LONGTIME BEHAVIOR AND WEAK-STRONG UNIQUENESS FOR A NONLOCAL POROUS MEDIA EQUATION

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ABSTRACT. In this manuscript we consider a non-local porous medium equation with non-local diffusion effects given by a fractional heat operator

\begin{align*}
\begin{aligned}
\partial_t u &= \text{div}(u\nabla p), \\
\partial_t p &= -(\Delta)^s p + u^2,
\end{aligned}
\end{align*}

in three space dimensions for $3/4 \leq s < 1$ and analyze the long time asymptotics. The proof is based on energy methods and leads to algebraic decay towards the stationary solution $u = 0$ and $\nabla p = 0$ in the $L^2(\mathbb{R}^3)$-norm. The decay rate depends on the exponent $s$. We also show weak-strong uniqueness of solutions and continuous dependence from the initial data. As a side product of our analysis we also show that existence of weak solutions, previously shown in [5] for $3/4 \leq s \leq 1$, holds for $1/2 < s \leq 1$ if we consider our problem in the torus.

1. Introduction

We consider the following porous medium equation with non-local diffusion effects:

\begin{align*}
\begin{aligned}
\partial_t u &= \text{div}(u\nabla p), \\
\partial_t p &= -(\Delta)^s p + u^2,
\end{aligned}
\end{align*}

(1)

For all $x \in \mathbb{R}^3$, the functions $u(x, t) \geq 0$ and $p(x, t) \geq 0$ denote respectively the density and the pressure. In a previous paper [5] we have introduced the model and showed existence of weak solutions. In the current manuscript we study the long time behavior and weak-strong uniqueness. The model describes the time evolution of a density function $u$ that evolves under the continuity equation

$$
\partial_t u = \text{div}(uv),
$$

where the velocity is conservative, $v = \nabla p$, and $p$ is related to $u^2$ by the inverse of the fractional heat operator $\partial_t + (\Delta)^s$. Equation (1) is the parabolic-parabolic version of a
problem recently studied in [3]:
\[ \partial_t u = \text{div} \left( |u|^{\alpha-1}(|u|^{m-2}u) \right). \]  
(2)

Note in fact that for \( m = 3 \) and \( \alpha = 2 - 2s \) equation (2) reduces to the parabolic-elliptic version of (1).

As we already mentioned, existence of weak solutions to (1) with \( 3/4 \leq s \leq 1 \) was recently studied in [5]. The introduction of \( \partial_t p \) introduced several complications due to the non-locality in time relation between \( u \) and \( p \). Consequence of this nonlocality is that techniques such as maximum principle and Stroock-Varopoulos inequality do not work in the current parabolic-parabolic setting. Existence results for \( s < 3/4 \) is still an open problem, except in the case when \( x \in \mathbb{T}^3 \) (see Theorem 4).

Existence of weak solutions, regularity and finite speed of propagation for a linear parabolic-elliptic version of (1)
\[ \partial_t u = \text{div} (u \nabla p), \quad p = (\Delta)^{-s}u, \quad 0 < s \leq 1, \]  
(3)

has been considered in [9, 8, 21, 6, 2] and long-time asymptotics in [7]. In [7] the authors perform a self-similar rescaling and rewrite (3) as a non-local Fokker-Planck equation with confinement potential. Entropy estimates lead to algebraic decay of the solution towards self-similar solutions called fractional Barenblatt functions. In this context we also recall a very recent result [1] that shows that solutions to the fractional drift-diffusion-Poisson model
\[ \partial_t u = -(\Delta)^{-\alpha} u + \text{div} (u \nabla p), \quad p = (\Delta)^{-1} u, \quad 0 < \alpha \leq 1, \]  

converge algebraically, as time grows, towards the fundamental solution to the linear fractional heat equation \( \partial_t u = -(\Delta)^{-\alpha} u \).

System (1) is also reminiscent to a well-studied macroscopic model proposed in [11] for phase segregation in particle systems with long range interaction:
\[ \begin{aligned} 
\partial_t u &= \Delta u + \text{div} (\sigma(u) \nabla p), \\
p &= K * u.
\end{aligned} \]  
(4)

Here \( \sigma(u) := u(1-u) \) denotes the mobility of the system and \( K \) a bounded, symmetric and compactly supported kernel. Several variants of (4) have been considered in the literature and we refer to [19, 11, 13, 12] and references therein for more detailed discussions on this topic. We also mention [16] for the study of a deterministic particle method for heat and FokkerPlanck equations of porous media type where the non-locality appears in the coefficients.

The main results of this manuscript are summarized in the following three theorems:

**Theorem 1 (Long-time behavior).** Let \( 3/4 \leq s < 1 \). Assume that \( u, p \) are weak solutions in the sense of Theorem 1 in [5] with initial data \( p_0 \) that satisfies \( \int_{\mathbb{R}^3} |x||\nabla p_0|^2 \, dx < \infty \). Let
\[ H[u, p] := \int_{\mathbb{R}^3} \left( u^2 + \frac{1}{2} |\nabla p|^2 \right) \, dx. \]
There exists a constant $C > 0$ such that
\[ H[u(t), p(t)] \leq Ct^{-\lambda}, \quad t > 1, \]
with
\[ \lambda = \frac{3(1 - s)}{2s(5 + 2s)} > 0. \]
Consequently we have strong convergence of $(u, \nabla p)$ towards $(0, 0)$ in $L^2(\mathbb{R}^3)$ with algebraic decay rate $t^{-\lambda/2}$.

The main idea of the proof of Theorem 1 relies on entropy methods. Throughout this entire paper we will denote with $C$ any generic positive constant independent of $T$. The functional $H[u, p] = \int_{\mathbb{R}^3} (u^2 + \frac{1}{2} |\nabla p|^2) \, dx$ is a Lyapunov functional for (1) and satisfies the bound
\[ \int_{\mathbb{R}^3} (u^2 + \frac{1}{2} |\nabla p|^2) \, dx + \int_0^T \int_{\mathbb{R}^3} |(-\Delta)^{s/2} \nabla p|^2 \, dx \, dt = \int_{\mathbb{R}^3} \left( u_{in}^2 + \frac{1}{2} |\nabla p_{in}|^2 \right) \, dx. \]

Indeed, formal computations show that
\[ \frac{d}{dt} \int_{\mathbb{R}^3} u^2 \, dx = \langle \text{div} (u \nabla p), 2u \rangle = -\int_{\mathbb{R}^3} \nabla u^2 \cdot \nabla p \, dx \]
\[ = -\frac{d}{dt} \int_{\mathbb{R}^3} \frac{|\nabla p|^2}{2} \, dx - \int_{\mathbb{R}^3} |(-\Delta)^{s/2} \nabla p|^2 \, dx, \]
after testing the equation for $p$ against $\Delta p$. This leads to
\[ \frac{d}{dt} H[u, p] + \int_{\mathbb{R}^3} |(-\Delta)^{s/2} \nabla p|^2 \, dx = 0, \quad t > 0. \]

The key (and new!) observation that leads to the proof of decay is that the expression $\int_0^T \int_{\mathbb{R}^3} \nabla (u^2) \cdot \nabla p \, dx \, dt$ defines a scalar product $A(\cdot, \cdot)$, namely
\[ \int_0^T \int_{\mathbb{R}^3} \nabla (u^2) \cdot \nabla p \, dx \, dt = A(\nabla (u^2), \nabla (u^2)), \]
and any sequence that is Cauchy in the $A$-norm converges almost everywhere. Moreover by writing $A(\nabla (u^2), \nabla (u^2))$ in terms of Fourier transform we get an improved bound $\|u\|_{L^2(0,T,L^2(\mathbb{R}^3))^2} \leq T^\alpha$ with $\alpha < 1$. This combined with a sharper estimate for $\|\nabla p\|_{L^2(0,T,L^2(\mathbb{R}^3))^2}$ yields our algebraic decay.

Our second main theorem concerns a weak-strong uniqueness result:

**Theorem 2** (Weak-strong uniqueness, continuous dependence on data). Let $\frac{3}{4} \leq s \leq 1$. Assume that $v$ is a strong solution to
\[ \partial_t v = \text{div}(v \nabla q), \quad \partial_t q + (-\Delta)^{s/2} q = v^2 \quad \text{in} \quad \mathbb{R}^3 \times (0, \infty), \]
\[ v(0) = v_0 \quad \text{in} \quad \mathbb{R}^3, \]
such that
\[ \exists \nu > 0 : \nabla v \in L^\infty(0, \infty; L^{\frac{12}{5\nu}}(\mathbb{R}^3)), \quad \sup_{\mathbb{R}^3 \times (0, \infty)} \Delta q < \infty. \]

Then there exists a constant \( K > 0 \) such that, for any \( u \) weak solution to (1) according to Theorem 1 in [5]:
\[ H[(u(t), p(t)|(v(t), q(t))] \leq e^{Kt}H[(u_0, p_0)|(v_0, q_0)], \quad t > 0, \]
where \( H[(u, p)|(v, q)] \) denotes the relative entropy between \( u \) and \( v \):
\[ H[(u, p)|(v, q)] = \int_{\mathbb{R}^3} \left( (u - v)^2 + \frac{1}{2}|\nabla (p - q)|^2 \right) dx. \]

In particular \( u \equiv v \) if \( u_0 \equiv v_0 \). This means that if there exists a strong solution, then any weak solution with the same initial data coincides with it.

The weak-strong uniqueness is a familiar concept in the field of fluid-dynamic equations and conservation laws [17, 18, 20, 10]. It is not a uniqueness result in the standard form: it states in fact that if strong solutions exist (still an open question for (1)), then they are unique even when compared to all weak solutions. As in the case of fluid-dynamic equations, our notion of weak solution includes an energy inequality and such energy inequality is fundamental for the proof of Theorem 2 (for the case of Navier-Stokes equation Scheffer and Shnirelman gave a counterexample to weak-strong uniqueness if bounds for the energy functional are removed).

Before stating our last result we recall for completeness the existence theorem for weak solutions proven in [5]:

**Theorem 3.** (Theorem 1 in [5]) Let \( 3/4 \leq s \leq 1 \) and \( u_0, p_0 : \mathbb{R}^3 \to (0, +\infty) \) be functions such that \( u_0, p_0 \in L^1(\mathbb{R}^3), \int_{\mathbb{R}^3} u_0^2 + |
abla p_0|^2 \, dx < +\infty \) and \( \int_{\mathbb{R}^3} u_0 \gamma(x) \, dx < +\infty \), with \( \gamma(x) := \sqrt{1 + |x|^2} \). Let \( q > 3/s \). There exist functions \( u, p : \mathbb{R}^3 \times [0, \infty) \to [0, +\infty) \) such that for every \( T > 0 \)
\[ u \in L^\infty(0, T, L^1 \cap L^2(\mathbb{R}^3)), \quad p \in L^\infty(0, T, H^1 \cap L^1(\mathbb{R}^3)), \]
\[ p \in L^2(0, T, H^{s+1}(\mathbb{R}^3)), \quad \sup_{[0, T]} \int_{\mathbb{R}^3} u \gamma \, dx < +\infty, \]
\[ \partial_t u \in L^2(0, T, (W^{1,q}(\mathbb{R}^3))'), \quad \partial_t p \in L^2(0, T, (L^2 \cap L^4(\mathbb{R}^3)))', \]
which satisfy the following weak formulation of (1):
\[ \int_0^T \langle \partial_t u, \phi \rangle dt + \int_0^T \int_{\mathbb{R}^3} u \nabla \cdot \nabla \phi \, dx dt = 0 \quad \forall \phi \in L^2(0, T; W^{1,q}(\mathbb{R}^3)), \]
\[ \int_0^T \langle \partial_t p, \psi \rangle dt + \int_0^T \int_{\mathbb{R}^3} ((-\Delta)^s p - u^2) \psi \, dx dt = 0 \quad \forall \psi \in L^2(0, T; L^2 \cap L^4(\mathbb{R}^3)), \]
\[ \lim_{t \to 0^+} u(t) = u_0 \quad \text{in } W^{1,q}(\mathbb{R}^3)', \quad \lim_{t \to 0^+} p(t) = p_0 \quad \text{in } (L^2 \cap L^4(\mathbb{R}^3)'). \]

Here is our extension of Theorem 3 to the torus case:
Theorem 4 (Existence of solutions, torus case). Same assumptions as in Theorem 3 with the exception that $1/2 < s \leq 1$. Then there exists $u : \mathbb{T}^3 \times [0, \infty) \to [0, \infty)$ weak solution to (1) with $\mathbb{R}^3$ replaced by $\mathbb{T}^3$.

The rest of the manuscript is divided into three sections: Section 2 contains proof of Theorem 1, Section 3 the one of Theorem 2 and Section 4 the proof of Theorem 4.

2. Proof of Theorem 1

We first show the following auxiliary result.

Lemma 1. Under the same assumptions of Theorem 1 there exists a constant $C > 0$ such that

$$\sup_{t \in [0, T]} \int_{\mathbb{R}^3} |x| \left( u^2(x, t) + \frac{1}{2} |\nabla p(x, t)|^2 \right) dx \leq CT^{1/2}, \quad T > 1.$$ 

Proof of Lemma 1. The proof is divided into several steps.

Step 1: bound for $\|p(t)\|_{L^2(\mathbb{R}^3)}$. By using $p$ as a test function in the second equation of (1) we get

$$\frac{1}{2} \|p(T)\|_{L^2(\mathbb{R}^3)}^2 - \frac{1}{2} \|p_0\|_{L^2(\mathbb{R}^3)}^2 + \int_0^T \int_{\mathbb{R}^3} |(-\Delta)^{s/2} p|^2 dx dt = \int_0^T \int_{\mathbb{R}^3} u^2 p dx dt$$

$$\leq \int_0^T \|p(t)\|_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} u^2 dx dt \leq C \int_0^T \|p(t)\|_{L^\infty(\mathbb{R}^3)} dt,$$

since $u \in L^\infty(0, \infty; L^2(\mathbb{R}^3))$ thanks to the entropy inequality (1). Moreover, given that $6/(3 - 2s) > 3$, the following Gagliardo-Nirenberg inequality holds

$$\|p\|_{L^\infty(\mathbb{R}^3)} \leq C \|p\|_{L^1(\mathbb{R}^3)}^{1-\xi} \|\nabla p\|_{L^{6/(3-2s)}(\mathbb{R}^3)}^\xi,$$

for some exponent $\xi \in (0, 1)$. On the other hand, integrating the second equation in (1) in $\mathbb{R}^3 \times [0, T]$ yields

$$\|p(T)\|_{L^1(\mathbb{R}^3)} - \|p_0\|_{L^1(\mathbb{R}^3)} = \int_0^T \int_{\mathbb{R}^3} u^2 dx dt,$$

which implies (again thanks to the entropy inequality (5))

$$\|p(T)\|_{L^1(\mathbb{R}^3)} \leq CT, \quad T > 1.$$ 

From (7) and (8) it follows

$$\int_0^T \|p(t)\|_{L^\infty(\mathbb{R}^3)} dt \leq CT^{1-\xi} \int_0^T \|\nabla p\|_{L^{6/(3-2s)}(\mathbb{R}^3)}^{\xi} dt.$$

By applying Hölder’s inequality we get

$$\int_0^T \|p(t)\|_{L^\infty(\mathbb{R}^3)} dt \leq CT^{1-\xi} \left( \int_0^T \|\nabla p\|_{L^{6/(3-2s)}(\mathbb{R}^3)}^{2} dt \right)^{\xi/2} T^{1-\xi/2}.$$
However, Sobolev’s embedding $H^s(\mathbb{R}^3) \hookrightarrow L^{6/(3-2s)}(\mathbb{R}^3)$ and the entropy inequality (5) allow us to write

$$\int_0^T \|\nabla p\|^2_{L^{6/(3-2s)}(\mathbb{R}^3)} dt \leq C \int_0^T \|\nabla (-\Delta)^{s/2} p\|^2_{L^2(\mathbb{R}^3)} dt \leq C,$$

which implies

$$\int_0^T \|p(t)\|_{L^\infty(\mathbb{R}^3)} dt \leq C T^{2-3s/2} \leq C T^2, \quad T > 1.$$ 

The above inequality and (6) lead to

$$\|p(T)\|_{L^2(\mathbb{R}^3)} \leq C T, \quad T > 1.$$ 

**Second step: bound for $\int_0^T \int_{\mathbb{R}^3} u^3 dx dt$.** It was already shown in (5) that $\int_0^T \int_{\mathbb{R}^3} u^3 dx dt \leq C(T)$. Now we need a more accurate estimate on the generic constant $C(T)$. For that consider

$$\int_0^T \int_{\mathbb{R}^3} u^3 dx dt = \int_{\mathbb{R}^3} u(T) p(T) dx - \int_{\mathbb{R}^3} u_0 p_0 dx + \int_0^T \int_{\mathbb{R}^3} u |\nabla p|^2 dx dt$$

$$+ \int_0^T \int_{\mathbb{R}^3} u (-\Delta)^s p dx dt.$$

From (5), (10) we get

$$\int_{\mathbb{R}^3} u(T) p(T) dx \leq \|u\|_{L^\infty(0,\infty; L^2(\mathbb{R}^3))} \|p(T)\|_{L^2(\mathbb{R}^3)} \leq C T, \quad T > 1.$$ 

Hölder inequality yields

$$\int_0^T \int_{\mathbb{R}^3} u |\nabla p|^2 dx dt \leq \int_0^T \left( \int_{\mathbb{R}^3} u^{\frac{2}{3-s}} dx \right)^{\frac{3-s}{2}} \left( \int_{\mathbb{R}^3} |\nabla p|^{\frac{2(2-s)}{3-s}} dx \right)^{1-\frac{2}{3-s}} dt.$$ 

The mass conservation and (5) yields $u \in L^\infty(0,\infty; L^1 \cap L^2(\mathbb{R}^3))$; given that $3/4 \leq s < 1$, an interpolation argument implies $u \in L^\infty(0,\infty; L^{3/2s}(\mathbb{R}^3))$. This relation, together with (9) and (13), yields

$$\int_0^T \int_{\mathbb{R}^3} u |\nabla p|^2 dx dt \leq C.$$ 

Let us apply Hölder inequality to

$$\int_0^T \int_{\mathbb{R}^3} u (-\Delta)^s p dx dt \leq C \left( \int_0^T \|u\|^2_{L^{\frac{6}{1+2s}}(\mathbb{R}^3)} dt \right)^{\frac{1}{2}} \left( \int_0^T \|(-\Delta)^s p\|^2_{L^{\frac{6}{1+2s}}(\mathbb{R}^3)} dt \right)^{\frac{1}{2}}.$$ 

Using Sobolev’s embedding $H^{1+s}(\mathbb{R}^3) \hookrightarrow W^{2s,6/(1+2s)}(\mathbb{R}^3)$ and (5) we can write

$$\int_0^T \|(-\Delta)^s p\|^2_{L^{6/(1+2s)}(\mathbb{R}^3)} dt \leq C \int_0^T \|\nabla (-\Delta)^{s/2} p\|^2_{L^2(\mathbb{R}^3)} dt \leq C.$$
On the other hand, since $1 \leq \frac{6}{5-2s} \leq 2$ and $u \in L^\infty(0, \infty; L^1 \cap L^2(\mathbb{R}^3))$, we deduce by interpolation

$$
\int_0^T \|u\|^2_{L^{\frac{6}{5-2s}}(\mathbb{R}^3)} \, dt \leq CT,
$$

which, together with (15) and (16), yields

$$(17) \quad \int_0^T \int_{\mathbb{R}^3} u(-\Delta)^s p \, dx \, dt \leq CT^{1/2}, \quad T > 1.
$$

From (11), (12), (14), (17) we conclude

$$(18) \quad \int_0^T \int_{\mathbb{R}^3} u^3 \, dx \, dt \leq CT, \quad T > 1.
$$

**Step 3: bound for $\int_{\mathbb{R}^3} |x| \left( u^2 + \frac{1}{2} |\nabla p|^2 \right) \, dx$.** Let us compute

$$
\frac{d}{dt} \int_{\mathbb{R}^3} \frac{|x|}{2} |\nabla p|^2 \, dx = \int_{\mathbb{R}^3} \frac{1}{2} \frac{x}{|x|} \cdot u^2 \nabla p \, dx - \int_{\mathbb{R}^3} |x| \nabla u^2 \cdot \nabla p \, dx,
$$

$$
\frac{d}{dt} \int_{\mathbb{R}^3} \frac{|x|}{2} |\nabla p|^2 \, dx = -\int_{\mathbb{R}^3} |x| \nabla p \cdot \nabla (-\Delta)^s p \, dx + \int_{\mathbb{R}^3} |x| \nabla u^2 \cdot \nabla p \, dx.
$$

Adding the above identities and integrating the resulting equation in the time interval $[0,T]$ we get

$$(19) \quad \int_{\mathbb{R}^3} |x| \left( u^2(T) + \frac{1}{2} |\nabla p(T)|^2 \right) \, dx = \int_{\mathbb{R}^3} |x| \left( u^2_0 + \frac{1}{2} |\nabla p_0|^2 \right) \, dx + I_1 + I_2,
$$

$$
I_1 = -2 \int_0^T \int_{\mathbb{R}^3} \frac{x}{|x|} \cdot u^2 \nabla p \, dx \, dt, \quad I_2 = -\int_0^T \int_{\mathbb{R}^3} |x| \nabla p \cdot \nabla (-\Delta)^s p \, dx \, dt.
$$

Let us now estimate $I_1$. Hölder inequality yields

$$
\frac{1}{2} I_1 \leq \int_0^T \int_{\mathbb{R}^3} u^2 |\nabla p| \, dx \, dt \leq \int_0^T \|\nabla p\|_{L^{\frac{6}{5-2s}}(\mathbb{R}^3)} \|u\|^2_{L^{\frac{12}{5+2s}}(\mathbb{R}^3)} \, dt
$$

$$
\leq \left( \int_0^T \|\nabla p\|^2_{L^{\frac{6}{5-2s}}(\mathbb{R}^3)} \, dt \right)^{1/2} \left( \int_0^T \|u\|^4_{L^{\frac{12}{5+2s}}(\mathbb{R}^3)} \, dt \right)^{1/2}.
$$

Thanks to (9) we deduce

$$
I_1 \leq C \left( \int_0^T \|u\|^4_{L^{\frac{12}{5+2s}}(\mathbb{R}^3)} \, dt \right)^{1/2} \leq C \left( \int_0^T \|u\|^{6-4s}_{L^3(\mathbb{R}^3)} \, dt \right)^{1/2},
$$

using the interpolation

$$
\|u\|_{L^{\frac{12}{5+2s}}(\mathbb{R}^3)} \leq \|u\|^{1-\omega}_{L^2(\mathbb{R}^3)} \|u\|^{\omega}_{L^3(\mathbb{R}^3)}, \quad \omega = \frac{3 - 2s}{2}.
$$

Note that $6-4s \leq 3$ if $s \geq 3/4$. Therefore from Hölder’s inequality and (18) it follows

$$(20) \quad I_1 \leq C \left( \int_0^T \|u\|^{3}_{L^3(\mathbb{R}^3)} \, dt \right)^{\frac{3-2s}{3}} T^{\frac{4s-3}{6}} \leq CT^{\frac{3-2s}{3}} T^{\frac{4s-3}{6}} = CT^{1/2}.
$$
Let us now consider $I_2$:

\begin{equation}
I_2 = -\int_0^T \int_{\mathbb{R}^3} |x|\nabla p \cdot \nabla (-\Delta)^{s/2} p dxdt
\end{equation}

Let us compute

\begin{align*}
(-\Delta)^{s/2}(|x|\nabla p) &= \int_{\mathbb{R}^3} \frac{|x|\nabla p(x) - |y|\nabla p(y)}{|x - y|^{3+s}} dy \\
&= |x| \int_{\mathbb{R}^3} \frac{\nabla p(x) - \nabla p(y)}{|x - y|^{3+s}} dy + \int_{\mathbb{R}^3} \frac{|x| - |y|}{|x - y|^{3+s}} \nabla p(y) dy \\
&= |x|(-\Delta)^{s/2}\nabla p + \int_{\mathbb{R}^3} \frac{|x| - |y|}{|x - y|^{3+s}} \nabla p(y) dy.
\end{align*}

Therefore (21) becomes

\begin{align*}
I_2 &= -\int_0^T \int_{\mathbb{R}^3} |x||(-\Delta)^{s/2}\nabla p|^2 dxdt \\
&\quad - \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|x| - |y|}{|x - y|^{3+s}} \nabla p(y) dy \cdot (-\Delta)^{s/2}\nabla p(x) dxdt.
\end{align*}

By Young’s inequality,

\begin{align*}
I_2 + \int_0^T \int_{\mathbb{R}^3} |x||(-\Delta)^{s/2}\nabla p|^2 dxdt \\
&\quad \leq \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\nabla p(y)}{|x - y|^{2+s}} dy \cdot |(-\Delta)^{s/2}\nabla p(x)| dxdt \\
&\quad \leq \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} (1 + |x||(-\Delta)^{s/2}\nabla p|^2 dxdt + \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\nabla p(y)}{|x - y|^{2+s}} dy \left|\frac{\nabla p(y)}{|x - y|^{2+s}}\right|^2 \frac{dxdt}{1 + |x|}.
\end{align*}

Thanks to (9) we deduce

\begin{align*}
I_2 &\leq C + C \int_0^T \int_{\mathbb{R}^3} |f_1 \ast \nabla p|^2 \frac{dxdt}{1 + |x|} + C \int_0^T \int_{\mathbb{R}^3} |f_2 \ast \nabla p|^2 \frac{dxdt}{1 + |x|},
\end{align*}

where $B$ is the ball of center 0 and radius 1. For $\varepsilon > 0$ small enough and $i = 1, 2$ Hölder’s inequality yields

\begin{align*}
\int_{\mathbb{R}^3} |f_i \ast \nabla p|^2 \frac{dx}{1 + |x|} &\leq \|f_i \ast \nabla p\|_{L^{3-s} (\mathbb{R}^3)}^2 \|1 + |\cdot|\|_{L^{\frac{3}{3-s}} (\mathbb{R}^3)}^{-1} \\
&\leq C \|f_i \ast \nabla p\|_{L^{3-s} (\mathbb{R}^3)}^2.
\end{align*}
Therefore

\[(22) \quad I_2 \leq C + C \int_0^T \sum_{i=1}^2 \| f_i * \nabla p \|_{L^{3-\varepsilon}(\mathbb{R}^3)}^2 dt. \]

Let us bound the term

\[ \| f_1 * \nabla p \|_{L^{3-\varepsilon}(\mathbb{R}^3)} \leq C \| f_1 \|_{L^q(\mathbb{R}^3)} \| \nabla p \|_{L^{6/(3-2s)}(\mathbb{R}^3)}, \]

where \( q = q(\varepsilon) \geq 1 \) satisfies

\[ \frac{1}{q(\varepsilon)} + \frac{3 - 2s}{6} = 1 + \frac{1}{3 - \varepsilon}. \]

Note that \( q(0) = \frac{6}{5+2s} \), so \((2 + s)q(0) < 3\). By continuity it follows that \((2 + s)q(\varepsilon) < 3\) for \( \varepsilon > 0 \) small enough. As a consequence \( f_1 \in L^p(\mathbb{R}^3) \) and

\[ \exists \varepsilon > 0 : \quad \int_0^T \| f_1 * \nabla p \|_{L^{3-\varepsilon}(\mathbb{R}^3)}^2 dt \leq C \int_0^T \| \nabla p \|_{L^{6/(3-2s)}(\mathbb{R}^3)}^2 dt \leq C \]

thanks to (9). Let us now consider, for \( \varepsilon > 0 \) small enough,

\[ \| f_2 * \nabla p \|_{L^{3-\varepsilon}(\mathbb{R}^3)} \leq C \| f_2 \|_{L^{\frac{4+s}{2+s}}(\mathbb{R}^3)} \| \nabla p \|_{L^{\kappa}(\mathbb{R}^3)}, \]

where \( \kappa = \kappa(\varepsilon) \) satisfies

\[ \frac{2 + s}{3 + \varepsilon} + \frac{1}{\kappa(\varepsilon)} = 1 + \frac{1}{3 - \varepsilon}. \]

Given that \( \varepsilon > 0 \), we have \( f_2 \in L^{\frac{4+s}{2+s}}(\mathbb{R}^3) \), so

\[ \| f_2 * \nabla p \|_{L^{3-\varepsilon}(\mathbb{R}^3)} \leq C \| \nabla p \|_{L^{\kappa}(\mathbb{R}^3)}. \]

We observe that \( \kappa(0) = \frac{3}{2-s} \). Gagliardo-Nirenberg inequality yields

\[ \| \nabla p \|_{L^{\kappa(0)}(\mathbb{R}^3)} \leq C \| p \|_{L^{1-\eta}} \| \nabla p \|_{L^{6/(3-2s)}(\mathbb{R}^3)}^{\eta}, \quad \eta = \frac{4 + 2s}{5 + 2s}. \]

From the above inequality and (8) it follows

\[ \int_0^T \| \nabla p \|_{L^{\kappa(0)}(\mathbb{R}^3)}^2 dt \leq CT^{2(1-\eta)} \int_0^T \| \nabla p \|_{L^{6/(3-2s)}(\mathbb{R}^3)}^{2\eta} dt, \]

and by applying Hölder’s inequality and (9) we get

\[ \int_0^T \| \nabla p \|_{L^{\kappa(0)}(\mathbb{R}^3)}^2 dt \leq CT^{3(1-\eta)} \left( \int_0^T \| \nabla p \|_{L^{6/(3-2s)}(\mathbb{R}^3)}^2 dt \right)^{\eta} \leq CT^{3(1-\eta)}. \]

It holds \( 3(1 - \eta) = \frac{3}{5+2s} \leq \frac{6}{13} < \frac{1}{2} \) since \( s \geq \frac{3}{4} \). If \( \varepsilon > 0 \) is small enough, by arguing in the same way one can show

\[ \int_0^T \| \nabla p \|_{L^{\kappa(\varepsilon)}(\mathbb{R}^3)}^2 dt \leq CT^{3(1-\eta(\varepsilon))}. \]
for some $\eta(\varepsilon) \in [0, 1]$ such that $3(1 - \eta(\varepsilon)) < 1/2$ (since $\eta(\varepsilon)$ is continuous). We conclude
\begin{equation}
\exists \varepsilon > 0 : \int_0^T \|f_2 * \nabla p\|_{L^6(K)}^2 dt \leq C \int_0^T \|\nabla p\|_{L^6(K)}^2 dt \leq CT^{1/2}.
\end{equation}
From (22)–(23) we obtain
\begin{equation}
I_2 \leq C T^{1/2}, \quad T > 1.
\end{equation}
The Lemma’s statement follows from (19), (20), (24). This finishes the proof. \hfill \Box

We are now ready to prove Theorem 1.

**Proof of Theorem 1.** By using $2u$ as a test function in the density equation we get that
\begin{equation*}
\int_{\mathbb{R}^3} u^2 dx|_{t=T} - \int_0^T \int_{\mathbb{R}^3} \nabla (u^2) \cdot \nabla p dx dt = 0.
\end{equation*}
In [5] it was shown that
\begin{equation}
\int_0^T \int_{\mathbb{R}^3} \nabla (u^2) \cdot \nabla p dx dt = A[\nabla (u^2), \nabla (u^2)],
\end{equation}
where $A(\cdot, \cdot)$ defines a scalar product, and any sequence that is Cauchy in the $A$-norm converges almost everywhere. By applying the representation of $A$ in terms of the Fourier transform, we get that
\begin{equation}
T \sum_{m=0}^{\infty} \int_{\mathbb{R}^3} \frac{|k|^{2s}(1 - e^{-|k|^2 T})}{|k|^{4s} + m^2/T} |w_m(k)|^2 dk \leq C,
\end{equation}
where $w_m(k) = [\mathcal{F}_t \mathcal{F}_x \nabla (u^2)]_m (k)$, where $\mathcal{F}_x$ denotes the Fourier-transform in space and $\mathcal{F}_t$ the Fourier-transform in time. For $m = 0$ in (25) we get
\begin{equation}
T \int_{\mathbb{R}^3} \frac{1 - e^{-|k|^2 T}}{|k|^{2s}} |w_0(k)|^2 dk \leq C,
\end{equation}
where
\begin{equation*}
w_0(k) = \frac{1}{\sqrt{2T}} \int_{-T}^{T} ik u^2(k, t) dk = \sqrt{\frac{2}{T}} ik \hat{U}(k, T),
\end{equation*}
where $U(k, t) := \int_0^t u^2(x, \tau) d\tau$ and $\hat{\cdot} = \mathcal{F}_x$ denotes the Fourier transform with respect to $x$, where we extended $u$ as even function of $t$. Thus, (26) implies
\begin{equation}
\int_{\mathbb{R}^3} \frac{1 - e^{-|k|^2 T}}{|k|^{2s-2}} |\hat{U}(k, T)|^2 dk \leq C.
\end{equation}
Now we consider
\begin{equation*}
\int_{\mathbb{R}^3} e^{-|k|^2 T} |\hat{U}(k, T)|^2 dk \leq \|\hat{U}\|_{L^\infty(\mathbb{R}^3)}^2 \int_{\mathbb{R}^3} e^{-|k|^2 T} |k|^{2s-2} dk.
\end{equation*}
However, it holds
\[ \|\hat{U}(T)\|_{L^\infty(\mathbb{R}^3)}^2 \leq C\|U(T)\|_{L^1(\mathbb{R}^3)}^2 = C \left( \int_0^T \int_{\mathbb{R}^3} u^2 \, dx \, dt \right)^2 \leq CT^2 \]
thanks to the entropy inequality (5). Consequently, it follows by using the rescaling \( \tilde{k} = kT^{1/2s} \)
\[ \int_{\mathbb{R}^3} \frac{e^{-|k|^{2s}T}}{|k|^{2s-2}} |\hat{U}(k, T)|^2 \, dk \leq CT^2 \int_{\mathbb{R}^3} \frac{e^{-|k|^{2s}T}}{|k|^{2s-2}} \, dk = CT^{3-\frac{s}{2}} \int_{\mathbb{R}^3} \frac{e^{-|\tilde{k}|^{2s}T}}{|\tilde{k}|^{2s-2}} \tilde{d\tilde{k}} \leq CT^{3-\frac{s}{2}}. \]
We deduce from (27) that
\[ (28) \quad \int_{\mathbb{R}^3} |k|^{2(1-s)}|\hat{U}(k, T)|^2 \, dk \leq CT^{3-\frac{s}{2}}. \]
Net now \( R = R(T) > 0 \) be a generic constant depending on \( T \). Then (28) implies
\[ R^{2(1-s)} \int_{B_R} |\hat{U}(k, T)|^2 \, dk \leq CT^{3-\frac{s}{2}}. \]
On the other hand, it holds for a ball \( B_R \) of radius \( R > 0 \) centered around the origin
\[ \int_{B_R} |\hat{U}(k, T)|^2 \, dk \leq |B_R| \|\hat{U}(T)\|_{L^\infty(\mathbb{R}^3)}^2 \leq CR^3T^2. \]
Thus, it follows
\[ \int_{\mathbb{R}^3} |\hat{U}(k, T)|^2 \, dk \leq C(R^{-2(1-s)}T^{3-\frac{s}{2}} + R^3T^2). \]
We now minimize the right-hand side: we choose \( R = R(T) > 0 \) such that
\[ 0 = \frac{d}{d\rho} (\rho^{-2(1-s)}T^{3-\frac{s}{2}} + \rho^3T^2) \big|_{\rho = R} = -2(1-s)R^{-3+2s}T^{3-\frac{s}{2}} + 3R^2T^2 \]
that is
\[ R = cT^{-1/2s}. \]
It follows
\[ \int_{\mathbb{R}^3} |\hat{U}(k, T)|^2 \, dk \leq CT^{2-3/2s}. \]
The fact that the Fourier transform is an isometry \( L^2 \to L^2 \) and the definition of \( U \) imply
\[ (29) \quad \int_{\mathbb{R}^3} \left( \int_0^T u^2(x, t) \, dt \right)^2 \, dx \leq CT^{2-3/2s}. \]
From (29) and Jensen’s inequality it follows
\[ \frac{CT^{2-3/2s}}{|B_R|} \geq \frac{1}{|B_R|} \int_{B_R} \left( \int_0^T u^2(x, t) \, dt \right)^2 \, dx \geq \left( \frac{1}{|B_R|} \int_{B_R} \int_0^T u^2(x, t) \, dt \, dx \right)^2. \]
and so

\[
\int_0^T \int_{B_R} u^2(x,t) \, dx \, dt \leq C R^{3/2} T^{1-3s} \quad \forall R > 0.
\]

However, Lemma 1 implies that

\[
\int_0^T \int_{\mathbb{R}^3 \setminus B_R} u^2(x,t) \, dx \, dt \leq \frac{T}{R} \sup_{t \in [0,T]} \int_{\mathbb{R}^3 \setminus B_R} |x| \left( u^2(x,t) + \frac{1}{2} |\nabla p(x,t)|^2 \right) \, dx \leq \frac{T^{3/2}}{R}.
\]

Summing (30) and (31) leads to

\[
\int_0^T \int_{\mathbb{R}^3} u^2 \, dx \, dt \leq C \left( R^{3/2} T^{1-3s} + R^{-1} T^{3/2} \right) \quad T \geq 1, \quad R > 0.
\]

Again, we choose \( R = R(T) \) such that the right-hand side of (32) is minimal, which yields \( R = c T^{1/5 + 3s/10} \). It follows

\[
\int_0^T \int_{\mathbb{R}^3} u^2 \, dx \, dt \leq C T^{13-3s/10}, \quad T \geq 1.
\]

Let us now find a similar estimate for \( \nabla p \). Gagliardo-Nirenberg inequality leads to

\[
\| \nabla p \|_{L^2(\mathbb{R}^3)} \leq C \| p \|_{L^1(\mathbb{R}^3)}^{1-\theta} \| \nabla p \|_{L^{6/(3-25)}(\mathbb{R}^3)}^\theta, \quad \theta = \frac{5}{5 + 2s}.
\]

Taking the power \( 2/\theta \) of both members of the above inequality and integrating it in time yield

\[
\int_0^T \| \nabla p \|_{L^2(\mathbb{R}^3)}^{2/\theta} \, dt \leq C \int_0^T \| p \|_{L^1(\mathbb{R}^3)}^{2(1-\theta)/\theta} \| \nabla p \|_{L^{6/(3-25)}(\mathbb{R}^3)}^2 \, dt
\]

\[
\leq C \| p \|_{L^{\infty}(0,T;L^1(\mathbb{R}^3))}^{2(1-\theta)/\theta} \int_0^T \| \nabla p \|_{L^{6/(3-25)}(\mathbb{R}^3)}^2 \, dt.
\]

From (33) it follows

\[
\int_0^T \| \nabla p \|_{L^2(\mathbb{R}^3)}^{2/\theta} \, dt \leq C T^{2(1-\theta)/\theta} \int_0^T \| \nabla p \|_{L^{6/(3-25)}(\mathbb{R}^3)}^2 \, dt.
\]

Sobolev’s embedding \( H^s(\mathbb{R}^3) \hookrightarrow L^{6/(3-25)}(\mathbb{R}^3) \) leads to

\[
\int_0^T \| \nabla p \|_{L^2(\mathbb{R}^3)}^{2/\theta} \, dt \leq C T^{2(1-\theta)/\theta} \int_0^T \| \nabla (-\Delta)^{s/2} p \|_{L^2(\mathbb{R}^3)}^2 \, dt,
\]

while (33) allows us to deduce

\[
\int_0^T \| \nabla p \|_{L^2(\mathbb{R}^3)}^{2/\theta} \, dt \leq C T^{2(1-\theta)/\theta} \quad T > 1.
\]

The definition (34) of \( \theta \) implies

\[
\int_0^T \| \nabla p \|_{L^2(\mathbb{R}^3)}^{(10+4s)/5} \, dt \leq C T^{4s/5} \quad T > 1.
\]
By summing (33) and (35) and noticing that \( \frac{13-3s}{10} > \frac{4s}{5} \) for \( \frac{3}{4} \leq s \leq 1 \) we obtain
\[
\int_0^T \left( \|u\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla p\|_{L^2(\mathbb{R}^3)}^{(10+4s)/5} \right) dt \leq C(T^{\frac{13-3s}{10}} + T^{\frac{4s}{5}}) \leq CT^{\frac{13-3s}{10}} \quad T > 1.
\]
Since \( (10 + 4s)/5 \geq 2 \) and \( u \in L^\infty(0, \infty; L^2(\mathbb{R}^3)) \) it follows that
\[
\int_0^T \left( \|u\|_{L^2(\mathbb{R}^3)}^{(10+4s)/5} + \|\nabla p\|_{L^2(\mathbb{R}^3)}^{(10+4s)/5} \right) dt \leq CT^{\frac{13-3s}{10}} \quad T > 1.
\]
By a convexity argument
\[
\int_0^T H[u(t), p(t)]^{1+2s/5} dt = \int_0^T \left( \|u\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{2}\|\nabla p\|_{L^2(\mathbb{R}^3)}^2 \right)^{1+2s/5} dt \\
\leq C \int_0^T \left( \|u\|_{L^2(\mathbb{R}^3)}^{(10+4s)/5} + \|\nabla p\|_{L^2(\mathbb{R}^3)}^{(10+4s)/5} \right) dt \\
\leq C T^{\frac{13-3s}{10}} \quad T > 1.
\]
On the other hand \( t \mapsto H[u(t), p(t)] \) is non-increasing in time, so
\[
\int_0^T H[u(t), p(t)]^{1+2s/5} dt \geq \int_0^T H[u(T), p(T)]^{1+2s/5} dt = TH[u(T), p(T)]^{1+2s/5}.
\]
Putting the two previous inequalities together leads to
\[
TH[u(T), p(T)]^{1+2s/5} \leq CT^{\frac{13-3s}{10}} \quad T > 1,
\]
which yields the statement of the Theorem. This finished the proof. \( \square \)

\section{3. Proof of Theorem 2}

\textbf{Proof of Theorem 2} Let us compute the time derivative of \( H[(u(t), p(t)); (v(t), q(t))] \):
\[
(36) \quad \frac{d}{dt} H[(u(t), p(t)); (v(t), q(t))] = -2 \int_{\mathbb{R}^3} \nabla (u - v) \cdot (u \nabla p - v \nabla q) dx \\
- \int_{\mathbb{R}^3} \frac{1}{s/2} \nabla (p - q)^2 dx \\
+ \int_{\mathbb{R}^3} \nabla (p - q) \cdot \nabla (u^2 - v^2) dx
\]
Let us consider the term
\[
-2 \int_{\mathbb{R}^3} \nabla (u - v) \cdot (u \nabla p - v \nabla q) dx \\
= - \int_{\mathbb{R}^3} (\nabla u^2 \cdot \nabla p + \nabla v^2 \cdot \nabla q - 2u \nabla v \cdot \nabla p - 2v \nabla u \cdot \nabla q) dx
\]
\[
\begin{align*}
\int_{\mathbb{R}^3} (\nabla(u^2 - v^2) \cdot \nabla(p - q) + \nabla v^2 \cdot \nabla p + \nabla u^2 \cdot \nabla q - 2u \nabla v \cdot \nabla p - 2v \nabla u \cdot \nabla q) \, dx \\
= - \int_{\mathbb{R}^3} \nabla(u^2 - v^2) \cdot \nabla(p - q) \, dx - 2 \int_{\mathbb{R}^3} ((v - u) \nabla v \cdot \nabla p + (u - v) \nabla u \cdot \nabla q) \, dx \\
= - \int_{\mathbb{R}^3} \nabla(u^2 - v^2) \cdot \nabla(p - q) \, dx \\
- 2 \int_{\mathbb{R}^3} (u - v) (\nabla(u - v) \cdot \nabla q - \nabla v \cdot \nabla(p - q)) \, dx \\
= - \int_{\mathbb{R}^3} \nabla(u^2 - v^2) \cdot \nabla(p - q) \, dx + \int_{\mathbb{R}^3} (u - v)^2 \Delta q \, dx + 2 \int_{\mathbb{R}^3} (u - v) \nabla(p - q) \cdot \nabla v \, dx
\end{align*}
\]

Therefore (36) becomes
\[
\begin{align*}
\frac{d}{dt} H[(u(t), p(t))(v(t), q(t))] + \int_{\mathbb{R}^3} |(-\Delta)^{s/2} \nabla(p - q)|^2 \, dx \\
= \int_{\mathbb{R}^3} (u - v)^2 \Delta q \, dx + 2 \int_{\mathbb{R}^3} (u - v) \cdot \nabla(p - q) \nabla v \, dx.
\end{align*}
\]

Let us bound the right-hand side of (37). It holds trivially
\[
\int_{\mathbb{R}^3} (u - v)^2 \Delta q \, dx \leq \left( \sup_{\mathbb{R}^3} \Delta q \right) H[(u, p)(v, q)].
\]

Let us now consider
\[
\begin{align*}
2 \int_{\mathbb{R}^3} (u - v) \nabla(p - q) \cdot \nabla v \, dx \leq \int_{\mathbb{R}^3} (u - v)^2 \, dx + \int_{\mathbb{R}^3} \nabla v^2 |\nabla(p - q)|^2 \, dx.
\end{align*}
\]

Let \(2 \lambda < \frac{6}{3 - 2s}, \lambda' = \frac{\lambda}{3 - 2s}\). H"older inequality allows us to write
\[
\int_{\mathbb{R}^3} |\nabla v|^2 |\nabla(p - q)|^2 \, dx \leq C \|\nabla v\|_{L^{2\lambda'}(\mathbb{R}^3)}^2 \|\nabla(p - q)\|_{L^{2\lambda}(\mathbb{R}^3)}^2.
\]

By interpolation
\[
\int_{\mathbb{R}^3} |\nabla v|^2 |\nabla(p - q)|^2 \, dx \leq C \|\nabla v\|_{L^{2\lambda'}(\mathbb{R}^3)}^2 \|\nabla(p - q)\|_{L^2(\mathbb{R}^3)}^2 \|\nabla(p - q)\|_{L^{2\lambda}(\mathbb{R}^3)}^{2(1 - \rho)} \|\nabla(p - q)\|_{L^{2\lambda}(\mathbb{R}^3)}^{2(1 - \rho)},
\]

for some \(\rho \in (0, 1)\). Moreover, thanks to the Sobolev embedding \(H^s(\mathbb{R}^3) \hookrightarrow L^{\frac{6}{3 - 2s}}(\mathbb{R}^3)\) it follows
\[
\int_{\mathbb{R}^3} |\nabla v|^2 |\nabla(p - q)|^2 \, dx \leq C \|\nabla v\|_{L^{2\lambda'}(\mathbb{R}^3)}^2 \|\nabla(p - q)\|_{L^2(\mathbb{R}^3)}^2 \|\nabla(p - q)\|_{L^{2\lambda}(\mathbb{R}^3)}^{2(1 - \rho)} \|(-\Delta)^{s/2} \nabla(p - q)\|_{L^2(\mathbb{R}^3)}^{2(1 - \rho)}.
\]

Young’s inequality yields
\[
\int_{\mathbb{R}^3} |\nabla v|^2 |\nabla(p - q)|^2 \, dx \leq C \|\nabla v\|_{L^{2\lambda'}(\mathbb{R}^3)}^{2\rho} \|\nabla(p - q)\|_{L^2(\mathbb{R}^3)}^{2\rho} + \frac{1}{2} \|(-\Delta)^{s/2} \nabla(p - q)\|_{L^2(\mathbb{R}^3)}^2.
\]
From the above inequality and (39) we deduce

\[(40) \ 2 \int_{\mathbb{R}^3} (u - v) \nabla (p - q) \cdot \nabla v \, dx \]

\[
\leq \int_{\mathbb{R}^3} (u - v)^2 \, dx + C \| \nabla v \|_{L^{2\nu}(\mathbb{R}^3)}^{2/\nu} \int_{\mathbb{R}^3} |\nabla (p - q)|^2 \, dx + \frac{1}{2} \| (-\Delta)^{s/2} \nabla (p - q) \|_{L^2(\mathbb{R}^3)}^2
\]

\[
\leq C(1 + \| \nabla v \|_{L^{2\nu}(\mathbb{R}^3)}^{2/\nu}) H[(u, p)](v, q) + \frac{1}{2} \| (-\Delta)^{s/2} \nabla (p - q) \|_{L^2(\mathbb{R}^3)}^2
\]

Adding (37), (38), (40) yields

\[
\frac{d}{dt} H[(u, p)](v, q) \leq C \left( 1 + \sup_{\mathbb{R}^3} \Delta q + \| \nabla v \|_{L^{2\nu}(\mathbb{R}^3)}^{2/\nu} \right) H[(u, p)](v, q).
\]

Since \(2 < \lambda < \frac{6}{3 - 2s}\), then \(\lambda' = \frac{\lambda}{\lambda - 1} > \frac{6}{3 + 2s}\). Therefore it is possible to choose \(\lambda\) such that \(2\lambda' = \frac{12}{3 + 2s} + \nu\). As a consequence \(\nabla v \in L^\infty(0, \infty; L^{2\lambda'}(\mathbb{R}^3))\). Gronwall’s Lemma allows us to obtain the theorem’s statement with

\[
K = C \left( 1 + \sup_{\mathbb{R}^3 \times (0, \infty)} \Delta q + \| \nabla v \|_{L^\infty(0, \infty; L^{\frac{12}{3 + 2s} + \nu}(\mathbb{R}^3))}^{2/\nu} \right).
\]

This finishes the proof. \(\square\)

4. Proof of Theorem \(\mathbb{I}\)

Proof. The only point of the existence proof where the assumption \(s \geq 3/4\) is used is the proof that \(u^{(\rho)}\) is bounded in \(L^3(\mathbb{R}^3 \times (0, T))\), which is in turn required to show that

\[(41) \quad \int_0^T \int_{\mathbb{R}^3} ((u^{(\rho)})^2 - v)(p^{(\rho)} - p) \, dx \, dt \to 0 \quad \text{as} \ \rho \to 0,
\]

where \(v\) is the weak limit of \((u^{(\rho)})^2\) in \(L^{3/2}(\mathbb{R}^3 \times (0, T))\). However, in the case where \(\mathbb{R}^3\) is replaced by \(\mathbb{T}^3\), it is possible to show the analogue of (41) without employing the bound for \(u^{(\rho)}\) in \(L^{3/2}(\mathbb{R}^3 \times (0, T))\). Indeed, under the assumption \(s > 1/2\), it holds that \(H^{s+1}(\mathbb{T}^3) \hookrightarrow L^\infty(\mathbb{T}^3)\) compactly. This fact, combined with the strong convergence of \(p^{(\rho)}\) in \(L^1(\mathbb{T}^3 \times (0, T))\), yields the strong convergence of \(p^{(\rho)}\) in \(L^1(0, T; L^\infty(\mathbb{T}^3))\). In particular \(p^{(\rho)}\) is Cauchy in \(L^1(0, T; L^\infty(\mathbb{T}^3))\), i.e. for every \(\varepsilon > 0\) there is \(\delta > 0\) such that

\[
\| p^{(\rho)} - p^{(\sigma)} \|_{L^1(0, T; L^\infty(\mathbb{T}^3))} < \varepsilon \quad \text{for} \ \rho, \sigma < \delta.
\]

Since \(u^{(\rho)}\) is bounded in \(L^\infty(0, T; L^2(\mathbb{T}^3))\) (thanks to the entropy inequality), it follows

\[
\int_0^T \int_{\mathbb{T}^3} ((u^{(\rho)})^2 - (u^{(\sigma)})^2)(p^{(\rho)} - p^{(\sigma)}) \, dx \, dt \leq C \| p^{(\rho)} - p^{(\sigma)} \|_{L^1(0, T; L^\infty(\mathbb{T}^3))} < C \varepsilon \quad \rho, \sigma < \delta.
\]

This means that \((u^{(\rho)})^2\) is Cauchy with respect to the norm \(\| \cdot \|_\mathcal{A}\). We point out that the positivity of the quadratic form \(\mathcal{A}\) can also be showed through energy methods (by testing the second equation in (I) against \(p\) under assumption that \(p(0) = 0\), so it holds also in the torus case.
At this point, one proceeds like in the $\mathbb{R}^3$ case to show that from the property that $(u^{(\rho)})^2$ is Cauchy with respect to $|\cdot|_A$ it follows that $u^{(\rho)}$ is (up to subsequences) a.e. convergent to $u$ in $T^3 \times (0,T)$, and therefore $v = u^2$. This finishes the proof. □

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