Robust state estimation and control for nonlinear system with uncertain parameters

Mariusz Buciakowski\textsuperscript{1}, Marcin Witczak\textsuperscript{1}, Didier Theilliol\textsuperscript{2}

\textsuperscript{1}Institute of Control and Computation Engineering, University of Zielona Góra, ul. Podgórna 50, 65-246 Zielona Góra, Poland
\textsuperscript{2}Centre de Recherche en Automatique de Nancy CRAN-UMR 7039, Nancy-Universite, CNRS, F-54506 Vandoeuvre-les-Nancy Cedex, France

E-mail: \textsuperscript{1}\{m.buciakowski, m.witczak\}@issi.uz.zgora.pl, \textsuperscript{2}didier.theilliol@univ-lorraine.fr

Abstract. The paper focuses on robust state estimation and control for nonlinear system with uncertain parameters. In particular, the problem is oriented towards a practical application for a laboratory three-tank system. The proposed approach starts with a general description of the system and assumptions regarding uncertain parameters and a nonlinear function. The subsequent part of the paper is concerned with the design of the robust observer and controller for nonlinear systems. To confirm the performance of the proposed approaches, the final part presents results obtained for the laboratory three-tank system.

1. Introduction

The problem of designing a robust observer and a state feedback control to guarantee the global stability of a nonlinear system with uncertain parameters has received considerable attention during the last three decades. There are many technique for state estimation and control for nonlinear system, for example: extended Kalman filter, adaptive technique, sliding mode, LPV technique, neural-networks, fuzzy rules base on Takagi–Sugeno etc. \cite{10, 7, 11, 3, 16, 22, 20, 13, 9, 2}. Irrespective of the large spectrum of available methods non of them is universal to be suitable for arbitrary nonlinear systems. Thus, the complexity of the nonlinear systems, in particular multi-input multioutput system forces the development of new methods. It can be perceived as a generalization of the approaches proposed for Lipschitz systems as well as its recent one-sided counterpart. The proposed approach provides a novel output feedback control strategy that is divide into two traditional steps, i.e., observer design and controller design. Firstly, a robust observer design strategy is described that can tackle both parametric uncertainty and additive external disturbances. Subsequently, the robust control strategy is portrayed that follows general ideas employed for the observer design strategy.

Note that the proposed strategy is a preliminary step to designing robust fault-tolerant control scheme\cite{22}. However, this is beyond the scope of this paper.

The paper is organised as follows. Section 2 introduces a description of the system along with assumptions regarding the nonlinearities. Whilst section 3 describes the concept of designing the robust observer. The subsequent section 4 presents the robust control framework. The final part of the paper is concerned with an illustrative example and conclusions.
2. A general description of system

Let us consider a non-linear system:

\[ \begin{align*}
    x_{k+1} &= (A + \Delta A)x_k + Bu_k + g(x_k) + W_1w_k, \\
    y_k &= Cx_k + W_2w_k
\end{align*} \]  

(1)

(2)

where matrices \( A, B \) define the nominal model and \( \Delta A \) represent admissible uncertainties which are assumed to be of the form

\[ [\Delta A] = H\mathcal{F}[E_A] \]

(3)

where \( H, E_A \) are now constant matrices of compatible dimensions, \( \mathcal{F} \) is an unknown matrix satisfies

\[ \mathcal{F}^T\mathcal{F} \preceq I, \]

(4)

where \( x_k \in \mathbb{R}^n \) is the state, \( u_k \in \mathbb{R} \) stands for the input, \( y_k \in \mathbb{R}^m \) denotes the output, \( g(x_k) \) is non-linear function with respect to \( x_k \) and \( u_k, w_k \in l_2 \) is a an exogenous disturbance vector and \( W_1 \in \mathbb{R}^{n \times n}, W_2 \in \mathbb{R}^{m \times n} \) stand for its distribution matrices while

\[ l_2 = \{ w \in \mathbb{R}^n | \| w \|_{l2} < +\infty \}, \| w \|_{l2} = \left( \sum_{k=0}^{\infty} \| w_k \|^2 \right)^{\frac{1}{2}}. \]

(5)

Moreover, the following set of assumptions are imposed:

**Assumption 1:**

\[ g(x)^T x \leq x^TMx, \quad M \in \mathcal{M}. \]

(6)

**Assumption 2:**

\[ g(x)^T g(x) \leq x^TM^TMx, \quad M \in \mathcal{M}. \]

(7)

**Assumption 3:** There exists a matrix \( M \) such that

\[ (g(a) - g(b))^T (a - b) \leq (a - b)^T M(a - b), \quad \forall a, b \in X. \]

(8)

**Assumption 4:** There exists a matrix \( M \) such that

\[ (g(a) - g(b))^T (g(a, u) - g(b, u)) \leq (a - b)^T M^TM(a - b), \quad \forall a, b \in X. \]

(9)

The paper extends the idea of one-sided Lipschitz condition [19, 25] by imposing Assumption 3. Assumption 4 extends the usual Lipschitz condition. Using the Differential Mean Value Theorem (DMVT) [24], it is possible to describe Lipschitz constant matrices by supremum of the norm of Jacobian of the function.

\[ g(a) - g(b) = M_x(a - b), \]

(10)

with

\[ M_{x,u} = \left[ \begin{array}{c}
    \left( \frac{\partial g_1}{\partial x}(c_1, u) \right) \\
    \vdots \\
    \left( \frac{\partial g_n}{\partial x}(c_n, u) \right)
\end{array} \right]^T, \]

(11)

where \( c_1, \ldots, c_n \in Co(a, b), c_i \neq a, c_i \neq b, i = 1, \ldots, n \). Assuming that

\[ \bar{a}_{i,j} \geq \frac{\partial g_i}{\partial x_j} \geq a_{i,j}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, n, \]

(12)

it is clear that:

\[ M_x = \left\{ M \in \mathbb{R}^{n \times n} | \bar{a}_{i,j} \geq m_{x,i,j} \geq \bar{a}_{i,j}, \quad i, j = 1, \ldots, n \right\}. \]

(13)

It is worthwhile that, if \( M^TM = \gamma^2 I \), then Assumption 3 becomes a usual Lipschitz condition [1, 15, 18, 17], with \( \gamma \) being a Lipschitz constant. This property makes the proposed employed strategy more general than those presented in the literature [1, 15, 18, 17].
3. Design of the state observer
The main objective of this section is to present the design procedure of the robust observer that will be able to estimate all state vector for proposed nonlinear system. For that purpose, the following structure is proposed

\[
\hat{x}_{k+1} = A\hat{x}_k + Bu_k + g(\hat{x}_k) + K_o(y_k - C\hat{x}_k),
\]

while the state estimation error is given by

\[
e_{k+1} = x_{k+1} - \hat{x}_{k+1} = \tilde{A}e_k + \tilde{G}x_k + s_k + \tilde{W}w_k,
\]

where

\[
\tilde{A} = (A - K_oC), \quad \tilde{G} = (HFE_A), \quad s_k = g(x_k) - g(\hat{x}_k)
\]

### Theorem 1
For a prescribed disturbance attenuation level \( \mu > 0 \), the observer design for the system (15) is solvable if there exist \( N, U, P > 0, \alpha > 0, \beta > 0, \epsilon > 0 \) such that the following condition is satisfied:

\[
\begin{bmatrix}
I - P + \alpha(M + MT) & 0 & 0 & 0 & 0 \\
0 & -P + \epsilon E_A^T E_A & 0 & 0 & 0 \\
-\alpha I & 0 & -\beta I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
U A - NC & 0 & U & UW_1 - NW_2 & 0 \\
UM & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-\epsilon I & 0 & 0 & 0 & 0 \\
0 & -\epsilon I & 0 & 0 & 0 \\
0 & 0 & -\epsilon I & 0 & 0 \\
0 & 0 & 0 & -\epsilon I & 0 \\
0 & 0 & 0 & 0 & -\epsilon I
\end{bmatrix}
\prec 0.
\]

### Proof
The problem of \( H_\infty \) observer design is to determine the matrix \( K_o \) such that\cite{12, 14, 23}

\[
\lim_{k \to \infty} e_k = 0 \quad \text{for} \quad w_k = 0
\]

\[
\|e_k\|_2 \leq \mu \|w_k\|_2 \quad \text{for} \quad w_k \neq 0, \quad e_0 = 0.
\]
In order to settle the above problem it is sufficient to find a Lyapunov function $V_k$ such that:

$$
\Delta V_k + e_k^T e_k - \mu^2 w_k^T w_k < 0, \ k = 0, \ldots, \infty,
$$

(20)

where

$$
\Delta V_k = V_{k+1} - V_k,
$$

(21)

$$
V_k = e_k^T P e_k + x_k^T P x_k,
$$

(22)

$$
\Delta V_k = e_{k+1}^T P e_{k+1} - e_k^T P e_k - x_k^T P x_k.
$$

(23)

Consequently, using (15)

$$
\Delta V_k + e_k^T e_k - \mu^2 w_k^T w_k =
\begin{align*}
e_k^T \left( \tilde{A}^T \tilde{P} \tilde{A} + I - P \right) e_k + e_k^T \left( \tilde{A}^T \tilde{P} \tilde{G} \right) x_k + e_k^T \left( \tilde{A}^T P \right) s_k + e_k^T \left( \tilde{A}^T \tilde{P} \tilde{W} \right) w_k + \\
x_k^T \left( \tilde{G}^T \tilde{P} \tilde{A} \right) e_k + x_k^T \left( \tilde{G}^T \tilde{P} \tilde{G} - P \right) x_k + x_k^T \left( \tilde{G}^T P \right) s_k + x_k^T \left( \tilde{G}^T \tilde{P} \tilde{W} \right) w_k + \\
s_k^T \left( P \tilde{A} \right) e_k + s_k^T \left( P \tilde{G} \right) x_k + s_k^T \left( P \right) s_k + s_k^T \left( P \tilde{W} \tilde{P} \tilde{W} - \mu^2 I \right) w_k
\end{align*}
$$

(24)

By defining

$$
\tilde{v}_k = [e_k^T, x_k^T, s_k^T, w_k^T]^T,
$$

(25)

it can be shown that (25) is equivalent to

$$
\tilde{v}_k^T \begin{bmatrix}
\tilde{A}^T \tilde{P} \tilde{A} + I - P & \tilde{A}^T \tilde{G} & \tilde{A}^T P & \tilde{A}^T \tilde{P} \tilde{W} \\
\tilde{G}^T \tilde{P} \tilde{A} & \tilde{G}^T \tilde{P} \tilde{G} - P & \tilde{G}^T P & \tilde{G}^T \tilde{P} \tilde{W} \\
\tilde{P} \tilde{A} & \tilde{P} \tilde{G} & P & \tilde{P} \tilde{W} \\
\tilde{W}^T \tilde{P} \tilde{A} & \tilde{W} \tilde{P} \tilde{G} & \tilde{W} P & \tilde{W}^T \tilde{P} \tilde{W} - \mu^2 I
\end{bmatrix} \tilde{v}_k < 0.
$$

(26)

Analyzing assumption of non-linear function, it can be shown that (8) is equivalent to

$$
s_k^T e_k \leq e_k^T M e_k = \frac{1}{2} e_k^T (M + M^T) e_k.
$$

(27)

Inequality (27) can be written as

$$
\frac{1}{2} e_k^T (M + M^T) e_k - s_k^T e_k \leq 0,
$$

which is equivalent to

$$
\frac{1}{2} e_k^T (M + M^T) e_k - \frac{1}{2} s_k^T e_k - \frac{1}{2} e_k^T s_k \geq 0.
$$

(28)

Thus, for any $\alpha > 0$

$$
\alpha \tilde{v}_k^T \begin{bmatrix}
(M + M^T) & 0 & -I & 0 \\
0 & 0 & 0 & 0 \\
-I & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \tilde{v}_k \geq 0.
$$

(29)
Similarly, for some $\beta$ inequality (9) can be written as
\[
\beta e_k^T M^T M e_k - \beta s_k^T s_k \geq 0.
\] (30)

Thus, for any $\beta > 0$
\[
\beta \bar{v}_k^T \begin{bmatrix} M^T M & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \bar{v}_k \geq 0.
\] (31)

Combining together (26), (29) and (31) gives
\[
\begin{bmatrix}
\tilde{A}^T P \tilde{A} + I - P + \alpha (M + M^T) + \beta M^T M & \tilde{A}^T \tilde{G} & \tilde{A}^T P - \alpha I & \tilde{A}^T P \tilde{W} \\
\tilde{G}^T P A & \tilde{G}^T P G - P & \tilde{G}^T P & \tilde{G}^T P \tilde{W} \\
\tilde{G}^T P A - \alpha I & \tilde{P} G & P - \beta I & \tilde{P} W \\
\tilde{W}^T \tilde{G} & \tilde{W} P \tilde{G} & \tilde{W}^T P & \tilde{W}^T P \tilde{W} - \mu^2 I
\end{bmatrix} \prec 0.
\] (32)

**Lemma 1.** The following statements are equivalent[6]:

(i) There exists $X > 0$ such that
\[
V^T X V - W \prec 0
\] (33)

(ii) There exists $X > 0$ such that
\[
\begin{bmatrix}
-W & V^T U^T \\
U V & X - U - U^T
\end{bmatrix} \prec 0.
\] (34)

Rewriting (32) as
\[
\begin{bmatrix}
\tilde{A}^T \\
\tilde{G}^T \\
\tilde{W}^T
\end{bmatrix} P \begin{bmatrix}
\tilde{A} & \tilde{G} & I & \tilde{W}
\end{bmatrix} + \begin{bmatrix}
I - P + \alpha (M + M^T) + \beta M^T M & 0 & -\alpha I & 0 \\
0 & -P & 0 & 0 \\
-\alpha I & 0 & -\beta I & 0 \\
0 & 0 & 0 & -\mu^2 I
\end{bmatrix} \prec 0.
\] (35)

and using **Lemma 1** to (35) yields
\[
\begin{bmatrix}
I - P + \alpha (M + M^T) & 0 & -\alpha I & 0 & \tilde{A}^T U^T & M^T U^T \\
0 & -P & 0 & 0 & \tilde{G}^T U^T & 0 \\
-\alpha I & 0 & -\beta I & 0 & U^T & 0 \\
0 & 0 & 0 & -\mu^2 I & \tilde{W}^T U^T & 0 \\
U \tilde{A} & U \tilde{G} & U & U \tilde{W} & P - U - U^T & 0 \\
U M & 0 & 0 & 0 & \beta I - U - U^T
\end{bmatrix} \prec 0.
\] (36)
with \( \bar{A} = UA - UK, C = UA - NC, \bar{W} = UW_1 - UK, W_2 = UW_1 - NW_2 \) and \( \bar{G} = UHFE_A \). Inequality (36) can be written in equivalent form

\[
\begin{bmatrix}
    I - P + \alpha (M + M^T) & 0 & -\alpha I & 0 & A^T U^T - C^T N^T & M^T U^T \\
    0 & -P + \epsilon E_A^T E_A^T & 0 & 0 & 0 & 0 \\
    -\alpha I & 0 & -\beta I & 0 & U^T & 0 \\
    0 & 0 & 0 & -\mu^2 I & \bar{W}^T U^T - \bar{W}_2^T N^T & 0 \\
    U A - N C & 0 & U & U W_1 - N W_2 & P - U - U^T & 0 \\
    U M & 0 & 0 & 0 & 0 & \beta I - U - U^T \\
\end{bmatrix} +
\]

\[
\begin{bmatrix}
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \prec 0,
\]

\[
(37)
\]

Lemma 2. Let \( H, E \) be given real matrices of appropriate dimensions and \( F \) satisfy \( F^T F \preceq I \). Then for any \( \epsilon > 0 \) the following holds

\[
H F E + E^T F^T H \preceq \epsilon H H^T + \frac{1}{\epsilon} E^T E.
\]

(38)

Applying Lemma 2 to (37), we obtain

\[
\begin{bmatrix}
    I - P + \alpha (M + M^T) & 0 & -\alpha I & 0 & A^T U^T - C^T N^T & M^T U^T \\
    0 & -P + \epsilon E_A^T E_A^T & 0 & 0 & 0 & 0 \\
    -\alpha I & 0 & -\beta I & 0 & U^T & 0 \\
    0 & 0 & 0 & -\mu^2 I & \bar{W}^T U^T - \bar{W}_2^T N^T & 0 \\
    U A - N C & 0 & U & U W_1 - N W_2 & P - U - U^T & 0 \\
    U M & 0 & 0 & 0 & 0 & \beta I - U - U^T \\
\end{bmatrix} +
\]

\[
\begin{bmatrix}
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \prec 0.
\]

(39)

Applying the Schur complements leads to (17), which completes the proof.

4. Design of the state feedback controller

The main objective of this section is to present the design procedure of the robust controller for the proposed system. The controller will be designed in such a way that a predefined disturbance
attenuation level with respect to the state of the system is achieved. To solve the above problem the following control scheme is proposed

\[ u_k = -K c x_k. \]  

(40)

Substituting (40) into (1) gives

\[ x_{k+1} = \bar{A} x_k + g(x_k) + \bar{W} w_k \]  

(41)

where

\[ \bar{A} = (A - BK_c + H F E A), \quad \bar{W} = W_1. \]  

(42)

**Theorem 2.** For a prescribed disturbance attenuation level \( \mu > 0 \), the controller design for the system (41) is solvable if there exist \( N, U, P > 0, \alpha > 0, \beta > 0, \epsilon > 0 \) such that the following condition is satisfied:

\[
\begin{bmatrix}
I - P + \alpha(M + M^T) + H H^T & -\alpha I & 0 & A U - B N & M U \\
-\alpha I & -\beta I & 0 & U & 0 \\
0 & 0 & -\mu^2 I & W_1 U & 0 \\
U^T A^T - N^T B^T & U^T & U^T W_1^T & P - U - U^T & 0 \\
U^T M^T & 0 & 0 & E_A U & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
U^T E_A^T & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-\epsilon I & 0 & 0 & 0 & 0 \\
0 & -\epsilon I & 0 & 0 & 0 \\
0 & 0 & -\epsilon I & 0 & 0 \\
0 & 0 & 0 & -\epsilon I & 0 \\
0 & 0 & 0 & 0 & -\epsilon I \\
\end{bmatrix} < 0. \]  

(43)

**Proof.** The problem of \( H_\infty \) controller design is to determine the matrix \( K_c \) such that\[21, 4\]

\[
\lim_{k \to \infty} x_k = 0 \quad \text{for} \quad w_k = 0 
\]

(44)

\[
\|x_k\|_2 \leq \mu \|v_k\|_2 \quad \text{for} \quad w_k \neq 0, \quad x_0 = 0. \]  

(45)

In order to settle the above problem it is sufficient to find a Lyapunov function \( V_k \) such that:

\[
\Delta V_k + x_k^T x_k - \mu^2 w_k^T w_k < 0, \quad k = 0, \ldots, \infty, \]  

(46)

where

\[
\Delta V_k = V_{k+1} - V_k, \]  

(47)

\[
V_k = x_k^T P x_k, \]  

(48)

\[
\Delta V_k = x_{k+1}^T P x_{k+1} - x_k^T P x_k, \]  

(49)
Consequently, using (41)
\[ \Delta V_k + x_k^T x_k - \mu^2 w_k^T w_k = x_k^T (A^T PA - P + I) x_k + x_k^T (A^T P) g(x_k) + x_k^T (A^T PW) w_k + g(x_k)^T (PA) x_k + g(x_k)^T (P) g(x_k) + g(x_k)^T (PW) w_k + w_k^T (W^T PA) x_k + w_k^T (W^T P) g(x_k) + w_k^T (W^T PW - \mu^2 I) w_k < 0. \]

By defining
\[ v_k = [x_k^T, g(x_k)^T, w_k^T]^T, \] (50)

it can be shown that (50) is equivalent to
\[ v_k^T \begin{bmatrix} A^T PA + I - P & \tilde{A}^T P & A^T PW \\ PA & P & PW \\ W^T PA & W^T P & W^T PW - \mu^2 I \end{bmatrix} v_k \preceq 0. \] (51)

Analyzing assumption of non-linear function, it can be show that (6) is equivalent to
\[ g(x)^T x \leq x^T M x = \frac{1}{2} x^T (M + M^T) x, \quad M \in \mathbb{M}. \] (52)

Inequality (52) can be written as
\[ \frac{1}{2} x_k^T (M + M^T) x_k - g(x_k)^T x_k \geq 0, \quad M \in \mathbb{M}, \] (53)

which is equivalent to
\[ \frac{1}{2} x_k^T (M + M^T) x_k - \frac{1}{2} g(x_k)^T x_k - \frac{1}{2} x_k^T g(x_k) \geq 0. \] (54)

Thus, for any \( \alpha > 0 \)
\[ \alpha v_k^T \begin{bmatrix} (M + M^T) & -I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} v_k \succeq 0, \quad \alpha > 0, \quad M \in \mathbb{M}. \] (55)

Similarly, for some \( \beta \) inequality (7) can be written as
\[ \beta x_k^T M^T M x_k - \beta g(x_k)^T g(x_k) \geq 0, \quad \beta > 0. \] (56)

\[ \beta v_k^T \begin{bmatrix} M^T M & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & 0 \end{bmatrix} v_k \succeq 0. \] (57)

Combining together (51), (55) and (57) gives
\[ \begin{bmatrix} A^T PA + I - P + \alpha (M + M^T) & \tilde{A}^T P - \alpha I & A^T PW \\ PA - \alpha I & P - \beta I & PW \\ W^T PA & W^T P & W^T PW - \mu^2 I \end{bmatrix} \prec 0 \] (58)
Rewriting (58) as
\[
\begin{bmatrix}
    \bar{A}^T \\
    I^T \\
    W^T
\end{bmatrix}
P
\begin{bmatrix}
    \bar{A} & I & W
\end{bmatrix}
\begin{bmatrix}
    I - P + \alpha(M + M^T) + \beta M^T M - \alpha I & 0 & 0 \\
    -\alpha I & -\beta I & 0 \\
    0 & 0 & -\mu^2 I
\end{bmatrix} < 0,
\] (59)
and using a transposed version Lemma 1 to (59) yields
\[
\begin{bmatrix}
    I - P + \alpha(M + M^T) - \alpha I & 0 & \bar{A}U & MU \\
    -\alpha I & -\beta I & 0 & U \\
    0 & 0 & -\mu^2 I & \bar{W}U \\
    U^T \bar{A}^T & U^T & U^T \bar{W}^T & P - U - U^T \\
    0 & 0 & 0 & \beta I - U - U^T
\end{bmatrix} < 0,
\] (60)
with \( \bar{A}U = AU - BK_U + HFE_A U = AU - BN + HFE_A U \) and \( \bar{W}U = W_1 U \). Written (60) similarly as inequality (36) and then applying Lemma 2 and Schur complements leads to (43), which completes the proof.

5. Case study
To verify the proposed approach, it was implemented for real three-tank system portrayed in Fig. 1 [5, 14, 8]. The system is composed of three separate tanks with variable cross-area sections, where nonlinearities are imposed from shape of the second and third tank. The model was identified and following system matrices were obtained as well as nonlinear function of the system:
\[
A = \begin{bmatrix}
    0.9982 & 0 & 0 \\
    0.0018 & 0.9973 & 0 \\
    0 & 0.0025 & 0.9972
\end{bmatrix}, \quad
B = \begin{bmatrix}
    11.1400 \\
    0 \\
    0
\end{bmatrix}, \quad
W_1 = \begin{bmatrix}
    0.0100 & 0 & 0 \\
    0 & 0.0100 & 0 \\
    0 & 0 & 0.0100
\end{bmatrix},
\]
\[
H = \begin{bmatrix}
    0.1 \\
    0.1 \\
    0
\end{bmatrix}, \quad
E = \begin{bmatrix}
    0.2 \\
    0 \\
    0
\end{bmatrix}, \quad
W_1 = 0.05C, \quad
g(x_k) = \begin{bmatrix}
    \frac{1}{\beta_1(x_1)} C_1 x_{1,k}^{\alpha_1} \\
    \frac{1}{\beta_2(x_2,k)} C_1 x_{1,k}^{\alpha_1} \\
    \frac{1}{\beta_3(x_3,k)} C_2 x_{2,k}^{\alpha_2}
\end{bmatrix} \frac{1}{\beta_1(x_1)} C_1 x_{1,k}^{\alpha_1} \frac{1}{\beta_2(x_2,k)} C_2 x_{2,k}^{\alpha_2} \frac{1}{\beta_3(x_3,k)} C_3 x_{3,k}^{\alpha_3}.
\]
where: \( \beta_i(x_i) \) is cross-sectional area of \( i \)-th tank, \( \alpha_i \) denotes the flow coefficient parameter and \( C_i \) stands for \( i \)-th valve open level, the last two parameters were obtain through identification.

\[ \text{Figure 1. Three tank system} \]
process. The numerical parameters obtained from identification along with operating ranges are: $0 \leq u \leq 1.5e^{-4} \frac{m^3}{s}$, $0.01 \leq x_{1-3} \leq 0.35[m]$ with $C_1 = 1.6721e^{-4}$, $C_2 = 1.8804e^{-4}$, $C_3 = 1.6637e^{-4}$, $\alpha_1 = 0.4334$, $\alpha_2 = 0.4134$, $\alpha_2 = 0.3673$.

By varying each component of $x_k$ within its possible domain, defined by the physical constraints of the three-tank system, it is possible to obtain the matrices

$$M_{\text{min}} = \begin{bmatrix}
-1.0094 & 0 & 0 \\
-0.0010 & -1.0457 & 0 \\
0 & -0.0014 & -1.0593
\end{bmatrix}, \quad M_{\text{max}} = \begin{bmatrix}
-0.9997 & 0 & 0 \\
0.0238 & -0.9968 & 0 \\
0 & 0.0224 & -0.9979
\end{bmatrix}$$

which contain all elements $a_{i,j}$ and $\bar{a}_{i,j}$, $i,j = 1,\ldots,n$ defining (13) respectively. Afterwards, taking into account all the possible combinations of elements $M_{\text{min}}$ and $M_{\text{max}}$, 32 matrices $M_i$ are obtained.

For the purpose of further comparative study, two cases were employed:

**Case SC1:** It will be assumed that the whole state vector is directly measured

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad K_o = \begin{bmatrix} 0.7382 & -0.0008 & -0.0000 \\ 0.0012 & 0.7402 & 0.0005 \\ -0.0000 & 0.0030 & 0.7405 \end{bmatrix}, \quad \mu = 0.5623, \quad x_0 = 0, \quad z_0 = 0$$

**Case SC2:** It will be assumed that some state variables are not directly measured

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad K_o = \begin{bmatrix} 0.6977 & -0.0000 \\ 0.0002 & 0.0026 \\ -0.0000 & 0.7056 \end{bmatrix}, \quad \mu = 0.5623, \quad x_0 = 0, \quad z_0 = 0$$

Figure 2–3 present the results obtained for the proposed observer structure for case SC1. From these results, it is evident that the state estimation is performed with a good quality. Figure 3 also shows the state estimation for second state which is not directly measured. This corresponds to case SC2. From obtained results, it is evident that the state estimation is performed with a good quality. Of course there are differences between the quality of the estimation for the case SC1 and SC2 for state $x_2$. Finally, Fig. 4 shows the system performance for the proposed control scheme. The goal is to control the liquid level in an upper tank. The reference signal (red solid line) is the target that has to be achieved by the controller. The robust controller was designed with (43) and $\mu = 0.5623$. The obtained robust controller is $K_c = [0.05731, 0.00076, 0.000001]$. The state vector is derived by the observer for SC2. From these results it can be seen that the state converges to the required setpoint.

6. Conclusions

The main objective of this paper was to propose problem of robust state estimation and robust control for nonlinear systems. The proposed design procedure for robust observer and robust controller is relatively simple and boils down to solving a set of linear matrix inequalities. The proposed approach can be efficiently implemented to the real-time system. The final part presents comprehensive case study of the three-tank system. The results clearly exhibit the high performance of the proposed scheme.

Acknowledgments

The work was supported by the National Science Centre of Poland under grant: 2013/11/B/ST7/01110
Figure 2. SC1: State variable $x_1$ (left) $x_2$ (right) (black line) and its estimates (red line) for $t = 0, \ldots, 700 [s]$

Figure 3. SC1(left): State variable $x_3$ (black line) and its estimates (red line) for $t = 0, \ldots, 700 [s]$. SC2(right): State variable $x_2$ (black line) and its estimates (red line) for $t = 0, \ldots, 700 [s]$

Figure 4. Performance of the system for $t = 0, \ldots, 100 [s]$

[1] M. Abbaszadeh and H.J. Marquez. Lmi optimization approach to robust $h_{\infty}$ observer design and static output feedback stabilization for non-linear uncertain systems. *International Journal of Robust and Nonlinear Control*, 19(3):313–340, 2008.

[2] Stanislaw Banika, Pawel Dworak, and Krzysztof Jaroszewski. Design of a multivariable neural controller for control of a nonlinear mimo plant. *International Journal of Applied Mathematics and Computer Science*, 24(2):357–369, 2014.
[3] Ali Ben Brahim, Slim Dhahri, Fayçal Ben Hmida, and Anis Sellami. An h inf sliding mode observer for takagi-sugeno nonlinear systems with simultaneous actuator and sensor faults. *International Journal of Applied Mathematics and Computer Science*, 25(3):547–559, 2015.

[4] M. Buciakowski, M. de Rozprza Faygel, J. Ochalek, and M. Witzczak. Actuator fault diagnosis and fault-tolerant control: application to the quadruple-tank process. In *11th European Workshop on Advanced Control and Diagnosis - ACD 2014*, pages 93–102, Berlin, Niemcy, 2014. [Berlin], [brak wydawcy].

[5] M. Buciakowski, M. Witzczak, and M. Luzar. Robust fault-tolerant control for a multi-tank system. In *11th International Conference on Diagnostics of Processes and Systems - DPS 2013*, pages 1–12, Lagw Lubuski, Polska, 2013. [B. m.], –.

[6] M.C. de Oliveira, J. Bernussou, and J.C. Geromel. A new discrete-time robust stability condition. *Systems and Control Letters*, 37(4):261–265, 1999.

[7] S. Gillijns and B. De Moor. Unbiased minimum-variance input and state estimation for linear discrete-time systems. *Automatica*, 43:111–116, 2007.

[8] INTECO. *Multitank System - User’s manual*. INTECO, www.inteco.com.pl, 2013.

[9] Cédric Join, J-C Ponsart, Dominique Sauter, and Didier Theilliol. Nonlinear filter design for fault diagnosis: application to the three-tank system. *IEEE Proceedings-Control Theory and Applications*, 152(1):55–64, 2005.

[10] J. Y. Keller and M. Darouach. Two-stage Kalman estimator with unknown exogenous inputs. *Automatica*, 35(2):339–342, 1999.

[11] K. Kemir, F. Ben Hmida, J. Ragot, and M. Gossa. Novel optimal recursive filter for state and fault estimation of linear systems with unknown disturbances. *International Journal of Applied Mathematics and Computer Science*, 21(4):629–638, 2011.

[12] H. Li and M. Fu. A linear matrix inequality approach to robust h∞ filtering. *IEEE Trans. Signal Processing*, 45(9):2338–2350, 1997.

[13] Francisco-Ronay López-Estrada, Jean-Christophe Ponsart, Carlos-Manuel Astorga-Zaragoza, Jorge-Luis Canas-Anzueto, and Didier Theilliol. Robust sensor fault estimation for descriptor-lpv systems with unmeasurable gain scheduling functions: Application to an anaerobic bioreactor. *International Journal of Applied Mathematics and Computer Science*, 25(2):233–244, 2015.

[14] M. Luzar, M. Witzczak, M. Buciakowski, and V. Puig. Robust h infinity actuator fault diagnosis and fault-tolerant control for a multi-tank system. In *Conference on Control and Fault-Tolerant Systems - SysTol 2013*, pages 758–763, Nice, Francja, 2013. [B. m.].

[15] H.J. Marquez. *Nonlinear control systems. Analysis and Design*. Wiley, New Jersey, 2003.

[16] E.G. Nobrega, M.O. Abdalla, and K.M. Grigoriadis. Robust fault estimation of uncertain systems using an lmi-based approach. *International Journal of Robust and Nonlinear Control*, 18(7):1657–1680, 2008.

[17] A. M. Pertew, H. J. Marquez, and Q. Zhao. H∞ synthesis of unknown input observers for non-linear lipschitz systems. *International Journal of Control*, 78(15):1155–1165, 2005.

[18] R. Rajamani. Observers for Lipschitz non-linear systems. *IEEE Transactions on Automatic Control*, 43(3):397–401, 1998.

[19] D.M. Stipanovic and D.D. Siljak. Robust stability and stabilization of discrete-time non-linear: the lmi approach. *International Journal of Control*, 74(5):1115–1165, 2005.

[20] Zhenhua Wang, Mickael Rodrigues, Didier Theilliol, and Yi Shen. Actuator fault estimation observer design for discrete-time linear parameter-varying descriptor systems. *International Journal of Adaptive Control and Signal Processing*, 29(2):242–258, 2015.

[21] M. Witzczak, M. Buciakowski, and Ch. Aubrun. Predictive actuator fault-tolerant control under ellipsoidal bounding. *International Journal of Adaptive Control and Signal Processing*, 2015.

[22] M. Witzczak, J. Korbič, and R. Jozefowicz. Design of unknown input observers for non-linear stochastic systems and their application to robust fault diagnosis. *Control and Cybernetics*, 42(1):in print, 2013.

[23] Marcin Witzczak, Mariusz Buciakowski, and Marcin Mrugalski. An h∞ approach to fault estimation of nonlinear systems: Application to one-link manipulator. In *Methods and Models in Automation and Robotics (MMAR), 2014 19th International Conference On*, pages 456–461. IEEE, 2014.

[24] A. Zemouche and M. Boutayeb. Observer design for Lipschitz non-linear systems: the discrete time case. *IEEE Trans. Circuits and Systems - II:Express Briefs*, 53(8):777–781, 2006.

[25] A. Zemouche, M. Boutayeb, and G. Jiliana Bara. Observer for a class of Lipschitz systems with extension to H∞ performance analysis. *Systems and Control Letters*, 57(1):18–27, 2008.