TWISTED BHARGAVA CUBES

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1. Introduction

In a seminal series of papers ([1], [2], [3]), Bhargava has extended Gauss’s composition law for binary quadratic forms to far more general situations. The key step in his extension is the investigation of the integral orbits of a group over \( \mathbb{Z} \) on a lattice in a prehomogeneous vector space. A prehomogeneous vector space which plays a particularly important role in elucidating the nature of Gauss’s composition is the natural action of \( \text{SL}_2(F)^3 \) on \( F^2 \otimes F^2 \otimes F^2 \) where \( F \) is a field. One of Bhargava’s achievements is the determination of the corresponding integral orbits, i.e. the determination of the \( \text{SL}_2(\mathbb{Z})^3 \)-orbits on \( \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2 \). In particular, he discovered that in this case, the generic integer orbits are in bijection with isomorphism classes of tuples \((A,I_1,I_2,I_3)\) where

- \( A \) is an order in an étale quadratic \( \mathbb{Q} \)-algebra;
- \( I_1, I_2 \) and \( I_3 \) are elements in the narrow class group of \( A \) such that \( I_1 \cdot I_2 \cdot I_3 = 1 \).

In this way, one sees that the arithmetic information of quadratic rings and their ideal class group is captured by this prehomogeneous vector space. Since Bhargava regards an element of \((F^2)^\otimes 3\) or \((\mathbb{Z}^2)^\otimes 3\) as a cube whose vertices are labelled by elements of \( F \) or \( \mathbb{Z} \) respectively, we shall call this prehomogeneous vector space (or its elements) Bhargava’s cube.

In fact, Bhargava’s cube arises naturally in the structure theory of the linear algebraic group of type \( D_4 \). More precisely, let \( G \) be a simply connected Chevalley group of this type. It has a maximal parabolic subgroup \( P = M \cdot N \) whose Levi factor \( M \) has a derived group \( M_{\text{der}}(F) \cong \text{SL}_2(F)^3 \) and whose unipotent radical \( N \) is a Heisenberg group. The adjoint action of \( M_{\text{det}}(F) \) on \( V = N(F)/[N(F),N(F)] \) is isomorphic to Bhargava’s cube.

Now the group \( G \) is exceptional in the sense that its outer automorphism group is isomorphic to \( S_3 \): no other absolutely simple linear algebraic group has such a large outer automorphism group. Let \( \tilde{G} \) be the semidirect product of \( G \) with the group of outer automorphisms. In particular, since \( S_3 \) is also the group of automorphisms of the split étale cubic \( F \)-algebra \( F \times F \times F \), we see that every étale cubic \( F \)-algebra \( E \) determines a quasi-split form \( \tilde{G}_E = G_E \rtimes S_E \), where \( S_E \) is a twisted form of \( S_3 \). This quasi-split group contains a maximal parabolic subgroup \( \tilde{P}_E = \tilde{M}_E \cdot N_E \) whose identity component \( P_E = M_E \cdot N_E \) is a twisted form of the parabolic \( P \) mentioned above. In particular, the derived group \( M_{E,\text{det}} \) of \( M_E \) is isomorphic to \( \text{SL}_2(E) \). Thus the adjoint action of \( M_{E,\text{det}}(F) \) on \( N_E(F)/[N_E(F),N_E(F)] \) is a twisted form of Bhargava’s cube. We shall call this prehomogeneous vector space (or its elements) the \( E \)-twisted Bhargava cube.

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The purpose of this paper is to classify the generic $\tilde{M}_E$-orbits (or equivalently $M_E$-orbits) on the $E$-twisted Bhargava’s cube. The main result is:

**Theorem 1.1.** $F$ is a field of characteristic not 2 or 3. Fix an étale cubic $F$-algebra $E$.

(i) There are natural bijections between the following sets:

(a) Generic $\tilde{M}_E$-orbits $O$ on the $E$-twisted Bhargava cube.
(b) $F$-isomorphism classes of $E$-twisted composition algebras $(C, Q, \beta)$ over $F$ which are of $E$-dimension 2.
(c) $F$-isomorphism classes of pairs $(J, i)$ where $J$ is a Freudenthal-Jordan algebra over $F$ of dimension 9 and

$$i : E \hookrightarrow J$$

is an $F$-algebra homomorphism. Here an $F$-isomorphism from $(J, i)$ to $(J', i')$ is a commutative diagram

$$
\begin{array}{ccc}
E & \xrightarrow{i} & J \\
\downarrow & & \downarrow \\
E & \xrightarrow{i'} & J'
\end{array}
$$

where the two vertical arrows are isomorphisms of $F$-algebras.

(i') Similarly, there are natural bijections between the following sets:

(a') Generic $M_E$-orbits $O$ on the $E$-twisted Bhargava cube.
(b') $E$-isomorphism classes of $E$-twisted composition algebras $(C, Q, \beta)$ over $F$ which are of $E$-dimension 2.
(c') $E$-isomorphism classes of pairs $(J, i)$, where an $E$-isomorphism is a commutative diagram as in (c) above but with the first vertical arrow equal to the identity map.

(ii) The bijections in (i) and (i') identifies

$$\text{Stab}_{\tilde{M}_E}(O) \cong \text{Aut}_F(C, Q, \beta) \cong \text{Aut}_F(i : E \hookrightarrow J).$$

and

$$\text{Stab}_{M_E}(O) \cong \text{Aut}_E(C, Q, \beta) \cong \text{Aut}_E(i : E \hookrightarrow J).$$

Moreover, there is a short exact sequence of algebraic groups:

$$1 \rightarrow \text{Stab}_{M_E}(O) \rightarrow \text{Stab}_{\tilde{M}_E}(O) \rightarrow S_E \rightarrow 1$$

(iii) There is a natural map (to be described later) from the set of objects in (a) (or equivalently (b) or (c)) to the set of isomorphism classes of étale quadratic $F$-algebras. If $O$ gives rise to the étale quadratic algebra $K$, then the group $\text{Stab}_{M_E}(O)$ in (ii) sits in a short exact sequence of algebraic groups:

$$1 \rightarrow T_{E,K} \rightarrow \text{Stab}_{M_E}(O) \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 1$$

where

$$T_{E,K} = \{ x \in (E \otimes_F K)^\times : N_{L/E}(x) = 1 = N_{L/K}(x) \},$$

and where the conjugation action of the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ on $T_{E,K}$ is given by $x \mapsto x^{-1}$. 
We note that the short exact sequences in (ii) and (iii) of Theorem 1.1 need not split. Further, it is possible that some non-identity fibers of the projection to $S_E$ or $\mathbb{Z}/2\mathbb{Z}$ may have no $F$-rational points, i.e. the sequence obtained by taking $F$-rational points need not be exact on the right.

In the course of studying the stabilizer group in (iii), we were led to another description of $T_{E,K}$, which we view as an exceptional Hilbert 90 theorem. More precisely, the étale algebras $E$ and $K$ correspond to conjugacy classes of homomorphisms

$$\rho_E : \text{Gal}(\overline{F}/F) \to \text{Aut}(\overline{F}^3) \quad \text{and} \quad \rho_K : \text{Gal}(\overline{F}/F) \to \text{Aut}(\overline{F}^2),$$

respectively. Define an action of $\text{Gal}(\overline{F}/F)$ on the algebra $\overline{F}^3 \otimes \overline{F}^2$, such that the usual action of $\text{Gal}(\overline{F}/F)$ on coordinates is twisted by $\rho_E \times \rho_K$. Under this action, the set of $F$-rational points in the algebra is $E \otimes_F K$. The torus $T_{E,K}$ is a subgroup of the multiplicative group $(E \otimes_F K)^\times$.

On the other hand, let $K_E$ be the quadratic discriminant algebra of $E$. Let $K_J$ be the étale quadratic algebra such that $[K_E] : [K] \cdot [K_J] = 1$ in $H^1(F, \mathbb{Z}/2\mathbb{Z})$. Let

$$\rho_{K_J} : \text{Gal}(\overline{F}/F) \to \text{Aut}(\overline{F}^2)$$

be the homomorphism corresponding to $K_J$. Define an action of $\text{Gal}(\overline{F}/F)$ on the algebra $\overline{F}^3 \otimes \overline{F}^2$, such that the usual action of $\text{Gal}(\overline{F}/F)$ on coordinates is twisted by $\rho_E \times \rho_{K_J}$. Under this action, the set of $F$-rational points in the algebra is $E \otimes_F K_J$. We define two tori,

$$\tilde{T}_{E,K}' = \{x \in (E \otimes_F K_J)^\times : N_{E \otimes_F K_J/E}(x) = 1 \}$$

and

$$\tilde{T}_{E,K}' = \tilde{T}_{E,K}' / K_J^\times.$$

**Theorem 1.2.** (Exceptional Hilbert 90) Let $\sigma \in \text{Aut}(\overline{F}^3) \cong S_3$ be an element of order 3. Then $x \mapsto \sigma(x)/\sigma^2(x)$ defines a homomorphism $\tilde{T}_{E,K}' \to T_{E,K}$ that descends to an isomorphism

$$\tilde{T}_{E,K}' \to T_{E,K}.$$

This theorem may be known, and is anyhow not difficult to show. But it is something a bit surprising to us, so we highlight it here. For example, taking $K = K_E$, so that $K_J = F^2$, we obtain

$$\tilde{T}_{E,K} = E^\times / F^\times \cong T_{E,K} = \{x \in E \otimes_F K^\times : N_{E \otimes K/E}(x) = 1 = N_{E \otimes K_E/E}(x)\}.$$

When $E$ is Galois so that $K_E = F^2$, this is the usual Hilbert’s Theorem 90.

In the main body of the paper, we shall introduce and explain the various classes of objects which appear in the above theorems. At this point, it suffices to illustrate the Theorem 1.1(iii) in the case when $E = F^3$ is split. In the context of (c), a Jordan algebra $J$ of dimension 9 contains $F^3$ if and only if it has an orthogonal system of primitive idempotents. Such a $J$ is isomorphic to the Jordan algebra of $3 \times 3$-Hermitian matrices with entries in some étale quadratic $F$-algebra. This gives the map in statement (iii) of the theorem (when $E = F^3$).
2. Étale cubic algebras

Let $F$ be a field of characteristic 0 with absolute Galois group $\text{Gal}(\overline{F}/F)$.

2.1. Étale cubic algebras. An étale cubic algebra is an $F$-algebra $E$ such that $E \otimes_F F \cong F^3$. More concretely, an étale cubic $F$-algebra is of the form:

$$E = \begin{cases} F \times F \times F; \\ F \times K, \text{ where } K \text{ is a quadratic field extension of } F; \\ \text{a cubic field.} \end{cases}$$

Since the split algebra $F \times F \times F$ has automorphism group $S_3$ (the symmetric group on 3 letters), the isomorphism classes of étale cubic algebras $E$ over $F$ are naturally classified by the pointed cohomology set $H^1(F, S_3)$, or more explicitly by the set of conjugacy classes of homomorphisms $\rho_E : \text{Gal}(\overline{F}/F) \to S_3$.

2.2. Discriminant algebra of $E$. By composing the homomorphism $\rho_E$ with the sign character of $S_3$, we obtain a quadratic character (possibly trivial) of $\text{Gal}(\overline{F}/F)$ which corresponds to an étale quadratic algebra $K_E$. We call $K_E$ the discriminant algebra of $E$. To be concrete,

$$K_E = \begin{cases} F \times F, \text{ if } E = F^3 \text{ or a cyclic cubic field}; \\ K, \text{ if } E = F \times K; \\ \text{the unique quadratic subfield in the Galois closure of } E \text{ otherwise.} \end{cases}$$

2.3. Twisted form of $S_3$. Fix an étale cubic $F$-algebra $E$. Then, via the associated homomorphism $\rho_E$, $\text{Gal}(\overline{F}/F)$ acts on $S_3$ (by inner automorphisms) and thus define a twisted form $S_E$ of the finite constant group scheme $S_3$. For any commutative $F$-algebra $A$, we have

$$S_E(A) = \text{Aut}_A(E \otimes_F A).$$

3. Twisted Composition Algebras

In this section, we introduce the $E$-twisted composition algebra of dimension 2 over $E$. The canonical reference, covering many topics of this paper, is *The Book of Involutions* [4]. Twisted composition algebras are treated in Chapter VIII, §36.

3.1. Twisted composition algebras. A twisted composition algebra over $F$ is a quadruple $(E, C, Q, \beta)$ where

- $E$ is an étale cubic $F$-algebra;
- $C$ is an $E$-vector space equipped with a nondegenerate quadratic form $Q$, with associated symmetric bilinear form $b_Q(v_1, v_2) = Q(v_1 + v_2) - Q(v_1) - Q(v_2)$;
- $\beta : C \to C$ is a quadratic map such that $\beta(xv) = x^\# \cdot \beta(v)$ and $Q(\beta(v)) = Q(v)^\#$, for every $x \in E$ and $v \in C$, where $x^\# \in E$ such that $x \cdot x^\# = N_{E/F}(x)$. 


• if we set
  \[ N(v) := b_Q(v, \beta(v)), \]
  then \( N(v) \in F \), for every \( v \in C \).

For a fixed \( E \), we shall call \((C, Q, \beta)\) an \( E\)-twisted composition algebra (over \( F \)), and the cubic form \( N \) the norm form of \( C \). Frequently, for ease of notation, we shall simply denote this triple by \( C \), suppressing the mention of \( Q \) and \( \beta \).

### 3.2. Morphisms

Given twisted composition algebras \((E, C, Q, \beta)\) and \((E', C', Q', \beta')\), a morphism of twisted composition algebras is a pair \( (\phi, \sigma) \in \text{Hom}_F(C, C') \times \text{Hom}_F(E, E') \) such that

\[ \phi(av) = \sigma(a) \cdot \phi(v) \]

for \( v \in C \) and \( a \in E \), and

\[ \phi \circ \beta = \beta' \circ \phi \quad \text{and} \quad \sigma \circ Q = Q' \circ \phi. \]

In particular, we have the automorphism group \( \text{Aut}_F(E, C, Q, \beta) \). The second projection gives a natural homomorphism

\[ \text{Aut}_F(E, C, Q, \beta) \to S_E. \]

The kernel of this map is the subgroup \( \text{Aut}_E(C, Q, \beta) \) consisting of those \( \phi \) which are \( E \)-linear.

### 3.3. Dimension 2 case

It is known, by Corollary 36.4 in [4], that for any \( E\)-twisted composition algebra \((C, Q, \beta)\), \( \dim_E C = 1, 2, 4 \) or \( 8 \). We shall only be interested in the case when \( \dim_E C = 2 \).

We give an example that will feature prominently in this paper. We set \( C_E = E \oplus E \), and define \( Q \) and \( \beta \) by

\[ Q(x, y) = x \cdot y \quad \text{and} \quad \beta(x, y) = (y^#, x^#) \]

for every \((x, y) \in E \oplus E \). It is easy to check that this defines an \( E\)-twisted composition algebra over \( F \), with norm form

\[ N(x, y) = N_{E/F}(x) + N_{E/F}(y). \]

The group of automorphisms of this \( E\)-twisted composition algebra is easy to describe. Let \( E^1 \) be the set of elements \( e \) in \( E \) such that \( N(e) = e \cdot e^# = 1 \). For very element \( e \in E^1 \) we have an \( E \)-automorphism \( i_e \) defined by \( i_e(x, y) = (ex, e^#y) \). We also have an \( E \)-automorphism \( w \) defined by \( w(x, y) = (y, x) \). The group \( E \)-automorphisms is

\[ \text{Aut}_E(C_E, Q, \beta) = E^1 \rtimes \mathbb{Z}/2\mathbb{Z}, \]

and the group of \( F \)-automorphisms is

\[ \text{Aut}_F(C_E, Q, \beta) = (E^1 \rtimes \mathbb{Z}/2\mathbb{Z}) \rtimes S_E = E^1 \rtimes (\mathbb{Z}/2\mathbb{Z} \times S_E). \]

If \( E = F \times F \times F \), we denote the corresponding twisted composition algebra by \( C_0 = (C_0, Q_0, \beta_0) \) and refer to it as the split twisted composition algebra. In this case, \( E^1 \) consist of \((t_1, t_2, t_3) \) such that \( t_1t_2t_3 = 1 \), so that

\[ \text{Aut}_E(C_0, Q_0, \beta_0) \cong \mathbb{G}_m^2 \rtimes \mathbb{Z}/2\mathbb{Z}. \]

Observe that there is a natural splitting

\[ S_3 \times \mathbb{Z}/2\mathbb{Z} \to \text{Aut}_F(C_0, Q_0, \beta_0). \]

(3.1)
3.4. Cohomological description. In view of the above, the set of isomorphism classes of twisted composition algebras over \( F \) is classified by the pointed cohomology set

\[
H^1(F, \text{Aut}_F(F^3, C_0, Q_0, \beta_0)),
\]

and there is a natural map

\[
H^1(F, \text{Aut}_F(F^3, C_0, Q_0, \beta_0)) \longrightarrow H^1(F, \mathbb{Z}/2\mathbb{Z}) \times H^1(F, S_3).
\]

Composing this with the first or second projection, we obtain natural maps

\[
H^1(F, \text{Aut}_F(F^3, C_0, Q_0, \beta_0)) \longrightarrow H^1(F, \mathbb{Z}/2\mathbb{Z}) = \{\text{étale quadratic } F\text{-algebras}\},
\]

and

\[
H^1(F, \text{Aut}_F(F^3, C_0, Q_0, \beta_0)) \longrightarrow H^1(F, S_3).
\]

All these projection maps are surjective, because of the natural splitting in (3.1). Indeed, (3.1) endows each fiber of the maps in (3.2), (3.3) and (3.4) with a distinguished point. Moreover, the map in (3.3) is the map alluded to in Theorem 1.1 (iii).

For an étale cubic \( F\)-algebra \( E \) with associated cohomology class \([E] \in H^1(F, S_3)\), the fiber of (3.4) over \([E]\) is precisely the set of isomorphism classes of \( E\)-twisted composition algebras. Moreover, a Galois descent argument shows that the distinguished point in this fiber furnished by the splitting (3.1) is none other than the \( E\)-twisted composition algebra \( C_E \) constructed in §4.4. Moreover, the quadratic algebra associated to \( C_E \) by (3.3) is the split algebra \( F \times F \).

Using \( C_E \) as the base point, the fiber in question is identified naturally with the set \( H^1(F, \text{Aut}_E(C_E, Q, \beta)) \) modulo the natural action of \( S_E(F) \) (by conjugation). The cohomology set \( H^1(F, \text{Aut}_E(C_E, Q, \beta)) \) classifies the \( E\)-isomorphism classes of \( E\)-twisted composition algebras \( C \) over \( F \), and the action of \( S_E(F) \) is given by

\[
\sigma : (C, Q, \beta) \mapsto (C \otimes_{E, \sigma} E, \sigma \circ Q, \beta)
\]

for \( \sigma \in S_E(F) \).

3.5. Tits construction. Given an element

\[
([E], [K]) \in H^1(F, S_3) \times H^1(F, \mathbb{Z}/2\mathbb{Z}),
\]

we describe the composition algebras in the fiber of (3.2) over \(([E], [K])\). Note that by (3.1), we have a distinguished point in this fiber. Now we have:

**Proposition 3.5.** If \( C \) is an \( E\)-twisted composition algebra, with the associated étale quadratic algebra \( K \), then we may identify \( C \) with \( E \otimes_F K \), such that

\[
Q(x) = e \cdot N_{E/F, K/E}(x) \quad \text{for some } e \in E^\times
\]

and

\[
\beta(x) = \overline{\nu}^# \cdot e^{-1} \cdot \overline{\nu} \quad \text{for some } \nu \in K
\]

where \( x \mapsto \overline{\nu} \) is induced by the nontrivial automorphism of \( K \) over \( F \). Moreover, we have:

\[
N_{E/F}(e) = N_{K/F}(\nu).
\]

The distinguished point in the fiber of (3.2) over \(([E], [K])\) corresponds to taking \((e, \nu) = (1, 1)\).
Proof. This proposition essentially follows by Galois descent. Indeed, a Galois descent argument, starting from the algebra $C_E$ introduced in §4.4, shows that $C$ can be identified with $E \otimes_F K$. Then $Q$ is necessarily of the form $e \cdot N_{E \otimes F K/E}$ for some $e \in E^\times$. On the other hand, we claim that for $x \in E \otimes_F K$ and $x_0 \in C$, one has
\[
\beta(x \cdot x_0) = \overline{x} \cdot \beta(x_0).
\]
Indeed, one can check this by going to $F$, whence one is reduced to checking this identity in the split algebra $C_0$, where it is straightforward. This shows that $\beta$ is determined by $\beta(1) = \nu \cdot e^{-1}$ for some $\nu \in E \otimes F K$. However, the identity $Q(1) = Q(\beta(1))$ implies that
\[
\nu \cdot \nu = N_{E \otimes F K/E}(\nu) = N_{E/F}(e) \in F.
\]
The requirement that $N(x) \in F$ for all $x \in E \otimes_F K$ implies that
\[
T_{\tau_{E \otimes F K/E}}(\nu \cdot N_{E \otimes F K/K}(x)) \in F.
\]
In particular, taking $x = 1$ and then a trace zero element $\delta \in K$ one obtains, respectively:
\[
\nu + \overline{\nu} \in F \quad \text{and} \quad \nu \delta + \overline{\nu} \delta \in F.
\]
All these conditions imply that $\nu \in K$.

Finally, it is easy to see by Galois descent that the distinguished point in the fiber over $([E], [K])$ corresponds to $(e, \nu) = (1, 1)$.

The description of twisted composition algebras given in the above proposition is sometimes referred to as a Tits construction (though usually this terminology is reserved for the Jordan algebra associated to the above twisted composition algebra by Springer’s construction, which is the subject matter of the next section).

3.6. Automorphism group. Using Proposition 3.5 it is not difficult to determine the automorphism group of any twisted composition algebra $C$. Indeed, if $C = E \otimes_F K$ as in the proposition, then the special orthogonal group
\[
SO(C, Q) = \{ \lambda \in E \otimes_F K : N_{E \otimes K/E}(\lambda) = 1 \}
\]
acts $E \otimes K$-linearly on $C$ by multiplication and preserves $Q$. An element $\lambda \in SO(C, Q)$ preserves $\beta$ if and only if
\[
\overline{\lambda} = \lambda.
\]
But $\lambda^\# = \lambda^{-1}$ since $N_{E \otimes K/E}(\lambda) = \lambda \cdot \lambda^\# = 1$. So
\[
\text{Aut}_E(C, Q, \beta) \cap SO(C, Q) = \{ \lambda \in L = E \otimes K : N_{L/E}(\lambda) = 1 = N_{L/K}(\lambda) \} = T_{E, K},
\]
which is a 2-dimensional torus. Since we know the automorphism group of the split twisted composition algebra $(C_0, Q_0, \beta_0)$, we see that
\[
\text{Aut}_F(C, Q, \beta) = T_{E, K}
\]
and $\text{Aut}_F(C, Q, \beta)$ and $\text{Aut}_E(C, Q, \beta)$ sit in short exact sequences of algebraic groups as in Theorem 1.1(ii) and (iii).
3.7. Cohomology of $T_{E,K}$. Using Proposition 3.5 and the above description of $\text{Aut}_E(C, Q, \beta)^0$, we can describe the fiber of the natural map

$$H^1(F, \text{Aut}_E(F^3, C_0, Q_0, \beta_0)) \rightarrow H^1(F, \mathbb{Z}/2\mathbb{Z}) \times H^1(F, S_3)$$

over the element $([K], [E]) \in H^1(F, \mathbb{Z}/2\mathbb{Z}) \times H^1(F, S_3)$. Indeed, this fiber is equal to $H^1(F, T_{E,K})$ modulo the action of $S_E(F) \times \mathbb{Z}/2\mathbb{Z}$.

The cohomology group $H^1(T_{E,K})$ classifies twisted composition algebras with fixed $E$ and $K$, up to $E \otimes_F K$-linear isomorphism. With $L = E \otimes_F K$, one has a short exact sequence of algebraic tori

$$1 \rightarrow T_{E,K} \rightarrow L^\times \xrightarrow{N_{L/E}N_{L/K}} (E^\times \times K^\times)^0 \rightarrow 1$$

where

$$(E \times K)^0 = \{ (e, \nu) \in E^\times \times K^\times : N_{E/F}(e) = N_{K/F}(\nu) \}.$$

The associated long exact sequence gives

$$H^1(F, T_{E,K}) \cong (E^\times \times K^\times)^0 / \text{Im} L^\times.$$  

This isomorphism is quite evident in the context of Proposition 3.5. Indeed, Proposition 3.5 tells us that any twisted composition algebra $C$ with invariants $(E, K)$ is given by an element $(e, \nu) \in (E^\times \times K^\times)^0$. Any $L$-linear map from $C$ to another twisted composition algebra $C'$ with associated pair $(e', \nu')$ is given by multiplication by an element $a \in L^\times$, and this map is an isomorphism of twisted composition algebras if and only if

$$(e, \nu) = (e' \cdot N_{L/E}(a), \nu' \cdot N_{L/K}(a)).$$

This is precisely what (3.6) expresses.

4. Springer’s construction.

We can now relate twisted composition algebras to Freudenthal-Jordan algebras. This construction is due to Springer. Our exposition follows §38A, page 522, in [4].

4.1. Freudenthal-Jordan algebra of dimension 9. A Freudenthal-Jordan algebra $J$ of dimension 9 over $F$ is a Jordan algebra which is isomorphic over $F$ to the Jordan algebra $J_0$ associated to the associative algebra $M_3(F)$ of $3 \times 3$-matrices, with Jordan product

$$a \circ b = \frac{1}{2} \cdot (ab + ba).$$

An element $a \in J$ satisfies a characteristic polynomial

$$X^3 - T_J(a)X^2 + S_J(a)X - N_J(a) \in F[X].$$

The maps $T_J$ and $N_J$ are called the trace and norm maps of $J$ respectively. The element

$$a^# = a^2 - T_J(a)a + S_J(a)$$

is called the adjoint of $a$. It satisfies $a \cdot a^# = N_J(a)$. The cross product of two elements $a, b \in J$ is defined by

$$a \times b = (a + b)^# - a^# - b^#.$$
4.2. Cohomological description. The automorphism group of \( J_0 \) is \( \text{PGL}_3 \times \mathbb{Z}/2\mathbb{Z} \), with \( g \in \text{PGL}_3 \) acting by conjugation and the nontrivial element of \( \mathbb{Z}/2\mathbb{Z} \) acting by the transpose: \( a \mapsto a^t \). Thus, the isomorphism classes of Freudenthal-Jordan algebra of dimension 9 is parametrized by the pointed set \( H^1(F, \text{PGL}_3 \times \mathbb{Z}/2\mathbb{Z}) \), and there is an exact sequence of pointed sets

\[
H^1(F, \text{PGL}_3) \xrightarrow{f} H^1(F, \text{PGL}_3 \times \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\pi} H^1(F, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\{\text{étale quadratic } F\text{-algebras}\}}.
\]

The map \( \pi \) is surjective and the fiber of \( \pi \) over the split quadratic algebra \( F \) is the image of \( f \). By Proposition 39(ii) and Corollary 1 in [5], page 52, the image of \( f \) is \( H^1(F, \text{PGL}_3) \) modulo a natural action of \( \mathbb{Z}/2\mathbb{Z} \) (see [5], page 52). Now the set \( H^1(F, \text{PGL}_3) \) parametrizes the set of central simple \( F \)-algebras \( B \) of degree 3, and the \( \mathbb{Z}/2\mathbb{Z} \) action is \( B \mapsto B^\text{op} \). Then the map \( f \) sends \( B \) to the associated Jordan algebra.

In general, for any étale quadratic \( F \)-algebra \( K \), an element in the fiber of \( \pi \) over \( [K] \in H^1(F, \mathbb{Z}/2\mathbb{Z}) \) is the Jordan algebra \( J_3(K) \) of \( 3 \times 3 \)-Hermitian matrices with entries in \( K \). The automorphism group of \( J_3(K) \) is an adjoint group \( \text{PGU}_3^K \). Using \( J_3(K) \) as the base point, the fiber of \( \pi \) over \( [K] \) can then be identified with \( H^1(F, \text{PGU}_3^K) \) modulo the action of \( \mathbb{Z}/2\mathbb{Z} \) (by [5], pages 50 and 52). By [4], page 400, \( H^1(F, \text{PGU}_3^K) \) has an interpretation as the set of isomorphism classes of pairs \((B_K, \tau)\) where

- \( B_K \) is a central simple \( K \)-algebra of degree 3,
- \( \tau \) is an involution of the second kind on \( B_K \).

Moreover the action of the non-trivial element \( \tau_K \in \text{Aut}(K/F) = \mathbb{Z}/2\mathbb{Z} \) is via the Galois twisting action: \( B \mapsto B \otimes_{K, \tau_K} K \), so that

\[
H^1(F, \text{PGU}_3^K)/\mathbb{Z}/2\mathbb{Z} \leftrightarrow \{\text{\{\text{isomorphism classes of } (B_K, \tau)\}}\}.
\]

Then the map \( f \) sends \((B_K, \tau)\) to the Jordan algebra \( B_K^\tau \) of \( \tau \)-symmetric elements in \( B_K \).

If \( J \) is a Freudenthal-Jordan algebra of dimension 9, we will write \( K_J \) for the étale quadratic algebra corresponding to \( \pi(J) \).

4.3. Relation with twisted composition algebras. Fix an étale cubic \( F \)-algebra \( E \) and a Freudenthal-Jordan algebra \( J \). Suppose we have an algebra embedding \( i : E \hookrightarrow J \). Then, with respect to the trace form \( T_J \), we have an orthogonal decomposition

\[
J = i(E) \oplus C
\]

where \( C = i(E)^\perp \). We shall identify \( E \) with its image under \( i \). Then for \( e \in E \) and \( v \in C \), one can check that \( e \times v \in C \). Thus, setting

\[
e \circ v := -e \times v
\]
equips \( C \) with the structure of an \( E \)-vector space. Moreover, writing

\[
v^\# = (-Q(v), \beta(v)) \in E \oplus C = J
\]
for \( Q(v) \in E \) and \( \beta(v) \in V \), we obtain a quadratic form \( Q \) on \( C \) and a quadratic map \( \beta \) on \( C \). Then, by Theorem 38.6 in [4], the triple \((C, Q, \beta)\) is an \( E \)-twisted composition algebra over \( F \).

Conversely, given an \( E \)-twisted composition algebra \( C \) over \( F \), Theorem 38.6 in [4] says that the space \( E \oplus C \) can be given the structure of a Freudenthal-Jordan algebra over \( F \). In particular, we have described the bijective correspondence between the objects in (b) and (c) of the main theorem:

\[
\{E\text{-twisted composition algebras over } F\} \\
\uparrow \\
\{i : E \rightarrow J \text{ with } J \text{ Freudenthal-Jordan of dimension } 9\}.
\]

It is also clear that under this identification, one has

\[
\text{Aut}_F(i : E \rightarrow J) = \text{Aut}_F(i(E)^\perp).
\]

4.4. Example. Let \( K \) be an étale quadratic \( F \)-algebra and consider the Jordan algebra \( J_3(K) \) of \( 3 \times 3 \) Hermitian matrices with entries in \( K \). Let \( E = F \times F \times F \) be the subalgebra of \( J_3(K) \) consisting of diagonal matrices. Then \( C \) consists of matrices

\[
v = \begin{pmatrix} 0 & x_3 & x_2 \\ x_3 & 0 & \bar{x}_1 \\ \bar{x}_2 & x_1 & 0 \end{pmatrix}.
\]

Thus \( C = K \times K \times K \), and one checks that

\[
Q(x_1, x_2, x_3) = (x_1 \bar{x}_1, x_2 \bar{x}_2, x_3 \bar{x}_3)
\]

and

\[
\beta(x_1, x_2, x_3) = (\bar{x}_2 \bar{x}_3, \bar{x}_3 \bar{x}_1, \bar{x}_1 \bar{x}_2).
\]

The algebra \( C \) is is the distinguished point in the fiber of \(([E^3], [K])\), in the sense of Proposition 3.5. The automorphism group of \( C \) is given by

\[
\text{Aut}_F(C, Q, \beta) = (K^1 \times K^1 \times K^1)^0 \times (\mathbb{Z}/2\mathbb{Z} \times S_3)
\]

where \( K^1 \) denotes the torus of norm 1 elements in \( K \) and \((K^1 \times K^1 \times K^1)^0 \) denotes the subgroup of triples \((t_1, t_2, t_3)\) such that \( t_1 t_2 t_3 = 1 \).

4.5. The quadratic algebra associated to \( i : E \rightarrow J \). If an \( E \)-twisted composition algebra \( C \) corresponds to a conjugacy class of embeddings \( i : E \rightarrow J \), then we may ask how the quadratic algebra \( K_C \) associated to \( C \) can be described in terms of \( i : E \rightarrow J \). In this case, \( C = E^\perp \) is an \( E \)-twisted composition algebra, and so \( C = E \otimes K_C \) for a quadratic algebra \( K_C \) as in Proposition 3.5. On the other hand, we know that \( J \) is associated to a pair \((B_K, \tau)\), where \( B_K \) is a central simple algebra over an étale quadratic \( F \)-algebra \( K_j \) and \( \tau \) is an involution of the second kind. Now, Examples (5) and (6) on page 527 in [4] show that

\[
[K_C] \cdot [K_E] \cdot [K_J] = 1 \in H^1(F, \mathbb{Z}/2\mathbb{Z}) = F^\times / F^x^2.
\]
5. Quasi-split Groups of type $D_4$

In this section, we shall introduce the $E$-twisted Bhargava’s cube by way of the quasi-split groups of type $D_4$.

5.1. Root system. Let $\Psi$ be a root system of type $D_4$, and $\Pi = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$ be a set of simple roots such that the corresponding Dynkin diagram is

```
   2
  /|
 / |
/  |
  0---1
     |
     3
```

The group of diagram automorphisms $\text{Aut}(\Pi)$ is identified with $S_3$, the group of permutations of $\{1, 2, 3\}$. We denote the highest root by $\beta_0 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_0$.

5.2. Quasi-split groups of type $D_4$. Let $G$ be a split, simply connected Chevalley group of type $D_4$. We fix a maximal torus $T$ contained in a Borel subgroup $B$ defined over $F$. The group $G$ is then generated by root groups $U_{\alpha} \cong \mathbb{G}_a$, where $\alpha \in \Psi$. Steinberg showed that one can pick the isomorphisms $x_{\alpha} : \mathbb{G}_a \rightarrow U_{\alpha}$ such that

$$[x_{\alpha}(u), x_{\alpha'}(u')] = x_{\alpha + \alpha'}(\pm uu')$$

whenever $\alpha + \alpha'$ is a root. Fixing such a system of isomorphisms is fixing an épalinge (or pinning) for $G$. As Kac noted, a choice of signs corresponds to an orientation of the Dynkin diagram. As one can pick an orientation of the Dynkin diagram which is invariant under $\text{Aut}(\Pi)$, the group of automorphisms of $\Pi$ can be lifted to a group of automorphisms of $G$. Thus, we have a semi-direct product

$$\tilde{G} = G \rtimes \text{Aut}(\Pi) = G \rtimes S_3,$$

where the action of $S_3$ permutes the root subgroups $U_{\alpha}$ and the isomorphisms $x_{\alpha}$.

Since the outer automorphism group $S_3$ of $G$ is also the automorphism group of the split étale cubic $F$-algebra $F^3$, we see that every cubic étale algebra $E$ defines a simply-connected quasi-split form $G_E$ of $G$, whose outer automorphism group is the finite group scheme $S_E$. Thus,

$$\tilde{G}_E = G_E \rtimes S_E$$

is a form of $\tilde{G}$, and it comes equipped with a pair $B_E \supset T_E$ consisting of a Borel subgroup $B_E$ containing a maximal torus $T_E$, both defined over $F$, as well as a Chevalley-Steinberg system of épalinge relative to this pair.
5.3. \textbf{$G_2$ root system.} The subgroup of $G_E$ fixed pointwise by $S_E$ is isomorphic to the split exceptional group of type $G_2$.

Observe that $B = G_2 \cap B_E$ is a Borel subgroup of $G_2$ and $T = T_E \cap G_2$ is a maximal split torus of $G_2$. Via the adjoint action of $T$ on $G_E$, we obtain the root system $\Psi_{G_2}$ of $G_2$, so that

\[ \Psi_{G_2} = \Psi|_T. \]

We denote the short simple root of this $G_2$ root system by $\alpha$ and the long simple root by $\beta$. Then

\[ \beta = \alpha_0|_T \quad \text{and} \quad \alpha = \alpha_1|_T = \alpha_2|_T = \alpha_3|_T. \]

Thus, the short root spaces have dimension 3, whereas the long root spaces have dimension 1. For each root $\gamma \in \Psi_{G_2}$, the associated root subgroup $U_\gamma$ is defined over $F$ and the Chevalley-Steinberg system of épiméthane gives isomorphisms:

\[ U_\gamma \cong \begin{cases} \text{Res}_{E/F}G_\alpha, & \text{if } \gamma \text{ is short;} \\ G_\alpha, & \text{if } \gamma \text{ is long.} \end{cases} \]

5.4. \textbf{The parabolic subgroup $P_E$.} The $G_2$ root system gives rise to 2 parabolic subgroups of $G_E$. One of these is a maximal parabolic $P_E = M_EN_E$ known as the Heisenberg parabolic. Its unipotent radical $N_E$ is a Heisenberg group with center $Z = U_{\beta_0}$. Moreover,

\[ N_E/Z = U_\beta \times U_{\beta+\alpha} \times U_{\beta+2\alpha} \times U_{\beta+3\alpha} \cong F \times E \times E \times F. \]

The Levi factor $M_E$ is given by:

\[ M_E(F) \cong \text{GL}_2(E)^0 = \{ g \in \text{GL}_2(E) : \det(g) \in F^\times \} \]

and we have

\[ \tilde{M}_E = M_E \rtimes S_E \hookrightarrow \tilde{G}_E. \]

5.5. \textbf{$E$-twisted Bhargava cube.} The natural adjoint action of $M_E(F)$ on

\[ V_E = N_E(F)/Z_E(F) \]

is the space of $E$-twisted Bhargava's cubes. It is a prehomogeneous vector space, in the sense that there is an essentially unique quasi-invariant polynomial $\Delta \in \text{Sym}^*V^*_E$, and there is a unique open $M_E$-orbit on $V_E$ given by the open subvariety on which $\Delta \neq 0$. We would like to relate the generic $\tilde{M}_E$-orbits of $V_E$ to the $E$-twisted composition algebras introduced earlier.

6. \textbf{Bhargava’s Cube}

In this section, we shall examine the case when $E = F^3$ in detail. In particular, we shall recall certain basic results of Bhargava [1].
6.1. Bhargava’s cube. When $E = F^3$, then $M_{\text{der}} = M_{E,\text{der}}$ can be identified with $\text{SL}_2(F)^3$ so that the representation of $M_{\text{der}}$ on $V = V_E$ is isomorphic to the natural action of $\text{SL}_2(F)^3$ on $F^2 \otimes F^2 \otimes F^2$. An element $v \in V$ is represented by a cube

\[
\begin{array}{c}
\begin{array}{ccc}
& e_1 & \\
& &  \\
e_3 & & \\
& & f_3 \\
e_2 & & \\
& 1 & \\
\end{array}
\end{array}
\]

where $a, \ldots, b \in F$, and the vertices correspond to the standard basis in $F^2 \otimes F^2 \otimes F^2$. More precisely, we fix this correspondence so that

\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

correspond to the vertices marked with letters $a$ and $b$, respectively. We note that elementary matrices in $\text{SL}_2(F)^3$ act on the space of cubes by the following three types of “row-column” operations on cubes:

- add or subtract the front face from the rear face of the cube, and vice-versa.
- add or subtract the top face from the bottom face of the cube, and vice-versa.
- add or subtract the right face from the left face of the cube, and vice-versa.

The vertices correspond to root groups. We shall assume that the vertex marked by $a$ corresponds to $\alpha_0$, and that the vertex marked by $b$ corresponds to $\beta_0 - \alpha_0$. The group $\text{Aut}(\Pi)$ acts as the group of symmetries of the cube fixing these two vertices. We shall often write the cube as a quadruple

\[(a, e, f, b)\]

where $e = (e_1, e_2, e_3)$ and $f = (f_1, f_2, f_3) \in F^3$. The group $\text{Aut}(\Pi)$ acts on the entries of $e$ and $f$ by permutations.

Steinberg’s elements $h_{\alpha_0}(t) \in T$, where $t \in F^\times$, act on the vertices of the cube by $t^m$, where $m$ are given by

\[
\begin{array}{c}
\begin{array}{ccc}
& 1 & \\
& &  \\
& & 1 \\
& & 0 \\
& 0 & \\
\end{array}
\end{array}
\]
By comparing the root data, we see that the group \( M(F) \) is isomorphic to the subgroup \( \text{GL}_2(F^3)^0 \) of \( \text{GL}_2(F^3) \) consisting of elements \((g_1, g_2, g_3)\) such that \( \det(g_1) = \det(g_2) = \det(g_3) \). Moreover, under this isomorphism, the representation of \( M \) on \( V \) is given by the standard action of \( \text{GL}_2(F^3)^0 \) on \( F^2 \otimes F^2 \otimes F^2 \).

6.2. **Reduced and distinguished cube.** It is not hard to see that, using the action of \( M(F) \), every cube can be transformed into a cube of the form \((1, 0, f, b)\):

We shall call such a cube a *reduced cube*. In particular, we call the cube \( v_0 = (1, 0, 0, 1) \) the *distinguished cube*.

6.3. **Stabilizer of distinguished cube.** Let \( \text{Stab}_M(v_0) \) and \( \text{Stab}_{\tilde{M}}(v_0) \) be the respective stabilizers in \( M \) and \( \tilde{M} \) of the distinguished cube \( v_0 \in V \). Since \( \text{Aut}(\Pi) \) stabilizes \( v_0 \), the group \( \text{Stab}_{\tilde{M}}(v_0) \) is a semi direct product of \( \text{Stab}_M(v_0) \) and \( \text{Aut}(\Pi) \). We shall now compute \( \text{Stab}_M(v_0) \). Let \( g = (g_1, g_2, g_3) \in M(F) \) where

\[
g_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}.
\]

Since

\[
v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

and

\[
g \cdot v_0 = \left( \begin{array}{c} a_1 \\ c_1 \end{array} \right) \otimes \left( \begin{array}{c} a_2 \\ c_2 \end{array} \right) \otimes \left( \begin{array}{c} a_3 \\ c_3 \end{array} \right) + \left( \begin{array}{c} b_1 \\ d_1 \end{array} \right) \otimes \left( \begin{array}{c} b_2 \\ d_2 \end{array} \right) \otimes \left( \begin{array}{c} b_3 \\ d_3 \end{array} \right)
\]

\( g \cdot v_0 = v_0 \) if and only if eight equations hold. Six of these equations are homogeneous. They are

\[
a_1c_2a_3 + b_1d_2b_3 = 0
\]

\[
a_1c_2c_3 + b_1d_2d_3 = 0
\]

and additional four obtained by cyclicly permuting the indices. If we multiply the first equation by \( d_3 \), the second by \( b_3 \), and subtract them, then

\[
0 = a_1c_2a_3d_3 - a_1c_2c_3b_3 = a_1c_2(a_3d_3 - c_3b_3).
\]

Since \( a_3d_3 - c_3b_3 \neq 0 \), we have \( a_1c_2 = 0 \). A similar manipulation of these two equations gives \( b_1d_2 = 0 \). By permuting the indices, we have \( a_ic_j = b_id_j = 0 \) for all \( i \neq j \). This implies that all \( g_i \) are simultaneously diagonal or off diagonal. Now it it easy to see that the remaining two equations imply that \( \text{Stab}_M(v_0) \) has two connected components, and the identity component
consists of \( g = (g_1, g_2, g_3) \) such that \( g_i \) are diagonal matrices, \( a_i d_i = 1 \), and \( a_1 a_2 a_3 = 1 \). The other component of \( \text{Stab}_M(v_0) \) contains an element \( w = (w_1, w_2, w_3) \) of order 2, where

\[
  w_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

We now have a complete description of \( \text{Stab}_M(v_0) \) (and of \( \text{Stab}_{\widetilde{M}}(v_0) \)):

\[
\text{Stab}_M(v_0) \cong \{(a_1, a_2, a_3) \in G^3_m : a_1a_2a_3 = 1\} \rtimes \mathbb{Z}/2\mathbb{Z} \cong G^2_m \rtimes \mathbb{Z}/2\mathbb{Z}.
\]

In particular, we have shown:

**Proposition 6.1.** The stabilizer \( \text{Stab}_{\widetilde{M}}(v_0) \) in \( \tilde{M} \) of the distinguished cube \( v_0 = (1, 0, 0, 1) \) is isomorphic to the group of \( F \)-automorphisms of the split twisted composition algebra \( C_0 \).

6.4. Three quadratic forms. One key observation in \cite{[1]} is that one can slice the cube (given in the picture in \S 6.1) in three different ways, giving three pairs of matrices:

\[
  A_1 = \begin{pmatrix} a & e_2 \\ e_3 & f_1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} e_1 & f_3 \\ f_2 & b \end{pmatrix}.
\]

\[
  A_2 = \begin{pmatrix} a & e_3 \\ e_1 & f_2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} e_2 & f_1 \\ f_3 & b \end{pmatrix}.
\]

\[
  A_3 = \begin{pmatrix} a & e_1 \\ e_2 & f_3 \end{pmatrix}, \quad B_3 = \begin{pmatrix} e_3 & f_2 \\ f_1 & b \end{pmatrix}.
\]

Note that the pairs \((A_2, B_2)\) and \((A_3, B_3)\) are obtained by rotating the pair \((A_1, B_1)\) about the axis passing through \(a\) and \(b\). For each pair \((A_i, B_i)\), Bhargava defines a quadratic binary form by

\[
Q_i = -\det(A_i x - B_i y).
\]

**Proposition 6.2.** Given a cube \( v \), the three forms \( Q_1, Q_2 \) and \( Q_3 \) have the same discriminant \( \Delta = \Delta(v) \).

**Proof.** We may assume the cube is reduced. Now an easy computation show that the three forms are

\[
\begin{align*}
Q_1(x, y) &= -f_1 x^2 + bxy + f_2 f_3 y^2 \\
Q_2(x, y) &= -f_2 x^2 + bxy + f_3 f_1 y^2 \\
Q_3(x, y) &= -f_3 x^2 + bxy + f_1 f_2 y^2.
\end{align*}
\]

These forms have the same discriminant \( \Delta = b^2 + 4 f_1 f_2 f_3 \). \( \square \)
6.5. **Quartic invariant.** To every cube $v \in V$, the discriminant $\Delta(v)$ described in the previous proposition is a homogeneous quartic polynomial in $v$, which is invariant under the action of $SL_2(F)^3$. This describes the quartic invariant of the prehomogeneous vector space $V$. An explicit computation gives the following formula:

$$
\Delta = a^2b^2 - 2ab(e_1f_1 + e_2f_2 + e_3f_3) + c_1^2f_1^2 + c_2^2f_2^2 + c_3^2f_3^2 + 44f_1f_2f_3 + 4bf_2e_2e_3 - 2(e_1e_2f_1f_2 + e_2e_3f_2f_3 + e_3e_1f_3f_1).
$$

If $v$ is reduced, then this simplifies to $\Delta(v) = b^2 + 4f_1f_2f_3$. It is easy to check that for $g \in M$, one has

$$
\Delta(g \cdot v) = \det(g)^2 \cdot \Delta(v).
$$

Thus, we see that $\Delta$ gives a well-defined map

$$
\Delta : \{\text{generic} \, \tilde{M}(F)\text{-orbits on } V(F)\} \rightarrow F^\times / F^\times 2 = \{\text{étale quadratic } F\text{-algebras}\}.
$$

7. **E-twisted Bhargava Cube**

Now we can extend the discussion of the previous section to the case of general $E$, where $V_E = F \oplus E \oplus E \oplus F$ and $M_E = GL_2(E)^0$, via a Galois descent using a cocycle in the class of $[E] \in H^1(F, Aut(\Pi)) = H^1(F, S_3)$.

A cube is a quadruple $v = (a, e, f, b)$, where $e, f \in E$. As in the split case, we shall call the cubes of the form $v = (1, 0, f, b)$ reduced, and the vector $v_{0,E} = (1, 0, 0, 1)$ the $E$-distinguished cube.

7.1. **Quartic invariant.** By Galois descent, we see that the basic polynomial invariant $\Delta$ is given by

$$
\Delta(a, e, f, b) = a^2b^2 - 2abTr_{E/F}(ef) + Tr_{E/F}(e^2f^2) + 4aN_{E/F}(f) + 4bN_{E/F}(e) - 2Tr_{E/F}(e^#f^#).
$$

If $v$ is reduced, then this simplifies to:

$$
\Delta(1, 0, f, b) = b^2 + 4 \cdot N_{E/F}(f).
$$

7.2. **Group action.** It is useful to note the action of certain elements of $GL_2(E)^0$ on $V_E$. Specifically, the diagonal torus elements

$$
t_{\alpha, \beta} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}
$$

acts on

$$(a, e, f, b) \mapsto (\alpha^#\beta^{-1}a, \alpha^#\alpha^{-1}e, \beta^#\beta^{-1}f, \beta^#\alpha^{-1}b).$$

It is easy to check that

$$
\Delta(t_{\alpha, \beta} \cdot v) = (\alpha\beta)^2 \cdot \Delta(v).
$$

Since the action of $SL_2(E)$ and $S_E$ preserve $\Delta$, we see that

$$
\Delta(g \cdot v) = (\det g)^2 \cdot \Delta(v)
$$

so that $\Delta$ induces a map

$$
\{\tilde{M}_E\text{-orbits on } V_E\} \rightarrow F^\times / F^\times 2 = \{\text{étale quadratic algebras}\}.
$$
In addition, the standard Weyl group element
\[ w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \text{GL}_2(E)^0 \]
acts on
\[ w : (a, e, f, b) \mapsto (b, f, e, a). \]

### 7.3. Stabilizer of distinguished \(E\)-cube

We can readily determine the stabiliser of the \(E\)-distinguished cube. Namely, under the action described in §7.2, it is easy to see that the subgroup
\[ E^1 = \{ \begin{pmatrix} \alpha & \alpha^{-1} \\ \alpha^{-1} & \alpha \end{pmatrix} : \alpha \in E^1 \} \subset \text{SL}_2(E) \]
fixes the \(E\)-distinguished cube \(v_{0,E}\). So does the Weyl group element \(w\). Thus we see that
\[ \text{Stab}_{M_E}(v_{0,E}) \cong E^1 \rtimes \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad \text{Stab}_{\tilde{M}_E}(v_{0,E}) = E^1 \rtimes (\mathbb{Z}/2\mathbb{Z} \times S_E). \]

In particular, we have shown:

**Proposition 7.1.** The stabilizer \(\text{Stab}_{\tilde{M}_E}(v_{0,E})\) in \(\tilde{M}_E\) of the \(E\)-distinguished cube \((1, 0, 0, 1)\) is isomorphic to the group of \(F\)-automorphisms of the twisted composition algebra \(C_E\) introduced in §4.4.

### 8. Generic Orbits

We come now to the main result of this paper: the determination of the generic \(\tilde{M}_E(F)\)-orbits in \(V_E(F)\).

#### 8.1. A commutative diagram

We have the following key commutative diagram
\[ \begin{array}{ccc}
H^1(F, \text{Stab}_{\tilde{M}}(v_0)) & \longrightarrow & H^1(F, \tilde{M}) \\
\downarrow & & \downarrow \\
H^1(F, \text{Aut}_F(C_0, Q_0, \beta_0)) & \longrightarrow & H^1(F, S_3)
\end{array} \quad (8.1) \]

We make several observations about this commutative diagram.

**Lemma 8.2.** (i) The first vertical arrow is bijective.

(ii) The second vertical arrow is bijective.

(iii) The horizontal arrows are surjective.

**Proof.** (i) This follows by Proposition 6.1.

(ii) Since \(\tilde{M}\) is a semi-direct product of \(M\) and \(S_3\), the map \(\psi\) is surjective. For injectivity, we shall use the exact sequence of pointed sets:
\[ 1 \to H^1(F, M) \to H^1(F, \tilde{M}) \to H^1(F, S_3) \to 1. \]

Let \(c \in H^1(F, S_3)\) and let \(E\) be the étale cubic algebra corresponding to \(c\). Then \(M_E\) is the twist of \(M\) by \(c\). In order to prove that \(\psi^{-1}(c)\) consists of one element, it suffices to show
that $H^1(F, M_E)$ is trivial, by the twisting argument on page 50 in [5]. We have an exact sequence of algebraic groups
\[ 1 \to M_{E,\text{der}} \to M_E \to \text{GL}_1 \to 1, \]
where $M_{E,\text{der}} \cong \text{Res}_{E/F} \text{SL}_2$. By Hilbert's Theorem 90, $H^1(F, \text{GL}_1)$ is trivial. Since
\[ H^1(F, \text{Res}_{E/F} \text{SL}_2) = H^1(E, \text{SL}_2) = 0, \]
see [5], page 130, it follows that $H^1(F, M_E)$ is trivial.

(iii) This follows because $\text{Stab}_{\tilde{M}_E}(v_0) = \text{Stab}_M(v_0) \rtimes \text{Aut}(\Pi)$, hence
\[ H^1(F, \text{Stab}_{\tilde{M}_E}(v_0)) \to H^1(F, \text{Aut}(\Pi)) \]
has a natural splitting. \hfill \Box

8.2. Determination of orbits. We can now determine the generic $\tilde{M}_E(F)$-orbits on $V_E(F)$.

Theorem 8.3. Fix an étale cubic $F$-algebra $E$.

(i) The generic $\tilde{M}_E(F)$-orbits on $V_E(F)$ are in bijective correspondence with the set of $F$-isomorphism classes of $E$-twisted composition algebras over $F$, with the orbit of $v_{0,E} = (1,0,0,1)$ corresponding to the twisted composition algebra $C_E$ introduced in 4.4.

(ii) The generic $M_E(F)$-orbits on $V_E(F)$ are in bijective correspondence with the set of $E$-isomorphism classes of $E$-twisted composition algebras over $F$.

(iii) There is a commutative diagram
\[ \{ E\text{-twisted composition algebras} \} \xrightarrow{\Delta} \{ \text{étale quadratic } F\text{-algebras} \} \]
\[ \downarrow \quad \downarrow \]
\[ \{ \text{generic } \tilde{M}_E\text{-orbits on } V_E \} \quad \xrightarrow{\Delta} \quad F^\times / F^\times 2 \]
where the bottom arrow is the map induced by $\Delta$ (see §7.2).

Proof. (i) Given a cohomology class $[E] \in H^1(F, S_3)$ corresponding to an étale cubic $F$-algebra, we consider the fibers of the two horizontal arrows in the commutative diagram over $[E]$. Since the map $\text{Stab}_{\tilde{M}_E}(v_0) \to S_3$ splits, the fiber of the second horizontal arrow has a distinguished element which corresponds to the twisted composition algebra $C_E$. Similarly, the fiber over $[E]$ of the first horizontal arrow has a distinguished point which corresponds to the orbit of $v_{0,E} = (1,0,0,1)$. Moreover, these two distinguished points correspond under the first vertical arrow.

By the twisting argument, see [5], page 50, we see that both fibers in question are naturally identified with
\[ \text{Ker}(H^1(F, \text{Stab}_{\tilde{M}_E}(v_{0,E})) \to H^1(F, \tilde{M}_E)). \]
Thus, the fiber of the first horizontal map over $[E]$ are the generic $\tilde{M}_E$-orbits in $V_E$, while the fibers of the second map are $F$-isomorphism classes of $E$-twisted composition algebras.

(ii) The bijection follows because both sets are in natural bijection with $H^1(F, \text{Stab}_{M_E}(v_{0,E}) = H^1(F, \text{Aut}_E(C_E))$. 

(iii) Suppose an $E$-twisted composition algebra is represented by a cocycle
$$(a_\sigma) \in H^1(F, \text{Stab}_{M_E}(v_{0,E})).$$
Then the associated étale quadratic $F$-algebra $K$ corresponds to the group homomorphism
$$\eta_K : \text{Gal}(\bar{F}/F) \longrightarrow \text{Stab}_{M_E}(v_{0,E})(\bar{F}) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$
given by $\sigma \mapsto \pi(a_\sigma)$, where $\pi : \text{Stab}_{M_E}(v_{0,E}) \to \mathbb{Z}/2\mathbb{Z}$ is the natural projection. In fact, regarding $\text{Stab}_{M_E}(v_{0,E}) \subseteq M_E$ as described in §7.3, we see that the map $\pi$ is simply given by the determinant map on $M_E = \text{GL}_2(E)$. On the other hand, the cocycle splits in $H^1(F, M_E) = 0$, so that we may write
$$a_\sigma = g^{-1} \cdot \sigma(g) \quad \text{for some } g \in M_E(\bar{F}).$$
Then the $M_E$-orbit associated to $(a_\sigma)$ is that of $g \cdot v_{0,E}$. Now we have:
$$\Delta(g \cdot v_{0,E}) = \det(g)^2 \cdot \Delta(v_{0,E}) = \det(g)^2$$
and
$$\eta_K(\sigma) = \det(a_\sigma) = \det(g)^{-1} \cdot \sigma(\det(g))$$
for any $\sigma \in \text{Gal}(\bar{F}/F)$. This shows that $\det(g)$ is a trace zero element in $K$, so that $K$ is represented by the square class of $\det(g)^2 \in F^\times$, as desired.

The theorem is proved. \hfill \square

In particular, we have established Theorem 1.1.

9. Gauss Law

Though we have classified the generic orbits in the twisted Bhargava cube by Galois cohomological arguments, it is useful in practice to have an explicit description of the correspondence between the orbits and the twisted composition algebras. In this section, we examine the case when $E = F^3$ is split.

We first review briefly Bhargava’s results and, following him, we shall work over $\mathbb{Z}$. Note that we have an action of the group $\text{SL}_2(\mathbb{Z})^3$ on the set of integer valued cubes, by the “row-column” operations as described in §6.1.

9.1. Bhargava’s result. In order to state the main result of Bhargava, we need a couple of definitions. Fix a discriminant $\Delta$. Let $K = \mathbb{Q}(\sqrt{\Delta})$ and $R$ the unique order of discriminant $\Delta$. A module $M$ is a full lattice in $K$. In particular, it is a $\mathbb{Z}$-module of rank 2. We shall write $M = \{u, v\}$ if $u$ and $v$ span $M$. For example,
$$R = \{1, \frac{\Delta + \sqrt{\Delta}}{2}\}.$$ 
By fixing this basis of $R$, we have also fixed a preferred orientation of bases of modules. An oriented module is a pair $(M, \epsilon)$ where $\epsilon$ is a sign. If $M = \{u, v\}$, then $M$ becomes an oriented module $(M, \epsilon)$, where $\epsilon = 1$ if and only if the orientation of $\{u, v\}$ is preferred. The norm of an oriented module $(M, \epsilon)$ is $N(M) = \epsilon \cdot [R : M]$.

Then:
• a triple of oriented modules \((M_1, M_2, M_3)\) with \(R\) as the multiplier ring, is said to be \textit{colinear}, if there exists \(\delta \in K^\times\) such that the product of the three oriented modules is a principal oriented ideal \(((\delta), \epsilon)\) where \(\epsilon = \text{sign}(N(\delta))\), i.e., \(M_1M_2M_3 = (\delta)\), as ordinary modules, and \(N(M_1)N(M_2)N(M_3) = N(\delta)\).

• a cube is \textit{projective} of discriminant \(\Delta\) if the three associated forms are primitive and have the discriminant \(\Delta\).

• two triples of oriented modules \((M_1, M_2, M_3)\) and \((M_1', M_2', M_3')\) are equivalent if there exist \(\mu_1, \mu_2, \mu_3\) in \(K^\times\) such that \(M'_i = \mu_i M_i\) and \(\epsilon'_i = \text{sign}(N(\mu_i))\epsilon_i\) for \(i = 1, 2, 3\).

Then Bhargava \cite{Bhargava2002} showed:

\textbf{Theorem 9.1.} There is a bijection, to be described in the proof, between the equivalence classes of oriented colinear triples of discriminant \(\Delta\) and \(SL_2(\mathbb{Z})^3\)-equivalence classes of projective cubes of discriminant \(\Delta\).

Sketch of proof: Let \(v\) be a projective cube. Again, without any loss of generality we can assume that the cube is reduced and that the numbers \(f_1, f_2\) and \(f_3\) are nonzero. Define three modules by

\[ M_1 = \{1, \frac{b - \sqrt{\Delta}}{2f_1}\}, \quad M_2 = \{1, \frac{b - \sqrt{\Delta}}{2f_2}\} \quad \text{and} \quad M_3 = \{1, \frac{b - \sqrt{\Delta}}{2f_3}\}. \]

The norms of the three modules are \(-1/f_1, -1/f_2\) and \(-1/f_3\), respectively, if we take the given bases to be proper. For \(\delta\) we shall take

\[ \delta = \frac{2}{b + \sqrt{\Delta}}, \]

which has the correct norm \(-1/f_1f_2f_3\).

The modules \(M_i\), with given oriented bases, correspond to the quadratic forms \(Q_i\). More precisely, if

\[ z_i = x_i - y_i \frac{b - \sqrt{\Delta}}{2f_i} \in M_i \]

then

\[ -f_iN(z_i) = Q_i(x_i, y_i) = -f_ix_i^2 + bx_iy_i + f^#_iy_i^2 \]

where \(f^# = (f_2f_3, f_3f_1, f_1f_2)\).

\textbf{9.2. Integral Twisted Composition algebras.} We describe a relationship of Barghava’s triples \((M_1, M_2, M_3)\) of oriented modules and integral twisted composition algebras. Assume the notation from the previous subsection. In particular, \(M_1M_2M_3 = (\delta)\). Let

\[ C = M_1 \oplus M_2 \oplus M_3, \]

Define a quadratic form \(Q : C \to \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}\) by

\[ Q(z_1, z_2, z_3) = (-f_1N(z_1), -f_2N(z_2), -f_3N(z_3)) = -f \cdot (N(z_1), N(z_2), N(z_3)). \]

and define a quadratic map \(\beta : C \to C\) by

\[ \beta(z_1, z_2, z_3) = \delta(f_2f_3\bar{z}_2\bar{z}_3, f_3f_1\bar{z}_3\bar{z}_1, f_1f_2\bar{z}_1\bar{z}_2) = \delta \cdot f^# \cdot (\bar{z}_1, \bar{z}_2, \bar{z}_3)^\#. \]
The relation $M_1M_2M_3 = (\delta)$ and $M\tilde{M} = N(M)$ imply that $\beta$ is well defined. Moreover, using $N(\delta) = -1/f_1f_2f_3$, one checks that

$$Q(\beta(z_1, z_2, z_3)) = Q(z_1, z_2, z_3)^\#$$

and

$$N(z_1, z_2, z_3) = Tr(\frac{z_1z_2z_3}{\delta})$$

e.i. the triple $(C, Q, \beta)$ is a twisted composition algebra.

Recall that $Q_i(z_i) = -f_iN(z_i)$ is written explicitly using the coordinates $(x_i, y_i)$ given by

$$z_i = x_i - y_i - \frac{b - \sqrt{\Delta}}{2f_i}.$$

We shall now do the same for $\beta$. Write $\beta(z_1, z_2, z_3) = (z'_1, z'_2, z'_3)$, and let $(x'_i, y'_i)$ be the coordinates of $z'_i$. A short calculation shows that

$$x'_1 = \begin{pmatrix} x_3 & -y_3 \end{pmatrix} \begin{pmatrix} 0 & f_3 \\ f_2 & b \end{pmatrix} \begin{pmatrix} x_2 \\ -y_2 \end{pmatrix}$$

$$y'_1 = \begin{pmatrix} x_3 & -y_3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & f_1 \end{pmatrix} \begin{pmatrix} x_2 \\ -y_2 \end{pmatrix}$$

while the expressions for $(x'_2, y'_2)$ and $(x_3, y_3)$ are obtained by cyclicly permuting the indices. There are two important observations to be made here. Firstly, these formulas make sense for any triple $(f_1, f_2, f_3)$ and any $b$. The axioms of twisted composition algebra are satisfied for formal reasons. For example, if $(f_1, f_2, f_3) = (0, 0, 0)$ and $b = 1$, we get the split algebra. Secondly, the two matrices are two opposite faces of the cube. So this gives a hint how to directly associate a composition algebra to any cube. This is the subject of the next section.

9.3. From cubes to twisted composition algebras. We can now define a twisted composition algebra over $F \times F \times F$, starting with any cube $v$. (The following construction works over $\mathbb{Z}$.) Let $C = F^2 \times F^2 \times F^2$. An element $z \in C$ is a triple $(z_1, z_2, z_3)$ of column vectors $z_i = (\begin{pmatrix} x_i \\ y_i \end{pmatrix})$. Slice a cube into three pairs of $2 \times 2$-matrices $(A_i, B_i)$, as before and let

$$Q_i(z_i) = -\det(A_i x_i - B_i y_i).$$

Let $Q : C \to F \times F \times F$ be defined by

$$Q(z_1, z_2, z_3) = (Q_1(z_1), Q_2(z_2), Q_3(z_3)).$$

Let $\beta : C \to C$ be defined by

$$\beta(z_1, z_2, z_3) = (z'_1, z'_2, z'_3)$$

where $z'_i = (\begin{pmatrix} x'_i \\ y'_i \end{pmatrix}),$

$$x'_1 = z_3^\top B_1 z_2, \quad x'_2 = z_1^\top B_2 z_3, \quad x'_3 = z_2^\top B_3 z_1$$

and

$$y'_1 = z_3^\top A_1 z_2, \quad y'_2 = z_1^\top A_2 z_3, \quad y'_3 = z_2^\top A_3 z_1.$$

These formulae define a twisted composition algebra. To see this, observe that the action of $\text{SL}_2(F)^3$ on the cube $v$ corresponds to change of coordinates in $C$. Indeed, if we replace the coordinate $x_1$ by $u = x_1 + ty_1$ and keep all other coordinates the same, then the resulting formulas for $\beta$ are the same as those obtained from the cube $v'$ such that the pair of faces
$$(A', B'_1)$$ is $$(A_1, B_1 + tA_1)$$. If we replace the coordinate $y_1$ by $u = y_1 + tx_1$ and keep all other coordinates the same, then the resulting formulas for $\beta$ are the same as those obtained from the cube $v'$ such that $$(A'_1, B'_1) = (A_1 + tB_1, A_1)$$. Thus, it suffices to check that $C$ is a composition algebra when $v$ is reduced, where it is known by §9.2.

As an example, assume that $K = F(\sqrt{\Delta})$ and consider the composition algebra corresponding to the cube

![Graph of a cube with coordinates labeled 0, 1, and 4]

This is the distinguished point in the fiber of $([F^3], [K])$, as in Proposition §3.5. (See also the example in §1.1)

10. Explicit Parametrization

Using the description of the previous section, and Galois descent, we can now give an explicit description of the correspondence between the generic $\tilde{M}_E$-orbits and the set of $E$-twisted composition algebras.

Let $v = (1, 0, f, b) \in V_E$ be a reduced cube. Then, to this cube, we attach

- an étale quadratic algebra

$$K = F(\sqrt{\Delta}) = F(\sqrt{b^2 + 4N_{E/F}(f)}).$$

- an $E$-twisted composition algebra $(C, Q, \beta)$, where

$$C = E \oplus E$$

is equipped with the quadratic form

$$Q(x, y) = -fx^2 + bxy + f^\# y^2$$

and the quadratic map

$$\beta(x, y) = (by^\# - \frac{1}{2}(fx) \times y, x^\# + fy^\#).$$

If $f \in E^\times$, we can relate this construction to Proposition §3.5. Identify $E \oplus E$ with $E \otimes K$ using the $E$-linear isomorphism given by

$$(x, y) \mapsto x \otimes 1 - \frac{y}{f} \otimes \frac{b - \sqrt{\Delta}}{2} = x - y\frac{b - \sqrt{\Delta}}{2f}$$
where, in the last expression, we omitted tensor product signs for readability. Then $Q$ can be written as

$$Q(x - y \frac{b - \sqrt{\Delta}}{2f}) = -f \cdot N_{E \otimes K/E}(x - y \frac{b - \sqrt{\Delta}}{2f})$$

and $\beta$ as

$$\beta(x - y \frac{b - \sqrt{\Delta}}{2f}) = \left( \frac{2f^#}{b + \sqrt{\Delta}} \right) (x - y \frac{b + \sqrt{\Delta}}{2f})^#$$

Hence, this is the composition algebra attached to the pair $(e, \nu)$, as in Proposition 3.5, where $e = -f$ and $\nu = b + \frac{\sqrt{\Delta}}{2}$.

(Note that $e^{-1} \cdot \nu \cdot \bar{\nu} = e^#$, since $N_{E/F}(e) = N_{K/F}(\nu)$.) Conversely, a composition algebra given by a pair $(e, \nu)$, as in Proposition 3.5, arises from the cube $(1,-e,b)$ where $b = Tr_{K/F}(\nu)$.

11. Exceptional Hilbert 90

Assume that $E$ is an étale cubic $F$-algebra field with corresponding étale quadratic discriminant algebra $K_E$ and let $K$ be an étale quadratic $F$-algebra. Recall that

$$T_{E,K} = \{x \in E \otimes_F K : N_{E/F}(x) = 1 = N_{K/F}(x)\}.$$

Suppose, for example, that $[K_E] = [K] = 1$, so $E$ is a Galois extension, and $T_{E,K}$ is the group of norm one elements in $E^\times$. Let $\sigma$ be a generator of the Galois group $G_{E/F}$. Then Hilbert’s Theorem 90 states that the map

$$x \mapsto \sigma(x) / \sigma^2(x)$$

induces an isomorphism of $E^\times / F^\times$ and $T_{E,K}(F)$. Our goal in this section is to generalize this statement to all tori $T_{E,K}$, thus obtaining an exceptional Hilbert’s Theorem 90. As an application, we give an alternative description of $H^1(F, T_{E,K})$.

11.1. The torus $T_{E,K}$.

We first describe the torus $T_{E,K}$ by Galois descent. Over $\mathcal{F}$, we may identify

$$T_{E,K}(\mathcal{F}) = \{(a, b) \in \mathcal{F}^3 \otimes \mathcal{F}^2 : a_i b_i = 1 \text{ for all } i \text{ and } a_1 a_2 a_3 = 1\}.$$

The $F$-structure is given by the twist of the Galois action on coordinates by the cocycle

$$\rho_E \times \rho_K : Gal(\mathcal{F}/F) \to Aut(\mathcal{F}^3) \times Aut(\mathcal{F}^2) \cong S_3 \times \mathbb{Z}/2\mathbb{Z},$$

where $S_3$ (respectively $\mathbb{Z}/2\mathbb{Z}$) acts on $\mathbb{Z}^3$ (respectively $\mathbb{Z}^2$) by permuting the coordinates.

We may describe $T_{E,K}$ using its cocharacter lattice $X$. We have:

$$X = \{(a, -a) \in \mathbb{Z}^3 \otimes \mathbb{Z}^2 : a_1 + a_2 + a_3 = 0\},$$

equipped with the Galois action given by

$$\rho_E \otimes \rho_K : Gal(\mathcal{F}/F) \to S_3 \times \mathbb{Z}/2\mathbb{Z}.$$
11.2. The torus $T'_{E,K}$. Now we introduce another torus $T'_{E,K}$ over $F$. Let $K_J$ be the étale quadratic $F$-algebra such that $[K_J] \cdot [K] \cdot [K_E] = 1$ in $H^1(F, \mathbb{Z}/2\mathbb{Z})$. We define the torus

$$\tilde{T}'_{E,K} = \{ x \in E \otimes_F K_J : N_{E \otimes K_J/E}(x) \in F^\times \},$$

and

$$T'_{E,K} = \tilde{T}'_{E,K}/K^\times_J$$

where the last quotient is taken in the sense of algebraic groups.

We may again describe these tori by Galois descent. Over $\overline{F}$, we may identify

$$\tilde{T}'_{E,K}(\overline{F}) = \{ (a, b) \in (\overline{F}^\times)^3 \otimes (\overline{F}^\times)^2 : a_1b_1 = a_2b_2 = a_3b_3 \},$$

and $T'_{E,K}(\overline{F})$ is the quotient of this by the subgroup consisting of the elements $(a \cdot 1, b \cdot 1)$. The action of $\text{Gal}(\overline{F}/F)$ which gives the $F$-structure of $\tilde{T}'_{E,K}$ is then described as follows. Let $\rho_E : \text{Gal}(\overline{F}/F) \to S_3$ be the cocycle associated to $E$, so that $\text{sign} \circ \rho_E : \text{Gal}(\overline{F}/F) \to \mathbb{Z}/2\mathbb{Z}$ is the homomorphism associated to $K_E$. On the other hand, we let $\rho_K$ be the homomorphism associated to $K_J$, so that $(\text{sign} \circ \rho_E) \cdot \rho_K : \text{Gal}(\overline{F}/F) \to S_3 \times \mathbb{Z}/2\mathbb{Z}$.

As before, we may describe the tori $\tilde{T}'_{E,K}$ and $T'_{E,K}$ by their cocharacter lattice. The cocharacter lattice $\tilde{Y}$ of $\tilde{T}'_{E,K}$ is given by

$$\tilde{Y} = \{ (a, b) \in \mathbb{Z}^3 \otimes \mathbb{Z}^2 : a_1 + b_1 = a_2 + b_2 = a_3 + b_3 \},$$

equipped with the Galois action given by

$$\rho_E \times (\text{sign} \circ \rho_E) \cdot \rho_K : \text{Gal}(\overline{F}/F) \to S_3 \times \mathbb{Z}/2\mathbb{Z}.$$ 

This contains the Galois-stable sublattice

$$Z = (1,1,1) \otimes \mathbb{Z}^2$$

so that $Y = \tilde{Y}/Z$ is the cocharacter lattice of $T'_{E,K}$.

11.3. A homomorphism. We are going to construct a morphism of tori from $\tilde{T}'_{E,K}$ to $T_{E,K}$. We shall first define this morphism over $\overline{F}$ and then shows that it descends to $F$.

Now we may define a morphism over $\overline{F}$:

$$f : \tilde{T}'_{E,K}(\overline{F}) \to T_{E,K}(\overline{F})$$

by

$$f : \left( \begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{array} \right) \mapsto \left( \begin{array}{ccc} a_2/a_3 & a_3/a_1 & a_1/a_2 \\ b_2/b_3 & b_3/b_1 & b_1/b_2 \end{array} \right)$$

It is easy to see that this defines an $\overline{F}$-isomorphism of tori

$$f : T'_{E,K}(\overline{F}) \cong T_{E,K}(\overline{F}).$$
Moreover, if $\sigma \in S_e(F) = S_3$ is the cyclic permutation
\[(a_1, a_2, a_3) \mapsto (a_2, a_3, a_1),\]
then the map $f$ is given by
\[f(x) = \sigma(\alpha)/\sigma^2(\alpha).\]

Now the morphism $f$ induces a map
\[f_\ast : \tilde{Y} \to X\]
given by
\[
\begin{pmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3
\end{pmatrix} \mapsto
\begin{pmatrix}
  a_2 - a_3 & a_3 - a_1 & a_1 - a_2 \\
  b_2 - b_3 & b_3 - b_1 & b_1 - b_2
\end{pmatrix}.
\]
This induces an isomorphism of $\mathbb{Z}$-modules $Y \cong X$.

11.4. Exceptional Hilbert 90. Now the main result of this section is:

**Theorem 11.1.** The isomorphism $f : T'_{E,K} \times_F \overline{F} \to T_{E,K} \times_F \overline{F}$ is defined over $F$, and thus gives an isomorphism of tori
\[T'_{E,K} \to T_{E,K}\]
given by
\[x \mapsto \sigma(x)/\sigma^2(x).\]

**Proof.** It remains to prove that $f$ is defined over $F$. For this, we may work at the level of cocharacter lattices and we need to show that $f_\ast$ is Galois-equivariant. For this, regard $\mathbb{Z}^3 \otimes \mathbb{Z}/2\mathbb{Z}$ as a $S_3 \times \mathbb{Z}/2\mathbb{Z}$-module with the permutation of the coordinates in $\mathbb{Z}^3$ and $\mathbb{Z}/2\mathbb{Z}$. Then observe that $f_\ast$ is not equivariant with respect to $S_3 \times \mathbb{Z}/2\mathbb{Z}$. On the other hand, we have the automorphism of $S_3 \times \mathbb{Z}/2\mathbb{Z}$ given by
\[(g,h) \mapsto (g, \text{sign}(g) \cdot h)\]
If we twist the $S_3 \times \mathbb{Z}/2\mathbb{Z}$-module structure on the domain of $f_\ast$ by this automorphism, then $f_\ast$ is easily seen to be equivariant. Together with our description of the $\text{Gal}(\overline{F}/F)$-actions on the domain and codomain of $f_\ast$, the desired $\text{Gal}(\overline{F}/F)$-equivariance follows. \[\square\]

11.5. Cohomology of $T_{E,K}$. As an application of the exceptional Hilbert 90, we may give an alternative description of the cohomology group $H^1(F, T_{E,K})$ which classifies twisted composition algebras with fixed invariants $(E, K)$, up to $E \otimes_F K$-linear isomorphisms.

In order to state results, we need additional notation. For every quadratic extension $K'$ of $F$, let $\text{Res}^1_{K'/F} \mathbb{G}_m$ be the 1-dimensional torus defined by the short exact sequence of algebraic tori:
\[
1 \to \text{Res}^1_{K'/F} \mathbb{G}_m \to \text{Res}^1_{K'/F} \mathbb{G}_m \to \mathbb{G}_m \to 1.
\]
By the classical Hilbert Theorem 90, the associated long exact sequence gives the exact sequence:
\[
1 \to H^2(F, \text{Res}^1_{K'/F} \mathbb{G}_m) \to H^2(K', \mathbb{G}_m) \to H^2(F, \mathbb{G}_m)
\]
where the last map is the corestriction. By a theorem of Albert and Albert-Riehm-Scharlau, Theorem 3.1 in [1], the kernel of the corestriction map is the set of Brauer equivalence classes
of central simple algebras over $K'$ that admit an involution of the second kind, and so we can view $H^2(F, \text{Res}_{K'/F}^1 \mathbb{G}_m)$ as the set of Brauer equivalence classes of such algebras.

Now we have:

**Proposition 11.2.** Let $K_J$ be the étale quadratic algebra such that $[K_J] \cdot [K] \cdot [K_E] = 1$ and set $M = E \otimes_F K_J$.

(i) If $K_J$ is a field, then we have an exact sequence

$$1 \rightarrow E^\times / F^\times N_{M/E}(M^\times) \rightarrow H^1(F, T_{E,K}) \rightarrow H^2(F, \text{Res}_{K_J/F}^1 \mathbb{G}_m) \rightarrow H^2(E, \text{Res}_{M/E}^1 \mathbb{G}_m)$$

The image of $H^1(F, T_{E,K})$ consists of those central simple algebras $B$ over $K_J$ which contain $M$ as a $K_J$-subalgebra and which admit an involution of the second kind fixing $E$ (or equivalently, restricting to the nontrivial automorphism of $M$ over $E$).

(ii) If $K_J = F^2$, then we have a simplified version of the above sequence

$$H^1(F, T_{E,K}) = \text{Ker}(H^2(F, \mathbb{G}_m) \rightarrow H^2(E, \mathbb{G}_m)).$$

**Proof.** (i) By the exceptional Hilbert Theorem 90, we have a short exact sequence of algebraic tori:

$$1 \rightarrow \text{Res}_{K_J/F}^1 \mathbb{G}_m \rightarrow \text{Res}_{E/F} \text{Res}_{M/E}^1 \mathbb{G}_m \rightarrow T_{E,K} \rightarrow 1.$$ 

Now (i) follows from the associated long exact sequence, using

$$H^1(F, \text{Res}_{K_J/F}^1 \mathbb{G}_m) = F^\times / N_{K_J/F} K_J^\times \quad \text{and} \quad H^1(E, \text{Res}_{M/E}^1 \mathbb{G}_m) = E^\times / N_{M/E} M^\times.$$

(ii) One argues as above, except that since $K_J = F^2$, we have:

$$1 \rightarrow \mathbb{G}_m \rightarrow \text{Res}_{E/F} \mathbb{G}_m \rightarrow T_{E,K} \rightarrow 1.$$ 

Thus the long exact sequence gives

$$1 \rightarrow H^1(F, T_{E,K}) \rightarrow H^2(F, \mathbb{G}_m) \rightarrow H^2(E, \mathbb{G}_m) \rightarrow$$



11.6. **Interpretation.** The above description of $H^1(F, T_{E,K})$ fits beautifully with the correspondence between $E$-twisted composition algebras and conjugacy classes of embeddings $E \hookrightarrow J$ where $J$ is a Freudenthal-Jordan algebra of dimension 9.

More precisely, Proposition 11.2 exhibits $H^1(F, T_{E,K})$ as the set of isomorphism classes of triples $(B, \tau, i)$ where

- $B$ is a central simple $K_J$-algebra of degree 3;
- $\tau$ is an involution of the second kind on $B$;
- $i : E \rightarrow B^\tau$ is an $F$-algebra embedding, or equivalently a $K_J$-algebra embedding $i : M = E \otimes_F K_J \rightarrow B$ such that $\tau$ pulls back to the nontrivial element of $\text{Aut}(M/E)$.

The map $\pi : H^1(F, T_{E,K}) \rightarrow H^2(F, \text{Res}_{K_J/F}^1 \mathbb{G}_m)$ sends $(B, \tau, i)$ to $B$. For a fixed

$$[B] \in \text{Ker}(H^2(F, \text{Res}_{K_J/F}^1 \mathbb{G}_m) \rightarrow H^2(E, \text{Res}_{M/E}^1 \mathbb{G}_m)),$$

so that $B$ contains $M = E \otimes_F K_J$ as an $K_J$-subalgebra, the fiber of $\pi$ over $[B]$ is the set of $\text{Aut}_{K_J}(B)$-conjugacy classes of pairs $(\tau, i)$. The Skolem-Noether theorem says that any two
embeddings $M \hookrightarrow B$ are conjugate, and on fixing an embedding $i : M \hookrightarrow B$, the fiber of $\pi$ over $[B]$ is then the set of $\text{Aut}_{K_J}(B,i)$-conjugacy classes of involutions of the second kind on $B$ which restricts to the nontrivial automorphism of $M$ over $E$. Therefore, the exact sequence in Proposition 11.2(i) says that the set of such $\text{Aut}_{K_J}(B,i)$-conjugacy classes of involutions is identified with $E^\times/F^\times N_{M/E}(M^\times)$. One has a natural map on the fiber $\pi^{-1}([B])$ sending a $\text{Aut}_{K_J}(B,i)$-conjugacy class of involutions to its $\text{Aut}_{K_J}(B)$-conjugacy class. This is the surjective map described in Corollary 19.31 in [4].

On the other hand, the map sending the triple $(B,\tau,i)$ to the pair $(B,\tau)$ is the natural map

$$H^1(F, T_{E,K}) \to H^1(F, \text{PGU}_3^{K_J})$$

induced by the map $T_{E,K} \hookrightarrow \text{PU}_3^{K_J}$ where $\text{PGU}_3^{K_J}$ is the identity component of the automorphism group of the Freuthendal-Jordan algebra associated to the distinguished twisted composition algebra with invariants $(E, K)$.

12. Local Fields

In this section, we specialize and explicate the main result in the case of local fields.

12.1. Local Fields. Let $F$ be a local field, $E$ an étale cubic $F$-algebra, and $K_E$ the corresponding discriminant algebra. Let $K$ be an étale quadratic $F$-algebra. We consider

$$\Omega_{E,K} = \{\text{generic } M_E\text{-orbits on } V_E \text{ with associated quadratic algebra } K\}$$

and

$$\Omega_{E,K} = \{\text{generic } M_E\text{-orbits on } V_E \text{ with associated quadratic algebra } K\}$$

We have seen that $\Omega_{E,K}$ has a distinguished element: this is the distinguished point of $H^1(T_{E,K})$ which is fixed by $S_E(F) \times \mathbb{Z}/2\mathbb{Z}$. Moreover, by Galois cohomological arguments,

$$\bar{\Omega}_{E,K} = H^1(F, T_{E,K})/S_E(F) \times \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad \Omega_{E,K} = H^1(F, T_{E,K})/\mathbb{Z}/2\mathbb{Z}$$

We would like to explicate the sets $\bar{\Omega}_{E,K}$ and $\Omega_{E,K}$.

12.2. Cohomology of tori. Recall that in (3.6), we have shown

$$H^1(F, T_{E,K}) = (E^\times \times K^\times)^0/\text{Im}(L^\times)$$

where $L = E \otimes_F K$,

$$(E^\times \times K^\times)^0 = \{(e,\nu) \in E^\times \times K^\times : N_{E/F}(e) = N_{K/F}(\nu)\}$$

and the map from $L^\times$ to $(E^\times \times K^\times)^0$ is given by

$$a \mapsto (N_{L/E}(a), N_{L/K}(a)).$$

This description of $H^1(F, T_{E,K})$ is natural but may not be so explicit. When $F$ is a local field, we can further explicate this description.

Since the case when $E$ or $K$ is not a field is quite simple, we consider the case when $E$ and $K$ are both fields. In that case, the norm map induces an isomorphism

$$E^\times/N_{L/E}(L^\times) \to F^\times/N_{K/F}(K^\times) \cong \mathbb{Z}/2\mathbb{Z},$$
so that any $(e, \nu) \in (E^\times \times K^\times)^0$ has $e = N_{L/E}(a)$ for some $a \in L^\times$. Hence any element in $H^1(F, T_{E,K})$ is represented by $(1, \nu)$ for some $\nu \in K^1 = \{\nu \in K^\times : N_{K/F}(\nu) = 1\}$. We thus deduce that, with $L^1 = \{a \in L^\times : N_{L/E}(a) = 1\}$,

$$H^1(F, T_{E,K}) = K^1/N_{L/K}(L^1) \cong K^\times/F^\times N_{L/K}(L^\times),$$

where the last isomorphism is induced by the usual Hilbert Theorem 90. Using this last expression, we easily see that

$$H^1(F, T_{E,K}) = \begin{cases} 1 & \text{if } K \neq K_E; \\ \mathbb{Z}/3\mathbb{Z}, & \text{if } K = K_E. \end{cases}$$

Exchanging the roles of $E$ and $K$ in the above argument, one also has:

$$H^1(F, T_{E,K}) = E^1/N_{L/E}(L_1)$$

where now $L_1 = \{a \in L^\times : N_{L/K}(a) = 1\}$. If $E/F$ is Galois (and $K$ is a field), it follows by the usual Hilbert Theorem 90 that

$$H^1(F, T_{E,K}) = E^1/N_{L/E}(L_1) \cong E^\times/F^\times N_{L/K}(E^\times) = 1,$$

thus partially recovering the result of the last paragraph.

Alternatively, we could use Proposition 11.2 to compute $H^1(F, T_{E,K})$. If $K_J$ is a field, then the only central simple $K_J$-algebra which admits an involution of the second kind is the split algebra $M_3(K_J)$. Thus we deduce from Proposition 11.2(i) that

$$H^1(F, T_{E,K}) \cong E^\times/F^\times N_{M/E}(M^\times)$$

where $M = E \otimes_F K_J$. On the other hand, if $K_J$ is split, then Proposition 11.2(ii) gives

$$H^1(F, T_{E,K}) \cong \ker(H^2(F, \mathbb{G}_m) \to H^2(E, \mathbb{G}_m))$$

which is $\mathbb{Z}/3\mathbb{Z}$ when $E$ is a field.

12.3. Fibers. With the various computations of $H^1(F, T_{E,K})$ given above, it is now not difficult to show the following proposition which determines $|\tilde{\Omega}_{E,K}|$ and $|\Omega_{E,K}|$.

**Proposition 12.1.** We have

| $E$          | $K$          | $T_{E,K}$ | $H^1(F, T_{E,K})$ | $|\tilde{\Omega}_{E,K}|$ | $|\Omega_{E,K}|$ |
|--------------|--------------|-----------|------------------|--------------------------|-----------------|
| $F \times K_E$ | $K = K_E$   | $K^\times$ | 1                | 1                        | 1               |
| $F \times K_E, K_E a field$ | $\text{field} \neq K_E$ | $(K \otimes K_E)^\times/K_E^\times$ | $\mathbb{Z}/2\mathbb{Z}$ | 2                  | 2               |
| $F \times K_E, K_E a field$ | $F \times F$ | $K_E^\times$ | 1                | 1                        | 1               |
| $F^3$ field | $K = K_E$   | $E^\times/F^\times$ | $\mathbb{Z}/3\mathbb{Z}$ | 2            | 2               |
| $\text{field}$ | $K \neq K_E$ | $E^\times/F^\times$ | $\mathbb{Z}/3\mathbb{Z}$ | 2            | 2               |

Here, the difference in the last two columns reflects the fact that $S_E(F)$ acts trivially on $H^1(F, T_{E,K})$ except when $E = F^3$ and $K$ is a field.
12.4. **Embeddings into** $J$. The main theorem says that the elements of $\Omega_{E,K}$ are in bijection with the conjugacy classes of embeddings

$$E \hookrightarrow J$$

where $J$ is a 9-dimensional Freudethal-Jordan algebra associated to a pair $(B,\tau)$ where $B$ is a central simple algebra over the quadratic algebra $K_J$ and $\tau$ is an involution of the second kind on $B$. We now describe the elements of $\Omega_{E,K}$ in terms of such embeddings.

- when $F$ is p-adic and $K = K_E$ so that $K_J = F \times F$ is split, then

  $$(B,\tau) = (D \times D^{op}, sw)$$

where $D$ is a central simple $F$-algebra of degree 3, and $sw$ denotes the involution which switches the two factors. Thus, there are 2 possible $J$'s in this case: the Jordan algebra $J^+$ attached to $M_3(F)$ or the Jordan algebra $J^−$ attached to a cubic division $F$-algebra (and its opposite). In either case, the set of embeddings $E \hookrightarrow J$ is either empty or a single conjugacy class, and it is empty if and only if $J = J^−$ and $E$ is not a field. Thus when $K = K_E$, we have:

$$\tilde{\Omega}_{E,K} = \Omega_{E,K} = \begin{cases} \{E \hookrightarrow J^+, E \hookrightarrow J^−\} & \text{if } E \text{ is a field;} \\ \{E \hookrightarrow J^+\} & \text{if } E \text{ is not a field.} \end{cases}$$

On the other hand, when $K_J$ is a field, then $B = M_3(K_J)$, and there is a unique isomorphism class of involution of the second kind on $B$, given by conjugation by a nondegenerate hermitian matrix, so that $J$ is isomorphic to the Jordan algebra of $3 \times 3$-Hermitian matrices with entries in $K_J$. According the the proposition, there is a unique conjugacy class of embedding $E \hookrightarrow J$ unless $E = F \times K_E$ and $K$ is a field with $K \neq K_E$. In the exceptional case, there are two subalgebras $E \subset J$ up to conjugacy. We may write down the 2 non-$F$-isomorphic twisted composition algebras corresponding to these. The twisted composition algebra can be realised on

$$E \otimes_F K = K \times (K_E \otimes K).$$

Let $\{1, \alpha\}$ denote representatives of $F^\times/NK^\times$. Then the 2 twisted composition algebras correspond to

$$(e, \nu) = ((1, 1), 1) \quad \text{or} \quad ((1, \alpha), \alpha) \in (F \times K_E)^\times \times K^\times.$$ 

We see that these two twisted composition algebras are not isomorphic because they are not isomorphic as quadratic spaces over $E$ (even allowing for twisting by $S_E(F)$).

Further, when $E = F^3$, there are in fact 4 conjugacy classes of embeddings $E \hookrightarrow J$. This corresponds to the fact that the $F$-isomorphism class of the twisted composition algebras associated to $((1, \alpha), \alpha)$ above breaks into 3 $E$-isomorphism classes. These are associated to

$$(e_1, \nu_1) = ((1, \alpha, \alpha), \alpha), \quad e_2 = ((\alpha, 1, \alpha), \alpha), \quad e_2 = ((\alpha, \alpha, 1), \alpha).$$

- when $F = \mathbb{R}$, then $E = \mathbb{R}^3$ or $\mathbb{R} \times \mathbb{C}$. When $K_J = \mathbb{R}^2$ is split, then there is a unique $J$, namely the one associated to $M_3(\mathbb{R})$, and there is a unique conjugacy class of embeddings $E \hookrightarrow J$. 

When $K_J = \mathbb{C}$, then there are two possible $J$'s, associated to $B = M_3(\mathbb{C})$ and the involution $\tau$ given by the conjugation action of two Hermitian matrices with signature $(1,2)$ and $(3,0)$. We denote these two Jordan algebras by $J_{1,2}$ and $J_{3,0}$.

When $E = \mathbb{R}^3$ and $K = \mathbb{C}$, we have $|\Omega_{E,K}| = 2$. However, the two elements in question correspond to embeddings $$\mathbb{R}^3 \hookrightarrow J_{3,0} \quad \text{and} \quad \mathbb{R}^3 \hookrightarrow J_{1,2}.$$ Thus, we see that these subalgebras are unique up to conjugacy. When $E = \mathbb{R} \times \mathbb{C}$ and $K = \mathbb{R}^2$, we have $|\Omega_{E,K}| = 1$. This reflects the fact that there is no embedding $\mathbb{R} \times \mathbb{C} \hookrightarrow J_{3,0}$, and there is a unique conjugacy class of embeddings $\mathbb{C} \hookrightarrow J_{1,2}$.

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