The coherent information of Pauli channels with coded inputs

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Abstract

The calculating of the coherent information is a fundamental step in obtaining the quantum capacity of a quantum channel. We introduce orthogonal and complete code basis to evaluate the coherent information per channel use when the input is the maximal mixture of stabilizer codewords. In the code basis, the output density matrix is diagonal, the joint output of the system and the auxiliary is block diagonal. The coherent information is worked out by counting the weights of error operators.

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1 Introduction

The basic issue in quantum information theory is quantum coding theorem. After ten year’s efforts, quantum coding theorem had at last been proven. The rate of faithfully transmitting quantum information per use of quantum channel is limited by quantum capacity, the capacity is asymptotically achievable [1] [2] [3] [4]. Quantum capacity is the maximization of coherent information over all input states. Unfortunately, since coherent information is non-additive [5], quantum capacity in single letter form is not available except for degradable [6] or anti-degradable channels. Regulation is need, that is, block input with infinitive number of qubits should be used to calculate quantum capacity in general. We may only obtain the lower bound of quantum capacity. The obstacle of obtaining the coherent information with other multipartite input state is obvious, the dimension of the state increases exponentially with the number of the input qubits, making the calculation of the output entropy and the entropy exchange (thus the coherent information) an aweful work. We will greatly reduce the complexity of diagonalizing the output density matrix by introducing quantum error-correcting code (QECC) as the input state.

2 QECC and Pauli Channels

The theory of QECCs was established more than a decade ago as the tool for fighting decoherence in quantum computers and quantum communication systems [8]. Maybe the most impressive development in quantum error-correction theory is the use of the stabilizer formalism [9] [10] [11] [12]. The power of the stabilizer formalism comes from the clever use of group theory. The $n$-fold Pauli operators $\{I, X, Y, Z\}^\otimes_n$ together with the possible overall factors $\pm 1, \pm i$ form a group $G_n$ under multiplication, the $n$-fold Pauli group. Suppose $S$ is an abelian subgroup of $G_n$. Stabilizer coding space $T$ is the simultaneous +1 eigenspace of all elements of $S$, $T = \{|\psi\rangle : M |\psi\rangle = |\psi\rangle, \forall M \in S\}$. For an $[n, k, d]$ stabilizer code, which encodes $k$ logical qubits into $n$ physical qubits, $T$ has dimension $2^k$ and $S$ has $2^{n-k}$ elements. The generators of $S$ are denoted as $M_i$ ($i = 1, \ldots, n-k$) which are Hermitian. There are many elements in $G_n$ that commute with every elements of $S$ but not actually in $S$. The set of elements in $G_n$ that commute with all of $S$ is defined as the centralizer $C(S)$ of $S$ in $G_n$. Clearly $S \subset C(S)$.

Denote $\Omega = \prod_i (I + M_i)$. Due to the properties that the elements of stabilizer group $S$ commute and $M_i^2 = I$, it follows $M_i \prod_i (I + M_i) = \prod_i (I + M_i)$. Further, we have $\Omega M_i \Omega = \Omega^2 = 2^{n-k} \Omega$. For error operator $E_o$ that anti-commutes with at least one of the generators $M_i$, we have

$$\Omega E_o \Omega = 0,$$

(1)

this is due to $(I + M_i)(I - M_i) = I - M_i^2 = 0$, the operator factor $(I - M_i)$ comes from $E_o(I + M_i) = (I - M_i)E_o$ if $E_o$ anti-commute with $M_i$.

In Krauss representation, Pauli channel map $E$ acting on qubit state $\rho$ can be written as $E(\rho) = f \rho + p_x X \rho X + p_y Y \rho Y + p_z Z \rho Z$, where $p_x(y, z) \in [0, 1]$ are the probabilities, $f = 1 - p_x - p_y - p_z \in [0, 1]$ is the fidelity of the channel. For depolarizing channel, $p_x = p_y = p_z = p$, $f = 1 - 3p$. The total error probability is $3p$. For $n$ use of depolarizing channels with $n$ qubits input state $\rho$, we have the output state $\rho' = E^{\otimes n}(\rho) = \sum_{\eta} \eta |E_o\rangle \langle E_o|$, with $\eta = f^{n-i-j-l} p_x^{i} p_y^{j} p_z^l$ for $E_o = X^i Y^j Z^l$. The purification of $\rho$ is $|\Psi\rangle$, we have the
joint output state $\rho_c = (\mathcal{E}^{\otimes n} \otimes I^{\otimes n})(|\Psi \rangle \langle \Psi|)$, whose entropy is the entropy exchange. The coherent information is $I_c = S(\rho) - S(\rho_c)$, where $S(\cdot)$ is the von Neumann entropy.

3 The coherent information of depolarizing channel with coded input

3.1 The $[[5,1,3]]$ code

To illustrate non-zero capacity even for zero fidelity of Pauli channel, we first consider the example of $[[5,1,3]]$ QECC as the input state to the depolarizing channel. The stabilizer code has four generators $M_1 = X_1Z_2Z_3X_4$, $M_2 = X_2Z_3Z_4X_5$, $M_3 = X_1X_2Z_3Z_5$, $M_4 = Z_1X_2X_4Z_5$. The codewords are $|0\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|1\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, where $X = XXXXX$, $X = C(S)/S$. Another useful operator in $C(S)|S\rangle$ is $\mathcal{Z} = ZZZZZ$, which anti-commutes with $X$. The input state $\rho$ is chosen to be $\tfrac{1}{2}(|0\rangle \langle 0 | + |1\rangle \langle 1 |)$ with maximal input entropy in the logical qubit basis $|0\rangle$ and $|1\rangle$. To evaluate the eigenvalues of output state and joint output state of the system and the auxiliary, we introduce the code basis: $|J\rangle = (2^5)^{\otimes m}$ $(J = 0, \ldots, 31)$, where $X = I, Z_m$ $(m = 1, \ldots, 5), Y_m, Z_m$. Note that $[[5,1,3]]$ can correct any single qubit error, the basis are orthonormal. For input code state $|J\rangle$, the elements of output density matrix are $\langle J|\sum_a \eta_a E_a |J\rangle$. Denote $|0\rangle = |00000\rangle$, the factor $\langle J|\sum_a \eta_a E_a |J\rangle = \tfrac{1}{16} \Omega_\lambda J E_a \Omega |0\rangle$ is 0 when $\Lambda_J E_a \not\in S + S\mathcal{Z}$ (up to $\pm 1, \pm \delta$ factors, which have no effect in channel mapping and are omitted hereafter), and is 1 when $\Lambda_J E_a \in S + S\mathcal{Z}$, according to the following reasons. We have three cases: (i) $\Lambda_J E_a \not\in C(S)$, then $\Lambda_J E_a$ anti-commutes with some generators of $S$, by Eq. (1), so $\langle J|E_a |J\rangle = 0$; (ii) $\Lambda_J E_a \in C(S)/(S + S\mathcal{Z})$, equivalently $\Lambda_J E_a \in \mathcal{X}(S + S\mathcal{Z})$. Then there is some $M \in S$ such that $\tfrac{1}{\Omega^2} \Omega_\lambda J E_a \Omega |0\rangle = \tfrac{1}{\Omega^2} \Omega M \Omega_\lambda J E_a \Omega |0\rangle = \tfrac{1}{\Omega^2} \Omega M \Omega_\lambda J E_a \Omega |0\rangle = \tfrac{1}{\Omega^2} \Omega M \Omega_\lambda J E_a \Omega |0\rangle = 0$; (iii) $\Lambda_J E_a \in S + S\mathcal{Z}$, then there is some $M \in S$ such that $\langle J|E_a |J\rangle = \tfrac{1}{\Omega^2} \Omega M \Omega_\lambda J E_a \Omega |0\rangle = \tfrac{1}{\Omega^2} \Omega M \Omega_\lambda J E_a \Omega |0\rangle = \tfrac{1}{\Omega^2} \Omega M \Omega_\lambda J E_a \Omega |0\rangle = 1$. If $\Lambda_J E_a \in S + S\mathcal{Z}$ and $E_a \Lambda_J \not\in S + S\mathcal{Z}$, we have their product $\Lambda_J E_a \Lambda_J \not\in S + S\mathcal{Z}$, which is only possible when $J = K$, since our $\Lambda_J$ is so chosen that each of which is the head of one of cosets $G_5/(S + S\mathcal{Z})$. Thus the state $\mathcal{E}^{\otimes n}(|J\rangle \langle J|)$ is diagonal in the code basis. Similar result can be found for $\mathcal{E}^{\otimes n}(|1\rangle \langle 1|)$. The output state in the representation of the code basis is diagonalized.

$$
\rho'_{JK} = \frac{1}{2} \delta_{JK} \left( \sum_a \eta_a + \sum_{a'} \eta_a' \right),
$$

with the conditions of $\Lambda_J E_a \in S + S\mathcal{Z}$ for the first term, and $\mathcal{X}\Lambda_J E_a \in S + S\mathcal{Z}$ for the second term at the right hand side.

The purification of the input state $\rho$ could be $|\psi\rangle = \sqrt{2} (|\overline{0}\rangle \langle \overline{0}| + |\overline{1}\rangle \langle \overline{1}|)$, where the first logical qubit is for the system, the second logical qubit is for the auxiliary and denoted by the subscript $a$. The joint output state of the system and the auxiliary is $\rho_c = (\mathcal{E}^{\otimes n} \otimes I^{\otimes n})(|\psi\rangle \langle \psi|)$. An obvious basis for the joint output density matrix is $|J\rangle \langle J|, |J\rangle \langle 1|$. The non-zero elements are $\langle J| \langle 0\rangle \rho_{aJ} |J\rangle \langle J|, \langle J| \langle 0\rangle \rho_{aJ} |J\rangle \langle 1|$. The reason can be seen from the calculating $\langle J| \langle 0\rangle \rho_{aJ} |J\rangle \langle J|, \langle J| \langle 0\rangle \rho_{aJ} |J\rangle \langle 1|$. We have $\langle J| \langle 0\rangle \rho_{aJ} |J\rangle \langle J| = 1/2 \langle J| \mathcal{E}^{\otimes n}(|\overline{0}\rangle \langle \overline{0}|) |J\rangle = 1/2 \sum_a \eta_a \langle J| E_a |J\rangle$, which is nonzero when $\Lambda_J E_a \in S + S\mathcal{Z}$ and $E_a \Lambda_J \not\in S + S\mathcal{Z}$, hence $K = \text{mod}(J + 16, 32)$. The reason can be seen from the calculating $\langle J| \langle 0\rangle \rho_{aJ} |J\rangle \langle 1|$. In the basis of $|J\rangle \langle 0\rangle, |J\rangle \langle 1|$, by rearranging the subscripts, the matrix $\rho_c$ can be decomposed to the direct summation of 32 matrices. $\rho_c = \otimes_{J=0}^{15} \rho_{aJ}$, with

$$
\rho_{aJ} = \left[ \begin{array}{cc}
\langle J| \langle 0\rangle \rho_{aJ} |J\rangle \langle J| & \langle J| \langle 0\rangle \rho_{aJ} |J\rangle \langle 1| \\
\langle J| \langle 0\rangle \rho_{aJ} \rangle |1\rangle \langle J| & \langle J| \langle 0\rangle \rho_{aJ} \rangle |1\rangle \langle 1| \\
\end{array} \right].
$$

We have $\langle J| \langle 0\rangle \rho_{aJ} |J\rangle \langle J| = 1/2 \langle J| \mathcal{E}^{\otimes n}(|\overline{0}\rangle \langle \overline{0}|) |J\rangle = 1/2 \sum_a \eta_a \langle J| E_a |0\rangle \langle 0| E_a |J\rangle = 1/2 \sum_a \eta_a \langle J| E_a |0\rangle \langle 0| E_a |J\rangle = 1/2 \sum_a \eta_a \langle J| E_a |0\rangle \langle 0| E_a |J\rangle$.

Notice that $\langle J| \langle 0\rangle \rho_{aJ} |J\rangle \langle 1| = \sum_a \eta_a \langle J| E_a |0\rangle \langle 0| E_a |J\rangle = 1/2 \sum_a \eta_a \langle J| E_a |0\rangle \langle 0| E_a |J\rangle$. Thus

$$
\rho_{aJ} = \left[ \begin{array}{cc}
\eta_{aJ} + \eta_{0J} & \eta_{aJ} - \eta_{0J} \\
\eta_{0J} - \eta_{aJ} & \eta_{0J} + \eta_{aJ} \\
\end{array} \right],
$$

with

$$
\eta_{aJ} = \sum_{E_a \in A_J} \eta_a,
$$

$$
\eta_{0J} = \sum_{E_a \in A_J} \eta_a.
$$

Thus the eigenvalues of $\rho_{aJ}$ are $\eta_{aJ}$ and $\eta_{0J}$. We can obtain from (2) that the eigenvalues of $\rho'$ are

$$
\lambda_J = \frac{1}{2} (\eta_{0J} + \eta_{aJ} + \eta_{0J} + \eta_{aJ}).
$$
The average coherent information per channel use thus is
\[
T_c = \frac{1}{5} \sum_{j=0}^{32} (-\lambda_j \log_2 \lambda_j + \eta_{oJ} \log_2 \eta_{oJ} + \eta_{eJ} \log_2 \eta_{eJ}).
\]

The eigenvalues \( \eta_{oJ}, \eta_{oJ} \) can be obtained by counting the weights of all the operators of \( \Lambda_J S \) and \( \Lambda_J S^Z \), respectively. The weight of an error operator is the number of qubits on which it differs from the identity. An error \( E_o \in \Lambda_J S \) with weight \( j \) will contribute \( \eta_o = f^{n-j}p^j \) to the eigenvalue of \( \eta_{oJ} \), similarly. An error \( E_o \in \Lambda_J S^Z \) with weight \( j \) will contribute \( \eta_o = f^{n-j}p^j \) to the eigenvalue of \( \eta_{oJ} \). Thus \( \eta_{oJ} \) and \( \eta_{oJ} \) can be written as \( \sum_{j=0}^{32} c_j^o f^{n-j}p^j \) and \( \sum_{j=0}^{32} c_j^o f^{n-j}p^j \), respectively. Furthermore, they are characterized by vector \( c_0 \) and \( c_0 \). A detailed counting shows \( c_0 = (0, 15, 0, 0, 0, 1) \), \( c_0 = (0, 15, 0, 0, 0, 0) \). The eigenvalues of the subscripts \( J = 1, 2, 3, 4, 5, 22, 23, 24, 25, 26, 11, 12, 13, 14, 15 \) are degenerated and represented by \( c_i = (3, 8, 4, 0, 1, 0) \) and \( c_i = (4, 6, 4, 2, 0, 0) \); also the eigenvalues are degenerate for \( J = 17, 18, 19, 20, 21, 6, 7, 8, 9, 10, 27, 28, 29, 30, 31 \) and represented by \( c_i \). The average coherent information per channel use then is
\[
T_c = \frac{1}{5} \left[ -\frac{(\eta_o + 3\eta_h o)}{2} \log_2 \frac{(\eta_o + 3\eta_h o)}{2} + \eta_o \log_2 \eta_o + 3\eta_h o \log_2 \eta_h o \right] + 3\left[ -\frac{(\eta_o + 3\eta_h o)}{2} \log_2 \frac{(\eta_o + 3\eta_h o)}{2} + \eta_o \log_2 \eta_o + 3\eta_h o \log_2 \eta_h o \right].
\]

3.2 The \([7,1,3]\) code

The six stabilizer generators of \([7,1,3]\) code are \( M_1 = X_1 X_2 X_3 X_4, M_2 = X_1 X_2 X_5 X_6, M_3 = X_1 X_3 X_5 X_7, M_4 = Z_1 Z_2 Z_4 Z_5, M_5 = Z_1 Z_2 Z_5 Z_6, M_6 = Z_1 Z_3 Z_3 Z_7 \). The bit flip and phase flip operators for the encoded (logical) qubit are \( \overline{X} = X_5 X_6 X_7 \) and \( \overline{Z} = Z_5 Z_6 Z_7 \), respectively. Since the code can correct any single qubit errors of \( X, Y, Z \) types and meanwhile it can correct errors \( X_i Z_j \) \((i \neq j)\) type, it is convenient to choose the 128 coset heads as \( \Lambda_J (J = 0, \ldots, 127) = I, X_k, (k = 1, \ldots, 7), Y_k, Z_k, X_i Z_j \) \((i \neq j; i, j = 1, \ldots, 7)\), \( \overline{X} \), \( X_k \overline{X}, (k = 1, \ldots, 7), Y_k \overline{X}, Z_k \overline{X}, X_i \overline{X}, Z_j \overline{X} \) \((i \neq j; i, j = 1, \ldots, 7)\). With \( \eta_{oJ} \) and \( \eta_{oJ} \) being written as \( \sum_{j=0}^{7} c_j^{oJ} f^{n-j}p^j \) and \( \sum_{j=0}^{7} c_j^{oJ} f^{n-j}p^j \), the non-zero eigenvalues of the joint output state \( \rho_c \) now can be characterized by vectors \( c_j^o \) and \( c_j^o \). They are \((0, 15, 0, 0, 0, 0, 1, 1)\), (ii) 7 fold degenerate represented by \( c_i = (6, 24, 21, 8, 4, 0, 1, 0) \), (iii) 7 fold degenerate represented by \( c_i = (11, 16, 18, 16, 3, 0, 0, 0) \), (iv) 7 fold degenerate represented by \( c_i = (8, 19, 24, 10, 0, 3, 0, 0) \), (v) 42 fold degenerate represented by \( c_i = (8, 21, 20, 10, 4, 1, 0, 0) \), \( c_i = (9, 20, 18, 12, 5, 0, 0, 0) \). The eigenvalues of the output state \( \rho'_c \) are \( \lambda_j = \frac{1}{2} (\eta_{oJ} + \eta_{oJ} + \eta_{oJ} + \eta_{oJ} + \eta_{oJ} + \eta_{oJ}) \), with \( J' = \text{mod}(J + 64, 128) \). The average coherent information per channel use then is
\[
T_c = \frac{1}{5} \left[ -\frac{(\eta_o + 3\eta_h o)}{2} \log_2 \frac{(\eta_o + 3\eta_h o)}{2} + \eta_o \log_2 \eta_o + 3\eta_h o \log_2 \eta_h o \right] + 3\left[ -\frac{(\eta_o + 3\eta_h o)}{2} \log_2 \frac{(\eta_o + 3\eta_h o)}{2} + \eta_o \log_2 \eta_o + 3\eta_h o \log_2 \eta_h o \right].
\]
tors are $\overline{X}_j$. Denote the elements of the cost head as $\Lambda_j (J = 0, \ldots, 255)$. Suppose the input state is 
\[ \rho = \frac{1}{2} \sum_{k_1, k_2, k_3 = 0}^{1} |k_1 k_2 k_3 \rangle \langle k_1 k_2 k_3 |, \]
then the purification of $\rho$ is $|\Psi\rangle = \frac{1}{2} \sum_{k_1, k_2, k_3 = 0}^{1} |k_1 k_2 k_3 \rangle \langle k_1 k_2 k_3 | A_{\Lambda_j} 000 \rangle$. The joint output of the system and auxiliary is $\rho_c = (E^{\otimes 8} \otimes I^{\otimes 8}) (|\Psi\rangle \langle \Psi|)$. In the basis of $|J\rangle = \Lambda_j |000\rangle$, we have 
\[ \langle K | \langle k_1' k_2' k_3' \rangle \rangle |J_{j_1 j_2 j_3}\rangle A_{\Lambda_j} = \sum_a \eta_a (K | E_a | k_1 k_2 k_3 \rangle \langle j_1 j_2 j_3 | E^a_{J} | J\rangle, \tag{11} \]
which is nonzero when $\overline{X}_1^j \overline{X}_2^j \overline{X}_3^j \Lambda_k^j E_a \in S \times \overline{Z}$ and $E^a_{J} \Lambda_j \overline{X}_1 \overline{X}_2 \overline{X}_3 \in S \times \overline{Z}$, thus $\overline{X}_1 \overline{X}_2 \overline{X}_3 \Lambda_k^j E_a = \sum_{k_1, k_2, k_3} ^{1} \overline{X}_1 \overline{X}_2 \overline{X}_3 \Lambda_k \overline{X}_1 \overline{X}_2 \overline{X}_3 \in S \times \overline{Z}$. However, $\Lambda_k$ and $\Lambda_j$ are coset heads, so we have $\overline{X}_1 \overline{X}_2 \overline{X}_3 \Lambda_k \overline{X}_1 \overline{X}_2 \overline{X}_3 = \sum_{k_1, k_2, k_3} ^{1} \overline{X}_1 \overline{X}_2 \overline{X}_3 \Lambda_{k_1 k_2 k_3} \overline{X}_1 \overline{X}_2 \overline{X}_3 = |000\rangle$. Then $E_a \Lambda_k \overline{X}_1 \overline{X}_2 \overline{X}_3 S_b \overline{Z}_1 \overline{Z}_2 \overline{Z}_3 \overline{Z}_4 = \Lambda_j \overline{X}_1 \overline{X}_2 \overline{X}_3 \Lambda_k \overline{X}_1 \overline{X}_2 \overline{X}_3 \overline{Z}_1 \overline{Z}_2 \overline{Z}_3 \overline{Z}_4$

We have 
\[ \langle K | E_a | k_1 k_2 k_3 \rangle \rangle = \sum_a \eta_a (K | E_a | k_1 k_2 k_3 \rangle \langle j_1 j_2 j_3 | E^a_{J} | J\rangle = \sum_a \eta_a (K | E_a | k_1 k_2 k_3 \rangle \langle j_1 j_2 j_3 | E^a_{J} | J\rangle = \sum_{a'} \sum_{a''} (-1)^{\sum_{i=1}^{3} (k_i + j_i) + c_i} \sum_{a''} \eta_{a''} \eta_{a'}. \tag{12} \]

Where $E_{a'} \in \Lambda_j \overline{X}_1 \overline{X}_2 \overline{X}_3 \Lambda_k \overline{X}_1 \overline{X}_2 \overline{X}_3 \overline{Z}_1 \overline{Z}_2 \overline{Z}_3 \overline{Z}_4$, and $mod(K + 32(4k_3 + 2k_2 + k_1), 256) = mod(J + 32(4j_3 + 2j_2 + j_1), 256)$. By rearranging the basis, the joint output state $\rho_c$ can be written in a block diagonalized form with each block being a $8 \times 8$ submatrix. A detail analysis shows that each $8 \times 8$ submatrix can be diagonalized with Hadamard transformation. The eigenvalues of $\rho_c$ are 
\[ \xi_i = \sum_{j=0}^{8} c_{i,j+1} f^{n-j} p^j \]
with 
\[ c = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 28 & 0 & 3 \\ 0 & 0 & 0 & 0 & 6 & 0 & 20 & 0 & 6 \\ 0 & 0 & 0 & 2 & 0 & 12 & 0 & 18 & 0 \\ 0 & 0 & 1 & 0 & 1 & 8 & 11 & 8 & 3 \\ 0 & 0 & 0 & 0 & 4 & 8 & 8 & 4 & 3 \\ 0 & 0 & 0 & 2 & 2 & 4 & 12 & 10 & 2 \\ 0 & 0 & 0 & 1 & 3 & 6 & 10 & 9 & 3 \\ 0 & 1 & 0 & 0 & 0 & 7 & 14 & 8 & 2 \\ 0 & 0 & 0 & 1 & 2 & 8 & 10 & 7 & 4 \\ 0 & 0 & 0 & 2 & 1 & 6 & 12 & 8 & 3 \\ 0 & 0 & 1 & 0 & 2 & 6 & 11 & 10 & 2 \\ 0 & 0 & 0 & 1 & 4 & 4 & 10 & 11 & 2 \\ 0 & 0 & 0 & 0 & 3 & 10 & 8 & 6 & 5 \\ 0 & 0 & 0 & 0 & 5 & 6 & 8 & 10 & 3 \\ \end{bmatrix} \]
The degeneracy vector is $d = (d_1, \ldots, d_{14}) = (1, 35, 28, 112, 168, 56, 112, 24, 504, 168, 336, 168, 168)$. The entropy exchange is 
\[ S(\rho_c) = -\frac{14}{i=1} d_i log_2(\xi_i). \tag{13} \]
The eigenvalues of the output state $\rho'$ are 
\[ \lambda_1 = \frac{1}{8} (\xi_1 + 35 \xi_2 + 28 \xi_3), \]
\[ \lambda_2 = 4(\xi_4 + \xi_5), \]
\[ \lambda_3 = \frac{1}{2} (\xi_4 + 8 \xi_5 + 7 \xi_6), \]
\[ \lambda_4 = \frac{1}{2} (\xi_4 + \xi_5 + 14 \xi_7), \]
\[ \lambda_5 = \frac{1}{8} (8 \xi_8 + 7(3 \xi_9 + \xi_{10}) + \xi_{11} + 2 \xi_{12} + \xi_{13} + \xi_{14}) \tag{14} \]
with degeneracy vector $d_i' = 8d_i'$, and 
\[ d = (d_1', \ldots, d_{14}') = (1, 3, 2, 2, 24). \]
Thus the entropy of $\rho'$ is 
\[ S(\rho') = -8 \sum_{i=1}^{14} d_i' \lambda_i log_2(\lambda_i). \tag{15} \]
The average coherent information per channel use is 
\[ T_c = -\frac{14}{i=1} d_i' \lambda_i log_2(\lambda_i) + \frac{1}{8} \sum_{i=1}^{14} d_i \lambda_i log_2(\lambda_i). \tag{16} \]

4 General Pauli channels

For Pauli channel with 3 error probabilities $p_x, p_y, p_z$, the coherent information of coded input can be evaluated in the same way as depolarizing channel. The
only difference is that now we should count the numbers of the each type of error separately in calculating $\sum \eta_a$. Define the functions

$$g_1(t, z, x, y) = t^5 + 5t(x^2 y^2 + x^2 z^2 + z^2 y^2),$$

$$g_2(t, z, x, y) = t^4 x + 2t^4 x(z^2 + y^2) + 2t y(2x^2 + z^2 + y^2) + x(2x^2 + x^2 z^2 + z^2 y^2).$$

Then the eigenvalues of the joint output state $\rho_e$ are given in Table 1, where the eigenvalues are expressed as $\xi_{ij}$ with degeneracy $d_i$. The entropy exchange then is

$$S(\rho_e) = -\sum_{i=1}^{4} d_i \sum_{j=1}^{4} \xi_{ij} \log_2 \xi_{ij}. \quad (17)$$

The eigenvalues of $\rho'$ are $\lambda_i = \frac{1}{2} \sum_{j=1}^{4} \xi_{ij}$ with degeneracy $2d_i$, thus we have

$$S(\rho') = -\sum_{i=1}^{4} 2d_i \lambda_i \log_2 \lambda_i. \quad (18)$$

The coherent information per channel use for Pauli channel with $[[5, 1, 3]]$ code as input is

$$I_e = -\frac{1}{5} \sum_{i=1}^{4} d_i [(\sum_{j=1}^{4} \xi_{ij}) \log_2 (\frac{1}{2} \sum_{j=1}^{4} \xi_{ij})]$$

$$- \sum_{j=1}^{4} \xi_{ij} \log_2 \xi_{ij}. \quad (19)$$

5 Discussions and Conclusions

There are bounds on the coding rate of QECC, the quantum Hamming bound [13], Knill-Laflamme (quantum Singleton) bound [14], Gottesman bound and so
The first two give rather tight upper bounds on some of additive quantum codes. The quantum Hamming bound (hashing bound) is a strict upper bound for non-degenerate (pure) quantum code, as it is seen from figure 2, where all three average coherent information calculated are upper bounded by the hashing bound. However, it has been known that quantum Hamming bound can be violated by degenerate (im-pure) quantum codes [6]. They obtain the result by calculating the coherent information of depolarizing channel with repetition quantum codes. We have introduced a systematical way of calculating the coherent information of Pauli channel with quantum code as input state. The main finding of this paper is that the channel output density matrix as well as the density matrix of the joint output of the system and the auxiliary can be diagonalized for Pauli environment with quantum code as input, the eigenvalue problem is reduced to counting the weight of the error operators in the coset. We have presented the input of [[8, 3, 3]] code as an example of calculating the coherent information of input state with multiple logical qubits. It is anticipated that our method should promote the way of violating quantum Hamming bound by calculating the coherent information of depolarizing channel with quantum code of encoding multiple logical qubit. Meanwhile, our method provide the way of calculating the lower bound for the distillable entanglement of quantum code state passing through Pauli channel, according to hashing inequality[3].

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