ON SUM SETS OF SETS, HAVING SMALL PRODUCT SET

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Annotation.

We improve a result of Solymosi on sum–products in \( \mathbb{R} \), namely, we prove that
\[
\max \{ |A + A|, |AA| \} \gg |A|^{\frac{4}{3} + c},
\]
where \( c > 0 \) is an absolute constant. New lower bounds for sums of sets with small product set are found. Previous results are improved effectively for sets \( A \subset \mathbb{R} \) with \( |AA| \leq |A|^{4/3} \).

1 Introduction

Let \( A, B \subset \mathbb{R} \) be finite sets. Define the sum set, the product set and quotient set of \( A \) and \( B \) as
\[
A + B := \{ a + b : a \in A, b \in B \},
\]
\[
AB := \{ ab : a \in A, b \in B \},
\]
and
\[
A/B := \{ a/b : a \in A, b \in B, b \neq 0 \},
\]
correspondingly. The Erdös–Szemerédi conjecture \cite{1} says that for any \( \epsilon > 0 \) one has
\[
\max \{ |A + A|, |AA| \} \gg |A|^{2-\epsilon}.
\]
Roughly speaking, it asserts that an arbitrary subset of real numbers (or integers) cannot have good additive and multiplicative structure, simultaneously. At the moment the best result in this direction is due to Solymosi \cite{2}.

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Theorem 1 Let $A \subset \mathbb{R}$ be a set. Then

$$|A + A|^2 |A/A|, \quad |A + A|^2 |AA| \geq \frac{|A|^4}{4|\log |A||}.$$  \hfill (1)

In particular

$$\max \{|A + A|, |AA|\} \gg \frac{|A|^{4/3}}{\log^{1/3} |A|}.$$  \hfill (2)

Here and below we suppose that $|A| \geq 2$.

It is easy to see that bound (1) is tight up to logarithmic factors if the size of $A + A$ is small relatively to $A$. The first part of the paper concerns the case where the product $AA$ is small. We will write $a \lesssim b$ or $b \gtrsim a$ if $a = O(b \cdot \log^c |A|)$, $c > 0$. In these terms inequality (1) implies the following.

Corollary 2 Let $A \subset \mathbb{R}$ be a finite set and $K \geq 1$ be a real number. Suppose that $|A/A| \leq K|A|$ or $|AA| \leq K|A|$. Then

$$|A + A| \gtrsim |A|^{\frac{5}{3}} K^{-\frac{1}{2}}.$$  \hfill (3)

Estimate (3) was improved for small $K$, see e.g. references in paper [11] (sharper bounds for difference of two sets, having small multiplicative doubling can be found in [8]). Here we give a result from [11].

Theorem 3 Let $A \subset \mathbb{R}$ be a finite set and $K \geq 1$ be a real number. Suppose that $|A/A| \leq K|A|$ or $|AA| \leq K|A|$. Then

$$|A + A| \gtrsim |A|^{\frac{58}{37}} K^{-\frac{42}{37}}.$$  \hfill (4)

It is easy to check that the bound of Theorem 3 is better than Corollary 2 for $K \lesssim |A|^{\frac{3}{7}}$.

Let us formulate the first result of the article (its refined version is contained in Theorem 11 and Theorem 13 below).
Theorem 4 Let $A \subset \mathbb{R}$ be a finite set and $K \geq 1$ be a real number. Suppose that $|A/A| \leq K|A|$ or $|AA| \leq K|A|$. Then

$$|A + A| \gtrsim |A|^{12} K^{-\frac{5}{6}}$$

and

$$|A + A| \gtrsim |A|^{\frac{5}{12}} K^{-\frac{5}{12}}.$$ 

Theorem 4 is stronger than Theorem 3 and refines estimate (3) for $K \lesssim |A|^{1/3}$.

In Theorem 15 we improve bound (2).

Theorem 5 Let $A \subset \mathbb{R}$ be a set. Then

$$\max \{|A + A|, |AA|\} \gg |A|^{\frac{4}{3} + c},$$

where $c > 0$ is an absolute constant.

Besides, a "critical" case of Solymosi's theorem, i.e. the situation where the reverse inequality to (1) takes place is considered in the paper, see Proposition 14.

We use a combination of methods from [12] and [7] in our arguments.

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2 Definitions and preliminary results

The additive energy $E^+(A, B)$ between two sets $A$ and $B$ is the number of the solutions of the equation (see [13])

$$E^+(A, B) = |\{a_1 + b_1 = a_2 + b_2 : a_1, a_2 \in A, b_1, b_2 \in B\}|.$$

The multiplicative energy $E^*(A, B)$ between two sets $A$ and $B$ is the number of the solutions of the equation (see [13])

$$E^*(A, B) = |\{a_1 b_1 = a_2 b_2 : a_1, a_2 \in A, b_1, b_2 \in B\}|.$$
In the case $A = B$ we write $E^+ (A)$ for $E^+ (A, A)$ and $E^x (A)$ for $E^x (A, A)$. Having $\lambda \in A / A$, we put $A_\lambda = A \cap \lambda A$. Clearly, if $0 \not\in A$ then

$$E^x (A) = \sum_{\lambda \in A / A} |A_\lambda|^2$$

and, similarly, for the energy $E^+ (A)$. Finally, the Cauchy–Schwarz inequality implies for $0 \not\in A$, $A_1 \subset A$, $A_2 \subset A$ that

$$E^x (A_1, A_2) |A / A| \geq |A_1|^2 |A_2|^2, \quad E^x (A_1, A_2) |AA| \geq |A_1|^2 |A_2|^2.$$  \hfill (6)

In particular

$$E^x (A) |A / A| \geq |A|^4, \quad E^x (A) |AA| \geq |A|^4.$$  \hfill (7)

Solymosi’s Theorem \ref{1} can be derived from a slightly delicate result on an upper bound for the multiplicative energy of a set via its sum set, see \cite{12}. Estimation of the cardinality of the set from the left hand side of (8) is the main task of our crucial Lemma 10.

**Theorem 6** Let $A, B \subseteq \mathbb{R}$ be a finite sets with $\min\{|A|, |B|\} \geq 2$ and $\tau \geq 1$ be a real number. Then

$$|\{x : |A \cap xB| \geq \tau\}| \ll \frac{|A + A||B + B|}{\tau^2}.$$ \hfill (8)

In particular

$$E^x (A, B) \ll |A + A||B + B| \cdot \log (\min\{|A|, |B|\}).$$ \hfill (9)

We need in Lemma 7 from \cite{5}. In paper \cite{7}, see Lemma 27, the same result was obtained with the additional factor $\log^2 d(A)$.

**Lemma 7** Let $A \subset \mathbb{R}$ be a finite set. Then for any finite set $B \subset \mathbb{R}$ and an arbitrary real number $\tau \geq 1$ one has

$$|\{x \in A + B : |A \cap (x - B)| \geq \tau\}| \ll d(A) \cdot \frac{|A||B|^2}{\tau^3},$$ \hfill (10)

where

$$d(A) := \min_{C \neq \emptyset} \frac{|AC|^2}{|A||C|}.$$
Obviously, if $|A/A| \leq K|A|$ or $|AA| \leq K|A|$ then $d(A)$ does not exceed $K^2$. The quantity $d(A)$ is a more delicate characteristic of a set than $|A/A|/|A|$ or $|AA|/|A|$. For example, rough estimate (11) can be derived from a stronger one

$$|A + A| \gtrsim |A|^{2}d(A)^{-2},$$

(11)

see [11].

Lemma 7 implies the following result.

**Corollary 8** Let $A_1, A_2, A_3 \subset \mathbb{R}$ be any finite sets and $\alpha_1, \alpha_2, \alpha_3$ be arbitrary nonzero numbers. Then the number of the solutions of the equation

$$\sigma(\alpha_1 A_1, \alpha_2 A_2, \alpha_3 A_3) := |\{\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 = 0 : a_1 \in A_1, a_2 \in A_2, a_3 \in A_3\}|$$

(12)
does not exceed $O(d^{1/3}(A_1)|A_1|^{1/3}|A_2|^{2/3}|A_3|^{2/3})$.

**Proof of the corollary.** Without loosing of generality, we can suppose that $\alpha_1 = 1$. Then the number of the solutions of equation (12) is

$$\sigma := \sum_{x \in (-\alpha_3 A_3)} |A_1 \cap (x - \alpha_2 A_2)|.$$  

(13)

Let us arrange the values of $|A_1 \cap (x - \alpha_2 A_2)|$ in decreasing order, that is

$$|A_1 \cap (x_1 - \alpha_2 A_2)| \geq |A_1 \cap (x_2 - \alpha_2 A_2)| \geq \ldots$$

Using Lemma 7, we obtain

$$|A_1 \cap (x_j - \alpha_2 A_2)| \ll d^{1/3}(A_1)|A_1|^{1/3}|A_2|^{2/3}j^{-1/3}.$$  

Substituting the last bound in (13), we get

$$\sigma \ll d^{1/3}(A_1)|A_1|^{1/3}|A_2|^{2/3}|A_3|^{2/3}$$

as required.

The last result of the section connects the quantity $E^+(A)$ with $|A/A|$ and $|AA|$. We follow the arguments from [2] in the proof.

**Theorem 9** Let $A \subset \mathbb{R}$ be a finite set. Then

$$|A/A||A|^{10} \log |A| \gg (E^+(A))^4, \quad |AA||A|^{10} \log |A| \gg (E^+(A))^4.$$  

(14)
**Proof of the theorem.** Without losing of generality, we can suppose that all elements of $A$ are positive. For $x \in \mathbb{R}$ put

$$N(x) = |A \cap (x - A)|.$$  

We have

$$\sum_{x \in A + A} N(x) = |A|^2, \quad \sum_{x \in A + A} N^2(x) = E^+(A). \quad \text{(15)}$$

Let

$$F = \left\{ x \in A + A : N(x) > \frac{E^+(A)}{2|A|^2} \right\}.$$  

Then

$$\sum_{x \notin F} N^2(x) \leq \sum_{x \notin F} N(x) \cdot \frac{E^+(A)}{2|A|^2}.$$  

Using this and the first formula of (15), we obtain

$$\sum_{x \notin F} N^2(x) \leq |A|^2 \cdot \frac{E^+(A)}{2|A|^2} = \frac{E^+(A)}{2}. \quad \text{(16)}$$

Applying (15) once more time, we get

$$\sum_{x \in F} N^2(x) \geq \frac{E^+(A)}{2}. \quad \text{(16)}$$

Put

$$U = \sum_{x \in F} N(x).$$

Because of (16) and a trivial bound $N(x) \leq |A|$, we have

$$U \geq \frac{E^+(A)}{2|A|}. \quad \text{(17)}$$

Further, by the definition of the set $F$

$$|F| \leq \frac{2|A|^2 U}{E^+(A)}. \quad \text{(17)}$$

Using this and inequality (17), we obtain

$$|F| + |A| \leq \frac{4|A|^2 U}{E^+(A)}. \quad \text{(18)}$$
Let us consider the set of points from $\mathbb{R}^2$:

$$P = (A \cup F) \times (A \cup F)$$

and let us estimate the number of collinear triples $T$ from $P$ (points in a triple are not necessarily distinct). On the one hand, a general upper bound for the number of such triples in Cartesian products ([13], Corollary 8.9) gives us

$$T \ll |A \cup F|^4 \log |A|.$$ 

Because of (18), it implies

$$T \ll |A|^8 U^4 \log |A| \quad \text{(19)}.$$ 

On the other hand, for $x \in A$ put

$$F(x) = \{ y \in A : x + y \in F \}.$$ 

Fixing $e, f \in A$, we have by (6) that there are at least

$$T(e, f) = F^2(e) F^2(f) / \min\{|AA|, |A/A|\}$$

quadruples $(a, b, c, d)$ such that $ab = cd$, $a, c \in F(e)$, $b, d \in F(f)$. They form at least $T(e, f)$ collinear triples

$$(e, f), (e + a, f + d), (e + c, f + b).$$

It follows that

$$T \geq \min(|AA|, |A/A|)^{-1} \sum_{e, f \in A} F^2(e) F^2(f) = \min(|AA|, |A/A|)^{-1} \left( \sum_{e \in A} F^2(e) \right)^2.$$ 

By the Cauchy–Schwarz inequality

$$\sum_{e \in A} F^2(e) \geq \left( \sum_{e \in A} F(e) \right)^2 |A|^{-1} = U^2 |A|^{-1}.$$ 

Whence

$$T \geq \min(|AA|, |A/A|)^{-1} U^4 |A|^{-2} \quad \text{(20)}.$$ 

Combining estimates (19) and (20), we obtain the required result.
3 The proof of the main results

We begin with a technical lemma.

Let $A \subset \mathbb{R}$, $0 \notin A$ be a finite set and $\tau > 0$ be a real number. Let also $S'_\tau$ be a set
\[ S'_\tau \subset S_\tau := \{ \lambda : \tau < |A_\lambda| \leq 2\tau \} \subseteq A/A \]
and for any nonzero $\alpha_1, \alpha_2, \alpha_3$ and different $\lambda_1, \lambda_2, \lambda_3 \in S'_\tau$ one has
\[ \sigma(\alpha_1 A_{\lambda_1}, \alpha_2 A_{\lambda_2}, \alpha_3 A_{\lambda_3}) \leq \sigma. \]

Lemma 10 Let $A \subset \mathbb{R}$, $0 \notin A$ be a finite set, $\tau > 0$ be a real number,
\[ 32\sigma \leq \tau^2 \leq |A + A|\sqrt{\sigma}, \]
and $S'_\tau$, $\sigma$ are defined above. Then
\[ |A + A|^2 \geq \frac{\tau^3 |S'_\tau|}{128\sqrt{\sigma}}. \]

Proof of the lemma. We follow the arguments from [12]. Without losing of generality, one can suppose that $A \subset \mathbb{R}^+$. Consider the Cartesian product $A \times A$ and the lines $l_\lambda$ of the form $y = \lambda x$, where $\lambda \in A/A$. Clearly, any line $l_\lambda$ intersects $A \times A$ under the points $(x, \lambda x)$, $x \in A_\lambda$. Put $A_\lambda = l_\lambda \cap (A \times A)$.

Let $2 \leq M \leq |S'_\tau|$ be an integer parameter, which we will choose later. Arrange the elements of the set $S'_\tau$ in increasing order and split it onto the groups of consecutive elements, each group has the size $M$. We get $k \geq \left[ \frac{|S'_\tau|}{M} \right] \geq \frac{|S'_\tau|}{2M}$ such groups $U_j$. Take the sets $A_\lambda$ from each of the group and consider all its sums. Clearly, the sums belong $(A + A) \times (A + A)$ and thus its total number does not exceed $|A + A|^2$. On the other hand, by the inclusion–exclusion principle the number of such sums in any fixed group $U_j$ is at least
\[ \rho_j := \tau^2 \binom{M}{2} - \sum_{\lambda_1, \ldots, \lambda_4 \in U_j, \lambda_1 \neq \lambda_2, \lambda_3 \neq \lambda_4, \{\lambda_1, \lambda_2\} \neq \{\lambda_3, \lambda_4\}} \left| \{ z : z \in (A_{\lambda_1} + A_{\lambda_2}) \cap (A_{\lambda_3} + A_{\lambda_4}) \} \right| \]
\[ = \tau^2 \binom{M}{2} - \sum_{\lambda_1, \ldots, \lambda_4 \in U_j, \lambda_1 \neq \lambda_2, \lambda_3 \neq \lambda_4, \{\lambda_1, \lambda_2\} \neq \{\lambda_3, \lambda_4\}} \varepsilon(\lambda_1, \ldots, \lambda_4). \]
Fix $\lambda_1, \ldots, \lambda_4$ and prove that the quantity $\mathcal{E}(\lambda_1, \ldots, \lambda_4)$ does not exceed $\sigma$.

Either all the numbers $\lambda_1, \ldots, \lambda_4$ are distinct or two of them coincide but the other two are different and differ from the first two numbers. In any case there is a number, which differs from all of them. Without losing of generality, we can suppose that it is $\lambda_4$. If 

$$z = (z_1, z_2) \in (A_{\lambda_1} + A_{\lambda_2}) \cap (A_{\lambda_3} + A_{\lambda_4})$$

then $z_1 = a_1 + a_2 = a_3 + a_4$, $z_2 = \lambda_1 a_1 + \lambda_2 a_2 = \lambda_3 a_3 + \lambda_4 a_4$ for some $a_j \in A_{\lambda_j}$ ($j = 1, 2, 3, 4$). It follows that 

$$0 = \lambda_1 a_1 + \lambda_2 a_2 - \lambda_3 a_3 - \lambda_4 a_4 - \lambda_4 (a_1 + a_2 - a_3 - a_4)$$

whence 

$$(\lambda_1 - \lambda_4) a_1 + (\lambda_2 - \lambda_4) a_2 - (\lambda_3 - \lambda_4) a_3 = 0.$$ 

The number of tuples $(a_1, a_2, a_3)$ satisfying the equation is 

$$\sigma((\lambda_1 - \lambda_4) A_{\lambda_1}, (\lambda_2 - \lambda_4) A_{\lambda_2}, (\lambda_4 - \lambda_3) A_{\lambda_3}) \leq \sigma.$$ 

Returning to formula (23) and using bound $\mathcal{E}(\lambda_1, \ldots, \lambda_4) \leq \sigma$, we get 

$$\rho_j \geq \tau^2 \left( \frac{M}{2} \right) - \sigma M^4.$$ 

Hence 

$$|A + A|^2 \geq \frac{|S'_{r}|}{2M} \left( \tau^2 \left( \frac{M}{2} \right) - \sigma M^4 \right) \geq \frac{|S'_{r}|}{2M} \left( \frac{\tau^2 M^2}{4} - \sigma M^4 \right).$$ 

Put $M = \lceil \sqrt{\tau^2/8\sigma} \rceil$. The required inequality $M \geq 2$ follows from the first condition of (21). Besides, if we have $M \leq |S'_{r}|$ then 

$$|A + A|^2 \geq \frac{M \tau^2 |S'_{r}|}{16} \geq \frac{\tau^3 |S'_{r}|}{128 \sqrt{\sigma}}$$

as required. In contrary, suppose that $M > |S'_{r}|$ and assume that inequality (22) fails. Then 

$$|A + A|^2 < \frac{\tau^3 |S'_{r}|}{128 \sqrt{\sigma}} < \frac{\tau^3 M}{128 \sqrt{\sigma}} < \frac{\tau^4}{256 \sigma}$$

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with a contradiction to the RHS condition (21). This concludes the proof of the lemma.

Let us prove the first part of Theorem 4 which is our main result on sets with small product set. It is easy to see, that theorem below refines Solymosi’s estimate (3) for $K \lesssim |A|^{1/4}$.

**Theorem 11**  Let $A \subset \mathbb{R}$ be a finite set and $K \geq 1$ be a real number. Suppose that $|AA| \leq K|A|$ or $|A/A| \leq K|A|$. Then

$$E^x(A) \ll K^{1/2}|A|^{3/4} |A + A|^{3/4} (\log |A|)^{3/4}.$$

In particular

$$|A + A| \gg |A|^{10/9} K^{-7/5} (\log |A|)^{-1/2}.$$

**Proof of the theorem.** Estimate (25) follows from (24) via inequality (7) thus it is sufficient to prove (24).

Without loosing of generality, we can suppose that $0 \notin A$. Let $L = \log |A|$. In the light of inequality (9) it is sufficient to check bound (24) just for $K^2 \leq L^2 |A + A|^4 |A|^{-5}$. From this bound and Solymosi’s estimate (1), we derive

$$|A + A| \gg |A|^{10/9} K^{-7/5} (\log |A|)^{-1/2}.$$

Further, because of $d(A) \leq K^2$, we have

$$d(A) \ll L^2 |A + A|^4 |A|^{-5}.$$

Take a parameter $\Delta = CL^{3/4} d^{1/8} |A + A|^3 |A|^{-11/8}$, where $C > 0$ is an absolute constant which we will choose later. The constant $C$ depends on another constant $C_1 > 0$ which we will choose later as well. By (27)

$$d(A)|A| \ll L^{3/2} d^{1/4} (A)|A + A|^3 |A|^{-11/4}$$

and we have for sufficiently large $C$ that

$$C_1 d(A)|A| \leq \Delta^2.$$

Further

$$E^x(A) = \sum_x |A \cap xA|^2 \leq \Delta |A|^2 + \sum_{j \geq 1} \sum_{x : \Delta 2^{j-1} < |A \cap xA| \leq \Delta 2^j} |A \cap xA|^2.$$

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Let us note that in formula (29) for large enough $|A|$ it is sufficient to consider $j$ satisfying inequality
\[2^j \leq |A|^{11/8}|A + A|^{-3/4}.\] (30)

Indeed, suppose in contrary that $2^j > |A|^{11/8}|A + A|^{-3/4}$. Then by inequality (26), we get
\[|A| \geq \Delta 2^j > CL^{3/4} d^{1/8}(A)|A + A|^{3/4} \geq CL^{3/4}|A + A|^{3/4} \gg C|A|^{33/32} L^{\frac{3}{8}}\]
with a contradiction for large $|A|$. Let $\tau = \Delta 2^j - 1$ and $\sigma = \sigma(S_\tau)$. Take an arbitrary $\lambda \in S_\tau$. By the definition of the set $S_\tau$, we get $d(A_\lambda) \leq |A|\tau^{-1}d(A)$. Applying Corollary 8 and using the definition of the set $S_\tau$ once more time, we get for any nonzero numbers $\alpha_1, \alpha_2, \alpha_3$
\[
\sigma(\alpha_1 A_\lambda, \alpha_2 A_\lambda, \alpha_3 A_\lambda) \leq \sigma,
\]
where
\[\sigma \ll (|A|\tau^{-1}d(A))^{1/3} \tau^{5/3}\] (31)
and we can take $\sigma = Md^{1/3}|A|^{1/3}\tau^{4/3}$, where $M > 0$ is some constant. Put $C_1 = (32M)^3$, and the constant $C$ has chosen such that inequality (28) takes place. It follows that
\[\Delta^{2/3} \geq 32Md^{1/3}|A|^{1/3}.\]
Hence for $\tau \geq \Delta$, we have
\[\tau^2 \geq 32Md^{1/3}|A|^{1/3} \tau^{4/3}.\]
Thus the first condition of (21) takes place.

For any $j$ and sufficiently large $|A|$ in view of inequality (30), we obtain
\[\tau = \Delta 2^{j-1} \leq CL^{3/4} d^{1/8}(A)|A + A|^{3/4} \leq M^{3/8}|A|^{1/8} d^{1/8}(A)|A + A|^{3/4}.
\]
It follows that
\[\tau^2 \leq M^{1/2}|A|^{1/6} d^{1/6}(A)|A + A|^{\tau^{2/3}}\]
and thus the second inequality of (21) holds.
So, both conditions (21) for $\tau = \Delta^{2^j - 1}$ take place and we can apply inequality (22) of the lemma to estimate the cardinality of the set $S_{\Delta^{2j-1}}$. Using (22), (31), we get

$$E^x(A) \ll \Delta |A|^2 + \sum_{j \geq 1} \frac{d^{1/6}(A)|A|^{1/6}|A + A|^2}{2^{j/3} \Delta^{1/3}} \ll \Delta |A|^2.$$  

It follows that

$$E^x(A) \ll L^{3/4}d^\frac{1}{4}(A)|A|^\frac{3}{4}|A + A|^\frac{1}{4} \leq L^{3/4}K^\frac{1}{3} |A|^\frac{1}{3}|A + A|^\frac{1}{3}.$$  

This completes the proof of the theorem.

In the next result we suppose that Solymosi’s inequality (1) cannot be improved. We will show that the assumption implies lower bound for the additive energy of a set and its product set $AA$.

**Lemma 12** Let $A \subset \mathbb{R}$, $0 \notin A$ be a finite set and $L \geq 1$ be a real number. Suppose that

$$|A + A|^2 |A/A| \leq L |A|^4.$$  

Then there is $\tau \geq E^x(A)/(2|A|^2)$ and some sets $S'_\tau \subseteq S_\tau \subseteq A/A$, $|S_\tau| \tau^2 \geq E^x(A)$, $|S'_\tau| \geq |S_\tau|/2$ such that for any element $\lambda$ from $S'_\tau$ one has

$$E^+(A_\lambda) \gtrsim \tau^3 L^{-4}$$  

and

$$|A_\lambda/A_{\lambda}| \gtrsim \tau^2 L^{-16}.$$  

Similarly, if

$$|A + A|^2 |AA| \leq L |A|^4$$  

then there exists $\tau \geq E^x(A)/(2|A|^2)$ and some sets $S'_\tau \subseteq S_\tau \subseteq A/A$, $|S_\tau| \tau^2 \geq E^x(A)$, $|S'_\tau| \geq |S_\tau|/2$ such that for any $\lambda \in S'_\tau$, we have (33) and

$$|A_\lambda A_{\lambda}| \gtrsim \tau^2 L^{-16}.$$  

Proof of the lemma. We consider the set $A/A$ because the arguments in
the case of the set $AA$ are similar. One can assume

$$L = \max(1, |A + A|^2 |A/A| |A|^{-1}).$$

By Dirichlet principle there is $\tau \geq E \times (A)/2|A|^2)$ such that $|S_\tau|^2 \gtrsim E \times (A)$.

From (7), we have

$$|S_\tau|^2 \gtrsim |A|^4 |A/A|. \tag{37}$$

If $|S_\tau| \geq 2$ then by $S''_\tau$ denote the set of cardinality $\lfloor |S_\tau|/2 \rfloor$ consisting all
$\lambda \in S_\tau$ with the minimal additive energy $E^+(A_\lambda)$ and put $S'_\tau = S_\tau \setminus S''_\tau$. It is
sufficient to check that for some $\lambda \in S''_\tau$ one has

$$E^+(A_\lambda) \gtrsim \tau^3 L^{-1}. \tag{38}$$

In the case $|S_\tau| = 1$ we put $S'_\tau = S''_\tau = S_\tau$ and it is sufficient to check
inequality (38) again.

Put $\sigma := \max_{\lambda \in S''_\tau} \sqrt{2 \tau E^+(A_\lambda)}$. Bound (33) follows from the inequality

$$\sigma \gtrsim \tau^2 L^{-2}, \tag{39}$$

which is aim of our proof.

By the Cauchy–Schwarz inequality for any $\alpha, \beta \neq 0$ and arbitrary sets $A_{\lambda_1}, A_{\lambda_2}, A_{\lambda_3}, \lambda_1, \lambda_2, \lambda_3 \in S''_\tau$ one has

$$\sigma(A_{\lambda_1}, \alpha A_{\lambda_2}, \beta A_{\lambda_3}) \leq |A_{\lambda_1}|^{1/2} (E^+(A_{\lambda_1}, \beta A_{\lambda_3}))^{1/2} \leq \leq (2 \tau)^{1/2} E^+(A_{\lambda_1})^{1/4} E^+(A_{\lambda_3})^{1/4} \leq \sigma.$$ 

If both conditions (21) of Lemma 10 (with $S''_\tau$ instead of $S'_\tau$) take place then we have

$$|A + A|^2 \geq \frac{\tau^3 |S''_\tau|}{128 \sqrt{\sigma}} \geq \frac{\tau^3 |S_\tau|}{384 \sqrt{\sigma}}.$$ 

Using condition (32), we get

$$\sigma^{1/2} \gg \frac{|S_\tau| \tau^3 |A/A|}{A/4 L}. \tag{40}$$

Substituting inequality (37) into (40), we get (39).

If the first condition (21) does not hold then we obtain (39) immediately. Suppose that the second condition (21) fails, that is $\tau^2 > |A + A| \sqrt{\sigma}$. By
inequality (7) for sums, we have a lower bound for $\sigma$, namely, $\sigma^2 \geq 2\tau^5 |A + A|^{-1}$. But then
\[
\tau^8 > |A + A|^4 \cdot 2\tau^5 |A + A|^{-1}
\]
with a contradiction, because, clearly, the parameter $\tau$ does not exceed the size of $A$.

Thus, we have proved inequality (33). Using Theorem 9 we obtain inequality (34). This concludes the proof of the lemma.

Now let us obtain the second main result of the paper, concerning the sets with small product set. It is easy to see that we improve inequality (3) for $K \lesssim |A|^{1/3}$.

Theorem 13 Let $A \subset \mathbb{R}$ be a finite set and $K \geq 1$ be a real number. Suppose that $|AA| \leq K|A|$ or $|A/A| \leq K|A|$. Then
\[
|A + A| \gtrsim |A|^{4/3} K^{-1/3}.
\] (41)

Proof of the theorem. Consider the situation where $|A/A| \leq K|A|$. The case $|AA| \leq K|A|$ is similar. One can suppose that $0 \notin A$. Let us apply Lemma 12 where
\[
L = \max(1, |A + A|^2 |A/A| |A|^{-1}).
\]
Take any $\lambda$ from $S'$ and use inequality (34) combining with the lower bound for $\tau$. It gives us
\[
|A/A| \gtrsim |A\lambda/A\lambda| \gtrsim \tau^2 L^{-16} \geq (E^\times(A))^2 |A|^{-4} L^{-16}.
\]
Further, because of (7), we have
\[
|A/A| \gtrsim |A|^4 |A/A|^{-2} L^{-16}.
\]
It follows that
\[
L \gtrsim |A|^{1/4} |A/A|^{-3/16}.
\]
After some simple calculations we obtain the result.

Theorem 13 improves Theorem 11 for $K \gtrsim |A|^{5/23}$.

Let us obtain a result on multiplicative energies of $A/A$, $AA$ in "critical case".
Proposition 14 Let $A \subset \mathbb{R}$ be a finite set. If condition (32) takes place then
\[
E^\times(A/A) \gtrsim \frac{(E^\times(A))^3}{L^{32}|A|^4}.
\] (42)

If condition (35) holds then
\[
E^\times(AA) \gtrsim \frac{(E^\times(A))^3}{L^{32}|A|^4}.
\] (43)

Proof of the proposition. Without losing of generality, we can suppose that $0 \notin A$. Let us begin with inequality (42). Put $\Pi = A/A$. Using Lemma 12, we find the number $\tau$ and the set $S'\tau$ satisfying all implications of the lemma. By the Katz–Koester inclusion (see [4]), namely $A_{\lambda}A_{\lambda} \subseteq \Pi \cap \lambda \Pi$, we see that for all $\lambda \in S'\tau$ the following holds
\[
|\Pi \cap \lambda \Pi| \geq |A_{\lambda}/A_{\lambda}| \gtrsim \tau^2 L^{-16}.
\]

Hence
\[
\sum_{\lambda \in S'\tau} |\Pi \cap \lambda \Pi| \gtrsim L^{-16} \tau^2 |S'\tau| \gtrsim L^{-16}E^\times(A).
\] (44)

Using the last bound as well as the Cauchy–Schwarz inequality, we get (42).

Now put $\Pi' = AA$. Then by the Katz–Koester inclusion, we have $A_{\lambda}A_{\lambda} \subseteq \Pi' \cap \lambda \Pi'$ and the previous arguments can be applied. This completes the proof of the proposition.

Thus, if $|A/A| \lesssim |A|^{4/3}$ and $L \lesssim 1$ then inequality (12) and bound (47) imply $E^\times(A/A) \gtrsim L^{-32}|A/A|^3 \gtrsim |A/A|^3$. In other words the multiplicative energy of the set $A/A$ is close to its maximal possible value. We use the observation in the proof of the final result of the paper.

Theorem 15 Let $A \subset \mathbb{R}$ be a set. Then for any $c < \frac{1}{25908}$ one has
\[
\max \{|A + A|, |A/A|\} \gg |A|^\frac{4}{3} + c
\] (45)

and
\[
\max \{|A + A|, |AA|\} \gg |A|^\frac{4}{3} + c.
\] (46)
Proof of the theorem. We prove estimate (45) because inequality (46) can be obtained similarly. Without losing of generality, suppose that $0 \not\in A$. Now assume that inequality (32) holds with some parameter $\tau$. Using Lemma 12, we have $|A/A|^{\lambda} \leq L'|A|^4$. Our task is to find a lower bound for quantities $L, L'$. Using this as well as the Katz–Koester inclusion, we obtain

\[
\sum_{x \in AA/A} |S'_{\tau} \cap x(A/A)| = \sum_{\lambda \in S'_{\tau}} |A/A \cap \lambda(A/A)| \geq \sum_{\lambda \in S'_{\tau}} |A\lambda/A\lambda| \geq L^{-16}\tau^2|S_{\tau}|.
\]

In view of the last bound and the Cauchy–Schwarz inequality, we get

\[
|A/A|E^\times(S'_{\tau}, A/A) = |A/A| \sum_{x} |S'_{\tau} \cap x(A/A)|^2 \geq L^{-32}\tau^4|S_{\tau}|^2.
\]

Applying the Cauchy–Schwarz inequality once more time, we obtain

\[
E^\times(S'_{\tau}) \gtrsim L^{-64}\tau^{8}|S_{\tau}|^4|A/A|^{-2}(E^\times(A/A))^{-1} \gtrsim L^{-64}E^\times(A)\tau^6|A/A|^{-5}|S_{\tau}|^3 = \eta|S_{\tau}|^3,
\]

where $\eta = L^{-64}E^\times(A)\tau^6|A/A|^{-5}$. We have

\[
\eta \gg L^{-64}E^\times(A)\left(E^\times(A)|A|^{-2}\right)^6|A/A|^{-5} = L^{-64}E^\times(A)^7|A|^{-12}|A/A|^{-5} \geq L^{-64}\left(|A|^4|A/A|^{-1}\right)^7|A|^{-12}|A/A|^{-5} = L^{-64}|A|^16|A/A|^{-12} \geq L^{-64}(L')^{-4}.
\]

In other words

\[
E^\times(S'_{\tau}) \gtrsim L^{-64}(L')^{-4}|S_{\tau}|^3.
\]

By Balog–Szemerédi–Gowers Theorem [1] (see also [6]) there is a set $S''_{\tau} \subseteq S'_{\tau}, |S''_{\tau}| \gtrsim \eta|S_{\tau}|$ such that $|S''_{\tau}/S''_{\tau}| \lesssim \eta^{-4}|S''_{\tau}|^3|S_{\tau}|^{-2}$.

Because of $S''_{\tau} \subseteq S_{\tau}$, we obtain

\[
\sum_{a \in A} |A \cap aS''_{\tau}| = \sum_{\lambda \in S''_{\tau}} |A \cap \lambda A| \gg \tau|S''_{\tau}|
\]

and hence there is $a \in A$ such that for the set $A' := A \cap aS''_{\tau}$ one has

\[
|A'| \gg \tau|S''_{\tau}||A|^{-1}.
\]
It follows that
\[ d(A') \leq \frac{|A'|^2}{|A'||S''_r|} \cdot \frac{|S''_r/A|}{\tau^2} \leq \eta^{-8} \frac{|A'|}{\tau} \cdot \frac{|S''_r|^4}{|S_r|^4}. \]

Using inequalities (7), (11) and the estimate for \( d(A') \), we get
\[ |A + A| \geq |A' + A'| \gtrsim |A'|^{\frac{58}{37}} d(A')^{-\frac{21}{37}} \gtrsim (\tau |S''_r||A|^{-1})^{\frac{58}{37}} (\eta^8 \tau |A|^{-1}|S_r|^4 |S''_r|^{-4})^{\frac{21}{37}} \]
\[ \gtrsim |S_r|^\frac{58}{37} (\tau |A|^{-1})^{\frac{21}{37}} \eta^{-\frac{21}{37}} \approx (E^x(A))^{\frac{58}{37}} |A|^{-\frac{78}{37}} \eta^{-\frac{21}{37}} \tau^{-1} \]
\[ = (E^x(A))^{\frac{58}{37}} |A|^{-\frac{78}{37}} \eta^{-\frac{21}{37}} (L^{-64} E^x(A)|A/A|^{-5})^{\frac{1}{3}} \]
\[ \gtrsim (E^x(A))^{\frac{58}{37}} |A|^{-\frac{78}{37}} (L^{-64} (L')^{-4})^{\frac{53}{37}} (L^{-64} E^x(A)|A/A|^{-5})^{\frac{1}{3}} \]
\[ = L^{-10772} (E^x(A))^{\frac{58}{37}} |A|^{-\frac{78}{37}} (L')^{-\frac{53}{37}} |A/A|^{-\frac{95}{37}} \]
\[ \gtrsim L^{-10772} (L')^{-\frac{1942}{37}} |A|^{-\frac{53}{37}} |A/A|^{-\frac{95}{37}} \gtrsim L^{-10772} (L')^{-\frac{1942}{37}} |A|^{-\frac{53}{37}} ((L')^{1/3}) |A|^{-4/3} \]
\[ = |A|^{\frac{51}{37}} L^{-10772} (L')^{-\frac{679}{37}}. \]

The last estimate is greater than \( |A|^{4/3} \) by some power of \( |A| \). Easy calculations show that one can take any number less than \( \frac{1}{20598} \) for the constant \( c \). This concludes the proof.

**Remark 16** It seems likely that the arguments of the proof of Theorem 13 allow to improve slightly the lower bound for the size of \( A + A \) of Theorem 13 in the regime where \( K \lesssim |A|^{1/3} \). We did not make such calculations.

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