Strong Converse Exponent for Entanglement-Assisted Communication

Ke Li, Yongsheng Yao

Abstract

We determine the exact strong converse exponent for entanglement-assisted classical communication of a quantum channel. Our main contribution is the derivation of an upper bound for the strong converse exponent which is characterized by the sandwiched Rényi divergence. It turns out that this upper bound coincides with the lower bound of Gupta and Wilde (Commun. Math. Phys. 334:867–887, 2015). Thus, the strong converse exponent follows from the combination of these two bounds. Our result has two implications. Firstly, it implies that the exponential bound for the strong converse property of quantum-feedback-assisted classical communication, derived by Cooney, Mosonyi and Wilde (Commun. Math. Phys. 344:797–829, 2016), is optimal. This answers their open question in the affirmative. Hence, we have determined the exact strong converse exponent for this problem as well. Secondly, due to an observation of Leung and Matthews, it can be easily extended to deal with the transmission of quantum information under the assistance of entanglement or quantum feedback, yielding similar results. The above findings provide, for the first time, a complete operational interpretation to the channel’s sandwiched Rényi information of order $\alpha > 1$.

Index Terms

quantum channel, strong converse exponent, entanglement-assisted communication, sandwiched Rényi information, quantum feedback

I. INTRODUCTION

Understanding the ultimate limit that the laws of quantum mechanics impose on our capability to communicate information is a topic of great significance. Started in the 1970s, the investigation of the quantum Shannon theory has made huge achievements, yet lots of important problems remain unsolved [1]. The setting of entanglement-assisted classical communication of quantum channels turns out to be of particular interest. It is regarded as the most natural quantum generalization of the classical channel coding problem considered by Shannon [2].

The entanglement-assisted classical capacity $C_E(N)$ of a quantum channel $N$ quantifies the maximal rate of classical information the channel $N$ can reliably transmit, when encoding among multiple uses of $N$ is permitted, and unlimited entanglement shared by the sender and the receiver are available [3]. The formula of $C_E(N)$ was derived by Bennett, Shor, Smolin and Thapliyal more than two decades ago [4], and it is given by the maximal quantum mutual information that the channel $N$ can generate, in formal analogy to Shannon’s formula for the capacity of classical channels. Later on, Bowen showed that feedback provides no increase to the entanglement-assisted capacity [5], paralleling the fact that classical feedback does not change the capacity of classical channels [6]. The quantum reverse Shannon theorem established by Bennett, Devetak, Harrow, Shor and Winter in [7] further gives that the quantity $C_E(N)$ is the minimal rate of classical communication needed to simulate the channel $N$ with the assistance of unlimited entanglement (see also [8] for an alternative proof).

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Ke Li is with the Institute for Advanced Study in Mathematics, Harbin Institute of Technology, Nangang District, Harbin 150001, China (e-mail:carl.ke.lee@gmail.com, keli@hit.edu.cn).

Yongsheng Yao is with the Institute for Advanced Study in Mathematics and School of Mathematics, Harbin Institute of Technology, Nangang District, Harbin 150001, China (e-mail:yongsh.yao@gmail.com).
As a result of the quantum reverse Shannon theorem, we know that the entanglement-assisted classical capacity of a quantum channel satisfies the strong converse property [7], [8]. That is, if one transmits classical information at a rate larger than the capacity $C_E(N)$, then the probability of successfully decoding the messages must decay to 0 exponentially fast, as the number of channel uses increases. This fact implies that $C_E(N)$ is a critical changing point for entanglement-assisted communication. The exact rate of this exponential decay that can be achieved by the best strategy is called the strong converse exponent.

The formula of the strong converse exponent of quantum channels for entanglement-assisted classical communication was unknown. This is in contrast to the fact that the strong converse exponents of classical channels [9], [10], [11] and classical-quantum channels [12], have been well understood. In [13], Gupta and Wilde have derived a lower bound in terms of the sandwiched Rényi divergence [14], [15], exploiting the multiplicativity of the completely bounded $p$-norms [16] as well as the “entanglement-assisted meta-converse” property [17]. Subsequently, Cooney, Mosonyi and Wilde [18] have strengthened this result, showing that the lower bound obtained in [13] still holds even if additional classical or quantum feedback from the receiver to the sender is allowed. Note that a lower bound for the strong converse exponent translates to an upper bound for the success probability.

In this paper, we derive an upper bound for the strong converse exponent of entanglement-assisted classical communication. Our upper bound coincides with the lower bound of Gupta and Wilde [13]. Thus, the combination of these two bounds lets us completely determine the exact strong converse exponent of entanglement-assisted classical communication. We construct a two-step proof, following the work of Mosonyi and Ogawa [12]. In the first step, we prove a Dueck-Körner-type [10] upper bound which is expressed using the quantum relative entropy, and we show that it is equivalent to an Arimoto-type [9] bound as a transform of the log-Euclidean Rényi information. In the second step, we employ multiple techniques to turn the suboptimal bound derived in the first step into the final one, which is of the similar form, but now the log-Euclidean Rényi information is replaced by the sandwiched Rényi information. Compared to the work [12] which deals with the classical-quantum channels, we need to develop new methods to overcome the technical difficulties in coping with entanglement-assisted communication over general quantum channels. On the one hand, we introduce a type of the log-Euclidean Rényi information of a quantum channel, which is defined with respect to an ensemble of input states, and this quantity will play an important role. One the other hand, we treat the eigenvalues and the eigenvectors of the input density matrices separately and we make several subtle uses of the minimax theorem.

As a corollary, we conclude that the lower bound of Cooney, Mosonyi and Wilde [18] for the strong converse exponent of quantum-feedback-assisted classical communication is optimal. So, we have determined the exact strong converse exponent for this problem as well. This shows that additional feedback does not affect the strong converse exponent of entanglement-assisted classical communication.

While we have focused on the transmission of classical information, our results can be easily extended to deal with the transmission of quantum information. When free entanglement is available, Leung and Matthews [19] have shown a quantitative equivalence between noisy classical communication and noisy quantum communication, generalizing the interchange between perfect classical communication and perfect quantum communication via teleportation [20] and dense coding [21]. This lets us obtain the strong converse exponent for quantum communication under the assistance of entanglement or quantum feedback.

We point out that entanglement-assisted communication has drawn much attention in the community of quantum information. In addition to the results mentioned above, we review some recent developments. One-shot characterizations of the entanglement-assisted capacity of a quantum channel were given, e.g., in [22], [17], [23]. The second-order asymptotics was addressed in [24], where an achievability bound was obtained. Progress on the achievability part of the direct error exponent can be seen in [25], [26]. The moderate deviation expansion in the high-error regime has been derived in [27]. The full resolutions of these asymptotic characterizations are interesting open problems.

The remainder of this paper is organized as follows. In Section II we introduce the necessary notation, definitions and some basic properties. In Section III we present the main problems and results.
Section IV we prove an intermediate upper bound for the strong converse exponent of entanglement-assisted classical communication. Then in Section V we improve it to obtain the final result. At last, in Section VI we conclude the paper with some discussion.

II. PRELIMINARIES

A. Notation and basic properties

For a Hilbert space $\mathcal{H}$, we denote by $\mathcal{L}(\mathcal{H})$ the set of linear operators on it, and we use $\mathcal{L}(\mathcal{H})_+$ for the set of positive semidefinite operators. Density operators, or quantum states, are positive semidefinite operators with trace 1. Let $\mathcal{H}_A$ be the Hilbert space associated with a quantum system $A$. The set of quantum states on $\mathcal{H}_A$ is denoted by $\mathcal{S}(\mathcal{H}_A)$, or $\mathcal{S}(A)$ for short. Pure states are denoted by $\mathcal{S}_1(\mathcal{H}_A)$ or $\mathcal{S}_1(A)$. $\mathbb{1}_A$ and $\pi_A$ are the identity operator and the maximally mixed state on $\mathcal{H}_A$, respectively. The support of a positive semidefinite operator $X$ is denoted by $\text{supp}(X)$. For $\rho \in \mathcal{S}(A)$, $\mathcal{S}_\rho(A)$ represents the set of quantum states whose support are included in $\text{supp}(\rho)$. Throughout this paper, we are restricted to quantum systems of finite dimension. We denote as $|A|$ the dimension of $\mathcal{H}_A$.

Let $\mathcal{X}$ denote a finite alphabet set. For the sequence $x^n \in \mathcal{X}^n$, the type [28] $P_{x^n}$ is defined as the empirical distribution of $x^n$, i.e.,

$$P_{x^n}(a) = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i,a}, \quad \forall a \in \mathcal{X}. \tag{1}$$

The notation $\mathcal{T}_n$ is used for the set of all types. We can bound the number of types as

$$|\mathcal{T}_n| \leq (n+1)^{|\mathcal{X}|}. \tag{2}$$

For $t \in \mathcal{T}_n$, we denote by $T^t_n$ the set of all sequences of type $t$, i.e.,

$$T^t_n := \{ x^n \mid P_{x^n} = t \}. \tag{3}$$

For a pure state $|\psi\rangle_{AA'}$ with Schmidt decomposition $|\psi\rangle_{AA'} = \sum_{x=1}^{|A|} \sqrt{p(x)} |a_x\rangle_{A} \otimes |a_x\rangle_{A'}$, we can write $|\psi\rangle_{AA'}^{\otimes n}$ as

$$|\psi\rangle_{AA'}^{\otimes n} = \sum_{t \in \mathcal{T}_n} \sqrt{p^n(t)} |\Psi^t\rangle_{A^nA'^n}, \tag{4}$$

where $\mathcal{T}_n$ is the set of types with respect to the alphabet set $\mathcal{X} = \{1, 2, \ldots, |A|\}$,

$$p^n(t) = \sum_{x^n \in T^t_n} p(x_1)p(x_2) \cdots p(x_n), \quad \text{and} \tag{5}$$

$$|\Psi^t\rangle_{A^nA'^n} = \frac{1}{\sqrt{|T^t_n|}} \sum_{x^n \in T^t_n} |a_{x^n}\rangle_{A^n} \otimes |a_{x^n}\rangle_{A'^n} \tag{6}$$

with $|a_{x^n}\rangle = |a_{x_1}\rangle \otimes |a_{x_2}\rangle \otimes \cdots \otimes |a_{x_n}\rangle$.

A quantum channel (or quantum operation) $\mathcal{N}$ is a linear, completely positive and trace-preserving (CPTP) map. We denote by $\mathcal{N}_{A\rightarrow B}$ a quantum channel $\mathcal{N} : \mathcal{L}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_B)$. A quantum measurement is represented by a set of positive semidefinite operators $\{M_x\}_x$ such that $\sum_x M_x = \mathbb{1}$. When making this measurement on a system in the state $\rho$, we get outcome $x$ with probability $\text{Tr} \rho M_x$.

Let $\sigma \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator. We denote the number of distinct eigenvalues of $\sigma$ as $v(\sigma)$. Let the spectral projections of $\sigma$ be $P_1, \ldots, P_{v(\sigma)}$. The pinching map $P_\sigma$ associated with $\sigma$ is given by

$$P_\sigma : X \mapsto \sum_{i=1}^{v(\sigma)} P_i X P_i. \tag{7}$$

The pinching inequality [29] says that for any positive semidefinite operator $X$, we have

$$X \leq v(\sigma) P_\sigma(X). \tag{8}$$
Let $S_n$ be the group of permutations over a set of $n$ elements. The natural representation of $S_n$ on $\mathcal{H}^{\otimes n}$ is given by the unitary transformations

$$W_{B^n}^\dagger : |\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle \mapsto |\psi_{\tau(1)}\rangle \otimes \cdots \otimes |\psi_{\tau(n)}\rangle, \quad |\psi_i\rangle \in \mathcal{H}_B, \tau \in S_n.$$ \hspace{1cm} (9)

We denote by $S_{\text{sym}}(B^n)$ the set of symmetric states on $\mathcal{H}^{\otimes n}$. That is,

$$S_{\text{sym}}(B^n) := \{ \sigma_{B^n} \in S(B^n) \mid W_{B^n}^\dagger \sigma_{B^n} W_{B^n}^\dagger = \sigma_{B^n}, \forall \tau \in S_n \},$$ \hspace{1cm} (10)

where $W_{B^n}^\dagger$ is the adjoint of $W_{B^n}$. There exists a symmetric state that dominates all the others, as stated in Lemma 1 below. Two different constructions are given by [30] and [31], respectively. Detailed arguments can be found in [32, Lemma 1] and [12, Appendix A].

**Lemma 1:** For every finite-dimensional system $B$ and every $n \in \mathbb{N}$, there exists a symmetric state $\sigma_{B^n}^\dagger \in S_{\text{sym}}(B^n)$ such that every symmetric state $\sigma_{B^n} \in S_{\text{sym}}(B^n)$ is dominated by $\sigma_{B^n}^\dagger$ as

$$\sigma_{B^n} \leq v_{n, |B|} \sigma_{B^n}^\dagger,$$ \hspace{1cm} (11)

where $v_{n, |B|} \leq (n + 1)^{(\frac{|B|+2}{|B|}-1)}$ is a polynomial of $n$. The number of different eigenvalues of $\sigma_{B^n}^\dagger$ is also upper bounded by $v_{n, |B|}$.

**B. Quantum Rényi divergences**

The classical Rényi divergence [33] is a crucial information quantity. In the quantum domain, there can be many inequivalent generalizations of the Rényi divergence. In this paper, we are concerned with the **sandwiched Rényi divergence** and the **log-Euclidean Rényi divergence**. The sandwiched Rényi divergence was introduced in [14] and [15], and it is one of the only two quantum Rényi divergences that have found operational meanings (the other one is Petz’s Rényi divergence [34]).

**Definition 2:** Let $\alpha \in (0, 1) \cup (1, +\infty)$, and let $\rho \in S(\mathcal{H})$ be a quantum state and $\sigma \in \mathcal{L}(\mathcal{H})_+$ be positive semidefinite. When $\alpha > 1$ and $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ or $\alpha \in (0, 1)$ and $\text{supp}(\rho) \not\subseteq \text{supp}(\sigma)$, the sandwiched Rényi divergence of order $\alpha$ is defined as

$$D_\alpha^s(\rho\|\sigma) := \frac{1}{\alpha - 1} \log \text{Tr} \left( \sigma^{\frac{1-\alpha}{\alpha}} \rho \sigma^{\frac{1}{\alpha}} \right)^\alpha;$$ \hspace{1cm} (12)

otherwise, we set $D_\alpha^s(\rho\|\sigma) = +\infty$.

The expression of the log-Euclidean Rényi divergence appeared already in [35] and [36], for certain range of the Rényi parameter. However, its significance has been recognized only quite recently in [12], where it has been thoroughly studied and was for the first time employed as an intermediate quantity in the derivation of the strong converse exponent for classical-quantum channels. This name was first called in [37].

**Definition 3:** Let $\alpha \in (0, 1) \cup (1, +\infty)$, and let $\rho \in S(\mathcal{H})$ be a quantum state and $\sigma \in \mathcal{L}(\mathcal{H})_+$ be positive semidefinite. The log-Euclidean Rényi divergence of order $\alpha$ is defined as

$$D_\alpha^L(\rho\|\sigma) := \frac{1}{\alpha - 1} \log \text{Tr} \left( 2^{\alpha \log \rho + (1-\alpha) \log \sigma} \right)^\alpha;$$ \hspace{1cm} (13)

if $\rho$ and $\sigma$ are of full rank; otherwise, we replace the expression in the logarithm by

$$\lim_{\epsilon \searrow 0} \text{Tr} 2^{\alpha \log(\rho + \epsilon 1) + (1-\alpha) \log(\sigma + \epsilon 1)}.$$ \hspace{1cm} (14)

**Remark 4:** The existence of the limit in Eq. (14) was shown in [12, Lemma 3.1], where an alternative expression was given. The condition for $D_\alpha^L(\rho\|\sigma) = +\infty$ is $\alpha > 1$ and $\text{supp}(\rho) \not\subseteq \text{supp}(\sigma)$ or $\alpha \in (0, 1)$ and $\text{supp}(\rho) \cap \text{supp}(\sigma) = \{0\}$; see [12, Remark 3.4].

For a bipartite state $\rho_{AB} \in S(AB)$ and $\alpha \in (0, 1) \cup (1, +\infty)$, the sandwiched Rényi mutual information is defined as [15], [38]

$$I_\alpha^s(A : B)_\rho = \min_{\sigma_B} D_\alpha^s(\rho_{AB}\|\rho_A \otimes \sigma_B).$$ \hspace{1cm} (15)
When \( \alpha \) goes to 1, both \( D^*_\alpha \) and \( D^\alpha \) converge to the quantum relative entropy [39]

\[
D(\rho \| \sigma) := \begin{cases} 
\text{Tr}(\rho (\log \rho - \log \sigma)) & \text{if } \text{supp}(\rho) \subseteq \text{supp}(\sigma), \\
+\infty & \text{otherwise}.
\end{cases}
\]

Similarly, the quantum mutual information for \( \rho_{AB} \in \mathcal{S}(AB) \), defined as

\[
I(A : B)_\rho := D(\rho_{AB} \| \rho_A \otimes \rho_B),
\]

is the limiting case of the sandwiched Rényi mutual information. So, we extend the definition of the Rényi information quantities to include the case \( \alpha = 1 \) by taking the limits. We refer the readers to [14], [15], [12] for the computation of these limits.

An ensemble of quantum states is a set of pairs \( \{p_x, \rho_x\}_{x \in X} \), where \( \{p_x\}_{x \in X} \) is a probability distribution and \( \{\rho_x\}_{x} \) are quantum states. Its Holevo information is defined as

\[
\chi(\{p_x, \rho_x\}_{x}) := \sum_x p_x D(\rho_x \| \sum_x p_x \rho_x).
\]

In the following proposition, we collect some properties of the Rényi information quantities.

**Proposition 5:** Let \( \rho \in \mathcal{S}(\mathcal{H}) \) be a quantum state and \( \sigma \in \mathcal{L}(\mathcal{H})_+ \) be positive semidefinite. The sandwiched Rényi divergence and the log-Euclidean Rényi divergence satisfy the following properties.

(i) Monotonicity in Rényi parameter [38], [14], [12]: if \( 0 \leq \alpha \leq \beta \), then \( D^\alpha (\rho \| \sigma) \leq D^\beta (\rho \| \sigma) \), for \( (t) = * \) and \( (t) = \flat \);

(ii) Monotonicity in \( \sigma \) [14], [12]: if \( \sigma' \geq \sigma \), then \( D^\alpha (\rho \| \sigma') \leq D^\alpha (\rho \| \sigma) \), for \( (t) = * \), \( \alpha \in \left[ \frac{1}{2}, +\infty \right) \) and for \( (t) = \flat \), \( \alpha \in [0, +\infty) \);

(iii) Variational representation [12]: the log-Euclidean Rényi divergence has the following variational representation

\[
D^\alpha (\rho \| \sigma) = s(\alpha) \max_{\tau \in \mathcal{S}_\rho (\mathcal{H})} s(\alpha) \{ D(\tau \| \sigma) - \frac{\alpha}{\alpha - 1} D(\tau \| \rho) \},
\]

where \( s(\alpha) = 1 \) for \( \alpha \in (1, +\infty) \) and \( s(\alpha) = -1 \) for \( \alpha \in (0, 1) \), and \( \mathcal{S}_\rho (\mathcal{H}) \) denotes the set of states whose supports are contained in the support of \( \rho \);

(iv) Additivity of sandwiched Rényi mutual information [32]: for two states \( \rho_{AB} \in \mathcal{S}(AB) \) and \( \sigma_{A'B'} \in \mathcal{S}(A'B') \), and for any \( \alpha \in \left[ \frac{1}{2}, +\infty \right) \), we have

\[
I^\alpha (AA' : BB')_{\rho \otimes \sigma} = I^\alpha (A : B)_\rho + I^\alpha (A' : B')_{\sigma};
\]

(v) Convexity in \( \sigma \) [12]: the function \( \sigma \mapsto D^\alpha (\rho \| \sigma) \) is convex for \( (t) = * \), \( \alpha \in \left[ \frac{1}{2}, +\infty \right) \) and for \( (t) = \flat \), \( \alpha \in [0, +\infty) \);

(vi) Approximation by pinching [40], [32]: for \( \alpha \geq 0 \), we have

\[
D^\alpha (\mathcal{P}_\sigma (\rho) \| \sigma) \leq D^\alpha (\rho \| \sigma) \leq D^\alpha (\mathcal{P}_\sigma (\rho) \| \sigma) + 2 \log v(\sigma).
\]

### III. Problems and Results

#### A. Entanglement-assisted classical communication

Suppose that the sender Alice and the receiver Bob are connected by a quantum channel \( \mathcal{N}_{A \rightarrow B} \), and they share arbitrary entangled quantum states. Alice wants to send classical messages to Bob. We start with a description of a code for a single use of the channel \( \mathcal{N}_{A \rightarrow B} \). Let \( \mathcal{M} = \{1, \ldots, M\} \) be the set of messages to be transmitted. Let \( \rho_{\overline{A}B} \) be the entangled state shared by Alice (\( \overline{A} \)) and Bob (\( B \)). To send the message \( m \in \mathcal{M} \), Alice applies a CPTP map \( \mathcal{E}^m_{\overline{A} \rightarrow A} \) to her half of the state \( \rho_{\overline{A}B} \). Then she inputs the \( A \) system to the channel \( \mathcal{N}_{A \rightarrow B} \). After receiving the channel output \( B \), Bob performs a decoding measurement \( \{\Lambda^m_{BB}\}_{m \in \mathcal{M}} \) on the system \( BB \) to recover the classical message. The collection

\[
\mathcal{C} \equiv \{\mathcal{E}^m_{\overline{A} \rightarrow A}\}_{m \in \mathcal{M}}, \{\Lambda^m_{BB}\}_{m \in \mathcal{M}, \rho_{\overline{A}B}}
\]
is called a code for $\mathcal{N}_{A\to B}$. The size of the code is $|C| := M$, and the success probability of $C$ is

$$P_s(\mathcal{N}_{A\to B}, C) := \frac{1}{M} \sum_{m=1}^{M} \text{Tr} \mathcal{N}_{A\to B} \circ \mathcal{E}_{A\to A}^m(\rho_{AB}) \Lambda_{B\bar{B}}^m.$$  \hspace{1cm} (23)

The art in information theory is to make codes for multiple uses of the channel. We denote a code for $\mathcal{N}_{A\to B}^\otimes n$ by $C_n$. The channel capacity is the maximal rate of information transmission that can be achieved by a sequence of codes $\{C_n\}_{n\in\mathbb{N}}$, under the condition that the success probability goes to 1 asymptotically. Formally, we define the entanglement-assisted classical capacity as

$$C_E(\mathcal{N}_{A\to B}) := \sup \left\{ \liminf_{n\to\infty} \frac{1}{n} \log |C_n| \bigg| \lim_{n\to\infty} P_s(\mathcal{N}_{A\to B}^\otimes n, C_n) = 1 \right\}. \hspace{1cm} (24)$$

It has been established in [4] that

$$C_E(\mathcal{N}_{A\to B}) = \max_{\psi_{A'A} \in \mathcal{S}_1(A'A)} I(A' : B)_{\mathcal{N}_{A\to B}(\psi_{A'A})}. \hspace{1cm} (25)$$

Note that $\mathcal{S}_1(A'A)$ is the set of pure states, and without loss of generality, we can assume that $|A'| = |A|$ in Eq. (25).

### B. Main result

It is known that when the transmission rate is larger than the capacity $C_E(\mathcal{N}_{A\to B})$, the success probability $P_s(\mathcal{N}_{A\to B}^\otimes n, C_n)$ inevitably converges to 0 as $n \to \infty$ [7]. This is called the strong converse property. The strong converse exponent characterizes the speed of this convergence. It is defined as the best rate of exponential decay of the success probability, for a fixed transmission rate $R$:

$$sc(\mathcal{N}_{A\to B}, R) := \inf \left\{ -\liminf_{n\to\infty} \frac{1}{n} \log P_s(\mathcal{N}_{A\to B}^\otimes n, C_n) \bigg| \liminf_{n\to\infty} \frac{1}{n} \log |C_n| \geq R \right\}. \hspace{1cm} (26)$$

Gupta and Wilde [13] have proven that

$$sc(\mathcal{N}_{A\to B}, R) \geq \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left( R - I^*_\alpha(\mathcal{N}_{A\to B}) \right), \hspace{1cm} (27)$$

where $I^*_\alpha(\mathcal{N}_{A\to B})$ is the sandwiched Rényi information of the channel $\mathcal{N}_{A\to B}$, defined as

$$I^*_\alpha(\mathcal{N}_{A\to B}) := \max_{\psi_{A'A} \in \mathcal{S}_1(A'A)} I^*_\alpha(A' : B)_{\mathcal{N}_{A\to B}(\psi_{A'A})}. \hspace{1cm} (28)$$

Without loss of generality, we assume that $|A'| = |A|$ in Eq. (28). The main contribution of the present paper is to prove the other direction. As a result, we have established the following equality.

**Theorem 6:** Let $\mathcal{N}_{A\to B}$ be a quantum channel. For any transmission rate $R > 0$, the strong converse exponent for entanglement-assisted classical communication is

$$sc(\mathcal{N}_{A\to B}, R) = \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left( R - I^*_\alpha(\mathcal{N}_{A\to B}) \right). \hspace{1cm} (29)$$

Theorem 6 has two implications as illustrated in the next two subsections. This enables us to also determine the strong converse exponents for quantum-feedback-assisted communication and for transmitting quantum information in the same settings. The proof of Theorem 6 is given in Section IV and Section V.
C. Implication to quantum-feedback-assisted communication

For a quantum channel \( \mathcal{N}_{A\to B} \), quantum-feedback-assisted classical communication is the transmission of classical information with the assistance of noiseless quantum feedback from the receiver to the sender [5], [18]. Since quantum feedback can generate arbitrary bipartite entangled state, it is a stronger resource than shared entanglement. Indeed, using quantum teleportation [20], we can easily see that quantum feedback is equivalent to shared entanglement plus classical feedback.

The quantum-feedback-assisted classical capacity of a quantum channel \( \mathcal{N}_{A\to B} \) is still \( C_E(\mathcal{N}_{A\to B}) \) [5]. When the transmission rate is larger than \( C_E(\mathcal{N}_{A\to B}) \), the strong converse property was proved in [7] via the technique of channel simulation. The exact strong converse exponent was not known, however. We point out that, a general quantum-feedback-assisted code can be very complicated, where the encoding can adaptively depend on previous feedbacks. We refer to [18, Section 5] for an explicit description of the quantum-feedback-assisted codes. The strong converse exponent in this setting is defined similar to Eq. (26), and we denote it by \( sc^c(\mathcal{N}_{A\to B}, R) \).

Cooney, Mosonyi and Wilde have proved in [18] that the right hand side of Eq. (29) is a lower bound for \( sc^c(\mathcal{N}_{A\to B}, R) \) as well. As an open question, they ask whether it is optimal. Having shown that it is already achievable under the assistance of shared entanglement, we answer their question in the affirmative.

**Corollary 7:** Let \( \mathcal{N}_{A\to B} \) be a quantum channel. For any transmission rate \( R > 0 \), the strong converse exponent for quantum-feedback-assisted classical communication is

\[
sc^c(\mathcal{N}_{A\to B}, R) = \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left( R - I_\alpha^* (\mathcal{N}_{A\to B}) \right).
\]

**Corollary 7** is an immediate consequence of [18, Theorem 4] and our Theorem 6, noting that by definition \( sc^c(\mathcal{N}_{A\to B}, R) \leq sc(\mathcal{N}_{A\to B}, R) \) is obvious.

D. Implication to quantum information transmission

The above results can be extended to the case of quantum information transmission. Let us recall an entanglement-assisted code for a single use of the channel \( \mathcal{N}_{A\to B} \), for transmitting quantum information that is stored in a quantum system \( M \). It consists of using a shared entangled state \( \rho_{\tilde{A}\tilde{B}} \) between the sender Alice and the receiver Bob, applying local operation \( \mathcal{E}_{M\tilde{A}\to A} \) at Alice’s side, feeding the system \( A \) into \( \mathcal{N}_{A\to B} \), and at last applying local operation \( \mathcal{D}_{B\tilde{B}\to M} \) at Bob’s side. We denote by

\[
\mathcal{C} \equiv (\mathcal{E}_{M\tilde{A}\to A}; \mathcal{D}_{B\tilde{B}\to M}; \rho_{\tilde{A}\tilde{B}})
\]

the code for \( \mathcal{N}_{A\to B} \). Its size is \( |\mathcal{C}| := |M| \), and its performance is quantified by the entanglement fidelity

\[
P_f(\mathcal{N}_{A\to B}, \mathcal{C}) := F(\mathcal{D}_{B\tilde{B}\to M} \circ \mathcal{N}_{A\to B} \circ \mathcal{E}_{M\tilde{A}\to A} (\Psi_{M\tilde{M}} \otimes \rho_{\tilde{A}\tilde{B}}), \Psi_{M\tilde{M}}),
\]

where \( \Psi_{M\tilde{M}} \) is a maximally entangled pure state and \( F(\rho, \sigma) = \| \sqrt{\rho} \sqrt{\sigma} \|_1 \) is the fidelity. We remark that the entanglement fidelity \( P_f \) can be regarded as an analogue of the success probability of Eq. (23). Denote a code for \( \mathcal{N}_{A\to B} \) by \( \mathcal{C}_n \). Then the strong converse exponent for entanglement-assisted quantum communication is defined similar to Eq. (26)—with \( \mathcal{C}_n \) replaced by \( \mathcal{C}_n \) and \( P_s \) replaced by \( P_f \)—and we denote it as \( sc^e(\mathcal{N}_{A\to B}, R) \).

Let \( P_s^e(\mathcal{N}_{A\to B}, k) \) be the optimal performance among all entanglement-assisted codes for quantum information transmission with size \( k \), and \( P_s^e(\mathcal{N}_{A\to B}, k) \) be the optimal success probability among all entanglement-assisted codes for classical information transmission with size \( k \). Leung and Matthews [19, Appendix B] have derived the exact relation

\[
(P_f^e(\mathcal{N}_{A\to B}, k))^2 = P_s^e(\mathcal{N}_{A\to B}, k^2).
\]

The combination of Theorem 6 and Eq. (33) directly gives the formula for \( sc^e(\mathcal{N}_{A\to B}, R) \).
Corollary 8: Let $\mathcal{N}_{A\rightarrow B}$ be a quantum channel. For any transmission rate $R > 0$, the strong converse exponent for entanglement-assisted quantum communication is

$$sc^q(\mathcal{N}_{A\rightarrow B}, R) = \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left( R - \frac{1}{2} I^*_\alpha(\mathcal{N}_{A\rightarrow B}) \right).$$

(34)

If the shared entanglement is replaced by quantum feedback, the strong converse exponent for quantum information transmission is still given by the right hand side of Eq. (34), which can be verified by a similar argument.

IV. AN UPPER BOUND VIA THE LOG-EUCLIDEAN RÉNYI INFORMATION

In this section, we derive an intermediate upper bound for the strong converse exponent of entanglement-assisted classical communication. It employs a type of Rényi information of quantum channels, in terms of the log-Euclidean Rényi divergence. This upper bound serves as a first step for the proof of Theorem 6. The full proof will be accomplished in Section V.

For brevity of expression, from now on we change the notation a little to use $A'$ to label the input system of the channel $\mathcal{N}$. The symbol $A$ is instead used to label a bipartite state $\psi_{AA'}$ whose $A'$ part is to be acted on by the channel $\mathcal{N}_{A\rightarrow B}$, resulting in a bipartite state on systems $A$ and $B$.

Definition 9: For the quantum channel $\mathcal{N}_{A\rightarrow B}$ and an arbitrary ensemble of bipartite quantum states $\{q(t), \psi_{AA'}^t\}_{t \in T}$ with $q$ a probability distribution, we define

$$I^*_\alpha(\mathcal{N}_{A\rightarrow B}, \{q(t), \psi_{AA'}^t\}_t) := \sum_{t \in T} q(t) \min_{\sigma_B \in S(B)} D^\alpha(\mathcal{N}_{A\rightarrow B}(\psi_{AA'}^t) \| \psi_A \otimes \sigma_B).$$

(35)

When the quantum state ensemble consists of only a single state $\psi_{AA'}$, we also write

$$I^*_\alpha(\mathcal{N}_{A\rightarrow B}, \psi_{AA'}) := \min_{\sigma_B \in S(B)} D^\alpha(\mathcal{N}_{A\rightarrow B}(\psi_{AA'}) \| \psi_A \otimes \sigma_B).$$

(36)

We always use $\Psi$ to denote the maximally entangled state on two isomorphic Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, whose vector form can be written as

$$|\Psi\rangle_{\mathcal{H}\mathcal{H}'} = \frac{1}{\sqrt{|\mathcal{H}|}} \sum_{x=1}^{|\mathcal{H}|} |a_x\rangle_{\mathcal{H}} \otimes |a_x\rangle_{\mathcal{H}'}$$

(37)

where $\{|a_x\rangle\}_x$ is a pre-fixed orthonormal basis.

We first prove the following upper bound. Then we will further improve it to obtain the main result of this section (Theorem 14).

Proposition 10: Let $\mathcal{H}_A \cong \mathcal{H}_{A'}$ and let $\mathcal{H}_A = \bigoplus_{t \in T} \mathcal{H}_A^t$ and $\mathcal{H}_{A'} = \bigoplus_{t \in T} \mathcal{H}_{A'}^t$ be decompositions of $\mathcal{H}_A$ and $\mathcal{H}_{A'}$ into orthogonal subspaces with $\mathcal{H}_A^t \cong \mathcal{H}_{A'}^t$. Let $\{q(t), \Psi_{AA'}^t\}_{t \in T}$ be an ensemble of quantum states with $\Psi_{AA'}^t$ the maximally entangled state on $\mathcal{H}_A^t \otimes \mathcal{H}_{A'}^t$. For the channel $\mathcal{N}_{A\rightarrow B}$ and any $R > 0$, the strong converse exponent for entanglement-assisted classical communication satisfies

$$sc(\mathcal{N}_{A\rightarrow B}, R) \leq \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - I^*_\alpha(\mathcal{N}_{A\rightarrow B}, \{q(t), \Psi_{AA'}^t\}_t) \right\}. $$

(38)

Before proving Proposition 10, we derive a variational expression for the right hand side of Eq. (38). This is given in Proposition 11 in a more general form. For brevity, we introduce the shorthand

$$F(\mathcal{N}_{A\rightarrow B}, R, \{q(t), \psi_{AA'}^t\}_t) \equiv \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - I^*_\alpha(\mathcal{N}_{A\rightarrow B}, \{q(t), \psi_{AA'}^t\}_t) \right\}. $$

(39)
**Proposition 11:** For a channel $\mathcal{N}_{A'\rightarrow B}$, any ensemble of quantum states $\{q(t), \psi_{AA'}^t\}_{t \in T}$ and any $R \geq 0$, it holds that

$$F(\mathcal{N}_{A'\rightarrow B}, R, \{q(t), \psi_{AA'}^t\})$$

$$= \inf_{\{\tau_{AB}^t\}_{t \in T} \in \mathcal{F}} \left\{ \left( R - \sum_{t \in T} q(t) D(\tau_{AB}^t \| \psi_{A}^t \otimes \tau_{B}^t) \right) + \sum_{t \in T} q(t) D(\tau_{AB}^t \| \mathcal{N}(\psi_{AA'}^t)) \right\},$$

(40)

where $\mathcal{F} := \{\{\tau_{AB}^t\}_{t \in T} \mid \tau_{AB}^t \in \mathcal{S}_{\mathcal{N}(\psi_{AA'}^t)}(AB), \forall t \in T\}$, and $x_+ := \max\{0, x\}$ for a real number $x$.

**Proof:** Define the following sets

$$\mathcal{O}_t := \{\sigma_B \mid \sigma_B \in \mathcal{S}(B), \supp(\sigma_B) \supseteq \supp(\mathcal{N}(\psi_{A}^t))\},$$

(41)

$$\mathcal{F}_t := \{\tau_{AB} \mid \tau_{AB} \in \mathcal{S}_{\mathcal{N}(\psi_{AA'}^t)}(AB)\}$$

(42)

for all $t \in T$. The minimization in Eq. (35) can be restricted to those $\sigma_B$ whose support contains the support of $\mathcal{N}(\psi_{A}^t)$, because otherwise $D_{\alpha}(\mathcal{N}(\psi_{AA'}^t)\|\psi_{A}^t \otimes \sigma_B) = +\infty$. Thus we have

$$F(\mathcal{N}_{A'\rightarrow B}, R, \{q(t), \psi_{AA'}^t\})$$

$$= \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - \sum_{t \in T} q(t) \inf_{\sigma_B \in \mathcal{O}_t} D_{\alpha}(\mathcal{N}_{A'\rightarrow B}(\psi_{AA'}^t)\|\psi_{A}^t \otimes \sigma_B) \right\}$$

$$= \sup_{\alpha > 1} \sum_{t \in T} q(t) \sup_{\sigma_B \in \mathcal{O}_t} \inf_{\tau_{AB} \in \mathcal{F}_t} \left\{ \frac{\alpha - 1}{\alpha} \left( R - D(\tau_{AB} \| \psi_{A}^t \otimes \sigma_B) \right) + D(\tau_{AB} \| \mathcal{N}(\psi_{AA'}^t)) \right\},$$

(43)

where the last equality follows from the variational expression of $D_{\alpha}(\rho \| \sigma)$ (see Proposition 5 (iii)). Let

$$f_t(\sigma_B, \tau_{AB}) := \frac{\alpha - 1}{\alpha} \left( R - D(\tau_{AB} \| \psi_{A}^t \otimes \sigma_B) \right) + D(\tau_{AB} \| \mathcal{N}(\psi_{AA'}^t))$$

$$= \frac{\alpha - 1}{\alpha} \left( R + \text{Tr}[\tau_{AB} \log(\psi_{A}^t \otimes \sigma_B)] \right) - \text{Tr}[\tau_{AB} \log \mathcal{N}(\psi_{AA'}^t)] - \frac{1}{\alpha} H(\tau_{AB}),$$

(44)

where $H(\rho) := -\text{Tr}(\rho \log \rho)$ is the von Neumann entropy. The relative entropy is convex with respect to both arguments, and the von Neumann entropy is concave. So, we can easily verify that, for any $\alpha > 1$, $f_t$ satisfies the following properties:

(i) for fixed $\tau_{AB}$, $f_t(\cdot, \tau_{AB})$ is concave and continuous on $\mathcal{O}_t$, and $\mathcal{O}_t$ is convex;

(ii) for fixed $\sigma_B$, $f_t(\sigma_B, \cdot)$ is convex and continuous on $\mathcal{F}_t$, and $\mathcal{F}_t$ is a compact convex set.

So, Sion’s minimax theorem (see Lemma 19 in the Appendix) applies. This lets us proceed as

$$F(\mathcal{N}_{A'\rightarrow B}, R, \{q(t), \psi_{AA'}^t\})$$

$$= \sup_{\alpha > 1} \sum_{t \in T} q(t) \inf_{\tau_{AB} \in \mathcal{F}_t} \inf_{\sigma_B \in \mathcal{O}_t} \left\{ \frac{\alpha - 1}{\alpha} \left( R - D(\tau_{AB} \| \psi_{A}^t \otimes \sigma_B) \right) + D(\tau_{AB} \| \mathcal{N}(\psi_{AA'}^t)) \right\}$$

$$= \sup_{\alpha > 1} \sum_{t \in T} q(t) \inf_{\tau_{AB} \in \mathcal{F}_t} \left\{ \frac{\alpha - 1}{\alpha} \left( R - D(\tau_{AB} \| \psi_{A}^t \otimes \tau_{B}^t) \right) + D(\tau_{AB} \| \mathcal{N}(\psi_{AA'}^t)) \right\}$$

$$= \sup_{\lambda \in (0, 1)} \inf_{\{\tau_{AB}^t\}_{t \in T} \in \mathcal{F}} \left\{ \lambda (R - \sum_{t \in T} q(t) D(\tau_{AB}^t \| \psi_{A}^t \otimes \tau_{B}^t)) + \sum_{t \in T} q(t) D(\tau_{AB}^t \| \mathcal{N}(\psi_{AA'}^t)) \right\}$$

$$= \inf_{\{\tau_{AB}^t\}_{t \in T} \in \mathcal{F}} \left\{ \lambda (R - \sum_{t \in T} q(t) D(\tau_{AB}^t \| \psi_{A}^t \otimes \tau_{B}^t)) + \sum_{t \in T} q(t) D(\tau_{AB}^t \| \mathcal{N}(\psi_{AA'}^t)) \right\},$$

(45)
where for the fourth equality, we have used Sion’s minimax theorem again. It can be verified similarly that the conditions for Sion’s minimax theorem are satisfied. The convexity of the expression under optimization as a function of \( \{ \tau_{tAB}^i \}_t \) is not obvious. To see this, we write it as

\[
\sum_{t \in T} q(t) \left\{ \lambda R - \lambda D(\tau_{tAB}^i | \psi_A^t \otimes \tau_B^t) + D(\tau_{tAB}^i | \mathcal{N}(\psi_{AA'})^t) \right\} = \sum_{t \in T} q(t) \left\{ \lambda R - (1 - \lambda) H(\tau_{tAB}^i) - \lambda H(\text{Tr}_A \tau_{tAB}^i) + \text{Tr} \left[ \tau_{tAB}^i (\lambda \log \psi_A^t - \log \mathcal{N}(\psi_{AA'})^t) \right] \right\}.
\] (46)

The convexity follows from the fact that for any \( t \in T \), \(- (1 - \lambda) H(\tau_{tAB}^i) \) and \(- \lambda H(\text{Tr}_A \tau_{tAB}^i) \) are convex as functions of \( \tau_{tAB}^i \), and \( \text{Tr} \left[ \tau_{tAB}^i (\lambda \log \psi_A^t - \log \mathcal{N}(\psi_{AA'})^t) \right] \) is linear.

To prove Proposition 10, we introduce

\[
F_1(\mathcal{N}, R, \{ q(t), \Psi_{AA'}^t \}_t) := \inf_{\{ \tau_{tAB}^i \}_t \in \mathcal{F}_1} \sum_{t \in T} q(t) D(\tau_{tAB}^i | \mathcal{N}(\psi_{AA'}^t)),
\] (47)

\[
F_2(\mathcal{N}, R, \{ q(t), \Psi_{AA'}^t \}_t) := \inf_{\{ \tau_{tAB}^i \}_t \in \mathcal{F}_2} \left\{ R - \sum_{t \in T} q(t) D(\tau_{tAB}^i | \pi_A^t \otimes \tau_B^t) + \sum_{t \in T} q(t) D(\tau_{tAB}^i | \mathcal{N}(\psi_{AA'}^t)) \right\}
\] (48)

with

\[
\mathcal{F}_1 := \left\{ \{ \tau_{tAB}^i \}_t \in T \mid (\forall t \in T) \tau_{tAB}^i \in \mathcal{S}_{\mathcal{N}(\psi_{AA'}^t)}(AB), \sum_{t \in T} q(t) D(\tau_{tAB}^i | \pi_A^t \otimes \tau_B^t) > R \right\},
\] (49)

\[
\mathcal{F}_2 := \left\{ \{ \tau_{tAB}^i \}_t \in T \mid (\forall t \in T) \tau_{tAB}^i \in \mathcal{S}_{\mathcal{N}(\psi_{AA'}^t)}(AB), \sum_{t \in T} q(t) D(\tau_{tAB}^i | \pi_A^t \otimes \tau_B^t) \leq R \right\},
\] (50)

where \( \pi_A^t \) is the maximally mixed state on \( \mathcal{H}_A \). It is obvious that

\[
F(\mathcal{N}, R, \{ q(t), \Psi_{AA'}^t \}_t) = \min \left\{ F_1(\mathcal{N}, R, \{ q(t), \Psi_{AA'}^t \}_t), F_2(\mathcal{N}, R, \{ q(t), \Psi_{AA'}^t \}_t) \right\}.
\] (51)

So, it suffices to show that \( sc(\mathcal{N}_A \rightarrow_B, R) \) is upper bounded by both \( F_1 \) and \( F_2 \), given in Eq. (47) and Eq. (48), respectively.

**Proof of Proposition 10:** Thanks to Proposition 11, it follows from Lemma 12 and Lemma 13 below.

**Lemma 12:** Let \( \{ q(t), \Psi_{AA'}^t \}_t \in T \) be any ensemble of quantum states as specified in Proposition 10. For the channel \( \mathcal{N}_A \rightarrow_B \) and any \( R > 0 \) we have

\[
sc(\mathcal{N}_A \rightarrow_B, R) \leq F_1(\mathcal{N}_A \rightarrow_B, R, \{ q(t), \Psi_{AA'}^t \}_t).
\] (52)

**Proof:** If \( \mathcal{F}_1 = \emptyset \), the statement is trivial because the right hand side is \(+\infty\). So we suppose that \( \mathcal{F}_1 \neq \emptyset \). The definition of \( F_1 \) in Eq. (47) and Eq. (49) implies that for any \( \delta > 0 \), there exists a set of states \( \{ \tau_{tAB}^i \}_t \in \mathcal{F}_1 \) such that \( \tau_{tAB}^i \in \mathcal{S}_{\mathcal{N}(\psi_{AA'}^t)}(AB) \) and

\[
\sum_{t \in T} q(t) D(\tau_{tAB}^i | \pi_A^t \otimes \tau_B^t) > R,
\] (53)

\[
\sum_{t \in T} q(t) D(\tau_{tAB}^i | \mathcal{N}(\psi_{AA'}^t)) \leq F_1(\mathcal{N}_A \rightarrow_B, R, \{ q(t), \Psi_{AA'}^t \}_t) + \delta.
\] (54)

We will employ the Heisenberg-Weyl operators. Defined on a \( d \)-dimensional Hilbert space \( \mathcal{H} \) with an orthonormal basis \( \{|x\rangle\}_{x=0}^{d-1} \), they are a collection of unitary operators

\[
V_{y,z} = \sum_{x=0}^{d-1} e^{2\pi i y \frac{x}{d}} |x + y \mod d\rangle \langle x|.
\] (55)
where $y, z \in \{0, 1, \ldots, d-1\}$. Let $\mathcal{V}'$ be the set of Heisenberg-Weyl operators on $\mathcal{H}_A$, defined with respect to the basis $\{|a_{x}^{t}\rangle \rangle_{x}$ for which $|\Psi\rangle_{AA'} = \frac{1}{\sqrt{|\mathcal{H}_A|}} \sum_{x} |a_{x}^{t}\rangle \rangle_{A} \otimes |a_{x}^{t}\rangle \rangle_{A'}$. This ensures that for any $U^t \in \mathcal{V}'$, 

$$
U^t \otimes 1 |\Psi\rangle_{AA'} = 1 \otimes (U^t)^T |\Psi\rangle_{AA'},
$$

where $(U^t)^T$ is the transpose of $U^t$ and it acts on $\mathcal{H}_{A'}$. Let

$$
\mathcal{U} := \left\{ \bigoplus_{t \in T} U_t \mid (\forall t) U^t \in \mathcal{V}' \right\}
$$

be a set of unitary operators on $\mathcal{H}_A$. We consider the ensemble of quantum states

$$
\left\{ \frac{1}{|\mathcal{U}|} U_A (\sum_{t \in T} q(t) \tau_{AB}^t) U_A^\dagger \right\}_{U_A \in \mathcal{U}}
$$

Its Holevo information is evaluated as

$$
\sum_{U_A \in \mathcal{U}} \frac{1}{|\mathcal{U}|} D \Bigg( U_A (\sum_{t \in T} q(t) \tau_{AB}^t) U_A^\dagger \Bigg) := \frac{1}{|\mathcal{U}|} \sum_{U_A \in \mathcal{U}} \frac{1}{|\mathcal{U}|} U_A (\sum_{t \in T} q(t) \tau_{AB}^t) U_A^\dagger 
$$

$$
= \sum_{U_A \in \mathcal{U}} \frac{1}{|\mathcal{U}|} D \Bigg( U_A (\sum_{t \in T} q(t) \tau_{AB}^t) U_A^\dagger \Bigg) := \frac{1}{|\mathcal{U}|} \sum_{U_A \in \mathcal{U}} \frac{1}{|\mathcal{U}|} U_A (\sum_{t \in T} q(t) \tau_{AB}^t) U_A^\dagger 
$$

$$
= D \Bigg( \sum_{t \in T} q(t) \tau_{AB}^t \Bigg) := \sum_{t \in T} q(t) D(\tau_{AB}^t).
$$

(59)

So, for the ensemble given in Eq. (58) and $R$ satisfying Eq. (53), we are able to apply Lemma 20. Thus for $n \in \mathbb{N}$ one can construct a set of signal states

$$
\mathcal{O}_n \equiv \left\{ \mathcal{E}_{A^n}^m \left( \left( \sum_{t \in T} q(t) \tau_{AB}^t \right)^{\otimes n} \right) \right\}_{m \in \mathcal{M}_n}
$$

encoding messages $m \in \mathcal{M}_n$ of size $|\mathcal{M}_n| = 2^{nR}$, such that there exists a decoding measurement

$$
\mathcal{D}_n \equiv \left\{ \Lambda_{A^n B^n}^m \right\}_{m \in \mathcal{M}_n}
$$

(61)

with success probability

$$
\tilde{P}_n(\mathcal{O}_n, \mathcal{D}_n) := \frac{1}{|\mathcal{M}_n|} \sum_{m \in \mathcal{M}_n} \text{Tr} \mathcal{E}_{A^n}^m \left( \left( \sum_{t \in T} q(t) \tau_{AB}^t \right)^{\otimes n} \right) \Lambda_{A^n B^n}^m 
$$

$$
\rightarrow 1, \quad \text{as } n \rightarrow \infty.
$$

(62)

Moreover, $\mathcal{E}_{A^n}^m(\cdot) = U_m(\cdot) U_m^\dagger$ is a unitary operation on $A^n$ of the form

$$
U_m = \left\{ \bigoplus_{t \in T} U_1^t (m) \right\} \otimes \cdots \otimes \left\{ \bigoplus_{t \in T} U_n^t (m) \right\}
$$

(63)

where $U_1^t (m) \in \mathcal{V}'$ is a Heisenberg-Weyl operator on $\mathcal{H}_A$.

Now, given the message set $\mathcal{M}_n$, we design a code $\mathcal{C}_n$ for $\mathcal{N}_{A^n \rightarrow B^n}$ as follows.

1. Alice and Bob share the entangled state $\rho_{A^n A'^n} = \left( \sum_{t \in T} q(t) |\Psi\rangle_{AA'}^{\otimes n} \right)$. Alice holds the $A^n$ part, and Bob holds the $A'$ part.

2. Alice encodes the classical message $m \in \mathcal{M}_n$ by applying the map $(\mathcal{E}_{A^n}^m)^T$ to the $A^n$ part of the state $\rho_{A^n A'^n}$. Here $(\mathcal{E}_{A^n}^m)^T(\cdot) = (U_m)^T(\cdot)((U_m)^T)^\dagger$ and

$$
(U_m)^T = \left( \bigoplus_{t \in T} (U_1^t (m))^T \right) \otimes \cdots \otimes \left( \bigoplus_{t \in T} (U_n^t (m))^T \right).
$$

(64)
is the transpose of $U_m$. Then she sends the $A_m^n$ part to Bob through the channel $N_{A' \to B}^{\otimes m}$.

3. Bob performs the decoding measurement $\{A_{A'B^n}^m\}_{m \in \mathcal{M}_n}$ of Eq. (61), aiming to recover the message $m$.

The success probability of the code $C_n$ can be evaluated as

$$P_s(N_{A' \to B}^{\otimes m} : C_n) = \frac{1}{|M_n|} \sum_{m \in \mathcal{M}_n} \text{Tr}(N_{A' \to B}^{\otimes m})^T (\rho A_m^n A_m^n) A_{A'B^n}^m$$

where the second equality is by the relation of Eq. (66).

To proceed, we set

$$\alpha = \sum_{t \in T} q(t) D(\tau_{AB}^t \| \mathcal{N}(\psi_{AA'}^t)) + \delta. \quad (66)$$

Then Eq. (62) and Eq. (65) together lead to

$$\widetilde{P}_s((O_n, D_n)) - 2^{na} P_s(N_{A' \to B}^{\otimes m} : C_n)$$

$$= \frac{1}{|M_n|} \sum_{m \in \mathcal{M}_n} \text{Tr}(E_{A_m^n}^m \left( \left( \sum_{t \in T} q(t) \tau_{AB}^t \right)^{\otimes n} - 2^{na} \left( \sum_{t \in T} q(t) \mathcal{N}(\psi_{AA'}^t) \right)^{\otimes n} \right) \Lambda_{A_m^n B_m^n})$$

$$= \frac{1}{|M_n|} \sum_{m \in \mathcal{M}_n} \text{Tr} \left( \left( \sum_{t \in T} q(t) \tau_{AB}^t \right)^{\otimes n} - 2^{na} \left( \sum_{t \in T} q(t) \mathcal{N}(\psi_{AA'}^t) \right)^{\otimes n} \right) \left( U_m^\dagger \Lambda_{A_m^n B_m^n} U_m \right)$$

$$\leq \text{Tr} \left( \left( \sum_{t \in T} q(t) \tau_{AB}^t \right)^{\otimes n} - 2^{na} \left( \sum_{t \in T} q(t) \mathcal{N}(\psi_{AA'}^t) \right)^{\otimes n} \right)^+ \quad (67)$$

where $X_+ := (|X| + X)/2$ is the positive part of a self-adjoint operator $X$. It has been proved in [36] (cf. Theorem 1 and discussions in Section III therein) that when $r > D(\rho \| \sigma)$,

$$\lim_{n \to \infty} \text{Tr}(\rho^{\otimes n} - 2^{nr} \sigma^{\otimes n})_+ = 0. \quad (68)$$

Thus, when $n \to \infty$, the limit of the last line of Eq. (67) is 0. Therefore, we combine Eq. (62) and Eq. (67) to get

$$\liminf_{n \to \infty} 2^{na} P_s(N_{A' \to B}^{\otimes m} : C_n) \geq 1. \quad (69)$$

Since the code $C_n$ has size $|C_n| = |M_n| = 2^{nR}$, by the definition of $sc(N_{A' \to B}, R)$, we have

$$\text{sc}(N_{A' \to B}, R) \leq - \liminf_{n \to \infty} \frac{1}{n} \log P_s(N_{A' \to B}^{\otimes m} : C_n)$$

$$\leq \sum_{t \in T} q(t) D(\tau_{AB}^t \| \mathcal{N}(\psi_{AA'}^t)) + \delta$$

$$\leq F_1(N_{A' \to B}, R, \{q(t), \psi_{AA'}^t\}) + 2\delta. \quad (70)$$

where the second inequality is by Eq. (66) and Eq. (69), and the last inequality is by Eq. (54). At last, because $\delta > 0$ is arbitrary, we are done. 

\[\blacksquare\]
Lemma 13: Let \( \{q(t), \Psi_{AA'}^t\}_{t \in T} \) be any ensemble of quantum states as specified in Proposition 10. For the channel \( \mathcal{N}_{A' \rightarrow B} \) and any \( R > 0 \) we have
\[
sc(\mathcal{N}_{A' \rightarrow B}, R) \leq F_2(\mathcal{N}_{A' \rightarrow B}, R, \{q(t), \Psi_{AA'}^t\}_t).
\] (71)

Proof: According to the definition of \( F_2 \) in Eq. (48) and Eq. (50), there exists a set of states \( \{\tau_{AB}^t\}_{t \in T} \) such that \( \tau_{AB}^t \in \mathcal{S}_{\mathcal{N}(\Psi_{AA'}^t)}(AB) \) and
\[
\sum_{t \in T} q(t) D(\tau_{AB}^t \| \pi_A^t \otimes \tau_B^t) \leq R,
\] (72)
\[
R - \sum_{t \in T} q(t) D(\tau_{AB}^t \| \pi_A^t \otimes \tau_B^t) + \sum_{t \in T} q(t) D(\tau_{AB}^t \| \mathcal{N}(\Psi_{AA'}^t)) = F_2(\mathcal{N}_{A' \rightarrow B}, R, \{q(t), \Psi_{AA'}^t\}_t).
\] (73)

Let \( R' := \sum_{t \in T} q(t) D(\tau_{AB}^t \| \pi_A^t \otimes \tau_B^t) - \delta \) with an arbitrary \( \delta > 0 \). Following the proof of Lemma 12, we easily see that there exists a sequence of codes
\[
C_n' = \left( \left\{ \mathcal{E}_m^m \right\}_m, \left\{ \Lambda_{A^n B^n}^m \right\}_m, \left( \sum_{t \in T} q(t) \Psi_{AA'}^t \right)^{\otimes n} \right), \quad n \in \mathbb{N}
\] (74)
with \( |C_n'| = \lfloor 2^{n R'} \rfloor \) such that
\[
- \liminf_{n \to \infty} \frac{1}{n} \log P_s(\mathcal{N}_{A' \rightarrow B}, C_n') \leq \sum_{t \in T} q(t) D(\tau_{AB}^t \| \mathcal{N}(\Psi_{AA'}^t)).
\] (75)

We point out that in the code \( C_n' \), the sender holds the \( A^n \) system and the receiver holds the \( A^n \) system of the entangled state \( \left( \sum_{t \in T} q(t) \Psi_{AA'}^t \right)^{\otimes n} \), and the encoding operation \( \mathcal{E}_m^m \) is a unitary map. Now we extend \( C_n' \) to construct a new code \( C_n \) with size \( |C_n| = \lfloor 2^{n R} \rfloor \), using the simple strategy:
(i) for \( m \in \{1, 2, \ldots, |C_n'| \} \), we set \( \mathcal{E}_m^m = \mathcal{E}_m^m \) and \( \Lambda_{A^n B^n}^m = \Lambda_{A^n B^n}^m \);
(ii) for \( m \notin \{1, 2, \ldots, |C_n'| \} \), we set \( \mathcal{E}_m^m = I_{A^n} \) as the identity map and \( \Lambda_{A^n B^n}^m = 0 \).

The success probability of the code
\[
C_n = \left( \left\{ \mathcal{E}_m^m \right\}_m, \left\{ \Lambda_{A^n B^n}^m \right\}_m, \left( \sum_{t \in T} q(t) \Psi_{AA'}^t \right)^{\otimes n} \right)
\] (76)
is calculated as
\[
P_s(\mathcal{N}_{A' \rightarrow B}, C_n)
= \frac{1}{|C_n|} \sum_{m=1}^{\lfloor 2^{n R} \rfloor} \text{Tr} \left( \mathcal{N}^{\otimes n} \circ \mathcal{E}_m^m \left( \left( \sum_{t} q(t) \Psi_{AA'}^t \right)^{\otimes n} \right) \Lambda_{A^n B^n}^m \right)
\]
\[
= \frac{|C_n'|}{|C_n|} \frac{1}{|C_n'|} \sum_{m=1}^{\lfloor 2^{n R} \rfloor} \text{Tr} \left( \mathcal{N}^{\otimes n} \circ \mathcal{E}_m^m \left( \left( \sum_{t} q(t) \Psi_{AA'}^t \right)^{\otimes n} \right) \right) \Lambda_{A^n B^n}^m
\]
\[
= \frac{|C_n'|}{|C_n|} P_s(\mathcal{N}_{A' \rightarrow B}, C_n').
\] (77)

Thus we have
\[
sc(\mathcal{N}_{A' \rightarrow B}, R) \leq - \liminf_{n \to \infty} \frac{1}{n} \log P_s(\mathcal{N}_{A' \rightarrow B}, C_n)
\leq R - R' + \sum_{t \in T} q(t) D(\tau_{AB}^t \| \mathcal{N}(\Psi_{AA'}^t))
= F_2(\mathcal{N}_{A' \rightarrow B}, R, \{q(t), \Psi_{AA'}^t\}_t) + \delta,
\] (78)
where the second inequality is by Eq. (75) and Eq. (77), as well as the choice of \(|C_n^C|\) and \(|C_n^C|\). At last, because \(\delta > 0\) is arbitrary, the proof is complete.

The bound in Proposition 10 depends on the quantum state set \(\{\Psi_{AA'}^t\}\), as well as the probability distribution \(q\) over it. We optimize it over all probability distribution \(q\) to obtain the following result.

**Theorem 14:** Let \(\mathcal{H}_A \cong \mathcal{H}_A'\), and let \(\mathcal{H}_A = \bigoplus_{t \in T} \mathcal{H}_A^t\) and \(\mathcal{H}_A' = \bigoplus_{t \in T} \mathcal{H}_A'^t\) be decompositions of \(\mathcal{H}_A\) and \(\mathcal{H}_A\) into orthogonal subspaces with \(\mathcal{H}_A^t \cong \mathcal{H}_A'^t\). Let \(\Psi_{AA'}^t\) be the maximally entangled state on \(\mathcal{H}_A^t \otimes \mathcal{H}_A'^t\). Then for the channel \(\mathcal{N}_{A^3 \rightarrow B}\) and any \(R > 0\), the strong converse exponent for entanglement-assisted classical communication satisfies

\[
sc(\mathcal{N}_{A^3 \rightarrow B}, R) \leq \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - \max_{t \in T} I_\alpha^b \left( \mathcal{N}_{A^3 \rightarrow B}, \Psi_{AA'}^t \right) \right\},
\]

where \(I_\alpha^b \left( \mathcal{N}_{A^3 \rightarrow B}, \Psi_{AA'}^t \right)\) is defined in Eq. (36).

**Proof:** Proposition 10 holds for any ensemble \(\{q(t), \Psi_{AA'}^t\}_{t \in T}\). So, we can take the infimum in Eq. (83), over all possible \(q\) in the probability simplex \(Q(T)\). Therefore,

\[
sc(\mathcal{N}_{A^3 \rightarrow B}, R) \leq \inf_{q \in Q(T)} \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - I_\alpha^b \left( \mathcal{N}_{A^3 \rightarrow B}, \{q(t), \Psi_{AA'}^t\}_t \right) \right\} = \inf_{q \in Q(T)} \sup_{\lambda \in (0,1)} G(\lambda, q),
\]

where

\[
G(\lambda, q) := \lambda \left\{ R - \sum_{t \in T} q(t) \min_{\sigma_B} D_{\text{b}}^{\lambda} \left( \mathcal{N}(\Psi_{AA'}^t), \|\mathcal{N}(\Psi_{AA'}^t)\| \right) \right\},
\]

and for the equality we use the definition of \(I_\alpha^b \left( \mathcal{N}_{A^3 \rightarrow B}, \{q(t), \Psi_{AA'}^t\}_t \right)\) and set \(\lambda = \frac{\alpha - 1}{\alpha}\). Following the steps in the proof of Proposition 11, we can also write \(G(\lambda, q)\) as

\[
G(\lambda, q) = \inf_{\{\tau_{AB}^t\}_{t \in T} \in F} \left\{ \lambda \left( R - \sum_{t \in T} q(t) D(\tau_{AB}^t, \|\mathcal{N}(\Psi_{AA'}^t)\|) + \sum_{t \in T} q(t) D(\tau_{AB}^t, \|\mathcal{N}(\Psi_{AA'}^t)\|) \right) \right\},
\]

where \(F := \{\{\tau_{AB}^t\}_{t \in T} \mid \tau_{AB}^t \in \mathcal{S}_{\mathcal{N}(\Psi_{AA'}^t)}(AB), \forall t \in T\}\). We claim that

(i) for fixed \(\lambda\), the function \(q \mapsto G(\lambda, q)\) is linear and continuous, on the compact and convex set \(Q(T)\); and

(ii) for fixed \(q\), the function \(\lambda \mapsto G(\lambda, q)\) is concave and upper semi-continuous, on the interval \((0,1)\).

Claim (i) is obviously seen from Eq. (81). For claim (ii), we can easily check that \(x \mapsto \inf_{q} f(x, y)\) is concave and upper semi-continuous if the function \(f(x, y)\) is linear with \(x\) and continuous with both arguments, and then we apply this observation to Eq. (82). Now we can invoke Sion’s minimax theorem to obtain

\[
\inf_{q \in Q(T)} \sup_{\lambda \in (0,1)} G(\lambda, q) = \sup_{\lambda \in (0,1)} \inf_{q \in Q(T)} G(\lambda, q) = \sup_{\lambda \in (0,1)} \inf_{q \in Q(T)} G(\lambda, q) = \sup_{\lambda \in (0,1)} \left\{ \lambda \left( R - \sum_{t \in T} q(t) D(\tau_{AB}^t, \|\mathcal{N}(\Psi_{AA'}^t)\|) + \sum_{t \in T} q(t) D(\tau_{AB}^t, \|\mathcal{N}(\Psi_{AA'}^t)\|) \right) \right\} = \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - \max_{t \in T} I_\alpha^b \left( \mathcal{N}_{A^3 \rightarrow B}, \Psi_{AA'}^t \right) \right\}, \tag{83}
\]

and we are done.
Remark 15: To achieve the bound of Theorem 14, we can employ a sequence of codes
\[ \left\{ \left\{ \mathcal{E}^m_{A'} \right\}_m; \left\{ \Lambda^m_{A'B'B} \right\}_m; \left( \sum_{t \in T} q(t) \Psi(t)^{i}_{A'} \right)^{\otimes n} \right\}_n, \tag{84} \]
where the encoding operation \( \mathcal{E}^m_{A'} \) is a unitary map, and the probability distribution \( q(t) \) in the shared entangled state is an optimizer of Eq. (80). This can be seen from the construction of codes in the proofs of Lemma 12 and Lemma 13. Note that the infima in Eq. (80) can be replaced by minima (cf. Sion’s minimax theorem in Lemma 19).

V. ACHIEVING THE STRONG CONVERSE EXPONENT

In this section, we complete the proof of the achievability of the strong converse exponent for entanglement-assisted classical communication. This is based on the result obtained in the previous section, namely, Theorem 14.

Theorem 16: Let \( \mathcal{N}_{A' \rightarrow B} \) be a quantum channel. For any transmission rate \( R > 0 \), the strong converse exponent for entanglement-assisted classical communication satisfies
\[ sc(\mathcal{N}_{A' \rightarrow B}, R) \leq \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left( R - I^*_\alpha(\mathcal{N}_{A' \rightarrow B}) \right). \tag{85} \]

Proof: We fix \( m \in \mathbb{N} \), and consider the channel \( \mathcal{N}^{(m)}_{A'm \rightarrow B'm} := \mathcal{P}_{\sigma^m_{B'm}} \circ \mathcal{N}^m_{A'A' \rightarrow B'B} \), where \( \sigma^m_{B'm} \) is the universal symmetric state described in Lemma 1. Let \( A \) be such that \( \mathcal{H}_A \cong \mathcal{H}_A' \). We further fix an arbitrary pure state \( \psi_{A'A'} \) on \( \mathcal{H}_A \otimes \mathcal{H}_A' \). The tensor product state \( \psi^m_{A'A'} \) can be expressed in the form of Eq. (4), i.e.,
\[ |\psi\rangle_{A'A'}^m = \sum_{t \in T^m} \sqrt{p^m(t)} |\Psi(t)^i\rangle_{A'A'}^m, \tag{86} \]
where \( |T^m| \leq (m + 1)^{|A|} \) and the set of states \( \{ |\Psi(t)^i\rangle_{A'A'}^m \}_t \) satisfies the condition of Theorem 14 for the channel \( \mathcal{N}^{(m)}_{A'm \rightarrow B'm} \). Let \( \pi^m_{A'm} = \Psi(t)^{i}_{A'm} \) be the maximally mixed state on the subspace \( \mathcal{H}^t_{A'm} := \text{supp}(\Psi(t)^{i}_{A'm}) \). By Theorem 14, we know that there exists a sequence of codes \( \{ C^{(m)}_k \}_{k \in \mathbb{N}} \), such that
\[ \liminf_{k \to \infty} \frac{1}{k} \log |C^{(m)}_k| \geq mR \tag{87} \]
and
\[ -\liminf_{k \to \infty} \frac{1}{k} \log P_s \left( (\mathcal{N}^{(m)}_{A'm \rightarrow B'm})^{\otimes k}, C^{(m)}_k \right) \leq \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ mR - \max_{t \in T^m} \bar{P}_\alpha \left( \mathcal{N}^{(m)}_{A'm \rightarrow B'm}, \Psi(t)^{i}_{A'A'}^m \right) \right\} \]
\[ = \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ mR - \max_{t \in T^m, \sigma^m_{B'm} \in S(B')} D_\alpha \left( \mathcal{N}^{(m)}(\Psi(t)^{i}_{A'A'}^m) \parallel \pi^m_{A'm} \otimes \sigma^m_{B'm} \right) \right\}. \tag{88} \]
As explained in Remark 15, the code \( C^{(m)}_k \) can be written as
\[ C^{(m)}_k = \left\{ \left\{ \mathcal{E}^{i}_{A'mk} \right\}_i; \left\{ \Lambda^i_{A'mkB'mk} \right\}_i; \rho^i_{A'mkB'mk} \right\}, \tag{89} \]
where $\mathcal{E}_{A'B'}^{i}$ is a unitary operation on $A'B'$, and the state $\rho_{A'mkA'mk}$ shared by the sender (with $A'mk$) and the receiver (with $A'mk$) is a $k$-fold tensor product of a mixture of the states $\{\Psi_{A'mA'm}\}_t$. With this, the success probability of $C_k^{(m)}$ can be written as

$$P_s\left(\left(\Lambda_{A'mB'}^{(m)}\right)^{\otimes k}, C_k^{(m)}\right)$$

$$= \frac{1}{|C_k^{(m)}|} \sum_{i=1}^{|C_k^{(m)}|} \text{Tr}\left(\left(\Lambda_{A'mB'}^{(m)}\right)^{\otimes k} \circ \mathcal{E}_{A'mk}^{i} (\rho_{A'mkA'mk})\right) \Lambda_{A'mkB'mk}^{i}$$

$$= \frac{1}{|C_k^{(m)}|} \sum_{i=1}^{|C_k^{(m)}|} \text{Tr}\left(\mathcal{P}_{\sigma_{Bm}^{\otimes k}}^{\otimes k} \circ \mathcal{N}_{A'mk}^{\otimes mk} \circ \mathcal{E}_{A'mk}^{i} (\rho_{A'mkA'mk})\right) \Lambda_{A'mkB'mk}^{i}$$

$$= \frac{1}{|C_k^{(m)}|} \sum_{i=1}^{|C_k^{(m)}|} \text{Tr}\left(\mathcal{N}_{A'B'}^{\otimes n} \circ \mathcal{E}_{A'mk}^{i} (\rho_{A'mkA'mk})\right) \mathcal{P}_{\sigma_{Bm}^{\otimes k}}^{\otimes k} (\Lambda_{A'mkB'mk}^{i}). \quad (90)$$

Now, for any integer $n$, we construct for the channel $\mathcal{N}_{A'B'}^{\otimes n}$ a code $C_n$, adapted from $\{C_k^{(m)}\}_k$. Write $n = mk + l$ with $0 \leq l < m$. We divide the channel $\mathcal{N}_{A'B'}^{\otimes n} = \mathcal{N}_{A'B'}^{\otimes mk} \otimes \mathcal{N}_{A'B'}^{\otimes l}$ into the first $mk$ copies and the last $l$ copies. The code $C_n$ is performed effectively on the first $mk$ copies of $\mathcal{N}_{A'B'}$, specified by

$$\left(\{\mathcal{E}_{A'mk}^{i}\}_i, \{\mathcal{P}_{\sigma_{Bm}^{\otimes k}}^{\otimes k} (\Lambda_{A'mkB'mk}^{i})\}_i, \rho_{A'mkA'mk}\right). \quad (91)$$

That is, we use the same entangled state and employ the same encoding maps with $C_k^{(m)}$, and we replace the decoding measurement operator by $\mathcal{P}_{\sigma_{Bm}^{\otimes k}}^{\otimes k} (\Lambda_{A'mkB'mk}^{i})$ (cf. Eq. (90)). On the last $l$ copies, we input an arbitrary state and we do not touch the output in the decoding (equivalently, all of the decoding measurement operators are tensored with an identity operator on this part). The codes $C_n$ and $C_k^{(m)}$ have the same size. So it follows from Eq. (87) that

$$\liminf_{n \to \infty} \frac{1}{n} \log |C_n| = \frac{1}{m} \liminf_{k \to \infty} \frac{1}{k} \log |C_k^{(m)}| \geq R. \quad (92)$$

On the other hand, it is obvious that the success probability of $C_n$ is given by Eq. (90) as well. So,

$$P_s\left(\mathcal{N}_{A'B'}^{\otimes n}, C_n\right) = P_s\left(\left(\Lambda_{A'mB'}^{(m)}\right)^{\otimes k}, C_k^{(m)}\right). \quad (93)$$

By the definition of $sc(\mathcal{N}_{A'B'}, R)$, we combine Eq. (88), Eq. (93) and Eq. (92) to get

$$sc(\mathcal{N}_{A'B'}, R) \leq - \liminf_{n \to \infty} \frac{1}{n} \log P_s\left(\mathcal{N}_{A'B'}^{\otimes n}, C_n\right)$$

$$= - \frac{1}{m} \liminf_{k \to \infty} \frac{1}{k} \log P_s\left(\left(\Lambda_{A'mB'}^{(m)}\right)^{\otimes k}, C_k^{(m)}\right)$$

$$\leq \sup_{\alpha > 1} \left\{ R - \frac{1}{m} \max_{t \in T_m} \min_{\sigma_{Bm} \in S(B')} D^\alpha (\mathcal{P}_{\sigma_{Bm}^{\otimes k}} \circ \mathcal{N}_{A'B'}^{\otimes m} (\Psi_{A'mA'm}) \| \pi_{A'm}^{t} \otimes \sigma_{B'm}) \right\}. \quad (94)$$
Next, we further bound \( sc(\mathcal{N}_{A'\rightarrow B}, R) \) based on Eq. (94). We have

\[
\frac{1}{m} \max_{t \in T_m} \min_{\sigma_{B^m} \in S(B^m)} D_{\alpha}^b \left( \mathcal{P}_{\sigma_{B^m}} \circ \mathcal{N}_{A^m \rightarrow A^m}^\otimes \left( \Psi_{A^m A^m}^t \right) \right) \left\| \pi_{A^m}^t \otimes \sigma_{B^m}^u \right\|
\]

\[
\geq \left( a \right) \frac{1}{m} \max_{t \in T_m} D_{\alpha}^b \left( \mathcal{P}_{\sigma_{B^m}} \circ \mathcal{N}_{A^m \rightarrow A^m}^\otimes \left( \Psi_{A^m A^m}^t \right) \right) \left\| \pi_{A^m}^t \otimes \sigma_{B^m}^u \right\| - \log v_{m,B^m} - \frac{\log v_{m,B^m}}{m}
\]

\[
\geq \left( b \right) \frac{1}{m} \max_{t \in T_m} D_{\alpha}^b \left( \mathcal{P}_{\sigma_{B^m}} \circ \mathcal{N}_{A^m \rightarrow A^m}^\otimes \left( \Psi_{A^m A^m}^t \right) \right) \left\| \pi_{A^m}^t \otimes \sigma_{B^m}^u \right\| - \log v_{m,B^m} - \frac{\log v_{m,B^m}}{m}
\]

\[
\geq \left( c \right) \frac{1}{m} \max_{t \in T_m} D_{\alpha}^b \left( \mathcal{P}_{\sigma_{B^m}} \circ \mathcal{N}_{A^m \rightarrow A^m}^\otimes \left( \Psi_{A^m A^m}^t \right) \right) \left\| \pi_{A^m}^t \otimes \sigma_{B^m}^u \right\| - \log v_{m,B^m} - \frac{\log v_{m,B^m}}{m}
\]

\[
\geq \left( d \right) \frac{1}{m} \max_{t \in T_m} D_{\alpha}^b \left( \mathcal{P}_{\sigma_{B^m}} \circ \mathcal{N}_{A^m \rightarrow A^m}^\otimes \left( \Psi_{A^m A^m}^t \right) \right) \left\| \pi_{A^m}^t \otimes \sigma_{B^m}^u \right\| - \log v_{m,B^m} - \frac{\log v_{m,B^m}}{m}
\]

where \((a)\) is essentially due to Proposition 5 (v) and is shown in Lemma 17, \((b)\) is by Lemma 1 and Proposition 5 (ii), \((c)\) is because \(\mathcal{P}_{\sigma_{B^m}} \circ \mathcal{N}_{A^m \rightarrow A^m}^\otimes \left( \Psi_{A^m A^m}^t \right) \) and \(\pi_{A^m} \otimes \sigma_{B^m}^u\) commute, and \((d)\) can be easily verified from the definition of \(D_{\alpha}^b\). To go ahead, we introduce

\[
\mathcal{P}_{T_m} \left( \cdot \right) = \sum_{t \in T_m} \Pi_t \left( \cdot \right) \Pi_t
\]

with \(\Pi_t\) being the projection onto the subspace \(\mathcal{H}_{A^m}^t\), and further bound Eq. (95) as follows.

\[
\frac{1}{m} D_{\alpha}^b \left( \mathcal{P}_{\sigma_{B^m}} \circ \mathcal{N}_{A^m \rightarrow A^m}^\otimes \left( \Psi_{A^m A^m}^t \right) \right) \left\| \pi_{A^m}^t \otimes \sigma_{B^m}^u \right\| - \log v_{m,B^m} - \frac{\log v_{m,B^m}}{m}
\]

\[
= \frac{1}{m} D_{\alpha}^b \left( \left( \mathcal{P}_{T_m} \otimes \mathcal{P}_{\sigma_{B^m}} \right) \left( \mathcal{N}_{A^m \rightarrow A^m}^\otimes \left( \Psi_{A^m A^m}^t \right) \right) \right) \left\| \pi_{A^m}^t \otimes \sigma_{B^m}^u \right\| - \frac{\log v_{m,B^m}}{m}
\]

\[
\geq \frac{1}{m} D_{\alpha} \left( \mathcal{N}_{A^m \rightarrow A^m}^\otimes \left( \Psi_{A^m A^m}^t \right) \right) \left\| \pi_{A^m}^t \otimes \sigma_{B^m}^u \right\| - \frac{3 \log v_{m,B^m}}{m} - \frac{3 \log v_{m,B^m}}{m} - \frac{2 \log (m+1)^{|A|}}{m}
\]

\[
\geq \frac{1}{m} \min_{\sigma_{B^m}} D_{\alpha} \left( \mathcal{N}_{A^m \rightarrow A^m}^\otimes \left( \Psi_{A^m A^m}^t \right) \right) \left\| \pi_{A^m}^t \otimes \sigma_{B^m}^u \right\| - \frac{3 \log v_{m,B^m}}{m} - \frac{3 \log v_{m,B^m}}{m} - \frac{2 \log (m+1)^{|A|}}{m}
\]

where \(I_{\alpha}(\mathcal{N}_{A'\rightarrow B}, \psi_{A'A'}) := I_{\alpha}(A:B|\mathcal{N}_{A'\rightarrow A'})\). In Eq. (97), \((a)\) is a result of Lemma 21, which generalizes Proposition 5 (vi), and \((b)\) comes from the additivity property of Proposition 5 (iv). Now, combining Eq. (94), Eq. (95) and Eq. (97) together, and letting \(m \rightarrow \infty\), we arrive at

\[
sc(\mathcal{N}_{A'\rightarrow B}, R) \leq \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left( R - I_{\alpha}(\mathcal{N}_{A'\rightarrow B}, \psi_{A'A'}) \right) + \frac{3 \log v_{m,B^m}}{m} + \frac{2 \log (m+1)^{|A|}}{m}
\]

\[
\Rightarrow \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left( R - I_{\alpha}(\mathcal{N}_{A'\rightarrow B}, \psi_{A'A'}) \right).
\]

At last, noticing that Eq. (98) holds for arbitrary pure state \(\psi_{A'A'}\), we can optimize it over all pure states

\[
\psi(\rho)_{A'A'} := (I_A \otimes \sqrt{\rho_A})(|A\rangle \otimes \sqrt{\rho_A})(1_A \otimes \sqrt{\rho_A}), \quad \text{with } \rho \in S(A').
\]

Therefore,

\[
sc(\mathcal{N}_{A'\rightarrow B}, R) \leq \inf_{\rho \in S(A')} \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left( R - I_{\alpha}(\mathcal{N}_{A'\rightarrow B}, \psi(\rho)_{A'A'}) \right)
\]

\[
\Rightarrow \inf_{\rho \in S(A')} \sup_{0 < \lambda < 1} \frac{\lambda}{1 - \lambda} \left( R - I_{\lambda}(\mathcal{N}_{A'\rightarrow B}, \psi(\rho)_{A'A'}) \right).
\]
The function
\[
    f(\lambda, \rho) = \lambda \left\{ R - I_{1/\lambda}^{\ast} \left( \mathcal{N}_{A' \rightarrow B}, \psi(\rho)_{A'A'} \right) \right\}
\]
(101)
is concave and continuous in $\lambda$ on the interval $(0, 1)$ by Lemma 18, and as shown in [41] it is convex and continuous in $\rho$ on the compact convex set $\mathcal{S}(A')$ for any $\lambda \in (0, 1)$. Thus Sion’s minimax theorem applies again. This lets us obtain
\[
    sc(\mathcal{N}_{A' \rightarrow B}, R) \leq \sup_{0 < \lambda < 1} \inf_{\rho \in \mathcal{S}(A')} \lambda \left\{ R - I_{1/\lambda}^{\ast} \left( \mathcal{N}_{A' \rightarrow B}, \psi(\rho)_{A'A'} \right) \right\}
\]
\[
    = \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - I_{\alpha}^{\ast}(\mathcal{N}_{A' \rightarrow B}) \right\},
\]
(102)
and we are done.

The following Lemma 17 and Lemma 18 are used in the proof of Theorem 16.

**Lemma 17:** The first equality of Eq. (95) holds: using the same notation as there, we have
\[
    \min_{\sigma_{Bm} \in \mathcal{S}(B^m)} D^{\alpha}_{\lambda} \left( \mathcal{P}_{\sigma_{Bm}} \circ \mathcal{N}_{A'm}^{\otimes m} \right) (\psi_{A'm}^{t} \otimes \sigma_{Bm})
\]
\[
    = \min_{\sigma_{Bm} \in \mathcal{S}(B^m)} D^{\alpha}_{\lambda} \left( \mathcal{P}_{\sigma_{Bm}} \circ \mathcal{N}_{A'm}^{\otimes m} \right) (\psi_{A'm}^{t} \otimes \sigma_{Bm}).
\]
(103)

**Proof:** Since $\mathcal{S}_{\text{sym}}(B^m) \subset \mathcal{S}(B^m)$, the “<” part is obvious. In the following, we prove the opposite direction. For a permutation $\iota \in S_{m}$, let $\mathcal{W}_{A'm}^{\iota}$ be the unitary operation that acts as $\mathcal{W}_{A'm}^{\iota}(X_{A'm}) = W_{A'm}^{\iota}X_{A'm}W_{A'm}^{\iota\dagger}$, where $W_{A'm}^{\iota}$ is given in Eq. (9). Also let $\mathcal{W}_{A'm}^{\iota}$ and $\mathcal{W}_{Bm}^{\iota}$ be defined similarly. It is easy to see that
\[
    \mathcal{P}_{\sigma_{Bm}} \circ \mathcal{N}_{A'm}^{\otimes m} \circ \mathcal{W}_{A'm}^{\iota} = \mathcal{W}_{Bm}^{\iota} \circ \mathcal{P}_{\sigma_{Bm}} \circ \mathcal{N}_{A'm}^{\otimes m}
\]
holds for any permutation $\iota$. This, together with the fact that $\psi_{A'm}^{t} \in \mathcal{S}_{\text{sym}}(A'm A'm)$, ensures that
\[
    \mathcal{P}_{\sigma_{Bm}} \circ \mathcal{N}_{A'm}^{\otimes m} (\psi_{A'm}^{t} \otimes \sigma_{Bm}) \in \mathcal{S}_{\text{sym}}(A'm B^m).
\]
Eq. (105) and the fact that $\pi_{A'm}^{t} \in \mathcal{S}_{\text{sym}}(A'm)$ lead to
\[
    D^{\alpha}_{\lambda} \left( \mathcal{P}_{\sigma_{Bm}} \circ \mathcal{N}_{A'm}^{\otimes m} \right) (\psi_{A'm}^{t} \otimes \sigma_{Bm})
\]
\[
    = D^{\alpha}_{\lambda} \left( \mathcal{W}_{A'm}^{\iota} \circ \mathcal{P}_{\sigma_{Bm}} \circ \mathcal{N}_{A'm}^{\otimes m} \right) (\psi_{A'm}^{t} \otimes \mathcal{W}_{Bm}^{\iota} \circ \mathcal{W}_{Bm}^{\iota\dagger} \circ \sigma_{Bm})
\]
\[
    = D^{\alpha}_{\lambda} \left( \mathcal{P}_{\sigma_{Bm}} \circ \mathcal{N}_{A'm}^{\otimes m} \right) (\psi_{A'm}^{t} \otimes \mathcal{W}_{Bm}^{\iota} \circ \mathcal{W}_{Bm}^{\iota\dagger} \circ \sigma_{Bm})
\]
(106)
for any $\iota \in S_{m}$ and $\sigma_{Bm} \in \mathcal{S}(B^m)$. Proposition 5 (v) and Eq. (106) let us obtain
\[
    D^{\alpha}_{\lambda} \left( \mathcal{P}_{\sigma_{Bm}} \circ \mathcal{N}_{A'm}^{\otimes m} \right) (\psi_{A'm}^{t} \otimes \sigma_{Bm})
\]
\[
    = \sum_{\iota \in S_{m}} D^{\alpha}_{\lambda} \left( \mathcal{P}_{\sigma_{Bm}} \circ \mathcal{N}_{A'm}^{\otimes m} \right) (\psi_{A'm}^{t} \otimes \mathcal{W}_{Bm}^{\iota} \circ \mathcal{W}_{Bm}^{\iota\dagger} \circ \sigma_{Bm})
\]
\[
    \geq D^{\alpha}_{\lambda} \left( \mathcal{P}_{\sigma_{Bm}} \circ \mathcal{N}_{A'm}^{\otimes m} \right) (\psi_{A'm}^{t} \otimes \mathcal{W}_{Bm}^{\iota} \circ \mathcal{W}_{Bm}^{\iota\dagger} \circ \sigma_{Bm})
\]
(107)
At last, noticing that $\sum_{\iota \in S_{m}} \frac{1}{|S_{m}|} \mathcal{W}_{Bm}^{\iota} \circ \mathcal{W}_{Bm}^{\iota\dagger} \circ \sigma_{Bm} \in \mathcal{S}_{\text{sym}}(B^m)$, we complete the proof.

**Lemma 18:** For any state $\rho_{AB} \in \mathcal{S}(AB)$, the function
\[
    g(\lambda) = \lambda I_{\frac{1}{1-\lambda}}^{\ast} (A : B)_{\rho}
\]
(108)
is convex and continuous on $(-1, 1)$.

**Proof:** Let $\rho \in \mathcal{S}(H)$ and $\omega \in \mathcal{S}(H)$ be two states that are commutative. We consider
\[
    g_{\rho,\omega}(\lambda) := \lambda I_{\frac{1}{1-\lambda}}^{\ast} (\rho \| \omega).
\]
(109)
\( g_{\theta,\omega}(\lambda) \) is obviously continuous on \((-1, 1)\). Since \( \rho \) and \( \omega \) commute, we have \( D_{\alpha}^*(\rho\|\omega) = D_{\alpha}^*(\rho\|\omega) \) and the variational expression of Proposition 5 (iii) applies. So

\[
g_{\theta,\omega}(\lambda) = s(\lambda) \lambda \max_{\tau \in \mathcal{S}_\theta(H)} s(\lambda) \left\{ D(\tau\|\omega) - \frac{1}{\lambda} D(\tau\|\rho) \right\} = \max_{\tau \in \mathcal{S}_\theta(H)} \left\{ \lambda D(\tau\|\omega) - D(\tau\|\rho) \right\},
\]

where \( s(\lambda) = 1 \) for \( \lambda \in (0, 1) \) and \( s(\lambda) = -1 \) for \( \lambda \in (-1, 0) \). When \( \lambda = 0 \), Eq. (110) still holds without the intermediate step. From Eq. (110), it is easy to verify that \( g_{\theta,\omega}(\lambda) \) is convex on \((-1, 1)\).

Now, we turn to the consideration of \( g(\lambda) \). It is proved in [32, Proposition 8] that for \( \alpha \in [\frac{1}{2}, \infty) \),

\[
I_\alpha^*(A : B)_\rho = \frac{1}{n} D_{^*_{1-\alpha}} \left( \mathcal{P}_{\rho_A^\otimes^n \otimes \sigma_B^n} (\rho_{AB}^\otimes^n) \right| \rho_A^\otimes^n \otimes \sigma_B^n ) + O\left( \frac{\log n}{n} \right),
\]

where the underlying constants in the term \( O\left( \frac{\log n}{n} \right) \) are independent of \( \alpha \). Thus, we have

\[
g(\lambda) = \frac{1}{n} \lambda D_{^*_{1-\alpha}} \left( \mathcal{P}_{\rho_A^\otimes^n \otimes \sigma_B^n} (\rho_{AB}^\otimes^n) \right| \rho_A^\otimes^n \otimes \sigma_B^n ) + O\left( \frac{\log n}{n} \right) \lambda.
\]

The second term of the right hand side of Eq. (112) vanishes uniformly in \( \lambda \), as \( n \) increases. So, the result obtained above for \( g_{\theta,\omega}(\lambda) \) lets us complete the proof.

VI. CONCLUSION AND DISCUSSION

We have determined the strong converse exponents for both entanglement-assisted and quantum-feedback-assisted communication over quantum channels, building on previous works of [13] and [18]. The formulas, being the same for these two tasks, are expressed in terms of the sandwiched Rényi information \( I_\alpha^*(\mathcal{N}_{A\rightarrow B}) \) of the channel \( \mathcal{N}_{A\rightarrow B} \) with \( \alpha > 1 \), providing a complete operational interpretation for this quantity. We point out that in [41], an operational interpretation for \( I_\alpha^*(\mathcal{N}_{A\rightarrow B}) \) with \( \alpha \in (1, 2] \) has been found by the authors, in characterizing the reliability function of quantum channel simulation.

Our results reinforce the viewpoint that the theory of entanglement-assisted communication is the natural quantum generalization of Shannon’s theory of classical communication. On the one hand, the obtained strong converse exponents take the form similar to Arimoto’s exponent for classical channels [9]. On the other hand, these results imply that additional classical or quantum feedback does not change the strong converse exponents of entanglement-assisted communication, in full analogy to the classical situation [42], [11], [10], too.

At last, we comment that, to achieve the strong converse exponents for entanglement-assisted communication, the shared entanglement can be restricted to the form of maximally entangled states. This is in agreement with the fact that maximally entangled states suffice to achieve the entanglement-assisted capacities [4]. Inspecting our proof, we see that the entanglement we used is an ensemble of maximally entangled states. However, for any fixed transmission rate \( R \), a particular one from the ensemble works, although the ensemble and the particular maximally entangled state vary with \( R \).

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APPENDIX

Auxiliary Lemmas

The following is Sion’s minimax theorem [43], [44].

Lemma 19: Let $X$ be a compact convex set in a topological vector space $V$ and $Y$ be a convex subset of a vector space $W$. Let $f : X \times Y \to \mathbb{R}$ be such that

(i) $f(x, \cdot)$ is quasi-concave and upper semi-continuous on $Y$ for each $x \in X$, and
(ii) $f(\cdot, y)$ is quasi-convex and lower semi-continuous on $X$ for each $y \in Y$.

Then, we have

$$\inf_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \inf_{x \in X} f(x, y),$$

and the infima in Eq. (113) can be replaced by minima.

The following lemma is adapted from [45] and [46], and it constitutes the main technical part of the Holevo-Schumacher-Westmoreland theorem (see also, e.g., [1, Chapter 20]).

Lemma 20: Let $\{p(x), \rho_x\}_{x \in X}$ be an ensemble of quantum states on a Hilbert space of finite dimension, where $p$ is a probability distribution on $X$. Let

$$R < \chi(\{p(x), \rho_x\}_{x}) \equiv \sum_x p(x) D(\rho_x \parallel \sum_x p(x) \rho_x)$$

be fixed. Denote $S_n = \{\rho_{x_1} \otimes \cdots \otimes \rho_{x_n}\}_{x^n \in X^n}$ and let $p^n(x^n) = \prod_{i=1}^n p(x_i)$ be the product distribution on $X^n$. For each $n \in \mathbb{N}$, there exist a set of quantum states $\mathcal{O}_n = \{\omega^{(m)}_n\}_{m=1}^M \subseteq S_n$ and a quantum measurement $\mathcal{D}_n = \{\Lambda^{(m)}_n\}_{m=1}^M$ such that $M = \lfloor 2^{nR} \rfloor$ and

$$\tilde{P}_n((\mathcal{O}_n, \mathcal{D}_n)) := \frac{1}{M} \sum_{m=1}^M \Tr \omega^{(m)}_n \Lambda^{(m)}_n \to 1, \quad \text{as } n \to \infty.$$ (115)

The set $\mathcal{O}_n$ can be constructed by randomly choosing each $\omega^{(m)}_n$ from $S_n$ according to the probability distribution $p^n$.

The following lemma is a trivial generalization of [32, Lemma 3], where the case $\mathcal{P} = \mathcal{P}_\sigma$ was proven.

Lemma 21: let $\rho \in \mathcal{S}(\mathcal{H})$ be a quantum state and $\sigma \in \mathcal{L}(\mathcal{H})_+$ be positive semidefinite. Let $\{P_i\}_{i=1}^M$ be a set of projections such that $\sum_i P_i = \mathbb{1}$ and for each $i$, $\text{supp}(P_i)$ is contained in an eigenspace of $\sigma$. Then for $\alpha \geq 0$ and the CPTP map $\mathcal{P} : X \mapsto \sum_i P_i X P_i$, we have

$$D^*_\alpha(\rho || \sigma) \leq D^*_\alpha(\mathcal{P}(\rho) || \sigma) + f_\alpha(M),$$

where

$$f_\alpha(M) = \begin{cases} \log M, & \text{if } \alpha \in [0, 2], \\ 2\log M, & \text{if } \alpha > 2. \end{cases}$$

The proof of [32, Lemma 3] works here, Simply using the pinching inequality $\rho \leq M \mathcal{P}(\rho)$ instead of $\rho \leq v(\sigma) \mathcal{P}_\sigma(\rho)$.

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