Bivectorial Mesoscopic Nonequilibrium Thermodynamics: Potentials of Continuous Markov Process and Random Perturbations

Ying-Jen Yang* and Yu-Chen Cheng†
Department of Applied Mathematics, University of Washington, Seattle, 98195, USA
(Dated: July 20, 2020)

With a scalar potential and a bivectorial potential, the drift of a diffusion is decomposed into the sum of a generalized gradient field, a field perpendicular to the gradient, and a divergence-free field. We give such decompositions a probabilistic interpretation by introducing cycle velocity from a bivectorial formalism of nonequilibrium thermodynamics. New understandings on the mean rates of thermodynamic quantities are presented. Scalar potentials of deterministic dynamics, obtained from three different random perturbations, are compared in terms of their Lyapunov and geometric properties.

Introduction In mathematics, one often can gain a deeper understanding of an equation when it is expressed in terms of its solutions. While this might not sound meaningful in engineering, where the goal is to find the solution(s), it has been extremely fruitful in theoretical science. A case in point is to re-write the polynomial equation \( x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0 \) into \( (x - x_0)(x - x_1) \cdots (x - x_n) = 0 \). In a sense, one could say the set of \( n \) roots, collectively, defines the algebraic equation! Indeed, providing the setting for the existence and uniqueness of such a re-writing became one of the most significant chapters of mathematics [1].

In stochastic thermodynamics, the aforementioned philosophy translates to rewriting dynamic equations in terms of their time-invariant solution, the stationary probability density \( \pi \). For equilibrium dynamics with detailed balance, this has yielded Boltzmann’s law that relates the equilibrium distribution to a potential function, as well as the celebrated fluctuation-dissipation theorem. For nonequilibrium dynamics, the nonequilibrium potential \( \Phi \) as the negative logarithm of \( \pi \) was discussed in 1970s [2–4] and more recently in [5–7]. It has led to a general force field decomposition [8], has provided a notion of “energy” in general nonequilibrium systems [6], and has identified two sources of entropy production [5, 7, 9]. In this Letter, we derive a physically-meaningful further decomposition of the force field based on the recently-revealed bivectorial structure of nonequilibrium steady state (NESS) [10].

The fundamental roles of kinetic cycles of NESS in continuous Markov processes without detailed balance lead to a bivectorial formalism for stationary continuous processes represented by anti-symmetric matrices [10]. In the present work, we introduce cycle velocity \( Q \) based on the cycle flux \( A \) [10] and demonstrate its importance in the thermodynamics of continuous Markov processes. The resulting force decomposition surprisingly coincides with the decomposition proposed by P. Ao from a rather different narrative [11, 12]. Our work shows the generality of the decomposition, its probabilistic origin, and a cycle interpretation. Novel expressions on the mean rates of thermodynamic quantities are derived based on the bivectorial representation. The notion of cycle velocity further allows us to organize the random perturbation method of finding a potential function for a dynamical system. Potentials from three known perturbations [11–14] are compared based on their Lyapunov properties, shape, and smoothness. Thermodynamics for continuous Markov processes involves not only the nonequilibrium potential \( \Phi \) and the diffusion matrix \( D \), but also the bivector cycle velocity \( Q \).

Two Representations of Continuous Markov Processes A continuous Markov process on \( \mathbb{R}^n \) has two quite different representations [15]. One is based on the (transition) probability density \( p(x, t) \) with the Fokker-Planck equation (FPE)

\[
\partial_t p = -\nabla \cdot [b(x)p - D(x)\nabla p]
\]

(1)

where \( b(x) \) is the drift of the diffusion and \( D(x) \) is the diffusion matrix. The other is based on stochastic trajectories \( X_t \) with its probability measure. The trajectories satisfy a stochastic differential equation (SDE)

\[
dX_t = [b(X_t) + \nabla \cdot D(X_t)] dt + \sqrt{2D(X_t)} dW_t.
\]

(2)

\( \langle \nabla \cdot D \rangle_j = \sum_{i=1}^n \partial_i D_{ij} \) with \( \partial_i \) denotes partial derivative with respect to \( x_i \). \( W_t \) is the \( n \)-D Brownian motion, and \( \sqrt{2D} \) is understood as the matrix \( \Gamma \) such that \( 2D = \Gamma \Gamma^\top \). Correspondence between the two representations can be established by Ito’s calculus [16].

With Eq. (1) as the continuity equation of \( p \), the probability flux is given by

\[
J(x, t) := b(x)p(x, t) - D(x)\nabla p(x, t).
\]

(3)

This gives a notion of (Eulerian) probability velocity as

\[
v(x, t) := \frac{1}{p}J = b(x) + D(x)\nabla S(x, t)
\]

(4)

where \( S(x, t) = -\ln p(x, t) \) is the (stochastic) Shannon entropy. Following Ref. [17], the total heat dissipation in a infinitesimal time interval \( (t, t + dt) \) is given by

\[
dQ = D^{-1}(X_t) b(X_t) \circ dX_t
\]

(5)

where \( \circ \) indicates Stratonovich mid-point integration. The vector field \( D^{-1}v \) gives the thermodynamic force of the (stochastic) total entropy production [7, 10],

\[
dS_{\text{tot}} = dS + dQ
\]

(6a)

\[
= \partial_t S(X_t, t) dt + D^{-1}(X_t)v(X_t, t) \circ dX_t.
\]

(6b)
The decomposition in Eq. (4) reflects the two origins of the thermodynamic force $D^{-1} v$: the force $D^{-1} b$ of total heat dissipation [17] and the entropic force $\nabla S$ in the entropy change $dS = \partial_t S dt + \nabla S \circ dX_t$.

*Entropy Production Decomposition* We assume the system has a steady state with an invariant density $\pi(x)$. The divergence-free stationary probability flux is

$$J^*(x) = b(x)\pi(x) - D(x)\nabla \pi(x).$$  \hspace{1cm} (7)

And, the stationary probability velocity is

$$\nu^*(x) = b(x) + D(x)\nabla \Phi(x)$$  \hspace{1cm} (8)

where $\Phi(x) = -\ln \pi(x)$ is the *nonequilibrium potential* in stochastic thermodynamics [5, 7]. This leads to a decomposition of $D^{-1} b$ [8],

$$D^{-1} b(x) = -\nabla \Phi + D^{-1} \nu^*$$  \hspace{1cm} (9)

and a decomposition of the thermodynamic force

$$D^{-1}(x)\nu(x, t) = -\nabla F(x, t) + D^{-1}(x)\nu^*(x)$$  \hspace{1cm} (10)

where $F(x, t) = \Phi(x) - S(x, t)$ is understood as the “free energy” in nonequilibrium systems [6, 18]. This decomposition in Eq. (10) corresponds to the (stochastic) total entropy production decomposition $dS_{\text{tot}} = df + dQ_{\text{hk}}$ where $Q_{\text{hk}}$ is the housekeeping heat dissipation. See Ref. [7] and the references within for a recent synthesis. We note a recent study showing rigorously how the housekeeping heat dissipation in a compact, driven process can be mapped to the energy dissipation of a lifted, detailed-balanced process[19].

The decomposition in Eq. (10) can be interpreted as a decomposition of the FPE generator [20–22],

$$\partial_t \rho = -\nabla \cdot (p \nu^*) - \nabla \cdot [p (-D \nabla F)] .$$  \hspace{1cm} (11)

The former on the right-hand-side of Eq. (11) corresponds to a Liouville equation

$$\partial_t \rho = -\nabla \cdot (p \nu^*)$$  \hspace{1cm} (12)

of a measure-preserving deterministic dynamical system $x'(t) = \nu^*(x)$ with $e^{-\Phi}$ as an invariant measure. The latter corresponds to a detailed-balanced diffusion process with the same invariant density $e^{-\Phi}$ described by

$$\partial_t \rho = -\nabla \cdot [p (-D \nabla F)].$$  \hspace{1cm} (13)

Therefore, every diffusion process can be regarded as a deterministic dynamical system coupled with the “randomly-damping”, detailed-balanced environment [21, 22]. The fact that $e^{-\Phi}$ is the invariant measure before and after the coupling is considered as a generalization of the *zeroth law of thermodynamics* [21].

**Nonequilibrium potential as a Potential Function and the Maxwell-Boltzmann Equilibrium** In equilibrium physics, a “potential function” has many important features. Among those stand the following two prominent ones [23]: 1) It is related to the equilibrium probability distribution via the Maxwell-Boltzmann (M-B) distribution; 2) It is a Lyapunov function of the underlying deterministic dynamics. The potential $\Phi$ satisfies the first feature by it’s definition and achieved the second feature by the equilibrium condition $v^* = 0$: it is the Lyapunov function of $x'(t) = b(x) = -D(x) \nabla \Phi(x)$.

For general nonequilibrium systems, however, $\Phi$ is not always the Lyapunov function of the deterministic dynamics $x'(t) = b(x)$. The necessary and sufficient condition of the Lyapunov property is

$$\nabla \Phi \cdot \nu^* \leq \nabla \Phi \cdot D \nabla \Phi.$$  \hspace{1cm} (14)

This can be further rewritten as

$$\nabla \cdot \nu^* \leq \nabla \Phi \cdot D \nabla \Phi$$  \hspace{1cm} (15)

with the stationary FPE

$$\nabla \cdot (\pi \nu^*) = 0 \iff \nu^* \cdot \nabla \Phi = \nabla \cdot \nu^*.$$  \hspace{1cm} (16)

In particular, a sufficient condition of the Lyapunov property of $\Phi$ is

$$\nabla \cdot \nu^* = \nu^* \cdot \nabla \Phi = 0.$$  \hspace{1cm} (17)

Nonequilibrium systems with Eq. (17) is said to admit *Maxwell-Boltzmann (M-B) equilibrium* [22]. In such systems, $\Phi$ is a Lyapunov function of $x'(t) = b(x)$, and the vector field $b$ can be decomposed into a generalized gradient term $-\nabla \Phi$ and a divergent-free term $\nu^*$, akin to the Helmholtz decomposition in $\mathbb{R}^3$. Furthermore, with M-B equilibrium, the deterministic dynamics $x'(t) = \nu^*(x)$ corresponding to the Liouville equation in Eq. (12) has a divergent-free vector field $\nabla \cdot \nu^* = 0$ and a conserved quantity $\Phi(x)$. It is a generalization of Hamiltonian systems with energy $\Phi$ [21, 22].

**Bivectorial Decomposition with Cycle Velocity** The divergent-free stationary current $\nu^*$ can be furthered expressed as the $\nu$-D “curl” of a *bivector* $A$, an anti-symmetric matrix that represents cycle flux [10],

$$J^* = \nabla \times A$$  \hspace{1cm} (18)

where $(\nabla \times A)_{ij} = \sum_{k=1}^n \partial_j A_{ik}$. The stationary probability velocity $\nu^*$ can then be expressed as

$$\nu^* = e^\Phi \nabla \times A = \nabla \times (e^\Phi A) - A \nabla e^\Phi .$$  \hspace{1cm} (19)

We then introduce the *cycle velocity*

$$Q = \frac{1}{\pi} A = e^\Phi A ,$$  \hspace{1cm} (20)

which is also a bivector. The stationary velocity $\nu^*$ then has the decomposition

$$\nu^* = -Q \nabla \Phi + \nabla \times Q$$  \hspace{1cm} (21)
where the former is perpendicular to $\nabla \Phi$ and the latter is divergence-free. This decomposition was first proposed by Ao from a different derivation [11, 12]. Our derivation shows its generality and reveals its probabilistic origin with a cyclic interpretation. The vector field $\mathbf{b}$ now has a decomposition in terms of $\Phi, \mathbf{Q}$, and $\mathbf{D}$,

$$\mathbf{b} = -\mathbf{D} \nabla \Phi - \mathbf{Q} \nabla \Phi + \nabla \times \mathbf{Q},$$  

(22)

leading to the following decomposition of probability flux

$$\mathbf{J} = p \mathbf{v} = -p (\mathbf{D} + \mathbf{Q}) \nabla \mathbf{F} + \nabla \times (p \mathbf{Q})$$  

(23)

where the product rule $\nabla \times (p \mathbf{Q}) = \mathbf{Q} \nabla p + p \nabla \times \mathbf{Q}$ is used.

**Mean Rate of Thermodynamic Quantities.** The decomposition in Eq. (23) can lead to new understanding on the mean rates of various thermodynamic quantities. Following Ref. [10], various mean rates can be derived by considering a general work-like quantity $\mathcal{W}$ whose infinitesimal change satisfies $d \mathcal{W} = \mathbf{f}(\mathbf{x}_t, t) \circ d \mathbf{x}_t$ with a force field $\mathbf{f}(\mathbf{x}, t)$. With $\mathbb{E} [\cdot]$ denoting expectation with respect to $p(\mathbf{x}, t)$, the mean rate of $\mathcal{W}$ has the following decomposition

$$\bar{\mathcal{W}} := \frac{\mathbb{E}[d \mathcal{W}]}{dt} = \mathbb{E} [(-\mathbf{D} \nabla \mathbf{F}) \cdot \mathbf{f}] + \mathbb{E} [\mathbf{v}^* \cdot \mathbf{f}].$$  

(24)

The second term, by integration by part, can be rewritten as

$$\mathbb{E} [\mathbf{v}^* \cdot \mathbf{f}] = \mathbb{E} [\mathbf{Q} \cdot \nabla \mathbf{f}] + \mathbb{E} [\mathbf{Q} \cdot \mathbf{f} \wedge (-\nabla \mathbf{F})].$$  

(25)

Both wedge terms are bivectors with components $(u_i \wedge w_j)_{ij} = u_i w_j - u_j w_i$ for $1 \leq i < j \leq n$. The scalar products in Eq. (25) between two bivectors are the half of the Forbenius products between matrices [10]. Since $||u \wedge w|| = \sqrt{(u \wedge w) \cdot (u \wedge w)}$ is the area of the parallelogram of $u$ and $w$ in $\mathbb{R}^n$, a (simple) bivector $u \wedge w$ can be understood as a generalized “signed” area of in $\mathbb{R}^n$. See supplemental material for a brief introduction.

The decomposition in Eq. (25) shows the fundamental roles of cycles in nonequilibrium thermodynamics with nonzero $\mathbf{v}^*$. Both terms in Eq. (25) are cyclic averages of bivectors, both averaged over the “cyclic flux” $p \mathbf{Q} \mathcal{I}_i$ in each infinitesimal plane $dx_i \wedge dx_j$. This is an extension to Ref. [10] where cycle flux $\mathbf{A} = \pi \mathbf{Q}$ was first introduced in NESS. We note that as the system approaches NESS as $t \to \infty$, the term $\mathbb{E} [\mathbf{Q} \cdot \nabla \mathbf{f}]$ persists whereas $\mathbb{E} [\mathbf{Q} \cdot \mathbf{f} \wedge (-\nabla \mathbf{F})]$ is zero since $F \to 0$.

The two terms in Eq. (25) have the following physical interpretations. With $\nabla \wedge \mathbf{f}$ being the $n$-D “curl” of vectors, its cyclic average $\mathbb{E} [\mathbf{Q} \cdot \nabla \wedge \mathbf{f}]$ is the mean circulation of the force $\mathbf{f}$. Hence, if a force is a gradient vector field $\mathbf{f} = -\nabla U$, then $\nabla \wedge (-\nabla U) = 0$. This implies $\mathbb{E} [\mathbf{Q} \cdot \nabla \wedge \mathbf{f}]$ would be zero in the mean rate of state observables: $\mathcal{S}, \Phi$, and $F$. On the other hand, the wedge product $\mathbf{f} \wedge (-\nabla \mathbf{F})$ is the generalized “signed” area spanned by the two vectors $\mathbf{f}$ and $-\nabla \mathbf{F}$. $\mathbb{E} [\mathbf{Q} \cdot \mathbf{f} \wedge (-\nabla \mathbf{F})]$ is thus a “torque-like” quantity representing the mean area between the force $\mathbf{f}$ and $-\nabla \mathbf{F}$ averaged over all its planar components. It would be zero when $\mathbf{f}$ is parallel to $-\nabla \mathbf{F}$.

From Eq. (25), we see that the average orthogonality between $\nabla \mathbf{F}$ and $\mathbf{v}^*$ discussed in Ref. [10] is due to both $\nabla \mathbf{F}$ being a gradient field and parallel to $-\nabla \mathbf{F}$. The mean rate of free energy then has the expression

$$\dot{\mathcal{S}} = \mathbb{E} [- (\nabla \mathbf{F}) \cdot \mathbf{D} \nabla \mathbf{F}] \leq 0.$$  

(26)

where the first term is the average cycle affinity [10]. Eq. (26) is purely cyclic, reflecting the measure-preserving deterministic dynamics of Eq. (12) hidden behind.

The mean rates of entropy and nonequilibrium potential now have the following expressions:

$$\dot{\mathcal{S}} = \mathbb{E} [\nabla \mathbf{S} \cdot \mathbf{D} \nabla \mathbf{S}] - r + w$$  

(27a)

$$\dot{\Phi} = - \mathbb{E} [\mathbf{v}^* \cdot \mathbf{D} \nabla \Phi] + r + w$$  

(27b)

where $w = \mathbb{E} [\mathbf{Q} \cdot (\nabla \Phi \wedge \nabla \mathbf{S})]$ denotes a wedge product term and $r = \mathbb{E} [\nabla \mathbf{S} \cdot \mathbf{D} \nabla \Phi]$ denotes a scalar product term. Besides the source terms, the two rates have two common contributions: a “curl” $w$ measuring the perpendicularity between $\nabla \Phi$ and $\nabla \mathbf{S}$ and an inner product $r$. As the system approaches NESS, $\mathcal{S} \to \Phi$ and the wedge product term $w \to 0$. Both $\mathbb{E} [\nabla \mathbf{S} \cdot \mathbf{D} \nabla \mathbf{S}]$ and $r$ converge to $\mathbb{E} [\nabla \Phi \cdot \mathbf{D} \nabla \Phi]$, canceling each other out.

**Random Perturbations in a Bivectorial Formalism.** The random perturbation method of finding a Lyapunov function of deterministic dynamics [11–14, 24] can be organized with the decomposition in Eq. (22). We consider a system parameter $\epsilon$ and denote the $\epsilon$-dependence of variables with a subscript $\epsilon$, e.g., $b_\epsilon(x)$. A variable without subscript $\epsilon$ is $\epsilon$-independent.

With a general drift $b_0$ and a diffusion matrix $D_0$, the resulting $\Phi_\epsilon$ and $Q_\epsilon$ are both $\epsilon$-dependent in general. We set $D_\epsilon = \epsilon D_0$ to make the $\epsilon$-tuning a random perturbation of the deterministic dynamics $\mathcal{X}'(\epsilon) = b_0(\mathbf{x})$. The perturbed dynamics would then satisfy the SDE: $d \mathbf{X}_\epsilon = [b_\epsilon(\mathbf{X}_t) + \epsilon \nabla \cdot \mathbf{D}(\mathbf{X}_t)] d\mathbf{t} + \sqrt{2\epsilon} \mathbf{D}(\mathbf{X}_t) d\mathbf{W}_\epsilon$ with $\mathbf{b}_\epsilon \to \mathbf{b}_0$. Then by Eq. (22), we have a generally-valid decomposition

$$b_\epsilon = -\epsilon \mathbf{D} \nabla \Phi_\epsilon - \mathbf{Q}_\epsilon \nabla \Phi_\epsilon + \nabla \times \mathbf{Q}_\epsilon.$$  

(28)

If $\Phi_\epsilon = O(1/\epsilon)$, then $Q_\epsilon = O(\epsilon)$ such that $b_\epsilon \to b_0$ as $\epsilon \to 0$. $Q_\epsilon$ can only have $O(1)$ term if $\Phi_\epsilon = O(1)$.

The three random perturbations of $\mathcal{X}'(\epsilon) = b_0(x)$ studied in the past [12–14] correspond to different choices of $b_1$ in $b_\epsilon = b_0 + \epsilon b_1$. Those random perturbations can then be described by the FPE,

$$\partial_t p = \nabla \cdot [\epsilon \mathbf{D} \nabla p - (b_0 + \epsilon b_1) p],$$  

(29)
Freidlin-Wentzell’s (FW’s) random perturbation in Ref. [13] corresponds to \( b_1 = -\nabla \cdot D \), and Qian-Cheng-Yang’s (QCY’s) random perturbation corresponds to \( b_1 = 0 \) [14]. The two are equivalent when the diffusion matrix \( D \) is a constant matrix. On a contrary, the random perturbation introduced by Ao corresponds to \( D_x = \epsilon D \), \( Q_x = \epsilon Q \) and \( \Phi_x = \phi/\epsilon \) [11, 12]. This gives \( b_0 = -D \nabla \phi - Q \nabla \phi + \epsilon \nabla \times Q \). The underlying deterministic dynamics of Ao’s perturbation has the form

\[
x'(t) = -D(x)\nabla \phi(x) - Q(x)\nabla \phi(x).
\]

(31)

The differences between the three perturbations can be studied by applying the WKB ansatz on the invariant density \( \pi_\epsilon \). This leads to an asymptotic series of \( \Phi_\epsilon \) [14],

\[
\Phi_\epsilon(x) = \frac{\phi(x)}{\epsilon} - \ln \omega(x) - a \ln \epsilon + O(\epsilon).
\]

Equations for \( \phi \) and \( \omega \) can be obtained by plugging this back to the stationary FPE \( \nabla \cdot J_\epsilon = 0 \):

\[
0 = -\gamma \cdot \nabla \phi
\]

(33a)

\[
\nabla \cdot (\omega \gamma) = -\nabla \phi \cdot [D \nabla \omega - b_1]
\]

(33b)

where \( \gamma(x) = b_0(x) + D(x)\nabla \phi(x) \). Eq. (33a) is also known as the invariant Hamilton-Jacobi equation (HJE) of \( \phi(x) \) [13]. It leads to a decomposition of the vector field \( b_0 \),

\[
b_0(x) = -D(x)\nabla \phi(x) + \gamma(x)
\]

(34)

with \( \gamma \cdot \nabla \phi = 0 \) and \( \phi \) guaranteed to be the Lyapunov function of \( x'(t) = b_0(x) \) [14]. The decomposition was connected to the entropy production decomposition in the limit \( \epsilon \to 0 \) as discussed in Ref. [14].

Notice that Eqs. (33a) and (34) are independent to \( b_1 \). Thus, potentials from the three perturbations all satisfy the same invariant HJE. However, this alone does not tell us in which domains the three potentials have the same shape. It is FW’s uniqueness theorem of the orthogonal decomposition [13] that guarantees the potential \( \phi(x) \) from either of the three perturbations to have the same shape as the quasipotential in the domain where \( \phi(x) \) is smooth and strictly convex. With respect to a fixed point \( x_0 \) of \( x' = b_0(x) \), the quasipotential is defined as

\[
\psi(x; x_0) := \inf_{T > 0} \inf_{\xi} \{ S_{0,T}(\xi) : \xi_0 = x_0, \xi_T = x \}.
\]

(35)

\( \xi \) is the set of smooth paths on the interval \([0, T]\), and \( S_{0,T}(\xi) = \frac{1}{\epsilon} \int_0^T [\xi_s - b(\xi_s)]D^{-1}(\xi_s)[\xi_{s+} - b(\xi_s)]ds \) is the action functional [13]. This definition is from a trajectory point of view. It has a corresponding representation in terms of the conditional probability density [24]

\[
\psi(x; x_0) = -\lim_{t \to \infty} \lim_{\epsilon \to 0} \epsilon \ln p_\epsilon(x, t|x_0, 0).
\]

(36)

We note that the potential in Ao’s perturbation has a different order of limits in terms of the probability density,

\[
\phi(x) = -\lim_{\epsilon \to 0} \lim_{t \to \infty} \epsilon \ln p_\epsilon(x, t).
\]

(37)

It gives a globally smooth potential for deterministic dynamics that can be written as Eq. (31). A detailed discussion on the domain of smooth \( \phi \) can be found in the supplementary material where the notion of viscous HJE is introduced.

In the domains where \( \phi(x) \) from the three perturbations match, the leading order differences of the three \( \Phi \) are in their prefactors \( \omega(x) \), which satisfy different Eq. (33b) with different \( b_1 \). In Ao’s perturbation, we have \( \gamma(x) = -Q(x)\nabla \phi(x) \) and \( \phi(x) = \phi(x) \). Therefore, Eq. (33b) in Ao’s perturbation becomes

\[
\nabla \omega(x) \cdot [D(x) - Q(x)] \nabla \phi(x) = 0.
\]

(38)

Ao’s perturbation corresponds to the particular solution \( \omega = 1 \) [25]. In comparison, in both QCY’s and FW’s perturbation, \( \omega \) is non-uniform in general. In fact, \( \omega = 1 \) is a sufficient condition for \( \nabla \cdot \gamma = 0 \) in QCY’s perturbation [14] but not in Ao’s perturbation.

**Conclusion and Discussion** In this study, we extend the decomposition of continuous Markov process associated with the total entropy production decomposition by introducing the bivectorial cycle velocity \( Q \). Hidden structures in the mean rate of thermodynamic quantities are revealed and differences between three random perturbations are discussed with the bivectorial formalism.

The introduction of cycle velocity organizes the two notions of equilibrium in thermodynamics: 1) Equilibrium in classical thermodynamics is detailed balanced with zero stationary velocity \( \nu^* = 0 \) and zero housekeeping heat dissipation \( Q_{hk} = 0 \). The cycle velocity satisfies \( \nabla \times Q = Q \nabla \Phi \) in equilibrium. 2) The notation of M-B equilibrium introduced by Qian [21, 22] is an extension of the notion of equilibrium, from \( \nu^* = 0 \) to \( \nu^* \perp \nabla \Phi \). The orthogonality \( \nu^* \perp \nabla \Phi \) is equivalent to \( (\nabla \times Q) \perp \nabla \Phi \) in the bivectorial formalism since \( \nabla \Phi \cdot Q \nabla \Phi = 0 \). Therefore, all systems with constant cycle velocity \( Q \) are in M-B equilibrium, e.g. the Ornstein-Uhlenbeck process [26]. For systems with M-B equilibrium, the potential \( \Phi \) is the Lyapunov function of \( x'(t) = b(x) \).

Our probabilistic derivation shows that Ao’s decomposition of the vector field \( b \) in Eq. (22) is indeed general. The orthogonality condition discussed in [21] is not needed. It could be interesting for future studies to explore the existence of an orthogonal decomposition \( -D \nabla \Phi - Q \nabla \Phi \) or a divergence-free decomposition \( -D \nabla \Phi + \nabla \times Q \) in curvilinear coordinates where \( \nabla \) is the new operator defined accordingly. Two theorems connecting Ao’s perturbations with FW’s perturbations of deterministic dynamics via coordinate transformations are presented in the supplementary materials.

In Ao’s perturbation, the re-scaled nonquilibrium potential \( \phi = \Phi_\epsilon/\epsilon \) is a globally smooth Lyapunov function of \( x'(t) = b_0(x) \) but is a Lyapunov function of \( x'(t) = b_0(x) \) only in the domain where \( \epsilon (\nabla \times Q) \cdot \nabla \phi \leq \nabla \phi \cdot D \nabla \phi \). The domain
can be enlarged with smaller $\epsilon$. And the domain is the whole space if a Ao’s perturbation has M-B equilibrium.

Last but not least, the notion of cycle velocity gives us a nice physical picture of the conjugate process. The conjugate process, as a notion of time reversal [7], corresponds to the transpose of the matrix $Q$, which is equivalent to reversing the cycle velocity, $Q \mapsto -Q$. Conjugate process is the process with all cycle velocities reversed.

The authors thank Dr. Hong Qian for his guidance and the many helpful discussions he had with the authors.

---

* yangji@uw.edu

† yuchench@uw.edu

[1] D. Berlinski, *Infinite Ascent: A Short History of Mathematics*, reprint edition ed. (Modern Library, New York, NY, 2008).

[2] Ryogo Kubo, Kazuhiro Matsuo, and Kazuo Kitahara, “Fluctuation and relaxation of macrovariables,” J Stat Phys. 9, 51–96 (1973).

[3] R. Graham and A. Schenzle, “Non-equilibrium potentials and stationary probability distributions of some dissipative models without manifest detailed balance,” Z. Physik B - Condensed Matter 52, 61–68 (1983).

[4] G. Nicolis and R. Lefever, “Comment on the kinetic potential and the maxwell construction in non-equilibrium chemical phase transitions,” Physics Letters A 62, 469–471 (1977).

[5] U. Seifert, “Stochastic thermodynamics, fluctuation theorems and molecular machines,” Rep. Prog. Phys. 75, 126001 (2012).

[6] L. F. Thompson and H. Qian, “Nonlinear Stochastic Dynamics of Complex Systems, II: Potential of Entropic Force in Markov Systems with Nonequilibrium Steady State, Generalized Gibbs Function and Criticality,” Entropy 18, 309 (2016).

[7] Y.-J. Yang and H. Qian, “Unified formalism for entropy production and fluctuation relations,” Phys. Rev. E 101, 022129 (2020).

[8] J. Wang, L. Xu, and E. Wang, “Potential landscape and flux framework of nonequilibrium networks: Robustness, dissipation, and coherence of biochemical oscillations,” PNAS 105, 12271–12276 (2008).

[9] H. Ge, “Extended forms of the second law for general time-dependent stochastic processes,” Phys. Rev. E 80, 021137 (2009).

[10] Y.-J. Yang and H. Qian, “Bivectorial Mesoscopic Nonequilibrium Thermodynamics: Landauer-Bennett-Hill Principle, Cycle Affinity and Vorticity Potential,” arXiv:2004.08677 [cond-mat] (2020).

[11] P. Ao, “Potential in stochastic differential equations: novel construction,” J. Phys. A: Math. Gen. 37, L25–L30 (2004).

[12] L. Yin and P. Ao, “Existence and construction of dynamical potential in nonequilibrium processes without detailed balance,” J. Phys. A: Math. Gen. 39, 8593–8601 (2006).

[13] M. I. Freidlin and A. D. Wentzell, *Random Perturbations of Dynamical Systems*, Grundlehren der mathematischen Wissenschaften (Springer-Verlag, New York, 1984).