Notes on the integrability of Virasoro generators for axisymmetric Killing horizons

Lin-Qing Chen
Centre for Quantum Information and Communication, Ecole polytechnique de Bruxelles, CP 165, Université Libre de Bruxelles, 1050 Brussels, Belgium
Perimeter Institute for Theoretical Physics, 31 Caroline St. N, Waterloo, Canada
E-mail: linqing.nehc@gmail.com

ABSTRACT: Through the analysis of null symplectic structure, we derived the conditions for the well-defined Virasoro generators on the covariant phase space of the axisymmetric Killing horizons. Various subtleties of the boundary terms are also discussed.
1 Introduction

This note is written to provide justifications for a substantial assumption made in [1]: the integrability of Virasoro charges on the gravitational phase space. In our previous work, by generalizing the construction of [2] to generic axisymmetric Killing horizons, we have shown that the central charges from the 2d conformal algebra of near-horizon phase space are proportional to the area of bifurcation surface: \( c_R = c_L = \frac{3A}{2\pi G(\alpha + \beta)} \). The canonical form of the Cardy entropy coincides with the Bekenstein-Hawking entropy.

\[
S_{\text{Cardy}} = \frac{\pi^2}{3} (c_RT_R + c_LT_L) = \frac{A}{4G} = S_{BH},
\]

Early pioneer works that have greatly facilitated our investigation includes [3–8]. However, our central charge results are based on the assumption that the Virasoro generators could actually be well-defined on the covariant phase space. Its rigorous justification is the quest that the present notes aiming to address: by analyzing the symplectic structure of the Killing horizons, we derived the conditions to ensure that the Virasoro generators could be well-defined on any spatial cross section \( S \) of the Killing horizon. The large diffeomorphisms change both the surface gravity and the component of the null generator which is parallel to \( S \), but keep the horizon null. To prove the integrability is to properly subtract all obstruction term which prevent the \( \delta H_m \) to be an exact form on the field space, finally only the central extension term is unremovable.

\footnote{The present notes are still in the preliminary form. It is posted on arXiv to match the releasing day with a separate investigation on the same topic [28] and to compare independent results. A complete reference list, introduction and more discussions will be improved in the final version. Comments are welcome.}
2 Charges on the covariant phase space

In this section, I will review the covariant phase space formalism with special emphasis on the integrability, central extension and the discussion of closed ambiguities. Various literature see [9–14].

2.1 General formalism

Starting from a classical action\(^2\): \( S = \int_M L + \int_{\partial M} I \), the variation of the action tells us about both of the equation of motion and the symplectic structure of the phase space:

\[
\delta S = \int_M E(g) \delta g + \int_{\partial M} (\theta + \delta l)
\]  

(2.1)

In our case \( E(g) \) is the Einstein equation, \( \partial M \) is the Killing horizon \( \mathcal{H} \). \( \theta(g, \delta g) \) is the symplectic potential density, which is a spacetime \( d-1 \)-form, field space 1-form. Here all the tensor indices are omitted. The Lagrangian boundary term \( I \) should be determined by well-defined variational principle. When \( \partial M \) is a time-like boundary (a comprehensive analysis for example, see [14]), \( I \) shall be the Gibbons-Hawking boundary term. When \( \partial M \) is null boundary, there are various proposals in the literature. Such boundary term corresponds to a change of polarization in the canonical analysis: \( p \wedge \delta q \rightarrow -\delta p \wedge q \)

The presymplectic 2-form is an integration on a Cauchy surface \( \Sigma \),

\[
\Omega(g, \delta_1 g, \delta_2 g) = \int_\Sigma \delta_1 \theta(g, \delta_2 g) - \delta_2 \theta(g, \delta_1 g).
\]  

(2.2)

We are studying the Hamiltonian (charges) which generates nontrivial transformation on the covariant phase space induced by diffeomorphisms. From all the metric variation generated by diffeomorphisms, \( \delta g_{ab} = \mathcal{L}_\xi g_{ab} \), the trivial one corresponds to the degenerate direction of the symplectic form: \( \Omega(g, \delta_1 g, \mathcal{L}_\xi g_{ab}) = 0 \), while the nontrivial one will have corresponding charges:

\[
\delta H_\xi = \Omega(\delta g_{ab}, \mathcal{L}_\xi g_{ab}) = \int_{\partial \Sigma} (\delta Q_\xi - \iota_\xi \theta(g_{ab}, \delta g_{ab}) - \iota_\xi \delta I)
\]  

(2.3)

where \( Q_\xi \) is the Noether potential \((d-2)\) form, or the Komar charge in gravity context:

\[
dQ_\xi = \theta(g_{ab}, \mathcal{L}_\xi g_{ab}) - \iota_\xi L
\]  

(2.4)

Here the notation \( \delta H_\xi \) is trying to emphasize that the Hamiltonian could only be well-defined if it is an exact form on the field space, i.e. \( \delta \delta H = 0 \), which is a nontrivial condition to satisfy. Otherwise the Hamiltonian isn’t really well-defined on the phase space. The reason is that to determine \( H_\xi \) from its variation (4.9), it need to be independent of the path of integration between any reference spacetime metric \( g_o \) to \( g \):

\[
H_\xi(g_o, g) = \int_{g_o}^g \delta H_\xi + N_\xi(g_o)
\]  

(2.5)

\(^2\)In this notes, we use bold letter to denote densities.
which is only true if $\partial H_{\xi}$ is a field-space exact form – then we say that the Hamiltonians (charges) are integrable. In the above equation, $N_{\xi}(g_o)$ is an integration constant and will be relevant for the later discussion.

The most direct approach to check the integrability is by the following condition:

$$\delta(\delta H_{\xi}) = -\delta \int_{\partial \Sigma} \iota_{\xi}(g, \delta g) = - \int_{\partial \Sigma} \iota_{\xi} \omega(g, \delta g, \delta_1 g, \delta_2 g) \equiv 0 \quad (2.6)$$

however sometimes this expression could be cumbersome as the symplectic form has double times of the field variation then the symplectic potential. Probably an easier approach is analyzing the obstruction terms within the (4.9) directly.

2.2 Central extension and boundary ambiguities

If the Hamiltonian is integrable, then given a set of vector fields $\xi_m$ (we assume that they are field-independent in the scope of this work, i.e. $\delta \xi_m = 0$), by the charge representation theorem [12][15], the algebra formed by the Hamiltonians under Poisson brackets is isomorphic to the Lie algebra of diffeomorphisms defined by the vector fields up to a central extension

$$\{ H_{\xi_m}, H_{\xi_n} \} = H_{[\xi_m, \xi_n]} + K_{m,n} \quad (2.7)$$

where $K_{m,n}$ is given by

$$K_{m,n}(g) = \int_{\partial \Sigma} \iota_{\xi_m} \theta(g, \mathcal{L}_{\xi_n} g) - \iota_{\xi_n} \theta(g, \mathcal{L}_{\xi_m} g) + \iota_{\xi_n} \iota_{\xi_m} \mathcal{L} \quad (2.8)$$

It is a Casimir on the phase space – does not generate any flow. It forms the two-cocycle on the Lie algebra of diffeomorphisms

$$K_{[m,n],k} + K_{[k,m],n} + K_{[n,k],m} = 0 \quad (2.9)$$

When the Hamiltonian get shifted by $\tilde{N}_{\xi}$ in eq.(2.5), then the central extension will be shifted as

$$H_{\xi} \rightarrow H_{\xi} + \tilde{N}_{\xi}, \quad K_{m,n} \rightarrow K_{m,n} - \tilde{N}_{[\xi_m, \xi_n]} \quad (2.10)$$

The existence of such shift comes from the fact as the symplectic potential density $\theta$ is only defined by the action up to a total derivative, which is part of the so-called JKM ambiguity [24]. More precisely, if we shift the symplectic potential density by a spacetime exact form:

$$\theta \rightarrow \theta + db(g, \delta g). \quad (2.11)$$

Such transformation would leave the equation of motion unimpacted, but shift the variation of the Hamiltonian eq.(4.9) by

$$\delta H_{\xi} \rightarrow \tilde{\delta} H_{\xi} + \int_{\partial \Sigma} \delta \frac{b(g, \mathcal{L}_{\xi g})}{\text{shift} \ Q_{\xi}} + \iota_{\xi} b(g, \delta g) \quad (2.12)$$

If the Hamiltonian is integrable, i.e. $\delta \tilde{\delta} H_{\xi} = 0$, then the above $\tilde{N}_{\xi} = \int_{\partial \Sigma} b(g, \mathcal{L}_{\xi g})$, and the central extension is shifted by

$$K_{m,n} \rightarrow K_{m,n} - \tilde{N}_{[\xi_m, \xi_n]} + \int_{\partial \Sigma} \iota_{\xi_m} b(g, \mathcal{L}_{\xi_n} g) - \iota_{\xi_n} b(g, \mathcal{L}_{\xi_m} g) \quad (2.13)$$
By exploiting this ambiguity in the symplectic potential, one could design the so-called Wald-Zoupus counterterm to substract an obstruction term which prevents the Hamiltonian from being an exact form \([10, 11]\). Part of this ambiguity also corresponds to the shift of zero energy in \([27]\).

Finally a word of caution here is that the field variation doesn’t change the boundary, hence in that case the following operation commutes\(^3\):
\[
\delta \int_{\partial \Sigma} \ldots \neq \int_{\partial \Sigma} \delta \ldots
\]
which is not the case in our situation, as the diffeomorphisms we are studying will move the boundary:
\[
\delta (\partial \Sigma) \neq 0 \quad (2.14)
\]

3 The null symplectic structure

3.1 Review of the symplectic structure on the null hypersurface

The symplectic structure of gravity on null hypersurface has a very elegant and simple form in terms of geometrical quantities. Relevant literatures please see \([16–23]\). This section provides as a brief review, mostly following the work of \([17, 20]\). Let us start from the symplectic potential density \((d - 1)\)-form from the Einstein-Hilbert action with cosmological constant:
\[
\theta(g_{ab}, \delta g_{ab}) = \frac{1}{16\pi G} \epsilon^a \left( \nabla^b \delta g_{ab} - g^{bc} \nabla_a \delta g_{bc} \right). \quad (3.1)
\]
where \(\epsilon\) denotes the spacetime volume form, and \(\epsilon_a = \iota_a \epsilon\) is the directed (D-1) form. Its pullback onto the null hypersurface \(\mathcal{H}\) with boundary \(\partial \mathcal{H}\) can be written as a bulk piece and a boundary piece:
\[
\Theta_{\mathcal{H}}(g_{ab}, \delta g_{ab}) = -\int_{\mathcal{H}} \ell^a \left( \nabla^b \delta g_{ab} - g^{bc} \nabla_a \delta g_{bc} \right) \epsilon_{\mathcal{H}} = \Theta_{\mathcal{H}}^{\text{bulk}} + \Theta_{\partial \mathcal{H}} \quad (3.2)
\]
The bulk piece \(\Theta_{\mathcal{H}}^{\text{bulk}}\) only depends on the intrinsic and extrinsic geometry of the null hypersurface \(\mathcal{H}\), and can be written in the form of canonical pairs:
\[
\Theta_{\mathcal{H}}^{\text{bulk}} = \frac{1}{2} \int_{\mathcal{H}} \left( \sigma^{ab} \delta \gamma_{ab} - 2 \omega_a \delta \ell^a - (\kappa + \theta) \delta \tilde{q} \right) \epsilon_{\mathcal{H}} + \int_{\mathcal{H}} \delta (\kappa \epsilon_{\mathcal{H}}) \quad (3.3)
\]
in which \(\epsilon_{\mathcal{H}}\) denotes the volume form of the null hypersurface. \(\ell^a\) is a properly defined null normal vector and we will provide more detail in our context in the next section. The canonical pairs are three terms: the spin-2 part is given by the conformal metric \(\gamma_{ab}\) of spacial cross-section and shear \(\sigma^{ab}\). In the spin-1 term, the configuration variable is the generator \(\ell^a\) of the null hypersurface, while the conjugate momenta is the twist \(\omega_a\) 1-form. In the spin-0 term, \(\tilde{q}_{ab}\) is the induced metric on the codimensional-2 surface and \(\tilde{q}\) is the trace. The momenta here is a combination of the expansion \(\theta\) and the surface gravity \(\kappa\).

The last term \(\delta (\kappa \epsilon_{\mathcal{H}})\) is a total variation on the field space. It can be subtracted by adding a boundary term \(\ell\) to the Lagrangian, see (2.1). In literature, Gibbons-Hawking like boundary term has been proposed to substract a total variation in this nature \([18, 19]\).
There are various forms to sort the boundary piece $\Theta_{\partial H}$ depending on the geometrical feature one wants to study. One convenient expression is the following

$$\Theta_{\partial H} = \frac{1}{2} \int_{\partial H} \left( (1 + \frac{1}{2} h) \delta \ell^a + (1 + h) \delta \ell^a \right) \iota_a \epsilon_H - \frac{1}{2} \int_{\partial H} \delta (h \ell^a \iota_a \epsilon_H)$$  \hspace{1cm} (3.4)

where the factor $h$ is defined by the normal volume element:

$$e^h := \sqrt{|g|}/\sqrt{\tilde{q}}$$  \hspace{1cm} (3.5)

The factor $h$ can be identified as a red shift factor of light rays parallel to a horizon measured by geodesic observer crossing the horizon. Please see the section VA in [17] on more physical discussion regarding of this factor.

3.2 Analysis in the context of axisymmetric Killing horizons

We are looking at the spacetime metric near a stationary, bifurcate, axisymmetric Killing horizon in $d \geq 3$. The horizon generators $\chi^a = t^a + \Omega H \psi^a$ are the linear combination of the timelike killing vector $t^a$ and the rotational Killing vector $\psi^a$. In the Rindler form of coordinate, the metric can be expressed as

$$ds^2 = -\kappa^2 x^2 dt^2 + dx^2 + \psi^2 d\phi^2 + q_{AB} d\theta^A d\theta^B - 2x^2 dt (\kappa N_\phi d\phi + \kappa N_A d\theta^A) + O(x^4)$$  \hspace{1cm} (3.6)

where $x$ is proper distance to the bifurcation surface at the leading order, $\phi$ is the comoving angular coordinate corresponds to the rotational killing vector and $\theta^A$ lables all the rest of angular coordinates. Note that (3.6) omitted certain irrelevant terms at the order of $O(x^2)$, the full detail see appendix of [1]. In the Kruskal coordinate $(U, V)$:

$$U = xe^{\kappa t}, \ V = xe^{-\kappa t}$$  \hspace{1cm} (3.7)

the same metric has the following form,

$$ds^2 = dUdV - VdU (N_\phi d\phi + N_A d\theta^A) + UdV (N_\phi d\phi + N_A d\theta^A) + \psi^2 d\phi^2 + q_{AB} d\theta^A d\theta^B + ...$$  \hspace{1cm} (3.8)

The past horizon $H^-$ and future horizon $H^+$ locate at $U, V = 0$ respectively, with bifurcation surface $B$ connecting $H^+_+$ and $H^-_-$.  

To match with the notation in our previous work, we will proceed the analysis in the Rindler form first. Although the redshift factor $h = \ln \left( \sqrt{|g|}/\sqrt{\tilde{q}} \right)$ is infinite on the horizon, the coordinate $(t, x)$ can still play the role of a good foliation function as $dt, dx \neq 0$. Given any spacelike cross section $S$ of the horizon, the normal derivatives covariant under diffeomorphisms of $S$ is defined as

$$D_i := \partial_i + A^a_i \partial_a, i \in \{t, x\}, \sigma \in \{\phi, \theta^A\};$$  \hspace{1cm} (3.9)

where $\sigma$ labels the coordinate indices on the cross section $S$. In our case

$$A^\phi_i = \kappa x^2 N_\phi/\psi^2, \ A^\theta_i = \kappa x^2 N_A q^{AB}, \ A^\phi_x = A^\theta_x = 0$$  \hspace{1cm} (3.10)
For any codimensional-two cross section $S$ of the horizon with the induced metric $\tilde{q}_{ab}$, following the light rays, there are two null normals $\ell^a q_{ab} = 0$. One is parallel to the horizon and its normalization defines a notion "clock" along the null ray: $\ell[t] = \text{constant}$; the other null normal is transverse to the horizon and we use the $\bar{\ell}$ to label it. The two null normals $\ell, \bar{\ell}$ are defined as

$$\ell^a \partial_a = \frac{1}{2} (D_t + \kappa x D_x), \quad \bar{\ell}_a dx^a = dt + \frac{1}{\kappa x} dx$$

(3.11)

The induced metric $\tilde{q}_{ab}$ on the codimensional-2 cross section can be expressed in terms of its two null normals and the spacetime metric $\tilde{q}_{ab} = g_{ab} - \ell^a \bar{\ell}_b - \bar{\ell}^a \ell_b$ (3.12)

One could readily check that actually $\ell^a q_{ab} \sim O(x^4)$. The null vector and null form $\bar{\ell}$ satisfies a convention of normalization $\ell^a \bar{\ell}_a = 1$. Note that in general such normalization fixed the full normals up to a rescaling of an arbitrary function $e^{\epsilon} \ell^a, e^{-\epsilon} \bar{\ell}_a$, which is called boost gauge in literature\(^4\). The choice of the definition (3.11) is to make sure each term in the symplectic potential is boost gauge invariant respectively [17]. The area form $\epsilon_S = \sqrt{\tilde{q}} d(D-2) \sigma = \sqrt{q_{AB}} |\psi| d\phi \wedge d\theta^A$.

Now let us look at the symplectic potential of the Killing horizons. The simplification comes from the vanishing of expansion $\theta$ and shear $\sigma_{ab}$. (However one need to be careful as their variation might not vanish.) The bulk piece of the symplectic potential (3.3) from the Einstein-Hilbert action becomes

$$\Theta_{H}^{bulk} = \int_H ( -\omega_a \delta l^a + \delta \kappa ) \epsilon_H. \quad (3.13)$$

in which $\omega_a$ is the twist and it describes how a surface twist inside the horizon if we let it move along the integral curve of the normal vector $\ell$

$$\omega_a := -\tilde{q}_a \delta a_{\nabla} \ell_b$$

(3.14)

Here $\tilde{q}_a$ is the projector onto the $S$. Evaluating on the horizon, the twist $\omega_a dx^a$ has the simple form:

$$\omega_a dx^a = N_\phi d\phi + N_A d\theta^A$$

(3.15)

Now let us look at the boundary term (3.4). We will study the class of metric variation $\delta g_{ab}$ which keeps the horizon null, then the integration vanishes $\int_S \delta \ell^a \iota_a \epsilon_H = 0$ hence the boundary piece of the symplectic potential simplified to be

$$\Theta_{S}^{bdry} = \frac{1}{2} \int_S (1 + \frac{h}{2}) \delta q \ell^a \iota_a \epsilon_H - \frac{1}{2} \int_S \delta (h \ell^a \iota_a \epsilon_H)$$

(3.16)

now making use of $2\delta (\iota_a \epsilon_H) = \delta \tilde{q}(\iota_a \epsilon_H)$, the above expression becomes

$$\Theta_{S}^{bdry} = \frac{1}{2} \int_S (\delta \tilde{q} - \delta h) \iota_a \epsilon_H$$

(3.17)

Hence the full symplectic potential of the Einstein-Hilbert action with cosmological constant when pull back to the killing horizon becomes a very compact form:

$$\Theta_H = \int_H ( -\omega_a \delta l^a + \delta \kappa ) \epsilon_H + \frac{1}{2} \int_S (\delta \tilde{q} - \delta h) \iota_a \epsilon_H$$

(3.18)

\(^4\)Special thanks to Antony Speranza on emphasizing this $\epsilon^a$ ambiguity to me in our previous project.
4 The Virasoro hairs

In our previous work [1], the following conformal coordinate transformation was designed to bring out the AdS$_3$ folia within the near horizon geometry:

$$w^+ = xe^{\alpha \phi + \kappa t} \quad (4.1)$$
$$w^- = xe^{\beta \phi - \kappa t} \quad (4.2)$$
$$y = e^{\alpha + \beta \phi} \quad (4.3)$$

which was inspired by the construction in [2][8]. $\alpha, \beta$ are two arbitrary parameters, which have an interpretation of the putative CFT temperatures ($\alpha = 2\pi T_R$, $\beta = 2\pi T_L$). The vector fields for the asymptotic AdS$_3$ folia are the Brown-Henneaux vector fields without the field-dependent term [2]:

$$\zeta^a = \varepsilon (w^+) \partial^a + \frac{1}{2} \varepsilon' (w^+) y \partial_y^a \quad (4.4)$$
$$\xi^a = \bar{\varepsilon} (w^-) \partial^a - \frac{1}{2} \bar{\varepsilon}' (w^-) y \partial_y^a \quad (4.5)$$

From the periodicity condition of $\phi$, the basis for the mode expansions are $\varepsilon_n(w^+) = \alpha$, $(w^+)^{1 + \frac{im}{\alpha}} \varepsilon_n(w^-) = -\beta (w^-)^{1 + \frac{im}{\alpha}}$. The corresponding generators are labeled as $\zeta_n^a$, $\xi_n^a$ respectively. Their Lie algebras form two commuting copies of the Witt algebra,

$$[\zeta_m, \zeta_n] = i(n-m)\zeta_{m+n}, \quad [\xi_m, \xi_n] = i(n-m)\xi_{m+n}. \quad (4.6)$$

The vector field could be expressed in terms of the component parallel, normal to the horizon, and parallel to the horizon cross section $S$: $\zeta_n^a = f_m \ell^a + \bar{f}_m \bar{\ell}^a + v^a$, in which $v^a//S$

$$f_m = \alpha(U)^{\frac{im}{\alpha}} e^{i m \phi} \frac{\beta - im}{\kappa (\alpha + \beta)}$$
$$\bar{f}_m = -\beta(U)^{\frac{im}{\alpha}} e^{i m \phi} \frac{\kappa x^2 (im + \alpha)}{2(\alpha + \beta)}$$
$$v = (U)^{\frac{im}{\alpha}} e^{i m \phi} \frac{im + \alpha}{\alpha + \beta} \partial_\phi \quad (4.7)$$

Here we simply use the Kruskal $U = x e^{\kappa t}$ as a clock along the past horizon. One could immediately see that the transverse component of the vector $\bar{f}_m \to 0$ when $x \to 0$ while keeping $U$ finite. In the above expression, all the common prefactors come from $(w^+)^{im/\alpha}$. On the past horizon, the vector fields become a $U$-dependent translation $f_m \ell^a$ plus a $U$-dependent rotation $v^a$. $\zeta_n$ are well-defined on the horizon, but when approaching the bifurcation surface, $U^{im}$ has singular limit when $U \to 0$. Vice versa, the other copy of the vector fields $\xi_n$ has singular limit when approaching bifurcation surface from the future horizon. As it has been observed in [2], the nontrivial central extension essentially comes from the $1/x$ pole (or $1/w^+$ pole) in the diffeomorphisms, more precisely

$$m(x e^{\kappa t})^{\frac{im}{\alpha}} e^{im \phi} \frac{m - i \alpha}{x \alpha (\alpha + \beta)} \subset L_{\zeta_m, g_{ab}} \quad (4.8)$$
4.1 The integrability conditions

Now our goal is to identify the Hamiltonian $H_m$ on the covariant phase space which generates the large diffeomorphisms corresponding to $\zeta_m$. In the previous work, we have assumed that the vector fields will be unchanged with variation of the metric. We will leave the general case $\delta \zeta_m \neq 0$ to be addressed in the future. Since $\zeta_m$ has singular limit when approaching the bifurcation surface from the past horizon, the Hamiltonian shall be defined on the cross section $S$ of the horizon for finite $U = xe^{ct}$, but not on the bifurcation surface.

$$\delta H_m = \int_S \delta Q_{\zeta_m} - \iota_{\zeta_m} \Theta_H$$  \hspace{1cm} (4.9)

The diffeomorphisms from $U$-dependent translation will move the surface $\delta \zeta S \neq 0$, however as it does not change the integration result on $S$, we will proceed without extra relabeling. We need to analyze the second term in eq. (4.9) to see whether some component is still in the form of $\delta(\ldots)$, which part is the obstacle to prevent the $\delta H_m$ being a field-space exact form, explicitly,

$$\iota_{\zeta_m} \Theta_H = \frac{1}{2} \int_S (-\omega_a \delta \ell^a + \delta \kappa) f_m \epsilon_S + \frac{1}{2} \int_S (\delta \tilde{q} - \delta h) \iota_{\zeta_m} \iota_t \epsilon_H$$  \hspace{1cm} (4.10)

The first factor of $1/2$ comes from our normalization convention $\ell^a \partial_a dt = 1/2$. As we can see here that due to the interior product of the volume form $\iota_{\zeta_m} \iota_t \epsilon_H$, the boundary piece vanishes when integrate on $S$. Hence we can now identify the obstacle term which needs to be treated in order to have a well-defined Hamiltonian: we denote its density using bold letter $\delta O(\phi, \theta^A)$:

$$\delta \int_S O(\phi, \theta^A) \epsilon_S = \frac{1}{2} \int_S F_{\ell}(\delta g) f_m$$  \hspace{1cm} (4.11)

in which $F_{\ell}(\delta g)$ is the gravitational flux density crossing through $S$ with the null normal $\ell$ in the canonical form, which was discussed in [20]:

$$F_{\ell}(\delta g) := (-\omega_a \delta \ell^a + \delta \kappa) \iota_t \epsilon_H.$$  \hspace{1cm} (4.12)

There are at least two ways to eliminate the obstacle terms to well-define the Virasoro charges:

1) **Imposing a boundary condition** to constrain the allowed metric variation $\delta g_{ab}$:

We can impose a boundary condition to require that the integrated gravitational flux density to be an axisymmetric function, i.e.

$$\partial_{\phi} \int F_{\ell}(\delta g) d\theta^A = 0$$  \hspace{1cm} (4.13)

On the “hard mode” side, such integrated condition constraints what kind of metric perturbation is allowed for well-defining the Virasoro charges. On the “soft mode” side, the constraints on $\delta \zeta g = L_{\zeta} g$ will select a class of vector fields within (4.4): i.e. provide a relationship between $\alpha$, $\beta$ and the spin $J_H$ of the axisymmetric Killing horizon, as the spin
coming from the integration of $N_\phi$ over the transverse direction $J_H = \frac{1}{8\pi} \int d\theta^A \sqrt{g} |\psi| N_\phi$.

On the first sight, one might worry that whether such boundary condition will eliminate the central extension as well. However, notice that

$$F_\ell (L_{\zeta_{-m}} g_{ab}) \propto e^{-im\phi}, \quad f_m \propto e^{im\phi} \tag{4.14}$$

The only obstruction term that such boundary condition will not annihilate is the following:

$$\iota_{\zeta_{-m}} \Theta_H (g, L_{\zeta_{m}} g_{ab}) \neq 0 \tag{4.15}$$

as it does not contain the phase term $e^{im\phi}$ to make the integration of an axisymmetric function to vanish. The commutator of (4.15) precisely gives the central charge in the Virasoro algebra.

2) Using Wald-Zoupus counterterm to compensate the gravitational flux through $S$ and keep the system integrable

Wald-Zoupus counterterm techniques were first discussed in [10, 11] and it plays a crucial role in the investigation of Kerr case [2]. As the definition of symplectic potential embodies the closed ambiguity e.q.(2.11), if we modified the full symplectic potential by a corner term, the dynamics doesn’t change. Hence such techniques could be used to subtract the non-integrable piece of the Hamiltonian instead of imposing the axisymmetric boundary condition.

Let us introduce a boundary quantity $F(\phi, \delta g)$ on $S$ to measure the integrated gravitational flux density over the transverse direction $\theta^A$ on the crosssection $S$. Hence it is a $\phi$-dependent density on the circle:

$$F_\ell (\phi, \delta g) d\phi := \int_S F_\ell (\delta g) d\theta^A \tag{4.16}$$

then a possible introduction of a Wald-Zoupus counterterm could be

$$\delta Q_{\zeta_{-m}} (\delta g) = \int_S \iota_{\zeta} (\ast F_\ell (\phi, \delta g) dx) . \tag{4.17}$$

Now if we look at the obstruction term, only the $[\zeta_{m}, \zeta_{-m}]$ component are left, which gives rise to the central extension. Adding the counterterm will shift the central charge by (2.13).

In a sequential work, we will show its precise relation with the holographic gravitational anomalies [25], which can be viewed as a measure of the lack of diffeomorphisms in the bulk of AdS$_3$ folia: as the counterterm measures the integrated gravitational symplectic flux through $\theta^A$, which can be viewed as the integrated flux injected into the bulk from the transverse direction.

4.2 Details on the field variations

This section will provide details of the field variations which was used above. Here we are looking at two type of variations: the standard $\delta$ refers to a general field variation, with
δg_{ab} satisfying the linearized Einstein equation. Here we use δξ to refer to a field variation induced by the diffeomorphisms generated by the vector field ζ^a, i.e. for any tensor T(g_{ab})

$$\delta_\xi T(g_{ab}) = \frac{\partial T(g_{ab})}{\partial g_{ab}} \mathcal{L}_\xi g_{ab}$$ \hspace{1cm} (4.18)

Here it is convenient to introduce \[ Ξ := \delta_\xi - \mathcal{L}_\xi \] \hspace{1cm} (4.19)

and it commutes with covariant derivatives \[ \Delta_\xi \nabla_a T(g_{ab}) = \nabla_a \Delta_\xi T(g_{ab}) \], which could be proven by directly checking the variation of the Christoffel symbols.

The variation of the null normal vector induced by diffeomorphisms

$$\delta_\xi \ell^a = -\bar{q}^{ab} e^c \nabla_c g_{bc} = \delta_\xi A^b$$ \hspace{1cm} (4.20)

Hence with the vector field (4.4), our null normal varies as the following

$$\delta_\zeta \ell^a \ell^b = (xe^{\kappa t}) \frac{im}{\alpha} e^{im\phi} \frac{m\kappa(m - i\alpha)}{\alpha(\alpha + \beta)} \partial_\phi$$ \hspace{1cm} (4.21)

After evaluating \[ \Delta_\zeta \ell^a \ell^b \], we realized that the angular component of \[ \Delta_\zeta \ell^a \partial_a \] cancels, and the difference lies in the time component:

$$\Delta_\zeta \ell^a \partial_a = (xe^{\kappa t}) \frac{im}{\alpha} e^{im\phi} \frac{m(m(\alpha - \beta) + 2i\alpha\beta)}{2\alpha(\alpha + \beta)} \partial_t$$ \hspace{1cm} (4.22)

which we will use later.

Now let us look at the field variation of surface gravity δξκ. It needs to be analyzed with special care, as we have known, both the metric and null normal \[ \ell \] varies. Directly varying its definition \[ \ell^a \nabla_a \ell^b = \kappa \ell^b \], at the same time making use of the property that the quantity (4.19) commutes with covariant derivatives, we arrive at

$$\delta_\zeta \kappa = \left( \delta_\xi \ell^a \nabla_a \ell^b + \ell^a \nabla_a (\Delta_\xi \ell^b) + \ell^a \nabla_a \ell^b (\Delta_\xi \ell^b) - \kappa \delta_\xi \ell^b \right) \bar{\ell}_b$$ \hspace{1cm} (4.23)

It looks complicated at the first sight, but actually quite simple to evaluate as we have calculated all the quantities in each term.

4.3 Virasoro charges and the possible shift of zero modes

After properly treating the obstruction term (4.11), now we are ready to evaluate the Virasoro generators. The Hamiltonian \[ H_m \] which generates the large diffeomorphisms on the phase space (induced by the vector fields \[ \zeta_m \]) transforms phase space function \[ F \] in the following way:

$$\{ H_m, F \} = \delta_\zeta_m F$$ \hspace{1cm} (4.24)

where \[ F \] is any arbitrary phase space function. After evaluating (4.9), we find that the Hamiltonian density has the following form

$$H_m = (xe^{\kappa t}) \frac{im}{\alpha} e^{im\phi} \frac{(m - i\alpha)(m + i(N_\phi + 2\beta))}{16\pi G(\alpha + \beta)} \epsilon_S$$ \hspace{1cm} (4.25)

\[ \epsilon_S \] – 10 –
in which the area form is \( \epsilon_S = \sqrt{\bar{q}|\psi|} d\phi \wedge d\theta^A \) in our coordinate. When integrate the Hamiltonian density \( H_m \) on the crosssection \( S \), all higher mode \( m \neq 0 \) expectation value vanishes apart from the zero mode:

\[
H_0 = \frac{1}{8\pi G} \int_S \frac{\alpha(N_\phi + 2\beta)}{2(\alpha + \beta)} \epsilon_S = \frac{\alpha}{\alpha + \beta} \left( \frac{\beta A}{8\pi G} + J_H \right)
\]

(4.26)

where \( A = 2\pi \int d\theta^A \sqrt{\bar{q}|\psi|} \) is the area of bifurcation surface, the spin \( J_H = \frac{1}{8\pi G} \int d\theta^A \sqrt{\bar{q}|\psi|} N_\phi \) is the Noether charge of the rotational Killing vector. As the charges on the future horizon is a completely equivalent calculation, we will not provide the detail here. The vanishing of the integration on \( S \) doesn’t impact \( H_m \) to be well-defined generator, as we only need its spacetime derivative and field space variation to study the flow on the phase space. As we have known that sending a shock wave into the horizon will implant supertranslation hair and transform the spacetime into an inequivalent vacua \[26\], it will be interesting to study the implantation of Virasoro hair on the horizons by such process and its physical consequence, which we will leave for the future work.

By exploiting the closed ambiguity e.q.(2.11), one could also add non-flux boundary term to the symplectic potential and shift the zero mode. Such type of ambiguity appeared in the microcanonical Cardy formula was discussed in \[27\]. In the work of \[20\], the author proposed the substraction of a codimensional two corner term, then the symplectic potential enjoys the character that it only depends on the intrinsic and extrinsic geometry of the null hypersurface, also with this prescription, the divergence of the boundary charge \( dQ_\zeta \) matches with the sum of both matter and gravitational energy momentum flux. In our context, such type of corner term is

\[
\Theta_H \to \Theta_H - \frac{1}{2} \int_S (\delta \tilde{q} - \delta h) \iota_\ell \epsilon_H.
\]

(4.27)

With such an additional corner term, the full symplectic potential becomes a pure bulk term on the horizon. Such an additional corner term will shift the density of all the Virasoro charges. For the zero mode \( H_0 \to H_0 + \Delta H_0 \), it has a shift of

\[
\Delta H_0 = \frac{1}{8\pi G} \int_S \frac{\alpha N_\phi}{2(\alpha + \beta)} \epsilon_S = \frac{\alpha}{\alpha + \beta} J_H
\]

(4.28)

Subtracting a corner term of this nature shifts the zero mode, and it will shift the central charge as well. However, it will create an extra term in the obstruction e.q.(4.11) in the form of \( \Delta \zeta_\ell \delta \ell^a \iota_\alpha \epsilon_H \), hence one need to modify the Wald-Zoupus counterterm at the same time – which in turn shifts the central charge again. Whether this whole process of adding closed ambiguity would leave the initial result \( c_R = c_L = \frac{3\pi A}{2\pi G(\alpha + \beta)} \) unimpacted or not, we will leave to the future study.

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