OPTIMAL HÖLDER EXPONENT FOR THE SLE PATH

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Abstract. We prove an upper bound on the optimal Hölder exponent for the chordal SLE path parameterized by capacity and thereby establish the optimal exponent as conjectured by J. Lind. We also give a new proof of the lower bound. Our proofs are based on the sharp estimates of moments of the derivative of the inverse map. In particular, we improve an estimate of the second author.

1. Introduction

The Schramm-Loewner evolution, or SLE(κ), is a one-parameter family of random fractal curves that was introduced by O. Schramm in [Sch00] as a candidate for the scaling limit of the loop-erased random walk. Since then, SLE has been shown to describe the scaling limits of a number of discrete models from statistical physics and to provide tools for their rigorous understanding. The properties of SLE curves have been studied by a number of authors. For instance, S. Rohde and Schramm proved in [RS05] the existence and continuity of the path and gave an upper bound on the Hausdorff dimension. V. Beffara [Bef08] proved a lower bound on Hausdorff dimension and thus showed that the dimension is almost surely the minimum of $1 + \kappa/8$ and 2. J. Lind [Lin08] improved the estimates by Rohde and Schramm and proved that the SLE(κ) path is almost surely Hölder continuous. She conjectured that the Hölder exponent she obtained is the optimal one, that is, that the exponent is the largest possible. In this paper we prove this conjecture. More precisely, we prove the following theorem. Let $\kappa \geq 0$ and set

$$\alpha_* = \alpha_*(\kappa) = 1 - \frac{\kappa}{24 + 2\kappa - 8\sqrt{8} + \kappa}, \quad \alpha_0 = \min\left\{\frac{1}{2}, \alpha_*\right\}.$$ 

Theorem 1.1. Let $\gamma(t)$ be the chordal SLE(κ) path parameterized by half-plane capacity. With probability one the following holds.

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• $\gamma(t)$, $t \in [0,1]$, is Hölder continuous of order $\alpha$ for $\alpha < \alpha_0$ and is not Hölder continuous of order $\alpha$ for $\alpha > \alpha_0$.

• For every $0 < \epsilon < 1$, $\gamma(t)$, $t \in [\epsilon,1]$, is Hölder continuous of order $\alpha$ for $\alpha < \alpha^*$ and is not Hölder continuous of order $\alpha$ for $\alpha > \alpha^*$.

As mentioned, the lower bound on the optimal exponent was proved in [Lin08]. The upper bound is new and we also give a new proof of the lower bound. The phase transition of $\alpha_0(\kappa)$ at $\kappa = 1$ is due to the geometry of the path at the base; roughly, in the capacity parameterization the path is Hölder-$1/2$ at $t = 0$ for all $\kappa \geq 0$.

The main tool needed is sharp estimates for the moments of the derivative close to the preimage of the tip of the growing curve. We will use the work in [Law09] and also [JL09], but we will have to extend one of these results in this paper. We will use these results as a “black box”, more or less, in the main part of the paper and then give a derivation in the final section, building on the argument in [Law09]. To be more specific, we need to control the growth of the derivative as the preimage of the tip is approached radially. To get the lower bound on the exponent estimates on the derivative from above are needed, and to get the upper bound on the optimal exponent, one has to control second moments and time correlations.

Our paper is organized as follows. In Section 2 we give definitions and discuss some well-known results. In Section 3 we prove a number of results about Loewner chains that are not particular to SLE, but rather to Loewner chains driven by functions which are weakly Hölder-$1/2$,
see (1.3) for a definition. Notably, we give estimates on the modulus of continuity of the curve in terms of the radial growth of the derivative. In the Section 4 we state the basic moment estimates and use these together with the results from Section 3 to prove Theorem 1.1. Finally, Section 5 proves the estimates on the derivative. Here we review without proof the main results from [Law09] that we need, and then establish the new estimates.

2. Preliminaries

Let $U_t$, $t \geq 0$ be a continuous real-valued function. We shall consider chordal Loewner chains $(g_t)$, that is, solutions to the chordal Loewner equation

$$\partial_t g_t = \frac{a}{g_t - U_t}, \quad t > 0, \quad g_0(z) = z,$$

for $a > 0$ fixed. We define the associated continuously growing hull

$$K_t = \{ z \in \mathbb{H} : \tau(z) \leq t \},$$

where $\tau(z)$ is the blow-up time of (1). For each $t > 0$ the function $z \mapsto g_t(z)$ maps $H_t := \mathbb{H} \setminus K_t$ conformally onto $\mathbb{H}$ and the inverse mapping $f_t := g_t^{-1}$ satisfies the partial differential equation

$$\partial_t f_t = -f_t' \frac{a}{z - U_t}, \quad f_0(z) = z.$$

Throughout the paper we will use the notation

$$\hat{f}_t(z) := f_t(U_t + z),$$

for $z \in \mathbb{H}$.

The time-reversed Loewner equation

$$\partial_t F_t = -\frac{a}{F_t - U_{T-t}}, \quad t \in (0,T], \quad F_0(z) = z$$

is often useful to avoid dealing directly with (2): it is easy to see that if $F$ is a solution (3) and $f$ a solution to (2) then

$$F_T(z) = f_T(z).$$

If there is a curve $\gamma(t)$ such that $H_t$ is the unbounded connected component of $\mathbb{H} \setminus \gamma[0,t]$ we say that the Loewner chain $(g_t)$ is generated by a curve. This property is known [RS05, Theorem 4.1] to be equivalent to the existence of the radial limit

$$\lim_{y \to 0} \hat{f}_t(iy) =: \gamma(t).$$
for each \( t > 0 \) together with continuity of \( t \mapsto \gamma(t) \). Loewner chains corresponding to driving functions with strictly higher regularity than Hölder-1/2 are always generated by a simple curve; in fact Hölder-1/2 continuity with a sufficiently small norm is sufficient to guarantee a simple curve. Conversely, there are examples of Loewner chains corresponding to Hölder-1/2 functions (with large norm) that are not generated by a curve, see [MR05] for these results.

In particular we will be interested in Schramm-Loewner evolution, \( \text{SLE}(\kappa) \), defined as the Loewner chain corresponding to \( a = 2 \) and \( U_t = \sqrt{\kappa}B_t \), where \( B \) is standard Brownian motion and \( \kappa \geq 0 \). SLE is known to be generated by a curve. We note that the \( \text{SLE}(\kappa) \) path is simple if and only if \( 0 \leq \kappa \leq 4 \) and space filling for \( \kappa \geq 8 \), see [RS05] for proofs of these facts. It may be noted that there is presently no known direct proof of that \( \text{SLE}(8) \) is generated by a curve, see [LSW04] for an indirect proof.

**Definition 2.1.** A (positive) subpower function is a continuous, non-decreasing function \( \varphi : [0, \infty) \to (0, \infty) \) that satisfies
\[
\lim_{x \to \infty} x^{-\nu} \varphi(x) = 0
\]
for all \( \nu > 0 \).

3. Deterministic results

In this section we prove a number of results about Loewner chains and associated curves that are not special to SLE. In particular we consider Loewner chains corresponding to driving functions which are weakly Hölder-1/2, see (13).

For convenience we state the following well-known result, see [Pom92] for a proof.

**Lemma 3.1** (The Koebe distortion and one-quarter theorems). Suppose \( f : D \to \mathbb{C} \) is a conformal map and set \( d = \text{dist}(z, \partial D) \) for \( z \in D \). Then
\[
\frac{1 - r}{(1 + r)^3} |f'(z)| \leq |f'(w)| \leq \frac{1 + r}{(1 - r)^3} |f'(z)|, \quad |z - w| \leq rd, \quad (5)
\]
and
\[
B(f(z), d|f'(z)|/4) \subset f(D), \quad (6)
\]
where \( B(w, \rho) \) denotes the open disk of radius \( \rho \) around \( w \).
Lemma 3.2. Let $S$ be the rectangle $S = \{x + iy : -1 < x < 1, 0 < y < 1\}$. There exist $c, \alpha < \infty$ such that if $f$ is a conformal map defined on $2S$, $z, w \in S$ and $\text{Im} \ z, \text{Im} \ w \geq 1/r$, then

$$|f'(z)| \leq c r^{\alpha} |f'(w)|.$$ (7)

Proof. One way to prove this is to take a conformal transformation of $2S$ onto the unit disk with $f(z) = 0$ and using the distortion theorem. A more direct approach is as follows. We may assume that $\text{Im} \ z = 1/r$. Take a Whitney decomposition of $S$, that is, a partition of $S$ into dyadic rectangles $\{S_{j,k}\}$ where

$$S_{j,k} = \{x + iy : j2^{-k} \leq x \leq (j + 1)2^{-k}, 2^{-(k+1)} \leq y \leq 2^{-k}\},$$

for $j = 0, \ldots, 2^k - 1$, and $k \in \mathbb{N}$. Let $u, v$ be in the same rectangle. Then by iterating (5) it follows that there exists a constant $c_1$ (uniform for all rectangles; $12^5$ would work for instance) such that

$$c_1^{-1} \leq |f'(u)|/|f'(v)| \leq c_1.$$ (8)

Suppose that $z \in S_{j,k}$. Then $k \leq \log r \leq k + 1$. It follows that there exists a path in $S$ (the hyperbolic geodesic, for instance) that connects $z$ to $w$ and intersects at most $2[\log r] + 2$ rectangles. Hence, there is a constant $c_2$ such that by iterating (8) along the path at most $c_2(\log r + 1)$ times

$$|f'(z)| \leq c_2^{\alpha} r^{\alpha} |f'(w)| = cr^{\alpha} |f'(w)|,$$

for $\alpha = c_2 \log c_1$ and $c = c_1^2$. \hfill \Box

A proof of the next lemma may be found in [Law05]. For a set $K$, we let $\text{diam}(K)$ denotes the diameter of $K$ and $\text{height}(K) := \sup \{\text{Im} \ z : z \in K\}$.

Lemma 3.3. Suppose $K_t$ is the hull obtained by solving (4) with $U_t$ as driving function. Let $R(t) = \sqrt{t} + \sup_{0 \leq s \leq t} \{|U_s - U_0|\}$. Then there exists a constant $c < \infty$ such that

$$c^{-1} R(t) \leq \text{diam}(K_t) \leq c R(t).$$ (9)

Lemma 3.4. Let $K$ be a hull. There exists a constant $c < \infty$ such that

$$\text{hcap}(K) \leq c \text{diam}(K) \text{ height}(K).$$ (10)

Proof. Let $d = \text{diam}(K)$ and $h = \text{height}(K)$. We shall only consider the case when $h < d$ (the other case is easily verified by considering the map $z \mapsto z + d^2/z$). By scaling, translation invariance, and monotonicity
of $\text{hcap}$ we may assume that $d = 1$ and that $K$ is contained in the rectangle $R = \{ z : \text{Re } z \leq 1/2, 0 \leq \text{Im } z \leq h \}$, so that $\text{hcap}(K) \leq \text{hcap}(R)$. It is known that
\[
\text{hcap}(R) = \lim_{y \to \infty} y \mathbb{E}^{iy}(\text{Im } B(T)),
\]
where $B$ denotes a complex Brownian motion and $T$ denotes the hitting time of $\partial(\mathbb{H} \setminus R)$, see [Law05]. Hence
\[
\text{hcap}(R) \leq \lim_{y \to \infty} y \omega(iy, \partial R, \mathbb{H} \setminus R) \cdot h,
\]
where $\omega$ denotes harmonic measure. Note that $R$ can be covered by $O(h^{-1})$ discs of radius $2h$ centered on the real line. The harmonic measure from $iy$ of any such disc is bounded from above by the harmonic measure of the disc centered at the origin. Since
\[
\omega(iy, \partial(2hD), \mathbb{H} \setminus (2hD)) = (4/\pi) \arctan(2h/y)
\]
for large $y$, the lemma follows from the maximum principle. □

Lemma 3.5. Suppose $f_t$ satisfies (2) and $z = x + iy \in \mathbb{H}$, then for $s \geq 0$
\[
e^{-5as/y^2} |f_t'(z)| \leq |f_{t+s}'(z)| \leq e^{5as/y^2} |f_t'(z)|.
\]
In particular, if $s \leq y^2$,
\[
e^{-5a} |f_t'(z)| \leq |f_{t+s}'(z)| \leq e^{5a} |f_t'(z)|.
\]
Also, if $0 \leq s \leq y^2$,
\[
|f_{t+s}(z) - f_t(z)| \leq \frac{y}{5} [e^{5a} - 1] |f_t'(z)|.
\]

Proof. Without loss of generality assume that $a = 1$. Differentiating (2) yields
\[
\partial_t f_t'(z) = -f_t''(z) \frac{1}{z - V_t} + f_t'(z) \frac{1}{(z - V_t)^2}.
\]
Note that $|z - V_t| \geq y$. Applying Bieberbach’s theorem (the $n = 2$ case of the Bieberbach conjecture) to the disk of radius $y$ about $z$, we can see that
\[
|f_t''(z)| \leq 4 y^{-1} |f_t'(z)|,
\]
and hence
\[
|\partial_t f_t'(z)| \leq 5 y^{-2} |f_t'(z)|,
\]
which implies (11). Returning to (2), we see that
\[
|\partial_t f_t(z)| \leq |f_t'(z)| \frac{a}{\text{Im}(z)}.
\]
Using (11), we see that
\[
|f_{t+s}(z) - f_t(z)| \leq \int_0^s |\partial_s f_{t+s}(z)| \, ds \\
\leq \frac{a|f'_t(z)|}{y} \int_0^s e^{5as/y^2} \, ds = \frac{y}{5} (e^{5a} - 1) |f'_t(z)|. 
\]

□

We shall consider Loewner chains corresponding to functions that are Hölder continuous of order \( \alpha \) for any \( \alpha < 1/2 \). We say that \( U \) is \textit{weakly Hölder-1/2} if there exists a subpower function \( \varphi \) such that \( r^{1/2}\varphi(1/r) \) is a modulus of continuity for \( U \), that is,
\[
\sup_{|s| \leq r} |U_{t+s} - U_t| \leq r^{1/2}\varphi(1/r). \tag{13}
\]

By P. Lévy’s theorem the sample paths of Brownian motion are almost surely weakly Hölder-1/2 with subpower function \( c\sqrt{\log(r)} \), \( c > \sqrt{2} \), see [RY99, Theorem I.2.7]. Therefore all results for Loewner chains corresponding to functions that satisfy (13) hold for SLE(\( \kappa \)) with probability one.

**Lemma 3.6.** There exist constants \( c, \alpha < \infty \) such that the following holds. Let \( (g_t) \) be a Loewner chain corresponding to the continuous function \( U_t \). Let \( s \in [0, y^2] \) for \( y > 0 \). Then
\[
|f'_{t+s}(U_{t+s} + iy)| \leq cM^\alpha |f'_{t+s}(U_t + iy)|, \tag{14}
\]
where \( M = \max\{|U_{t+s} - U_t|/y, 1\} \). In particular, if \( U_t \) satisfies (13), then there exists a subpower function \( \varphi \) such that for all \( t \) and all \( s \in [0, y^2] \),
\[
|f'_{t+s}(U_{t+s} + iy)| \leq \varphi(1/y)|f'_{t+s}(U_t + iy)|. 
\]

**Proof.** We first rescale by \( |U_{t+s} - U_t| \), and then apply Lemma [3.2] with \( r = \max\{|U_{t+s} - U_t|/y, 1\} \) to get the conclusion. □

The last two lemmas immediately imply the following result, which we record as a lemma. This essentially says that for weakly Hölder-1/2 Loewner chains, it is enough to consider the derivative at dyadic times.

**Lemma 3.7.** Let \( (g_t) \) be the Loewner chain corresponding to \( U_t \) satisfying (13). Suppose that there exist constants \( c \) and \( \beta \) such that for all \( n \geq 1 \)
\[
|\hat{f}'_{t_k}(i2^{-n})| \leq c2^{n\beta}, \tag{15}
\]
where \( t_k = k2^{-2n}, \ k = 0, 1, \ldots, 2^{2n}. \) Then for every \( \beta_1 > \beta, \) there exists a constant \( c_1 < \infty \) such that

\[
|f'_{t}(i2^{-n})| \leq c_1 2^{n\beta_1},
\]

for \( t \in [0, 1]. \)

Let \((g_t)\) be the Loewner chain corresponding to a function \( U_t \) satisfying (13) that is generated by a curve \( \gamma(t) \). We want to estimate the modulus of continuity of \( \gamma \). The following quantity will be useful

\[
v(t, y) := \int_0^y |f'_{t}(U_t + ir)| \, dr, \quad y > 0.
\]

The geometrical interpretation is of course the length (if it exists) of the image of the segment \([U_t, U_t + iy]\) under \( f_t \). For a given \( t \), the limit

\[
\gamma(t) = \lim_{y \to 0+} f_t(U_t + iy)
\]

exists if \( v(t, y) \) is finite for some \( y > 0 \). By integration we have

\[
|\gamma(t) - f_t(U_t + iy)| \leq v(t, y).
\]

Using the Koebe one-quarter theorem, we can see that

\[
v(t, y) \geq y |f'_{t}(iy)|/4.
\]

The next result shows that Loewner chains corresponding to weakly Hölder-1/2 functions are generated by a curve if \( v(t, y) \) decays polynomially in \( y \). We also get an estimate of the modulus of continuity of the curve.

**Proposition 3.8.** Let \((g_t)\) be the Loewner chain corresponding to \( U_t \) satisfying (13). Then there exists a subpower function \( \varphi \) such that if \( 0 \leq t \leq t + s \leq 1 \) and \( s \in [0, y^2] \)

\[
|\gamma(t + s) - \gamma(t)| \leq \varphi(1/y) [v(t + s, y) + v(t, y)].
\]

**Remark.** We have not assumed existence of the curve in this proposition. If \( v(t + s, y), v(t, y) < \infty \), then we know that radial limit (4) exists, so we can write \( \gamma(t), \gamma(t + s) \). If one of the radial limits does not exist we can define \( \gamma \) any way that we want since the right hand side of (19) is infinite.

**Proof.** We start by writing

\[
|\gamma(t + s) - \gamma(t)| \leq |\gamma(t + s) - f_{t+s}(U_{t+s} + iy)| + |\gamma(t) - f_t(U_t + iy)|
\]

\[
+ |f_{t+s}(U_{t+s} + iy) - f_{t+s}(U_t + iy)| + |f_{t+s}(U_t + iy) - f_t(U_t + iy)|.
\]

We have to estimate the last two terms. Note that

\[
|f_{t+s}(U_{t+s} + iy) - f_{t+s}(U_t + iy)| \leq |U_{t+s} - U_t| \max |f'_{t+s}(w)|,
\]
where the maximum is over all \( w \) on the line segment connecting \( U_{t+s} + iy \) and \( U_t + iy \). By Lemma 3.6 using assumption (13) and then by (18), we see that

\[
|f_{t+s}(U_{t+s} + iy) - f_{t+s}(U_t + iy)| \leq c \varphi(1/y) y |f'_{t+s}(U_{t+s} + iy)| \\
\leq c \varphi(1/y) v(t + s, y).
\]

By (12) and (18)

\[
|f_{t+s}(U_t + iy) - f_t(U_t + iy)| \leq \frac{y}{5} [e^{5a} - 1] |f'_t(U_t + iy)| \\
\leq c [e^{5a} - 1] v(t, y).
\]

Proposition 3.9. Let \((g_t)\) be the Loewner chain corresponding to \( U_t \) satisfying (13). Suppose that \((g_t)\) is generated by a curve \( \gamma(t) \). Then for each \( t \), there exist \( t_1, t_2 \in [t, t + y^2] \), and constants \( c, \delta > 0 \) such that

\[
|\gamma(t_1) - \gamma(t_2)| \geq c |f'_t(iy)| y \varphi(1/y)^\delta.
\]

Proof. Let \( \beta := g_t(\gamma[t, t + y^2]) \). Then \( \beta \) is the curve obtained by solving (11) with \( U_{t+r}, r \in [0, y^2] \), as driving function. Hence \( \text{heap}(\beta) = ay^2 \), and by Lemma 3.3 together with the assumption (13) on \( U \),

\[
\rho_d := \text{diam}(\beta) \leq y \varphi(1/y) .
\]

Next, we combine (20) with Lemma 3.4 to find

\[
\rho_h := \text{height}(\beta) \geq c y \varphi(1/y)^{-1} ,
\]

for some constant \( c \). Let \( z \in \beta \) satisfy \( \text{Im} z \geq \rho_h \). Note that \( |\text{Re} z - U_t| \leq \text{diam}(\beta) \leq \rho_d \). By scaling by \( \rho_d^{-1} \) and then using Lemma 3.2 with \( r = \varphi(1/y)^2 \) we get

\[
|f'_t(z)| \geq c \varphi(1/y)^{-2\alpha} |f'_t(iy)| ,
\]

where \( \alpha \) is the exponent from Lemma 3.2. Let \( w \) be a point on \( \beta \) such that \( |w - z| = \rho_h/2 \). Since \( z, w \in \beta \) there are \( t_1, t_2 \in [t, t + y^2] \) such that \( \gamma(t_1) = f_t(z) \) and \( \gamma(t_2) = f_t(w) \). In view of (16) we have

\[
B(f_t(z), |f'_t(z)| \rho_h/16) \subseteq f_t(B(z, \rho_h/4)) ,
\]

where \( B(z, \rho) \) denotes the open ball of radius \( \rho \) around \( z \). Hence, using (23) and (22), we conclude that

\[
|\gamma(t_1) - \gamma(t_2)| = |f_t(z) - f_t(w)| \\
\geq |f'_t(z)| \rho_h/16 \\
\geq c |f'_t(iy)| y \varphi(1/y)^{-(2\alpha + 1)} ,
\]

and this completes the proof. \( \square \)
Although we do not use it in this paper, we will give some results about existence of the curve for continuous driving functions that are not necessarily weakly Hölder-$1/2$.

**Proposition 3.10.** There exists $c < \infty$ such that the following is true. Suppose $\delta > 0$ and $(g_t)$ is a Loewner chain with driving function $U_t$. Suppose that $0 \leq s, y < \infty$ and

$$|U_{t+s} - U_s| \leq \delta.$$  

Then

$$|\gamma(t + s) - \gamma(t)| \leq c e^{5a} [v(t, \delta) + v(t + s, \delta)].$$  

**Proof.** We use the triangle inequality on $|\gamma(t + s) - \gamma(t)|$ as in the beginning of the proof of Proposition 3.8 with $y = \delta$. The first two terms are bounded by $v(t+s, \delta)$ and $v(t, \delta)$, respectively, and the fourth term is bounded in the same way. The distortion theorem, (24), and (18) imply

$$|f_{t+s}(U_{t+s} + i\delta) - f_{t+s}(U_t + i\delta)| \leq c \delta |f'_{t+s}(U_{t+s} + i\delta)|$$

and we get the desired estimate. \hfill \Box

**Corollary 3.11.** Suppose $(g_t)$ is a Loewner chain with continuous driving function $U_t$. Suppose that for each $0 < t_1 < t_2 < \infty$,

$$\lim_{y \to 0^+} v(t, y) = 0$$

uniformly for $t \in [t_1, t_2]$. Then $(g_t)$ is generated by a curve.

**Proof.** Since $\gamma$ is a uniform limit of a sequence of continuous functions on $[t_1, t_2]$, $\gamma$ is continuous on $[t_1, t_2]$. One can check directly from the Loewner equation that $\gamma$ is right continuous at 0 and hence $\gamma$ is continuous. \hfill \Box

**Remark.** This result also gives an estimate for the modulus of continuity of $\gamma$. However, the assumptions are very strong. If we do not assume that $U_t$ is weakly Hölder-$1/2$, we do not have Lemma 3.7.

### 4. Proof of Theorem 1.1

We now turn to the proof of our main result. The proof of Theorem 1.1 is split into two parts: the lower bound (which requires derivative estimates from above) is proven in Subsection 4.1 and the upper bound (which requires estimates from below and control of correlations) in Subsection 4.2. The SLE moment estimates (Lemma 4.1 and Lemma 4.4)
we need for the proof are only stated in this section. The proofs of
the estimates build on those in [Law09] and are discussed in the last
section.

To state the lemmas it is convenient to introduce a number of \( \kappa \)-
dependent parameters. Suppose

\[-\infty < r < r_c := \frac{1}{2} + \frac{4}{\kappa}.\]

The significance of \( r_c \) is discussed in Section 5. Let

\[
\lambda = \lambda(r) = r \left( 1 + \frac{\kappa}{4} \right) - \frac{\kappa r^2}{8},
\]

\[
\zeta = \zeta(r) = r - \frac{\kappa r^2}{8},
\]

and

\[
\beta = \beta(r) = -1 + \frac{\kappa}{4 + \kappa - \kappa r}.
\]

Note that \( \beta \) and \( \lambda \) strictly increase with \( r \) for \(-\infty < r < r_c \) and hence,
we could alternatively consider either of them as the free parameter.

Let \( r_+ \) be the larger root to \( \lambda \beta + \zeta = 2 \) and let \( \beta_+, \lambda_+ \) and \( \zeta_+ \) be the
corresponding values of \( \beta, \lambda \) and \( \zeta \) respectively. Note that if \( \kappa \neq 8 \),
then \( r_+ < r_c \) and

\[-1 = \beta(-\infty) < \beta_+ = -1 + \frac{\kappa}{12 + \kappa - 4\sqrt{8 + \kappa}} < \beta(r_c) = 1.\]

Also,

\[\alpha_* = \frac{1 - \beta_+}{2}.\]

4.1. **Lower bound.** The following is the main moment estimate for
the lower bound. This was proved in [Law09] for a certain range of \( r \)
including \( r = 1, \kappa < 8 \) which was most important for that paper. We
give a different proof here that is valid for all \( r < r_c \).

**Lemma 4.1.** Suppose \( r < r_c \). Then there exists \( c < \infty \) such that for
all \( t \geq 1 \)

\[\mathbb{E} \left[ |\hat{f}_{iz}(i)|^\lambda \right] \leq c t^{-\zeta}. \quad (25)\]

**Proof.** See Section 5 \( \square \)

In the proof of this result one also finds that the expectation in (25),
roughly speaking, is carried on an event on which \( |\hat{f}_{iz}(i)| \approx t^\beta \) and
this has probability of order \( t^{-(\zeta + \lambda \beta)} \). From this lemma we can derive
the following uniform estimate from which the lower bound will follow.
Recall that \( \hat{f}_0'(iy) = 1 \) for all \( y > 0 \), so we have to restrict our attention to positive \( \beta \).

**Proposition 4.2.** Suppose \( \beta > \max\{0, \beta_+\} \). With probability one there exists \( y_0 > 0 \), such that for all \( t \in [0, 1] \) and all \( y < y_0 \),

\[
|\hat{f}'_t(iy)| \leq y^{-\beta}.
\]

**Proof.** For \( \beta > \max\{0, \beta_+\} \) we have \( \lambda > 0, \zeta > 0 \) and \( \beta \lambda + \zeta > 2 \). By choosing \( \beta \) smaller if necessary, but still larger than \( \max\{0, \beta_+\} \), we can guarantee that \( \zeta < 2 \). We write

\[
\hat{f}_{j,n} = \hat{f}_{(j-1)2^{-n}}, \quad j = 1, \ldots, 2^{2n}.
\]

By Lemma 3.7 and the distortion theorem it suffices to prove that for all \( \beta > \max\{0, \beta_+\} \), with probability one there exists \( N < \infty \) such that for \( n \geq N \),

\[
|\hat{f}'_{j,n}(i2^{-n})| \leq 2^{\beta n}, \quad j = 1, 2, \ldots, 2^{2n}.
\]  \( \text{(26)} \)

Note that scale invariance implies

\[
\mathbb{P}\left\{ |\hat{f}'_{j,n}(i2^{-n})| \geq 2^{\beta n} \right\} = \mathbb{P}\left\{ |\hat{f}'_{j-1}(i)| \geq 2^{\beta n} \right\}.
\]

By Lemma 4.1 and Chebyshev’s inequality,

\[
\mathbb{P}\left\{ |\hat{f}'_{j-1}(i)| \geq 2^{\beta n} \right\} \leq 2^{-\lambda \beta n} \mathbb{E}\left[ |\hat{f}'_{j-1}(i)|^\lambda \right] \\
\leq c j^{-\frac{\zeta}{2}} 2^{-\lambda \beta n} = c \left( \frac{j}{2^{2n}} \right)^{-\frac{\zeta}{2}} 2^{-n(\lambda \beta + \zeta)}.
\]

Since \( \lambda \beta + \zeta > 2 \) and \( \zeta < 2 \), we can sum over \( j \) to get

\[
\sum_{j=1}^{2^{2n}} \mathbb{P}\left\{ |\hat{f}'_{j-1}(i)| \geq 2^{\beta n} \right\} \leq c 2^{-n(\lambda \beta + \zeta - 2)}, \quad \text{(27)}
\]

and hence

\[
\sum_{n=1}^{\infty} \sum_{j=1}^{2^{2n}} \mathbb{P}\left\{ |\hat{f}'_{j-1}(i)| \geq 2^{\beta n} \right\} < \infty.
\]

The Borel-Cantelli lemma now implies \( \text{(26)} \). \( \square \)

**Proposition 4.3.** Suppose \( \beta_+ < \beta < 0 \). With probability one, for every \( \epsilon > 0 \), there exists \( y_\epsilon > 0 \) such that for all \( t \in [\epsilon, 1] \) and all \( y < y_\epsilon \),

\[
|f'_t(iy)| \leq y^{-\beta}.
\]
Proof. For $\beta_+ < \beta < 0$, we have $\lambda \beta + \zeta > 2$ and $\lambda > 0$. The proof is identical to the previous proposition except that we do not have $\zeta > 0$. We replace (27) with

$$\sum_{\epsilon^{2n} \leq j \leq 2^{2n}} \mathbb{P} \left\{ |\hat{f}_{j-1}'(i)| \geq 2^{3n} \right\} \leq c_* 2^{-n(\lambda \beta + \zeta - 2)}.$$

We can now easily prove the lower bound in Theorem 1.1. Recall that $\alpha^* = \min \{1/2, (1 - \beta_+)/2\}$.

Proof of lower bound for Theorem 1.1. Let $1 \geq \beta > \tilde{\beta} > \max \{0, \beta_+\}$; recall that $\beta_+ = 1$ if and only if $\kappa = 8$. Almost surely, by Proposition 4.2, we have for all $t \in [0, 1]$

$$v(t, y) = \int_0^y |\hat{f}_t'(ir)| \, dr \leq y^{1-\tilde{\beta}} \quad (28)$$

if $y$ is small enough. The last inequality together with Proposition 3.8 show that the SLE($\kappa$) Loewner chain is generated by a curve when $\kappa \neq 8$ and imply the following modulus of continuity

$$|\gamma(t + s) - \gamma(t)| \leq c s^{(1-\beta)/2}$$

for all $s$ small enough since $\tilde{\beta} > \beta$. The lower bound follows.

If $\beta_+ < \beta < 0$, it suffices to prove the result for each fixed $\epsilon$. The argument is the same using Proposition 4.3.

4.2. Upper bound. In this subsection we shall use the notation

$$\hat{f}_{j,n} = \hat{f}_{(j-1)/n^2}, \quad j = n^2/2, \ldots, n^2.$$

Lemma 4.4. Suppose $r < r_c$. Then there exist $0 < c_1, c_2 < \infty$, a subpower function $\varphi$, and events

$$E_{j,n}, \quad n = 1, 2, \ldots, \quad j = 1, \ldots, n^2$$

such that the following hold. Let $E(j, n) = 1_{E_{j,n}}$ and

$$F(j, n) = n^{\tilde{c}-2} |\hat{f}_{j,n}'(i/n)|^{\lambda} E(j, n).$$

- If $n^2/2 \leq j \leq n^2$, then on the event $E_{j,n}$,
  $$|\hat{f}_{j,n}'(i/n)| \geq \varphi(n)^{-1} n^\beta. \quad (29)$$

- If $n^2/2 \leq j \leq n$,
  $$c_1 n^{-2} \leq \mathbb{E} [F(j, n)] \leq c_2 n^{-2}. \quad (30)$$
If \( n^2/2 \leq j \leq k \leq n^2 \),

\[
\mathbb{E}[F(j,n) F(k,n)] \leq n^4 \left( \frac{n^2}{k-j+1} \right)^{\lambda \beta + \zeta} \varphi \left( \frac{n^2}{k-j+1} \right).
\]  

(31)

**Proof.** See Section 5.2. \( \square \)

For fixed \( \beta \), we let \( A_n = A_{n,\beta} \) denote the event that there exists an integer \( j \) with \( 1 \leq j \leq n^2 \),

\[
|\hat{f}_{j,n}(i/n)| \geq \varphi(n)^{-1} n^\beta,
\]  

(32)

where \( \varphi \) is as in (29). We then have the following.

**Lemma 4.5.** Suppose \( \lambda \beta + \zeta < 2 \). Then there exist \( c > 0 \) such for all \( n \) sufficiently large,

\[
\mathbb{P}(A_n) \geq c.
\]

In particular,

\[
\mathbb{P}\{A_n \text{ i.o.}\} \geq c.
\]

**Proof.** Let \( F(j,n) \) be as in Lemma 4.4 and let

\[
Y_n = \sum_{j=n^2/2}^{n^2} F(j,n).
\]

Note that \( A_n \supset \{Y_n > 0\} \). The estimates from Lemma 4.4 show that there exist \( 0 < c < c_2 < \infty \) such that

\[
\mathbb{E}[Y_n] \geq c, \quad \mathbb{E}[Y_n^2] \leq c_2^2.
\]

(This uses \( \beta \lambda + \zeta < 2 \).) Therefore, a standard second moment argument gives

\[
\mathbb{P}(A_n) \geq \mathbb{P}\{Y_n > 0\} \geq \frac{\mathbb{E}[Y_n]^2}{\mathbb{E}[Y_n^2]} \geq \frac{c^2}{c_2^2}.
\]

\( \square \)

We can now prove the upper bound for Theorem 1.1 and thereby complete the proof. Notice that Proposition 3.9 immediately implies that \( \gamma[0,t] \) cannot be Hölder continuous of order \( > 1/2 \), since \( \hat{f}_0(z) = 1 \).

**Proof of upper bound for Theorem 1.1.** Let \( \beta < \tilde{\beta} < \beta_+ \). Proposition 3.9 implies that on the event \( A_n = A_{n,\beta} \) there exist times \( t_1, t_2 \in [0,1] \) such that

\[
|\gamma(t_1) - \gamma(t_2)| \geq cn^{\beta-1} = c(n^{-2})^{(1-\beta)/2}, \quad |t_1 - t_2| \leq n^{-2}.
\]
Consequently on the event \( \{ A_n \text{ i.o.} \} \), the curve \( \gamma(t), t \in [0, 1] \), is not Hölder continuous of order \((1 - \beta)/2\). To show that this happens almost surely, let \( A_r \) be the event that \( \gamma(t), t \in [0, r] \), is not Hölder continuous of order \((1 - \beta)/2\). By Lemma 4.5 we have
\[
P(A_1) =: c_0 > 0,
\]
and by scale invariance
\[
P(A_r) = c_0, \quad r > 0.
\]
Note that \( A_r \subset A_1 \) if \( r \leq 1 \) so that
\[
P \left( \bigcap_{r > 0} A_r \right) = c_0.
\]
Since \( c_0 > 0 \), it now follows from the Blumenthal zero-one law (see [RY99, Theorem III.2.15]) that \( c_0 = 1 \) and this completes the proof. \( \square \)

5. Moments of derivatives

In this section we review some results from [Law09] and extend one result. We fix \( \kappa \) and we let \( a = 2/\kappa \). We also fix a real number \( r \) such that
\[
r < r_c = \frac{1 + 4a}{2}.
\]
All constants and parameters in this section depend on \( a \) and \( r \). We let
\[
q = r_c - r = 2a + \frac{1}{2} - r > 0.
\]
The positivity of \( q \) is important for the arguments in this section and this is why \( r \) must be less than \( r_c \).

A useful tool for estimating moments of \( |\hat{f}'| \) is the reverse Loewner flow (see, e.g., [Law09, Section 10.3]). Suppose \( U_t \) is a standard Brownian motion and \( h_t(z) \) is the solution to the reverse-time Loewner equation
\[
\partial_t h_t(z) = \frac{a}{U_t - h_t(z)}, \quad h_0(z) = z. \tag{33}
\]
For fixed \( T \), the distribution of \( h_T(z) - U_T \) is the same as that of \( \hat{f}_T(z) \) and hence, \( h_T'(z) \) has the same distribution as \( \hat{f}_T'(z) \). Indeed, suppose \( g_t \) is the solution of the forward-time equation for some continuous \( V \) with \( V_0 = 0 \):
\[
\partial_t g_t(z) = \frac{a}{g_t(z) - V_t},
\]
and we let $V_t^{(T)} = V_{T-t} - V_T$. Then if $h$ is a solution to (33) with driving function $V_t^{(T)}$ we have

$$g^{-1}_T(z + V_T) = h_T(z) + V_T = h_T(z) - V_T^{(T)}.$$ 

It remains to note that $V_t^{(T)}, 0 \leq t \leq T$, is a standard Brownian motion starting at 0 if $V$ is. If $S < T$ and $h^{(S)}, h^{(T)}$ denote the solutions to (33) with driving functions $U_t^{(S)} = U_{S-t} - U_S$ and $U_t^{(T)} = U_{T-t} - U_T$ then $h_t^{(T)}, 0 \leq t \leq T - S$ and $h_t^{(S)}, 0 \leq t \leq S$ are independent. Note that $f_S^t(z) f_T^t(z)$ has the same distribution as $(h_S^{(s)})'(z) (h_T^{(T)})'(z)$.

Let $Z_t = Z_t(i) = X_t + iY_t = h_t(i) - U_t$ and $S_t = \sin[\text{arg} Z_t] = [1 + X_t^2/Y_t^2]^{-1/2}$. To study the behavior of $h'_t(i)$, it is useful to do a time-change so that the logarithm of the imaginary part grows linearly. Let

$$\sigma(t) = \inf \{ s : Y_s(z) = e^{ot} \}.$$ 

The following lemma can be deduced from the Loewner equation and the analogous time change in a stochastic differential equation.

**Lemma 5.1.** [Law09, Section 5] Suppose $J_t$ satisfies

$$dJ_t = -r_c \tanh J_t \, dt + dW_t, \quad J_0 = 0,$$ 

where $W_t$ is standard Brownian motion. Let

$$L_t = t - \int_0^t \frac{2 \, ds}{\cosh^2 J_s},$$

$$\sigma(t) = \int_0^t e^{2as} \cosh^2 J_s \, ds, \quad \eta(s) = \sigma^{-1}(s).$$

Then the joint distribution of

$$e^{o_L^{\sigma(s)}}, \quad e^{o_{\eta(s)}}, \quad \cosh J_{\eta(s)}, \quad 0 \leq s < \infty$$

is the same as that of

$$|h'_s(i)|, \quad Y_s, \quad S_s^{-1}, \quad 0 \leq s < \infty.$$ 

Let $\lambda, \zeta, \beta$ be as in Section 4 we can write

$$\lambda = r \left( 1 + \frac{1}{2a} \right) - \frac{r^2}{4a}, \quad \zeta = \lambda - \frac{r}{2a},$$

$$\beta = \frac{1 - 2q}{1 + 2q} = \frac{r - 2a}{1 - 2r + 2a}.$$ 

We comment here that the $\beta$ of this paper is the same as $\mu$ in [Law09]; the $\beta$ in that paper is half this value.
Note that \((34)\) can be written as

\[
dJ_t = -(q + r) \tanh J_t \, dt + dW_t, \quad J_0 = 0.
\]

Using Itô’s formula, one can see that

\[
N_t = e^{a \lambda t} \cosh^{\alpha t} [\cosh J_t]^r
\]

is a martingale satisfying

\[
dN_t = r \left[ \tanh J_t \right] N_t \, dW_t.
\]

Let \(P_*\) denote the probability measure obtained by weighting by the martingale \(N_t\), that is, if \(E\) is an event measurable with respect to \(\{W_s : 0 \leq s \leq t\}\),

\[
P_*(E) = N_0^{-1} \mathbb{E}[N_t 1_E] = \mathbb{E}[N_t 1_E].
\]

The Girsanov theorem (see, e.g., [RY99, Chapter VIII] and [Law09, Appendix A]) implies that

\[
dW_t = r \tanh J_t \, dt + dB_t,
\]

\[
dJ_t = -q \tanh J_t \, dt + dB_t,
\]

(36)

where \(B_t\) is a standard Brownian motion with respect to the measure \(P_*\). We write \(\mathbb{E}_*\) for expectations with respect to \(P_*\).

**Lemma 5.2.** ([Law09] Lemma 7.1) Suppose \(J_t\) satisfies (36).

- \(J_t\) is a positive recurrent diffusion (with respect to the measure \(P_*\)) with invariant density

\[
v(x) = \frac{\Gamma(q + \frac{1}{2})}{\Gamma(\frac{1}{2}) \Gamma(q)} \frac{1}{cosh^{2q} x}.
\]

- \[
\int_{-\infty}^{\infty} \left[ 1 - \frac{2}{cosh^2 x} \right] v(x) \, dx = \frac{1 - 2q}{1 + 2q} = \beta.
\]

- Assume \(J_0 = 0\). There exists \(c < \infty\) such that if \(k \geq 0, u \geq 0\),

\[
P_* \{ \cosh J_t \geq u \text{ for some } k \leq t \leq k + 1 \} \leq cu^{-2q}.
\]

(37)

**Lemma 5.3.** There exists \(c < \infty\) such that the following holds. Suppose \(J_t\) satisfies (36) with \(J_0 = x > 0\).

- If \(y > x\),

\[
P_* \left\{ \max_{0 \leq s \leq 1} |J_s| \geq y \right\} \leq c \exp \left\{ -\frac{(y - x)^2}{2} \right\}.
\]

(38)

- If \(y < x\) and \(1 \leq t < (x - y)/q\),

\[
P_* \left\{ \min_{0 \leq s \leq t} J_s \leq y \right\} \leq c \exp \left\{ -\frac{(x - qt - y)^2}{2t} \right\}.
\]

(39)
Proof. Since the drift of $J_t$ points towards the origin, the distribution of $|J_t|$ is stochastically dominated by the absolute value of a Brownian motion. Hence by the reflection principle (see, e.g., [RY99, Proposition III.3.7]) if $V$ is a standard Brownian motion

$$
P_* \left\{ \max_{0 \leq s \leq 1} |J_s| \geq y \right\} \leq P_* \left\{ \max_{0 \leq s \leq 1} |V_s| \geq y - x \right\} \leq 4 P_* \{ V_1 \geq y - x \},$$

which gives the first estimate. Also, $J_t \geq \tilde{J}_t$ where $\tilde{J}_t$ satisfies

$$d \tilde{J}_t = -q dt + dB_t.$$

Note that $W_t := \tilde{J}_t + tq - x$ is a standard Brownian motion starting at the origin. We get for $y < x$ and $1 \leq t < (x - y)/q$, again using the reflection principle,

$$P_* \left\{ \min_{0 \leq s \leq t} J_s \leq y \right\} \leq P_* \left\{ \max_{0 \leq s \leq t} W_s \geq x - y - qt \right\} = 2 P_* \{ W_t \geq x - y - qt \},$$

and the second estimate follows. □

5.1. Proof of Lemma 4.1. Lemma 4.1 extends a result in [Law09] where the result was proved for a certain range of $r$ (including $r = 1, a > 1/4$ which was most important for that paper). We start by restating it in terms of $h$ with an added lower bound (and upgrading it to a theorem).

**Theorem 5.4.** If $r < r_c$, there exists $0 < c_1, c_2 < \infty$ such that for all $t \geq 1$,

$$c_1 t^{-\zeta} \leq \mathbb{E} \left[ |h'_{\sigma(t)}(i)|^\lambda \right] \leq c_2 t^{-\zeta}.$$

The heuristic argument is fairly straightforward, so let us consider that first. Consider the martingale $N_t$ which we can write as

$$N_t = |h'_{\sigma(t)}(i)|^\lambda e^{at \zeta} S_{\sigma(t)}^{-r}.$$

We know that $\mathbb{E}[N_t] = 1$. If $r < r_c$, then under $\mathbb{P}_*$, since $J$ is positive recurrent, $S_{\sigma(s)}$ tends to be of order 1 for $s < t$. Hence we would expect that $\sigma(t) \approx e^{2at}$ and hence we would expect

$$\mathbb{E} \left[ |h'_{\sigma(t)}(i)|^\lambda e^{at \zeta} \right] \approx 1.$$

We will only prove the upper bound in Theorem 5.4 in this subsection; the lower bound follows from the work of the next subsection, which uses the upper bound. The next lemma gives a quantitative bound on
the assertion $\sigma(t) \approx e^{2at}$. As a slight abuse of notation, in this section, if $E$ is an event we also write $E$ for the indicator function of the event.

**Lemma 5.5.** There exists $c < \infty$ such that for all $u > 0$,

$$\mathbb{P}_s \{ \sigma(t) \geq u^2 e^{2at} \} \leq c u^{-2q}. \quad (40)$$

**Proof.** Suppose $\cosh J_s \leq u e^{a(t-s)} (t-s+1)^{-1}$ for all $0 \leq s \leq t$. Then,

$$\sigma(t) = \int_0^t e^{2as} \cosh^2 J_s ds \leq u^2 e^{2at} \int_0^t (t-s+1)^{-2} ds < u^2 e^{2at}.$$ 

Therefore,

$$\mathbb{P}_s \{ \sigma(t) \geq u^2 e^{2at} \} \leq \mathbb{P}_s \left[ \bigcup_{k=1}^{\infty} K_k \right] \leq \sum_{k=1}^{\infty} \mathbb{P}_s(K_k),$$

where $K_k = K_{k,t}$ denotes the event

$$K_k = \{ \cosh J_{t-s} \geq u e^{a(k-1)} (k+1)^{-1} \text{ for some } k-1 \leq s < k \}. \quad (41)$$

By (37),

$$\mathbb{P}_s(K_k) \leq c u^{-2q} e^{-2aqk} k^{2q}, \quad (42)$$

and hence we can sum over $k$ to get the result. \qed

The next lemma is a useful “smoothing” result that lets us consider the average of $\mathbb{E}[|h_s'(i)|^\lambda]$ over $t^2 \leq s \leq 2t^2$ instead of $\mathbb{E}[|h_s'(i)|^\lambda]$.

**Lemma 5.6.** There exists $c < \infty$ such that

$$\mathbb{E} \left[ \left| h_{t^2}(i) \right|^\lambda \right] \leq \frac{c}{t^2} \int_0^{2t^2} e^{2as} \mathbb{E} \left[ (\cosh^2 J_s) |h_{\sigma(s)}'(i)|^\lambda \right] I_s \, ds$$

$$= \frac{c}{t^2} \int_0^{2t^2} e^{as(2-\zeta)} \mathbb{E}_s \left[ (\cosh J_s)^{2-\zeta} |I_s| \right] \, ds,$$

where $I_s = I_{s,t}$ denotes the event

$$I_s = \{ t^2 \leq \sigma(s) \leq 2t^2 \}.$$ 

**Proof.** By scaling and the distortion theorem, if $t^2 \leq u \leq 2t^2$,

$$\mathbb{E} \left[ \left| h_u'(i) \right|^\lambda \right] = \mathbb{E} \left[ \left| h_{t^2}(it/\sqrt{u}) \right|^\lambda \right] \geq c \mathbb{E} \left[ \left| h_{t^2}(i) \right|^\lambda \right].$$

Therefore,

$$\mathbb{E} \left[ \left| h_{t^2}(i) \right|^\lambda \right] \leq \frac{c}{t^2} \int_{t^2}^{2t^2} \mathbb{E} \left[ \left| h_u'(i) \right|^\lambda \right] \, du.$$ 

If we let $s = \sigma^{-1}(u)$, we can change variables and write

$$\int_{t^2}^{2t^2} \mathbb{E} \left[ \left| h_u'(i) \right|^\lambda \right] \, du = \int_0^{\infty} \mathbb{E} \left[ h_{\sigma(s)}'(i) \right]^\lambda I_s du = \int_0^{\infty} e^{2as} (\cosh^2 J_s) |h_{\sigma(s)}'(i)|^\lambda I_s \, ds.$$
By taking expectations we get the inequality in the lemma, and the equality follows from the definition of $E_\ast$.  

Since $\sigma(t) \geq \int_0^t e^{2as} ds = \frac{1}{2a} [e^{2at} - 1]$, there is a $c$ such that $I_{s,e^{at}}$ is empty if $s \geq t + c$. Hence the previous proposition implies that there exists a $c < \infty$ such that

$$E[|h_{e^{2at}}(i)|^3] \leq c e^{-2at} \int_0^{t+c} e^{as(2-c)} E_\ast [(\cosh J_s)^{2-r}; e^{2at} \leq \sigma(s) \leq 2e^{2at}] \, ds. \quad (43)$$

**Lemma 5.7.** There exists $c < \infty$ such that if $u \geq 1$,

$$E_\ast [(\cosh J_t)^{2-r}; u^2 \leq e^{-2at} \sigma(t) \leq 2u^2] \leq c \omega^{(2-r)e^{-2at}}.$$

**Remark.** Roughly speaking, we expect that if $\sigma(t) \approx u^2 e^{2at}$, then $\cosh J_t \approx u$. Hence, we would guess

$$E_\ast [(\cosh J_t)^{2-r}; u^2 \leq e^{-2at} \sigma(t) \leq 2u^2] \approx u^{2-r} \mathbb{P}_\ast \{u^2 \leq e^{-2at} \sigma(t) \leq 2u^2\} \leq c \omega^{2r-2q},$$

where the probability is estimated by (40). This lemma makes the argument rigorous but only gives a weaker result for $r > 2$. Lemma 5.8 below establishes the stronger result for some values of $r > 2$.

**Proof.** Let $I_t = I_{t,u}$ be the the event $\{u^2 \leq e^{-2at} \sigma(t) \leq 2u^2\}$. We claim that $I_t$ is contained in the event

$$A = A_{t,u} = \left\{ \min_{(t-1) \vee 0 \leq s \leq t} \cosh^2 J_s \leq 2u^2 e^{2at} \right\}.$$ 

Indeed, this is obvious for $t \leq 1$ since $\cosh J_0 = 1$ and if $t > 1$ and $\cosh^2 J_s > 2e^{2at} u^2$ for $t - 1 \leq s \leq t$, then

$$\sigma(t) \geq \int_{t-1}^t e^{2as} \cosh^2 J_s ds > 2u^2 e^{2at}.$$ 

Let

$$V_k = V_{k,t,u} = I_t \cap \{u e^{(k-1)} \leq \cosh J_t < u e^k\}.$$ 

Then,

$$E_\ast [(\cosh J_t)^{2-r} I_t] = \sum_{k=-\infty}^{\infty} E_\ast [(\cosh J_t)^{2-r} V_k].$$

Note that

$$E_\ast [(\cosh J_t)^{2-r} V_k] \approx u^{2-r} e^{k(2-r)} \mathbb{P}_\ast (V_k). \quad (44)$$
We will first show that
\[ \sum_{k=1}^{\infty} \mathbb{E}_*[\text{cosh}^2 J_t V_k] \leq c u^{2-r-2q}. \] (45)

Let \( k_0 \) be an integer such that \( e^{k-1} > 2e^{2a} \) for \( k \geq k_0 \). Then,
\[ \sum_{k=1}^{k_0} \mathbb{E}_*[\text{cosh}^2 J_s V_k] \leq u^{2-r} \sum_{k=1}^{k_0} \mathbb{P}_*[V_k] \leq u^{2-r} \mathbb{P}_*[\text{cosh}^2 J_t \geq u^2] \leq c u^{2-r-2q}. \]
The last inequality uses (37).

If \( k > k_0 \), let
\[ \eta_t = \inf \{ s \geq (t-1) \lor 0 : \text{cosh}^2 J_s = 2e^{2a} u^2 \}. \]
Since \( V_k \subset A \) and \( \text{cosh}^2 J_t > 2e^{2a} u^2 \) on \( V_k \), we know that on the event \( V_k, t-1 \leq \eta_t < t \). Hence we can estimate
\[ \mathbb{P}_*[V_k] \leq \mathbb{P}_*\{\eta_t < t\} \mathbb{P}_*[\text{cosh}^2 J_t \geq u^2 e^{k-1} \mid \eta_t < t]. \]
By (37),
\[ \mathbb{P}_*\{\eta_t < t\} \leq \mathbb{P}_*\{\text{cosh}^2 J_s \geq 2e^{2a} u^2 \text{ for some } t-1 \leq s \leq t\} \leq c u^{-2q}. \]
Using (38), we can see that there exist \( c, \alpha \) such that
\[ \mathbb{P}_*\{\text{cosh}^2 J_t \geq u^2 e^{k-1} \mid \eta_t < t\} \leq c e^{-\alpha k^2}. \]
Hence, plugging into (44), we have
\[ \mathbb{E}_*[\text{cosh}^2 J_t V_k] \leq c u^{2-r-2q} e^{k(2-r)} e^{-\alpha k^2}. \]
Therefore,
\[ u^{2q+r-2} \sum_{k=1}^{\infty} \mathbb{E}_*[\text{cosh}^2 J_t V_k] \leq c \sum_{k=0}^{\infty} e^{k(2-r)} e^{-\alpha k^2 < \infty}. \]
This proves (45).

Let \( \tilde{V} = \bigcup_{k=0}^{\infty} V_k \). On the event \( \tilde{V} \),
\[ 1 \leq \text{cosh} J_t \leq u. \]
Hence \( \text{cosh}^2 J_t \leq u^{2-r} \) and, using (40),
\[ \mathbb{E}_*[\text{cosh}^2 J_t \tilde{V}] \leq u^{2-r} \mathbb{P}_*(\tilde{V}) \leq u^{2-r} \mathbb{P}_*[I_t] \leq c u^{2-r} u^{-2q}. \]

**Lemma 5.8.** If \( r > a+1 \), there exists \( c < \infty \) such that for all \( u \),
\[ \mathbb{E}_*[\text{cosh}^2 J_t; u^2 \leq e^{-2at} \sigma(t) \leq 2u^2] \leq c u^{2-r-2q}. \]
Proof. We use the notation from the proof of Lemma 5.7 and let $V^j = V_{-j}$. We note that (45) holds for all values of $r$. Hence, we only need to show
\[ \sum_{j=0}^{\infty} \mathbb{E}_s[(\cosh J_t)^{2-r} V^j] \leq cu^{2-r-2q}. \]
Let $K_k$ be as in (41). Then
\[ \sum_{j=0}^{\infty} \mathbb{E}_s[(\cosh J_t)^{2-r} V^j] \leq \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \mathbb{E}_s[(\cosh J_t)^{2-r} V^j K_k]. \]
Note that
\[ u^{2q+r-2} \mathbb{E}_s[(\cosh J_t)^{2-r} V^j K_k] \leq u^{2q} e^{j(r-2)} \mathbb{P}_s(V^j \cap K_k) \]
\[ = u^{2q} \mathbb{P}_s(K_k) e^{j(r-2)} \mathbb{P}_s(V^j \mid K_k). \]
By (42), we know that
\[ u^{2q} \mathbb{P}_s(K_k) \leq ce^{2aqk} k^{2q}. \]
Hence, it suffices to prove that
\[ \sum_{k=1}^{\infty} e^{-2aqk} \sum_{j=0}^{\infty} e^{j(r-2)} \mathbb{P}_s(V^j \mid K_k) < \infty. \] (46)
Note that for $1 + a < r < r_c$, we have $a > q$ and
\[ -2aq + \frac{(r-2)^2}{2} + (2-r)(a-q) < 0, \]
Choose $\hat{a}$ satisfying $q < \hat{a} < a$ and
\[ -2\hat{a}q + \frac{(r-2)^2}{2} + (2-r)(\hat{a}-q) < 0, \] (47)
We will show
\[ \sum_{k=1}^{\infty} e^{-2\hat{a}qk} \sum_{j=0}^{\infty} e^{j(r-2)} \mathbb{P}_s(V^j \mid K_k) < \infty. \]
On the event $K_k$, let $s$ be the largest number less than or equal to $k$ such that
\[ \cosh J_{t-s} \geq u e^{a(k-1)} (k+1)^{-1}. \]
By the definition of $K_k$, we know that $k-1 \leq s \leq k$. Also, we can find $c_1$ such that
\[ J_{t-s} \geq \log u + a(k-1) - \log(k+1) \geq \log u + \hat{a}k - c_1 + \log 2. \]
On the event $V^j$, $\cosh J_t \leq e^{-j} u$ which implies
\[ J_t \leq \log u - j + \log 2. \]
We estimate \( \mathbb{P}_s( V^j | K_k) \) from above by
\[
sup_{k-1 \leq s \leq k} \mathbb{P}_s \{ J_t \leq \log u - j + \log 2 \mid J_{t-s} = \log u + \hat{a}k - c_1 + \log 2 \}.
\]
As in (39), this probability is bounded above by the corresponding probability if \( J_t \) is a Brownian motion with drift \(-q\). In particular, (39) shows that there exists \( M \) and \( c_2 \) such that for \( j > M \),
\[
\mathbb{P}_s( V^j | K_k) \leq C \exp \left\{ -\frac{1}{2k} \left[ (\hat{a} - q)k + j - c_1 \right]^2 \right\}.
\]
Note that
\[
\sum_{k=1}^{\infty} e^{-2\hat{a}qk} \sum_{j=0}^{M} e^{j(r-2)} \mathbb{P}_s( V^j | K_k) \leq \sum_{k=1}^{\infty} e^{-2\hat{a}qk} \sum_{j=0}^{M} e^{j(r-2)} < \infty.
\]
Also, (48) gives
\[
\sum_{j=1}^{\infty} e^{j(r-2)} \mathbb{P}_s( V^j | K_k) \leq C \sum_{j=1}^{\infty} \exp \left\{ j(r - 2) - \frac{1}{2k} |j + b|^2 \right\},
\]
where
\[
b = (\hat{a} - q)k - c_2.
\]
We bound the right hand side from above by a constant times
\[
\int_{-\infty}^{\infty} e^{x(r-2)} e^{-\frac{(x+b)^2}{2k}} dx = e^{b(2-r)} \int_{-\infty}^{\infty} e^{y(r-2)} e^{-\frac{y^2}{2k}} dy
\]
\[
\leq c \sqrt{k} \exp \left\{ b(2-r) + \frac{k(r-2)^2}{2} \right\}
\]
\[
\leq c \sqrt{k} \exp \left\{ (2-r)(\hat{a} - q)k + \frac{k(r-2)^2}{2} \right\}
\]
Hence
\[
e^{-2\hat{a}qk} \sum_{j=M+1}^{\infty} e^{j(r-2)} \mathbb{P}_s( V^j | K_k) \leq C \sqrt{k} e^{k\xi}
\]
where
\[
\xi = -2\hat{a}q + (2-r)(\hat{a} - q) + \frac{r-2)^2}{2}.
\]
Recalling from (47) that \( \xi < 0 \), we conclude
\[
\sum_{k=1}^{\infty} e^{-2\hat{a}qk} \sum_{j=M+1}^{\infty} e^{j(r-2)} \mathbb{P}_s( V^j | K_k) < \infty.
\]
\[\square\]
Lemma 5.9. Suppose $r < r_c$. Then there exists $\theta < 2 - \zeta$ and $c < \infty$ such that for all $u$

$$\mathbb{E}_s[(\cosh J_t)^{2-r}; u^2 \leq e^{-2at} \sigma(t) \leq 2u^2] \leq cu^\theta. \tag{49}$$

Proof. We set

$$\theta = \begin{cases} 
2 - r - 2q, & \text{if } r \leq 2 \text{ or } r \geq a + \frac{1}{2} \\
-2q & \text{otherwise}.
\end{cases}$$

The estimate (49) follows from the previous two lemmas so we only need to verify that $\theta < 2 - \zeta$. It is easy to see that $1 - r - 2q < 2 - \zeta$ for all $r$. Also, one can check that $-2q < 1 - 4a + r$ provided that $a < 3/5$ or $a \geq 3/5$ and

$$r < 2a \left[3 - \sqrt{5 - (3/a)}\right]. \tag{50}$$

If $a \leq 4$, then one can show that (50) holds for all $r < r_c$. Hence we only need to consider $a \geq 4$ and $r < r_c$ that do not satisfy (50). Such an $r$ satisfies

$$r \geq 2a \left[3 - \sqrt{5 - (3/a)}\right] \geq 2a \left[3 - \sqrt{17/4}\right] \geq 9a/5.$$ 

In particular, $r > a + 1$. □

Proof of Theorem 5.4. Let $\theta$ be as in the previous lemma. Then by (49),

$$\mathbb{E}_s\left[(\cosh J_s)^{2-r}; e^{2at} \leq \sigma(s) \leq 2e^{2at}\right] \leq ce^{(t-s)a\theta}.$$ 

Therefore, by (43),

$$\mathbb{E}\left[|h'_{c,2at}(i)|^\lambda\right] \leq ce^{-2at} \int_0^{t+c} e^{2as} e^{-as\zeta} e^{(t-s)a\theta} ds$$

$$\leq ce^{-a\zeta t} \int_0^{t+c} e^{a(t-s)[-2+\zeta+\theta]} ds.$$ 

$$\leq ce^{-a\zeta t} \int_{-\infty}^{\infty} e^{ay[-2+\zeta+\theta]} dy \leq ce^{-a\zeta t}.$$ 

The last inequality uses $\theta < 2 - \zeta$. □
5.2. Proof of Lemma 4.4. We essentially follow the proof in [Law09]. In that paper, it was assumed that $r \geq 0$, which we do not want to assume here, but with the upper bound of Theorem 5.4, we can do the argument for all $r < r_c$. We emphasize that the positive recurrence of $J_t$ is critical for this argument and hence we need $q = r_c - r > 0$. Since $J_t$ is positive recurrent, we expect that $J_t = O(1)$ and that (approximately)

$$L_t = \beta t + O(t^{1/2}).$$

The next lemma gives an estimate of this type.

Lemma 5.10. [Law09, Proposition 7.3] For each $u, t > 0$, let $E_{t,u}$ be the event that the following holds for all $0 \leq s \leq t$:

$$|J_s| \leq u \log \min\{s + 2, t - s + 2\},$$

$$|L_s - s\beta| \leq u \sqrt{s} \log(s + 2),$$

$$|L_t - L_s - (t - s)\beta| \leq u \sqrt{t - s} \log(t - s + 2).$$

Then,

$$\lim_{u \to \infty} \inf_{t > 0} \mathbb{P}_*(E_{t,u}) = 1.$$

By integrating, we can see there exists $c_0$ and $c(u)$ such that for all $t \geq 1$, on the event $E_{t,u}$

$$c_0 e^{2at} \leq \sigma(t) \leq c(u) e^{2at}. \quad (51)$$

Note that the lower bound does not depend on $u$ and follows from the estimate $\cosh^2 J_s \geq 1$. An event such as this is used to define the event $E_t = E_{t,u}$ in Lemma 4.2 where $u$ is sufficiently large so that

$$\mathbb{P}_*(E_t) = \mathbb{E} [N_t 1_{E_t}] \geq \frac{1}{2}.$$

Note that on the event $E_t$,

$$|h'_{\eta(t)}(i)| \asymp e^{at\beta},$$

and using (51), one can show that $|h'_{e^{2at}}(i)| \asymp (e^{at})^\beta$. In fact, there is a function $\psi$ with $\psi(s) = o(s)$ as $s \to \infty$, such that for all $0 \leq s \leq t$,

$$e^{a\beta s - \psi(s)} \leq |h'_{\eta(s)}(i)| \leq e^{a\beta s + \psi(s)},$$

$$e^{a\beta (t-s) - \psi(t-s)} \leq \frac{|h'_{\eta(t)}(i)|}{|h'_{\eta(s)}(i)|} \leq e^{a\beta (t-s) + \psi(t-s)}.$$

As in (51), we can see that this implies (with perhaps a different $o(s)$ function $\psi$)

$$e^{a\beta s - \psi(s)} \leq |h'_{e^{2as}}(i)| \leq e^{a\beta s + \psi(s)},$$

$$e^{a\beta (t-s) - \psi(t-s)} |h'_{e^{2as}}(i)| \leq |h'_{e^{2at}}(i)| \leq e^{a\beta (t-s) + \psi(t-s)} |h'_{e^{2as}}(i)|.$$
We get

**Proposition 5.11.** If \( r < r_c \), there is a \( c < \infty \) and a subpower function \( \varphi \) such that if \( E_t = E_{t, \beta, \psi} \) denotes the event that for \( 1 \leq s \leq t \),

\[
\begin{align*}
S^\beta \varphi(s)^{-1} &\leq |h'_s(i)| \leq S^\beta \varphi(s), \quad (52) \\
(t/s)^\beta \varphi(t/s)^{-1} |h'_s(i)| &\leq |h'_t(i)| \leq (t/s)^\beta \varphi(t/s) |h'_t(i)|, \quad (53)
\end{align*}
\]

then

\[
\mathbb{E} \left[ |h'_t(i)|^\lambda 1_{E_t} \right] \geq c.
\]

Note that

\[
|h'_s(i)|^\lambda 1_{E_t} \leq |h'_s(i)|^\lambda (t/s)^{\beta \lambda} \varphi(t/s)^\lambda.
\]

By the definition of the event, we also get \( Y_{c_2t} \approx c_\alpha t \) and \( S_{c_2t} \approx 1 \), so that

\[
\mathbb{E} \left[ |h'_{c_2t}(i)|^\lambda \right] \approx [c_\alpha t]^{-\zeta}.
\]

This is how (29) and (30) are derived.

To get (31) we have to estimate the correlations. Suppose that \( S, T \) are positive integers with \( n^2/2 \leq S \leq T \leq n^2 \). Let \( h^{(S)}, h^{(T)} \) be defined as in the beginning of the section. Recall that \( f^S_S(z) f^T_T(z) \) has the same distribution as \( (h^S_S)'(z)(h^T_T)'(z) \). Set \( h := h^{(S)} \) and \( \hat{h} := h^{(T)} \). Let \( E(S), E(T) \) denote the indicator function of the event from Proposition 5.11 with \( t = n \) and with \( h \) being \( h \) or \( \hat{h} \), respectively. We have already noted that \( \hat{h}_t, 0 \leq t \leq T - S \) is independent of \( h_s, 0 \leq s \leq S \). Using (53) we see that

\[
|h'^S_S(i)|^\lambda E(T) \leq |h'^S_{T-S}(i)|^\lambda \left( \frac{n^2}{T-S+1} \right)^{\frac{\beta \lambda}{2}} \varphi \left( \frac{n^2}{T-S+1} \right).
\]

(Here we use \( \varphi \) for a subpower function whose exact value may change from line to line.) The random variable on the right hand side is independent of \( h \). Therefore, using Theorem 5.4,

\[
\begin{align*}
\mathbb{E} \left[ |h'^S_S(i)|^\lambda |h'^T_T(i)|^\lambda E(T) \right] &
\leq \mathbb{E} \left[ |h'^S_S(i)|^\lambda \right] \mathbb{E} \left[ |h'^T_{T-S}(i)|^\lambda \right] \left( \frac{n^2}{T-S+1} \right)^{\frac{\beta \lambda}{2}} \varphi \left( \frac{n^2}{T-S+1} \right) \\
&\leq c n^{-\zeta} (T-S+1)^{-\zeta/2} \left( \frac{n^2}{T-S+1} \right)^{\frac{\beta \lambda}{2}} \varphi \left( \frac{n^2}{T-S+1} \right) \\
&= c n^{-2\zeta} \left( \frac{n^2}{T-S+1} \right)^{\frac{\beta \lambda + \zeta}{2}} \varphi \left( \frac{n^2}{T-S+1} \right),
\end{align*}
\]

and (31) follows.
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