ONE LIE GROUP TO DEFINE THEM ALL

ANNALISA CONVERSANO AND MARCELLO MAMINO

Abstract. We produce a connected real Lie group that, as a first order structure in the group language, interprets the real field expanded with a predicate for the integers. Moreover, the domain of our interpretation is definable in the group.

1. Introduction

The purpose of this note is to exhibit a connected Lie group which is, from the point of view of model theory, as badly behaved as it possibly can. In particular, we produce a Lie group – which happens to be connected, nilpotent and non-compact – that, seen as a first order structure in the group language, interprets the real field expanded with a predicate for the integers \((\mathbb{R}, +, \times, \mathbb{Z})\). This, in turn, can be considered a wild structure from the point of view of model theory, and, in particular, every Lie group is interpretable in it.

Indeed, we claim a slightly stronger statement, namely that there is a connected Lie group that defines the real field with a predicate for the integers, and thus every Lie group. To clarify this second claim, we must specify what is intended for a structure \(A\) to define a structure \(B\). A structure is a collection of sets: the domain, the relations, the functions in its signature. Hence, it makes sense to say that \(B\) is definable in \(A\) whenever the domain of \(B\) is a definable set in \(A\), and all relations and functions in the signature of \(B\) are definable sets in \(A\). More generally, we say that \(B\) is definable in \(A\) when there is a structure \(B'\) isomorphic to \(B\) which is definable in \(A\) in the sense above. In other words, \(A\) defines \(B\) precisely when \(A\) interprets \(B\) in such a way that the domain of the interpretation is a definable set in \(A\). Thus, for instance, we can say that the group \(SO_2(\mathbb{R})\) is definable in \((\mathbb{R}, +, \cdot)\), meaning that there is a set \(G\) definable in \((\mathbb{R}, +, \cdot)\) and a function \(\cdot : G \times G \to G\) such that \((G, \cdot)\) is isomorphic, as a first order structure (or, equivalently, as a group) to \(SO_2(\mathbb{R})\).

We mention here that definable or interpretable, in this work, always means definable or interpretable with parameters.

The immediate motivation for this note has been a question by Antongiulio Fornasiero, about the possibility to interpret every connected Lie group in a d-minimal structure (for results in this context see [5]), which is ruled out by our example. More in general, there is a growing number of classes of Lie groups known to enjoy nice model-theoretical properties, such as: Nash groups [16], algebraic groups [14], compact [12, 9] or semisimple [13] Lie groups, and covers of all the above [2, 10]. At the same time, model theoretic methods are being used to attack classical problems [3]. Our result provides a negative example, suggesting that there are non-compact Lie groups that, despite being connected and nilpotent, are intractable for model theory. This contrasts with the compact case, in fact every
compact Lie group is isomorphic to a group definable in the real field, and, in turn, the real field can be recovered from any semisimple Lie group \[13\]. To study the class of Lie groups up to first order definability in more detail is a complicated and possibly interesting task; it is, however, beyond the scope of this note: all we intend to say is that this class has a maximal element.

2. Construction of the group

Recall that a real Lie group – for short Lie group – is a real analytic manifold equipped with a real analytic group operation. Real manifolds are assumed to be second countable and Hausdorff.

**Fact 2.1.** Identify the field \(\mathbb{R}\) with the subset \([0] \times \mathbb{R}\) of \(\mathbb{R}^2\). Then the field operations of \(\mathbb{R}\) can be defined using only the incidence graph of the straight lines in \(\mathbb{R}^2\). More precisely, the field operations are first order definable in the structure \((\mathbb{R}^2, \text{coll})\) where \(\text{coll}(p, q, r)\) denotes the ternary collinearity relation.

**Proof.** First observe that, working in \((\mathbb{R}^2, \text{coll})\), we can identify a straight line by a pair of different points, and we can tell whether the straight lines identified by two pairs of points coincide, meet, or are parallel, by means of first order formulas. We can thus implement constructions based on the notions of incidence and parallelism.

It is an ancient observation that the arithmetic of segments can be defined geometrically: geometrical constructions to this effect can be found, for instance, in the works of Descartes [5] and Hilbert [8 § 15]. We need constructions, however, that rely solely on incidence and parallelism. These are classically called von Staudt constructions, with reference to [17] § 19, despite the fact that, formally, this work is set in the context of projective geometry. The affine versions of the von Staudt constructions are depicted in the figure below: the reader will work out the details without any difficulty.

Let \((\mathbb{R}, +, \cdot, \mathbb{Z})\) denote the first order structure whose domain is the set of real numbers, with the field operations and a predicate for the subset of the integers. It is well known that the definable sets in \((\mathbb{R}, +, \cdot, \mathbb{Z})\) coincide with the projective sets (see, for instance, [11] exercise 37.6). Therefore we get immediately the following fact.

**Fact 2.2.** All Lie groups are definable in \((\mathbb{R}, +, \cdot, \mathbb{Z})\).

**Proof.** Given a Lie group \(G\) of dimension \(n\), by Whitney’s embedding theorem, there is an embedding of \(G\) in \(\mathbb{R}^{2n+1}\) as a closed subset \(C\). The group operation of \(G\) induces an operation \(\cdot\) : \(C \times C \to C\), which is continuous, hence a closed subset of \(\mathbb{R}^{3(2n+1)}\). Therefore \(C\) and \(\cdot\) are projective sets, hence definable in \((\mathbb{R}, +, \cdot, \mathbb{Z})\). 

\(\square\)
We show now that one needs the full power of \((\mathbb{R}, +, \cdot, \mathbb{Z})\) to be able to define all connected Lie groups. Namely, there is a connected Lie group \(G\) such that \((\mathbb{R}, +, \cdot, \mathbb{Z})\) itself is definable in the group structure \((G, \cdot)\).

**Theorem 2.3.** There is a connected nilpotent Lie group that interprets \((\mathbb{R}, +, \cdot, \mathbb{Z})\). Moreover, there is a connected Lie group that defines \((\mathbb{R}, +, \cdot, \mathbb{Z})\).

**Proof.** First, we will construct the connected nilpotent Lie group \(G\) in which \((\mathbb{R}, +, \cdot, \mathbb{Z})\) is interpretable. After that we will show how this group can be modified to obtain a group \(G'\), so that the domain \(\mathbb{R}\) is a definable set in \(G'\).

Consider the following nilpotent group

\[
G = H_3(\mathbb{R})/\Gamma
\]

Where \(H_3(\mathbb{R})\) is the Heisenberg group

\[
H_3(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\} < \text{GL}_3(\mathbb{R})
\]

and

\[
\Gamma = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{Z} \right\} \subset H_3(\mathbb{R})
\]

is a discrete subgroup of the center of \(H_3\)

\[
Z(H_3) = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : c \in \mathbb{R} \right\}
\]

(we choose a particular \(\Gamma\); however, up to isomorphism, \(G\) does not depend on this choice). For ease of notation, write \([a, b, c]\) to denote the class of the element

\[
\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}
\]

of \(H_3(\mathbb{R})\) in the quotient. One can check directly that \([a, b, c] = [a', b', c']\) if and only if \(a = a', b = b',\) and \(c - c' \in \mathbb{Z}\). It follows that the centralizer of \([a, b, c]\) in \(G\) is

\[
C([a, b, c]) = \left\{ [a', b', c'] : ab' - ba' \in \mathbb{Z}\right\}.
\]

For any \(a, b \in \mathbb{R}\) we can define the subgroup

\[
L_{a,b} = C([a, b, 0]) \cap C([\sqrt{2}a, \sqrt{2}b, 0])
\]

where \(\sqrt{2}\) can be replaced by any irrational number, as this is enough to ensure that

\[
L_{a,b} = \left\{ [a', b', c'] : ab' - ba' = 0\right\}.
\]

In particular, the following are definable subgroups of \(G\):

\[
A \overset{\text{def}}{=} L_{0,1} = \left\{ [0, b, c] : b, c \in \mathbb{R}\right\}
\]

\[
B \overset{\text{def}}{=} L_{0,1} \cap C([1, 0, 0]) = \left\{ [0, b, c] : b \in \mathbb{Z}, c \in \mathbb{R}\right\}
\]

Hence the following groups are interpretable in \(G\):

\[
E \overset{\text{def}}{=} G/Z(G) \quad R \overset{\text{def}}{=} A/Z(G) \quad Z \overset{\text{def}}{=} B/Z(G)
\]

Clearly \(E > R > Z\). Observe that two elements \([a, b, c]\) and \([a', b', c']\) of \(G\) are equivalent modulo \(Z(G)\) if and only if \(a = a'\) and \(b = b'\). It follows that \(E, R,\)
and $Z$ are isomorphic respectively to $\mathbb{R}^2$, $\{0\} \times \mathbb{R}$, and $\{0\} \times Z$ through the map $t: [a, b, c] \mapsto (a, b) \in \mathbb{R}^2$. These are the ingredients of our interpretation.

The group $E$ is the domain of the interpretation of $(\mathbb{R}, +, \cdot, Z)$, and $Z$ is the interpretation of $\mathbb{Z}$. To define the field operations, we will make use of Fact 2.1. Let $\mathcal{L}$ denote the set $\{L_{a,b}: (a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}\}$ of subgroups of $G$. Clearly the family $\{\ell(L)\}_{L \in \mathcal{L}}$ spans all the straight lines through $(0, 0)$, so, if the set $\mathcal{L}$ happens to be the image of a uniform family of definable subgroups, then we can define the collinearity relation over $E \sim \mathbb{R}^2$ as follows

$$\text{coll}(p, q, r) \overset{\text{def}}{=} \exists L \in \mathcal{L} \ p^{-1}q \in L \land p^{-1}r \in L$$

then we conclude by Fact 2.1. To write $\mathcal{L}$ as a uniform family, recall that the subgroups $L_{a,b}$ are intersections of pairs of centralizers, therefore, it suffices to find a definable predicate that tells whether, given $g_1, g_2 \in G \setminus Z(G)$, there are $a, b$ such that $C(g_1) \cap C(g_2) = L_{a,b}$. Indeed, this happens if and only if the group $C(g_1) \cap C(g_2)$ is divisible by $Z$, i.e., for all $x \in C(g_1) \cap C(g_2)$ there is $y \in C(g_1) \cap C(g_2)$ such that $x = y \cdot y$.

Now, to get a group in which $(\mathbb{R}, +, \cdot, Z)$ is definable, as opposed to interpretable, we replace $H_3 (\mathbb{R})$ with the group

$$H' = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & x & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c, x \in \mathbb{R} \land x \in \mathbb{R}^+ \right\} < \text{GL}_3 (\mathbb{R})$$

Again $\Gamma$ is central in $H'$, and we can construct $G'$ as

$$G' = H'/\Gamma$$

Now, using the notation $[a, b, c, x]$ for elements of $G'$, it is easy to recover the group $G$ as the product of the centralizers of $[0, 1, 0, 1]$ and $[1, 0, 0, 1]$. In fact

$$C([0, 1, 0, 1]) = \{[0, b, c, 1] : b, c \in \mathbb{R}\}$$
$$C([1, 0, 0, 1]) = \{[a, 0, c, 1] : a, c \in \mathbb{R}\}$$
$$[0, b, c_1, 1] \cdot [a, 0, c_2, 1] = [a, b, c_1 + c_2, 1]$$

Therefore we can carry out the construction of $E$, $R$, and $Z$ as before and get an interpretation of $(\mathbb{R}, +, \cdot, Z)$. To turn this into an actual definition, we only need a choice of representatives for the elements of $R = A/Z(G)$. To this aim, let $O$ be the orbit of $[0, 1, 0, 1]$ under conjugation by elements of $C([0, 0, 0, 2])$. An easy computation shows that

$$C([0, 0, 0, 2]) = \{[0, 0, c, x] : c \in \mathbb{R} \land x \in \mathbb{R}^+ \}$$
$$O = \{[0, b, 0, 1] : b \in \mathbb{R}^+ \}$$

Hence $O \cup O^{-1} \cup \{[0, 0, 0, 1]\}$ intersects each equivalence class of $A$ modulo $Z(G)$ in a single point. ☐

3. Additional remarks

The group $G$ of Theorem 2.3 has minimal dimension, since all connected Lie groups of dimension up to 2 are definable in the real field: up to Lie isomorphism, connected 1-dimensional groups are $SO_2(\mathbb{R})$ and $\mathbb{R}$, 2-dimensional groups are $\mathbb{R} \times \mathbb{R}$, $\mathbb{R}^+ \times \mathbb{R}$, $SO_2(\mathbb{R}) \times \mathbb{R}$, and $SO_2(\mathbb{R}) \times SO_2(\mathbb{R})$ (see for instance [13] p.36)). We don’t know whether $G'$ could be replaced by a group of dimension 3.

The group $G$, defined as the quotient of a connected real algebraic group by a discrete subgroup, could also be obtained as the quotient of a definably
connected semialgebraic group by a definably connected \( \lor \)-definable subgroup (see [4] Example 5.9).

The group \( G \) is not linear (for instance, by a Theorem of Gotô [7] Theorem 5] the derived subgroup of a connected solvable linear Lie group needs to intersect trivially any maximal compact subgroup, but the derived subgroup of \( G \) coincides with a maximal compact subgroup). In a private communication, Ya’acov Peterzil observed that, by a modification of [15] example on p. 5], there is a linear group interpreting \( (C,+,\cdot,Z) \)—in fact, it suffices to repeat the same construction of the example with \( \mathbb{R} \) replaced by \( C \). This raises the question of whether there is a linear group interpreting \( (\mathbb{R},+,\cdot,Z) \).

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