Diffusion effects in a superconductive model

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Abstract

A superconductive model characterized by a third order parabolic operator $L_{\varepsilon}$ is analysed. When the viscous terms, represented by higher-order derivatives, tend to zero, a hyperbolic operator $L_{0}$ appears. Furthermore, if $P_{\varepsilon}$ is the Dirichlet initial boundary-value problem for $L_{\varepsilon}$, when $L_{\varepsilon}$ turns into $L_{0}$, $P_{\varepsilon}$ turns into a problem $P_{0}$ with the same initial-boundary conditions as $P_{\varepsilon}$. The solution of the nonlinear problem related to the remainder term $r$ is achieved, as long as the higher-order derivatives of the solution of $P_{0}$ are bounded. Moreover, some classes of explicit solutions related to $P_{0}$ are determined, proving the existence of at least one motion whose derivatives are bounded. The estimate shows that the diffusion effects are bounded even when time tends to infinity.

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1 Introduction

We deal with the following third order parabolic equation:

$$\left( \partial_{xx} - \lambda \partial_{x} \right) (\varepsilon u_{t} + u) - \partial_{t}(u_{t} + \alpha u) = F(x,t,u)$$

which characterizes a lot of phenomena (see, for instance, [1] and references therein). In particular, when $F = \sin u - \gamma$ ($\gamma = const$), the semilinear equation (1) describes the evolution of the phase $u$ inside an exponentially shaped Josephson junction (ESJJ) [2, 3].

The (ESJJ) provides several advantages in comparison with rectangular junctions and the related bibliography is large and extensive. [1–3, 16–21]. Besides, interesting applications of Josephson junctions are SQUIDs which can be applied in a lot of fields [4]. Indeed, they are very versatile and they are used in medicine for

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the Magnetoencephalography (MEG) and for Magnetic resonance imaging (MRI) or in cosmology, being SQUIDs extremely sensitive detectors of magnetic and electric signals from any origin. Moreover, in geophysics they are used as magnetometers to discover mineral deposits, or as X-ray detectors to screen radioactive materials. [5, 6]. Furthermore, they are suitable in the development of smart materials such as superconductor cables, used to improve power grids. [7, 8]

Equation (1) is equivalent to the equation:

$$\varepsilon u_{xxt} - u_{tt} + u_{xx} - (\alpha + \frac{\varepsilon \lambda^2}{4}) u_t = \frac{\varepsilon \lambda^2}{4} u + e^{-\frac{\lambda}{2} x} F(x, t, e^{\frac{\lambda}{2} x} u)$$

(2)

and from this, letting

$$a = \alpha + \frac{\varepsilon \lambda^2}{4} - \frac{1}{\varepsilon} \quad \text{and} \quad b = \frac{\lambda^2 \varepsilon}{4} - \frac{a}{\varepsilon},$$

it is possible to obtain the following system:

$$\begin{cases}
\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} - v - au - \varepsilon e^{-\frac{\lambda}{2} x} F(x, t, e^{\frac{\lambda}{2} x} u) \\
\frac{\partial v}{\partial t} = bu - \frac{1}{\varepsilon} v - \varepsilon u_t F_t (x, t, e^{\frac{\lambda}{2} x} u)
\end{cases}$$

(3)

which is related to biological dynamical systems such as, for instance, the Fitzhugh Nagumo model (FHN) which reproduces the neurocomputational features of the neuron [9, 10].

So, as it has already been underlined in [11–15], by means of equation (1) it is possible to treat both superconductive problems and neuroscientific ones.

In this paper, the Dirichlet initial boundary value problem for equation (1) is considered and, according to the theory of the phenomena related to (ESJJ), as source term, it is assumed that $F = \sin u - \gamma$ ($\gamma = \text{const}$).

Equation (1) is a semilinear hyperbolic equation, perturbed by viscous terms which are described by higher-order derivatives with small diffusion coefficients $\varepsilon$. When $\varepsilon \equiv 0$, the parabolic equation (1) turns into a hyperbolic equation:

$$\left( \partial_{xx} - \lambda \partial_x \right) u_0 - \partial_t (\partial_t + \alpha) u_0 = F(x, t, u_0),$$

(4)

and the influence of the dissipative terms $\varepsilon \left( u_{xxt} - \lambda u_{xt} \right)$ on the wave behaviour of $u_0$ can be estimated when the difference

$$r(x, t, \varepsilon) = u(x, t, \varepsilon) - u_0(x, t)$$

(5)

is evaluated.

A similar problem has already been studied in [22], when the coefficient $\lambda$ in (1) is equal to zero and for finite intervals of time. Moreover, in [23, 24], for $\lambda = 0$ and $a =$
0, an asymptotic approximation is established by means of the two characteristic times: slow time \( \tau = \varepsilon t \) and fast time \( \theta = t/\varepsilon \).

In this paper, in order to evaluate the diffusion effects, a rigorous estimate of the remainder term \( r \), defined in (5), is done. Indeed, by means of a Fourier series with properties of rapid convergence, the solution in both linear and non linear case of the problem related to \( r \), is determined. Furthermore, some classes of solutions of the hyperbolic equation have been determined, proving that there exists almost a solution whose derivatives are bounded. Finally, as soon as the hyperbolic equation admits solutions with limited derivatives, an estimate for the remainder term is achieved by proving that the diffusion effects are both bounded when \( \varepsilon \) goes to zero and when the time \( t \) tends to infinity.

2 Statement of the problem

If \( T, l \) are two positive constants and

\[
\Omega_T = \{(x, t) : 0 < x < l, \ 0 < t \leq T\},
\]

let us consider the initial boundary value problem \( P_\varepsilon \):

\[
\begin{array}{l}
(\partial_{xx} - \lambda \partial_x) (\varepsilon \partial_t + u) - \partial_t (u_t + \alpha u) = F(x, t, u), \ (x, t) \in \Omega_T, \\
u(x, 0) = h_0(x), \ u_t(x, 0) = h_1(x), \ x \in [0, l], \\
u(0, t) = \varphi_0, \ u(l, t) = \varphi_1, \ 0 < t \leq T.
\end{array}
\]

(6)

where \( F, h_i, \varphi_i (i = 0, 1) \) are arbitrary specified functions and \( \varepsilon, \alpha, \) and \( \lambda \) are positive constants.

If \( \varepsilon \equiv 0 \), the parabolic equation (6) turns into the hyperbolic equation (4) and the new problem \( P_0 \) for \( u_0(x, t) \) has the same initial boundary conditions of \( P_\varepsilon \).

To evaluate the influence of the dissipative terms, letting

\[
\bar{F}(x, t, r) = F(x, t, r + u_0) - F(x, t, u_0) - \varepsilon \partial_{x t} (\partial_x - \lambda) u_0,
\]

(7)

the following problem \( P \) related to the remainder term \( r \), defined in (5) has to be considered:

\[
\begin{array}{l}
(\partial_{xx} - \lambda \partial_x) (\varepsilon \partial_t + 1) r - \partial_t (\partial_x + \alpha) r = \bar{F}(x, t, r), \ (x, t) \in \Omega_T, \\
r(x, 0) = 0, \ r_t(x, 0) = 0, \ x \in [0, l], \\
r(0, t) = 0, \ r(l, l) = 0, \ 0 < t \leq T.
\end{array}
\]

(8)

As for the model of superconductivity, the problem (8) is characterized by the source force:

3
\[ \bar{F} = \sin(r + u_0) - \sin u_0 - \varepsilon \partial_{xt} (\partial_x - \lambda)u_0. \] (9)

### 3 The Green function and its properties

Let us introduce the linear third order operator:

\[ \mathcal{L}_\varepsilon = (\partial_{xx} - \lambda \partial_x)(\varepsilon \partial_t + 1) - \partial_t(\partial_t + \alpha). \] (10)

The Green function of \( \mathcal{L}_\varepsilon \) has already been determined in [1] by means of Fourier method. Let

\[ \gamma_n = \frac{n\pi}{l}, \quad b_n = (\gamma_n^2 + \lambda^2/4), \quad g_n = \frac{1}{2}(\alpha + \varepsilon b_n), \quad \omega_n = \sqrt{g_n^2 - b_n} \] (11)

and

\[ G_n(t) = \frac{1}{\omega_n} e^{-g_n t} \sinh(\omega_n t), \] (12)

standard techniques allow to obtain the Green function as:

\[ G(x,t,\xi) = 2 e^{\frac{\lambda}{2} x} \sum_{n=1}^{\infty} G_n(t) \sin\gamma_n \xi \sin\gamma_n x. \] (13)

The following theorem, proved in [1], ensures that this series is characterized by properties of rapid convergence and that it is exponentially vanishing as \( t \) tends to zero.

**Theorem 1.** Whatever the constants \( \alpha, \varepsilon, \lambda \) may be in \( \mathbb{R}^+ \), denoting by \( a_\lambda = \alpha + \varepsilon \lambda^2/4 \) and with

\[ p_\lambda = \frac{\pi^2}{\varepsilon \pi^2 + a_\lambda l^2}, \quad q_\lambda = \frac{a_\lambda + \varepsilon (\pi/l)^2}{2}, \quad \delta \equiv \min(p_\lambda, q_\lambda), \] (14)

the function \( G(x,\xi,t) \) defined in (13) and all its time derivatives, are continuous functions in \( \Omega_T \) and it results:

\[ |G(x,\xi,t)| \leq Me^{-\delta t}, \quad \left| \frac{\partial^j G}{\partial \xi^j} \right| \leq N_j e^{-\delta t}, \quad j \in \mathbb{N} \] (15)

where \( M, N_j \) are constants depending on \( \alpha, \lambda, \varepsilon \).

Moreover, one has:

\[ |\partial_x^{(i)}(\varepsilon G_t + G)| \leq A_i e^{-\delta t}, \quad (i = 0, 1, 2) \] (16)

where \( A_i \) \((i = 0, 1, 2)\) are constants depending on \( a, \varepsilon, \lambda \). \( \square \)
The uniform convergence proved in theorem 1 allows to prove that [1]:

\[ \mathcal{L}_\varepsilon G = (\partial_{xx} - \lambda \partial_x)(\varepsilon G_t + G) - \partial_t(G_t + \alpha G) = 0. \]  

(17)

4 The remainder term

To make an estimate of the solution to the problem \( P \) related to the remainder term \( r \), two cases should be distinguished according to the source term being known or depending by the unknown function.

Let us assume \( \bar{F}(x, r, t) = f(x, t) \). In this circumstance the remainder term \( r \) can be expressed by an explicit form. Indeed, standard computations (see f.i. [25]) ensure that the solution of the linear problem (8) is given by

\[ r_f(x, t) = - \int_0^t d\tau \int_0^l G(x, \xi, t - \tau) f(\xi, \tau) d\xi. \]  

(18)

Otherwise, if the source term \( \bar{F} \) is non linear, an integro differential formulation of the problem can be obtained.

Let us introduce the following

**Conditions A:** Let us assume that function \( F(x, t, u) \) is defined and continuous on the set

\[ D_T \equiv \{(x, t, u) : (x, t) \in \Omega_T, \ -\infty < u < \infty \} \]  

(19)

and that it is Lipschitz continuous in \( u \) i.e. there exists a constant \( K \) such that the inequality

\[ |F(x, t, u_1) - F(x, t, u_2)| \leq K |u_1 - u_2| \]  

(20)

holds for all \((u_1, u_2)\) and \((x, t) \in \Omega_T\).

Under Assumption A, by means of the fixed point theorem, it is possible to prove [26–28] that the non linear problem \( P \) admits a solution given by

\[ r(x, t) = \int_0^l d\tau \int_0^l G(x, \xi, t - \tau) \bar{F}(\xi, \tau, r(\xi, \tau)) d\xi. \]  

(21)

5 Explicit solutions of the hyperbolic equation

Let us consider the semilinear second order equation:

\[ u_{0,xx} - u_{0,tt} - \alpha u_{0,t} - \lambda u_{0,x} = \sin u_0 - \gamma \]  

(22)

When \( \lambda = 0 \), (22) represents the perturbed sine - Gordon equation (PSGE) and there is plenty of literature about its classes of solution. (e.g. [29–31] and bibliography therein.)
Now, let $\psi$ be an arbitrary function and let us consider $\Pi(\psi)$ as:

$$\Pi(\psi) = 2 \arctan e^\psi$$

such that

$$\sin \Pi(\psi) = \frac{1}{\cosh(\psi)}, \quad \cos \Pi(\psi) = -\tanh(\psi).$$

By means of function (23) it is possible to find a class of solutions of equation (22). Indeed, it is possible to verify that the following function:

$$u_0 = 2 \Pi[f(\xi)] \quad \text{with} \quad \xi = \frac{x-t}{\alpha - \lambda}$$

is a solution of (22) provided that one has:

$$(\lambda - \alpha) u_{0,t} = \sin u_0 - \gamma.$$  \hspace{1cm} (26)

Moreover, since (24) and being $\dot{\Pi} = \frac{1}{\cosh f}$, it results:

$$(\lambda - \alpha) u_{0,t} = 2 \frac{f'}{\cosh f}$$

$$\sin u_0 = 2 \sin \Pi \cos \Pi = -2 \frac{\tanh f}{\cosh f},$$

so, from (26), one deduces that function $f$ must satisfy the following equation:

$$\frac{df}{\tanh f + \gamma/2 \cosh f} = -d\xi.$$  \hspace{1cm} (27)

Since physical experiences show that generally $0 \leq \gamma \leq 1$, we point our attention to those cases in which it results:

$$u_0 = 4 \arctan(y + \sqrt{y^2 + 1}).$$  \hspace{1cm} (28)

So, let $h$ be an arbitrary constant of integration, one obtains:

$$y = h e^{-\xi} \quad \text{when} \quad \gamma = 0,$$  \hspace{1cm} (29)

$$y = \frac{1 - (\xi - h)}{1 + \xi - h} \quad \text{when} \quad \gamma = 1.$$  \hspace{1cm} (30)

Moreover, assuming $\gamma < 1$, let

$$A = \pm \sqrt{1 - \gamma^2} \quad \text{and} \quad \beta = 1 + A.$$

For $\gamma^2 \neq \beta^2$, it results
\[ y = \frac{h \gamma e^\xi A - \beta}{1 - h e^\xi A}. \]  

**Remark 1:** In the case \( \gamma = 0 \), if denoted by

\[ z(x,t) = e^{-\frac{\xi - t}{\alpha - \lambda}} + \sqrt{e^{-\frac{2\xi - t}{\alpha - \lambda}} + 1}, \]

equation (22) admits the solution

\[ u_0(x,t) = 4 \arctan[z(x,t)]. \]

and it is possible to prove that

\[
\begin{align*}
& \left| u_{0,xt}(x,t) \right| \leq \frac{1}{2(\alpha - \lambda)^2} \frac{y(y^2 - 1)}{(y^2 + 1)^2}, \\
& \left| u_{0,xxt}(x,t) \right| \leq \frac{1}{2(\alpha - \lambda)^3} \frac{y(y^6 - 5y^5 - 5y^2 + 1)}{(y^2 + 1)^4}
\end{align*}
\]

which are bounded for all \((x,t) \in \Omega_T\).

### 6 Estimates for the remainder term

Let us assume \( \varepsilon = 0 \), and let \( u_0 \) be a solution of the problem \( \mathcal{P}_0 \).

**Theorem 2.** If there exist two positive constants \( h, k \) depending only on \( \alpha \) and \( \lambda \) such that

\[ |u_{0,xt}(x,t)| \leq h; \quad |u_{0,xxt}(x,t)| \leq k, \]  

denoting by

\[ S(t) = \sup_{\Omega_T} |r(x,t,\varepsilon)|, \]

the following inequality holds:

\[ 0 \leq S(t) \leq \frac{Ml(h + k)\varepsilon}{\delta} e^{(1 - e^{-\delta t})} \]

beeing \( M \) a positive constant depending on \( \alpha, \lambda, \varepsilon \) and \( \delta \) is defined by (14).

Proof: Let us consider function \( \bar{F} \) defined in (9):

\[ \bar{F} = \sin(r + u_0) - \sin u_0 - \varepsilon \partial_{xt}(\partial_x - \lambda)u_0. \]

That source term satisfies conditions A and so, by means of hypotheses (35), it results:
\[ |r(x,t, \varepsilon)| \leq \int_0^t d\tau \int_0^t G(x, \xi, t - \tau) \left[ |r(\xi, t, \varepsilon)| + \varepsilon (h + k) \right] d\xi. \tag{39} \]

By properties of the Green function \( G \), and in particular since (15)\textsubscript{1}, one obtains:

\[ 0 \leq S(t) \leq M l \int_0^t e^{-\delta (t-\tau)} \left[ S(\tau) + \varepsilon (h + k) \right] d\tau \tag{40} \]

So, by means of Gronwall Lemma, inequality (37) holds.

Inequality (37) shows that, under assumptions of theorem 2, in the evolution of \( u(x,t) \) the correction due to the \( \varepsilon \)-terms is bounded even when \( t \) tends to infinity, and it vanishes as soon as \( \varepsilon \) tends to zero.

**Remark 2** Formulas (34) show that the class of functions satisfying hypotheses of Theorem 2 is not empty.

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