Compatibility of Riemannian structures and Jacobi structures

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Abstract

We give a notion of compatibility between a Riemannian structure and a Jacobi structure. We prove that in case of fundamental examples of Jacobi structures: Poisson structures, contact structures and locally conformally symplectic structures, we get respectively Riemann-Poisson structures in the sense of M. Boucetta, $\frac{1}{2}$-Kenmotsu structures and locally conformal Kähler structures.

Key Words. Jacobi manifold, Riemannian Poisson manifold, contact Riemannian manifold, Kenmotsu manifold, locally conformal symplectic manifold, locally conformal Kähler manifold, Lie algebroid.

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Introduction

Jacobi manifolds, introduced by A. Lichnerowicz [5], generalize Poisson manifolds, contact manifolds and locally conformally symplectic manifolds. We ask the natural question of the existence of a notion of compatibility between a Jacobi structure and pseudo-Riemannian structure, a compatibility for which particular Jacobi structures give arise to remarkable geometric structures. In this work, we introduce such a notion which in the case of a Poisson manifold gives a pseudo-Riemannian Poisson structure in the sense of M. Boucetta. We prove that for a contact Riemannian structure, with this notion of compatibility we get a $\frac{1}{2}$-Kenmotsu structure, and that in the case of a locally conformally symplectic structure with an "associated" metric, we get a locally conformal Kähler structure.

Let $M$ be a smooth manifold. In this work, we consider on $M$ a bivector field $\pi$, a vector field $\xi$ and a 1-form $\lambda$, and associate with the triple $(\pi, \xi, \lambda)$ a skew algebroid $(T^*M, \sharp_{\pi, \xi}, [,],_{\pi, \xi})$ on $M$. We prove that if the pair $(\pi, \xi)$ is a Jacobi structure and that
\( \pi \neq 0 \), the skew algebroid \((T^*M, \sharp_{\pi, \xi}, [.,.]_{\pi, \xi})\) is an almost Lie algebroid if and only if \( \sharp_{\pi, \xi}(\lambda) = \xi \). In case \( \xi = \lambda = 0 \), it is the cotangent algebroid of the Poisson manifold \((M, \pi)\). We also prove that in case \((\pi, \xi)\) is the Jacobi structure associated with a contact form \( \eta \), respectively with a locally conformally symplectic structure \((\omega, \theta)\), the skew algebroid \((T^*M, \sharp_{\pi, \xi}, [.,.]_{\pi, \xi}^\theta)\), respectively \((T^*M, \sharp_{\pi, \xi}, [.,.]_{\pi, \xi}^\theta)\), is a Lie algebroid isomorphic to the tangent algebroid of \( M \).

Next, for a triple \((\pi, \xi, g)\) consisting of a bivector field \( \pi \), a vector field \( \xi \) and a pseudo-Riemannian metric \( g \) on \( M \), we put \( \lambda = g(\xi, \xi)\) and \( [.,.]_{\pi, \xi}^g = [.,.]_{\pi, \xi}^\lambda \), where \( \sharp_g : TM \to T^*M \) and \( \sharp_g^{-1} \) are the musical isomorphisms of \( g \) and where \( J \) is the endomorphism of the tangent bundle \( TM \) given by \( \pi(\alpha, \beta) = g(J\sharp_g(\alpha), \sharp_g(\beta)) \), and define a contravariant derivative \( D \) to be the unique contravariant symmetric derivative compatible with \( g \). If \((\pi, \xi)\) is Jacobi, and if \( \sharp_{\pi, \xi} \) is an isometry, a condition that is satisfied in the particular cases of a contact and of a locally conformally symplectic structure, we prove that \( D \) is related to the (covariant) Levi-Civita connection \( \nabla \) of \( g \) by \( \sharp_{\pi, \xi}(D\alpha\beta) = \nabla\sharp_{\pi, \xi}(\alpha)\sharp_{\pi, \xi}(\beta) \).

Finally, with the use of the contravariant Levi-Civita derivative \( D \) we introduce a notion of compatibility of the triple \((\pi, \xi, g)\). In case \( \xi = 0 \), it is just the compatibility of the pair \((\pi, g)\) introduced by M. Boucetta \((2)\). In the case of a Jacobi structure \((\pi, \xi)\) associated with a contact metric structure \((\eta, g)\), the triple \((\pi, \xi, g)\) is compatible if and only if the structure \((\eta, g)\) is \( 1/2 \)-Kenmotsu. In case \((\pi, \xi)\) is the Jacobi structure associated with a locally conformally symplectic structure \((\omega, \theta)\), if \( g \) is a somehow associated metric, the triple \((\pi, \xi, g)\) is compatible if and only if the structure \((\omega, \theta, g)\) is locally conformally Kähler.

1 Almost Lie algebroids associated with a Jacobi manifold

1.1 Almost Lie algebroids associated with a Jacobi manifold

Throughout this paper \( M \) is a smooth manifold, \( \pi \) a bivector field and \( \xi \) a vector field on \( M \).

The pair \((\pi, \xi)\) defines a Jacobi structure on \( M \) if we have the relations

\[
[\pi, \pi] = 2\xi \wedge \pi \quad \text{et} \quad [\xi, \pi] := \mathcal{L}_\xi \pi = 0, \tag{1.1}
\]

where \([.,.]\) is the Schouten-Nijenhuis bracket. We say that \((M, \pi, \xi)\) is a Jacobi manifold. In the case \( \xi = 0 \), the relations above are reduced to \([\pi, \pi] = 0 \) that corresponds to a Poisson structure \((M, \pi)\).

Recall on the other hand, see for instance \([4]\), that a skew algebroid over \( M \) is a triple \((E, \sharp_E, [.,.]_E)\) where \( E \) is the total space of a vector bundle on \( M \), \( \sharp_E \) is a vector bundle morphism from \( E \) to \( TM \), called the anchor map, and \([.,.]_E : \Gamma(E) \times \Gamma(E) \to \Gamma(E), (s, t) \mapsto [s, t]_E \), is an alternating \( \mathbb{R} \)-bilinear map over the space \( \Gamma(E) \) of sections of \( E \).
verifying the Leibniz identity:

\[ [s, \varphi t]_E = \varphi [s, t]_E + \sharp_E(s)(\varphi)t, \quad \forall \varphi \in C^\infty(M), \forall s, t \in \Gamma(E). \]

A skew algebroid \((E, \sharp_E, [., .]_E)\) is an almost Lie algebroid if

\[ \sharp_E ([s, t]_E) = [\sharp_E(s), \sharp_E(t)], \quad \forall s, t \in \Gamma(E), \]

and a Lie algebroid if \((\Gamma(E), [., .]_E)\) is a Lie algebra, i.e., if

\[ [s, [t, r]]_E + [t, [s, r]]_E + [r, [s, t]]_E = 0, \quad \forall s, t, r \in \Gamma(E). \]

Note that a Lie algebroid is an almost Lie algebroid and that, on the other hand, an almost Lie algebroid \((E, \sharp_E, [., .]_E)\) such that the anchor map \(\sharp_E\) is an isomorphism is a Lie algebroid isomorphic to the tangent algebroid \((TM, \text{id}_M, [., .])\) of \(M\).

Let \(\sharp_\pi : T^*M \rightarrow TM\) be the vector bundle morphism defined by \(\beta ([\pi, \alpha]) = \pi(\alpha, \beta)\) and let \([., .]_\pi : \Omega^1(M) \times \Omega^1(M) \rightarrow \Omega^1(M)\) be the map defined by

\[ [\alpha, \beta]_\pi := L_\pi(\beta) \alpha - L_\pi(\alpha) \beta - d(\pi(\alpha, \beta)), \]

called the Koszul bracket. Consider the morphism of vector bundles \(\sharp_\pi, \xi : T^*M \rightarrow TM\) defined by

\[ \sharp_\pi, \xi(\alpha) = \sharp_\pi(\alpha) + \alpha(\xi)\xi \]

and, for a 1-form \(\lambda \in \Omega^1(M)\), the map \([., .]_{\pi, \xi} : \Omega^1(M) \times \Omega^1(M) \rightarrow \Omega^1(M)\) defined by

\[ [\alpha, \beta]_{\pi, \xi}^\lambda := [\alpha, \beta]_\pi + \alpha(\xi) (L_\xi \beta - \beta(\xi) L_\xi \alpha - \pi(\alpha, \beta) \lambda. \]

The triple \((T^*M, \sharp_\pi, \xi, [., .]_{\pi, \xi})\) associated with \((\pi, \xi, \lambda)\) is a skew algebroid on \(M\).

In case \(\xi = \lambda = 0\), the triple \((T^*M, \sharp_\pi, [., .]_{\pi, \xi})\) is just the skew algebroid \((T^*M, \sharp_\pi, [., .]_{\pi})\) associated with the bivector field \(\pi\). Recall that for any differential forms \(\alpha, \beta, \gamma \in \Omega^1(M)\) we have

\[ \gamma ([\pi, [\alpha, \beta]]_\pi) - [\pi, [\alpha, \beta]]_\pi = \frac{1}{2} [\pi, [\alpha, \beta, \gamma]], \quad (1.2) \]

and for any \(\varphi, \psi, \phi \in C^\infty(M)\) we have

\[ [d\varphi, [d\psi, d\phi]]_\pi + [d\psi, [d\phi, d\varphi]]_\pi + [d\phi, [d\psi, d\varphi]]_\pi = -\frac{1}{2} d ([\pi, [\alpha, \beta, \gamma]) \]

Thus, \((T^*M, \sharp_\pi, [., .]_{\pi})\) is a Lie algebroid if and only if \(\pi\) is a Poisson tensor. If \(\pi\) is a Poisson tensor on \(M\), the triple \((T^*M, \sharp_\pi, [., .]_{\pi})\) is called the cotangent algebroid of the Poisson manifold \((M, \pi)\). In the case of a Jacobi structure we have the following result

**Theorem 1.1.** Assume that \((\pi, \xi)\) is a Jacobi structure on \(M\) and let \(\lambda \in \Omega^1(M)\). We have

\[ \sharp_{\pi, \xi}^\lambda ([\alpha, \beta]_{\pi, \xi}) - [\pi, [\alpha, \beta]]_{\pi, \xi} = \pi(\alpha, \beta) (\xi - \sharp_{\pi, \xi}(\lambda)), \]

for any differential forms \(\alpha, \beta \in \Omega^1(M)\).
Proof. We have
\[
\#_{\pi,\xi}([\alpha,\beta]^\lambda_{\pi,\xi}) = \#_{\pi}([\alpha,\beta]_\pi) - \alpha(\xi)\#_{\pi}(\beta) + \beta(\xi)\#_{\pi}(\alpha) - \pi(\alpha,\beta)\#_{\pi,\xi}(\lambda) + \alpha(\xi)\#_{\pi}(\xi) - \beta(\xi)\#_{\pi}(\alpha) + [\#_{\pi,\xi}(\alpha)(\beta(\xi))] - \alpha(\xi)\#_{\pi,\xi}(\beta(\xi))\xi
\]
and on the other hand
\[
\left[\#_{\pi,\xi}(\alpha),\#_{\pi,\xi}(\beta)\right] = \left[\#_{\pi}(\alpha),\#_{\pi}(\beta)\right] + \alpha(\xi)\#_{\pi}(\beta(\xi)) - \beta(\xi)\#_{\pi}(\alpha(\xi)) + [\#_{\pi,\xi}(\alpha)(\beta(\xi)) - \#_{\pi,\xi}(\beta(\alpha(\xi)))\xi.
\]
Therefore, using the identity (1.2), we deduce that
\[
\#_{\pi,\xi}([\alpha,\beta]^\lambda_{\pi,\xi}) - \left[\#_{\pi,\xi}(\alpha),\#_{\pi,\xi}(\beta)\right] = \left(\frac{1}{2}[\pi,\pi] - \xi \wedge \pi\right)\alpha(\beta,\cdot) - \alpha(\xi)\#_{\pi}(\beta) + \beta(\xi)\#_{\pi}(\alpha) - \left[\#_{\pi,\xi}(\alpha)(\beta(\xi))\right] + \alpha(\xi)\#_{\pi,\xi}(\beta(\xi))\xi + \pi(\alpha,\beta)(\xi - \#_{\pi,\xi}(\lambda)).
\]

Now use the relations (1.1).

Corollary 1.2. Assume that \((\pi,\xi)\) is a Jacobi structure on \(M\) and let \(\lambda \in \Omega^1(M)\). If \(\#_{\pi,\xi}(\lambda) = \xi\), the skew algebroid \((T^*M,\#_{\pi,\xi},\ldots\),\(\,\,\lambda_{\pi,\xi}\)) associated with the triple \((\pi,\xi,\lambda)\) is an almost Lie algebroid, i.e.
\[
\#_{\pi,\xi}([\alpha,\beta]_{\pi,\xi}^\lambda) = \left[\#_{\pi,\xi}(\alpha),\#_{\pi,\xi}(\beta)\right],
\]
for any differential forms \(\alpha,\beta \in \Omega^1(M)\). The converse is also true if \(\pi \neq 0\).

Proof. It is a direct consequence of the theorem above.

1.2 Cotangent algebroid of a contact manifold

Assume that \(M\) is of odd dimension \(2n + 1, n \in \mathbb{N}^*\). Recall that a contact form on \(M\) is a differential 1-form \(\eta\) on \(M\) such that the form \(\eta \wedge (d\eta)^{\wedge n}\) is a volume form. Assume that the pair \((\pi,\xi)\) is the Jacobi structure associated with a contact form \(\eta\) on \(M\), i.e. we have
\[
\pi(\alpha,\beta) = d\eta(\varpi_{\eta}(\alpha),\varpi_{\eta}(\beta)),
\]
where \(\varpi_{\eta}\) is the inverse isomorphism of the isomorphism of vector bundles \(\varpi_{\eta} : TM \to T^*M\), \(\varpi_{\eta}(X) = -i_X d\eta + \eta(X)\eta\), and \(\xi = \varpi_{\eta}(\eta)\). The vector field \(\xi\) is called the Reeb field associated with the contact structure \((M,\eta)\), it is characterized by the formulae
\[
i_\xi d\eta := d\eta(\xi,\cdot) = 0 \quad \text{and} \quad i_\xi \eta := \eta(\xi) = 1.
\]
Proposition 1.3. The skew algebroid \((T^*M, \sharp_{\pi,\xi}, [\cdot, \cdot]_{\pi,\xi})\) is a Lie algebroid isomorphic to the tangent algebroid of \(M\).

Proof. Let us show that \(\sharp_{\pi,\xi}\) is equal to the isomorphism \(\sharp_\eta\), inverse of the isomorphism \(\flat_\eta\). Let \(\alpha, \beta \in \Omega^1(M)\), and let \(X, Y\) be such that \(\alpha = \flat_\eta(X)\) and \(\beta = \flat_\eta(Y)\). First, notice that \(\alpha(\xi) = \flat_\eta(X)(\xi) = \eta(X)\) and \(\beta(\xi) = \eta(Y)\). Therefore, we have

\[
\begin{align*}
\beta(\sharp_{\pi,\xi}(\alpha)) &= \pi(\alpha, \beta) + \eta(X)\eta(Y) \\
&= (-i_Y d\eta + \eta(Y)\eta)(X) \\
&= \flat_\eta(Y)(X) \\
&= \beta(\sharp_\eta(\alpha)).
\end{align*}
\]

Thus \(\sharp_{\pi,\xi} = \sharp_\eta\) and in particular \(\sharp_{\pi,\xi}(\eta) = \sharp_\eta(\eta) = \xi\). We see that the proposition is a consequence of Corollary 1.2 and the fact that \(\sharp_{\pi,\xi}\) is an isomorphism. \(\square\)

Hence, if \((M, \eta)\) is a contact manifold and if \((\pi, \xi)\) is the associated Jacobi structure, by the proposition above we have \(\sharp_{\pi,\xi} = \sharp_\eta\). If we put \([\cdot, \cdot]_\eta = [\cdot, \cdot]_{\pi,\xi}\), then we have a Lie algebroid \((T^*M, \sharp_\eta, [\cdot, \cdot]_\eta)\) associated naturally with the contact manifold \((M, \eta)\). This Lie algebroid may henceforth be called the cotangent algebroid of the contact manifold \((M, \eta)\).

1.3 Cotangent algebroid of a locally conformally symplectic manifold

A locally conformally symplectic structure on \(M\) is a pair \((\omega, \theta)\) of a differential closed 1-form \(\theta\) and a nondegenerate differential 2-form \(\omega\) on \(M\) such that

\[
d\omega + \theta \wedge \omega = 0.
\]

In the particular case where \(\theta\) is exact, i.e. \(\theta = df\), we say that \((\omega, df)\) is conformally symplectic, it is equivalent to \(e^f \omega\) being symplectic and this justifies the terminology.

The next proposition shows that it is equivalent to give a locally conformally symplectic manifold and to give a Jacobi manifold such that the underlying bivector field is nondegenerate (see also \(\text{[5, \S 2.3, ex. 4]}\)). Having not found a proof in the literature, we give one here.

Assume that \(\omega \in \Omega^2(M)\) is a nondegenerate 2-form and let \(\theta \in \Omega^1(M)\). Assume that the pair \((\pi, \xi)\) is associated with the pair \((\omega, \theta)\), i.e. \(i_\pi(\alpha)\omega = -\alpha\) for any \(\alpha \in \Omega^1(M)\), and that \(i_\xi\omega = -\theta\). This means that

\[
\pi(\alpha, \beta) = \omega(\sharp_\omega(\alpha), \sharp_\omega(\beta))
\]

where \(\sharp_\omega\) is the inverse isomorphism of the vector bundle isomorphism \(\flat_\omega: TM \rightarrow T^*M\), \(\flat_\omega(X) = -i_X\omega\), and that \(\xi = \sharp_\omega(\theta)\). We have the following

Lemma 1.4. Let \(X, Y, Z\) be vector fields on \(M\) and let \(\alpha, \beta, \gamma\) be the differential 1-forms such that \(X = \sharp_\pi(\alpha)\), \(Y = \sharp_\pi(\beta)\) and \(Z = \sharp_\pi(\gamma)\). We have
1. \((dw + \theta \wedge \omega)(X, Y, Z) = \left(\frac{1}{2} [\pi, \pi] - \xi \wedge \pi\right)(\alpha, \beta, \gamma)\).

2. \(\mathcal{L}_\xi \omega(X, Y) = -\mathcal{L}_\xi \pi(\alpha, \beta)\).

Proof. Using the identity \(\pi(\alpha, \beta) = \omega(X, Y)\) and the identity \([1, 2]\), we get

\[\omega([X, Y], Z) = \gamma([X, Y]) = -\frac{1}{2} [\pi, \pi] (\alpha, \beta, \gamma) + \pi((\alpha, \beta)_\pi, \gamma),\]

therefore, with a direct calculation, we deduce

\[d\omega(X, Y, Z) = \frac{1}{2} [\pi, \pi] (\alpha, \beta, \gamma).\] (1.3)

On the other hand, notice that \(\theta(X) = -i_\xi \omega(X) = i_X \omega(\xi) = i_{\xi^\sharp(\alpha)} \omega(\xi) = -\alpha(\xi)\), likewise \(\theta(Y) = -\beta(\xi)\) and \(\theta(Z) = -\gamma(\xi)\), thus \(\theta \wedge \omega(X, Y, Z) = -\xi \wedge \pi(\alpha, \beta, \gamma)\). Hence, with (1.3), we get the first assertion of the lemma. For the second assertion, is suffices to notice that

\[\pi(\mathcal{L}_\xi \alpha, \beta) = -\mathcal{L}_\xi \alpha(Y) = -\xi(\alpha(Y)) + \alpha(\mathcal{L}_\xi Y) = \xi(\omega(X, Y)) - \omega(X, \mathcal{L}_\xi Y).\]

\[\square\]

**Proposition 1.5.** The pair \((\omega, \theta)\) is a locally conformally symplectic structure if and only if the pair \((\pi, \xi)\) is a Jacobi structure.

Proof. From the first assertion of Lemma [1, 4] we deduce that the identity \(d\omega + \theta \wedge \omega = 0\) is satisfied if and only if the identity \([\pi, \pi] = 2\xi \wedge \pi\) is, and if one of the two is satisfied then, using the Cartan formula, we get

\[\mathcal{L}_\xi \omega = d(i_\xi \omega) + i_\xi d\omega = -d\theta - i_\xi (\theta \wedge \omega) = -d\theta,\]

and then, with the assertion 2. of Lemma [1, 4], that \(\mathcal{L}_\xi \pi = 0\) if and only if \(d\theta = 0\). \[\square\]

**Proposition 1.6.** Assume that \((M, \omega, \theta)\) is a locally conformally symplectic manifold and let \((\pi, \xi)\) be the associated Jacobi structure. The skew algebroid \((T^*M, \sharp_{\pi, \xi}, [\cdot, \cdot]_{\pi, \xi}^\theta)\) is a Lie algebroid isomorphic to the tangent algebroid of \(M\).

Proof. Since \(\sharp_{\pi, \xi}(\theta) = \sharp_{\pi}(\theta) + \theta(\xi) = \sharp_{\pi}(\theta) = \xi\). Then, by Corollary [1, 2] the triple \((T^*M, \sharp_{\pi, \xi}, [\cdot, \cdot]_{\pi, \xi}^\theta)\) is an almost Lie algebroid. It remains to prove that \(\sharp_{\pi, \xi}\) is an isomorphism. It suffices to prove that it is injective. Since we have \(\sharp_{\pi, \xi}(\alpha) = \sharp_{\pi}(\alpha + \alpha(\xi)\theta)\) and that the bivector field \(\pi\) is nondegenerate, then \(\sharp_{\pi, \xi}(\alpha) = 0\) implies \(\alpha = -\alpha(\xi)\theta\), thus \(\alpha(\xi) = -\alpha(\xi)\theta(\xi) = 0\), and therefore \(\alpha = 0\). \[\square\]

Hence, if \((M, \omega, \theta)\) is a locally conformally symplectic manifold and \((\pi, \xi)\) the associated Jacobi structure, by the above proposition, if we put \(\sharp_{\omega, \theta} := \sharp_{\pi, \xi}\) and \([\cdot, \cdot]_{\omega, \theta} := [\cdot, \cdot]_{\pi, \xi}^\theta\), then we have a Lie algebroid \((T^*M, \sharp_{\omega, \theta}, [\cdot, \cdot]_{\omega, \theta})\) associated naturally with the locally conformally symplectic manifold \((M, \omega, \theta)\). This Lie algebroid may be called the cotangent algebroid of the locally conformally symplectic manifold \((M, \omega, \theta)\).
2 Levi-Civita contravariant derivative associated with the triple \((\pi, \xi, g)\)

2.1 Definition and properties

In all what follows, we denote by \(g\) a pseudo-Riemannian metric on \(M\), by \(\flat_g : T^*M \to TM\) the vector bundle isomorphism such that \(\flat_g(X)(Y) = g(X,Y)\), by \(\sharp_g\) the inverse isomorphism of \(\flat_g\), and by \(g^*\) the cometric of \(g\), i.e. the tensor field defined by \(g^*(\alpha, \beta) := g(\sharp_g(\alpha), \sharp_g(\beta))\).

With the pair \((\pi, g)\) we associate the vector field endomorphisms \(J\) of \(TM\) and \(J^*\) of \(T^*M\) defined respectively by

\[
g(J\sharp_g(\alpha), \sharp_g(\beta)) = \pi(\alpha, \beta) \quad \text{and} \quad g^*(J^*\alpha, \beta) = \pi(\alpha, \beta). \tag{2.1}
\]

We have \(J = \sharp_g \circ J^* \circ \flat_g\). With the triple \((\pi, \xi, g)\) we associate the differential 1-form \(\lambda\) defined by

\[
\lambda = g(\xi, \xi)\flat_g(\xi) - \flat_g(J\xi),
\]

and we use the notation \([.,.]_g^{\pi,\xi}\) instead of \([.,.]_g^\pi\).

We call the contravariant Levi-Civita derivative associated with the triple \((\pi, \xi, g)\) the unique derivative \(D : \Omega^1(M) \times \Omega^1(M) \to \Omega^1(M)\), symmetric with respect to the bracket \([.,.]_g^{\pi,\xi}\) and compatible with the metric. It is entirely characterized by the formula :

\[
2g^*(D\alpha\beta, \gamma) = \sharp_g(\alpha) \cdot g^*(\beta, \gamma) + \sharp_g(\beta) \cdot g^*(\alpha, \gamma) - \sharp_g(\gamma) \cdot g^*(\alpha, \beta) - g^*([\alpha, \beta]_g^{\pi,\xi}, \gamma) + g^*([\alpha, \gamma]_g^{\pi,\xi}, \beta) + g^*([\beta, \gamma]_g^{\pi,\xi}, \alpha). \tag{2.2}
\]

In the case \(\xi = 0\), the derivative \(D\) is just the Levi-Civita contravariant derivative associated in \([2]\) with the pair \((\pi, g)\).

**Proposition 2.1.** Assume that the skew algebroid \((T^*M, \sharp_g^{\pi,\xi}, [.,.]_g^{\pi,\xi})\) is an almost Lie algebroid and that the anchor map \(\sharp_g^{\pi,\xi}\) is an isometry. Then

\[
\sharp_g^{\pi,\xi}(D\alpha\beta) = \nabla_{\sharp_g^{\pi,\xi}(\alpha)}\sharp_g^{\pi,\xi}(\beta),
\]

where \(\nabla\) is the Levi-Civita (covariant) connection associated with \(g\).

**Proof.** Since we have assumed that \((T^*M, \sharp_g^{\pi,\xi}, [.,.]_g^{\pi,\xi})\) is an almost Lie algebroid, we have

\[
\sharp_g^{\pi,\xi}(\alpha) \cdot \sharp_g^{\pi,\xi}(\beta) = [\sharp_g^{\pi,\xi}(\alpha), \sharp_g^{\pi,\xi}(\beta)],
\]

for every \(\alpha, \beta \in \Omega^1(M)\). Since we have also assumed that \(\sharp_g^{\pi,\xi}\) is an isometry, from the formula (2.2) and the Koszul formula relative to the Levi-Civita connection \(\nabla\) of \(g\) we deduce that

\[
g^*(\sharp_g^{\pi,\xi}(D\alpha\beta), \sharp_g^{\pi,\xi}(\gamma)) = g(\nabla_{\sharp_g^{\pi,\xi}(\alpha)}\sharp_g^{\pi,\xi}(\beta), \sharp_g^{\pi,\xi}(\gamma))
\]

for any differential 1-forms \(\alpha, \beta, \gamma \in \Omega^1(M)\).
2.2 Skew algebroid associated with an almost contact Riemannian manifold

Let $(\Phi, \xi, \eta)$ be a triple consisting of a 1-form $\eta$, a vector field $\xi$ and $(1,1)$-tensor field $\Phi$ on $M$. The triple $(\Phi, \xi, \eta)$ defines an almost contact structure on $M$ if $\Phi^2 = -\text{Id}_{TM} + \eta \otimes \xi$ and $\eta(\xi) = 1$. From what we deduce, see for instance \[1, \text{Th. 4.1}], that $\Phi(\xi) = 0$ and $\eta \circ \Phi = 0$.

We say that the metric $g$ is associated with the triple $(\Phi, \xi, \eta)$ if the following identity is verified

$$g(\Phi(X), \Phi(Y)) = g(X, Y) - \eta(X)\eta(Y).$$

(2.3)

We say that the manifold $(M, \Phi, \xi, \eta, g)$ is almost contact pseudo-Riemannian if the triple $(\Phi, \xi, \eta)$ is an almost contact structure and $g$ an associated metric. If in addition the metric $g$ is positive definite, we say that $(M, \Phi, \xi, \eta, g)$ is an almost contact Riemannian manifold.

Notice that if we set $Y = \xi$ in the formula (2.3), we deduce that if $(\Phi, \xi, \eta, g)$ is an almost contact pseudo-Riemannian structure then

$$g(X, \xi) = \eta(X),$$

for any $X \in \mathfrak{X}(M)$, i.e. $b_g(\xi) = \eta$. In particular $g(\xi, \xi) = 1$.

**Proposition 2.2.** If $(\Phi, \xi, \eta, g)$ is an almost contact pseudo-Riemannian structure on $M$, then the map $\pi : \Omega^1(M) \times \Omega^1(M) \to C^\infty(M)$ defined by

$$\pi(\alpha, \beta) = g(\sharp_g(\alpha), \Phi(\sharp_g(\beta)))$$

is a bivector field on $M$ and the vector bundle morphism $\sharp_{\pi, \xi}$ is an isometry.

**Proof.** Let $\alpha \in \Omega^1(M)$ and put $X = \sharp_g(\alpha)$. Using (2.3) and $\eta \circ \Phi = 0$, we get $\pi(\alpha, \alpha) = g(\Phi(X), \Phi^2(X))$, and since $\Phi^2 = -\text{Id}_{TM} + \eta \otimes \xi$ and $g(\Phi(X), \xi) = \eta \circ \Phi(X) = 0$, we get

$$\pi(\alpha, \alpha) = -g(\Phi(X), X) + \eta(X)g(\Phi(X), \xi) = -g(\Phi(X), X) = -\pi(\alpha, \alpha),$$

and then, that $\pi$ is a bivector field. Let us prove that $\sharp_{\pi, \xi}$ is an isometry. Let $\alpha \in \Omega^1(M)$. Recall that by definition, we have $\sharp_{\pi, \xi}(\alpha) = \sharp_\pi(\alpha) + \alpha(\xi)\xi$. Since we have on one hand $\alpha(\xi) = g(\sharp_g(\alpha), \xi) = \eta(\sharp_g(\alpha))$ and on the other hand, for any $\beta \in \Omega^1(M)$,

$$\beta(\sharp_\pi(\alpha)) = g(\sharp_g(\alpha), \Phi(\sharp_g(\beta))) = -g(\Phi(\sharp_g(\alpha)), \sharp_g(\beta)) = -\beta(\sharp_g(\alpha)),$$

i.e. $\sharp_\pi(\alpha) = -\Phi(\sharp_g(\alpha))$, we deduce that

$$\sharp_{\pi, \xi}(\alpha) = -\Phi(\sharp_g(\alpha)) + \eta(\sharp_g(\alpha))\xi.$$  \hspace{2cm} (2.4)

Let $\alpha, \beta \in \Omega^1(M)$. From the formula (2.4) and the fact that $g(\Phi(X), \xi) = \eta \circ \Phi(X) = 0$ and $g(\xi, \xi) = 1$, we deduce that

$$g(\sharp_{\pi, \xi}(\alpha), \sharp_{\pi, \xi}(\beta)) = g(\Phi(\sharp_g(\alpha)), \Phi(\sharp_g(\beta))) + \eta(\sharp_g(\alpha))\eta(\sharp_g(\beta)).$$

By using the formula (2.3), we get $g(\sharp_{\pi, \xi}(\alpha), \sharp_{\pi, \xi}(\beta)) = g(\sharp_g(\alpha), \sharp_g(\beta)) = g^*(\alpha, \beta).$ \hfill $\Box$
Corollary 2.3. Assume \((\Phi, \xi, \eta, g)\) is an almost contact pseudo-Riemannian structure on \(M\) and let \(\pi\) be the associated bivector field, i.e. the one defined in Proposition 2.2. If the skew algebroid \((T^*M, \sharp_{\pi, \xi}, \lbrack\lbrack ., .\rbrack_{\pi, \xi})\) is an almost Lie algebroid, then
\[
\sharp_{\pi, \xi}(\mathcal{D}_{\alpha \beta}) = \nabla_{\sharp_{\pi, \xi}(\alpha)}\sharp_{\pi, \xi}(\beta),
\]
for every \(\alpha, \beta \in \Omega^1(M)\).

Proof. This is a direct consequence of Propositions 2.2 and 2.1.

Assume that \(\eta\) is a contact form on \(M\). We say that the manifold \((M, \eta, g)\) is contact pseudo-Riemannian, or that the metric \(g\) is associated with the contact form \(\eta\), if there exists a vector field endomorphism \(\Phi\) of \(TM\) such that \((\Phi, \xi, \eta, g)\) is an almost contact pseudo-Riemannian structure and that
\[
g(X, \Phi(Y)) = d\eta(X, Y).
\] (2.5)
If in addition \(g\) is positive definite, we say that \((M, \eta, g)\) is a contact Riemannian manifold.

Theorem 2.4. Assume that \((M, \eta, g)\) is a contact pseudo-Riemannian manifold. We have
\[
\sharp_{\eta}(\mathcal{D}_{\alpha \beta}) = \nabla_{\sharp_{\eta}(\alpha)}\sharp_{\eta}(\beta),
\]
for every \(\alpha, \beta \in \Omega^1(M)\).

Proof. Let \((\Phi, \xi, \eta, g)\) be the almost contact pseudo-Riemannian structure associated with the contact pseudo-Riemannian manifold \((M, \eta, g)\). Let \((\pi, \xi)\) be the Jacobi structure associated with \(\eta\), then \(\sharp_{\eta} = \sharp_{\pi, \xi}\). By Proposition 1.3 and the corollary above, we need only to prove that \(\pi\) is associated with \((\Phi, \xi, \eta, g)\) and that \(\eta = \lambda\). Let \(\alpha \in \Omega^1(M)\) and put \(X = \sharp_{\eta}(\alpha)\). By using (2.5), we have
\[
\sharp_{g}(\alpha) = \sharp_{g}(b_{\eta}(X)) = -\sharp_{g}(i_X d\eta) + \eta(X)\xi = \Phi(X) + \eta(X)\xi.
\]
Therefore, applying \(\Phi\),
\[
\Phi(\sharp_{g}(\alpha)) = \Phi^2(X) = -X + \eta(X)\xi = -\sharp_{\eta}(\alpha) + \alpha(\xi)\xi = -\sharp_{\pi}(\alpha).
\]
We deduce that \(\pi(\alpha, \beta) = g(\sharp_{g}(\alpha), \Phi(\sharp_{g}(\beta)))\) for any \(\alpha, \beta \in \Omega^1(M)\), and that \(\Phi = -J\), where \(J\) is the field of endomorphisms associated with the pair \((\pi, g)\). Hence, \(J\xi = 0\), and since \(g(\xi, \xi) = 1\) it follows that \(\lambda = b_{g}(\xi) = \eta\).
2.3 Riemannian metric associated with a locally conformally symplectic structure

Assume that \( \omega \in \Omega^2(M) \) is a nondegenerate 2-form and let \( \theta \in \Omega^1(M) \). Assume that the pair \((\pi, \xi)\) is associated with the pair \((\omega, \theta)\). We say that the pseudo-Riemannian metric \( g \) is associated with the pair \((\omega, \theta)\) if \( \#_{\omega, \theta} := \#_{\pi, \xi} \) is an isometry, i.e., if

\[
g(\#_{\omega, \theta}(\alpha), \#_{\omega, \theta}(\beta)) = g^*(\alpha, \beta),
\]

(2.6)

for every \( \alpha, \beta \in \Omega^1(M) \).

If \( \theta = 0 \), then \( \xi = 0 \) and \( \#_{\omega, \theta} = \#_{\omega} \), and if \( J \) and \( J^* \) are the fields of endomorphisms defined by the formulae (2.1), then

\[
g(\#_{\omega, \theta}(\alpha), \#_{\omega, \theta}(\beta)) = g(\#_{\omega}(\alpha), \#_{\omega}(\beta))
= g^*(b_g(\#_{\omega}(\alpha)), b_g(\#_{\omega}(\beta)))
= g^*(J^*\alpha, J^*\beta),
\]

for any \( \alpha, \beta \in \Omega^1(M) \). Hence, in the case \( \theta = 0 \), the relation (2.6) is equivalent to

\[
g^*(J^*\alpha, J^*\beta) = g^*(\alpha, \beta).
\]

If moreover \( g \) is positive definite, this last identity means that the pair \((\omega, g)\) is an almost Hermitian structure on \( M \) and that \( J \) is the associated almost complex structure, i.e., we have

\[
g(JX, JY) = g(X, Y) \quad \text{and} \quad \omega(X, Y) = g(JX, Y),
\]

for every \( X, Y \in \mathfrak{X}(M) \).

**Theorem 2.5.** Assume that \((\omega, \theta)\) is a locally conformally symplectic structure and that \( g \) is an associated metric. We have

\[
\#_{\omega, \theta}(\mathcal{D}_\alpha \beta) = \nabla_{\#_{\omega, \theta}(\alpha)} \#_{\omega, \theta}(\beta)
\]

for every \( \alpha, \beta \in \Omega^1(M) \).

**Proof.** By Propositions 2.1 and 1.6, we need only to prove that \( \lambda = \theta \). On one hand, we have \( \#_{\pi, \xi}(\theta) = \xi \). On the other hand, for any \( \alpha \in \Omega^1(M) \), we have

\[
g(\#_{\pi, \xi}(\lambda), \#_{\pi, \xi}(\alpha)) = g(\#_g(\lambda), \#_g(\alpha))
= g(\xi, \xi)\alpha(\xi) + g(\xi, J\#_g(\alpha))
= g(\xi, \xi)\alpha(\xi) + g(\xi, \#_\pi(\alpha))
= g(\xi, \#_{\pi, \xi}(\alpha)).
\]

Since \( \#_{\pi, \xi} \) is an isometry, hence an isomorphism, then \( \#_{\pi, \xi}(\lambda) = \xi \). \( \square \)
Corollary 2.6. Under the same hypotheses of the theorem above, we have
\[ D_{\pi}(\alpha, \beta, \gamma) = \nabla \omega(\sharp_{\omega, \theta}(\alpha), \sharp_{\omega, \theta}(\beta), \sharp_{\omega, \theta}(\gamma)). \]

Proof. We have \( \omega(\xi, \sharp_{\pi}(\alpha)) = -i_{\sharp_{\pi}(\alpha)} \omega(\xi) = \alpha(\xi) \) and likewise \( \omega(\xi, \sharp_{\pi}(\beta)) = \beta(\xi) \), consequently
\[ \omega(\sharp_{\pi, \xi}(\alpha), \sharp_{\pi, \xi}(\beta)) = \pi(\alpha, \beta). \] (2.7)
It suffices now to compute \( \nabla \omega(\sharp_{\pi, \xi}(\alpha), \sharp_{\pi, \xi}(\beta), \sharp_{\pi, \xi}(\gamma)) \) and use the theorem above. \( \square \)

3 Compatibility of the triple \((\pi, \xi, g)\)

3.1 Definition
We say that the metric \( g \) is compatible with the pair \((\pi, \xi)\) or that the triple \((\pi, \xi, g)\) is compatible if
\[ D_{\pi}(\alpha, \beta, \gamma) = \frac{1}{2} (\gamma(\xi)\pi(\alpha, \beta) - \beta(\xi)\pi(\alpha, \gamma) - J^{\ast} \gamma(\xi)g^{\ast}(\alpha, \beta) + J^{\ast} \beta(\xi)g^{\ast}(\alpha, \gamma)), \] (3.1)
for every \( \alpha, \beta, \gamma \in \Omega^{1}(M) \). The formula (3.1) can also be written in the form
\[ (D_{\alpha}J^{\ast}) \beta = \frac{1}{2} (\pi(\alpha, \beta)\varphi_{g}(\xi) - \beta(\xi)J^{\ast} \alpha + g^{\ast}(\alpha, \beta)J^{\ast} \varphi_{g}(\xi) + J^{\ast} \beta(\xi)\alpha), \] (3.2)
for any \( \alpha, \beta \in \Omega^{1}(M) \).
The compatibility in the case \( \xi \) is the zero vector field means that \((M, \pi, g)\) is a pseudo-Riemannian Poisson manifold, and Riemannian Poisson if moreover the metric \( g \) is positive definite, see [2, 3].

3.2 \( \frac{1}{2} \)-Kenmotsu manifolds
Recall, see for instance [1, §6.6], that an almost contact Riemannian structure \((\Phi, \xi, \eta, g)\) on \( M \) is said to be \( \frac{1}{2} \)-Kenmotsu if we have
\[ (\nabla_{X}\Phi)(Y) = \frac{1}{2} (g(\Phi(X), Y)\xi - \eta(Y)\Phi(X)), \]
for any \( X, Y \in \mathfrak{X}(M) \).

Lemma 3.1. Assume that \((\Phi, \xi, \eta, g)\) is an almost contact pseudo-Riemannian structure on \( M \) and let \( \pi \) be the associated bivector field. If the skew algebroid \((T^{\ast}M, \sharp_{\pi, \xi}, [, ,]_{\pi, \xi})\) is an almost Lie algebroid, then
\[ \sharp_{\pi, \xi}((D_{\alpha}J^{\ast}) \beta) = - (\nabla_{\sharp_{\pi, \xi}(\alpha)} \Phi)(\sharp_{\pi, \xi}(\beta)), \]
for every \( \alpha, \beta \in \Omega^{1}(M) \).
Proof. By using the formula \( (2.4) \) and the fact that we have \( \sharp_g \circ J^* = J \circ \sharp_g \), we deduce that \( \sharp_{\pi,\xi}(J^* \alpha) = \Phi(\sharp_g(J^* \alpha)) + \eta(\sharp_g(J^* \alpha)) \xi = -\Phi(\sharp_{\pi,\xi}(\alpha)) = -\Phi(\sharp_{\pi,\xi}(\alpha)). \) Therefore

\[
\sharp_{\pi,\xi} \circ J^* = -\Phi \circ \sharp_{\pi,\xi}. 
\]

Hence, with Corollary \( 2.3 \) we have

\[
\sharp_{\pi,\xi}((\mathcal{D}_\alpha J^*) \beta) = \sharp_{\pi,\xi}(\mathcal{D}_\alpha (J^* \beta)) - (\sharp_{\pi,\xi} \circ J^*) (\mathcal{D}_\alpha \beta),
\]

\[
= \nabla_{\sharp_{\pi,\xi}(\alpha)} (\sharp_{\pi,\xi} (J^* \beta)) + \Phi (\sharp_{\pi,\xi}(\alpha)),
\]

\[
= -\nabla_{\sharp_{\pi,\xi}(\alpha)} (\Phi (\sharp_{\pi,\xi}(\beta))) + \Phi (\nabla_{\sharp_{\pi,\xi}(\alpha)} \sharp_{\pi,\xi}(\beta)),
\]

\[
= - (\nabla_{\sharp_{\pi,\xi}(\alpha)} \Phi) (\sharp_{\pi,\xi}(\beta)).
\]

Proposition 3.2. Under the same hypotheses of the above lemma, the compatibility of the triple \((\pi, \xi, g)\) is equivalent to

\[
(\nabla_X \Phi)(Y) = \frac{1}{2} (g(\Phi(X), Y)\xi - \eta(Y)\Phi(X)),
\]

for any \( X, Y \in \mathfrak{X}(M) \), and if moreover the metric \( g \) is positive definite, then the triple \((\pi, \xi, g)\) is compatible if and only if the almost contact Riemannian manifold \((M, \Phi, \xi, \eta, g)\) is \(\frac{1}{2}\)-Kenmotsu.

Proof. Since we have \( J^* b_g(\xi) = b_g(J\xi) = -b_g(\Phi \xi) = 0 \) and

\[
J^* \beta(\xi) = J^* \beta(\sharp_g(\eta)) = \eta(\sharp_g(J^* \beta)) = \eta(J^* \sharp_g(\beta)) = -\Phi(\sharp_g(\beta)) = 0,
\]

then the formula \( (3.2) \) becomes

\[
(\mathcal{D}_\alpha J^*) \beta = \frac{1}{2} (\pi(\alpha, \beta)\eta - \beta(\xi) J^* \alpha).
\]

Applying \( \sharp_{\pi,\xi} \) which by Proposition \( 2.2 \) is an isometry and hence an isomorphism, this last formula is equivalent to

\[
\sharp_{\pi,\xi}((\mathcal{D}_\alpha J^*) \beta) = \frac{1}{2} (\pi(\alpha, \beta)\sharp_{\pi,\xi}(\eta) - \beta(\xi)\sharp_{\pi,\xi}(J^* \alpha)).
\]

Now, by Formula \( (2.4) \), we have \( \sharp_{\pi,\xi}(\eta) = \xi \), and if we put \( X = \sharp_{\pi,\xi}(\alpha) \) \( Y = \sharp_{\pi,\xi}(\beta) \), then we have \( \beta(\xi) = \eta(Y) \), also using \( (3.3) \), we have \( \sharp_{\pi,\xi}(J^* \alpha) = -\Phi(X) \) and

\[
\pi(\alpha, \beta) = g(\sharp_{\pi}(\alpha), \Phi(\sharp_{\pi}(\beta)))
= -g(\sharp_{\pi}(\alpha), \sharp_{\pi}(J^* \beta))
= -g^*(\alpha, J^* \beta)
= g(X, \Phi(Y)).
\]

It remains to use the lemma above. \( \square \)
**Theorem 3.3.** Assume that \((\eta, g)\) is a contact Riemannian structure on \(M\) and let \((\Phi, \xi, \eta, g)\) be the associated almost contact Riemannian structure. Assume that \((\pi, \xi)\) is the Jacobi structure associated with the contact form \(\eta\). Then the triple \((\pi, \xi, g)\) is compatible if and only if \((M, \Phi, \xi, \eta, g)\) is \(1/2\)-Kenmotsu.

*Proof.* We have proved that \(\pi\) is the bivector field of the proposition above and that \(\lambda = \eta\), see the proof of Theorem 2.4. \(\square\)

### 3.3 Locally conformally Kähler manifolds

Recall that if \(\omega\) is a nondegenerate 2-form and \(g\) an associated Riemannian metric, the almost Hermitian structure \((\omega, g)\) is Hermitian if the associated almost complex structure is integrable, and Kähler if moreover \(\omega\) is closed. Recall also that if \((\omega, g)\) is almost Hermitian, then it is Kähler if and only if the 2-form \(\omega\) is parallel for the Levi-Civita connection of \(g\).

If \((\omega, \theta)\) is a locally conformally symplectic structure and \((\omega, g)\) a Hermitian structure, we say that the triple \((\omega, \theta, g)\) is a locally conformally Kähler structure.

We shall prove that if \((\omega, \theta)\) is a locally conformally symplectic structure on \(M\) and that \((\pi, \xi)\) is the associated Jacobi structure, if \(g\) is a Riemannian metric associated with \(\omega\) and with \((\omega, \theta)\), the compatibility of the triple \((\pi, \xi, g)\) induces a locally conformally Kähler structure on \(M\).

**Lemma 3.4.** Assume that \(\omega \in \Omega^2(M)\) is a nondegenerate differential 2-form and let \(\theta \in \Omega^1(M)\). Assume that \((\pi, \xi)\) is the pair associated with \((\omega, \theta)\). If the pseudo-Riemannian metric \(g\) is associated with the 2-form \(\omega\) and with the pair \((\omega, \theta)\), then we have

\[ J \circ \sharp_{\pi, \xi} = \sharp_{\pi, \xi} \circ J^* \]

*Proof.* Since the metric \(g\) is assumed to be associated with \(\omega\) and using (2.7), we get

\[ g(J\sharp_{\pi, \xi}(\alpha), \sharp_{\pi, \xi}(\beta)) = \omega(\sharp_{\pi, \xi}(\alpha), \sharp_{\pi, \xi}(\beta)) = \pi(\alpha, \beta) = g^*(J^* \alpha, \beta) \]

and since \(g\) is also assumed to be associated with the pair \((\omega, \theta)\), i.e. \(\sharp_{\pi, \xi}\) is an isometry, then

\[ g(J\sharp_{\pi, \xi}(\alpha), \sharp_{\pi, \xi}(\beta)) = g(\sharp_{\pi, \xi}(J^* \alpha), \sharp_{\pi, \xi}(\beta)). \]

Finally, since \(\sharp_{\pi, \xi}\) is an isometry, hence an isomorphism, the result follows. \(\square\)

**Theorem 3.5.** Assume that \((\omega, df)\) is a conformally symplectic structure on \(M\) and that \((\pi, \xi)\) is the associated Jacobi structure. If \(g\) is a Riemannian metric associated with \(\omega\) and with \((\omega, df)\), then the triple \((\pi, \xi, g)\) is compatible if and only if the triple \((\omega, df, g)\) is a conformally Kähler structure.
Proof. We need to prove that the triple \((\pi, \xi, g)\) is compatible if and only if the pair \((e^f\omega, e^f g)\) is compatible, i.e., if and only if the 2-form \(e^f\omega\) is parallel with respect to the Levi-Civita connection \(\nabla^f\) associated with the metric \(g^f = e^f g\). As the connections \(\nabla\) and \(\nabla^f\) are related by the formula

\[
\nabla^f_X^Y = \nabla_X^Y + \frac{1}{2} (X(f)^Y + Y(f)^X - g(X, Y)\text{grad}_g f),
\]

where \text{grad}_g f = \#_g(df), we deduce that

\[
\nabla^f\omega(X, Y, Z) = \nabla\omega(X, Y, Z) - X(f)^\omega(Y, Z) - \frac{1}{2} Y(f)^\omega(X, Z) + \frac{1}{2} Z(f)^\omega(X, Y) + \frac{1}{2} (g(X, Y)\omega(\text{grad}_g f, Z) - g(X, Z)\omega(\text{grad}_g f, Y)),
\]

and then that

\[
\nabla^f(e^f\omega)(X, Y, Z) = e^f (X(f)^\omega(Y, Z) + \nabla f\omega(X, Y, Z)) = e^f \Lambda_f(X, Y, Z),
\]

where we have set

\[
\Lambda_f(X, Y, Z) = \nabla\omega(X, Y, Z) - \frac{1}{2} (Y(f)^\omega(X, Z) - Z(f)^\omega(X, Y)) + \frac{1}{2} (g(X, Y)\omega(\text{grad}_g f, Z) - g(X, Z)\omega(\text{grad}_g f, Y)).
\]

It follows that \(\nabla^f(e^f\omega) = 0\) if and only if \(\Lambda_f = 0\), hence that the pair \((e^f\omega, e^f g)\) is compatible if and only if

\[
\nabla\omega(X, Y, Z) = \frac{1}{2} (Y(f)^\omega(X, Z) - Z(f)^\omega(X, Y) - g(X, Y)\omega(\text{grad}_g f, Z) + g(X, Z)\omega(\text{grad}_g f, Y)).
\]

Let us prove now that this last identity is equivalent to the formula \((3.1)\). Let \(\alpha, \beta, \gamma \in \Omega^1(M)\) be such that \(X = \#_\pi,\xi(\alpha), Y = \#_\pi,\xi(\beta)\) and \(Z = \#_\pi,\xi(\gamma)\). By Corollary \(2.6\) we have \(\nabla\omega(X, Y, Z) = D\pi(\alpha, \beta, \gamma)\). On the other hand, setting \(\theta = df\), we have \(Y(f) = \theta(Y) = \theta(\#_\pi(\beta)) + \beta(\xi)\theta(\xi) = -\beta(\#_\pi(\xi)) = -\beta(\xi)\) and likewise \(Z(f) = -\gamma(\xi)\). Also, by \((2.7)\), we have \(\omega(X, Y) = \pi(\alpha, \beta)\) and \(\omega(X, Z) = \pi(\alpha, \gamma)\). Finally, since the metric \(g\) is associated with \(\omega\), it follows that

\[
\omega(\text{grad}_g f, Y) = -\omega(Y, \#_g(\theta)) = -g(JY, \#_g(\theta)) = -\theta(JY) = \omega(\xi, JY) = \omega(\#_\pi,\xi(\theta), J\#_\pi,\xi(\beta)),
\]

and since \(g\) is associated with \(\omega\) and with \((\omega, \theta)\), by using the lemma above and \((2.7)\), we get

\[
\omega(\text{grad}_g f, Y) = \omega(\#_\pi,\xi(\theta), \#_\pi,\xi(J^*\beta)) = \pi(\theta, J^*\beta) = J^*\beta(\#_\pi(\theta)) = J^*\beta(\xi)
\]

and likewise \(\omega(\text{grad}_g f, Z) = J^*\gamma(\xi)\).
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