Abstract

We continue the development of a so called contextual statistical model (here context has the meaning of a complex of physical conditions). It is shown that, besides contexts producing the conventional trigonometric cos-interference, there exist contexts producing the hyperbolic cos-interference. Starting with the corresponding interference formula of total probability we represent such contexts by hyperbolic probabilistic amplitudes or in the abstract formalism by normalized vectors of a hyperbolic analogue of the Hilbert space. There is obtained a hyperbolic Born’s rule. Incompatible observables are represented by noncommutative operators. This paper can be considered as the first step towards hyperbolic quantum probability. We also discuss possibilities of experimental verification of hyperbolic quantum mechanics: in physics of elementary particles, string theory as well as in experiments with nonphysical systems, e.g. in psychology, cognitive sciences and economy.

1 Introduction

In Ref. 1 there was presented a so called contextual viewpoint of the origin of quantum (conditional) probabilities (here a context has the mean-
ing of a complex of physical conditions). Such an approach gives the possibility to unify classical Kolmogorov (measure theoretical) and quantum (Hilbert space) probability theories by constructing a natural representation of the Kolmogorov model in a complex Hilbert space. Thus in the contextual approach quantum probabilistic behavior (in particular, *interference of probabilities*) is simply a consequence of a very special representation of Kolmogorov probabilities – by complex amplitudes (vectors in a complex Hilbert space). Each representation is based on a fixed pair of observables (Kolmogorov random variables) $a$ and $b$ – *reference observables* – which produce the contextual image of a Kolmogorov probability space in a complex Hilbert space. The crucial point is that all Kolmogorov probabilities should be considered as conditional (or better to say contextual) probabilities, cf. L. Accardi$^{2-4}$, L. Ballentine$^{5,6}$, W. De Muynck$^{7,8}$, S. Gudder$^{9,10}$, A. Lande$^{11}$, G. Mackey$^{12}$.

In Ref. 1 we introduced a class $\mathcal{C}_{\text{tr}}$ of contexts (“trigonometric contexts”) which can be represented by complex probabilistic amplitudes inducing the representation in the complex Hilbert space. The $\mathcal{C}_{\text{tr}}$ consists of context producing the conventional trigonometric cos-interference. However, in general the set of contexts is not reduced to the class of trigonometric contexts $\mathcal{C}_{\text{tr}}$. There exist contexts producing the hyperbolic cosh-interference. The set of hyperbolic contexts is denoted by the symbol $\mathcal{C}_{\text{hyp}}$.

In this paper we show that it is possible to represent contexts belonging to $\mathcal{C}_{\text{hyp}}$ by so called *hyperbolic amplitudes*. Such amplitudes take values in the set of “hyperbolic numbers” (two dimensional Clifford algebra). It will be demonstrated that in the hyperbolic framework we can proceed quite far in the same directions as in the trigonometric framework. We obtain hyperbolic analogues of the interference of probabilities, probability amplitudes, Born’s rule, representation of incompatible observables by noncommuting operators... The crucial difference between two representations is that in the hyperbolic case the principle of superposition is violated.

2 Contextual viewpoint to the Kolmogorov model and interference of probabilities

In this section we repeat the main points of contextual measure-theoretical approach to interference of probabilities, see Ref. 1 for details.
Let \((\Omega, F, P)\) be a Kolmogorov probability space: \(\Omega\) is an arbitrary set, \(F\) is a \(\sigma\)-field of subsets of \(\Omega\) and \(P\) is a countably additive measure on \(F\) taking values in \([0, 1]\) and normalized by one (Kolmogorov probability). By the standard Kolmogorov axiomatics sets \(A \in F\) represent events. In our simplest model of contextual probability (which can be called the Kolmogorov contextual space) the same system of sets, \(F\), is used to represent complexes of experimental physical conditions – contexts.

Thus depending on circumstances a set \(O \in F\) will be interpreted either as event or as context. We shall sharply distinguish events and contexts on phenomenological level, but we shall use the same mathematical object \(F\) to represent both events and contexts in a mathematical model. In principle, in a mathematical model events and contexts can be represented by different families of sets, e.g., in Renye’s model. We will not do this from the beginning. But later we will fix families of contexts, e.g., \(C^{ct}\) or \(C^{hyp}\), which are proper subfamilies of \(F\).

The conditional probability is mathematically defined by the Bayes’ formula:

\[
P(A/C) = \frac{P(AC)}{P(C)}, P(C) \neq 0. \]

In our contextual model this probability has the meaning of the probability of occurrence of the event \(A\) under the complex of physical conditions \(C\). Thus it would be more natural to call \(P(A/C)\) a contextual probability and not conditional probability. Roughly speaking to find \(P(A/C)\) we should find parameters \(\omega^A\) favoring for the occurrence of the event \(A\) among parameters \(\omega^C\) describing the complex of physical conditions \(C\).

Let \(A = \{A_n\}\) be finite or countable complete group of disjoint contexts (or in the event-terminology – complete group of disjoint events):

\[
A_iA_j = \emptyset, i \neq j, \quad \cup_i A_i = \Omega.
\]

Let \(B \in F\) be an event and \(C \in F\) be a context and let \(P(C) > 0\). We have the standard formula of total probability:

\[
P(B/C) = \sum_n P(A_n/C)P(B/A_nC).
\]

Let \(a = a_1, \ldots, a_n\) and \(b = b_1, \ldots, b_n\) be discrete random variables. Then

\[
P(b = b_i/C) = \sum_n P(a = a_n/C)P(b = b_i/a = a_n, C). \tag{1}
\]

We remark that sets

\[
B_x = \{\omega \in \Omega : b(\omega) = x\} \quad \text{and} \quad A_y = \{\omega \in \Omega : a(\omega) = y\} \tag{2}
\]

have two different interpretations. On the one hand, these sets represent events corresponding to occurrence of the values \(b = x\) and \(a = y\), respectively. On the other hand, they represent contexts (complexes of physical conditions) corresponding to selections of physical systems with respect to values \(b = x\) and \(a = y\), respectively. The main problem with the formula of total probability is that in
general it is impossible to construct a context “A\(y\)C” corresponding to a selection with respect to the value \(a = y\) which would not disturb systems prepared by the context C. But only in the absence of disturbance we can use the set theoretical operation of intersection. I would like to modify the formula of total probability by eliminating sets “A\(y\)C” which in general do not represent physically realizable contexts.

A set C belonging to \(\mathcal{F}\) is said to be a non degenerate context with respect to \(A = \{A_n\}\) if \(P(A_nC) \neq 0\) for all \(n\). We denote the set of such contexts by the symbol \(C_{A,\text{nd}}\).

Let \(A = \{A_n\}\) and \(B = \{B_n\}\) be two complete groups of disjoint contexts. They are said to be incompatible if \(P(B_nA_k) \neq 0\) for all \(n\) and \(k\). Random variables \(a\) and \(b\) inducing, see (2), incompatible complete groups \(A = \{A_n\}\) and \(B = \{B_k\}\) of disjoint contexts are said to be incompatible random variables.

**Theorem 2.1.** (Interference formula of total probability) Let \(A = \{A_1, A_2 = \Omega \setminus A_1\}\) and \(B = \{B_1, B_2 = \Omega \setminus B_1\}\) be incompatible and let a context \(C \in C_{A,\text{nd}}\). Then, for any \(B \in \mathcal{B}\):

\[
P(B/C) = \sum_{j=1}^{2} P(A_j/C)P(B/A_j) + 2\lambda(B/A, C) \sqrt{\prod_{j=1}^{2} P(A_j/C)P(B/A_j)}
\]

where

\[
\lambda(B/A, C) = \frac{P(B/C) - \sum_{j=1}^{2} P(B/A_j)P(A_j/C)}{2\sqrt{P(A_1/C)P(B/A_1)P(A_2/C)P(B/A_2)}}
\]

To prove Theorem, we put the expression for \(\lambda\) into the sum and obtain identity. In fact, this formula is just a representation of the probability \(P(B/C)\) in a special way. The \(\lambda(B/A, C)\) are called coefficients of statistical disturbance. We shall use at few occasions teh following result:

**Lemma 2.1.** Let conditions of Theorem 2.1 hold true. Then

\[
\sum_k \lambda(B_k/A, C) \sqrt{P(A_1/C)P(A_2/C)P(B_k/A_1)P(B_k/A_2)} = 0 \quad (3)
\]

**Proof.** We have

\[
1 = \sum_k P(B_k/C) = \sum_k \sum_n P(A_n/C)P(B_k/A_n)
\]

\[
+ \sum_k \lambda(B_k/A, C) \sqrt{P(A_1/C)P(A_2/C)P(B_k/A_1)P(B_k/A_2)}.
\]
But $\sum_n (\sum_k P(B_k/A_n))P(A_n/C) = 1$.

1) Suppose that for every $B \in B$, $|\lambda(B/A, C)| \leq 1$. In this case we can introduce new statistical parameters $\theta(B/A, C) \in [0, 2\pi]$ and represent the coefficients of statistical disturbance in the trigonometric form: $\lambda(B/A, C) = \cos \theta(B/A, C)$. Parameters $\theta(B/A, C)$ are said to be relative phases of an event $B$ with respect to $A$ (in the context $C$). We have the following interference formula of total probability:

$$P(B/C) = \sum_{j=1}^{2} P(A_j/C)P(B/A_j) + 2 \cos \theta(B/A, C) \prod_{j=1}^{2} P(A_j/C)P(B/A_j)$$

This is nothing other than the famous formula of interference of probabilities.

In Ref. 1 there was shown that by starting with this formula we can construct the representation of the set of trigonometric contexts

$$C^\text{tr} = \{ C \in C_{a,nd} : |\lambda(B_j/a, c)| \leq 1, j = 1, 2 \}$$

in the complex Hilbert space, obtain Born’s rule and represent incompatible variables $a$ and $b$ by (noncommutative) operators.

2) Suppose that for every $B \in B$, $|\lambda(B/A, C)| \geq 1$. In this case we can introduce new statistical parameters $\theta(B/A, C) \in (-\infty, +\infty)$ and represent the coefficients of statistical disturbance in the trigonometric form: $\lambda(B/A, C) = \pm \cosh \theta(B/A, C)$. Parameters $\theta(B/A, C)$ are said to be hyperbolic relative phases. In this case we obtain the formula of total probability with hyperbolic cosh-interference:

$$P(B/C) = \sum_{j=1}^{2} P(A_j/C)P(B/A_j) \pm 2 \cosh \theta(B/A, C) \prod_{j=1}^{2} P(A_j/C)P(B/A_j)$$

The aim of this paper is to show that by starting with this formula we can construct the representation of the set of hyperbolic contexts

$$C^\text{hyp} = \{ C \in C_{a,nd} : |\lambda(B_j/a, c)| \geq 1, j = 1, 2 \}$$

in the hyperbolic Hilbert space, obtain an analogue of Born’s rule and represent incompatible variables $a$ and $b$ by (noncommutative) operators.

We can also consider the case of mixed hyper-trigonometric behavior: one of coefficients is larger than 1 and one is smaller than 1. However, in this paper we shall discuss only the case of the hyperbolic interference.

In our further considerations the complete groups of disjoint contexts $A$ and $B$ will correspond to some incompatible random variables $a$ and $b$. We shall use the symbols $\lambda(b = x/a, C)$ instead of $\lambda(b = x/A, C)$.
3 Representation of contexts by hyperbolic amplitudes, hyperbolic Hilbert space representation.

Everywhere below we study contexts producing the hyperbolic interference for incompatible dichotomous random variables $a = a_1, a_2, b = b_1, b_2$. This pair of variables will be fixed. We call such variables reference variables. For each pair $a, b$ of reference variables we construct a representation of the set of contexts $\mathcal{C}^\text{hyp}$ in hyperbolic Hilbert space ("quantum-like representation").

3.1. Hyperbolic algebra. Instead of the field complex numbers $\mathbb{C}$, we shall use so called hyperbolic numbers, namely the two dimensional Clifford algebra, $\mathbb{G}$. We call this algebra hyperbolic algebra.

Remark 3.1. Of course, it is rather dangerous to invent an own name for a notion established almost as firm as complex numbers. We use a new name, hyperbolic algebra, for the well known algebraic object, the two dimensional Clifford algebra, by following reasons. First we explain why we dislike to use the standard notion Clifford algebra in this particular case. The standard Clifford machinery was developed around noncommutative features of general Clifford algebras. The two dimensional Clifford algebra, hyperbolic algebra in our terminology, is commutative. Commutativity of $\mathbb{G}$ is very important in our considerations. We now explain why we propose the name hyperbolic algebra. Hyperbolic functions are naturally related to the algebraic structure of $\mathbb{G}$ through a hyperbolic generalization of Euler’s formula for the complex numbers. This is the crucial point of our considerations - the possibility to use this algebraic structure to represent some special transformations for hyperbolic functions.

Denote by the symbol $j$ the generator of the algebra $\mathbb{G}$ of hyperbolic numbers:

$$j^2 = 1.$$ 

The algebra $\mathbb{G}$ is the two dimensional real algebra with basis $e_0 = 1$ and $e_1 = j$. Elements of $\mathbb{G}$ have the form $z = x + jy$, $x, y \in \mathbb{R}$. We have $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$ and $z_1 z_2 = (x_1 x_2 + y_1 y_2) + j(x_1 y_2 + x_2 y_1)$. This algebra is commutative. It is not a field - not every element has the inverse one.

We introduce an involution in $\mathbb{G}$ by setting $\bar{z} = x - jy$ and set $|z|^2 = z \bar{z} = x^2 - y^2$. We remark that $|z| = \sqrt{x^2 - y^2}$ is not well defined for an arbitrary $z \in \mathbb{G}$. We set $\mathbb{G}_+ = \{ z \in \mathbb{G} : |z|^2 \geq 0 \}$. We remark that $\mathbb{G}_+$ is a multiplicative semigroup as follows from the equality $|z_1 z_2|^2 = |z_1|^2 |z_2|^2$.

Thus, for $z_1, z_2 \in \mathbb{G}_+$, we have that $|z_1 z_2|$ is well defined and $|z_1 z_2| = |z_1| |z_2|$. We define a hyperbolic exponential function by using a hyperbolic analogue of the
Euler’s formula:
\[ e^{j\theta} = \cos \theta + j \sin \theta, \ \theta \in \mathbb{R}. \]

We remark that
\[ e^{j\theta_1} e^{j\theta_2} = e^{j(\theta_1 + \theta_2)}, e^{j\theta} = e^{-j\theta}, |e^{j\theta}|^2 = \cosh^2 \theta - \sinh^2 \theta = 1. \]

Hence, \( z = \pm e^{j\theta} \) always belongs to \( G_+ \). We also have
\[ \cosh \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}, \ \sinh \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}. \]

We set \( G_+^* = \{ z \in G_+ : |z|^2 > 0 \} \). Let \( z \in G_+^* \). We have
\[ z = |z| \left( \frac{x}{|z|} + j \frac{y}{|z|} \right) = \text{sign} x |z| \left( \frac{\text{xsign} x}{|z|} + j \frac{\text{ysign} x}{|z|} \right). \]

As \( \frac{x^2}{|z|^2} - \frac{y^2}{|z|^2} = 1 \), we can represent \( x \ \text{sign} x = \cosh \theta \) and \( y \ \text{sign} x = \sinh \theta \), where the phase \( \theta \) is unequally defined. We can represent each \( z \in G_+^* \) as
\[ z = \text{sign} x |z| e^{j\theta}. \]

By using this representation we can easily prove that \( G_+^* \) is a multiplicative group. Here \( \frac{1}{z} = \frac{\text{sign} x}{|z|} e^{-j\theta} \). The unit circle in \( G \) is defined as \( S_1 = \{ z \in G : |z|^2 = 1 \} = \{ z = \pm e^{j\theta}, \theta \in (-\infty, +\infty) \} \). It is a multiplicative subgroup of \( G_+^* \).

To construct a \( G \)-linear representation of the set \( C_{\text{hyp}} \) of hyperbolic contexts, we shall use the following elementary formulas:
\[ D = A + B \pm 2AB \cosh \theta = |\sqrt{A} \pm e^{j\theta} \sqrt{B}|^2, \quad \text{(5)} \]

for real coefficients \( A, B > 0 \).

3.2. Hyperbolic probability amplitude, hyperbolic Born’s rule. We set \( Y = \{ a_1, a_2 \}, X = \{ b_1, b_2 \} \) (“spectra” of random variables \( a \) and \( b \)). Let \( C \in C_{\text{hyp}} \). We set
\[ p_C^a(y) = P(a = y/C), p_C^b(x) = P(b = x/C), p(x/y) = P(b = x/a = y), \]
\( x \in X, y \in Y \). The interference formula of total probability \( \square \) can be written in the following form:
\[ p_C^b(x) = \sum_{y \in Y} p_C^a(y)p(x/y) \pm 2 \cosh \theta_C(x) \sqrt{\prod_{y \in Y} p_C^a(y)p(x/y)} , \quad \text{(6)} \]

where \( \theta_C(x) = \theta(b = x/a, C) = \pm \arccosh |\lambda(b = x/a, C)|, x \in X, C \in C_{\text{hyp}} \). Here the coefficient \( \lambda \) is defined by
\[ \lambda(b = x/a, C) = \frac{p_C^b(x) - \sum_{y \in Y} p_C^a(y)p(x/y)}{2 \sqrt{\prod_{y \in Y} p_C^a(y)p(x/y)}}. \quad \text{(7)} \]

By using \( \square \), we can represent the probability \( p_C^b(x) \) as the square of the hyperbolic amplitude:
\[ p_C^b(x) = |\varphi_C(x)|^2, \quad \text{(8)} \]
where
\[ \varphi(x) \equiv \varphi_C(x) = \sqrt{p_C^a(a_1)p(x/a_1)} + \epsilon_C(x)e^{ipC(x)} \sqrt{p_C^a(a_2)p(x/a_2)}. \] (9)

Here \( \epsilon_C(x) = \text{sign} \lambda(x/a, C) \).

Thus we have a hyperbolic generalization of Born’s rule for the \( b \)-variable, see (9).

### 3.3. Hyperbolic Hilbert space.

Hyperbolic Hilbert space is \( G \)-linear space (module) \( E \) with a \( G \)-linear scalar product: a map \((\cdot, \cdot): E \times E \to G\) that is
1) linear with respect to the first argument:
\[ (az + bw, u) = a(z, u) + b(w, u), a, b \in G, z, w, u \in E; \]
2) symmetric: \((z, u) = (u, z)\);
3) nondegenerate: \((z, u) = 0\) for all \( u \in E \) iff \( z = 0 \).

**Remark 3.2.** If we consider \( E \) as just a \( R \)-linear space, then \((\cdot, \cdot)\) is a bilinear form which is not positive defined. In particular, in the two dimensional case we have the signature: \((+, -, +, -)\).

### 3.4. Hyperbolic Hilbert space representation.

We introduce on the space \( \Phi(X, G) \) of functions: \( \varphi: X \to G \). Since \( X = \{b_1, b_2\} \), the \( \Phi(X, G) \) is the two dimensional \( G \)-module. We define the \( G \)-scalar product by
\[ (\varphi, \psi) = \sum_{x \in X} \varphi(x) \bar{\psi}(x). \] (11)

with conjugation in the algebra \( G \). The system of functions \( \{e^b_x\}_{x \in X} \) is an orthonormal basis in the hyperbolic Hilbert space \( H^{hyp} = (\Phi(X, G), (\cdot, \cdot)) \). Thus we have the hyperbolic analogue of the Born’s rule in \( H^{hyp} \):
\[ p_C^b(x) = |(\varphi_C, e^b_x)|^2. \] (12)

Let \( X \subset R \). By using the hyperbolic Hilbert space representation (12) of the Born’s rule we obtain the hyperbolic Hilbert space representation of the expectation of the (Kolmogorovian) random variable \( b \):
\[ E(b/C) = \sum_{x \in X} x p_C^b(x) = \sum_{x \in X} x |\varphi_C(x)|^2 = \sum_{x \in X} x (\varphi_C, e^b_x) (\varphi_C, e^b_x) = (\hat{b} \varphi_C, \varphi_C), \] (13)

where the (self-adjoint) operator \( \hat{b} : H^{hyp} \to H^{hyp} \) is determined by its eigenvectors: \( \hat{b} e^b_x = x e^b_x, x \in X \). This is the multiplication operator in the space of \( G \)-valued functions \( \Phi(X, G) \):
\[ \hat{b} \varphi(x) = x \varphi(x) \]
By \((\text{14})\) the conditional expectation of the Kolmogorovian random variable \(b\) is represented with the aid of the self-adjoint operator \(\hat{b}\).

Thus we constructed a \(G\)-linear representation of the contextual Kolmogorov model:

\[
J^{b/a} : C^{\text{hyp}} \to H^{\text{hyp}}.
\]

We set \(S^{\text{hyp}} = J^{b/a}(C^{\text{hyp}})\). This is a subset of the unit sphere \(S\) of the Hilbert space \(H^{\text{hyp}}\). We introduce the coefficients

\[
u^a_j = \sqrt{p_b^a(c_j)}, \quad \nu^b_j = \sqrt{p_b^b(b_j)}, \quad p_{ij} = p(b_j/a_i), \quad u_{ij} = \sqrt{p_{ij}}, \quad \theta_j = \theta_C(b_j).
\]

and \(\epsilon_i = \epsilon(b_i)\). We remark that the coefficients \(\nu^a_j, \nu^b_j\) depend on a context \(C\); so \(\nu^a_j = \nu^a_j(C), \nu^b_j = \nu^b_j(C)\). We also consider the matrix of transition probabilities \(P^{b/a} = (p_{ij})\). It is always a stochastic matrix: \(p_{11} + p_{12} = 1, i = 1, 2\). In further considerations we shall also consider double stochastic matrices: \(p_{1j} + p_{2j} = 1, j = 1, 2\).

We represent a state \(\varphi_C\) by \(\varphi_C = \nu^b_1e_1 + \nu^b_2e_2\), where \(\nu^b_i = \nu^a_i u_{1i} + \epsilon_i u^a_2 u_{2i} e^{i\theta_i}\). So

\[
p^{b}_C(b_i) = |\nu^b_i|^2 = |u^a_1 u_{1i} + \epsilon_i u^a_2 u_{2i} e^{i\theta_i}|^2.
\]

This is the \(G\)-linear representation of the hyperbolic interference of probabilities. This formula can also be derived in the formalism of the hyperbolic Hilbert space, see section 4. We remark that here the \(G\)-linear combination \(u^a_1 u_{1i} + \epsilon_i u^a_2 u_{2i} e^{i\theta_i}\) belongs to \(G^*_+\).

Thus for any context \(C_0 \in C^{\text{hyp}}\) we can represent \(\varphi_{C_0}\) in the form:

\[
\varphi_{C_0} = u^a_1 e^a_1 + u^a_2 e^a_2,
\]

where

\[
\epsilon^a_1 = (u_{11}, u_{12}), \quad \epsilon^a_2 = (\epsilon_1 e^{i\theta_1} u_{21}, \epsilon_2 e^{i\theta_2} u_{22}).
\]

As in the \(C\)-case, we introduce the matrix \(V\) with coefficients \(v_{11} = u_{11}, v_{21} = u_{21}\) and \(v_{12} = \epsilon_1 e^{i\theta_1} u_{21}, v_{22} = \epsilon_2 e^{i\theta_2} u_{22}\). We remark that here coefficients \(v_{ij} \in G^*_+\).

In the same way as in the complex case the Born’s rule

\[
p^a_{C_0}(a_i) = |(\varphi_{C_0}, e^a_i)|^2
\]

holds true in the \(a\)-basis iff \(\{e^a_i\}\) is an orthonormal basis in \(H^{\text{hyp}}\). The latter is equivalent to the \(G\)-unitary of the matrix \(V\) (corresponding to the transition from \(\{e^b_i\}\) to \(\{e^a_i\}\)) : \(\overline{V}V = I\), or

\[
\bar{v}_{11} v_{11} + \bar{v}_{21} v_{21} = 1, \quad \bar{v}_{12} v_{12} + \bar{v}_{22} v_{22} = 1,
\]

where \(\bar{v}_{ij} = \epsilon_1 e^{i\theta_1} u_{21} v_{12} + \epsilon_2 e^{i\theta_2} u_{22} v_{22}\).
\[ \bar{v}_{11}v_{12} + \bar{v}_{21}v_{22} = 0. \]  
(17)

Thus 1 = \( u_{11}^2 + u_{21}^2 = p(b_1/a_1) + p(b_1/a_2) \) and 1 = \( u_{12}^2 + u_{22}^2 = p(b_2/a_1) + p(b_2/a_2) \). Thus the first two equations of the G-unitary are equivalent to the double stochasticity of \( P^{b/a} \) (as in the C-case\(^1\)). We remark that the equations (16) can be written as

\[ |v_{11}|^2 + |v_{21}|^2 = 1, |v_{12}|^2 + |v_{22}|^2 = 1, \]  
(18)

cf. section 4. The third unitarity equation (17) can be written as

\[ u_{11}^2 e^{-j\theta_2} + u_{21}^2 e^{-j\theta_2} = 0. \]  
(19)

By using double stochasticity of \( P^{a/b} \) we obtain \( e^{j\theta_1} = e^{j\theta_2} \). Thus

\[ \theta_1 = \theta_2. \]  
(20)

**Lemma 3.1.** Let a and b be incompatible random variables and let \( P^{b/a} \) be double stochastic. Then

\[ \cosh \theta_C(b_2) = \cosh \theta_C(b_1) \]  
(21)

for any context \( C \in \mathcal{C}^{hyp} \).

**Proof.** By Lemma 2.1 we have:

\[ \sum_x e(x) \cosh \theta_C(x) \sqrt{\Pi_y p_C^a(y)p(x/y)} = 0. \]

Double stochasticity of \( P^{b/a} \) implies (21).

The constraint (21) induced by double stochasticity can be written as the constraint to phases:

\[ \theta_C(b_2) = \pm \theta_C(b_1). \]  
(22)

To obtain unitary of the matrix \( V \) of transition \( \{e^b_j\} \to \{e^a_j\} \) we should choose phases according to (20). And by (22) we can always do this for a double stochastic matrix of transition probabilities.

By choosing such a representation we obtain the hyperbolic generalization of the Born’s rule for the \( a \)-variable:

\[ p_C^a(a_j) = |(\varphi, e^a_j)|^2. \]  
(23)

We now investigate the possibility to use one fixed basis \( \{e^a_j \equiv e^a_j(C_0)\}, C_0 \in \mathcal{C}^{hyp} \), for all states \( \varphi_C, C \in \mathcal{C}^{hyp} \). For any \( C \in \mathcal{C}^{hyp} \) we would like to have the representation:

\[ \phi_C = v_1^a(C)e^a_1(C_0) + v_2^a(C)e^a_2(C_0), \text{ where } |v_j^a(C)|^2 = p_C^a(a_j). \]  
(24)
We have
\[ \varphi_C(b_1) = u_1^\ast(C)v_{11}(C_0) + \epsilon_C(b_1)\epsilon_{C,b}(b_1)e^{j[\theta_C(b_1) - \theta_{C,b}(b_1)]}u_2^\ast(C)v_{12}(C_0) \]
\[ \varphi_C(b_2) = u_1^\ast(C)v_{21}(C_0) + \epsilon_C(b_2)\epsilon_{C,b}(b_2)e^{j[\theta_C(b_2) - \theta_{C,b}(b_2)]}u_2^\ast(C)v_{22}(C_0) \]
Thus to obtain (24) we should have
\[ \epsilon_C(b_1)\epsilon_{C,b}(b_1)e^{j[\theta_C(b_1) - \theta_{C,b}(b_1)]} = \epsilon_C(b_2)\epsilon_{C,b}(b_2)e^{j[\theta_C(b_2) - \theta_{C,b}(b_2)]} \]
Thus
\[ \theta_C(b_1) - \theta_{C,b}(b_1) = \theta_C(b_2) - \theta_{C,b}(b_2), \text{ or } \theta_C(b_1) - \theta_C(b_2) = \theta_{C,b}(b_1) - \theta_{C,b}(b_2). \]
By choosing the representation with (20) we satisfy the above condition.

**Theorem 3.1** We can construct the quantum-like (Hilbert space) representation of a contextual Kolmogorov space such that the hyperbolic Born’s rule holds true for both reference variables \(a\) and \(b\) iff the matrix of transition probabilities \(P^{b/a}\) is double stochastic.

We remark that basic contexts \(B_x = \{ \omega \in \Omega : b(\omega) = x \}, x \in X, \) always belong to \(C^{hyp}, \) so \(\varphi_B \in H^{hyp};\) and \(B_x \in C^{tr} \cap C^{hyp} \) iff \(a\) and \(b\) are uniformly distributed (\(P^{a/b}\) and \(P^{b/a}\) are double stochastic).

### 4 Hyperbolic quantum mechanics

As in the ordinary quantum formalism, we represent physical states by normalized vectors of a hyperbolic Hilbert space \(E : \varphi \in E\) and \((\varphi, \varphi) = 1.\) We shall consider only dichotomous physical variables and quantum states belonging to the two dimensional Hilbert space. Thus everywhere below \(E\) denotes the two dimensional space. Let \(a = a_1, a_2\) and \(b = b_1, b_2\) be two physical variables. We represent they by \(G\)-linear operators: \(\hat{a} = |a_1 > < a_1| + |a_2 > < a_2| \) and \(\hat{b} = |b_1 > < b_1| + |b_2 > < b_2|,\) where \(|a_i >\}_{i=1,2} \) and \(|b_i >\}_{i=1,2} \) are two orthonormal bases in \(E.\) The latter condition plays the fundamental role in hyperbolic quantum mechanics. This is an analogue of the representation of physical observables by self-adjoint operators in the conventional quantum mechanics (in the complex Hilbert space).

Let \(\varphi\) be a state (normalized vector belonging to \(E\)). We can perform the following operation (which is well defined from the mathematical point of view). We expand the vector \(\varphi\) with respect to the basis \(|b_i >\}_{i=1,2} :\)
\[ \varphi = v_1^b|b_1 > + v_2^b|b_2 >, \]  
(25)
where the coefficients (coordinates) \(v_i^b\) belong to \(G.\) We remark that we consider the two dimensional \(G\)-Hilbert space. There exists (by definition) a basis consisting of
two vectors. As the basis \( \{|b_i>\}_{i=1,2} \) is orthonormal, we have (as in the complex case) that:

\[
|v_1|^2 + |v_2|^2 = 1.
\]  

(26)

However, we could not automatically use Born’s probabilistic interpretation for normalized vectors in the hyperbolic Hilbert space: it may be that \( v_i^b \notin G_+ \) and hence \( |v_i^b|^2 < 0 \) (in fact, in the complex case we have \( C = C_+ \); thus there is no problem with positivity). Since we do not want to consider negative probabilities, in such a case we cannot use the hyperbolic version of Born’s probability interpretation.

**Definition 4.1.** A state \( \varphi \) is decomposable with respect to the system of states \( \{|b_i>\}_{i=1,2} \) (b-decomposable) if

\[
v_i^b \in G_+.
\]  

(27)

In such a case we can use generalization of Born’s probabilistic interpretation for a hyperbolic Hilbert space. Numbers

\[
p_{\varphi}^b(b_i) = |v_i^b|^2, i = 1, 2,
\]

are interpreted as probabilities for values \( b = b_i \) for the \( G \)-quantum state \( \varphi \).

We remark that in this framework (here we started with a hyperbolic Hilbert space and not with a contextual statistical model, cf. section 3) a hyperbolic generalization of Born’s rule is a postulate!

Thus decomposability is not a mathematical notion. This is not just linear algebraic decomposition of a vector with respect a basis. This is a physical notion describing the possibility of probability interpretation of a measurement over a state. As it was already mentioned, in hyperbolic quantum mechanics a state \( \varphi \in E \) is not always decomposable. Thus for an observable \( b \) there can exist a state \( \varphi \) such that the probabilities \( p_{\varphi}^b(b_i) \) are not well defined. One of reasons for this can be the impossibility to perform the \( b \)-measurement for systems in the state \( \varphi \). Such a situation is quite natural from the experimental viewpoint. Moreover, it looks surprising that in ordinary quantum (as well as classical) theory we can measure any observable in any state. I think that this is just a consequence of the fact that there was fixed the set of states corresponding to a rather special class of physical observables. Thus in the hyperbolic quantum formalism for each state \( \varphi \in E \) there exists its own set of observables \( O(\varphi) \). And in general \( O(\varphi) \neq O(\psi) \).

We cannot exclude another possibility. The set of observables \( O \) does not depend on a state \( \varphi \). And the result of an individual measurement of any \( b \in O \) is well defined for any state \( \varphi \). But relative frequencies of realizations of the value \( b = b_k \) do not converge to any limit. Therefore probabilities are not well defined. Thus the principle of the statistical stabilization should be violated, cf. Ref 13.
Let $\mathcal{K}$ be a Kolmogorov probability model and let $\varphi \in \mathcal{S}_{\text{hyp}}$. Thus $\varphi = \varphi_C$ for some context $C \in \mathcal{C}_{\text{hyp}}$. Let the matrix of transition probabilities $P^{b/a}$ be double stochastic. Then $\varphi$ is decomposable with respect to both reference variables $b$ and $a$. Moreover, basis vectors $\psi_i^b = |b_i>$ are $a$-decomposable and vice versa.

We now start the derivation of the hyperbolic probabilistic rule by using the hyperbolic Hilbert space formalism. Suppose that a state $\varphi \in \mathcal{E}$ is $a$-decomposable:

$$\varphi = \psi_1^a |a_1> + \psi_2^a |a_2>$$

and the coefficients $\psi_i^a \in \mathcal{G}_+$.

We also suppose that each state $|a_i>$ is decomposable with respect to the system of states $\{|b_i>\}_i=1,2$. We have:

$$|a_1> = v_{11}|b_1> + v_{12}|b_2>, \quad |a_2> = v_{21}|b_1> + v_{22}|b_2>,$$

where the coefficients $v_{ik}$ belong to $\mathcal{G}_+$. We have (since both bases are orthonormal):

$$|v_{11}|^2 + |v_{12}|^2 = 1, \quad |v_{21}|^2 + |v_{22}|^2 = 1,$$

cf. (15). We can use the probabilistic interpretation of numbers $p_{ik} = |v_{ik}|^2$, namely $p_{ik} = p_{|a_i>,|b_k>}$ is the probability for $b = b_k$ in the state $|a_i>$. Let us consider matrix $V = (v_{ik})$. As in the complex case, the matrix $V$ is unitary, since vectors $|a_1> = (v_{11}, v_{12})$ and $|a_2> = (v_{21}, v_{22})$ are orthonormal. Hence we have normalization conditions (29) and the orthogonality condition:

$$v_{11}v_{21} + v_{12}v_{22} = 0,$$

cf. (17). It must be noticed that in general unitarity does not imply that $v_{ik} \in \mathcal{G}_+$. The latter condition is the additional constraint on the unitary matrix $V$. Let us consider the matrix $P^{b/a} = (p_{ik})$. This matrix is double stochastic (since $V$ is unitary).

By using the $\mathcal{G}$-linear space calculation (the change of the basis) we get $\varphi = \psi_1^b |b_1> + \psi_2^b |b_2>$, where $\psi_1^b = \psi_1^a v_{11} + \psi_2^a v_{21}$ and $\psi_2^b = \psi_1^a v_{12} + \psi_2^a v_{22}$.

We remark that decomposability is not transitive. In principle $\varphi$ may be not decomposable with respect to $\{|b_i>\}_i=1,2$, despite the decomposability of $\varphi$ with respect to $\{|a_i>\}_i=1,2$ and the decomposability of the latter system with respect to $\{|b_i>\}_i=1,2$.

The possibility of decomposability is based on two (totally different) conditions: normalization, and positivity. Any $\mathcal{G}$-unitary transformation preserves the normalization condition. Thus we get automatically that $|\psi_1^b|^2 + |\psi_2^b|^2 = 1$. However, the condition of positivity in general is not preserved: it can be that $\psi_i^b \notin \mathcal{G}_+$ even if we have $\psi_i^a \in \mathcal{G}_+$ and the matrix $V$ is $\mathcal{G}$-unitary.
Finally, suppose that $\varphi$ is decomposable with respect to $\{|b_i>\}_{i=1,2}$. Thus $v_k^b \in \mathbf{G}_+$. Therefore coefficients $p_k^b(b_i) = |v_k^b|^2$ can be interpreted as probabilities for $b = b_k$ for the $\mathbf{G}$-quantum state $\varphi$.

Let us consider states such that coefficients $v_i^a, v_{ik}$ belong to $\mathbf{G}_{+}^*$. We can uniquely represent them as

$$v_i^a = \pm \sqrt{p_\varphi^a(a_i)} e^{i\xi_i}, v_{ik} = \pm \sqrt{p_\varphi^{ik} e^{i\gamma_{ik}}}, i, k = 1, 2.$$  

We find that

$$p_k^b(b_1) = p_\varphi^a(a_1)p_{11} + p_\varphi^a(a_2)p_{21} + 2\epsilon_1 \cosh \theta_1 \sqrt{p_\varphi^a(a_1)p_{11}p_\varphi^a(a_2)p_{21}},$$

$$p_k^b(b_2) = p_\varphi^a(a_1)p_{12} + p_\varphi^a(a_2)p_{22} + 2\epsilon_2 \cosh \theta_2 \sqrt{p_\varphi^a(a_1)p_{12}p_\varphi^a(a_2)p_{22}},$$

where $\theta_i = \gamma_i + \xi_i$ and $\eta = \xi_1 - \xi_2, \gamma_1 = \gamma_{11} - \gamma_{21}, \gamma_2 = \gamma_{12} - \gamma_{22}$ and $\epsilon_i = \pm$. To find the right relation between signs of the last terms in equations (31), (32), we use the normalization condition

$$|v_2^b|^2 + |v_2^b|^2 = 1$$

(which is a consequence of the normalization of $\varphi$ and orthonormality of the system $\{|b_i>\}_{i=1,2}$).

We remark that the normalization condition (33) can be reduced to relations between coefficients of the transition matrix $V$. So it does not depend on the original $a$-decomposition of $\varphi$, namely coefficients $v_i^a$. Condition of positivity, $|v_k^b|^2 \geq 0$, could not be written by using only coefficients of $V$. We also need to use coefficients $v_i^a$. Therefore it seems to be impossible to find such a class of linear transformations $V$ that would preserve condition of positivity, “decomposition-group” of operators.

Equation (33) is equivalent to the equation:

$$\sqrt{p_{12}p_{22}} \cosh \theta_2 \pm \sqrt{p_{11}p_{21}} \cosh \theta_2 = 0.$$  

Thus we have to choose opposite signs in equations (31), (32). Unitarity of $V$ also implies that $\theta_1 - \theta_2 = 0$, so $\gamma_1 = \gamma_2$. We recall that in the ordinary quantum mechanics we have similar conditions, but trigonometric functions are used instead of hyperbolic and phases $\gamma_1$ and $\gamma_2$ are such that $\gamma_1 - \gamma_2 = \pi$.

Finally, we get that unitary linear transformations in the $\mathbf{G}$-Hilbert space (in the domain of decomposable states) represent the following transformation of probabilities:

$$p_k^b(b_1) = p_\varphi^a(a_1)p_{11} + p_\varphi^a(a_2)p_{21} \pm 2\epsilon_1 \cosh \theta_1 \sqrt{p_\varphi^a(a_1)p_{11}p_\varphi^a(a_2)p_{21}},$$

$$p_k^b(b_2) = p_\varphi^a(a_1)p_{12} + p_\varphi^a(a_2)p_{22} \pm 2\epsilon_2 \cosh \theta_2 \sqrt{p_\varphi^a(a_1)p_{12}p_\varphi^a(a_2)p_{22}}.$$
This is hyperbolic interference. In section 2 it was derived from the contextual statistical model and then in section 3 by using interference formulas we obtained the hyperbolic Hilbert space representation for contexts. In this section we started directly from the hyperbolic Hilbert space representation and derived interference of probabilities.

5 Experimental verification of hyperbolic quantum mechanics

This paper contains an important experimental prediction:

\textit{In statistical experiments with physical (micro as well as macro) systems there could be produced not only the ordinary trigonometric, but also the hyperbolic interference picture.}

We start with the general description of interference experiments for discrete observables. There are considered two dichotomous observables: \( a \) - “slit number”, and \( b \) - “position of a particle on the registration screen.” The observable \( a \) is measured in the following way. There are placed particle detectors behind the screen having two open slits. The observable \( a = j \) if the detector behind the \( j \)th slit clicks. To define another observable, we choose some domain \( D \) on the registration screen and we set \( b = 1 \) if a particle is registered inside \( D \) and \( b = 0 \) if outside. The complex of physical conditions under consideration (context) \( C \) is screen with two open slits and the registration screen. We find frequency probabilities \( p^b_C(1) \) and \( p^b_C(0) \) by counting the numbers of particles inside and outside the domain \( D \) on the registration screen. Then we perform the measurement of the \( a \)-variable by placing detectors behind the first screen. We find frequency probabilities \( p^a_C(1) \) and \( p^a_C(2) \) by counting the numbers of particles passing through the first screen and the second screen, respectively (if the source is located symmetrically with respect to screens, then \( p^a_C(a = 1) = p^a_C(a = 2) = 1/2 \)). We also find transition probability \( p^{b/a}(i/j) \) by closing the \( j \)th slit and performing the \( b \)-measurement under this complex of physical conditions. For systems described by classical (noncontextual) probability theory we get the well known formula of total probability:

\[
p^b_C(x) = p^{b/a}(x/1)p^a_C(1) + p^{b/a}(x/2).
\]

Here the coefficient of statistical disturbance \( \lambda(b = x/a, C) = 0 \). For systems described by quantum probability, we get the interference formula:

\[
p^b_C(x) = p^{b/a}(x/1)p^a_C(1) + p^{b/a}(x/2) + 2 \cos \theta \sqrt{p^{b/a}(x/1)p^a_C(1)p^a_C(2)p^{b/a}(x/2)}.
\]
This formula is usually derived in the Hilbert space formalism. In the book of Feynman and Hibbs\((14)\) violation of the formula of total probability was considered as the most important exhibition of difference between probabilistic laws for classical and quantum systems. However, in papers of some authors e.g. Ref. 2, 5, 6, 7, 8, 13 there was pointed out that violation of the formula of total probability is just an exhibition of contextuality of quantum probabilities.

In this paper we predict that contextual statistics produced by experiments of two slit type is not reduced to classical and quantum. Besides the absence of interference and the quantum trigonometric interference, we predict a new type of interference – the hyperbolic interference. In our approach it is very easy to find the type of interference of probabilities. In a statistical test for some context \(C\) we calculate the coefficient \(\lambda(a = x/b, C)\)

\[
\lambda(a = x/b, C) = \frac{p^b_C(x) - p^{b/a}(x/1)p^a_C(1) - p^a_C(2)p^{b/a}(x/2)}{2\sqrt{p^{b/a}(x/1)p^a_C(1)p^a_C(2)p^{b/a}(x/2)}}.
\]

An empirical situation with \(\lambda(a = x/b, C) > 1\) would yield evidence for quantum-like hyperbolic behavior. The coefficient \(\lambda(a = x/b, C)\) can be easily calculated on the basis of statistical data.

Thus our hyperbolic quantum mechanics predicts a testable result, namely the hyperbolic interference, that ordinary quantum mechanics does not!

We wrote about experiments of “two slit type”. They need not be precisely experiments with space-variables. The \(a\) and \(b\) can be any pair of incompatible observables. Incompatibility is understood as the impossibility to escape mutual disturbances in the process of measurement. The coefficient \(\lambda(a = x/b, C)\) gives the measure of statistical disturbance. Classical measurements are characterized by (statistically) negligibly small mutual disturbances, so here \(\lambda(b = x/a, C) = 0\) (and we have the conventional formula of total probability). Quantum measurements are characterized by mutual disturbances which are not negligible (statistically). Here \(\lambda(b = x/a, C) \in (0, 1]\). The conventional formula of total probability is violated and we have the conventional trigonometric interference. However, the quantum case, i.e., \(\lambda(b = x/a, C) \in (0, 1]\), does not describe all nonclassical measurements. There can exist incompatible observables which produce mutual disturbances which are (statistically) essentially larger than the conventional quantum disturbances. In such a case \(\lambda(a = x/b, C) > 1\). As in the quantum case, the conventional formula of total probability is violated, but we have nonconventional hyperbolic interference.

Thus hyperbolic interference might be found in experiments with systems which are essentially more sensitive to disturbance effects of measurement devices than quantum systems. So to find such an interference we should go to new
scales of space, time and energy: distances and time intervals which are essentially smaller than approached in the conventional quantum experiments. One may speculate that there can be some connections with string theory and cosmology. It may be that quantum mechanics for string theory and cosmology is hyperbolic quantum mechanics.

Another possibility to find hyperbolic interference (which looks more realizable at the present technological level) is to look for observables on ordinary quantum or classical systems which would produce very strong statistical disturbances.

Since we derived the hyperbolic (as well as the conventional trigonometric) interference in the general contextual probabilistic approach, our formalism can be applied to any kind of systems, for example cognitive systems. Experiments of the two slit type can be done for cognitive systems, e.g. human beings. Here observables $a$ and $b$ are given in the form of questions. It might be that cognitive systems can produce hyperbolic interference and should be described by hyperbolic quantum mechanics.

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