Quark number densities at imaginary chemical potential
in $N_f = 2$ lattice QCD with Wilson fermions and its model analyses

Junichi Takahashi, Hiroaki Kouno, and Masanobu Yahiro

1Department of Physics, Graduate School of Sciences, Kyushu University, Fukuoka 812-8581, Japan
2Department of Physics, Saga University, Saga 840-8502, Japan
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We investigate the chemical-potential ($\mu$) dependence of quark number densities ($n_q$) at both imaginary and real $\mu$ by using $N_f = 2$ lattice QCD and the hadron resonance gas (HRG) model. Quark number densities are first calculated at imaginary $\mu$ with lattice QCD on an $8^3 \times 16 \times 4$ lattice with the clover-improved $N_f = 2$ Wilson fermion action and the renormalization-group-improved Iwasaki gauge action. The present results are consistent with the previous results of the staggered-type quark action. The $n_q$ thus obtained are extrapolated to real $\mu$ by assuming the Fourier series for the confinement region and the polynomial series for the deconfinement region. The extrapolated results are also consistent with the previous results of the Taylor expansion method for the reweighting factor. The upper bound $\langle \mu/T \rangle_{\text{max}}$ of the reliable extrapolation is estimated for each temperature $T$. We deduce nucleon and $\Delta$-resonance masses in the vicinity of the pseudocritical temperature ($T_c$) with the HRG model from LQCD data on $n_q$ at imaginary $\mu$. The resulting nucleon and $\Delta$-resonance masses reduce by about 10% as $T$ increases from $T = 0.93T_c$ to $0.99T_c$.

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I. INTRODUCTION

There are many interesting topics on quantum chromodynamics (QCD) at high density. The observation [1] of a two-solar-mass neutron star has an impact on the equation of state (EoS) of dense matter and the QCD phase diagram at high density. The experiments of relativistic heavy-ion collisions, for example the Beam Energy Scan experiments, are exploring QCD not only at finite temperature $T$ but also at finite quark-chemical potential $\mu$ [2,3]. Lattice QCD (LQCD) is the first-principle calculation to study QCD, but has the serious sign problem at finite $\mu$.

In LQCD, the fermion determinant $\det M(\mu/T)$ becomes complex for finite real $\mu$, because

$$ (\det M(\mu/T))^* = \det M(-\mu^*/T) = \det M(-\mu/T). \tag{1} $$

This interferes with the use of Monte-Carlo simulations based on the importance sampling. For this reason, several methods were proposed so far in order to avoid the sign problem [4,5]. Very recently, the complex Langevin method [6,9] and the Lefschetz thimble theory [10,11] have attracted much attention as the method to go beyond $\mu/T = 1$.

One of the methods to avoid the sign problem is the imaginary-$\mu$ approach. For pure imaginary chemical potential $\mu = i\mu_1$, it is convenient to introduce the dimensionless chemical potential $\theta = \mu_1/T$. The first equality of Eq. (1) shows that the fermion determinant $\det M(i\theta)$ is real for imaginary $\mu$. This makes LQCD simulations feasible there. Observables at real $\mu$ are extracted from those at imaginary $\mu$ by assuming functional forms for the observables.

In the imaginary $\mu$ region, QCD has two characteristic properties: one is the Roberge-Weiss (RW) periodicity and the other is the RW phase transition [12]. Figure 1 shows a schematic graph for the QCD phase diagram in the $T$–$\theta$ plane. The QCD grand partition function $Z(\theta)$ has a periodicity of $2\pi/N_c$ in $\theta$:

$$ Z(\theta) = Z(\theta + 2\pi k/N_c), \tag{2} $$

where $N_c$ is the number of colors and $k = 1, \cdots, N_c$. This is a remnant of $Z_{N_c}$ symmetry in pure gauge theory and is now called the RW periodicity. Meanwhile, the RW transition is the first-order phase transition appearing at $T$ higher than some temperature $T_{\text{RW}}$ and $\theta = \pi/N_c$. This transition line and its $Z_{N_c}$ images are plotted by the solid lines in Fig. 1, whereas the confinement/deconfinement crossover is drawn by the dashed lines. The point located at $(T, \theta) = (T_{\text{RW}}, \pi/N_c)$ is called the RW endpoint. The critical temperature $T_{\text{RW}}$ is larger than the pseudocritical temperature $T_c$ of the deconfinement crossover at zero $\mu$ [13]; $T_{\text{RW}}$ is located between $1.08T_c$ and $1.20T_c$, as shown later. The order parameter of the RW transition is a $C$-odd quantity such as the phase of the Polyakov loop or the quark-number density [14], where $C$ means charge conjugation. The existence of the RW transition and the RW periodicity is numerically confirmed with LQCD simulations [15–20] and the underlying mechanism is clearly understood with the effective model [14,21,22] by introducing a new concept of extended $Z_{N_c}$ symmetry.

The quark number density $n_q$ is a fundamental quantity to study high-density physics and important in determining the EoS at finite real $\mu$. The EoS plays an essential role in investigating the structure of neutron stars. Moreover, $n_q$ is useful to determine the strength of vector-type interaction in the effective model [23,24]. For small real $\mu/T$, the quark number density was calculated with the Taylor expansion method for the reweighting factor in which either the staggered-type [25] or the Wilson-type quark action [26] is
taken. The quark number density is also computed at imaginary $\mu$ in Refs. \[17, 18, 27, 28\] with the staggered-type quark action, and $n_q$ at real $\mu$ is deduced from that at imaginary $\mu$ by assuming analytic forms for $n_q$.

In this paper, we investigate the $\mu$ dependence of $n_q$ at both imaginary and real $\mu$. We first perform LQCD simulations at imaginary $\mu$ with the Wilson-type quark action, since the quark number density was never calculated with the Wilson-type quark action at imaginary $\mu$. LQCD simulations at imaginary $\mu$ do not require any special prescription in numerical calculations, since there is no sign problem there. The $n_q$ thus obtained at imaginary $\mu$ are extrapolated to the real $\mu$ region by assuming functional forms for $n_q$. This analytic-continuation is confirmed to be reliable by comparing the extrapolated results with the previous results \[26\] based on the Taylor expansion method for the reweighting factor. The upper bound $(\mu/T)_{\text{max}}$ of the reliable extrapolation is estimated for each $T$.

The hadron resonance gas (HRG) model is reliable in the confinement region. For the the 2+1 flavor case with no $\mu$, in fact, it is shown that the model well reproduces LQCD data on pressure at $T < 1.2T_c$ \[29\]. As shown in Fig. 1 the pseudocritical temperature of deconfinement transition goes down from $T = T_{\text{RW}}$ to $T_c$ as $(\mu/T)^2$ increases from $-(\pi/N_c)^2$. When we are interested in the vicinity of $T_c$, the HRG model is more reliable at imaginary $\mu$ than at real $\mu$. Using this property, we deduce the $T$ dependence of nucleon and $\Delta$-resonance masses in the vicinity of $T_c$ with the HRG model from LQCD data on $n_q$ at imaginary $\mu$.

Actual LQCD simulations are done on an $8^2 \times 16 \times 4$ lattice with the clover-improved two-flavor Wilson fermion action and the renormalization-group-improved Iwasaki gauge action. We consider two temperatures $T/T_c = 0.93$ and 0.99 in the confinement region and four temperatures $T/T_c = 1.08$, 1.20, 1.35 and 2.07 in the deconfinement region. Following the previous LQCD simulation \[26\], we compute $n_q$ along the line of constant physics at $m_{PS}/m_V = 0.80$, where $m_{PS}$ and $m_V$ are pseudoscalar- and vector-meson masses, respectively. This corresponds to the case of the pion mass $m_\pi \sim 616$ MeV and the quark mass $m_q \sim 130$ MeV \[24\] for $T_c \sim 171$ MeV \[30\]. The analytic continuation is carried out with the Fourier series for $T < T_c$ and the polynomial series for $T > T_{\text{RW}}$.

This paper is organized as follows. In Sec. II, we explain the lattice action, the quark number density and analytic continuation. In Sec. III, we show our simulation parameters and numerical results for $n_q$ at both imaginary and real $\mu$. In Sec. IV, we deduce nucleon and $\Delta$-resonance masses from $n_q$ at imaginary $\mu$ by using the HRG model. Section V is devoted to a summary.

II. FORMULATION

A. Lattice action

We use the renormalization-group-improved Iwasaki gauge action $S_g$ \[31\] and the clover-improved two-flavor Wilson quark action $S_q$ \[32\] defined by

\[
S = S_g + S_q, \tag{3}
\]

\[
S_g = -\beta \sum_x \left( c_0 \sum_{\mu<\nu} W^{1\times1}_{\mu\nu}(x) + c_1 \sum_{\mu\neq\nu} W^{1\times2}_{\mu\nu}(x) \right), \tag{4}
\]

\[
S_q = \sum_f \sum_{x,y} \bar{\psi}_f(x) M_{f,x,y} \psi_f(y), \tag{5}
\]

where $\beta = 6/g^2$ for the gauge coupling $g$, $c_1 = -0.331$, $c_0 = 1 - 8c_1$, and

\[
M_{x,y} =
\]

\[
\delta_{xy} - \kappa \sum_{i=1}^3 \{(1 - \gamma_i) U_{x,i} \delta_{x+1,y} + (1 + \gamma_i) U_{y,i}^\dagger \delta_{x,y+1} + (1 - \gamma_i) U_{y,i} \delta_{x+1,y} + (1 + \gamma_i) U_{x,i}^\dagger \delta_{x,y+1} \}
\]

\[
- \kappa \{ e^{\tilde{\mu}} (1 - \gamma_4) U_{x,4} \delta_{x+4,y} + e^{-\tilde{\mu}} (1 + \gamma_4) U_{y,4}^\dagger \delta_{x,y+4} \}
\]

\[
- \delta_{xy} c_{\text{SW}} \kappa \sum_{\mu<\nu} \sigma_{\mu\nu} F_{\mu\nu}. \tag{6}
\]

Here $\kappa$ is the hopping parameter, $\tilde{\mu}$ is the quark chemical potential in lattice units, and the lattice field strength $F_{\mu\nu}$ is defined as $F_{\mu\nu} = (f_{\mu\nu} - f_{\mu}f_{\nu})/(8t)$ with $f_{\mu\nu}$ the standard clover-shaped combination of gauge links. For the clover coefficient $c_{\text{SW}}$, we take the mean-field value estimated from $W^{1\times1}$ in the one-loop level: $c_{\text{SW}} = (W^{1\times1})^{-3/4} = (1 - 0.8412/3^{-1})^{-3/4} \[31\]$. The value of $\kappa$ is determined at $\mu = 0$ for each $\beta$ along the line of constant physics with $m_{PS}/m_V = 0.80$ \[30, 33, 34\].

![Fig. 1: QCD phase diagram in the imaginary $\mu$ region. The solid and dashed lines stand for the RW phase transition and the deconfinement crossover, respectively.](image)
B. Quark number density

The quark number density \( n_q \) is defined as

\[
\frac{n_q}{T^3} = \frac{1}{V T^2} \frac{\partial}{\partial \mu} \ln Z
\]

\[= \frac{N_f N_s^2}{N_V} \text{tr} \left[ M^{-1} \frac{\partial M}{\partial \mu} \right], \tag{8}
\]

where \( V \) is the volume, \( N_f \) is the number of flavors, \( N_s \) is the temporal lattice size, \( N_V \) is the lattice volume and \( M \) is the fermion matrix. We apply the random-noise method for the trace in Eq. (8). The number of noise vectors is about 4,000. The partition function \( Z \) is \( \mu \)-even (\( C \)-even), so that \( n_q \) is \( \mu \)-odd (\( C \)-odd) from Eq. (7). This means that \( n_q \) is pure imaginary for imaginary \( \mu \); actually,

\[
n_q^* = \left( \frac{1}{V} \frac{\partial \ln Z}{\partial (i\theta)} \right)^* = -\frac{1}{V} \frac{\partial \ln Z}{\partial (-i\theta)} = -n_q. \tag{9}
\]

We have confirmed in our LQCD simulations that the real part of \( n_q \) is zero at imaginary \( \mu \). For later convenience, we represent the imaginary part of \( n_q \) by \( n_q^* \); \( n_q^* = \Im(n_q) \).

C. Analytic continuation

Our final interest is \( n_q \) at real \( \mu \). We then extrapolate the \( n_q \) calculated at imaginary \( \mu \) with LQCD to the real \( \mu \) region, assuming some functional forms for \( n_q \).

In the imaginary-\( \mu \) region, \( n_q \) is a \( \theta \)-odd function with the RW periodicity. We then consider only a period \(-\pi/3 < \theta < \pi/3 \) for simplicity. In the confinement region at \( T < T_c \), the quark number density is smooth for any \( \theta \), indicating that \( n_q = 0 \) at \( \theta = 0, \pm \pi/3 \) [14, 21, 22]. Hence \( n_q \) can be described with good accuracy by a partial sum \( S^n_p(T, \theta) \) of the Fourier series:

\[
\frac{n_q(T, i\theta)}{T^3} \approx iS^n_p(T, \theta) = i \sum_{k=1}^{n} a_F^{(k)}(T) \sin(3k\theta). \tag{10}
\]

where the superscript \( n \) of \( S^n_p(T, \theta) \) represents the highest order in the partial sum. The coefficients \( a_F^{(k)}(T) \) are obtained by fitting the function (10) to LQCD data at imaginary \( \mu = i\mu_1 \). The analytic continuation from \( \mu = i\mu_1 \) to \( \mu = \mu_R \) can be made by replacing \( i\mu_1/T \) by \( \mu_R/T \) in Eq. (10):

\[
\frac{n_q(T, \mu_R/T)}{T^3} \approx g^n_p \left( T, \frac{\mu_R}{T} \right)
\]

\[= \sum_{k=1}^{n} a_F^{(k)}(T) \sinh(3k\frac{\mu_R}{T}). \tag{11}
\]

Here note that the coefficients \( a_F^{(k)}(T) \) have already been determined at imaginary \( \mu \).

In the region \( T_c < T < T_{RW} \), the system is in the deconfinement region for small \( \theta \) but in the confinement region for large \( \theta \) near \( \pi/3 \), as shown in Fig. [1]. Because of this property, the \( \theta \) dependence of \( n_q \) is complicated and makes the analytic continuation difficult. We then do not perform the analytic continuation in this region.

In the deconfinement region at \( T > T_{RW} \), the quark number density is discontinuous at \( \theta = \pm \pi/3 \) where the first-order RW phase transition takes place; note that \( n_q \) is the order parameter of the RW first-order transition [14]. Owing to this property, \( n_q \) monotonically increases with \( \theta \), as shown later in Fig. [2]. This suggests that \( n_q \) can be described with good accuracy by a partial sum \( S^n_p(T, \theta) \) of the polynomial series:

\[
\frac{n_q(T, i\theta)}{T^3} \approx iS^n_p(T, \theta) = i \sum_{k=1}^{n} a_p^{(2k-1)}(T) \theta^{2k-1}, \tag{12}
\]

where the superscript \( n \) of \( S^n_p(T, \theta) \) represents the highest order in the partial sum. Again, the analytic continuation is made by replacing \( i\mu_1/T \) by \( \mu_R/T \) in Eq. (12):

\[
\frac{n_q(T, \mu_R/T)}{T^3} \approx g^n_p \left( T, \frac{\mu_R}{T} \right)
\]

\[= \sum_{k=1}^{n} (-1)^{(k-1)} a_p^{(2k-1)}(T) \left( \frac{\mu_R}{T} \right)^{2k-1}. \tag{13}
\]

III. NUMERICAL RESULTS

Full QCD configurations with \( N_f = 2 \) dynamical quarks were generated with the hybrid Monte-Carlo algorithm on a lattice of \( N_x \times N_y \times N_z \times N_t = 8^3 \times 16 \times 4 \). The step size of the molecular dynamics is \( \delta t = 0.02 \) and the step number is \( N_t = 50 \). The acceptance ratio is more than 95%. We generated about 32,000 trajectories and removed the first 4,000 trajectories for the thermalization of all the parameters, and measured \( n_q \) at every 100 trajectories. The relation of parameters \( \kappa \) and \( \beta \) to the corresponding \( T/T_c \) was determined in Refs. [30, 33, 34]; see Table I for the relation.

| \( \kappa \) | \( \beta \) | \( T/T_c \) |
|---|---|---|
| 0.141139 | 1.80 | 0.93(5) |
| 0.140070 | 1.85 | 0.99(5) |
| 0.138817 | 1.90 | 1.08(5) |
| 0.137716 | 1.95 | 1.20(6) |
| 0.136931 | 2.00 | 1.35(7) |
| 0.135010 | 2.20 | 2.07(10) |

TABLE I: Summary of the simulation parameter sets determined in Refs. [30, 33, 34]. Here, \( T_c \) is the pseudocritical temperature of deconfinement transition at \( \mu = 0 \). In the parameter setting, the lattice spacing \( a \) is about 0.14 ~ 0.2 fm.
A. Quark number density at imaginary $\mu$

Figure 2 shows $n_q^1/T^3$ as a function of $\theta$ for all the temperatures we consider. The LQCD data are plotted by symbols with error bars, although the error bars are quite small. The number density $n_q^1$ should be zero at $\theta = \pi/3$ below $T_{\text{RW}}$ but finite above $T_{\text{RW}}$, since $n_q^1$ is the order parameter of the first-order RW phase transition. One can see from the fact that $T_{\text{RW}}$ is located between $1.08T_c$ and $1.2T_c$. The quark number density $n_q^1/T^3$ behaves as the sine function for $T < T_c$, but monotonically increases up to $\theta = \pi/3$ for $T > T_{\text{RW}}$. As for $T = 1.08T_c$, the system is in the deconfinement region for $\theta < 0.8$ but in the confinement region for $0.8 < \theta < \pi/3$, since $n_q^1/T^3$ increases monotonically up to $\theta \sim 0.7$ but decreases to zero for $\theta > 0.8$. This result is consistent with that of the staggered-type fermion in Ref. [28] The present result is thus independent of the fermion action taken.

![Figure 2: $\mu/T$ dependence of $n_q^1/T^3$ at various values of $T$. The LQCD data are shown by symbols with error bars, although the error bars are quite small.](image)

First we consider the case of $T < T_c$ and determine the coefficients $a_F^{(k)}(T)$ of the Fourier series from the $n_q$ calculated at imaginary $\mu$ with LQCD. In Fig. 3 the results of the $\chi^2$ fittings (curves) are compared with the LQCD results (symbols with error bars). Two cases of $S_p^1$ and $S_p^2$ are plotted by dashed and solid curves, respectively. The two results well reproduce the LQCD data. The coefficients thus obtained are tabulated in Table III together with the $\chi^2$ per degree of freedom (d.o.f). As for $T = 0.93T_c$, the fitting quality is almost same between the two cases of $S_p^1$ and $S_p^2$, as shown by the $\chi^2$/d.o.f. Furthermore, $a_F^{(2)}$ is much smaller than $a_F^{(1)}$ in the case of $S_p^2$. These results show that the coefficients $a_F^{(1)}$ and $a_F^{(2)}$ are determined with high accuracy in the case of $S_p^2$. Also for $T = 0.99T_c$, two cases of $S_p^1$ and $S_p^2$ are shown in Table III. The deviation of LQCD data from $S_p^2$ has a $\theta$ dependence different from the next-order term $\sin(6\theta)$, so we stopped the $\chi^2$ fitting with $S_p^2$.

Next we consider the case of $T > T_{\text{RW}}$ and determine the coefficients $a_F^{(2k-1)}(T)$ of the polynomial series. In Fig. 4 the fitting results (curves) are compared with LQCD data (symbols with error bars) for the case of $S_p^3$ in panel (a) and for the case of $S_p^5$ in panel (b). For each temperature, the solid line denotes the $\chi^2$-minimum result, whereas two dotted lines show the upper and lower bounds of the $\chi^2$ fitting. The resulting coefficients $a_F^{(2k-1)}(T)$ are tabulated in Table III for two cases of $S_p^3$ and $S_p^5$.

![Figure 3: Results of $\chi^2$ fitting to LQCD data for the case of $T < T_c$. The results of $S_p^1$ and $S_p^2$ are plotted by dashed and solid lines, respectively. LQCD data are shown by symbols with error bars.](image)

| $T/T_c$ | $a_F^{(1)}$ | $a_F^{(2)}$ | $\chi^2$/d.o.f | $\mu/T$ (fitting range) |
|---------|------------|------------|--------------------|-------------------------|
| 0.93    | 0.250(2)   |            | 5.937              | 0 $\sim$ $\pi/3$       |
| 0.93    | 0.251(2)   | $-0.00457(216)$ | 6.084              | 0 $\sim$ $\pi/3$       |
| 0.99    | 0.718(2)   |            | 11.06              | 0 $\sim$ $\pi/3$       |
| 0.99    | 0.728(3)   | $-0.0179(26)$ | 7.453              | 0 $\sim$ $\pi/3$       |

TABLE II: Coefficients of the Fourier series for $S_p^1$ and $S_p^2$.
Results of the $\chi^2$ fitting to LQCD data for the case of $T > T_{\text{RW}}$. Panels (a) and (b) show the results of $S^3_p$ and $S^5_p$, respectively. For each temperature, the solid line represents the $\chi^2$-minimum result, whereas two dotted lines correspond to the upper and lower bounds of the $\chi^2$ fitting. LQCD data are shown by symbols with error bars.

| $T/T_c$ | $a_p^{(1)}$ | $a_p^{(3)}$ | $a_p^{(5)}$ | $\chi^2$/d.o.f | $\mu/T$ (fitting range) |
|---------|-------------|-------------|-------------|----------------|-------------------------|
| 1.20    | 4.437(4)    | -1.214(7)   |             | 13.66         | 0 ~ 1                  |
| 1.20    | 4.407(5)    | -1.024(27)  | -0.1935(260)| 8.472         | 0 ~ 1                  |
| 1.35    | 4.675(3)    | -0.9973(49) |             | 6.036         | 0 ~ 1                  |
| 1.35    | 4.662(5)    | -0.9223(233)| -0.06736(1956) | 5.308     | 0 ~ 1                  |
| 2.07    | 5.174(2)    | -0.8904(40) |             | 9.161         | 0 ~ 1                  |
| 2.07    | 5.177(4)    | -0.9056(177)| 0.01356(1531)| 10.21      | 0 ~ 1                  |

Table III: Coefficients of the polynomial series for $S^3_p$ and $S^5_p$.

B. Quark number density at real $\mu$

First we consider the case of $T/T_c < 1$. Figure 5 shows the $\mu/T$ dependence of $n_q/T^3$ for $T = 0.99T_c$. As the extrapolation from imaginary $\mu$ to real $\mu$, we consider two cases of $g_F^2$ and $g_F^2$. For each case, the solid line represents the result calculated from the mean value of $a_p^{(k)}$, whereas two dotted lines correspond to the results from the upper and lower bounds of $a_p^{(k)}$. The $g_F^2$ case has a larger error than the $g_F^2$ case, because the former error comes from both of $a_p^{(1)}$ and $a_p^{(2)}$ but the latter error does from $a_p^{(1)}$ only. The result of $g_F^2$ well reproduces the previous LQCD result [26] (symbols with error bars) based on the Taylor expansion method for the reweighting factor in which $g_F^2$ is assumed for $n_q/T^3$ but the coefficients $a_p^{(k)}$ are calculated at $\mu = 0$. The previous result thus has contributions up to $(\mu/T)^3$. Meanwhile, the present result of $g_F^2$ includes the higher-order contributions in addition to contributions up to $(\mu/T)^3$. When the higher-order contributions are switched off in the result of $g_F^2$, we can get the dot-dashed line together with two dotted lines calculated from the upper and lower bounds of $a_p^{(k)}$. The dot-dashed line agrees with the previous result based on the Taylor expansion method for the reweighting factor. The difference between the dot-dashed line and the result of $g_F^2$ thus represents higher-order corrections to the previous result. From the fact that the difference is small at $\mu/T < 0.9$, we can conclude that both the previous result based on the Taylor expansion method for the reweighting factor and the present result of $g_F^2$ are reliable at $\mu/T < 0.9$.

Figure 6 shows the $\mu/T$ dependence of $n_q/T^3$ at $T = 0.93T_c$. The definition of lines is the same as in Fig. 5. The difference between two results of $g_F^2$ and $g_F^2$ reduces as $T$ decreases from 0.99Tc to 0.93Tc. The extrapolation of $g_F^2$ and $g_F^2$ thus becomes more reliable as $T$ decreases. As mentioned above, the $g_F^2$ extrapolation is reliable at $T = 0.99T_c$ in the range $\mu/T < 0.9$. This means that the $g_F^2$ extrapolation is reliable also at $T = 0.93T_c$ at least in the range $\mu/T < 0.9$.
Fig. 5: \( \mu/T \) dependence of \( n_q/T^3 \) at \( T = 0.99T_c \). Two cases of \( g_F^1 \) and \( g_F^2 \) are plotted. For each case, the solid line stands for the result calculated from the mean value of \( a_p^{(k)} \), whereas two dashed lines denote the results from the upper and lower bounds of \( a_p^{(k)} \). In the dot-dashed line, higher-order contributions than \((\mu/T)^3\) are switched off in \( g_F^2 \). The symbols denote LQCD results of the Taylor expansion method for the reweighting factor in Ref. [26].

Figures 6, 7, and 8 show the \( \mu/T \) dependence of \( n_q/T^3 \) at \( T = 1.20T_c, 1.35T_c \), and \( 2.07T_c \), respectively. Two cases of \( g_p^3 \) and \( g_p^5 \) are plotted. For each case, the solid line represents the result calculated from the mean value of \( a_p^{(k)} \), whereas two dotted lines correspond to the results from the upper and lower bounds of \( a_p^{(k)} \). For each temperature, the present result of \( g_p^3 \) agrees with the previous LQCD result [26] based on the Taylor expansion method for the reweighting factor, because the previous study assumes \( g_p^3 \) for \( n_q/T^3 \). The difference between \( g_p^3 \) and \( g_p^5 \) becomes small in the range \( \mu/T < 1 \) as \( T \) increases. Higher-order contributions than \((\mu/T)^3\) thus become less important as \( T \) increases.

The relative difference between two results of \( g_p^3 \) and \( g_p^5 \) exceeds 10% at \( \mu/T \approx 0.72 \) for \( T = 1.20T_c \), 1.2 for \( T = 1.35T_c \) and 2.6 for \( T = 2.07T_c \). The upper bound \((\mu/T)_{\max}\) of the reliable extrapolation is plotted as a function of \( T/T_c \) in Fig. 10. The upper bound goes up as \( T \) increases, indicating that higher-order contributions than \((\mu/T)^3\) become less important. The \( g_p^3 \) extrapolation, i.e., the previous result [26] based on the Taylor expansion method for the reweighting factor thus becomes more reliable as \( T \) increases.

Fig. 6: \( \mu/T \) dependence of \( n_q/T^3 \) at \( T = 0.93T_c \). See Fig. 5 for the definition of lines. Two cases of \( g_F^1 \) and \( g_F^2 \) are plotted.

Fig. 7: \( \mu/T \) dependence of \( n_q/T^3 \) at \( T = 1.20T_c \). Two cases of \( g_p^3 \) and \( g_p^5 \) are plotted. For each case, the solid line represents the result calculated from the mean value of \( a_p^{(k)} \), whereas two dashed lines denote the results from the upper and lower bounds of \( a_p^{(k)} \). The symbols denote LQCD results of the Taylor expansion method for the reweighting factor in Ref. [26].

Fig. 8: \( \mu/T \) dependence of \( n_q/T^3 \) at \( T = 1.35T_c \). Two cases of \( g_p^3 \) and \( g_p^5 \) are plotted. See Fig. 7 for the definition of lines. The symbols denotes LQCD results of the Taylor expansion method for the reweighting factor in Ref. [26].
Fig. 9: $\mu/T$ dependence of $n_i/T^3$ at $T = 2.07T_c$. Two cases of $g_p^3$ and $g_n^3$ are plotted. See Fig. 5 for the definition of lines. The symbols denotes LQCD results of the Taylor expansion method for the reweighting factor in Ref. [26].

Fig. 10: $T$ dependence of the upper bound $(\mu/T)_{\text{max}}$ of the reliable extrapolation for the case of $T > T_{\text{EW}}$.

IV. HADRON RESONANCE GAS MODEL

Now we consider the confinement region at $T < T_c$ for the case of imaginary $\mu$, and analyze the present LQCD results with the hadron resonance gas (HRG) model in order to understand properties of $n_q$ and deduce the $T$ dependence of baryon mass in the vicinity of $T_c$. The HRG model considers non-interacting hadrons and resonances, each classified with species $i$, i.e., with mass $m_i$, baryon number $B_i$ and isospin $I_{3i}$. For the 2+1 flavor case at zero chemical potential, the HRG model well reproduces LQCD data on pressure at $T < 1.2T_c$ [29]. This means that the HRG model is applicable at $T < T_c$ and imaginary $\mu$, because the pseudocritical temperature of deconfinement transition becomes higher as $\mu_1$ increases. The pressure of the model is obtained by

$$p^{HRG} = \frac{T}{V} \sum_{i} \ln Z_i^M(T, V, \mu_i) - \frac{T}{V} \sum_{i} \ln Z_i^B(T, V, \mu_i)$$

with

$$\ln Z_i^{M/B} = \pm \frac{V g_i}{2\pi^2} \int_0^\infty dp p^2 \ln \left(1 \mp z_i e^{-\epsilon_i/T}\right)$$

for the energy $\epsilon_i = \sqrt{p^2 + m_i}$, the degeneracy factor $g_i$ and the fugacity

$$z_i = e^{\mu_i/T} = \exp \left(\frac{B_i \mu_B + 2I_{3i} \mu_{iso}}{T}\right),$$

where $\mu_B (= 3\mu)$ and $\mu_{iso}$ are the baryon and isospin chemical potentials, respectively. Here we consider only the case of $\mu_{iso} = 0$. The baryon number density is easily obtained as

$$n_B^{HRG} = -\frac{\partial}{\partial \mu_B} p^{HRG}.$$  

There are lattice artifacts in LQCD simulations. These were already discussed in Refs. [25, 24, 35, 37]. For small $N_f$, for example, thermodynamic quantities exceed the Stefan-Boltzmann (SB) limit. Since it is not easy to eliminate the lattice artifact exactly, we take the following simple prescription. We consider the lattice SB limit that is defined by the lattice action with massless and free quarks, and normalize LQCD results with the values in the lattice SB limit. We regard these normalized values as the lattice-artifact free values; see Appendix A for the quark number density in the lattice SB limit. For the HRG model, meanwhile, the quark number density is normalized by the value in the continuum SB limit. In the HRG model, we assume that nucleon mass $m_N$ and $\Delta$-resonance mass $m_\Delta$ depend on $T$ only. This assumption works well in the $\chi^2$ fitting of $m_N$ and $m_\Delta$ to LQCD data on $n_q$ at imaginary $\mu$, as shown later in Fig 11. The baryon masses are determined so as to reproduce the LQCD result. Here we assume that the masses of 24 resonance states above the mass threshold $m_{cut} = 1.8$ GeV, following Ref. [35]. But the contribution of 24 states to $n_q$ is small.

Figure 11 shows the $T$ dependence of the normalized quark number density $n_q/n_{SB}$ at $T = 0.93T_c$ and $0.99T_c$. The HRG model (solid line) reproduces the $T$ dependence of LQCD results (symbols with error bars) for both cases of $T = 0.93T_c$ and $0.99T_c$. This means that $m_N$ and $m_\Delta$ little depend on $T$. The resulting nucleon and $\Delta$-resonance masses are summarized in Table IV. The resulting masses are heavier than the corresponding physical values, because the quark mass is much heavier than the physical value in our simulations. As shown in Table IV both $m_N$ and $m_\Delta$ decrease by about 10% as $T$ increases from $0.93T_c$ to $0.99T_c$.

Finally we consider the case of real $\mu$. The HRG model becomes less reliable as $\mu/T$ increases, because the pseudocritical temperature of deconfinement transition goes down from...
The HRG model is thus reliable only at the range, the HRG model overestimates the LQCD results. Beyond the Taylor expansion method for the reweighting factor in Ref. [26], the HRG model result (solid line) is consistent with the previous result of $\mu/T < 0.9$. The present analytic continuation based on $g_P^2$ is thus reliable at $\mu/T < 0.9$ for $T = 0.99T_c$. Furthermore, the difference between the two results of $g_P^2$ and $g^2_q$ reduces as $T$ decreases from $T_c$, indicating that higher-order contributions become less important as $T$ decreases. Therefore, the present extrapolation based on $g_P^2$ is reliable at $\mu/T < 0.9$ for any $T$ less than $T_c$.

For $T > T_{RW}$, the previous study based on the Taylor expansion method for the reweighting factor has contributions up to $(\mu/T)^3$, but the present $g_P^5$ extrapolation retains contributions up to $(\mu/T)^5$. Using this advantage of the present method from the previous method, we have estimated to what extent the Taylor expansion or the $g_P^2$ extrapolation works. The upper bound $(\mu/T)_{\text{max}}$ of the reliable extrapolation goes up as $T$ increases from $T_{RW}$, because higher-order contributions become less important.

The HRG model is reliable in the confinement region. For the 2+1 flavor case at zero chemical potential, in fact, the HRG model well reproduces LQCD data on pressure at $T < 1.2T_c$. [29]. The pseudocritical temperature of deconfinement transition goes down from $T = T_{RW}$ as $(\mu/T)^2$ increases from $-(\pi/3)^2$ to plus values. This means that the HRG model is more reliable at imaginary $\mu$ than real $\mu$ in the vicinity of $T_c$. Using this property, we have deduced the $T$ dependence of $m_N$ and $m_\Delta$ with the HRG model from the $n_q$ calculated with LQCD simulations at imaginary $\mu$. The HRG model well reproduces the LQCD results, when $m_N$ and $m_\Delta$ are assumed to depend on $T$ only. This indicates that $m_N$ and $m_\Delta$ little depend on $\theta (= \mu_1/T)$. For the $T$ dependence, $m_N$ and $m_\Delta$ decrease by about 10% when $T$ increases from 0.93$T_c$ to 0.99$T_c$. This is a handy way of deducing the $T$ dependence of $m_N$ and $m_\Delta$ in the vicinity of $T_c$. This method is quite practical, since it is not easy to measure the $T$ dependence of the pole masses of nucleon and $\Delta$-resonance directly with LQCD simulations.

**TABLE IV:** $m_N$ and $m_\Delta$ at $T = 0.93T_c$ and 0.99$T_c$.

| $T/T_c$ | $m_N$ [MeV] | $m_\Delta$ [MeV] |
|---------|-------------|------------------|
| 0.93    | 1091        | 1547             |
| 0.99    | 940         | 1385             |

Fig. 11: $\theta$ dependence of $n_q/n_{SB}$ at $T = 0.93T_c$ and 0.99$T_c$. The cross and square symbols with error bars represent the LQCD results at $T = 0.93T_c$ and 0.99$T_c$, respectively, whereas the solid lines stand for the HRG model results.

Fig. 12: $\mu/T$ dependence of $n_q/n_{SB}$ in the real $\mu$ region for $T = 0.99T_c$. The symbols with error bars stand for LQCD results of the Taylor expansion method for the reweighting factor in Ref. [26], while the solid line is the result of the HRG model.

V. SUMMARY

We have investigated the $\mu$ dependence of $n_q$ at imaginary and real $\mu$, performing LQCD simulations at imaginary $\mu$ and extrapolating the results to the real-$\mu$ region by assuming functional forms for $n_q$. LQCD calculations were done on an $8^3 \times 16 \times 4$ lattice with the clover-improved two-flavor Wilson fermion action and the renormalization-group-improved Iwasaki gauge action. We considered two temperatures below $T_c$ and four temperatures above $T_c$. The quark number density was computed along the line of constant physics at $m_{PS}/m_N = 0.80$.

For imaginary $\mu$, the quark number density calculated with the Wilson-type fermion action is consistent with the previous result [28] based on the staggered-type fermion action. The LQCD results thus do not depend on the fermion action taken.

We have extrapolated $n_q$ at imaginary $\mu$ to real $\mu$, assuming the Fourier series $g_P^2$ at $T < T_c$ and the polynomial series $g_P^n$ at $T > T_{RW}$. Here the superscript $n$ denotes the highest order taken in the partial sum. As for $T = 0.99T_c$, the present result of $g_P^2$ is consistent with the previous result [26] based on the Taylor expansion method for the reweighting factor in the range $\mu/T < 0.9$. The present analytic continuation based on $g_P^2$ is thus reliable at $\mu/T < 0.9$ for $T = 0.99T_c$. Furthermore, the difference between the two results of $g_P^2$ and $g^2_q$ reduces as $T$ decreases from $T_c$, indicating that higher-order contributions become less important as $T$ decreases. Therefore, the present extrapolation based on $g_P^2$ is reliable at $\mu/T < 0.9$ for any $T$ less than $T_c$.

For $T > T_{RW}$, the previous study based on the Taylor expansion method for the reweighting factor has contributions up to $(\mu/T)^3$, but the present $g_P^5$ extrapolation retains contributions up to $(\mu/T)^5$. Using this advantage of the present method from the previous method, we have estimated to what extent the Taylor expansion or the $g_P^2$ extrapolation works. The upper bound $(\mu/T)_{\text{max}}$ of the reliable extrapolation goes up as $T$ increases from $T_{RW}$, because higher-order contributions become less important.

The HRG model is reliable in the confinement region. For the 2+1 flavor case at zero chemical potential, in fact, the HRG model well reproduces LQCD data on pressure at $T < 1.2T_c$. [29]. The pseudocritical temperature of deconfinement transition goes down from $T = T_{RW}$ as $(\mu/T)^2$ increases from $-(\pi/3)^2$ to plus values. This means that the HRG model is more reliable at imaginary $\mu$ than real $\mu$ in the vicinity of $T_c$. Using this property, we have deduced the $T$ dependence of $m_N$ and $m_\Delta$ with the HRG model from the $n_q$ calculated with LQCD simulations at imaginary $\mu$. The HRG model well reproduces the LQCD results, when $m_N$ and $m_\Delta$ are assumed to depend on $T$ only. This indicates that $m_N$ and $m_\Delta$ little depend on $\theta (= \mu_1/T)$. For the $T$ dependence, $m_N$ and $m_\Delta$ decrease by about 10% when $T$ increases from 0.93$T_c$ to 0.99$T_c$. This is a handy way of deducing the $T$ dependence of $m_N$ and $m_\Delta$ in the vicinity of $T_c$. This method is quite practical, since it is not easy to measure the $T$ dependence of the pole masses of nucleon and $\Delta$-resonance directly with LQCD simulations.
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Appendix A: Quark number density for the Wilson fermion in the massless free-gas limit

We consider \( n_q \) for the Wilson fermion in the lattice SB limit (the massless free-gas limit). In the high-\( T \) limit, we can consider a quark as a massless and non-interacting particle, since the effects of finite quark mass and interactions between quarks are negligible there. In Appendix of Ref. [26], the lattice SB limit is discussed except for the quark number density.

The partition function with free Wilson fermions is given by

\[
Z(\kappa, \hat{\mu}) = (\det \bar{M})^{N_f},
\]

\[
M_{xy} = \delta_{xy} - \kappa \sum_{i=1}^3 \left[ (1 - \gamma_i)\delta_{x+i,y} + (1 + \gamma_i)\delta_{x-i,y} \right]
- \kappa \left[ e^{+\hat{\mu}}(1 - \gamma_4)\delta_{x+4,y} + e^{-\hat{\mu}}(1 + \gamma_4)\delta_{x-4,y} \right],
\]

on an \( N_x \times N_y \times N_z \times N_t \) lattice. After the unitary transformation to momentum space, we obtain

\[
Z(1/8, \hat{\mu}) = \left( \prod_k \det \bar{M}(k) \right)^{N_x N_y N_z N_t}
\]

\[
\det \bar{M}(k) = \frac{16}{8^4} \left[ \sum_i \sin^2 k_i + \left\{ 2 \sum_i \sin^2 \left( \frac{k_i}{2} \right) \right\}^2 + 4 \left\{ 2 \sum_i \sin^2 \left( \frac{k_i}{2} \right) + 1 \right\} \sin^2 \left( \frac{k_t - i\hat{\mu}}{2} \right) \right]^2
\]

in the massless quark limit \( \kappa = 1/8 \), where \( \bar{M} \) is the fermion matrix in momentum space, and

\[
k_i = \frac{2\pi j_i}{N_i}, \quad j_i = 0, \pm 1, \ldots, N_i/2
\]

for the spatial components \((i = x, y, z)\) and

\[
k_t = \frac{2\pi (j_t + 1/2)}{N_t}, \quad j_t = 0, \pm 1, \ldots, N_t/2
\]

for the time component. The quark number density in the lattice SB limit is then obtained as

\[
\frac{n_q}{T^3} = \frac{N^3_v}{N^2_v} \frac{\partial}{\partial \hat{\mu}} \ln Z(1/8, \hat{\mu})
= N_v N_f N^3_v \sum_k \frac{\partial \det \bar{M}(k)}{\partial \hat{\mu}} \left[ \det \bar{M}(k) \right]^{-1}
\]

with

\[
\frac{\partial \det \bar{M}(k)}{\partial \hat{\mu}} = -\frac{1}{8^2} \left\{ 2 \sum_i \sin^2 \left( \frac{k_i}{2} \right) + 1 \right\} \times (\sinh \hat{\mu} \cos k_t + i \cosh \hat{\mu} \sin k_t).
\]

The quark-number density at imaginary \( \mu \) is obtained by replacing \( \hat{\mu} \) by \( i\hat{\mu_1} \) in Eqs. (A7) and (A8).

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