Stability analysis under dilatation of coordinates

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Abstract. The stability of periodic orbits under translation and also dilatation (or contraction) of coordinates is analysed using a $q$-Jacobian. Specifically we analyse the stability of period-1 and period-2 orbits of the quadratic map and show that the dilatation changes the parameter values for which bifurcations take place. The dilatation ($|q| > 1$) or contraction ($|q| < 1$) can be used to stabilize (destabilize) the unstable (stable) periodic orbits. In addition, dilatation effects on the stability depend only on the period of the orbit, and not on the location of orbital points in phase-space. This is shown to be true for any period-$n$ orbit of the quadratic map. As a possible practical application, we suggest that the dilatation (contraction) considered here could represent temperature variations in physical systems.

1. Introduction
The study of basic hypergeometric series (also called $q$-hypergeometric series, where $q$ is a parameter) has essentially started in 1748 by Euler [1]. These series have the additional parameter $q$ which deforms the usual hypergeometric functions. The connection between the basic hypergeometric series and the usual hypergeometric functions is nicely shown in Ref. [2]. Although in the last decades many works considered the subject of $q$-deformations, ranging from mathematical physics, statistical physics, quantum physics to relativistic mechanics, it is still not clear what is the physical interpretation of the parameter $q$. The related literature is quite extensive and in this introduction we attain us to mention just some classical $q$-deformations in the context of the present work. From the point of view of nonlinear dynamical systems it was shown that the $q$-deformed analysis constitutes a natural way to treat the fractal properties of functions [3]. The $q$-deformed exponential function was also used to show a relation between the fractal dimension of an attractor and the parameter $q$ [4], and to show the power-law sensitivity to the initial conditions at the edge of chaotic motion [5, 6]. More recently, $q$-deformation has been proposed to study the properties of the $q$-deformed quadratic map [7] and also the $q$-deformed Gaussian map [8]. In both cases the $q$ deformation enters in the map itself and they observed the co-existence of attractors in the one-dimensional maps. In this paper we use the $q$-deformed derivative [3]

$$\partial_x^{(q)} f(x; y; ... ) \equiv \frac{f(qx; y; ... ) - f(x; y; ... )}{(q - 1)x},$$

(1)
to analyse stability in the quadratic map. Our quadratic map is the usual one but the derivative, used to determine the stability via the Jacobian of the map, is deformed. The derivative (1) measures the rate of changes of the function \( f(x; y; \ldots) \) with respect to a dilatation of its argument by a factor of \( q \). In the context of the linear stability analysis in dynamical systems, we can use Eq. (1) to study the stability of periodic points under a dilatation of the dynamical variable \( x \). In the limit of \( q \to 1 \), the usual derivative is recovered.

The usual way \[9\] to study the stability of periodic orbits in one-dimensional maps is to analyse the eigenvalues of the Jacobian given by

\[
J = \frac{\partial f(x)}{\partial x},
\]

where \( f(x) \) is the function which describes the dynamics of the map. The stability index \( m \) from an \( n \)-periodic orbit can be calculated from the equation \[10, 11, 12\]

\[
W_m^n = J_n \times J_{n-1} \times \ldots \times J_1 - m = 0.
\]

For \(|m| < 1\) the orbits are stable, for \(|m| > 1\) the orbits are unstable, and for \(|m| = 1\) there is a bifurcation point. In this paper we use the \( q \)-derivative (1) to determine the \( q \)-Jacobian and to analyse the stability of the periodic points under dilatation (\(|q| > 1\)) or contraction (\(|q| < 1\)) of the argument of the map. Using the \( q \)-derivative defined in \[3\], the \( q \)-Jacobian is calculated from

\[
J^q = \partial^q_x f(x) = [\psi]_q \frac{f(x)}{x},
\]

where

\[
[\psi]_q = \frac{q^\psi - 1}{q - 1},
\]

and \( \psi \) is the degree of the homogeneous function \( f(x) \). The deformed stability index \( m_q \) is determined from

\[
W_m^{m_q} = J_n^q \times J_{n-1}^q \times \ldots \times J_1^q - m_q = 0.
\]

In the limit \( q \to 1 \) we expect to obtain \( J^q \to J \). The deformed stability index \( m_q \) recovers the usual stability index \( m \) when \( q \to 1 \). Also in this \( q \)-analysis we expect to have \(|m_q| < 1\) for \( q \)-stable orbits, \(|m_q| > 1\) for \( q \)-unstable orbits and \(|m_q| = 1\) for \( q \)-bifurcating orbits. It is important to mention that \( q \)-stable and \( q \)-unstable means that the periodic orbits are respectively stable and unstable under simultaneous translation and dilatation. Each time we just mention the word stable or unstable, we are referring to the stability of the periodic orbits without dilatation. We also use the expression \( q \)-bifurcation when \( q \neq 1 \), otherwise we use bifurcation.

This work analyses the implication of dilatation or contraction of coordinates on the stability of a physical model, namely the quadratic map. Since such simple map presents the main features of more complicated dynamical systems, its dynamics under dilatation (contraction) of coordinates should also present the main features observed in more complicated dilated (contracted) systems. We believe that our method can be very useful to describe the stability of systems when the dynamical variable experiments dilatation. It may also help us to elucidate the physical relevance of the parameter \( q \) in dynamical systems.

2. Quadratic Map

As a prototype model we use the quadratic map

\[
f(x_n) = x_{n+1} = a - x_n^2,
\]

where
which experiences bifurcations at the parameter values \( a = -1/4 \) (born of period-1 orbits), \( a = 3/4 \) (bifurcation 1 \( \rightarrow \) 2), \( a = 5/4 \) (bifurcation 2 \( \rightarrow \) 4), \( a = 7/4 \) (bifurcation 4 \( \rightarrow \) 8), and so on. The usual Jacobian from Eq. (7) is given by

\[
J_n = \frac{\partial f}{\partial x_n} = -2x_n,
\]

while the \( q \)-Jacobian is

\[
J_n^q = \frac{\partial (q^2 f)}{\partial x_n} = -\frac{(q^2 - 1)}{(q - 1)} x_n.
\]

In the limit \( q \to 1 \) we obtain \( J_n^q \to J_n \). To understand in details the effects of the dilatation on the stability analysis, we determine now the \( (a, q) \) dependence of the deformed stability index \( m_q \) for the simplest cases of period-1 and period-2 orbits.

\[\textbf{2.1. Fixed points}\]

The two fixed points are obtained from Eq. (7) using \( n = 0 \) and the condition \( x_0 = x_1 \). The results are

\[
x_1^+ = -\frac{1}{2} (1 \pm \eta),
\]

where \( \eta = \sqrt{1 + 4a} \). Using the deformed Jacobian (9) the condition for the stability index \( m_q \) is obtained from Eq. (6) and given by

\[
W_{m_q}^m = m_q - \frac{1}{2} (q + 1) (1 \pm \eta) = 0,
\]

and the solutions are

\[
m_q^\pm = \frac{(q + 1)}{2} (1 \pm \eta).
\]

When \( q \to 1 \), Eq. (12) reduces to

\[
m_1^\pm = m_1^\pm = (1 \pm \eta),
\]

which is the usual stability index under translation. Using \( m = 1 \) in this equation we obtain the condition \( a = -1/4 \), which is the value of the parameter where both fixed points are born. When \( a = -1 \) we recover the 1 \( \rightarrow \) 2 bifurcation point which occurs at \( a = 3/4 \). For \(-1/4 < a < 3/4 \) only the fixed point \( x_1^- \) is stable. The fixed point \( x_1^+ \) is unstable. The relevant point here is that Eq. (12) imposes a relation between the parameter \( a \) and the deformation \( q \). It means that when we consider \( q \neq 1 \) the dilatation changes the values of \( a \) where the \( q \)-bifurcation occurs.

We can also vary \( q \) so that the fixed points change their stability. We will analyse this explicitly next.

\[\textbf{2.1.1. Changing the stability of the fixed points using dilatation (contraction)}\]

Looking at Eq. (12) it is easy to observe that the stability for a given fixed point may change by varying the parameter \( q \). As an example, let us discuss the stable fixed point \( x_1^- \) for \( a = 1/2 \). Using this fixed point, the deformed stability index is \( m_q^- = (q + 1)(1 - \sqrt{3})/2 \). We can now “destabilize” the point \( x_1^- \) by making an appropriate dilatation (contraction). Imposing the instability condition \( |m_q^-| > 1 \) we find the values \( q < -3.732 \) and \( q > 1.732 \). For these values of \( q \) the stable point \( x_1^- \) becomes \( q \)-unstable. It is also possible to “stabilize” the unstable fixed point \( x_1^+ \). Using \( m_q^+ = (q + 1)(1 + \sqrt{3})/2 \) and forcing \( |m_q^+| < 1 \) we obtain the interval \(-1.732 < q < -0.268 \).
Such (de)stabilization condition can be generalized for any value of $a$. Forcing $|m_q^-| > 1$ and $|m_q^+| < 1$ in Eq. (12) we obtain the “destabilization” condition for the stable point $x^-_1$

$$q < -\frac{1 + \eta}{-1 + \eta}, \quad q > \frac{3 - \eta}{-1 + \eta},$$

and the “stabilization” condition for the unstable point $x^+_1$

$$-\frac{3 - \eta}{1 + \eta} < q < \frac{1 - \eta}{1 + \eta}.$$

2.1.2. Case $m_q = 1$ (Born of q-fixed points) Here and in Sec. 2.1.3 we analyse the effect of the dilatation and contraction from another perspective. The question is what happens at the q-bifurcation points defined through $|m_q| = 1$?

We start with $m_q^\pm = 1$, where the q-fixed points $x^\pm_1$ are born. Solving Eq. (12) for $q$ we obtain

$$q^\pm = -\frac{1}{2a}(1 + 2a \mp \eta).$$

(16)

For each value of $a$ there is a given dilatation (or contraction) $q^\pm$ for which the q-stable and q-unstable fixed points are born. From Eq. (16) we determine

$$q^+ - q^- = 2\sqrt{1 + 4a},$$

(17)

which says that the dilatation for both fixed points is different at the q-bifurcation points. The only exception occurs at the bifurcation point $a = -1/4$, where $x^+_1 = x^-_1$ and $q^+ = q^- = 1$.

It is also easy to verify that

$$q^- x^-_1 = x^+_1 \quad \text{and} \quad q^+ x^+_1 = x^-_1,$$

(18)

which is independent of $a$. This means that the stable (unstable) point $x^-_1$ ($x^+_1$) when multiplied by the dilatation factor $q^-$ ($q^+$), will be located at the unstable (stable) fixed point $x^+_1$ ($x^-_1$). There is an interchange between the fixed points. This is a more or less obvious result since the stable point $x^-_1$ becomes unstable when it is located at the unstable point $x^+_1$, and vice-versa. For later purposes we call to attention that in this section we found a relation between stable and unstable fixed points.

2.1.3. Case $m_q = -1$ (q-Bifurcation 1 $\rightarrow$ 2) Now imposing $m_q^\pm = -1$ in Eq. (12) and solving in $q$ we obtain

$$q^\pm = \frac{1}{2a}(-1 + 2a \mp \eta),$$

(19)

which again connects the parameter $a$ and the dilatation $q$. It is easy to verify that, independent of $a$, we have

$$q^- x^-_1 = 2 + x^+_1, \quad \text{and} \quad q^+ x^+_1 = 2 + x^-_1.$$ 

(20)

Different from Eq. (18) here the dilatation factors do not transform the stable (unstable) fixed point to the unstable (stable) fixed point. There is a factor 2 in between. Equation (20) gives a relation between a q-bifurcating point and an unstable fixed point, and also says that by a convenient dilatation (or contraction), the dilated fixed point $q^- x^-_1$ bifurcates at $2 + x^+_1$, while the dilated fixed point $q^+ x^+_1$ bifurcates at $2 + x^-_1$. 
2.2. Period-2 orbits

Iterating twice the map (7) we find the points of the period-2 orbit, namely

\[ x_2^{(1)} = \frac{1}{2} (1 + \beta), \quad \text{and} \quad x_2^{(2)} = \frac{1}{2} (1 - \beta), \tag{21} \]

where \( \beta = \sqrt{-3 + 4a} \). This orbit bifurcates from the stable fixed point \( x_1^- \) at \( a = 3/4 \). The deformed stability index is calculated from Eq. (6), and gives

\[ m_q = J_2^q \times J_1^q = \left[ -\frac{q^2 - 1}{q - 1} x_2^{(1)} \right] \left[ -\frac{q^2 - 1}{q - 1} x_2^{(2)} \right]. \tag{22} \]

Using the expression for \( x_2^{(1)} \) and \( x_2^{(2)} \) from Eq. (21) we obtain

\[ m_q = (q + 1)^2 (1 - a), \tag{23} \]

which is the deformed stability index for the period-2 orbit. In the limit \( q \to 1 \) we obtain from Eq. (23) the condition \( m_1 = m = 4 - 4a \). Using \( m = 1 \) we get \( a = 3/4 \), which is the value where the bifurcation \( 1 \to 2 \) occurs. For \( m = -1 \) we obtain \( a = 5/4 \), where the bifurcation \( 2 \to 4 \) occurs. In the interval \( 3/4 < a < 5/4 \) both points \( x_2^{(1)} \) and \( x_2^{(2)} \) are stable. For \( a > 5/4 \) both points are unstable.

2.2.1. Changing the stability by dilatation (contraction)  

As discussed in Sec. 2.1.1, also here the stability of the period-2 orbit can be changed by varying \( q \), as can be observed from Eq. (23). Since the orbit is stable, we can only destabilize it. The “destabilization” condition for the period-2 orbits in the interval \( 3/4 < a < 5/4 \) is obtained by imposing \(|m_q| > 1 \) to Eq. (23), to obtain

\[ \sqrt{\frac{1 - a}{1 - a}} - 1 < q < \frac{\sqrt{a - 1}}{(a - 1)} - 1. \tag{24} \]

2.2.2. Case \( m_q = 1 \) (\( q \)-Bifurcation \( 1 \to 2 \))  

Making \( m_q = 1 \) in Eq. (23) we obtain

\[ q = \frac{\sqrt{1 - a}}{1 - a} - 1, \tag{25} \]

which is real for \( a < 1 \). At the original bifurcation point \( a = 3/4 \) we obtain \( q = 1 \), which is equal to \( q^- \) obtained in last section for the bifurcation \( 1 \to 2 \) (\( m_q^- = -1 \)). Therefore, the points \( q \) and \( q^- \) give correctly the same value at the bifurcation \( 1 \to 2 \).

In order to observe the effect of the dilatation \( q \) we determine

\[ q x_2^{(1)} = x_2^{(2)} + \frac{(1 + \beta)\sqrt{1 - a}}{2(1 - a)} - 1, \tag{26} \]

and

\[ q x_2^{(2)} = x_2^{(1)} + \frac{(1 - \beta)\sqrt{1 - a}}{2(1 - a)} - 1, \tag{27} \]

which gives a relation between the two originally stable points. At the bifurcation point \( a = 3/4 \) we obtain \( x_2^{(1)} = x_2^{(2)} \) as expected.
2.2.3. Case \( m_q = -1 \) (q-Bifurcation \( 2 \to 4 \)) Now we use \( m_q = -1 \) in (23) to obtain

\[
q = \frac{\sqrt{a-1}}{a-1} - 1,
\]

which is real for \( a > 1 \). It is also straightforward to obtain

\[
q x_2^{(1)} = x_2^{(2)} + \frac{(1 + \beta)\sqrt{a-1}}{2(a-1)} - 1,
\]

and

\[
q x_2^{(2)} = x_2^{(1)} + \frac{(1 - \beta)\sqrt{a-1}}{2(a-1)} - 1.
\]

Observe that these relations between the points are not valid for \( a < 1 \). For the point \( a = 5/4 \) (bifurcation \( 2 \to 4 \)) we obtain from the above equations \( x_2^{(1)} = x_2^{(2)} + \sqrt{2} \) and \( x_2^{(2)} = x_2^{(1)} - \sqrt{2} \), respectively.

2.3. Period-n orbits

From Eq. (22) we can observe that the dilatation effect on the deformed stability index \( m_q \) is a global factor. For a period-n orbit we obtain from Eqs. (6) and (9)

\[
m_q = J_n^q \times \ldots J_2^q \times J_1^q = \left[ - (q + 1) \right]^n x^{(n)} x^{(n-1)} \times \ldots \times x^{(2)} x^{(1)},
\]

where \( x^{(i)}, i = 1, \ldots, n \) are the orbital points. The term \( \left[ - (q + 1) \right]^n \) is a global factor, independent of the orbital points. Taking the limit \( q \to 1 \) in the above equation we observe that the stability index \( m \) can be written as

\[
m = J_n \times \ldots J_2 \times J_1 = \left[ -2 \right]^n x^{(n)} x^{(n-1)} \times \ldots \times x^{(2)} x^{(1)}.
\]

From Eqs. (31) and (32) we obtain the interesting relation

\[
\frac{m_q}{m} = \left[ \frac{(q + 1)}{2} \right]^n = e^n \ln \left[ \frac{q + 1}{2} \right].
\]

This shows us the connection between the stability index \( m \) and the q-deformed stability index \( m_q \) for any period-n orbit. Some interesting conclusions can be obtained from Eq. (33):

- For \( q = 1 \) we have \( m_q = m \), as expected. For \( q = -1 \) we have \( m_q = 0 \), a superstable orbit.
- For \( -1 < q < 1 \) (contraction) we obtain \( |m_q| < |m| \). Therefore the contraction tends to stabilize the periodic orbits. For \( n \to \infty \) we obtain \( m_q \to 0 \) (superstability).
- For \( |q| > 1 \) (dilatation) we obtain \( |m_q| > |m| \). The dilatation tends to destabilize the periodic orbits. For \( n \to \infty \) we obtain \( m_q \to \infty \) (a very large instability).
- From the last two conclusions we can conjecture that at the border line between periodic and chaotic motion (assuming \( n \) is very large), while contractions leave to superstable orbits, dilatation leaves to very unstable orbits.
- At the bifurcation points \( m_q = m = \pm 1 \) (or more generally at \( m_q = m \)) we always obtain \( q = 1 \). This means that, keeping \( a \) fixed, changes in the stability of the periodic orbits will only occur when \( q \neq 1 \).
- Changes in the stability (\( m_q \neq m \)) of periodic orbits via dilatation depend only on the period of the orbit, and not on their location in phase-space.
3. Summary
This paper proposes to use a $q$-deformed derivative [3] to analyse the stability of periodic points under simultaneous dilatation and translation of coordinates. Different from recent works [7, 8], where $q$-deformed maps were proposed, here the map is the usual one but the derivative utilized to determine the Jacobian of the map is deformed. Specifically we analyse the stability of period-1 and period-2 orbits of the quadratic map. We show that the dilatation can change the parameter values $a = -1/4, 3/4$, and $5/4$, where bifurcations of the low periodic orbits from the quadratic map occur. The $q$-deformation can be used to stabilize (destabilize) the unstable (stable) periodic orbits under dilatation ($|q| > 1$) or contraction ($|q| < 1$). We also show that the effect of dilatation and contraction on the stability depends only on the period of the orbit, and not on the location of the orbital points in phase-space. This occurs for any period-$n$ orbit of the quadratic map and suggests that at the border line between periodic and chaotic motion (assuming $n$ very large), contractions leave to superstable orbits while dilatation to very unstable orbits. More generally, the stability analysis proposed here, via the $q$-derivative, can be applied to any dynamical system, discrete or continuous, and explores a new approach to analyse the stability of physical systems. To finish we would like to make an attempt to give a physical interpretation for the dilatation and contraction: consider a one-dimensional monoatomic ideal gas with temperature defined by $\frac{1}{2} kT = \frac{1}{2} \langle mv^2 \rangle$. By making the dilatation (contraction) $x \to qx$, and therefore $v \to qv$, we obtain $\frac{1}{2} kT = \frac{1}{2} q^2 \langle mv^2 \rangle$, which shows that dilatation ($q^2 > 1$) are related to a temperature increase and contractions ($q^2 < 1$) are related to a temperature decrease. Obviously that our quadratic map did not represent the dynamics of an ideal gas, but the above arguments suggest that dilatation (contraction) of coordinates could be related to a temperature increase (decrease). The dynamics of a particle moving on a metallic plate which changes its temperature, or a particle moving on the surface of a balloon which inflates are just another examples of the dynamics on dilatated coordinates. The key point is that since the quadratic map presents the main features of more complicated dynamical systems, its stability under dilatation (contraction) of coordinates should also present the main features observed in more complicated dilatated (contracted) systems, whatever induces the dilatation (contraction).

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[1] Euler L 1748 Introduction in Analysis Infinitorum (Bousquet: Lausanne)
[2] Gasper G and Rahman M Basic 1997 Hypergeometric Series (University Press: Cambridge)
[3] Ayse E 1997 Phys. Rev. Lett. 78 3245
[4] Costa U M S, Lyra M L, Plastino A R and Tsallis C (1997) Phys. Rev. E 56 245
[5] Tsallis T, Plastino A R and Zheng W M 1997 Chaos Solitons Fractals 8 885
[6] Lyra M L and Tsallis C 1998 Phys. Rev. Lett. 80 53
[7] Jaganathan R and Sinha S 2005 Phys. Lett. A 338 277
[8] Patidar V and Sud K K 2009 Commun. Nonlinear Sci. Numer. Simulat. 14 827
[9] Lichtenberg A J and Lieberman M A 1992 Regular and Chaotic Dynamics (Springer-Verlag: New York)
[10] Rech P C, Beims M W and Gallas J A C 2005 Phys. Rev. E 71 17202
[11] Beims M W and Gallas J A C 1997 Physica A 238 225
[12] Rech P C, Beims M W and Gallas J A C 2000 Europhys. Lett. 49 702