Finite subset spaces of graphs and punctured surfaces

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Abstract The $k$th finite subset space of a topological space $X$ is the space $\exp_k X$ of non-empty finite subsets of $X$ of size at most $k$, topologised as a quotient of $X^k$. The construction is a homotopy functor and may be regarded as a union of configuration spaces of distinct unordered points in $X$. We calculate the homology of the finite subset spaces of a connected graph $\Gamma$, and study the maps $(\exp_k \phi)_*$ induced by a map $\phi: \Gamma \to \Gamma'$ between two such graphs. By homotopy functoriality the results apply to punctured surfaces also. The braid group $B_n$ may be regarded as the mapping class group of an $n$–punctured disc $D_n$, and as such it acts on $H_*(\exp_k D_n)$. We prove a structure theorem for this action, showing that the image of the pure braid group is nilpotent of class at most $\lfloor (n-1)/2 \rfloor$.

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1 Introduction

1.1 Finite subset spaces

Let $X$ be a topological space and $k$ a positive integer. The $k$th finite subset space of $X$ is the space $\exp_k X$ of nonempty subsets of $X$ of size at most $k$, topologised as a quotient of $X^k$ via the map

$$(x_1, \ldots, x_k) \mapsto \{x_1\} \cup \cdots \cup \{x_k\}.$$

The construction is functorial: given a map $f: X \to Y$ we obtain a map $\exp_k f: \exp_k X \to \exp_k Y$ by sending $S \subseteq X$ to $f(S) \subseteq Y$. Moreover, if $\{h_t\}$ is a homotopy between $f$ and $g$ then $\{\exp_k h_t\}$ is a homotopy between $\exp_k f$ and $\exp_k g$, so that $\exp_k$ is in fact a homotopy functor.
The first finite subset space is of course simply \( X \), and the second finite subset space coincides with the second symmetric product \( \text{Sym}^2 X = X^2/S_2 \). However, for \( k \geq 3 \) we have a proper quotient of the symmetric product as \( \exp_k X \) is unable to record multiplicities: both \((a, a, b)\) and \((a, b, b)\) in \( X^3 \) map to \( \{a, b\} \) in \( \exp_3 X \). As a result there are natural inclusion maps

\[
\exp_j X \hookrightarrow \exp_k X: S \mapsto S
\]

whenever \( j \leq k \), stratifying \( \exp_k X \). We define the full finite subset space \( \exp X \) to be the direct limit of this system of inclusions,

\[
\exp X = \lim_{\to} \exp_k X.
\]

If \( X \) is Hausdorff then the subspace topology on \( \exp_j X \subseteq \exp_k X \) coincides with the quotient topology it receives from \( X^j \) [11]. In this case each stratum \( \exp_j X \setminus \exp_{j-1} X \) is homeomorphic to the configuration space of sets of \( j \) distinct unordered points in \( X \), so that \( \exp_k X \) may be regarded as a union over \( 1 \leq j \leq k \) of these spaces. Moreover \( \exp_k X \) is compact whenever \( X \) is, in which case it gives a compactification of the corresponding configuration space. Such spaces and their compactifications have been of considerable interest recently in algebraic topology: see, for example, Fulton and MacPherson [10] and Ulyanov [22].

For each \( k \) and \( \ell \) the isomorphism \( X^k \times X^\ell \to X^{k+\ell} \) descends to a map

\[
\cup: \exp_k X \times \exp_\ell X \to \exp_{k+\ell} X
\]

sending \((S, T)\) to \( S \cup T \). This leads to a form of product on maps \( g: Y \to \exp_k X, \ h: Z \to \exp_\ell X \), and we define \( g \cup h: Y \times Z \to \exp_{k+\ell} X \) to be the composition

\[
Y \times Z \xrightarrow{g \times h} \exp_k X \times \exp_\ell X \xrightarrow{\cup} \exp_{k+\ell} X.
\]

Clearly \((f \cup g) \cup h = f \cup (g \cup h)\). Given a point \( x_0 \in X \) we obtain as a special case a map

\[
\cup \{x_0\}: \exp_k X \to \exp_{k+1} X
\]

taking \( S \subseteq X \) to \( S \cup \{x_0\} \). The image of \( \cup \{x_0\} \) is the subspace \( \exp_{k+1}(X, x_0) \) consisting of the \( k+1 \) or fewer element subsets of \( X \) that contain \( x_0 \). In contrast to the symmetric product, where the analogous map plays the role of (1.1), the spaces \( \exp_k X \) and \( \exp_{k+1}(X, x_0) \) are in general topologically different. The map \( \cup \{x_0\} \) is one-to-one at the point \( \{x_0\} \in \exp_1 X \) and on the top level stratum \( \exp_k X \setminus \exp_{k-1} X \), but is two-to-one elsewhere, as \( S \) and \( S \cup \{x_0\} \) have the same image for \(|S| < k, x_0 \notin S\). Nevertheless the based finite subset spaces \( \exp_k(X, x_0) \) frequently act as a stepping stone in understanding \( \exp_k X \), often being topologically simpler.
1.2 History

The space $\exp_k X$ was introduced by Borsuk and Ulam [6] in 1931 as the symmetric product, and since then appears to have been studied at irregular intervals, under various notations, and principally from the perspective of general topology. In their original paper Borsuk and Ulam showed that $\exp_k I \cong I^k$ for $k = 1, 2, 3$, but that $\exp_k I$ cannot be embedded in $\mathbb{R}^k$ for $k \geq 4$. In 1957 Molski [15] proved similar results for $I^2$ and $I^n$, namely that $\exp_2 I^2 \cong I^4$ but that neither $\exp_k I^2$ nor $\exp_2 I^k$ can be embedded in $\mathbb{R}^{2k}$ for any $k \geq 3$. The last was done by showing that $\exp_2 I^k$ contains a copy of $S^k \times \mathbb{R}^{P^{k-1}}$.

Other authors including Curtis [8], Curtis and To Nhu [9], Handel [11], Illanes [12] and Macías [14] have established general topological and homotopy-theoretic properties of $\exp_k X$ and $\exp X$, and Beilinson and Drinfeld [2, sec. 3.5.1] and Ran [17] have used these spaces in the context of mathematical physics and algebraic geometry. The set $\exp X$ has also been studied extensively under a different topology as the Pixley-Roy hyperspace of finite subsets of $X$; the two topologies are surveyed in Bell [3]. We mention some results on $\exp_k X$ of a homotopy-theoretic nature. In 1999 Macías showed that for compact connected metric $X$ the first Čech cohomology group $\check{H}^1(\exp_k X; \mathbb{Z})$ vanishes for $k \geq 3$, and in 2000 Handel proved that for closed connected $n$–manifolds, $n \geq 2$, the singular cohomology group $H^i(\exp_k M^n; \mathbb{Z}/2\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ for $i = nk$, and 0 for $i > nk$. In addition, Handel showed that the inclusion maps $\exp_k(X, x_0) \hookrightarrow \exp_{2k-1}(X, x_0)$ and $\exp_k X \hookrightarrow \exp_{2k+1} X$ induce the zero map on all homotopy groups for path connected Hausdorff $X$.

However, although these and other properties of $\exp_k$ have been established, it appears that until recently the only homotopically non-trivial space for which $\exp_k X$ was at all well understood for $k \geq 3$ was the circle. In 1952 Bott [7] proved the surprising result that $\exp_3 S^1$ is homeomorphic to the three-sphere, correcting Borsuk’s 1949 paper [5], and Shchepin (unpublished; for three different proofs see [16] and [19]) later proved the even more striking result that $\exp_2 S^1$ inside $\exp_3 S^1$ is a trefoil knot. An elegant geometric construction due to Mostovoy [16] in 1999 connects both of these results with known facts about the space of lattices in the plane, and in our previous paper [19] we showed that Bott’s and Shchepin’s results can be viewed as part of a larger pattern: $\exp_k S^1$ has the homotopy type of an odd dimensional sphere, and $\exp_k S^1 \setminus \exp_{k-2} S^1$ that of a $(k-1, k)$–torus knot complement. This paper aims to increase the list of spaces for which $\exp_k$ is understood by using the techniques of [19] to study the finite subset spaces of connected graphs. The results apply to punctured surfaces too, by homotopy equivalence, and represent a step towards under-
standing finite subset spaces of closed surfaces, as they may be used to study these via Mayer-Vietoris type arguments. Further steps towards this goal are taken in our dissertation [20], in which this paper also appears.

Various different notations have been used for $\exp_k X$, including $X(k)$, $X^{(k)}$, $\mathcal{F}_k(X)$ and $Sub(X,k)$. Our notation follows that used by Mostovoy [16] and reflects the idea that we are truncating the (suitably interpreted) series

$$\exp X = \emptyset \cup X \cup \frac{X^2}{2!} \cup \frac{X^3}{3!} \cup \ldots$$

at the $X^k/k! = X^k/S_k$ term. The name, however, is our own. There does not seem to be a satisfactory name in use among geometric topologists—indeed, recent authors Mostovoy and Handel do not use any name at all—and while symmetric product has stayed in use among general topologists we prefer to use this for $X^k/S_k$. We therefore propose the descriptive name “$k$th finite subset space” used here and in our previous paper.

1.3 Summary of results

We study the finite subset spaces of a connected graph $\Gamma$ using techniques from our previous paper [19] on $\exp_k S^1$. Since $\exp_k$ is a homotopy functor we may reduce to the case where $\Gamma$ has a single vertex, and accordingly define $\Gamma_n$ to be the graph with one vertex $v$ and $n$ edges $e_1, \ldots, e_n$. Our first result is a complete calculation of the homology of $\exp_k(\Gamma_n,v)$ and $\exp_k \Gamma_n$ for each $k$ and $n$:

**Theorem 1** The reduced homology groups of $\exp_k(\Gamma_n,v)$ vanish outside dimension $k - 1$ and those of $\exp_k \Gamma_n$ vanish outside dimensions $k - 1$ and $k$. The non-vanishing groups are free. The maps

$$i: \exp_k(\Gamma_n,v) \hookrightarrow \exp_k \Gamma_n$$

and

$$\cup\{v\}: \exp_k \Gamma_n \rightarrow \exp_{k+1}(\Gamma_n,v)$$

induce isomorphisms on $H_{k-1}$ and $H_k$ respectively while

$$\exp_k \Gamma_n \hookrightarrow \exp_{k+1} \Gamma_n$$

is twice $(i \circ \cup\{v\})_*$ on $H_k$. The common rank of

$$H_k(\exp_k \Gamma_n) \cong H_k(\exp_{k+1} \Gamma_n) \cong H_k(\exp_{k+1}(\Gamma_n,v))$$

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is given by

\[ b_k(\exp_k \Gamma_n) = \sum_{j=1}^{k} (-1)^{j-k} \binom{n+j-1}{n-1} = \begin{cases} \sum_{\ell=1}^{\ell} \binom{n+2j-2}{n-2} & \text{if } k = 2\ell \text{ is even}, \\ n + \sum_{j=1}^{\ell} \binom{n+2j-1}{n-2} & \text{if } k = 2\ell + 1 \text{ is odd}. \end{cases} \] (1.2)

A list of Betti numbers \( b_k(\exp_k \Gamma_n) \) for \( 1 \leq k \leq 20 \) and \( 1 \leq n \leq 10 \) appears as table 1 in the appendix on page 903.

In the case of a circle the homology and fundamental group of \( \exp_k S^1 \) were enough to determine its homotopy type completely. The argument no longer applies to \( \exp_k \Gamma_n, \ n \geq 2 \), and its applicability for \( n = 1 \) is perhaps properly regarded as being due to a “small numbers co-incidence”, the vanishing of \( H_{2\ell}(\exp_2 \Gamma_1) \). However, the argument does apply to \( \exp_k(\Gamma_n, v) \), and for \( k \geq 2 \) we have the following:

**Theorem 2** For \( k \geq 2 \) the space \( \exp_k(\Gamma_n, v) \) has the homotopy type of a wedge of \( b_{k-1}(\exp_k(\Gamma_n, v)) \) \((k-1)\)-spheres.

Having calculated the homology of \( \exp_k \Gamma_n \) we turn our attention to the maps \((\exp_k \phi)_*\) induced by maps \( \phi: \Gamma_n \to \Gamma_m \). Our main result is to reduce the problem of calculating such maps to one of finding images of chains under maps

\[ \exp_1 S^1 \to \exp_1 S^1 \]

and

\[ \exp_2 S^1 \to \exp_2 \Gamma_2 \]

induced by maps \( S^1 \to S^1 \) and \( S^1 \to \Gamma_2 \) respectively. The reduction is achieved by defining a ring without unity structure on a subgroup \( \tilde{C}_* \) of the cellular chain complex of \( \exp \Gamma_n \). The subgroup carries the top homology of \( \exp_k \Gamma_n \) and is preserved by chain maps of the form \((\exp \phi)_*\), and the ring structure is defined in such a way that these chain maps are ring homomorphisms. The ring \( \tilde{C}_* \otimes \mathbb{Z} \mathbb{Q} \) is generated over \( \mathbb{Q} \) by cells in dimensions one and two, leaving a mere \( 2n \) cells whose images must be found directly.

As an application of these results and as an illustration of how much \((\exp_k \phi)_*\) remembers about \( \phi \) we study the action of the braid group \( B_n \) on \( H_k(\exp_k \Gamma_n) \).

The braid group may be regarded as the mapping class group of a punctured disc.
and as such it acts on the graph $\Gamma_n$ via homotopy equivalence. We show that, for a suitable choice of basis, the braid group acts by block upper-triangular matrices whose diagonal blocks are representations of $B_n$ that factor through $S_n$. Consequently, the image of the pure braid group consists of upper-triangular matrices with ones on the diagonal and is therefore nilpotent. The number of blocks depends mildly on $k$ and $n$ but is no more than about $n/2$, and this leads to a bound on the length of the lower central series.

We remark that the main results of this paper may be used to study the finite subset spaces of a closed surface $\Sigma$ via Mayer-Vietoris type arguments. This may be done by constructing a cover of $\exp_k \Sigma$ such that each element of the cover and each $m$–fold intersection is a finite subset space of a punctured surface, as follows. Choose $k+1$ distinct points $p_1, \ldots, p_{k+1}$ in $\Sigma$ and let $U_i = \exp_k(\Sigma \setminus \{p_i\})$. The $U_i$ form an open cover of $\exp_k \Sigma$, since each $\Lambda \in \exp_k \Sigma$ must omit at least one of the $p_i$, and moreover each $m$–fold intersection has the form

$$\bigcap_{j=1}^m U_{i_j} = \exp_k(\Sigma \setminus \{p_{i_1}, \ldots, p_{i_m}\}),$$

a finite subset space of a punctured surface as desired. The results of this paper may then be used to calculate the homology of each intersection and the maps induced by inclusion, leading to a spectral sequence for $H_*(\exp_k \Sigma)$.

In [21] this idea is used to prove two vanishing theorems for the homotopy and homology groups of the finite subset spaces of a connected cell complex.

1.4 Outline of the paper

The calculation of the homology of $\exp_k \Gamma_n$ and $\exp_k(\Gamma_n, v)$ is the main topic of section 2. We find explicit cell structures for these spaces in section 2.2 and use them to calculate their fundamental groups. We then show that the reduced chain complex of $\exp(\Gamma_n, v)$ is exact in section 2.3, and use this to prove Theorems 1 and 2 in section 2.4. We give an explicit basis for $H_*(\exp_k \Gamma_n)$ in section 2.5 and close with generating functions for the Betti numbers $b_k(\exp_k \Gamma_n)$ in section 2.6. A table of Betti numbers for $1 \leq k \leq 20$ and $1 \leq n \leq 10$ appears in the appendix on page 903.

We then turn to the calculation of induced maps in section 3. The ring structure on $\tilde{C}_*$ is motivated and defined in section 3.2 and we show that maps $\phi: \Gamma_n \to \Gamma_m$ induce ring homomorphisms in section 3.3. As illustration of the ideas we calculate two examples in section 3.4, the first reproducing a result from [19]...
and the second relating to the generators of the braid groups. We then state and prove the structure theorem for the braid group action in sections 4.1 and 4.2, and conclude by looking at the action of $B_3$ on $H_3(\exp_3 \Gamma_3)$ in some detail in section 4.3.

1.5 Notation and terminology

We take a moment to fix some language and notation that will be used throughout.

We will work exclusively with graphs having just one vertex, so as above we define $\Gamma_n$ to be the graph with one vertex $v$ and $n$ edges $e_1, \ldots, e_n$. Write $I$ for the interval $[0, 1]$, and for each non-negative integer $m$ let $[m] = \{i \in \mathbb{Z} | 1 \leq i \leq m\}$. We parameterise $\Gamma_n$ as the quotient of $I \times [n]$ by the subset $\{0, 1\} \times [n]$, sending $\{0, 1\} \times [n]$ to $v$ and $[0, 1] \times \{i\}$ to $e_i$. This directs each edge, allowing us to order any subset of its interior, and we will use this extensively.

Associated to a finite subset $\Lambda$ of $\Gamma_n$ is an $n$–tuple $J(\Lambda) = (j_1, \ldots, j_n)$ of non-negative integers $j_i = |\Lambda \cap \text{int} e_i|$. Given an $n$–tuple $J = (j_1, \ldots, j_n)$ we define its support $\text{supp} J$ to be

$$\text{supp} J = \{i \in [n] | j_i \neq 0\}$$

and its norm $|J|$ by

$$|J| = \sum_{i=1}^{n} j_i.$$

Note that

$$|J(\Lambda)| = \begin{cases} |\Lambda| & \text{if } v \notin \Lambda, \\ |\Lambda| - 1 & \text{if } v \in \Lambda. \end{cases}$$

In addition we define the mod 2 support and norm by

$$\text{supp}_2(J) = \{i \in [n] | j_i \neq 0 \text{ mod } 2\}$$

and

$$|J|_2 = |\text{supp}_2(J)|.$$

Bringing two points together in the interior of $e_i$ or moving a point to $v$ decreases $J(\Lambda)_i$ by one. It will be convenient to have some notation for this, so we define

$$\partial_i(J) = (j_1, \ldots, j_i - 1, \ldots, j_n)$$

provided $j_i \geq 1$. Lastly, for each subset $S$ of $[n]$ and $n$–tuple $J$ we write $J|_S$ for the $|S|$–tuple obtained by restricting the index set to $S$. 

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2 The homology of finite subset spaces of graphs

2.1 Introduction

Our first step in calculating the homology of a finite subset space of a connected graph $\Gamma$ is to find explicit cell structures for $\exp_k(\Gamma_n, v)$ and $\exp_k \Gamma_n$. The approach will be similar to that taken in [19], and we will make use of the boundary map calculated there. However, we will adopt a different orientation convention, with the result that some signs will be changed.

Our cell structure for $\exp_k(\Gamma_n, v)$ will consist of one $j$–cell $\sigma^J$ for each $n$–tuple $J$ such that $|J| = j \leq k - 1$, the interior of $\sigma^J$ containing those $\Lambda \in \exp_k(\Gamma_n, v)$ such that $J(\Lambda) = J$. A cell structure for $\exp_k \Gamma_n$ will be obtained by adding additional cells $\tilde{\sigma}^J$ for each $J$ with $|J| = j \leq k$; the interior of $\tilde{\sigma}^J$ will contain those $\Lambda \subseteq \Gamma_n \setminus \{v\}$ such that $J(\Lambda) = J$. By a “stars and bars” argument there are \[ \binom{n+j-1}{n-1} \] solutions to \[ j_1 + \cdots + j_n = j \] in non-negative integers (count the arrangements of $j$ ones and $n - 1$ pluses), so that

\[
\chi(\exp_k(\Gamma_n, v)) = \sum_{j=0}^{k-1} (-1)^j \binom{n+j-1}{n-1}
\]  

(2.1)

and

\[
\chi(\exp_k \Gamma_n) = 1 + 2 \left[ \sum_{j=1}^{k-1} (-1)^j \binom{n+j-1}{n-1} \right] + (-1)^k \binom{n+k-1}{n-1}.
\]

A cell structure may be found in a similar way for an arbitrary connected graph $\Gamma$, with up to $2^{|V(\Gamma)|}$ $j$–cells for each $|E(\Gamma)|$–tuple $J$ with $|J| = j$.

In these cell structures the spaces $\exp_{k+1}(\Gamma_n, v)$ and $\exp_{k+1} \Gamma_n$ are obtained from $\exp_k(\Gamma_n, v)$ and $\exp_k \Gamma_n$ by adding cells in dimensions $k$ and $k + 1$. This has the following consequence for their homotopy groups. The $(k - 1)$–skeleta of $\exp_k(\Gamma_n, v)$ and $\exp_k \Gamma_n$ co-incide for $\ell \geq k$, and this means that the map on $\pi_i$ induced by the inclusion $\exp_k(\Gamma_n, v) \hookrightarrow \exp_\ell(\Gamma_n, v)$ is an isomorphism for $i \leq k - 2$. By Handel [11] this map is zero for $\ell = 2k - 1$, implying that $\exp_k(\Gamma_n, v)$ (and by a similar argument $\exp_k \Gamma_n$) is $(k-2)$–connected. It follows immediately that the augmented chain complex of $\exp(\Gamma_n, v)$ is exact, and in conjunction with the Euler characteristic (2.1) and the boundary maps (2.2) and (2.3) this is enough to prove Theorem 1. We nevertheless show directly
that this chain complex is exact in section 2.3 in order to find bases for the homology groups in section 2.5.

The fact that the $k$th finite subset space of a connected graph is $(k - 2)$–connected can be used to show that the same conclusion holds for the $k$th finite subset space of a connected cell complex [21].

### 2.2 Cell structures for $\exp_k \Gamma_n$ and $\exp_k(\Gamma_n, v)$

We now proceed more concretely. Following the strategy of [19], each element $\Lambda \in \exp_j e_i$ has at least one representative $(x_1, \ldots, x_j) \in [0, 1]^j$ such that $x_1 \leq \cdots \leq x_j$. Define simplices

$$\Delta_j = \{(x_1, \ldots, x_{j+1})|0 \leq x_1 \leq \cdots \leq x_j \leq x_{j+1} = 1\}$$

for each $j \geq 0$, and

$$\tilde{\Delta}_j = \{(x_1, \ldots, x_j)|0 \leq x_1 \leq \cdots \leq x_j \leq 1\}$$

for each $j \geq 1$. There are surjections $\Delta_j \to \exp_{j+1}(e_i, v)$, $\tilde{\Delta}_j \to \exp_{j+1}(e_i)$, and we let $\sigma^j_i$ be the composition

$$\Delta_j \to \exp_{j+1}(e_i, v) \hookrightarrow \exp_{j+1}(\Gamma_n, v),$$

$\tilde{\sigma}^j_i$ the composition

$$\tilde{\Delta}_j \to \exp_{j+1}(e_i) \hookrightarrow \exp_{j+1}(\Gamma_n).$$

We give $\Delta_j$ and $\tilde{\Delta}_j$ each the orientation $[x_1, \ldots, x_j]$, a convention that disagrees with the one used in [19] for some $j$. There $\tilde{\Delta}_j$ was oriented by letting its $i$th vertex be

$$v_i = (0, \ldots, 0, 1, \ldots, 1)$$

for $i = 0, \ldots, j$, and the sign of this orientation relative to the standard one on $\mathbb{R}^j$ is given by

$$\det[(v_1 - v_0)^T, \ldots, (v_j - v_0)^T] = \begin{cases} +1 & j \equiv 0, 1 \mod 4, \\ -1 & j \equiv 2, 3 \mod 4. \end{cases}$$

A similar calculation shows the same conclusion holds for $\Delta_j$. To account for this difference we should insert a minus sign in the boundary map calculated in [19] precisely when it is applied to $\sigma = \sigma^j_i \leq \partial \sigma$ for $j$ even, since then exactly one of $\sigma$ and $\partial \sigma$ has been given the opposite orientation. Note however that $\partial \tilde{\sigma}^j_i = \partial \sigma^j_i = 0$ for $j$ odd, simplifying the matter and allowing us to simply insert a minus everywhere.
Returning to the discussion at hand, given an \( n \)-tuple \( J \) let
\[
\Delta_J = \Delta_{j_1} \times \cdots \times \Delta_{j_n},
\]
\[
\tilde{\Delta}_J = \tilde{\Delta}_{j_1} \times \cdots \times \tilde{\Delta}_{j_n},
\]
omitting any empty factor \( \tilde{\Delta}_0 \) from this last product. Finally let
\[
\sigma^J = \sigma^{j_1} \cup \cdots \cup \sigma^{j_n} : \Delta_J \to \exp_{|J|+1}(\Gamma_n, v),
\]
\[
\tilde{\sigma}^J = \tilde{\sigma}^{j_1} \cup \cdots \cup \tilde{\sigma}^{j_n} : \tilde{\Delta}_J \to \exp_{|J|}(\Gamma_n),
\]
again omitting any factor with \( j_i = 0 \) from \( \tilde{\sigma}_J \). Each of \( \sigma^J \vert_{\int \Delta_J} \), \( \tilde{\sigma}^J \vert_{\int \tilde{\Delta}_J} \) is a homeomorphism of an open \( |J| \)-ball onto its image, and we claim:

**Lemma 1** The spaces \( \exp_k(\Gamma_n, v) \) and \( \exp_k \Gamma_n \) have cell structures consisting respectively of \( \{ \sigma^J \vert_{|J|} \leq k-1 \} \) and of \( \{ \sigma^J \vert_{|J|} \leq k-1 \} \cup \{ \tilde{\sigma}^J \vert_{|J|} \leq k \} \).

The boundary maps are given by
\[
\partial \sigma^J = - \sum_{i \in \supp J} \frac{1 + (-1)^{j_i}}{2} (-1)^{|J|_{i-1}} \sigma_{\partial_i(J)}
\]
and
\[
\partial \tilde{\sigma}^J = \sum_{i \in \supp J} \frac{1 + (-1)^{j_i}}{2} (-1)^{|J|_{i-1}} \left( \tilde{\sigma}^h(J) - 2 \sigma_{\partial_i(J)} \right).
\]

Notice that the behaviour of a cell under the boundary map depends only on the support and parity pattern of \( J \). This fact will be of importance in understanding the chain complexes in section 2.3.

**Proof** Each element \( \Lambda \in \exp_k \Gamma_n \) lies in the interior of the image of precisely one cell \( \sigma^J \) or \( \tilde{\sigma}^J \), namely \( \sigma^{\mathcal{J}(\Lambda)} \) if \( v \in \Lambda \) and \( \tilde{\sigma}^{\mathcal{J}(\Lambda)} \) if \( v \notin \Lambda \). The image of \( \sigma^J \) is contained in \( \exp_{|J|+1}(\Gamma_n, v) \) and that of \( \tilde{\sigma}^J \) in \( \exp_{|J|} \Gamma_n \), so we may set the \( j \)-skeleton of \( \exp_k(\Gamma_n, v) \) equal to \( \exp_{j+1}(\Gamma_n, v) \) and the \( j \)-skeleton of \( \exp_k \Gamma_n \) equal to \( (\exp_j \Gamma_n) \cup (\exp_{j+1}(\Gamma_n, v)) \) for \( j < k \) and \( \exp_k \Gamma_n \) for \( j = k \). The boundary of \( \tilde{\Delta}_J \) is found by replacing one or more inequalities in \( 0 \leq x_1 \leq \cdots \leq x_j \leq 1 \) with equalities, resulting in fewer points in the interior of \( e_i \); thus the image of the boundary of \( \tilde{\Delta}_J \) under \( \tilde{\sigma}^J \) is contained in \( \exp_{|J|-1}(\Gamma_n, v) \cup \exp_{|J|}(\Gamma_n, v) \). Similarly, \( \sigma^J \) maps the boundary of \( \Delta_J \) into \( \exp_{|J|}(\Gamma_n, v) \). So the boundary of a \( j \)-cell is contained in the \((j-1)\)-skeleton, and the \( \sigma^J, \tilde{\sigma}^J \) form cell structures as claimed.
To calculate the boundary map we use Lemma 1 of [19], which with our present notation and orientation convention says
\[ \partial \sigma^j_i = -\frac{1}{2} + \left(-\frac{1}{2}\right)^j \sigma^{j-1}_i, \]
\[ \partial \tilde{\sigma}^j_i = \frac{1}{2} \left(\frac{-1}{2}\right)^j (\sigma^{j-1}_i - 2\sigma^{j-1}_i), \] (2.4)
together with the relation \( \partial (\sigma \times \tau) = (\partial \sigma) \times \tau + (-1)^{\dim \sigma} \sigma \times (\partial \tau) \). Calculating the boundary of \( \tilde{\sigma}^1_1 \times \cdots \times \tilde{\sigma}^n_n \) and then applying \( \cup \) it follows that
\[ \partial \tilde{\sigma}^J = \sum_{i \in \text{supp } J} (-1)^{|J|\{i-1\}} \tilde{\sigma}^j_1 \cup \cdots \cup \partial \tilde{\sigma}^j_i \cup \cdots \cup \tilde{\sigma}^j_n. \]
Substituting (2.4) and observing that \( \tilde{\sigma}^j_1 \cup \cdots \cup \sigma^j_i \cup \cdots \cup \tilde{\sigma}^j_n = \sigma^J \) gives (2.3), and (2.2) follows by a similar argument or by using \( \sigma^J = (\cup\{v\})_2 \tilde{\sigma}^J \).

Let \( C_* \) be the free abelian group generated by the \( \sigma^J, 0 \leq |J| < \infty \), and \( \tilde{C}_* \) the free abelian group generated by the \( \tilde{\sigma}^J, 1 \leq |J| < \infty \), each graded by degree. Then
\[ H_*(\exp_k \Gamma_n) = H_*(C_{\leq k-1}) \]
and
\[ H_*(\exp_k \Gamma_n) = H_*(C_{\leq k-1} \oplus \tilde{C}_{\leq k}). \]
As discussed at the end of section 2.1 we know a priori that \( (C_*, \partial) \) is exact except at \( C_0 \). We nevertheless give a direct proof of this in section 2.3, with a view to constructing explicit bases for the homology groups in section 2.5 after calculating their ranks in section 2.4. Before doing so however we use Lemma 1 to calculate the fundamental groups of \( \exp_k \Gamma_n \) and \( \exp_k (\Gamma_n, v) \) for each \( k \) and \( n \), showing directly that \( \exp_k \Gamma_n \) and \( \exp_k (\Gamma_n, v) \) are simply connected for \( k \geq 3 \).

**Theorem 3**  The fundamental group of \( \exp_k \Gamma_n \) is

1. free of rank \( n \) if \( k = 1 \);
2. free abelian of rank \( n \), containing \( i_* \pi_1(\exp_1 \Gamma_n) \) as a subgroup of index \( 2^n \), if \( k = 2 \); and
3. trivial if \( k \geq 3 \).

The fundamental group of \( \exp_k (\Gamma_n, v) \) is free of rank \( n \) if \( k = 2 \) and trivial otherwise.

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Figure 1: Relations in $\pi_1(\exp_k \Gamma_n)$ arising from the 2–cells. The boundary of a cell is found by moving a point in the interior of an edge to $v$, or bringing two points in the interior into co-incidence. The first gives an untilded cell and the second a cell of the same kind as the interior. In (a) we see a torus killing the commutator of $[\sigma^i_1]$ and $[\sigma^i_1]$; in (b) a Möbius strip with fundamental group generated by $[\sigma^i_1]$ and boundary $[\tilde{\sigma}^i_1]$; and in (c) a dunce cap killing $[\sigma^i_1]$.

Proof In the unbased case $\exp_k \Gamma_n$ the result is obvious for $k = 1$ so consider $k = 2$. The group $\pi_1(\exp_2 \Gamma_n)$ is generated by $[\sigma^i_1]$, $[\tilde{\sigma}^i_1]$, $1 \leq i \leq n$, with relations arising from the $\tilde{\sigma}^2_i$ and $\tilde{\sigma}^i_1 \cup \tilde{\sigma}^j_1$, $i \neq j$. The image of $\tilde{\sigma}^i_1 \cup \tilde{\sigma}^j_1$ is a torus with meridian $[\sigma^i_1]$ and longitude $[\sigma^j_1]$, while the image of $\tilde{\sigma}^2_i$ is a Möbius strip that imposes the relation $[\tilde{\sigma}^i_1] = [\sigma^i_1]^2$ (see figures 1(a) and (b)). It follows that $\pi_1(\exp_2 \Gamma_n)$ is free abelian with generators $[\sigma^i_1]$, $1 \leq i \leq n$, and that $i_*\pi_1(\exp_1 \Gamma_n) = \langle [\tilde{\sigma}^1_1], \ldots, [\tilde{\sigma}^n_1] \rangle$ has index $2^n$. When $k \geq 3$ there are no new generators and additional relations $[\sigma^i_1] = 1$ from each $\sigma^2_i$ (see figure 1(c)), so that $\pi_1(\exp_k \Gamma_n) = \{1\}$.

In the based case $\exp_1(\Gamma_n, v) = \{v\}$, the map $\cup \{v\}: \exp_1 \Gamma_n \to \exp_2(\Gamma_n, v)$ is a homeomorphism, and for $k \geq 3$ the relations $[\sigma^i_1] = 1$ from the $\sigma^2_i$ apply as above.

2.3 Direct proof of the exactness of $C_{\geq 1}$

We show directly that $C_\ast$ is exact at each $\ell > 0$ by expressing it as a sum of finite subcomplexes and showing that each summand is exact. This decomposition will be used to construct explicit bases for the homology in section 2.5.

As a first reduction, for each subset $S$ of $[n]$ let $C^S_\ast$ be the free abelian group generated by $\{\sigma^J|\text{supp } J = S\}$. Since $\partial \sigma^J$ is a linear combination of the cells $\sigma^{\partial_i(J)}$ with $i \in \text{supp } J$ and $j_i \equiv 0 \mod 2$, each $C^S_\ast$ is a subcomplex and we have

$$C_\ast = \bigoplus_{S \subseteq [n]} C^S_\ast.$$
Finite subset spaces of graphs and punctured surfaces

Note that \( C_s^0 = C_0 \). Clearly \( C_s^S \cong C_s^T \) if \( |S| = |T| \) so we will show \( C_s^{[m]} \) is exact for each \( m > 0 \).

We claim that \( C_s^{[m]} \) may be regarded as a sum of many copies of a single finite complex, the \( m \)-cube complex. For each \( m \)-tuple \( L \) with all entries odd let \( C_s^{[m]}(L) \) be the subgroup of \( C_s^{[m]} \) generated by \( \{ \sigma^J | j_i - \ell_i \in \{0,1\} \} \). Again the fact that \( \partial \sigma^J \) is a linear combination of \( \{ \sigma^\partial(J) | i \in \text{supp} J, j_i \equiv 0 \mod 2 \} \) implies \( C_s^{[m]}(L) \) is a subcomplex, and moreover that

\[
C_s^{[m]} = \bigoplus_{L:|L|=m} C_s^{[m]}(L).
\]

Further, on translating each \( m \)-tuple by \( (\ell_1 - 1, \ldots, \ell_m - 1) \) each \( C_s^{[m]}(L) \) can be seen to be isomorphic to \( C_s^{[m]}((1,\ldots,1)) \) with its grading shifted by \( |L| - m \).

We call this common isomorphism class of complex the \( m \)-cube complex \( C_s^{[m]} \), and, replacing \( J \) with the set of indices of its even entries, will take the free abelian group generated by the power set of \( [m] \), graded by cardinality and with boundary map

\[
\partial S = \sum_{i \in S} (-1)^{|i|-1}\{S \setminus \{i\}\}
\]

to be its canonical representative; for aesthetic purposes we are dropping the minus sign outside the sum. The name \( m \)-cube complex comes from the fact that the lattice of subsets of \( [m] \) forms an \( m \)-dimensional cube, and that \( \partial S \) is a signed sum of the neighbours of \( S \) of smaller degree. See figure 2 for the case \( m = 3 \).
Let
\[ V_j = \{ S \subseteq [m] \mid |S| = j \text{ and } 1 \in S \}. \]

The exactness of \( C^m_{>j} \) follows from the first statement of the following lemma; the second statement will be used in section 2.5 to construct explicit bases for \( H_*(\exp_k \Gamma_n) \).

**Lemma 2** The \( m \)-cube complex \( C^m_* \) is exact. The homology of the truncated complex \( C^m_{<j} \) is free of rank \( \binom{m-1}{j} \) in dimension \( j \), with basis \( \{ \partial S \mid S \in V_{j+1} \} \), and zero otherwise.

**Proof** We claim that \( V_j \cup \partial V_{j+1} \) forms a basis for \( C^m_{>j} \), from which the lemma follows. Since \( V_j \cup \partial V_{j+1} \) has at most \( \binom{m-1}{j-1} + \binom{m-1}{j} = \binom{m}{j} = \text{rank} C^m_j \) elements we simply check that \( V_j \cup \partial V_{j+1} \) spans \( C^m_j \). It suffices to show that \( S \in \text{span} V_j \cup \partial V_{j+1} \) for each subset \( S \) of size \( j \) not containing 1; this follows from
\[ \partial(S \cup \{1\}) = S - \sum_{i \in S} (-1)^{[i-1]\backslash S} S \cup \{1\} \setminus \{i\} \]
if \( 1 \notin S \).

### 2.4 The homology groups of \( \exp_k(\Gamma_n, v) \) and \( \exp_k \Gamma_n \)

We calculate the homology groups of \( \exp_k(\Gamma_n, v) \) and \( \exp_k \Gamma_n \) using the exactness of \( C_{>1} \), the Euler characteristic (2.1), and the boundary maps (2.2) and (2.3). Explicit bases are found in section 2.5 using the decomposition of \( C_* \) into subcomplexes.

**Proof of Theorem 1** Since \( \exp_k(\Gamma_n, v) \) is path connected with homology equal to that of \( C_{<k-1} \), its reduced homology vanishes except perhaps in dimension \( k - 1 \), by the exactness of \( C_{>1} \). Moreover \( H_{k-1} \) is equal to \( \ker \partial_{k-1} \) and is therefore free; its rank may be found using \( \chi(\exp_k(\Gamma_n, v)) = b_0 + (-1)^{k-1} b_{k-1} \) and equation (2.1), yielding
\[ b_{k-1}(\exp_k(\Gamma_n, v)) = \sum_{j=1}^{k-1} (-1)^{k-j-1} \binom{n+j-1}{n-1}. \]

This may be expressed as a sum of purely positive terms by grouping the summands in pairs, starting with the largest, and using \( \binom{p}{q} - \binom{p-1}{q} = \binom{p-1}{q-1} \). Doing this for \( b_k(\exp_{k+1}(\Gamma_n, v)) \) gives the expression in equation (1.2).
Now consider \( H_*(\exp^k \Gamma_n) = H_*(C_{\leq k-1} \oplus C_{\leq k}) \). Write \( C_j \) for the \( j \)th chain group of \( C_{\leq k-1} \oplus \tilde{C}_{\leq k} \), \( \tilde{Z}_j \) for the \( j \)–cycles in \( C_j \) and \( Z_j \) for the \( j \)–cycles in \( C_j \). Extending \( \sigma^j \mapsto \tilde{\sigma}^j \) linearly to a group isomorphism from \( C_j \) to \( \tilde{C}_j \) for each \( j \geq 1 \), the boundary maps (2.2) and (2.3) give

\[
\partial \tilde{c} = 2\partial c - \partial c
\]

for each chain \( c \in C_{>1} \). It follows that \( Z_j = Z_j \oplus \tilde{Z}_j \) for \( 1 \leq j \leq k-1 \) and that \( H_k(\exp^k \Gamma_n) = Z_k \) is equal to \( \tilde{Z}_k \). Moreover

\[
\partial C_k = \{2z - \tilde{z} | z \in Z_{k-1}\}
\]

by the exactness of \( C_* \) at \( C_{k-1} \), so that \( H_{k-1}(\exp^k \Gamma_n) = Z_{k-1} \). We show that the remaining reduced homology groups vanish.

Let \( z = z_1 \oplus \tilde{z}_2 \in Z_j \) for some \( 1 \leq j \leq k-2 \). By the exactness of \( C_{\geq 1} \) there are \( w_1, w_2 \in C_{j+1} \) such that

\[
\partial w_1 = z_1 + 2\tilde{z}_2,
\]

\[
\partial w_2 = \tilde{z}_2.
\]

Since \( j \leq k-2 \) we have \( w_1 - \tilde{w}_2 \in C_{j+1} \), and

\[
\partial(w_1 - \tilde{w}_2) = \partial w_1 - \partial \tilde{w}_2
\]

\[
= z_1 + 2\tilde{z}_2 - 2\partial w_2 + \tilde{\partial}w_2
\]

\[
= z_1 + \tilde{z}_2,
\]

so that \( C_* \) is exact at \( C_j \) as claimed.

It remains to determine the maps induced by

\[
i: \exp^k(\Gamma_n, v) \leftrightarrow \exp^k \Gamma_n,
\]

\[
\cup\{v\}: \exp^k \Gamma_n \rightarrow \exp^{k+1}(\Gamma_n, v)
\]

and

\[
\exp^k \Gamma_n \leftrightarrow \exp^{k+1} \Gamma_n
\]

on homology. In each case there is only one dimension in which the induced map is not trivially zero. We have \( H_{k-1}(\exp^k(\Gamma_n, v)) = Z_{k-1} = H_{k-1}(\exp^k \Gamma_n) \), so that \( i_* \) is an isomorphism on \( H_{k-1} \), and adding \( v \) to each element of \( \exp^k \Gamma_n \) sends \( \tilde{z} \) to \( z \) for each \( z \in Z_k \), inducing an isomorphism on \( H_k \). Lastly, \( \exp^k \Gamma_n \leftrightarrow \exp^{k+1} \Gamma_n \) sends \( [\tilde{z}] \) to \( [\tilde{z}] = 2[z] = 2(i \circ \cup\{v\})_*[\tilde{z}] \) for each \( z \in Z_k \), inducing two times \( (i \circ \cup\{v\})_* \) as claimed.

The homology and fundamental group of \( \exp^k(\Gamma_n, v) \) are enough to determine its homotopy type completely. When \( k = 1 \) it is a single point \( \{\{v\}\} \), and when \( k \geq 2 \) we have:
Corollary 1 (Theorem 2) For \( k \geq 2 \) the space \( \exp_k(\Gamma_n, v) \) has the homotopy type of a wedge of \( b_{k-1}(\exp_k(\Gamma_n, v)) \) \((k - 1)\)-spheres.

Proof Since \( \exp_2(\Gamma_n, v) \) is homeomorphic to \( \Gamma_n \) we may assume \( k \geq 3 \). But then \( \exp_k(\Gamma_n, v) \) is a simply connected Moore space \( M(\mathbb{Z}^{b_{k-1}}, k - 1) \) and the result follows from the Hurewicz and Whitehead theorems. ☐

2.5 A basis for \( H_k(\exp_k \Gamma_n) \)

We use the decomposition of \( C_* \) as a sum of subcomplexes to give an explicit basis for \( H_k(\exp_k \Gamma_n) \).

Theorem 4 The set

\[
\mathcal{B}(k, n) = \left\{ \partial \sigma | |J| = k + 1 \text{ and } j_i \equiv 0 \text{ mod } 2 \text{ for } i = \min(\text{supp } J) \right\}
\]

is a basis for \( H_k(\exp_k \Gamma_n) \).

Proof It suffices to find a basis for \( \mathcal{Z}_k \) and map it across to \( \tilde{\mathcal{Z}}_k \). Extending notation in obvious ways we have

\[
\mathcal{Z}_k = \bigoplus_{S \subseteq [n]} \mathcal{Z}_k^S = \bigoplus_{S \subseteq [n]} \bigoplus_{L: \supp L = S, |L| = |S|} \mathcal{Z}_k^S(L).
\]

Each \( \mathcal{Z}_k^S(L) \) in this sum is isomorphic to \( \mathcal{Z}_j^{[S]} \) for some \( j \), and tracing back through this isomorphism we see that \( \mathcal{V}_{j+1}(L) \) is carried up to sign to

\[
\mathcal{V}_{j+1}(L) = \{ \sigma | |J| = k + 1, j_i - \ell_i \in \{0, 1\}, j_i \equiv 0 \text{ mod } 2 \text{ for } i = \min(\text{supp } J) \}.
\]

By Lemma 2 \( \{ \partial \sigma | \sigma \in \mathcal{V}_{j+1}^S(L) \} \) is a basis for \( \mathcal{Z}_k^S(L) \), and taking the union over \( S \) and \( L \) completes the proof. ☐

As an exercise in counting we check that \( \mathcal{B}(k, n) \) has the right cardinality. This is equivalent to showing that the number \( s(k, n) \) of non-negative integer solutions to

\[
j_1 + \cdots + j_n = k
\]

in which the first non-zero summand is odd is given by equation (1.2). We do this by induction on \( k \), inducting separately over the even and odd integers.
In the base cases \( k = 1 \) and \( 2 \) there are clearly \( n \) and \( \binom{n}{2} \) solutions respectively. It therefore suffices to show that \( s(k, n) - s(k - 2, n) = \binom{n+k-2}{n-2} \). Adding two to the first non-zero summand gives an injection from solutions with \( k = \ell - 2 \) to solutions with \( k = \ell \), hitting all solutions except those for which the first non-zero summand is one. If \( j_{n-i} = 1 \) is the first non-zero summand then what is left is an unconstrained non-negative integer solution to

\[
j_{n-i+1} + \cdots + j_n = \ell - 1,
\]

of which there are \( \binom{\ell+i-2}{\ell-1} \), so that

\[
s(k, n) - s(k - 2, n) = \sum_{i=1}^{n-1} \binom{k + i - 2}{k - 1}.
\]

This is a sum down a diagonal of Pascal’s triangle and as such is easily seen to equal \( \binom{k+n-2}{k} = \binom{k+n-2}{n-2} \).

### 2.6 Generating functions for \( b_k(\exp_k \Gamma_n) \)

We conclude this section by giving generating functions for the Betti numbers \( b_k(\exp_k \Gamma_n) \).

**Theorem 5** The Betti number \( b_k(\exp_k \Gamma_n) \) is the co-efficient of \( x^k \) in

\[
\frac{1 - (1 - x)^n}{(1 + x)(1 - x)^n}.
\]

**Proof** The co-efficient of \( x^j \) in

\[
\frac{1}{(1 - x)^n} = (1 + x + x^2 + x^3 + \cdots)^n
\]

is the number of non-negative solutions to \( j_1 + \cdots + j_n = j \), in other words \( \binom{n+j-1}{n-1} \). Multiplication by \( (1 + x)^{-1} = 1 - x + x^2 - x^3 + \cdots \) has the effect of taking alternating sums of co-efficients, so we subtract 1 first to remove the unwanted degree zero term from \( (1 - x)^{-n} \), arriving at (2.5). \( \square \)
3 The calculation of induced maps

3.1 Introduction

Given a pointed map \( \phi: (\Gamma_n, v) \to (\Gamma_m, v) \) there are induced maps

\[
(\exp_k \phi)_*: H_{k-1}(\exp_k (\Gamma_n, v)) \to H_{k-1}(\exp_k (\Gamma_m, v))
\]

and

\[
(\exp_k \phi)_*: H_p(\exp_k \Gamma_n) \to H_p(\exp_k \Gamma_m)
\]

for \( p = k - 1, k \). In view of the commutative diagrams

\[
\begin{array}{ccc}
\exp_k (\Gamma_n, v) & \xrightarrow{\exp_k \phi} & \exp_k (\Gamma_m, v) \\
\downarrow i & & \downarrow i \\
\exp_k \Gamma_n & \xrightarrow{\exp_k \phi} & \exp_k \Gamma_m
\end{array}
\]

and

\[
\begin{array}{ccc}
\exp_{k-1} \Gamma_n & \xrightarrow{\exp_{k-1} \phi} & \exp_{k-1} \Gamma_m \\
\downarrow \cup \{v\} & & \downarrow \cup \{v\} \\
\exp_k (\Gamma_n, v) & \xrightarrow{\exp_k \phi} & \exp_k (\Gamma_m, v)
\end{array}
\]

and the isomorphisms induced by \( i \) and \( \cup \{v\} \) on \( H_{k-1} \) and \( H_k \) respectively it suffices to understand just one of these, and we will focus our attention on \( H_k(\exp_k \Gamma_n) \to H_k(\exp_k \Gamma_m) \). The purpose of this section is to reduce the problem of calculating this map to the problem of finding the images of the basic cells \( \tilde{\sigma}_1^j, \tilde{\sigma}_2^j \) under the chain map \( (\exp \phi)_* \). The reduction will be done by defining a multiplication on \( \tilde{C}_* \) giving it the structure of a ring without unity generated by the \( \tilde{\sigma}_i^j \). The multiplication will be defined in such a way that the cellular chain map \( (\exp \phi)_*|_{\tilde{C}_*} \) is a ring homomorphism, reducing calculating \( (\exp_k \phi)_* \) to calculations in the chain ring once the \( (\exp \phi)_* \tilde{\sigma}_i^j \) are found. The reduction to just the cells \( \tilde{\sigma}_1^1, \tilde{\sigma}_2^2 \) is achieved by working over the rationals, as \( \tilde{C}_* \otimes_{\mathbb{Z}} \mathbb{Q} \) will be generated over \( \mathbb{Q} \) by just these \( 2n \) cells.

In what follows we will assume that \( \phi \) is smooth, in the sense that \( \phi \) is smooth on the open set \( \phi^{-1}(\Gamma_m \setminus \{v\}) \). This ensures that \( \exp_j \phi \) is smooth off the preimage of the \((j - 1)\)–skeleton \( (\exp_j \Gamma_n)^{j-1} = (\exp_{j-1} \Gamma_n) \cup (\exp_j (\Gamma_n, v)) \), allowing us to use smooth techniques on the manifold \( \exp_j \Gamma_n \setminus (\exp_j \Gamma_n)^{j-1} \).

Smoothness of \( \phi \) may be ensured by homotoping it to a standard form defined as follows. The restriction of \( \phi \) to each edge \( e_i \) is an element of \( \pi_1(\Gamma_m, v) \), and as such is equivalent to a reduced word \( w_i \) in the \( \{e_a\} \cup \{\tilde{e}_a\} \). We consider \( \phi \) to be in standard form if \( \phi|_{e_i} \) traverses each letter of \( w_i \) at constant speed.
3.2 The chain ring

We observe that the operation \((g, h) \mapsto g \cup h\) suggests a natural way of multiplying cells and we study it with an eye to applying the results to maps of the form \((\exp_k \phi) \circ \tilde{\sigma}^J\).

A map of pairs \(g: (B^j, \partial B^j) \to (\exp_j \Gamma_n, (\exp_j \Gamma_n)^{j-1})\) induces a map

\[ g_*: H_j(B^j, \partial B^j) \to H_j(\exp_j \Gamma_n, (\exp_j \Gamma_n)^{j-1}), \]

and the homology group on the right is canonically isomorphic to the cellular chain group \(\tilde{C}_j\). Writing \(\varepsilon^j\) for the positive generator of \(H_j(B^j, \partial B^j)\), if \(g\) is smooth on the open set \(g^{-1}(\exp_j \Gamma_n \setminus (\exp_j \Gamma_n)^{j-1})\) then this map is given by

\[ g_* \varepsilon^j = \sum_{|J|=j} \langle g, \tilde{\sigma}^J \rangle \tilde{\sigma}^J, \]

in which \(\langle g, \tilde{\sigma}^J \rangle\) is the signed sum of preimages of a generic point in the interior of \(\tilde{\sigma}^J\). If \(h\) is a second map of pairs \((B^\ell, \partial B^\ell) \to (\exp_\ell \Gamma_n, (\exp_\ell \Gamma_n)^{\ell-1})\) then \(g \cup h\) is a map of pairs

\[ (B^{j+\ell}, \partial B^{j+\ell}) \to (\exp_{j+\ell} \Gamma_n, (\exp_{j+\ell} \Gamma_n)^{j+\ell-1}) \]

also. The following lemma shows that \(\cup\) behaves as might be hoped on the chain level.

**Lemma 3** Given two maps of pairs \(g: (B^j, \partial B^j) \to (\exp_j \Gamma_n, (\exp_j \Gamma_n)^{j-1}), h: (B^\ell, \partial B^\ell) \to (\exp_\ell \Gamma_n, (\exp_\ell \Gamma_n)^{\ell-1})\), each smooth off the preimage of the codimension one skeleton, we have

\[ (g \cup h)_* = \sum_{|J|=j, |L|=\ell} \langle g, \tilde{\sigma}^J \rangle \langle h, \tilde{\sigma}^L \rangle (\tilde{\sigma}^J \cup \tilde{\sigma}^L)_*. \tag{3.1} \]

**Proof** Fix an \(n\)-tuple \(M\) such that \(|M| = j + \ell\) and let \(\Lambda\) be a generic point in the interior of \(\tilde{\sigma}^M\). It suffices to check that \(g \times h\) and \(\sum_{J,L} \langle g, \tilde{\sigma}^J \rangle \langle h, \tilde{\sigma}^L \rangle (\tilde{\sigma}^J \times \tilde{\sigma}^L)\) have the same signed sum of preimages at each point \((\Lambda', \Lambda'') \in \exp_J \Gamma_n \times \exp_L \Gamma_n\) such that \(\Lambda' \cup \Lambda'' = \Lambda\); note that for cardinality reasons \((\Lambda', \Lambda'')\)
forms a partition of \( \Lambda \). For \( g \times h \) this signed sum is given by

\[
\langle g \times h, \tilde{\sigma}^J(\Lambda) \times \tilde{\sigma}^J(\Lambda') \rangle = \sum_{p \in g^{-1}(\Lambda)} \sum_{q \in h^{-1}(\Lambda')} \text{sign}(\det D(g \times h)(p, q))
\]

\[
= \sum_{p \in g^{-1}(\Lambda)} \sum_{q \in h^{-1}(\Lambda')} \text{sign}(\det Dg(p)) \text{sign}(\det Dh(q))
\]

\[
= \left[ \sum_{g^{-1}(\Lambda')} \text{sign}(\det Dg(p)) \right] \left[ \sum_{h^{-1}(\Lambda')} \text{sign}(\det Dh(q)) \right]
\]

\[
= \langle g, \tilde{\sigma}^J(\Lambda) \rangle \langle h, \tilde{\sigma}^J(\Lambda') \rangle.
\]

The lemma follows from the fact that \( \langle \tilde{\sigma}^J \times \tilde{\sigma}^L, \tilde{\sigma}^J(\Lambda) \times \tilde{\sigma}^J(\Lambda') \rangle \) is zero unless \( J = J(\Lambda) \) and \( L = J(\Lambda') \), in which case it is one.

Since \( \langle \tilde{\sigma}^J \cup \tilde{\sigma}^L \rangle \) is a multiple of \( \tilde{\sigma}^{J+L} \) the next step is to understand the pairings \( \langle \tilde{\sigma}^J \cup \tilde{\sigma}^L, \tilde{\sigma}^{J+L} \rangle \). Interchanging adjacent factors \( \tilde{\sigma}^a_b \) and \( \tilde{\sigma}^s_b \) in the product \( \tilde{\sigma}^J \cup \tilde{\sigma}^L \) simply introduces a sign \((-1)^{rs}\), so we may gather basic cells from the same edge together and consider pairings of the form

\[
\langle (\tilde{\sigma}^j_1 \cup \tilde{\sigma}^{j_1}), \ldots, (\tilde{\sigma}^j_n \cup \tilde{\sigma}^{j_n}), \tilde{\sigma}^{J+L} \rangle = \prod_{i=1}^n \langle \tilde{\sigma}^{j_i} \cup \tilde{\sigma}^{\ell_i}, \tilde{\sigma}^{j_i+\ell_i} \rangle.
\]

The quantity \( \langle \tilde{\sigma}^r \cup \tilde{\sigma}^r, \tilde{\sigma}^{r+s} \rangle \) is equal to \( \binom{r+s}{r}_{-1} \), the \( q \text{-binomial co-efficient} \) specialised to \( q = -1 \). The correspondence can be seen as follows. Take \( r+s \) objects, numbered from 1 to \( r+s \) and laid out in order, and paint \( r \) of them blue and the rest red. Shuffle them so that the blue ones are at the front in ascending order, followed by the red ones in ascending order, giving an element of the symmetric group \( S_{r+s} \). Then \( \binom{r+s}{r}_{-1} \) is the number of ways of choosing \( r \) objects from \( r+s \) in this way, counted with the sign of the associated permutation, and is equal to \( \langle \tilde{\sigma}^r_a \cup \tilde{\sigma}^r_a, \tilde{\sigma}^{r+s}_a \rangle \): the blue and red points represent the elements of a generic point in \( \exp_{r+s} e_a \) coming from \( \tilde{\sigma}^r_a \) and \( \tilde{\sigma}^s_a \) respectively, and the derivative at this preimage is the matrix of the associated permutation.

The calculation of \( \binom{r+s}{r}_{-1} \) is the subject of the following lemma. The result, which we might call “Pascal’s other triangle”—being a much less popular model than the one we know and love—appears in figure 3. For further information on \( q \)-binomial co-efficients and related topics see for example Stanley [18], Kac and Cheung [13], and Baez [1, weeks 183–188].

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The first ten rows of Pascal’s other triangle, enough so that the pattern is clear. Each row of Pascal’s triangle appears twice; on the first occurrence zeros are inserted between each entry, and on the second each entry appears twice.

\[ \begin{array}{ccccccccc}
1 \\
1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 1 & 1 \\
1 & 0 & 2 & 0 & 1 \\
1 & 1 & 2 & 2 & 1 & 1 \\
1 & 0 & 3 & 0 & 3 & 0 & 1 \\
1 & 1 & 3 & 3 & 3 & 3 & 1 & 1 \\
1 & 0 & 4 & 0 & 6 & 0 & 4 & 0 & 1 \\
1 & 1 & 4 & 4 & 6 & 6 & 4 & 4 & 1 & 1 \\
\end{array} \]

Figure 3: The first ten rows of Pascal’s other triangle, enough so that the pattern is clear. Each row of Pascal’s triangle appears twice; on the first occurrence zeros are inserted between each entry, and on the second each entry appears twice.

**Lemma 4** The value of the signed binomial co-efficient \( \binom{m}{r} \) is given by

\[
\binom{m}{r}_{-1} = \frac{1 + (-1)^{r(m-r)}}{2} \left( \frac{\lfloor m/2 \rfloor}{\lfloor r/2 \rfloor} \right).
\]  

(3.2)

**Proof** The \( q \)-binomial co-efficient \( \binom{m}{r} \) satisfies \( \binom{m}{0} = \binom{m}{m} = 1 \) and

\[
\binom{m}{r} = \binom{m-1}{r-1} + q^r \binom{m-1}{r},
\]

both easily seen for \( q = -1 \) from the definition above: the recurrence relation is proved analogously to the familiar one for Pascal’s triangle, the sign \((-1)^r\) arising from shuffling the first object into place if it is red instead of blue. The equality (3.2) is then readily proved by induction on \( m \), by considering in turn the four possibilities for the parities of \( m \) and \( r \).

We remark that the equality \( \binom{r+s}{r} = \binom{r+s}{s} \) can be seen for \( q = -1 \) from the fact that \( \lfloor r/2 \rfloor + \lfloor s/2 \rfloor = \lfloor (r+s)/2 \rfloor \) unless both \( r \) and \( s \) are odd, in which case the co-efficient \( \binom{r+s}{r}_{-1} \) vanishes. Additionally \( \binom{r+s}{r}_{-1} \binom{\lfloor r+s/2 \rfloor}{\lfloor r/2 \rfloor} \) and \( \binom{r+s+t}{r}_{-1} \binom{\lfloor r+s+t/2 \rfloor}{\lfloor r/2 \rfloor} \) are both equal to

\[
\frac{(r+s+t)/2)!}{\lfloor r/2 \rfloor! \lfloor s/2 \rfloor! \lfloor t/2 \rfloor!}
\]
if no more than one of \( r, s, t \) is odd, and zero otherwise.

We now define the chain ring of \( \exp \Gamma_n \) to be the ring without unity generated over \( \mathbb{Z} \) by the set \( \{ \tilde{\sigma}_i^j | 1 \leq i \leq n, j \geq 1 \} \), with relations

\[
\tilde{\sigma}_a^j \tilde{\sigma}_b^\ell = (-1)^{j\ell} \tilde{\sigma}_b^\ell \tilde{\sigma}_a^j
\]

\[
\tilde{\sigma}_a^j \tilde{\sigma}_a^\ell = [j + \ell]_{-1} \tilde{\sigma}_a^{j+\ell}
\]

for all \( 1 \leq a, b \leq n \) and \( j, \ell \geq 1 \). This definition is again a specialisation to \( q = -1 \) of a construction that applies more generally, and is chosen so that the conclusion (3.1) of Lemma 3 may be rewritten as

\[(g \cup h)_* e^{j+\ell} = (g_* e^j)(h_* e^\ell), \tag{3.3}\]

in which the multiplication on the right hand side takes place in the chain ring. The lack of an identity could be easily remedied but we have chosen not to so that all elements of the ring are chains. We shall denote the chain ring simply by \( \tilde{C}_* \), or \( \tilde{C}_*(\Gamma_n) \) in case of ambiguity.

### 3.3 Calculating induced maps

We now have all the machinery required to state and prove our main result on the calculation of \( (\exp_k \phi)_* \), namely that the cellular chain map \((\exp \phi)_|_{\tilde{C}_*}\) is a ring homomorphism. This reduces the calculation of the chain map to the calculation of the images of the basic cells \( \tilde{\sigma}_i^j \), each an exercise in counting points with signs, and multiplication and addition in the chain ring.

**Theorem 6** If \( \phi: (\Gamma_n, v) \to (\Gamma_m, v) \) is smooth on \( \phi^{-1}(\Gamma_m \setminus \{v\}) \) then the cellular chain map \((\exp \phi)_|_{\tilde{C}_*}: \tilde{C}_*(\Gamma_n) \to \tilde{C}_*(\Gamma_m)\) is a ring homomorphism.

**Proof** If \( \phi \) is smooth off \( \phi^{-1}(v) \) then as noted at the end of section 3.1 the map \( \exp_j \phi = \exp \phi|_{\exp_j \Gamma_n} \) is smooth off the codimension one skeleton for each \( j \), so we may use Lemma 3. Thus

\[
(\exp \phi)_* (\tilde{\sigma}_i^j \tilde{\sigma}_i^\ell) = (\exp_{j+\ell} \phi)_* ((\tilde{\sigma}_i^j \cup \tilde{\sigma}_i^\ell)_* e^{j+\ell})
\]

by (3.3)

\[
= (\exp_{j+\ell} \phi) \circ (\tilde{\sigma}_i^j \cup \tilde{\sigma}_i^\ell)_* e^{j+\ell}
\]

\[
= ((\exp_j \phi) \circ \tilde{\sigma}_i^j) \cup ((\exp_\ell \phi) \circ \tilde{\sigma}_i^\ell)_* e^{j+\ell}
\]

\[
= ((\exp_j \phi) \circ \tilde{\sigma}_i^j)_* e^j \circ ((\exp_\ell \phi) \circ \tilde{\sigma}_i^\ell)_* e^\ell
\]

by (3.3)

\[
= ((\exp_j \phi)_* \tilde{\sigma}_i^j e^j) (\exp_\ell \phi)_* \tilde{\sigma}_i^\ell e^\ell
\]

from which the result follows. □

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As an immediate consequence we have

$$(\exp \phi)_{2 \tilde{\sigma}^j} = \prod_{i=1}^{n} (\exp \phi)_{2 \tilde{\sigma}^j_i},$$

so that $$(\exp \phi)_{2 \tilde{\sigma}^j}$$ may be found knowing just the images of the basic cells $\tilde{\sigma}^j_i$ as claimed. To reduce the number of cells $\tilde{\sigma}^j_i$ for which $$(\exp \phi)_{2 \tilde{\sigma}^j}$$ must be calculated directly even further, observe that

$$\tilde{\sigma}^j_i = \begin{cases} \frac{1}{\ell} \left( \tilde{\sigma}^1_i \right)^{\ell} & \text{if } j = 2\ell, \\ \frac{1}{\ell} \tilde{\sigma}^1_i \left( \tilde{\sigma}^2_i \right)^{\ell} & \text{if } j = 2\ell + 1, \end{cases}$$

so that $\mathcal{C}_* \otimes \mathbb{Q}$ is generated over $\mathbb{Q}$ by the set $\{ \tilde{\sigma}^j_i | 1 \leq i \leq n, j = 1, 2 \}$, subject only to the relations that the $\tilde{\sigma}^1_i$ anti-commute with each other and the $\tilde{\sigma}^2_i$ commute with everything. In essence this reduces calculating $$(\exp \phi)_{2 \tilde{\sigma}^j}$$ to understanding the behaviour of chains under maps

$$\exp_1 S^1 \to \exp_1 S^1$$

and

$$\exp_2 S^1 \to \exp_2 \Gamma_2$$

induced by maps $S^1 \to S^1$ and $S^1 \to \Gamma_2$ respectively. These are both simple exercises in counting points with signs and we give the answers, which are easily checked.

Let $w$ be a reduced word in $\{ e_1, e_2, \bar{e}_1, \bar{e}_2 \}$, and let $\phi$ from $S^1 = \Gamma_1$ to $\Gamma_2$ send $e_1$ to $w$. Then $\langle (\exp \phi) \circ \tilde{\sigma}^1_i, \tilde{\sigma}^1_i \rangle$ and $\langle (\exp \phi) \circ \tilde{\sigma}^2_i, \tilde{\sigma}^2_i \rangle$ are both given by the winding number of $w$ around $e_i$, and $\langle (\exp \phi) \circ \tilde{\sigma}^{(1,1)}_i, \tilde{\sigma}^{(1,1)}_i \rangle$ is the number of pairs of letters $(a_1, a_2)$ in $w$, $a_i \in \{ e_i, \bar{e}_i \}$, counted with the product of a minus one for each bar and a further minus one if $a_2$ occurs before $a_1$.

### 3.4 Examples

As an illustration of the ideas in this section we calculate two examples. The first reproduces a result from [19] on maps $S^1 \to S^1$, and the second will be useful in understanding the action of the braid group.

Let $\phi: S^1 \to S^1$ be a degree $d$ map. By Theorem 1 of this paper or Theorem 4 of [19] we have

$$H_k(\exp_k S^1) \cong \begin{cases} 0 & k \text{ even}, \\ \mathbb{Z} & k \text{ odd}, \end{cases}$$

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so the only map of interest is (exp_{2\ell-1})_\tau on H_{2\ell-1}. The homology group 
H_{2\ell-1}(exp_{2\ell-1}S^1) is generated by \hat{\sigma}_1^{2\ell-1}, and by (3.4) and the discussion at the 
end of section 3.3 we have

\[(\exp_{2\ell-1}\phi)_{\tau} \hat{\sigma}_1^{2\ell-1} = \frac{1}{(\ell - 1)!} (\exp_{2\ell-1}\phi)_{\tau} (\hat{\sigma}_1 (\hat{\sigma}_1)^{\ell-1})\]

\[= \frac{1}{(\ell - 1)!} (d\hat{\sigma}_1)(d\hat{\sigma}_1^2)^{\ell-1}\]

\[= \frac{d^\ell}{(\ell - 1)!} \hat{\sigma}_1^{\ell-1}\]

\[= d^\ell \hat{\sigma}_1^{2\ell-1}.\]

Thus \exp_{2\ell-1}\phi is a degree \(d^\ell\) map, as found in Theorem 7 of [19].

For the second example consider the map \(\tau: \Gamma_2 \to \Gamma_2\) sending \(e_1\) to \(e_2\) and \(e_2\) to \(e_2\). We shall compare this with the map \(\bar{\tau}\) that simply switches \(e_1\) and \(e_2\). These two maps are homotopic through a homotopy that drags \(\tau\) around \(e_2\), so we expect the same induced map once we pass to homology, but an understanding of the chain maps will be useful in section 4 when we study the action of the braid group. Clearly (exp \(\bar{\tau}\))_2 simply interchanges \(\hat{\sigma}_1^j\) and \(\hat{\sigma}_2^j\), and likewise (exp \(\tau\))_2 \hat{\sigma}_1^j = \hat{\sigma}_2^j for each \(j\). The only difficulty therefore is in finding (exp \(\tau\))_2 \hat{\sigma}_2^j, and we shall do this in two ways, first by calculating it directly and then by working over \(Q\) using (3.4).

To calculate (exp \(\tau\))_2 \hat{\sigma}_2^j directly let \(p = (x_1, \ldots, x_{\ell}, y_1, \ldots, y_m)\) be a generic point in \(\Delta_\ell \times \Delta_m\), \(\ell + m = j\), \(m \geq 1\). Preimages of \(p\) come in pairs that differ only in whether the preimage of \(y_j\) comes from the letter \(e_2\) or \(\bar{e}_2\) of \(w\). In the first case the \((\ell + 1)\)th row of the derivative has a 3 in the first column and zeros elsewhere, and in the second it has a \(-3\) in the last column and zeros elsewhere. Thus each pair contributes \((-1)^j + (-1)^m\) times the sign of the corresponding preimage in \(\hat{\sigma}_2^{j-1}\) of \((x_1, \ldots, x_{\ell}, y_2, \ldots, y_m) \in \Delta_\ell \times \Delta_{m-1}\), and consequently

\[\langle (\exp \tau) \circ \hat{\sigma}_2^j, \hat{\sigma}^{(\ell,m)} \rangle = \langle (-1)^{\ell} + (-1)^m \rangle \langle (\exp \tau) \circ \hat{\sigma}_2^{j-1}, \hat{\sigma}^{(\ell,m-1)} \rangle.\]

This recurrence relation is easily solved to give

\[\langle (\exp \tau) \circ \hat{\sigma}_2^j, \hat{\sigma}^{(\ell,m)} \rangle = \begin{cases} 
1 & \text{if } m = 0, \\
-2 & \text{if } \ell \text{ odd, } m = 1, \\
0 & \text{otherwise},
\end{cases}\]

so that

\[\langle (\exp \tau) \circ \hat{\sigma}_2^j, \hat{\sigma}^{(\ell,m)} \rangle = \begin{cases} 
\hat{\sigma}_1^j & \text{if } j \text{ is odd}, \\
\hat{\sigma}_1^j - 2\hat{\sigma}^{(j-1,1)} & \text{if } j > 0 \text{ is even}.
\end{cases}\]
To find \( (\exp \tau) \tilde{\sigma}_2^{2\ell} \) using (3.4) we first need \((\exp \tau) \tilde{\sigma}_1 = \tilde{\sigma}_1^1 \) and \((\exp \tau) \tilde{\sigma}_2^2 = \tilde{\sigma}_1^2 - 2\tilde{\sigma}_1^1 \tilde{\sigma}_2^1 \), each easily found directly. Then

\[
(\exp \tau) \tilde{\sigma}_2^{2\ell} = \frac{1}{\ell!} (\tilde{\sigma}_1^2 - 2\tilde{\sigma}_1^1 \tilde{\sigma}_2^1)^\ell.
\]

The cell \( \tilde{\sigma}_1^2 \) commutes with everything, so the binomial theorem applies, but \( \tilde{\sigma}_1^1 \) and \( \tilde{\sigma}_2^1 \) square to zero, so only two terms are nonzero. We get

\[
(\exp \tau) \tilde{\sigma}_2^{2\ell} = \frac{1}{\ell!} (\tilde{\sigma}_1^2)^\ell - 2 \ell \tilde{\sigma}_1^1 \tilde{\sigma}_2^1 (\tilde{\sigma}_1^2)^{\ell-1}
\]

the even case of (3.5). Multiplying by \( (\exp \tau) \tilde{\sigma}_2^1 = \tilde{\sigma}_1^1 \) kills the second term and we get the odd case also.

To complete the calculation of \((\exp \tau) \tilde{\sigma}_2^\ell \) we find the image of the cells \( \tilde{\sigma}^{(\ell, m)} \). If \( m = 2p + 1 \) is odd we have simply

\[
(\exp \tau) \tilde{\sigma}^{(\ell, 2p+1)} = \tilde{\sigma}_2^\ell \tilde{\sigma}_1^{2p+1} = (\exp \bar{\tau}) \tilde{\sigma}^{(\ell, 2p+1)},
\]

while if \( m = 2p > 0 \) is even we get

\[
(\exp \tau) \tilde{\sigma}^{(\ell, 2p)} = \tilde{\sigma}_2^\ell \left( \tilde{\sigma}_1^{2p} - 2\tilde{\sigma}_1^{2p-1} \tilde{\sigma}_2 \right)
\]

\[
= \tilde{\sigma}_2^\ell \tilde{\sigma}_1^{2p} - 2(-1)^{\ell+1} \tilde{\sigma}_1^{2p-1} \tilde{\sigma}_2^{\ell+1}
\]

\[
= \begin{cases} 
\tilde{\sigma}_2^\ell \tilde{\sigma}_1^{2p} & \text{if } \ell \text{ is odd,} \\
\tilde{\sigma}_2^\ell \tilde{\sigma}_1^{2p} - 2\tilde{\sigma}_1^{2p-1} \tilde{\sigma}_2^{\ell+1} & \text{if } \ell \text{ is even.}
\end{cases}
\]

Thus

\[
(\exp \tau) \tilde{\sigma}^{(\ell, m)} = \begin{cases} 
(\exp \bar{\tau}) \tilde{\sigma}^{(\ell, m)} - 2\tilde{\sigma}^{(m-1, \ell+1)} & \ell, m \text{ both even, } m > 0, \\
(\exp \bar{\tau}) \tilde{\sigma}^{(\ell, m)} & \text{otherwise.}
\end{cases}
\]

(3.6)

Since elements of homology are linear combinations of cells each having at least one odd index we have \( (\exp_k \tau)_* = (\exp_k \bar{\tau})_* \) as expected.

4 The action of the braid group

4.1 Introduction

The braid group on \( n \) strands \( B_n \) may be defined as the mapping class group of an \( n \)-punctured disc \( D_n \), or more precisely as the group of homeomorphisms of \( D_n \) that fix \( \partial D_n \) pointwise, modulo those isotopic to the identity rel \( \partial D_n \).
As such it acts on $H_k(\exp_k D_n)$, and since $D_n \simeq \Gamma_n$ we may regard this as an action on $H_k(\exp_k \Gamma_n)$. We prove the following structure theorem for this action.

**Theorem 7** The image of the pure braid group $P_n$ under the action of $B_n$ on $H_k(\exp_k \Gamma_n)$ is nilpotent of class at most $\min\{ (k-1)/2, [(n-1)/2] \}$ if $k$ is odd, or $\min\{ (k-2)/2, [(n-2)/2] \}$ if $k$ is even.

For the above and other definitions of the braid group see Birman [4].

Recall that the pure braid group $P_n$ is the kernel of the map $B_n \to S_n$ sending each braid $\beta$ to the induced permutation $\bar{\beta}$ of the punctures. Consider the subgroup of $P_n$ consisting of braids whose first $n-1$ strands form the trivial braid. The $n$th strand of such a braid may be regarded as an element of $\pi_1(D_{n-1})$, and doing so gives an isomorphism from this subgroup to the free group $F_{n-1}$. This shows that $P_n$ is not nilpotent for $n \geq 3$. The group $P_2$ inside $B_2$ is isomorphic to $2\mathbb{Z}$ inside $\mathbb{Z}$, and is therefore nilpotent of class 1; however the bound for $n = 2$ in Theorem 7 is zero, implying $P_2$ acts trivially, and in fact this follows from the second example of section 3.4. Thus we have in particular that the action of $B_n$ on $H_k(\exp_k \Gamma_n)$ is unfaithful for all $k$ and $n \geq 2$.

There is an obvious action of $S_n$ on $H_k(\exp_k \Gamma_n)$, induced by permuting the edges of $\Gamma_n$. The theorem will be proved by relating the action of each braid $\beta$ to that of $\bar{\beta}$. Note that there is again nothing lost by considering only the action on $H_k(\exp_k \Gamma_n)$, because of the isomorphisms induced by $i$ and $\cup\{ v \}$.

For brevity, in what follows we shall simply write $H_k$ for $H_k(\exp_k \Gamma_n)$.

### 4.2 Proof of the structure theorem

We fix a representation of $D_n$ and a homotopy equivalence $\Gamma_n \to D_n$, the embedding shown in figure 4(a). $B_n$ is generated by the “half Dehn twists” $\tau_1, \ldots, \tau_{n-1}$, where $\tau_i$ interchanges the $i$th and $(i+1)$th punctures with an anti-clockwise twist. The effect of $\tau_i$ on the embedded graph is shown in figure 4(b), and we see that it induces the map

$$e_a \mapsto \begin{cases} 
  e_{i+1} & \text{if } a = i, \\
  e_{i+1}e_i\overline{e}_{i+1} & \text{if } a = i+1, \\
  e_a & \text{if } a \neq i, i+1,
\end{cases}$$

on $\Gamma_n$; regarding the $e_i$ as generators of the free group $F_n$ this is the standard embedding $B_n \hookrightarrow \text{Aut}(F_n)$. We will call the induced map $\Gamma_n \to \Gamma_n \tau_i$ also,
and our goal is to understand the \((\exp_k \tau_i)_*\) well enough to show that the pure braid group acts by upper-triangular matrices.

To this end we define a filtration of \(H_k\). Each homology class has precisely one representative in \(\tilde{C}_k\), since there are no boundaries, so we may regard \(H_k\) as a subspace of \(\tilde{C}_k\) and work unambiguously on the level of chains. Let

\[
F_j = \{ \sum_J c_J \tilde{\sigma}^J \in H_k | c_J = 0 \text{ if } |J|_2 < n - j \},
\]

so that \(F_j\) is the subspace spanned by cells having no more than \(j\) even indices. Clearly \(\{0\} \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = H_k\), so the \(F_j\) form a filtration. Moreover \(F_j\) has \(F_j \cap B(k, n)\) as a basis: each element of \(B(k, n)\) has the form

\[
\tilde{\partial} \sigma^J = \sum_{i:j_i > 0 \text{ even}} c_{ji} \tilde{\sigma}^{\partial_i(J)}
\]

for some \(J\) with \(|J| = k + 1\), and the mod 2 norm of every term in this sum is \(|J|_2 + 1\).

We now describe when \(F_i = F_j\) for \(i \neq j\), as the indexing has been chosen to be uniform and meaningful at the expense of certain a priori isomorphisms among the \(F_j\), arising from the parity and size of \(k\). Since \(|J|_2 = k\) for each \(k\)-cell \(\tilde{\sigma}^J\) we have in particular \(|J|_2 \equiv k \mod 2\). Thus \(F_{j+1} = F_j\) if \(n-j\) and \(k\) have the same parity. Next, \(|J|_2 \leq |J| = k\), so \(F_j = \{0\}\) if \(n-j > k\), or in other words if \(j < n-k\). Lastly, no cell for which \(|J|_2 = 0\) is a summand of an element of \(H_k\) (recall that \(C_\ast\) is exact, and that such a cell sits at the top of an \(m\)-cube complex and so is not part of any boundary), so that either \(F_{n-2}\) or \(F_{n-1}\) is all of \(H_k\), depending on the parity of \(k\). To see that these are the only circumstances in which \(F_i\) and \(F_j\) can coincide suppose that \(i < j \leq n-1\),

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4}
\caption{The punctured disc \(D_n\) and the generators \(\tau_i\) for \(n = 5\) and \(i = 3\). We embed \(\Gamma_5\) in \(D_5\) as shown in (a); the effect of \(\tau_3\) on the embedding is shown in (b).}
\end{figure}
n − i and n − j have the same parity as k, n − j ≤ k, and let
\[ J = (0, \ldots, 0, 1, \ldots, 1, k + j + 1 - n). \]

Every non-zero index of \( J \) is odd so \( \tilde{\sigma}^J \) is a cycle. It has \( j > i \) even indices and is therefore non-trivial in the quotient \( \mathcal{F}_j/\mathcal{F}_i \).

The significance of the filtration is given by the following lemma, which is the main step in proving the structure theorem for the action.

**Lemma 5** Each \( \mathcal{F}_j \) is an invariant subspace for the action of \( B_n \) on \( H_k \). Moreover, the action on \( \mathcal{F}_j/\mathcal{F}_{j-1} \) factors through the symmetric group \( S_n \).

**Proof** The lemma is proved by relating the action of \( B_n \) to that of \( S_n \) induced by permuting the edges of \( \Gamma_n \). Much of the work has been done already in section 3.4, since \( \tau_i \) and \( \overline{\tau}_i \) act as the maps \( \tau \) and \( \tilde{\tau} \) considered there on \( e_i \cup e_{i+1} \), and as the identity on the remaining edges.

For each \( n \)-tuple \( J \) let
\[ \delta_i(J) = (j_1, \ldots, j_i + 1, \ldots, j_n) \]
so that
\[ \partial_i \circ \delta_{i+1} \circ \overline{\tau}_i(J) = (j_1, \ldots, j_{i-1}, j_{i+1} - 1, j_i + 1, j_{i+2}, \ldots, j_n). \]

By (3.6) we may write
\[
(\exp \tau_i)_\# \tilde{\sigma}^J = \begin{cases} 
(\exp \overline{\tau}_i)_\# \tilde{\sigma}^J - 2\tilde{\sigma}^J \partial_i \circ \delta_{i+1} \circ \overline{\tau}_i(J) & \text{if } j_i, j_{i+1} \text{ both even, } j_{i+1} > 0, \\
(\exp \overline{\tau}_i)_\# \tilde{\sigma}^J & \text{otherwise}.
\end{cases}
\]

If \( j_i \) and \( j_{i+1} \) are both even then
\[ |\partial_i \circ \delta_{i+1} \circ \overline{\tau}_i(J)|_2 = |J|_2 + 2 \]
and it follows that
\[ (\exp_k \tau_i)_* c \in (\exp_k \overline{\tau}_i)_* c + \mathcal{F}_{j-1} \]
for each \( c \in \mathcal{F}_j \) and \( i \in [n - 1] \). Since the \( \tau_i \) generate \( B_n \) we get
\[ (\exp_k \beta)_* c \in (\exp_k \overline{\beta})_* c + \mathcal{F}_{j-1} \]
for all braids \( \beta \) and \( c \in \mathcal{F}_j \), and the lemma follows immediately.
Proof of Theorem 7  By Lemma 5, with respect to a suitable ordering of $B(k,n)$ the braid group acts by block upper-triangular matrices. The diagonal blocks are the matrices of the action on $\mathcal{F}_j/\mathcal{F}_{j-1}$, and since this factors through $S_n$ we have that the pure braids act by upper-triangular matrices with ones on the diagonal. It follows immediately that the image of $P_n$ is nilpotent.

To bound the length of the lower central series we count the number of nontrivial blocks, as the class of the image is at most one less than this. By the discussion following the definition of $\{\mathcal{F}_j\}$ this is the number of $0 \leq j \leq n-1$ such that $n-j \leq k$ and $n-j \equiv k \mod 2$; letting $\ell = n-j$ this is the number of $1 \leq \ell \leq \min\{n,k\}$ such that $\ell \equiv k \mod 2$. There are $\lfloor (m + 1)/2 \rfloor$ positive odd integers and $\lfloor m/2 \rfloor$ positive even integers less than or equal to a positive integer $m$, and the given bounds follow.

4.3 The action of $B_3$ on $H_3(\exp_3 \Gamma_3)$

We study the action of $B_3$ on $H_3(\exp_3 \Gamma_3)$, being the smallest non-trivial example, and show that $P_3$ acts as a free abelian group of rank two.

For simplicity we will simply write $J$ for the cell $\tilde{\sigma}^J$. From Theorem 4 we obtain the basis

\begin{align*}
  u_1 &= (3,0,0) & w_1 &= (0,1,2) + (0,2,1) \\
  u_2 &= (0,3,0) & w_2 &= (1,0,2) + (2,0,1) \\
  u_3 &= (0,0,3) & w_3 &= (1,2,0) + (2,1,0) \\
  v &= (1,1,1)
\end{align*}

for $H_3(\exp_3 \Gamma_3)$ (in fact this is the negative of the basis given there). Let $U$ be the span of $\{u_1,u_2,u_3\}$, $V$ the span of $\{v\}$, and $W$ the span of $\{w_1,w_2,w_3\}$. The subspaces $U$, $V$ and $V \oplus W$ are easily seen to be invariant using equation (4.1), and moreover the action on $U$ is simply the permutation representation of $S_3$. We therefore restrict our attention to $V \oplus W$. The actions on $V$ and $(V \oplus W)/V$ are the sign and permutation representations of $S_3$ respectively,
and with respect to the basis \( \{ v, w_1, w_2, w_3 \} \) we find that

\[
(\exp_{3\tau_1})_* |_{V \oplus W} = T_1 = \begin{bmatrix}
-1 & -2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

\[
(\exp_{3\tau_2})_* |_{V \oplus W} = T_2 = \begin{bmatrix}
-1 & 0 & -2 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}.
\]

In each case the inverse is obtained by moving the \(-2\) one place to the right.

A product of \( T_1, T_2 \) and their inverses has the form

\[
P = \begin{bmatrix}
\det \bar{P} & p \\
0 & \bar{P}
\end{bmatrix}
\]

where \( \bar{P} \) is a permutation matrix and \( p = (p_1, p_2, p_3) \) is a vector of even integers. Consider \( \Sigma(P) = p_1 + p_2 + p_3 \). Multiplying \( P \) by \( T_1^{\pm 1} \) we see that \( \Sigma(T_1^{\pm 1} P) = -\Sigma(P) - 2 \), and it follows that \( \Sigma(P) \) is zero if \( \bar{P} \) is even and \(-2\) if \( \bar{P} \) is odd. In particular \( \Sigma(P) \) is zero if \( P \) is the image of a pure braid.

If \( P = QR \) where \( \bar{Q} = \bar{R} = I \) then \( p = q + r \) and it follows that the pure braid group \( P_3 \) acts as a free abelian group of rank at most two. That the rank is in fact two can be verified by calculating \( T_1^2 \) and \( T_2^2 \) and checking that the corresponding vectors are independent.

### A Table of Betti numbers

Table 1 lists the Betti numbers

\[
b_k(\exp_k \Gamma_n) = \sum_{j=1}^{k} (-1)^{j-k} \binom{n+j-1}{n-1}
\]

\[
= \begin{cases}
\sum_{j=1}^{\ell} \binom{n+2j-2}{n-2} & \text{if } k = 2\ell \text{ is even}, \\
n + \sum_{j=1}^{\ell} \binom{n+2j-2}{n-2} & \text{if } k = 2\ell + 1 \text{ is odd},
\end{cases}
\]

for \( 1 \leq k \leq 20, 1 \leq n \leq 10 \). To find the other non-vanishing Betti numbers recall that \( b_{k-1}(\exp_k \Gamma_n) = b_{k-1}(\exp_k(\Gamma_n, v)) = b_{k-1}(\exp_{k-1} \Gamma_n) \) for \( k \geq 2 \).
## Table 1: Betti numbers $b_k(\exp_k \Gamma_n)$ for $1 \leq k \leq 20$ and $1 \leq n \leq 10$.

| $k$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-----|----|----|----|----|----|----|----|----|----|----|
| 1   |  1 |  2 |  3 |  4 |  5 |  6 |  7 |  8 |  9 | 10 |
| 2   |  0 |  1 |  3 |  6 | 10 | 15 | 21 | 28 | 36 | 45 |
| 3   |  1 |  3 |  7 | 14 | 25 | 41 | 63 | 92 |129 |175 |
| 4   |  0 |  2 |  8 | 21 | 45 | 85 |147 |238 |366 |540 |
| 5   |  1 |  4 | 13 | 35 | 81 |167 |315 |554 |921 |1462 |
| 6   |  0 |  3 | 15 | 49 |129 |295 |609 |1162|2082|3543 |
| 7   |  1 |  5 | 21 | 71 |201 |497 |1107|2270|4353|7897 |
| 8   |  0 |  4 | 24 | 94 |294 |790 |1896|4165|8517|16413|
| 9   |  1 |  6 | 31 |126 |421 |1212 |3109|7275|15793|32207|
| 10  |  0 |  5 | 35 |160 |580 |1791 |4899 |12173 |27965 |60171 |
| 11  |  1 |  7 | 43 |204 |785 |2577 |7477 |19651 |47617 |107789 |
| 12  |  0 |  6 | 48 |251 |1035 |3611 |11087 |30737 |78353 |186141 |
| 13  |  1 |  8 | 57 |309 |1345 |4957 |16045 |46783 |125137 |311279 |
| 14  |  0 |  7 | 63 |371 |1715 |6671 |22715 |69497 |194633 |505911 |
| 15  |  1 |  9 | 73 |445 |2161 |8833 |31549 |101047 |295681 |801593 |
| 16  |  0 |  8 | 80 |524 |2684 |11516 |43064 |144110 |439790 |1241382 |
| 17  |  1 | 10 | 91 |616 |3301 |14818 |57883 |201994 |641785 |1883168 |
| 18  |  0 |  9 | 99 |714 |4014 |18831 |76713 |278706 |920490 |2803657 |
| 19  |  1 | 11 |111 |826 |4841 |23673 |100387 |379094 |1299585 |4103243 |
| 20  |  0 | 10 |120 |945 |5785 |29457 |129843 |508936 |1808520 |5911762 |

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