Asymptotic behaviour of first passage time distributions for subordinators

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Abstract

In this paper we establish local estimates for the first passage time of a subordinator under the assumption that it belongs to the Feller class, either at zero or infinity, having as a particular case the subordinators which are in the domain of attraction of a stable distribution, either at zero or infinity. To derive these results we first obtain uniform local estimates for the one dimensional distribution of such a subordinator, which sharpen those obtained by Jain and Pruitt [5]. In the particular case of a subordinator in the domain of attraction of a stable distribution our results are the analogue of the results obtained by the authors in [3] for non-monotone Lévy processes. For subordinators an approach different to that in [3] is necessary because the excursion techniques are not available and also because typically in the non-monotone case the tail distribution of the first passage time has polynomial decrease, while in the subordinator case it is exponential.

Keywords and phrases: Increasing Lévy processes, first passage time distribution, local limit theorems, Feller class.

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1 Introduction and main results

Let $X$ be a subordinator, a stochastic process with non-decreasing càdlàg paths with independent and stationary increments, with Laplace exponent $\psi$, 

$$-\frac{1}{t} \log (E(\exp\{-\lambda X_t\})) =: \psi(\lambda) = b\lambda + \int_{(0,\infty)} (1-e^{-\lambda x})\Pi(dx), \quad \lambda \geq 0,$$

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where $b$ denotes the drift and $\Pi$ the Lévy measure of $X$. We are interested in determining the local asymptotic behaviour of the distribution of $T_x = \inf\{t > 0 : X_t > x\}$. More precisely, we would like to establish estimates for the density function $h_x(t)$, (if it exists: it does if $b = 0$), or more generally of

$$P(T_x \in (t, t + \Delta]),$$

uniformly for $\Delta$ in bounded sets and uniformly for $x$ in certain regions, both as $t \to \infty$ or as $t \to 0$.

This is a continuation of recent research in [3], where the same problem, in the $t \to \infty$ case, has been solved for Lévy processes, excluding subordinators, that are in the domain of attraction of a stable law without centering. The reasons for excluding subordinators from that research were that the techniques used there rely heavily on excursion theory for the reflected process, which in this case does not make sense, and that in the subordinators case the rate of decrease of the tail distribution of the first passage time is typically exponential, while for other Lévy processes it is polynomial.

As can be seen in the paper [3], and in the present case, the distribution of the first passage time has different behaviour according to whether the process first crosses the barrier by a jump or continuously, that is by creeping. So, our results will describe the contributions of these events to the first passage time distribution separately. Of course if a subordinator has zero drift, it cannot creep, and moreover the distribution of $T_x$ is absolutely continuous, so our results become somewhat simpler in that case.

In the present work we allow a more general behaviour than that of being in the domain of attraction of a stable law, namely for most of our results we only require $X$ to be in the Feller class, said otherwise to be stochastically compact, either at infinity or at zero depending on whether $x/t$ tends to $b$ from above, or to $E(X_1)$ from below, or is bounded away from $b$ and $E(X_1)$. A further difference from our work in [3] is that the results here obtained apply equally to subordinators which are stochastically compact with or without centering, while in [3] the assumption that the Lévy process is in the domain of attraction of a stable law without centering is in force.

In order to provide precise definitions of these notions we start by introducing some notation.

We will write

$$H(u) = \psi(u) - u\psi'(u), \quad \sigma^2(u) = \int_0^\infty y^2e^{-uy}\Pi(dy), \quad u \geq 0, \quad (1)$$

and for $x > 0$,

$$\overline{\Pi}(x) = \Pi(x, \infty), \quad K_\Pi(x) = x^{-2}\int_{y \in (0, x)} y^2\Pi(dy), \quad (2)$$

$$Q_\Pi(x) = \overline{\Pi}(x) + K_\Pi(x). \quad (3)$$

An elementary verification shows that

$$Q_\Pi(z) = 2z^{-2}\int_0^z y\overline{\Pi}(y)dy, \quad z > 0, \quad (4)$$
and that $Q_{\Pi}$ is a non-increasing function. We define $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ via the relation
\[
\psi'(\rho(s)) = s, \quad 0 \leq b = \psi'(\infty) < s < \psi'(0+) = E(X_1) =: \mu \leq \infty.
\]
From this relation it is easily seen that $\rho(\cdot)$ is a non-increasing function.

For notational convenience for $b < x < t < \mu$, we will write $\rho_t := \rho(x_t)$.

Note that $\rho_t \downarrow 0$ when $x_t \uparrow \mu$ and $\rho_t \uparrow \infty$ when $x_t \downarrow b$.

We will say that $X$ is in a Feller class or is stochastically compact at infinity, respectively at 0, if
\[
[SC] \limsup \frac{\Pi(y)}{K_{\Pi}(y)} < \infty \text{ as } y \to \infty, \text{ respectively as } y \to 0+.\]

It is known that this condition is equivalent to
\[
[SC'] \exists \alpha \in (0, 2] \text{ and } c \geq 1 \text{ such that } \limsup \frac{\lambda z y \Pi(y)dy}{z \Pi(y)dy} \leq c \lambda^{2-\alpha} \text{ for } \lambda > 1, \text{ as } z \to \infty, \text{ respectively as } z \to 0+;\]

see [6] for a proof of this equivalence and background on the study of the Feller class for general Lévy processes. In this case we will say that the condition $SC_\infty$, respectively $SC_0$, holds.

For subordinators, there is a pioneering work by Jain and Pruitt [5], which is one of the main sources of this research, and where estimates for $P(T_x > t) = P(X_t \leq x)$ are obtained. Their main result will be recalled later, but first we quote some facts from the work by Maller and Mason in [4] and [3].

In the case where $X$ is stochastically compact at infinity (respectively at zero), Maller and Mason proved that there exist functions $c : [0, \infty) \rightarrow (0, \infty)$ and $b : [0, \infty) \rightarrow [0, \infty)$ such that for any sequence $(t_k, k \geq 0)$ tending towards infinity (respectively; towards 0) there is a subsequence $(t'_{k}, k \geq 0)$ such that
\[
\frac{X_t - b(t')}{{c(t')} \rightarrow Y',} (5)
\]
where $Y'$ is a real valued non-degenerate random variable, whose law may depend on the subsequence taken. A standard representation of the functions $c$ and $b$ are
\[
tQ_{\Pi}(c(t)) = 1, \quad b(t) = t \left( b + \int_0^t c(t) y \Pi(dy) \right), \quad t > 0. \quad (6)
\]

If in addition to the condition $SC_\infty$ (respectively $SC_0$) the condition
\[
\limsup_{y \to \infty} \frac{y(b + \int_0^y z \Pi(dz))}{y^2 \int_0^y z^2 \Pi(dz)} < \infty, \quad (7)
\]
holds, then the above defined functions satisfy
\[
\limsup_{t \to \infty} \frac{b(t)}{{c(t)} < \infty,} \quad (8)
\]
so that the normalizing function $b$ is not needed and hence can be assumed to be 0. In this case it is said that the process $X$ is stochastically compact at zero (respectively at infinity) without centering. In all other cases,

$$\limsup_{t \to \infty} \frac{b(t)}{c(t)} = \infty.$$  \hspace{1cm} (9)$$

Throughout the paper we will work in one of the following frameworks on $\Pi$, $t$ and $x$: always $b < x_t := x/t < \mu$ and

$(SC_0-I)$ the Lévy measure $\Pi$ satisfies the condition $SC_0$, $t \to \infty$, $x_t \to b$;

$(SC_0-II)$ the Lévy measure $\Pi$ satisfies the condition $SC_0$, $t \to 0$, $x_t \to b$, and

$$\frac{x - b(t)}{c(t)} \to -\infty \quad \text{as} \quad t \to 0$$

if (9) fails; or

$$\frac{x - bt}{c(t)} \to 0 \quad \text{as} \quad t \to 0$$

if (9) holds.

$(SC_\infty-I)$ the Lévy measure $\Pi$ satisfies the condition $SC_\infty$, $t \to \infty$, $x_t \to \mu$ and

$$\frac{x - b(t)}{c(t)} \to -\infty \quad \text{as} \quad t \to \infty$$

if (9) fails; or

$$\frac{x - bt}{c(t)} \to 0 \quad \text{as} \quad t \to \infty$$

if (9) holds.

$(G)$ $t \to \infty$ and $b < \liminf_{t \to \infty} \frac{x_t}{t} \leq \limsup_{t \to \infty} \frac{x_t}{t} < \mu$, and $X$ is non-lattice.

We start by providing some local estimates of the distribution of $X$. The following two results are an improvement of the main result by Jain and Pruitt for subordinators in the sense that we recover the precise estimate for $P(X_t \leq x)$ as $t \to \infty$, obtained in [5], but we also prove that the estimate is uniform in $x$, and furthermore we provide a precise estimate for $P(X_t \in (x - u, x])$ which holds uniformly in $x$ and $u$. The technique we use is also different to that of Jain and Pruitt [5], though both techniques involve normal approximations.

Throughout this note $\phi : \mathbb{R} \to \mathbb{R}^+$, will denote the standard normal density.

**Theorem 1** Suppose that $X$ is a subordinator which has drift $b \geq 0$ and Lévy measure $\Pi$. For $b < x/t < \mu := E(X_1)$, define $x_t := x/t$ and $\rho_t := \rho(x/t)$, that is $\psi'(\rho_t) = x/t$, $H(u) = \psi(u) - u\psi'(u)$, and $\sigma^2(u) = \int_0^\infty y^2 e^{-uy}\Pi(dy)$.

(i) If $X$ is stochastically compact at 0, the unidimensional law of $X$ admits a density, say $P(X_t \in dy) = f_t(y)dy$, $y \geq 0$, and such that $f_t \in C^\infty(\mathbb{R})$. 


(ii) In the settings \((SC_0\text{-}(I-II))\) we have the estimate
\[ \sqrt{\sigma(\rho_t)} f_t(z) = \phi((z - x)/\sqrt{\sigma(\rho_t)}) + o(1), \]
uniformly in \(z > 0 \) and \(x\).

**Theorem 2** In the settings \((SC_0\text{-}(I-II)), (SC_\infty)\) and \((G)\) the following estimates
\[
P(X_t \in (x - u, x]) = e^{-tH(\rho_t)} \left( \int_0^u e^{-\rho_tv} \phi \left( \frac{v}{\sqrt{\sigma(\rho_t)}} \right) dv + o(1) \right),
\]
\[P(X_t \leq x) \sim e^{-tH(\rho_t)} \sqrt{2\pi t \sigma(\rho_t)} \rho_t^2,\]
hold uniformly in \(u < x\) and uniformly in \(x\).

From this, we deduce corresponding results for the passage time. Here
\[h_J^x(t) = \int_0^x P(X_t \in dy) \Pi(x - y), \quad t > 0,\]
denotes the density function of the first passage time on the event \(X_T > x\), see \([3]\) Lemma 1, and
\[h_C^x(t, \Delta) = \int_t^{t+\Delta} P(T_x \in [t, t+\Delta], X_T = x) dt, \quad t > 0,\]

**Theorem 3** Let \(\Delta_0 > 0\) fixed. In the settings \((SC_0\text{-}(I-II)), (SC_\infty)\) and \((G)\), the following estimates
\[h_J^x(t) \sim \frac{e^{-tH(\rho_t)}}{\sqrt{2\pi t \sigma(\rho_t)}} \phi(1) \rho_t^2, \]
\[h_C^x(t, \Delta) \sim b \int_t^{t+\Delta} \frac{e^{-sH(\rho_s)}}{\sqrt{2\pi s \sigma(\rho_s)}} ds, \]
hold uniformly in \(0 < \Delta < \Delta_0\) and in \(x\). Furthermore, under the settings \((SC_0\text{-}(I-II))\) the more precise estimate
\[P(T_x \in dt, X_T = x) = b \left( \frac{e^{-tH(\rho_t)}}{\sqrt{2\pi t \sigma(\rho_t)}} + o(1) \right) dt,\]
hold uniformly in \(x\).

When specialised to the case that \(\Pi\) is regularly varying, at infinity or zero, this gives the following.

**Corollary 4** Let \(\Delta_0 > 0\) fixed. Then the following estimates,
\[h_J^x(t) \sim \frac{e^{-tH(\rho_t)}}{\sqrt{2\pi t \sigma(\rho_t)}} \Pi(1/\rho_t), \]
\[h_C^x(t, \Delta) \sim \frac{b\Delta e^{-tH(\rho_t)}}{\sqrt{2\pi t \sigma(\rho_t)}}, \]
hold uniformly in \(0 < \Delta < \Delta_0\) and in \(x\) such that either of the following conditions hold.
(i) \( t \to \infty, \frac{t}{x} \to b, \) and \( \Pi(c(t)) \in RV(-\alpha) \) at 0 with \( \alpha \in (0, 1) \)

(ii) \( t \to \infty, x/t \to \infty \) and \( x/c(t) \to 0 \), when \( \Pi(c(t)) \in RV(-\alpha) \) at \( \infty \) with \( \alpha \in (0, 1) \) and the function \( c \) is determined by the relation \( \Pi(c(t)) = 1, t > 0 \).

Remark 5 If (i) holds and \( b > 0 \), we see that \( h^2_x(t, \Delta)/h^C_x(t, \Delta) \to 0 \), but note in this scenario \( X_t \) is not in the domain of attraction of an \( \alpha \)-stable subordinator without centering as \( t \to 0 \). If (ii) holds, and \( b > 0 \), \( X_t \) is in the domain of attraction of an \( \alpha \)-stable subordinator without centering as \( t \to \infty \), and this ratio \( \to \infty \). In this situation, our forthcoming final result shows that it is possible for polynomial, rather than exponential decay to occur, but again this ratio \( \to \infty \).

Proposition 6 Suppose now that \( X \) is a non-lattice subordinator which has drift \( b \geq 0 \) and \( \Pi(c(t)) \in RV(-\alpha) \) at \( \infty \) with \( \alpha \in (0, 1) \). Define \( c \) by \( \Pi(c(t)) = 1 \), so that \( \{X(t)/c(t)\} \) converges weakly to \( S \), a stable subordinator of index \( \alpha \). Let \( \tilde{g}_t(\cdot) \) and \( h_x(\cdot) \) denote the density functions of \( S_t \) and \( T_x^S := \inf\{t : S_t > x\} \) respectively. Then for any fixed \( D > 1 \), uniformly for \( y_t := x/c(t) \in (D^{-1}, D) \)
\[
\tilde{h}^2_x(t) = \tilde{h}_{y_t}(1) + o(1) \quad \text{as} \quad t \to \infty, \quad (17)
\]
and, if \( b > 0 \), uniformly for \( y_t \in (D^{-1}, D) \) and \( 0 < \Delta < \Delta_0 \),
\[
c(t)h^C_x(t, \Delta) = \frac{ab\Delta}{1-\alpha}(\tilde{g}_t(y_t) + o(1)) \quad \text{as} \quad t \to \infty. \quad (18)
\]

2 Preliminaries

Most of our calculations involve an exponential change of measure, which we introduce now. For \( \psi'(\infty) = b < \frac{1}{\Delta} := x_t < \mu = \psi'(0+) \) we denote by \( (Y_s, s \geq 0) \), a subordinator whose Laplace exponent is given by \( \psi_{\mu} \),
\[
\psi_{\mu}(\lambda) = \psi(\mu + \lambda) - \psi(\mu) = b\lambda + \int_{(0, \infty)} (1 - e^{-\lambda y}) e^{-\mu y} \Pi(dy), \quad \lambda \geq 0. \quad (19)
\]
In particular we have the following relation:
\[
P(Y_t \in dy) = e^{\Delta H(\mu)} e^{-\mu(y - xt)} P(X_t \in dy), \quad y \in \mathbb{R}^+. \quad (20)
\]
Observe that in the above definition of \( Y \) we are deliberately excluding the dependence in \( x_t \) of \( Y \). We do this for notational convenience and also because we will mainly use the equality of measures in \( (20) \).

The proof of our main results rely on the following technical results. The first of them is a consequence of Lemma 1, P109, of Petrov.

Lemma 7 Let \( Z_1, Z_2, \ldots, Z_n \) be independent rvs having finite 3rd moments, write \( EZ_r = \mu_r, VarZ_r = \sigma_r^2, \) and \( E|Z_r - \mu_r|^3 = \nu_r, \) and put \( W = \sum^n_i Z_r, m = EW = \sum^n_i \mu_r, \) and \( s^2 = VarW = \sum^n_i \sigma_r^2. \) Assume further that \( \int_{-\infty}^\infty |\Psi(u)| du < \infty \), where \( \Psi(u) = Ee^{iuW}, \) and denote by \( f \) and
\( \phi \) the pdf of \( W \) and the standard Normal pdf. Then there is an absolute constant \( A \) such that
\[
\sup_y |f(y) - s^{-1} \phi(\frac{y-m}{s})| \leq AL + d, \tag{21}
\]
where \( L = \sum_i^N \nu_i/s^4 \) and, with \( l = (ALs^2)^{-1} \),
\[
d = 2 \int_{l}^{\infty} |\Psi(u)|du.
\]

**Proof.** Essentially same as Lemma 3 of [2]. ■

**Remark 8** Our use of this result exploits the fact that, for any Lévy process, any \( t > 0 \), and any \( n \geq 1 \), \( X_t \) is the sum of \( n \) independent and identically distributed summands.

The second relates the various quantities we will consider.

**Lemma 9** We have the following relations

(a) \( \frac{1}{2e} Q(1/u) \leq H(u) \leq Q(1/u) \), for \( u > 0 \),
(b) \( u^2 \sigma^2(u) \leq 2H(u) \) for \( u \geq 0 \).
(c) \( \frac{u^2 \sigma^2(u)}{H(u)} \geq \frac{e^{-1}}{1 + \frac{\pi^2(1/u)}{\gamma(1/u)}} \), for \( u > 0 \). In particular, if \( X \) is stochastically compact at infinity, respectively at 0, then
\[
\liminf u^2 \sigma^2(u)/H(u) > 0,
\]
as \( u \to 0 \), respectively as \( u \to \infty \).

**Proof.** Just proceed as in Lemma 5.1 in [5]. ■

The proof of Theorem 2 relies on the following proposition

**Proposition 10** In the settings \((SC_0-I-II)\), \((SC_\infty)\) and \((G)\), the estimate
\[
\frac{1}{2h} P(Y_t \in (y-h, y+h)) = \frac{e^{H(\rho_t)}}{2h} \int_{y-h}^{y+h} e^{\rho_t(x-z)} P(X_t \in dz)
\]
\[
= \frac{1}{\sqrt{t \sigma(\rho_t)}} \left\{ \phi \left( \frac{x-y}{\sqrt{t \sigma(\rho_t)}} \right) + o(1) \right\}. \tag{22}
\]
holds uniformly in \( 0 < h \leq h_0, y > h, \) and \( x \).

The proof of Theorem 1 and Proposition 10 uses among other things the following Lemma.

**Lemma 11** For \( t > 0, b < x_t < \mu \), we have for any \( s > 0 \)
\[
E(Y_s) = sx_t =: \mu_s, \tag{23}
\]
\[
E(Y_s - \mu_s)^2 = s \int_0^\infty y^2 e^{-\rho_t y} \Pi(dy) = s \sigma^2(\rho_t), \tag{24}
\]
\[
E(Y_s - \mu_s)^3 = s \int_0^\infty y^3 e^{-\rho_t y} \Pi(dy), \tag{25}
\]
\[
E|Y_s - \mu_s|^3 \leq 6s(\rho_t)^{-3} Q(1/\rho_t) + 2\mu_s s \sigma^2(\rho_t). \tag{26}
\]
Proof. The first three identities are proved by bare hands calculations, while the claimed upper bound is obtained as follows
\[
E|Y_s - \mu_s|^3 = E(Y_s - \mu_s)^3 + 2E ((\mu_s - Y_s)^3 : Y_s \leq \mu_s) \\
\leq E(Y_s - \mu_s)^3 + 2\mu_s E ((\mu_s - Y_s)^2 : Y_s \leq \mu_s) \\
\leq s \int_0^\infty y^3 e^{-\rho t y} \Pi(dy) + 2\mu_s \sigma^2(\rho_t) \\
= s \int_{\{\rho_t \leq 1\}} y^3 e^{-\rho t y} \Pi(dy) + s \int_{\{\rho_t > 1\}} y^3 e^{-\rho t y} \Pi(dy) + 2\mu_s \sigma^2(\rho_t) \\
\leq s(\rho_t)^{-1} \int_{\{\rho_t \leq 1\}} y^2 \Pi(dy) + 6s(\rho_t)^{-3} \Pi(1/\rho_t) + 2\mu_s \sigma^2(\rho_t) \\
\leq 6s(\rho_t)^{-3} \Pi(1/\rho_t) + 2\mu_s \sigma^2(\rho_t)
\]

Lemma 12 In the settings \((SC_0-(I-II))\), \((SC_\infty)\) and \((G)\), we have that \(tH(\rho_t) \to \infty\) uniformly in \(x\).

Proof. The proof of the case \((G)\) is a straightforward consequence of the fact that in this setting \(t \to \infty\) and
\[
0 < \liminf_{t \to \infty} H(\rho_t) \leq \limsup_{t \to \infty} H(\rho_t) < \infty,
\]
as can be seen below, in the proof of Proposition \([12]\) under the present assumptions. To deal with the cases \((SC_0-(I-II))\), \((SC_\infty)\) we use the Theorem 5.1 of Jain and Pruitt \([5]\) which establishes that the condition \(tH(\rho_t) \to \infty\) is equivalent to \(P(X_t \leq x) \to 0\), as \(t \to \infty\) or 0. For the case \((SC_0-I)\) when \([7]\) fails, the equality
\[
P(X_t \leq x) = P \left( \frac{X_t}{t} - b \leq \frac{(x_t - b)}{c(t)} \right),
\]
and an application of the weak law of large numbers for subordinators gives the result. To deal with the case \((SC_0-II)\), we use the equality
\[
P(X_t \leq x) = P \left( \frac{X_t - b(t)}{c(t)} \leq \frac{x - b(t)}{c(t)} \right),
\]
which together with the sequential convergence in \([5]\) and the assumption that \(\frac{x - b(t)}{c(t)} \to -\infty\) as \(t \to 0\), lead to \(P(X_t \leq x) \to 0\) as \(t \to 0\). The case when \([7]\) holds as well as the cases \((SC_\infty)\) are proved with a similar argument. To show the uniformity observe that the function
\[
\lambda \mapsto H(\lambda) = \psi(\lambda) - \lambda \psi'(\lambda) = \int_{(0,\infty)} (1 - e^{-\lambda x} - \lambda xe^{-\lambda x}) \Pi(dx),
\]
is increasing because the function \(z \mapsto 1 - e^{-z} - ze^{-z}\) is so. This implies that the function \(\lambda \mapsto H(\rho(\lambda))\) is decreasing. The uniformity in the cases \((G)\) and \((SC_0-(I-II))\) follows easily from this fact. Indeed, it is enough to observe that \(tH(\rho_t)\) tends towards \(\infty\) as soon as we take a \(x_0\) such that \(x_0 > x\) and \(tH(\rho(\frac{c(x)}{x})) \to \infty\). To establish the uniformity in the case...
(SC$_{\infty} - I$) when (7) holds, we observe that the hypotheses imply that there is a function $D$ such that $x \leq b(t) - D(t)$, and $D(t)/c(t) \to \infty$ as $t \to \infty$. The function $D$ is such that

$$tH(\rho t) \geq tH\left(\rho \left(\frac{b(t) - D(t)}{t}\right)\right) \overset{t \to \infty}{\longrightarrow} \infty,$$

because by the assumption of stochastic compactness at $\infty$ we have that

$$P(X_t < b(t) - D(t)) = P\left(\frac{X_t - b(t)}{c(t)} \leq -\frac{D(t)}{c(t)}\right) \overset{t \to \infty}{\longrightarrow} 0.$$

In the case (SC$_{\infty} - I$) when (7) does not hold we proceed as above but using that there is a function $j$ such that $x \leq bt + j(t)$ and $j(t)/c(t) \to 0$ as $t \to \infty$.

We have all the elements to prove Theorem 1.

2.1 Proof of (i) in Theorem 1

Let $U(x) = \int_0^x y\Pi(y)dy$, $x \geq 0$, and

$$\hat{U}(s) = \int_0^\infty e^{-sy}U(dy) = s\int_0^\infty e^{-sy}U(y)dy, \quad s > 0,$$

be its Laplace transform. By hypothesis we have that the condition (SC'$_0$) is satisfied, which implies that $U$ has bounded increase at 0, see [1] page 68. So by Theorem 2.10.2 in [1], actually from its proof, we know that there are constants $0 < c_1 \leq c_2 < \infty$ such that for small $s$, $c_1 U(s) \leq \hat{U}(1/s) \leq c_2 U(s)$. From this it follows that $\hat{U}$ has bounded decrease at infinity. Indeed, taking $\lambda$ as in (SC'$_0$) we have for $\lambda > 1$

$$\liminf_{s \to \infty} \frac{\hat{U}(s\lambda)}{U(s)} \geq c_3 \liminf_{s \to \infty} \frac{\hat{U}(\frac{1}{s})}{U(\frac{1}{s})} = c_3 \left(\limsup_{v \to 0^+} \frac{U(\lambda v)}{U(v)}\right)^{-1} \geq c_3 \frac{1}{\lambda^{2 - \alpha}}.$$

Proposition 2.2.1 in [1] implies that for any $\beta < -(2 - \alpha)$ there exist constants $c_4 > 0$ and $\bar{\ell} > 0$, such

$$\frac{\hat{U}(y)}{U(x)} \geq c_4 \left(\frac{y}{x}\right)^\beta, \quad y \geq x \geq \bar{\ell}. \quad (27)$$

Also, an easy integration by parts implies the identity

$$\hat{U}(s) = -s\left(\frac{\psi(s)}{s}\right)' = \frac{H(s)}{s^2}, \quad s > 0. \quad (28)$$

So by (27) we have

$$\frac{H(y)}{H(x)} \geq c_4 \left(\frac{y}{x}\right)^{2 + \beta}, \quad y \geq x \geq \bar{\ell}, \quad (29)$$

where $2 + \beta < \alpha \leq 2$. Since $0 < \alpha$ there exists $\beta_0$ and positive constants $c_5$ and $\bar{\ell}$ such that $0 < 2 + \beta_0 < \alpha \leq 2$ and

$$H(y) \geq y^{2 + \beta_0} c_5, \quad y \geq \bar{\ell}.$$
To conclude we observe that the following inequalities hold
\[
\int_0^\infty (1 - \cos(\theta y))\Pi(dy) \geq c_7 \theta^2 \int_0^{1/\theta} y^2 \Pi(dy) \\
= c_7 K_{11}(1/\theta) = c_7 \frac{K_{11}(1/\theta)}{Q_{11}(1/\theta)} Q_{11}(1/\theta) \\
\geq c_9 Q_{11}(1/\theta) \\
= 2c_8 \theta^2 U(1/\theta) \\
\geq c_9 H(\theta)
\]
for \(\theta\) large enough; here we used the assumption \((SC_0)\) and the equality \((I)\). We infer that for \(\theta > 0\) large enough
\[
\Re(\psi(i\theta)) = \int_0^\infty (1 - \cos(\theta y))\Pi(dy) \geq c_{10} \theta^{\rho_0 + 2}.
\]
As a consequence, for \(n \geq 0\)
\[
\int_\mathbb{R} |\theta|^n |E(e^{i\theta X_t})|d\theta = \int_\mathbb{R} |\theta|^n \exp\{-t\Re(\psi(i\theta))\}d\theta < \infty,
\]
and the conclusion follows from standard results.

2.2 Proof of (ii) in Theorem 1 and of Proposition 10 in the cases \((SC_0-\text{I-II})\)

Given that the result in (ii) in Theorem 1 is more precise than the one in Proposition 10 it will be enough to prove the former.

We observe first that the assumption that \(b < x_t < \mu\) and \(x_t \to b\) implies that \(\rho_t \to \infty\), irrespective of whether \(t \to 0\) or \(t \to \infty\). We next establish that these conditions on \(x_t\), the fact that \(tH(\rho_t) \to \infty\), and the stochastic compactness at 0, imply that \(x_{\rho_t} \to \infty\), again irrespective of whether \(t \to 0\) or \(t \to \infty\). Indeed, the identities
\[
\frac{tH(\rho_t)}{x_{\rho_t}} = \frac{t\psi(\rho_t) - \rho_t \psi'(\rho_t)}{\rho_t \psi''(\rho_t)} = \frac{\psi(\rho_t)}{\rho_t \psi''(\rho_t)} - 1,
\]
allow us to ensure that it is enough to justify that \(0 < \liminf_{z \to \infty} \frac{z\psi'(z)}{\psi(z)}\).
If the drift of \(X\) is positive this is straightforward. If the drift is zero this holds whenever \(\limsup_{z \to 0} \frac{z^2 \Pi(z)}{\int_0^z y\Pi(dy)} < \infty\), which in turn holds by stochastic compactness at zero,
\[
\lim_{z \to 0} \frac{z^2 \Pi(z)}{\int_0^z y\Pi(dy)} \leq \limsup_{z \to 0} \frac{z^2 \Pi(z)}{\int_0^z y\Pi(dy)} < \infty.
\]
The former claim is an easy consequence of the following inequalities
\[
\frac{\lambda \psi'(\lambda)}{\psi(\lambda)} = \frac{\int_0^\infty ye^{-\lambda y}\Pi(dy)}{\int_0^\infty e^{-\lambda y}\Pi(y)dy} \\
\geq \frac{\int_0^{1/\lambda} ye^{-\lambda y}\Pi(dy)}{\int_0^{1/\lambda} \Pi(y)dy + (1/\lambda)\Pi(1/\lambda)} \\
= e^{-1} \frac{\int_0^{1/\lambda} y\Pi(dy) + (2/\lambda)\Pi(1/\lambda)}{\int_0^{1/\lambda} \Pi(dy) + (2/\lambda)\Pi(1/\lambda)}
\]
which are obtained by barehand calculations. It is important to remark that the above facts and the Lemma 12 imply that \( x_\rho t \to \infty \) uniformly in \( x \). Furthermore, our previous remarks allow us to provide a unified proof of the cases \( t \to 0 \) or \( t \to \infty \).

We will apply Lemma 7 with \( n = \lfloor x_\rho t \rfloor \) and \( W = Y_t = \sum_{k=1}^n Z_k \) with \( Z_k = \frac{Y_{t,k} - Y_{t,(k-1)}}{n} \), for \( k \in \{1, \ldots, n\} \). We use the estimate (23) with \( s = t/n \), thus \( \mu = x/n \), which together with our choice of \( n \) lead to the approximation

\[
n\nu := nE|Z_1 - x/n|^3 \leq t \left\{ 6(\rho_t)^{-3}Qn(1/\rho_t) + 2\frac{x}{n}\sigma^2(\rho_t) \right\} \approx t \left\{ 6(\rho_t)^{-3}Qn(1/\rho_t) + 2\frac{\rho_t^2\sigma^2(\rho_t)}{\rho_t^3} \right\},
\]

for \( \rho_t \) large enough. It is then immediate from the definition of \( L \) that for \( \rho_t \) large enough

\[
\sqrt{t}\sigma(\rho_t)L = \frac{n\nu}{\{t\sigma^2(\rho_t)\}^{1/2}} \lesssim \frac{1}{(t\rho_t^2\sigma^2(\rho_t))^{1/2}} \left( \frac{6Qn(1/\rho_t)}{\rho_t^2\sigma^2(\rho_t)} + 2 \right);
\]

which because of the assumption of stochastic compactness at 0 and Lemma 9 is

\[
\sqrt{t}\sigma(\rho_t)L \lesssim \frac{k_1}{\sqrt{tH(\rho_t)}}.
\]

So the lemma tells us that (10) holds provided that

\[
\gamma := \sqrt{t}\sigma(\rho_t)\int_{1}^{\infty} e^{-t\Re(\psi_{\rho_t}(i\theta))}d\theta \to 0.
\]

To prove that this is indeed the case, observe the above estimate for \( L \) gives

\[
\ell = \frac{1}{4L^2} = \frac{t\sigma^2(\rho_t)}{4n\nu} \gtrsim k_2 \left( \frac{t\rho_t^2\sigma^2(\rho_t)}{(t\sigma^2(\rho_t))^{1/2}} \right)^{1/2} = k_2\rho_t,
\]

for \( \rho_t \) large enough. Applying the inequality (30) we obtain that for \( \theta \geq \ell \geq k_3\rho_t \)

\[
\Re(\psi_{\rho_t}(i\theta)) = \int_{0}^{\infty} (1 - \cos(\theta y))e^{-\rho_{\psi_t}(y)}\Pi(dy) \geq k_3 e^{-1/k_4} \int_{0}^{1/\theta} y^2\Pi(dy) \geq k_4 H(\theta).
\]

It follows from the above and the estimate (29) that for any \( 0 < \alpha_0 < \)
\( \alpha \leq 2 \), with \( \alpha \) as in \((SC_0)\), and for \( \rho_t \) large enough

\[
\sqrt{t} \sigma(\rho_t) \int_{\ell}^{\infty} e^{-tH(\rho_t)(\theta)} \, d\theta \leq \sqrt{\mathcal{I}}(\rho_t) \int_{\ell}^{\infty} \exp \left\{ -k_5 tH(\rho_t) \left( \frac{\theta}{\rho_t} \right)^{a_0} \right\} \, d\theta
\]

\[
\leq k_6 \sqrt{t} \rho_t \sigma(\rho_t) \int_{k_2}^{\infty} \exp \left\{ -k_4 tH(\rho_t) \theta^{a_0} \right\} \, d\theta
\]

\[
\leq k_7 \frac{\sqrt{t} \rho_t \sigma(\rho_t)}{(tH(\rho_t))^{1/\alpha_0}}
\]

\[
\leq k_8 \frac{1}{(tH(\rho_t))^{1/\alpha_0 - 1/2}} \to 0,
\]

where in the last inequality we used Lemma 7. Observe that the uniformity follows from Lemma 12 and the fact that \( \rho_t \) tends to infinity uniformly as well because it is non-increasing.

### 2.3 Proof of Proposition 10 in the case \((SC_{\infty})\)

We will apply the Lemma 7 to \( W = Y_t + U_h + \Delta_a = \sum_1^n Z_r \), where \( n = \lfloor x \rho_t \rfloor + 2 \), \( Z_r := Y_{\lfloor x \rho_t \rfloor} - Y_{\lfloor x \rho_t - 1 \rfloor} \) for \( 1 \leq r < n - 1 \), and the two independent random variables \( U_h \) and \( \Delta_a \) are independent of \( Y_t \), \( U_h \) has a uniform distribution on \([-h, h]\), and \( \Delta_a \) has \( EW = t\sigma^2(\rho_t) + c_1 h^2 + c_2 a^2 \). Also in virtue of the identity

\[
E(e^{i\theta W}) = E(e^{i\theta Y_t}) \cdot \frac{\sin \theta h}{\theta h} \cdot \left( \frac{\sin \theta a}{\theta a} \right)^2
\]

we see that \( |E(e^{i\theta W})| \) is in \( L_1 \), so that \( W \) has a density, \( n_t(\cdot) \).

Observe that the arguments in the above proof apply equally well to establish that \( x \rho_t \to \infty \), uniformly in \( x \), even though \( \rho_t \to 0 \), because in the present setting we assume stochastic compactness at infinity. Arguing exactly as in the proof of the case \((SC_0)\) we deduce that for \( 0 < h < h_0 \), \( 0 < a < a_0 \), and \( \rho_t \) small enough

\[
\sqrt{t} \sigma(\rho_t) L \leq \sum_{1}^{n-2} E[Y_1 - \mu_1]^3 + c_3 h^3 + c_4 a^3
\]

\[
(\tau \sigma^2(\rho_t) + c_1 h^2 + c_2 a^2)^{1/2}
\]

\[
\leq k_1 \frac{c_3 h_0 + c_4 a_0}{(\tau \sigma^2(\rho_t))^{1/2}}
\]

\[
= k_1 \frac{c_3 h_0 + c_4 a_0}{(\tau \sigma^2(\rho_t))^{1/2}}
\]

\[
\leq k_2 \frac{c_3 h_0 + c_4 a_0}{(\tau \sigma^2(\rho_t))^{1/2}}
\]

where in the last inequality we used Lemma 9 and that \( \rho_t \to 0 \). So the lemma tells us that

\[
\sqrt{t} \sigma(\rho_t) n_t(z) = \phi((z - x)/\sqrt{t} \sigma(\rho_t)) + o(1)
\]
provided
\[ \gamma := \sqrt{t} \sigma(\rho_t) \int_1^\infty e^{-\tau R(\psi_{\rho_t}(i\theta))} d\theta \to 0. \]

Observe that as in the proof of the case \((SC_0)\) we have that there is a constant \(k_3\) such that \(t \geq k_3 \rho_t\) for \(\rho_t\) small enough.

The hypothesis of stochastic compactness at infinity \((SC_\infty)\), and Proposition 2.2.1 in [1] imply that for any \(\alpha_0 \in (2 - \alpha, 2)\) there are constants \(k_4\) and \(k_5\) such that
\[ \int_0^u z \frac{1}{\Pi(z)} dz = k_4 \left( \frac{u}{v} \right)^{\alpha_0}, \quad u \geq v \geq k_5, \]
and thus
\[ \frac{Q_{\Pi}(u)}{Q_{\Pi}(v)} \leq k_4 \left( \frac{u}{v} \right)^{\alpha_0 - 2}, \quad u \geq v \geq k_5. \quad (35) \]

We fix \(\alpha_0 \in (2 - \alpha, 2)\), take \(\overline{\rho} > \sup_{(1, 1)} \rho_t\), and choose \(v_0 > 1\) such that \(k_5 \overline{\rho} \left( \frac{1}{v_0} \right) < v_0\). Now, making a change of variables we bound \(\gamma\) as follows
\[ \sqrt{t} \sigma(\rho_t) \int_1^\infty e^{-\tau R(\psi_{\rho_t}(i\theta))} \left| \frac{\sin \theta h}{\theta h} \right| \left( \frac{\sin \theta a}{\theta a} \right)^2 d\theta \]
\[ = \sqrt{t} \sigma(\rho_t) \rho_t \int_{1/\rho_t}^\infty e^{-\tau R(\psi_{\rho_t}(i\theta \rho_t))} \left| \frac{\sin \rho_t \theta h}{\rho_t \theta h} \right| \left( \frac{\sin \rho_t \theta a}{\rho_t \theta a} \right)^2 d\theta \]
\[ \leq \sqrt{t} \sigma(\rho_t) \rho_t \int_{1/\rho_t}^{(v_0/k_5)^{-1}} e^{-\tau R(\psi_{\rho_t}(i\theta \rho_t))} \left| \frac{\sin \rho_t \theta h}{\rho_t \theta h} \right| \left( \frac{\sin \rho_t \theta a}{\rho_t \theta a} \right)^2 d\theta \]
\[ + \sqrt{t} \sigma(\rho_t) \rho_t \int_{(1/\rho_t, k_5^{-1})}^{v_0^{-1}} e^{-\tau R(\psi_{\rho_t}(i\theta \rho_t))} \left| \frac{\sin \rho_t \theta h}{\rho_t \theta h} \right| \left( \frac{\sin \rho_t \theta a}{\rho_t \theta a} \right)^2 d\theta \equiv: I_1 + I_2. \]

To describe the behaviour of \(I_1\) we start by lower bounding the exponent of the integrand as follows. For \(\theta \in ((v_0)^{-1}, (L_{\alpha_0} \rho_t)^{-1})\), or equivalently \(L_{\alpha_0} < (\theta \rho_t)^{-1} < v_0/\rho_t\)
\[ \Re \psi_{\rho_t}(i\theta \rho_t) = \int_0^{v_0^{-1}} (1 - \cos(\theta \rho_t y)) e^{-\rho_t y} \Pi(dy) \]
\[ \geq k_6 e^{-1/\theta} K_{\Pi}(1/(\theta \rho_t)) \]
\[ \geq k_6 e^{-v_0} \inf_{u \geq 1/(v_0 \rho_t)} \left\{ \frac{K_{\Pi}(u)}{Q_{\Pi}(u)} \right\} \frac{Q_{\Pi}(1/(\theta \rho_t))}{Q_{\Pi}(v_0/\rho_t)} Q_{\Pi}(v_0/\rho_t) \]
\[ \geq k_7 \inf_{u \geq 1/(v_0 \rho_t)} \left\{ \frac{K_{\Pi}(u)}{Q_{\Pi}(u)} \right\} \left( \theta v_0 \right)^{-2\alpha_0} Q_{\Pi}(v_0/\rho_t), \]
where in the last inequality we used (35). The later together with the inequality
\[ v_0^2 Q_{\Pi}(v_0/\rho_t) = \rho_t^2 \int_0^{v_0/\rho_t} y \Pi(y) dy \geq \rho_t^2 \int_0^{1/\rho_t} y \Pi(y) dy = Q_{\Pi}(1/\rho_t) \geq H(\rho_t), \]

imply
\[ \Re \psi_{\rho_t}(i\theta \rho_t) \geq k_8 \theta^{2 - \alpha_0} (v_0)^{-\alpha_0} H(\rho_t), \]
for \( t \) large enough, uniformly in \( x \). Applying this in \( I_1 \) and the results from Lemma 9 we obtain

\[
I_1 \leq \sqrt{\sigma(\rho_1)} \rho_1 \int_{1/\rho_1}^{1/\rho_1 k_4} \exp\{-k_9 \theta^{2-\alpha_0}(v_0)^{-\alpha_0} t H(\rho_1)\} d\theta \\
\leq \sqrt{2 t H(\rho_1)} \int_{1/\rho_1}^{1/\rho_1 k_4} \exp\{-k_9 \theta^{2-\alpha_0} t H(\rho_1)\} d\theta,
\]

where \( k_9 = k_8 v_0^{-\alpha} \). Recall that \( t H(\rho_1) \to \infty \), so that putting \( \theta^{2-\alpha_0} t H(\rho_1) = z^{2-\alpha_0} \) gives

\[
I_1 \leq \left( \sqrt{2 t H(\rho_1)} \right)^{1/(2-\alpha_0)} \int_{(t H(\rho_1)/v_0)^{2-\alpha_0}}^{\infty} \exp\{-k_9 z^{2-\alpha_0}\} dz \to 0.
\]

We next deal with the term \( I_2 \). Proceeding as above we easily get that for \( \theta > 1/\rho_1 k_5 \)

\[
\Re \psi_{\rho_1}(i\theta \rho_1) \geq k_{10} e^{-v_0 Q_\Pi(1/\theta \rho_1)} \geq k_{10} e^{-v_0 Q_\Pi(L_{\alpha_0})} := k_{11}.
\]

Applying this estimate to \( I_2 \) we get

\[
I_2 \leq \left( \sqrt{\sigma(\rho_1)} \rho_1 \right) \int_{1/\rho_1 k_5}^{\infty} e^{-k_{11} t} \frac{d\theta}{\theta} \\
\leq \frac{\sqrt{\sigma(\rho_1)} \rho_1 e^{-k_{11} t} k_5}{a^2 \rho_1^2} \leq k_{12} \sqrt{t H(\rho_1)} e^{-k_{11} t} \cdot (36)
\]

The argument is concluded by using the fact that \( t H(\rho_1) \to \infty \) to deduce from (35) that for all large enough \( t \)

\[
t \rho_1^{2-\alpha_0} \geq k_{13} Q_\Pi(1/\rho_1) \geq k_{13} t H(\rho_1) \geq k_{13} > 0,
\]

that is

\[
\rho_t \geq k_{14} t^{-1/(2-\alpha_0)}, \quad \text{for } t \text{ large enough.}
\]

We infer therefore that

\[
I_2 \leq a^{-2} k_{15} t^{1/(2-\alpha_0)} e^{-k_{11} t} = o(1),
\]

as \( t \to \infty \).

We have completed the proof that \( n_t \), the pdf of \( Y_1 + U_h + \Delta_x \), satisfies (34), uniformly for \( 0 < h < h_0 \) and \( \varepsilon < a < \alpha_0 \). However, we also have

\[
n_t(z) = (2h)^{-1} P(Y_1 + \Delta_a \in (z-h, z+h)),
\]

so it follows by choosing \( a \) small, that uniformly for \( \varepsilon' < h < h_0 \), for any \( h > 0 \),

\[
\sqrt{\sigma(\rho_t)}(2h)^{-1} P(Y_1 \in (z-h, z+h)) = \phi((z-x)/\sqrt{\sigma(\rho_t)}) + o(1),
\]

and the proposition is proved. As in the previous proof the uniformity follows from Lemma 12 and the monotonicity of \( \rho_t \).
2.4 Proof of Proposition 10 in the case \((G)\)

The proof of this result follows the same line of argument as that of the previous section, so we will just point out the changes needed for that proof to apply in this setting. Observe that the function \(\psi'\) is strictly decreasing and continuous, and hence under our assumptions

\[
0 < \rho := \liminf_{t \to \infty} \rho_t \leq \limsup_{t \to \infty} \rho_t =: \bar{\rho} < \infty.
\]

This implies in turn that

\[
0 < \liminf_{t \to \infty} \int_0^\infty y^2 e^{-\rho_t y} \Pi(dy) \leq \limsup_{t \to \infty} \int_0^\infty y^2 e^{-\rho_t y} \Pi(dy) < \infty,
\]

and

\[
0 < \liminf_{t \to \infty} \int_0^\infty y^3 e^{-\rho_t y} \Pi(dy) \leq \limsup_{t \to \infty} \int_0^\infty y^3 e^{-\rho_t y} \Pi(dy) < \infty.
\]

Next we define \(n\) and \(W\) as in the previous proof and recall that

\[
E(W) = x, \quad \text{Var}(W) = t\sigma^2(\rho_t) + c_1 h^2 + c_2 a^2 \sim t\sigma^2(\rho_t),
\]

where the above estimate is uniform in \(x\), \(0 < h < h_0\) and \(0 < a < a_0\). Using the above facts about \(\rho\) and arguing as in the previous proofs it is easily seen that

\[
x \rho_t \to \infty, \quad \text{and} \quad \sqrt{\sigma}(\rho_t)L \leq \frac{k_1 H(\rho)}{\sqrt{H(\rho)}} \to 0,
\]

uniformly in \(x\), \(0 < h < h_0\), \(0 < a < a_0\). Next we prove that

\[
\tilde{\gamma} := \sqrt{\sigma}(\rho_t) \int_t^\infty e^{-\sigma(\rho_t)(\theta h)} \sin \theta h \cdot \left(\frac{\sin \theta a}{\theta a}\right)^2 d\theta \to 0,
\]

uniformly in \(x\). The properties listed at the beginning of the proof and the definition of \(l\) imply that it can be bounded by below by a strictly positive constant, say \(l^*\). Also, as we have assumed \(X\) non-lattice, and since this is a property that is preserved under change of measure we have that

\[
\liminf_{\theta \to \infty} R(\psi_{\rho_t}(i\theta)) = \liminf_{\theta \to \infty} \int_0^\infty (1 - \cos(\theta y)) e^{-\rho_t y} \Pi(dy) \geq \liminf_{\theta \to \infty} \int_0^\infty (1 - \cos(\theta y)) e^{-\theta y} \Pi(dy) > 0.
\]

We denote \(\tilde{\psi}(\theta) = \int_0^\infty (1 - \cos(\theta y)) e^{-\theta y} \Pi(dy)\), and \(m(s) = \inf_{\theta \geq s} \tilde{\psi}(\theta)\). The above observations and the continuity of \(\tilde{\psi}(\theta)\) imply that \(m(s) > 0\), for all \(s > 0\). It follows that for \(t > t_0\)

\[
\sqrt{\sigma}(\rho_t) \int_t^\infty e^{-\sigma(\rho_t)(\theta h)} \sin \theta h \cdot \left(\frac{\sin \theta a}{\theta a}\right)^2 d\theta \leq \sqrt{\sigma}(\rho_t) e^{-(t-t_0)m(l^*)} \int_{t_0}^\infty e^{-\frac{1}{3} \tilde{\psi}(\theta)} \sin \theta h \cdot \left(\frac{\sin \theta a}{\theta a}\right)^2 d\theta \rightarrow 0,
\]

uniformly in \(x\), \(0 < h < h_0\) and \(0 < \epsilon < a < a_0\), for any \(\epsilon > 0\). The rest of the proof follows just as before.
3 Proof of Theorem 2

The proof of Theorem 2 relies on Proposition 10 and it is the same for the three cases $(SC_0)$, $(SC_\infty)$ and $(G)$. We recall the identity, for $t \geq 0$,

$$P(Y_t \in dy) = e^{tH(\rho_t)} e^{-\rho_t(y-z)} P(X_t \in dy), \quad y \in \mathbb{R}^+.$$ (38)

The estimate in Proposition 10 implies that

$$\sqrt{\sigma(\rho_t)} \frac{1}{2h} e^{tH(\rho_t)} \int_{x-h}^{x+h} e^{-\rho_t(y-z)} P(X_t \in dy) = \phi \left( \frac{x-z}{\sqrt{\sigma(\rho_t)}} \right) + o(1),$$

uniformly in $x$, $z > h$ and $0 < h < h_0$. By making a change of variables $x - z = v$ the latter becomes

$$\sqrt{\sigma(\rho_t)} \frac{1}{2h} e^{tH(\rho_t)} \int_{x-v-h}^{x-v+h} e^{-\rho_t(y-z)} P(X_t \in dy) = \phi \left( \frac{v}{\sqrt{\sigma(\rho_t)}} \right) + o(1),$$

uniformly in $x$ and $h \leq h_0$ and $v < x - h$. In particular, for $0 < h < 2h_0 / x$, we get by taking $v = h$, using the uniform continuity of the normal density and making elementary manipulations that

$$\int_{x-h}^{x} e^{-\rho_t(y-z)} P(X_t \in dy) = \frac{he^{-tH(\rho_t)}}{\sqrt{\sigma(\rho_t)}} \left( \phi \left( \frac{h}{2\sqrt{\sigma(\rho_t)}} \right) + o(1) \right)$$

= $$\frac{e^{-tH(\rho_t)}}{\sqrt{\sigma(\rho_t)}} \left( \int_{0}^{h} dv \left( \phi \left( \frac{v}{\sqrt{\sigma(\rho_t)}} \right) + o(1) \right) \right)$$

(41)

uniformly in $x$ and $0 < h < 2h_0 / x$. More generally, uniformly in $x$

$$\int_{x-u}^{x} P(X_t \in dy) e^{-\rho_t(y-z)} = \frac{e^{-tH(\rho_t)}}{\sqrt{\sigma(\rho_t)}} \left( \int_{0}^{u} dy \left( \phi \left( \frac{y}{\sqrt{\sigma(\rho_t)}} \right) + o(1) \right) \right)$$

(42)

uniformly in $u < x$. This estimate follows from (10) by splitting the interval $(x - u, x]$ into disjoints intervals of length $\leq 1 := h_0$, and using again the uniform continuity of the normal density. We omit the details.

Now, Fubini’s theorem implies that the following identities hold for $u < x$,

$$P(X_t \in (x-u, x)) = \int_{x-u}^{x} P(X_t \in dy) e^{-\rho_t(y-z)} \int_{-\infty}^{y-z} \rho_t e^{\rho_t z} dz$$

= $$\int_{-\infty}^{0} dz \rho_t e^{\rho_t z} \int_{(x-u)^{\vee}(z+x)}^{x} P(X_t \in dy) e^{-\rho_t(y-z)}$$

= $$\int_{0}^{u} dz \rho_t e^{-\rho_t z} \int_{z-x}^{x} P(X_t \in dy) e^{-\rho_t(y-z)}$$

+ $$e^{-\rho_t x} \int_{x-u}^{x} P(X_t \in dy) e^{-\rho_t(y-z)}$$

(43)
Applying the estimate in (42) into the first and second term of the latter expression, respectively, we obtain

\[
\int_0^u dz \rho t e^{-\rho z} = e^{-u \rho t} \int_0^x dy \left( \frac{y}{\sqrt{t} \sigma(t)} + o(1) \right) \left( e^{-\rho y} - e^{-\rho u} \right),
\]

and

\[
e^{-u \rho t} \int_x^u P(X_t \in dy) e^{-\rho y} = e^{-u \rho t} \frac{e^{-tH(\rho)}}{\sqrt{2 \pi t} \sigma(\rho)} \left( \int_0^u dy \left( \phi \left( \frac{y}{\sqrt{t} \sigma(\rho)} \right) + o(1) \right) \right).\]

Adding the two terms above we get the claimed estimate.

We now get the estimate for \(P(X_t \leq x)\). For \(\epsilon > 0\) there is a \(\delta > 0\) such that for \(0 \leq y < \delta \sigma(t) \sqrt{t}\), the inequality \(1 - \epsilon < e^{-\left(y^2 / 2t \sigma^2(\rho)\right)} \leq 1\), holds. It follows from (41) that

\[
(1 - \epsilon + o(1)) \frac{e^{-tH(\rho)}}{\sqrt{2 \pi t} \sigma(\rho)} \left( 1 - e^{-\rho \delta \sigma(\rho) \sqrt{t}} \right)
\leq P(X_t \in (x - \delta \sigma(\rho) \sqrt{t}, x]) \leq (1 + o(1)) \frac{e^{-tH(\rho)}}{\sqrt{2 \pi t} \sigma(\rho)} \left( e^{-\rho \delta \sigma(\rho) \sqrt{t}} \right).
\]

Now, the identity (45) can be used to obtain the inequality

\[
P(X_t \in (0, x - \delta \sigma(\rho) \sqrt{t}]) = e^{-tH(\rho)} \int_0^{x - \delta \sigma(\rho) \sqrt{t}} e^{tH(\rho)} e^{-\rho y} e^{-\rho t(y - x)} P(X_t \in dy) \leq e^{-tH(\rho)} e^{-\rho \delta \sigma(\rho) \sqrt{t}} P(Y_t \in (0, x - \delta \sigma(\rho) \sqrt{t})).
\]

From where it follows that

\[
\sqrt{t} \rho t \sigma(\rho) e^{H(\rho)} P(X_t \in (0, x - \delta \sigma(\rho) \sqrt{t}]) \leq \sqrt{t} \rho t \sigma(\rho) e^{-\rho \delta \sigma(\rho) \sqrt{t}} \quad \text{as} \quad t \to \infty,
\]

since the fact that \(t H(\rho) \to \infty\) and Lemma 3 imply that \(\rho t \sigma(\rho) \sqrt{t} \to \infty\), uniformly. The estimates in (46) and (48) lead to (12).

4 Proof of estimate (13) in Theorem 3

We only prove here the estimate in (13) in the case where \(X\) is stochastically compact at 0. In the other two cases the proof is similar. The proof of the estimate (14) is given later.

We start by observing that
\[
\sqrt{2\pi \sigma(\rho_t)} e^{\lambda H(\rho_t)} h_x(t) = \sqrt{2\pi \sigma(\rho_t)} e^{\lambda H(\rho_t)} \int_0^x \Pi(dy) P(X_t \in (x - y, x)) + \Pi(x) \sqrt{2\pi \sigma(\rho_t)} e^{\lambda H(\rho_t)} P(X_t < x)
\]

\[
\sim \sqrt{2\pi} \int_0^x \Pi(dy) \left( \int_y^x e^{-\rho_t v} \frac{v}{\sqrt{4\pi \sigma(\rho_t)}} dv + o(1) \frac{1 - e^{-\rho_t y}}{\rho_t} \right) + (1/\rho_t) \Pi(x)
\]

\[
\leq \int_0^x \Pi(dy) \left( \int_0^y e^{-\rho_t v} dv + o(1) \frac{1 - e^{-\rho_t y}}{\rho_t} \right) + (1/\rho_t) \Pi(x) = \left( \frac{\psi(\rho_t)}{\rho_t} - b \right) + o(1) \left( \frac{\psi(\rho_t)}{\rho_t} - b \right) + (1/\rho_t) \int_x^\infty e^{-\rho_t y} \Pi(dy).
\]

The upper bound is thus obtained from the following estimate of the final term above

\[
\frac{(1/\rho_t) \int_x^\infty e^{-\rho_t y} \Pi(dy)}{\frac{\psi(\rho_t)}{\rho_t} - b} \leq \frac{\Pi(x) e^{-\rho_t x}}{\int_0^\infty (1 - e^{-\rho_t y}) \Pi(dy)} \leq \frac{\Pi(x) e^{-\rho_t x}}{(1 - e^{-\rho_t x}) \Pi(x)} = \frac{1}{e^{\rho_t x} - 1},
\]

when we recall that in (31) we proved that \( x\rho_t \to \infty \), uniformly either as \( t \to \infty \) or \( t \to 0 \).

To establish a lower bound, we note that the stochastic compactness of \( X \) allows to ensure that

\[
\limsup_{z \to 0} \frac{z \Pi(z)}{\int_0^z \Pi(u) du} < 1.
\]

This is due to the inequality

\[
\frac{1}{z} \int_0^z u^2 \Pi(u) du + z \Pi(z) \leq \int_0^z du \Pi(u), \quad z > 0,
\]

which implies:

\[
\limsup_{z \to 0} \frac{z \Pi(z)}{\int_0^z du \Pi(u)} \leq \frac{1}{1 + \liminf_{z \to 0} \frac{\int_0^z u^2 \Pi(u) du}{z^2 \Pi(z)}} < 1.
\]

The property (50) is known to imply that there is a \( \kappa_1 \in [0, 1) \) such that

\[
\frac{\int_0^z \Pi(u) du}{\int_0^z \Pi(u) du} \leq \lambda^{\kappa_1}, \quad \lambda > 1,
\]

for all \( z \) small enough. Using this we get
\[
\frac{\Pi(x)}{\int_0^\infty (1 - e^{-\rho y}) \Pi(dy)} \leq \frac{1}{x \rho_t} \frac{\rho_t \int_0^x \Pi(u)du}{\int_0^\infty (1 - e^{-\rho y}) \Pi(dy)} = \frac{1}{x \rho_t} \frac{\rho_t \int_0^x \Pi(u)du}{\int_0^\infty \rho e^{-\rho y} \Pi(y)dy} \leq C \frac{\rho_t \int_0^x \Pi(u)du}{x \rho_t \int_0^{1/\rho_t} \Pi(u)du} \leq C(x \rho_t)^{\kappa_1 - 1} \xrightarrow{t \to \infty} 0.
\]

(52)

Next we use that for \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that if \( v \leq \delta \sqrt{\lambda} (\rho_t) \) we have \( \sqrt{2 \pi} \left( \frac{x}{\sqrt{\lambda} (\rho_t)} \right) \geq 1 - \varepsilon \). So if \( x \geq \delta \sqrt{\lambda} (\rho_t) \)

\[
\sqrt{2 \pi} \int_0^x \Pi(dy) \left( \int_0^y e^{-\rho y} \phi \left( \frac{v}{\sqrt{\lambda} (\rho_t)} \right) dv + o(1) \frac{(1 - e^{-\rho y})}{\rho_t} \right) + (1/\rho_t) \Pi(x) \\
\geq (1 - \varepsilon) \int_0^\delta \sqrt{\lambda} (\rho_t) \Pi(dy) \frac{1 - e^{-\rho y}}{\rho_t} + o(1) \left( \frac{\psi(\rho_t)}{\rho_t} - b \right) \\
= (1 - \varepsilon + o(1)) \left( \frac{\psi(\rho_t)}{\rho_t} - b \right) - (1 - \varepsilon + o(1)) \int_\delta^{\infty} \Pi(dy) \frac{(1 - e^{-\rho y})}{\rho_t}.
\]

Clearly the integral term here is bounded above by \( \Pi(\delta \sqrt{\lambda} (\rho_t))/\rho_t \). On the other hand, if \( x < \delta \sqrt{\lambda} (\rho_t) \), we note that

\[
\frac{\Pi(x)}{\rho_t} \left( \frac{\psi(\rho_t)}{\rho_t} - b \right) \to 0.
\]

But the previous calculation, with \( x \) replaced by \( x^* \), gives an upper bound of \( C(x^* \rho_t)^{\kappa_1 - 1} \) for the LHS, and we know that \( x^* \rho_t \to \infty \) uniformly, and so the result follows.

5 Proof of estimates (14) and (15) and Corollary 4

For the behaviour of the first passage time distribution on the event of creeping,

\[ P(T_x \leq t, X_{T_x} = x), \]

we recall that the creeping probability is strictly positive if the drift \( b \) of \( X \) is strictly positive and \( P(X_{T_x} = x) = bu(x) \), where \( u : \mathbb{R} \to \mathbb{R}^+ \) denotes the density of the potential measure of \( X \). From a result by Winkel [5], and generalized by Griffin and Maller [4], we know that in the case where \( b > 0 \) we have the identity

\[ P(T_x \leq t, X_{T_x} = x) = b \partial_x \int_0^t ds P(X_s \leq z)_{|x = x}, \quad x > 0, t \geq 0. \]
In the case where the one dimensional distribution of $X$ has a density, say $f_t$, that is
\[ P(X_t \in dy) = f_t(y), \quad y \geq 0, \]
the former expression takes the form
\[ P(T_x \in dt, X_{T_x} = x) = b f_t(x) dt, \quad x > 0, t \geq 0. \] (53)
The estimate (15) follows immediately from this and Theorem 1.
We can also use this result and the former estimates to prove the estimate (14) in a straightforward way.

**Proof of the estimate (14) in the cases (SC$_\infty$) and (G).** We know that
\[ h_C^C(t, \Delta) = b \int_t^{t+\Delta} ds P(X_s \leq z) \bigg|_{z=x} \]
\[ = \lim_{h \to 0^+} \frac{b}{2h} \int_t^{t+\Delta} ds P(X_s \in (x-h, x+h)). \] (54)
Following the arguments in the proof of Theorem 2 it is easy to prove that
\[ P(X_s \in (x-h, x+h]) = 2h \frac{1}{\sqrt{2s\pi \sigma(x)}} e^{-sH(\rho(x))} (1 + o(1)), \] (55)
uniformly in $0 \leq h < x$, and we recall that $\rho(x)$ is determined by the relation $\psi'(\rho(x)) = \lambda$, $\lambda > 0$. We deduce therefrom the equality
\[ h_C^C(t, \Delta) = b \int_t^{t+\Delta} ds \frac{1}{\sqrt{2s\pi \sigma(x)}} e^{-sH(\rho(x))} (1 + o(1)). \]
Since $s \in (t, t+\Delta)$ and the error term is uniform in $x/s$, the claim follows.

We will next establish Corollary 4. Since the arguments used to prove (i) and (ii) are rather similar we will only provide those needed for the latter.

**Proof of (ii) in Corollary 4** In order to sharpen the estimate in (14) we will start by estimating the difference
\[ sH \left( \rho \left( \frac{x}{s} \right) \right) - tH \left( \rho \left( \frac{x}{t} \right) \right) = x \left( \frac{\psi'(\rho(x)) \psi(\rho(x)) - \psi(\rho(x)) \psi'(\rho(x))}{\psi'(\rho(x)) \psi'(\rho(x))} \right) \]
\[ - x \left( \rho \left( \frac{x}{s} \right) - \rho \left( \frac{x}{t} \right) \right), \] (56)
for $t \leq s \leq t + \Delta$. Observe that $\psi'$ is decreasing and regularly varying at 0 with index $\alpha - 1$, which implies that its inverse, $\rho$, is non-increasing and regularly varying at $\infty$ with index $1/(\alpha - 1)$. Also, an easy calculation shows that the function $\lambda \mapsto \frac{\psi'(\rho(\lambda))}{\psi'(\rho(\lambda))}$, $\lambda > 0$ is a non-decreasing function and regularly varying at 0 with index 1. So the function $G(\lambda) := \frac{\psi(\rho(\lambda))}{\psi'(\rho(\lambda))}$, $\lambda > 0$ is regularly varying at infinity with index $1/(\alpha - 1)$ and non-increasing.
In particular we have the inequalities
\[ -x \left( \frac{x}{t + \Delta_0} - \frac{\rho(\frac{x}{t})}{\rho(\frac{x}{t})} \right) \geq x \left( \frac{\psi(\rho(\frac{x}{t}))}{\psi'(\rho(\frac{x}{t}))} - \frac{\psi(\rho(\frac{s}{t}))}{\psi'(\rho(\frac{s}{t}))} \right) \]
\[ \geq \frac{\psi(\rho(\frac{x}{s}))}{\psi'(\rho(\frac{x}{s}))} - \frac{\psi(\rho(\frac{x}{t}))}{\psi'(\rho(\frac{x}{t}))}, \]
(57)

for \( t \leq s \leq t + \Delta, \Delta < \Delta_0 \). So it will be enough to prove that the right and left terms in the above inequality tend towards 0, as \( x/t \to \infty \). We will next prove that the rightmost term in the above inequality tends to 0, the proof that the leftmost term tends to 0 is analogous and so we omit it. To that end we recall that the function \( G \) can be written as \( G(z) = z^{1-\alpha}G(z) \) for some slowly varying function \( \ell \), and as a consequence
\[ x \left( G \left( \frac{x}{t + \Delta_0} \right) - G \left( \frac{x}{t} \right) \right) = xG \left( \frac{x}{t} \right) \left[ 1 - \frac{(t + \Delta_0)^{1-\alpha}G(x)}{(x)^{1-\alpha}G(x)} \right] \]
\[ = xG \left( \frac{x}{t} \right) \left[ 1 - \left( 1 + \frac{\Delta_0}{t} \right)^{1-\alpha} \right] + xG \left( \frac{x}{t} \right) \left( 1 + \frac{\Delta_0}{t} \right)^{1-\alpha} \left[ 1 - \frac{\ell \left( \frac{x}{t + \Delta_0} \right)}{\ell \left( \frac{x}{t} \right)} \right]. \]
(58)
The first term in the rightmost term in the above identity tends towards zero as \( t \) tends to infinity as the following elementary calculation shows
\[ xG \left( \frac{x}{t} \right) \left[ 1 - \left( 1 + \frac{\Delta_0}{t} \right)^{1-\alpha} \right] \sim - \frac{\Delta_0}{1 - \alpha} \frac{x}{t} G \left( \frac{x}{t} \right) \]
\[ = - \frac{\Delta_0}{1 - \alpha} \left( \frac{x}{t} \right)^{1-\alpha} \ell \left( \frac{x}{t} \right) \xrightarrow{t \to \infty} 0, \]
(59)
due to the fact that \( x/t \to \infty \). To estimate the second term we use Potter's bounds for regularly varying functions to ensure that for any \( \delta > 0 \) there is a \( T \) such that if \( \frac{x}{t + \Delta_0}, \frac{x}{t} > T \) then
\[ \left( \frac{t + \Delta_0}{t} \right)^{-\delta} \leq \frac{\ell \left( \frac{x}{t + \Delta_0} \right)}{\ell \left( \frac{x}{t} \right)} \leq \left( \frac{t + \Delta_0}{t} \right)^{\delta}. \]

We proceed as in the latter estimate to prove that both the lim inf and lim sup of \( xG \left( \frac{x}{t} \right) \left( 1 + \frac{\Delta_0}{t} \right)^{1-\alpha} \left[ 1 - \frac{\ell \left( \frac{x}{t + \Delta_0} \right)}{\ell \left( \frac{x}{t} \right)} \right] \) are equal to zero, as \( t \to \infty \).

Having proved that the difference in (58) tends to 0 uniformly in \( t \leq s \leq t + \Delta, \Delta < \Delta_0 \), we are just left to prove that
\[ \frac{\sqrt{\sigma(\rho(\frac{x}{t}))}}{\sqrt{\sigma(\rho(\frac{x}{t}))}} \xrightarrow{t \to \infty} 1 \]
(60)
uniformly as above. But this is a straightforward consequence of the uniformity properties of regularly varying functions because the function \( \lambda \mapsto \sigma(\rho(\lambda)) \) is regularly varying at infinity with index \( \frac{\alpha^2 - 1}{\alpha - 1} > 0 \) and hence

\[
\frac{\sigma(\rho(\lambda c))}{\sigma(\rho(\lambda))} \xrightarrow{\lambda \to \infty} c^{\frac{\alpha^2 - 2}{\alpha - 1}},
\]

uniformly for \( c \) in \((0, A)\), for any \( A > 0 \). Putting the pieces together we conclude that

\[
h_c(t, \Delta) = b \Delta \frac{1}{\sqrt{2\pi t}} e^{-\frac{t}{2} H(\rho t)} (1 + o(1)),
\]

(61)

uniformly in \( x/t \to \infty \) and \( 0 < \Delta < \Delta_0 \).

### 6 Proof of Proposition 6

**Proof of estimate (17), the jump term.** Write the LHS as \( I_1 + I_2 \), where

\[
I_1 = t \int_0^{x-c(t)} P(X_t \in dy) \Pi(x - y)
\]

\[
= \int_0^{y_t - \varepsilon} P(X_t \in c(t)dz) \frac{\Pi(c(t)(y_t - z))}{\Pi(c(t))}
\]

\[
= \int_0^{y_t - \varepsilon} P(X_t \in c(t)dz)(y_t - z)^{-\alpha} + o(1),
\]

where we use the fact that the previous fraction converges uniformly to \((y_t - z)^{-\alpha}\) on the range of integration. Since this function and its derivative are bounded on the range, we can integrate by parts and use the central limit theorem to see that

\[
\lim_{\varepsilon \to 0} \lim_{t \to \infty} |I_1 - \int_0^{y_t - \varepsilon} P(S_1 \in dz)(y_t - z)^{-\alpha}| = 0.
\]

An integration by parts and the local limit theorem shows that

\[
I_2 = t \int_0^{c(t)} P\{X_t \in (x - z, x]\} \Pi(dz) + t\Pi(\varepsilon c(t))P\{X_t \in (x - \varepsilon c(t), x]\}
\]

\[
= \frac{t}{c(t)} \{\tilde{g}_1(y_t) + o(1)\} \int_0^{c(t)} z\Pi(dz) + \frac{t\Pi(\varepsilon c(t))}{c(t)}\varepsilon c(t)\{\tilde{g}_1(y_t) + o(1)\},
\]

which gives \( \lim_{\varepsilon \to 0} \limsup_{t \to \infty} I_2 = 0 \). Since

\[
\tilde{h}_c(1) = \int_0^{y_t} P(S_1 \in dz)\Pi^S(y_t - z),
\]

the conclusion follows because the normalisation \( t\Pi(c(t)) = 1 \) implies that \( \Pi^S(x) = x^{-\alpha} \); this can be seen by the fact that as \( t, x \to \infty \), \( P(X_t > c(t)x) \sim t\Pi(c(t)x) \sim x^{-\alpha} \).
Remark 13 Suppose now that $X$ is in the domain of attraction of a stable subordinator of index $\alpha$, at 0, with norming function $\gamma$, and put $y'_t = x/\gamma(t)$. Then the same result holds as $t \to 0$, uniformly for $y'_t \in (D^{-1}, D)$.

Proof of estimate (15), the creeping term. As in the previous proof we have

$$h^C_x(t, \Delta) = \lim_{h \to 0} \frac{b}{2h} \int_t^{t+\Delta} ds P(X_s \in (x-h, x+h)).$$

(62)

So, using the local limit theorem we get that for large $t$

$$h^C_x(t, \Delta) = b \int_t^{t+\Delta} ds \frac{1}{c(s)} \left( \tilde{g}_1 \left( \frac{x}{c(s)} \right) + o(1) \right).$$

(63)

where $o(1)$ is uniform in $x$. Using that $c(s)/c(t) \sim 1$ uniformly in $t \leq s \leq t + \Delta_0$ and the uniform continuity of $g$ it is deduced therefrom that

$$h^C_x(t, \Delta) = \left( \tilde{g}_1 \left( \frac{x}{c(t)} \right) + o(1) \right) \int_t^{t+\Delta} ds \frac{1}{c(s)}. \quad (64)$$

Now, observe that the function $r \to \int_r^{\infty} \frac{ds}{c(s)}$ is regularly varying at infinity with index $1 - \frac{1}{\alpha}$ and by Karamata's theorem it satisfies that

$$\int_r^{\infty} \frac{ds}{c(s)} \sim \frac{\alpha}{1-\alpha} \frac{r}{c(r)}, \quad r \to \infty.$$ 

So, arguing as in the proof of Corollary [3] it is proved that

$$\int_t^{t+\Delta} ds \frac{1}{c(s)} \sim \frac{\alpha \Delta}{(1-\alpha)c(t)},$$

uniformly in $0 < \Delta < \Delta_0$. The claim follows. 

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