COMPACTIFICATION OF CERTAIN KÄHLER MANIFOLDS WITH NONNEGATIVE RICCI CURVATURE

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Abstract. We prove compactification theorems for some complete Kähler manifolds with nonnegative Ricci curvature. Among other things, we prove that a complete noncompact Kähler Ricci flat manifold with maximal volume growth and quadratic curvature decay is a crepant resolution of a normal affine algebraic variety. Furthermore, such affine variety degenerates in two steps to the unique metric tangent cone at infinity.

1. Introduction

In [31], Yau proposed the uniformization conjecture which states that a complete noncompact Kähler manifold with positive bisectional curvature is biholomorphic to $\mathbb{C}^n$. In [17]-[22], we studied the uniformization conjecture and its related problems. One of the main tools is the Gromov-Hausdorff convergence theory developed by Cheeger-Colding [2]-[5] and Cheeger-Colding-Tian [6].

In this paper, we extend some techniques in [17]-[22] to study the compactification of certain complete Kähler manifolds with nonnegative Ricci curvature.

Definition 1.1. [23] [30] On a Kähler manifold $M^n$, we say the bisectional curvature is greater than or equal to $K$ ($BK \geq K$), if

$$(1.1) \quad \frac{R(X, \overline{X}, Y, \overline{Y})}{||X||^2||Y||^2 + ||(X, Y)||^2} \geq K$$

for any two nonzero vectors $X, Y \in T^{1,0}M$.

The bisectional curvature lower bound condition is weaker than the sectional curvature lower bound, while stronger than the Ricci curvature lower bound.

The main result is

Theorem 1.1. Let $(M^n, p)$ ($n \geq 2$) be a complete noncompact Kähler manifold with nonnegative Ricci curvature and maximal volume growth. Let $r(x) = d(x, p)$. Then

(I) $M$ is biholomorphic to a Zariski open set of a Moishezon manifold, if for some $\epsilon > 0$, the bisectional curvature $BK \geq -\frac{C}{r^2}$. If fact, on $M$, the ring of polynomial growth holomorphic functions is finitely generated.

(II) If $BK \geq -\frac{C}{r^2}$ and $M$ has a unique tangent cone at infinity, then $M$ is biholomorphic to a Zariski open set of a Moishezon manifold.

(III) $M$ is quasiprojective, if the Ricci curvature is positive and $|Rm| \leq \frac{C}{r^2}$.

Combining part (II) with some argument in [14] or [12], we obtain

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Corollary 1.1. Let $M$ be a complete noncompact Kähler-Ricci flat manifold with maximal volume growth. Assume the curvature has quadratic decay. Then $M$ is a crepant resolution of a normal affine algebraic variety. Furthermore, there exist two step degenerations from that affine variety to the unique metric tangent cone of $M$ at infinity.

Remark 1.1. It is desirable to remove the uniqueness of the tangent cone in part II.

Remark 1.2. A conjecture of Yau [32] states that if a complete Ricci flat Kähler manifold has finite topological type, then it can be compactified complex analytically. Corollary 1.1 supports the conjecture, at least in this very special setting.

Another conjecture of Yau (question 71 in [33], page 304) states that complete noncompact Kähler manifolds with positive Ricci curvature is biholomorphic to a Zariski open set of a compact Kähler manifold. Part III of theorem 1.1 supports this conjecture.

Theorem 1.1 is a generalization of several known results. For instance, theorem 1.1 part I is a generalization of the following

Theorem 1.2. [18] Let $M^n$ be a complete noncompact Kähler manifold with nonnegative bisectional curvature and maximal volume growth. Then $M$ is biholomorphic to an affine algebraic variety $\mathbb{C}^n$.

In a series of papers, R. Conlon and H. Hein [8]-[10] systematically studied asymptotically conical Calabi-Yau manifolds. In some sense, corollary 1.1 is a generalization of some of their results.

In [25], Mok proved the following

Theorem 1.3. Let $M^n$ be a complete noncompact Kähler manifold with positive Ricci curvature and maximal volume growth. Assume $|Rm| \leq \frac{C}{r^2}$ and $\int_M Ric^n < +\infty$, then $M$ is biholomorphic to a quasi-projective variety.

Theorem 1.1 part III removes the assumption that $\int_M Ric^n < +\infty$.

The strategy of part I and II is to consider polynomial growth holomorphic functions. We shall follow the argument in [19]. However, there are some differences.

1. In this paper, we construct plurisubharmonic functions by using elliptic method (see proposition 2.3). In [19], the parabolic method of Ni-Tam [29] was adopted.

2. The original three circle theorem in [17] does not work in this paper. In part I, we just use the extended version in [17] (proposition 2.1). In part II, we apply Donaldson-Sun’s three circle theorem [14] (lemma 4.1).

3. In the setting of [19], polynomial growth holomorphic functions separate points and tangents. This is no longer true in part I and II, due to the possibility of compact subvarieties.

In some sense, part III resembles part I and II. However, the argument is very different. We basically follow the argument of Mok [25]. The strategy is to consider pluri-anticanonical sections with polynomial growth. The key new result is a uniform multiplicity estimate for pluri-anticanonical sections (proposition 2.10). This provides the dimension estimate for polynomial growth pluri-anticanonical sections, without the extra assumption $\int_M Ric^n < +\infty$ (compare theorem 2.2 of [25]).

1Recently, in [34], it was proved that $M$ is in fact biholomorphic to $\mathbb{C}^n$. 

This paper is organized as follows: In section 2, we prove results which are essential to all three parts of theorem 1.1. The proof of part I is presented in section 3. The main point is to prove a properness result, which is a modification of theorem 6.1 of [19]. In section 4, we prove part II of theorem 1.1 and corollary 1.1. We shall follow part I, [14] and [12]. In the last section, we prove part III of theorem 1.1. We follow Mok’s argument [25], with indication of the differences.

Here are some conventions in this paper. Let $e_\alpha$ be a local unitary frame of $T^{1,0}M$ and $s$ be a smooth tensor on $M$. Define $\Delta s = s_{\alpha\beta} + s_{\alpha\beta}$. Note this is twice the Laplacian defined in [29]. Also define $|\nabla u|^2 = 2u_{\alpha\beta}g^{\alpha\beta}$. Let $\frac{1}{\int}$ be the average. We will denote by $\Phi(u_1,\ldots,u_k|$ any nonnegative functions depending on $u_1,\ldots,u_k$ and some additional parameters such that when these parameters are fixed, 

$$\lim_{u_i \to 0} \cdots \lim_{u_1 \to 0} \Phi(u_1,\ldots,u_k) = 0.$$ 

Let $C(\cdot,\cdot,\ldots,\cdot)$ and $c(\cdot,\cdot,\ldots,\cdot)$ be large and small positive constants respectively, depending only on the parameters. The values might change from line to line.

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2. Preliminary results

In this section, we prove several results which are essential for the proof of theorem 1.1.

First we need the following version of three circle theorem:

**Proposition 2.1.** Let $M^p$ be a complete noncompact Kähler manifold, $p \in M$, $r(x) = \text{dist}(p,x)$. Suppose there exist constants $\epsilon, A > 0$ such that for any $e_i \in T^{1,0}_x M$ with unit length, $R_{\alpha\beta} \geq -\frac{4}{(r+1)^2}$. Then for any holomorphic function $f$ on $M$, $\log M(r)$ is convex in terms of $h(r)$, where $M(r) = \sup_{B(p,r)} |f|$ and

$$h(r) = \int_1^r e^{\frac{\epsilon}{t}} dt$$

for $r \geq 1$. In particular, $\dim(O_d(M)) \leq Cd^p$, where $O_d(M) = \{f \text{ holomorphic on } M \mid |f(x)| \leq C(1 + r(x))^{d^p} \}$ for any $\epsilon > 0$.

**Proof.** This is a combination of theorem 8 and theorem 11 in [17]. Notice that we have multiplied the original $h(r)$ in (8) of [17] by $e^{\frac{\epsilon}{r}}$.

It is clear that there exists a constant $b$ so that

$$\lim_{r \to \infty} (h(r) - \log r - b) = 0.$$ 

Given $d > 0$, $R > 2$, define $d_R = \frac{\log 2}{h(R) - \log \frac{R}{2}}$. Then $d_R \to d$ as $R \to \infty$. We have the following

**Corollary 2.1.** Under the assumption of proposition 2.1, let $f$ be a holomorphic function on $B(p,R)$ so that $\frac{M(R)}{M(p)} \leq 2^d$. Assume $M(p) = 1$ for some $1 < p < \frac{R}{10}$. Then for any $2\rho \leq r \leq R$, $M(r) \leq r^d$. 

Proof. From the assumption and proposition 2.1, we see that \( \log M(r) - d_k h(r) \) is non-increasing for \( r \leq \frac{R}{2} \). In particular,

\[
\log M(r) - d_k h(r) \leq \log M(\rho) - d_k h(\rho) \leq 0.
\]

Thus by (2.2),

\[
\log M(r) - d_k \log r \leq 0.
\]

This completes the proof of the corollary. \( \square \)

The following proposition could be found in [24] or [22]. For reader’s convenience, we include the proof.

**Proposition 2.2.** [Mok] Let \( f, g \) be polynomial growth holomorphic functions on a complete Kähler manifold \( M \) with \( \text{Ric} \geq 0 \). Suppose \( h = \frac{f}{g} \) is holomorphic, then \( h \) is of polynomial growth.

**Proof.** Let us say \( f(p), g(p) \neq 0 \). Set \( F_1(x) = \log |f(x)|^2 + \int_{B(p, R)} G_R(x, y) \Delta \log |f(y)|^2, F_2(x) = \log |g(x)|^2 + \int_{B(p, R)} G_R(x, y) \Delta \log |g(y)|^2 \).

**Claim 2.1.** For large \( R \) and \( i = 1, 2 \), on \( B(p, \frac{R}{2}) \), \(-C \log R \leq F_i(x) \leq C \log R \).

**Proof.** Note that \( F_i(x) \) is harmonic on \( B(p, R) \). Now maximum principle says that \( F_i(x) \leq C \log R \) on \( B(p, R) \). Let \( H_i = C \log R - F_i \geq 0 \). Then gradient estimate implies that on \( B(p, \frac{3}{2}R) \), \( |\nabla \log H_i| \leq \frac{C}{R^2} \). Observe \( H_i(p) \leq C \log R \). Then the harnack inequality implies that \( H_i \leq C_2 \log R \) on \( B(p, \frac{R}{2}) \). This completes the proof of the claim. \( \square \)

It is clear that on \( B(p, \frac{R}{2}) \), \( \log |h(x)|^2 \leq F_1(x) - F_2(x) \leq C \log R \) (\( C \) is independent of \( R \)).

The proof of the proposition is complete. \( \square \)

In the rest of this section, unless otherwise stated, we assume \( (M^n, p) \) is a complete noncompact Kähler manifold with nonnegative Ricci curvature, maximal volume growth and the bisectional curvature \( BK \geq \frac{c}{r} \) where \( r(x) = \text{dist}(x, p) \). Given any sequence \( \mu_i \rightarrow \infty \), set \( (M_i, p_i, d_i) = (M, p, \frac{r_i}{2}) \). Set \( r_i \) be the distance to \( p_i \). By passing to subsequence, we assume \( (M_i, p_i, d_i) \rightarrow (M_\infty, p_\infty, d_\infty) \) in the pointed Gromov-Hausdorff sense. Cheeger-Colding theorem [2] says \( (M_\infty, p_\infty, d_\infty) \) is a metric cone. We have the following

**Proposition 2.3.** Given any \( R > 100 \), for sufficiently large \( i \), we can find a plurisubharmonic function \( u \) on \( B(p_i, R) \) with

\[
|u - r_i^2| \leq \Phi(\frac{1}{i})R^2,
\]

\[
|\nabla u|^2 - 4r_i^2 \leq \Phi(\frac{1}{i})R^2.
\]

Furthermore, if \( \omega_i \) is the Kähler form on \( M_i \), then on \( B(p_i, R) \backslash B(p_i, \Phi(\frac{1}{i})R) \),

\[
\sqrt{-1} \omega_i \geq (1 - \Phi(\frac{1}{i})) \omega_i.
\]

**Remark 2.1.** In [19] and [20], we applied the parabolic method of Ni-Tam [29] to obtain such function. As we shall see in the proof below, elliptic method suffices.

**Corollary 2.2.** There exists some \( R > 0 \) so that any compact subvariety of positive dimension in \( M \) is contained in \( B(p, R) \).
Proof. Assume that there exists a sequence of compact subvarieties \( V_i \) of positive dimension so that \( \mu_i = d(p_i, V_i) \to \infty \). By the scaling as in the proposition above, we may assume that \( V_i \) is a subvariety of \( M_i \) and \( d(p_i, V_i) = 1 \). Then for sufficiently large \( i \), there exists a psh function \( u \) on \( B(p_i, 200) \) satisfying
\[
(2.8) \quad |u - r_i^2| \leq \Phi\left(\frac{1}{i}\right).
\]
Furthermore, on \( B(p_i, 200) \setminus B(p_i, \Phi\left(\frac{1}{i}\right)) \),
\[
(2.9) \quad \sqrt{-1} \partial \bar{\partial} u \geq (1 - \Phi\left(\frac{1}{i}\right))\omega_i.
\]
Then the maximum of \( u \) on \( V_i \) is achieved somewhere on \( B(p_i, 200) \setminus B(p_i, 0.2) \), where \( u \) is strictly plurisubharmonic. This is a contradiction.

Now we prove proposition 2.3.

Proof. Following [2], we solve the equation
\[
(2.10) \quad \Delta \hat{u} = 2n
\]
on \( B(p_i, 2R) \setminus B(p_i, 1) \) with boundary value \( \hat{u} = 4R^2 \) on \( \partial B(p_i, 2R) \), \( \hat{u} = 1 \) on \( \partial B(p_i, 1) \). Laplace comparison implies that
\[
(2.11) \quad \hat{u} \leq r_i^2.
\]
Let \( G_i(x, y) \) be the Green function on \( B(p_i, 2R) \setminus B(p_i, 1) \). Then
\[
(2.12) \quad v(x) = \hat{u}(x) + \int_{B(p_i, 2R) \setminus B(p_i, 1)} G_i(x, y) \Delta \hat{u}(y) dy
\]
is harmonic. By checking the boundary values, we find
\[
(2.13) \quad |v| \leq 4R^2
\]
on \( B(p_i, 2R) \setminus B(p_i, 1) \). Now from the maximum principle, for \( x, y \in B(p_i, 2R) \setminus B(p_i, 1) \),
\[
(2.14) \quad G_i(x, y) \leq \hat{G}_i(x, y),
\]
where \( \hat{G}_i(x, y) \) is the Green function on \( B(p_i, 3R) \). Since the manifold has maximal volume growth, we find
\[
(2.15) \quad \int_{B(p_i, 2R)} \hat{G}_i(x, y) dy \leq C
\]
where \( C \) is independent of \( i \) and \( x \in B(p_i, 2R) \). Then from (2.12), we find that
\[
(2.16) \quad |\hat{u}| \leq C(R).
\]
Following proposition 4.35 of [2], we have
\[
(2.17) \quad \int_{B(p_i, 2R) \setminus B(p_i, 1)} |\nabla \hat{u} - \nabla r_i^2|^2 < \Phi\left(\frac{1}{i}\right).
\]
Then similar as proposition 4.50 of [2], on \( B(p_i, 1.75R) \setminus B(p_i, 1.25) \),
\[
(2.18) \quad |\hat{u} - r_i^2| < \Phi\left(\frac{1}{i}\right).
\]
The argument uses Poincare inequality and the gradient estimate of Cheng-Yau. As in (4.56) of [2], the Bochner formula for \( \Delta |\nabla \hat{u}|^2 \) gives that
\[
(2.19) \quad \int_{B(p_i, 1.5R) \setminus B(p_i, 2)} |\nabla^2 \hat{u} - g|^2 < \Phi\left(\frac{1}{i}\right).
\]
This of course implies that

\[ \int_{B(p_i, 1.5R) \setminus B(p_i, 2)} |\hat{u}_{\alpha \beta} - g_{\alpha \beta}|^2 < \Phi\left( \frac{1}{i} \right). \]

Claim 2.2. On \( B(p_i, 1.5R) \setminus B(p_i, 2) \), \( \Delta |\hat{u}_{\alpha \beta} - g_{\alpha \beta}| \geq -C|\hat{u}_{\alpha \beta} - g_{\alpha \beta}|, \) \( |\nabla \hat{u}| \leq 2r_i + \Phi\left( \frac{1}{i} \right) \).

Remark 2.2. As we will see, the bisectional curvature lower bound can be replaced by the quadratic orthogonal bisectional curvature lower bound.

Proof. Set

\[ \hat{v}_{\alpha \beta} = \hat{u}_{\alpha \beta} - g_{\alpha \beta}. \]

Let us diagonalize \( \hat{v} \) at a point so that

\[ \hat{v}_{\beta \beta} = a_{\alpha \beta}. \]

As \( BK(M) \geq -\frac{C}{r_i^2} \), on \( B(p_i, 2R) \setminus B(p_i, 1) \), the bisectional curvature has a lower bound \(-C\). The calculation of (26) on page 186 and 187 ((iv) and (vi)) gives

\[ \Delta (\hat{v})^2 = |\hat{v}_{\alpha \beta}|^2 + |\hat{v}_{\alpha \beta}|^2 + 2R \cdot \Delta (a_{\alpha \beta} - a_{\beta \alpha})^2 \geq |\hat{v}_{\alpha \beta}|^2 + |\hat{v}_{\alpha \beta}|^2 - C, \]

Note

\[ |\nabla |\hat{v}||^2 \leq |\hat{v}_{\alpha \beta}|^2 + |\hat{v}_{\alpha \beta}|^2. \]

The first statement follows from (2.23) and (2.24). For the second statement, we use an argument similar to Cheeger-Naber [7], page 1116-1117. By the Bochner formula,

\[ \Delta (|\nabla \hat{u}|^2 - 4\hat{u}) \geq 0. \]

Notice (2.17) and (2.18) imply

\[ \int_{B(p_i, 1.6R) \setminus B(p_i, 1.5)} |\nabla \hat{u}|^2 - 4\hat{u} < \Phi\left( \frac{1}{i} \right). \]

Therefore, by applying the mean value inequality to \( \max(|\nabla \hat{u}|^2 - 4\hat{u}, 0) \), we finish the proof of the second statement.

\[ \square \]

Now by applying the parabolic mean value inequality to (2.20) and claim 2.2 we find that

\[ \hat{u}_{\alpha \beta} \geq (1 - \Phi\left( \frac{1}{i} \right)) g_{\alpha \beta}. \]

on \( B(p_i, 1.2R) \setminus B(p_i, 3) \). Set \( u = \max(\hat{u} - 5, 0) \). Then for large \( i \), \( u \) is a psh function on \( B(p, R) \) with

\[ \sqrt{-1} \partial \bar{\partial} u = (1 - \Phi\left( \frac{1}{i} \right)) \omega_i \]

on \( B(p_i, R) \setminus B(p_i, 10) \). Moreover,

\[ |u - r_i^2| \leq 5 + \Phi\left( \frac{1}{i} \right), |\nabla u|^2 - 4r_i^2 \leq \Phi\left( \frac{1}{i} \right). \]

Now we can solve new equations \( \Delta \hat{u} = 2n \) on \( B(p_i, 2R) \setminus B(p_i, \Phi\left( \frac{1}{i} \right)) \) where inner radius \( \Phi\left( \frac{1}{i} \right) \) goes to zero sufficiently slow. Then we can apply the same argument as before. Finally, if we set \( u = \max(\hat{u} - \Phi\left( \frac{1}{i} \right), 0) \) for some other \( \Phi\left( \frac{1}{i} \right) \) going to zero sufficiently slow, then \( u \) satisfies (2.5), (2.6) and (2.7). This completes the proof of proposition 2.3.

\[ \square \]
Now let us adopt the assumption in proposition 2.3. Recall \((M_i, p_i, d_i) = (M, p, \frac{d}{\mu}) \to (M_\infty, p_\infty, d_\infty)\). Pick any points \(q_1 \neq q_2 \in \partial B(p_\infty, 5)\). Let \(q_j^i \in M_i\) be so close to \(q_j\). According to proposition 6.1 in [19] (see also proposition 5.2 in [20]), there exists \(\delta > 0\) independent of \(i\) so that for all sufficiently large \(i\), we can find a smooth function \(v_j^i\) with isolated singularity at \(q_j^i\) and supported on \(B(q, \delta)\). Moreover, \(e^{-v_j^i}\) is not locally integrable at \(q_j^i\) and \(\sqrt{-1}\partial\bar{\partial}v_j^i \geq -C\omega_i\). Let \(u\) be the psh function constructed in proposition 2.3. Thus there exists \(C > 0\) independent of \(i\) so that \(\sqrt{-1}\partial\bar{\partial}(Cu + v_1^i + v_2^i) \geq \omega_i\) on \(B(p_i, R)\backslash B(p_i, 1)\) and \((Cu + v_1^i + v_2^i)\) is psh on \(B(p_i, R)\). By using the same argument as in proposition 5.2 of [20], we can holomorphically separate points on \(B(p_i, 2)\backslash B(p_i, 1)\). Also we can separate \(p_i\) from \(B(p_i, 2)\backslash B(p_i, 1)\). We have the following

**Proposition 2.4.** There exist \(N \in \mathbb{N}, A > 5\) depending only on \(M\) so that for all sufficiently large \(i\), there exist holomorphic functions \(g_1^i, \ldots, g_N^i\) on \(B(p_i, 6A)\)

\[
(2.30) \quad g_j^i(p_i) = 0, \quad \sup_{B(p_i, 1)} \sum_{j=1}^N |g_j^i|^2 = 1.
\]

\[
(2.31) \quad \min_{x \in \partial B(p_i, 3A)} \sum_{j=1}^N |g_j^i(x)|^2 > 4 \sup_{x \in B(p_i, 1)} \sum_{j=1}^N |g_j^i(x)|^2 = 4.
\]

Furthermore, for all \(j\),

\[
(2.32) \quad \frac{\sup_{x \in B(p_i, 3A)} |g_j^i(x)|^2}{\sup_{x \in B(p_i, 2A)} |g_j^i(x)|^2} \leq C = C(M).
\]

**Proof.** Given proposition 2.3, the argument is almost the same as in proposition 6.1 of [19]. One point is that the three circle theorem in (6.32) of [19] should be replaced by the following

**Lemma 1.1.** Let \(R_0 > 2\) be a constant. For sufficiently large \(i\), if \(h\) is a holomorphic function on \(B(p_i, R_0)\) satisfying

\[
\sup_{B(p_i, R_0)} |h| \leq C \text{ and } \sup_{B(p_i, 1)} |h| = 1, \text{ then } \sup_{B(p_i, R_0)} |h| \leq C(R_0).
\]

**Proof.** Let \(i \to \infty\). If we first normalize \(h_i = c_i h\) so that \(\sup_{B(p_i, R_0)} |h_i| = 1\), then by passing to subsequence, \(h_i\) converges uniformly on each compact set to a harmonic function \(h_\infty\) on \(B(p_\infty, R_0)\). Then \(\sup_{B(p_\infty, R_0)} |h_\infty| \geq \frac{1}{C}\) and

\[
(2.33) \quad \frac{\int_{B(p_\infty, 0.8R_0)} |h_\infty|^2}{\int_{B(p_\infty, 0.6R_0)} |h_\infty|^2} \leq C.
\]

By three circle theorem of harmonic functions over metric cones, we obtain that \(\sup_{B(p_\infty, 1)} |h_\infty| \geq c > 0\). This complete the proof of the lemma. \(\square\)

This suffices to prove proposition 2.4. \(\square\)

The above argument shows that \(|g_j^i| \leq C(M)\) on \(B(p_i, 3A)\). Set \(G_i = (g_1^i, \ldots, g_N^i)\). Let \(B(p_i, 1) \subset \Omega_{i} \subset \subset B(p_i, 2A)\) be the connected component of \(G_i^{-1}(B_{C^\alpha}(0, 1))\) containing \(p_i\).
Then \( G_i : \Omega'_i \to B_{C^0}(0, 1) \) is proper. Let \( V_i = G_i(\Omega'_i) \subset B_{C^0}(0, 1) \) be the subvariety. Note \( V_i \) has dimension \( n \). To see this, take \( x \in \partial B_{C^0}(0, \frac{1}{i}) \cap V_i \). Then \( G_i^{-1}(x) \cap \Omega'_i \) is a compact subvariety of \( \Omega'_i \), in particular, a compact subvariety \( W \) of \( M \). As \( i \) is large, each irreducible component of \( W \) must be far away from \( p \). According to corollary 2.2, \( W \) is zero dimensional. Hence, \( V_i \) has dimension \( n \).

According to local parametrization theorem 4.19 in [13], given any analytic variety \( 0 \in V^n \subset B_{C^0}(0, 1) \), we can locally properly project \( (V^n, 0) \) to a complex \( n \)-dimensional plane through 0. Since the volume of \( V_i \) is uniformly bounded from above, Bishop’s theorem [11] says \( V_i \) form a relatively compact set of the subvarieties of dimension \( n \), through 0 in unit ball of \( \mathbb{C}^N \). Then we can find \( c_1 > 0, c_2 > 0 \), a compact set \( K \) of \( GL(N, \mathbb{C}) \) depending only on \( M \) so that the following hold for all large \( i \):

1. \( B' \in K \)
2. Set \( \pi'_i : (g^1_i, ..., g^N_i) \to (h^1_i, ..., h^N_i) \) be given by \( h^k_i = \sum_{j=1}^{N} B'_{jk} g^j_i. \)
3. For \( (g^1_i, ..., g^N_i) \in V_i \), if \( \sum_{j=1}^{N} |g^j_i|^2 = c_1 < 1, \sum_{k=1}^{N} |h^k_i|^2 \geq c_2 \sum_{j=1}^{N} |g^j_i|^2. \)

**Claim 2.3.** Set \( \pi_i = (h^1_i, ..., h^N_i) : B(p_i, 3A) \to \mathbb{C}^n \). Set \( r_0 = \frac{1}{4 \sqrt{c_1 c_2}} \). Let \( \Omega_i \) be the connected component of \( \pi_i^{-1}(B_{C^0}(0, 6r_0)) \) containing \( p_i \). Then for all sufficiently large \( i \), \( \Omega_i \subset B(p_i, 2A) \).

**Proof.** It is clear that \( \pi_i = \pi'_i \circ G_i \). Assume the claim is false. Then there exists a curve \( \gamma_i \subset \Omega_i \) connecting \( p_i \) and some point \( q_i \) on \( \partial B(p_i, 2A) \). Observe that by (2.31), \( |G_i(q_i)| \geq 2 \). As \( |G_i(p_i)| = 0 \), we can find a point \( t_i \in \gamma_i \) so that \( |G_i(t_i)|^2 = c_1 \). Then according to item 3 above, \( |\pi_i(t_i)| \geq \sqrt{c_1 c_2} > 6r_0 \). This is a contradiction. \( \square \)

Since the matrix \( B' \) and \( g^j_i \) are bounded, there exists a constant \( r_1 \) depending only on \( M \) so that \( \pi_i(B(p_i, 2r_1)) \subset B_{C^0}(0, r_0) \). Let us summarize the results as

**Proposition 2.5.** There exist positive constants \( r_0, r_1 \) depending only on \( M \) so that for all large \( i \), \( \pi_i(B(p_i, 2r_1)) \subset B_{C^0}(0, r_0), \Omega_i \subset B(p_i, 2A) \).

Proposition 2.5 will be used later.

**Proposition 2.6.** Any tangent cone of \((M, p)\) at infinity is homeomorphic to a normal affine algebraic variety.

**Proof.** Let \((M_i, p_i, d_i) = (M, p, \frac{d}{\mu})\) for some sequence \( \mu_i \to \infty \). Assume \((M_i, p_i, d_i)\) converges in the pointed Gromov-Hausdorff limit \((M_\infty, p_\infty, d_\infty)\). By proposition 2.4, we can find \( g^1_i, ..., g^N_i \) satisfying (2.30), (2.31) and (2.32). Then we can apply the same argument as in [20] to prove the proposition. \( \square \)

**Proposition 2.7.** \( \int_{B(p, r)} S \leq C(M) \) for any \( r > 0 \), where \( S \) is the scalar curvature.

**Proof.** Let \((M_i, p_i, d_i) = (M, p, \frac{d}{\mu})\) for some sequence \( \mu_i \to \infty \). Then for large \( i \), \( B(p_i, 1) \) is sufficiently close to a metric cone. Hence, we can apply proposition 2.3. By Gromov compactness theorem and Cheeger-Colding theory, we can find a point \( q \in \partial B(p_i, \frac{1}{i}) \) so that \( B(q, \delta) = e_0 \delta \)-Gromov-Hausdorff close to \( B_{C^0}(0, \delta) \), where \( \delta > 0 \) is independent of \( i \) and \( e_0 \) is a very small number depending only on \( n \). According to [20], there exists a
holomorphic chart \((z_1, \ldots, z_n)\) on \(B(q, \varepsilon_1 \delta)\), where \(\varepsilon_1\) depends only on \(n\). Furthermore,

\[(2.34) \int_{B(q, \varepsilon_1 \delta)} |dz_k \overline{dz_l} - \delta_{kl}|^2 < \phi(\varepsilon_1).\]

Then for sufficiently small \(\varepsilon_1\), there exists a point \(q' \in B(q, \frac{\delta}{\lambda})\) so that \(|dz_1 \wedge \cdots \wedge dz_n(q')| \geq \frac{1}{2}\).

Set \(\Omega = dz_1 \wedge \cdots \wedge dz_n\). We can solve \(\overline{\partial} s' = \overline{\partial} (v^i \Omega)\), where \(v^i\) is analogous to the cut-off function \(v_i\) above proposition [2.4]. Then we find \(s \in \Gamma(B(p, 1), K)\) \((K\text{ is the canonical line bundle})\) so that

\[(2.35) |s(q')| \geq \frac{1}{2}\]

and on \(B(p, 0.9)\),

\[(2.36) \sup |s| \leq C,\]

where \(C\) is independent of \(i\). The Poincare-Lelong equation says

\[(2.37) \frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} \log |s|^2 = [D] + \text{Ric}(M),\]

where \(D\) is the divisor of \(s\). Let \(G(x, y)\) be the Green function on \(B(p, 0.8)\). Let \(S_i\) be the scalar curvature on \(M_i\). By taking the trace and integrate, we find

\[(2.38) \int_{B(p, 0.8)} G(q', y) \Delta \log |s(y)|^2 dy \geq 2\pi \int_{B(p, 0.8)} G(q', y) S_i(y) dy \geq c(M) \int_{B(p, \frac{1}{2})} S_i(y) dy.\]

Define

\[(2.39) F(x) = \log |s(x)|^2 + \int_{B(p, 0.8)} G(x, y) \Delta \log |s(y)|^2 dy.\]

Then \(F(x)\) is harmonic. Maximum principle says \(F\) bounded from above by \(C\) on \(B(p, 0.8)\). Therefore

\[(2.40) \int G(q', y) \Delta \log |s(y)|^2 dy + \log |s(q')|^2 \leq C.\]

As \(|s(q')| \geq \frac{1}{2}\), we find \(\int_{B(p, \frac{1}{2})} S_i \leq C\). The proposition is proved.

\[
\square
\]

**Corollary 2.3.** Let \((X^n, p)\) be a complete noncompact Kähler manifold with nonnegative bisectional curvature and maximal volume growth. Then there exists some \(C = C(X) > 0\) so that for any \(q \in X\), any \(r > 0\), \(r^{2-2n} \int_{B(q, r)} S \leq C\).

**Remark 2.3.** This result was proved in [27] when \(X\) has bounded curvature. In [18], the result was proved for \(C\) depending on \(q\).

**Proof.** Let \(v = \lim_{r \to \infty} \frac{\text{vol}(B(p, r))}{p^n} > 0\). By volume comparison, we can find \(N \in \mathbb{N}\) so that for any \(q \in M, r > 0\), there exists \(1 \leq l \leq N\) and that \(B(q, 2^l r)\) is \(\varepsilon r\)-Gromov-Hausdorff close to a metric cone. Here \(\varepsilon = \varepsilon(n, v)\) is so small that the argument in proposition [2.7] can be applied. Then we find that

\[(2.41) (2^{l-1} r)^{2-2n} \int_{B(q, 2^{l+1} r)} S \leq C.\]

Note by Gromov compactness theorem, such \(C\) depends only on \(n, v\). This concludes the proof of the corollary.

\[
\square
\]

Recall theorem 1.1 by Ni-Shi-Tam [28]:
Proposition 2.8. Let \((X^n, \rho)\) be a complete noncompact Kähler manifold with nonnegative Ricci curvature and maximal volume growth. Let \(f\) be a smooth nonnegative function on \(X\). Set \(k(x, r) = \int_{B_{\rho}(x, r)} f\) and \(k(r) = \int_{B_{\rho}(p, r)} f\). Suppose \(\int_0^\infty k(t)dt < +\infty\), then there exists a solution \(u\) to \(\Delta u = f\) so that

\[
(2.42) - C(r \int_0^\infty k(t)dt + \int_0^r t k(x, t)dt + c \int_0^{2r} t k(t)dt) \leq u(x) \leq C(r \int_0^\infty k(t)dt + \int_0^{2r} t k(t)dt),
\]

where \(C, c\) are positive constant independent of \(x\).

We set \(f\) to be the scalar curvature on \(M\). Proposition 2.7 says \(k(t) \leq \frac{C}{1 + t^2}\). Then by proposition 2.8, we obtain

Proposition 2.9. There exists a function \(\rho'\) so that \(\Delta \rho' = \pi S\) and \(\rho' \leq C \log(r + 2)\). If in addition, \(S \leq \frac{r_0}{r}\), then \(C \log(r + 2) \geq \rho' \geq c \log(r + 1) - C\).

The following proposition plays an important role in part II and III. It is an improvement of theorem 2.3 in [25].

Proposition 2.10. Let \(\rho'\) be defined as in proposition 2.9. Let \(f\) be a holomorphic section in \(K^{-q}(M)\) and \(V\) be the divisor of \(f\). Assume that \(|f e^{\rho'}| \leq C(1 + r)^d\) on \(M\). Then for \(x\) not on compact subvarieties of positive dimension, \(\text{Mul}_x(V) \leq C d\), where \(C\) depends only on \(M\), \(\text{Mul}_x(V)\) is the multiplicity of \(V\) at \(x\).

Proof. Define

\[
(2.43) M'(r) = \sup_{B(r)} |f| e^{\rho'}. \]

Let \(A\) be the constant in proposition 2.4. \(r_0, r_1\) be the constants in proposition 2.5. Both constants depend only on \(M\). Since \(|f e^{\rho'}| \leq C(1 + r)^d\), we can find a sequence \(\mu_i \to \infty\) with

\[
(2.44) M'(20A \mu_i) M'(r_1 \mu_i) \leq \left(\frac{20A}{r_1}\right)^{2d}. \]

Set \((M_i, p_i, d_i) = (M, p, \frac{\mu_i}{r_1})\). Let \(g_i\) be as in proposition 2.4. Let \(\pi_i, \Omega_i\) be defined as in claim 2.3. From now on, we shall restrict \(\pi_i\) to \(\Omega_i\). Then \(\pi_i : \Omega_i \to B_{\rho}(0, 2r_0)\) is a proper map.

According to proposition 2.5 for large \(i\),

\[
(2.45) \pi_i^{-1}(B_{\rho}(0, 5r_0) \setminus \Omega_i) \subset B(p_i, 2A) \setminus B(p_i, 2r_0). \]

Recall \(V\) is the divisor of \(f\). Let \(V'\) be the union of irreducible components of \(V\) containing \(x\). Then \(\text{Mul}_x(V') = \text{Mul}_i(V)\). As \(x\) is a fixed point on \(M\), \(x \in B(p, \frac{r_0}{r_1})\) for \(i\) large. Let \(x_i = \pi_i(x) \to 0 \in \mathbb{C}^n\). Since \(x\) is not on any positive dimensional compact subvarieties and \(\pi_i\) is proper, each irreducible component of \(\pi_i(V')\) has dimension \(n - 1\). The multiplicity of \(\pi_i(V')\) at \(x_i\) will be at least \(m = \text{Mul}_i(V') = \text{Mul}_i(V)\) (just apply the line test, consider a line intersecting \(\pi_i(V')\) at \(x_i\), then pull back). The standard Lelong monotonicity implies that

\[
(2.46) \text{vol}(\pi_i(V') \cap B(x_i, 4r_0) \setminus B(x_i, 2r_0)) \geq c(r_0)m > 0. \]

Recall \(x_i \to 0 \in \mathbb{C}^n\). Then (2.45) implies that

\[
(2.47) \pi_i^{-1}(B(x_i, 4r_0) \setminus B(x_i, 2r_0)) \subset B(p_i, 2A) \setminus B(p_i, 2r_1). \]
Recall $|g^i_j| \leq C(M)$ on $B(p_i, 3A)$. By the gradient estimate for $g^i_j$,

\begin{equation}
\text{vol}(V' \cap B(p_i, 2A) \setminus B(p_i, 2r)) \geq c(M, A, r_i)m > 0.
\end{equation}

Therefore,

\begin{equation}
\text{vol}(V' \cap B(p, 2A\mu_i) \setminus B(p, 2\mu_ir_i)) \geq c(M, A, r_0, r_i)m\mu_i^{2n-2}.
\end{equation}

Poincare-Lelong equation says

\begin{equation}
\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |f|^2 = -q\text{Ric} + [V].
\end{equation}

Let $G_{20A\mu_i}(z, y)$ be the Green function on $B(p, 20A\mu_i)$. Define $F(y) = |f(y)|e^{\rho'(y)}$. As $\Delta \rho' = \pi\mathcal{S}$,

\begin{equation}
\frac{1}{\pi} \int_{\partial B(p, 2A\mu_i) \setminus \partial B(p, 2r\mu_i)} G_{20A\mu_i}(z, y)\Delta \log F(y)dy \
\geq \int_{V \cap B(p, 2A\mu_i) \setminus B(p, 2r\mu_i)} G_{20A\mu_i}(z, y)dy.
\end{equation}

Now we apply similar argument as in proposition 2.2. Let $H(z)$ be the harmonic function defined by

\begin{equation}
H(z) = \log F(z) + \int_{B(p, 20A\mu_i)} G_{20A\mu_i}(z, y)\Delta \log F(y)dy.
\end{equation}

According to maximum principle and (2.44-1 on $B(p, 20A\mu_i)$,

\begin{equation}
H_r \leq dC\left(\frac{20A}{r_1}\right) + \log M(r_1\mu_i).
\end{equation}

Let $z^i_0 \in \partial B(p, r_1\mu_i)$ be so that $F(z^i_0) = M'(r_1\mu_i)$. Set $z = z^i_0$ in (2.52). Then (2.51), (2.52) and (2.53) imply that

\begin{equation}
\int_{V \cap B(p, 2A\mu_i) \setminus B(p, 2r_1\mu_i)} G_{20A\mu_i}(z^i_0, y)dy \
\leq \frac{1}{\pi} \int_{B(p, 20A\mu_i)} G_{20A\mu_i}(z^i_0, y)\Delta \log F(y)dy \
\leq dC\left(\frac{20A}{r_1}\right).
\end{equation}

Notice for $y \in B(p, 2A\mu_i) \setminus B(p, 2r_1\mu_i)$, $G_{20A\mu_i}(z^i_0, y) \geq \frac{C(M, A, r_0)}{\mu_i}$. Then we find that

\begin{equation}
\text{vol}(V \cap B(p, 2A\mu_i) \setminus B(p, 2r_1\mu_i)) \leq C(M, A, r_1)\mu_i^{2n-2}.
\end{equation}

The proposition follows from (2.49) and (2.55).

\[\square\]

**Definition 2.1.** Let $\mathcal{H}^d(M) = \{f \in K^{-q}(M)||f(x)e^{\theta(x)}| \leq C(1 + r(x)^d e^{-\epsilon})\}$ for any $\epsilon > 0$. Let $\mathcal{O}_d(M) = \{f \text{ holomorphic on } M||f(x)| \leq C(1 + r(x)^d e^{-\epsilon})\}$ for any $\epsilon > 0$. For a polynomial growth holomorphic function $f$, let $\text{deg}(f) = \min_{f \in \mathcal{O}_d(M)} d$.

**Corollary 2.4.** $\dim(\mathcal{H}^d(M)) \leq Cd^q$, where $C$ is independent of $q$ and $d$. In particular, if we set $q = 0$, then $\dim(\mathcal{O}_d(M)) \leq Cd^q$. 
3. Proof of theorem \[1.1\] part I

The crucial proposition is the following:

**Proposition 3.1.** Under the assumption of part I, we can find polynomial growth holomorphic functions \(g_1, \ldots, g_N\) so that \((g_1, \ldots, g_N)\) is a \(a\) proper map to \(\mathbb{C}^N\).

**Proof.** The argument is almost the same as theorem 6.1 of [19]. Let us sketch the argument, for completeness. For any sequence \(\mu_i \to \infty\) let \((M_i, p_i, d_i) = (M, p, \frac{d}{\mu_i})\). By passing to subsequence, we may assume \((M_i, p_i, d_i) \to (M_\infty, p_\infty, d_\infty)\) in pointed Gromov-Hausdorff sense. By proposition 2.4 there exist \(K, 0 < a < \frac{1}{10}\) independent of \(i\) and holomorphic function \(f_i^j (1 \leq j \leq K)\) on \(B(p_i, 6)\) so that

\[
(3.1) \quad f_i^j (p_i) = 0, \quad \max_j \sup_{B(p_i, a)} |f_i^j| = 1.
\]

Furthermore, for all \(j\),

\[
(3.2) \quad \min_{x \in \partial B(p_i, 1)} \sum_{j=1}^K |f_i^j(x)|^2 > 2 \sup_{x \in B(p_i, a)} \sum_{j=1}^K |f_i^j(x)|^2.
\]

Lemma 2.1 says

\[
(3.3) \quad \sup_{x \in B(p_i, 3)} |f_i^j(x)|^2 \leq C = C(M).
\]

\(\text{Lemma2.1}\) says

\[
(3.4) \quad |f_i^j| \leq C
\]

on \(B(p_i, 3)\). Let \(\lambda_i\) be a sequence tending to \(\infty\) very slowly. Let \(w_i = \log(u + 1)\), where \(u\) was defined in proposition 2.3. We assume \(u\) is defined on \(B(p_i, \lambda_i)\). Then

\[
(3.5) \quad \sqrt{-1} \partial \bar{\partial} w_i = \frac{(u + 1) \partial \bar{\partial} u}{(u + 1)^2}.
\]

According to proposition 2.3 since \(\lambda_i\) is increasing so slowly, for all large \(i\), \(w_i\) is strictly plurisubharmonic on \(B(p_i, \lambda_i) \setminus B(p_i, 1)\). Furthermore, outside \(B(p_i, \frac{1}{10})\),

\[
(3.6) \quad |w_i - \log(r_i^2 + 1)| < \Phi_i \left(\frac{1}{i}\right).
\]

Also, there exists \(c > 0\) independent of \(i\) so that on \(B(p_i, 10) \setminus B(p_i, \frac{1}{10})\),

\[
(3.7) \quad \sqrt{-1} \partial \bar{\partial} w_i \geq cw_i.
\]

Now consider a cut off function \(\eta_i\) depending only on \(r_i\) so that \(\eta_i = 1\) on \(B(p_i, 2)\), \(\eta_i\) has compact support on \(B(p_i, 2.5)\). By using the weight \(Cw_i\) \((C\) is large independent of \(i)\), we can solve the \(\partial\) equation \(\overline{\partial} v_i = \overline{\partial}(\eta_i f_i^j)\), as on page 303 of [19]. Then we obtain holomorphic functions \(f_i^j\) on \(B(p_i, \lambda_i)\) which are close \(\tilde{f}_i^j\) on \(B(p_i, 2)\). Furthermore,

\[
(3.8) \quad |f_i^j(x)| \leq C(1 + d(x, p_i))^{d_0}
\]

for some \(d_0\) depending only on \(M\). By shifting \(f_i^j\) by small constants, we may assume

\[
(3.9) \quad f_i^j(p_i) = 0, \quad \sum_{j=1}^k |f_i^j|^2 \geq c > 0
\]
on \( \partial B(p, 1) \). By passing to subsequence, we assume \( f^j_i \to f^j \) uniformly on each compact set of \( M_\infty \).

Let \( V = \{ g \in O_d(M) | g(p) = 0 \} \). Let \( g_s \), \( (s = 1, \ldots, N) \) be an orthonormal basis of \( V \) with respect to the \( L^2 \) inner product on \( B(p, 1) \). Assume \( (g_1, \ldots, g_N) \) is not proper. Then there exists a constant \( C \) and a sequence \( \mu_i \to \infty \) so that

\[
\min_{\partial B(p,\mu_i)} \sum |g_j|^2 \leq C. 
\] (3.10)

For each \( i \), there exists a basis \( g^j_s \) of \( V \) with

\[
\int_{B(p,1)} g^j_s \overline{g^j_t} = \delta_{st}; \quad \int_{B(p,1)} g^j_s \overline{g^j_t} = \lambda^j_{ss} \delta_{st}. 
\] (3.11)

Then

\[
\sum |g_j|^2 = \sum |g^j_s|^2. 
\] (3.12)

As \( g^j_s(p) = 0 \), \( \sup_{B(p,1)} |d g^j_s| \geq c > 0 \) for any \( i \). Gradient estimate implies that

\[
\lambda_{ss}^j > c \mu^2_i. 
\] (3.13)

Now for the sequence \( \mu_i \to \infty \), we can find a basis \( h^j_1, \ldots, h^j_N \) of \( V \) so that \( \int_{B(p,1)} h^j_s \overline{h^j_t} = \delta_{st} \). Mean value inequality and corollary 2.1 imply that we can pass \( h^j_j \) to \( M_\infty \), after taking subsequence. Say \( h^j_j \to h_j \). Then \( h_j \in O_d(M_\infty) \).

Claim 3.1. \( \text{Span}\{f^k \} \subset \text{Span}\{h_j\} \).

Proof. The argument is the same as claim 6.1 in [19], except that we replace three circle theorem by corollary 2.1. We skip the details. \( \square \)

Claim 3.1 implies that on \( B(p, 1) \), \( f^k \) is almost in the span of \( h^j_j \). More precisely,

\[
\lim_{i \to \infty} \sup_{B(p,1)} |f^j_i(x) - \sum c^j_i h^j_i| = 0
\] (3.14)

for \( c^j_i = \int_{B(p,1)} f^j_i h^j_i \). In particular, \( |c^j_i| \leq C(M) \). By (3.9),

\[
\min_{\partial B(p,1)} \sum_{j=1}^{K} |f^j_i(x)|^2 \geq c > 0. 
\] (3.15)

By (3.13),

\[
|h^j_i|^2 = \frac{|g^j_i|^2}{\lambda_{ss}^j} \leq \frac{|g^j_i|^2}{c \mu^2_i}. 
\] (3.16)

Then from (3.12),

\[
\min_{\partial B(p,\mu_i)} \sum |g_j|^2 = \min_{\partial B(p,\mu_i)} \sum |g^j_s|^2 = \min_{\partial B(p,\mu_i)} \sum |g^j_s|^2 \geq c \mu^2_i > 0. 
\] (3.17)

This contradicts (3.10). \( \square \)

Let \( F = (g_1, \ldots, g_N) \). Proper mapping theorem says the image \( F(M) \) is an irreducible analytic subvariety \( X \subset \mathbb{C}^N \). The preimage of a point of \( X \) is a compact subvariety. By corollary 2.2, we can pick a point on \( X \) far away so that the its preimage consists of finitely many points. Therefore, \( \text{dim}(X) = n \). Combining with proposition 2.1 that \( \text{dim}(O_d(M)) \leq Cd^n \), we find the transcendental dimension of \( O_d(M) \) over \( \mathbb{C} \) is \( n \). Let \( Y \subset \mathbb{C}^N \) be the affine
algebraic variety determined by the integral ring $\mathbb{C}[g_1, \ldots, g_N]$. Then $Y$ has dimension $n$ and $X$ is a subvariety of $Y$. As $Y$ is irreducible and $\dim(X) = n$, $X = Y$. Thus $X$ is an affine algebraic variety. The function $v = \log(1 + |g_1|^2 + \cdots + |g_N|^2)$ is psh with logarithmic growth on $M$. Now pick a generic point $q \in X$ so that $F^{-1}(q) = \{q_1, \ldots, q_m\} \in M$. We further require that $F$ is a local biholomorphism near $q_1, \ldots, q_m$. Then $v$ is strictly psh near $q_1, \ldots, q_m$. By solving the equation with weight function $Cv$ ($C$ is large), we obtain polynomial growth holomorphic functions which separate $q_1, \ldots, q_m$. By adding these functions to $F$, we find that $F$ is generically one to one. Recall $X$ is the image of $F$ which is affine. Let us replace $X$ by the normalization $\bar{X}$. According to proposition 2.2, it suffices to add finitely many polynomial growth holomorphic functions to $F$, in order to achieve $\bar{X}$. We still denote $\bar{X}$ by $X$, for simplicity.

Then $F$ is a biholomorphism from $M$ to $X$ outside the compact subvarieties $Z_1, \ldots, Z_k$, which are contracted to points $z_1, \ldots, z_k$ on $X$. Let $R(M)$ be the field of polynomial growth holomorphic functions. As the transcendental dimension of $R(M)$ over $\mathbb{C}$ is $n$, $R(M)$ can be generated by $(n + 1)$ polynomial growth holomorphic functions. Say $g_1', \ldots, g_{n+1}'$. By adding these functions to the map $F$, we may assume $g_1', \ldots, g_{n+1}'$ are regular functions on $X$. Therefore, any polynomial growth holomorphic functions $f$ on $M$ is a rational function on $X$. Since $X$ is normal, $f$ is regular [15]. Thus the ring of polynomial growth holomorphic functions on $M$ can be identified with the affine coordinate ring of $X$. Hence, it is finitely generated.

Now $X$ can be compactified as a projective manifold by attaching some ample divisor $D$ at infinity. Since $F$ is a biholomorphism outside compact set, we can also attach $D$ to $M$ at infinity. The resulting manifold is Moishezon, since it contains algebraic functions $g_1, \ldots, g_N$ which have transcendental dimension $n$. Thus $M$ is biholomorphic to a Zariski open set of a Moishezon manifold.

4. Proof of theorem [1.1] part II

Let $C(M)$ be the unique tangent cone of $M$ at infinity. Let $D = \{\gamma \geq 0\}$ there exists a nonzero harmonic function $f$ on $C(M)$ so that $f(\lambda x) = \lambda^\gamma f(x)$, where $\lambda$ is the homothety map on the metric cone $C(M)$. Let us call $D$ the spectrum of harmonic functions on $C(M)$. Assume $\alpha$ is not contained in the spectrum of harmonic functions on $C(M)$. This is always possible, since the values in $D$ are discrete (they are related to the spectrum of harmonic functions on the cross section).

We need a result of Donaldson-Sun (proposition 3.23 of [14]):

**Lemma 4.1.** Let $(M, p)$ be a complete noncompact Kähler manifold with nonnegative Ricci curvature and maximal volume growth. Assume $\alpha$ is not contained in the spectrum of harmonic functions on any tangent cone of $M$. Then for $R$ large enough, for any holomorphic function $f$ on $B(p, 4R)$, if $\frac{\|f\|_{L^2}}{\|f\|_{L^2}} \leq 2^n$, then $\frac{\|f\|_{L^2}}{\|f\|_{L^2}} < 2^n$, where $I(f, R) = \int_{B(p, R)} |f|^2$.

**Proposition 4.1.** Under the assumption of theorem [1.1] part II, there exist holomorphic functions of polynomial growth $g_1, \ldots, g_N$ so that the map $(g_1, \ldots, g_N)$ is proper.

**Proof.** The argument is almost the same as proposition [3.1]. We only indicate the minor differences.

1. In [3.3], by increasing $d_0$ if necessary, we may assume $d_0 \notin D$, the spectrum of harmonic functions on $C(M)$. Recall between [3.13] and claim [3.1] when we pass $h'_j$ to $M_{\infty}$, corollary [2.1] is used. Now we replace corollary [2.1] by lemma [4.1]
2. We still have $\dim(O_d(M)) \leq Cd^s$, by corollary 4.3.

3. In the proof of claim 6.1 of [19], the three circle theorem used between (6.90) and (6.91) should be replaced by lemma 4.1.

By exactly the same argument as in part I, we can prove that $M$ is biholomorphic to a Zariski open set of a Moishezon manifold.

Now we prove corollary 1.1. In the Ricci flat case, we can apply the result of Colding-Minicozzi [11] to see the uniqueness of the tangent cone. Hence, part II of theorem 1.1 can be applied. Alternatively, we can also adapt an argument of Donaldson-Sun. Basically from proposition 2.21 of [14], we see that the holomorphic spectrum on tangent cone must be algebraic numbers. Hence lemma 4.1 can still be applied. The first statement of corollary 1.1 is a consequence of theorem 1.1 part II.

The degeneration argument follows from [14] and [12]. For reader’s convenience, let us give a sketch. Let $V_j = O_{d_j}(M)$, where $0 = d_0 < d_1 < d_2 < \ldots$ and for any $\alpha$ strictly between $d_j$ and $d_{j+1}$, $\alpha \alpha(M) = O_{d_j}(M)$. Pick basis $f_1, \ldots, f_{m_j}$ of $V_j$. Inductively, we add basis $f_{m_j+1}, \ldots, f_{m_{j+1}}$ to $V_j$. Let $V_j$ be the span of $(f_{m_j+1}, \ldots, f_{m_{j+1}})$.

**Lemma 4.2.** $\dim(O_d(M)) = \dim(O_d(C(M)))$ for any $d$.

**Proof.** The proof is the same as in proposition 3.26 of [14]. □

**Lemma 4.3.** $\oplus_{d \geq 0} O_{d}(M) / O_{d+1}(M)$ is finitely generated.

**Proof.** The argument is the same as in proposition 3.14 and lemma 3.15 of [14]. □

Choose $j$ large so that $\oplus_{d \geq 0} O_{d}(M) / O_{d+1}(M)$ is generated by $\oplus_{d \geq 0} O_{d}(M) / O_{d+1}(M)$. Furthermore, we require that $F = (f_1, \ldots, f_{m_j}, \ldots, f_{m_{j+1}}, \ldots, f_{m_{k+1}})$ is a biholomorphism outside compact set of $M$ to its image in $\mathbb{C}^N$, where $N = m_j$. Let $X = F(M)$. As we see before, $X$ is a normal affine algebraic variety. $F$ just contracts finitely many compact subvarieties to points on $X$.

Recall definition 2.1. Consider a degeneration of $X$ in $\mathbb{C}^N$ given by $\sigma_t(f_k) = e^{lt} f_k$, where $t \to 0^+$ as positive real numbers. Then $X$ degenerates to the affine algebraic variety $W = \text{Spec} \oplus_{d \geq 0} O_{d}(M) / O_{d+1}(M)$.

Take a sequence $r_i \to \infty$. Define $(M_i, p_i, d_i) = (M, p, \frac{1}{r_i})$. Assume $(M_i, p_i, d_i)$ converges in the pointed Gromov-Hausdorff sense to a tangent cone $C(M) = (M_\infty, p_\infty, d_\infty)$ at infinity.

**Proposition 4.2.** For any $g \in W_j$, $\lim_{i \to \infty} \frac{\log \| \delta_k \||_{g, R_{j+1}}}{\log \| \delta_k \||_{g, R_j}} = 2d_i$.

**Proof.** Let $\epsilon > 0$ be a small number. Since $g \in W_j$, we can find a sequence $R_{i, \epsilon} \to \infty$ so that $\frac{\log \| \delta_k \||_{g, R_{j+1}}}{\log \| \delta_k \||_{g, R_j}} < 2d_i + \epsilon$. As $g \notin O_{d_i - \epsilon}(M)$, we can find a sequence $R'_{i, \epsilon} \to \infty$ so that $\frac{\log \| \delta_k \||_{g, R'_{j+1}}}{\log \| \delta_k \||_{g, R'_{j}}} > 2d_i - \epsilon$. By the discreteness of the spectrum of the tangent cone, $d_i - \epsilon, d_i + \epsilon$ are not on the spectrum of harmonic functions of $C(M)$. By letting $\epsilon \to 0$, we obtain the proposition from lemma 4.1.

For any $k \in \mathbb{N}$, let $f_{m_j+1}', \ldots, f_{m_{k+1}}'$ be a basis of $W_k$ so that $\int_{B(p_j, 1)} f_{m_j}' f_{m_{k+1}}' = \delta_{sl}$, where $m_k + 1 \leq s, l \leq m_{k+1}$. Proposition 4.2 and lemma 3.6 of [14] imply that after passing to subsequence, $f_{m_j}'$ converges, uniformly in each compact set of $C(M)$, to a homogeneous polynomial growth holomorphic function $f_{m_j}'$ of degree $d_k$. Furthermore, since any homogeneous holomorphic function with different degree are $L^2$ orthogonal on $B(p_\infty, 1)$, we
conclude for any $1 \leq a, b \leq m_j$, \[ f^\infty_u \hat{f}^\infty_b = \delta_{ab}. \] Lemma 4.2 implies that $f^\infty_1, ..., f^\infty_m$ is a basis of $O_d(C(M))$.

The tangent cone of $C(M)$ admits a natural $\mathbb{T}$ action generated by $r\frac{\partial}{\partial r}$ and $\sqrt{-1} r J \frac{\partial}{\partial \bar{r}}$. For any homogeneous polynomial growth holomorphic function $f$ of degree $a$, $r \frac{\partial}{\partial r} f = af$, $r J \frac{\partial}{\partial \bar{r}} f = \sqrt{-1} af$. Spec $C(M)$ has a natural grading by the degree of homogeneous polynomial growth holomorphic functions. Note by lemma 4.2, the grading is the same as Spec $W$. Thus there is also a natural $\mathbb{T}$ action on $W$, with the same Hilbert function as $C(M)$.

According to [14] [16], there is a multi-graded Hilbert scheme $\text{Hilb}$, which is a projective scheme parametrizing polarized affine schemes in $\mathbb{C}^N$ invariant under the $\mathbb{T}$ action with fixed Hilbert function. We can consider the embedding of $W$ and $C(M)$ to $\mathbb{C}^N$ by $([f^M_1], ..., [f^M_m])$ and $([f^\infty_1], ..., [f^\infty_m])$, where $[-]$ means the quotient in $O_d(M)/O_{d-1}(M)$. These define points $[W]$ and $[C(M)]$ in $\text{Hilb}$.

Let $G$ be the linear transformations of $\mathbb{C}^N$ that commutes with $\mathbb{T}$ action. Let $K = G \cap U(N)$, where $U(N)$ is the unitary group. With the same proof as in proposition 3.16 of [14], we obtain

**Proposition 4.3.** $[W]$ converges to $[C(M)]$, up to $K$ action.

By Matsushima’s theorem (proposition 4.9 of [14]), $Aut(C(M))$ preserving $\mathbb{T}$ is reductive. By general theory, there exists a degeneration from $W$ to $C(M)$. This completes the proof of corollary 4.1.

**Remark 4.1.** We can apply the Luna slice theorem as in [14] to prove that all tangent cones of $M$ at infinity are isomorphic as affine algebraic varieties. By the uniqueness of the Kähler-Einstein metric, we find that the tangent cone is unique. This recovers Colding-Minicozzi’s result [11] in the Kähler case.

## 5. Proof of theorem 1.1 part III

In this section, we assume the Kähler manifold $M^n (n \geq 2)$ has positive Ricci curvature, maximal volume growth and $|RM| \leq \frac{C}{r^2}$. The only difference from [25] is the absence of $\int_{M} Ric^n < \infty$. We follow Mok’s approach in [25]. In fact, the argument is almost the same as in [25]. For completeness, we shall include some details, with emphasis on the difference.

The main strategy is to consider the embedding by pluri-anticanonical sections with polynomial growth.

**Definition 5.1.** Let $K$ be the canonical line bundle of $M$. Define $P^d_d(M) = \{ f \in \Gamma(M, K^{-d}) | |f| \leq C(1 + r)^d \}$ for some $C > 0$. Set $P(M) = \cup_{d \geq 0} P^d_d(M)$.

Following Mok [25], let us divide the proof into several steps:

**Step 1:** Uniform multiplicity estimate for pluri-anticanonical system

**Proposition 5.1.** Let $(M, p)$ be a complete noncompact Kähler manifold with nonnegative Ricci curvature and maximal volume growth. Assume further that $|RM| \leq \frac{C}{r^2}$. For any nonzero $s \in P^d_d(M)$, let $V$ be the zero divisor of $s$. Then for any $x$ which is not on any compact subvariety of positive dimension, $\text{Mul}_s(V) \leq C(d + q)$ for some constant $C$ independent of $s$ and $x$. 
Remark 5.1. Compare theorem 2.3 of [25], where it was proved that the multiplicity estimate holds except for $x$ on a disjoint union of compact subvarieties depending on $s$. Notice that the union of these compact subvarieties could be zero dimensional and noncompact.

Proof. This is just a combination of proposition 2.9 and proposition 2.10. □

Step 2: Quasi-embedding into a projective variety

Let $R(M)$ be the degree zero part (rational functions) of the quotient field of $P(M)$. Similar as proposition 3.2 of [25], we have

Proposition 5.2. $R(M)$ has transcendental dimension $n$ over $C$. Moreover, $R(M)$ separates points and tangents on $M$. In particular, $R(M)$ a finite extension over $C(f_1, \ldots, f_n)$ for some algebraically independent rational functions $f_1, \ldots, f_n \in R(M)$.

Remark 5.2. In [25], the argument requires theorem 2.2 of [25], which involves the finiteness of $\int_M Ric^\gamma$.

Proof. By the argument on page 383 of [25], for large $q$, we can find nontrivial $L^2$ holomorphic sections of the pluri-anticanonical bundle $K^{-q}$. Furthermore, standard $\partial\bar{\partial}$ estimates imply that these sections separates points and tangents of $M$. Let us verify that these $L^2$ sections are in $P(M)$. Say $s \in \Gamma_{L^2}(M, K^{-q})$. Then $s$ satisfies

\begin{equation}
\sqrt{-1} \frac{1}{2\pi} \partial\bar{\partial} \log |s|^2 = [D] - qRic,
\end{equation}

where $D$ is the divisor of $s$. By taking the trace, we obtain that

\begin{equation}
\frac{1}{2\pi} \Delta (2q\rho' + \log |s|^2) \geq 0,
\end{equation}

where $\rho'$ is in proposition 2.9. Therefore, $|s|^2 e^{2q\rho'}$ is a subharmonic function on $M$. Note

\begin{equation}
\int_{B(p,2r)} |s|^2 e^{2q\rho'} \leq \sup_{B(p,2r)} e^{2q\rho'} \int_{B(p,2r)} |s|^2 \leq C(1 + r)^{\rho C}.
\end{equation}

Mean value inequality and proposition 2.9 imply that $s \in P(M)$. Now assume $s_1, \ldots, s_{n+1} \in P(M)$ and they are algebraically independent. By taking some power if necessary, we may assume $s_1, \ldots, s_{n+1} \in P_{d_0}^0$. According to proposition 2.9 and corollary 2.4, $\dim P_{d_0}^0(M) \leq Ck^n$, where $C$ is independent of $k$. A simple linear algebra argument yields a contradiction to the assumption that $s_1, \ldots, s_{n+1}$ are algebraically independent. □

By a meromorphic map of $M$ to $\mathbb{P}^N$, we mean a holomorphic map into $\mathbb{P}^N$ on $M - Q$ for some subvariety $Q$ of codimension $\geq 2$. By a quasi-embedding of $M$ into $\mathbb{P}^N$, we mean a meromorphic map $F$ for which there exists a subvariety $Q$ of $M$ such that $F|_{M - Q}$ is a holomorphic embedding into $\mathbb{P}^N$.

According to proposition 5.2, we can assume $R(M) = \mathbb{C}(f_1, \ldots, f_n, g)$, where $g$ is algebraic over $\mathbb{C}(f_1, \ldots, f_n)$ and $f_1, \ldots, f_n \in R(M)$ are algebraically independent. We may assume $f_1, \ldots, f_n, g$ have common denominator $s_0 \in P(M)$. Say $f_i = \frac{s_i}{s_0}, g = \frac{s}{s_0}$. Let $Q$ be the common zero of $s_1, \ldots, s_n, u$. Then $F = [s_0 : s_1 : \cdots : s_n : u]$ defines a holomorphic map from $M - Q_0$ to $\mathbb{P}^{n+1}$, where $Q_0$ is a subvariety of $Q$ of codimension at least 2 in $M$.

Proposition 5.3. The meromorphic map $F: M \to \mathbb{P}^{n+1}$ is a quasi-embedding into some irreducible hypersurface $Z$ of $\mathbb{P}^{n+1}$.
The proof is the same as proposition 3.3 of [25].

Step 3: Almost surjectivity of quasi-embedding

The rough idea is this: given any $\xi \in Z - S - F(M - W)$ where $S$ is certain proper subvariety of $Z$, construct $g \in P(M)$ which extends to a meromorphic section on $Z$ and have pole at $\xi$. Then prove $Z - S - F(M - W)$ is given by union of finitely many divisors. The main tool is Skoda’s $L^2$ estimate.

Let us summarize some construction in section 4 of [25]. Let $F = [x_0 : s_1 : \cdots : s_n : u]$ be as in [25]. Recall $f_i = \frac{z_i}{w}$. Let $W$ be the union of zero set $W_0$ of $s_0$ and the branching locus of the holomorphic map $\rho_0(x) = (f_1(x), \ldots, f_n(x))$ defined on $M - W_0$. Then $M - W$ is a Stein manifold and $\rho = \rho_0|_{M-W}$ realizes $M - W$ as a Riemann domain of holomorphy over $\mathbb{C}^n$. It turns out that there is a positive integer $N$ so that for any $z \in \mathbb{C}^n$, $\rho^{-1}(z)$ contains at most $N$ points. On the Riemann domain of holomorphy $\rho : M - W \to \mathbb{C}^n$, let $\delta$ denote the distance to the boundary as on page 386 of [25]. The following proposition is the key ingredient of section 4 of [25]. Let us provide a variant proof.

**Proposition 5.4.** There exists $s \in P(M)$ so that $\delta(x) \geq c_1|s|^n|x_0|^n(1 + r)^{-c}$, where $a, b, c, c_1$ are some positive constants.

**Proof.** The following is lemma 4.3 of [25].

**Lemma 5.1.** For any $s \in P(M)$, $|\nabla s|$ has polynomial growth.

**Proof.** From the assumption of $M$, it is easy to see that the injectivity radius has a lower bound $c > 0$. As $|Rm| \leq C$, at each point $x \in M$, there exists a $C^{1,\omega}$ holomorphic chart $(z_1, \ldots, z_n)$ containing $B(x, r_0)$ for some fixed $r_0 > 0$. In particular, the Christoffel symbol is $C^\omega$ continuous. Since $0 < cI \leq \frac{|g_{ij}|}{C} \leq CI$, locally we can write $s = s_k(\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n})^n$, where $s_k$ is a holomorphic function near $x$ with polynomial bound of $r(x)$. Then $|\nabla s| = \|(\nabla s_k)(\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n})^n + s_k\nabla(\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n})^n)\|$ has polynomial growth.

Notice $\frac{\partial}{\partial z_k} = s_0\nabla s_k - s_k\nabla s_0$. Then according to lemma 5.1,

$$\left|\frac{\partial s_k}{\partial z_k}\right| \leq \frac{C(r + 1)^d}{|s_0|^2}.$$  \hfill (5.4)

We also obtain that $s : = s_0^2\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n} \in P(M)$. Since $|\nabla s|$ has polynomial bound, if $|s(x)| \neq 0$, there exists $c > 0, d > 0$ so that on $B(x, \frac{r_0}{1+c})|s(x)|$,

$$|s| \geq \frac{1}{2}|s(x)|.$$  \hfill (5.5)

Similarly, on $B(x, \frac{r_0}{1+c})|s_0(x)|$,

$$|s_0| \geq \frac{1}{2}|s_0(x)|.$$  \hfill (5.6)

Let $\mu = \min(\frac{1}{|s(x)|}, \frac{1}{|s_0(x)|}, r_0)$ (recall $r_0$ is the size of the holomorphic chart in lemma 5.1). This implies that on $B(x, \mu)$, in terms of the holomorphic chart $(z_1, \ldots, z_n)$ in lemma 5.1,

$$\left|\frac{\partial f_k}{\partial z_i}\right| \leq \frac{C(r + 1)^d}{|s_0(x)|^2} \cdot \left|\det\frac{\partial f_k}{\partial z_i}\right| \geq \frac{c|s(x)|}{|s_0(x)|^2c}.$$  \hfill (5.7)

Let $q_k = (f_1(x), \ldots, f_n(x))$. By standard Cauchy estimate and integration (see also the ODE argument in section 4 of [25]), we conclude $(f_1, \ldots, f_n)|_{B(x, \mu)}$ is a holomorphic chart.
and the image contains $B_{C^0}(q_0, c_1 |s|^b |s_0|^{b-\epsilon})$ for some $a, b, c, c_1 > 0$. This concludes the proof of proposition 5.4.

□

One can apply Skoda estimate as in [25]. With exactly the same argument, we have

**Proposition 5.5.** There exists a subvariety $T$ of $Z$ such that $F(M - W) = Z - T$.

**Step 4:** Completion of the proof of theorem 1.1, part III

We shall add more polynomial growth sections to $F$ so that $F^j = [s_0 : \cdots : s_{N_i}] : M \to \mathbb{P}^{N_i}$ is a holomorphic embedding onto its image which is quasi-projective. It suffices to solve two problems:

(a) Prove that the base locus of $s_0, \ldots, s_{N_i}$ is empty.

(b) Prove that there is no branch points for $F^j$.

**Proposition 5.6.** There exists a quasi-embedding $F' : M \to Z' \subset \mathbb{P}^N$ into a normal projective variety such that for some disjoint union $G$ of discrete points of $M$, $F'|_{M - G}$ is a biholomorphism from $M - G$ onto a Zariski open subset of $Z'$.

The argument is exactly the same as proposition 5.2 of [25]. Notice that by corollary 2.2 any compact subvariety outside a compact set must be zero dimensional. Also the uniform multiplicity estimate, proposition 5.1 plays an essential role.

We claim that $F' : M \to Z'$ is holomorphic except finite many points. Let $q \in G$ be so that $F'$ does not extend holomorphically through $q$. As remarked on page 395 of [25], there exists $U \ni q$ so that $U - q$ is biholomorphic to $V - K$ where $V$ is an open set of $Z'$ and $K$ is a compact subvariety of $V$ of positive dimension. By a Mayer-Vietories sequence argument, we find that such $K$ gives a nontrivial contribution to $H_2(Z', \mathbb{R})$. For different $q$ in $G$, the contributions to $H_2(Z', \mathbb{R})$ are linear independent. As $Z'$ has finite topological type, the number of such $q \in G$ must be finite.

Then we just add finitely polynomial growth sections to desingularize and separate these finitely many points. Theorem 1, part III is concluded.

**Remark 5.3.** In [25], there is a section on Bezout estimate, where the extra condition $\int Ric^n < \infty$ was used. This section is unnecessary for us, due the fact that there is no compact subvariety of positive dimension outside certain compact set (corollary 2.2).

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