the previous statement is equivalent to
\[
\exists (z_n)_{n \in \mathbb{N}} \subseteq \mathcal{Y}, \forall k \geq 1 \exists N_k \in \mathbb{N} : \| (T - \lambda I) z_{N_k} \| < \frac{1}{k},
\]
or again,
\[
\forall k \geq 1, \exists x \in \mathcal{Y} : \| T x - \lambda x \| < \frac{1}{k}.
\]
Therefore,
\[
A = \bigcap_{k \geq 1} \bigcup_{x \in \mathcal{Y}} A^x_k
\]
with
\[
A^x_k = \{(T, \lambda) \in L(X) \times \mathbb{D} : \| T x - \lambda x \| < \frac{1}{k} \}.
\]
Since \(L(X)\) is equipped with the strong operator convergence \(S_{op}\), it follows that \(A^x_k\) are open sets. Hence, \(A\) is a Borel set.

On the other hand, “\((T - \lambda I)X\) is not dense in \(X\)” is equivalent to
\[
\exists y \in S_X \text{ and } \exists k \geq 1 \text{ such that } \forall x \in X : \| y - (T - \lambda I)x \| \geq \frac{1}{k},
\]
or again,
\[
\exists y \in \mathcal{Y} \text{ and } \exists k \geq 1 \text{ such that } \forall n \in \mathbb{N} : \| y - (T - \lambda I)x_n \| \geq \frac{1}{k}.
\]
Therefore
\[
B = \bigcup_{y \in \mathcal{Y}} \bigcup_{k \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} B^y_{k,n}
\]
with
\[
B^y_{k,n} = \{(T, \lambda) \in L(X) \times \mathbb{D} : \| y - (T - \lambda I)x_n \| \geq \frac{1}{k} \}.
\]
Similarly to \(A^x_k\), it is not difficult to see that the sets \(B^y_{k,n}\) are Borel sets. Hence \(B\) is also a Borel set. This proves the claim and ends the proof of the theorem 3.1.

\[\square\]

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