On the automorphism group of a Johnson graph

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Abstract

The Johnson graph \( J(n, i) \) is defined to be the graph whose vertex set is the set of all \( i \)-element subsets of \( \{1, \ldots, n\} \), and two vertices \( A, B \) are said to be adjacent in this graph whenever \( |A \cap B| = i - 1 \). In Ramras and Donovan [SIAM J. Discrete Math, 25(1): 267-270, 2011], it is conjectured that if \( n = 2i \), then the automorphism group of the Johnson graph \( J(n, i) \) is \( S_n \times \langle T \rangle \), where \( T \) is the complementation map \( A \mapsto A^c \) and \( A^c := \{1, \ldots, n\} \setminus A \). We resolve this conjecture in the affirmative. The proof uses only elementary group theory and is based on an analysis of the clique structure of the graph.

Index terms — Johnson graph, automorphism group, cliques

1. Introduction

The Johnson graph \( J(n, i) \) is defined to be the graph whose vertex set is the set of all \( i \)-element subsets of \( \{1, \ldots, n\} \), and two vertices \( A, B \) are said to be adjacent in this graph whenever \( |A \cap B| = i - 1 \). This graph has been well-studied in the literature (cf. [1] [2] [3] [4] [5] [8] [9] [10]). The automorphism group of a graph is the set of all permutations of the vertex set of the graph that preserves adjacency [6]. In [10], it is proved that if \( n \neq 2i \), then the automorphism group of the Johnson graph \( J(n, i) \) is isomorphic to \( S_n \). In [10, Conjecture 1, p. 269] it is conjectured that if \( n = 2i \), then the automorphism group of \( J(n, i) \) is isomorphic to \( S_n \times \langle T \rangle \), where \( T \) is the complementation map \( A \mapsto A^c \) and \( A^c := \{1, \ldots, n\} \setminus A \). In the present paper, this conjecture is resolved in the affirmative.

Actually, the automorphism group of \( J(n, i) \) for both the \( n \neq 2i \) and \( n = 2i \) cases was already determined in [7], but the proof given there uses heavy group-theoretic machinery. The main result of [10] was to provide a proof for the \( n \neq 2i \) case that uses only elementary group theory; the proof is based on an analysis of the clique structure of the graph. In [10] the authors leave the \( n = 2i \) case open but make a conjecture for this case. We resolve this conjecture in the affirmative by providing a proof that again uses only elementary group theory and a similar analysis of the clique structure of the graph.

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We first recall some basic facts about the Johnson graphs $J(n,i)$. Two vertices $A, B$ are adjacent in this graph iff their intersection $A \cap B$ has cardinality $i - 1$, and this occurs exactly when the cardinality of their symmetric difference is 2. The Johnson graph $J(n,i)$ is isomorphic to the Johnson graph $J(n,n-i)$; an explicit bijection between their vertex sets that preserves adjacency is the complementation map $T: A \mapsto A^c$. Hence, without loss of generality we shall restrict our study of the Johnson graphs $J(n,i)$ to the case where $i \leq n/2$. Also, the graphs $J(n,1)$ are the complete graphs and hence are not very interesting. The graphs $J(n,2)$ are the line graphs of complete graphs, and their automorphism groups are known. Thus, when studying $J(n,i)$ henceforth, it is assumed that $i \geq 3$.

Each permutation in $S_n$ acts in a natural way on the set of $i$-element subsets of $\{1, \ldots, n\}$, and this induced action on the vertices of $J(n,i)$ is an automorphism of the graph. Also, distinct permutations in $S_n$ induce distinct automorphisms of the $i$-element subsets. Hence $S_n$ is isomorphic to a subgroup of the automorphism group of $J(n,i)$. In some cases, $S_n$ happens to be the (full) automorphism group of $J(n,i)$. A special case of the results in [7] Theorem 2(a)(c)] is that when $n \neq 2i$, the automorphism group of $J(n,i)$ is isomorphic to $S_n$; a special case of the results in [7] Theorem 2(e)] is that when $n = 2i$, the automorphism group of $J(n,i)$ is isomorphic to $S_n \times S_2$. The proofs given in [7] use heavy group-theoretic machinery. An elementary combinatorial proof of the former result is given in [10], and an elementary combinatorial proof of the latter result is given in the present paper.

The following is the main result proved in the present paper:

**Theorem 1.** If $n = 2i$, then the automorphism group of the Johnson graph $J(n,i)$ is $S_n \times \langle T \rangle$, where $T$ is the complementation map $A \mapsto A^c$.

For $\theta \in S_n$, let $\rho_\theta$ denote the permutation of the vertex set of $J(n,i)$ induced by $\theta$. It is clear that $\{\rho_\theta : \theta \in S_n\}$ is a subgroup of the automorphism group of $J(n,i)$. When $n = 2i$, the subgroup $\langle T \rangle$ also acts as a group of automorphisms of $J(n,i)$:

**Lemma 2.** Suppose $n = 2i$. Then the complementation map $T: A \mapsto A^c$ is an automorphism of the Johnson graph $J(n,i)$.

**Proof:** Let $A$ and $B$ be two vertices in $J(n,i)$. We show that $A$ and $B$ are adjacent in $J(n,i)$ iff $A^c$ and $B^c$ are adjacent in $J(n,i)$. Recall that two vertices are adjacent in $J(n,i)$ iff their intersection has cardinality $i - 1$. The cardinality $|A^c \cap B^c| = n - |A \cup B| = n - (|A| + |B| - |A \cap B|) = n - 2i + |A \cap B|$, which equals $|A \cap B|$ since $n = 2i$. Since $A \cap B$ and $A^c \cap B^c$ have the same cardinality, the complementation map preserves adjacency and nonadjacency in $J(n,i)$. The group $\{\rho_\theta : \theta \in S_n\}\langle T \rangle$ of automorphisms of $J(n,i)$ obtained so far can be expressed as a direct product:

**Lemma 3.** Let $T$ denote the complementation map $A \mapsto A^c$. The group $H := \{\rho_\theta : \theta \in S_n\}\langle T \rangle$ of automorphisms of $J(2i,i)$ is isomorphic to the direct product $S_n \times \langle T \rangle \cong S_n \times S_2$.  

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Proof: Observe that if $A$ is any $i$-element subset of $\{1, \ldots, n\}$, then $[\theta(A)]^c = \theta(A^c)$, whence $T$ and $\rho_\theta$ commute. It follows that $\{\rho_\theta : \theta \in S_n\}/T$ is a group and its two factors are normal subgroups. It remains to show that the two factors $\{\rho_\theta : \theta \in S_n\}$ and $\langle T \rangle$ have a trivial intersection. By way of contradiction, suppose $T = \rho_\theta$ for some $\theta \in S_n$. Then $\theta$ takes $\{1, \ldots, i - 1, i\}$ to its complement $\{i + 1, \ldots, 2i\}$, and $\{1, \ldots, i - 1, i + 1\}$ to its complement $\{i, i + 2, \ldots, 2i\}$. Hence $\theta$ takes the common elements $\{1, \ldots, i - 1\}$ to $\{i + 2, \ldots, 2i\}$, and hence the remaining elements $\{i, i + 1\}$ to $\{i, i + 1\}$. Take $A = \{2, \ldots, i - 1, i, i + 1\}$. Then $A^{\rho_\theta} \supseteq \{i, i + 1\}$. Thus $A^{\rho_\theta} \neq A^c$, a contradiction. 

**Notation.** Fix a vertex $X$ of the graph $J(n, i)$. Let $L_i$ denote the set of vertices of $J(n, i)$ whose distance to $X$ is exactly $i$. Thus, $L_0 = \{X\}$, and $L_1$ is the set $N(X)$ of neighbors of $X$. Let $G$ denote the automorphism group of $J(n, i)$. The stabilizer of $X$ in $G$ is denoted $G_X$.

We use the following additional notation from [10]. Each neighbor of a vertex $X$ in $J(n, i)$ is of the form $(X - \{p\}) \cup \{q\}$ for some $p \in X, q \notin X$. We denote this neighbor by $Y_{p,q}$. For each $p \in X$, the set of neighbors $\{Y_{p,q} : q \notin X\}$ forms a clique, denoted by $Y_p$. The set $\{Y_p : p \in X\}$ is a partition of the set $N(X)$ of neighbors of $X$ into $i$ cliques, each of cardinality $n - i$. Similarly, for each $q \notin X$, the set $\{Y_{p,q} : p \in X\}$ forms a clique, denoted by $Z_q$. The set $\{Z_q : q \notin X\}$ is a partition of $N(X)$ into $n - i$ cliques, each of cardinality $i$. Each maximal clique in $J(n, i)$ that contains the vertex $X$ is of the form $\{X\} \cup Y_p$ for some $p \in X$ or of the form $\{X\} \cup Z_q$ for some $q \notin X$ (cf. [10] Lemma 1).

We call each clique $Y_p$ a clique of the first kind. Similarly, each clique $Z_q$ is a clique of the second kind. When $n \neq 2i$, the cardinality of a clique of the first kind is not equal to the cardinality of a clique of the second kind; thus, any automorphism of the graph that fixes the vertex $X$ must permute the set of cliques of the first kind in $N(X)$ amongst themselves. On the other hand, when $n = 2i$, the cliques in $N(X)$ of the first and second kind have the same cardinality, and so it is possible that there is an automorphism in $G_X$ that takes a clique of the first kind to a clique of the second kind. Indeed, we show below that such an automorphism exists and can be expressed in terms of the complementation map.

**Proposition 4.** Suppose $n = 2i$, and let $X$ be a vertex of $J(n, i)$ and let $g \in G_X$. Then there exist $\theta \in S_n$ and $i \in \{0, 1\}$ such that the actions of $g$ and $\rho_\theta T^i$ on $L_0 \cup L_1$ are identical.

Proof: Let $g \in G_X$. Then $g$ acts on the set $N(X)$ of neighbors of $X$, and hence permutes the maximal cliques in $N(X)$ amongst themselves. Recall that these maximal cliques are either of the first kind or the second kind. We consider two cases.

First suppose that $g$ permutes the set of cliques in $N(X)$ of the first kind amongst themselves. Since $g \in G_X$, $g$ acts bijectively on the set of all maximal cliques in $N(X)$, and so $g$ also permutes the set of cliques of the second kind amongst themselves. Hence $g : Y_p \mapsto Y_{\theta_1(p)}, Z_q \mapsto Z_{\theta_2(q)}$ for some $\theta_1 \in \text{Sym}(X), \theta_2 \in \text{Sym}(X^c)$. Define $\theta \in S_n$ to be the map that takes $j$ to $\theta_1(j)$ if $j \in X$ and that takes $j$ to $\theta_2(j)$ if $j \in X^c$. As shown in [10] p. 268, the actions of $g$ and $\rho_\theta$ on $L_0 \cup L_1$ are identical.
For the rest of the proof, suppose that \( g \) takes some clique of the first kind to a clique of the second kind. So there exist \( p' \in X, q' \notin X \) such that \( g : Y_{p'} \mapsto Z_{q'} \). We show that \( g \) takes every clique of the first kind to some clique of the second kind. Observe that \( Z_{q'} \) contains exactly one vertex from \( Y_{p'} \), for each \( p \in X \). Any two cliques of the first kind are disjoint, and \( g \) must map disjoint cliques to disjoint cliques. Also, any two cliques of the second kind are disjoint, whereas a clique of the first kind and a clique of the second kind meet: \( Y_p \cap Z_q \neq \emptyset \) since it contains \( Y_{p,q} \). Thus, if \( g \) takes a clique of the first kind to a clique of the second kind, then \( g \) takes each clique of the first kind to some clique of the second kind. Hence \( g \) interchanges the set of cliques of the first kind and the set of cliques of the second kind.

Thus \( g : Y_p \mapsto Z_{\theta_1(p)}, Z_q \mapsto Y_{\theta_2(q)} \) for some \( \theta_1 : X \mapsto X^c \) and \( \theta_2 : X^c \mapsto X \). Define \( \theta \in S_n \) to be the map that takes \( j \) to \( \theta_1(j) \) if \( j \in X \) and that takes \( j \) to \( \theta_2(j) \) if \( j \in X^c \). Recall that \( \rho_\theta \) is defined as \( \rho_\theta \) induced on the vertex set of \( J(n,i) \) and that \( T \) denotes the complementation map \( A \mapsto A^c \).

We show that the actions of \( g \) and \( \rho_\theta T \) on \( L_0 \cup L_1 \) are identical. It is clear that both the actions fix \( L_0 = \{ X \} \). For \( g \in G_X \) implies \( g \) fixes \( X \). And \( X^{\rho_\theta T} = (X^c)^T = X \). Let \( Y_{p,q} \) be a vertex in \( L_1 \), and consider the action of \( g \) and \( \rho_\theta T \) on this vertex. Recall that \( Y_{p,q} \) is the unique vertex in the intersection \( Y_p \cap Z_q \). We have that \( (Y_p \cap Z_q)^g = Z_{\theta_1(p)} \cap Y_{\theta_2(q)} = Y_{\theta_2(q),\theta_1(p)} \). The vertex \( Y_{p,q} \) has the same image under \( \rho_\theta T \) as under \( g \): \( (Y_{p,q})^{\rho_\theta T} = ((X - \{ p \}) \cup \{ q \})^{\rho_\theta T} = [(X^{\theta_1} - \{ \theta_1(p) \}) \cup \{ \theta_2(q) \}]^T = [(X^c - \{ \theta_1(p) \}) \cup \{ \theta_2(q) \}]^T = (X - \{ \theta_2(q) \}) \cup \{ \theta_1(p) \} = Y_{\theta_2(q),\theta_1(p)} \). Thus, \( g \) and \( \rho_\theta T \) act identically on \( L_1 \).

The following result, which is proved in [10] Lemma 2 and Proposition 1], does not use the condition that \( n \neq 2i \) and hence also applies when \( n = 2i \):

**Lemma 5.** In the Johnson graph \( J(n,i) \), if an automorphism \( g \) fixes a vertex \( X \) and each of its neighbors, then it is the trivial automorphism.

We now complete the proof of the main theorem.

**Corollary 6.** If \( n = 2i \), then the automorphism group of the Johnson graph \( J(n,i) \) is \( S_n \times \langle T \rangle \), where \( T \) is the complementation map \( X \mapsto X^c \).

**Proof:** Let \( g \in G_X \). By Proposition [\ref{prop:automorphism}] there exist \( \theta \in S_n \) and \( i \in \{0, 1\} \) such that the action of \( g \) and \( \rho_\theta T^i \) are identical on \( L_0 \cup L_1 \). Hence, \( g^{-1} \rho_\theta T^i \) acts trivially on \( L_0 \cup L_1 \). By Lemma [\ref{lem:trivialautomorphism}], \( g^{-1} \rho_\theta T^i \) is the trivial automorphism of \( J(n,i) \). Hence \( g = \rho_\theta T^i \). This proves that every element in \( G_X \) is one of the \( 2i! \) automorphisms specified in the proof above, i.e. every element in \( G_X \) is either one of the \( i! \) elements in \( G_X \) that permutes the \( i \) cliques of the first kind amongst themselves and the \( i \) cliques of the second kind amongst themselves, or is one of the \( i! \) elements in \( G_X \) that interchanges the set of cliques of the first kind and the set of cliques of the second kind. Hence \( |G_X| = 2i! \). Finally, since the graph \( J(n,i) \) is vertex-transitive, the automorphism group \( G \) has order \( |G_X| \binom{n}{i} = 2i! \cdot \binom{n}{i} = 2i! \). Hence the group of automorphisms \( S_n \times \langle T \rangle \) obtained above is the (full) automorphism group of \( J(n,i) \). 

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